Subordination algebras in modal logic

Laurent De Rudder, Georges Hansoul and Valentine Stenetfeld

The aim of this paper is to show that even if the natural algebraic semantic for modal (normal) logic is modal algebra, the more general class of subordination algebras (roughly speaking, the non symmetric contact algebras) is adequate too - so leading to completeness results. This motivates for an algebraic (in the sense of universal algebra) study of those relational structures that are subordinate algebras.

1 Dualities

Modal algebra is a powerful tool of investigation of normal modal logics, dual in the sense of Stone Duality to the Kripke semantic.

Apparently far from the modal land, a contact algebra (see for instance [11], [14] or [18]) is a hybrid structure (both algebraic and relational) useful in the spatial reasoning in terms of regions rather than in terms of points.

The common point between modal and contact algebra is clear at the dual level : both duals are Stone spaces with a closed relation on them. In particular, their non topological parts are Kripke frames and their canonical extensions are in both cases complete modal algebras.

Of course contact algebras are symmetric by nature while modal algebras are not (in the sense that the accessibility relation in a Kripke frame may be any binary relation). Fortunately, non-symmetric versions of contact algebras had been introduced by Esakia in [15] under the name of subordination algebras, and we shall therefore adopt Esakia’s terminology - although subordination algebras had been reintroduced by contact algebraist under the name of pre-contact algebras or proximity algebras (see [12] and [13]).

We use classical notations. In particular Boolean operations are denoted \( \land, \lor, \neg, 0 \) and \( 1 \). For an order \( \leq \), we note

\[
a \uparrow = \{ b \mid a \leq b \}
\]

(and a similar definition for \( a \downarrow \)). For a binary relation \( R \), we note

\[
R(a, \_ \_ \_ ) = \{ b \mid a R b \}
\]

(and a similar definition for \( R(\_ \_ \_, a) \)). And finally, our notations \( T \) concerning elements of a set are freely extended to subsets of the set by

\[
T(A) = \bigcup \{ T(a) \mid a \in A \}.
\]

Definition 1.1 ([15]). A subordination algebra is a structure \( \mathbf{B} = (B, \prec) \) where \( B \) is a Boolean algebra and \( \prec \) a subordination on \( B \), that is a binary relation subject to the following axioms:

1. Reflexivity: \( a \prec a \) for all \( a \in B \).
2. Transitivity: \( a \prec b \) and \( b \prec c \) implies \( a \prec c \) for all \( a, b, c \in B \).
3. Upper transitivity: \( a \prec b \) implies \( a \leq c \) implies \( c \prec b \) for all \( a, b, c \in B \).
4. Lower transitivity: \( a \prec b \) implies \( c \leq a \) implies \( c \prec b \) for all \( a, b, c \in B \).

These axioms ensure that the relation \( \prec \) is a pre-order on \( B \).

Definition 1.2 ([15]). A subordination algebra is complete if for every subset \( A \) of \( B \), the least upper bound (LUB) of \( A \), denoted \( \bigvee A \), exists in \( B \).

Definition 1.3 ([15]). A subordination algebra is compact if every subset \( A \) that is bounded above has a least upper bound (LUB) in \( B \).
(S1) $0 \prec 0$ and $1 \prec 1$,
(S2) $a \prec b, c$ implies $a \prec b \land c$,
(S3) $b, c \prec a$ implies $b \lor c \prec a$,
(S4) $a \leq b \prec c \leq d$ implies $a \prec d$.

Equivalently, for each $b \in B$, $\prec (b, -)$ is a filter and $\prec (-, b)$ is an ideal of $B$.

An example of subordination algebras is given by the contact algebras
(11): those satisfying the axioms (S5) (of extensionality), (S6) (reflexivity) and (S7) (symmetry):
(S5) $a \neq 0$ implies $b \prec a$ for some $b \neq 0$,
(S6) $a \prec b$ implies $a \leq b$,
(S7) $a \prec b$ implies $\neg b \prec \neg a$.

And we recall that de Vries algebras (see 2 and 10) are (Boolean) complete contact algebras satisfying axiom (S8) (of transitivity):
(S8) $a \prec b$ implies $a \prec c \prec b$ for some $c$.

Investigations on subordination algebras have to be done in a suitable categorical environment. In the realm of contact algebra, the first morphisms have been introduced in 1962 by de Vries in 10 for the particular case of de Vries algebras. From the algebraic point of view, these are very weak morphisms since they are not even Boolean algebra homomorphisms.

Of course, from the model point of view, morphisms should be those Boolean algebra homomorphisms that respect the subordination relation $\prec$. They have been taken into account in various papers (see for instance 5 or [Düntsh]), and we shall consider them in this paper too, but under the name of weak morphism, as a common denominator of three other kinds of morphism we shall introduce now, and whose justification - as we shall soon see - is to reduce to modal algebra homomorphisms when applied to modal algebras. We arrive at different categories whose objects are the subordination algebras.

Definition 1.2. Let $B, C$ be subordination algebras and let $f$ be a map $B \rightarrow C$. We consider the following axioms:

(w) $a \prec b$ implies $f(a) \prec f(b)$,
(♦) $f(a) \prec c$ implies $a \prec b$ and $f(b) \leq c$ for some $b$,
(♠) $c \prec f(a)$ implies $b \prec a$ and $c \leq f(b)$ for some $b$.

Boolean algebra homomorphisms satisfying (w) will be called weak morphisms, giving rise to the category wSub. Those satisfying (w) and (♦) are the $\Diamond$-morphisms or white morphisms, or simply morphisms, and give rise to the white category $\Diamond$Sub, or more simply Sub. Those satisfying (w) and (♠) are the $\spadesuit$-morphisms or black morphisms and give rise to the black category $\spadesuit$Sub. Finally, those satisfying all three axioms are called strong morphisms. They give rise to the strong category sSub.
The usefulness of all these four categories, the choice of modal denominations, and more strangely of two different ones (the categories \( \diamondsuit \text{Sub} \) and \( \check{\bullet} \text{Sub} \) are trivially isomorphic, so what is the point of introducing two copies of a given category) will clearly appear in the next section. But we can say something now.

For completeness, we recall that a modal algebra \( [7] \) is an algebra \( B = (B, \diamondsuit) \) where \( \diamondsuit \) is an operator on \( B \), that is a map \( \diamondsuit : B \rightarrow B \) with \( \diamondsuit 0 = 0 \) and \( \diamondsuit (a \lor b) = \diamondsuit a \lor \diamondsuit b \). Any modal algebra may be considered as a subordination algebra by defining

\[
a \prec\diamondsuit b \quad \text{if} \quad \diamondsuit a \leq b.
\]

(1)

By (1), the category \( \text{MA} \) (of modal algebras with usual modal algebra morphisms) becomes a full subcategory of \( \check{\bullet} \text{Sub} \). But there is another way to consider a modal algebra, let’s write it \( B = (B, \check{\bullet}) \), as a subordination algebra. Indeed, we can define

\[
a \prec\check{\bullet} b \quad \text{if} \quad b \leq \neg \diamondsuit \neg a,
\]

(2)

where, as usual, \( \neg \) denotes \( \neg \diamondsuit \neg \). By this formula, the category of modal algebras with usual modal algebras morphisms becomes a full subcategory of \( \check{\bullet} \text{Sub} \). We will call it \( \check{\bullet} \text{MA} \) in this case.

There is also the weak category \( \text{wMA} \) of modal algebras with Boolean algebras homomorphisms \( f \) satisfying \( \diamondsuit f(a) \leq f(\diamondsuit a) \), corresponding to axiom (w) by way of (1). The strong category, the one of tense algebras, will be considered in the next section.

It is now time to say that subordination algebras have been studied by Celani under the name of quasi-modal algebras in [5] and [6]. Among other things he has extended Stone duality for modal algebras to the category that correspond to \( \check{\bullet} \text{Sub} \) in our setting. We present his results (adapted to the category \( \check{\bullet} \text{Sub} \)) and adopt his definition of subordination algebra because it is particularly suited for introducing canonical extensions.

**Remark 1.3.** Adapting definition of Celani in [5], let us call **multi-operator** on a Boolean algebra \( B \) a map \( \check{\bullet} \) from \( B \) into the filter lattice \( \mathcal{F}(B) \) of \( B \) such that \( \check{\bullet} 0 = B \) and \( \check{\bullet} (a \lor b) = \check{\bullet} a \cap \check{\bullet} b \) (if \( \mathcal{F}(B) \) is ordered by reverse inclusion, this can be written as \( \check{\bullet} 0 = 0 \) and \( \check{\bullet} (a \lor b) = \check{\bullet} a \lor \check{\bullet} b \), whence the name of multi-operator).

Then a subordination algebra may be defined as a structure \( B = (B, \check{\bullet}) \) where \( B \) is a Boolean algebra and \( \check{\bullet} \) a multi-operator on \( B \) (it suffices to define \( \check{\bullet} b = \prec \check{\bullet} (b, -) \)).

Duality comes as no surprise (see [5] or also [3]) and we have the following.

**Definition 1.4.**

1. A **subordination space** is a topological structure \( X = (X, R) \) where \( X \) is a Boolean space and \( R \), the **accessibility relation**, a closed binary relation on \( X \). Such a structure is called descriptive quasi-modal space by Celani in [5].

2. Let \( X, Y \) be subordination spaces and let \( h \) be a map \( X \rightarrow Y \). We consider the following axioms:

   (w) \( x R y \) implies \( h(x) R h(y) \),

   (\( \diamondsuit \)) \( h(x) R y \) implies for some \( z \in X, y = h(z) \) and \( x R z \),
(♦) \( y R h(x) \) implies for some \( z \in X \), \( y = h(z) \) and \( z R x \).

As in Definition 1.2, continuous maps satisfying (w) are the weak morphisms and give rise to the weak category \( \text{wSubS} \). Those satisfying (w) and (♦) are the white, or ♦, morphisms, or simply morphisms and lead to the category \( \text{♦SubS} \), also denoted \( \text{SubS} \). Those satisfying (w) and (♠) are the black morphisms or ♦ morphisms and are the arrows of the category \( \text{♦SubS} \). Finally, those satisfying all three axioms are called strong morphisms. They give rise to the strong category \( \text{sSubS} \).

**Definition 1.5.** Let \( B \) be a subordination algebra. Its dual is \( X = (X, R) \) where \( X = \text{Ult}(B) \) is the ultrafilter space of \( B \) with the topology generated by (the clopen sets)

\[
  r(b) = \{ x \in X \mid x \ni b \}, \ b \in B
\]

and \( R \) is the binary relation on \( X \) defined by

\[
  x R y \iff (y, -) \subseteq x.
\]

Let \( X = (X, R) \) be a subordination space. Its dual is \( B = (B, \prec) \) where \( B = \text{Clop}(X) \) is the Boolean algebra of all clopen subsets of \( X \) and \( \prec \) is the binary relation on \( B \) defined by

\[
  O \prec U \iff R(\neg, O) \subseteq U.
\]

If \( f : B \to C \) is a morphism in \( \text{wSub} \), then its dual is

\[
  \text{Ult}(f) : \text{Ult}(C) \to \text{Ult}(B) : y \mapsto f^{-1}(y).
\]

On the other hand, if \( h : X \to Y \) is a morphism in \( \text{wSubS} \), then its dual is

\[
  \text{Clop}(h) : \text{Clop}(Y) \to \text{Clop}(X) : O \mapsto h^{-1}(O).
\]

**Theorem 1.6.** ([5]) The functors \( \text{Ult} \) and \( \text{Clop} \) establish a dual equivalence between \( \text{wSub} \) and \( \text{wSubS} \). Their restriction also induce a dual equivalence between \( \text{Sub} \) and \( \text{SubS} \), ♦\text{Sub} and ♦\text{SubS} and finally between \( \text{sSub} \) and \( \text{sSubS} \).

**Proof.** To extend Celani’s proof to the strong category, it is important to observe that the duality between ♦\text{Sub} and ♦\text{SubS}, established in [5] thanks to functors slightly different from ours, may also be realized by the functors \( \text{Ult} \) and \( \text{Clop} \) of Definition 1.5, so that the duality \( \text{sSub} - \text{sSubS} \) is just a superposition if the dualities \( \text{Sub} - \text{SubS} \) and ♦\text{Sub} - ♦\text{SubS}.

We now give the discrete version of these dualities. The canonical extension of a subordination algebra will be realized as the "composition" of the topological duality with the discrete one.

**Definition 1.7.** Let \( B \) be a subordination algebra. then \( B \) is said to be complete atomic if \( B \) is a complete Boolean algebra and \( \prec \) is a complete subordination, that is it satisfies, for any family \( (b_i \mid i \in I) \) in \( B \) :

\[
  (S'2) \quad a \prec b_i \text{ for all } i \in I \text{ implies } a \prec \bigwedge b_i,
\]

\[
  (S'3) \quad b_i \prec \text{ for all } i \in I \text{ implies } \bigvee b_i \prec a.
\]
We obtain again four categories whose objects are the complete atomic subordination algebras whose morphisms are complete Boolean homomorphisms. The categories only differ on the choice of the axioms for morphisms as far as the subordination relation is concerned: axiom (w) of Definition 1.2 will give the subcategory $wCM$ of $wSub$, axioms (w) and ($\diamond$) the subcategory $CM$ of $Sub$, axioms (w) and ($\lozenge$) the subcategory $\diamond CM$ of $\diamond Sub$ and finally, taking all axioms of Definition 1.2 will give rise to the subcategory $sCM$ of $sSub$.

Our definition of atomic completeness is given as the natural one in the subordination world, but it fails to reveal an obvious, though important, property of complete atomic subordination algebras: they are in fact - in two different ways - complete modal algebras.

Indeed, (S’2) means that $\preceq (a, -)$ is a principal filter so that we may write

$$\preceq (a, -) = (\lozenge a)^\uparrow$$

and (S’3) means that the map $a \mapsto \lozenge a$ is a complete operator, that is, commutes with supremum of arbitrary families of elements of $B$. Order, dually (S’3) means that $\preceq (a, -)$ is a principal ideal, so that we may write

$$\preceq (a, -) = (\blacksquare a)^\downarrow$$

and (S’2) means that the map $a \mapsto \blacksquare a$ is a complete dual operator, that is, commutes with infimum of arbitrary families of elements of $B$. It follows from this that we may consider $wCM$ as a subcategory of $wMA$, $CM$ as a subcategory of $MA$, $\diamond CM$ as a subcategory of $\diamond MA$ and $sCM$ as a subcategory of $sMA$.

Since complete atomic subordination algebras and complete modal algebras have been shown to be the same objects, there is no specific discrete duality in the subordination case. We recall the duality for latter uses.

**Definition 1.8.** Let $\mathcal{B}$ be a complete atomic subordination algebra. Its dual is the Kripke frame $\mathcal{X} = \text{At}(\mathcal{B}) = (X, R)$, where $X = \text{At}(B)$ is the set of atoms of $B$ and the accessibility relation $R$ is given by

$$\alpha R \beta \text{ if } (\forall b \in B)(\beta \preceq b \Rightarrow \alpha \leq b).$$

Conversely, if $\mathcal{X} = (X, R)$ is a Kripke frame, its dual is $\mathcal{B} = \mathcal{P}(X) = (B, \prec)$ where $B = \mathcal{P}(X)$, the power set of $X$, and for $E, F \subseteq X$,

$$E \prec F \text{ if } R(\neg, E) \subseteq F.$$
Theorem 1.9. The functors $\text{At}$ and $\mathcal{P}$ establish a dual equivalence between the categories $\text{wCM}$ and $\text{wKF}$, $\text{CM}$ and $\text{KF}$, $\text{CM}$ and $\text{KF}$, and finally between $\text{sCM}$ and $\text{sKF}$.

Proof. This result is folklore and easily checked. □

Definition 1.10. The composition $\cdot^\delta = \mathcal{P} \text{Ult}$, where $F$ forget the topology, is called the canonical extension functor, and $\mathcal{B}^\delta = \mathcal{P}(\text{Ult}(\mathcal{B}))$ is the canonical extension of $\mathcal{B}$. The functor $\cdot^\delta$ may be considered as a functor from $\text{wSub}$ to $\text{wCM}$, from $\text{Sub}$ to $\text{CM}$, from $\text{Sub}$ to $\text{CM}$, or from $\text{sSub}$ to $\text{sCM}$.

To complete the picture, we mention that the natural map $r : \mathcal{B} \to \mathcal{B}^\delta$, as defined in (3), is usually not a morphism in $\text{Sub}$, but it is a little more than a weak morphism, as we can see now.

Definition 1.11. Let $\mathcal{B}, \mathcal{C}$ be subordination algebras. Then a map $f : \mathcal{B} \to \mathcal{C}$ is said to be a weak embedding if it is a one to one weak morphism such that $f(b) \prec f(c)$ implies $b \prec c$.

Proposition 1.12. 1. The natural map $r : \mathcal{B} \to \mathcal{B}^\delta$ is a weak embedding.

2. For each morphism $f$ in $\text{wSub}$ from $\mathcal{B}$ into a complete atomic subordination algebra $\mathcal{C}$, there is a unique morphism $g : \mathcal{B}^\delta \to \mathcal{C}$ in $\text{wCM}$ such that $g \circ r = f$.

Proof. The first assertion is clear. Note that the second assertion does not follow from the functoriality of $\cdot^\delta$ because $\mathcal{C}^\delta$ does not necessarily coincide with $\mathcal{C}$ in case the latter is complete atomic. Anyway, the result is well known at the Boolean level. To reach the subordination level, the easiest way is to notice that $g$ is necessarily the dual (in the discrete duality $\text{wCM} \dashv \text{wKF}$) of the composition $h = \text{Ult}(f) \circ j$, where $j$ is the natural weak embedding $\text{At}(\mathcal{C}) \to \text{Ult}(\mathcal{C}) : \alpha \mapsto \alpha \uparrow$, and $\text{Ult}(f)$ is the dual (in the duality $\text{wSub} \dashv \text{wSubS}$) of $f : \mathcal{B} \to \mathcal{C}$. So $g$ is weak whenever $f$ is weak since $j$ is weak.

Note that the second assertion of 1.12 does not extend to morphisms in $\text{Sub}$, meaning that $g$ is not necessarily a morphism in $\text{Sub}$ in this case, as seen in the case $\mathcal{B} = \mathcal{C}$ and $f$ is the identity.

Our definition of canonical extension is not the only possible one. Often, canonical extensions of structures with additional operations are obtained in a two-step construction. First, consider an already made canonical extension for the structure, then device a formula to extend the additional operations to the canonical extension of the structure. This could have been done here too. In two ways in fact subordination algebras are Boolean algebras with an additional relation. The canonical extension $\mathcal{B}^\delta$ of a Boolean algebra $\mathcal{B}$ is well known: $\mathcal{B}^\delta$ is the power set of the dual $X$ of $\mathcal{B}$. There are two ways to consider $\prec$.

Either as in [3], as the binary operation

$$\prec : B \times B \to B : (a,b) \mapsto \left\{ \begin{array}{ll}
1 & \text{if } a \prec b \\
0 & \text{otherwise}
\end{array} \right.$$ 

As shown in [1], this gives a non smooth extension (see [27] for definitions).

Or, as in [5] or as in Remark 1.3, as the multi-operator $\bowtie : a \mapsto \bowtie (a,-)$ (we could have chosen the black version too). The formulas giving the upper
(f^n) and the lower (f^\circ) extensions of a n-ary multi-operation are easily adapted from the univalued case (see for instance \[16\] or \[27\]) and give for an increasing map f : B^n → (F(B), ≥) :

\[
f^n(x) = \bigvee \{v \{f(b) \mid b ∈ B, b ≥ c\} \mid c \text{ closed, } c ≤ x\}
\]

\[
f^\circ(x) = \bigwedge \{v \{f(b) \mid b ∈ B, b ≤ o\} \mid o \text{ open, } o ≥ x\}
\]

The advantage of the latter point of view is that unary multi-operators are smooth, as shown in the following result.

**Proposition 1.13.** The subordination relation in a subordination algebra is smooth. And its extension to the canonical extension of its Boolean part coincide with the canonical extension functor of Definition 1.10.

**Proof.** The canonical extension functor gives \(B^\delta = \mathcal{P}(F(\text{Ult}(B)), \preceq)\) where \(E \prec F\) if \(R(−, E) \subseteq F\), so its associated modal operator is \(\lozenge^\delta E = R(−, E)\).

We show now that \(\lozenge^\sigma = \lozenge^\delta = R(−, E)\). Indeed, let \(\mathcal{O}(X)\) denotes the open subsets of \(X\) and \(\mathcal{C}(X)\) its closed subsets, we obtain

\[
\lozenge^\sigma(E) = \cap \{\{R(−, V) \mid \text{Clop}(X) \ni V \subseteq O\} \mid \mathcal{O}(X) \ni O \ni E\}
\]

\[
= \cap \{R(−, O) \mid \mathcal{O}(X) \ni O \ni E\}
\]

\[
\ni \cup \{R(−, F) \mid \mathcal{C}(X) \ni F \subseteq E\}
\]

(6)

\[
= \cup \{\{R(−, V) \mid \text{Clop}(X) \ni V \ni F\} \mid \mathcal{C}(X) \ni F \ni E\} = \lozenge^\sigma E
\]

(7)

\[
\ni \cup \{R(−, O) \mid \mathcal{O}(X) \ni O \ni E\} = \lozenge^\delta(E)
\]

where \(\lambda = \{f \mid f \text{ choice function } : F \text{ closed } \subseteq E \mapsto \alpha(F) \text{ clopen } \ni F\}\) and the equality (6) = (7) is obtained by Esakia’s lemma (\[15\] or \[23\]).

### 2 Modal logic of subordination algebras

We now define validity of a modal formula in a subordination algebra, taking into account the fact that \(B^\delta\) is a modal algebra. In fact, we noticed that any complete atomic subordination algebra may be endowed with two modalities \(\lozenge\) and \(\blacklozenge\) (see Definition \[17\]). In case of \(B^\delta\), the first as been evaluated in Proposition \[17,18\]: if \(B^\delta = \mathcal{P}(\text{Ult}(X))\), then, for every \(E ∈ B^\delta\)

\(\lozenge E = R(−, E)\).

The other one is its order dual, and we have, for every \(E ∈ B^\delta\)

\(\blacklozenge E = R(E, −)\).

So, our formulas will be bimodal formulas, that is, terms over the language \((∨, ∧, −, ⊤, ⊥, \lozenge, \blacklozenge)\). Those formulas not using \(\blacklozenge\) will be called simply modal formulas, or white formulas if needed, while formulas not using \(\lozenge\) will be called black formulas. Of course, \(B^\delta\), as a bimodal algebra, satisfies formulas not valid in all bimodal algebras since \(\lozenge\) and \(\blacklozenge\) are both induced by a single accessibility relation. They are what is called in the literature tense algebras. We recall basic facts about tense logic and tense algebra.
Definition 2.1. A tense bimodal logic (see for instance [20]) is a provably closed set of bimodal formulas containing all tautologies, the white and black versions of axiom $K$

\[(K\Diamond) \quad (\Box \phi \to \psi) \to (\Box \phi \to \Box \psi),\]

\[(K\Box) \quad (\Box \phi \to \psi) \to (\Box \phi \to \Box \psi),\]

and the following axioms

\[(T_1) \quad \phi \to \Box \Diamond \phi,\]

\[(T_2) \quad \Diamond \phi \to \phi.\]

It was shown in [25] that a bimodal algebra satisfies $T_1$ and $T_2$ (so is a tense algebra) if and only if the accessibility relation associated to $\Diamond$ is the converse of the accessibility relation associated to $\Box$. This shows that complete atomic subordination algebras are tense algebras, and their strong category $sCM$ is a subcategory of the category $TA$ of tense algebras (which is also the strong category of modal algebras as initiated in Definition 1.2).

Definition 2.2. Let $B$ be a subordination algebra and let $\varphi$ be a bimodal formula. A valuation on $B$ is a map $v : \text{Var} \to B$. Using the map $r$ of Definition 1.10, the composition $r \circ v$ is a map $\text{Var} \to B^b$ and as such, extends to a unique homomorphism, also denoted $v$, from the algebra of all bimodal formulas into $B^b$. We say that $\varphi$ is valid in $B$ under the valuation $v$, denoted $B \models v \varphi$

, if $v(\varphi) = 1$. Also, as usual, $B \models \varphi$

means $B \models v \varphi$ for all valuations $v$. If $\mathcal{K}$ is a class of subordination algebras and $L$ a set of bimodal formulas, $\mathcal{K} \models \varphi$, $B \models L$ and $\mathcal{K} \models L$ receive their usual meaning. At the dual level, things are rather natural. Indeed, let $X = (X,R)$ be the dual of $B$. Then $B \models \varphi$ is equivalent to $(X,R) \models v \varphi$ in the classical Kripke semantic meaning, for all valuations $v$ with values in the clopen subsets of $X$.

Before turning to completeness results, the following observations is in order. For a set $L$ of formulas, denote by $\text{Thm}(L)$ the logic axiomatized by $L$

\[\text{Thm}(L) = \{ \varphi \mid L \vdash \varphi \}.\]

For a class $\mathcal{K}$ of (subordination) algebras, or spaces, denote by $\text{Log}(\mathcal{K})$ the logic of $\mathcal{K}$

\[\text{Log}(\mathcal{K}) = \{ \varphi \mid \mathcal{K} \models \varphi \}.\]

Then, a completeness theorem identifies syntactic truth with semantic truth, that is, is of the form $\text{Thm}(L) = \text{Log}(\mathcal{K})$ for some $L$ and some $\mathcal{K}$. This is an impossible challenge in our case since $\text{Thm}(L)$ is always a normal modal logic, while this is not always the case for $\text{Log}(\mathcal{K})$. If $\text{Log}(\mathcal{K})$ is closed under modus ponens and necessitation, it is not always closed under substitution, as proved by the following example.
Example 2.3. Let $X$ be a Boolean space with an accumulation point $x$. Define $R \subseteq X \times X$ by $y R z$ if $y = x$ or $y = z$. Then $\overline{X} = (X, R)$ is a subordination space whose logic is not a normal modal logic.

Proof. We first show that $\overline{X} \models \varphi \equiv p \to \Diamond \Box p$. Indeed it is not difficult to show that if $O$ is clopen in $X$, then $O = \Diamond \Box O$, except when $O \neq \emptyset$ and $O \not\ni x$, in which case, $\Diamond \Box O = O \cup \{x\}$. In all case, $O \subseteq \Diamond \Box O$.

To complete the proof, it suffices to give an instance of $\varphi$ which is not valid in $\overline{X}$. Let $\psi \equiv p \land \neg \Box p$. Then, $\overline{X} \not\models \psi \to \Diamond \Box \psi$, as seen when $p$ is evaluated at a proper clopen subset containing $x$ (in this case, $\psi = \{x\}$ and $\Diamond \Box \psi = \emptyset$).

Notation 2.4. We shall see in Theorem 3.15 conditions under which $\text{Log}(K)$ is a normal modal logic but an immediate observation is that the substitution rule may be replaced by the use of schemes. To distinguish the formula $\varphi(\overline{\psi})$ from its associated scheme, we shall write the latter $\varphi(\psi)$, this expression denotes the collection of formulas $\varphi(\psi)$ when $\psi$ ranges over all modal (or bimodal if needed) tuples of formulas. We arrive at the following completeness results.

Theorem 2.5. Let $L$ be a set of schemes of modal formulas, and let $\varphi$ be a modal formula. Then the following are equivalent:

1. $L \vdash \varphi$,
2. for any modal algebra $\mathfrak{B}, \mathfrak{B} \models L$ implies $\mathfrak{B} \models \varphi$,
3. for any subordination algebra $\mathfrak{B}, \mathfrak{B} \models L$ implies $\mathfrak{B} \models \varphi$.

Proof. This is not really a new completeness theorem since 3. $\Rightarrow$ 2. is obvious while 1. $\Leftrightarrow$ 2. is well known. What is new is the soundness part 1. $\Rightarrow$ 3., whose proof is by induction on the length of a proof of $\varphi$. Here of course, the fact that $L$ is a set of schemes in essential.

Theorem 2.6. Let $L$ be a set of schemes of bimodal formulas containing the least tense bimodal logic, and let $\varphi$ be a bimodal formula. Then the following are equivalent:

1. $L \vdash \varphi$,
2. for any modal algebra $\mathfrak{B}, \mathfrak{B} \models \varphi$,
3. for any black modal algebra $\mathfrak{B}, \mathfrak{B} \models L$ implies $\mathfrak{B} \models \varphi$,
4. for any subordination algebra $\mathfrak{B}, \mathfrak{B} \models L$ implies $\mathfrak{B} \models \varphi$.

Proof. Since any tense bimodal algebra is in particular a modal algebra, and, that, clearly, validity qua bimodal algebra is equivalent to validity qua modal algebra as far as bimodal algebra are concerned, 2. implies

2’. for any tense bimodal algebra $\mathfrak{B}, \mathfrak{B} \models L$ implies $\mathfrak{B} \models \varphi$.

Then 2’, is equivalent to 1. by classical completeness theorem and 1. implies 4. which implies 2. as in the preceding Theorem.

Finally, 2. $\Leftrightarrow$ 3. by order duality.
It follows from Theorem 2.5 that under the hypothesis of the theorem, if there exists a subordination algebra $B \models L$ with $B \not\models \varphi$, then there exists some modal algebra $C \models L$ with $C \not\models \varphi$. An analogue observation can be made about Theorem 2.6. We give a functorial way to pass from $B$ to $C$.

**Definition 2.7.** Let $B$ be a subordination algebra and let $r$ be the weak embedding $B \rightarrow B^\delta$. The (white) modal subalgebra of $B^\delta$ generated by $r(B)$ is called the **modalisation** of $B$ and it is denoted by $B^m$ and we have the following result.

**Proposition 2.8.** The object mapping $B \rightarrow B^m$ can be extended to a functor $^m : \text{Sub} \rightarrow \text{MA}$. The natural map $r : B \rightarrow B^m$ is a weak embedding.

**Proof.** Suppose $f : B \rightarrow C$ is a morphism in $\text{Sub}$. By the canonical extension functor, $f$ lifts to $f^\delta : B^\delta \rightarrow C^\delta$ in $\text{CM}$. Let $f^m$ be the restriction of $f^\delta$ to $B^m$. It suffices now to show that $f^m$ takes value into $C^m$.

If $b \in B^m$, there are $b_1, ..., b_n \in B$ and a modal formula $\varphi$ with $b = \varphi(b_1, ..., b_n)$. Then,

$$f^\delta(b) = f^\delta(\varphi(b_1, ..., b_n)) = \varphi(f^\delta(b_1), ..., f^\delta(b_n)) \in C^m$$

as required.

Of course, next to the modalisation functor, there is the **black modalisation functor** $\blm : \text{Sub} \rightarrow \text{MA}$ ($B^\blm$ is the black modal algebra generated by $r(B)$) and the **bimodalisation functor** $\blm : \text{sSub} \rightarrow \text{TA}$ ($B^\blm$ is the least tense algebra generated by $r(B)$).

The relevance of these concepts is that they preserve, not validity of formulas, but validity of schemes, as shown in the following result. We give result in case of modalisation, but of course, there is an analogue result for black and bimodalisation.

**Proposition 2.9.** Let $B$ be a subordination algebra. Then for any scheme $\varphi(\overline{v})$ of modal formulas,

$$B \models \varphi(\overline{v}) \iff B^m \models \varphi(\overline{v}).$$

**Proof.** Since $B^m$ is submodal algebra of $B^\delta$, the if part directly follows from the definition of validity in 2.2.

Suppose $\overline{v} = (\psi_1, ..., \psi_n)$ and $B \models \varphi(\overline{v})$. We have to show $B^m \models \varphi(\overline{v})$. Let $v$ be a valuation $\text{Var} \rightarrow B^m$. There are $b_1, ..., b_r \in B$ and formulas $\varphi_1, ..., \varphi_n$ in the variables $q_1, ..., q_r$ such that $v(\psi_i) = \varphi_i(b_1, ..., b_r)$ for all $i$. Let $v'$ be the valuation $\overline{v} \rightarrow B^m$ such that $v'(q_i) = b_i$ for all $i$. Then, $B \models v(\varphi_1, ..., \varphi_n)$ and it follows

$$1 = v'(\varphi(\varphi_1, ..., \varphi_n))$$

$$= \varphi(v'(\varphi_1), ..., v'(\varphi_n))$$

$$= \varphi(\varphi(\psi_1(b_1, ..., b_r), ..., \varphi_n(b_1, ..., b_r)))$$

$$= \varphi(v(\psi_1), ..., v(\psi_n))$$

$$= v(\varphi(\overline{v})),$$

as required.
Note that Example 2.3 shows that a single modal formula may fail to be preserved by modalisation: if $B$ is the dual of $X$ as described in 2.3, then $B \models p \rightarrow \Diamond \Box p$. Otherwise, by modal logic, $B^m \not\models p \rightarrow \Diamond \Box p$ for all modal formula $\psi$ and thus, by Proposition 2.9, $B \models \psi \rightarrow \Diamond \Box \psi$, which is not the case.

To find constructs that preserve validity of formulas, we turn to universal algebra. Since the validity of $\varphi$ is equivalent to the validity of the equation $\varphi = 1$, one is tempted to look at homomorphic images, subalgebras and products.

Homomorphic images, or quotient, of subordination algebras have been introduced and dualized by Celani in [6]. We recall and improve his results using a slightly different terminology, following the spirit of universal algebra.

**Definition 2.10.** Let $B$ be a subordination algebra. We say that an equivalence relation $\theta$ on $B$ is a congruence if it is the kernel of a morphism (in $\text{Sub}$), that is there is a morphism $f : B \rightarrow C$ such that

$$\theta = \ker(f) = \{(a, b) \mid f(a) = f(b)\}.$$ 

Of course, a congruence is necessarily a Boolean congruence and as such, characterised by its 0-class (its 0-kernel), namely the ideal

$$I = \{a \mid f(a) = 0\}$$

or by its 1-class (its 1-kernel), namely the filter

$$F = \{a \mid f(a) = 1\}.$$ 

Remember that $\theta, I$ and $F$ are linked by the formulas: $a \theta b$ if and only if there is $i \in I$ with $a \lor i = b \lor i$, if and only if there is $f \in F$ with $a \land f = b \land f$. Remember also from [2] that an ideal is round if $a \in I$ implies $a \prec b$ for some $b \in I$, and that a filter is round if $a \in F$ implies $b \prec a$ for some $b \in F$. Implicit in [6] is the following result.

**Proposition 2.11.** Let $B$ be a subordination algebra and let $\theta$ be a Boolean congruence on $B$. With the notation of Definition 2.10, the following are equivalent:

1. $\theta$ is a congruence,
2. $\theta$ satisfies $a \theta b \prec c$ implies $a \prec d \theta c$ for some $d$,
3. $I$ is a round ideal,
4. $F$ satisfies: $a \in F$ implies $\lnot a \prec \lnot b$ for some $b \in F$.

The dual objects of congruences are easy to characterise: they are the closed subsets $C$ of the dual space which are $R$-increasing, that is, such that $x \in C$ and $x R y$ imply $y \in C$. The correspondence, described in [6], is obviously the restriction to congruences of the correspondence between Boolean congruences (or ideals) and closed subsets of the dual:

$$I \text{ ideal of } B \mapsto C = \{x \in \text{Ult}(B) \mid x \cap I = \emptyset\}.$$ 

As a corollary, we have the following result which is a small improvement of Lemma 9 of [6].
Corollary 2.12. Let $B$ be a subordination algebra and let us denote by $\text{Con}(B)$ the ordered set of all congruences on $B$. Then $\text{Con}(B)$ is a frame (complete Heyting algebra) in which finite meet is intersection while arbitrary joint is joint in the equivalence lattice.

Proof. By Proposition 2.11 it suffices to argue on round ideals. Let $I$ and $J$ be round ideals. If $a \in I \cap J$, there are $b \in I$ and $c \in J$ such that $a \prec b$ and $a \prec c$. Then $a \prec b \wedge c$ and $b \wedge c \in I \cap J$ (of course this argument does not work for infinitely many round ideals).

Let now $(I_l \mid l \in \Lambda)$ be round ideals and let $I$ be the joint of the $I_l$ computed in the ideal lattice. If $a \in I$, there are $l_1, ..., l_n \in \Lambda$ and $a_{l_1}, ..., a_{l_n} \in I_{l_i}$ with $a \leq a_{l_1} \vee ... \vee a_{l_n}$. And there are $b_{l_1} \in I_{l_1}, ..., b_{l_n} \in I_{l_n}$ with $a_{l_1} \prec b_{l_1}, ..., a_{l_n} \prec b_{l_n}$. It follows that $a \prec b = b_{l_1} \vee ... \vee b_{l_n} \in I$.

It follows that $\text{Con}(B)$ is a frame, being a subset of the ideal lattice of $B$, closed under finite meets and arbitrary joints. □

To continue to show that subordination algebras are more on the side of universal algebra than relational structures, it is time to adapt the three classical isomorphism theorem. We omit the classical proof.

Proposition 2.13. Suppose $B = (B, \preceq)$ is a subordination algebra and $\theta$ is a congruence on $B$. The structure $B/\theta = (B/\theta, \preceq^\theta)$ with

$$a^\theta \preceq^\theta b^\theta \text{ if } \exists c, a \preceq c^\theta b$$

is a subordination algebra such that the canonical projection $\pi : B \rightarrow B/\theta$ is a morphism.

Proposition 2.14. Suppose $A$ and $B$ are subordination algebras.

1. Let $f : A \rightarrow B$ be a morphism and let $\theta$ be a congruence on $A$. Then, there is a morphism $g : A/\theta \rightarrow B$ such that $g \circ \pi = f$ if and only if $\theta \subseteq \ker(f)$. In particular, $A/\ker(f)$ is isomorphic with the range of $f$, which appears to be a subordination subalgebra of $B$ (see [7]).

2. Suppose $A$ is a subalgebra of $B$ and let $\theta$ be a congruence on $B$. Then the restriction of $\theta$ to $A$ is a congruence on $A$, the saturation

$$A^\theta := \{b \in B \mid \exists a \in A, a \theta b\}$$

is a subalgebra of $B$ and $A/\theta|_A$ is isomorphic with $A^\theta/\theta$.

3. Let $\theta$ be a congruence on $B$. Then the congruence lattice of $B/\theta$ is isomorphic with the principal filter of $\text{Con}(B)$ generated by $\theta$.

Of course, analogue to Propositions 2.12 and 2.14 which are relative to $\text{Sub}$, there are corresponding results in the categories $\text{Sub}$ and $\text{sSub}$. So we have the black congruences (those Boolean congruence which are kernel of morphisms in $\text{Sub}$ and which are characterised by round filter as a 1-kernel) and the strong congruences (kernels of strong morphisms, whose 0-kernel are round ideals and 1-kernel are round filters). We freely use these facts and examine validity.
Proposition 2.15. If $\varphi$ is a modal formula (resp. a black modal formula or a bimodal formula), $B$ a subordination algebra and $\theta$ a congruence (resp. a black congruence or a strong congruence), then

$$B \models \varphi \Rightarrow B/\theta \models \varphi.$$ 

In other words, validity of a formula is preserved by morphic image of its language.

Proof. It suffices to examine the white language. Let $f$ be an onto morphism (in Sub) from $B$ to $C$. Suppose $B \models \varphi$. We want to prove that $C \models \varphi$. Let $v : \text{Var} \rightarrow C$ be a valuation and let $v_1$ be any valuation $\text{Var} \rightarrow B$ such that $f \circ v_1 = v$. Then $v_1$ extend to a modal homomorphism $v_1 : \text{Form} \rightarrow B^\delta$ such that $v_1(\varphi) = 1$. Now, applying the canonical extension functor, we have a morphism $f^\delta : B^\delta \rightarrow C^\delta$, so $f^\delta \circ v_1$ is the extension to $\text{Form} \rightarrow C^\delta$ of $v$. And we have $v(\varphi) = f^\delta(v_1(\varphi)) = 1$ as required.

We now turn to subalgebras (subobjects in Sub), a topic not examined in [5] and [6].

Definition 2.16. Let $B$ be a subordination algebra and $A \subseteq B$. Then $A$ is a subalgebra of $B$ if $A$ is a sub-Boolean algebra of $B$ and

$$\text{if } a \in A, b \in B \text{ and } a \prec b, \text{ there is } a \in A \text{ with } a \prec c \leq b. \quad (8)$$

Of course the associate subordination algebra is $A = (A, \prec)$ where $\prec$ is the restriction to $A$ of the subordination relation of $B$. We also say that $A$ is a subalgebra of $B$. This is the appropriate concept since an equivalence $\theta$ on $X$ is a congruence if and only if the inclusion mapping $i : A \rightarrow B$ is a morphism. And moreover, if $B$ is a modal algebra, the concepts of subalgebra of $B$, qua modal algebra qua subordination algebra, coincide.

However, this concept of subalgebra lacks many usual properties of subalgebras of universal algebra. We shall see in [2,20] that the intersection of two subordination subalgebras may not be a subordination subalgebra. So it is even not clear that the set Sub($B$) of all subalgebras of $B$ forms a lattice. The only obvious positive result we have in this direction is the following.

Proposition 2.17. If $(A_i \mid i \in I)$ is a directed family of subalgebras of $B$, so is its union. Hence, Sub($B$) is a dcpo in the sense of [17].

Definition 2.18. Let $X = (X, R)$ be a subordination space. A congruence on $X$ (or more precisely a white congruence) is an equivalence relation $\theta$ on $X$ such that:

1. $\theta$ is a congruence of Boolean spaces (that is non equivalent points can be separated by clopen $\theta$-saturated sets),

2. for $x, y, z \in X$

$$x \theta y \Rightarrow \exists u \in X : x R u \theta z. \quad (9)$$

Here again this is the appropriate concept since an equivalence $\theta$ on $X$ is a congruence if and only if there is a, necessarily unique, subordination structure on $X/\theta$ such that the natural projection is a morphism. For $x, y \in X$, we have
\(x^0 R^0 y^0\) if there is \(y^0 \in X\) such that \(x^0 R y^0 \theta y\), or equivalently, if for any \(x^0 \theta x\), there is \(y^0 \in X\) such that \(x^0 R y^0 \theta y\).

Let us denote by \(\text{Con}(X)\) the ordered set of congruences on \(X\). The duality \(\text{Sub} \leftrightarrow \text{Subs}\) of Theorem 1.10 exchanges one to one maps with onto ones, so exchanges subalgebras with congruences, and we have the following result.

**Proposition 2.19.** Let \(B\) be a subordination algebra and let \(X\) be its dual. Then \(\text{Sub}(B)\) is anti-isomorphic (that is isomorphic to the order dual of) to the set of congruences on \(X\).

**Proof.** This is a direct byproduct of the duality. It suffices to check that the well-known anti-isomorphism between sub-Boolean algebras of \(B\) and Boolean congruences of \(X\) exchanges subordination subalgebras with congruence.

So let \(A\) be a subordination subalgebra of \(B\) and let \(\theta\) be its dual Boolean equivalence, that is

\[x \theta y\text{ if and only if }x \cap A = y \cap A.\]

We show that \(\theta\) satisfies 9. If \(x \theta y R z\), we have to show that \(z \in (R(x, -))^\theta\). If not, since \(\theta\) is Boolean there is a clopen \(\theta\)-saturated set \(O\) with \(O \ni z\) and \(O \cap R(x, -) = \emptyset\). It follows from \(z \in O\) that \(y \in R(-, O)\) and from \(O \cap R(x, -) = \emptyset\) that \(x \notin R(-, O)\) such that there is a clopen set \(U\) with \(x \notin U\) and \(R(-, O) \subseteq U\). Since \(O\) is a clopen \(\theta\)-saturated set, \(O = r(a)\) with \(a \in A\). If \(b \in B\) is such that \(r(b) = U\), then we have \(a \prec b\). If \(A\) is a subordination subalgebra, there is \(a_1 \in A\) with \(a \prec a_1 \leq b\). Let \(U_1 = r(a_1)\), we have

\[R(-, O) \subseteq U_1 \subseteq U.\]

So \(y \in U_1\) while \(x \notin U_1\). This is impossible since \(U_1\) is \(\theta\)-saturated.

Conversely, let \(\theta\) be a congruence on \(X\). Suppose \(a \in A\) and \(a \prec b\). Let \(O = r(a)\) and \(U = r(b)\). We first show that \(R(-, O)\) is \(\theta\)-saturated. If \(x \theta y R z \in O\), then for some \(u, x R u \theta z\). Now \(O\) is \(\theta\)-saturated, so \(u \in O\) and \(x \in R(-, O)\). It follows that \(R(-, O)\) is the intersection of all the clopen saturated sets containing it, and one of them has to be in \(U\) by compactness. \(\square\)

**Example 2.20.** Let \(\omega\) be the set of natural numbers and \(\omega^+\) be its successor ordinal (topologically, \(\omega^+\) is the Alexandroff compactification of \(\omega\)). We consider \(\omega^+\) as a subordination space with the relation \(R\) defined by \(x R y\) if \(y = x\) or \(x = \omega\) or \(x = x\). It is not difficult to show that an equivalence is a congruence if and only if all its classes are closed and the class containing \(\omega\) is either \(\{\omega\}\) or \(\omega^+\). Let \(\theta\) be the equivalence whose classes are \(\{2i, 2i + 1\}\) for \(i \in \omega\) and \(\{\omega\}\); let \(\xi\) be the equivalence whose classes are \(\{0, 1\}, \{2\}, \{2i + 3, 2i + 4\}\) for \(i \in \omega\) and \(\{\omega\}\). Then, the supremum of \(\theta\) and \(\xi\) in the lattice of Boolean congruences of \(\omega^+\) has two classes \(\{0, 1\}\) and its complement, hence it is not a congruence. By Proposition 2.19, this shows that the intersection of two subalgebras of a subordination algebra may fail to be a subalgebra.

As was the case for congruences, there are **black subalgebras** \(A\) of \(B\) \((a \prec b\) and \(a \in A\) implies \(b \leq c \prec a\) for some \(c \in A\)) with dual black congruences on \(X\) satisfying the first condition of Definition 2.18 and

\[2'\text{ For }x, y, z \in X\]

\[x R y \theta z \Rightarrow \exists u \in X : x \theta u R z;\]
and there are strong subalgebras (both black and white subalgebras) and strong congruences of subordination spaces (both white and black congruences).

**Proposition 2.21.** If \( \varphi \) is a modal formula (resp. a black modal formula or a bimodal formula), \( B \) a subordination algebra and \( A \) a subalgebra (resp. a black subalgebra or a strong subalgebra) then

\[
B \models \varphi \Rightarrow A \models \varphi.
\]

**Proof.** As in Proposition 2.15, it suffices to examine the white language. The inclusion morphism \( i : A \rightarrow B \) lifts to a CM-morphism \( i^\delta : A^\delta \rightarrow B^\delta \) which is one-to-one since \( \cdot^\delta = \mathcal{P}F \text{Ult} \) and both \( \text{Ult} \) and \( \mathcal{P} \) exchange onto with one-to-one.

We suppose \( B \models \varphi \) and want to prove \( A \models \varphi \). So let \( v \) be a valuation \( \text{Var} \rightarrow A \). It extends to a modal homomorphism \( v : \text{Form} \rightarrow A^\delta \) and therefore \( i^\delta v : \text{Form} \rightarrow B^\delta \) is the homomorphic extension of the valuation \( v \) considered as a valuation on \( B \). Since \( B \models \varphi \), we have \( i^\delta (v(\varphi)) = 1 \), whence \( v(\varphi) = 1 \), as required.

To end with subalgebras, just note that (as a slight improvement of 2.9), we have the implication \( B^m \models \varphi \) implies \( B \models \varphi \) for formulas and not only for schemes.

We now turn to products, concentrating on the Cartesian product (defined pointwise).

**Proposition 2.22.** Let \( (A_j \mid j \in J) \) be a family of subordination algebras and let \( P = \prod_{j \in J} A_j \) be its Cartesian product. Then

1. \( P \) is the categorical product in the weak category \( w\text{Sub} \),
2. the projections are morphisms in the strong category \( s\text{Sub} \),
3. a finite product is a categorical one in the categories \( \text{Sub} \), \( \Diamond \text{Sub} \) and \( s\text{Sub} \).

**Proof.** Assertions 1. and 2. follow from direct calculations and we prove assertion 3. for the category \( \text{Sub} \).

Suppose \( B \) is a subordination algebra and \( f_j \) are morphisms \( B \rightarrow A_j \) for \( j \in J \). The Cartesian product \( f : b \mapsto (f_j(b) \mid j \in J) \) is a weak morphism and we show it is a morphism in \( \text{Sub} \), that is, we prove axiom (2) of [12]. So let \( f(b) \prec c \), we have \( f_j(b) \prec c_j \) for all \( j \) and there are \( d_j \) in \( B \) with \( b \prec d_j \) and \( f_j(d_j) \leq c_j \). Since, \( J \) is finite, \( d = \wedge_{j \in J} d_j \) exists in \( B \) and we have \( b \prec d \) and \( f(d) \leq c \), as required.

The following result shows that for a Cartesian product, being a categorical product - in \( \text{Sub} \), \( \Diamond \text{Sub} \) or \( s\text{Sub} \) - is a very strong property, except for finite products and the trivial case where the \( A_j \)’s are modal algebras.

**Proposition 2.23.** If \( A \) is a subordination algebra which is not a modal algebra, some Cartesian power of \( A \) is not a power in the category \( \text{Sub} \).

**Proof.** Let \( a \in A \) be such that \( \prec (a, -) \) is not a principal filter, and let \( (b_j \mid j \in J) \) be another notation for the set \( \prec (a, -) \). Then \( A^J \) is not a product in \( \text{Sub} \). Too see this, let \( A_j = A \) for any \( j \in J \) (so that \( A^J = \prod_{j \in J} A_j \)) and let
$f_j : A \rightarrow A_j$ be the identity map for any $j \in J$. The Cartesian product $f$ of the $f_j$ is the diagonal map

$$a \mapsto (a_j \mid j \in J)$$

with $a_j = a$ for all $j \in J$. This is not a morphism : let $c \in A^J$ be defined by $c_j = b_j$ for $j \in J$. Then $f(a) \prec c$ but there is no $b \in A$ such that we have $a \prec b$ and $f(b) \leq c$. \[\square\]

The following results examine validity of formulas in products.

**Proposition 2.24.** Let $\varphi$ be a bimodal formula. If $(A_j \mid j \in J)$ is a finite family of subordination algebras then

$$\prod_{j \in J} A_j \models \varphi \iff A_j \models \varphi \ \forall j.$$  

**Proof.** The if part follows from 2.15 and 2.22.

Suppose now $A_j = \varphi$ for all $j$. We have to prove $\prod A_j \models \varphi$. So let $v$ be a valuation on $\prod A_j$. Then, if $p_j$ denotes the projection from $\prod A_j$ into $A_j$, $p_j \circ v$ is a valuation on $A_j$ and $p_j(v(\varphi)) = 1$ in $A_j$. It follows $v(\varphi) = 1$ in $\prod A_j$ and since

$$\prod A_j \models (\prod A_j)^\beta,$$

$v(\varphi) = 1$ in $(\prod A_j)^\beta$, that is $\prod A_j \models \varphi$. \[\square\]

For infinite products, the proof does not work since $(\prod A_j)^\beta$ is not isomorphic with $\prod A_j^\beta$, in general.

**Lemma 2.25.** Let $A_j$ ($j \in J$) be a subordination algebra, with dual $X_j = (X_j, R_j)$. Then the dual of $\prod A_j$ is

$$\beta(X) = (\beta(\Sigma X_j), \Sigma R_j)$$

where $\Sigma$ is cardinal sum and $\beta$ Stone-Čech compactification.

**Proof.** The theorem is well known at the Boolean level. It remains to prove that the accessibility relation - let us denote it by $R^\beta$ - in the dual of $\prod A_j$ is the closure of $\Sigma R_j$. Since clearly $\Sigma R_j \subseteq R^\beta$, we have $(\Sigma R_j)^- \subseteq \Sigma R_j$. Let us prove the converse inclusion : suppose $x R^\beta y$ but $(x, y) \not\in (\Sigma R_j)^-$. There are clopen sets $U$ and $V$ in $\Sigma X_j$ with $(x, y) \in U \times V$ and $U \times V \cap \Sigma R_j = \emptyset$. We may suppose $A_j = \text{Clop}(X_j)$, so that the latter assertions mean $U \in x, V \in y$ and $V \not\subseteq \neg U$. By $\Pi$, $x R^\beta y$ implies $\neg U \subseteq x$, contradicting $U \subseteq x$. \[\square\]

**Lemma 2.26.** For any family $(A_j \mid j \in J)$ of subordination algebras, there is a unique onto morphism in $sCM$ $f : (\prod A_j)^\delta \rightarrow (\prod A_j)^\delta$ such that $\id \circ f = p$, where $\id$ is the identity map $\prod A_j \rightarrow (\prod A_j)^\delta$ and $p$ is the canonical weak embedding $\prod A_j \rightarrow \prod A_j^\delta$.

**Proof.** Each projection $p_j : \prod A_j \rightarrow A_j$ lifts to a morphism $p_j^\delta$ in $sCM$ by 2.22, $(\prod A_j)^\delta \rightarrow A_j^\delta$, and the product of the $p_j^\delta$s is the required morphism $f$. If the dual of $A_j$ is $X_j$, then by 2.25 we may write $(\prod A_j)^\delta = \mathcal{P}(\beta(\Sigma X_j))$, $\prod A_j^\delta = \mathcal{P}(\Sigma X_j)$ and $f$ is the map $E \mapsto E \cap \Sigma X_j$ which is clearly onto. \[\square\]
Definition 2.27. The map of Lemma 2.26 is the canonical epimorphism \( (\prod A_j)^\delta \to \prod A_j \delta \). Its restriction \( f^m \) to \( (\prod A_j)^m \) clearly takes its values in \( \prod A_j^m \). We call it the canonical morphism (in \( MA \)) \( (\prod A_j)^m \to \prod A_j^m \). We say that \( (A_j \mid j \in J) \) is a good family if the canonical map \( f^m \) is an embedding.

Proposition 2.28. Let \( (\Delta_j \mid j \in J) \) be a family of subordination algebras. Then \( A_j \models \varphi \) for all \( j \) implies \( \prod A_j \models \varphi \) for all modal formulas \( \varphi \) if and only if \( (\Delta_j \mid j \in J) \) is a good family.

Proof. The elements of \( (\prod A_j)^m \) are of the form \( \varphi(a_1, \ldots, a_n) \) for some modal formula \( \varphi \) and elements \( a_1, \ldots, a_n \) of \( \prod A_j \). Since \( f^m \) is a modal algebra homomorphism, then

\[
\varphi(a_1, \ldots, a_n) = ( \varphi(a_{1j}, \ldots, a_{nj}) \mid j \in J).
\]

And \( f^m \) is an embedding if and only if its 1-kernel is reduced to \( \{1\} \), that is, \( \varphi(a_1, \ldots, a_n) = 1 \) if and only if \( \varphi(a_{1j}, \ldots, a_{nj}) = 1 \) for all \( j \), for all \( \varphi \) and all \( a_1, \ldots, a_n \). Now \( \varphi(a_1, \ldots, a_n) = 1 \) means \( \prod A_j \models_v \varphi \) for the valuation \( v \) sending the variable \( p_i \) to \( a_i \), \( i = 1, \ldots, n \), and the proposition is proved.

Remark 2.29. Of course by using black modalisation or bimodalisation, we may adapt the proposition to black modal or bimodal formulas.

3 Correspondence theory

We now turn to the problem of correspondence theory. In our context of subordination algebra, several aspects may be studied.

The classical one is concerned with the translation of modal (or bimodal) equation on a subordination algebra into first-order properties of the accessibility relation on its dual. But another kind of correspondence arises when one wants to translate modal or bimodal equations on a subordination algebra into first-order properties in the language of subordination algebra (i.e., using the Boolean connective and the subordination \( \prec \)).

We will not consider translations from first-order properties of the subordination language into first order properties of the accessibility relation (see however [1] and [24] for instance), but give some results on the comparative expressivity of the three modal languages (white, black and bicolour).

We first begin with some specific examples of translation, to get the flavour of more general results.

Our examples will be given in the realm of bimodal formulas. Recall from [24] that \( \varphi(\overline{\psi}) \) is the scheme associated to the formula \( \varphi(\overline{\psi}) \) and that \( B \models \varphi(\overline{\psi}) \) means \( B \models \varphi(\overline{\psi}) \) for every tuple \( \overline{\psi} \). We begin by an idyllic example.

Example 3.1. For a subordination algebra \( B \) with dual X and for any \( k, l, m, n \in \mathbb{N} \), the following are equivalent:

1. \( B \models \Diamond^k \Box^l p \rightarrow \Box^m \Diamond^n p \),
2. \( B \models \Diamond^k \Diamond^l p \rightarrow \Diamond^m \Diamond^n p \),
3. \( B \models \Diamond^k \Box^l \psi \rightarrow \Box^m \Diamond^n \psi \),
4. \( B \models \Diamond^k \Diamond^l \psi \rightarrow \Diamond^m \Diamond^n \psi \),
5. \( X \models (x R^k y \text{ and } x R^m z) \to (\exists u)(y R^l u \text{ and } z R^m u) \)

6. \( B \models (\neg b \not\sim \neg a \text{ and } a \not\sim\neg c) \to (\exists d)(b \not\sim^k d \text{ and } c \not\sim^m d) \),

where \( a \not\sim b \) is a shortcut for \( a \not\sim \neg b \).

**Proof.** Of course, 3. \( \Rightarrow \) 1. and 4. \( \Rightarrow \) 2.

We know consider the following sequence of equivalences, in which \( A, B, C \) and \( D \) are clopen:

(a) \( R^k(-, \neg R^l(-, \neg A)) \subseteq \neg R^m(-, \neg R^n(-, A)) \),

(b) \( R^k(-, \neg R^l(-, \neg A)) \cap R^m(-, \neg R^n(-, A)) = \emptyset \),

(c) For all \( B \) and \( C \) such that \( B \subseteq \neg R^l(-, \neg A) \) and \( C \subseteq \neg R^n(-, A) \), \( R^k(-, B) \cap R^m(-, C) = \emptyset \),

(d) For all \( B \) and \( C \) such that \( B \subseteq \neg R^l(-, \neg A) \) and \( C \subseteq \neg R^n(-, A) \), there exists \( D \) such that \( R^k(-, B) \subseteq D \) and \( R^m(-, C) \subseteq \neg D \),

(e) For all \( B \) and \( C \), \( R^k(-, \neg A) \subseteq \neg B \) and \( C \times A \cap R^m = \emptyset \) imply there exists \( D \) such that \( R^k(-, B) \subseteq D \) and \( D \times C \cap R^m = \emptyset \).

This sequence shows that 1. (i.e. (a)) and 6. (i.e. (e)) are equivalent.

We now prove 6. \( \Rightarrow \) 5. Suppose \( x R^k y \) and \( x R^m z \) while for no \( u, y R^l u \) and \( z R^m u \). Then \( R^l(y, -) \cap R^m(z, -) = \emptyset \). So there is \( A \) with \( R^l(y, -) \subseteq \neg A \) and \( R^m(z, -) \subseteq A \). And there \( B \supseteq y \) and \( C \supseteq z \) such that \( R^l(B, -) \subseteq \neg A \) and \( R^m(z, -) \subseteq A \), in other words with \( B \subseteq \neg R^l(-, A) \) and \( C \subseteq \neg R^n(-, A) \). By (d) (which is equivalent to 6.), there is \( D \) with \( R^k(-, B) \subseteq D \) and \( R^m(-, C) \subseteq \neg D \). Since \( x R^k y \), we have \( x \in D \) and since \( y R^m z \), we have \( x \in \neg D \), which is impossible.

Finally, by modal logic (see [8]), 5. is equivalent to

\[ X \models \Diamond^k \Box^l p \to \Box^m \Diamond^n p \]

that is to

\[ B^\delta \models \Diamond^k \Box^l p \to \Box^m \Diamond^n p \]

whence to

\[ B^\delta \text{ schm} \models \Diamond^k \Box^l \psi \to \Box^m \Diamond^n \psi \]

which implies 3..

We have proved 3. \( \Rightarrow \) 1. \( \Rightarrow \) 6. \( \Rightarrow \) 5. \( \Rightarrow \) 3.. One proves 4. \( \Rightarrow \) 2. \( \Rightarrow \) 6. \( \Rightarrow \) 5. \( \Rightarrow \) 4. in a similar way.

In the example, the characterisation of modal formulas in term of the accessibility relation is exactly the same as in the purely modal case. We give a two variable example of this phenomenon (without proof since this example, as well as in 3.11 is taken into account in Theorems 3.5 and 3.13).

**Example 3.2.** For a subordination algebra \( B \) with dual \( X = (X, R) \) the following are equivalent:

1. \( B \models \Box(\Box p \to q) \lor \Box(q \to p) \),
2. \((X, R) \models (x R y \text{ and } x R z) \rightarrow (y R z \text{ or } z R y)\),

3. \(B \models (a \perp b \text{ and } b \perp a) \rightarrow ((\exists c)(a < c \text{ and } b \perp c))\).

We now give an analogue of Sahlqvist theorem, that is, give a set of modal formulas that are first-order expressible in a uniform way. In particular, those formulas are subordination canonical, in a natural sense given in 3.7. The obtained set of Sahlqvist formulas for subordination algebras is definitely smaller than the set of Sahlqvist formulas for modal algebras. This is justified by the fact that there exists (see 3.9) Sahlqvist formulas which are not subordination canonical.

**Definition 3.3.** A bimodal formula \(\phi\) is **closed** (resp. **open**) if it is obtained from constants \(\top, \bot\), propositional variables and their negations, by applying \(\lor, \land, \lozenge\) and \(\blacksquare\) (resp. \(\lor, \land, \boxdot\) and \(\blacklozenge\)).

A bimodal formula \(\phi\) is **positive** (resp. **negative**) if it is obtained from constants \(\top, \bot\) and propositional variables (resp. and negations of propositional variables) by applying \(\land, \lor, \lozenge, \square\) and \(\blacksquare\) (resp. \(\lor, \land, \lozenge, \boxdot\)).

A bimodal formula \(\phi\) is **s-positive** (resp. **s-negative**) if it is obtained from closed positive formulas (resp. open negative formulas) by applying \(\lor, \land, \square\) and \(\blacksquare\) (resp. \(\lor, \land, \lozenge\)).

A bimodal formula \(\phi\) is **g-closed** (resp. **g-open**) (g for generalized) if it is obtained from closed (resp. open) formulas by applying \(\lor, \land, \square\) and \(\blacksquare\) (resp. \(\lor, \land, \lozenge\)).

To obtain the analogue of Sahlqvist result, we need two more ingredients.

**Definition 3.4.** A **strongly positive** bimodal formula is conjunction of formulas of the form \(\square\langle \mu \rangle p := \square \mu_1 \blacksquare \mu_2 \ldots \square \mu_k p\), where \(p \in \text{Var}\) and \(\mu \in \mathbb{N}^k\) for some \(k \in \mathbb{N}\).

A **s-untied** bimodal formula is a formula obtained from strongly positive and s-negative formulas by applying only \(\land, \lozenge\) and \(\blacksquare\).

Finally, a formula \(\varphi\) is said to be **s-Sahlqvist** if of the form \(\varphi = \square^{(\mu)}(\varphi_1 \rightarrow \varphi_2)\) where \(\varphi_1\) is s-untied and \(\varphi_2\) s-positive. By definition any s-Sahlqvist formula \(\varphi\) is a Sahlqvist formula and by Sahlqvist’s theorem ([21], [23], adapted for bimodal formulas in [9]), there is a first order formula \(f(\varphi)\) in the language of a binary relation such that for any bimodal algebra \(B\) with dual \(X\), \(B \models \varphi\) if and only if \(X \models f(\varphi)\).

**Theorem 3.5.** Let \(\varphi\) be a s-Sahlqvist bimodal formula and let \(f(\varphi)\) be its associated first-order formula as defined in Definition 3.4. Then for any subordination algebra \(B\) with dual \(X\), we have
\[B \models \varphi \text{ if and only if } X \models f(\varphi).\]

**Proof.** We prove Sahlqvist theorem in the generalised context of subordination algebras simply by following the topological proof of Sambin and Vacca in [23]. In almost all places only the closedness of the accessibility relation in needed. The only place where the extra assumption that \(R(O, -)\) is open when \(O\) is open is necessary is the intersection lemma. This explains our definition of s-Sahlqvist formulas. The intersection lemma we use is then the following. \(\Box\)
Lemma 3.6 (Intersection lemma). Let $\varphi(p_1, ..., p_k)$ be a s-positive bimodal formula and $X$ a subordination space. For every $A \subseteq X$ and for every $C_1, ..., C_{k-1}$ closed sets of $X$

$$\varphi(C_1, ..., \text{cl}(A), ..., C_{k-1}) = \cap\{\varphi(C_1, ..., O, ..., C_{k-1}) \mid A \subseteq O \in \text{Clop}(X)\},$$

where $\text{cl}(A)$ denotes the topological closure of $A$.

Proof. The proof is done by induction on the complexity of $\varphi$. We note that, $\varphi$ being s-positive, $\varphi \equiv \Diamond \psi$ or $\Diamond \psi$ implies that $\psi$ does not contain any $\Box$ or $\lozenge$. Hence, $(\psi(C_1, ..., O, ..., C_{k-1}) \mid A \subseteq O \in \text{Clop}(X))$ is a filtered family of closed sets. This allows us to use the Esakia’s Lemma as in [23] and conclude the proof.

Definition 3.7. A bimodal formula $\varphi$ is s-canonical if $B \models \varphi$ implies $B^m \models \varphi$ for any subordination algebra $B$. It is said to be scheme-extensible if $B \models \varphi$ implies $B^m \models \varphi$. Hence, being s-canonical implies being scheme-extensible (since $B^m$ is a subalgebra of $B^\delta$).

Corollary 3.8. Any s-Sahlqvist bimodal formula is s-canonical, hence scheme-extensible.

Proof. Let $\varphi$ be an s-Sahlqvist formula. Then $\varphi$ is a Sahlqvist formula and $(X, R) \models \varphi$ if and only if $(X, R) \models f(\varphi)$ for any Kripke frame $(X, R)$. Therefore, $B \models \varphi$ if and only if $X \models f(\varphi)$ (if $X = (X, R, \tau)$ is the dual of $B$) by Theorem 3.5. The latter being equivalent to $(X, R) \models \varphi$ if and only if $B^m \models \varphi$.

Example 3.9. Let us have a look at the formula

$$\varphi \equiv p \rightarrow \Diamond \Box p,$$

already examined in Example [23]. It is a Sahlqvist formula, but not a s-Sahlqvist formula. On modal algebras, it is equivalent to the formula

$$f(\varphi) \equiv (\forall x)(\exists y)(x R y \text{ and } R(y, -) \subseteq \{x\}).$$

This fact is no longer true on subordination algebras: the subordination space of Example [23] satisfies $\varphi$ but not $f(\varphi)$.

Finally, the formula $\varphi$ is not scheme-extensible, as shown in [23] hence $\varphi$ is an example of canonical formula which is not s-canonical.

We now study our second kind of correspondence theory, namely the translation of a bimodal formula into the subordination algebra language. Examples of such translations have already been given in Examples 3.1 and 3.2. As promised, we generalise these examples in the next results.

Definition 3.10. A bimodal formula $\varphi = \varphi(p)$ is said to be s-definable (resp. $\leq$-definable; $\geq$-definable) if there is an effectively produced first order formula $\xi = \xi(\varphi) = \xi(p)$ (resp. $\xi_\leq = \xi_\leq(\varphi) = \xi_\leq(p, q)$ and $\xi_\geq = \xi_\geq(\varphi) = \xi_\geq(p, q)$) such that for any subordination algebra $B$ and any valuation $v : \text{Var} \rightarrow B$, one has:
Here again, we use induction to conclude.

Clearly, if $\phi$ is $\geq$-definable (resp. $\leq$-definable), then $\phi$ (resp. $\neg\phi$) is s-definable. Also $\phi$ is $\geq$-definable if and only if $\neg\phi$ is $\leq$-definable.

**Theorem 3.11.** If $\phi$ is an open or a closed formula, then both $\phi$ and $\neg\phi$ are both $\leq$ and $\geq$-definable.

**Proof.** We begin by the following general remark, that will help to facilitate computation. We may assume that our working subordination algebra is $B = \text{Clop}(X)$ where $X = (X, R)$ is the dual of $B = (B, \prec)$. Under a valuation $v$, variables $p$ and their negations $\neg p$ are therefore clopen subsets of $X$ and more generally, formulas are subsets of $X$. Also, $\Box \phi = \neg R(\neg, \neg\phi)$, $\square \phi = \neg R(\neg\phi, -)$, $\Diamond \phi = R(-, \phi)$ and $\lozenge \phi = R(\phi, -)$. On the subordination side, remember that $\phi \times \psi$ is equivalent to each of the following: $R(-, \phi) \subseteq \psi$; $\neg \psi \times \phi \cap R = 0$ and $\phi \subseteq \neg R(\neg\psi, -)$. Hence, each of these expressions, when restricted to clopen subsets $p^+$ (see the second point of Definition 3.10) of $X$, corresponds to an atomic formula in the first order language of subordination algebra. Finally, we make use of the following topological remarks, in which $A \subseteq X$, $O$ is an open subset and $F$ closed subset of $X$:

1. $O \subseteq A$ if and only if for all variables $p$, $p \subseteq O$ implies $p \subseteq A$,
2. $A \subseteq F$ if and only if for all variables $p$, $F \subseteq p$ implies $A \subseteq p$,
3. $F \subseteq O$ if and only if for some variable $p$, $F \subseteq p \subseteq O$,
4. $R(-, F) \subseteq O$ if and only if for some variables $p, q$, one has $F \subseteq p$, $q \subseteq O$ and $R(-, p) \subseteq q$.

We are ready for the proof, that is done by induction on the complexity of $\phi$. We only consider the case where $\phi$ is open since $\phi$ is closed if and only if $\neg\phi$ is open.

If $\phi$ is a constant, a variable or the negation of a variable, the result is clear by our beginning remark.

Suppose now $\phi \equiv \theta \lor \psi$. Then $\phi \rightarrow q^\pm$ is equivalent to $(\theta \rightarrow q^\pm) \land (\psi \rightarrow q^\pm)$ and the result follows by the induction hypothesis. Also, $q^\pm \rightarrow \theta \lor \psi$ is equivalent to

$$(\exists r, s)((r \rightarrow \theta) \land (s \rightarrow \psi) \land (q^\pm \rightarrow r \lor s)).$$

Here again, we use induction to conclude.

If $\phi \equiv \theta \land \psi$, then $q^\pm \rightarrow \phi$ is equivalent to $(q^\pm \rightarrow \theta) \land (q^\pm \rightarrow \psi)$ while $\phi \rightarrow q^\pm$ is equivalent to

$$(\forall r)(((r \rightarrow \theta) \land (r \rightarrow \psi)) \rightarrow (r \rightarrow q^\pm)).$$

Finally, we consider the case $\phi \equiv \Box \psi$. Then $\phi \rightarrow q^\pm$ is equivalent to

$$(\forall r)(r \subseteq \neg R(-, \neg\psi) \rightarrow r \subseteq q^\pm),$$

21
which is in turn equivalent to

\((\forall r)((\exists s)((\neg \psi \subseteq s \text{ and } R(-, s) \subseteq -r) \rightarrow (r \rightarrow q^\pm))\).

And \(q^\pm \rightarrow \Box \psi\) is equivalent to \(q^\pm \subseteq -R(-, -\psi)\), that is \(R(-, -\psi) \rightarrow q^\mp\), which is equivalent to

\((\exists r)(-\psi \subseteq r \text{ and } R(-, r) \subseteq q^\mp)\).

\[\Box\]

We now need an analogue of the intersection lemma.

**Lemma 3.12.** If \(\varphi\) is positive open and \(\overline{\psi}\) is closed, then

\[\varphi(\overline{\psi}) = \cap \{\varphi(\overline{\mathfrak{p}}) \mid \mathfrak{p} \text{ open and } \mathfrak{p} \geq \overline{\psi}\}\].

**Proof.** Since \(\varphi\) is positive, we have \(\varphi(\overline{\psi}) \subseteq \cap \{\varphi(\overline{\mathfrak{p}}) \mid \mathfrak{p} \geq \overline{\psi}\}\) and we prove the opposite inclusion \(\supseteq\) by induction on the complexity of \(\varphi\).

This is clear when \(\varphi\) is a variable, because \(\overline{\psi}\) is closed.

Consider the case \(\varphi \equiv \xi \lor \theta\). If \(x \in \varphi(p)\) for all \(p \geq \overline{\psi}\) but \(x \notin \varphi(\overline{\psi})\), then \(x \notin \xi(\overline{\psi})\) and \(x \notin \theta(\overline{\psi})\). By induction, there \(\overline{\mathfrak{p}} \geq \overline{\psi}\) with \(x \notin \xi(\overline{\mathfrak{p}})\) and \(\overline{\mathfrak{q}} \geq \overline{\psi}\) with \(x \notin \theta(\overline{\mathfrak{q}})\). Then \(\overline{\mathfrak{p}} \cap \overline{\mathfrak{q}}\) is clopen and \(\overline{\psi} \leq \overline{\mathfrak{p}} \cap \overline{\mathfrak{q}}\) so that

\(x \in \varphi(\overline{\mathfrak{p}} \cap \overline{\mathfrak{q}}) = \xi(\overline{\mathfrak{p}} \cap \overline{\mathfrak{q}}) \cap \theta(\overline{\mathfrak{p}} \cap \overline{\mathfrak{q}}) \subseteq \xi(\overline{\mathfrak{p}}) \cap \theta(\overline{\mathfrak{q}})\),

a contradiction.

If \(\varphi \equiv \xi \land \theta\), then

\[\varphi(\overline{\psi}) = \xi(\overline{\psi}) \cap \theta(\overline{\psi}) = \cap \{\xi(\overline{\mathfrak{p}}) \mid \mathfrak{p} \geq \overline{\psi}\} \cap \{\theta(\overline{\mathfrak{p}}) \mid \mathfrak{p} \geq \overline{\psi}\} = \cap \{\varphi(\overline{\mathfrak{p}}) \mid \mathfrak{p} \geq \overline{\psi}\}\].

Finally, suppose \(\varphi \equiv \Box \theta\). Then,

\[\varphi(\overline{\psi}) = -R(-, \neg \theta(\overline{\psi}))\]
\[= -R(-, \cup \{\neg \theta(\overline{\mathfrak{p}}) \mid \mathfrak{p} \geq \overline{\psi}\})\]
\[= \cap \{-R(-, \neg \theta(\overline{\mathfrak{p}}) \mid \mathfrak{p} \geq \overline{\psi}\}\}
\[= \cap \{\varphi(\overline{\mathfrak{p}}) \mid \mathfrak{p} \geq \overline{\psi}\}\]

as required. \[\Box\]

**Theorem 3.13.** If \(\xi\) is a g-closed formula, then \(\xi\) is \(\geq\)-definable, hence s-definable.

**Proof.** If \(\xi\) is g-closed, there is a positive open formula \(\varphi\) and a tuple of closed formulas \(\overline{\psi}\) such that \(\xi = \varphi(\overline{\psi})\). Then, the formula \(q^\pm \rightarrow \varphi(\overline{\psi})\) is equivalent, by Lemma 3.12 to

\(\forall \mathfrak{p} \geq \overline{\psi}, q^\pm \rightarrow \varphi(\overline{\mathfrak{p}})\).

And both formulas \(\mathfrak{p} \geq \overline{\psi}\) and \(q^\pm \rightarrow \varphi(\overline{\mathfrak{p}})\) are s-definable by Theorem 3.11. \[\Box\]

As announced, we now compare the three modal languages one with another. The comparison is first done semantically by establishing analogues of Birkhoff’s characterisation of varieties for each modal language. Specific examples are then derived.
In universal algebra, Birkhoff theorem is twofold. First, a characterisation of those sets of identities which are true in a class of algebras in term of a provability system. And then, a characterisation of those classes of algebras that satisfy some set of identities in terms of semantic constructs.

In modal algebra, where identities may be assimilated to formulas, our provability system always gives a normal modal logic as set of theorems and as discussed in Definition 2.2 this is not always the case for the logic of a class $K$ of subordination algebras. So our first result will be a criterion to ensure that $\text{Log}(K)$ is a normal modal logic and, then, give a characterisation of those classes of subordination algebras that satisfy some normal modal logic.

Proposition 3.14. If $K$ is a class of subordination algebras, then $L = \text{Log}(K)$ is a normal modal logic if and only if $B \in K$ implies $B^m \in K$.

Proof. Suppose $L = \text{Log}(K)$ is a normal modal logic and $B \in K$. Since $L$ may be axiomatized by schemes, this follows directly from Proposition 2.9.

Suppose now $B \in K$ implies $B^m \in K$. We have to prove that $L$ is closed under substitution, that is $\varphi \in L$ implies $\varphi(\psi) \in L$. Let $B \in K$. Then, $B^m \in K$ and so $B^m \models \varphi$, whence $B^m \models \varphi(\psi)$ as $B^m$ is a modal algebra, and it follows that $B \models \varphi(\psi)$ as proved in 2.9.

Theorem 3.15. Let $K$ be a class of subordination algebras. Then the following are equivalent:

1. $K = \text{mod}(L)$ for some modal normal logic $L$,
2. $K$ is definable by schemes of modal formulas,
3. $K$ is closed under subalgebras and morphic images (in $\text{Sub}$), products of good families and modalisations, and reflects modalisation, that is $B^m \in K$ implies $B \in K$,
4. $K$ is closed under subalgebras and morphic images (in $\text{Sub}$), and for any family $(B_i \mid i \in I)$, one has $\prod B^m_i \in K$ if and only if $\forall i \in I, B_i \in K$.

Proof. The equivalence 1. $\iff$ 2. is clear. Both implications 2. $\Rightarrow$ 3. and 2. $\Rightarrow$ 4. follows for 2.9, 2.13, 2.21 and 2.28.

Let us prove 3. $\Rightarrow$ 2. (one proves 4. $\Rightarrow$ 2. in a similar way). Let $M = \{B \in \text{MA} \mid B \in K\}$. Then $M$ is a class of modal algebras closed under $H,S$ and $P$ and is therefore an equational class by Birkhoff classical theorem. Let $L$ be an axiomatisation of $M$ by schemes. All we have to prove is $K = \text{mod}(L)$.

If $B \in \text{mod}(L)$, then $B^m \in \text{mod}(L)$ by Proposition 2.23, so that $B^m \in M \subseteq K$. Since $K$ reflects modalisation, it follows that $B \in K$. Conversely, if $B \in K$, then $B^m \in K$ and, being a modal algebra, $B^m \in M$. Hence, $B^m \in \text{mod}(L)$. It follows from 2.9 that $B \in \text{mod}(L)$.

Of course, there is a black and a bimodal version of this theorem. We only present the bimodal version. A family $(B_i \mid i \in I)$ of subordination algebras is said to be a $\textbf{good}$ if the canonical morphism $f^{\text{bim}}$ (in $\text{sSub}$) (the restriction $(\prod B_i)^{\text{bim}} \rightarrow \prod B^m_i$ of the canonical epimorphism $(\prod B_i)^{\text{bim}} \rightarrow \prod B^m_i$) is an embedding. Of course, this is stronger than being good.
Theorem 3.16. Let $K$ be a class of subordination algebras. Then, the following are equivalent:

1. $K = \text{mod}(L)$ for some bimodal tense logic $L$.
2. $K$ is closed under strong subalgebras, strong morphic images, product of s-good families, bimodalisations and reflects bimodalisation.

This leaves open the non-scheme versions of the two theorems.

Problems 3.17.

1. Characterise the sets of formulas of the form $\text{Log}(K)$ where $K$ is a class of subordination algebras in term of provability - and give the associated completeness theorem.

2. Characterise semantically the equational classes of subordinations algebras, that is, the classes $\text{mod}(L)$ where $L$ is an arbitrary set of modal formulas (not necessarily closed under substitution).

A third kind of correspondence can be realized within the realm of modal formulas, if we remember that, for unimodal formulas, three different languages may be adopted: the white language, the black one and the bicolour (bimodal) one. An example of this phenomenon is given in example 3.1: the bicolour formula $\Box p \rightarrow \Box \Box p$ is equivalent to the white formula $\Box \Box p \rightarrow p$, and to the black formula $\Box \Box p \rightarrow p$ (all are equivalent to the symmetry of $R$). At the theoretical level, everything is settled by the (white, black and bicolour) Birkhoff theorems 3.15 and 3.16, and we just give here some examples and counterexamples of correspondences between these three languages.

Example 3.18. Seriality ($D : \forall x \exists y : x R y$) is expressible in the bicolour language by $p \rightarrow \Box \Box p$, in the white language by $\Box p \rightarrow p$, but not in the black language.

Proof. The first two assertions are particular instance of 3.1. To prove the third one, it suffices by Corollary, to give a subordination space $X$ which is serial but admits a subobject in $\Box \text{SubS}$ which is not. This is easy: take $X = \{0, 1\}$ with $R = \{(0, 1), (1, 1)\}$ and take $\{0\}$ as subobject.

Example 3.19. The axiom $\Box \Box \Box p \rightarrow \Box p$ is not expressible by a unicolour axiom.

Proof. It is not difficult to see that the mentioned axiom correspond to the first order property

$$x R y, z R y, z R u \rightarrow x R u.$$  \hspace{1cm} (10)

Let $X = \langle X, R \rangle$ be the subordination space with $X = \{a, b, c, d, e\}$ and $R = \{(a, b), (c, d), (c, e)\}$. Then, $X \models (10)$ (vacuously). Now, the equivalence $\theta$ generated by $\{(b, d)\}$ is a $\text{SubS}$-congruence. The quotient $X/\theta$ is $(X/\theta, R/\theta)$ where $X/\theta = \{a^\theta, b^\theta = d^\theta, e^\theta\}$ and $R/\theta = \{(a^\theta, b^\theta), (e^\theta, b^\theta), (e^\theta, c^\theta)\}$ and clearly $X/\theta \not\models (10)$. This shows that axiom (10) is not expressible in the white language. One prove in a similar way (consider $(X, R^\theta)$) that axiom (10) is not expressible in the black language.

Example 3.20. The scheme $\psi \rightarrow \Box \Box \psi$ has the black equivalent $\psi \rightarrow \Box \Box \psi$ and has the first order correspondent

$$\forall x \exists y : x R y \text{ and } R(y, -) = \{x\}.$$  \hspace{1cm} (11)
Proof. Since we have a defining scheme, the class of models of the given axiom is closed under modalisation: if $B \models \psi \rightarrow \diamond \Box \psi$, then $B^m \models \psi \rightarrow \diamond \Box \psi$, so that $X^m \models \Box$ (where $X^m$ denotes the dual of $B^m$) by modal logic.

References

[1] Balbiani, P., Kikot, S.: Sahlqvist theorems for precontact logics. Advances in modal logic 9, 55–70 (2012)

[2] Bezhanishvili, G., Bezhanishvili, N. and Harding, J.: Modal compact Hausdorff spaces. Journal of Logic and Computation 25(1), 1–35 (2015)

[3] Bezhanishvili, G., Bezhanishvili, N., Sourabh, S. and Venema, Y.:Irreducible Equivalence Relations, Gleason Spaces, and de Vries Duality. Appl Categ Struct 25, 381–401 (2017)

[4] Burgess, J.P.: Handbook of Philosophical Logic. 7. 2nd ed. Cambridge Studies in Advanced Mathematics, vol.3. University Press, Cambridge (1982)

[5] Celani, S.: Quasi-modal algebras. Mathematica Bohemica 126(4),721–736 (2001)

[6] Celani, S.: Subdirectly irreducible quasi-modal algebras. Acta Math. Univ. Comenian. 74, 219 –228 (2005)

[7] Chagrov, A., Zakharyaschev M.: Modal Logic. Oxford logic guides, vol.35. Clarendon Press, Oxford (1997)

[8] Chellas, B.: Modal Logic : An introduction. Cambridge University Press, Cambridge (1980).

[9] de Rijke, M., Venema, Y.: Sahlqvist’s theorem for Boolean algebras with operators with an application to cylindric algebras. Studia Logica 54, 61–78 (1995)

[10] de Vries, H.: Compact spaces and compactifications. An algebraic approach. Ph. D. thesis, Amsterdam (1962)

[11] Dimov, G., Vakarelov, D.: Contact Algebras and Region-based Theory of Space: A Proximity Approach - I.Fundamenta Informaticae 74, 209–249 (2006)

[12] Dimov, G., Vakarelov, D.: Topological representation of precontact algebras. MacCaull, Winter, and Düntsch, editors, Relational Methods in Computer Science, Lecture notes in in Computer Science 3929,1–16 (2006)

[13] Düntsch, I., Vakarelov, D.: Region based theory of discrete spaces: A proximity approach. Ann. Math. Artif. Intell. 49, 5–14 (2007)

[14] Düntsch, I., Winter, M.: A representation theorem for Boolean contact algebras. Theoretical Computer Science 347, 492–512 (2005)

[15] Esakia, L.:Topological Kripke models. Soviet Math. Dokl. 15,147–151 (1974)
[16] Gerhke, M., Jönsson, B.: Bounded distributive lattice expansions. Mathematica Scandinavica 94, 13–45 (2004)

[17] Gierz, G., Hofmann, K., Keimel, K., Lawson, J., Mislove, M. and Scott, D.: Continuous lattices and domains. Encyclopedia of Mathematics and its Applications, vol. 93. Cambridge University Press, Cambridge (2003)

[18] Koppelberg, S., Düntsch, I. and Winter, M.: Remarks on contact relations on Boolean algebras. Algebra Universalis 68, 353–366 (2012)

[19] Prior, A.: Past, Present and Future. Oxford University Press, Oxford (1967)

[20] Rescher, N., Urquhart, A.: Temporal Logic. Library of Exact Philosophy, vol. 3. Springer-Verlag, New York (1971)

[21] Sahlqvist, H.: Completeness and correspondence in the first and second order semantics for modal logic. Kanger editor, Proceedings of the third Scandinavian logic symposium, 110–143. North-Holland, Amsterdam (1975)

[22] Sambin, G., Vaccaro, V.: Topology and duality in modal logic. Annals of Pure and Applied Logic 37, 249–296 (1988)

[23] Sambin, G., Vaccaro, V.: A new proof of Sahlqvist’s theorem on modal definability and completeness. The Journal of Symbolic Logic 54(3), 992–999 (1989)

[24] Santoli, T.: Logics for Compact Hausdorff Spaces via de Vries Duality. Ph. D. thesis, Amsterdam (2016)

[25] Unterholzner, P.: Algebraic and relational semantics for tense logics. Rend. Sem. Mat. Univ. Padova 65, 119–128 (1981)

[26] Vakarelov, D.: Region-based theory of space: algebras of regions, representation theory and logics. In: Gabbay, D., Goncharov, S. and Zakharyaschev, M. (eds) Mathematical problems from Applied logic II. Logics for the XXIst century, 267-348. Springer, New-York (2007)

[27] Venema, Y.: Algebras and Co-algebras. In: Blackburn, P., Van Benthem, J. and Wolter, F. (eds.) Handbook of Modal Logic. Studies in logic and practical reasoning, vol. 3, 331–426. Elsevier, Amsterdam (2007)