Transmission and Reflection in a Double Potential Well: Doing it the Bohmian Way

Regien G. Stomphorst *
Molecular Physics Group, Dept. of Biomolecular Sciences, Wageningen University, Dreijenlaan 3, 6703 HA Wageningen The Netherlands
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Abstract

The Bohm interpretation of quantum mechanics is applied to a transmission and reflection process in a double potential well. We consider a time dependent periodic wave function and study the particle trajectories. The average time, eventually transmitted particles stay inside the barrier is the average transmission time, which can be defined using the causal interpretation. The question remains whether these transmission times can be experimentally measured.

keywords
Bohm interpretation; transmission times. 03.65 BZ

1 Introduction

It is well known that (under certain circumstances) the causal interpretation is empirically equivalent to the orthodox Copenhagen interpretation 1, 2. However, there are a number of physical problems for which this orthodox approach provides no clear-cut

*E-mail: regien@stomphorst.net
answers. One of such problems is the question of a time observable for a tunneling process.

Tunneling is the quantum mechanical phenomenon that a particle can cross a barrier with potential \( V \), even if its energy is strictly less than \( V \). It is a natural question to ask how long it takes on average for particles to cross such a barrier. Unfortunately quantum theory does not provide a suitable time operator, whose expectation value for a given wave packet can be compared with experiment. Time enters quantum mechanics as a parameter, not as an operator. Therefore, this question about tunneling time is not an easy one.

Tunneling processes may be classified in two types: scattering type, where a wave packet is initially incident on a barrier, and then partly transmitted; and decay type when, the particle is initially in a bound state, surrounded by a barrier, and subsequently leaks out of this confinement. Many authors have addressed the duration of tunneling processes, in case of scattering processes [3, 4, 5] and in case of a decay process [6].

In Bohm’s causal interpretation of quantum mechanics various concepts of tunneling times for scattering processes can be distinguished. The most well-known time is dwell time, i.e. the time particles spend inside the barrier. For an ensemble of particles, we can determine the average dwell time. This average dwell time \( \langle t_d \rangle \) can be decomposed into an average transmission times \( \langle t_t \rangle \), i.e. the time spent inside the barrier by those particles, which eventually cross the barrier, and average reflection time \( \langle t_r \rangle \), i.e. the time consumed by reflected particles [2]:

\[
\langle t_d \rangle = |T|^2 \langle t_t \rangle + |R|^2 \langle t_r \rangle
\]

where \( |T|^2 \) and \( |R|^2 \) are the transmission and reflection probabilities respectively. This relation consists of:

\[
\langle t_d \rangle = |T|^2 \langle t_t \rangle + |R'|^2 \langle t_{r'} \rangle + |R''|^2 \langle t_{r''} \rangle
\]

where \( R' \) and \( t_{r'} \) refer to particles, which penetrate the barrier but re-emerge on the same side and \( R'' \) and \( t_{r''} \) to particles that do not enter the barrier. This relation assumes that particles are either transmitted or reflected, i.e. they do not remain inside the barrier.

In this paper, we consider a decay type of transmission and reflection by applying the causal interpretation to a double potential well. The decomposition of the dwell time
into average transmission and reflection time according to relation (1) in a decay type of tunnelling is not as straightforward as for the scattering cases. Definitions for transmission and reflection probabilities are not common for this decay type of transmission and reflection. However, we shall propose natural definitions for these concepts and use these to define average transmission and reflection time. For reflection we will concentrate on particles, which penetrate the barrier but re-emerge on the same side. It will again appear to be convenient to obtain the average transmission time by using the concept of the average arrival time.

The aim of this paper is firstly, to investigate whether the causal interpretation of quantum mechanics provides a straightforward way to define transmission times in double potential wells. For simplicity, we will study periodic wave functions. Secondly, the question whether it is necessary to adopt the causal interpretation to give meaning to the thus obtained transmission times is addressed. The paper is organised as follows. In section 2 we apply the causal interpretation method to transmission and reflection in a double potential well. In section 3 we define transmission times in terms of the probability density of the wave function. The last section is devoted to a discussion about the question whether or not the causal interpretation provides a clear way to define transmission times in a double potential well and about the necessity to rely on the causal interpretation for this definition.

2 The causal interpretation applied to a double potential well

In this section, the causal interpretation is applied to transmission and reflection in a double potential well. We discuss the possibilities to define transmission and reflection coefficients, average dwell time, average transmission time and reflection time according to this method.

2.1 Description of the double potential well

To model the decay type of transmission processes we consider a wave packet in a double potential well. The potential well is described by a one-dimensional box, defined from \(-a\)
Figure 1: The system under consideration: a double potential well. The total length of the box is $2a$, the potential at $a$ and $-a$ is infinite. The barrier is situated from $-b$ to $b$ and has a constant height $V$. to $a$ (see Fig. 1). At these points, the walls are infinitely high. In the middle of the box a barrier from $-b$ to $b$ is situated ($b < a$).

$$V(x) = \begin{cases} \infty & \text{if } |x| \geq a, \\ 0 & \text{if } b \leq |x| \leq a, \\ V & \text{if } 0 \leq |x| \leq b \end{cases} \quad (3)$$

We consider a wave packet which is, initially, concentrated on one side of the barrier only, and in the course of time moves to the other side of the well.

To obtain such a wave packet, we calculate, according to standard procedure (see Appendix) the energy eigenvalues and eigenfunctions for this double potential well. A linear combination of the lowest even energy eigenfunction ($f_e(x, t)$) and the lowest odd energy eigenfunction ($f_o(x, t)$) is the wave function considered in this paper:

$$\Psi^{(0)}(x, t) = \frac{1}{\sqrt{2}} f_e(x, t) + \frac{1}{\sqrt{2}} f_o(x, t) \quad (4)$$

The choice of these constants assures a high probability of finding the electron at $t = 0$ between $-a$ and $-b$.

Eqn. (4) provides a wave packet with (approximately) maximum extinction at one side. However, this extinction is not complete and can never be complete. We could enhance this extinction by using a linear combination of more than two eigenfunctions. But this

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1 Note that in the causal interpretation a particle described by a stationary wave function does not move, and hence no tunneling occurs.
might introduce recurrent trajectories, which we want to avoid.

Our wave packet shows periodic behavior, and transmission from one side to the other takes place in half a period. This period time is inversely proportional to \( E_o^{(0)} - E_e^{(0)} \), where \( E_o^{(0)} \) and \( E_e^{(0)} \) are proportional to the odd and even energy eigenvalues respectively. From the wave packet we can calculate trajectories. The particle velocity in the causal interpretation is given by [8, 9]:

\[
\dot{x} = \frac{\hbar}{m} \text{Im}(\Psi^*(x, t) \frac{\partial}{\partial x} \Psi(x, t)) |\Psi(x, t)|^2
\]

where the right-hand side denotes the probability current density over the probability density function. Analytical expressions of the velocity \( \dot{x} \) can be obtained by using the expressions given in the Appendix.

The trajectories are obtained as the solutions to equation (5), subject to the specification of the initial condition \( x_0 \). An ensemble of possible trajectories associated with the same wave is generated by varying \( x_0 \). Hence, to obtain the trajectories \( x(t) \) the differential equation (Eqn. 5) should be solved after a starting value \( (x_0) \) is chosen. We solved this differential equation numerically [10]. The trajectories are given in Figs. 2 and 3. In the last figure trajectories are weighted by the initial particle density \( \rho(x, 0) = |\Psi(x, 0)|^2 \).

2.2 A classification of the trajectories

Let us first note some salient aspects of the form of the possible trajectories. Fig. 2 shows various trajectories, that are generated by varying the choice of an initial position \( (x_0) \). One clearly sees their periodic behavior, and that the trajectories do not cross. Note that at any given time, all trajectories move in the same direction, i.e., to the right during the first half period, and towards the left during the second half. The figure shows that at very left and at the very right there are trajectories, which remain on the same side of the barrier. This occurs for all trajectories with a starting point left of point \( s_1 \) and to the right of \( b \). The area indicated with R (returners) contains trajectories, which reach inside the barrier but do not pass it, they return to their original area. Their starting positions at \( t = 0 \) is between \( s_1 \) and \( s_2 \). These trajectories correspond to reflection. The trajectories starting in area T (travellers) are trajectories, which pass the barrier and end up, after half a period, at the other side of the barrier. At \( t = 0 \) the starting point of the travellers is between \( s_2 \) and \( -b \). This behavior corresponds to transmission. The trajectories
between $-b$ and $b$ are already inside and between $b$ and $a$ already past the barrier at $t = 0$.

In Fig. 3 the probability density $|\Psi(x,0)|^2$ according to the wave function (4) is taken into account (the meaning of the symbols $a$, $b$, $s_1$, $s_2$ and $t_{1/2}$ are given in Figs. 2 and 3). Along the time axis we marked some special times, $t_p$, $t_m$, and $t_n$. Time $t_p$ is the instant at which the trajectory, starting at $t = 0$ in $s_2$ passes at $-b$. This trajectory marks the bifurcation between the reflection and transmission area. Time $t_m$ indicates the time the trajectory starting at $t = 0$ in $-b$ arrives at $b$. This is the time needed to deliver the trajectories, which at $t = 0$ already are inside the barrier, to the right hand side of the barrier. Although time $t_n$ lies outside half a period of time, we marked this time because it gives information about reflection times. All trajectories starting at $t = 0$ between $s_1$ and $s_2$ are at $t = t_{1/2}$ inside the barrier. Between $t = t_{1/2}$ and $t = t_n$ they leave the barrier and hence, reflected particles are inside the barrier between time $t_p$ and time $t_n$.

### 2.3 Definitions of transmission and reflection coefficients

In view of the above classification, the most straightforward way to define transmission and reflection seems to reserve the term transmission for travellers (T) and reflection for returners (R). Indeed, only travellers are actively involved in the transmission process. A probability for transmission, the transmission coefficient $|T|^2$, can then be defined as:

$$|T|^2 = \int_{s_2}^{-b} dx |\Psi(x,0)|^2$$

(6)

The reflection coefficient can be defined likewise:

$$|R|^2 = \int_{s_1}^{s_2} dx |\Psi(x,0)|^2$$

(7)

Note that this reflection coefficient is in fact related to $|R'|^2$ in relation (2). The transmission and reflection coefficients (6 and 7) do not add up to unity: $|T|^2 + |R|^2 \neq 1$. Of course, this is due to the fact that we did not include $|R''|^2$ and that there is a finite probability that the trajectories are already inside or past the barrier. These last possibilities are usually excluded in discussions of tunneling in one-dimensional scattering processes [11].
Figure 2: Bohm trajectories to show the different starting point $x_0$ possibilities at $t = 0$. Left of $s_2$ and right of $b$ trajectories do not leave their own area. $R$=reflection area, starting positions between $s_1$ and $s_2$. $T$=transmission area starting positions between $s_2$ and $-b$. 
Figure 3: Bohm trajectories for a one dimensional double potential well. We represent the probability density function $t = 0$ by $N$ points. These points are spaced with equal probability density intervals. $\int_{-a}^{x_N} dx |\Psi(x,0)|^2 = \frac{N-\frac{1}{2}}{N_{tot}}. x_N$ is the position of $x_0$ dependent on $N$. $N_{tot}$ is the total number of points. The value of $-\frac{1}{2}$ is arbitrarily chosen. Any other value (between 0 and 1) gives an equally valid description. $t_p$ is the time the trajectory, starting at $t = 0$ in $s_2$, passes at $-b$. $t_m = t_{1/2} - t_p$ and $t_n = t_{1/2} + t_m$. In this example $N = 15$. 
2.4 Definitions of dwell time, transmission and reflection times

Average dwell time is the average time that particles spend inside the barrier region:

$$\langle t_d \rangle = \int_{0}^{\infty} dt \int_{-b}^{b} dx |\Psi(x, t)|^2$$  \hspace{1cm} (8)

The probability to encounter a particle inside the barrier ($\int_{-b}^{b} dx |\Psi(x, t)|^2$) is for the wave function given in Eqn. (4) independent of time and hence, the average dwell time inside the barrier in half a period is the probability to encounter a particle inside the barrier multiplied by half a period:

$$\langle t_d \rangle = t_{\frac{1}{2}} \int_{-b}^{b} dx |\Psi(x, t)|^2$$  \hspace{1cm} (9)

To define average transmission and reflection times, we have to go back to the trajectories because trajectories provide information about the position $x$ at each instant of time $t$, which, under the assumption that each trajectory passes $x$ only once, can be inverted to give the function $t(x)$. It is convenient to express the transmission and reflection times in terms of arrival time distributions. In particular, let the instant at which a trajectory, starting in $x_0$ at $t = 0$, arrives at $x_1$, be denoted as $t(x_0; x_1)$. Averaging over the probability density $|\Psi(x_0, 0)|^2$ that a particle starts at $x_0$, we obtain the arrival time distributions ($\Pi(t)$):

$$\Pi_{x_1}(t) = \frac{\int dx_0 |\Psi(x_0, 0)|^2 \delta(t - t(x_0; x_1)) \int dx_0 |\Psi(x_0, 0)|^2}{\int dx_0 |\Psi(x_0, 0)|^2}$$  \hspace{1cm} (10)

where the integration limits should be chosen in such a way as to fulfill the assumption that each trajectory that passes $x_1$ does so only once in half a period.

Hence, the arrival time distribution of the to-be-transmitted particles at the exit of the barrier ($b$) is:

$$\Pi_b(t) = \frac{\int_{s_2}^{-b} dx_0 |\Psi(x_0, 0)|^2 \delta(t - t(x_0; b)) \int_{s_2}^{-b} dx_0 |\Psi(x_0, 0)|^2}{\int_{s_2}^{-b} dx_0 |\Psi(x_0, 0)|^2}$$  \hspace{1cm} (11)

The denominator is the transmission coefficient (see formula [3]). The starting points $x_0$ are within the transmission area, between $-b$ to $s_2$.

Similarly, the arrival time distribution of the to-be-transmitted particles at the entrance...
of the barrier \((-b)\) is:
\[
\Pi_{-b}(t) = \frac{\int_{s_2}^{-b} dx_0 |\Psi(x_0, 0)|^2 \delta(t - t(x_0; -b))}{\int_{s_2}^{-b} dx_0 |\Psi(x_0, 0)|^2}
\]
(12)

As long as the trajectories cross a particular point \((x_1)\) only once, Leavens \cite{7} showed that:
\[
\int dx_0 |\Psi(x_0, 0)|^2 \delta(t(x_1) - t(x_1; x)) = j(x_1, t(x_1))
\]
(13)
which gives us the arrival time distributions in term of probability current densities. Hence, the average arrival time at point \(x_1\) \(\langle t_a(x_1) \rangle\) under the same conditions, is:
\[
\langle t_a(x_1) \rangle = \frac{\int dt t j(x_1, t)}{\int dt j(x_1, t)}
\]
(14)

In our case, the probability current density in the double potential well is unidirectional for half a period and hence, we can determine the average arrival time at the entrance and the exit of the barrier. Taking the time boundaries from Fig. \cite{3} in consideration the average arrival time at the entrance of the barrier is:
\[
\langle t_a(-b) \rangle = \frac{\int_{t_p}^{\frac{t_m}{2}} dt t j(-b, t)}{\int_{t_p}^{t_m} dt j(-b, t)}
\]
(15)
and at the exit:
\[
\langle t_a(b) \rangle = \frac{\int_{\frac{t_m}{2}}^{t_1} dt t j(b, t)}{\int_{\frac{t_m}{2}}^{t_1} dt j(b, t)}
\]
(16)
and hence, the average transmission time, the average arrival time at the exit of the barrier minus the average arrival time at the entrance of the barrier, reads:
\[
\langle t_t \rangle = \langle t_a(b) \rangle - \langle t_a(-b) \rangle = \frac{\int_{\frac{t_m}{2}}^{t_1} dt t j(b, t)}{\int_{\frac{t_m}{2}}^{t_1} dt j(b, t)} - \frac{\int_{t_p}^{\frac{t_m}{2}} dt t j(-b, t)}{\int_{t_p}^{t_m} dt j(-b, t)}
\]
(17)

To determine reflection times, we have to extend the observation time to \(t_n\) (see Fig. \cite{3}). To-be reflected particles enter the barrier between time \(t_p\) and \(\frac{t_m}{2}\) they pass the barrier again on their way back between time \(\frac{t_m}{2}\) and \(t_n\). Hence, the average reflection time is:
\[
\langle t_r' \rangle = \frac{\int_{\frac{t_m}{2}}^{t_1} dt t j(-b, t)}{\int_{\frac{t_m}{2}}^{t_1} dt j(-b, t)} - \frac{\int_{t_p}^{\frac{t_m}{2}} dt t j(-b, t)}{\int_{t_p}^{t_m} dt j(-b, t)}
\]
(18)
In this definition we use time \( t_n \), which does not fall inside the range defined as half a period (from \( t = 0 \) to \( t_\frac{1}{2} \)). This might seem in contradiction to Eqn (9), the definition of average dwell time, which uses \( t_\frac{1}{2} \) as its upper bound limit. However, the time taken by the reflecting particles on their way back to their original place (\( t_\frac{1}{2} \) to \( t_n \)) is equal to the time taken by particles already inside the barrier at \( t = 0 \), to leave the barrier. The last mentioned time was not accounted for as belonging to average transmission time. Hence, the integration limits of the addition of average transmission and reflection times and the total average dwell times are consistent.

3 Transmission time in terms of probability density of the wave function

In the previous section, the definitions of the average transmission and reflection times, using the average arrival time distribution were based on trajectories from the causal interpretation of quantum mechanics. However, the final expressions (Eqns. 17 and 18) do no longer depend on the trajectories but on probability current densities. This suggests the possibility to define average transmission (and reflection times) without explicit use of the trajectories. In this section, we will show, that average transmission times can, indeed be obtained without explicit reference to trajectories. Instead we only rely on the non-crossing property of the causal interpretation and the unidirectionality between \( t = 0 \) and \( t = t_\frac{1}{2} \) of the current density flow and the periodicity of the wave functions. Average reflection times will not be considered, because we are interested in the transmission process. The question whether, the fact that the trajectories are not needed explicitly in order to determine the transmission times implies that the causal interpretation is superfluous for this purpose is left for the next section.

In order to show how average transmission times for a periodic wave function can be determined from the probability density, we refer to Figs. 4 and 5. Here we have partitioned the interior of the double well in 5 areas, as shown in the figures. In Fig. 4 the probability densities at \( t = 0 \) and in Fig. 5 at \( t = t_\frac{1}{2} \) are given. One can see that \(^2\)Actually, we chose the constants (see Eqn. (8)) in such a way that the probability to encounter particles at \( t = 0 \) at the right-hand side of the barrier is large enough to be visible in graphical presentations.
Figure 4: Probability density at $t = 0$. The transmission area ($T$) and reflection area ($R$, between $s_1$ and $s_2$) are indicated. Their destinations are given in Fig. 5. $-a$ and $a$ are box boundaries and $-b$ and $b$ the barrier boundaries.
Figure 5: Probability density at $t = t_\frac{1}{2}$. Compare to Fig. 4. $T$ is past the barrier and $R$ inside the barrier.

The probability density goes from left to right during half a period. The probability densities in Figs. 4 and 5 are mirror images. We will exploit this symmetry in our calculations.

The 5 areas indicated in Figs. 4 and 5 have the following meaning:

- The probability in area at the utmost left-hand side, between $-a$ and $s_1$, remains in its own domain. $s_1$ can be found by the condition that the probability between $-a$ and $s_1$ should be equal to the probability between $-a$ and $-b$ in Fig. 5. Because of symmetry the probability density between $-a$ and $s_1$ is equal to the probability...
density between $b$ and $a$ at $t = 0$ (Fig. 4). Hence,
\[ \int_{-a}^{s_1} dx \left| \Psi(x,0) \right|^2 = \int_{b}^{a} dx \left| \Psi(x,0) \right|^2 \]  \hspace{1cm} (19)

- The area which is indicated with an $R$ in Fig. 4 is the reflection area (between $s_1$ and $s_2$). In Fig. 4 the probability within that area has moved inside the barrier. The probability density inside the barrier is constant at all times and hence is the same as the probability inside the barrier at $t = 0$:
\[ \int_{s_1}^{s_2} dx \left| \Psi(x,0) \right|^2 = \int_{b}^{a} dx \left| \Psi(x,0) \right|^2 \]  \hspace{1cm} (20)

- We indicated a $T$ in Fig. 4 for the transmission area (between $s_2$ and $-b$). The probability density from this area arrives at $t = t_{1/2}$, Fig. 4, at the opposite side of the barrier. Because the probability density of all other areas are known, this probability density can be determined by:
\[ \int_{s_2}^{-b} dx \left| \Psi(x,0) \right|^2 = 1 - 2 \int_{-b}^{a} dx \left| \Psi(x,0) \right|^2 \]  \hspace{1cm} (21)

To determine the average transmission time, we need the average arrival time of the to-be transmitted particles at the entrance of the barrier and at the exit of the barrier. We use the assumption that the flux is unidirectional between $t = 0$ and $t = t_{1/2}$.

- At the entrance of the barrier the transmission flux starts at $t = 0$ and should be stopped at the time, when the left-hand side of the barrier is emptied of all travelling probability. The probability left behind at the left-hand side of the barrier at $t = t_p$ is equal to the probability between $-a$ and $s_2$ at $t = 0$. Hence, this time, $t_p$, can be found implicitly by:
\[ \int_{-a}^{-b} dx \left| \Psi(x,t_p) \right|^2 = \int_{s_2}^{s_1} dx \left| \Psi(x,0) \right|^2 \]  \hspace{1cm} (22)

- At the exit of the barrier, we must wait until all the probability initially inside the barrier has passed before the travelling part arrives. This happens between time $t = 0$ and $t = t_m$. After half a period the travelling part is inside the right-hand side well. Between time $t_m$ to $t_{1/2}$ the transmission part passes at the exit of the
barrier. \( t_m \) starts when the right hand side of the barrier contains the probability of the right-hand side and the probability under the barrier:

\[
\int_b^a dx |\Psi(x, t_m)|^2 = \int_{-b}^a dx |\Psi(x, 0)|^2
\]

(23)

The right-hand side terms of Eqns. (22) and (23) are equal. The half period evolution \((t_{1/2})\) is \(\frac{\pi}{E_1 - E_0}\) (see the Appendix for symbols).\(^3\)

Now, Eqn. (17) can be filled in, which gives us the average transmission time of the double potential well, without using the trajectories.

4 Discussion and conclusions

In this section, we discuss the question whether or not the causal interpretation provides an unambiguous way to define average transmission times in a double potential well. Secondly, we discuss the necessity to adopt the causal interpretation to define this transmission time. We also discuss the experimental accessibility of transmission times.

In the causal interpretation of quantum mechanics, the position of an individual particle travelling along a particular trajectory is determined at each instant of time. The trajectories show the possible behavior of particles inside the barrier and hence, the causal interpretation creates the possibility (at least numerically) to discriminate between trajectories inside the barrier, whose fate is transmission and whose fate is reflection. Although definitions for transmission and reflection coefficients are not common practice in a double potential well, the definitions, given in Eqns. (6) and (7) provide a useful tool to differentiate between these two possibilities. The wave functions were made out of two eigenfunctions and hence the trajectories show that the current density flow, between \( t = 0 \) and \( t = t_{1/2} \), is unidirectional. Hence, a straightforward way to define transmission is provided. We took the lowest energy level eigenfunctions but any pair of eigenfunctions would give an equally straightforward way to define transmission. However, the addition...\(^3\)

\[^3\text{Using the probability current density the times } t_p \text{ and } t_m \text{ can also be found by:}\]

\[
|T|^2 = \int_0^{t_p} dt j(-b, t) = \int_{t_m}^{t_{1/2}} dt j(b, t)
\]

(24)
of more eigenfunctions to create wavefunctions would cause recurrent trajectories. In case of recurrent trajectories the labeling of transmitted and reflected trajectories is not straightforward.

Similar to the more well-known case of scattering processes, the transmission times can be conveniently expressed in terms of the arrival time distributions. Average arrival times at the entrance and exit of the barrier are used to express the transmission time. Hence, the average time spent inside the barrier by eventually transmitted particles can (for the above mentioned wave functions) unambiguously be defined and hence the answer to the question whether the causal interpretation can give an unambiguous definition of average transmission time in a double potential well is "yes".

Next, let us discuss the question whether or not the causal interpretation of quantum mechanics is needed to define transmission time. Transmission times for a double potential well are defined in the causal interpretation of quantum mechanics. However, the causal interpretation also implies a different world view than the standard, orthodox interpretation. The trajectories describe the way particles move. The initial position of the particle, although unknown to us, fixes its future path completely and hence, in contrast to the orthodox interpretation, the causal interpretation theory is a deterministic theory.

In section 3 we showed how the definition of transmission times could be obtained, without trajectories, from the probability density of the wave function. This may suggest that the causal interpretation is superfluous for the determination of transmission times. However, transmission times are obtained under the assumption that the flow of probability density follows the non-crossing property of the trajectories of causal interpretation. Outside the causal interpretation, the justification for this assumption is not clear. Hence, our discussion of the definition of average transmission time in a double potential is dependent on this aspect of the causal interpretation of quantum mechanics for its justification.

Finally, we pose the question of the experimental accessibility of average transmission times. For a double potential well model, a definition for the average transmission time is offered in this paper but the question whether an experimental set-up to measure average transmission times can be devised is open and hence, the question remains whether the
average transmission time for a double potential well can be verified by experiments. The usefulness of the definition of average transmission times would be greatly enhanced if average transmission times can be experimentally measured.

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**Appendix**

We take atomic units, i.e. we put \( m = 1 \) and \( \hbar = 1 \). Taking the boundary conditions in consideration gives us even (symmetric) and odd (asymmetric) solutions. For the even solutions we find:

\[
f_e(x, t) = N_e i e^{−iE_e t} \begin{cases}
    \sin(k_e(x + a)) & \text{if } -a \leq x \leq -b, \\
    \sin(k_e(b - a)) \frac{\cosh(\alpha_e x)}{\cosh(\alpha_e b)} & \text{if } -b \leq x \leq b, \\
    -\sin(k_e(x - a)) & \text{if } b \leq x \leq a
\end{cases}
\]

Similarly, for the odd solutions, one obtains:

\[
f_o(x, t) = N_o i e^{−iE_o t} \begin{cases}
    \sin(k_o(x + a)) & \text{if } -a \leq x \leq -b, \\
    \sin(k_o(b - a)) \frac{\sinh(\alpha_o x)}{\sinh(\alpha_o b)} & \text{if } -b \leq x \leq b, \\
    \sin(k_o(x - a)) & \text{if } b \leq x \leq a
\end{cases}
\]

where \( a, b \) and \( V \) are explained in Fig 1. \( N_{e,o} \) are (complex)normalisation factors, \( k_{e,o} = \sqrt{2E_{e,o}}, \alpha_{e,o} = \sqrt{2(V - E_{e,o})}, \) and \( E_{e,o} < V \). Further, for even, \( E_e \) is the solution of the equation:

\[
\arctan(\frac{\sqrt{E_e}}{\sqrt{V - E_e}} \coth(b\sqrt{2(V - E_e)}) = n\pi - (a - b)\sqrt{2E_e}
\]

and for odd, \( E_o \) is determined by

\[
\arctan(\frac{\sqrt{E_o}}{\sqrt{V - E_o}} \tanh(b\sqrt{2(V - E_o)}) = n\pi - (a - b)\sqrt{2E_o}.
\]
References

[1] J.T. Cushing. The Causal Quantum Theory Program. In J.T. Cushing, A. Fine, and S. Goldstein, editors, *Bohmian Mechanics and Quantum Theory : an Appraisal*. Kluwer Academic Publishers, 1996.

[2] C.R. Leavens. Bohmian Mechanics and the Tunneling Time Problem for Electrons. *Proceedings of the Adriatico Research Conference on Tunneling and its Implications*, ed. D. Mugnai and A. Rafani and L.S. Schulman:100–120, 1996.

[3] E.H. Hauge and J.A. Stovneng. *Rev. Mod. Phys.*, 61(4):917–936, 1989.

[4] V.S. Olkhovsky and E. Recami. *Phys. Rep.*, 214(6):339–356, 1992.

[5] R. Landauer and Th. Martin. *Rev. Mod. Phys.*, 66(1):217–228, 1994.

[6] Y. Nogami and F.M. Toyama and W. van Dijk. *Phys. Lett. A.*, 270:279–287, 2000.

[7] C.R. Leavens. *Phys. Let. A*, 178:27–32, 1993.

[8] D. Bohm. *Phys. Rev.*, 85:166–197, 1952.

[9] R.P. Holland. *The quantum theory of motion*. Cambridge University Press, 1993.

[10] W.H. Press, S.A. Teukolsky, W.T. Vetterling, and B.P. Flannery. *Numerical Recipes in C*. Cambridge University Press, 1992.

[11] C.R. Leavens. *Solid State Communications*, 74(9):923–928, 1990.

Figure captions

**Fig. 1**

The double potential well. The total length of the box is $2a$, the potential at $a$ and $-a$ is infinite. The barrier is situated from $-b$ to $b$ and has a constant height $V$.  

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**Fig. 2**

Bohm trajectories showing the different types of behaviour, depending on starting point $x_0$ at $t = 0$. To the left of $s_2$, and right of $b$, trajectories do not leave their own area. $R =$ reflection area, starting positions between $s_1$ and $s_2$. $T =$ transmission area starting positions between $s_2$ and $-b$.

**Fig. 3**

Bohm trajectories for a one-dimensional double potential well. The probability density function is represented at $t = 0$ by $N$ points. These points are spaced with equal probability density intervals. $\int_{x_a}^{x_N} dx |\Psi(x, 0)|^2 = \frac{N - \frac{1}{2}}{N_{\text{tot}}}$. $x_N$ is the position of $x_0$ dependent on $N$. $N_{\text{tot}}$ is the total number of points. The value of $-\frac{1}{2}$ is arbitrarily chosen. Any other value (between 0 and 1) gives an equally valid description. $t_p$ is the time the trajectory, starting at $t = 0$ in $s_2$, passes at $-b$. $t_m = t_{1/2} - t_p$ and $t_n = t_{1/2} + t_m$. In this example $N = 15$.

**Fig. 4**

The probability density at $t = 0$. The transmission area ($T$) and reflection area ($R$, between $s_1$ and $s_2$) are indicated. Their destinations are given in Fig. 5. $-a$ and $a$ are box boundaries and $-b$ and $b$ the barrier boundaries.

**Fig. 5**

The probability density at $t = t_{1/2}$. In comparison with Fig. 4: $T$ is now past the barrier and $R$ inside the barrier.