Information Leakage Games: Exploring Information as a Utility Function

MÁRIO S. ALVIM*, UFMG, Brazil
KONSTANTINOS CHATZIKOKOLAKIS, University of Athens, Greece and CNRS, France
YUSUKE KAWAMOTO, AIST, Japan
CATUSCIA PALAMIDESSI, Inria Saclay, France and École Polytechnique, France

A common goal in the areas of secure information flow and privacy is to build effective defenses against unwanted leakage of information. To this end, one must be able to reason about potential attacks and their interplay with possible defenses. In this paper, we propose a game-theoretic framework to formalize strategies of attacker and defender in the context of information leakage, and provide a basis for developing optimal defense methods. A novelty of our games is that their utility is given by information leakage, which in some cases may behave in a non-linear way. This causes a significant deviation from classic game theory, in which utility functions are linear with respect to players' strategies. Hence, a key contribution of this paper is the establishment of the foundations of information leakage games. We consider two kinds of games, depending on the notion of leakage considered. The first kind, the QIF-games, is tailored for the theory of quantitative information flow (QIF). The second one, the DP-games, corresponds to differential privacy (DP).

CCS Concepts: • Security and privacy → Formal security models; Information flow control; Data anonymization and sanitization; • Theory of computation → Theory of database privacy and security; Convex optimization; • Mathematics of computing → Information theory; • Networks → Network privacy and anonymity.

Additional Key Words and Phrases: information leakage, quantitative information flow, differential privacy, game theory, convex-concave optimization

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1 INTRODUCTION

A fundamental problem in computer security is the leakage of sensitive information due to the correlation of secret information with observable information publicly available, or in some way accessible, to the attacker. Typical examples are side-channel attacks, in which (observable) physical aspects of the system, such as the execution time of a decryption algorithm, may be exploited by the attacker to restrict the range of the possible (secret) decryption keys. The branch of security

* All authors contributed equally to this research, hence are listed in alphabetical order

Authors’ addresses: Mário S. Alvim, UFMG, Av. Antônio Carlos, 6627, Belo Horizonte, Brazil; Konstantinos Chatzikokolakis, University of Athens, 15784, Ilisia, Athens, Greece, CNRS, Gif-sur-Yvette, France; Yusuke Kawamoto, AIST, 2-4-7 Aomi, Koto-ku, Tokyo, Japan; Catuscia Palamidessi, Inria Saclay, Bâtiment Alan Turing, rue Honoré d’Estienne d’Orves, Campus de l’École Polytechnique, Palaiseau, France, École Polytechnique, Palaiseau, France.

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that studies the amount of information leaked by a system is called \textit{quantitative information flow} (QIF), and it has seen growing interest over the past decade. See, for instance, [9, 30, 49, 63].

Another prominent example is the privacy breach in public databases, where even if personal identifiers such as an individual’s name and date of birth are removed, the information stored in the records may be enough to uniquely re-identify a person. The problem of privacy has become more prominent with the advent of Big Data technologies, and the study of attacks and protection methods is a very active field of research. We mention in particular the \textit{differential privacy} approach [32, 35], which has become extremely popular in the last decade.

It has been recognized that randomization can be very useful to obfuscate the link between secrets and observables. Examples include various anonymity protocols (for instance, the dining cryptographers [25] and Crowds [59]), and the typical mechanisms for differential privacy such as Laplace and geometric noise [35]. The \textit{defender} (the system designer or the user) is, therefore, typically probabilistic. As for the \textit{attacker} (the party interested in inferring sensitive information), most works in the literature on quantitative information flow consider only \textit{passive attacks}, limited to observing the system’s behavior. Notable exceptions are the works of Boreale and Pampaloni [17] and of Mardziel et al. [53], which consider \textit{adaptive attackers} who interact with and influence the system. We note, however, that [17] does not consider probabilistic strategies for the attacker. As for [53], although their model allows them, none of their extensive case studies uses probabilistic attack strategies to maximize leakage. This may seem surprising, since, as mentioned before, randomization is known to be helpful (and, in general, crucial) for the defender to undermine the attack and protect the secret. Thus there seems to be an asymmetry between attacker and defender with respect to probabilistic strategies. Our thesis is that there is indeed an asymmetry, but that does not mean that the attacker has nothing to gain from randomization: when the defender can change his own strategy according to the attacker’s actions, it becomes advantageous for the attacker to try to be \textit{unpredictable} and, consequently, adopt a probabilistic strategy. For the defender, while randomization is useful for the same reason, it is also useful because \textit{it potentially reduces information leakage}, which constitutes the utility of the attacker. This latter aspect introduces the asymmetry mentioned above.

In the present work, we consider scenarios in which both attacker and defender can make choices that influence the system during an attack. We aim, in particular, at analyzing the attacker’s strategies that can maximize information leakage, and the defender’s most appropriate strategies to counter the attack and keep the system as secure as possible. As argued before, randomization can help both the attacker and defender by making their moves unpredictable. The most suitable framework for analyzing this kind of interplay is, naturally, game theory. In particular, we consider zero-sum games, and employ measures of \textit{information leakage} as the \textit{utility} function (with the positive sign for the attacker and the negative one for the defender). In this context, randomization is naturally captured by the notion of \textit{mixed strategies}, and the optimal strategies for the defender and attacker are expressed by the notion of \textit{Nash equilibria}.

However, it is important to note that the notion of information is fundamentally different from the typical utility functions modeled in game theory, such as money, time, or similar resources. These traditional quantities are linear with respect to the combination of strategies; hence the utility of a mixed strategy can be soundly defined as expectation. In classical game theory, this concept of utility has been formalized by the axioms of von Neumann and Morgenstern [65]. For instance, consider a lottery where the prize is stashed in two identical treasure chests, one containing 100 gold coins, and the other containing 200 gold coins, and assume that when a participant wins, a hidden fair coin is flipped to decide what chest he should get his prize from. The winner expects to get 150 golden coins, and the corresponding benefit from that does not depend on his exactly knowing from which chest the prize is coming. Note that that is still true even if he never gets to
explicity know the result of the coin flip, nor of what was the chosen chest: no matter whence the golden coins came, the corresponding (monetary) utility to him is the same.

On the other hand, the concept of information has a rather different nature, which relates to the asymmetry mentioned before. To understand this point, consider a scenario in which we can pose “yes”/“no” questions to one of two oracles and our utility is the amount of information we can extract from that oracle. Assume we know that one of the oracles always tells the truth, while the other always lies. For instance, if we ask “Is Alice single?” to the truthful oracle and get “yes” as an answer, we know for sure that Alice is single, and if we get the same answer from the lying oracle, we know for sure that Alice has a partner. Note that the information utility of the answers is maximum, as long as we know the oracle they came from, because they entirely determine the truth or falsehood of the fact we asked about. Now suppose that we cannot directly query any of these perfectly informative oracles, but instead we have to ask our question to some intermediate person who will first flip a hidden, fair coin to decide which oracle to pose the question to. The person then poses the question to the oracle, hidden from us, obtains an answer, and lets us know that the answer was, say, “yes”. How useful is this answer? Not useful at all, indeed, as we don’t know what oracle it came from! The critical insight is that, contrarily to what happens with quantities like money, time, or other similar resources, the “average” of two highly informative answers is not necessarily a highly informative answer. More precisely, information is non-linear with respect to hidden choice (in this example, the hidden coin flip that decides among strategies). For this reason, traditional game theory as formalized by the von Neumann and Morgenstern’s axioms fails to adequately capture games in which utility is information.

Consequently, in this paper, we consider a new kind of games, which we call information leakage games. In these games, the defender can mix two (pure) strategies, by choosing one strategy or the other, without necessarily revealing his choice to the attacker. As seen in the above example, the lack of knowledge of the defender’s choice may reduce the amount of information that the attacker can infer. Namely, the information leaked with a mixed strategy is never larger than the leakage of each original pure strategy, which means that utility is a quasi-convex function of the defender’s strategy. As for the attacker’s strategies, the properties of the utility depend on the particular definition of leakage, and we will discuss the various instances in the next section. In any case, our setting departs from the standard game theory, in which utility is a linear function on the strategies of both players. Nevertheless, we show that our games still have Nash equilibria, namely pairs of defender-attacker strategies where neither player has anything to gain by unilaterally changing his own strategy. This means that there exists an “optimal” balance of strategies between defender and attacker, and we will propose algorithms to compute these equilibria using a more efficient variant of the well known subgradient method for two variables. The efficiency improvement is achieved thanks to the specificity of our problem, and it can be considered our original contribution. Specifically, we are able to rewrite the minimax problem in a more convenient form (c.f. Proposition 3.3), involving a function which can be locally maximized easily. We then prove that the two standard sufficient conditions for the convergence of the subgradient method (bounded subgradients and bounded distance between the initial and optimal points) hold in our case. Finally, we show that not only our method converges to a saddle point, but also that an approximate solution for the strategy provides an “approximate saddle point” (c.f. Theorem 3.4).

1.1 Specific information-leakage frameworks

The literature on information leakage in the probabilistic setting can be divided into two main lines of research: quantitative information flow (QIF) [9, 10, 49, 63] and differential privacy (DP) [33–35]. Our game-theoretic approach encompasses both of them, and we will refer to them specifically as QIF-games and DP-games, respectively. For reasoning about information leakage, we adopt the
information-theoretic view, which is very popular in the QIF community. In this setting, a system or
a mechanism is modeled as an information-theoretic channel, where the secrets are the inputs and
the observables are the outputs. We will use this model both for QIF-games and for DP-games. We
assume that both attacker and defender can influence with their actions the choice of the channel
used in the game. In other words, we assume that channels are determined by two parameters that
represent the actions, or strategies, of the two players.

In the case of QIF, the typical measures of leakage are based on the concept of vulnerability, which
quantifies how easily the secret can be discovered (and exploited) by the attacker. There are various
models of attackers and corresponding notions of vulnerability proposed in the literature, but for
the sake of generality, here we abstract from the specific model. Following [8, 17], we adopt the view
that vulnerability is any convex and continuous function. Such a class of functions has been shown
in [8] to be the unique family presenting a set of fundamental information-theoretic properties,
and subsuming most previous measures of the QIF literature, including Bayes vulnerability (a.k.a.
min-vulnerability [24, 63]), Shannon entropy [61], guessing entropy [54], and g-vulnerability [10].
It is important to note that quasi-convexity is implied by convexity, and therefore this definition
of vulnerability respects the assumption of our games. As for the attacker, he knows his own
choice, thus the information that he can infer by mixing two strategies is the average of that of the
individual strategies, weighted on the percentages by which they appear in the mix. This means that
utility is a linear (more precisely, affine) function of his strategies. In conclusion, in the QIF-games
the utility is convex on the defender and linear on the attacker.

On the other hand, the property of DP is usually defined for a (probabilistic) mechanism associated
with a specific query on datasets. We say that a mechanism \( \mathcal{K} \) is \( \varepsilon \)-differentially private if for any
two neighbor datasets \( x \) and \( x' \) that differs only for one record, the ratio of the probabilities that
\( \mathcal{K} \), applied respectively on \( x \) and \( x' \), reports the same answer \( y \), is bounded by \( e^\varepsilon \). Here we adopt
a more abstract view: Following the approach in [23, 48], we assume that \( x \) and \( x' \) range over a
generic domain \( X \) endowed with a binary relation \( \sim \) that generalizes the neighborhood relation.
For the sake of uniformity, we will also model the mechanisms as information-theoretic channels,
whose typical element is the probability of reporting a certain \( y \) when applied to a given \( x \). In
the differential-privacy literature, the parameter \( \varepsilon \) is used to control the privacy provided by a
mechanism: the higher its value, the more secret information the mechanism can reveal. Here we
follow that approach and adopt as the measure of payoffs the minimum parameter \( \varepsilon \) for which a
mechanism is \( \varepsilon \)-differentially private.

For what concerns the composition of strategies, we remark an important difference between
the QIF and DP settings. While the measure of leakage in QIF can be –and often is– defined as an
average on all secrets and observables, in DP it is a worst-case measure, in the sense that only the
worst possible case counts (namely, the minimum \( \varepsilon \)). In line with this philosophy, the composition
of two mechanisms in DP is not linear with respect to the leakage, even in the case of visible choice:
the leakage is determined by the worst of the two, at least for non-trivial combinations. We will call
this property quasi-max and we will see that it is a particular case of quasi-convexity. Because of
this general non-linearity, it makes sense to explore for DP also the case in which the defender uses
visible choice (i.e., he makes known to the attacker which mechanism he is using). In the case of
DP, we therefore consider two cases for the defender’s choice: visible and invisible. For the attacker,
on the other hand, we consider only visible choice, as it is natural to assume that he knows what
choices he has made. We show that also for DP, and in all these possible scenarios, Nash equilibria
exist, and we provide algorithms to compute them.

Note that we do not consider the case of visible choice for the defender in QIF because in that
setting, for visible choice, the leakage is linear. Since the leakage would be linear in both the
attacker’s and defender’s strategies, the games would be just ordinary games, and the existence of
Nash equilibria and their computation would derive from standard game theory. The interested reader can find an analysis of this kind of game in [6]. Table 1 summarizes the various scenarios.

### 1.2 Contributions

The main contributions of this paper are the following:

- We define a general framework of information leakage games to reason about the interplay between attacker and defender in QIF and DP scenarios. The novelty w.r.t. standard game theory is that the utility of mixed strategies is not the expected utility of pure strategies.
- We extend the existence of Nash equilibria to DP-games (their existence for QIF-games is proved in a preliminary version [5] of this paper).
- We provide methods for finding Nash equilibria of QIF-games by solving a convex optimization problem, and for computing optimal strategies for DP-games by solving a sequence of linear programs. Our implementation of those algorithms for solving QIF-games and DP-games is freely available online as part of the LIBQIF tool [1].
- We examine the difference between our information leakage games and standard game theory, from a principled point of view, by analyzing the axioms of von Neumann and Morgenstern that constitute the foundations of the latter, and show why they cannot encompass the notion of utility as information.
- As a case study on QIF-games, we consider the Crowds protocol in a MANET (Mobile Ad-hoc NETwork). We study the case in which the attacker can add a corrupted node as an attack, the defender can add an honest node as a countermeasure, and we compute the defender component of the Nash equilibrium.
- As a case study on DP-games, we illustrate how to design a local DP mechanism using a real dataset on criminal records. Specifically, we consider a DP-game between a defender who discloses his non-sensitive attributes to a data curator, and an attacker (a data analyst) who queries the data curator about one of the non-sensitive information to extract the defender’s secret that is correlated with the disclosed non-sensitive information. By computing the Nash equilibrium of this DP-game, we construct a privacy mechanism for the defender.

### 1.2.1 Relation with preliminary version

A preliminary version of this work, considering only the QIF-games, appeared in [5]. The main novelties of this paper to that version are:

- We revise and present in more detail the algorithms for computing the Nash equilibria (Section 3.3).
- We introduce the notion of DP-games (Section 4.1) and study its theory, showing the existence of Nash equilibria (Sections 4.2 to 4.4).
- We provide an algorithm for computing optimal strategies for DP-games (Section 4.5).
- We show a case study on DP-games to illustrate how to design a local DP mechanism for an adaptive disclosure of non-sensitive information correlated with a secret (Section 6.2).
• We present proofs for our technical results (Appendix A).

1.2.2 Relation with our subsequent work. There are two subsequent papers based on the preliminary conference version [5] of this paper. These were published in [7] (preliminary conference version) and in [6] (full journal version). In these two papers, the focus was on the relation between the QIF-games that we consider here and other kinds of games (sequential games, games with visible choice for the defender). In particular, we built a taxonomy of the various kinds of games.

In the present paper, on the contrary, the focus is on the foundational aspects of a class of games that subsumes QIF and differential privacy (DP) and the contrast with the foundations of traditional game theory (as established by the von Neumann-Morgenstern axioms). Furthermore, the extension to the case of DP is new to this paper, and was not present in the previous papers [5], [7], and [6].

1.3 Related work
There is extensive literature on game theory models for security and privacy in computer systems, including network security, physical security, cryptography, anonymity, location privacy, intrusion detection, and economics of security and privacy. See [52] for a survey.

In many studies, security games have been used to model and analyze utilities between interacting agents, especially attackers and defenders. In particular, Korzhyk et al. [51] present a theoretical analysis of security games and investigate the relation between Stackelberg and simultaneous games under various forms of uncertainty. In application to network security, Venkitasubramaniam [64] investigates anonymous wireless networking, formalized as a zero-sum game between the network designer and the attacker. The task of the attacker is to choose a subset of nodes to monitor so that the anonymity of routes is minimum whereas the task of the designer is to maximize anonymity by choosing nodes to evade flow detection by generating independent transmission schedules.

Khouzani et al. [47] present a framework for analyzing a trade-off between usability and security. They analyze guessing attacks and derive the optimal policies for secret picking as Nash/Stackelberg equilibria. Khouzani and Malacaria [43–45] study the problem of optimal channel design in presence of hard and soft constraints. They employ entropies (the dual of vulnerability) and, in order to capture the widest class of leakages, they consider the property of core-concavity, a generalization of concavity which includes entropies which are not concave (like for instance the Rényi entropies when $\alpha > 1$). They show the existence of universally optimal strategies in the context of a two-player zero-sum game with incomplete information and that the defender’s Bayes-Nash equilibrium strategies are solutions of convex programming problems. Furthermore, they show that for any choice of the entropy measure, this problem can be solved via convex programming with zero duality gap, for which the Karush-Kuhn-Tucker (KKT) conditions can be used. In summary, their work uses game theory to support the design of optimal channels within operational constraints, while ours focuses on modeling the interplay between attacker and defender regarding information leakage of given channels, and to reason about their optimal strategies.

Concerning costs of security, Yang et al. [68] propose a framework to analyze user behavior in anonymity networks. The utility is modeled as a combination of weighted cost and anonymity utility. They also consider incentives and their impact on users’ cooperation.

Some security games have considered leakage of information about the defender’s choices. For example, Alon et al. [2] present two-player zero-sum games where a defender chooses probabilities of secrets while an attacker chooses and learns some of the defender’s secrets. Then they show how the leakage on the defender’s secrets influences the defender’s optimal strategy. Xu et al. [67] present zero-sum security games where the attacker acquires partial knowledge on the security resources the defender is protecting, and show the defender’s optimal strategy under such attacker’s knowledge. More recently, Farhang et al. [37] present two-player games where utilities are defined...
taking account of information leakage, although the defender’s goal is different from our setting. They consider a model where the attacker incrementally and stealthily obtains partial information on a secret, while the defender periodically changes the secret after some time to prevent a complete compromise of the system. In particular, the defender is not attempting to minimize the leak of a certain secret, but only to make it useless (for the attacker). Hence their model of defender and utility is totally different from ours.

The research area of quantitative information flow (QIF) has been dedicated to the development of theories to quantify and mitigate the amount of information leakage, including foundational works [12, 18, 29, 30, 49, 63], verification of QIF properties [16, 27, 28, 31, 50], and mitigation of information leakage [11, 44–46]. In this research area, several studies [4, 17, 40, 53, 56] have quantitatively modeled and analyzed the information leakage in interactive systems by using different approaches than ours. Boreale and Pampaloni [17] model adaptive attackers interacting with systems and investigate the relationship between adaptive and non-adaptive adversaries, although they do not deal with probabilistic strategies for the attacker. Mardziel et al. [53] formalize adaptive attackers using probabilistic finite automata and analyze the information leakage of several case studies using a probabilistic programming language, whereas they do not show any extensive case studies that use probabilistic attack strategies to maximize leakage. Alvim et al. [4] show that interactive systems can be modeled as information-theoretic channels with memory and feedback without using game theory to model the adaptiveness of agents. Kawamoto and Given-Wilson [40] show a quantitative model of information leakage in scheduler-dependent systems, although they assume only a passive observer who receives traces interleaved by some scheduler.

To the best of our knowledge, however, no other work has explored games with utilities defined as information-leakage measures, except the preliminary version of this paper [5], which introduced the QIF-games (called information-leakage games there), and our subsequent work [6, 7], which extended the QIF-games to sequential games, and considered also the case in which the defender’s choice is visible to the attacker.

Finally, in game theory, Matsui [55] uses the term “information leakage game” with a meaning different than ours, namely, as a game in which (part of) the strategy of one player may be leaked in advance to the other player, and the latter may revise his strategy based on this knowledge.

1.4 Plan of the paper
In Section 2, we review fundamental concepts from game theory, quantitative information flow, and differential privacy. In Section 3, we formalize the concept of QIF-games, in which utility is the vulnerability of a channel, and provide a series of examples of such games. We then derive the existence of Nash equilibria for these games and provide a method for computing them. In Section 4, we formalize the concept of DP-games, in which utility is the level $\varepsilon$ of differential privacy provided by a system. We demonstrate that such utility functions are quasi-convex or quasi-max, depending, respectively, on whether the defender’s action is hidden from or visible to the attacker. We derive the existence of Nash equilibria for these games and provide a method to compute them. In Section 5, we compare our information leakage games with the axiomatic formulation of standard game theory, and show why the latter does not capture our games. In Section 6, we apply our framework to two case studies: a version of the Crowds protocol and a design of local DP mechanisms for an adaptive disclosure of information correlated with secrets. Finally, in Section 7, we present our final remarks. The proofs of the formal results are in Appendix A.

2 PRELIMINARIES
In this section, we review some basic notions from game theory, quantitative information flow (QIF), and differential privacy (DP).
We recall basic definitions from two-player games. We recall that a set \( S \subseteq \mathbb{R}^n \) is convex if \( p s_0 + (1 - p)s_1 \in S \) for all \( s_0, s_1 \in S \) and \( p \in [0, 1] \). Let \( S \) be a convex set. A function \( f : S \rightarrow \mathbb{R} \) is convex if \( f(p s_0 + (1 - p)s_1) \leq p f(s_0) + (1 - p) f(s_1) \) for all \( s_0, s_1 \in S \) and \( p \in [0, 1] \), and it is concave if \( -f \) is convex. We say that \( f \) is quasi-convex if \( f(p s_0 + (1 - p)s_1) \leq \max\{f(s_0), f(s_1)\} \) for all \( s_0, s_1 \in S \) and \( p \in [0, 1] \), and it is quasi-concave if \( -f \) is quasi-convex. It is easy to see that if \( f \) is convex then it is also quasi-convex, and if it is concave then it is also quasi-concave.

2.1 Convexity and quasi-convexity

We recall basic definitions from two-player games, a model for reasoning about the behavior of two strategic players. We refer to [58] for more details.

In a game, each player disposes of a set of actions that he can perform, and obtains some payoff (gain or loss) depending on the outcome of the actions chosen by both players. The payoff’s value to each player is evaluated using a utility function. Each player is assumed to be rational, i.e., his choice is driven by the attempt to maximize his own utility. We also assume that the set of possible actions and the utility functions of both players are common knowledge.

In this paper, we only consider finite games, namely the cases in which the set of actions available to each player is finite. Furthermore, we only consider simultaneous games, in which each player chooses actions without knowing the actions chosen by the other.

Formally, a finite simultaneous game between a defender and an attacker is defined as a tuple \((\mathcal{D}, \mathcal{A}, u_d, u_a)\), where \( \mathcal{D} \) is a nonempty set of the defender’s actions, \( \mathcal{A} \) is a nonempty set of the attacker’s actions, \( u_d : \mathcal{D} \times \mathcal{A} \rightarrow \mathbb{R} \) is the defender’s utility function, and \( u_a : \mathcal{D} \times \mathcal{A} \rightarrow \mathbb{R} \) is the attacker’s utility function.

Each player may choose an action deterministically or probabilistically. A pure strategy of the defender (resp. attacker) is a deterministic choice of an action, i.e., an element \( d \in \mathcal{D} \) (resp. \( a \in \mathcal{A} \)). A pair \((d, a)\) of the defender’s and attacker’s pure strategies is called a pure strategy profile. The values \( u_d(d, a) \) and \( u_a(d, a) \) respectively represent the defender’s and the attacker’s utilities.

A mixed strategy of the defender (resp. attacker) is a probabilistic choice of an action, defined as a probability distribution \( \delta \in \mathcal{D} \) (resp. \( \alpha \in \mathcal{A} \)). A pair \((\delta, \alpha)\) of the defender’s and attacker’s mixed strategies is called a mixed strategy profile. The defender’s and the attacker’s expected utility functions for mixed strategies are respectively defined as:

\[
U_d(\delta, \alpha) \overset{\text{def}}{=} \mathbb{E}_{d\sim\delta, a\sim\alpha} u_d(d, a) = \sum_{d \in \mathcal{D}} \delta(d) \alpha(a) u_d(d, a), \quad U_a(\delta, \alpha) \overset{\text{def}}{=} \mathbb{E}_{d\sim\delta, a\sim\alpha} u_a(d, a) = \sum_{d \in \mathcal{D}} \delta(d) \alpha(a) u_a(d, a).
\]

A defender’s mixed strategy \( \delta \in \mathcal{D} \) is a best response to an attacker’s mixed strategy \( \alpha \in \mathcal{A} \) if \( U_d(\delta, \alpha) = \max_{\delta' \in \mathcal{D}} U_d(\delta', \alpha) \). Symmetrically, \( \alpha \in \mathcal{A} \) is a best response to \( \delta \in \mathcal{D} \) if \( U_a(\delta, \alpha) = \max_{\alpha' \in \mathcal{A}} U_a(\delta, \alpha') \). A mixed strategy Nash equilibrium is a profile \((\delta^*, \alpha^*)\) such that \( \delta^* \) is a best response to \( \alpha^* \) and vice versa. This means that in a Nash equilibrium, no unilateral deviation by any single player provides better utility to that player. If \( \delta^* \) and \( \alpha^* \) are point distributions concentrated on some pure strategies \( d^* \in \mathcal{D} \) and \( a^* \in \mathcal{A} \), respectively, then \((\delta^*, \alpha^*)\) is a pure strategy Nash equilibrium, and is denoted by \((d^*, a^*)\). While every finite game has a mixed strategy Nash equilibrium, not all games have a pure strategy Nash equilibrium.
2.3 Zero-sum games and minimax theorem

A game \((\mathcal{D}, \mathcal{A}, u_d, u_a)\) is zero-sum if for any \(d \in \mathcal{D}\) and any \(a \in \mathcal{A}\), \(u_d(d, a) = -u_a(d, a)\), i.e., the defender’s loss is equivalent to the attacker’s gain. For brevity, in zero-sum games we denote by \(u\) the attacker’s utility function \(u_a\), and by \(U\) the attacker’s expected utility \(U_a\). Consequently, the goal of the defender is to minimize \(U\), and the goal of the attacker is to maximize it.

In simultaneous zero-sum games, a Nash equilibrium corresponds to a solution of the minimax problem, i.e., a mixed strategy profile \((\delta^*, \alpha^*)\) such that \(U(\delta^*, \alpha^*) = \min_\delta \max_\alpha U(\delta, \alpha)\). (This is equivalent to the maximin problem, which looks for \((\delta^*, \alpha^*)\) such that \(U(\delta^*, \alpha^*) = \max_\delta \min_\alpha U(\delta, \alpha)\).) Von Neumann’s minimax theorem ensures that such a solution always exists and is stable.

**Theorem 2.1** (von Neumann’s minimax theorem). Let \(\Delta \subset \mathbb{R}^m\) and \(A \subset \mathbb{R}^n\) be compact convex sets, and \(U : \Delta \times A \rightarrow \mathbb{R}\) be a continuous function such that \(U(\delta, \alpha)\) is convex in \(\delta \in \Delta\) and concave in \(\alpha \in A\). Then it is the case that

\[
\min_{\delta \in \Delta} \max_{\alpha \in A} U(\delta, \alpha) = \max_{\alpha \in A} \min_{\delta \in \Delta} U(\delta, \alpha) =: U^*(\alpha) =: U^*(\delta).
\]

Moreover, under the conditions of Theorem 2.1, there exists a saddle point \((\delta^*, \alpha^*)\) s.t., for all \(\delta \in \Delta\) and \(\alpha \in A\), \(U(\delta^*, \alpha) \leq U(\delta^*, \alpha^*) \leq U(\delta, \alpha^*)\).

2.4 Quantitative information flow (QIF)

Next we recall the standard framework of quantitative information flow, which is used to measure the amount of information leakage in a system [9]. Notably, we define notions of “secret”, an adversary’s “prior knowledge” about the secret (or simply, “prior”), and a “vulnerability measure” to gauge how well the adversary can exploit her knowledge about the secret. We also define “channels”, probabilistic mappings used to model systems with observable behavior that changes the adversary’s probabilistic knowledge, making the secret more vulnerable and hence causing information “leakage”. Throughout this paper, we remain faithful to terminology and a presentation style that is well established in the field of quantitative information flow [9].

A secret is some piece of sensitive information the defender wants to protect, such as a user’s password, social security number, or current geo-location. The attacker often has some partial knowledge about the secret values, which is represented as a probability distribution on secrets called a prior distribution (or simply a prior). We denote by \(X\) the set of all possible secrets, and we typically use \(\pi\) to denote a prior that belongs to the set \(\mathcal{D}X\) of distributions over \(X\).

The vulnerability of a secret is a measure of the usefulness of the attacker’s knowledge about the secret. In this paper, we consider a very general notion of vulnerability, following [8], and define a vulnerability measure \(\mathcal{V}\) to be any continuous and convex function of type \(\mathcal{D}X \rightarrow \mathbb{R}\). It has been shown in [8] that these functions coincide with the set of \(g\)-vulnerabilities, and are, in a precise sense, the most general information measures w.r.t. a set of basic axioms.

Systems can be modeled as information-theoretic channels. A channel \(C : X \times Y \rightarrow \mathbb{R}\) is a function in which \(X\) is a finite set of input values, \(Y\) is a finite set of output values, and \(C(x, y)\) represents the conditional probability of the channel producing output \(y \in Y\) when input \(x \in X\) is provided. Every channel \(C\) satisfies \(0 \leq C(x, y) \leq 1\) for all \(x \in X\) and \(y \in Y\), and \(\sum_{y \in Y} C(x, y) = 1\) for all \(x \in X\).

A prior distribution \(\pi \in \mathcal{D}X\) and a channel \(C\) with inputs \(X\) and outputs \(Y\) induce a joint distribution \(p_{X,Y}(x, y) = \pi(x)C(x, y)\) on \(X \times Y\), with marginal probabilities \(p_X(x) = \sum_y p_{X,Y}(x, y)\) and

\[p_Y(y) = \sum_x p_{X,Y}(x, y)\]

1Conventionally in game theory the utility \(u\) is set to be that of the first player, but we prefer to look at the utility from the point of view of the attacker to be in line with the definition of utility as vulnerability, which we will define in Section 2.4.

2 More precisely, if posterior vulnerability is defined as the expectation of the vulnerability of posterior distributions, the measure respects the data-processing inequality and yields non-negative leakage iff vulnerability is convex.
\( p_Y(y) = \sum_x p_{X,Y}(x, y), \) and conditional probabilities \( p_{X|Y}(x|y) = p_{X,Y}(x,y) / p_Y(y) \) if \( p_Y(y) \neq 0. \) For a given \( y \) (s.t. \( p_Y(y) \neq 0 \)), the conditional probability distribution \( p_{X|Y} \) is called the posterior distribution on \( X \) given \( y. \) When clear from the context, we may omit subscripts on probability distributions. Writing, e.g., \( p(y), p(x,y), \) and \( p(x|y) \) for \( p_Y(y), p_{X,Y}(x,y), \) and \( p_{X|Y}(x|y) \), respectively.

A channel \( C \) in which \( X \) is a set of secret values and \( Y \) is a set of observable values produced by a system can be used to model computations on secrets. Assume that the attacker has prior knowledge \( \pi \) about the secret value, knows how a channel \( C \) works, and can observe the channel’s outputs. Then the effect of the channel is to update the attacker’s knowledge from the prior \( \pi \) to a collection of posteriors \( p_{X|y} \), each occurring with probability \( p(y) \).

Given a vulnerability measure \( V \), a prior \( \pi \), and a channel \( C \), the posterior vulnerability \( V[\pi, C] \) is the vulnerability of the secret after the attacker has observed the output of \( C. \) Formally,

\[
V[\pi, C] \overset{\text{def}}{=} \mathbb{E}_{y \sim p_Y} V[p_{X|y}].
\]

The information leakage of a channel \( C \) under a prior \( \pi \) is a comparison between the vulnerability of the secret before the system was run, called the prior vulnerability, and the posterior vulnerability of the secret. The leakage reflects how much the observation of the system’s outputs increases the utility of the attacker’s knowledge about the secret. It can be defined either additively (by \( V[\pi, C] - V[\pi] \)), or multiplicatively (by \( V[\pi, C] / V[\pi] \)). Additive leakage is a measure of an adversary’s absolute gain of information, and the resulting value retains the same meaning as the original vulnerability functions, e.g., the adversary’s “monetary gain”, or “damage caused”. Multiplicative leakage, on the other hand, is a dimensionless scalar representing the adversary’s relative gain of information, usually interpreted as a percentage.

### 2.5 Differential privacy (DP)

Next we briefly recall the notion of differential privacy [32, 35]. Consider a system that takes inputs from a set \( X \), endowed with an adjacency relation \( \sim \subseteq X \times X \) representing which pairs of inputs should be considered (almost) “indistinguishable”. We say that \( x \) and \( x' \) are adjacent if \( x \sim x' \). Intuitively, in the discrete case, a system (or mechanism) that takes inputs from \( X \) is differentially-private if the probability of its producing any output given input \( x \in X \) is roughly the same as the probability of its producing the same output given any other input \( x' \in X \) adjacent to \( x \). Using the channel notation to represent a system/mechanism, this intuition is formalized as follows.

**Definition 1** \((\epsilon\text{-differential privacy}).\) Let \( \epsilon \geq 0 \) and \( \sim \subseteq X \times X \) be an adjacency relation of secrets. A channel \( C : X \times Y \rightarrow \mathbb{R} \) provides \( \epsilon \text{-differential privacy} \) if for any pair of values \( x, x' \) s.t. \( x \sim x' \), and for any output value \( y \in Y \),

\[
C(x, y) \leq e^\epsilon C(x', y),
\]

where \( C(x, y) \) (resp. \( C(x', y) \)) is the probability of producing \( y \) on input \( x \) (resp. \( x' \)).

Note that our definition of differential privacy is in line with the influential Pufferfish framework [48] by Kifer and Machanavajjhala, in which a mechanism’s (secret) input domain can be arbitrary, and the adjacency relation can be customized to explicitly define what is considered to be secret. Traditional differential privacy is a particular instantiation of this definition in which the (secret) input set consists in datasets, and the adjacency relation is defined to reflect the usual distinction between the presence or absence of any single individual in a dataset. It is noteworthy that our abstract definition of differential privacy also encompasses local differential privacy, first formalized by Kasiviswanathan et al. [38]. More precisely, one only needs to define the set of secrets to be, e.g., values of an attribute for a single individual, and the adjacency relation to be that in which any two distinct values are adjacent.
In this section, we investigate QIF-games, in which the utility function is the information leakage of a channel. We demonstrate that the utility function is convex on mixed strategies, and that there exist Nash equilibria for these games. We then show how to compute these equilibria.

3.1 Definition of QIF-games

We introduce our games through a couple of examples to motivate the use of game theory. In the first example (the two-millionaires problem) the utility happens to behave like in standard game theory, but it is just a particular case. In general, there is a fundamental difference, and we will illustrate this with our second example (the binary-sum game).

3.1.1 The two-millionaires problem. The two-millionaires problem was introduced by Yao in [69]. In the original formulation, there are two “millionaires”, Alex and Don, who want to discover who is the richest among them, but neither wants to reveal to the other the amount of money they have.

We consider a (conceptually) asymmetric variant of this problem, where Alex is the attacker and Don is the defender. Don wants to learn whether or not he is richer than Alex, but does not want Alex to learn anything about the amount \( x \) of money he has. For this purpose, Don sends \( x \) to a trusted server, Jeeves, who in turn asks Alex, privately, what is her amount \( a \) of money. Jeeves then checks which among \( x \) and \( a \) is greater, and sends the result \( y \) back to Don.\(^3\) However, Don is worried that Alex may intercept Jeeves' message containing the result of the comparison, and exploit it to learn more accurate information about \( x \) by tuning her answer \( a \) appropriately (since, given \( y \), Alex can deduce whether \( a \) is an upper or lower bound on \( x \)). Note that this means, effectively, that \( a \) is the choice of action given to Alex in the game. We assume that Alex may get to know Jeeves’ reply, but not the messages from Don to Jeeves.\(^4\) See Fig. 1.

We will use the following information-flow terminology: the information that should remain secret (to the attacker) is called high, and what is visible to (and possibly controllable by) the attacker is called low. Hence, in the program run by Jeeves \( a \) is a low input and \( x \) is a high input. The result \( y \) of the comparison (since it may be intercepted by the attacker) is a low output. The problem is to avoid the flow of information from \( x \) to \( y \) (given \( a \)).

One way to mitigate this problem is to use randomization. Assume that Jeeves provides two different programs to ensure the service. Then, when Don sends his request to Jeeves, he can make a random choice \( d \) among the two programs 0 and 1, sending \( d \) to Jeeves along with the value \( x \). Now, if Alex intercepts the result \( y \), it will be less useful to her since she does not know which of

\(^3\)The reason to involve Jeeves is that Alex may not want to reveal \( a \) to Don, either.

\(^4\)That may be the case because, say, although all messages are exchanged using encryption, Jeeves’ secret key has been hacked by Alice, whereas Don’s remains protected.
\[
\begin{array}{c|cc}
 & y = T & y = F \\
\hline
\text{C}_{00} & x = 0 & 1 \\
 & x = 1 & 0 \\
\text{C}_{01} & x = 0 & 1 \\
 & x = 1 & 0 \\
\text{C}_{10} & x = 0 & 1 \\
 & x = 1 & 0 \\
\text{C}_{11} & x = 0 & 0 \\
 & x = 1 & 1 \\
\end{array}
\]

Fig. 3. The four channels \( C_{da} \) representing all possible behaviors of the two-millionaires system, from the point of view of the attacker.

the two programs has been run. As Don, of course, knows which program was run, the result \( y \) will still be just as useful to him.\(^5\)

In order to determine the best probabilistic strategy that Don should apply to select the program, we analyze the problem from a game-theoretic perspective. For simplicity, we assume that \( x \) and \( a \) both range in \( \{0, 1\} \). The two alternative programs that Jeeves can run are shown in Fig. 2.

The combined choices of Alex and Don determine how the system behaves. Let \( \mathcal{D} = \{0, 1\} \) represent Don’s possible choices, i.e., the program to run, and \( \mathcal{A} = \{0, 1\} \) represent Alex’s possible choices, i.e., the value of the low input \( a \). We shall refer to the elements of \( \mathcal{D} \) and \( \mathcal{A} \) as actions. For each possible combination of actions \( d \) and \( a \), we can construct a channel \( C_{da} \) with inputs \( \mathcal{X} = \{0, 1\} \) (the set of possible high-input values) and outputs \( \mathcal{Y} = \{T, F\} \) (the set of possible low-output values), modeling the behavior of the system from the point of view of the attacker. Intuitively, each channel entry \( C_{da}(x, y) \) is the probability that the program run by Jeeves (which is determined by \( d \)) produces output \( y \in \mathcal{Y} \) given that the high input is \( x \in \mathcal{X} \) and that the low input is \( a \). Fig. 3 presents the four channel matrices representing all possible behaviors of the program. Note that channels \( C_{01} \) and \( C_{10} \) do not leak any information about the input \( x \) (output \( y \) is constant), whereas channels \( C_{00} \) and \( C_{11} \) completely reveal \( x \) (output \( y \) is in a bijection with \( x \)).

We now investigate how the defender’s and the attacker’s strategies influence the leakage of the system. For that, we can consider the (simpler) notion of posterior vulnerability, since, for any given prior, the value of leakage is in a one-to-one, monotonic correspondence with the value of posterior vulnerability.\(^6\) For this example, we consider posterior Bayes vulnerability [24, 63], defined as

\[
\mathbb{V}_{\text{Bayes}}^\pi[C] \overset{\text{def}}{=} \mathbb{E}_{\mathcal{Y} \leftarrow \pi_{xy}} \max_x \max_y \pi(x) \mathbb{C}(x, y),
\]

which measures the expected probability of the attacker guessing the secret correctly in one try after having observed the output of the system. It can be shown that \( \mathbb{V}_{\text{Bayes}}^\pi[C] \) coincides with the converse of the Bayes error.

For simplicity, we assume a uniform prior distribution \( \pi_u \). It has been shown that, in this case, the posterior Bayes vulnerability of a channel \( C \) can be computed as the sum of the greatest elements

\(^5\)Note that \( d \) should not be revealed to the attacker: although \( d \) is not sensitive information in itself, knowing it would help the attacker figure out the value of \( x \).

\(^6\)More precisely, for any fixed prior \( \pi \), the additive leakage of a channel \( C \) is obtained by a simple shifting of the posterior vulnerability \( \mathbb{V}[\pi, C] \) by a term of \( -\mathbb{V}[\pi] \) corresponding to the prior vulnerability. Similarly, the multiplicative leakage of a channel \( C \) is obtained by a simple scaling of the posterior vulnerability \( \mathbb{V}[\pi, C] \) by a factor of \( \mathbb{V}[\pi] \) corresponding to the prior vulnerability. Both transformations consist in a monotonic bijection between the corresponding leakage notions and posterior vulnerability.
Jeeves will send $T$ i.e., a probabilistic choice between the channels representing the defender’s actions. The overall vulnerability of the system is then given by the vulnerability of $C_{\delta a}$, averaged over all attacker’s actions.

We now define formally the ideas illustrated above using a simultaneous game (Section 2.2).

**Definition 2 (QIF-game).** A QIF-game is a simultaneous game $(\mathcal{D}, \mathcal{A}, C, \mathbb{V})$ where $\mathcal{D}$, $\mathcal{A}$ are the finite sets of actions of the attacker and the defender, respectively, $C = \{C_{da}\}_{da}$ is a family of channel matrices of same type indexed on pairs of actions $d \in \mathcal{D}, a \in \mathcal{A}$, and $\mathbb{V}$ is a vulnerability measure. The game is zero-sum, and for a prior $\pi$, the utility for the attacker of a pure strategy profile $(d, a)$ is given by $\mathbb{V}[\pi, C_{da}]$. The utility $\mathbb{V}(\delta, \alpha)$ for the attacker of a mixed strategy profile $(\delta, \alpha)$ is defined as

$$\mathbb{V}(\delta, \alpha) \overset{\text{def}}{=} \mathbb{E}_{d \sim \delta} \mathbb{V}[\pi, C_{\delta a}] = \sum_{a} \alpha(a) \mathbb{V}[\pi, C_{\delta a}],$$

where $C_{\delta a}$ is the hidden-choice channel composition defined as in (2).

In our example, $\delta$ can be represented by a single number $p$: the probability that the defender chooses $d = 0$ (i.e., Program 0). From the point of view of the attacker, once he has chosen $a$, the system will look like a channel $C_{\delta a} = p C_{0a} + (1 - p) C_{1a}$. For instance, in the case $a = 0$, if $x$ is 0 Jeeves will send $T$ with probability 1, but, if $x$ is 1, Jeeves will send $F$ with probability $p$ and $T$ with



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7Note that two channel matrices with different column indices can always be made compatible by adding appropriate columns with 0-valued cells in each of them.
The utility is obtained as the expectation with respect to the strategy of the attacker, hence the same utility strategies we obtain the utility table shown in Fig. 8. This means that all pure strategies have the same utility 1 and therefore they are all equivalent. In standard game theory this would mean probability 1 − p. Similarly for a = 1. Fig. 5 summarizes the various channels modeling the attacker’s point of view. It is easy to see that \( V^\text{Bayes}[\pi_u, C_{p0}] = (1+p)/2 \) and \( V^\text{Bayes}[\pi_u, C_{p1}] = (2−p)/2 \). In this case \( V^\text{Bayes}[\pi_u, C_{pa}] \) coincides with the expected utility with respect to \( p \), i.e., \( V^\text{Bayes}[\pi_u, C_{pa}] = p \, V^\text{Bayes}[\pi_u, C_{0a}] + (1−p) \, V^\text{Bayes}[\pi_u, C_{1a}] \).

Assume now that the attacker chooses \( a = 0 \) with probability \( q \) and \( a = 1 \) with probability \( 1−q \). The utility is obtained as the expectation with respect to the strategy of the attacker, hence the total utility is: \( V^\text{Bayes}(p, q) = q \, (1+p)/2 + (1−q) \, (2−p)/2 \), which is affine in both \( p \) and \( q \). By applying standard game-theoretic techniques, we derive that the optimal strategy is \((p^*, q^*) = (1/2, 1/2)\).

In the above example, things work just like in standard game theory. However, the next example fully exposes the difference of our games with respect to those of standard game theory.

**3.1.2 The binary-sum game.** The previous example is an instance of a general scenario in which a user, Don, delegates to a server, Jeeves, a certain computation that also requires some input from other users. Here we will consider another instance, in which the function to be computed is the binary sum \( \oplus \). We assume Jeeves provides the programs in Fig. 6. The resulting channel matrices are represented in Fig. 7.

![Fig. 6. The programs for computing binary-sum, and for computing its complement.](image)

![Fig. 7. The four channels \( C_{da} \) representing all possible behaviors of the binary-sum system, from the attacker’s point of view.](image)
Theorem 3.1 Convexity of vulnerability w.r.t. channel composition [6]. Let \( C_\{i\} \subset I \) be a family of channels of the same type, all with domain \( X \). Then, for every prior distribution \( \pi \in \mathbb{D}X \), every vulnerability \( \nabla \), and any probability distribution \( \mu \) on \( I \), we have that
\[
\nabla[\pi, \frac{\mathbb{E}}{\mu} C_\{i\}] \leq \frac{\mathbb{E}}{\mu} \nabla[\pi, C_\{i\}] .
\]

As formalized in the next result, the existence of Nash equilibria for QIF-games immediately follows from the above theorem.

Corollary 3.2 (Existence of Nash-equilibrium for QIF-games). For any (zero-sum) QIF-game there exists a Nash equilibrium, which in general is given by a mixed strategy.

3.2 Existence of Nash equilibrium for QIF-games

In [6] it was proved that the posterior vulnerability of a convex combination of channels is smaller than or equal to the convex combination of their vulnerabilities. As a consequence, posterior vulnerability is a convex function of the strategy of the defender, and using the von Neumann’s minimax theorem we are able to derive the existence of the Nash equilibrium for QIF-games.

Theorem 3.1 (Convexity of vulnerability w.r.t. channel composition [6]). Let \( \{C_\{i\}\}_{i \in I} \) be a family of channels of the same type, all with domain \( X \). Then, for every prior distribution \( \pi \in \mathbb{D}X \), every vulnerability \( \nabla \), and any probability distribution \( \mu \) on \( I \), we have that

\[
\nabla[\pi, \frac{\mathbb{E}}{\mu} C_\{i\}] \leq \frac{\mathbb{E}}{\mu} \nabla[\pi, C_\{i\}] .
\]

As formalized in the next result, the existence of Nash equilibria for QIF-games immediately follows from the above theorem.

Corollary 3.2 (Existence of Nash-equilibrium for QIF-games). For any (zero-sum) QIF-game there exists a Nash equilibrium, which in general is given by a mixed strategy.

3.3 Computing equilibria for QIF-games

Recall that the utility of a mixed strategy profile \((\delta, \alpha)\), given by Definition 2, is

\[
\nabla(\delta, \alpha) = \frac{\mathbb{E}}{\alpha} \nabla[\pi, \frac{\mathbb{E}}{d} C_{\{a\}}],
\]

and that \( \nabla(\delta, \alpha) \) is convex on \( \delta \) and affine on \( \alpha \). Theorem 2.1 guarantees the existence of an equilibrium (i.e., a saddle-point) \((\delta^*, \alpha^*)\), which is a solution of both the minimax and the maximin problems. The goal in this section is to compute: (i) a \( \delta^* \) that is part of an equilibrium, which is

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\(^3\)The convexity of posterior vulnerability w.r.t. the strategy of the defender is formally shown in Theorem 3.1 ahead.
important in order to optimize the defense, and (ii) the utility $V(\delta^*, \alpha^*)$, which is important to provide an upper bound on the effectiveness of an attack when $\delta^*$ is applied.

This is a convex-concave optimization problem for which various methods have been proposed in the literature. If $V$ is twice differentiable (and satisfies a few extra conditions) then the Newton method can be applied [20]; however, many $V$’s of interest, and notably the Bayes vulnerability, one of the most popular vulnerability measures, are not differentiable. For non-differentiable functions, [57] proposes an approximate iterative method that computes a subgradient and updates both $\alpha$ and $\delta$ at each step in the direction of the subgradient, moving towards the point of equilibrium. The process stops when the change in the value of $V(\delta, \alpha)$ induced by the update of $\alpha$ and $\delta$ is “small enough”, meaning that we are close to $V(\delta^*, \alpha^*)$, and the current values of $\alpha$ and $\delta$ are returned as approximations of $\delta^*$ and $\alpha^*$. Some precautions must be taken to ensure stability, i.e., to avoid swinging around the point of equilibrium: the size of the updates needs to become smaller as we get closer to it. Clearly, there is a trade-off between the precision of the approximation and the convergence speed.

In case we are interested in computing only the optimal strategy of the defender, however, we propose a method more efficient than the one of [57]. Our method is iterative like the one in [57], but, in contrast to [57], at every step it performs the update only on the $\delta$ component, while it computes the exact argmax on the $\alpha$ component, thus improving efficiency. For this purpose, we exploit the fact that $V(\delta, \alpha)$ is affine on $\alpha$ (not just concave). This means that, for a fixed $\delta$, maximizing $E_{a\leftarrow \alpha} V[\pi, E_{d\leftarrow \delta} C_{da}]$ simply involves selecting an action $a$ with the highest $V[\pi, E_{d\leftarrow \delta} C_{da}]$ and assigning probability 1 to it.

**Proposition 3.3.** The minimax problem $\delta^* = \arg \min_{\delta} \max_{\alpha} V(\delta, \alpha)$ is equivalent to:

$$\delta^* = \arg \min_{\delta} f(\delta) \quad \text{where} \quad f(\delta) = \max_{a} V[\pi, E_{d\leftarrow \delta} C_{da}]. \quad (3)$$

It is important to point out that solving (3) produces a $\delta^*$ that is the defender’s mixed strategy of an equilibrium, but in general it does not produce the attacker’s mixed strategy $\alpha^*$ of the equilibrium. For the latter, we would need to solve the maximin problem.

Note that $f$ is a function of a single variable $\delta$, and our problem reduces to computing its minimum value. We now proceed to apply the subgradient descent method for this task. We start by recalling the notions of subgradient and projection on the simplex.

**Definition 3 (Subgradient).** Let $U$ be a convex open set in the Euclidean space $\mathbb{R}^n$. A subgradient of a convex function $f : U \rightarrow \mathbb{R}$ at a point $x_0$ in $U$ is any vector $v \in \mathbb{R}^n$ such that, for all $x \in U$,

$$f(x) - f(x_0) \geq v \cdot (x - x_0),$$

where “$\cdot$” denotes the dot product between vectors.

**Definition 4 (Projection).** The $n$-simplex $\mathbb{S}^n$ is the set of all vectors in $\mathbb{R}^n$ that represent probability distributions. Given a vector $v \in \mathbb{R}^n$, the projection of $v$ on $\mathbb{S}^n$, which we will denote by $P(v)$, is the probability distribution $\delta \in \mathbb{S}^n$ that is closest, in the Euclidean sense, to $v$.

Note that if $v \in \mathbb{S}^n$ then $P(v) = v$. To compute the projection we can use the efficient algorithms proposed in [66].

We are now ready to define our iterative method for the approximation of $\delta^*$ and $V(\delta^*, \alpha^*)$. The idea is to start from an arbitrary fully supported distribution over $\mathcal{D}$, for instance the uniform distribution $u_\mathcal{D}$, and then compute the next $\delta$ from previous one following (3). More precisely, at
step \( k + 1 \) we compute \( \delta^{(k+1)} \) from \( \delta^{(k)} \) according to the following inductive definition:

\[
\delta^{(1)} = u_D,
\delta^{(k+1)} = P (\delta^{(k)} - s_k h^{(k)})
\]

where \( s_k = 0.1/\sqrt{k} \) and \( h^{(k)} \) is any subgradient of \( f(\delta) = \max_{\pi} V[\pi, \mathbb{E}_{d \sim \delta} C_{da}] \) at point \( \delta^{(k)} \).

The step size \( s_k \) has to be decreasing in \( k \) in order to avoid the swinging effect, as explained above. There are various choices for \( s_k \) that guarantee convergence [19]. We have chosen \( s_k = 0.1/\sqrt{k} \) because we have determined heuristically that it gives a good performance.

We use the same stopping criterion of [19, Section 3.4]. Namely, at each step \( k \), we compute the following value \( l^{(k)} \), which is guaranteed to be a lower bound on \( f(\delta^*) \):

\[
l^{(k)} = \frac{2 \sum_{i=1}^{k} s_i f(\delta^{(i)}) - \| \delta - \delta^{(i)} \|_2^2}{2 \sum_{i=1}^{k} s_i}.
\]

Hence, we can stop the process when \( f(\delta^{(k)}) - l^{(k)} \leq \epsilon \), for some given \( \epsilon > 0 \), and then the value returned is \( \hat{\delta} = \delta^{(k)} \).\(^9\) The parameter \( \epsilon \) determines the desired level of approximation, as formally stated in Theorem 3.4 below. Before stating the result we need to recall the Lipschitz property.

**Definition 5** (Lipschitz). Let \((A, d_A)\) and \((B, d_B)\) be two metric spaces and let \( G \) be a non-negative real constant. A function \( F : A \rightarrow B \) is \( G \)-Lipschitz if for all \( a, a' \in A \),

\[
d_B(F(a), F(a')) \leq G \cdot d_A(a, a') .
\]

We also say that \( F \) is Lipschitz if it is \( G \)-Lipschitz for some \( G \).

**Theorem 3.4** (Convergence). If \( V \) is Lipschitz then the sequence \( \{\delta^{(k)}\}_k \) in (4) converges to a \( \delta^* \) that is part of an equilibrium. Moreover, when the stopping condition is met, i.e., \( f(\delta^{(k)}) - l^{(k)} < \epsilon \), then the approximate solution \( \hat{\delta} = \delta^{(k)} \) returned by the algorithm, and for an equilibrium \((\delta^*, \alpha^*)\),

\[
V(\hat{\delta}, \alpha) - \epsilon \leq V(\hat{\delta}, \alpha^*) \leq V(\delta, \alpha^*) + \epsilon \quad \forall \delta, \alpha .
\]

This implies that the computed strategy is \( \epsilon \)-close to the optimal one, namely:

\[
|V(\hat{\delta}, \alpha^*) - V(\delta^*, \alpha^*)| \leq \epsilon .
\]

The convergence crucially depends on the \( G \)-Lipschitzness assumption, which ensures the boundness of the subgradients and therefore that there is no "swinging effect" around the point of equilibrium. One may be tempted to also find a relation between the constant \( G \) and the convergence rate, but, unfortunately, there is no such relation in general. Essentially, this is because \( G \) is a global bound, while the convergence rate depends on the steepness around the point of equilibrium and how well it combines with the rate by which the step size diminishes.

Notably, the Lipschitz condition is satisfied by the large class of the \( g \)-vulnerability measures [10] when the "set of guesses" (or "actions") is finite. The set of guesses \( \mathcal{W} \) is a parameter of a \( g \)-vulnerability measure, along with the gain function \( g : \mathcal{W} \times \mathcal{X} \rightarrow \mathbb{R} \). One example of \( g \)-vulnerability with finite \( \mathcal{W} \) is that of Bayes vulnerability: in this case \( \mathcal{W} = \mathcal{X} \) and \( g(w, x) = 1 \) if \( w = x \), and \( g(w, x) = 0 \) otherwise.

In general, if \( \mathcal{W} \) is finite, the posterior \( g \)-vulnerability \( V \) is given by:

\[
V[\pi, C] = \sum_y \max_w \sum_x \pi(x) C(x, y) g(w, x) .
\]

\(^9\)Note that \( f(\delta^{(k)}) \) is not necessarily decreasing, so an optimization would be to keep track of the best values of \( f(\delta^{(k)}) \) and \( l^{(k)} \), stop when \( f(\delta^{(\text{best})}) - l^{(\text{best})} < \epsilon \), and return \( \hat{\delta} = \delta^{(\text{best})} \).

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The above $V$ is piecewise linear, and therefore $G$-Lipschitz, where $G$ is the maximum of the norms of the subgradients at the point distributions. In general, a subgradient vector $h^{(k)}$ is given by:

$$h_d^{(k)} = \delta^{(k)}(d) \sum_y \pi(x^*_y) C_{d a^*}(x^*_y, y),$$

where $a^*, x^*_y$ are (any of) the ones giving the max in the branches of $f(\delta^{(k)})$.

Note that if $V$ is piecewise linear then also $f$ is piecewise linear. Hence, in the case of $g$-vulnerability, the convex optimization problem could be transformed into a linear one using a standard technique, and then solved by linear programming. However, typically $V$ has a large number of max branches, and consequently this conversion can generate a huge number of constraints. In our experiments, we found that the subgradient method described above is significantly more efficient than linear programming (although the latter has the advantage of giving an exact solution).

On the other hand, not all vulnerabilities are Lipschitz; Shannon entropy, for instance, is not. Note that Shannon entropy can be expressed as $g$-vulnerability, but the corresponding set of guesses $W$ is infinite [8].

Finally, although the subgradient method is general, it might be impractical in applications where the number of attacker or defender actions is very large. Application-specific methods could offer better scalability in such cases; we leave the development of such methods as future work.

4 DP-GAMES

In this section, we investigate DP-games, in which the utility function is the level of privacy of a channel, namely the minimum $\epsilon$ for which the channel is $\epsilon$-differentially private.

We demonstrate that such utility functions are quasi-convex or quasi-max (rather than linear) on mixed strategies, depending, respectively, on whether the defender’s action is hidden from or visible to the attacker, and derive the existence of Nash equilibria for these games. We then show how to compute these equilibria.

4.1 Definition of DP-games

We begin by formalizing the level of differential privacy of a channel, following an alternative characterization of differential privacy based on max-divergence [36]. Let $X, Y$ be two sets (the domains of “secrets” and “observables”, respectively), and let $\sim$ be a symmetric binary relation on $X$, i.e., $\sim \subseteq X \times X$ and such that $x \sim x'$ iff $x' \sim x$. We say that a channel $C : X \times Y \rightarrow \mathbb{R}$ is conforming to $\sim$ if for all $x, x' \in X$ and $y \in Y, x \sim x'$ implies that $C(x, y) = 0$ iff $C(x', y) = 0$.

Note that the notions of domain and adjacency relation are more abstract than in the typical definition of differential privacy. This is because we want to capture both the cases of the standard (central) DP and the local DP. In the central DP, the domain of secrets consists of datasets, and the adjacency relation $x \sim x'$ holds if and only if $x$ and $x'$ differ for one record. In the local DP, the domain of secrets consists of all possible values of a record, and the adjacency relation $x \sim x'$ holds for all values $x$ and $x'$ different from each other.

**Definition 6 (Differential-privacy level of a channel).** Let $C : X \times Y \rightarrow \mathbb{R}$ be a channel and let $\sim$ be a binary symmetric relation on $X$. We say that $C$ is differentially private if it is conforming to $\sim$, and, in this case, its differential-privacy level $V^{DP}[C]$ is defined as

$$V^{DP}[C] = \max_{y, x, x'} \ln \frac{C(x, y)}{C(x', y)},$$

subject to $x \sim x'$.
As shown in [36], the relation with the standard notion of differential privacy (cf. Definition 1) is that a channel $C$ provides $\varepsilon$-differential privacy iff $\mathcal{V}^{\text{DP}}[C] \leq \varepsilon$. In other words, $\mathcal{V}^{\text{DP}}[C]$ is the least value $\varepsilon$ for which $C$ is $\varepsilon$-differentially private. This presentation eases mathematical treatment, and effectively means that: (i) $0 \leq \mathcal{V}^{\text{DP}}[C] < \infty$; and (ii) $\mathcal{V}^{\text{DP}}[C] = \infty$ if and only if $C$ is not conforming to $\sim$. Moreover, as usual, it is the case that the higher $\mathcal{V}^{\text{DP}}[C]$, the less private channel $C$ is.

We will use the level of privacy (or more appropriately, non-privacy, or leakage) $\mathcal{V}^{\text{DP}}$ as the definition of utility in DP-games. In the next sections, we will see that the properties of $\mathcal{V}^{\text{DP}}$ are rather different from those of vulnerability in QIF-games.

### 4.2 DP-games vs. QIF-games

Intuitively, DP and QIF have a similar “philosophy”: in both cases the attacker is trying to infer information about the secrets from the observables.\(^\text{10}\) The notion of attacker in QIF-games and DP-games differ significantly. In both cases, the attacker is trying to gain some knowledge about the secrets from the observables, but the modalities are very different. Specifically:

- (A) In DP-games, the prior does not appear in the definition of the payoff like it does in QIF-games. In DP-games, the payoff is simply the minimal $\varepsilon$ such that $\varepsilon$-DP holds.
- (B) In DP-games, the attacker is not trying to maximize her expected posterior probability of success on all observables like in QIF-games, but rather the maximum distinguishability (i.e., the ratio of the likelihoods) of two secrets for an observable whatsoever. This is what we call “worst-case principle.”
- (C) From the game-theoretic point of view, this worst-case principle makes a big difference. In fact, we show that it implies that the leakage is not a convex function as in QIF-games, but only quasi-convex. Furthermore, even in the case of visible choice, the leakage is not linear as in QIF-games, but only quasi-max, which is a particular case of quasi-convexity.

### 4.3 Formal definition of DP-games

Recall that for QIF-games we defined (cf. (2)) the hidden-choice composition $C_{\delta,a}$ resulting from a mixed strategy $\delta$ by the defender and a pure strategy $a$ by the attacker as:

$$C_{\delta,a} \overset{\text{def}}{=} \mathbb{E}_{d \sim \delta} C_{d,a},$$

In our DP-games we will also consider the visible-choice composition modeling the case in which the attacker is able to identify the action taken by the defender. This is because even in the case of visible choice, the level of privacy is not linear w.r.t. the channel composition. Hence visible-choice DP-games are not captured by standard game theory.

In order to define the latter formally, let us introduce some notation. Given a family $\{M_i\}_{i \in I}$ of compatible matrices s.t. each $M_i$ has type $X \times Y_i \to \mathbb{R}$, their concatenation $\bigotimes_{i \in I} M_i$ is the matrix having all columns of every matrix in the family, in such a way that every column is tagged with the matrix it came from. Formally, $((\bigotimes_{i \in I} M_i)(x, (y, j)) = M_j(x, y)$, if $y \in Y_j$, and the resulting matrix has type $X \times (\bigsqcup_{j \in I} Y_j) \to \mathbb{R}$.\(^\text{11}\) Fig. 10 provides an example of the concatenation of two matrices.

We now will define the “visible choice” among channel matrices, representing the situation in which a channel is probabilistically picked from a family of channels, and the choice of channel is revealed to the attacker. Formally, given $\{C_i\}_{i \in I}$ of compatible channels s.t. each $C_i$ has type...

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\(^{\text{10}}\)The interpretation of DP in terms of inference can be found, for instance, in [3, 15, 22, 35, 48].

\(^{\text{11}}\)For $I = \{i_1, i_2, \ldots, i_n\}$, $\bigsqcup_{i \in I} Y_i = Y_{i_1} \sqcup Y_{i_2} \sqcup \ldots \sqcup Y_{i_n}$ denotes the disjoint union $\{(y, i) | y \in Y_i, i \in I\}$ of the sets $Y_{i_1}, Y_{i_2}, \ldots, Y_{i_n}$. 
where 

$$C\in\mathcal{A}$$

are the finite sets of actions of the attacker and the defender, respectively, and 

$$C = \{C_{da}\}_{da}$$

is a family of compatible channel matrices, all conforming to a symmetric binary relation \(\sim\) on their inputs, indexed on pairs of actions \(d \in D, a \in A\). The game is zero-sum, and the utility for the attacker of a pure strategy profile \((d, a)\) is given by \(\mathcal{V}^{\text{DP}}[C_{da}]\).

When the defender’s choice is hidden from the attacker, the attacker’s utility of a mixed strategy profile \((\delta, \alpha)\) is defined as

$$\mathcal{V}^{\text{DP}}(\delta, \alpha) \overset{\text{def}}{=} \mathcal{V}^{\text{DP}} \left[ \left| \right|_{i\sim\mu} C_{\delta a} \right],$$

where \(C_{\delta a} \overset{\text{def}}{=} \mathbb{E}_{d\sim\delta} C_{da}\) is the hidden choice as defined in (2).

When the defender’s choice is visible to the attacker, the attacker’s utility of a mixed strategy profile \((\delta, \alpha)\) is defined as

$$\mathcal{V}^{\text{DP}}(\delta, \alpha) \overset{\text{def}}{=} \mathcal{V}^{\text{DP}} \left[ \left| \right|_{a\sim\alpha} \left| \right|_{d\sim\delta} C_{da} \right].$$

Fig. 10. Example of the concatenation of two matrices \(M_1\) and \(M_2\).

It can be shown that the resulting matrix \(\left| \right|_{i\sim\mu} C_i\) has type \(X \times (\left| \right|_{i\in I} Y_i) \to \mathbb{R}\), and it is also a channel [7]. Fig. 11 depicts the visible choice among two channel matrices \(C_1\) and \(C_2\), with probability \(1/3\) and \(2/3\), respectively.

It is important to notice that both the result of a visible choice composition as in (2) and of a visible choice composition as in (6) cannot increase the maximum level of differential privacy provided by the operand channels, as formalized in Theorem 4.1 ahead. Intuitively, that is a consequence of the fact that both operators produce a new channel matrix only by concatenating, scaling and summing up columns of the original channel matrices, and these three transformations cannot increase the maximum ratio between any two elements in a same column of the resulting matrix.

We are now ready to formalize the concept of DP-games.

**Definition 7** (DP-game). A DP-game is a simultaneous game \((D, A, C)\) where \(D, A\) are the finite sets of actions of the attacker and the defender, respectively, and \(C = \{C_{da}\}_{da}\) is a family of compatible channel matrices, all conforming to a symmetric binary relation \(\sim\) on their inputs, indexed on pairs of actions \(d \in D, a \in A\). The game is zero-sum, and the utility for the attacker of a pure strategy profile \((d, a)\) is given by \(\mathcal{V}^{\text{DP}}[C_{da}]\).

When the defender’s choice is hidden from the attacker, the attacker’s utility of a mixed strategy profile \((\delta, \alpha)\) is defined as

$$\mathcal{V}^{\text{DP}}(\delta, \alpha) \overset{\text{def}}{=} \mathcal{V}^{\text{DP}} \left[ \left| \right|_{i\sim\mu} C_{\delta a} \right],$$

where \(C_{\delta a} \overset{\text{def}}{=} \mathbb{E}_{d\sim\delta} C_{da}\) is the hidden choice as defined in (2).

When the defender’s choice is visible to the attacker, the attacker’s utility of a mixed strategy profile \((\delta, \alpha)\) is defined as

$$\mathcal{V}^{\text{DP}}(\delta, \alpha) \overset{\text{def}}{=} \mathcal{V}^{\text{DP}} \left[ \left| \right|_{a\sim\alpha} \left| \right|_{d\sim\delta} C_{da} \right].$$

Note that in our definition of DP-games, the defender is interested in minimizing maximum leakage, whereas the adversary is interested in maximizing this same quantity. Note that it is reasonable for the defender always to try to minimize maximum leakage, as the guarantees of differential privacy are intended to hold independently of the adversary. However, it would be reasonable to consider scenarios in which the adversary would not be necessarily interested in maximizing the channel’s maximum leakage, e.g., because he adopts a different notion of relation \(\sim\) than that used by the defender. In such a case, the resulting game would not be, in general,
Fig. 12. Four channels $C_{da}$ representing all possible behaviors of a DP-game.

zero-sum. A study of such games, however, is beyond the scope of this paper, and we therefore assume here that defender and attacker agree on the utility measure to be minimized/maximized.

Now we present an example of a DP-game and show that $\nabla_{\text{DP}}$ is not convex w.r.t. hidden choice.

**Example 8 ($\nabla_{\text{DP}}$ is not convex w.r.t. hidden choice).** Let us consider a DP-game comprised of the action sets $D = \{0, 1\}$, $A = \{0, 1\}$, and four channels $C_{da}$ shown in Fig. 12 over an input domain $X = \{x_0, x_1\}$ and an output domain $Y = \{y_0, y_1\}$. Assume also that $x_0 \sim x_1$. It is easy to see that the differential-privacy levels of these channels are $\nabla_{\text{DP}}[C_{00}] = \nabla_{\text{DP}}[C_{11}] = \ln \max\{0.90/0.10, 0.90/0.10\} = 2.197$, and $\nabla_{\text{DP}}[C_{01}] = \nabla_{\text{DP}}[C_{10}] = \ln \max\{0.03/0.97, 0.90/0.10\} = 1.099$. Now suppose that the attacker chooses $a = 0$, and that the defender chooses either $d = 0$ or $d = 1$ with probability $1/2$. By Definition 6,

$$\nabla_{\text{DP}}[1/2 C_{00} + 1/2 C_{01}] = \max\{\ln 0.455/0.065, \ln 0.935/0.545\} = 1.946$$

$$1/2 \nabla_{\text{DP}}[C_{00}] + 1/2 \nabla_{\text{DP}}[C_{01}] = 1/2 \cdot 2.197 + 1/2 \cdot 1.099 = 1.648.$$  

Hence we obtain $\nabla_{\text{DP}}[\sum_d 1/2 C_{ad}] = 1.946 > 1.648 = \sum_d 1/2 \nabla_{\text{DP}}[C_{0d}]$, which implies that $\nabla_{\text{DP}}$ is not a convex function w.r.t. the defender’s hidden choice.

### 4.4 Existence of Nash equilibrium for DP-games with hidden choice

We first show that the utility functions of DP-games are quasi-convex w.r.t. hidden choice composition, and quasi-max w.r.t. visible choice (the term quasi-max is just our way of indicating the property expressed in (2) of the following theorem).

**Theorem 4.1 ($\nabla_{\text{DP}}$ is quasi-convex/quasi-max on channel composition).** Let $\{C_i\}_{i \in I}$ be a family of compatible channels such that each $C_i$ is conforming to a symmetric binary relation $\sim$ over their input set, and $\mu$ be a probability distribution on $I$. Then:

1. $\nabla_{\text{DP}}$ is quasi-convex w.r.t. hidden choice:

$$\nabla_{\text{DP}}\left[\mathbb{E}_{i \sim \mu} C_i\right] \leq \max_{i \in \text{supp}(\mu)} \nabla_{\text{DP}}[C_i].$$

2. $\nabla_{\text{DP}}$ is quasi-max w.r.t. visible choice:

$$\nabla_{\text{DP}}\left[\mathbb{E}_{i \sim \mu} C_i\right] = \max_{i \in \text{supp}(\mu)} \nabla_{\text{DP}}[C_i].$$

We are now ready to provide an upper bound on the utility under any strategy in a DP-game with hidden choice.

**Proposition 4.2 (Upper bound on the utility for hidden choice).** A DP-game provides $(\max_{d,a} \nabla_{\text{DP}}(d, a))$-differential privacy for any mixed strategy profile $(\delta, \alpha)$,

$$\nabla_{\text{DP}}\left[\mathbb{E}_{a \sim \alpha} \mathbb{E}_{d \sim \delta} C_{da}\right] \leq \max_{a \in \text{supp}(\alpha)} \nabla_{\text{DP}}[C_{da}].$$

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Now we prove the existence of a Nash equilibrium in DP-games as follows. Interestingly, there is an optimal strategy for the attacker that is independent of the defender’s strategy due to the fact that the attacker’s strategy is visible to himself.

**Theorem 4.3 (Nash-equilibria for DP-games with hidden choice).** For any DP-game with hidden choice, there exists a Nash equilibrium. Moreover, an arbitrary mixed strategy $\alpha^*$ such that $\text{supp}(\alpha^*) = \mathcal{A}$ is an optimal strategy for the attacker.

### 4.5 Computing equilibria for DP-games with hidden choice

In this section, we show how to compute optimal strategies in DP-games with hidden choice of the defender’s action. Recall that by Theorem 4.3, an optimal strategy for the attacker is an arbitrary mixed strategy $\alpha^*$ such that $\text{supp}(\alpha^*) = \mathcal{A}$. To compute an optimal strategy for the defender, we use the following result.

**Proposition 4.4 (Optimal strategy for DP-games with hidden choice).** For any DP-game with hidden choice, an optimal strategy for the defender can be obtained by linear programming.

**Example 9 (Optimal strategy for a DP-game with hidden choice).** We use the DP-game with hidden choice in Example 8 to illustrate the differential-privacy level $\mathcal{V}_{\text{DP}}$ when changing the defender’s mixed strategy $\delta$ in Fig. 13. Let $\alpha^*$ be an arbitrary mixed strategy such that $\text{supp}(\alpha^*) = \mathcal{A}$. Then the level of $\mathcal{V}_{\text{DP}}$ for $\alpha^*$ is the maximum of those for the pure strategies $a = 0, 1$; hence $\alpha^*$ is an optimal strategy for the attacker. The defender’s optimal strategy $\delta^*$ is given by $\delta^*(0) \approx 0.14$ and $\delta^*(1) \approx 0.86$.

### 4.6 Nash equilibria for DP-games with visible choice

Analogously, we obtain the following theorem under a visible choice of the defender’s action.

**Theorem 4.5 (Nash equilibrium for DP-games with visible choice).** For any DP-game with visible choice, there exists a Nash equilibrium. Moreover, an arbitrary mixed strategy $\alpha^*$ such that $\text{supp}(\alpha^*) = \mathcal{A}$ is an optimal strategy for the attacker, and a pure strategy $d^* \in \arg\min_d \max_a (\mathcal{V}_{\text{DP}}[C_{da}])$ is an optimal strategy for the defender.

**Example 10 (Optimal strategy for a DP-game with visible choice).** To illustrate the Nash equilibria for DP-games with visible choice, we again use the DP-game in Example 8. By Definition 6, $\mathcal{V}_{\text{DP}}[C_{00}] \approx 2.20$, $\mathcal{V}_{\text{DP}}[C_{01}] \approx 1.10$, $\mathcal{V}_{\text{DP}}[C_{10}] \approx 1.10$, and $\mathcal{V}_{\text{DP}}[C_{11}] \approx 1.95$. Hence the defender’s optimal strategy is the pure strategy $d^* = \arg\min_a \max_{\alpha} \mathcal{V}_{\text{DP}}[C_{da}] = 1$.

Finally, we compare the DP-games with visible choice with those with hidden choice in terms of $\mathcal{V}_{\text{DP}}$ as follows.

**Proposition 4.6 (Visible choice $\geq$ hidden choice).** The DP-games with hidden choice of the defender’s action do not leak more than those with visible choice, i.e.,

$$\min_{\delta} \max_{\alpha} \mathcal{V}_{\text{DP}} \left[ \left| \frac{\cdot}{\delta} \right| \mathcal{C}_{da} \right] \geq \min_{\delta} \max_{\alpha} \mathcal{V}_{\text{DP}} \left[ \left| \frac{\cdot}{\delta} \right| \mathcal{E} \mathcal{C}_{da} \right].$$

### 5 INFORMATION-LEAKAGE GAMES VS. STANDARD GAME THEORY MODELS

In this section, we elaborate on the differences between our information-leakage games and standard approaches to game theory. In particular, we discuss: (1) why the use of vulnerability as a utility function makes QIF-games and DP-games non-standard w.r.t. the von Neumann-Morgenstern’s treatment of utility; (2) why the use of concave utility functions to model risk-averse players does not capture the behavior of the attacker in QIF-games; and (3) how QIF-games differ from traditional convex-concave games.
5.1 The von Neumann-Morgenstern’s treatment of utility

In their treatment of utility, von Neumann and Morgenstern [65] demonstrated that the utility of a mixed strategy equals the expected utility of the corresponding pure strategies when a set of axioms is satisfied for player’s preferences over probability distributions (a.k.a. lotteries) on payoffs. Since in our information-leakage games the utility of a mixed strategy is not the expected utility of the corresponding pure strategies, it is relevant to identify how exactly our framework fails to meet von Neumann and Morgenstern (vNM) axioms.

Let us first introduce some notation. Given two mixed strategies \( \sigma, \sigma' \) for a player, we write \( \sigma \preceq \sigma' \) (or \( \sigma' \succeq \sigma \)) when the player prefers \( \sigma' \) over \( \sigma \), and \( \sigma \sim \sigma' \) when the player is indifferent between \( \sigma \) and \( \sigma' \). Then, the vNM axioms can be formulated as follows [60]. For every mixed strategies \( \sigma, \sigma' \) and \( \sigma'' \):

- **A1** Completeness: it is either the case that \( \sigma \preceq \sigma' \), \( \sigma \succeq \sigma' \), or \( \sigma \sim \sigma' \).
- **A2** Transitivity: if \( \sigma \preceq \sigma' \) and \( \sigma' \preceq \sigma'' \), then \( \sigma \preceq \sigma'' \).
- **A3** Continuity: if \( \sigma \preceq \sigma' \leq \sigma'' \), then there exist \( p \in [0, 1] \) s.t. \( p \sigma + (1 - p) \sigma'' \sim \sigma' \).
- **A4** Independence: if \( \sigma \preceq \sigma' \) then for any \( \sigma'' \) and \( p \in [0, 1] \) we have \( p \sigma + (1 - p) \sigma'' \preceq p \sigma' + (1 - p) \sigma'' \).

The utility function \( V[\pi, C] \) for QIF-games (for any fixed prior \( \pi \) on secrets), and the utility function \( V^{\text{DP}}[C] \) for DP-games are both total functions on \( C \) ranging over the reals, and therefore satisfy axioms A1, A2 and A3 above. However, neither satisfy A4, as the next example illustrates.

**Example 11.** Consider the following three channel matrices from input set \( X = \{ 0, 1 \} \) to output set \( Y = \{ 0, 1 \} \), where \( \delta \) is a small positive constant.

| \( C_1 \) | \( y = 0 \) | \( y = 1 \) | \( C_2 \) | \( y = 0 \) | \( y = 1 \) | \( C_3 \) | \( y = 0 \) | \( y = 1 \) |
|----------|-----------|-----------|----------|-----------|-----------|----------|-----------|-----------|
| \( x = 0 \) | 1 - 2\( \delta \) | \( 2\delta \) | \( x = 0 \) | 1 - \( \delta \) | \( \delta \) | \( x = 0 \) | \( \delta \) | 1 - \( \delta \) |
| \( x = 1 \) | \( 2\delta \) | 1 - 2\( \delta \) | \( x = 1 \) | \( \delta \) | 1 - \( \delta \) | \( x = 1 \) | 1 - \( \delta \) | \( \delta \) |

In a QIF-game, if we focus on posterior Bayes vulnerability on a uniform prior \( \pi_u \), it is clear that an attacker would prefer \( C_2 \) over \( C_1 \), because

\[
V^{\text{Bayes}}[\pi_u, C_1] = 1 - 2\delta < 1 - \delta = V^{\text{Bayes}}[\pi_u, C_2].
\]

Similarly, in a DP-game the attacker would also prefer \( C_2 \) over \( C_1 \), since

\[
V^{\text{DP}}[C_1] = (1 - 2\delta)/2\delta < (1 - \delta)/\delta = V^{\text{DP}}[C_2].
\]

However, for the probability \( p = 1/2 \) we would have the following hidden composition:

| \( p C_1 + (1 - p) C_3 \) | \( y = 0 \) | \( y = 1 \) | \( p C_2 + (1 - p) C_3 \) | \( y = 0 \) | \( y = 1 \) |
|---------------------------|-----------|-----------|---------------------------|-----------|-----------|
| \( x = 0 \) | \( (1 - \delta)/2 \) | \( (1 + \delta)/2 \) | \( x = 0 \) | \( 1/2 \) | \( 1/2 \) |
| \( x = 1 \) | \( (1 + \delta)/2 \) | \( (1 - \delta)/2 \) | \( x = 1 \) | \( 1/2 \) | \( 1/2 \) |

But notice that the channel \( p C_1 + (1 - p) C_3 \) clearly reveals no less information about the secret than channel \( p C_2 + (1 - p) C_3 \), and the utility of the corresponding QIF-game would be

\[
V^{\text{Bayes}}[\pi_u, p C_1 + (1 - p) C_3] = (1+\delta)/2 > 1/2 = V^{\text{Bayes}}[\pi_u, p C_2 + (1 - p) C_3].
\]

Similarly, for a DP-game we would have

\[
V^{\text{DP}}[p C_1 + (1 - p) C_3] = (1+\delta)/(1-\delta) > 1 = V^{\text{DP}}[p C_2 + (1 - p) C_3].
\]

Hence, in both kinds of games we have \( C_1 \preceq C_2 \), but \( p C_1 + (1 - p) C_3 \preceq p C_2 + (1 - p) C_3 \), and the axiom of independence is not satisfied.
It is actually quite natural that neither vulnerability nor differential privacy satisfies independence: a convex combination of two “informative” channels (i.e., high-utility outcomes) can produce a “non-informative” channel (i.e., a low-utility outcome) whereas we can never obtain a “more informative” channel from a convex combination of “non-informative” channels. As a consequence, the traditional game-theoretic approach to the utility of mixed strategies does not apply to our information-leakage games.

5.2 Risk functions
At first glance, it may seem that our QIF-games could be expressed with some clever use of the concept of a risk-averse player (which, in our case, would be the attacker), which is also based on convex utility functions (cf. [58]). There is, however, a crucial difference: in the models of risk-averse players, the utility function is convex on the payoff of an outcome of the game, but the utility of a mixed strategy is still the expectation of the utilities of the pure strategies, i.e., it is linear on the distributions. On the other hand, the utility of mixed strategies in our QIF-games is convex on the distribution of the defender. This difference arises precisely because in QIF-games utility is defined as the vulnerability of the channel perceived by the attacker, and, as we discussed, this creates an extra layer of uncertainty for the attacker.

5.3 Convex-concave games
Another well-known model from standard game theory is that of convex-concave games, in which each of two players can choose among a continuous set of actions yielding convex utility for one player, and concave for the other. In this kind of game, the Nash equilibria are given by pure strategies for each player.

A natural question would be why not represent our systems as convex-concave in which the pure actions of players are the mixed strategies of our QIF-games. Namely, the real values \( p \) and \( q \) that uniquely determine the defender’s and the attacker’s mixed strategies, respectively, in the two-millionaires game of Section 3.1, could be taken to be the choices of pure strategies in a convex-concave game in which the set of actions for each player is the real interval \([0, 1]\).

This mapping from QIF-games to convex-concave, however, would not be natural. One reason is that utility is still defined as expectation in the standard convex-concave, in contrast to QIF-games. Consider two strategies \( p_1 \) and \( p_2 \) with utilities \( u_1 \) and \( u_2 \), respectively. If we mix them using the coefficient \( q \in [0, 1] \), the resulting strategy \( q p_1 + (1 - q) p_2 \) will have utility \( u = q u_1 + (1 - q) u_2 \) in the standard convex-concave game, while in our case the utility would in general be strictly smaller than \( u \). The second reason is that a pure action corresponding to a mixed strategy may not always be realizable. To illustrate this point, consider again the two-millionaires game, and the defender’s mixed strategy consisting in choosing Program 0 with probability \( p \) and Program 1 with probability \( 1 - p \). The requirement that the defender has a pure action corresponding to \( p \) implies the existence of a program (on Jeeves’ side) that internally makes a probabilistic choice with bias \( p \) and, depending on the outcome, executes Program 0 or Program 1. However, it is not granted that Jeeves disposes of such a program. Furthermore, Don would not know what choice has actually been made, and thus the program would not achieve the same functionality, i.e., let Don know who is the richest. (Note that Jeeves should not communicate to Don the result of the choice, because of the risk that Alice intercepts it.) This latter consideration underlines a key practical aspect of QIF-games, namely, the defender’s advantage over the attacker due to his knowledge of the result of his own random choice (in a mixed strategy). This advantage would be lost in a convex-concave representation of the game since the random choice would be “frozen” in its representation as a pure action.
who can access the cellular network, but do not trust the network provider. The goal is to send a message on the global network (e.g., Onion Routing), the simplicity of Crowds makes it particularly appealing for MANETs. The location for such a node. The answer is far from trivial: on the one side being connected to many users in a 1km×1km area.

![MANET diagram](image)

**Fig. 14.** A MANET with 30 users in a 1km×1km area.

### Table 2. Utility (Bayes vulnerability) for each pure strategy profile (%).

| Attacker’s action | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|------------------|---|---|---|---|---|---|---|---|---|
| Defender's action |   |   |   |   |   |   |   |   |   |
| 1                | 7.38 | 6.88 | 6.45 | 6.23 | 7.92 | 6.45 | 9.32 | 7.11 | 6.45 |
| 2                | 9.47 | 6.12 | 6.39 | 6.29 | 7.93 | 6.45 | 9.32 | 7.11 | 6.45 |
| 3                | 9.50 | 6.84 | 5.46 | 6.29 | 7.94 | 6.45 | 9.32 | 7.11 | 6.45 |
| 4                | 9.44 | 6.92 | 6.45 | 5.60 | 7.73 | 6.45 | 9.03 | 7.11 | 6.45 |
| 5                | 9.48 | 6.91 | 6.45 | 6.09 | 6.90 | 6.13 | 9.32 | 6.92 | 6.44 |
| 6                | 9.50 | 6.92 | 6.45 | 6.29 | 7.61 | 5.67 | 9.32 | 7.11 | 6.24 |
| 7                | 9.50 | 6.92 | 6.45 | 5.97 | 7.94 | 6.45 | 7.84 | 7.10 | 6.45 |
| 8                | 9.50 | 6.92 | 6.45 | 6.29 | 7.75 | 6.45 | 9.32 | 6.24 | 6.45 |
| 9                | 9.50 | 6.92 | 6.45 | 6.29 | 7.92 | 6.24 | 9.32 | 7.11 | 5.68 |

### 6 CASE STUDIES

#### 6.1 The Crowds protocol as a QIF-game

In this section, we apply our game-theoretic analysis to the case of anonymous communication on a mobile ad-hoc network (MANET). In such a network, nodes can move in space and communicate with other nearby nodes. We assume that nodes can also access some global (wide area) network, but such connections are neither anonymous nor trusted. Consider, for instance, smartphone users who can access the cellular network, but do not trust the network provider. The goal is to send a message on the global network without revealing the sender’s identity to the provider. For that, users can form a MANET using some short-range communication method (e.g., Bluetooth), and take advantage of the local network to achieve anonymity on the global one.

Crowds [59] is a protocol for anonymous communication that can be employed on a MANET for this purpose. Note that, although more advanced systems for anonymous communication exist (e.g., Onion Routing), the simplicity of Crowds makes it particularly appealing for MANETs. The protocol works as follows: the *initiator* (i.e., the node who wants to send the message) selects some other node connected to him (with uniform probability) and forwards the request to him. A *forwarder*, upon receiving the message, performs a probabilistic choice: with probability $p_f$ he keeps forwarding the message (again, by selecting uniformly a user among the ones connected to him), while with probability $1 - p_f$ he delivers the message on the global network. Replies, if any, can be routed back to the initiator following the same path in reverse order.

Anonymity comes from the fact that the *detected* node (the last in the path) is most likely not the initiator. Even if the attacker knows the network topology, he can infer that the initiator is most likely a node close to the detected one, but if there are enough nodes we can achieve some reasonable anonymity guarantees. However, the attacker can gain an important advantage by deploying a node himself and participating to the MANET. When a node forwards a message to this *corrupted* node, this action is observed by the attacker and increases the probability of the forwarding node being the initiator. Nevertheless, the node can still claim that he was only forwarding the request for someone else; hence we still provide some level of anonymity. By modeling the system as a channel, and computing its posterior Bayes vulnerability [63], we get the probability that the attacker correctly guesses the identity of the initiator, after performing his observation.

In this section, we study a scenario of 30 nodes deployed in an area of 1km×1km, in the locations illustrated in Fig. 14. Each node can communicate with others up to a distance of 250 meters, forming the network topology shown in the graph. To compromise the anonymity of the system, the attacker plans to deploy a corrupted node in the network; the question is which is the *optimal location* for such a node. The answer is far from trivial: on the one side being connected to many...
nodes is beneficial, but at the same time these nodes need to be “vulnerable”, being close to a highly connected clique might not be optimal. At the same time, the administrator of the network is suspecting that the attacker is about to deploy a corrupted node. Since this action cannot be avoided (the network is ad-hoc), a countermeasure is to deploy a deliverer node at a location that is most vulnerable. Such a node directly delivers all messages forwarded to it on the global network; since it never generates messages, its own anonymity is not an issue; it only improves the anonymity of the other nodes. Moreover, since it never communicates in the local network, its operation is invisible to the attacker. But again, the optimal location for the new deliverer node is not obvious, and most importantly, the choice depends on the choice of the attacker.

To answer these questions, we model the system as a QIF-game where the actions of attacker and defender are the locations of newly deployed corrupted and honest nodes, respectively. We assume that the possible locations for new nodes are the nine ones shown in Fig. 14. For each pure strategy profile \((d, a)\), we construct the corresponding network and use the PRISM model checker to construct the corresponding channel \(C_{da}\) using a model similar to the one of [62]. Note that the computation considers the specific network topology of Fig. 14, which reflects the positions of each node at the time when the attack takes place; the corresponding channels need to be recomputed if the network changes in the future. As a leakage measure, we use the posterior Bayes vulnerability (with uniform prior \(\pi\)), which is the attacker’s probability of correctly guessing the initiator given his observation in the protocol. According to Definition 2, for a mixed strategy profile \((\delta, \alpha)\) the utility is \(V(\delta, \alpha) = E_{\alpha \rightarrow a} V[\pi, C_{\delta a}]\).

The utilities (posterior Bayes vulnerability %) for each pure profile are shown in Table 2. Note that the attacker and defender actions substantially affect the effectiveness of the attack, with the probability of a correct guess ranging between 5.46% and 9.5%. Based on Section 3.3, we can compute the best strategy for the defender, which turns out to be (probabilities expressed as %):

\[
\delta^* = (34.59, 3.48, 3.00, 10.52, 3.32, 2.99, 35.93, 3.19, 2.99)
\]

This strategy is part of an equilibrium and guarantees that for any choice of the attacker the vulnerability is at most 8.76%, and is substantially better than the best pure strategy (location 1) which leads to the worst vulnerability of 9.32%. As expected, \(\delta^*\) selects the most vulnerable locations (1 and 7) with the highest probability. Still, the other locations are selected with non-negligible probability, which is important for maximizing the attacker’s uncertainty about the defense. Finally, note that for this specific \(\delta^*\), the adversary’s best response is the pure strategy 1.

### 6.2 Design of an LDP mechanism using a DP-game

In this section, we illustrate how to use our DP-game framework in the design of a privacy mechanism. We use a real dataset related to the COMPAS (Correctional Offender Management Profiling for Alternative Sanctions) risk assessment instrument developed by Equivant (former Northpointe). The tool provides several scores for individuals, including recidivism risk and violent recidivism risk, calculated using criminal history, jail and prison time, and demographics. The data we use in our case study is taken from the Propublica’s study evaluating COMPAS’ accuracy and bias on data from individuals from Broward County, in the USA, between the years of 2013 and 2014 [13].

In our case study, we consider that a defender is an individual who may participate in an independent study about criminal recidivism. The defender has access to his own COMPAS data, and is willing to disclose to a data curator the following non-sensitive attributes for the purpose of research: his \(\text{ethnicity} (z_1), \text{gender} (z_2), \text{language} (z_3),\) and \(\text{marital status} (z_4)\). On the other hand, he wishes to hide the status of his own \(\text{agency text} \) attribute \(x\), taken from the COMPAS dataset, which is considered sensitive information. This attribute can take one of four values: \(BC\) (Broward County), \(DDRD\) (indicating that the individual is under the Day Report Rentry Division, which helps
the following protocol:

1. The defender obfuscates his own record \((z_1, z_2, z_3, z_4)\) using one of four available privacy mechanisms \(M_1, M_2, M_3, M_4\), and sends it to the data curator. Each mechanism obfuscates the whole record \((z_1, z_2, z_3, z_4)\) at once, producing a new record \((z'_1, z'_2, z'_3, z'_4)\) of randomized values to be reported to the data curator. Hence, the defender’s action \(d\) is taken from the index set \(D = \{1, 2, 3, 4\}\) of the available mechanisms \(M_1, M_2, M_3, M_4\). We denote by \(M_d\) the mechanism that the defender uses.

2. The data curator receives and stores the randomized record \((z'_1, z'_2, z'_3, z'_4)\) from the defender.

3. The attacker (i.e., the data analyst) can ask for the data curator to reveal only one of the four sanitized values \(z'_{1}, z'_{2}, z'_{3}\) or \(z'_{4}\), and we denote the disclosed value by \(y\). The attacker’s action \(a\) is taken from the set \(A = \{1, 2, 3, 4\}\) of the attribute indices of \(z'_{1}, z'_{2}, z'_{3}, z'_{4}\).

We assume that when the defender selects a privacy mechanism, he does not know which attribute \(z_a\) the attacker will request. Symmetrically, the attacker does not know what privacy

| \(\text{Attribute} \) | \(\text{BC} \) | \(\text{DRRD} \) | \(\text{Pretrial} \) | \(\text{Probation} \) |
|----------------------|-------------|---------------|-----------------|-----------------|
| \(\text{African} \)   | 0.5366      | 0.7234        | 0.4956          | 0.3267          |
| \(\text{Caucasian} \) | 0.3171      | 0.1436        | 0.3509          | 0.3799          |
| \(\text{Hispanic} \)  | 0.1463      | 0.0957        | 0.0909          | 0.2590          |
| \(\text{Other} \)     | 0           | 0.0266        | 0.0545          | 0.0176          |
| \(\text{Arabic} \)    | 0           | 0.0053        | 0.0009          | 0.0017          |
| \(\text{Native} \)    | 0           | 0.0053        | 0.0034          | 0.0009          |
| \(\text{Asian} \)     | 0           | 0             | 0.0032          | 0.0039          |
| \(\text{Oriental} \)  | 0           | 0             | 0.0004          | 0.0101          |

(a) The conditional probabilities \(p(z_1 \mid x)\) of ethnicity \(z_1\) given sensitive information \(x\).

| \(\text{Gender} \) | \(\text{Male} \) | \(\text{Female} \) |
|-------------------|---------------|-----------------|
| \(\text{BC} \)    | 0.731707      | 0.268293        |
| \(\text{DRRD} \)  | 0.87234       | 0.12766         |
| \(\text{Pretrial} \) | 0.802482  | 0.197518        |
| \(\text{Probation} \) | 0.732053 | 0.267947        |

(b) The conditional probabilities \(p(z_2 \mid x)\) of gender \(z_2\) given sensitive information \(x\).

| \(\text{Marital Status} \) | \(\text{Single} \) | \(\text{Married} \) | \(\text{Divorced} \) | \(\text{Separated} \) | \(\text{Widowed} \) | \(\text{Unknown} \) |
|-----------------------------|-------------------|------------------|-----------------|----------------|----------------|----------------|
| \(\text{BC} \)              | 0.9268            | 0.0000           | 0.0488          | 0.0244         | 0.0000         | 0.0000         |
| \(\text{DRRD} \)            | 0.9149            | 0.0479           | 0.0106          | 0.0053         | 0.0053         | 0.0160         |
| \(\text{Pretrial} \)        | 0.7664            | 0.1201           | 0.0482          | 0.0258         | 0.0050         | 0.0295         |
| \(\text{Probation} \)       | 0.6820            | 0.1685           | 0.0990          | 0.0387         | 0.0094         | 0.0020         |

(c) The conditional probabilities \(p(z_3 \mid x)\) of language \(z_3\) given sensitive information \(x\).

Fig. 15. Correlation between the non-sensitive attributes \(z_1, z_2, z_3, z_4\) and the sensitive attribute \(x\). In each table, each row corresponds to a distribution on non-sensitive values \(z_i\) given a particular sensitive value \(x\).

reintegrate offenders back into the community following release from jail), pretrial, or probation. However, the defender is aware that there are correlations between the non-sensitive values \(z_1, z_2, z_3, z_4\) and the sensitive value \(x\), as shown in Fig. 15. Therefore, the defender is interested in obfuscating his non-sensitive values before revealing them, in order to mitigate any possible leakage of information about \(x\).

To do that, the defender may employ a local DP mechanism and add noise to the values of \(z_1, z_2, z_3, z_4\) before reporting them to a data curator. On the other hand, we assume that a data analyst –here taking the role of the attacker– is allowed to query the data curator about only one of the four attributes \(z_1, z_2, z_3, z_4\). This assumption is justified by the fact that the combination of all four attributes could reveal too much information about \(x\). We can capture this scenario with the following protocol:

1. The defender obfuscates his own record \((z_1, z_2, z_3, z_4)\) using one of four available privacy mechanisms \(M_1, M_2, M_3, M_4\), and sends it to the data curator. Each mechanism obfuscates the whole record \((z_1, z_2, z_3, z_4)\) at once, producing a new record \((z'_1, z'_2, z'_3, z'_4)\) of randomized values to be reported to the data curator. Hence, the defender’s action \(d\) is taken from the index set \(D = \{1, 2, 3, 4\}\) of the available mechanisms \(M_1, M_2, M_3, M_4\). We denote by \(M_d\) the mechanism that the defender uses.

2. The data curator receives and stores the randomized record \((z'_1, z'_2, z'_3, z'_4)\) from the defender.

3. The attacker (i.e., the data analyst) can ask for the data curator to reveal only one of the four sanitized values \(z'_{1}, z'_{2}, z'_{3}\) or \(z'_{4}\), and we denote the disclosed value by \(y\). The attacker’s action \(a\) is taken from the set \(A = \{1, 2, 3, 4\}\) of the attribute indices of \(z'_{1}, z'_{2}, z'_{3}, z'_{4}\).

We assume that when the defender selects a privacy mechanism, he does not know which attribute \(z_a\) the attacker will request. Symmetrically, the attacker does not know what privacy
mechanism $M_d$ the defender uses. By using our framework of DP-games, we design an optimal privacy mechanism by computing an optimal strategy for the defender.

Let $Z_i$ be the range of the attribute $z_i$’s possible values, and $Z = \prod_{i=1}^{4} Z_i$. To construct the defender’s privacy mechanism, we define $(e, Z_i)$-randomized response $RR^Z_{i,e} : Z_i \rightarrow \mathbb{D}Z_i$ by:

$$RR^Z_{i,e}(y \mid z) = \begin{cases} \frac{e^x}{|Z_i|+e^x-1} & \text{if } y = z \\ \frac{1}{|Z_i|+e^x-1} & \text{if } y \in Z_i \setminus \{z\} \end{cases}$$

Then, the defender’s actions are realized by mechanisms $M_d : Z \rightarrow \mathbb{D}Z$ that apply $(0.1, Z_d)$-randomized response $RR^Z_{0.1}$ to $z_d$ and $(2.0, Z_j)$-randomized response $RR^Z_{2.0}$ to $z_j$ for all $j \neq d$. Note that each mechanism $M_d$ adds significantly more noise to attribute $z_d$ than to all other attributes. For instance, $M_3(z_1, z_2, z_3, z_4) = (RR^Z_{0.1}(z_1), RR^Z_{0.1}(z_2), RR^Z_{0.1}(z_3), RR^Z_{2.0}(z_4))$.

To design an optimal privacy mechanism for the defender, we consider a DP-game. To set the game properly, we need to consider the full mechanisms, namely those that operate on the secrets. Remember that the domain of secrets consists of the four possible values of the agency-text attribute (and the adjacency relation is the usual one for local differential privacy: $x \sim x'$ for all pairs of different values $x$ and $x'$). For each choice $a$ of one of the $z_i$’s attributes, and each choice $d$ of the mechanism $M_d$ that obfuscates the $z_i$’s, the corresponding full mechanism $C_{da}$ is obtained by composing the probabilistic relation from $x$ to $z_a$ with $M_d$. Note that this is similar to the addition of noise to the result of a query in standard differential privacy. The main difference is that a query is deterministic, while in our case the relation between $x$ and $z_a$ is probabilistic. For this reason, the addition of the noise $M_d$ has to be done by multiplying the channel representing the probabilistic relation and $M_d$. This kind of channel composition is called cascade.

More formally: For a pure strategy profile $(d, a) \in \mathcal{D} \times \mathcal{A}$, the channel $C_{da} : X \times Z_a \rightarrow \mathbb{R}$ is the cascade composition of the channels representing the correlation $p(z_i \mid x)$ and the randomized response $p(y \mid z_i)$:

$$C_{da}(x, y) = \begin{cases} \sum_{z \in Z_a} RR^Z_{0.1}(y \mid z) p(z \mid x) & \text{if } d = a \\ \sum_{z \in Z_a} RR^Z_{2.0}(y \mid z) p(z \mid x) & \text{otherwise.} \end{cases}$$

Note that the output ranges over $Z_a$, since the data curator provides the attacker with the data of only one of the four attributes.

For each channel $C_{da}$, the DP level $\mathcal{V}_{\text{DP}}[C_{da}]$ is given by Definition 6. When the defender’s choice is hidden, we compute a solution for the DP-game using the algorithm shown in Proposition 4.4. We obtain the following optimal mixed strategy $\delta^* \in \mathbb{D} \mathcal{D}$ for the defender:

$$\delta^* = (0.5714, 0.0183, 0.0000, 0.4103).$$

Then the defender’s optimal privacy mechanism $M^* : Z \rightarrow \mathbb{D}Z$ is given by $M^* = \mathcal{E}_{d \leftarrow \delta^*}[M_d]$, and its differential privacy level is $\mathcal{V}_{\text{DP}}[M^*] = 0.3892$.

By using $M^*$, each attribute is obfuscated by the following mechanism:

$$p(y \mid z_1) = 0.5714 \cdot RR^Z_{0.1}(y \mid z_1) + (1 - 0.5714) \cdot RR^Z_{2.0}(y \mid z_1)$$

$$p(y \mid z_2) = 0.0183 \cdot RR^Z_{0.1}(y \mid z_2) + (1 - 0.0183) \cdot RR^Z_{2.0}(y \mid z_2)$$

$$p(y \mid z_3) = RR^Z_{0.1}(y \mid z_3)$$

$$p(y \mid z_4) = 0.4103 \cdot RR^Z_{0.1}(y \mid z_4) + (1 - 0.4103) \cdot RR^Z_{2.0}(y \mid z_4).$$

Then the optimal mechanism $M^*$ adds less perturbation to the attributes $z_2$ (gender) and $z_3$ (language). This can be explained by observing $p(z_2 \mid x)$ and $p(z_3 \mid x)$ in Figs. 15b and 15c. Clearly, $z_2$ and $z_3$ leak little information on the secret $x$, so the defender need not add much noise to $z_2$ or $z_3$. 

Note that to construct this optimal mechanism $M^*$, the defender needs to know the correlation $p(z_a \mid x)$ for each $a \in A$. (This is reasonable in our scenario, since the defender has access to the original COMPAS dataset and can compute the correlations from it.) On the other hand, the attacker need not know the correlation $p(z_a \mid x)$ at all, since an arbitrary mixed strategy $\alpha^*$ with $\text{supp}(\alpha^*) = A$ is optimal for the attacker. (Similarly, note that this is reasonable in our scenario, and this lack of knowledge is one of the reasons why the data analyst is interested in collecting data from individuals in the COMPAS dataset in the first place.)

When the defender’s choice is visible, we compute a solution for the DP-game using Theorem 4.5. The pure strategy $d = 4$ is optimal for the defender, while an arbitrary mixed strategy $\alpha^*$ with $\text{supp}(\alpha^*) = A$ is optimal for the attacker. The DP level of the equilibrium is given by $0.5994$.

### 7 Conclusion and Future Work

In this paper, we explored information-leakage games, in which a defender and an attacker have opposing goals in optimizing the amount of information revealed by a system. In particular, we discussed QIF-games, in which utility is information leakage of a channel, and we introduced DP-games, in which utility is the level of differential privacy provided by the channel. In contrast to standard game theory models, in our games the utility of a mixed strategy is either a convex, quasi-convex, or quasi-max function of the distribution of the defender’s actions, rather than the expected value of the utilities of the pure strategies in the support. Nevertheless, the important properties of game theory, notably the existence of a Nash equilibrium, still hold for our zero-sum information-leakage games, and we provided algorithms to compute the corresponding optimal strategies for the attacker and the defender.

As future research, we would like to extend information-leakage games to scenarios with repeated observations, i.e., when the attacker can repeatedly observe the outcomes of the system in successive runs, under the assumption that both the attacker and the defender may change the channel at each run. This would incorporate the sequential and parallel compositions of quantitative information flow [39] in information-leakage games. Furthermore, we would like to consider the possibility of adapting the defender’s strategy to the secret value, as we believe that in some cases this would provide a significant advantage to the defender. We would also like to deal with the cost of attack and of defense, which would lead to non-zero-sum games. Finally, we are also interested in incorporating our information leakage games in signaling games [26], which model asymmetric information among players and are applied to economics and behavioral biology.

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A PROOFS

This appendix contains the proof of the formal results of this paper.

Proof for Corollary 3.2. Given a mixed strategy profile \((\delta, \alpha)\), the utility \(V(\delta, \alpha)\) given in Definition 2 is affine (hence concave) on \(\alpha\). Furthermore, by Theorem 3.1, \(V(\delta, \alpha)\) is convex on \(\delta\). Hence we can apply the von Neumann’s minimax theorem (Section 2.3), which ensures the existence of a saddle point, i.e., a Nash equilibrium. □

Proof for Proposition 3.3. It is sufficient to show that for all mixed strategy \(\delta\) for the defender:

\[
\max_\alpha V(\delta, \alpha) = \max_a V[\pi, E_{d \sim \delta} C_{da}].
\]

Let \(\alpha\) be an arbitrary mixed strategy for the attacker. Then we have that:

\[
V(\delta, \alpha) = \sum_a \alpha(a) V[\pi, E_{d \sim \delta} C_{da}]
\]

\[
\leq \sum_a \alpha(a) \left( \max_a V[\pi, E_{d \sim \delta} C_{da}] \right) 
\]

\[
= \left( \max_a V[\pi, E_{d \sim \delta} C_{da}] \right) \left( \sum_a \alpha(a) \right)
\]

\[
= \max_a V[\pi, E_{d \sim \delta} C_{da}].
\]

This holds for any \(\alpha\), hence \(\max_\alpha V(\delta, \alpha) \leq \max_a V[\pi, E_{d \sim \delta} C_{da}]\); the \((\geq)\) case is trivial since we can take \(\alpha\) to be the point distribution on any \(a\). □

Proof for Theorem 3.4. Our iterative method for computing the minimum value \(f^* = \min_\delta f(\delta)\) is an application of the subgradient descent method, that can be applied to any convex function.

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We start by showing that \( f(\delta) \) is convex. Let \( \delta_1, \delta_2 \in \mathbb{D} \), and \( c \in [0, 1] \) be a convex coefficient.

\[
f(c\delta_1 + (1-c)\delta_2) = \max \nabla [\pi, \sum_d (c\delta_1(d) + (1-c)\delta_2(d)) C_{da}]
\leq \max \nabla [\pi, \sum_d \delta_1(d) C_{da}] + (1-c) \max \nabla [\pi, \sum_d \delta_2(d) C_{da}] \\
= c f(\delta_1) + (1-c)f(\delta_2) .
\]

(by (3))

The proof of convergence of the subgradient decent method applied to \( f \) is provided by [19, Sections 3 and 5]. We only need to ensure that the two crucial conditions of this proof, stated in [19, Section 3.1], are met.

First, the proof requires that the norm of all subgradients \( \|h^{(k)}\|_2 \) is bounded by some constant \( G \). For this, it is sufficient to show that \( f(\delta) \) is Lipschitz; the latter follows from the fact that \( \nabla (\pi) \) itself is assumed to be Lipschitz.

Second, the proof requires that there is a known number \( R \) bounding the distance between the initial point \( \delta^{(1)} \) and the optimal one, namely \( \|\delta^{(1)} - \delta^*\|_2 \leq R \). Since we start from the uniform distribution \( u_D \), we take \( R \) to be the distance between \( \delta^{(1)} \) and the point in the simplex maximally distant from it, which is a point distribution \( \delta^{\text{point}} = (1, 0, \ldots) \). That is, for \( n = |X| \), we take:

\[
R = \|\delta^{(1)} - \delta^{\text{point}}\|_2 = \sqrt{\left(1 - 1/n\right)^2 + (0 - 1/n)^2 (n-1)} = \sqrt{(n-1)/n} .
\]

Note also that the “diminishing step size” \( s_k \) that we use is among those known to guarantee convergence [19, Sections 3 and 5].

The fact that \( l^{(k)} \) is a lower bound on \( f^* \) comes from [19, Section 3.4], using the \( R \) above. As a consequence, the stopping criterion guarantees that

\[
f(\hat{\delta}) - f(\delta^*) \leq \epsilon .
\]

(by (7))

Let \( \alpha \) be an arbitrary mixed strategy for the attacker. From Proposition 3.3 and the above:

\[
\nabla (\hat{\delta}, \alpha) - \epsilon \leq f(\hat{\delta}) - \epsilon \quad \text{(by def. of } f \text{ in (3)})
\leq f(\delta^*) \quad \text{(by (7))}
= \nabla (\delta^*, \alpha^*) \quad \text{(by (3))}
\leq \nabla (\hat{\delta}, \alpha^*) .
\]

(since \( \delta^*, \alpha^* \) is a saddle point)

Similarly for \( \nabla (\hat{\delta}, \alpha^*) \leq \nabla (\delta, \alpha^*) + \epsilon \), which concludes the proof. \( \square \)

**Proof for Theorem 4.1.** (1) Since each \( C_i \) is conforming to \( \sim \), so is \( \sum_i \mu(i) C_i \). Then quasi-convexity of \( \nabla^{DP} \) w.r.t. hidden choice is derived as follows.

\[
\nabla^{DP} \left[ \sum_{i=\mu} C_i \right] = \max_{y, x', x : x' \sim x} \frac{\ln \sum_i \mu(i) C_i(x, y)}{\sum_i \mu(i) C_i(x', y)} \quad \text{(by def. of } \nabla^{DP} \text{)}
\leq \max_{y, x', x : x' \sim x} \frac{\mu(i) C_i(x, y)}{\mu(i) C_i(x', y)} \quad \text{(\ast)}
= \max_{i \in \supp(\mu)} \frac{\ln C_i(x, y)}{C_i(x', y)} \quad \text{(\ast\ast)}
= \max_{i \in \supp(\mu)} \nabla^{DP} [C_i] \quad \text{(by def. of } \nabla^{DP} \text{)}
\]

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To complete the proof, we derive the inequality \(^{12}\). Let \( y \in \mathcal{Y} \) and \( x, x' \in \mathcal{X} \) such that \( x \sim x' \) and \( \sum_i \mu(i) C_i(x, y) > 0 \). Then there is an \( i \in \text{supp}(\mu) \) such that \( C_i(x, y) > 0 \). Since \( C \) is conforming to \( \sim \) and \( x \sim x' \), it holds for any \( j \) that \( C_j(x, y) > 0 \) if \( C_j(x', y) > 0 \). Hence, for any \( j \in \text{supp}(\mu) \),

\[
\mu(j) C_j(x, y) \leq \mu(j) C_j(x', y) \cdot \max_{i \in \text{supp}(\mu), C_i(x, y) > 0} \frac{\mu(i) C_i(x, y)}{\mu(i) C_i(x', y)}.
\]

Therefore, \( (*) \) follows from:

\[
\sum_j \mu(j) C_j(x, y) \leq \max_{i \in \text{supp}(\mu), C_i(x, y) > 0} \frac{\mu(i) C_i(x, y)}{\mu(i) C_i(x', y)} \cdot \sum_j \mu(j) C_j(x', y).
\]

(2) Let \( C = \{ x \}_{i=\mu} C_i \). Since each \( C_i \) is conforming to \( \sim \), so is \( C \). Then the quasi-max property of \( \mathbb{V}^{\text{DP}} \) w.r.t. visible choice is derived as follows.

\[
\mathbb{V}^{\text{DP}} \left[ \frac{\left| \right|}{\mu} C_i \right] = \max_{i, y, x, x': x \sim x', C(x, (y, i)), \neq 0, i \in \text{supp}(\mu)} \ln \frac{C(x, (y, i))}{C(x', (y, i))} \quad \text{(by def. of \( \mathbb{V}^{\text{DP}} \) and vis. choice)}
\]

\[
= \max_{i, y, x, x': x \sim x', C(x, (y, i)), \neq 0, i \in \text{supp}(\mu)} \ln \frac{\mu(i) C_i(x, y)}{\mu(i) C_i(x', y)} \quad \text{(by def. of visible choice)}
\]

\[
= \max_{i \in \text{supp}(\mu)} \mathbb{V}^{\text{DP}}[C_i] \quad \text{(by def. of \( \mathbb{V}^{\text{DP}} \))}
\]

\[\square\]

**Proof for Proposition 4.2.** The proof is based on the quasi-convexity and quasi-max of \( \mathbb{V}^{\text{DP}} \).

\[
\mathbb{V}^{\text{DP}} \left[ \frac{\left| \right|}{a_\alpha} C_{\delta a} \right] = \max_{a \in \text{supp}(\alpha)} \mathbb{V}^{\text{DP}} [C_{\delta a}] \leq \max_{a \in \text{supp}(\alpha)} \max_{d \in \text{supp}(\delta)} \mathbb{V}^{\text{DP}} [C_{da}] .
\]

(by Theorem 4.1 (2))

(by Theorem 4.1 (1))

\[\square\]

**Proof for Theorem 4.3.** By Theorem 4.1 (2), it holds for any mixed strategy profile \( (\delta, \alpha) \) that:

\[
\mathbb{V}^{\text{DP}} (\delta, \alpha) \overset{\text{def}}{=} \mathbb{V}^{\text{DP}} \left[ \frac{\left| \right|}{a_\alpha} C_{\delta a} \right] = \max_{a \in \text{supp}(\alpha)} \mathbb{V}^{\text{DP}} [C_{\delta a}] \leq \max_{a \in \mathcal{A}} \mathbb{V}^{\text{DP}} [C_{\delta a}] .
\]

Hence for any \( \delta \), an arbitrary mixed strategy \( \alpha^* \) s.t. \( \text{supp}(\alpha^*) = \mathcal{A} \) maximizes \( \mathbb{V}^{\text{DP}} (\delta, \alpha) \). Therefore:

\[
\min_{\delta} \max_{\alpha} \mathbb{V}^{\text{DP}} (\delta, \alpha) = \min_{\delta} \mathbb{V}^{\text{DP}} (\delta, \alpha^*) = \max_{\alpha} \min_{\delta} \mathbb{V}^{\text{DP}} (\delta, \alpha),
\]

i.e., there exists a Nash equilibrium. \[\square\]

**Proof for Proposition 4.4.** Let \( \alpha^* \) be an arbitrary mixed strategy such that \( \text{supp}(\alpha^*) = \mathcal{A} \). By Theorem 4.3,

\[
\min_{\delta} \max_{\alpha} \mathbb{V}^{\text{DP}} (\delta, \alpha) = \max_{\alpha} \min_{\delta} \mathbb{V}^{\text{DP}} (\delta, \alpha)
\]

\[
= \min_{\delta} \mathbb{V}^{\text{DP}} (\delta, \alpha^*)
\]

\[
= \min_{\delta} \max_{a \in \mathcal{A}} \mathbb{V}^{\text{DP}} (\delta, a)
\]

\[
= \min_{\delta} \max_{a \in \mathcal{A}} \max_{x, x', y} \sum_d \delta(d) C_{da}(x, y)
\]

\[
= \sum_d \delta(d) C_{\delta a}(x, y).
\]

\[^{12}\text{In [41, 42], a similar inequality is used to prove their main theorems.}\]
We can solve this program using the Dinkelbach-type algorithm [14] as follows. Define

\[
\delta_k = \max_{\delta \in \mathcal{D}} f_j(\delta),
\]

where \( J = \mathcal{A} \times \mathcal{X} \times \mathcal{X} \times \mathcal{Y} \),

\[
f_j(\delta) = \sum_d \delta(d) C_{da}(x, y),
\]

\[
g_j(\delta) = \sum_d \delta(d) C_{da}(x', y), \quad j = (a, x, x', y) \in J.
\]

We can solve this program using the Dinkelbach-type algorithm [14] as follows. Define

\[
\lambda_k = \max_{\delta \in \mathcal{D}} f_j(\delta),
\]

\[
F_k(\delta) = \max_{\delta \in \mathcal{D}} \left[ f_j(\delta) - \lambda_k g_j(\delta) \right],
\]

\[
\delta_k = \arg \min_{\delta \in \mathcal{D}} F_k(\delta).
\]

We start from a uniform \( \delta_0 \), and iterate using the above formulas, until \( F_k(\delta_k) = 0 \) for some \( k \geq 1 \), in which case \( \delta_k \) is guaranteed to be the optimal solution with value \( \lambda_k \). Note that for each iteration \( k \geq 1 \), \( \lambda_k \) is constant (computed from \( \delta_{k-1} \)). Moreover, the optimization problem for \( \delta_k \), which we need to solve in every iteration, requires to minimize the max of linear functions, and can be transformed into the following linear program:

\[
\begin{align*}
\text{variables} \; & \delta(d), z \\
\text{minimize} \; & z \\
\text{s.t.} \; & 1 = \sum_d \delta(d) \quad \text{and} \quad z \geq f_j(\delta) - \lambda_k g_j(\delta) \quad \forall j \in J,
\end{align*}
\]

since \( f_j(\delta), g_j(\delta) \) are linear on \( \delta \). Therefore, an optimal strategy for the DP-game is obtained by solving a sequence of linear programs.

\[\square\]

**Proof for Theorem 4.5.** Similarly to Theorem 4.3, an arbitrary mixed strategy \( \alpha^* \) such that \( \text{supp}(\alpha^*) = \mathcal{A} \) maximizes the attacker’s payoff independently of \( \delta \). Hence, for any \( \delta \) and \( \alpha \),

\[\mathbb{V}^{\text{DP}}(\delta, \alpha) = \max_{\delta \in \text{supp}(\delta)} \max_{a \in \mathcal{A}} \mathbb{V}^{\text{DP}}[C_{da}].\]

This is minimized for a point distribution \( \delta^* \) s.t. \( \delta^*(d^*) = 1 \) for \( d^* \in \arg \min_{d} \max_{a} \mathbb{V}^{\text{DP}}[C_{da}] \). \[\square\]

**Proof for Proposition 4.6.** For any \( a \in \mathcal{A} \) and any \( \delta \in \mathcal{D} \) we have:

\[
\mathbb{V}^{\text{DP}}[\frac{1}{d-\delta} C_{da}] = \max_{\delta \in \text{supp}(\delta)} \mathbb{V}^{\text{DP}}[C_{da}] \tag{by Theorem 4.1 (2)}
\]

\[
\geq \mathbb{V}^{\text{DP}}[\mathbb{E}_{d-\delta} C_{da}]. \tag{by Theorem 4.1 (1)} \tag{8}
\]

Then for any \( \alpha \in \mathcal{D}, \delta \in \mathcal{D}, \) and any \( a^+ \in \text{argmax}_{a \in \text{supp}(a)} \mathbb{V}^{\text{DP}}[\mathbb{E}_{d-\delta} C_{da}] \), we obtain:

\[
\mathbb{V}^{\text{DP}}[\frac{1}{d-\delta} C_{da}] = \max_{a \in \text{supp}(a)} \mathbb{V}^{\text{DP}}[\frac{1}{d-\delta} C_{da}] \tag{by Theorem 4.1 (2)}
\]

\[
\geq \mathbb{V}^{\text{DP}}[\frac{1}{d-\delta} C_{da^+}] \tag{by (8)}
\]

\[
\geq \mathbb{V}^{\text{DP}}[\mathbb{E}_{d-\delta} C_{da}] \tag{by def. of \( a^+ \))
\]

\[
= \max_{a \in \text{supp}(a)} \mathbb{V}^{\text{DP}}[\mathbb{E}_{d-\delta} C_{da}] \tag{by Theorem 4.1 (2)}
\]

Therefore the theorem follows immediately. \[\square\]
B A BAYESIAN INTERPRETATION OF DIFFERENTIAL PRIVACY

Differential privacy has been interpreted in terms of Bayesian inference [3, 15, 22, 35, 48]. Here we present two alternative versions of this view.

In the following, we assume that a mechanism (or channel) $C$ satisfies $\epsilon$-differential privacy as in Definition 1. The mechanism has an input set $\mathcal{X}$, an output set $\mathcal{Y}$, and is defined w.r.t. an adjacency relation $\sim$ on $\mathcal{X}$. Recall from Section 2.4 that given a prior distribution $\pi$ on inputs to the mechanism, we can derive a joint distribution $p_{X,Y}$ on $\mathcal{X} \times \mathcal{Y}$ in the usual way, and from that marginal distributions $p_X$ and $p_Y$, as well as conditional distributions $p_{X|Y}$ and $p_{Y|X}$ for particular non-zero values of $y \in \mathcal{Y}$ and $x \in \mathcal{X}$. When clear from the context, we may omit subscripts on probability distributions, writing, e.g., $p(y)$, $p(x,y)$, and $p(x \mid y)$ for $p_Y(y)$, $p_{X,Y}(x,y)$, and $p_{X|Y}(x \mid y)$, respectively.

Using this notation, the $\epsilon$-differential privacy property can be rewritten as:

$$e^{-\epsilon} \leq \frac{p(y \mid x)}{p(y \mid x')} \leq e^\epsilon, \quad \text{for all } y \in \mathcal{Y}, x, x' \in \mathcal{X} \text{ s.t. } x \sim x'. \quad (9)$$

B.1 Interpretation of differential privacy in terms of hypothesis testing

A first interpretation presents differential privacy in terms of a bound on the adversary’s success in performing hypothesis testing on the secret value [22, 48]. More precisely, it notes that differential privacy ensures that the adversary’s a priori capability of distinguishing between two adjacent secrets (e.g., two neighboring datasets differing only by the presence/absence of a single individual) does not change significantly after an observation of the mechanism’s output (e.g., the answer to a query in that dataset). This guarantee (which, of course, depends on the value of the parameter $\epsilon$) holds for every prior —and that is the reason DP is said to be “independent from the adversary’s knowledge.”

Formally, the guarantee that a mechanism satisfies

$$\frac{p(y \mid x)}{p(y \mid x')} \leq e^\epsilon, \quad \text{for all } y \in \mathcal{Y}, x, x' \in \mathcal{X} \text{ s.t. } x \sim x', \quad (10)$$

where $\sim$ is a symmetric adjacency relation, is equivalent to

$$\frac{p(x \mid y)}{p(x' \mid y)} \leq e^\epsilon \frac{p(x)}{p(x')}, \quad \text{for all } y \in \mathcal{Y}, x, x' \in \mathcal{X} \text{ s.t. } x \sim x', \text{ and all priors } p(\cdot) \text{ on } \mathcal{X}. \quad (11)$$

Note that in Equation (11), the term $p(x)/p(x')$ represents the adversary’s capability of distinguishing between adjacent secrets $x$ and $x'$ a priori (i.e., before the output of the mechanism is observed), whereas the term $p(x|y)/p(x'|y)$ represents his capability of distinguishing between the same adjacent secrets after an output $y$ of the mechanism is observed. Because of the universal quantification on all adjacent secret inputs $x, x'$, outputs $y$, and prior distribution $p(\cdot)$ on inputs, Equation (11) exactly states differential privacy’s guarantees against maximum distinguishability of two secrets.

The proof of the equivalence of Equations (10) and (11) follows immediately by the Bayes Theorem, which we recall here:

\[ \text{Bayes Theorem} \quad p(x \mid y) = \frac{p(y \mid x) p(x)}{p(y)}. \]

Hence, we have:

$$\frac{p(x \mid y)}{p(x' \mid y)} = \frac{p(y \mid x) p(x)}{p(y)} \frac{p(y)}{(y \mid x') p(x')} = \frac{p(y \mid x)}{p(y \mid x')} \frac{p(x)}{p(x')} \leq e^\epsilon \frac{p(x)}{p(x')}.$$
B.2 Interpretation of differential privacy in terms of increase in information

A second interpretation presents differential privacy in terms of a bound on the adversary’s increase in information about the secret value [3, 15, 35]. In this case, however, the exact formulation depends on the nature of the adjacency relation \( \sim \). We present here the examples of local differential privacy [38], and of the standard differential privacy on datasets [32, 35].

B.2.1 Local differential privacy. In local differential privacy, the relation \( \sim \) holds between every two secrets values \( x \) and \( x' \). In this case, the Bayesian formulation is expressed by the following bound, for some real number \( \alpha \geq 0 \):

\[
e^{-\alpha} \leq \frac{p(x | y)}{p(x)} \leq e^{\alpha} \quad \text{for all } y \in \mathcal{Y}, x \in \mathcal{X}, \text{ and all priors } p(\cdot) \text{ on } \mathcal{X}.
\]

(12)

Essentially, inequality (12) means that the prior and posterior probabilities of the secret taking a particular value do not differ too much. We now state precisely and prove the correspondence between DP and this Bayesian formulation.

**Theorem 12.** Given a certain mechanism \( C \):

1. If \( C \) satisfies inequality (9), then it satisfies inequality (12) with \( \alpha = \epsilon \).
2. If \( C \) satisfies inequality (12), then it satisfies inequality (9) with \( \epsilon = 2 \alpha \).

**Proof.**

(1) Assume that (9) is satisfied. Then:

\[
p(x | y) p(y) = p(y | x) p(x) = \frac{p(y | x)}{\sum_{z \in \mathcal{X}} p(y, z)} \quad \text{(by the Bayes theorem)}
\]

\[
= \frac{p(y | x) p(y | z) p(z)}{\sum_{z \in \mathcal{X}} p(y | z) p(z)} \quad \text{(marginals)}
\]

\[
\leq \frac{e^{-\epsilon} p(y | x) p(z)}{\sum_{z \in \mathcal{X}} e^{-\epsilon} p(y | x) p(z)} \quad \text{(by (9))}
\]

\[
= e^\epsilon
\]

(2) Assume that (12) is satisfied. Then:

\[
p(y | x) = p(y | x') p(x') p(y) = \frac{p(x | y) p(y)}{p(x)} \quad \text{(by the Bayes theorem)}
\]

\[
= \frac{p(x | y) p(y')}{p(x')} \frac{p(x')}{p(x | y)}
\]

\[
\leq e^\alpha \frac{p(x')}{p(x | y)} \quad \text{(by (12))}
\]

\[
= e^{2 \alpha}
\]
B.2.2 Standard differential privacy on datasets. Standard differential privacy is also called, nowadays, differential privacy in the central model. In this case, the relation $\sim$ holds between every two datasets that differ for the presence or absence of a single record. We will indicate by $x$ the presence of the target record and by $\neg x$ its absence. We will indicate by $s$ the rest of the dataset. By using this notation, the differential privacy property can be rewritten as:

$$e^{-\varepsilon} \leq \frac{P(y \mid x, s)}{P(y \mid \neg x, s)} \leq e^\varepsilon \quad \text{for all } y \in \mathcal{Y}, x \in \mathcal{X}, \text{ and datasets } s.$$  \hfill (13)

The Bayesian formulation is expressed by the following bounds, for some real number $\alpha \geq 0$:

$$e^{-\alpha} \leq \frac{p(x \mid y, s)}{p(x \mid s)} \leq e^\alpha \quad \text{for all } y \in \mathcal{Y}, x \in \mathcal{X}, \text{ datasets } s, \text{ and all priors } p(\cdot) \text{ on } \mathcal{X}.$$  \hfill (14)

and

$$e^{-\alpha} \leq \frac{p(\neg x \mid y, s)}{p(\neg x \mid s)} \leq e^\alpha \quad \text{for all } y \in \mathcal{Y}, x \in \mathcal{X}, \text{ datasets } s, \text{ and all priors } p(\cdot) \text{ on } \mathcal{X}.$$  \hfill (15)

This inequality (14) means that the prior and posterior probabilities of the secret taking a particular value do not differ too much. We now state precisely and prove the correspondence between DP and this Bayesian formulation.

**Theorem 13.** Given a certain mechanism $C$:

1. If $C$ satisfies inequality (13), then it satisfies inequalities (14) and (15) with $\alpha = \varepsilon$.
2. If $C$ satisfies inequalities (14) and (15), then it satisfies inequality (13) with $\varepsilon = 2\alpha$.

The proof is analogous to that of Theorem 12; we just need to replace $p(x), p(y), p(x \mid y)$ and $p(y \mid x)$ by $p(x \mid s), p(y \mid s), p(x \mid y, s)$ and $p(y \mid x, s)$ respectively, and similarly for $\neg x$. Also, we use the fact that $x$ and $\neg x$ are complementary events, i.e., $p(x) + p(\neg x) = 1$, and the same holds for the conditional probabilities on $x$ and $\neg x$. 

□