Windowed Radon Transforms, Analytic Signals and the Wave Equation

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Abstract

The act of measuring a physical signal or field suggests a generalization of the wavelet transform that turns out to be a windowed version of the Radon transform. A reconstruction formula is derived which inverts this transform. A special choice of window yields the “Analytic–Signal transform” (AST), which gives a partially analytic extension of functions from $\mathbb{R}^n$ to $\mathbb{C}^n$. For $n = 1$, this reduces to Gabor’s classical definition of “analytic signals.” The AST is applied to the wave equation, giving an expansion of solutions in terms of wavelets specifically adapted to that equation and parametrized by real space and imaginary time coordinates (the Euclidean region).

1. Introduction

The ideas presented here originated in relativistic quantum theory [13, 14, 15], where a method was developed for extending arbitrary functions from $\mathbb{R}^n$ to $\mathbb{C}^n$ in a semi–analytic way. This gave rise to the “Analytic–Signal transform” (AST) [16]. Later it was realized that the AST has a natural generalization to what we have called a Windowed X–Ray transform [17], and the latter is a special case of a Windowed Radon transform, to be introduced below. For $n = 1$, these transforms reduce to the (continuous) Wavelet transform. In the general case, they retain many of the properties of the Wavelet transform.

In Section 2 we motivate and define the d–dimensional Windowed Radon transform in $\mathbb{R}^n$ for $1 \leq d \leq n$ and derive reconstruction formulas which can be used to invert it. In Section 3 we define the AST in $\mathbb{R}^n$ and give some of its applications. In Section 4 we develop a new application of the AST by generalizing a construction in [16] to the wave equation in $\mathbb{R}^2$. This results in a representation of solutions of the wave equation as combinations of “dedicated” wavelets that are especially customized to that equation. In particular, these wavelets are themselves solutions and represent coherent wave packets, being well-localized in both space (at any particular time) and frequency, within the limitations of the uncertainty principle. The parameters labeling these wavelets (i.e., the variables on which the AST depends) have a direct geometrical significance: They give the initial position, direction of motion and average frequency or color of the wavelets. The representation of a
solution in terms of these wavelets therefore gives a geometrical–optics (ray) picture of the solution. It is suggested that this could be of considerable practical value in signal analysis, since many naturally occurring signals (e.g., sound waves, electromagnetic waves) satisfy the wave equation away from sources and the geometrical–optics picture gives a readily accessible display of their informational contents.

2. Windowed Radon Transforms

2.1. The Windowed X–Ray Transform

Suppose we wish to measure a physical field distributed in $\mathbb{R}^n$. This field could be a “signal,” such as an electromagnetic field or the pressure distribution due to a sound wave. For simplicity, we assume to begin with that it is real–valued, such as pressure. (Our considerations easily extend to complex–valued, vector–valued or tensor–valued signals, such as electromagnetic fields; we shall indicate later how this is done.) The given field is therefore a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$. We may think of $\mathbb{R}^n$ as physical space (so that $n = 3$), in which case the field is time–independent, or as space–time (so that $n = 4$), in which case the field may be time–dependent. In the former case, $\mathbb{R}^n$ is endowed with a Euclidean metric, while in the latter case the appropriate metric is Lorentzian, as mandated by Relativity theory.

Actual measurements are never instantaneous, nor do they take place at a single point in space. A measurement is performed by reading an instrument, and the instrument necessarily occupies some region in space and must interact with the field for some time–interval before giving a meaningful reading. Let us assume, to begin with, that the spatial extension of the instrument is negligible, so that it can be regarded as being concentrated at a single point at any time. We allow our instrument to be in an arbitrary state of uniform motion, so that its position is given by $x(t) = x + vt$, where $t \in \mathbb{R}$ is a “time” parameter and $x, v \in \mathbb{R}^n$. Note that $t$ need not be the physical time. For example, if $\mathbb{R}^n$ is space–time, then each “point” $x$ represents an event, i.e. a particular location in space at a particular time. In that case, the line $x(t)$ is called a world–line and represents the entire history of the point–instrument. The “velocity” vector $v$ then has one too many components and may be regarded as a set of homogeneous coordinates for the physical velocity. Note that in this case $v$ cannot vanish, since this would correspond to an instrument not subject to the flow of time. Even if $\mathbb{R}^n$ is space, the case $v = 0$ is not interesting since then the instrument can only measure the field at a single point. We therefore assume that $v \neq 0$, hence $v \in \mathbb{R}^n_\times \equiv \mathbb{R}^n \setminus \{0\}$.

Let us assume that the reading registered by the instrument at time $s$ gives a weight $h(t–s)$ to the value of the field passed by the instrument at time $t$. (For motivational purposes we note that causality would demand that $h(t–s) = 0$ for $t > s$; moreover, $h(t–s)$ should be concentrated in some interval $s – \tau \leq t \leq s$, where $\tau$ is a “response time” or memory characteristic of the instrument. However, the results below do not depend on these assumptions.) Our model for the observed value of the field at the “point” $x$, as measured by the instrument traveling with uniform velocity $v$, is then
\[ f_h(x, v) \equiv \int_{-\infty}^{\infty} dt \, h(t) f(x + vt). \]  

To accommodate complex-valued signals, we allow the weight function \( h \) to be complex-valued. \( h(t)^* \) will denote the complex conjugate of \( h(t) \). In order to minimize analytical subtleties, we assume that \( h \) is smooth and bounded, and that \( f \) is smooth with rapid decay (say, a Schwartz test function).

**Definition 1.** The Windowed X-Ray Transform of \( f : \mathbb{R}^n \to \mathbb{C} \) is the function \( f : \mathbb{R}^n \times \mathbb{R}^n_* \to \mathbb{C} \) given by

\[ f_h(x, v) = \int_{-\infty}^{\infty} dt \, h(t)^* f(x + vt). \]  

**Remarks.**

1. In the special case \( h(t) \equiv 1 \) and \( |v| = 1 \), \( f_h \) is known as the (ordinary) X-Ray transform of \( f \) (Helgason [11]), due to its applications in tomography. We may then regard \( f_h \) as being defined on the set of all lines in \( \mathbb{R}^n \), independent of their parametrization. In the general case, we think of the function \( h(t) \) as a window, which explains our terminology.

2. Some work along related lines was recently done by Holschneider [12]. He considers a two-dimensional wavelet transform which is covariant under translations, rotations and and dilations of \( \mathbb{R}^2 \). When the window function is supported on a line, say \( h(t_1, t_2) = \delta(t_2) \), this becomes an X-Ray transform in \( \mathbb{R}^2 \). His inversion method is less direct than ours in that it involves a limiting process.

3. Note that \( f_h \) has the following dilation property for \( a \neq 0 \):

\[ f_h(x, av) = \int_{-\infty}^{\infty} dt \, |a|^{-1} h(t/a)^* f(x + tv) = f_{h_a}(x, v) \]  

where \( h_a(t) \equiv |a|^{-1} h(t/a) \). This may be used to study the behavior of \( f_h \) as \( v \to 0 \). For the “forbidden” value \( v = 0 \), the transform becomes \( f_h(x, 0) = \hat{h}(0)^* f(x) \), where \( \hat{h} \) is the Fourier transform of \( h \). (We shall see that \( \hat{h}(0) = 0 \) for “admissible” \( h \).)

4. For \( n = 1 \) and \( v \neq 0 \), a change of variables gives

\[ f_h(x, v) = |v|^{-1} \int_{-\infty}^{\infty} dt' \ h \left( \frac{t' - x}{v} \right)^* f(t') \]  

\[ = |v|^{-1/2} Wf(x, v), \]  

where \( Wf \) is the usual wavelet transform of \( f \) [5, 7, 23], with \( v \) playing the role of a dilation factor and the window function \( h(t) \) playing the role of a basic wavelet.

5. All our considerations extend to vector-valued signals. The cleanest approach is to let the window function \( h^* \) assume values in the dual vector space, so that \( h(t)^* f(x + vt) \) and \( f_h(x, v) \) are scalars. More than one window needs to be used (or rotated versions of a
single window), in order to ‘probe’ the different components of $f$. The same applies to
tensor–valued signals such as electromagnetic fields, since they may be regarded as being
valued in a higher–dimensional vector space. However, a more correct way to measure a
vector– or tensor–valued field is to use an instrument which is not rotationally invariant,
and that implies that the instrument has some spatial extension. This is done in Section
2.3.

It will be useful to write $f_h$ in another form by substituting the Fourier representation of
$f$ into $f_h$. Formally, this gives

$$f_h(x, v) = \int_{-\infty}^{\infty} dt \int_{\mathbb{R}^n} dp \: e^{2\pi i p \cdot (x + tv)} \: h(t)^* \: \hat{f}(p)$$

$$= \int_{\mathbb{R}^n} dp \: e^{2\pi i p \cdot x} \: \hat{h}(p \cdot v)^* \: \hat{f}(p)$$

$$\equiv \langle \hat{h}_{x,v} \: \hat{f} \rangle_{L^2(dp)} = \langle h_{x,v} \: f \rangle_{L^2(dx)},$$

where $\hat{h}_{x,v}$ is defined by

$$\hat{h}_{x,v}(p) = e^{-2\pi i p \cdot x} \: \hat{h}(p \cdot v),$$

so that

$$h_{x,v}(x') = \int_{\mathbb{R}^n} dp \: e^{2\pi i p \cdot (x' - x)} \: \hat{h}(p \cdot v).$$

(We have adopted the convention used in the physics literature, where complex inner
products are linear in the second factor and antilinear in the first factor.)

The functions $h_{x,v}$ are $n$–dimensional “wavelets” and will be used in the next subsection
to reconstruct the signal $f$. Note that $\hat{h}_{x,v}$ (hence also $h_{x,v}$) is not square–integrable for
$n > 1$, since its modulus is constant along directions orthogonal to $v$. But eq. (5) still
makes sense provided $\hat{f}$ is sufficiently well–behaved. (This is one of the reasons we have
assumed that $f$ is a test function.)

A common method for the construction of $n$–dimensional wavelets consists of taking tensor
products of one–dimensional wavelets. However, this means that not all directions in $\mathbb{R}^n$
are treated equally, and consequently the set of wavelets does not transform “naturally”
(in a sense to be explained below) under the affine group $G$ of $\mathbb{R}^n$, which consists of all
transformations of the form

$$x \mapsto g(A, b)x \equiv Ax + b$$

with $A$ a non–singular $n \times n$ matrix and $b \in \mathbb{R}^n$. Each such $g(A, b)$ defines a unitary
operator on $L^2(\mathbb{R}^n)$, given by

$$(U(A, b)f)(x) \equiv |A|^{-\frac{n}{2}} f \left(A^{-1}(x - b)\right),$$

where $|A|$ denotes the absolute value of the determinant of $A$. The map
$g(A, b) \mapsto U(A, b)$ forms a representation of $G$ on $L^2(\mathbb{R}^n)$, meaning that it preserves
the group structure of $G$ under compositions. To see how $h_{\mathbf{x},\mathbf{v}}$ transforms under $U$, note that the unitarity of $U$ implies

$$\langle U(A, b) h_{\mathbf{x},\mathbf{v}}, f \rangle = \langle h_{\mathbf{x},\mathbf{v}}, U(A, b)^{-1} f \rangle = \int_{-\infty}^{\infty} dt \ h(t)^* |A|^{\frac{1}{2}} f (A(x + tv) + b) = |A|^{\frac{1}{2}} \langle h_{A\mathbf{x}+b,Av}, f \rangle.$$  

(10)

Hence

$$U(A, b) h_{\mathbf{x},\mathbf{v}} = |A|^{\frac{1}{2}} h_{A\mathbf{x}+b,Av},$$  

(11)

which states that affine transformations take wavelets to wavelets. Thus, for example, translations, rotations and dilations merely translate, rotate and dilate the labels $\{\mathbf{x}, \mathbf{v}\}$, while the factor $|A|^{\frac{1}{2}}$ preserves unitarity. By contrast, tensor products of one-dimensional wavelets are not transformed into one another by rotations.

### 2.2. A Reconstruction Formula

A reconstruction consists of a recovery of $f$ from $f_h$ or its restriction to some subset. In the one-dimensional case, for example, $f$ can be reconstructed using all of $\mathbb{R} \times \mathbb{R}_*$ or (for certain choices of $h$) just a discrete subset $[2, 6, 18, 21, 22]$. For general $n$, the choice of reconstructions becomes even richer since various new possibilities arise. For example, $h$ may have symmetries which imply that $f_h$ is determined by its values on some lower-dimensional subsets of $\mathbb{R}^n \times \mathbb{R}_*^n$, making integration over the whole space unnecessary and, moreover, undesirable since it may lead to a divergent integral. Furthermore, $f$ may satisfy some partial differential equation which implies that it is determined by its values on subsets of $\mathbb{R}^n$. For example, if $\mathbb{R}^n$ is space–time and $f$ represents a pressure wave or an electromagnetic potential, it satisfies the wave equation away from sources, hence is determined by initial data on a Cauchy surface in $\mathbb{R}^n$, and it becomes both unnecessary and undesirable to use all of $\mathbb{R}^n \times \mathbb{R}_*^n$ in the reconstruction (cf. Sections 3.2 and 4).

The reconstruction to be developed in this subsection is “generic” in that it does not assume any particular forms for $h(t)$ or $f(x)$. It uses all of $\mathbb{R}^n \times \mathbb{R}_*^n$, so it breaks down for certain choices of $h$ or $f$. Again we emphasize that this is far from the only way to proceed; other types of reconstruction will be discussed below and elsewhere. The present reconstruction formula is interesting in part because it generalizes the one for the ordinary continuous wavelet transform ($n = 1$).

To reconstruct $f$, we look for a resolution of unity in terms of the vectors $h_{\mathbf{x},\mathbf{v}}$. This means we need a measure $d\mu(x, v)$ on $\mathbb{R}^n \times \mathbb{R}_*^n$ such that

$$\int_{\mathbb{R}^n \times \mathbb{R}_*^n} d\mu(x, v) |f_h(x, v)|^2 = \int_{\mathbb{R}^n} df(x) |f(x)|^2 \equiv \|f\|_{L^2}^2.$$  

(12)

(Such an identity is sometimes called a “Plancherel formula.”) For then the map $T: f \mapsto f_h$ is an isometry from $L^2(dx)$ onto its range $\mathcal{H} \subset L^2(d\mu)$, and polarization gives

$$\langle g, T^*Tf \rangle_{L^2(dx)} \equiv \langle Tg, Tf \rangle_{\mathcal{H}} = \langle g, f \rangle_{L^2(dx)}.$$  

(13)
This shows that \( f = T^* T f = T^* f_h \) in \( L^2(dx) \), which is the desired reconstruction formula. (Cf. [16] for background on resolutions of unity, generalized frames and related subjects.) To obtain a resolution of unity, note that

\[
f_h(x, v) = \left( \hat{h}(p \cdot v)^* \hat{f}(p) \right) \hat{\cdot}(x), \tag{14}\]

where \( \hat{\cdot} \) denotes the inverse Fourier transform, so by Plancherel’s theorem,

\[
\int_{\mathbb{R}^n} dx |f_h(x, v)|^2 = \int_{\mathbb{R}^n} dp |\hat{h}(p \cdot v)|^2 |\hat{f}(p)|^2. \tag{15}\]

We therefore need a measure \( d\rho(v) \) on \( \mathbb{R}^n_* \) such that

\[
H(p) \equiv \int_{\mathbb{R}^n_*} d\rho(v) |\hat{h}(p \cdot v)|^2 \equiv 1 \quad \text{for almost all } p, \tag{16}\]

since then \( d\mu(x, v) = dx d\rho(v) \) has the desired property. The solution is simple: Every \( p \neq 0 \) can be transformed to \( q \equiv (1, 0, \ldots, 0) \) by a dilation and rotation of \( \mathbb{R}^n \). That is, the orbit of \( q \) (in Fourier space) under dilations and rotations is \( \mathbb{R}^n_* \). Thus we choose \( d\rho \) to be invariant under rotations and dilations, which gives

\[
d\rho(v) = N |v|^{-n} dv, \tag{17}\]

where \( N \) is a normalization constant, \( |v| \) is the Euclidean norm of \( v \) and \( dv \) is Lebesgue measure in \( \mathbb{R}^n \). Then for \( p \neq 0 \),

\[
H(p) = H(q) = N \int |v|^{-n} dv |\hat{h}(v_1)|^2
= N \int_{-\infty}^{\infty} dv_1 |\hat{h}(v_1)|^2 \int_{\mathbb{R}^{n-1}} dv_2 \cdots dv_n \frac{dv_2 \cdots dv_n}{(v_1^2 + \cdots v_n^2)^{n/2}}. \tag{18}\]

Now a straightforward computation gives

\[
\int_{\mathbb{R}^{n-1}} \frac{dv_2 \cdots dv_n}{(v_1^2 + \cdots v_n^2)^{n/2}} = \frac{\pi^{n/2}}{|v_1| \Gamma(n/2)}. \tag{19}\]

This shows that the measure \( d\mu(x, v) \equiv dx d\rho(v) \) gives a resolution of unity if and only if

\[
c_h \equiv \int_{-\infty}^{\infty} d\xi |\hat{h}(\xi)|^2 < \infty, \tag{20}\]

which is precisely the admissibility condition for the usual (one-dimensional) wavelet transform [5]. (As mentioned above, admissibility implies that \( \hat{h}(0) = 0 \).) If \( h \) is admissible, the normalization constant is given by

\[
N = \frac{\Gamma(n/2)}{\pi^{n/2} c_h} \tag{21}\]

and the reconstruction formula is
\[ f(x') = (T^* f_h)(x') = N \int_{\mathbb{R}^n \times \mathbb{R}^n_0} |v|^{-n} d\mathbf{v} h_{x,\mathbf{v}}(x') f_h(x, \mathbf{v}). \]  

The sense in which this formula holds depends on the behavior of \( f \). The class of possible \( f \)'s, in turn, depends on the choice of \( h \). Note that in spite of the factor \(|v|^n\) in the denominator, there is no problem at \( \mathbf{v} = \mathbf{0} \) since \( f_h(x, 0) = \hat{h}(0)^* f(x) = 0 \) by the admissibility condition, and a similar analysis can be made for small \(|\mathbf{v}|\) by using the dilation property (eq.(3)).

### 2.3. The \( d \)-Dimensional Windowed Radon Transform

Next, we allow the instrument to extend in \( k \geq 0 \) spatial dimensions. For example, \( k = 1 \) for a wire antenna whereas \( k = 2 \) for a dish antenna. If \( \mathbb{R}^n \) is space–time, then \( k \leq n-1 \). When moving through space with a uniform velocity, the instrument sweeps out a \( d \)-dimensional surface in \( \mathbb{R}^n \), where \( d = k + 1 \) if the motion is transverse to its spatial extension and \( d = k \) if it is not. If \( k < n \), then the set of non–transversal motions is “non–generic” (has measure zero) and can thus be ignored; we therefore set \( d = k + 1 \) in that case. If \( k = n \), then necessarily \( d = n \). In either case, we represent the moving instrument by a window function \( h : \mathbb{R}^d \to \mathbb{C} \).

The parameter \( t \in \mathbb{R} \) has thus been replaced by \( t \in \mathbb{R}^d \). The velocity vector \( \mathbf{v} \), which may be regarded as a linear map \( t \to t \mathbf{v} \) from \( \mathbb{R} \) to \( \mathbb{R}^n \), is now replaced by a linear map \( A : \mathbb{R}^d \to \mathbb{R}^n \), which we call a motion of the instrument in \( \mathbb{R}^n \). Denote the set of all such maps by \( L(\mathbb{R}^d, \mathbb{R}^n) \). Later, when seeking reconstruction, we shall need to restrict ourselves to subsets of \( L(\mathbb{R}^d, \mathbb{R}^n) \) (“rigid” motions); but this need not concern us presently.

**Definition 2.** The \( d \)-dimensional Windowed Radon Transform of \( f : \mathbb{R}^n \to \mathbb{C} \) is the function \( f_h : \mathbb{R}^n \times L(\mathbb{R}^d, \mathbb{R}^n) \to \mathbb{C} \) given by

\[ f_h(x, A) = \int_{\mathbb{R}^d} dt \, h(t)^* f(x + At). \]  

Upon substituting the Fourier representation of \( f \), a computation similar to the above yields the expression

\[ f_h(x, A) = \int_{\mathbb{R}^d} dt \int_{\mathbb{R}^n} d\mathbf{p} \, e^{2\pi i \mathbf{p} \cdot (x + At)} h(t)^* \hat{f}(\mathbf{p}) 
= \int_{\mathbb{R}^n} d\mathbf{p} \, e^{2\pi i \mathbf{p} \cdot x} \hat{h}(A'\mathbf{p})^* \hat{f}(\mathbf{p}) 
\equiv \langle \hat{h}_{x, A}, \hat{f} \rangle_{L^2(d\mathbf{p})} = \langle h_{x, A}, f \rangle_{L^2(dx)}, \]  

where \( A' : \mathbb{R}^n \to \mathbb{R}^d \) is the map dual to \( A \). (For given bases in \( \mathbb{R}^d \) and \( \mathbb{R}^n \), \( A \) is represented by an \( n \times d \) matrix; then \( A' \) is the transposed \( d \times n \) matrix.) In the above equation we have set

\[ \hat{h}_{x, A}(\mathbf{p}) = e^{-2\pi i \mathbf{p} \cdot x} \hat{h}(A'\mathbf{p}), \]  

which gives the generalized wavelets.
\[ h_{x,A}(x') = \int_{\mathbb{R}^n} d\mathbf{p} e^{2\pi i \mathbf{p} \cdot (x' - x)} \hat{h}(A' \mathbf{p}). \] (26)

Let us now attempt to reconstruct \( f \) from \( f_h \) by generalizing the procedure in Section 2.2. Eq. (15) now becomes

\[ \int_{\mathbb{R}^n} dx \left| f_h(x,A) \right|^2 = \int_{\mathbb{R}^n} d\mathbf{p} \left| \hat{h}(A' \mathbf{p}) \right|^2 |\hat{f}(\mathbf{p})|^2. \] (27)

Again we need a measure \( d\rho(A) \) which is invariant under dilations and rotations of \( \mathbb{R}^n \). Now the largest set of maps \( A \) which can be considered consists of all those with rank \( d \). (Otherwise the instrument sweeps out a surface of dimension lower than \( d \).) Let us call this set \( L_d(\mathbb{R}^d, \mathbb{R}^n) \). Then a measure on \( L_d(\mathbb{R}^d, \mathbb{R}^n) \) which is invariant with respect to rotations and dilations of \( \mathbb{R}^n \) has the form

\[ d\tilde{\rho}(A) = |\det(A' A)|^{-n/2} dA, \] (28)

where \( dA \) is the Haar measure on \( L(\mathbb{R}^d, \mathbb{R}^n) \approx \mathbb{R}^{nd} \) as an additive group. However, no reconstruction is possible using the measure \( dx d\tilde{\rho}(A) \) on \( \mathbb{R}^n \times L_d(\mathbb{R}^d, \mathbb{R}^n) \), because no admissible window exists in general when \( d > 1 \). (This can be easily verified when \( d = n = 2 \).) Thus \( L_d(\mathbb{R}^d, \mathbb{R}^n) \) is too large, and we return to our imaginary measuring process for inspiration. On physical grounds, we are interested in rigid motions of the instrument. A map corresponding to such a motion must have the form \( A = vRJ \), where \( J : \mathbb{R}^d \to \mathbb{R}^n \) is the canonical inclusion map, \( R \) is a rotation of \( \mathbb{R}^n \) (\( RJ \) gives the orientation of the instrument as well as its direction of motion), and \( v > 0 \) is the speed. (If \( \mathbb{R}^n \) is space–time, then “rotations” involving the time axis are actually Lorentz transformations! For the present, assume that \( \mathbb{R}^n \) is space, so \( R \) is a true rotation.) We therefore parametrize the set of permissible \( A \)'s by \((v, R) \in \mathbb{R}^+ \times SO(n) \equiv G\), where \( SO(n) \) is the group of unimodular orthogonal \( n \times n \) matrices, which represent rotations in \( \mathbb{R}^n \). This parametrization is redundant because two rotations of \( \mathbb{R}^n \) which have the same effect on the subspace \( \mathbb{R}^d \) give the same motion. A non–redundant parametrization of rigid motions is given by \( \mathbb{R}^+ \times (SO(n)/SO(n - d)) \). However, we use the redundant one here for simplicity. (We shall need the Haar measure on \( SO(n) \).) Note that for \( d = 1 \), \( J \) is represented by the vector \( \mathbf{q} = (1, 0, \cdots, 0) \) and the set of all maps \( A = vRJ \) as above coincides with the set \( \mathbb{R}^n_0 \) of non–zero velocities considered in Sections 2.1 and 2.2. A measure on \( G \) which is invariant under rotations and dilations of \( \mathbb{R}^n \) (i.e., under \( G \) itself) has the form

\[ d\rho(A) = N v^{-1} dv dR, \] (29)

where \( N \) is a normalization constant and \( dv dR \) is the Haar measure on \( SO(n) \). Thus for all \( \mathbf{p} \neq 0 \),

\[ H(\mathbf{p}) = \left| \int_G d\rho(A) \left| \hat{h}(A' \mathbf{p}) \right|^2 \right| = H(\mathbf{q}). \] (30)

Now
\[ A' q = v' R' q = v' R_1', \]  
(31)

where \( R_1 \) is the first row of \( R \), which is a unit vector, and \( J' \) is the projection of \( R^n \) onto \( R^d \). The admissibility condition therefore reads

\[
N^{-1} \equiv \int_0^\infty v^{-1} dv \int_{SO(n)} dR |\hat{h}(vJ'R_1')|^2 < \infty.
\]  
(32)

For \( d = 1 \), this reduces to eq. (20). If \( h \) is admissible, we obtain the reconstruction formula

\[
f(x') = \int_{R^n} dx \int_G d\rho(A) h_{x,A}(x') f_{h}(x, A).
\]  
(33)

### 3. Analytic–Signal Transforms

#### 3.1. Analytic Signals in One Dimension

Suppose we are given a (possibly complex–valued) one–dimensional signal \( f : \mathbb{R} \to \mathbb{C} \). For simplicity, assume that \( f \) is smooth with rapid decay. Consider the positive– and negative– frequency parts of \( f \), defined by

\[
f^+(x) \equiv \int_0^{\infty} dp e^{2\pi ipx} \hat{f}(p) \\
f^-(x) \equiv \int_{-\infty}^0 dp e^{2\pi ipx} \hat{f}(p),
\]  
(34)

Then \( f^+ \) and \( f^- \) extend analytically to the upper–half and lower–half complex planes, respectively; i.e.,

\[
f^+(x + iy) = \int_0^{\infty} dp e^{2\pi ip(x+iy)} \hat{f}(p), \quad y > 0 \\
f^-(x + iy) = \int_{-\infty}^0 dp e^{2\pi ip(x+iy)} \hat{f}(p), \quad y < 0,
\]  
(35)

since the factor \( e^{-2\pi py} \) decays rapidly for \( p \to \pm \infty \) in the respective integrals. \( f^+(z) \) and \( f^-(z) \) are just the (inverse) Fourier–Laplace transforms of the restrictions of \( \hat{f} \) to the positive and negative frequencies. If \( f \) is complex–valued, then \( f^+ \) and \( f^- \) are independent and the original signal can be recovered from them as

\[
f(x) = \lim_{y \downarrow 0} [f^+(x + iy) + f^-(x - iy)].
\]  
(36)

If \( f \) is real–valued, then

\[
\hat{f}(p) = \hat{f}(-p)^*.
\]  
(37)
In that case, $f^+$ and $f^-$ are related by reflection,
\begin{equation}
 f^+(x + iy) = f^-(x - iy)^*, \quad y > 0,
\end{equation}
and
\begin{equation}
 f(x) = 2 \lim_{y \downarrow 0} \Re f^+(x + iy) = 2 \lim_{y \downarrow 0} \Re f^-(x - iy).
\end{equation}

When $f$ is real, the function $f^+(z)$ is known as the analytic signal associated with $f(x)$. Such functions were first introduced and applied extensively by Gabor [8]. A complex–valued signal would have two independent associated analytic signals $f^+$ and $f^-$. What significance do $f^\pm$ have? For one thing, they are regularizations of $f$. Eq. (36) states that $f$ is jointly a “boundary–value” of the pair $f^+$ and $f^-$. As such, $f$ may actually be quite singular while remaining the boundary–value of analytic functions. Also, $2f^\pm$ provide a kind of “envelope” description of $f$ (cf. Born and Wolf [4], Klauder and Sudarshan [19]). For example, if $f(x) = \cos x$, then $2f^\pm(x) = e^{\pm ix}$.

In order to extend the concept of analytic signals to more than one dimension, let us first of all unify the definitions of $f^+$ and $f^-$ by setting
\begin{equation}
 \tilde{f}(x + iy) \equiv \int_{-\infty}^{\infty} dp \theta(py) e^{2\pi ip(x+iy)} \hat{f}(p)
\end{equation}
for arbitrary $x + iy \in \mathbb{C}$, where $\theta$ is the unit step function, defined by
\begin{equation}
 \theta(u) = \begin{cases} 
 0, & u < 0 \\
 \frac{1}{2}, & u = 0 \\
 1, & u > 0.
\end{cases}
\end{equation}

Then we have
\begin{equation}
 \tilde{f}(x + iy) = \begin{cases} 
 f^+(x + iy), & y > 0 \\
 \frac{1}{2} f(x), & y = 0 \\
 f^-(x + iy), & y < 0.
\end{cases}
\end{equation}

Although this unification of $f^+$ and $f^-$ may at first appear to be somewhat artificial, it turns out to be quite natural, as will now be seen. Note first of all that for any real $u$, we have
\begin{equation}
 \theta(u) e^{-2\pi u} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{d\tau}{\tau - i} e^{2\pi i \tau u},
\end{equation}
since the contour on the right–hand side may be closed in the upper–half plane when $u > 0$ and in the lower–half plane when $u < 0$. For $u = 0$, the equation states that
\[ \theta(0) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{(\tau + i) d\tau}{\tau^2 + 1} = \frac{1}{2}, \quad (44) \]

in agreement with our definition, if we interpret the integral as the limit when \( L \to \infty \) of the integral from \(-L\) to \(L\). Therefore

\[ \theta(\pi y) e^{2\pi i p(x+iy)} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{d\tau}{\tau - i} e^{2\pi i p(x+\tau y)}. \quad (45) \]

If this is substituted into our expression for \( \tilde{f}(z) \) and the order of integrations on \( \tau \) and \( p \) is exchanged, we obtain

\[ \tilde{f}(x + iy) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{d\tau}{\tau - i} f(x + \tau y) \quad (46) \]

for arbitrary \( x + iy \in \mathbb{C} \). We have referred to the right-hand side as the Analytic-Signal transform of \( f(x) \) \[16, 17\]. It bears a close relation to the Hilbert transform, which is defined by

\[ Hf(x) = \frac{1}{\pi} \text{PV} \int_{-\infty}^{\infty} \frac{du}{u} f(x - u), \quad (47) \]

where PV denotes the principal value of the integral. Consider the complex combination

\[ f(x) - iHf(x) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{du}{u} \left[ \pi i\delta(u) + \text{PV} \frac{1}{u} \right] f(x - u) \]

\[ = \frac{1}{\pi i} \lim_{\epsilon \downarrow 0} \int_{-\infty}^{\infty} \frac{du}{u - i\epsilon} f(x - u) \]

\[ = \frac{1}{\pi i} \lim_{\epsilon \downarrow 0} \int_{-\infty}^{\infty} \frac{d\tau}{\tau - i} f(x - \tau \epsilon) \]

\[ = 2 \lim_{\epsilon \downarrow 0} \tilde{f}(x - i\epsilon). \quad (48) \]

Similarly,

\[ f(x) + iHf(x) = 2 \lim_{\epsilon \downarrow 0} \tilde{f}(x + i\epsilon). \quad (49) \]

Hence

\[ Hf(x) = i \lim_{\epsilon \downarrow 0} [\tilde{f}(x - i\epsilon) - \tilde{f}(x + i\epsilon)], \quad (50) \]

which for real-valued \( f \) reduces to

\[ Hf(x) = 2 \lim_{\epsilon \downarrow 0} \Im \tilde{f}(x + i\epsilon) = -2 \lim_{\epsilon \downarrow 0} \Im \tilde{f}(x - i\epsilon). \quad (51) \]
3.2. Generalization to n Dimensions

We are now ready to generalize the idea of analytic signals to an arbitrary number of dimensions. Again we assume initially that $f(x)$ belongs to the space of Schwartz test functions $S(\mathbb{R}^n)$, although this assumption proves to be unnecessary.

**Definition 3.** The Analytic–Signal Transform (AST) of $f \in S(\mathbb{R}^n)$ is the function $\tilde{f} : \mathbb{C}^n \to \mathbb{C}$ defined by

$$\tilde{f}(x + iy) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{d\tau}{\tau - i} f(x + \tau y).$$

(52)

The same argument as above shows that for $z = x + iy \in \mathbb{C}^n$,

$$\tilde{f}(z) = \int_{\mathbb{R}^n} d^np \ \theta(p \cdot y) e^{2\pi i p \cdot z} \hat{f}(p)$$

$$= \int_{M_y} dp e^{2\pi i p \cdot z} \hat{f}(p),$$

(53)

where $M_y$ is the half–space

$$M_y \equiv \{p \in \mathbb{R}^n \text{ such that } p \cdot y \geq 0\}, \quad y \neq 0.$$

(54)

We shall refer to the right–hand side of eq. (53) as the (inverse) Fourier–Laplace transform of $\hat{f}$ in $M_y$. The integral converges absolutely whenever $\hat{f} \in L^1(\mathbb{R}^n)$, since $|e^{2\pi i p \cdot z}| \leq 1$ on $M_y$, defining $\tilde{f}$ as a function on $\mathbb{C}^n$, although not an analytic one in general (see below).

This shows that $\tilde{f}(z)$ can actually be defined for some distributions $f$, not only for test functions.

**Note:** In spite of the appearance of expressions such as $p \cdot y$, we have not assumed any particular metric structure in $\mathbb{R}^n$. The Fourier transform naturally takes functions on $\mathbb{R}^n$ (considered as an abelian group) to functions on the dual space $\mathbb{R}_n^* \equiv (\mathbb{R}^n)^*$ of linear functionals, and $p \cdot y$ merely denotes the value $p(y)$. (See Rudin [25].) This remark becomes especially important when considering time–dependent signals, so that $\mathbb{R}^n$ is space–time, for then the natural structure on $\mathbb{R}^n$ is a Lorentzian metric rather than a Euclidean metric (cf. [16], Section 1.1.)

For $n = 1$, $\tilde{f}(z)$ was analytic in the upper– and lower–half planes. In more than one dimension, $\tilde{f}(z)$ need not be analytic, even though, for brevity, we still write it as a function of $z$ rather than $z$ and its complex conjugate $z^*$. However, $\tilde{f}(z)$ does in general possess a partial analyticity which reduces to the above when $n = 1$. Consider the partial derivative of $\tilde{f}(z)$ with respect to $z_k^* \equiv x_k - iy_k$, defined by

$$2\partial_k \tilde{f} \equiv 2 \frac{\partial \tilde{f}}{\partial z_k^*} = \frac{\partial \tilde{f}}{\partial x_k} + i \frac{\partial \tilde{f}}{\partial y_k}.$$  

(55)

Then $\tilde{f}$ is analytic at $z$ if and only if $\partial_k \tilde{f} = 0$ for all $k$. But using our definition of $\tilde{f}(z)$, we find that
\[ 2\bar{\partial}_k \tilde{f}(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau \frac{\partial f}{\partial x_k}(x + \tau y). \]  

(56)

It follows that the complex $\bar{\partial}$–derivative in the direction of $y$ vanishes, i.e.

\[ 4\pi y \cdot \bar{\partial} \tilde{f}(z) \equiv 4\pi \sum_k y_k \bar{\partial}_k \tilde{f}(z) = \int_{-\infty}^{\infty} d\tau \sum_k y_k \frac{\partial f}{\partial x_k}(x + \tau y) = \int_{-\infty}^{\infty} d\tau \frac{\partial}{\partial \tau} f(x + \tau y) = 0, \]

(57)

if $f$ decays for large $|x|$ (for example, if $f$ is a test function, as we have assumed). Equivalently, using

\[ \begin{align*}
2\bar{\partial}_k \left[ \theta(p \cdot y) e^{2\pi i p \cdot z} \right] &= 2\bar{\partial}_k \left[ \theta(p \cdot y) \right] e^{2\pi i p \cdot z} \\
&= i \frac{\partial \theta(p \cdot y)}{\partial y_k} e^{2\pi i p \cdot z} \\
&= ip_k \delta(p \cdot y) e^{2\pi i p \cdot z} \\
&= ip_k \delta(p \cdot y) e^{2\pi i p \cdot x},
\end{align*} \]

(58)

we have for $y \neq 0$

\[ 2y \cdot \bar{\partial} \tilde{f}(z) = i \int_{\mathbb{R}^n} d^n p \ (p \cdot y) \delta(p \cdot y) e^{2\pi i p \cdot x} \tilde{f}(p) = 0. \]

(59)

Thus $\tilde{f}(z)$ is analytic in the direction $y$. In the one–dimensional case, this reduces to

\[ \frac{\partial \tilde{f}(z)}{\partial z^*} = 0 \quad \forall y \neq 0, \]

(60)

which states that $\tilde{f}(z)$ is analytic in the upper– and lower–half planes. In one dimension, there are only two imaginary directions, whereas in $n$ dimensions, every $y \neq 0$ defines an imaginary direction.

The multivariate AST is related to the Hilbert transform in the direction $y$ (cf. [26], p. 49), defined as

\[ H_y f(x) = \frac{1}{\pi} \text{PV} \int_{-\infty}^{\infty} \frac{du}{u} f(x - uy), \quad x, y \in \mathbb{R}^n, \ y \neq 0. \]

(61)

(Usually, it is assumed that $y$ is a unit vector; we do not make this assumption.) Namely, an argument similar to the above shows that

\[ f(x) \pm iH_y f(x) = 2 \lim_{\epsilon \downarrow 0} \tilde{f}(x \pm i\epsilon y), \]

(62)

hence

13
\[ H_y f(x) = i \lim_{\epsilon \downarrow 0} [\tilde{f}(x - i\epsilon y) - \tilde{f}(x + i\epsilon y)]. \] (63)

For \( n = 1 \) and \( y > 0 \), this reduces to the previous relation with the ordinary Hilbert transform.

As in the one–dimensional case, \( f(x) \) is the boundary–value of \( \tilde{f}(z) \) in the sense that

\[ f(x) = \lim_{\epsilon \to 0} [\tilde{f}(x - i\epsilon y) - \tilde{f}(x + i\epsilon y)]. \] (64)

For real–valued \( f \), these equations become

\[ f(x) = 2 \lim_{\epsilon \to 0} \Re \tilde{f}(x + i\epsilon y) \]
\[ H_y f(x) = 2 \lim_{\epsilon \downarrow 0} \Im \tilde{f}(x + i\epsilon y). \] (65)

Two unresolved yet fundamental questions are:

(a) For what classes of ‘functions’ (possibly distributions) can the AST be defined, apart from \( S(\mathbb{R}^n) \); i.e., what is the domain of the AST?

(b) Given a vector space \( \mathcal{H} \) of ‘functions’ on \( \mathbb{R}^n \) for which the AST is defined, what is the range of the AST on \( \mathcal{H} \)? That is, given a function \( F \) on \( \mathbb{C}^n \), how can we tell whether \( F \) is the transform of some \( f \in \mathcal{H} \)?

A necessary, though probably not sufficient, condition for \( F = \tilde{f} \) is that \( F \) satisfy the directional Cauchy–Riemann equation \( y \cdot \bar{\partial} F(x, y) = 0 \). Complete answers to the above questions can be given in some specific cases: When \( f \) is a solution of the Klein–Gordon equation, then \( \tilde{f} \) must satisfy a certain consistency condition (the reproducing property of the associated wavelets). This condition, when satisfied by \( F \), also guarantees that \( F = \tilde{f} \) for some \( f \) (cf. [16], Chapters 1 and 4). A similar result will be obtained in Section 4 for solutions of the wave equation in two space–time dimensions. The comments below apply to the general case and are, consequently, informal.

The most direct way to find if \( F \) is the AST of some \( f \) is to construct \( f \) from \( F \) and then check that \( \tilde{f} = \tilde{f} \). The first part has already been done formally, since \( f \) has been shown to be the boundary–value of \( \tilde{f} \). Here we suggest an alternative method which can be used to construct \( \tilde{f}(p) \) instead of \( f(x) \). Assume that the Fourier transform is defined on \( \mathcal{H} \). If \( F(x, y) = \tilde{f}(x + iy) \) for some \( f(x) \in \mathcal{H} \), then the \( 2n \)–dimensional Fourier transform of \( F \) is seen (formally) to be

\[
\hat{F}(p, q) \equiv \int_{\mathbb{R}^{2n}} dx \, dy \, e^{-2\pi i(p \cdot x + q \cdot y)} \, F(x, y) \\
= \hat{f}(p) \int_{\mathbb{R}^n} dy \, \theta(p \cdot y) \, e^{-2\pi i(q - i\epsilon p) \cdot y} \\
= \hat{f}(p) \tilde{\delta}(q - i\epsilon p),
\] (66)

where \( \tilde{\delta} \) is the AST, in Fourier space, of the Dirac measure \( \delta(q) \). (This suggests that the AST, like the Fourier transform, exhibits some symmetry between space and Fourier
space.) \(\tilde{\delta}\) can be shown to be invariant under real rotations \((q - ip \rightarrow R(q - ip)),\) with \(R \in SO(n)\) and to transform under dilations as

\[
\tilde{\delta}(\lambda(q - ip)) = \lambda^{-n} \tilde{\delta}(q - ip), \quad \lambda \neq 0.
\]

Given \(p \neq 0,\) let \(R\) be a rotation such that \(p = |p| Re,\) where \(e = (1, 0, \cdots, 0),\) and let \(k \equiv |p|^{-1}R^{-1}q,\) so that \(q - ip = |p| R(k - ie).\) Then

\[
\tilde{\delta}(q - ip) = |p|^{-n} \tilde{\delta}(k - ie)
\]

\[
= |p|^{-n} \int_{\mathbb{R}^n} dy \theta(y_1) e^{-2\pi i (k - ie) \cdot y}
\]

\[
= |p|^{-n} \int_{0}^{\infty} dy_1 e^{-2\pi (1 + ik_1)y_1} \int_{\mathbb{R}^{n-1}} dy_2 \cdots dy_n e^{-2\pi i (k_2y_2 + \cdots + k_ny_n)}
\]

\[
= \frac{\delta(k_2, \cdots, k_n)}{2\pi |p|^n(1 + ik_1)}.
\]

Let \(P : \mathbb{R}^n \rightarrow \mathbb{R}\) and \(Q : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}\) denote the projections \(Pk = k_1\) and \(Qk = (k_2, \cdots, k_n).\) Then the numerator in the last expression is

\[
\delta(Qk) = \delta(Q|p|^{-1}R^{-1}q)| = |p|^{n-1} \delta(QR^{-1}q),
\]

hence

\[
\tilde{\delta}(q - ip) = \frac{\delta(QR^{-1}q)}{2\pi(|p| + iP|R^{-1}q|)}, \quad p \neq 0.
\]

Together with eq. (66), this gives an explicit formal condition for \(F\) to be the AST of \(f\) and, if so, to determine \(\hat{f}(p).\) When \(n = 1,\) \(\tilde{\delta}\) takes the simple form

\[
\tilde{\delta}(q - ip) = \frac{\text{sgn}(p)}{2\pi(p + iq)}, \quad p \neq 0.
\]

### 3.3. Some Applications of the AST

The AST is an example of a Windowed X–Ray transform, introduced in Section 2.1, with the window function

\[
h(\tau)^* = \frac{1}{2\pi i(\tau - i)}.
\]

(This window function is not “admissible” in the sense of Section 2.2, hence the reconstruction formula developed there fails. However, that reconstruction was based on some assumptions which may not be appropriate in general; see the comments below Eq. (88).)

We now give two examples of the usefulness of the AST. An extensive use of this transform will also be made in Section 4.
Example 1: Hardy spaces

Suppose that \( \hat{f}(p) \) vanishes outside of some closed convex cone \( V_+ \). The cone \( V_+ \) dual to \( V_+ \) is defined as

\[
V_+ = \{ y \in \mathbb{R}^n \text{ such that } p \cdot y > 0 \ \forall p \in V_+ \},
\]

and it is clearly an open convex cone. Note that for \( y \in V_+^\prime \), \( \theta(p \cdot y) \equiv 1 \) on the support of \( \hat{f} \) (except at \( p = 0 \), which has measure 0), hence if \( f \in L^2(\mathbb{R}^n) \) and \( y \in V_+^\prime \), then

\[
\hat{f}(x + iy) = \int_{V_+} dp \, e^{2\pi i p \cdot (x + iy)} \hat{f}(p)
\]

and it follows (Stein and Weiss [27]) that \( \tilde{f}(z) \) is analytic in the tube domain

\[
T^+ \equiv \{ x + iy \in \mathbb{C}^n \text{ such that } y \in V_+^\prime \}.
\]

The set \( H^2 \equiv \{ \tilde{f} \text{ such that } \hat{f} \in L^2(V_+) \} \) is known as a Hardy space. Note also that \( \tilde{f} \) vanishes in the tube

\[
T^- \equiv \{ x + iy \in \mathbb{C}^n \text{ such that } -y \in V_+^\prime \},
\]

since there \( p \cdot y < 0 \) for all \( 0 \neq p \in V_+ \). Eq. (74) gives

\[
f(x) = \lim_{\epsilon \downarrow 0} \tilde{f}(x + i\epsilon y), \quad y \in V_+^\prime,
\]

which states that \( f \) is a boundary–value of \( \tilde{f} \). Since \( \tilde{f}(z) \) is analytic, it may be regarded as a regularization of \( f(x) \) (the latter, being merely square–integrable, is a distribution). Eq. (77) can be viewed as a “reconstruction” of \( f \) from \( \tilde{f} \), albeit a somewhat trivial one.

Example 2: The Klein–Gordon Equation

An important application of the AST is to signals that satisfy some partial differential equations. (In fact, it was in this context that the transform originated.) Suppose that \( f \) is a solution of the Klein–Gordon equation in \( \mathbb{R}^n \),

\[
\Box f + m^2 c^4 f = 0,
\]

where

\[
\Box \equiv \frac{\partial^2}{\partial x_1^2} - c^2 \frac{\partial^2}{\partial x_2^2} - \cdots - c^2 \frac{\partial^2}{\partial x_n^2}
\]

is the D’Alembertian or wave operator for waves with propagation speed \( c \). Here \( \mathbb{R}^n \) is interpreted as space–time, with \( x_1 \) the time coordinate and \( (x_2, \cdots, x_n) \) the space coordinates, and \( m > 0 \) is a mass parameter. This equation describes free relativistic particles in quantum mechanics. The limit \( m \to 0 \) gives the wave equation, which will be discussed below. Define the solid light cone in Fourier space by

\[
V = \{ p \in \mathbb{R}^n \text{ such that } p^2 \equiv p_1^2 - c^2 p_2^2 - \cdots - c^2 p_n^2 \geq 0 \}.
\]
(Note that we are now using a Lorentz metric!) \( V \) is the union of the forward and backward light cones \( V_+ \) and \( V_- \), where \( p_1 \geq 0 \) and \( p_1 \leq 0 \), respectively. Note that \( V_\pm \) are convex but \( V \) is not. The fact that \( f \) satisfies the Klein–Gordon equation means that its Fourier transform \( \hat{f}(p) \) is supported on the double mass hyperboloid

\[
\Omega_m = \{ p \in \mathbb{R}^n \text{ such that } p^2 = m^2 c^4 \} = \Omega_m^+ \cup \Omega_m^-,
\]

where \( \Omega_m^\pm \subset V_\pm \). Thus \( \hat{f} = \hat{f}^+ + \hat{f}^- \), where \( \hat{f}^\pm \) are distributions supported on \( \Omega_m^\pm \). Since \( \Omega_m^\pm \subset V_\pm \), an argument similar to that used for Hardy spaces shows that the corresponding solutions \( f^\pm(x) \) have AST’s \( \tilde{f}^\pm(z) \) which are analytic in \( \mathcal{T}^\pm \) and vanish in \( \mathcal{T}^\mp \), where

\[
\mathcal{T}^\pm = \{ x + iy \in \mathbb{C}^n \text{ such that } y \in V_\pm' \}
\]
and \( V_\pm' \) are the cones dual to \( V_\pm \), which can be seen to be

\[
V_\pm' = \{ y \in \mathbb{R}^n \text{ such that } y^2 \equiv c^2 y_1^2 - y_2^2 - \cdots - y_n^2 > 0, \quad \pm y_1 > 0 \}.
\]

Note that while \( V \) is a cone in Fourier space (i.e., \( p_1 \) is a frequency and \( p_2, \cdots, p_n \) are wave numbers per unit length), \( V' \equiv V'_+ \cup V'_- \) is a cone in space–time. Technically, these two spaces are dual and should not be identified with one another.

The AST of \( f \), given by \( \tilde{f}(z) = \tilde{f}^+(z) + \tilde{f}^-(z) \), is therefore analytic in the double tube \( \mathcal{T} \equiv \mathcal{T}^+ \cup \mathcal{T}^- \), with \( \mathcal{T}^+ \) and \( \mathcal{T}^- \) containing only the positive– and negative–frequency parts of \( f \), respectively. This “polarization” of frequencies is important because it makes it possible to reconstruct the solution \( f \) from \( \tilde{f} \) without approaching the singular boundary \( \mathbb{R}^n \ (y \to 0) \). Eq. (74) shows that \( \tilde{f}^\pm \) have the form

\[
\tilde{f}^\pm(z) = \int_{\Omega_m^\pm} d\tilde{p} \, \hat{e}_z^\pm(p)^* a^\pm(p), \quad z \in \mathcal{T}^\pm,
\]

where

\[
d\tilde{p} = \frac{dp_2 dp_3 \cdots dp_n}{2 |p_1|} \equiv \frac{dp_2 dp_3 \cdots dp_n}{2c \sqrt{m^2 c^2 + p_2^2 + \cdots + p_n^2}}
\]

is the induced measure on \( \Omega_m \) and

\[
\hat{e}_z^\pm(p)^* \equiv e^{2\pi i p \cdot z}, \quad z \in \mathcal{T}^\pm, \quad p \in \Omega_m^\pm.
\]

The corresponding expression \( e_z^\pm \) in the space–time domain, defined by

\[
e_z^\pm(x') = \int_{\Omega_m^\pm} d\tilde{p} \, e^{2\pi i p \cdot (z-x')},
\]

is a solution of the Klein–Gordon equation which can be shown to be a coherent wave–packet whose parameters \( z = x + iy \) have a direct geometric interpretation: \( x \) is a point in space–time about which \( e_z^\pm \) is “focused” (i.e., \( e_z^\pm(x') \) converges toward the point \((x_2, \cdots, x_n)\) in space for times \( x'_1 < x_1 \) and diverges away from it for times \( x'_1 > x_1 \)), and \( y \) is a set of homogeneous coordinates for the average velocity at which \( e_z^\pm \) is traveling.
Furthermore, the invariant $\lambda > 0$ defined by $\lambda^2 \equiv y^2$ can be interpreted as a scale parameter which, roughly speaking, measures the spread (resolution) of the wave packet at the instant of its maximal focus ($x'_1 = x_1$) in its rest-frame (the coordinate system in which $y_2 = y_3 = \cdots = y_n = 0$). For small $\lambda$, $e^\pm_z(x')$ is localized near $x'_k = x_k$ ($k = 2, 3, \cdots, n$) at time $x'_1 = x_1$, whereas for large $\lambda$, it is spread out in space. (In Fourier space, on the other hand, it is $\lambda^{-1}$ which measures the spread.) The positive- and negative-frequency packets $e^+_z$ and $e^-_z$ are interpreted in quantum theory as particles and antiparticles, respectively. (This agrees with the usual observation that antiparticles “go backward in time.” Cf. [16], Chapter 5.)

Since the window function $h(\tau)$ used in the AST has Fourier transform $\hat{h}(\xi) = \theta(\xi) e^{-2\pi \xi}$, it is not admissible in the general sense developed in Section 2.2; i.e., we have

$$\int_{-\infty}^{\infty} \frac{d\xi}{|\xi|} |\hat{h}(\xi)|^2 = \infty.$$  

(88)

However, the rules of the game have changed. Eq. (88) was associated with a reconstruction formula which represents $f$ as an integral of generalized wavelets parametrized by all of $\mathbb{R}^n \times \mathbb{R}_+^n$, i.e., all $x + iy$ with $y \neq 0$. This was acceptable when considering general functions $f(x)$ in $L^2(\mathbb{R}^n)$, since then we could define a representation of the affine group on such functions. But now we are dealing with a Hilbert space $\mathcal{H}$ of solutions of the Klein–Gordon equation,

$$f(x) = \int_{\Omega_m} d\tilde{p} e^{2\pi i p \cdot x} a(p),$$  

(89)

with the Sobolev norm

$$\|f\|^2 \equiv \int_{\Omega_m} d\tilde{p} |a(p)|^2$$  

(90)

rather than the $L^2$ norm used in Section 2.2. General affine transformations no longer map solutions to solutions, i.e., they no longer define operators on $\mathcal{H}$. The mass $m$ spoils the invariance of the equation under dilations. Only the subgroup $P$ of translations together with Lorentz transformations (i.e., linear maps $y \mapsto Ay$ which preserve the Lorentz norm $y^2$) maps solutions to solutions. $P$ is called the inhomogeneous Lorentz or Poincaré group. Recall that the measure used in the reconstruction formula of Section 2.2 was chosen to be invariant under dilations and rotations. Since dilations no longer define operators on $\mathcal{H}$, this measure is no longer appropriate. Rather, we now expect to reconstruct $f$ by integrating in $y$ over the double hyperboloid

$$\Omega_\lambda = \{ y \in V' \text{ such that } y^2 = \lambda^2 \} = \Omega^+_\lambda \cup \Omega^-_\lambda$$  

(91)

for an arbitrary fixed $\lambda > 0$. Furthermore, we do not expect to integrate over all $x \in \mathbb{R}^n$, since a solution is determined by its data on any Cauchy surface $S \subset \mathbb{R}^n$. For simplicity, take $S$ to be the hyperplane $x_1 = t$ for fixed $t \in \mathbb{R}$, though any Cauchy surface ($n-1$)-dimensional submanifold of $\mathbb{R}^n$ will do ([16], Section 4.5). Thus consider the $(2n-2)$-dimensional submanifold
\[ \sigma = \{ \mathbf{x} + i \mathbf{y} \in T \text{ such that } x_1 = t, \mathbf{y} \in \Omega_\lambda \} = \sigma_+ \cup \sigma_- \]  

(92)

where \( \mathbf{y} \in \Omega^\pm_\lambda \) in \( \sigma^\pm \). \( \sigma \) parametrizes all possible locations and velocities of a classical particle at the fixed time \( t \), i.e. it is a phase space. A reconstruction formula has been obtained in the form

\[ f(x') = \int_{\sigma} d\mu(\mathbf{z}) e_{\mathbf{z}}(x') \tilde{f}(\mathbf{z}), \]  

(93)

where \( e_{\mathbf{z}} \equiv e^\pm_{\mathbf{z}} \) on \( \sigma^\pm, \sigma^\pm \) is parametrized by \((x_2, \cdots, x_n, y_2, \cdots, y_n)\), and

\[ d\mu(\mathbf{z}) = A(\lambda, m)^{-1} dx_2 \cdots dx_n dy_2 \cdots dy_n. \]  

(94)

\( A(\lambda, m) \) is a certain constant related to the admissibility of the wavelets \( e_{\mathbf{z}} \) with respect to the measure \( dx_2 \cdots dy_n \). Note that this differs from the usual construction of a solution from its initial data, which uses the values of both \( f \) and \( \partial f / \partial x_1 \) on \( S \). The intuitive explanation is that the dependence of \( \tilde{f}(x + iy) \) on \( \mathbf{y} \in \Omega_\lambda \), for fixed \( \mathbf{x} \in S \), gives the equivalent ”velocity” information. The independence of the reconstruction from the choice of Cauchy surface is due to a conservation law satisfied by solutions (cf. [16]).

The above reconstruction formula bears a close resemblance to the standard representation of a function in terms of wavelets, for the following reason: In the hyperbolic geometry of spacetime, a moving object undergoes a Lorentz contraction, i.e. it shrinks in its direction of motion. Since \( \mathbf{y} \) represents a velocity, eq. (93) expresses \( f \) as a linear combination of ”wavelets” centered about all possible points in space (at the given time \( t \)) and in various states of compression. However, the analogy is incomplete since the \( e_{\mathbf{z}} \)’s can only contract and not dilate. (That is, they have a minimum width in their rest frames, determined by the choice of \( \lambda \).) Their contraction is due to Lorentz transformations rather than ordinary dilations of the form \( \mathbf{x} \mapsto a \mathbf{x}, a \neq 0 \). As noted earlier, the Klein–Gordon equation is not invariant under such dilations, due to the presence of \( m > 0 \). On the other hand, the wave equation \( (m \to 0) \) is invariant under dilations, hence the analogy with wavelets can be expected to be closer. This is the subject of our next section.

4. Wavelets and The Wave Equation

4.1. Introduction

As explained in Section 1, a representation of solutions of the wave equation similar to that given for the Klein–Gordon equation in (93) should be of some interest in the analysis of naturally occurring signals, since it would automatically display much of their informational contents. Unfortunately, the reconstruction formula in (93) fails when \( m \to 0 \) because \( A(\lambda, m) \) diverges in that limit. (The wavelet representation is no longer square–integrable in that limit.) This is probably related to the fact, well–known in quantum mechanics, that the Klein–Gordon equation with \( m > 0 \) has a very different group–theoretical structure from the wave equation. The symmetry group of the Klein–Gordon equation is the Poincaré group \( \mathcal{P} \), while the symmetry group of the wave equation is the conformal group \( \mathcal{C} \), which contains the Poincaré group as well as dilations and uniform accelerations.
The fundamental difference between massive particles (such as electrons) and massless particles (such as photons) is that the former can be at rest while the latter necessarily travel at the speed of light. It may well be that once the conformal group is taken into account, an appropriate reconstruction formula can be found. In this section we confirm this hypothesis in two-dimensional space–time.

4.2. Symmetries of the Wave and Dirac Equations in $\mathbb{R}^2$

In this subsection, we examine some group-theoretical aspects of solutions of the wave equation in $\mathbb{R}^2$. To simplify the notation, we choose the units of length and time such that the propagation speed $c = 1$. The wave equation then reads

$$0 = -\Box f(x,t) \equiv (\partial^2_x - \partial^2_t) f(x,t) = (\partial_x + \partial_t)(\partial_x - \partial_t) f(x,t),$$

and we consider solutions which are possibly complex–valued. In terms of the light–cone coordinates

$$u = x + t, \quad v = x - t,$$

the equation becomes $\partial_u \partial_v f = 0$, hence the general solution has the form first given by D’Alembert,

$$f(x,t) = f_+(u) + f_-(v).$$

$f_+(t + x)$ is a left–moving wave since it is constant on the characteristics $x = x_0 - t$, and similarly $f_-(t - x)$ is a right–moving wave. Later we shall find an appropriate family of Hilbert spaces $\mathcal{H}_s (s \geq 0)$ to which $f_\pm$ will be required to belong, so we now write $f_\pm \in \mathcal{H}_s$ without being specific. Note that we can let $f_\pm \to f_\pm \pm c$, where $c$ is a constant, without affecting $f$. This ambiguity will not be a problem since $\mathcal{H}_s$ contains no non–zero constants. Hence the solutions are in one–to–one correspondence with the elements of the orthogonal sum $\mathcal{D}_s = \mathcal{H}_s \oplus \mathcal{H}_s$, whose elements can be written in the vector form

$$\psi = \begin{pmatrix} f_+ \\ f_- \end{pmatrix}.$$  

The wave equation can be restated as $\partial_v f_+ = \partial_u f_- = 0$, or

$$\begin{pmatrix} 0 & \partial_t + \partial_x \\ \partial_t - \partial_x & 0 \end{pmatrix} \psi \equiv (\gamma_t \partial_t + \gamma_x \partial_x) \psi \equiv \partial \psi = 0,$$

where

$$\gamma_t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_x = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$  

Eq. (99) is known in quantum mechanics as the (two–dimensional, massless) Dirac equation; $\psi$ is a spinor, and $\gamma_t, \gamma_x$ are Dirac matrices. The Dirac equation may be viewed as a particular system of first–order equations equivalent to the wave equation which is,
moreover, especially suited to the symmetries of the latter. The Dirac operator \( \partial \) is a “square root” of the wave operator in the sense that \( \partial^2 = -\Box I \), where \( I \) is the \( 2 \times 2 \) identity matrix. (This is related to the Clifford algebra associated with the Lorentz metric.)

In more than one space dimension, there is no such simple relation between scalar–valued solutions \( f \) and spinor–valued solutions \( \psi \).

We denote by \( D_{s+} \) and \( D_{s-} \) the subspaces of \( D_s \) with vanishing second and first components, respectively. Thus \( D_{s\pm} \approx \mathcal{H}_s \).

A symmetry of the wave equation is a transformation which maps solutions to solutions. As symmetries can be composed and inverted, they form a group. We shall be particularly interested in “geometric” symmetries, which are induced from mappings on the underlying space–time that respect the wave equation. Some obvious ones are:

Translations: For each \( (x_0, t_0) \in \mathbb{R}^2 \), the map \((x, t) \to (x + x_0, t + t_0) \) induces a symmetry transformation \( f(x, t) \to f(x - x_0, t - t_0) \). This maps the right– and left–moving waves by

\[
T(u_0, v_0) : f_+(u) \to f_+(u - u_0), \quad f_-(v) \to f_-(v - v_0),
\]

where \( u_0 = x_0 + t_0 \) and \( v_0 = x_0 - t_0 \). Hence translations can be made to act independently on the left and right parts of solutions.

Lorentz transformations: For any real \( \theta \), the map

\[
x \to x \cosh \theta + t \sinh \theta, \quad t \to x \sinh \theta + t \cosh \theta
\]

preserves the Lorentz metric \( x^2 - t^2 = uv \). (It is a “rotation” by the imaginary angle \(-i\theta \) and represents the space–time coordinates as measured by an observer moving with velocity \(-\tanh \theta \).) This map has a particularly simple form in terms of the light–cone coordinates:

\[
u \to e^\theta u \equiv \lambda u, \quad v \to e^{-\theta} v \equiv \lambda^{-1} v.
\]

Since this leaves the wave operator \( \Box = -4\partial_u \partial_v \) invariant, it induces a symmetry on solutions. The simplest such transformation acts on \( f_{\pm} \) by \( f_+(u) \to f_+(\lambda^{-1} u), \ f_-(v) \to f_-(\lambda v) \). We shall need a more general induced transformation, given by

\[
L(\lambda) : f_+(u) \to S_+(\lambda) f_+(\lambda^{-1} u), \quad f_-(v) \to S_-(\lambda) f_-(\lambda v),
\]

where \( S_\pm : \mathbb{R}^+ \to \mathbb{C}_* \equiv \mathbb{C}\setminus\{0\} \) (called “multipliers”) must satisfy

\[
S_\pm(\lambda^{-1}) = S_\pm(\lambda)^{-1}, \quad S_\pm(\lambda \lambda') = S_\pm(\lambda) S_\pm(\lambda'),
\]

in order that Lorentz transformations form a group. (This means that \( S_\pm \) are group homomorphisms.) Continuity in \( \lambda > 0 \) then implies that \( S_+(\lambda) = \lambda^{-j} \) and \( S_-(\lambda) = \lambda^{-j'} \) for some \( j, j' \in \mathbb{C} \).

The three–dimensional symmetry group of all maps \( T(u_0, v_0) L(\lambda) \) is the restricted Poincaré group \( \mathcal{P}_0 \) in \( 1 + 1 \) space–time dimensions. Note that \( \mathcal{P}_0 \) leaves invariant the subspaces \( D_{s\pm} \) of right– and left–moving waves. Since we shall be interested in irreducible representations of the symmetry group which characterize the complete wave equation, it is desirable to include a symmetry which mixes these two subspaces.
Space reflection: The map \((x,t) \rightarrow (-x,t)\) is a discrete symmetry of the wave equation, corresponding to \((u,v) \rightarrow (-v,-u)\). We take the induced mapping on solutions to be

\[
P : f_+(u) \rightarrow f_-(u), \quad f_-(v) \rightarrow f_+(v).
\]  

(106)

Thus, \(P\) interchanges right and left waves, as desired. The idea that right and left be on equal footing is expressed as

\[
L(\lambda) P = P L(\lambda^{-1}),
\]

(107)

which implies that

\[
S_-(\lambda^{-1}) = S_+(\lambda),
\]

(108)

hence \(j' = -j\). We call \(j\) the Lorentz weight of the solution. (In more than one space dimension, it is related to spin.)

The group \(P^\uparrow\) obtained from \(P_0\) by adjoining the space reflection is called the orthochronous Poincaré group in the physics literature, since it still leaves the direction of time invariant. The full Poincaré group \(P\) is obtained by further adjoining time reversal. However, the latter must be antilinear for reasons which need not concern us here (cf. Streater and Wightman [28]). The fact that \(P^\uparrow\) leaves the direction of time invariant implies that the subspaces of positive–and negative–frequency solutions are invariant under it. To mix them, we introduce the following.

Total reflection: The map \((x,t) \rightarrow (-x,-t)\) is another discrete symmetry of the wave equation, corresponding to \((u,v) \rightarrow (-u,-v)\). We take the induced mapping on solutions to be

\[
R : f_+(u) \rightarrow f_+(u), \quad f_-(v) \rightarrow f_-(v).
\]

(109)

Note that unlike translations, Lorentz transformations do not act independently on the left– and right–moving waves. This will be remedied by including the next symmetry.

Dilations: Since the wave equation is homogeneous in \(x\) and \(t\), it is invariant under the map \((x,t) \rightarrow (\alpha x, \alpha t)\), for any \(\alpha \neq 0\). Equivalently, \((u,v) \rightarrow (\alpha u, \alpha v)\). It suffices to confine our attention to \(\alpha > 0\), since dilations with \(\alpha < 0\) can be obtained by combining \(D(\alpha)\) with \(R\). As with Lorentz transformations, we shall allow a multiplier \(M : \mathbb{R}^+ \rightarrow \mathbb{C}_+\) in the induced mapping. Thus \(f(x,t) \rightarrow M(\alpha) f(\alpha^{-1} x, \alpha^{-1} t)\), or

\[
D(\alpha) : f_+(u) \rightarrow M(\alpha) f_+(\alpha^{-1} u), \quad f_-(v) \rightarrow M(\alpha) f_-(\alpha^{-1} v).
\]

(110)

In order that dilations form a group, we must have \(M(\alpha^{-1}) = M(\alpha)^{-1}\) and \(M(\alpha \alpha') = M(\alpha) M(\alpha')\). Again, continuity implies that \(M(\alpha) = \alpha^{-\kappa}\) for some \(\kappa \in \mathbb{C}\), called the conformal weight of the solution.

The following simple argument should convince the reader of the need to include non–trivial multipliers in eq. (110), i.e., to consider conformal weights other than zero: Suppose that \(f(x,t)\) is a solution with conformal weight \(\kappa\) and let \(g(x,t) = (a\partial_x + b\partial_t) f(x,t)\), where \(a\)}
and $b$ are constants. Then $g$ is also a solution of the wave equation, and it is easily seen to have conformal weight $\kappa + 1$. In this way we can shift the conformal weight of a solution by any positive integer $n$ by applying a homogeneous partial differential operator of order $n$ with constant coefficients. (In three space dimensions, the electromagnetic potentials satisfy the wave equation in free space; the electromagnetic fields are obtained from them by applying first–order partial differential operators, hence are solutions but with higher weight.)

Note that dilations commute with Lorentz transformations. Composing $L(\lambda)$ and $D(\alpha)$, we obtain

$$f_+(u) \rightarrow \alpha^{-\kappa} \lambda^{-j} f_+(\lambda^{-1} \alpha^{-1} u), \quad f_-(v) \rightarrow \alpha^{-\kappa} \lambda^j f_-(\lambda \alpha^{-1} v).$$

(111)

In order for the combination of dilations and Lorentz transformation to act independently on $f_+$ and $f_-$, we must therefore require that $j = \kappa$. We shall refer to $\kappa \equiv j \equiv -j'$ simply as the weight of $f$. Setting $\beta = \alpha \lambda$ and $\gamma = \alpha / \lambda$, we then obtain

$$f_+(u) \rightarrow \beta^{-\kappa} f_+(\beta^{-1} u), \quad f_-(v) \rightarrow \gamma^{-\kappa} f_-(\gamma^{-1} v).$$

(112)

Consequently, the semi–direct product $\mathbb{R}^+ \times \mathcal{P}_0$ acts on $f$ by

$$f_+(u) \rightarrow \beta^{-\kappa} f_+(\beta^{-1} (u - u_0)), \quad f_-(v) \rightarrow \gamma^{-\kappa} f_-(\gamma^{-1} (v - v_0)).$$

(113)

This means that the group $\mathcal{G}_0 \equiv \mathbb{R}^+ \times \mathcal{P}_0$ generated by the continuous transformations $T, L$ and $D$ can be represented as a direct product $\mathcal{A} \times \mathcal{A}$ of two copies of the affine group $\mathcal{A}$ acting independently on the right–moving and left–moving waves. We denote by $\mathcal{G}_1$ the group obtained from $\mathcal{G}_0$ by adjoining $R$, and by $\mathcal{G}_2$ the group obtained by further adjoining $P$. $\mathcal{G}_1$ contains negative as well as positive dilations, but the signs of the dilations in the two components are equal. (They can be made independent by adjoining yet another discrete symmetry, namely $(x, t) \rightarrow (t, x)$, but we resist the temptation.) Note that the subspaces of right– and left–moving waves are invariant under $\mathcal{G}_1$ but not under $\mathcal{G}_2$.

Since the affine group is closely related to wavelets, the decomposition $\mathcal{G}_0 \approx \mathcal{A} \times \mathcal{A}$ suggests that we apply separate wavelet analyses to $f_+$ and $f_-$. Actually, we shall see that more can be done, due to the fact that the wave equation has another, quite unexpected, symmetry.

### 4.3. Hilbert Structures on Solutions

So far we have dealt exclusively with solutions in the space–time domain, where symmetries have a direct and intuitive meaning. We now wish to introduce inner products on solutions which make the above symmetry transformations unitary. For this purpose, it is necessary to venture into the Fourier domain and, later, also into the domain of complex space–time, invoking the Analytic–Signal transform introduced earlier.

Let $f(x, t)$ be a solution transforming under $\mathcal{G}_2$ with weight $\kappa$. Representing $f_\pm$ formally by Fourier integrals, we obtain

$$f(x, t) = \int_{-\infty}^{\infty} dp e^{2\pi i p u} \hat{f}_+(p) + \int_{-\infty}^{\infty} dp e^{2\pi i p v} \hat{f}_-(p).$$

(114)
The symmetry operations defined earlier can now be represented in Fourier space. Translations act by

\[ T(u_0, v_0) : \hat{f}_\pm(p) \to e^{-2\pi i p u_0} \hat{f}_\pm(p), \quad \hat{f}_\pm(p) \to e^{-2\pi i p v_0} \hat{f}_\pm(p), \]  

(115)

Lorentz transformations act by

\[ L(\lambda) : \hat{f}_+(p) \to \lambda^{1-\kappa} \hat{f}_+(\lambda p), \quad \hat{f}_-(p) \to \lambda^{\kappa-1} \hat{f}_-(\lambda^{-1} p), \]  

(116)

space reflection acts by

\[ P : \hat{f}_+(p) \to \hat{f}_-(p), \quad \hat{f}_-(p) \to \hat{f}_+(p), \]  

(117)

total reflection acts by

\[ R : \hat{f}_\pm(p) \to \hat{f}_\mp(-p) \]  

(118)

and dilations act by

\[ D(\alpha) : \hat{f}_\pm(p) \to \alpha^{1-\kappa} \hat{f}_\pm(\alpha p). \]  

(119)

Let \( s \equiv 2\Re(\kappa) - 1 \geq 0 \) and define \( \mathcal{H}_s \) to be the Hilbert space of all ‘functions’ \( g(x) \) such that

\[ \|g\|_s^2 \equiv \int_{-\infty}^{\infty} dp \ |p|^{-s} |g(p)|^2 < \infty. \]  

(120)

In the terminology of Battle [3], this is the “massless Sobolev space of degree \(-s/2\).” The norm of a solution \( f(x, t) \) (which may be identified with the corresponding spinor \( \psi \in \mathcal{D}_s \)) is then given by

\[ \|f\|_s^2 \equiv \|\psi\|_s^2 = \|f_+\|_s^2 + \|f_-\|_s^2. \]  

(121)

From the above actions it easily follows that the transformations \( T, L, R, P \) and \( D \) act unitarily on \( \mathcal{D}_s \), thus giving a unitary representation of \( G_2 \). Denote by \( \mathcal{H}_s^+ \) and \( \mathcal{H}_s^- \) the subspaces of \( \mathcal{H}_s \) with support on \([0, \infty)\) and \((-\infty, 0]\), respectively. (Note that since \( s \geq 0 \), the value of \( \hat{f}_\pm \) at \( p = 0 \) is unimportant; when \( s = 0 \), the origin has zero measure, and when \( s > 0 \), \( f_\pm \in \mathcal{H}_s \) implies \( f_\pm(0) = 0 \).) The continuous symmetries \( T, L \) and \( D \) all preserve the sign of \( p \), hence the group \( G_0 \) generated by them leaves invariant the subspaces of solutions

\[ f_+(x, t) = \int_0^{\infty} dp \ e^{2\pi i p u} \hat{f}_+(p), \]  

(122)

\[ f_-(x, t) = \int_{-\infty}^0 dp \ e^{2\pi i p u} \hat{f}_+(p) \]  

\[ f_-(x, t) = \int_{-\infty}^{\infty} dp \ e^{2\pi i p v} \hat{f}_-(p), \]  

\[ f_+(x, t) = \int_0^{\infty} dp \ e^{2\pi i p v} \hat{f}_-(p), \]
where the superscripts denote positive– and negative–frequency components, as in Section 3. The complete solution is $f = f_+^+ + f_+^- + f_-^- + f_-^+$. Note that $\hat{f}_-(p)$ with $p > 0$ represents a negative–frequency component since $v = x - t$. Thus $f_+^+ \in \mathcal{H}_s^+$ but $f_+^- \in \mathcal{H}_s^-$. The decomposition $\mathcal{H}_s = \mathcal{H}_s^+ \oplus \mathcal{H}_s^-$ therefore gives a corresponding decomposition

$$
\mathcal{D}_s = \mathcal{D}_{s+}^+ \oplus \mathcal{D}_{s+}^- \oplus \mathcal{D}_{s-}^+ \oplus \mathcal{D}_{s-}^-
$$

(123)

where $\mathcal{D}_{s+}^\pm = \mathcal{H}_{s+}^\pm \oplus \{0\}$ and $\mathcal{D}_{s-}^\pm = \{0\} \oplus \mathcal{H}_{s-}^\pm$. Let $\mathcal{D}_s^\pm = \mathcal{D}_{s+}^\pm \oplus \mathcal{D}_{s-}^\pm$. The restriction of the representation of $\mathcal{G}_2$ to $\mathcal{G}_0$ leaves invariant all four subspaces $\mathcal{D}_{s,\sigma}^\tau$ ($\sigma, \tau = \pm$), hence it is reducible. When $R$ is included, only the subspaces $\mathcal{D}_{s+}$ and $\mathcal{D}_{s-}$ remain invariant. When $P$ is further included, no invariant subspaces remain and we have an irreducible unitary representation of $\mathcal{G}_2$ on $\mathcal{D}_s$. This shows that the discrete symmetries $P$ and $R$ serve to ‘weave’ the four representations on $\mathcal{D}_{s,\pm}^\pm$, $\mathcal{D}_{s,\pm}^\mp$ (which are associated with the affine group rather than the wave equation) into a single representation characterising the wave equation as a whole. This was the purpose for which they were introduced.

However, distinct values of $\kappa$ are not significantly different at this point. For if $f$ has weight $\kappa$, let $\kappa' \in \mathbb{C}$ with $s' \equiv 2\mathcal{R}(\kappa') - 1$ and

$$
\hat{g}_\pm(p) \equiv |p|^{\kappa' - \kappa} \hat{f}_\pm(p).
$$

(124)

Then the corresponding solution $g(x, t)$ has weight $\kappa'$ and, furthermore, the map $f \to g$ is unitary from $\mathcal{D}_s$ onto $\mathcal{D}_{s'}$. This shows that the UIR’s with any two values of $\kappa$ are unitarily equivalent. If no more could be said, we might as well restrict ourselves to a single value of $\kappa$, since solutions of arbitrary weight can be obtained by the above method. (Recall that we could shift the weight up by a positive integer by applying a differential operator; the above generalizes this process to arbitrary shifts.) Actually, it turns out that for certain special values of $\kappa$, the group of symmetries can be significantly enlarged, and distinct values of $\kappa$ then give unitarily inequivalent representations of the larger group. Unlike $\mathcal{A}$, however, the larger group no longer acts simply on the Fourier space; instead, its natural domain of action is the complexified space–time associated with the AST. For this reason it becomes necessary to re–express the norm of $\mathcal{D}_s$ in terms of the AST $\hat{f}$ of $f$, as will be done next.

### 4.4. Norms in terms of Analytic Signals

We wish to give the norms defined above expressions which are local in space, that is, involve only values of $f(x, t)$ at an arbitrary fixed time $t$, say $t = 0$. Since a knowledge of both $f(x, 0)$ and $\partial_t f(x, 0)$ is required to determine $f$, we cannot expect such expressions to characterize the complete solution $f$. On the other hand, the positive– and negative–frequency parts $f_\pm$ of $f$ satisfy the first–order pseudo–differential equation

$$
-i\partial_t f_\pm = \pm \sqrt{-\partial_x^2} f_\pm,
$$

(125)

hence are determined by their initial values $f_\pm(x, 0)$. (Thus, instead of using $f(x, 0)$ and $\partial_t f(x, 0)$ as initial data, we use the symmetric pair $f_\pm(x, 0)$.) We therefore consider separately the norms on $\mathcal{H}_s^+$ and $\mathcal{H}_s^-$, given respectively by
\[ \|f^+\|^2 = \int_{-\infty}^{\infty} dp |p|^{-s} \left[ \theta(p) |\hat{f}_+(p)|^2 + \theta(-p) |\hat{f}_-(p)|^2 \right] \]
\[ = \int_0^\infty dp \ p^{-s} \left[ |\hat{f}_+(p)|^2 + |\hat{f}_-(p)|^2 \right], \]  
(126)
\[ \|f^-\|^2 = \int_{-\infty}^{\infty} dp |p|^{-s} \left[ \theta(-p) |\hat{f}_+(p)|^2 + \theta(p) |\hat{f}_-(p)|^2 \right] \]
\[ = \int_0^\infty dp \ p^{-s} \left[ |\hat{f}_+(p)|^2 + |\hat{f}_-(p)|^2 \right]. \]

The separation of positive and negative frequencies is characteristic of the AST; hence let us consider the AST of the complete solution \( f \):
\[ \tilde{f}(x + ix', t + it') \equiv \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{d\tau}{\tau - i} f(x + \tau x', t + \tau t') \]
\[ = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{d\tau}{\tau - i} \left[ f_+(u + \tau u') + f_-(v + \tau v') \right] \]
\[ = \tilde{f}_+(u + iv') + \tilde{f}_-(v + iv'), \]  
(127)
where the imaginary parts of the space–time and light–cone coordinates are related by \( u' = x' + t', \ v' = x' - t' \) and \( \tilde{f}_\pm \) are the AST's of \( f_\pm \). Recall from Section 3 that the AST’s \( \tilde{f}_\pm \) of \( f_\pm(x, t) \) are obtained by restricting \( \tilde{f} \) to the forward and backward tubes \( T^\pm \), where \( \pm t' > |x'| \). This suggests that the AST is well–suited for the analysis of the above norms, since it naturally breaks \( f \) up into its four components.

The symmetry operations on real space–time extend to the complexified space–time by \( \mathbb{C} \)–linearity, and these extensions induce transformations on \( \tilde{f} \) in the obvious way. For example,
\[ T(u_0, v_0) : \tilde{f}_+(u + iv') \to \tilde{f}_+(u - u_0 + iv'), \ \tilde{f}_-(v + iv') \to \tilde{f}_+(v - v_0 + iv'). \]  
(128)
(Note that the parameters of the symmetries are still real; e.g., we do not consider complex translations.)

Since only the combinations \((x + ix') \pm (t + it')\) enter into \( \tilde{f} \), it suffices to set \( x' = 0 \) and \( t = 0 \). This is called the Euclidean region in quantum field theory, since the Lorentzian metric \( x^2 - t^2 \) becomes \( x^2 + t'^2 \), which is Euclidean. (See Glimm and Jaffe [10].) Note that
\[ \tilde{f}(x, it') = \tilde{f}_+(x + it') + \tilde{f}_-(x - it') \equiv \tilde{f}_+(z) + \tilde{f}_-(z^*) \]
(129)
is now presented as a sum of analytic and anti–analytic functions of \( z \equiv x + it' \), as befits a solution of the “analytically continued” wave equation
\[ \left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial t'^2} \right] \tilde{f}(x, it') = 0, \]
(130)
which states that \( \tilde{f}(x, it') \) is harmonic. In terms of the Fourier transforms,
\[ \hat{f}(x, it') = \int_{-\infty}^{\infty} dp \left[ \theta(pt') e^{2\pi ipz} \hat{f}_+(p) + \theta(-pt') e^{2\pi ipz} \hat{f}_-(p) \right]. \]  

(131)

Hence the initial values of \( f^\pm \) are given by

\[
F^\pm(x) \equiv f^\pm(x,0) = \lim_{t' \to 0^+} \hat{f}(x, it') \equiv \hat{f}(x, \pm i0) = \int_{-\infty}^{\infty} dp e^{2\pi ipx} \left[ \theta(\pm p) \hat{f}_+(p) + \theta(\mp p) \hat{f}_-(p) \right],
\]

(132)

from which

\[
\hat{F}^\pm(p) = \theta(\pm p) \hat{f}_+(p) + \theta(\mp p) \hat{f}_-(p).
\]

(133)

Returning to the norms, consider first the simplest case \( s = 0 \) (i.e., \( \kappa = \frac{1}{2} + i\mu \), with \( \mu \) real):

\[
\|\| f^\pm \|\|^2 = \int_{-\infty}^{\infty} dp \left[ \theta(\pm p) |\hat{f}_+(p)|^2 + \theta(\mp p) |\hat{f}_-(p)|^2 \right]
\]

\[
= \int_{-\infty}^{\infty} dp |\hat{F}^\pm(p)|^2
\]

\[
= \int_{-\infty}^{\infty} dx |\hat{F}(x)|^2
\]

(134)

by Plancherel’s theorem. We have therefore proved

**Theorem 4.** The norms in the positive– and negative–frequency subspaces \( D^\pm_0 \) of \( D_0 \) can be be written as

\[
\|\| f^\pm \|\|^2 = \int_{-\infty}^{\infty} dx |\hat{f}(x, \pm i0)|^2,
\]

(135)

and the complete norm in \( D_0 \) is therefore

\[
\|\| f \|\|^2 = \int_{-\infty}^{\infty} dx \left[ |\hat{f}(x, i0)|^2 + |\hat{f}(x, -i0)|^2 \right].
\]

(136)

This is not quite local in space since it involves the boundary values \( \hat{f}(x, \pm i0) \). Rather, it is ‘local’ in the complex space generated by the AST. We refer to this property as *pseudo–locality*; it stands in the same relation to locality as the pseudo–differential equation (125) stands to the wave equation. Since \( f(x, 0) = \hat{f}(x, i0) + \hat{f}(x, -i0) \) and \( \hat{f}(x, it') \) may be regarded as a regularized version of \( f(x, 0) \), the term ‘pseudo–locality’ seems appropriate.

Next, fix \( \kappa \in \mathbb{C} \) with \( s > 0 \). Can we obtain pseudo–local expressions for the norms with \( s > 0 \)? We have
\[ \|f^\pm\|_s^2 = \int_{-\infty}^{\infty} dp \, |p|^{-s} \left[ \theta(\pm p) |\hat{f}_+(p)|^2 + \theta(\mp p) |\hat{f}_-(p)|^2 \right] \]
\[ = \int_0^{\infty} dp \, p^{-s} \left[ |\hat{f}_+(p)|^2 + |\hat{f}_-(p)|^2 \right]. \quad (137) \]

**Theorem 5.** For arbitrary \( s > 0 \), the norms \( \|f^\pm\|_s \) in \( D_s^\pm \) have the following pseudo–local expressions in terms of the restrictions of \( \tilde{f} \) to the Euclidean region:

\[ \|f^\pm\|_s^2 = N_s \int_{-\infty}^{\infty} dx \int_0^{\infty} dt' \, t'^{s-1} \left| \tilde{f}(x, \pm it') \right|^2, \quad (138) \]

where

\[ N_s = \frac{(4\pi)^s}{\Gamma(s)}. \quad (139) \]

Hence the norm in \( D_s \) is given by

\[ \|f\|_s^2 = N_s \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dt' \, |t'|^{s-1} \left| \tilde{f}(x, it') \right|^2. \quad (140) \]

**Proof:** We prove the theorem for \( \|f^+\|_s \). The proof for \( \|f^-\|_s \) is similar. By eq. (131), \( \tilde{f}(x, it') \) can be written as an inverse Fourier transform

\[ \tilde{f}(x, it') = \left[ \theta(pt') e^{-2\pi pt'} \hat{f}_+(p) + \theta(-pt') e^{2\pi pt'} \hat{f}_-(p) \right] \cdot (x), \quad (141) \]

hence by Plancherel’s theorem,

\[ \int_{-\infty}^{\infty} dx \left| \tilde{f}(x, it') \right|^2 = \int_{-\infty}^{\infty} dp \left[ \theta(pt') e^{-4\pi pt'} |\hat{f}_+(p)|^2 + \theta(-pt') e^{4\pi pt'} |\hat{f}_-(p)|^2 \right] \]
\[ = \int_{-\infty}^{\infty} dp \, \theta(pt') e^{-4\pi pt'} \left[ |\hat{f}_+(p)|^2 + |\hat{f}_-(p)|^2 \right]' \quad (142) \]

When \( t' > 0 \), then \( p > 0 \) in the last integral. Integrating over \( t' \) with the factor \( t'^{s-1} \), exchanging the order of integration on the right–hand side and using

\[ \int_0^{\infty} dt' \, t'^{s-1} e^{-4\pi pt'} = (4\pi p)^{-s} \Gamma(s) = N_s^{-1} p^{-s} \quad (143) \]

for \( s > 0 \), we obtain

\[ N_s \int_{-\infty}^{\infty} dx \int_0^{\infty} dt' \, t'^{s-1} \left| \tilde{f}(x, it') \right|^2 = \int_{0}^{\infty} dp \, p^{-s} \left[ |\hat{f}_+(p)|^2 + |\hat{f}_-(p)|^2 \right] \]
\[ = \|f^+\|_s^2 \quad (144) \]

as claimed. \( \Box \)
4.5. The Wavelets $e_z$ and their Mother

Here we show that Theorem 5 establishes a resolution of the identity $I_s$ in $\mathcal{D}_s$ ($s > 0$) in terms of wavelets $e_z \in \mathcal{D}_s$ parametrized by $z = x + it'$, $t' \neq 0$, in the manner described in Sections 2.2 and 3.3. We shall define $e_z$ as an element of $\mathcal{D}_s$ whose inner product with $f \in \mathcal{D}_s$ is the value of the AST $\hat{f}$ at $(x, it')$, i.e.,

$$\hat{f}(x, it') = \langle \langle e_z, f \rangle \rangle_s \equiv \langle e_{z+}, f_+ \rangle_s + \langle e_{z-}, f_- \rangle_s,$$  \hspace{1cm} (145)

where $\langle \cdot, \cdot \rangle_s$ and $\langle \langle \cdot, \cdot \rangle \rangle_s$ denote the inner products in $\mathcal{H}_s$ and $\mathcal{D}_s$, respectively. Since $e_z$ is to belong to $\mathcal{D}_s$, it must itself be a solution. Eq. (131) shows that its components in Fourier space are

$$\hat{e}_{z+}(p) = \theta(p) |p|^s e^{-2\pi p z^*},$$

$$\hat{e}_{z-}(p) = \theta(-p) |p|^s e^{-2\pi p z^*}. \hspace{1cm} (146)$$

(Recall that according to our convention, $\langle g, f \rangle_s$ is anti–linear in $g$.) Note that $e_{z-} = e_{z+}^*$ as elements of $\mathcal{H}_s$, and that $e_{z-}$ and $e_{z+}$ are orthogonal not only as elements of the direct sum $\mathcal{D}_s = \mathcal{D}_{s+} \oplus \mathcal{D}_{s-}$ (since $e_{z\pm} \in \mathcal{D}_{s\pm}$) but also as elements of $\mathcal{H}_s$. General solutions clearly do not share this property. The positive– and negative–frequency components $e_{z\pm}$ of $e_z$ are obtained by choosing $t' > 0$ and $t' < 0$, respectively:

$$e_z = \begin{cases} 
    e_{z+} & \text{if } t' > 0 \\
    e_{z-} & \text{if } t' < 0.
\end{cases} \hspace{1cm} (147)$$

Hence

$$\hat{e}_{z+}(p) = \theta(p) |p|^s e^{-2\pi p z^*}, \hspace{1cm} t' > 0$$

$$\hat{e}_{z-}(p) = \theta(-p) |p|^s e^{-2\pi p z^*}, \hspace{1cm} t' > 0$$

$$\hat{e}_{z+}^-(p) = \theta(-p) |p|^s e^{-2\pi p z^*}, \hspace{1cm} t' < 0$$

$$\hat{e}_{z-}^-(p) = \theta(p) |p|^s e^{-2\pi p z^*}, \hspace{1cm} t' < 0. \hspace{1cm} (148)$$

For $t' > 0$, we have

$$\|e_{z\pm}^+\|^2_s = \|e_{z+}^+\|^2_s + \|e_{z-}^+\|^2_s$$

$$= \int_{-\infty}^{\infty} dp \ |p|^s \left[ \theta(p) e^{-4\pi pt'} + \theta(-p) e^{4\pi pt'} \right]$$

$$= 2 \int_{0}^{\infty} dp \ p^s e^{-4\pi pt'} \frac{2\Gamma(s+1)}{(4\pi t')^{s+1}} \hspace{1cm} (149)$$

Since $e_{z-} = e_{z+}^*$ in $\mathcal{H}_s$, we have for $t' < 0$

$$\|e_{z-}\|^2_s = \frac{2\Gamma(s+1)}{(-4\pi t')^{s+1}} \hspace{1cm} (150)$$

hence
Thus for translations, To see how of the Klein–Gordon equation. Fouri er domain, can be made manifest in the complex space–time domain by introducing $x_t$ time, with $\varphi$ normalized factor for $e_{z^*}$ analytic in $z$ and $z^*$, respectively. The wavelet $e_z$ can be regarded as a regularization of the point $x$ in space, with $|\varphi'\rangle$ as a measure of its diffusion. Hence $e_z$ transforms naturally under those symmetry operations which leave the Euclidean region invariant. These consist of space translations $T(x_0)$, dilations $D(\alpha)$ and the reflections $P$ and $R$. Clearly the Euclidean region is not invariant under Lorentz transformations and time translations. (It is not difficult to extend the family of wavelets to one parametrized by general points $(x + ix', t + it')$ in complex space–time, with $x' \neq \pm t'$. This extended family is then invariant under $G_2$, and the above inner products are obtained by simply integrating over the submanifold defined by $x' = t = 0$. Then the invariance of the inner product under the full group $G_2$, known to hold in the Fourier domain, can be made manifest in the complex space–time domain by introducing conserved currents and using Stokes’ theorem. Cf. [16], Section 4.5 for a related treatment of the Klein–Gordon equation.)

To see how $e_z$ transforms under a symmetry operation, we need only apply eq. (145). Thus for translations,

$$
\langle T(x_0) e_z, f \rangle_s = \langle e_z, T(-x_0)f \rangle_s = \left( T(-x_0) \tilde{f} \right) (x, it')
$$

$$
= \tilde{f}(x + x_0, it') = \langle e_{z+x_0}, f \rangle_s , \quad (155)
$$

By simple integration over the one–dimensional subspace of $e_z$ in $D_s$. For $\varphi' \neq 0$,
hence
\[ T(x_0) e_z = e_{z+x_0}. \]  
(156)

Similarly,
\[ \langle D(\alpha) e_z, f \rangle_s = \langle e_z, D(\alpha^{-1}) f \rangle_s \]
\[ = \alpha^\kappa \tilde{f}(\alpha x, i\alpha t') = \alpha^\kappa \langle e_{\alpha z}, f \rangle_s, \]
(157)

hence
\[ D(\alpha) e_z = \alpha^\kappa e_{\alpha z}, \quad \alpha > 0. \]  
(158)

Since \( P^* = P^{-1} = P \) and \( (P \tilde{f})(x, it') = \tilde{f}(-x, it') \), we have
\[ Pe_z = e_{-z^*}. \]  
(159)

Finally, \( R^* = R^{-1} = R \) and \( (R \tilde{f})(x, it') = \tilde{f}(-x, -it') \) implies
\[ Re_z = e_{-z}. \]  
(160)

Thus \( T(x_0) \) and \( D(\alpha) \) generate a single copy of \( \mathcal{A} \) (as opposed to \( \mathcal{G}_0 \approx \mathcal{A} \times \mathcal{A} \)), and \( P \) and \( R \) extend this to include space reflections and negative dilations. We may therefore begin with a single basic wavelet, say \( \phi = e_{i+} \in D_{s+} \) (left–moving wavelet with \( z = i \)). Its components in Fourier space are then
\[ \hat{\phi}_+(p) = \theta(p) p^s e^{-2\pi p}, \quad \hat{\phi}_-(p) \equiv 0. \]  
(161)

For \( z = x + it' \) with \( t' > 0 \) we obtain all the left–moving wavelets by applying \( T, D \) and \( R \) to \( \phi \):
\[ e_{z+}^+ = (t')^{-\kappa^*} T(x) D(t') \phi \]
\[ e_{z-}^+ = (t')^{-\kappa} T(x) RD(t') \phi, \]  
(162)

and the right–moving wavelets are obtained in the same way from \( P\phi = e_{i-} \). Note that the choice of \( \phi \) depends only on \( s \) and not on the imaginary part of \( \kappa \). We may regard eqs. (162) as providing a construction of the UIR with weight \( \kappa \) by giving all of its wavelets. Modified versions of the left component \( \phi_+ \) of \( \phi \) have already appeared in the literature, in connection with the representation theory of the affine group. Namely, if we absorb the Sobolev weight function \(|p|^{-s}\) into the elements \( g \) of \( \mathcal{H}_s \) by defining \( \tilde{G}(p) = |p|^{-s/2} \hat{g}(p) \) (so that \( \tilde{G} \in L^2(\mathbb{R}, dp) \)), then \( \phi \) becomes
\[ \hat{\Phi}_+(p) = \theta(p) p^{s/2} e^{-2\pi p}, \quad \hat{\Phi}_-(p) \equiv 0. \]  
(163)

The function \( \hat{\Phi}_+(p) \) with \( s = 1 \) first appeared in the classical papers of Aslaksen and Klauder [1], where it was used as a ‘fiducial vector’ to obtain a ‘continuous representation’ of the affine group. Since those papers contain the first instance of what is now called
continuous wavelet analysis, we suggest that \( \phi \) with \( s = 1 \) deserves to be called the *Mother of all Wavelets*. For \( s = 1, 2, 3, \ldots \), \( \hat{\Phi}_{\pm}(p) \) also appears in the work of Paul [24] in connection with representations of the affine group and their extension to \( SL(2, \mathbb{R}) \) (see below). It must be noted, however, that in our case the wavelets are prescribed by the problem at hand (solving the wave equation and extending the solutions to complex space–time, via the AST) rather than chosen arbitrarily as a convenient family of functions to be used in expansions. This is further discussed in Section 5.

Having established a resolution of unity in terms of the wavelets \( e_z \), let us now investigate the associated reconstruction (cf. Section 2.2). The formula corresponding to eq. (22) is

\[
f(x, t) = N_s \int_{\mathbb{R}^2} dx_1 dt_1' |t_1'|^{s-1} e_{x_1 + it_1'}(x, t) \tilde{f}(x, it_1').
\]  

(164)

This gives an expansion of an arbitrary solution \( f \in \mathcal{D}_s \) in terms of the wavelets \( e_{z_1} \), this time expressed in the real space–time domain. Let us therefore compute the left–moving wavelet \( e_{z_1+}(x, t) \). For simplicity, choose \( x_1 = 0 \) and \( t_1' > 0 \); the other cases \((x_1 \neq 0, t_1' < 0 \) and right–moving wavelets) can be obtained easily by using \( T(x_1) \), \( R \) and \( P \). By eq. (166),

\[
e_{it_1'}(x, t) = \int_0^\infty dp \ p^s e^{2\pi ip(u + it_1')} = \frac{\Gamma(s + 1)}{(2\pi(t_1' - iu))^{s+1}}.
\]  

(165)

Hence

\[
|e_{it_1'}(x, t)|^2 = \frac{\Gamma(s + 1)^2}{(2\pi)^2s+2 (t_1'^2 + u^2)^{s+1}}.
\]  

(166)

At time \( t \), this solution is localized in space around \( x = -t \), and its width is proportional to \( t_1' \). As \( t_1' \to 0 \), the wavelet becomes an infinitely sharp spike at \( x = -t \). This is not surprising, since \( t_1' \) acts as a scale parameter for the affine group. Similarly, \( e_{x_1 + it_1'}(x, t) \) is centered near \( x = x_1 - t \).

A direct and important interpretation of \( t' \) is that \( 1/t' \) is proportional to the average frequency \( \nu_{s\pm} \), or color, in the frequency spectrum of \( e_{z\pm} \). This can be seen most easily by noting that the function \( p^s \exp(-2\pi pt') \) has a maximum at \( p = s/(2\pi t') \). A more precise argument is based on the quantum mechanical notion of expectation values, where \( |\hat{e}_{z\pm}(p)|^2 \) is viewed as an (unnormalized) probability distribution for \( p \) with respect to the measure \( |p|^{-s} \ dp \). Remembering that the frequency in \( \mathcal{D}_s\pm \) is \( \pm p \), this gives

\[
\nu_{s\pm} = \frac{\int_0^\infty |p|^{-s} dp \ (\pm p) |\hat{e}_{z\pm}(p)|^2}{\int_{-\infty}^\infty |p|^{-s} dp |\hat{e}_{z\pm}(p)|^2} = \pm \left(-\frac{1}{4\pi \partial t'}\right) \log \|e_{z\pm}\|^2_s = \frac{s + 1}{4\pi t'}.
\]  

(167)

(Note that \( \nu_{s+} = \nu_{s-} = \nu_s \) and, as expected, \( \nu_s \) has the same sign as \( t' \).) Similarly, one computes the standard deviation in the frequency to be

\[
\Delta\nu_s = \sqrt{\left(-\frac{1}{4\pi \partial t'}\right)^2 \log \|e_{z\pm}\|^2_s} = \frac{\sqrt{s + 1}}{4\pi |t'|}.
\]  

(168)
Note that eq. (154) assumes an especially simple form in terms of the variables \((x, \nu_s)\):

\[
\frac{2s}{s+1} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} d\nu_s P \left( x + i \frac{s+1}{4\pi\nu_s} \right) = I_s,
\]

which gives a resolution of the identity in \(\mathcal{D}_s\) in terms of the orthogonal projections onto the wavelet subspaces parametrized by initial location and color, with all locations and colors given equal a priori probability; since the measure is Lebesgue! (The appearance of Lebesgue measure is not an accident. The variables \((x, \nu_s)\) are phase-space coordinates, also famous in symplectic geometry as Darboux canonical coordinates, and \(dx\,d\nu_s\) is the corresponding Liouville measure. For a similar analysis of the wavelets associated with the Klein–Gordon equation, see ref. [16], Section 4.4.)

We are now in a position to answer a question which was posed in a more general context in Section 3.2: Given a function \(F(x, t')\), how can we tell whether \(F = \tilde{f}(x, it')\) for some \(f \in \mathcal{D}_s\)? Suppose this were the case for some positive integer \(s\). From the polarized version of eq. (140), i.e.,

\[
\langle \langle g, f \rangle \rangle_s = N_s \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dt' |t'|^{s-1} \tilde{g}(x, it')^* \tilde{f}(x, it'),
\]

we obtain by choosing \(g = e_{z_1}\) with \(z_1 = x_1 + it'_1, z = x + it'\):

\[
\tilde{f}(x_1, it'_1) = N_s \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dt' |t'|^{s-1} \langle \langle e_{z_1}, e_z \rangle \rangle_s \tilde{f}(x, it').
\]

The function

\[
K(z_1, z) \equiv \langle \langle e_{z_1}, e_z \rangle \rangle_s
\]

is a reproducing kernel for the function space \(\mathcal{F}_s \equiv \{ \tilde{f} \) such that \( f \in \mathcal{D}_s \}\}. A computation similar to that for \(\|e_z\|_s\) gives

\[
K(z_1, z) = \frac{2\theta(t'_1 + t')}{(2\pi|t'_1 + t'|)^{s+1}} \Re \left[ \left( 1 - i \frac{x_1 - x}{t'_1 + t'} \right)^{-s-1} \right].
\]

The integral operator defined by \(K\) acts as the orthogonal projection from \(L^2(|t'|^{s-1} dx\,dt')\) to \(\mathcal{F}_s\). The given function \(F\) is therefore the AST of some \(f \in \mathcal{D}_s\) if and only if it satisfies the consistency condition

\[
F(x_1, t'_1) = N_s \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dt' |t'|^{s-1} K(z_1, z) F(x, t')
\]

(cf. [16], Chapter 1 for an exposition of this general idea).

4.6. Extension to \(SL(2, \mathbb{R})\)

We are at last ready to extend the symmetry group to a larger one that will ‘break’ the unitary equivalence of the UIR’s of \(\mathcal{A}\) with different weights. We do so by showing that
for $2\kappa = 1, 2, 3, \cdots$, the unitary action of $A$ on each of the subspaces $D_{s\pm}$ extends to the group of real $2 \times 2$ matrices of unit determinant, and that this extension preserves the positive– and negative–frequency subspaces $D_{s+}$ and $D_{s-}$ and is therefore irreducible when restricted to these subspaces. The extension can be implemented most easily on the AST’s of solutions in $D_s$. We concentrate on $D_{s+} \approx H_s$ for simplicity and denote its elements by $g$ rather than $f_+$ to avoid a proliferation of indices.

The action of the affine group on $H_s$ is induced by the map $u \to au + \beta$ on $\mathbb{R}$, which is but a very special case of an order–preserving diffeomorphism $h : \mathbb{R} \to \mathbb{R}$; i.e., an invertible map $h$ such that both $h$ and $h^{-1}$ are $C^\infty$ and $dh/du > 0$. Under composition, the set of all such maps forms a group $\text{Diff}_+(\mathbb{R})$, sometimes called the pseudo–conformal group, of which $A$ is a subgroup. Note that whereas it takes only two parameters to specify an affine map, an infinite number of parameters are needed to specify a general diffeomorphism. Nevertheless, for $\kappa = \frac{1}{2}$, the action of $A$ on $H_0 = L^2(\mathbb{R})$ can be extended to $\text{Diff}_+(\mathbb{R})$ by defining

$$A(h) : g(u) \to \sqrt{\frac{dh^{-1}}{du}} \, g(h^{-1}(u)),$$

which is unitary and gives a representation since $A(h_1) A(h_2) = A(h_1 \circ h_2)$. However, this representation is problematic since general elements of $\text{Diff}_+(\mathbb{R})$ do not preserve the positive– and negative–frequency subspaces of $H_0$. To see this, recall that $g$ is the boundary–value of its AST $\tilde{g}(z)$ and $\tilde{g} = 0$ in $\mathbb{C}^\mp$ if $g \in H_s^{\pm}$. Hence elements of $H_s^{\pm}$ are boundary–values of functions analytic in $\mathbb{C}^\pm$. When composed with a diffeomorphism $h^{-1}$ which is not the boundary–value of a function analytic in $\mathbb{C}^\pm$, the result cannot be the boundary–value of a function analytic in $\mathbb{C}^\pm$. ($\text{Diff}_+(\mathbb{R})$ plays important roles in string theory and quantum field theory; there, the fact that it mixes positive and negative frequencies leads to a difficulty known as the “conformal anomaly.”) Since the decomposition into positive– and negative–frequency components is fundamental to our approach, the representation theory of $\text{Diff}_+(\mathbb{R})$ will not be pursued further. However, the above discussion suggests that we confine ourselves to diffeomorphisms of $\mathbb{R}$ which extend ‘naturally’ to $\mathbb{C}$ and whose restrictions to $\mathbb{C}^\pm$ are holomorphisms (analytic diffeomorphisms) of $\mathbb{C}^\pm$.

That is, we consider symmetry operations on $H_s$ which are induced from geometric maps of the complex space $\mathbb{C}^+ \cup \mathbb{C}^-$ associated with the AST rather than geometric maps of the real space $\mathbb{R}$. Since only the analytic functions $\tilde{g}(z)$ or their boundary–values $\tilde{g}(x \pm i0)$ enter into the pseudo–local inner products of $H_s$, we might even get away with maps which have singularities on $\mathbb{R}$. Thus we look for the group of holomorphisms of $\mathbb{C}^\pm$. Now it is well–known that the group of holomorphisms of the Riemann sphere $\hat{\mathbb{C}} \equiv \mathbb{C} \cup \{\infty\}$ is $\text{SL}(2, \mathbb{C})$, the set of all complex $2 \times 2$ matrices with unit determinant, acting by fractional–linear (Möbius) transformations. The subgroup preserving $\mathbb{C}^\pm$, hence also the one–point compactification $\hat{\mathbb{R}} \equiv \mathbb{R} \cup \{\infty\}$, then consists of all such real matrices, and is denoted $\text{SL}(2, \mathbb{R})$. We shall write $G \equiv \text{SL}(2, \mathbb{R})$ throughout this subsection. An element of $G$, being a diffeomorphism of $\hat{\mathbb{R}}$ rather than $\mathbb{R}$, will have a singularity in $\mathbb{R}$ (the inverse image of $\infty$) unless it has $\infty$ as a fixed point.

We begin by constructing two inequivalent UIR’s of $G$ which extend the UIR’s of $A$ on $H_0^\pm$. The matrix
\[ \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1, \] (176)

acts on \( C^+ \cup C^- = \hat{C} \setminus \hat{R} \) by the fractional-linear transformation

\[ \sigma : z \mapsto \sigma(z) \equiv \frac{az + b}{cz + d} \equiv w. \] (177)

(This action will be derived below.) From the easily derived identity

\[ w - w^* = \frac{z - z^*}{|cz + d|^2} \] (178)

it follows that \( \sigma \) leaves \( C^\pm \) invariant. Note that the boundary map \( \sigma(u) \) on \( R \) has a singularity at \( u = -d/c \), in agreement with the above discussion. Since \( d\sigma(u)/du = (cu + d)^{-2} \), the induced map on boundary-values in \( \mathcal{H}_0 \) is, according to eq. (175),

\[ A(\sigma^{-1}) : g(u) \rightarrow |cu + d|^{-1} g \left( \frac{au + b}{cu + d} \right). \] (179)

The singularity at \( u = -d/c \) poses no problem, since \( C^\infty \) functions of compact support \( g \in \mathcal{D}(\mathbb{R}) \) vanish at infinity and \( \mathcal{D}(\mathbb{R}) \) is dense in \( \mathcal{H}_0 \). The set of \( \sigma \)'s whose boundary maps are non-singular is, in fact, just the affine group \( \mathcal{A} \), since \( c = 0 \) implies

\[ \sigma(u) = a^2 u + ab \equiv \alpha u + \beta, \] (180)

giving the relation between the parameters of \( \mathcal{A} \) and those of \( G \). To represent \( A(\sigma^{-1}) \) on \( \tilde{g}(z) \), we first replace the multiplier \( |cu + d|^{-1} \) with \( (cu + d)^{-1} \), which also leads to a representation and, unlike the former, can be extended analytically. This does not affect the norm as \( z \rightarrow u \in \mathbb{R} \). Thus define

\[ B(\sigma^{-1}) : \tilde{g}(z) \rightarrow (cz + d)^{-1} \tilde{g} \left( \frac{az + b}{cz + d} \right). \] (181)

For \( z \in C^\pm \), we obtain the actions on \( \mathcal{H}_0^\pm \), which are unitary by the same argument as used for \( \text{Diff}_+(\mathbb{R}) \). This gives inequivalent UIR's of \( SL(2, \mathbb{R}) \) on \( \mathcal{H}_0^\pm \). Comparison with Bargmann’s classification shows that these coincide with the two representations of the “mock discrete series” with \( s = 0 \) (Lang [20], pp. 120 and 123).

We can arrive at the action of \( G \) on \( C^\pm \) (eq. (177)) by the method of cosets. Consider the subgroups \( T \) and \( H \) of \( G \) given by

\[ T = \left\{ t(u) \equiv \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \text{ such that } u \in \mathbb{R} \right\} \] (182)

\[ H = \left\{ h(c,d) \equiv \begin{pmatrix} d^{-1} & 0 \\ c & d \end{pmatrix} \text{ such that } d \neq 0 \right\}. \]

Then \( T \) is isomorphic to \( \mathbb{R} \), and \( H \) is isomorphic to \( \mathcal{A} \). Moreover, any element of \( G \) with \( d \neq 0 \) can be written uniquely in the form
\[ \sigma \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & b/d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d^{-1} & 0 \\ c & d \end{pmatrix} = t(b/d) h(c,d). \quad (183) \]

Hence we may coordinatize \( G \) (except for a set of Haar measure zero) by the product of sets \( TH \). (This does not mean that \( G \) is a direct product since \( T \) and \( H \), as subgroups of \( G \), do not commute.) The space \( U \equiv G/H \) of right cosets is then parametrized by \( u \in \hat{\mathbb{R}} \), as follows: If \( d \neq 0 \), then

\[ \sigma H \equiv \{ \sigma k \text{ such that } k \in H \} = \{ t(b/d) h(c,d) k \text{ such that } k \in H \} = t(b/d) H \equiv H_{b/d}. \quad (184) \]

The set of matrices with \( d = 0 \) forms a single coset, which will be denoted by \( H_\infty \). Thus \( U \approx \hat{\mathbb{R}} \). Now \( G \) acts on \( U \) by left multiplication, and this translates into an action of \( G \) on \( \hat{\mathbb{R}} \) as follows: If \( u \neq \infty \), then

\[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} H_u = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} H = \begin{pmatrix} a & au+b \\ c & cu+d \end{pmatrix} H = H_{\sigma(u)}, \quad (185) \]

with \( \sigma(u) \) as given by eq. (177). The action on \( H_\infty \) is similarly found to be

\[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} H_\infty = H_{a/c}. \quad (186) \]

(If \( c = 0 \), then \( H_\infty \) is invariant under \( \sigma \).)

Thus we see that the action of \( G \) on \( \mathbb{R} \) (more precisely, on \( \hat{\mathbb{R}} \)) can be derived from the action of \( G \) on itself. Similarly, the representation of \( G \) on \( L^2(\mathbb{R}) \) obtained above can be viewed as a group–theoretical construction: The Lebesgue measure \( du \) on \( \mathbb{R} \) can be extended to \( U = \hat{\mathbb{R}} \) by letting \( \{ \infty \} \) have measure zero; this extension \( d\hat{u} \) is quasi–invariant under the action of \( G \), and functions in \( L^2(\mathbb{R}) = L^2(U, d\hat{u}) \) may be interpreted as functions on \( G \) which are constant on each coset \( H_u \). The above representation can then be interpreted as being “induced” (Lang [20], chapter 3) from the identity representation of \( H \). Since \( H \) is not compact, non–zero functions on \( G \) which are constant on the cosets \( H_u \) are not square–integrable on \( G \). This proves that the above representation is not square–integrable on \( G \). That means that we cannot use it to obtain a resolution of the identity in \( L^2(G) \).

We now construct some other representations of \( G \), with weights \( \kappa \) other than \( \frac{1}{2} \), which do turn out to be square–integrable on \( G \). Let \( \kappa \) be real, so that \( s = 2\kappa - 1 \). Define the action of \( G \) on \( H_s \) by

\[ B(\sigma^{-1}) : \tilde{g}(z) \rightarrow (cz+d)^{-s-1} \tilde{g} \left( \frac{az+b}{cz+d} \right). \quad (187) \]

We must have \( s \in \mathbb{Z} \) in order to preserve analyticity. For \( \kappa = \frac{1}{2} \), this coincides with the earlier action. Note that for \( a = 1/\sqrt{\alpha} \), \( b = -\beta/\sqrt{\alpha} \), \( c = 0 \) and \( d = \sqrt{\alpha} \) we obtain

\[ B(\sigma^{-1}) : \tilde{g}(z) \rightarrow \alpha^{-\kappa} \tilde{g}(\alpha^{-1}(z - \beta)), \quad (188) \]
which shows that the action restricts to that of the affine group on $\mathcal{H}_s$.

To show that $B(\sigma^{-1})$ acts unitarily, we need to prove that it preserves the pseudo-local norm introduced in Theorem 5. There we found that for $G$ replaced by its subgroup $A$, we needed $s > 0$. Since $s$ must now, in addition, be an integer, we therefore have $2\kappa - 1 = s = 1, 2, 3, \ldots$. We must show that

$$
\int_{\mathbb{C}} d^2 z \, |\Im(z)|^{s-1} |cz + d|^{-2s-2} \left| \tilde{g} \left( \frac{az + b}{cz + d} \right) \right|^2 = \int_{\mathbb{C}} d^2 z \, |\Im(z)|^{s-1} |\tilde{g}(z)|^2, \quad (189)
$$

where $d^2 z$ denotes Lebesgue measure in $\mathbb{C}$. Let $w = \sigma(z) = (az + b)/(cz + d)$. Then $dw/dz = (cz + d)^{-2}$, hence

$$
d^2 w = |cz + d|^{-4} d^2 z, \quad \text{and} \quad \Im(w) = |cz + d|^{-2} \Im(z), \quad (190)
$$

where the second equality follows from eq. (178). Thus

$$
\int_{\mathbb{C}} d^2 z \, |\Im(z)|^{s-1} |cz + d|^{-2s-2} |\tilde{g}(w)|^2 = \int_{\mathbb{C}} d^2 w \, |\Im(w)|^{s-1} |\tilde{g}(w)|^2 = \|g\|_s^2, \quad (191)
$$

proving the result. The unitary representation on $\mathcal{H}_s$ decomposes into a direct sum of UIR’s on $\mathcal{H}_s^\pm$. All these representations are inequivalent, and they are known collectively as the discrete series (Gelfand et al. [9], Lang [20]).

The action of $\text{SL}(2, \mathbb{R})$ on the wavelets $e_{z+} \in \mathcal{D}_{s+} \approx \mathcal{H}_s$ is easily computed. By the unitarity of $B(\sigma)$, we have

$$
\langle B(\sigma) e_{z+}, g \rangle_s = \langle e_{z+}, B(\sigma^{-1}) g \rangle_s = (B(\sigma^{-1}) \tilde{g})(z) = (cz + d)^{-s-1} \tilde{g}(\sigma(z)) = (cz + d)^{-s-1} \langle e_{\sigma(z)+}, g \rangle_s, \quad (192)
$$

hence

$$
B(\sigma) e_{z+} = (cz^* + d)^{-s-1} e_{\sigma(z)+}. \quad (193)
$$

The representations of the discrete series are square-integrable over $G$. The subgroup of $G$ which leaves $z = \pm i$ invariant is $SO(2)$, hence $\mathbb{C}^\pm \approx G/SO(2)$. Since $SO(2)$ is compact (unlike its counterpart $H$ for $s = 0$), the norms in $\mathcal{H}_s^\pm$ can be rewritten as integrals over all of $G$ rather than just the homogeneous spaces $\mathbb{C}^\pm$.

Returning to the wave equation, we obtain mutually inequivalent UIR’s of what we shall call the restricted conformal group $\mathcal{C}_0 \equiv G \times G$ on $\mathcal{D}_{s+}^+, \mathcal{D}_{s-}^+, \mathcal{D}_{s-}^-$ and $\mathcal{D}_{s-}^+$ ($s = 0, 1, 2, \ldots$). When the total reflection $R$ is included, we obtain inequivalent UIR’s of the resulting group $\mathcal{C}_1$ on $\mathcal{D}_{s+}$ and $\mathcal{D}_{s-}$. When the space reflection $P$ is further included, we obtain a single set of mutually inequivalent UIR’s of the resulting group $\mathcal{C}_2$ on $\mathcal{D}_s$. As they did for $\mathcal{G}_0$, the reflections unify the four subspaces $\mathcal{D}_{s\pm}^\pm$ into a single one representing the wave equation as a whole.
5. Concluding Remarks: Dedicated Wavelets

The wavelet analysis developed in [16] for the Klein–Gordon equation has the interesting feature that the wavelets are “dedicated” to the equation rather than being merely a convenient set of functions to be used in expansions. (This is somewhat reminiscent of the situation in the spectral theorem, where expansions are customized to a given operator.) The reward for such dedication is that symmetry operations (such as translations, rotations, Lorentz transformations, and even time evolution) take wavelets to wavelets. This has the practical consequence of making the description economical and precise. For example, wavelets obtained by taking tensor products of one–dimensional wavelets cannot be rotated; consequently, a function consisting of but a few wavelets in one coordinate system is represented (inefficiently) by a combination of many wavelets in a rotated coordinate system. Similar considerations apply to the dedicated wavelets associated with the wave equation in \( \mathbb{R}^2 \) developed in the last section. But since there is now only one space dimension, the results are somewhat less dramatic: There are only two directions, left and right, giving rise to the labeling \( e_{z+} \) and \( e_{z-} \). We believe that the results of Section 4 generalize to \( d > 1 \) space dimensions, where the set of directions is parametrized by \( S^{d-1} \). (The case \( d = 1 \) is degenerate since \( S^0 = \{ \pm 1 \} \) is disconnected.) Work on this is in progress.

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