A rigorous equation for the Cole-Hopf solution of the conservative KPZ dynamics

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Abstract

A rigorous equation is stated and it is shown that the spatial derivative of the Cole-Hopf solution of the KPZ dynamics is a solution of this equation. The approximation of the Cole-Hopf solution by the density fluctuations in $\sqrt{\varepsilon}$-weakly asymmetric exclusion is used in conjunction with a weak resolvent method instead of a Boltzmann-Gibbs principle.

KEY WORDS stochastic partial differential equation, interacting particle system, martingale problem, generalized functions

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1 Motivation and Summary

A very nice motivation of the problem which in particular answers the question of why there is need for a rigorous equation describing the KPZ dynamics can be found in [JG2010]. Nevertheless the present article starts a little bit as a critique of the notion of solution of the KPZ equation introduced in [JG2010] simply to justify why the author thinks that there is still need to establish a rigorous equation in the first place.

The formal equation discussed in this paper is

$$\frac{\partial}{\partial t} Y = \frac{\partial^2}{\partial u^2} Y + \tilde{\gamma} \frac{\partial}{\partial u} (Y^2) + \sqrt{2} \frac{\partial^3 B}{\partial t \partial u^2}$$

where $\tilde{\gamma}$ is a real-valued parameter and $B$ stands for a Brownian sheet thus $\frac{\partial^3 B}{\partial t \partial u^2}$ can be interpreted as the spatial derivative of a space-time white noise driving force. There is a candidate for a stationary solution of this equation which was constructed in [BG1997]. But this stationary solution is only a generalized and NOT a regular function of $(t, u)$ hence one has to give meaning to the non-linear term $\frac{\partial}{\partial u} (Y^2)$ in the above equation. The straightforward approach would be to explain $\frac{\partial}{\partial u} (Y^2)$ by the limit $\frac{\partial}{\partial u} [(Y_t \ast d_N)^2]$, $N \to \infty$, for every fixed $t \geq 0$ using a mollifier $d$ to approximate the identity, that is $Y_t \ast d_N$ is the convolution of
the generalized function $Y_t$ and the smooth function $d_N(u) = N d(uN)$, $u \in \mathbb{R}$. It turned out that it is hard to make sense of this limit in an appropriate space. The author only achieved to get convergence in a rather artificial space of so-called generalized random variables which made it kind of impossible to understand ($\ast$) as a PDE and the notion of solution was based on a generalized martingale problem (compare [A2002]). It even remains to be shown that the above mentioned candidate for a stationary solution is indeed a solution of this generalized martingale problem.

The difficulty seems to be that, as far as we know, there is no control of moments higher than two. Very good if not the best second order moment estimates for the stationary candidate solution can be found in [BQS2011] but the authors themselves remark that their method cannot be applied to moments of higher order.

On the other hand, for this candidate of a solution, the convergence of time integrals $\int_s^t \frac{\partial}{\partial u}[(Y_r * d_N)^2] dr, N \to \infty, s \leq t$ fixed, is much more regular and the notion of solution introduced in [JG2010] is based on the existence of such a limit. However, in [JG2010] it is not shown that the time integrals $\int_s^t \frac{\partial}{\partial u}[(Y_r * d_N)^2] dr$ converge in a suitable function space. It is only shown that

$$-\lim_{N \to \infty} \int_s^t \int_{\mathbb{R}} (Y_r * d_N)^2(u) \frac{G(u + 1/N) - G(u)}{1/N} dudr$$

exists as a limit in mean square for every $s \leq t$ and every test function $G$ in the Schwartz space $\mathcal{S}(\mathbb{R})$ and one issue of the present paper, but the smaller one, is to go a little bit beyond this.

The main message from [A2002] is that it is not possible to interchange the above $\lim_{N \to \infty}$ and the time integration. But, using $\ast$ to denote convolution with respect to the space parameter, one can rewrite

$$-\int_s^t \int_{\mathbb{R}} (Y_r * d_N)^2(u) G'(u) du dr = \langle 1_{[s,t]} \otimes G, \frac{\partial}{\partial u}[(Y \ast_2 d_N)^2] \rangle$$

thinking of $1_{[s,t]} \otimes G$ as a test function and of $\langle , \rangle$ as a dual pairing. This triggers the idea to explain $\frac{\partial}{\partial u}(Y^2)$ by a function $F(Y) \in \mathcal{D}'((0, T) \times \mathbb{R})$ where $Y$ is considered a random element with state space $D([0, T]; \mathcal{D}'(\mathbb{R}))$ as described in [BG1997]. Thus the function $F$ one wants to define should map

$$F : D([0, T]; \mathcal{D}'(\mathbb{R})) \to \mathcal{D}'((0, T) \times \mathbb{R})$$

turning the equation ($\ast$) into an SPDE with a stationary weak solution in the classical sense, that is

$$\langle \phi, \frac{\partial}{\partial t} Y - \frac{\partial^2}{\partial u^2} Y - \sqrt{2} \frac{\partial^3 B}{\partial t \partial u^2} \rangle = 0 \text{ for all } \phi \in \mathcal{D}((0, T) \times \mathbb{R}) \text{ a.s.}$$

for the candidate solution $Y$ constructed in [BG1997] and some Brownian sheet $B$ both given on the same probability space.

The wanted function $F$ is defined in the present paper and it is shown that the candidate solution $Y$ constructed in [BG1997] is a stationary weak solution of ($\ast$) in the above sense. But the second and real issue of the present paper is that the proof only uses the replacement result stated in [A2011], Corollary 2, which is weaker than the so-called ‘Second-order
Boltzmann-Gibbs principle’ used in [JG2010]. As a consequence, showing how the equation follows from Corollary 2 in [A2011] requires more effort than usually is required for showing how a limiting equation follows from a Boltzmann-Gibbs principle.

Altogether the present paper demonstrates that the general method laid out in Section 1 of [A2011] works in the case of an important example. The ingredients of this general method are the tightness of a sequence of approximating fields and resolvent estimates of appropriate functionals with respect to the symmetric part of the underlying particle system. The weak convergence hence the tightness of the approximating fields in the case of (⋆) was already shown in [BG1997] using a Cole-Hopf-type transform. Equation (⋆) is the equation the spatial derivative of a solution of the KPZ equation for growing interfaces would formally satisfy and the main result in [BG1997] is actually an approximation scheme for the KPZ equation. The limiting field of this approximating scheme became important and the community started to call it the Cole-Hopf solution of the KPZ equation. Taking the spatial derivative of the KPZ equation turns it into a conservative system with an invariant state and that’s why some people call (⋆) the conservative KPZ equation. Remark that the existence of an invariant state is important for the resolvent method hence, while the approximation scheme introduced in [BG1997] also provides non-stationary candidate solutions for (⋆), the present paper only concentrates on the stationary candidate solution. Nevertheless, it is worth to be mentioned that showing the tightness of the approximating fields in the stationary case can also be reduced to resolvent estimates of the symmetric system.

Finally a remark on the spaces used. The weak convergence towards candidate solutions of (⋆) shown in [BG1997] works for processes taking values in $D([0, T]; D'(\mathbb{R}))$, the space of cadlag functions mapping $[0, T]$ into the space of distributions $D'(\mathbb{R})$ equipped with the Skorokhod topology. In the case of the stationary candidate solution, in particular since the invariant state is Gaussian, this can be relaxed to $D([0, T]; S'(\mathbb{R}))$ with $S'(\mathbb{R})$ being the space of tempered distributions. However, in the non-stationary case, the growth conditions implied by the theorems in [BG1997] would not allow for $S'(\mathbb{R})$ without further analysis. As a consequence the function $F$ used to explain the non-linearity of (⋆) is defined to map into $D'((0, T) \times \mathbb{R})$ to leave room for non-stationary solutions.

2 Notation and Results

Fix $p, q \geq 0$ such that $p + q = 1$ and let $(\Omega, \mathcal{F}, \mathbb{P}_\eta; \eta \in \{0, 1\}^\mathbb{Z}, (\eta_t)_{t \geq 0})$ be the strong Markov Feller process the generator $L$ of which acts on local functions $f: \{0, 1\}^\mathbb{Z} \to \mathbb{R}$ as

$$Lf(\eta) = \sum_{x \in \mathbb{Z}} (2p \eta(x)(1 - \eta(x + 1)))[f(\eta^{x,x+1}) - f(\eta)] + 2q \eta(x)(1 - \eta(x - 1))[f(\eta^{x,x-1}) - f(\eta)]$$

where the operation

$$\eta^{x,y}(z) = \begin{cases} 
\eta(z) & z \neq x, y \\
\eta(x) & z = y \\
\eta(y) & z = x 
\end{cases}$$
exchanges the “spins” at \(x\) and \(y\). Remark that this process describes infinitely many particles moving on \(\mathbb{Z}\) like nearest neighbor random walks obeying an exclusion rule: when a particle attempts to jump onto a site occupied by another particle the jump is suppressed. Such processes are called nearest neighbor simple exclusion processes, compare [L1999] for a good account on the existing theory.

Denote by \(\nu_{1/2}\) the Bernoulli product measure on \(\{0, 1\}^\mathbb{Z}\) satisfying \(\nu_{1/2}(\eta(x) = 1) = 1/2\) for all \(x \in \mathbb{Z}\) and define

\[
P = \int P_\eta \, d\nu_{1/2}(\eta) \quad \text{as well as} \quad \xi_t(x) = \frac{\eta_t(x) - \mathbb{E}\eta_t(x)}{\sqrt{\text{Var}(\eta_t(x))}}
\]

where \(\mathbb{E}\) and \(\text{Var}\) stand for the expectation and variance with respect to \(P\), respectively. The process \((\xi_t)_{t \geq 0}\) is a stationary process on \((\Omega, \mathcal{F}, P)\) which takes values in \(\{-1, 1\}^\mathbb{Z}\) and has the push forward of \(\nu_{1/2}\) with respect to the map \(\eta \mapsto \xi\) given by

\[
\eta \mapsto \xi \quad \text{given by} \quad \xi(x) = \frac{\eta(x) - 1/2}{\sqrt{1/4}}, \quad x \in \mathbb{Z},
\]

as its invariant measure.

Denote by \(\delta_{\varepsilon x}\) the Dirac measure concentrated in the macroscopic point \(\varepsilon x\), define the measure-valued density fluctuation field

\[
Y^\varepsilon_t = \sqrt{\varepsilon} \sum_{x \in \mathbb{Z}} \xi_{\varepsilon t-2}(x) \delta_{\varepsilon x}, \quad t \geq 0,
\]

with respect to a scaling parameter \(\varepsilon > 0\) and fix a smooth test function \(G \in D(\mathbb{R})\) with compact support. Then

\[
M^G_{t, \varepsilon} = Y^\varepsilon_t(G) - Y^\varepsilon_0(G) - \int_0^t \varepsilon^{-2} L Y^\varepsilon_s(G) \, ds, \quad t \geq 0,
\]

is a martingale on \((\Omega, \mathcal{F}, P)\) by standard theory on strong Markov processes and

\[
\int_0^t \varepsilon^{-2} L Y^\varepsilon_s(G) \, ds = \int_0^t \varepsilon^{-3} \sum_{x \in \mathbb{Z}} G(\varepsilon x) L \xi_{\varepsilon x-2}(x) \, ds, \quad t \geq 0,
\]

(2)

where

\[
L \xi_{\varepsilon x-2}(x) = \left[ (\xi_{\varepsilon x-2}(x-1) - 2\xi_{\varepsilon x-2}(x) + \xi_{\varepsilon x-2}(x+1))
\right.
\]

\[
+ \gamma \left( \xi_{\varepsilon x-2}(x) \xi_{\varepsilon x-2}(x+1) - \xi_{\varepsilon x-2}(x+1) \xi_{\varepsilon x-2}(x) \right) \]

(3)

follows from (1). Here

\[
\gamma = p - q
\]

is the mean velocity of the moving particles described by the process \((\eta_t)_{t \geq 0}\). Substituting (3) into (2), performing a summation by parts and approximating by Taylor expansion implies

\[
\int_0^t \varepsilon^{-2} L Y^\varepsilon_s(G) \, ds = \int_0^t Y^\varepsilon_s(G') \, ds - \frac{\gamma}{\sqrt{\varepsilon}} \int_0^t \sum_{x \in \mathbb{Z}} G'(\varepsilon x) \xi_{\varepsilon x-2}(x) \xi_{\varepsilon x-2}(x+1) \, ds
\]

\[
+ \frac{\gamma \sqrt{\varepsilon}}{2} \int_0^t \sum_{x \in \mathbb{Z}} G''(\varepsilon x) \xi_{\varepsilon x-2}(x) \xi_{\varepsilon x-2}(x+1) \, ds + R^G_\varepsilon(t)
\]
with
\[ |R_\varepsilon^G(t)| \leq (2 + \gamma) \sqrt{\varepsilon} c_G \|G''\|_\infty \cdot t, \quad t \geq 0, \] (4)
if \( c_G \) is chosen such that \( \text{supp}G \subseteq [-c_G + 1, c_G - 1] \). At this and in what follows, \( \|H\|_r \)
denotes the norm of a test function \( H \) in \( L^r(\mathbb{R}) \), \( 1 \leq r \leq \infty \).

Now assume that for fixed \( \varepsilon > 0 \) the jump probabilities \( p \) and \( q \) also depend on \( \varepsilon \) in such a way that
\[ \gamma = \gamma_{\varepsilon} = \tilde{\gamma} \cdot \sqrt{\varepsilon} \]
for some \( \tilde{\gamma} \neq 0 \); hence the underlying process is now a weakly \( \sqrt{\varepsilon} \)-asymmetric simple exclusion process. As a consequence \( L, P \) and \( E \) introduced above are denoted by \( L_\varepsilon, P_\varepsilon \) and \( E_\varepsilon \) in what follows. Furthermore
\[ M_t^{G,\varepsilon} + R_\varepsilon^G(t) = Y_t^\varepsilon(G) - Y_0^\varepsilon(G) - \int_0^t \left[ Y_s^\varepsilon(G'') + \gamma V_\varepsilon^G(\xi_{s\varepsilon^{-2}}) - \frac{\tilde{\gamma} \varepsilon}{2} V_\varepsilon^{G'}(\xi_{s\varepsilon^{-2}}) \right] \, ds \] (5)
for all \( t \geq 0 \) where
\[ V_\varepsilon^G(\xi) = - \sum_{x \in \mathbb{Z}} G'(\varepsilon x)\xi(x)\xi(x + 1) \]
is a local function for fixed \( \varepsilon \) and \( G \).

Fix a finite time horizon \( T \) and let \( D([0, T]; \mathcal{D}'(\mathbb{R})) \) be the space of all cadlag functions mapping \( [0, T] \) into the space of distributions \( \mathcal{D}'(\mathbb{R}) \). Equip \( D([0, T]; \mathcal{D}'(\mathbb{R})) \) with the Skorokhod topology \( J_1 \) and let \( Y \) be the notation for both an element in and the identity map on \( D([0, T]; \mathcal{D}'(\mathbb{R})) \). Furthermore, regard \( Y^\varepsilon = (Y_t^\varepsilon)_{t \in [0, T]} \) as a random variable taking values in \( D([0, T]; \mathcal{D}'(\mathbb{R})) \) and denote by \( \hat{P}_\varepsilon \) the push forward of \( P_\varepsilon \) with respect to the map \( Y^\varepsilon \). Then, by Theorem B.1 in [BG1997], the probability measures \( \hat{P}_\varepsilon, \varepsilon \downarrow 0 \), converge weakly to a probability measure on \( D([0, T]; \mathcal{D}'(\mathbb{R})) \) which is denoted by \( P_\gamma \) in what follows.

Remark 1 This result in [BG1997] is stronger than tightness of the measures \( \hat{P}_\varepsilon, \varepsilon > 0 \), since the tightness would only give the weak convergence with respect to certain subsequences \( \varepsilon_k, \varepsilon_k \downarrow 0 \), with possibly different limit measures. So Theorem B.1 means that the density fluctuations in \( \sqrt{\varepsilon} \)-weakly asymmetric exclusion must converge in law to the Cole-Hopf solution of the conservative KPZ dynamics. However it was also shown in [BG1997] that both
\begin{enumerate}
\item[a)] the support of the measure \( P_\gamma \) is a subset of \( C([0, T]; \mathcal{D}'(\mathbb{R})) \) and
\item[b)] the process \( Y \) is stationary under \( P_\gamma \) satisfying \( Y_t \sim \mu, t \in [0, T] \), where \( \mu \) is the mean zero Gaussian white noise measure with covariance \( \mathbb{E}_\gamma Y_t^\varepsilon(G)Y_t^\varepsilon(H) = \int_{\mathbb{R}} GH \, du \)
\end{enumerate}
hold true but the present paper’s task is to find more structure of the paths in the support of the measure \( P_\gamma \).

A possible approach is based on Corollary 2 in [A2011] which allows to replace \( V_\varepsilon^G(\xi_{s\varepsilon^{-2}}) \) in (5) by \( F_N(Y_s^\varepsilon, G) \) where
\[ F_N(\mathcal{Y}, G) = - \int_{\mathbb{R}} G'(u)(\mathcal{Y} \ast dN)^2(u) \, du, \quad \mathcal{Y} \in \mathcal{D}'(\mathbb{R}). \]
Here, for every $N \geq 1$, $d_N$ denotes the function $u \mapsto N d(N u)$ given by a symmetric nonnegative mollifier $d \in \mathcal{D}(\mathbb{R})$. However, as this replacement is of rather weak type, this approach requires several technical steps which are outlined below.

**Proposition 1** Let $\mathcal{F}_t^Y = \sigma(\{ Y_s(G) : s \leq t, G \in \mathcal{D}(\mathbb{R}) \})$ and define the map

$$\mathcal{M}_N : \mathcal{D}([0, T]; \mathcal{D}'(\mathbb{R})) \to \mathcal{D}([0, T]; \mathcal{D}'(\mathbb{R}))$$

by

$$\mathcal{M}_N(Y)_t^G = Y_t(G) - Y_0(G) - \int_0^t Y_s(G^\prime) \, ds - \gamma \int_0^t F_N(Y_s, G) \, ds.$$  

Then, for every $G \in \mathcal{D}(\mathbb{R})$, there exists a $\mathcal{B}([0, T]) \otimes \mathcal{F}_t^Y$-measurable process

$$\tilde{M}^G : [0, T] \times \mathcal{D}([0, T]; \mathcal{D}'(\mathbb{R})) \to \mathbb{R}$$

such that

$$\int_0^T \, dt \, \mathbb{E}_\gamma \left[ (\tilde{M}^G_t - \mathcal{M}_N(Y)_t^G)^2 \right] \leq C_d \gamma^2 e^T \left( N^{-2} \cdot c_G^2 \|G^\prime\|_\infty^2 + N^{-1} \cdot c_G^2 \|G^\prime\|_\infty^2 + N^{-1/3} \cdot c_G \|G^\prime\|_\infty \right)$$

for a constant $C_d$ which only depends on the choice of the mollifier $d$ hence

$$\int_0^T \, dt \, \mathbb{E}_\gamma \left[ (\tilde{M}^G_t - \mathcal{M}_N(Y)_t^G)^2 \right] \to 0, \quad N \to \infty.$$  

Denote by $\mathbb{F}$ the filtration $(\mathcal{F}_t)_{t \in [0, T]}$ with $\mathcal{F}_t = \sigma(\mathcal{F}_t^Y \cup \mathcal{N})$ where $\mathcal{N}$ is the collection of all $\mathbb{P}_\gamma$-null sets in $\mathcal{F}_t^Y$ and let $\ell$ be the Lebesgue measure on $[0, T]$.

**Proposition 2** For every $G \in \mathcal{D}(\mathbb{R})$, there exists a continuous $\mathbb{F}$-adapted process $M^G = (M^G_t)_{t \in [0, T]}$ on $(\mathcal{D}([0, T]; \mathcal{D}'(\mathbb{R})), \mathcal{F}_t^Y, \mathbb{P}_\gamma)$ which is a version of $\tilde{M}^G$ in the following sense: there is a measurable subset $T_G \subseteq [0, T]$ with $\ell(T_G) = T$ such that $\tilde{M}^G_t = M^G_t$ a.s. for all $t \in T_G$. For every positive $T' < T$, when restricted to $[0, T']$, the process $M^G$ is a square integrable $\mathbb{F}$-martingale.

**Corollary 1** (i) For every $G \in \mathcal{D}(\mathbb{R})$, the process $M^G = (M^G_t)_{t \in [0, T]}$ is an $\mathbb{F}$-Brownian motion with variance $2\|G^\prime\|_2^2$ on the probability space $(\mathcal{D}([0, T]; \mathcal{D}'(\mathbb{R})), \mathcal{F}_t^Y, \mathbb{P}_\gamma)$.

(ii) It holds that

$$M^G_{t_1}\gamma_{1}M^G_{t_2} = a_1 M^G_t + a_2 M^G_t$$

for every $t \in [0, T]$, $a_1, a_2 \in \mathbb{R}$ and $G_1, G_2 \in \mathcal{D}(\mathbb{R})$.

(iii) The process $M^G_t$ indexed by $t \in [0, T]$ and $G \in \mathcal{D}(\mathbb{R})$ is a centred Gaussian process on $(\mathcal{D}([0, T]; \mathcal{D}'(\mathbb{R})), \mathcal{F}_t^Y, \mathbb{P}_\gamma)$ with covariance

$$\mathbb{E}_\gamma M^G_{t_1} M^G_{t_2} = 2(t_1 \wedge t_2) \int \mathbb{E}_\gamma G_{t_1}^G(u) G_{t_2}^G(u) \, du$$

hence there is a Brownian sheet $B(t, u)$, $t \in [0, T]$, $u \in \mathbb{R}$, on $(\mathcal{D}([0, T]; \mathcal{D}'(\mathbb{R})), \mathcal{F}_t^Y, \mathbb{P}_\gamma)$ such that

$$M^G_t = \sqrt{2} \int \mathbb{E}_\gamma B(t, u) G^\prime(u) \, du$$

for every $t \in [0, T]$ and $G \in \mathcal{D}(\mathbb{R})$.  


In what follows let $M = (M_t)_{t \in [0, T]}$ denote the continuous $\mathcal{D}'(\mathbb{R})$-valued process defined by

$$M_t(G) = \sqrt{2} \int_\mathbb{R} B(t, u)G''(u) \, du, \quad t \in [0, T], \ G \in \mathcal{D}(\mathbb{R}). \quad (6)$$

Remark that, by Schwartz’ kernel theorem, $M$ and $Y$ can also be considered random variables taking values in $\mathcal{D}'((0, T) \times \mathbb{R})$ such that

$$\int_0^T dt \ g'(t) \left[-Y_t(G) + Y_0(G) + \int_0^t Y_s(G'') \, ds + M_t(G)\right] = \langle g \otimes G, \frac{\partial}{\partial t} Y - \frac{\partial^2}{\partial u^2} Y - \frac{\partial}{\partial t} M \rangle$$

for all $g \in \mathcal{D}((0, T)), \ G \in \mathcal{D}(\mathbb{R})$ where $\langle \cdot, \cdot \rangle$ denotes the dual pairing between $\mathcal{D}'((0, T) \times \mathbb{R})$ and $\mathcal{D}'((0, T) \times \mathbb{R})$. Also

$$\int_0^T dt \ g'(t) \int_0^t F_N(Y_s, G) \, ds = -\int_0^T dt \ g(t) \ F_N(Y_t, G)$$

thus Propositions (together with Cauchy-Schwarz) implies

$$E_{\tilde{\gamma}} \left| \int_0^T dt \ g(t) \ F_N(Y_t, G) - \langle g \otimes G, \frac{\partial}{\partial t} Y - \frac{\partial^2}{\partial u^2} Y - \frac{\partial}{\partial t} M \rangle / \tilde{\gamma} \right| \leq C_d \ e^T \left\| g' \right\|^2_{L^2[0, T]} \left( N^{-2} \cdot c_G^2 \left\| G'' \right\|^2_\infty + N^{-1} \cdot c_G^2 \left\| G''' \right\|^2_\infty + N^{-1/3} \cdot c_G \left\| G'' \right\|^2_\infty \right)$$

$$\leq 2C_d \ e^T \left\| g' \right\|^2_{L^2[0, T]} \left( N^{-2} \cdot c_G^3 \left\| G^{(4)} \right\|^2_2 + N^{-1} \cdot c_G^3 \left\| G''' \right\|^2_2 + N^{-1/3} \cdot c_G^2 \left\| G'' \right\|^2_2 \right) \quad (7)$$

where the sup-norm is estimated by the $L^2$-norm in a straightforward way.

The above inequality is the type of inequality Jara/Goncalves used in [JG2010] on page 10 to define what they call ‘energy solution’. Indeed, if $g = 1_{[s, t]}$ then the process $A_{s, t}(G)$ introduced in that paper, when multiplied by the analogon to $\tilde{\gamma}$, is indistinguishable of

$$Y_t(G) - Y_0(G) - \int_s^t Y_r(G'') \, dr = M_t(G) + M_s(G) = \langle g \otimes G, \frac{\partial}{\partial t} Y - \frac{\partial^2}{\partial u^2} Y - \frac{\partial}{\partial t} M \rangle$$

whereupon the dual pairing $\langle \cdot, \cdot \rangle$ on the right-hand side makes sense for this $g$ since $Y$ and $M$ both take values in $C([0, T]; \mathcal{D}'(\mathbb{R}))$. Of course, the corresponding inequality in [JG2010] is stronger than (7) since the former allows for $g = 1_{[s, t]}$ while the latter does not. This is due to the resolvent method which requires an extra time integration. The differences of the bounds in $G$ and $N$ are not intrinsic. The author just took what he had used in [A2007] in the symmetric case and this can be improved.

However, for the purpose of defining a function $F(Y)$ explaining $\frac{\partial}{\partial t} (Y^2)$, both inequalities are worth about the same since, as already pointed out in Section 11 there seems to be no meaningful notion of convergence which could be used to interchange $\lim_{N \to \infty}$ and the time integration in

$$\lim_{N \to \infty} \int_0^T dt \ g(t) \ F_N(Y_t, G)$$
neither if \( g \) was differentiable. So both inequalities should eventually lead to \( F(Y) \) being constructed as an element of \( \mathcal{D}'((0, T) \times \mathbb{R}) \) and this construction is going to be carried out using (7) in what follows.

The idea is to choose ONBs \((g_m)_{m=1}^\infty\) and \((G_n)_{n=1}^\infty\) of \( L^2[0, T] \) and \( L^2(\mathbb{R}) \), respectively, and to construct a negative order Sobolev space \((\mathcal{H}, \| \cdot \| )\) such that \( \mathcal{D}((0, T) \times \mathbb{R}) \subseteq \mathcal{H} \subseteq L^2([0, T] \times \mathbb{R}) \subseteq \mathcal{H} \) and

\[
E_{\tilde{\gamma}} \| F_N(Y) - (\frac{\partial}{\partial t} Y - \frac{\partial^2}{\partial u^2} Y - \frac{\partial}{\partial t} M)/\tilde{\gamma} \|_\pi = O(N^{-1/3}), \quad N \to \infty, \quad (8)
\]
uniformly in \( \tilde{\gamma} \) where \( F_N(Y) \) is defined by the Fourier-type series

\[
\sum_{m,n} (g_m \otimes G_n) \cdot \int_0^T dt g_m(t) F_N(Y, G_n) \quad \text{in} \quad L^2([0, T] \times \mathbb{R}).
\]

It is obvious from (7) that such a construction leading to (8) can be achieved but, for completeness, some technical details are sketched in the Appendix’ Remark 4.

Now, when setting \( N_k = k^{a+\delta} \), \( k \geq 1 \), for an arbitrarily small \( a > 0 \), from (8) follows that

\[
\sum_{k=1}^\infty P_{\gamma}(\| F_{N_k}(Y) - (\frac{\partial}{\partial t} Y - \frac{\partial^2}{\partial u^2} Y - \frac{\partial}{\partial t} M)/\tilde{\gamma} \|_\pi \geq \delta) < \infty
\]

for all \( \delta > 0 \) hence

\[
\| F_{N_k}(Y) - (\frac{\partial}{\partial t} Y - \frac{\partial^2}{\partial u^2} Y - \frac{\partial}{\partial t} M)/\tilde{\gamma} \|_\pi \rightarrow 0, \quad k \rightarrow \infty, \quad (9)
\]
for all \( Y \in \Omega_\gamma \) for some \( \Omega_\gamma \in \mathcal{F}_\gamma \) with \( P_\gamma(\Omega_\gamma) = 1 \). Since the above convergence is stronger than the weak convergence in \( \mathcal{D}'((0, T) \times \mathbb{R}) \) one obtains that \( \lim_{k \to \infty} \langle \phi, F_{N_k}(Y) \rangle \) exists for all \( \phi \in \mathcal{D}((0, T) \times \mathbb{R}) \) and \( Y \in \Omega_\gamma \) where of course

\[
\langle \phi, F_{N_k}(Y) \rangle = - \int_0^T \int_\mathbb{R} \frac{\partial}{\partial u} \phi(t, u) (Y_t \ast d_{N_k})^2(u) dudt \quad (10)
\]
proving the following proposition.

**Proposition 3** There exists a subsequence \((N_k)_{k=1}^\infty\) such that for every \( \tilde{\gamma} \neq 0 \) there is a set \( \Omega_{\tilde{\gamma}} \in \mathcal{F}_\gamma \) with \( P_\gamma(\Omega_{\tilde{\gamma}}) = 1 \) such that

\[
\lim_{k \to \infty} \int_0^T \int_\mathbb{R} \frac{\partial}{\partial u} \phi(t, u) (Y_t \ast d_{N_k})^2(u) dudt \in \mathbb{R}
\]
for all \( \phi \in \mathcal{D}((0, T) \times \mathbb{R}) \) and \( Y \in \Omega_{\tilde{\gamma}} \).

From this proposition immediately follows that the function

\[
F : D([0, T]; \mathcal{D}'(\mathbb{R})) \to \mathcal{D}'((0, T) \times \mathbb{R})
\]

which is defined by

\[
\langle \phi, F(Y) \rangle = \lim_{k \to \infty} \int_0^T \int_\mathbb{R} \frac{\partial}{\partial u} \phi(t, u) (Y_t \ast d_{N_k})^2(u) dudt
\]
if this limit exists for all \( \phi \in \mathcal{D}((0, T) \times \mathbb{R}) \) and which is set to be zero otherwise has an \( \mathcal{F}_\gamma' / \mathcal{B}(\mathcal{D}'((0, T) \times \mathbb{R})) \)-measurable \( P_\gamma \)-version for every \( \tilde{\gamma} \).
Remark 2 (i) This function $F$ is defined by the above limits for at least all $Y \in \bigcup_{\gamma \neq 0} \Omega_{\gamma}$. There is an indication that all measures $P_{\gamma}, \gamma \neq 0$, are singular to each other hence the set $\bigcup_{\gamma \neq 0} \Omega_{\gamma}$ might be quite large.

(ii) The function $F$ actually maps into the space $\mathcal{H} \subseteq \mathcal{D}'((0,T) \times \mathbb{R})$ which is determined by the estimate (7), compare with the construction of $\mathcal{H}$ in the Appendix. Using $F$ with $\mathcal{H}$ to make (*) work would also require to show that $\partial \partial Y, \partial^2 \partial Y, \partial M$ are in $\mathcal{H}$. This seems to be possible but using $\mathcal{D}'((0,T) \times \mathbb{R})$ instead has the advantage that $\partial \partial Y, \partial^2 \partial Y, \partial M \in \mathcal{D}'((0,T) \times \mathbb{R})$ immediately follows from Schwartz’ kernel theorem.

Combining the definition of $F$ and (6), (9), (10) gives the main result of the present paper.

Theorem 1 For every $\gamma \neq 0$, there exists a Brownian sheet $B(t,u), t \in [0,T], u \in \mathbb{R}$, on $(D([0,T]; \mathcal{D}'(\mathbb{R})), \mathcal{F}_t^Y, P_{\gamma})$ such that

$$\frac{\partial}{\partial t} Y = \frac{\partial^2}{\partial u^2} Y + \gamma F(Y) + \sqrt{2} \frac{\partial^3 B}{\partial t \partial u^2}$$

in the sense of

$$\langle \phi, \frac{\partial}{\partial t} Y - \frac{\partial^2}{\partial u^2} Y - \gamma F(Y) - \sqrt{2} \frac{\partial^3 B}{\partial t \partial u^2} \rangle = 0, \quad \phi \in \mathcal{D}((0,T) \times \mathbb{R}),$$

for $P_{\gamma}$-a.e. $Y \in D([0,T]; \mathcal{D}'(\mathbb{R}))$ where $D([0,T]; \mathcal{D}'(\mathbb{R})) \subseteq \mathcal{D}'((0,T) \times \mathbb{R})$ by Schwartz’ kernel theorem and

$$\langle \phi, \frac{\partial^3 B}{\partial t \partial u^2} \rangle = - \int_0^T \int_\mathbb{R} B(t,u) \frac{\partial^3}{\partial t \partial u^2} \phi(t,u) \, dudt, \quad \phi \in \mathcal{D}((0,T) \times \mathbb{R}).$$

3 Proofs

In what follows it is frequently used that, when setting $H = -G'$, one can split

$$\int_0^t F_N(Y_s^\varepsilon, G) \, ds - \int_0^t V_s^G(\xi, \varepsilon^2 \varepsilon) \, ds \quad \text{into} \quad \sum_{i=1}^4 \varepsilon^2 \int_0^{t\varepsilon^{-2}} V_{\varepsilon,N}^H(\xi) \, ds$$

using further fluctuation fields $V_{\varepsilon,N}^{H,1}, V_{\varepsilon,N}^{H,2}, V_{\varepsilon,N}^{H,3}, V_{\varepsilon,N}^{H,4}$ given by

$$V_{\varepsilon,N}^{H,1}(\xi) = \sum_{x \in \mathbb{Z}} \int_\mathbb{R} [H(u) - H(\varepsilon x)] d_N(u - \varepsilon x) \sum_{\tilde{x} \in \mathbb{Z}} \varepsilon d_N(u - \varepsilon \tilde{x}) du \xi(x) \xi(\tilde{x}),$$

$$V_{\varepsilon,N}^{H,2}(\xi) = \varepsilon \sum_{x \in \mathbb{Z}} H(\varepsilon x) \int_\mathbb{R} d_N^2(u - \varepsilon x) du \xi(x) [\xi(x) - \xi(x + 1)],$$

$$V_{\varepsilon,N}^{H,3}(\xi) = \varepsilon \sum_{x \notin \mathbb{Z}} H(\varepsilon x) \int_\mathbb{R} d_N(u - \varepsilon x) d_N(u - \varepsilon \tilde{x}) du \xi(x) [\xi(\tilde{x}) - \xi(x + 1)],$$

$$V_{\varepsilon,N}^{H,4}(\xi) = \sum_{x \in \mathbb{Z}} H(\varepsilon x) \int_\mathbb{R} d_N(u - \varepsilon x) \left[ \sum_{\tilde{x} \in \mathbb{Z}} \varepsilon d_N(u - \varepsilon \tilde{x}) - 1 \right] du \xi(x) \xi(x + 1).$$
Proof of Proposition 1. It suffices to show that the sequence of processes $(\mathfrak{M}_N (Y)^G_t)_{t \in [0, T]}$, $N = 1, 2, \ldots$, is a Cauchy sequence in $L^2 (\ell \otimes \mathbb{P}_\gamma)$ where $\ell$ denotes the Lebesgue measure on $[0, T]$. But, for $\bar{N} > N \geq 1$ arbitrarily chosen, one obtains

$$
\int_0^T dt \mathbb{E}_{\bar{N}} \left[ |\mathfrak{M}_{\bar{N}}(Y)^G_t - \mathfrak{M}_N(Y)^G_t|^2 \right] = \bar{\gamma}^2 \int_0^T dt \mathbb{E}_{\bar{N}} \left[ \left( \int_0^t \int_{\mathbb{R}} G'(u) \left( (Y_s * d_{\bar{N}})^2(u) - (Y_s * d_N)^2(u) \right) du ds \right)^2 \right]
$$

where by Lemma in the Appendix

$$
\mathbb{E}_{\bar{N}} \left( (Y_s * d_{\bar{N}})^2(u_1) - (Y_s * d_N)^2(u_1) \right) \left( (Y_s * d_{\bar{N}})^2(u_2) - (Y_s * d_N)^2(u_2) \right)
$$

such that

$$
|\mathbb{E}_{\bar{N}} \left( (Y_s * d_{\bar{N}})^2(u_1) - (Y_s * d_N)^2(u_1) \right) \left( (Y_s * d_{\bar{N}})^2(u_2) - (Y_s * d_N)^2(u_2) \right)|^2 \leq \bar{f}(\|d_{\bar{N}}\|^2, \|d_N\|^2)
$$

for all $\varepsilon \leq 1$, $0 \leq s_1, s_2 \leq T$ and $u_1, u_2 \in \mathbb{R}$. Hence, by dominated convergence, for an arbitrarily small $\delta > 0$ it follows that

$$
\int_0^T dt \mathbb{E}_{\bar{N}} \left[ |\mathfrak{M}_{\bar{N}}(Y)^G_t - \mathfrak{M}_N(Y)^G_t|^2 \right] \leq \bar{\gamma}^2 \left( \delta + \int_0^T dt \mathbb{E}_\varepsilon \left[ \left( \int_0^t \int_{\mathbb{R}} G'(u) \left( (Y_s * d_{\bar{N}})^2(u) - (Y_s * d_N)^2(u) \right) du ds \right)^2 \right] \right) \quad (11)
$$

if $\varepsilon = \varepsilon_{\bar{N}, N} > 0$ is chosen to be sufficiently small. Assuming $\varepsilon^T \geq 1$, the last term can be further estimated by

$$
\leq \bar{\gamma} e^T \left( \delta + \int_0^\infty dt e^{-t} \mathbb{E}_\varepsilon \left[ \left( \int_0^t \int_{\mathbb{R}} G'(u) \left( (Y_s * d_{\bar{N}})^2(u) - (Y_s * d_N)^2(u) \right) du ds \right)^2 \right] \right)
$$

$$
\leq 8\bar{\gamma} e^T \left( \delta + \int_0^\infty dt e^{-t} \sum_{i=1}^4 \mathbb{E}_\varepsilon \left[ \varepsilon^2 \int_0^t e^{-2s} V_{\varepsilon, N}^{G, i}(\xi_s) ds \right]^2 \right)
$$

where

$$
\int_0^\infty dt e^{-t} \sum_{i=1}^4 \mathbb{E}_\varepsilon \left[ \varepsilon^2 \int_0^t e^{-2s} V_{\varepsilon, N}^{G, i}(\xi_s) ds \right]^2 \leq C_d \left( N^{-2} \cdot c^2_G \|G''\|_\infty^2 + N^{-1} \cdot c^2_G \|G''\|_\infty^2 + N^{-1/3} \cdot c_G \|G'\|_1^2 \right)
$$

$$
+ C_d \left( \varepsilon^2 N^2 \cdot \|G''\|_\infty^2 + \varepsilon^2 N^2 \cdot c_G \|G''\|_\infty^2 + \varepsilon^2 N^4 \cdot \|G'\|_1^2 \right)
$$
by (9) and (7) in [A2011] for a constant $C_d$ which only depends on the choice of the mollifier $d$. Of course, the same inequality holds if $N$ is replaced by $\tilde{N}$ above. Hence, if $N_\delta$ is chosen such that
\[
C_d \cdot \left( N_\delta^{-2} \cdot c_G \|G''\|^2_\infty + N_\delta^{-1} \cdot c_G \|G''\|^2_\infty + N_\delta^{-1/3} \cdot c_G \|G'\|^2_\infty \right) < \delta/4
\]
then
\[
\int_0^T dt \mathbb{E}_\tilde{\gamma} \left[ \mathcal{M}_N(Y)_t^G - \mathcal{M}_N(Y)_t^G \right]^2 < 16\tilde{\gamma} e^T : \delta \quad \text{for all } \tilde{N} > N \geq N_\delta
\]
proving the proposition. Indeed, in the above calculations, one only has to choose $\varepsilon = \varepsilon_{N,\tilde{N}}$ for the corresponding $N, \tilde{N}$ small enough such that both (11) holds true and
\[
C_d \cdot \left( \varepsilon^2 \tilde{N}^2 \cdot \|G''\|^2_\infty + \varepsilon \tilde{N}^2 \cdot c_G \|G''\|^2_\infty + \varepsilon^2 \tilde{N}^4 \cdot \|G'\|^2_1 \right) < \delta/4.
\]
\[\square\]

Proof of Proposition 2 Fix $G \in \mathcal{D}(\mathbb{R})$. Applying Proposition 1 there exists a subsequence $(N_k)_{k=1}^\infty$ and a measurable subset $\mathcal{T}_G \subseteq [0, T]$ with $\ell(\mathcal{T}_G) = T$ such that
\[
\lim_{k \to \infty} \mathbb{E}_\tilde{\gamma} \left[ \tilde{M}_{t_n}^G - \mathcal{M}_{N_k}(Y)_t^G \right]^2 = 0 \tag{12}
\]
for all $t \in \mathcal{T}_G$. For technical reasons assume $T \notin \mathcal{T}_G$ and let $\{t_1, t_2, \ldots\} \subseteq \mathcal{T}_G$ be a dense subset of $[0, T]$.

At first one observes that $\tilde{M}_{t_n}^G$ is $\mathcal{F}_{t_n}$-measurable for all $n = 1, 2, \ldots$ Then one shows the following $\mathcal{F}_t^\gamma$-martingale property
\[
\mathbb{E}_\tilde{\gamma} X [\tilde{M}_{t_n}^G - \tilde{M}_{t_n'}^G] = 0
\]
for $t_n', t_n \in \{t_1, t_2, \ldots\}$ satisfying $t_n' < t_n$ and an arbitrary random variable $X$ of the form $X = f(Y_{s_1}(H_1), \ldots, Y_{s_p}(H_p))$ where $f : \mathbb{R}^p \to \mathbb{R}$ is a bounded continuous function, $H_i \in \mathcal{D}(\mathbb{R})$ and $0 \leq s_i \leq t_n', 1 \leq i \leq p$. Of course, this martingale property holds true if there exists $\text{const} > 0$ such that
\[
\left( \mathbb{E}_\tilde{\gamma} X [\tilde{M}_{t_n}^G - \tilde{M}_{t_n'}^G] \right)^2 \leq \text{const} \cdot \delta \quad \text{for all } \delta > 0. \tag{13}
\]

In this proof the notation $\text{const}$ is used when a notation for a constant is needed thus $\text{const}$ can take different values depending on the situation.

In order to prove (13), fix an arbitrary $\delta > 0$. Denote
\[
R_{\varepsilon,N}^{G',0}(t) = \frac{-\varepsilon^3}{2} \int_0^{t \varepsilon^2} V_{\varepsilon} G' (\xi_s) \, ds = \frac{-\varepsilon^3}{2} \int_0^{t \varepsilon^2} \sum_{x \in \mathbb{Z}} G''(\varepsilon x) \xi_s(x) \xi_s(x + 1) \, ds, \quad t \geq 0,
\]
and observe that
\[
\int_0^T dt \mathbb{E}_\varepsilon [R_{\varepsilon,N}^{G',0}(t)]^2 = O(\varepsilon^2) \quad \text{uniformly in } N \tag{14}
\]
by feeding the estimates obtained in [A2007] for these quadratic fluctuations but with respect to the symmetric exclusion process into the resolvent method as demonstrated in Section 3 of [A2011]. Furthermore denote

\[ R_{\varepsilon,N}^{G,i}(t) = \varepsilon^2 \int_0^{t \varepsilon^{-2}} V_{\varepsilon,N}^{G,i}(\xi_s) \, ds, \quad t \geq 0, \quad i = 1, 2, 3, 4, \]

and remark that (9) in [A2011] implies

\[ \int_0^T dt \, E_\varepsilon [R_{\varepsilon,N}^{G,1}(t)]^2 = O(N^{-1}) \quad \text{as well as} \quad \int_0^T dt \, E_\varepsilon [R_{\varepsilon,N}^{G,3}(t)]^2 = O(N^{-1/3}) \]

uniformly in \( \varepsilon > 0 \). Hence, for some \( \tau > 0 \) satisfying \( t_n + 2\tau < T \), one can choose \( k \) big enough such that both

\[ \ell(\{ t \in [0, T] : E_\varepsilon [R_{\varepsilon,N}^{G,1}(t)]^2 + E_\varepsilon [R_{\varepsilon,N}^{G,3}(t)]^2 \geq \delta \}) \leq \tau/2 \quad \text{for all} \quad \varepsilon > 0 \]  

and

\[ E_\gamma \left[ \bar{M}_{t_n}^G - \mathfrak{M}_{N_k}(Y)^G_{t_n} \right]^2 + E_\gamma \left[ \bar{M}_{t_{n'}}^G - \mathfrak{M}_{N_k}(Y)^G_{t_{n'}} \right]^2 < \delta \]  

hold true. It is this \( k = k_\delta \) which is chosen and fixed during the rest of the proof.

Of course, applying Cauchy-Schwarz, (16) implies

\[ \left( E_\gamma X[\bar{M}_{t_n}^G - \mathfrak{M}_{N_k}(Y)^G_{t_n}] \right)^2 \leq \text{const} \left\{ \delta + \left( E_\gamma X[\mathfrak{M}_{N_k}(Y)^G_{t_n} - \mathfrak{M}_{N_k}(Y)^G_{t_{n'}}] \right)^2 \right\}. \]  

(17)

Now, substituting the definition of \( \mathfrak{M}_{N_k} \), one obtains that

\[ \left( E_\gamma X[\mathfrak{M}_{N_k}(Y)^G_{t_n} - \mathfrak{M}_{N_k}(Y)^G_{t_{n'}}] \right)^2 = \left( E_\gamma X[Y_{t_n}(G) - Y_{t_{n'}}(G)] - \int_{t_{n'}}^{t_n} E_\gamma X Y_s(G''') \, ds \right)^2 \]

\[ + \tilde{\gamma} \int_{t_{n'}}^{t_n} \int_R G'(u) E_\gamma X (Y_s \ast d_{N_k})^2(u) \, duds \]

where

\[ E_\gamma X (Y_s \ast d_{N_k})^2(u) = \lim_{\varepsilon \downarrow 0} \hat{E}_\varepsilon X (Y_s \ast d_{N_k})^2(u) \]

such that

\[ |\hat{E}_\varepsilon X (Y_s \ast d_{N_k})^2(u)| \leq \| f \|^2_{\infty} \hat{f}_d(\| d_{N_k} \|_1^2) \]

for all \( \varepsilon > 0 \), \( s \in [0, T] \) and \( u \in \mathbb{R} \) by Lemma 2 in the Appendix. Here \( f \) is the function defining \( X \) while \( \hat{f}_d \) corresponds to Lemma 2 applied to \( (Y_s \ast d_{N_k})^2(u) \) and does not depend on \( u \). So

\[ \int_{t_{n'}}^{t_n} \int_R G'(u) E_\gamma X (Y_s \ast d_{N_k})^2(u) \, duds = \lim_{\varepsilon \downarrow 0} \int_{t_{n'}}^{t_n} \int_R G'(u) \hat{E}_\varepsilon X (Y_s \ast d_{N_k})^2(u) \, duds \]
by dominated convergence and, as similar estimates can be obtained for the remaining but easier terms, one arrives at

\[ \left( E_\varepsilon X \left[ \mathcal{M}_{N_k}(Y)^G_{t_n} - \mathcal{M}_{N_k}(Y)^G_{t_n'} \right] \right)^2 \]

\[ = \lim_{\varepsilon \to 0} \left( E_\varepsilon X^\varepsilon \left[ Y^\varepsilon_{t_n}(G) - Y^\varepsilon_{t_n'}(G) - \int_{t_n'}^{t_n} \left\{ Y_s^\varepsilon(G^\varepsilon) - \gamma \int_{G^\varepsilon} (Y_s^\varepsilon * d_{N_k})^2(u) \, du \right\} \, ds \right] \right)^2 \]

\[ = \lim_{\varepsilon \to 0} \left( E_\varepsilon X^\varepsilon \left[ M^{G,\varepsilon}_{t_n} - M^{G,\varepsilon}_{t_n'} + R^G_{\varepsilon}(t_n) - R^G_{\varepsilon}(t_n') + \gamma \sum_{i=0}^{4} \left( R^{G,i}_{\varepsilon,N_k}(t_n) - R^{G,i}_{\varepsilon,N_k}(t_n') \right) \right] \right)^2 \]

using (5) for the last equality and writing \( X^\varepsilon \) as a substitute for \( f(Y^\varepsilon_{t_1}(H_1), \ldots, Y^\varepsilon_{t_p}(H_p)) \). Since \( E_\varepsilon X^\varepsilon[M^{G,\varepsilon}_{t_n} - M^{G,\varepsilon}_{t_n'}] \) will disappear by the martingale property, if \( \varepsilon_0 \) is chosen small enough, then

\[ \left( E_\varepsilon X \left[ \mathcal{M}_{N_k}(Y)^G_{t_n} - \mathcal{M}_{N_k}(Y)^G_{t_n'} \right] \right)^2 \leq \text{const} \left\{ \delta + \sum_{t \in \{t_n, t_n'\}} \left( E_{\varepsilon_0} \left[ R^G_{\varepsilon_0}(t) \right]^2 + \sum_{i=1}^{4} E_{\varepsilon_0} \left[ R^{G,i}_{\varepsilon_0,N_k}(t) \right]^2 \right) \right\} \quad (18) \]

by Cauchy-Schwarz. Also, choose \( \varepsilon_0 \) small enough such that

\[ E_{\varepsilon_0} \left[ R^G_{\varepsilon_0}(t_n) \right]^2 + E_{\varepsilon_0} \left[ R^G_{\varepsilon_0}(t_n') \right]^2 < \delta \]

which is possible by (4). The next lemma will provide estimates for the remaining summands.

**Lemma 1** Fix \( 0 \leq i \leq 4 \), \( t \in \{t_n, t_n'\} \) and \( \tau > 0 \) satisfying \( t_n + 2\tau < T \). If

\[ \ell(\{ t \in [0, T] : E_\varepsilon \left[ R^{G,i}_{\varepsilon,N}(t) \right]^2 \geq \delta \}) \leq \tau / 2 \]

then there exists \( \tilde{t} \in [t, t + 2\tau] \) such that

\[ E_\varepsilon \left[ R^{G,i}_{\varepsilon,N}(\tilde{t}) \right]^2 < \delta \quad \text{and} \quad E_\varepsilon \left[ R^{G,i}_{\varepsilon,N}(\tilde{t}) - R^{G,i}_{\varepsilon,N}(t) \right]^2 < \delta. \]

Indeed, observe that if \( \tilde{t} \geq t \) then

\[ E_\varepsilon \left[ R^{G,i}_{\varepsilon,N}(\tilde{t}) - R^{G,i}_{\varepsilon,N}(t) \right]^2 = E_\varepsilon \left[ R^{G,i}_{\varepsilon,N}(\tilde{t} - t) \right]^2 \]

by stationarity and the Markov property. Now assume the contrary of the lemma’s assertion, hence

\[ [t, t + 2\tau] \subseteq \{ \tilde{t} \in [t, t + 2\tau] : E_\varepsilon \left[ R^{G,i}_{\varepsilon,N}(\tilde{t}) \right]^2 \geq \delta \} \]

\[ \cup \{ \tilde{t} \in [t, t + 2\tau] : E_\varepsilon \left[ R^{G,i}_{\varepsilon,N}(\tilde{t}) - R^{G,i}_{\varepsilon,N}(t) \right]^2 \geq \delta \} \]

\[ = \{ \tilde{t} \in [t, t + 2\tau] : E_\varepsilon \left[ R^{G,i}_{\varepsilon,N}(\tilde{t}) \right]^2 \geq \delta \} \]

\[ \cup \{ \tilde{t} \in [t, t + 2\tau] : E_\varepsilon \left[ R^{G,i}_{\varepsilon,N}(\tilde{t} - t) \right]^2 \geq \delta \}. \]
Thus, as the Lebesgue measures of each of the sets on the above equality’s right-hand side are bounded by $\tau /2$, one obtains that $2\tau \leq \tau$ which is a contradiction proving the lemma.

Since for fixed $N_k$ it holds that

$$\int_0^T dt \, E_\varepsilon \left[ R_{\varepsilon, N_k}^{G,2}(t) \right]^2 = \mathcal{O}(\varepsilon)$$

as well as

$$\int_0^T dt \, E_\varepsilon \left[ R_{\varepsilon, N_k}^{G,4}(t) \right]^2 = \mathcal{O}(\varepsilon^2)$$

by (9) and (7) in [A2011] but also taking into account (14) one has

$$\ell (\{t \in [0, T] : E_\varepsilon \left[ R_{\varepsilon, N_k}^{G,0}(t) \right]^2 + E_\varepsilon \left[ R_{\varepsilon, N_k}^{G,2}(t) \right]^2 + E_\varepsilon \left[ R_{\varepsilon, N_k}^{G,4}(t) \right]^2 \geq \delta \}) \leq \tau /2$$

for a sufficiently small $\varepsilon > 0$. Thus, because (15) holds for all $\varepsilon > 0$ and for $\varepsilon_0$ in particular, one can estimate

$$E_\varepsilon \left[ R_{\varepsilon, N_k}^{G,i}(t) \right]^2 \leq 2 E_\varepsilon \left[ R_{\varepsilon, N_k}^{G,i}(\tilde{t}) - R_{\varepsilon, N_k}^{G,i}(t) \right]^2 + 2 E_\varepsilon \left[ R_{\varepsilon, N_k}^{G,i}(\tilde{t}) \right]^2 \leq 2\delta + 2\delta$$

using Lemma 1 for each $i = 0, 1, 2, 3, 4$ and $t = t_n, t_n'$ where $\tilde{t}$ of course depends on the chosen $i$ and $t$. So, when $\varepsilon_0$ in (18) is replaced by the minimum of $\varepsilon_0$ and $\varepsilon_1$, it follows that

$$\left( E_{\tilde{\gamma}} X \left[ |M_{N_k}(Y)_{t_n}^{G} - M_{N_k}(Y)_{s_n}^{G}| \right] \right)^2 \leq \text{const} \cdot \delta$$

which together with (17) proves (13). Altogether $(M_{s_j}^{G})_{j=1}^m$ is an $(F_j^{s_j})_{j=1}^m$-martingale for every finite ordered subset $\{s_1, \ldots, s_m\}$ of $\{t_1, t_2, \ldots\}$.

Now, choose arbitrary $s, t \in \mathcal{T}_G$ and fix $a > 0$. Without restricting the generality one can assume for a moment that $s, t$ play the role of $t_n', t_n$ chosen in the first part of this proof. Combining Chebyshev’s inequality and (16) yields

$$P_{\tilde{\gamma}}(|M_{t_n}^{G} - \bar{M}_s^{G}| > a) \leq \frac{\text{const}}{a^2} \cdot \delta + P_{\tilde{\gamma}}(|M_{N_k}(Y)_{t_n}^{G} - M_{N_k}(Y)_{s_n}^{G}| > a / 3)$$

for the corresponding $k = k_\delta$. Remark that the set $\{|M_{N_k}(Y)_{t_n}^{G} - M_{N_k}(Y)_{s_n}^{G}| > a / 3\}$ is open in $D([0, T]; \mathcal{D}'(\mathbb{R}))$ with respect to the uniform topology and that convergence in $J_1$ to elements of $C([0, T]; \mathcal{D}'(\mathbb{R}))$ is equivalent to uniform convergence. Thus, by Remark (19), the weak convergence of the measures $P_{\varepsilon}, \varepsilon \downarrow 0$, implies

$$P_{\tilde{\gamma}}(|M_{N_k}(Y)_{t_n}^{G} - M_{N_k}(Y)_{s_n}^{G}| > a / 3) \leq \lim_{\varepsilon \downarrow 0} P_{\varepsilon}(|M_{N_k}(Y)_{t_n}^{G} - M_{N_k}(Y)_{s_n}^{G}| > a / 3)$$

where the $\lim inf$ on the right-hand side is equal to

$$\lim_{\varepsilon \downarrow 0} P_{\varepsilon} \left( \left| Y_{t_n}^{\varepsilon}(G) - Y_{s_n}^{\varepsilon}(G) \right| - \int_{s_n}^{t_n} \left( Y_{t_n}^{\varepsilon}(G') - \tilde{\gamma} \int_{t_n}^{\varepsilon} G'(u) (Y_{t_n}^{\varepsilon} * d_{N_k})^2(u) du \right) \, dt \right)$$

$$\leq \lim_{\varepsilon \downarrow 0} P_{\varepsilon} \left( \left| M_{t_n}^{G,\varepsilon} - M_{s_n}^{G,\varepsilon} + R_{\varepsilon}^{G}(t) - R_{\varepsilon}^{G}(s) + \tilde{\gamma} \sum_{i=0}^{4} \left( R_{\varepsilon, N_k}^{G,\varepsilon}(t) - R_{\varepsilon, N_k}^{G,\varepsilon}(s) \right) \right| > a / 3 \right)$$

$$\leq \lim_{\varepsilon \downarrow 0} \left( P_{\varepsilon}(|M_{t_n}^{G,\varepsilon} - M_{s_n}^{G,\varepsilon}| > a / 6) + \frac{36}{a^2} E_{\varepsilon} \left[ R_{\varepsilon}^{G}(t) - R_{\varepsilon}^{G}(s) + \tilde{\gamma} \sum_{i=0}^{4} \left( R_{\varepsilon, N_k}^{G,\varepsilon}(t) - R_{\varepsilon, N_k}^{G,\varepsilon}(s) \right) \right]^2 \right)$$

14
where

$$E_\varepsilon [R_\varepsilon^G(t) - R_\varepsilon^G(s) + \gamma \sum_{i=0}^{4} \left( R_{\varepsilon,N_i}^G(t) - R_{\varepsilon,N_i}^G(s) \right)]^2 \leq \text{const} \cdot \delta \quad \text{for all } \varepsilon < \varepsilon_0 \wedge \varepsilon_1$$

as in the proof of (13). Altogether, there is a constant const such that for all $\delta > 0$

$$P_\gamma (|\tilde{M}^G_t - \tilde{M}^G_s| > a) \leq \lim_{\varepsilon \to 0} \frac{\text{const}}{\varepsilon^4} \left( \frac{\varepsilon}{a} \cdot \left( |M_{G,\tilde{\varepsilon}}^G - M_{s,\tilde{\varepsilon}}^G| > a/6 \right) + \frac{\text{const}}{\varepsilon^4} \cdot \delta \right)$$

that is

$$P_\gamma (|\tilde{M}^G_t - \tilde{M}^G_s| > a) \leq \lim_{\varepsilon \to 0} P_\gamma (|M_{G,\tilde{\varepsilon}}^G - M_{s,\tilde{\varepsilon}}^G| > a/6).$$

Now recall that $s, t \in T_G$ were arbitrarily chosen and observe that

$$P_\gamma (|M_{G,\tilde{\varepsilon}}^G - M_{s,\tilde{\varepsilon}}^G| > a/6) \leq \frac{6^4}{a^4} E_{\varepsilon} \cdot \frac{|M_{G,\tilde{\varepsilon}}^G - M_{s,\tilde{\varepsilon}}^G|}{a^4} \leq \frac{6^4 C_4}{a^4} E_{\varepsilon} \cdot \left( |M_{G,\tilde{\varepsilon}}^G| - |M_{G,\tilde{\varepsilon}}^G|_{s} \right)^2$$

by applying Chebyshev’s and then Burkholder-Davis-Gundy’s inequality with constant $C_4$. Furthermore

$$E_{\varepsilon} \cdot \left( |M_{G,\tilde{\varepsilon}}^G| - |M_{G,\tilde{\varepsilon}}^G|_{s} \right)^2 \leq C(T,G) \{\varepsilon^2 + (t-s)^2\}$$

by standard theory on exclusion processes (see Lemma 3 in the Appendix) hence there is a constant const which only depends on $T$ and $G$ such that

$$P_\gamma (|\tilde{M}^G_t - \tilde{M}^G_s| > a) \leq \text{const} \cdot a^{-4} (t-s)^2 \quad (19)$$

for all $a > 0$ and $s, t \in T_G$.

The next step in the proof is to construct a continuous process $(M_{t}^{G})_{t \in [0,T]}$ such that $\tilde{M}^G_t = M_{t}^{G}$ $\mathbb{P}_\gamma$-a.s. for all $t \in T_G$. But such a construction can be achieved almost the same way the continuous version of a process is constructed in the proof of the Kolmogorov-Chentsov theorem (see [KS1991] for example). As in this proof it follows from (19) that, for a dense subset $D$ of $[0, T]$, $\{M_{t}^{G}(\omega); t \in D\}$ is uniformly continuous in $t$ for every $\omega \in \Omega^*$ where $\Omega^*$ is an event in $\mathcal{F}_T^2$ of $\mathbb{P}_\gamma$-measure one. But in difference to [KS1991], $D$ should not be the set of dyadic rationals in $[0, T]$ but rather an appropriate subset of the set $\{t_1, t_2, \ldots\}$ chosen in the beginning of the present proof. Then one can define $M_{t}^{G}(\omega) \equiv 0, 0 \leq t \leq T,$ for $\omega \not\in \Omega^*$ while, for $\omega \in \Omega^*$, $M_{t}^{G}(\omega) = \tilde{M}^G_{t}(\omega)$ if $t \in D$ and $M_{t}^{G}(\omega) = \lim_n M_{s_n}^{G}(\omega)$ for some $(s_n)_{n=1}^\infty \subset D$ with $s_n \to t$ if $t \in [0, T] \setminus D$ which indeed gives a continuous process. To see that indeed $\tilde{M}^G_t = M_{t}^{G}$ a.s. for all $t \in T_G$ one splits $T_G$ into $D$ and $T_G \setminus D$. For $t \in D$ one has $\tilde{M}^G_t = M_{t}^{G}$ a.s. since $P_\gamma (\Omega^*) = 1$. For $t \in T_G \setminus D$ and $(s_n)_{n=1}^\infty \subset D$ with $s_n \to t$ one has $\tilde{M}^G_t = \lim_n M_{s_n}^{G}$ a.s. by construction as well as $\tilde{M}^G_t = \lim_n M_{s_n}^{G}$ in probability by (19) which also gives $\tilde{M}^G_t = M_{t}^{G}$ a.s.

Realize that, without restricting the generality, both $T_G$ and $D$ can be chosen to contain zero as $\mathbb{M}_N(Y)_0^G = 0$ for all $N$ by definition. Furthermore, since $D \subseteq \{t_1, t_2, \ldots\}$ and $\tilde{M}^G_n$ is $\mathcal{F}_t^G$-measurable for all $n$ and $\Omega^* \subseteq \mathcal{F}_t^G$, if $t \in D$ then $M^G_t$ is $\mathcal{F}_t$-measurable. Hence $(M_{t}^{G})_{t \in [0,T]}$ is $\mathcal{F}_t$-adapted since it is left-continuous and $D$ is dense in $[0, T]$.

Finally, from the $\mathcal{F}_t^G$-martingale property of $\tilde{M}_t^G$, $n = 1, 2, \ldots,$ shown in the first part of this proof follows that $(M_{s_j}^{G})_{j=1}^m$ is an $(\mathcal{F}_{s_j})_{j=1}^m$-martingale for every finite ordered subset
\{s_1, \ldots, s_m\} of D. All these martingales are square integrable since \(E_\gamma(\tilde{M}_n^G)^2 < \infty\) by the choice of \(t_n, n = 1, 2, \ldots,\) at the beginning of this proof. But \(T\) is not an element of \(D\) so choose an arbitrary positive \(T' < T\). Then \((M_t^G)_{t \in [0,T]}\) is a square integrable \(\mathbb{F}\)-martingale as the limits used to construct it can be interchanged with both expectations and conditional expectations by Doob’s maximal inequality for martingales since there must be an element of \(D\) between \(T'\) and \(T\).

\[\textbf{Proof} \quad \text{of Corollary}\] For part (i) fix a test function \(G \in \mathcal{D}(\mathbb{R})\). Since \((M_t^G)_{t \in [0,T]}\) is a continuous \(\mathbb{F}\)-adapted process it suffices to show that for every positive \(T' < T\), when restricted to \([0, T']\), the process \(M^G\) is an \(\mathbb{F}\)-Brownian motion with variance \(2\|G'\|_2^2\). So, in what follows, \(T\) is identified with some positive \(T' < T\) to simplify notation.

Obviously it only remains to show that \((M_t^G)^2 - 2\|G'\|_2^2 \cdot t, t \in [0, T]\), is an \(\mathbb{F}\)-martingale. Recalling the construction of \(M^G\) in the proof of Proposition 2, the \(\mathbb{F}\)-martingale property already follows from

\[E_\gamma X[(M_t^G)^2 - 2\|G'\|_2^2 \cdot t - (M_t^G)^2 + 2\|G'\|_2^2 \cdot t'] = 0\]

for all \(t, t' \in D\) such that \(t' < t\) and \(X = f(Y_{s_1}(H_1), \ldots, Y_{s_p}(H_p))\) where \(f : \mathbb{R}^p \to \mathbb{R}\) is a bounded continuous function, \(H_i \in \mathcal{D}(\mathbb{R})\) and \(0 \leq s_i \leq t', 1 \leq i \leq p\), which will be verified by showing that there is a \(const > 0\) such that

\[\left(E_\gamma X[(M_t^G)^2 - 2\|G'\|_2^2 \cdot t - (M_t^G)^2 + 2\|G'\|_2^2 \cdot t']\right)^2 \leq const \cdot \delta \quad \text{for all } \delta > 0. \quad (20)\]

So fix \(t, t' \in D\) such that \(t' < t\) and observe that

\[
\left(E_\gamma X[(M_t^G)^2 - 2\|G'\|_2^2 \cdot t - (M_t^G)^2 + 2\|G'\|_2^2 \cdot t']\right)^2 \\
\leq \quad \text{const} \quad \left\{\delta + \left(E_\gamma X[(\mathcal{M}_{N_k}(Y)_t^G)^2 - (\mathcal{M}_{N_k}(Y)_t^G)^2 - 2\|G'\|_2^2 \cdot (t - t')]\right)^2\right\}
\]

for some \(k = k_\delta\) big enough since the inequality

\[
\left(E_\gamma[(M_t^G)^2 - (\mathcal{M}_{N_k}(Y)_t^G)^2]\right)^2 \leq 2 \ E_\gamma[M_t^G - \mathcal{M}_{N_k}(Y)_t^G]^2 \left(E_\tilde{\gamma}(M_t^G)^2 + E_\tilde{\gamma}(\mathcal{M}_{N_k}(Y)_t^G)^2\right)
\]

holds for \(t\) and \(t'\). Furthermore

\[
E_\gamma X(\mathcal{M}_{N_k}(Y)_t^G)^2 = \lim_{\varepsilon \downarrow 0} E_\varepsilon X(Y_t(G) - Y_0(G) - \int_0^t \left\{Y_s(G''(u) - \tilde{\gamma} \int_R G''(u) \cdot dN_k\right\} du) ds)^2
\]

again by Lemma 2 in the Appendix which simplifies to

\[
= \lim_{\varepsilon \downarrow 0} E_\varepsilon X^\varepsilon \left(M_t^{G,\varepsilon} + R_\varepsilon^{G,\varepsilon}(t) + \tilde{\gamma} \sum_{i=0}^4 R_{\varepsilon, N_k}^{G,\varepsilon}(t)\right)^2 \quad \text{with } X^\varepsilon = f(Y_{s_1}^\varepsilon(H_1), \ldots, Y_{s_p}^\varepsilon(H_p)).
\]
Since the same equality holds for $t'$, one obtains that
\[
\left( E_\delta X \left[ (M_t^G)^2 - 2\|G'\|^2 \cdot t - (M_{t'}^G)^2 + 2\|G'\|^2 \cdot t' \right] \right)^2
\leq \text{const} \left\{ \delta + \left( E_\delta X \left[ (M_t^G)^2 - (M_{t'}^G)^2 - 2\|G'\|^2 \cdot (t-t') \right] \right)^2 \right\}
\]
for a sufficiently small $\varepsilon > 0$ by estimating
\[
E_\varepsilon M_t^{G,\varepsilon}\left( R^G_{\varepsilon}(t) + \sum_{i=0}^{4} R^{G,\varepsilon}_{\varepsilon,N_k}(t) \right) \quad \text{and} \quad E_\varepsilon \left( R^G_{\varepsilon}(t) + \sum_{i=0}^{4} R^{G,\varepsilon}_{\varepsilon,N_k}(t) \right)^2
\]
for $t$ and $t'$ as in the proof of Proposition 2.

Now $(M_t^{G,\varepsilon})^2, t \geq 0$, is a submartingale in the class $(\mathfrak{D}L)$ hence $(M_t^{G,\varepsilon})^2 - \langle M^{G,\varepsilon} \rangle_t, t \geq 0$, is a martingale thus
\[
\left( E_\delta X \left[ (M_t^G)^2 - 2\|G'\|^2 \cdot t - (M_{t'}^G)^2 + 2\|G'\|^2 \cdot t' \right] \right)^2
\leq \text{const} \left\{ \delta + \left( E_\delta X \left[ \langle M^{G,\varepsilon} \rangle_t - \langle M^{G,\varepsilon} \rangle_{t'} - 2\|G'\|^2 \cdot (t-t') \right] \right)^2 \right\}.
\]
Finally $E_\varepsilon [\langle M^{G,\varepsilon} \rangle_t - \langle M^{G,\varepsilon} \rangle_{t'} - 2\|G'\|^2 \cdot (t-t')]^2$ can be made arbitrarily small by choosing a suitable $\varepsilon$ which proves 20 hence part (i) of the corollary. The last argument is standard, for completeness its proof is sketched in the Appendix (see Lemma 3).

For part (ii) fix $a_1, a_2 \in \mathbb{R}$ and $G_1, G_2 \in \mathcal{D}(\mathbb{R})$ The wanted linearity holds for $\mathfrak{M}_N(Y)$ and, because of $\mathfrak{M}_N(Y)$ being an approximation for $(\bar{M}^G)_{G \in \mathcal{D}(\mathbb{R})}$, it should also hold for the version $(\bar{M}^G)_{G \in \mathcal{D}(\mathbb{R})}$ of $(\bar{M}^G)_{G \in \mathcal{D}(\mathbb{R})}$. But some care has to be taken since the construction of $(\bar{M}^G)_{G \in \mathcal{D}(\mathbb{R})}$ depends on the choice of subsequences and also since the notion of version used in this paper is special as not all $t \in [0, T]$ are covered. So recall the meaning of the set $\mathcal{T}_G$ from Proposition 2 and remark that the Lebesgue measure of
\[
\mathcal{T} = \mathcal{T}_{G_1} \cap \mathcal{T}_{G_2} \cap \mathcal{T}_{a_1 G_1 + a_2 G_2}
\]
is still $T$ hence $\mathcal{T}$ is dense in $[0, T]$ and one only has to show
\[
M_t^{a_1 G_1 + a_2 G_2} = a_1 M_t^{G_1} + a_2 M_t^{G_2} \quad \text{a.s.}
\]
for $t \in \mathcal{T}$ because the processes $M_t^{a_1 G_1 + a_2 G_2}, M_t^{G_1}, M_t^{G_2}$ are continuous. Fix $t \in \mathcal{T}$ and realise that, by Proposition 2, the above equality is equivalent to
\[
\bar{M}_t^{a_1 G_1 + a_2 G_2} = a_1 \bar{M}_t^{G_1} + a_2 \bar{M}_t^{G_2} \quad \text{a.s.}
\]
which would follow from 12 if the subsequence $(N_k)_{k=1}^\infty$ was independent of the chosen test function $G$. Indeed, applying 12 in the case of $G = a_1 G_1 + a_2 G_2$ gives
\[
\bar{M}_t^{a_1 G_1 + a_2 G_2} = a_1 \lim_{j \to \infty} \mathfrak{M}_{k_j} (Y)^{G_1}_t + a_2 \lim_{j \to \infty} \mathfrak{M}_{k_j} (Y)^{G_2}_t \quad \text{a.s.}
\]
for a subsequence \((k_j)_{j=1}^\infty\) of \((N_k)_{k=1}^\infty\). But if \((N_k)_{k=1}^\infty\) can be used in (12) for \(G = G_1\) and \(G = G_2\) as well then the above right-hand side would converge a.s. to \(a_1M_t^{G_1} + a_2M_t^{G_2}\) at least for a subsequence of \((k_j)_{j=1}^\infty\).

It remains to show that there is a subsequence \((N_k)_{k=1}^\infty\) such that (12) holds true for all \(G \in D(\mathbb{R})\). But, when choosing \(N_k = k^{3+a}, k \geq 1\), for an arbitrarily small \(a > 0\), Proposition 1, together with Chebyshev’s inequality implies that

\[
\sum_{k=1}^\infty \ell(\{t \in [0, T] : \mathbb{E}_{\tilde{\epsilon}} [\tilde{M}_t^G - \mathcal{M}_{N_k}(Y)_t^G]^2 \geq \delta\}) < \infty
\]

for all \(\delta > 0\) and \(G \in D(\mathbb{R})\). Thus, by a standard Borel-Cantelli argument, the above chosen subsequence \(N_k = k^{3+a}, k \geq 1\), has the desired property proving part (ii) of the corollary.

Remark that part (iii) would not follow from part (i) alone but, together with part (ii), it is straight forward to check both the Gaussian distribution as well as the covariance structure of the process \(M_t^G\) indexed by \(t \in [0, T]\) and \(G \in D(\mathbb{R})\). Of course, from the covariance structure follows that the index set of the process can be extended to \(t \in [0, T]\) and absolutely continuous functions \(G\) on \(\mathbb{R}\) with density \(G' \in L^2(\mathbb{R})\) without changing the underlying probability space. Hence

\[
\tilde{B}(t, u) = M_t^{G_u}/\sqrt{2}, \quad t \in [0, T], \quad u \in \mathbb{R},
\]

is properly defined using test functions \(G_u(\tilde{u}), \tilde{u} \in \mathbb{R}\), given by

\[
G_u(\tilde{u}) = \begin{cases} 
0 \lor (u \land \tilde{u}) : & u \geq 0; \\
0 \land (u \lor \tilde{u}) : & u < 0.
\end{cases}
\]

Obviously, \(\tilde{B}(t, u), t \in [0, T], u \in \mathbb{R}\), is a centred Gaussian process on \((D([0, T]; D'(\mathbb{R})), \mathcal{F}_T, \mathbb{P}_\gamma)\) with covariance \(\mathbb{E}_\gamma \tilde{B}(t, u)\tilde{B}(t', u') = (t \lor t')(|u| \lor |u'|)\) if \(u, u'\) have the same sign and \(\mathbb{E}_\gamma \tilde{B}(t, u)\tilde{B}(t', u') = 0\) otherwise. So, as in the proof of the Kolmogorov-Chentsov theorem, one can construct a version \(B(t, u)\) of \(\tilde{B}(t, u)\) on the same probability space which is continuous in \(t\) and \(u\) hence a Brownian sheet. By standard theory on random linear functionals, see [W1986] for a good reference, there is an \(S'(\mathbb{R})\)-valued version of the process \(M_t^G\) which is of course indistinguishable of

\[
\sqrt{2} \int_\mathbb{R} B(t, u)G''(u) \, du \quad t \in [0, T], \quad G \in S(\mathbb{R}),
\]

finally proving part (iii) of the corollary.

\section{Appendix}

Recall that \(\hat{\mathbb{P}}_\varepsilon\) is the push forward of \(\mathbb{P}_\varepsilon\) with respect to the map \(Y^\varepsilon\) and denote by \(\hat{\mathbb{E}}_\varepsilon\) the expectation when integrating against \(\hat{\mathbb{P}}_\varepsilon\). Then it is a consequence of Remark 1 that weak convergence still implies

\[
\hat{\mathbb{E}}_\varepsilon X \overset{\varepsilon \downarrow 0}{\to} \mathbb{E}_\gamma X, \quad (21)
\]

for \(X = f(Y_{s_1}(H_1), \ldots, Y_{s_p}(H_p))\) defined by bounded continuous maps \(f : \mathbb{R}^p \to \mathbb{R}\) and \(H_i \in D(\mathbb{R}), 0 \leq s_i \leq T, 1 \leq i \leq p\), although such functions \(X\) are not \(J_1\)-continuous on the space \(D([0, T]; D'(\mathbb{R}))\).
Lemma 2  The convergence (21) remains true for \(X\) defined by continuous functions \(f\) with polynomial growth and

\[
\sup_{\varepsilon \leq 1} |\hat{E}_\varepsilon X|^2 + |E_\gamma X|^2 \leq \hat{f}(\|H_1\|_2^2, \ldots, \|H_p\|_2^2)
\]

where \(\hat{f}\) is a polynomial not depending on the time points \(s_1, \ldots, s_p\) defining \(X\).

Proof. It suffices to show the lemma for polynomials \(f\). The convergence claim follows from Remark 1b). Indeed, as the one-dimensional marginal distributions of \(Y\) under \(P_\gamma\) are Gaussian, one can cut-off \(f\) turning it into a bounded continuous function for which (21) holds and estimate the remainder using the exponential decay of the tails of the Gaussian distribution. The uniform bound \(\hat{f}(\|H_1\|_2^2, \ldots, \|H_p\|_2^2)\) also follows from Remark 1b) by successively applying Hölder’s inequality and estimating moments of Gaussian distributions by powers of the variances.

Lemma 3  Let \(M^\varepsilon\) denote the martingale given by

\[
M_t^\varepsilon = Y_t^\varepsilon(G) - Y_0^\varepsilon(G) - \int_0^t LY_r^\varepsilon(G) \, dr, \quad t \geq 0,
\]

where \(G \in \mathcal{D}(\mathbb{R})\) is a test function and denote by \(\langle M^\varepsilon \rangle\) the compensator of the square bracket \([M^\varepsilon]\). Then:

(i) \(E_\varepsilon (\langle M^\varepsilon \rangle_t - 2\|G'\|_2 \cdot t)^2 \longrightarrow 0, \varepsilon \rightarrow 0, \) for all \(t \geq 0\);

(ii) there is a constant \(C(T, G)\) such that

\[
E_\varepsilon (\langle M^\varepsilon \rangle_t - \langle M^\varepsilon \rangle_s)^2 \leq C(T, G)\{\varepsilon^2 + (t - s)^2\}
\]

for all \(0 \leq s, t \leq T\) and \(\varepsilon > 0\).

Proof of (i) (cf. proof of (3.2) in [CLO2001] for example). At first observe that

\[
\langle M^\varepsilon \rangle_t = \int_0^t \Gamma_{2}^\varepsilon(r) \, dr, \quad t \geq 0,
\]

where

\[
\Gamma_2^\varepsilon(r) = \varepsilon^{-2}L \left[ Y_r^\varepsilon(G) \right]^2 - 2Y_r^\varepsilon(G)\varepsilon^{-2}LY_r^\varepsilon(G)
\]

\[
= \varepsilon \sum_{|x| \leq c_0/\varepsilon} 2p_{e}(\xi_{r\varepsilon^{-2}}(x) + 1)(1 - \xi_{r\varepsilon^{-2}}(x + 1)) \left[ G(\varepsilon(x + 1)) - G(\varepsilon x) \right]^2 \varepsilon
\]

\[
+ \varepsilon \sum_{|x| \leq c_0/\varepsilon} 2q_{e}(\xi_{r\varepsilon^{-2}}(x) + 1)(1 - \xi_{r\varepsilon^{-2}}(x - 1)) \left[ G(\varepsilon(x - 1)) - G(\varepsilon x) \right]^2 \varepsilon.
\]

So part (i) of the lemma follows since \((\xi_t)_{t \geq 0}\) is stationary, \(E_\varepsilon [\Gamma_2^\varepsilon(r) - E_\varepsilon \Gamma_2^\varepsilon(r)]^2\) converges to zero and \(E_\varepsilon \Gamma_2^\varepsilon(r)\) converges to \(2\|G'\|_2^2\) if \(\varepsilon\) tends to zero.
Proof of (ii) (cf. derivation of bound for (5.8) in [CLO2001] for example). As $\langle M^\varepsilon \rangle$ is the compensator of the square bracket $[M^\varepsilon]$ there is a local martingale $Z^\varepsilon$ such that

$$[M^\varepsilon]_t = Z^\varepsilon_t + \int_0^t \Gamma^\varepsilon_2(r) \, dr, \quad t \geq 0, \ P_\varepsilon\text{-a.s.}$$

From this equation directly follows that

$$( [M^\varepsilon]_t - [M^\varepsilon]_s )^2 \leq 3(Z^\varepsilon_t)^2 + 3(Z^\varepsilon_s)^2 + 3(t-s)^2 \|\Gamma^\varepsilon_2\|_{L^\infty[0,T]}^2 \quad P_\varepsilon\text{-a.s.}$$

with

$$\|\Gamma^\varepsilon_2\|_{L^\infty[0,T]}^2 \leq 16c_G\|G''\|_\infty^2$$

hence

$$E( [M^\varepsilon]_t - [M^\varepsilon]_s )^2 \leq 6C_2 E[Z^\varepsilon]_T + 48c_G\|G''\|_\infty^2 (t-s)^2$$

by Burkholder-Davis-Gundy’s inequality with constant $C_2$. The local martingale $Z^\varepsilon$ is of bounded variation and cannot have a continuous part which implies

$$[Z^\varepsilon]_T = \langle (Z^\varepsilon)^4 \rangle_T + \sum_{r \leq T} (\Delta Z^\varepsilon_r)^2 = \sum_{r \leq T} (\Delta Z^\varepsilon_r)^2 = \sum_{r \leq T} (\Delta [M^\varepsilon]_r)^2 \quad P_\varepsilon\text{-a.s.}$$

By the same argument

$$[M^\varepsilon]_r = \sum_{r \leq T} (\Delta M^\varepsilon_r)^2 = \sum_{r \leq T} (\Delta Y^\varepsilon_r(G))^2, \quad r \geq 0, \ P_\varepsilon\text{-a.s.},$$

thus

$$[Z^\varepsilon]_T = \sum_{r \leq T} (\Delta Y^\varepsilon_r(G))^4 \quad P_\varepsilon\text{-a.s.}$$

so one has to estimate jump size and number of jumps of the density fluctuation field.

By Lemma[3] it holds a.s. that for all $r \geq 0$ there is at most one bond $\{x^{(r)}, x^{(r)} + 1\}$ such that a particle jumps at time $r$ either from $x^{(r)}$ to $x^{(r)} + 1$ or from $x^{(r)} + 1$ to $x^{(r)}$; thus

$$|\Delta Y^\varepsilon_r(G)| \leq \sqrt{\varepsilon} \cdot 2|G(\varepsilon(x^{(r)} + 1)) - G(\varepsilon x^{(r)})| \leq 2\varepsilon^{3/2}\|G''\|_\infty, \quad r \geq 0, \ P_\varepsilon\text{-a.s.}$$

For the number of jumps one has the estimate

$$\sum_{x < -c_G/\varepsilon} N^x_{T_\varepsilon-2} 1_{\{N^x_{T_\varepsilon-2} > -\frac{c_G}{\varepsilon} - x\}} + \sum_{x \leq c_G/\varepsilon} N^x_{T_\varepsilon-2} + \sum_{x > c_G/\varepsilon} N^x_{T_\varepsilon-2} 1_{\{N^x_{T_\varepsilon-2} > x - \frac{c_G}{2}\}}$$

where $N^x$, $x \in \mathbb{Z}$, is a family of independent Poisson processes with intensity $2p_\varepsilon + 2q_\varepsilon = 2$.

But

$$E_\varepsilon \sum_{x > c_G/\varepsilon} N^x_{T_\varepsilon-2} 1_{\{N^x_{T_\varepsilon-2} > x - \frac{c_G}{2}\}} = e^{-2T\varepsilon^{-2}} \sum_{k > x - \frac{c_G}{\varepsilon}} k \frac{(2T\varepsilon^{-2})^k}{k!}$$

$$\leq e^{-2T\varepsilon^{-2}} \sum_{k=1}^\infty (k+1)k \frac{(2T\varepsilon^{-2})^k}{k!} \sim 4T^2\varepsilon^{-4}$$

1Compare with the explicit form of $\Gamma^\varepsilon_2(r)$ given in the proof of the first part of this lemma.
and the same bound holds for the expectation of the sum over \( x < -c_G/\varepsilon \). Altogether the expected number of jumps is bounded above by

\[
2 \cdot 4T^2\varepsilon^{-4} + \frac{2c_G}{\varepsilon} \cdot 2T\varepsilon^{-2}
\]

which gives

\[
E_{\varepsilon}[Z_{\varepsilon}]_T = E_{\varepsilon} \sum_{r \leq T} (\Delta Y_r^{\varepsilon}(G))^4 \leq (2 \cdot 4T^2\varepsilon^{-4} + \frac{2c_G}{\varepsilon} \cdot 2T\varepsilon^{-2}) \cdot (2\varepsilon^{3/2}\|G'\|_{\infty})^4
\]

hence the lemma follows from (22).

**Lemma 4** Consider the exclusion process on \((\Omega, \mathcal{F}, P)\) as introduced at the beginning of Section 2 and let \(X[i, j]\) denote the position of the particle \(i\) after the \(j\)th move where by ‘move’ is meant that a particle jumps or tries to jump. Denote by \(\tau_{ij}\) the random times at which these moves happen. Then

\[
P\left( \bigcup_{i \neq k} \bigcup_{j,l} \{\tau_{ij} = \tau_{kl}\} \right) = 0.
\]

**Proof.** Under the condition of \(X[i, 0] = x_0, X[i, 1] = x_1, \ldots, X[i, j] = x_j\) one knows that \(\tau_{ij}\) is the sum of \(j\) independent exponentially distributed random variables all of which have the same rate 2. Hence \(\tau_{ij}\) conditioned on a path is a continuous random variable with a pdf \(f_j(t), t \geq 0\), which does not depend on the specific path but only on the number of moves \(j\). Additionally conditioning on the path of another particle \(k\) one even knows that \(\tau_{kl}\) and \(\tau_{ij}\) are independent. Thus

\[
P\left( \bigcup_{i \neq k} \bigcup_{j,l} \{\tau_{ij} = \tau_{kl}\} \right) = E_P \left( \bigcup_{i \neq k} \bigcup_{j,l} \{\tau_{ij} = \tau_{kl}\} \mid X[\cdot, \cdot] \right)
\]

\[
\leq \sum_{i \neq k} \sum_{j,l} \int_0^\infty \int_0^\infty 1_{\{t=s\}} f_j(t) f_l(s) \, ds \, dt = 0
\]

proving the lemma.

**Remark 3 on how to construct the Sobolev space** \((\mathcal{H}, \| \cdot \|_{-})\). A natural choice for \((g_m)_{m=1}^{\infty}\) would be the eigenbasis of the one-dimensional Laplacian on \([0, T]\) with Dirichlet boundary conditions and a natural choice for \((G_n)_{n=1}^{\infty}\) would be the collection of Hermite functions.

Of course, the Dirichlet eigenbasis of the Laplacian on \([0, T]\) would not be a subset of \(\mathcal{D}(\mathbb{R})\) and neither would the Hermite functions be in \(\mathcal{D}(\mathbb{R})\). But it is very clear from how the test functions \(g\) were used that one just needs \(g \in C^1[0, T]\) with \(g(T) = 0\) hence choosing the Dirichlet eigenbasis of the Laplacian on \([0, T]\) works.

But choosing the Hermite functions \((G_n)_{n=1}^{\infty}\) must be treated with slightly more care as, although they are in \(\mathcal{S}(\mathbb{R})\), they do not have compact support. At first one observes that the measures \(P_\varepsilon\) can be considered on \(D([0, T]; \mathcal{S}'(\mathbb{R}))\). Second, in the case of the initial condition \(\nu_{1/2}\) introduced on page 4 of the present paper, the limit measure \(P_\gamma\) can also be obtained on \(D([0, T]; \mathcal{S}'(\mathbb{R}))\) following the ideas of the proof of Theorem B.1 in [BG1997]. Indeed, in
the case of this initial condition, (2.13) on page 578 of \[BG1997\] is satisfied for \(m \equiv 0\) and one can rule out that the functions \(f_X\) used in the proof of Theorem B.1 have exponential growth. As a consequence all proofs before \([7]\) work for \(G \in \mathcal{S}(\mathbb{R})\), too, since one only has to deal with the constant \(c_G\) as in the proof of part (ii) of Corollary 2 in \([A2011]\), eventually leading to the following rough modification

\[
E_{\tilde{\gamma}} \left| \int_0^T dt \, g(t) \, F_N(Y_t, G) - \langle g \otimes G, \frac{\partial}{\partial t} Y - \frac{\partial^2}{\partial u^2} Y - \frac{\partial}{\partial t} M \rangle / \tilde{\gamma} \right|^2 
\]

\[
\leq 32 C_d e^T \| g' \|^2_{L^2(0,T)} \left( N^{-2} \| G'' \|^2_{\infty} + \| u^2 G''' \|^2_{\infty} \right) + N^{-1} \left( \| G'' \|^2_{\infty} + \| u^2 G''' \|^2_{\infty} \right)
\]

\[
\leq 128 C_d e^T N^{-1/3} \| g'' \|^2_{L^2(0,T)} \left( \| G'' \|^2_{2} \| G' \|^2_{2} + \| u^2 G'' \|^2_{2} \| u^2 G' \|^2_{2} + \| G''' \|^2_{2} \| G'' \|^2_{2} + \| u^2 G''' \|^2_{2} \| u^2 G' \|^2_{2} + \| G'' \|^2_{2} \| G''' \|^2_{2} + \| G'' \|^2_{2} \| u^2 G' \|^2_{2} \right)
\]

of the estimate \([7]\).

So, recalling the definition of \(F_N(Y)\) underneath \([8]\), one observes that at least

\[
E_{\tilde{\gamma}} \left| \langle g_m \otimes G_n, F_N(Y) - \langle \frac{\partial}{\partial t} Y - \frac{\partial^2}{\partial u^2} Y - \frac{\partial}{\partial t} M \rangle \rangle / \tilde{\gamma} \right|^2 \leq \text{const.} \cdot N^{-1/3} m^2 n^6
\]

where the constant \(\text{const}\) does not depend on the choice of \(m\) and \(n\). Of course the factor \(m^2\) goes back to the eigenvalue associated with \(g_m\) and, using the combinatorial properties of the Hermite functions, \(O(n^6)\) is a quite crude estimate of all the norms involving \(G_n\) and its derivatives in the above modification of \([7]\). However the last inequality justifies \([8]\) if the Sobolev space \(\mathcal{H}\) is taken to be the completion of \(\mathcal{D}'((0, T) \times \mathbb{R})\) with respect to the norm \(\| \cdot \|_{-}\) given by

\[
\| \phi \|^2 = \sum_{m,n} \left[ (m^3 + n^3) m^2 n^6 \right]^{-1} \langle g_m \otimes G_n, \phi \rangle^2
\]

Needless to say that this construction of \(\mathcal{H}\) is not optimal but it definitely serves its purpose.

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