MULTI-FOLD SUMS FROM A SET WITH FEW PRODUCTS

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Abstract. In this paper we show that for any \(k \geq 2\), there exist two universal constants \(C_k, D_k > 0\), such that for any finite subset \(A\) of positive real numbers with \(|AA| \leq M|A|\), \(|kA| \geq C_k M^{D_k} \cdot |A|^\log_4 2k\).

1. Introduction

We begin with some notation: Given a finite subset \(A\) of some commutative ring, we let \(A \star A\) denote the set \(\{a \star b : a, b \in A\}\), where \(\star\) is a binary operation on \(A\). When three or more summands or multiplicands are used, we let \(kA\) denote the \(k\)-fold sum-set \(A + A + \cdots + A\), and let \(A^{(k)}\) denote the \(k\)-fold product-set \(AA \cdots A\).

Erdős and Szemerédi ([8]) once conjectured that for any \(\alpha < 2\), there exists a universal constant \(C_\alpha > 0\), such that for finite subset \(A\) of real numbers,

\[\max\{|A + A|, |AA|\} \geq C_\alpha |A|^\alpha.\]

Non-trivial lower bounds for \(\alpha\) were achieved by many authors such as by Erdős and Szemerédi ([8], qualitatively), Nathanson ([14], 32/31), Ford ([9], 16/15), Chen ([3], 6/5), Elekes ([5], 5/4), and Solymosi ([17], 14/11 − o(1); [18], 4/3 − o(1)).

Another type of question than one can attack regarding sums and products is to either assume that the sum-set \(A + A\) is very small, and then to show that the product-set \(AA\) is very large, or to suppose that \(AA\) is very small, and then to show that \(A + A\) is very large. The best two results toward this question are respectively due to Elekes and Ruzsa ([7]), who fully confirmed the first part of the question, and by Chang ([2]), who solved the second part of the question in the setting of integers.

Similarly, one can consider multi-fold sums and products, but very few results are known especially in the setting of reals. Let \(B\) be a finite subset of integers, then Chang ([2]) showed that if \(|BB| \leq |B|^{1+\epsilon}\), then the multi-fold sum-set \(|kB| \gg_{\epsilon,k} |B|^n\delta\), where \(\delta \to 0\) as \(\epsilon \to 0\); and Bourgain and Chang ([1]) proved that for any \(b \geq 1\), there exists \(k \in \mathbb{N}\) independent of \(B\) such that \(|kB| \cdot |B^{(k)}| \geq |B|^b\). At the moment how to extend these results to the real numbers is not known yet. Recently, Croot and Hart established in [4] the following interested result:

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**Theorem 1.1.** For all $k \geq 2$ and $\epsilon \in (0, \epsilon_0(k))$ we have that the following property holds for all $n > n_0(k, \epsilon)$: If $A$ is a set of $n$ real numbers and $|AA| \leq n^{1+\epsilon}$, then

$$|kA| \geq n^{\log_4 k - f_k(\epsilon)},$$

where $f_k(\epsilon) \to 0$ as $\epsilon \to 0$.

Croot and Hart also remarked that they have several different approaches to proving a theorem of the quality of Theorem 1.1.

The purpose of the present paper is to give the following slight improvement of the above Croot-Hart theorem in a rather elementary way. Our idea comes from Solymosi’s wonderful proof ([18]) of the best currently known sum-product estimates of real numbers mentioned earlier. Solymosi’s idea has appeared elsewhere in [11] and [13].

**Theorem 1.2.** For any $k \geq 2$, there exist three positive universal constants $C_k, D_k, \Psi_k$, such that for any finite subset $A$ of positive real numbers with $|AA| \leq M|A|$, $|kA| \geq C_k M D_k \cdot |A|^{\Psi_k}$. With $\Psi_1 \triangleq 1$, the constants $\{\Psi_k\}_{k \geq 2}$ can be generated in any of the following way:

$$\Psi_k = \frac{1 + \Psi_{k_1} + \Psi_{k_2}}{2} \quad (k_1 + k_2 = k).$$

Particularly, we can take $\Psi_k = \log_4 2k$.

There are some other interested estimates on sum-sets and product-sets in the reals. For example, see [6], [12], [15] and [16].

2. **Proof of the main theorem**

We will prove Theorem 1.2 for all $k \in \mathbb{N}$ by induction. Obviously, one can choose $D_1 = 0$, $C_1 = \Psi_1 = 1$. Next for any $k \geq 2$, we assume the existences of positive universal constants $C_i, D_i$ and $\Psi_i$ for all $i \in [2, k)$. Our purpose is to find $C_k, D_k$ and $\Psi_k$ satisfying the required property. Let $k_1, k_2$ be any two natural numbers such that $k_1 + k_2 = k$.

By the Ruzsa triangle inequality, $|A/A| \leq M^2|A|$. For any $s \in A/A$, let $A_s \triangleq \{(x, y) \in A \times A : y = sx\}$. Let $D = \{s : |A_s| \geq \frac{|A|}{M^2}\}$, and let $s_1 < s_2 < \cdots < s_m$ denote the elements of $D$, labeled in increasing order. Obviously,

$$\sum_{s \in D} |A_s| \geq \frac{|A|^2}{2},$$

which implies $m \geq \frac{|A|}{2}$. Let $A_{m+1} = \Pi : \mathbb{R}^2 \to \mathbb{R}$ be the projection map from $\mathbb{R}^2$ onto the vertical axis. It is geometrically evident that $\{k_1 A_j + k_2 A_{j+1}\}_{j=1}^m$ are mutually disjoint. Thus

$$|(kA) \times (kA)| \geq \sum_{j=1}^m |k_1 A_j + k_2 A_{j+1}| = \sum_{j=1}^m |k_1 A_j| \cdot |k_2 A_{j+1}| = \sum_{j=1}^m |k_1 \Pi(A_j)| \cdot |k_2 \Pi(A_{j+1})|. $$
Note
\[ |\Pi(A_j)\Pi(A_j)| \leq |AA| \leq M|A| \leq 2M^3|\Pi(A_j)|. \]
Applying induction to all of the \(\Pi(A_j)\)'s,
\[ |kA|^2 \geq \frac{|A|}{2} \cdot \frac{C_{k_1}}{(2M^3)^{D_{k_1}}} \cdot \frac{|A|}{(2M^2)^{\Psi_{k_1}}} \cdot \frac{C_{k_2}}{(2M^3)^{D_{k_2}}} \cdot \frac{|A|}{(2M^2)^{\Psi_{k_2}}}, \]
which yields
\[ |kA| \geq \left( \frac{C_{k_1} \cdot C_{k_2}}{2 \cdot (2M^3)^{D_{k_1}+D_{k_2}} \cdot (2M^2)^{\Psi_{k_1}+\Psi_{k_2}}} \right)^{1/2} \cdot |A|^{\frac{1+\Psi_{k_1}+\Psi_{k_2}}{2}}. \]
Thus one can let \( \Psi_k \triangleq \frac{1+\Psi_{k_1}+\Psi_{k_2}}{2} \) and define \( C_k, D_k \) in a similar way.

Finally, let \( z \triangleq \lfloor \log_2 k \rfloor \). Then
\[ \Psi_{2^z} \geq \frac{1}{2} + \Psi_{2^{z-1}} \geq \cdots \geq \frac{z}{2} + \Psi_1 = \frac{z+2}{2} \geq \log_4 2k. \]
Consequently,
\[ |kA| \geq |2^z A| \geq \frac{C_{2^z}}{M^{D_{2^z}}} \cdot |A|^{\Psi_{2^z}} \geq \frac{C_{2^z}}{M^{D_{2^z}}} \cdot |A|^{|\log_4 2k|. \]
This concludes the whole proof.

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References

[1] J. Bourgain, M.-C. Chang, On the size of \(k\)-fold sum and product sets of integers, J. Amer. Math. Soc. 17 (2003) 473–497.
[2] M.-C. Chang, The Erdős-Szemerédi problem on sum set and product set, Ann. Math. 157 (2003) 939–957.
[3] Y. G. Chen, On sums and products of integers, Proc. Amer. Math. Soc. 127 (1999) 1927–1933.
[4] E. Croot, D. Hart, \(h\)-fold sums from a set with few products, SIAM J. Discrete Math. 24 (2010) 505–519.
[5] Gy. Elekes, On the number of sums and products, Acta Arith. 81 (1997) 365–367.
[6] Gy. Elekes, M. B. Nathanson, I. Z. Ruzsa, Convexity and sumsets, J. Number Theory 83 (1999) 194–201.
[7] Gy. Elekes, I. Z. Ruzsa, Few sums, many products. Studia Sci. Math. Hungar. 40 (2003) 301–308.
[8] P. Erdős and E. Szemerédi, On sums and products of integers. In: Studies in Pure Mathematics (Birkhauser, Basel, 1983) 213–218.
[9] K. Ford, Sums and products from a finite set of real numbers, Ramanujan J. 2 (1998) 59–66.
[10] M. Z. Garaev, C.-Y. Shen, On the size of the set \(A(A+1)\), Math. Z. 265 (2010) 125–132.
[11] D. Hart, A. Niziolek, Some results on the size of sum and product sets of finite sets of real numbers, Involve 2 (2009) 603–609.
[12] A. Iosevich, O. Roche-Newton, M. Rudnev, On an application of Guth-Katz theorem, arXiv:1103.1354, accepted by Math. Research Letters, 2011.
[13] L. Li, J. Shen, A sum-division estimate of reals, Proc. Amer. Math. Soc. 138 (2010) 101–104.
[14] M. B. Nathanson, On sums and products of integers, Proc. Amer. Math. Soc. 125 (1997) 9–16.
[15] T. Schoen, I. D. Shkredov, On sumsets of convex sets, arXiv:1105.3542, 2011.
[16] C.-Y. Shen, Algebraic methods in sum-product phenomena, arXiv:0911.2627, to appear in Israel J. Math., 2009.
[17] J. Solymosi, On the number of sums and products, Bull. London Math. Soc. 37 (2005) 491–494.
[18] J. Solymosi, Bounding multiplicative energy by the sumset, Adv. Math. 222 (2009) 402–408.
