Attitude Estimation From Vector Measurements: Necessary and Sufficient Conditions and Convergent Observer Design

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Abstract—In this article, we address the problem of attitude estimation for rigid bodies using (possibly time-varying) vector measurements, for which we provide a necessary and sufficient condition of distinguishability. Such a condition is shown to be strictly weaker than those previously used for attitude observer design. Thereafter, we show that even for the single vector case, the resulting condition is sufficient to design almost globally convergent attitude observers, and two explicit designs are obtained. To overcome the weak excitation issue, the first design employs to make full use of historical information, whereas the second scheme dynamically generates a virtual reference vector, which remains noncollinear to the given vector measurement. Simulation results illustrate the accurate estimation despite noisy measurements.

Index Terms—Attitude estimation, nonlinear system, observability, observer design.

I. INTRODUCTION

The attitude of a rigid body is its orientation with respect to an inertial reference frame. Attitude estimation is an essential element in a wide range of robotics and aerospace applications, in particular control, navigation, and localization tasks. Many common sensor types, e.g., magnetometers, accelerometers, and monocular cameras, provide body-fixed measurements of quantities with known inertial values, e.g., the Earth’s magnetic field and gravitational force, or the bearing to certain known landmarks. These are known as complementary measurements [20]. In some less common scenarios, a set of known vectors in the body-fixed frame is measured in the inertial frame, e.g., measurements from two GPS receivers attached to the rigid body with a known baseline. These are known as compatible measurements [20].

Estimation of attitude from multiple noncollinear vector measurements was formulated as a total least-squares problem over rotation matrices by Wahba [21]. Several efficient algorithms exist for its solution, including singular value decomposition methods, TRIAD, and QUEST [19]. However, when estimating a time-varying attitude variable, it is often beneficial to fuse the vector measurements with information from gyroscopes using a dynamical model. The resulting dynamic estimator is commonly known as a filter or observer. These approaches can significantly reduce the impact of high-frequency measurement noise. Furthermore, in many applications, there is only a single vector available for attitude estimation, and in this case, the attitude cannot be completely determined at a single moment. Applications for estimation from a single vector measurement include Sun sensors in eclipse periods [12], improving reliability with redundant measurements and simplifying designs [16], as well as visual-inertial navigation with only two feature points visible during some periods.

Among filtering approaches, extended Kalman filter is the most widely applied for attitude estimation. However, the domain of attraction is intrinsically local since the filter is based on first-order linearization; see [7] for a recent review. Alternatively, interest in nonlinear attitude observers was spurred by Salcudean’s seminal work [17], having achieved significant progress since then. There are many nonlinear attitude observers making direct use of vector measurements, e.g., with multiple measurements [10], [20], [28] or single vector measurements [1], [2], [8], [9]. The latter works impose a persistently nonconstant condition on the single reference vector, or similar conditions in which the uniformity of excitation with respect to time plays an essential role to guarantee asymptotic convergence. Trumpf et al. [20] provided a comprehensive treatment of observability of a rigid-body attitude kinematic model with vectorial outputs. However, as illustrated in [20, Remark 3.9], the condition is only sufficient but not necessary for distinguishability, a specific type of observability for nonlinear dynamical systems [4], [5]. Recently, stochastic attitude observability was studied in [23], showing that estimation is possible with a single vector measurement provided that the angular velocity and the direction measurements are resolved in appropriate frames. Its connection to the proposed condition will be briefly discussed in Section III. In this article, we revisit the problem of observability analysis and propose two novel attitude observers. To be precise, the main contributions of the note are twofold.

C1: For the problem of attitude estimation from vector measurements, we provide a necessary and sufficient condition for the distinguishability of initial states of the associated dynamical model, which is equivalent to the ability to reconstruct the attitude over time for a deterministic model.

C2: We show that the resulting distinguishability condition is also sufficient to design a continuous-time attitude observer. By focusing on single vector measurements, we provide two novel almost globally convergent attitude observers, which require significantly weaker conditions than the existing methods.

The constructive tool we adopt in observer design is the parameter estimation-based observer (PEBO), which was recently proposed in Euclidean space [13], [14], and extended to matrix Lie groups by the authors in [24] and [25]. Its basic idea is to translate system state observation into the estimation of certain constant quantities. The interested reader may refer to [26] for a geometric interpretation to PEBOs. In contrast to the case with at least two noncollinear vectors in [24] and [25], in this article, we consider a more challenging scenario with only a single vector measurement available under a weak excitation condition. We are unaware of any previous results capable of dealing with such a case. In the first observer design, after translating the
problem into online parameter identification, we propose a mechanism to integrate both the historical and current information to achieve uniform convergence. The second proposed scheme uses a filter to generate a “virtually” measurable vector, which remains noncollinear with the given reference vector.

Notations: \( I_n \in \mathbb{R}^{n \times n} \) represents the identity matrix of dimension \( n \), and \( 0_n \in \mathbb{R}^n \) and \( 0_{n \times m} \in \mathbb{R}^{n \times m} \) denote the zero column vector of dimension \( n \) and the zero matrix of dimension \( n \times m \), respectively. We use \( \mathbb{N} \) to represent the set of all natural integers, and \( \mathbb{N}_0 \) for the set of positive integers. We also define the skew-symmetric matrix \( J := [0, -1, 1, 0] \). Given a square matrix \( A \in \mathbb{R}^{n \times n} \) and a vector \( x \in \mathbb{R}^n \), the Frobenius norm is defined as \( ||A|| = \sqrt{\text{tr}(A^T A)} \), and \( |x| \) represents the standard Euclidean norm. The n-sphere is defined as \( S^n := \{ x \in \mathbb{R}^{n+1} : |x| = 1 \} \), and we use \( SO(3) \) to represent the special orthogonal group, and \( so(3) \) is the associated Lie algebra as the set of skew-symmetric matrices satisfying \( SO(3) := \{ R \in \mathbb{R}^{3 \times 3} : R^T R = I_3, \det(R) = 1 \} \). Given a variable \( R \in SO(3) \), we use \( R_l \) to represent the normalized distance to \( I_3 \) on \( SO(3) \) with \( |R_l^2| := 1/\text{tr}(I_3 - R) \). The operator skew(·) is defined as \( \text{skew}(A) := 1/2(A - A^T) \) for a square matrix \( A \). Given \( a \in \mathbb{R}^3 \), we define \( a_x := \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \in so(3) \) and its inverse operator is defined as \( \text{vex}(a_x) = a \).

The conference version [27] contains the main results in Section III and parts of Section IV-A. In this journal version, extra technical details, discussions, and simulations have been added in Sections IV–VI, as well as in the Appendices.

II. Problem Formulation

The aim of this note is to study observability and observer design of the rotation matrix representing the coordinates of the body-fixed frame \( \{B\} \) with respect to the coordinates of the inertial frame \( \{I\} \), which lives in the group \( SO(3) \). Its dynamics is given by

\[
\dot{R} = R \omega_x, \quad R(0) = R_0
\]

with the rotational velocity \( \omega \in \mathbb{R}^3 \) measured in the body-fixed coordinate. A complementary measurement \( y_B \) corresponds to a known vector \( g \in S^2 \) in the inertial frame, which is measured in the body-fixed frame

\[
y_B = R \cdot g
\]

and examples include magnetometers and rate gyros. We also consider the compatible measurement \( y_I \), i.e., a known vector \( b \in S^2 \) in the body-fixed frame, which is measured in the inertial frame

\[
y_I = Rb
\]

with \( y_B, y_I \in S^2 \), e.g., the vector difference derived from two GPS receivers with known baselines in the body frame. The reader is referred to [20, Sec. III] for more details about “complementary” and “compatible” measurements, and their practical motivation.

Before closing this section, let us recall some definitions used throughout this article.

Definition 1: (Distinguishability [4]) Consider an open set \( X \subset \mathbb{R}^n \) and a forward-complete nonlinear system

\[\dot{x} = f(x, t), \quad y = h(x, t)\]

with state \( x \in \mathbb{R}^n \) and output \( y \in \mathbb{R}^m \). The system \( (4) \) is distinguishable on \( X \) if for all \( (x_a, x_b) \in X \times X \)

\[h(X(t; t_0, x_a), t) = h(X(t; t_0, x_b), t) \quad \forall t \geq t_0 \Rightarrow x_a = x_b\]

in which \( X(t; t_0, x_a) \) represents the solution at time \( t \) of \( (4) \) through \( x_a \) at time \( t_0 \). In this article, without loss of generality, we take \( t_0 = 0 \).

Definition 2: (Persistent and interval excitation [115]) Given a bounded signal \( \phi: \mathbb{R}_+ \to \mathbb{R}^n \), it is persistently excited (PE) if

\[\int_{t}^{t+T} \phi(s) \phi^T(s) ds \geq \delta I_n, \forall t \geq 0 \text{ for some } T > 0, \delta > 0; \text{ or intervally excited (IE), if there exists } T \geq 0 \text{ such that } \int_{0}^{T} \phi(s) \phi^T(s) ds \geq \delta I_n, \text{ for some } \delta > 0.\]

III. NECESSARY AND SUFFICIENT CONDITIONS TO OBSERVABILITY

We first consider the general case where there may be multiple measurements of both compatible and complementary types

\[y_{B_i} = R^T g_i, \quad i \in \ell_1 := \{1, \ldots, n_1\}\]

\[y_{I_j} = Rb_j, \quad j \in \ell_2 := \{1, \ldots, n_2\}\]

with \( n_1, n_2 \in \mathbb{N} \). It is clear that the single measurement is corresponding to the case \( n_1 + n_2 = 1 \), for which we will construct two asymptotically convergent observers in the next section.

In the following proposition, we uncover a necessary and sufficient condition for distinguishability in attitude estimation.

Proposition 1: The time-varying system \( (1) \) with the output \( (5) \) and \( n := n_1 + n_2 \geq 1 \), is distinguishable if and only if there exist \( t_1, t_2 \geq 0 \) such that

\[
\delta > 0
\]

\[
\sum_{i,j,k,l \in \ell_1, \ell_2} |g_i(1) - g_j(1)| + |g_i(1) - R_0 b_j(t_2)| + |b_j(t_1) - \Phi(t_1, b_j(t_2))| > 0
\]

in which \( \Phi(t, s) \) is the state transition matrix of the time-varying system matrix \( -\omega_x(t) \) from \( s \) to \( t \).

Proof: The state transition matrix \( \Phi(t, s) \) of the linear time-varying (LTV) system \( \dot{x} = -\omega_x x \) with \( x \in \mathbb{R}^3 \) is defined as

\[
\frac{\partial}{\partial t} \Phi(t, s) = -\omega_x(t) \Phi(t, s), \quad \Phi(s, s) = I_3.
\]

It is equivalent to define \( \Phi(t, s) = Q(t)^{-1} Q(s) \), in which \( Q \in SO(3) \) is generated by the dynamics

\[
\dot{Q} = Q \omega_x, \quad Q(0) = I_3
\]

with \( Q \in SO(3) \). From the fact \( Q^T = Q^{-1} \) and

\[
\dot{R}Q^T = RQ^T - RQ^T Q^{-1} = 0
\]

we have for all \( t, s \geq 0 \)

\[
R(t)Q(t)^{-1} = R(0)Q(0)^{-1} \quad \Leftrightarrow \quad R(t) = R_0 Q(t)
\]

with \( R_0 := R(0) \).

Now we collect all the measured outputs in the vector

\[\bar{y} = \text{col}(y_{B_1}, \ldots, y_{B_{n_1}}, y_{I_1}, \ldots, y_{I_{n_2}})\]

With a slight abuse of notation, we denote the output signal \( \bar{y} \) from the initial condition \( R_0 \in SO(3) \) as \( \bar{y}(t; R_0) \). In terms of Definition 1, the system is distinguishable from \( t = 0 \) if and only if

\[\bar{y}(t; R_a) = \bar{y}(t; R_b) \quad \forall t \geq 0 \Rightarrow R_a = R_b\]

for any \( R_a, R_b \). Clearly, the abovementioned condition (12) is equivalent to the identifiability of the constant matrix \( R_0 \in SO(3) \) from the nonlinear regressor equation

\[\bar{y} = h(R_0, t)\]

\[\text{if } n_i = 0 \quad (i = 1, 2), \text{ then the set } \ell_i \text{ is defined as the empty set } \emptyset.\]
with the equation

$$h(R_0, t) := \begin{bmatrix} Q^T(t) R_0^T g_1(t) \\ \vdots \\ Q^T(t) R_0^T g_n(t) \\ R_0 Q(t) b_1(t) \\ \vdots \\ R_0 Q(t) b_n(t) \end{bmatrix}.$$  

The regressor (13) can be equivalently rewritten as

$$Y(t) = R_0^T \phi(t), \quad R_0 \in SO(3)$$  

with $Y \in \mathbb{R}^{3 \times n}$ and $\phi \in \mathbb{R}^{3 \times n}$ given by

$$Y := Q [y_1, \ldots, y_n, b_1, \ldots, b_n]: \quad \phi := [g_1, \ldots, g_n, y_1, \ldots, y_n].$$  

Hence, the identifiability of the constant matrix $R_0$ on SO(3) from the nonlinear regression model (13) is equivalent to the unique solvability of $R_0$ on SO(3) for the regression model (14) with infinite numbers of equations over time—involving that (14) holds for all $t \geq 0$.

For the first case, (16) is equivalent to

$$\sum_{i, \ell \in \ell_1} |g_i(t_1) \cdot g_i(t_2)| > 0.$$  

Recalling (3), the second case is equivalent to for some $j, k \in \ell_2$

$$y_{1,n} \cdot y_{1,k} \neq 0 \quad \iff \quad R(t_1) b_j(t_1) \neq R(t_2) b_k(t_2) \neq 0.$$

where we use, in the second implication the identity $(Rb)_s = Rb_s R^T$, the full rankness of $R(t)$ in the third implication, and in the last

$$R(t_1) R(t_2) = Q(t_1) R_0 Q(t_2) = \Phi(t_1, t_2).$$

The last line of the condition (18) can be compactly written as

$$\sum_{j, k \in \ell_2} |b_j(t_1) \cdot \Phi(t_1, t_2) b_k(t_2)| > 0.$$  

Similarly, we get that for the third case the condition (16) is equivalent to

$$\sum_{i, \ell \in \ell_1, j \in \ell_2} |g_i(t_1) \cdot R_0 \Phi(0, t_2) b_j(t_2)| > 0.$$  

Combining these three cases, it is sufficient to obtain (6). On the other hand, since each term in (6) is nonnegative, if the condition (6) holds,

$$|g_i(t_1) \cdot g_i(t_2)| + |b_j(t_1) \cdot \Phi(t_1, t_2) b_k(t_2)| > 0.$$  

If there are two types of measurements, the identifiability condition (6) is dependent of the initial attitude $R_0$, and this implies that some region in SO(3) may be not distinguishable for a given specific trajectory. However, the following corollary shows that such a region has zero Lebesgue measure in the group SO(3). Note that the following condition does not rely on the initial attitude $R_0$.

**Corollary 1**: If the condition (6) is replaced by the initial attitude-independent term

$$\sum_{i, \ell \in \ell_1, j, k \in \ell_2} |g_i(t_1) \cdot g_i(t_2)| + |b_j(t_1) \cdot \Phi(t_1, t_2) b_k(t_2)| + |b_j(t_1) \cdot \Phi(t_1, t_2) b_k(t_2)| > 0$$

the distinguishability is guaranteed almost surely.\(^4\)

**Remark 1**: Trumpf et al. [20] proposed the following sufficient (but not necessary, cf., [20, Remark 3.9]) condition for distinguishability of the given system

$$\lambda_2 \left( \int_0^T g_1(s) g_1^T(s) ds \right)^{1/2} + \int_0^T \sum_{j \in \ell_2} \left( \omega(s), g_j(s) + \frac{d}{ds} b_j(s) \right) ds > 0$$

for some $T > 0$, with $\lambda_2$ being the second largest eigenvalue of a square matrix. Note that in the abovementioned condition, it is necessary to impose (piecewise) smoothness of the signals $b_j$. In the following corollary, we show that (23) is sufficient for the proposed necessary and sufficient condition (6) to hold.

**Corollary 2**: Consider the time-varying system (1) with the output (5), and $n := n_1 + n_2 \geq 1$. If (23) holds, then the condition in Proposition 1 is also verified.

It is interesting to note that the condition for the second item (with the body velocity and a constant $b_j$) in [23, Th. VI.2] is sufficient to guarantee the term $\int_0^T \omega(s), b_j(s) ds > 0$ in (23)—invoking the abovementioned corollary—which yields the positiveness of the last term in the proposed condition (6).

**IV. ATTITUDE OBSERVER FOR A SINGLE VECTOR MEASUREMENT**

In this section, we show that the distinguishability condition—identified in Proposition 1—is sufficient to design a continuous-time observer with almost globally asymptotically convergent estimate to the unknown attitude.

Since the scenario with only a single vector measurement is more challenging than the multiple vector case, we focus on the former in this section. The main results can be extended to the case with multiple vector measurements in a straightforward manner.

**A. Attitude Observer Using Integral Correction Term**

Let us consider the observer design with a single complementary measurement (2). In the first observer design, we construct a dynamic extension—following the PBO methodology [14]—in order to reformulate attitude estimation as an online consistent parameter

\(^2\)The estimation of constant $R_0$ is in the Wahba problem formulation, i.e., estimating a fixed rotation matrix from a set of vector observation, though $Y$ and $\phi$ are sampled from different time instants.

\(^3\)We do not distinguish the order of $i$ and $j$.

\(^4\)We refer to the initial attitude set which makes the system lose distinguishability having zero Lebesgue measure in the entire state space.
identification problem. By adding a novel “integral”-type correction term, we are able to achieve asymptotic stability of the observer.

**Proposition 2:** For the system (1) with the complementary output (2), we assume that all signals are piecewisely continuous and the reference satisfies the distinguishability condition, i.e.,

\[ \exists t_1, t_2 > 0, \quad |g(t_1) - g(t_2)| > 0 \quad (24) \]

with a known bound \( T > 0 \) on the distinguishability interval.\(^5\) The attitude observer

\[ \hat{Q} = Q \omega_c \quad (25) \]

and

\[ \dot{\hat{Q}}_c = \eta_c \hat{Q}_c, \quad \hat{R} = \hat{Q}^\top_c Q \quad (26) \]

with \( Q(0) \in \text{SO}(3) \) and \( \hat{Q}_c(0) \in \text{SO}(3) \)

\[ \eta = \gamma \eta (\hat{Q}_c, g, (Qy_b) + \gamma_1 \xi_c \quad \xi = 2 \text{vex}(A \hat{Q}_c) \quad (27) \]

and the gains \( \gamma, \gamma_1 > 0 \) and \( A(0) = 0_{3 \times 3} \), guarantees \( \hat{R} \in \text{SO}(3) \), and the convergence \( \lim_{t \to \infty} | \hat{R}(t) - R(t) | = 0 \) almost globally.

**Proof:** Let us consider the dynamic extension \( Q = Q \omega_c \) for the initial condition \( Q(0) \in \text{SO}(3) \). By defining a variable \( E(R, Q) = QR^\top \) which also lives in \( \text{SO}(3) \)—we have

\[ \dot{E} = \dot{Q} R^\top - QR^\top \dot{R} R^\top = 0. \quad (28) \]

Therefore, there exists a constant matrix \( Q_0 \in \text{SO}(3) \) such that

\[ Q(t) R^\top (t) = Q_c \quad \forall t \in [0, +\infty). \quad (29) \]

Note that \( Q(t) \) is an available signal by construction, and \( Q_c \) is unknown. Invoking (29) and the full rankness of \( Q_0 \), the estimation of \( R \) is equivalent to the one of \( Q_c \).

Based on the abovementioned idea, we construct the auxiliary system

\[ \Sigma_c : \dot{Q}_c = 0, \quad y_c = Q_c b_c \quad (30) \]

in which \( Q_c(0) \in \text{SO}(3) \) is constant and the output satisfies

\[ y_c(t) := Q(t) y_b(t) \quad (31) \]

and the “body-fixed coordinate” reference \( b_c := g \). It is clear that the system \( \Sigma_c \) is exactly in the same form as the kinematic model with a compatible measurement (1) and (3), and zero velocity.

We now define the estimation error of \( \hat{Q}_c := Q_c \hat{Q}_c \), the dynamics of which is given by

\[ \dot{\hat{Q}}_c = \mathcal{A} \hat{Q}_c - Q_c \mathcal{A} \hat{Q}_c^{-1} \hat{Q}_c^{-1} \hat{Q}_c^{-1} = \eta_c \hat{Q}_c. \quad (32) \]

The term \( \eta \) satisfies

\[ \eta_c = \gamma \eta (\hat{Q}_c, g, (Qy_b) + \gamma_1 \xi_c \quad \xi = 2 \text{vex}(A \hat{Q}_c) \quad (33) \]

in which for \( t \in [0, T] \), we have \( A(t) = \int_0^t Q(s) y_b(s) g(s) g(s)^\top ds \). Then, substituting into the second equation of (27) yields

\[ \xi_c(t) = \int_0^t \left[ Q(s) y_b(s) \left( \hat{Q}_c(t) g(s) \right)^\top - \hat{Q}_c(t) (g(s) g(s)^\top Q(s) y_b(s) \right) \right] ds \]

\[ = 2 \text{vex} \left( \int_0^t y_b(s) y_b(s)^\top ds \cdot \hat{Q}_c(t) \right) \quad (34) \]

and for \( t > T \), we have \( \xi(t) = \xi(T) \).

Consider the candidate Lyapunov function \( V(\hat{Q}_c) = 3 - \text{tr}(\hat{Q}_c) \), which has its minimal value at \( \hat{Q}_c = I_3 \). It yields

\[ \dot{V} = -\text{tr}(\eta_c(t) \hat{Q}_c(t)) \]

\[ = -\gamma \text{tr} \left( y_c(t) y_b(t) - \hat{Q}_c(t) y_c(t) y_b(t)^\top \hat{Q}_c(t) \right) \]

\[ - \gamma \int_0^t \text{tr} \left( y_c(s) y_b(s) - \hat{Q}_c(s) y_c(s) y_b(s)^\top \hat{Q}_c(s) \right) ds \]

\[ = -\gamma y_b(t) - (I - \hat{Q}_c^2(t)) y_c(t) - \gamma \int_0^t y_b(s) - (I - \hat{Q}_c^2(s)) y_c(s) ds \]

\[ = -2 \text{vex} \left( \text{skew}(\hat{Q}_c(t)) \right)^\top \Gamma(t) \text{vex} \left( \text{skew}(\hat{Q}_c(t)) \right) \]

\[ \leq -\lambda_{\text{skew}}(\Gamma) ||\text{skew}(\hat{Q}_c)||^2 \quad (35) \]

where in the fourth equation, we have used \( 2||v||^2 = ||v_x||^2 \) for any \( v \in \mathbb{R}^3 \), and the definition of \( \Gamma \) as

\[ \Gamma = \Gamma + \Gamma \quad (36) \]

with

\[ \Gamma(t) := \gamma \left( I - y_c(t) y_b(t)^\top \right) \]

\[ \Gamma(t) := \left\{ \begin{array}{ll} \gamma \int_0^t (I - y_c(s) y_b(s)^\top) ds, & t \in [0, T] \\ \Gamma(T), & t > T. \end{array} \right. \]

Let us study the property of the matrix \( \Gamma \in \mathbb{R}^{3 \times 3} \). From the assumption \( |g(t_1) - g(t_2)| > 0 \) for some \( t_1, t_2 \leq T \), we have

\[ \begin{array}{c} |y_c(t_1) - y_c(t_2)| \end{array} = \left| (Q_c g(t_1)) - (Q_c g(t_2)) \right| \]

\[ = ||Q_c g(t_1) - Q_c g(t_2)|| > 0. \quad (37) \]

It implies that

\[ \delta_{\gamma} := 2 I - y_c(t_1) y_b(t_1)^\top - y_c(t_2) y_b(t_2)^\top > 0 \quad (38) \]

where we have used the fact that for \( a, b \in \mathbb{R}^3 \), \( |a \cdot b| > 0 \) implies the positiveness of \( (2I - aa^\top - bb^\top) \); see [20, Lemma A.2]. On the other hand, using piecewise continuity of \( y_c \) and (38), we have

\[ \int_{t_1}^{t_1 + \Delta} I - y_c(s) y_b(s)^\top ds + \int_{t_2}^{t_2 + \Delta} I - y_c(s) y_b(s)^\top ds = \delta_{\gamma} + \Delta(s^2) > 0 \quad (39) \]

for sufficiently small \( \varepsilon > 0 \) with a high-order term \( \Delta(s^2) \), and thus

\[ \int_0^T I - y_c(s) y_b(s)^\top ds > 0 \Rightarrow \Gamma(t) > 0 \]

\[ \Rightarrow \lambda_{\text{skew}}(\Gamma(t)) > \delta_{\gamma} \quad \forall t \geq T \quad (40) \]

for some \( \delta_{\gamma} > 0 \). Note that in the second implication, we use (36) and positive semidefiniteness of \( \Gamma \). From \( \dot{V} \leq -\lambda_{\text{skew}}(\Gamma)||\text{skew}(\hat{Q}_c)||^2 \), we get global stability of the closed loop, and \( 0 \leq V(\hat{Q}_c(t)) \leq V(\hat{Q}_c(0)) \)

\[ V(\hat{Q}_c(t)) - V(\hat{Q}_c(0)) < -\int_0^t \lambda_{\text{skew}}(\Gamma(s)) ||\text{skew}(\hat{Q}_c)||^2 ds \]

By taking \( t \to \infty \), we have \( \int_0^\infty \lambda_{\text{skew}}(\Gamma(s)) ||\text{skew}(\hat{Q}_c)||^2 ds \leq V(\hat{Q}_c(0)) - V(\hat{Q}_c(\infty)) < +\infty \), thus \( ||\text{skew}(\hat{Q}_c)|| \in L_2 \). From (32), it yields the boundedness of the time derivatives of \( \hat{Q}_c \) and \( ||\text{skew}(\hat{Q}_c)|| \).

Using Barbalat’s lemma, we have \( ||\text{skew}(\hat{Q}_c)|| \to 0 \). The set \( \{ \hat{Q}_c \in \text{SO}(3) : ||\text{skew}(\hat{Q}_c)|| = 0 \} \) only contains the stable equilibrium \( I_3 \) and the unstable equilibria \( U \) \diag(1, -1, 1) for some \( U \in \text{SO}(3) \). For the latter case, the Lyapunov function \( V(\hat{Q}_c) \) equals to its maximal \( 0 \).
value, thus having zero Lebesgue measure. Therefore, the dynamics (32) is almost globally asymptotically stable. Invoking the algebraic relation $R = Q^T \Gamma Q$, we complete the proof.

Remark 2: The observer error term $\eta$ contains two parts
\[
\eta = \gamma_R (\hat{Q} g) (Q g) + \gamma_t \xi
\]
which may be viewed as an observer design using a “proportional + integral”-type error term. The first term only utilizes the current information, making it behave as an online design. The second “integral” term, which may be written as
\[
\xi(t) = \int_0^t [Q_s(t) g(s)] ds, \quad t \in [0, T]
\]
for some $T > 0$. The bound $T$ is used in the dynamics of the variable $\eta$ in order to be able to guarantee the boundedness of the observer internal states, and verification of the condition within $[0, T]$. Indeed, the bound $T$ is not necessarily known a priori, since the distinguishability condition (24) is an easily-checkable condition on measured quantities. The proposed scheme may be modified as an adaptive design in which such a condition is checked online continuously, and the dynamics of $\eta$ simply changes when the condition is satisfied. It is also natural to replace the condition (24) by $g(t_i) \cdot g(t_{i+1}) > \delta$ for some $\delta > 0$, to deal with sensor noise.

Remark 4: As shown previously, the “integral” term only accumulates information in the interval $[0, T]$, which, however, does not have the sort of “fading memory” property on past measurements. As long as the excitation condition is satisfied, which can be easily monitored online, the observer performance can be improved considering the moving interval $[t - T, t]$ rather than $[0, T]$ in Proposition 2.

B. Attitude Observer Using Virtual Vectors

In this section, we provide an alternative observer design, which does not need the information of $T$. The basic idea is to generate a new “virtual” vector measurement
\[
y_v = Q_s b_v
\]
from the real measurement (30), such that
\[
b_v \times b_c \neq 0
\]
uniformly after some moment with $b_v = g$. Then, it becomes the well-studied attitude observer design problem with (not less than) two noncollinear vectors, which has been well addressed in the literature.

Proposition 3: For the system (1) with the output (2), we assume that all signals are continuous and satisfy that $\exists t_i > 0 \ (i = 1, 2, 3)$
\[
\det \begin{bmatrix} g(t_1) & g(t_2) & g(t_3) \end{bmatrix} \neq 0.
\]
Consider the dynamic extension (25) and the LTV filter
\[
\begin{aligned}
\dot{Z} &= \gamma_t (Q y_b) (Q g) - g^T Z, \ Z(0) = Z_0 \\
\dot{\Omega} &= -\gamma_t g g^T \Omega, \ \Omega(0) = I_3 \\
\dot{P} &= \gamma_t (I_3 - \Omega)^T [Z - \Omega Z_0 - (I_3 - \Omega) P]
\end{aligned}
\]
with $P(0) \in \mathbb{R}^{3 \times 3}, \gamma_t, \gamma_v > 0, Z_0 \in \mathbb{R}^{3 \times 3}$, and the filtering outputs
\[
\begin{aligned}
b_v &= U g, \ y_v &= P^T U g
\end{aligned}
\]
in which $U := \text{diag} (J, 1) \text{diag} (J, J)$. Then, the observer (26) with
\[
\begin{aligned}
\eta &= \gamma_t (Q y_b) (Q g) + \gamma_v (\hat{Q} b_v) \times y_v
\end{aligned}
\]
and the gains $\gamma_t, \gamma_v > 0$ guarantees $\hat{R}(t) \in \text{SO}(3)$ for all $t \in [0, \infty)$ and the convergence $\hat{R} \to R$ as $t \to \infty$ almost globally.

Proof: First, let us study the property of the LTV filter (46). Note that $U$ can be decomposed as the product of three basic (element) rotations $U = R_x(\theta_1) R_y(\theta_2) R_z(\theta_3)$ with $\theta_1 = \theta_3 = \frac{\pi}{2}$ and $\theta_2 = 0$. Hence, $g \times (Ug) \neq 0$, verifying (44). Then, we need to verify (43) in an asymptotic sense.

According to the proof in Proposition 2, we may reformulate the estimation of $R$ as the one of $Q_v$. From the dynamics of $Z$, one has
\[
\frac{d}{dt} (Z - Q_v^T) = \gamma_t g (y_v - g^T Z) = -\gamma_t g g^T (Z - Q_v^T)
\]
and thus
\[
Z - Q_v^T = \Omega (Z_0 - Q_v^T) \Rightarrow Z - Z_0 = (I - \Omega) Q_v.
\]
It yields $\frac{d}{dt} (P - Q_v^T) = -\gamma_v \psi^T (P - Q_v^T)$ with $\psi := I_3 - \Omega$. From the condition (45), for any nonzero $x \in \mathbb{R}^3$, it may always be represented as
\[
x = c_1 g(t_1) + c_2 g(t_2) + c_3 g(t_3)
\]
with at least one of $c_i \ (i = 1, 2, 3)$ nonzero. Hence,
\[
x^T \sum_{i=1}^3 g(t_i) g(t_i)^T x > 0 \Rightarrow \sum_{i=1}^3 g(t_i) g(t_i)^T > 0
\]
\[
= \int_{t_i}^{t_{i+3}} \sum_{i=1}^3 g(s) g(s)^T ds > 0
\]
\[
= \int_0^T g(s) g(s)^T ds > 0
\]
for some $T > t_i$ and some sufficiently small $\epsilon > 0$. It implies that the vector signal $g$ is IE. As a result, the matrix $\psi = I_3 - \Omega$ is PE [22, Prop. 2]. Following [18, Th. 2.5.1], we are able to derive
\[
\lim_{t \to \infty} \|P - Q_v^T\| = 0 \Rightarrow \lim_{t \to \infty} |P^T U g - Q_v g| = 0
\]
with an exponentially decaying term $\epsilon_t$.

The last step is to study the convergence of the estimation error of $Q_v$, which is defined as $\hat{Q}_v := Q_v - Q_v^\perp$. The observer $\hat{Q}_v = \eta_v \hat{Q}_v$ with the output error term (48) may be written as the “standard” compatible observer
\[
\hat{Q}_v = \hat{Q}_v \left[ \gamma_t \left( \hat{Q} g(b_v) \times g_v + \gamma_v (\hat{Q} b_v + \epsilon_v) \times b_v \right) \right]_x
\]
for the auxiliary dynamics (30) with $\omega_c = 0$. If the term $\epsilon_v$ was zero, by using [20, Th. 4.3] and the uniform noncollinear relation $b_v \times b_v \neq 0$, we would obtain $\hat{Q}_v \to \hat{Q}_v$ as $t \to \infty$ almost globally. However, the term $\epsilon_v$ is exponentially decaying to zero, and we may follow the perturbation analysis in [24, Prop. 6] using a time-varying Lyapunov function to obtain the almost global asymptotic stability. We omit the details here. Invoking the algebraic relation (29), we complete the proof.

Remark 5: The condition (45) with three noncollinear moments is slightly stronger than the one (24). However, it removes the necessity of
having a known bound $T > 0$ for the observer design in Proposition 3. The key idea to construct a new “virtual” noncollinear reference vector $b_v$ is similar to some recent results on generation of PE regressors from those only satisfying IE [6], [22], [24]. We refer the reader to the monograph [5] for a discussion of excitation conditions in observer design.

Remark 6: For the case with a single compatible measurement (3), we may still get the auxiliary model (30) by designing the dynamic extension $\tilde{Q} = Q_\omega \tau$, but with the new definitions of $y_e := y_t$ and $g_e := Q b$. Then, the abovementioned two designs are capable of solving the problem with slight modifications accordingly.

V. DISCUSSIONS

In this section, we show some practical issues in attitude estimation—intermittent, delayed, and biased measurements [1], [3]—can be easily (and even trivially) tackled by the proposed methodology.

Remark 7: (Delayed measurement) Time delay in attitude estimation is generally unavoidable [1]. A common scenario is that a known delay $\tau$ appears in the single vector measurement, i.e., $y(t) = y_t(t - \tau) = R^T(t - \tau)g(t - \tau)$, in which $\tau(t)$ may be constant or time-varying. After designing the dynamic extension (25), the delayed output can be rewritten as $g(t) = \tilde{Q} (t - \tau)\hat{Q}_t g(t - \tau)$. Then, we are still able to get the auxiliary model (30) with $y_e := Q(t - \tau)y(t), \ b_e := g(t - \tau)$.

Since $Q_e$ is a constant matrix on the special orthogonal group, the observers in Propositions 2 and 3 can provide asymptotically convergent attitude estimation by modifying the “reference vector” $b_e$ and the vector $y_e$ as (56).

Remark 8: (Intermittent measurement) Some sensors only provide intermittent measurement $y_t$ at some instants of time $t_k$ ($k \in \mathbb{N}_\times$). Let sequence $(t_k)_{k \in \mathbb{N}_\times}$ be strictly increasing, and $|t_k|$ and $|t_{k+1} - t_k|$ ($k \geq 1$) are upper and lower bounded by two positive constants. In the proposed attitude PEBO framework, we have translated the estimation of the variable $R(t)$ into that of constant $Q_e$, and thus, the intermittent measurement does not bring any difficulty in observer design. Define the vectors

$$y_e(t) := Q(t) y(t), \ b_e(t) := g(t) \quad \forall t \in [t_k, t_{k+1})$$

(57)

Again, we get the auxiliary model (30) with the modified reference vectors in (57). The proposed two continuous-time observers can solve attitude determination under intermittent measurements.

Remark 9: (Robustness) In practice, the measurement vector is perturbed by sensor noise, i.e., $y_t = y_t + n_r$ with a bounded term $n_r$. Then, the correction term $\eta$ of the observer in Proposition 2 becomes $\eta_t := \eta + \Delta_y$, in which $\eta$ is the nominal part defined in Proposition 2 and $\Delta_y$ is the additive term stemmed from the noise term $n_r$. Since $Q_e, Q, g, \text{and } y_t$ all live in some compact sets, as well as the variable $A$ being the integral over a finite interval $[0, T]$, there exists a constant $k > 0$ such that $|\Delta_y(t)| \leq k|n_r|_{\infty}$. For this case, the time derivative of the Lyapunov function becomes $V \leq -\lambda_{\text{min}}(\Gamma)[\text{skew}(\dot{Q}_t)]^2 + |\text{tr}((\Delta_y), \dot{Q}_t)|$. Hence, we are able to establish the robustness of the observer in the bounded-input-bounded-output sense. This is also verified via noisy simulations in the coming section.

Remark 10: (Biased velocity) The proposed schemes can be extended to the case with biased velocity, i.e., $\omega = \omega' + \theta$, with $\omega' \in \mathbb{R}^3$ the true value and $\theta \in \mathbb{R}^3$ the constant gyro bias. By calculating the time derivative of $y_b$ and applying the stable filter $\frac{\alpha + p}{\alpha + p + \lambda_p} [\dot{y}_b] + \frac{1}{\alpha + p} \omega^T \omega_b = -\frac{\alpha}{\alpha + p} [\dot{y}_b] \theta$. We may use this regressor to estimate $\theta$ online under some excitation conditions. By cascading this estimate to the proposed observers, we are able to solve the adaptive attitude estimation problem.

VI. SIMULATIONS

Example 1: We consider a single time-varying inertial vector

$$g(t) = \begin{cases} e_1, & t \in [0, 5) s \\ e_3, & t \geq 5 s \end{cases}$$

(58)

in which $e_i$ represents the $i$th standard Euclidean basis in $\mathbb{R}^3$. Clearly, it satisfies our excitation condition (24), but not the persistently nonconstant reference vector assumption in many works [2]. The attitude of the rigid body starts from the initial condition $R(0) = \text{diag}(-1, -1, 1)$ under the rotational velocity $\omega = [0.25 - 0.50, 15]^{T}$ (rad/s). We added measurement noise to both the angular velocity readings and the vector measurements, generated as a uniform random number with the amplitude ranges $[-0.02, 0.02]$ (rad/s) and $[-0.1, 0.1]$, respectively.

First, we evaluate the performance of the scheme in Proposition 2 using Matlab/Simulink with the solver ODE 4 (Runge–Kutta). The observer is initialized from $Q(0) = \hat{Q}_0(0) = I_3$, with the gains $\gamma_3 = 3$ and $\gamma_1 = 1$. It corresponds to the initial yaw, pitch, and roll estimates all being 0°. The results of simulations are shown in Fig. 1 in the form of Euler angles, and also see the norm of the estimation error $|\vec{R}|$ in Fig. 2, which is drawn in a logarithmic scale for the $y$-axis. During $[0, 5] s$, the error $\vec{R}$ is converging to some nonzero constant under a constant vector measurement. This is because a single vector output makes two of three Euler angles partially observable [11]. After 5 s, the model satisfies the distinguishability, and then, all Euler angles converge to their true values. Note that the proposed scheme is robust vis-à-vis measurement noise. Then, we test the second observer design in Proposition 3, with the simulation results presented in the same figure. The gain parameters were selected as $\gamma_3 = \gamma_c = 3, \gamma_1 = 1$, and $\gamma = 10$. Although the reference vector $g$ in (58) does not satisfy the sufficient condition (45), it is interesting to observe that all the Euler angles converge to zero asymptotically for the same reference vector $g$. This implies that the condition (45) is not necessary for
We consider the helicopter trajectory, as shown in Fig. 4. To make the simulation more realistic, the gyros and the accelerometer provide data at 100 Hz, and the GPS receiver is at 10 Hz—all with high-frequency noise. In order to evaluate robustness of the proposed design, we consider acceleration bias $[0.01, 0.01, 0.01] \, \text{m/s}^2$ from the sensor, but do not make any compensation. The first observer design was implemented at 1000 Hz using the solver “ODE 4 (Runge–Kutta)” in Matlab/Simulink, with $\gamma_1 = 1$, $\gamma_p = 5$, and $\alpha = 1$, and the initial conditions $Q(0) = \dot{Q}(0) = I_3$. The simulation results are given in Fig. 3 and 5, which illustrates its good robustness, although it brings additional errors in the steady-state stage. We compare it to the design in [8] using $H_1(p)$ with $\alpha = 8$ to approximate the differentiator. The phase lag from the filter leads to the offset in estimates observed in Fig. 3. This effect can be reduced by increasing $\alpha$, but at the expense of higher sensitivity to noise. As discussed previously, using (59) the proposed design does not suffer from this issue.

VII. CONCLUDING REMARKS

In this article, we studied the observability and observer design for the attitude estimation problem with vectorial measurements. By translating the attitude state observation problem into one of online parameter identification, we provided a necessary and sufficient condition to the distinguishability for the dynamical model on $SO(3)$, which is complementary to the existing necessary conditions in the literature. As is shown later, although the resulting distinguishability condition is quite weak, we are still able to use it to derive a continuous-time attitude observer with almost global asymptotic stability guaranteed for the single vector case. Finally, simulation results demonstrated accurate estimation performance in the presence of measurement noise. As future work, we will study the necessary condition to convergence in the second observer design.

APPENDIX A

PROOF OF COROLLARY 1

Proof: Compared with Proposition 1, the only difference relies on the second term, which corresponds to the third case in the proof. The modified condition assumes the existence of two indices $i \in \ell_1$ and $j \in \ell_2$ such that $|g_i(t_1) \cdot \Phi(0, t_2) b_j(t_2)| > 0$. Since $\Phi(0, t_2) b_j(t_2) \in \mathbb{S}^2$, we have

$$|g_i(t_1) \cdot [R_0 \Phi(0, t_2) b_j(t_2)]| > 0.$$  \hspace{1cm} (60)

if $R_0$ is not in an inadmissible initial set

$$E := \{ R_0 \in SO(3) \mid R_0 v = \pm w \}$$  \hspace{1cm} (61)

with $v := \Phi(0, t_2) b_j(t_2)$ and $w = g_i(t_1)$ both in $\mathbb{S}^2$. For a given rotational velocity $\omega$ and the references $b_j$ and $g_i$, two of three Euler angles of the initial rotation matrix $R_0$ are uniquely determined by the equality in (61). Hence, the inadmissible initial set $E$ has zero Lebesgue measure in the group $SO(3)$. As a result, we guarantee the condition (21) from the modified assumption almost surely. \hfill $\square$

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**Fig. 3.** Simulation results for helicopter attitude estimation using the observers in Proposition 2 and in [8] (Example 2).

**Fig. 4.** Trajectory of the helicopter (Example 2).

**Fig. 5.** Comparison of the norms of estimation errors $|\tilde{R}|_j$ among the proposed observer and the design in [8] (Example 2).
Our task is to show that both of these two cases imply (6).

In case (i), note that for a single vector $g_i(t) \in \mathbb{S}^2$, the matrix $g_i(t) g_i^\top(t)$ has rank one at any instance $t \geq 0$. Hence, a necessary condition to (62) is the existence of $t_1, t_2 \geq 0$ and $i, l \in \ell_1$ (i and $l$ may be the same) such that

$$\lambda_2 \left( \sum_{i \in \ell_1} \int_0^T g_i(s) g_i^\top(s) ds \right) > 0 \quad (62)$$

which implies $\sum_{i \in \ell_1} g_i(t_1) \cdot g_i(t_2) > 0$. Indeed, this is the first term in (6), thus guaranteeing the condition (6).

For the case (ii), we consider (piecewisely) smooth outputs $y_{1,j}$ with $j \in \ell_2$. Its dynamics is given by

$$\dot{y}_{1,j} = R(\omega, b_j + \dot{b}_j) \quad (65)$$

We will use the abovementioned dynamics and the condition (63) to show the variation of $y_{1,j}$ over time, with $y_{1,j} \in \mathbb{S}^2$. A necessary condition to (63) is that there exist $j$ and moment $t_1 > 0$ such that

$$\omega(t_1) \cdot b_j(t_1) + \dot{b}_j(t_1) \neq 0 \quad \Rightarrow \quad y_{1,j}(t_1) \neq 0 \quad (66)$$

Let us select a sufficiently small $\Delta t > 0$, and define $t_2 := t_1 + \Delta t$. It yields

$$y_{1,j}(t_2) = y_{1,j}(t_1) + y_{1,j}(t_1)\Delta t + o(\Delta t^2) \quad (67)$$

in which $o(\Delta t^2)$ represents the high-order remainder term, with the constraint $y_{1,j}(t_2) \in \mathbb{S}^2$. Now, we show $y_{1,j}(t_1) \cdot \frac{\partial}{\partial x} y_{1,j}(t_1) \neq 0$ by contradiction. If this cross product is equal to zero, invoking $\frac{\partial}{\partial x} y_{1,j}(t_1) \neq 0$ from (66), we have $y_{1,j}(t_2) = ay_{1,j}(t)$ for some nonzero $a \in \mathbb{R}$ independent of $\Delta t$. Then, we have

$$|y_{1,j}(t_2)| = \left| (1 + a\Delta t) y_{1,j}(t_1) + o(\Delta t^2) \right|$$

$$= |1 + a\Delta t| + o(\Delta t^2). \quad (68)$$

With a sufficiently small $\Delta t$, we have $1 + a\Delta t > 0$, then $|y_{1,j}(t_2)| = 1 + |a|\Delta t + o(\Delta t^2)$, which contradicts with the fact $y_{1,j}(t_2) \in \mathbb{S}^2$. As a consequence, we have verified $y_{1,j}(t_1) \cdot \frac{\partial}{\partial x} y_{1,j}(t_1) \neq 0$, and thus, obtain that $y_{1,j}(t_1) \cdot y_{1,j}(t_2) \neq 0$ for a sufficiently small $\Delta t > 0$. Invoking the equivalence (18), we have

$$(63) \quad \sum_{j,k \in \ell_2} |b_j(t_1) \cdot \Phi(t_1, t_2, b_k(t_2))| > 0 \quad (69)$$

thus verifying (6). It completes the proof.

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