A calculation of the shear viscosity in SU(3) gluodynamics

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We perform a lattice Monte-Carlo calculation of the two-point functions of the energy-momentum tensor at finite temperature in the SU(3) gauge theory. Unprecedented precision is obtained thanks to a multi-level algorithm. The lattice operators are renormalized non-perturbatively and the classical discretization errors affecting the correlators are corrected for. A robust upper bound for the shear viscosity to entropy density ratio is derived, \( \eta/s < 1.0 \), and our best estimate is \( \eta/s = 0.134(33) \) at \( T = 1.65T_c \) under the assumption of smoothness of the spectral function in the low-frequency region.

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Introduction.— Models treating the system produced in heavy ion collisions at RHIC as an ideal fluid have had significant success in describing the observed flow phenomena\(^1\)\(^2\). Subsequently the leading corrections due to a finite shear viscosity were computed\(^3\), in particular the flattening of the elliptic flow coefficient \( v_2(p_T) \) above 1GeV. It is therefore important to compute the QCD shear and bulk viscosities from first principles to establish this description more firmly. Small transport coefficients are a signature of strong interactions, which lead to efficient transmission of momentum in the system. Strong interactions in turn require non-perturbative computational techniques. Several attempts have been made to compute these observables on the lattice in the SU(3) gauge theory\(^4\)\(^5\). The underlying basis of these calculations are the Kubo formulas, which relate each transport coefficient to a spectral function

\[
C(x_0) = L_0^5 \int d^3x \langle T_{12}(0) T_{12}(x_0, x) \rangle.
\]

In this Letter we present a new calculation which dramatically improves on the statistical accuracy of the Euclidean correlator relevant to the shear viscosity through the use of a two-level algorithm\(^16\). This allows us to derive a robust upper bound on the viscosity and a useful estimate of the ratio \( \eta/s \), which has acquired a special significance since its value 1/4\(\pi \) in a class of strongly coupled supersymmetric gauge theories\(^7\) was conjectured to be an absolute lower bound for all substances\(^8\).

Methodology.— In the continuum, the energy-momentum tensor \( \hat{T}_{\mu\nu}(x) = F_{\mu\rho} F_{\nu\sigma} - \frac{1}{4} \delta_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \), being a set of Noether currents associated with translations in space and time, does not renormalize. With \( L_0 = 1/T \) the inverse temperature, we consider the Euclidean two-point function (0 < \( x_0 < L_0 \))

\[
C(x_0) = L_0^5 \int_0^{\infty} \rho(\omega) \frac{\cosh(\frac{\omega}{4} L_0 - x_0)}{\sinh(\frac{\omega}{4} L_0 x_0)} d\omega.
\]

The shear viscosity is given by\(^9\)\(^13\)

\[
\eta(T) = \pi \frac{d\rho}{d\omega} \bigg|_{\omega=0}.
\]

Important properties of \( \rho \) are its positivity, \( \rho(\omega)/\omega \geq 0 \) and parity, \( \rho(-\omega) = -\rho(\omega) \). The spectral function that reproduces \( C^{1,1}(x_0) \) is

\[
\rho^{1,1}(\omega) = \frac{A_{1,1} \omega^4}{\tanh(\frac{\omega}{4} L_0)} + B L_0^{-4} \omega \delta(\omega),
\]

\[
A_{1,1} = \frac{1}{10} \frac{d_A}{(4\pi)^2}, \quad B = \left(\frac{2\pi}{15}\right)^2 d_A.
\]

While the \( \omega^4 \) term is expected to survive in the interacting theory with only logarithmic corrections, the \( \delta \)-function at the origin corresponds to the fact that gluons...
are asymptotic states in the free theory and implies an infinite viscosity.

On the lattice, translations only form a discrete group, so that a finite renormalization is necessary, \( T^{(\text{bare})}_{\mu
u}(g_0) = Z(g_0)T^{(\text{bare})}_{\mu
u} \). We employ the Wilson action \( S_8 = \frac{1}{\beta L^4} \sum_{x,\mu \neq \nu} \text{Tr} \{ 1 - P_{\mu\nu}(x) \} \), on an \( L_0 \cdot L^3 \) hypertoroidal lattice, and the following discretized expression of the Euclidean energy:

\[
T^{(\text{bare})}_{00}(x) \equiv \frac{2}{a^2 g_0^2} \left[ \sum_{k<l} \text{Re Tr} P_{kl}(x) - \sum_k \text{Re Tr} P_{0k}(x) \right]
\]

One of the lattice sum rules \( 10 \) can be interpreted as a non-perturbative renormalization condition for this particular discretization, from which we read off \( Z(g_0) = 1 - \frac{1}{7}g_0^2 (c_\sigma - c_\gamma) \). The definition of the anisotropy coefficients \( c_\sigma, c_\gamma \) can be found in \( 12 \), where they are computed non-perturbatively. With a precision of about 1%, a Padé fit constrained by the one-loop result \( 11 \) yields

\[
Z(g_0) = 1 - 1.0225 g_0^2 + 0.1305 g_0^4 \frac{1}{1 - 0.8557 g_0^2}, \quad (6/g_0^2 \geq 5.7). \tag{6}
\]

**Numerical results.** We report results obtained on a \( \beta = 6/65 \) \( 6.2 \times 40^3 \) lattice and on a \( \beta = 6.408, 8 \times 28^3 \) lattice. The first is thus at a temperature of 1.24\( T_c \), the second at \( T = 1.65 T_c \). We use the results for \( aT_c \) obtained in \( 14 \) and the non-perturbative lattice \( \beta \)-function of \( 13 \) to determine this. We employ the two-level algorithm described in \( 16 \). The computing time invested into the 1.65\( T_c \) simulation is about 860 PC days. Following \( 14 \), we discretize \( \frac{1}{2}(T_{11} - T_{22})(T_{11} + T_{22}) \) instead of \( T_{12} T_{21} \) (the two are equal in the continuum) to write \( C(x_0) = \frac{L^4}{T^2} (O_0(0)O_0(x_0)) + O(a^2) \), where

\[
O_0(x_0) \equiv \frac{1}{a^2 g_0^2} \sum_x \{ T_{11} - T_{22} \}(g_0, x)
\]

\[
= \frac{2Z(g_0)}{a^2 g_0^2} \sum_x \text{Re Tr} \{ P_{10} + P_{13} - P_{20} - P_{23} \}(x).
\]

The three electric-electric, magnetic-magnetic and electric-magnetic contributions to \( C(x_0) \) are computed separately and shown on Fig. 1. We apply the following technique to remove the tree-level discretization errors \( 17 \) separately to \( C_{BB} \), \( C_{EE} \) and \( C_{EB} \). Firstly, \( \tilde{x}_0 \) is defined such that \( C_{\text{cont}}^{-1}(\tilde{x}_0) = C_{\text{lat}}^{-1}(x_0) \). The improved correlator is defined at a discrete set of points through \( \tilde{C}(\tilde{x}_0) = C(x_0) \), and then augmented to a continuous function via \( \tilde{C}(\tilde{x}_0) = C + \beta C_{\text{cont}}(x_0) \), \( i = 1, 2 \), where \( \tilde{x}_0 \) and \( \tilde{x}_0 \) correspond to two adjacent measurements.

The resulting improved correlator, normalized by the continuum tree-level result, is shown on Fig. 2. One observes that the deviations from the tree-level result are surprisingly small, while deviations from conformality are visible. The latter is not unexpected at these temperatures, where \( p/T^4 \) is still strongly rising \( 18 \). Finite-volume effects on the \( T = 1.65 T_c \) lattice are smaller than one part in \( 10^3 \) at tree-level. Non-perturbatively, at the same temperature with resolution \( L_0/a = 6 \), increasing \( L/a \) from 20 to 30 reduces \( \tilde{C}(L_0/2) \) by a factor 0.922(73). While not statistically compelling as it stands, the effect deserves further investigation.

The entropy density is obtained from the relation \( s = (\epsilon + p)/T \) and the standard method to compute \( \epsilon + p \) \( 12 \), Eq.1.14. We find \( s/T^3 = 4.72(3)(5) \) and 5.70(2)(6) respectively at \( T/T_c = 1.24 \) and 1.65 (the first error is statistical and the second is the uncertainty on \( Z(g_0) \)). The Stefan-Boltzmann value is 32\( \pi^2/45 \) in the continuum and 1.0867 times that value \( 12 \) at \( L_0/a = 8 \). Unsatisfactory attempts to extract the viscosity. In order to compare with previous studies \( 4, 5 \), we fit \( C(x_0) \)
with a Breit-Wigner ansatz

$$\rho(\omega)\omega = \frac{F}{1 + b^2(\omega - \omega_0)^2} + \frac{F}{1 + b^2(\omega + \omega_0)^2}, \quad (7)$$

although it clearly ignores asymptotic freedom, which implies that $\rho(\omega) \sim \omega^4$ at $\omega \gg T$ [6]. The result of a correlated fit at $T = 1.65T_c$ using the points at $T x_0 = 0.5$, 0.35 and 0.275 is $a^2F = 0.78(4)$, $(b/\alpha)^2 = 240(30)$ and $\omega_0 = 2.36(4)$, and hence $\eta/s|_{T=1.65T_c} = 0.33(3)$. A comparison of this to the results of Ref. [3] illustrates the progress made in statistical accuracy.

An ansatz motivated by the hard-thermal-loop framework is [6]

$$\rho(\omega)/\omega = \frac{\eta/\pi}{1 + b^2\omega^2} + \theta(\omega - \omega_1) \frac{A\omega^3}{\tanh(\omega/4T)}, \quad (8)$$

It is capable of reproducing the tree-level prediction, Eq. [4] and it allows for a thermal broadening of the delta function at the origin. Fitting the $T = 1.65T_c$ points shown on Fig. 2 the $\chi^2$ is minimized for $b = 0$ (effectively eliminating a free parameter), $A/A_{1.1} = 0.996(8)$, $\omega_1/T = 7.5(2)$ and $\eta/s = 0.25(3)$, with $\chi^2_{\text{min}} = 4.0$. Thus while the ansatz is hardly compatible with the data, it shows that the data tightly constrains the coefficient $A$ to assume its tree-level value.

A bound on the viscosity.— The positivity property of $\rho(\omega)$ allows us to derive an upper bound on the viscosity, based on the following assumptions:

1. the contribution to the correlator from $\omega > \Lambda$ is correctly predicted by the tree-level formula
2. the width of any potential peak in the region $\omega < T$ is no less than $O(T)$.

The standard QCD sum rule practice is to use perturbation theory from the energy lying midway between the lightest state and the first excitation. With this in mind we choose $\Lambda = \max(\frac{1}{2}M_0 + M_2, \ldots ) \approx 2.66$GeV, 57T, where $M_2$ are the masses of the two lightest tensor glueballs. Perturbation theory predicts a Breit-Wigner centered at the origin of width $\Gamma = 2\gamma$ [9], where $\gamma \approx \alpha_s N T$ is the gluon damping rate. To derive the upper bound we conservatively assume that for $\omega < \sqrt{2T}$, $\rho(\omega)/\omega$ is a Breit-Wigner of width $\Gamma = T$ centered at the origin. From $C(\frac{1}{2} L_0) \geq L_0 \int_0^{\frac{T}{2}} d\omega \rho(\omega)/\omega$ we obtain (with $90\%$ statistical confidence level)

$$\eta/s < \begin{cases} 0.96 & (T = 1.65T_c) \\ 1.08 & (T = 1.24T_c). \end{cases} \quad (9)$$

The result for $\rho(\omega)$. The meaning of the error bands and the curves is described in the text. The area under them equals $C(L_0/2) = 8.09(31)$ and $9.35(42)$ for $1.24T_c$ and $1.65T_c$ respectively.

![Figure 3](image)

FIG. 3: The result for $\rho(\omega)$. Where the high-frequency behavior of $\rho(\omega)$ and correspondingly define $\bar{K}(x_0, \omega) = \cosh(\omega(x_0 - \frac{1}{2}L_0)/\cosh(\frac{1}{2}\omega L_0)$ and $M(x_0, \omega)$ to $m(\omega)$ already is a smooth approximate solution to $\bar{\rho}(\omega)$; inserting (10) into (Eq. 2), one requires that $\bar{\rho}(\omega) = \sum a_{\ell} a(\omega)$, with $\{a_{\ell}\}$ a basis of functions which is as sensitive as possible to the discrepancy between the lattice correlator and the correlator generated by $m(\omega)$. These are the eigenfunctions of largest eigenvalue of the symmetric kernel $G(\omega, \omega) = \int_0^{\frac{T}{2}} d\omega \rho(\omega)/\omega \int_0^{\frac{T}{2}} d\omega \rho(\omega)/\omega$, where $M(x_0, \omega) = \bar{K}(x_0, \omega)m(\omega)$. These functions satisfy $\int_0^{\frac{T}{2}} d\omega \rho(\omega)/\omega = \delta_{\ell \ell}$ and have an increasing number of nodes as their eigenvalue decreases. Thus more data points available, the larger the basis and the finer details of the spectral function one is able to determine.

To determine the spectral function from $N$ points of the correlator, we proceed by first discretizing the $\omega$ variable into an $N_\omega$-vector. The final spectral function is given by the last member $\rho(N)$ of a sequence whose first member is $\rho(0) = m$ and whose general member $\rho(n)$ reproduces $n$ points (or linear combinations) of the lattice correlator. For $n \geq 1$, $\rho(n) = \rho(n-1)[1 + \sum_{\ell=1} a_{\ell}^{(n)}]$ and the functions $a_{\ell}^{(n)}(\omega)$ are found by the SVD decomposition [10] of the $N_\omega \times n$ matrix $M^{(n)}$, where $M_{ij}^{(n)} \equiv \bar{K}(x_i^{(j)}, \omega_j)$. The ‘model’ $m(\omega)$ is thus updated and agrees with $\rho(\omega)$ at the end of the procedure. We first performed this procedure on coarser lattices with $L_0/a = 6$ at the same temperatures, starting from $m(\omega) = A_{1.1} \omega^4/(\tanh(\frac{1}{2}\omega L_0)\tanh(\frac{1}{2}\omega L_0)\tanh^2(\omega L_0))$ with $\frac{1}{4} \leq c \leq \frac{1}{2}$, and then recycled the output as seed for
the $L_0/a = 8$ lattices. On the latter we used the $N = 4$ points shown on Fig. 2.

The next question to address is the uncertainty on $\rho(\omega)$. It is important to realize that even in the absence of statistical errors, a systematic uncertainty subsists due to the finite number of basis functions we can afford to describe $\rho(\omega)$ with. A reasonable measure of this uncertainty is by how much $\rho(\omega)$ varies if one doubles the resolution on $C(x_0)$. This can be estimated by ‘generating’ new points by using the computed $\rho^{(N)}(\omega)$. On the other hand we perform a two-point interpolation in $x_\alpha$ (we chose the form $(\alpha + \beta(x_0 - \frac{1}{4}L_0)^2)/\sin^5(\pi x_0/L_0)$), and take the difference between these and the generated ones as their systematic uncertainty. In practice this difference is added in quadrature with the statistical uncertainty. Next we repeat the procedure to find $\rho$ described above with $N \to 2N$: if we use as seed $\rho^{(N)}$, then by construction it is left invariant by the iterative procedure, but the derivatives of $\rho^{(2N)}$ with respect to the $2N$ points of the correlator can be evaluated. The error on $\rho(\omega)$ is then obtained from a formula of the type $(\delta \rho)^2 = \sum_{\omega_0=1}^{2N} (\frac{\partial \rho}{\partial C_{\omega}})^2 (\delta C_{\omega})^2$ which however keeps track of correlations in $x_0$ and Monte-Carlo time. This is the error band shown on Fig. 3 and the corresponding shear viscosity values are

$$\eta/s = \begin{cases} 0.134(33) & (T = 1.65T_c) \\ 0.102(56) & (T = 1.24T_c). \end{cases} \quad (11)$$

It is also interesting to check for the stability of the solution under the use of a larger basis of functions. If instead of starting from $\rho^{(N)}(\omega)$ we restart from $\rho^{(0)}$ (the output of the $L_0/a = 6$ lattice) and fit the $2N$ (dependent) points using $2N$ basis functions $\{a_\ell\}$, we obtain the curves drawn on Fig. 3. As one would hope, the oscillations of $\rho^{(2N)}(\omega)$ are covered by the error band.

**Conclusion.** — Using state-of-the-art lattice techniques, we have computed the correlation functions of the energy-momentum tensor to high accuracy in the SU(3) pure gauge theory. We have calculated the leading high-temperature cutoff effects and removed them from the correlator relevant to the shear viscosity, and we normalized it non-perturbatively, exploiting existing results. We obtained the entropy density with an accuracy of 1%. The most robust result obtained on the shear viscosity is the upper bound Eq. 9, which comes from lumping the area under the curve on Fig. 3 in the interval [0, 6T] into a peak of width $\Gamma = T$ centered at the origin. Secondly, our best estimate of the shear viscosity is given by Eq. 11, using a new method of extraction of the spectral function. The errors contain an estimate of the systematic uncertainty associated with the limited resolution in Euclidean time. We are extending the calculation to finer lattice spacings and larger volumes to further consolidate our findings.

The values 11 are intriguingly close to saturating the KSS bound 8 $\eta/s \geq 1/4\pi$. We note that in perturbation theory the ratio $\eta/s$ does not depend strongly on the number of quark flavors 20. Our results thus corroborate the picture of a near-perfect fluid that has emerged from the RHIC experiments, with the magnitude of the anisotropic flow incompatible with $\eta/s \gtrsim 0.2$ 3.

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