A Novel Approach for Fast Detection of Multiple Change Points in Linear Models

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Abstract A change point problem occurs in many statistical applications. If there exist change points in a model, it is harmful to make a statistical analysis without any consideration of the existence of the change points and the results derived from such an analysis may be misleading. There are rich literatures on change point detection. Although many methods have been proposed for detecting multiple change points, using these methods to find multiple change points in a large sample seems not feasible. In this article, a connection between multiple change point detection and variable selection through a proper segmentation of data sequence is established, and a novel approach is proposed to tackle multiple change point detection problem via the following two key steps: (1) apply the recent advances in consistent variable selection methods such as SCAD, adaptive LASSO and MCP to detect change points; (2) employ a refine procedure to improve the accuracy of change point estimation. Five algorithms are hence proposed, which can detect change points with much less time and more accuracy compared to those in literature. In addition, an optimal segmentation algorithm based on residual sum of squares is given. Our simulation study shows that the proposed algorithms are computationally efficient with improved change point estimation accuracy. The new approach is readily generalized to detect multiple change points in other models such as generalized linear models and nonparametric models.

KEY WORDS: Adaptive LASSO; Asymptotic normality; Least squares; Linear model; MCP;
1. Introduction

The most popular statistical model used in practice is a linear model, which has been extensively studied in the literature. This model is simple and can be used to approximate a nonlinear function locally. However, there may be change points in a linear model such that the regression parameters may change at these points. Thus, if there do exist change points in a linear model, the linear model is actually a segmented linear model.

A change point problem occurs in many statistical applications in the areas including medical and health sciences, life science, meteorology, engineering, financial econometrics and risk management. To detect all change points are of great importance in statistical applications. If there exists a change point, it is harmful to make a statistical analysis without any consideration of the existence of this change point and the results derived from such an analysis may be misleading. There are rich literatures on change point detection, see, e.g., Csörgő and Horváth (1997) and Chen and Gupta (2000).

Compared with the detection of one change point, to locate all change points is a very challenge problem. Although, it has been studied in literature (see Davis, Lee, and Rodriguez-Yam (2006), Pan and Chen (2006), and Kim, Yu and Feuer (2009), and Loschi, Pontel and Cruz (2010) among others), a powerful and efficient method still needs to be explored. Thus this paper is mainly concerned with the multiple change point detection problem in linear regression.

Consider a linear model with \( K_0 \leq K_U < \infty \) multiple change points located at \( a_{1,n}^{(0)}, \ldots, a_{K_0,n}^{(0)} \):

\[
y_{i,n} = \sum_{j=1}^{q} x_{i,j,n} \beta_{j,0} + \sum_{\ell=1}^{K_0} \sum_{j=1}^{q} x_{i,j,n} \delta_{j,0}^{(\ell)} I(a_{\ell,n}^{(0)} < i \leq n) + \varepsilon_{i,n}
\]

\[
= x_{i,n}^T \left[ \beta_0 + \sum_{\ell=1}^{K_0} \delta_{\ell,0} I(a_{\ell,n}^{(0)} < i \leq n) \right] + \varepsilon_{i,n}, \quad i = 1, \ldots, n,
\]

(1)
where $\{x_{i,n} = (x_{i,1,n}, \ldots, x_{i,q,n})^T\}$ is a sequence of $q$-dimensional predictors, $\beta_0 = (\beta_{1,0}, \ldots, \beta_{q,0})^T \neq 0$ is unknown $q$-dimensional vector of regression coefficients, $K_0$ is unknown number of change points, $a_{1,n}^{(0)}$, $\ldots$, and $a_{K_0,n}^{(0)}$ are unknown change point locations (or change points), $\delta_{\ell,0}$, $1 \leq \ell \leq K_0$, denote unknown amounts of changes in regression coefficient vectors at change points, and $\varepsilon_{1,n}, \ldots, \varepsilon_{n,n}$ are random errors. In this paper, we assume that $K_U$ is an upper bound of $K_0$. Set $a_{K_0+1,n}^{(0)} = n$. If there is no change point, $K_0 = 0$ and the model (1) becomes

$$y_{i,n} = \sum_{j=1}^{q} x_{i,j,n} \beta_{j,0} + \varepsilon_{i,n}, \quad i = 1, \ldots, n.$$  

Otherwise, $K_0 \geq 1$, and we assume that

$$0 < a_{\ell,n}^{(0)}/n \to \tau_{\ell} < 1, \quad \text{for } 1 \leq \ell \leq K_0.$$  

If $K_0 \geq 2$, we assume that

$$\min_{1 \leq \ell \leq K_0-1} (\tau_{\ell+1} - \tau_{\ell}) > 0$$  

is unknown. The problem studied in this paper is to estimate $K_0$, $a_{1,n}^{(0)}$, $\ldots$, and $a_{K_0,n}^{(0)}$ or in other words to detect multiple change points. If there is no confusion, the superscript “(0)”, subscript “0”, and subscript $n$ will be suppressed.

For detecting multiple change points, it may be convenient to consider the following linear model with probable multiple change points located at $1 < a_{1,n} < \cdots < a_{K,n} < n$

$$y_i = x_i^T \left[ \beta + \sum_{\ell=1}^{K} \delta_{\ell} I(a_{\ell,n} < i \leq n) \right] + \varepsilon_i, \quad i = 1, \ldots, n,$$  

where $\beta$, $\delta_1$, $\ldots$, $\delta_K$ are unknown $q$-dimensional parameter vectors. We can instead test the following null hypothesis:

$$H_0: \quad \text{There is no change point, i.e., for any } 1 < a_{1,n} < \cdots < a_{K,n} < n,$$

$$\delta_{\ell} = (\delta_{1}^{(\ell)}, \ldots, \delta_{q}^{(\ell)})^T = 0 \quad \text{for any } \ell \in \{1, \ldots, K\}, \quad \text{where } 1 \leq K \leq K_U$$
versus the alternative hypothesis:

\[ H_1 : \quad \text{There exist } 1 \leq K \leq K_U \text{ change points, i.e., there exist } 1 < a_{1,n} < \cdots < a_{K,n} < n \]

such that \( \mathbf{\delta}_\ell = (\delta^{(\ell)}_1, \ldots, \delta^{(\ell)}_q)^T \neq \mathbf{0} \) for any \( \ell \in \{1, \ldots, K\} \).

Many classical methods have been given in literature for detecting change points, which include the popular model selection based change point detection method and the well known cumulative sum (CUSUM) method. However the amounts of computing time required by these two typical change point detection methods are respectively \( O(2^n) \) and \( O(n^2) \). When \( n \) is very large, using these methods to find multiple change points seems not feasible.

If the set of all true change points in the model (1) is a subset of \( \{a_{\ell,n}, \ 1 \leq \ell \leq K\} \), it is easy to see that \( a_{j,n} \) is a change point if and only if \( \delta_j \neq \mathbf{0} \). We rewrite (1) as follows:

\[ \mathbf{y}_n = \mathbf{X}_n \tilde{\mathbf{\beta}} + \mathbf{\varepsilon}_n, \quad (5) \]

where \( \mathbf{y} = (y_1, y_2, \ldots, y_n)^T, \tilde{\mathbf{\beta}} = (\mathbf{\beta}^T, \mathbf{\delta}_1^T, \ldots, \mathbf{\delta}_K^T)^T, \mathbf{\varepsilon}_n = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)^T, \) and

\[ \mathbf{X}_n = \begin{pmatrix} X_{(1,1)} & 0_{(1,1)} & 0_{(0,1)} & \cdots & 0_{(0,1)} \\ X_{(1,2)} & X_{(1,2)} & 0_{(1,2)} & \cdots & 0_{(1,2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ X_{(K,K+1)} & X_{(K,K+1)} & X_{(K,K+1)} & \cdots & X_{(K,K+1)} \end{pmatrix}_{n \times (K+1)q} \]

with \( 0_{(j-1,j)} \) is a zero matrix of dimension \((a_{j,n} - a_{j-1,n}) \times q\), and \( a_{0,n} = 0 \),

\[ \mathbf{X}_{(j-1,j)} = \begin{pmatrix} x_{a_{j-1,n}+1,1} & \cdots & x_{a_{j-1,n}+1,q} \\ \vdots & \ddots & \vdots \\ x_{a_{j,n},1} & \cdots & x_{a_{j,n},q} \end{pmatrix}_{(a_{j,n} - a_{j-1,n}) \times q} \quad \text{for } j = 1, \ldots, K + 1. \]

Thus to detect all the true change points and remove the pseudo change points in (1) can be considered as a variable selection problem for the linear regression model (5), and we may tackle the problem by employing variable selection methods. This leads us to explore a possibility by first properly segmenting data sequence and then applying variable selection methods and/or other methods for detecting probable multiple change points.
The paper is arranged as follows. The segmentation of data sequence and multiple change point estimation are discussed in Section 2. Five algorithms for detecting probable multiple change points are proposed in Section 3. Simulation studies and practical recommendations are given in Section 4. Two real data examples are provided in Section 5.

Throughout the rest of the paper, $1_q = (1, \ldots, 1)^T$ is the $q$-dimensional vector, $I_q$ is the $q \times q$ identity matrix, an indicator function is written as $I(\cdot)$, the transpose of a matrix $A$ is denoted by $A^T$, and $\lfloor c \rfloor$ is the integer part of a real number $c$. For a vector $a$, $a^T$ is its transpose, $a(j)$ is its $j$th component, $|a|$, $\|a\|$ and $\|a\|_{\infty}$ are respectively its $L_1$-norm, $L_2$-norm (Euclidean norm) and $L_\infty$ norm. If $\mathcal{A}$ is a set, its complement and its size are denoted by $\bar{\mathcal{A}}$ and $|\mathcal{A}|$, respectively. In addition, the notations $\rightarrow_p$ and $\rightarrow_d$ denote convergence in probability and convergence in distribution, respectively. Furthermore, the $(1 - \alpha)$th quantile of the chi-square distribution with $\ell$ degrees of freedom is denoted by $\chi^2_{\alpha,\ell}$.

2. Segmentation and Change Point Estimation

For a multiple change point detection problem, the multiple change point locations are unknown and in practice their approximate locations within a permissible range is main concern, which inspires us to partition the data sequence to search for change points. We thus divide the data sequence into $p_n + 1$ segments. Let $m = m_n = \lfloor n/(p_n + 1) \rfloor$. The segmentation is such that the first segment has length $0 < m \leq n - p_n m \leq c_0 m$ with some $c_0 \geq 1$ and each of the rest $p_n$ segments has length $m$. Without loss of generality, we assume that $p_n \rightarrow \infty$ as $n \rightarrow \infty$. The partition of the data sequence yields the following segmented regression model:

$$
y_i = x_i^T \left[ \beta + \sum_{\ell=1}^{p_n} d_\ell I(n - (p_n - \ell + 1)m < i \leq n) \right. $$
$$
+ \sum_{\ell=1}^{p_n} \omega_\ell(i) I(n - (p_n - \ell + 1)m < i \leq n - (p_n - \ell)m) \left. \right] + \varepsilon_i, \quad i = 1, \ldots, n, \quad (6)
$$

where two sets $\{d_1, \ldots, d_{p_n}\}$ and $\{0, \delta_1, \ldots, \delta_{K_0}\}$ are equal, and $\{\omega_\ell\}$ are defines as follows: if there is a change point located in $\{n - (p_n - \ell + 1)m + 1, \ldots, n - (p_n - \ell)m - 1\}$, say $a_{k,n}$,
then
\[ \omega_\ell(i) = \begin{cases} -\delta_k, & n - (p_n - \ell + 1)m < i \leq a_{k,n} < n - (p_n - \ell)m, \\ 0, & \text{otherwise;} \end{cases} \]

otherwise,
\[ \omega_\ell(i) = 0, \quad i = 1, \ldots, n. \]

The model (6) can be written as
\[ y_n = \tilde{X}_n \theta_n + X_\omega \sum_{\ell=1}^{p_n} \bar{\omega}_\ell + \varepsilon_n, \]  
(7)

where \( y_n \) and \( \varepsilon_n \) are defined in Section 1, \( \theta_n = (\theta_1, \ldots, \theta_{q(p_n+1)})^T = (\beta^T, d_1^T, \ldots, d_{p_n}^T)^T \),
\[ d_r = (d_{r1}, \ldots, d_{rq})^T, \quad r = 1, \ldots, p_n; \]
\[ \tilde{X}_n = \begin{pmatrix} X_{(1)} & 0_{m \times q} & 0_{m \times q} & \cdots & 0_{m \times q} \\ X_{(2)} & X_{(2)} & 0_{m \times q} & \cdots & 0_{m \times q} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ X_{(p_n+1)} & X_{(p_n+1)} & X_{(p_n+1)} & \cdots & X_{(p_n+1)} \end{pmatrix} = (X_{n(1)}, \ldots, X_{n(p_n+1)})^{(p_n+1)q} \]
with \( X_{n(j)} = (0_{q \times m}, \ldots, 0_{q \times m}, X_{(j)}, \ldots, X_{(p_n+1)})^T \),
\[ X_{(1)} = \begin{pmatrix} x_{1,1} & \cdots & x_{1,q} \\ \vdots & \vdots & \vdots \\ x_{n-p_n,m,1} & \cdots & x_{n-p_n,m,q} \end{pmatrix}_{(n-p_n,m) \times q}, \]
\[ X_{(j)} = \begin{pmatrix} x_{n-(p_n-j+2)m+1,1} & \cdots & x_{n-(p_n-j+2)m+1,q} \\ x_{n-(p_n-j+1)m,1} & \cdots & x_{n-(p_n-j+1)m,q} \end{pmatrix}_{m \times q}, \quad \text{for} \ j = 2, \ldots, p_n + 1, \]
\[ X_\omega = \text{diag} (x_1^T, \ldots, x_n^T), \quad \text{and} \quad \bar{\omega}_\ell = (\omega_\ell^T(1), \ldots, \omega_\ell^T(n))^T. \]

It is easy to see that \( x_\omega \equiv X_\omega \sum_{\ell=1}^{p_n} \bar{\omega}_\ell \) is an \( n \) dimensional vector and all its elements excluding at most \( K_0(m-1) \) of them are zeros.

It is noted that in Harchaoui and Levy-Leduc (2008), the mean-shift model is considered and the length of each of their segments is only 1.

Consider a special case that each true change point is at an end of a segment. Then an end of a segment is a true change point if and only if the corresponding \( d_r \neq 0 \). Thus to locate
all the true change points in (1) is equivalent to carry out variable selection. Since \( p_n \to \infty \), we may take advantage of the recent advances in consistent variable selection methods for a linear regression model as (7) with a large number of regression coefficients, which include the SCAD (Fan and Li (2001)), the adaptive LASSO (Zhou (2006)), and the MCP (Zhang (2010)) among others.

Let us examine the relationship between the models (1) and (7). It can be seen that under the null hypothesis \( H_0, \beta = \beta_0 \), and \( d_{r_k} = 0 \), \( k \in \{1, \cdots, p_n\} \). We now assume that \( H_1 \) hold. Thus, there exist \( \{r_k, k = 1, \cdots, K_0\} \) such that \( a_{k,n} \in \{n - p_n m + (r_k - 1)m, \cdots, n - p_n m + r_k m - 1\} \). Since \( K_0 \) is finite with an upper bound \( K_U \), in view of (2) and (3), it follows that

\[
\beta = \beta_0, \quad d_{r_k - 1} = 0, \quad d_{r_k} = \delta_k \neq 0, \quad \text{and} \quad d_{r_k + 1} = 0
\]

for large \( n \). Thus in order to detect all the change points \( \{a_{1,n}, \cdots, a_{K_0,n}\} \), we may estimate \( \{d_i\} \) in advance.

The following assumptions are made for investigating the asymptotic properties of the estimates of \( \{d_i\} \):

**Assumption C1.** \( \sum_{i=s}^t x_i x_i^T / (t - s) \to W > 0 \) as \( t - s \to \infty \).

It is noted that Assumption C1 is a common assumption made in change point analysis for a mean shift model. Under Assumption C1, it can be shown that \( X_{(1)}^T X_{(1)} / (n - p_n m) \to W > 0 \), and \( X_{(i)}^T X_{(i)} / m \to W > 0 \) for \( i \in \{2, \cdots, p_n + 1\} \).

**Remark 1.** Assumption C1 is similar to Condition (b) in Zhou (2006). If we only consider the consistency of change point estimators, Assumption C1 can be relaxed to the following weaker one: For \( b_1, b_2 > 0, b_1 I_q \leq \sum_{i=s}^t x_i x_i^T / (t - s) \leq b_2 I_q \) when \( t - s \) is large enough.

**Assumption C2.** \( \{\varepsilon_i, \quad i = 1, 2, \ldots\} \) is a sequence of independently and identically distributed (i.i.d.) random variables with mean 0 and variance \( \sigma^2 \).

**Remark 2.** This assumption can be replaced by a weaker assumption of the strong mixing condition in (2.1) in Kuelbs and Philipp (1980), which adapts to the autoregressive models in
Davis, Huang and Yao (1995) and Wang, Li and Tsai (2007). Let \( \{ \varepsilon_i, i = 1, 2, \ldots \} \) be a weak sense stationary sequence of random variables with mean 0 and \((2+\delta)\)th moments for \( 0 < \delta \leq 1 \) that are uniformly bounded by some positive constant. Suppose that \( \{ \varepsilon_i, i = 1, 2, \ldots \} \) satisfies the strong mixing condition \(|P(AB) - P(A)P(B)| \leq \rho(n) \downarrow 0 \) for all \( n, s \geq 1 \), all \( A \in \mathcal{M}_1^s \) and \( B \in \mathcal{M}_s^\infty \), where \( \mathcal{M}_a^b \) is the \( \sigma \)-field generated by the random vectors \( \varepsilon_a, \varepsilon_{a+1}, \ldots, \varepsilon_b \), and \( \rho(n) \ll n^{-(1+t)(1+2/\delta)} \) for some \( t > 0 \). Then Theorem 4 and Lemma 3.4 in Kuelbs and Philipp (1980) warrant the same results as given in Theorems 1-3 below.

For simple presentation below, we assume that each of \( \{ X_r \} \) is of full rank in this paper. If a \( X_r \) is not of full rank, Moore-Penrose matrix inverse can be used instead of the matrix inverse.

2.1. Estimate \( \{ d_i \} \) by least squares

By least squares method, we estimate \( d_r, r = 1, \ldots, p_n \), as follows:

\[
\hat{d}_r = \left( X_{(r+1)}^T X_{(r+1)} \right)^{-1} X_{(r+1)}^T y^{(r+1)} - \left( X_r^T X_r \right)^{-1} X_r^T y^{(r)}, \quad r = 1, \ldots, p_n, \tag{10}
\]

where \( y^{(1)} = (y_1, \ldots, y_{n-p_n m})^T \), and \( y^{(r)} = (y_{n-(p_n-r+2) m+1}, \ldots, y_{n-(p_n-r+1) m})^T, \quad r = 2, \ldots, p_n + 1 \). It is easy to see that

\[
\hat{d}_r + \hat{d}_{r+1} = \left( X_{(r+2)}^T X_{(r+2)} \right)^{-1} X_{(r+2)}^T y^{(r+2)} - \left( X_{(r)}^T X_{(r)} \right)^{-1} X_{(r)}^T y^{(r)}.
\]

It is obvious that under \( H_0 \), for any \( \ell \in \{ 1, \ldots, p_n \} \) and any \( i \in \{ n - p_n m + 1, \ldots, n \} \),

\[
\omega_\ell(i) = 0 \quad \text{and} \quad d_\ell = 0.
\]

We have the following theorem.

**Theorem 1.** Assume that \( m \to \infty \) as \( n \to \infty \). If \( H_0 \) holds, under the assumptions C1-C2, it follows that

\[
\sqrt{m} \hat{d}_i \to_d N \left( 0, 2 \sigma^2 W^{-1} \right), \quad i = 1, \ldots, p_n.
\]
We now assume that $H_1$ holds. In view of (9), it follows that $d_{r_k} + d_{r_k+1} = \delta_k$. By the definition of $\{\omega_\ell(i)\}$, we have
\[
\sum_{\ell=1}^{p_n} \omega_\ell(i) I(n - (p_n - \ell + 1)m < i \leq n - (p_n - \ell)m) = \begin{cases} 
-\delta_k, & \text{if } \exists r_k \text{ such that } n - (p_n - r_k + 1)m < a_{k,n} < n - (p_n - r_k)m, \\
0, & \text{otherwise}. 
\end{cases} 
\]
(11)

It can also be verified that
\[
\sum_{\ell=1}^{p_n} d_\ell I(n - (p_n - \ell + 1)m < i \leq n) = \begin{cases} 
\sum_{\ell=1}^{r_k-1} d_\ell, & \text{if } n - (p_n - r_k + 2)m < i \leq n - (p_n - r_k + 1)m, \\
\sum_{\ell=1}^{r_k+1} d_\ell, & \text{if } n - (p_n - r_k)m < i \leq n - (p_n - r_k - 1)m. 
\end{cases} 
\]
(12)

Thus, we have the following theorem:

**Theorem 2.** If Assumptions C1-C2 hold, then under $H_1$,
\[
\sqrt{m} \left( \hat{d}_{r_k} + \hat{d}_{r_k+1} - \delta_k \right) \rightarrow_d N \left( 0, 2\sigma^2 W^{-1} \right), \quad k = 1, \ldots, K_0.
\]

The proofs of Theorems 1-2 follow from the least squares theory. The details are omitted.

2.2. Estimate $\{d_i\}$ by recent advances in consistent variable selection methods

2.2.1. Estimate $\{d_i\}$ by the adaptive LASSO

The adaptive LASSO, extending the LASSO in Tibshirani (1996), was proposed in Zhou (2006) and possesses oracle properties for fixed number of regression coefficients.

In light of Zhou (2006), the adaptive LASSO type estimator of $\theta_n$ for the model (7) is defined by
\[
\hat{\theta}_n = \arg\min_{\theta_n} \left\{ ||y - X_n \theta_n||^2 + \lambda_n \sum_{r=1}^{p_n} \frac{1}{|d_r|^{\nu}} |d_r| \right\},
\]
(13)
where $\nu > 0, \lambda_n$ is a thresholding parameter and $\tilde{d}_r \{r = 1, \cdots, p_n\}$ are initial estimators satisfying certain conditions.

**Remark 3.** The adaptive LASSO estimate of $\theta_n$ may also be defined by

$$
\tilde{\theta}_n = \arg\min_{\theta_n} ||y - X_n\theta_n||^2 + \lambda_n \sum_{r=1}^{p_n} \sum_{i=1}^{q} |d_{ri}|^\nu + \gamma_n \sum_{i=1}^{q} \frac{1}{|\beta_{0i}|^{\nu}} |\beta_{0i}|,
$$

where $\mu > 0$, $\lambda_n$ and $\gamma_n$ are thresholding parameters satisfying certain conditions. The difference between (13) and (14) is that the variable selection in addition to the multiple change point detection is also considered in (14). Due to the similarity in the techniques for finding the asymptotic behavior of both $\tilde{\theta}_n$ and $\hat{\theta}_n$, we only consider $\tilde{\theta}_n$ in this paper for simple presentation.

Since the dimension of $\theta_n$ increases with $n$ in (7), the asymptotic results in Zhou (2006) are not applicable here. In the following we will investigate the limiting behavior of those $d_i$s associated with change points under the condition that $K_0 \geq 1$, i.e., there exists at least one change point in the model (1). As stated before, the subscript $n$ may be suppressed for convenience if there is no confusion.

Before we proceed, we define some notations as follows: Let $B = \{\kappa_1, \kappa_2, \ldots, \kappa_\iota\} \subset \{2, \ldots, p_n + 1\}$ such that $\kappa_1 < \ldots < \kappa_\iota$. Denote $\theta_B = (d_{\kappa_1}^T, \cdots, d_{\kappa_\iota}^T)^T, X_B = (X_{n(\kappa_1)}, \ldots, X_{n(\kappa_\iota)})$, where $\{X_{n(i)}\}$ are given in (8).

Recall that for each $\delta_k$ in (1), there exists $r_k$ such that $d_{r_k} = \delta_k$, or equivalently there exists a change point within $\{n - (p_n - r_k + 1)m, \ldots, n - (p_n - r_k)m - 1\}$ for $k = 1, \ldots, K_0$.

Define

$$
A_c = \{i: d_{i-1} = 0, d_i \neq 0, d_{i+1} = 0\}, \quad A_1 = \{i: d_{i-1} \neq 0, d_i = 0, d_{i+1} = 0\},
$$

$$
A_2 = \{i: d_{i-1} = 0, d_i = 0, d_{i+1} \neq 0\}, \quad A_3 = \{i: d_{i-1} = 0, d_i = 0, d_{i+1} = 0\}.
$$

It is easy to see that for large $n$, $\bar{A}_c = A_1 \cup A_2 \cup A_3$. 

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In view of Zhou (2006) and Huang, Ma and Zhang (2008), we need to make some assumptions on the initial estimators \( \{ \tilde{d}_i \} \) used in (13) for investigating the asymptotic properties of \( \tilde{\theta}_n \). By the remark 1 of Zhou (2006), one might assume that for any \( i \), there is a sequence of \( \{ a_n \} \) such that \( a_n \to \infty \) and \( a_n(\tilde{d}_i - d_i) = O_p(1) \). But \( p_n \) is fixed in Zhou (2006). Huang, Ma and Zhang (2008) allows \( p_n \to \infty \) as \( n \to \infty \). Thus a stronger assumption like that \( r_n \max_i |\tilde{d}_i - d_i| = O_p(1) \) as \( r_n \to \infty \) (see (A2) of Huang, Ma and Zhang 2008) might be made. However such assumptions may not be enough for the multiple change point detection problem. A careful study shows that we need put some lower bound on \( |\tilde{d}_i| \) for \( i \in A_c \) such that they are not close to 0. Hence we make the following assumption on \( \{ \tilde{d}_r \} \):

**Assumption C3.** There exists a constant \( a > 0 \) such that for large \( n \),

\[
|\tilde{d}_i| \begin{cases} 
\geq a > 0, & \text{for } i \in A_c, \\
= O_p\left(1/\sqrt{m}\right), & \text{for } i \notin A_c.
\end{cases}
\]

To obtain \( \{ \tilde{d}_r \} \) in practice, we can estimate the set \( A_c \) first, which, for example, may be estimated by the least squares based multiple change point detection algorithm given in Subsection 3.1. After we obtain the estimate \( \hat{A}_c \) of \( A_c \), we can set \( \tilde{d}_i = c \) for \( i \in \hat{A}_c \), and \( 1_q/\sqrt{m} \) otherwise.

To study the asymptotic behavior of \( \hat{\theta} \), the following three Lemmas are necessary.

**Lemma 1.** Under Assumption C1, there exists positive definite matrix \( W_{A_c} \) (defined in (A.4) in the appendix) such that \( X_{A_c}^T X_{A_c} / n \to W_{A_c} \).

**Remark 4.** One can not replace \( X_{A_c}^T X_{A_c} \) by \( \tilde{X}_n^T \tilde{X}_n \) above since the minimum eigenvalue may converge to 0 in consideration of the fact that \( p_n \to \infty \) (see Condition (b) in Zou (2006) and (2.13) in Zhang and Huang (2008)). Thus if they allow \( p_n \to \infty \), their conditions no longer hold and may be strengthened as Assumption C1.

**Lemma 2.** Under Assumption C1, for large \( n \) elements of \( \tilde{X}_n^T x_\omega / m \) are uniformly bounded.

**Lemma 3.** Under Assumptions C1-C2, for large \( n \) elements of \( \tilde{X}_n^T \varepsilon_n / \sqrt{n} \) is uniformly bounded in probability.
If there exists at least one change point, i.e., $K_0 \geq 1$, the limiting behavior of the adaptive LASSO estimator $\tilde{\theta}_n$ is given in the following theorem.

**Theorem 3.** Assume that $\lambda_n/\sqrt{n} \to 0$, $m/\sqrt{n} \to 0$ and $\lambda_n(n/p_n)^{\nu/2}/\sqrt{n} \to \infty$ for $\nu > 0$ as $n \to \infty$. If Assumptions C1-C3 hold, then

$$\sqrt{n}(\tilde{\theta}_{Ac} - (\theta_n)_{Ac}) \to_d N(0, \sigma^2 W_{Ac}^{-1}).$$

**Remark 5.** If we replace the weight $1/|x|^{\nu}$ by $\exp(-1/|x|)$ in (13), the condition $\lambda_n(n/p_n)^{\nu/2}/\sqrt{n} \to \infty$ can be relaxed to the weaker condition: $\lambda_n \exp\left(\sqrt{n/p_n}\right)/\sqrt{n} \to \infty$. Although it may result in an absorbing state in $x = 0$ (see Fan and Lv (2008)), it has not occurred in simulations.

**Remark 6.** By (13), $\tilde{\theta}$ is a unique solution of a convex optimization problem and hence the Karush-Kuhn-Tucker condition holds. For any vector $b = (b_1, \ldots, b_p)^T$, denote its sign vector by $\text{sgn}(b) = (\text{sgn}(b_1), \ldots, \text{sgn}(b_p))^T$, with the convention $\text{sgn}(0) = 0$. As in Zhao and Yu (2006), we say that $\tilde{\theta}_n = s\theta$ if and only if $\text{sgn}(\tilde{\theta}_n) = \text{sgn}(\theta)$. If the condition $p_n/n^{\nu/(2+\nu)} = o(1)$ is further assumed to hold, by Lemma 1-3 and Theorem 3, it can be shown that

$$P(\tilde{\theta}_n = s\theta) \to 1, \quad \text{as } n \to \infty.$$

The proof is similar to the proof of Theorem 1 in Huang, Ma and Zhang (2008) and hence omitted.

### 2.2.2. Estimate $\{d_i\}$ by the SCAD or MCP

SCAD (Fan and Li (2001)) and MCP (Zhang (2010)) are two popular recent consistent variable selection methods. They can also be employed to solve the multiple change point detection problem.

Consider the following estimator of $\theta_n$:

$$\hat{\theta}^\nu = \arg\min_{\theta} \left\{ ||y - X_n\theta||^2 + n \sum_{r=1}^{p_n} p_{\lambda, \gamma}(|d_r|) \right\},$$

12
where \( p_{\lambda,\gamma} \) is the penalty function with tuning parameters \( \lambda > 0 \) and \( \gamma > 0 \). If

\[
p_{\lambda,\gamma}(x) = \begin{cases} \lambda x, & \text{if } x \leq \lambda, \\ \frac{\gamma \lambda x - 0.5(x^2 + \lambda^2)}{\gamma - 1}, & \text{if } \lambda < x \leq \gamma \lambda, \\ \frac{\lambda^2(\gamma + 1)}{2}, & \text{if } x > \gamma \lambda, \end{cases}
\]

(15)

the SCAD penalty function proposed by Fan and Li (2001), \( \hat{\theta}^p \) is the SCAD type estimator of \( \theta_n \). Denote it by \( \hat{\theta}^{scad} \). Instead, let

\[
p_{\lambda,\gamma}(x) = \begin{cases} \lambda x - \frac{x^2}{2\gamma}, & \text{if } x \leq \gamma \lambda, \\ \frac{1}{2} \gamma \lambda^2, & \text{if } x > \gamma \lambda, \end{cases}
\]

(16)

the MCP penalty function proposed by Zhang (2010), \( \hat{\theta}^p \) becomes the MCP type estimator of \( \theta_n \). Denote it by \( \hat{\theta}^{mcp} \).

Under certain conditions, the asymptotic properties of both \( \hat{\theta}^{scad} \) and \( \hat{\theta}^{mcp} \) are similar to the asymptotic properties of \( \hat{\theta} \). Since the emphasis of this paper is on the algorithms for detecting multiple change points, their asymptotic properties will not be discussed here.

3. Multiple change points detection algorithms

For a given \( p_n \) or \( m \), we divide the data sequence into \( p_n + 1 \) segments such that the first segment has the length between \( m \) and \( c_0 m \) with \( c_0 \geq 1 \) and the rest \( p_n \) segments are all of length \( m \), and we have the model (6). Define

\[
\hat{\sigma}_n^2 = \sum_{\ell=1}^{n-p_n m} (y_\ell - x_\ell^T \hat{\beta})^2 / (n - p_n m - q)
\]

(17)

with \( \hat{\beta} = (X_{(1)}^T X_{(1)})^{-1} X_{(1)}^T y_{(1)} \). Given a significance level \( \alpha \), five multiple change point detection algorithms are proposed in this section.

3.1 Least squares based multiple change point detection algorithm

In light of Theorems 1-2, the least squares based multiple change point detection algorithm is given as follows:
Least squares based multiple change points detection algorithm (LSMCPDA):

**Step 1.** Set \( i = 1, j = 1 \) and \( \hat{K} = 0 \).

**Step 2.** If \( i \geq p_n - 3 \), go to Step 3. Otherwise, we test the hypothesis \( H_{0,i} : \mathbf{d}_i = \mathbf{0} \) by checking if

\[
\hat{d}_i^T X_{(i+1)}^T X_{(i+1)} \hat{d}_i / (2q\hat{\sigma}_n^2) \geq \chi^2_{\alpha,q},
\]

where \( \hat{d}_i \) is given in (10). If the test is significant, set \( i = i + 1 \) and repeat Step 2, otherwise we test the hypothesis \( H_{0,(i+1),(i+2)} : \mathbf{d}_{i+1} + \mathbf{d}_{i+2} = \mathbf{0} \) by checking if

\[
\left( \mathbf{d}_{i+1} + \hat{d}_{i+2} \right)^T X_{(i+1)}^T X_{(i+1)} \left( \mathbf{d}_{i+1} + \hat{d}_{i+2} \right) / (2q\hat{\sigma}_n^2) \geq \chi^2_{\alpha,2q}.
\]

If the test is not significant, set \( i = i + 1 \) and repeat Step 2, otherwise, a change point estimate is \( n - p_n m + i m \). Set \( \hat{r}_j = n - p_n m + i m, j = j + 1, i = i + 2, \) and \( \hat{K} = \hat{K} + 1 \). Then repeat Step 2.

**Step 3.** If \( \hat{K} = 0 \), then go to the next step. Otherwise, we use the CUSUM to improve the accuracy of the multiple change point detection as follows: We search for the change points within the \( \hat{K} \) sets: \( \{ n - p_n m + (\hat{r}_j - 1)m, \ldots, n - p_n m + (\hat{r}_j + 1)m \}, j = 1, \ldots, \hat{K} \} \) by the CUSUM. An estimate of the change point within the \( j \)th set is given by

\[
\hat{a}_{j,n} = \arg\max_{\ell} \left[ \min_{\beta} \sum_{j=n-p_n m+(\hat{r}_j-1)m}^{\ell} (y_j - x_j^T \beta)^2 + \min_{\beta} \sum_{j=\ell+1}^{n-p_n m+(\hat{r}_j+1)m} (y_j - x_j^T \beta)^2 \right].
\]

**Step 4.** If \( \hat{K} = 0 \), there is no change points. Otherwise, there are \( \hat{K} \) change points and they are \( \hat{a}_{1,n}, \ldots, \hat{a}_{\hat{K},n} \).

If in the algorithm above, the chi-square tests in Step 2 are replaced by the CUSUM tests (see Appendix A.1) and Step 3 is replaced by Steps 3-5 of the SMCPDA with \( \{\hat{r}_{\text{scad}}\}, \{\hat{d}_{\text{scad}}\}, \hat{K}_{\text{scad}} \) and \( \{\hat{a}_{\text{scad}}\} \) replaced by \( \{\hat{r}_j\}, \{\hat{d}_j\}, \hat{K}, \) and \( \{\hat{a}_{j,n}\} \) respectively, the new algorithm is named as CLSMCPDA, where “C” is the first letter of “CUSUM”.

3.2 Adaptive LASSO based multiple change points detection algorithm
In light of Theorems 3, the adaptive LASSO based multiple change point detection algorithm is given as follows:

Adaptive Lasso based multiple change points detection algorithm (ALMCPDA):

**Step 1.** Set \( i = 1 \), \( j = 1 \) and \( \hat{K} = 0 \). Execute the algorithm LSMCPDA and obtain \( \hat{K} \). If \( \hat{K} > 0 \), we also obtain \( \hat{a}_{1,n}, \ldots, \hat{a}_{K,n} \).

**Step 2.** If \( \hat{K} = 0 \), set \( \tilde{d}_{1} = \cdots = \tilde{d}_{p_n} = 1_q / \sqrt{m} \), otherwise, set

\[
\tilde{d}_{\ell} = \begin{cases} 
\ell \in \{ r_k, r_km < \hat{a}_{k,n} - n + p_n m \leq (r_k + 1)m \}, \\
1_q / \sqrt{m}, & \text{elsewhere;}
\end{cases}
\]

where \( r_k \) is an integer such that \( r_km < \hat{a}_{k,n} - n + p_n m \leq (r_k + 1)m \) and \( c \) is a prechosen constant. Select \( \lambda > 0 \) and \( \nu > 0 \). Find the adaptive LASSO estimate \( \tilde{\theta} \) of \( \theta \) via

\[
\tilde{\theta} = \arg \min_{\theta} \left\{ ||y - X_n \theta||^2 + \lambda \sum_{r=1}^{p_n} \frac{1}{|d_r|^\nu} |d_r| \right\},
\]

and we obtain \( \tilde{d}_{\ell} \) for \( 1 \leq \ell \leq p_n \).

**Step 3.** We compute \( z_{\ell} = ||\tilde{d}_\ell||_\infty \) for \( 1 \leq \ell \leq p_n \). If \( z_1 = z_2 = \cdots = z_{p_n} = 0 \), go to Step 5. Otherwise, we treat \( \{z_{\ell}\} \) as random variables from the model \( z = \mu + \epsilon \) with \( \mu = (\mu_1, \ldots, \mu_{p_n})^T \) and \( \epsilon \sim N(0, I_{p_n}) \). Use LASSO, SCAD or MCP among other recent advances in variable selection to perform variable selection based on \( \{z_{\ell}\} \). We obtain the estimates \( \{\tilde{\mu}_{\ell}\} \). If \( \tilde{\mu}_{\ell} \), \( 1 \leq \ell \leq p_n \), are all zeros, set \( \hat{K} = 0 \) and go to Step 6. Otherwise, let \( \mathcal{I} \) be the subset of \( \{1, \ldots, p_n\} \) such that \( \ell \in \mathcal{I} \) if and only if \( \tilde{\mu}_{\ell} \neq 0 \). Write \( \mathcal{I} = \{s_1, \ldots, s_{|\mathcal{I}|}\} \) such that \( s_1 < \ldots < s_{|\mathcal{I}|} \).

**Step 4.** If \( i > |\mathcal{I}| \), go to Step 5. Otherwise, we test the hypothesis \( H_{0,s_i} : d_{s_i} = 0 \) by checking if

\[
(p_n - s_i)\tilde{d}^T_{s_i}X^T_{(s_i+1)}X_{(s_i+1)}\tilde{d}_{s_i} / (q\hat{s}_n^2) \geq \chi^2_{\alpha,q},
\]

where \( \hat{s}_n^2 \) is given in (17). If the test is not significant, set \( i = i + 1 \) and repeat Step 4.
Otherwise, a change point estimate is \( n - p_n m + (s_i - 1)m \). Set \( \tilde{r}_j = n - p_n m + (s_i - 1)m \), \( j = j + 1 \), \( i = i + 2 \), and \( \tilde{K} = \tilde{K} + 1 \). Then repeat Step 4.

**Step 5.** If \( \tilde{K} = 0 \), then go to the next step. Otherwise, we use the CUSUM to improve the accuracy of the multiple change point detection as follows: We search for the change points within the \( \tilde{K} \) sets: \( \{ n - p_n m + \tilde{r}_1 m, \ldots, n - p_n m + \tilde{r}_j m \} \), \( j = 1, \ldots, \tilde{K} \) by the CUSUM. An estimate of the change point for the \( j \)th set is given by

\[
\hat{a}_{j,n} = \arg \max_{\ell} \left[ \min_{\beta} \sum_{j=n-p_n m + (\tilde{r}_j - 1)m}^{\ell} (y_j - x_j^T \beta)^2 \right. + \min_{\beta} \left. \sum_{j=\ell+1}^{n-p_n m + (\tilde{r}_j + 1)m} (y_j - x_j^T \beta)^2 \right].
\]

**Step 6.** If \( \tilde{K} = 0 \), there is no change points. Otherwise, there are \( \tilde{K} \) change points and they are \( \hat{a}_{1,n}, \ldots, \hat{a}_{\tilde{K},n} \).

If the algorithm above, the chi-square test is replaced by the CUSUM test in Step 4, the new algorithm is named as CALMCPDA, where “C” is also the first letter of “CUSUM”. Denote all the estimates based on CALMCPDA by adding a superscript “C” to the corresponding estimates based on ALMCPDA. For example, the estimate of \( K_0 \) based on CALMCPDA is denoted by \( \hat{K}_0^C \).

### 3.3 SCAD based multiple change points detection algorithm

Similar to the ALMCPDA, the SCAD based multiple change point detection algorithm is given as follows:

**SCAD based multiple change points detection algorithm (SMCPDA):**

**Step 1.** Set \( i = 1, j = 1 \) and \( \hat{K}^{\text{scad}} = 0 \).

**Step 2.** Select \( \lambda > 0 \) and \( \gamma > 0 \). Find the SCAD estimate \( \hat{\theta}^{\text{scad}} = \left( \left( \hat{\beta}^{\text{scad}} \right)^T, \left( \hat{d}_1^{\text{scad}} \right)^T, \ldots, \right)^T \) of \( \theta \) via

\[
\hat{\theta}^{\text{scad}} = \arg \min_{\theta} \left\{ \| y - X_n \theta \|^2 + n \sum_{r=1}^{p_n} p_{\lambda,\gamma}(\| d_r \|) \right\},
\]

16
where \( p_{\lambda, \gamma} \) is given in (15) and we obtain \( \hat{d}_{\ell}^{scad} \) for \( 1 \leq \ell \leq p_n \).

**Step 3.** It is same as Step 3 of ALMCPDA with \( z_\ell = \|d_\ell\|_\infty \) replaced by \( \hat{d}_{\ell}^{scad} \) and \( \bar{K} = 0 \) is replaced by \( \hat{K}^{scad} \).

**Step 4.** If \( i > |I| \), go to Step 5. Otherwise, we test the hypothesis \( H_{0,s_i} : d_{s_i} = 0 \) by CUSUM. If the test is not significant, set \( i = i + 1 \) and repeat Step 4. Otherwise, a change point estimate is \( n - p_n m + (s_i - 1)m \). Set \( \hat{r}_{j}^{scad} = n - p_n m + (s_i - 1)m, j = j + 1, i = i + 2, \) and \( \hat{K}^{scad} = \hat{K}^{scad} + 1 \). Then repeat Step 4.

**Step 5.** If \( \hat{K}^{scad} = 0 \), then go to the next step. Otherwise, we use the CUSUM to improve the accuracy of the multiple change point detection as follows: We search for the change points within the \( \hat{K}^{scad} \) sets: \( \{n - p_n m + (\hat{r}_j^{scad} - 1)m, \ldots, n - p_n m + (\hat{r}_j^{scad} + 1)m \}, j = 1, \ldots, \hat{K}^{scad} \} \) by the CUSUM. An estimate of the change point for the \( j \)th set is given by

\[
\hat{a}_{j,n}^{scad} = \arg \max_{\ell} \left[ \min_{\beta} \sum_{j=n-p_n m+(\hat{r}_j^{scad}-1)m}^{\ell} (y_j - x_j^T \beta)^2 + \min_{\beta} \sum_{j=\ell+1}^{n-p_n m+(\hat{r}_j^{scad}+1)m} (y_j - x_j^T \beta)^2 \right].
\]

**Step 6.** If \( \hat{K}^{scad} = 0 \), there is no change points. Otherwise, there are \( \hat{K}^{scad} \) change points and they are \( \hat{a}_{1,n}^{scad}, \ldots, \hat{a}_{\hat{K}^{scad},n}^{scad} \).

### 3.4 MCP based multiple change points detection algorithm

The differences between the SMCPDA and the MCP based multiple change point detection algorithm (MMCPDA) are as follows:

1. The superscript “\( scad \)” in the SMCPDA is replaced by the superscript “\( mcp \)” in the MMCPDA.

2. The step 2 in the SMCPDA is modified to the following step 2 in the MMCPDA:

   **Step 2.** Select \( \lambda > 0 \) and \( \gamma > 0 \). Find the MCP estimate \( \hat{\theta}^{mcp} = \left( (\hat{\beta}^{mcp})^T, (\hat{d}_1^{mcp})^T \right) \),
. . ., $\left(\hat{d}_{n}^{mcp}\right)^{T}$ of $\theta$ via

$$
\hat{\theta}^{mcp} = \arg \min_{\theta} \left\{ \|y - X_{n}\theta\|^2 + n \sum_{r=1}^{p_{n}} p_{\lambda, \gamma}(|d_{r}|) \right\},
$$

where $p_{\lambda, \gamma}$ is given in (16).

Remark 7. The use of CUSUM in these algorithms is for improving the change point estimation accuracy. The amounts of computing time required by these algorithms are all $O(n) + O(m)$, where $O(m)$ corresponds to the time required for using CUSUM method. If a segmentation satisfies that $m = o(n)$, $O(n) + O(m) = O(n)$, which is computationally more efficient than the existing multiple change point detection methods in literature.

4. Simulation study

In this section, we present simulation studies of multiple change point analysis. Since the time for finding the multiple change points in a large sample by the algorithms proposed in Section 3 is significantly reduced compared to the existing multiple change point detection methods in the literature, such comparison studies are omitted in this section. We will only compare the number of times of selecting the true number of change points and the accuracy of change point estimation by the algorithms proposed in Section 3 based on 1000 simulation. A Dell server (two E5520 Xeon Processors, two 2.26GHz 8M Caches, 16GB Memory) is used in the simulation.

It is noted that the LARS algorithm (Efron, Hastie, Johnstone, and Tibshirani 2004) is used to compute $\tilde{\theta}_{n}$ defined in (13) with $\nu = 1$ and an optimal $\lambda_{n}$ selected by the BIC. For applying LARS, the added penalty on $\beta$ is set as $1/|1_{q}|$, which will not affect the multiple change-point detection results as $\beta \neq 0$. The PLUS algorithm (Zhang, 2010) with the added penalty $np_{\lambda, \gamma}(|\beta|)$ on $\beta$ is used to compute $\hat{\theta}_{n}^{scad}$ defined in (15) or $\hat{\theta}_{n}^{mcp}$ defined in (16), which also do not affect the multiple change point detection results as $\beta \neq 0$. Let $\sigma_{n}^{2}$ be given in
We use $\lambda = \hat{\sigma}_n \sqrt{2 \log p_n / n}$ in the PLUS algorithm as suggested in Zhang (2010). In all of our numerical examples, we set $\gamma = 3.7$ for SCAD by following the recommendation of Fan and Li (2001), but set $\gamma = 2.4$ for MCP based on some preliminary simulation studies. It is noted that in the step 3 of the algorithms ALMCPDA, CALMCPDA, SMCPDA, and MMCPDA, we use SCAD to perform variable selection for model $z = \mu + \epsilon$ by applying the PLUS algorithm with $\lambda = 0.02$. To use such small $\lambda$ is for avoiding the possibility of overestimation of the number of multiple change points.

Throughout this section, $\alpha = 0.05$.

4.1. The case that there is no change point in the data sequence of size 5000

In this subsection, we consider the case that there is no change point in the data sequence. We will examine the performance of the proposed algorithms to see if they do claim that there is no change point.

Consider the following linear model

$$y_i = x_i^T \beta_0 + \epsilon_i, \quad i = 1, \ldots, n,$$

where $\beta_0$ is a $q \times 1$ parameter vector. Set $n = 5000$, $q = 3$, $\beta_0 = (1, 1.4, 0.7)^T$, and $x_{i,1,n} = 1$ for $i = 1, \ldots, 5000$. Generate $\epsilon_i$, $i = 1, \ldots, 5000$, such that they are i.i.d. $N(0,1)$ distributed, and generate two sequences $x_{i,2,n}$, $1, \ldots, n$, and $x_{i,3,n}$, $1, \ldots, 5000$, such that they are i.i.d. $N(1,2)$ distributed. For demonstration, a sample scatter plot of simulated data is given in Figure 1.

We compare the following five algorithms: LSMCPDA, both ALMCPDA and CALMCPDA with $c = 1$, SMCPDA and MMCPDA. Recall that all the tests used in the algorithms CALMCPDA, SMCPDA, and MMCPDA are based on CUSUM. The number of correct detection and average computation time in second based on 1000 simulations are given Table 1.
From Table 1, it can be seen that all algorithms perform very well. The average detection time required by CALMCPDA for a sample of size 5000 is more than other proposed algorithms but only 6.78 seconds.

4.2. The case that there are nine change points in the data sequence of size 5000

In this subsection, we consider a case that there are nine change points in the data sequence of size 5000. We will examine the performance of the proposed algorithms via the rate for correctly estimating the number of change points and the accuracy of change point estimation. The average computation time for multiple change point detection is also given for each algorithm.
Consider the model (1), i.e.,

\[ y_{i,n} = \sum_{j=1}^{q} x_{i,j,n} \beta_{j,0} + \sum_{\ell=1}^{K_0} \sum_{j=1}^{q} x_{i,j,n} \delta_{j,0}^{(\ell)} I(a_{\ell,n}^{(0)} < i \leq n) + \varepsilon_{i,n} \]

\[ = x_{i,n}^T \left[ \beta_0 + \sum_{\ell=1}^{K_0} \delta_{\ell,0} I(a_{\ell,n}^{(0)} < i \leq n) \right] + \varepsilon_{i,n}, \quad i = 1, \ldots, n. \]

As in Subsection 4.1, set \( n = 5000, q = 3, \beta_0 = (1, 1.4, 0.7)^T \), choose \( p_n = \lfloor n/50 \rfloor \) and \( m = \lfloor n/(p_n + 1) \rfloor \), and generate \( \{x_{i,j,n}\} \) and \( \{\varepsilon_i\} \) in the same way as in Subsection 4.1. Set \( K_0 = 9, \delta_1 = \delta_3 = \delta_5 = \delta_7 = \delta_9 = (0.5, -0.7, 0.4)^T \), and \( \delta_2 = \delta_4 = \delta_6 = \delta_8 = -\delta_1 \). Consider the following two change point location settings:

**CPL1.** \( a_i = 500 \times i \), for \( i = 1, \ldots, 9 \);

**CPL2.** \( a_1 = 503, a_2 = 923, a_3 = 1471, a_4 = 2077, a_5 = 2334, a_6 = 2890, a_7 = 3410, a_8 = 3909, \) and \( a_9 = 4546 \).

For demonstration, two scatter plots of simulated data for the settings CPL1 and CPL2 are given respectively in Figures 2-3. One can hardly find any change points from these two figures.

We compare the following five algorithms: LSMCPDA, ALMCPDA, CALMCPDA, SMCPDA and MMCPDA. Let \( \hat{a}_i \) stand for \( \hat{a}_i, \tilde{a}_i, \tilde{a}_i^C, \tilde{a}_i^{scad} \) or \( \tilde{a}_i^{mcp} \) for \( i = 1, \ldots, 9 \). We check the accuracy of multiple change point estimation based on each algorithm by examining the
distance between $\tilde{a}_i$ and $a_i$ for $i = 1, \ldots, 9$. We only consider such distance to be equal to 0 or less than or equal to 5 or 10. The simulation results for the two change point location settings CPL1 and CPL2 are presented in Tables 2-3.

From both tables, it can be seen that all algorithms perform well in terms of accuracy of multiple change point estimation and the rate for correctly estimating the number of change points. The ALMCPDA and CALMCPDA are compatible and in generally outperform others. The average detection time required by CALMCPDA for a sample of size 5000 is more than all other algorithms, which is 8.20 seconds for CPL1 and 8.65 seconds for CPL2. In contrast, the average detection time required by ALMCPDA is only 5.61 seconds for CPL1 and 5.97 seconds for CPL2.

4.3. Practical recommendation of $p_n$

It is clear that the choice of $p_n$ will affect the performance of the proposed algorithms. Too
large $p_n$ may tend to underestimate the true number of multiple change points and increase biases in change point estimation while may cut down the computation time. Hence a care must be taken in choosing a proper $p_n$, and we propose the following algorithm:

*Step 1.* We choose an initial set $\mathcal{B}$ containing probable values of $p_n$.

*Step 2.* For each $p_n$ in the set $\mathcal{B}$, we obtain an estimate of $\theta_n$ in (14) by using an algorithm, say ALMCPDA. We can then calculate the residual sum of squares, denoted by $RSS(p_n)$.

*Step 3.* The optimal $p_n$ is chosen as $\arg\min_{p_n \in \mathcal{B}} RSS(p_n)$.

5. Empirical applications

In this section, we consider empirical applications of the multiple change point detection methods proposed in this paper by analyzing the U.S. Ex-Post Real Interest Rate (Garcia and Perron, 1996) and Gross domestic product in U.S.A (Maddala, 1977).
5.1. The U.S. Ex-Post Real Interest Rate

Garcia and Perron (1996) considered the time series behavior of the U.S. Ex-Post real interest rate (constructed from the three-month treasury bill rate deflated by the CPI inflation rate taken from the Citibase data base). The data are quarterly series from January, 1961 to March, 1986, which is plotted in Figure 4. We are interested in finding out if there are change points in the mean of the series. Thus we apply the proposed algorithms to the mean shift model. It is noted that by Remark 2, the algorithms are applicable even if there exists potential serial correlation.

![U.S. Ex-Post Real Interest Rate, the first quarter of 1961 – the third quarter of 1986](image)

Figure 4: U.S. Ex-Post Real Interest Rate, the first quarter of 1961 – the third quarter of 1986

First, we need to select a $p_n$. Following the recommendations in Subsection 4.3, we will choose an optimal $p_n$ from the range 3 to 13. For each $p_n \in \{3, 4, \ldots, 13\}$, we obtain $\hat{\theta}_n$ by the ALMCPDA, and calculate the corresponding $RSS(p_n)$. Choose $\arg \min_{3 \leq p_n \leq 13} RSS(p_n)$ as the optimal $p_n$, which is 5. See Figure 5.

Based on the first step, we set $p_n = 5$ and apply the five algorithms given in Section 3 to
the data. Two change points are found based on the ALMCPDA and the CALMCPDA, which are located at 47 and 79 (see Figure 4) with RSS=455.95 corresponding to the third quarter of 1972 and the third quarter of 1980. These results are consistent with those of Garcia and Perron (1996). However the other three algorithms LSMCPDA, SMCPDA and MMCPDA only detect one change point located at 47 with RSS=1214.89. By comparing their RSSs, it is clear that both ALMCPDA and CALMCPDA have better performance than the other three algorithms.

![Graph](image)

Figure 5: $RSS(p_n)$ against $p_n$ for the U.S. ex-post real interest rate data

5.2. Gross domestic product in U.S.A

The data presented in Maddala (1977, Table 10.3) gives the gross domestic product ($G$), the labor input index ($L$) and the capital input index ($C$) in the United States for the years 1929-1967. $\log G$ is modeled as a linear function of $\log L$ and $\log C$. The $\log G$, $\log L$ and $\log C$ are plotted over time given in Figure 6. Worsley (1983) used the likelihood ratio method to search for change points in this data set and pointed out that the data contained two change points.
located at 1942 and 1946 (RSS = 0.011). Caussinus and Lyazrhi (1997) used Bayes invariant optimal multi-decision procedure to detect change points in the data series and claimed three change points located at 1938, 1944 and 1948 (RSS = 0.01).

![Graph of logG, logL, and logC](image)

**Figure 6**: Logarithms of Gross domestic product (logG), labor-input index (logL) and capital-input index (logC) in U.S.A. for the years 1929-1967.

Since the sample size is only 39, the proposed algorithms employing least squares or the CUSUM test may not work. Thus we only apply the first two steps of the SMCPDA or the MMCPDA to carry out multiple change point analysis. As in the previous example, we need to select a $p_n$. Following the recommendations in Subsection 4.3, we will choose an optimal $p_n$ from 13 to 17. For each $p_n \in \{13, \ldots, 17\}$, we obtain $\hat{\theta}_n^{scad}$ by the SMCPDA, and calculate the corresponding $RSS(p_n)$. Choose $\arg\min_{13 \leq p_n \leq 17} RSS(p_n)$ as the optimal $p_n$, which is 17. With $p_n = 17$, four change points detected by applying the SMCPDA are located at 1936, 1942, 1946 and 1950 with $RSS = 0.0054$. With the same $p_n$, two change points detected by applying the MMCPDA are located at 1942 and 1958 with $RSS = 0.015$. Thus, in terms of the RSSs, the SMCPDA has a better performance.
6. Conclusion

By properly segmenting the data sequence, we proposed five multiple change point detection algorithms. The proposed approach is based on the following reasons. On the one hand, a proper segmentation can isolate the finite change points such that each change point is only located in one segment, and a connection between multiple change point detection and variable selection can be established. Thus the recent advances in consistent variable selection methods such as SCAD, adaptive LASSO and MCP can be used to detect these change points simultaneously. On the other hand, a refining procedure using a method such as CUSUM can improve the accuracy of change point estimates. Compared with other change point detection methods, which is very time consuming, the newly proposed algorithms are much faster, more effective, and have strong theoretical backup. The proposed approach can be extended to detect multiple change points in other models such as generalized linear models and nonparametric models without any extra difficulties.

Appendix

A.1. CUSUM test for a single change point

Consider the following model

\[ y_i = x_i^T \beta_1 I(n_\ell \leq i \leq k) + x_i^T \beta_2 I(k < i \leq n_{\ell+1}) + \varepsilon_i, \quad n_\ell \leq i \leq n_{\ell+1}, \]  

where \( y_\ell = (y_{n_\ell}, \ldots, y_{n_{\ell+1}})^T, x_{n_\ell}, x_{n_\ell+1}, \ldots, x_{n_{\ell+1}} \) are \( q \)-dimensional predictors, \( \beta_1 \) and \( \beta_2 \) are unknown \( q \)-dimensional vectors of regression coefficients, and \( \varepsilon_n = (\varepsilon_{n_\ell}, \ldots, \varepsilon_{n_{\ell+1}})^T \). If \( n_\ell \leq k < n_{\ell+1} \) and \( \beta_1 \neq \beta_2 \), there is a change point at \( k \).

Let \( N_\ell = n_{\ell+1} - n_\ell + 1 \). Define

\[ \hat{\sigma}^2_{\ell,k} = \frac{1}{N_\ell} \left[ \min_{\beta} \sum_{i=n_\ell}^{k} (y_i - x_i^T \beta)^2 + \min_{\beta} \sum_{i=k+1}^{n_{\ell+1}} (y_i - x_i^T \beta)^2 \right], \]

and \( \hat{\sigma}^2_\ell = \min_{\beta} \sum_{i=n_\ell}^{n_{\ell+1}} (y_i - x_i^T \beta)^2 / N_\ell \). By Theorem 3.1.1 of Csörgő and Horvath (1997), it
follows that
\[
\lim_{N_\ell \to \infty} P \left[ a_\ell \Lambda_\ell^{1/2} \leq x/2 + b_{\ell, q} \right] = \exp \left( -2e^{-x/2} \right),
\] (A.2)
for all \( x \), where \( a_\ell = (2 \log \log N_\ell^{1/2})^{1/2} \), \( b_{\ell, q} = 2 \log \log N_\ell + q \log \log N_\ell/2 - \log \Gamma(q/2), \Gamma(x) \) is the Gamma function, \( \Lambda_\ell = \max_{n_\ell + q \leq k \leq n_\ell + 1 - q} \left[ -2 \log \left( \frac{\hat{\sigma}_2^2}{\hat{\sigma}_{\ell,k}^2} \right) \frac{N_\ell}{2} \right] \).

In light of the proof of Corollary 2.1 of Hušková, Prášková and Steinebach (2007), it can be shown that
\[
\lim_{N_\ell \to \infty} P \left[ a_\ell \Lambda_\ell^{1/2} \leq x/2 + b_{\ell, q} \right] = \lim_{N_\ell \to \infty} P \left[ (\Lambda_\ell - \tilde{b}_{\ell, q})/\tilde{a}_{\ell, q} \leq x \right],
\]
where \( \tilde{b}_{\ell, q} = (b_{\ell, q}/a_\ell)^2 \) and \( \tilde{a}_\ell = b_{\ell, q}/a_\ell^2 \), which jointly with (A.2) implies that
\[
\lim_{N_\ell \to \infty} P \left[ (\Lambda_\ell - \tilde{b}_{\ell, q})/\tilde{a}_{\ell, q} \leq x \right] = \exp \left( -2e^{-x/2} \right).
\]

By Lemma 3.1.9 of Csörgő and Horvath (1997), it can be shown that
\[
\lim_{N_\ell \to \infty} P \left[ \left( \frac{1}{\hat{\sigma}_2^2} \max_{n_\ell + q \leq k \leq n_\ell + 1 - q} N_\ell (\hat{\sigma}_2^2 - \hat{\sigma}_{\ell,k}^2) - \tilde{b}_{\ell, q} \right)/\tilde{a}_{\ell, q} \leq x \right] = \exp \left( -2e^{-x/2} \right).
\]

Let \( T_{\ell, k} = N_\ell (\hat{\sigma}_2^2 - \hat{\sigma}_{\ell,k}^2) \) and \( T_\ell = \max_{n_\ell + q \leq k \leq n_\ell + 1 - q} T_{\ell, k} \). Given a significant level \( \alpha \), the CUSUM test for testing if there is a change point in the model (A.1) is given in the following: If
\[
T_\ell > \left( \hat{b}_{\ell, q} + 2\hat{\sigma}_{\ell, q} \log(-2/\log(1 - \alpha)) \right) \hat{\sigma}_2^2,
\]
there exists a \( k \in \{n_\ell + q, \ldots, n_\ell + 1 - q\} \) such that \( \beta_1 \neq \beta_2 \) in the model (A.1).

Denote \( C_\ell = \sum_{i=n_\ell}^{n_\ell+1} x_i x_i^T, \hat{\beta}_\ell = C_\ell^{-1} \sum_{i=n_\ell}^{n_\ell+1} x_i y_i, C_{\ell, k} = \sum_{i=n_\ell}^{k} x_i x_i^T, C_{\ell,k}^0 = C_\ell - C_{\ell,k}, S_{\ell, k} = \sum_{i=n_\ell}^{k} x_i (y_i - x_i^T \hat{\beta}_\ell) \) for \( k = n_\ell + q, \ldots, n_\ell + 1 - q \). By Hušková, Prášková and Steinebach (2007),
\[
T_\ell = \max_{n_\ell + q \leq k \leq n_\ell + 1 - q} S_{\ell, k}^T C_{\ell,k}^{-1} C_{\ell,k}^0 (C_{\ell,k}^0)^{-1} S_{\ell, k}.
\] (A.3)
Since $S_{\ell,k}$ and $C_{\ell,k}$ can be computed recursively, the computing time of $T_{\ell}$ is reduced to $O(n_{\ell+1} - n_{\ell})$ from $O((n_{\ell+1} - n_{\ell})^2)$ by using (A.3).

A.2. Proof of Lemma 1

Denote the elements of $A_c$ by $A_c = \{r_1, r_2, \ldots, r_{K_0}\}$. In view of $|n - p_n m + r_km - a_{k,n}| \leq m$, $a_{k,n}/n \to \tau_i$, for $k = 1, \ldots, K_0$ and $m = o(n)$, by Assumption C1, it follows that

$$\frac{1}{n} \sum_{i=r_1}^{r_2} X_{(i)}^T X_{(i)} \to (\tau_2 - \tau_1)W, \quad \ldots, \quad \frac{1}{n} \sum_{i=r_{K_0}}^{p_{n+1}} X_{(i)}^T X_{(i)} \to (1 - \tau_{K_0})W.$$ 

Hence,

$$\frac{1}{n} X_{A_c}^T X_{A_c} = U^T \begin{pmatrix}
\frac{1}{n} \sum_{i=r_1}^{r_2-1} X_{(i)}^T X_{(i)} & 0 & \cdots & 0 \\
0 & \frac{1}{n} \sum_{i=r_2}^{r_3-1} X_{(i)}^T X_{(i)} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \frac{1}{n} \sum_{i=r_{K}}^{p_{n+1}} X_{(i)}^T X_{(i)}
\end{pmatrix} U$$

$$\to U^T \begin{pmatrix}
(\tau_2 - \tau_1)W & 0 & \cdots & 0 \\
0 & (\tau_3 - \tau_2)W & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & (1 - \tau_K)W
\end{pmatrix} U \succeq \mathcal{W}_{A_c} > 0, \quad \text{(A.4)}$$

where

$$U = \begin{pmatrix}
I_q & 0 & \cdots & 0 \\
I_q & I_q & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
I_q & I_q & \cdots & I_q
\end{pmatrix}.$$ 

A.3. Proof of Lemma 2

As in the proof of Lemma 1, denote $A_c = \{r_1, \ldots, r_{K_0}\}$. It is easy to see that

$$\tilde{X}_n^T x_\omega / m = \frac{1}{m} \sum_{\ell=1}^{p_n} \begin{pmatrix}
\sum_{i=1}^{n} \mathbf{x}_i^T \omega_\ell(i) \\
\sum_{i=n-p_n+1}^{n} \mathbf{x}_i^T \omega_\ell(i) \\
\sum_{i=-(p_n-1)m+1}^{n-m+1} \mathbf{x}_i^T \omega_\ell(i) \\
\vdots \\
\sum_{i=n-m+1}^{n} \mathbf{x}_i^T \omega_\ell(i)
\end{pmatrix}.$$
Consider the first row of $\tilde{X}_{n}^T \mathbf{x}_\omega / m$. By Assumption C1, $\sum_{i=n-(p_n-r_j)m}^{n} x_i \mathbf{x}_i^T / m \to W$. Hence for large $n$,

$$\left\| \frac{1}{m} \sum_{\ell=1}^{p_n} \sum_{i=1}^{n} x_i \mathbf{x}_i^T \omega_\ell (i) \right\| \leq \frac{1}{m} \sum_{j=1}^{K_0} \sum_{i=n-(p_n-r_j)m+1}^{a_{j,n}} x_i \mathbf{x}_i^T \delta_j \leq \frac{1}{m} \sum_{j=1}^{K_0} \| \delta_j \| \sum_{i=n-(p_n-r_j)m+1}^{n-(p_n-r_j)m} x_i \mathbf{x}_i^T \leq 2K_0 \| W \| \max_{1 \leq i \leq K_0} \| \delta_j \| \tag{A.5}$$

Similarly, it can be shown that for any $\varepsilon > 0$ such that $\sum_{i=1}^{n} x_i \mathbf{x}_i^T \omega_\ell (i) \leq 2\| W \| \max_{1 \leq i \leq K_0} \| \delta_j \|$, the proof is complete.

A.4. Proof of Lemma 3

By the definition of $\tilde{X}_n$, it follows that

$$\tilde{X}_n^T \mathbf{e}_n / \sqrt{n} = \frac{1}{\sqrt{n}} \left( \begin{array}{cc} \sum_{i=1}^{n} x_i \mathbf{x}_i & \mathbf{x}_i \varepsilon_i \\ \sum_{i=n-p_n,m+1}^{n} x_i \mathbf{x}_i & \mathbf{x}_i \varepsilon_i \\ \sum_{i=n-(p_n-1)m+1}^{n} x_i \mathbf{x}_i & \mathbf{x}_i \varepsilon_i \\ \vdots & \vdots \\ \sum_{i=n-m+1}^{n} x_i \mathbf{x}_i & \mathbf{x}_i \varepsilon_i \end{array} \right).$$

Consider the first element of $\tilde{X}_n^T \mathbf{e}_n / \sqrt{n}$. By Assumption C1, for $j = 1, \ldots, q$, $\sum_{i=1}^{n} x_{i,j}^2 / n \to W_{jj}$. By applying Markov’s inequality, we have $\sum_{i=1}^{n} x_{i,j} \varepsilon_i / \sqrt{n} = O_p(1)$.

In the following, we show that for any $\varepsilon > 0$, there exists an $M_\varepsilon$ such that

$$p_{n,j} \overset{\text{P}}{=} \max_{1 \leq k \leq p_n} \left( \frac{1}{\sqrt{n}} \sum_{i=n-(p_n-k+1)m+1}^{n} x_{i,j} \varepsilon_i \right) > M_\varepsilon < \varepsilon.$$
Denote \( \eta_{\ell,j} = \sum_{n-(p_n-\ell)m}^{n-(p_n-\ell+1)m+1} x_{i,j} \epsilon_i \). Then we have

\[
p_{n,j} = P \left( \frac{1}{\sqrt{n}} \max_{1 \leq t \leq p_n} \left| \sum_{\ell=1}^{t} \eta_{p_n-\ell+1,j} \right| > M_\epsilon \right).
\]

Note that for any \( v > u > 0 \), by Assumption C1, we have

\[
\text{Var} \left( \sum_{\ell=u}^{v} \frac{\eta_{\ell,j}}{\sqrt{m}} \right) \leq 2(v-u)W_{jj}\sigma^2 \leq 2(v-u)\sigma^2 \max_{1 \leq j \leq q} W_{jj},
\]

when \( n \) is large enough. By Lemma 2.1 of Lavielle (1999), it follows that

\[
p_{n,j} = P \left( \max_{1 \leq t \leq p_n} \left| \sum_{\ell=1}^{t} \eta_{p_n-\ell+1,j} \sqrt{m} \right| > M_\epsilon \frac{n}{m} \right) \leq \frac{cP_n}{M_\epsilon^2 n/m} \leq c/M_\epsilon^2 < \epsilon,
\]

which means that each element of vector \( \tilde{X}_n^T \epsilon_n / \sqrt{n} \) is bounded uniformly in probability. The proof of Lemma 3 is complete.

A.5. Proof of Theorem 3

Let \( \mathbf{u} = (u_0^T, u_1^T, \ldots, u_{p_n}^T)^T \) be bounded. Put \( \mathbf{\theta} = \mathbf{\theta}_n + \frac{\mathbf{u}}{\sqrt{n}} \) and

\[
\psi_n(\mathbf{u}) = \left\| \mathbf{y} - X_n^{(1)} \left( \mathbf{\beta}_0 + \frac{\mathbf{u}_0}{\sqrt{n}} \right) - \sum_{j=1}^{p_n} X_n^{(j+1)} \left( d_j + \frac{\mathbf{u}_j}{\sqrt{n}} \right) \right\|^2 + \lambda_n \sum_{r=1}^{p_n} \left| d_r \right| \sqrt{n} \left| d_r + \frac{\mathbf{u}_r}{\sqrt{n}} \right|.
\]

Let \( \mathbf{\hat{u}}_n = \arg \min \psi_n(\mathbf{u}) = \arg \min (\psi_n(\mathbf{u})_n - \psi_n(0)) \). Thus \( \mathbf{\hat{\theta}} = \mathbf{\theta}_n + \mathbf{\hat{u}}_n / \sqrt{n} \), and we only need to investigate the limiting behavior of \( \mathbf{\hat{u}}_n \). Write \( \psi_n(\mathbf{u}) - \psi_n(0) = \mathcal{V}_n(\mathbf{u}) \), which can be expressed as

\[
\mathcal{V}_n(\mathbf{u}) = \left( \mathbf{u}^T \left( \frac{\bar{X}_n^T \bar{X}_n}{n} \right) \mathbf{u} - 2 \mathbf{u}^T \frac{\bar{X}_n^T \epsilon_n}{\sqrt{n}} - 2 \mathbf{u}^T \frac{\bar{X}_n^T \mathbf{x}_\omega}{\sqrt{n}} \right) + \lambda_n \sum_{r=1}^{p_n} \left| d_r \right| \sqrt{n} \left( \left| d_r + \frac{\mathbf{u}_r}{\sqrt{n}} \right| - \left| d_r \right| \right).
\]

Consider the following two cases:

Case I. For any \( r \notin A_c, \mathbf{u}_r = 0; \)

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Case II: There are some \( r \notin \mathcal{A}_c \) such that \( \mathbf{u}_r \neq \mathbf{0} \). Denote the number of such \( rs \) as \( n_c \).

We first consider the case I. By Lemmas 1-2 and the assumption that \( m/\sqrt{n} \to 0 \), it can be shown that as \( n \to \infty \),

(A1) \( \mathbf{u}^T \left( \frac{1}{n} \tilde{X}_n^T \tilde{X}_n \right) \mathbf{u} = \mathbf{u}^T_{\mathcal{A}_c} \left( \frac{1}{n} \tilde{X}_{n\mathcal{A}_c}^T \tilde{X}_{n\mathcal{A}_c} \right) \mathbf{u}_{\mathcal{A}_c} \to \mathbf{u}^T_{\mathcal{A}_c} \mathbf{W}_{\mathcal{A}_c} \mathbf{u}_{\mathcal{A}_c} \);

(A2) \( \mathbf{u}^T \tilde{X}_n^T \varepsilon / \sqrt{n} = \mathbf{u}^T_{\mathcal{A}_c} (\tilde{X}_{n\mathcal{A}_c}^T \varepsilon) / \sqrt{n} \to_d \mathbf{u}^T_{\mathcal{A}_c} \mathbf{w}_{\mathcal{A}_c} \), where \( \mathbf{w}_{\mathcal{A}_c} = \mathcal{N}(0, \sigma^2 \mathbf{W}_{\mathcal{A}_c}) \);

(A3) \( \mathbf{u}^T \tilde{X}_n^T \mathbf{x}_\omega / \sqrt{n} \to 0 \).

Note that for any \( r \notin \mathcal{A}_c \), the second term of \( V_n(\mathbf{u}) \) equals to 0. Let \( r \in \mathcal{A}_c \). By Assumption C3, it follows that \( 1/|d_r|^{\nu} \leq c^{-\nu} \) in probability. Since \( \sqrt{n} \left| d_r + \frac{u_r}{\sqrt{n}} - |d_r| \right| \leq |u_r| \), and \( |\mathcal{A}_c| = K_0 \), by the assumption that \( \lambda_n/\sqrt{n} \to 0 \), we have

\[
\frac{\lambda_n}{\sqrt{n}} \sum_{r=1}^{p_n} \frac{1}{|d_r|^{\nu}} \sqrt{n} \left( \left| d_r + \frac{u_r}{\sqrt{n}} \right| - |d_r| \right) \to_p 0,
\]

which, jointly with (A1)-(A3) above, implies that \( V_n(\mathbf{u}) \to_p \mathbf{u}^T_{\mathcal{A}_c} \mathbf{W}_{\mathcal{A}_c} \mathbf{u}_{\mathcal{A}_c} - 2 \mathbf{u}^T_{\mathcal{A}_c} \mathbf{w}_{\mathcal{A}_c} \), as \( n \to \infty \).

We now consider the case II. By Lemmas 2-3 and the assumption that \( m/\sqrt{n} \to 0 \), it can be shown that

(B1) \( \mathbf{u}^T \left( \frac{1}{n} \tilde{X}_n^T \tilde{X}_n \right) \mathbf{u} \geq 0 \);

(B2) \( \mathbf{u}^T \tilde{X}_n^T \varepsilon_n / \sqrt{n} = O_p(n_c) \);

(B3) \( \frac{1}{n_c} \mathbf{u}^T \tilde{X}_n^T \mathbf{x}_\omega / \sqrt{n} \to 0 \).

As argued previously, it can also be shown that

(B4) \( \frac{\lambda}{\sqrt{n}} \sum_{r \in \mathcal{A}_c} \frac{1}{|d_r|^{\nu}} \sqrt{n} \left( \left| d_r + \frac{u_r}{\sqrt{n}} \right| - |d_r| \right) \to 0 \).
Now let \( r \notin A_c \). Since \( \{|r, d_r = 0, u_r \neq 0\} = n_c \), by Assumption C3 and the assumption that 
\[ \lambda_n(n/p_n)^{\nu/2}/\sqrt{n} \to \infty, \] it follows that 
\[ \frac{1}{n_c} \sum_{r \notin A_c, d_r = 0, u_r \neq 0} \frac{\lambda_n}{\sqrt{n} |d_r|^\nu} \sqrt{n} \left( |d_r + \frac{u_r}{\sqrt{n}}| - |d_r| \right) \]
\[ = \frac{1}{n_c} \sum_{r \notin A_c, d_r = 0, u_r \neq 0} \frac{\lambda_n}{\sqrt{n} (n/p_n)^{\nu/2}} |u_r| \times \left| \sqrt{\frac{n}{p_n} d_r} \right|^{-\nu} \to_p \infty, \]
which, jointly with (B1)-(B4), implies that \( V_n(u) \to_p \infty \).

So far we have showed that 
\[ V_n(u) \to_p V(u) = \begin{cases} u^T A_c W_{A_c} u_{A_c} - 2 u^T A_c w_{A_c}, & \text{Case I}, \\ \infty, & \text{Case II}. \end{cases} \quad (A.7) \]

It can be seen that \( V \) is a convex function and has a unique minimum at \( \tilde{u} \) such that \( \tilde{u}_{A_c} = 0 \) and \( \tilde{u}_{A_c} = W_{A_c}^{-1} w_{A_c} \). Since \( V_n(\cdot) \) is also a convex function and has a unique minimum denoted by \( \tilde{u}_n \), by (A.7), 
\[ \tilde{u}_n = \arg \min V_n(u) \to_p \arg \min V(u) = \tilde{u}, \]
and hence, 
\[ (\tilde{u}_n)_{A_c} \to_p \tilde{u}_{A_c} = W_{A_c}^{-1} w_{A_c} \quad \text{and} \quad (\tilde{u}_n)_{\bar{A}_c} \to_p \tilde{u}_{\bar{A}_c} = 0. \]

In view of the fact that \( w_{A_c} \sim N(0, \sigma^2 W_{A_c}) \), the proof is complete.

**Acknowledgements**

This work was supported by the Natural Sciences and Engineering Research Council of Canada. The authors thank Professor Pierre Perron for his kindly sharing the U.S. ex-post real interest rate data with them.

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Table 2: The entries are the numbers of $\tilde{a}_i$ such that $|\tilde{a}_i - a_{i,n}| \leq 0, 5, 10$ for $i = 1, \ldots, 9$, the number of correctly estimating the number of change points and the corresponding average computation time by each of the five algorithms LSMCPDA, ALMCPDA, CALMCPDA, SMCPDA and MMCPDA based on 1000 simulations for the change point location setting CPL1.

|                | LSMCPDA | ALMCPDA | CALMCPDA | SMCPDA | MMCPDA |
|----------------|---------|---------|----------|--------|--------|
| $|\tilde{a}_{1,n} - a_1| = 0$ | 208     | 215     | 215      | 212    | 212    |
| $|\tilde{a}_1 - a_1| \leq 5$ | 958     | 973     | 973      | 973    | 974    |
| $|\tilde{a}_1 - a_1| \leq 10$ | 990     | 993     | 993      | 992    | 992    |
| $|\tilde{a}_2 - a_2| = 0$ | 489     | 532     | 532      | 520    | 525    |
| $|\tilde{a}_2 - a_2| \leq 5$ | 924     | 939     | 939      | 918    | 922    |
| $|\tilde{a}_2 - a_2| \leq 10$ | 979     | 982     | 982      | 960    | 964    |
| $|\tilde{a}_3 - a_3| = 0$ | 263     | 262     | 262      | 253    | 253    |
| $|\tilde{a}_3 - a_3| \leq 5$ | 806     | 807     | 807      | 773    | 792    |
| $|\tilde{a}_3 - a_3| \leq 10$ | 972     | 977     | 977      | 932    | 952    |
| $|\tilde{a}_4 - a_4| = 0$ | 162     | 174     | 174      | 157    | 174    |
| $|\tilde{a}_4 - a_4| \leq 5$ | 810     | 806     | 806      | 773    | 786    |
| $|\tilde{a}_4 - a_4| \leq 10$ | 961     | 959     | 959      | 921    | 939    |
| $|\tilde{a}_5 - a_5| = 0$ | 716     | 726     | 726      | 694    | 703    |
| $|\tilde{a}_5 - a_5| \leq 5$ | 961     | 975     | 975      | 931    | 947    |
| $|\tilde{a}_5 - a_5| \leq 10$ | 986     | 998     | 998      | 953    | 970    |
| $|\tilde{a}_6 - a_6| = 0$ | 210     | 223     | 223      | 215    | 218    |
| $|\tilde{a}_6 - a_6| \leq 5$ | 980     | 985     | 985      | 941    | 956    |
| $|\tilde{a}_6 - a_6| \leq 10$ | 993     | 1000    | 1000     | 955    | 971    |
| $|\tilde{a}_7 - a_7| = 0$ | 201     | 219     | 219      | 195    | 204    |
| $|\tilde{a}_7 - a_7| \leq 5$ | 824     | 876     | 876      | 814    | 844    |
| $|\tilde{a}_7 - a_7| \leq 10$ | 928     | 973     | 973      | 904    | 937    |
| $|\tilde{a}_8 - a_8| = 0$ | 455     | 511     | 511      | 460    | 474    |
| $|\tilde{a}_8 - a_8| \leq 5$ | 893     | 978     | 978      | 897    | 927    |
| $|\tilde{a}_8 - a_8| \leq 10$ | 907     | 991     | 991      | 911    | 942    |
| $|\tilde{a}_9 - a_9| = 0$ | 240     | 277     | 277      | 276    | 279    |
| $|\tilde{a}_9 - a_9| \leq 5$ | 786     | 935     | 936      | 922    | 918    |
| $|\tilde{a}_9 - a_9| \leq 10$ | 822     | 980     | 981      | 966    | 961    |

No. of Correct Detection | 818 | 950 | 987 | 898 | 920 |

Average Computation Time | 2.23 | 5.61 | 8.20 | 2.88 | 2.98 |
Table 3: The entries are the numbers of $\tilde{a}_i$ such that $|\tilde{a}_i - a_{i,n}| \leq 0,5,10$ for $i = 1, \ldots, 9$, the number of correctly estimating the number of change points and the corresponding average computation time by each of the five algorithms LSMCPDA, ALMCPDA, CALMCPDA, SMCPDA and MMCPDA based on 1000 simulations for the change point location setting CPL2.

| $|\tilde{a}_1 - a_1| = 0$ | LSMCPDA | ALMCPDA | CALMCPDA | SMCPDA | MMCPDA |
|--------------------------|---------|---------|-----------|---------|---------|
| $|\tilde{a}_1 - a_1| \leq 5$ | 362     | 378     | 378       | 377     | 381     |
| $|\tilde{a}_1 - a_1| \leq 10$ | 955     | 961     | 961       | 955     | 960     |

| $|\tilde{a}_2 - a_2| = 0$ | 270     | 276     | 275       | 271     | 274     |
| $|\tilde{a}_2 - a_2| \leq 5$ | 858     | 872     | 869       | 861     | 865     |
| $|\tilde{a}_2 - a_2| \leq 10$ | 975     | 991     | 988       | 976     | 981     |

| $|\tilde{a}_3 - a_3| = 0$ | 426     | 522     | 522       | 522     | 523     |
| $|\tilde{a}_3 - a_3| \leq 5$ | 767     | 952     | 952       | 957     | 958     |
| $|\tilde{a}_3 - a_3| \leq 10$ | 811     | 982     | 982       | 987     | 988     |

| $|\tilde{a}_4 - a_4| = 0$ | 195     | 194     | 194       | 115     | 150     |
| $|\tilde{a}_4 - a_4| \leq 5$ | 892     | 911     | 911       | 525     | 714     |
| $|\tilde{a}_4 - a_4| \leq 10$ | 955     | 970     | 970       | 562     | 766     |

| $|\tilde{a}_5 - a_5| = 0$ | 272     | 295     | 294       | 169     | 249     |
| $|\tilde{a}_5 - a_5| \leq 5$ | 910     | 980     | 978       | 578     | 834     |
| $|\tilde{a}_5 - a_5| \leq 10$ | 921     | 997     | 995       | 582     | 845     |

| $|\tilde{a}_6 - a_6| = 0$ | 793     | 795     | 795       | 783     | 779     |
| $|\tilde{a}_6 - a_6| \leq 5$ | 967     | 971     | 968       | 954     | 946     |
| $|\tilde{a}_6 - a_6| \leq 10$ | 987     | 993     | 988       | 972     | 964     |

| $|\tilde{a}_7 - a_7| = 0$ | 293     | 317     | 315       | 309     | 309     |
| $|\tilde{a}_7 - a_7| \leq 5$ | 922     | 941     | 939       | 932     | 931     |
| $|\tilde{a}_7 - a_7| \leq 10$ | 973     | 991     | 989       | 984     | 986     |

| $|\tilde{a}_8 - a_8| = 0$ | 197     | 210     | 196       | 211     | 206     |
| $|\tilde{a}_8 - a_8| \leq 5$ | 836     | 899     | 899       | 904     | 910     |
| $|\tilde{a}_8 - a_8| \leq 10$ | 891     | 968     | 968       | 969     | 975     |

| $|\tilde{a}_9 - a_9| = 0$ | 305     | 298     | 298       | 304     | 304     |
| $|\tilde{a}_9 - a_9| \leq 5$ | 927     | 924     | 924       | 934     | 932     |
| $|\tilde{a}_9 - a_9| \leq 10$ | 974     | 977     | 977       | 982     | 982     |

| No. of Correct Detection | 895     | 947     | 964       | 572     | 759     |
| Average Computation Time | 2.29    | 5.97    | 8.65      | 3.00    | 2.98    |