Improved Global Guarantees for the Nonconvex Burer–Monteiro Factorization via Rank Overparameterization

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Abstract We consider minimizing a twice-differentiable, $L$-smooth, and $\mu$-strongly convex objective $\phi$ over an $n \times n$ positive semidefinite matrix $M \succeq 0$, under the assumption that the minimizer $M^*$ has low rank $r^* \ll n$. Following the Burer–Monteiro approach, we instead minimize the nonconvex objective $f(X) = \phi(XX^T)$ over a factor matrix $X$ of size $n \times r$. This substantially reduces the number of variables from $O(n^2)$ to as few as $O(n)$ and also enforces positive semidefiniteness for free, but at the cost of giving up the convexity of the original problem. In this paper, we prove that if the search rank $r \geq r^*$ is overparameterized by a constant factor with respect to the true rank $r^*$, namely as in $r > \frac{1}{4}(L/\mu - 1)^2 r^*$, then despite nonconvexity, local optimization is guaranteed to globally converge from any initial point to the global optimum. This significantly improves upon a previous rank overparameterization threshold of $r \geq n$, which we show is sharp in the absence of smoothness and strong convexity, but would increase the number of variables back up to $O(n^2)$. Conversely, without rank overparameterization, we prove that such a global guarantee is possible if and only if $\phi$ is almost perfectly conditioned, with a condition number of $L/\mu < 3$. Therefore, we conclude that a small amount of overparameterization can lead to large improvements in theoretical guarantees for the nonconvex Burer–Monteiro factorization.

1 Introduction

Consider minimizing a convex objective $\phi$ over an $n \times n$ positive semidefinite matrix $M \succeq 0$, as in

$$M^* = \text{minimize} \ \phi(M) \ \text{over} \ M \succeq 0, \quad (P)$$

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In practice, the matrix order $n$ is often so large—from tens of thousands to hundreds of millions—that even explicitly forming the $n^2$ elements of the $n \times n$ matrix variable $M$ would be intractable. Where $r^* = \text{rank}(M^*)$ is known a priori to be small, the standard approach is the nonconvex Burer–Monteiro factorization [6], which rewrites $M = XX^T$ and then directly optimizes over $X$ using a local optimization algorithm:

$$X^* = \minimize_{X} f(X) \overset{\text{def}}{=} \phi(XX^T) \quad \text{where } X \in \mathbb{R}^{n \times r} \text{ and } r \geq r^*. \quad \text{(BM)}$$

This way, the semidefinite constraint $M = XX^T \succeq 0$ is automatically enforced for free. Moreover, by using a small search rank $r \ll n$, the number of variables is reduced to $O(n)$. However, this improved scalability comes at the cost of giving up the convexity of (P). In principle, local optimization can fail by getting stuck at a spurious local minimum of (BM)—a local minimum that is strictly worse than that of the global minimum. In practice, however, there seems to be many situations where this failure mode does not occur [8, 31, 41].

In pursuit of a theoretical explanation for the empirical effectiveness of the Burer–Monteiro approach, it was discovered that if the convex function $\phi$ is sufficiently well-conditioned, then $f$ has no spurious local minima [2, 18, 21, 29, 44]. To state this precisely, suppose that $\phi$ is twice-differentiable, $\mathcal{L}$-smooth, and $\mu$-strongly convex over the entire set of $n \times n$ real symmetric matrices $\mathbb{S}^n$, as in

$$\mu \|E\|_F^2 \leq \left\langle \nabla^2 \phi(M)[E], E \right\rangle \leq \mathcal{L} \|E\|_F^2 \quad \text{for all } M \in \mathbb{S}^n,$$

where $\langle E, F \rangle \overset{\text{def}}{=} \text{tr}(E^T F)$ and $\|E\|_F \overset{\text{def}}{=} \sqrt{\langle E, E \rangle}$ denote the matrix Euclidean inner product and norm respectively. Bhojanapalli et al. [2] pointed out that if $\phi$ has condition number $\mathcal{L}/\mu < 3/2$, then the usual second-order necessary conditions for local optimality in $f$ are also sufficient for global optimality, meaning that

$$\nabla f(X) = 0, \quad \nabla^2 f(X) \succeq 0 \quad \iff \quad f(X) = \min_U f(U).$$

Moreover, satisfying these conditions to $\epsilon$-accuracy will yield a point within a $\rho$-neighborhood of a globally optimal solution. Where this guarantee holds, local optimization cannot fail by getting stuck, because every local minimum is also a global minimum, and every saddle-point has sufficiently negative curvature to allow escape. (See also [18, 21, 29, 44] for improvements and extensions on this result.)

While powerful, this global guarantee is unfortunately also very conservative. Practical choices of $\phi$ can be assumed to have a finite condition number $\mathcal{L}/\mu$, because they tend to be $\mathcal{L}$-smooth by formulation, and $\mu$-strong convexity can be made to hold by adding a small regularizer, as in $\phi_{\mu}(M) = \phi(M) + \frac{\mu}{2} \|M\|_F^2$. But adding a regularizer would also worsen the accuracy and quality of the solution, so it is unrealistic to assume condition numbers as small as $\mathcal{L}/\mu < 3/2$. On the other hand, Zhang et al. [40] stated a counterexample $\phi$ with $\mathcal{L}/\mu = 3$ but whose $f$ admits a spurious second-order point (see also [21, 42, 44]). For more realistic condition numbers $\mathcal{L}/\mu \geq 3$, the Burer–Monteiro approach may continue to work well in practice, but one can no longer rule out failure by getting stuck at a spurious local minimum.
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Fig. 1 Overparameterization eliminates spurious local minima. Stochastic gradient descent (SGD) with Nesterov momentum [33] applied to an $f(X) \overset{\text{def}}{=} \phi(XX^T)$ with a spurious second-order point $X_{\text{spur}}$ for $r = 3$: (Left) With search rank $r = 3$, GD remains stuck at $X \approx X_{\text{spur}}$, resulting in 55 failures out of 100 trials. (Right) Overparameterizing to $r = 4$ eliminates $X_{\text{spur}}$ as a spurious second-order point, and GD now succeeds in all 100 trials.

1.1 Main result

Surprisingly, we show in this paper that overparameterizing the search rank $r > r^*$ can improve existing global guarantees to all finite values of the condition number $L/\mu$—far beyond the apparently fundamental barrier of $L/\mu < 3$. We are inspired by parallel work on the Burer–Monteiro approach for semidefinite programming (see our literature review in Section 1.3.1), where it was discovered that progressively overparameterizing the search rank $r$ makes the nonconvexity increasingly benign. The same empirical observation is easily replicated in our unconstrained setting: for a fixed unconstrained convex $\phi$, the corresponding nonconvex $f$ admits progressively fewer spurious local minima as its search rank $r$ is progressively increased past $r^*$ (see Fig. 1). Once the overparameterization ratio $r/r^*$ exceeds some constant threshold, local optimization consistently succeeds at globally minimizing $f$.

Our main result rigorously justifies the above empirical observation: so long as the convex function $\phi$ has a bounded condition number $L/\mu = O(1)$ with respect to $n$, the corresponding nonconvex function $f$ is guaranteed to have no spurious local minima as its search rank $r$ is progressively increased past $r^*$ (see Theorem 1.1). To the best of our knowledge, this is the first rigorous proof that a constant-factor overparameterization ratio $r/r^*$ can eliminate spurious local minima.

**Theorem 1.1 (Overparameterization)** Let $\phi : \mathbb{S}^n \to \mathbb{R}$ be twice-differentiable, $L$-smooth and $\mu$-strongly convex, let the minimizer $M^* = \arg\min_{M \succeq 0} \phi(M)$ have true rank $r^* = \text{rank}(M^*)$.

- (Sufficiency) If $r > \frac{1}{4}(L/\mu - 1)^2 r^*$ and $r^* \leq r < n$, then the Burer–Monteiro function $f : \mathbb{R}^{n \times r} \to \mathbb{R}$ defined as $f(U) \overset{\text{def}}{=} \phi(UU^T)$ has no spurious local minima:
  \[ \nabla f(X) = 0, \quad \nabla^2 f(X) \succeq 0 \quad \iff \quad f(X) = \min_U f(U). \]
Once the search rank is large enough to satisfy $r \geq \frac{1}{4}(L/\mu - 1)^2 r^*$ and $r^* \leq r < n$, then there exists an $L$-smooth and $\mu$-strongly convex quadratic counterexample $\phi_0$ whose minimizer $M_0^\star = \arg\min_{M \succeq 0} \phi_0(M)$ has rank($M_0^\star$) = $r^*$, but whose $f_0(U) \overset{\text{def}}{=} \phi_0(UU^T)$ admits an $n \times r$ spurious second-order point $X_{\text{spur}}$:

$$\nabla f_0(X_{\text{spur}}) = 0, \quad \nabla^2 f_0(X_{\text{spur}}) \succeq 0, \quad f_0(X_{\text{spur}}) - \min_U f_0(U) > \frac{\mu}{2} \cdot \|M_0^\star\|^2_F.$$ 

Once the search rank is large enough to satisfy $r \geq n$, one can easily adapt existing results to prove that $f$ has no spurious local minima; see Journèe et al. [20, Corollary 8] and Boumal et al. [5, Corollary 3.2]. Note that this guarantee does not require $\phi$ to be strongly convex, nor uniformly $L$-smooth over its entire domain. Of course, setting $r \geq n$ would also force us to optimize over $O(n^2)$ matrix elements in $X$, thereby obviating the computational advantages of the Buré–Monteiro factorization in the first place.

In fact, the necessary condition in Theorem 1.1 shows that the overparameterization rank threshold $r \geq n$ is sharp. Without smoothness and strong convexity, it is inherently impossible to make a global guarantee for a search rank $r < n$, due to existence of a counterexample (Example 6.2). The sufficient condition in Theorem 1.1 is able to improve upon the sharp threshold of $r \geq n$ only because it further keeps the condition number $L/\mu$ bounded. Our key insight is that highly overparameterized counterexamples $\phi$ do exist, but their condition numbers $L/\mu$ must necessarily diverge to infinity. If instead the condition number $L/\mu$ can be kept constant—which is indeed the case if the convex function $\phi$ is held fixed—then we can progressively increase the overparameterization ratio $r/r^*$, and use the inexistence of a well-conditioned overparameterized counterexample to prove that $f$ has no spurious local minima.

Without explicitly accounting for overparameterization, the best global guarantee that we can make is for almost-perfect choices of $\phi$ with conditions numbers $L/\mu < 3$. Indeed, we obtain the following by allowing the true rank $r^*$ in Theorem 1.1 to take on all values smaller than the search rank $r$ (i.e. by substituting the upper-bound $r^* \leq r$).

It provides an affirmative answer to a conjecture first posed in Zhang et al. [40] that is widely believed to be true in the existing literature [21, 42, 44].

**Corollary 1.2 (Exact parameterization)** Let $\phi : S^0 \to \mathbb{R}$ be twice-differentiable, $L$-smooth and $\mu$-strongly convex, and let the search rank $r$ satisfy $r \geq \text{rank}(M^\star)$ where $M^\star = \arg\min_{M \succeq 0} \phi(M)$.

- If $L/\mu < 3$, then $f(U) \overset{\text{def}}{=} \phi(UU^T)$ for $U \in \mathbb{R}^{n \times r}$ has no spurious local minima.
- If $L/\mu \geq 3$, then nothing can be said due to the existence of a counterexample.

Comparing Theorem 1.1 and Corollary 1.2, we conclude that a small amount of overparameterization can lead to large improvements in global guarantees based on the condition number $L/\mu$. As an important future work, we expect that explicitly accounting for rank overparameterization should also greatly improve other theoretical guarantees, including local guarantees based on a specific initial point and choice of algorithm [37, 42], and also application-specific global guarantees, based on narrower structures like incoherence [17, 18, 23], low-rank measurements [8, 10, 11], and structured sparsity [26]. In this regard, an important contribution of this paper is to provide a set of tools to reason about the inexistence of overparameterized counterexamples.

Finally, the global guarantee in Theorem 1.1—though improved—is unfortunately still quite conservative. For example, with a condition number of $L/\mu \approx 200$, the overparameterization ratio $r/r^* = \frac{1}{4}(L/\mu - 1)^2 \approx 10^3$ required by Theorem 1.1 would
be far too large for practical use. On the other hand, this conservatism appears fundamental; Theorem 1.1 is already sharp, and as we explain in Section 1.3.2 below, cannot be improved by adopting the rank-restricted versions of $\mu$-strong convexity and $L$-smoothness. As explained earlier, it is generally undesirable to use a regularizer to improve the condition number, as it would also worsen accuracy. A better approach is to adopt a preconditioner, as is typically done to improve the convergence rate of algorithms, but this is not a general-purpose approach and would require further investigations. It also remains an important open question whether there exists even stronger structures than $L$-smoothness and $\mu$-strong convexity, that can better control the benign nonconvexity of the Burer–Monteiro factorization.

1.2 Algorithmic implications

While Theorem 1.1 guarantees global optimality for a point $X$ that exactly satisfies the second-order conditions for local optimality, as in $\nabla f(X) = 0$ and $\nabla^2 f(X) \succeq 0$, practical algorithms like gradient descent [19] and trust-region methods [9, 27] are only able to compute a point $\tilde{X}$ that approximately satisfies these conditions up to some prescribed accuracy $\epsilon > 0$, as in $\|\nabla f(\tilde{X})\|_F \leq \epsilon$ and $\nabla^2 f(\tilde{X}) \succeq -\epsilon I$. In order to guarantee global convergence towards global optimality, we additionally require the strict saddle property, which says that every $\epsilon$ approximate second-order point $\tilde{X}$ is also guaranteed to be $\rho$-close to an exact second-order point $X$ [16, 18, 19]. Fortunately within our setting, this strengthened property comes essentially for free.

Proposition 1.3 (Strict saddle property) Let $\phi : S^n \to \mathbb{R}$ be twice-differentiable, $L$-smooth and $\mu$-strongly convex, and let the search rank $r$ satisfy $r \geq \text{rank}(M^*)$ where $M^* = \arg \min_{M \succeq 0} \phi(M)$. If the Burer–Monteiro function $f(U) \overset{\text{def}}{=} \phi(UU^T)$ has no spurious local minima

$$\nabla f(X) = 0, \quad \nabla^2 f(X) \succeq 0 \quad \iff \quad f(X) = \min_U f(U)$$

then $f$ also satisfies the strict saddle property:

$$\|\nabla f(\tilde{X})\|_F \leq \epsilon(\delta), \quad \nabla^2 f(\tilde{X}) \succeq -\epsilon(\delta)I \quad \implies \quad f(X) - \min_U f(U) < \delta$$

where $\epsilon$ is a nondecreasing function that satisfies $0 < \epsilon(\delta) \leq 1$ for all $\delta > 0$.

Note that Proposition 1.3 promises only global convergence towards global optimality in finite time, and not necessarily in reasonable time. Nevertheless, once convergence can be rigorously established, we expect that a convergence rate should readily follow with a more careful analysis. In the existing literature, once it became clear that $f$ has no spurious local minima with $L/\mu < 3/2$, subsequent refinements quickly showed that every $\epsilon$ approximate second-order point $\tilde{X}$ is guaranteed to be $O(\epsilon)$-close to an exact second-order point, and therefore at most $O(\epsilon^2)$-globally suboptimal [18, 21, 44]. With exact rank parameterization $r = r^*$, gradient descent globally converges at a linear rate to a $\delta$-globally suboptimal point $\tilde{X}$ satisfying $f(\tilde{X}) - \min_U f(U) < \delta$ in $O(\log(1/\delta))$ iterations, as if $f$ were strongly convex [18, 19]. With rank over-parameterization $r > r^*$, gradient descent slows down to a sublinear rate; instead, Zhang et al. [36, 38] recently proposed a preconditioned gradient descent algorithm and proved that it restores global linear convergence. We expect that all of these
results can be generalized to the overparameterized regime of $r/r^* > \frac{1}{4}(L/\mu - 1)^2$, by suitably extending the proof of Theorem 1.1 to approximate second-order points. We highlight this as another important future direction.

1.3 Related work

1.3.1 Rank overparameterization for semidefinite programs

Rank overparameterization for the Burer–Monteiro factorization is currently best understood in the context of solving semidefinite programs

$$\minimize \quad \langle C, XX^T \rangle \quad \text{subject to} \quad \mathcal{A}(XX^T) = b, \quad X \text{ is } n \times r, \tag{SDP}$$

in which $\mathcal{A} : \mathbb{S}^n \to \mathbb{R}^m$ models a set of $m < \frac{1}{4}n(n+1)$ linear constraints with right-hand side $b \in \mathbb{R}^m$, and $C \in \mathbb{S}^n$ models a linear cost. The critical insight (which also applies to our setting) is that a spurious local minimum at search rank $r$ must correspond to a strict saddle-point at search rank $r + 1$ that can be escaped [4, 5, 20]. The algorithm obtained by incrementally overparameterizing the search rank $r$ this way—known as the staircase method [4, 5]—often terminates at $r \approx 10$ with the global solution [13, 31].

To provide a correctness guarantee, Journée et al. [20, Corollary 8] proved that (SDP) has no spurious local minima if $r \geq n$, so the staircase method must terminate. Boumal et al. [4, 5] improved this threshold to $r > \sqrt{2m + \frac{3}{2} - \frac{1}{2}}$ for a generic instance of (SDP), which Waldspurger and Waters [35] showed is essentially sharp due to the existence of counterexamples. More recently, O’Carroll et al. [28] established, in the non-generic worst case, that the threshold $r \geq n$ is essentially sharp.

Although (SDP) and (BM) consider two very different settings, a connection between them can be established through the augmented Lagrangian

$$f_\beta(X) \overset{\text{def}}{=} \phi_\beta(XX^T, y), \quad \phi_\beta(M; y) \overset{\text{def}}{=} \left\langle C - \mathcal{A}^T(y), M \right\rangle + \frac{\beta}{2} \|A(M) - b\|^2.$$

In one direction, the original papers of Burer and Monteiro [6, 7] proposed to solve (SDP) via this particular instance of (BM). However, $f_\beta$ can never be strongly convex, so the best global guarantee for $f_\beta$ is the same rank threshold $r \geq n$, which is also sharp within our context via Theorem 1.1. Imposing a strongly convex regularizer, as in $f_{\beta,\mu}(X) \overset{\text{def}}{=} f_\beta(X) + \frac{\mu}{2} \|X^TX\|^2_F$, could potentially allow global guarantees for $r \ll n$ via Theorem 1.1 to be eventually extended to (SDP), but this could again come at the cost of introducing substantial errors.

In reverse, the counterexamples originally formulated for (SDP) can be adapted to (BM) under second-order sufficiency conditions. The following lemma establishes this classical connection (see e.g. [1, Section 2.2.2]) while quotienting out the rotational invariance $f_\beta(X) = f_\beta(XR)$ for $RR^T = I_r$.

**Lemma 1.4** Let $\mathcal{V} = \{V \in \mathbb{R}^{n \times r} : XV^T + VX^T \neq 0\}$. If $X \in \mathbb{R}^{n \times r}, y \in \mathbb{R}^m$ satisfy the second-order sufficient conditions for local optimality in (SDP):

$$\begin{align*}
[C - \mathcal{A}^T(y)]X &= 0, \quad \mathcal{A}(XX^T) = b, \tag{1.1a} \\
V \in \mathcal{V}, \quad \mathcal{A}(XX^T + VX^T) &= 0 \implies \left\langle C - \mathcal{A}^T(y), VV^T \right\rangle > 0, \tag{1.1b}
\end{align*}$$
then there exists $\beta \geq 0$ such that $X$ satisfies $\nabla f_\beta(X) = 0$, $\nabla^2 f_\beta(X) \succeq 0$, and $\langle \nabla^2 f_\beta(X)[V], V \rangle > 0$ for all $V \in \mathcal{V}$.

Proof: Explicitly write $\nabla f_\beta(X) = 0$ and $\langle \nabla^2 f_\beta(X)[V], V \rangle > 0$ for $V \in \mathcal{V}$ as

$$
[C - A^T(y_+)]X = 0 \quad \text{where } y_+ = y + \beta[b - A(XX^T)],
$$

$$(1.2a)

\langle C - A^T(y_+), VV^T \rangle + \frac{\beta}{2} \|A(XV^T + VX^T)\|^2 > 0 \quad \text{for all } V \in \mathcal{V}.
$$

Clearly, (1.1a) implies (1.2a) because $y_+ = y$. Observe that $\mathcal{V}$ is a linear subspace; we evoke the homogeneous nonstrict version of the S-lemma [30, Proposition 3.2] to conclude that there exists no $V \in \mathcal{V}$ that jointly satisfies $\|A(XV^T + VX^T)\|^2 \leq 0$ and $\langle C - A^T(y), VV^T \rangle \leq 0$ if and only if (1.2b) holds for some $\beta \geq 0$. Finally, any $U \notin \mathcal{V}$ can be written as $U = XA$ for $A = -A^T$, and in this case, $\nabla^2 f_\beta(X)[U] = [C - A^T(y_+)]XA = 0$ via (1.2a). Therefore, we conclude that $\nabla^2 f_\beta(X) \succeq 0$ holds.

The second-order sufficient conditions are satisfied by the counterexamples of Waldspurger and Waters [35] for the MAXCUT problem (where it is called nondegeneracy; see [35, Definition 2.7]), so applying Lemma 1.4 results in a counterexample $\phi$ with true rank $r^* = 1$ but a spurious second-order point at $r \geq \sqrt{2n}$. The counterexamples of O’Carroll et al. [28] do not satisfy these conditions, but nevertheless hint at the existence of a counterexample $\phi$ with $r^* = 1$ and $r = \Omega(n)$. Bhaskara et al. [3, Theorem 5] stated a counterexample $\phi$ with $r^* = 1$ and $r = n - 1$ that fills this gap. Finally, Theorem 1.1 states the optimal counterexample $\phi^* \succeq 1$ and $r = n - 1$, and whose finite condition number $L/\mu = 2\sqrt{n - 1} + 1$ is in fact as small as possible.

### 1.3.2 Matrix sensing

Our results are closely related to a problem known in the literature as matrix sensing, which considers a nonlinear least-squares objective like $f(X) = 1/2 \|A(XX^T - M^*)\|^2$, in which the measurement operator $A$ is assumed to satisfy the $(\delta, r + r^*)$-restricted isometry property or $(\delta, r + r^*)$-RIP, with constant $\delta \in [0, 1]$:

$$(1 - \delta)\|E\|^2_F \leq \|A(E)\|^2 \leq (1 + \delta)\|E\|^2_F \quad \text{for all } \text{rank}(E) \leq r + r^*.$$

The matrix sensing problem has drawn considerable interest because it gives some of the most convincing and unambiguous demonstration of a nonconvex problem with a “benign landscape” [2, 12, 18]. In practice, applying simple gradient descent to $f$ consistently results in rapid linear convergence to the global optimum, as if it were strongly convex [19, 34, 43]. Rigorously, it is known that $\delta < 1/5$ is sufficient [2, 18, 21, 29, 44] and $\delta < 1/2$ is necessary [21, 40, 44] for the nonconvex function $f$ to satisfy the strict saddle property and have no spurious local minima.

A twice-differentiable, $L$-smooth, and $\mu$-strongly convex function $\phi$ by definition satisfies $(\delta, r + r^*)$-RIP with $\delta = L/\mu + 1$, so existing results can be adapted to imply that a condition number of $L/\mu < 3/2$ is sufficient and $L/\mu < 3$ is necessary for spurious local minima not to exist. However, note that RIP is a more general assumption than smoothness and strong convexity, so our global guarantee in Theorem 1.1 does not imply a similar result under RIP. Fortunately, its proof can be strengthened to cover this more general case, although it does not come with any substantial improvements.
Corollary 1.5 (Restricted isometry property) Let $M^* \succeq 0$ have true rank $r^* = \text{rank}(M^*)$ and let $A : \mathbb{S}^n \to \mathbb{R}^m$ satisfy $(\delta, r + r^*)$-RIP.

- (Sufficiency) If $\delta < 1/(1 + \sqrt{r^*/r})$, then $f(X) \overset{\Delta}{=} \|A(XX^T - M^*)\|_2^2$ has no spurious local minima.
- (Necessity) If $\delta \geq 1/(1 + \sqrt{r^*/r})$, then there exists a counterexample $A_0$ that satisfies $(\delta, r + r^*)$-RIP, but whose $f_0(X) \overset{\Delta}{=} \|A_0(XX^T - M^*)\|_2^2$ admits a spurious second-order point.

Proof Repeat the proof of Theorem 1.1 to minimize the function $\delta(X, Z)$ as posed in [42, Theorem 8]. We defer the specific details of the sufficient condition to the technical report [39], and prove necessity using Example 6.2 in Section 6.

Without overparameterization, an RIP constant of $\delta < 1/2$ is both necessary and sufficient for global recovery. It is in fact possible to prove global guarantees for RIP with all values of $\delta < 1$ by incorporating the overparameterization ratio $r/r^*$ as a “problem structure”. However, from the necessary condition, we see that the rank parameter of the RIP assumption must also increase. Here, we mention that rank overparameterization has only been recently studied for matrix sensing [22, 45]; its impact on the optimization landscape had been previously unknown.

1.3.3 Low-rank matrix recovery

Our work is more distantly related to the more general problem of low-rank matrix recovery, of which matrix sensing is a particular variant; see e.g. [12] and the references therein. The more general problem seeks to recover an unknown $M^* \succeq 0$ of low-rank $r^* \ll n$ by minimizing an objective like $f(X) = \frac{1}{2}\|A(XX^T - b\|_2^2$ given possibly noisy measurements $b = A(M^*) + \varepsilon$, but the choice of measurement operator $A$ usually does not satisfy $(\delta, r + r^*)$-RIP. For certain instances like matrix completion and robust PCA [18], the standard approach is to design a regularized objective $f_R(X) = f(X) + R(X)$ whose global minimizer $X^*_R = \arg\min f_R(X)$ is guaranteed to recover $M^*$ up to a statistical error bound. In theory, if the regularized objective can be written as $f_R(X) = \phi_R(XX^T)$ with respect to some $\phi_R$ that satisfies $(\delta, r + r^*)$-RIP, then our results in this paper could be applied to guarantee global optimality, and hence global recovery. Other instances like phase retrieval are designed to work without a regularizer; without $(\delta, r + r^*)$-RIP, our results are not applicable to these cases.

Recent works have studied overparameterization in the context of robust low-rank matrix recovery [14, 24], wherein the loss function is typically a nonsmooth $\ell_1$-based norm like $f(X) = \|A(XX^T - b\|_{\ell_1} = \sum_i |\langle A_i, XX^T \rangle - b_i|$. A reviewer points out that, in the nonsmooth setting, a standard measure of conditioning is the ratio of the Lipschitz constant over the modulus of sharp growth. It is an important future direction to see if the proof technique presented in this paper could be extended to the nonsmooth setting.

2 Notations

Basic linear algebra. Lower-case letters are vectors and upper-case letters are matrices. We use “MATLAB notation” in concatenating vectors and matrices:
The sufficient condition follows because there exists no improved global guarantees via rank overparameterization. We will infer their dimensions from context. Denote \( \vec(\cdot) \) as the usual column-stacking vectorizing operator, and \( \mathbf{1} \) as the projection onto the positive orthant. Denote \( \mathbf{1} = [1, 1, \ldots, 1]^T \) as the vector-of-ones and \( I = \text{diag}(\mathbf{1}) \) as the identity matrix; we will infer their dimensions from context. Denote \( \langle X, Y \rangle = \text{tr}(X^TY) \) and \( \|X\|_F^2 = \langle X, X \rangle \) as the Frobenius inner product and norm. Denote \( \text{nnz}(x) \) as the number of nonzero elements in \( x \). The sets \( S^n \supset S^n_+ \) are the \( n \times n \) real symmetric matrices, and the corresponding positive semidefinite cone.

\textbf{Positive cones and projections.} The sets \( \mathbb{R}^n \supset \mathbb{R}^n_+ \) are the \( n \) vectors, and the corresponding positive orthant. Denote \( x_+ = \max\{0, x\} = \arg\min\{\|x - y\| : y \in \mathbb{R}^n_+\} \) as the projection onto the positive orthant.

\textbf{Vectorization and Kronecker product.} Denote \( \vec(X) \) as the usual column-stacking vectorizing operator, and \( \text{mat}(x) \) as its inverse (the dimensions of the matrix are inferred from context). The Kronecker product \( \otimes \) is defined to satisfy the identity
\[
\vec(AXB^T) = (B \otimes A)\vec(X).
\]

\textbf{Pseudoinverse.} Denote the (Moore–Penrose) pseudoinverse \( A^+ \overset{\text{def}}{=} VS^{-1}U^T \) of matrix \( A = USV^T \) where \( U^TU = V^TV = I_r \) and \( S \succ 0 \). Define \( 0^+ \overset{\text{def}}{=} 0 \).

\textbf{Error vector \( e \) and Jacobian matrix \( J_X \).} Given fixed \( X \in \mathbb{R}^{n \times r} \) and \( Z \in \mathbb{R}^{n \times r} \), we denote \( e = \vec(XX^T - ZZ^T) \) and implicitly define \( J_X \in \mathbb{R}^{n \times n \times r} \) to satisfy
\[
J_X \vec(V) = \vec(XV^T + YX^T).
\]

Note that the associate adjoint operator satisfies \( J_X^\dagger \vec(M) = \vec((M + M^T)X) \).

3 Proof of the main result

We will establish “no spurious local minima” guarantees by showing that there cannot exist a counterexample that is simultaneously highly overparameterized, with a large value of \( r/r^* \), and also well-conditioned, with a small value of \( \kappa \).

\textbf{Definition 3.1} The function \( \phi : S^n \to \mathbb{R} \) is said to be a \((\kappa, r, r^*)\)-counterexample if it satisfies the following:

\begin{itemize}
  \item The function \( \phi \) is twice-differentiable, \( L \)-smooth, \( \mu \)-strongly convex, and \( L/\mu \leq \kappa \).
  \item The minimizer \( M^* = \arg\min_{M \succeq 0} \phi(M) \) satisfies \( \text{rank}(M^*) = r^* \leq r \).
  \item There exists an \( n \times r \) spurious point \( X \) with \( XX^T \neq M^* \) that satisfies \( \nabla f(X) = 0 \) and \( \nabla^2 f(X) \succeq 0 \) for \( f(U) \overset{\text{def}}{=} \phi(UU^T) \).
\end{itemize}

For a fixed \( r \) and \( r^* \), our central claim in Theorem 1.1 is that the condition number \( \kappa^* \) of the best-conditioned counterexample is exactly determined as
\[
\kappa^* = \inf_{\phi : S^n \to \mathbb{R}} \{ \kappa : \phi \text{ is a } (\kappa, r, r^*)\text{-counterexample} \} = 1 + 2\sqrt{r/r^*}.
\]

The sufficient condition follows because there exists no \((\kappa, r, r^*)\)-counterexample with \( \kappa < 1 + 2\sqrt{r/r^*} \). And so for a fixed \( \kappa \), we can always overparameterize \( r/r^* > \frac{1}{4}(\kappa - \ldots
1)^2 to eliminate all counterexamples. The neccessary condition holds because a best-conditioned \((\kappa^*, r, r^*)\)-counterexample does exist.

To prove that \(\kappa^*\) is equal to the value stated in (3.1), we proceed by reformulating (3.1) into two-stage optimization problem

\[
\kappa^* = \inf_{X, Z \in \mathbb{R}^{n \times r}} \{\kappa(X, Z) : \text{rank}(Z) = r^*, XX^T \neq ZZ^T\}
\]

in which \(\kappa(X, Z)\) corresponds to the best-conditioned counterexample associated with a fixed minimizer \(ZZ^T\) and a fixed spurious point \(X\):

\[
\kappa(X, Z) = \inf_{\phi : \mathbb{S}^n \to \mathbb{R}} \left\{ \phi : ZZ^T = \arg \min_{M \succeq 0} \phi(M), \nabla f(X) = 0, \nabla^2 f(X) \succeq 0 \text{ where } f(U) \equiv \phi(UU^T). \right\} \tag{3.2}
\]

This two-stage reformulation is inspired by Zhang et al. [40, 42], who showed for a fixed \(X, Z\) that finding the best-conditioned quadratic counterexample \(\phi\) that satisfies \(\nabla \phi(ZZ^T) = 0\) can be exactly posed as a standard-form semidefinite program or SDP. Our proof extends this SDP formulation to a nonquadratic \(\phi\) with a possibly nonzero \(\nabla \phi(ZZ^T) \succeq 0\) at optimality by introducing a slack variable.

**Lemma 3.2 (SDP formulation)** Define \(\kappa(X, Z)\) as in (3.2). We have \(\kappa_{\text{ub}}(X, Z) \geq \kappa(X, Z) \geq \kappa_{\text{lb}}(X, Z)\) where

\[
\kappa_{\text{ub}}(X, Z) \equiv \min_{\kappa, H} \left\{ \kappa : I \preceq H \preceq \kappa I, J_X^T H e = 0, -2I_r \otimes \text{mat}(He) \preceq J_X^T H J_X \right\}
\]

\[
\kappa_{\text{lb}}(X, Z) \equiv \min_{\kappa, H} \left\{ \kappa : \begin{array}{l}
I \preceq H \preceq \kappa I,
\text{mat}(s) \succeq 0,
J_Z^T s = 0,
\end{array}
\begin{array}{l}
J_X^T (He + s) = 0,
-2I_r \otimes \text{mat}(He + s) \preceq \kappa J_X^T J_X
\end{array} \right\}
\]

and \(e = \text{vec}(XX^T - ZZ^T)\) and \(J_U \text{vec}(V) = \text{vec}(UV^T + VU^T)\) for all \(V\).

**Proof** Denote \(S_M\) as the convex gradient \(\nabla \phi(M)\) evaluated at \(M\), and denote \(H_M\) as the matrix representation of the convex Hessian operator \(\nabla^2 \phi(M)\) evaluated at \(M\). It follows from \((\nabla^2 \phi(M))[E, E] = \text{vec}(E)^T H_M \text{vec}(E)\) that:

\[
\phi \text{ is 1-strongly convex and } \kappa \text{-smooth} \iff I \preceq H_M \preceq \kappa I \quad \text{for all } M \in \mathbb{S}^n. \tag{3.3}
\]

Additionally, the Karush–Kuhn–Tucker conditions at \(M^* = ZZ^T\) read:

\[
ZZ^T = \arg \min_{M \succeq 0} \phi(M) \iff S_{ZZ^T} \succeq 0, \quad S_{ZZ^T} Z = 0. \tag{3.4}
\]

Note that strong duality holds in the above because \(ZZ^T + \epsilon I\) is a strictly feasible point with a bounded objective due to \(\kappa\)-smoothness. Next, we observe that the two gradient evaluations \(S_{XX^T}\) and \(S_{ZZ^T}\) are related via the fundamental theorem of calculus:

\[
\nabla \phi(XX^T) = \nabla \phi(ZZ^T) + \int_0^1 \nabla^2 \phi((1-t)ZZ^T + txX^T)[XX^T - ZZ^T] \, dt
\]

\[
\iff \text{vec}(S_{XX^T}) = \text{vec}(S_{ZZ^T}) + \left[ \int_0^1 H_{M(t)} \, dt \right] \text{vec}(XX^T - ZZ^T).
\]
Denoting \( s = \text{vec}(S_{ZZ^T}) \) and \( H_{av} = \int_0^1 H_M(t) \, dt \) and \( e = \text{vec}(XX^T - ZZ^T) \) allows us to rewrite the directional derivatives of \( f(U) \) as follows:

\[
\langle \nabla f(X), V \rangle = \langle \text{vec}(V)^T J^T_X(H_{av}e + s), \nabla^2 f(X)[V], V \rangle = \langle \text{vec}(V)^T [2I_r \otimes \text{mat}(H_{av}e + s) + J^T_X H_{XX^T}J_X] \text{vec}(V) \rangle.
\]

(3.5)

Substituting (3.3), (3.5), and (3.4) into (3.2) yields exactly

\[
\kappa(X, Z) = \inf_{\kappa, \phi, S^n \to \mathbb{R}} \left\{ \kappa : \begin{array}{l}
I \preceq H_M \preceq \kappa I \\
\text{for all } M \in S^n,
\end{array}
\quad \begin{array}{l}
\kappa : \text{mat}(s) \geq 0, \\
J^T_X s = 0, \\
J^T_X (H_{av}e + s) = 0,
\end{array}
\quad -2I_r \otimes \text{mat}(H_{av}e + s) \preceq J^T_X H_{XX^T}J_X. \right\}
\]

(3.6)

We obtain the upper-bound \( \kappa_{ub}(X, Z) \) from \( \kappa(X, Z) \) in (3.6) by fixing \( H_M = H \) for all \( M \) and \( S_{ZZ^T} \equiv \text{mat}(s) = 0 \). We obtain the lower-bound \( \kappa_{lb}(X, Z) \) from \( \kappa(X, Z) \) in (3.6) by substituting the relaxation

\[
-2I_r \otimes \text{mat}(H_{av}e + s) \preceq J^T_X H_{XX^T}J_X \preceq \kappa \cdot J^T_XJ_X,
\]

in order to avoid explicitly optimizing over \( H_{XX^T} \):

\[
\kappa(X, Z) \geq \inf_{\kappa, \phi, S^n \to \mathbb{R}} \left\{ \kappa : \begin{array}{l}
I \preceq H_M \preceq \kappa I \\
\text{for all } M \in S^n,
\end{array}
\quad \begin{array}{l}
\kappa : \text{mat}(s) \geq 0, \\
J^T_X s = 0, \\
J^T_X (H_{av}e + s) = 0,
\end{array}
\quad -2I_r \otimes \text{mat}(H_{av}e + s) \preceq \kappa \cdot J^T_XJ_X. \right\}
\]

(3.7)

Finally, we impose \( I \preceq H_M \preceq \kappa I \) over the average Hessian \( H_{av} = \int_0^1 H_M(t) \, dt \) and then relax this constraint over \( H_M \) for all other values of \( M \).

Remark 3.3 We derived \( \kappa_{ub}(X, Z) \) by restricting the candidate counterexamples \( \phi \) in (3.2) to quadratics like \( \phi(M) = \frac{1}{2} \text{vec}(M - M^*)^T H \text{vec}(M - M^*) \). Note that we also set \( \nabla \phi(M^*) = 0 \) to avoid having to optimize over it.

Remark 3.4 We derived \( \kappa_{lb}(X, Z) \) by substituting the following relaxation, which is implied by the \( L \)-smoothness of \( \phi \) into (3.2):

\[
\langle \nabla^2 f(X)[V], V \rangle \geq 0 \implies 2 \langle \nabla \phi(XX^T), VV^T \rangle + L \cdot \|XX^T + VV^T\|^2_F \geq 0.
\]

After making this relaxation, the best-conditioned \( \phi \) turns out to be a quadratic like \( \phi(M) = s^T \text{vec}(M - M^*) + \frac{1}{2} \text{vec}(M - M^*)^T H \text{vec}(M - M^*) \). However, \( \phi \) might not be a valid counterexample; due to the relaxation, it is not necessarily feasible for (3.2).

We prove \( \kappa^* \leq 1 + 2\sqrt{r/r^*} \) by stating an explicit choice of \( X \) and \( Z \) that heuristically minimizes the upper-bound \( \kappa_{ub}(X, Z) \) in Lemma 3.2 over \( n \times r \) matrices \( X \) and full-rank \( n \times r^* \) matrices \( Z \). We defer its proof to Section 6.

Lemma 3.5 (Heuristic upper-bound) Let \( [Q_1, Q_2] \) have orthonormal columns with \( Q_1 \in \mathbb{R}^{n \times r} \) and \( Q_2 \in \mathbb{R}^{n \times r^*} \). Then, we have

\[
\kappa_{ub}(X, Z) \leq 1 + 2\sqrt{r/r^*} \quad \text{where} \quad X = Q_1, \quad Z = \sqrt{1 + \sqrt{r/r^*}Q_2}.
\]

We prove \( \kappa^* \geq 1 + 2\sqrt{r/r^*} \) by deriving a closed-form lower-bound on \( \kappa_{lb}(X, Z) \) in Lemma 3.2, and then analytically minimizing it over \( n \times r \) matrices \( X \) and full-rank \( n \times r^* \) matrices \( Z \). We defer the derivation of this lower-bound to Section 5.
Lemma 3.6 (Closed-form lower-bound) For $X, Z \in \mathbb{R}^{n \times r}$ such that $XX^T \neq ZZ^T$, let $Z_\perp = (I - XX^T)Z$. We have

$$\kappa_{lb}(X, Z) \geq \begin{cases} 1 + \sqrt{1 - \alpha^2} & \text{if } \beta \geq \frac{\alpha}{1 + \sqrt{1 - \alpha^2}}, \\ 1 - \frac{\alpha}{\sqrt{1 - \alpha^2}} & \text{if } \beta \leq \frac{\alpha}{1 + \sqrt{1 - \alpha^2}}, \end{cases}$$

where

$$\alpha = \frac{\|Z_\perp Z_\perp^T\|_F}{\|XX^T - ZZ^T\|_F}, \quad \beta = \frac{\lambda_{\min}(X^T X) \cdot \text{tr}(Z_\perp Z_\perp^T)}{\|XX^T - ZZ^T\|_F \|Z_\perp Z_\perp^T\|_F}.$$

Remark 3.7 The definition of $\beta$ becomes ambiguous when $Z_\perp = 0$. This is without loss of precision, because $Z_\perp = 0$ implies $\alpha = 0$, and therefore $\kappa_{lb}(X, Z) = 1$ regardless of the value of $\beta$.

Our main difficulty in our proof is the need to minimize the closed-form lower-bound in Lemma 3.6 over all possible choices of $X$ and $Z$. In the rank-1 case, this easily follows by substituting the symmetry invariants $\rho = \|x\|/\|z\|$ and $\varphi = \arccos\left(\frac{x^T z}{\|x\|\|z\|}\right)$, as in

$$\alpha = \frac{\sin^2 \varphi}{\sqrt{(1 - \rho^2)^2 + 2 \rho^2 \sin^2 \varphi}}, \quad \beta = \frac{\rho^2}{\sqrt{(1 - \rho^2)^2 + 2 \rho^2 \sin^2 \varphi}},$$

and then explicitly minimizing the lower-bound with respect to $\rho$ and $\varphi$; see Zhang et al. [42, Theorem 3]. In the possibly overparameterized rank-$r$ case, however, the same approach would force us to solve a nonconvex optimization problem over up to $rr^* + r + r^* - 1$ symmetry invariants. Moreover, it is unclear how we can preserve the nonconvex rank equality constraint $\text{rank}(Z) = r^*$ once the problem is reformulated into symmetry invariants.

The main innovation in our proof is to relax the explicit dependence of $\alpha$ and $\beta$ on their arguments $X, Z$, and to minimize the lower-bound in Lemma 3.6 directly over $\alpha, \beta$ as variables. Adopting a classic strategy from integer programming, we tighten this relaxation by introducing a valid inequality: an inequality constraint on $\alpha, \beta$ that remains valid for all choices of $X, Z \in \mathbb{R}^{n \times r}$ with $\text{rank}(Z) = r^*$. Below, we write $(\cdot)_+ = \max\{\cdot, 0\}$.

Lemma 3.8 (Valid inequality) For all $X, Z \in \mathbb{R}^{n \times r}$ with $\text{rank}(Z) = r^*$, the parameters $\alpha$ and $\beta$ defined in Lemma 3.6 satisfy $\alpha^2 + (r/r^*) \beta^2 \leq 1 + [(\beta - \alpha)_+]^2$.

We will soon prove Lemma 3.8 in Section 4 below. Surprisingly, this very simple valid inequality is all that is needed to minimize our simple lower-bound $\kappa_{lb}(X, Z)$ to global optimality. The proof of Theorem 1.1 quickly follows from Lemma 3.5, Lemma 3.6, and Lemma 3.8.

Proof (Theorem 1.1) For a fixed $r$ and $r^*$, recall that we have defined $\kappa^*$ as the condition number of the best-conditioned counterexample, and formulated it as the optimal value for a two-stage optimization problem of the following form

$$\kappa^* = \inf_{\phi \in \mathbb{R}^{n \times r}} \{\kappa : \phi \text{ is } (\kappa, r, r^*)\text{-counterexample}\}$$

$$= \inf_{X, Z \in \mathbb{R}^{n \times r}} \{\kappa(X, Z) : \text{rank}(Z) = r^*, \ XX^T \neq ZZ^T\}$$
We claim that the minimum of the following expression over \( \alpha, \beta \geq 0 \)

\[
\gamma(\alpha, \beta) \overset{\text{def}}{=} \begin{cases} 
\frac{1 + \sqrt{1 - \alpha^2}}{1 - \alpha^2} & \text{if } \beta \geq \frac{\alpha}{1 + \sqrt{1 - \alpha^2}}, \\
\frac{1}{1 - \frac{(\alpha - \beta)^2}{\beta}} & \text{if } \beta \leq \frac{\alpha}{1 + \sqrt{1 - \alpha^2}}, 
\end{cases}
\]

subject to the valid inequality from Lemma 3.8 is given:

\[
1 + 2 \sqrt{r/r^*} = \min_{\alpha, \beta \geq 0} \{ \gamma(\alpha, \beta) : \alpha^2 + (r/r^*) \min(\alpha^2, \beta^2) \leq 1 \}. \tag{3.8}
\]

Taking (3.8) to be true, substituting into Lemma 3.6 and evoking Lemma 3.5 yields

\[
1 + 2 \sqrt{r/r^*} \leq \inf_{X \in \mathbb{R}^n, Z \in \mathbb{R}^n} \kappa_{1b}(X, Z) \leq \kappa^* \leq \kappa_{1b}(X, Z) \leq 1 + 2 \sqrt{r/r^*}.
\]

The upper-bound proves necessity: if \( r/r^* \leq \frac{1}{2}(\kappa - 1)^2 \), then there exists a \((\kappa, r, r^* )\)-counterexample. The lower-bound proves sufficiency: if \( r \geq \frac{1}{2}(\kappa - 1)^2 r^* \) and \( r^* \leq r < n \), then there exists no \((\kappa, r, r^* )\)-counterexample to refute a "no spurious local minima" guarantee.

We now prove the claim in (3.8). We first minimize \( \gamma(\alpha, \beta) \) subject to \( \beta \geq \frac{\alpha}{1 + \sqrt{1 - \alpha^2}} \) and find via monotonicity with respect to \( \alpha \) that

\[
\min_{0 \leq \alpha \leq 1} \left\{ \frac{1 + \sqrt{1 - \alpha^2}}{1 - \alpha^2} : \beta \geq \frac{\alpha}{1 + \sqrt{1 - \alpha^2}} \right\} = \frac{1 + \alpha \beta}{(\alpha - \beta) \beta}.
\]

Indeed, the minimizer \( \alpha \) lies at the boundary \( \beta = \frac{\alpha}{1 + \sqrt{1 - \alpha^2}} = \frac{1 - \sqrt{1 - \alpha^2}}{\alpha} \), where

\[
\frac{1 + \sqrt{1 - \alpha^2}}{1 - \sqrt{1 - \alpha^2}} = \frac{1 - \alpha^2 + (1 - \alpha^2)}{1 - \alpha^2 - (1 - \alpha^2)} = \frac{1 + \sqrt{1 - \alpha^2} - \alpha^2}{\alpha^2 - (1 - \sqrt{1 - \alpha^2})} = \frac{(1 - \alpha^2)^{-1} - \alpha}{\alpha - \beta}.
\]

Next, we minimize \( \gamma(\alpha, \beta) \) subject to \( \beta \leq \frac{\alpha}{1 + \sqrt{1 - \alpha^2}} \leq \alpha \) and the valid inequality:

\[
\min_{0 \leq \beta \leq \alpha} \left\{ \frac{1 - \alpha \beta}{(\alpha - \beta) \beta} : \alpha^2 + \frac{r}{r^*} \beta^2 \leq 1 \right\} = \min_{\rho \geq 1} \left\{ \frac{1 - \rho \beta^2}{(\rho - 1) \beta^2} : (\rho^2 + \frac{r}{r^*}) \beta^2 \leq 1 \right\} = \min_{\rho \geq 1} \left\{ \rho + \frac{1}{r^*} \rho \beta^2 \right\} = 1 + 2 \sqrt{r/r^*}.
\]

We obtain the second line from the first by substituting \( \alpha = \rho \beta \) for \( \rho \geq 1 \). The minimizers are \( \beta^2 = (\rho^2 + \frac{r}{r^*})^{-1} \) and \( \rho = 1 + \sqrt{r/r^*} \).

\[\square\]

4 Proof of the valid inequality over \( \alpha \) and \( \beta \) (Lemma 3.8)

The main innovation in our proof is Lemma 3.8, which provided a valid inequality that allowed us to relax the dependence of the parameters \( \alpha \) and \( \beta \) in our closed-form lower-bound (Lemma 3.6) with respect to \( X, Z \), while simultaneously capturing the rank equality constraint \( r^* = \text{rank}(Z) \). The lemma in turn crucially depends on the following generalization of the classic result of Eckart and Young [15].
Theorem 4.1 (Regularized Eckart–Young) Given \( A \in \mathbb{S}_+^n \) and \( B \in \mathbb{S}_+^r \) with \( r \leq n \), let \( A = \sum_{i=1}^n s_i u_i u_i^T \) and \( B = \sum_{i=1}^r d_i v_i v_i^T \) denote the usual orthonormal eigendecompositions with \( s_1 \geq \cdots \geq s_n \geq 0 \) and \( 0 \leq d_1 \leq \cdots \leq d_r \). Then,

\[
\min_{Y \in \mathbb{R}^{n \times r}} \left\{ \|A - YY^T\|_F^2 + 2(B, Y^T Y) \right\} = \sum_{i=1}^n s_i^2 - \sum_{i=1}^r [(s_i - d_i)_+]^2 \tag{4.1}\]

with minimizer \( Y^* = \sum_{i=1}^r u_i v_i^T (s_i - d_i)_+ \) where \( (\cdot)_+ \overset{\text{def}}{=} \max\{0, \cdot\} \).

Setting \( B = 0 \) in Theorem 4.1 recovers the original Eckart–Young Theorem: The best rank-\( r \) approximation \( YY^T \approx A \) in Frobenius norm is the truncated singular value decomposition \( YY^T = \sum_{i=1}^r s_i u_i u_i^T \), with approximation error \( \|A - YY^T\|_F^2 = \sum_{i=r+1}^n s_i^2 \). Hence, \( B \neq 0 \) may be viewed as a regularizer that prevents \( Y^* \) from becoming excessively large.

Our proof of Theorem 4.1 follows a trace inequality of Ruhe [32] (see also [25, Chapter 9]). The inequality states that, for \( n \times n \) positive semidefinite matrices \( A, B \), we have

\[
\sum_{i=1}^n \lambda_i(A) \lambda_{n-i+1}(B) \leq \langle A, B \rangle \leq \sum_{i=1}^n \lambda_i(A) \lambda_i(B) \tag{4.2}
\]

where the eigenvalues are ordered \( \lambda_1(A) \geq \cdots \geq \lambda_n(A) \) and \( \lambda_1(B) \geq \cdots \geq \lambda_n(B) \).

Proof (Theorem 4.1) Recall that \( A = \sum_{i=1}^n s_i u_i u_i^T \) and \( B = \sum_{i=1}^r d_i v_i v_i^T \) with \( s_1 \geq \cdots \geq s_n \geq 0 \) and \( 0 \leq d_1 \leq \cdots \leq d_r \). It follows from (4.2) that

\[
\langle A, YY^T \rangle \leq \sum_{i=1}^r \lambda_i(YY^T) \cdot \lambda_i(A) = \sum_{i=1}^r s_i \sigma_i^2(Y),
\]

\[
\langle B, Y^T Y \rangle \geq \sum_{i=1}^r \lambda_i(Y^T Y) \cdot \lambda_{n-i+1}(B) = \sum_{i=1}^r d_i \sigma_i^2(Y).
\]

Substituting yields the following lower-bound, which is valid for all choices of \( Y \):

\[
\|A - YY^T\|_F^2 + 2 \langle B, Y^T Y \rangle = \|A\|_F^2 + \|YY^T\|_F^2 - 2 \langle A, YY^T \rangle + 2 \langle B, Y^T Y \rangle \\
\geq \sum_{i=1}^n s_i^2 + \sum_{i=1}^r \sigma_i^4(Y) - 2 \sum_{i=1}^r s_i \sigma_i^2(Y) + 2 \sum_{i=1}^r d_i \sigma_i^2(Y) \\
\geq \sum_{i=1}^n s_i^2 - \sum_{i=1}^r [(s_i - d_i)_+]^2.
\]

The final line applied the following result as a lower-bound:

\[
\sigma_i^4(Y) - 2(s_i - d_i) \sigma_i^2(Y) \geq \min_{x_i \geq 0} x_i^2 - 2(s_i - d_i)x_i \\
= \min_{x_i \geq 0} [(s_i - d_i) + (s_i - d_i)]^2 - (s_i - d_i)^2 \\
= [(s_i - d_i)_+ - (s_i - d_i)]^2 - (s_i - d_i)^2 = -(s_i - d_i)_+[s_i - d_i]^2.
\]

Finally, we verify that \( Y^* = \sum_{i=1}^r u_i v_i^T (s_i - d_i)_+ \) attains the lower-bound derived above. \( \square \)

\(^1\) This simplified proof via Ruhe’s inequality is due to an anonymous reviewer.
The proof of Lemma 3.8 quickly follows from Theorem 4.1. We will also need the following two technical lemmas.

**Lemma 4.2** If \( x \geq 0 \) and \( 1^T x \leq \|x\|^2 \), then \( 1^T \left( I - xx^T / \|x\|^2 \right) 1 \geq \|(1 - x)\|^2 \).

**Proof** Define \( u = \alpha x \), where \( \alpha = 1^T x / \|x\|^2 \) is chosen so that \( 1^T u = \|u\|^2 \). First, verify that \( 1^T \left( I - xx^T / \|x\|^2 \right) 1 = \|1 - u\|^2 \geq \|(1 - u)\|^2 \). Next, observe that \( \psi(\alpha) \equiv \|(1 - \alpha t)\|^2 \) is a decreasing function of \( \alpha \geq 0 \) when \( t \geq 0 \), and therefore \( \|(1 - \alpha x)\|^2 \geq \|(1 - x)\|^2 \) for \( x \geq 0 \) and \( 0 \leq \alpha \leq 1 \). Finally, the hypotheses \( x \geq 0 \) and \( 1^T x \leq \|x\|^2 \) ensure that \( 0 \leq \alpha \leq 1 \). \( \square \)

**Lemma 4.3** Given \( s, d \in \mathbb{R}^n_+ \), let \( s_i \geq s_{lb} \geq 0 \) for all \( i \). Then we have

\[
\|s\|^2 - \|(s - d)\|^2 \geq \begin{cases} 
2 \frac{t^2}{s_{lb}} & \text{if } s_{lb} 1^T d \leq \|d\|^2, \\
2 s_{lb} 1^T d - \|d\|^2 & \text{if } s_{lb} 1^T d \geq \|d\|^2.
\end{cases}
\]

**Proof** First, we observe that \( \psi(t) \equiv t^2 - [(t - \alpha)_]^2 \) is an increasing function of \( t \geq 0 \). It follows from element-wise monotonicity that

\[
\|s\|^2 - \|(s - d)\|^2 \geq \|s_{lb} 1\|^2 - \|(s_{lb} 1 - d)\|^2 = 2 s_{lb} \left( \|1\|^2 - \|(1 - d/s_{lb})\|^2 \right)
\]

If \( s_{lb} 1^T d \leq \|d\|^2 \), then write \( x = d/s_{lb} \geq 0 \), and observe that \( 1^T x \leq \|x\|^2 \) holds. Therefore, applying Lemma 4.2 yields

\[
s_{lb}^2 \left( \|1\|^2 - \|(1 - d/s_{lb})\|^2 \right) = s_{lb}^2 \left( \|1\|^2 - \|(1 - x)\|^2 \right) \geq s_{lb} 1^T \left( xx^T / \|x\|^2 \right) 1 = s_{lb} \left( 1^T (d/s_{lb})^2 / \|d/s_{lb}\|^2 \right) = s_{lb} (1^T d)^2 / \|d\|^2.
\]

If otherwise \( s_{lb} 1^T d \geq \|d\|^2 \), then we have

\[
s_{lb}^2 \left( \|1\|^2 - \|(1 - d/s_{lb})\|^2 \right) \geq s_{lb} \left( \|1\|^2 - \|1 - d/s_{lb}\|^2 \right) = 2 s_{lb} 1^T d - \|d\|^2.
\]

\( \square \)

Instead of proving Lemma 3.8, we will prove the following equivalent statement.

**Lemma 4.4** For \( X, Z \in \mathbb{R}^{n \times s} \), define the parameters

\[
\alpha = \frac{\|Z^T Z\|^2_F}{\|X X^T - ZZ^T\|^2_F}, \quad \beta = \frac{\lambda_{\min}(X^T X)}{\|X X^T - ZZ^T\|^2_F \cdot \|Z^T Z\|^2_F}.
\]

Then, \( \alpha^2 + (r/p) \beta^2 \leq 1 + ((\beta - \alpha)\_)^2 \) holds for all \( p \) satisfying \( \text{rank}(Z) \leq p \leq r \).

Assuming the above to be true, setting \( p = \text{rank}(Z) \) yields exactly Lemma 3.8. Conversely, if the above is true for \( p = \text{rank}(Z) \), then it is obviously true for any \( p \geq \text{rank}(Z) \), given that the left-hand side decreases monotonically with increasing \( p \).
Proof The two parameters $\alpha$ and $\beta$ are unitarily invariant with respect to $X, Z$, so we may assume without loss of generality that $X, Z$ are partitioned as

$$X = \begin{bmatrix} X_1 \\ 0 \end{bmatrix} \quad Z = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$$

where $X_1 \in \mathbb{R}^{r \times r}$, $Z_1 \in \mathbb{R}^{r \times p}$, $Z_2 \in \mathbb{R}^{(n-r) \times p}$.

(Otherwise, take the QR decomposition $X = QR$ and note that $\alpha, \beta$ remain unchanged with $X \leftarrow Q^TX$ and $Z \leftarrow Q^TZ$.) Denote $s \in \mathbb{R}^r$ as the $r$ eigenvalues of $X_1^TX_1$ and $d \in \mathbb{R}^p$ as the $p$ eigenvalues of $Z_2^TZ_2$.

$$s_i \overset{\text{def}}{=} \lambda_i(X_1X_1^T) = \lambda_i(XX^T), \quad s_1 \geq s_2 \geq \cdots \geq s_r \geq 0,$$

$$d_i \overset{\text{def}}{=} \lambda_i(Z_2^TZ_2) = \lambda_i(Z_2Z_2^T), \quad 0 \leq d_1 \leq d_2 \leq \cdots \leq d_p.$$

Writing $e = \text{vec}(XX^T - ZZ^T)$ yields

$$\alpha = \frac{\|Z_2Z_2^T\|_F}{\|XX^T - ZZ^T\|_F}, \quad \beta = \frac{\lambda_{\min}(X_1^TX_1)}{\|XX^T - ZZ^T\|_F} \cdot \frac{\text{tr}(Z_2Z_2^T)}{\|ZZ_2\|_F} = \frac{s_r \cdot 1^Td}{\|e\| \|d\|}.$$

We expand $\|XX^T - ZZ^T\|_F^2$ block-wise and evoke our Regularized Eckart–Young Theorem (Theorem 4.1) over the $r \times p$ matrix block $Z_1$:

$$\begin{align*}
\|XX^T - ZZ^T\|_F^2 &= \|X_1X_1^T - Z_1Z_1^T\|_F^2 + 2(Z_1^T Z_2, Z_1^T Z_1) + \|Z_2Z_2^T\|_F^2 \\
&\geq \min_{Z_1 \in \mathbb{R}^{r \times r}} \{\|X_1X_1^T - Z_1Z_1^T\|_F^2 + 2(Z_1^T Z_2, Z_1^T Z_1) + \|Z_2Z_2^T\|_F^2 \}
\end{align*}$$

where $s' = (s_1, s_2, \ldots, s_p)$. Now, evoking our technical lemma (Lemma 4.3) with $s_1 \equiv s_r$ yields the following two lower-bounds on $\|XX^T - ZZ^T\|_F^2 = \|e\|^2$:

$$\begin{align*}
\text{(4.3a)} & \quad \text{if } s_r 1^Td \leq \|d\|^2 \quad \Rightarrow \quad \|e\|^2 \geq \frac{2}{s_r} \frac{(1^Td)^2}{\|d\|^2} + (r-p)s_r^2 + \|d\|^2, \\
\text{(4.3b)} & \quad \text{if } s_r 1^Td \geq \|d\|^2 \quad \Rightarrow \quad \|e\|^2 \geq 2\|1^Td - \|d\|^2 + (r-p)s_r^2 + \|d\|^2.
\end{align*}$$

Substituting $s_r = \beta \cdot \|e\|\|d\|/(1^Td)$ yields

$$\begin{align*}
(r-p)s_r^2 &= (r-p)\frac{\|d\|^2}{(1^Td)^2} \beta^2 \|e\|^2 \geq (r-p)\frac{1}{\min(d)} \beta^2 \|e\|^2 \geq \frac{(r-p)}{p} \beta^2 \|e\|^2,
\end{align*}$$

and substituting into (4.3) yields

$$\begin{align*}
\beta \leq \alpha & \quad \Rightarrow \quad 1 \geq \beta^2 + \frac{(r-p)}{p} \beta^2 = \alpha^2 + \frac{(r-p)}{p} \beta^2, \\
\beta \geq \alpha & \quad \Rightarrow \quad 1 \geq 2\alpha \beta + \frac{(r-p)}{p} \beta^2 = \alpha^2 + \frac{(r-p)}{p} \beta^2 - (\beta - \alpha)^2,
\end{align*}$$

as desired. \qed
5 Proof of the closed-form lower-bound (Lemma 3.6)

Given fixed $X, Z \in \mathbb{R}^{n \times r}$, define the error vector $e = \text{vec}(XX^T - ZZ^T)$. Recall that we define $J_X$ with respect to a given $X \in \mathbb{R}^{n \times r}$ to implicitly satisfy

$$J_X \text{vec}(V) = \text{vec}(XV^T + VX^T), \quad J_X^T \text{vec}(M) = \text{vec}((M + M^T)X) \quad (5.1)$$

for all $V \in \mathbb{R}^{n \times r}$ and $M \in \mathbb{R}^{n \times n}$.

**Lemma 5.1** For any $X \in \mathbb{R}^{n \times r}$, the matrix $J_X$ defined in (5.1) satisfies $I - J_X J_X^T = (I - XX^T) \otimes (I - XX^T)$ where $\dagger$ denotes the pseudoinverse.

**Proof** Let $X = QR$ be the usual QR decomposition with orthogonal complement $Q_\perp$, so that $XX^T = QQ^T$ and $I - XX^T = Q_\perp Q_\perp^T$. Observe that for any $M \in \mathbb{R}^{n \times n}$:

$$J_X^T \text{vec}(M) = 0 \iff (M + M^T)Q = 0 \iff M = Q_\perp M Q_\perp^T.$$

Vectorizing the final equality as $\text{vec}(M) = (Q_\perp \otimes Q_\perp) \text{vec}(M)$ yields

$$(I - J_X J_X^T)b = \arg \min_v \{ ||v - b|| : J_X^T v = 0 \} = \begin{cases} \arg \min_v \{ ||v - b|| : v = (Q_\perp \otimes Q_\perp)\hat{v} \} \\ \arg \min_v \{ ||v - b|| : (Q_\perp \otimes Q_\perp)\hat{v} = v \} \end{cases} = (Q_\perp \otimes Q_\perp)(Q_\perp \otimes Q_\perp)^Tb = (Q_\perp Q_\perp^T \otimes Q_\perp Q_\perp^T)b. \quad \Box$$

To lower-bound $\kappa_0(X, Z)$ as defined in Lemma 3.2, we will plug a carefully-chosen heuristic solution into its Lagrangian dual. This dual problem is stated in the lemma below; see Appendix A for its derivation.

**Lemma 5.2** We have

$$\kappa_0(X, Z) = \max_{t \geq 0} \frac{1 + \cos \theta(t)}{2t + 1 - \cos \theta(t)},$$

where $\cos \theta(t)$ is defined as follows

$$\cos \theta(t) = \max_{y, W_{i,j}} \left\{ f \in \mathbb{R}^{n \times r} : \begin{array}{l} f \equiv J_X y - \sum_{i=1}^r \text{vec}(W_{i,j}) \quad \|e\|_{\text{vec}} = 1, \\ (I - ZZ^T) \text{mat}(f) (I - ZZ^T) \succeq 0, \\ (J_X^T J_X, W) = 2t, \\ W = [W_{i,j}]_{i,j=1}^r \succeq 0, \end{array} \right\} \quad (5.2)$$

with respect to $y \in \mathbb{R}^n$ and $W_{i,j} \in \mathbb{R}^{n \times n}$ for $i, j \in \{1, 2, \ldots, r\}$.

Our heuristic choice of $y$ and $W_{i,j}$ is motivated by the optimal solution to the much simpler rank-1 case previously derived by Zhang et al. [42]. This choice is not necessarily optimal for the rank-$r$ case, but it does remain simple enough for the final maximization over $t \geq 0$ in Lemma 5.2 to be solved in closed-form.

We begin by recalling the core insights used to derive the rank-1 solution. Our goal of maximizing $e^T(J_X y - w)$ where $w = \sum_{i=1}^r \text{vec}(W_{i,i})$ can equivalently be understood as minimizing $\|J_X y - w - e\|$ over $y$ and $w$, subject to $\text{mat}(w) \succeq 0$. This is a projection problem, so a basic idea is to decompose $e$ into orthogonal components:

$$J_X y = \gamma_1 \cdot (J_X J_X^T)e, \quad w = -\gamma_2 \cdot (I - J_X J_X^T)e. \quad (5.3)$$
We verify that \( \text{mat}(w) \succeq 0 \) indeed holds for \( \gamma_2 \geq 0 \) via Lemma 5.1

\[
\text{mat}((I - J_X J_X^\dagger) e) = (I - XX^\dagger)(XX^T - Z Z^T)(I - XX^\dagger) = -Z \perp Z^\perp
\]

where we have written \( Z \perp = (I - XX^\dagger) Z \).

Equation (5.3) specifies a unique choice of \( y \), but leaves several possible valid choices of \( W_{i,j} \). We need the following claim to make a concrete choice.

**Claim** Let \( W = \sum_{j=1}^m \text{vec}(V_j) \text{vec}(V_j)^T \) where \( V_1, \ldots, V_m \in \mathbb{R}^{n \times r} \). Partition \( W \) into \( r \times r \) blocks of \( n \times n \), as in \( W = [W_{i,j}]_{i,j=1}^r \) for \( W_{i,j} \in \mathbb{R}^{n \times n} \). Then,

\[
\sum_{i=1}^r W_{i,i} = \sum_{j=1}^m V_j V_j^T, \quad \left\langle J_X^T J_X, W \right\rangle = \sum_{j=1}^m \|X V_j^T + V_j X^T\|_F^2.
\]

Following the claim, the obvious first attempt is to pick \( W = \gamma_2 \cdot \text{vec}(Z \perp) \text{vec}(Z \perp)^T \).

Indeed, this yields \( \sum_{i=1}^r W_{i,i} = \gamma_2 \cdot Z \perp Z \perp^T \) as specified in (5.3), and is in fact optimal for the rank-1 case. In the rank-\( r \) case, however, the choice of \( W \) can be improved by making \( \left\langle J_X^T J_X, W \right\rangle \) as small as possible. A more subtle second attempt is to pick

\[
W = \gamma_2 \cdot \sum_{j=1}^r \text{vec}(Z \perp_v e_j v_r^T) \text{vec}(Z \perp_v e_j v_r^T)^T
\]

where \( v_r \) is the \( r \)-th unit eigenvector of \( X^T X \). Indeed, we can verify that (5.3) is also satisfied:

\[
\sum_{i=1}^r W_{i,i} = \sum_{j=1}^r \gamma_2 \cdot Z \perp e_i e_j^T Z \perp^T = \gamma_2 \cdot Z \perp \left( \sum_{i=1}^r e_i e_i^T \right) Z \perp^T = \gamma_2 \cdot Z \perp Z \perp^T.
\]

Moreover, we have

\[
\left\langle J_X^T J_X, W \right\rangle = \gamma_2 \cdot \sum_{i=1}^r \|J_X \text{vec}(Z \perp_v e_i v_r^T)\|^2 = \gamma_2 \cdot \sum_{i=1}^r \|X v_r e_i^T Z \perp^T + Z \perp e_i v_r^T X^T\|_F^2
\]

\[
= \gamma_2 \cdot \sum_{i=1}^r 2\|Z \perp e_i v_r^T X^T\|_F^2 = 2\gamma_2 \cdot \lambda_{\min}(X^T X) \|Z \perp\|_F^2
\]

We now plug our choice of \( y \) and \( W_{i,j} \) specified in (5.3) and (5.5) into Lemma 5.2, along with a good choice of coefficients \( \gamma_1 \) and \( \gamma_2 \).

**Lemma 5.3** We have \( \cos \theta(t) \geq \alpha \cdot (t/\beta) + \sqrt{1 - \alpha^2} \sqrt{1 - t^2/\beta^2} \) for \( 0 \leq t \leq \alpha \beta \) where \( Z \perp = (I - XX^\dagger) Z \) and

\[
\alpha = \frac{\|Z \perp Z \perp^T\|_F}{\|XX^T - ZZ^T\|_F}, \quad \beta = \frac{\lambda_{\min}(X^T X) \cdot \text{tr}(Z \perp Z \perp^T)}{\|XX^T - ZZ^T\|_F \|Z \perp Z \perp^T\|_F}.
\]

**Proof** Write \( H = I - ZZ^\dagger \) and \( e = \text{vec}(XX^T - ZZ^T) \), and decompose \( e_1 = (J_X J_X^\dagger) e \) and \( e_2 = (I - J_X J_X^\dagger) e \). We verify that the following heuristic choice

\[
y = \frac{\sqrt{1 - (t/\beta)^2}}{\|e\| \cdot \|e_1\|} J_X^T e, \quad W = \frac{t/\beta}{\|e\| \cdot \|e_2\|} \sum_{i=1}^r \text{vec}(Z \perp_v e_i v_r^T) \text{vec}(Z \perp_v e_i v_r^T)^T,
\]
is feasible for \(5.2\). Indeed, \(W \geq 0\) holds, and we have from \(5.4\), \(5.6\), and \(5.7\):

\[
f = J_X y - \sum_{i=1}^{r} \text{vec}(W_{i,i}) = \gamma_1 \cdot J_X J_X^T e - \gamma_2 \text{vec}(Z \perp Z \perp^T) = \gamma_1 \cdot e_1 + \gamma_2 e_2,
\]

\[
\|e\|^2 \|f\|^2 = \gamma_1^2 \cdot \|e\|^2 \|e_1\|^2 + \gamma_2^2 \cdot \|e\|^2 \|e_2\|^2 = (1 - (t / \beta)^2) + (t / \beta)^2 = 1,
\]

\[
\langle T_{X} J_X, W \rangle = 2 \left( \frac{t / \beta}{\|e\| \|e_2\|} \right) \frac{\lambda_{\min}(X^T Y)}{\|e\|^2} = \frac{2t}{\beta} \frac{\|e_2\|^2}{\|e\| \|Z \perp Z \perp^T\|_F} = 2t.
\]

To verify that \(II \text{ mat}(f) II \geq 0\), note that \(t \leq \alpha \beta\) implies \(t / \beta \leq \alpha \leq 1\), so it follows from \(\alpha = \|e_2\| / \|e\|\) and \(\sqrt{1 - \alpha^2} = \|e_1\| / \|e\|\) that

\[
\gamma_1 = \frac{\sqrt{1 - (t / \beta)^2}}{\|e_1\| / \|e\|} \geq \frac{\sqrt{1 - \alpha^2}}{\|e_1\| / \|e\|} = \frac{\|e_1\|^2}{\|e\|^2} \geq \frac{\|e_2\|^2}{\|e\|^2} = \gamma_2.
\]

In turn, \(\gamma_1 \geq \gamma_2\) implies

\[
f = \gamma_1 \cdot e_1 + \gamma_2 \cdot e_2 = \gamma_1 \cdot e - (\gamma_1 - \gamma_2) e_2,
\]

\[
\text{mat}(f) = \gamma_1 \cdot (XX^T - ZZ^T) + (\gamma_1 - \gamma_2) \cdot Z \perp Z \perp^T.
\]

\(II \text{ mat}(f) II = \gamma_1 \cdot II(XX^T) II + (\gamma_1 - \gamma_2) \cdot II(Z \perp Z \perp^T) II \geq 0\).

Having verified \(y\) and \(W\) as feasible for \(5.2\), we use its objective value to heuristically lower-bound the maximum

\[
\cos(\theta) \geq e^T f = \gamma_1 \cdot \|e_1\|^2 + \gamma_2 \cdot \|e_2\|^2 = \sqrt{1 - (t / \beta)^2} \sqrt{1 - \alpha^2} + (t / \beta) \cdot \alpha.
\]

\[\square\]

Finally, we evoke the following lemma, which was previously used in the proof of Zhang et al. [42, Theorem 12] to fully solve the rank-1 case in closed-form.

**Lemma 5.4** Given \(0 \leq \alpha < 1\) and \(\beta \geq 0\), let \(\gamma_{\alpha, \beta} : \mathbb{R}_+ \rightarrow [0, 1]\) satisfy \(\gamma_{\alpha, \beta}(t) = \alpha \cdot (t / \beta) + \sqrt{1 - \alpha^2} \sqrt{1 - t^2 / \beta^2}\) for \(t \leq \alpha \beta\). Then, we have

\[
\max_{0 \leq \gamma \leq \alpha \beta} \frac{1 + \gamma_{\alpha, \beta}(t)}{2t + 1 - \gamma_{\alpha, \beta}(t)} = \begin{cases} 
1 + \sqrt{1 - \alpha^2} & \text{if } \beta \geq \frac{\alpha}{1 + \sqrt{1 - \alpha^2}} \\
1 - \sqrt{1 - \alpha^2} & \text{if } \beta \leq \frac{\alpha}{1 + \sqrt{1 - \alpha^2}} \\
\frac{\beta}{\alpha - \beta} \left(1 - \frac{\alpha}{\beta(1 - \alpha)}\right) & \text{if } \beta = \frac{\alpha}{1 + \sqrt{1 - \alpha^2}} \\
1 & \text{if } \beta = \frac{\alpha}{1 + \sqrt{1 - \alpha^2}}
\end{cases}
\]

The proof of Lemma 3.6 immediately follows by substituting Lemma 5.3 into Lemma 5.2 and then solving the resulting optimization over \(t\) using Lemma 5.4.

### 6 Counterexample that attains the heuristic upper-bound (Lemma 3.5)

Finally, we give a constructive proof for the heuristic upper-bound in Lemma 3.5. First, recall from Lemma 3.2 that our SDP upper-bound is defined as the following

\[
\kappa_{\text{ub}}(X, Z) = \min_{\kappa, H} \left\{ \kappa : I \preceq H \preceq \kappa I, J_X^T H e = 0, -2I_{r} \otimes \text{mat}(He) \preceq J_X^T H J_X \right\} \quad (6.1)
\]

where \(e = \text{vec}(X X^T - ZZ^T)\) and its Jacobian \(J_X\) is defined to satisfy \(J_X \text{ vec}(V) = \text{vec}(XV^T + V X^T)\) for all \(V\). Below, we derive a set of sufficient conditions for a given \((\kappa, H)\) to be feasible for (6.1).
Lemma 6.1 Let \([Q_1, Q_2]\) have orthonormal columns with \(Q_1 \in \mathbb{R}^{n \times r}\) and \(Q_2 \in \mathbb{R}^{n \times r'}\). If \(\kappa = 1 + 2\sqrt{r/r'}\), and \(H\) is any \(n^2 \times n^2\) matrix that satisfies \(I \preceq H \preceq \kappa I\) and the following \(rr^* + 2\) eigenvalue equations

\[
\begin{align*}
H \text{vec} \left( Q_1 Q_1^T + \sqrt{r/r'} Q_2 Q_2^T \right) &= \kappa \cdot \text{vec} \left( Q_1 Q_1^T + \sqrt{r/r'} Q_2 Q_2^T \right), \\
H \text{vec} \left( Q_1 Q_1^T - \sqrt{r/r'} Q_2 Q_2^T \right) &= \cdot \text{vec} \left( Q_1 Q_1^T - \sqrt{r/r'} Q_2 Q_2^T \right), \\
H \text{vec} \left( Q_1 V Q_2^T + Q_2 V^T Q_1^T \right) &= \kappa \cdot \text{vec} \left( Q_1 V Q_2^T + Q_2 V^T Q_1^T \right) \quad \text{for all } V \in \mathbb{R}^{r \times r'},
\end{align*}
\]

then \((\kappa, H)\) is a feasible point for (6.1) with \(X = Q_1\) and \(Z = \sqrt{1 + \sqrt{r/r'} Q_2}\). Moreover, \(e^T H e = (1 + 2\sqrt{r/r'}) \cdot (1 + \sqrt{r/r'}) \cdot r^*\).

Proof Write \(\rho \equiv \sqrt{r/r'}\), and take \([Q_1, Q_2] = I_{r+r^*}\) without loss of generality. Decomposing \(e = \text{vec}(XX^T - ZZ^T)\) into eigenvectors and applying \(H\) yields

\[
\begin{align*}
\text{mat}(e) &= \begin{bmatrix} I_r & 0 \\ 0 & -(1 + \rho)I_{r^*} \end{bmatrix} = \begin{bmatrix} -1/2\rho & I_r & 0 & 0 \\ 0 & \rho I_{r^*} & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 + 2\rho & I_r & 0 \\ 0 & -\rho I_{r^*} & 0 & 0 \end{bmatrix}, \\
\text{mat}(He) &= -\frac{1 + 2\rho}{2\rho} \begin{bmatrix} I_r & 0 \\ 0 & \rho I_{r^*} \end{bmatrix} + \frac{1 + 2\rho}{2\rho} \begin{bmatrix} I_r & 0 \\ 0 & -\rho I_{r^*} \end{bmatrix} = \begin{bmatrix} 0_r & 0 \\ 0 & 0 -\kappa I_{r^*} \end{bmatrix}.
\end{align*}
\]

Clearly, \((\text{mat}(e), \text{mat}(He)) = \kappa(1 + \rho)r^*\). For \(v = \text{vec}(V) = \text{vec}(\left[ V_1; V_2 \right])\) with \(V_1 \in \mathbb{R}^{r \times r}\) and \(V_2 \in \mathbb{R}^{r^* \times r}\), observe that

\[
\begin{align*}
v^T J_X H e &= \left[ \begin{bmatrix} V_1 + V_1^T V_2^T \\ V_2 \end{bmatrix} \right] \begin{bmatrix} 0_r & 0 \\ 0 & -\kappa I_{r^*} \end{bmatrix} = 0, \\
v^T [I_r \otimes \text{mat}(He)] v = \langle \text{mat}(He), V V^T \rangle &= \left[ \begin{bmatrix} 0_r & 0 \\ 0 & -\kappa I_{r^*} \end{bmatrix} \right] \left[ \begin{bmatrix} V_1 V_1^T & V_1 V_2^T \\ V_2 V_1^T & V_2 V_2^T \end{bmatrix} \right] = \kappa \|V_2\|^2, \\
v^T J_X^T H J_X v &= \left\| H^{1/2} \text{vec} \left[ \begin{bmatrix} V_1 + V_1^T V_2^T \\ V_2 \end{bmatrix} \right] \right\|^2 \geq \left\| H^{1/2} \text{vec} \left[ \begin{bmatrix} 0_r & V_2^T \\ V_2 & 0_r \end{bmatrix} \right] \right\|^2 = 2\kappa \|V_2\|^2.
\end{align*}
\]

Therefore, \(v^T J_X H e = 0\) and \(2v^T [I_r \otimes \text{mat}(He)] v \geq -v^T J_X^T H J_X v\) hold for all \(v\). \(\Box\)

We now state an explicit counterexample that satisfies the conditions in Lemma 6.1. Without loss of generality, take \(u_1, \ldots, u_n\) to be the standard basis for \(\mathbb{R}^n\). The counterexample is constructed by taking the rescaled standard basis \(A^{(i,j)} = \sqrt{\kappa} u_i u_j^T\) for \(\mathbb{R}^{n \times n}\), and then replacing the elements \(\{A^{(i,j)}\}_{i=1}^n\) with a new orthogonal basis for \(\text{diag}(\mathbb{R}^n)\) whose first two elements \(A^{(1,1)}\) and \(A^{(2,2)}\) are exactly the first two eigenvectors in Lemma 6.1. The ordering \(\pi\) for the Gram–Schmidt orthogonalization below is chosen specifically to ensure that \(\text{span}\{A^{(i,j)}\}_{i=1}^n = \text{diag}(\mathbb{R}^n)\).

Example 6.2 \((\kappa = 1 + 2\sqrt{r/r^*} \text{ counterexample})\) Given \(r, r^*, n\) satisfying \(1 \leq r^* \leq r < n\), let \(u_1, u_2, \ldots, u_n\) be an orthonormal basis for \(\mathbb{R}^n\). Define the quadratic function

\[
\phi_0(M) = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \langle A^{(i,j)}, M - M^* \rangle^2 \quad \text{where } M^* = \sum_{k=r+1}^{r+r^*} u_k u_k^T.
\]
and $A^{(i,j)} = \sqrt{\kappa} \cdot u_i u_j^T$ for all $1 \leq i, j \leq n$ except the following

$$A^{(1,1)} = \sqrt{\frac{\kappa}{2r}} \sum_{j=1}^r u_j u_j^T + \sqrt{\frac{\kappa}{2r^*}} \sum_{k=r+1}^{r+r^*} u_k u_k^T,$$

$$A^{(2,2)} = \frac{1}{\sqrt{2r}} \sum_{j=1}^r u_j u_j^T - \frac{1}{\sqrt{2r^*}} \sum_{k=r+1}^{r+r^*} u_k u_k^T,$$

$$A^{(i,i)} = u_{\pi(i)} u_{\pi(i)}^T - \sum_{j=1}^{i-1} \frac{A^{(j,j)}(\pi(i), \pi(i))}{\|A(U(j))\|_F^2} \langle A(U(j), u_{\pi(i)} u_{\pi(i)}^T) \rangle$$

for $i \in \{3, \ldots, r + r^*\}$, with the ordering $\pi = (1, r + 1, 2, 3, \ldots, r, r + 2, r + 3, \ldots, r + r^*)$. Note that $\{A^{(i,j)}\}$ is an orthogonal basis for $\mathbb{R}^{n \times n}$ with $1 \leq \|A^{(i,j)}\|_F^2 \leq \kappa$, so the quadratic function $\phi_0$ is twice-differentiable, 1-strongly convex and $\kappa$-smooth,

$$\|E\|_F^2 \leq \langle \nabla^2 \phi_0(M)[E], E \rangle \leq \kappa \|E\|_F^2$$

for all $M \in \mathbb{R}^{n \times n}$.

Its minimizer $M^* = \arg \min_M \phi_0(M)$ satisfies $r^* = \text{rank}(M^*)$, but its corresponding $f_0(U) \overset{\text{def}}{=} \phi_0(UU^T)$ admits the $n \times r$ matrix $X = \frac{1}{\sqrt{1 + \sqrt{r/r^*}}} [u_1, \ldots, u_r]$ as a second-order critical point

$$f_0(X) - \min_U f_0(U) = \left(1 + \frac{1}{1 + \sqrt{r/r^*}}\right) \frac{\|M^*\|_F^2}{2}, \quad \nabla f_0(X) = 0, \quad \nabla^2 f_0(X) \geq 0.$$

\[\square\]

### 7 Proof of the strict saddle property (Proposition 1.3)

We now prove that, in our setting of an $L$-smooth and $\mu$-strongly convex $\phi$, the absence of spurious local minima in $f(U) = \phi(UU^T)$ implies the strict saddle property [16, 18, 19]. Let $S(\epsilon, \rho)$ denote the set of spurious $\epsilon$ second-order critical points for its corresponding Burer–Monteiro function $f(U) \overset{\text{def}}{=} \phi(UU^T)$ that lie more than a distance of $\rho$ from the minimizer:

$$S(\epsilon, \rho) = \{X \in \mathbb{R}^{n \times r} : \|\nabla f(X)\|_F \leq \epsilon, \quad \nabla^2 f(X) \succeq -\epsilon I, \quad \|XX^T - M^*\|_F \geq \rho\}.$$

Our crucial observation is that $S(\epsilon, \rho)$ is a compact set: (i) it is by definition closed; (ii) it is bounded because any $X \in S(\epsilon, \rho)$ with $\|X\|_F \to \infty$ would imply $\|\nabla f(X)\|_F \to \infty$ via Lemma 7.1 below.

**Lemma 7.1** Let $\phi$ be $L$-smooth and $\mu$-strongly convex, and let $M^* = \arg \min_M \phi(M)$ be attained. Then, the gradient norm $\|\nabla f(X)\|_F$ of the Burer–Monteiro function $f(X) \overset{\text{def}}{=} \phi(XX^T)$ is a coercive function, meaning that $\|\nabla f(X)\|_F \to \infty$ as $\|X\|_F \to \infty$.

**Proof** It follows from the $L$-smoothness and $\mu$-strong convexity of $\phi$ that its Hessian approximately preserves inner products (see e.g. Li et al. [21, Proposition 2.1])

$$\frac{2}{\mu + L} \langle \nabla^2 \phi(M)[E], F \rangle \geq \langle E, F \rangle - \frac{L - \mu}{L + \mu} \|E\|_F \|F\|_F.$$
Moreover, the fact that $M^* = \arg\min_{M \succeq 0} \phi(M)$ implies that $\nabla \phi(M^*) \succeq 0$. It follows from the fundamental theorem of calculus that
\[
\langle \nabla f(X), X \rangle = 2 \left\langle \nabla \phi(X X^T), X X^T \right\rangle = 2 \left( \int_0^1 \nabla^2 \phi(M(t))[X X^T - M^*] \, dt, X X^T \right) + 2 \left\langle \nabla \phi(M^*), X X^T \right\rangle \\
\geq (L + \mu) \left\langle X X^T - M^*, X X^T \right\rangle - (L - \mu)\|X X^T - M^*\|_F \|X X^T\|_F \\
\geq 2\mu\|X X^T\|_F^2 - 2L\|X X^T\|_F\|M^*\|_F \geq 2\|X\|_F^2 \left( \frac{\mu}{r} \|X\|_2^2 - L\|M^*\|_F \right)
\]
where we used $\sqrt{r}\|X X^T\|_F \geq \text{tr}X X^T$ for $X \in \mathbb{R}^{n \times r}$ and $n \geq r$. Therefore, once $\|X\|_F^2 > \frac{L\|M^*\|_F}{\mu}$ holds, we have $\|\nabla f(X)\|_F \|X\|_F \geq \langle \nabla f(X), X \rangle \geq C\|X\|_F^2$ for some $C > 0$. \hfill $\Box$

The strict saddle property then follows immediately from compactness.\footnote{This proof was first suggested by a reviewer of the paper \cite{Zhang18} at \url{https://openreview.net/forum?id=5-Of1F1qkmnoteId=FWhbQu0GxmD}}

**Lemma 7.2** Let $\phi : \mathbb{S}^n \to \mathbb{R}$ be twice-differentiable, $L$-smooth and $\mu$-strongly convex, and let the minimizer $M^* = \arg\min_{M \succeq 0} \phi(M)$ exist. If the function $f(U) \overset{\text{def}}{=} \phi(U U^T)$ satisfies the following
\[
\nabla f(X) = 0, \quad \nabla^2 f(X) \succeq 0 \iff X X^T = M^*
\]
then it also satisfies the strict saddle property
\[
\|\nabla f(X)\|_F \leq \epsilon(\rho), \quad \nabla^2 f(X) \succeq -\epsilon(\rho) I \implies \|X X^T - M^*\|_F < \rho
\]
where the function $\epsilon$ is nondecreasing and satisfies $0 < \epsilon(\rho) \leq 1$ for all $\rho > 0$.

**Proof** For an arbitrary $\rho > 0$, define the sequence of compact subsets $S_1 \supseteq S_2 \supseteq S_3 \supseteq \cdots$ where $S_k = \mathcal{S}(\frac{1}{k}, \rho)$. Given that $f$ has no spurious local minima, the intersection of all $\{S_k\}_{k=1}^\infty$ must be empty. It follows from the finite open cover property of the compact set $S_1$ that there exists some finite $N \geq 1$ such that $S_N = \emptyset$. Therefore, any $X$ that satisfies $\|\nabla f(X)\|_F \leq \frac{1}{N}$ and $\nabla^2 f(X) \succeq -\frac{1}{N} I$ must also satisfy $\|X X^T - M^*\|_F < \rho$. Finally, we repeat this for all $\rho > 0$ and define $\epsilon(\rho) = \frac{1}{\rho}$. We observe that the function $\epsilon(\cdot)$ satisfies $\epsilon(\rho) \in (0, 1]$ because $N \geq 1$ is finite, and that it is nondecreasing because $S_{k+1} \subseteq S_k$ for all $k$. \hfill $\Box$

We conclude the proof by using the convexity and $L$-smoothness of $\phi$ to show that an $\epsilon$-second-order point $X$ that is also $\rho$-close is guaranteed to be $\delta$-globally suboptimal.

**Proof (Proposition 1.3)** Write $M^* = ZZ^T$, and let $X \in \mathbb{R}^{n \times r}$ be an $\epsilon$-second-order point satisfying $\|\nabla f(X)\|_F \leq \epsilon$ and $\nabla^2 f(X) \succeq -\epsilon I$. If $f$ has no spurious local minima, then Lemma 7.2 says that $X$ is also $\rho(\epsilon)$-close, as in $\|X X^T - ZZ^T\|_F < \rho(\epsilon)$ where $\rho$ is a nondecreasing function such that $\rho(\epsilon) \to 0$ as $\epsilon \to 0$. If overparameterized $r > \text{rank}(Z)$, we use the rank deficiency of $X$ to certify global optimality [36, Proposition 10]
\[
f(X) - f(Z) \leq \frac{1}{2} \|X\|_F \|\nabla f(X)\|_F - \frac{1}{2} \|Z\|_F^2 \lambda_{\min}(\nabla^2 f(X)) + 2L\|Z\|_F^2 \lambda_{\min}(X^T X) \\
\leq \left[ \frac{1}{2} \left(\|Z\|_F^2 + \rho \right) + \frac{1}{2} \|Z\|_F^2 \right] \epsilon + 2L\|Z\|_F^2 \rho(\epsilon) \overset{\text{def}}{=} \varphi(\epsilon).
\]
Note that the near rank deficiency of $X$ is due to Weyl’s inequality $\lambda_r(XX^T) \leq \lambda_r(ZZ^T) + \|XX^T - ZZ^T\|_F \leq 0 + \rho(\epsilon)$. The function $\varphi(\epsilon)$ is nondecreasing and satisfies $\varphi(\epsilon) \to 0$ as $\epsilon \to 0$. We define $\epsilon(\delta) = \varphi^{-1}(\delta)$ to ensure that an $\epsilon(\delta)$-second-order point $X$ will satisfy $f(X) - f(Z) \leq \delta$.

In the exactly parameterized regime $r = \text{rank}(Z)$, we use the displacement vector $\Delta = X - ZR$ where $R = \arg\min_{R^T = RT = I} \|X - ZR\|_F$; see also [12, 18, 21, 34, 44]. Within a small neighborhood $\rho(\epsilon) \leq \frac{1}{2}\lambda_{\min}(ZZ^T)$, the norm of the displacement vector scales with the error norm [34, Lemma 5.4]:

$$\|\Delta\|_F^2 \leq \frac{\|XX^T - ZZ^T\|_F^2}{2(\sqrt{2} - 1)\lambda_{\min}(XX^T)} \leq \frac{\rho(\epsilon)^2}{2[\lambda_{\min}(ZZ^T) - \rho(\epsilon)]} \leq \frac{4\rho(\epsilon)^2}{\lambda_{\min}(ZZ^T)}.$$  

The error vector decomposes as $XX^T - ZZ^T = X\Delta + \Delta X^T - \Delta \Delta^T$, and this yields via the convexity of $\phi$:

$$f(X) - f(Z) = \phi(XX^T) - \phi(ZZ^T) \leq \left\langle \nabla \phi(XX^T), XX^T - ZZ^T \right\rangle$$

$$= \left\langle \nabla \phi(XX^T), X\Delta + \Delta X^T \right\rangle - \left\langle \nabla \phi(XX^T), \Delta \Delta^T \right\rangle$$

$$\leq (\nabla f(X), \Delta) - \frac{1}{2} \left[ \|\nabla^2 f(X)(\Delta, \Delta) - L\|X\Delta^T + \Delta X^T\|_F^2 \right]$$

$$\leq \epsilon \|\Delta\|_F^2 + \frac{1}{2} \|\Delta\|_F^2 + L\lambda_{\max}(XX^T)\|\Delta\|_F^2$$

$$\leq \frac{2\epsilon}{\sqrt{\lambda_{\min}(ZZ^T)}} \rho(\epsilon) + \left[ \frac{2\epsilon + 4L\lambda_{\max}(ZZ^T) + \rho(\epsilon)}{\lambda_{\min}(ZZ^T)} \right] \rho(\epsilon)^2 \overset{\text{def}}{=} \varphi(\epsilon).$$

The function $\varphi(\epsilon)$ is nondecreasing and satisfies $\varphi(\epsilon) \to 0$ as $\epsilon \to 0$. We define $\epsilon(\delta) = \min\{\varphi^{-1}(\delta), \rho^{-1}(\frac{1}{2}\lambda_{\min}(ZZ^T))\}$ to ensure that an $\epsilon(\delta)$-second-order point $X$ will satisfy $f(X) - f(Z) \leq \delta$.

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Conflicts of interest/Competing interests

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A Derivation of the dual for the lower-bound problem (Proof of Lemma 5.2)

We will derive the Lagrangian dual for the following

\[
\kappa_{lb}(X, Z) = \min_{s, H, \lambda} \left\{ \lambda : \begin{array}{l}
I \preceq H \preceq \lambda I, \\
J^T_X s = 0, \\
s \in \text{vec}(\mathcal{S}_r^+)
\end{array} \right\}
\]

Recall that \( e = \text{vec}(XX^T - ZZ^T) \) and \( J_X \text{vec}(V) = \text{vec}(XX^T + VX^T) \) with respect to fixed \( X, Z \in \mathbb{R}^{n \times n} \). Observe that

\[
J^T_Z \text{vec}(S) = 0 \iff (S + S^T)Z = 0 \iff S = Q_\perp S \perp^T \iff \text{vec}(S) = Q_\perp \text{vec}(S_\perp)
\]

where the orthogonal complement \( Q_\perp \) of \( Z \) is such that \( Q_\perp^T Z^T = I_n - ZZ^T \), and \( Q_\perp = Q_\perp \otimes Q_\perp \). Substituting and taking the Lagrangian dual yields

\[
\kappa_{lb}(X, Z) = \min_{s, H, s_\perp} \left\{ \kappa : \begin{array}{l}
I \preceq H \preceq \kappa I, \\
J^T_X (He + s) = 0, \\
s \in \text{vec}((\mathcal{S}_r^+)_{\perp}), \\
J^T_X (He + s_\perp) = 0
\end{array} \right\}
\]

(\ref{eq:lb_dual_lemma})

in which \( y \in \mathbb{R}^{nr} \) and \( W_{i,j} \in \mathbb{R}^{n \times n} \) for \( i, j \in \{1, 2, \ldots, r\} \). Strong duality holds here because \( y = 0, W = \epsilon \cdot J_{nr}, U = \frac{1}{\epsilon} I_n, \) and \( V = \frac{1}{\epsilon} I_n + \epsilon \cdot r \cdot \text{vec}(I_n)^T + \text{vec}(I_n)^T \) is strictly feasible for a sufficiently small \( \epsilon > 0 \).

The dual problem (\ref{eq:lb_dual_lemma}) has a closed-form solution over \( U \) and \( V \) [\ref{bertsekas04}, Lemma 13]

\[
\frac{\text{tr}(M)}{\text{tr}(M^+)} = \min_{t, U, V} \left\{ \text{tr}(U) + t \cdot U, \quad \frac{t}{\text{tr}(V)} : \begin{array}{l}
U \geq 0, \\
V \geq 0
\end{array} \right\}
\]

and \( f = \epsilon T + \epsilon f^T \) has exactly two nonzero eigenvalues \( \epsilon T \pm \|\epsilon\| \|f\| \) [\ref{bertsekas04}, Lemma 14]. Substituting this solution yields exactly

\[
\kappa_{lb}(X, Z) = \max_{y, W_{i,j}} \left\{ \frac{\|e\| \|f\| + \epsilon T \|f\| + \epsilon T f^T + (J^T_X J_X, W)}{\|e\| \|f\| + \epsilon T f^T + (J^T_X J_X, W)} : \begin{array}{l}
y \equiv J_X y - \sum_{i=1}^{r} \text{vec}(W_{i,i}), \\
W = [W_{i,j}]_{i,j=1}^{r} \geq 0, \\
Q_\perp^T f \in \text{vec}(\mathcal{S}_r^+) \end{array} \right\}
\]

(\ref{eq:lb_dual_lemma})

where \( \cos(\theta(t)) \) is itself defined as the following maximization

\[
\cos(\theta(t)) = \max_{y, W_{i,j}} \left\{ \frac{f \equiv J_X y - \sum_{i=1}^{r} \text{vec}(W_{i,i}), \quad \|e\| \|f\| = 1}{(J^T_X J_X, W) = 2t, \quad W = [W_{i,j}]_{i,j=1}^{r} \geq 0} \right\}
\]

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