RAPIDLY CONVERGING APPROXIMATIONS AND
REGULARITY THEORY

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Abstract. We consider distributions on a closed compact manifold $M$ as maps on smoothing operators. Thus spaces of maps between $\Psi^{-\infty}(M) \to \mathcal{C}^\infty(M)$ are considered as generalized functions. For any collection of regularizing processes we produce an algebra of generalized functions and a diffeomorphism equivariant embedding of distributions into this algebra. We provide examples invariant under certain group actions. The regularity for such generalized functions is provided in terms of a certain tameness of maps between graded Fréchet spaces. This notion of regularity implies the regularity in Colombeau algebras in the $G^\infty$ sense.

1. Introduction

Regularization of nonsmooth structures such as distributions and discontinuous metrics by smooth approximates has been an important ingredient of many problems in mathematics and physics. The choice of regularizing process is often dictated by their interaction with different operators involved and their symmetries. In numerical processes one desires that the regularizations converge optimally. By a regularizing approximation we mean a net of smoothing operator $T_\varepsilon$ such that $T_\varepsilon u$ is a smooth function and $\lim_{\varepsilon \to 0} T_\varepsilon u = u$. By fixing the asymptotic properties of a regularizing approximation one can study the regularity and singularity of the nonsmooth objects in terms of asymptotic behaviour of the approximation.

Here we shall consider regularizing approximations that are in a sense optimal in view of Lorant Schwartz’s theorem that states the impossibility of constructing an associative product on distributions consistent with continuous functions. We shall call such approximations rapidly converging approximations.

Let $M$ be a compact Riemannian manifold. The Weyl’s asymptotic formula provides an asymptotic estimate on the growth of eigenvalues of the associated Laplace operator $\Delta$ on $M$. For a Schwartz function $F(x) \in \mathcal{S}(\mathbb{R})$ let $F_\varepsilon(x) = F(\varepsilon x)$. Then for a suitable choice of Schwartz function $F$ the net of smoothing operator $T_\varepsilon = F_\varepsilon(\Delta)$ provides the

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basic example of rapidly converging approximations in view of Weyl’s theorem. In the special case of compact Lie groups $T_\varepsilon$ can be obtained in terms of convolution with characters of irreducible representations directly from Peter-Weyl theorem.

In order to study regularity of a distribution $u \in \mathcal{D}'(M)$ in terms of the approximation $T_\varepsilon u$ one is naturally led to consider it as a map from smoothing operators $\Psi^{-\infty}(M)$ to $\mathcal{C}^\infty(M)$,

$$\Theta_u : \Psi^{-\infty}(M) \to \mathcal{C}^\infty(M) \quad \Theta_u(T) := T(u) \forall T \in \Psi^{-\infty}(M).$$

Here the Sobolev regularity of $u$ can be interpreted in terms of degree of tameness of the map $\Theta_u$ with respect to certain grading on the two Frechét spaces $\mathcal{C}^\infty(M)$ and $\Psi^{-\infty}(M)$. For example a smooth function $f$ provides a map $\Theta_f$ which is tame for all possible degrees of tameness and this character classifies all smooth maps. This point of view immediately provides us with a natural way to study local regularity of $u$. For instance one can easily identify singular support and the wavefront set of $u$ either from $\Theta_u$ or $T_\varepsilon u$.

This leads one to consider a general Frechét smooth map

$$\phi : \Psi^{-\infty}(M) \to \mathcal{C}^\infty(M)$$

as a generalized function space for nonlinear operators. The action of diffeomorphisms and pseudodifferential operators on $\mathcal{D}'(M)$ can naturally be extended to the space $E$ of all such maps. The pointwise algebra structure on $E$ obtained from the algebra $\mathcal{C}^\infty(M)$ provides meaningful subalgebras which correspond to regularity of a generalized function $\phi \in E$.

For instance the regularity of such generalized functions can be measured in two different ways. Either we can modify the notion of degree of tameness in which case pseudodifferential operator change the degree of tameness analogous to their mapping properties on Sobolev spaces. Or else we can consider maps with certain asymptotic behaviour for the smooth functions $\phi(T_\varepsilon)$ where we choose the rapidly converging approximations $T_\varepsilon$ to belong to a fixed set $\mathcal{L}$ invariant under symmetries of a given problem.

The fact that the net of operators $T_\varepsilon$ induce a sheaf-morphism from the sheaf of distributions $\mathcal{D}'(M)$ to the sheaf $\mathcal{G}^s(M)$ of special algebra of Colombeau with smooth dependence of parameter plays an important role in the present interpretation of local regularity properties of distributions and more generally for generalized functions in $E$.

In Euclidian space rapidly converging approximations can be constructed from convolution with a net of mollifier converging appropriately to the delta distributions (see [1]). Another such construction
can be carried out on $\mathbb{R}^{2n+1}$ by convolution by a mollifier $\rho_\varepsilon$ on the Heisenberg group. These further yield rapidly converging approximations on $\mathbb{R}^n$ by apply Fourier transform with respect to the Schrödinger representation to each $\rho_\varepsilon$. We note that although each of the above approximations characterize local regularity of distributions in much the same way, they tend to isolate different properties of global regularity and the growth at infinity. This is because they preserve different large scale structure on $\mathbb{R}^n$. In the present article we shall study only the local behaviour of regularity.

2. Preliminaries

Let $X$ be a locally convex (Hausdorff) topological vector space then one can associate a generalized locally convex space $\mathcal{G}_X$ (see [3]) as follows. Let $I$ be the interval $(0, 1)$. Define the smooth moderate nets on $X$ to be smooth maps

$$I \rightarrow X \quad \varepsilon \rightarrow x_\varepsilon$$

such that for all continuous semi-norms $\rho$ there exists an integer $N$ such that

$$|\rho(x_\varepsilon)| \sim O(\varepsilon^N) \quad \text{as} \quad \varepsilon \rightarrow 0$$

Here as usual by $f(\varepsilon) \sim O(g(\varepsilon))$ as $\varepsilon \rightarrow 0$ we mean there exists an $\varepsilon_0 > 0$ and a constant $C > 0$ such that $f(\varepsilon) < Cg(\varepsilon)$ for $\varepsilon < \varepsilon_0$. We denote the set of all moderate smooth nets on $X$ by $E(X)$. Similarly we can define the negligible nets to be the smooth maps $x_\varepsilon$ such that for all continuous seminorms $\rho$ and for all $N$ Equation (1) holds. We shall denote the set of all smooth negligible nets by $N(X)$.

The generalized locally convex space over $X$ is then defined to be the quotient,

$$\mathcal{G}_X := E(X)/N(X).$$

One notes that in defining $E(X)$ and $N(X)$ it suffices to restrict to a family of seminorms that generate the locally convex topology on $X$. If $x_\varepsilon$ is a moderate net in $E(X)$ then the element it represents in the quotient $\mathcal{G}_X$ shall be represented by $\langle x_\varepsilon \rangle$.

When $X = C^\infty(M)$ is the space of smooth functions on a manifold $M$ then we also represent $\mathcal{G}^*(M) := \mathcal{G}_{C^\infty(M)}$. Also for $X = \mathbb{C}$ the space $\mathcal{G}_{\mathbb{C}}$ inherits a ring structure from $\mathbb{C}$ and we call it the space of generalized numbers and denote it by $\hat{\mathbb{C}}$. Every $\mathcal{G}_X$ is naturally a $\hat{\mathbb{C}}$ module, and hence is often referred to as the $\hat{\mathbb{C}}$ module associated with $X$. The sharp topology on $\hat{\mathbb{C}}$ is the topology generated by sets of the form $U_{x,p}$.
where \( x \in \tilde{\mathbb{C}} \) and \( p \) is an integer and
\[
U_{x,p} := \{ \langle x, e^p, x, e^p \rangle | \langle x \rangle = x \}.
\]
Any continuous seminorm \( \rho \) on a locally convex spaces \( X \) by definition provides a map \( \tilde{\rho} : E(X) \to E(\mathbb{C}) \) by applying \( \rho \) to each component.

In fact \( \tilde{\rho} \) descend to a map from \( \mathcal{G}_X \to \tilde{\mathbb{C}} \). The sharp topology on any \( \tilde{\mathbb{C}} \) module \( \mathcal{G}_X \) shall be defined to be the weakest topology that makes each \( \tilde{\rho} \) above continuous.

We recall the functoriality of the above construction [11].

**Lemma 2.1.** If \( \phi : X \to Y \) is a continuous linear map between locally convex spaces \( X \) and \( Y \) then there is a natural induced map \( \phi_* : \mathcal{G}_X \to \mathcal{G}_Y \) defined on the representatives as \( \phi_* (\langle x \rangle) = \langle \phi (x) \rangle \). Further \( \phi_* \) is continuous with respect to sharp topology.

**Proof.** A continuous linear map \( \phi \) maps smooth net \( x_\varepsilon \) to smooth net \( \phi_* (x_\varepsilon) \). If \( \tau \) is a continuous seminorm on \( Y \) then \( \tau \circ \phi \) is a continuous seminorm on \( X \). Thus if \( x_\varepsilon \) satisfies an estimate with \( \tau \circ \phi \) in \( E(X) \) or \( N(X) \) then \( \phi (x_\varepsilon) \) satisfies the exact same estimates with respect to \( \tau \) in \( E(Y) \) or \( N(Y) \). Thus \( \phi_* \) is well-defined. Since basic open set \( U \) in \( \mathcal{G}_Y \) are pull-back of open sets in \( \tilde{\mathbb{C}} \) by some seminorm \( \rho \), then \( \phi_*^{-1}(U) \) is a pullback of an open set with respect to \( \rho \circ \phi \).

For example any smooth map between two manifold \( f : M \to N \) gives rise to a pull back map \( f^* : \mathcal{G}^*(N) \to \mathcal{G}^*(M) \). As a consequence we can define a presheaf of algebras on \( M \) by assigning to any open set \( U \subseteq M \to \mathcal{G}^*(U) \). The restriction maps are given by the pull back under inclusions, that is \( i : U \leftarrow V \) is an inclusion of open sets then \( i^* : \mathcal{G}^*(V) \to \mathcal{G}^*(U) \) is the restriction map. This presheaf is in fact a fine sheaf. Thus in particular we can define the support of a global section \( x \in \mathcal{G}^*(M) \) as usual to be the complement of the biggest open subset of \( M \) on which \( x \) restricts to 0.

For any locally convex space \( X \) we can also define a subalgebra \( \mathcal{G}_X^\infty \) of regular elements of \( \mathcal{G}_X \). These are all elements in \( \mathcal{G}_X \) such that there exists an integer \( N \) so that (1) holds independent of the seminorm \( \rho \) chosen. Again we shall denote by \( \mathcal{G}^\infty (M) \) the algebra \( \mathcal{G}_\mathcal{C}^\infty (M) \). The algebra \( \mathcal{G}^\infty (M) \) provides regularity features for analysis of generalized functions and operation in \( \mathcal{G}^*(M) \) in a way that \( \mathcal{C}^\infty (M) \) provides these features in \( \mathcal{D}'(M) \). For instance:

(a) **Singular support:** For \( \phi \in \mathcal{G}^*(M) \) the singular support is defined as the complement of largest open set \( U \) on which the restriction \( \phi_U \) is in \( \mathcal{G}^\infty (M) \).
(b) Wavefront set: Let $P$ be an order 0 classical pseudodifferential operator and let $\operatorname{char}(P) \subseteq T^*M$ be the characteristic set of $P$, that is the 0 set of its principal symbol. Then we can define the generalized wavefront set of a generalized function $\phi$ as:

$$\operatorname{WF}_g(\phi) := \cap_{P\phi \in \mathcal{G}^\infty(M)} \operatorname{char}(P) \quad P \in \Psi^0(M).$$

(c) Hypoellipticity: We can define an operator $P$ to be $\mathcal{G}^\infty(M)$ hypoelliptic if

$$Pu \in \mathcal{G}^\infty(M) \implies u \in \mathcal{G}^\infty(M).$$

Under appropriate circumstances the above notions are extensions of the same in the distributional sense. Lemma 3.5 provides a general example.

3. Moderate approximate units

Let $M$ be a closed manifold. Let $\Omega$ be the bundle of 1-densities on $M$. By distributions on $M$ we mean the dual space $\mathcal{D}'(M) = \mathcal{C}^\infty(M : \Omega)'$. A continuous linear operator $\mathcal{D}'(M) \to \mathcal{C}^\infty(M)$ is called a smoothing operator. The space of all smoothing operators shall be denoted by $\Psi^{-\infty}(M)$ and its forms an ideal in the algebra of pseudodifferential operators $\Psi^\infty(M)$. By identifying an operator to its kernel, smoothing operators can be viewed as a Frechét space of smooth sections of a vector bundle namely,

$$\Psi^{-\infty}(M) = \Gamma^\infty(M \times M : \pi_2^*\Omega),$$

where $\pi_2 : M \times M \to M$ is the projection on the second component.

Now we introduce certain nets of smoothing operators that shall play the role of delta nets generated from a mollifier. These are the regularizing processes we are interested in.

**Definition 3.1.** A net of smoothing operators $T_\varepsilon$ is called a moderate approximate unit if:

(a) Its a moderate net that is $T_\varepsilon \in E_{\Psi^{-\infty}(M)}$. That is $T_\varepsilon$ satisfies (1) with respect to any seminorm on $\Psi^{-\infty}(M)$.

(b) For any $u \in \mathcal{D}'(M)$

$$\lim_{\varepsilon \to 0} T_\varepsilon u = u.$$

(c) For any smooth function $f \in \mathcal{C}^\infty(M)$ the approximation $T_\varepsilon f \to f$ converges rapidly in the sense that given a seminorm $\rho$ on smooth functions

$$\rho(T_\varepsilon f - f) \sim O(\varepsilon^N) \text{ for all } N \in \mathbb{Z}.$$
The set of all Moderate approximate units shall be denoted by \( \mathcal{U}(M) \) or sometimes with \( \mathcal{U} \) for simplicity. The set \( \mathcal{U} \) is is closed under the obvious action of the diffeomorphism group.

**Lemma 3.2.** Let \( \chi \) be a diffeomorphism. Let \( T_\varepsilon \in \mathcal{U} \) be a moderate approximate unit then the \( \varepsilon \)-wise push forward \( \chi_* T_\varepsilon \) is also a moderate approximate unit.

**Proof.** Since the push forward map \( \chi_* : \Psi^{-\infty}(M) \rightarrow \Psi^{-\infty}(M) \) is defined as

\[
\chi_*(T)(f) := \chi^*(T(\chi^{-1} f)),
\]

where \( \chi^* \) is a pull back of functions. It is clear that \( \chi_* \) is continuous linear map, by Lemma 2.1 it maps moderate nets to moderate nets. In particular \( \chi_*(T_\varepsilon) \) is moderate net in \( E_{\Psi^{-\infty}(M)} \).

For the same reason \( \chi_* \) maps all negligible nets in \( C^\infty(M) \) to negligible nets. First we observe that

\[
\chi_* T(f) - f = \chi^*(T \chi^{-1} f) - f = \chi^*(T \chi^{-1} f - \chi^{-1} f)
\]

Therefore

\[
T_\varepsilon \chi^{-1} f - \chi^{-1} f \in N_{C^\infty(M)} \Rightarrow \chi^*(T_\varepsilon \chi^{-1} f - \chi^{-1} f) \in N_{C^\infty(M)}.
\]

By continuity of \( \chi \)

\[
\lim_{\varepsilon \to 0} \chi_* T_\varepsilon u = \lim_{\varepsilon \to 0} \chi^*(T_\varepsilon \chi^{-1} u)
\]

\[
= \chi^*(\lim_{\varepsilon \to 0} T_\varepsilon \chi^{-1} u) = u
\]

Thus \( \chi_* T_\varepsilon \) satisfies Definition 3.1. \( \square \)

The following proposition follows directly form the definition of moderate approximate units and underlines one of the reasons for the definition.

**Proposition 3.3.** A moderate approximate unit \( T_\varepsilon \) provides an embedding of the distributions \( \mathcal{D}'(M) \) into the space of smooth special algebra \( \mathcal{G}^s(M) \) by \( u \rightarrow T_\varepsilon u \). This maps restrict to an algebra homomorphism on \( C^\infty(M) \).

**Definition 3.4.** Fix a Riemannian metric on \( M \times M \). We call a net \( T_\varepsilon \in \Psi^{-\infty}(M) \) a local moderate approximate unit or a rapidly converging approximation if

1. It is a moderate approximate unit satisfying the Definition 3.1.
2. The following transfer of regularity holds:

\[
T_\varepsilon(\mathcal{D}'(M)) \cap \mathcal{G}^\infty(M) = C^\infty(M).
\]
(3) There is no propagation of support that is, for any \( \delta > 0 \) there exists a decomposition of the form

\[
T_\varepsilon = L_\varepsilon + N_\varepsilon,
\]

such that \( N_\varepsilon \) is a negligible net of operators, and \( L_\varepsilon \) is supported in a \( \delta \) neighbourhood of the diagonal in \( M \times M \).

Thus for a local moderate approximate unit \( \text{supp}(T_\varepsilon(u)) = \text{supp}(u) \).

The local moderate units are also preserved under diffeomorphisms. We shall refer to the condition (2) on a moderate approximate unit as tameness condition.

**Proposition 3.5.** If \( T_\varepsilon \) is local then the map \( u \to T_\varepsilon u \) is a sheaf-morphism on the sheaf \( \mathcal{D}(M) \to \mathcal{G}^\ast(M) \). In particular local units preserve supports and singular supports of distributions.

**Proof.** This is accomplished as usual by covering by precompact open sets and cut offs. Here are the details for completeness.

Let \( U \subseteq M \) be an open subset. Now cover \( U \) by an open cover \( U_\lambda, \lambda \in \Lambda \) such that the closure \( \overline{U}_\lambda \) is compact in \( U \) and let \( \phi_\lambda \in C^\infty_c(U) \) be such that \( \phi_\lambda \equiv 1 \) on some neighbourhood of \( U_\lambda \). Define the map \( T_U : \mathcal{D}'(U) \to \mathcal{G}^\ast(U_\lambda) \) by \( T_\lambda(u) := T_\varepsilon(\phi_\lambda u)|_{U_\lambda} \). Then we check that:

\[
T_\lambda(w)|_{U_\lambda \cap U_\mu} = T_\mu(w)|_{U_\lambda \cap U_\mu}.
\]

This follows immediately as \( (\phi_\lambda - \phi_\mu)w \) is supported away from \( U_\lambda \cap U_\mu \) implies that \( T_\varepsilon((\phi_\lambda - \phi_\mu)w) \) is also supported away from \( U_\lambda \cap U_\mu \) by Definition 3.4. Thus there exists a \( T_U(w) \in \mathcal{G}^\ast(U) \) such that \( T_U(w)|_{U_\lambda} = T_\lambda(w) \). The map \( w \to T_U(w) \) by a similar argument is independent of the covering \( U_\lambda \) and the cut off functions \( \phi_\lambda \) and provide the required sheaf morphism.

For a distribution \( u \in \mathcal{D}'(M) \) and an open set \( U \subset M \) it follows then by (2) that \( T_\varepsilon(u)|_U \) is in \( \mathcal{G}^\infty(U) \) precisely if \( u|_U \in C^\infty(U) \). Thus \( T_\varepsilon \) preserves the singular support. \( \Box \)

We shall denote the set of local moderate approximate units on \( M \) by \( \mathcal{U}_{\text{loc}}(M) \) or simply \( \mathcal{U}_{\text{loc}} \). There is of course a plentiful supply of moderate approximate units. All the examples given below are of local moderate approximate units.

**Example 3.6.** Let \( \Delta \) be the Laplace operator associated to a Riemannian manifold \( M \). Let \( F \in \mathcal{S}(\mathbb{R}) \) be a Schwartz function on the reals such that \( F \) is identically 1 near origin. Let \( F_\varepsilon(x) := F(\varepsilon x) \). Then by applying standard functional calculus \( F_\varepsilon(\Delta) \) is a moderate approximate unit. All the asymptotic properties follow from Weyl’s estimate
on eigenvalues of $\Delta$. In addition $F_\varepsilon(\Delta)$ is invariant under isometries. (See [2] for details.)

As a special case of the above example consider a compact Lie group $G$ and let $\hat{G}$ denote the set of all irreducible representations of $G$. Let $\pi \in \hat{G}$ be an irreducible representation and let

$$\chi_\pi(g) = \text{tr}[\pi(g)],$$

be the character of $\pi$, and is well defined as $\pi$ is necessarily finite dimensional with dimension denoted by $d_\pi$. As a consequence of Peter Weyl theorem one can obtain (see [12]),

$$f = \sum_{\pi \in \hat{G}} d_\pi \chi_\pi * f \quad f \in L^2(G).$$

Let $\Delta_G$ denote the Laplace operator on $G$, obtained from a basis of the Lie algebra $\mathcal{G}$ then $\chi_\pi$ are eigenfunctions of $\Delta_G$ with eigenvalues $\lambda_\pi$. Hence if $F$ is a Schwartz function as in the above example then

$$F_\varepsilon(\Delta_G) f = \sum_{\pi \in \hat{G}} d_\pi F_\varepsilon(\Delta_G)(\chi_\pi * f)$$

$$= \sum_{\pi \in \hat{G}} d_\pi F_\varepsilon(\Delta_G)(\chi_\pi * f)$$

$$= \sum_{\pi \in \hat{G}} d_\pi F(\varepsilon \lambda_\pi)(\chi_\pi * f)$$

here we have used the left invariance of $F_\varepsilon(\Delta_G)$.

We now move to some noncompact examples of Lie group where convolution shall play an important part.

**Example 3.7.** Due to noncompactness of $\mathbb{R}^n$ one has to modify the above notions slightly and work with compactly supported distributions.

**Definition 3.8.** A rapidly converging approximate unit on $\mathbb{R}^n$ is a net of operators $T_\varepsilon$ such that their kernels $\ker T_\varepsilon \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$, and the following holds.

(a) For any $u \in \mathcal{E}'(\mathbb{R}^n)$ the net $T_\varepsilon(u)$ is a moderate net in $\mathcal{C}^\infty(\mathbb{R}^n)$, and

$$\lim_{\varepsilon \to 0} T_\varepsilon(u) = u.$$

That is $T_\varepsilon : \mathcal{E}'(\mathbb{R}^n) \to \mathcal{G}^*(\mathbb{R}^n)$ is a well defined injective map.

(b) For a compactly supported function $f \in \mathcal{C}^\infty_c(M)$ the regularization converges rapidly that is $T_\varepsilon(f) - f$ is a negligible net in special colombeau algebra $\mathcal{G}^*(\mathbb{R}^n)$. 
(c) The regularization preserves supports. That is for $u \in \mathcal{E}'(\mathbb{R}^n)$, $\text{supp } u = \text{supp}_g \mathcal{T}_\varepsilon(u)$

As in the original construction of Colombeau [1] a moderate approximate unit can be constructed from a mollifier $\rho \in \mathcal{S}(\mathbb{R}^n)$ satisfying the following conditions:

$$\int_{\mathbb{R}^n} \rho(x) dx = 1 \quad \int_{\mathbb{R}^n} x^\alpha \rho(x) dx = 0 \quad \alpha \in \mathbb{N}^n_.$$

(3) Then the net of functions $\rho_\varepsilon(x) := \frac{1}{\varepsilon^n} \rho(\frac{x}{\varepsilon})$ is a delta net and convolution with such a delta net provides an example of moderate approximate unit. An important characteristic of these approximate units is their equivariance with respect to the Euclidian translations.

Note that on $\mathbb{R}$ a mollifier satisfying (3) can be obtained as the Fourier transform of the function $F$ used in Example 3.6.

**Example 3.9.** The previous example can be modified in many ways. For instance let $\rho$ be a Schwartz function on $\mathbb{R}^2$ satisfying (3). Let $\tilde{\rho}_\varepsilon(x,y) := \frac{1}{\varepsilon^3} \rho(\varepsilon^{-1} x, \varepsilon^{-2} y)$. Then convolution with this new delta net continues to provide an moderate approximate unit.

On $\mathbb{R}^{2n+1}$ convolution in Heisenberg group with a delta net provides moderate approximate units. We provide an elementary construction in all detail in the following subsection.

### 3.1. The Heisenberg group

Let $H_n := \mathbb{R}^{2n} \times \mathbb{R}$ be the Heisenberg group with the usual composition:

$$(x, \xi, t) \circ (y, \eta, s) = \left( x + y, \xi + \eta, t + s + \frac{1}{2} (x.\eta - y.\xi) \right).$$

The usual volume form $dxd\xi dt$ is invariant under both left and right translations that is:

$$L_p^* dxd\xi dt = R_p^* dxd\xi dt = dxd\xi dt \quad p \in H_n.$$

The Lie algebra $\mathfrak{h}_n$ of the Heisenberg group is generated by left invariant vector fields:

$$X_i = \frac{\partial}{\partial x_i} - \frac{\xi_i}{2} \frac{\partial}{\partial t},$$

$$\Xi_i = \frac{\partial}{\partial \xi_i} + \frac{x_i}{2} \frac{\partial}{\partial t},$$

$$T = \frac{\partial}{\partial t}.$$
Let $\Delta$ be the associated Laplace operator,
\[
\Delta := T^2 + \sum_{i} X_i^2 + \Xi_i^2.
\]

Then the metric associated to it is a left invariant metric on $H_n$. For example in case of $H_1 = \mathbb{R}^3$ it can be given by:
\[
G = \begin{pmatrix}
1 + \frac{y^2}{2} & -\frac{xy}{2} & \frac{y}{2} \\
-\frac{xy}{2} & 1 + \frac{x^2}{2} & -\frac{x}{2} \\
\frac{y}{2} & -\frac{x}{2} & 1
\end{pmatrix}
\]

We shall need the operator $\Delta$ apply standard elliptic regularity argument to our constructions.

Now to construct moderate approximate units on $H_n$ we can proceed like in the Euclidian case discussed before. We shall need the following:

**Proposition 3.10.** Let $\rho \in \mathcal{S}(\mathbb{R}^{2n+1})$ be a mollifier that satisfies Equation (3). Let $\rho_\varepsilon(x) := \frac{1}{\varepsilon^n}\rho(\frac{x}{\varepsilon})$. Then $T_\varepsilon(u) = \rho_\varepsilon * u$ is a rapidly converging approximation.

(a) By elliptic regularity of $\Delta$ on any bounded domain given any compactly supported distribution $u \in \mathcal{E}'(H_n)$ there exists a compactly supported continuous function $f$ supported in any neighbourhood of $\text{supp} \ u$ such that for some constants $c_j$'s
\[
\sum_j c_j \Delta^j f = u.
\]

Thus for any smooth $\rho$
\[
\rho_\varepsilon * u = \sum_j c_j \Delta^j f \ast \rho_\varepsilon
\]
\[
= \sum_j c_j \Delta^j (f \ast \rho).
\]

Or it suffices to check the asymptotics for the case when $u = f$ is continuous. Now the estimate can be directly be obtained from the integral formula:
\[
\rho_\varepsilon \ast f(p) = \int_{G} \rho_\varepsilon(q)f(pq^{-1})dq.
\]

(b) Let $f$ be a compactly supported smooth function. then:
Let $\text{dist}^H_\varepsilon(p, q) = p - (\varepsilon q^{-1})p$ be the euclidean difference. then
\[
\text{dist}^H_\varepsilon(p, q) = -\varepsilon q + \frac{\varepsilon}{2} \omega(\bar{p}, \bar{q})
\]
where \((x, \xi, t) = (x, \xi)\) and \(\omega\) is the standard symplectic form on \(\mathbb{R}^{2n}\). Now remembering that \(\rho\) satisfies (3)

\[
\int f \ast_{H_n} \rho_\varepsilon(p) - f(p) = \int f(q^{-1}p)\rho_\varepsilon(q)dq - f(p) \\
= \int f((\varepsilon q^{-1})p)\rho(q)dq - f(p) \\
= \int f(p - \text{dist}^H_\varepsilon(p, q))\rho(q)dq - f(p) \\
= \int (f(p - \text{dist}^H_\varepsilon(p, q)) - f(p))\rho(q)dq.
\]

Applying Taylor expansion we get

\[
f \ast_{H_n} \rho_\varepsilon(p) - f(p) = \int \sum_{|\alpha| < N} \frac{\text{dist}^H_\varepsilon(p, q)}{\alpha!} \partial^\alpha f(p) \rho(q)dq + C\varepsilon^N \\
\sim O(\varepsilon^N).
\]

(c) One only needs to observe that given \(\delta > 0\) one can decompose \(\rho_\varepsilon\) into \(\rho_\varepsilon = w_\varepsilon + v_\varepsilon\) with \(w_\varepsilon\) supported in a \(\delta\) neighbourhood of the origin and \(v_\varepsilon\) is negligible. Thus support of \(u_H \ast_{H_n} \rho_\varepsilon\) is contained in every \(\delta\) neighbourhood of \(\text{supp} \ u\).

Remark 3.11. Given a smooth “delta net” the proof of the above proposition holds for any Lie group. But for a general Lie group there is no ”easy” choice of a smooth net of functions suitably approximating the delta distribution at identity similar to \(\rho_\varepsilon\).

We gather all the facts together to obtain,

**Proposition 3.12.** For a schwart function \(\rho\) satisfying (3) the map

\[
\rho : \mathcal{E}'(H_n) \to \mathcal{G}^s(H_n) \quad \rho(u) := \rho_\varepsilon \ast_{H_n} u,
\]

extends to a sheaf morphism \(\mathcal{D}'(H_n) \to \mathcal{G}(H_n)\) and is an algebra homomorphism on \(\mathcal{C}^\infty(H_n)\).

**Example 3.13.** For \(x, \xi \in \mathbb{R}^n\) let \(T_x\) and \(M_\xi\) be the operation of translation and modulation on \(\mathbb{R}^n\). That is

\[
T_x f(t) := f(t - x), \\
M_\xi f(t) := e^{2\pi i \xi \cdot t} f(t).
\]

We recall that the Schrödinger representation is the unitary representation of the Heisenberg group on \(L^2(\mathbb{R}^n)\) given by

\[
\pi(x, \xi, \tau) := e^{2\pi i r} e^{\pi i x \cdot \xi} T_x M_\xi \in B(L^2(\mathbb{R}^n)).
\]
The Fourier transform of a Schwartz function $F \in \mathcal{S}(\mathbb{R}^{2n+1}) \subset L^1(H_n)$ is a bounded operator on $L^2(M)$ given by:

$$\hat{\pi}(F) := \int_{H_n} F(\alpha) \pi(\alpha^{-1}) d\alpha.$$ 

In fact one computes the kernel to be

$$\ker \hat{\pi}(F)(x, y) = F_{2, 3} F(y - x, \frac{y + x}{2}, 1) \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n).$$

and hence $\hat{\pi}(F)$ defines a smoothing operator on $L^2(\mathbb{R}^n)$.

We wish to claim that the Fourier transform with respect to the Schrödinger representation maps a delta net $\rho_\varepsilon$ to a rapidly converging approximate unit.

We shall need the following result.

**Lemma 3.14.** Given $f \in C^\infty_c(\mathbb{R}^n)$ there exists there exists a $g \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\hat{\pi}(F) g = f.$$ 

**Proof.** Let $\phi(\tau) \in \mathcal{S}(\mathbb{R})$ be such that $\int \phi(\tau) d\tau = 1$ and let $g \in \mathcal{S}(\mathbb{R}^n)$ such that $\|g\|_2 = 1$ then set

$$F(x, \xi, \tau) : = e^{2\pi i \tau} \phi(\tau) e^{i x \cdot \xi} \int f(t) g(t + x) e^{2\pi i \xi \cdot t} dt.$$

$$= e^{2\pi i \tau} \phi(\tau) e^{i x \cdot \xi} V_g f(-x, -\xi).$$

here $V_g f(x, \xi)$ is the Short Time Fourier Transform(STFT). The inversion formula for STFT is given by (see [4]) :

$$f(t) = \frac{1}{\langle g, g \rangle} \int V_g f(x, \xi) M_\xi T_x g dx d\xi.$$

Then as an immediate consequence of the inversion formula we have

$$\hat{\pi}(F) g(t) = \int F(x, \xi, \tau) e^{-2\pi i \tau} e^{-i x \cdot \xi} M_{-\xi} T_{-x} g(t) dx d\xi d\tau$$

$$= \int V_g f(-x, -\xi) M_{-\xi} T_{-x} g(t) dx d\xi = f(t).$$

\[\square\]

**Proposition 3.15.** Let $\rho \in \mathcal{S}(\mathbb{R}^{2n+1})$ be a Schwartz function that satisfies (3) and let $\rho_\varepsilon(x) = \frac{1}{\varepsilon^{n+1}} \rho(x \varepsilon)$. Then $\hat{\pi} \rho_\varepsilon$ is a rapidly converging approximation on $\mathbb{R}^n$. 

Proof. We already know that the kernel of \( \hat{\pi}(\rho_z) \) is in \( \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n) \). We shall here only show that for a compactly supported smooth function \( f \in \mathcal{E}'(\mathbb{R}^N) \) the approximation \( \hat{\pi}(\rho_z)f \) converges to \( f \) rapidly. The proof of other properties is routine. By lemma 3.14 we have

\[
f = \hat{\pi}(F)g \quad \exists F \in \mathcal{S}(\mathbb{R}^{2n+1}), \; g \in \mathcal{S}(\mathbb{R}^n).
\]

Therefore

\[
\hat{\pi}(\rho_z)f - f = \hat{\pi}(\rho_z)\hat{\pi}(F)g - \hat{\pi}(F)g,
\]

\[
\hat{\pi}(\rho_z * H_n F - F)g.
\]

Hence the result follows from Proposition 3.12. \( \square \)

4. Global algebras of generalized functions

In this section we shall construct various candidates for algebras of generalized functions. We shall generally refer to all of them as “full type algebras”. They shall depend on choice of a set of regularizing processes. For a particular problem such a set of regularizing process might be chosen depending on the symmetries involved. We shall provide a few toy examples in Section 5. On closed manifold \( M \) both \( C^\infty(M) \) and \( \Psi^\infty(M) \) forms nuclear Fréchet space with jointly continuous multiplication. Let \( E^\infty \) be the set of all maps

\[
\phi : \Psi^\infty(M) \to C^\infty(M) \quad \phi \text{ a Fréchet smooth map}.
\]

Being a space of maps into a commutative algebra \( E^\infty \) is an algebra under pointwise operations.

By evaluation on operators a distribution defines a map

\[
(4) \quad \Theta_u : \Psi^\infty(M) \to C^\infty(M) \quad \Theta_u(T) := T(u) \forall T \in \Psi^\infty(M).
\]

If \( K_T(x,y) \) denotes the integral kernel of \( T \) then the evaluation on \( u \in \mathcal{D}'(M) \) is given by

\[
\Theta_u(T) = T(u) = \langle u(y), k_T(x,y) \rangle.
\]

Thus \( u \to \Theta_u \) is a map from \( \mathcal{D}'(M) \) into \( \mathfrak{L}(\Psi^\infty(M), C^\infty(M)) \) the space of continuous linear maps between these two Fréchet spaces. This map is clearly an injective map as a section \( f \in C^\infty(M, \Omega) \) can be used to define a smoothing map \( u \to u(f) \) which separates distributions.

The map \( u \to \Theta_u \) extends to a map on the tensor algebra \( \mathfrak{T}\mathcal{D}'(M) \to E^\infty \) by

\[
\rho(u_o \otimes u_1 \otimes \ldots \otimes u_r)(T) = T(u_0)T(u_1) \ldots T(u_r).
\]

The restriction to smooth functions \( f \to \Theta_f f \in C^\infty(M) \) is however not an algebra homomorphism. By choosing a subalgebra of \( E^\infty \) and
quotienting by an ideal containing $f - \Theta f$ for all $f \in \mathcal{C}^\infty(M)$ one can easily obtain an algebra homomorphism on $\mathcal{C}^\infty(M)$. There is a choice of such a morphism for any collection of regularizing processes.

Let $\mathcal{L} \subset \mathcal{U}$ be a set of moderate approximate units or regularizing processes (see Definition 3.1). We say that a smooth map $\phi : \Psi^\infty(M) \to \mathcal{C}^\infty(M)$ is moderate over $\mathcal{L}$ if for all $T_\varepsilon \in \mathcal{L}$ the evaluation $\phi(T_\varepsilon) \in E_{\mathcal{C}^\infty(M)}$ is a moderate net. Or more elaborately $\phi$ is moderate over $\mathcal{L}$ if given an approximate unit $T_\varepsilon \in \mathcal{L}$ and a continuous seminorm $\rho$ on $\mathcal{C}^\infty(M)$ there exists an integer $N$ such that:

$$\rho(\phi(T_\varepsilon)) \sim O(\varepsilon^N).$$

The set of all moderate maps over $\mathcal{L}$ shall be denoted by $E_{\mathcal{L}}(M)$.

Similarly $\phi \in E^\infty(M)$ is said to be negligible over $\mathcal{L}$ if for any approximate unit $T_\varepsilon$ in $\mathcal{L}$ $\phi(T_\varepsilon)$ is a negligible net of smooth functions. That is for all any seminorm $\rho$ on $\mathcal{C}^\infty(M)$,

$$\rho(\phi(T_\varepsilon)) \sim O(\varepsilon^N) \text{ for all } N \in \mathbb{Z}.$$

The set of all negligible maps over $\mathcal{L}$ shall be denoted by $N_{\mathcal{L}}(M)$. Of course it suffices to check the estimates for $E_{\mathcal{L}}(M)$ and $N_{\mathcal{L}}(X)$ only for a family of seminorm that generate the locally convex topology on $\mathcal{C}^\infty(M)$.

One can readily check that $N_{\mathcal{L}}(M)$ is an ideal in $E_{\mathcal{L}}(M)$. One way to see this is that given a differential operator $D$ there exists differential operators $P_i, Q_i$ such that for any two smooth functions $g, h \in \mathcal{C}^\infty(M)$

$$D(hg) = \sum_i P_i(h)Q_i(g).$$

Thus for instance

$$\|D(hg)\|_{L^2(M)} \leq \sum_i \|P_i(h)\|_{L^\infty(M)}\|Q_i(g)\|_{L^2(M)}.$$

Let $D$ be an invertible elliptic operator. Let $\phi \in E_{\mathcal{L}}(M)$ and $\psi \in N_{\mathcal{L}}(M)$ then for any $T_\varepsilon \in \mathcal{L}$

$$\|D(\phi, \psi(T_\varepsilon))\|_{L^2(M)} \leq \sum_i \|P_i(\phi(T_\varepsilon))\|_{L^\infty(M)}\|Q_i(\psi(T_\varepsilon))\|_{L^2(M)}.$$

which proves that the product is in $N_{\mathcal{L}}(M)$

**Definition 4.1.** Let $\mathcal{L}$ be a set of moderate approximate units. The full algebra of generalized functions over $\mathcal{L}$ is defined as

$$\mathcal{G}_{\mathcal{L}}(M) := \frac{E_{\mathcal{L}}(M)}{N_{\mathcal{L}}(M)}.$$
We note that if $\mathcal{M} \hookrightarrow \mathcal{L}$ is a subset of moderate approximate units then $\mathcal{G}_\mathcal{L}(M) \hookrightarrow \mathcal{G}_\mathcal{M}(M)$ thus provides a contravariant functor from subsets of regularizing processes to generalized functions.

Of course any distribution defines a map from $\Psi^{-\infty}(M)$ to $\mathcal{C}^\infty(M)$ by evaluation (4).

**Lemma 4.2.** For any distribution $u$ the map $\Theta_u \in E(\mathcal{U})$. Hence $\Theta_u \in E_{\mathcal{L}}(M)$ for any set of moderate approximate units $\mathcal{L}$. Thus we have an embedding of $\mathcal{D}'(M) \hookrightarrow \mathcal{G}_\mathcal{L}(M)$. This embedding restricts on $\mathcal{C}^\infty(M)$ to an algebra homomorphism.

**Proof.** Since any $T_\varepsilon \in \mathcal{U}$ is by definition a moderate net of smoothing operators and a distribution $u : \Psi^{-\infty}(M) \to \mathcal{C}^\infty(M)$ is a continuous map, hence by Lemma 2.1 $\Theta_u(T_\varepsilon) \in E_{\mathcal{C}^\infty(M)}$. This embedding restricts on $\mathcal{C}^\infty(M)$ to an algebra homomorphism because for any $T_\varepsilon \in \mathcal{U}$ and any $f \in \mathcal{C}^\infty(M)$

$$\Theta_f(T_\varepsilon) - f = T_\varepsilon f - f \in N_{\mathcal{C}^\infty(M)}, $$

again by Definition 3.1. □

4.1. **Operations of generalized functions.** Next we move to action of diffeomorphisms on these algebras and we check that the embedding of distributions is equivariant with respect to diffeomorphism action.

4.1.1. **Action of diffeomorphism.** Let $\chi : M \to M$ be a diffeomorphism on $M$. Then $\chi$ acts on $\Psi^{-\infty}(M)$ by push forward of operators as:

$$\chi^*(T)f := \chi^*T(\chi^{-1}*f).$$

Let $\mu_y \Omega$ be a nonzero section of the density bundle on $M$. Let $T$ be given by a kernel $k_T(x,y)\mu_y \in \Gamma(M \times M, \pi_2^*\Omega)$ then the kernel of $\chi^*T$ is given by $k_T(\chi x, \chi y)\chi^{-1}*\mu_y$.

We extend the action of Diff$(M)$ on $\phi \in E^\infty$ by

$$\chi^*\phi(T) := \chi^*(\phi(\chi^*(T))).$$

The composition can be seen as the following diagram,

$$
\begin{array}{ccc}
\Psi^{-\infty}(M) & \xrightarrow{\chi^* \phi} & \mathcal{C}^\infty(M) \\
\downarrow{\chi^*} & & \downarrow{\chi^*} \\
\Psi^{-\infty}(M) & \xrightarrow{\phi} & \mathcal{C}^\infty(M)
\end{array}
$$

**Lemma 4.3.** The embedding of $\mathcal{D}'(M)$ in $E^\infty$ is equivariant under diffeomorphisms that is

$$\chi^*(\Theta_u) = \Theta_{\chi^* u}.$$
Proof. Let $T$ be a smoothing operator with kernel $k_T(x, y)\mu_y$ then
\[
\Theta_{x^*u}(T) = x^*u(T) = \langle u(x^*u(y), k_T(x, y)\mu_y) \\
= \langle u(y), k_T(x, x(y))x^{-1^*}\mu_y \rangle \\
= x^{-1^*}\langle u(y), k_T(x(x), x(y))x^{-1^*}\mu_y \rangle = x^*\Theta_u(T).
\]

\[
\square
\]

Corollary 4.4. Let $X \subset U$ be a set of moderate units. Now let $\chi$ be a diffeomorphism then $\chi^* : G_X(M) \to G_{\chi^*(X)}(M)$ and hence if $\chi^*(X) = X$ then the action of $\chi$ descends naturally to an action on $G_X(M)$.  

4.1.2. Pseudodifferential operators. The smoothing operators $\Psi^{-\infty}(M)$ form an ideal in the algebra of pseudodifferential operators $\Psi^\infty(M)$ hence it is very easy to define an action of pseudodifferential operator $P$ on the space $E^\infty$ that extends their action on $D'(M)$. We define the operator on $\phi \in E^\infty$ by,
\[
\tag{5} P\phi(T) := \phi(TP), \quad \phi \in E^\infty, T \in \Psi^{-\infty}(M).
\]
This is indeed an extension of the operators on $D'(M)$ as $P\Theta_u(T) = \Theta_u(TP) = TP(u) = \Theta_Pu(T)$.  

Let $\hat{E}$ be the ring of polynomially bounded smooth maps on the complex plane. More precisely the set of all maps
\[
u : \mathbb{C} \to \mathbb{C} \quad \exists p \in \mathbb{C}[z] \mid |u(z)| \leq |p(z)| \forall z.
\]
Every element $u \in \hat{E}$ induces a map $u_* : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. Let $\hat{N}$ be all elements in $\hat{E}$ such that the induce map is 0 map or $\hat{N} = \ker(u \to u_*)$. We define
\[
\hat{\mathbb{C}} := \frac{\hat{E}}{\hat{N}}.
\]
Then it is clear that $\hat{\mathbb{C}}$ is a ring and all the algebras $G_{\mathbb{C}}(M)$ are algebras over the ring $\hat{\mathbb{C}}$. by the action
\[
u.\phi : \Psi^\infty(M) \to \mathcal{C}^\infty(M) \quad \nu.\phi(T) = u(\text{Tr}(T))\phi(T).
\]
Here $\text{Tr}(T) = \int_M k_T(y, y)\mu_y$ is the operator trace of $T$.

5. Examples

5.1. Riemannian manifolds. Let $M$ be a closed Riemannian manifold and let $\Delta$ be the associated scalar Laplace operator. Let $f \in \mathcal{S}(\mathbb{R})$ be a Schwartz function with $f \equiv 1$ near the origin. Then $T_\varepsilon := f_\varepsilon(\Delta)$ is a moderate approximate unit and let $X_f = \{T_\varepsilon\}$ be the singleton set. Since the Laplace operator is invariant under isometries $X_f$ is invariant under the group of isometries $\Gamma := \text{Iso}(M)$. Therefore $\tilde{G}_{X_f}(M)$
has a natural action of $\Gamma$ which is equivariant under the embedding of the distributions in $G_X f(M)$.

**Proposition 5.1.** Let $f \in \mathcal{S}(\mathbb{R})$ be a real valued Schwartz function with $f \equiv 1$ near the origin and $f(x)$ is monotone on $(0, \infty)$ and strictly monotonically decreasing on $(1, \infty)$. Let $G^s(M)$ denote the special Colombeau algebra with smooth nets then $G_X f(M)$ is naturally algebra isomorphic to $G^s(M)$.

**Proof.** Let $\phi : \Psi^\infty(M) \to C^\infty(M)$ represent $\alpha \in G_X f(M)$. The map $\rho(\alpha) := [\phi(T_\varepsilon)] \in G^s(M)$, shall be the required isomorphism.

It is evident that $\rho : G_X f(M) \to G^s(M)$ is well defined, injective algebra morphism. To check the surjectivity of $\rho$ we first note that by our assumption on $f$ it follows that $\varepsilon \to T_\varepsilon$ is a smooth embedded curve in $\Psi^\infty(M)$. To see this we check,

- The map $\varepsilon \to T_\varepsilon$ is injective: If $\varepsilon \neq \delta$ then we have $f_\varepsilon(x) \neq f_\delta(x)$ for all $x$ large enough and hence $f_\varepsilon(\Delta) \neq f_\delta(\Delta)$.
- It is an immersion as $\frac{d}{d\varepsilon} T_\varepsilon = F_\varepsilon'(\Delta) \neq 0$.
- It homeomorphism on its image: Let $\lambda_n, \phi_n$ be respectively the eigenvalues and eigenvectors of the Laplace (say ordered and counted with multiplicity.) For any $n \in \mathbb{N}$ the map $j_n : \Psi^\infty(M) \to \mathbb{C}$ $j_n(T) := \langle T(\phi_n), \phi_n \rangle$, is a continuous map such that $j_n(F(\lambda)) = F(\lambda \lambda_n)$, At each $\delta \in (0, 1)$ there exists $k$ such that $F(\lambda \lambda_k) = C_\delta \leq \frac{1}{2}$. By monotonicity of $F$ there is an interval $(\delta - \tau, \delta + \tau)$ such that The inverse image of an open ball, of radius $r < \frac{C_\delta}{2}$ around $C_\delta$ in $\mathbb{C}$ under $j_k$ intersects $T_\varepsilon$ in an interval. Since $\Psi^\infty(M)$ is a metrizable this is enough to proof that the image is homeomorphic to $(0, 1)$.

Now given an element $\beta \in G^s(M)$ we pick a smooth representative $u_\varepsilon$. The map $\phi(T_\varepsilon) = u_\varepsilon$ is a smooth map from an embedded submanifold and hence can be extended to all of $\Psi\infty(M)$. This is because $\Psi^\infty(M)$ is a nuclear Frechét space and hence is $C^\infty$-paracompact (see [8] Theorem 16.10). We call one such extension $\phi$ Then $\rho([\phi]) = \beta$. □

5.2. **Symplectic manifolds.** Let $(M, \omega)$ be a closed symplectic manifold. Recall that an almost complex structure on $M$ is an is a bundle map $J : TM \to TM$ such that $J^2 = -\text{Id}$.
A metric $\mathcal{G}$ is compatible with $\omega$ if there exists an almost complex structure $J$ such that

$$\omega(Jx, Jy) = \mathcal{G}(X, Y), \quad X, Y \text{ vectorfields.}$$

By polar decomposition on any metric one knows that compatible metrics and almost complex structures always exist. Let $\mathcal{G}(\omega)$ be the set of all compatible Riemannian metrics and let $\Delta(\omega)$ be the set of all scalar Laplaces associated with compatible Riemannian metrics. Let $f \in \mathcal{S}(\mathbb{R})$ be a Schwartz function $f \equiv 1$ near origin. Let

$$X(\omega, f) := \{ f_\varepsilon(\Delta) | \Delta \in \Delta(\omega) \}.$$

Then $X(\omega, f)$ is a set of moderate approximate units.

**Lemma 5.2.** The set of approximate units $X(\omega, f)$ is invariant under the group of symplectic diffeomorphism $\text{Symp}(M)$.

**Proof.** Let $\Delta$ be the scalar Laplace operator associated with the metric $\mathcal{G}$ and let $\phi$ be a diffeomorphism then the push forward operator $\phi_* (\Delta)$ is the scalar Laplace associated to the metric $\phi^{-1*}(\mathcal{G})$.

Also for any $f \in \mathcal{S}(\mathbb{R})$ a Schwartz function

$$f(\phi_* (\Delta)) = \phi_* (f(\Delta)).$$

Thus the desired result follows from the fact that if $\phi$ is a symplectic diffeomorphism then $\phi$ preserves the space of compatible metrics $\mathcal{G}(\omega)$.

Thus combining with Corollary 4.4 we have the following theorem.

**Theorem 5.3.** The full type algebra defined by the set of approximate units $X(\omega, f)$ has an action of $\text{Symp}(M)$ and the natural embedding of distributions is equivariant under this action.

### 6. Regularity

In this section we study the regularity structure of distributions within the settings of our generalized functions.

We first consider regularity of maps between Frechét spaces. Recall that a grading on a Frechét space $X$ is a sequence of seminorms $\| \|_n$ that is increasing (that is $\| \|_1 \leq \| \|_2 \leq \ldots$) and generates the locally convex topology on $X$. We refer to [6] for further study.

**Definition 6.1.** Let $X$ and $Y$ be graded Frechét spaces. We denote by $\| \|_n$ and $\| \|'_n$ the $n$-th graded norm on $X$ and $Y$ respectively. We say that a Frechét smooth map $\phi : X \to Y$ is polynomially tame if there exist $b, k \in \mathbb{N}$ and some $r \in \mathbb{Z}$ such that

$$\| \phi(x) \|'_n \leq C \| x \|'_n^{k+r} \quad \text{for all } n \geq b + |r|. \quad (6)$$


Here $C > 0$ is a constant that depends only on $n$. The number $r$ is called the degree of tameness and the set of all maps of tameness degree $r$ is denoted by $\PT^r(X,Y)$. Let $\PT(X,Y) := \cup_r \PT^r(X,Y)$.

A polynomially tame map is called regular if there exists a $k$ and there exists a $b$ such that for any degree of tameness $r \in \mathbb{Z}$ the condition (6) holds. It is clear that

$$\Reg(X,Y) \subseteq \cap_r \PT^r(X,Y).$$

We would also require the following tameness property for an associative multiplication on a Fréchet space.

**Definition 6.2.** We say that $X$ is a Frechet algebra if it is a Frechet space with an associative product and the multiplication is jointly continuous. A graded Frechet algebra is a Frechet algebra such that the multiplication satisfies a tameness condition namely there exist $b, r_1, r_2 \in \mathbb{N}$ such that

$$\|x \cdot y\|_n \leq C \|x\|_{n+r_1} \|y\|_{n+r_2} \ orall \ n \geq b.$$

With the above definition the following lemma is self-evident.

**Lemma 6.3.** Let $Y$ be a graded Frechet algebra. And let $X$ be any graded Frechet space. then the space of polynomially tame maps from $X$ to $Y$ is an algebra under pointwise operations. The regular maps form a subalgebra.

The algebra of polynomially tame maps $\PT(X,Y)$ and the regular maps as $\Reg(X,Y)$ depend on not just the topology of $X$ and $Y$ but also on the choice of the grading structures on them. Equivalent gradings provide the same algebras. We shall grade $C^\infty(M)$ with Sobolev norms. Let $\Delta$ denote the Laplace operator on $M$ associated to a metric then,

$$\|f\|_n := \|(1 + \Delta)^{\frac{n}{2}} f\|_{L^2(M)}.$$  

With the above grading $C^\infty(M)$ is a graded Frechet algebra in the sense of Definition 6.2.

With the metric we identify $\Psi^{-\infty}(M) = C^\infty(M \times M)$. Define a grading on $\Psi^{-\infty}(M)$ by:

$$\|T\|_n := \sum_{q+p=n, q,p \geq -n} \|(1 + \Delta)^{\frac{q}{2}} (1 + \Delta)^{\frac{p}{2}} T\|_{\text{HS}}.$$  

For any operator $D : L^2(M) \rightarrow L^2(M)$ we denote by $\|D\|_{\text{HS}}$ its Hilbert-Schmidt norm. With the above choice of gradings we shall denote $\PT(M) := \PT(\Psi^{-\infty}(M), C^\infty(M))$ and $\Reg(M) := \Reg(\Psi^{-\infty}(M), C^\infty(M))$.

**Proposition 6.4.** Let $\Psi^{-\infty}(M)$ and $C^\infty(M)$ be graded as above then:
(1) A smooth map \( \phi : \Psi^{-\infty}(M) \to \mathcal{C}^\infty(M) \) is a regular map then \( \phi(T_\varepsilon) \in \mathcal{G}^\infty(M) \) for any moderate approximate unit \( T_\varepsilon \).

(2) All polynomially tame maps \( \phi \in \text{PT}(M) \) belong to \( E(U) \).

(3) For any distribution \( u \in \mathcal{D}'(M) \) the image \( \Theta_u : \Psi^{-\infty}(M) \to \mathcal{C}^\infty(M) \) is a polynomially tame map. In fact
\[
u \in H^k(M) \leftrightarrow \Theta_u \in \text{PT}^{-k}(M).
\]

Proof. We recall that \( \mathcal{G}^\infty(M) \) is a subalgebra of the the special algebra \( \mathcal{G}^s(M) \) such that an element \( x \) is moderate of same order with respect to all seminorms on \( \mathcal{C}^\infty(M) \).

(1) By assumption \( T_\varepsilon \) is a moderate net of smoothing operator hence for any \( b \) there exists an \( M \) such that \( \| T_\varepsilon \|^b_k \sim O(\varepsilon^M) \). Since \( \phi \in \text{Reg}(M) \) by definition there exist \( k, b \) satisfying (6) for any \( r \). In particular set \( r = b - n \) for \( n \) large enough we have
\[
\| \phi(T_\varepsilon) \|^n \leq C\| T_\varepsilon \|^b_k \sim O(\varepsilon^M).
\]
Thus the net \( \phi(T_\varepsilon) \) is in \( \mathcal{G}^\infty(M) \).

(2) Again follows from moderateness of \( T_\varepsilon \in U \).

(3) Let \( u \in H^k(M) \) be a distribution then:
\[
\| \Theta_u(T) \|^n = \| T(u) \|^n = \| (1 + \Delta)^{\frac{n}{2}} T(u) \|^{|L^2(M)}
= \| (1 + \Delta)^{\frac{n}{2}} T(1 + \Delta)^{\frac{k}{2}} (1 + \Delta)^{\frac{k}{2}} (u) \|^{|L^2(M)}
\leq \| (1 + \Delta)^{\frac{n}{2}} T(1 + \Delta)^{\frac{k}{2}} \|^{|H_S}|(1 + \Delta)^{\frac{k}{2}} (u) \|^{|L^2(M)}
= C\| T \|^{|n-k|}.
\]

Corollary 6.5. The only distributions which give rise to regular maps are smooth functions. That is
\[
\mathcal{D}'(M) \cap \text{Reg}(M) = \mathcal{C}^\infty(M).
\]

Proof. Let \( T_\varepsilon = F_\varepsilon(\Delta) \) then we know that (see[2])
\[
F_\varepsilon(\Delta)(\mathcal{D}'(M)) \cap \mathcal{G}^\infty(M) = \mathcal{C}^\infty(M).
\]
Hence by part (1) of Proposition 6.4 if \( \Theta_u \) is in \( \text{Reg}(M) \) then \( u \) must be smooth. Also from the proof of part (2) it is obvious that any smooth function defines a map in \( \text{Reg}(M) \).}

In view of the above result we regard the subalgebra \( \text{Reg}(M) \) as an analogue of Oberguggenberger’s algebra \( \mathcal{G}^\infty(M) \). It provides some regularity features to the space \( \mathcal{G}^s(M) \) consistant with the regularity
of distributions. It is clear that PT(M) and Reg(M) are modules over \( \mathcal{C}^\infty(M) \) with the natural action as

\[
f.\psi(T) := \Theta f \cdot \psi(T) \quad \forall f \in \mathcal{C}^\infty(M) \ \psi \in PT(M).
\]

Note that this module action restricted to \( \mathcal{D}'(M) \hookrightarrow PT(M) \) is not the usual module action of \( \mathcal{C}^\infty(M) \) on distributions. Let \( M_f \) denote multiplication by a smooth function \( f \) on space of distributions \( \mathcal{D}'(M) \). Then we define

\[
M_f.\phi(T) := \phi(T.M_f) \quad \phi \in E(U) \quad T \in \Psi^{-\infty}(M).
\]

**Definition 6.6.** The support of \( \phi \in E(U_{\text{loc}}) \) is the complement of the biggest open set \( U \subseteq M \) such that \( f.\phi \in N(U_{\text{loc}}) \) for any function \( f \) supported in \( U \). The singular support of \( \phi \in E(U) \) is the complement of the biggest open set \( U \subseteq M \) such that \( M_f.\phi \in \text{Reg}(M) \) for any function \( f \) supported in \( U \).

**Lemma 6.7.** For a distribution \( u \in \mathcal{D}'(M) \) and any local moderate approximate unit \( T_\varepsilon \), the following are true.

(a) \( \text{supp } u = \text{supp } \Theta u = \text{supp } T_\varepsilon u \).

(b) \( \text{singsupp } u = \text{singsupp } \Theta u = \text{singsupp } T_\varepsilon u \).

**Proof.** The statements \( \text{supp } u = \text{supp } T_\varepsilon \) and \( \text{singsupp } u = \text{singsupp } T_\varepsilon u \) follow from Lemma 3.5. By definition \( \text{supp } \Theta u = \cap_{T_\varepsilon \in U_{\text{loc}}} \text{supp } T_\varepsilon u \).

Also it is immediate from the definition that \( M_f \Theta u = \Theta f u \). Hence the equality of singular support follows from Corollary 6.5.\( \square \)

Now if \( P \) is a pseudodifferential operator of order \( m \) then we study the action of \( P \) on polynomially tame maps. First a quick observation that right multiplication by \( P \) on \( \Psi^{-\infty}(M) \) namely the map \( T \rightarrow TP \) is tame of tameness \( m \). To see this one notes that the operator \( (1 + \Delta)^{m/2} \) generates \( \Psi^m(M) \) as a left-module (and also right-module) over \( \Psi^0(M) \). Thus \( P = P_0(1 + \Delta)^{m/2} \). Therefore we set

\[
P = P_0(1 + \Delta)^{m/2} \quad P_0 \in \Psi^0(M).
\]

By the same token for \( k \) an integer multiple of \( \frac{1}{2} \) we find an order 0 operator \( T_k \) such that

\[
[P_0, (1 + \Delta)^k] = (1 + \Delta)^{k-1}T_k.
\]
Put this together we have
\[ \|TP\|_{p,q} = \|(1 + \Delta)^{\frac{q}{2}} TP(1 + \Delta)^{\frac{q}{2}}\|_{HS} = \|(1 + \Delta)^{\frac{q}{2}} TP_0(1 + \Delta)^{\frac{p+m}{2}}\|_{HS} \]
\[ \leq \|(1 + \Delta)^{\frac{q}{2}} T(1 + \Delta)^{\frac{p+m}{2}} P_0\|_{HS} + \|(1 + \Delta)^{\frac{q}{2}} T [P_0, (1 + \Delta)^{\frac{p+m}{2}}]\|_{HS} \]
\[ \leq \|P_0\|\|(1 + \Delta)^{\frac{q}{2}} T(1 + \Delta)^{\frac{p+m}{2}}\|_{HS} + \|T_{p+m}\|\|(1 + \Delta)^{\frac{q}{2}} T(1 + \Delta)^{\frac{p+m-1}{2}}\|_{HS} \]
\[ \leq C\|T\|_{p+q+m} \]
In particular this implies that
\[ \|TP\|_n \leq C\|T\|_{n+m}. \]

If \( \phi \in PT^r(X,Y) \) and \( \tau \in PT^m(Y,Z) \) then it is obvious that \( \tau \circ \phi \in OpPT^{r+m}(X,z) \) Hence we have the following result.

**Proposition 6.8.** Let \( P \) be an order \( m \) pseudodifferential operator than it extends to a map \( PPT^r(M) \to PT^{r+m}(M,M) \) by the map \( P(\phi)(T) := \phi(TP) \) as defined in (5).

Now in exact analogy with classical notion of wavefront set of a distribution to the wavefront set of a generalized function as follows. Recall that for a pseudodifferential operator \( P \) let \( \sigma(P) \) denote the principal symbol of \( P \). Then we denote as usual the characteristic set of \( P \) by \( \text{Char}(P) = \sigma^{-1}\{0\} \subseteq T^*M \). We denote by \( \Psi^M_d(M) \) all classical pseudodifferential operator of order \( M \).

**Definition 6.9.** The wavefront set of a generalized function \( \phi \in E(U) \) would be given by:
\[ WF_R(\phi) := \bigcap_{P \in \text{Reg}(M)} \text{Char}(P), \quad P \in \Psi^0_d(M). \]

**Proposition 6.10.** Let \( u \in \mathcal{D}'(M) \) be a distribution then
\[ WF_R(\Theta u) = WF u. \]

**Proof.** Since \( P\Theta u = \Theta P u \) it follows from Corollary 6.5 that
\[ P\Theta u \in \text{Reg} \iff \Theta P u \in \text{Reg}(M) \iff P u \in C^\infty(M). \]

Hence
\[ WF_R(\Theta u) = \bigcap_{P \Theta u \in \text{Reg}(M)} \text{Char}(P) \]
\[ = \bigcap_{P u \in C^\infty(M)} \text{Char}(P) = WF(u). \]
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