Embedding Complete Bipartite Graph into Quadrilateral Snake

Leo Joshwa P 1*, A. Berin Greeni1

1School of Advanced Sciences, Vellore Institute of Technology, Chennai – 600127.

E-mail: joseleowa18@gmail.com

Abstract. This paper deals with the technique that maps a guest graph into a host graph, known as Graph embedding. This tool is used in the simulation of interconnection networks. The exact wirelength of embedding complete bipartite graphs into quadrilateral snakes is considered as a research problem in this paper.

Keywords: complete bipartite graph, quadrilateral snakes, wirelength.

1. Introduction
The computing architecture utilizes interconnection network as the communication between the processor cores and memory modules. Thus an interconnection network is nothing but a graph. Thus the process of data exchange between parallel computers is made feasible in a computer network. Distinctive feature of graph embedding is of major concern in the design of the interconnection network. It is a strategy where the guest graph $D$ maps to a host graph $F$ [1]. Numerous applications are likely to be modeled as graph embedding problem. For instance, one architecture can be simulated into another by modelling as an embedding problem [2]. Dilation, congestion and wirelength are important measure of an embedding which we compute in this paper for embedding complete bipartite graph into quadrilateral snakes.

2. Basic concepts

Definition 2.1. [1]
An one - one mapping $\mathcal{E}$ of the vertex set of a graph $D$ into the vertex set of a graph $F$ is called an embedding if every edges $(x, y)$ in $D$ is mapped to a path between $\mathcal{E}(x)$ and $\mathcal{E}(y)$ in $F$.

Definition 2.2. [1]
Let $\mathcal{E} : D \rightarrow F$ be an embedding. For $e \in E(F)$, the number of path in $\{ p_\mathcal{E}(u, v) : p_\mathcal{E}(u, v)$ is a path between $\mathcal{E}(u)$ and $\mathcal{E}(v)$ in $F$ for $(u, v) \in E(F)$ that contains $e \}$ is called the congestion on $e$ with respect to $\mathcal{E}$ and is defined by $c_\mathcal{E}(e)$. For $S \subseteq E(M)$, we define $c_\mathcal{E}(S) = \sum_{e \in S} c_\mathcal{E}(e)$.
**Definition 2.3.** [1]
Let $E : D \rightarrow F$ be an embedding. Then the congestion of $E : D \rightarrow F$ is given by $c_E(D,F) = \max_{e \in E} c_E(e)$. The congestion problem is to find $\min_{E} c_E(D,F)$ where the minimum is taken over all embedding $E : D \rightarrow F$.

**Definition 2.4.** [3]
The wirelength of an embedding $E : D \rightarrow F$ is given by $WL_E(D,F) = \sum_{e \in E(F)} c_E(e)$. The wirelength of embedding $D$ into $F$ denoted as $WL(D,F) = \min_{E} WL_E(D,F)$ where the minimum is taken over all embedding $E : D \rightarrow F$.

The wirelength problem [1, 3] of a graph $D$ into $F$ is to determine $WL_E(D,F)$. The wirelength problem is NP-complete [4].

The wirelength problem of various architectures such as hypercube into grid [3], complete bipartite graph into grid [5], complete bipartite graph into necklace graph [6] and complete bipartite graph into sibling trees [7] are studied.

**Definition 2.5.** [8]
A subgraph $K$ of a graph $D$ is said to be convex if all the shortest paths between any two vertices of $K$ lie in $K$. An edge cut $M$ of $D$ is said to be a convex cut if $D \setminus M$ splits into two components, each of which is convex.

**Lemma 2.6. (Congestion Lemma)** [9]
For an embedding $E$ of a graph $D$ into $F$, let $M$ be a convex edge cut of $F$ such that $F \setminus M$ splits into components $F_1$ and $F_2$ and let $D_1 = \overline{E^{-1}(F_1)}$ and $D_2 = \overline{E^{-1}(F_2)}$. Suppose $D_1$ and $D_2$ are maximal subgraph of $D$ and $P_E(u,v)$ with $u \in D_1$ and $v \in D_2$ contains exactly one edge in $M$ for every $(u,v) \in E(D)$. Further

$$c_E(M) = \sum_{v \in V(D_1)} d_G(v) - 2|E(D_1)|$$
$$= \sum_{v \in V(D_2)} d_G(v) - 2|E(D_2)|$$

**Lemma 2.7. ($k$ - Partition Lemma)** [3, 10]
Let $E : D \rightarrow F$ be an embedding. Let $E^k(F)$ represent the collection of edges of $F$, where each edge of $F$ is repeated exactly $k$ times. Let $\{X_1, X_2, \ldots, X_p\}$ be a partition of $E^k(F)$ such that each $X_i$ is an edge cut of $F$. Then the wirelength of embedding $D$ into $F$ with respect to $E$ is given by

$$WL_E(D,F) = \frac{1}{k} \sum_{i=1}^{p} c_E(X_i).$$

3 Quadrilateral Snakes

**Definition 3.1.** [11]
Complete bipartite graph denoted as $K_{m,n}$, is a special type of bipartite graph whose every vertex of one set $X$ is connected to every vertex of other set $Y$, with $|X| = m$ and $|Y| = n$.

**Lemma 3.2.** [5] $K_{\lceil l \rceil \lfloor l \rceil}$ is the maximum subgraph of $K_{m,n}$ on $l$ vertices.

*Even-odd labeling of $K_{m,n}$*
If \((X, Y)\) is the partition of \(V(K_{m,n})\), with \(|X| = m\) and \(|Y| = n, m \geq n\). The vertices of \(X\) and \(Y\) are labeled as \(1, 3, ..., 2m - 1\) and \(2, 4, ..., 2n\) respectively.

**Definition 3.3.** [12]
A quadrilateral snake \(Q(n)\) is obtained from a path \(v_1, v_2, ..., v_{n+1}\) by joining vertices \(v_i\) and \(v_{i+1}\) to new vertices \(x_i\) and \(y_i\) respectively and adding edges \(x_iy_i\) for \(1 \leq i \leq n\). See Figure 1.

**Remark 3.4.** The quadrilateral snake \(Q(n)\) has \(3n+1\) vertices and \(4n\) edges.

**Figure 1:** Edge cut of Quadrilateral snake \(Q(3)\)

**Algorithm A**

**Input:** The complete bipartite graph \(K_{\frac{3n+1}{2}, \frac{3n+1}{2}}\) and the quadrilateral snake \(Q(n)\).

**Algorithm:** Using Even-odd labelling, label \(V\left(K_{\frac{3n+1}{2}, \frac{3n+1}{2}}\right)\). Label \(V(Q(n))\) as follows: the vertex \(v_k\) is labeled as \(3k - 2\) for \(1 \leq k \leq n\) and \(u_i, w_i\) as \(3k - 1, 3k\) respectively for \(k = 1, 2, ..., n - 1\).

**Output:** An embedding \(E : K_{\frac{3n+1}{2}, \frac{3n+1}{2}} \rightarrow Q(n)\) given by \(E(y) = y\) gives the wirelength minimum.

**Proof of correctness:** For \(1 \leq k \leq n\), let \(M_{3k-1} = \{(3k - 2, 3k + 1), (3k - 1, 3k)\}\) be the edge cut of the quadrilateral snake \(Q(n)\). The edge cut \(M_{3k-1}, 1 \leq k \leq n\) disconnects \(Q(n)\) into components: \(F_{k1}, F_{k2}\). Clearly \(V(F_{k1})\) is labeled consecutively. See Figure 1. By Lemma 3.2, \(D_{k1} = E^{-1}(F_{k1})\) and \(D_{k2} = E^{-1}(F_{k2})\) are maximum subgraph of \(K_{\frac{3n+1}{2}, \frac{3n+1}{2}}\). Thus the edge cut \(M_{3k-1}, 1 \leq k \leq n\) satisfies conditions (1) and (2) of the Congestion Lemma. Clearly for \(k = 1, 2, ..., n\), \(C_e(M_{3k-1})\) is minimum. Similarly, for \(1 \leq k \leq n\), let \(M_k = \{(3k-1,3k), (3k-2,3k+1)\}\) be the edge cut of the quadrilateral snake \(Q(n)\). The edge cut \(M_k, 1 \leq k \leq n\) disconnects \(Q(n)\) into components: \(F_{k1}, F_{k2}\). Clearly \(V(F_{k1})\) is labeled consecutively. See Figure 1. By Lemma 3.2, \(D_{k1} = E^{-1}(F_{k1})\) and \(D_{k2} = E^{-1}(F_{k2})\) are maximum subgraph of \(K_{\frac{3n+1}{2}, \frac{3n+1}{2}}\). Thus the edge cut \(M_k, 1 \leq k \leq n\) satisfies conditions (1) and (2) of the Congestion Lemma. Clearly for \(k = 1, 2, ..., n\), \(C_e(M_k)\) is minimum. Therefore, using Partition lemma the wirelength is obtained.
Theorem 3.5.
The wirelength of the complete bipartite graph $K_{\frac{(3n+1)}{2},\frac{(3n+1)}{2}}$ denoted as $D$ and the quadrilateral snake $Q(n)$, denoted by $F$ is given by

$$WL(D,F)=\frac{1}{2}\left[\sum_{k=1}^{n} (3k-1)(3(n-k)+2)\right] + \sum_{k=2}^{n} 9k(n+1-k)-(3n+1) + n(3n-1): \text{n is odd}$$

$$WL(D,F)=\frac{1}{2}\left[\sum_{k=1}^{n} (3k-1)(3(n-k)+2)\right] + \sum_{k=2}^{n} 9k(n+1-k)-(3n+2) + n(3n-1): \text{n is even}$$

Proof. Following the Algorithm A, we split the proof into two cases.

Case 1 (n is odd) By Congestion Lemma,

$$C_e(M_{3k-1}) = \left(\frac{3n+1}{2}\right)(3k-1) - 2I_G(3k-1) \text{ for } 1 \leq k \leq n$$

and $C_e(M_k) = \left(\frac{3n+1}{2}\right)(2) - 2 = 3n-1$, for $1 \leq k \leq n$. Clearly,

$$C_e(M_{3k-1}) = \left\{\begin{array}{ll}
\frac{1}{2} [(3k-1)(3(n-1))] & \text{: } k \text{ is odd} \\
\frac{1}{2} [9k(n+1-k)-(3n+2)] & \text{: } k \text{ is even}
\end{array}\right.$$

Using Partition Lemma, we obtain,

$$WL(D,F) = \frac{1}{2}\left[\sum_{k=1}^{n} C_e(M_{3k-1})\right] + \sum_{k=1}^{n-1} C_e(M_k)$$

Case 2 (n is even) By Congestion Lemma,

$$C_e(M_{3k-1}) = \left(\frac{3n+1}{2}\right)(3k-1) - 2I_G(3k-1) \text{ for } 1 \leq k \leq n$$

and $C_e(M_k) = \left(\frac{3n+1}{2}\right)(2) - 2 = 3n-1$, for $1 \leq k \leq n$. Clearly,

$$C_e(M_{3k-1}) = \left\{\begin{array}{ll}
\frac{1}{2} [(3k-1)(3(n-1))] & \text{: } k \text{ is odd} \\
\frac{1}{2} [9k(n+1-k)-(3n+2)] & \text{: } k \text{ is even}
\end{array}\right.$$

Using Partition Lemma, we obtain,

$$WL(D,F) = \frac{1}{2}\left[\sum_{k=1}^{n} E_G(M_{3k-1})\right] + \sum_{k=1}^{n-1} E_G(M_k)$$

Definition 3.6. [12]
The quadrilateral snake $pQ(n)$ has $n$ blocks of quadrilateral, where each block contains $p$ number of quadrilaterals, with one edge common. When $n = 2$, it is called double quadrilateral.

Remark 3.7. The quadrilateral snake $pQ(n)$ has $(2pn + n) + 1$ vertices and $(3pn + n)$ edges.

Algorithm B

Input : $K_{\frac{(2pn+n)}{2},\frac{(2pm+n)}{2}}$, the complete bipartite graph and $pQ(n)$, $p > 1$, the quadrilateral snake.

Algorithm : Using Even-odd labelling, label $V\left(K_{\frac{(2pn+n)}{2},\frac{(2pm+n)}{2}}\right)$ label $V(pQ(n))$ as follows:
Let $L_i$ denote the path induced by the vertices in $Q(n)$, $1 \leq i \leq 3n + 1$. Sequentially, label the quadrilateral snake $L_i$, $1 \leq i \leq p$ of $pQ(n)$ with numbers $1, 2, ..., p(2n + 1) + 1$ alternately from left to right and from right to left, beginning with $L_i$. 
Output: An embedding $\mathcal{E} : K_{\left\lfloor \frac{(2pn+n+1)}{2} \right\rfloor, \left\lceil \frac{(2pn+n+1)}{2} \right\rceil} \rightarrow pQ(n)$, $p > 1$ given by $\mathcal{E}(y) = y$ gives wirelength minimum. See Figure 2.

Figure 2: Edge cut of Quadrilateral snake $2Q(3)$.

Proof of correctness: Let $M_k$, $1 \leq k \leq n$ be the edge cut sets which cuts the $k$-blocks of the quadrilateral snake $pQ(n)$ such that $M_k$ disconnects $pQ(n)$ into two components $F_{k1}$ and $F_{k2}$ where $V(F_{k1})$ contains $k + p(2k-1) + 1$ vertices with atmost equal number of odd and even labels. By Lemma 3.2, $D_{k1} = \mathcal{E}^{-1}(F_{k1})$ and $D_{k2} = \mathcal{E}^{-1}(F_{k2})$ are maximum subgraph of $K_{\left\lfloor \frac{(2pn+n+1)}{2} \right\rfloor, \left\lceil \frac{(2pn+n+1)}{2} \right\rceil}$. Thus the edge cut $M_k$, $1 \leq k \leq n$ satisfies conditions (1) and (2) of the Congestion Lemma. Therefore $C_{\mathcal{E}}(M_k)$ is minimum for $k = 1, 2, ..., n$. For $1 \leq k \leq n$, $p > 1$, let $M'_{pk}$ be the edge cut which cuts the quadrilateral snake $pQ(n)$ horizontally in each block as algorithm A. Clearly for $k = 1, 2, ..., n$, $C_{\mathcal{E}}(M'_{pk})$ is minimum. Therefore, using Partition lemma the wirelength is obtained.

Theorem 3.7.
The embedding of complete bipartite graph $K_{\left\lfloor \frac{(2pn+n+1)}{2} \right\rfloor, \left\lceil \frac{(2pn+n+1)}{2} \right\rceil}$ into the quadrilateral snake $pQ(n)$, $p > 1$ yields minimum wirelength.

4 Conclusion
The wirelength of embedding complete bipartite graph into quadrilateral snake is studied in this paper.

References
[1] S.L. Bezrukov, J.D. Chavez, L.H. Harper, M. R’ottger and U.P. Schroeder, 1998. Embedding of hypercubes into grids, Mathematical Foundations of Computer Science, Lecture Notes in Computer Science, 1450, 693 – 701.
[2] S.L. Bezrukov, J.D. Chavez, L.H. Harper, M. R’ottger and U.P. Schroeder, 2000. The congestion of n-cube layout on a rectangular grid, Discrete Mathematics, 213, 13 – 19.
[3] P. Manuel, I. Rajasingh, B. Rajan and H. Mercy, 2009. Exact wirelength of hypercube on a grid, Discrete Applied Mathematics, 157( 7), 1486 – 1495.
[4] M.R. Garey and D.S. Johnson, 1979. Computers and intractability: a guide to the theory of NP completeness, Freeman, San Francisco, California.
[5] A.B. Greeni and I. Rajasingh, 2017. Embedding complete bipartite graph into Grid with optimum congestion and Wirelength, International Journal of Network and Virtual
Organisation, 17, 64 – 75.

[6] A.B. Greeni, 2020. Embedding complete bipartite graphs into necklace graphs, Procedia Computer Science, 172, 199-203.

[7] A.B. Greeni, 2020. Embedding Complete Bipartite Graph into Sibling Tree with Optimum Wirelenth, Journal of Combinatorial Mathematics and Combinatorial Computing, 112, 115-126.

[8] S.L. Bezrukov, 1999. Edge isoperimetric problems on graphs, Graph Theory and Combinatorial Biology, Bolyai Soc. Math. Stud. 7, L. Lovasz, A. Gyarfás, G.O.H. Katona, A. Recski, L. Szekely eds., Budapest, 157 – 197.

[9] M. Miller, R.S. Rajan, N. Parthiban and I. Rajasingh, 2015. Minimum linear arrangement of incomplete hypercubes, The Computer Journal, 58(2), 331 – 337.

[10] M. Arockiaraj, P. Manuel, I. Rajasingh and B. Rajan, 2011. Wirelength of 1-fault hamiltonian graphs into wheels and fans, Information Processing Letters, 111, 921–925.

[11] J.M. Xu, 2001. Topological structure and analysis of interconnection networks, Kluwer Academic Publishers, Boston.

[12] T.Manjula, R.Rajeswari, 2016. Dominator coloring of Quadrilateral snake, Triangle snake and Barbell graph, in Second International Conference on Science Technology Engineering and Management (ICONSTEM), Chennai, 115-119, doi: 10.1109/ICONSTEM.2016.7560934.