We present the results of three dimensional post-Newtonian calculation of coalescing binary neutron stars including gravitational radiation reaction. We use the finite different method (FDM) and the hydrodynamics equations are integrated using van Leer's scheme with the total variation diminishing (TVD) condition. The calculation is started from a binary system consisting of two simple spherical polytropes with $\gamma=2$. We perform also a calculation from the same initial condition but neglecting the post-Newtonian terms and compare it with the post-Newtonian calculation in order to estimate effects of general relativity. We find that shock wave is induced by the first centrifugal bounce in the post-Newtonian calculation while it is not in the Newtonian one. It is because general relativity effectively increases the gravitational force. The shock wave makes the matter reexpand and the contraction-expansion oscillation lasts longer in the post-Newtonian case. As a result the total energy of the gravitational radiation increases. Although the evolution of the density distribution looks so different, the general relativistic effect on the waveform is small.

§ 1. Introduction

In previous papers1)-4) (hereinafter referred to as Papers I-IV), we reported the results of Newtonian calculations of a coalescing neutron-star binary. In these calculations we used Newtonian hydrodynamics equations which include gravitational radiation reaction. The emission of gravitational radiation was followed using the quadrupole approximation. As a result, we found that the energy of the gravitational radiation can be approximately 3% of the total rest mass and the maximal amplitude is as much as $h \sim 5 \times 10^{-21} (d/10\text{Mpc})$, where $d$ is the distance between the observer and the place of coalescence. These results mean that coalescence of a neutron-star binary is a promising source of gravitational radiation for laser-interferometric gravitational-wave detectors.

In Papers III and IV, calculations started from hydrostatic equilibrium models of close neutron-star binaries in synchronized circular orbits. Rasio and Shapiro found that these hydrostatic equilibrium models become dynamically unstable slightly before the surfaces of the two stars come into contact.5) They showed the final merging of two stars even if the radiation damping is not considered, using smoothed particle hydrodynamics (SPH). However, although the dynamical instability makes the onset of coalescence of two star earlier, it must produce very little effect on the overall evolution of the binary and the emission of gravitational radiation, if the radiation damping is considered. We also performed a similar calculation to that of Rasio and Shapiro using our code without the radiation reaction and obtained essentially the same results as they did.6) If the radiation reaction is included, the results are completely different from those without the radiation reaction as shown in
Papers III and IV.

As pointed out in Papers II~IV, coalescence of a neutron-star binary of total mass \( \leq 3M_\odot \) is likely to lead to gravitational collapse to a black hole. Then effects of general relativity must be important. However we neglected them except for emission of gravitational radiation and radiation reaction. Indeed \( (v/c)^2 \sim GM/c^2R \leq 0.25 \) at the beginning of coalescence and general relativistic effects are rather small when gravitational radiation is explosively emitted, but the effects may not be ignored when the gravity becomes so strong that gravitational collapse to a black hole may occur. Evolution of the binary and characteristics of emitted gravitational radiation must change if we solve fully general relativistic hydrodynamics equations. Unfortunately, however, a three dimensional general relativistic code has not yet been implemented. On the other hand, Blanchet, Damour and Schäfer presented post-Newtonian hydrodynamics equations suitable for numerical calculations.\(^7\) In their formalism, the equations of motion include general relativistic effects of order \( (v/c)^2 \), the first post-Newtonian corrections, and gravitational damping effects of order \( (v/c)^3 \), which are second-and-a-half post-Newtonian terms. Now general relativistic effects might be estimated comparing the results of calculations using a post-Newtonian hydrodynamics code with those using a Newtonian code.

In this paper, we report the results of a calculation of coalescence of two neutron stars using \((1+2.5)\)-post-Newtonian hydrodynamics equations. We also follow the evolution from the same initial condition neglecting first post-Newtonian terms in hydrodynamics equations in order to estimate general relativistic effects. The basic equations are described in \( \S \) 2. For numerical calculation we use the finite difference method (FDM) with a uniform Cartesian grid. The numerical results are presented in \( \S \) 3. Section 4 is devoted to discussion.

\( \S \) 2. \((1+2.5)\) post-Newtonian hydrodynamics equations

The equations of motion including general relativistic effects up to order \( (v/c)^2 \) are called the first post-Newtonian (1PN) approximation. However, gravitational damping effects start at order \( (v/c)^3 \), second-and-a-half post-Newtonian (2.5PN) terms, and the formulation for adding the effects to the equation of motion depends on the gauge condition. For example, the radiation reaction is represented by adding to the standard Newtonian potential a “radiation-reaction” potential proportional to the fifth time derivatives of the quadrupole moments.\(^8\) In numerical calculations, however, it is rather hard to calculate the fifth time derivatives with satisfactory accuracy. Blanchet, Damour and Schäfer have presented \((1+2.5)\)PN hydrodynamics equations including radiation damping effects,\(^7\) where we need up to the third time derivatives of the quadrupole moments. Following them, we use the evolution equations given by

\[
\begin{align*}
\partial_t \rho + \partial_i (\rho v^i) &= 0, \tag{2.1} \\
\partial_t (\rho v^i) + \partial_i (\rho v^j v^j) &= F_i^{\text{press}} + F_i^{\text{1PN}} + F_i^{\text{ren}}, \tag{2.2} \\
\partial_t (\rho \varepsilon) + \partial_i (\rho \varepsilon v^i) &= -\rho \partial_t v^i, \tag{2.3}
\end{align*}
\]

\(^7\) Blanchet, Damour, and Schäfer.

\(^8\) Blanchet, Damour, and Schäfer.
where \( \rho, \varepsilon \) and \( p \) are the coordinate rest-mass density, the internal energy density and the pressure, respectively. In order to express a hard equation of state for neutron star, we use a polytropic equation of motion

\[
p = (\gamma - 1) \rho \varepsilon
\]

with \( \gamma = 2 \). The quantity \( u^i \) denotes the 3-velocity and \( w_i \) is the "momentum per unit rest-mass." In terms of \( w_i \), the 3-velocity \( v_i \) is given by

\[
v^i = \left( 1 - \frac{\beta}{c^2} \right) w_i + \frac{1}{c^2} A_i + \frac{4G}{5c^5} w_j Q^{[j]}_i.
\]

The forces are given by

\[
F^{\text{press}}_i = -\partial_i \left[ \left( 1 + \frac{\alpha}{c^2} \right) \rho \right],
\]

\[
F^{\text{PN}}_i = -\rho \left[ \left( 1 + \frac{\delta}{c^2} \right) \partial_i \phi + \frac{1}{c^2} \partial_i \phi_2 + \frac{1}{c^2} w_j \partial_i A_j \right],
\]

\[
F^{\text{rec}}_i = -\frac{1}{c^2} \rho \partial_i \phi_r.
\]

Here \( \phi \) is the Newtonian potential \( \alpha, \beta, \delta, \phi_2 \) and \( A_i \) are 1PN quantities and \( \phi_r \) is the radiation-reaction potential (a 2.5PN quantity):

\[
\alpha = (3\gamma - 2) \phi - \frac{1}{2} \gamma w^2,
\]

\[
\beta = \frac{1}{2} w^2 + \gamma \varepsilon - 3 \phi,
\]

\[
\delta = \frac{3}{2} w^2 + (3\gamma - 2) \varepsilon + \phi,
\]

\[
A_i = U_i - \frac{1}{2} x^i \phi_t,
\]

\[
\phi_r = \frac{2}{5} G [R - Q^{[j]}_i x^j \partial_i \phi],
\]

where \( w^2 = \delta^i w_i w_j \) and \( \phi_t \) is the time derivative of \( \phi \) with the addition of corrections of order \( O(c^{-5}) \). To compute \( \phi, \phi_t, \phi_2, R \) and \( U_i \), we must solve seven Poisson equations at each time step:

\[
\Delta \phi = 4\pi G \rho,
\]

\[
\Delta \phi_t = -4\pi G \partial_i (\rho w_i),
\]

\[
\Delta \phi_2 = 4\pi G \rho \delta,
\]

\[
\Delta R = 4\pi G Q^{[i]}_j x^j \partial_i \rho,
\]

\[
\Delta U_i = -4\pi G \left( 4\rho w_i + \frac{1}{2} x^j \partial_j (\rho w_i) \right).
\]
The quantity $Q_{ij}^{\text{J}}$ is given by

$$Q_{ij}^{\text{J}} = \text{STF}\left\{ 2\int\left[ \rho \partial_i \phi_j - 2\rho w^i \partial_j \phi + x^i \partial_j \phi \partial_k (\rho w^k) - \rho x^i \partial_j \phi \right] dV \right\}, \quad (2.13)$$

which differs from the third time derivative of the reduced quadrupole moment of the system by corrections of order $O(c^{-2})$. Here $\text{STF}$ is the operator which takes the symmetric, trace-free part of any two-index object $A^{ij}$:

$$\text{STF}(A^{ij}) = \frac{1}{2} A^{ij} + \frac{1}{2} A^{ji} - \frac{1}{3} \delta^{ij} A^{kk}. \quad (2.14)$$

The energy flux of the gravitational waves is given by

$$\frac{dE}{dt} = -\frac{G}{5c^5} Q_{ij}^{\text{J}} \frac{d}{dt} I_{ij}, \quad (2.15)$$

where

$$I_{ij} = \text{STF}\left\{ 2\int\rho [w_i w_j - x^i \partial_j \phi] dV \right\}, \quad (2.16)$$

which is the second time derivative of the reduced quadrupole moment with the addition of corrections of order $O(c^{-2})$. The standard quadrupole formula gives the amplitude $h_{ij}$ of the gravitational wave in the transverse-traceless gauge (Misner, Thorne and Wheeler, 1973)

$$h_{ij} = \frac{2}{\gamma} \left( P_{mi} P_{nj} - \frac{1}{2} P_{mn} P_{ij} \right) \frac{d^2 Q_{mn}}{dt^2}, \quad (2.17)$$

where $Q_{mn}$ is the reduced mass quadrupole moment

$$Q_{mn} = \text{STF}\left\{ \int \rho x^m x^n dV \right\} \quad (2.18)$$

and $P_{ij} = \delta_{ij} - n_i n_j$ is the projection operator onto the plane transverse to the outgoing wave direction, $n_i = x^i / r$. For consistency with post-Newtonian accuracy, we must take into account the relativistic corrections up to the relative order $(v/c)^2$. This requires the time derivatives of the mass octupole, mass 24-pole, current quadrupole and current octupole. However we neglect these corrections since the gravitational wave amplitude is evaluated with quantities at each time step and small truncation errors above 'Newtonian' accuracy do not accumulate. With this accuracy, we can use $I_{ij}$ defined by Eq.(2.16) in place of $Q_{ij}$. From Eq.(2.17), two polarizations are given by

$$h_+ = \frac{1}{r} (I_{\hat{z}\hat{z}} - I_{\hat{x}\hat{x}}),$$

$$h_\times = \frac{2}{r} I_{\hat{x}\hat{x}}, \quad (2.19)$$

where $I_{\hat{i}\hat{j}}$ is the quantity in the orthonormal basis.
\[ I_{\theta \phi} = (I_{xx} \cos^2 \phi + I_{yy} \sin^2 \phi + 2I_{xy} \sin \phi \cos \phi) \cos^2 \theta + I_{xx} \sin^2 \theta - 2(I_{xx} \cos \phi + I_{x\theta} \sin \phi) \sin \theta \cos \theta, \]
\[ I_{\phi \phi} = I_{xx} \sin^2 \phi + I_{yy} \cos^2 \phi - 2I_{xy} \sin \phi \cos \phi, \]
\[ I_{\phi \phi} = (I_{yy} - I_{xx}) \cos \theta \sin \phi \cos \phi + I_{xy} \cos \theta (\cos^2 \phi - \sin^2 \phi) + I_{xx} \sin \theta \sin \phi - I_{x\theta} \sin \phi \cos \phi. \]

The equations are discretized by the finite difference method (FDM) with a uniform Cartesian grid. The evolution equations are integrated using van Leer's scheme\textsuperscript{10} with second-order accuracy in space. In order to make the scheme stable, a monotonicity condition, the so-called TVD (Total Variation Diminishing) limiter,\textsuperscript{11} is imposed on this scheme. In order to achieve second-order accuracy in time we adopt a two-step procedure.\textsuperscript{12}

In order to treat shock waves, we use a tensor artificial viscosity given by
\[
p_{ij} = \begin{cases} 
\rho l^2 (\partial_i v^k) \text{STF}(2\partial_i u^r) & \text{if } \partial_i u^k < 0, \\
0 & \text{otherwise,}
\end{cases}
\]

where \( l \) is an appropriate number with units of length. The gas pressure \( p \) is replaced by \( p_{ij} = -\rho \partial_i u^r + p_{ij} \). The Poisson equations are solved using the MICCG (Modified Incomplete Cholesky-decomposition and Conjugate Gradient) method,\textsuperscript{13\textendash}17 which is fully vectorized using the hyperplane method proposed by Ushiro.\textsuperscript{17}

§ 3. Results

We start calculations from two spherical relativistic polytropes with \( \gamma = 2 \) whose surfaces are in contact. We performed two calculations with the same initial conditions; one is a calculation using the full set of \((1 + 2.5)\)-post-Newtonian hydrodynamics equations described the preceding section and the other is a calculation neglecting the first post-Newtonian terms which are of order \((v/c)^3\) while the radiation damping term of order \((v/c)^5\) is included. The mass and the radius of each star in the initial condition is \(0.62M_\odot\) and \(15 \text{ km}\), respectively. The two stars are in synchronized circular orbits with angular velocity \( \Omega = 2.0 \times 10^9 \text{ sec}^{-1} \). The total angular momentum \( J_t \) of the system is \(1.6GM_\odot^2/c\).

We use a \(171 \times 171 \times 86\) Cartesian uniform grid. Since initial data has reflection symmetry with respect to the \( z = 0 \) plane, we consider the region \( z \geq 0 \) only. Figure 1 shows the evolution of the density and the velocity on the \( x \cdot y \) plane and Fig. 2 shows the emitted luminosity \( L \) and the central density \( \rho_c \) as functions of time. The result of the post-Newtonian(PN) calculation is shown on the right of Fig. 1 and as solid lines in Fig. 2, while the result of the Newtonian(N) calculation is on the left and as dashed lines. Comparing PN with N, coalescence begins more rapidly in PN since general relativity effectively increases the gravitational force (Figs. 1 b2\textendash}c2). In both cases, a centrifugal bounce occurs after coalescence and a shock wave appears (Figs. 1 b1, b2\textendash}e2). The shock makes the matter reexpand but it is so weak that the
Fig. 1. Density and velocity on the \(x\)-\(y\) plane. The left and right figures are for the Newtonian (N) and post-Newtonian (PN) calculations, respectively. The time in units of milliseconds is shown. Solid lines are drawn in steps of a tenth of the maximum density and the inner and the outer dashed lines indicate \(19/20\) and \(1/20\) of maximum density. Arrows indicate the velocity vectors of the matter.
Gravitational Radiation from Coalescing Binary Neutron Stars. V

Fig. 2. Luminosity (in units of erg/sec) and central density (in units of g/cm³) as functions of time (in units of millisecond). The solid and dashed lines are for PN and N, respectively.

Fig. 3. The ratio of kinetic energy $T$ to gravitational binding energy $|W|$ as a function of time (in units of millisecond).

Fig. 4. Wave forms of $h_+$ and $h_\times$ observed on the $z$-axis at 10 Mpc. The solid and dashed lines are for PN and N, respectively.

matter may turn back to contraction (Figs. 1 c1~d1, f2~h2). That leads to a contraction-expansion oscillation (Figs. 1 e1~i1, i2~l2). The shock in the PN case is much stronger than in the N case. A local peak in the flux $L$ at $t \approx 0.3$ msec (Fig. 2) corresponds to appearance of the shock. Indeed the shock is rather weak even in the PN case but it excites a radial oscillation of the matter with large amplitude. In the N case, by contrast, only small oscillation is excited. Thus the contraction-expansion oscillation lasts longer in the PN case. The energy flux of the gravitational radiation decreases more slowly as shown in Fig. 2. As a result, the total energy emitted in the gravitational radiation increases in comparison with the N case; $3.4 \times 10^{50}$ erg in PN and $2.2 \times 10^{50}$ erg in N up to 8.6 msec. In Fig. 3, time evolution of the ratio of kinetic energy $T$ to gravitational binding energy $|W|$ is shown for PN (solid line) and N (dashed line). A bend in the solid line at $t \approx 0.3$ msec for PN is caused by appearance of the shock. Increase in the effective gravitational force by general relativity steepens the initial gradient of $T/|W|$. This figure also shows the contraction-expansion oscillation with a large amplitude in the PN case and it is not damped yet at $t = 8$ msec while in the N case the amplitude of the oscillation is small and it is damped soon.

Figure 4 shows the wave forms $h_+$ and $h_\times$ observed along the $z$-axis, which are given

$$h_+ = \frac{1}{\rho} (I_{xx} - I_{yy}),$$

$$h_\times = \frac{1}{\rho} I_{xy},$$

$$h_\times = \frac{1}{\rho} I_{yx}.$$
\[ h_x = \frac{2}{\gamma} I_{xy}, \tag{3.1} \]

where \( r \) is the distance between the event and the observer. Solid lines and dashed lines in Fig. 4 are for PN and for N; respectively. Although \( h \) is damped faster in N than in PN, the maximum amplitudes are almost the same. The appearance of the strong shock in the PN calculation has little effect on the wave form.

§ 4. Discussion

The difference between PN and N is caused principally by the shock which appears in the initial stage. Thus if the initial data consists of two stars in rotational equilibrium and coalescence proceeds more slowly, the difference between PN and N may be less pronounced. We considered in this paper less relativistic stars than in previous papers. For more massive stars, such as two 1.4\( M_\odot \) stars, gravitational collapse occurs in the central region at the coalescence stage in a post-Newtonian calculation. The post-Newtonian terms effectively increases the gravitational force while effective pressure is decreased (see Eqs. (2.6), (2.7)). Thus the effective gravitational potential becomes so deep that the centrifugal force cannot prevent the collapse. This means that the post-Newtonian approximation breaks down for massive stars and we must solve a full set of Einstein equations. In a fully relativistic calculation, the effects of red-shift and the quasi-normal modes, which are neglected in a post-Newtonian calculation, are automatically included. Therefore it is urgent to construct a fully general relativistic 3D numerical code for the evolution of matter and the gravitational field.

Calculations were performed on a supercomputer HITAC S820/80 at National Laboratory for High Energy Physics (KEK). For construction of the code as well as analysis of numerical results, graphical displays of large data sets are necessary. In particular, it is very efficient to make an animation from the numerical results. We made a system for making video movies on a graphic workstation IRIS 4D/25G; the data sets created by numerical simulations on the supercomputer are transferred to the workstation through the LAN at KEK, graphical images are made on the workstation and then they are recorded frame by frame on a video tape or a laser video disc. In analysis of numerical results in this paper, it was very useful to show the evolution of the density using a video movie. For example, it would be very hard to find out the contraction-expansion oscillation if we followed only the sequence of snapshots like Fig. 1. However, even in the N calculation, we can easily recognize it from video movies.

Acknowledgements

This work was supported in part by a Grant-in-Aid for Scientific Research (B) of the Ministry of Education, Science and Culture (02452051) and a Grant-in-Aid for Scientific Research on Priority Areas of the Ministry of Education, Science and Culture (03250103).
Gravitational Radiation from Coalescing Binary Neutron Stars. V

References

1) K. Oohara and T. Nakamura, Prog. Theor. Phys. 82 (1989), 535 (Paper I).
2) T. Nakamura and K. Oohara, Prog. Theor. Phys. 82 (1989), 1066 (Paper II).
3) K. Oohara and T. Nakamura, Prog. Theor. Phys. 83 (1990), 906 (Paper III).
4) T. Nakamura and K. Oohara, Prog. Theor. Phys 86 (1991), 73 (Paper IV).
5) F. A. Rasio and S. L. Shapiro, Preprint CRSR 993 (1992).
6) K. Oohara and T. Nakamura, in preparation.
7) L. Blanchet, T. Damour and G. Schäfer, Mon. Not. R. Astron. Soc. 241 (1990), 289.
8) C. W. Misner, K. S. Thorne and J. A. Wheeler, Gravitation (W. H. Freeman and Company, San Francisco, 1973), p. 993.
9) L. S. Finn, in Frontiers of Numerical Relativity, ed. C. R. Evans, L. S. Finn and D. W. Hobill (Cambridge University Press, Cambridge, 1989), p. 128.
10) B. van Leer, J. Comput. Phys. 23 (1977), 276.
11) A. Harten, J. Comput. Phys. 49 (1983), 357.
12) K. Oohara and T. Nakamura, in Approaches to Numerical Relativity, ed. C. Clarke and R. d’Inverno (Cambridge University Press, Cambridge, 1992), in press.
13) J. A. Meijerink and H. A. van der Vorst, Math. Comp. 31 (1977), 148.
14) I. Gustafsson, BIT 18, 142.
15) Y. Ushiro, Kokyuroku of RIMS 514 (1984), 110 (in Japanese).
16) H. A. van der Vorst, Comput. Phys. Comm. 53 (1989), 223.
17) K. Murata, T. Oguni and Y. Karaki, Supercomputer (Maruzen, Tokyo, 1985), in Japanese.
18) R. F. Stark and T. Piran, in Proceedings of the Fourth Marcel Grosmann Meeting on General Relativity, ed. R. Ruffini (Elsevier, Amsterdam, 1986), p. 327.
19) T. Nakamura, K. Oohara and Y. Kojima, Prog. Theor. Phys. Suppl. No. 90 (1987).