Superconductivity of electron-hole pairs in a bilayer graphene system in a quantizing magnetic field

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Abstract

The state with a spontaneous interlayer phase coherence in a graphene based bilayer quantum Hall system is studied. This state can be considered as a gas of superfluid electron-hole pairs with the components of the pair belonging to different layers. Superfluid flux of such pairs is equivalent to two electrical supercurrents in the layers. It is shown that the state with the interlayer phase coherence emerges in the graphene system if a certain imbalance of the Landau level filling factors of the layers is created. We obtain the temperature of transition into the superfluid state, the maximum interlayer distance at which the phase coherence is possible, and the critical values of the supercurrent. The advantages of use of graphene systems instead of GaAs heterostructures for the realization of the bilayer electron-hole superconductivity is discussed.

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INTRODUCTION

The experimental discovery of graphene [1] served as a stimulus for performing large-scale investigations of the electronic properties of conductors with a Dirac dispersion law for charge carriers. On the one hand such investigations are certainly fundamental. On the other hand the uniqueness of the properties of graphene makes it an extremely interesting material from the standpoint of different practical applications.

In the present article we shall examine the prospects for using graphene to obtain superconductivity of bound electron-hole pairs in a system of two two-dimensional conductors separated by a dielectric. The possibility of such superconductivity was suggested quite a long time ago [2, 3]. The system considered was a sandwich consisting of a two-dimensional n-type semiconductor – dielectric – two-dimensional p-type semiconductor. Somewhat later it was determined [4, 5] that a similar phenomenon can be obtained in a semiconductor heterostructure with two quantum wells (quasi-two-dimensional electronic layers). If such a system is placed in the quantizing magnetic field oriented perpendicular to the layers (a quantum Hall effect regime is realized) and the total filling factor of the Landau levels is 1 (ν₁ + ν₂ = 1), pairing of electrons in one layer with holes (unfilled states in the lowest Landau level) in the other layer occurs. These pairs are bosons; as temperature decreases, such a pair gas can transition into a superfluid state. Because pair gas is two-dimensional the transition will occur by the Berezinskii-Kosterlitz-Thouless mechanism. Since the electron and hole components of the pairs belong to different layers, the superfluid flow of the electrically neutral pairs is equivalent to two electric supercurrents flowing in opposite directions in neighboring layers.

The analogy with a quantum ferromagnet is widely used to describe such a state (which is called a state with spontaneous interlayer phase coherence) [6, 7, 8, 9, 10]. This analogy makes it possible to study, within the framework of the same approach, the case of zero imbalance of the filling factors of the layers (ν₁ = ν₂ = 1/2) as well as a situation with nonzero imbalance (ν₁ = ν, ν₂ = 1 − ν, ν ≠ 1/2), which arises, for example, because of the external electrostatic field of a gate. For a very large imbalance (ν ≪ 1) the system can be described as a rarefied gas of bosons (magnetoexcitons) with a dipole-dipole interaction. Such an approach was developed in Refs. [11, 12, 13, 14, 15]. We note that experimental investigations of two-layer electron-hole superconductivity have been performed primarily on quantum Hall system [16, 17, 18, 19, 20, 21, 22, 23].

The superconductivity of electron-hole pairs in a graphene-dielectric-graphene system was studied in Refs. [24, 25, 26, 27, 28, 29, 30]. In Ref. [24, 25, 26, 27] and [29] the possibility of such pairing was studied in the absence of a magnetic field perpendicular to the layers (i.e. not in the quantum Hall effect regime). In the absence of a magnetic field the condition for BCS pairing of electrons and holes is that their Fermi surfaces must coincide. For semiconductors with a quadratic dispersion law for the charge carriers the last condition requires that the electron and hole effective masses be the same. For a Dirac carrier spectrum (which obtains in graphene) the required condition on the Fermi surface is satisfied automatically. Neglecting screening the theory predicts a very high (hundreds of degrees) temperature for the transition into the superconducting state in such a system [24, 29]. Taking account of screening the estimate for the transition temperature is much less optimistic and lies in the millikelvin range [27, 28].
In a quantum Hall system the screening effects should not be so strong, which gives hope of attaining high temperatures for the transition into the superconducting state. In addition, the quantum Hall effect itself in graphene is observed at high temperatures (right up to room temperatures) \[31\]. The question of the superconductivity of a gas of magnetoejectons in a two-dimensional graphene system was examined in Refs. \[28\] and \[30\]. The approach of Refs. \[28, 30\] can be use only for low magnetoexciton densities (\(\nu \ll 1\)). But the low-density case is not optimal from the standpoint of reaching high transition temperatures. More likely, the maximum critical temperature will obtain for half-filling of the Landau levels in each layer. The present article is devoted to analyzing this problem.

**STATE WITH INTERLAYER PHASE COHERENCE IN A BILAYER GRAPHENE SYSTEM**

The starting point for studying electron-hole pairing in a quantum Hall system is the Coulomb interaction Hamiltonian written in the lowest (active) Landau level approximation. Consequently, we shall begin by obtaining the desired Hamiltonian for the graphene system.

To describe the properties of the electronic subsystem of graphene in a magnetic field we shall follow the approach presented in detail in the review \[32\] (see also Ref. \[33\]). Graphene possesses a honeycombed crystal structure, which can be represented as two simple triangular lattices A and B inserted into one another. The distance a between the nearest carbon atoms (which belong to different sublattices) is 1.42 Å. In the tight-binding approximation and taking account of tunneling only between the nearest sites the Hamiltonian of the system has the form

\[
H = -t \sum_{\langle i,j \rangle, \sigma} (a_{i,\sigma}^+ b_{j,\sigma} + h.c),
\]

where \(a_{i,\sigma}^+\), \(b_{i,\sigma}^+\) (\(a_i, b_i\)) are, respectively, operators creating (annihilating) electrons on the i-th sites of the sublattices A and B, \(t \approx 2.8\) eV is the tunneling amplitude, and \(\sigma\) is the spin index.

Having written the Hamiltonian (1) in the momentum approximation it is easy to verify that the energy band is divided into two sub-bands which possess only two nonequivalent points of contiguity \(K = (2\pi/3a, 2\pi/3\sqrt{3}a)\) and \(K' = (2\pi/3a, -2\pi/3\sqrt{3}a)\) (the x axis is directed along the line connecting any pair of nearest-neighbor sites). In undoped graphene (containing one free electron per site) the Fermi level passes through the point of contiguity of these sub-bands. Correspondingly, the sub-bands can be interpreted as electron and hole bands. In connection with such a structure of the spectrum the low-energy excitations in undoped and weakly doped graphene can be described by adding a pseudospin index \(\alpha = \pm 1\), corresponding to states whose quasimomenta lie close to \(K\) and \(K'\). For low energies the Hamiltonian is diagonal with respect to the pseudospin indices. It is also assumed that there are no interactions which destroy the diagonality with respect to spin. The characteristic quantum numbers of such a Hamiltonian possess a definite spin and pseudospin.

For such states the Schrödinger equation written in the coordinate representation has the form

\[
-\frac{i}{\hbar}v_F \left( \frac{\partial}{\partial x} + i\alpha \frac{\partial}{\partial y} \right) \begin{pmatrix} \Psi_A \\ \Psi_B \end{pmatrix} = E \begin{pmatrix} \Psi_A \\ \Psi_B \end{pmatrix},
\]

where \(v_F = 3ta/2\hbar\). For graphene the indicated parameter equals \(\approx 10^8\) cm/s. An electronic state is a spinor whose components correspond to the sublattices A and B. Switching in Eq. (2) to the momentum representation we find the spectrum in the form \(E = \pm \hbar v_F k\) (\(k\) is the modulus of the wave vector, measured from the point \(K\) or \(K'\)). Evidently, \(v_F\) is the velocity of the electrons on the Fermi surface. For fermions with such a spectrum this velocity does not depend on the carrier concentration.

In a magnetic field perpendicular to graphene Schrödinger equation (2) assumes the form

\[
-\frac{i}{\hbar}v_F \left( \frac{\partial}{\partial x} - \frac{\alpha eBx}{\hbar c} + i\alpha \frac{\partial}{\partial y} \right) \begin{pmatrix} \Psi_A \\ \Psi_B \end{pmatrix} = E \begin{pmatrix} \Psi_A \\ \Psi_B \end{pmatrix}.
\]

The solution of Eq. (3) gives the Landau energy levels in graphene: \(E_0 = 0\), \(E_{\pm N} = \pm (\hbar v_F / k)\sqrt{2N}\), where \(k = \sqrt{\hbar c / eB}\) is the magnetic length and \(N = 1, 2, \ldots\). In Eq. (3) we neglected the Zeeman terms, since for reasonable values of the magnetic field the Zeeman splitting is much less than the spacing between the Landau levels. The eigenfunctions \(\Psi_{N,\alpha,k}\) corresponding to the zeroth and \(\pm N\)-th Landau levels have the following form:

\[
\Psi_{0,-1,k}(x, y) = \frac{e^{-iky}}{\pi^{1/4} \sqrt{L_y}} e^{-(x-x_0)^2 / 2\ell^2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \Psi_{0,+1,k}(x, y) = \frac{e^{-iky}}{\pi^{1/4} \sqrt{L_y}} e^{-(x-x_0)^2 / 2\ell^2} \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]
The filling factor is \( \pm 1 \), \( \pm 2 \), \( \pm 3 \) for positive \( \nu \), \( \nu \) and negative \( \nu \) respectively. If the separation of the Landau levels is much greater than the Coulomb energy \( \Delta \), \( \Delta \lesssim 4 \), in undoped graphene the chemical potential corresponds to the zeroth Landau level, so

\[
\nu = 0, \quad \rho_i(r) = \sum_{a} \hat{\Psi}_{x,a}^{\pm}(r) \hat{\Psi}_{i,a}(r)
\]

is the electron density operator, \( i = 1, 2 \) is the index of the layer, and \( \hat{\Psi}_{x,a}(r) \) and \( \hat{\Psi}_{x,a}^{\pm}(r) \) are operators creating and annihilating at the point \( r \) an electron with prescribed spin and pseudospin.

We shall represent the operators \( \hat{\Psi}_{x,a}^{\pm}, \hat{\Psi}_{x,a} \) in terms of the operators \( (4) \) and \( (5) \) creating and annihilating electrons in the quantum states, respectively. If the separation of the Landau levels is much greater than the Coulomb energy \( \varepsilon^2/(\varepsilon \ell^2) \), then it is sufficient to retain in this expansion only a single active (partially filled) Landau level

\[
\hat{\Psi}_{x,a}(x,y) = \sum_{\lambda,k} \Psi_{x,a,k}(x,y) a_{i,k,a}, \quad \hat{\Psi}_{x,a}^{\pm}(x,y) = \sum_{k} \Psi_{x,a,k}(x,y) a_{i,k,a}^{\pm},
\]

where \( \lambda \) is the number of the active level, \( \Psi_{x,a,k}^{\pm} \) are two-component vector-matrices determined by Eqs. \( (4) \) and \( (5) \), \( \Psi_{x,a,k}^{\pm} \) are the hermitian-conjugate matrices, and \( a_{i,k,a}^{\pm} \) and \( a_{i,k,a} \) are operators creating and annihilating electrons in a state with the corresponding quantum numbers in the level \( \lambda \).

In this approximation the Hamiltonian \( (6) \) written in the Fourier representation has the form

\[
H_C = \frac{1}{2} \sum_{i,i'} \sum_{\mathbf{q}} V_{i,i'}(\mathbf{q}) \hat{\rho}_i(\mathbf{q}) \hat{\rho}_{i'}(-\mathbf{q}),
\]

where \( V_{i,i'}(\mathbf{q}) = (2\pi e^2/\varepsilon q) \exp(-qd|i-i'|) \) is the Fourier component of the Coulomb potential. For \( \lambda = 0 \) the Fourier component of the electron density operator is

\[
\hat{\rho}_i(\mathbf{q}) = \sum_{a} \sum_{k} a_{i,k+q_{y}/2,a}^{\pm} a_{i,k-q_{y}/2,a} \exp \left( -i q x \ell^2 - \frac{q^2 \ell^2}{4} \right).
\]

For \( \lambda \neq 0 \)

\[
\hat{\rho}_i(\mathbf{q}) = \sum_{a} \sum_{k} a_{i,k+q_{y}/2,a}^{\pm} a_{i,k-q_{y}/2,a} \exp \left( -i q x \ell^2 - \frac{q^2 \ell^2}{4} \right) L_{\lambda} \left( \frac{q^2 \ell^2}{2} \right) + L_{\lambda-1} \left( \frac{q^2 \ell^2}{2} \right),
\]

where \( L_{\lambda}(x) \) is a Laguerre polynomial.

If the system is composed of two undoped graphene layers, then in each layer the half-filled zeroth Landau level will be active. If an electrostatic field is applied in a direction perpendicular to the layers, then an imbalance of the filling factor can arise, i.e. the filling factors of the layers will become equal to \( \nu_1 = 2 + \bar{\nu} \) and \( \nu_2 = 2 - \bar{\nu} \) (0 < \( \bar{\nu} \) ≤ 2).
In stronger fields the active levels will become \( \lambda = +1 \) in layer 1 and \( \lambda = -1 \) in the layer 2, then +2 and -2, and so on.

We shall examine first the case where the zeroth levels are the active levels. The Coulomb interaction Hamiltonian differs from the Hamiltonian of the usual bilayer quantum Hall system with total filling factor \( \nu = 1 \) only by the fact that it contains an interaction of not one but four components, corresponding to the quantum numbers \( (\alpha = \pm 1, \sigma = \uparrow, \downarrow) \). We shall enumerate these sets of quantum numbers by the index \( \beta = 1, 2, 3, 4 \).

By analogy with the single-component case \([3, 4]\) we write the trial multi-particle wave function of the system in the form
\[
|\Psi\rangle = \prod_k \prod_\beta \left( \cos \frac{\theta_\beta}{2} a_{1, k\beta}^+ + e^{i\nu_\beta} \sin \frac{\theta_\beta}{2} a_{2, k\beta}^+ \right) |0\rangle.
\]

The function \([11]\) describes a state where interlayer phase coherence arises for each component. It is easily shown (see, for example, Refs. \([34, 35]\)) that the state \([11]\) can be represented in the form of a BCS wave function describing the pairing of electrons of one layer with holes of the other layer with the paired electrons and holes corresponding to the same state \( \beta \). The parameter \( \theta_\beta \) is related with the magnitude of the imbalance for this component by the relation \( \tilde{\nu}_\beta = \cos \theta_\beta/2 \); the filling factors for this component are \( \nu_1(2)\beta = 1/2 \pm \tilde{\nu}_\beta \), and the total imbalance is
\[
\nu = \sum_\beta \tilde{\nu}_\beta.
\]

The energy of the system in the state \([11]\) is given by the expression \( E = \langle \Psi | H_C | \Psi \rangle + E_g + E_{bg} \), where \( E_g \) is the interaction energy of the electrons with the electrostatic field produced by the external gate, and \( E_{bg} \) takes account of the Coulomb interaction with the positive core. A direct calculation (see, for example, Ref. \([36]\)) gives
\[
E = \frac{S}{2\pi \ell^2} \left( W \left( \sum_\beta \tilde{\nu}_\beta \right)^2 - J_0 \left( 1 + \sum_\beta \tilde{\nu}_\beta^2 \right) - J_1 \left( 1 - \sum_\beta \tilde{\nu}_\beta^2 \right) - eV \sum_\beta \tilde{\nu}_\beta \right),
\]

where \( \text{erfc}(x) \) is the complementary error function and \( V \) is the potential difference produced between the layers by the gate field.

The first term in Eq. \([12]\) is identical to the energy of a plane capacitor and represents the contribution of the direct Coulomb interaction to the energy. The second and third terms contribute to the exchange interaction inside and between the layers, respectively. We note that \( J_0 > J_1 \), i.e. the in-layer exchange energy constant is greater than the interlayer exchange energy constant.

To find the parameters \( \theta_\beta \) appearing in the trial function \([11]\) it is necessary to find \( \tilde{\nu}_\beta \) corresponding to the minimum of the energy \([12]\) taking account of the constraint \( |\tilde{\nu}_\beta| \leq 1/2 \) (which follows from the definition of \( \tilde{\nu}_\beta \)).

In a non-graphene bilayer quantum Hall system with \( \nu_1 + \nu_2 = 1 \) there is only one component, and the energy minimum in the absence of a gate field corresponds to \( \nu = 0 \), i.e. \( \nu_1 = \nu_2 = 1/2 \). The situation is more complicated in a multi-component system. For \( J_0 > J_1 \) the part quadratic in \( \tilde{\nu}_\beta \) in Eq. \([12]\) is not positive-definite. Moreover, it is not positive-definite with one and two \( \tilde{\nu}_\beta \) fixed. Therefore the minimum of the energy \([12]\) is attained at the boundary of the range of \( \tilde{\nu}_\beta \), and in addition at least three of the four \( \tilde{\nu}_\beta \) must assume limiting values, equal \( \pm 1/2 \). For \( V = 0 \) the minimum is reached for
\[
\sum_\beta \tilde{\nu}_\beta = 0,
\]
but when two \( \tilde{\nu}_\beta = \pm 1/2 \) and the remaining other two \( \tilde{\nu}_\beta = -1/2 \). Physically, this result is understandable. Since \( J_0 > J_1 \), a gain in the energy exchange is attained for maximum imbalance of a given component. In a one-component system a direct exchange interaction impedes this. In a multi-component system the latter effect is absent, since the magnitudes of the imbalance of the components can be of opposite sign. The values \( \tilde{\nu}_\beta = \pm 1/2 \) correspond to \( \theta_\beta = 0, \pi \) which means absence of interlayer phase coherence (i.e. absence of a superconducting state). We note that this situation is analogous to the one arising in a non-graphene bilayer quantum Hall system with total filling factor \( \nu_1 + \nu_2 = 2 \).
We shall show that for \( V \neq 0 \) it becomes possible for a state with interlayer phase coherence to appear. For definiteness, we shall assume that \( V > 0 \). We shall also assume that the parameters determined by Eqs. (13) satisfy the inequality \( W - J_0 + J_1 > 0 \). Analysis of Eq. (12) shows that for \( eV \leq J_0 - J_1 \) an energy minimum corresponds to the same state as for \( V = 0 \). If the potential difference satisfies the condition

\[
J_0 - J_1 < eV < 2W - J_0 + J_1,
\]

then the minimum energy is attained in the state

\[
\nu_1 = \nu_2 = \frac{1}{2}, \quad \nu_3 = \frac{eV - W}{2(W - J_0 + J_1)}, \quad \nu_4 = -\frac{1}{2}.
\]

The state (15) is degenerate with respect to transposition of the indices \( \beta \). Most likely, such degeneracy is removed as a result of weaker interactions which are not taken into account in Eq. (12). This is not essential for our analysis. In the parameter range

\[
2W - J_0 + J_1 \leq eV \leq 2W + J_0 - J_1
\]

the minimum corresponds to the state

\[
\nu_1 = \nu_2 = \nu_3 = \frac{1}{2}, \quad \nu_4 = -\frac{1}{2}.
\]

If \( V \) lies in the range

\[
2W + J_0 - J_1 \leq eV \leq 4W - J_0 + J_1,
\]

then the minimum is reached when

\[
\nu_1 = \nu_2 = \nu_3 = \frac{1}{2}, \quad \nu_4 = \frac{eV - 3W}{2(W - J_0 + J_1)}.
\]

As \( V \) increases further, the zeroth Landau level will become inactive: it will be completely filled in layer 1 and empty in layer 2.

In the next section we shall see that the superfluid density and, correspondingly, the transition temperature are proportional to \( \sqrt{\nu_\beta(1 - \nu_\beta)} = \sqrt{1/4 - \nu_\beta^2} \), i.e. different from zero for \( \nu_\beta \neq \pm 1/2 \). As follows from the expressions presented above, pairing will arise when the external potential difference lies in the range (14) and (18). Then the maximum superfluid density (corresponding to \( \nu_\beta = 0 \)) will correspond to \( eV = W \) and \( eV = 3W \). To estimate the required field strength in the gate we note that the condition \( eV = W \) corresponds to field strength \( E \approx 2 \cdot 10^4 \varepsilon^{-1} B V/cm \), where \( B \) is taken in T.

We also note that for \( eV = W \) the filling factors of the layers are \( \nu_1 = 5/2 \) and \( \nu_2 = 3/2 \), and for \( eV = 3W \) they assume the values \( \nu_1 = 7/2 \) and \( \nu_2 = 1/2 \). In both cases only one component has a nonzero imbalance between the layers, and it is this (active) component that is responsible for the pairing of electrons and holes.

As \( V \) increases further, the electrons will transition from the level \( \lambda = -1 \) of layer 2 to the level \( \lambda = +1 \) of layer 1, and pairing of electrons in the \( \lambda = +1 \) level with holes in the \( \lambda = -1 \) level will become possible. According to Eq. (10), the expressions for the Fourier components of the density operator for the levels \( \lambda = \pm 1 \) have the form

\[
\hat{\rho}_i(q) = \sum_{\alpha,\sigma} \sum_k a_{i,k+q/2,\alpha,\sigma} a_{i,k-\sigma} \left( 1 - \frac{q^2 k^2}{4} \right) \exp \left( -i q x_k \right).
\]

The state with pairing of electrons in level \( \lambda = +1 \) with holes in level \( \lambda = -1 \) can be described by a wave function of the form (11), in which the operators creating electrons in the layer 1 belong to the Landau level \( \lambda = +1 \) and the operators creating electrons in the layer 2 belong to the level \( \lambda = -1 \).

The Coulomb interaction energy in such a state, taking account of the imbalance of the components of the level \( \lambda = 0 \), assumes the form

\[
E = \frac{S}{2 \pi \ell^2} \left( W \left( \frac{1}{2} + \nu_\gamma \right) \right)^2 - 2J_0 - J_2 \left( 1 + \sum_\gamma \nu_\gamma^2 \right).
\]
where

\[
J_2 = \frac{11}{16} \sqrt{\frac{\pi}{2}} \frac{e^2}{\varepsilon \varepsilon'}, \quad J_3 = J_2 \left[ \frac{11 - 2d^2 + \tilde{d}^4}{11} \exp \left( \frac{\tilde{d}^2}{2} \right) \text{erfc} \left( \frac{\tilde{d}}{\sqrt{2}} \right) + \frac{2\tilde{d}(3 - \tilde{d}^2)}{11\sqrt{2\pi}} \right]
\]

(22)

are the in-layer and interlayer exchange interaction constants for the levels \( |\lambda| = 1 \) (\( \tilde{d} = d/\ell \)). As in the case of an active zeroth level, the in-layer exchange constant \( J_2 \) is greater than the interlayer exchange constant \( J_3 \). In Eq. (21) \( \Omega = 2\sqrt{2\varepsilon_F} / \ell \) is the splitting between the levels \( \lambda = +1 \) and \( \lambda = -1 \). \( \gamma \) denote the same quantum numbers as \( \beta \) but for the levels \( |\lambda| = 1 \), and \( \tilde{\nu}_\gamma \) determine the imbalance of the components of \( \gamma \).

We find from Eq. (21) that for

\[
4W + \Omega + J_2 - J_3 \leq eV \leq 6W + \Omega - J_2 + J_3
\]

(23)

the minimum energy corresponds to the state

\[
\tilde{\nu}_1 = \frac{eV - \Omega - 5W}{2(W - J_2 + J_3)}, \quad \tilde{\nu}_2 = \tilde{\nu}_3 = \tilde{\nu}_4 = -\frac{1}{2},
\]

(24)

i.e. the inequalities (23) determine the next range of \( V \) where a state with interlayer phase coherence can arise but this time between the electrons belonging to the levels \( \lambda = +1 \) and \( \lambda = -1 \). Specifically, for \( eV = \Omega + 5W \) the component in which such coherence will arise will have zero imbalance, i.e. the maximum superfluid density will be reached.

**CRITICAL PARAMETERS FOR ELECTRON-HOLE SUPERCONDUCTIVITY IN GRAPHENE**

As shown in the preceding section, a state with interlayer phase coherence can arise in a bilayer graphene system in a quantizing magnetic field and an electric field perpendicular to the layers. We established that even though four components are present in a Landau level in graphene, only one component can be active. Depending on the magnitude of the gate field interlayer phase coherence can arise between the electrons in the zeroth Landau level or between the electrons in the \( \lambda = \pm 1 \) levels. In the first case the critical parameters of electron-hole superconductivity in graphene will be described by the same equations as in bilayer quantum Hall systems in GaAs heterostructures. In the second case the equations will be somewhat different. In the present section we shall compare two such cases. We shall use for the analysis the approach of Ref. [10], which was further elaborated in Ref. [38], as well as in the present work.

We shall examine a state where \( V \) gives interlayer phase coherence of one of the components, and in which the parameters \( \theta \) and \( \varphi \) for this component depend on \( k \):

\[
|\Psi\rangle = \prod_k \left( \cos \frac{\theta_k}{2} a_{1,k}^+ + e^{i\varphi_k} \sin \frac{\theta_k}{2} a_{2,k}^+ \right) |0\rangle.
\]

(25)

Here and below the creation and annihilation operators as well as the functions \( \theta_k \) and \( \varphi_k \) refer to the active component. The quantum index \( k \) determines the coordinate \( x \) of the center of the electron orbit: \( X = k\ell^2 \). Consequently, the function (25) describes a state in which the order parameter \( \langle \Psi | a_{1,X}^+ a_{2,X}^+ |\Psi\rangle = (1/2) \sin \theta_X e^{i\varphi_X} \) for electron-hole pairing varies along the \( x \) axis.

The energy of the system in the state (25) is

\[
E = E_0 - e\bar{V} \sum_X \cos \theta_X + \frac{1}{2L_y} \sum_{X,X'} \left\{ \begin{array}{c} \left[ H(X - X') - F_S(X - X') \right] \cos \theta_X \cos \theta_{X'}, \\
- F_D(X - X') \sin \theta_X \sin \theta_{X'} \cos(\varphi_X - \varphi_{X'}) \end{array} \right\},
\]

(26)

where \( \bar{V} = V - V_0 \) is the potential difference corresponding to zero imbalance of this component and \( E_0 \) is the Coulomb energy of the inactive components. To abbreviate the notations appearing in Eq. (25) we shall express the functions
of $X - X'$ in terms of the relation $A(X) = \ell^2 \int dq A(q) e^{i q X}$, where $A(X) = H(X), F_S(X)$ and $F_D(X)$ and the explicit form the Fourier component of these quantities is as follows:

$$\mathcal{H}(q) = \frac{e^2}{2\varepsilon \ell^2} e^{-\frac{q^2}{2\varepsilon \ell^2}} \frac{1 - e^{-d|q|}}{|q|} f_\lambda(q\ell), \quad (27)$$

$$\mathcal{F}_S(q) = \frac{e^2}{2\varepsilon \ell} \int_0^\infty dk e^{-\frac{k^2}{2\ell^2}} J_0(kq\ell) f_\lambda(k), \quad (28)$$

$$\mathcal{F}_D(q) = \frac{e^2}{2\varepsilon \ell} \int_0^\infty dk e^{-\frac{k^2}{2\ell^2}} J_0(kq\ell) e^{-kd} f_\lambda(k). \quad (29)$$

In Eqs. (27)-(29) $J_0(q)$ is the Bessel function of order zero, and

$$f_\lambda(k) = \begin{cases} 1 & \text{for } \lambda = 0, \\ \left(1 - \frac{k^2}{\ell^2}\right)^2 & \text{for } |\lambda| = 1 \end{cases} \quad (30)$$

(the active levels are $\lambda = 0$ or $\lambda = \pm 1$).

In the uniform state with nonzero flux of electron-hole pairs in the $x$ direction the phase of the order parameter is linear in $X$ ($\varphi_X = QX$) and $\theta_X = 0$ is independent of $X$. The energy of this state is

$$E_{mf} = E_0 - \frac{eV S \cos \theta_0}{2\pi \ell^2} + \frac{S}{4\pi \ell^2} \left[ \mathcal{H}(0) - \mathcal{F}_S(0) \cos^2 \theta_0 - \mathcal{F}_D(Q) \sin^2 \theta_0 \right]. \quad (31)$$

Since the imbalance $\tilde{\nu}_\beta = \cos \theta_0/2$ and $2\mathcal{H}(0) = W$, $2\mathcal{F}_S(0) = J_0 (J_2)$, and $2\mathcal{F}_D(0) = J_1 (J_3)$ (for $\lambda = 0$ ($\pm 1$) respectively), it is evident that there is complete correspondence between the expression (31) at $Q = 0$ and the expressions (12) and (21).

For small $Q$ the energy (31) can be represented as

$$E = S(\text{const} + \frac{1}{2} \rho_{s0} Q^2), \quad (32)$$

where

$$\rho_{s0} = \sin^2 \theta_0 \frac{E_c}{16\pi} \hat{\rho} \left( \frac{d}{\ell} \right), \quad (33)$$

$E_c = e^2/\varepsilon \ell$ is the Coulomb energy, and the function

$$\hat{\rho}(x) = \begin{cases} \sqrt{\frac{x^2}{\ell^2}} \text{erfc} \left( \frac{x}{\sqrt{2k}} \right) \left( 1 + \frac{x^2}{\ell^2} \right) - x, & \text{for } \lambda = 0, \\ \sqrt{\frac{x^2}{\ell^2}} \text{erfc} \left( \frac{x}{\sqrt{2k}} \right) \frac{7+13x^2+7x^4+x^6}{16} - \frac{x(3+x^2)^2}{16}, & \text{for } |\lambda| = 1 \end{cases} \quad (34)$$

gives the dependence of this quantity on the distance between the layers.

The quantity $\rho_{s0}$ is called the superfluid stiffness. More precisely, the expression (33) gives the value of this quantity at $T = 0$ in the mean-field approximation. A plot of the function $\rho_{s0}(d/\ell)$ is displayed in Fig. 1. The quantity $\pi \rho_{s0}/2$ gives an estimate of the temperature of the transition into the superfluid state. This estimate neglects the temperature correction to $\rho_s$ and is valid in a bounded interval of $d$. We shall discuss this question in greater detail below.

To find the critical current and the transition temperature it is necessary to obtain the spectrum of the collective modes. We shall use the approach of Refs. [10] and [38], which is based on the quantization of the energy of small fluctuations of $\theta_X$ and $\varphi_X$. First we shall require the spectrum for wave vectors $q$ directed along the gradient of the phase $\varphi$ (which was obtained in Ref. [38]). The general case is analyzed in the Appendix. According to Ref. [38], the expression for the spectrum can be written in a form that is formally identical to the expression for the Bogolyubov spectrum of quasiparticles in a moving condensate:

$$E(q) = \sqrt{\varepsilon (\varepsilon + 2\gamma)} + \hbar q v. \quad (35)$$
The following notations have been introduced in Eq. (35)

\[ \epsilon = 2F_D(Q) - F_D(q + Q) - F_D(q - Q), \]  

(36)

\[ \gamma = \sin^2 \theta_0 \left[ H(q) - F_S(q) + \frac{F_D(q + Q) + F_D(q - Q)}{2} \right], \]  

(37)

\[ v = \frac{F_D(q + Q) - F_D(q - Q)}{\hbar q} \cos \theta_0. \]  

(38)

The quantity (36) is the kinetic energy of pairs. In the limits \( q \to 0 \) and \( Q \to 0 \) the expression (36) reduces to

\[ \epsilon = \frac{\hbar^2 q^2}{2M}, \]  

(39)

where \( M \) is the magnetic mass of a pair

\[ M = \frac{2\hbar^2 \varepsilon}{e^2 \ell} \frac{1}{\tilde{\rho}(d/\ell)}, \]  

and \( \tilde{\rho}(x) \) is determined by Eq. (34). Since the mass \( M \) is inversely proportional to \( \rho_s \), according to Fig. 1 it increases with \( d \), and the magnetic mass for the active levels \( \lambda = \pm 1 \) is greater than that for the active level \( \lambda = 0 \).

The expression (37) in the limits \( q \to 0 \) and \( Q \to 0 \) goes to a constant

\[ \gamma_0 = \frac{E_c}{2} \sin^2 \theta_0 \tilde{\gamma} \left( \frac{d}{\ell} \right), \]  

(40)

where

\[ \tilde{\gamma}(x) = \begin{cases} 
\frac{x - \sqrt{x^2 - 1} - e^{2x^2} \text{erfc}(\frac{x}{\sqrt{2}})}{x^2} & \text{for } \lambda = 0 \\
\frac{1}{x} - e^{2x^2} \text{erfc}(\frac{x}{\sqrt{2}}) & \text{for } |\lambda| = 1 
\end{cases} \]  

(41)

The quantity (40) determines the pair interaction energy per pair. The function \( \gamma_0(\ell/d) \) is presented in Fig. 2. Evidently, the interaction energy for \( \lambda = \pm 1 \) is greater than that for \( \lambda = 0 \). Moreover, for small \( d \) the energy \( \gamma_0 \propto d \) for \( \lambda = \pm 1 \), while for \( \lambda = 0 \) the function \( \gamma_0(d) \) is quadratic.

In the long-wavelength limit the spectrum is linear:

\[ E = \hbar s q \ (Q = 0), \]  

where the sound velocity equals

\[ s = \sqrt{\gamma_0/M} = \frac{e^2}{\hbar \sin \theta_0 \tilde{s} \left( \frac{d}{\ell} \right)}, \]  

(42)

\[ s(x) = \frac{1}{2} \sqrt{\tilde{\gamma}(x)} \rho(x). \]

A plot of the function \( s(d/\ell) \) is presented in Fig. 3. According to Fig. 3, the maximum sound velocity (which is reached at \( d/\ell \approx 1 \) and zero imbalance of the active component) does not depend on the magnetic field and for \( \epsilon \approx 3.9 \) (SiO₂) it is \( \approx 10^7 \) cm/s.
The expression (38) in the limits $q \to 0$, $Q \to 0$ assumes the form $v = (\hbar Q/M) \cos \theta_0$. Since $Q = \partial \varphi / \partial X$, the quantity $v$ is the product of the superfluid velocity ($v_s = \hbar \nabla \varphi / M$) and an additional factor that depends on the imbalance. Specifically, for zero imbalance $v = 0$. As discussed in Ref. [38], this is a consequence of the electron-hole symmetry. In the present work we want to elucidate this feature on the basis of several other arguments.

The superfluidity of electron-hole pairs in a bilayer quantum Hall system can be viewed as an analog of the so-called counterflow superfluidity [39]. Counterflow superfluidity arises in a system in which type I and II bosons on a lattice are present, the total filling factor of the lattice sites is 1, and strong same-site repulsion forbids two bosons from occupying the same site. An elementary act of boson motion in such a system is a type-I boson hopping onto a site with a type-II boson and the type-II boson simultaneously hopping onto the site occupied by the type-I boson. The fluxes of type-I and -II bosons in such a system are equal in modulus and are oppositely directed. The ratio of the velocity of the components I and II depends on the concentration ratios of the components. For equal concentrations the moduli of the velocities are likewise the same.

In the bilayer quantum Hall system considered here the number of electrons of the top layer is equal to the number of holes in the bottom layer, and likewise the number of holes in the top layer is equal to the number of electrons in the bottom layer. The pairing of electrons and holes from neighboring layers can be regarded as the formation of two kinds of pairs differing by the direction of the dipole moment (upwards or downwards). In such an interpretation pair motion is an exchange process between different kinds of pairs. In other words, the flux of pairs of one kind is accompanied by a counterflux of pairs of the other kind. In the absence of imbalance the concentrations of different kinds of pairs is the same.

Counterflow superfluidity is a particular case of a two-component superfluid [40, 41, 42]. In this particular case additional conditions are imposed—equality of the counterfluxes and conservation of the locally total density of the components (for this reason there is only one and two oscillation modes). The spectrum of the collective modes in the two-component system [41, 42] possesses the feature that it does not contain terms which are linear in the gradient of the phase, if the velocities of the components are equal in magnitude and opposite in direction. At the same time this spectrum contains a quadratic dependence on the gradient of the phase. The spectrum [35] demonstrates similar properties, and vanishing of $v$ with zero imbalance is a consequence of the counterflow character of the superfluidity in a bilayer quantum Hall system.

For finite values of $q$ the spectrum deviates considerably from the Bogolyubov spectrum. Specifically, for sufficiently
large $d$ a roton-like minimum appears in the spectrum; the depth of this minimum increases with increasing $d$ (Fig. 4). For $d > d_c$ the spectrum becomes imaginary for finite $q$ and an instability relative to the formation of a charge density wave arises in the system. The dependence of the critical distance $d_c$ between the layers on the imbalance of the filling factors of the active component is presented in Fig. 5, whence it is evident that the component with pairing of the electrons and holes on the Landau level $\lambda = \pm 1$ is unstable for smaller $d$ than the same state arising in a situation where the active level is $\lambda = 0$.

The critical values of the supercurrents can likewise be determined by analyzing the spectrum (35). As already discussed in Ref. [38], the limitation on the currents in the present case is not associated with the magnetic field which the currents in the layers generate and is oriented parallel to the layers. The maximum value of the current corresponds to the maximum value of $Q$ for which the spectrum (35) is positive (Landau’s superfluidity criterion) and real (stability criterion). The dependence of the densities of the supercurrents in the layers on $Q$ is given by the expression (38)

$$j_1 = -j_2 = \frac{e}{\hbar} \frac{1}{4\pi l^2} \sin^2 \theta_0 \frac{dF_D(Q)}{dQ}. $$

The computational results for the critical current are presented in Fig. 6. The maximum critical current is proportional to the quantizing magnetic field. For $B = 1$ T and $\varepsilon = 3.9$ the maximum critical current density equals approximately $\approx 1$ A/m.

In closing this section we shall estimate the dependence of the temperature of the transition into the superfluid state on $d$. Since the system considered here is two-dimensional, the transition into the superfluid state is a Berezinskii-Kosterlitz-Thouless transition. The critical transition temperature $T_c$ is determined by the equation $T_c = \pi \rho_s(T_c)/2$, where $\rho_s(T)/2$ is the superfluid stiffness at finite temperature. To find $\rho_s(T)$ we note that the superfluid flux density $j_s$ is related with the gradient of the phase $Q$ by the relation $j_s = \rho_s(T)Q/\hbar$ (for small $Q$). On the other hand the flux density can be found from the relation $j_s = (1/\hbar)\partial F/\partial Q$, where

$$F = E_{mf} + E_{zp} + T \sum_q \ln \left(1 - \exp \left(-\frac{E(q)}{T}\right)\right) .$$
FIG. 6: Critical current density (in the units $e^3/\hbar \ell^2$) versus the distance between the layers for different magnitudes of the imbalance. Solid line - $\nu = 0$; dashed line - $\nu = 0.17$; dot-dash line - $\nu = 0.25$; dashed line - $\nu = 0.4$ for $\lambda = 0$ (a) and $|\lambda| = 1$ (b).

is the free energy. Here $E_{zp}$ is the energy of zero-point vibrations. The collective and single-particle excitations must be taken into account in the energy of the zero-point vibrations. As shown in Ref. [8], for $d$ not too close to the critical value the contribution of zero-point vibrations to the renormalization of the superfluid stiffness is negligible. Consequently, we shall neglect $E_{zp}$ when calculating the superfluid stiffness. To take account of the temperature correction to $\rho_s$ it is sufficient to include only the collective modes in the entropy term in Eq. (44) (the single-particle excitation spectrum has a gap of the order of $E_c$). As a result we have

$$\rho_s(T) = \rho_{s0} + \frac{1}{S} \lim_{Q \to 0} \frac{1}{Q} \sum_q \frac{\partial E(q)}{\partial Q} N_B(E(q)),$$

where $N_B(E) = (\exp(E/T) - 1)^{-1}$ is the Bose distribution function. For a single-component superfluid system with the dispersion law $E(q) = E_0(q) + \hbar q_x v_s$ (where $v_s = \hbar Q/M$ is the superfluid velocity, and $E_0(q)$ is the spectrum as $v_s = 0$) the expression (45) reduces to the standard form $\rho_s(T) = \rho_{s0} - \rho_n$, where

$$\rho_n = -\frac{\hbar^2}{MS} \sum_q \frac{\hbar^2 q_x^2}{M} \left( \frac{\partial N_B(E)}{\partial E} \right)_{E=E_0(q)}$$

is the normal density. In our case the expression for the normal density will be different, since the superfluidity is of a counterflow character.

Here we shall focus on the calculation of the transition temperature with zero imbalance. In this case the spectrum for small $Q$ can be written in the form

$$E(q) = E_0(q) + \frac{1}{2} \alpha(q) Q^2 + \ldots$$

which gives

$$\rho_s(T) = \rho_{s0} + \frac{1}{S} \sum_q \alpha(q) N_B(E_0(q)).$$

To calculate the coefficient $\alpha(q)$ in the expansion we shall require the spectrum of excitations for an arbitrary direction of the wave vector. A method for calculating the desired spectrum, generalizing the approach used in Ref. [38], is described in the Appendix. According to the results obtained in the Appendix, for $\theta_0 = \pi/2$ the spectrum assumes the form

$$E(q) = \sqrt{\epsilon_{q,Q}(\epsilon_{q,Q} + 2\gamma_{q,Q})},$$

where

$$\epsilon_{q,Q} = 2F_D(Q) - F_D(|q + Q\hat{x}|) - F_D(|q - Q\hat{x}|),$$

$$\gamma_{q,Q} = \left[ H(q,Q) - F_S(q) + \frac{F_D(|q + Q\hat{x}|) + F_D(|q - Q\hat{x}|)}{2} \right].$$
FIG. 7: Critical temperature versus the distance between the layers. The solid curves 1 and 2 correspond to $\lambda = 0$ and $|\lambda| = 1$, respectively. The dashed curves show the critical temperature in the mean-field approximation (neglecting the temperature renormalization of the superfluid stiffness).

the functions $\mathcal{F}_S(q)$ and $\mathcal{F}_D(q)$ are determined by Eq. (28) and (29), $\hat{x}$ is a unit vector in the direction $x$, and

$$\mathcal{H}(q, Q) = \frac{e^2}{2\varepsilon l^2} e^{-\frac{2\varepsilon^2}{l^2}} \left( 1 - e^{-|q|} \cos(q_y Q \ell^2) \right) f_{\lambda}(q \ell).$$

(51)

Using Eqs. (48-51) to calculate the function $\alpha(q)$ we arrive at the following expression for $\rho_s(T)$:

$$\rho_s(T) = \rho_{s0} + \int_0^\infty dq \frac{q N_0(E_0(q))}{2\pi E_0(q)} \left( 2(\gamma_q + \epsilon_q) \mathcal{F}_D''(0) - \left( \gamma_q + \frac{\epsilon_q}{2} \right) \left( \mathcal{F}_D''(q) + \frac{\mathcal{F}_D'(q)}{q} \right) + \epsilon_q D(q) \right),$$

(52)

where

$$D(q) = \frac{e^2}{4\varepsilon} e^{-\frac{2\varepsilon^2}{l^2}} q \ell^2 e^{-|q|} f_{\lambda}(q \ell).$$

$\epsilon_q = \epsilon_{q,0}$ and $\gamma_q = \gamma_{q,0}$. The dependence of the critical temperature on the distance between the layers, as calculated from Eq. (52), is presented in Fig. 7.

According to the results obtained, for intermediate values of $d$ the expression $T_c = \pi \rho_{s0}/2$ is a good approximation for the transition temperature; it is important to take account of the temperature correction for small $d/\ell$ and for $d$ close to the critical values. The maximum critical temperature is attained for intermediate values of $d$. The dependence $T_c(d/\ell)$ is qualitatively similar to the dependence of the critical temperature on $d/\ell$. In a state where electrons and holes from the zeroth Landau level form pairs the critical temperature is higher than in a state where electrons in the level $\lambda = +1$ form pairs with holes in the level $\lambda = -1$.

We note that the results of the present section, which were obtained for the zeroth level, also describe the situation in bilayer quantum Hall systems in semiconductor heterostructures based on GaAs (and, correspondingly, partially repeat the results of Refs. [6, 7, 8, 10, 38]).

**DISCUSSION**

The essential result following from the preceding section is that the maximum parameters of electron-hole superconductivity in a bilayer graphene system in a quantizing magnetic field can be attained if interlayer phase coherence arises between electrons belonging to the zeroth Landau level. In this connection there arises the question of whether or not there are any advantages to using graphene instead of GaAs heterostructures.

To answer this question we recall the general conditions under which a superfluid state of bound electron-hole pairs can be expected to appear in a bilayer quantum Hall system. Three characteristic energies can be identified in the present problem: the splitting $\omega_0$ between the Landau levels, the Coulomb energy $E_c$, and the interlayer tunneling amplitude $t$. The analysis performed in the present work is based on the assumption that the following strong inequalities hold: $t \ll E_c$ and $E_c \ll \omega_0$.

The first inequality makes it possible to neglect tunneling when analyzing collective modes. In addition, even weak tunneling results in the formation of a system of vortices which are similar to Josephson vortices. If a bilayer system is used to transfer current from source to load (this is the configuration realized in Refs. [20, 21, 22]), the vortices...
will move, which will result in energy dissipation\cite{43} whose magnitude dissipation is proportional to the squared tunneling amplitude. We note that a configuration where the vortices are stationary and dissipation is absent can be realized\cite{44} but such configurations cannot be used to transfer energy from source to load\cite{45}. Consequently, systems with a negligibly small interlayer tunneling amplitude are needed. In a double quantum well the tunneling is quite large because of the relative narrowness and smallness of the barrier.

The second inequality is the condition for using the so-called lowest Landau level approximation (where transitions to inactive levels are neglected). In this case the electrons and holes in the active level are well-determined quasiparticles. In systems with a quadratic dispersion law for charge carriers the splitting between the Landau levels $\omega_0 = \frac{h e B}{m^*}$ ($m^*$ is the effective mass of the carriers) is directly proportional to the magnetic field and the Coulomb energy $E_c = \frac{e^2}{\varepsilon \ell}$ is proportional to the square root of the magnetic field. Consequently, the inequality $E_c < \omega_0$ holds only in quite strong fields ($B \gtrsim 10$ T for GaAs). The magnetic length is small, i.e. the critical value of $d$ is small. Correspondingly, large values of $d$ cannot be used to suppress interlayer tunneling.

For a bilayer graphene system $\omega_0 = \sqrt{2} h v_F/\ell$, i.e. the condition $E_c < \omega_c$ is equivalent to the condition $\varepsilon > e^2/(\sqrt{2} h v_F) \approx 1.5$. For an appropriate choice of $\varepsilon$ the condition is satisfied for arbitrary magnetic fields and large values of $d$ can be used (the ratio $d/\ell$ can be made less than 1 by an appropriate choice of the magnetic length). Since graphene is a monolayer and not a sufficiently wide quantum well, the interlayer tunneling amplitude will be much less than for the same values of $d$.

We note that the inequality $E_c \ll \omega_0$ is not so fundamental for a graphene system as for a bilayer electronic system in a GaAs heterostructure. In the second case, in the limit of weak magnetic fields we switch to the case of two layers with carriers of the same type, and in the first case (in a gate field) we arrive at an electron-hole graphene system in a zero magnetic field\cite{24, 25, 26, 27}. For $\varepsilon = 1$ and when a large number of positive Landau levels are filled in a single layer (and the same number of empty negative layers in the other layer) the approach used in the present work, strictly speaking, is not applicable. Nonetheless, the pairing effect apparently remains. The question of the transition temperature in this case remains open for the time being.

In closing, we shall discuss the question of which maximum superconducting transition temperatures can be reached in a bilayer quantum Hall system based on graphene. Let us assume that we have created a system with a prescribed value of $d$ and we start to vary the magnetic field (at the same time adjusting the gate field so as to remain in the regime with zero imbalance of the active component). To determine the maximum transition temperature we shall represent the plot shown in Fig. 7 in a somewhat different form: specifically, we shall choose the quantity $e^2/\varepsilon d$ as the unit of temperature. The results are displayed in Fig. 8. As one can see from the dependences presented, the maximum temperature is $T_c \approx 0.01 e^2/\varepsilon d$ and is reached for $d/\ell = 0.5 \div 1$ in the case where carriers in the zeroth level participate in the pairing. For $d = 50$ A and $\varepsilon = 3.9$ the maximum temperature $T_c \approx 8$ K. The required magnetic fields $B \gtrsim 25$ T. A transition temperature of the order of 1 K can be attained with $d \approx 400$ A (for the same $\varepsilon$) in sufficiently strong fields $B \gtrsim 0.4$ T.

APPENDIX. Spectrum of collective oscillations with arbitrary direction of the wave vector

The problem of the collective modes in a bilayer quantum system with interlayer phase coherence is examined in Refs.\cite{8, 4, 10} and [38]. In Ref. [38] the approach of Ref. [10] (based on the quantization of the energy of the phase fluctuations and local imbalance) is extended to the case of an arbitrary average imbalance. It is shown in [38] that

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig8.png}
\caption{Critical temperature with fixed distance between the layers versus the reciprocal of the magnetic length. Solid line – $\lambda = 0$; dashed line – $|\lambda| = 1$.}
\end{figure}
the spectrum possesses electron-hole symmetry and becomes a Bogolyubov spectrum in the low-density limit. The case where the direction of the wave vector coincides with the direction of the phase gradient has been examined in Ref. [9]. In the present Appendix we extend the approach used in Ref. [38] to the case of an arbitrary direction of the wave vector.

Using the analogy with a quantum ferromagnet [6] we shall examine the Fourier components of the pseudospin density operators

$$\hat{m}(q) = \sum_{k_1,k_2} (k_1|e^{-iqr}|k_2) a_{i_{k_1},k_1}^+ \sigma_{ij} a_{j,k_2}$$  \hspace{1cm} (A.1)$$

where $\sigma_{ij}$ are Pauli matrices. The state with nonzero phase gradient can be interpreted as a helical state with $m_z = \cos \theta_0$, $m_x = \sin \theta_0 \cos(Qr)$, $m_y = \sin \theta_0 \sin(Qr)$. In such a state the Fourier series expansion of the averages $\langle \hat{m}_{x(y)}(q)|\Psi \rangle$ contains terms with $q = \pm Q$.

The wave function

$$|\Psi\rangle = \prod_k \left[ \cos \frac{\theta_0}{2} a_{1,k}^+ + e^{iQk} \sin \frac{\theta_0}{2} a_{2,k}^+ \right] |0\rangle$$  \hspace{1cm} (A.2)$$
describes the helical state with $Q = Q\hat{x}$, where $\hat{x}$ is a unit vector along the $x$ axis. To describe the helical state with arbitrary $Q = (Q_x, Q_y)$ the wave function must be chosen in the form

$$|\Psi\rangle = \prod_k \left[ \cos \frac{\theta_k}{2} a_{1,k}^+ + e^{iQ_x k + i\tilde{\varphi}} \sin \frac{\theta_k}{2} a_{2,k-\varphi} - Q_y \right] |0\rangle.$$  \hspace{1cm} (A.3)$$

It is easily verified that the energy is independent of the direction of $Q$: in the state (A.3), just as in the state (A.2), the expression for the energy has the form (A.9). Indeed, the transition from (A.2) to (A.3) is simply a rotation of the coordinate system, and the energy should not change under such a transformation. We shall examine the fluctuations of $\theta$ and $\varphi$ along the $x$ axis in a new coordinate system. To take account of such fluctuations we write the wave function in the form

$$|\Psi\rangle = \prod_k \left[ \cos \frac{\theta_k}{2} a_{1,k}^+ + e^{iQ_x k + i\tilde{\varphi}} \sin \frac{\theta_k}{2} a_{2,k-\varphi} - Q_y \right] |0\rangle.$$  \hspace{1cm} (A.4)$$

A calculation of the energy in the state (A.4) gives

$$E = E_0 - e\bar{V} \sum_X \cos \theta_X + \frac{1}{2L} \sum_{X,X'} \left\{ \left( H_{Q_y}(X - X') - F_S(X - X') \right) \cos \theta_X \cos \theta_{X'} - F_{D,Q_y}(X - X') \sin \theta_X \sin \theta_{X'} \right\} \cos(Q_x(X - X') + \tilde{\varphi}_X - \tilde{\varphi}_{X'}),$$  \hspace{1cm} (A.5)$$

where

$$H_{Q_y}(X) = \frac{\epsilon^2}{2e} \int_{-\infty}^{\infty} dq \left[ 1 - e^{-iq(\cos(Q_y q)^2)} \right] e^{iqX - \frac{q^2}{2\lambda^2}} f_q(\sqrt{q^2 + X^2/\lambda^2})$$

$$F_S(X) = \frac{\epsilon^2}{2e} e^{-\frac{X^2}{2\lambda^2}} \int_{-\infty}^{\infty} dq \frac{dq}{\sqrt{q^2 + X^2/\lambda^2}} e^{-\frac{q^2}{2\lambda^2}} f_q \left( \sqrt{q^2 + X^2/\lambda^2} \right)$$

$$F_{D,Q_y}(X) = \frac{\epsilon^2}{2e} e^{-\frac{X^2}{2\lambda^2}} \int_{-\infty}^{\infty} dq \frac{dq}{\sqrt{q^2 + X^2/\lambda^2}} e^{-\frac{q^2}{2\lambda^2}} e^{iqQ_y} f_q \left( \sqrt{q^2 + X^2/\lambda^2} \right).$$  \hspace{1cm} (A.6)$$

Performing the expansion (A.6) with respect to small fluctuations $\tilde{\varphi}_X$ and $\tilde{m}_z(X) = \cos \theta_X - \cos \theta_0$ and switching to the Fourier components

$$\tilde{m}_z(q) = \frac{2\pi l^2}{S} \sum_X \tilde{m}_z(X) e^{-iqX}, \quad \tilde{\varphi}(q) = \frac{2\pi l^2}{S} \sum_X \tilde{\varphi}(X) e^{-iqX},$$  \hspace{1cm} (A.7)$$

we obtain the following expression for the energy of the fluctuations:

$$E_{\tilde{m}} = \frac{S}{4\pi l^2} \sum_q \left[ \tilde{m}_z(-q)K_{zz}(q)\tilde{m}_z(q) + \tilde{\varphi}(-q)K_{\varphi\varphi}(q)\tilde{\varphi}(q) - (i\tilde{m}_z(-q)K_{z\varphi}(q)\tilde{\varphi}(q) + c.c.) \right].$$  \hspace{1cm} (A.8)$$
where
\[ \mathcal{K}_{zz}(q) = \mathcal{H}_{Q_y}(q) - \mathcal{F}_S(q) + \mathcal{F}_D(Q) + \left( \mathcal{F}_D(Q) - \frac{\mathcal{F}_D(|q\hat{x} + Q|) + \mathcal{F}_D(|q\hat{x} - Q|)}{2} \right) \cot^2 \theta_0, \]
(A.9)

\[ \mathcal{K}_{z\varphi}(q) = \cos \theta_0 \frac{\mathcal{F}_D(|q\hat{x} + Q|) - \mathcal{F}_D(|q\hat{x} - Q|)}{2}, \]
(A.10)

\[ \mathcal{K}_{\varphi\varphi}(q) = \sin^2 \theta_0 \left[ \mathcal{F}_D(Q) - \frac{\mathcal{F}_D(|q\hat{x} + Q|) + \mathcal{F}_D(|q\hat{x} - Q|)}{2} \right]. \]
(A.11)

\[ \mathcal{H}_{Q_y}(q) = \frac{\epsilon^2}{2\pi l^2} e^{-\frac{2q^2}{l^2}} \left( 1 - e^{-d|q|} \cos(Q_y q l^2) \right) f_\lambda(q\ell), \]
(A.12)

and the functions \( \mathcal{F}_S, D(q) \) are determined by Eqs. (28) and (29). To quantize the energy (A.8) we take account of the fact that \( \hat{m}_z \) and \( \hat{\varphi} \) are canonically conjugate and the commutator of the operators of these quantities equals

\[ [\hat{m}_z(q), \hat{\varphi}(q')] = -2i \frac{2\pi l^2}{S} \delta_{q,-q'}. \]
(A.13)

Expressing the operators \( \hat{m}_z(q) \) and \( \hat{\varphi}(q) \) in terms of Bose creation and annihilation operators

\[ \hat{m}_z(q) = A_q \sqrt{\frac{2\pi l^2}{S}} (b_q + b^\dagger_{-q}), \]
\[ \hat{\varphi}(q) = i \frac{1}{2q} \sqrt{\frac{2\pi l^2}{S}} (b_q - b^\dagger_{-q}), \]
replacing in Eq. (A.8) the quantities \( \hat{m}_z \) and \( \hat{\varphi} \) by the operators (A.14), and requiring the terms containing two creation or two annihilation operators to vanish (which makes it possible to determine \( A_q \)), we arrive at the equation

\[ H_\lambda = \sum_q E(q) \left( b^\dagger(q)b(q) + \frac{1}{2} \right), \]
(A.15)

where

\[ E(q) = 2 \left( \sqrt{\mathcal{K}_{\varphi\varphi}(q)\mathcal{K}_{zz}(q) + \mathcal{K}_{z\varphi}(q)} \right). \]
(A.16)

The spectrum (A.16) of the collective modes depends on \( q, Q \), and the angle between the \( x \) axis (which is chosen in the direction of the wave vector \( q \)) and the vector \( Q \). Rotating the coordinate system to that the \( x \) axis is oriented along \( Q \), we obtain the spectrum of the collective modes for an arbitrary direction of the wave vector:

\[ E(q) = \sqrt{\epsilon_{q,Q}^2 + 2\gamma_{q,Q} \sin^2 \theta_0 + \cos \theta_0 \left( \mathcal{F}_D(|q + Q\hat{x}|) - \mathcal{F}_D(|q - Q\hat{x}|) \right)}, \]
(A.17)

where the quantities \( \epsilon_{q,Q} \) and \( \gamma_{q,Q} \) are determined by the expressions (49) and (50), respectively.

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