Integral equations for the correlation functions of the quantum one-dimensional Bose gas

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Abstract
The large time and long distance behavior of the temperature correlation functions of the quantum one-dimensional Bose gas is considered. We obtain integral equations, which solutions describe the asymptotics. These equations are closely related to the thermodynamic Bethe Ansatz equations. In the low temperature limit the solutions of these equations are given in terms of observables of the model.

1 Introduction
This paper continues the series of works [1]–[5], devoted to the study of the correlation functions of the quantum one-dimensional Bose gas with delta-function interaction. Here we consider the problems related to the large time and long distance asymptotics of the correlation functions. The main result of this paper is a system of integral equations (see section 4), which solutions describe the asymptotics. The form of these equations is close to the thermodynamic Bethe Ansatz equations, and apparently their solutions are closely related to observables of the one-dimensional Bose gas model. Anyhow such the relationship does exist in the low temperature limit.

The main object of our investigation is a temperature correlation functions of local fields

$$\langle \Psi(0, 0) \Psi\dagger(x, t) \rangle_T = \frac{\text{tr} \left( e^{-\frac{H}{T}} \Psi(0, 0) \Psi\dagger(x, t) \right)}{\text{tr} e^{-\frac{H}{T}}}.$$  (1.1)

Here $T$ is a temperature, $H$ is the Hamiltonian

$$H = \int dx \left( \partial_x \Psi\dagger(x) \partial_x \Psi(x) + c \Psi\dagger(x) \Psi\dagger(x) \Psi(x) \Psi(x) - h \Psi\dagger(x) \Psi(x) \right),$$  (1.2)

where $c > 0$ is the coupling constant, $h$—chemical potential. The operators $\Psi(x, t)$ and $\Psi\dagger(x, t)$ are the canonical Bose fields

$$[\Psi(x, t), \Psi\dagger(y, t)] = \delta(x - y).$$  (1.3)

The equation of motion, corresponding to the Hamiltonian (1.2) is refered as quantum Nonlinear Schrödinger equation (quantum NLS).
For the reader’s convenience we present here the list of the basic equations for observables of the one-dimensional Bose gas at a finite temperature. The energy of elementary excitation (particle) satisfies the Yang–Yang equation
\[
\varepsilon(\lambda) = \lambda^2 - h - \frac{T}{2\pi} \int_{-\infty}^{\infty} K(\lambda, \mu) \ln \left(1 + e^{-\varepsilon(\mu)/T} \right) d\mu, \quad K(\lambda, \mu) = \frac{2c}{c^2 + (\lambda - \mu)^2}.
\] (1.4)

In the case of positive chemical potential the function \(\varepsilon(\lambda)\) has two real roots \(\varepsilon(\pm q_T) = 0\). The total spectral density of the vacancies in the gas is given by the equation
\[
2\pi \rho_t(\lambda) = 1 + \int_{-\infty}^{\infty} K(\lambda, \mu) \vartheta(\mu) \rho_t(\mu) d\mu,
\] (1.5)
where
\[
\vartheta(\lambda) = \left(1 + \exp \left[\frac{\varepsilon(\lambda)}{T}\right]\right)^{-1}
\] (1.6)
plays the role of the Fermi weight.

The momentum of particle, being a function of the spectral parameter, also can be found from the corresponding integral equation. However below we shall need only the derivative of the momentum with respect to the spectral parameter. This quantity coincides up to a coefficient with the total density of the vacancies:
\[
\frac{\partial k(\lambda)}{\partial \lambda} = 2\pi \rho_t(\lambda).
\] (1.7)

The velocity of the particle \(v(\lambda)\) is
\[
v(\lambda) = \frac{\partial \varepsilon}{\partial k} = \frac{\varepsilon'(\lambda)}{k'(\lambda)} = \frac{\varepsilon'(\lambda)}{2\pi \rho_t(\lambda)},
\] (1.8)

Finally, we present also the integral equation for the scattering phase
\[
2\pi F(\lambda, \nu) = \int_{-\infty}^{\infty} K(\lambda, \mu) \vartheta(\mu) F(\mu, \nu) d\mu + i \ln \left(\frac{ic + \lambda - \nu}{ic + \nu - \lambda}\right).
\] (1.9)

Here and further the branch of a logarithm is chosen such that \(-\pi < \arg \ln z \leq \pi\). The quantity \(S(\lambda, \nu) = \exp\{2\pi i F(\lambda, \nu)\}\) is the scattering matrix of two particles possessing momenta \(k(\lambda)\) and \(k(\nu)\).

All the thermodynamics observables listed here will be needed later on for the description of the correlation function asymptotics.

We shall consider the behavior of the correlation function \([\underline{11}]\) at \(x \to \infty, \ t \to \infty\) and fixed ratio \(\lambda_0 = x/2t\). The one-dimensional Bose gas is massless model, therefore the correlation function \([\underline{11}]\) decay as a power at zero temperature
\[
\langle \Psi(0, 0) \Psi^\dagger(x, t) \rangle_0 \longrightarrow Ct^{-\Delta_0}.
\]
The exponent $\Delta_0$ can be computed in the framework of the conformal field theory \([7]\). At finite temperature the asymptotics replaces by the exponential one

$$\langle \Psi(0,0)\Psi^\dagger(x,t) \rangle_T \rightarrow Ct^{-\Delta}e^{-t/r}. \quad (1.10)$$

In other words the zero temperature turns out to be the point of phase transition. The methods of the conformal field theory can be applied at a finite temperature only approximately, for instance, for calculations of the low temperature contributions into the correlation radius \([7]\).

The determinant representation method, used in \([1]–[5]\), allows one in principle to consider correlation functions at arbitrary temperature. This method is especially powerful for study of their asymptotics. In the present paper we focus our attention at the main, exponential law of decay in the equation (1.10). We shall not concern neither the pre-exponential factor $t^{-\Delta}$, nor the constant coefficient $C$ in (1.10). In other words our goal is to compute the correlation radius.

Let us give now the general content of the paper. In the next section we present a brief description of the method, based on the representation of the correlation functions in terms of Fredholm determinants. Here we also discuss the questions related to the construction of the dual fields. In particular we explain the cause of these auxiliary quantum operators appearance and their role in the calculation of the correlation functions. On the whole the section 2 is a sort of review. The reader can find the details in the papers \([1]–[5]\), mentioned above. In the section 3 the methods of the averaging of the operators, depending on dual fields, is developed. Here we define the ‘averaging mapping’, associating quantum operators with certain classical functions. This mapping allows us to compute the asymptotics of the correlation function (1.1). This is done in the section 4. Here the system of integral equations, describing the asymptotics, is given. In the last section we consider some specific cases of the correlation function (1.1). In particular we prove that in the low temperature limit our results exactly coincide with the results of the paper \([7]\).

## 2 Dual fields

The method of the Fredholm determinant representation for evaluation of the correlation functions is described in details, for example, in the book \([8]\). At the first stage of this method one need to represent a correlation function in terms of a determinant of an integral operator, which kernel depends on distance and time (and other physical parameters of the model). For the correlation function (1.1) such the representation was found in \([1]\). Further investigation of the Fredholm determinant obtained, can be performed via the methods of classical exactly solvable equations. The matter is that the determinants mentioned above turns out to be $\tau$-functions of classical integrable systems. In particular, for the model of the quantum one-dimensional Bose gas they can be expressed in terms of solutions of classical forms NLS (in the simplest case of two-points field correlation function in the free fermionic limit the corresponding differential equation is scalar Nonlinear Schrödinger equation). In turn the solutions of classical integrable equations can
be found by means of the Riemann–Hilbert problem approach. In particular, for the calculation of the large time and long distance asymptotics the nonlinear steepest descent method is used \[9\], allowing in principle to obtain the complete asymptotic expansion for correlation functions. Thus, the computation of the correlation function asymptotics in the framework of the method described, consists of two stages:

1) representation of the correlation function in the Fredholm determinant form;

2) computation of the asymptotics of the determinant obtained via the methods of the Riemann–Hilbert problem and classical exactly solvable equations.

However that is how the matter stands only for the free fermionic models (in the quantum NLS this corresponds to the limit of infinite coupling constant: \(c \rightarrow \infty\)). Out of the free fermionic point the situation is more complicated, and one more step should be added to the scheme described above:

3) averaging with respect to an auxiliary vacuum.

The present paper is devoted just to the last stage of calculations, therefore we consider this problem more detailed.

In order to obtain the Fredholm determinant representation for the correlation functions of non-free fermionic models (finite coupling constant for one-dimensional Bose gas), one need to introduce auxiliary quantum operators—Korepin’s dual fields \[10\]. This necessity is caused by the existence of a non-trivial \(S\)-matrix. Consider the main idea in brief. For evaluation of a correlation function of an operator \(\mathcal{O}\) with respect to a state \(|a\rangle\) one can use the form factor decomposition

\[
\langle a|\mathcal{O}(x,t)\mathcal{O}(0,0)|a\rangle = \sum_{|b\rangle} |\langle a|\mathcal{O}(0,0)|b\rangle|^2 e^{D_{ab}(x,t)}.
\] (2.1)

Here the state \(|b\rangle\) runs through a complete set, the function \(D_{ab}(x,t)\) describes the dispersion law of the model. The quantity \(|\langle a|\mathcal{O}(0,0)|b\rangle|\) is refered as a form factor of the operator \(\mathcal{O}\), and, for example, in the quantum NLS it can be computed explicitly via the algebraic Bethe Ansatz. Thus, for \(\mathcal{O} = \Psi\) the form factor is proportional to a certain determinant:

\[
\langle a|\Psi(0,0)|b\rangle \sim \det M.
\] (2.2)

We would like to emphasize that in distinction of the quantum field theory models here the form factors are constructed in the bare Fock vacuum, but not in the physical one. If the states \(|a\rangle\) and \(|b\rangle\) are parameterized by the spectral parameters \(\{\lambda\}\) and \(\{\mu\}\) respectively, then the entries \(M_{jk}\) are functions of these spectral parameters

\[
M_{jk} = M_{jk}(\{\lambda\}, \{\mu\}).
\]

The type of dependency of \(M_{jk}\) on the spectral parameters is extremely important. In fact, in order to obtain a determinant representation for the mean value \(\langle 2.1 \rangle\), we need to reduce the
infinite sum of determinants in the r.h.s. (2.1) to a single determinant. This can be done only for very special form of matrices \( M \), namely, if \( M_{jk} = M(\lambda_j, \mu_k) \), i.e. the entries are parameterized by a single two-variable function \( M(x, y) \), taken in the points \( x = \lambda_j \) and \( y = \mu_k \). These are so-called local matrices. In the reality the entries of the matrix \( M_{jk} \), describing the form factor (2.2), contain the products of two-particle \( S \)-matrices

\[
S(\lambda, \mu) = -\frac{ic + \lambda - \mu}{ic + \mu - \lambda}.
\]  

(2.3)

with respect to all parameters \( \{\lambda\} \) and \( \{\mu\} \). This leads to the effect that for interacting fermions the entries \( M_{jk} \) depend on all spectral parameters \( M_{jk} = M(\{\lambda\}, \{\mu\}) \) (non-local matrices). As a result, it becomes impossible to reduce the sum of determinants (2.1) to a single determinant. On the other hand, in the free fermionic limit \( (c \to \infty) \) the \( S \)-matrix becomes trivial, and, as a consequence, the entries \( M_{jk} \) depend only on two variables \( M_{jk} = M(\lambda_j, \mu_k) \). The details and all explicit equations can be found in [1].

The introducing of dual fields updates the situation. In order to reduce the matrix \( M \) to the local form it is sufficient to factorize the \( S \)-matrix

\[
S(\lambda, \mu) = f_1(\lambda)f_2(\mu),
\]  

(2.4)

where \( f_j \) are some functions. Of course, such a factorization is impossible for the \( S \)-matrix (2.3), if \( f_j \) are classical functions. However one can easily achieve this by means of quantum objects.

Consider two creation operators \( q_\psi(\lambda) \) and \( q_\phi(\lambda) \), and two annihilation operators \( p_\psi(\lambda), p_\phi(\lambda) \), acting in an auxiliary Fock space as

\[
(0|q_\psi(\lambda) = (0|q_\phi(\lambda) = 0, \quad p_\psi(\lambda)|0) = p_\phi(\lambda)|0) = 0.
\]  

(2.5)

The non-vanishing commutation relations are

\[
[p_\phi(\lambda), q_\psi(\mu)] = -[p_\psi(\lambda), q_\phi(\mu)] = \ln \left( \frac{ic + \lambda - \mu}{ic + \mu - \lambda} \right).
\]  

(2.6)

The dual fields are linear combinations

\[
\psi(\lambda) = q_\psi(\lambda) + p_\psi(\lambda), \quad \phi(\lambda) = q_\phi(\lambda) + p_\phi(\lambda).
\]  

(2.7)

Obviously, due to (2.6) the dual fields commute with each other

\[
[\psi(\lambda), \psi(\mu)] = [\phi(\lambda), \phi(\mu)] = [\psi(\lambda), \phi(\mu)] = 0.
\]  

(2.8)

On the other hand, the vacuum expectation value of the expressions, containing the dual fields, might be non-trivial

\[
S(\lambda, \mu) = -\frac{ic + \lambda - \mu}{ic + \mu - \lambda} = -(0|e^{\phi(\lambda)}e^{\psi(\mu)}|0).
\]  

(2.9)
Thus, the $S$-matrix can be presented as vacuum expectation value of the factorized expression of the form (2.4). Due to the property (2.8), the determinants, depending on the dual fields, are well defined. As a result the form factors can be presented in the form

$$
\langle a|O(0,0)|b\rangle \sim \det M = (0|\det \tilde{M}(\phi, \psi)|0),
$$

(2.10)

where matrix $\tilde{M}$ is now local one, i.e. $\tilde{M}_{jk} = \tilde{M}(\lambda_j, \mu_k)$.

Similar representation can be obtained for the square of the absolute value of the form factor (2.1), and the summation with respect to complete set $\{|b\rangle\}$ can be performed under the symbol of the vacuum expectation value. Eventually we arrive at the representation for the mean value (2.1) in terms of the vacuum expectation value in the auxiliary Fock space of a determinant of a certain matrix, depending on the dual fields. In the thermodynamic limit this matrix turns into an integral operator, and we obtain

$$
\langle O(x,t)O(0,0)\rangle \sim (0|\det(I + V)|0).
$$

(2.11)

Here the kernel of the operator $V$ depends on the dual fields $\phi(\lambda)$ and $\psi(\lambda)$, however due to (2.8), the Fredholm determinant is well defined object.

We would like to emphasize that here we have described only the main idea of the dual fields method. In the specific cases some modifications are possible, in particular, sometimes one need to introduce additional dual fields. The reader can find the detailed formulæ in [1]. However the basic idea of the method always consists of the applying of factorization (2.4), which allows to reduce the determinants of non-local matrices to the local ones.

Thus the correlation function of the quantum one-dimensional Bose gas turns out to be proportional to the vacuum expectation value of the Fredholm determinant, functionally depending on the dual fields. Nevertheless the asymptotic analysis of this determinant can be performed in the same manner, as for free fermionic limit. Due to the commutation relations (2.8) at the certain stage of the calculations one can consider the dual fields as some classical functions, which are analytical in the strip $|3\lambda| < c/2$. The last property follows from the representation of the dual fields in terms of the canonical Bose fields

$$
\phi(\lambda) = \frac{1}{\sqrt{2\pi}} \int \ln \left( \frac{i\nu + \nu - \lambda}{i\nu - \nu + \lambda} \right) (\varphi_1^\dagger(\nu) + \varphi_1(\nu)) \, d\nu;
$$

$$
\psi(\lambda) = \frac{c}{\sqrt{2\pi}} \int \frac{\varphi_2^\dagger(\nu) + \varphi_2(\nu)}{(\nu - \lambda)^2 + \frac{c^2}{4}} \, d\nu,
$$

(2.12)

where

$$
[\varphi_1(\lambda), \varphi_2^\dagger(\mu)] = [\varphi_2(\lambda), \varphi_1^\dagger(\mu)] = \delta(\lambda - \mu), \quad \varphi_j(\lambda)|0\rangle = 0, \quad (0|\varphi_j^\dagger(\lambda) = 0.
$$

(2.13)

It is easy to check, that commutation relations between the operators $p_{\psi,\phi}$ and $q_{\psi,\phi}$, defined by (2.12), exactly coincide with (2.6).
Via the Riemann–Hilbert problem and classical integrable equations methods one can find the large time and long distance behavior of the Fredholm determinant (2.11) (see [2]-[5]). The asymptotic expression obtained is a functional of the dual fields, which up to this moment are considered as some \textit{classical} functional parameters. In order to calculate the asymptotics of the correlation function one need now to remember about quantum nature of these operators and to average the obtained expression with respect to auxiliary vacuum.

Here, however, one important moment exists, which had been discussed in details in [5]. The matter is, that the procedures of averaging and calculation of the asymptotics, strictly speaking, do not commute with each other. The asymptotics of the correlation function is equal to the asymptotics of the vacuum expectation value (2.11), but not to the vacuum expectation value of the asymptotics. Just because of this reason the results of the work [11], where some questions of averaging of the expressions containing dual fields, were considered, can be applied to the evaluation of the correlation functions only in some measure. Nevertheless it was proved in the mentioned above paper [5] that there exist asymptotic representations, which are stable with respect to the procedure of averaging, i. e.

\[
\text{asymptotics(vacuum expectation value)} = \text{vacuum expectation value(asymptotics)}.
\]

Such the representation for the Fredholm determinant was obtained in [5], and further we shall deal just with this one.

3 Averaging mapping

The asymptotic formulæ, obtained in [5] for the Fredholm determinant in terms of the dual fields, are rather bulky (see (4.1), (4.2) below). Therefore before averaging of these expressions we consider several more simple, but at the same time quite general examples. The technique, developed in the present section, will allow us to average the equations (4.1), (4.2) easily.

Consider two dual fields \( \phi \) and \( \psi \), defining by (2.7). Let the commutation relations between the creation and annihilation operators are

\[
[p_\phi(\lambda), q_\phi(\mu)] = \xi(\lambda, \mu), \quad [p_\psi(\lambda), q_\psi(\mu)] = \xi(\mu, \lambda),
\]

where \( \xi(\lambda, \mu) \) is a two-variable function. Here we do not need the explicit form of this function, however it is easy to see that putting \( \xi(\lambda, \mu) = \ln(ic + \mu - \lambda) - \ln(ic + \lambda - \mu) \), we reproduce the commutation relations (2.6). Besides the relations (3.1) we introduce an additional one

\[
[p_\psi(\lambda), q_\psi(\mu)] = \eta(\lambda, \mu) = \eta(\mu, \lambda),
\]

where \( \eta(\lambda, \mu) \) is a symmetric two-variable function. Below we shall see that the additional relation (3.2) does not affect essentially on the final result, however for some time we do not put the function \( \eta(\lambda, \mu) \) equal to zero.
It is easy to see that the main property of the dual fields (2.8) is still valid. However the vacuum expectation values may be non-trivial as before.

The main goal of this section is evaluation of the mean value of the following form:

$$(0|e^{\mathcal{F}(\psi,\phi)} F(\phi)|0).$$

(3.3)

Here $\mathcal{F}(\psi, \phi)$ and $F(\phi)$ are operator-valued functionals (later on functionals for brief) of the dual fields. Functional $F$ does not depend on the field $\psi$, while $\mathcal{F}$ depends on this field linearly

$$\frac{\delta^2 \mathcal{F}}{\delta \psi^2} = 0.$$  

(3.4)

All the functionals (and functions) of the dual fields should be understood in the sense of formal series.

We are starting with the formula, obtained in [11]

$$(0|\exp \left\{ \sum_{k=1}^{N} \alpha \psi(\lambda_k) f_k(\phi(\mu_k)) \right\} F(\phi(\nu))|0) = F \left( \sum_{m=1}^{N} z_m \xi(\lambda_m, \nu) \right) \frac{E(\eta)}{\det g},$$

(3.5)

where

$$E(\eta) = \exp \left\{ \frac{1}{2} \sum_{n,m=1}^{N} z_n z_m \eta(\lambda_n, \lambda_m) \right\}.$$  

(3.6)

Here in the l.h.s. $F(\phi(\nu))$ is a functional (or a function) of the field $\phi$, $\alpha$ is a complex parameter, functions $f_k(z)$ assumed to be holomorphic in a vicinity of the origin. In the r.h.s. the quantities $z_j$ are the roots of the system

$$z_j - \alpha f_j \left( \sum_{m=1}^{N} z_m \xi(\lambda_m, \mu_j) \right) = 0.$$  

(3.7)

If the system has several solutions, then one should choose the single one, approaching zero at $\alpha \to 0$ (see [11]). Finally, in the denominator of (3.5) the Jacobian of the system (3.7) is placed

$$g_{jk} = \frac{\partial}{\partial z_k} \left[ z_j - \alpha f_j \left( \sum_{m=1}^{N} z_m \xi(\lambda_m, \mu_j) \right) \right].$$  

(3.8)

It is suitable to modify the result (3.5). Namely, let us introduce a function $w(u)$

$$w(u) = \sum_{m=1}^{N} z_m \xi(\lambda_m, u).$$

(3.9)

Multiplying each of the equations (3.7) by $\xi(\lambda_j, u)$ and summing up all of them, we obtain

$$G(w) \equiv w(u) - \alpha \sum_{m=1}^{N} f_m \left( w(\mu_m) \right) \xi(\lambda_m, u) = 0.$$  

(3.10)
Thus instead of the system (3.7) we obtain only one equation (3.10) for the function \( w(u) \), which has to be valid, however, for any value of \( u \). The roots of the system \( z_j \) can be expressed in terms of \( w(u) \) as follows

\[
z_j = \alpha f_j(w(\mu_j)). \tag{3.11}
\]

It is easy to see also that

\[
det \ g_{jk} = det \left[ \frac{\delta G(w(u))}{\delta w(u')} \right]. \tag{3.12}
\]

Thus the result (3.5) turns into

\[
\langle 0 | \exp \left\{ \sum_{k=1}^{N} \alpha \psi(\lambda_k)f_k(\phi(\mu_k)) \right\} F(\phi(\nu))|0 \rangle = F(w(\nu)) \frac{E(\eta)}{det \ \delta G/\delta w}. \tag{3.13}
\]

Let now the functional \( F(\phi) \) is simultaneously a function of some parameter \( t \): \( F(\phi) = F(\phi|t) \). This parameter will play the role of time in the asymptotic formulae for the correlation function. Then the equation (3.13) can be written in the form

\[
\langle 0 | \exp \left\{ \sum_{k=1}^{N} \alpha \psi(\lambda_k)f_k(\phi(\mu_k)) \right\} F(\phi(\nu)|t)|0 \rangle = CF(w(\nu)|t), \tag{3.14}
\]

where \( C \) is a constant, does not depending on \( t \). Constant factors of the such type will be not interesting for us in computation of the correlation function asymptotics. Just because of this reason one can put the function \( \eta(\lambda, \mu) \) equal to zero, since it contributes only into the common constant multiplier and does not affect on the dependency of the r.h.s. of (3.14) on \( t \).

Thus we see that under the averaging of the expressions of the form (3.5) a mapping of the operator \( \phi \) into the classical function \( w \) takes place

\[
\phi(\nu) \rightarrow w(\nu), \quad F(\phi(\nu)) \rightarrow F(w(\nu)), \tag{3.15}
\]

where \( w(\nu) \) is the solution of the equation (3.10). We call the mapping (3.13) (as well as more general equations (3.17), (3.18)) averaging mapping.

Consider now a generalization of the functional

\[
\mathcal{F}(\psi, \phi) = \alpha \sum_{k=1}^{N} \psi(\lambda_k)f_k(\phi(\mu_k)), \tag{3.16}
\]

standing in the exponent of (3.14). It is clear, that one can easily include the derivatives of the fields \( \psi^{(m)}(\lambda) \) and \( \phi^{(m)}(\lambda) \) into this functional. One can consider also the continuous limit, when the sum with respect to \( k \) turns into an integral (this was done in [11]). Besides it is possible to consider infinite series instead of finite sum with respect to \( k \). In all listed cases the result of averaging can be easily obtained by the corresponding limiting procedure in the equation (3.14) (or in (3.13), if more precise results are needed). It is important only that \( \mathcal{F}(\psi, \phi) \) would be always
linear with respect to the field $\psi$ and its derivatives, while the dependency of $F$ and $F$ on the field $\phi$ might be rather complicated.

Thus we arrive at the following formula

$$0\left|e^{\mathcal{F}(\psi,\phi)}\mathcal{F}(\phi(\nu)|t)\right|0 = CF\left(w(\nu)|t\right),$$

(3.17)

where the function $w(u)$ satisfies the following equation

$$w(u) - F\left(\xi(\cdot, u), w\right) = 0.$$  

(3.18)

As for functionals, depending on the field $\psi$ in non-linear way, such a simple formula, generally speaking, does not exist. However in scattered instances it is possible to generalize the averaging mapping for more complicated functionals. Below we consider an example of functionals, which are ‘asymptotically linear’ with respect to the field $\psi(\lambda)$.

Consider a mean value

$$0\left|e^{\mathcal{F}(\psi,\phi,\hat{\Lambda})}\mathcal{F}_{\epsilon}(\phi(\nu), \hat{\Lambda})e^{tF_{\epsilon}(\phi(\nu), \hat{\Lambda})}|0\right),$$

(3.19)

We are interesting of the behavior of this mean value at $t \to \infty$. Let the explicit dependency of the functional $\mathcal{F}(\psi, \phi, \hat{\Lambda})$ on the field $\psi$ is linear as before. Besides let $\mathcal{F}$ and $F$ depend on $\psi$ implicitly through an operator $\hat{\Lambda}$, which is defined as the ‘root’ of the equation

$$\hat{\Lambda} = \lambda_0 + \frac{i}{2t}\psi'(\hat{\Lambda}), \quad \hat{\Lambda} \to \lambda_0 \quad \text{at} \quad t \to \infty.$$  

(3.20)

Here $\lambda_0$ is a constant, prime means the derivative with respect to the argument. The definition (3.21) is formal, and one should understand it in the sense of a series with respect to $t^{-1}$. If we replace in (3.21) the operator $\psi$ by a complex function holomorphic in a vicinity of $\lambda_0$, then for arbitrary function $f(z)$, holomorphic in the same vicinity, we have under the corresponding choice of the contour

$$f(\hat{\Lambda}) = \frac{1}{2\pi i} \oint dz f(z) \frac{1 - \frac{i}{2t}\psi''(z)}{z - \lambda_0 - \frac{i}{2t}\psi'(z)}.$$  

(3.21)

For large enough $t$ one can expand the denominator in (3.21) into the absolutely convergent series, what gives

$$f(\Lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{i}{2t} \right)^n \frac{d^n}{d\lambda_0^n} \left( 1 - \frac{i}{2t}\psi''(\lambda_0) \right) \left[ (\psi'(\lambda_0))^n f(\lambda_0) \right].$$  

(3.22)

In particular

$$\Lambda = \lambda_0 + \sum_{n=1}^{\infty} \frac{1}{n!} \left( \frac{i}{2t} \right)^n \frac{d^{n-1}}{d\lambda_0^{n-1}} \left[ (\psi'(\lambda_0))^n \right].$$  

(3.23)

We accept the formula (3.23) as the definition of the operator $\hat{\Lambda}$. Accordingly functions (and functionals) of this operator should be understood in the sense (3.22). Obviously, for large value of $t$ the difference between $\lambda_0$ and $\hat{\Lambda}$ behaves as $t^{-1} \to 0$. Therefore we call the functional
\( \mathcal{F}(\psi(\cdot), \phi, \hat{\Lambda}) \) asymptotically linear with respect to \( \psi \). Nevertheless the replacement \( \hat{\Lambda} \) by \( \lambda_0 \) in (3.19) is illegal, since \( \psi \) is unbounded operator.

Using (3.23) one can rewrite the mean value (3.19) in the form

\[
(0 | \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{i}{2t} \right)^n \frac{\partial^{2n}}{\partial \lambda_0^2 \partial \beta^n} \left[ e^{\beta \psi'(\lambda_0) + \mathcal{F}(\psi(\cdot), \phi, \lambda_0)} \left( 1 - \frac{i \psi''(\lambda_0)}{2t} \right) F_c \left( \phi(\nu), \lambda_0 \right) e^{t F_c \left( \phi(\nu), \lambda_0 \right)} | 0 \right] \right)_{\beta=0}
\]

what in turn reduces to the double integral

\[
-\frac{2t}{\pi} \int_0^\infty \rho d\rho e^{2i(t+i0)\rho^2} \int \frac{dz}{z} e^{\rho z^{-1} \psi'(\lambda_0 + \rho z) + \mathcal{F}(\psi(\cdot), \phi, \lambda_0 + \rho z)}
\]

\[
\times \left( 1 - \frac{i}{2t} \psi''(\lambda_0 + \rho z) \right) F_c \left( \phi(\nu), \lambda_0 + \rho z \right) e^{t F_c \left( \phi(\nu), \lambda_0 + \rho z \right)} | 0 \right),
\]

where the integral over \( z \) is taken with respect to a small contour around the origin. Recall that we are interested in the asymptotic behavior of this mean value at \( t \to \infty \). It is not difficult to check that in this case the term, containing the second derivative \( \psi'' \), contributes only into a constant factor and hence, it can be removed. Then the integrand takes the form considered above, and we can average it. We have

\[
-\frac{2t}{\pi} \int_0^\infty \rho d\rho e^{2i(t+i0)\rho^2} \int \frac{dz}{z} C(\rho, z, w) e^{t F_c \left( w(\nu), \lambda_0 + \rho z \right)},
\]

where

\[
w(u, \rho, z) - \rho z^{-1} \xi'(\lambda_0 + \rho z, u) - \mathcal{F}(\xi(\cdot, u), w, \lambda_0 + \rho z) = 0.
\]

It remains now to take the integral in (3.24). For \( t \to \infty \) it can be estimated via the steepest descent method, and we arrive at

\[
(0 | e^{\mathcal{F}(\psi(\cdot), \phi, \hat{\Lambda})} F_c \left( \phi(\nu) \right) e^{t F_c \left( \phi(\nu) \right)} | 0 \right) \to C e^{2it(\Lambda - \lambda_0)\beta + t F_c \left( w(\nu, \Lambda, \beta) \right)}.
\]

Here the function \( w(u) \) satisfies the equation

\[
w(u, \Lambda, \beta) - \beta \xi'\left( \Lambda, u \right) - \mathcal{F}(\xi(\cdot, u), w, \Lambda) = 0,
\]

where numbers \( \Lambda \) and \( \beta \) are defined by the equations of the saddle point:

\[
2i\beta + \frac{\partial}{\partial \Lambda} F_c \left( w, \Lambda \right) = 0,
\]

\[
2i(\Lambda - \lambda_0) + \frac{\partial}{\partial \beta} F_c \left( w, \Lambda \right) = 0.
\]

Thus, in the example considered we again deal with the averaging mapping, however of more general form

\[
\phi(\nu) \to w(\nu), \quad \hat{\Lambda} \to \Lambda, \quad F(\phi, \hat{\Lambda}) \to F(w, \Lambda).
\]
4 Asymptotics of the correlation function

Now we are ready to consider the asymptotics of the correlation function \((1.1)\). There are two different phases in the model of the one-dimensional Bose gas, corresponding to the negative and positive chemical potential \(h\) in the Hamiltonian \((1.2)\). While at \(h < 0\) the ground state coincides with the bare Fock vacuum, at \(h > 0\) the ground state is the Fermi sphere. At a finite temperature the difference is not so significant, but nevertheless the sign of the chemical potential affects on the asymptotics of the correlation function. The corresponding formulæ were obtained in the paper \([5]\). Let us present here the explicit expressions.

\[
\langle \Psi(0,0)\Psi(x,t) \rangle_T \to (0|C_-(\phi,\tilde{\phi}|\lambda_0)(2t) - \frac{(\phi(\lambda)+1)^2}{2} e^{\phi(\lambda)+i\lambda^2-i\lambda t-ih} \times \exp \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( x - 2\lambda t + i\psi'(\lambda) \right) \text{sign}(\hat{\Lambda} - \lambda) \ln \left( \frac{e^{\frac{\epsilon(\lambda)}{2}} - e^{\phi(\lambda)\text{sign}(\lambda-\hat{\Lambda})}}{e^{\frac{\epsilon(\lambda)}{2}} + 1} \right) d\lambda \right\} |0\rangle, \quad (4.1)
\]

for \(h < 0\), and

\[
\langle \Psi(0,0)\Psi(x,t) \rangle_T \to (0|C_+(\phi,\tilde{\phi}|\lambda_0)(2t) - \frac{\epsilon(\lambda)}{2} e^{\frac{1}{2}\psi(\hat{\Lambda}_1)+\frac{1}{2}\psi(\hat{\Lambda}_2)+\frac{1}{4}(\hat{\Lambda}_1^2+\hat{\Lambda}_2^2)-\frac{i}{2}(\hat{\Lambda}_1+\hat{\Lambda}_2)-ih} \times \exp \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( x - 2\lambda t + i\psi'(\lambda) \right) \text{sign}(\hat{\Lambda} - \lambda) \ln \left( \frac{e^{\frac{\epsilon(\lambda)}{2}} - e^{\phi(\lambda)\text{sign}(\lambda-\hat{\Lambda})}}{e^{\frac{\epsilon(\lambda)}{2}} + 1} \omega(\lambda) \right) d\lambda \right\} |0\rangle, \quad (4.2)
\]

for \(h > 0\).

Let us make necessary explanations. The equations \((4.1), (4.2)\) were obtained under the assumption that \(t \to \infty, x \to \infty\), while the ratio \(x/2t = \lambda_0\) remains fixed. Function \(\epsilon(\lambda)\) is the solution of the Yang–Yang equation \((1.4)\).

The most important objects for us are three dual fields \(\psi(\lambda), \phi(\lambda)\) and \(\tilde{\phi}(\lambda)\). The commutation relations between corresponding creation and annihilation operators are

\[
[p_\psi(\lambda), q_\phi(\mu)] = -[p_\phi(\lambda), q_\psi(\mu)] = \xi(\lambda, \mu) \equiv \ln \left( \frac{ic + \mu - \lambda}{ic + \lambda - \mu} \right),
\]

\[
[p_\psi(\lambda), q_\psi(\mu)] = \bar{p}_\phi(\lambda), q_\psi(\mu) = [p_\psi(\lambda), \bar{q}_\phi(\mu)] = \eta(\lambda, \mu) \equiv \ln \left( \frac{c^2}{(\lambda - \mu)^2 + c^2} \right). \quad (4.3)
\]

As usual all the dual fields commute with each other. Recall that just this property allows one at the certain stage of calculations to consider these objects as classical functions. The equations \((1.1), (1.2)\) were obtained in \([3]\) just under such treatment of the dual fields, although some operator features were taken into account. In particular, a special attention were paid to the stability of the asymptotic formulæ under the averaging procedure with respect to the auxiliary vacuum, i.e. the corrections, depending on the dual fields, would remain small after their averaging. The presence in the asymptotic formulæ of the value \(\hat{\Lambda}\), which we dealt already in the previous section,

\[
\hat{\Lambda} = \lambda_0 + \frac{i}{2t} \psi'(\hat{\Lambda}), \quad (4.4)
\]
is caused just by this reason. All the expressions, containing the operator $\hat{\Lambda}$, should be understood in the sense of the series (3.22), (3.23), in particular, the integrals, depending on the sign-function, are equal by definition

$$\int_{-\infty}^{\infty} f(\lambda, \text{sign}(\hat{\Lambda} - \lambda)) \, d\lambda = \int_{-\infty}^{\hat{\Lambda}} f(\lambda, 1) \, d\lambda + \int_{\hat{\Lambda}}^{\infty} f(\lambda, -1) \, d\lambda. \quad (4.5)$$

One can use (3.22) for each of the integrals in the r.h.s. of (4.5). In fact one can say that $\hat{\Lambda}$ arises as the result of partial summation of the asymptotic series for the correlation function.

For operators $\hat{\Lambda}_1$ and $\hat{\Lambda}_2$ the situation is quite analogous. If we consider dual fields as classical functions, then $\hat{\Lambda}_i$ are the roots of the equation

$$\varepsilon(\hat{\Lambda}_i) = T \text{sign}(\hat{\Lambda} - \hat{\Lambda}_i) \phi(\hat{\Lambda}_i), \quad \hat{\Lambda}_1 \to -q_T, \quad \hat{\Lambda}_2 \to q_T, \quad \text{at} \quad T \to 0, \quad (4.6)$$

where $\pm q_T$—zeros of the Yang–Yang function: $\varepsilon(\pm q_T) = 0$. In other words the numbers $\hat{\Lambda}_i$ correspond to the zeros of the expression, standing under the symbol of the logarithm in the integral (4.2). In the real situation, when $\phi(\lambda)$ is the dual field, $\hat{\Lambda}_i$ become operators, defining by the series similar to (3.23). We shall simply demand the operator equation (4.6) to transform to the usual equation under the averaging mapping

$$\varepsilon(\Lambda_i) = T \text{sign}(\Lambda - \Lambda_i) w(\Lambda_i), \quad \Lambda_1 \to -q_T, \quad \Lambda_2 \to q_T, \quad \text{at} \quad T \to 0, \quad (4.7)$$

Function $\omega(\lambda)$, entering (4.2), is equal

$$\omega(\lambda) = \text{sign}(\lambda - \hat{\Lambda}_1) \text{sign}(\lambda - \hat{\Lambda}_2).$$

These sign-functions are defined in the same manner as the sign-functions of $\hat{\Lambda}$ (4.3).

The equations (4.4), (4.2) contain also a function $\nu(\hat{\Lambda})$, defining the power law of $t$

$$\nu(\hat{\Lambda}) = \frac{1}{2\pi} \ln \left[ \left( \frac{e^{\xi(\hat{\Lambda})} - e^{-\phi(\hat{\Lambda})}}{e^{\xi(\hat{\Lambda})} + 1} \right) \left( \frac{e^{\xi(\hat{\Lambda})} - e^{\phi(\hat{\Lambda})}}{e^{\xi(\hat{\Lambda})} + 1} \right) \right]. \quad (4.8)$$

Finally the explicit form of factors $C_{\pm}(\phi, \tilde{\phi}|\lambda_0)$ is unknown (see [4]). It is known, however, that they do not depend on the field $\psi$, and they depend on the distance $x$ and time $t$ only through the ratio $x/2t = \lambda_0$, which, recall, remains fixed.

In spite of the asymptotic formulæ (4.1), (4.2) look rather complicated, nevertheless they belong to the class of functionals of the dual fields, considered in the previous section. They are asymptotically linear with respect to the field $\psi(\lambda)$. The operators $\hat{\Lambda}_i$ are some (implicit) functionals of the field $\phi(\lambda)$. The presence of the third field $\tilde{\phi}(\lambda)$, as well as existence of unknown factors $C_{\pm}$, should not worry us, since due to the trivial commutation relations

$$[\tilde{p}_\phi(\lambda), q_\phi(\mu)] = [p_\phi(\lambda), \tilde{q}_\phi(\mu)] = 0$$

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one can easily see, that the averaging of the field $\bar{\phi}$ may contribute only into the common constant factor. Since our goal is to compute the correlation radius, this factor is not essential, and we even can put $\bar{\phi} = 0$.

Thus, applying the averaging mapping (3.29) to the equations (4.1), (4.2) and neglecting the common constant factor and power law dependency on $t$, we have

$$
\langle \Psi(0,0) | \Psi(t,x) \rangle_T \rightarrow e^{-t/r_{\pm}}, \quad (4.9)
$$

where $r_\pm$ correspond to the positive and negative chemical potential respectively. We shall call $r_\pm$ correlation radii, although, strictly speaking, they are not necessary real.

For $h < 0$

$$
-t/r_- = 2it(\Lambda - \lambda_0)\beta + it\Lambda^2 - ix\Lambda - iht + \frac{1}{2\pi} \int_{-\infty}^{\infty} (x - 2\lambda t) \text{sign}(\Lambda - \lambda) \ln \left[ \frac{e^{\xi(\lambda)} - e^{w(\lambda)\text{sign}(\lambda - \Lambda)}}{e^{\xi(\lambda)} + 1} \right] d\lambda. \quad (4.10)
$$

Here function $w(u)$, depending on parameters $\Lambda$ and $\beta$, is the solution of the integral equation

$$
w(u) = \xi(\Lambda, u) + i\beta K(\Lambda, u) - \frac{1}{2\pi} \int_{-\infty}^{\infty} K(u, \lambda) \text{sign}(\Lambda - \lambda) \ln \left[ \frac{e^{\xi(\lambda)} - e^{w(\lambda)\text{sign}(\lambda - \Lambda)}}{e^{\xi(\lambda)} + 1} \right] d\lambda, \quad (4.11)
$$

where

$$
\xi(\lambda, \mu) = \ln \left( \frac{i\mu + \lambda - \mu}{i\mu + \mu - \lambda} \right), \quad K(\lambda, \mu) = -i \frac{\partial}{\partial \lambda} \xi(\lambda, \mu) = \frac{2c}{(\lambda - \mu)^2 + c^2}. \quad (4.12)
$$

The parameters $\Lambda$ and $\beta$ in turn are defined by two additional equations

$$
i(\Lambda - \lambda_0) + \frac{1}{2\pi} \frac{\partial}{\partial \beta} \int_{-\infty}^{\infty} (\lambda_0 - \lambda) \text{sign}(\Lambda - \lambda) \ln \left[ \frac{e^{\xi(\lambda)} - e^{w(\lambda)\text{sign}(\lambda - \Lambda)}}{e^{\xi(\lambda)} + 1} \right] d\lambda = 0, \quad (4.13)
$$

$$
i(\beta + \Lambda - \lambda_0) + \frac{1}{2\pi} \frac{\partial}{\partial \Lambda} \int_{-\infty}^{\infty} (\lambda_0 - \lambda) \text{sign}(\Lambda - \lambda) \ln \left[ \frac{e^{\xi(\lambda)} - e^{w(\lambda)\text{sign}(\lambda - \Lambda)}}{e^{\xi(\lambda)} + 1} \right] d\lambda = 0. \quad (4.14)
$$

We also took into account that $x = 2t\lambda_0$.

For positive chemical potential formulæ are rather similar, but there is a set of differences. The correlation radius is given by

$$
-t/r_+ = 2it(\Lambda - \lambda_0)\beta + \frac{it}{2}(\Lambda_1^2 + \Lambda_2^2 - 2h) - \frac{ix}{2}(\Lambda_1 + \Lambda_2)
$$

$$
+ \frac{1}{2\pi} \int_{-\infty}^{\infty} (x - 2\lambda t) \text{sign}(\Lambda - \lambda) \ln \left[ \frac{e^{\xi(\lambda)} - e^{w(\lambda)\text{sign}(\lambda - \Lambda)}}{e^{\xi(\lambda)} + 1} \omega(\lambda) \right] d\lambda. \quad (4.15)
$$

Hereby the function $w(u)$ satisfies the integral equation

$$
w(u) = \frac{1}{2}(\xi(\Lambda_1, u) + \xi(\Lambda_2, u)) + i\beta K(\Lambda, u)
$$

$$
- \frac{1}{2\pi} \int_{-\infty}^{\infty} K(u, \lambda) \text{sign}(\Lambda - \lambda) \ln \left[ \frac{e^{\xi(\lambda)} - e^{w(\lambda)\text{sign}(\lambda - \Lambda)}}{e^{\xi(\lambda)} + 1} \omega(\lambda) \right] d\lambda, \quad (4.16)
$$
where $\Lambda_1$ and $\Lambda_2$ are the roots of the equation (4.7)

$$\varepsilon(\Lambda_i) = T \text{ sign}(\Lambda - \Lambda_i) w(\Lambda_i), \quad \Lambda_1 \to -q_T, \quad \Lambda_2 \to q_T, \quad \text{at} \quad T \to 0,$$

and

$$\omega(\lambda) = \text{sign}(\lambda - \Lambda_1) \text{sign}(\lambda - \Lambda_2).$$

The parameters $\Lambda$ and $\beta$ can be found from additional equations

$$i(\Lambda - \lambda_0) + \frac{1}{2\pi} \frac{\partial}{\partial \beta} \int_{-\infty}^{\infty} (\lambda_0 - \lambda) \text{sign}(\Lambda - \lambda) \ln \left[ \frac{e^{\varepsilon(\lambda)} - e^{w(\lambda) \text{sign}(\lambda - \Lambda)}}{e^{\varepsilon(\lambda)} + 1} \omega(\lambda) \right] d\lambda = 0,$$  

$$i\beta + \frac{1}{2\pi} \frac{\partial}{\partial \Lambda} \int_{-\infty}^{\infty} (\lambda_0 - \lambda) \text{sign}(\Lambda - \lambda) \ln \left[ \frac{e^{\varepsilon(\lambda)} - e^{w(\lambda) \text{sign}(\lambda - \Lambda)}}{e^{\varepsilon(\lambda)} + 1} \omega(\lambda) \right] d\lambda = 0.$$  

Thus, the exponential decay of the correlation function (1.1) is defined by the function $w(u)$. This function can be found from the system of integral equations, which structure is rather similar to the Yang–Yang equation (1.4). Therefore it is quite possible that $w(u)$ can be expressed in terms of observables of the one-dimensional Bose gas. In the next section we shall demonstrate this for some limiting cases.

In the conclusion of this section we consider free fermionic limit: $c \to \infty$. Then $K(\lambda, \mu) = \xi(\lambda, \mu) = 0$ (see (4.12)), and hence, $w(u) = 0$. Besides $\varepsilon(\lambda) = \lambda^2 - h$, $\Lambda = \lambda_0$, $\beta = 0$ and $\Lambda_{1,2} = \pm \sqrt{h}$. The integral term in the equations (4.10), (4.15) turns into

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |x - 2\lambda t| \ln \left| \frac{e^{\varepsilon(\lambda)} - 1}{e^{\varepsilon(\lambda)} + 1} \right| d\lambda.$$  

In the case of positive chemical potential this expression completely describes the asymptotic behavior of the correlation function, since $\Lambda_1^2 + \Lambda_2^2 - 2h = 0$. For $h < 0$ we have additional oscillating term $-it(\Lambda_0^2 + h)$. These formulæ exactly reproduce the results of [12].

However the free fermionic limit is not too representative. Indeed, in this case all the commutation relations between operators $p$ and $q$ (1.3) become trivial, therefore we could put from the very beginning all the dual fields equal to zero in the equations (1.1), (1.2).

## 5 Limiting cases

In this section we consider some limiting cases of the correlation function (1.1). The first case corresponds to the autocorrelation. In spite of we considered the limit $x \to \infty$, $t \to \infty$, in fact all the asymptotic analysis performed in [3] was based on the assumption that the $x/2t = \lambda_0$ is finite. In particular $\lambda_0$ may be equal to zero, what describes the case $x = 0$.

Consider, for instance, the case $h < 0$. It is easy to check that for $\lambda_0 = 0$ one can put $\Lambda = \beta = 0$ in the equations (4.11)–(4.14). Indeed, it follows from (4.11) that for zero values of
these parameters \( w(u) \) is an odd function. Differentiating (4.11) with respect to \( \Lambda \) and \( \beta \) at zero point we find that the derivatives \( \partial w/\partial \Lambda \) and \( \partial w/\partial \beta \) are even functions of \( u \). Then the equations (4.13), (4.14) are valid automatically, and the result slightly simplifies

\[
 r_{-1}^{-1} = i h - \frac{2}{\pi} \int_{0}^{\infty} \lambda \ln \left[ \frac{e^{\frac{\varepsilon(\lambda)}{T}} - e^{\frac{w(\lambda)}{T}}}{e^{\frac{\varepsilon(\lambda)}{T}} + 1} \right] d\lambda, \tag{5.1}
\]

where \( w(u) \) is the solution of the equation

\[
 w(u) = -\xi(u, 0) + \frac{1}{2\pi} \int_{0}^{\infty} \left( K(\lambda, u) - K(\lambda, -u) \right) \ln \left[ \frac{e^{\frac{\varepsilon(\lambda)}{T}} - e^{\frac{w(\lambda)}{T}}}{e^{\frac{\varepsilon(\lambda)}{T}} + 1} \right] d\lambda. \tag{5.2}
\]

Similarly for positive chemical potential we obtain

\[
 r_{+1}^{-1} = i(h - \Lambda_{1}^{2}) - \frac{2}{\pi} \int_{0}^{\infty} \lambda \ln \left[ \frac{e^{\frac{\varepsilon(\lambda)}{T}} - e^{\frac{w(\lambda)}{T}}}{e^{\frac{\varepsilon(\lambda)}{T}} + 1} \omega(\lambda) \right] d\lambda, \tag{5.3}
\]

and

\[
 w(u) = \frac{1}{2} (\xi(\Lambda_{1}, u) - \xi(\Lambda_{1}, -u)) + \frac{1}{2\pi} \int_{0}^{\infty} \left( K(\lambda, u) - K(\lambda, -u) \right) \ln \left[ \frac{e^{\frac{\varepsilon(\lambda)}{T}} - e^{\frac{w(\lambda)}{T}}}{e^{\frac{\varepsilon(\lambda)}{T}} + 1} \omega(\lambda) \right] d\lambda. \tag{5.4}
\]

Hereby it turns out that \( \Lambda_{2} = -\Lambda_{1} \).

The most interesting is the low temperature limit for the case \( h > 0 \). In this limit we have a possibility to compare our results with the known ones. Indeed, as we have mentioned already, at \( T = 0 \) the asymptotics of the correlation function must be power-like. This means that the correlation radius \( r_{+} \to \infty \). Besides, the correlation radius was computed for small temperature in the linear approximation in [7]. Thus, one can consider the low temperature limit as a good test for our results. Besides we have a possibility to make more precise estimates of the results obtained previously.

First, let us present the system of equations, describing the ground state for \( h > 0 \) [13]. Recall the Yang–Yang equation (1.4)

\[
 \varepsilon(\lambda) = \lambda^{2} - h - \frac{T}{2\pi} \int_{-\infty}^{\infty} K(\lambda, \mu) \ln \left( 1 + e^{-\frac{\varepsilon(\mu)}{T}} \right) d\mu. \tag{5.5}
\]

The function \( \varepsilon(\lambda) \) has two real roots \( \varepsilon(\pm q_{T}) = 0 \), and \( \varepsilon(\lambda) > 0 \), if \( |\lambda| > q_{T} \), and \( \varepsilon(\lambda) < 0 \) if \( |\lambda| < q_{T} \). At \( T \to 0 \) the roots \( \pm q_{T} \) tend to some fixed values (corresponding to the boundaries of the Fermi sphere): \( q_{T} \to q \). It is easy to see that

\[
 \exp \left\{ -\frac{\varepsilon(\mu)}{T} \right\} \xrightarrow{T \to 0} \begin{cases} 0, & |\mu| > q, \\ \infty, & |\mu| < q. \end{cases}
\]
Thus the equation (5.6) turns into
\[ \varepsilon_0(\lambda) = \lambda^2 - h + \frac{1}{2\pi} \int_{-q}^{q} K(\lambda, \mu) \varepsilon_0(\mu) \, d\mu, \quad \varepsilon_0(\pm q) = 0. \] (5.6)

Here and further we denote observables at zero temperature by the subscript 0. Similarly the equations for the density and the scattering phase have the form
\[ \rho_0(\lambda) = \frac{1}{2\pi} + \frac{1}{2\pi} \int_{-q}^{q} K(\lambda, \mu) \rho_0(\mu) \, d\mu, \] (5.7)
\[ F_0(\lambda, \nu) = \frac{1}{2\pi} \int_{-q}^{q} K(\lambda, \mu) F_0(\mu, \nu) \, d\mu + \frac{1}{2\pi i} \xi(\lambda, \nu). \] (5.8)

The derivative of the momentum and the velocity as before are equal to
\[ \frac{\partial k_0(\lambda)}{\partial \lambda} = 2\pi \rho_0(\lambda), \quad v_0(\lambda) = \frac{\varepsilon'_0(\lambda)}{k'_0(\lambda)}. \] (5.9)

Let us give also the equation for the resolvent of the integral operator, entering the equations (5.6)–(5.8): \((I + R)(I - \frac{1}{2\pi} K) = I\)
\[ R(\lambda, \nu) = \frac{1}{2\pi} \int_{-q}^{q} K(\lambda, \mu) R(\mu, \nu) \, d\mu + \frac{1}{2\pi} K(\lambda, \nu). \] (5.10)

It is easy to see that \( R(\lambda, \mu) = -\partial_\mu F_0(\lambda, \mu) \). Besides, comparing the equations (5.7) and (5.8) and using \( K(\lambda, \mu) = -i\partial_\lambda \xi(\lambda, \mu) \), we find:
\[ 2\pi \rho_0(\lambda) = 1 + F_0(\lambda, -q) - F_0(\lambda, q). \] (5.11)

Similarly to the Yang–Yang equation the integral equation for the function \( w(u) \) becomes linear in the limit \( T \to 0 \). Let, for example, \( \Lambda_1 < \Lambda_2 < \Lambda \). Then instead of (4.16) we have at \( T = 0 \)
\[ w_0(u) = \frac{1}{2}(\xi(-q, u) + \xi(q, u)) + i\beta K(u, \Lambda) + \frac{1}{2\pi} \int_{-q}^{q} K(u, \lambda) w_0(\lambda) \, d\lambda. \] (5.12)

Here we have used that \( \Lambda_2 = -\Lambda_1 = q \) for \( T = 0 \). Comparing the equation (5.12) with (5.8) and (3.11), we find
\[ w_0(u) = -i\pi (F_0(u, -q) + F_0(u, q)) + 2\pi i \beta R(u, \Lambda). \] (5.13)

The restrictions for the numbers \( \Lambda \) and \( \beta \) have the following form:
\[ i(\Lambda - \lambda_0) - \frac{1}{2\pi} \frac{\partial}{\partial \beta} \int_{-q}^{q} (\lambda_0 - \lambda) w_0(\lambda) \, d\lambda = 0, \] (5.14)
\[ i\beta - \frac{1}{2\pi} \frac{\partial}{\partial \Lambda} \int_{-q}^{q} (\lambda_0 - \lambda) w_0(\lambda) \, d\lambda = 0. \] (5.15)
It follows immediately from (5.13) and (5.15) that \( \beta = 0 \), and hence,

\[
w_0(u) = -i\pi (F_0(u, -q) + F_0(u, q))
\]

Substituting this into (4.15) (where the logarithm also turns into linear function), we find

\[
-t/r_+ = it(q^2 - h) + \frac{i}{2} \int_{-q}^q (x - 2\lambda t)(F_0(\lambda, q) + F_0(\lambda, -q)) d\lambda.
\]

It is easy to see that the r.h.s. of (5.17) is equal to zero identically. Indeed, due to (5.8) the integral in the r.h.s. of (5.17) can be represented in the form

\[
\int_{-q}^q (x - 2\lambda t)(\delta(\lambda - \mu) + R(\lambda, \mu))(\xi(\mu, q) + \xi(\mu, -q)) d\lambda d\mu.
\]

Acting by the resolvent to the left and using (5.6)–(5.9), we obtain

\[
\frac{1}{4\pi} \int_{-q}^q (xk'_0(\lambda) - t\varepsilon'_0(\lambda))(\xi(\lambda, q) + \xi(\lambda, -q)) d\lambda.
\]

Since \( k'_0(\lambda) \) is an even function, then the coefficient at \( x \) is equal to zero. The remaining integral after the integration by parts gives

\[
\frac{it}{4\pi} \int_{-q}^q \varepsilon_0(\lambda)(K(\lambda, q) + K(\lambda, -q)) d\lambda.
\]

Now one can use the equation (5.6) and the condition \( \varepsilon_0(\pm q) = 0 \), after that we obtain

\[
r_{-1} = 0, \quad \text{for} \quad T = 0.
\]

Thus our formulæ do give the correct result at zero temperature.

Recall that the introducing of the dual fields is equivalent in some sense to factorization of the \( S \)-matrix. It is natural to expect that under the averaging of the dual fields the objects have to appear, having direct relation to the scattering matrix. However, while the factorization takes place at the bare Fock vacuum, the averaging is performed already in the thermodynamic limit, i.e. at the physical vacuum. It is remarkable that at this rate instead of the ‘bare’ \( S \)-matrix (2.3) the function \( w_0(u) \) arises, which can be expressed in terms of the ‘dressed’ scattering phase (5.8) at the boundary of the Fermi sphere.

Observe also, that in the calculations presented above, we did not need the explicit value of \( \Lambda \). However, it is not difficult to find it. It follows from the equation (5.17) that

\[
\Lambda - \lambda_0 - \int_{-q}^q (\lambda_0 - \lambda)R(\lambda, \Lambda) d\lambda = 0,
\]

and we immediately find

\[
\lambda_0 k'_0(\Lambda) - \frac{1}{2} \varepsilon'_0(\Lambda) = 0, \quad \text{or} \quad v_0(\Lambda) = \frac{x}{t}.
\]
Thus the quantity \( \Lambda \) has a pure physical sense: a particle possessing velocity \( v_0(\Lambda) \), passes the distance \( x \) spending the time \( t \).

In order to compute the correlation radius for small, but not zero temperature, more accurate estimates of the integrals containing logarithmic function are necessary. The example of such calculations is given in the Appendix A. It is clear, however, that all low temperature corrections to the correlation radius, as well as to the integral equation for the function \( w(u) \), are provided by vicinities of the points \( \pm q \), defining the boundary of the Fermi sphere. Let us present here the integral equation for the function \( w(u) \), arising in the first order with respect to \( T \). We shall look for the function \( w(u) \) as a series

\[
 w(u) = w_0(u) + Tw_1(u) + T^2w_2(u) + \ldots
\]

Here \( w_0(u) \) is given by (5.16). For \( w_1(u) \) we have

\[
 w_1(u) = \frac{1}{2\pi} \int_{-q}^{q} K(u, \lambda) w_1(\lambda) d\lambda = -\frac{K(u, q)}{4\pi \varepsilon_0(q)} (w_0(q) + i\pi)^2 - \frac{K(u, -q)}{4\pi \varepsilon_0(q)} (w_0(-q) - i\pi)^2. \tag{5.20}
\]

Hence

\[
 w_1(u) = \frac{\pi^2}{2\varepsilon_0(q)} (1 - F_0(q, q) - F_0(q, -q))^2 (R(u, q) + R(u, -q)). \tag{5.21}
\]

Here we took into account, first, that \( F_0(-\lambda, -\mu) = -F_0(\lambda, \mu) \). Second, we have put already the parameter \( \beta = 0 \). On can easily check that it is always so in the framework of the low temperature approximation. Indeed, for any order of \( T \) the dependency of \( w(u) \) on the sign-functions \( \text{sign}(\Lambda - u) \) disappears. The parameters \( \Lambda \) and \( \beta \) enter the equation for \( w(u) \) only in the combination \( \beta K(\Lambda, u) \) (see (A.15)). Therefore the derivative \( \partial w/\partial \Lambda \) always turns out to be proportional to \( \beta \), and hence, the equation (4.19) is valid for \( \beta = 0 \).

Substituting \( w_1(u) \) into the answer for the correlation radius, after simple algebra we arrive at

\[
 r_+^{-1} = \frac{\pi T}{4v_0} (1 - F_0(q, q) - F_0(q, -q))^2 (|x - v_0t| + |x + v_0t|). \tag{5.22}
\]

Here \( v_0 = v_0(q) \). This result is given for arbitrary location of the roots \( \Lambda_1 \) and \( \Lambda_2 \) with respect to the point \( \Lambda \). In the case \( \Lambda_1 < \Lambda_2 < \Lambda \), which we considered from the very beginning, one can remove the symbols of the absolute value.

Comparing (5.22) with the result obtained in [7]

\[
 r_+^{-1} = \frac{\pi T}{4v_0} \left( \frac{|x - v_0t| + |x + v_0t|}{(2\pi \rho_0(q))^2} \right), \tag{5.23}
\]

we obtain an identity

\[
 \frac{1}{2\pi \rho_0(q)} = 1 - F_0(q, q) - F_0(q, -q). \tag{5.24}
\]

At the first sight such the non-linear relationship between the solutions of the linear integral equations looks rather strange. However, taking into account a non-linear identity for the scattering phase [14], [15]

\[
 (1 - F_0(q, q))^2 - F_0^2(q, -q) = 1 \tag{5.25}
\]
and the equation (5.11), we make sure of the validity of the identity (5.24). Thus, in the linear approximation with respect to the temperature our result for the correlation radius exactly coincides with the result obtained in the framework of the conformal field theory.

The computation of higher temperature corrections does not meet serious difficulties. In the all orders \( T^n \) the equations for \( w_n(u) \) remain linear (see (A.8)), and their solutions can be written explicitly in terms of the resolvent of the operator \( I - \frac{1}{2\pi} K \) and its derivatives. Respectively the correlation radius is expressed in terms of functions \( k_0'(q), \varepsilon_0'(q), F_0(q, \pm q) \) and their derivatives. In particular the correction to \( r_{+-}^{-1} \) of the order \( T^2 \) vanishes. Thus the result (5.22) is valid up to the terms of order \( T^3 \).

**Conclusion**

We have considered the large time and long distance asymptotic behavior of the correlation function of the quantum one-dimensional Bose gas. As we have shown, the asymptotics can be expressed in terms of the integral equation solutions, which are closely related to the equations of the thermodynamic Bethe Ansatz. In the low temperature limit we were able even to describe the solutions of mentioned equations in terms of observables of the model. We would like to draw attention of the reader to the fact, that these integral equations arise under the averaging of the dual fields. While in the earlier works these quantum operators played purely auxiliary role and mostly were used for solving of certain combinatorial problems, the results obtained now allows us to have a look at these objects from a new point of view. In particular, it is hardly accidentally that at the zero temperature the solutions of the integral equations depend on the scattering phase. Apparently the role of the dual fields is much more fundamental, than it was understood before.

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**A Appendix**

As an example of the low temperature expansion we consider the integral, entering the Yang–Yang equation

\[
I = T \int_{-\infty}^{\infty} K(\lambda, \mu) \ln \left( 1 + e^{-\frac{\varepsilon(\mu)}{T}} \right) d\mu. \tag{A.1}
\]

Obviously, at \( T = 0 \) only the interval \([-q, q]\) contributes into this integral

\[
I_{T=0} = -\int_{-q}^{q} K(\lambda, \mu) \varepsilon(\mu) d\mu, \quad \varepsilon(\pm q) = 0 \quad T = 0. \tag{A.2}
\]

At small, but finite temperature the corrections to this expression arise due to the contributions of vicinities of the points \( \pm q \). Consider, for instance, the integral

\[
I_+ = T \int_{qT}^{\infty} K(\lambda, \mu) \ln \left( 1 + e^{-\frac{\varepsilon(\mu)}{T}} \right) d\mu, \tag{A.3}
\]
where \( \varepsilon(q_T) = 0 \). Making the replacement of variables \( \varepsilon(\mu) = Tz \), we obtain

\[
I_+ = T^2 \int_0^\infty \varphi(\lambda, Tz) \ln \left(1 + e^{-z}\right) dz,
\]

(A.4)

where \( \varphi(\lambda, Tz) = K(\lambda, \mu)/\varepsilon'(\mu) \). The integral (A.4) can be expanded now into the series with respect to \( T \). We have

\[
I_+ = \sum_{n=0}^{\infty} T^{n+2} \varphi^{(n)}(\lambda, 0)(1 - 2^{-n-1})\zeta(n + 2),
\]

(A.5)

or in terms of the functions \( K(\lambda, \mu) \) and \( \varepsilon(\mu) \)

\[
I_+ = \sum_{n=0}^{\infty} T^{n+2}(1 - 2^{-n-1})\zeta(n + 2)D_n^{q_T} \frac{K(\lambda, q_T)}{\varepsilon'(q_T)}, \quad \text{where} \quad D_\lambda = \frac{1}{\varepsilon'(\lambda)} \frac{\partial}{\partial \lambda}.
\]

(A.6)

Similarly all the remaining contributions from vicinities of the points \( \pm q_T \) can be computed. Then one need to substitute the expansion obtained into the Yang–Yang equation and to put

\[
\varepsilon(\lambda) = \sum_{n=0}^{\infty} T^n \varepsilon_n(\lambda),
\]

(A.7)

\[
q_T = q + \sum_{n=1}^{\infty} T^n q_n.
\]

The arising infinite set of equations can be solved via recursion. As a result the functions \( \varepsilon_n \) and parameters \( q_n \) are expressed in terms of \( \varepsilon'_0(q) \) and the resolvent \( R(\lambda, \pm q) \).

Similarly all the integrals of the section 4 can be decomposed into series. Let us present the complete expansion with respect to the temperature of the integral equation (4.16) for the function \( w(u) \)

\[
w(u) - \frac{1}{2\pi} \int_{-q}^{q} K(u, \lambda) w(\lambda) d\lambda = \frac{1}{2} \left(\xi(-q, u) + \xi(q, u)\right) + i\beta K(A, u)
\]

\[-\frac{1}{2\pi} \sum_{\gamma = \pm 1} \sum_{n=2}^{\infty} \frac{(-2\pi i\gamma)^n T^{n-1}}{n!} D_{q_T}^{n-2} K(u, \gamma q_T) \frac{B_n \left(1 + \gamma w(\gamma q_T) \right)}{2\pi i} \left(1 - B_n(1/2)\right). \]

(A.8)

Here \( B_n(z) \) are the Bernulli polynomials. It is yet necessary to substitute into this equation decompositions (A.7), after that functions \( w_n(u) \) can be evaluated in terms of \( \varepsilon'_0(q) \), \( R(\lambda, \pm q) \) and their derivatives.

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