ON THE RELATION BETWEEN STRONG BALLISTIC TRANSPORT AND EXPONENTIAL DYNAMICAL LOCALIZATION

ILYA KACHKOVSKII

Abstract. We establish strong ballistic transport for a family of discrete quasiperiodic Schrödinger operators as a consequence of exponential dynamical localization for the dual family. The latter has been, essentially, shown by Jitomirskaya and Krüger in the one-frequency setting and by Ge–You–Zhou in the multi-frequency case. In both regimes, we obtain strong convergence of \( \frac{1}{T} X(T) \) to the asymptotic velocity operator \( Q \), which improves recent perturbative results by Zhao and provides the strongest known form of ballistic motion. In the one-frequency setting, this approach allows to treat Diophantine frequencies non-perturbatively and also consider the weakly Liouville case.

1. Introduction and main results

In this paper, we consider the following class of multi-frequency quasiperiodic operators on \( \ell^2(\mathbb{Z}) \):

\[
(H(x)\psi)(n) = \psi(n + 1) + \psi(n - 1) + \varepsilon v(x + n\alpha)\psi(n), \quad x, \alpha \in \mathbb{T}^d, \quad n \in \mathbb{Z}.
\]

where

\[ n\alpha = \{n\alpha_1\}, \ldots, \{n\alpha_d\} \in \mathbb{T}^d. \]

We identify \( \mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d \) with \([0, 1]^d\) and assume that \( v \in C^\omega(\mathbb{T}^d; \mathbb{R}) \) is a real analytic potential (considered also as a \( \mathbb{Z}^d \)-periodic function on \( \mathbb{R}^d \)). Here \( \alpha = (\alpha_1, \ldots, \alpha_d) \) is the frequency vector such that \( \{1, \alpha_1, \ldots, \alpha_d\} \) are independent over \( \mathbb{Q} \). Whenever we introduce an abstract concept that does not use quasiperiodic specifics, we will use the notation \( H \) for the Schrödinger operator.

The position operator is defined on the natural domain of definition in \( \ell^2(\mathbb{Z}) \) by

\[
(X\psi)(n) = n\psi(n),
\]

and its Heisenberg evolution can be represented as

\[
X(T) = e^{iT H} X e^{-iT H} = X + \int_0^T e^{i t H} A e^{-i t H} dt, \quad T \in \mathbb{R},
\]

where

\[
A\psi(n) = i(\psi(n + 1) - \psi(n - 1)).
\]

Since \( A \) is bounded, (1.3) implies that \( X = X(0) \) and \( X(T) \) have the same domain. We will be interested in computing the limits

\[
\lim_{T \to +\infty} \frac{1}{T} X(T) \psi_0.
\]
where \( \psi_0 \in \text{Dom}(X) \). One can consider the limit (1.5), if it exists, as the “asymptotic velocity” of the state \( \psi_0 \) at infinite time. The asymptotic velocity operator is defined as

\[
Q = \lim_{T \to +\infty} \frac{1}{T} X(T) = \lim_{T \to +\infty} \frac{1}{T} \int_0^T e^{itH} A e^{-itH} dt.
\]  

The first limit is only defined on a dense set, but it is natural to remove the term \( \frac{1}{T} X(0) \) and consider only the right hand side. We say that a Schrödinger operator \( H \) demonstrates strong ballistic transport, if the right hand side of (1.6) converges on \( \ell^2(\mathbb{Z}) \) and \( \ker Q = \{0\} \). Strong ballistic transport immediately implies

\[
\|X(T)\psi_0\| \geq c(\psi_0)|T|, \quad |T| \gg 1, \quad \psi \in \text{Dom}(X).
\]

Ballistic motion is recognized as one of the manifestations of absolutely continuous spectrum. Originally, it was studied in the Cesàro averaged sense, see, for example [14] and references therein. The non-averaged lower bounds for \( X(T) \) were first found in [1] for periodic operators in the continuum. Later, they were extended in [5] to the discrete Jacobi matrix case, motivated by applications to XY spin chains. Some anomalous bounds for Fibonacci type Hamiltonians were found in [1]. In the quasiperiodic case, an \( x \)-averaged version of ballistic transport was obtained in [13] by the duality method based on [10]. As a consequence, one can still obtain lower bounds on Lieb–Robinson velocity for the XY chain, but the actual ballistic transport would only be proved for a sequence of time scales. In the same year, a different approach was developed in [16] in order to obtain bounds of type (1.7) in the perturbative setting. It does not require considering a sequence of time scales, but fall short of (1.5). The KAM method of [16] was later developed in [15] to treat the one-frequency Liouvillian case, by further weakening (1.7) to a bound on the transport exponent. The limit-periodic case was studied in [6] where an analogue of (1.5) was obtained.

While (1.7) is already a very strong condition, the convergence statement (1.5) is more desirable, since it shows that the wavepacket takes a particular asymptotic shape at large times, assuming it is properly rescaled. One can compare this process with localization. In the quasiperiodic case, one of the results of [13] is the calculation of the asymptotic velocity operator \( Q(x) \), but, since it is only obtained on a sequence of time scales, one cannot exclude the possibility of large oscillations. Moreover, [13] predicts a possible mechanism of convergence: after applying duality, it becomes a procedure of diagonal truncation of an operator dual to (1.4) in the basis of the eigenvectors of the dual Hamiltonian with point spectrum. The convergence of the truncation is only obtained in the dual \( L^2 \) direct integral space (in other words, averaged over \( \theta \)), which is not enough to guarantee pointwise strong convergence in the original direct integral space. A natural question arises: can we improve it? In order to obtain a pointwise bound (say, \( L^\infty \) in the \( x \) variable), can try to obtain an \( \ell^1 \) bound in the dual \( \mathbb{Z}^d \) variable. Clearly, if we truncate the dual \( \mathbb{Z}^d \) space, then \( \ell^1 \) bound would follow from \( \ell^2 \) bound, which is already obtained in [13]. It turns out that the missing ingredient is a uniform \( \ell^1 \) bound on the tails, which is an extra property that we require from the dual model. This property follows from exponential dynamical localization and has been established in [11] and [7].

As usual, we call a frequency vector \( \alpha \) Diophantine (denoted \( \alpha \in \text{DC}(c, \tau) \) for some \( c, \tau > 0 \)) if

\[
\text{dist}(k \cdot \alpha, \mathbb{Z}) \geq c|k|^{-\tau}, \quad \forall k \in \mathbb{Z}^d \setminus \{0\}.
\]
We also use the notation $\text{DC} = \bigcup_{c,\tau \geq 0} \text{DC}(c, \tau)$. For $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, denote also

$$\beta(\alpha) = \limsup_{k \to \infty} \frac{\ln q_{k+1}}{q_k},$$

where $\{q_k\}$ is the sequence of continued fraction approximants of $\alpha$. Note that $\alpha \in \text{DC}$ implies $\beta(\alpha) = 0$, but not vice versa. The main result of the paper is Theorem 2.4 which establishes strong ballistic transport as a consequence of exponential dynamical localization for the dual operator. We postpone the complete setup to Section 2, and formulate two main corollaries.

**Corollary 1.1.** Suppose that $d = 1$, $v \in C^\omega(\mathbb{T})$, $0 < \beta(\alpha) < +\infty$. There exists $\varepsilon_0 = \varepsilon_0(v, \beta) > 0$ such that, for $0 < \varepsilon < \varepsilon_0$, the operator $H(x)$ (1.1) has strong ballistic transport for a.e. $x \in \mathbb{T}$.

**Corollary 1.2.** Suppose that $v \in C^\omega(\mathbb{T}^d)$, $\alpha \in \text{DC}$. There exists $\varepsilon_0 = \varepsilon_0(v, \alpha) > 0$ such that, for $0 < \varepsilon < \varepsilon_0$, the operator $H(x)$ (1.1) has strong ballistic transport for a.e. $x \in \mathbb{T}^d$.

A version of Corollary 1.2 with (1.7) instead of (1.5) was obtained in [16].

### 2. Preliminaries and the main result

The proof will refine convergence bounds from [13], part of which is based on Aubry duality. Let $(\hat{\ast} \cdot ) : \mathbb{Z}^d \to \mathbb{Z}^d$ be the convolution operator

(2.1) $$(\hat{\ast} \psi)(n) = \sum_{m \in \mathbb{Z}^d} \hat{\psi}(n - m) \psi(m),$$

where

$$v(x) = \sum_{m \in \mathbb{Z}^d} \hat{\psi}(m) e^{2\pi im \cdot x}$$

is the usual Fourier series. The dual operator family $\tilde{H}(\theta)$ is defined by

(2.2) $$(\tilde{H}(\theta) \psi)(m) = \varepsilon(\hat{\ast} \psi)(m) + 2 \cos 2\pi(\theta + m \cdot \alpha) \psi(m), \quad \theta \in \mathbb{T}^1 = [0, 1), \ m \in \mathbb{Z}^d.$$  

Denote the corresponding direct integral spaces (for $H$ and $\tilde{H}$ respectively) by

$$\mathcal{H} := \int_{\mathbb{T}^d} \ell^2(\mathbb{Z}) \, dx, \quad \tilde{\mathcal{H}} := \int_{\mathbb{T}} \ell^2(\mathbb{Z}_d) \, d\theta.$$  

The unitary duality operator $\mathcal{U} : \mathcal{H} \to \tilde{\mathcal{H}}$ is defined on functions $\Psi = \Psi(x, n)$ as

(2.3) $$(\mathcal{U}\Psi)(\theta, m) = \tilde{\Psi}(m, \theta + \alpha \cdot m),$$  

where $\tilde{\Psi}$ denotes the Fourier transform over $x \in \mathbb{T}^d \to m \in \mathbb{Z}^d$ combined with the inverse Fourier transform $n \in \mathbb{Z} \to \theta \in \mathbb{T}$:

$$\tilde{\Psi}(m, \theta) = \sum_{n \in \mathbb{Z}} \int_{\mathbb{T}^d} e^{2\pi in\theta - 2\pi imx} \Psi(n, x) \, dx.$$  

Let also

$$\mathcal{H} := \int_{\mathbb{T}^d} H(x) \, dx, \quad \tilde{\mathcal{H}} := \int_{\mathbb{T}} \tilde{H}(\theta) \, d\theta.$$  

Moreover, the operator \( \tilde{\mathcal{A}}(\theta) \psi(m) = 2\sin 2\pi(m \cdot \alpha + \theta)\psi(m), \quad m \in \mathbb{Z}^d. \)

In fact, any operator on \( \mathcal{H} \) has a dual counterpart, defined in a similar way. The dual version of the operator \( A \) is a decomposable operator:

\[
(\tilde{A}(\theta)\psi)(m) = 2\sin 2\pi(m \cdot \alpha + \theta)\psi(m), \quad m \in \mathbb{Z}^d.
\]

2.1. Strong ballistic transport in expectation. Denote by

\[
Q(x, T) = \frac{1}{T} \int_0^T e^{iH(x)t} A e^{-iH(x)t} \, dt, \quad \tilde{Q}(\theta, T) = \frac{1}{T} \int_0^T e^{i\tilde{H}(\theta)t} \tilde{A}(\theta)e^{-i\tilde{H}(\theta)t} \, dt,
\]

and the corresponding direct integrals

\[
Q(T) = \int_{\mathbb{T}^d} Q(x, T) \, dx, \quad \tilde{Q}(T) = \int_{\mathbb{T}} \tilde{Q}(\theta, T) \, d\theta.
\]

The following result is, essentially, established in \([13]\).

**Proposition 2.1.** Suppose that the family \( \tilde{H}(\theta) \) has purely point spectrum for a.e. \( \theta \). Then, for a.e. \( \theta \), the following limit exists:

\[
\tilde{Q}(\theta) = \mathbf{s}\text{-}\lim_{T \to +\infty} \frac{1}{T} \int_0^T e^{i\tilde{H}(\theta)t} \tilde{A}(\theta)e^{-i\tilde{H}(\theta)t} \, dt.
\]

Moreover, the operator \( \tilde{Q}(\theta) \) is the diagonal part of \( \tilde{A}(\theta) \) with respect to any orthonormal basis of eigenfunctions \( \{\psi_k(\theta)\} \) of \( \tilde{H}(\theta) \):

\[
\tilde{Q}(\theta)\psi_k(\theta) = \langle \tilde{A}(\theta)\psi_k(\theta), \psi_k(\theta) \rangle \psi_k(\theta),
\]

and \( \text{ker} \tilde{Q} \neq \{0\} \) for a.e. \( \theta \). As a consequence, there exist decomposable operators \( Q, \tilde{Q} : \)

\[
Q = \mathbf{s}\text{-}\lim_{T \to +\infty} Q(T), \quad \tilde{Q} = \mathbf{s}\text{-}\lim_{T \to +\infty} \tilde{Q}(T); \quad \text{ker} \ Q = \text{ker} \ \tilde{Q} = \{0\}.
\]

**Remark 2.2.** In \([13]\), Proposition 2.1 was formulated in a slightly different setting, assuming that the family \( H(x) \) satisfies \( L^2 \) degree zero reducibility condition. However, one can check that the same proof follows through. In particular, the proof of the fact \( \text{ker} \tilde{Q} = \{0\} \) is done exactly the same way as in Appendix C of \([3]\), see also Remark 5.1 in \([10]\).

Once the convergence in \( L^2 \) is obtained, one can apply a diagonal procedure to establish the following fact: there is a sequence of time scales \( T_k \to \infty \) as \( k \to \infty \) such that, for almost every \( x \in \mathbb{T}^d \), \( Q(x, T_k) \) converges to \( Q(x) \) strongly.

2.2. Main results. Our goal is to improve the convergence of \( \tilde{Q}(T) \). This requires some additional information about the dual operator. Denote by \( \{\delta_k : k \in \mathbb{Z}^d\} \) the standard basis in \( \mathbb{Z}^d \).

**Definition 2.3.** We say that the family \( \{\tilde{H}(\theta)\} \) satisfies **exponential dynamical localization in expectation** if the spectra of \( \tilde{H}(\theta) \) are purely point for a.e. \( \theta \in \mathbb{T} \), and the following bound holds with some constants \( C, \gamma > 0 \):

\[
\int_T \sup_{t \in \mathbb{R}} |\langle \delta_k, e^{-it\tilde{H}(\theta)}\delta_l \rangle| \, d\theta \leq Ce^{-\gamma|k-l|}.
\]
It turns out that EDL is the missing ingredient for establishing “true” strong ballistic transport \((1.5)\). The following is the main result of the present paper.

**Theorem 2.4.** Suppose that the family \(\{\tilde{H}(\theta)\}\) satisfies EDL. Then, for almost every \(x \in \mathbb{T}^d\), the operator \(H(x)\) has strong ballistic transport.

There are two cases in which EDL is established. The first one is the weakly Liouvillean one-frequency case. In [11], it is obtained for the almost Mathieu operator. However, the proof relies on Theorem 5.1 from [2] which has, conveniently, been obtained for the general non-local case, and the rest of the argument can be repeated verbatim. See also [8] for earlier application of the method and [12] for a significantly refined result for the almost Mathieu operator.

**Proposition 2.5.** Fix \(v \in C^\omega(\mathbb{T})\) and \(\beta > 0\). There exists \(\varepsilon_0 = \varepsilon_0(v, \beta) > 0\) such that the operator family
\[
(\tilde{H}(\theta)\psi)(m) = \varepsilon(\hat{v} \ast \psi)(m) + 2\cos 2\pi(\theta + m \cdot \alpha)\psi(m), \quad m \in \mathbb{Z}.
\]
satisfies EDL.

Recently, a multi-dimensional analogue has been obtained in [7]:

**Proposition 2.6.** Fix \(v \in C^\omega(\mathbb{T}^d)\) and suppose that \(\alpha \in \text{DC}\). There exists \(\varepsilon_0 = \varepsilon_0(v, \alpha) > 0\) such that the operator family
\[
(\tilde{H}(\theta)\psi)(m) = \varepsilon(\hat{v} \ast \psi)(m) + 2\cos 2\pi(\theta + m \cdot \alpha)\psi(m), \quad m \in \mathbb{Z}^d
\]
satisfies EDL.

We should note that [7] also contains a version of Proposition 2.5 in the Diophantine setting, obtained by a different method from the “reducibility” side.

### 3. Proof of Theorem 2.4

Suppose that \(\{\theta_j\}_{j \in \mathbb{Z}^d} \subseteq \mathbb{T}^1\) is some fixed sequence of phases. Denote by \(L^2_{\text{dual}}\) the space of functions \(\Psi\) on \(\mathbb{T}^1 \times \mathbb{Z}^d\) with the norm
\[
\|\Psi\|_{L^2_{\text{dual}}} = \left\{ \int_{\mathbb{T}^1} \left( \sum_{m \in \mathbb{Z}^d} |\Psi(\theta + \theta_m; m)| \right)^2 \ d\theta \right\}^{1/2}.
\]
The definition resembles the vector-valued space \(L^2(\mathbb{T}; \ell^1(\mathbb{Z}^d))\). However, before calculating \(\ell^1\)-norm, we shear the argument of the \(m\)th component by \(\theta_m\). Let also \(P_N\) be the orthogonal projection onto \(\text{Span}\{\delta_n: n \in \mathbb{Z}^d, |n| \leq N\}\) in \(\ell^2(\mathbb{Z}^d)\), and \(P_N^\perp = I - P_N\).

**Lemma 3.1.** Under the assumptions of Theorem 2.4 define \(\tilde{Q}(\theta, T)\) by \((2.6)\). Then, the following bound holds:
\[
\left( \int_{\mathbb{T}} \|P_N^\perp \tilde{Q}(\theta, T)\delta_k\|^2_{\ell^1(\mathbb{Z})} d\theta \right)^{1/2} \leq C_1 e^{-C_2 |N - |k||}.
\]
Moreover, the norm in the left hand side can be replaced by the norm in \(L^2_{\text{dual}}\) for any choice of \(\{\theta_m\}\), with the same bounds.
Proof. We will prove a stronger statement: a uniform bound of (3.2) without Cesàro averaging; that is, with \( \tilde{Q}(\theta, T) \) replaced by \( e^{i\tilde{H}(\theta)t} A(\theta)e^{-i\tilde{H}(\theta)t} \). Using the triangle inequality applied to the \( \| \cdot \|_{L^2_{\text{dual}}} \) norm of the sum

\[
P_N^+ e^{i\tilde{H}(\theta)t} A(\theta)e^{-i\tilde{H}(\theta)t} \delta_k = \sum_{|n|>N} \left\langle \delta_n, e^{i\tilde{H}(\theta)t} A(\theta)e^{-i\tilde{H}(\theta)t} \delta_k \right\rangle \delta_n,
\]

we can estimate (3.2):

\[
\left( \int_T \| P_N^+ e^{i\tilde{H}(\theta)t} A(\theta)e^{-i\tilde{H}(\theta)t} \delta_k \|^2_{L^2(\mathbb{Z}^d)} d\theta \right)^{1/2} \leq \sum_{|n|>N} \left( \int_T \left| \left\langle \delta_n, e^{i\tilde{H}(\theta)t} A(\theta)e^{-i\tilde{H}(\theta)t} \delta_k \right\rangle \right|^2 d\theta \right)^{1/2}
\]

\[
\leq 2 \sum_{|n|>N} \left( \int_T \left| \langle e^{-i\tilde{H}(\theta)t} \delta_n, \tilde{A}(\theta)e^{-i\tilde{H}(\theta)t} \delta_k \rangle \right| \right) \right)^{1/2} \leq 2 \sum_{|n|>N} \left( \int_T \left| \langle e^{-i\tilde{H}(\theta)t} \delta_n, \tilde{A}(\theta)\delta_\ell \rangle \langle \tilde{A}(\theta)\delta_\ell, e^{-i\tilde{H}(\theta)t} \delta_k \rangle \right| \right) ^{1/2}.
\]

In the second inequality, we used the fact that \( \left| \left\langle \delta_n, e^{i\tilde{H}(\theta)t} \tilde{A}(\theta)e^{-i\tilde{H}(\theta)t} \delta_k \right\rangle \right| \leq 2 \). Moreover, \( \tilde{A}(\theta) \) is a self-adjoint operator of multiplication by \( 2 \sin(m \cdot \alpha + \theta) \), and therefore it can be removed from the last expression with an extra factor of 2. We will also use the bound \( \left| \langle \delta_k, e^{-i\tilde{H}(\theta)t} \delta_\ell \rangle \right| \leq 1 \) several times in the continued estimates:

\[
\leq 4 \sum_{|n|>N} \left( \sum_{\ell \in \mathbb{Z}^d} \int_T \left| \langle e^{-i\tilde{H}(\theta)t} \delta_n, \delta_\ell \rangle \langle \delta_\ell, e^{-i\tilde{H}(\theta)t} \delta_k \rangle \right| \right) ^{1/2}
\]

\[
\leq 4 \sum_{|n|>N} \sum_{\ell \in \mathbb{Z}^d} \left( \int_T \left| \langle e^{-i\tilde{H}(\theta)t} \delta_n, \delta_\ell \rangle \langle \delta_\ell, e^{-i\tilde{H}(\theta)t} \delta_k \rangle \right| \right)^{1/2}
\]

\[
\leq 4 \sum_{|n|>N} \sum_{\ell \in \mathbb{Z}^d} \left\{ \int_T \left| \langle e^{-i\tilde{H}(\theta)t} \delta_n, \delta_\ell \rangle \left| \langle e^{i\tilde{H}(\theta)t} \delta_\ell, \delta_k \rangle \right| \right|^{1/2} \right\}^{1/2}
\]

\[
\leq 4 \sum_{|n|>N} \sum_{\ell \in \mathbb{Z}^d} \left\{ \int_T \left| \langle \delta_\ell, e^{-i\tilde{H}(\theta)t} \delta_n \rangle \right| d\theta \int_T \left| \langle \delta_\ell, e^{-i\tilde{H}(\theta)t} \delta_k \rangle \right| d\theta \right\}^{1/4}
\]

\[
\leq C_1 \sum_{|n|>N} \sum_{\ell \in \mathbb{Z}^d} e^{-C_2|n-\ell|} e^{-|n-\ell-k|} \leq C_3 e^{-C_4|N-|k||}.
\]

The last inequalities follow from the EDL property. The proof for arbitrary \( \theta_k \) is similar: note that we immediately use the triangle inequality, after which one can change variable in the integrand for each \( n \) separately. \( \blacksquare \)

**Corollary 3.2.** Let \( \varphi_q(\theta) = e^{2\pi i q \theta} \), \( \Psi = \varphi_q \delta_k \in L^2_{\text{dual}}. \) Then \( \tilde{Q} \Psi \in L^2_{\text{dual}} \), and

\[
\| \tilde{Q}(T) \Psi - \tilde{Q} \Psi \|_{L^2_{\text{dual}}} \to 0 \quad \text{as} \quad T \to +\infty.
\]
Proof. First, let us note that $\tilde{Q}\Psi$ is well defined as an element of $L^2(\mathbb{T}; \ell^2(\mathbb{Z}^d))$, since $\tilde{Q}$ is a bounded operator. Now, from Lemma 3.1 (the factor $\varphi_q$ does not change (3.2)), assuming, say, $N > 2k$:

$$\|\tilde{Q}\Psi\|_{L^2_{\text{dual}}} \leq \|P_N\tilde{Q}\Psi\|_{L^2_{\text{dual}}} + \|(1-P_N)\tilde{Q}\Psi\|_{L^2_{\text{dual}}} \leq N^{1/2}\|\tilde{Q}\Psi\|_{L^2(\mathbb{T}; \ell^2(\mathbb{Z}^d))} + C_1(\Psi)e^{-C_2(\Psi)N} < +\infty.$$  

Similarly, one can prove second claim:

$$\|\tilde{Q}(T)\Psi - \tilde{Q}\Psi\|_{L^2_{\text{dual}}} \leq N^{1/2}\|\tilde{Q}(T)\Psi - \tilde{Q}\Psi\|_{L^2(\mathbb{T}; \ell^2(\mathbb{Z}^d))} + C_1(\Psi)e^{-C_2(\Psi)N}.$$  

The first term in the right hand side converges to zero, since Proposition 2.1 guarantees convergence in $L^2(\mathbb{T}; \ell^2(\mathbb{Z}))$.  

Proof of Theorem 2.4. Recall the definition:

$$Q(x, T) = \frac{1}{T} \int_0^T e^{iH(x)t} A e^{-iH(x)t} dt.$$  

Since $\|A\| \leq 2$, we have $\|Q(x, T)\|_{\ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})} \leq 2$. Hence, it would be sufficient to show

$$Q(x, T)\delta_p \to Q(x)\delta_p, \quad p \in \mathbb{Z}^d,$$  

for all basis elements and for almost every $x \in \mathbb{T}^d$. Let

$$w_T(x) = Q(x, T)\delta_p \in \ell^2(\mathbb{Z}), \quad w(x) = Q(x)\delta_p \in \ell^2(\mathbb{Z}),$$  

where we are assuming that $x$ belongs to the full measure set of $\mathbb{T}^d$ on which $Q(x)$ exists (as a fiber of $Q$). Let $w_T(x; n)$ denote the $n$th component of $w_T$, $n \in \mathbb{Z}$. Consider the Fourier transforms

$$\hat{w}_T(x; \theta) = \sum_{n \in \mathbb{Z}} e^{2\pi in\theta} w_T(x; n), \quad \hat{w}(x; \theta) = \sum_{n \in \mathbb{Z}} e^{2\pi in\theta} w(x; n).$$  

Perform also the inverse Fourier transform in the first argument, and denote the results by $\tilde{w}$:

$$\tilde{w}_T(m; \theta) = \int_{\mathbb{T}^d} \sum_{n \in \mathbb{Z}} e^{-2\pi im \cdot x} e^{2\pi in\theta} w_T(x; n) \, dx, \quad \tilde{w}(m; \theta) = \int_{\mathbb{T}^d} \sum_{n \in \mathbb{Z}} e^{-2\pi im \cdot x} e^{2\pi in\theta} w(x; n) \, dx.$$  

Here $m \in \mathbb{Z}^d$, $\theta \in \mathbb{T}$. In the next computation, sup denotes ess sup. We have

$$\text{(3.4)}$$  

$$\sup_{x \in \mathbb{T}^d} \|w_T(x) - w(x)\|_{\ell^2(\mathbb{Z})}^2 = \sup_x \sum_{n \in \mathbb{Z}} |w_T(x; n) - w(x; n)|^2 = \sup_x \int_{\mathbb{T}} \left| \hat{w}_T(x; \theta) - \hat{w}(x; \theta) \right|^2 \, d\theta$$  

$$\leq \int_{\mathbb{T}} \left( \sup_x \left| \hat{w}_T(x, \theta) - \hat{w}(x, \theta) \right| \right)^2 \, d\theta \leq \int_{\mathbb{T}} \left( \sum_{m \in \mathbb{Z}^d} \left| \tilde{w}_T(m, \theta) - \tilde{w}(m, \theta) \right| \right)^2 \, d\theta$$  

$$= \int_{\mathbb{T}} \left( \sum_m \left| (\mathcal{U}w_T)(\theta + m\alpha, m) - (\mathcal{U}w)(\theta + m\alpha, m) \right| \right)^2 \, d\theta$$  

$$= \|U_{\mathcal{U}w_T} - Uw\|_{L^2_{\text{dual}}},$$  

where $\mathcal{U}$ denotes the duality transformation (2.3) and the phases $\theta_n$ in the definition of $\tilde{L}^{21}$ are chosen in the form $\theta_n = m \cdot \alpha$. However,

$$(\mathcal{U}w)(\theta, m) = \tilde{Q}(\theta)e^{2\pi i\theta m \alpha} \delta_q(m), \quad (\mathcal{U}w_T)(\theta, m) = \tilde{Q}(\theta, T)e^{2\pi i\theta m \alpha} \delta_q(m).$$
Hence, the right hand side of (3.4) converges to zero from Corollary 3.2.

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Department of Mathematics, Michigan State University, Wells Hall, 619 Red Cedar Road, East Lansing, MI 48824, United States of America

E-mail address: ikachkov@msu.edu