A SUBLUMINOUS SCHRÖDINGER EQUATION

Philip Rosenau and Zeev Schuss*

July 13, 2010

*Department of Applied Mathematics, Tel-Aviv University, Tel-Aviv 69978, Israel, e-mail: rosenau@post.tau.ac.il,schuss@post.tau.ac.il
Abstract

The standard derivation of Schrödinger’s equation from a Lorentz-invariant Feynman path integral consists in taking first the limit of infinite speed of light and then the limit of short time slice. In this order of limits the light cone of the path integral disappears, giving rise to an instantaneous spread of the wave function to the entire space. We ascribe the failure of Schrödinger’s equation to retain the light cone of the path integral to the very nature of the limiting process: it is a regular expansion of a singular approximation problem, because the boundary conditions of the path integral on the light cone are lost in this limit. We propose a distinguished limit, which produces an intermediate model between non-relativistic and relativistic quantum mechanics: it produces Schrödinger’s equation and preserves the zero boundary conditions on and outside the original light cone of the path integral. These boundary conditions relieve the Schrödinger equation of several annoying, seemingly unrelated unphysical artifacts, including non-analytic wave functions, spontaneous appearance of discontinuities, non-existence of moments when the initial wave function has a jump discontinuity (e.g., a collapsed wave function after a measurement), the EPR paradox, and so on. The practical implications of the present formulation are yet to be seen.

Keywords: Lorentz-invariant path integral, subluminous propagation, boundary conditions

PACS numbers 03.65.-w, 03.65.Ta

1 Introduction

The original derivation of Schrödinger’s equation from Feynman’s path integral postulates a classical non-Lorentz-invariant action [1], [2], [3], [4]. The Lorentz-invariant (discrete) Feynman path integral [5], which propagates within the light cone emanating from the support of the initial wave function, is usually approximated by expanding the Lorentz-invariant action in powers of $\dot{x}/c$ first ($c$ is the speed of light) and then expanding the path integral in powers of the time slice $\Delta t$. In this approximation the light-cone disappears and Schrödinger’s equation in the entire space is recovered, as in the non-relativistic case (see details below). If the order of limits is reversed,
the discrete path integral converges to the initial wave function and does not propagate. This anomaly in the approximation of the path integral can be ascribed to the observation that the aforementioned power series is a regular expansion of a two-parameter singular perturbation problem. Therefore, the construction of a uniform asymptotic approximation to the Lorentz-invariant Feynman path integral calls for a singular perturbation approach.

In this paper we construct a uniform asymptotic approximation to the Lorentz-invariant Feynman path integral by identifying a distinguished limit of large $c$ and small time slice $\Delta t$, such that $c\Delta t \to 0$, but $c\sqrt{\Delta t} \to \infty$. More specifically, as the speed of light $c$ is constant, and explicitly present in the problem, the time slice $\Delta t$ cannot be assumed arbitrarily small, as explained above. In fact, the distinguished limit requires that $\Delta t$ be bounded below by $\Delta t \gg \frac{\hbar}{mc^2} \approx 6.2 \times 10^{-22}$ sec (for an electron). In this distinguished limit the leading order approximation to the path integral is a solution of the initial value problem for Schrödinger’s equation with zero boundary conditions on and outside the light-cone emanating from the support of the initial wave function (see Figure 1). This subluminous Schrödinger equation can be considered an intermediate model between non-relativistic and relativistic quantum mechanics and it is valid for the above mentioned time resolution. We stress, however, that it is not our aim to construct a comprehensive relativistic quantum mechanics theory (in, say, the sense of [6]).

Notably, the emergence of the light cone relieves Schrödinger’s equation of several annoying, seemingly unrelated mathematical and unphysical artifacts. These include non-analytic wave functions, spontaneous appearance of discontinuities [7], non-existence of moments when the initial wave function has a jump discontinuity (e.g., a collapsed wave function after a measurement) [8], the EPR paradox [9] (relativistic quantum mechanics theory not withstanding), and so on. On the other hand, conventional quantum mechanics is recovered inside the light cone after an appropriate relativistic delay. The practical implications of all this are yet to be seen.

2 Superluminous propagation

The Feynman path integral formulation of quantum mechanics in one dimension postulates that the propagation of a free particle is defined by the action
functional

$$S(x(t)) = \int_0^t \frac{m x^2(t)}{2} dt \quad (2.1)$$

as a convolution with the propagator

$$\exp \left\{ \frac{i}{\hbar} S(x(t)) \right\}. \quad (2.2)$$

This means that the propagation of the time-sliced wave function at time \( t + \Delta t \) is given by [4]

$$\psi(x, t + \Delta t) = \sqrt{\frac{m}{2\pi i\hbar \Delta t}} \int_{-\infty}^{\infty} \exp \left\{ \frac{i m (x - y)^2}{2 \Delta t} \right\} \psi(y, t) dy. \quad (2.3)$$

The limit \( \Delta t \to 0 \) is found [1], [4] by changing the variable of integration to \( y = x - z \sqrt{\hbar \Delta t/m} \), which converts (2.3) to

$$\psi(x, t + \Delta t) = \sqrt{\frac{1}{2\pi i}} \int_{-\infty}^{\infty} \exp \left\{ \frac{iz^2}{2} \right\} \psi \left( x - z \sqrt{\frac{\hbar \Delta t}{m}}, t \right) dz \quad (2.4)$$

$$= \sqrt{\frac{1}{2\pi i}} \int_{-\infty}^{\infty} \exp \left\{ \frac{iz^2}{2} \right\} \times$$

$$\left[ \psi(x, t) - z \sqrt{\frac{\hbar \Delta t}{m}} \psi_x(x, t) + \frac{1}{2} z^2 \frac{\hbar \Delta t}{m} \psi_{xx}(x, t) + \cdots \right] dz$$

$$= \psi(x, t) - \frac{i}{2} \frac{\hbar \Delta t}{m} \psi_{xx}(x, t) + O \left( \left( \frac{\hbar \Delta t}{m} \right)^2 \right),$$

hence Schrödinger’s equation follows.

If instead of the non-relativistic Lagrangian \( L(z) = z^2/2 \) in (2.1), we use the unique relativistic Lagrangian of a free particle in one dimension \( L(z) = -\sqrt{1 - z^2} \) [5], it leads to the Lorentz invariant action

$$S(x(t)) = \int_0^t mc^2 L \left( \frac{\dot{x}(t')}{c} \right) dt'.$$  \quad (2.5)
Now, the Lorentz invariant propagation is given by

\[ \psi(x, t + \Delta t) = N^{-1} \times \int_{-\infty}^{\infty} \exp \left\{ i \left[ 1 - \sqrt{1 - \left( \frac{x - y}{c \Delta t} \right)^2} \right] mc^2 \Delta t \frac{1}{\hbar} \right\} H \left[ c^2 \Delta t^2 - (x - y)^2 \right] \psi(y, t) \, dy, \tag{2.6} \]

where the normalization constant \( N \) is given by

\[ N = c \Delta t \int_{-\infty}^{\infty} \exp \left\{ \frac{imc^2 \Delta t}{\hbar} L \left( \frac{x - y}{c \Delta t} \right) \right\} H \left[ c^2 \Delta t^2 - (x - y)^2 \right] \, dy. \tag{2.7} \]

Note that modifying \( L(z) \) to \( L(z) = 1 - \sqrt{1 - z^2} \) does not change the integral equation (2.6). Obviously, the wave function \( \psi(x, t) \) defined by (2.6) vanishes on and outside the light cone.

Taking the first the limit \( c \to \infty \) converts (2.6) to (2.3), and then the limit \( \Delta t \to 0 \), as in (2.4), recovers the Schrödinger equation in the entire space and the zero boundary conditions on light cone disappear and so does the light cone. Evidently, the wave function (2.3) spreads to the entire line instantaneously, thus rendering Schrödinger’s equation superluminous. Taking the limits in the reverse order does not lead to a wave equation.

The decay of the integral in (2.3) for large \( |x| \), similarly to the Fourier transform, is determined by the regularity of \( \psi(y, t) \). Thus jump discontinuities in \( \psi(y, t) \) lead to a decay of \( \psi(x, t + \Delta t) \) as \( |x|^{-1} \), which prevents the existence of moments, energy, and other artifacts [10].

### 3 Subluminous propagation

From the mathematical point of view, the approximation of the integral (2.6) by taking limits in the order described in Section 2 is a regular expansion of a singular approximation problem. The singularity of this expansion is manifested in the loss of the boundary conditions on the light cone. We propose here a distinguished limit that removes the singularity and the ensuing artifacts. Specifically, we consider a general Lorentz-invariant action functional (2.5), where the Lagrangian \( L(z) \) is an analytic function near the origin with

\[ L(0) = 0, \quad L'(0) = 0, \quad L''(0) = 1. \tag{3.1} \]
The assumption that the action (2.5) is Lorentz-invariant implies that the
time-sliced wave function it defines cannot propagate outside the light cone
emanating from the support of the initial wave function. We postulate, as
above, the integral equation

$$
\psi(x, t + \Delta t) = N^{-1} \times \int_{-\infty}^{\infty} \exp \left\{ \frac{imc^2\Delta t}{\hbar} \mathcal{L} \left( \frac{x - y}{c\Delta t} \right) \right\} H \left[ c^2 \Delta t^2 - (x - y)^2 \right] \psi(y, t) \, dy,
$$

with the normalization factor $N$ given by

$$
N = c\Delta t \int_{-\infty}^{\infty} \exp \left\{ \frac{imc^2\Delta t}{\hbar} \mathcal{L} \left( \frac{x - y}{c\Delta t} \right) \right\} H \left[ c^2 \Delta t^2 - (x - y)^2 \right] \, dy
= c\Delta t \int_{-1}^{1} \exp \{ i\mathcal{L}(\xi) \} \, dz.
$$

where

$$
\xi = \frac{mc^2\Delta t}{\hbar}.
$$

To investigate the behavior of $\psi(x, t)$ on the boundary of the light cone
emanating from the initial support, we rewrite (3.2) in the explicit form

$$
\psi(x, t + \Delta t) = N^{-1} \int_{x-c\Delta t}^{x+c\Delta t} \exp \left\{ \frac{imc^2\Delta t}{\hbar} \mathcal{L} \left( \frac{x - y}{c\Delta t} \right) \right\} \psi(y, t) \, dy.
$$

If $x_0$ is a point on the boundary of the initial support, we assume the light
cone emanating from it at time $\Delta t$ is $x = x_0 - c\Delta t$. At this point (3.5) is

$$
\psi(x_0 - c\Delta t, \Delta t) = N^{-1} \times \int_{x_0 - 2c\Delta t}^{x_0} \exp \left\{ \frac{imc^2\Delta t}{\hbar} \mathcal{L} \left( \frac{x_0 - c\Delta t - y}{c\Delta t} \right) \right\} \psi(y, 0) \, dy = 0,
$$
because the entire (open) interval of integration, \((x_0 - 2c\Delta t, x_0)\), is outside the support of \(\psi(y, 0)\). Note that \(\psi(x_0 - c\Delta t, \Delta t) = 0\), even if \(\psi(x_0, 0) \neq 0\). This boundary condition persists as long as the line \(x = x_0 - ct\) is on the light cone (see Figure 1). Thus, if \((a, b)\) is a finite interval on the \(x\)-axis outside the support of \(\psi(x, 0)\), then the wave function \(\psi(x, t)\) vanishes in the triangle in the \((x, t)\) plane, whose vertices are the points \([a, 0]\), \([b, 0]\), and \([\frac{a+b}{2}, \frac{b-a}{2c}]\). This implies, for example, that if in the two slits experiment \((a, b)\) is the interval separating the two slits, there will be no interference pattern between the slits before time \((b - a)/2c\). Furthermore, the influence on the wave function of every finite interval in the support of the initial wave function is confined to the light cone emanating from the interval (see Figure 1).

It is easily seen that the solution \(\psi(x, t)\) of (3.5) is Lorentz-invariant on the discrete time grid and it is supported on the light cone emanating from the initial support.

Figure 1: The thick line is the initial wave function, normalized in \(L^2(\mathbb{R})\). The light cone for \(c = 1\) and \(t < 2\) is the domain above the \(x\)-axis and above the black region in the \((x, t)\) plane.
3.1 The distinguished limit

Equations (3.3) and (3.4) suggest the distinguished limit of large $c$ and small $\Delta t$ such that $c \Delta t \to 0$, but $c \sqrt{\Delta t} \to \infty$ (in appropriate dimensionless units).

In more precise mathematical terminology the meaning of this distinguished limit is the assumption that $\xi \to \infty$, or more specifically,

$$\frac{\hbar}{mc^2} \ll \Delta t \ll \frac{L^2 m}{\hbar}, \quad (3.7)$$

where $L$ is a characteristic length, e.g., the width of the initial support of the wave function (the lower bound is due to the presence of $c$, and the corresponding characteristic time and the upper is needed for the approximation of a finite difference quotient by a derivative, see below).

For a point $(x, t)$ inside the light cone emanating from the initial support, we change the variable of integration to $y = x - z c \Delta t$ and write (3.2) as

$$\psi(x, t + \Delta t) = N^{-1} c \Delta t \times \int_{-1}^{1} \exp \left\{ i L(z) \frac{mc^2 \Delta t}{\hbar} \right\} \psi(x - c \Delta t (1 + z), t) \, dz. \quad (3.8)$$

If the light cone containing $(x, t)$ is the interval $x_0 - ct < x < x_0 + L + ct$, we assume that $x_0 - ct + \varepsilon = x < x_0 + L + ct$ for some positive $\varepsilon$ independent of $\Delta t$. Then integration in (3.8) extends over the interval $-1 < z < -1 + \varepsilon/c \Delta t$, which contains the entire interval $-1 < z < 1$ if $c \Delta t$ is sufficiently small. Now we expand in Taylor’s series to convert (3.8) to

$$\psi(x, t + \Delta t) = N^{-1} c \Delta t \int_{-1}^{1} \exp \left\{ i L(z) \frac{mc^2 \Delta t}{\hbar} \right\} \times$$

$$\left[ \psi(x, t) - z c \Delta t \psi_x(x, t) + \frac{1}{2} z^2 c^2 \Delta t^2 \psi_{xx}(x, t) + \cdots \right] \, dz. \quad (3.9)$$

The left hand side of the inequality (3.7) ensures that $\xi \gg 1$ (see (3.4)), so the integrals in (3.9) can be evaluated by the stationary phase method \[11\]. The right hand side of (3.7) serves to ensure that $\psi(x, t + \Delta t) - \psi(x, t) = \Delta t \psi_t(x, t) + o(\Delta t)$. Thus the expansion (3.9) with the assumptions (3.1), (3.7) gives

$$i \psi_t(x, t) = -\frac{\hbar}{2m} \psi_{xx} + O \left( \xi^{-1/2} \right), \quad (3.10)$$
which in the (formal) limit $\xi \to \infty$ is Schrödinger’s equation in every interior point of the light cone.

Because the support of $\psi(x, t + \Delta t)$ cannot exceed the support of $\psi(x, t)$ by more than $c\Delta t$ on either side, Schrödinger’s equation (3.11) with a finitely supported initial wave function has to be solved inside the light cone emanating from the initial support (see Figure 1) with vanishing boundary conditions. In particular, zero boundary conditions have to be imposed on the triangles mentioned above. Thus, for example, if the initial wave function vanishes on a finite number of finite intervals and outside the minimal finite interval containing its support, the zero boundary conditions on the triangles are given only for a finite time, but the zero boundary condition on the light cone emanating from the minimal interval containing the support are given for all times. The Schrödinger equation has to be solved piecemeal outside the triangles for finite time intervals and then in the entire light cone emanating from the above mentioned minimal interval. Obviously, the initial conditions at times corresponding to every apex of a triangle, as mentioned above, is the wave function obtained from solving the Schrödinger equation in the light cones emanating from the support up to that time. Thus, after the time corresponding to the highest triangle the Schrödinger equation has to be solved in the light cone emanating from the minimal interval, with the initial value of the wave function constructed up to that time. An important result of this structure is that at all times the wave function is analytic inside its (finite) support and has finite moments at all times.

We apply the above procedure to the case discussed in Section 2 $\mathcal{L}(z) = 1 - \sqrt{1 - z^2}$. In this case all integrals can be expressed explicitly in terms of Bessel and Struve functions [12] without invoking the saddle point expansion. The Lorentz invariant propagation is now given by (2.6), so the change the variable of integration to $y = x - zc\Delta t$ and expansion in Taylor’s series converts (2.6) to

$$
\psi(x, t + \Delta t) = \mathcal{N}^{-1}c\Delta t \int_{-1}^{1} \exp \left\{ -i \left[ \sqrt{1 - z^2} - 1 \right] \frac{mc^2\Delta t}{\hbar} \right\} \times (3.11) \\
\left[ \psi(x, t) - zc\Delta t\psi_x(x, t) + \frac{1}{2}z^2c^2\Delta t^2\psi_{xx}(x, t) + \cdots \right] dz.
$$

The coefficient of $\frac{h}{m}\psi_{xx}(x, t)$ in (3.11) is shown in Figures 2 and 3. In
Figure 2: The lower curve is the real part and the upper is the imaginary part of the coefficient of $\frac{\hbar}{m} \psi_{xx}(x,t)$ in (3.11) for $0 < \xi < 20$.

Figure 3: The lower curve is the real part, oscillating around its limit 0 and upper curve is the imaginary part, oscillating around the limit value of the coefficient $\frac{1}{2}$ of $\frac{\hbar}{m} \psi_{xx}(x,t)$ in (3.11) for $1000 < \xi < 1200$. 
the limit $\Delta t \to 0$, that is, $\xi \to 0$, all coefficients in the expansion (3.11) (after dividing by $\Delta t$) converge to 0, giving $\psi_t(x,t) = 0$ so the path integral converges to the initial wave function. However, the limit $c\Delta t \to 0$ and $\xi \to \infty$ in the expansion (3.11) yields (3.10). It is evident that the path integral converges to the solution of the subluminous Schrödinger equation as $\Delta t \to 0$, except for a narrow range of $\Delta t$, where the restriction (3.7) is violated.

4 Quantization

In the presence of a potential $\Phi(x)$ the action (2.5) is replaced with

$$S(x(t)) = \int_0^t \left[ mc^2L\left(\frac{\dot{x}(t')}{c}\right) - \Phi(x(t'))\right] dt', \quad (4.1)$$

and the exponent in (3.2) becomes

$$i\frac{\hbar}{\hbar}\left\{ mc^2L\left(\frac{x-y}{c\Delta t}\right) - \Phi(y) \right\} \Delta t. \quad (4.2)$$

The Schrödinger approximation is now

$$i\hbar \psi_t(x,t) = -\frac{\hbar^2}{2m} \psi_{xx}(x,t) + \Phi(x)\psi(x,t) + O(\xi^{-1/2}) \quad (4.3)$$

within the appropriate light cones confines. Quantized energy levels appear when the potential has a proper local minimum. To see how discrete energies emerge in the subluminous Schrödinger equation, we consider the elementary case of the harmonic oscillator with potential $\Phi(x) = \frac{1}{2}m\omega^2x^2$ and assume that the support of $\psi(x,0) = \varphi(x)$ is the interval $(-x_0, x_0)$ and that the boundary conditions are $\psi(\pm(x_0 + ct), t) = 0$. First, we nondimensionalize equation (4.3) by setting

$$x = y\sqrt{\frac{h}{m\omega}}, \quad t = \frac{\tau}{\omega}, \quad \psi(x,t) = U(y,\tau), \quad \varepsilon = \frac{1}{c}\sqrt{\frac{\omega\hbar}{m}}, \quad (4.4)$$
which converts the initial and boundary value problem for Schrödinger’s equation (4.3) into

$$2iU_\tau(y,\tau) = -U_{yy}(y,\tau) + y^2U(y,\tau) \quad \text{for } |y| < y_0 + \frac{\tau}{\varepsilon} \quad (4.5)$$

$$U(y,0) = \varphi\left(y\sqrt{\frac{\hbar}{m\omega}}\right), \quad U\left(\pm \left(y_0 + \frac{\tau}{\varepsilon}\right),\tau\right) = 0. \quad (4.6)$$

Introducing the fast and slow time scales $\sigma = \tau/\varepsilon$ and $\tau$, respectively, and setting $U(y,\tau) = V(y,\sigma,\tau)$, we rewrite (4.5) as

$$2i(V_\tau + \varepsilon^{-1}V_\sigma) = V_{yy} + y^2V \quad (4.7)$$

and assume the regular outer expansion $[11]$

$$V = V^0 + \varepsilon V^1 + \cdots. \quad (4.8)$$

The resulting hierarchy of equations

$$V^1_\sigma = 0, \quad 2iV^1_\sigma = 2iV^0_\tau - V^0_{yy} + y^2V^0, \ldots \quad (4.9)$$

implies that $V^0$ is independent of $\varepsilon$ and $\sigma$ and thus cannot satisfy the boundary condition on the light cone $y = \pm (y_0 + \sigma)$. The solvability condition for (4.9), obtained from averaging with respect to the fast variable $\sigma$, is the Schrödinger’s equation

$$2iV^0_\tau - V^0_{yy} + y^2V^0 = 0, \quad (4.10)$$

which has to be solved on the entire line with the initial condition (4.6). The first approximation $V^0$ is thus merely the scaled classical quantum harmonic oscillator

$$V^0(y,\tau) = \psi^0(x,t) = \sum_{n=0}^{\infty} a_n e^{-\i E_n t/\hbar} \psi_n(x), \quad (4.11)$$

where $E_n$ are the energies, $\psi_n(x)$ are the (real valued) eigenfunctions of the harmonic oscillator on the entire line and $a_n$ are the coefficients in the expansion of the initial wave function $V^0(y,0) = \varphi(x)$ in the eigenfunctions $\psi_n(x)$.
Higher order terms in the outer expansion (4.8) cannot satisfy the boundary conditions (4.6), so a boundary layer correction \[11\] is needed at \( y = \pm (y_0 + \tau/\varepsilon) \). We thus introduce the boundary layer variable and function near the right boundary

\[
\eta = \frac{y_0 + \frac{\tau}{\varepsilon} - y}{\varepsilon}, \quad U(y, \tau) = W(\eta, \tau) \tag{4.12}
\]

Schrödinger’s equation (4.3) gives

\[
2i\varepsilon^2 W_{\tau\tau} + 2iW_{\eta} = -W_{\eta\eta} + \left(\varepsilon y_0 + \tau - \varepsilon^2 \frac{\eta}{\varepsilon}\right)^2 W. \tag{4.13}
\]

Expanding \( W = W^0 + \varepsilon W^1 + \cdots \), we obtain the boundary layer equation

\[
2iW^0_{\eta\eta} = -W^0_{\eta\eta} + \tau^2 W^0 \tag{4.14}
\]

and the boundary and matching conditions

\[
W^0(0, \tau) = -V^0\left(y_0 + \frac{\tau}{\varepsilon}, \tau\right), \quad W^0(\infty, \tau) = 0. \tag{4.15}
\]

The solution is given by

\[
W^0(\eta, \tau) = -V^0\left(y_0 + \frac{\tau}{\varepsilon}, \tau\right) \exp\left\{ -\eta \left[ i + \sqrt{\tau^2 - 1} \right] \right\}. \tag{4.16}
\]

With a similar construction at the left end of the interval, we get the uniform expansion for \( \varepsilon \ll 1 \)

\[
U(y, \tau) \sim H \left[ (y - y_0)^2 - \frac{\tau^2}{\varepsilon^2} \right] \left[ V^0(y, \tau) \right. \\
\left. -V^0\left(y_0 + \frac{\tau}{\varepsilon}, \tau\right) \exp\left\{ -\left[ i + \sqrt{\tau^2 - 1} \right] \frac{\tau + \varepsilon(y_0 - y)}{\varepsilon^2} \right\} \right. \\
\left. -V^0\left(-y_0 - \frac{\tau}{\varepsilon}, \tau\right) \exp\left\{ -\left[ i + \sqrt{\tau^2 - 1} \right] \frac{\tau + \varepsilon(y_0 + y)}{\varepsilon^2} \right\} \right]. \tag{4.17}
\]

Equation (4.17) indicates that, for large \( \tau/\varepsilon \), the expansion of the wave function in the eigenmodes of Schrödinger’s equation on the line is recovered inside the light cone, away from the boundary. To regain a single mode, the initial wave function has to be of the boundary layer form (4.17) at time
\( \tau = 0 \) and \( V^0(y,0) \) has to be the an eigenfunction of Schrödinger’s equation on the entire line.

Obviously, energy is preserved, because boundary conditions and integration by parts of the energy integral give

\[
\frac{dE(t)}{dt} = -ic\psi\bar{\psi_t}(x_0 + ct) + ic\psi\bar{\psi_t}(-x_0 - ct) - i \int_{-x_0 - ct}^{x_0 + ct} [\psi_t\bar{\psi_t} + \psi\bar{\psi_{tt}}] dx = 0.
\]

The energy at time \( t \) is up to errors of order of magnitude of the contribution of the boundary layer

\[
E(t) \sim \int_{-x_0 - ct}^{x_0 + ct} \sum_{m,n=0}^{\infty} \left[ -a_m\bar{a}_n\hbar \frac{\partial}{\partial t} e^{-iE_m t/\hbar} c^{iE_n t/\hbar} \psi_m(x)\bar{\psi_n}(x) \right] dx
\]

\[
= \sum_{m,n=0}^{\infty} a_m\bar{a}_n E_m e^{i(E_n - E_m) t/\hbar} \int_{-x_0 - ct}^{x_0 + ct} \psi_m(x)\bar{\psi_n}(x) dx
\]

\[
\rightarrow \sum_{m=0}^{\infty} |a_m|^2 E_m \quad \text{as } c t \rightarrow \infty. \quad (4.18)
\]

## 5 Discussion

The standard non-relativistic Schrödinger model of quantum mechanics consists of the initial value problem for Schrödinger’s equation in the entire space. In the conventional interpretation of this model any boundary conditions imposed on the wave function can be represented in terms of appropriate potentials in Schrödinger’s equation. In this paper we adopt an alternative model, which consists of a Lorentz-invariant Feynman path integral that force vanishes on and outside the light cone emanating from the support of the initial wave function. This boundary condition does not require any potentials, rather, it is an inseparable part of this model of quantum mechanics. As explained in this paper, the path integral converges to a limit as the time slice \( \Delta t \rightarrow 0 \), except in a narrow range where the restriction (3.7) is violated. The distinguished limit \( c\Delta t \rightarrow 0 \) and \( \xi \rightarrow \infty \) of the path integral is the solution of the subluminous Schrödinger equation (the solution that satisfies zero boundary conditions on and outside the original light cone of the path...
integral). We can view, therefore, the subluminous Schrödinger equation as an intermediate model between non-relativistic and relativistic quantum mechanics. The results of the standard non-relativistic quantum theory are recovered in the intermediate model, albeit after a relativistic delay.

To clear the issue of how the results of this paper relate to existing theories, we point that

1. This paper mainly addresses issues related to the original (non-relativistic) Schrödinger equation.

2. Ours is not a relativistic quantum theory, but rather a conventional one that upholds the basic tenets of physics: all propagation must be confined within a relevant light-cone.

3. Why should one care about such matters as light-cones in the non-relativistic regime? The quick answer is that neglecting the speed of light and the resulting boundary conditions on light-cones is, mathematically speaking, a singular perturbation, which results in unphysical artifacts such as the disappearance of moments and energy when the initial wave function is discontinuous (e.g., after a collapse due to a measurement) or the instantaneous spread of information into the entire space. These artifacts merely reflect the mathematical shortcomings ingrained in the original Schrödinger formulation.

4. Recall that similar difficulties occur in diverse physical theories. For instance, air viscosity may be practically negligible, but if it is neglected altogether, gas dynamics predicts no drag force on airplanes wings. For no matter how irrelevant it is elsewhere, close to a moving body there is a boundary layer, where viscous effects are dominant. Thus, while viscosity manifests itself locally, its effect is global!

5. A closer kin to our problem is the heat equation and its unphysical artifact of instantaneous spreading of initially localized perturbation to the entire line. This artifact can be attributed to the disregard of the acoustic speed limitation in the derivation of the diffusion equation. If the acoustic bound is enforced on the process, finite propagation fronts emerge and the unphysical artifact of instantaneous propagation disappears [13].
6. Perhaps the most succinct summary of our discussion would be to say that it is not enough that we know the limitation of our model; the model itself should contain its own limitations, as is the case at hand: the Schrödinger equation with boundary conditions contains the limitation on propagation.

7. The non-relativistic propagator of Schrödinger’s equation has some of the properties of the Fourier transform: the decay at infinity of the wave function reflects the smoothness of its initial value. Thus a jump discontinuity is converted by the propagator into a decay of the wave function at infinity as $O(|x|^{-1})$, which although square integrable, has no moments \[\text{[10]}\]. In contrast, none of this anomaly exists for Schrödinger’s equation with zero boundary conditions on and outside the light cone.

To conclude, while confining the Schrödinger equation within a relevant light cone may only mildly extend the scope of its applicability, which is yet to be seen, it definitely relieves it of annoying paradoxes and artifacts. Given the central position of Schrödinger’s equation in modern physics, our extension should be expected to have implications beyond mere scientific esthetics. The basic predicament of relativity manifests itself in our theory only in crucial ‘junctions’ of space-time: the theory limits the domain of propagation, endows the wave function with a definite front, thus eliminating the unlimited precursors and causing a time delay for the conventional quantum effects to re-emerge.

Acknowledgments. We thank A. Marchewka for useful discussions. The work of the first author was supported by the ISF grant no.801/07.

References

[1] R.P. Feynman, ”Space-time approach to a non-relativistic quantum mechanics,” Rev. Mod. Phys. 20, pp.367-387 (1948).

[2] L.S. Schulman, Techniques and Applications of Path Integrals, Dover NY 2005.

[3] H. Kleinert, Path Integrals in Quantum Mechanics, Statistics, and Polymer Physics, World Scientific, NY 1994
[4] J.B. Keller and D.W. McLaughlin. “The Feynman integral.” Amer. Math. Month., 82 (5), 451–576 (1975).

[5] T. Miura, ”Relativistic Path Integrals,” Progress of Theoretical Physics 61 (5), pp. 1521-1535 (1979).

[6] V.B. Berestetskii, E.M. Lifshits, and L.P. Pitaevskii, Relativistic Quantum Theory (Course of Theoretical Physics Volume 4 Part 1), Elsevier 1971.

[7] A. Peres, Quantum Theory: Concepts and Methods, Kluwer Academic Publishers, 1995.

[8] A. Marchewka and Z. Schuss, “Path integral approach to the Schrödinger current,” Phys. Rev. A, 61, 052107 (2000).

[9] A. Einstein, B. Podolsky, N. Rosen, ”Can quantum-mechanical description of physical reality be considered complete?” Physical Review 41, p.777 (1935).

[10] A. Marchewka and Z. Schuss, ”Schrödinger propagation of initial discontinuities leads to divergence of moments,” Phys. Lett. A 240, 177 (1998).

[11] C.M. Bender and S.A. Orszag. Advanced Mathematical Methods for Scientists and Engineers. McGraw-Hill, New York, 1978.

[12] M. Abramowitz and I. A. Stegun. Handbook of Mathematical Functions. Dover, New York, 1972.

[13] P. Rosenau, ”Tempered diffusion: A transport process with propagating fronts and inertial delay.” Phys. Rev. A, 46, R7371(1992).