SOME RESULTS ON SIMPSON TYPE CONFORMABLE FRACTIONAL INEQUALITIES

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ABSTRACT. In this paper we established a new Simpson type conformable fractional integral equality for convex functions. Based on this identity, some results related to Simpson-like type inequalities are obtained. These results are then applied to some special means of real numbers and two special functions, modified Bessel function and q–digamma function, respectively.

1. INTRODUCTION

We will start with the following inequality is well known in the literature as Simpson’s inequality.

**Theorem 1.** Let \( \psi : [\gamma, \delta] \to \mathbb{R} \) be a four times continuously differentiable mapping on \((\gamma, \delta)\) and \( \|\psi^{(4)}\|_\infty = \sup |\psi^{(4)}(\varepsilon)| < \infty \). Then, the following inequality holds:

\[
\left| \int_\gamma^\delta \psi(\varepsilon) \, d\varepsilon - \frac{\delta - \gamma}{3} \left[ \frac{\psi(\gamma) + \psi(\delta)}{2} + 2\psi\left(\frac{\gamma + \delta}{2}\right) \right] \right| \leq \frac{1}{2880} \|\psi^{(4)}\|_\infty (\delta - \gamma)^4. \tag{1.1}
\]

This inequality (1.1) has been studied by several authors, these papers can be seen in [2, 4, 7, 8, 9, 10, 11, 13, 14, 16, 18].

In [13], Sarikaya et al. obtained the following inequality for differentiable convex functions on Simpson’s inequality and used the following lemma to show this.

**Lemma 1.** Let \( \psi : I \subset \mathbb{R} \to \mathbb{R} \) be an absolutely continous mapping on \(I^o\) such that \( \psi' \in L^1(\gamma, \delta) \), where \( \gamma, \delta \in I^o \) with \( \gamma < \delta \). Then, the following equality holds:

\[
\frac{1}{6} \left[ \psi(\gamma) + 4\psi\left(\frac{\gamma + \delta}{2}\right) + \psi(\delta) \right] - \frac{1}{\delta - \gamma} \int_\gamma^\delta \psi(\varepsilon) \, d\varepsilon = \frac{\delta - \gamma}{2} \int_0^1 \left[ \left( \frac{w}{2} - \frac{1}{3} \right) \psi'\left(\frac{1 + w}{2} \delta + \frac{1 - w}{2} \gamma\right) + \left( \frac{1}{3} - w \right) \psi'\left(\frac{1 - w}{2} \gamma + \frac{1 - w}{2} \delta\right) \right] \, dw. \tag{1.2}
\]

The main theorem in [13] is as follows.

**Theorem 2.** Let \( \psi : I \subset \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \(I^o\) such that \( \psi' \in L^1[a, b] \), where \( \gamma, \delta \in I^o \) with \( \gamma < \delta \). If \( |\psi'|^q \) is convex on \([\gamma, \delta] \), \( q > 1 \), then the following inequality holds:

\[
\left| \frac{1}{6} \left[ \psi(\gamma) + 4\psi\left(\frac{\gamma + \delta}{2}\right) + \psi(\delta) \right] - \frac{1}{\delta - \gamma} \int_\gamma^\delta \psi(\varepsilon) \, d\varepsilon \right| \leq \frac{\delta - \gamma}{12} \left( \frac{1 + 2^q + 1}{3 (p + 1)} \right)^{\frac{1}{q}} \left[ \left( \frac{3 |\psi'(\delta)|^q + |\psi'(\gamma)|^q}{4} \right)^{\frac{1}{q}} + \left( \frac{3 |\psi'(\gamma)|^q + 3 |\psi'(\delta)|^q}{4} \right)^{\frac{1}{q}} \right]. \tag{1.3}
\]

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where \( \frac{1}{p} + \frac{1}{q} = 1 \).

**Definition 1.** A function \( \psi : [\gamma, \delta] \rightarrow \mathbb{R} \) is said to be convex on \([\gamma, \delta]\) if the inequality

\[
\psi(wa + (1 - w)b) \leq w\psi(a) + (1 - w)\psi(b)
\]

holds for all \( a, b \in [\gamma, \delta] \) and \( w \in [0, 1] \). If \( (-\psi) \) is convex, \( \psi \) is concave.

Convex functions are important for mathematical inequalities. Many authors obtained several inequalities for convex functions [3, 5, 15, 12]. The most famous inequality has been used with convex functions is Hermite-Hadamard, which is stated as follows:

Let \( \psi : [\gamma, \delta] \rightarrow \mathbb{R} \) be a convex function and \( a, b \in [\gamma, \delta] \) with \( a < b \). Then the following double inequalities hold:

\[
\psi \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b \psi(\varepsilon) \, d\varepsilon \leq \frac{\psi(a) + \psi(b)}{2}.
\]

The aim of this paper is to establish Simpson type conformable fractional integral inequalities based on convexity.

2. **Preliminaries**

In this section, we give some definitions and basic results we will use.

**Definition 2.** Let \( \gamma, \delta \in \mathbb{R} \) with \( \gamma < \delta \) and \( f \in L[\gamma, \delta] \). The left and right Riemann-Liouville fractional integrals \( J_{\gamma+}^\tau f \) and \( J_{\delta-}^\tau f \) of order \( \tau > 0 \) are defined by

\[
J_{\gamma+}^\tau f = \frac{1}{\Gamma(\tau)} \int_{\gamma}^{\varepsilon} (\varepsilon - w)^{\tau-1} \psi(w) \, dw, \ \varepsilon > \gamma
\]

and

\[
J_{\delta-}^\tau f = \frac{1}{\Gamma(\tau)} \int_{\varepsilon}^{\delta} (w - \varepsilon)^{\tau-1} \psi(w) \, dw, \ \varepsilon < \delta
\]

respectively, where \( \Gamma(\tau) \) is the Gamma function defined by \( \Gamma(\tau) = \int_0^{\infty} e^{-w} t^{\tau-1} \, dw \) (see [4], p. 69).

The following definition of conformable fractional integrals could be found in [11, 15].

**Definition 3.** Let \( \tau \in (m, m + 1] \), \( m = 0, 1, 2, ..., \beta = \tau - m \), \( \gamma, \delta \in \mathbb{R} \) with \( \gamma < \delta \) and \( \psi \in L[\gamma, \delta] \). The left and right conformable fractional integrals \( I_{\gamma}^\tau \psi \) and \( I_{\delta}^\tau \psi \) of order \( \alpha > 0 \) are defined by

\[
I_{\gamma}^\tau \psi = \frac{1}{m!} \int_{\gamma}^{\varepsilon} (\varepsilon - w)^m (w - \gamma)^{\beta-1} \psi(w) \, dw, \ \varepsilon > \gamma
\]

and

\[
I_{\delta}^\tau \psi = \frac{1}{m!} \int_{\varepsilon}^{\delta} (w - \varepsilon)^m (\delta - w)^{\beta-1} \psi(w) \, dw, \ \varepsilon < \delta
\]

respectively.

It is easily seen that if one takes \( \tau = m + 1 \) in the Definition 3 (for the left and right conformable fractional integrals), one has the Definition 2 (the left and right Riemann-Liouville fractional integrals) for \( \tau \in \mathbb{N} \).
Lemma 2. Let \( \psi : I \subset (0, \infty) \to \mathbb{R} \), be a differentiable function on \( I^0, \gamma, \delta \in I^0 \) and \( \gamma < \delta \). If \( \psi' \in L[\gamma, \delta] \), then the following equality holds:

\[
\begin{align*}
I_1 & = \frac{1}{m!} \left[ \int_0^1 \left( \frac{1}{3} \beta (m + 1, \tau - m) - \frac{1}{2} \beta_w (m + 1, \tau - m) \right) \psi' \left( \frac{1+w}{2} \gamma + \frac{1-w}{2} \delta \right) \, dw \\
& - \frac{1}{2m!} \int_0^1 \left( \frac{1}{2} \beta_w (m + 1, \tau - m) \psi' \left( \frac{1+w}{2} \gamma + \frac{1-w}{2} \delta \right) \, dw \right)^2 \\
& - \frac{1}{2m!} \int_0^1 \left( \int_0^w \left( \frac{1}{2} \beta_w (m + 1, \tau - m) \psi' \left( \frac{1+w}{2} \gamma + \frac{1-w}{2} \delta \right) \, dw \right)^2 \right) \\
& - \frac{1}{2m!} \int_0^1 \left( \frac{1}{2} \beta_w (m + 1, \tau - m) \psi' \left( \frac{1+w}{2} \gamma + \frac{1-w}{2} \delta \right) \, dw \right)^2 \\
& = \frac{2}{\gamma - \delta} \left[ \int_0^w \left( \int_0^1 (1 - \varepsilon)^{\tau - m - 1} \, d\varepsilon \right) \psi' \left( \frac{1+w}{2} \gamma + \frac{1-w}{2} \delta \right) \, dw \right] \\
& + \int_0^w \left( \int_0^1 (1 - \varepsilon)^{\tau - m - 1} \, d\varepsilon \right) \psi' \left( \frac{1+w}{2} \gamma + \frac{1-w}{2} \delta \right) \, dw \\
& = \frac{2}{\gamma - \delta} \left[ \int_0^w \left( \int_0^1 (1 - \varepsilon)^{\tau - m - 1} \, d\varepsilon \right) \psi' \left( \frac{1+w}{2} \gamma + \frac{1-w}{2} \delta \right) \, dw \right] \\
& + \int_0^w \left( \int_0^1 (1 - \varepsilon)^{\tau - m - 1} \, d\varepsilon \right) \psi' \left( \frac{1+w}{2} \gamma + \frac{1-w}{2} \delta \right) \, dw \\
& = \frac{2}{\gamma - \delta} \left[ \int_0^w \left( \int_0^1 (1 - \varepsilon)^{\tau - m - 1} \, d\varepsilon \right) \psi' \left( \frac{1+w}{2} \gamma + \frac{1-w}{2} \delta \right) \, dw \right] \\
& + \int_0^w \left( \int_0^1 (1 - \varepsilon)^{\tau - m - 1} \, d\varepsilon \right) \psi' \left( \frac{1+w}{2} \gamma + \frac{1-w}{2} \delta \right) \, dw \\
& \text{and similarly}
\end{align*}
\]

Proof. We start by considering the following computations which follows from change of variables and using the definition of the conformable fractional integrals.
\[ I_2 = \frac{1}{m!} \int_0^1 \left( \frac{1}{3} \beta (m + 1, \tau - m) - \frac{1}{2} \beta_w (m + 1, \tau - m) \right) \psi' \left( \frac{1+w}{2} \gamma + \frac{1-w}{2} \delta \right) \, dw \]

\[ = \frac{2}{\delta - \gamma} \frac{\Gamma (\tau - m)}{\Gamma (\tau + 1)} \left( -\frac{1}{6} \psi (\delta) - \frac{1}{3} \psi \left( \frac{\gamma + \delta}{2} \right) \right) - \frac{1}{2} \left( \frac{2}{\delta - \gamma} \right)^{\tau+1} \delta I_\tau \psi \left( \frac{\gamma + \delta}{2} \right). \]

Thus, we can write

\[ I_1 - I_2 = \frac{1}{3(\delta - \gamma)} \frac{\Gamma (\tau - m)}{\Gamma (\tau + 1)} \left[ \psi (\gamma) + \psi (\delta) + 4\psi \left( \frac{\gamma + \delta}{2} \right) \right] - \frac{1}{2} \left( \frac{2}{\delta - \gamma} \right)^{\tau+1} \left[ I_\tau \psi \left( \frac{\gamma + \delta}{2} \right) + \delta I_\tau \psi \left( \frac{\gamma + \delta}{2} \right) \right], \]

which is completes the proof.

\[ \square \]

**Remark 1.** If we take \( \tau = m + 1 \) in Lemma 4 we have the following equality

\[ I_1 = \frac{1}{6} \left[ \psi (\gamma) + 4\psi \left( \frac{\gamma + \delta}{2} \right) + \psi (\delta) \right] - \frac{2^{\tau-1}}{(\delta - \gamma)} \frac{\Gamma (\tau - m)}{\Gamma (\tau + 1)} \left[ I_{\tau+} \psi \left( \frac{\gamma + \delta}{2} \right) + I_{\tau-} \psi \left( \frac{\gamma + \delta}{2} \right) \right] \]

which is proved by Matloka in [8, Lemma 5].

**Remark 2.** If we take \( \tau = 1 \) in Remark 7 we have the following equality

\[ \frac{1}{6} \left[ \psi (\gamma) + 4\psi \left( \frac{\gamma + \delta}{2} \right) + \psi (\delta) \right] - \frac{1}{\delta - \gamma} \int_\gamma^\delta \psi (\varepsilon) \, d\varepsilon \]

\[ = \frac{\delta - \gamma}{2} \int_0^1 \left( 1 - \frac{w}{2} \right) \psi' \left( \frac{1+w}{2} \gamma + \frac{1-w}{2} \delta \right) \, dw \]

in Theorem 4.

**Theorem 3.** Let \( \psi : I \subset (0, \infty) \to \mathbb{R} \), be a differentiable function on \( I^o, \gamma, \delta \in I^o \) and \( \gamma < \delta \). If \( \psi' \in L [\gamma, \delta] \) and \( |\psi'| \) is a convex function on \( [\gamma, \delta] \), then the following inequality holds:

\[ \frac{1}{6} \left[ \psi (\gamma) + 4\psi \left( \frac{\gamma + \delta}{2} \right) + \psi (\delta) \right] - \frac{2^{\tau-1}}{(\delta - \gamma)^{\tau}} \frac{\Gamma (\tau + 1)}{\Gamma (\tau - m)} \left[ I_{\tau-} \psi \left( \frac{\gamma + \delta}{2} \right) + \delta I_\tau \psi \left( \frac{\gamma + \delta}{2} \right) \right] \leq \frac{\delta - \gamma}{2m!} \frac{\Gamma (\tau - m)}{\Gamma (\tau + 1)} Z_1 (\tau, m) \left( |\psi' (\gamma)| + |\psi' (\delta)| \right) \]

where

\[ Z_1 (\tau, m) = \int_0^1 \left| \frac{1}{3} \beta (m + 1, \tau - m) - \frac{1}{2} \beta_w (m + 1, \tau - m) \right| \, dw \]

with \( m = 0, 1, 2, \ldots \) and \( \tau \in (m, m + 1) \).
Proof. From Lemma 2 and \(|\psi'|\) is convex, we have
\[
\left| \frac{1}{6} \left[ \psi (\gamma) + 4 \psi \left( \frac{\gamma + \delta}{2} \right) + \psi (\delta) \right] - \frac{2^{r-1}}{(\delta - \gamma)^r} \frac{\Gamma (\tau + 1)}{\Gamma (\tau - m)} \left[ I_{\tau}^\gamma \psi \left( \frac{\gamma + \delta}{2} \right) + \delta I_{\tau} \psi \left( \frac{\gamma + \delta}{2} \right) \right] \right|
\]
\[
\leq \frac{b - a}{2m!} \frac{\Gamma (\tau + 1)}{\Gamma (\tau - m)} \int_0^1 \left| \left( \frac{1}{2} \beta (m + 1, \tau - m) - \frac{1}{2} \beta_w (m + 1, \tau - m) \right) \right| \left| \psi' \left( \frac{1 + w}{2} \gamma + \frac{1 - w}{2} \delta \right) \right| + \left| \psi' \left( \frac{1 - w}{2} \gamma + \frac{1 + w}{2} \delta \right) \right| \, dw
\]
\[
\leq \frac{b - a}{2m!} \frac{\Gamma (\tau + 1)}{\Gamma (\tau - m)} \int_0^1 \left| \left( \frac{1}{2} \beta (m + 1, \tau - m) - \frac{1}{2} \beta_w (m + 1, \tau - m) \right) \right| \left| \psi' (\gamma) \right| + \left| \psi' (\delta) \right| \, dw
\]
\[
= \frac{b - a}{2m!} \frac{\Gamma (\tau + 1)}{\Gamma (\tau - m)} \left[ \int_0^1 \frac{1}{3} \beta (m + 1, \tau - m) - \frac{1}{2} \beta_w (m + 1, \tau - m) \right]
\]
\[
\times \left( \left| \psi' (\gamma) \right| + \left| \psi' (\delta) \right| \right) \, dw
\]
\[
= \frac{b - a}{2m!} \frac{\Gamma (\tau + 1)}{\Gamma (\tau - m)} Z_1 (\tau, m) \left( \left| \psi' (\gamma) \right| + \left| \psi' (\delta) \right| \right).
\]
This completes the proof. \(\square\)

Remark 3. If we take \(\tau = m + 1\), after that if we take \(\tau = 1\) in Theorem 3, we obtain the following inequality

(3.5) \[
\left| \frac{1}{6} \left[ \psi (\gamma) + 4 \psi \left( \frac{\gamma + \delta}{2} \right) + \psi (\delta) \right] - \frac{1}{\delta - \gamma} \int_\gamma^\delta \psi (\varepsilon) \, d\varepsilon \right|
\]
\[
\leq \frac{5 (b - a)}{72} (|f'(a)| + |f'(b)|)
\]
where
\[
\int_0^1 \left| \frac{1}{2} w - \frac{1}{3} \right| \, dw = \frac{5}{36}.
\]
See also (2, Corollary 1).

Theorem 4. Let \(\psi : I \subset (0, \infty) \to \mathbb{R}\), be a differentiable function on \(I^\circ, \gamma, \delta \in I^\circ\) and \(\gamma < \delta\). If \(\psi' \in L [\gamma, \delta]\) and \(|\psi'|^q\) is a convex function on \([\gamma, \delta]\) for \(q > 1\) and \(\frac{1}{p} + \frac{1}{q} = 1\), then the following inequality holds:

(3.6) \[
\left| \frac{1}{6} \left[ \psi (\gamma) + 4 \psi \left( \frac{\gamma + \delta}{2} \right) + \psi (\delta) \right] - \frac{2^{r-1}}{(\delta - \gamma)^r} \frac{\Gamma (\tau + 1)}{\Gamma (\tau - m)} \left[ I_{\tau}^\gamma \psi \left( \frac{\gamma + \delta}{2} \right) + \delta I_{\tau} \psi \left( \frac{\gamma + \delta}{2} \right) \right] \right|
\]
\[
\leq \frac{\delta - \gamma}{m!} \frac{\Gamma (\tau + 1)}{\Gamma (\tau - m)} \left( \frac{1}{2} \right)^{2q+1} \left( Z_2 (\tau, m) \right)^\frac{1}{2}
\]
\[
\left[ \left( 3 \left| \psi' (\gamma) \right|^q + \left| \psi' (\delta) \right|^q \right)^\frac{1}{q} \right] + \left( \left| \psi' (\gamma) \right|^q + \left| \psi' (\delta) \right|^q \right)^\frac{1}{q}.
\]
where
\[
Z_2 (\tau, m) = \int_0^1 \left| \frac{1}{3} \beta (m + 1, \tau - m) - \frac{1}{2} \beta_w (m + 1, \tau - m) \right|^p \, dw
\]
Proof. From Lemma 2 and using the Hölder’s integral inequality and the convexity of $|\psi'|^q$, we have

\[
\frac{1}{6} \left[ \psi(\gamma) + 4 \psi \left( \frac{\gamma + \delta}{2} \right) + \psi(\delta) \right] - \frac{2^{\tau-1}}{(\delta - \gamma)^\tau} \frac{\Gamma(\tau + 1)}{\Gamma(\tau - m)} \left[ I_1 \psi \left( \frac{\gamma + \delta}{2} \right) + \delta I_\tau \psi \left( \frac{\gamma + \delta}{2} \right) \right]
\]

\[
\leq \frac{\delta - \gamma}{2. m! \Gamma(\tau - m)} \left\{ \left( \int_0^1 \left| \frac{1}{3} \beta (m + 1, \tau - m) - \frac{1}{2} \beta_w (m + 1, \tau - m) \right|^p \, dw \right) \right\}^{\frac{1}{p}}
\times \left\{ \left( \int_0^1 \left| \psi' \left( \frac{1+w}{2} \gamma + \frac{1-w}{2} \delta \right) \right|^q \, dw \right) \right\}^{\frac{1}{q}}
\]

\[
\leq \frac{\delta - \gamma}{2. m! \Gamma(\tau - m)} \left\{ \left( \int_0^1 \left| \psi' \left( \frac{1+w}{2} \gamma + \frac{1-w}{2} \delta \right) \right|^q \, dw \right) \right\}^{\frac{1}{q}}
\times \left\{ \left( \int_0^1 \left| \psi' \left( \frac{1+w}{2} \gamma + \frac{1-w}{2} \delta \right) \right|^q \, dw \right) \right\}^{\frac{1}{q}}
\]

\[
\leq \frac{\delta - \gamma}{2. m! \Gamma(\tau - m)} \left\{ \left( \int_0^1 \left| \psi' \left( \frac{1+w}{2} \gamma + \frac{1-w}{2} \delta \right) \right|^q \, dw \right) \right\}^{\frac{1}{q}}
\times \left\{ \left( \int_0^1 \left| \psi' \left( \frac{1+w}{2} \gamma + \frac{1-w}{2} \delta \right) \right|^q \, dw \right) \right\}^{\frac{1}{q}}
\]

\[
\leq \frac{\delta - \gamma}{m! \Gamma(\tau - m)} \left( \frac{1}{2} \right)^{2q+1} (Z_2(\tau, m))^{\frac{1}{p}} \left[ \left( 3 \left| \psi'(\gamma) \right|^q + \left| \psi'(\delta) \right|^q \right)^\frac{1}{q} \right]^{\frac{1}{q}}
\]

This completes the proof.
Remark 4. If we take \( \tau = m + 1 \), after that if we take \( \tau = 1 \) in Theorem 4, we obtain the following inequality

\[
\left| \frac{1}{6} \left[ \psi(\gamma) + 4\psi \left( \frac{\gamma + \delta}{2} \right) + \psi(\delta) \right] - \frac{1}{\delta - \gamma} \int_{\gamma}^{\delta} \psi(\varepsilon) \, d\varepsilon \right| \\
\leq \frac{\delta - \gamma}{12} \left( \frac{2^{p+1} + 1}{3(p + 1)} \right)^{\frac{1}{q}} \left( \frac{1}{4} \right)^{\frac{1}{q}} \\
\times \left[ (3 \left| \psi'(\gamma) \right|^q + \left| \psi'(\delta) \right|^q) \right]^{\frac{1}{q}} \\
+ \left[ (3 \left| \psi'(\gamma) \right|^q + 3 \left| \psi'(\delta) \right|^q) \right]^{\frac{1}{q}}
\]

with

\[
\int_{0}^{1} \left| \frac{1}{2} - \frac{1}{3} \right|^p \, dw = \frac{2^{p+2} + 2}{(p + 1)6^{p+1}}.
\]

See also ([13], Theorem 4).

Theorem 5. Let \( \psi : I \subset (0, \infty) \rightarrow \mathbb{R} \) be a differentiable function on \( I^\circ, \gamma, \delta \in I^\circ \) and \( \gamma < \delta \). If \( \psi' \in L[\gamma, \delta] \) and \( |\psi'|^q \) is a convex function on \( [\gamma, \delta] \) for \( q \geq 1 \), then the following inequality holds:

\[
(3.7) \left| \frac{1}{6} \left[ \psi(\gamma) + 4\psi \left( \frac{\gamma + \delta}{2} \right) + \psi(\delta) \right] - \frac{2^{r-1} \Gamma(\tau + 1)}{(\delta - \gamma)^2 \Gamma(\tau - m)} \left[ I_\tau^* \psi \left( \frac{\gamma + \delta}{2} \right) + \delta I_\tau \psi \left( \frac{\gamma + \delta}{2} \right) \right] \right| \\
\leq \frac{\delta - \gamma}{(2m) \Gamma(\tau - m)} \left( Z_1(\tau, m) \right)^{1-\frac{1}{q}} \left[ (Z_3(\tau, m) \left| \psi'(\gamma) \right|^q + Z_4(\tau, m) \left| \psi'(\delta) \right|^q) \right]^{\frac{1}{q}} \\
+ \left[ (Z_4(\tau, m) \left| \psi'(\gamma) \right|^q + Z_3(\tau, m) \left| \psi'(\delta) \right|^q) \right]^{\frac{1}{q}}
\]

where

\[
Z_3(\tau, m) = \int_{0}^{1} \left| \frac{1}{3} \beta (m + 1, \tau - m) - \frac{1}{2} \beta (m + 1, \tau - m) \right| \frac{1+w}{2} \, dw
\]

\[
Z_4(\tau, m) = \int_{0}^{1} \left| \frac{1}{3} \beta (m + 1, \tau - m) - \frac{1}{2} \beta (m + 1, \tau - m) \right| \frac{1-w}{2} \, dw
\]

and \( Z_1(\tau, m) \) is defined as in the Theorem 3 with \( m = 0, 1, 2, \ldots \) and \( \tau \in (m, m+1] \).
Proof. From Lemma 2 and using the power mean inequality, we have that the following inequality holds:

\[
\left| \frac{1}{6} \left[ \psi(\gamma) + 4\psi \left( \frac{\gamma + \delta}{2} \right) + \psi(\delta) \right] - \frac{2^{\tau-1}}{(\delta - \gamma)^\tau} \Gamma(\tau + 1) \left[ I_{\tau} \psi \left( \frac{\gamma + \delta}{2} \right) + \delta I_{\tau} \psi \left( \frac{\gamma + \delta}{2} \right) \right] \right|
\]

\[
\leq \frac{\delta - \gamma}{\Gamma(\tau + 1)} \frac{\Gamma(\tau + 1)}{(\tau - m)} \left\{ \int_0^1 \left| \frac{1}{2} \beta(m + 1, \tau - m) - \frac{1}{2} \beta_w(m + 1, \tau - m) \right| \left| \psi' \left( \frac{1+\mu}{2} \gamma + \frac{1-\mu}{2} \delta \right) \right|^q dw \right\} \frac{1}{q}
\]

By the convexity of \(|\psi'|^q|

\[
\int_0^1 \left| \frac{1}{2} \beta(m + 1, \tau - m) - \frac{1}{2} \beta_w(m + 1, \tau - m) \right| \left| \psi' \left( \frac{1+\mu}{2} \gamma + \frac{1-\mu}{2} \delta \right) \right|^q dw
\]

\[
\leq \left| \psi'(\gamma) \right|^q \int_0^1 \left| \frac{1}{2} \beta(m + 1, \tau - m) - \frac{1}{2} \beta_w(m + 1, \tau - m) \right| \frac{1+\mu}{2} dt
\]

\[
+ \left| \psi'(\delta) \right|^q \int_0^1 \left| \frac{1}{2} \beta(m + 1, \tau - m) - \frac{1}{2} \beta_w(m + 1, \tau - m) \right| \frac{1-\mu}{2} dw
\]

and

\[
\int_0^1 \left| \frac{1}{2} \beta(m + 1, \tau - m) - \frac{1}{2} \beta_w(m + 1, \tau - m) \right| \left| \psi' \left( \frac{1-\mu}{2} \gamma + \frac{1+\mu}{2} \delta \right) \right|^q dt
\]

\[
\leq \left| \psi'(\gamma) \right|^q \int_0^1 \left| \frac{1}{2} \beta(m + 1, \tau - m) - \frac{1}{2} \beta_w(m + 1, \tau - m) \right| \frac{1-\mu}{2} dt
\]

\[
+ \left| \psi'(\delta) \right|^q \int_0^1 \left| \frac{1}{2} \beta(m + 1, \tau - m) - \frac{1}{2} \beta_w(m + 1, \tau - m) \right| \frac{1+\mu}{2} dw
\]

Using the last two inequalities we obtain
\[\frac{1}{6} \left[ \psi \left( \gamma + \frac{\delta}{2} \right) + \psi \left( \delta \right) \right] - \frac{2^{\gamma-1}}{(\delta - \gamma)^{\gamma}} \frac{\Gamma(\tau + 1)}{\Gamma(\tau - m)} \left[ I^\gamma_\tau \psi \left( \frac{\gamma + \delta}{2} \right) + \delta I^\gamma_\tau \psi \left( \frac{\gamma + \delta}{2} \right) \right] \]

\[\leq \frac{\delta - \gamma}{2m!} \frac{\Gamma(\tau + 1)}{\Gamma(\tau - m)} \left( \int_0^1 \left[ \frac{1}{3} \beta (m + 1, \tau - m) - \frac{1}{2} \beta_w (m + 1, \tau - m) \right] dw \right) \]

\[\leq \frac{\delta - \gamma}{2m!} \frac{\Gamma(\tau + 1)}{\Gamma(\tau - m)} \left( \int_0^1 \left[ \frac{1}{3} \beta (m + 1, \tau - m) - \frac{1}{2} \beta_w (m + 1, \tau - m) \right] dw \right) \]

\[\leq \frac{\delta - \gamma}{2} \left( \frac{5}{36} \right)^{1-\frac{\tau}{\gamma}} \left( \frac{1}{648} \right)^{\frac{\tau}{2}} \left[ \left( 61 \right)^\frac{\gamma}{2} \right] \left[ \left( 29 \right)^\frac{\gamma}{2} \right] \left[ \left( 61 \right)^\frac{\gamma}{2} \right] \left[ \left( 29 \right)^\frac{\gamma}{2} \right] \]

See also (II, Remark 2.7).

Remark 5. If we take \( \tau = m + 1 \), after that if we take \( \tau = 1 \) in Theorem 5, we obtain the following inequality

\[\text{(3.8)} \left\| \frac{1}{6} \left[ \psi \left( \gamma + \frac{\delta}{2} \right) + \psi \left( \delta \right) \right] - \frac{1}{\delta - \gamma} \int_0^\gamma \psi (\varepsilon) \, d\varepsilon \right\| \leq \frac{\delta - \gamma}{2} \left( \frac{5}{36} \right)^{1-\frac{\tau}{\gamma}} \left( \frac{1}{648} \right)^{\frac{\tau}{2}} \left[ \left( 61 \right)^{1-\frac{\tau}{\gamma}} \left( 29 \right)^{1-\frac{\tau}{\gamma}} \right] \left[ \left( 61 \right)^{1-\frac{\tau}{\gamma}} \left( 29 \right)^{1-\frac{\tau}{\gamma}} \right] \]

Theorem 6. Let \( \psi : I \subset (0, \infty) \to \mathbb{R} \), be a differentiable function on \( I^\gamma, \gamma, \delta \in I^\gamma \) and \( \gamma < \delta \). If \( f' \in L \left( [\gamma, \delta] \right) \) and \( |\psi'|^q \) is a convex function on \( [\gamma, \delta] \) for \( q > 1 \) and \( \frac{1}{p} + \frac{1}{q} = 1 \), then the following inequality holds:

\[\text{(3.9)} \left\| \frac{1}{6} \left[ \psi \left( \gamma + \frac{\delta}{2} \right) + \psi \left( \delta \right) \right] - \frac{2^{\gamma-1}}{(\delta - \gamma)^{\gamma}} \frac{\Gamma(\tau + 1)}{\Gamma(\tau - m)} \left[ I^\gamma_\tau \psi \left( \frac{\gamma + \delta}{2} \right) + \delta I^\gamma_\tau \psi \left( \frac{\gamma + \delta}{2} \right) \right] \right\| \leq \frac{\delta - \gamma}{2m!} \frac{\Gamma(\tau + 1)}{\Gamma(\tau - m)} \left( Z_2 (\tau, m) \right) \]

where \( Z_2 (\tau, m) \) is defined as in Theorem 4 with \( m = 0, 1, 2, \ldots \) and \( \tau \in (m, m + 1) \).
Proof. From Lemma 2 and using the Hölder’s inequality, we have
\[
\left| \frac{1}{6} \left[ \psi(\gamma) + 4\psi \left(\frac{\gamma + \delta}{2}\right) + \psi(\delta) \right] - \frac{2^{r-1}}{(\delta - \gamma)^r} \frac{\Gamma(\tau + 1)}{\Gamma(\tau - m)} \left[ I^\gamma \psi \left(\frac{\gamma + \delta}{2}\right) + \delta I^\gamma \psi \left(\frac{\gamma + \delta}{2}\right) \right] \right|
\]
\[
\leq \delta - \gamma \frac{\Gamma(\tau + 1)}{2. m! \Gamma(\tau - m)} \left\{ \left( \int_0^1 \left| \frac{1}{3} \beta (m + 1, \tau - m) - \frac{1}{2} \beta_w (m + 1, \tau - m) \right|^p \, dw \right)^{\frac{1}{p}} \times \left( \int_0^1 \left| \psi' \left(\frac{1+w}{2} \gamma + \frac{1-w}{2} \delta\right) \right|^q \, dw \right)^{\frac{1}{q}} \right\}^{\frac{1}{p}}
\]
\[
+ \left( \int_0^1 \left| \frac{1}{3} \beta (m + 1, \tau - m) - \frac{1}{2} \beta_w (m + 1, \tau - m) \right|^p \, dw \right)^{\frac{1}{p}} \times \left( \int_0^1 \left| \psi' \left(\frac{1+w}{2} \gamma + \frac{1-w}{2} \delta\right) \right|^q \, dw \right)^{\frac{1}{q}}
\]
\[
\leq \delta - \gamma \frac{\Gamma(\tau + 1)}{2. m! \Gamma(\tau - m)} \left\{ \left( \int_0^1 \left| \frac{1}{3} \beta (m + 1, \tau - m) - \frac{1}{2} \beta_w (m + 1, \tau - m) \right|^p \, dw \right)^{\frac{1}{p}} \times \left( \int_0^1 \left| \psi' \left(\frac{1+w}{2} \gamma + \frac{1-w}{2} \delta\right) \right|^q \, dw \right)^{\frac{1}{q}} \right\}^{\frac{1}{p}}
\]
\[
\times \left[ \int_0^1 \left| \psi' \left(\gamma\right) \right|^q \, dw \right]^\frac{1}{q} + \left[ \int_0^1 \left| \psi' \left(\frac{\gamma + \delta}{2}\right) \right|^q \, dw \right]^\frac{1}{q}
\]
Since \( |\psi'|^q \) is convex by 1.3, we have
\[
\int_0^1 \left| \psi' \left(\frac{1+w}{2} \gamma + \frac{1-w}{2} \delta\right) \right|^q \, dw \leq \left| \psi' \left(\gamma\right) \right|^q + \left| \psi' \left(\frac{\gamma + \delta}{2}\right) \right|^q,
\]
and
\[
\int_0^1 \left| \psi' \left(\frac{1-w}{2} \gamma + \frac{1+w}{2} \delta\right) \right|^q \, dw \leq \frac{\left| \psi' \left(\frac{\gamma + \delta}{2}\right) \right|^q + |\psi'(\delta)|^q}{2}.
\]
So, we obtain
\[
\left| \frac{1}{6} \left[ \psi(\gamma) + 4\psi \left(\frac{\gamma + \delta}{2}\right) + \psi(\delta) \right] - \frac{2^{r-1}}{(\delta - \gamma)^r} \frac{\Gamma(\tau + 1)}{\Gamma(\tau - m)} \left[ I^\gamma \psi \left(\frac{\gamma + \delta}{2}\right) + \delta I^\gamma \psi \left(\frac{\gamma + \delta}{2}\right) \right] \right|
\]
\[
\leq \delta - \gamma \frac{\Gamma(\tau + 1)}{2. m! \Gamma(\tau - m)} \left( \int_0^1 \left| \frac{1}{3} \beta (m + 1, \tau - m) - \frac{1}{2} \beta_w (m + 1, \tau - m) \right|^p \, dx \right)^{\frac{1}{p}} \times \left[ \left| \psi' \left(\gamma\right) \right|^q + \left| \psi' \left(\frac{\gamma + \delta}{2}\right) \right|^q \right]^\frac{1}{q}
\]
\[
+ \left[ \left| \psi' \left(\frac{\gamma + \delta}{2}\right) \right|^q + |\psi'(\delta)|^q \right]^\frac{1}{q}
\]
\(\square\)
Remark 6. If we take $\tau = m + 1$, after that if we take $\tau = 1$ in Theorem 4, we obtain the following inequality

$$\left| \frac{1}{6} \left[ \psi(\gamma) + 4\psi\left(\frac{\gamma + \delta}{2}\right) + \psi(\delta) \right] - \frac{1}{\delta - \gamma} \int_{\gamma}^{\delta} \psi(\varepsilon) \, d\varepsilon \right| \leq \frac{\delta - \gamma}{12} \left( \frac{2^{p+1} + 1}{3(p+1)} \right)^{\frac{1}{p}} \left( \frac{1}{4} \right)^{\frac{1}{q}} \times \left[ \left( |\psi'(\gamma)|^q + 3 |\psi'\left(\frac{\gamma + \delta}{2}\right)|^q \right) \right] \times \left[ + (3 |\psi'\left(\frac{\gamma + \delta}{2}\right)|^q + |\psi'(\delta)|^q \right]$$

with

$$\int_{0}^{1} \left| \frac{1}{2} w - \frac{1}{3} \right|^p \, dw = \frac{2^{p+2} + 2}{(p+1)6^{p+1}}.$$

See also ([13], Theorem 9 (for $s = 1$)).

4. Estimation Results

In the function $\psi'$ is bounded from below and above, then we have the following result.

Theorem 7. Let $\psi : [\gamma, \delta] \rightarrow \mathbb{R}$ be differentiable and continuous mappings on $(\gamma, \delta)$ and let $\psi' \in L[\gamma, \delta]$. Assume that there exist constants $k < K$ such that $-\infty < k \leq \psi' \leq K < +\infty$. Then,

$$\left| \frac{1}{6} \left[ \psi(\gamma) + 4\psi\left(\frac{\gamma + \delta}{2}\right) + \psi(\delta) \right] - \frac{2^{p-1}}{(\delta - \gamma)} \frac{\Gamma(\tau+1)}{\Gamma(\tau-m)} \left[ I_\tau^\gamma \psi\left(\frac{\gamma + \delta}{2}\right) + \delta I_\tau^\gamma \psi\left(\frac{\gamma + \delta}{2}\right) \right] \right| \leq \frac{K-k}{2m!} \frac{\Gamma(\tau+1)}{\Gamma(\tau-m)} (\delta - \gamma) Z_1(\tau, m),$$

where

$$h(w) = \frac{1}{3} \int_{0}^{1} \beta(m+1, \tau-m) \, dw - \frac{1}{2} \int_{0}^{1} \beta_w(m+1, \tau-m) \, dw.$$
Proof. From Lemma 2, we have that
\[
\frac{1}{6} \left[ \psi (\gamma) + 4\psi \left( \frac{\gamma + \delta}{2} \right) + \psi (\delta) \right] - \frac{2^{\tau-1}}{(\delta - \gamma) \Gamma (\tau + 1)} \Gamma (\tau - m) \left[ I_{\gamma}^\tau \psi \left( \frac{\gamma + \delta}{2} \right) + \delta I_{\tau}^\gamma \psi \left( \frac{2 + \delta}{2} \right) \right]
\]
\[
= \delta - \gamma \frac{\Gamma (\tau + 1)}{2m! \Gamma (\tau - m)} \left\{ \int_0^1 h(w) \left[ \psi' \left( \frac{1+w}{2} \gamma + \frac{1-w}{2} \delta \right) - \frac{K+k}{2} \right] dw \right\}
\]
\[
+ \frac{K+k}{2} \left( \delta - \gamma \right) \int_0^1 h(w) dw.
\]
So
\[
\left| \frac{1}{6} \left[ \psi (\gamma) + 4\psi \left( \frac{2 + \delta}{2} \right) + \psi (\delta) \right] - \frac{2^{\tau-1}}{(\delta - \gamma) \Gamma (\tau + 1)} \Gamma (\tau - m) \left[ I_{\gamma}^\tau \psi \left( \frac{2 + \delta}{2} \right) + \delta I_{\tau}^\gamma \psi \left( \frac{2 + \delta}{2} \right) \right] \right|
\]
\[
\leq \delta - \gamma \frac{\Gamma (\tau + 1)}{2m! \Gamma (\tau - m)} \left\{ \int_0^1 \left| h(w) \right| \left[ \psi' \left( \frac{1+w}{2} \gamma + \frac{1-w}{2} \delta \right) - \frac{K+k}{2} \right] dw \right\}
\]
\[
+ \frac{K+k}{2} \left( \delta - \gamma \right) \int_0^1 \left| h(w) \right| dw.
\]
Since \( \psi' \) satisfies \(-\infty < k \leq \psi' \leq K < +\infty \), we have that
\[
k - \frac{K+k}{2} \leq \psi' - \frac{K+k}{2} \leq K - \frac{K+k}{2},
\]
which implies that
\[
\left| \psi' - \frac{K+k}{2} \right| \leq \frac{K-k}{2}.
\]
Hence,
\[
\left| \frac{1}{6} \left[ \psi (\gamma) + 4\psi \left( \frac{2 + \delta}{2} \right) + \psi (\delta) \right] - \frac{2^{\tau-1}}{(\delta - \gamma) \Gamma (\tau + 1)} \Gamma (\tau - m) \left[ I_{\gamma}^\tau \psi \left( \frac{2 + \delta}{2} \right) + \delta I_{\tau}^\gamma \psi \left( \frac{2 + \delta}{2} \right) \right] \right|
\]
\[
\leq \frac{K-k}{2} \Gamma (\tau + 1) \left( \delta - \gamma \right) \int_0^1 \left| h(w) \right| dw
\]
\[
\leq \frac{K-k}{2} \Gamma (\tau + 1) \left( \delta - \gamma \right) Z_1 (\tau, m),
\]
where

\[ \int_{0}^{1} |h(w)| \, dw = Z_1(\tau, m). \]

\(Z_1(\tau, m)\) is defined as in Lemma 2. This ends the proof. □

**Remark 7.** If we take \(\tau = m + 1\), after that if we take \(\tau = 1\) in Theorem 7, we obtain the following inequality

\[
\left| \frac{1}{6} \left[ \psi(\gamma) + 4\psi\left(\frac{\gamma + \delta}{2}\right) + \psi(\delta) \right] - \frac{1}{\delta - \gamma} \int_{\gamma}^{\delta} \psi(\varepsilon) \, d\varepsilon \right| \\
- \frac{K + k}{2} (\delta - \gamma) \int_{0}^{1} h(w) \, dw \\
\leq \frac{5}{72} (K - k) (\delta - \gamma).
\]

Our next aim is an estimation-type result considering the Simpson-like type conformable fractional integral inequality when \(\psi'\) satisfies a Lipschitz condition.

**Theorem 8.** Let \(\psi : [\gamma, \delta] \to \mathbb{R}\) be differentiable and continuous mappings on \((a, b)\) and let \(\psi' \in L[a, b]\). Assume that \(\psi'\) satisfies the Lipschitz condition for some \(L > 0\). Then,

\[
\left| \frac{1}{6} \left[ \psi(\gamma) + 4\psi\left(\frac{\gamma + \delta}{2}\right) + \psi(\delta) \right] - \frac{2^{\tau-1}}{(\delta - \gamma)} I_{[\gamma, \delta]} (\frac{\gamma + \delta}{2}) + \delta I_{[\gamma, \delta]} (\frac{\gamma + \delta}{2}) \right| \\
- (\delta - \gamma) \psi'\left(\frac{\gamma + \delta}{2}\right) \int_{0}^{1} h(w) \, dw \\
\leq L \frac{(\delta - \gamma)^2}{2m!} \frac{\Gamma(\tau + 1)}{\Gamma(\tau - m)} Z_5(\tau, m),
\]

where

\[ Z_5(\tau, m) = \int_{0}^{1} |h(w)| \, wdw \]

and \(h(w)\) is defined as in Theorem 7.
Proof. From Lemma 2, we have that

\[
\frac{1}{6} \left[ \psi(\gamma) + 4\psi\left(\frac{\gamma + \delta}{2}\right) + \psi(\delta) \right] - \frac{2^{\tau-1}}{(\delta - \gamma)^{\tau}} \frac{\Gamma(\tau + 1)}{\Gamma(\tau - m)} \left[ I^\tau_\gamma \psi\left(\frac{\gamma + \delta}{2}\right) + \delta I^\tau_\gamma \psi\left(\frac{\gamma + \delta}{2}\right) \right]
\]

So

\[
\frac{1}{6} \left[ \psi(\gamma) + 4\psi\left(\frac{\gamma + \delta}{2}\right) + \psi(\delta) \right] - \frac{2^{\tau-1}}{(\delta - \gamma)^{\tau}} \frac{\Gamma(\tau + 1)}{\Gamma(\tau - m)} \left[ I^\tau_\gamma \psi\left(\frac{\gamma + \delta}{2}\right) + \delta I^\tau_\gamma \psi\left(\frac{\gamma + \delta}{2}\right) \right]
\]

\[
+ (\delta - \gamma) \psi\left(\frac{\gamma + \delta}{2}\right) \int_0^1 h(w) dw.
\]

\[
\leq \frac{\delta - \gamma}{2m!} \frac{\Gamma(\tau + 1)}{\Gamma(\tau - m)} \left\{ \int_0^1 |h(w)| \left| \psi\left(\frac{1+w}{2}\gamma + \frac{1-w}{2}\delta\right) - \psi\left(\frac{\gamma + \delta}{2}\right) \right| dw \right\}.
\]

Since \(\psi\) satisfies Lipschitz conditions for some \(L > 0\), we have that

\[
| \psi\left(\frac{1+w}{2}\gamma + \frac{1-w}{2}\delta\right) - \psi\left(\frac{\gamma + \delta}{2}\right) | \leq L \left| \frac{1-w}{2}\gamma + \frac{1+w}{2}\delta - \frac{\gamma + \delta}{2} \right|
\]

\[
\leq \frac{\delta - \gamma}{2} L |w|
\]

and

\[
| \psi\left(\frac{1-w}{2}\gamma + \frac{1+w}{2}\delta\right) - \psi\left(\frac{\gamma + \delta}{2}\right) | \leq L \left| \frac{1-w}{2}\delta + \frac{1+w}{2}\gamma - \frac{\gamma + \delta}{2} \right|
\]

\[
\leq \frac{\delta - \gamma}{2} L |w|.
\]
Hence,
\[
\left| \frac{1}{6} \left[ \psi'(\gamma) + 4\psi'\left(\frac{\gamma + \delta}{2}\right) + \psi'(\delta) \right] - \frac{2^{\tau-1}}{(\delta - \gamma)^\tau} \frac{\Gamma(\tau+1)}{\Gamma(\tau - m)} \left[ I_\tau \psi'\left(\frac{\gamma + \delta}{2}\right) + 4I_{\tau-1} \psi'\left(\frac{\gamma + \delta}{2}\right) \right] - (\delta - \gamma) \psi'\left(\frac{\gamma + \delta}{2}\right) \int h(w) \, dw \right|
\]
\[
\leq L \frac{(\delta - \gamma)^2}{2.\tau!} \frac{\Gamma(\tau + 1)}{\Gamma(\tau - m)} \int_0^1 |h(w)| \, dw 
\]
\[
\leq L \frac{(\delta - \gamma)^2}{2.\tau!} \frac{\Gamma(\tau + 1)}{\Gamma(\tau - m)} Z_5(\tau, m),
\]
where
\[
\int_0^1 |h(w)| \, dw = Z_5(\tau, m).
\]
This ends the proof.

**Remark 8.** If we take \( \tau = m + 1 \), after that if we take \( \tau = 1 \) in Theorem 8, we obtain the following inequality
\[
\left| \frac{1}{6} \left[ \psi'(\gamma) + 4\psi'\left(\frac{\gamma + \delta}{2}\right) + \psi'(\delta) \right] - \frac{1}{\delta - \gamma} \int_{\gamma}^{\delta} \psi'(\varepsilon) \, d\varepsilon \right| - (\delta - \gamma) \psi'(\frac{\gamma + \delta}{2}) \int_0^1 h(w) \, dw 
\]
\[
\leq L \frac{2(\delta - \gamma)^2}{81}.
\]
with
\[
\int_0^1 \left| \frac{1}{3} - \frac{w}{2} \right| \, dw = \frac{4}{81}.
\]

5. **Applications**

5.1. **Special Means.** For \( 0 \leq \gamma < \delta \), we consider the following special means:

**Theorem 9.**
(i) The arithmetic mean: \( A(\gamma, \delta) = \frac{\gamma + \delta}{2} \),

(ii) The geometric mean: \( G(\gamma, \delta) = \sqrt{\gamma \delta} \),

(iii) The logarithmic mean: \( L(\gamma, \delta) = \frac{\delta - \gamma}{\ln \delta - \ln \gamma}, \gamma \delta \neq 0 \),

(iv) The logarithmic mean: \( L_s(\gamma, \delta) = \left( \frac{\delta^{s+1} - \gamma^{s+1}}{(s+1)(\gamma - \delta)} \right)^{1/s}, s \in \mathbb{Z} \setminus \{0, 1\} \).

Next, using the main results obtained in Section 3, we give some applications to special means of real numbers.

**Proposition 1.** Let \( 0 < \gamma < \delta, s \in \mathbb{N} \). Then
\[
| \frac{1}{3} A(\gamma^s, \delta^s) + \frac{2}{3} A^s(\gamma, \delta) - L^s(\gamma, \delta) | \leq \frac{5(\delta - \gamma)}{72} \gamma^{s-1} + \delta^{s-1}.
\]

**Proof.** The proof is obvious from Remark 3 applied \( \psi'(\varepsilon) = \varepsilon^s \).
Remark 9. If we take \( s = 1 \) in (5.1) we obtain the following inequality
\[
|A(\gamma, \delta) - L(\gamma, \delta)| \leq \frac{5(\delta - \gamma)}{36}.
\]

See also ([2], Page 13).

Proposition 2. Let \( 0 < \gamma < \delta, s \in \mathbb{N} \). Then
\[
|A(\gamma, \delta) - L(\gamma, \delta)| \leq \frac{\delta - \gamma}{2} \left( \frac{2p+1}{3(p+1)} \right)^{\frac{1}{q}} \left( \frac{1}{4} \right)^{\frac{1}{q}} \left[ \left( (s\gamma^{s-1})^{q} + 3(s\delta^{s-1})^{q} \right)^{\frac{1}{q}} + \left( (3s\gamma^{s-1})^{q} + (3s\delta^{s-1})^{q} \right)^{\frac{1}{q}} \right].
\]

Proof. The proof is obvious from Remark 4 applied \( \psi(\varepsilon) = \varepsilon^s \). \( \square \)

Remark 10. If we take \( s = 1 \) in (5.3) we obtain the following inequality
\[
|A(\gamma, \delta) - L(\gamma, \delta)| \leq (\delta - \gamma) \left( \frac{2p+1}{3(p+1)} \right)^{\frac{1}{q}}.
\]

Proposition 3. Let \( 0 < \gamma < \delta, s \in \mathbb{N} \) and \( 0 < q < 1 \). Then
\[
|A(\gamma, \delta) - L(\gamma, \delta)| \leq \frac{\delta - \gamma}{2} \left( \frac{5}{36} \right)^{1-\frac{1}{q}} \left( \frac{1}{648} \right)^{\frac{1}{q}} \left[ (61(s\gamma^{s-1})^{q} + 29(s\delta^{s-1})^{q})^{\frac{1}{q}} + (29(s\gamma^{s-1})^{q} + 61(s\delta^{s-1})^{q})^{\frac{1}{q}} \right].
\]

Proof. The proof is obvious from Remark 5 applied \( \psi(\varepsilon) = \varepsilon^s \). \( \square \)

Remark 11. If we take \( s = 1 \) in (5.5) we obtain the following inequality
\[
|A(\gamma, \delta) - L(\gamma, \delta)| \leq \frac{5(\delta - \gamma)}{36}.
\]

See also ([2], Page 13).

Proposition 4. Let \( 0 < \gamma < \delta, s \in \mathbb{N} \). Then
\[
|A(\gamma, \delta) - L(\gamma, \delta)| \leq \frac{\delta - \gamma}{12} \left( \frac{2p+1}{3(p+1)} \right)^{\frac{1}{q}} \left( \frac{1}{4} \right)^{\frac{1}{q}} \left[ \left( (s\gamma^{s-1})^{q} + 3 \left( s \left( \frac{\gamma + \delta}{2} \right)^{s-1} \right)^{q} \right)^{\frac{1}{q}} + \left( 3 \left( s \left( \frac{\gamma + \delta}{2} \right)^{s-1} \right)^{q} \right)^{\frac{1}{q}} \right].
\]

Remark 12. If we take \( s = 1 \) in (5.7) we obtain the following inequality
\[
|A(\gamma, \delta) - L(\gamma, \delta)| \leq \frac{\delta - \gamma}{6} \left( \frac{2p+1}{3(p+1)} \right)^{\frac{1}{q}}.
\]

Proposition 5. Let \( 0 < \gamma < \delta \). Then
\[
|A(\alpha, \beta) + \frac{2}{3} G(\alpha, \beta) - L(\alpha, \beta)| \leq (\ln \beta - \ln \alpha) \frac{5}{36} A(\alpha, \beta).
\]

Proof. The proof is obvious from the Remark 3 applied \( f(\varepsilon) = e^{\varepsilon}, \varepsilon > 0 \) and \( \alpha = e^{\gamma}, \beta = e^{\delta} \). \( \square \)

5.2. Inequalities for some special functions.
5.2.1. Modified Bessel function. Recall the first kind modified Bessel function \( I_{\rho} \), which has the series representation \( \left[17\right], \text{p.77} \)

\[
I_{\rho}(\varepsilon) = \sum_{n \geq 0} \frac{\left(\frac{\varepsilon}{2}\right)^{\rho+2n}}{n! \Gamma(\rho + n + 1)},
\]

where \( \varepsilon \in \mathbb{R} \) and \( \rho > -1 \), while the second kind modified Bessel function \( K_{\rho} \) \( \left[17\right], \text{p.78} \) is usually defined as

\[
K_{\rho}(\varepsilon) = \frac{\pi}{2} \frac{I_{-\rho}(\varepsilon) - I_{\rho}(\varepsilon)}{\sin \rho \pi}.
\]

Here, we consider the function \( \Psi_{\rho} : \mathbb{R} \to [1, \infty) \) defined by

\[
\Psi_{\rho}(\varepsilon) = 2\rho \Gamma(\rho+1) \varepsilon^{-\rho} I_{\rho}(\varepsilon),
\]

where \( \Gamma \) is the Gamma function.

**Proposition 6.** Let \( \rho > -1 \) and \( 0 < \gamma < \delta \). Then

\[
| \frac{1}{6} \left[ \phi_{\rho}(\gamma) + 4\phi_{\rho} \left( \frac{\gamma + \delta}{2} \right) + \phi_{\rho}(\delta) \right] - \frac{1}{\delta - \gamma} \int_{\gamma}^{\delta} \phi_{\rho}(\varepsilon) \, d\varepsilon | \leq \frac{1}{2} \gamma \frac{1}{\rho + 1} \left( \frac{5}{36} \right)^{\frac{1}{\rho + 1}} \left( \frac{1}{648} \right)^{\frac{1}{\rho + 1}} \left[ (61 \gamma |\phi_{\rho+1}(\gamma)|^q + 29 \delta |\phi_{\rho+1}(\delta)|^q \right]^{\frac{1}{\rho + 1}}.
\]

Specially, if \( \rho = \frac{1}{2} \), then

\[
| \frac{1}{6} \left[ \cosh(\gamma) + 4 \cosh \left( \frac{\gamma + \delta}{2} \right) + \cosh(\delta) \right] - \frac{1}{\delta - \gamma} \int_{\gamma}^{\delta} \cosh(\varepsilon) \, d\varepsilon | \leq \frac{1}{2} \gamma \frac{1}{\rho + 1} \left( \frac{5}{36} \right)^{\frac{1}{\rho + 1}} \left( \frac{1}{648} \right)^{\frac{1}{\rho + 1}} \left[ (61 \gamma |\sinh(\gamma)|^q + 29 \delta |\sinh(\delta)|^q \right]^{\frac{1}{\rho + 1}}.
\]

**Proof.** Apply inequality \( 3.8 \) to the mapping \( f(\varepsilon) = \phi_{\rho}(\varepsilon), \varepsilon > 0 \) and \( \phi_{\rho}'(\varepsilon) = \frac{\varepsilon}{\rho+1} \phi_{\rho+1}(\varepsilon) \). Now taking into account the relations \( \phi_{\frac{1}{\rho}}(\varepsilon) = \cosh(\varepsilon) \) and \( \phi_{\frac{1}{\rho}}(\varepsilon) = \sinh(\varepsilon) \), we have that inequality \( 5.10 \) is reduced to inequality \( 5.11 \). \( \square \)

5.2.2. \( q \)-digamma function. Let \( 0 < q < 1 \). The \( q \)-digamma function \( \Psi_q \) is the \( q \)-analogue of the \( \Psi \) or digamma function \( \Psi \) defined by

\[
\Psi_q(\eta) = -\ln(1-q) + \ln q \sum_{u=0}^{\infty} \frac{q^{u+\eta}}{1-q^{u+\eta}} = -\ln(1-q) + \ln q \sum_{u=0}^{\infty} \frac{q^{u\eta}}{1-q^{u\eta}}.
\]

For \( q > 1 \) and \( \eta > 0 \), the \( q \)-digamma function \( \Psi_q \) is defined by

\[
\Psi_q(\eta) = -\ln(q-1) + \ln q \sum_{u=0}^{\infty} \frac{q^{-(u+\eta)}}{1-q^{-(u+\eta)}} = -\ln(q-1) + \ln q \sum_{u=0}^{\infty} \frac{q^{-u\eta}}{1-q^{-u\eta}}.
\]
Proposition 7. Let $0 < \gamma < \delta, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$
\left| \frac{1}{6} \left[ \Psi_q(\gamma) + 4\Psi_q \left( \frac{\gamma + \delta}{2} \right) + \Psi_q(\delta) \right] - \frac{1}{\delta - \gamma} \int_{\gamma}^{\delta} \Psi_q(\varepsilon) d\varepsilon \right| \\
\leq \frac{\delta - \gamma}{12} \left( \frac{2^{p+1} + 1}{3(p+1)} \right) \left( \frac{1}{4} \right) \left[ \left( \frac{1}{q} \left| \Psi_q(\gamma) \right|^q + 3 \left| \Psi_q' \left( \frac{\gamma + \delta}{2} \right) \right|^q \right)^{\frac{1}{q}} \right].
$$

(5.12)

Proof. The assertion can be obtained immediately by using Remark 6 to $\psi(\varepsilon) = \Psi_q(\varepsilon)$, and $\varepsilon > 0$, since $\psi'(x) = \Psi_q'(\varepsilon)$ is convex on $(0, +\infty)$.

6. Conclusion

In this paper, using a new identity of Simpson-like type for conformable fractional integral, we obtained some new Simpson type conformable fractional integral inequalities. Furthermore, some interesting applications were examined, for example, we applied the investigated results to special means of real numbers and two special functions named modified Bessel function and $q$–digamma function. So, this paper is a detailed examination of the Simpson-like type conformable fractional integral inequalities.

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