Virtual homological eigenvalues and the Weil-Petersson translation length

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Abstract For any pseudo-Anosov automorphism on an orientable closed surface, an inequality is established by bounding certain growth of virtual homological eigenvalues with the Weil-Petersson translation length. The new inequality fits nicely with other known inequalities due to Kojima and McShane (2018) and Lê (2014). The new quantity to be considered is the square sum of the logarithmic radii of the homological eigenvalues (with multiplicity) outside the complex unit circle, called the homological Jensen square sum. The main theorem is as follows. For any cofinal sequence of regular finite covers of a given surface, together with lifts of a given pseudo-Anosov, the homological Jensen square sum of the lifts grows at most linearly fast compared with the covering degree, and the square root of the growth rate is at most $1/\sqrt{4\pi}$ times the Weil-Petersson translation length of the given pseudo-Anosov.

Keywords homological eigenvalue, finite cover, Weil-Petersson metric, translation length

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1 Introduction

Let $S$ be a connected closed orientable surface of genus at least 2. Let $f: S \to S$ be a pseudo-Anosov automorphism. In [10], Kojima and McShane established an elegant inequality, which we can rearrange as

$$\frac{\text{Vol}_h(M_f)}{6\pi} \leq \frac{\ell_{WP}(f)}{\sqrt{4\pi}} \times \sqrt{\text{genus}(S) - 1}. \quad (1.1)$$

In this inequality, $\text{Vol}_h(M_f)$ denotes the hyperbolic volume of the mapping torus $M_f$ of $f$ with respect to its hyperbolic metric which is unique up to isometry, and $\ell_{WP}(f)$ denotes the Weil-Petersson translation length of $f$, as a mapping class acting on the Teichmüller space of $S$.

The left-hand side of (1.1) is equal to the logarithmic $L^2$ torsion of the orientable closed 3-manifold $M_f$ (see [16, Chapter 3]) (with a different sign convention). This quantity is a known upper bound for the superior linear growth rate of the virtual homological logarithmic Mahler measures with respect to any cofinal sequence of regular finite covers $(S'_n)_{n \in \mathbb{N}}$ of $S$ and lifts $(f'_n)_{n \in \mathbb{N}}$ of $f$. To be precise, if $\kappa_n: S'_n \to S$ are cofinal regular finite covering maps (i.e., the corresponding finite-index normal subgroups of $\pi_1(S)$...
have trivial common intersection), and \( f'_n : S'_n \to S'_n \) are coverings of \( f \) (i.e., \( \kappa_n \circ f'_n = f \circ \kappa_n \)), then the following inequality holds:

\[
\lim_{n \to \infty} \frac{\mathfrak{m}(P_n)}{[S'_n : S]} \leq \frac{\text{Vol}_h(M_f)}{6\pi}.
\] (1.2)

In this inequality, \([S'_n : S]\) denotes the covering degree, and \(\mathfrak{m}(P_n)\) denotes the logarithmic Mahler measure of the characteristic polynomial \(P_n\) of the induced linear automorphism \(f'_n : H_1(S'_n; \mathbb{C}) \to H_1(S'_n; \mathbb{C})\). In fact, (1.2) follows easily from an inequality due to Lê [13] combined with [12, Theorem 1].

We recall that for any nonzero complex polynomial \(Q = Q(z)\), the *logarithmic Mahler measure of \(Q\) refers to the non-negative quantity

\[
\mathfrak{m}(Q) = \frac{1}{2\pi} \int_0^{2\pi} \log |Q(e^{i\theta})| d\theta.
\]

When \(Q\) is monic of degree \(d\) with complex roots \(\lambda_1, \ldots, \lambda_d\), the Jensen formula implies that \(\mathfrak{m}(Q)\) measures exactly the total logarithmic radii of those roots outside the complex unit circle, namely,

\[
\mathfrak{m}(Q) = \sum_{j=1}^{d} \log \max(1, |\lambda_j|)
\]

(see [3, Chapter 1]).

For any monic polynomial \(Q\) of degree \(d\) with roots \(\lambda_1, \ldots, \lambda_d\), we introduce another non-negative quantity

\[
\mathfrak{w}(Q) = \sum_{j=1}^{d} \log^2 \max(1, |\lambda_j|).
\] (1.3)

In general, \(\mathfrak{w}(Q)\) is very different from the average of \(\log^2 |Q|\) over the unit circle. The latter has been called the second higher Mahler measure (compare [11]). Let us call \(\mathfrak{w}(Q)\) the *Jensen square sum of \(Q\).*

The polynomials \(P_n\) in (1.2) are monic of degree \(2 \times \text{genus}(S'_n)\). Moreover, they are all palindromic. In particular, at most half the complex roots lie outside the unit circle. On applying the Cauchy-Schwarz inequality, one obtains the following inequality:

\[
\frac{\mathfrak{m}(P_n)}{[S'_n : S]} \leq \left( \frac{\mathfrak{w}(P_n)}{[S'_n : S]} \right)^{1/2} \left( \frac{\text{genus}(S'_n)}{[S'_n : S]} \right)^{1/2},
\]

where the limit of the last factor is clear, i.e.,

\[
\lim_{n \to \infty} \sqrt{\frac{\text{genus}(S'_n)}{[S'_n : S]}} = \sqrt{\text{genus}(S)} - 1.
\]

In light of (1.1) and (1.2), there seems to be some unrevealed inequality between the growth of \(\mathfrak{w}(P_n)\) and \(\ell_{QP}(f)\). In this paper, we establish that inequality, exactly in the predicted form.

This is our main theorem.

**Theorem 1.1.** Let \(f : S \to S\) be a pseudo-Anosov automorphism on a connected closed orientable surface of genus at least 2. Then for any cofinal sequence of regular finite covers \((S'_n)_{n \in \mathbb{N}}\) of \(S\) with lifts \((f'_n)_{n \in \mathbb{N}}\) of \(f\), the following inequality holds:

\[
\lim_{n \to \infty} \sqrt[4]{\frac{\mathfrak{w}(P_n)}{[S'_n : S]}} \leq \frac{\ell_{QP}(f)}{\sqrt{4\pi}},
\]

where \(P_n\) denotes the characteristic polynomial of \(f'_n\) on \(H_1(S'_n; \mathbb{C})\).

With notations of Theorem 1.1, denote by \(h(P_n)\) the maximum of \(\log \max(1, |\lambda|)\), where \(\lambda\) ranges over all the complex roots of \(P_n\). Denote by \(\ell_T(f)\) the Teichmüller translation length of \(f\), which is equal to the entropy of \(f\). In [18], McMullen showed that the well-known inequality \(\lim_{n \to \infty} h(P_n) \leq \ell_T(f)\) must
be strict if the invariant foliations of $f$ have a prong singularity of odd order, and moreover, in that case, there is a uniform gap that depends only on $f$. It seems reasonable to conjecture that the inequality in Theorem 1.1 is strict for many, if not all, pseudo-Anosov $f$, and there is a uniform gap that depends only on $f$.

For any pseudo-Anosov $(S, f)$, one can always find a sequence $(S'_n, f'_n)_{n \in \mathbb{N}}$ as in Theorem 1.1 such that $h(P_n) > 0$ holds for each $f'_n$ (see [15]). Hence, $m(P_n) > 0$ and $w(P_n) > 0$ also hold. Note that $\lim_n w(P_n)/[S'_n : S] > 0$ holds if and only if $\lim_n m(P_n)/[S'_n : S] > 0$ holds. Although it appears very possible, there are no known examples of a pseudo-Anosov $(S, f)$ and a sequence $(S'_n, f'_n)_{n \in \mathbb{N}}$ as in Theorem 1.1 such that $w(P_n)$ or $m(P_n)$ grows strictly linearly fast compared with the covering degree.

We also point out the following trivial comparison about virtual homological eigenvalues:

$$\sqrt{\frac{w(P_n)}{[S'_n : S]}} \leq h(P_n) \times \sqrt{\frac{\text{genus}(S_n)}{[S'_n : S]}}.$$ 

This is analogous to another well-known inequality

$$\frac{\ell_{WP}(f)}{\sqrt{4\pi}} \leq \ell_T(f) \times \sqrt{\text{genus}(S) - 1},$$

which follows immediately from Linch's comparison between the Weil-Petersson and the Teichmüller metrics [14].

For general dynamical systems, the idea of comparing entropy and invariants of the induced action on the homology can be traced back at least to works of Fried [4], Manning [17], Yomdin [22] and many others. For pseudo-Anosov surface automorphisms, the entropy inequality takes a sharper form as McMullen's gap theorem [18]. Our main theorem, together with other inequalities mentioned above, suggests that the virtual homological invariants $m(P_n)/[S'_n : S]$, $w(P_n)/[S'_n : S]$ and $h(P_n)$ can be thought of as suitable analogues of the geometrical invariants Vol$(M)/6\pi$, $\ell_{WP}(f)/\sqrt{4\pi}$ and $\ell_T(f)$, respectively.

Below, we outline the proof of Theorem 1.1. Our argument is inspired by [18]. The main difference is that we make use of invariant metrics on the Teichmüller space related to the Weil-Petersson metric, rather than the Teichmüller metric.

Using the Bergman metric on Riemann surfaces $X$ marked by $S$, one obtains an invariant Riemannian metric on the Teichmüller space $\text{Teich}(S)$, which we call the Habermann-Jost metric. The way is just the same as obtaining the Weil-Petersson metric on $\text{Teich}(S)$ from the conformal hyperbolic metric on $X$. Using the natural map $\text{Teich}(S) \rightarrow \mathfrak{H}(S)$ of the Teichmüller space to the Siegel space of Hodge structures on $H^1(S; \mathbb{C})$, one obtains an invariant Riemannian pseudometric on $\text{Teich}(S)$ pulling back the Siegel metric on $\mathfrak{H}(S)$, which we call the Royden-Siegel metric. Adopting suitable normalization, we see that the $L^2$ norms of these metrics satisfy the comparison $2 \times \|\xi\|_{\mathfrak{H}_1} \geq \|\xi\|_{\mathfrak{H}}$ for any tangent vector $\xi \in T_X\text{Teich}(S)$.

Moreover, virtual versions of the above metrics on $\text{Teich}(S)$ can be defined by using the regular finite covering $S' \rightarrow S$ and the natural embedding $\text{Teich}(S) \rightarrow \text{Teich}(S')$. Adopting suitable normalization, we see that the $L^2$ norms of the virtual Habermann-Jost metrics satisfy the convergence $\lim_{n \rightarrow \infty} \|\xi\|_{\mathfrak{H}_{1, n}} = \|\xi\|_{\mathfrak{H}}/\sqrt{4\pi}$ for any sequence of covers $(S'_n)_{n \in \mathbb{N}}$ as assumed. This is basically because the virtual Bergmann metrics $g_{\mathfrak{H}_{1, n}}$ on $X$ converge to the rescaled conformal hyperbolic metric $g_{\mathfrak{H}}/4\pi$.

With the above facts, consider any point on the Weil-Petersson axis in $\text{Teich}(S)$ of a given pseudo-Anosov $f$ on $S$. Then essentially speaking, we can obtain comparison of translation lengths (see Definition 2.2) as follows:

$$\frac{\ell_{WP}(f)}{\sqrt{4\pi}} \geq \lim_{n \rightarrow \infty} \ell_{\mathfrak{H}_{1, n}}(f) \geq \lim_{n \rightarrow \infty} \frac{\ell_{\mathfrak{H}_{1, n}}(f)}{2} \geq \lim_{n \rightarrow \infty} 2 \times \ell_{\mathfrak{H}}(f'_n) \geq \frac{\ell_{\mathfrak{H}}(f'_n)}{2}.$$ 

The nominator $\ell_{\mathfrak{H}}(f'_n)$ in the last expression refers to the Siegel translation length of $f'_n \in \text{Sp}(H_1(S'_n; \mathbb{R}))$ acting on $\mathfrak{H}(S'_n)$. On the other hand, we can establish the formula $\ell_{\mathfrak{H}}(f'_n) = 2 \times \sqrt{w(P_n)}$ by studying the Siegel geometry on the generalized upper half plane model. Then the desired inequality follows.
The rest of this paper is organized as follows. In Section 2, we recall structures associated with the Teichmüller space and general properties of the translation length. In Section 3, we determine the Siegel translation length of a symplectic linear transformation. In Section 4, we recall several invariant Riemannian metrics on the Teichmüller space, as mentioned above, and discuss their virtual versions. In Section 5, we prove two technical lemmas regarding comparison and convergence of virtual metrics. In Section 6, we prove our main result (see Theorem 1.1).

2 Preliminaries

In this section, we recall background materials needed for our discussion. For general reference books, see [7] for the Teichmüller theory, and [20] for the Siegel geometry, and [2, Chapter II.6] about the translation length of isometries on metric spaces. We only consider closed surfaces of hyperbolic type, as it suffices for our purpose.

2.1 The Teichmüller space

Let $S$ be an oriented closed surface of genus at least 2. The Teichmüller space $\text{Teich}(S)$ consists of all the isotopy classes of complex structures on $S$, compatible with the fixed orientation. Equivalently, we think of any point of $\text{Teich}(S)$ as represented by a Riemann surface $X$ together with an orientation-preserving homeomorphism $S \to X$, called the marking of $X$, and denote the point by $X$ with the marking implicit.

We think of $\text{Teich}(S)$ as a smooth (real) manifold, diffeomorphic to an open cell of dimension $6 \times (\text{genus}(S) - 1)$. There are natural identifications of the tangent and the cotangent spaces (as real vector spaces) $T_X \text{Teich}(S) \cong Q(X)$ and $T_X \text{Teich}(S) \cong B(X)/Q(X)^\perp$ at any point $X \in \text{Teich}(S)$. Here, $Q(X)$ denotes the space of all the holomorphic quadratic differentials on $X$, and $B(X)$ denotes the space of all the $L^\infty$ Beltrami differentials on $X$. These spaces pair naturally as

$$Q(X) \times B(X) \to \mathbb{R}: (q, \mu) \mapsto \int_X q \mu,$$

so $Q(X)^\perp$ refers to the subspace of $B(X)$ annihilated by $Q(X)$, i.e., $\mu \in B(X)$ lies in $Q(X)^\perp$ if and only if $\int_X q \mu = 0$ holds for all $q \in Q(X)$.

The mapping class group $\text{Mod}(S)$ of $S$, consisting of all the isotopy classes of orientation-preserving self-homeomorphisms, acts properly and diffeomorphically on $\text{Teich}(S)$, transforming the markings of points.

There are many natural differential metrics (or pseudometrics) on $\text{Teich}(S)$ that are invariant under the action of $\text{Mod}(S)$. For example, the Teichmüller metric $d_T$ is an invariant Finsler metric determined uniquely by the formula $d_T(X,Y) = \frac{\log K(X,Y)}{2}$ for any $X, Y \in \text{Teich}(S)$, where $K(X,Y) > 1$ denotes the quasi-conformality constant of the Teichmüller extremal map $X \to Y$ that commutes homotopically with the markings. The infinitesimal forms of $d_T$ are the norms $\|q\|_T = \int_X |q|$ for any $q$ in $Q(X) \cong T_X \text{Teich}(S)$, and

$$\|\xi\|_T = \sup \left\{ \left| \int_X q \mu \right| : \int_X |q| \leq 1 \right\}$$

for any $\xi = \mu + Q(X)^\perp$ in $B(X)/Q(X)^\perp \cong T_X \text{Teich}(S)$.

We recall the Weil-Petersson metric $d_{WP}$ and other invariant Riemannian metrics on $\text{Teich}(S)$ in Section 4 with more discussion.

2.2 The Siegel space

Let $V$ be a real vector space of dimension $2p$ equipped with a symplectic form $\omega$. Then $\omega$ extends complex-linearly over the complex vector space $V \otimes_\mathbb{R} \mathbb{C}$ as a complex-valued alternating 2-form, and determines a Hermitian form of signature $(p, p)$ on $V \otimes_\mathbb{R} \mathbb{C}$, i.e.,

$$\langle z, w \rangle = \frac{\sqrt{-1}}{2} \cdot \omega(z, \bar{w}).$$
The Siegel space $\mathcal{S}(V)$ consists of all the Hermitian orthogonal splittings $V \otimes \mathbb{C} = V^{1,0} \oplus V^{0,1}$ such that $(\cdot, \cdot)$ is positive definite on $V^{1,0}$, and the complex conjugation $z \mapsto \bar{z}$ maps $V^{1,0}$ isometrically onto $V^{0,1}$. We call any point of the Siegel space $\mathcal{S}(V)$ a Hodge structure on $V \otimes \mathbb{C}$ with respect to $\omega$.

The group $\text{Sp}(V) \cong \text{Sp}(2p, \mathbb{R})$ of symplectic linear transformations acts transitively on $\mathcal{S}(V)$ such that any $\varphi \in \text{Sp}(V)$ extends complex-linearly over $V \otimes \mathbb{C}$, and takes any Hodge structure $V^{1,0} \oplus V^{0,1}$ to $\varphi(V^{1,0}) \oplus \varphi(V^{0,1})$. The isotropy group at $V^{1,0} \oplus V^{0,1}$ is isomorphic to the unitary group $U(V^{1,0}) \cong U(p)$, acting simultaneously on the summands $V^{1,0}$ and $V^{0,1}$ as canonical unitary transformations and their complex conjugates, respectively. In particular, $\mathcal{S}(V)$ is the homogenous space associated with $\text{Sp}(V)$, diffeomorphic to an open cell of dimension $p^2 + p$.

The most important example in what follows is the Siegel space of Hodge structures on a surface. If $S$ is an oriented closed surface of genus at least 2, then the first real cohomology $H^1(S; \mathbb{R})$ is equipped with a natural symplectic form, evaluating the cup product of any pair of 1-classes on the fundamental class. In this case, we simply denote the Siegel space of Hodge structures on $H^1(S; \mathbb{C})$ by $\mathcal{S}(S)$. For any Teichmüller point $X \in \text{Teich}(S)$, we obtain a unique Hermitian orthogonal splitting

$$H^1(S; \mathbb{C}) \cong H^1(X; \mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X) \cong \Omega^1(X) \oplus \Omega^1(X),$$

where the complex linear subspaces $H^{1,0}(X)$ and $H^{0,1}(X)$ of $H^1(X; \mathbb{C})$ are naturally identified as the spaces $\Omega^1(X)$ and $\Omega^1(X)$ of the holomorphic and the anti-holomorphic differentials on $X$, respectively, and $H^1(S; \mathbb{C})$ is identified with $H^1(X; \mathbb{C})$ via the marking. Therefore, the construction determines a natural map

$$J: \text{Teich}(S) \to \mathcal{S}(S),$$

which is equivariant with respect to the natural group homomorphism $\text{Mod}(S) \to \text{Sp}(H^1(S; \mathbb{Z}))$. It is known that the group homomorphism is surjective, and $J$ is smooth, and the tangent map of $J$ is injective except along the hyperelliptic locus (see [19, Section 3] for more information).

Back to the general setting with $(V, \omega)$, there is a unique invariant Riemannian metric on $\mathcal{S}(V)$ with respect to $\text{Sp}(V)$, up to normalization. In fact, $(\mathcal{S}(V), \text{Sp}(V))$ is a simple Riemannian symmetric space of noncompact type (and moreover, a Hermitian symmetric domain). This metric has been studied by Siegel systematically in his expository book [20]. Fixing a symplectic basis of $V$, we can identify $V$ as the real linear space spanned by $\tilde{x}_1, \ldots, \tilde{x}_p, \tilde{y}_1, \ldots, \tilde{y}_p$ and equipped with the standard symplectic form $\tilde{x}_1 \wedge \tilde{y}_1 + \cdots + \tilde{x}_p \wedge \tilde{y}_p$. Then the transformational geometry of the Siegel space $(\mathcal{S}(V), \text{Sp}(V))$ can be identified with Siegel’s generalized upper half plane model, on which the invariant metric can be described explicitly. We recall the explicit description below, following Siegel’s normalization in [20].

### 2.3 The generalized upper half plane

For any natural number $p$, the generalized upper half plane $\mathcal{S}_p$ of rank $p$ refers to the open subset of symmetric $(p \times p)$-matrices with complex entries such that the imaginary parts are positive definite. Namely,

$$\mathcal{S}_p = \{ Z \in \text{Sym}_{p \times p}(\mathbb{C}) : \exists Z > 0 \},$$

where we define $\text{Sym}_{p \times p}(\mathbb{C}) = \{ Z \in \text{Mat}_{p \times p}(\mathbb{C}) : Z^\dagger = Z \}$ († means transpose) and $\exists Z = (Z - \bar{Z})/2\sqrt{-1}$ (see [20, Chapter I, Section 2]).

The Riemannian metric tensor of the Siegel metric at any point $Z \in \mathcal{S}_p$ is defined as

$$g_Z = \text{tr}(Y^{-1} \cdot dZ \cdot Y^{-1} \cdot d\bar{Z}),$$

where we define $Y = \exists Z$ and treat the matrix entries of the real and the imaginary parts as coordinates (see [20, Chapter III, Section 11]). The Siegel metric $g_Z$ has nonpositive sectional curvature everywhere (see [20, Chapter III, Section 17]). In other words, $(\mathcal{S}_p, g_Z)$ forms a Hadamard manifold (by definition, simply connected, metric complete and nonpositively curved).

For any pair of points $Z, W \in \mathcal{S}_p$, the generalized cross-ratio matrix

$$R = R(Z, W) = (Z - W) \cdot (Z - W)^{-1} \cdot (\bar{Z} - W) \cdot (\bar{Z} - W)^{-1}$$
is a well-defined element in $\text{Mat}_{p \times p}(\mathbb{C})$ with all the characteristic roots contained in $[0, 1)$. Hence, the series expression
\[\log^2 \left( \frac{1 + R^{1/2}}{1 - R^{1/2}} \right) = 4R \cdot \left( \sum_{m=0}^{\infty} \frac{R^m}{2m + 1} \right)^2\]
converges absolutely to a matrix in $\text{Mat}_{p \times p}(\mathbb{C})$. The Siegel distance between the pair of points $Z, W \in \mathfrak{d}_p$ can be calculated as
\[d_s(Z, W) = \sqrt{\text{tr} \left( \log^2 \left( \frac{1 + R^{1/2}}{1 - R^{1/2}} \right) \right)} \quad (2.4)\]
(see [20, Chapter III, Section 13]).

The symplectic group $\text{Sp}(2p, \mathbb{R})$ acts on $\mathfrak{d}_p$ by generalized fractional linear transformations, as follows. Any matrix $\varphi \in \text{Sp}(2p, \mathbb{R})$ can be divided into four blocks $A, B, C, D \in \text{Mat}_{p \times p}(\mathbb{R})$ such that the basis is transformed as
\[
\varphi^*(\vec{x}_1, \ldots, \vec{x}_p, \vec{y}_1, \ldots, \vec{y}_p) = (\vec{x}_1, \ldots, \vec{x}_p, \vec{y}_1, \ldots, \vec{y}_p) \begin{pmatrix} A & B \\ C & D \end{pmatrix}.
\]
Then $\varphi$ being symplectic is equivalent to the conditions $A^\dagger C = C^\dagger A$, $D^\dagger B = B^\dagger D$ and $A^\dagger D - C^\dagger B = I$ altogether ($I$ denotes the identity matrix). The action of $\text{Sp}(2p, \mathbb{R})$ on $\mathfrak{d}_p$ is explicitly
\[\varphi: Z \mapsto (AZ + B) \cdot (CZ + D)^{-1}\]
for any $\varphi \in \text{Sp}(2p, \mathbb{R})$ and $Z \in \mathfrak{d}_p$ (see [20, Chapter II, Sections 4 and 5]).

**Remark 2.1.** For any real vector space $V$ of dimension $2p$ equipped with a symplectic form $\omega$, an identification $(\mathfrak{d}(V), \text{Sp}(V)) \cong (\mathfrak{d}_p, \text{Sp}(2p, \mathbb{R}))$ is canonically determined by fixing a symplectic basis of $V$. Since the canonical identifications $V \cong \mathbb{R}^{2p}$ and $\text{Sp}(V) \cong \text{Sp}(2p, \mathbb{R})$ are obvious, it remains to identify any point $Z \in \mathfrak{d}_p$ with a Hodge structure on $V \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}^{2p}$. We first require the distinguished point $Z = \sqrt{-1} \cdot I$ to be identified with the Hodge structure $V^{1,0} \oplus V^{0,1}$, where $V^{1,0}$ is the real linear subspace spanned by the complex vectors $\vec{z}_1 = \vec{x}_1 + \sqrt{-1} \cdot \vec{y}_1$, $\ldots$, $\vec{z}_p = \vec{x}_p + \sqrt{-1} \cdot \vec{y}_p$. Hence, $V^{0,1} = \overline{V^{1,0}}$. In general, any point $X + \sqrt{-1} \cdot Y$ in $\mathfrak{d}_p$ is the transformation image of $\sqrt{-1} \cdot I$ under the symplectic matrix
\[M = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \cdot \begin{pmatrix} Y^{1/2} & 0 \\ 0 & Y^{-1/2} \end{pmatrix} = \begin{pmatrix} Y^{1/2} \cdot X \cdot Y^{-1/2} \end{pmatrix}.
\]
Therefore, the point $\tilde{Z} = X + \sqrt{-1} \cdot Y$ is identified with the Hodge structure $\tilde{V}^{1,0} \oplus \tilde{V}^{0,1}$, where $\tilde{V}^{1,0}$ is spanned by $M \vec{z}_1, \ldots, M \vec{z}_p$.

### 2.4 The translation length

**Definition 2.2.** For any metric space $(\mathcal{X}, d_{\mathcal{X}})$, the translation length of an isometry $\sigma: \mathcal{X} \to \mathcal{X}$ is defined as $\ell_{\mathcal{X}}(\sigma) = \inf_{x \in \mathcal{X}} d_{\mathcal{X}}(x, \sigma x)$.

The translation length can be thought of as a function on the isometry group of $\mathcal{X}$:
\[\ell_{\mathcal{X}}: \text{Isom}(\mathcal{X}) \to [0, +\infty).
\]

This function is clearly invariant under the conjugations and inversion. Moreover, it satisfies the following upper semi-continuity property with respect to sequences of pointwise convergence.

**Lemma 2.3.** If $(\sigma_m)_{m \in \mathbb{N}}$ is a sequence of isometries on a metric space $(\mathcal{X}, d_{\mathcal{X}})$ that converges to an isometry $\varphi_{\infty}$ everywhere in $\mathcal{X}$, then $\liminf_{m \to \infty} \ell_{\mathcal{X}}(\sigma_m) \leq \ell_{\mathcal{X}}(\varphi_{\infty})$.

**Proof.** For any $x \in \mathcal{X}$ and $\epsilon > 0$, we obtain $d_{\mathcal{X}}(x, \sigma_m x) \leq d_{\mathcal{X}}(x, \varphi_{\infty} x) + \epsilon$ for all sufficiently large $m$. Then $\ell_{\mathcal{X}}(\sigma_m) \leq d_{\mathcal{X}}(x, \varphi_{\infty} x) + \epsilon$, and then $\liminf_{m \to \infty} \ell_{\mathcal{X}}(\sigma_m) \leq d_{\mathcal{X}}(x, \varphi_{\infty} x) + \epsilon$. Taking the infimum over all $x \in \mathcal{X}$, we obtain $\liminf_{m \to \infty} \ell_{\mathcal{X}}(\sigma_m) \leq \ell_{\mathcal{X}}(\varphi_{\infty}) + \epsilon$. Let $\epsilon$ tend to $0+$, and the asserted inequality follows. \(\Box\)
3 The Siegel translation length of symplectic linear transformations

In this section, we determine the translation length of a symplectic matrix with respect to its fractional linear transformation on the generalized upper half plane and the Siegel metric. We are able to reduce the task to explicit simple computations, thanks to an available classification of symplectic linear transformations. In fact, conjugacy classes in \( \text{Sp}(2p, \mathbb{R}) \) have been classified, and there is a complete list of normal forms. The theory is similar to the well-known Jordan normal form theory for \( \text{GL}(n, \mathbb{R}) \), although notationally more involved. For details of this theory, we refer to Gutt [5]. However, we need a few ingredients from [5] in our proof of Theorem 3.1, and we elaborate when we use them.

Theorem 3.1. Let \((V, \omega)\) be a (finite-dimensional real) symplectic vector space. Then for any symplectic linear transformation \( \varphi \in \text{Sp}(V) \), the Siegel translation length of \( \varphi \) satisfies the formula

\[
\ell_{\text{S}}(\varphi) = 2 \times \sqrt{w(P_{\varphi})},
\]

where \( P_{\varphi} \) denotes the characteristic polynomial of \( \varphi \), and where \( w \) denotes the Jensen square sum as defined in (1.3).

The rest of this section is devoted to the proof of Theorem 3.1.

Lemma 3.2. If \( V = V_1 \oplus \cdots \oplus V_k \) is a symplectic orthogonal decomposition of \( V \) into symplectic summands \( V_i \) that are invariant under \( \varphi \), then

\[
\ell_{\text{S}}(\varphi) = \sqrt{\ell_{\text{S}}(\varphi_1)^2 + \cdots + \ell_{\text{S}}(\varphi_k)^2},
\]

where \( \varphi_i \in \text{Sp}(V_i) \) denotes the restricted transformation of \( \varphi \).

Proof. The decomposition induces a natural inclusion \( \mathcal{H}(V_1) \times \cdots \times \mathcal{H}(V_k) \) into \( \mathcal{H}(V) \). With respect to the Cartesian product of Siegel metrics on \( \mathcal{H}(V_1) \times \cdots \times \mathcal{H}(V_k) \) and the Siegel metric on \( \mathcal{H}(V) \), the inclusion is an isometric embedding with the totally geodesic image, by (2.3) and a simple observation. Since the Siegel space \( \mathcal{H}(V) \) is Hadamard with respect to the Siegel metric, the nearest point projection \( \mathcal{H}(V) \to \mathcal{H}(V_1) \times \cdots \times \mathcal{H}(V_k) \) is well defined and distance non-increasing (see [2, Chapter II, Corollary 2.5]). It follows that \( \ell_{\text{S}}(\varphi) \) is witnessed by some sequence of points \((x_m)_{m \in \mathbb{N}}\) in \( \mathcal{H}(V_1) \times \cdots \times \mathcal{H}(V_k) \), namely, \( \ell_{\text{S}}(\varphi) = \lim_m d_{\mathcal{S}}(x_m, \varphi \cdot x_m) \). Therefore, \( \ell_{\text{S}}(\varphi) \) agrees with the translation length of \( \varphi \) acting on \( \mathcal{H}(V_1) \times \cdots \times \mathcal{H}(V_k) \). The latter is evidently equal to \( (\ell_{\text{S}}(\varphi_1)^2 + \cdots + \ell_{\text{S}}(\varphi_k)^2)^{1/2} \). \( \Box \)

Let \((V, \omega)\) be a symplectic vector space. For any element \( \varphi \in \text{Sp}(V) \), it follows from the Jordan-Chevalley decomposition that there is a unique, commutative factorization of \( \varphi \) in \( \text{Sp}(V) \), \( \varphi = \varphi_{\text{ss}} \varphi_{\text{n}} = \varphi_{\text{n}} \varphi_{\text{ss}} \), such that the eigenvectors of \( \varphi_{\text{ss}} \) span \( V \otimes_{\mathbb{R}} \mathbb{C} \), and the only eigenvalue of \( \varphi_{\text{n}} \) is 1. In fact, the Jordan-Chevalley decomposition in the semisimple real Lie algebra \( \mathfrak{sp}(V) \) is the same as inherited from \( \mathfrak{gl}(V) \), and it determines the factorization in the linear group \( \text{Sp}(V) \) (see [8, Chapter II, Subsection 6.4]).

More explicitly, if one represents \( \varphi \) as a complex general linear matrix in the Jordan normal form over some basis of \( V \otimes_{\mathbb{R}} \mathbb{C} \), then \( \varphi_{\text{ss}} \) can be obtained by erasing the off-diagonal entries, and \( \varphi_{\text{n}} \) can be obtained by multiplying each \( \lambda \)-Jordan block by a scalar \( 1/\lambda \), just as usual, while the above fact guarantees that \( \varphi_{\text{ss}} \) and \( \varphi_{\text{n}} \) stay in \( \text{Sp}(V) \). By similar means, one may read off \( \varphi_{\text{ss}} \) and \( \varphi_{\text{n}} \) from Gutt’s list of symplectic normal forms (see [5, Theorem 1.1]), although a full description is slightly lengthy and not mentioned therein.

We refer to \( \varphi_{\text{ss}} \) and \( \varphi_{\text{n}} \) as the semisimple factor and the unipotent factor of \( \varphi \), respectively.

Lemma 3.3. Theorem 3.1 holds if \( \varphi = \varphi_{\text{ss}} \).

Proof. We say that \((V, \varphi)\) is symplectic simple if \( \varphi = \varphi_{\text{ss}} \), and if \( V \) contains no nontrivial \( \varphi \)-invariant symplectic subspaces. In this case, \((V, \varphi)\) decomposes as a direct sum of symplectic simple transformations \((V_i, \varphi_i)\). In view of the Pythagorean formula (see Lemma 3.2) for \( \ell_{\text{S}}(\varphi) \) and the obvious similar formula for \( 2 \times \sqrt{\mathfrak{w}(P_{\varphi})} \), it suffices to prove the lemma for any symplectic simple transformation.

Any symplectic simple transformation \((V, \varphi)\) is either 2-dimensional central (with a unique eigenvalue \( \lambda = \pm 1 \) of multiplicity 2), or 2-dimensional hyperbolic (with distinct real eigenvalues \( \{\lambda, 1/\lambda\} \)),
or 2-dimensional elliptic (with distinct unimodular eigenvalues \( \{ \lambda, \bar{\lambda} \} \)), or otherwise, 4-dimensional ‘loxodromic’ (with distinct complex eigenvalues \( \{ \lambda, \bar{\lambda}, 1/\lambda, 1/\bar{\lambda} \} \) off the unit circle and the real axis) (see [5, Theorem 1.1] for general normal forms of matrices in \( \text{Sp}(2p, \mathbb{R}) \)).

When \( V \) is 2-dimensional, the formula for \( \ell_\delta(\varphi) \) is actually well known. In this case, the Siegel metric on the Siegel space \( S_\delta(V) \) agrees with the hyperbolic metric on the upper-half complex plane (of constant curvature \(-1\)), and \( \text{Sp}(V) \cong \text{SL}(2, \mathbb{R}) \) acts the same way as the fractional linear transformations, so

\[
\ell_\delta(\varphi) = \begin{cases} 
0, & \varphi \text{ central/elliptic}, \\
2 \log |\lambda|, & \varphi \text{ hyperbolic}.
\end{cases}
\]

In any of the subcases, \( \ell_\delta(\varphi) \) is equal to \( 2 \times \sqrt{w(\varphi)} \), as asserted.

When \( V \) is 4-dimensional, the Siegel space \( S_\delta(V) \) can be identified with the generalized upper half plane \( H_2 \), consisting of all the symmetric complex \( 2 \times 2 \)-matrices with positive definite imaginary parts. Any symplectic transformation with distinct eigenvalues \( \{ \lambda, \bar{\lambda}, 1/\lambda, 1/\bar{\lambda} \} \) can be conjugated in \( \text{Sp}(4, \mathbb{R}) \) into the normal form

\[
\varphi = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \rho \cdot \begin{pmatrix} \cos \theta - \sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} & \rho^{-1} \cdot \begin{pmatrix} \cos \theta - \sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \end{pmatrix}
\]

by denoting \( \rho = |\lambda| \) and \( \theta = \arg \lambda \). Recall that \( \varphi \) acts isometrically on \( H_2 \) as \( Z \mapsto (AZ + B)(CZ + D)^{-1} \). The purely positive imaginary scalar \( 2 \times 2 \)-matrices form a \( \varphi \)-invariant geodesic \( l \) in the Hadamard manifold \( H_2 \). As the nearest point projection \( H_2 \to l \) is distance non-increasing (see [2, Chapter II, Corollary 2.5]), the translation length of \( \varphi \) is realized at every point on \( l \). On \( l \), we can apply (2.4) and directly compute

\[
\ell_\delta(\varphi) = \sqrt{2} \cdot |2 \log |\lambda||, \quad \text{which is also } 2 \times \sqrt{w(\varphi)}.
\]

**Lemma 3.4.** Under the hypothesis of Theorem 3.1, \( \ell_\delta(\varphi) = \ell_\delta(\varphi_{ss}) \).

**Proof.** To see one direction, we observe that symplectic linear transformations with no redundant characteristic roots form a dense open subset of \( \text{Sp}(V) \). In fact, the discriminant of the characteristic polynomial being zero determines an algebraic, proper subset of the real algebraic group \( \text{Sp}(V) \), whose interior has to be empty.

Therefore, we can take a sequence \((\psi_m)_{m \in \mathbb{N}} \) in \( \text{Sp}(V) \), converging to \( \varphi \), such that each \( \psi_m \) has no redundant characteristic roots. In particular, \( \psi_m \) are all semisimple. It follows from Lemma 3.3 that the formula in Theorem 3.1 holds for all \( \ell_\delta(\psi_m) \) and for \( \ell_\delta(\varphi_{ss}) \). Note that \( \varphi_{ss} \) has the same characteristic polynomial as that of \( \varphi \), so the characteristic roots of \( \psi_m \) converge to those of \( \varphi_{ss} \). By Lemma 2.3, we obtain \( \ell_\delta(\varphi_{ss}) = \lim_{m \to \infty} \ell_\delta(\psi_m) \leq \ell_\delta(\varphi) \).

To see the other direction, we consider another sequence \((\psi_{ss}^r \varphi_u^1)^{1/m})_{m \in \mathbb{N}} \), which converges to \( \varphi_{ss} \) in \( \text{Sp}(V) \). Note that the element \( \varphi_u \in \text{Sp}(V) \) is well defined for all \( r \in \mathbb{R} \), via the series expansion

\[
(1 + x)^r = \sum_{j=0}^{\infty} \binom{r}{j} x^j,
\]

because \( \varphi_u - 1 \) is nilpotent in \( \text{End}_\mathbb{R}(V) \). Moreover, one can show that \( \varphi_{ss} \varphi_u^r \) is conjugate to \( \varphi \) in \( \text{Sp}(V) \) for all \( r > 0 \). This is probably well known to experts, and an elementary proof is included below, based on Gutt’s characterization of conjugacy classes in \( \text{Sp}(V) \), so let us assume it for the moment. Since \( \ell_\delta \) is constant on conjugacy classes of \( \text{Sp}(V) \), we can apply Lemma 2.3 again, and obtain

\[
\ell_\delta(\varphi) = \lim_{m \to \infty} \ell_\delta(\varphi_{ss} \varphi_u^r)^{1/m} \leq \ell_\delta(\varphi_{ss}).
\]

Then we can conclude \( \ell_\delta(\varphi) = \ell_\delta(\varphi_{ss}) \) as desired.

For completeness, we provide a proof of the aforementioned fact: The conjugacy class of \( \varphi(r) = \varphi_{ss} \varphi_u^r \) in \( \text{Sp}(V) \) is invariant for all \( r \).
We first recall Gutt’s characterization of conjugacy classes in $\text{Sp}(V)$ as follows. For any element $\varphi \in \text{Sp}(V)$, the set of complex eigenvalues of $\varphi$ is invariant under the complex conjugation $\lambda \mapsto \bar{\lambda}$ and the reciprocal involution $\lambda \mapsto 1/\lambda$. We denote by $E_{m}^{\lambda}$ the $m$-th generalized complex eigenspace associated with any complex eigenvalue $\lambda$, namely, the kernel of $(\varphi - \lambda \cdot \text{id})^m \in \text{End}_{\mathbb{C}}(V \otimes_{\mathbb{R}} \mathbb{C})$. When $|\lambda| = 1$ and $\lambda \neq \pm 1$, there are Hermitian pairings $\hat{Q}_{m}^{\lambda} : E_{m}^{\lambda} \times E_{m}^{\lambda} \to \mathbb{C}$ defined, according to the parity of $m$, as either

$$
\hat{Q}_{2j+1}^{\lambda}(v, w) = \lambda : \omega((\varphi - \lambda \cdot \text{id})^{j}(v), (\varphi - \bar{\lambda} \cdot \text{id})^{j}(\bar{w}))
$$

or

$$
\hat{Q}_{2j+2}^{\lambda}(v, w) = \sqrt{-1} : \omega((\varphi - \lambda \cdot \text{id})^{j}(v), (\varphi - \bar{\lambda} \cdot \text{id})^{j}(\bar{w}))
$$

where $\omega$ denotes the complex-linear extension of the symplectic form over $V \otimes_{\mathbb{R}} \mathbb{C}$. When $\lambda = \pm 1$, the pairings $\hat{Q}_{m}^{\lambda}$ defined with the same expressions restrict to the real subspaces $E_{m}^{\lambda} \cap V$ as real bilinear pairings, and the restricted pairings are symmetric (and real-valued) for $m = 2j$, or skew-symmetric (and purely imaginary-valued) for $m = 2j + 1$.

According to Gutt’s characterization (see [5, Theorem 1.2]), the conjugacy class of any $\varphi \in \text{Sp}(V)$ is uniquely determined by the following numerical data, indexed by the eigenvalues $\lambda$ of $\varphi$ and positive integers $m$: the complex dimension of $E_{m}^{\lambda}$, and the rank and the signature of the Hermitian pairings $\hat{Q}_{m}^{\lambda}$ ($|\lambda| = 1$ and $\lambda \neq \pm 1$), and the rank and the signature of all the real symmetric pairings $\hat{Q}_{m}^{\lambda}$ ($\lambda = \pm 1$ and $m$ is even).

Consider the family $\varphi(r) = \varphi_{\pm}^{\pm} \varphi_{\pm}^{\pm}$, parametrized by $r > 0$. By the standard Jordan normal form theory, the set of eigenvalues $\Lambda = \Lambda(\varphi(r))$ and the dimensions of the generalized eigenspaces $d_{m}^{\lambda} = \dim_{\mathbb{C}} E_{m}^{\lambda}(\varphi(r))$ are invariant for all $r > 0$, as these quantities depend only on the Jordan normal form of $\varphi(r)$. It remains to show the invariance of the rank and the signature of any $Q_{m}^{\lambda}(\varphi(r))$ from the above list. To this end, we argue by induction on the rank minimum $k_{m}^{\lambda} = \min\{\text{rank}(Q_{m}^{\lambda}(\varphi(r))) : r > 0\}$.

For any $(\lambda, m)$ with $k_{m}^{\lambda} = d_{m}^{\lambda}$, the (Hermitian or real symmetric) pairing $\hat{Q}_{m}^{\lambda}(\varphi(r))$ is nondegenerate. In this case, the rank and the signature of $\hat{Q}_{m}^{\lambda}(\varphi(r))$ are invariant under small perturbation of $r > 0$, and by induction, they are also invariant for all $r > 0$. For any $(\lambda, m)$ with $k_{m}^{\lambda} = d_{m}^{\lambda} - 1$, we infer from the previous case that the rank of $\hat{Q}_{m}^{\lambda}(\varphi(r))$ cannot increase under small perturbation of $r > 0$, so the rank, and hence, the signature of $\hat{Q}_{m}^{\lambda}(\varphi(r))$ have to be invariant under small perturbation of $r > 0$, and by continuation, they are again invariant for all $r > 0$. Proceeding successively for any $(\lambda, m)$ with $k_{m}^{\lambda} = d_{m}^{\lambda} - 2, d_{m}^{\lambda} - 3, \ldots, 0$, we see that the rank and the signature of any $\hat{Q}_{m}^{\lambda}(\varphi(r))$ are invariant for all $r > 0$.

By Gutt’s characterization, the conjugacy class of $\varphi(r)$ in $\text{Sp}(V)$ is invariant for all $r > 0$.

Theorem 3.1 follows from Lemmas 3.3 and 3.4.

### 4 Invariant Riemannian metrics on the Teichmüller space

In this section, we recall some invariant Riemannian metrics or pseudometrics on the Teichmüller space, and also consider similar constructions associated with regular finite covers of the surface. As some of the metrics have name collision in the literature, we rename a few in this paper, not necessarily suggesting an accurate attribution.

#### 4.1 Ordinary versions

##### 4.1.1 The Weil-Petersson metric

By the uniformization theorem, there is a unique conformal hyperbolic metric $g_{h}$ on $X$ (i.e., a Riemannian metric of constant curvature $-1$ conformal to the complex charts). Applying the Gauss-Bonnet theorem yields

$$
\text{Area}_{g_{h}}(X) = -2\pi \chi(S) = 4\pi \times (\text{genus}(S) - 1).
$$

With respect to $g_h$, any tangent vector $\xi \in T_X\text{Teich}(S)$ is represented by a unique harmonic Beltrami differential $\mu \in B(X)$, i.e., $\mu = \overline{q/g_h}$ for some holomorphic quadratic differential $q \in Q(X)$. The Weil-Petersson metric on $\text{Teich}(S)$ is the Riemannian metric determined by

$$\|\xi\|_{\text{WP}}^2 = \int_X |\mu|^2 \text{dArea}_h = \int_X \frac{\overline{q q}}{g_h}$$

for any $\xi \in T_X\text{Teich}(S) = B(X) / Q(X)^{\perp}$ and the hyperbolic harmonic representative $\mu = \overline{q/g_h}$.

The Weil-Petersson metric on $\text{Teich}(S)$ has nonpositive sectional curvature. It is not complete, but any pair of points in $\text{Teich}(S)$ can be connected by a unique geodesic segment. The mapping class group $\text{Mod}(S)$ acting on $\text{Teich}(S)$ preserves the Weil-Petersson metric. Moreover, any pseudo-Anosov mapping class $f \in \text{Mod}(S)$ has an axis, namely, a unique invariant complete geodesic, along which the Weil-Petersson translation length $\ell_{\text{WP}}(f)$ is realized (see the survey [21] of Wolpert for more information).

4.1.2 The Habermann-Jost metric

Using the Bergman metric instead of the hyperbolic metric, we obtain what we call the Habermann-Jost metric on $\text{Teich}(S)$. This metric has been studied by Habermann and Jost [6] in detail.

For any $X \in \text{Teich}(S)$, the Bergmann metric $g_h$ on $X$ refers to the pullback of the canonical flat metric on $\text{Jac}(X) = \Omega^1(X)^* / H_1(X; \mathbb{Z})$ via the Abel-Jacobi map $X \to \text{Jac}(X)$, which is independent of the auxiliary base point. To avoid confusion about the normalization, we make it explicit that it satisfies the following relation:

$$\text{Area}_h(X) = \text{genus}(S).$$

The Habermann-Jost metric on $\text{Teich}(S)$ is the Riemannian metric determined by

$$\|\xi\|_{\text{HJ}}^2 = \int_X |\mu|^2 \text{dArea}_h = \int_X \frac{\overline{q q}}{g_h}$$

for any $\xi \in T_X\text{Teich}(S) = B(X) / Q(X)^{\perp}$ and its Bergman harmonic representative $\mu = \overline{q/g_h} \in B(X)$ for some unique $q \in Q(X)$.

**Lemma 4.1.** Let $\theta_1, \ldots, \theta_p$ be a Hermitian orthonormal basis of $\Omega^1(X)$, where $p = \dim_{\mathbb{C}} \Omega^1(X)$ is equal to the genus of $S$. Then the following formulas hold:

$$g_h = \sum_{j=1}^{p} \theta_j \overline{\theta}_j,$$

and for any $\xi \in T_X\text{Teich}(S)$ represented uniquely as $\mu = \overline{q/g_h}$,

$$\|\xi\|_{\text{HJ}}^2 = \sum_{j=1}^{p} \int_X \mu \theta_j \overline{\mu \overline{\theta}_j}.$$

**Proof.** In fact, the formula for $g_h$ in Lemma 4.1 is a usual definition of the Bergman metric on $X$, and its geometric interpretation is our above description with the Abel-Jacobi map. The formula for $\|\cdot\|_{\text{HJ}}$ in Lemma 4.1 follows immediately from the formula for $g_h$ and the defining expression in (4.2). Note that $\overline{\mu \overline{\theta}_j}$’s are differentials of type $(1, 0)$, but not holomorphic in general. \qed

4.1.3 The Royden-Siegel metric

The natural map $J : \text{Teich}(S) \to \mathcal{H}(S)$ and the Siegel metric on $\mathcal{H}(S)$ induce a canonical Riemannian pseudometric on $\text{Teich}(S)$, degenerating exactly on the hyperelliptic locus. This construction results in what we call the Royden-Siegel metric. In the survey [19], Royden discussed several such pullback metrics on $\text{Teich}(S)$ using canonical differential metrics on $\mathcal{H}(S)$.

For any $\xi \in T_X\text{Teich}(S)$, the Royden-Siegel metric is the Riemannian pseudometric determined by

$$\|\xi\|_{\text{RS}}^2 = \|J_*(\xi)\|_0^2,$$

according to our notations in (2.1) and (2.3).
Lemma 4.2. Let $\theta_1, \ldots, \theta_p$ be a Hermitian orthonormal basis of $\Omega^1(X)$, where $p = \dim\mathbb{C}\Omega^1(X)$ is equal to the genus of $S$. Then for any $\xi \in T_X\text{Teich}(S)$ represented by any $\mu \in B(X)$, the following formula holds:

$$\|\xi\|_{\text{He}}^2 = 4 \times \sum_{j=1}^{p} \sum_{k=1}^{p} \left( \int_X \mu \theta_j \theta_k \right)^2.$$ 

Proof. The formula is equivalent to [19, Theorem 2], up to a factor 4 which is due to normalization. To be precise, Royden used a rescaled metric on the generalized upper half plane for which distances are exactly half of those for the Siegel metric (compare [19, p. 397, (15)] and [20, p. 3, (2)]). With our notations (following Siegel’s normalization), [19, Theorem 2] can be rewritten as

$$\|\xi\|_{\text{He}}^2 = 4 \times \int_X \mu(x')K(x', x'')\mu(x''),$$

where $x'$ parametrizes the first $X$ and $x''$ parametrizes the second $X$. The kernel $K(x', x'')$ is a holomorphic quadratic differential in the first variable and an anti-holomorphic quadratic differential in the second variable, and is Hermitian symmetric under switching of the variables. It is explicitly constructed as

$$K(x', x'') = \sum_j \theta_j(x')\overline{\theta}_j(x''),$$

the indices all ranging over $\{1, \ldots, p\}$. Plug the last expression into the above double integral, and move the summations out of the integrations. Then the variables separate, and the asserted formula follows. 

4.2 Virtual versions

Any regular finite covering map $S' \to S$ induces a canonical embedding $\text{Teich}(S) \to \text{Teich}(S')$ such that any Riemann surface with the marking $S \to X$ goes to $S' \to X'$, where $X' \to X$ is holomorphic and equivariant with $S' \to S$.

The Royden-Siegel metric on $\text{Teich}(S')$ induces a pseudometric on $\text{Teich}(S)$ by restriction. To normalize, we introduce the virtual Royden-Siegel metric on $\text{Teich}(S)$ with respect to the regular finite cover $S'$ as the Riemannian pseudometric such that for any $\xi \in T_X\text{Teich}(S)$, we define

$$\|\xi\|_{\text{He}}^2 = \|\xi'\|_{\text{He}}^2 \times \frac{1}{\|S' : S\|} = \|J'(\xi')\|_{\text{He}}^2 \times \frac{1}{\|S' : S\|},$$

where $\xi' \in T_{X'}\text{Teich}(S')$ denotes the tangent map image of $\xi$, and $J'$ denotes the natural map $\text{Teich}(S') \to \mathfrak{H}(S')$ as in (2.1).

We introduce the virtual Habermann-Jost metric on $\text{Teich}(S)$ with respect to any regular finite cover $S'$ as follows. For any $X \in \text{Teich}(S)$, since the deck transformations on the corresponding $X'$ are all biholomorphic, they all preserve the Bergman metric $g_0'$ on $X'$. Therefore, $g_0'$ is the pullback of a unique conformal metric $g_0$ on $X$. We notice the relation

$$\text{Area}_{g_0}(X) = \frac{\text{Area}_{g_0}(X')}{\|S' : S\|} = \text{genus}(S) - 1 + \frac{1}{\|S' : S\|},$$

(the second equality follows quickly from the fact that the Euler characteristic is proportional to the covering degree, i.e., $2 \cdot (1 - \text{genus}(S')) = 2 \cdot (1 - \text{genus}(S)) \cdot (\|S' : S\|)$. For any $\xi \in T_X\text{Teich}(S)$, we define

$$\|\xi\|^2_{g_0'} = \int_X |\mu|^2 d\text{Area}_{g_0'} = \int_X \frac{q q'}{g_0'},$$

where $\mu = \overline{q}/(g_0)$ denotes the Beltrami differential representative of $\xi$ that is harmonic with respect to $g_0$. The last expression implies a relation $\|\xi\|^2_{g_0'} = \|\xi'\|^2_{g_0'} \times \frac{1}{\|S' : S\|}$, where $\xi' \in T_{X'}\text{Teich}(S')$ denotes the tangent map image of $\xi$. 
There is no need to introduce any “virtual Weil-Petersson metric”, as it would be the same thing as the Weil-Petersson metric. This is because the conformal hyperbolic metric $g'_h$ on $X'$ agrees with the pullback of $g_h$ on $X$. We only mention the obvious relation $||\xi||_{WP}' = ||\xi'||_{WP} \times \frac{1}{\sqrt{4\pi n}}$, where $\xi' \in TX'\text{Teich}(S')$ denotes the tangent map image of $\xi \in TX\text{Teich}(S)$.

5 Comparison and convergence of virtual metrics

In this section, we prove two lemmas regarding the relation of virtual metrics on the Teichmüller space. The first is an inequality due to Habermann and Jost [6] in the case of ordinary metrics, and we only carry it onto the virtual version (see Lemma 5.1). The second is an application of a well-known theorem due to Kazhdan [9], regarding convergence of the Bergmann metrics on finite coverings of Riemann surfaces (see Lemma 5.2).

Lemma 5.1. Let $S$ be a closed orientable surface of genus at least 2. With respect to any finite regular cover $S'$ over $S$, and for all $X \in \text{Teich}(S)$ and $\xi \in TX\text{Teich}(S)$, the following inequality holds:

$$||\xi||_{WP'} \leq 2 \times ||\xi||_{WP}.$$  

Proof. In the ordinary case, the inequality $||\xi||_{WP} \leq 2 \times ||\xi||_{WP}$ is the same as [6, Lemma 6.3], except a constant factor due to normalization (check the formula (6.5) therein, which has to agree with Royden’s normalization of the Siegel metric [19, p. 397, (15)], rather than Siegel’s [20, p. 3, (2)]). For the reader’s convenience, below we derive the inequality quickly from Lemmas 4.1 and 4.2, following the same idea as in [6].

Note that it suffices to prove the ordinary case $S' = S$, otherwise arguing with $S'$ and dividing both sides by $[S' : S]$. Fix an orthonormal basis of holomorphic differentials $\theta_1, \ldots, \theta_p$ in $\Omega^1(X)$. Let $\mu = q/g_h$ be a Bergmann harmonic Beltrami differential representative of $\xi$. We denote by $\eta_j(\mu) \in \Omega^1(X)$ the orthogonal summand of the $(1,0)$-differential $\overline{\mu \theta_j} \in A^1_0(X)$, namely, $\eta_j(\mu) = \sum_{l=1}^p \theta_l \int_X \mu \theta_j \theta_l$ for any $j = 1, \ldots, p$. Since $\theta_1, \ldots, \theta_j$ are orthonormal, we obtain that by Lemma 4.2,

$$\sum_j \int_X \eta_j(\mu)\overline{\eta_j(\mu)} = \sum_j \sum_k \left( \int_X \mu \theta_j \theta_k \right) \left( \int_X \mu \theta_j \theta_k \right) = ||\xi||^2_{WP} \times 1/4,$$

and by Lemma 4.1,

$$\sum_j \int_X \eta_j(\mu)\overline{\eta_j(\mu)} \leq \sum_j \int_X \mu \theta_j \mu \overline{\theta_j} = ||\xi||^2_{WP}.$$

Then the asserted inequality follows. \qed

Lemma 5.2. Let $S$ be a closed orientable surface of genus at least 2. Suppose that $(S'_n \to S)_{n \in \mathbb{N}}$ is a sequence of finite regular covers converging to the universal cover. Then for all $X \in \text{Teich}(S)$ and $\xi \in TX\text{Teich}(S)$, the following convergence holds:

$$\lim_{n \to \infty} ||\xi||_{WP'} = \frac{1}{\sqrt{4\pi n}} \times ||\xi||_{WP},$$

where the notation $n$ in the subscript indicates the respective cover $S'_n$. Moreover, the convergence is uniform on any compact set of tangent vectors $(X, \xi)$ on $\text{Teich}(S)$.

Proof. For any compact Riemann surface $X$, Kazhdan [9] showed that the sequence of virtual Bergmann metrics $g_h$, on $X$ converges uniformly everywhere to the hyperbolic metric $g_h$ rescaled by $1/4\pi$ (i.e., by $1/\sqrt{4\pi}$ in the length) (see [18, Appendix] for a clear statement and a short proof, and see also Remark 5.3 regarding the assumption on the covering sequence).

For cotangent vector $q$ in $TX\text{Teich}(S) = Q(X)$, the dual norm squares of $q$ converge as

$$\lim_{n \to \infty} ||q||^2_{WP'} = \lim_{n \to \infty} \int_X \frac{q(q)}{g_{h_n}} = 4\pi \times \int_X \frac{q(q)}{g_h} = 4\pi \times ||q||^2_{WP}.$$
This implies the asserted convergence of norms \( \lim_{n \to \infty} \|\xi\|_{HJ_n} = \frac{1}{\sqrt{4\pi}} \times \|\xi\|_{\wp} \) for any tangent vector \( \xi \in T_{X} \text{Teich}(S) \).

Moreover, the convergence is uniform on any compact subsets of tangent vectors, because the norms \( \|\cdot\|_{HJ_n} \) and \( \|\cdot\|_{\wp} \) are continuous functions on the Teichmüller tangent bundle. \( \square \)

**Remark 5.3.** In the literature, Kazhdan’s theorem is classically stated for cofinal *towers* of regular finite covers of a compact Riemann surface \( X \). However, we can replace the ‘tower’ with the ‘sequence’. In fact, as is already evident from the proof in [18, Appendix], the argument only relies on the properties that \( \pi_1(X) \) acts on all the covers \( X_n \), the covers \( X_n \) are all compact, and the hyperbolic injectivity radii of \( X_n \) tend to infinity.

In a recent work [1], Baik et al. obtained a generalization of Kazhdan’s theorem for any sequence of regular finite covers that converges to any infinite regular cover of a compact Riemann surface. We refer the readers to that paper for more background and discussion.

### 6 The proof of the main theorem

This section is devoted to the proof of Theorem 1.1.

Let \( f : S \to S \) be a pseudo-Anosov automorphism on a connected closed orientable surface of genus at least 2. Suppose that \( (S_n', f'_n)_{n \in \mathbb{N}} \) is a cofinal sequence of regular finite covers \( (S_n')_{n \in \mathbb{N}} \) of \( S \) together with pseudo-Anosov automorphisms \( f'_n : S'_n \to S'_n \) lifting \( f \).

Let \( (S', f') = (S'_n, f'_n) \) be any term of the above sequence. Denote by \( J' \) the canonical composite map \( \text{Teich}(S) \to \text{Teich}(S') \to \mathcal{H}(S') \), where the first map is the embedding induced by \( S' \to S \). By definition, the action of \( f' \in \text{Mod}(S') \) on \( \text{Teich}(S') \) preserves \( \text{Teich}(S) \), and extends the action of \( f \in \text{Mod}(S) \) on \( \text{Teich}(S) \).

For any smooth path \( \gamma : [0,1] \to \text{Teich}(S) \) with the property \( \gamma(1) = f' \gamma(0) \), it follows that the induced action \( f'_n \) on \( \mathcal{H}(S') \) also moves \( J'(\gamma(0)) \) to \( J'(\gamma(1)) \). We obtain that by Theorem 3.1, the length of the path \( J' \circ \gamma \) with respect to the Siegel metric on \( J'(S') \) (without normalization) can be estimated as

\[
\text{Length}_{J'}(J' \circ \gamma) \geq \ell_{\wp}(f'_n) = 2 \times \sqrt{w(P)}
\]

where \( P \) denotes the characteristic polynomial of \( f'_n \) on \( H_1(S'; \mathbb{C}) \). On the other hand, we obtain that by Lemma 5.1 and (4.3),

\[
\text{Length}_{J'}(J' \circ \gamma) = \text{Length}_{\wp}(\gamma) \times \sqrt{[S' : S]} \leq 2 \times \text{Length}_{\wp}(\gamma) \times \sqrt{[S' : S]}
\]

Retaining the subscript \( n \), we see that the above inequalities yield

\[
\sqrt{\frac{w(P_n)}{[S'_n : S]}} \leq \text{Length}_{\wp}(\gamma).
\]

Passing to the limit, we obtain that by Lemma 5.2,

\[
\lim_{n \to \infty} \sqrt{\frac{w(P_n)}{[S'_n : S]}} \leq \frac{\text{Length}_{\wp}(\gamma)}{\sqrt{4\pi}}.
\]

Apply the above inequality to a smooth path \( \gamma \) along the Weil-Petersson axis of \( f \). The Weil-Petersson translation length is realized as \( \ell_{\wp}(f) = d_{\wp}(\gamma(0), \gamma(1)) = \text{Length}_{\wp}(\gamma) \). In particular, we obtain the desired inequality, i.e.,

\[
\lim_{n \to \infty} \sqrt{\frac{w(P_n)}{[S'_n : S]}} \leq \frac{\ell_{\wp}(f)}{\sqrt{4\pi}}.
\]

This completes the proof of Theorem 1.1.

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References

1. Baik H, Shokrieh F, Wu C X. Limits of canonical forms on towers of Riemann surfaces. J Reine Angew Math, 2020, 764: 287–304
2. Bridson M R, Haefliger A. Metric Spaces of Non-Positive Curvature. Grundlehren der mathematischen Wissenschaften, vol. 319. Berlin: Springer-Verlag, 1999
3. Everest G, Ward T. Heights of Polynomials and Entropy in Algebraic Dynamics. London: Springer-Verlag, 1999
4. Fried D. Entropy and twisted cohomology. Topology, 1986, 25: 455–470
5. Gutt J. Normal forms for symplectic matrices. Port Math, 2014, 71: 109–139
6. Habermann L, Jost J. Riemannian metrics on Teichmüller space. Manuscripta Math, 1996, 89: 281–306
7. Hubbard J H. Teichmüller Theory and Applications to Geometry, Topology, and Dynamics: Volume 1: Teichmüller Theory. Ithaca: Matrix Editions, 2006
8. Humphreys J E. Introduction to Lie Algebras and Representation Theory. Graduate Texts in Mathematics, vol. 9. New York-Berlin: Springer-Verlag, 1972
9. Kazhdan D A. On arithmetic varieties. In: Lie Groups and Their Representations. New York: Halsted, 1975, 151–217
10. Kojima S, McShane G. Normalized entropy versus volume for pseudo-Anosovs. Geom Topol, 2018, 22: 2403–2426
11. Kurokawa N, Lalín M, Ochiai H. Higher Mahler measures and zeta functions. Acta Arith, 2008, 135: 269–297
12. Lê T T Q. Homology torsion growth and Mahler measure. Comment Math Helv, 2014, 89: 719–757
13. Lê T T Q. Growth of homology torsion in finite coverings and hyperbolic volume. Ann Inst Fourier (Grenoble), 2018, 68: 611–645
14. Linch M. A comparison of metrics on Teichmüller space. Proc Amer Math Soc, 1974, 43: 349–352
15. Liu Y. Virtual homological spectral radii for automorphisms of surfaces. J Amer Math Soc, 2020, 33: 1167–1227
16. Lück W. $L^2$-Invariants: Theory and Applications to Geometry and $K$-Theory. Berlin-Heidelberg: Springer-Verlag, 2002
17. Manning A. Topological entropy and the first homology group. In: Dynamical Systems. Lecture Notes in Mathematics, vol. 468. Berlin: Springer, 1975, 185–190
18. McMullen C T. Entropy on Riemann surfaces and the Jacobians of finite covers. Comment Math Helv, 2013, 88: 953–964
19. Royden H L. Invariant metrics on Teichmüller space. In: Contributions to Analysis. A Collection of Papers Dedicated to Lipman Bers. New York-London: Academic Press, 1974, 393–399
20. Siegel C L. Symplectic Geometry. New York-London: Academic Press, 1964
21. Wolpert S A. The Weil-Petersson metric geometry. In: Handbook of Teichmüller Theory, Vol. II. IRMA Lectures in Mathematics and Theoretical Physics, vol. 13. Zürich: Eur Math Soc, 2009, 47–64
22. Yomdin Y. Volume growth and entropy. Israel J Math, 1987, 57: 285–300