EXPLICIT GENERATORS AND RELATIONS FOR
THE CENTRE OF THE QUANTUM GROUP

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Abstract. For the standard Drinfeld-Jimbo quantum group $U_q(g)$ associated with a simple Lie algebra $g$, we construct explicit generators of the centre $Z(U_q(g))$, and determine the relations satisfied by the generators. For $g$ of type $A_n (n \geq 2)$, $D_{2k+1} (k \geq 2)$ or $E_6$, the centre $Z(U_q(g))$ is isomorphic to a quotient of a polynomial algebra in multiple variables, which is described in a uniform manner for all cases. For $g$ of any other type, $Z(U_q(g))$ is generated by $n = \text{rank}(g)$ algebraically independent elements.

1. Introduction

Let $g$ be a finite dimensional simple complex Lie algebra of rank $n$. In the literature [Tan92, Jan96], there are two different versions of the Drinfeld-Jimbo quantum group $U_q(g)$, which are denoted by $\overline{U}_q(g)$ and $U_q(g)$ respectively with $q$ being an indeterminate. The former contains among generators $K_\lambda$ with $\lambda$ in the weight lattice $P$ of $g$, while the latter contains those $K_\alpha$ with $\alpha$ in the root lattice $Q$ of $g$. We will focus on the latter quantum group $U_q(g)$ (see Section 2.1) and study the structure of its centre.

Drinfeld [Dri90] and Reshetikhin [Res90] constructed explicitly a natural isomorphism from the representation ring to the centre of the quantum group $\overline{U}_q(g)$. Their method exploits the quasi-triangular structure of $U_q(g)$ and can be generalised to the quantum affine algebras [Eng95]. It turns out that the centre $Z(\overline{U}_q(g))$ is a polynomial algebra generated by $n$ algebraically independent central elements associated to certain representations. Algebraically independent explicit generators of $Z(\overline{U}_q(g))$ have been constructed in [Dai].

In contrast to the case of $\overline{U}_q(g)$, the centre $Z(U_q(g))$ of the quantum group $U_q(g)$ is not necessarily a polynomial algebra, and much remains to be understood about its algebraic structure.

A fundamental problem, analogous to the first and second fundamental theorems of classical invariant theory, is to describe explicit generators of $Z(U_q(g))$ and the relations which they obey. The generators of $Z(U_q(g))$ are elements of $U_q(g)$ which commute with all elements of $U_q(g)$. They are usually referred to as quantum Casimir operators, and play important roles in studying symmetries of physical systems.

We give a complete solution of this problem for all $g$ in Theorem 2.5.

Now we briefly describe the key ingredients used in the proof of Theorem 2.5.

We point out here that [LXZ10] proved to be very useful for our study, and will make comments later on results of op. cit..

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Given any finite dimensional $\mathfrak{u}_q(\mathfrak{g})$-module $V$ of type-1 with some conditions on the weights, we employ the quasi $R$-matrix of $\mathfrak{u}_q(\mathfrak{g})$ (see e.g., [KS97 §8.3.3] for an explicit formula) to construct an infinite set of explicit central elements $C^{(k)}_V$ for $k = 1, 2, \ldots$ in Definition 2.2 by following a method developed in [ZGB91a, ZGB91b]. Our main theorem (i.e., Theorem 2.5) states that there exists a finite set $\Sigma$ of $\mathfrak{u}_q(\mathfrak{g})$-modules such that $C_V = C^{(1)}_V$ for $V \in \Sigma$ generate the centre $Z(\mathfrak{u}_q(\mathfrak{g}))$.

We determine the set $\Sigma$ and obtain the relations satisfied by the generators by making essential use of the quantised Harish-Chandra isomorphism of $\mathfrak{u}_q(\mathfrak{g})$, which is an isomorphism from the centre $Z(\mathfrak{u}_q(\mathfrak{g}))$ to the Weyl group $W$ invariant subalgebra $(U^0_{ev})^W$ [Jan96], where $U^0_{ev}$ is spanned by the even elements $K_{2\alpha}$ for $\lambda \in M := \frac{1}{2}Q \cap P$ with $\frac{1}{2}Q := \{ \frac{1}{2}\alpha | \alpha \in Q \}$ the half root lattice. In particular, we require a specific representation-theoretical description of the isomorphism. For this, we consider the Grothendieck algebra $S(\mathfrak{u}_q(\mathfrak{g}))$ of the category of finite dimensional $\mathfrak{u}_q(\mathfrak{g})$-modules whose weights are contained in $M$. Then the quantised Harish-Chandra isomorphism leads to an isomorphism from $S(\mathfrak{u}_q(\mathfrak{g}))$ to $Z(\mathfrak{u}_q(\mathfrak{g}))$, sending each isomorphism class $[V]$ to $C_V$.

To gain a conceptual understanding of the Grothendieck algebra $S(\mathfrak{u}_q(\mathfrak{g}))$, we bring the monoid algebra $\mathbb{C}[M^+]$ into the picture [LXZ16], where $M^+ := \frac{1}{2}Q \cap P^+$ denotes the additive monoid consisting of dominant weights in the half root lattice $\frac{1}{2}Q$. We describe the Hilbert basis $\text{Hilb}(M^+)$, a minimal generating set of $M^+$, and then split the simple Lie algebras into two types (see (3.1)). In the case of type I, the set $\text{Hilb}(M^+)$ comprises exactly all fundamental weights of $\mathfrak{g}$ by straightforward calculation, and hence the associated monoid algebra $\mathbb{C}[M^+]$ is a polynomial algebra. In the case of type II, where $\mathfrak{g}$ is of $A_n(n \geq 2)$, $D_{2k+1}(k \geq 2)$ or $E_6$, the automorphism of the corresponding Dynkin diagram (see Figure 1) induces an involution of the monoid $M^+$, which permits us to describe generators and relations of the monoid algebra $\mathbb{C}[M^+]$ in a unified way.

We prove that there is a natural isomorphism between $S(\mathfrak{u}_q(\mathfrak{g}))$ and the monoid algebra $\mathbb{C}(q)[M^+] := \mathbb{C}(q) \otimes_\mathbb{C} \mathbb{C}[M^+]$ over the field $\mathbb{C}(q)$ of rational functions, and therefore obtain $Z(\mathfrak{u}_q(\mathfrak{g})) \cong S(\mathfrak{u}_q(\mathfrak{g})) \cong \mathbb{C}(q)[M^+]$. By means of these isomorphisms, for each generator of $\mathbb{C}(q)[M^+]$ we construct an explicit generator $C_T$ of $Z(\mathfrak{u}_q(\mathfrak{g}))$ associated to a certain tensor product $T$ of fundamental representations of $\mathfrak{g}$. Using the presentation of $\mathbb{C}[M^+]$, we determine relations among these generators $C_T$.

We must point out that the isomorphism $Z(\mathfrak{u}_q(\mathfrak{g})) \cong \mathbb{C}(q)[M^+]$ and a presentation of the monoid algebra $\mathbb{C}(q)[M^+]$ were previously obtained in [LXZ16] by a case by case study. Here we have developed a new method for deriving these results, which is conceptual and uniform. Also the presentation of $\mathbb{C}(q)[M^+]$ given in [LXZ16] has different (but equivalent) relations from ours despite the fact that the generating set is the same.

We should emphasise the difference between the present paper and [LXZ16]. While we have given explicit generators and relations of the centre $Z(\mathfrak{u}_q(\mathfrak{g}))$, the authors of [LXZ16] gave a presentation for the isomorphic algebra $\mathbb{C}(q)[M^+]$ instead. Their results, while being interesting in their own right, do not help in constructing explicit generators of $Z(\mathfrak{u}_q(\mathfrak{g}))$, which is one of our main concerns in this paper.

We note that the eigenvalues of higher order central elements $C^{(k)}_V$ are computed explicitly in [LZ98, DGL05] for quantum supergroups $U_q(\mathfrak{gl}_{m|n})$ and $U_q(\mathfrak{osp}_{m|2n})$. 
where $V$ is the natural representation. With the eigenvalue formula, it is shown in [L10] that the centre of $U_q(g_n)$ is generated by $C^{(k)}_V$ for $1 \leq k \leq n$. In a sequel to this paper, we will prove an analogue of this result for quantum groups of types $B$, $C$ and $D$.

This paper is organised as follows. In Section 2 we construct an explicit central element $C_V$ from any finite dimensional $U_q(g)$-module $V$ whose weights are contained in $M$, and then state our main theorem (Theorem 2.5), which will be proved in later sections. In Section 3 we describe the Hilbert basis Hilb$(M^+)$, and give a presentation of the monoid algebra $\mathbb{C}[M^+]$ in Theorem 3.15 and Theorem 3.16 which correspond to the Lie algebras of type I and II respectively. In Section 4 we prove our main theorem by showing that the centre $Z(U_q(g))$ is isomorphic to the monoid algebra $\mathbb{C}(q)[M^+]$ from a representation theoretical point of view.

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2. Construction of central elements

We first recall the definition of quantum groups. Given any finite dimensional $U_q(g)$-module $V$ whose weights are contained in $M$, we construct explicitly a central element $C_V$ by using the quasi $R$-matrix of $U_q(g)$. Finally, we state the main theorem of this paper.

2.1. Quantum groups. Let $g$ be a finite dimensional simple Lie algebra of rank $n$ over the complex field $\mathbb{C}$. Let $\mathfrak{h}$ be the Cartan subalgebra of $g$, and let $\Phi \subseteq \mathfrak{h}^*$ be the set of roots. Fix a set $\Phi^+$ of positive roots, and denote by $\Pi = \{\alpha_1, \ldots, \alpha_n\} \subseteq \Phi^+$ the set of corresponding simple roots.

Let $(\cdot, \cdot)$ be a non-degenerate invariant symmetric bilinear form on $\mathfrak{h}^*$. The Cartan matrix $A = (a_{ij})$ is the $n \times n$ matrix with $a_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$. The fundamental weights $\varpi_i$ of $g$ are defined by $2(\varpi_i, \alpha_j)/(\alpha_j, \alpha_j) = \delta_{ij}$ for $1 \leq i, j \leq n$. We define

$$
P = \bigoplus_{i=1}^n \mathbb{Z}\varpi_i, \quad Q = \bigoplus_{i=1}^n \mathbb{Z}\alpha_i$$

to be the weight lattice and root lattice, respectively. Let $P^+ \subseteq P$ be the set of dominant weights, i.e., weights that are non-negative integer combinations of $\varpi_i$.

Throughout, let $q$ be an indeterminate and $\mathbb{C}(q)$ the field of rational functions. The quantum group $U_q(g)$ [Jan96] is the unital associative algebra over $\mathbb{C}(q)$ generated by $E_i, F_i$ and $K_i := K_{\alpha_i}$ for $1 \leq i \leq n$, subject to the following relations:

$$
K_i K_i^{-1} = 1 = K_i^{-1} K_i, \quad K_i K_j = K_j K_i, \quad K_i E_j K_i^{-1} = q^{(\alpha_i, \alpha_j)} E_j, \quad K_i F_j K_i^{-1} = q^{-(\alpha_i, \alpha_j)} F_j,
$$

$$
E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},
$$

$$
\sum_{s=0}^{1-a_{ij}} (-1)^s \binom{1-a_{ij}}{s} q^s q^{s-a_{ij}} E_i^{-a_{ij}-s} E_j E_i^s = 0, \quad i \neq j,
$$



\[
\sum_{s=0}^{1-a_{ij}} (-1)^s \left[ \frac{1-a_{ij}}{s} \right] F_i^{1-a_{ij}-s} F_j F_s = 0, \quad i \neq j,
\]
where \( q_i = q^{(\alpha_i, \alpha_i)/2} \), and for any \( m \in \mathbb{N} \)
\[
[m]_q = q^m - q^{-m}, \quad [m]_{q, 1} = [1]_q [2]_q \cdots [m]_q, \quad \left[ \frac{m}{k} \right]_q = \frac{[m]_q!}{[m-k]_q! [k]_q!}.
\]
It is well known that \( U_q(\mathfrak{g}) \) is a Hopf algebra with co-multiplication \( \Delta \), co-unit \( \varepsilon \) and antipode \( S \) given by
\[
\Delta(K_i) = K_i \otimes K_i, \quad \Delta(E_i) = K_i \otimes E_i + E_i \otimes K_i, \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i,
\]
\[
\varepsilon(K_i) = 1, \quad \varepsilon(E_i) = 0, \quad \varepsilon(F_i) = 0,
\]
\[
S(K_i) = K_i^{-1}, \quad S(E_i) = -K_i^{-1} E_i, \quad S(F_i) = -F_i K_i.
\]
Write \( U = U_q(\mathfrak{g}) \). The quantum group is graded by the root lattice \( Q \), i.e.,
\[
U = \bigoplus_{\nu \in Q} U_{\nu},
\]
where \( U_{\nu} = \{ u \in U \mid K_i u K_i^{-1} = q^{(\nu, \alpha_i)} u, \forall i = 1, \ldots, n \} \).

Define \( U^+ \) (resp. \( U^- \)) to be the subalgebra generated by all \( E_i \) (resp. \( F_i \)), and introduce \( U^+_\nu = U_\nu \cap U^+ \) (resp. \( U^-_\nu = U_\nu \cap U^- \)).

The representation theory of \( U_q(\mathfrak{g}) \) is parallel to that of the Lie algebra \( \mathfrak{g} \) [Hum72, Jan96]. Throughout, we are concerned with finite dimensional \( U_q(\mathfrak{g}) \)-modules of type 1. Each \( U_q(\mathfrak{g}) \)-module \( V \) admits the weight space decomposition \( V = \bigoplus_{\nu \in \Pi(V)} V_{\nu} \), where \( V_{\nu} \) is the weight space of \( V \) and \( \Pi(V) \subset P \) is the set of weights. Define \( m_V(\mu) := \dim V_{\mu} \). The dominant weights \( \lambda \in P^+ \) are in bijection with the simple \( U_q(\mathfrak{g}) \)-modules \( L(\lambda) \) with the highest weight \( \lambda \). For any two weights \( \lambda, \mu \) we have the partial ordering \( \lambda > \mu \) if and only if \( \lambda - \mu \) is a sum of positive roots.

2.2. Central elements. Let \( V \) be an arbitrary finite dimensional \( U_q(\mathfrak{g}) \)-module. Let \( T_1 \) denote the partial trace on the first tensor factor of \( \text{End}(V) \otimes U_q(\mathfrak{g}) \), i.e., \( T_1(\xi \otimes x) = \text{Tr}(\xi)x \) for any \( \xi \in \text{End}(V) \) and \( x \in U_q(\mathfrak{g}) \). Note that \( T_1(\xi \otimes x) \) is an element of \( U_q(\mathfrak{g}) \).

The following crucial lemma is essentially from [ZGB91a], where the setting is slightly different from ours. We include a proof in Appendix [A].

Lemma 2.1. [ZGB91a] Proposition 1] Given an operator \( \Gamma_V \in \text{End}(V) \otimes U_q(\mathfrak{g}) \) satisfying
\[
[\Gamma_V, \Delta(K_i^{\pm 1})] = [\Gamma_V, \Delta(E_i)] = [\Gamma_V, \Delta(F_i)] = 0, \quad \forall i,
\]
the elements \( C_V^{(k)} \) for \( k = 1, 2, \ldots \) defined by
\[
C_V^{(k)} := \text{Tr}_1((K_{2\rho} \otimes 1)(\Gamma_V)^k)
\]
are central in \( U_q(\mathfrak{g}) \), where \( \rho \) denotes the half sum of positive roots of \( \mathfrak{g} \).

Using the quasi \( R \)-matrix of \( U_q(\mathfrak{g}) \), we shall construct an explicit operator \( \Gamma_V \) satisfying (2.1).

Recall that the Drinfeld version of quantum group defined over formal power series \( \mathbb{C}[[\hbar]] \) admits a universal \( R \) matrix [Dri80, ZGB91a], which is absent for the quantum group \( U_q(\mathfrak{g}) \) considered here. But a quasi \( R \)-matrix \( \mathcal{R} \) exists for \( U_q(\mathfrak{g}) \), which can be described as follows [Lus10].
Let $U_q(g)\otimes U_q(g)$ be a completion of the tensor product $U_q(g) \otimes U_q(g)$. There is an algebra automorphism $\phi$ of $U_q(g) \otimes U_q(g)$ defined by
\[
\phi(K_i \otimes 1) = K_i \otimes 1, \quad \phi(E_i \otimes 1) = E_i \otimes K_i^{-1}, \quad \phi(F_i \otimes 1) = F_i \otimes K_i,
\]
and $\phi$ can be extended to $U_q(g)\otimes U_q(g)$. The quasi $R$-matrix $\mathcal{R}$ is an element of $U_q(g)\otimes U_q(g)$ which has the form
\[
\mathcal{R} = 1 \otimes 1 + \sum_{\nu > 0} \Theta_{\nu} \in U_q(g)\otimes U_q(g),
\]
where $\Theta_{\nu} \in U_{-\nu} \otimes U_{\nu}$ for positive $\nu \in Q$. An explicit formula for $\mathcal{R}$ can be found in, e.g., $[KS97, \S8.3.3]$ (see Example $2.4$ for case of $U_q(sl_2)$). The quasi $R$-matrix satisfies the following relations
\[
(2.3) \quad \mathcal{R} \Delta(x) = \phi(\Delta'(x))\mathcal{R}, \quad \mathcal{R}^T \Delta'(x) = \phi(\Delta(x))\mathcal{R}^T,
\]
where $\mathcal{R}^T = T(\mathcal{R})$ with $T$ being the linear map defined by $T(x \otimes y) = y \otimes x$ for $x, y \in U_q(g)$ (see, e.g., $[Tan92, \S4.3]$).

In what follows, we assume that the $U_q(g)$-module $V$ satisfies
\[
(2.4) \quad \Pi(V) \subset M = \frac{1}{2} Q \cap P,
\]
where $\Pi(V)$ is the set of all weights of $V$. Denote by $\zeta_V : U_q(g) \to GL(V)$ the linear representation. We define the following elements of $\text{End}(V) \otimes U_q(g)$:
\[
(2.5) \quad \mathcal{R}_V := (\zeta_V \otimes \text{id})(\mathcal{R}), \quad \tilde{\mathcal{R}}_V^T := (\zeta_V \otimes \text{id})\phi(\mathcal{R}^T)
\]
On the other hand, we define the diagonal part by
\[
(2.6) \quad \mathcal{K}_V := \sum_{\eta \in \Pi(V)} P_\eta \otimes K_{2\eta},
\]
where $P_\eta$ is the linear projection from $V$ to its weight space $V_\eta$. Note that the condition $(2.3)$ guarantees that $2\eta \in Q$ for any weight $\eta$ of $V$ and hence $\mathcal{K}_V \in \text{End}(V) \otimes U_q(g)$.

**Definition 2.2.** Given a finite dimension $U_q(g)$-module $V$ whose weights are contained in $M = \frac{1}{2} Q \cap P$, we define the operator
\[
\Gamma_V := \mathcal{K}_V \tilde{\mathcal{R}}_V^T \mathcal{R}_V \in \text{End}(V) \otimes U_q(g),
\]
where $\mathcal{R}_V$ and $\tilde{\mathcal{R}}_V^T$ are given in $(2.5)$ and $\mathcal{K}_V$ is defined by $(2.6)$. Let
\[
C^{(k)}_V := \text{Tr}_1((K_{2\rho} \otimes 1)(\Gamma_V)^k), \quad k = 1, 2, \ldots,
\]
and write $C_V = C^{(1)}_V$.

By Lemma $2.1$ and Proposition $2.3$ below, $C^{(k)}_V$ are central elements of $U_q(g)$.

**Proposition 2.3.** The element $\Gamma_V$ given in Definition $2.2$ satisfies the commutative relations $(2.1)$, i.e., $[\Gamma_V, \Delta(x)] = 0$ for any $x \in U_q(g)$.

A proof of Proposition $2.3$ is given in Appendix $A$.

The following is an example of our construction.
Example 2.4. Let \( g = \mathfrak{sl}_2 \). The quasi \( R \)-matrix of \( U_q(\mathfrak{sl}_2) \) is given by

\[
R = \sum_{n=0}^{\infty} q^{\frac{(-1)^n n(n+1)}{2}} (1 - q^{-2})^n F_n \otimes E^n.
\]

Let \( V \) be the 2-dimensional simple module, then the weights of \( V \) are contained in \( M = \frac{1}{2} Q \cap P = \mathbb{Z} \). Denote by \( \zeta : U_q(g) \rightarrow \text{End}(V) \) the representation corresponding to the standard basis \( \{ e_1, e_2 \} \) of \( V \), we have

\[
\zeta(E) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \zeta(F) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \zeta(K) = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}.
\]

Now

\[
\mathcal{R}_V = 1 \otimes 1 + (q - q^{-1})\zeta(F) \otimes E,
\]

\[
\tilde{\mathcal{R}}_V = 1 \otimes 1 + (q - q^{-1})\zeta(EK) \otimes K^{-1} F,
\]

\[
K_V = P_1 \otimes K + P_{-1} \otimes K^{-1},
\]

where \( P_1 \) (resp. \( P_{-1} \)) is the linear projection from \( V \) onto the weight space \( \mathbb{C}(q)e_1 \) (resp. \( \mathbb{C}(q)e_2 \)), and we have \( P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \) (res. \( P_{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \)). We have

\[
\Gamma_V = K_V \tilde{\mathcal{R}}_V \mathcal{R}_V
\]

\[
= P_1 \otimes K + P_{-1} \otimes K^{-1} + (q - q^{-1})\zeta(F) \otimes K^{-1} E
\]

\[
+ (1 - q^{-2})\zeta(E) \otimes F + (q - q^{-1})^2 q^{-1} P_1 \otimes FE.
\]

Note that in \( \Gamma_V \) the first tensor factors \( \zeta(E) \) and \( \zeta(F) \) have no contributions to the partial trace \( \text{Tr}_1 \). Using \( K_{2p} P_{\pm 1} = KP_{\pm 1} = q^{\pm 1} P_{\pm 1} \), we obtain the following central element associated to \( V \):

\[
C_V = \text{Tr}_1((K_{2p} \otimes 1) \Gamma_V) = qK + q^{-1} K^{-1} + (q - q^{-1})^2 FE.
\]

By similar straightforward calculation, one can also express the higher order central elements \( C^{(k)} \) as \( \mathbb{C}(q) \)-linear combinations of the powers \( C_V^k \):

\[
\begin{align*}
C_V^{(2)} &= q^{-1} C_V^2 - q^{-1} - q^{-3}, \\
C_V^{(3)} &= q^{-2} C_V^3 - (2q^{-2} + q^{-4}) C_V, \\
C_V^{(4)} &= q^{-3} C_V^4 - (3q^{-3} + q^{-5}) C_V^2 + q^{-3} + q^{-5}.
\end{align*}
\]

2.3. The main theorem. We shall state our main theorem which exhibits explicit generators and relations of the centre of the quantum group.

Recall from Definition 2.2 that the central elements are associated with \( U_q(g) \)-modules \( V \) whose weights are contained \( M \). In particular, the highest weights of these modules are contained in \( M^+ = \frac{1}{2} Q \cap P^+ \), which is an additive monoid, i.e., a commutative semigroup with the identity 0. An element \( x \in M^+ \) is said to be irreducible if \( x = y + z \) implies either \( y = 0 \) or \( z = 0 \). The Hilbert basis \( \text{Hilb}(M^+) \) of \( M^+ \) is a minimal set of generators given by its irreducible elements.

The generators of the centre \( Z(U_q(g)) \) can be chosen in bijection with the elements of \( \text{Hilb}(M^+) \). Given any \( \lambda = \sum_{i=1}^{n} a_i \omega_i \in \text{Hilb}(M^+) \), we define the tensor module

\[
T(\lambda) := \bigotimes_{i=1}^{n} L(\omega_i)^{\otimes a_i},
\]
where $L(\varpi_i)$ is the fundamental representation of $U_q(\mathfrak{g})$. Note that for any weight $\mu \in \Pi(T(\lambda))$, we have $\lambda - \mu \in Q \subseteq \frac{1}{2}Q \cap P$. Therefore, $\Pi(T(\lambda)) \subseteq M$ and we may define the associated central element $C_{T(\lambda)}$.

Particularly, if $\mathfrak{g}$ is of type $A_1$, $B_n(n \geq 2)$, $C_n(n \geq 3)$, $D_{2k}(k \geq 2)$, $E_7$, $E_8$, $F_4$, and $G_2$, we will show that the Hilbert basis $\text{Hilb}(M^+)$ consists of all fundamental weights of $\mathfrak{g}$. Hence we obtain $n$ generators $C_{L(\varpi_i)}$ of $Z(U_q(\mathfrak{g}))$, which will be shown to be algebraically independent.

To describe relations among central elements $C_{T(\lambda)}$ for $\mathfrak{g}$ of one of the remaining types $A_n(n \geq 2)$, $D_{2k+1}(k \geq 2)$ and $E_6$, we introduce the automorphism $\sigma$ of the corresponding Dynkin diagram. This is depicted as in Figure 1, where each pair of vertices which are connected by a curved double arrow means they are swapped by the involution $\sigma$ and the rest vertices are fixed by $\sigma$. For instance, $\sigma(i) = n + 1 - i, 1 \leq i \leq n$ for type $A_n(n \geq 2)$.

![Figure 1. The involutions $\sigma$ of types $A$, $D$ and $E_6$.](image)

The automorphism $\sigma$ induces an involution $\sigma_{M^+}$ of the monoid $M^+$. Precisely, if $\lambda = \sum_{i=1}^{n} a_i \varpi_i \in \text{Hilb}(M^+)$, then we define $\overline{\lambda} := \sigma_{M^+}(\lambda) = \sum_{i=1}^{n} a_{\sigma(i)} \varpi_i$; refer to Lemma 3.6. The element $\lambda$ is said to be self-conjugate if $\lambda = \overline{\lambda}$; otherwise, it is called non-self-conjugate. These elements will be characterised explicitly in Lemma 3.8 and Lemma 3.10.

The following is our main theorem of this paper.

**Theorem 2.5.** Let $\mathfrak{g}$ be a complex simple Lie algebra of rank $n$, and let $\text{Hilb}(M^+)$ be the Hilbert basis of the monoid $M^+ = \frac{1}{2}Q \cap P^+$, where $\frac{1}{2}Q$ denotes the half root lattice and $P^+$ is the monoid of dominant weights of $\mathfrak{g}$.

1. If $\mathfrak{g}$ is one of the types $A_1$, $B_n(n \geq 2)$, $C_n(n \geq 3)$, $D_{2k}(k \geq 2)$, $E_7$, $E_8$, $F_4$, and $G_2$, then the centre $Z(U_q(\mathfrak{g}))$ of the quantum group $U_q(\mathfrak{g})$ is generated by $n$ algebraically independent elements $C_{L(\varpi_1)}, \ldots, C_{L(\varpi_n)}$, where $L(\varpi_i)$ are simple modules corresponding to the fundamental weights $\varpi_i$;

2. If $\mathfrak{g}$ is one of the types $A_n(n \geq 2)$, $D_{2k+1}(k \geq 2)$ and $E_6$, then the centre $Z(U_q(\mathfrak{g}))$ of the quantum group $U_q(\mathfrak{g})$ is generated by $C_{T(\lambda)}, \lambda \in \text{Hilb}(M^+)$, subject to the following relations:

$$C_{T(\lambda)} C_{T(\lambda)} = \prod_{i : i < \sigma(i)} C_{T(\mu_i)}^{\max\{a_i, a_{\sigma(i)}\}},$$

$$C_{T(\lambda)}^{\ell(\lambda)} = \prod_{i=1}^{n} C_{T(\nu_i)}^{\ell(\lambda) a_i / s_i} \text{ with } \lambda \neq \nu_i, 1 \leq i \leq n$$

for each non-self-conjugate pair $\{\lambda, \overline{\lambda}\}$ of $\text{Hilb}(M^+)$ with $\lambda = \sum_{i=1}^{n} a_i \varpi_i$ and $\overline{\lambda} = \sum_{i=1}^{n} a_{\sigma(i)} \varpi_i$, where $\sigma$ is the involution of the Dynkin diagram given by Figure 1. $\mu_i \in \text{Hilb}(M^+)$ are self-conjugate elements given by Lemma 3.8.
\( \nu_i = s_i \varpi_i \in \text{Hilb}(M^+) \) are scalar multiples of the fundamental weights \( \varpi_i \) with \( s_i \) given by Lemma 3.13 and \( \ell(\lambda) \) is a positive integer defined by (3.3).

The remainder of the paper is on the proof of Theorem 2.5. The main idea of the proof is to show that the centre \( Z(U_q(g)) \) is isomorphic to the monoid algebra \( \mathbb{C}(q)[M^+] \), which is more conceptual and will be studied systematically in Section 3. The actual proof of Theorem 2.5 is given in Section 4.

3. The monoid \( M^+ \)

In this section we describe the Hilbert basis \( \text{Hilb}(M^+) \) for the monoid \( M^+ \) associated to a simple Lie algebra \( g \). Using the automorphism of the Dynkin diagram, we give a presentation of the monoid algebra of \( M^+ \).

3.1. The Hilbert basis of \( M^+ \). Some results in this subsection can be found in [LXZ16]. We include proofs for them to make the paper more accessible.

In the sequel, for explicit formulae of fundamental weights of \( g \) we refer to [Hum72] §13.2, Table 1, and for Dynkin diagrams we refer to [Hum72] §11, Theorem 11.4.

**Lemma 3.1.** [LXZ16] Lemma 3.4] For each simple Lie algebra \( g \), the Hilbert basis \( \text{Hilb}(M^+) \) is finite.

**Proof.** Observe from [Hum72] §13.2, Table 1] that for every fundamental weight \( \varpi_i \), there exists a minimal positive integer \( s_i \) such that \( s_i \varpi_i \in M^+ \). Given any \( \lambda = \sum_{i=1}^{n} a_i \varpi_i \in \text{Hilb}(M^+) \), we just need to prove that \( 0 \leq a_i \leq s_i \) for all \( i = 1, \ldots, n \).

Assuming for contradiction that there exists an index \( i_0 \) such that \( a_{i_0} > m_{i_0} \), we have

\[
\lambda = \sum_{i \neq i_0} a_i \varpi_i + (\alpha_{i_0} - s_{i_0}) \varpi_{i_0} + s_{i_0} \varpi_{i_0}.
\]

Let \( \mu = \sum_{i=1}^{n} a_i \varpi_i + (\alpha_{i_0} - s_{i_0}) \varpi_{i_0} \). Then we have \( \mu \in P^+ \). As \( \lambda \in \frac{1}{2}Q \) and \( s_{i_0} \varpi_{i_0} \in \frac{1}{2}Q \), we also have \( \mu = \lambda - s_{i_0} \varpi_{i_0} \in \frac{1}{2}Q \) and hence \( \mu \in M^+ = \frac{1}{2}Q \cap P^+ \). It follows that \( \lambda \) is a sum of two nonzero elements \( \mu \) and \( s_{i_0} \varpi_{i_0} \) of \( M^+ \), contrary to the irreducibility of \( \lambda \). \( \square \)

**Lemma 3.2.** [LXZ16] Lemma 3.5 Let \( \varpi_i \) be the fundamental weights of \( g \).

1. If \( g \) is of one of types \( A_1, B_n(n \geq 2), C_n(n \geq 3), D_{2k}(k \geq 2), E_7, E_8, F_4, \) and \( G_2 \), we have

\[
\text{Hilb}(M^+) = \{ \varpi_1, \cdots, \varpi_n \}.
\]

2. If \( g \) is of type \( D_{2k+1}(k \geq 2) \), we have

\[
\text{Hilb}(M^+) = \{ \varpi_1, \cdots, \varpi_n-2, 2\varpi_{n-1}, 2\varpi_n, \varpi_{n-1} + \varpi_n \},
\]

where \( n = 2k + 1 \).

3. If \( g \) is of type \( E_6 \), we have

\[
\text{Hilb}(M^+) = \{ 3\varpi_1, \varpi_2, 3\varpi_3, \varpi_4, 3\varpi_5, 3\varpi_6, \varpi_1 + \varpi_3, \varpi_1 + \varpi_6, \varpi_3 + \varpi_5, \varpi_5 + \varpi_6, \varpi_1 + 2\varpi_5, 2\varpi_1 + \varpi_5, \varpi_3 + 2\varpi_6, 2\varpi_3 + 2\varpi_6 \}.
\]

**Proof.** Part (1) can be checked case by case by using [Hum72] §13.2, Table 1]. One has \( M^+ = \frac{1}{2}Q \cap P^+ = P^+ \) in these cases, and hence \( \text{Hilb}(M^+) \) consists of all fundamental weights.
For part (2), let $s_i$ be the smallest positive integer such that $s_i \omega_i \in \frac{1}{2}Q$ for $1 \leq i \leq n$. Using [Hum72, §13.2, Table 1], it is easy to see that

$$s_i = 1, \quad 1 \leq i \leq n - 2, \quad \text{and} \quad s_{n-1} = s_n = 2,$$

where $n \geq 3$ is an odd integer. It follows that $\omega_i \in \text{Hilb}(M^+)$ for $1 \leq i \leq n - 2$ and also $2 \omega_{n-1}, 2 \omega_n \in \text{Hilb}(M^+)$. Next we consider the irreducible element of $\text{Hilb}(M^+)$ which can be written as a sum of fundamental weights. Assume that $\lambda = \sum_{i=1}^n a_i \omega_i \in \text{Hilb}(M^+)$ is not a multiple of some fundamental weight. Then by the proof of Lemma 3.1, we have $a_i \leq s_i$ for each $i$. It is verified readily that $\omega_{n-1} + \omega_n \in \frac{1}{2}Q$, while $\omega_i + \omega_j \notin \frac{1}{2}Q$ for any $i \in \{1, \ldots, n-2\}$ and $j \in \{n-1, n\}$. Therefore, the only irreducible element which is a sum of fundamental weights is $\omega_{n-1} + \omega_n$. This completes the proof.

Part (3) can be done similarly with a suitable computer program.

Now we consider the case of type $A_n$ for $n \geq 2$. The following lemma is useful.

**Lemma 3.3.** [LXZ16, Lemma 4.3] Let $\lambda = \sum_{i=1}^n a_i \omega_i \in P^+$ be a dominant weight of the Lie algebra of type $A_n$. Then $\lambda \in M^+$ if and only if $\sum_{i=1}^n ia_i \in r_{n+1} \mathbb{Z}$, where $r_{n+1} = \frac{n+1}{\gcd(n+1,2)}$.

**Proof.** We need to show that $\lambda \in \frac{1}{2}Q$ if and only if the given condition holds. Note that in the case of type $A$ we have $\omega_i = \omega_1 - (\alpha_i - 1 + 2\alpha_{i-2} + \ldots (i-1)\alpha_{i-1})$ for $2 \leq i \leq n$. Then we have the expression

$$\lambda = a_1 \omega_1 + \sum_{i=2}^n a_i (i \omega_1 - (\alpha_i - 1 + 2\alpha_{i-2} + \ldots (i-1)\alpha_{i-1}))$$

$$= (\sum_{i=1}^n ia_i) \omega_1 - \sum_{i=2}^n a_i (\alpha_i - 1 + 2\alpha_{i-2} + \ldots (i-1)\alpha_{i-1}).$$

It follows that $\lambda \in \frac{1}{2}Q$ if and only if $(\sum_{i=1}^n ia_i) \omega_1 \in \frac{1}{2}Q$. Recall that $\omega_1 = \frac{1}{n+1} \omega_{n+1}$. We have $(\sum_{i=1}^n ia_i) \omega_1 \in \frac{1}{2}Q$ if and only if $r_{n+1} = \frac{n+1}{\gcd(n+1,2)}$ divides $\sum_{i=1}^n ia_i$.

**Corollary 3.4.** In the case of type $A_n$, assume that $\lambda = \sum_{i=1}^n a_i \omega_i \in \text{Hilb}(M^+)$. Then we have $\sum_{i=1}^n a_i \leq r_{n+1}$, where $r_{n+1} = \frac{n+1}{\gcd(n+1,2)}$.

**Proof.** For convenience, we denote $\lambda$ by $(a_1, a_2, \ldots, a_n) \in \mathbb{N}^n$, where $\mathbb{N}$ is the set of non-negative integers. Let $\prec$ be the lexicographical order on $\mathbb{N}^n$. Without loss of generality, we may assume $a_1 \neq 0$. Then there is a strictly increasing sequence:

$$\lambda_1 < \lambda_2 < \cdots < \lambda_p = \lambda,$$

where $p = \sum_{i=1}^n a_i$, $\lambda_1 = (1, 0, \ldots, 0)$, and $\lambda_{i+1} - \lambda_i = \omega_j$ for some $j \geq i$. For any $\lambda_i = (a_1, \ldots, a_i)$, we define

$$b_i \equiv \sum_{j=1}^i ja_{ij} \mod r_{n+1}, \quad 1 \leq i \leq p.$$

Assume for contradiction that $p > r_{n+1}$. Then by the pigeonhole principle, there exists a pair $i_1 < i_2$ such that $b_{i_1} = b_{i_2}$, which implies that $r_{n+1}$ divides $\sum_{j=1}^i j(a_{ij} - a_{i_2j})$. It follows from Lemma 3.3 that $\lambda_{i_2} - \lambda_{i_1} \in M^+$, and hence $\lambda = (\lambda_{i_2} - \lambda_{i_1}) + (\lambda - \lambda_{i_2} + \lambda_{i_1})$, contrary to the irreducibility of $\lambda$. □
In accord with the previous lemmas, we split simple Lie algebras into the following two types:

Type I: \( A_1, B_n(n \geq 2), C_n(n \geq 3), D_{2k}(k \geq 2), E_7, E_8, F_4, \) and \( G_2 \).

Type II: \( A_n(n \geq 2), D_{2k+1}(k \geq 2) \) and \( E_6 \).

For each Lie algebra \( \mathfrak{g} \) of type I, the Hilbert basis \( \text{Hilb}(M^+) \) comprises exactly all fundamental weights of \( \mathfrak{g} \). For type II, we will focus on the symmetry property of \( M^+ \) derived from the involution of the corresponding Dynkin diagram. This is treated in the following subsection.

Remark 3.5. The classification (3.1) is also in accord with the fact that \(-1 \in W\) if and only if \( \mathfrak{g} \) is of type I (see, e.g., [Bou68, Chapter V, §6.2, Corollary 3]). In this case, \(-1\) is the longest element of \( W\).

3.2. The involution of \( M^+ \). For the purpose of this paper, we are only concerned with the involutions corresponding to Lie algebras of type II.

Lemma 3.6. Let \( \mathfrak{g} \) be a simple Lie algebra of type II, and let \( \sigma \) be the automorphism of the Dynkin diagram given in Figure 4. Then we have the involution of \( M^+ \):

\[
\sigma_{M^+} : M^+ \to M^+, \quad \lambda = \sum_{i=1}^{n} a_i \omega_i \mapsto \bar{\lambda} = \sum_{i=1}^{n} a_{\sigma(i)} \omega_i
\]

such that \( \sigma_{M^+}^2 = 1 \). In particular, if \( \lambda \in \text{Hilb}(M^+) \), then \( \bar{\lambda} \in \text{Hilb}(M^+) \).

Proof. We only give the proof for type \( A \), and the other cases can be treated similarly. By definition \( \sigma \) sends the simple root \( \alpha_i \) to \( \alpha_{\sigma(i)} = \alpha_{n+1-i} \), and hence \( \sigma \) gives rise to an involution \( \psi \) of the vector space \( \mathbb{Q}Q := \mathbb{Q} \otimes \mathbb{Z} Q \) over \( \mathbb{Q} \). In particular, \( \psi \) restricts to an involution of the half root lattice \( \frac{1}{2}Q \). On the other hand, recall that the fundamental weights are given by

\[
\omega_i = \sum_{j=1}^{i} \frac{(n+1-i)j}{n+1} \alpha_j + \sum_{j=i+1}^{n} \frac{(n+1-j)i}{n+1} \alpha_j \in \mathbb{Q}Q, \quad 1 \leq i \leq n.
\]

It is straightforward to check that

\[
\psi(\omega_i) = \sum_{j=1}^{i} \frac{(n+1-i)j}{n+1} \alpha_{n+1-j} + \sum_{j=i+1}^{n} \frac{(n+1-j)i}{n+1} \alpha_{n+1-j} = \sum_{k=1}^{n+1-i} \frac{ik}{n+1} \alpha_k + \sum_{k=n+2-i}^{n} \frac{(n+1-i)(n+1-k)}{n+1} \alpha_k = \omega_{n+i-1}.
\]

It follows that \( \psi \) induces an involution of the monoid \( P^+ \) with \( \psi(\omega_i) = \omega_{n+1-i} \) for \( 1 \leq i \leq n \). Therefore, the restriction \( \sigma_{M^+} := \psi|_{M^+} \) is an involution satisfying

\[
\sigma_{M^+} \left( \sum_{i=1}^{n} a_i \omega_i \right) = \sum_{i=1}^{n} a_i \omega_{\sigma(i)} = \sum_{i=1}^{n} a_{\sigma(i)} \omega_i,
\]

and \( \sigma_{M^+}^2 = 1 \). For the last assertion, it is clear that \( \lambda \) is irreducible if and only if its image \( \bar{\lambda} \) is irreducible. \( \square \)

Definition 3.7. The elements \( \lambda \in M^+ \) satisfying \( \bar{\lambda} = \lambda \) are said to be self-conjugate; the other elements of \( M^+ \) are called non-self-conjugate.
Now we can split the finite generating set $\text{Hilb}(M^+)$ into two disjoint subsets, depending on whether they are self-conjugate. In the following we figure out the self-conjugate elements of $\text{Hilb}(M^+)$.  

**Lemma 3.8.** Let $\sigma$ be the involution of the Dynkin diagram given in Figure 1 and let $\lambda \in \text{Hilb}(M^+)$. Then $\lambda = \bar{\lambda}$ if and only if $\lambda$ is of the following form:

1. $\mu_i = \varpi_i + \varpi_{\sigma(i)}$ for all $i$ with $i < \sigma(i)$;
2. $\mu_i = \varpi_i$ for all $i$ with $\sigma(i) = i$.

**Proof.** Assume that $\lambda = \sum_{i=1}^n a_i \varpi_i \in \text{Hilb}(M^+)$ satisfies $\overline{\lambda} = \lambda$. Then we have $a_i = a_{\sigma(i)}$ for $1 \leq i \leq n$. It follows that

$$\lambda = \sum_{i: i < \sigma(i)} a_i \varpi_i + \sum_{i: \sigma(i) = i} a_i \varpi_i.$$  

It suffices to show that $\varpi_i + \varpi_{\sigma(i)} \in \text{Hilb}(M^+)$ whenever $i < \sigma(i)$ and $\varpi_i \in \text{Hilb}(M^+)$ whenever $\sigma(i) = i$.

We only do it for type $A$, and the other two cases can be treated similarly. For $\varphi$ of type $A_n$, we first claim that $\varpi_i \not\in M^+$ and hence $\varpi_i \not\in \text{Hilb}(M^+)$ whenever $i < \sigma(i)$. By definition (refer to Figure 1) $i < \sigma(i)$ if and only if either $n$ is even or $n$ is odd and $i \neq \frac{n+1}{2}$. Using Lemma 3.3 we obtain that $\varpi_i \not\in M^+$ for $1 \leq i \leq n$ if $n$ is even, and $\varpi_i \not\in M^+$ for $i \neq \frac{n+1}{2}$ if $n$ is odd. Thus our claim follows. Secondly, by Lemma 3.3 we have $\varpi_i + \varpi_{\sigma(i)} \in M^+$ for all $i$ with $i < \sigma(i)$. Moreover, by our claim $\varpi_i + \varpi_{\sigma(i)}$ is irreducible and hence $\varpi_i + \varpi_{\sigma(i)} \in \text{Hilb}(M^+)$. It remains to deal with the case $\sigma(i) = i$. This happens if and only if $n$ is odd and $i = \frac{n+1}{2}$. In this case, $\varpi_{\frac{n+1}{2}} \in \text{Hilb}(M^+)$ by Lemma 3.3. \hfill $\square$

**Example 3.9.** Using Lemma 3.8 we can write out all self-conjugate elements $\mu_i$ of $\text{Hilb}(M^+)$ explicitly as follows (the indices of fundamental weights $\varpi_i$ are in accord with the labellings of Dynkin diagrams Figure 1).

1. **Type $A_n (n \geq 2)$.** If $n$ is even, we have
   $$\mu_i := \varpi_i + \varpi_{n+1-i}, \quad 1 \leq i \leq \frac{n}{2}.$$  
   If $n$ is odd, we have
   $$\mu_i := \varpi_i + \varpi_{n+1-i}, \quad 1 \leq i \leq \frac{n+1}{2} - 1,$$
   $$\mu_{\frac{n+1}{2}} := \varpi_{\frac{n+1}{2}}.$$  
2. **Type $D_{2k+1} (k \geq 2)$.** We have
   $$\mu_i := \varpi_i, \quad 1 \leq i \leq n-2, \quad \mu_{n-1} := \varpi_{n-1} + \varpi_n,$$
   where $n = 2k+1$. This can also be verified directly by Lemma 3.2.
3. **Type $E_6$.** We have
   $$\mu_1 := \varpi_1 + \varpi_6, \quad \mu_2 = \varpi_2, \quad \mu_3 := \varpi_3 + \varpi_5, \quad \mu_4 = \varpi_4.$$  

For the non-self-conjugate elements of $\text{Hilb}(M^+)$, we have the following.

**Lemma 3.10.** Let $\lambda = \sum_{i=1}^n a_i \varpi_i \in \text{Hilb}(M^+)$. For each $i = 1, \ldots, n$, we have

1. If $i \neq \sigma(i)$, then either $a_i a_{\sigma(i)} = 0$ or $a_i = a_{\sigma(i)} = 1$. In the latter case, all other $a_i$ are 0, i.e. $\lambda = w_i + w_{\sigma(i)}$.
(2) If \( i = \sigma(i) \), then either \( a_i = 0 \) or \( a_i = 1 \). In the latter case, all other \( a_i \) are 0, i.e. \( \lambda = \varpi_i \).

Therefore, for any non-self-conjugate element \( \lambda \in \text{Hilb}(M^+) \), we have \( a_i a_{\sigma(i)} = 0 \) for \( i \neq \sigma(i) \), and \( a_i = 0 \) for \( i = \sigma(i) \).

Proof. For part (1), we assume that \( i \neq \sigma(i) \). If \( a_i = a_{\sigma(i)} \), then we may write

\[
\lambda = a_i (\varpi_i + \varpi_{\sigma(i)}) + \sum_{j:j \neq i, \sigma(i)} a_j \varpi_j.
\]

It follows from Lemma 3.8 that \( \varpi_i + \varpi_{\sigma(i)} \in \text{Hilb}(M^+) \). By the irreducibility of \( \lambda \), we have \( a_i = a_{\sigma(i)} = 1 \) and all other \( a_i \) are 0, i.e. \( \lambda = \varpi_i + \varpi_{\sigma(i)} \). If \( a_i \neq a_{\sigma(i)} \), we need to prove that one of \( a_i \) and \( a_{\sigma(i)} \) is 0, that is, \( a_i a_{\sigma(i)} = 0 \). Assume for contradiction that \( 0 < a_i < a_{\sigma(i)} \). Then we may write

\[
\lambda = a_i (\varpi_i + \varpi_{\sigma(i)}) + (a_{\sigma(i)} - a_i) \varpi_{\sigma(i)} + \sum_{j:j \neq i, \sigma(i)} a_j \varpi_j.
\]

By the irreducibility of \( \lambda \) we have \( a_i = a_{\sigma(i)} = 1 \), which is a contradiction.

Part (2) can be proved similarly, by using the fact from Lemma 3.8 that \( w_i \in \text{Hilb}(M^+) \) if \( i = \sigma(i) \).

Lemma 3.11. For any non-self-conjugate element \( \lambda = \sum_{i=1}^n a_i \varpi_i \in \text{Hilb}(M^+) \), we have the relation

\[
\lambda + \overline{\lambda} = \sum_{i:i < \sigma(i)} \max\{a_i, a_{\sigma(i)}\} \mu_i.
\]

Proof. As \( \lambda + \overline{\lambda} \) is fixed by \( \sigma_{M^+} \), it can be expressed linearly by elements \( \mu_i \) given in Lemma 3.8. On the other hand, since \( \overline{\lambda} \neq \lambda \) we have \( a_i = 0 \) for \( i = \sigma(i) \) and \( a_i a_{\sigma(i)} = 0 \) for \( i \neq \sigma(i) \) by Lemma 3.10. It follows that

\[
\lambda + \overline{\lambda} = \sum_{i:i < \sigma(i)} (a_i + a_{\sigma(i)}) \mu_i = \sum_{i:i < \sigma(i)} \max\{a_i, a_{\sigma(i)}\} \mu_i,
\]

where the second equation holds since one of \( a_i \) and \( a_{\sigma(i)} \) is zero.

Example 3.12. For convenience, we denote \( \lambda = \sum_{i=1}^n a_i \varpi_i \) by \( (a_1, a_2, \ldots, a_n) \). Then in type \( A \) case \( \overline{\lambda} = (a_n, a_{n-1}, \ldots, a_1) \).

(1) Type \( A_2 \). Self-conjugate: \((1, 1)\), and non-self-conjugate: \((3, 0)\) and \((0, 3)\).

(2) Type \( A_3 \). Self-conjugate: \((1, 0, 1)\), \((0, 1, 0)\), and non-self-conjugate: \((2, 0, 0)\), \((0, 0, 2)\).

(3) Type \( A_4 \). Self-conjugate: \((1, 0, 0, 1)\), \((0, 1, 1, 0)\), and non-self-conjugate:

\[
(5, 0, 0, 0), (0, 5, 0, 0), (2, 0, 1, 0), (1, 2, 0, 0), (3, 1, 0, 0), (1, 0, 3, 0),
(0, 0, 0, 5), (0, 0, 5, 0), (0, 1, 0, 2), (0, 0, 2, 1), (0, 0, 1, 3), (0, 3, 0, 1).
\]

We proceed to explore other relations among elements of \( \text{Hilb}(M^+) \). Note that for each fundamental weight \( \varpi_i \) there exists a minimal positive integer \( s_i \) such that \( s_i \varpi_i \in M^+ \). Since \( s_i \) is minimal, we have \( s_i \varpi_i \in \text{Hilb}(M^+) \). Therefore, we may form a sequence \( (s_1, s_2, \ldots, s_n) \), which is determined for each Lie algebra of type II as follows (note that for type I all \( s_i \) are equal to 1).

Lemma 3.13. Let \( (s_1, s_2, \ldots, s_n) \) be a sequence of minimal positive integers such that \( s_i \varpi_i \in \text{Hilb}(M^+) \) for each \( i \). Then
(1) For type $A_n (n \geq 2)$, we have
\[ s_i = s_{n+1-i} = \frac{n+1}{\gcd(n+1, 2i)}, \quad 1 \leq i \leq n. \]

(2) For type $D_{2k+1} (k \geq 2)$, we have
\[ s_1 = \cdots = s_{n-1} = 1, \quad s_{n-1} = s_n = 2. \]

(3) For type $E_6$, we have
\[ s_1 = s_3 = s_5 = s_6 = 3, \quad s_2 = s_4 = 1. \]

Recalling that $\sigma$ is the involution of the Dynkin diagram given in Figure 1, we have $s_i = s_{\sigma(i)}$ for all $i$. In particular, $s_i = s_{\sigma(i)} = 1$ if and only if $i = \sigma(i)$.

**Proof.** In the case of type $A_n (n \geq 2)$, recall from Lemma 3.3 that $s_i \varpi_i \in M^+$ if and only if $r_{n+1}|i s_i$, where $r_{n+1} = (n+1)/\gcd(n+1, 2)$. By the minimality we have $s_i = (n+1)/\gcd(r_{n+1}, i) = (n+1)/\gcd(n+1, 2i)$. The other two cases can be verified directly by Lemma 3.2.

For any $\lambda = \sum_{i=1}^n a_i \varpi_i \in M^+$ we may define
\begin{equation}
\ell(\lambda) := \text{lcm}\{s_i \mid 1 \leq i \leq n \text{ and } a_i \neq 0\},
\end{equation}
i.e., $\ell(\lambda)$ is the least common multiple of $s_i$ for all $i$ for which $a_i$ is nonzero. The following lemma is trivial.

**Lemma 3.14.** Let $(s_1, \ldots, s_n)$ be the sequence as defined in Lemma 3.13 and let $\nu_i = s_i \varpi_i$ for $1 \leq i \leq n$. For any $\lambda = \sum_{i=1}^n a_i \varpi_i \in M^+$, we have the relation
\[ \ell(\lambda) \lambda = \sum_{i=1}^n \frac{\ell(\lambda) a_i}{s_i} \nu_i, \]
i.e., $\ell(\lambda) \lambda$ is an integer combination of $\nu_i$.

We have obtained relations among generators of Hilb($M^+$) from Lemma 3.11 and Lemma 3.14. Next we will show that these relations are enough in the sense that they give rise to complete relations among generators of the monoid algebra of $M^+$.

### 3.3. The monoid algebra

Given the monoid $M^+$ of any simple Lie algebra, we may form the monoid algebra $\mathbb{C}[M^+]$. As a $\mathbb{C}$-vector space, $\mathbb{C}[M^+]$ has a basis consisting of symbols $X^\lambda, \lambda \in M^+$, with multiplication given by the bilinear extension of $X^\lambda X^\mu = X^{\lambda+\mu}$. We agree that $X^0 = 1$.

Clearly, $\{X^\lambda \mid \lambda \in \text{Hilb}(M^+)\}$ is a generating set of $\mathbb{C}[M^+]$. In what follows, we shall determine relations among these generators, and hence obtain a presentation of the monoid algebra $\mathbb{C}[M^+]$.

Recall that for the simple Lie algebra of type I considered in Lemma 3.2 the Hilbert basis Hilb($M^+$) consists of all fundamental weights $\varpi_i$ of $\mathfrak{g}$.

**Theorem 3.15.** Let $\mathfrak{g}$ be a simple Lie algebra of type I with rank $n$, and let $M^+$ be the monoid associated with $\mathfrak{g}$. Then the monoid algebra $\mathbb{C}[M^+]$ is isomorphic to the polynomial algebra $\mathbb{C}[x_1, \ldots, x_n]$ in $n$ variables $x_i$.

**Proof.** Since $\mathfrak{g}$ is of type I, the monoid algebra $\mathbb{C}[M^+]$ is generated by $X^\varpi_i$ for $1 \leq i \leq n$. As the fundamental weights $\varpi_i$ are linearly independent, $\mathbb{C}[M^+]$ is isomorphic to the polynomial algebra $\mathbb{C}[x_1, \ldots, x_n]$ in $n$ variables $x_i$, with each $X^\varpi_i$ assigned to $x_i$.

$\square$
We consider the type II case, i.e., $\mathfrak{g}$ is one of the types $A_n (n \geq 2)$, $D_{2k+1} (k \geq 2)$ and $E_6$. In this case, Hilb($M^+$) is a disjoint union of self-conjugate elements and non-self-conjugate elements. The non-self-conjugate elements appear in pairs; we use $\{\lambda, \bar{\lambda}\}$ to indicate that $\lambda$ and $\bar{\lambda}$ are conjugate to each other.

**Theorem 3.16.** Let $\mathfrak{g}$ be a simple Lie algebra of type II with rank $n$, and let $\sigma$ be the involution of the corresponding Dynkin diagram given by Figure 4. Let Hilb($M^+$) be the Hilbert basis of the monoid $M^+$ associated with $\mathfrak{g}$. Then the monoid algebra $\mathbb{C}[M^+]$ is isomorphic to $A = \mathcal{P}/\mathcal{I}$, where $\mathcal{P}$ is the polynomial algebra over $\mathbb{C}$ in variables $x_{\lambda}, \lambda \in \text{Hilb}(M^+)$, and $\mathcal{I}$ is the ideal of $\mathcal{P}$ generated by

$$x_{\lambda} \bar{x}_{\lambda} = \prod_{i: i \leq \sigma(i)} x_{\mu_i}^{\max\{a_i, a_{\sigma(i)}\}},$$

$$x_{\lambda}^{(\lambda)} = \prod_{i=1}^n x_{\mu_i}^{(\mu_i)_{\sigma(i)/\nu_i}} \text{ with } \lambda \neq \nu, 1 \leq i \leq n$$

for each non-self-conjugate pair $\{\lambda, \bar{\lambda}\}$ of Hilb($M^+$) with $\lambda = \sum_{i=1}^n a_i \omega_i$ and $\bar{\lambda} = \sum_{i=1}^n a_{\sigma(i)} \omega_i$, where $\mu_i \in \text{Hilb}(M^+)$ are self-conjugate elements given by Lemma 3.13, $\nu_i = s_i \omega_i \in \text{Hilb}(M^+)$ are scalar multiples of the fundamental weights $\omega_i$ with $s_i$ given by Lemma 3.13 and $\ell(\lambda)$ is defined by (3.3).

Before embarking on the proof, let us illustrate this theorem with examples.

**Example 3.17.**

1. Type $A_2$. Hilb($M^+$) consists of the following elements:

   $$\mu_1 = \omega_1 + \omega_2, \ {\nu}_1 = 3\omega_1, \ {\nu}_2 = 3\omega_2.$$

   The ideal $\mathcal{I}$ is generated by $x_{\nu_1} x_{\nu_2} - x_{\mu_1}^3$. Ideals for type $A_3$ and $A_4$ can be obtained by using Example 3.12.

2. Type $D_{2k+1} (k \geq 2)$. Hilb($M^+$) consists of the following elements:

   $$\mu_i = \nu_i = \omega_i, \ 1 \leq i \leq n-2,$$

   $$\mu_{n-1} = \omega_{n-1} + \omega_n, \ \{\nu_{n-1} = 2\omega_{n-1}, \nu_n = 2\omega_n\},$$

   where $n = 2k + 1$. The ideal $\mathcal{I}$ is generated by $x_{\nu_{n-1}} x_{\nu_n} - x_{\mu_{n-1}}^2$.

3. Type $E_6$. Hilb($M^+$) consists of the following elements:

   $$\mu_1 = \omega_1 + \omega_6, \ {\mu}_2 = \omega_2, \mu_3 = \omega_3 + \omega_5, \mu_4 = \omega_4,$$

   $$\nu_1 = 3\omega_1, \nu_6 = 3\omega_6, \{\nu_3 = 3\omega_3, \nu_5 = 3\omega_5\},$$

   $$\{\omega_1 + \omega_3, \omega_5 + \omega_6\}, \{\omega_1 + 2\omega_5, 2\omega_3 + \omega_6\}, \{2\omega_1 + \omega_5, \omega_3 + 2\omega_6\}.$$  

Hence the ideal $\mathcal{I}$ is generated by the following binomials:

$$x_{\nu_1} x_{\nu_6} - x_{\mu_1}^3, \ x_{\nu_3} x_{\nu_5} - x_{\mu_3}^3, \ x_{\nu_1 + \omega_3} x_{\nu_5 + \omega_6} - x_{\mu_1} x_{\mu_3},$$

$$x_{\nu_1 + 2\omega_3} x_{2\omega_3 + \omega_6} - x_{\mu_1} x_{\mu_3}^2, \ x_{\nu_1 + \omega_3} x_{\nu_5 + 2\omega_6} - x_{\mu_1} x_{\mu_3}^2, \ x_{\nu_1 + \omega_3} x_{\nu_5 + \omega_6} - x_{\mu_1} x_{\mu_3}^2.$$  

**Proof of Theorem 3.16.** We define the following surjective algebra homomorphism

$$\varphi : \mathcal{P} \rightarrow \mathbb{C}[M^+], \ x_{\lambda} \mapsto X^\lambda, \ \lambda \in \text{Hilb}(M^+).$$

Then by Lemma 3.13 and Lemma 3.14 we have $\mathcal{I} \subseteq \text{Ker} \varphi$. We need to prove that $\text{Ker} \varphi = \mathcal{I}$, whence $A = \mathcal{P}/\mathcal{I}$ is isomorphic to $\mathbb{C}[M^+]$. 


We start by defining a polynomial subalgebra \( R \subseteq \mathcal{P} \) and show that the restriction \( \varphi|_R \) is injective. Let \( I := \{ i \mid i < \sigma(i) \text{ for } 1 \leq i \leq n \} \) and
\[
\Upsilon := \{ \mu_i \mid i \in I \} \cup \{ \nu_i \mid i \in \{1, 2, \ldots, n\} \setminus I \}.
\]
are linearly independent. Then it can be verified case by case for all \( A_n(n \geq 2) \), \( D_{2k+1}(k \geq 2) \) and \( E_6 \) that \( \Upsilon \) is a linearly independent set. Denote by \( R := \mathbb{C}[X, \lambda \in \Upsilon] \) the polynomial subalgebra of \( \mathcal{P} \). By the linear independence of \( \Upsilon \), the image \( \varphi(R) \) is a polynomial subalgebra of \( \mathbb{C}[M^+] \) generated by the algebraically independent elements \( X^\lambda, \lambda \in \Upsilon \). Therefore, \( \varphi|_R \) is an isomorphism and \( R \cap \ker \varphi = 0 \). Define the multiplicatively closed sets \( S := R - \{0\} \) and \( \varphi(S) \).

Next we consider the rings of fractions \( S^{-1}\mathcal{P} \) and \( \varphi(S)^{-1}\mathbb{C}[M^+] \), and the induced surjective homomorphism
\[
\varphi_S : S^{-1}\mathcal{P} \to \varphi(S)^{-1}\mathbb{C}[M^+]
\]
given by \( \varphi_S(s^{-1}a) = \varphi(s)^{-1}\varphi(a) \) for \( a \in \mathcal{P} \) and \( s \in S \). Clearly, \( \ker \varphi \subseteq \ker \varphi_S \cap \mathcal{P} \). We claim that \( S^{-1}\mathcal{I} \) is a maximal ideal of \( S^{-1}\mathcal{P} \). If the claim is done, then \( \ker \varphi_S = S^{-1}\mathcal{I} \) since \( S^{-1}\mathcal{I} \subseteq \ker \varphi_S \). Moreover, \( \mathcal{I} \) is a prime ideal since \( S^{-1}\mathcal{I} \) is maximal (hence a prime ideal) and \( \mathcal{I} \cap S = 0 \) (follows from \( R \cap \ker \varphi = 0 \)). It follows that \( S^{-1}\mathcal{I} \cap \mathcal{P} = \mathcal{I} \). Combining all above together, we obtain
\[
\ker \varphi \subseteq \ker \varphi_S \cap \mathcal{P} = S^{-1}\mathcal{I} \cap \mathcal{P} = \mathcal{I}
\]
as required.

It remains to prove our claim, which is equivalent to showing that \( S^{-1}\mathcal{P} / S^{-1}\mathcal{I} \cong S^{-1}\mathcal{A} \) is a field. Consider the fraction field \( F = S^{-1}R \) of \( R \). We will adjoin extra elements \( x_\lambda, \lambda \in \text{Hilb}(M^+) \backslash \Upsilon \) to \( F \), and then \( S^{-1}\mathcal{A} \) is equal to the resulting extension field of \( F \). Note that by Lemma \( \text{3.10} \) and Lemma \( \text{3.13} \) we have \( x_{\mu_i} = x_{\nu_i} = x_{\lambda_i} \in F \) for all \( i = \sigma(i) \). For any \( i < \sigma(i) \), we define \( F_1 := F[x_{\nu_i}] \cong F[t]/(t - x_{\nu_i}^{-1}\prod_{i \in \sigma(i)} x_{\nu_i}^{\mu_i}) \). Since \( x_{\nu_i}^{-1}\prod_{i \in \sigma(i)} x_{\nu_i}^{\mu_i} \in F \), we have \( F_1 = F \) and hence \( x_{\nu_i} \in F \). Therefore, \( x_{\mu_i}, \lambda, x_{\nu_i} \), and \( x_{\lambda_i} \) belong to the fraction field \( F \). Now taking an arbitrary non-self-conjugate pair \( (\lambda, \overline{\lambda}) \) of \( \text{Hilb}(M^+) \) with \( \lambda \neq \nu_i \), we define \( F_2 = F[x_\lambda] \cong F[t]/(t - x_\lambda^{-1}\prod_{i \in \sigma(i)} x_\lambda^{\max\{X_{\nu_i}^{\lambda}, X_{\nu_i}^{\overline{\lambda}}\}}) \). Since \( x_\lambda \) is algebraic over \( F \), \( F[x_\lambda] = F(x_\lambda) \) is a field. Similarly, define \( F_3 = F_2[x_\lambda] \cong F_2[t]/(t - x_\lambda^{-1}\prod_{i \in \sigma(i)} x_\lambda^{\max\{X_{\nu_i}^{\lambda}, X_{\nu_i}^{\overline{\lambda}}\}}) \). Since \( x_\lambda^{-1}\prod_{i \in \sigma(i)} x_{\mu_i}^{\max\{X_{\nu_i}^{\lambda}, X_{\nu_i}^{\overline{\lambda}}\}} \in F_2 \), we have \( F_2 = F_3 \) is a field. Repeating the above step for each non-self-conjugate pair \( (\lambda, \overline{\lambda}) \), we obtain an extension field \( F' = S^{-1}\mathcal{A} \) of \( F \). This completes the proof. \( \square \)

4. Proof of the main theorem

This section is devoted to proving Theorem 2.5. We will review the quantised Harish-Chandra theorem, which allows us to construct explicitly an isomorphism between the centre \( Z(U_q(\mathfrak{g})) \) and the Grothendieck algebra \( S(U_q(\mathfrak{g})) \) of the category of finite dimensional \( U_q(\mathfrak{g}) \)-modules whose weights are contained in \( M \). Then we show that \( S(U_q(\mathfrak{g})) \) is isomorphic to the monoid algebra \( \mathbb{C}[M^+] \) over \( \mathbb{C}(q) \), and hence \( Z(U_q(\mathfrak{g})) \cong \mathbb{C}(q)[M^+] \). Combining with the presentation of \( \mathbb{C}[M^+] \), we obtain explicit generators and relations of \( Z(U_q(\mathfrak{g})) \) as given in Theorem 2.5.
4.1. The Harish-Chandra isomorphism. We will follow [Jan96] Chapter 6 and retain notation from Section 2.1. Write \( U = U_q(\mathfrak{g}) \). Recall that the quantum group \( U \) is graded by the root lattice \( Q \), i.e., \( U = \bigoplus_{\nu \in Q} U_{\nu} \). In particular, \( U_0 = U^0 \oplus \bigoplus_{\nu > 0} U_{-\nu} U^0 U_{\nu} \), where \( U^0 \) denotes the subalgebra generated by all \( K_i^{\pm 1} \). It is known that the projection 
\[
\pi : U_0 \to U^0
\]
is an algebra homomorphism, and the centre \( Z(U_0(\mathfrak{g})) \) is contained in \( U_0 \).

The Harish-Chandra isomorphism identifies \( Z(U_q(\mathfrak{g})) \subseteq U_0 \) with a \( W \)-invariant subalgebra of \( U^0 \). Precisely, we define an algebra automorphism of \( U^0 \) by 
\[
\gamma_{-\rho} : U^0 \to U^0, \quad K_\alpha \mapsto q^{(-\rho,\alpha)} K_\alpha,
\]
for any \( \alpha \in Q \), where \( \rho \) denotes the half sum of positive roots of \( \mathfrak{g} \). Then the composite \( \gamma_{-\rho} \circ \pi \) is called the Harish-Chandra homomorphism, under which the image of \( Z(U) \) can be described as follows.

Recall that \( \frac{1}{2}Q := \{ \frac{1}{2} \alpha \mid \alpha \in Q \} \) and \( M = \frac{1}{2}Q \cap P \). We define
\[
U_{ev}^0 := \langle K_{2\lambda} \mid \lambda \in M \rangle
\]
to be the subalgebra of \( U^0 \) spanned by \( K_{2\lambda} \) for all \( \lambda \in M \). Recall that the Weyl group \( W \) of \( \mathfrak{g} \) acts naturally on \( U^0 \) via \( w.K_\alpha = K_{\alpha} \) for any \( w \in W \) and \( \alpha \in Q \). This action carries over to \( U_{ev}^0 \), and we denote by \( (U_{ev}^0)^W \) the \( W \)-invariant subalgebra.

**Theorem 4.1.** [Jan96] Theorem 6.25] The Harish-Chandra homomorphism 
\[
\gamma_{-\rho} \circ \pi : Z(U_q(\mathfrak{g})) \to (U_{ev}^0)^W
\]
is an isomorphism.

Note that for each \( \lambda \in M = \frac{1}{2}Q \cap P \), there exists a unique \( w \in W \) such that \( w\lambda \in M^+ = \frac{1}{2}Q \cap P^+ \).

**Lemma 4.2.** The elements \( av(\lambda) = \sum_{w \in W} K_{2w\lambda}, \lambda \in M^+ \) form a basis of \( (U_{ev}^0)^W \).

**Proof.** Clearly, \( av(\lambda) \in (U_{ev}^0)^W \). If the finite sum \( f = \sum_\lambda c_\lambda K_{2\lambda} \) is \( W \)-invariant, then 
\[
f = \frac{1}{|W|} \sum_\lambda \sum_{w \in W} c_\lambda w.K_{2\lambda} = \frac{1}{|W|} \sum_\lambda c_\lambda \left( \sum_{w \in W} K_{2w\lambda} \right).
\]
Since \( \sum_{w \in W} K_{2w\lambda} = av(\lambda_0) \) for a unique dominant weight \( \lambda_0 \in M^+ \), the element \( f \) can be expressed uniquely as a linear combination of some \( av(\lambda), \lambda \in M^+ \). Therefore, \( av(\lambda), \lambda \in M^+ \) form a basis for \( (U_{ev}^0)^W \). \( \square \)

4.2. Representation theoretical viewpoint. Recall from Section 2.2 our construction of the central element \( CV \). The image of \( CV \) under the Harish-Chandra isomorphism can be calculated as follows.

**Lemma 4.3.** Let \( V \) be a finite dimensional \( U_q(\mathfrak{g}) \)-module with \( \Pi(V) \subseteq M \), and let \( CV \) be the associated central element as defined in Definition 2.2. Then we have 
\[
\gamma_{-\rho} \circ \pi(CV) = \sum_{\mu \in \Pi(V)} m_V(\mu) K_{2\mu} \in (U_{ev}^0)^W,
\]
where \( m_V(\mu) = \dim L(\lambda)_\mu \).
Proof. Recall that \( \pi \) is an algebra homomorphism from \( U_0 \) to \( U^0 \), where the latter is a subalgebra generated by all \( K_i^{\pm 1} \). Hence we have

\[
\pi(C_V) = \pi(\text{Tr}_1((K_{2p} \otimes 1)K_V R_k^T R_V)) = \text{Tr}_1((K_{2p} \otimes 1)K_V) = \sum_{\mu \in \Pi(V)} q^{(\mu, 2\rho)} m_V(\mu)K_{2\mu}.
\]

By the definition of \( \gamma - \rho \) we have \( \gamma - \rho \circ \pi(C_V) = \sum_{\mu \in \Pi(V)} m_V(\mu)K_{2\mu} \) as desired. \( \square \)

Recall that the character of \( V \) is defined by \( \chi(V) = \sum_{\mu \in \Pi(V)} m_V(\mu)e^\mu \). \[ \text{Jan96} \]. Hence \( \gamma - \rho \circ \pi(C_V) \) is equal to the character \( \chi(V) \) with \( e^\mu \) replaced with \( K_{2\mu} \), and we may study the centre \( Z(U_q(\mathfrak{g})) \) from a representation-theoretic point of view.

We define \( \tilde{R}(U_q(\mathfrak{g})) := \mathbb{C}(q) \otimes_\mathbb{Z} R(U_q(\mathfrak{g})) \), where \( R(U_q(\mathfrak{g})) \) is the Grothendieck ring of the category of finite dimensional \( U_q(\mathfrak{g}) \)-modules of type 1. It is well known that the isomorphism classes \( \{L(\lambda)\}, \lambda \in \mathcal{P}^+ \) form a basis for \( \tilde{R}(U_q(\mathfrak{g})) \). Define \( S(U_q(\mathfrak{g})) \) to be the subalgebra of \( \tilde{R}(U_q(\mathfrak{g})) \) which has a basis consisting of isomorphism classes \( \{L(\lambda)\}, \lambda \in \mathcal{M}^+ \).

Note that for any \( \lambda, \mu \in \mathcal{M}^+ \) there is a unique decomposition \( L(\lambda) \otimes L(\mu) \cong \bigoplus_{\nu} c_{\lambda, \mu}^\nu L(\nu) \), where \( \nu \in \mathcal{M}^+ \) and \( c_{\lambda, \mu}^\nu \) is the multiplicity of \( L(\nu) \). Therefore, given any isomorphism class \( [V] \in S(U_q(\mathfrak{g})) \), we have \( \Pi(V) \subseteq \mathcal{M} \) and hence can define a corresponding central element \( C_V \). This gives rise to the following commutative diagram:

\[
\begin{array}{ccc}
S(U_q(\mathfrak{g})) & \xrightarrow{\xi} & (U_{ev})^W \\
\downarrow & & \downarrow \\
Z(U_q(\mathfrak{g})) & \xrightarrow{\gamma - \rho \circ \pi} & (U_{ev})^W.
\end{array}
\]

Lemma 4.4. There is an algebra isomorphism \( \xi : S(U_q(\mathfrak{g})) \to (U_{ev})^W \) defined by

\[ \xi([V]) = \sum_{\mu \in \Pi(V)} m_V(\mu)K_{2\mu}, \]

where \( m_V(\mu) = \dim L(\lambda)_\mu \).

Proof. First, since \( \xi([V][W]) = \xi([V \otimes W]) = \xi([V])\xi([W]) \) (see, e.g., \[ \text{Hum72} \rightleftharpoons \text{§22.5, Proposition B} \]) \( \xi \) is an algebra homomorphism. If \( \xi([V]) = \xi([W]) \), then \( V \) and \( W \) have the same character and hence \( V \cong W \). Therefore \( \xi \) is injective. It remains to show the surjectivity.

Recall from Lemma 4.2 that \( (U_{ev})^W \) has a basis \( \text{av}(\lambda) = \sum_{\mu \in \mathcal{M}} K_{2\mu, \lambda}, \lambda \in \mathcal{M}^+ \). It suffices to show that for any \( \lambda \in \mathcal{M}^+ \) there exists a \( U_q(\mathfrak{g}) \)-module \( V \) such that \( \xi([V]) = \text{av}(\lambda) \). We use induction on \( \lambda \).

For the base case, if \( \lambda \in \mathcal{M}^+ \) and there are no dominant weights lower than \( \lambda \), we have

\[
\sum_{\mu \in \mathcal{W}} K_{2\mu, \lambda} = \frac{|W|}{|W_{\lambda}|} \sum_{\mu \in W_{\lambda}} K_{2\mu} = \frac{|W|}{|W_{\lambda}|} \xi([L(\lambda)]),
\]
where $W\lambda := \{ w\lambda \mid w \in W \}$ denotes the $W$-orbit of $\lambda$. For general $\lambda \in M^+$, we have

$$
\sum_{w \in W} K_{2w\lambda} = \frac{|W|}{|W\lambda|} \xi([L(\lambda)]) - \sum_{\mu \in \Pi(\lambda)/W\lambda} \dim L(\lambda)_{\mu} K_{2\mu},
$$

$$
= \frac{|W|}{|W\lambda|} \xi([L(\lambda)]) - \sum_{\mu < \lambda, \mu \in P^+} \frac{|W\mu|}{|W\lambda|} \dim L(\lambda)_{\mu} \sum_{w \in W} K_{2w\mu},
$$

where $\mu < \lambda$ and $\mu \in P^+$ imply that $\mu \in M^+$. By induction hypothesis, each sum $\sum_{w \in W} K_{2w\mu}$ has a preimage in $S(U_q(\mathfrak{g}))$ and so does $av(\lambda) = \sum_{w \in W} K_{2w\lambda}$. This completes the proof. \hfill \Box

The following is a consequence of the commutative diagram (4.2) and Lemma 4.4

**Corollary 4.5.** The algebra $S(U_q(\mathfrak{g}))$ is isomorphic to $Z(U_q(\mathfrak{g}))$, with each isomorphism class $[V] \in S(U_q(\mathfrak{g}))$ assigned to the central element $C_V$.

### 4.3. Proof of the main theorem.

Recall from Section 3.3 the monoid algebra $C[q][M^+]$ generated by $X^\lambda, \lambda \in \operatorname{Hilb}(M^+)$. The explicit relations among these generators are given in Theorem 3.15 and Theorem 3.16 for the Lie algebra $\mathfrak{g}$ of type I and II, respectively. In the sequel, we shall consider the monoid algebra $C[q][M^+] = C[q] \otimes \mathbb{C}[M^+]$, of which the generators and relations remain the same.

**Lemma 4.6.** The monoid algebra $C[q][M^+]$ is isomorphic to $S(U_q(\mathfrak{g}))$, with each generator $X^\lambda$ mapped to the isomorphism class $[T(\lambda)]$ for $\lambda \in \operatorname{Hilb}(M^+)$.\hfill \Box

**Proof.** Recall that as a vector space $C[q][M^+]$ has a basis $X^\lambda$ with $\lambda \in M^+$. If $S(U_q(\mathfrak{g}))$ has a basis $[T(\lambda)], \lambda \in M^+$ then there exists a bijective linear map sending $X^\lambda$ to $[T(\lambda)]$ for all $\lambda \in M^+$. Moreover, this linear map preserves the algebra structure since $[T(\lambda)][T(\mu)] = [T(\lambda) \otimes T(\mu)] = [T(\lambda + \mu)]$.

Now it suffices to show that $[T(\lambda)], \lambda \in M^+$ make up a basis for $S(U_q(\mathfrak{g}))$. Note that there is the following decomposition

$$
T(\lambda) \cong L(\lambda) \oplus \bigoplus_{\mu < \lambda, \mu \in M^+} L(\mu)^{\otimes m_{\lambda, \mu}}.
$$

Recall that by definition $[L(\lambda)], \lambda \in M^+$ form a basis for $S(U_q(\mathfrak{g}))$. For any $\gamma \in M^+$, the set $I(\gamma) = \{ \mu \in M^+ \mid \mu \leq \gamma \}$ is finite. Then the elements $[T(\lambda)], \lambda \in I(\gamma)$ are linearly independent, since we may arrange $I(\gamma)$ non-decreasingly under the partial order and the transformation matrix between $[T(\lambda)], \lambda \in I(\gamma)$ and $[L(\lambda)], \lambda \in I(\gamma)$ is non-singular by (4.3). Moreover, $[T(\lambda)], \lambda \in M^+$ is a spanning set of $S(U_q(\mathfrak{g}))$, since each element of $S(U_q(\mathfrak{g}))$ is a finite linear combination of $[L(\lambda)], \lambda \in M^+$ and hence a finite linear combination of $[T(\lambda)], \lambda \in M^+$ by using (4.3). Therefore, $[T(\lambda)], \lambda \in M^+$ form a basis for $S(U_q(\mathfrak{g}))$. \hfill \Box

**Corollary 4.7.** We have the following:

1. The isomorphism classes $[T(\lambda)], \lambda \in \operatorname{Hilb}(M^+)$ generate the algebra $S(U_q(\mathfrak{g}))$.
2. The elements $C_{T(\lambda)}, \lambda \in \operatorname{Hilb}(M^+)$ generate the centre $Z(U_q(\mathfrak{g}))$.

**Proof.** Part (1) is a consequence of Lemma 4.6 and part (2) follows from the combination of part (1) and Corollary 4.5. \hfill \Box

We are in a position to prove our main theorem.
Proof of Theorem 2.5. By Corollary 1.7, \( Z(U_q(g)) \) is generated by \( C_{T(\lambda)} \) for all \( \lambda \in \mathrm{Hilb}(M^+) \). Combining Corollary 1.5 and Lemma 4.6, we have the algebra isomorphism \( Z(U_q(g)) \cong C[q][M^+] \), with each generator \( C_{T(\lambda)} \) assigned to \( X^\lambda \) for \( \lambda \in \mathrm{Hilb}(M^+) \). Now the theorem follows from Theorem 3.1 and Theorem 3.10 by replacing the ground field \( C \) with \( C(q) \).

Remark 4.8. In part (2) of Theorem 2.5, one can show that \( C_{T(\lambda)}, \lambda \in \mathrm{Hilb}(M^+) \) also form a generating set of \( Z(U_q(g)) \), but they do not obey the same relations. For examples of these relations, refer to Example 3.17, where \( x_\lambda \) should be replaced with \( C_{T(\lambda)} \) for \( \lambda \in \mathrm{Hilb}(M^+) \).

Appendix A. Proofs of commutative relations

In this appendix, we shall prove commutative relations in Lemma 2.1 and Proposition 2.3. First we prove Lemma 2.7.

Proof of Lemma 2.7. For succinctness, we just prove that \( C_{V}^{(k)} \) commutes with \( E_i \) and the other cases can be treated similarly. Since \( [\Gamma_V, \Delta(x)] = 0 \), then we have \( [[\Gamma_V]^{k}, \Delta(x)] = 0 \) for any \( x \in U \). Assuming that \( (\Gamma_V)^k = \sum_j A_j \otimes B_j \), we have

\[
0 = \text{Tr}_1((K_i^{-1}K_{2p} \otimes 1)[(\Gamma_V)^k, \Delta(E_i)])
= \text{Tr}_1(\sum_j (K_i^{-1}K_{2p} \otimes 1)[A_j \otimes B_j, K_i \otimes E_i + E_i \otimes 1])
= \text{Tr}_1(\sum_j (K_i^{-1}K_{2p} \otimes 1)(A_jK_i \otimes B_jE_i + A_jE_i \otimes B_j - K_iA_j \otimes E_iB_j - E_iA_j \otimes B_j)).
\]

This can be written as a sum of two terms: the first term is

\[
\text{Tr}_1(\sum_j (K_i^{-1}K_{2p} \otimes 1)(A_jK_i \otimes B_jE_i - K_iA_j \otimes E_iB_j))
= \sum_j \text{Tr}(K_{2p}A_j)(B_jE_i - E_iB_j)
= [C_V, E_i],
\]

and the second term is

\[
\text{Tr}_1(\sum_j (K_i^{-1}K_{2p} \otimes 1)(A_jE_i \otimes B_i - E_iA_j \otimes B_j))
= \sum_j (\text{Tr}(K_i^{-1}K_{2p}A_jE_i) - \text{Tr}(K_i^{-1}K_{2p}E_iA_j))B_j,
\]

which is equal to 0 since

\[
\text{Tr}(K_i^{-1}K_{2p}A_jE_i) = \text{Tr}(E_iK_i^{-1}K_{2p}A_j) = q^{-(2p-\alpha_i, \alpha_i)}\text{Tr}(K_i^{-1}K_{2p}E_iA_j) = \text{Tr}(K_i^{-1}K_{2p}E_iA_j).
\]

Therefore, we have \( [C_V^{(k)}, E_i] = 0 \). \( \square \)

Now we turn to prove Proposition 2.3. Let us start with the following lemma.

Lemma A.1. Let \( \zeta = \zeta_V : U_q(g) \to \mathrm{GL}(V) \) be the linear representation associated to \( V \), and let \( K_V \) be as defined in (2.6).
(1) For $1 \leq i, j \leq n$, we have
\[
\mathcal{K}_V(\zeta(K_i^\pm) \otimes K_j^{\pm 1}) = (\zeta(K_i^\pm) \otimes K_j^{\pm 1})\mathcal{K}_V \\
\mathcal{K}_V(\zeta(E_i) \otimes 1) = (\zeta(E_i) \otimes K_i^2)\mathcal{K}_V, \\
\mathcal{K}_V(1 \otimes E_i) = (\zeta(K_i^2) \otimes E_i)\mathcal{K}_V, \\
\mathcal{K}_V(\zeta(F_i) \otimes 1) = (\zeta(F_i) \otimes K_i^{-2})\mathcal{K}_V, \\
\mathcal{K}_V(1 \otimes F_i) = (\zeta(K_i^{-2}) \otimes F_i)\mathcal{K}_V.
\]

(2) For any $x \in U_q(g)$, we have
\[
\mathcal{K}_V\phi^2(\Delta(x)) = \Delta(x)\mathcal{K}_V,
\]
where the first tensor factors in both $\Delta(x)$ and $\phi^2(\Delta(x))$ are regarded as elements in $\text{End}(V)$ via the linear representation $\zeta$.

Proof. For part (1), we only prove the second equation; the others can be treated similarly. Recall that $P_\eta : V \rightarrow V_\eta$ is the linear projection onto the weight space $V_\eta$ of $V$. It can be verified easily that
\[
P_\eta\zeta(K_i^{\pm 1}) = \zeta(K_i^{\pm 1})P_\eta, \\
P_\eta(\zeta(E_i)) = \zeta(E_i)P_{\eta-\alpha_i}, \\
P_\eta(\zeta(F_i)) = \zeta(F_i)P_{\eta+\alpha_i},
\]
where $P_{\eta-\alpha_i} := 0$ (resp. $P_{\eta+\alpha_i} := 0$) if $\eta - \alpha_i \notin \Pi(V)$ (resp. $\eta + \alpha_i \notin \Pi(V)$). Using the second equality, we have
\[
\mathcal{K}_V(\zeta(E_i) \otimes 1) = \sum_{\eta \in \Pi(V)} P_\eta^V \zeta(E_i) \otimes K_{2\eta} \\
= \sum_{\eta \in \Pi(V)} \zeta(E_i)P_{\eta+\alpha_i} \otimes K_{2\eta} \\
= (\zeta(E_i) \otimes K_i^2)\mathcal{K}_V.
\]

Part (2) is a direct consequence of part (1), and we take $x = E_i$ as an example. Using $\phi^2(\Delta(E_i)) = K_i^{-1} \otimes E_i + E_i \otimes K_i^{-2}$, we have
\[
\mathcal{K}_V\phi^2(\Delta(E_i)) = \mathcal{K}_V(\zeta(K_i^{-1}) \otimes E_i + \zeta(E_i) \otimes K_i^{-2}) \\
= (\zeta(K_i) \otimes E_i + \zeta(E_i) \otimes 1)\mathcal{K}_V \\
= \Delta(x)\mathcal{K}_V.
\]
This completes the proof. \qed

Now we are in a position to prove Proposition 2.3.

Proof of Proposition 2.3. By (2.3) there are equations $R_V\Delta(x) = \phi(\Delta'(x))R_V$ and $R^T\Delta'(x) = \phi(\Delta(x))R^T$. Applying the algebra homomorphism $\phi$ to the latter, we have $\phi(R^T\Delta'(x)) = \phi^2(\Delta(x))\phi(R^T_V)$. Then it follows that
\[
\Gamma_V\Delta(x) = \mathcal{K}_V\phi(R^T_V)R_V\Delta(x) \\
= \mathcal{K}_V\phi(R^T_V)\phi(\Delta'(x))R_V \\
= \mathcal{K}_V\phi^2(\Delta(x))\phi(R^T_V)R_V \\
= \Delta(x)\Gamma_V,
\]
where the last equation follows from part (2) of Lemma 4.1. \qed
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