On the inductive blockwise Alperin weight condition for type $A$

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Abstract

In this paper we prove the blockwise Alperin weight conjecture for finite special linear and unitary groups, for finite groups with abelian Sylow 3-subgroups, and verify the inductive blockwise Alperin weight condition for certain cases of groups of type $A$. We also give a classification for the 2-blocks of special linear and unitary groups.

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1 Introduction

The famous Alperin weight conjecture relates for a prime $\ell$ information about irreducible $\ell$-Brauer characters of a finite group $G$ to properties of $\ell$-local subgroups of $G$. We call an $\ell$-weight a pair $(R, \varphi)$, where $R$ is an $\ell$-subgroup of $G$ and $\varphi \in \text{Irr}(N_G(R))$ with $R \subseteq \ker \varphi$ is of $\ell$-defect zero viewed as a character of $N_G(R)/R$. When such a character $\varphi$ exists, $R$ is necessarily a radical $\ell$-subgroup of $G$ (i.e. $R = O_{\ell}(N_G(R))$). For an $\ell$-block $B$ of $G$, a weight $(R, \varphi)$ is called a $B$-weight if $\text{bl}(\varphi)^G = B$, where $\text{bl}(\varphi)$ is the $\ell$-block of $N_G(R)$ containing $\varphi$. We denote by $\text{Alp}_G(B)$ the set of all $G$-conjugacy classes of $B$-weights. In [1] p. 371, J. L. Alperin states the following (blockwise Alperin weight) conjecture.

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Conjecture (Alperin, 1986). Let $G$ be a finite group, $\ell$ a prime. If $B$ is an $\ell$-block of $G$, then $|\text{Alp}_\ell(B)| = |\text{IBr}_\ell(B)|$.

While the Alperin weight conjecture was subsequently checked for several families of finite groups, a significant breakthrough in the case of general groups was achieved by Navarro and Tiep [35] in 2011, where they reduced the block-free version of Alperin’s weight conjecture to a question on simple groups. In 2013, the blockwise version was also reduced by Späth [37]. The blockwise Alperin weight conjecture holds for all finite groups at the prime $\ell$, if all finite non-abelian simple groups satisfy the so-called inductive blockwise Alperin weight (iBAW) condition at the prime $\ell$.

This paper is a continuation of our previous paper [22]. In that paper, the inductive condition for the block-free version of Alperin weight conjecture from [35] has been verified for all simple groups of type $A$.

We also focus on simple groups of type $A$ in this paper. Let $q = p^f$ be a power of a prime $p$, $\eta = \pm 1$ and $\text{SL}_n(\eta q)$ be the finite special linear or unitary group. Here, $\text{SL}_n(-q)$ is understood as $\text{SU}_n(q)$. We also denote $\text{PSL}_n(\eta q) = \text{SL}_n(\eta q)/Z(\text{SL}_n(\eta q))$. We remark that some progress has been made for type $A$; for example, the (iBAW) condition has been verified for $\text{PSL}_n(\eta q)$ with $\gcd(n, q-\eta) = 1$ in [31] [32], for unipotent blocks of $\text{SL}_n(\eta q)$ with $\ell \nmid q \cdot \gcd(n, q-\eta)$ in [20], and for blocks of $\text{SL}_n(\eta q)$ with abelian defect groups when $\ell \nmid 3q(q-\eta)$ by Brough and Späth [9]. For a list about other types and more cases for which the inductive conditions has been verified, see for example [22] and the references therein.

Using the results in [22], we will verify the (iBAW) condition for a system of certain blocks of $\text{SL}_n(\eta q)$; see Theorem 4.3. We give some consequences of this result as follows. First, we achieve the blockwise Alperin weight conjecture for $\text{SL}_n(\eta q)$, which generalises [20] Thm. 1.2).

**Theorem 1.** The blockwise Alperin weight conjecture holds for all finite special linear and unitary groups and all primes.

The (iBAW) condition is verified for the unipotent blocks of $\text{SL}_n(\eta q)$ for $\ell \nmid \gcd(n, q-\eta)$ in [20]. The following result generalized this work to all primes and more blocks.

**Theorem 2.** Let $G = \text{SL}_n(\eta q)$, $\ell$ a prime not dividing $q$ and $B$ an $\ell$-block of $G$. If $B$ is either a unipotent block or a block of maximal defect, then the inductive blockwise Alperin weight condition holds for $B$.

For general blocks of special linear and unitary groups, we prove the following theorem which generalized the work for simple groups of type $A$ with trivial Schur multiplier in [31] [32].
**Theorem 3.** Let $S = \text{PSL}_n(\eta q)$ be a simple group and $\ell$ be a prime not dividing $q$. Assume that $\gcd(n, q - \eta)\ell'$ is square-free. Then the inductive blockwise Alperin weight condition holds for $S$ and $\ell$.

From this, we prove that the (iBAW) condition holds for simple groups $\text{PSL}_n(q)$ and $\text{PSU}_n(q)$ for $n \leq 7$ (see Proposition 4.6).

In [37], Späth proved the blockwise Alperin weight conjecture holds for finite groups with abelian Sylow 2-subgroups via the (iBAW) condition. We will consider finite groups with abelian Sylow 3-subgroups and prove in this paper the following result.

**Theorem 4.** Assume that $G$ is a finite group with abelian Sylow 3-subgroups. Then the blockwise Alperin weight conjecture holds for $G$ and any prime.

The last section of this paper is a continuation of [20, §4]. There is a classification of the $\ell$-blocks of $\text{SL}_n(\eta q)$ in [20, §4] when $\ell \nmid q$ is odd, using the labelling set of $d$-Jordan-cuspidal pairs given by Cabanes–Enguehard [12] and Kessar–Malle [26]. The results related to $d$-Jordan-cuspidal pairs in [12, 26] are restricted to odd primes. In Section 6 of this paper, we consider the 2-blocks of $\text{SL}_n(\eta q)$ (with odd $q$). Our result relies on the classification of the Brauer pairs of $\text{GL}_n(\eta q)$ by Broué [7] and the description of the radical subgroups of $\text{SL}_n(\eta q)$ in [22]. The number of 2-blocks of $\text{SL}_n(\eta q)$ covered by a given 2-blocks of $\text{GL}_n(\eta q)$ is determined in Remark 6.9 and in this way we obtain a parametrization for the 2-blocks of finite special linear and unitary groups, which complements the result of [20, §4].

We begin Section 2 by recalling the criterion for the (iBAW) condition given by Brough-Späth and give some notation and preliminaries for linear and unitary groups in Section 3. Using the results of [22], it suffices to consider only one condition for type $A$, and from this we verify the (iBAW) condition for certain cases of groups of type $A$ and prove Theorem 1–3 in Section 4. The blockwise Alperin weight conjecture is verified to hold for finite groups with abelian Sylow 3-subgroups in Section 5. In Section 6 we give a classification of the 2-blocks of finite special linear and unitary groups; see Remark 6.9.

## 2 A criterion for the inductive blockwise Alperin weight condition

All groups considered in this paper are finite. For the notation for the block and character theory, we mainly follow [25] [34], except that we denote the restriction
of \( \chi \in \text{Irr}(G) \cup \text{IBr}(G) \) to some subgroup \( H \leq G \) by \( \text{Res}^G_H \chi \), while \( \text{Ind}^G_H \psi \) denotes the character induced from \( \psi \in \text{Irr}(H) \cup \text{IBr}(H) \) to \( G \).

If a group \( A \) acts on a finite set \( X \), we denote by \( A \times X \) the stabilizer of \( x \in X \) in \( A \), analogously we denote by \( A_X \) the setwise stabilizer of \( X' \subseteq X \). Let \( \chi \in \text{Irr}(G) \cup \text{IBr}(G) \) given by \( a^\chi(g) = \chi^a(g) = \chi(g^a) \) for every \( g \in G \), \( a \in A \) and \( \chi \in \text{Irr}(G) \cup \text{IBr}(G) \). For \( P \leq G \) and \( \chi \in \text{Irr}(G) \cup \text{IBr}(G) \), we denote by \( A_{P,\chi} \) the stabilizer of \( \chi \) in \( A_P \). By \( \chi' \in A \) and \( A_X \) the stabilizer of \( X \) in \( A_P \).

For \( N \leq G \) G we sometimes identify the characters of \( G/N \) with the characters of \( G \) whose kernel contains \( N \). If \( \chi \in \text{Irr}(G) \), then \( \chi^0 \) is used for the restriction to the \( \ell \)-regular elements of \( G \).

If \( G \) is abelian, we also write \( \text{Lin}(G) = \text{Irr}(G) \) since all irreducible characters of \( G \) are linear. Let \( \text{Lin}_\ell(G) \) denote the element of \( \text{Lin}(G) \) of \( \ell \) order. Then the map \( \text{Lin}_\ell(G) \to \text{IBr}(G), \chi \mapsto \chi^0 \) is bijective. From this, we always identify \( \text{IBr}(G) \) with \( \text{Lin}_\ell(G) \) when \( G \) is abelian.

For a finite group \( G \), we denote by \( \text{diz}_\ell(G) \) the set of all \( \ell \)-defect zero characters of \( G \). If \( \chi \in \text{Irr}(G) \cup \text{IBr}(G) \), we write \( \text{bl}_\ell(\chi) \) for the \( \ell \)-block of \( G \) containing \( \chi \). If \( R \) is a radical \( \ell \)-subgroup of \( G \) and \( B \) is an \( \ell \)-block of \( G \), then we define the set
\[
\text{diz}_\ell(N_G(R)/R, B) := \{ \varphi \in \text{diz}_\ell(N_G(R)/R) \mid \text{bl}_\ell(\varphi)^G = B \},
\]
where we regard \( \varphi \) as an irreducible character of \( N_G(R) \) containing \( R \) in its kernel when considering the induced block \( \text{bl}_\ell(\varphi)^G \).

The inductive blockwise Alperin weight (IBAW) condition can be stated using the notion of modular character triples and isomorphisms between them (for background on modular character triples, see, e.g., \([34, \S 8]\)). By \([38, \text{Thm. 4.4}]\), an \( \ell \)-block \( B \) of \( G \) satisfies the (IBAW) condition from \([37, \text{Def. 4.1}]\) if for \( \Gamma := \text{Aut}(G)_B \) there exists a \( \Gamma \)-equivariant bijection \( \Omega : \text{IBr}(B) \to \text{Alp}_\ell(B) \) such that for every \( \psi \in \text{IBr}(B) \) and \( \Omega(\psi) = (R, \varphi) \), one has
\[
\text{(2.1)} \quad (G \rtimes \Gamma_{R, \varphi}, G, \psi) \succeq_b (N_G(Q) \rtimes \Gamma_{R, \varphi}, N_G(Q), \varphi^0).
\]
For the definition of the relation \( \succeq_b \), which is called the block isomorphism of modular character triples, see \([38, \text{Def. 3.2}]\).

Recently, J. Brough and B. Spáth \([10, \text{Thm. 4.5}]\) gave a criterion for the inductive Alperin weight condition adapted to quasi-simple groups of Lie type with abelian outer automorphism groups. Here we rewrite it and give a new version suitable for quasi-simple groups with possibly non-abelian outer automorphism groups. In fact, conditions (i) – (iv) are the same, and condition (v) which considers relations of blocks for irreducible constituents of (Brauer) characters is altered.

**Theorem 2.2.** Let \( S \) be a finite non-abelian simple group and \( \ell \) a prime dividing \( |S| \). Let \( G \) be the universal \( \ell \)-covering group of \( S \), \( B \) a union of \( \ell \)-blocks of \( G \).
and assume there are groups \( \bar{G}, D \) such that \( G \leq \bar{G} \rtimes D \), \( B \) is a \( \bar{G} \)-orbit and the following hold.

(i) (a) \( G = [\bar{G}, \bar{G}] \) and \( D \) is abelian,
(b) \( C_D(G) = Z(\bar{G}) \) and \( \bar{G}D/Z(\bar{G}) \cong \text{Aut}(G) \),
(c) any element of \( \text{IBr}_\ell(B) \) extends to its stabilizer in \( \bar{G} \),
(d) for any radical \( \ell \)-subgroup \( R \) of \( G \) and any \( B \in \mathcal{B} \), any element of 
\( \text{dz}_\ell(N_G(R)/R | B) \) extends to its stabilizer in \( N_G(R)/R \).

(ii) Let \( \bar{B} \) be the union of \( \ell \)-blocks of \( \bar{G} \) covering \( \mathcal{B} \). There exists a \( \text{Lin}_\ell(\bar{G}/G) \times D_{\bar{G}} \)-equivariant bijection \( \Omega: \text{IBr}_\ell(\bar{B}) \to \text{Alp}_\ell(\bar{B}) \) such that
\[
\begin{align*}
\text{(a) } & \Omega(\text{IBr}_\ell(\bar{B})) = \text{Alp}_\ell(\bar{B}) \text{ for every } \bar{B} \in \mathcal{B} , \\
\text{(b) } & J_G(\psi) = J_G(\Omega(\psi)) \text{ for every } \psi \in \text{IBr}_\ell(\bar{G}).
\end{align*}
\]

(iii) For every \( \psi \in \text{IBr}_\ell(\bar{B}) \), there exists some \( \psi_0 \in \text{IBr}_\ell(G | \psi) \) such that
\[
\begin{align*}
\text{(a) } & (\bar{G} \rtimes D)_{\psi_0} = \bar{G}_{\psi_0} \rtimes D_{\psi_0} , \\
\text{(b) } & \psi_0 \text{ extends to } G \rtimes D_{\psi_0} .
\end{align*}
\]

(iv) For every \( (\bar{R}, \bar{\varphi}) \in \text{Alp}_\ell(\bar{B}) \), there is an \( \ell \)-weight \( (R, \varphi_0) \) of \( G \) covered by 
\( (\bar{R}, \bar{\varphi}) \) such that
\[
\begin{align*}
\text{(a) } & (\bar{G}D)_{R,\varphi_0} = \bar{G}_{R,\varphi_0} (GD)_{R,\varphi_0} , \\
\text{(b) } & \varphi_0 \text{ extends to } (G \rtimes D)_{R,\varphi_0} .
\end{align*}
\]

(v) If the \( \ell \)-Brauer character \( \tilde{\psi} \) in (iii) and the weight \( (\bar{R}, \bar{\varphi}) \) in (iv) satisfy \( (\bar{R}, \bar{\varphi}) = \Omega(\bar{\psi}) \), then the \( \ell \)-Brauer character \( \psi_0 \) and the weight \( (R, \varphi_0) \) can be chosen in the same block \( B \in \mathcal{B} \) and satisfy
\[
\text{(2.3) } bl_\ell(\tilde{\psi}) = bl_\ell(\hat{\varphi})^{\bar{G}}
\]
where \( \tilde{\psi} \in \text{IBr}_\ell(\bar{G} \psi | \psi) \) is the Clifford correspondent of \( \tilde{\psi} \), \( \hat{\varphi} \) is an extension of \( \varphi_0 \) to \( N_G(R)_{\psi}/R \) such that via induction and the map \( \Delta_\varphi \) from \( \text{Thm. 2.10} \) it corresponds to \( (\bar{R}, \bar{\varphi}) \).

Then the inductive blockwise Alperin weight \( \text{(iBAW)} \) condition holds for every block \( B \in \mathcal{B} \).

For the definition of \( J_G(\psi) \), see \( \text{[10, §2]} \).

Proof. We use the construction in \( \text{[10, Thm. 4.5]} \). In fact, according to its proof, the conditions (i)–(iv) already give a \( (\bar{G} \rtimes D)_{\bar{G}} \)-equivariant bijection \( \Omega: \text{IBr}_\ell(\mathcal{B}) \to \text{Alp}_\ell(\mathcal{B}) \). Similar as the proof of \( \text{[10, Lemma 4.6]} \), we can show that \( (2.1) \) holds.
via $\Omega$. Thus it suffices to show that $\Omega$ preserves blocks. For $\hat{\psi} \in \text{IBr}_i(\hat{B})$ and weight $(\hat{R}, \hat{\varphi}) \in \text{Alp}_i(\hat{B})$ satisfying $(\hat{R}, \hat{\varphi}) = (\hat{\Omega}(\hat{\psi}))$, we let $\psi_0$ and $(R, \varphi_0)$ satisfy (iii) and (iv) respectively. Then $\psi_0$ and $(R, \varphi_0)$ have the same stabilizer in $\hat{G} \rtimes D$ and by the construction of $\Omega$ in [10], we have $\Omega(\psi_0) = (R, \varphi_0)$. From this, if $\psi_0$ and $(R, \varphi_0)$ can be chosen in the same block of $G$, then $\Omega$ preserves blocks, as desired.

Let $S$ be a non-abelian finite simple group, $\ell$ a prime dividing $|S|$ and $G$ the universal $\ell'$-covering group of $S$. We say that the (iBAW) condition holds for $S$ and $\ell$ if the (iBAW) condition holds for every $\ell$-block of $G$. Moreover, we say the (iBAW) condition holds for $S$ if the (iBAW) condition holds for $S$ and any prime $\ell$ dividing $|S|$.

**Lemma 2.4.** Let $\tilde{G}$ be an arbitrary finite group and $G \triangleleft \tilde{G}$ with abelian quotient $\tilde{G}/G$. Let $\hat{\psi} \in \text{IBr}_i(\tilde{G})$, $\psi \in \text{IBr}_i(G \mid \hat{\psi})$, $\tilde{B} = \text{bl}_i(\hat{\psi})$ and $B = \text{bl}_i(\psi)$. Assume that $\psi$ extends to $\tilde{G}_\varphi$ and $\gcd(|\tilde{G}_\varphi|/|G|, |\tilde{G}/G|) = 1$. Then there is a unique block $\tilde{B}$ of $\tilde{G}_\varphi$ such that $\tilde{B}$ covers $\tilde{B}$ and $\tilde{B}$ covers $B$.

**Proof.** Note that $\text{IBr}_1(G \cap Z(\tilde{G}) \mid \psi) = \{\tau\}$. Then the blocks of $\text{IBr}_1(\tilde{G})$ covering $B$ are parametrized by $\text{IBr}_1(Z(\tilde{G}) \mid \tau)$ and $\text{IBr}_1(B_\lambda) = \{\lambda \cdot \chi \mid \chi \in \text{IBr}_1(B)\}$ where $\lambda \cdot \chi(zg) = \lambda(z)\chi(g)$ for the $\ell$-regular elements $z \in Z(\tilde{G})$ and $g \in G$. Any two of those blocks are not $\tilde{G}$-conjugate. For every subgroup $\text{IBr}_1(G) \leq \tilde{G} \leq \tilde{G}$ and every block $B_1$ of $G_1$, we let $\lambda_1 \in \text{IBr}_1(Z(\tilde{G}) \mid \chi_1)$ for $\chi_1 \in \text{IBr}_1(B_1)$. Then $B_1$ covers $B$ if and only if $B_1$ covers $B_\lambda$. So we can assume that $G = \text{IBr}(\tilde{G})$ without loss of generality.

Let $\tilde{G} = \tilde{G}_\varphi$ and let $\hat{\psi} \in \text{IBr}_i(\tilde{G})$ be the Clifford correspondent of $\hat{\psi}$. Then $\hat{\psi}$ is an extension of $\psi$. Let $\hat{B}_0 = \text{bl}_i(\hat{\psi})$. Then $\tilde{B}$ covers $\hat{B}_0$ and $\hat{B}_0$ covers $B$. Let $\mathcal{L} = \{\hat{\psi}^x \mid g \in \tilde{G}\}$ and $\mathcal{S} = \{\lambda_1 \hat{\psi}^x \mid \lambda \in \text{IBr}_i(\tilde{G}/G)\}$. Then $\mathcal{L} = \text{IBr}_i(\tilde{G} \mid \hat{\psi})$ and $\mathcal{S} = \text{IBr}_i(\tilde{G} \mid \psi)$. Moreover, $\mathcal{S}$ consists of the extensions of $\psi$ to $\tilde{G}$ and $\mathcal{L} \cap \mathcal{S} = \{\psi\}$.

Let $\mathcal{A} = \{\lambda_1 \hat{\psi}^x \mid g \in \tilde{G}, \lambda \in \text{IBr}_i(\tilde{G}/G)\}$ be the $\text{IBr}_i(\tilde{G}/G) \times (\tilde{G}/G)$-orbit on $\text{IBr}_i(\tilde{G})$ containing $\hat{\psi}$. Note that $\text{IBr}_i(\tilde{G}/G)$ permutes the blocks of $\tilde{G}$ covering $B$. Now $\gcd(|\tilde{G}_\varphi|/|G|, |\tilde{G}/\tilde{G}_{\varphi}|) = 1$, so every subgroup of $\text{IBr}_i(\tilde{G}/G) \times (\tilde{G}/G)$ is of form $H \times K$ with $H \leq \text{IBr}_i(\tilde{G}/G)$ and $K \leq \tilde{G}/\tilde{G}$. In particular, the stabilizer of $\hat{B}_0$ in $\text{IBr}_i(\tilde{G}/G) \times (\tilde{G}/\tilde{G})$ is $I \times (\tilde{G}/\tilde{G})$ where $I$ is the stabilizer of $\hat{B}_0$ in $\text{IBr}_i(\tilde{G}/G)$. Thus $\text{IBr}_i(\tilde{B}_0) \cap \mathcal{A} = \{\lambda_1 \hat{\psi}^x \mid g \in \tilde{G}_{\varphi} / \tilde{G}, \lambda \in I\}$.

Assume that $\tilde{B}$ is a block of $\tilde{G}$ such that $\tilde{B}$ covers $\hat{B}$ and $\hat{B}$ covers $B$. Then there exists $g \in \tilde{G}$ such that $\tilde{B} = \tilde{B}_{\varphi}$. In particular, $\hat{\psi}^x \in \text{IBr}_i(\hat{B})$. Since $\tilde{B}$ covers $B$, one has that $\tilde{B} = \lambda \otimes \hat{B}_0$ with $\lambda \in \text{IBr}_i(\tilde{G}/G)$. Then $\hat{\psi}^x = \lambda \hat{\psi}^x \in \text{IBr}_i(\hat{B}_0)$. It follows that $\hat{\psi}^x = \lambda^{-1} \hat{\psi}^x \in \mathcal{A}$. Let $\hat{\psi}' = \lambda' \hat{\psi}^x$ with $\lambda' \in I$ and $g' \in \tilde{G}_{b_0}$. Then $\hat{\psi}'^x = \lambda' \lambda^{-1} \hat{\psi}^x$. This implies that $\lambda' \lambda^{-1} \in I_{\tilde{G}}$ and $gg'^{-1} \in \tilde{G}_{b_0}$ because $\mathcal{L} \cap \mathcal{S} = \{\hat{\psi}\}$. So $g \in \tilde{G}_{b_0}$ and $\hat{B} = \hat{B}_0$, which complete the proof.
Proposition 2.5. In Theorem 2.2 if moreover one of the following holds:

(i) if $B'$ is a block of some group $J$ with $G \leq J \leq \bar{G}$ that is covered by a block of $\bar{B}$, then $\bar{G}_{B'} = \bar{G}$,

(ii) there is some $\chi \in \text{Irr}(\bar{B}) \cup \text{IBr}(\bar{B})$ such that $\text{Res}_{\bar{G}}^{\bar{G}} \chi$ is irreducible,

(iii) $|\bar{G}/GZ(\bar{G})|_{\ell}$ is square-free,

then (2.3) holds.

Proof. Let $\psi_0$ and $(R, \varphi_0)$ be as in Theorem 2.2 (v). We will prove that $\text{bl}(\hat{\psi}) = \text{bl}(\hat{\varphi})^{ar{G}_\varphi}$, where $\hat{\psi} \in \text{IBr}(G_\psi | \psi)$ is the Clifford correspondent of $\hat{\psi}$, and $\bar{G}$ is an extension of $\varphi_0$ to $N_\bar{G}(R)/R$ such that via induction and the map $\Delta_\varphi$ (from [10] Thm. 2.10) it corresponds to $(\bar{R}, \bar{\varphi})$. Since $\Omega$ and $\bar{\Omega}$ respect blocks, by [27] Lemma 2.3, one has $\text{bl}(\hat{\varphi})^{ar{G}_\varphi}$ covers $\text{bl}(\bar{\psi})$. Also by [10] Thm. 2.10, $\text{bl}(\hat{\psi})$ covers $\text{bl}(\hat{\varphi})^{ar{G}_\varphi}$. For (i) or (iii), $\text{bl}(\hat{\psi})$ covers a unique block of $\bar{G}_\psi$. For (ii), Lemma 2.4 also implies $\text{bl}(\hat{\psi}) = \text{bl}(\hat{\varphi})^{ar{G}_\varphi}$. This completes the proof. \qed

3 Linear and unitary groups

We will follow the notation in [22] for linear and unitary groups. Much of this notation originally comes from [2, 3, 4, 5, 23].

Assume $q = p^f$ is a power of a prime $p$ and $n \geq 2$. Let $\text{GL}_n(q)$ be the group of all invertible $n \times n$ matrices over $\mathbb{F}_q$. Also we denote by $F_p$, $F_q$ and $\sigma_q$ the field automorphism, standard Frobenius endomorphism and graph automorphism respectively; see the definitions for example in [22] §2. Recall that $\text{GL}_n(-q)$ denotes the general unitary group

$$\text{GU}_n(q) = \{ A \in \text{GL}_n(q^2) | (F_q(A))^t A = I_n \},$$

where $I_n$ is the identity matrix of degree $n$. We will use the similar notation $\text{SL}_n(-q)$ ($\text{PSL}_n(-q)$) for $\text{SU}_n(q)$ ($\text{PSU}_n(q)$). Let $\bar{G} = \text{GL}_n(\eta q)$ and $G = \text{SL}_n(\eta q)$ for $\eta = \pm 1$. We define $D = \langle F_p, \sigma_q \rangle$ if $n \geq 3$ while $D = \langle F_p \rangle$ if $n = 2$. Then by [24] Thm. 2.5.1, $(\bar{G} \rtimes D)/Z(\bar{G}) \cong \text{Aut}(G)$.

We also recall the subset $\mathcal{F}$ of the set of monic irreducible polynomials from [23] §1. Denote by $\text{Irr}(\mathbb{F}_q[X])$ the set of all monic irreducible polynomials over the field $\mathbb{F}_q$. For $\Delta(X) = X^m + a_{m-1}X^{m-1} + \cdots + a_0$ in $\mathbb{F}_q[X]$, we define $\tilde{\Delta}(X) = X^m a_0^{-\eta} \Delta(X^{-1})$, where $\Delta(X)$ means the polynomial in $X$ whose coefficients are the $q$-th powers of the corresponding coefficients of $\Delta(X)$. Now, we denote by

$$\mathcal{F}_0 = \{ \Delta | \Delta \in \text{Irr}(\mathbb{F}_q[X]), \Delta \neq X \},$$

$$\mathcal{F}_1 = \{ \Delta | \Delta \in \text{Irr}(\mathbb{F}_q[X]), \Delta \neq X, \Delta = \tilde{\Delta} \},$$

$$\mathcal{F}_2 = \{ \Delta \tilde{\Delta} | \Delta \in \text{Irr}(\mathbb{F}_q[X]), \Delta \neq X, \Delta \neq \tilde{\Delta} \}.$$
and

$$\mathcal{F} = \begin{cases} \mathcal{F}_0 & \text{if } \eta = 1; \\ \mathcal{F}_1 \cup \mathcal{F}_2 & \text{if } \eta = -1. \end{cases}$$

We denote by $d_\ell$ the degree of any polynomial $\Gamma$. For any semisimple element $s$ of $\hat{G}$, we let $s = \prod_{\Gamma} s_{\Gamma}$ be its primary decomposition. We denote by $m_{\Gamma}(s)$ the multiplicity of $\Gamma$ in $s_{\Gamma}$. If $m_{\Gamma}(s)$ is not zero, we call $\Gamma$ an elementary divisor. Denote by $\mathcal{F}'$ the subset of $\mathcal{F}$ of those polynomials whose roots are of $\ell'$-order. For $\Gamma \in \mathcal{F}'$, denote by $e_{\Gamma}$ the multiplicative order of $(\eta q)^{\ell'}$ modulo $\ell$. Also, we define $e$ to be the multiplicative order of $\eta q$ modulo $\ell$. Note that $e = e_{\Gamma} = 1$ when $\ell = 2$.

Let $F_{\eta q} = \sigma_{\mu} F_q$. Then $F_{\eta q}$ acts on $\mathbb{F}_q^\times$ by $F_{\eta q}(\xi) = \xi^{\eta q}$. A polynomial $\Gamma \in \mathcal{F}$ can be identified with the set of roots of $\Gamma$, which can be again identified with an $F_{\eta q}$-orbit $(F_{\eta q}) \cdot \xi$ of this action, where $\xi$ is a root of $\Gamma$; see for example [17, §3.1]. Let $\mathcal{Z} = \{ z \in \mathbb{F}_q^\times | z^{e_{\eta q}} = 1 \}$. For any $z \in \mathcal{Z}$ and $\Gamma \in \mathcal{F}$, $z \cdot \Gamma$ is defined to be the polynomial in $\mathcal{F}$ whose roots are the roots of $\Gamma$ multiplied by $z$, defining an action of $\mathcal{Z}$ on $\mathcal{F}$. Note that we can identify $Z(\hat{G})$ with $\mathcal{Z}$.

For the representations of finite groups of Lie type, see for example [13]. Let $\hat{G} = \text{GL}_n(\mathbb{F}_q)$, $F = F_{\eta q}$, then $\hat{G} = \hat{G}_F$. If $\mathcal{L}$ is a Levi subgroup of a reductive group $\hat{G}$ with the Frobenius map $F$, then by the fact that $Z(\hat{G})$ is connected, there is an isomorphism (see for example [13] (8.19))

$$Z(\mathcal{L}_F) \rightarrow \text{Irr}(\mathcal{L}_F / [\mathcal{L}_F, \mathcal{L}_F]), \quad z \mapsto \hat{z}.$$  

If $s$ is a semisimple element of $\hat{G}$, then $C_G(s)$ is a Levi subgroup of $\hat{G}$.

4 The inductive blockwise Alperin weight conditions for simple groups of type A

Given a semisimple element $s$ of $\hat{G} = \text{GL}_n(\eta q)$, let $\prod_{\Gamma \in \mathcal{F}} s_{\Gamma}$ be the primary decomposition of $s$. Here $s_{\Gamma}$ is conjugate to $m_{\Gamma}(s)(\Gamma)$, where $(\Gamma)$ is the companion matrix of $\Gamma$. Thus $n = \sum_{\Gamma \in \mathcal{F}} m_{\Gamma}(s)d_\ell$. Jordan decomposition gives a bijection between the irreducible characters of $\hat{G} = \text{GL}_n(\eta q)$ and the $\hat{G}$-conjugacy classes of pairs $(s, \mu)$, where $s = \prod_{\Gamma \in \mathcal{F}} m_{\Gamma}(s)(\Gamma)$ is a semisimple element of $\hat{G}$ and $\mu = \prod_{\Gamma \in \mathcal{F}} \mu_{\Gamma}$ with $\mu_{\Gamma} = m_{\Gamma}(s)$. See for instance [23] §1 and [13] Chap. 8.

The blocks of $\hat{G} = \text{GL}_n(\eta q)$ have been classified in [23] [23]: the $\ell$-blocks of $\hat{G}$ are in bijection with the set of $\hat{G}$-conjugacy classes of pairs $(s, \lambda)$, where $s$ is a semisimple $\ell'$-element of $\hat{G}$ and $\lambda = \prod_{\Gamma} \lambda_{\Gamma}$ with $\lambda_{\Gamma}$ the $e_{\Gamma}$-core of a partition of
Recall that $e_{\Gamma}$ is the multiplicative order of $(\eta q)^{\ell}$ modulo $\ell$. Note that, for $\ell = 2$, $(s, \lambda)$ is always $(s, -)$ (here, − denotes the empty partition), which means that $E_2(\tilde{G}, s)$ is a single 2-block of $\tilde{G}$. Let $\tilde{B}$ be an $\ell$-block of $\tilde{G}$ corresponding to $(s, \lambda)$. Then an irreducible $\ell$-Brauer character with labeling $(s', \lambda') \in i\text{IBr}(\tilde{G})$ (see [31] for the notation) is in the block $\tilde{B}$ if and only if $s'$ is $\tilde{G}$-conjugate to $s$ and $\lambda'$ has $e_{\Gamma}$-core $\lambda$ for every $\Gamma$.

On the other hand, the $\tilde{B}$-weights are classified in [2, 3, 4, 5] and we will use the explicit labelling $(s, \lambda, K) \in i\text{Alp}(\tilde{G})$ from [22, §3]. In addition, an $\ell$-weight with labeling $(s', \lambda', K')$ is in the block $\tilde{B}$ with label $(s, \lambda)$ if and only if $(s', \lambda')$ is $\tilde{G}$-conjugate to $(s, \lambda)$.

The first consequence of the results in [22] is that Theorem 1 holds.

Proof of Theorem 1. Thanks to [11], we only need to consider the non-defining characteristic. For $G = SL_n(\eta q)$ and $\tilde{G} = GL_n(\eta q)$, we let $\tilde{B}$ be an $\ell$-block of $\tilde{G}$ and $B$ the union of $\ell$-blocks of $G$ covered by $\tilde{B}$. By the correspondence between $i\text{IBr}(\tilde{B})$ and $i\text{Alp}(\tilde{B})$ in [2, 3, 4, 5], the proof of the main theorem of [22] indeed obtained that $|i\text{IBr}(\tilde{B})| = |i\text{Alp}(\tilde{B})|$, which implies that $|i\text{IBr}(B)| = |i\text{Alp}(B)|$ for every $B \in B$ immediately since the blocks in $B$ are $\tilde{G}$-conjugate. □

Now we consider the (iBAW) condition for the blocks of $G = SL_n(\eta q)$.

Theorem 4.1. Assume that $G = SL_n(\eta q)$ is the universal covering of the finite simple group $S = PSL_n(\eta q)$. Let $\ell$ be a prime not dividing $q$, $\tilde{G} = GL_n(\eta q)$ and $B$ be a $\tilde{G}$-orbit of $\ell$-blocks of $G$. If furthermore the condition (v) of Theorem 2.2 holds for the bijection $\tilde{\Omega}$ in [22, §6], then the (iBAW) condition holds for any $B \in B$.

Proof. By the above observations, the bijection $\tilde{\Omega}$ used in [22] perserves blocks, i.e., condition (ii)(c) of Theorem 2.2 holds. Thus by the proof of the main theorem of [22], conditions (i)–(iv) of Theorem 2.2 hold. From this, if condition (v) of Theorem 2.2 holds, then the (iBAW) condition holds for any $B \in B$. □

Remark 4.2. Keep the notation of Theorem 4.1, we know from its proof that if (2.3) holds and there is a $(\tilde{G} \rtimes D)_{\tilde{\beta}}$-equivariant bijection between $i\text{IBr}(B)$ and $i\text{Alp}(B)$ which preserves blocks, then the (iBAW) condition from holds for every block $B \in B$.

Now we consider the (iBAW) condition for certain blocks of groups of type A.

Theorem 4.3. Let $G = SL_n(\eta q)$, $\tilde{G} = GL_n(\eta q)$ and $\ell$ a prime not dividing $q$. Let $B$ be a $\tilde{G}$-orbit of $\ell$-blocks of $G$. If (2.3) holds, then the (iBAW) condition holds for every $B \in B$ if one of the following is satisfied.

(i) $\tilde{G}_B = \tilde{G}$ for $B \in B$. 

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(ii) If \( \psi \in \text{IBr}_l(B) \) satisfies that \((\tilde{G} \times D)_\psi = \tilde{G}_\psi \times D_\psi \), then for any \( B \in B \) and any \( g \in \tilde{G} \setminus \tilde{G}_B \), there exists \( g_0 \in g\tilde{G}_B \) such that either \([\langle \tilde{G}_\psi, g_0 \rangle, D_\psi] \subseteq \tilde{G}_\psi \) or \( \tilde{G}_\psi[\langle \tilde{G}_\psi, g_0 \rangle, D_\psi] \nsubseteq \tilde{G}_B \).

(iii) If \( \psi \in \text{IBr}_l(B) \) satisfies that \((\tilde{G} \times D)_\psi = \tilde{G}_\psi \times D_\psi \), then for any \( B \in B \) and any \( g \in \tilde{G} \setminus \tilde{G}_B \), there exists \( g_0 \in g\tilde{G}_B \) such that \( \langle \tilde{G}_\psi, g_0 \rangle \cap \tilde{G}_B = \tilde{G}_\psi \).

(iv) \( \tilde{G}(\tilde{G} \times D)_B / GZ(\tilde{G}) \) is abelian.

(v) \( \gcd(\tilde{G} : \tilde{G}_B, |\tilde{G}_B : \tilde{G}_\psi|) = 1 \) for any \( B \in B \) and \( \psi \in \text{IBr}_l(B) \).

(vi) \( \tilde{G}_B = \tilde{G}_\psi \), for any \( B \in B \) and \( \psi \in \text{IBr}_l(B) \).

Proof. We use Theorem 4.1 and then (i) follows immediately. For (ii), we assume that \( \tilde{G}_B < \tilde{G} \) for \( B \in B \). Let \( \psi, (R, \varphi) \) satisfy condition (iii) and (iv) of Theorem 2.2 respectively and let \( B \in B \) and \( g \in \tilde{G} \) such that \( \psi \in \text{IBr}_l(B) \) and \((R, \varphi) \in \text{Alp}_l(B^\sigma)\). If \( B^\sigma = B \), then the assertion holds by Theorem 4.1. Thus we assume that \( g \notin \tilde{G}_B \).

We claim that \( \tilde{G}_\psi[\langle \tilde{G}_\psi, g' \rangle, D_\psi] \subseteq \tilde{G}_B \) for any \( g' \in g\tilde{G}_B \). Obviously, \( B^\sigma = B^g \). By the construction in [10], \( (\tilde{G} \times D)_\psi = \tilde{G}_\psi \times D_\psi \). Since \((R, \varphi) \in \text{Alp}_l(B^\sigma)\), we know \( \tilde{G}_\psi \times D_\psi = (\tilde{G} \times D)_{\varphi^R} \leq (\tilde{G} \times D)_{B^\sigma} \). Since \( \tilde{G} = G / G \) is cyclic, any subgroup of \( \tilde{G} \) containing \( G \) is normal in \( \tilde{G} \). Also \((\tilde{G} \times D)_{B^\sigma} = \tilde{G}_\psi D_\psi^\sigma \leq (\tilde{G} \times D)_{B^\sigma} \) because \( D_\psi \leq (\tilde{G} \times D)_{B^\sigma} \). So \( \tilde{G}_\psi[\langle \tilde{G}_\psi, g' \rangle, D_\psi] \subseteq \tilde{G}_B \) and the claim holds.

Therefore, \( \tilde{G}_\psi[\langle \tilde{G}_\psi, g' \rangle, D_\psi] \subseteq \tilde{G}_B \) for some \( g_0 \in g\tilde{G}_B \) by the hypothesis. Then \( (\tilde{G} \times D)_{\varphi_{g_0}} = \tilde{G}_\psi D_\psi^{\sigma_{g_0}} \leq \tilde{G}_\psi D_\psi \). From this we have \((\tilde{G} \times D)_{\varphi_{g_0}} = \tilde{G}_{\varphi_{g_0}} \times D_{\varphi_{g_0}} \). Thus both the Brauer character \( \psi^{g_0} \) and the weight \( (R, \varphi) \) lie in the block \( B^{g_0} = B^g \) and satisfy conditions (iii) and (iv) of Theorem 2.2. So the assertion follows by Theorem 4.1.

Now we consider (iii). Note that \([\langle \tilde{G}_\psi, g' \rangle, D_\psi] \leq \langle \tilde{G}_\psi, g' \rangle \). By the hypothesis, for any \( B \in B \) and any \( g \in \tilde{G} \setminus \tilde{G}_B \), there exists \( g_0 \in g\tilde{G}_B \) such that \([\langle \tilde{G}_\psi, g_0 \rangle \cap \tilde{G}_B = \tilde{G}_\psi \). So either \([\langle \tilde{G}_\psi, g_0 \rangle, D_\psi] \subseteq \tilde{G}_\psi \) or \( \tilde{G}_\psi[\langle \tilde{G}_\psi, g_0 \rangle, D_\psi] \nsubseteq \tilde{G}_B \). For the rest, we mention that the implications (ii) \( \Rightarrow \) (iv) and (iii) \( \Rightarrow \) (v) \( \Rightarrow \) (vi) \( \Rightarrow \) (vii) are direct and we complete the proof.

Proof of Theorem 2 When \( B \) is a unipotent block, we have \( \tilde{G}_B = \tilde{G} \) by [20], Remark 4.13], where \( \tilde{G} = \text{GL}_n(\eta q) \). Thus the assertion follows from Proposition 2.5 (ii) and Theorem 4.3 (i).

If \( B \) is of maximal defect, then by [14, Prop. 5.4], one has that Proposition 2.5 (i) is satisfied. So the assertion follows from Theorem 4.3 (i).
Let \( S \in \{ \text{PSL}_n(q), \text{PSU}_n(q) \} \) be a simple group of type A and \( G \) be the universal covering group of \( S \). We first consider the exceptional covering cases; see [24] Table 6.1.3 for the list of \( S \). Note that the (iBAW) condition has been verified for the alternating groups in [33], for simple groups of Lie type in defining characteristic in [37], and for cyclic blocks in [28, 29]. Then by a similar argument in [22] §8, the only prime we need to consider for the simple group \( \text{PSL}_3(4), \text{PSU}_4(3), \text{PSU}_6(2) \) is just 3, 2, 3, respectively. These cases are settled in [18, 19]. We mention that the paper [19] dealt with the the blocks of the universal covering groups of \( \text{PSU}_4(3) \) and \( \text{PSU}_6(2) \) which dominate no block of \( \text{SU}_4(3) \) and \( \text{SU}_6(2) \), while the blocks of special unitary groups are considered in this paper.

**Proof of Theorem 3.** By the above arguments, we may assume that \( G = \text{SL}_n(\eta q) \) is the universal covering group of \( S \). According to Proposition 2.5 (iii), the block correspondence property (2.3) holds. For an \( \ell \)-block \( B \) of \( G \), we have \( \ell \nmid |\tilde{G} : \tilde{G}_B| \), where \( \tilde{G} = \text{GL}_n(\eta q) \). On the other hand \( |\tilde{G} : \tilde{G}_\psi| = \prod p_i^{e_i} \) is a product of pairwise distinct primes. Thus \( \gcd(|\tilde{G} : \tilde{G}_B|, |\tilde{G}_B : \tilde{G}_\psi|) = 1 \) and the assertion follows by Theorem 4.3 (v). □

**Corollary 4.4.** Assume that \( n \) is square-free. Then the (iBAW) condition holds for the simple group \( S = \text{PSL}_n(\eta q) \).

**Proof.** By [37] Thm. C, we only need to consider the non-defining characteristic case, which follows from Theorem 3 immediately. □

Now we consider the simple groups of type A of small rank. We first have the following.

**Lemma 4.5.** The (iBAW) condition holds for \( S = \text{PSL}_4(\eta q) \) and any prime.

**Proof.** As above, we assume that \( G = \text{SL}_4(\eta q) \) is the universal covering group of \( S \). Let \( \tilde{G} = \text{GL}_4(\eta q) \). Also by [37] Thm. C we assume that \( \ell \nmid q \). If \( \ell = 2 \), then \( \tilde{G}_B = \tilde{G} \) for every 2-block \( B \) of \( G \) and then the (iBAW) condition holds by Theorem 3.

Assume that \( \ell \) is odd. By Theorem 3, we only need to consider the case \( 4 \mid q - \eta \). If \( \ell \nmid q - \eta \), then we can check directly that the Sylow \( \ell \)-subgroups of \( G \) are cyclic if \( e > 2 \). In addition, if \( e = 2 \), then \( G \) has an abelian defect group, and any \( \ell \)-block of \( G \) is either a cyclic block or of maximal defect. Recall that \( e \) denotes the multiplicative order of \( \eta q \) modulo \( \ell \). Then the lemma follows from [28] and Theorem 2. Now we assume that \( \ell \mid q - \eta \). Let \( B \) be an \( \ell \)-block of \( G \) and \( \tilde{B} \) an \( \ell \)-block of \( \tilde{G} \) covering \( B \). By [23], we may assume that \( \tilde{B} = E_\ell(G, s) \) for some semisimple \( \ell \)-element \( s \) of \( \tilde{G} \).

If \( s \) has an elementary divisor of degree 4, then it can be checked that \( \tilde{B} \) is a cyclic block, and so is \( B \). If \( s \) has an elementary divisor of degree 3, then there
is an irreducible character $\chi \in \mathcal{E}(\tilde{G}, s)$ such that $\text{Res}^{\tilde{G}}_{G}\chi$ is irreducible, and thus the (iBAW) condition holds for $B$ by Proposition 2.5(ii) and Theorem 4.3(i). If $s$ has two (possibly the same) elementary divisors of degree 2, then using the structure of defect groups of $\tilde{B}$ giving in [23] and the determinant of radical subgroups in [22], we know $B$ is a cyclic block. Now we assume that $s$ has no elementary divisors of degree $\geq 3$ and has an elementary divisor of degree 1. By [20, Remark 4.13], $\tilde{B}$ covers only one block of $G$. Blocks of maximal defect are dealt with in Theorem 2. Let $D$ be a defect group of $B$ and $\tilde{D}$ be a defect group of $\tilde{B}$ with $D \cap G = D$. If $B$ is not of maximal defect, then direct calculation shows that $\tilde{D}$ is abelian. By [22, Prop. 4.24], $C_G(D) = C_{\tilde{G}}(\tilde{D})$. If $G \leq J \leq \tilde{G}$ and $B_1$ is a block of $J$ covered by $\tilde{B}$, then $\tilde{D} \leq C_{\tilde{G}}(C_{\tilde{G}}(\tilde{D})) = C_{\tilde{G}}(C_G(D)) \leq C_{\tilde{G}}(C_J(\tilde{D} \cap J))$. So $C_{\tilde{G}}(C_J(\tilde{D} \cap J))G \geq \tilde{D}G = \tilde{G}$. By [14, Prop. 5.2], $C_{\tilde{B}_1} = \tilde{G}$. Thus the (iBAW) condition holds for $B$ by Proposition 2.5(i) and Theorem 4.3(i) and this completes the proof.

By Corollary 4.4 and Lemma 4.5 we have a consequence for simple groups of type $A$ with small rank immediately.

**Proposition 4.6.** Let $S \in \{\text{PSL}_n(q), \text{PSU}_n(q) \mid n \leq 7\}$ be a simple group. Then the (iBAW) condition holds for $S$.

## 5 The blockwise Alperin weight conjecture for finite groups with abelian Sylow 3-subgroups

We consider finite groups with abelian Sylow 3-subgroups and prove Theorem 4.4.

**Proof of Theorem 4.4** According to the reduction theorem [37, Thm. A], it suffices to prove that any non-abelian simple group $S$ with order dividing by $\ell$ involved in $G$ satisfies the (iBAW) condition. If $S$ is an alternating group, then this follows by [33, Thm. 1.1]. If $S$ is one of the sporadic simple groups, then the (iBAW) condition has been checked in [6] except when $S$ is one of $J_4$, $\text{Fi}_{24}'$, $\text{B}$ and $\text{M}$. These four sporadic simple groups are not involved in $G$ since their Sylow 3-subgroups are non-abelian (cf. [24, §5.3]).

Now we assume that $S$ is of Lie type. Then by [21, Lemma 2.2], $S$ is $\text{PSL}_n(\eta q)$ with $n \leq 5$, a Suzuki group, or $\text{PSp}_4(q)$ ($q > 2, 3 \mid q$). The case of defining characteristic has been verified in [37, Thm. C]. If $S$ is a Suzuki group, then $S$ satisfies the (iBAW) condition by [33, Thm. 1.1]. The simple group $\text{PSp}_4(q)$ is verified in [36] for even $q$ and in [8] or [50] for odd $q$, while the simple groups $\text{PSL}_n(\eta q)$ with $n \leq 5$ satisfy the (iBAW) condition by Proposition 4.6. This completes the proof.

□


6 2-blocks of special linear and unitary groups

This section is a continuation of [20 §4], focusing on the blocks of special linear and unitary groups. In [20 §4], the author gives a classification of the blocks of \( SL_n(q) \) for odd prime \( \ell \mid q \), using the labelling set of \( d \)-Jordan-cuspidal pairs given in [12, 26]. More precisely, for a given \( \ell \)-block \( B \) of \( G = GL_n(q) \), [20 Remark 4.13] gives the number of \( \ell \)-blocks of \( G = SL_n(q) \) covered by \( B \) when \( \ell \) is odd. If \( \ell = 2 \), [20 Remark 4.13] also gives an upper bound for this number. In this section we compute this number.

6.A Basic results  For arbitrary finite groups \( K \leq H \) and \( \chi \in \text{Irr}(H) \), we denote by \( \kappa^H_k(\chi) \) the number of irreducible constituents of \( \text{Res}^H_k(\chi) \) forgetting multiplicities. If \( \tilde{B} \) is a block of \( H \), then we denote the number of blocks of \( K \) covered by \( \tilde{B} \) by \( \kappa^H_K(\tilde{B}) \).

Let \( G \leq \hat{G} \leq \tilde{G} \). Since \( \tilde{G} / G \) is cyclic, by Clifford theory (see for example [25 Chap. 6] or [20 Lemma 2.1]) we have

\[
\kappa^G_{\chi}(\tilde{G} / G) = ||z \in \mathbb{Z} | z(s, \lambda) = (s, \lambda) \tilde{G}, o(z) \mid |\tilde{G} / \hat{G}||].
\]

Note that the group \( \mathbb{Z} \) is defined as on page \( 8 \).

Let \( G, \hat{G}, \tilde{G} \) be as above. By [15 Remark 4.7], for any \( \tilde{\chi} \in \text{Irr}(\tilde{G}) \), there is a \( \hat{\chi}_0 \in \text{Irr}(\hat{G} \mid \tilde{\chi}) \) such that \( (\hat{G} \times D)\hat{\chi}_0 = \hat{G} \hat{\chi}_0 \times D\hat{\chi}_0 \). Inspired by this, we introduce the following.

Definition 6.2. Let \( H, V \) be finite groups such that \( V \) normalises \( H \). Assume that finite groups \( K \) is a normal subgroup of \( HV \) satisfy that \( H \cap V \) acts on \( K \) via inner automorphisms. For any \( \tilde{\chi} \in \text{Irr}(H) \), a character \( \chi \in \text{Irr}(K \mid \tilde{\chi}) \) is called \( V \)-split if \( (HV)_\chi = H_\chi V_\chi \).

An easy and immediate property is as follows.

Lemma 6.3. Keep the assumptions in Definition 6.2

(i) If \( \chi \in \text{Irr}(K \mid \tilde{\chi}) \) is \( V \)-split, then \( V_\chi \leq V_\tilde{\chi} \).

(ii) Assume furthermore that \( V \) acts trivially on \( H / K \). If one character in \( \text{Irr}(K \mid \tilde{\chi}) \) is \( V \)-split, then so is any one in \( \text{Irr}(K \mid \chi) \).

Proof. (i) is obvious. For (ii), we assume \( \chi \in \text{Irr}(K \mid \tilde{\chi}) \) is \( V \)-split. Any character in \( \text{Irr}(K \mid \tilde{\chi}) \) is of the form \( \chi^h \) for some \( h \in H \). Then \((HV)_\chi^h = (HV)_\chi^h \) and \( H_\chi^h = H_\chi^h \). Assume \( v \in V_\chi \), then \((\chi^h)^v = \chi^{v^{-1}hv} \). Since \( V \) acts trivially on \( H / K \), there is \( k \in K \) such that \( v^{-1}hv = kh \). Thus \((\chi^h)^v = \chi^h \). So \( V_\chi \leq V_\chi^h \) and the equality holds by interchanging the role of \( \chi \) and \( \chi^h \). Consequently, \( H_\chi^h V_\chi \leq (HV)_\chi^h \). Comparing the order, we have \((HV)_\chi^h = H_\chi^h V_\chi = H_\chi^h V_\chi^h \). \( \square \)
The following lemma is a generalisation of [20, Lemma 4.7].

**Lemma 6.4.** Let $H$, $\bar{K}$, $\bar{H}$ be finite groups such that $\bar{H} = H\bar{K}$ and $\bar{K} \leq \bar{H}$. Assume that $\bar{K} = \prod_{i=1}^n \bar{K}_i$ with $\bar{K}_i = \mathrm{GL}_{m_i}((\eta q)^{d_i})$. Assume that $t_i : \bar{K}_i \to \bar{H}$ is a group homomorphism for every $i$ and set $t = \prod_{i=1}^n t_i : \bar{K} \to \bar{H}$. Let $K_i = \ker(t_i)$ and $K = \ker(t)$. The group $H = \prod_{i=1}^n H_i \rtimes \varepsilon_i$, satisfies that $H_i$ stabilizes $\bar{K}_i$ and $H_i = K_i\langle \sigma_i \rangle$, where $\sigma_i$ acts on $\bar{K}_i$ as $F_{\eta q}$.

Let $\bar{\chi} \in \mathrm{Irr}(\bar{K})$. For any $\chi \in \mathrm{Irr}(K \mid \bar{\chi})$, $\bar{\chi} \in \bar{K}$, $h \in H$, one can have $\chi^h = \bar{\chi^h}$ only if $\chi^h = \chi^h = \chi$.

**Proof.** It is equivalent to proving that every character $\chi \in \mathrm{Irr}(K \mid \bar{\chi})$ is $H$-split. First we consider the case that $u = 1$ and $t_1 = 1$, namely, $\bar{K} = \mathrm{GL}_{m_1}((\eta q)^{d_1})$. Let $\bar{\chi} = \bar{\chi}_1 \times \cdots \times \bar{\chi}_n$, where $\bar{\chi}_i \in \mathrm{Irr}(\bar{K}_i)$ for $1 \leq i \leq t_1$. For $\chi \in \mathrm{Irr}(K \mid \bar{\chi})$, we let $\chi_0 \in \mathrm{Irr}(K_i \mid \chi)$ with $\chi_0 = \chi_{0,1} \times \cdots \times \chi_{0,t_1}$, where $\chi_{0,i} \in \mathrm{Irr}(K_i)$ for $1 \leq i \leq t_1$. Let $h \in H$ with $\chi^h \in \mathrm{Irr}(K \mid \bar{\chi})$. Then $\chi_0^h \in \mathrm{Irr}(K_i \mid \chi)$, and without loss of generality, we may assume that $h = (h_1, \ldots, h_t, \tau)$ with $h_i \in H_i$ and $\tau = (1, \ldots, t_1)$ is a $t_1$-cycle. So $\chi_0^h = \chi_{0,1}^{h_1} \times \chi_{0,1}^{h_2} \times \cdots \times \chi_{0,1}^{h_{t_1}}$. Hence there exist $\bar{\tau}_1, \ldots, \bar{\tau}_{t_1} \in \bar{K}$ such that $\chi_{0,1}^{\bar{\tau}_1} = \chi_{0,1}^{h_1}$ and $\chi_{0,1}^{\bar{\tau}_{t_1}} = \chi_{0,1}^{h_{t_1}}$. Since the outer automorphisms induced by $H \bar{K}_1$ on $K_1$ form an abelian group, one has that $\chi_{0,1}^{\bar{\tau}_1} = \chi_{0,1}^{h_{t_1}}$, and then by the above paragraph $\bar{\tau}_1 \cdots \bar{\tau}_{t_1} \in \bar{K}_1 \chi_{0,1}^{h_{t_1}}$. By replacing $\bar{\tau}_1$ with $(\bar{\tau}_1 \cdots \bar{\tau}_{t_1})^{-1} \bar{\tau}_1$, we can assume that $\bar{\tau}_1 \cdots \bar{\tau}_{t_1} = 1$. Let $\bar{\tau} = (\bar{\tau}_1, \ldots, \bar{\tau}_{t_1})$. Then $\bar{\tau} \in K$ and $\chi^\bar{\tau} = \chi^\bar{\tau}$. So $\chi_0^h \in \mathrm{Irr}(K_i \mid \chi)$. Hence $\chi_i, \chi^h \in \mathrm{Irr}(K \mid \bar{\chi}) \cap \mathrm{Irr}(K \mid \chi_0)$. By [22, Lemma 7.2], $\chi^h = \chi$. This proves that $\chi$ is $H$-split.

The assertion in the general case now follows by reduction to the preceding cases.

Assume that $p$ is odd and $\ell = 2$ and let $\alpha$ be a non-negative integer, then the sets $\mathcal{F}_{\alpha,0}, \mathcal{F}_{\alpha,1}, \mathcal{F}_{\alpha,2}, \mathcal{F}_{\alpha}, \mathcal{F}_\alpha'$ are defined similarly as $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}, \mathcal{F}'$ with $q$ replaced by $(\eta q)^{2\ell}$ respectively. We have a surjective map

$$
\Phi_\alpha : \mathcal{F}_\alpha \to \mathcal{F}, \langle F_{\eta q}^2 \rangle \cdot \xi \mapsto \langle F_{\eta q} \rangle \cdot \xi.
$$

The inverse images of $\langle F_{\eta q} \rangle \cdot \xi$ under $\Phi_\alpha$ are the $\langle F_{\eta q} \rangle$-orbits of the set $\langle F_{\eta q}^2 \rangle \cdot \xi$.

Let $\Gamma \in \mathcal{F}$ and $\xi \in \mathcal{F}_{\eta q}^\times$ be a root of $\Gamma$. We define $\Gamma_{(\alpha)}$ to be a polynomial in $\mathcal{F}_\alpha$ which has a root $\xi$. Then the set of roots of $\Gamma_{(\alpha)}$ is an element in the inverse images of $\langle F_{\eta q} \rangle \cdot \xi$ under $\Phi_\alpha$. Conversely, every element in the inverse images of $\langle F_{\eta q} \rangle \cdot \xi$ under $\Phi_\alpha$ provides a choice of $\Gamma_{(\alpha)}$. 

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Lemma 6.6. Let \( z \in \mathcal{Z} \) and \( \Gamma \in \mathcal{F}' \) (i.e., the roots of \( \Gamma \) have 2'-order). Then \( z.\Gamma = \Gamma \) if and only if \( z.\Gamma_{(\alpha)} = \Gamma_{(\alpha)} \) for any non-negative integer \( \alpha \).

Proof. Let \( \xi \) be a root of both \( \Gamma \) and \( \Gamma_{(\alpha)} \), and \( o(\xi) = p_1^{\nu_1} \cdots p_u^{\nu_u} \), where \( p_1, \ldots, p_u \) are distinct odd primes. Also we write \( d = d_{\Gamma} = m_{\Gamma}2^{\nu} \) with odd \( m_{\Gamma} \). We denote by \( v_i \) the discrete valuation such that \( v_i(p_i) = 1 \), by \( e_i \) the multiplicative order of \( \eta q \) modulo \( p_i \), and by \( a_i = v_i((\eta q)^{\nu_i} - 1) \) for \( 1 \leq i \leq u \). Let \( \alpha_i = \max\{t_i - a_i, 0\} \). Then by [22] (3A) or [22] Lemma 5.3, \( d \) is the least common multiple of the \( e_i p_i^{\nu_i} \), where \( i \) runs through the integers between 1 and \( u \). So there exists \( 1 \leq i_0 \leq u \) such that \( 2^{\nu} \mid e_{i_0} \). Also, the polynomial \( \Gamma_{(\alpha)} \) has degree \( d/\gcd(d, 2^{\nu}) \). Thus \( \Gamma_{(\alpha)} = \Gamma \) if \( \alpha = 0 \) and \( \Gamma_{(\alpha)} = \Gamma_{(\sigma_\alpha)} \) if \( \alpha > 0 \). So it suffices to let \( 0 < \alpha \leq \alpha_\Gamma \). In particular \( e_{i_0} > 1 \).

We may assume that \( z \) has 2'-order. It is obvious that \( z.\Gamma_{(\alpha)} = \Gamma_{(\alpha)} \) implies \( z.\Gamma = \Gamma \). Now we assume that \( z.\Gamma = \Gamma \), then \( z^\xi = \xi^{(\eta q)^{\nu}} \) and then \( z = \xi^{(\eta q)^{\nu-1}} \) for some \( k \geq 1 \). Thus \( o(\xi) \mid ((\eta q)^{\nu-1})(\eta q-1) \) and in particular \( p_{i_0}^{\nu_0} \mid ((\eta q)^{\nu-1})(\eta q-1) \). Note that \( p_{i_0} \mid (\eta q-1) \) since \( e_{i_0} > 1 \). So \( p_{i_0}^{\nu_0} \mid ((\eta q)^{\nu-1}) \) and thus \( e_{i_0} \mid k \). In particular, \( 2^{\nu} \mid k \). By the definition of \( \Gamma_{(\alpha)} \), \( \xi^{(\eta q)^{\nu}} \) is a root of \( \Gamma_{(\alpha)} \), and so \( z.\Gamma_{(\alpha)} = \Gamma_{(\alpha)} \). This completes the proof. \( \square \)

6.B 2-blocks of special linear and unitary groups. We will follow the notation for basic 2-subgroups and radical 2-subgroups of linear and unitary groups in [22] and always consider the 2-modular representations from now on.

Let \( \tilde{B} \) be a 2-block of \( \tilde{G} \) with a maximal Brauer pair \((\tilde{R}, \tilde{b})\). Let \( \tilde{\theta} \) be the canonical character of \( \tilde{b} \) and \( R = \tilde{R} \cap G \). Note that the defect groups are radical subgroups.

First we recall the structure of \( \tilde{R} \). Set \( 2^{a+1} = (q^2 - 1)_2 \), thus \( a \geq 2 \). Recall that for \( G = \text{GL}_n(\eta q) \) with odd \( q \), by [7], there is a semisimple 2'-element \( s \) of \( G \) associated with \( \tilde{B} \), namely, \( \tilde{B} = C^e_2(\tilde{G}, s) = \bigcup_t E(\tilde{G}, st) \), where \( t \) runs through the semisimple 2-elements of \( \tilde{G} \) commuting with \( s \). Let \( s = \prod_{\Gamma \in \mathcal{F}} m_{\Gamma}(s)(\Gamma) \) be the primary decomposition. Here \( m_{\Gamma}(s) \) is the multiplicity of \( \Gamma \) in \( s \) as an elementary divisor and \( (\Gamma) \) is the companion matrix of \( \Gamma \). By [7] again, a Sylow 2-subgroup \( \tilde{R} \) of \( C_\tilde{G}(s) \) is a defect group of \( \tilde{B} \). Now \( C_\tilde{G}(s) = \prod_\Gamma C_\Gamma(s) \) with \( C_\Gamma(s) = \text{GL}_{m_{\Gamma}(s)}((\eta q)^{d_{\Gamma}}) \).

Then \( \tilde{R} = \prod_\Gamma \tilde{R}_\Gamma \), where \( \tilde{R}_\Gamma \) is a Sylow 2-subgroup of \( C_\Gamma(s) \). For each \( \Gamma \), let \( m_{\Gamma} \) and \( \alpha_\Gamma \) be such that \( m_{\Gamma}2^{\alpha_\Gamma} = d_{\Gamma} \) with odd \( m_{\Gamma} \). Then by [16], we have the following.

1. If \( 4 \mid q - \eta \) or \( \alpha_\Gamma \geq 1 \), set \( m_{\Gamma}(s) = 2^{\beta_{\Gamma,1}} + \cdots + 2^{\beta_{\Gamma,\nu}} \) with \( 0 \leq \beta_{\Gamma,1} < \cdots < \beta_{\Gamma,\nu} \).

Then \( \tilde{R}_\Gamma = \prod_{\beta_{\Gamma,j}} \tilde{R}_{m_{\Gamma}(s), 0, \beta_{\Gamma,j}} \), where \( \tilde{R}_{m_{\Gamma}(s), 0, \beta_{\Gamma,j}} \) is defined as in [22] §4 and \( \tilde{b}_{\Gamma,j} = (1, 1, \ldots, 1) \) with \( \beta_{\Gamma,j} \) one’s.
(2) If $4 \mid q + \eta$ and $\alpha_\Gamma = 0$, then $d_\Gamma = m_\Gamma$ is odd.

(2i) If furthermore $m_\Gamma(s)$ is even, set $m_\Gamma(s) = 2^{2^i,1} + \cdots + 2^{2^i,2^r}$ with $0 < \beta_{1,1} \cdots < \beta_{1,r}$. Then $\bar{R}_\Gamma = \prod_{i=1}^{r} S_{m_i,1,0,2^{i,1-1}}$, where $S_{m_i,1,0,2^{i,1-1}}$ is defined as in [22, §4].

(2ii) If furthermore $m_\Gamma(s)$ is odd, set $m_\Gamma(s) = 1 + 2^{2^i,1} + \cdots + 2^{2^i,2^r}$ with $0 < \beta_{1,1} \cdots < \beta_{1,r}$. Then $\bar{R}_\Gamma = \bar{R}_{m_i,0} \times \prod_{i=1}^{r} S_{m_i,1,0,2^{i,1-1}}$. Note that $\bar{R}_{m_i,0} = \{ \pm I_{m_i} \}$. Let $\tilde{C} = C_G(\bar{R})$. Then $\tilde{C} \equiv \prod_i \prod_{\ell = 1}^{m_i} \text{GL}_{m_i}((\eta \gamma)^{2^{i,1}}) \otimes I_{2^{i,1}}$, and $\tilde{\theta} = \prod_{i=1}^{r} \tilde{\theta} \otimes I_{2^{i,1}}$, where $\tilde{\theta} \otimes I_{2^{i,1}}$ is defined as in [22, §3E]. Note that $\tilde{\theta}$ is the character of $\text{GL}_{m_i}((\eta \gamma)^{2^{i,1}})$ corresponding to the semisimple $2'$-element with the only elementary divisor $\Gamma_{(a_\Gamma)}$ of multiplicity 1.

Now we recall the normaliser $\tilde{N} = N_G(\bar{R})$. First note that it is possible that some component of $R_\Gamma$ is isomorphic to a component of $\bar{R}_\Gamma$, for different $\Gamma$, $\Gamma'$ with $d_\Gamma = d_{\Gamma'}$. From this we rewrite $\tilde{R} = \prod_{i=1}^{r} \bar{R}_i$, where $R_i$'s are the components of $\bar{R}$ such that $\tilde{R}_i \neq \bar{R}_j$ if $i \neq j$. Then $\tilde{C} = \prod_{i=1}^{r} \tilde{C}_i$ and $\tilde{N} = \prod_{i=1}^{r} \tilde{N}_i \tilde{S}_u$, where $\tilde{S}_u$ is the symmetric group on $u_i$ symbols. Now we assume that $\tilde{R}_i$ is a component of $R_\Gamma$ for some $\Gamma$. By [3, 4], then $\tilde{N}_i/\tilde{R}_i$ is a direct product of $\tilde{C}_i \tilde{R}_i/\tilde{R}_i$ and a subgroup of $\tilde{N}_i/\tilde{R}_i$ if $\alpha_\Gamma = 0$. If $\alpha_\Gamma > 0$, then $\tilde{C}_i \equiv \text{GL}_{m_i}((\eta \gamma)^{2^{i,1}})$. Then $\tilde{N}_i/\tilde{R}_i$ is a direct product of $\tilde{N}_{i,0}/\tilde{R}_i$ and a subgroup of $\tilde{N}_i/\tilde{R}_i$, where $\tilde{N}_{i,0} \leq \tilde{N}_i$ such that $\tilde{N}_i = \tilde{N}_{i,0}(\tau)$. Here $\tau$ acts on $\tilde{C}_i \equiv \text{GL}_{m_i}((\eta \gamma)^{2^{i,1}})$ as $F_{\eta \gamma}$.

Let $R = \tilde{R} \cap G$ and $R' = O_2(N_G(R))$. Then $R \leq R' \leq \bar{R}$ and $N_G(R') = N_G(R)$ by [20, Lemma 2.2]. By [22, Prop. 4.37 and 4.47], $C_G(R') = C_G(R)$.

First we consider the case that $R' = \tilde{R}$. We view $\tilde{\theta}$ as a character of $C_G(\bar{R})/Z(\bar{R})$ and let $\theta \in \text{Irr}(C_G(\bar{R})/Z(\bar{R}))$. Then $\theta$ is also of $2$-defect zero. We also view $\theta$ as a character of $RC_G(\bar{R})$ whose kernel contains $R$. Let $b$ be the $2$-block of $RC_G(\bar{R})$ containing $\theta$. Then $\theta$ is the canonical character of $b$. Let $B = b^\theta$. By [27, Lemma 2.3], $\tilde{B}$ covers $B$. Conversely, every $2$-block of $G$ covered by $\tilde{B}$ can be obtained by the above process from a suitable $\theta \in \text{Irr}(C_G(\bar{R})/Z(\bar{R}))$. Then $\theta$ acts on $\tilde{C}_i \equiv \text{GL}_{m_i}((\eta \gamma)^{2^{i,1}})$ as $F_{\eta \gamma}$.

If $\tilde{R}' \neq \tilde{R}$, then according to [22, §5D], exactly one of the following cases occurs. We set $\tilde{C}' = C_G(\bar{R}')$ and $\tilde{N}' = N_G(\bar{R}')$.

**Case (i).** $4 \mid q - \eta$, $\tilde{R} = \tilde{R}' = \tilde{R}_{m_i,\alpha}$ and $\tilde{\theta} = \tilde{\theta}_i$ for some $\Gamma \in \mathcal{F}'$ with $\alpha_\Gamma > 0$. This means that $m_\Gamma(s) = 1$ and $d_\Gamma = 1$. Up to conjugacy, we may take $\tilde{R}' = \tilde{R}_{m_i,\alpha'}$, where $\alpha' = \min(a, \alpha_\Gamma)$. Let $\tilde{\theta}' = \tilde{\theta}_i = \tilde{\theta}_i^{\alpha'}$, where $\tilde{\theta}_i^{\alpha'}$ is defined as in [22, §5C]. Also, $\tilde{C}' \equiv \text{GL}_{m_i,\alpha'}((\eta \gamma)^{2^{i,1}})$ and $\tilde{N}' = \tilde{C}'(\tau)$, where $\tau$ acts on $\tilde{C}'$ as $F_{\eta \gamma}$.

**Case (ii).** $4 \mid q - \eta$, $\alpha = 2$, $\tilde{R} = \tilde{R}_\Gamma = \tilde{R}_{m_i,0,0,1}$ and $\tilde{\theta} = \tilde{\theta}_{\Gamma,1}$ for some $\Gamma \in \mathcal{F}'$ with $\alpha_\Gamma = 0$, where $\tilde{\theta}_{\Gamma,1}$ is defined as in [22, §3E]. This means that $m_\Gamma(s) = 2$
and \( n = 2d_1 \). Up to conjugacy, we may take \( \tilde{R}' = \tilde{R}_{mr,0,1} \). By \([22] \S 3B\), \( \tilde{N}'/\tilde{R}' = \tilde{C}'/\tilde{R}' \times \text{Sp}_2(2) \) and \( \tilde{C}' \cong \text{GL}_{m_1}(\eta \theta) \otimes I_2 \). Let \( \tilde{\theta}' = \tilde{\theta}_{I} \otimes I_2 \).

Case (iii). \( 4 | q + \eta \), \( \tilde{R} = \tilde{R}_{1} = \tilde{R}_{m_0,0,1} \) and \( \tilde{\theta} = \tilde{\theta}_{I} \) for some \( \Gamma \in \mathcal{F}' \) with \( \alpha_{\Gamma} > 1 \). This means that \( m_{T}(s) = 1 \) and \( n = d_{T} \). Up to conjugacy, we take \( \tilde{R}' = \tilde{R}_{2m_0,0,1} \) and \( \tilde{\theta}' = \tilde{\theta}'_{I} = \tilde{\theta}_{I}^{(1)} \). Also, \( \tilde{C}' \cong \text{GL}_{2m_1}(\eta \theta \tau) \) and \( \tilde{N}' = \tilde{C}(\tau) \), where \( \tau \) acts on \( \tilde{C}' \) as \( F_{\eta \theta} \).

Case (iv). \( 4 | q + \eta \), \( a = 2 \) and \( \tilde{R} = \tilde{R}_{1} = \tilde{S}_{m_0,1,0} \) for some \( \Gamma \in \mathcal{F}' \) with \( \alpha_{\Gamma} = 0 \). This means that \( m_{T}(s) = 2 \) and \( n = 2d_{T} \). Up to conjugacy, we may take \( \tilde{R}' = \tilde{R}_{m_0,0,1} \) where \( \tilde{R}_{m_0,0,1} \) is defined as in \([22] \S 3C\). Also \( \tilde{N}'/\tilde{R}' = \tilde{C}'/\tilde{R}' \times \text{GO}_{2}(2) \) and \( \tilde{C}' \cong \text{GL}_{m_1}(\eta \theta) \otimes I_2 \). Let \( \tilde{\theta}' = \tilde{\theta}_{I} \otimes I_2 \).

For the case (i)–(iv) above, the Brauer pair \((\tilde{R}', \tilde{\theta}')\) is a \( \tilde{B}\)-pair of \( \tilde{G} \) by \([7]\); see also \([22] \S 5D\). We view \( \tilde{\theta}' \) as a character of \( C_{G}(\tilde{R}')/Z(\tilde{R}') \) and let \( \theta \in \text{Irr}(C_{G}(R)/Z(R) | \tilde{\theta}') \). Then \( \theta \) is also of 2-defect zero by \([22] \S 5C\). We also view \( \theta \) as a character of \( RC_{G}(R) \) whose kernel contains \( R \). Let \( b \) be the 2-block of \( RC_{G}(R) \) containing \( \theta \). Then \( \theta \) is the canonical character of \( b \). Let \( B = b \tilde{C} \). By \([27] \text{Lemma 2.3}\), \( \tilde{B} \) covers \( B \). Conversely, every 2-blocks of \( G \) covered by \( \tilde{B} \) can be obtained through the above process from a suitable \( \theta \in \text{Irr}(C_{G}(R)/Z(R) | \tilde{\theta}') \) with \( \tilde{n} \in N_{G}(\tilde{R}') \).

In order to deal with the two cases that \( \tilde{R}' = \tilde{R} \) and \( \tilde{R}' \neq \tilde{R} \) simultaneously, we also use the notation \( \tilde{C}', \tilde{N}', \tilde{\theta}' \) for \( \tilde{C}, \tilde{N}, \tilde{\theta} \) if \( \tilde{R}' = \tilde{R} \). Also, we let \( C = C_{G}(R), N = N_{G}(R) \).

**Lemma 6.7.** Keep the hypotheses and setup above. Then \( k_{G}^{\tilde{C}}(\tilde{B}_{s}) = k_{G}^{\tilde{C}'}(\tilde{B}') \).

**Proof.** As above, every 2-block of \( G \) covered by \( \tilde{B} \) can be obtained through the above process from a suitable \( \theta \in \text{Irr}(C_{G}(R)/Z(R) | \tilde{\theta}') \) with \( \tilde{n} \in N_{G}(\tilde{R}') \). If we are neither in the case (ii) nor (iv), then using \([22] \text{Prop. 4.34 and 4.43}\), direct calculation case-by-case shows that \( \tilde{N}' = \tilde{R}' C' N \), and then we may assume that \( \tilde{n} \in N \). If we are in the case (ii) or (iv), then \( \tilde{N}' \) acts on \( \tilde{C}' \) by inner automorphisms. Hence if we fix a \( \tilde{\theta}' \), then \( \theta \) gives all 2-blocks of \( G \) when running through \( \text{Irr}(C_{G}(R)/Z(R) | \tilde{\theta}') \). From this it remains to prove that the characters in \( \text{Irr}(C_{G}(R)/Z(R) | \tilde{\theta}') \) are not \( N \)-conjugate.

By the above arguments, the group \( \tilde{C}' = \prod_{i}(\tilde{C}_{i})^{u_{i}} \) such that each component \( \tilde{C}_{i} \) is a general linear or unitary group and \( \tilde{N}' = \prod_{i} \tilde{N}'_{i} \otimes u_{i} \). Furthermore, the action of any element of \( \tilde{N}' \) on \( \tilde{C}' \) (via conjugation) is a product of an inner automorphism and a field automorphism which commute. Thus this assertion follows by Lemma 6.4.

**Theorem 6.8.** \( k_{G}^{\tilde{C}'}(\tilde{B}_{s}) \) is the number of \( x \in O_{2}(\mathcal{A}) \) satisfying \( x.\Gamma = \Gamma \) for every elementary divisor \( \Gamma \) of \( s \).

**Proof.** According to \([22] \text{Lemma 4.35 and 4.45}\), \( O_{2}(\tilde{C}'/C) \cong O_{2}(\tilde{G}/G) \). By \([20] \text{Lemma 2.1}\) and Lemma 6.7, \( k_{G}^{\tilde{C}'}(\tilde{B}_{s}) \) is the number of \( z \in \mathcal{A} \) satisfying \( z\tilde{\theta}' = \tilde{\theta}' \).
Then the proof is similar with [20, Cor. 4.12], and then \( \hat{\sigma'} = \sigma' \) if and only if \( z \Gamma_{(k)} = \Gamma_{(k)} \) for every elementary divisor \( \Gamma \) of \( s \), where \( k \Gamma \) is an integer such that \( 0 \leq k \Gamma \leq \alpha \Gamma \). Thus by Lemma 6.6, \( \hat{\sigma'} = \sigma' \) if and only if \( z \in O_{2}(3) \) and \( z \Gamma = \Gamma \) for every elementary divisor \( \Gamma \) of \( s \).

**Remark 6.9.** Analogously with [20, Remark 4.13], we give a description for the \( 2 \)-blocks of \( G = \text{SL}_{n}(\eta q) \) by summarizing the argument above.

For \( \sigma \in \overline{F}_{q}^{\times} \), we denote by \([\sigma]\) the set of all roots of the polynomial in \( F \) which has \( \sigma \) as a root. Thus \([\sigma]\) is a single \( F_{\eta q}\)-orbit. Denote by deg(\( \sigma \)) the cardinality of \([\sigma]\). Then deg(\( \sigma \)) is the minimal integer \( d \) such that \( \sigma^{(\eta q)^{d-1}} = 1 \) and

\[
[\sigma] = \{ \sigma, \sigma^{(\eta q)}, \sigma^{(\eta q)^{2}}, \ldots, \sigma^{(\eta q)^{\text{deg}(\sigma)-1}} \}.
\]

We call an \( a \)-tuple

\[
(([\sigma_{1}], m_{1}), \ldots, ([\sigma_{a}], m_{a}))
\]

of pairs an \((n, 2)\)-admissible block tuple, if

- for every \( 1 \leq i \leq a \), \( \sigma_{i} \in \overline{F}_{q}^{\times} \) is a \( 2' \)-element, and \( m_{i} \) is a positive integer,

- \( [\sigma_{i}] \neq [\sigma_{j}] \) if \( i \neq j \), and

- \( \sum_{i=1}^{a} m_{i}\text{deg}(\sigma_{i}) = n \).

An equivalence class of an \((n, 2)\)-admissible block tuple

\[
(([\sigma_{1}], m_{1}), \ldots, ([\sigma_{a}], m_{a}))
\]

up to a permutation of pairs \( ([\sigma_{1}], m_{1}), \ldots, ([\sigma_{a}], m_{a}) \) is called an \((n, 2)\)-admissible block symbol and is denoted as

\[
(6.10) \quad b = \{ ([\sigma_{1}], m_{1}), \ldots, ([\sigma_{a}], m_{a}) \}.
\]

The set of \((n, 2)\)-admissible block symbols is in bijection with conjugacy classes of semisimple \( 2' \)-elements of \( \tilde{G} = \text{GL}_{n}(\eta q) \), and then is a labeling set for \( 2 \)-blocks of \( \tilde{G} \) by [7]. Denote by \( \tilde{B}_{b} \) the \( 2 \)-block of \( \tilde{G} \) corresponding to the \((n, 2)\)-admissible block symbol \( b \) as in (6.10). Now we define \( \kappa(b) \) to be the cardinality of the set \( \{ z \in O_{2}(3) \mid [z\sigma_{i}] = [\sigma_{i}] \text{ for all } 1 \leq i \leq a \} \).

By Theorem 6.8, \( \kappa_{G}^{G}(\tilde{B}_{b}) = \kappa(b) \) (i.e. the number of \( 2 \)-blocks of \( G \) covered by \( \tilde{B}_{b} \) is \( \kappa(b) \)). For two \((n, 2)\)-admissible block symbols \( b \) and \( b' \), if they are in the same \( O_{2}(3) \)-orbit, then the sets of \( 2 \)-blocks of \( G \) covered by \( \tilde{B}_{b} \) and \( \tilde{B}_{b'} \) are the same.

If moreover, we let \( (B_{b})_{1}, (B_{b})_{2}, \ldots, (B_{b})_{\kappa(b)} \) be the \( 2 \)-blocks of \( G \) covered by \( \tilde{B}_{b} \), then the set \( \{ (B_{b})_{j} \} \), where \( b \) runs through the \( O_{2}(3) \)-orbit representatives of \((n, 2)\)-admissible block symbols and \( j \) runs through the integers between 1 and \( \kappa(b) \), is a complete set of \( 2 \)-blocks of \( G \).
Remark 6.11. We mention that the case that $\ell$ is odd can be also obtained by the same method as above, which has been dealt with in [20, Remark 4.13] via the $d$-Jordan-cuspidal pairs.

Remark 6.12. In [20, Remark 4.13], the author gives an upper bound for the number of 2-blocks of $\text{SL}_\eta(q)$ covered by a given 2-block of $\text{GL}_\eta(q)$. In fact, if $4 \mid q - \eta$, then that upper bound is just the $\kappa(b)$ defined in Remark 6.9 while if $4 \mid q + \eta$, that upper bound may be bigger than $\kappa(b)$ above. This provides an example to show that the assumption of odd primes is necessary in [26, Thm. A (e)], which states that the $d$-Jordan-cuspidal pairs form a labeling set for the blocks of a finite groups of Lie type.

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References

[1] J. L. Alperin, Weights for finite groups. In: The Arcata Conference on Representations of Finite Groups, Arcata, Calif. (1986), Part I. Proc. Sympos. Pure Math., vol. 47, Amer. Math. Soc., Providence, 1987, pp. 369–379.
[2] J. L. Alperin, P. Fong, Weights for symmetric and general linear groups. J. Algebra 131 (1990), 2–22.
[3] J. An, 2-weights for general linear groups. J. Algebra 149 (1992), 500–527.
[4] J. An, 2-weights for unitary groups. Trans. Amer. Math. Soc. 339 (1993), 251–278.
[5] J. An, Weights for classical groups. Trans. Amer. Math. Soc. 342 (1994), 1–42.
[6] T. Breuer, Computations for some simple groups. http://www.math.rwth-aachen.de/~Thomas.Breuer/ctblocks/doc/overview.html
[7] M. Broué, Les $\ell$-blocs des groupes $\text{GL}(n, q)$ et $\text{U}(n, q^2)$ et leurs structures locales. Séminaire Bourbaki, Astérisque 640 (1986), 159–188.
[8] J. Brough, A. A. Schaeffer Fry, Radical subgroups and the inductive blockwise Alperin weight conditions for $\text{PSp}_4(q)$. Rocky Mountain J. Math. 50 (2020), 1181–1205.
[9] J. Brough, B. Späth, On the Alperin–McKay conjecture for simple groups of type $A$. J. Algebra 558 (2020), 221–259.
[10] J. Brough, B. Späth, A criterion for the inductive Alperin weight condition. Bull. London Math. Soc. 54 (2022), 466–481.
[11] M. Cabanes, Brauer morphism between modular Hecke algebras. J. Algebra 115 (1988), 1–31.
[12] M. Cabanes, M. Enguehard, On blocks of finite reductive groups and twisted induction. Adv. Math. 145 (1999), 189–229.
[13] M. Cabanes, M. Enguehard, Representation Theory of Finite Reductive Groups. New Math. Monogr., vol. 1, Cambridge University Press, Cambridge, 2004.
[14] M. Cabanes, B. Späth, On the inductive Alperin–McKay condition for simple groups of type $A$. J. Algebra 442 (2015), 104–123.
[15] M. Cabanes, B. Späth, Equivariant character correspondences and inductive McKay condition for type $A$. J. reine angew. Math. 728 (2017), 153–194.
[16] R. Carter, P. Fong, The Sylow 2-subgroups of the finite classical groups. J. Algebra 1 (1964), 139–151.
[17] D. Denoncin, Stable basic sets for finite special linear and unitary groups. Adv. Math. 307 (2017), 344–368.
[18] Y. Du, The inductive blockwise Alperin weight condition for $\text{PSL}(3, 4)$. Comm. Algebra 49 (2021), 292–300.
[19] Y. Du, The inductive blockwise Alperin weight condition for $\text{PSU}(4, 3)$ and $\text{PSU}(6, 2)$. Comm. Algebra 50 (2022), 54–72.
[20] Z. Feng, The blocks and weights of finite special linear and unitary groups. J. Algebra 523 (2019), 53–92.
[21] Z. Feng, C. Li, Z. Li, The McKay conjecture for finite groups with abelian Sylow 3-subgroups. Algebra Colloq. 24 (2017), 181–194.
[22] Z. Feng, C. Li, J. Zhang, Equivariant correspondences and the inductive Alperin weight condition for type $A$. Trans. Amer. Math. Soc. 374 (2021), 8365–8433.
[23] P. Fong, B. Srinivasan, The blocks of finite general linear and unitary groups. Invent. Math. 69 (1982), 109–153.
[24] D. Gorenstein, R. Lyons, R. Solomon, The Classification of the Finite Simple Groups, Number 3. Math. Surveys Monogr., vol. 40, American Mathematical Society, Providence, RI, 1998.
[25] I. M. Isaacs, Character Theory of Finite Groups. Pure Appl. Math., vol. 69, Academic Press, New York, 1976.
[26] R. Kessar, G. Malle, Lusztig induction and $\ell$-blocks of finite reductive groups. Pacific J. Math. 279 (2015), 269–298.
[27] S. Koshitani, B. Späth, Clifford theory of characters in induced blocks. Proc. Amer. Math. Soc. 143 (2015), 3687–3702.
[28] S. Koshitani, B. Späth, The inductive Alperin–McKay and blockwise Alperin weight conditions for blocks with cyclic defect groups and odd primes. J. Group Theory 19 (2016), 777–813.
[29] S. Koshitani, B. Späth, The inductive Alperin–McKay condition for 2-blocks with cyclic defect groups. Arch. Math 106 (2016), 107–116.
[30] C. Li, Z. Li, The inductive blockwise Alperin weight condition for $\text{PSP}_6(q)$. Algebra Colloq. 26 (2019), 361–386.
[31] C. Li, J. Zhang, The inductive blockwise Alperin weight condition for $\text{PSL}_6(q)$ and $\text{PSU}_6(q)$ with cyclic outer automorphism groups. J. Algebra 495 (2018), 130–149.
[32] C. Li, J. Zhang, The inductive blockwise Alperin weight condition for $\text{PSL}_n(q)$ with $(n, q - 1) = 1$. *J. Algebra* **558** (2020), 582–594.

[33] G. Malle, On the inductive Alperin–McKay and Alperin weight conjecture for groups with abelian Sylow subgroups. *J. Algebra* **397** (2014), 190–208.

[34] G. Navarro, *Characters and Blocks of Finite Groups*. London Mathematical Society Lecture Note Series, vol. **250**, Cambridge University Press, Cambridge, 1998.

[35] G. Navarro, P. H. Tiep, A reduction theorem for the Alperin weight conjecture. *Invent. Math.* **184** (2011), 529–565.

[36] A. A. Schaeffer Fry, $\text{Sp}_6(2^a)$ is “good” for the McKay, Alperin weight, and related local-global conjectures. *J. Algebra* **401** (2014), 13–47.

[37] B. Späth, A reduction theorem for the blockwise Alperin weight conjecture. *J. Group Theory* **16** (2013), 159–220.

[38] B. Späth, Inductive conditions for counting conjectures via character triples. In: *Representation Theory – Current Trends and Perspectives*. Series of Congress Reports, European Mathematical Society, Zürich, 2017, pp. 665–680.