Orthogonal Decomposition of Symmetric Tensors

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Abstract

A real symmetric tensor is orthogonally decomposable (or odeco) if it can be written as a linear combination of symmetric powers of \( n \) vectors which form an orthonormal basis of \( \mathbb{R}^n \). Motivated by the spectral theorem for real symmetric matrices, we study the properties of odeco tensors. We give a formula for all of the eigenvectors of an odeco tensor. Moreover, we give a set of polynomial equations that vanish on the odeco variety and we conjecture that these polynomials generate its prime ideal. We prove this conjecture in some cases and give strong evidence for its overall correctness.

1 Introduction

The spectral theorem states that every \( n \times n \) real symmetric matrix \( M \) possesses \( n \) real eigenvectors \( v_1, \ldots, v_n \) which form an orthonormal basis of \( \mathbb{R}^n \). Moreover, one can express \( M \) as \( M = \sum_{i=1}^{n} \lambda_i v_i v_i^T \), where \( \lambda_1, \ldots, \lambda_n \) are the corresponding eigenvalues. In this paper we investigate when such a decomposition is possible for real symmetric tensors. We address the following two questions.

Question 1. Can all symmetric tensors \( T \) be decomposed as \( T = \lambda_1 v_1^{\otimes d} + \cdots + \lambda_n v_n^{\otimes d} \), where \( v_1, \ldots, v_n \) form an orthonormal basis of \( \mathbb{R}^n \)? If not, can we find equations in the entries of \( T \) that cut out the set of tensors for which such a decomposition exists?

Question 2. Given that a tensor \( T \) can be decomposed as \( T = \lambda_1 v_1^{\otimes d} + \cdots + \lambda_n v_n^{\otimes d} \), where \( v_1, \ldots, v_n \in \mathbb{R}^n \) are orthonormal, can we express the eigenvectors of \( T \) (to be defined) in terms of \( v_1, \ldots, v_n \) ?

Let \( S^d(\mathbb{R}^n) \) denote the space of \( n \times n \times \cdots \times n \) (\( d \) times) symmetric tensors, i.e. tensors whose entries are real numbers \( T_{i_1 \cdots i_d} \) invariant under permuting the indices: \( T_{i_1 \cdots i_d} = T_{i_{\sigma(1)} \cdots i_{\sigma(d)}} \) for all permutations \( \sigma \) of the set \( \{1, 2, \ldots, d\} \). For example, when \( d = 2 \), the space \( S^2(\mathbb{R}^n) \) consists of all \( n \times n \) real symmetric matrices. We study the elements \( T \in S^d(\mathbb{R}^n) \) which can be written as \( T = \lambda_1 v_1^{\otimes d} + \cdots + \lambda_n v_n^{\otimes d} \), where \( v_1, \ldots, v_n \in \mathbb{R}^n \) form an orthonormal basis of \( \mathbb{R}^n \). We call such tensors \( T \) orthogonally decomposable or, for short, odeco.
The notion of eigenvectors of matrices was extended to symmetric tensors independently by Lim and Qi [11, 15] in 2005. A vector \( w \in \mathbb{C}^n \) is an eigenvector of \( T \in S^d(\mathbb{R}^n) \) if there exists \( \lambda \in \mathbb{C} \), the corresponding eigenvalue, such that

\[
Tw^{d-1} := \left[ \sum_{i_2,\ldots,i_d=1}^n T_{i_1,i_2,\ldots,i_d}w_{i_2}\ldots w_{i_d} \right]_i = \lambda w.
\]

Two eigenpairs \((w, \lambda)\) and \((w', \lambda')\) are equivalent if there exists \( t \neq 0 \) such that \( w = tw' \) and \( \lambda = t^{d-2}\lambda' \). When \( d = 2 \), these definitions agree with the usual definitions of eigenvectors, eigenvalues, and equivalence of eigenpairs for matrices.

The spectral theorem answers both Question 1 and Question 2 in the case \( d = 2 \): every symmetric matrix \( M \in S^2(\mathbb{R}^n) \) can be written as \( M = \sum_{i=1}^n \lambda_i v_i v_i^T = \sum_{i=1}^n \lambda_i v_i \otimes v_i \), where \( v_1,\ldots,v_n \) are orthonormal. Moreover, if \( M \) is generic (in the sense that its eigenvalues are distinct), then \( v_1,\ldots,v_n \) are all of the eigenvectors of \( M \) up to scaling.

In Section 2 we give an explicit algebraic formula of all of the eigenvectors of an odec tensor \( T = \lambda_1 v_1^{\otimes d} + \cdots + \lambda_r v_r^{\otimes d} \) in terms of \( v_1,\ldots,v_n \), answering Question 2 above. It easily follows from the definition of eigenvectors that \( v_1,\ldots,v_n \) are eigenvectors of \( T \). These are not all of the eigenvectors of \( T \), but it turns out that one can explicitly express the rest of them in terms of \( v_1,\ldots,v_n \).

For general \( d \), not all tensors \( T \in S^d(\mathbb{R}^n) \) are odec. In Section 3, we address Question 1. We study the set of all odec tensors and find equations that vanish on this set. In Conjecture 3.2 we claim that these define the prime ideal of the variety of odec tensors. In Proposition 3.6 we prove Conjecture 3.2 for the special case \( n = 2 \). In Section 3.1 we conclude the paper by giving evidence for the correctness of this conjecture.

In the remainder of this section we review symmetric tensor decomposition as well as the equivalent characterization of symmetric tensors as homogeneous polynomials. We conclude the section by describing an algorithm, called the tensor power method, which finds the orthogonal decomposition of an odec tensor.

### 1.1 Symmetric tensor decomposition

Orthogonal decomposition is a special type of symmetric tensor decomposition which has been of much interest in the recent years; references include [3, 10, 12, 13], and many others. Given a tensor \( T \in S^d(\mathbb{R}^n) \), the aim is to decompose it as

\[
T = \sum_{i=1}^r \lambda_i v_i^{\otimes d},
\]

where \( v_1,\ldots,v_r \) are any vectors. The smallest \( r \) for which such a decomposition exists is called the (symmetric) rank of \( T \). Finding the symmetric tensor decomposition of a given tensor \( T \) is an NP hard problem [6] and algorithms for it have been proposed by several authors, for example [3, 12].

**Remark 1.1.** Orthogonal tensor decomposition has also been studied in the non-symmetric case [7, 8]. An odec tensor as defined in the present work is also orthogonally decomposable according to the definition in the non-symmetric case.
1.2 Symmetric tensors as homogeneous polynomials

An equivalent way to think about a symmetric matrix $M \in S^2(\mathbb{R}^n)$ is via its corresponding quadratic form $f_M \in \mathbb{R}[x_1, \ldots, x_n]$ given by

$$f_M(x_1, \ldots, x_n) = x^T M x = \sum_{i,j} M_{ij} x_i x_j.$$ 

More generally, a tensor $T \in S^d(\mathbb{R}^n)$ can equivalently be represented by a homogeneous polynomial $f_T \in \mathbb{R}[x_1, \ldots, x_n]$ of degree $d$ given by

$$f_T(x_1, \ldots, x_n) = T \cdot x^d := \sum_{i_1, \ldots, i_d=1}^n T_{i_1 \ldots i_d} x_{i_1} x_{i_2} \cdots x_{i_d}.$$ 

Given $T \in S^d(\mathbb{R}^n)$, we can describe the notions of eigenvectors, eigenvalues, and symmetric and orthogonal decomposition in terms of the corresponding polynomial $f_T \in \mathbb{R}[x_1, \ldots, x_n]$ as follows.

A vector $x \in \mathbb{C}^n$ is an eigenvector of $T$ with eigenvalue $\lambda$ if and only if

$$\nabla f_T(x) = \lambda dx.$$ 

The tensor $T$ can be decomposed as $T = \sum_{i=1}^r \lambda_i v_i \otimes v_i$ if and only if the corresponding polynomial $f_T$ can be decomposed as

$$f_T(x_1, \ldots, x_n) = \sum_{i=1}^r \lambda_i (v_i x_1 + \cdots + v_i x_n)^d.$$ 

Similarly, $T$ is orthogonally decomposable with $T = \lambda_1 v_1 \otimes v_1 + \cdots + \lambda_n v_n \otimes v_n$, where $v_1, \ldots, v_n$ are orthonormal, if and only if $f_T(x_1, \ldots, x_n) = \lambda_1 (v_1 \cdot x)^d + \cdots + \lambda_n (v_n \cdot x)^d$.

This equivalent characterization of symmetric tensors as homogeneous polynomials proves to be quite useful in the sequel.

1.3 Finding an orthogonal decomposition

Finding the symmetric decomposition of a given $T \in S^d(\mathbb{R}^n)$ is NP hard. However, there are simple algorithms that recover the orthogonal decomposition of an odec tensor $T \in S^d(\mathbb{R}^n)$. One such algorithm is the tensor power method [1].

Let $T \in S^d(\mathbb{R}^n)$. If $T$ is orthogonally decomposable, i.e. $T = \sum_{i=1}^k \lambda_i v_i \otimes v_i$ with $v_1, \ldots, v_k$ orthonormal, then

$$T \cdot v_j^{d-1} = \sum_{i=1}^r \lambda_i (v_i \cdot v_j)^{d-1} v_i = \lambda_j v_j,$$

for all $j = 1, 2, \ldots, n$. Thus, $v_1, \ldots, v_k$ are eigenvectors of $T$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_k$. 

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Definition 1.2. A unit vector $u \in \mathbb{R}^n$ is a robust eigenvector of $T$ if there exists $\epsilon > 0$ such that for all $\theta \in \{u' \in \mathbb{R}^n : ||u - u'|| < \epsilon\}$, repeated iteration of the map

$$\bar{\theta} \mapsto \frac{T\bar{\theta}^{d-1}}{||T\bar{\theta}^{d-1}||},$$

starting from $\theta$ converges to $u$.

The following theorem shows that if $T$ has an orthogonal decomposition $T = \sum_{i=1}^k \lambda_i v_i \otimes d_i$, then the set of robust eigenvectors of $T$ is precisely the set $\{v_1, v_2, ..., v_k\}$, implying that the orthogonal decomposition is unique up to the obvious reordering.

Theorem 1.3 (Theorem 4.1, [1]). Let $T$ have an orthogonal decomposition $T = \sum_{i=1}^k \lambda_i v_i \otimes d_i$, where $v_1, ..., v_k$ are orthonormal.

1. The set of $\theta \in \mathbb{R}^n$ which do not converge to some $v_i$ under repeated iteration of (1.1) has measure 0.

2. The set of robust eigenvectors of $T$ is equal to $\{v_1, v_2, ..., v_k\}$.

Therefore, to recover the orthogonal decomposition of $T$, one needs to find the robust eigenvectors. The definition of robust eigenvectors suggests an algorithm to compute them, using repeated iteration of the map (1.1) starting with random vectors $u \in \mathbb{R}^n$.

Algorithm 1 The Tensor Power Method

1: **Input:** an orthogonally decomposable tensor $T$.
2: Set $i = 1$.
3: **Repeat** until $T = 0$.
4: Choose random $u \in \mathbb{R}^n$.
5: Let $v_i$ be the result of repeated iteration of (1.1) starting with $u$.
6: Compute the eigenvalue $\lambda_i$ corresponding to $v_i$, from the equation $Tv_i^{d-1} = \lambda_i v_i$.
7: Set $T = T - \lambda_i v_i \otimes d_i$.
8: $i \leftarrow i + 1$.
9: **Output** $v_1, ..., v_k$ and $\lambda_1, ..., \lambda_k$.

In certain cases, this algorithm can be used to find the symmetric decomposition of a given tensor. For example, the authors of [1] consider a class of statistical models, such as the exchangeable single topic model, in which one observes tensors $T_2$ and $T_3$, where $T_d = \sum_{i=1}^k \omega_i \mu_i \otimes d$ for $d = 2, 3$ and the aim is to recover the unknown parameters $\omega = (\omega_1, ..., \omega_k) \in \mathbb{R}^k$ and $\mu_1, ..., \mu_k \in \mathbb{R}^n$. (Note that $T_2$ and $T_3$ have decompositions using the same vectors and observing both of them gives more information than observing only $T_3$). This is done by transforming $T_2$ and $T_3$ (in an invertible way) into orthogonally decomposable tensors $\bar{T}_2$ and $\bar{T}_3$, where $\bar{T}_d = \sum_{i=1}^k \bar{\omega}_i \bar{\mu}_i \otimes d$ and $\bar{\mu}_1, ..., \bar{\mu}_k$ are orthonormal, $d = 2, 3$. Then, they use the tensor power method to find $\bar{\mu}_1, ..., \bar{\mu}_k$ and $\bar{\omega}_1, ..., \bar{\omega}_k$ and use the inverse transformation to recover the original $\mu_1, ..., \mu_k$ and $\omega_1, ..., \omega_k$. 
2 The Variety of Eigenvectors of a Tensor

In this section, we are going to study the set of all eigenvectors of a given orthogonally decomposable tensor.

As we mentioned in the introduction, a symmetric tensor \( T \in S^d(\mathbb{R}^n) \) can equivalently be represented by a homogeneous polynomial \( f_T \in \mathbb{R}[x_1, ..., x_n] \) of degree \( d \). Indeed, given \( T \), we obtain \( f_T \) by

\[
f_T(x_1, ..., x_n) = \sum_{i_1, ..., i_d} T_{i_1, ..., i_d} x_{i_1} \cdots x_{i_d}.
\]

Then, for \( x \in \mathbb{C}^n \), \( Tx^{d-1} = \lambda x \) is equivalent to \( \nabla f_T(x) = d\lambda x \), i.e. \( \nabla f_T(x) \) and \( x \) are parallel to each other. This is equivalent to the vanishing of the \( 2 \times 2 \) minors of the \( n \times 2 \) matrix \( [\nabla f_T(x) | x] \).

**Definition 2.1.** The variety of eigenvectors \( V_T \) of a given symmetric tensor \( T \) with corresponding polynomial \( f_T \) is the zero set of the \( 2 \times 2 \) minors of the matrix \( [\nabla f_T(x) | x] \).

**Remark 2.2.** Consider the gradient map as a map on projective spaces:

\[
\nabla f_T : \mathbb{P}^{n-1} \to \mathbb{P}^{n-1}
\]

\[
[x] \mapsto [\nabla f_T(x)]
\]

Then, the eigenvectors of \( f_T \) are precisely the fixed points of \( \nabla f_T \). This map is well-defined provided the hypersurface \( \{f_T = 0\} \) has no singular points.

The aim of this section is to prove the following theorem.

**Theorem 2.3.** Let \( T \in S^d(\mathbb{R}^n) \) be decom with \( f_T(x) = \sum_{i=1}^n \lambda_i (v_i \cdot x)^d \), where \( v_1, ..., v_n \) are an orthonormal basis for \( \mathbb{R}^n \). Assume that \( \lambda_1, ..., \lambda_n \neq 0 \). Then, \( T \) has exactly \( \frac{(d-1)^{n-1}}{d-2} \) eigenvectors, given explicitly in terms of \( v_1, ..., v_n \) and the \( (d-2) \)-nd roots of \( \lambda_1, ..., \lambda_n \). Let \( V = \begin{bmatrix} \cdots & -v_1 & \cdots \\ \cdots & \vdots & \cdots \\ \cdots & -v_n & \cdots \end{bmatrix} \). Namely, for any \( 1 \leq k \leq n \), any \( \mathcal{I} = \{i_1, i_2, ..., i_k\} \subseteq [n] \) and any \( (k-1) \)-tuple \( \eta_1, ..., \eta_{k-1} \) of \( (d-2) \)-nd roots of unity, there is one eigenvector \( w \), up to scaling, where \( w = V^T(y_1, ..., y_n)^T \) and

\[
y_i = \begin{cases} 
\eta_j \lambda_{i_j}^{-\frac{1}{d-2}} & \text{if } i = i_j \text{ and } j \in \{1, ..., k-1\} \\
\lambda_{i_k}^{-\frac{1}{d-2}} & \text{if } i = i_k \\
0 & \text{if } i \notin \mathcal{I}.
\end{cases}
\]

**Remark 2.4.** It is known by [4] that if a tensor \( T \in S^d(\mathbb{R}^n) \) has finitely many equivalence classes of eigenpairs \( (x, \lambda) \) over \( \mathbb{C} \), then their number, counted with multiplicity, is equal to \( \frac{(d-1)^{n-1}}{d-2} \). If the entries of \( T \) are sufficiently generic, then all multiplicities are equal to 1, so there are exactly \( \frac{(d-1)^{n-1}}{d-2} \) equivalence classes of eigenpairs.
Figure 1: This figure shows the structure of the eigenvectors of an odeco tensor $T \in S^3(\mathbb{R}^3)$ such that $T = \lambda_1 v_1 \otimes^3 + \lambda_2 v_2 \otimes^3 + \lambda_3 v_3 \otimes^3$ inside $\mathbb{CP}^2$.

Therefore, Theorem 2.3 shows that every odeco tensor $T$ with orthogonal decomposition $T = \lambda_1 v_1 \otimes^d + \cdots + \lambda_n v_n \otimes^d$, such that $\lambda_1, \ldots, \lambda_n \neq 0$, has finitely many equivalence classes of eigenpairs, i.e. is entries are sufficiently generic in the above sense.

If some of the $\lambda_i$ in the expression for $T$ are equal to 0, then all nonzero vectors in the span of all the corresponding $v_i$’s will be eigenvectors of $T$ with eigenvalue 0. In this case, it will be harder to give a characterization of all eigenvectors of $T$.

Figure 1 shows what the eigenvectors look like geometrically in the case $d = n = 3$. We illustrate Theorem 2.3 by two simple concrete examples.

**Example 2.5.** Let $d = n = 3$ and consider the odeco tensor $T$ with polynomial form

$$f_T(x, y, z) = ax^3 + by^3 + cz^3.$$  

This type of polynomial is called a Fermat polynomial. In this case $v_1 = (1, 0, 0), v_2 = (0, 1, 0), v_3 = (0, 0, 1)$, the matrix $V = I$, and $\lambda_1 = a, \lambda_2 = b, \lambda_3 = c$. Since $d - 2 = 1$, taking the $(d - 2)$-nd root is the identity map. Thus, the eigenvectors of $T$ are as follows.

When $k = 1$, $I = \{1\}, \{2\},$ or $\{3\}$. The corresponding three eigenvectors are

$$\left(\frac{1}{a}, 0, 0\right)^T, \left(0, \frac{1}{b}, 0\right)^T, \left(0, 0, \frac{1}{c}\right)^T.$$  

When $k = 2$, $I = \{1, 2\}, \{1, 3\},$ or $\{2, 3\}$. The corresponding eigenvectors are

$$\left(\frac{1}{a}, \frac{1}{b}, 0\right)^T, \left(\frac{1}{a}, 0, \frac{1}{c}\right)^T, \left(0, \frac{1}{b}, \frac{1}{c}\right)^T.$$  

When $k = 3$, $I = \{1, 2, 3\}$ and the corresponding eigenvector is

$$\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)^T.$$
Lemma 2.8. Theorem 2.3 is true in the case \( \lambda \) in the polynomial ring \( \mathbb{C} \). In this case, \( v_1 = \left( \frac{3}{5}, \frac{4}{5}, 0 \right), v_2 = \left( \frac{4}{5}, -\frac{3}{5}, 0 \right), v_3 = (0, 0, 1) \), the matrix \( V = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} & 0 \\ -\frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \), and \( \lambda_1 = a, \lambda_2 = b, \lambda_3 = c \).

According to Theorem 2.3, the eigenvectors of \( T \) are as follows.

When \( k = 1, I = \{1\}, \{2\}, \{3\} \). The corresponding three eigenvectors are

\[
V^T(\frac{1}{a}, 0, 0)^T = \left( \frac{3}{5a}, \frac{4}{5a}, 0 \right)^T, \quad V^T(0, \frac{1}{b}, 0)^T = \left( \frac{4}{5b}, -\frac{3}{5b}, 0 \right)^T, \quad V^T(0, 0, \frac{1}{c})^T = (0, 0, \frac{1}{c})^T.
\]

When \( k = 2, I = \{1, 2\}, \{1, 3\}, \{2, 3\} \). The corresponding eigenvectors are

\[
V^T(\frac{1}{a}, \frac{1}{b}, 0)^T = \left( \frac{3}{5a} + \frac{4}{5b}, \frac{4}{5a} - \frac{3}{5b}, 0 \right)^T, \quad V^T(\frac{1}{a}, 0, \frac{1}{c})^T = \left( \frac{3}{5a}, \frac{4}{5a}, \frac{1}{c} \right)^T, \quad V^T(0, \frac{1}{b}, \frac{1}{c})^T = \left( \frac{4}{5b}, -\frac{3}{5b}, \frac{1}{c} \right)^T.
\]

When \( k = 3, I = \{1, 2, 3\} \) and the corresponding eigenvector is

\[
V^T(\frac{1}{a}, \frac{1}{b}, \frac{1}{c})^T = \left( \frac{3}{5a} + \frac{4}{5b}, \frac{4}{5a} - \frac{3}{5b}, \frac{1}{c} \right).
\]

In the rest of this section we prove Theorem 2.3. We proceed as follows. First we show that the theorem is valid when \( f_T = \lambda_1 x_1^d + \cdots + \lambda_n x_n^d \), where \( \lambda_1, \ldots, \lambda_n \neq 0 \). This is done in Lemma 2.8. For the general case, \( f_T = \lambda_1 (v_1 \cdot x)^d + \cdots + \lambda_n (v_n \cdot x)^d \), where \( \lambda_1, \ldots, \lambda_n \neq 0 \) and \( v_1, \ldots, v_n \) are orthonormal, we observe that setting \( y_i = v_i \cdot x \) the eigenvectors of the Fermat polynomial tensor \( \lambda_1 y_1^d + \cdots + \lambda_n y_n^d \) are in a 1-to-1 correspondence with the eigenvectors of \( T \) via the transformation given by the matrix \( V \) with rows \( v_1, \ldots, v_n \). This is how we recover the formula in Theorem 2.3.

Definition 2.7. Given \( f(x_1, \ldots, x_n) = \lambda_1 x_1^d + \cdots + \lambda_n x_n^d \), \( I = \langle i_1, \ldots, i_k \rangle \subseteq \{1, 2, \ldots, n\} \), and \( \eta = \{\eta_1, \ldots, \eta_{k-1}\} \) such that \( \eta_1, \ldots, \eta_{k-1} \) are \( (d-2) \)-nd roots of unity, and define the ideal

\[
I_{\mathcal{I}, \eta} = \langle \lambda_1^{\frac{1}{\eta_1}} x_{i_1} - \eta_1 \lambda_{i_2}^{\frac{1}{\eta_2}}, \ldots, \lambda_{i_{k-1}}^{\frac{1}{\eta_{k-2}}} x_{i_{k-1}} - \eta_{k-1} \lambda_{i_k}^{\frac{1}{\eta_k}} x_{i_k} \rangle + \langle x_j \mid j \notin \mathcal{I} \rangle
\]

in the polynomial ring \( \mathbb{C}[x_1, \ldots, x_n] \).

Lemma 2.8. Theorem 2.3 is true in the case \( f_T(x_1, \ldots, x_n) = \lambda_1 x_1^d + \lambda_2 x_2^d + \cdots + \lambda_n x_n^d \), where \( \lambda_1, \ldots, \lambda_n \neq 0 \). In particular, the primary decomposition of the ideal \( I \) is as follows.

\[
I = \bigcap_{\mathcal{I} \subseteq [n], \eta = \{\eta_1, \ldots, \eta_{|\mathcal{I}|-1}\}} I_{\mathcal{I}, \eta}.
\]
where \( \eta_1, ..., \eta_{k-1} \) are \((d-2)\)-nd roots of unity. In particular, there are \( \binom{n}{k} \) \((d-2)^{k-1}\) homogeneous prime ideals \( I_{\mathcal{I}, \eta} \) with \(|\mathcal{I}| = k\). Each ideal \( I_{\mathcal{I}, \eta} \) has exactly one solution in \( \mathbb{C}P^{n-1} \), representing one eigenvector, namely \( \mathbf{w} = (w_1 : ... : w_n) \) such that

\[
\mathbf{w}_i = \begin{cases} 
\eta_{\mathbf{w}_i} \sqrt{x_{i_1}} & \text{if } i = i_1 \text{ and } l \leq k - 1, \\
\lambda_{i_k} \sqrt{x_{i_k}} & \text{if } i = i_k, \\
0 & \text{if } i \notin \mathcal{I}.
\end{cases}
\]

The total number of such solutions is \( \frac{(d-1)^{n-1}}{d^2} \).

**Proof.** Note that in this case, up to a factor of \( d \) in the first row, we have that

\[
\nabla f(x) | x = \begin{bmatrix} 
\lambda_1 x_1^{d-1} & x_1 \\
\lambda_2 x_2^{d-1} & x_2 \\
\vdots & \vdots \\
\lambda_n x_n^{d-1} & x_{n-1}
\end{bmatrix}
\]

Therefore, the ideal of \( 2 \times 2 \) minors is given by

\[
I = \langle x_i x_j (\lambda_i x_i^{d-2} - \lambda_j x_j^{d-2}) : i \neq j \rangle.
\]

We would like to decompose this ideal. Note that any primary ideal \( P \supseteq I \) would either contain \( x_i x_j \) or \( \lambda_i x_i^{d-2} - \lambda_j x_j^{d-2} \) for all \( i \neq j \). Suppose that for a given \( P \), \( \sqrt{P} \) contains exactly \( n-k \) of the variables \( x_1, ..., x_n \). Let \( \mathcal{I} = \{i_1, ..., i_k\} \subseteq [n] \) and \( \sqrt{P} \) contains \( x_i \) for all \( i \notin \mathcal{I} \). Thus, \( \sqrt{P} \) also contains \( \lambda_i x_i^{d-2} - \lambda_j x_j^{d-2} \) for \( i \neq j, i, j \in \mathcal{I} \). Then, we have that \( \sqrt{P} \) equals \( (x_i : i \notin \mathcal{I})+ \) an associated prime of

\[
I_{\mathcal{I}} = \langle \lambda_i x_i^{d-2} - \lambda_j x_j^{d-2} : i \neq j, i, j \in \mathcal{I} \rangle = \langle \lambda_i x_i^{d-2} - \lambda_{j+1} x_{j+1}^{d-2} : j = 1, ..., k-1 \rangle.
\]

The decomposition of \( I_{\mathcal{I}} \) is the same as the decomposition of the lattice ideal (cf. [5]) associated to the lattice \( L_\rho = \langle (d-2)(e_{i_j} - e_{i_k}) : j = 1, ..., k-1 \rangle \) with partial character \( \rho : L_\rho \to \mathbb{C}^* \) given by

\[
\rho((d-2)(e_{i_j} - e_{i_k})) = \frac{\lambda_{i_k}}{\lambda_{i_j}}.
\]

By [Theorem 2.1.(d), [5]], we know that \( I(\rho) = \langle x^m - \rho(m) : m \in L_\rho \rangle = \langle x_{i_j}^{d-2} x_{i_k}^{-(d-2)} - \frac{\lambda_{i_j}}{\lambda_{i_k}} \rangle \) has the following decomposition

\[
I(\rho) = \bigcap_{\rho' \text{ extends } \rho \text{ to } L} I(\rho'),
\]

where \( L_\rho \subseteq L \subseteq \mathbb{Z}^n \) and \(|L/L_\rho|\) is finite. In this case, we can choose

\[
L = \langle e_{i_j} - e_{i_k} : j = 1, ..., k-1 \rangle.
\]
Then, $|L/L_\rho| = (d - 2)^{k-1}$. Moreover, by the same theorem, the number of $\rho'$ extending $\rho$ is exactly $|L/L_\rho| = (d - 2)^{k-1}$. Also, note that each such $\rho' : L \to \mathbb{C}^*$ is uniquely defined by the values

$$\eta_j(\frac{\lambda_{i_k}}{\lambda_{i_j}})^{\frac{1}{d-2}} := \rho'(e_{i_j} - e_{i_k})$$

and each $\eta_j$ is a $(d - 2)$-nd root of unity. Therefore,

$$I(\rho') = \langle x_{i_j} - \eta_j(\frac{\lambda_{i_k}}{\lambda_{i_j}})^{\frac{1}{d-2}} x_{i_k} : j = 1, 2, ..., k - 1 \rangle.$$

This gives an explicit description of the set

$$V(I(\rho')) = \{(\eta_1 \lambda_{i_1}^{-\frac{1}{d-2}} : \eta_2 \lambda_{i_2}^{-\frac{1}{d-2}} : \cdots : \eta_{k-1} \lambda_{i_{k-1}}^{-\frac{1}{d-2}} : \lambda_{i_k}^{-\frac{1}{d-2}})\}.$$

We have that

$$I_\mathcal{I} = I(\rho) \cap \mathbb{C}[x_{i_1}, ..., x_{i_k}] = \bigcap_{\eta_1, ..., \eta_{k-1}} \langle x_{i_j} - \eta_j(\frac{\lambda_{i_k}}{\lambda_{i_j}})^{\frac{1}{d-2}} x_{i_k} : j = 1, 2, ..., k - 1 \rangle,$$

where $\eta_1, ..., \eta_{k-1}$ vary over the $(d - 2)$-nd roots of unity. Note that $|V(I(\rho))| = |\{\rho' : \rho' extends \rho\}| = (d - 2)^{k-1}$, because there are $d - 2$ options for each of the $(d - 2)$-nd roots of unity $\eta_i$ for $i = 1, 2, ..., k - 1$. Each element of $V(I(\rho))$ defines one eigenvector of $\rho$, namely $w = (w_1 : \cdots : w_n)$, where

$$w_i = \begin{cases} 
\eta_i \lambda_{i_l}^{-\frac{1}{d-2}} & \text{if } i = i_l \text{ and } l \leq k - 1, \\
\lambda_{i_k}^{-\frac{1}{d-2}} & \text{if } i = i_k, \\
0 & \text{if } i \notin \mathcal{I}
\end{cases}.$$

Moreover, $I(\rho) = I_{\mathcal{I}, \eta}$, where $\mathcal{I} = \{k + 1, ..., n\}$. The other ideals in the decomposition are obtained in exactly the same way by choosing different $\mathcal{I} \subseteq [n]$.

To sum up, we have that

$$I = \bigcap_{\mathcal{I} = \{i_1, ..., i_k\} \subseteq [n], \eta} \langle \lambda_{i_j}^{\frac{1}{d-2}} x_{i_j} - \eta_j \lambda_{i_k}^{\frac{1}{d-2}} x_{i_k} : j = 1, ..., k - 1 \rangle + \langle x_i : i \notin \mathcal{I} \rangle = \bigcup_{\mathcal{I} \subseteq [n], \eta} I_{\mathcal{I}, \eta}.$$

Each such ideal $I_{\mathcal{I}, \eta}$ is zero-dimensional and corresponds to one eigenvector.

Moreover, since there are $\binom{n}{k}$ options for choosing $\mathcal{I} \subseteq [n]$ with $|\mathcal{I}| = k$ and $(d - 2)^{k-1}$ options for choosing $\eta = (\eta_1, ..., \eta_{k-1})$, the total number of eigenvectors of $\rho$ is

$$\sum_{k=1}^{n} \binom{n}{k} (d - 2)^{k-1} = \frac{1}{d - 2} \sum_{k=1}^{n} \binom{n}{k} (d - 2)^{k} = \frac{1}{d - 2} ((d - 2 + 1)^n - 1) = \frac{(d - 1)^n - 1}{d - 2},$$

recovering the formula expected by [4].
Now, we proceed with the proof of Theorem 2.3.

**Proof of Theorem 2.3.** Let $V = \begin{bmatrix} -v_1 & \ldots & -v_n \end{bmatrix} \in \mathbb{R}^{n \times n}$ be an orthogonal matrix, i.e. the vectors $v_1, \ldots, v_n$ form an orthonormal basis of $\mathbb{R}^n$. Let

$$f(x) = \sum_{i=1}^{n} \lambda_i (v_i \cdot x)^d$$

and

$$\frac{1}{d} \nabla f(x) = \sum_{i=1}^{n} \lambda_i (v_i \cdot x)^{d-1} v_i.$$ 

If $x \in \mathbb{C}^n$ is an eigenvector, then

$$\frac{1}{d} \nabla f(x) = \sum_{i=1}^{n} \lambda_i (v_i \cdot x)^{d-1} v_i = \lambda x.$$ 

But note that $v_1, v_2, \ldots, v_n$ are an orthonormal basis of $\mathbb{R}^n$ and, thus, they are also a basis of $\mathbb{C}^n$. Moreover, $x = \sum_{i=1}^{n} (v_i \cdot x) v_i$, where $v_i \cdot x = \sum_j v_{ij} x_j$ is still the usual dot product on $\mathbb{R}^n$. Since the $v_i$ form a basis of $\mathbb{C}^n$ and

$$\sum_{i=1}^{n} \lambda_i (v_i \cdot x)^{d-1} v_i = \lambda \sum_{i=1}^{n} (v_i \cdot x) v_i,$$

then $x$ is an eigenvector if and only if the vectors $(\lambda_1 (v_1 \cdot x)^{d-1}, \ldots, \lambda_n (v_n \cdot x)^{d-1})$ and $(v_1 \cdot x, \ldots, v_n \cdot x)$ are parallel. Let

$$y_i = (v_i \cdot x),$$

i.e. $y = V x$.

Then, an equivalent description of $x$ being an eigenvector is that $(\lambda_1 y_1^{d-1}, \ldots, \lambda_n y_n^{d-1})$ and $y$ are parallel. In other words, the ideal of $2 \times 2$ minors of the matrix

$$\begin{bmatrix} \lambda_1 y_1^{d-1} & \cdots & \lambda_n y_n^{d-1} \\ y_1 & \cdots & y_n \end{bmatrix}$$

is precisely the ideal defining the set of eigenvectors of $f$

$$I = \langle \lambda_i y_i^{d-1} y_j - \lambda_j y_j^{d-1} y_i : i \neq j \rangle.$$ 

By Lemma 2.8, the primary decomposition of this ideal

$$I = \bigcap_{I \subseteq [n], \eta} I_{I, \eta}$$

and each ideal $I_{I, \eta}$ with $I = \{i_1, \ldots, i_k\}$ has the form

$$I_{I, \eta} \langle \lambda_i^{\frac{1}{d-2}} y_{i_1} - \eta_1 \lambda_i^{\frac{1}{d-2}} y_{i_1}, \ldots, \lambda_{i_{k-1}}^{\frac{1}{d-2}} y_{i_{k-1}} - \eta_{k-1} \lambda_{i_{k-1}}^{\frac{1}{d-2}} y_{i_{k-1}}, y_{i_k} \rangle + \langle y_{i} : i \notin I \rangle, \quad (2.1)$$

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where $\eta_1, ..., \eta_{k-1}$ are $(d-2)^{nd}$ roots of unity. Each such ideal has exactly one solution in $\mathbb{CP}^n$, representing one eigenvector $(y_1, ..., y_n)$ such that

$$y_i = \begin{cases} \eta^{\frac{1}{d-2}}_{i_l} - \frac{1}{d-2} & \text{if } i = i_l \text{ and } l \leq k - 1, \\ \lambda^{-\frac{1}{d-2}}_{i_k} & \text{if } i = i_k, \\ 0 & \text{if } i \not\in \mathcal{I}. \end{cases} \tag{2.2}$$

Note that $y = Vx$ and $V$ is an orthogonal matrix. Therefore,

$$x = V^T y.$$

By Lemma 2.8, we know that for each $k$ there are $\binom{n}{k}(d-2)^{k-1}$ eigenvectors with $k$ nonzero entries, which makes for a total of $\frac{(d-1)^{n-1}}{d-2}$ eigenvectors. \hfill \Box

3 The Odeco Variety

The *odeco variety* is the Zariski closure in $S^d(\mathbb{C}^n)$ of the set of all tensors $T \in S^d(\mathbb{R}^n)$ which are orthogonally decomposable. If a tensor is odeco, then, in particular, its corresponding polynomial $f_T$ is decomposable as a sum of $n$ $d$-th powers of linear forms, i.e. it lies in the $n$-th secant variety of the $d$-th Veronese variety, denoted by $\sigma_n(v_d(\mathbb{C}^n))$.

When $d = n = 3$, there is one equation defining $\sigma_3(v_3(\mathbb{C}^3))$, called the Aronhold invariant [9], and it is given by the Pfaffian of a certain skew-symmetric matrix. The corresponding odeco variety in $S^3(\mathbb{C}^3)$ has codimension 4 and its prime ideal is generated by six quadrics, defined in Example 3.4. For higher $d$ and $n$, the equations defining $\sigma_n(v_d(\mathbb{C}^n))$ are much harder to compute. However, the odeco variety is smaller than $\sigma_n(v_d(\mathbb{C}^n))$ and we believe that the defining equations of its prime ideal are quadrics that are easy to write down. They are shown in Conjecture 3.2.

**Lemma 3.1.** The dimension of the odeco variety in $S^d(\mathbb{C}^n)$ is $\binom{n+1}{2}$.

**Proof.** Consider the map

$$\phi : \mathbb{R}^n \times SO_n \to S^d(\mathbb{R}^n) \subset S^d(\mathbb{C}^n)$$

given by

$$(\lambda_1, ..., \lambda_n), V \mapsto \sum_{i=1}^n \lambda_i v_i^{\otimes d},$$

where $v_i$ is the $i$th row of the orthogonal matrix $V$. The image $\text{Im}(\phi)$ of this map is precisely the set of orthogonally decomposable tensors in $S^d(\mathbb{R}^n)$. The odeco variety is $\overline{\text{Im}(\phi)} \subset S^d(\mathbb{C}^n)$. Note that by Theorem 1.3, $\phi$ has a finite fiber (up to permutations of the input). Then, $\dim(\text{Im}(\phi)) = \dim(\mathbb{R}^n \times SO_n) = n + \binom{n}{2} = \binom{n+1}{2}$. Therefore, the dimension of the odeco variety is $\dim(\overline{\text{Im}(\phi)}) = \binom{n+1}{2}$. \hfill \Box
We are going to conjecture what the defining equations of the odeco variety are. In Proposition 3.6 we prove the result for the case \( n = 2 \).

Consider a tensor \( T \in S^d(\mathbb{C}^n) \) and the corresponding homogeneous polynomial \( f_T(x_1, x_2, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n] \) of degree \( d \). To define our equations, it is more convenient to work with the polynomial version of the tensor. As mentioned before, given \( T \in S^d(\mathbb{C}^n) \), the corresponding polynomial can be rewritten as

\[
f_T(x_1, \ldots, x_n) = \sum_{j_1, \ldots, j_d} T_{j_1 \ldots j_d} x_{j_1} \cdots x_{j_d}
\]

\[
= \sum_{i_1 + \cdots + i_n = d} \left( \begin{array}{c} d \\ i_1, \ldots, i_n \end{array} \right) T_{i_1 \ldots i_n}^{\frac{1}{i_1 \cdots i_n}} x_1^{i_1} \cdots x_n^{i_n} = \sum_{i_1 + \cdots + i_n = d} \left( \frac{d}{i_1 \cdots i_n} \right) u_{i_1, \ldots, i_n} x_1^{i_1} \cdots x_n^{i_n},
\]

where

\[
u_{i_1, \ldots, i_n} = d! T_{i_1 \ldots i_n}^{\frac{1}{i_1 \cdots i_n}}.
\]

We write the equations defining the odeco variety in terms of the variables \( u_{i_1, \ldots, i_n} \). Note that for all such variables \( i_1 + \cdots + i_n = d \).

**Conjecture 3.2.** The prime ideal of the odeco variety inside \( S^d(\mathbb{C}^n) \) is generated by

\[
\sum_{s=1}^n u_{y+s, v+w, z} - u_{w+e, z+e, s} = 0,
\]

where \( y, v, w, z \in \mathbb{Z}_{\geq 0}^n \) are such that such that \( \sum_i y_i = \sum_i v_i = \sum_i z_i = \sum_i w_i = d - 1 \) and \( y + v = z + w \).

**Example 3.3.** When \( d = 2 \) the elements of \( S^2(\mathbb{R}^n) \) are symmetric matrices and the set of equations (3.1) is empty, which is equivalent to the fact all symmetric matrices are odeco.

In essence, the ideal defined by (3.1) is a lifting of the toric ideal defining the Veronese variety \( v_{d-1}(\mathbb{C}^n) \subset S^{d-1}(\mathbb{C}^n) \) to non-toric equations on \( S^d(\mathbb{C}^n) \).

**Example 3.4.** Let \( d = n = 3 \). We will illustrate how to obtain the equations (3.1) of the odeco variety in \( S^3(\mathbb{C}^3) \) from the equations of the Veronese variety \( v_{d-1}(\mathbb{C}^n) = v_2(\mathbb{C}^3) \).

Consider the Veronese embedding \( v_2 : \mathbb{C}^3 \to S^2(\mathbb{C}^3) \) given by \( x \mapsto x^\otimes 2 \). The image \( v_2(\mathbb{C}^3) \) is the set of rank one \( 3 \times 3 \) symmetric matrices. The space \( S^2(\mathbb{C}^3) \) has coordinates \( u_{i_1i_2i_3} \), where \( i_1 + i_2 + i_3 = 2 \). There are 6 equations that define the prime ideal of the Veronese variety \( v_2(\mathbb{C}^3) \subset S^2(\mathbb{C}^3) \) and they are

\[
\begin{align*}
& u_{200}u_{020} - u_{110}^2 = 0, \quad u_{200}u_{011} - u_{110}u_{010} = 0, \\
& u_{200}u_{002} - u_{010}^2 = 0, \quad u_{110}u_{002} - u_{010}u_{001} = 0, \\
& u_{011}u_{100} - u_{110}u_{001} = 0, \quad u_{020}u_{002} - u_{001}^2 = 0.
\end{align*}
\]

Each of these equations has the form \( u_yu_v - u_wu_z = 0 \), where \( y, v, w, z \in \mathbb{Z}_{\geq 0}^3 \), \( \sum_i y = \sum_i v = \sum_i w = \sum_i z = 2 \), and \( y + v = w + z \). Each such equation leads to one of the equations in (3.1) as follows

\[
u_yu_v - u_wu_z \mapsto u_{y+e_1}u_{v+e_1} - u_{w+e_1}u_{z+e_1} + u_{y+e_2}u_{v+e_2} - u_{w+e_2}u_{z+e_2} + u_{y+e_3}u_{v+e_3} - u_{w+e_3}u_{z+e_3}.
\]
Thus, at the point $T$, the equations (3.1) vanish on the odec variety.

**Lemma 3.5.** The equations (3.1) vanish on the odec variety.

**Proof of Lemma 3.5.** Let $T = \sum_i \lambda_i v_i^{(d)}$ be odec. Then, by definition of the $u$-variables, at the point $T$

$$ u_{y_1 \ldots y_n} = d! \sum_{i=1}^n \lambda_i v_i^{y_1} \cdots v_i^{y_n} = d! \sum_{i=1}^n \lambda_i v_i^y. $$

Thus, at the point $T$, the equations (3.1), for $y, v, w, z \in \mathbb{Z}_{\geq 0}^n$ with $y + v = w + z$ and $\sum_i y = \sum_i v = \sum_i w = \sum_i z = d - 1$, have the form

$$ \sum_{s=1}^n u_{y+e_s}u_{v+e_s} - u_{w+e_s}u_{z+e_s} = $$

$$ = (d!)^2 \sum_{s=1}^n \left( \sum_{i=1}^n \lambda_i v_i^{y+e_s} \right) \left( \sum_{j=1}^n \lambda_j v_j^{v+e_s} \right) - \left( \sum_{i=1}^n \lambda_i u_i^{w+e_s} \right) \left( \sum_{j=1}^n \lambda_j u_j^{z+e_s} \right) $$

$$ = (d!)^2 \sum_{s=1}^n \left( \sum_{i=1}^n \lambda_i^2 \left( v_i^{y+e_s} - v_i^{w+e_s} \right) \right) + \left( \sum_{i \neq j} \lambda_i \lambda_j \left( v_i^{y+e_s} v_j^{v+e_s} - v_i^{w+e_s} v_j^{z+e_s} \right) \right) $$

$$ = (d!)^2 \sum_{i \neq j} \lambda_i \lambda_j \left( v_i^y v_j^v - v_i^w v_j^z \right) \sum_{s=1}^n v_is v_js = 0, $$

where the last row is 0 since $v_i$ and $v_j$ are orthogonal and $\sum_{s=1}^n v_is v_js = v_i \cdot v_j = 0$.

Therefore, (3.1) vanish on the odec variety. 

We are going to select a subset of the equations (3.1) that spans the vector space defined by (3.1). More precisely, consider

$$ f_{y,v,i,j} = \sum_{s=1}^n u_{y+e_s}u_{v+e_s} - u_{y+e_i-e_j+e_s}u_{v+e_i+e_j+e_s}; \quad (3.3) $$

for all $i \neq j \in \{1,2,\ldots,n\}$ and all $y,v \in \mathbb{Z}_{\geq 0}^n$ whose entries sum to $d - 1$ and $y_j \geq 1$, $v_i \geq 1$.

We now prove Conjecture 3.2 for the case $n = 2$.

**Proposition 3.6.** When $n = 2$, the equations (3.3) form a Gröbner basis and the dimension of the variety they cut out is $\binom{n+1}{2} = 3$. The ideal defined by (3.3) is the prime ideal of the Odeco variety.
Proof. We are going to work over the polynomial ring
\[ \mathbb{C}[u] := \mathbb{C}[u_{i_1 i_2} | i_1, i_2 \geq 0 \text{ and } i_1 + i_2 = d] \]

Then, the equations (3.3) are
\[ f_{y,v,1,2} = u_{y+e_1} u_{v+e_1} - u_{y+e_1-e_2+e_1} u_{v-e_1+e_2+e_1} + u_{y+e_2} u_{v+e_2} - u_{y+e_1-e_2+e_2} u_{v-e_1+e_2+e_2}, \]

where \( y, v \in \mathbb{Z}_{\geq 0}^2 \), the sum of the entries of each of \( y \) and \( v \) is \( d-1 \) and \( y_2 \geq 1, v_1 \geq 1 \). Let the ideal they generate be
\[ I = \langle f_{y,v,1,2} | y, v \in \mathbb{Z}_{\geq 0}^2, \sum y_i = \sum v_i = d-1, y_2 \geq 1, v_1 \geq 1 \rangle. \]

We introduce the following weights on our variables.
\[ \text{weight}(u_{i(d-i)}) = i, \]

for all \( i = 0, 1, \ldots, d \).

Consider the lexicographic weighted term order on monomials \( \prec \) given by the above weights and by the order of the variables \( u_{d0}, u_{(d-1)1}, \ldots, u_{0d} \) in case of equal weights.

We first show that the equations (3.3) form a Gröbner basis. Using Macaulay2, we have shown that they form a Gröbner basis for \( d = 1, 2, \ldots, 9 \). Now, consider any \( d > 9 \). Take \( f_{y',v',1,2} \) and \( f_{y'',v'',1,2} \). By Buchberger’s second criterion, we only need to consider the two polynomials when their initial terms have a common variable. Then, the two polynomials \( f_{y',v',1,2} \) and \( f_{y'',v'',1,2} \) contain \( l \leq 9 \) different variables in total. If we restrict our generators (3.3) to these \( l \) variables only, the weighted term order is the same as in the case \( d = l-1 \), and we have shown that in this case, the restricted generators form a Gröbner basis. Therefore, we can reduce the S-pair of \( f_{y',v',1,2} \) and \( f_{y'',v'',1,2} \) to 0 using the generators (3.3). Thus, the equations (3.3) form a Gröbner basis.

Next, we show that the ideal \( I \) generated by (3.3) has dimension 3. One way to see this is to use Lemma 3.7 together with the fact that \( I \) is prime, which is proven below. Another way to see that \( \dim I = 3 \) is to reason with standard monomials as follows.

Note that because of our choice of term order, the initial term of every \( f_{u,v,1,2} \) is square-free. The reason is that if \( u_{y+e_1} u_{v+e_2} \) then, weight(\( u_{y+e_1} u_{v+e_2} \)) = weight(\( u_{y+e_1-e_2+e_1} u_{v-e_1+e_2+e_1} \)) > weight(\( u_{y+e_2} u_{v+e_2} \)) = weight(\( u_{y+e_1-e_2+e_2} u_{v-e_1+e_2+e_2} \)), but \( u_{y+e_1-e_2+e_2} \) appears first in the variable order, so, \( u_{y+e_1-e_2+e_1} u_{v-e_1+e_2+e_1} \) is the leading term. The reasoning is similar if \( u_{y+e_1-e_2+e_2} = u_{v-e_1+e_2+e_2} \). Therefore, \( \text{in}_<I \) (and thus \( I \)) is a radical ideal.

To show that \( \dim I = 3 \), let \( S = \{ u_{i_1(d-i_1)}, u_{i_2(d-i_2)}, u_{i_3(d-i_3)}, u_{i_4(d-i_4)} \} \) be a set of four variables, where \( i_1 > i_2 > i_3 > i_4 \). We will show that there is a monomial with only variables from \( S \) which is not standard. This would mean that \( \dim I \leq 3 \). Indeed, consider
\[ f_{(i_1-1,d-i_1+1),(i_3+1,d-i_3-1),1,2} = u_{(i_1-1)(d-i_1+1)} u_{(i_3+1)(d-i_3+1)} - u_{i_1(d-i_1)} u_{i_3(d-i_3)} \]
\[ + u_{(i_1-2)(d-i_1+2)} u_{i_2(d-i_2)} - u_{(i_1-1)(d-i_1+1)} u_{(i_2-1)(d-i_2+1)}. \]
Since $i_1 - 2 \geq i_3$, the initial term is $u_{i_1(d-i_1)}u_{i_3(d-i_3)}$. Therefore, $\dim I \leq 3$.

Now, consider the set $S = \{u_{2(d-2)}, u_{1(d-1)}, u_{0d}\}$. Suppose there exists

$$f_{g,v,1,2} = u_{y+e_1}u_{v+e_1} - u_{y+e_1-e_2+e_1}u_{v-e_1+e_2+e_1} + u_{y+e_2}u_{v+e_2} - u_{y+e_1-e_2+e_2}u_{v-e_1+e_2+e_2},$$

such that $\text{in}_\prec(f)$ has both of its variables in $S$. We know that $\text{in}_\prec(f) = u_{y+e_1}u_{v+e_1}$ or $\text{in}_\prec(f) = u_{y+e_1-e_2+e_1}u_{v-e_1+e_2+e_1}$. Moreover, if $y = (y_1, y_2)$ and $v = (v_1, v_2)$, then, $y_2, v_1 \geq 1$ and $y_1, v_2 \leq d - 2$. Thus, if $\text{in}_\prec(f) = u_{y+e_1}u_{v+e_1}$ and $u_{y+e_1}, u_{v+e_1} \in S$, then, $v = (1, d - 2)$ and $y = (1, d - 2)$ or $y = (0, d - 1)$. Since $f_{g,v,1,2}$ is not the trivial polynomial 0, then, $y \neq (0, d - 1)$. Thus, $y = (1, d - 2)$. But this is impossible since $\text{in}_\prec(f)$ is square-free for every generator $f$. If $\text{in}_\prec(f) = u_{y+e_1-e_2+e_1}u_{v-e_1+e_2+e_1}$ and $u_{y+e_1-e_2+e_1}, u_{v-e_1+e_2+e_1} \in S$, then, $u_{(y_1+2,y_2-1)} \in S$. But $y_1 \geq 1$, so, $y_1 + 2 \geq 3$, therefore, $u_{(y_1+2,y_2-1)} \notin S$. In any case, there can’t be a monomial with only variables in $S$, which is a leading term of an element in $I$. Thus, $\dim I = 3$.

Another way to see that $\dim I \geq 3$ is by noting that $V(I)$ contains the Odeco variety, which has dimension 3 in this case.

Finally, we show that the ideal generated by (3.3) is prime. Let $J$ be the ideal generated by the leading binomials of the elements in (3.3) with respect to the weight order (without considering the refinement given by the order of the variables). Denote by $g_w$ the leading term of a polynomial just with respect to the weight order. Then, $(f_{g,v,1,2})_w = u_{y+e_1}u_{v+e_1} - u_{y+e_1-e_2+e_1}u_{v-e_1+e_2+e_1}$, and $J = \langle u_{y+e_1}u_{v+e_1} - u_{y+e_1-e_2+e_1}u_{v-e_1+e_2+e_1} \rangle : y,v \in \mathbb{Z}_{\geq 0}, y_1 + y_2 = v_1 + v_2 = d - 1, y_2, v_1 \geq 1 \rangle$. The ideal $J$ is the prime ideal of the rational normal curve; in particular, it is prime. Moreover, by Proposition 1.13 in [14], $\text{in}_\prec(I) = \text{in}_\prec(J)$. Therefore, $\text{in}_\prec(I)$ is an initial ideal of both $I$ and $J$. In the following paragraph, we show that $J$ is the initial ideal of $I$ with respect to the weight order $w$. Then, since $J$ is prime, it follows that $I$ is prime.

Suppose $J$ is not initial, i.e. there exists $g \in I$ such that $g_w \notin J$. Choose $g$ with $\text{in}_\prec(g)$ as small as possible. Since the elements $f_{g,v,1,2}$ form a Gröbner basis of $I$, then, there exist $y, v$ such that $\text{in}_\prec(g)$ is divisible by $\text{in}_\prec(f_{g,v,1,2})$. Then, $g = \alpha_{y,v}f_{g,v,1,2} + g_1$, where $\alpha_{y,v}$ is a monomial and $\text{in}_\prec(g_1) \prec \text{in}_\prec(g)$. But note that then, $g_w = \alpha_{y,v}(u_{y+e_1}u_{v+e_1} - u_{y+e_1-e_2+e_1}u_{v-e_1+e_2+e_1}) + (g_1)_w$. Since $u_{y+e_1}u_{v+e_1} - u_{y+e_1-e_2+e_1}u_{v-e_1+e_2+e_1} \in J$ and $g_w \notin J$, then, $(g_1)_w \notin J$. But this is a contradiction since $\text{in}_\prec(g_1) \prec \text{in}_\prec(g)$ and we chose in $\text{in}_\prec(g)$ to be as small as possible such that $g_w \notin J$.

Therefore, $J$ is initial. Since it is prime, then, $I$ is also prime. By Lemma 3.7, the dimension of the Odeco variety for $n = 2$ is 3. Moreover, it is contained in $V(I)$. Since $V(I)$ is also irreducible and has dimension 3, then, $I$ is exactly the prime ideal of the Odeco variety. 

3.1 Evidence for Conjecture 3.2

Lemma 3.7. The Odeco variety is an irreducible component of $V(I)$, where $I$ is the ideal generated by the equations (3.1).

Proof. We show that the dimension of the component of $V(I)$ containing the Odeco variety is equal to $\binom{n+1}{2}$. This equals the dimension of the Odeco variety. Since it is irreducible, then it is an irreducible component of $V(I)$. 

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Thus, the rank of the Jacobian at a smooth point in the irreducible component of $T$ is 1.

We can select generators $f_{v,w}$ for $I$ such that $v, w \in \mathbb{Z}_{\geq 0}^n$ with $\sum_i v_i = \sum_i w_i = d - 1$ and

$$f_{v,w} = \sum_{i=1}^{s} u_{v+e_s} u_{w+e_s} - u_{\text{sort}(v,w)1+e_s} u_{\text{sort}(v,w)2+e_s},$$

where $\text{sort}(v,w)_1$ and $\text{sort}(v,w)_2$ are defined as follows. Given $v$ and $w$, form the corresponding sequences $t(v) = 1 \ldots 1 2 \ldots 2 \ldots n \ldots n$ and $t(w) = 1 \ldots 1 2 \ldots 2 \ldots n \ldots n$. Let $t(v,w) = \text{sort}(t(v) \cup t(w))$ be the sequence obtained by concatenating $t(v)$ and $t(w)$ and then sorting. Let $t(v,w)_1$ be the subsequence of elements in odd positions and $t(v,w)_2$ the subsequence of elements in even positions. Define $u_{\text{sort}(v,w)_1}$ and $u_{\text{sort}(v,w)_2}$ be the corresponding $u$ variables. The fact that the polynomials $f_{v,w}$ generate $I$ follows from Theorem 14.2 in [14].

We form the Jacobian $J$ of $I$ at the point $T$. Index the rows of $J$ by the generators $f_{v,w}$ and index the columns by the variables $u_{i_1,\ldots,i_n}$. Note that $\frac{\partial f}{\partial u_{de_i}}|T = 0$ since the monomials in $f_{v,w}$ containing $u_{de_i}$ contain another variable $u_{i_1,\ldots,i_n} \neq u_{de_j}$ for all $j = 1,\ldots,n$. Therefore, the column corresponding to $u_{de_i}$ is zero.

Note that the monomials $u_{\text{sort}(v,w)_1+e_s} u_{\text{sort}(v,w)_2+e_s}$ cannot contain a variable $u_{de_i}$ for any $v$ and $w$ that give a nontrivial $f_{v,w}$, so they don’t matter in the Jacobian analysis.

Now, the column of $J$ corresponding to the variable $u_{(d-1)e_i+e_j}$ for $i \neq j$ has 1 only in the rows corresponding to $f_{(d-1)e_i,(d-1)e_j}$ and so does the variable $u_{(d-1)e_i+e_j}$. Therefore, the variables $u_{(d-1)e_i+e_j}$ and the polynomials $f_{(d-1)e_i,(d-1)e_j}$ form a block in $J$ of rank $\binom{n}{2}$, which equals the number of pairs $i \neq j$.

For any other variable $u_{i_1,\ldots,i_n}$, such that $(i_1,\ldots,i_n) \neq de_i$ or $(d-1)e_i+e_j$, its corresponding column is nonzero only at the rows corresponding to the polynomials $f_{(i_1,\ldots,i_s)\cdots e_s,(d-1)e_s}$ for all $s$ such that $i_s > 0$. Each such polynomial has no other 1’s in its row except for the one at $u_{i_1,\ldots,i_n}$. Therefore, each variable $u_{i_1,\ldots,i_n}$, such that $(i_1,\ldots,i_n) \neq de_i$ or $(d-1)e_i+e_j$, contributes a size $1 \times \{ s : i_s > 0 \}$ nonzero block to $J$, so it contributes 1 to the rank. Therefore, the rank of $J$ is

$$\# \text{ variables} - \# \{ u_{de_i} \} - \# \{ u_{(d-1)e_i+e_j; i \neq j} \} + \binom{n}{2}$$

$$= \# \text{ variables} - n - n(n-1) + \binom{n}{2} = \# \text{ variables} - \binom{n+1}{2}.$$ 

Thus, the rank of the Jacobian at a smooth point in the irreducible component of $T$ is at least $\# \text{ variables} - \binom{n+1}{2}$, so the dimension of an irreducible component containing $T$ is at most $\binom{n+1}{2}$.

Since the odeco variety is irreducible, has dimension $\binom{n+1}{2}$, contains $T$, and is contained in $\mathcal{V}(I)$, then it is one of the irreducible components of $\mathcal{V}(I)$. □

Lemma 3.7 shows that one only needs to show that the ideal $I$ is prime in order to confirm Conjecture 3.2.
Computations

We show a table with some computational checks of the conjecture.

| $n$ | $d$ | dimension | degree | # min. gens. | conjecture check |
|-----|-----|-----------|--------|-------------|-----------------|
| 3   | 3   | 6         | 10     | 6           | True            |
| 3   | 4   | 6         | 35     | 27          | True            |
| 3   | 5   | 6         | 84     | 75          |                 |
| 4   | 3   | $\geq 10$ |        | 20          |                 |
| 4   | 4   | $\geq 10$ |        | 126         |                 |
| 5   | 3   | $\geq 15$ |        | 50          |                 |

Figure 2: A table of what can be found computationally about the ideal $I$ generated by the equations in (3.1).

Since the ideal $I$ becomes quite large, as $n$ and $d$ grow, it soon becomes hard to check its primality. It was easy to check the conjecture was correct in the case $n = d = 3$ using Macaulay2. The case $n = 3, d = 4$ was checked using the numerical homotopy software Bertini. We were unable to confirm the rest of the results using (short) computations.

Acknowledgements

I would like to thank my advisor Bernd Sturmfels for his great help in this project. I would also like to thank Kaie Kubjas and Luke Oeding for helpful comments and Matthew Niemerg for his help with the software Bertini. The author was supported by a UC Berkeley Graduate Fellowship and by the National Institute of Mathematical Sciences (NIMS) in Daejeon, Korea.

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