Finite multiple zeta values, formal algebraic relations and the fundamental group of $M_{0,4}$ and $M_{0,5}$ - I

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Abstract

This is the first part of a study of relations satisfied by finite multiple zeta values and their analogues, by means of the motivic fundamental group of the moduli spaces $M_{0,4}$ and $M_{0,5}$. We define notions of double shuffle relations and associator relations for finite multiple zeta values. We give new proofs to two results of "depth drop" phenomena of multiple zeta values. We prove an explicit $p$-adic lift of a family of congruences among finite multiple zeta values, which has applications to $p$-adic zeta values.

We refer to the equations that appear as formal algebraic because they are infinite sums of algebraic relations, defining formal schemes.

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1 Introduction

1.1 Definitions

Multiple zeta values are the real numbers defined as follows, for $(s_d, \ldots, s_1) \in (\mathbb{N}^*)^d$ with $s_d \geq 2$:

$$\zeta(s_d, \ldots, s_1) = \sum_{0 < n_1 < \ldots < n_d} \frac{1}{n_1^{s_1} \cdots n_d^{s_d}}$$ (1)
The integer \( n = s_d + \ldots + s_1 \) is called the \textit{weight} ; the integer \( d \) is called the \textit{depth}. Let us denote by \( \omega_0 \) and \( \omega_1 \) the differential forms \( \frac{dz}{z} \) and \( \frac{dz}{z-1} \), and by \( (\omega_{i_n}, \ldots, \omega_{i_1}) \) the sequence \( \omega_0, \ldots, \omega_0, \omega_1, \ldots, \omega_0, \omega_0, \ldots, \omega_0, \omega_1, \ldots, \omega_0, \omega_0, \ldots, \omega_0 \); then we have an \textit{iterated integral} formula:

\[
\zeta(s_d, \ldots, s_1) = (-1)^d \int_0^1 \omega_{i_n}(t_n) \int_0^{t_n} \omega_{i_{n-1}}(t_{n-1}) \ldots \int_0^{t_2} \omega_{i_1}(t_1)
\]

These two formulas permit to extend the definition of multiple zeta values in two different ways to the case where \( s_d = 1 \), using regularizations. The iterated integral formula shows that multiple zeta values are periods of the Betti-de Rham comparison of the motivic fundamental groupoid of \( M_{0,4} \approx \mathbb{P}^1 \setminus \{0, 1, \infty\} \), defined by Deligne in [D] and by Deligne and Goncharov in [DG] (see §2.1, §2.2).

\textit{p-adic multiple zeta values} are numbers indexed in the same way, defined geometrically. They express the action of Frobenius (or its inverse, depending on the conventions) on the \( \mathbb{Q}_p \)-points of the de Rham fundamental group of \( \mathbb{P}^1 \setminus \{0, 1, \infty\} \), with base points at 0 and 1 ([DG], §5.28).

For the purposes of the present paper, it is natural to consider more generally, for each \( k \in \mathbb{Z} \setminus \{0\} \) and especially for \( k < 0 \), the variant of \( p \)-adic multiple zeta values corresponding to the \( k \)th power of Frobenius, which we denote by:

\[
\zeta_p(k)(s_d, \ldots, s_1) \in \mathbb{Q}_p
\]

Another version of \( p \)-adic multiple zeta values has been defined by Furusho in [F1], [F2] : it expresses the path invariant under the action of Frobenius. All variants of \( p \)-adic multiple zeta values share conjecturally the same algebraic relations with real multiple zeta values, plus the analogue of "\( 2i\pi = 0 \)", i.e. \( \zeta_p(2) = 0 \).

\textit{Multiple harmonic sums} are the following rational numbers : for \( N, s_1, \ldots, s_d \in \mathbb{N}^* \), let:

\[
H_N(s_d, \ldots, s_1) = \sum_{0 < n_1 < \ldots < n_d < N} \frac{1}{n_1^{s_1} \ldots n_d^{s_d}}
\]

They have been studied since a decade, in particular by analogy with multiple zeta values. A special role is played by the multiple harmonic sums \( H_p(s_d, \ldots, s_1) \) taken modulo \( p \), where \( p \) is a prime number.

\textit{Finite multiple zeta values} have been defined more recently by Zagier, and they make precise the analogy between harmonic sums \( H_p \) modulo \( p \) and multiple zeta values. Let \( \mathcal{A} \) be the ring of \textit{integers modulo infinitely large primes}, which appears for the first time in [Ko] :

\[
\mathcal{A} = \left( \prod_{p \text{ prime}} \mathbb{Z}/p\mathbb{Z} \right) \otimes_{\mathbb{Z}} \mathbb{Q} = \left( \prod_{p \text{ prime}} \mathbb{Z}/p\mathbb{Z} \right) / \left( \bigoplus_{p \text{ prime}} \mathbb{Z}/p\mathbb{Z} \right)
\]

It is also the quotient of the adelic ring \( \prod_p \mathbb{Q}_p = \{(x_p) \in \prod_p \mathbb{Q}_p \mid v_p(x_p) \geq 0 \text{ for } p \text{ large}\} \) by the ideal \( \{(x_p) \in \prod_p \mathbb{Q}_p \mid v_p(x_p) \geq 1 \text{ for } p \text{ large}\} \).
Finite multiple zeta values are the numbers

\[ \zeta_{\mathcal{A}}(s_d, \ldots, s_1) = (H_p(s_d, \ldots, s_1))_{p \text{ prime}} \in \mathcal{A} \quad (6) \]

**Conjecture 1.1**: 1) (Zagier) Let \( Z_s \) the \( \mathbb{Q} \)-sub-vector space of \( \mathcal{A} \) generated by the finite multiple zeta values of weight \( s \). Then

\( s \) i) The sum of the \( Z_s \)'s is direct.

\( s \) ii) We have \( \sum_{n \geq 0} \dim(Z_s)x^n = \frac{1}{1-x} \).

2) (Kaneko-Zagier) The following correspondence defines an isomorphism of \( \mathbb{Q} \)-algebras from the algebra generated by the \( \zeta_s \)'s to the algebra generated by the \( \zeta_s \)'s modulo \( \zeta(2) \) (the right hand side is independent of the chosen regularization):

\[ \zeta_{\mathcal{A}}(s_d, \ldots, s_1) \mapsto \sum_{n=0}^{d} (-1)^{s_{n+1}+\ldots+s_d} \zeta(s_{n+1}, \ldots, s_d) \zeta(s_n, \ldots, s_1) \pmod{\zeta(2)} \quad (7) \]

Conjecture 1.1 has led us to compute explicitly \( p \)-adic multiple zeta values and relate them to finite multiple zeta values [J1]. We have found formulas for \( p \)-adic multiple zeta values as sums of series whose terms involve multiple harmonic sums, and lower bounds for the valuation of \( p \)-adic multiple zeta values. A joint study of \( p \)-adic and finite multiple zeta values is explained in [J1], [J2], and uses also the results of the present paper.

The computations have suggested:

**Conjecture 1.2.** [J1] The valuation of each \( p \)-adic multiple zeta value \( p^{-(s_d+\ldots+s_1)} \zeta_{p(-1)}(s_d, \ldots, s_1) \)

is nonnegative for \( p \geq p_0(s_d, \ldots, s_1) \) (with \( p_0(s_d, \ldots, s_1) \) explicit in terms of \( (s_d, \ldots, s_1) \)) ; and we have for \( p \geq p_0(s_d, \ldots, s_1) :

\[ p^{-(s_d+\ldots+s_1)} \sum_{n=0}^{d} (-1)^{s_{n+1}+\ldots+s_d} \zeta_{p(-1)}(s_{n+1}, \ldots, s_d) \zeta_{p(-1)}(s_n, \ldots, s_1) \equiv H_p(s_d, \ldots, s_1) \pmod{p} \]

Moreover, reduction modulo \( p \) big enough induces an isomorphism from the \( \mathbb{Q} \)-algebra of \( p \)-adic multiple zeta values, included in \( \prod_p \mathbb{Q}_p \), to the \( \mathbb{Q} \)-algebra of finite multiple zeta values in \( \mathcal{A} \).

These facts have also been conjectured recently, with additional information coming from rigid cohomology, by S. Yasuda and K. Akagi [Y2].

We have also introduced in [J5] :

**Definition 1.3.** For each \( k \in \mathbb{N}^* \), let us call finite multiple zeta values the following numbers, for \( d \in \mathbb{N}^* \), \( (s_d, \ldots, s_1) \in (\mathbb{N}^*)^d \):

\[ \zeta_{f(-k)}(s_d, \ldots, s_1) = (p^k)^{s_d+\ldots+s_1} H_p(s_d, \ldots, s_1) \in \prod_p \mathbb{Q}_p \]

For all \( k \in \mathbb{N}^* \) and for all indices \( (s_d, \ldots, s_1) \in (\mathbb{N}^*)^d \), \( d \in \mathbb{N}^* \), we have :

\[ \zeta_{f(-k)}(s_d, \ldots, s_1) - \zeta_{f(-1)}(s_d, \ldots, s_1) \in \prod_{p \text{ prime}} p^{s_d+\ldots+s_1+1} \mathbb{Z}_p \quad (8) \]

In particular the \( p \)-adic valuations of the components of a \( \zeta_{f(-k)}(s_d, \ldots, s_1) \) are lower bounded by
its weight \( s_d + \ldots + s_1 \), and we have:

\[
\zeta_A(s_d, \ldots, s_1) = \left( p^{-(s_d+\ldots+s_1)} \zeta_f(-k) (s_d, \ldots, s_1) \mod p \right)_{\text{prime}}
\]  

(9)

The justification of this terminology follows from the following theorem [J5].

Notations : to a sequence \((s_d, \ldots, s_1) \in (\mathbb{N}^*)^d\) is associated the word \( e_0^{s_d} \cdot e_1 \cdot \ldots \cdot e_0^{s_1} e_1 \) in two letters \( e_0, e_1 \). The ring of formal power series in the two non-commutative variables \( e_0, e_1 \) with coefficients in a \( \mathbb{Q} \)-algebra \( R \) is denoted by \( R(\langle e_0, e_1 \rangle) \); an element \( f \in R(\langle e_0, e_1 \rangle) \) can be written in a unique way as \( f = \sum f[w]w \) where the sum is on the words in two letters \( e_0, e_1 \). To real and \( p \)-adic multiple zeta values are associated their generating series \( \Phi \in R(\langle e_0, e_1 \rangle) \) and \( \Phi_{p(\ast)} \in \mathbb{Q}_p(\langle e_0, e_1 \rangle) \).

We have:

**Theorem 1.4.** [J5] For all \( k \in \mathbb{N}^*, d \in \mathbb{N}^*, (s_d, \ldots, s_1) \in (\mathbb{N}^*)^d \), we have:

\[
\zeta_f(-k) (s_d, \ldots, s_1) = \left( \Phi_{p(-k)}^{-1} e_1 \Phi_{p(-k)} \left[ \frac{1}{1-e_0 e_1^{s_d} e_1^{s_1}} \right] \right)_{\text{prime}}
\]  

(10)

Moreover, this tends when \( k \to +\infty \) to the same series for the Ihara inverse of Furusho’s version of \( p \)-adic multiple zeta values.

**Sketch of the proof.** (The explicit version is given in [J5]). The proof is very short, although it uses in some way the explicit computations of \( p \)-adic multiple zeta values [J1]. It consists in considering the equation of horizontality of Frobenius (§2.1.3), which is the equality among \( p \)-adic multiple polylogarithms in [F2], theorem 2.14; translating it on its Taylor coefficients at \( z = 0 \) (not only of order \( p^k \); this gives additional information on more general multiple harmonic sums), and the coefficient of a word denoted by \( e_0^{l_0} e_1^{s_d} e_1^{s_1} \); finally, taking \( \to +\infty \) : almost all terms tend to 0 because of (rough) lower bounds of the valuation of \( p \)-adic multiple polylogarithms in [J1], §4; this gives the desired equality.

The \( k = 1 \) case of this theorem has been conjectured by S.Yasuda and M.Hirose [Y2], under the form stated below (the conventions of [Y2] are slightly different). For \( k = 1 \) and \( d = 1 \) it is a known equality concerning the classical \( p \)-adic zeta values. For \( k = 1 \) and \( d = 2 \) it has been proved by M. Hirose [Y2].

Conjecture (Yasuda-Hirose, [Y2]) For all \( d \in \mathbb{N}^*, (s_d, \ldots, s_1) \in (\mathbb{N}^*)^d \), and all primes \( p \):

\[
(-1)^d \sum_{n=0}^d (-1)^{\sum_{j=n+1}^d s_j} \left( \sum_{l_n+\ldots+l_d \geq i=n+1} \prod_{i=n+1}^d p^{l_i} \binom{l_i + s_i - 1}{l_i} \zeta_{p(-1)} (s_{n+1} + l_{n+1}, \ldots, s_d + l_d) \right) \times \zeta_{p(-1)} (s_n, \ldots, s_1) = p^{s_d+\ldots+s_1} H_p(s_d, \ldots, s_1)
\]  

(11)

**1.2 Contents of this paper**

In [J3] we have sketched a study of algebraic relations among finite multiple zeta values in \( A \) and their analogues of conjecture 1.1; it as well entirely based on the motivic fundamental group. Indeed, one can deal in a geometric way with the passage from multiple zeta values to their variants of conjecture 1.1 (which we call symmetrized multiple zeta values), and with finite multiple zeta
values in $\mathcal{A}$, in order to prove algebraic relations that both families satisfy.

This has led us in [J3] to a natural notion of double shuffle equations for finite multiple zeta values in $\mathcal{A}$ and symmetrized multiple zeta values (and their $p$-adic and motivic analogues). Kaneko and Zagier have proven similar facts, but their statement and proof for symmetrized multiple zeta values are entirely different. We also have reproved Hoffman’s duality theorem for finite multiple zeta values [H], and found its analogue for multiple zeta values, which appears to be a consequence of Drinfeld’s 2-cycle and 3-cycle equations. These equations permit to understand how the numbers of conjecture 1.1 can be expressed explicitly in terms of multiple zeta values of depth $\leq d - 1$.

The purpose of the present paper is to give the details of [J3], and to extend this approach by taking into account theorem 1.4 as well: thus we study the two sides of theorem 1.4 as variants of multiple zeta values, which englobes the results of [J3].

Our primary goal is to define and prove the variant of double shuffle equations and associator equations, for finite multiple zeta values $\zeta_{f(-k)}$, and their geometric analogues arising from the theorem 1.4. This is essentially achieved in this paper and will be completed in the future part two of this paper. We also derive from it other consequences.

It is known that all proved congruences between the $\zeta_A$’s should come from $p$-adic identities involving infinite sums of the numbers $p^{s_d + \ldots + s_1} H_p(s_d, \ldots, s_1)$. More precisely, it has been conjectured by Rosen [Ro2] that such lifts always exist in $\prod_p \mathbb{Z}/p^n\mathbb{Z}/p^n\mathbb{Z}$, i.e. the ring of $p$ infinitely large $p$-adic integers.

In most cases, actually, the congruences arise primarily as equalities between multiple harmonic sums in $\mathbb{Q}$; and one sometimes obtains quantities of the form $(p - n)^{-s}$ with $0 < n < p$, $s \in \mathbb{N}^*$ which can be expressed as $\sum_{l \geq 0} \left(\frac{-s}{l}\right) \frac{p^l}{n^l}$ in $\mathbb{Q}_p$, giving the desired type of sums of series. The explicit computation of $p$-adic multiple zeta values gives also a similar type of sums of series. In [J3], we have stated the double shuffle relations as an equality of this type involving the $\zeta_{f(-1)}$’s in $\prod_p \mathbb{Q}_p$.

Because the powers of $p$ are the integers such that, for $n \in \mathbb{N}^*$, one has the implication $0 < n < p^k \Rightarrow v_p(n) < v_p(p^k)$, the proofs involving $p$-adic series expansions of $(p - n)^{-s}$ with $0 < n < p$ adapt to the more general $H_{p^k}$’s.

The theorem 1.4 enables to give a geometric meaning to those identities involving infinite sums, as being equal to infinite sums of algebraic relations between ($p$-adic) multiple zeta values, and enables also to recast Rosen’s conjecture. We also discuss, and prove in the first case, a result of lift of congruences, which holds in restricted products $\prod_{p > \theta} \mathbb{Q}_p$. As an application, it gives a new way to write the series expansion of the classical $p$-adic zeta values.

These identities involving sums of series lead to a richer framework. Before all, they remain true for some power series whose coefficients are multiple zeta values, which generalize numbers of theorem 1.4, and which we call "formal symmetrized multiple zeta values". This follows from the weight homogeneity of all known, and conjecturally all, algebraic relations, which is also a motivic property.

Substituting to the formal variable a complex or $p$-adic one, one obtains analytic functions which are partly classical, essentially known, and variants are also known in the $p$-adic case (see [J4]).
The two parallel families of identities that are derived define in a natural way formal schemes over, respectively, $\mathbb{Q}[\![T]\!]$ (in the case of series of multiple zeta values), and $\mathbb{Z}_p$ (in the case of finite multiple zeta values). The type of formal schemes that is involved is subjacent to the usual pro-unipotent schemes defined by algebraic relations, where the completion is relative to the weight. Thus we refer to those identities as "formal algebraic relations" - we could also say "pseudo-algebraic relations".

The plan goes as follows.

In §2 we establish the geometric interpretation and the principles to study the numbers of conjecture 1.1 and theorem 1.4.

In §3 we establish pseudo-algebraic double shuffle equations for the (real, $p$-adic, motivic) formal symmetrized multiple zeta values and, independently, specifically for finite multiple zeta values with another proof:

**Theorem 1 : formal double shuffle relations**: for $\zeta_f(-k)$ and formal symmetrized multiple zeta values.

The shuffle relation permits to recast (§3.3) the "asymptotic reflexion theorem" of [Ro2]. We give other consequences of the double shuffle relation; for example, an analog of the "cyclic sum formula" for finite multiple zeta values (§3.5).

In §4 we do the same for an associator relation arising from $M_{0,4}$:

**Theorem 2 : formal associator relations, arising from $M_{0,4}$**: for $\zeta_f(-k)$ and formal symmetrized multiple zeta values

This gives a new proof to Hoffman’s duality theorem, and an alternative version to Rosen’s "asymptotic duality theorem" [Ro2] which is different combinatorially. Our results in §4 also have applications for the "de Rham-de Rham" (by contrast with Betti-de Rham, or rigid-de Rham) periods of the fundamental group, in particular, to the *monovalued* multiple zeta values studied by Brown in [Br1], [Br2] and by Furusho in [F2] - they correspond to the action of Frobenius at infinity.

In §5, using results of §4, we find new proofs to two phenomena of depth drop:

**Theorem 3 : depth-drop phenomena**:

(3-a) (§5.1) for symmetrized multiple zeta values
(3-b) (§5.2) for multiple zeta values such that the difference weight - depth is odd

The first result implies a new proof to a result of Yasuda ([Y1], proposition 3.1). The second result had proven before by Tsumura [Ts] and by Ihara-Kaneko-Zagier [IKZ]; this last proof has been corrected by Brown [Br3].

In §6, we retrieve in the first case the depth drop of symmetrized multiple zeta values as a
consequence of §3 and §4. We show that it lifts explicitly to a formal relation: on the level of finite multiple zeta values, the identities \( \zeta_A(s) = 0 \) admit explicit lifts to \( \prod_{p>s+1} \mathbb{Q}_p \). It has application to \( p \)-adic multiple zeta values, which is detailed in the appendix A of [J1].

**Theorem 4 : lift of congruences :** for \( \zeta_f(-k) \) and formal symmetrized multiple zeta values; it implies a \( p \)-adic lift of the congruence \( \zeta_A(s) = 0 \) to an explicit equality in \( \prod_{p>s+1} \mathbb{Q}_p \) with rational coefficients in \( \mathbb{Z}[\frac{1}{2}, \ldots, \frac{1}{s+1}] \).

In §7 we make additional comments.

In this first paper, we essentially limit ourselves to the properties arising from \( M_{0,4} \), except, of course, for the series shuffle equation, which is related to \( M_{0,5} \). The consequences of the 5-cycle equation will appear in the second part.

### 2 Reminders and a geometric interpretation

In §2.1 and §2.2, we recall the definition of the motivic fundamental groupoid of \( M_{0,4} \cong \mathbb{P}^1 \setminus \{0, 1, \infty\} \); in §2.3, we recall the definitions of the associator relations and regularized double shuffle relations of multiple zeta values. In §2.4 and §2.5 we define the geometric interpretation and formalism for the variants of \( (p \)-adic) multiple zeta values appearing in conjecture 1.1 and theorem 1.4.

#### 2.1 The motivic fundamental groupoid of \( \mathbb{P}^1 \setminus \{0, 1, \infty\} \)

The various realizations of the motivic fundamental group, as in §2.1.2, and Frobenius action on the de Rham realization of §2.1.3 have been defined by Deligne in [D]. As an object in a category of motives as in §2.1.1, the fundamental group has been defined by Deligne and Goncharov [DG]. The rigid version of the fundamental group of §2.1.3 is defined by Chiarellotto-Le Stum [CL] Shiho [Sh], and the existence and unicity of Frobenius invariant paths is established by Besser [Bes] and Vologodsky [V].

**The Tannakian category of mixed Tate motives over \( \mathbb{Z} \).**

Let \( \text{MT}(\mathbb{Q}) \) be the Tannakian category, defined by Levine [Le], of mixed Tate motives over \( \mathbb{Q} \). It is the rigid abelian tensor category over \( \mathbb{Q} \) of iterated extensions of the Tate motives \( \mathbb{Q}(n) \), \( n \in \mathbb{Z} \). Its simple objects are the \( \mathbb{Q}(n) \)'s, and each object has a weight filtration \( W \) such that the graded piece of weight \(-2n\) is a direct sum of \( \mathbb{Q}(n) \). It is equipped with the natural fiber functor \( \omega : M \mapsto \oplus_n \text{Hom}(\mathbb{Q}(n), \text{Gr}^W_{-2n}(M)) \).

Let \( \text{MT}(\mathbb{Z}) \) be its full subcategory of objects which are non-ramified at each finite place \( p \) of \( \mathbb{Q} \), i.e., all their \( l \)-adic realizations for \( l \neq p \) are non ramified at \( p \). The category \( \text{MT}(\mathbb{Z}) \) is determined by: \( \text{Ext}^1_{\text{MT}(\mathbb{Z})}(\mathbb{Q}(0), \mathbb{Q}(n)) = \mathbb{Q} \) if \( n \geq 3 \) and \( n \) is odd, \( \text{Ext}^1_{\text{MT}(\mathbb{Z})}(\mathbb{Q}(0), \mathbb{Q}(n)) = 0 \) otherwise, and \( \text{Ext}^2_{\text{MT}(\mathbb{Z})}(\mathbb{Q}(0), \mathbb{Q}(n)) = 0 \) for all \( n \).

The Galois group \( \Gamma' \) associated to the fiber functor \( \omega \) of \( \text{MT}(\mathbb{Z}) \) is described as follows. Its action on \( \omega(\mathbb{Q}(1)) = \mathbb{Q} \) defines a morphism \( \Gamma' \to \mathbb{G}_m \). Let \( U' = \text{ker} \omega \) be its kernel. There is an action of \( \mathbb{G}_m \) on \( \omega \) defined by \( \lambda \mapsto (\text{scalar multiplication by } \lambda^n) \) in motivic weight \(-2n\). This defines
a section to $G^\omega \to G_m$ and we have a semi-direct product decomposition $G^\omega = G_m \rtimes U^\omega$. The group $U^\omega$ acts trivially on the $\omega(Gr^W(M))$ for each object $M$, and turns out to be a pro-unipotent affine group scheme. It is then characterized as the exponential of its Lie algebra, which is the pro-nilpotent free Lie algebra with generators $\{\sigma_{2n+1}, n \geq 1\}$ having degree $-(2n+1)$, corresponding to $\text{Ext}^1((\mathbb{Q}(0), \mathbb{Q}(2n+1))$.

Let $\omega_{dR}, \omega_B : \text{MT}(\mathbb{Z}) \to \text{Vect}_\mathbb{Q}$ be, respectively, the de Rham and Betti realization functors. There is a canonical isomorphism $\omega_{dR} \simeq \omega$.

The motivic fundamental groupoid of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$

Let $X$ be the variety $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. The rational points of $X$, as well as the rational points of $T_z \setminus \{0\}$ where $z \in \{0, 1, \infty\}$, can be taken as base-points - in the second case they are called tangential base-points: for $x, y$ base-points, there exists an affine scheme $yP_x$ in the category $\text{MT}(\mathbb{Q})$, called the motivic space of paths from $x$ to $y$. When $x$ and $y$ are the tangential base-points $\bar{1}, -\bar{1}$ at $0, 1, \infty$, the corresponding $yP_x$ is in $\text{MT}(\mathbb{Z})$. We have morphisms $zP_y \times yP_x \to zP_x$ which make it into a groupoid, and $yP_x$ is a bitorsor under $(xP_x, yP_y)$. The groupoid is $\pi^{\text{mot}}(X)$, the motivic fundamental groupoid of $X$.

2.1.2 Betti and de Rham realizations

Betti realization

Let $\pi^B(X/\mathbb{C})$ be the fundamental groupoid associated to the Tannakian category over $\mathbb{Q}$ of local systems over $X(\mathbb{C})$ which are unipotent (i.e. iterated extensions of the trivial object). This is also the category of functors $C \to \text{Vect}_\mathbb{Q}$ where $C$ is the category having as objects the base-points of $X$, as sets of morphisms from $x$ to $y$, for base-points $x, y$, the set $y\pi^1_{\text{top}}(X(\mathbb{C}))_x$ where $\pi^1_{\text{top}}$ is the topological fundamental groupoid. The case of tangential base-points involves paths whose extremities may tend to $0, 1, \infty$, with prescribed tangent vectors at these extremities.

Let $H$ be a group; the Malcev completion $H^\text{un}$ of $H$ is the universal (pro-)unipotent affine group scheme $U$ over $\mathbb{Q}$ with a morphism $H \to U(\mathbb{Q})$. Its Hopf algebra is $\mathcal{O}(H^\text{un}) = \lim_{\text{lin}} \text{Hom}_{\mathbb{Q}[H]}(Q[H]/I^{N+1}, \mathbb{Q})$, where $I$ is the augmentation ideal of the $\mathbb{Q}[H]$.

Applying the Malcev completion to the groups $x\pi^1_{\text{top}}(X(\mathbb{C}))_x$, and then to the sets $y\pi^1_{\text{top}}(X(\mathbb{C}))_x$ using that they are $x\pi^1_{\text{top}}(X(\mathbb{C}))_y$-torsors, the previous definition amounts then to say that $\pi^B(X)$ is the Malcev pro-unipotent completion of the topological fundamental groupoid of $X(\mathbb{C})$.

de Rham realization

Let $K$ a field of characteristic $0$; let $\pi^{dR}(X/K)$ be the fundamental groupoid associated to the tannakian category over $K$ of vector bundles on $\tilde{X} = \mathbb{P}^1$ equipped with an (integrable) connection having logarithmic singularities at $0, 1, \infty$, and which are unipotent.

The functor which maps an object $(E, \nabla)$ to the vector space of global sections of $E$ on $\mathbb{P}^1$ is a tensor functor, because $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0$. It defines a canonical base point $\eta$, and canonical isomorphisms $\eta^\pi^dR \simeq y^\pi^dR$, $f \to yf_x$, for all base-points $x, y$, which are compatible to the groupoid structure.

The Lie algebra of $\eta^\pi^dR$ is the pro-nilpotent free Lie algebra generated by $H^1.dR(X)^\vee = \text{Vect}(\text{res}_0, \text{res}_1, \text{res}_\infty)/(\text{res}_0 + \text{res}_1 + \text{res}_\infty)$, where res refers to the residues. The classes of the $\text{res}_x$'s are denoted by $e_x$. Concretely, we have $\eta^\pi^dR = \text{Spec}(\mathcal{H}_m)$ where $\mathcal{H}_m$ is the shuffle Hopf algebra.
over \( \mathbb{Q} \) in the variables \( e_0, e_1 \); its precise description is recalled in §2.3.2; we have:

\[
\text{Lie}(\eta^{\pi\mathbb{Q}^dR}_n) : R \mapsto \{ f \in R(\langle e_0, e_1 \rangle) \text{ s.t. } \Delta_m(f) = f \otimes 1 + 1 \otimes f \}
\]

\[
y^{\pi\mathbb{Q}^dR}_n : R \mapsto \{ f \in R(\langle e_0, e_1 \rangle) \text{ s.t. } \Delta_m(f) = f \otimes f, \epsilon(f) = 1 \}
\]

Notations: \( R(\langle e_0, e_1 \rangle) \) is the non-commutative algebra of power series in \( e_0, e_1, \epsilon : R(\langle e_0, e_1 \rangle) \to R \) is the augmentation morphism and \( \Delta_m \) is the unique linear multiplicative map, continuous for the \( \ker(\epsilon) \)-adic topology, satisfying \( \Delta_m(e_i) = e_i \otimes 1 + 1 \otimes e_i, i = 0, 1. \)

For \( f \in R(\langle e_0, e_1 \rangle) \) and \( w \) a word in \( e_0, e_1 \), the coefficient of \( w \) in \( f \) is denoted by \( f[w] \), i.e. we have: \( f = \sum_w f[w]w \). This notation extends to the whole of \( \mathbb{Q}(e_0, e_1) \) by linearity. Moreover, for simplicity the various \( y(e_z)_x \) are simply denoted by \( e_z \).

The de Rham fundamental group is itself equipped with the universal unipotent connection called the KZ (Knizhnik-Zamolodchikov) connection. On the torsor \( (\pi^{\mathbb{D}R}_0) \), of paths starting at the tangential base-point \( \bar{1}_0 \) (i.e. \( \bar{1} \) at \( 0 \)), trivialized at \( \bar{1}_0 \), the connection is \( \nabla_{KZ} = -e_0 \frac{dz}{z} - e_1 \frac{dz}{z-1} \), and the horizontal sections are called multiple polylogarithms.

**Betti-de Rham comparison**

The Riemann-Hilbert correspondence induces an equivalence between the Tannakian categories associated respectively to the Betti and de Rham realizations; it corresponds to Chen’s theorem on iterated integration (with regularization in the case of tangential base-points). This gives an isomorphism of comparison:

\[
\text{comp} : y^{\pi\mathbb{Q}^B}_x(\mathbb{C}) \xrightarrow{\sim} y^{\pi\mathbb{Q}^dR}_x(\mathbb{C})
\]

**Multiple polylogarithms over \( \mathbb{C} \)**

One can define multivalued holomorphic functions on \( \mathbb{C}\setminus\{0,1\} \), by iterated integration of \( \frac{dz}{z} \) and \( \frac{dz}{z-1} \). We will denote by \( \text{Li}_{s_d, \ldots, s_1} \), or \( \text{Li}_w \) with \( w = e_0^{s_d-1} e_1 \ldots e_0^{s_1-1} e_1 \), the single valued holomorphic function on \( \mathbb{C}\setminus[\infty,0[\cup]1, +\infty[ \) which extends (via the iterated integral formula) the power series:

\[
\text{Li}_{s_d, \ldots, s_1}(z) = \sum_{n_d>0} \frac{H_{n_d}(s_d-1, \ldots, s_1)}{n_d^{s_d}} z^{n_d} = \int_0^z \frac{\omega_d^{s_d-1} \ldots \omega_1^{s_1-1} \omega_1}{\omega_d \ldots \omega_1} |z| < 1
\]

This extends all words \( w \) in \( e_0, e_1 \) by \( \text{Li}_{e_0} = 0 \) and the formula (17) recalled below.

We denote the Taylor coefficients at 0 of multiple polylogarithms by: \( \text{Li}_w(z) = \sum_{n \geq 0} c_n(\text{Li}_w) z^n \).

### 2.1.3 Rigid fundamental group and Frobenius action on the de Rham realization

Let \( K \) a non-archimedean local field of characteristic 0, whose residue field \( k \) is finite of characteristic \( p \).

**Direct construction of Frobenius on \( \pi^{dR} \)**

It is defined in [D] §11, a Frobenius auto-equivalence \( F^* \) on the de Rham category of the previous paragraph over \( K \), thus an action on the groupoid \( \pi^{dR}(X/K) \). In this setting, we have a kind
of "comparison" isomorphism

\[ F_* : (\pi^{dR}, \nabla_{KZ}) \sim \rightarrow F^*(\pi^{dR}, \nabla_{KZ}) \]

Moreover, \( F_* \) is horizontal with respect to the KZ connection [D], §11.11, §11.12. This is one of the starting points of the strategy to compute \( p \)-adic multiple zeta values: the very first example appears in [D], §19.6.

**Rigid fundamental group [CL]**

Let \( \pi^{rig}(X/k) \) be the fundamental groupoid associated with the Tannakian category over \( K \) of unipotent (thus necessarily overconvergent) isocrystals over \( X/k \).

This category is endowed in a natural way with an auto-equivalence of Tannakian categories \( F^* \) induced by the Frobenius of \( k \).

In terms of modules with connections on rigid analytic spaces: let \( V \) be the valuation ring of \( K \), \( P \) be the formal completion of \( \bar{X}/V \) along \( \bar{X}\setminus \{x\} \) where \( x \in \{0,1,\infty\} \), and \( P^n = \mathbb{P}^{1,an}\setminus \{x\} \) its generic fiber. Then, this category is equivalent to the one of unipotent coherent \( \tilde{\Omega}^1_{\bar{X}/\bar{X}} \), (integrable and) overconvergent along \( \bar{X}\setminus X \) [Ber].

**Rigid-de Rham comparison**

There exists an equivalence of Tannakian categories between the de Rham and the rigid categories above [CL]. For \( x,y \) base-points, including tangential base-points, of \( X/K \), we have isomorphisms compatible to the groupoid structure:

\[ y \pi^{dR}_x \times K[T] \sim \rightarrow \bar{y} \pi^{rig}_x \times K[T] \]

where \( K[T] \) reflects a choice of a branch of the \( p \)-adic logarithm [V], and \( \bar{x}, \bar{y} \) are reductions over \( k \).

In the case where \( \bar{x}, \bar{y} \) are base-points of \( X/k \) again in the same sense that we took in §2.1, we obtain more simply

\[ y \pi^{dR}_x \sim \rightarrow \bar{y} \pi^{rig}_x \]

### 2.2 Multiple zeta values and the fundamental group

In what follows, the tangential base-points \( \mathbf{1}_0 \) and \(-\mathbf{1}_1\) are simply denoted by 0 and 1.

Let \( \zeta_*(s_d, \ldots, s_1) \) and \( \zeta_{\text{un}}(s_d, \ldots, s_1) \) be the numbers indexed by \((\mathbb{N}^*)^d\), and obtained for \( s_d = 1 \) by regularizing multiple zeta values as, respectively, iterated sums and iterated integrals. There exists a simple formula relating \( \zeta_* \) and \( \zeta_{\text{un}} \). See [C] for details.

#### 2.2.1 Real multiple zeta values

Let \( \Phi \) be the element of \( \pi_0^{dR}(\mathbb{R}) \) corresponding, via the Betti-de Rham comparison, to the \textit{droit chemin}, \( \text{dch} \in \pi_0^B(\mathbb{R}) \), i.e. the element induced by the straight path from 0 to 1 ([DG] §5.16):

\[ \Phi := \text{comp}(\text{dch}) \in \pi_0^{dR}(X)(\mathbb{R}) \]

We have

\[ \zeta_{\text{un}}(s_d, \ldots, s_1) = (-1)^d \Phi|e^{s_d-1}e_1 \cdots e^{s_1-1}e_1] \]

(13)
2.2.2 $p$-adic multiple zeta values

The Frobenius action induces a map $F_* : 1\pi_0^{dR}(X)(\mathbb{Q}_p) \to 1\pi_0^{dR}(X)(\mathbb{Q}_p)$.

For $k \in \mathbb{N}^*$, let (as in [DG] §5.28 for $k = 1$ with other notations):

$$
\Phi_{p(-k)} = F_*^{-k}(110)(p^ke_0, p^ke_1) \in 1\pi_0^{dR}(X)(\mathbb{Q}_p)
$$

(14)

Let, for $d \in \mathbb{N}^*$, $(s_d, \ldots, s_1) \in (\mathbb{N}^*)^d$,

$$
\zeta_{p(-k)}(s_d, \ldots, s_1) = (-1)^d\Phi_{p(-k)}[e_0^{s_d-1}e_1 \ldots e_0^{s_1-1}e_1]
$$

(15)

The action of $F_*^{-k}$ on $1\pi_0^{dR}(\mathbb{Q}_p)$ is then described as

$$
F_*^{-k} : u \mapsto \Phi_{p(-k)}(p^{-k}e_0, p^{-k}e_1) \circ u(p^{-k}e_0, p^{-k}e_1)
$$

where, for $f, g$ points of $1\pi_0^{dR}$, $g \circ f = g.f(e_0,g^{-1}e_1g)$. This operation is the Ihara group law ; it enables to express the action of the pro-unipotent motivic Galois group $U^\omega$ on $1\pi_0^{dR}$.

In depth 1 the $\zeta_{p(-1)}$'s coincide with classical $p$-adic zeta values.

We will state our results for the families of $\zeta_{p(-k)}$'s, $k \in \mathbb{N}^*$. They remain true for the analogous $\Phi_{p(k)}$ with $k > 0$, which are the respective inverses of the $\Phi_{p(-k)}$ for $\circ$, for the associator $\Phi_{pKZ}^\ast$ defined by Furusho and its inverse for $\circ$ (see [J5]).

2.3 Algebraic relations

2.3.1 Drinfeld associator relations

The series $\Phi$ had been previously defined by Drinfeld. In [Dr], it is shown that $\Phi$ is an associator ; this has a meaning in terms of monoidal categories, and is equivalent to algebraic equations involving both $\pi^{dR}(M_{0,4})$ and $\pi^{dR}(M_{0,5})$. Let us recall from the description of $\eta_0\pi_0^{dR}(M_{0,4})$ in §2.1.2 that it involves formal variables $e_0, e_1, e_\infty$ such that $e_0 + e_1 + e_\infty = 0$. The analogous $\eta_0\pi_0^{dR}(M_{0,5})$, which exists for the same reasons, is the exponential of the pro-nilpotent Lie algebra generated by $e_{i,j}, 0 \leq i, j \leq 4$, with relations : $e_{ii} = 0$, $e_{ji} = e_{ij}$, $\sum_i e_{ij} = 0$, and $[e_{ij}, e_{kl}] = 0$ if $i, j, k, l$ are distinct.

The fact that $\Phi$ is an associator is the equivalent to the following set of equations, with, in this case, $m = 2i\pi :$ they are called, respectively, 2-cycle, 3-cycle or hexagon, and 5-cycle or pentagon equation.

$$
\Phi(e_0, e_1)\Phi(e_1, e_0) = 1
$$

$$
e^{\frac{m}{m}}e_0\Phi(e_\infty, e_0)e^{\frac{m}{m}}e_\infty\Phi(e_1, e_\infty)e^{\frac{m}{m}}e_1\Phi(e_0, e_1) = 1
$$

$$
\Phi(e_{23}, e_{34})\Phi(e_{40}, e_{01})\Phi(e_{12}, e_{23})\Phi(e_{34}, e_{40})\Phi(e_{01}, e_{12}) = 1
$$

Ünver has shown that $\Phi_{p(-1)}^\ast$ satisfies Drinfeld’s relations with $m = 0$ [U2] ; it implies the same for all versions of $p$-adic multiple zeta values. Furusho’s definition of a version $\Phi_{pKZ}^\ast$ is actually a $p$-adic analog of Drinfeld’s definition [F1]. It is conjectured that Drinfeld’s relations imply all algebraic relations over $\mathbb{Q}$ between multiple zeta values.
Moreover, Furusho has shown the general fact that the 5-cycle equation actually implies the 2-cycle and 3-cycle equations [F3].

2.3.2 Regularized double shuffle relations

2.3.2.a Integral shuffle relations

The shuffle graded Hopf algebra, $\mathcal{H}_m$

Let the ring monoid $\mathbb{Q}\langle e_0, e_1 \rangle$ over $\mathbb{Q}$ associated to the monoid of words in letters $e_0, e_1$, graded by the length of words (the "weight") ; it can be endowed with the shuffle product defined by

$$(u_1 \ldots u_r)\mathcal{W}(u_{r+1} \ldots u_{r+s}) = \sum_{\sigma \text{ permutation of } \{1, \ldots, r+s\}} u_{\sigma^{-1}(1)} \ldots u_{\sigma^{-1}(r+s)},$$

the deconcatenation coproduct $\Delta_{\text{dec}} : u_1 \ldots u_r \mapsto \sum_{k=0}^{r} u_1 \ldots u_k \otimes u_{k+1} \ldots u_r$; the counit $\epsilon$ equal to the augmentation morphism; and the antipode $S : u_r \ldots u_1 \mapsto (-1)^r u_1 \ldots u_r$. This defines the shuffle graded Hopf algebra $\mathcal{H}_m$.

The description of $\text{Spec}(\mathcal{H}_m)$ as a functor in §2.1.2 has involved the completion of the graded dual, $\overline{\mathcal{H}_m'}$ of $\mathcal{H}_m$; it is the non-commutative algebra of power series $\mathbb{Q}(\langle e_0, e_1 \rangle)$ equipped with the coproduct $\Delta_m$ defined by $\Delta_m(e_i) = e_i \otimes 1 + 1 \otimes e_i$, for $i = 0, 1$, and the corresponding counit and antipode.

Shuffle product and derivations $\partial_{e_i}$ and $\tilde{\partial}_{e_i}$

Let $\partial_{e_0}$ and $\partial_{e_1}$, (resp. $\tilde{\partial}_{e_0}$ and $\tilde{\partial}_{e_1}$) be the linear applications $\mathbb{Q}\langle e_0, e_1 \rangle \to \mathbb{Q}\langle e_0, e_1 \rangle$ defined on words $w$ by $\partial_{e_i}(e_i w) = w$ and $\partial_{e_i}(e_{1-i} w) = 0$ (resp. $\tilde{\partial}_{e_i}(w e_i) = w$ and $\tilde{\partial}_{e_i}(w e_{1-i}) = 0$). We have for all $w \in \mathbb{Q}\langle e_0, e_1 \rangle$,

$$w = e_0 \partial_{e_0}(w) + e_1 \partial_{e_1}(w) = \tilde{\partial}_{e_0}(w) e_0 + \tilde{\partial}_{e_1}(w) e_1$$

The shuffle product is the unique bilinear map which makes $\partial_{e_0}$ and $\partial_{e_1}$, (resp. $\tilde{\partial}_{e_0}$ and $\tilde{\partial}_{e_1}$) into derivations: this amounts to define $\mathfrak{m}$ by induction on the weight, by, for example for $\partial_{e_0}$ and $\partial_{e_1}$,

$$(e_i, w_1)\mathfrak{W}(e_i, w_2) = e_{i_1}(w_1 \mathfrak{W} e_{i_2} w_2) + e_{i_2}(e_{i_1} w_1 \mathfrak{W} w_2)$$

The shuffle equation for multiple zeta values

The iterated integral formula of multiple zeta values, combined with a way of expressing, a product of simplexes $\{(t_1, \ldots, t_n) \in \mathbb{R}^n \mid 0 \leq t_1 \leq \ldots \leq t_n \leq 1\} \times \{(t'_1, \ldots, t'_{n'}) \in \mathbb{R}^{n'} \mid 0 \leq t'_1 \leq \ldots \leq t'_{n'} \leq 1\}$ as a disjoint union of simplexes in $\mathbb{R}^{n+n'}$ up to sets of measure 0, implies the shuffle equation:

$$\zeta_m(w) \zeta_m(w') = \zeta_m(w \mathfrak{W} w')$$

For example, in depth (1,1), we have, for all $s, s' \in \mathbb{N}^*$:

$$\zeta_m(s) \zeta_m(s') = \sum_{k=0}^{s-1} \binom{s-1+k}{k} \zeta_m(s-k, s'+k) + \sum_{k=0}^{s'-1} \binom{s'-1+k}{k} \zeta_m(s'-k, s+k)$$

Various properties

We will use specifically that, for $f$ a point of $\text{Spec}(\mathcal{H}_m)$ we have $S^\vee(f) = f^{-1}$, and that for $h$
If \( f \) is a function on \( \mathbb{Q}(e_0, e_1) \) satisfying for all words \( w \), that \( f[wmc_0] = f[w]f[e_0] \), then, for all \( s_d, \ldots, s_1 \in \mathbb{N}^* \) and \( T, U_1, \ldots, U_d \) formal variables:

\[
\Phi_\ast(e_0) = \frac{e_0^{s_d-1}}{(1 - U_d e_0)^{s_d}} e_1 \cdots \frac{e_0^{s_1-1}}{(1 - U_1 e_0)^{s_1}} e_1 \frac{1}{1 - T e_0} \]

\[
= f\left[ \frac{e_0^{s_d-1}}{(1 - (U_d - T) e_0)^{s_d}} e_1 \cdots \frac{e_0^{s_1-1}}{(1 - (U_1 - T) e_0)^{s_1}} e_1 \right] e^f T \tag{17}
\]

2.3.2.b Series shuffle - "stuffle" - relations

The stuffle graded Hopf algebra \( \mathcal{H}_\ast \) (see [H] for more details)

Let the ring monoid \( \mathbb{Q}(\langle y_s \rangle_{s \in \mathbb{N}^*}) \) over \( \mathbb{Q} \) associated to the monoid of words in letters \( y_s \), graded by the length of words, and we denote by \( y_0 = 1 \) : it is endowed with the "stuffle" product \( \ast \), defined recursively by, for \( w_1, w_2 \) words, and \( s, s' \in \mathbb{N}^* \),

\[ y_s w_1 \ast y_{s'} w_2 = y_s(w_1 \ast y_{s'} w_2) + y_{s'}(y_s w_1 \ast w_2) + y_{s+s'}(w_1 \ast w_2) \]

the deconcatenation coproduct \( \Delta_{\text{dec}} \) relative to words in the \( y_s \)'s, the counit \( \epsilon \) equal to the augmentation morphism, and the antipode given by the two following formulas: let

\[ z_{s_d, \ldots, s_1} = \sum_{1 \leq i_1 \leq d} \sum_{1 \leq i_2 \leq d} \cdots \sum_{1 \leq i_d \leq d} y_{\sum_{i=1}^{i_1} s_i} \cdots y_{\sum_{i=1}^{i_d} s_i} \]

Then

\[ S(y_{s_d} \cdots y_{s_1}) = (-1)^d z_{s_1, \ldots, s_d} = \sum_{y_{s_d} \cdots y_{s_1} \ast w_1 \cdots w_1} (-1)^d w_1 \ast \cdots \ast w_1 \]

This defines the stuffle graded Hopf algebra \( \mathcal{H}_\ast \). The completed dual \( \mathcal{H}_\ast^{\vee} \) is the non-commutative algebra of series \( \mathbb{Q}(\langle \langle y_s \rangle \rangle_{s \in \mathbb{N}^*}) \), equipped with the (continuous) coproduct \( \Delta_\ast \), satisfying \( \Delta_\ast(y_n) = \sum_{k=0}^n y_k \ast y_{n-k} \).

The series shuffle relation for multiple zeta values

We identify a sequence \( (s_d, \ldots, s_1) \) to \( y_{s_d} \cdots y_{s_1} \) : the iterated sum formula for multiple zeta values, combined with a way of expressing a product \( \{(n_1, \ldots, n_d) \in (\mathbb{N}^*)^d | 0 < n_1 < \ldots < n_d \} \times \{(n_1', \ldots, n_d') \in (\mathbb{N}^*)^d | 0 < n_1' < \ldots < n_d' \} \) as a disjoint union of subsets of \((\mathbb{N}^*)^{d+d'}\) defined by strict or large inequalities between the \( n_i \)'s and the \( n_i' \)'s, implies:

\[ \zeta_\ast(w) \zeta_\ast(w') = \zeta_\ast(w \ast w') \]

For example, in depth \((1,1)\), for all \( s, t \in \mathbb{N}^* \):

\[ \zeta_\ast(s) \zeta_\ast(t) = \zeta_\ast(s + t) + \zeta_\ast(s, t) + \zeta_\ast(t, s) \]

Equivalently, the series \( \Phi_\ast = 1 + \sum_{s_d, \ldots, s_1 \geq 1} \zeta_\ast(s_d, \ldots, s_1) y_{s_d} \cdots y_{s_1} \) satisfies \( \Delta_\ast(\Phi_\ast) = \Phi_\ast \otimes \Phi_\ast \).
2.3.2.c Regularized double shuffle relations

The stuffle and shuffle relations, the fact that \( \zeta_m[e_1] = \zeta_s[e_1] = 0 \), and that for all \( w \in e_0 \mathbb{Q}(e_0, e_1)e_1 \) we have:

\[
\zeta(e_1 mw - e_1 * w) = 0
\]

(note that \( e_1 mw - e_1 * w \in e_0 \mathbb{Q}(e_0, e_1)e_1 \) form the regularized double shuffle relations. They imply conjecturally all algebraic relations between multiple zeta values, and are conjecturally equivalent to Drinfeld’s relations. Furusho, Besser and Jafari have shown that \( p \)-adic multiple zeta values satisfy the regularized double shuffle relations [BF], [FJ]. Furusho has shown that Drinfeld’s relations imply the regularized double shuffle relations [F3].

2.4 The map \( \Phi \mapsto \Phi^{-1}e_1\Phi \)

We now interpret geometrically the passage from \( (p\text{-}adic) \) multiple zeta values to their variants of theorem 1, and we explain a framework to deal with the numbers of theorem 1.1. Let us fix \( (s_d, \ldots, s_1) \in (\mathbb{N}^*)^d \) and \( w = e_0^{s_d-1}e_1 \ldots e_0^{s_1-1}e_1 \).

2.4.1 The numbers of (7), (11) and \( \Phi^{-1}e_1\Phi \)

We have used implicitly in the introduction that:

**Fact 2.1.** We have:

\[
\sum_{k=0}^{d} (-1)^{s_{k+1} + \ldots + s_d} \zeta_m(s_{k+1}, \ldots, s_d)\zeta_m(s_k, \ldots, s_1) = (-1)^d(\Phi^{-1}e_1\Phi)[e_1w]
\]

We can then define \( p \)-adic analogues of the numbers of (7) as \( (-1)^d((\Phi^{-1}_{p(-k)})^{-1}e_1\Phi_p(-k))[e_1w] \).

**Fact 2.2.** Similarly, we have:

\[
\sum_{n=0}^{d} (-1)^{\sum_{j=n+1}^{d} s_j} \left( \sum_{l_n, \ldots, l_d \geq 0} \prod_{i=n+1}^{d} \left( l_i + s_i - 1 \right) \zeta_p(-k)(s_{n+1+l_n+1}, \ldots, s_d+l_d) \right) \zeta_p(-k)(s_n, \ldots, s_1)
\]

\[
= \sum_{l_2 \geq 0} \left( (\Phi^{-1}_{p(-k)})^{-1}e_1\Phi_p(-k)[e_2w] = ((\Phi^{-1}_{p(-k)})^{-1}e_1\Phi_p(-k))[\frac{1}{1-e_0}e_1w] \right)
\]

The real analogues can be defined by regularization, using §2.5.

2.4.2 The motivic map \( \Phi \mapsto \Phi^{-1}e_1\Phi \) in the fundamental group of \( \mathbb{P}^1 \setminus \{0, 1, \infty\} \)

**Remark 2.3.** It is natural to consider that

\[
\Phi^{-1}e_1\Phi \in \text{Lie}(\sigma_0^{dR})(\mathbb{R}) \quad (\text{resp. } \Phi^{-1}e_1\Phi_p \in \text{Lie}(\sigma_0^{dR})(\mathbb{Q}_p)) \quad (18)
\]

More generally, any \( f \in yPi_x^{dR}(R) \) defines an application \( \text{Lie}(yPi_y^{dR})(R) \xrightarrow{\sim} \text{Lie}(xPi_x^{dR})(R) \) given by conjugation by \( f \), i.e.: \( u \mapsto f^{-1}uf \) - this follows from the multiplicativity of \( \Delta_m \) and the characterization in terms of \( y\sigma_\eta^{dR} \) and its Lie algebra in terms of \( \Delta_m \).

The \( \mathbb{Q} \)-algebras of coefficients \( \Phi \) and \( \Phi^{-1}e_1\Phi \) are the same: this is stated precisely and explicitly
as a lemma in §5.1.2. Here, let us explain simply that $\Phi$ is the unique grouplike series $f$ satisfying $f[e_1] = 0$ and $f^{-1}e_1f = \Phi^{-1}e_1\Phi$.

**Remark 2.4.** $\Phi \mapsto \Phi^{-1}e_1\Phi$ is injective.

Let $u \in R\langle\langle e_0, e_1\rangle\rangle$. Then : $u$ commutes to $e_1$ (resp. commutes to $e_1$ and is a grouplike series) if and only if $u \in R\langle\langle e_1\rangle\rangle$ (resp. is in $\exp(Re_1)$).

Indeed, let $w$ a word not of the form $e_1^n$, $n \geq 1$. We have $u[w] = (ue_1)[we_1] = (e_1u)[we_1] = u(\partial_{e_1}(w)e_1)$. Because of the hypothesis on $w$, this shows $u[w] = 0$ by induction on the index of nilpotence of $w$ relatively to $\partial_{e_1}$.

**Remark 2.5.** $\Phi \mapsto \Phi^{-1}e_1\Phi$ is motivic.

This means that the Ihara product defined by $v \circ u = u \cdot v = u(e_0, v^{-1}e_1v)$ on $1\pi_0^{dR}$ - which is known to preserve the double shuffle and associator relations - admits a version on $\text{Lie}(0\pi_0^{dR})$, namely $h_2 \circ h_1 = h_1(e_0, h_2)$, such that we have a commutative diagram where the horizontal arrows are $(f, g) \mapsto (f^{-1}e_1f, g^{-1}e_1g)$ and $h \mapsto h^{-1}e_1h$ :

$$
\begin{array}{ccc}
1\pi_0^{dR} \times 1\pi_0^{dR} & \rightarrow & \text{Lie}(0\pi_0^{dR}) \times \text{Lie}(0\pi_0^{dR}) \\
\downarrow \circ & & \downarrow \circ \\
1\pi_0^{dR} & \rightarrow & \text{Lie}(0\pi_0^{dR})
\end{array}
$$

(19)

In this paper, we will use systematically $\Phi^{-1}e_1\Phi$ and the map $\Phi \mapsto \Phi^{-1}e_1\Phi$, as in [J3] which is motivated by the previous facts. In [J3], we have suggested the following terminology :

**Definition 2.6.** We call $\Phi^{-1}e_1\Phi$ a symmetrized version of $\Phi$ and we call *symmetrized multiple zeta values* the numbers

$$
\zeta_{\text{sym}}(l; s_d, \ldots, s_1; r) = (\Phi^{-1}e_1\Phi)[e_0^l e_1 e_0^{s_d-1} e_1 \ldots e_0^{s_1-1} e_1 e_0^r]
$$

(20)

and in particular,

$$
\zeta_{\text{sym}}(s_d, \ldots, s_1) = \zeta_{\text{sym}}(0; s_d, \ldots, s_1; 0)
$$

Their "symmetry" property is that, using (16),

$$
\zeta_{\text{sym}}(l; s_d, \ldots, s_1; r) = (-1)^{s_1 + \ldots + s_d + l + r} \zeta_{\text{sym}}(r; s_1, \ldots, s_d; l)
$$

(21)

### 2.5 Formalism for completed variables

#### 2.5.1 Consequences of weight homogeneity

The action of $G_m$, as a subgroup of the motivic Galois group $G^\omega$, on $1\pi_0^{dR}$, can be seen as a map :

$$
T: \mathcal{O}(1\pi_0^{dR}) \hookrightarrow \mathcal{O}(1\pi_0^{dR})[T]
$$

defined by $w \mapsto T^{\text{weight}(w)}w$ where $T$ is a formal variable (this map associates to a word its orbit).

We denote it below by $w \mapsto T.w$. The conjecture of weight homogeneity of algebraic relations amounts to say that it descends to a map defined on the $\mathbb{Q}$-algebra of multiple zeta values. It
induces a map \( \widehat{\mathcal{O}}(\pi_0^{dR}) \hookrightarrow \mathcal{O}(\pi_0^{dR})[[T]] \), which fits into a natural commutative diagram:

\[
\begin{array}{ccc}
\mathcal{O}(\pi_0^{dR}) & \hookrightarrow & \mathcal{O}(\pi_0^{dR})[[T]] \\
\downarrow & & \downarrow \\
\mathcal{O}(\pi_0^{dR}) & \hookrightarrow & \mathcal{O}(\pi_0^{dR})[T]
\end{array}
\] (22)

Since each number \( ((\Phi_p^{-1})^{-1} e_1 \Phi_p^{-1})[\frac{1}{1-e_0} e_1 w] \) involves different weights, let us replace it by its \( \mathbb{G}_m \)-orbit

\[
((\Phi_p^{-1})^{-1} e_1 \Phi_p^{-1})[\frac{1}{1-e_0} T e_1 T \text{weight}(w) w] \in \mathbb{Q}_p[[T]]
\]

It does not alter the known algebraic relations, and conjecturally none. It is this object, and its complex analogue, that we will study.

It is also possible to substitute to \( T \) a complex or \( p \)-adic variable \( t \). This gives back some essentially known, partly classical, analytic functions and there exists variants in the \( p \)-adic case. The first application of it is to obtained a definition of the real analogues \( (\Phi^{-1} e_1 \Phi)[\frac{1}{1-e_0} e_1 w] \) of those numbers as mentioned below. Note that the pseudo-algebraic relations will then appear as algebraic equations depending on parameters, and which remain true for the values of these analytic functions and their meromorphic continuations.

### 2.5.2 Maps from shuffle Hopf algebras to their completions

The series shuffle Hopf algebra \( \mathcal{H}_* \) is also graded by the weight, and the analogous diagram as (22) holds.

The notation \( T^{-1} \) will refer below to the inverse action which multiplies words by \( T^{-\text{weight}} \).

**Notations : composition and conjugation by the weight action.** These will appear throughout the paper. For \( i \) a linear map from the series of integral shuffle Hopf algebra to its completion, we denote by

\[
\tilde{i}_T = T \circ i \\
i_T = T \circ i \circ T^{-1}
\]

For maps which are adapted to the Lie algebra of \( \mathfrak{g}_0^{dR} \), which will be signaled by the exponent \( \text{Lie} \), we will use for convenience the different notations

\[
\tilde{j}_T^{\text{Lie}} = \frac{1}{T} \times (T \circ j^{\text{Lie}}) \\
j_T^{\text{Lie}} = \frac{1}{T} \times (T \circ j^{\text{Lie}} \circ T^{-1})
\]

The multiplication by \( \frac{1}{T} \) corresponds to the shift of 1 in the weight existing between the coefficients of \( \Phi \) and the ones of \( \Phi^{-1} e_1 \Phi \).

**Maps for \( \pi_0^{dR} \)**

Let

\[
i : \mathcal{H}_m \longrightarrow \mathcal{H}_m, \quad w \longmapsto \frac{1}{1-e_0}
\]
The map \( \iota \) induces 
\[
\iota_* : \mathcal{H}_* \longrightarrow \widehat{\mathcal{H}}_* = w(e_0, e_1) \mapsto w\left(\frac{1}{1 + e_0}, \frac{1}{1 + e_0} e_1\right)
\]
as equal to \( \mathcal{H}_* \hookrightarrow \widehat{\mathcal{H}}_* \xrightarrow{pr} \widehat{\mathcal{H}}_* \) where the quotient \( pr \) corresponds to the relations arising from (17), satisfied by points \( f = f_0 e_0 = 0 : \) we have then \( f([w]) = f([\iota_*([w])]) \) for \( w \in \mathcal{H}_*. \) Then \( \iota_* \) is the unique morphism of concatenation algebras satisfying: \( \iota_* : y_s \mapsto \left(\frac{1}{1 + e_0}\right)^s y_s = \sum_{l \geq 0} (-1)^l \left(l + s - 1\right) y_{s+l}. \)

**Maps for Lie \( \pi_0^{dR} \)**

Let:
\[
\varphi_{\text{Lie}} : \mathcal{H}_m \longrightarrow \widehat{\mathcal{H}}_m = w(e_0, e_1) \mapsto \frac{1}{1 - e_0} e_1 \end{array}
\]

We have, for all point \( f \) of \( \pi_0^{dR} \) satisfying \( f[e_0] = 0, \) and \( w \in \mathcal{H}_*, \) that \( (f \circ \iota_*)[\varphi_{\text{Lie}}([w])] = (f \circ \iota_* f)[[w]]. \) Here \( \varphi : \mathcal{H}_* \rightarrow \mathcal{H}_* \) is the unique anti-morphism of concatenation algebras satisfying \( \varphi(y_n) = (-1)^n y_n \) and \( \phi \rightarrow \phi^{\text{inv}} \) is its dual.

The two families of numbers considered in §1.1 and §2.4 are
\[
\left(\zeta^{\text{inv}} \circ \iota_* [T, T] = 0\right) \zeta([w]) = (-1)^{\text{depth}([w])} \left(\Phi^{-1} e_1 \Phi\right)[e_1 w]
\]
\[
\left(\zeta_r^{\text{inv}} \circ \iota_* [T, T = 1]\right) \zeta_r^{(-1)} ([w]) = (-1)^{\text{depth}([w])} \left(\Phi_p^{(-1)} e_1 \Phi_p^{(-1)}\right)[e_1 w]
\]
We note that, for \( s \geq 0, \) and \( w = y_s w': \)
\[
\frac{1}{(s - 1)!} \left(\frac{\partial}{\partial T}\right)^{s-1} \iota_*^{\text{Lie}}(w') = \iota_* ([y_s] w)
\]

We will sometimes use:
\[
\iota_0^{\text{sym}} : \mathcal{H}_m \rightarrow \mathcal{H}_m \left[[U, T]\right], \quad w \mapsto \frac{1}{1 - Te_0} e_1 w \frac{1}{1 - U e_0}
\]

Let, for \( w \in \mathcal{H}_*, \)
\[
\zeta_{\text{sym}} (T, U) (w) = \sum_{l \geq 0} \sum_{r \geq 0} T^{l} U^{r} \zeta_{\text{sym}} (l; w; r), \text{ where } \zeta_{\text{sym}} (l; w; r) \text{ is as in (20).}
\]
The symmetry property (21) rewrites as
\[
\zeta_{\text{sym}} (T, U) = \zeta_{\text{sym}} (-U, -T) \circ \text{inv}
\]

**Maps for analytic functions**

Let also, for later reference, since the analytic functions evoked above can be expressed as having several variables:
\[
\iota_{\text{sym}}^{\text{an}} : \left\{w \in \mathcal{H}_* \mid \text{depth}([w]) = d\right\} \longrightarrow W e_1 \left[[T_1, \ldots, T_d]\right] \left[y_{s_d} \ldots y_{s_1} \mapsto \sum_{l_1, \ldots, l_d \geq 0} \prod_{i=1}^{d} T_i \left(l_i + s_i - 1\right) y_{s_d+l_d} \ldots y_{s_1+l_1}\right]
\]
The restriction of \( \iota_* \) to words of a given depth is equal to \( \iota_* [T, \ldots, T] \) of this depth.

### 2.5.3 Interpretation in terms of the shuffle algebra

The goal of this paragraph is to interpret some of the combinatorics of this paper in terms of the shuffle Hopf algebra.
Role of the words of weight 1

For all words \( w \) in \( \mathcal{H} \), we have:

\[
\frac{1}{1 + e_0} \, \mathfrak{m} \, w = w \left( \frac{1}{1 + e_0} e_0, \frac{1}{1 + e_0} e_1 \right) \frac{1}{1 + e_0}
\]

This implies a formula for \( \iota_* \):

\[
\iota_* : w \mapsto \left( \frac{1}{1 + e_0} \, \mathfrak{m} \, w \right) (1 + e_0)
\]

Let \( W_1 = T_0 e_0 + T_1 e_1 \), where \( T_0, T_1 \) are formal variables, i.e. \( W_1 \) is the universal word of weight one.

**Proposition 2.7.** \((1 - W_1)^{-1}\) is the universal non-zero solution to the following equation relative to \( x \), which has to be valid for all words \( w \):

\[
x \, \mathfrak{m} \, w (e_0, e_1) = w (xe_0, xe_1) x
\]

**Proof.** Taking \( w = e_0 \) and \( w = e_1 \) we see that (25) implies

\[
x \, \mathfrak{m} \, W_1 = x W_1 x
\]

Let \( x \) satisfying (26). For \( k \in \mathbb{N} \), let \( x_k \) be the weight \( k \) part of \( x \) satisfying (25); we have \( x = \sum_{k=0}^{\infty} x_k \). Considering the weight 1 part of (26) gives \( x_0 = 0 \) or \( x_0 = 1 \).

If \( x_0 = 0 \), we obtain by induction on \( k \), using (26), that \( x_0 = \ldots = x_{k-1} = 0 \); this is because of the implication \( y \, \mathfrak{m} \, W_1 = 0 \Rightarrow y = 0 \), which itself follows from Radford’s theorem that the shuffle algebra is a free polynomial algebra.

If \( x_0 = 1 \), we obtain by induction on \( k \), using (26), that \( x_k = x_1^k \); indeed the weight \( k + 1 \) part of (26) gives by induction that \( x_k \) is both of the form \( x_1^{k-1} u \) and \( u' x_1^{k-1} \), hence \( x_k = x_1^k \). This implies \( x = (1 - x_1)^{-1} \).

Conversely, a word of this form clearly satisfies (25), which is, in fine, equivalent to (26).

As a corollary, a significant part of the combinatorics of this paper remains true when we replace \((1 - T e_0)^{-1}\) by \((1 - W_1)^{-1}\), as we will explain in our next paper. For later reference, we note that:

\[
(1 - W_1)^{-1} = (1 - T_0 e_0)^{-1} + (1 - T_0 e_0)^{-1} T_1 e_1 (1 - W_1)^{-1}
\]

\[
= (1 - T_1 e_1)^{-1} + (1 - T_1 e_1)^{-1} T_0 e_0 (1 - W_1)^{-1}
\]

This implies a "base-change" formula between our maps \( j_T^{\text{lie}} \) and some variants obtained by replacing \( T e_0 \) by more general \( T_0 e_0 T_1 e_1 \). This change of variables contains as a particular case the exchange of \( e_0 \) and \( e_1 \).

The analog \( \iota_* T \) obtained by replacing \((1 + e_1)^{-1}\) by \((1 + e_0)^{-1}\), seen with a very different point of view, appears in [IKZ].

We hope that the point of view of this paper provides, ultimately, a natural geometric notion related to some of the operations of [IKZ].

*Shuffle exponential*
Let \( \exp_m : \mathcal{H}_m \rightarrow \mathcal{H}_m \) be the exponential map associated to the shuffle product. We have \( \frac{1}{1+e_0} = \exp_m(-e_0) \). Since \( \exp_m \) commutes with the action of \( T \):

\[
\frac{\partial}{\partial T}(T \frac{1}{1+e_0}) = -e_0 \quad \text{III} \quad (T \frac{1}{1+e_0})
\]

\[
(T.E_0) \quad \text{III} \quad (U \frac{1}{1+e_0}) = (T + U) \frac{1}{1+e_0}
\]

Using the shuffle equation (17), we obtain

\[
\ast_{T,U} = \ast_{T+U}
\]

### 2.5.4 Language of commutative variables

A slightly different presentation of some computations of multiple zeta values, introduced at first by Ecalle, uses that the \( \mathbb{Q} \)-vector space \( \mathcal{H}_m \) is isomorphic in a natural way to a space of commutative polynomials:

\[
\mathcal{H}_m \xrightarrow{\sim} \mathbb{Q}[[x_i]_{i \in \mathbb{N}^*}]
\]

\[
e_0^{s_0-1}e_1 \cdots e_0^{s_0-1}e_1 e_0^{s_0-1} \leftrightarrow x_0^{s_0-1} \cdots x_1^{s_1-1} x_0^{s_0-1}
\]

To a function \( f \) over \( \mathcal{H}_m \) one associates its commutative counterpart:

\[
f \leftrightarrow \left( f(x_d, \ldots, x_1, x_0) = \sum_{s_d, \ldots, s_1, s_0 \geq 1} f[e_0^{s_0-1} e_1 \cdots e_0^{s_0-1} e_1 e_0^{s_0-1}] x_d^{s_d-1} \cdots x_1^{s_1-1} x_0^{s_0-1} \right)_{d \in \mathbb{N}^*}
\]

Let us denote by \( \tilde{f} \) its restriction to \( \mathcal{H}_s \):

\[
\tilde{f}(x_d, \ldots, x_1) = \sum_{s_d, \ldots, s_1 \geq 1} f[e_0^{s_0-1} e_1 \cdots e_0^{s_0-1} e_1] x_d^{s_d-1} \cdots x_1^{s_1-1}
\]

The shuffle formula (17) can be expressed in this language, as well as the maps of §2.5.2: the most general statement is that:

\[
(\ast_{T_1, \ldots, T_d}) \vee : \tilde{f}(x_d, \ldots, x_1) \mapsto \tilde{f}(x_d + T_d, \ldots, x_1 + T_1)
\]

### 3 Double shuffle equations

We derive formal equations which are counterparts to standard families of relations, in the two parallel situations of formal series of multiple zeta values and finite multiple zeta values as explained in §1.2.

#### Theorem 1: in the multiple zeta values setting

Let \( R \) be a \( \mathbb{Q} \)-algebra.

1) **series shuffle**

If \( f : \mathcal{H}_s \rightarrow R \) satisfies the series shuffle relation, then so does \((f^{\text{inv}} \circ \ast_{T,U}) f \).

2) **integral shuffle**

Let \( f : \mathcal{H}_m \rightarrow R \) such that \( f[e_0^n] = 0 \) for all \( n \geq 1 \). We have an equivalence between:

- \( f \) is a grouplike series
- \( h = f^{-1} e_1 f \) is a Lie series
iii) $h$ satisfies (17) and one of the following conditions, and these are equivalent

a) For all $w, w' \in \mathbb{Q}(e_0, e_1)_{e_1}, s \in \mathbb{N}^*$, $h \left[ f_T^{\text{Lie}} \left( e_0^{-s} e_1 w w' w \right. \right. - w w \left. \left. (i_{s,T} \circ \text{inv}) (e_0^{-s} e_1 w') \right) \right] = 0.$

b) For all $u, w, w' \in \mathbb{Q}(e_0, e_1)_{e_1}, h \left[ f_T^{\text{Lie}} \left( uv \ w w' - w w \ (i_{s,T} \circ \text{inv})(u) w' \right) \right] = 0$

c) For all $w, w' \in \mathbb{Q}(e_0, e_1)_{e_1}, h \left[ f_T^{\text{Lie}} \left( w \ w w' - (i_{s,T} \circ \text{inv})(w) w' \right) \right] = 0$

2') symmetry

Let $h \in R(\langle e_0, e_1 \rangle)$ be a Lie series. Then we have :

$$h \left[ f_T^{\text{Lie}} \left( i_{s,T} \circ \text{inv} \right)(w) w' \right] = h \left[ f_T^{\text{Lie}} \left( i_{s,T} \circ \text{inv} \right)(w) w' \right]$$

3) comparison of regularizations

Let $f_s : \mathcal{H}_s \to R, f_m : \mathcal{H}_m \to R$ such that $(f_s, f_m)$ forms a solution to the regularized double shuffle equations. The families $(f_m^{-1} e_1 f_m) [f_T^{\text{Lie}}(w)]$ and $((f_m^{-1} i_{s,T} f_s)[w], w \in \mathcal{H}_s$, are equal modulo $f(e_0 e_1)$.

**Theorem 1 :** in the finite multiple zeta values setting

Finite multiple zeta values $\zeta_f(\{-k\})$ in $\prod_p \mathbb{Q}_p$ satisfy :

1) series shuffle

$$\zeta_f(\{-k\}) (w * w') = \zeta_f(\{-k\}) (w) \zeta_f(\{-k\}) (w')$$

2) integral shuffle

$$\zeta_f(\{-k\}) (w \ w') = \zeta_f(\{-k\}) ((i_{s,T=1} \circ \text{inv})(w') w)$$

2') symmetry

$$\zeta_f(\{-k\}) ((i_{s,T=1} \circ \text{inv})(w') w) = \zeta_f(\{-k\}) ((i_{s,T=1} \circ \text{inv})(w) w')$$

3) comparison of regularizations

The definition of finite multiple zeta values is independent of the choice of a regularization.

**Appendix to theorem 1** Let $\partial : \mathcal{H}_s \to \mathcal{H}_s$ be the linear map which sends 1 to 0 and each monomial $y_{a_1} \ldots y_{a_k}$ to $y_{a_{k+1}} \cdots y_{a_k}$.

Let $w, w' \in \mathcal{H}_s$, which we write under the form $w = y_{a_1} \partial w$ and $w' = y_{b_1} \partial w'$ ; then :

$$\zeta_f(\{-k\}) ((i_{s,T=1} \circ \text{inv})(w) w') = \left( \sum_{n=1}^{p^k-1} H_n(\partial w) H_{p^k-n}(w') \right)_{p \text{ prime}} = \left( \sum_{n=1}^{p^k-1} H_n(w) H_{p^k-n}(\partial(w')) \right)_{p \text{ prime}}$$

**Related work.** The formula $H_p(s_1, \ldots, s_d) = (-1)^{s_1+\cdots+s_d} H_p(s_1, \ldots, s_d) \mod p$ is standard and appears in Hoffman’s paper on multiple harmonic sums ([H], theorem 4.5). Its $p$-adic lift has been written explicitly by Rosen and named the "asymptotic reflexion theorem" ([Ro2]). This is the symmetry equation 2') for $w = 0$ or $w' = 0$. It follows from the shuffle relation, and, in the multiple zeta values setting, it also follows more specifically from the action of the antipode on the Hopf algebra of the fundamental group on its Lie algebra.

In the multiple zeta values setting, the $T = 0$ case of the formulation iii-c) of the shuffle relation has been proven by a very different method by Kaneko and Zagier which doesn’t involve $\Phi^{-1} e_1 \Phi$ neither the others formulations.

In the finite multiple zeta values setting, for $k = 1$, and for the reduction modulo $p$, the shuffle
relation is also known to Kaneko and Zagier, and proved by the same method. We had initially proven a slightly different result in [J3].

3.1 Series shuffle equation

3.1.1 Multiple zeta values setting

Proof. (of 1) of the theorem. It follows from the multiplicativity of $\Delta_*$ and the following lemma.

Lemma 3.1. i) $\text{inv} : \mathcal{H}_* \to \mathcal{H}_*$ is a anti-morphism of series shuffle algebras. 
ii) $\tau_* : \mathcal{H}_* \to \mathcal{H}_*[T]$ is a morphism of series shuffle algebras 

Proof. i) Clear. ii) The dual of $\iota_*$ is the concatenation algebra morphism $\iota'_*$ defined by

$$y_s \mapsto T^s \sum_{i=0}^{s-1} T^i \left( \frac{s-1}{l} \right)^{-1} y_{s-l} = T^s \sum_{i=1}^{s} T^{s-i} (-1)^i y_i \left( \frac{s-1}{s-l} \right)$$

We have $(\iota' \otimes \iota') \Delta_*(y_s) = 1 \otimes \iota'(y_s) + \iota'(y_s) \otimes 1 + \sum_{k=1}^{s-1} \iota'(y_k) \otimes \iota'(y_{s-k})$; the third term of this sum is

$$T^s \sum_{k=1}^{s-1} \left( \sum_{i=1}^{k} T^{k-i} \left( \frac{k-1}{k-l} \right)^{-1} y_l \right) \otimes \left( \sum_{i=1}^{s-k} T^{s-k-i} \left( \frac{s-k-1}{s-k-l'} \right)^{-1} y_{l'} \right)$$

$$= T^s \sum_{L=2}^{s} T^{s-L} (-1)^{s-L} \sum_{l,l' \geq 1 \atop l + l' = L} y_l \otimes y_{l'} \sum_{k \leq 1 \atop s-L+1 \leq L+1} \left( \frac{k-1}{k-l} \right) \left( \frac{s-k-1}{s-k-l'} \right)$$

For all $l, l'$ such that $l + l' = L$,

$$\sum_{l \leq k \leq s-L+l} \left( \frac{k-1}{k-l} \right) \left( \frac{s-k-1}{s-k-l'} \right) = \sum_{k' = 0}^{s-L} \left( \frac{k' + l - 1}{k'} \right) \left( \frac{s-L - k' + l' - 1}{s-L - k'} \right) = \left( \frac{s-L + l - 1}{s-L} \right)$$

This gives the result.

3.1.2 Finite multiple zeta values setting

Proposition 3.2. (trivial) Finite multiple zeta values satisfy the shuffle relation.

Geometric point of view. The series shuffle formula for multiple polylogarithms expresses as :

$$\text{Li}_w(z) \text{Li}_w(z') = \text{Li}_{w(z) * w'(z')}(z, z')$$

where $w(z) * w'(z')$ is the generalized stuffle product $\mathcal{H}_*(z) \times \mathcal{H}_*(z') \to \mathcal{H}_*(z, z')$ permitting to keep track of the powers of $z, z'$, and $\text{Li}_{w(z) * w'(z')}(z, z')$ are multiple polylogarithms on a simply connected subset of $\mathcal{M}_{0,5}(\mathbb{C})$ which contains $(0, 0)$.

This gives a linear expression of the sum of Taylor coefficients $\sum_{n=1}^{p_k-1} \sum_{m=1}^{p_k-1} c_n(\text{Li}_w)c_m(\text{Li}_{w'})$ which is the series shuffle relation of finite multiple zeta values.
3.2 Integral shuffle equation

3.2.1 Multiple zeta values setting

Reformulation of the fact that a series $f$ is grouplike in terms of $f^{-1}e_1f$

**Proposition 3.3.** Let $f \in R(\langle e_0, e_1 \rangle)$ such that $f[0] = 1$.

i) (Reminder from §2.5.4) If $f$ satisfies the shuffle equation, then $f^{-1}e_1f$ satisfies the shuffle equation modulo products.

Geometrically, $f \mapsto f^{-1}e_1f$ defines an application $\pi_0^d(R) \to \text{Lie}(\pi_0^d(R))$. Moreover if $f$ maps to $h$, then the preimage of $h$ is $\exp(Re_1)f$.

ii) Conversely, if $f^{-1}e_1f$ satisfies the shuffle equation modulo products and $f[e_1^n] = 0$ for all $n \geq 1$ then $f$ satisfies the shuffle equation.

**Proof.** The fact that $f^{-1}e_1f$ is a Lie series is equivalent to say that $\Delta_m(f)(f \otimes f)^{-1}$ commutes to $\Delta_m(e_1)$. Thus the proposition comes from the following lemma.

\begin{lemma}
Let $u \in R(\langle e_0, e_1 \rangle) \otimes R(\langle e_0, e_1 \rangle)$.
Then: $u$ commutes to $\Delta_m(e_1)$ if and only if $u \in R(\langle e_1 \rangle) \otimes R(\langle e_1 \rangle)$ .

**Proof.** The implication $\Leftarrow$ is clear. Let’s prove $\Rightarrow$. For $u$ in $R(\langle e_1 \rangle) \otimes R(\langle e_1 \rangle)$, we have :

\[
(\Delta_m(e_1)u)[w \otimes w'] = u[\partial_{e_1}(w) \otimes w'] + u[w \otimes \partial_{e_1}(w')]
\]

\[
(u\Delta_m(e_1))[w \otimes w'] = u[\tilde{\partial}_{e_1}(w) \otimes w'] + u[w \otimes \tilde{\partial}_{e_1}(w')]
\]

Let $(w, w') \in W \times W$, where $W$ is the set of words in $e_0, e_1$, with at least one among $w, w'$ not of the form $e_0^N$, $N \geq 0$ - we can assume that it is $w$ - we show that $u[w \otimes w'] = 0$.

\[
u[w \otimes w'] = u[\tilde{\partial}_{e_1}(we_1) \otimes w']
\]

\[
= (u\Delta_m(e_1))[we_1 \otimes w'] - u[we_1 \otimes \tilde{\partial}_{e_1}(w')] = (\Delta_m(e_1)u)[we_1 \otimes w'] - u[we_1 \otimes \tilde{\partial}_{e_1}(w')]
\]

\[
= u[\partial_{e_1}(w)e_1' \otimes w'] + u[we_1 \otimes \partial_{e_1}(w')] - u[we_1 \otimes \tilde{\partial}_{e_1}(w')]
\]

Because of the hypothesis on $w$, the index of nilpotence for $\partial_{e_1}$ is strictly smaller for $\partial_{e_1}(w)e_1$ than for $w$. The result then follows by induction on $m + m' + k$ where $m, n, k$, are respectively the smallest integers satisfying :

\[
\partial_{e_1}^m(w) = 0, \partial_{e_1}^m(w') = 0, (\tilde{\partial}_{e_1})^k(w') = 0
\]

\end{lemma}

Maps $i$ and the shuffle product

**Lemma 3.5.** i) For all $w, w' \in \mathbb{Q}\langle e_0, e_1 \rangle$ :

\[
\mathcal{J}_T^{\text{Lie}}(w) \triangledown \mathcal{J}_U^{\text{Lie}}(w') = \mathcal{J}_T^{\text{Lie}}(w \triangledown \mathcal{J}_U^{\text{Lie}}(w')) + \mathcal{J}_T^{\text{Lie}}(w) \triangledown w'
\]

ii) For all $w, w' \in \mathbb{Q}\langle e_0, e_1 \rangle$ :

\[
i_T(w) \triangledown i_U(w') = i_T(w) \triangledown (\tilde{\partial}_{e_1}(w) \triangledown i_U(w') + i_U(w) \triangledown \tilde{\partial}_{e_1}(w'))
\]
Proof. i) We use the definition of \( \mathfrak{m} \) via derivations:

\[
\partial_{e_1}(j^{\text{Lie}}_T(w) \mathfrak{m} j^{\text{Lie}}_U(w')) = w \mathfrak{m} j^{\text{Lie}}_U(w') + j^{\text{Lie}}_T(w) \mathfrak{m} w'
\]

\[
\partial_{e_0}(j^{\text{Lie}}_T(w) \mathfrak{m} j^{\text{Lie}}_U(w')) = (T + U)e_0\left(j^{\text{Lie}}_T(w) \mathfrak{m} j^{\text{Lie}}_U(w')\right)
\]

ii) is equivalent to i). \( \square \)

**Remark 3.6.** Applying \( \left( \frac{\partial}{\partial T} \right)^{s-1} \left( \frac{\partial}{\partial w} \right)^{s-1} \) to lemma 3.6 i) and using (23) we obtain, for all \( w, w' \in Q(e_0, e_1) e_1 \) and \( s, s' \geq 1 \):

\[
\iota_{*, T}(y_s) w \mathfrak{m} \iota_{*, U}(y_{s'}) w' = \sum_{0 \leq k_1 \leq s-1 \atop 0 \leq k_2 \leq s'-1} \binom{k_1 + k_2}{k_1} \iota_{*, T+U}(e_0^{k_1+k_2}e_1) \times \left[w^2 \left(\iota_{*, U}(y_{s-k_1}) w'\right) + \left(\iota_{*, T}(y_{s-k_2}) w\right) w'\right]
\]

(31)

**Corollary 3.7.** (rough version) The map \( \iota_{*, T} \mathfrak{m} \iota_{*, U} \) admits an expression close to a factorization \( \iota_{*, T+U} \circ f \), yet not of this type, which involves \( \iota_{*, T+U}, \iota_{*, T}, \iota_{*, U} \).

Proof. This follows by induction on the depth using the previous remark and the details are left to the reader. \( \square \)

We now prove that all possibles shuffles arise as coefficients of words of the form \( j^{\text{Lie}}_T(w) \):

**Proposition 3.8.** We have:

\[
-j^{\text{Lie}}_T\left((e_0^{s-1} e_1 w) \mathfrak{m} w' - w \mathfrak{m} (\iota_{*, T}(e_0^{s-1} e_1) w')\right) = \sum_{k=0}^{s-1} (e_0^k e_1 w) \mathfrak{m} ((-1)^{s-k} \iota_{*, T}(e_0^{s-1-k} e_1) w')
\]

(32)

Proof. Let \( r \) be the right-hand side. It suffices to show that \( \partial_{e_1}(r) = - \left((e_0^{s-1} e_1 w) \mathfrak{m} w' - w \mathfrak{m} (\iota_{*, T}(e_0^{s-1} e_1) w')\right) \) and \( \partial_{e_0}(r) = Tr \). The first equality is clear, and we have

\[
\partial_{e_0}(r) = \sum_{k=1}^{s-1} (e_0^{k-1} e_1 w) \mathfrak{m} \frac{1}{(T e_0 - 1)^{s-k}} e_0^{s-1-k} e_1 w' + \sum_{k=0}^{s-2} (e_0^k e_1 w) \mathfrak{m} \frac{1}{(T e_0 - 1)^{s-k}} e_0^{s-2-k} e_1 w' + (e_0^{s-1} e_1 w) \mathfrak{m} \frac{T}{T e_0 - 1} e_1 w'
\]

(33)

The sum of the two first terms equals

\[
\sum_{k=0}^{s-2} (e_0^k e_1 w) \mathfrak{m} \left[1 + \frac{1}{T e_0 - 1}\right] \frac{1}{(T e_0 - 1)^{s-k}} e_0^{s-2-k} e_1 w' = T \sum_{k=0}^{s-2} (e_0^k e_1 w) \mathfrak{m} \frac{1}{(T e_0 - 1)^{s-k}} e_0^{s-1-k} e_1 w'
\]

This and the third term are, respectively, the \( 0 \leq k \leq s - 2 \) terms and the \( k = s - 1 \) term of \( Tr \). \( \square \)
Remark 3.9. The \( T = 0 \) case is

\[
\sum_{k=0}^{s-1} (e_0^k e_1 w) \text{mod} \((-1)^{s-1-k} e_0^{s-1-k} e_1 w') = e_1 ((e_0^{s-1} e_1 w) \text{mod} w' + (-1)^{s-1} w \text{mod} e_0^{s-1} e_1 w'))
\]

In this case, the same proof is that \( \partial_{e_0}(r) \) gives directly a telescopic sum which vanishes.

Various formulations of "h is a Lie series" in terms of coefficients of the form \( J^{\text{Lie}}(w) \)

Let \( W \) be the set of words in the letters \( e_0, e_1 \) of the form \( w_1 \text{ or empty, i.e. the kernel of } \bar{\partial}_{e_0}. \)

Proposition 3.10. Let \( h : \mathcal{H}_m \to R \). We have an equivalence between

i) \( h \) satisfies the shuffle equation modulo products

ii) \( h \) satisfies (17) and, for all \( w, w' \in W \) and \( s \in \mathbb{N}^* : \)

\[
h(i_{s-1}^{k} e_0^{k} e_1 w) \text{mod}((-1)^{s-k} e_1^{s-1-k} e_0^{s-1-k} e_1 w') = 0 \]

Proof. It suffices to show that the linear combinations of shuffles of the statement generate all shuffles of words of \( W \). The coefficient of \( T^i \) is of the form:

\[
(e_0^{s-1} e_1 w) \text{mod}(e_0^{s-1} e_1 w') + \sum_{0 \leq s' < s} c_{s'}(e_0^{s-1} e_1 w) \text{mod}(e_0^{l+s-s'} e_1 w') = 0 \text{ with } c_{s'} \in \mathbb{Q}. \]

This shows that for all \( z, z' \in W \), \( z \text{mod} z' \) is a linear combination of the shuffles of the statement, by induction on the index of nilpotence of \( z \) for \( \partial_{e_0}. \)

We will apply the following lemma to \( F : w \mapsto h[\mathcal{J}^{\text{Lie}}(w)] \) and \( \bar{i} = i_{s-1} \circ \text{inv.} \)

Lemma 3.11. Let \( F \) a function \( W \to R \) and \( \bar{i} \), a function \( W \to W[[T]] \) satisfying, for all \( a, b \in W, \)

\( \bar{i}(a b) = \bar{i}(b) \bar{i}(a). \)

We have an equivalence between:

i) \( \forall s \in \mathbb{N}^* \), \( \forall w, w' \in W, F((e_0^{s-1} e_1 w) \text{mod} w') = F(w \text{mod}(e_0^{s-1} e_1 w')) \)

ii) \( \forall w, u, w' \in W, F((w u) \text{mod} w') = F(w \text{mod}(u w')) \)

iii) \( \forall w, w' \in W, F(w \text{mod} w') = F(u(w) \text{mod} w'). \)

Proof. i) \( \Rightarrow \) ii) : we write \( u \) as a concatenation of words of the form \( e_0^{s-1} e_1 \) and we iterate iii).

ii) \( \Rightarrow \) iii) : we take \( w = \emptyset. \)

iii) \( \Rightarrow \) i) : we apply iii) to each member of i).

The combination of all the lemmas of this paragraph gives the multiple zeta values side of theorem 1.

3.2.2 Finite multiple zeta values setting

Let us now prove the finite multiple zeta values side of theorem 1, and, at the same time, the appendix to theorem 1.

Proof. Multiple polylogarithms in one variable as in §2.1.2 satisfy the shuffle equation, namely
Li_{wmw'} = Li_w Li_{w'}. This can be translated on the sums of Taylor coefficients \( \sum_{0 < n < p^k} c_n \) at 0 as

\[
\sum_{n=1}^{p^k-1} c_n(Li_{wmw'}) = \sum_{0 < m < n < p^k} c_m(Li_w) c_{n-m}(Li_{w'}) = \sum_{0 < m < p^k} c_m(w) \sum_{0 < m' < p^k-m} c_m'(w')
\]

We obtain, for words \( w = y_{s_d} \partial w \) and \( w' = y_{t_d} \partial w' \),

\[
H_{p^k}(w_{wmw'}) = \sum_{n=1}^{p^k-1} \frac{H_n(\partial w)}{n^{s_d}} H_{p^k-n}(w') = \sum_{n=1}^{p^k-1} H_n(w) \frac{H_{p^k-n}(\partial w')}{(p^k-n)^{t_d}}
\]

We are reduced to:

**Proof.** (of Appendix 1 :) this follows from a change of variable \( m_i \mapsto p^k - m_i \) and the series expansion of \( (p^k - m_i)^{-t_i} = (\frac{-1}{m_i})^{t_i} (1 - \frac{p^k}{m_i})^{-t_i} \) in \( \mathbb{Z}_p \)

### 3.3 A special consequence of the integral shuffle equation : the symmetry relation

At the view of the statement of the theorem, the symmetry equation 2') follows instantly from 2), since the shuffle product is commutative : \( w_{wmw'} = w'_{mw} \).

Nevertheless, it is interesting to give a direct proof of 2') in the multiple zeta values side : it enlightens more specifically the role of the action of the antipode of \( \mathcal{O}_{0, \mathbb{R}} \) on its Lie algebra.

**Proof.** The number

\[
(-1)^{s_1 + \ldots + s_d} h \left[ \frac{1}{1 - Te_0} e_1 \frac{e_0^{s_1-1}}{(1 - Te_0)^{s_1}} e_1 \frac{e_0^{s_2-1}}{(1 - Te_0)^{s_2}} e_1 \ldots e_1^{t_d-1} e_1 \right]
\]

is equal, because of (16), to

\[
= (-1)^{t_1 + \ldots + t_d'} h \left[ e_1 e_0^{t_1-1} e_1 \ldots e_0^{t_d-1} e_1 \frac{e_0^{s_1-1}}{(1 + Te_0)^{s_1}} e_1 \ldots e_0^{s_1-1} e_1 \right]
\]

and then, because (17) to

\[
(-1)^{t_1 + \ldots + t_d'} h \left[ \frac{1}{1 - Te_0} e_1 \frac{e_0^{t_1-1}}{(1 - Te_0)^{t_1}} e_1 \frac{e_0^{t_2-1}}{(1 - Te_0)^{t_2}} e_1 \ldots e_0^{t_d-1} e_1 \right]
\]

In the finite multiple zeta values side, the proof of the symmetry relation 2') is included in the one of appendix to theorem 1.

### 3.4 Relation between the two regularizations

We now prove 3) of theorem 1. We take \( (f_* , f_m) \) as in the statement.

**Proof.** Let \( f^-\) be the function \( \mathcal{H}_* \rightarrow R \) equal to the restriction of \( f \) to \( \mathbb{Q}(e_0, e_1)e_1 \) identified to \( \mathcal{H}_* \).
Then ([C], equation (162)) we have \( f_\ast = \Lambda \tilde{f}_m \) where
\[
\Lambda = \exp \left( \sum_{n \geq 2} (-1)^{n-1} \frac{f[y_n]}{n} y_1^1 \right)
\]
This implies that \( f_\ast \Lambda^{inv} \tilde{f}_m = \Lambda^{inv} \tilde{f}_m \), and \( \Lambda^{inv} \Lambda = \exp \left( \sum_{m \in \mathbb{N}^\ast} -2(f[y_{2m}]/2m)y_1^{2m} \right) \).
Finally, for all \( m \in \mathbb{N}^\ast \), we have \( f[y_{2m}] \in \mathbb{Q} f[y_2]^m \). This proves that
\[
 f_\ast \Lambda^{inv} \tilde{f}_m \equiv \Lambda^{inv} \tilde{f}_m \mod f[y_2]
\]
hence the result. \( \square \)

This proof implies also a relation between the two regularizations as follows:

**Corollary 3.12.** We have, where, for \( n \in \mathbb{N}^\ast \), \( \gamma_n \in \mathbb{Q} : \)
\[
(f_\ast \circ \iota_{s,T}) f_s \equiv \sum_{u,u' \in \mathcal{H}_s \cap \mathbb{N}^\ast} \gamma_n f[y_2]^n (\tilde{f}_m^{inv} \circ \iota_{s,T}) [u] \tilde{f}_m [u']
\]

**Proof.** Indeed, the left-hand side is equal to \( ((f_\ast^{inv} \circ \iota_{s,T})(\Lambda^{inv} \circ \iota_{s,T}) \Lambda \tilde{f}_m)[w] \) and, since \( \Lambda \) involves only powers of \( y_1 \), we have \( \Lambda^{inv} \circ \iota_{s,T} = \Lambda^{inv} \). \( \square \)

### 3.5 Other consequences of the shuffle equation

In the literature one can find many special identities among multiple zeta values, which have been proved by elementary ways, and which have been retrieved later as consequences of the double shuffle equations. In this paragraph we give an analog for finite multiple zeta values of the "cyclic sum formula" for multiple zeta values, which is an example of such identities.

The following equalities are obtained by comparing two ways of expressing the inverse of the formal series \( \text{Li}(z) \in z \pi_0^{HR} (\mathbb{C}) \); one involves the antipode \( S \) of the groupoid \( \pi_0^{HR} \), and the other one combines it with the expression of the usual inversion of a formal series in \( \mathbb{C}((e_0, e_1)) \).

**Notation:** For \( w \in \mathcal{H}_s \), let \( \iota_{s,T}(w)[T^k] \) be the coefficient of \( T^k \) in \( \iota_{s,T}(w) \).

**Proposition 3.13.** We have, for all \( d \in \mathbb{N}^\ast \), \((s_d, \ldots, s_1) \in (\mathbb{N}^\ast)^d : \)
\[
(-1)^{s_d + \ldots + s_1} \zeta_{f(-)}(1, s_1, \ldots, s_{d-1}) [T^{s_d-1}] \\
= -\zeta_{f(-)}(s_d, \ldots, s_1) - \sum_{n=1}^{d} \sum_{t_n=0}^{s_n-1} \zeta_{f(-)}(1, s_1, \ldots, s_n+1) [T^{s_n}], s_n-t_n+1, s_{n-1}, \ldots, s_1)
\]

(34)

**Proof.** We have \( \text{Li}^{-1}[e_0^{s_d-1} e_1 \ldots e_0^{s_1-1} e_1] = (-1)^{s_d + \ldots + s_1} \text{Li}[e_0^{s_0} \ldots e_0^{s_0-1}] \) and, on the other
Proof.

We translate this equality on the sums of Taylor coefficients of order \( n \), and we apply the appendix to theorem 1 to express the products of finite multiple zeta values which occur.

**Proposition 3.14.** We have an alternative symmetry formula:

\[
(−1)^{s_1+\ldots+s_d} \zeta_f(−k)(s_1,\ldots,s_d) − \zeta_f(−k)(s_d,\ldots,s_1) = \sum_{n=1}^{d} \sum_{t_n=1}^{s_n} \zeta_f(−k)\left(t_n,T=1(s_d,\ldots,s_{n+1},s_n−t_n),t_n+1,s_{n−1},\ldots,s_1\right)
\]

**Proof.** This is obtained by translating on the Taylor coefficients of order \( p^k \) the equality:

\[
(−1)^{\sum_{i=1}^{d} s_i+1} \text{Li}[e_0^{s_d−1} e_1 \ldots e_0^{s_1−1} e_1] = \text{Li}[e_1 e_0^{s_d−1} e_1 \ldots e_0^{s_1−1} e_1] = −\text{Li}[e_1 e_0^{s_d−1} e_1 \ldots e_0^{s_1−1} e_1]
\]

\[
− \sum_{n=1}^{d} \sum_{t_n=0}^{s_n−1} (−1)^{\sum_{i=1}^{d} s_i−t_i} \text{Li}[e_0^{s_n−1−t_i e_1 e_0^{s_{n+1}−1} e_1 \ldots e_0^{s_1−1} e_1}] \text{Li}[e_1 e_0^{s_n−1} e_1 \ldots e_0^{s_1−1} e_1]
\]

**Remark 3.15.** Dividing by \( p^{s_d+\ldots+s_1} \) and taking the reduction modulo \( p \), we obtain a proof of \( H_p(s_d,\ldots,s_1) \equiv (−1)^{s_d+\ldots+s_1} H_p(s_1,\ldots,s_d) \mod p \) which does not involve the change of variable \( (n_d,\ldots,n_1) \rightarrow (p−n_d,\ldots,p−n_1) \).

**Corollary 3.16.** Cyclic sum formula: For all \( s_d,\ldots,s_1 \in \mathbb{N}^* \), \( p \) prime:

\[
(−1)^{s_1+\ldots+s_d} \sum_{i=1}^{d} s_i H_p(s_1,\ldots,s_i+1,\ldots,s_d)
\]

\[
\equiv \sum_{n=1}^{d} \sum_{t_n=0}^{s_n−1} H_p(t_n,T=0(s_d,\ldots,s_{n+1},s_n−t_n),t_n+1,1,s_{n−1},\ldots,s_1) \mod p
\]

**Proof.** Compare the terms of weight \( s_d+\ldots+s_1 \) and \( s_d+\ldots+s_1+1 \) in the symmetry formula 2') of theorem 1 and proposition 3.14; divide by \( p^{s_d+\ldots+s_1+1} \) and take the reduction modulo \( p \).

**Remark 3.17.** Taking limits \( z \rightarrow 1 \) in the proofs of propositions 3.13 and 3.14 gives relations among multiple zeta values which are consequences of the shuffle relation.

4 **Associator equations in \( M_{0,4} \)**

Let \( C(x, y) = \log(e^x e^y) \in \mathbb{Q}([x,y]) \) be the Campbell-Haussdorff series. Let \( \tilde{C}(x, y) = C(x, y) − x − y \in \mathbb{Q}[[x,y]] \); \( \tilde{C} \) is made of iterated brackets of \( x \) and \( y \).
Let $R$ be a $\mathbb{Q}$-algebra, $f : \mathcal{H}_m \to R$, and $m \in R^\times$. Let $v_m(f)$ be the automorphism of $\text{Spec}(\mathcal{H}_m)/R$ given by:

$$
e^{e_1} \mapsto f(e_0, e_1)^{-1} e^{m e_1} f(e_0, e_1)
$$

$$
e^{m/2} \mapsto e^{(m/2) e_0} f(e_0, e_\infty)^{-1} e^{m e_\infty} f(e_0, e_\infty) e^{-(m/2) e_0}
$$

The first part of Theorem 2 is a new point of view on some known results.

**Theorem 2:** in the multiple zeta values setting

**residues equation on** $M_{0,4}$

Assume that $f$ satisfies Drinfeld’s 2-cycle and 3-cycle relation with parameter $m$ (§2.3.1). Then, with the notations above:

$$e_0 + f^{-1} e_1 f + e_{(m/2) e_0} f^{-1} (e_0, e_\infty) e_\infty f(e_0, e_\infty) e^{-(m/2) e_0} + \frac{1}{m} \text{Lie}(v_m(f))(\tilde{C}(m e_1, m e_\infty)) = 0 \quad (39)
$$

Assume that $f$ satisfies Drinfeld’s 2-cycle and 3-cycle relations with parameter 0. Then

$$e_0 + f(e_0, e_1)^{-1} e_1 f(e_0, e_1) + f(e_0, e_\infty)^{-1} e_\infty f(e_0, e_\infty) = 0 \quad (40)
$$

Let $h = f^{-1} e_1 f$ and $\tilde{h} = (-1)^{\text{depth} h}$; equation (40) is equivalent, respectively, to

i) for every $w \neq e_0$ in $\mathcal{H}_m$, $\tilde{h}[w] = -\tilde{h}[w(e_0 + e_1, -e_1)]$

ii) for every word $w \neq e_0$ in $\mathcal{H}_m$,

$$\tilde{h}[\text{Lie}_T(w)] = -\tilde{h}[\text{Lie}_T(\sum_{z \in \{y_{s_d} \cdots y_{s_1}, s_d, \ldots, s_1 \geq 0\}} (-1)^{\text{depth}(z)} z.w(e_0 + e_1, -e_1))] \quad (41)
$$

We call (39) and (40) **residues equations**, since they express the action of a group of special automorphisms on the relation of vanishing of the sum of residues, i.e.

$$e_0 + e_1 + e_\infty = 0
$$

**Theorem 2:** in the finite multiple zeta values setting

**relation arising from** $(z \mapsto \frac{z}{z-T})_*$ **on** $M_{0,4}$

For all $w \in \mathcal{H}_*$ we have:

$$\zeta_{f(-)}(w(e_0 + e_1, -e_1)) = -\sum_{z \in \{y_{s_d} \cdots y_{s_1}, s_d, \ldots, s_1 \geq 0\}} (-1)^{\text{depth}(z)} \zeta_{f(-)}(z, w) \quad (42)
$$

**Remark.** We also have, for $w = y_{s_d} \cdots y_{s_1}$,

$$\zeta_{f(-)}(w(e_0 + e_1, -e_1)) = \zeta_{f(-)}(y_{s_d+1} - y_{s_d}) (y_{s_{d-1}} \cdots y_{s_1}) \ast \frac{1}{1+y_1} \quad (43)
$$

and

$$\zeta_{f(-)}(w(e_0 + e_1, -e_1)) = \zeta_{f(-)}(w + (w \ast \frac{1}{1+y_1})) \quad (44)
$$

**Appendix to theorem 2.** For all $w$ and for all $T \in \mathbb{N}$, we have:

$$\zeta_{f(-)}(y_{T+1}(w \ast \frac{1}{1+y_1})) = -\sum_{d \geq 1, x_d \geq 1, z \in \{y_{s_d} \cdots y_{s_1}, s_d, \ldots, s_1 \geq 0\}} (-1)^{\text{depth}(z)} \zeta_{f(-)}(z, w)$$

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hence the equality between (42), (43), (44). However, those three formulas are quite different combinatorially. The particular simplicity of the new statement (42) enables us to lift congruences quite explicitly in §6.2.

**Related work.** The fact that (39) follows from the 2-cycle and 3-cycle equations has been showed by Alekseev-Enriquez-Torossian [AET].

We prove equation (40) in the general setting of mixed Tate motives over $\mathbb{Z}$, and apply it also to "single-valued" periods (see below). A slightly different statement for this equation, in the particular case of the $p$-adic Drinfeld associator, appears in [U1].

In the finite multiple zeta values setting, the $T = 0$ case has been proved by Hoffman in [H], using Newton series, and named the "duality theorem": $H_p(w(e_0, e_1)) = -H_p(w(e_0 + e_1, -e_1)) \mod p$. A $p$-adic lift of Hoffman's theorem has been given by Rosen [Ro2] and named the "asymptotic duality theorem". This is (44) for $k = 1$. We give an alternative proof to Hoffman's theorem using multiple polylogarithms.

### 4.1 Equation related to $z \mapsto \frac{z}{z-1}$ in $M_{0,4}$

#### 4.1.1 Multiple zeta values setting

**Introduction**

Let the six natural integral tangential base-points on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ :

$$B = \{ \tilde{1}_0, -\tilde{1}_0, \tilde{1}_1, -\tilde{1}_1, \tilde{1}_\infty, -\tilde{1}_\infty \}$$

where $\tilde{1}_0$ (resp. $\tilde{1}_1$, $\tilde{1}_\infty$) is the tangential base-point of length 1 at 0 (resp. 1, $\infty$) which points to 1 (resp. $\infty$, 0). Let $\mathbb{P}_{\mathbb{Z}}^d$ the set of the schemes $y \pi_x^d(\mathbb{P}^1 \setminus \{0, 1, \infty\})$ with $x, y \in B$, equipped with its structure of groupoid. The subset of elements of $\text{Aut}(\mathbb{P}^1)$ which preserve $\{0, 1, \infty\}$ identifies to $S_3$ and acts simply transitively on $B$. Its induced action on $\mathbb{P}_{\mathbb{Z}}^d$ is given, for $s \in S_3$ and $f(e_a, e_b) \in b \pi^d(\mathbb{P}^1)$, by $s(f) = f(e_{s(a)}, e_{s(b)}) \in s(b) \pi^d(\mathbb{P}^1)$. We denote such an element $s$ by $\sigma_{s(0), s(1)}$.

We have $\sigma_{0\infty} = (z \mapsto \frac{z}{z-1})$, which induces the isomorphism $\rho_{0\infty}^d : \mathcal{O}(\infty \pi^d_0) \sim \mathcal{O}(\pi^d_0)$ given by :

$$\sigma_{0\infty}^d : \mathcal{O}(\infty \pi^d_0) \rightarrow \mathcal{O}(\pi^d_0) \quad w(e_0, e_1) \mapsto w(e_0 - e_1, -e_1)$$

Since the formulae relating the associators to multiple zeta values involve a $(-1)^{\text{depth}}$ factor, we also consider $r_1 \circ \sigma_{0\infty}^d \circ r_\infty$, where $r_x : \mathcal{O}(\pi^d_0) \rightarrow \mathcal{O}(\pi^d_0)$, $(e_0, e_1) \mapsto (e_0, -e_1)$

$$r_1 \circ \sigma_{0\infty}^d \circ r_\infty : \mathcal{O}(\infty \pi^d_0) \rightarrow \mathcal{O}(\pi^d_0) \quad w(e_0, e_1) \mapsto w(e_0 + e_1, -e_1)$$

We will deal with special automorphisms of the restricted groupoid $\mathbb{P}_{\mathbb{Z}}^d$.

**Drinfeld and Kashiwara-Vergne setting**

A result which is equivalent to equation (39) for $m = 1$ is stated in several slightly different ways in [AET], in terms of an automorphism of the abstract free pro-nilpotent Lie algebra with two generators.
In order to rewrite it in terms of algebraic relations, and relate it to the setting of mixed Tate motives as below, let us reformulate it slightly differently.

We take the notations of the statement of theorem 2.

**Proposition 4.1.** (Alekseev-Enriquez-Torossian) Suppose that \( f \) satisfies Drinfeld’s 2-cycle and 3-cycle relations as written in §2.3.1. Then we have

\[
v_f(m)(e^{me_1}e^{me_\infty}) = e^{-me_0}
\]

**Proof.** (corresponding to [AET], §5.2, First proof) The 2-cycle relation implies

\[
v_f(m)(e^{me_1}e^{me_\infty}) = f(e_0, e_0) e^{m e_1} f(e_0, e_1) e^{(m/2)e_0} f(e_\infty, e_0) e^{me_\infty} f(e_0, e_\infty) e^{-(m/2)e_0}
\]

The 3-cycle relation implies

\[
e^{(m/2)e_1} f(e_0, e_1) e^{(m/2)e_0} f(e_\infty, e_0) e^{(m/2)e_\infty} = f(e_\infty, e_1)
\]

We obtain

\[
v_f(m)(e^{me_1}e^{me_\infty}) = f(e_0, e_0) e^{(m/2)e_1} f(e_\infty, e_1) e^{(m/2)e_\infty} f(e_0, e_\infty) e^{-(m/2)e_0}
\]

and we apply again the 3-cycle equation. \( \square \)

**Corollary 4.2.** (equivalent to work of [AET]) We have an equivalency between \( v_f(m)(e^{me_1}e^{me_\infty}) = e^{-me_0} \) and

\[
e_0 + f^{-1}e_1 f + e^{(m/2)e_0} f^{-1}(e_0, e_\infty) e^{e_\infty} f(e_0, e_\infty) e^{-(m/2)e_0} \frac{1}{m} \text{Lie}(v_f(m))(\hat{C}(me_1, me_\infty)) = 0
\]

**Proof.** Indeed, this last equality is \( \text{Lie}(v_f(m))(e_0 + e_1 + e_\infty) = 0. \)

Explicitly, \( \text{Lie}(v_f(m)) \) is

\[
e_0 \mapsto e_0 + \frac{1}{m} v_f(m)(\hat{C}(me_1, me_\infty))
\]

\[
e_1 \mapsto f(e_0, e_1)^{-1} e_1 f(e_0, e_1)
\]

\[
e_\infty \mapsto e^{(m/2)e_0} f(e_0, e_\infty)^{-1} e_\infty f(e_0, e_\infty) e^{-(m/2)e_0}
\]

\( \square \)

**Mixed Tate motives over \( \mathbb{Z} \) setting**

We now explain the \( m = 0 \) case.

Facts explained in [DG] §5 imply that the action of the pro-unipotent motivic Galois group \( U^\omega \) on \( \pi^{dR}_B(\mathbb{P}^1 - \{0, 1, \infty\}) \) factorizes through the following group of automorphisms:

**Definition 4.3.** Let \( \text{Aut} \) be the set of families of automorphisms of elements of \( B\pi_B^{dR} \) which are compatible with:

1) the local monodromies \( \pi^{dR}_B(T_x - \{0\}) \simeq \mathbb{G}_m \) \( \rightarrow \pi^{dR}_B(\mathbb{P}^1 - \{0, 1, \infty\}) \) where \( x = 0, 1, \infty \), and the trivial action on \( \pi^{dR}_B(\mathbb{G}_m) \) with \( y, z \in \{1, -1\} \).
2) the groupoid structure of $\pi^{dR}$. 
3) the automorphisms induced by the homographies of $\mathbb{P}^1$ which stabilize $\{0, 1, \infty\}$.

The condition of triviality of 1) follows from that $\xi_x \pi(x x m \mu)_{x}$, for $x$ a tangential base-point at 0 and $\xi$ a root of unity is a trivialized (Kummer) $\mathbb{Q}(1)$-torsor ([DG], §5.4).

**Lemma 4.4.** The application $a \mapsto a(\mu 1, 1 1 1 0)$ induces a isomorphism of schemes $\text{Aut} \rightarrow \mathcal{D}_{23}$ where $\mathcal{D}_{23}$ is the subscheme of $\pi_0^{dR}$ defined by the 2-cycle equation and the 3-cycle Drinfeld equation with parameter 0.

**Proof.** (Roughly.) The hypothesis 1) of definition 4.3 amounts to say that we can make as if we had three base-points $0, 1, \infty$, and canonical paths $y 1 x$ between them. 2) rewrites then as some cycle relations. 3) is then equivalent to formulas for all $a(1 1 1 0)$ in terms of $a(1 1 1 0)$, and the only cycles we have to consider are then the ones of the statement.

**Lemma 4.5.** Let $f$ be a point of $\pi_0^{dR}$ of the form $a(\mu 1, 1 1 1 0)$. Then $f$ satisfies the residue relation with parameter 0.

**Proof.** Apply the action of $\text{Aut}$ to $e_0 + e_1 + e_\infty = 0$.

**Application to Frobenius-twisted multiple zeta values.**

Drinfeld’s associator $\Phi$ is not a $\mathbb{R}$-point of $\text{Aut}$: it only induces, by moding out by $\zeta(2)$, a $\mathbb{Z}/\zeta(2)\mathbb{Z}$-point of $\text{Aut}$ where $\mathbb{Z}$ is the $\mathbb{Q}$-algebra of multiple zeta values. It is then natural to ask oneself whether $\text{Aut}$ has a natural $\mathbb{R}$-point or, at least, $\mathbb{C}$-point.

On the other hand, by definition, $\text{Aut}$ is made of automorphisms of the de Rham realization of $\pi^{mot}$, instead of involving the Betti-de Rham, or rigid-de Rham comparison. Natural examples of such automorphisms are given by the complex and $p$-adic Frobenius and their associated periods, which are "de Rham-de Rham" periods: for their definitions and study, see [F2] and, [Br1], [Br2].

The Frobenius at infinity $F_\infty$ induces an action on $\pi^{dR}(\mathbb{P}^1 - \{0, 1, \infty\})(\mathbb{C})$. This permits to define a variant of the Drinfeld associator: $\Phi^-(= (F_\infty)_*(1 1 0))$, which turns out to be $\Phi$ twisted by $F_\infty$, namely

$$\Phi = \Phi^-. \Phi(\mu 1, \mu(\mu^2)^{-1})$$

The complex and $p$-adic cases are different. Since $-1$ is a root of unity while $p$ is not, the $\mathbb{Q}$-algebra of coefficients of $\Phi^-$ is strictly included in the one of coefficients of $\Phi$, whereas in the $p$-adic case, the versions expressing the Frobenius-invariant path and expressing the action of Frobenius give conjecturally the same algebras of coefficients.

Similarly, $F_\infty$ is involutive whereas the $p$-adic Frobenius has infinite order.

**Remark 4.6.** $\Phi^-$ satisfies the motivic algebraic relations of $\Phi$ (since, in the sense of the Ihara group law, $\Phi^-= \Phi(\mu 1, \mu(\mu^2)^{-1})$ plus other relations; we think that these other relations are generated by the fact that $F_\infty$ is an involution, which implies $F_\infty^2(1 1 0) = 1 1 0$, which can be rewritten in a compact way involving only $\Phi^-$ as:

$$\Phi^-. \Phi^-(\mu 1, \mu(\mu^2)^{-1}) = 1$$
**Fact 4.7.** The following is clear geometrically: the $p$-adic Frobenius iterated $k$ times divided by $p^k$ (or $p^{-k}$ depending on the convention) and the complex Frobenius divided by $-1$ are points of $\text{Aut}$.

**Corollary 4.8.** We have

$$e_0 + ((\Phi^-)^{-1}e_1\Phi^-)(e_0, e_1) + ((\Phi^-)^{-1}e_1\Phi^-)(e_0, e_\infty) = 0$$

$$e_0 + (\Phi_p^{-1}(\Phi_p(-k)))(e_0, e_1) + (\Phi_p^{-1}(\Phi_p(-k)))(e_0, e_\infty) = 0$$

(In [U1] this last equation for a version $\Phi_p$ appears, stated slightly differently in terms of a variant $\Phi_p(\infty)$ (and not $\Phi_p(e_0, e_\infty)$)).

**Other remarks**

**Remark 4.9.** Residues equation and value of $\zeta(2k)$

In [Dr], Drinfeld proves that the 3-cycle equation permits to retrieve that $\zeta(2) = \frac{\pi^2}{6}$. Here, the residues equation for a series $f$ and $m = 0$ implies that $f[e_0^{2k-1}e_1] = 0$ for all $k \in \mathbb{N}^*$. Indeed, it implies that $f(e_0, e_\infty)$ vanishes in depth $\leq 1$ (see §5.1.2), hence for all $s \in \mathbb{N}^*$,

$$(1 + (-1)^s)f[e_0^{s-1}e_1] = (f^{-1}e_1f)[e_1e_0^{s-1}e_1] = 0,$$

i.e. for all $s' \in \mathbb{N}^*$, $\zeta(2s') = 0 \mod \zeta(2)$. A similar computation holds when $m = 2i\pi$.

**Remark 4.10.** The residues equation can also be written in a more symmetric, thus more conceptual way as:

$$\forall i \in \{0, 1, \infty\}, \sum_{j \in \{0, 1, \infty\}} (\Phi^{-1}e_1\Phi)(e_i, e_j) \equiv 0 \mod \zeta(2) \quad (46)$$

**The map $j^{\text{Lie}}_T$ and automorphisms of $\mathbb{P}^1 - \{0, 1, \infty\}$**

**Fact 4.11.** We have $(1 - T(e_0 + e_1))^{-1} = (1 - Te_0)^{-1} + (1 - Te_0)^{-1}e_1(1 - T(e_0 + e_1))^{-1}$, which implies:

$$(r_1 \circ \sigma_0^{-1} \circ r_\infty) \circ j^{\text{Lie}}_T = -j^{\text{Lie}}_T \left(-\text{id} + \circ (r_1 \circ \sigma_0^{-1} \circ r_\infty) \circ j^{\text{Lie}}_T\right)$$

This gives the translation on the numbers $(\Phi^{-1}e_1\Phi)[j^{\text{Lie}}_T(w)]$ of the residues equation of theorem 2.

**Translation of the 2-cycle and 3-cycle relation in terms of Lie series**

**Proposition 4.12.** Let $f$ be a point of $\mathbb{R}^{dR}_0$, satisfying $f[e_0] = f[e_1] = 0$. There is an equivalence between:

i) $f$ satisfies the 2-cycle and 3-cycle relation with parameter 0

ii) $f$ satisfies both

$$(f_0f^{-1})(e_0, e_1) = (f^{-1}e_1f)(e_1, e_0)$$

$$e_0 + (f^{-1}e_1f)(e_0, e_1) + (f^{-1}e_1f)(e_0, e_\infty) = 0$$

Note that $f_0f^{-1}$ is a point of $\text{Lie}(\mathbb{R}^{dR}_1)$.

**Proof.** i) $\Rightarrow$ ii) : we have shown that i) implies the second equation of ii), and the first one clearly follows from the 2-cycle equation.
\[ e_1 + (f^{-1} e_1 f)(e_1, e_0) + (f^{-1} e_1 f)(e_1, e_\infty) = 0 \]
\[ e_1 + f(e_0 f^{-1}(e_0, e_1) + f(e_0)(e_0, e_\infty)e_\infty f(e_0, e_\infty)(e_0, e_1) = 0 \]

We recall the injectivity of the map \( u \mapsto u^{-1} e_1 u \) defined on the set of grouplike series satisfying \( u[e_1] = 0 \) (Remark 2.4). In particular \( (f e_0 f^{-1})(e_0, e_1) = (f^{-1} e_1 f)(e_1, e_0) \) is actually equivalent to the 2-cycle equation, and the equality \( (f^{-1} e_1 f)(e_1, e_\infty) = f f^{-1}(e_0, e_\infty)e_\infty f(e_0, e_\infty) f^{-1} \) is equivalent to the 3-cycle relation. This shows the result. \( \square \)

### 4.1.2 Finite multiple zeta values setting

**Proof.** (Appendix to theorem 2) There are two expressions in terms of finite multiple zeta values of

\[ \sum_{n=1}^{p^k-1} \frac{H_n(w)}{n^k} (-1)^{n+1} \binom{p^k}{n} \]  

(47)

Firstly, as is pointed out in [Ro2] for \( k = 1 \), we have:

\[ (-1)^{n+1} \binom{p^k}{n} = \frac{p^k}{n} \prod_{i=1}^{n-1} \left( 1 - \frac{p^k}{i} \right) = \frac{p^k}{n} \sum_{l \geq 0, 0 < i_1 < \ldots < i_l < n} \left(-\frac{p^k}{i_1} \right) \cdots \frac{p^k}{i_1 \cdots i_l} = \frac{p^k}{n} \sum_{l \geq 0} (-p^k)^l H_n(1, \ldots, 1) \]  

(48)

This gives a first expression of (47) by applying the series shuffle equation.

Secondly, since \( \binom{p^k}{n} = \binom{p^k}{p^k-n} \), we also have

\[ (-1)^{n+1} \binom{p^k}{n} = (-1)^{p^k} \frac{p^k}{p^k-n} \sum_{l \geq 0} (-p^k)^l H_{p^k-n}(1, \ldots, 1) \]  

(49)

and applying the appendix of the theorem 1 gives another expression of (47). \( \square \)

**Proof.** (Theorem 2) 1) The automorphism \( (\sigma_\infty)_* \) is horizontal with respect to the connection \( \nabla_{KZ} \). Denoting by \( \tilde{L}_i = (-1)^{\text{depth}} \tilde{L}_i \) (i.e. changing \( \frac{dz}{z} \) to \( \frac{dz}{z-w} \)) this gives, for all \( w \):

\[ \tilde{L}_i-w(\frac{z}{z-1}) = \tilde{L}_i(\varphi_{e_0+e_1, -e_1}) (z) \]  

(50)

For a word \( w = y_{s_d} \partial w \), this is translated on the sum of Taylor coefficients \( \sum_{n=1}^{p^k-1} c_n \) at 0 as:

\[ H_{p^k}(w(e_0 + e_1, -e_1)) = \sum_{n=1}^{p^k-1} \frac{H_n(\partial w)}{n^{s_d}} (-1)^n \binom{p^k-1}{n} = \sum_{n=1}^{p^k-1} \frac{H_n(\partial w)}{n^{s_d}} (-1)^n \frac{p^k-n}{p^k} \binom{p^k}{n} \]

The proof of appendix 2 gives the result (42) and the first remark (43).

2) Variant : in [H], Hoffman proves with a very different method that

\[ H_{p^k}(w(e_0 + e_1, -e_1)) = \sum_{n=1}^{p^k} (-1)^{n+1} \binom{p^k}{n} H_n(w) \]  

(51)
This can be retrieved by translating the equation (50) for \( \tilde{L}_{e_1w} \) instead of \( \tilde{L}_w \), on the Taylor coefficients at 0 of order \( p^k \), instead of the sum \( \sum_{n=1}^{p^k-1} c_n \).

The proof of appendix 2 gives the result (42) and the second remark (44). This proof of (44) using Hoffman’s lemma is the one appearing in [Ro2].

5 Phenomena of depth drop

Let \( w \in \mathcal{H}_m \) of depth \( d \). The number \( \Phi[w] \) is said to admit a depth drop if it can be expressed as a \( \mathbb{Q} \)-polynomial of coefficients \( \Phi[w'] \) with \( w' \) of depth \( \leq d - 1 \). The first example, discovered by Euler, is that the double zeta values of odd weight are actually of depth one.

The numbers \( (\Phi^{-1} e_1 \Phi)[e_1 w e_1] \) are defined in terms of multiple zeta values of depth \( \leq d \). It is fundamental that they can be expressed in terms of multiple zeta values of depth \( \leq d - 1 \). We explain and reprove it by the results of §4 (§5.1), and we give for it new explicit formulas and algorithms.

Using again §4, we do the same thing for a second depth drop result (§5.2), which is that multiple zeta values of weight \( s \) and depth \( d \) such that \( d - s \) is odd can be expressed in depth \( \leq d - 1 \).

For the simplicity of the formulas, we place ourselves modulo \( \zeta(2) \). Because of the nature of the results, it implies depth drop in the general case, non moded out by \( \zeta(2) \). Only the explicit formulas are simpler modulo \( \zeta(2) \); the full formulas can also be obtained by the same methods.

5.1 A depth drop for symmetrized multiple zeta values

5.1.1 Statement and proof

Let \( f \) and \( f_\infty \), points of \( \text{Spec}(\mathcal{H}_m) \) satisfying \( f[e_0] = f[e_1] = f_\infty[e_0] = f_\infty[e_1] = 0 \) and:

\[
e_0 + f^{-1} e_1 f + f_\infty^{-1} e_\infty f_\infty = 0
\]

We don’t need to assume that \( f_\infty = f(e_0, e_\infty) \).

**Notation 5.1.** For \( u \) a point of \( \text{Spec}(\mathcal{H}_m) \), for \( R \) a ring, let \( R.u(\text{depth} \leq d) \), respectively \( R.u(\text{weight} n, \text{depth} \leq d) \), be the \( R \)-module generated by polynomials, with constant coefficient equal to zero, of numbers \( u[w_i] \) where \( w_i \) are non-empty words of depth \( \leq d \), respectively of weight \( n \) and depth \( \leq d \).

**Notation 5.2.** For the lisibility of the formulas let

\[
e_0^{t_0, \ldots, t_0} = e_0^{t_0} \cdot e_1 \cdot e_1 e_0^{t_0}
\]

The words \( e_0^{s_d-1, \ldots, s_0-1, 0} \) can thus also be denoted by \( e_0^{s_d-1, \ldots, s_0-1, e_1} \).

**Theorem 3-a:** depth drop of symmetrized multiple zeta values Let \( w = e_0^{s_d-1, \ldots, s_1-1} e_1 \in \mathcal{H}_m \), with \( s_d, \ldots, s_1 \in \mathbb{N}^* \).

1) The coefficients \( (f^{-1} e_1 f)[e_1 w] \) are in \( \mathbb{Z}.f(\text{depth} \leq d-1) \).

Let \( z = \partial_{e_1}(w) \) ; then

\[
(f^{-1} e_1 f)[e_1 w] \equiv f[z e_0] + f^{-1} e_0 z \equiv f[z e_0] - f[e_0 z] \mod \mathbb{Z}.f(\text{depth} \leq d-2)
\]
\[ (-1)^{\sum_{i=1}^{d}s_i} \sum_{l_1, \ldots, l_{d-1} \geq 0 \atop l_1 + \ldots + l_{d-1} = s_d} \prod_{i=1}^{d} (-1)^{l_i} \binom{l_i + s_i - 1}{l_i} f[e_0^{s_1+l_1-1, \ldots, s_d-1+l_{d-1}-1} e_1] \]

\[ + \sum_{l_1', \ldots, l_d' \geq 0 \atop l_1' + \ldots + l_d' = s_1} \prod_{i=2}^{d} (-1)^{l_i'} \binom{l_i' + s_i - 1}{l_i'} f[e_0^{s_d+l_d-1, \ldots, s_2+l_2-1} e_1] \mod Z.f \text{ (depth \leq d-2)} \] (54)

2) A full formula for \((f^{-1}e_1 f)[e_1 w]\) as an element of \(Z.f \text{ (depth \leq d-1)}\) can be written using:

i) the expression of \((f^{-1}e_1 f)[e_1 w]\) in terms of \(f_\infty\):

\[ (f^{-1}e_1 f)[e_1 w] = \sum_{k=0}^{d} (-1)^{s_k+1+\ldots+s_d} f_\infty[e_0^{s_k+1-1, \ldots, s_d-1} e_1] f_\infty[e_0^{s_k+1-1, \ldots, s_1-1} e_1] \]

\[ + \sum_{k=1}^{d-2} \sum_{j=1}^{s_k+1-1} (-1)^{s_k+1+\ldots+s_d-j-1} f_\infty[e_0^{s_k+1-j-1, \ldots, s_d-1} e_1] f_\infty[e_0^{s_k+1-j, \ldots, s_1-1} e_1] \] (55)

ii) the expression of \(f_\infty\) in all depth \(d'\) in terms of coefficients of \(f\) of depth \(\leq d' - 1\), given by induction on \(d'\) as in lemma 5.3.

**Related work** The fact that symmetrized multiple zeta values admit this depth drop has been announced by Zagier in 2013, and proved apparently with a different method.

Yasuda has proven ([Y1] §1-5) that, if \(f\) satisfies the double shuffle equations, then, in the case where \(d-s\) is odd, the part 1) of theorem 3 holds. This is the content of the key proposition 3.1 of [Y1].

To prove this theorem, we prove the following more general fact which is subjacent to it: the equation (52) gives a way to express fully \(f\) and \(f_\infty\) (and \(f^{-1}e_1 f\)) in terms of each other.

**Notation**: \(h = f^{-1}e_1 f\).

Thus we prove the clearer statement:

**Appendix to theorem 3**: comparison of \(f\), \(f_\infty\) and \(h\)

We have, for all \(n \in \mathbb{N}^*\), \(d \in \mathbb{N}^*\):

\[ Z.f \text{ (depth \leq d)} = Z.f_\infty \text{ (depth \leq d+1)} = Z.(f^{-1}e_1 f) \text{ (depth \leq d+1)} \]

More explicitly, for all \(d \in \mathbb{N}^*\), \((s_{d+1}, s_d, \ldots, s_1) \in (\mathbb{N}^*)^{d+1}\):

\[ h[e_0^{s_{d+1}, s_d-1, \ldots, s_1-1} e_1] \equiv f_\infty[e_0^{s_{d+1}-1, \ldots, s_1-1} e_1] \]

\[ \equiv (-1)^{s_1+\ldots+s_d+1} f[e_0^{s_{d+1}-1, \ldots, s_d-1, s_{d+1}}] \mod Z.f_\text{ (depth \leq d-1)} \] (56)

\[ -f[e_0^{s_d, s_{d-1}-1, \ldots, s_1-1} e_1] \equiv f_\infty[e_0^{s_d-1, \ldots, s_1-1} e_1 e_1] \mod Z.f_\text{ (depth \leq d-1)} \] (57)

**Proof.** Induction on the depth using the following more precise lemma. We use that, because both \(f\), \(f_\infty\) \(h\) all are points \(F\) of Spec(\(\mathcal{H}_w\)) which satisfy \(F[w e_0] = F[w e_1] = 0\) for all \(w \in \mathcal{H}_w\), the
associated modules of coefficients as above are generated by coefficients of the form \( F[e_0 \ldots e_1] \), i.e. \( y_{t_0} \ldots y_{t_1} \) with \( t_{d'} \geq 2 \).

The full comparison is given indeed by :

**Lemma 5.3.** We have, for all \( d \in \mathbb{N}^* \), \((s_d+1, s_d, \ldots, s_1) \in (\mathbb{N}^*)^{d+1} : \)

\[
h[e_0^{s_d+1}, e_0^{s_d-1}, \ldots, e_0^{s_1-1} e_1] = f_{\infty}[e_0^{s_d+1}, e_0^{s_d-1}, \ldots, e_0^{s_1-1} e_1] + \sum_{k=1}^{d} f_{\infty}^{-1}[e_0^{s_d+1}, e_0^{s_d-1}, \ldots, e_0^{s_k-1} e_0^{s_k}] f_{\infty}[e_0^{s_k-1}, \ldots, e_0^{s_1-1} e_1] + \sum_{k=2}^{d-1} \sum_{t_k=0}^{s_k-2} f_{\infty}^{-1}[e_0^{s_d+1}, e_0^{s_d-1}, \ldots, e_0^{s_k-1}, e_0^t e_0^{s_k}] f_{\infty}[e_0^{s_k-2}, e_0^{s_k-1}, \ldots, e_0^{s_1-1} e_1] = f_{\infty}^{-1}[e_0^{s_d+1}, e_0^{s_d-1}, \ldots, e_0^{s_1-1} e_1] + \sum_{k=2}^{d} f_{\infty}^{-1}[e_0^{s_d+1}, e_0^{s_d-1}, \ldots, e_0^{s_k-1} e_0^{s_k}] f[e_0^{s_k-1}, \ldots, e_0^{s_1-1} e_1] \quad (58) \]

\[
h[e_0^{s_d-1}, e_0^{s_d-1}, \ldots, e_0^{s_1-1} e_1 e_1] = f_{\infty}[e_0^{s_d-1}, e_0^{s_d-1}, \ldots, e_0^{s_1-1} e_1 e_1] + \sum_{k=1}^{d-1} f_{\infty}^{-1}[e_0^{s_d+1}, e_0^{s_d-1}, \ldots, e_0^{s_k-1}, e_0^t e_0^{s_k}] f_{\infty}[e_0^{s_k-1}, \ldots, e_0^{s_1-1} e_1 e_1] + \sum_{k=2}^{d-2} \sum_{t_k=0}^{s_k-2} f_{\infty}^{-1}[e_0^{s_d+1}, e_0^{s_d-1}, \ldots, e_0^{s_k-1}, e_0^t e_0^{s_k}] f_{\infty}[e_0^{s_k-2}, e_0^{s_k-1}, \ldots, e_0^{s_1-1} e_1 e_1] = f_{\infty}^{-1}[e_0^{s_d-1}, e_0^{s_d-1}, \ldots, e_0^{s_1-1} e_1 e_1] + \sum_{k=2}^{d} f_{\infty}^{-1}[e_0^{s_d+1}, e_0^{s_d-1}, \ldots, e_0^{s_k-1} e_0^{s_k}] f[e_0^{s_k-1}, \ldots, e_0^{s_1-1} e_1 e_1] \quad (59) \]

**Proof.** We write the coefficients \( h[e_0^{s_d+1}, e_0^{s_d-1}, \ldots, e_0^{s_1-1} e_1] \) and \( h[e_0^{s_d-1}, e_0^{s_d-1}, \ldots, e_0^{s_1-1} e_1 e_1] \) in terms of \( f \) and \( f_{\infty} \) using the assumption (52). We use that \( f^{-1} \equiv -f \mod \text{products of coefficients of} \ f. \)

**5.1.2 Details in depth 1 and 2**

In depth 1 and 2 the formulas are

\[
(f^{-1} e_1 f)[e_0^{s_1} e_1 e_0^{s_1-1} e_1] \equiv 0 \mod \zeta(2)
\]

\[
(f^{-1} e_1 f)[e_1 e_0^{s_2-1} e_0^{s_1-1} e_1] \equiv (-1)^{s_1} \left( s_1 + s_2 \right) f[e_0^{s_1} e_0^{s_2-1} e_1] \mod \zeta(2)
\]

Parallel results for finite multiple zeta values in \( A \) are now well-known :

It is very standard, and older than the theory of finite multiple zeta values, that :

\[
H_p(s) \equiv \left\{ \begin{array}{ll} -1 & \text{mod } p \text{ if } p - 1 \mid s \\ 0 & \text{mod } p \text{ otherwise} \end{array} \right. \quad (60)
\]

**Facts 5.4.** (Hoffman, Zagier, Zhao... )
For \( p > s_1 + s_2 \),
\[
H_p(s_2, s_1) \equiv (-1)^{s_1} \left( s_1 + s_2 \right) \frac{H_p(1, s_1 + s_2 - 1)}{s_1 + s_2} \equiv (-1)^{s_1} \left( s_1 + s_2 \right) \frac{B_{p-s_1-s_2}}{s_1 + s_2} \mod p
\]  
(61)

Hence
\[
\zeta_A(s) = 0
\]
(62)
\[
\zeta_A(s_2, s_1) = (-1)^{s_1} \left( s_1 + s_2 \right) \frac{\zeta_A(1, s_1 + s_2 - 1)}{s_1 + s_2} = (-1)^{s_1} \left( s_1 + s_2 \right) \left( \frac{B_{p-s_1-s_2}}{s_1 + s_2} \right)_p
\]
(63)
\[
\zeta_A(1, 2s - 1) = 0
\]
(64)

Proof. (following [H], 6.1) : let \( Q_u(T) = \sum_{k=1}^{u+1} \binom{u+1}{k} B_{u+1-k} T^k \) be the Bernoulli polynomial satisfying \( Q_u(n) = \sum_{0 \leq m \leq n} m^u \) for all \( n \in \mathbb{N}^* \). For \( p - 1 > s \), we have in \( \mathbb{F}_p \) that
\[
H_p(s_2, s_1) = \sum_{0 < n_2 < p} \frac{1}{n_2^2} Q_{p-1-s_1}(n_2) = \sum_{l=1}^{p-s_1} \frac{1}{p-s_1} \left( \frac{p-s_1}{l} \right) B_{p-s_1-l} H_p(s_2 - l)
\]
If \( p > s_2 + s_1 \), the only non-zero term is \( l = s_2 \).
\( \square \)

5.2 A depth drop for multiple zeta values with a parity condition

5.2.1 Statement and proof

We take again a point \( f \) of \( \text{Spec}(\mathcal{H}_m) \) satisfying \( f[e_0] = f[e_1] = 0 \).

It is well-known that if \( f \) satisfies the double shuffle equations, then, if \( n, d \in \mathbb{N}^* \) are such that \( d - n \) odd, we have \( f_{(\text{depth} \leq d)}^{(\text{weight} \leq n)} \subset \mathbb{Q}.f_{(\text{depth} \leq d-1)}^{(\text{weight} \leq n)} \) (see §1.2 for the references). We now prove that the 2-cycle and 3-cycle equations with \( m = 0 \) imply the same facts. The nature of the result implies that taking \( m = 0 \), i.e. modding out by \( \zeta(2) \), is sufficient for obtaining the result for multiple zeta themselves.

Theorem 3-b : depth drop for multiple zeta values with weight-depth odd

Suppose that \( f \) satisfies the 2-cycle and 3-cycle equation with parameter 0. Then, in the case where \( d - s \) odd, we have \( \mathbb{Z}.f_{(\text{depth} \leq d)}^{(\text{weight} \leq n)} \subset \mathbb{Z}.f_{(\text{depth} \leq d-1)}^{(\text{weight} \leq n)} \)

One can also write explicit formulas for it as in theorem 3-a.

Lemma 5.5. \( \mathbb{Z}.(f_0 f^{-1})_{(\text{depth} \leq d)}^{(\text{weight} \leq n+1)} = \mathbb{Z}.f_{(\text{depth} \leq d)}^{(\text{weight} \leq n)} \)

Proof. We recall that \( f^{-1} \equiv -f \) modulo products of coefficients of \( f \) and we have
\[
(f_0 f^{-1})[e_0^{s_0}, e_{s_0-1}, \ldots, e_{s_1-1}] = f_0^{-1}[e_0^{s_0-1}, \ldots, e_{s_1-1}, e_1] + \sum_{k=1}^{d-1} \sum_{t_k=0} \sum_{s_k-2} f^{-1}[e_0^{s_0-2}, \ldots, e_{s_k-1}, s_k-2-t_k].f_0^{t_k-1}s_{d-1-1}, \ldots, e_{s_1-1}] (65)
\]
\( \square \)
**Notation.** For $u$ a function on $\mathcal{H}_u$, $R$ a ring, we denote by $R.u^{\text{weight } n}_{\text{depth } \leq d}$ the $R$-module defined in a similar way as $R.u^{\text{weight } n}_{\text{depth } \leq d}$ of §5.1.1, but with words of depth $\geq d$ instead of $\leq d$.

**Lemma 5.6.** $\mathbb{Z}.u(e_0, e_1)^{\text{weight } n}_{\text{depth } \geq n-d} = \mathbb{Z}.u(e_1, e_0)^{\text{weight } n}_{\text{depth } \leq d}$.

**Proof.** Clear. □

**Lemma 5.7.** Assume the equation $e_0 + (f^{-1}e_1f)(e_0, e_1) + (f^{-1}e_1f)(e_0, e_\infty) = 0$. Then, if $d$ is even, $\mathbb{Z}。(f^{-1}e_1f)^{\text{weight } n}_{\text{depth } \geq d} \subset \mathbb{Z}。(f^{-1}e_1f)^{\text{weight } n}_{\text{depth } \geq d+1}$.

**Proof.** For all words $w \neq e_0$, we have $(f^{-1}e_1f)[w(e_0 - e_1, -e_1)] = -(f^{-1}e_1f)[w(e_0, e_1)]$; it implies that $((-1)^{\text{depth}(w)} + 1)(f^{-1}e_1f)[w] \in (f^{-1}e_1f)^{\text{weight } n}_{\text{depth } \geq d+1}$.

**Proof.** (of the theorem 3-b) We have shown in §4.1.1 that the 2-cycle plus 3-cycle equations rephrase as

$$(fe_0f^{-1})(e_0, e_1) = (f^{-1}e_1f)(e_1, e_0)$$

$$e_0 + (f^{-1}e_1f)(e_0, e_1) + (f^{-1}e_1f)(e_0, e_\infty) = 0$$

Applying the lemma 5.7, and combining the first of these equations with the lemma 5.6, we obtain, if $d$ is even:

$$\mathbb{Z}。(f^{-1}e_1f)^{\text{weight } n}_{\text{depth } \leq n-d} \subset \mathbb{Z}。(f^{-1}e_1f)^{\text{weight } n}_{\text{depth } \leq n-d-1}$$

Applying then the lemma 5.5 we have, if $d$ is even:

$$\mathbb{Z}。f^{\text{weight } n-1}_{\text{depth } \leq n-d} \subset \mathbb{Z}。f^{\text{weight } n-1}_{\text{depth } \leq n-d-1}$$

and, of course, $n - 1 - (n - d)$ is odd if and only if $d$ is even. □

### 5.2.2 Applications

The theorem 3-a and theorem 3-b combined imply:

**Corollary 5.8.** If $f$ satisfies the 2-cycle and 3-cycle relations with $m = 0$, then the numbers $(f^{-1}e_1f)[e_1e_0^{s_2-1}...e_1^{-1}]$, when $d - (s_1 + ... + s_d)$ is even, are in $\mathbb{Z}。f^{\text{weight } s_1+...+s_d}_{\text{depth } \leq d-2}$.

**Proof.** Because of theorem 2, the 2-cycle and 3-cycle relations imply the assumption (52) of theorem 3-a. Because of theorem 3-a, the numbers $(f^{-1}e_1f)[e_1e_0^{s_2-1}...e_1^{-1}]$ with $d - (s_1 + ... + s_d)$ even are in $\mathbb{Z}。f^{\text{weight } s_1+...+s_d}_{\text{depth } \leq d-1}$ and since $d - 1 - (s_1 + ... + s_d)$ is thus odd, we can apply theorem 3-b.

To obtain this result, one can also replace theorem 3-b by the following result, which is implicit in [H].

**Lemma 5.9.** Let $(h(s_d, ..., s_1))$ be a family numbers indexed by $\mathcal{H}_u$ satisfying the series shuffle equation and $h(s_d, ..., s_1) = (-1)^{s_1}h(s_1, ..., s_d)$. For example, $h(s_d, ..., s_1) = (\Phi^{-1}e_1\Phi)[e_1e_0^{s_d-1}...e_1^{-1}]$ modulo $\zeta(2)$, or $h(s_d, ..., s_1) = \zeta_4(s_d, ..., s_1)$. 

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Then if $s_d + \ldots + s_1 - d$ is even, we have

$$2h(s_d, \ldots, s_1) \equiv (-1)^d \sum_{i=1}^{d-1} h(s_1, \ldots, s_i + s_{i+1}, \ldots, s_d) \mod \mathbb{Z}.h(\text{depth} \leq d-2)$$

**Proof.** ([H], proof of theorem 6.2) The equality between the two formulas for the antipode of the series shuffle Hopf algebra implies that

$$(-1)^d y_{s_1} \ldots y_{s_d} + \sum_{i=1}^{d} (-1)^d y_{s_1} \ldots y_{s_i+1} \ldots y_{s_d} = -y_{s_d} \ldots y_{s_1} + y_{s_d}^* (y_{s_d-1} \ldots y_{s_1}) + (y_{s_d} \ldots y_{s_2})^* y_{s_1}$$

is a $*$-polynomial of elements of depth $\leq d - 2$ with coefficients in $\mathbb{Z}$. 

### 5.2.3 Details in depth 1 and 2

Let $s_1, s_2 \in \mathbb{N}^*$ such that $s_1 + s_2$ is even. Then we have

$$(f^{-1}e_1 f)[e_1 e_0^{s_2-1} e_1 e_0^{s_1-1} e_1] = 0$$

Similarly, the formulas of §5.1.2 for finite multiple zeta values in $\mathcal{A}$, imply, because of the vanishing of odd Bernoulli numbers, that

$$\zeta_\mathcal{A}(s_2, s_1) = 0$$

### 6 Lift of a family of congruences

The theorems 1 and 2 of formal relations enable to retrieve, in the case where $d = 1$, the depth drop of symmetrized multiple zeta values, and to obtain a $T$-adic (resp. $p$-adic) lift of it. Moreover, this lift is explicit, and one can control the denominators of the rational coefficients. All of this should be the first example of a more general phenomenon.

In this section, we consider finite multiple zeta values and their analogues as families of numbers indexed by $\mathcal{H}_+$, without reference to the existence of $\Phi$.

### 6.1 Statement and proof

Let us take a family of numbers $(h(s_d, \ldots, s_1))$, can be either the numbers $(\Phi^{-1} e_1 \Phi)[j_T^{\text{Lie}}(w)]$ modulo $\zeta(2)$ (or their $p$-adic variants) or the finite multiple zeta values $\zeta_f(-s)$, indexed by $\mathcal{H}_+$, which satisfy satisfy the relations of theorems 1 and 2 (see §7 for further formalization of this context).

We denote by $h(\text{weight} \geq s)$ as the $\mathbb{Q}$-vector space generated by polynomials, with constant coefficient 0, of infinite sums $\sum_n h(w_n)$ with for all $n$, $\text{weight}(w_n) \geq s$, and $\text{weight}(w_n) \to +\infty$ when $n \to +\infty$.

**Theorem 4** We have, for all $s \in \mathbb{N}^*$:

$$h(s) \equiv 0 \mod \mathbb{Z}[\frac{1}{2}, \ldots, \frac{1}{s + 1}], h(\text{weight} \geq s+1)$$

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and this can be lifted adically as follows.

By induction on \( N \), let us assume that, for \( N \in \mathbb{N}^* \), \( n \in \{1, \ldots, N-1\} \) and \( s \in \mathbb{N}^* \), there exists \( w^N_n(s) \in H_s \) of weight \( s + N \) such that

\[
h(s) \equiv \sum_{n=1}^{N-1} h(w^N_n(s)) \mod \mathbb{Z}[\frac{1}{2}, \ldots, \frac{1}{s+1}]h(\text{weight} \geq s+N)
\]

Then, for \( N \in \mathbb{N}^* \), \( n \in \{1, \ldots, N\} \), and \( s \in \mathbb{N}^* \), denoting by

\[
w^{N+1}_n(s) = -\frac{s}{s+1} \sum_{\text{z, word of weight } n} (-1)^{\text{depth}(z)} h(z, y_s) - \frac{s}{s+1} \sum_{2 \leq l \leq s} \sum_{n_1, \ldots, n_l \in \mathbb{N}^*_{k_1+\ldots+k_l=s}} \frac{1}{\prod_{i=1}^{l} k_i} w^N_{n_1}(k_1) \ast \ldots \ast w^N_{n_l}(k_l) \quad (66)
\]

we have

\[
h(s) \equiv \sum_{n=1}^{N} h(w^{N+1}_n(s)) \mod \mathbb{Z}[\frac{1}{2}, \ldots, \frac{1}{s'+1}]h(\text{weight} \geq s+N+1)
\]

**Related work.** The question of lifting \( p \)-adically the congruence \( \zeta_A(s) = 0 \) has been tackled by Rosen in [Ro2] §5. The goal in [Ro2] is to obtain "infinitely large \( p \)-adic lifts of congruences, i.e. congruences modulo higher powers \( p^n \), valid for all \( p \geq p_n \), but with the weaker assumption \( p_n \to \infty \) when \( n \to \infty \). For \( s \in \{1, 2\} \), the result of existence of such a lift is stated in (the first version only of) [Ro2], §5, and proven up to \( \mod p^7 \), with a reference to algorithms for higher congruences.

### 6.1.1 First example

In the multiple zeta values case, we have:

\[
(\Phi^{-1} e_1 \Phi)[{\text{Linv}}(e_0^{s-1} e_1)] = T \frac{s}{s+1} (\Phi^{-1} e_1 \Phi)[{\text{Linv}}(e_1 e_0^{s-1} e_1)] \mod T^2
\]

In the finite multiple zeta values case, we have:

**Lemma 6.1.** (Zhao) For \( p > s + 1 \), we have

\[
H_p(s) \equiv p \frac{s}{s+1} H_p(1, s) \mod p^2
\]

**Proof.** We have \( H_p(s) \equiv \sum_{n=1}^{p-1} n^{p(p-1)-s} \equiv Q_{p(p-1)-s}(p) \mod p^2 \). This equals

\[
\sum_{l=1}^{p(p-1)-s+1} \frac{1}{p(p-1)-s+1} \binom{p(p-1)-s+1}{l} B_{(p-1)p-s+1-l} \equiv \sum_{l=1}^{p(p-1)-s+1} \frac{1}{p(p-1)-s+1} \binom{p(p-1)-s+1}{l} B_{(p-1)p-s+1-l} \mod p.
\]

For \( p \) big enough this is \( B_{p(p-1)-s} \); by Kummer congruences, we have \( B_{p(p-1)-s} \equiv \frac{(p-1)p-s}{p-1-s} B_{p-1-s} \mod p \). \( \square \)
6.1.2 Proof

Lemma 6.2. It follows from the series shuffle equation that

\[ \sum_{w \text{ of weight } s} h(w) = \sum_{l=1}^{s} \sum_{\{(k_1, \ldots, k_l) \in (\mathbb{N}^*)^l| k_1 + \ldots + k_l = s\}} \prod_{i=1}^{l} \frac{h(k_i)}{k_i} \]

Proof. In [H], theorem 2.2, Hoffman proves that the series shuffle equation implies, for all \( s_d, \ldots, s_1 \in \mathbb{N}^* \),

\[ \sum_{\phi \text{ permutation of } \{1, \ldots, d\}} h(s_{\phi(d)}, \ldots, s_{\phi(1)}) = \sum_{\{B_1, \ldots, B_l\} \text{ partitions of } \{1, \ldots, d\}} (-1)^{d-l} \prod_{i=1}^{l} \left( (\#B_i - 1)h\left( \sum_{u \in B_i} s_u \right) \right) \]  
(67)

This implies that \( \sum_{w \text{ of weight } s} h(w) \) is equal to

\[ \sum_{l=1}^{s} \sum_{\{(k_1, \ldots, k_l) \in (\mathbb{N}^*)^l| k_1 + \ldots + k_l = s\}/S_l} \prod_{i=1}^{l} h(k_i)(-1)^l \sum_{d\geq l} \sum_{n_1, \ldots, n_l \geq 1} \prod_{i=1}^{l} \left( \frac{1}{n_i} \right) \left( \frac{k_i}{n_i} \right) (-1)^{n_i} \]

hence the result. \( \square \)

Corollary 6.3. With the assumption of theorem 4, denoting by \( S_l \) is the group of permutations of \( \{1, \ldots, l\} \) we have:

\[ -\sum_{\{(k_1, \ldots, k_l) \in (\mathbb{N}^*)^l| k_1 + \ldots + k_l = s\}/S_l} \prod_{i=1}^{l} \frac{h(k_i)}{k_i} = \frac{s + 1}{s} h(s) + \sum_{z \in W^* \setminus \{s\}} (-1)^{\text{depth}(z)} h(z, y_s) \]  
(68)

Proof. This combines the residues equation in depth one with the previous lemma. \( \square \)

Lemma 6.4. With the assumption of theorem 4, we have, for all \( s \in \mathbb{N}^* \),

\[ h(s) \equiv 0 \mod \mathbb{Z} \left[ \frac{1}{2}, \ldots, \frac{1}{s+1}, h(\text{weight } \geq s+1) \right] \]

Proof. Let us take \( s = 1 \) in the symmetry relation, i.e. equation 2') of theorem 1. This gives

\[ 2h(1) \equiv 0 \mod \mathbb{Z}[h(\text{weight } \geq 2)] \]

i.e. the result for \( s = 1 \). On the other hand, the previous corollary implies

\[ -\frac{1}{s} h(s) + (\mathbb{Z}-\text{linear combinations of products of } \frac{h(s')}{s'} \text{ with } s' < s) = h(s) \mod \mathbb{Z}[h(\text{weight } \geq s+1)] \]

Hence the result by induction on \( s \), by regrouping the two terms involving \( h(s) \) and obtaining a term \( \frac{s + 1}{s} h(s) \). \( \square \)
Lemma 6.5. Let us assume that, for a given \( N \in \mathbb{N}^* \), for all \( s' \in \mathbb{N}^* \), we have:

\[
h(s') \equiv \sum_{n=1}^{N-1} h(w_n^N(s')) \mod \mathbb{Z}\left[\frac{1}{2}, \ldots, \frac{1}{s'+1}\right] h(\text{weight} \geq s' + N)
\]

with, for all \( n = 1, \ldots, N - 1 \), weight\( (w_n^N(s')) = s' + n \).

Let us take \( s \in \mathbb{N}^*, (k_1, \ldots, k_l) \in (\mathbb{N}^*)^l \) with \( l \geq 2 \) and \( k_1 + \ldots + k_l = s \). Then:

\[
\prod_{i=1}^{l} \frac{h(k_i)}{k_i} = \prod_{i=1}^{l} \sum_{n_i=1}^{N-1} h(w_n^N(s')) \mod \mathbb{Z}\left[\frac{1}{2}, \ldots, \frac{1}{s'+1}\right] h(\text{weight} \geq s + N + 1)
\]

Proof. We apply the hypothesis to \( s' = k_1, \ldots, s = k_l \). First of all, for all \( i \), we have \( \mathbb{Z}\left[\frac{1}{2}, \ldots, \frac{1}{s'+1}\right] \subset \mathbb{Z}[\frac{1}{2}, \ldots, \frac{1}{s'+1}] \). The numbers \( \frac{1}{k_i} h(w_n^N(k_i)) \) are of weight \( \geq k_i + 1 \), and the elements of \( \mathbb{Z}[\frac{1}{2}, \ldots, \frac{1}{s'+1}] h(\text{weight} \geq k_i + N) \) are of weight \( \geq k_i + N \) : thus, the difference

\[
\prod_{i=1}^{l} \frac{h(k_i)}{k_i} - \prod_{i=1}^{l} \sum_{n_i=1}^{N-1} h(w_n^N(s'))
\]

is made of terms whose weight is superior or equal to \( \sum_{i=1}^{l} k_i + N + l - 1 \), which is itself superior or equal to \( \sum_{i=1}^{l} + N + 1 \) since \( l \geq 2 \).

Proof. (of theorem 4) Follows from this last lemma and the corollary 6.3.

6.2 Applications

6.2.1 The "star" variant (sums with large inequalities)

Remark 6.6. Variant for the multiple harmonic sums with large inequalities

One has also in [H], theorem 2.2, a variant of equation (67) used in the proof of lemma 6.3 ; it applies to the variant of multiple harmonic sums defined by large inequalities

\[
\sum_{0 \leq n_1 \leq \ldots \leq n_d < N} \frac{1}{n_1^{s_1} \ldots n_d^{s_d}}
\]

It is the same equation with (67) without the sign \((-1)^d\). It permits to show an analogous result for the associated variants of finite multiple zeta values ; sums of the type \( \sum_{v=1}^{u} (-1)^v \binom{u}{v} = -1 \) become \( \sum_{v=1}^{u} \binom{u}{v} = 2^u - 1 \), which slightly changes the computations.

The variants of multiple zeta values involving sums with large inequalities are usually referred to as "multiple zeta star values".

6.2.2 The depth two and even weight case

Let \( s_1, s_2 \in \mathbb{N}^* \) such that \( s_1 + s_2 \) is even. Then, with the hypothesis of theorem 4, because of the series shuffle relation:

\[
h(s_2, s_1) = \frac{1}{2} (h(s_2) h(s_1) - h(s_1 + s_2))
\]

Corollary 6.7. \( h(s_2, s_1) \) admits a \( p \)-adic expression valid for \( p > s_1 + s_2 + 1 \) given by the identity above and theorem 4.
6.2.3 Application to algebraic relations

Theorem 4 consists of an identity between formal series whose coefficients are multiple zeta values. By considering the coefficient of each \( T^l \), \( l \geq 0 \), this is equivalent to an infinite family of relations among multiple zeta values, which are consequences of the associator and double shuffle relations.

6.2.4 Application to \( p \)-adic zeta values

The \( p \)-adic multiple zeta values in depth one \( \zeta_{p^{(-k)}} \) admit expression as sums of series whose terms involve the multiple zeta values \( \zeta_{f^{(-k)}} \) in depth one as well; this is essentially a classical fact. Using theorem 4, one can rewrite those series expansions in a different way, for \( p \) sufficiently large. We conjecture a more general re-writing of the series expansions of \( p \)-adic multiple zeta values, related to the lift of congruences. See [J1] for details.

7 Final comments

7.1 Formal relations

When dealing with the numbers \( (\Phi^{-1}e_1\Phi)[\mathcal{H}_T^p(w)] \) and \( \zeta^{f^{(-k)}} \), one has to make a distinction between their relations, which involve sums of series and infinitely many values of the weight, and their congruences obtained when moding out by an ideal \((T^{s+1})\), resp \((p^{s+1})\). Thus, we have adopted a terminology which refers to formal schemes, in order to make a distinction with the context of usual algebraic relations of multiple zeta values, which define schemes involving completions, but are homogeneous for the weight.

More precisely, the numbers \( (\Phi^{-1}e_1\Phi)[\mathcal{H}_T^p(w)] \) and \( \zeta^{f^{(-k)}} \) lie, respectively, in the valuation rings \( \mathbb{R}[[T]] \), \( \mathbb{Z}_p \), and satisfy the inequality \( \text{valuation} \geq \text{weight} \). And, on the other hand, the relations of §3 and §4 define families of affine schemes \( X_s \) over \( \mathbb{Q}[T]/(T^{s+1}) \), \( s \in \mathbb{N}^* \), resp. \( \mathbb{Z}_p/(p^{s+1}) \), corresponding to moding out by the weight \( > s \) parts of the formal relations, and we have natural isomorphisms \( X_{s+1} \otimes_{\mathbb{Q}[T]/(T^{s+1})} \mathbb{Q}[T]/(T^{s+1}) = X_s \), resp. \( X_{s+1} \otimes_{\mathbb{Z}_p/(p^{s+2})} \mathbb{Z}_p/(p^{s+1}) = X_s \); this defines formal schemes over \( \mathbb{Q}[[T]] \) (resp. \( \mathbb{Z}_p \)).

7.2 Conjectures of generation of all relations

We conjecture that the formal double shuffle and associator equations for finite multiple zeta values and their analogues - those of §3 and §4, and the consequences of the 5-cycle equation which don’t appear in this first paper - imply respectively all relations of the following type: for \( k \in \mathbb{N}^* \), \( (w_n)_{n \in \mathbb{N}} \) a sequence of \( \mathcal{H}_m \) and \( (a_n)_{n \in \mathbb{N}^*} \), a sequence of rational numbers,

\[
\sum_{n \geq 0} a_n (p^k)^{\text{weight}(w_n)} H_{p^k}(w_n) = 0 \in \mathbb{Z}_p \tag{69}
\]

7.3 Conjectural algorithm for lifting congruences

Rosen has conjectured in [Ro2] that all algebraic relations among finite multiple zeta values in \( \mathcal{A} \) lift to equalities of the type analogous to (69) (for \( k = 1 \)) in the complete topological ring \( \lim \left( \Pi_p \mathbb{Z}_p^{n} \mathbb{Z} / \mathbb{P}_p^{n} \mathbb{Z} / \mathbb{P}_p \mathbb{Z} \right) = \left( \Pi_p \mathbb{Z}_p / \mathbb{P}_p \mathbb{Z}_p \right) \); \( \Pi_p \mathbb{Z}_p \) is equipped with the uniform topology relative to the \( p \)-adic topologies on the \( \mathbb{Z}_p \)'s, and the bar denotes the closure. Let us explain roughly a conjectural algorithm for this lift.
First of all, the conjecture 1.2 of §1.1 would imply that, for any relation among finite multiple zeta values in \( \mathcal{A} \), the same relation is true for the symmetrized \( p \)-adic multiple zeta values.

Secondly, let us take such a relation among symmetrized \( p \)-adic multiple zeta values, i.e. \( w_0 \in \mathcal{H}_* \) such that 
\[
\left( \Phi_{p^{-1} k}^{-1} e_1 \Phi_{p^{-1} k} \right) \left[ \mathcal{L}_{T}^\text{Lin} \left( w_0 \right) \right] = 0,
\]
for \( k \in \mathbb{N}^* \). Then, there exists a sequence \((a_n)_{n \in \mathbb{N}}\) of rational numbers and a sequence \((w_n)\) of elements of \( \mathcal{H}_* \), with weight\((w_n) \geq \text{weight}(w_0) + n\), such that
\[
\sum_{n \geq 0} a_n \left( \Phi_{p^{-1} k}^{-1} e_1 \Phi_{p^{-1} k} \right) \left[ \mathcal{L}_{T}^\text{Lin} \left( w_n \right) \right] = 0 \tag{70}
\]

Indeed, Yasuda’s result that multiple zeta values can be expressed in terms of symmetrized multiple zeta values ([Y1], §6), implies that any \( \left( \Phi_{p^{-1} k}^{-1} e_1 \Phi_{p^{-1} k} \right) [w], w \in \mathcal{H}_* \), can be written in the form \( \left( \Phi_{p^{-1} k}^{-1} e_1 \Phi_{p^{-1} k} \right) [w'] \) with \( w' \in \mathcal{H}_* \) a linear combination of words of the form \( e_1 e^{s_1\ldots s_{i-1} e_1} \). Thus, by induction on \( n \), one can construct \( a_m, w_m, m = 1, \ldots, n \) satisfying the equality (70) modulo weight > weight\((w_0) + n\). The result of [Y1] is applied at each step of the induction.

This actually shows a stronger statement: that any \( p \)-adic symmetrized multiple zeta value (thus any \( p \)-adic multiple zeta value) can be expressed in terms of numbers \( \Phi_{p^{-1} k}^{-1} e_1 \Phi_{p^{-1} k} \left[ \mathcal{L}_{T}^\text{Lin} \left( w \right) \right] \) - which are known to be finite multiple zeta values for \( T = 1 \) because of theorem 1.4. Of course the analogous result also works in the real case.

The fact that symmetrized multiple zeta values generate multiple zeta values, applied at each step of the induction on \( n \), can be made into an algorithm as follows: first of all, multiple zeta values of a given weight \( n \) and depth \( d - 1 \) can be expressed algorithmically in terms of the ones appearing in the formula (54) for the depth drop (§5.1.1, theorem 3-a, 1), modulo relations given by the depth graded double shuffle relations in this weight and depth. This consists in the inversion of a linear system, and that this is inversion is possible is equivalent to Yasuda’s result.

To pass from the depth graded formula to the general one, one can combine it, for example, with the full formula of depth drop in theorem 3-a 2) and Lemma 5.3, which seems to us the simplest full formula possible.

This gives the second step of the algorithm. One gets the desired lift of congruences by applying theorem 1.4.

### 7.4 Variants of the results

One can derive the following variants.

**Statements without the restriction modulo \( \zeta(2) \)**

In §5 and §6, we have considered the algebra of multiple zeta values moded out by the ideal generated by \( \zeta(2) \), which amounts to consider the \( p \)-adic case. Similar facts can be written without this restriction, although the formulas are more complicated since they involve a Campbell-Hausdorff series. This is also related to what follows.

**Version in the group \( 0\pi_0^{dR} \)**
Let us consider the series
\[ \Phi^{-1} e^{2i\pi e_1} \Phi \in \mathbb{R} \]
(It satisfies in particular the shuffle equation.) It enables to express the monodromy of multiple polylogarithms ([C] §3). The consequences of the 2-cycle and 3-cycle relations of §4 can be written as involving these series: this is in §4 and already in [AET], for the analog of the residues equation, the other equation is \( (\Phi^{-1} e^{2i\pi e_1} \Phi) (e_1, e_0) = (\Phi e^{2i\pi e_0} \Phi^{-1}) (e_1, e_0) \). We have:
\[ \Phi^{-1} e^{2i\pi e_1} \Phi = 2i\pi \Psi \]
with \( \Psi \) still having coefficients in the \( \mathbb{Q} \)-algebra generated by multiple zeta values and \( 2i\pi \), and
\[ \Psi \equiv \Phi^{-1} e_1 \Phi \mod 2i\pi \]
Thus, the analogous equations concerning \( \Phi^{-1} e^{2i\pi e_1} \Phi \) permit to retrieve the ones concerning \( \Phi^{-1} e_1 \Phi \).
The map \( f \mapsto f^{-1} e^{2i\pi e_1} f \), \( \mathbb{R}_0^d \rightarrow \mathbb{R}_0^d \) is also "motivic" as in §2.4.2, if we consider a variant of the Ihara action involving a composition with the action of multiplication by \( (2i\pi)^{\text{weight}} \); \( g \circ f = g \cdot f (2i\pi e_0, g^{-1} 2i\pi e_1 g) \).

\textit{Cyclotomic generalizations}

Let us replace the ring \( \mathcal{A} \) by:
\[ \prod_p \mathbb{F}_p / \oplus_p \mathbb{F}_p \]
Let \( N \in \mathbb{N}^* \); let \( \xi = (\xi_p)_{p \mid N} \) be a primitive \( N \)-th root of unity in \( \prod_{p \mid N} \mathbb{F}_p \), and \( \xi \) its image in \( \prod_{p \mid N} \mathbb{F}_p \); then let, for \( d \in \mathbb{N}^* \), \( (s_d, \ldots, s_1) \in (\mathbb{N}^*)^d \), \( (m_{d+1}, \ldots, m_1) \in (\mathbb{Z}/N\mathbb{Z})^{d+1} \), and for \( k \in \mathbb{N}^* \):
\[ \zeta_f (\mathbb{N}) (s_d, \ldots, s_1; m_{d+1}, \ldots, m_1) \]
\[ = \left( \prod_{n \mid N} \mathbb{F}_p / \oplus_p \mathbb{F}_p \right) \]
and
\[ \zeta_A (s_d, \ldots, s_1; m_{d+1}, \ldots, m_1) \]
\[ = \sum_{0 < n_1 < \ldots < n_d < p} \frac{e^{-n_1 m_1 + (n_1-n_2) m_2 + \ldots + (n_d-n_{d-1}) m_d + (n_{d-1}-n_d) m_{d+1}}}{n_1^{s_1} \ldots n_d^{s_d}} \in \prod_{p \mid N} \mathbb{Q}_p \]
These "cyclotomic finite multiple zeta values" give a geometric framework to deal with, in particular, the variant of finite multiple zeta values involving alternating sums with numerators \( \prod_i (\pm 1)^{n_i} \), already considered in the literature, which corresponds to the case \( N = 2 \).

It is also possible to consider the variant defined by sums involving congruences modulo \( N \) : \( \sum_{0 < n_1 < \ldots < n_d < p} \), which are related to cyclotomic finite multiple zeta values via a base change with coefficients in \( \mathbb{Q}(\xi) \). This corresponds to replacing iterated integrals of \( \frac{dz}{z^u} \), \( u \in \{0, \ldots, N-1\} \), and \( \frac{dz}{z^u} \), by iterated integrals of \( \frac{dz}{z^{u+N}} \), \( u \in \{0, \ldots, N-1\} \), and \( \frac{dz}{z^u} \).
On the level of $\pi^{dR}$, the variety $\mathbb{P}^1 \setminus \{0,1,\infty\}$ is replaced by $\mathbb{P}^1 \setminus (\mu_N \cup \{0,\infty\}) = \mathbb{G}_m \setminus \mu_N$, considered over a field which has a primitive $N$-th root of unity.

The description of the canonical scheme $\pi^{dR} (\mathbb{P}^1 \setminus \{0,1,\infty\})$ as in §2.1.2 adapts, by replacing $e_1$ by a collection of formal variables $e_{\xi^k}$, $k = 0, \ldots, N - 1$, and the relation $e_0 + e_1 + e_\infty = 0$ by $e_0 + \sum_{k=0}^{N-1} e_{\xi^k} + e_\infty = 0$ ([DG]).

The series $\Phi$ and $\Phi_p \{ -k \}$ are defined in the same way ([DG], §5.28) and, provided the computation of cyclotomic $p$-adic multiple zeta values gives similar lower bounds of valuations as for $N = 1$, one can consider the numbers

$$\Phi_{p(\cdot-k)}^{-1} e_1 \Phi_{p(-k)} \left[ \frac{1}{1 - e_0} e_{\xi^{s_{d+1}}} e_0 e_{\xi^{s_d}} \cdots e_0 e_{\xi^{s_1}} \right]$$

and prove the analogue of theorem 1.4.

The double shuffle relations of §3 are still valid for cyclotomic finite multiple zeta values and can be written in the same way.

The equation of §4 rewrites as

$$e_0 + \sum_{u=0}^{N-1} (z \mapsto \xi^u z)_* (\Phi^{-1} e_1 \Phi) + \Phi_{\infty}^{-1} e_\infty \Phi_{\infty} \equiv 0 \mod \zeta(2)$$

where $\Phi_{\infty}$ is as follows: assume that $\Phi = v(110)$ where $v \in Aut$ is an automorphism of the restricted groupoid $B$ as in Definition 4.3. Then we have, by composition of paths, $v(\infty 10) = v(\infty 1) v(110)$, and $v(\infty 11) = (z \mapsto \frac{z}{z-1})_* (v(110)^{-1})$ (we cannot apply $z \mapsto \frac{z}{z-1}$ which is not an automorphism of $\mathbb{G}_m \setminus \mu_N$).

A priori, a special role is played by the finite multiple zeta values such that $u_{d+1} = u_1$, and because of the automorphisms $(z \mapsto \xi^u z)_*$ we can restrict to the case where $u_1 = 1$.

Finally, in the cyclotomic case, one has additionally a "distribution relation" expressing a sum of cyclotomic finite multiple zeta values in terms of usual finite multiple zeta values, arising from the map $z \mapsto z^N$, $\mathbb{G}_m - \mu_N \to \mathbb{P}^1 \setminus \{0,1,\infty\}$.

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