Prescribed Q-curvature flow on closed manifolds of even dimension

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Abstract

On a closed Riemannian manifold \((M, g_0)\) of even dimension \(n \geq 4\), the well-known prescribed \(Q\)-curvature problem asks whether there is a metric \(g\) conformal to \(g_0\) such that its \(Q\)-curvature, associated with the GJMS operator \(P_g\), is equal to a given function \(f\). Letting \(g = e^{2u}g_0\), this problem is equivalent to solving

\[
P_{g_0}u + Q_{g_0} = fe^{nu},
\]

where \(Q_{g_0}\) denotes the \(Q\)-curvature of \(g_0\). The primary objective of the paper is to introduce the following negative gradient flow of the time dependent metric \(g(t)\) conformal to \(g_0\),

\[
\frac{\partial g(t)}{\partial t} = -2 \left( Q_{g(t)} - \frac{\int_M f Q_{g(t)} d\mu_{g(t)}}{\int_M f^2 d\mu_{g(t)}} \right) g(t) \quad \text{for } t > 0,
\]

to study the problem of prescribing \(Q\)-curvature. Since \(\int_M Q_g d\mu_g\) is conformally invariant, our analysis depends on the size of \(\int_M Q_{g_0} d\mu_{g_0}\) which is assumed to satisfy

\[
\int_M Q_0 d\mu_{g_0} \neq k(n - 1)! \text{vol}(\mathbb{S}^n) \quad \text{for all } k = 2, 3, \ldots
\]

The paper is twofold. First, we identify suitable conditions on \(f\) such that the gradient flow defined as above is defined to all time and convergent, as time goes to infinity, sequentially or uniformly. Second, we show that various existence results for prescribed \(Q\)-curvature problem can be derived from the convergence of the flow.

Keywords \(Q\)-curvature · Negative gradient flow · Closed manifolds · Even dimension

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1 Introduction

On a closed manifold \((M, g)\) of dimension \(n \geq 3\), a formally self-adjoint geometric differential operator \(A_g\) of the metric \(g\) is called conformally covariant of bidegree \((a, b)\) if

\[
A_{g_w}(\varphi) = e^{-bw} A_g(e^{aw} \varphi)
\]  

(1.1)

for all \(\varphi \in C^\infty(M)\), where \(g_w := e^{2w} g\) is a conformal metric to \(g\). A typical geometric differential operator in conformal geometry is the second-order conformal Laplacian which is defined by

\[
L_g(\varphi) := -\Delta_g \varphi + \frac{n-2}{4(n-1)} R_g \varphi,
\]

where \(R_g\) is the scalar curvature of \(g\). A well-known fact is that \(L_g\) is conformally covariant of bidegree \(((n-2)/2, (n+2)/2)\) in the sense of (1.1) since

\[
L_{g_w}(\varphi) = e^{-(n+2)w} L_g(e^{(n-2)w} \varphi), \quad \forall \varphi \in C^\infty(M).
\]  

(1.2)

If we write \(u^{4/(n-2)} = e^{2w}\), then Eq. (1.2) is changed into

\[
L_{g_u}(\varphi) = u^{\frac{n+2}{n-2}} L_g(u^{-1} \varphi), \quad \forall \varphi \in C^\infty(M).
\]

By setting \(\varphi \equiv 1\), we get, with the conformal metric \(g_u = u^{4/(n-2)} g\), the transformation law of the scalar curvature \(R_g\)

\[
-\frac{4(n-1)}{n-2} \Delta_g u + R_g u = R_{g_u} u^{\frac{n+2}{n-2}}.
\]  

(1.3)

In the literature, Eq. (1.3) is closely related to the prescribed scalar curvature problem which is to find a positive smooth function \(u\), for a given function \(f\), such that \(R_{g_u} = f\). This challenging problem has already captured much attention by many mathematicians during the past few decades. In the case that the function \(f\) is constant, the prescribed scalar curvature problem is widely known as Yamabe problem.

The first higher order example of conformal operators, discovered by Paneitz [29], is the fourth-order Paneitz operator \(P^4_g\) on a 4-manifold \((M, g)\). It is given by

\[
P^4_g(\varphi) = (-\Delta_g)^2 \varphi - \text{div}_g \left( \left( \frac{2}{3} R_g g - 2 \text{Ric}_g \right) d\varphi \right),
\]

where \(d\) is the differential and \(\text{Ric}_g\) denotes the Ricci tensor. The Paneitz operator \(P^4_g\) is conformally covariant of bidegree \((0, 4)\) in the sense of (1.1) because under the conformal change \(g_u = e^{2u} g\), there holds

\[
P^4_{g_u}(\varphi) = e^{-4u} P^4_g(\varphi), \quad \forall \varphi \in C^\infty(M).
\]  

(1.4)

By letting \(\varphi \equiv 1\), we deduce from (1.4) that

\[
Q^4_{g_u} = e^{-4u} (P^4_g(u) + Q^4_g),
\]

where the quantity \(Q^4_g\) is defined by

\[
Q^4_g = -\frac{1}{6} (\Delta R_g - R_g^2 + 3 |\text{Ric}_g|^2).
\]
The quantity $Q^4_g$, discovered by Branson [5], is called the $Q$-curvature associated with the Paneitz operator $P^4_g$ on $(M, g)$. One of interesting features of the $Q$-curvature is that it is closely related to the Gauss–Bonnet–Chern theorem via the identity

$$\int_M Q^4_g + \frac{1}{4} |W_g|^2_g d\mu_g = 8\pi^2 \chi(M),$$

where $\chi(M)$ is the Euler characteristic of $M$ and $W_g$ is the Weyl tensor. Since the Weyl tensor $W_g$ is conformally invariant, the identity above implies that the total $Q$-curvature $\int_M Q^4_g d\mu_g$ is also conformally invariant; that is the value $\int_M Q^4_g d\mu_g$ does not depend on the metric $g$ but its conformal class $[g]$. Taking this fact into account, one may split the argument for the existence of solutions to (1.4) into several cases according to the size of $\int_M Q^4_g d\mu_g$. Served as Aubin’s threshold in the Yamabe problem, there is a threshold $16\pi^2$ for $\int_M Q^4_g d\mu_g$ in the $Q$-curvature problem on the 4-manifold $M$. (The number $16\pi^2$, which is $3!\text{vol}(S^4)$, is exactly equal to $\int_{S^4} Q^4_{g_{S^4}} d\mu_{g_{S^4}}$ being calculated with respect to the standard metric $g_{S^4}$ on $S^4$.) The technical condition $\int_M Q^4_g d\mu_g < 16\pi^2$ is often used since the standard variational argument can be applied. Notice that such a condition is not so restrictive because there is a large class of satisfied manifolds; see [20].

For general even dimension $n \geq 4$, Graham et al. [19] discovered a similar operator $P^n_g$ which is also a conformally invariant, elliptic, self-adjoint operator of order $n$ with leading term $(-\Delta)^{n/2}$. This operator is commonly known as the GJMS operator. The work [19] was based on an earlier one by Fefferman and Graham [16], in which existence of scalar conformal invariants were derived via a new tool called ambient metric construction. Similar to (1.4), the GJMS operator is conformally covariant of bidegree $(0, n)$ in the sense that under the conformal change $g_u = e^{2u} g$, there holds

$$P^n_{g_u} (\varphi) = e^{-nu} P^n_g (\varphi), \quad \forall \varphi \in C^\infty (M).$$

As in the fourth order case, the higher order operator $P^n_g$ has an associated curvature quantity $Q^n_g$; see [6, page 11]. The quantity $Q^n_g$ obeys a similar transformation law as $Q^4_g$ does, namely

$$Q^n_{g_u} = e^{-nu} (P^n_{g_u} u + Q^n_g). \quad (1.5)$$

An immediate consequence of (1.5) is that $\int_M Q^n_{g_u} d\mu_{g_u}$ is conformal invariant.

Up to this point, we can describe the prescribed $Q$-curvature problem on a generic manifold of even dimension $n$. Assume that $(M, g_0)$ is a closed Riemannian manifold of even dimension $n$ with background metric $g_0$ and that $f$ is a smooth function on $M$. Then one may ask if there exists a pointwise conformal metric $g$, that is $g = e^{2u} g_0 \in [g_0]$ for some smooth function $u$, such that $f$ can be realized as the $Q$-curvature of $g$. Now, thanks to the rule (1.5), this geometric problem is equivalent to solving the higher order nonlinear equation

$$P_0 u + Q_0 = f e^{nu}, \quad (1.6)$$

where we simply set $P_0 := P^n_{g_0}$ and $Q_0 := Q^n_{g_0}$. In the sequel, we also set $P_g := P^n_g$ and $Q_g := Q^n_g$ for brevity. Back to (1.6), if the equation has a solution $u$, then integrating both sides of the equation gives

$$\int_M Q_0 d\mu_{g_0} = \int_M f e^{nu} d\mu_{g_0}.$$
Consequently, the following conditions, corresponding to the value of $\int_M Q_0 \, d\mu_{g_0}$, on the smooth function $f$ are necessary for the existence of solutions to (1.6)

\[
\begin{align*}
\sup_{x \in M} f(x) > 0 & \quad \text{if } \int_M Q_0 \, d\mu_{g_0} > 0, \\
\sup_{x \in M} f(x) > 0 \text{ and } \inf_{x \in M} f(x) < 0 & \quad \text{if } \int_M Q_0 \, d\mu_{g_0} = 0, \\
\inf_{x \in M} f(x) < 0 & \quad \text{if } \int_M Q_0 \, d\mu_{g_0} < 0.
\end{align*}
\] (1.7)

In the higher dimensional cases, the number $(n - 1)! \text{vol}(\mathbb{S}^n)$ is the threshold, which plays a similar role as $16\pi^2$ does on 4-manifolds. As a result, we call Eq. (1.6) in the subcritical case, critical case, or the supercritical case accordingly if $\int_M Q_0 \, d\mu_{g_0} <, = $, or $> (n - 1)! \text{vol}(\mathbb{S}^n)$. For convenience, we also call Eq. (1.6) in the negative case or the null case if $\int_M Q_0 \, d\mu_{g_0} <$ or $= 0$, respectively.

Within the last few decades, the class of lower-order problems has already been intensively studied and a general picture is now quite clear. People have already turned their attention to higher-order problems. There are many research works on the Eq. (1.6), of which some are listed as follows: subcritical case: [3,4,7]; critical case: [8,9,11,21,23,24,32]; and supercritical case: [4,13,25,26]. The majority of these works make use of an assumption on the operator $P_0$, that is,

the operator $P_0$ is positive with kernel consisting of the constant functions. (P)

It is possible to work with a weaker assumption on $P_0$, for instance, one may relax the positivity of $P_0$ as in [4]. In this sense, the operator $P_0$ has finitely many negative eigenvalues and the analysis is rather involved. For example, the Adams inequality (2.2) is only valid for any function $u$ with zero projection onto the eigenspace spanned by negative eigenvalues of $P_0$. In the present work, we do not pursue this direction and leave it for future research.

Among those works above, in particular, Brendle [7] considered the $L^2$-gradient flow

\[
\begin{align*}
\frac{\partial g(t)}{\partial t} &= -\left( Q_{g(t)} - \frac{\int_M Q_0 \, d\mu_{g_0}}{\int_M f \, d\mu_{g_0}} f \right) g(t) \quad \text{for } t > 0, \\
g(0) &= e^{2u_0} g_0,
\end{align*}
\] (1.8)

where $f$ is assumed to be positive everywhere and $u_0$ is a smooth initial data. Then he proved, in the subcritical case, that if the condition (P) holds, the flow (1.8) has a solution which is defined to all time and converges at infinity to a metric $g_\infty$ with

\[ Q_{g_\infty} = \frac{\int_M Q_0 \, d\mu_{g_0}}{\int_M f \, d\mu_{g_0}} f. \]

A direct conclusion of the above is that after shifting by a constant, Eq. (1.6) has a solution in the negative case (with $f$ replaced by $-f$ in Eq. (1.6) in this case) or if $0 < \int_M Q_0 \, d\mu_{g_0} < (n - 1)! \text{vol}(\mathbb{S}^n)$. Notice that by the positivity of $P_0$ mentioned in the hypothesis (P) we mean

\[
\int_M u \cdot P_0 u \, d\mu_{g_0} \geq \lambda_1 \int_M (u - \bar{u})^2 \, d\mu_{g_0} \geq 0
\] (1.9)

for all $u \in C^\infty(M)$. Here $\lambda_1 > 0$ is the first (non-trivial) eigenvalue of the operator $P_0$ and

\[ \bar{u} = \frac{1}{\text{vol}(M, g_0)} \int_M u \, d\mu_{g_0}. \]
denotes the average of \( u \) calculated with respect to \( g_0 \). (In the rest of our paper, by \( \text{vol}(K) \) we mean the volume of a subset \( K \) of \( M \) being calculated with respect to \( g_0 \). If we want to emphasize that the metric \( g \) is being used to calculate the volume of \( K \), we shall write \( \text{vol}(K, g) \). Furthermore, we also drop the symbol \( g_0 \) in any functional space if the background metric \( g_0 \) is being used, for examples, \( L^2(M) := L^2(M, g_0) \), \( H^{n/2}(M) := H^{n/2}(M, g_0) \), etc.)

Later on, Baird et al. [3] extended Brendle’s result to the case where the candidate curvature \( f \) is allowed to change sign when the underlying manifold has dimension 4. They adopted a \( H^2 \)-gradient flow which is different from (1.8). To be precise, they considered the functional

\[
J[u] = \int_M u \cdot P_0 u \, d\mu_{g_0} + 2 \int_M Q_0 u \, d\mu_{g_0},
\]

on the Sobolev space \( H^2(M) \) under the constraint

\[
u \in X = \{ u \in H^2(M) : \int_M f e^{\Delta u} \, d\mu_{g_0} = \int_M Q_0 \, d\mu_{g_0} \}.
\]

Then, they studied the following negative gradient flow of functional \( J \) with respective to hypersurface \( X \) of Sobolev space \( H \)

\[
\begin{align*}
\partial_t u &= -\nabla^X J(u) \\
u(0) &= u_0 \in X,
\end{align*}
\]

where \( \nabla^X J \) is the projection of \( \nabla J \) onto \( X \). If the flow (1.10) exists for all time and converges at infinity, then the limit function \( u_\infty \) yields a solution to (1.6) with \( n = 4 \).

The primary objective of the paper is to generalize the 4-dimensional results in [3] to ones of all even dimensions and to further extend the work [7] to the supercritical case. This means that we will relax the widely used restriction \( \int_M Q_0 \, d\mu_{g_0} \leq (n-1)\text{vol}(S^n) \) to consider, instead, that \( \int_M Q_0 \, d\mu_{g_0} \neq k(n-1)\text{vol}(S^n) \) for all \( k \in \{2, 3, \ldots\} \). We will use a similar \( L^2 \)-gradient flow as (1.8); see also [27]. To overcome the difficulty in the supercritical case, we shall exploit the connection between \( L^2 \)-bound and the finite energy of the flow, which has already been observed in [15] for the case of constant \( Q \)-curvature; see Sect. 3. For examples of manifolds having \( \int_M Q_0 \, d\mu_{g_0} > (n-1)\text{vol}(S^n) \), we refer the reader to the work of Djadli and Malchiodi [12].

Now, let us describe our flow method in detail and state the main results of the paper. As in [3], we associate any function \( u \in H^{n/2}(M) \) the following energy

\[
\mathcal{E}[u] = \frac{n}{2} \int_M u \cdot P_0 u \, d\mu_{g_0} + n \int_M Q_0 u \, d\mu_{g_0}.
\]

We assume the condition \( (P) \). Since \( P_0 \) has the leading term \( (-\Delta)^{n/2} \), by adopting the convention that \( (-\Delta_0)^{m+1/2} = \nabla_0 (-\Delta_0)^m \) for all integers \( m \), it is not hard to see that for any \( u \in H^{n/2}(M) \) we have

\[
\int_M (-\Delta_0)^{n/4} u^2 \, d\mu_{g_0} \lesssim \int_M u \cdot P_0 u \, d\mu_{g_0} \lesssim \int_M (-\Delta_0)^{n/4} u^2 \, d\mu_{g_0}.
\]

Here and thereafter, we sometimes use \( \lesssim \) to denote an inequality up to a uniform constant depending only on the geometry of \( (M, g_0) \). Notice that by [30, Proposition 2] we can define an equivalent norm for \( H^P(M) \)

\[
\|u\|^2_{H^P(M)} = \int_M (-\Delta_0)^{P/2} u^2 \, d\mu_{g_0} + \int_M u^2 \, d\mu_{g_0}.
\]
However, in view of (1.12), we have another equivalent and more convenient norm on $H^{n/2}(M)$
\[
\|u\|_{H^{n/2}(M)}^2 = \int_M u \cdot P_0 u \, d\mu_{g_0} + \int_M u^2 \, d\mu_{g_0}.
\] (1.14)

Now, we define our negative gradient flow for the functional (1.11) as follows
\[
\partial_t g(t) = -2(\mathcal{Q}_g(t) - \lambda(t) f) g(t),
\] (1.15)
where the parameter $\lambda$ is chosen to fix the quantity $\int_M f \, d\mu_{g(t)}$ for all time. A direct calculation for $\partial_t \int_M f \, d\mu_{g(t)} = 0$ shows that
\[
\int_M f(\lambda(t) f - \mathcal{Q}_g(t)) \, d\mu_{g(t)} = 0.
\] (1.16)

Hence, $\lambda$ is given by
\[
\lambda(t) = \frac{\int_M f \mathcal{Q}_g(t) \, d\mu_{g(t)}}{\int_M f^2 \, d\mu_{g(t)}}.
\] (1.17)

One may easily find that the fundamental difference between our flow (1.15) and Brendle’s (1.8) is the choice of parameter $\lambda(t)$. In the Brendle’s work [8], the volume $\int_M d\mu_{g(t)}$ remains constant, while, in the current one, we maintain the quantity $\int_M f \, d\mu_{g(t)}$ unchanged instead.

As a result, given an initial metric $g(0)$ one has
\[
\int_M f \, d\mu_{g(t)} = \int_M f \, d\mu_{g(0)} \quad \text{for all } t > 0.
\] (1.18)

Since the evolution equation (1.15) preserves the conformal structure, we may write $g(t) = e^{2u(t)} g_0$ for some real-valued function $u(t)$. Then, the evolution equation for metrics $g(t)$ can be transformed into one for $u(t)$, namely
\[
\begin{cases}
\frac{\partial u}{\partial t} = \lambda(t) f - \mathcal{Q}_g(t) & \text{for } t > 0, \\
u(0) = u_0,
\end{cases}
\] (F)

where and throughout this paper, by the initial data $u_0$ we mean a smooth function chosen in such a way that
\[
\begin{align*}
u_0 &\in Y & \text{if } & \int_M Q_0 \, d\mu_{g_0} \leq (n - 1)! \text{vol}(\mathbb{S}^n), \\
u_0 &\in C^\infty(M) & \text{if } & \int_M Q_0 \, d\mu_{g_0} > (n - 1)! \text{vol}(\mathbb{S}^n),
\end{align*}
\] (1.19)

where
\[
Y = \left\{ u \in H^2(M) : \int_M f e^{nu} \, d\mu_{g_0} = \int_M Q_0 \, d\mu_{g_0} \right\}.
\]

We are ready to state our first result concerning the long-time existence and convergence of the flow (F) in the non-critical case. In what follows, we set $h^+ = \max\{h, 0\}$ and $h^- = \max\{-h, 0\}$ for any function $h$ on $M$.

**Theorem 1.1** Let $(M, g_0)$ be a compact, oriented $n$-dimensional Riemannian manifold with $n$ even. Assume that the GJMS operator $P_0$ is positive with kernel consisting of constant functions and the metric $g_0$ satisfies
\[
\int_M Q_0 \, d\mu_{g_0} \neq k(n - 1)! \text{vol}(\mathbb{S}^n) \quad \text{for all } k \in \{1, 2, \ldots\}.
\]
Let the initial data $u_0$ satisfy (1.19). We also let $f$ be a non-trivial smooth function satisfying (1.7). In addition, we further assume that, in the negative case, there are positive constants $C_0$ depending on $f^-$ and $(M, g_0)$ and $\tau$ depending only on $(M, g_0)$ such that
\[
\sup_M f^+ \leq C_0 \exp \left( -\tau \|u_0\|_{H^{n/2}(M)}^2 \right)
\]
and that $f > 0$ in the supercritical case. Then, for some $u_0$, the flow $(F)$ has a smooth solution on $[0, +\infty)$ and converges sequentially in the sense that there exists a smooth function $u_\infty$ and a real number $\lambda_\infty$ such that for a suitable time sequence $(t_j)_j$ with $t_j \to +\infty$ as $j \to +\infty$, there hold
\[
\text{(i)} \quad \|u(t_j) - u_\infty\|_{C^\infty(M)} \to 0,
\]
\[
\text{(ii)} \quad |\lambda(t_j) - \lambda_\infty| \to 0,
\]
\[
\text{(iii)} \quad \|Qg(t_j) - \lambda_\infty f\|_{C^\infty(M)} \to 0
\]
as $j \to +\infty$. Furthermore, there holds $\lambda_\infty = 1$ in the subcritical but non-null case.

**Remark 1.2** We will actually prove that the initial data $u_0$ can be freely chosen in $Y$ when the subcritical case is considered. However, the convergence of the flow in the supercritical case can only be guaranteed for some suitable $u_0$.

Now, let us turn to the critical case, namely $\int_M Q_0 d\mu_{g_0} = (n - 1)!\text{vol}(S^n)$. In this scenario, we only consider the underlying manifold to be the standard $n$-sphere $S^n$, equipped with the standard metric $g_{S^n}$. Also, we let $G$ be a group of isometries of $S^n$. Then a function $f$ is said to be $G$-invariant if it satisfies
\[
f(\sigma x) = f(x) \quad \text{for all} \quad \sigma \in G \text{ and } x \in S^n.
\]
Furthermore, we say that a conformal metric $g$ to $g_{S^n}$ is $G$-invariant if $u$ is a $G$-invariant function when $g$ is written as $g = e^{2u} g_{S^n}$. Let $\Sigma$ be the set of fixed points of $G$, that is,
\[
\Sigma = \{ x \in S^n : \sigma x = x \text{ for all } \sigma \in G \}.
\]
Inspired by [2, Theorem 3] concerning the prescribed scalar curvature on the 2-sphere, our result for the prescribed $Q$-curvature on the $n$-sphere is as follows.

**Theorem 1.3** Let $f$ be a smooth and $G$-invariant function on $S^n$ with $\sup_{x \in S^n} f > 0$. Also, let $u_0 \in Y$ be a $G$-invariant initial data. If
\[
\text{(a)} \quad \text{either} \quad \Sigma = \emptyset
\]
\[
\text{(b)} \quad \text{or}
\]
\[
\sup_{x \in \Sigma} f(x) \leq (n - 1)! \exp \left( -\frac{\mathcal{E}[u_0]}{(n - 1)!\text{vol}(S^n)} \right),
\]
then the flow $(F)$ has a solution which is defined to all time. In addition, there exists a smooth function $u_\infty$ such that, for a suitable time sequence $(t_j)_j$ with $t_j \to +\infty$ as $j \to +\infty$, there hold
\[
\text{(i)} \quad \|u(t_j) - u_\infty\|_{C^\infty(M)} \to 0,
\]
\[
\text{(ii)} \quad |\lambda(t_j) - 1| \to 0,
\]
\[
\text{(iii)} \quad \|Qg(t_j) - f\|_{C^\infty(M)} \to 0
\]
as $j \to +\infty$. 

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Note that when $M = S^n$, in addition to the first condition in (1.7) there is another necessary condition stating that if (1.6) has a solution $u$, then $f$ must satisfy
\[
\int_{S^n} \langle \nabla f, \nabla x_j \rangle e^{nu} \, d\mu_{S^n} = 0
\] (1.21)
for all $j = 1, 2, \ldots, n+1$. This condition is essentially due to Chang and Yang [10, page 205], which is a higher dimensional analogue of the well-known Kazdan and Warner’s condition.

**Remark 1.4** In view of Theorems 1.1 and 1.3 above, we do not have the uniform but sequential convergence of the flow $(F)$. This is supported by the fact that “bubbling” phenomena of the flow $(F)$ may appear, resulting in many different sub-sequential limits of the flow as $t \to +\infty$. For interested readers, one may refer to the recent work [28] on such a situation on 4-manifolds in the null case, which is a higher-order analogue of the situation studied by Struwe [31] for the Gaussian curvature flow on the two dimensional torus.

Although the uniform convergence in time $t$ is not guaranteed in general, it indeed holds true in some special cases.

**Theorem 1.5** Assume all hypotheses in Theorems 1.1 and 1.3. If

(a) either $\int_M Q_0 \, d\mu_{g_0} \neq 0$ and the problem (1.6) has unique solution

(b) or $\int_M Q_0 \, d\mu_{g_0} < 0$ and $f \leq 0$ everywhere,

then all the convergences in Theorems 1.1 and 1.3 are uniform in time, namely

(i) $\|u(t) - u_\infty\|_{C^\infty(M)} \to 0$,

(ii) $|\lambda(t) - \lambda_\infty| \to 0$, and

(iii) $\|Q^{(t)} - \lambda_\infty f\|_{C^\infty(M)} \to 0$

as $t \to +\infty$, and the convergence corresponding to the case (b) is exponentially fast. Finally, there holds $\lambda_\infty = 1$ in the subcritical but non-null case or when $M = S^n$.

A direct consequence of the convergence of the flow $(F)$ is the existence of solutions to the prescribed $Q$-curvature equation (1.6).

**Corollary 1.6** Let $(M, g_0)$ be a compact, oriented $n$-dimensional Riemannian manifold with $n$ even. Assume that the GJMS operator $P_0$ is positive with kernel consisting of constant functions. Let $f$ be a non-constant, smooth function on $M$. Then, we have the following claims:

(i) Assume that

\[
\int_M Q_0 \, d\mu_{g_0} < 0.
\]

Then there is a positive constant $C_0$ depending on $f^-$ and $(M, g_0)$ such that there exists a conformal metric to $g_0$ with $Q$-curvature $f$ whenever

\[
\sup_M f^+ \leq C_0.
\]

(ii) Assume that

\[
\int_M Q_0 \, d\mu_{g_0} = 0.
\]
Then there exists a conformal metric to $g_0$ with $Q$-curvature $\alpha f$ for $\alpha \in \{-1, 1\}$ provided $f$ is sign-changing with $\int_M f \, d\mu_{g_0} \neq 0$. Moreover, if

$$\int_M f \, d\mu_{g_0} < 0,$$

then there exists a conformal metric to $g_0$ with $Q$-curvature $f$, namely $\alpha = 1$.

(iii) Assume that

$$0 < \int_M Q_0 \, d\mu_{g_0} < (n - 1)! \text{vol}(S^n).$$

Then there exists a conformal metric to $g_0$ with $Q$-curvature $f$ if and only if

$$\sup_M f > 0.$$ 

(iv) Let $M = S^n$ and $f$ be a $G$-invariant function on $S^n$ with $\sup_{S^n} f > 0$. If

(a) either $\Sigma = \emptyset$

(b) or there exist $y_0 \in \Sigma$ and $r_0 > 0$ such that

$$\sup_{x \in \Sigma} f(x) \leq \frac{1}{\text{vol}(S^n)} \max \left\{ \int_{S^n} f \circ \phi_{y_0, r_0} \, d\mu_{S^n}, 0 \right\}, \quad (1.22)$$

then there exists a $G$-invariant conformal metric to $g_0$ with $Q$-curvature $f$. Here $\phi_{y, r} : S^n \to S^n$ is a conformal diffeomorphism given by

$$\phi_{y, r}(x) = \pi_y^{-1}(r \pi_y(x)),$$

where $r > 0, y \in S^n$, and $\pi_y : S^n \to \mathbb{R}^n$ is the stereographic projection from the north pole $y$ to the equatorial plane of $S^n$. In particular, upon choosing $r = 1$ we conclude that if

$$\sup_{x \in \Sigma} f(x) \leq \frac{1}{\text{vol}(S^n)} \max \left\{ \int_{S^n} f \, d\mu_{S^n}, 0 \right\},$$

then there exists a $G$-invariant conformal metric to $g_0$ with $Q$-curvature $f$.

(v) Assume that $f > 0$ and that

$$(n - 1)! \text{vol}(S^n) < \int_M Q_0 \, d\mu_{g_0} \neq k(n - 1)! \text{vol}(S^n), \quad k \in \{2, 3, \ldots \}.$$ 

Then there always exists a conformal metric to $g_0$ with $Q$-curvature $f$.

Remark 1.7 We have the following remarks:

(1) Our results in Corollary 1.6 shares some similarities with other existing results in the literature such as those in [4,14] for manifolds of even dimension. Although the configurations, with a weaker condition on the operator $P_0$, in these two works are similar to ours, the prescribed function $f$ has a fixed sign. The work [4] also considers the supercritical case. However, some sort of symmetric assumption on $f$ has been made in [4] while our $f$ is a generic positive smooth function. Therefore, more or less, Corollary 1.6 generalizes certain results in [4,14] in several directions.
(2) In view of the result in Part (ii), there is a multiple constant \( \alpha \) which cannot be removed in general as the sign of \( \lambda_\infty \) cannot be determined. However, if limiting ourselves to 4-manifolds, then Baird et al. were able to determine the sign of \( \lambda_\infty \) under an extra condition; see [3, Corollary 2.4]. Unfortunately, their proof seems to be effective only for 4-dimension. Later on, Ge and Xu [18] used a variational approach to replace the extra condition in [3] by a natural condition \( \int_M f \, d\mu_{g_0} < 0 \). In “Appendix B”, we modify the argument of Ge and Xu to give a slightly different proof of existence for this case. Interestingly, this new proof allows us to recover the result of Ge and Xu by using our flow.

(3) In Part (iv), the conformal diffeomorphism \( \phi_{y,r} \) is well-studied; see [22, Section 5]. Also, the two inequalities in Part (iv) are sharp. Indeed, let us show the sharpness of

\[
\sup_{x \in S} f(x) \leq \frac{1}{\text{vol}(S^n)} \max \left\{ \int_{S^n} f \, d\mu_{S^n}, 0 \right\}.
\]

Let

\[
f(x) = \varepsilon x_{n+1}
\]

and consider \( G \) the group of isometries which fixes the north and south poles of \( S^n \), namely the two points \((0, \ldots, 0, \pm 1) \in \mathbb{R}^{n+1}\). Then the function \( f \) is \( G \)-invariant and

\[
\sup_{x \in S} f(x) = \varepsilon.
\]

On one hand, since the function \( f \) is odd with respect to the origin, we must have

\[
\int_{S^n} f \, d\mu_{S^n} = 0.
\]

On the other hand, the function \( f \) is not the \( Q \)-curvature of any metric conformal to \( g_{S^n} \) by the obstruction (1.21).

(4) The result in Part (v) can be regarded as a generalization of the existence result obtained in [15]. We note that in the critical and supercritical cases, the analysis is delicate since blow-up phenomena may occur; see [15].

To close this section, we would like to mention the organization of the paper. Section 2 is devoted to a flow of conformal factor \( u \) and the energy functional associated with the flow. Sections 3 and 4 are devoted to \( H^{n/2} \) and \( H^n \)-bounds of \( u(t) \) in the maximal interval of existence \([0, T)\). Putting these bounds together and by performing integral estimates, we obtain \( H^{2k} \)-bounds for \( u(t) \) in Sect. 5 for any \( k > 2 \). This helps us to conclude that the flow is defined to all time. To study the convergence of the flow, uniform boundedness of the flow is required. We spend Sect. 7 to prove certain global boundedness of the flow. Putting all these preparation together, we study convergence of the flow in Sect. 8. Proofs of Theorems 1.1 and 1.3 and Corollary 1.6 are put in the final section; see Sect. 9.

## 2 The flow equation and its energy functional

The purpose of this section is to derive a flow equation for the conformal factor \( u \). Using this flow equation, we shall show that the energy functional \( E \) is non-increasing along the flow. Other properties of the energy functional are also presented. Thanks to (1.5), we also obtain

\[
\partial_t u(t) = -P_{g(t)}u - e^{-nu} Q_0 + \lambda(t) f
\]
In particular, the energy functional

\[ E(u(t)) \]

for some constant \( C \) for some constant \( C \).

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Lemma 2.1 Suppose that \( u(t) \) is a solution to the flow (F). Then, we have

\[ \partial_t Q_g(t) = -P_g(t)(Q_g(t) - \lambda(t)f) + n Q_g(t)(Q_g(t) - \lambda(t)f) \]

and

\[ \lambda'(t) = \left( \int_M f^2 d\mu_{g(t)} \right)^{-1} \int_M (P_g(t)f - n\lambda(t)f^2) (\lambda(t)f - Q_g(t)) d\mu_{g(t)}. \]

where, as always, \( P_g(t) = e^{-nu}P_0 \) is the GJMS operator with respective to the conformal metric \( g(t) \).

Proof Since this is just a routine calculation, we leave the proof to the reader. \( \square \)

Lemma 2.2 Suppose that \( u(t) \) is a solution to the flow (F). Then, one has

\[ \frac{d}{dt} E[u] = -n \int_M (Q_g(t) - \lambda(t)f)^2 d\mu_{g(t)}. \]

In particular, the energy functional \( E[u] \) is non-increasing along the flow.

Proof It follows from the flow equation (F), (1.5), and (1.16) that

\[
\frac{d}{dt} E[u] = \frac{n}{2} \int_M (u_t \cdot P_0 u + u \cdot P_0 u_t) d\mu_{g_0} + n \int_M Q_0 u_t d\mu_{g_0} \\
= -n \int_M (Q_g(t) - \lambda(t)f) Q_g(t) d\mu_{g(t)} \\
= -n \int_M (Q_g(t) - \lambda(t)f)^2 d\mu_{g(t)} + n \lambda(t) \int_M f (\lambda(t)f - Q_g(t)) d\mu_{g(t)} \\
= -n \int_M (Q_g(t) - \lambda(t)f)^2 d\mu_{g(t)}. \\
\]

The proof is complete. \( \square \)

Before going further, let us recall some important inequalities that will be used in the paper. First, we recall the Gagliardo–Nirenberg interpolation inequality

\[
\int_M |\nabla^j \varphi|^p d\mu_{g_0} \leq \mathcal{C}^p \left( \int_M |\nabla^m \varphi|^r d\mu_{g_0} \right)^{ps/r} \left( \int_M |\varphi|^q d\mu_{g_0} \right)^{p(1-s)/q} \tag{2.1}
\]

for some \( \mathcal{C} > 0 \) with \( 0 < s < 1 \leq q, r \leq +\infty \) satisfying \( 1/p = j/n + (1/r - m/n)\alpha + (1 - s)/q \).

Since the operator \( P_0 \) is self-adjoint and positive with kernel consisting of constant functions, we can apply Adams’ inequality [1, Theorem 2] to get

\[
\int_M \exp \left( \frac{n! \text{vol}(\mathbb{S}^n)}{2} \frac{(u - \overline{u})^2}{\int_M u \cdot P_0 u d\mu_{g_0}} \right) d\mu_{g_0} \leq \mathcal{C}_A \tag{2.2}
\]

for some constant \( \mathcal{C}_A > 0 \). A detailed explanation for the validity of (2.2) can be found, for example, in [7, page 330]. As a consequence of (2.2), we obtain the following Trudinger-type inequality

\[
\int_M e^{\alpha(u - \overline{u})} d\mu_{g_0} \leq \mathcal{C}_A \exp \left( \frac{\alpha^2}{2n! \text{vol}(\mathbb{S}^n)} \int_M u \cdot P_0 u d\mu_{g_0} \right) \\
\]

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for all real number $\alpha$. An equivalent form of the Trudinger-type inequality is the following
\[
\int_M e^{\alpha u} \, d\mu_{g_0} \leq C_A \exp \left( \frac{\alpha^2}{2n! \vol(S^n)} \int_M u \cdot P_0 u \, d\mu_{g_0} + \frac{\alpha}{\vol(M)} \int_M u \, d\mu_{g_0} \right). \tag{2.3}
\]
Since
\[
\frac{\alpha}{\vol(M)} \int_M u \, d\mu_{g_0} \leq \frac{\alpha^2}{2n! \vol(S^n)} \int_M u^2 \, d\mu_{g_0} + \frac{n! \vol(S^n)}{2\vol(M)} \int_M u \, d\mu_{g_0},
\]
we obtain from (2.3) the following
\[
\int_M e^{\alpha u} \, d\mu_{g_0} \leq C_A \exp \left( \frac{n! \vol(S^n)}{2\vol(M)} \cdot \alpha \max \left\{ \left( \int_K e^{\alpha u} \, d\mu_{g_0} \right)^{\alpha}, 1 \right\} \right). \tag{2.4}
\]
Finally, when working on $S^n$, instead of using Adams’ inequality (2.2), we shall use the sharp Beckner’s inequality
\[
\frac{1}{\vol(S^n)} \int_{S^n} e^{nu} \, d\mu_{g_0} \leq \exp \left( \frac{n}{2(n-1)! \vol(S^n)} \int_{S^n} u \cdot P_0 u \, d\mu_{g_0} + \frac{n}{\vol(S^n)} \int_{S^n} u \, d\mu_{g_0} \right); \tag{2.5}
\]
see [10, Eq. (4.1’)]. Similar to Trudinger’s inequality (2.4), we obtain the following Beckner inequality in terms of norm
\[
\frac{1}{\vol(S^n)} \int_{S^n} e^{nu} \, d\mu_{g_0} \leq \exp \left( \frac{n!}{2} \exp \left( \frac{n}{2(n-1)! \vol(S^n)} \|u\|_{H^{n/2}(S^n)}^2 \right) \right). \tag{2.6}
\]

3 Boundedness of $u(t)$ in $H^{n/2}(M)$ for $0 \leq t < T$

In this section, we aim to show that $u(t)$ is bounded in $H^{n/2}(M, g_0)$ for all $t$ in the maximal interval of existence. We will split our argument into four cases depending on the size of $\int_M Q_0 \, d\mu_{g_0}$. As we shall see later, such a $H^{n/2}$-bound for $u(t)$ is uniform in the non-supercritical case while depends on the maximal time of existence in the supercritical case; see Proposition 3.13.

3.1 The negative case

To achieve our goal, we first need an analogue of [3, Lemma 4.1] whose proof is provided in “Appendix A” for completeness.

Lemma 3.1 Let $K$ be a measurable subset of $M$ with $\vol(K) > 0$. Then there exist two constants $\alpha > 1$ depending on $M$ and $g_0$ and $C_K > 1$ depending on $M$, $g_0$, and $\vol(K)$ such that
\[
\int_M e^{\alpha u} \, d\mu_{g_0} \leq C_K \exp \left( \alpha \|u_0\|_{H^{n/2}(M)}^2 \right) \cdot \left\{ \left( \int_K e^{\alpha u} \, d\mu_{g_0} \right)^{\alpha}, 1 \right\}.
\]

With help of this lemma, we are able to obtain a uniform bound for $\vol(M, g(t))$ along the flow.

Lemma 3.2 Suppose the flow $(F)$ with initial data $u_0$ is defined on the maximal interval of existence $[0, T)$ for some $T > 0$. Also, assume that there exists a constant $C_0$ depending on
\( f^- \) and \((M, g_0)\), and a constant \( \tau \) depending only on \((M, g_0)\) such that

\[
\sup_M f^+ \leq C_0 \exp\left(-\tau \|u_0\|_{H^{n/2}(M)}^2\right).
\]

Then, there exists a uniform constant \( \gamma > 0 \) such that

\[
\int_M e^{nu(t)} \, d\mu_{g_0} \leq \gamma
\]

for all \( t \in [0, T) \).

**Proof** Define

\[ K = \left\{ x \in M : f(x) \leq \frac{1}{2} \inf_M f \right\}. \]

Notice that \( K \neq \emptyset \) due to the fact that \( \inf_M f < 0 \). Hence, by the continuity of \( f \), we conclude \( \text{vol}(K) > 0 \). Since \( u_0 \in Y \) and \( f = f^+ - f^- \), we have

\[
- \int_M Q_0 \, d\mu_{g_0} = - \int_M f e^{nu_0} \, d\mu_{g_0} \leq (\sup_M f^-) \int_M e^{nu_0} \, d\mu_{g_0},
\]

which implies that

\[
\frac{1}{-\sup_M f^-} \int_M Q_0 \, d\mu_{g_0} \leq \int_M e^{nu_0} \, d\mu_{g_0}.
\]

From (2.4) we have

\[
\int_M e^{nu_0} \, d\mu_{g_0} \leq C \exp\left(C \|u_0\|_{H^{n/2}(M)}^2\right),
\]

where \( C > 1 \) is a constant depending on \( M \) and \( g_0 \). (Here, we can freely assume that \( C > 1 \).) Therefore, we get

\[
\frac{1}{-\sup_M f^-} \int_M Q_0 \, d\mu_{g_0} \leq C \exp\left(C \|u_0\|_{H^{n/2}(M)}^2\right).\]

To conclude the assertion, we shall show that the inequality (3.1) is stable along the flow in the sense that \( \int_M e^{nu} \, d\mu_{g_0} \) is bounded from above by some multiple of the right hand side of (3.1). With the set \( K \) chosen as above, let \( C_K \) and \( \alpha \) be the constants defined in Proposition 3.1. Set

\[
\gamma = 2C_K (8C)^\alpha \exp\left((C + 1)\alpha \|u_0\|_{H^{n/2}(M)}^2\right),
\]

Then we will show that this constant is exactly the upper bound for \( \int_M e^{nu} \, d\mu_{g_0} \). We choose the two constants \( \tau \) and \( C_0 \) in the lemma in such a way that

\[
C_0 \exp\left(-\tau \|u_0\|_{H^{n/2}(M)}^2\right) = 2C \gamma^{-1} \exp\left(C \|u_0\|_{H^{n/2}(M)}^2\right) \sup_M f^-,
\]

which can be computed precisely to get

\[
C_0 = 8^{-\alpha} C_K^{-1} C^{1-\alpha} \sup_M f^-.
\]

and

\[
\tau = \alpha(C + 1) - C.
\]
Our choice of \( C_0 \) and \( \tau \) is to ensure that
\[
\gamma \sup_M f^+ \leq 2C \exp(C \|u_0\|^2_{H^{n/2}(M)}), \tag{3.3}
\]
Now, in the rest of the proof, we claim that
\[
\int_M e^{nu(t)} d\mu_{g_0} \leq \gamma, \tag{3.4}
\]
for all \( t \in [0, T) \). To see this, we let
\[
I = \left\{ t \in [0, T) : \int_M e^{nu(s)} d\mu_{g_0} \leq \gamma \text{ for all } s \in [0, t] \right\}.
\]
From (3.1), we have that \( 0 \in I \). Set \( \beta = \sup I \) and suppose that \( \beta < T \). Then by continuity, we have
\[
\int_M e^{nu(\beta)} d\mu_{g_0} = \gamma. \tag{3.5}
\]
Notice that \( u(t) \in Y \) for all \( t \in [0, T) \). Hence, in particular, \( u(\beta) \in Y \). In the following, we split our argument into two cases.

**Case 1** Suppose that
\[
\int_M f^+ e^{nu(\beta)} d\mu_{g_0} \leq \frac{1}{2} \int_M f^- e^{nu(\beta)} d\mu_{g_0}.
\]
Then the fact that \( u(\beta) \in Y \) gives
\[
-\int_M Q_0 d\mu_{g_0} = \int_M f^- e^{nu(\beta)} d\mu_{g_0} - \int_M f^+ e^{nu(\beta)} d\mu_{g_0} \geq \frac{1}{2} \int_M f^- e^{nu(\beta)} d\mu_{g_0}.
\]
This together with the fact that \( 2f^{-}(x) \geq \sup_M f^{-} \) for all \( x \in K \) yields
\[
\int_K e^{nu(\beta)} d\mu_{g_0} \leq \frac{4}{-\sup_M f^{-}} \int_M Q_0 d\mu_{g_0}. \tag{3.6}
\]
Combining (3.2) and (3.6) gives
\[
\int_K e^{nu(\beta)} d\mu_{g_0} \leq 8C \exp \left( C \|u_0\|^2_{H^{n/2}(M)} \right).
\]
From Lemma 3.1, it follows that
\[
\int_M e^{nu(\beta)} d\mu_{g_0} \leq C_K \exp \left( \alpha \|u_0\|^2_{H^{n/2}(M)} \right) \max \left\{ \left( \int_K e^{nu(\beta)} d\mu_{g_0} \right)^\alpha, 1 \right\} \leq C_K \exp \left( \alpha \|u_0\|^2_{H^{n/2}(M)} \right) \max \left\{ (8C)^\alpha \exp \left( \alpha C \|u_0\|^2_{H^{n/2}(M)} \right), 1 \right\} \leq C_K (8C)^\alpha \exp \left( (C + 1)\alpha \|u_0\|^2_{H^{n/2}(M)} \right) = \gamma / 2,
\]
which contradicts (3.5).

**Case 2** Suppose that
\[
\int_M f^+ e^{nu(\beta)} d\mu_{g_0} > \frac{1}{2} \int_M f^- e^{nu(\beta)} d\mu_{g_0}.
\]
This situation corresponds to the case when \( f^+ \) plays some role and therefore we need certain smallness of \( \sup_M f^− \). Again by using the fact that \( 2 f^−(x) \geq \sup_M f^− \) for all \( x \in K \) and (3.5), we can estimate
\[
\int_K e^{n u(\beta)} d\mu_{g_0} \leq \frac{2}{\sup_M f^−} \int_M f^− e^{n u(\beta)} d\mu_{g_0} \leq \frac{4}{\sup_M f^−} \int_M f^+ e^{n u(\beta)} d\mu_{g_0} \leq \frac{4\gamma \sup_M f^+}{\sup_M f^−}.
\]
By (3.3), we then have
\[
\int_M e^{n u(\beta)} d\mu_{g_0} \leq 8C \exp\left(C \frac{\|u_0\|^2_{H^{n/2}(M)}}{\text{vol}(M)}\right).
\]
Thus, by arguing as in Case 1, we obtain a contradiction once more. Hence, the claim (3.4) holds and the lemma is proved.

Lemma 3.3 Suppose that the flow \((F)\) is defined on \([0, T)\). Then, there exists a uniform constant \( C > 0 \) such that
\[
\|u(t)\|_{H^{n/2}(M)} \leq C
\]
for all \( t \in [0, T) \).

Proof From Lemma 3.2 and Jensen’s inequality, it follows that
\[
\bar{u} \leq \frac{1}{n} \log \left(\frac{1}{\text{vol}(M)} \int_M e^{n u} d\mu_{g_0}\right) \leq \frac{1}{n} \log \frac{\gamma}{\text{vol}(M)}.
\]
By Lemma 2.2, we get
\[
\frac{n}{2} \int_M u \cdot P_0 u d\mu_{g_0} + n \int_M Q_0 (u - \bar{u}) d\mu_{g_0} + n\bar{u} \int_M Q_0 d\mu_{g_0} \leq \mathcal{E}[u_0] .
\]
Using the Poincaré-type inequality (1.9), we get
\[
\int_M (u - \bar{u})^2 d\mu_{g_0} \leq \frac{1}{\lambda_1} \int_M u \cdot P_0 u d\mu_{g_0} ,
\]
which together with Young’s inequality implies that
\[
\left| n \int_M Q_0 (u - \bar{u}) d\mu_{g_0} \right| \leq \frac{n}{4} \int_M u \cdot P_0 u d\mu_{g_0} + \frac{n}{\lambda_1} \int_M Q_0^2 d\mu_{g_0} .
\]
Substituting (3.10) into (3.8) yields
\[
\frac{n}{4} \int_M u \cdot P_0 u d\mu_{g_0} + n\bar{u} \int_M Q_0 d\mu_{g_0} \leq \mathcal{E}[u_0] + \frac{n}{\lambda_1} \int_M Q_0^2 d\mu_{g_0} .
\]
By (3.7) and (3.11), we have
\[
\int_M u \cdot P_0 u d\mu_{g_0} \leq -\frac{4}{n} \log \frac{\gamma}{\text{vol}(M)} \int_M Q_0 d\mu_{g_0} + \frac{4\mathcal{E}[u_0]}{n} + \frac{4}{\lambda_1} \int_M Q_0^2 d\mu_{g_0} .
\]
In view of (1.14), (3.12) and (3.9), it remains to get a lower bound for \( \bar{u} \). Indeed, from the positivity of \( P_0 \), (3.11), and \( \int_M Q_0 d\mu_{g_0} < 0 \), we know that
\[
\bar{u} \geq \frac{(n/\lambda_1) \int_M Q_0^2 d\mu_{g_0} + \mathcal{E}[u_0]}{n \int_M Q_0 d\mu_{g_0}} .
\]
We thus conclude that there exists a uniform constant $C_2 > 0$ such that
\[ \int_M u^2 d\mu_{g_0} \leq C_2. \quad (3.13) \]
Combining (3.12) and (3.13) yields the assertion.

\[ \square \]

### 3.2 The null case

In this case, because $u_0 \in Y$ and $\partial_t \int_M f d\mu_{g(t)} = 0$, we conclude that $\int_M f d\mu_{g(t)} = 0$ for all $t \geq 0$. Furthermore, in the null case, we have $\int_M Q_{g(t)} d\mu_{g(t)} = 0$ for all $t \geq 0$. Using these properties, we obtain the following result.

**Lemma 3.4** Along the flow $(F)$, the volume of $g(t)$ denoted by $\text{vol}(M, g(t))$ is preserved.

**Proof** Clearly, we have
\[ \frac{d}{dt} \int_M e^{nu(t)} d\mu_{g_0} = n \int_M u_t d\mu_{g(t)} = n \int_M (\lambda(t) f - Q_{g(t)}) d\mu_{g(t)} = 0. \]
Hence $\int_M e^{nu(t)} d\mu_{g_0}$ is constant along the flow; this proves the assertion.

\[ \square \]

Using the preceding lemma, we can control $u(t)$ in $H^{n/2}$-norm as shown below.

**Lemma 3.5** There exists a uniform constant $C > 0$ such that
\[ \|u(t)\|_{H^{n/2}(M)} \leq C \]
for all time $t$ in the maximal interval of existence.

**Proof** From Lemma 2.2, it follows that
\[ \frac{n}{2} \int_M u \cdot P_0 u d\mu_{g_0} + n \int_M Q_0 (u - \overline{u}) d\mu_{g_0} \leq \mathcal{E}[u_0]. \quad (3.14) \]

By substituting (3.10) into (3.14), we have
\[ \int_M u \cdot P_0 u d\mu_{g_0} \leq \frac{4}{n} \mathcal{E}[u_0] + \frac{4}{\lambda_1} \int_M Q_0^2 d\mu_{g_0}. \quad (3.15) \]
Hence, in view of (1.14), to bound $\|u(t)\|_{H^{n/2}(M)}$, it suffices to bound $\int_M u(t) d\mu_{g_0}$. By Jensen’s inequality and Lemma 3.4, we have that
\[ e^{n\overline{u}} \leq \frac{1}{\text{vol}(M)} \int_M e^{nu} d\mu_{g_0} = \frac{V^0}{\text{vol}(M)}. \]
Therefore, we can bound $\overline{u}$ from above as follows
\[ \overline{u} \leq \frac{1}{n} \log \left( \frac{V^0}{\text{vol}(M)} \right). \]
To bound $\overline{u}$ from below, we apply the Trudinger-type inequality (2.3) and (3.15) to get
\[ \int_M \exp(n(u - \overline{u})) d\mu_{g_0} \leq \mathcal{C}_A \exp \left( \frac{2\mathcal{E}[u_0]}{(n-1)! \text{vol}(\mathbb{S}^n)} + \frac{2n}{\lambda_1(n-1)! \text{vol}(\mathbb{S}^n)} \int_M Q_0^2 d\mu_{g_0} \right). \]
Since $\int_M e^{nu} d\mu_{g_0} = V^0$, we conclude that
$$
\exp \left( - n\bar{\nu} \right) \leq \frac{\mathcal{E}_A}{V^0} \exp \left( \frac{2\mathcal{E}[u_0]}{(n-1)! \text{vol}(S^n)} + \frac{2n}{\lambda_1(n-1)! \text{vol}(S^n)} \int_M Q_0^2 d\mu_{g_0} \right).
$$
Hence, we get a lower bound for $\bar{\nu}$ as follows
$$
\bar{\nu} \geq - \left( \frac{1}{n} \log \left( \frac{\mathcal{E}_A}{V^0} \right) + \frac{2}{\lambda_1(n-1)! \text{vol}(S^n)} \int_M Q_0^2 d\mu_{g_0} + \frac{2\mathcal{E}[u_0]}{n! \text{vol}(S^n)} \right).
$$
Our proof is thus complete by combining all estimates above. \qed

### 3.3 The case $0 < \int_M Q_0 d\mu_{g_0} < (n-1)! \text{vol}(S^n)$

Notice that $\int_M f e^{nu} d\mu_{g_0} = \int_M Q_0 d\mu_{g_0}$ for all $t \geq 0$. Therefore,
$$
\int_M Q_0 d\mu_{g_0} = \int_M f e^{nu} d\mu_{g_0} \leq (\sup_M f) \int_M e^{nu} d\mu_{g_0}.
$$
Hence,
$$
\int_M e^{nu} d\mu_{g_0} \geq \frac{1}{\sup_M f} \int_M Q_0 d\mu_{g_0}. \quad (3.16)
$$
Combining (2.3) and (3.16) gives
$$
\frac{n/2}{(n-1)! \text{vol}(S^n)} \int_M u \cdot P_0 u d\mu_{g_0} \geq \log \int_M \exp \left( n(u - \bar{\nu}) \right) d\mu_{g_0} - \log \mathcal{E}_A
\geq \log \left( \frac{1}{\mathcal{E}_A \sup_M f} \int_M Q_0 d\mu_{g_0} \right) - n\bar{\nu}.
$$
Thus, we have
$$
\bar{\nu} \geq - \frac{n/2}{n! \text{vol}(S^n)} \int_M u \cdot P_0 u d\mu_{g_0} + \frac{1}{n} \log \left( \frac{1}{\mathcal{E}_A \sup_M f} \int_M Q_0 d\mu_{g_0} \right). \quad (3.17)
$$
Using (3.17) and Lemma 2.2, we get
$$
\mathcal{E}[u_0] \geq \frac{n}{2} \int_M u \cdot P_0 u d\mu_{g_0} + n \int_M Q_0 (u - \bar{\nu}) d\mu_{g_0} + n\bar{\nu} \int_M Q_0 d\mu_{g_0}
\geq \frac{n}{2} \left( 1 - \int_M Q_0 d\mu_{g_0} \right) \int_M u \cdot P_0 u d\mu_{g_0} + n \int_M Q_0 (u - \bar{\nu}) d\mu_{g_0}
+ \int_M Q_0 d\mu_{g_0} \log \left( \frac{1}{\mathcal{E}_A \sup_M f} \int_M Q_0 d\mu_{g_0} \right). \quad (3.18)
$$
Since $\int_M Q_0 d\mu_{g_0} < (n-1)! \text{vol}(S^n)$, by Young’s inequality and (1.9) we have the following
$$
\left| n \int_M Q_0 (u - \bar{\nu}) d\mu_{g_0} \right| \leq \frac{n}{4} \left( 1 - \int_M Q_0 d\mu_{g_0} \right) \int_M u \cdot P_0 u d\mu_{g_0}
+ \frac{n}{\lambda_1} \left( \frac{1}{1 - (\int_M Q_0 d\mu_{g_0})/(n-1)! \text{vol}(S^n)} \right). \quad (3.19)
$$
By substituting (3.19) into (3.18), we conclude that there exists a uniform constant $C_1 > 0$ such that
$$
\int_M u \cdot P_0 u d\mu_{g_0} \leq C_1. \quad (3.20)$$
Now substituting (3.20) into (3.17) yields that
\[ u \geq C_2, \quad (3.21) \]
for some uniform constant \( C_2 \). To obtain an upper bound for \( u \), we notice that (3.10) still holds in this case. Hence, by substituting (3.10) into the first inequality in (3.18), the positivity of \( P_0 \), and the fact that \( \int_M Q_0^2 d\mu_{g_0} > 0 \), we get
\[ u \leq \left( \frac{n}{\lambda_1} \right) ^{n} \int_M Q_0^2 d\mu_{g_0} + \mathcal{E}[u_0]. \]
By plugging (3.20), (3.21), and (3.22) into (1.9), we conclude that there exists a uniform constant \( C_3 > 0 \) such that
\[ \int_M u^2 d\mu_{g_0} \leq C_3. \]
(3.23)
The assertion follows by combining (3.20) and (3.23).

3.4 The case \( M = \mathbb{S}^n \)

Now we consider the case where \((M, g_0)\) is the standard sphere \( \mathbb{S}^n \) equipped with the standard metric \( g_{\mathbb{S}^n} \). In particular, one has that \( Q_0 \equiv (n-1)! \). As we have already seen before, the strict inequality \( \int_M Q_0 d\mu_{g_0} < (n-1)\text{vol}(\mathbb{S}^n) \) is crucial in Sects. 3.1–3.3. However, this is no longer the case in this new setting. Therefore, to obtain the uniform boundedness of \( u(t) \) in \( H^{n/2}(\mathbb{S}^n) \), we need the following concentration-compactness lemma whose proof is provided in “Appendix A”.

**Lemma 3.6** Suppose that \([0, T)\) is the maximal interval of existence of the flow \((F)\). Then we have the following alternatives:

(i) either there exists a uniform constant \( C > 0 \) independent of \( T \) such that
\[ \|u(t)\|_{H^{n/2}(\mathbb{S}^n)} \leq C \]
for all \( t \in [0, T) \).

(ii) or there exists a sequence \( t_k \to T \) as \( k \to +\infty \) and a point \( x_\infty \in \mathbb{S}^n \) such that
\[ \lim_{k \to +\infty} \int_{B_r(x_\infty)} f e^{n u(t_k)} d\mu_{\mathbb{S}^n} = (n-1)!\text{vol}(\mathbb{S}^n), \]
for all \( r > 0 \). Here \( B_r(x_\infty) \) is the geodesic ball in \( \mathbb{S}^n \) with radius \( r \) and centered at \( x_\infty \). In addition, there holds
\[ \lim_{k \to +\infty} \int_{B_r(y)} f e^{n u(t_k)} d\mu_{\mathbb{S}^n} = 0, \]
for all \( y \in \mathbb{S}^n \setminus \{x_\infty\} \) and all \( 0 \leq r < \text{dist}(y, x_\infty) \).

Note that Lemma 3.6 is a higher dimensional analogue of [3, Lemma 4.2]. With help of this lemma and under the hypotheses of Theorem 1.3, we can conclude that \( u(t) \) is uniformly bounded in \( H^{n/2}(\mathbb{S}^n) \).

**Lemma 3.7** Suppose that the flow \((F)\) is defined on \([0, T)\). Then, under the hypotheses of Theorem 1.3, there exists a uniform constant \( C > 0 \) independent of \( T \) such that
\[ \|u(t)\|_{H^{n/2}(\mathbb{S}^n)} \leq C, \]
for all \( t \in [0, T) \).
Proof Since \( u_0 \) is \( G \)-invariant and the solution \( u(t) \) is unique, it is not hard to see that \( u(t) \) is also \( G \)-invariant. Moreover, the uniqueness of \( u(t) \) also helps us to assume that
\[
\mathcal{E}[u(t)] < \mathcal{E}[u_0] \tag{3.26}
\]
for \( t > 0 \). We now prove the lemma by way of contradiction. Suppose that \( u(t) \) is not uniformly bounded in \( H_n^{1/2} (\mathbb{S}^n) \) for \( t \in [0, T) \). It follows from Lemma 3.6 that there exists some point \( x_\infty \in \mathbb{S}^n \) satisfying (3.24) for all \( r > 0 \). We have the following two possible cases:

Case 1 The case \( \Sigma = \emptyset \). In this scenario, we can find some \( \sigma \in G \) such that \( \sigma(x_\infty) \neq x_\infty \). From the fact that \( u(t) \) is \( G \)-invariant we find that
\[
\lim_{k \to +\infty} \int_{B_r(\sigma(x_\infty))} f e^{nu(t_k)} d\mu_{\mathbb{S}^n} = \lim_{k \to +\infty} \int_{B_r(x_\infty)} f e^{nu(t_k)} d\mu_{\mathbb{S}^n} = (n-1)!\text{vol}(\mathbb{S}^n),
\]
which contradicts (3.25).

Case 2 The case \( \Sigma \neq \emptyset \). Observe that if \( x_\infty \notin \Sigma \), then we arrive at a contradiction in the same way as above. Therefore, we are left with \( x_\infty \in \Sigma \). In this scenario, we can estimate
\[
\int_{B_r(x_\infty)} f e^{nu(t_k)} d\mu_{\mathbb{S}^n} \leq \left( \sup_{x \in B_r(x_\infty)} f \right) \int_{B_r(x_\infty)} e^{nu(t_k)} d\mu_{\mathbb{S}^n} \leq \max \left( \sup_{x \in B_r(x_\infty)} f, 0 \right) \int_{\mathbb{S}^n} e^{nu(t_k)} d\mu_{\mathbb{S}^n} \tag{3.27}
\]
for all \( r > 0 \). From Beckner’s inequality (2.5) and Lemma 2.2, it follows that
\[
\frac{1}{\text{vol}(\mathbb{S}^n)} \int_{\mathbb{S}^n} e^{nu(t_k)} d\mu_{\mathbb{S}^n} \leq \exp \left( \frac{\mathcal{E}[u(t_1)]}{(n-1)!\text{vol}(\mathbb{S}^n)} \right). \tag{3.28}
\]

Plugging (3.28) into (3.27) and letting \( k \to +\infty \) yield
\[
(n-1)! \leq \max \left( \sup_{x \in B_r(x_\infty)} f, 0 \right) \exp \left( \frac{\mathcal{E}[u(t_1)]}{(n-1)!\text{vol}(\mathbb{S}^n)} \right).
\]

Letting \( r \searrow 0 \) in the above inequality gives
\[
(n-1)! \leq \max \left( f(x_\infty), 0 \right) \exp \left( \frac{\mathcal{E}[u(t_1)]}{(n-1)!\text{vol}(\mathbb{S}^n)} \right).
\]

Consequently, we have \( f(x_\infty) > 0 \) and therefore we get
\[
(n-1)! \leq f(x_\infty) \exp \left( \frac{\mathcal{E}[u(t_1)]}{(n-1)!\text{vol}(\mathbb{S}^n)} \right).
\]

Thanks to (3.26), we deduce from the preceding estimate that
\[
f(x_\infty) > (n-1)! \exp \left( - \frac{\mathcal{E}[u_0]}{(n-1)!\text{vol}(\mathbb{S}^n)} \right),
\]
which contradicts the hypothesis of Theorem 1.3. The proof is now complete. \( \Box \)

As can be seen from the proof of Lemma 3.7, the condition (1.20) for \( \sup_{\Sigma} f \) plays some role. It is natural to ask what would happen if we relax this condition. The following result provides an answer for this question.

Proposition 3.8 Suppose that the flow (F) is defined on \([0, T)\). Then there holds
\[
\inf_{0 \leq t < T} \mathcal{E}[u(t)] > -\infty.
\]
Proof We prove by way of contradiction. Suppose that

\[ \inf_{0 \leq t < T} E[u(t)] = -\infty. \]

It is easy to see that the case (i) in Lemma 3.6 cannot happen. Hence, there must exist a sequence of \((t_k)\) with \(t_k \to T\) as \(k \to +\infty\) and a point \(x_\infty \in \mathbb{S}^n\) such that (3.24) holds for all \(r > 0\). On the other hand, because the energy \(E[u(t_k)]\) is monotone decreasing, we then have

\[ \lim_{k \to +\infty} E[u(t_k)] = -\infty. \]

Since for the case \(\Sigma = \emptyset\) we simply argue as before to get a contradiction, we are left with the case \(\Sigma \neq \emptyset\). Furthermore, we are also left with \(x_\infty \in \Sigma\). Now we use Beckner’s inequality (2.5) in the following way

\[
\frac{1}{\text{vol}(\mathbb{S}^n)} \int_{\mathbb{S}^n} e^{nu(t_k)} d\mu_{\mathbb{S}^n} \leq \exp \left( \frac{E[u(t_k)]}{(n-1)! \text{vol}(\mathbb{S}^n)} \right).
\]

Then we combine this with (3.27) and letting \(k \to +\infty\) in the resulting inequality to get

\[
(n-1)! \leq \max \left( \sup_{x \in B_r(x_\infty)} f, 0 \right) \lim_{k \to +\infty} \exp \left( \frac{E[u(t_k)]}{(n-1)! \text{vol}(\mathbb{S}^n)} \right) = 0.
\]

This is a contradiction and Proposition 3.8 is proved.

As we shall see later, once the energy functional \(E\) has a lower bound, the time-depending \(H^{n/2}\)-bound will hold, and then follow Brendle’s argument in [7] to get the global existence of the flow. This means that the condition (1.20) is not necessary for the global existence. The said scheme will be carried out in the supercritical case in the next subsection.

3.5 The supercritical case

Now we turn our attention to the supercritical case. In this scenario, it seems difficult for us to get the uniform \(H^{n/2}\)-bound by direct estimates as we did before. Instead, as suggested above, our argument here is to carefully exploit the connection between energy bound and \(H^{n/2}(M)\)-norm of the flow and we can merely obtain the time-depending \(H^{n/2}\) bound in this subsection.

Recall that we have assumed that \(f > 0\), however, it is worth noting that the positivity of \(f\) is not necessary in Lemmas 3.9 and 3.12 below, where we only need to assume that \(\int_M f d\mu_{g(0)} > 0\).

As always, we let \(T\) be the maximal time of existence of the flow (F) and let \(T_0 \in [0, T)\) be arbitrary. Given any initial data \(u_0 \in C^\infty(M)\) we also let

\[
\tilde{\lambda}(t) = \lambda(t) - \frac{\int_M Q_0 d\mu_{g_0}}{\int_M f d\mu_{g(0)}}.
\]

where \(\lambda\) is given in (1.17).

Lemma 3.9 Given any \(\mathcal{U} > 0\), if

\[
\int_M e^{nu_0} d\mu_{g_0} \leq \mathcal{U}, \quad E[u_0] \leq \mathcal{U}, \quad \inf_{t \in [0, T_0]} E[u(t)] \geq -\mathcal{U},
\]

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then it follows that
\[ \int_0^{T_0} \left| \lambda(t) \right| dt \leq n \mathcal{U} (1 + T_0) \left( \int_M f d\mu_{g(0)} \right)^{-1}, \]
or equivalently
\[ \int_0^{T_0} |\lambda(t)| dt \leq \left( n \mathcal{U} + \left| \int_M Q_0 d\mu_{g_0} \right| \right) (1 + T_0) \left( \int_M f d\mu_{g(0)} \right)^{-1}. \]

**Proof** First, we notice the following estimate
\[ \frac{d}{dt} \int_M e^{nu} d\mu_{g_0} = n \int_M u_t e^{nu} d\mu_{g_0} \leq n \left( \int_M u_t^2 e^{nu} d\mu_{g_0} \right)^{1/2} \left( \int_M e^{nu} d\mu_{g_0} \right)^{1/2}. \]
Solving this differential inequality and using Lemma 2.2 we obtain
\[ \int_M e^{nu} d\mu_{g_0} \leq \left( \int_M e^{nu_0} d\mu_{g_0} \right)^{1/2} + \frac{n}{2} \int_0^{T_0} \left( \int_M u_t^2 e^{nu} d\mu_{g_0} \right)^{1/2} dt \]
\[ = 2 \mathcal{U} + \frac{n^2}{2} T_0 \int_0^{T_0} \int_M u_t^2 e^{nu} d\mu_{g_0} dt \]
\[ = 2 \mathcal{U} - \frac{n}{2} T_0 \int_0^{T_0} \frac{d}{dt} \mathcal{E}[u(t)] dt \]
\[ = 2 \mathcal{U} + \frac{n}{2} T_0 \left( \mathcal{E}[u_0] - \mathcal{E}[u(T_0)] \right) . \]

(3.29)

for any \( t \in [0, T_0] \). On the other hand, by the flow equation (F), (1.18) and the conformal invariant of \( \int_M Q_{g(t)} d\mu_{g(t)} \), we deduce that
\[ \int_M u_t e^{nu} d\mu_{g_0} = \lambda(t) \int_M f d\mu_{g(t)} - \int_M Q_{g(t)} d\mu_{g(t)} = \lambda(t) \int_M f d\mu_{g(0)}. \]
This together with Lemma 2.2, and (3.29) gives
\[ \left( \tilde{\lambda}(t) \int_M f d\mu_{g(0)} \right)^2 = \left| \int_M u_t e^{nu} d\mu_{g_0} \right|^2 \]
\[ \leq \int_M u_t^2 e^{nu} d\mu_{g_0} \int_M e^{nu} d\mu_{g_0} \]
\[ \leq -n \mathcal{U} (1 + T_0) \frac{d}{dt} \mathcal{E}[u(t)]. \]
Keep in mind that \( \int_M f d\mu_{g(0)} > 0 \). The estimate above then yields that
\[ \int_0^{T_0} \left| \tilde{\lambda}(t) \right| dt \leq T_0^{1/2} \left( \int_0^{T_0} \tilde{\lambda}^2(t) dt \right)^{1/2} \leq n \mathcal{U} (1 + T_0) \left( \int_M f d\mu_{g(0)} \right)^{-1}. \]
Now by using triangle inequality, we easily obtain the last inequality. This completes the proof.

The goal of our next step is to show that there exists a suitable initial data \( u_0 \in C^\infty(M) \) such that the energy has a lower bound along the flow (F) whenever it exists.
Proposition 3.10 Suppose

\[(n - 1)!\text{vol}(\mathbb{S}^n) < \int_M Q_0 \, d\mu_g \neq k(n - 1)!\text{vol}(\mathbb{S}^n), \quad k \in \{2, 3, \ldots\},\]

and assume that \(f > 0\) everywhere. Then there exists some \(u_0 \in C^\infty(M)\) such that the solution \(u(t)\) to the flow \((F)\) with the initial data \(u_0\) enjoys the following

\[\inf_{t \in [0, T]} \mathcal{E}[u(t)] > -\infty.\]

Proof We prove by contradiction. Given an initial data \(u \in C^\infty(M)\) and let \(\varphi(t, u)\) be the solution of \((F)\) with \(\varphi(0, u) = u\), namely

\[
\begin{cases}
\partial_t \varphi = -(P_0 \varphi + Q_0)e^{-\varphi} + \lambda \varphi(t) f, & t > 0, \\
\varphi(0, u) = u,
\end{cases}
\]

where

\[\lambda(t) = \frac{\int_M f (P_0 \varphi + Q_0) \, d\mu_g}{\int_M f^2 e^{\varphi} \, d\mu_g}.
\]

Let \([0, T_u]\) be the maximal interval of existence of \(\varphi(\cdot, u)\). We assume, by contradiction, that

\[\inf_{t \in [0, T_u]} \mathcal{E}[\varphi(t, u)] = -\infty \tag{3.30}\]

for any smooth initial data \(u \in C^\infty(M)\). Let \(E := C^\infty(M)\) be equipped with its natural \(C^\infty\)-topology. We also define the sub-level set

\[E_{-\gamma} := \left\{ u \in E : \mathcal{E}[u] \leq -\gamma \right\},\]

where \(\gamma\) is a positive constant. It follows from the decay property of \(\mathcal{E}\) that \(E_{-\gamma}\) is invariant under the flow \(\varphi(t)\), that is, if \(u \in E_{-\gamma}\), then \(\varphi(t, u) \in E_{-\gamma}\) for all \(t \in [0, T_u]\). Since \(f > 0\) everywhere, our functional \(\mathcal{E}\) is comparable with one in the constant \(Q\)-curvature case in [25]. This fact enables us to make use of [25, Proposition 4.4] to show that there exists a sufficiently large constant \(\gamma\) such that \(E_{-\gamma}\) is not contractible. Let us fix such a constant \(\gamma\).

Next, we show that the flow \(\varphi(t, u)\) defines a deformation retract from \(E\) to \(E_{-\gamma}\). Indeed, for each \(u \in E\), by (3.30) we can define

\[t_u = \min \left\{ t \in [0, T_u] : \mathcal{E}[\varphi(t, u)] \leq -\gamma \right\}.
\]

It follows from the continuity of \(\varphi\) that

\[\mathcal{E}[\varphi(t_u, u)] = -\gamma.
\]

We then extend \(\varphi\) on \([0, +\infty)\) by considering

\[\hat{\varphi}(t, u) = \begin{cases} \varphi(t, u) & \text{if } t \in [0, t_u), \\
\varphi(t_u, u) & \text{if } t \geq t_u. \end{cases}
\]

In view of the decay property of the energy along the flow, we can obtain that \(\hat{\varphi}\) is continuous on \([0, +\infty) \times E\). Now, we define the homotopy map \(H : [0, 1] \times E \to E\) by

\[H(t, u) = \begin{cases} \hat{\varphi}(\frac{t}{1-t}, u) & \text{if } t \in [0, 1), \\
\hat{\varphi}(t_u, u) & \text{if } t = 1. \end{cases}
\]
Let \( u \in E \) be arbitrary. Then it is easy to see that

\[
H(0, u) = \hat{\varphi}(0, u) = \varphi(0, u) = u
\]

and

\[
H(1, u) = \hat{\varphi}(t_u, u) = \varphi(t_u, u) \in E_{-\gamma}.
\]

Moreover, if we further let \( u \in E_{-\gamma} \), then by the decay property of the energy along the flow \( \varphi(\cdot, u) \) we deduce that \( \varphi(t, u) \in E_{-\gamma} \) for all \( t \). Thus, \( t_u = 0 \) and therefore by definition

\[
\hat{\varphi}(t, u) = \varphi(0, u) = u
\]

for any \( t \geq 0 \), giving the conclusion that \( H(t, u) = u \) for any \( t \in [0, 1] \). This shows that \( E_{-\gamma} \) is a strong deformation retract of \( E \), and we obtain a contradiction, since \( E \) is contractible as a topological vector space. \( \square \)

**Remark 3.11** We may achieve three goals by proposing the technical condition \( f > 0 \) everywhere: (1) it enables us to apply [25, Proposition 4.4]; (2) the initial data \( u_0 \) found above enjoys the requirement \( \int_M f d\mu_{g(0)} > 0 \) due to the assumption on \( \int_M Q_0 d\mu_{g_0} \); (3) the volume \( \text{vol}(M, g(t)) \) is bounded along the flow, which is crucial for the boundedness of \( \lambda(t) \); see Lemma 6.2 below.

To go further, we set

\[
A_t = \{ x \in M : u(x, t) \geq \alpha_0 \},
\]

where

\[
\alpha_0 = \frac{1}{n} \log \left( \frac{\int_M f d\mu_{g(0)}}{2 \text{vol}(M) \sup_M f} \right). \tag{3.31}
\]

Clearly, \( \alpha_0 \) is well-defined. We are going to estimate the size of \( A_t \).

**Lemma 3.12** Given any \( \gamma_0 > 0 \), there exists a constant \( C_1 > 0 \) depending only on \( \gamma_0 \), \( \sup_M |f|, n, \text{vol}(M), \int_M Q_0 d\mu_{g_0} \), and \( \int_M f d\mu_{g(0)} \) such that if

\[
\int_M e^{n u_0} d\mu_{g_0} \leq \gamma_0, \quad \| u_0 \|_{H^{n/2}(M)} \leq \gamma_0, \quad \inf_{t \in [0, T_0]} \mathcal{E}[u(t)] \geq -\gamma_0, \tag{3.32}
\]

then

\[
\text{vol}(A_t) \geq \exp \left( - C_1 e^{C_1 T_0} \right)
\]

for all \( t \in [0, T_0] \).

**Proof** By the assumption, we have

\[
\mathcal{E}[u] = \frac{n}{2} \int_M u \cdot P_0 u d\mu_{g_0} + n \int_M Q_0 u d\mu_{g_0} \geq -\gamma_0
\]
for all \( t \in [0, T_0] \). This together with the conformal invariant of \( \int_M Q_g(t) \, d\mu_g(t) \), the sign of \( \int_M f \, d\mu_g(0) \), and the positivity of the operator \( P_0 \) implies that

\[
\frac{d}{dt} \int_M u e^{nu} \, d\mu_{g_0} = \int_M u \, e^{nu} \, d\mu_{g_0} + n \int_M u e^{nu} \, u_t \, d\mu_{g_0} = \lambda(t) \int_M f \, e^{nu} \, d\mu_{g_0} - \int_M Q_0 \, d\mu_{g_0} + \lambda(t) \int_M f \, e^{nu} \, d\mu_{g_0} - n \lambda(t) \int_M f \, u \cdot P_0 u \, d\mu_{g_0} - \varepsilon[u] + n \lambda(t) \int_M f \, u e^{nu} \, d\mu_{g_0} \leq \lambda(t) \int_M f \, e^{nu} \, d\mu_{g_0} - \frac{n}{2} \int_M u \cdot P_0 u \, d\mu_{g_0} - \varepsilon[u] + n \lambda(t) \int_M f \, e^{nu} \, d\mu_{g_0} + \gamma_0. \tag{3.33}
\]

We split the term involving \( \int_M f \, u e^{nu} \, d\mu_{g_0} \) as follows

\[
\lambda(t) \int_M f \, u e^{nu} \, d\mu_{g_0} = \lambda(t) \int_M f \, u^+ e^{nu} \, d\mu_{g_0} - \lambda(t) \int_M f \, u^- e^{nu} \, d\mu_{g_0} =: I - II. \tag{3.34}
\]

**Estimate of I** To estimate this term, we first recall the elementary inequality \( se^{s} \geq -1 \), which holds for any real \( s \). From this we claim that there holds the pointwise estimate

\[ 0 \leq u^- e^{nu} \leq 1/n. \]

Indeed, for each fixed \( t \), if \( x \in \{ u \leq 0 \} \), then \( u^- e^{nu} = -ue^{nu} = -1/n[(nu)e^{nu}] \leq 1/n \); while \( x \in \{ u > 0 \} \), \( u^- e^{nu} = 0 \). On the other hand, one obviously has \( u^- e^{nu} \geq 0 \). We thus proved the claim. The pointwise estimate above together with the mean value theorem for integrals implies that

\[
I = n\lambda(t) f(p(t)) \int_M u^+ e^{nu} \, d\mu_{g_0}
= n\lambda(t) f(p(t)) \int_M u e^{nu} \, d\mu_{g_0} + n\lambda(t) f(p(t)) \int_M u^- e^{nu} \, d\mu_{g_0}
\leq n\lambda(t) f(p(t)) \int_M u e^{nu} \, d\mu_{g_0} + |\lambda(t)| \sup_M |f| \text{vol}(M). \tag{3.35}
\]

where \( p(t) \), depending on the time \( t \), is some point in \( M \).

**Estimate of II** As in (3.35), we have the estimate

\[
|II| \leq n|\lambda(t)| \int_M |f| u^- e^{nu} \, d\mu_{g_0} \leq |\lambda(t)| \sup_M |f| \text{vol}(M). \tag{3.36}
\]

Substituting (3.35) and (3.36) into (3.34) yields

\[
n\lambda(t) \int_M f \, u e^{nu} \, d\mu_{g_0} \leq n\lambda(t) f(p(t)) \int_M u e^{nu} \, d\mu_{g_0} + 2|\lambda(t)| \sup_M |f| \text{vol}(M).
\]

\[ \heartsuit \text{ Springer} \]
Putting (3.33), (3.34), (3.35), and (3.36) together we arrive at the differential inequality
\[
\frac{d}{dt} \int_M u e^{nu} d\mu_{\gamma_0} \leq n\lambda(t) f(p(t)) \int_M u e^{nu} d\mu_{\gamma_0} + \tilde{\lambda}(t) \int_M f d\mu_{\gamma(0)} + 2 \sup_M |f| \text{vol}(M) |\lambda(t)| + \gamma_0.
\]

We set
\[
\sigma(t) = \int_M u e^{nu} d\mu_{\gamma_0}.
\]

Then solving the differential inequality above gives
\[
\sigma(t) \leq \sigma(0) \exp(n \int_0^t \lambda(s) f(p(s)) \, ds)
+ \int_0^t \left[ \tilde{\lambda}(t) \int_M f d\mu_{\gamma(0)} + 2 \sup_M |f| \text{vol}(M) |\lambda(s)| + \gamma_0 \right] \exp(n \int_s^t \lambda(\tau) f(p(\tau)) \, d\tau) \, ds
\]

for all \( t \in [0, T_0] \). Observe that
\[
\sigma(0) = \int_M u_0 e^{nu_0} d\mu_{\gamma_0} \leq \int_M e^{(n+1)u_0} d\mu_{\gamma_0}.
\]

Thanks to (2.4), we conclude that \( \sigma(0) \) is bounded from above by some number depending only on \( \gamma_0 \). In addition, we also have
\[
n \int_s^t \lambda(s) f(p(s)) ds \leq (n \sup_M |f|) \int_0^{T_0} |\lambda(t)| dt
\]

for all \( 0 \leq s \leq t \leq T_0 \). Therefore, the above estimate for \( \sigma(t) \) can be simplified as
\[
\sigma(t) \leq C_0 \left( \int_M f d\mu_{\gamma(0)} \right) \int_0^{T_0} |\tilde{\lambda}(t)| dt + \int_0^{T_0} |\lambda(t)| dt + T_0 \exp(n \sup_M |f| \int_0^{T_0} |\lambda(t)| dt)
\]

for some constant \( C_0 > 0 \) depending only on \( \gamma_0, \sup_M |f|, n, \text{vol}(M) \). Combining Lemma 3.9 and (3.37) gives
\[
\sigma(t) \leq C_0 \exp(C_0 T_0)
\]

(3.38)

for all \( t \in [0, T_0] \) and for some new constant \( C_0 \) depending only on \( \gamma_0, \sup_M |f|, n, \text{vol}(M), \int_M Q_0 d\mu_{\gamma_0}, \) and \( \int_M f d\mu_{\gamma(0)} \). Now, it follows from the inequality \( se^t \geq -1 \) that
\[
\int_A u e^{nu} d\mu_{\gamma_0} = \int_M u e^{nu} d\mu_{\gamma_0} - \int_{M \setminus A} u e^{nu} d\mu_{\gamma_0} \leq \sigma(t) + \text{vol}(M)
\]

for all \( t \in [0, T_0] \) and for all \( A \subset M \). Combining this and (3.38) gives
\[
\int_A u e^{nu} d\mu_{\gamma_0} \leq C_0 \exp(C_0 T_0)
\]

(3.39)

for some new constant \( C_0 \) still depending only on \( \gamma_0, \sup_M |f|, n, \text{vol}(M), \int_M Q_0 d\mu_{\gamma_0}, \) and \( \int_M f d\mu_{\gamma(0)} \). Now, we define
\[\varphi : (0, +\infty) \to [-1/e, +\infty), \quad s \mapsto s \log s.\]
Then $\varphi$ is convex on $(0, +\infty)$ and it satisfies

$$s = \frac{\varphi(\mu s)}{\varphi(\mu)} - \frac{\varphi(s)}{\log \mu}$$

for $\mu > 0$ and $s > 0$. Since $\varphi(s) \geq -1$ for $s > 0$, we obtain

$$s \leq \frac{\varphi(\mu s)}{\varphi(\mu)} + \frac{1}{\log \mu}.$$  

(3.40)

Now we apply Jensen’s inequality together with (3.39) to get

$$\varphi\left(\frac{1}{\text{vol}(A_r)} \int_{A_r} e^{nu} d\mu_{g_0}\right) \leq \frac{1}{\text{vol}(A_r)} \int_{A_r} \varphi(e^{nu}) d\mu_{g_0} \leq \frac{C_0 \exp(C_0 T_0)}{\text{vol}(A_r)}.$$  

(3.41)

If $\text{vol}(A_r) \geq 1$, then the lemma is proved by taking any constant $C_0 > 0$. So, we are left with the case $\text{vol}(A_r) < 1$. Using (3.40) with $\mu = \frac{1}{\text{vol}(A_r)}$ and $s = \int_{A_r} e^{nu} d\mu_{g_0}$ and by (3.41) we have

$$\int_{A_r} e^{nu} d\mu_{g_0} \leq -\frac{\text{vol}(A_r)}{\log \text{vol}(A_r)} \varphi\left(\frac{1}{\text{vol}(A_r)} \int_{A_r} e^{nu} d\mu_{g_0}\right) - \frac{1}{\log \text{vol}(A_r)}.$$  

(3.42)

Keep in mind that $\int_{M} f d\mu_g(0) > 0$. Clearly,

$$\frac{1}{\sup_{M} f} \int_{M} f d\mu_g(0) = \frac{1}{\sup_{M} f} \int_{M} f e^{nu} d\mu_{g_0} \leq \int_{A_r} e^{nu} d\mu_{g_0} + \int_{M \setminus A_r} e^{nu} d\mu_{g_0}.$$  

However, it follows from the definition of $\omega_0$ in (3.31) that

$$\int_{M \setminus A_r} e^{nu} d\mu_{g_0} \leq e^{n\omega_0} \text{vol}(M) = \frac{1}{2 \sup_{M} f} \int_{M} f d\mu_g(0).$$  

Therefore, combining the previous two estimates gives

$$\int_{A_r} e^{nu} d\mu_{g_0} \geq \frac{1}{2 \sup_{M} f} \int_{M} f d\mu_g(0).$$  

This together with (3.42) yields

$$\log \frac{1}{\text{vol}(A_r)} \leq C_0 \exp(C_0 T_0)$$

or equivalently,

$$\text{vol}(A_r) \geq \exp\left(-C_0 e^{C_0 T_0}\right).$$

The proof is thus complete. \qed

To end this subsection, we try to get the time-depending bound for $u(t)$ in $H^{n/2}$-norm.

**Proposition 3.13** Assume that $(n - 1)! \text{vol}(\mathbb{S}^n) < \int_{M} Q_0 d\mu_{g_0} \neq k(n - 1)! \text{vol}(\mathbb{S}^n), \ k \in \{2, 3, \ldots\}$ and that $f > 0$ everywhere. Then there exist an initial data $u_0 \in C^\infty(M)$ and a positive constant $C_0$ depending only on $f$, $Q_0$, $n$, and $(M, g_0)$ such that

$$\sup_{t \in [0, T]} \|u(t)\|_{H^{n/2}(M)} \leq \exp(C_0 e^{C_0 T}).$$  

(3.43)
Proof. To prove the bound, we need to control \( \int_M u \cdot P_0 u \) and \( \|u\|_{L^2(M)} \). For any \( t \in [0, T_0] \subset [0, T) \) we have

\[
\left| \int_M u \, d\mu_{g_0} \right| \leq \left| \int_{A_t} u \, d\mu_{g_0} \right| + \left| \int_{M \setminus A_t} u \, d\mu_{g_0} \right| =: I + II. \tag{3.44}
\]

To estimate \( I \), we simply make use of the elementary inequality \( s \leq e^s \) and (3.29) to get

\[
\int_{A_t} u \, d\mu_{g_0} \leq \frac{1}{n} \int_{A_t} e^{nu} \, d\mu_{g_0} \leq \frac{1}{n} \int_M e^{nu} \, d\mu_{g_0} \leq n \gamma_0 (1 + T_0).
\]

On the other hand, by the definition of \( A_t \) we have

\[
\int_{A_t} u \, d\mu_{g_0} \geq -|\alpha_0| \text{vol}(M).
\]

Therefore, by letting \( C_1 = n \gamma_0 + |\alpha_0| \text{vol}(M) \), we get

\[
I \leq C_1 (1 + T_0).
\]

For the estimate of \( II \), we use Hölder’s inequality to get

\[
II \leq \text{vol}(M \setminus A_t)^{1/2} \|u(t)\|_{L^2(M)}.
\]

Plugging the estimates of \( I \) and \( II \) into (3.44) yields

\[
\left| \int_M u \, d\mu_{g_0} \right| \leq \text{vol}(M \setminus A_t)^{1/2} \|u\|_{L^2(M)} + C_1 (1 + T_0).
\]

This together with the Young’s inequality implies that for any \( \varepsilon > 0 \)

\[
\left( \int_M u \, d\mu_{g_0} \right)^2 \leq (1 + \varepsilon) \text{vol}(M \setminus A_t) \|u\|^2_{L^2(M)} + (1 + \varepsilon^{-1}) C_1 (1 + T_0)^2. \tag{3.45}
\]

Now, it follows from Poincaré’s inequality (1.9) that

\[
\|u\|^2_{L^2(M)} \leq \frac{1}{\lambda_1} \int_M u \cdot P_0 u \, d\mu_{g_0} + \frac{1}{\text{vol}(M)} \left( \int_M u \, d\mu_{g_0} \right)^2. \tag{3.46}
\]

Combining (3.45) and (3.46) gives

\[
\left( 1 - \frac{(1 + \varepsilon) \text{vol}(M \setminus A_t)}{\text{vol}(M)} \right) \|u\|^2_{L^2(M)} \leq \frac{1}{\lambda_1} \int_M u \cdot P_0 u \, d\mu_{g_0} + (1 + \varepsilon^{-1}) C_1 (1 + T_0)^2,
\]

that is,

\[
(\text{vol}(A_t) - \varepsilon \text{vol}(M \setminus A_t)) \|u\|^2_{L^2(M)} \leq \frac{\text{vol}(M)}{\lambda_1} \int_M u \cdot P_0 u \, d\mu_{g_0}
\]

\[
+ (1 + \varepsilon^{-1}) \text{vol}(M) C_1 (1 + T_0)^2.
\]

Notice that we can apply Proposition 3.10 to get \( \inf_{t \in [0, T_0]} e^t [u(t)] > -\infty \). This enables us to find some \( \gamma_0 > 0 \) such that (3.32) holds. Then by Lemma 3.12 we have \( \text{vol}(A_t) \geq \exp(-C_0 e^{C_0 T_0}) \). By choosing \( \varepsilon = \exp(-C_0 e^{C_0 T_0})/(2 \text{vol}(M)) \) and observing the fact that \( \text{vol}(M \setminus A_t) \leq \text{vol}(M) \) we get

\[
\text{vol}(A_t) - \varepsilon \text{vol}(M \setminus A_t) \geq \frac{1}{2} \exp(-C_0 e^{C_0 T_0}),
\]
which implies that
\[ \|u\|_{L^2(M)}^2 \leq C_2 \left( \int_M u \cdot P_0 u \, d\mu_{g_0} + 1 \right) \exp(C_0 e^{C_0 T_0}) \]  
for some constant \( C_2 > 0 \) depending only on \( C_0, C_1, T_0, \text{vol}(M) \), and \( \lambda_1 \). In view of the energy bound \( -\gamma_0 \leq \delta'[u(t)] \leq \delta[u_0] \) and Hölder’s inequality, we can bound
\[ \int_M u \cdot P_0 u \, d\mu_{g_0} \leq C_3 \|u\|_{L^2(M)} + C_3. \]  
for some constant \( C_3 > 0 \) depending only on \( \gamma_0 \) and \( \int_M Q_0^2 \, d\mu_{g_0} \). Substituting (3.48) into (3.47) gives
\[ \|u\|_{L^2(M)}^2 \leq C_2(C_3 + 1) \left( \|u\|_{L^2(M)} + 1 \right) \exp(C_0 e^{C_0 T_0}). \]  
However, this is enough to conclude that
\[ \|u\|_{L^2(M)} \leq \exp(C_0 e^{C_0 T_0}). \]  
for some constant \( C_0 > 0 \) depending only on \( \gamma_0, f, (M, g_0), \) and \( n \). Going back to (3.48), the bound in (3.49) immediately gives us a bound for \( \int_M u \cdot P_0 u \); hence concluding (3.43) as claimed. The proof is complete. \( \square \)

4 Boundedness of \( u(t) \) in \( H^n(M) \) for \( 0 \leq t < T \)

We will improve our boundedness of \( u(t) \) in \( H^{n/2}(M) \) established in Sect. 3 to the higher-order Sobolev space. In this section, we show the boundedness of \( u(t) \) in \( H^n(M) \). The first step in this procedure is to show that \( \lambda(t) \) is pointwise bounded on the maximal interval of existence of the flow. This can be regarded as an improvement of Lemma 3.9.

**Lemma 4.1** For the subcritical case or \( M = \mathbb{S}^n \), then there exists a universal constant \( \mathcal{C}_\ell > 0 \) such that
\[ |\lambda(t)| \leq \mathcal{C}_\ell. \]

While, for the supercritical case, there exists a constant \( \mathcal{C}_\ell(T) > 0 \) depending on the maximal time of existence \( T \), such that
\[ |\lambda(t)| \leq \mathcal{C}_\ell(T) \]
for all \( t \in [0, T) \).

**Proof** From the definition of \( \lambda(t) \) in (1.17), to bound \( \lambda(t) \) it suffices to bound \( \int_M Q d\mu_{g(t)} \) from above and \( \int_M P^2 d\mu_{g(t)} \) from below positively away from zero. By the expression of \( Q \)-curvature in (1.5) we conclude that
\[
\left| \int_M Q_{g(t)} d\mu_{g(t)} \right| = \left| \int_M (P_0 u + Q_0) d\mu_{g_0} \right| = \left| \int_M \left[ u \cdot P_0 f + f Q_0 \right] d\mu_{g_0} \right| \\
\leq \sup_M \left( |P_0 f| \right) \|u(t)\|_{H^{n/2}(M) \sqrt{\text{vol}(M)}} + \sup_M |f Q_0| \text{vol}(M). \]  
Thus (4.1) and the boundedness of \( u(t) \) in \( H^{n/2}(M) \) established in Sect. 3 provide us an upper bound for \( \int_M Q_{g(t)} d\mu_{g(t)} \). To bound \( \int_M P^2 d\mu_{g(t)} \) from below, first we denote \( \sigma = \int_M f^2 d\mu_{g_0} > 0 \). Then, by Jensen’s inequality we can estimate \( \mathcal{C} \) Springer
\[
\int_M f^2 d\mu_{g(t)} = \int_M e^{nu} f^2 d\mu_{g_0} = \sigma \left( \frac{1}{\sigma} \int_M e^{nu} f^2 d\mu_{g_0} \right) \\
\geq \sigma \exp \left( \frac{n}{\sigma} \int_M u f^2 d\mu_{g_0} \right) \\
\geq \sigma \exp \left( -\frac{n}{2\sigma} \int_M [u^2 + f^4] d\mu_{g_0} \right) \\
\geq \sigma \exp \left( -\frac{n}{2\sigma} (\|u\|_{H^{n/2}(M)}^2 + \sup_M (f^4) \text{vol}(M)) \right). \tag{4.2}
\]

Again the boundedness of \(u(t)\) in \(H^{n/2}(M)\) tells us that \(\int_M f^2 d\mu_{g(t)}\) is bounded from below. Now, the proof follows from (4.1) and (4.2).

Next, we follow the argument in [7] to bound \(\|u\|_{H^n(M)}\). To this end, first by standard elliptic estimates we know that
\[
\|u\|_{H^n(M)} \lesssim \|P_0 u\|_{L^2(M)} + \|u\|_{L^2(M)}.
\]
Using the continuous embedding \(H^{n/2}(M) \hookrightarrow L^2(M)\) together with the boundedness of \(u(t)\) in \(H^{n/2}(M)\) we deduce that
\[
\|u\|_{H^n(M)} \lesssim \|P_0 u\|_{L^2(M)} + 1.
\]
Hence, to bound \(\|u\|_{H^n(M)}\), it suffices to bound \(\int_M (P_0 u)^2 d\mu_{g_0}\) from above. Let \(T_0 \in (0, T)\) be a fixed, positive real number and \(u(t)\) defined on \([0, T_0]\). For brevity, we let
\[
v = \frac{e^{nu/2}}{\lambda(t)} (Q_{g(t)} - \lambda(t) f).
\]
Then it is not hard to verify that
\[
\begin{aligned}
\frac{\partial}{\partial t} u &= e^{-nu/2} v, \\
P_0 u &= e^{-nu/2} v - Q_0 + e^{nu} \lambda(t) f, \\
\frac{\partial}{\partial t} P_0 u &= P_0 \left( \frac{\partial}{\partial t} u \right) = P_0 (e^{-nu/2} v).
\end{aligned}
\]
Consequently, we can compute the time derivative of \(\int_M (P_0 u)^2 d\mu_{g_0}\) as follows
\[
\frac{d}{dt} \left( \int_M (P_0 u)^2 d\mu_{g_0} \right) = 2 \int_M P_0 u \cdot P_0 (e^{-nu/2} v) d\mu_{g_0} \\
= -2 \int_M e^{nu/2} v \cdot P_0 (e^{-nu/2} v) d\mu_{g_0} \\
- 2 \int_M Q_0 P_0 (e^{-nu/2} v) d\mu_{g_0} \\
+ 2\lambda(t) \int_M e^{nu} f P_0 (e^{-nu/2} v) d\mu_{g_0}. \tag{4.3}
\]
To be able to examine this time derivative, we further estimate (4.3) as follows
\[
\frac{d}{dt} \left( \int_M (P_0 u)^2 d\mu_{g_0} \right) = -2 \int_M (-\Delta_0)^{n/4} (e^{nu/2} v) (-\Delta_0)^{n/4} (e^{-nu/2} v) d\mu_{g_0} \\
- 2 \int_M (-\Delta_0)^{n/4} Q_0 (-\Delta_0)^{n/4} (e^{-nu/2} v) d\mu_{g_0} \\
+ 2\lambda(t) \int_M (-\Delta_0)^{n/4} (e^{nu} f) (-\Delta_0)^{n/4} (e^{-nu/2} v) d\mu_{g_0} \\
+ \text{lower order terms}.
\]
Note that the right-hand side of (4.3) involves derivatives of $u$ and $v$ of order at most $n/2$ and the total number of derivatives is at most $n$. In view of Lemma 4.1, we obtain

$$
\frac{d}{dt} \left( \int_M (\mathbf{P}_0 u)^2 \, d\mu_{g_0} \right) \leq -2 \int_M \left( (-\Delta_0)^{n/4} v \right)^2 \, d\mu_{g_0} + \mathcal{C} \sum_{k_1, \ldots, k_m} \int_M |\nabla_0^{k_1} v| |\nabla_0^{k_2} v| |\nabla_0^{k_3} u| \cdots |\nabla_0^{k_m} u| \, d\mu_{g_0}
$$

and

$$
+ \mathcal{C} \sum_{l_1, \ldots, l_m} \int_M |\nabla_0^{l_1} v| |\nabla_0^{l_2} u| \cdots |\nabla_0^{l_m} u| e^{\alpha(l_1, \ldots, l_m) u} \, d\mu_{g_0}.
$$

Note that in (4.4), the first sum is taken over all $m$-tuples $(k_1, \ldots, k_m)$ with $m \geq 3$ satisfying $0 \leq k_i \leq n/2$ for $1 \leq i \leq 2$ and $1 \leq k_i \leq n/2$ for $3 \leq i \leq m$ with $\sum_{i=1}^m k_i \leq n$ and the second sum is taken over all $m$-tuples $(l_1, \ldots, l_m)$ with $m \geq 1$ satisfying $0 \leq l_1 \leq n/2$ and $1 \leq l_i \leq n/2$ for $2 \leq i \leq m$ with $\sum_{i=1}^m l_i \leq n$.

Next, we estimate the right hand side of (4.4) term by term. Following [7, Section 4], we obtain the following two fundamental estimates

$$
- \frac{1}{2} \int_M \left( (-\Delta_0)^{n/4} v \right)^2 \, d\mu_{g_0} + \mathcal{C} \sum_{k_1, \ldots, k_m} \int_M |\nabla_0^{k_1} v| |\nabla_0^{k_2} v| |\nabla_0^{k_3} u| \cdots |\nabla_0^{k_m} u| \, d\mu_{g_0}
$$

$$
\leq C \|v\|_{L^2(M)}^2 \left( \|u\|_{H^n(M)}^2 + 1 \right)
$$

and

$$
- \frac{1}{2} \int_M \left( (-\Delta_0)^{n/4} v \right)^2 \, d\mu_{g_0} + \mathcal{C} \sum_{l_1, \ldots, l_m} \int_M |\nabla_0^{l_1} v| |\nabla_0^{l_2} u| \cdots |\nabla_0^{l_m} u| e^{\alpha u} \, d\mu_{g_0}
$$

$$
\leq C \left( \|v\|_{L^2(M)}^2 + 1 \right) \left( \|u\|_{H^n(M)}^2 + 1 \right)
$$

Then combining (4.4), (4.5), and (4.6) gives

$$
\frac{d}{dt} \left( \int_M (\mathbf{P}_0 u)^2 \, d\mu_{g_0} \right) \leq C \left( \|v\|_{L^2(M)}^2 + 1 \right) \left( \|u\|_{H^n(M)}^2 + 1 \right)
$$

for some constant $C$ independent of $t$. Observe that

$$
\|v\|_{L^2(M)}^2 = 4 \int_M e^{\alpha u} (Q_{g(t)} - \lambda(t) f)^2 \, d\mu_{g_0}
$$

and there holds $\|u\|_{H^n(M)}^2 \lesssim \int_M (\mathbf{P}_0 u)^2 \, d\mu_{g_0} + 1$. Hence,

$$
\frac{d}{dt} \left( \int_M (\mathbf{P}_0 u)^2 \, d\mu_{g_0} + 1 \right) \leq C \left( 1 + \int_M (Q_{g(t)} - \lambda(t) f)^2 \, d\mu_{g(t)} \right) \left( 1 + \int_M (\mathbf{P}_0 u)^2 \, d\mu_{g(t)} \right),
$$

which implies

$$
\frac{d}{dt} \log \left( \int_M (\mathbf{P}_0 u)^2 \, d\mu_{g_0} + 1 \right) \leq C \left( 1 + \int_M (Q_{g(t)} - \lambda(t) f)^2 \, d\mu_{g(t)} \right).
$$

Upon integrating both sides of the preceding inequality over $[0, T_0]$ and using Lemma 2.2, we conclude that

$$
\int_M (\mathbf{P}_0 u)^2 \, d\mu_{g_0} \leq C(T_0)
$$

for all $t \in [0, T_0]$. Thus, combining (1.9) and (4.7) gives the uniform boundedness of $u(t)$ in $H^n(M)$ as claimed. Thus, we have just proved the following result.
Lemma 4.2 For any solution \( u(t) \) to the flow \((F)\), there exists a constant \( \mathcal{C}(T) > 0 \) such that
\[
\sup_{0 \leq t < T} \| u(t) \|_{H^n(M)} \leq \mathcal{C}(T).
\]

Note that if the solution \( u(t) \) to the flow is uniformly bounded in \( H^n(M) \) for every fixed time interval \([0, T_0]\), then by Sobolev’s inequality it is easy to show that along the flow \( u(t) \) is also pointwise bounded in \([0, T_0]\). Therefore we have the following simple corollary.

Corollary 4.3 For any solution \( u(t) \) to \((F)\), there exists a constant \( \mathcal{C}(T) > 0 \) such that
\[
\sup_{0 \leq t < T} | u(t) | \leq \mathcal{C}(T).
\]

In the next section, we turn our \( H^n \)-boundedness into even higher-order Sobolev spaces. Having such a boundedness, we obtain a global existence of the flow \((F)\).

5 Boundedness of \( u(t) \) in \( H^{2k}(M) \) for \( 0 \leq t < T \) and the long time existence of the flow

The aim of this section is to strengthen the \( H^n \)-boundedness of \( u(t) \) obtained in Sect. 4 above. What we are going to prove is that \( u(t) \) is bounded in \( H^{2k}(M) \) for any \( k > 2 \), which is crucial to claim that the flow is defined to all time. Again, it is worth noting that the argument below does not depend on the size of \( \int_M Q_0 \, d\mu_{g_0} \). To realize this, we follow the argument in [7].

In view of (1.13), to estimate \( u(t) \) in \( H^{2k} \)-norm, we need to estimate \( \int_M |(-\Delta_0)^k u|^2 \, d\mu_{g_0} \).

Using the flow equation \( \partial_t u(t) = \lambda(t) f - Q_{g(t)} u \), for some constant \( \mathcal{C} > 0 \), we obtain
\[
\frac{d}{dt} \left( \int_M |(-\Delta_0)^k u|^2 \, d\mu_{g_0} \right) = 2 \int_M (-\Delta_0)^k u (-\Delta_0)^k (\partial_t u) \, d\mu_{g_0}
\]
\[
= -2 \int_M (-\Delta_0)^k u (-\Delta_0)^k (e^{-nu} P_0 u) \, d\mu_{g_0}
\]
\[
- 2 \int_M (-\Delta_0)^k u (-\Delta_0)^k (e^{-nu} Q_0 u) \, d\mu_{g_0}
\]
\[
+ 2\lambda(t) \int_M (-\Delta_0)^k u (-\Delta_0)^k f \, d\mu_{g_0}
\]
\[
\leq -2 \int_M e^{-nu} |(-\Delta_0)^{k+n/4} u|^2 \, d\mu_{g_0}
\]
\[
+ \mathcal{C} \sum_{k_1, \ldots, k_m} \int_M |\nabla^{k_1} u| \cdots |\nabla^{k_m} u| \, d\mu_{g_0} \tag{5.1}
\]
for all \( t \geq 0 \). Here the sum is taken for all tuples \((k_1, \ldots, k_m)\) with \( m \geq 3 \) satisfying \( 1 \leq k_i \leq 2k + n/2 \) and \( \sum_{i=1}^{m} k_i \leq 4k + n \). We now choose real numbers \( p_i \in [2, +\infty) \) such that \( k_i \leq 2k + n/p_i \) and \( \sum_{i=1}^{m} 1/p_i = 1 \). Then we set
\[
\theta_i = \max \left\{ \frac{k_i - n/p_i - n/2}{2k - n/2}, 0 \right\}
\]
for each \( i = 1, \ldots, m \). Since \( m \geq 3 \), we immediately have \( \sum_{i=1}^{m} \theta_i < 2 \). Now we estimate (5.1). First, in view of Corollary 4.3, we can further estimate the left hand side of (5.1) to
obtain
\[
\frac{d}{dt} \left( \int_M |(-\Delta_0)^k u|^2 \, d\mu_{g_0} \right) \leq -\mathcal{C} \int_M |(-\Delta_0)^{k+\frac{n}{4}} u|^2 \, d\mu_{g_0} + \mathcal{C} \sum_{k_1, \ldots, k_m} \int_M \|\nabla^{k_1} u\| \cdots \|\nabla^{k_m} u\| \, d\mu_{g_0}
\] (5.2)
for some new constant \(\mathcal{C}\). By repeatedly using (5.2) for suitable \(k\) and Hölder’s inequality, we arrive at
\[
\frac{d}{dt} \|u\|_{H^{2k}(M)}^2 \leq -\mathcal{C} \|u\|_{H^{2k+\frac{n}{2}}(M)}^2 + \mathcal{C} \sum_{k_1, \ldots, k_m} \|\nabla^{k_1} u\|_{L^{p_1}(M)} \cdots \|\nabla^{k_m} u\|_{L^{p_m}(M)}
\] (5.3)
for some new constant \(\mathcal{C}\). Thanks to (2.1), we obtain
\[
\|\nabla^{k_i} u\|_{L^{p_i}(M)} \leq \mathcal{C} \|u\|_{H^{k_i - \frac{n}{p_i} + \frac{n}{2}}(M)}.
\]
Again, we apply (2.1) to obtain
\[
\|u\|_{H^{k_i - \frac{n}{p_i} + \frac{n}{2}}(M)} \leq \mathcal{C} \|u\|_{H^{k_i + \frac{n}{2}}(M)}^{\theta_i},
\]
By combining the last two estimates above and the boundedness of \(u(t)\) in \(H^n(M)\) established in Lemma 4.2, we get
\[
\|\nabla^{k_i} u\|_{L^{p_i}(M)} \leq \mathcal{C} \|u\|_{H^{k_i + \frac{n}{2}}(M)}^{\theta_i}.
\]
This together with (5.3) implies that
\[
\frac{d}{dt} \|u\|_{H^{2k}(M)}^2 \leq -\mathcal{C} \|u\|_{H^{2k+\frac{n}{2}}(M)}^2 + \mathcal{C} \sum_{k_1, \ldots, k_m} \|u\|_{H^{2k+\frac{n}{2}}(M)}^{\theta_{1} + \cdots + \theta_{m}}.
\] (5.4)
Since \(\sum_{i=1}^{m} \theta_i < 2\), from (5.4) we have by Young’s inequality that
\[
\frac{d}{dt} \|u\|_{H^{2k}(M)}^2 \leq -\mathcal{C} \|u\|_{H^{2k+\frac{n}{2}}(M)}^2 + \mathcal{C} \leq -\mathcal{C} \|u\|_{H^{2k}(M)}^2 + \mathcal{C},
\]
where in the last estimate, we have used the embedding \(H^{2k+1}(M) \hookrightarrow H^{2k}(M)\). From this, it is routine to get
\[
\|u\|_{H^{2k}(M)} \leq \mathcal{C}_H(k, T)
\]
for some constant \(\mathcal{C}_H(k, T)\) as claimed. Thus, we have just finished proving the following result.

**Lemma 5.1** For any solution \(u(t)\) to the flow (F), there exists a constant \(\mathcal{C}_H(k, T)\) depending on \(k\) and \(T\) such that
\[
\sup_{0 \leq t < T} \|u(t)\|_{H^{2k}(M)} \leq \mathcal{C}_H(k, T)
\]
for any \(k > 1\).

An immediate consequence of Lemma 5.1 above is that the flow (F), with an appropriate initial data, cannot blow up in any finite time.
Proposition 5.2  Given a suitable initial data $u_0$, the flow $(F)$ has a smooth solution which is defined to all time.

With the help of the long time existence of the flow $(F)$, we claim that there is a universal constant $C > 0$ such that

$$\int_0^{+\infty} \int_M (\lambda(t) f - Q_{g(t)})^2 d\mu_{g(t)} dt \leq C. \quad (5.5)$$

Indeed, from the proofs in the Sect. 3 and Proposition 5.2, one can easily deduce that

$$\inf_{t \geq 0} E[u(t)] > -\infty.$$ 

This together with Lemma 2.2 implies (5.5).

6 Improved $H^{n/2}$-bound for $u(t)$ in the supercritical case

Up to the previous section, we have already seen that $u(t)$ is uniformly bounded in $H^{n/2}(M)$ in any fixed time interval and the flow is defined to all time. Furthermore, in the subcritical case or when $M = \mathbb{S}^n$, the $H^{n/2}$-bound of $u(t)$ does not depend on the length of the time interval; see Sect. 3. This section is devoted to showing a similar result in the supercritical case.

To this purpose, let us mention the following concentration-compactness result due to Fardoun, Regbaoui in [15]. Although the authors in [15] only consider 4-manifolds, their proof can be easily adapted to the general even dimensional case.

Proposition 6.1 (Fardoun and Regbaoui) Let $h \in C^0(M)$ and $(u_k, h_k)_k$ be a sequence in $H^{n/2}(M) \times L^1(M)$ satisfying the equation

$$P_0 u_k + h_k = f_k e^{nu_k}$$

with

$$\int_M f_k e^{nu_k} d\mu_{g_0} = \int_M h_k d\mu_{g_0}$$

and

$$h_k \to h$$

in $L^1(M)$ as $k \to +\infty$, where $f_k \in C^0(M)$ and $C^{-1} \leq f_k \leq C$ for some positive constant $C$. Then either

(i) the sequence $(e^{nu_k})_k$ is bounded in $L^p(M)$ for all $p \in [1, +\infty)$ or

(ii) for a subsequence, still denoted by $(u_k)_k$, there exist a finite number of points $x_1, \ldots, x_m \in M$ and integers $l_1, \ldots, l_m \in \mathbb{N}^*$ such that

$$\sum_{j=1}^m l_j = (n-1)! \text{vol}(\mathbb{S}^n) \int_M h d\mu_{g_0}$$

and as $k \to +\infty$

$$e^{nu_k} \to \frac{(n-1)! \text{vol}(\mathbb{S}^n)}{\int_M h d\mu_{g_0}} \sum_{j=1}^m l_j \delta_{x_j}$$

in the sense of measure, where $\delta_x$ is the Dirac mass at the point $x \in M$. 

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Using the above result, we are able to control \( u(t) \) in \( H^{n/2} \)-norm, which can be thought of as an improvement of Proposition 3.13.

**Lemma 6.2** Let \((n-1)!\text{vol}(S^n) < \int_M Q_0 \, d\mu_{g_0} \neq k(n-1)!\text{vol}(S^n), \ k \in \{2, 3, \ldots\}\). Then, there exists a uniform constant \( C_{n/2} > 0 \) such that

\[
\sup_{t \geq 0} \|u(t)\|_{H^{n/2}(M)} \leq C_{n/2}.
\]

**Proof** In view of (5.5), we have

\[
\int_0^{+\infty} \int_M u_t(t)^2 e^{nu(t)} \, d\mu_{g_0} \, dt < +\infty.
\]

This implies that there exists a sequence \( t_k \in [k, k+1) \) for all \( k \in \mathbb{N}^* \) such that

\[
\lim_{k \to +\infty} \int_M |u_t(t_k)|^2 e^{nu(t_k)} \, d\mu_{g_0} = 0.
\]

Since \( f > 0 \) everywhere, it is easy to see that

\[
\int_M e^{nu(t_k)} \, d\mu_{g_0} \leq \frac{1}{\inf_M f} \int_M f \, d\mu_{g(0)}.
\]

Besides, as in the proof of Lemma 3.9, there holds

\[
\left( \lambda(t_k) - \frac{\int_M Q_0 \, d\mu_{g_0}}{\int_M f \, d\mu_{g(0)}} \right)^2 \left( \int_M f \, d\mu_{g(0)} \right)^2 \leq \int_M |u_t(t_k)|^2 e^{nu(t_k)} \, d\mu_{g_0} \int_M e^{nu(t_k)} \, d\mu_{g_0}.
\]

Putting the above three estimates together, we deduce that

\[
\lambda(t_k) \to \frac{\int_M Q_0 \, d\mu_{g_0}}{\int_M f \, d\mu_{g(0)}}
\]

as \( k \to +\infty \). Now, if we set

\[
u_k = u(t_k), \ h_k = u_t(t_k) e^{nu_k} + Q_0, \ \text{and} \ f_k = \lambda(t_k) f,
\]

then we have

\[
P_0 u_k + h_k = f_k e^{nu_k}
\]

(6.1)

with \( f_k \in C^0(M) \) satisfying

\[
\frac{1}{2} \frac{\int_M Q_0 \, d\mu_{g_0}}{\int_M f \, d\mu_{g(0)}} \inf_M f \leq f_k \leq \frac{1}{2} \frac{\int_M Q_0 \, d\mu_{g_0}}{\int_M f \, d\mu_{g(0)}} \sup_M f
\]

and

\[
\int_M f_k e^{nu_k} \, d\mu_{g_0} = \int_M h_k \, d\mu_{g_0}.
\]

In addition, we obtain

\[
\|h_k - Q_0\|_{L^1(M)} \leq \left( \int_M e^{nu(t_k)} \, d\mu_{g_0} \right)^{1/2} \left( \int_M |u_t(t_k)|^2 e^{nu(t_k)} \, d\mu_{g_0} \right)^{1/2} \to 0
\]

as \( k \to +\infty \). Therefore, under the condition \( \int_M Q_0 \, d\mu_{g_0} \notin (n-1)!\text{vol}(S^n)\mathbb{N} \) we know that the alternative (ii) in Proposition 6.1 cannot occur. Hence, for any \( p \geq 1 \) but fixed, we know that

\[
\int_M e^{p|u_k|} \, d\mu_{g_0} \leq C_p
\]

(6.2)
for some constant $C_p > 0$ depending on $p$. Hence, it is not hard to verify that
\[ \|f_k\|_{L^p} \leq C_p \quad \text{and} \quad \|h_k\|_{L^p} \leq C_p, \quad (6.3) \]
for all $p \in [1, 2]$. Using (6.2) and (6.3) we apply the standard elliptic regularity theory to (6.1) and get that $(u_k)_k$ is bounded in $W^{n,p}(M)$ for all $p \in [1, 2)$. Keep in mind that Sobolev’s embedding theorem $W^{n,p}(M) \subset H^{n/2}(M) \cap C^\alpha(M)$ holds true for all $\alpha \in [0, 1 - 1/p]$. Hence, we have proved that $(u_k)_k$ is bounded in $H^{n/2}(M)$, that is,
\[ \|u_k\|_{H^{n/2}(M)} \leq C, \quad (6.4) \]
where $C$ is a positive constant depending only on $n, \gamma, u_0, Q_0$ and $M$. Now, we define
\[ v_k(t) := u(t + t_k). \]
Clearly, $v_k$ solves
\[ \begin{cases} 
 e^{ng_k} \partial_t v_k = -(P_0 v_k + Q_0) + \lambda(t + t_k) f e^{v_k}, \\
 v_k(0) = u_k. 
\end{cases} \]
In view of (6.4) we know that $\|v_k(0)\|_{H^{n/2}(M)} < +\infty$. From this and
\[ \inf_{t \in [0,1]} \delta'[v_k(t)] = \inf_{t \in [t_k, t_k+1]} \delta'[u(t)] > -\infty, \]
we can apply Proposition 3.13 with $T_0 = 1$ to get that
\[ \sup_{t \in [0,1]} \|v_k(t)\|_{H^{n/2}(M)} \leq \exp(C_0 e^{C_0}), \]
which is equivalent to
\[ \sup_{t \in [t_k, t_k+1]} \|u(t)\|_{H^{n/2}(M)} \leq \exp(C_0 e^{C_0}). \]
Since $t_k \in [k, k + 1]$ for all $k \in \mathbb{N}^*$, we have
\[ \sup_{t \in (0, +\infty)} \|u(t)\|_{H^{n/2}(M)} \leq \exp(C_0 e^{C_0}). \]
This conclude the supercritical case. \(\square\)

Once we have the time-independent $H^{n/2}$-bound for $u(t)$, we can follow the proof of Lemma 4.1 to bound $\lambda(t)$ for all time, leading us to the following result.

**Lemma 6.3** There exists a universal constant $C_\ell > 0$ such that
\[ |\lambda(t)| \leq C_\ell \]
for all $t \geq 0$.

### 7 Boundedness of $u(t)$ in $C^\infty(M)$ for all time

In this section, we shall derive the uniform boundedness of our flow in $C^\infty$-norm. The key step is to show the convergence of the quantity below
\[ F_2(t) = : \int_M (Q_{g(t)} - \lambda(t) f)^2 d\mu_{g(t)}. \]
as time goes to infinity. To do so, we first notice that the time-independent bound for our flow in $H^{n/2}$-norm, (2.4) and Proposition 5.2 gives the estimate: for any real number $\alpha$, there is a uniform constant $C(\alpha) > 0$ such that
\[
\int_M e^{\alpha u(t)} \, d\mu_{g_0} \leq C(\alpha). \tag{7.1}
\]
for all $t > 0$.

Recall that we have obtained the time-independent bound for $\lambda(t)$; see Lemmas 4.1 and 6.3. This uniform bound is crucial to the convergence of $F_2(t)$.

**Lemma 7.1** We have $F_2(t) \to 0$ as $t \to +\infty$.

**Proof** Let $0 < \epsilon \ll 1$ be arbitrary and choose $t_0 \geq 0$ in such a way that $F_2(t_0) \leq \epsilon$. We shall show that $F_2(t) \leq 3\epsilon$ for all $t \geq t_0$. By way of contradiction, we may assume that there is a finite number $t_1$ defined by
\[
t_1 := \inf\{t \geq t_0 : F_2(t) \geq 3\epsilon\}.
\]
Let us now consider the flow in $[0, t_1]$ and throughout this proof, by $C$ we mean a positive constant independent of $t_0$ and $t_1$ which may change from line to line. Then by using (1.5), Lemma 4.1, the fact that $F_2(t) \leq 3\epsilon$ for $t_0 \leq t \leq t_1$ and the estimate
\[|Qg(t)|^2 \leq 2|Qg(t) - \lambda(t)f|^2 + 2|\lambda(t)|^2 f^2\]
we deduce that
\[
\int_M e^{-nu} (Q_0 + P_0 u)^2 \, d\mu_{g_0} = \int_M |Qg(t)|^2 \, d\mu_{g(t)} \leq C \tag{7.2}
\]
for all $t_0 \leq t \leq t_1$. Notice that by (7.1) one has $\int_M e^{3nu} \, d\mu_{g_0} \leq C$ for all $t \geq 0$. Then Hölder’s inequality implies that
\[
\int_M |Q_0 + P_0 u|^{3/2} \, d\mu_{g_0} \leq \left( \int_M e^{-nu} (Q_0 + P_0 u)^2 \, d\mu_{g_0} \right)^{3/4} \left( \int_M e^{3nu} \, d\mu_{g_0} \right)^{1/4}.
\]
From all three estimates above, we deduce that
\[
\int_M |P_0 u|^{3/2} \, d\mu_{g_0} \leq C \tag{7.3}
\]
for all $t_0 \leq t \leq t_1$. By using Sobolev’s inequality, we obtain $|u(t)| \leq C$ for all $t_0 \leq t \leq t_1$. It follows from Lemma 2.1 that $Qg(t) - \lambda(t)f$ satisfies the evolution equation
\[
\frac{\partial}{\partial t} (Qg(t) - \lambda(t)f) = -Pg(t)(Qg(t) - \lambda(t)f) + nQg(t) (Qg(t) - \lambda(t)f) + \lambda'(t)f.
\]
From this we can estimate the time derivative of $F_2(t)$ as follows

$$\frac{d}{dt} F_2(t) = 2 \int_M \left( Q_{g(t)} - \lambda(t) f \right) \left( \frac{\partial}{\partial t} (Q_{g(t)} - \lambda(t) f) d\mu_{g(t)} + n \int_M (\lambda(t) f - Q_{g(t)})^2 d\mu_{g(t)} \right)$$

$$= -2 \int_M \left( Q_{g(t)} - \lambda(t) f \right) P_{g(t)} (Q_{g(t)} - \lambda(t) f) d\mu_{g(t)}$$

$$+ 2n \int_M (Q_{g(t)} - \lambda(t) f) Q_{g(t)} (Q_{g(t)} - \lambda(t) f) d\mu_{g(t)}$$

$$+ 2\lambda'(t) \int_M f (Q_{g(t)} - \lambda(t) f) d\mu_{g(t)} + n \int_M (\lambda(t) f - Q_{g(t)})^3 d\mu_{g(t)}$$

$$= -2 \int_M \left( Q_{g(t)} - \lambda(t) f \right) P_{g(t)} (Q_{g(t)} - \lambda(t) f) d\mu_{g(t)}$$

$$+ 2n\lambda(t) \int_M f (Q_{g(t)} - \lambda(t) f)^2 d\mu_{g(t)} + n \int_M (Q_{g(t)} - \lambda(t) f)^3 d\mu_{g(t)}.$$

Here we have used (1.16) to drop off the term involving $\lambda'(t)$. Next we apply the Gagliardo–Nirenberg interpolation inequality (2.1) to get

$$\int_M |\lambda(t) f - Q_{g(t)}|^3 d\mu_{g(t)} \leq C \left( \int_M (\lambda(t) f - Q_{g(t)})^2 d\mu_{g(t)} \right) \left\| \lambda(t) f - Q_{g(t)} \right\|_{H^{n/2}(M,g(t))},$$

where the norm can be taken with respect to $g(t)$ due to the uniform boundedness of $u$ on $[t_0, t_1]$. From this, we easily get

$$\int_M |\lambda(t) f - Q_{g(t)}|^3 d\mu_{g(t)} \leq C \left( \int_M (\lambda(t) f - Q_{g(t)})^2 d\mu_{g(t)} \right)$$

$$\times \left( \int_M (Q_{g(t)} - \lambda(t) f) P_{g(t)} (Q_{g(t)} - \lambda(t) f) d\mu_{g(t)} \right)^{1/2}$$

$$+ C \left( \int_M (\lambda(t) f - Q_{g(t)})^2 d\mu_{g(t)} \right)^{3/2}$$

for some constant $C$ independent of $t$. By using Young’s inequality, we further get

$$\int_M |\lambda(t) f - Q_{g(t)}|^3 d\mu_{g(t)} \leq C \left( \int_M (\lambda(t) f - Q_{g(t)})^2 d\mu_{g(t)} \right)^2$$

$$+ \frac{1}{n} \int_M (Q_{g(t)} - \lambda(t) f) P_{g(t)} (Q_{g(t)} - \lambda(t) f) d\mu_{g(t)}$$

$$+ C \left( \int_M (\lambda(t) f - Q_{g(t)})^2 d\mu_{g(t)} \right)^{3/2}$$

for some constant $C$ independent of $t$. Thus, we have just shown that

$$\frac{d}{dt} F_2(t) \leq CF_2(t)^2 + CF_2(t)^{3/2} + CF_2(t)$$

for all $t \in [t_0, t_1]$. Since $0 \leq \varepsilon < 1$, the preceding inequality gives $F_2'(t) \leq CF_2(t)$ for all $t \in [t_0, t_1]$. Integrating both sides of the inequality over $[t_0, t_1]$ gives

$$2\varepsilon \leq F_2(t_1) - F_2(t_0) \leq C \int_{t_0}^{t_1} F_2(t) dt.$$

Now by (5.5), we can select $t_0 \gg 1$ in such a way that $C \int_{t_0}^{t_1} F_2(t) dt < \varepsilon$. Hence we obtain $2\varepsilon < \varepsilon$, which is impossible. Thus we have just obtained $L^2$-convergence of the flow. \qed

\hspace{1cm} Springer
An immediate consequent of Lemma 7.1 is that $F_2(t)$ is bounded and this is exactly the same as that of (7.2), which is limited to a finite time interval, namely we have
\[ \int_M Q_2(t)^2 d\mu_{g(t)} \leq C \]
for all $t \geq 0$ and for some constant $C > 0$. Now combining Lemma 7.1 and (7.1) gives a uniform bound for $\int_M |Q_0 + P_0u|^{3/2} d\mu_{g_0}$. Consequently, we obtain a bound similar to (7.3), however, for all $t \geq 0$, namely we have
\[ \int_M |P_0u|^{3/2} d\mu_{g_0} \leq C \]
for all $t \geq 0$. Applying Green’s formula gives
\[ u(x) = \bar{u} + \int_M P_0u(z)G(x,z)d\mu_{g_0(z)}, \]
where $G(x,z)$ is Green’s function associated to the operator $P_0$. Green’s function $G$ is in $C^\infty(M \times M \setminus \text{diagonal})$ and in the asymptotic expansion of the kernel of $G$, the leading term coincides with that of the Green’s function for $(-\Delta)^{n/2}$ in $\mathbb{R}^n$. To be more precise, up to a constant multiple, we have the following asymptotic for $G$
\[ G(x,z) \sim \log |x-z| + R(x,z), \]
where $R \in C^\infty(M \times M)$. From this we easily verify that $u \in W^{n,3/2}(M)$. By Sobolev’s inequality, we obtain the following.

**Lemma 7.2** For any solution $u(t)$ to the flow (F), there holds
\[ \sup_{t \geq 0} |u(t)| < +\infty. \quad (7.4) \]

Thus, we have just shown that the solution $u(t)$ is uniformly bounded along the flow. A consequence of (7.4) is that
\[ \int_M (Q_0 + P_0u)^2 d\mu_{g_0} \leq C \]
for all $t \geq 0$. Hence, we have just improved Lemma 4.2 as follows.

**Lemma 7.3** For any solution $u(t)$ to the flow (F), there holds
\[ \sup_{t \geq 0} \|u(t)\|_{H^1(M)} \leq \mathcal{C}_n, \]
where $\mathcal{C}_n > 0$ is a uniform constant.

By repeating arguments used in Sect. 5, we conclude

**Lemma 7.4** For any solution $u(t)$ to the flow (F), there exists a constant $\mathcal{C}_{2k} > 0$ depending on $k$ such that
\[ \sup_{t \geq 0} \|u(t)\|_{H^{2k}(M)} \leq \mathcal{C}_{2k} \]
for any $k > 1$. 

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8 Convergence of the flow

This section is devoted to various convergences of the flow.

8.1 Sequential convergence of the flow

A direct consequence of the $H^{2k}$-boundedness of $u(t)$ obtained in Lemma 7.4 above is the following sequential convergence of the flow $(F)$ with an appropriate initial data $u_0$. Of course, we always assume that the hypotheses of Theorems 1.1 and 1.3 are fulfilled.

**Corollary 8.1** There exist a function $u_\infty \in C^\infty(M)$, a real number $\lambda_\infty$, and a time sequence $(t_j)_j$ with $t_j \to +\infty$ as $j \to +\infty$ such that

$$P_0u_\infty + Q_0 = \lambda_\infty fe^{nu_\infty}$$

and that the following claims hold:

(i) $\|u(t_j) - u_\infty\|_{C^\infty(M)} \to 0$,

(ii) $|\lambda(t_j) - \lambda_\infty| \to 0$, and

(iii) $\|Q g(t_j) - \lambda_\infty f\|_{C^\infty(M)} \to 0$

as $j \to +\infty$. Furthermore, if either $0 \neq \int_M Q_0 d\mu g_0 < (n-1)! \text{vol}(\mathbb{S}^n)$ or $M = \mathbb{S}^n$, then there holds $\lambda_\infty = 1$.

**Proof** To start our proof, we recall that $u(t)$ is bounded in $H^{2k}(M)$ for arbitrary but fixed $k$. Therefore, by Sobolev’s embedding theory, we deduce that there is some smooth function $u_\infty$ and a time sequence $(t_j)_j$ with $t_j \to +\infty$ as $j \to +\infty$ such that

$$\|u(t_j) - u_\infty\|_{C^\infty(M)} \to 0$$

as $j \to +\infty$. This establishes Part (i). The strong convergence $u(t_j) \to u_\infty$ as $j \to +\infty$ also implies that

$$e^{nu(t_j)} \to e^{nu_\infty}$$

as $j \to +\infty$. Put

$$g_\infty = e^{2u_\infty} g_0, \quad Q_\infty = e^{-nu_\infty}(P_0u_\infty + Q_0), \quad \lambda_\infty = \frac{\int_M F Q_\infty d\mu g_\infty}{\int_M f^2 d\mu g_\infty}.$$

Part (ii) follows easily because of the strong convergence $u(t_j) \to u_\infty$ as $j \to +\infty$. Now we establish Part (iii). To this purpose, we first establish the following weaker result

$$\|Q g(t_j) - \lambda_\infty f\|_{L^2(M)} \to 0$$

as $j \to +\infty$. Indeed, by writing

$$\left( Q g(t_j) - \lambda(t_j) f \right)^2 = \left( Q g(t_j) - \lambda(t_j) f \right)^2 e^{nu(t_j)} e^{-nu(t_j)}$$

and (7.4), we can estimate

$$\|Q g(t_j) - \lambda(t_j) f\|_{L^2(M)} \leq C \|Q g(t_j) - \lambda(t_j) f\|_{L^2(M,g(t_j))}.$$

This together with Lemma 7.1 gives

$$\|Q g(t_j) - \lambda(t_j) f\|_{L^2(M)} \to 0$$
as $j \to +\infty$. From this we apply triangle inequality and assertion (i) to get
\[
\|Q_g(t_j) - \lambda_\infty f\|_{L^2(M)} \leq \|Q_g(t_j) - \lambda(t_j) f\|_{L^2(M)} + |\lambda(t_j) - \lambda_\infty| \|f\|_{L^2(M)} \to 0
\]
as $j \to +\infty$. Now we apply the Gagliardo–Nirenberg inequality in the following way
\[
\|Q_g(t_j) - \lambda_\infty f\|_{H^k(M)} \leq C \|Q_g(t_j) - \lambda_\infty f\|_{H^{2k}(M)}^{1/2} \|Q_g(t_j) - \lambda_\infty f\|_{L^2(M)}^{1/2}
\]
for each $k$ fixed. If we rewrite
\[
Q_g(t_j) - \lambda_\infty f = e^{-nu(t_j)}(P_0 u + Q_0) - \lambda_\infty f,
\]
then for each $k$ fixed it follows from the boundedness of $u(t)$ in $H^{n+2k}(M)$ that $Q_g(t_j) - \lambda_\infty f$ is uniformly bounded in $H^{2k}(M)$. Hence applying Lemma 7.1 and Sobolev’s embedding theory gives us the conclusion. Finally, to show $u_\infty$ solves the equation, we note that
\[
Q_g(t_j) - Q_\infty = (P_0 u(t_j) - P_0 u_\infty)e^{-nu(t_j)} + P_0 u_\infty(e^{-nu(t_j)} - e^{-nu_\infty}) + (e^{-nu(t_j)} - e^{-nu_\infty})Q_0,
\]
which implies that
\[
\|Q_g(t_j) - Q_\infty\|_{H^k(M)} \to 0
\]
as $j \to +\infty$. Hence
\[
Q_\infty = \lambda_\infty f
\]
everywhere on $M$. This implies that
\[
P_0 u_\infty + Q_0 = \lambda_\infty f e^{nu_\infty}.
\]
Finally, we evaluate $\lambda_\infty$. Since $u_0 \in Y$ for either $\int_M Q_0 d\mu_{g_0} < (n-1)!\text{vol}(S^n)$ or $M = S^n$, we have
\[
\int_M f e^{u_\infty} d\mu_{g_0} = \int_M Q_0 d\mu_{g_0}.
\]
Hence, from this it is easy to conclude that $\lambda_\infty = 1$ provided $\int_M Q_0 d\mu_{g_0} \neq 0$. □

### 8.2 Uniform convergence of the flow

The main result of this subsection is the following.

**Lemma 8.2** If $\int_M Q_0 d\mu_{g_0} \neq 0$ and the prescribed $Q$-curvature equation
\[
P_0 u + Q_0 = f e^{nu}
\]
has a unique solution, the convergence in Corollary 8.1 is uniform in time, namely we have

(i) $\|u(t) - u_\infty\|_{C^\infty(M)} \to 0$,
(ii) $|\lambda(t) - \lambda_\infty| \to 0$, and
(iii) $\|Q_g(t) - \lambda_\infty f\|_{C^\infty(M)} \to 0$
as $t \to +\infty$. 

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Proof Let $u_\infty$ be found in Corollary 8.1. By way of contradiction, let us assume that there exist some $\varepsilon > 0$ and a time sequence $(t_j)_j \subset [0, +\infty)$ with $t_j \to +\infty$ as $j \to +\infty$ such that

$$\|u(t_j) - u_\infty\|_{C^\infty(M)} \geq \varepsilon$$

for $j$ large. Now, we consider the associated sequence of functions $u(t_j)$ and sequence of numbers $\lambda(t_j)$. By following the proof of Corollary 8.1 and up to a subsequence, still denoted by $(t_j)_j$ we obtain

(i) $\|u(t_j) - \bar{u}_\infty\|_{C^\infty(M)} \to 0$,
(ii) $|\lambda(t_j) - \bar{\lambda}_\infty| \to 0$, and
(iii) $\|Q_{g(t_j)} - \bar{\lambda}_\infty f\|_{C^\infty(M)} \to 0$

as $j \to +\infty$. Furthermore, the function $\bar{u}_\infty$ solves

$$P_0\bar{u}_\infty + Q_0 = \bar{\lambda}_\infty f e^{n\bar{u}_\infty}.$$

It is important to note that in the case $\int_M Q_0 d\mu_{g_0} \neq 0$ our choice of $u_0$ leads us to the positivity of both $\lambda_\infty$ and $\bar{\lambda}_\infty$. Consequently, the two functions $u_\infty + (1/n) \log \lambda_\infty$ and $\bar{u}_\infty + (1/n) \log \bar{\lambda}_\infty$ solve the prescribed $Q$-curvature equation. By the uniqueness property, we deduce that

$$\int_M f e^{n\bar{u}_0} d\mu_{g_0} = \int_M f e^{n\lambda_\infty} d\mu_{g_0} = e^{\log \bar{\lambda}_\infty - \log \lambda_\infty} \int_M f e^{n\bar{u}_\infty} d\mu_{g_0} = \frac{\bar{\lambda}_\infty}{\lambda_\infty} \int_M f e^{n\bar{u}_0} d\mu_{g_0}.$$

Hence $\bar{\lambda}_\infty = \lambda_\infty$, giving us $\bar{u}_\infty = u_\infty$. This is a contradiction. The proof is complete. \qed

8.3 Exponential convergence of the flow

In the final part of this section, we want to study the uniform convergence of the flow (1.15) and its rate under the assumptions $\int_M Q_0 d\mu_{g_0} < 0$ and $f \leq 0$. However, we try to prove a more general result which provides a sufficient condition on the exponentially fast convergence. We should point out that the result below is inspired by Struwe [31].

Theorem 8.3 Suppose that $u_\infty$ is a strictly relative minimizer in the sense that for some constant $c_0 > 0$ there holds

$$\int_M (h \cdot P_0 h - n\lambda_\infty f e^{n\lambda_\infty} h^2) d\mu_{g_0} \geq 2c_0 \|h\|_{H^n(M)}^2$$

for all $h \in T_{u_\infty}$, where we let

$$T_u = \left\{ h \in H^n(M) : \int_M f e^{n\lambda} h d\mu_{g_0} = 0 \right\}.$$

Then $u(t) \to u_\infty$ and $\lambda(t) \to \lambda_\infty$ exponentially fast as $t \to +\infty$ in the following sense: with a constant $C$ depending only on the initial data $u_0$, there holds

$$\|Q_{g(t)} - \lambda(t) f\|_{L^2(M, g(t))} \leq C e^{-2c_0 t}$$

for all time and hence

$$|\lambda(t) - \lambda_\infty| + \|u(t) - u_\infty\|_{C^\infty(M)} \leq C e^{-c_0 t / 2}$$

for all $t$. 

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\textbf{Proof} Set
\[w(t) = |\lambda(t) - \lambda_\infty| + \|u(t) - u_\infty\|_{H^n(M)}.
\]
Let \(0 < \varepsilon \ll 1\) be an arbitrary positive number. From Corollary 8.1, it follows that there exists some \(t_0 > 0\) such that \(w(t_0) \leq \varepsilon\). Then we claim that \(w(t) \leq 3\varepsilon\) for all \(t \geq t_0\). We may assume by contradiction that there exists a finite number \(t_1\) such that
\[t_1 := \inf\{t \geq t_0 : w(t) \geq 3\varepsilon\}.
\]
This implies that
\[w(t) \leq 3\varepsilon \quad \text{for} \quad t_0 \leq t \leq t_1. \tag{8.2}
\]
Now, we recall that
\[
\frac{1}{2} \frac{d}{dt} F_2(t) = -\int_M (Q_{g(t)} - \lambda(t)f) P_{g(t)}(Q_{g(t)} - \lambda(t)f) \, d\mu_{g(t)} \\
+ n\lambda(t) \int_M f(Q_{g(t)} - \lambda(t)f)^2 \, d\mu_{g(t)} + \frac{n}{2} \int_M (Q_{g(t)} - \lambda(t)f)^3 \, d\mu_{g(t)}.
\]
By Hölder's inequality, the embedding \(H^{n/2}(M) \hookrightarrow L^4(M)\), and the equivalence between \(g(t)\) and \(g_0\), we have
\[
\int_M |Q_{g(t)} - \lambda(t)f|^3 \, d\mu_{g(t)} = o(1) \\
\leq \left(\int_M |Q_{g(t)} - \lambda(t)f|^2 \, d\mu_{g(t)}\right)^{1/2} \left(\int_M |Q_{g(t)} - \lambda(t)f|^4 \, d\mu_{g(t)}\right)^{1/2} \\
\leq o(1) \int_M (Q_{g(t)} - \lambda(t)f) P_{g(t)}(Q_{g(t)} - \lambda(t)f) \, d\mu_{g(t)} \\
+ o(1) \int_M (Q_{g(t)} - \lambda(t)f)^2 \, d\mu_{g(t)}
\]
with error \(o(1) \to 0\) as \(t \to +\infty\). Therefore, the time derivative of \(F_2(t)\) can be estimated further as follows
\[
\frac{1}{2} \frac{d}{dt} F_2(t) \leq -(1 + o(1)) \int_M (Q_{g(t)} - \lambda(t)f) P_{g(t)}(Q_{g(t)} - \lambda(t)f) \, d\mu_{g(t)} \\
+ n\lambda(t) \int_M f(Q_{g(t)} - \lambda(t)f)^2 \, d\mu_{g(t)} + o(1) F_2(t) \\
= -(1 + o(1)) d^2 \mathcal{L}_u(u_t, u_t) + o(1) F_2(t),
\]
where
\[
d^2 \mathcal{L}_u(u_t, u_t) = \int_M (u_t P_0 u_t - n\lambda(t)f u_t^2 e^{nu}) \, d\mu_{g_0}.
\]
Notice that we can find \(\delta(t) \in \mathbb{R}\) such that \(u_t + \delta(t)f \in T_{u_\infty}\). In fact, one only needs to solve the following equation with \(g_\infty = e^{2u_\infty}\)
\[
0 = \int_M (u_t + \delta(t)f) \, d\mu_{g_\infty} = \int_M u_t f(e^{2u_\infty} - u) \, d\mu_{g(t)} + \delta(t) \int_M f^2 \, d\mu_{g_\infty}.
\]
Here we have used the fact that \(u_t \in T_{u(t)}\) due to (1.16). It follows from (8.2) and (4.2) that
\[
|\delta| \leq C \|u(t) - u_\infty\|_{L^\infty(M)} \sqrt{F_2(t)} \leq C \varepsilon \sqrt{F_2(t)}.
\]
for \( t_0 \leq t \leq t_1 \). In consequence, we set \( h_0 = u_t + \delta(t)f \) for brevity. Then we have
\[
d^2 L_u(u_t, u_t) = \int_M u_t P_0 u_t - n \lambda(t)f u_t^2 e^{nu} \, d\mu_{g_0} \\
= d^2 L_{u_\infty}(h_0, h_0) + I + II \\
\geq 2c_0\|h_0\|^2_{H^{n/2}(M)} + I + II
\]
with error terms
\[
I = \int_M (u_t P_0 u_t - h_0 P_0 h_0) \, d\mu_{g_0} \\
\quad = -\delta \int_M P_0 f (2u_t + \delta f) \, d\mu_{g_0} \\
\quad = -2\delta \int_M u_t P_0 f \, d\mu_{g_0} - \delta^2 \int_M f P_0 f \, d\mu_{g_0} = O(\varepsilon F_2(t))
\]
and
\[
II = n \int_M (\lambda_\infty f h_0^2 e^{nu_\infty} - \lambda(t)f u_t^2 e^{nu}) \, d\mu_{g_0} \\
\quad = n\lambda_\infty \int_M f (h_0^2 e^{nu_\infty} - u_t^2 e^{nu}) \, d\mu_{g_0} - n(\lambda(t) - \lambda_\infty) \int_M f u_t^2 e^{nu} \, d\mu_{g_0} \\
\quad = n\lambda_\infty \int_M f ((h_0^2 - u_t^2) + u_t^2(1 - e^{n(u - u_\infty)})d\mu_{g_\infty} + O(\varepsilon F_2(t)) \\
\quad = O(\varepsilon F_2(t))
\]
for \( t_0 \leq t \leq t_1 \). Moreover, similar computations and Lemma 7.4 yield
\[
\|h_0\|^2_{L^2(M)} = \|u_t\|^2_{L^2(M)} + O(\varepsilon F_2(t))
\]
for \( t_0 \leq t \leq t_1 \). Therefore, for sufficiently small \( \varepsilon > 0 \) and \( t_0 \leq t \leq t_1 \) we have
\[
\frac{1}{2} \frac{d}{dt} F_2(t) \leq -c_0 F_2(t) \quad (8.3)
\]
for some \( c_1 > 0 \). Hence we find that
\[
F_2(t) \leq F_2(t_0) \exp(-2c_0(t - t_0)) \quad (8.4)
\]
for \( t_0 \leq t \leq t_1 \). Now, from Lemma 2.1 we have
\[
\lambda'(t) = \left( \int_M f^2 d\mu_{g(t)} \right)^{-1} \left[ \int_M (P_{g(t)}f) u_t d\mu_{g(t)} + n \int_M \lambda(t)f^2 (Q_{g(t)} - \lambda(t)f) d\mu_{g(t)} \right].
\]
The term \( \int_M (P_{g(t)}f) u_t d\mu_{g(t)} \) can be estimated as follows
\[
\left| \int_M (P_{g(t)}f) u_t d\mu_{g(t)} \right| \leq C\|f\|_{C^a} \|u_t\|_{L^2(M, g(t))}.
\]
For the term \( \int_M \lambda(t)f^2 (Q_{g(t)} - \lambda(t)f) d\mu_{g(t)} \), we can use Hölder’s inequality and Lemma 4.1 to get
\[
\left| \int_M \lambda(t)f^2 (Q_{g(t)} - \lambda(t)f) d\mu_{g(t)} \right| \leq C \|\lambda(t)f - Q_{g(t)}\|_{L^2(M, g(t))}.
\]
Hence, from (8.3), we have
\[ |\lambda'(t)| \leq C \sqrt{F_2(t_0)} \exp(-c_0(t - t_0)), \]
and for \( t_0 < t \leq t_1 \) we obtain
\[ |\lambda(t) - \lambda(t_0)| \leq \int_{t_0}^{t} |\lambda'(s)|ds \leq C \sqrt{F_2(t_0)}. \quad (8.5) \]
To estimate \( \|u(t) - u_\infty\|_{L^2(M)} \), we use Lemma 7.4 to get
\[ \|u(t) - u(t_0)\|_{L^2(M)} \leq C \int_{t_0}^{t} \|u_t\|_{L^2(M,G(t))}ds \leq C \sqrt{F_2(t_0)}. \]
This together with the Gagliardo–Nirenberg inequality and Lemma 7.4 implies that
\[ \|u(t) - u(t_0)\|_{H^\alpha(M)} \leq C \|u(t) - u(t_0)\|^{1/2}_{H^{2\alpha}(M)} \|u(t) - u(t_0)\|^{1/2}_{L^2(M)} \leq CF_2(t_0)^{1/4}. \quad (8.6) \]
Combining (8.5) and (8.6) gives
\[ |\lambda(t) - \lambda(t_0)| + \|u(t) - u(t_0)\|_{H^\alpha(M)} \leq CF_2(t_0)^{1/4} \]
for \( t_0 \leq t \leq t_1 \). From this and the fact that \( w(t_0) \leq \varepsilon \), it follows that
\[ w(t) \leq CF_2(t_0)^{1/4} + \varepsilon \]
for \( t_0 \leq t \leq t_1 \). Now, choose \( t_0 \) sufficiently large such that \( CF_2(t_0)^{1/4} + \varepsilon \leq \varepsilon \). Then, by the definition of \( t_1 \), we obtain that \( 3\varepsilon \leq w(t_1) \leq 2\varepsilon \), which is a contradiction. Hence, we have \( w(t) \leq 3\varepsilon \) for all \( t > t_0 \). However, this will imply that (8.4) holds true for all \( t > t_0 \). Then, one has
\[ |\lambda(t) - \lambda(t_0)| + \|u(t) - u(t_0)\|_{L^2(M)} \leq C\sqrt{F(t_0)} \]
for all \( t > t_0 \). Now, by setting \( t = t_j \) in the above equality and then sending \( t_j \) to \( +\infty \), we see that
\[ |\lambda_\infty - \lambda(t_0)| + \|u_\infty - u(t_0)\|_{L^2(M)} \leq C\sqrt{F(t_0)} \]
for all sufficiently large \( t_0 > 0 \). By the Gagliardo–Nirenberg inequality and Lemma 7.4, we obtain
\[ \|u_\infty - u(t_0)\|_{H^k(M)} \leq C \|u_\infty - u(t_0)\|^{1/2}_{H^{2k}(M)} \|u_\infty - u(t_0)\|^{1/2}_{L^2(M)} \leq CF_2(t_0)^{1/4} \]
for all \( k \geq 1 \). Hence, we have
\[ |\lambda_\infty - \lambda(t_0)| + \|u_\infty - u(t_0)\|_{C^\infty(M)} \leq CF_2(t_0)^{1/4}. \]
Renaming \( t_0 \) as \( t \) and choosing \( t_0 > 0 \) such that (8.4) holds for all \( t > t_0 \), we then complete the proof. □

In general, the condition (8.1) is not easy to test; however, by a fairy simple argument we know that the inequality (8.1) holds true if \( f \) is non-vanishing with \( f \leq 0 \) everywhere; see [17, Lemma 2.2] for a similar argument. Hence, we have the following corollary

**Corollary 8.4** If \( \int_M Q_0 \, d\mu_{\mathcal{R}_0} < 0 \) and \( f \leq 0 \) everywhere, then we have

(i) \( \|u(t) - u_\infty\|_{C^\infty(M)} \to 0 \),
(ii) \(|\lambda(t) - \lambda_\infty| \to 0\), and
(iii) \(\|Q_g(t) - \lambda_\infty f\|_{C^\infty(M)} \to 0\)

exponentially fast as \(t \to +\infty\).

9 Proof of main results

9.1 Proof of Theorems 1.1, 1.3 and 1.5

Theorems 1.1 and 1.3 follows from Corollary 8.1 while Theorem 1.5 follows from Lemma 8.2 and Corollary 8.4. We thus complete the proof of the three theorems.

9.2 Proof of Corollary 1.6

For Part (i), one may pick some \(u_0 \in Y\) and fix it. Then, the term involving \(u_0\) on the right hand side of the following inequality

\[
\sup_M f^+ \leq C_0 \exp\left(-\tau \|u_0\|_{H^{n/2}(M)}^2\right)
\]

can be absorbed into the constant \(C_0\). And the existence result follows immediately from Theorem 1.1.

Clearly, Part (iii) is straightforward.

For Part (v), it follows from Corollary 8.1 that the limiting function \(u_\infty\) solves

\[
P_0u_\infty + Q_0 = \lambda_\infty fe^{nu_\infty}.
\]

Recall that our assumption on the initial data \(u_0\) gives \(\int_M fe^{nu_0} d\mu_{g_0} > 0\). Therefore, integrating both sides of the equation yields

\[
\int_M Q_0 d\mu_{g_0} = \lambda_\infty \int_M fe^{nu_\infty} d\mu_{g_0} = \lambda_\infty \int_M fe^{nu_0} d\mu_{g_0}.
\]

Hence \(\lambda_\infty > 0\). From this we know that the function

\[
u_\infty = u_\infty + (1/n) \log \lambda_\infty
\]
solves (1.6).

For Part (ii), by choosing a new background metric if necessary, we may assume that \(Q_0 \equiv 0\). Then \(u_\infty\) satisfies

\[
P_0u_\infty = \lambda_\infty fe^{nu_\infty}.
\]

First we rule out the possibility of \(\lambda_\infty = 0\). Indeed, if this is not the case, then we get from the preceding equation that \(P_0u_\infty = 0\). From this we conclude that \(u_\infty \equiv C\). Thus

\[
0 \neq e^{nuC} \int_M f d\mu_{g_0} = \int_M fe^{nuu_\infty} d\mu_{g_0} = \int_M Q_0 d\mu_{g_0} = 0,
\]

which is a contradiction. Hence, there holds \(\lambda_\infty \neq 0\). Now, if \(\lambda_\infty > 0\), then the new function

\[
v_\infty = u_\infty + \frac{1}{n} \log \lambda_\infty
\]
satisfies $P_0 v_0 = f e^{ru} v_0$; while if $\lambda_\infty < 0$, then the function 

$$v_\infty = u_\infty + \frac{1}{n} \log(-\lambda_\infty)$$

satisfies $P_0 v_\infty = -f e^{v_\infty}$. Thus, for either case, $e^{v_\infty} g_0$ is the metric we need. We now consider the case $\int_M f \, d\mu_{g_0} < 0$. In this scenario, we shall show that $\lambda_\infty > 0$. Indeed, the idea is to choose initial data $u_0$ carefully. Mimicking the idea of [18], we first let $g_0$ be such that $Q_0 \equiv 0$. Then as shown in “Appendix B” the minimizing problem

$$\inf_{w \in \tilde{H}} E[w]$$

over the set $\tilde{H}$, defined in Step 2 of “Appendix B”, admits a solution $w \in \tilde{H}$. Still by “Appendix B”, it is important to note that $w$ is also a minimizer of

$$\inf_{w \in H} E[w]$$

over the set $H$ defined in Step 1 of “Appendix B”. Therefore, we get

$$\int_M f e^{nu} u \, d\mu_{g_0} = 0$$

and

$$\int_M e^{nu} u \, d\mu_{g_0} = \text{vol}(M).$$

Let us consider our flow $(F)$ with the initial data $w$. Then the flow converges to some non-constant function $u_\infty$ which solves

$$P_0 u_\infty = \lambda_\infty f e^{nu} u_\infty.$$ 

for some $\lambda_\infty \neq 0$. Suppose that $\lambda_\infty < 0$, then it follows that

$$\int_M f u_\infty e^{nu} u_\infty \, d\mu_{g_0} = \frac{1}{\lambda_\infty} \int_M u_\infty \cdot P_0 u_\infty \, d\mu_{g_0} < 0.$$ 

Hence, following the idea in [18], because $\int_M f e^{nu} u \, d\mu_{g_0} \geq 0$ and $\int_M e^{nu} u \, d\mu_{g_0} = \text{vol}(M)$ there is some $t \in (0, 1)$ closed to 1 such that

$$\int_M f e^{ntu} u \, d\mu_{g_0} > 0$$

and that

$$\frac{\text{vol}(M)}{2} \leq \int_M e^{ntu} u \, d\mu_{g_0} \leq \frac{3\text{vol}(M)}{2}.$$ 

Putting these facts together, we deduce that $tu_\infty \in \tilde{H}$. However, it is not hard to see that

$$0 < E[tu_\infty] = t^2 E[u_\infty] < E[u_\infty] \leq E[w],$$

which contradicts the fact that $w$ is an minimizer of $E$ over $\tilde{H}$. Thus $\lambda_\infty > 0$.

Finally, we consider Part (iv). The idea is to show that there exists suitable initial data $u_0$ such that we can apply Theorem 1.3. If the set $\Sigma = \emptyset$, then the conclusion follows immediately from Corollary 8.1. Otherwise, the set $\Sigma \neq \emptyset$ and $f$ satisfies the inequality (1.22). We consider the following two cases:

**Case (a)** Suppose $\int_{\Sigma^c} f \circ \phi_{y_0, r_0} \, d\mu_{g^n} \leq 0$. In this case, the inequality (1.22) tells us that

$$\sup_{x \in \Sigma} f(x) \leq 0,$$

so the condition (b) in Theorem 1.3 obviously holds for any $G$-invariant initial data $u_0$. Hence the conclusion follows.
Case (a2) Suppose \( \int_{\mathbb{S}^n} f \circ \phi_{y_0, r_0} \, d\mu_{\mathbb{S}^n} > 0 \). In this scenario, we set

\[
u_0 = \frac{1}{n} \log(\det d\phi_{y_0, r_0}^{-1}) + C,
\]

where the constant \( C \) is chosen in such a way that

\[
\epsilon^n C \int_{\mathbb{S}^n} f \circ \phi_{y_0, r_0} \, d\mu_{\mathbb{S}^n} = (n - 1)!\operatorname{vol}(\mathbb{S}^n).
\]

With this choice of \( C \), we conclude that \( u_0 \in Y \). Since \( y_0 \in \Sigma \), it is not hard to see that \( u_0 \) is \( G \)-invariant. Moreover, the well-known fact

\[
\mathcal{E}\left[ \frac{1}{n} \log(\det d\phi_{y_0, r_0}^{-1}) \right] = 0
\]

implies that

\[
\mathcal{E}[u_0] = \mathcal{E}\left[ \frac{1}{n} \log(\det d\phi_{y_0, r_0}^{-1}) \right] + n!\operatorname{vol}(\mathbb{S}^n)C = n!\operatorname{vol}(\mathbb{S}^n)C.
\]

From this, it follows from the inequality (1.22) that

\[
\sup_{x \in \Sigma} f(x) \leq \frac{1}{\operatorname{vol}(\mathbb{S}^n)} \int_{\mathbb{S}^n} f \circ \phi_{y_0, r_0} \, d\mu_{\mathbb{S}^n}
\]

\[
= (n - 1)! \exp(-nC)
\]

\[
= (n - 1)! \exp\left(-\frac{\mathcal{E}[u_0]}{(n - 1)!\operatorname{vol}(\mathbb{S}^n)}\right).
\]

Hence the condition (b) in Theorem 1.3 holds and the conclusion follows from Corollary 8.1.

\( \square \)

**Remark 9.1** Before closing this work, we would like to summarize our major existence results for prescribing \( Q \)-curvature problem as follows:

| \( \int_M Q_0 \) | \( < 0 \) | \( = 0 \) | \( \epsilon (0, (n - 1)!\operatorname{vol}(\mathbb{S}^n)) \) | \( > (n - 1)!\operatorname{vol}(\mathbb{S}^n) \) |
| --- | --- | --- | --- | --- |
| n. cond. | \( \inf f < 0 \) | \( f \) changes sign | \( \sup f > 0 \) | \( \sup f > 0 \) |
| s. cond. | \( \sup f < C(f^-) \) | \( \int_M f \, d\mu_{g_0} < 0 \) | \( \sup f > 0 \) | \( f > 0 \) |

In the above table, ‘n. cond.’ and ‘s. cond.’ stand for necessary and sufficient conditions respectively. In the null case, as far as we know, there is no existence/non-existence result for the case \( \int_M f \, d\mu_{g_0} \geq 0 \). As already commented in [3,18], there is an example of 4-manifolds with non-negative total \( Q \)-curvature. In the supercritical case, the positivity of \( f \) seems to be technical and we do not know whether this is necessary.

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Appendix A: Proof of Lemmas 3.1 and 3.6

A.1. Proof of Lemma 3.1

We prove that for a measurable subset $K$ of $M$ with $\text{vol}(K) > 0$, there are two constants $\alpha > 1$ and $C_K > 1$ such that

$$\int_M e^{nu(t)} \, d\mu_{g_0} \leq C_K \exp \left( \alpha \|u_0\|_{H^{n/2}(M)}^2 \right) \max \left\{ \left( \int_K e^{nu(t)} \, d\mu_{g_0} \right)^\alpha, 1 \right\}. \quad (A.1)$$

Here $\alpha$ depends on $(M, g_0)$ and $C_K$ depends on $(M, g_0)$ and $\text{vol}(K)$. Fix $t > 0$ and denote $u = u(t)$ if no confusion occurs.

Step 1 Borrowing the idea in [3], in this step we prove that

$$\int_M u \, d\mu_{g_0} \leq \left| \mathcal{E}[u_0] \right| + \frac{C}{\text{vol}(K)} + \frac{4 \text{vol}(M)}{\text{vol}(K)} \max \left\{ \int_K u \, d\mu_{g_0}, 0 \right\} \quad (A.2)$$

for some (uniform) constant $C > 0$. Suppose that $\int_M u \, d\mu_{g_0} > 0$, otherwise (A.2) is trivial.

By the definition of $\mathcal{E}$ and Lemma 2.2, there holds

$$\int_M P_0 u \cdot u \, d\mu_{g_0} \leq 2 \mathcal{E}[u_0] - 2 \int_M Q_0 u \, d\mu_{g_0}.$$

One way to bound $\int_M u \, d\mu_{g_0}$ from above is to bound $\int_M u^2 \, d\mu_{g_0}$ from above. Following this strategy, by Poincaré’s inequality (1.9), we first obtain

$$\int_M u^2 \, d\mu_{g_0} \leq \frac{2}{n\lambda_1} \mathcal{E}[u_0] - \frac{2}{\lambda_1} \int_M Q_0 u \, d\mu_{g_0} + \frac{1}{\text{vol}(M)} \left( \int_M u \, d\mu_{g_0} \right)^2. \quad (A.3)$$

Depending on the sign of $\int_K u \, d\mu_{g_0}$, we consider the following two cases.

Case 1 Suppose that $\int_K u \, d\mu_{g_0} \leq 0$. Then it is easy to see that

$$\left( \int_M u \, d\mu_{g_0} \right)^2 \leq \left( \int_{M \setminus K} u \, d\mu_{g_0} \right)^2 \leq \text{vol}(M \setminus K) \int_M u^2 \, d\mu_{g_0}.$$

From this and (A.3) we obtain the following

$$\frac{\text{vol}(K)}{\text{vol}(M)} \int_M u^2 \, d\mu_{g_0} \leq \frac{2}{n\lambda_1} \mathcal{E}[u_0] - \frac{2}{\lambda_1} \int_M Q_0 u \, d\mu_{g_0}. \quad (A.4)$$

Applying Young’s inequality gives

$$\left| \frac{2}{\lambda_1} \int_M Q_0 u \, d\mu_{g_0} \right| \leq \frac{2 \text{vol}(M)}{\lambda_1^2 \text{vol}(K)} \int_M Q_0^2 \, d\mu_{g_0} + \frac{\text{vol}(K)}{2 \text{vol}(M)} \int_M u^2 \, d\mu_{g_0}.$$ 

Hence, combining this and (A.4) gives

$$\int_M u^2 \, d\mu_{g_0} \leq \frac{4 \text{vol}(M)}{n\lambda_1 \text{vol}(K)} \mathcal{E}[u_0] + \frac{4 \text{vol}(M)^2}{\lambda_1^2 \text{vol}(K)^2} \int_M Q_0^2 \, d\mu_{g_0}. $$
which then yields

\[
\left( \int_M u \, d\mu_{g_0} \right)^2 \leq \frac{4 \text{vol}(M)^2}{n\lambda_1 \text{vol}(K)} \|\mathcal{E}[u_0]\| + \frac{4}{\lambda_1^2} \frac{\text{vol}(M)^3}{\text{vol}(K)^2} \int_M Q_0^2 \, d\mu_{g_0},
\]

(A.5)

Thus we have just shown from (A.5) that

\[
\int_M u \, d\mu_{g_0} \leq \|\mathcal{E}[u_0]\| + \frac{2 \text{vol}(M)^2}{n\lambda_1 \text{vol}(K)} + \frac{2 \text{vol}(M)^{3/2}}{\lambda_1} \left( \int_M Q_0^2 \, d\mu_{g_0} \right)^{1/2}.
\]

This establishes (A.2) in the first case.

Case 2 Suppose that \( \int_K u \, d\mu_{g_0} > 0 \). In this scenario, we first rewrite (A.3) as follows

\[
\int_M u^2 \, d\mu_{g_0} \leq \frac{2}{n\lambda_1} \mathcal{E}[u_0] - \frac{2}{\lambda_1} \int_M Q_0 u \, d\mu_{g_0}
\]

\[
+ \frac{1}{\text{vol}(M)} \left( \left( \int_K u \, d\mu_{g_0} \right)^2 + \left( \int_{M \setminus K} u \, d\mu_{g_0} \right)^2 \right)
\]

\[
+ \frac{2}{\text{vol}(M)} \left( \int_K u \, d\mu_{g_0} \right) \left( \int_{M \setminus K} u \, d\mu_{g_0} \right).
\]

By Young’s inequality and the inequality

\[
\left( \int_{M \setminus K} u \, d\mu_{g_0} \right)^2 \leq \text{vol}(M \setminus K) \int_M u^2 \, d\mu_{g_0},
\]

we obtain

\[
\frac{2}{\text{vol}(M)} \left( \int_K u \, d\mu_{g_0} \right) \left( \int_{M \setminus K} u \, d\mu_{g_0} \right) \leq \frac{2 \text{vol}(M \setminus K)}{\text{vol}(K) \text{vol}(M)} \left( \int_K u \, d\mu_{g_0} \right)^2
\]

\[
+ \frac{\text{vol}(K)}{2 \text{vol}(M)} \left( \int_M u^2 \, d\mu_{g_0} \right).
\]

Hence

\[
\int_M u^2 \, d\mu_{g_0} \leq \frac{2}{n\lambda_1} \mathcal{E}[u_0] - \frac{2}{\lambda_1} \int_M Q_0 u \, d\mu_{g_0} + \frac{\text{vol}(K) + 2 \text{vol}(M \setminus K)}{\text{vol}(K) \text{vol}(M)} \left( \int_K u \, d\mu_{g_0} \right)^2
\]

\[
+ \frac{\text{vol}(K)}{2 \text{vol}(M)} \left( \int_M u^2 \, d\mu_{g_0} \right),
\]

which implies

\[
\frac{\text{vol}(K)}{2 \text{vol}(M)} \int_M u^2 \, d\mu_{g_0} \leq \frac{2}{n\lambda_1} \mathcal{E}[u_0] - \frac{2}{\lambda_1} \int_M Q_0 u \, d\mu_{g_0} + \frac{2}{\text{vol}(K)} \left( \int_K u \, d\mu_{g_0} \right)^2
\]

Again applying Young’s inequality gives

\[
\left| \frac{2}{\lambda_1} \int_M Q_0 u \, d\mu_{g_0} \right| \leq \frac{4 \text{vol}(M)}{\lambda_1^2 \text{vol}(K)} \int_M Q_0^2 \, d\mu_{g_0} + \frac{\text{vol}(K)}{4 \text{vol}(M)} \int_M u^2 \, d\mu_{g_0}.
\]

Therefore,

\[
\int_M u^2 \, d\mu_{g_0} \leq \frac{8}{n\lambda_1} \frac{\text{vol}(M)}{\text{vol}(K)} \mathcal{E}[u_0] + \frac{16 \text{vol}(M)^2}{\lambda_1^2 \text{vol}(K)^2} \int_M Q_0^2 \, d\mu_{g_0} + \frac{8 \text{vol}(M)}{\text{vol}(K)^2} \left( \int_K u \, d\mu_{g_0} \right)^2.
\]
which yields
\[
\left( \int_M u \, d\mu_{g_0} \right)^2 \leq \varepsilon [u_0]^2 + \frac{4\text{vol}(M)^2}{n\lambda_1 \text{vol}(K)} \varepsilon [u_0] \\
+ \frac{16\text{vol}(M)^3}{\lambda_1^2 \text{vol}(K)^2} \int_M Q_0^2 \, d\mu_{g_0} + \frac{8\text{vol}(M)^2}{\text{vol}(K)^2} \left( \int_K u \, d\mu_{g_0} \right)^2. \tag{A.6}
\]

Thus we have just shown from (A.6) that
\[
\int_M u \, d\mu_{g_0} \leq \varepsilon [u_0] + \frac{4\text{vol}(M)^2}{n\lambda_1 \text{vol}(K)} + \frac{4\text{vol}(M)^3/2}{\lambda_1 \text{vol}(K)} \left( \int_M Q_0^2 \, d\mu_{g_0} \right)^{1/2} \\
+ \frac{2\sqrt{2}\text{vol}(M)}{\text{vol}(K)} \left( \int_K u \, d\mu_{g_0} \right).
\]

This establishes (A.2) in the second case.

**Step 2** By the definition of \( \varepsilon \) and Lemma 2.2, there holds
\[
\int_M P_0 u \cdot u \, d\mu_{g_0} \leq \frac{2}{n} \varepsilon [u_0] - 2 \left( \int_M Q_0 \, d\mu_{g_0} \right) \overline{u} - 2 \int_M Q_0 (u - \overline{u}) \, d\mu_{g_0}.
\]

By making use of Young’s inequality and Poincaré’s inequality (1.9), we obtain
\[
\int_M P_0 u \cdot u \, d\mu_{g_0} \leq \frac{2}{n} \varepsilon [u_0] - 2 \left( \int_M Q_0 \, d\mu_{g_0} \right) \overline{u} \\
+ \frac{2}{\lambda_1} \left( \int_M Q_0^2 \, d\mu_{g_0} \right) + \frac{1}{2} \int_M P_0 u \cdot u \, d\mu_{g_0}.
\]

Thus, we have just proved that
\[
\frac{1}{2} \int_M P_0 u \cdot u \, d\mu_{g_0} \leq \frac{2}{n} \varepsilon [u_0] - 2 \left( \int_M Q_0 \, d\mu_{g_0} \right) \overline{u} + \frac{2}{\lambda_1} \left( \int_M Q_0^2 \, d\mu_{g_0} \right).
\]

Thanks to (3.1), we can estimate
\[
\int_M e^{nu} \, d\mu_{g_0} \leq C_A \exp \left( \frac{n}{2(n-1)! \text{vol}(\mathbb{S}^n)} \int_M u \cdot P_0 u \, d\mu_{g_0} + \frac{n}{\text{vol}(M)} \int_M u \, d\mu_{g_0} \right) \\
\leq C_A \exp \left[ \frac{2}{(n-1)! \text{vol}(\mathbb{S}^n)} \varepsilon [u_0] \\
+ \frac{n}{\text{vol}(M)} \left( 1 - \frac{2}{(n-1)! \text{vol}(\mathbb{S}^n)} \int_M Q_0 \, d\mu_{g_0} \right) \left( \int_M u \, d\mu_{g_0} \right) \right].
\]

Note that
\[
\varepsilon [u_0] \leq \frac{n}{2} \|u_0\|_{H^n(M)}^2 + \int_M Q_0^2 \, d\mu_{g_0}.
\]
Therefore, we can further estimate \( \int_M e^{nu} d\mu_{g_0} \) as follows

\[
\int_M e^{nu} d\mu_{g_0} \leq C_A \exp \left[ \frac{2}{(n-1)!\text{vol}(\mathbb{S}^n)} \left( 1 + \frac{n}{\lambda_1} \right) \int_M Q_0^2 d\mu_{g_0} \right. \\
+ \left. \frac{2}{(n-1)!\text{vol}(\mathbb{S}^n)} \int_M Q_0 d\mu_{g_0} \left( \int_M u d\mu_{g_0} \right) \right] \\
\leq C \exp \left( \frac{n}{(n-1)!\text{vol}(\mathbb{S}^n)} \left( u_0 \right)^2_{H^{n/2}(M)} + B \int_M u d\mu_{g_0} \right). \tag{A.7}
\]

Here \( B \) and \( C \) are positive constants depending only on \((M, g_0)\). Combining (A.2) and (A.7) gives

\[
\int_M e^{nu} d\mu_{g_0} \leq C'_K \exp \left( A_1 \|u_0\|^2_{H^{n/2}(M)} + \frac{B_1}{\text{vol}(K)} \max \left\{ \int_K u d\mu_{g_0}, 0 \right\} \right). 
\]

Again \( A_1 \) and \( B_1 \) are positive constants depending on \((M, g_0)\) and \( C'_K > 0 \) depends on \((M, g_0)\) and \( \text{vol}(K) \). Clearly, we may assume that \( B_1 > n \) by replacing it with \( \max\{n+1, B_1\} \), if necessary. Therefore, by letting \( \alpha = \max\{A_1, B_1/n\} > 1 \), we deduce that

\[
\int_M e^{nu} d\mu_{g_0} \leq C'_K \exp \left( \alpha \|u_0\|^2_{H^{n/2}(M)} \right) \exp \left( \frac{\alpha}{\text{vol}(K)} \max \left\{ \int_K nu d\mu_{g_0}, 0 \right\} \right). \tag{A.8}
\]

By Jensen’s inequality, we have

\[
\exp \left( \frac{\alpha}{\text{vol}(K)} \int_K nu d\mu_{g_0} \right) \leq \left( \frac{1}{\text{vol}(K)} \int_K e^{nu} d\mu_{g_0} \right)^\alpha.
\]

From this we know that

\[
\exp \left( \frac{\alpha}{\text{vol}(K)} \max \left\{ \int_K nu d\mu_{g_0}, 0 \right\} \right) = \max \left\{ \exp \left( \frac{\alpha}{\text{vol}(K)} \int_K nu d\mu_{g_0} \right), 1 \right\} \\
\leq \max \left\{ \left( \frac{1}{\text{vol}(K)} \int_K e^{nu} d\mu_{g_0} \right)^\alpha, 1 \right\}. \tag{A.9}
\]

Plugging (A.9) into (A.8) gives

\[
\int_M e^{nu} d\mu_{g_0} \leq C'_K \exp \left( \alpha \|u_0\|^2_{H^{n/2}(M)} \right) \\
\times \max \left\{ \left( \frac{1}{\text{vol}(K)} \right)^\alpha, 1 \right\} \max \left\{ \left( \int_K e^{nu} d\mu_{g_0} \right)^\alpha, 1 \right\} \\
= C_K \exp \left( \alpha \|u_0\|^2_{H^{n/2}(M)} \right) \max \left\{ \left( \int_K e^{nu} d\mu_{g_0} \right)^\alpha, 1 \right\},
\]

which is the desired estimate (A.1). The proof is complete. \( \square \)

A.2. Proof of Lemma 3.6

To prove the lemma, we follow the idea in [2,3] with some modifications. As shown in [9, Proposition 6] and in [11, page 939], for each smooth positive function \( u(t) \), there exist some point \( p(t) \in \mathbb{R}^{n+1} \) such that the normalized companion of \( u(t) \) defined by

\[
w_{p(t)} = u(t) \circ \phi_{p(t)} + \frac{1}{n} \log(\det d\phi_{p(t)}),
\]

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where \( \phi_{p(t)} : S^n \to S^n \) is the conformal diffeomorphism given by

\[
\phi_p(x) = \frac{1}{1 + 2 \langle p, x \rangle + \|p\|^2} \left[ (1 - |p|^2)x + 2(1 + \langle p, x \rangle)p \right]
\]

with \( p \in \mathbb{R}^{n+1} \), enjoys the following

\[
\int_{S^n} x_j e^{nw_{p(t)}} \, d\mu_{S^n} = 0 \tag{A.10}
\]

for all \( j = 1, 2, \ldots, n + 1 \). Furthermore, if \( u(t) \) depends smoothly on the time \( t \), then so does \( p(t) \). By conformal invariance there holds

\[
\mathcal{E}[w_{p(t)}] = \mathcal{E}[u(t)];
\]

see for instance [11, page 940]. To emphasize that we are working on \( S^n \), let us denote \( P_0 \) by \( P_{S^n} \). Then by Lemma 2.2 and the preceding identity, we have

\[
\frac{n}{2} \int_{S^n} w_{p(t)} \cdot P_{S^n} w_{p(t)} \, d\mu_{S^n} + n! \int_{S^n} w_{p(t)} \, d\mu_{S^n} \leq \mathcal{E}[u_0], \tag{A.11}
\]

which implies that

\[
\int_{S^n} w_{p(t)} \, d\mu_{S^n} \leq \frac{\mathcal{E}[u_0]}{n!}. \tag{A.12}
\]

Since \( u_0 \in Y \), we obtain

\[
\int_{S^n} f \circ \phi_{p(t)} e^{nw_{p(t)}} \, d\mu_{S^n} = \int_{S^n} f e^{nu(t)} \, d\mu_{S^n} = (n - 1)! \text{vol}(S^n).
\]

In particular, this gives

\[
\int_{S^n} e^{nw_{p(t)}} \, d\mu_{S^n} \geq \frac{(n - 1)! \text{vol}(S^n)}{\sup_{S^n} f}. \tag{A.13}
\]

Since \( w_{p(t)} \) satisfies (A.10), we can apply an improved Beckner inequality due to Wei and Xu in [32, Theorem 2.6] and use (A.11) and (A.13) to get

\[
\frac{(n - 1)!}{\sup_{S^n} f} \leq \frac{1}{\text{vol}(S^n)} \int_{S^n} e^{nw_{p(t)}} \, d\mu_{S^n} \leq \exp \left[ \frac{1}{(n - 1)! \text{vol}(S^n)} \left( \frac{an}{2} \int_{S^n} w_{p(t)} \cdot P_{S^n} w_{p(t)} \, d\mu_{S^n} + n! \int_{S^n} w_{p(t)} \, d\mu_{S^n} \right) \right] \leq \exp \left[ \frac{n(a - 1)}{2(n - 1)! \text{vol}(S^n)} \int_{S^n} w_{p(t)} \cdot P_{S^n} w_{p(t)} \, d\mu_{S^n} + \frac{\mathcal{E}[u_0]}{(n - 1)! \text{vol}(S^n)} \right] \tag{A.14}
\]

for some \( 0 < a < 1 \). From this it follows that there exists a uniform constant \( C_1 > 0 \) such that

\[
\int_{S^n} w_{p(t)} \cdot P_{S^n} w_{p(t)} \, d\mu_{S^n} \leq C_1. \tag{A.15}
\]

Plugging (A.15) into (A.14) yields

\[
\int_{S^n} w_{p(t)} \, d\mu_{S^n} \geq \frac{\text{vol}(S^n)}{n} \log \left( \frac{(n - 1)!}{\sup_{S^n} f} \right) - \frac{aC_1}{2(n - 1)!}.
\]

Combining (A.12) and (A.16), we obtain that there exists a uniform constant \( C_2 > 0 \) such that

\[
\left| \int_{S^n} w_{p(t)} \, d\mu_{S^n} \right| \leq C_2. \tag{A.17}
\]
It follows from (A.15), (A.17), and Poincaré’s inequality (1.9) that

$$\int_{\mathbb{S}^n} w_{p(t)}^2 \, d\mu_{\mathbb{S}^n} \leq C_3$$

for some uniform constant $C_3 > 0$, which together with (A.15) implies that there exists a uniform constant $C_4 > 0$ such that

$$\|w_{p(t)}\|_{H^{n/2}(\mathbb{S}^n)} \leq C_4.$$  \hspace{1cm} (A.18)

It is worth noticing that all constant $C_i$ with $1 \leq i \leq 4$ are independent of $T$. Now depending on the size of $\sup_{t \in [0, T]} |p(t)|$, we have two possibilities:

**Case 1** Suppose that $\sup_{t \in [0, T]} |p(t)| < 1$. We shall prove that Part (i) of the lemma occurs. Indeed, it follows from [11, page 943] that

$$\det(d\phi_p^{-1}(x)) = \left(\frac{1 - |p|^2}{1 - 2\langle p, x \rangle + |p|^2}\right)^n x \in \mathbb{S}^n,$$

which yields

$$\left(\frac{1 - |p|}{1 + |p|}\right)^n \leq \det(d\phi_p^{-1}) \leq \left(\frac{1 + |p|}{1 - |p|}\right)^n.$$

Therefore, under our hypothesis, there is a constant $C \gg 1$ independent of $T$ such that

$$0 < C^{-1} \leq \det(d\phi_{p(t)}) \leq C.$$

By Beckner’s inequality (2.6) and (A.18), there holds

$$\sup_{[0, T]} \int_{\mathbb{S}^n} e^{n|u(t)|} \, d\mu_{\mathbb{S}^n} \leq C_3.$$

Now we pass our bound for $w_{p(t)}$ in (A.18) to a similar bound for $u(t)$. To this purpose, we first note that

$$\int_{\mathbb{S}^n} e^{n|u(t)|} \, d\mu_{\mathbb{S}^n} = \int_{\mathbb{S}^n} \exp(n|u(t) \circ \phi_{p(t)}| + \log(\det d\phi_{p(t)})) \, d\mu_{\mathbb{S}^n}$$

$$\leq \int_{\mathbb{S}^n} \exp(n|w_{p(t)}| + 2\log(\det d\phi_{p(t)})) \, d\mu_{\mathbb{S}^n}$$

$$= \int_{\mathbb{S}^n} e^{n|w_{p(t)}|} |\det d\phi_{p(t)}|^2 \, d\mu_{\mathbb{S}^n}.$$  \hspace{1cm} (A.19)

From this and the bounds for $\det(d\phi_{p(t)})$, we deduce that

$$\sup_{[0, T]} \int_{\mathbb{S}^n} e^{n|u(t)|} \, d\mu_{\mathbb{S}^n} \leq C_4.$$  \hspace{1cm} (A.20)

In particular, applying Jensen’s inequality gives

$$\int_{\mathbb{S}^n} |u(t)| \, d\mu_{\mathbb{S}^n} \leq C_5,$$  \hspace{1cm} (A.19)

which together with Lemma 2.2 implies that

$$\int_{\mathbb{S}^n} u \cdot P_{\mathbb{S}^n} u \, d\mu_{\mathbb{S}^n} \leq C.$$  \hspace{1cm} (A.20)
Now it follows from Poincaré’s inequality, (A.19), and (A.20) that
\[ \|u(t)\|_{H^{n/2}(\mathbb{S}^n)} \leq C \]
for some constant \( C > 0 \) independent of \( T \).

**Case 2** Suppose that \( \sup_{t \in [0,T]} |p(t)| = 1 \). In this scenario, we can find a sequence \( t_k \uparrow T \) in such a way that \( |p(t_k)| \uparrow 1 \) and that \( p(t_k) \to x_\infty \) as \( k \to +\infty \) for some point \( x_\infty \in \mathbb{S}^n \).

In view of (A.18), there exists a subsequence of \( t_k \), still denoted by \( t_k \), and some function \( w_\infty \in H^{n/2}(\mathbb{S}^n) \) such that
\[ w_k =: w_{p(t_k)} \rightharpoonup w_\infty \]
weakly in \( H^{n/2}(\mathbb{S}^n) \) and strongly in \( L^2(\mathbb{S}^n) \).

Let \( r > 0 \) be arbitrary but fixed and set
\[ B_k = (\phi_{p(t_k)})^{-1}(B_r(x_\infty)). \]

Then by Hölder’s inequality, we get
\[ \left| \int_{\mathbb{S}^n \setminus B_k} f \circ \phi_{p(t_k)} e^{n[w_k]} \, d\mu_{\mathbb{S}^n} \right| \leq \left( \sup_{\mathbb{S}^n} |f| \right) \left( \vol(\mathbb{S}^n \setminus B_k) \int_{\mathbb{S}^n} e^{2n[w_k]} \, d\mu_{\mathbb{S}^n} \right)^{1/2} \leq C \sqrt{\vol(\mathbb{S}^n \setminus B_k)}. \] \hspace{1cm} (A.21)

To obtain the last inequality in (A.21), we have used (A.18) and Beckner’s inequality (2.6).

Notice that, one can easily verify that as \( k \to +\infty \) there holds
\[ \phi_{p(t_k)} \to \phi_{x_\infty} \equiv x_\infty \]
uniformly in \( \mathbb{S}^n \setminus B_\delta(x_\infty) \) with any sufficiently small \( \delta > 0 \). Next, we need to estimate \( \vol(B_k) \) for large \( k \).

**Claim** There holds
\[ \lim_{k \to +\infty} \vol(B_k) = \vol(\mathbb{S}^n). \]

**Proof of Claim** For each \( \varepsilon > 0 \) sufficiently small but fixed, because \( \phi_{p(t_k)} \to x_\infty \) uniformly in \( \mathbb{S}^n \setminus B_\varepsilon(x_\infty) \), there is some \( N > 1 \) independent of \( x \) such that
\[ d(\phi_{p(t_k)}(x), x_\infty) < r \]
for all \( k \geq N \) and all \( x \in \mathbb{S}^n \setminus B_\varepsilon(x_\infty) \). In particular, there holds
\[ \mathbb{S}^n \setminus B_\varepsilon(x_\infty) \subset \phi_{p(t_k)}^{-1}(B_r(x_\infty)) = B_k \]
for all \( k \geq N \). Thus, we have just shown that
\[ \mathbb{S}^n \setminus B_\varepsilon(x_\infty) \subset B_k \subset \mathbb{S}^n, \]
for large \( k \), which immediately implies
\[ \vol(\mathbb{S}^n \setminus B_\varepsilon(x_\infty)) \leq \liminf_{k \to +\infty} \vol(B_k) \leq \limsup_{k \to +\infty} \vol(B_k) \leq \vol(\mathbb{S}^n). \]

Letting \( \varepsilon \downarrow 0 \) gives the desired result. \( \square \)

As a consequence of the above Claim, we get
\[ \lim_{k \to +\infty} \vol(\mathbb{S}^n \setminus B_k) = 0. \]
Keep in mind that
\[ \int_{B_r(x_\infty)} f \ e^{nu(t_k)} \ d\mu_{\mathbb{S}^n} = \int_{B_r(x_\infty)} f \circ \phi_{p(t_k)} e^{nu_k} \ d\mu_{\mathbb{S}^n}. \]

This fact together with (A.21) and the fact \( u(t) \in Y \) implies that
\[ \lim_{k \to +\infty} \int_{B_r(x_\infty)} f \ e^{nu(t_k)} \ d\mu_{\mathbb{S}^n} = (n - 1)! \vol(\mathbb{S}^n) \]
for all \( r > 0 \). This establishes (3.24). Now, let \( y \in \mathbb{S}^n \setminus \{x_\infty\} \) and \( 0 \leq r < \text{dist}(y, x_\infty) \). Then we can choose some \( 0 < s < \text{dist}(y, x_\infty) - r \) and fix it. By the argument leading to (3.24), we know that
\[ \lim_{k \to +\infty} \vol(\mathbb{S}^n \setminus (\phi_{p(t_k)})^{-1}(B_s(x_\infty))) = 0. \]

However, we have \( B_s(x_\infty) \cap B_r(y) = \emptyset \), which implies that
\[ \phi_{p(t_k)}^{-1}(B_r(y)) \cap \phi_{p(t_k)}^{-1}(B_s(x_\infty)) = \emptyset. \]

Hence, there holds
\[ \lim_{k \to +\infty} \vol(\phi_{p(t_k)}^{-1}(B_r(y))) = 0. \]

However, as in (A.21), we can estimate
\[ \left| \int_{B_r(y)} f \ e^{nu(t_k)} \ d\mu_{\mathbb{S}^n} \right| = \left| \int_{\phi_{p(t_k)}^{-1}(B_r(y))} f \circ \phi_{p(t_k)} e^{nu_k} \ d\mu_{\mathbb{S}^n} \right| \]
\[ \leq (\sup_{\mathbb{S}^n} |f|) \left( \int_{\mathbb{S}^n} e^{2n|w_k|} \ d\mu_{\mathbb{S}^n} \right)^{1/2} \left( \int_{\phi_{p(t_k)}^{-1}(B_r(y))} d\mu_{\mathbb{S}^n} \right)^{1/2}. \]

Keep in mind that, by Beckner’s inequality (2.6), the term \( \int_{\mathbb{S}^n} e^{2n|w_k|} \ d\mu_{\mathbb{S}^n} \) is uniformly bounded. Therefore, we get
\[ \lim_{k \to +\infty} \int_{B_r(y)} f \ e^{nu(t_k)} \ d\mu_{\mathbb{S}^n} = 0. \]

This establishes (3.25). Hence Part (ii) of the lemma is proved. \( \square \)

**Appendix B: An alternative proof of Ge–Xu’s result**

In this appendix, we provide an alternative proof of [18, Theorem 3.2]. Using our notation and conventions, we shall reprove the following result.

**Theorem B.1** (Ge and Xu). Let \( (M, g_0) \) be a compact, oriented \( n \)-dimensional Riemannian manifold with \( n \) even. Assume that the GJMS operator \( P_0 \) is positive with kernel consisting of constant functions. Moreover, assume that the metric \( g_0 \) satisfies
\[ \int_M Q_0 \ d\mu_{g_0} = 0. \]

If \( f \) is a smooth function on \( M \) such that \( \int_M f \ d\mu_{g_0} < 0 \), then there exists a conformal metric \( g \in [g_0] \) such that \( Q_g = f \). 

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To prove Theorem B.1 above, we also use variational techniques; however, the difference between our approach and that of [18] is the constraint. Thanks to the resolution of the problem of prescribing constant $Q$-curvature [25], we can assume at the beginning that $g_0$ has $Q_0 \equiv 0$. From this fact, to conclude Theorem B.1, it is equivalent to solving

$$P_0 u = f e^{nu} \quad \text{(B.1)}$$

for $u$ under the condition $\int_M f \, d\mu_{g_0} < 0$.

**Step 1** Inspired by Lemma 3.4, we let

$$\mathcal{H} = \left\{ u \in H^{n/2}(M) : \int_M f e^{nu} \, d\mu_{g_0} = 0, \int_M e^{nu} \, d\mu_{g_0} = \text{vol}(M) \right\}. \quad \text{(B.2)}$$

As can be easily observed, the condition $\int_M u \, d\mu_{g_0} = 0$ in [18] is replaced by the condition $\int_M e^{nu} \, d\mu_{g_0} = \text{vol}(M)$ in (B.2).

As always, we shall minimize the following functional energy

$$\mathcal{E}[u] = \int_M u \cdot P_0 u \, d\mu_{g_0}$$

in $\mathcal{H}$. Following the method in [18], it is not hard to see that the set $\mathcal{H}$ is non-empty, thanks to $\int_M f \, d\mu_{g_0} < 0$. Since $\mathcal{H}$ is non-empty and $\mathcal{E}$ is bounded from below by 0, we know that

$$\mu = \inf_{u \in \mathcal{H}} \mathcal{E}(u)$$

is finite and non-negative. As in (1.14), we still use

$$\|u\|_{H^{n/2}(M)} = \left( \int_M u \cdot P_0 u \, d\mu_{g_0} + \int_M u^2 \, d\mu_{g_0} \right)^{1/2}$$

as an equivalent norm on $H^{n/2}(M)$. Suppose that $(u_k)_k \subset \mathcal{H}$ is a minimizing sequence for $\mu$. Clearly, $\int_M u_k \cdot P_0 u_k \, d\mu_{g_0}$ is bounded. Hence, to bound the sequence $(u_k)_k$ in $H^{n/2}(M)$, it suffices to bound $\int_M u_k^2 \, d\mu_{g_0}$. However, this can be easily obtained, in the same fashion of Lemma 3.5, since we can easily bound $\int_M u_k \, d\mu_{g_0}$. Therefore, up to subsequences, there exists some $u_\infty \in H^{n/2}(M)$ such that

- $u_k \rightharpoonup u_\infty$ weakly in $H^{n/2}(M)$ and
- $u_k \rightarrow u_\infty$ almost everywhere in $M$

as $k \rightarrow +\infty$. Clearly,

$$\mathcal{E}[u_\infty] \leq \liminf_{k \rightarrow +\infty} \mathcal{E}[u_k].$$

From this we conclude that $\mathcal{E}[u_\infty] = \mu$ and that $u_\infty$ is an optimizer for $\mu$ since $u_\infty \in \mathcal{H}$, thanks to Trudinger’s inequality. By Lagrange’s multiplier theorem, there exist two constants $\alpha$ and $\beta$ such that

$$\int_M \phi \cdot P_0 u_\infty \, d\mu_{g_0} = \alpha \int_M e^{nu_\infty} \phi \, d\mu_{g_0} + \beta \int_M f e^{nu_\infty} \phi \, d\mu_{g_0} \quad \text{(B.3)}$$

for any test function $\phi \in H^{n/2}(M)$. Testing (B.3) with $\phi \equiv 1$ gives $\alpha = 0$ and hence $\beta \neq 0$.

**Step 2** To realize that our equation (B.1) has a solution, it is necessary to prove that $\beta > 0$. To this purpose, we minimize $\mathcal{E}$ over the set

$$\mathcal{\hat{H}} = \left\{ u \in H^{n/2}(M) : \int_M f e^{nu} \, d\mu_{g_0} \geq 0, \frac{\text{vol}(M)}{2} \leq \int_M e^{nu} \, d\mu_{g_0} \leq \frac{3\text{vol}(M)}{2} \right\}.$$
Set

\[ \tilde{\mu} = \inf_{u \in \mathcal{H}} E(u). \]

Because \( \mathcal{H} \subset \tilde{\mathcal{H}} \) we clearly have \( \tilde{\mu} \leq \mu \). Using the same argument as before, there exists some function \( \tilde{u}_\infty \in \tilde{\mathcal{H}} \) such that \( \tilde{\mu} = E(\tilde{u}_\infty) \). (Note that the positive boundedness away from zero of \( \int_M e^{\mu u} \, d\mu_{g_0} \) will be enough to guarantee an upper bound for \( \tilde{u} \); see Lemma 3.5.) Suppose that \( \tilde{\mu} < \mu \), again by Lagrange’s multiplier theorem, there exists some function \( \tilde{u}_\infty \in \tilde{\mathcal{H}} \) such that

\[ \tilde{\mu} = E(\tilde{u}_\infty) \]

for any \( \varphi \in H^{n/2}(M) \). By testing (B.4) with \( \varphi \equiv \tilde{u}_\infty \) we deduce that \( \tilde{u}_\infty \) is constant. From this we obtain a contradiction because \( \tilde{u}_\infty \in \tilde{\mathcal{H}} \setminus \mathcal{H} \) and

\[ 0 \leq \int_M f e^{\tilde{\mu} u_\infty} \, d\mu_{g_0} = e^{\tilde{\mu} u_\infty} \int_M f \, d\mu_{g_0} < 0. \]

Thus, we have just shown that \( \tilde{\mu} = \mu \). By contradiction, suppose that \( \beta < 0 \). Then by (B.3), we have

\[ E[u_\infty] = \mu = \beta \int_M f e^{\mu u_\infty} \, d\mu_{g_0}, \]

which implies that \( \int_M f e^{\mu u_\infty} \, d\mu_{g_0} < 0 \). Set

\[ l(t) = \int_M f e^{t u_\infty} \, d\mu_{g_0}. \]

Clearly, \( l(1) = 0 \) and \( l'(1) < 0 \). Hence there exists some \( \lambda \in (0, 1) \) closed to 1 such that \( l(\lambda) > 0 \) and

\[ \frac{\text{vol}(M)}{2} \leq \int_M e^{\lambda u_\infty} \, d\mu_{g_0} \leq \frac{3\text{vol}(M)}{2}. \]

In other words, we conclude that \( \lambda u_\infty \in \tilde{\mathcal{H}} \) which implies that

\[ \tilde{\mu} \leq E[\lambda u_\infty] = \lambda^2 E[u_\infty] < E[u_\infty] = \tilde{\mu}. \]

This contradiction shows \( \beta > 0 \) as we wish.

\textit{Step 3} Once we can show that \( \beta > 0 \), it is immediate to verify that \( u_\infty + (\log \beta)/n \) solves (B.1). The proof is complete.

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