Bohr chaoticity of topological dynamical systems

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Abstract

We introduce the notion of Bohr chaoticity, which is a topological invariant for topological dynamical systems, and which means that the system is not orthogonal to any non-trivial weights. We prove the Bohr chaoticity for all systems which have a horseshoe and for all toral affine dynamical systems of positive entropy, some of which don’t have a horseshoe.

1 Introduction

A sequence of complex numbers \((w_n)_{n\geq 0} \in \ell^\infty(\mathbb{N})\) is called a (non-trivial) weight or weight sequence if it satisfies

\[
\limsup_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} |w_n| > 0.
\] (1.1)

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Let us consider a weight \((w_n)\) and a topological dynamical system \((X, T)\). For a given continuous function \(f \in C(X)\) and a given point \(x \in X\), we say that \((w_n)\) is orthogonal to the observation \((f(T^n x))_{n \geq 0}\) if
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} w_n f(T^n x) = 0. \tag{1.2}
\]
If it is the case for all \(f \in C(X)\) and all \(x \in X\), we say that \((w_n)\) is orthogonal to the dynamical system \((X, T)\). Sarnak’s conjecture states that the Möbius sequence \((\mu(n))\) is orthogonal to all topological dynamical systems of zero entropy [29]. This conjecture is still open. The following questions arise. Is there any non-trivial weight which is orthogonal to all systems of zero entropy? Which systems are orthogonal to a given weight or a family of weights? Which weights are orthogonal to a given system or a family of systems? The answer to the first question is affirmative and it is known that there do exist non-trivial weights (other than the Möbius sequence) which are orthogonal to all topological dynamical systems of zero entropy, for example the random Bernoulli sequence taking values \(-1\) and \(1\) with probability \(1/2\) is almost surely such a sequence [5]. It is also the case for \((e^{2\pi i \beta^n})\) for almost all \(\beta > 1\) [30].

Wiener-Wintner’s ergodic theorem states that for any measure-preserving dynamical system \((X, \mathcal{B}, \mu, T)\), the orthogonality (1.2) holds for all polynomial weights \(w_n = e^{2\pi i n t}\) (of order 1) and for \(\mu\) almost all \(x\) when \(f\) is orthogonal to the Kronecker factor of the system [36]. Lesigne [25] generalized Wiener-Winter theorem to the case of all polynomial weights \(w_n = e^{2\pi i P(n)}\) (\(P\) being a non-constant polynomial with real coefficients). This result was strengthened by Frantzikinakis [15]. A topological version of Wiener-Wintner’s ergodic theorem for unique ergodic systems is due to Robinson [28]. Combining the ideas of Lesigne and Robinson, Fan [11] proved that for any ergodic nilsystem, the orthogonality (1.2) holds for all \(x\) and all polynomial weights \(w_n = e^{2\pi i P(n)}\) when \(f\) is orthogonal to the eigenfunctions of the nilsystem. On the other hand, Fan and Jiang [13] proved that the so-called 1-oscillating weights are orthogonal to all circle homeomorphisms and all Feigenbaum interval maps of zero entropy.

A topological dynamical system \((X, T)\) is said to be Bohr chaotic if it is not orthogonal to any (non-trivial) weights. That is to say, for any non-trivial weight \((w_n)_{n \geq 0} \in \ell^\infty(\mathbb{N})\), there exist a continuous function \(f \in C(X)\) and a point \(x \in X\) such that
\[
\limsup_{N \to \infty} \frac{1}{N} \left| \sum_{n=0}^{N-1} w_n f(T^n x) \right| > 0. \tag{1.3}
\]
If (1.3) hold, then we say that \((f, x)\) is a pair correlated to \((w_n)\) for the system \((X, T)\).

It is easy to see that the Bohr chaoticity is a topological invariant, namely topologically conjugate systems share the Bohr chaoticity at the same time. It is even true that the extensions of a Bohr chaotic system are all Bohr chaotic (Proposition 3.1). In this paper we will provide some Bohr chaotic systems and some non Bohr chaotic systems among systems of positive entropy.

Bohr chaoticity is a kind of complexity.

A Bohr chaotic system must have positive entropy, according to the preceding discussion. But having positive entropy is not sufficient for having Bohr chaoticity. In other words, there are dynamical systems of positive entropy which are not Bohr-chaotic and it is the case for uniquely ergodic systems (see Theorem 1.3 below).
Recall that a topological dynamical system $(X, T)$ admits a two-sided horseshoe (resp. one-sided horseshoe) if there exists a subsystem $(\Lambda, T^N)$ of some power system $(X, T^N)$ ($N \geq 1$ being some integer), that is topologically conjugate to the two-sided full shift ($\{0, 1\}^\mathbb{Z}, \sigma$) (resp. one-sided full shift ($\{0, 1\}^N, \sigma$)). The subsystem $(\Lambda, T^N)$ is sometimes called a $N$-order horseshoe of the system $(X, T)$. If, furthermore, the horseshoe $(\Lambda, T^N)$ satisfies the extra condition

$$\Lambda \cap \bigcup_{k=1}^{N-1} T^k(\Lambda) = \emptyset,$$

we say that the $N$-order horseshoe $(\Lambda, T^N)$ has disjoint steps, meaning that the steps $T^k(\Lambda)$ ($1 \leq k \leq N - 1$) are disjoint from the initial set $\Lambda$.

We shall prove that systems having a horseshoe must have a horseshoe with disjoint steps (Theorem 5.1). The proof of this fact occupies an important part of the present paper. But it has a nice consequence on the Bohr chaoticity of the system.

**Theorem 1.1** Any system having a horseshoe is Bohr chaotic.

As corollaries of Theorem 1.1, all subshifts of finite type of positive entropy are Bohr-chaotic and all $\beta$-shifts are Bohr chaotic. Actually all piecewise monotonic interval maps of positive entropy are Bohr chaotic, because Young [34] proved the existence of subsystems which are subshifts of finite type of positive entropy.

By a result of Smale [31], on a manifold $X$ of dimension $\dim X \geq 2$ there are many diffeomorphisms which are Bohr Chaotic. Let us state Smale’s result precisely: there exists a non-empty open set $U$, in the $C^1$-topology, of the space of diffeomorphisms of $X$ such that for each $T \in U$ there exists a $T$-invariant Cantor set $K \subset X$ such that the subsystem $T : K \to K$ is topologically conjugate to a full shift. By Proposition 3.1 and Proposition 3.4, every $T \in U$ is Bohr chaotic. Here we don’t need Theorem 1.1, because it is easy to see that the full shift is Bohr chaotic.

For an individual smooth system, let us state that every $C^{1+\alpha}$ ($\alpha > 0$) diffeomorphism $T$ of a compact smooth manifold admitting an ergodic non-atomic Borel probability invariant measure with non-zero Lyapunov exponents is Bohr chaotic. Indeed, Katok [23] proved the existence of a closed invariant hyperbolic set $F$ such that the subsystem $T : F \to F$ is topologically conjugate to a subshift of finite type (called topological Markov chain in [23]) of positive entropy. In particular, Anosov systems are Bohr chaotic.

But a Bohr chaotic system doesn’t necessarily have a horseshoe. Such systems exist among toral automorphisms, as we shall see. Let us consider an affine map defined by $T(x) = Bx + b \mod \mathbb{Z}^d$ on the $d$-dimensional torus $\mathbb{T}^d$, where $b \in \mathbb{R}^d$ and $B$ is a $d \times d$ matrix of integral entries. We assume that $\det B \neq 0$. By Sinai’s theorem, the topological entropy $T$ is equal to $\sum_i \log |\lambda_i|$ where the sum is taken over all eigenvalues of $B$ such that $|\lambda_i| > 1$ (cf. Theorem 8.15, [33]).

**Theorem 1.2** All toral affine systems of positive entropy are Bohr-chaotic. Actually the set of $x \in \mathbb{T}^d$ such that (1.3) holds has Hausdorff dimension $d$.

According to Lind and Schmidt (cf. Example 3.4 in [27]), any ergodic irreducible partially hyperbolic toral automorphism $T_A$ defined by a matrix $A \in \text{GL}(d, \mathbb{Z})$ has no non-trivial
homoclinic point. Here is an example:

\[
A = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 3 & -3 & 3
\end{pmatrix}
\]  \hspace{1cm} (1.4)

which is the companion matrix of the polynomial \( z^4 - 3z^3 + 3z^2 - 3z + 1 \).

Since the fullshift \((\{0, 1\}^Z, \sigma)\) has a dense set of homoclinic points, automorphisms like \(T_A\) can not have horseshoe. But it is Bohr chaotic by Theorem 1.2.

In a forthcoming paper [14], the authors have studied the Bohr chaoticity for principal algebraic \(\mathbb{Z}^d\)-actions on \(T^\mathbb{Z}^d\). When \(d > 1\), a key point of proof is the existence of summable homoclinic points.

The weighted ergodic limit involved in (1.2) may be multifractal, meaning that not only non-zero limit is possible but a range of limits are possible and the set of \(x\) which realize a given limit has positive Hausdorff dimension or positive entropy. This is a strengthening of Bohr chaoticity. But the computation of these Hausdorff dimensions or entropies is a difficult problem for general systems. For the shift dynamical system, a systematic study was undertaken in [12].

Now we point out that the unique ergodicity is an obstruction for the Bohr chaoticity, as we state in the following theorem.

**Theorem 1.3** The uniquely ergodic dynamical systems are not Bohr chaotic.

Theorem 1.3 is a consequence of a result due to Robinson [28] about the uniform convergence of Wiener-Wintner ergodic averages. Recall that there exist uniquely ergodic dynamical systems having positive entropy on Cantor spaces according to [23]. See [1] for topological uniquely ergodic dynamical systems having positive entropy on smooth spaces. Actually, according to [2], the class of uniquely ergodic homeomorphisms of positive entropy is large. Systems with positive entropy and moderately large set of invariant measures would not be Bohr chaotic. More precisely, systems having at most countably many ergodic measures should not be Bohr chaotic. This is recently affirmed by Matan Tal [32] who used an argument of joining. Guohua Zhang [37] had given another proof in a personal communication by adopting a similar argument used in the present paper.

Theorem 1.2 will be proved by using Riesz product measures borrowed from harmonic analysis (cf. [18, 38]), in order to find points \(x\) satisfying (1.3). Let us first recall the following weighted ergodic theorem taken from [7]. Let \((X, B, \mu, T)\) be any measure-preserving dynamical system. Suppose that \((w_n)_{n \geq 0}\) be an oscillating weight of Davenport type in the sense that

\[
\sup_{0 \leq t < 2\pi} \left| \sum_{n=0}^{N-1} w_n e^{int} \right| = O \left( \frac{N}{\log^h N} \right) \quad \text{as} \quad N \to \infty
\]

for some \(h > \frac{1}{2}\). Then for any integrable function \(f \in L^1(\mu)\), for \(\mu\)-almost all \(x\) we have

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} w_n f(T^n x) = 0.
\]

We emphasize that the limit is zero. As we will see in the proof of Theorem 1.2, for any non-trivial weight, the weighted ergodic limit in (1.3) exists and is equal to a constant
different from zero for almost every point with respect to the Riesz product measure. It follows
that our Riesz product measures are not $T$-invariant. The positivity of entropy implies some
lacunarity of the affine map (cf. Lemmas 4.1 and 4.2) which effectively allows us to construct
such measures, and the convergence of the weighted ergodic averages is proved by using the
classical Menshov–Rademacher theorem ([38] vol. 2, p. 193). This method is also used in
[14].

Theorem 1.1 will be proved by a different method. From the given horseshoe $(\Lambda, T^N)$,
we shall construct a $T^\tau$-invariant set $K$ for some $\tau \geq 1$ such that $K$ is disjoint from
$\bigcup_{1 \leq j \leq \tau - 1} T^j(K)$. This system $(K, T^\tau)$ is Bohr chaotic and the Bohr chaoticity of $(X, T)$
follows (cf. Theorem 3.3). A little more can be proved. Actually the system $(K, T^\tau)$ that
we construct is a horseshoe with disjoint steps (cf. Theorem 5.1). This is the main part of
the proof: construct a horseshoe with disjoint steps from a horseshoe $(\Lambda, T^N)$. To do so, we
consider the extended horseshoe $T : \Lambda^* \rightarrow \Lambda^*$ where

$$\Lambda^* = \bigcup_{n=0}^{N-1} T^n(\Lambda).$$

This is a subsystem of the original system $(X, T)$. We shall use the fact that the extended
horseshoe doesn’t have root, that is to say, the equation $S^n = T$ has no continuous solution
$S$ for any $n \geq 2$ (cf. Proposition 5.2). This fact forces that $\Lambda \not\subset \bigcup_{j \in J} T^j(\Lambda)$ for some set
$J$ (cf. Proposition 5.3). We say that $\Lambda$ is displaced by $T^j$ with $j \in J$. Roughly speaking,
we can construct a better horseshoe in $\Lambda \setminus \bigcup_{j \in J} T^j(\Lambda)$. Here by “better” we mean that the
new horseshoe has “more” disjoint steps. The possibility to construct this new horseshoe is
based on the fact that in any cylinder of the full shift space we can construct a horseshoe with
disjoint steps (cf. Proposition 8.2). If the order $N$ of the horseshoe is equal to 2, this new
horseshoe is already what we need. But for general order $N$, we have to repeat this procedure
many times (cf. Propositions 5.5, 5.6, Lemma 5.7).

At the end of this introduction we mention some open questions.

Recall that semi-horseshoes are similarly defined as horseshoes, but the conjugation is
weakened to the semi-conjugation. Semi-horseshoes are well studied in [19, 20, 24, 26].

Question 1 Are systems having semi-horseshoes Bohr-chaotic?

Herman [17] constructed a real-analytic diffeomorphism on a compact, connected mani-
fold of dimension 4 that is minimal and has positive entropy. Because of the minimality,
Herman’s diffeomorphism doesn’t have horseshoe, so Theorem 1.1 doesn’t apply to it. Her-
man’s diffeomorphism is not uniquely ergodic, so Theorem 1.3 doesn’t apply either.

Question 2 Is Herman’s diffeomorphism not Bohr-chaotic?

Related to Question 2, Matan Tal [32] has constructed a minimal (non-unique ergodic)
dynamical system of positive entropy that is Bohr chaotic, using the idea from B. Weiss’s
example of a universal minimal system [35]. So the minimality is not an obstruction for Bohr
chaoticity. But Question 2 remains open.

Question 3 Is there any non Bohr chaotic system $(X, T)$ such that $(X, T^\tau)$ is Bohr chaotic
for some integer $\tau \geq 2$?
In other words, is the converse proposition of Proposition 3.2 not true?

**Question 4** Is it possible to describe the Bohr chaoticity (1.3) by using weights \((w_n)\) which only take 0 and 1 as values?

Finally let us explain the organization of the paper. We shall first prove Theorem 1.3 in Sect. 2, the proof of which is easy. In Sect. 3 we shall study the Bohr chaoticity of a system via its subsystems, Proposition 3.1 there shows that the Bohr chaoticity is a topological invariant and Theorem 3.3 there will serve as a tool to prove Theorem 1.1.

The Bohr-chaoticity of affine maps on torus (Theorem 1.2) will be proved in Sect. 4. Theorem 1.1 will be proved in Sect. 5 (the case of one-sided horseshoe) and Sect. 6 (the case of two-sided horseshoe). We put two technical results in Appendixes, which are possibly well-known to experts. Appendix A proves that any cylinder contains a horseshoe. Appendix B provides two proofs of the fact that the extended horseshoe doesn’t have a root, one self-contained proof for one-sided horseshoe and another proof for two-sided horseshoe which is based on the no existence of root for the fullshift.

## 2 Uniquely ergodic systems are not Bohr chaotic

We give here a quick proof of Theorem 1.3. It is a simple consequence of Robinson’s topological weighted ergodic theorem, which states as follows (in a weak form) [28, Theorem 1.1]. Let \((X, T)\) be a uniquely ergodic dynamical system. If \(\lambda \in \mathbb{S}^1 := \{z \in \mathbb{C} : |z| = 1\}\) is not an eigenvalue of the Koopman operator \(f \mapsto f \circ T\) acting on \(L^2(\mu)\) (\(\mu\) being the invariant measure), then for every continuous function \(f \in C(X)\) we have

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} \lambda^{-k} f(T^k x) = 0,
\]

where the limit is actually uniform in \(x \in X\). Note that there are at most countably many eigenvalues. Take \(\lambda \in \mathbb{S}^1\) which is not an eigenvalue. The nontrivial weight defined by \(w_n = \lambda^{-n}\) is then orthogonal to the system \((X, T)\). Hence, the uniquely ergodic dynamical system \((X, T)\) is not Bohr chaotic. Theorem 1.3 is thus proved.

Let give some comments on systems which are not Bohr chaotic.

Krieger [23] proved that every ergodic measure-preserving invertible transformation of a Lebesgue measure space is isomorphic to a uniquely ergodic subsystem, which is the closure of an orbit in the shift system on some symbolic space. So, by Theorem 1.3, every ergodic measure-preserving system has a topological model which is not Bohr chaotic. These models are Cantor sets.

Béguin, Crovisier and Le Roux [1] proved that any compact manifold of dimension \(d \geq 2\) which carries a minimal uniquely ergodic homeomorphism \(\mathcal{R}\) also carries a minimal uniquely ergodic homeomorphism \(T\) with positive topological entropy. Actually \(T\) is an extension of \(\mathcal{R}\). By Theorem 1.3, such extensions are not Bohr chaotic.

For any bounded, real-valued sequence \((w_n)\) with zero average along every infinite arithmetic progression (such sequences are said to be aperiodic), Downarowicz and Serafin [4] proved that there exists a subshift over \(N\) symbols to which \((w_n)\) is orthogonal and the entropy
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of the subshift can approach \( \log N \). By Theorem 4 in [10], every 1-oscillating sequence is aperiodic. Recall that \((w_n)\) is 1-oscillating means

\[
\forall t \in [0, 1), \quad \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} w_n e^{2\pi i n t} = 0.
\]

That means, in harmonic analysis, that the Fourier-Bohr spectrum of \((w_n)\) is empty. Note that for any real polynomial \( P(x) = \sum_{k=0}^{n} a_k x^k \in \mathbb{R}[x] \) with at least one irrational coefficient \( a_k \) \((k \geq 2)\), \( e^{2\pi i P(n)} \) is 1-oscillating.

Also recall that Karagulyan [21] proved that the Möbius sequence is not orthogonal to subshifts of finite type with positive topological entropy. But the Bohr-chaoticity requires more.

3 Subsystems of Bohr-chaotic systems

We present here a basic idea for proving the Bohr chaoticity. It is to find a Bohr chaotic factor.

3.1 Subsystems of Bohr-chaotic systems

A system \((Y, S)\) is a factor of a system \((X, T)\) if there exists a surjective continuous map \( \pi : X \to Y \) such that \( S \circ \pi = \pi \circ T \). In this case, \((X, T)\) is called an extension of \((Y, S)\).

The following proposition follows from the definition.

**Proposition 3.1** Any extension of a Bohr chaotic system is Bohr chaotic. Consequently, Bohr chaoticity is a topological invariant.

**Proof** Let \((Y, S)\) be a Bohr-chaotic factor of \((X, T)\) with factor map \( \pi \). Let \((w_n)\) be a non-trivial weight. Since \((Y, S)\) is Bohr-chaotic, there exists a pair \((g, y)\) with \( g \in C(Y) \) and \( y \in Y \) correlated to \((w_n)\) for the system \((Y, S)\). Let \( x \) be a pre-image of \( y \) under \( \pi \). Then \((g \circ \pi, x)\) is a pair correlated to \((w_n)\) for the system \((X, T)\). \(\square\)

The following proposition is also obvious.

**Proposition 3.2** Let \( \tau \geq 1 \) be an integer. If \((X, T)\) is Bohr-chaotic, so is \((X, T^\tau)\).

**Proof** Let \((w_n)\) be a non-trivial weight. Define a new one as follows: \( v_{\tau n} = w_n \) and \( v_j = 0 \) if \( j \) is not a multiple of \( \tau \). Suppose that \((f, x)\) is a pair correlated to \((v_j)\) for the system \((X, T)\). Then \((f, x)\) is a pair correlated to \((w_n)\) for the system \((X, T^\tau)\). \(\square\)

A partial inverse of the above proposition holds and it provides a criterion for proving Bohr chaoticity. Let \((X, T)\) be a given system and \( K \) be a \( T^\tau \)-invariant compact set, i.e. \( T^\tau K \subset K \). We say that \( \tau \) is the first return time of \( K \) if \( T^k x \notin K \) for all \( x \in K \) and all \( 1 \leq k < \tau \). Notice that \( T^k K \) with \( 1 \leq k < \tau \) are all compact sets disjoint from \( K \). In this case, it is convenient to say that the \( T^\tau \)-invariant set \( K \) has disjoint steps \( T^j K \) \((1 \leq j < \tau)\). When \( \tau = 1 \), it just means that \( K \) is \( T \)-invariant.

**Theorem 3.3** Let \((X, T)\) be a topological dynamical system. Let \( K \) be a \( T^\tau \)-invariant compact subset \((\tau \geq 1)\) having \( \tau \) as its first return time. If \((K, T^\tau)\) is Bohr-chaotic, so is \((X, T)\).
Proof Let \((w_n)\) be a weight. Assume that
\[
\limsup_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} |w_{\tau n+j_0}| > 0
\]
for some \(0 \leq j_0 < \tau\). Since \((K, T^\tau)\) is Bohr chaotic, there exists \(g \in C(K)\) and \(x_0 \in K\) such that
\[
\limsup_{N \to \infty} \frac{1}{N} \left| \sum_{n=1}^{N-1} w_{\tau(n-1)+j_0} g(T^{\tau n} x_0) \right| > 0. \tag{3.1}
\]
Since \(K^* := \bigcup_{k=1}^{\tau} T^k K\) is disjoint from \(K\), there exists a continuous function \(g^* \in C(K)\) such that \(g^*|_K = g\) and \(g^*|_{K^*} = 0\), by Urysohn’s theorem. It follows that for \(1 \leq j < \tau\), we have
\[
\limsup_{N \to \infty} \frac{1}{N} \left| \sum_{n=1}^{N-1} w_{\tau(n-1)+j_0+j} g^*(T^{\tau n+j} x_0) \right| = 0. \tag{3.2}
\]
From (3.1) and (3.2) we get
\[
\limsup_{N \to \infty} \frac{1}{N} \left| \sum_{n=0}^{\tau N-1} w_{n+j_0} g^*(T^{n+j_0+(\tau-j_0)} x_0) \right| > 0. \tag{3.3}
\]
So, the function \(f := g^*\) and the point \(x := T^{\tau-j_0} x_0\) satisfy the definition (1.3) of the Bohr chaoticity of \((X, T)\).

We shall prove Theorem 1.1 by using Theorem 3.3 and the Bohr chaoticity of the full shift.

3.2 Full shift

On the symbolic space \([0,1]^\mathbb{N}\) we define the shift map \(\sigma\) by \((x_n)_{n \geq 0} \mapsto (x_{n+1})_{n \geq 0}\).

Proposition 3.4 The one-sided full shift \(((0,1]^\mathbb{N}, \sigma)\) is Bohr chaotic.

Proof Let \((w_n)\) be a non-trivial weight. We can assume that \(w_n\)’s are real numbers. Choose the function \(f(x) = 1_{[0]}(x) - 1_{[1]}(x)\). Then choose the point \((x_n)\) defined by \(x_n = 0\) or 1 according to \(w_n \geq 0\) or \(w_n < 0\). Thus we have
\[
\limsup_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} w_n f(\sigma^n x) = \limsup_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} |w_n| > 0.
\]

The Bohr-chaoticity of the two-sided full shift follows from that of the one-sided full shift.

4 Affine maps on torus: Proof of Theorem 1.2

The proof of Theorem 1.2 was essentially contained in [8] in the context of uniform distribution on \(\mathbb{T}^d\) of sequences of the form \((B^n x)\) where \(B\) is an expanding matrix. It uses Riesz product measures on the group \(\mathbb{T}^d\) to find points \(x\) required by (1.3) and it is based on the lacunarity of the powers \(B^n\). We start with two combinatorial lemmas about the lacunarity.
4.1 Two lemmas

A sequence of vectors \( H = (h_n) \subset \mathbb{R}^d \) \((d \geq 1)\) is said to be **dissociate** if any vector in \( \mathbb{R}^d \) can be written in at most one way as a finite sum of the form

\[
\sum \epsilon_j h_j \quad \text{with} \quad \epsilon_j \in \{-1, 0, 1\}.
\]

This notion comes from Hewitt and Zuckermann [18] and it allows us to define the so-called Riesz product measures. For any sequence \( H \), we will use the following notation

\[
H^* = \left\{ \sum_{\text{finite}} \epsilon_j h_j : \epsilon_j \in \{-1, 0, 1\}; \ h_j \in H \right\}.
\]

A sequence of complex numbers \( \Lambda = (\lambda_n)_{n \geq 0} \subset \mathbb{C} \setminus \{0\} \) is said to be **\( \theta \)-lacunary** (à la Hadamard) for some \( \theta > 1 \) if \( |\lambda_{n+1}| \geq \theta |\lambda_n| \) for all \( n \geq 0 \). A \( \theta \)-lacunary sequence with \( \theta \geq 3 \) is dissociate (see a proof in [38], Vol. 1, p.208).

Given a lacunary sequence of complex numbers \( \Lambda = (\lambda_n) \) and an integer \( q \geq 2 \). We decompose \( \Lambda \) into \( q \) parts in the following manner:

\[
\Lambda = \bigcup_{k=0}^{q-1} \Lambda_k \quad \text{with} \quad \Lambda_k = (\lambda_{qn+k})_{n \geq 0}.
\]

Let \( E \) and \( F \) be two subsets of \( \mathbb{R}^n \). Denote by \( D(E, F) = \inf_{x \in E, y \in F} |x - y| \) the distance between \( E \) and \( F \).

**Lemma 4.1** ([8]) *Suppose that \( \Lambda = (\lambda_n) \) be a \( \theta \)-lacunary sequence of complex numbers with \( \theta > 1 \). Assume \( 0 < \delta < |\lambda_0|/(\theta - 1) \). Then there exists an integer \( q_0 \) such that for \( q \geq q_0 \) we have*

(i) \( \Lambda_0 \) is dissociate;

(ii) \( D(\Lambda_1 \cup \cdots \cup \Lambda_{q-1}, \Lambda_0) \geq \delta \);

(iii) \( D((\Lambda_p - \Lambda_q) \setminus \{0\}, \Lambda_0) \geq \delta \) for \( 1 \leq p \leq q - 1 \).

*The above statements remain true if \( \Lambda_0 \) is replaced by any other \( \Lambda_k \).*

Now let us consider a \( d \times d \) matrix \( B \) of entries in \( \mathbb{Z} \) such that \( \det B \neq 0 \). It defines an endomorphism on \( \mathbb{T}^d \). Let \( B^* \) denote the transpose of \( B \). The matrix \( B \) admits its complex Jordan normal form

\[
J = \begin{bmatrix}
J_1 \\
J_2 \\
\vdots \\
J_r
\end{bmatrix}
\]

with

\[
J_k = \begin{bmatrix}
\xi_k & 1 \\
\xi_k & \ddots \\
& \ddots & 1 \\
& & \xi_k
\end{bmatrix}
\]

where \( \xi_1, \xi_2, \ldots, \xi_r \) are the eigenvalues of \( B \) with their multiplicities \( d_1, d_2, \ldots, d_r \). Suppose that

\[
|\xi_1| \geq |\xi_2| \geq \cdots \geq |\xi_r|.
\]

We have

\[
B = T J T^{-1}
\]
where $T = (v_{1,1}, \ldots, v_{1,d_1}; \ldots; v_{r,1}, \ldots, v_{r,d_r})$ is a (complex) non singular matrix. The column vectors of $T$ are generalized eigenvectors of $B$:

$$Bv_k,1 = \xi_k v_k,1, \quad Bv_k,j = \xi_k v_{k,j} + v_{k,j-1} \quad (1 \leq k \leq r, 2 \leq j \leq d_k).$$

Choose a vector $h_0 \in \mathbb{Z}^d$ such that $\langle v_{1,1}, h_0 \rangle \neq 0$. We can choose it among in the canonical basis of $\mathbb{R}^d$. We are interested in the sequence $H = (B^n h_0)_{n \geq 0} \subset \mathbb{Z}^d$. Let $h_n = B^n h_0$. Notice that

$$T^n \sum_{j=0}^{n} (\epsilon_j - \epsilon'_j) h_j = \sum_{j=0}^{n} (\epsilon_j - \epsilon'_j) \text{diag}(J_1^{*j}, \ldots, J_r^{*j}) T^n h_0 \quad (4.1)$$

so that

$$\|T^n\| \|\sum_{j=0}^{n} (\epsilon_j - \epsilon'_j) h_j\| \geq |\langle v_{1,1}, h_0 \rangle| \|\sum_{j=0}^{n} (\epsilon_j - \epsilon'_j) \xi_1\|. \quad (4.2)$$

Here $T^*$ is considered as a linear operator on $\mathbb{C}^d$ which is equipped with the usual Hermite norm. To obtain the inequality (4.2), it suffices to estimate the first coordinate of the vector on the right hand side of (4.1).

Decompose $H$ into $q$ parts ($q \geq 2$):

$$H = \bigcup_{k=0}^{q-1} H_k, \quad H_k = (h_{qn+k})_{n \geq 0}.$$

From the estimation (4.2) and Lemma 4.1, we can get the following fact about the sequence of vectors $(h_n)$ which shares a refined dissociateness.

**Lemma 4.2** ([8]) Let $\rho$ be the spectral radius of $B$ and suppose $\rho > 1$. For $0 < \delta < (\rho - 1)|\langle v_{1,1}, h_0 \rangle|/\|T^*\|$, there exists an integer $q_0 \geq 2$ such that for $q \geq q_0$ we have

(i) $H_0$ is dissociate;

(ii) $D(H_1 \cup \cdots \cup H_{q-1}, H_q^n) \geq \delta$;

(iii) $D((H_p - H_q) \setminus \{0\}, H_q^n) \geq \delta$ for $1 \leq p \leq q - 1$.

The above statements remain true if $H_0$ is replaced by any $H_k$.

We will use the classical Mensov-Rademacher theorem concerning the almost everywhere convergence of orthogonal series, that we state as the following lemma. See [38] vol. 2, p. 193 (see also [9] for a case where the condition $\sum_{n=1}^{\infty} |a_n|^2 \log^2 n < \infty$ can be replaced by $\sum_{n=1}^{\infty} |a_n|^2 < \infty$).

**Lemma 4.3** Let $(X, B, \mu)$ be a probability space. Let $(\phi_n)_{n \geq 1}$ be an orthonormal system in $L^2(X, B, \mu)$. The series $\sum_{n=1}^{\infty} a_n \phi_n(x)$ with $\{a_n\} \subset \mathbb{C}$ converges $\mu$-almost everywhere if $\sum_{n=1}^{\infty} |a_n|^2 \log^2 n < \infty$.

### 4.2 Proof of Theorem 1.2

For $T x = B x + b$ we have

$$T^n x = B^n x + (B^{n-1} + \cdots + B + I) b.$$ 

Take $f(x) = e^{2\pi i \langle h_0, x \rangle}$. We have

$$f(T^n x) = e^{2\pi i \psi_n} e^{2\pi i \langle B^n h_0, x \rangle}$$
where $\psi_n = \langle h_0, (B^{n-1} + \cdots + B + I)b \rangle$.

Let $q$ be sufficiently large such that the statements in Lemma 4.2 hold. For any given non-trivial weight $(w_n)$, there exits $0 \leq k \leq q - 1$ such that

$$\limsup_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} |w_{qn+k}| > 0. \quad (4.3)$$

Without loss of generality, we assume that $k = 0$. Since $H_0$ is dissociate (Lemma 4.2 (i)), for any sequence of complex numbers $(a_n)$ such that $|a_n| \leq 1$, we can define the following Riesz product measure on $\mathbb{T}^d$ (see [18])

$$v_a = \prod_{n=0}^{\infty} \left(1 + \text{Re}(a_n e^{2\pi i \langle B^{*qn}h_0, x \rangle}) \right). \quad (4.4)$$

This Borel probability measure $v_a$ is characterized by its Fourier coefficients as follows

$$\hat{v}_a \left( \sum_{n} \epsilon_n B^{*qn}h_0 \right) = \prod_{n} a_n^{(\epsilon_n)} \quad (4.5)$$

where $a_n^{(0)} = 1$, $a_n^{(1)} = a_n/2$ and $a_n^{(-1)} = \overline{a}_n/2$; and

$$\hat{v}_a(n) = 0 \text{ if } n \notin H_0^g. \quad (4.6)$$

By the formula (4.5), $\{e^{2\pi i \langle B^{*qn}h_0, x \rangle} - \frac{\overline{a}_n}{2}\}$ is an orthogonal system in $L^2(v_a)$. Then, by the Menshov–Rademacher theorem, the series

$$\sum_{n} w_{qn} e^{2\pi i \psi_{qn}} \left( e^{2\pi i \langle B^{*qn}h_0, x \rangle} - \frac{\overline{a}_n}{2} \right)$$

converges $v_a$-almost everywhere. By Kronecker’s lemma, it follows that $v_a$-almost everywhere

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} w_{qn} e^{2\pi i \psi_{qn}} \left( e^{2\pi i \langle B^{*qn}h_0, x \rangle} - \frac{\overline{a}_n}{2} \right) = 0. \quad (4.8)$$

We will fix our choice for $a_n$ as follows

$$a_n = r e^{i \arg w_{qn} + 2\pi i \psi_{qn}}, \quad (4.7)$$

where $r$ is any fixed number such that $0 < r \leq 1$. So, for $v_a$-almost every $x$ we have

$$\limsup_{N \to \infty} \frac{1}{N} \left| \sum_{n=0}^{N-1} w_{qn} f(T^{qn}x) \right| = \frac{r}{2} \limsup_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} |w_{qn}| > 0. \quad (4.8)$$

Fix $1 \leq p < q$ and let $X_n = e^{2\pi i \langle B^{*qn+p}h_0, x \rangle}$. By the formula (4.6) and Lemma 4.2 (ii) and (iii), we have

$$E_{v_a} X_n = 0, \quad E_{v_a} X_n X_m = 0 \text{ if } n \neq m. \quad (4.9)$$

Again, by using the Menshov–Rademacher theorem to the orthogonal system $\{X_n\}$, for $v_a$-almost every $x$ we have

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} w_{qn+p} f(T^{qn+p}x) = 0. \quad (4.9)$$
From (4.8) and (4.9) we conclude that for $\nu_a$-almost every point $x$ we have

$$\limsup_{N \to \infty} \frac{1}{N} \left| \sum_{n=0}^{N-1} w_n f(T^n x) \right| > 0. \tag{4.10}$$

Thus we have proved the Bohr chaoticity of $(T^d, T)$.

The Hausdorff dimension of the set of $x$ such that (4.10) holds is not less than the Hausdorff dimension of the measure $\nu_a$. But $\dim \nu_a$ tends to $d$ when $r \to 0$ (cf. [6]).

We remark that if $(w_n)$ is aperiodic and if we take $a_n = re^{2\pi i \psi_n}$, the weighted ergodic limit exists:

$$\nu_a - a.e. \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} w_n f(T^n x) = \frac{r}{2} \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} w_{qn}.$$

Notice that the above limit on the right hand side exists since $(w_n)$ is aperiodic.

**4.3 Bohr chaotic automorphisms on the torus having no horseshoe**

Hyperbolic automorphisms on torus admit horseshoes. So, Theorem 1.1 covers the class of all hyperbolic automorphisms on torus. But it doesn’t cover all endomorphisms of positive entropy concerned by Theorem 1.2. For example, let $B \in \text{GL}(d, \mathbb{Z})$ with $\det B = \pm 1$.

Suppose that

(i) the characteristic polynomial $\chi_B$ of $B$ is irreducible over $\mathbb{Q}$;

(ii) some but not all of eigenvalues of $B$ is on the unit circle.

Then the automorphism on the torus $\mathbb{T}^d$ defined by $T x = B x \mod \mathbb{Z}^d$ has no homoclinic point (see Example 3.4. in [27]). Consequently, for any integer $m \geq 1$, the power $T^m$ has no homoclinic point. Since the full shift $([0, 1]^{\mathbb{Z}}, \sigma)$ has a dense set of homoclinic points, $T$ can not admit a horseshoe.

Therefore it is not possible to prove Theorem 1.2 by constructing a horseshoe as we do for proving Theorem 1.1. The family of systems admitting horseshoes is a proper sub-family of the family of Bohr chaotic systems.

However, the above automorphism $T$ admits a semi-horseshoe, because according to [19], any automorphism on a compact metric abelian group having positive topological entropy has a semi-horseshoe. The above automorphism also shows that a partially hyperbolic system can have a semi-horseshoe, but not a horseshoe.

**5 Proof of Theorem 1.1: case of one-sided horseshoe**

In order to prove Theorem 1.1 (the case of one-sided horseshoe), it suffices to prove, by Theorem 3.3, that a topological dynamical system having a one-sided horseshoe must have a horseshoe with disjoint steps, as we state as follows

**Theorem 5.1** Suppose that $(X, T)$ is a topological dynamical system admitting a one-sided $N$-order horseshoe $(\Lambda, T_N)$. Then there exist a closed subset $K \subset \Lambda$ and a positive integer $M$ such that $(K, T^{MN})$ is a one-sided horseshoe of $(X, T)$ with disjoint steps.

The strategy for proving Theorem 1.1 in the case of two-sided horseshoe is the same. But there are some differences in details. The proof of this case is postponed to the next section.
The proof of Theorem 5.1 is rather long. In the next, we first present the ideas of proof and then present the proof in several subsections.

5.1 Ideas of proof

Let \((\Lambda^*, T^N)\) be the given horseshoe. We consider the extended horseshoe \(T^*: \Lambda^* \to \Lambda^*\) where
\[
\Lambda^* = \bigcup_{n=0}^{N-1} T^n(\Lambda).
\]

It is a subsystem of \((X, T)\). One of key points for proving Theorem 5.1 is that \((\Lambda^*, T^N)\) has no roots, i.e. there is no continuous map \(S: \Lambda^* \to \Lambda^*\) such that \(S^n = T\) for some \(n \geq 2\). Its exact statement is as follows.

**Proposition 5.2** Suppose that \((X, T)\) is a topological dynamical system admitting a \(N\)-order one-sided horseshoes \((\Lambda^*, T^N)\) such that

(i) \((\Lambda^*, T^N)\) has disjoint steps;

(ii) every map \(T: T^j(\Lambda) \to T^{j+1}(\Lambda)\) is bijective for \(0 \leq j \leq N - 2\).

Then for any integer \(n \geq 2\), there is no continuous map \(S: \Lambda^* \to \Lambda^*\) such that \(S^n = T\).

The proof of this proposition will be given in Appendix A.

The second key point is that in any cylinder of \(\{0, 1\}^N\), there is a horseshoe with disjoint steps for the shift map. This will be proved in Appendix B. Then, since \((\Lambda, T^N)\) is conjugate to \(\{0, 1\}^N\), we can find horseshoes \(\Lambda'\) in any non-empty open set of \(\Lambda\). These horseshoes \(\Lambda'\), called prototypes of horseshoes, have not yet the required disjoint steps, but some steps are really disjoint from \(\Lambda'\). Actually, from the fact that \((\Lambda, T^N)\) is conjugate to the shift map, we can easily find our first prototype of horseshoe which is of the form \((\Lambda', T^N)\) for some integer \(M \geq 1\) such that \(T^k(\Lambda')\) for \(1 \leq k < M\) are disjoint from \(\Lambda'\) (a partial disjointness). This is the starting point of our construction by induction.

In general, given a horseshoe \((\Lambda', T^{pq})\) \((p, q\) being integers) such that
\[
\Lambda' \cap T^{kq}(\Lambda') = \emptyset \quad (1 \leq k < p),
\]
we have
\[
\Lambda' \not\subset \bigcup_{k=0}^{p-1} T^{kq+s}(\Lambda') \quad \text{for any } 1 \leq s < q \text{ with } s|q
\]
(cf. Proposition 5.3). This is a consequence of the fact that \(T\) has no roots. We could say that \(\Lambda'\) is displaced by \(T^{kq+s}\)'s. The second key point allows us to find a better horseshoe \(\Lambda'' \subset \Lambda'\) such that
\[
\Lambda'' \subset \Lambda' \setminus \bigcup_{k=0}^{p-1} T^{kq+s}(\Lambda').
\]

Here, by “better” we mean that \(\Lambda''\) has “more” disjointness. Indeed, from (5.1), (5.2) and (5.3) we get
\[
\Lambda'' \cap T^{kq}(\Lambda'') = \emptyset, \quad \Lambda'' \cap T^{kq+s}(\Lambda'') = \emptyset \quad (1 \leq k < p, 1 \leq s < q \text{ with } s|q).
\]
Basing on the decomposition of $N = p_1^{y_1} \cdots p_s^{y_s}$ into primes, recursively we can find finer and finer horseshoes to finally get what we want.

We invite the readers to read the isolated proofs for the case $N = 2$ (Sect. 5.3) and the case $N = 3$ (Sect. 5.4), which show some more details of the above ideas. The machinery of induction is explained in Proposition 5.5 and Proposition 5.6. The proof in the case $N = 2$ doesn’t need Proposition 5.5 and Proposition 5.6. The proof in the case $N = 3$ only need Proposition 5.5.

5.2 Displacement of horseshoe

Let $(\Lambda, T^N)$ be a $N$-order one-sided horseshoe. If $\Lambda \nsubseteq T^j(\Lambda)$ for some $1 \leq j < N$, we say that $\Lambda$ is displaced by $T^j$. We shall prove that the horseshoe $\Lambda$ is displaced by $T^j$ when $j | N$.

Actually the following more general Proposition 5.3, which is a consequence of Proposition 5.2.

**Proposition 5.3** Let $(X, T)$ be a topological dynamical system. Suppose that $(X, T)$ has a $pq$-order one-sided horseshoe $(\Lambda, T^{pq})$ for some integers $p \geq 1$ and $q \geq 1$ such that

(i) $\Lambda \cap T^{kq}(\Lambda) = \emptyset$ for $1 \leq k \leq p - 1$;

(ii) $T^{(p-1)q} : \Lambda \to T^{(p-1)q}(\Lambda)$ is a bijection.

Then for any integer $1 \leq s < q$ such that $s | q$ we have $\Lambda \nsubseteq \bigcup_{k=0}^{p-1} T^{kq+s}(\Lambda)$.

**Proof** We give a proof by contradiction. Suppose that $\Lambda \subseteq \bigcup_{k=0}^{p-1} T^{kq+s}(\Lambda)$ for some $s$ with $s | q$. Then, from the fact $\Lambda = T^{pq}(\Lambda)$, we get

$$T^{\ell q}(\Lambda) \subset T^{\ell q} \left( \bigcup_{k=0}^{p-1} T^{kq+s}(\Lambda) \right) = \bigcup_{k=0}^{p-1} T^{kq+s}(\Lambda) \quad \text{for } 1 \leq \ell \leq p - 1.$$ 

Therefore, $\bigcup_{k=0}^{p-1} T^{kq}(\Lambda) \subset \bigcup_{k=0}^{p-1} T^{kq+s}(\Lambda)$. Let

$$\Lambda^*_q = \bigcup_{k=0}^{p-1} T^{kq}(\Lambda).$$

We claim that $\Lambda^*_q = T^s(\Lambda^*_q)$. Indeed, from $\Lambda^*_q \subset T^s(\Lambda^*_q)$ which is just proved above, we have

$$\Lambda^*_q \subset T^s(\Lambda^*_q) \subset T^{2s}(\Lambda^*_q) \subset \cdots \subset T^{pq}(\Lambda^*_q) = \Lambda^*_q.$$ 

Now let us consider the dynamical system $(\Lambda^*_q, T^q)$ which admits a $p$-order horseshoe $(\Lambda^*_q, T^{pq})$ with disjoint steps. On one hand, $T^s : \Lambda^*_q \to \Lambda^*_q$ is a $q/s$-root of $T^q$. On the other hand, the following factor maps are bijective:

$$\Lambda \xrightarrow{T^q} T^q(\Lambda) \xrightarrow{T^q} T^{2q}(\Lambda) \xrightarrow{T^q} \cdots \xrightarrow{T^q} T^{(p-2)q}(\Lambda) \xrightarrow{T^q} T^{(p-1)q}(\Lambda)$$

because the map $T^{(p-1)q} : \Lambda \to T^{(p-1)q}(\Lambda)$ is supposed bijective. These two facts together contradict to Proposition 5.2 when we consider the system $(\Lambda^*_q, T^q)$.

Taking $p = 1$ and $N = q$, as an immediate consequence, we obtain that $\Lambda \nsubseteq T^s(\Lambda)$.

**Corollary 5.4** Let $(X, T)$ be a topological dynamical system admitting an $N$-order horseshoe $(\Lambda, T^N)$ with $N \geq 2$. Then for any positive integer $1 \leq s < N$ dividing $N$ we have $\Lambda \nsubseteq T^s(\Lambda)$. 

\(\square\) Springer
5.3 Proof of Theorem 5.1: $N = 2$

The proof will be based on Proposition 5.3 (displacement of horseshoe) and Proposition 8.2 (construction of horseshoe in a cylinder). The argument will be inductive and several constructions of horseshoe will be performed and the final horseshoe will have a high order. The proof will be relatively easy if the order $N$ is a power of prime i.e. $N = p^m$. In the following, we first give a proof for $N = 2$ which is rather direct and for $N = 3$ which shows the idea for the general case.

Proof for 2-order horseshoe. Suppose $(\Lambda, T^2)$ is a one-side horseshoe of the topological dynamical system $(X, T)$. By Corollary 5.4, $\Lambda$ is displaced by $T$. In other words, there exist a point $x \in \Lambda$ such that $x \notin T(\Lambda)$. Since $T$ is continuous and $\Lambda$ is compact, there exist a neighborhood $U \subset \Lambda$ of $x$ such that $U \cap T(\Lambda) = \emptyset$. Since $(\Lambda, T^2)$ is topologically conjugate to the full shift system $([0, 1]^N, \sigma)$, by Proposition 8.2, there exist an integer $M$ and a subset $K \subset U$ such that $(K, T^2M)$ is a horseshoe with $K \cap T^{2j}(K) = \emptyset$ for $1 \leq j \leq M - 1$. Notice that $T^{2j+1}(K) \subset T^{2j+1}(\Lambda) = T(\Lambda)$, because of $T^2(\Lambda) = \Lambda$. From $U \cap T(\Lambda) = \emptyset$, it follows that $K \cap T^{2j+1}(K) = \emptyset$ for $j \geq 0$.

Thus we have proved that $(K, T^{2M})$ is a horseshoe with disjoint steps of $(X, T)$.

5.4 Proof of Theorem 5.1: $N = 3$

Proof for 3-order horseshoe. Suppose $(\Lambda, T^3)$ is a one-side horseshoe of the topological dynamical system $(X, T)$. The proof is decomposed into two steps.

- By the same argument as in the case for 2-order horseshoe, there exist an integer $M_1$ and a subset $\Lambda_1$ such that $(\Lambda_1, T^{3M_1})$ is a horseshoe with $\Lambda_1 \cap T^{3j}(\Lambda_1) = \emptyset$ for $1 \leq j \leq M_1 - 1$, and $\Lambda_1 \cap T^{3j+1}(\Lambda_1) = \emptyset$ for $0 \leq j \leq M_1 - 1$.

- This second step is sketchy. Details are in the proof of Proposition 5.5 (corresponding to the special case $p = M_1, q = 3, J = \{1\}$). In general, $\Lambda_1$ may intersect $T^{3j+2}(\Lambda_1)$ for some $0 \leq j \leq M_1 - 1$. We claim the following displacement (see (5.5)):

$$\Lambda_1 \not\subset \bigcup_{j=0}^{M_1-1} T^{3j+2}(\Lambda_1).$$

Then we can construct a subset $\Lambda_2 \subset \Lambda_1$ such that $(\Lambda, T^{3M_1M_2})$ is a horseshoe for some integer $M_2$. It can be checked that the horseshoe $(\Lambda, T^{3M_1M_2})$ has disjoint steps.

Here, for the second step, we prove the displacement by using the simple arithmetic fact $2 + 2 \equiv 1 (\text{mod } 3)$ (taking $s = 2, n = 2$ and $q = 3$ in Proposition 5.5).
5.5 Construction of finer horseshoes

The following proposition improves the disjointness of the steps of a horseshoe, by constructing a smaller horseshoe.

**Proposition 5.5** Let $p, q$ be two positive integers and let $J \subset \{1, 2, \ldots, q\}$. Suppose that a topological dynamical system $(X, T)$ has a $T^{pq}$-invariant subset $\Lambda \subset X$ such that

(a) $(\Lambda, T^{pq})$ is a horseshoe;
(b) $\Lambda \cap T^k(\Lambda) = \emptyset$ for $1 \leq k \leq p - 1$;
(c) $\Lambda \cap T^{kq+j}(\Lambda) = \emptyset$ for $0 \leq k \leq p - 1$ and $j \in J$

Then for any integer $s \geq 1$ such that $ns \pmod{q} \in J$ for some positive integer $n$, there exists an integer $M \geq 1$ and a subset $\Lambda' \subset \Lambda$ such that

(A) $(\Lambda', T^{Mpq})$ is a horseshoe;
(B) $\Lambda' \cap T^k(\Lambda') = \emptyset$ for $1 \leq k \leq pM - 1$;
(C) $\Lambda' \cap T^{kq+j}(\Lambda') = \emptyset$ for $0 \leq k \leq pM - 1$ and $j \in J \cup \{s\}$.

**Proof** Let $\Lambda^* = \bigcup_{k=0}^{p-1} T^{kq}(\Lambda)$. We claim that

\[ \Lambda \not\subset T^s(\Lambda^*) . \]  

(5.5)

Otherwise, by the fact $T^{pq}(\Lambda) = \Lambda$, we have $T^q(\Lambda^*) = \Lambda^*$, which implies

\[ T^{kq}(\Lambda) \subset T^{kq+s}(\Lambda^*) = T^s(\Lambda^*) \text{ for } 0 \leq k \leq p - 1. \]

Hence, $\Lambda^* \subset T^s(\Lambda^*)$, and consequently

\[ \forall \ell \in \mathbb{N}, \quad \Lambda^* \subset T^{\ell s}(\Lambda^*) . \]  

(5.6)

However, by the assumption on $s$, there exists a positive integer $n$ such that

\[ T^{ns}(\Lambda^*) \subset \bigcup_{j \in J} T^j(\Lambda^*) . \]

This, together with (5.6), implies

\[ \Lambda^* \subset \bigcup_{j \in J} T^j(\Lambda^*) . \]

This contradicts the condition (c).

By (5.5), there exists a point $x \in \Lambda \setminus T^s(\Lambda^*)$. We take an open neighborhood $U$ of $x$ such that $U \cap T^s(\Lambda^*) = \emptyset$. By the condition (a) and Proposition 8.2, there exists a $T^{Mpq}$-invariant set $\Lambda' \subset \Lambda \cap U$ for some integer $M \geq 1$ such that $(\Lambda', T^{Mpq})$ is conjugated to $((0, 1)^{\mathbb{N}}, \sigma)$ and

\[ \Lambda' \cap T^{\ell pq}(\Lambda') = \emptyset \text{ for } 1 \leq \ell \leq M - 1. \]  

(5.7)

It remains to check (B) and (C).

Write $k = sp + r$ with $1 \leq s \leq M - 1$ and $0 \leq r \leq p - 1$. If $r = 0$, we get the property (B) from (5.7). If $r \neq 0$, we get the property (B) from (b) and the facts $\Lambda' \subset \Lambda$ and $T^{pq}\Lambda = \Lambda$. Indeed,

\[ \Lambda' \cap T^{kq}(\Lambda') = \Lambda' \cap T^{spq+rq}(\Lambda') \subset \Lambda' \cap T^{rq}(\Lambda) = \emptyset. \]

(B) is thus checked.
Now we first check (C) for $j \in J$, using (c). Write $k = sp + r$ with $0 \leq s \leq M - 1$ and $0 \leq r \leq p - 1$. Using $T^{pq}(\Lambda) = \Lambda$ and $\Lambda' \subseteq \Lambda$, we get
\[T^{kq+j}(\Lambda') \subset T^{spq+rq+j}(\Lambda) = T^{rq+j}(\Lambda).\]
The last set is disjoint from $\Lambda$, by (c). So, it is disjoint from $\Lambda'$. Then let us check (C) for $j = s$.

Note that
\[T^{kq+s}(\Lambda') \subset T^{kq+s}(\Lambda) \subset T^s(\Lambda^*).\]
The facts $\Lambda' \subset U$ and $U \cap T^s(\Lambda^*) = \emptyset$ imply that $\Lambda'$ is disjoint from $\bigcup_{k=0}^{p-1} T^{kq+s}(\Lambda)$. This, together with $T^{pq}(\Lambda) = \Lambda$, implies that $\Lambda'$ is disjoint from $\bigcup_{k=0}^{Mp-1} T^{kq+s}(\Lambda')$. \hfill \Box

The proof for the case of 3-order horseshoe uses Proposition 5.5. Using Proposition 5.5 inductively, we can give a complete proof of Theorem 5.1 for the case $N = p'$. However, for the general case, Proposition 5.5 is not enough for the proof of Theorem 5.1. For example, when $N = 6$, for any factor $j$ of 6, there exists a number $s \in \{2, 3\}$ such that $ns \not\equiv j \pmod{6}$ for any positive integer $n$.

The following proposition is another improvement of disjointness of steps of horseshoe.

**Proposition 5.6** Let $p$, $q$ be two positive integers and let $J \subset \{1, 2, \cdots, q\}$. Suppose that a topological dynamical system $(X, T)$ has a $T^{pq}$-invariant subset $\Lambda \subset X$ such that
(a) $(\Lambda, T^{pq})$ is a horseshoe;
(b) $T^{(p-1)q} : \Lambda \to T^{(p-1)q}(\Lambda)$ is a bijection;
(c) $\Lambda \cap T^{kq}(\Lambda) = \emptyset$ for $1 \leq k \leq p - 1$;
(d) $\Lambda \cap T^{kq+j}(\Lambda) = \emptyset$ for $0 \leq k \leq p - 1$ and $j \in J$.

Then for any integer $s \geq 1$ such that $s \mid q$, there exist an integer $M$ and a subset $\Lambda' \subset \Lambda$ such that
(a') $(\Lambda', T^{Mpq})$ is a horseshoe;
(b') $T^{(pM-1)q} : \Lambda' \to T^{(pM-1)q}(\Lambda')$ is a bijection;
(c') $\Lambda' \cap T^{kq}(\Lambda') = \emptyset$ for $1 \leq k \leq pM - 1$;
(d') $\Lambda' \cap T^{kq+s}(\Lambda') = \emptyset$ for $0 \leq k \leq pM - 1$ and $j \in J \cup \{s\}$.

**Proof** Let $\Lambda^* = \bigcup_{k=0}^{p-1} T^{kq}(\Lambda)$.

We have $T^s(\Lambda^*) = \bigcup_{k=0}^{p-1} T^{kq+s}(\Lambda)$. Under the condition (a), (b) and (c), we apply Proposition 5.3 to get the displacement $\Lambda \not\subseteq T^s(\Lambda^*)$. Choose $x \in \Lambda$ such that $x \not\in T^s(\Lambda^*)$ and then choose an open neighborhood $U$ of $x$ such that $U$ and $T^s(\Lambda^*)$ are disjoint. As $(\Lambda, T^{pq})$ is conjugate to the full shift $(\{0, 1\}^\mathbb{N}, \sigma)$, we can assume that $U = \pi(C)$ for some cylinder $C$, where $\pi : (\{0, 1\}^\mathbb{N}, \sigma) \to (\Lambda, T^{pq})$ is the conjugation. Applying Proposition 8.2 to $C$ to get a horseshoe contained in $C$ and then projecting the horseshoe by $\pi$. Thus we get a $T^{Mpq}$-invariant set $\Lambda' \subset U \cap \Lambda$ for some positive integer $M$, such that
(a') $(\Lambda', T^{Mpq})$ is a horseshoe;
(b') $T^{(pM-1)q} : \Lambda' \to T^{(pM-1)q}(\Lambda')$ is bijective;
(c') $\Lambda' \cap T^{kq}(\Lambda') = \emptyset$ for $1 \leq k \leq M - 1$;
(d') $\Lambda' \cap T^{kq+s}(\Lambda') = \emptyset$ for $0 \leq k \leq pM - 1$.

Let us check (C), (D) and then (B).

Since $\Lambda' \subset \Lambda$ and $T^{pq}(\Lambda) = \Lambda$, (C) follows from (c) and (C').
Since
\[ \Lambda' \cap T^{s}(\Lambda^*) = \emptyset, \]
(D) follows from (d).

Recall the facts \( \Lambda' \subset \Lambda \) and \( T^{(M-1)pq}(\Lambda) = \Lambda \). Also recall that \( T^{(p-1)q} \) is a bijection from \( \Lambda \) to \( T^{(p-1)q}(\Lambda) \), by (b). It follows that \( T^{(p-1)q} \) is a bijection from \( T^{(M-1)pq}(\Lambda') \) to \( T^{(M-1)pq}(\Lambda') \). Hence, (B) follows from (B').

**5.6 A simple arithmetic fact**

In the proof of Theorem 5.1, we shall use a simple arithmetic fact stated as Lemma 5.7 below. Let \( N \geq 2 \) be an integer and let us consider the cyclic group \( \mathbb{Z}/N\mathbb{Z} \) identified with \([0, 1, \cdots, N - 1]\). For each element \( a \in \mathbb{Z}/N\mathbb{Z} \), denote by \( \langle a \rangle \) the subgroup generated by \( a \).

Assume that \( N = p_1^{\gamma_1} p_2^{\gamma_2} \cdots p_s^{\gamma_s} \) is decomposed into primes with \( \gamma_j \geq 1 \) \( (j = 1, 2, \cdots, s) \).

It is easy to see that the subgroups \( \langle N/p_1 \rangle, \cdots, \langle N/p_s \rangle \) are exactly the smallest nontrivial subgroups of \( \mathbb{Z}/N\mathbb{Z} \). Here, by smallest nontrivial subgroup, we mean a subgroup which does not contain a non-trivial proper subgroup. It is also easy to see that each nontrivial subgroup of \( \mathbb{Z}/N\mathbb{Z} \) contains at least one subgroup \( \langle N/p_i \rangle \). These facts imply the following lemma.

**Lemma 5.7** Let \( N = p_1^{\gamma_1} p_2^{\gamma_2} \cdots p_s^{\gamma_s} \) be a positive integer decomposed into primes with \( \gamma_j \geq 1 \) \( (j = 1, 2, \cdots, s) \).

Then for each integer \( n \in \{1, 2, \cdots, N - 1\} \), there exist an integer \( x \in \{1, 2, \cdots, N - 1\} \) and a prime \( p_i \) \( (1 \leq i \leq s) \) such that
\[ nx \equiv \frac{N}{p_i} \mod N. \]

**Proof** For each given nonzero element \( n \in \mathbb{Z}/N\mathbb{Z} \), the group \( \langle n \rangle \) contains a subgroup \( \langle N/p_i \rangle \) for some \( 1 \leq i \leq s \). So, \( N/p_i \in \langle n \rangle \), which means that \( N/p_i = kn \mod N \) for some positive integer \( k \in \{1, 2, \cdots, N - 1\} \).

**5.7 Proof of Theorem 5.1: for general \( N \geq 1 \)**

There is nothing to prove if \( N = 1 \). Then assume \( N \geq 2 \). Write \( N = p_1^{\gamma_1} p_2^{\gamma_2} \cdots p_s^{\gamma_s} \), where \( p_1 > p_2 > \cdots > p_s \geq 2 \) are distinct prime numbers and \( \gamma_1 \geq 1, \cdots, \gamma_s \geq 1 \). For \( 1 \leq i \leq s \), let \( N_i = \frac{N}{p_i} \).

We shall divide our proof into two steps. Our machinery is to construct a sub-horseshoe whenever a horseshoe is displaced, using Proposition 5.5 or Proposition 5.6, which are based on Proposition 5.3 and Proposition 8.2 in Appendix B. The first step is to construct a horseshoe \( (\Lambda_s, T^{L_s N}) \), where \( \Lambda_s \subset \Lambda \) and \( L_s > 1 \) is an integer, such that

1. \( \Lambda_s \cap T^{nN}(\Lambda_s) = \emptyset \) for \( 1 \leq n \leq L_s - 1 \);
2. \( \Lambda_s \cap T^{nN+N_i}(\Lambda_s) = \emptyset \) for all \( 1 \leq i \leq s \) and all \( 0 \leq n \leq L_s - 1 \).

The second step is to repeat the same machinery to construct a subset \( \Lambda' \subset \Lambda_s \) such that for some integer \( M \) we have

3. \( (\Lambda', T^{M N}) \) is a horseshoe;
4. \( \Lambda' \cap T^{n}(\Lambda') = \emptyset \) for \( 1 \leq n \leq MN - 1 \).

This is just what we would like to prove. As we shall see, Lemma 5.7 will be useful in the second step.

**Step 1.** Take \( p = 1, N = q, s = N_1 \) and \( J = \emptyset \). By Proposition 5.6,
Furthermore, we have

(A1) $(\Lambda_1, T^{M_1 N})$ is conjugate to $([0, 1]^N, \sigma)$;
(B1) $T^{n N} : \Lambda_1 \to T^{n N}(\Lambda_1)$ is bijective for each $1 \leq n \leq M_1 - 1$;
(C1) $\Lambda_1, T^{N}(\Lambda_1), \ldots, T^{(M_1 - 1) N}(\Lambda_1)$ are pairwise disjoint.

Furthermore, we have

(D1) $\Lambda_1 \cap T^{n N + N_1}(\Lambda_1) = \emptyset$ for $0 \leq n \leq M_1 - 1$.

Indeed, on one hand, the fact $U \cap T^{N_1}(\Lambda) = \emptyset$ implies $\Lambda_1 \cap T^{N_1}(\Lambda) = \emptyset$. On the other hand, the fact $\Lambda = T^{N}(\Lambda)$ implies $T^{N_1} = T^{n N + N_1}(\Lambda)$ for all integers $n \geq 0$. These imply (D1) because $\Lambda_1 \subset \Lambda$.

If $N = 2$, we are done because both (C1) and (D1) shows that $\Lambda_1$ is a horseshoe with disjoint steps. If $N \geq 3$ is of the form $p^r$ ($p$ being a prime), we go directly to Step 2.

Using Proposition 5.6, by induction on $\ell \in \{1, 2, \ldots, s\}$, we get that there exist positive integers $M_\ell$ ($1 \leq \ell \leq s$) and a $T^{L_\ell N}$-invariant closed set $\Lambda_\ell$, where $L_\ell = M_1 M_2 \cdots M_\ell$, such that

(A_{\ell}) $(\Lambda_\ell, T^{L_\ell N})$ is conjugate to $([0, 1]^N, \sigma)$;
(B_{\ell}) $T^{n N} : \Lambda_\ell \to T^{n N}(\Lambda_\ell)$ is bijective for $1 \leq n \leq L_\ell - 1$;
(C_{\ell}) $\Lambda_\ell \cap T^{N_1}(\Lambda_\ell) = \emptyset$ for $1 \leq n \leq L_\ell - 1$;
(D_{\ell}) $\Lambda_\ell \cap T^{N_1 + N_1}(\Lambda_\ell) = \emptyset$ for all $1 \leq i \leq \ell$ and all $0 \leq n \leq L_\ell - 1$.

**Step 2.** Let $\{1, 2, \ldots, N - 1\} \setminus \{N_1, N_2, \ldots, N_s\} = \{k_1, k_2, \ldots, k_{(N - 1) - s}\}$. Lemma 5.7 says that for each $k_j$, there exist a positive integer $n$ such that $nk_j \equiv N_j (\text{mod } N)$ for some $N_j$.

Then, using Proposition 5.5, by induction on $k_1, k_2, k_3, \ldots, k_{N-s}$ we get an integer $M \geq 1$ and a $T^{MN}$-invariant set $\Lambda'$ such that

(A') $(\Lambda', T^{MN})$ is conjugated to $([0, 1]^N, \sigma)$,
(B') $\Lambda' \cap T^{N}(\Lambda') = \emptyset$ for $1 \leq n \leq M - 1$,
(C') $\Lambda' \cap T^{N + N_1}(\Lambda') = \emptyset$ for $1 \leq i \leq s$ and $0 \leq n \leq M - 1$,
(D') $\Lambda' \cap T^{N + k_j}(\Lambda') = \emptyset$ for $0 \leq n \leq M - 1$ and $1 \leq j \leq (N - 1) - s$.

Notice that the combination of (C') and (D') means $\Lambda' \cap T^n(\Lambda') = \emptyset$ for $1 \leq n \leq MN - 1$, which is the desired disjointness.

### 6 Proof of Theorem 1.1: case of two-sided Horseshoe

In this section, we shall prove Theorem 1.1 for topological dynamical systems $(X, T)$ having two-sided horseshoes $(\Lambda, T^N)$. As in the case of one-sided horseshoe, what we have to show is the following result, similar to Theorem 5.1.

**Theorem 6.1** Suppose that $(X, T)$ is a topological dynamical system admitting a two-sided $N$-order horseshoe $(\Lambda, T^N)$. Then there exist a closed subset $K \subset \Lambda$ and a positive integer $M$ such that $(K, T^{MN})$ is a two-sided horseshoe of $(X, T)$ with disjoint steps.

But the proof of Theorem 6.1 will be different from that of Theorem 5.1. The key is the non-existence of $n$-th root of the extended horseshoe, stated as follows

**Proposition 6.2** Suppose that $(X, T)$ is a topological dynamical system admitting an $N$-order two-sided horseshoe $(\Lambda, T^N)$ with disjoint steps. Let $\Lambda^* = \bigcup_{k=0}^{N-1} T^k(\Lambda)$. Then for any integer $n \geq 2$, there is no continuous map $S : \Lambda^* \to \Lambda^*$ such that $S^n = T$. 

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The proof of this proposition is given in Appendix A. The other parts of proof are almost the same as in the case of one-sided horseshoes (Theorem 5.1). We just sketch a quick proof by pointing out the differences from the case of one-sided horseshoe.

Notice that the maps $T^n \ (n \geq 1)$ are now bijective on $\Lambda$. This brings to something easier because the condition (ii) concerning the bijective property of $T$ in Proposition 5.2 and 5.3 is automatically satisfied.

About the displacement of horseshoe, we have the following result. Let $(\Lambda, T^N)$ be a $N$-order two-sided horseshoe. Recall that, if $\Lambda \not\subset T^j(\Lambda)$ for some $1 \leq j < N$, we say that $\Lambda$ is displaced by $T^j$. It is true that the horseshoe $\Lambda$ is displaced by $T^j$ when $j|N$. This is actually a special case of the following Proposition 6.3 ($p = 1, q = N$), whose proof is based on Proposition 6.2.

**Proposition 6.3** Let $(X, T)$ be a topological dynamical system. Suppose that $(X, T)$ has a $pq$-order two-sided horseshoe $(\Lambda, T^{pq})$ for some integers $p \geq 1$ and $q \geq 1$ such that

$\Lambda \cap T^{kq}(\Lambda) = \emptyset$ for $1 \leq k \leq p - 1$.

Then for any integer $1 \leq s < q$ such that $s \mid q$ we have $\Lambda \not\subset \bigcup_{k=0}^{p-1} T^{kq+s}(\Lambda)$.

We omit the proof of Proposition 6.3, since it is the same as the proof of Proposition 5.3. Note that in the present case, $T^n : \Lambda \to T^n(\Lambda)$ is bijective for every $n \geq 1$. Hence, the condition (ii) in Proposition 5.3 is automatically satisfied.

Taking $p = 1$ and $q = N$, we obtain the following immediate corollary. In particular, $\Lambda \not\subset T(\Lambda)$.

**Corollary 6.4** Let $(X, T)$ be a topological dynamical system admitting an $N$-order horseshoe $(\Lambda, T^N)$ with $N \geq 2$. Then for any positive integer $1 \leq s < N$ dividing $N$ we have $\Lambda \not\subset T^s(\Lambda)$.

**Proof of Theorem 6.1**: Based on Proposition 6.3 (displacement of horseshoe) and Proposition 8.4 (construction of horseshoe in a cylinder), the proof is the same as in the case of one-sided horseshoe. There is no need to repeat it.

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**7 Appendix A: Extended horseshoes have no roots**

The extended horseshoe has no root. A self-contained proof of this fact will be given below for the case of one-sided horseshoe. A different proof is given for the case of two-sided horseshoe and it relies on the no existence of root for the two-sided full shift, which is known ( [16]). It seems that the first proof can not be adapted to the case of two-sided horseshoe.
7.1 No root of $T : \Lambda^* \to \Lambda^*$ (one-sided case)

Here we give a proof of Proposition 5.2.

The proof is based on the investigation of the maximal entropy measure of the dynamical system $(\Lambda^*, T)$. Given a topological dynamical system $(X, T)$ and an integer $s \geq 2$. A $T$-invariant measure is $T^s$-invariant. If such a measure is $T^s$-ergodic, it is $T$-ergodic. But in general, a $T$-ergodic measure is not necessarily $T^s$-ergodic. That is the case for the invariant measure $\frac{1}{2} (\delta_{\frac{1}{4}} + \delta_{\frac{3}{4}})$ of the doubling dynamics $x \to 2x \mod 1$. The proof of Proposition 5.2 is based on the following lemma, which shows that the existence of a $T$-invariant and $T^s$-ergodic measure is an obstruction for $T^{ns}$ to be both 2-to-1 and surjective.

**Lemma 7.1** Let $(X, T)$ be a topological dynamical system and $s \geq 2$ an integer. Suppose that there exists a $T$-invariant measure which is $T^s$-ergodic. Then $T^{ns}$ cannot be both exactly 2-to-1 and surjective for any integer $n \geq 1$.

**Proof** Suppose that $T^{ns} : X \to X$ is 2-to-1 and surjective for some integer $n \geq 1$. Then the map $T : X \to X$ must be surjective and each point $x \in X$ has at most two pre-images by $T$, i.e. $1 \leq \#T^{-1}(x) \leq 2$. This allows us to decompose $X$ into two disjoint sets

$$A_0 = \{x \in X : \#T^{-1}(x) = 1\}, \quad B_0 = \{x \in X : \#T^{-1}(x) = 2\}.$$

We claim that these two sets are measurable. For any $\epsilon > 0$, let

$$B_0^\epsilon = \{x \in X : \#T^{-1}(x) = 2 \text{ and } |y_1 - y_2| \geq \epsilon \text{ for distinct } y_1, y_2 \in T^{-1}(x)\}.$$

Then $B_0^\epsilon$ is closed. Since $B_0 = \bigcup_{n \geq 1} B_0^{1/n}$, it is an $F_\sigma$ set. Hence, $B_0$ is measurable which implies $A_0$ is also measurable.

For $0 \leq k \leq ns$, let $B_k = T^{-k}(B_0)$. First notice that we have

$$T^{-1}(B_k) = B_{k+1}, \quad T(B_{k+1}) = B_k \quad (0 \leq k \leq ns - 1)$$

where the second equality is because of the surjectivity of $T$. We claim that the maps $T : B_{k+1} \to B_k$ are injective and then bijective for $1 \leq k \leq ns - 1$.

Indeed, otherwise, for some $k$ and some $u \in B_k$ there are two distinct points $u', u'' \in B_{k+1}$ such that $T(u') = T(u'') = v$. Let $w = T^{k-1}(v)$, which belongs to $B_1 = T^{-1}(B_0)$. Let $z = T(w)$, which belongs to $B_0$. By the definition of $B_0$, there exists a point $w^* \in B_1$ different from $w$ such that $T(w^*) = z$ and then there exists a point $u^* \in B_{k+1}$ different from $u'$, $u''$ such that $T^k(u^*) = w^*$. Therefore, $z$ has at least three $T^{k+1}$-preimages $u', u'', u^*$ and then at least three $T^{ns}$-preimages. This contradicts the fact that $T^{ns}$ is 2-to-1.

The above claim implies that $B_k \subset A_0$ for $1 \leq k \leq ns - 1$. Since $A_0$ and $B_0$ form a partition of $X$, we have $B_k \cap B_0 = \emptyset$ for $1 \leq k \leq ns - 1$. Consequently, all $B_j$'s for $0 \leq j \leq ns - 1$ are disjoint. We claim that all $B_j$'s for $0 \leq j \leq ns - 1$ form a partition of $X$. To that end, it suffices to prove

$$A_0 = \bigcup_{k=1}^{ns-1} B_k. \quad (7.1)$$

Suppose that (7.1) is not true, which means there exists a point $x \in A_0 \setminus \bigcup_{k=1}^{ns-1} B_k$. Hence $T^k(x) \in A_0$ for $0 \leq k \leq ns - 1$, by the definition of $B_k$ and the fact that $\{A_0, B_0\}$ is a partition of $X$. Therefore, the point $T^{ns-1}(x) \in A_0$ has a unique $T^{ns}$-preimage $T^{-1}(x)$, which contradicts to the assumption that $T^{ns}$ is 2-to-1.
Since $X = A_0 \cup B_0$, we get the decomposition
\[
X = \bigsqcup_{k=0}^{n-1} B_k = \bigsqcup_{j=0}^{s-1} T^{-j}(X') \quad \text{with} \quad X' = \bigsqcup_{k=0}^{n-1} B_{ks}.
\]

By the hypothesis, there exists a $T$-invariant measure $\mu$ which is $T^s$-ergodic. The $T$-invariance of $\mu$ implies that $\mu(X') = 1/s$.

Let $A_1 = T^{-1}(A_0)$. Recall that $B_1 = T^{-1}(B_0)$. Since $\{A_0, B_0\}$ is a partition of $X$, so is $\{A_1, B_1\}$. As $B_1 \subset A_0$ which is proved above, we have $B_0 \subset A_1$ and actually $B_0 = A_1 \setminus A_0$.

By (7.1) and the definition of $A_1$, we get $A_1 = \bigsqcup_{k=0}^{n^s} B_k$. Then from $A_0 \cup B_0 = A_1 \cup B_1$ we get $B_0 = B_{ns}$, in other words,
\[
T^{-ns}(B_0) = B_0.
\]

Consequently, $T^{-s}(X') = X'$. Then, by the $T^s$-ergodicity of $\mu$, we have $\mu(X') = 0$ or 1, which contradicts to $\mu(X') = 1/s$ with $s \geq 2$. \hfill \Box

Now we shall prove Proposition 5.2 by contradiction. Basic properties of entropy function will be used. We refer to Walters’ book [33] (cf. Theorem 4.13, Theorem 7.5, Theorem 7.10, Theorem 8.1).

**Proof** (Proof of Proposition 5.2)

Assume that $S : \Lambda^* \to \Lambda^*$ is a continuous map such that $S^n = T$ for some integer $n \geq 2$.

For $1 \leq j \leq N - 1$, we consider the dynamical system $(T^j(\Lambda), T^N)$. By the hypothesis, the map $T^j : \Lambda \to T^j(\Lambda)$ is a homeomorphism. So, the system $(T^j(\Lambda), T^N)$ is conjugate to $(\Lambda, T^N)$ with the conjugation $T^j$. Hence, the topological entropy of the system $(T^j(\Lambda), T^N)$ is equal to $\log 2$. It follows that $h_{\text{top}}(\Lambda^*, T^N) = \log 2$. Therefore,
\[
h_{\text{top}}(\Lambda^*, T) = \frac{1}{N} \log 2.
\]

It follows that the root dynamical system $(\Lambda^*, S)$ admits its topological entropy
\[
h_{\text{top}}(\Lambda^*, S) = \frac{1}{nN} \log 2.
\]

Let $\mu$ be a maximal entropy measure of the system $(\Lambda^*, S)$. It is also a maximal entropy measure of dynamical systems $(\Lambda^*, T)$ and $(\Lambda^*, T^N)$.

Let $\mu_j = \mu|_{T^j(\Lambda)}$ be the restriction for $0 \leq j \leq N - 1$. By the $T$-invariance of $\mu$ and the disjointness of $T^j(\Lambda)$’s ($0 \leq j \leq N - 1$), it is easy to get
\[
\mu_{j+1} = \mu_j \circ T^{-1} \quad (0 \leq j \leq N - 1)
\]
with the convention $\mu_N = \mu_0$. It follows that $\mu_j$ are all $T^N$-invariant. Let $v_j = N \cdot \mu_j$, which is a probability measure concentrated on $T^j(\Lambda)$. Since the systems $(T^j(\Lambda), v_j, T^N)$ are all conjugate, the measure-theoretic entropies $h_{v_j}(T^j(\Lambda), T^N)$ are equal and the common value is $h_{v_j}(\Lambda^*, T^N)$. From this, the relation $\mu = \frac{1}{N} \sum_{j=0}^{N-1} v_j$ and the affinity of entropy, we get
\[
\log 2 = h_{\mu}(\Lambda^*, T^N) = \frac{1}{N} \sum_{j=0}^{N-1} h_{v_j}(\Lambda^*, T^N) = h_{v_0}(\Lambda^*, T^N) = h_{v_0}(\Lambda, T^N).
\]

So, the measure $v_0$ is the unique maximal entropy measure of the horseshoe $(\Lambda, T^N)$, i.e. the Bernoulli $(\frac{1}{2}, \frac{1}{2})$-measure on $[0, 1]^N$. Since $v_0$ is $T^N$-ergodic, so are $v_j$’s. Now we claim
that $\mu$ is $T$-ergodic. In fact, assume that $A \subset \Lambda^*$ is a $T$-invariant set. Then $A \cap T^j(\Lambda)$ is $T^N$-invariant for every $0 \leq j \leq N - 1$. By the $T^N$-ergodicity of $v_j$, $v_j(A \cap T^j(\Lambda)) = 0$ or 1 for each $j$. But $v_j(A \cap T^j(\Lambda))$ are equal for different $j$'s. Then we get $\mu(A) = 0$ or 1, because

$$\mu(A) = \frac{1}{N} \sum_{j=0}^{N-1} v_j(A \cap T^j(\Lambda)).$$

The existence of $S$-invariant measure $\mu$ which is $T$-ergodic measure, contradicts the fact that $T^N: \Lambda^* \rightarrow \Lambda^*$ is 2-to-1, by Lemma 7.1. □

7.2 No root of $T: \Lambda^* \rightarrow \Lambda^*$ (two-sided case)

Here we prove Proposition 6.2. The proof is based on the fact that the shift map $\sigma : \{0, 1\}^Z \rightarrow \{0, 1\}^Z$ has no root, which is well known (cf. [16], Corollary 18.2, p.371).

Given a two-sided horseshoe $(\Lambda, T^N)$ with disjoint steps, it is convenient to identify the subsystem $(\Lambda^*, T)$ with the following system $((0, 1)^Z \times \mathbb{Z}/N\mathbb{Z}, \sigma_N)$, a tower of height $N$, where $\sigma_N$ is defined by

$$\sigma_N(\omega, k) = \begin{cases} (\sigma(\omega), 0), & \text{if } k = N - 1; \\ (\omega, k + 1), & \text{otherwise.} \end{cases} \quad (7.2)$$

The tower $\{0, 1\}^Z \times \mathbb{Z}/N\mathbb{Z}$ has $N$ floors $F_i = \{0, 1\}^Z \times \{i\}$ for $0 \leq i < N$. We extend $F_i$ for all integers $i \geq 0$ by defining $F_i = F_i \mod N$. Especially $F_N = F_0$. In the following we denote $\sigma_N$ by $T$.

Suppose that $T$ has a root $S$, i.e. $S^p = T$ for some $p \geq 2$. It is clear that $S$ is bijective and commutes with $\sigma_N$. We claim that $S$ permute floors, that is to say, for any $i$ there exists a $j$ such that $S(F_i) = F_j$. We shall prove this claim by the commutativity of $S$ with $T$. Assume this claim for the moment. Then each floor $F_i (0 \leq i < N)$ is mapped back to $F_i$ by $S^N$, that is to say,

$$\forall \omega, \quad S^N(\omega, i) = (R_i \omega, i), \quad (7.3)$$

where $R_i : \{0, 1\}^Z \rightarrow \{0, 1\}^Z$ is some map, which depends on $i$. It is easy to see that $R_i$ is continuous. Thus, on one hand, we have

$$\forall \omega, \quad S^{Np}(\omega, i) = (R_i^p \omega, i);$$

on the other hand, we have

$$\forall \omega, \quad S^{Np}(\omega, i) = T^N(\omega, i) = (\sigma \omega, i).$$

It follows that $\sigma = R_i^p$, which is impossible.

Now let us prove the claim. Let $\mu$ the ergodic probability measure of the maximal entropy of $T$ (its restriction on each floor is the symmetric Bernoulli measure). We have $\mu(F_i) = \frac{1}{N}$. Let $C_i = S(F_0) \cap F_i$, the portion of $SF_0$ contained in $F_i$, for $0 \leq i < N$. Suppose $0 < \mu(C_i) < \mu(F_i)$ for some $i$. There would be a contradiction. Indeed, we have the invariance

$$T \left( \bigcup_{j=0}^{N-1} T^j(C_i) \right) = \bigcup_{j=0}^{N-1} T^j(C_i).$$
This is because

\[ T^j(C_i) = T^j(S(F_0) \cap F_i) = ST^j(F_0) \cap T^j F_i = S(F_j) \cap F_{i+j} \]

and

\[ \bigcup_{j=1}^{N} S(F_j) \cap F_{i+j} = \bigcup_{j=0}^{N-1} S(F_j) \cap F_{i+j}. \]

The invariant set \( \bigcup_{j=0}^{N-1} T^j(C_i) \) has its measure between 0 and 1 because of \( 0 < \mu(C_i) < \frac{1}{N} \). This contradicts the ergodicity of \( \mu \). We have thus proved that for every \( i \), the measure \( \mu(S(F_0) \cap F_i) \) is equal to 0 or \( \mu(F_i) \). There exists one \( i \) such that \( \mu(S(F_0) \cap F_i) = \mu(F_i) \), otherwise \( \mu(S(F_0)) = 0 \) so that

\[ \mu(S(F_j)) = \mu(ST^j(F_0)) = \mu(T^j S(F_0)) = \mu(S(F_0)) = 0, \]

which implies that \( \mu \) is the null measure. There is at most one \( i \) such that \( \mu(S(F_0) \cap F_i) = \mu(F_i) \), otherwise \( \mu(S(F_0)) \geq \frac{2}{N} \), which implies that \( \mu \) has a total measure equal to at least 2. So, \( S(F_0) \) and \( F_i \) are equal almost everywhere. If we take into account the continuity of \( S \), we get that the two compact sets \( S(F_0) \) and \( F_i \) must be equal. In place of \( F_0 \), we can consider any \( F_j \). The same argument shows that \( S(F_j) \) must be equal to some \( F_k \). In this way, \( S \) defines a permutation on floors. Otherwise, under \( S \) we have a cycle

\[ F_{i_0} \rightarrow F_{i_1} \rightarrow \cdots \rightarrow F_{i_{\ell-1}} \rightarrow F_{i_0} \]

with \( \ell < N \). Then the union \( U \) of these \( \ell \) floors is \( S \)-invariant and it is also \( S^p \)-invariant, i.e. \( T \)-invariant, an obvious contradiction.

8 Appendix B: Any cylinder contains a horseshoe

8.1 Horseshoe with disjoint steps in any cylinder: one-sided case

Consider the one-sided full shift system \( (\{0, 1\}^N, \sigma) \). For any non-empty open set \( U \subset \{0, 1\}^N \), we shall show that there exists a horseshoe \( (\Lambda, \sigma^N) \) with disjoint steps for some sufficiently large integer \( N \geq 1 \) such that \( \Lambda \subset U \) and the map \( \sigma^{N-1} : \Lambda \rightarrow \sigma^{N-1}(\Lambda) \) is bijective. In other word, this horseshoe \( (\Lambda, \sigma^N) \) satisfies the conditions (i) and (ii) required by Proposition 5.2.

For a word \( a_0 \cdots a_{n-1} \) of length \( n \), denote by \( [a_0 \cdots a_{n-1}] \) the cylinder of rank \( n \)

\[ [a_0 \cdots a_{n-1}] = \{ y \in \{0, 1\}^N : y_i = a_i \text{ for } 0 \leq i \leq n-1 \}. \]

Denote the \( n \)-prefix of \( x = (x_j)_{j \geq 0} \) by \( x|_n \), i.e. \( x|_n = x_0 x_1 \cdots x_{n-1} \). Hence \( [x|_n] \) denote a cylinder of rank \( n \). Let \( uv \) denote the concatenation \( u_0 u_1 \cdots u_{n-1} v_0 \cdots v_{m-1} \) of two words \( u = u_0 u_1 \cdots u_{n-1} \) and \( v = v_0 \cdots v_{m-1} \). So, \( u^r \) denotes \( u \cdot u \cdots u \) (\( r \) times). In particular, \( 1^r \) means \( 1 \cdots 1 \) (\( r \) times). The following lemma is a preparation for proving the above announced existence of horseshoe in a given cylinder.

Lemma 8.1 For any cylinder \( C \) in \( \{0, 1\}^N \) of rank \( M \geq 1 \), there exists a sub-cylinder \( C' \subset C \) of rank \( N \) with \( N \geq M \) such that

(i) \( \sigma^N(C') = \{0, 1\}^N \);
(ii) $C' \cap \sigma^n(C') = \emptyset$ for $1 \leq n \leq N - 1$.

**Proof** Assume $C = [a_0a_1 \cdots a_{M-1}]$. We assume $a_0 = 0$, without loss of generality. Note that $\sigma^M(C) = \{0, 1\}^N$. Let $n_0 \geq 1$ be the minimal positive integer such that $C \subset \sigma^{n_0}(C)$, so that

\[
C \cap \sigma^n(C) = \emptyset \quad \text{for all } 1 \leq n \leq n_0 - 1.
\]

We restate these facts as follows:

\[
[0a_1 \cdots a_{M-1}] \cap [a_1 \cdots a_{M-1}] = \emptyset, \ldots, [0a_1 \cdots a_{M-1}] \cap [a_{n_0-1}an_0 \cdots a_{M-1}] = \emptyset; \quad (8.2)
\]

\[
[0a_1 \cdots a_{M-1}] \subset [a_{n_0} \cdots a_{M-1}]. \quad (8.3)
\]

We have $n_0 \leq M$. If $n_0 = M$, we are done and we can take $C' = C$. In the following, we assume $n_0 < M$.

The inclusion (8.3) means

\[
an_{n_0+j} = a_j \quad \text{for } 0 \leq j \leq M - n_0 - 1. \quad (8.4)
\]

Let $x = \overline{a_0a_1 \cdots a_{n_0-1}}$, which is $n_0$-periodic. By (8.4), $a_0a_1 \cdots a_{M-1}$ is a prefix of $x$, so $x$ is in $C$. By (8.2), $n_0$ is the minimal period of $x$. Define the sub-cylinder $C' = [0a_1a_2 \cdots a_{M-1}1^{n_0}]$, or more precisely

\[
C' = [(0a_1 \cdots a_{n_0-1})^q0a_1 \cdots a_{j-1}1^{n_0}]
\]

where $q \geq 0$ and $0 \leq j \leq n_0 - 1$ are determined by $M = qn_0 + j$. Now we shall check that the sub-cylinder $C'(of C)$ of rank $N = M + n_0$ has property (ii). We distinguish three cases.

**Case I.** $1 \leq n \leq n_0 - 1$. Since $C' \subset C$, we proved $C' \cap \sigma^n(C') = \emptyset$, by (8.1).

**Case II.** $n_0 \leq n \leq M - 1$. Suppose $C' \cap \sigma^n(C') \neq \emptyset$. Then $C \subset \sigma^n(C') \neq \emptyset$. Since $|\sigma^n(C')| \leq M = |C|$, we get $C \subset \sigma^n(C')$. Hence, the word $a_{n_0} \cdots a_{M-1}1^{n_0}$ defining the cylinder $\sigma^n(C')$ has $1^{n_0}$ as suffix, which is a word contained in $a_0a_1 \cdots a_{M-1}$ defining the cylinder $C$. But, on the other hand, any word of length $n_0$ contained in $a_0a_1 \cdots a_{M-1}$ contains 0, a contradiction.

**Case III.** $M \leq n \leq N - 1$. This case is evident because $\sigma^n(C') \subset [1]$, but $C' \subset [0]$. \(\square\)

Let us look at the cylinders $C$ of rank $M = 4$ and the cylinders $C'$ constructed in Lemma 8.1:

\[
C = [0000], \quad C' = [00001^3]
\]

\[
C = [0001], \quad C' = C
\]

\[
C = [0010], \quad C' = [00101^3]
\]

\[
C = [0011], \quad C' = C
\]

\[
C = [0100], \quad C' = [01001^3]
\]

\[
C = [0101], \quad C' = [01011^2]
\]

\[
C = [0110], \quad C' = [01101^3]
\]

\[
C = [0111], \quad C' = C
\]

where the exponent represents $n_0$.

We are now ready to prove the existence of horseshoe contained in a given cylinder.
Proposition 8.2 Let $C \subset [0, 1]^N$ be an arbitrary cylinder. There exists a $\sigma^N$-invariant closed subset $\Lambda \subset C$ for some integer $N \geq |C|$, such that

(i) The system $(\Lambda, \sigma^N)$ is topologically conjugate to the full shift $([0, 1]^N, \sigma)$;
(ii) The maps $\sigma^n : \Lambda \to \sigma^n(\Lambda)$ $(1 \leq n \leq N - 1)$ are bijections;
(iii) The sets $\Lambda$, $\sigma(\Lambda)$, $\ldots$, $\sigma^{N-1}(\Lambda)$ are disjoint.

Proof Assume $C = [y_0 y_1 \cdots y_{m-1}]$ be a cylinder of rank $m$. We can assume $y_0 = 0$ without loss of generality. By Lemma 8.1, there exists an integer $n_* \geq m$ and a sub-cylinder $C' \subset C$ of rank $n_*$ such that

$$C' \cap \sigma^n(C') = \emptyset \quad \text{for} \quad 1 \leq n \leq n_* - 1. \quad (8.5)$$

It is obvious that $\sigma^{n_*}(C') = [0, 1]^N$ and $\sigma^{n_*} : C' \to [0, 1]^N$ is bijective. Assume that $C' = [y_0 y_1 \cdots y_{m-1} y_m \cdots y_{n_*-1}]$. Notice that $y_{n_*-1} = 1$ because $C' \cap \sigma^{n_*-1}(C') = \emptyset$. Let $y = y_0 y_1 \cdots y_{n_*-1}$. This periodic point $y$ is the unique periodic point contained in $C'$ of exact period $n_*$, by (8.5). Let $n_1 = \min(0 \leq i \leq n_* - 1 : y_i = 1)$. Since $y_0 = 0$ and $y_{n_*-1} = 1$, we have $1 \leq n_1 \leq n_* - 1$.

Define two sub-cylinders of $C'$ of rank $n_* + n_1 + 2$:

- $C_1 = [y_0 y_1 \cdots y_{n_*-1} 10011] = [0^{n_1} y_{n_1} \cdots y_{n_*-1} 10011] = [0^{n_1} a 0^{n_1}]$,
- $C_2 = [y_0 y_1 \cdots y_{n_*-1} 11011] = [0^{n_1} y_{n_1} \cdots y_{n_*-1} 11011] = [0^{n_1} b 0^{n_1}]$

where $a = y_{n_1} \cdots y_{n_*-1} 10$ and $b = y_{n_1} \cdots y_{n_*-1} 11$.

Also observe that

(iv) for $1 \leq n \leq n_* - 1$, $\sigma^n(C_1 \cup C_2) \cap C' = \emptyset$ and $\sigma^n$ is injective on $C_1 \cup C_2$.

This follows from the relation $C_1 \cup C_2 \subset C'$, the disjointness (8.5) and the injectivity of $\sigma^{n_*} : C' \to [0, 1]^N$. By the definition of $n_1$, we have $C_1 \cup C_2 \subset [0^{n_1}1]$. Note that

$$\sigma^{n_*}(C_1) = [10011], \quad \sigma^{n_*+1}(C_1) = [0^{n_1}+1]; \quad \sigma^{n_*}(C_2) = [11011], \quad \sigma^{n_*+1}(C_2) = [10^{n_1}]$$

which are all disjoint from $[0^{n_1}1]$. Hence,

(v) for $n = n_* + 1$, $\sigma^n(C_1 \cup C_2) \cap (C_1 \cup C_2) = \emptyset$ and $\sigma^n$ is injective on $C_1 \cup C_2$.

Now take $N = n_* + 2$ and define

$$\Lambda = \{x \in [0, 1]^N : \forall k \geq 0, \sigma^k(x) \in C_1 \cup C_2\}.$$

Observe that both the words defining $C_1$ and $C_2$ have $0^{n_1}$ as their prefix and as well as their suffix. Therefore $\Lambda$ can be identified with the symbolic space $[0^{n_1} a, 0^{n_1} b]^N$ and $(\Lambda, \sigma^N)$ is conjugate to the shift map on $[0^{n_1} a, 0^{n_1} b]^N$. This is the property (i) of $(\Lambda, \sigma^N)$. The required properties (ii) and (iii) follow from (iv) and (v).

8.2 Horseshoe with disjoint steps in any cylinder: two-sided case

Consider the full shift system $([0, 1]^Z, \sigma)$. For any non-empty cylinder $C \subset [0, 1]^Z$, we shall show that there exists a horseshoe $(\Lambda, \sigma^N)$ with disjoint steps for some sufficiently large integer $N \geq 1$, where $\Lambda \subset C$. We start with the following preparative lemma, the counterpart of Lemma 8.1.

Lemma 8.3 For any cylinder $C$ in $[0, 1]^Z$ of rank $M \geq 1$, there exists a sub-cylinder $C' \subset C$ of rank $N$ with $N \geq M$ such that
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(i) $\sigma^N(C') \cap C' \neq \emptyset$;
(ii) $C' \cap \sigma^n(C') = \emptyset$ for $1 \leq n \leq N - 1$.

Proof The proof is almost the same as that of Lemma 8.1. We assume $C = [a_0 a_1 \cdots a_{M-1}]$, and $d_0 = 0$ without loss of generality. Note that $C \cap \sigma^M(C) \neq \emptyset$. Let $n_0 \geq 1$ be the minimal positive integer such that $C \cap \sigma^{n_0}(C) \neq \emptyset$, so that

$$C \cap \sigma^n(C) = \emptyset \quad \text{for all } 1 \leq n \leq n_0 - 1.$$  \hfill (8.6)

We restate these facts as follow

$$[0 a_1 \cdots a_{M-1}]_0 \cap [a_1 \cdots a_{M-1}]_0 = \emptyset, \cdots, [0 a_1 \cdots a_{M-1}]_0 \cap [a_{n_0-1} a_{n_0} \cdots a_{M-1}]_0 = \emptyset;$$  \hfill (8.7)

$$[0 a_1 \cdots a_{M-1}]_0 \subset [a_{n_0} \cdots a_{M-1}]_0.$$  \hfill (8.8)

The rest of the proof is the same if we replace cylinders of the form $\lfloor \ast \rfloor$ by $\lfloor \ast \rfloor_0$ where $\ast$ represents a word.

We are now ready to prove the existence of horseshoe contained in a given cylinder.

Proposition 8.4 Let $C \subset \{0, 1\}^N$ be an arbitrary cylinder. There exists a $\sigma^N$-invariant closed subset $\Lambda \subset C$ for some integer $N \geq |C|$, such that

(i) The system $(\Lambda, \sigma^N)$ is topologically conjugate to the full shift $(\{0, 1\}^N, \sigma)$;
(ii) The sets $\Lambda, \sigma(\Lambda), \ldots, \sigma^{N-1}(\Lambda)$ are disjoint.

Proof It is exactly the same proof as that of Proposition 8.2, if we replace cylinders of the form $\lfloor \ast \rfloor$ by $\lfloor \ast \rfloor_0$.

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