NUMERICAL FLATNESS AND PRINCIPAL BUNDLES ON FUJIKI MANIFOLDS

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Abstract. Let \( M \) be a compact connected Fujiki manifold, \( G \) a semisimple affine algebraic group over \( \mathbb{C} \) with one simple factor and \( P \) a fixed proper parabolic subgroup of \( G \). For a holomorphic principal \( G \)-bundle \( E_G \) over \( M \), let \( E_P \) be the holomorphic principal \( P \)-bundle \( E_G \to E_G/P \) given by the quotient map. We prove that the following three statements are equivalent: (1) \( \text{ad}(E_G) \) is numerically flat, (2) the holomorphic line bundle \( \bigwedge^{\text{top}} \text{ad}(E_P)^* \) is nef, and (3) for every reduced irreducible compact complex analytic space \( Z \) with a Kähler form \( \omega \), holomorphic map \( \gamma : Z \to M \), and holomorphic reduction of structure group \( E_P \subset \gamma^* E_G \) to \( P \), the inequality degree(\( \text{ad}(E_P) \)) \( \leq 0 \) holds.

1. Introduction

A basic theorem of Miyaoka says that a vector bundle \( E \) on a smooth complex projective curve \( C \) is semistable if and only if the relative anticanonical line bundle for the natural projection \( \mathbb{P}(E) \to C \) is nef \([Mi]\). A holomorphic vector bundle \( E \) on \( C \) is semistable if and only if the vector bundle \( \text{ad}(E) \to C \) of trace zero endomorphisms is numerically flat. The very useful notion of numerically flat vector bundles was introduced by Demailly, Peternell and Schneider in \([DPS]\); we recall that a holomorphic vector bundle \( V \) is numerically flat if both \( V \) and \( V^* \) are nef. Therefore, a reformulation of Miyaoka’s theorem says that \( \text{ad}(E) \) is numerically flat if and only if the relative anticanonical line bundle on \( \mathbb{P}(E) \) is nef.

For \( E \) as above, fix any \( 1 \leq r < \text{rank}(E) \). Let \( \text{Gr}(r, E) \to C \) be be the Grassmann bundle that parametrizes the \( r \)-dimensional quotients of the fibers of \( E \). Bruzzo and Hernández Ruipérez proved the following big generalization of the above theorem of Miyaoka: The relative anticanonical line bundle on \( \text{Gr}(r, E) \) is nef if and only if \( \text{ad}(E) \) is numerically flat \([BH]\).

Let \( X \) be a smooth complex projective variety and \( E_G \) a holomorphic principal \( G \)-bundle on \( X \), where \( G \) is a simple affine algebraic group without center over \( \mathbb{C} \). Fix a parabolic subgroup \( P \subseteq G \). In \([BB]\), the following generalization of the above result of Bruzzo and Hernández Ruipérez was proved: The adjoint vector bundle \( \text{ad}(E_G) \) is numerically flat if and only if the relative anticanonical line bundle for the natural projection \( E_G/P \to X \) is nef.

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Our aim here is to investigate the principal bundles on a compact Fujiki manifold from the above point of view. We recall that a Fujiki manifold is a compact complex manifold which is the image of a bimeromorphic surjective map from a compact Kähler manifold [Fu1], [Fu2], or equivalently, the image of a holomorphic surjective map from a compact Kähler manifold [Va].

Let \( G \) be a semisimple affine algebraic group, over \( \mathbb{C} \), with one simple factor and \( P \subset G \) a fixed parabolic subgroup. Let \( M \) be a compact connected Fujiki manifold and \( E_G \) a holomorphic principal \( G \)-bundle on \( M \). The quotient map \( E_G \to E_G/P \) defines a holomorphic principal \( P \)-bundle on \( E_G/P \). The top exterior product of the adjoint bundle for this principal \( P \)-bundle is the relative canonical bundle for the natural projection \( E_G/P \to M \).

We prove the following (see Theorem 3.1 and Theorem 4.1):

**Theorem 1.1.** Let \( E_G \) be a holomorphic principal \( G \)-bundle on a compact connected Fujiki manifold \( M \). Then the following three statements are equivalent:

1. The holomorphic vector bundle \( \text{ad}(E_G) \) is numerically flat.
2. The relative anticanonical bundle for the natural projection \( E_G/P \to M \) is nef.
3. For every quadruple of the form \((Z, \omega, \gamma, E_P)\), where \( Z \) is a reduced irreducible compact complex analytic space equipped with a Kähler form \( \omega \),

\[
\gamma : Z \to M
\]

is a holomorphic map, and \( E_P \subset \gamma^*E_G \) is a holomorphic reduction of structure group of the principal \( G \)-bundle \( \gamma^*E_G \) to the subgroup \( P \), the inequality

\[
\text{degree}(\text{ad}(E_P)) \leq 0
\]

holds.

Note that the first statement of Theorem 1.1 does not involve the parabolic subgroup \( P \). Therefore, Theorem 1.1 has the following corollary:

**Corollary 1.2.** Let \( E_G \) be a holomorphic principal \( G \)-bundle on a compact connected Fujiki manifold \( M \). If the second and third statements in Theorem 1.1 hold for one parabolic subgroup \( P \subset G \), then they hold for every proper parabolic subgroup of \( G \).

### 2. Numerically flat bundles

Let \( M \) be a compact connected complex manifold. Fix a Hermitian structure \( H_0 \) on \( M \). Let \( \omega_{H_0} \) be the corresponding positive \((1, 1)\)-form on \( M \). A holomorphic line bundle \( L \) on \( M \) is called numerically effective (nef for short) if for every \( \epsilon > 0 \), there is a Hermitian structure \( h_\epsilon \) on \( L \) such that

\[
\Theta_{h_\epsilon}(L) \geq -\epsilon \cdot \omega_{H_0},
\]

where \( \Theta_{h_\epsilon}(L) \) is the curvature of the Hermitian complex connection on \( L \) corresponding to \( h_\epsilon \) [DPS, p. 299, Definition 1.2]. We note that while the definition of nefness uses \( H_0 \), the nefness of any given line bundle is actually independent of the choice of \( H_0 \).
A holomorphic vector bundle \( V \) on \( M \) is called nef if the tautological line bundle \( \mathcal{O}_{\mathbb{P}(V)}(1) \) on \( \mathbb{P}(V) \) is nef \([\text{DPS}, \text{p. 305, Definition 1.9}]\). A holomorphic vector bundle \( V \) on \( M \) is called numerically flat if both \( V \) and its dual \( V^* \) are nef \([\text{DPS}, \text{p. 311, Definition 1.17}]\).

A compact connected complex manifold \( M \) is called a Fujiki manifold if there exists a surjective bimeromorphic map \( f: Y \rightarrow M \) where \( Y \) is a compact Kähler manifold \([\text{Fu1}, \text{Fu2}]\). A basic theorem of Varouchas says that \( M \) is a Fujiki manifold if it is the image of a compact Kähler manifold by a surjective morphism (it need not be a bimeromorphism) \([\text{Va}, \text{p. 51, Theorem 5}]\).

**Proposition 2.1.** Let \( M \) be a compact connected Fujiki manifold, and let \( f: Y \rightarrow M \) be a surjective bimeromorphic map, where \( Y \) is a compact connected Kähler manifold. Then a holomorphic vector bundle \( E \) on \( M \) is numerically flat if and only if the pulled back vector bundle \( f^*E \) is numerically flat.

**Proof.** Given any nef vector bundle \( F \) on \( M \), the vector bundle \( f^*F \) is nef \([\text{DPS}, \text{p. 305, Proposition 1.10}]\). Applying this to both \( E \) and \( E^* \) we conclude that \( f^*E \) is numerically flat if \( E \) is numerically flat.

To prove the converse, assume that \( f^*E \) is numerically flat. A structure theorem of \([\text{DPS}]\) says that \( f^*E \) admits a filtration of holomorphic subbundles
\[
0 = V_0 \subset V_1 \subset \cdots \subset V_{\ell - 1} \subset V_{\ell} = f^*E
\] (2.1)
such that the holomorphic vector bundle \( V_i/V_{i-1} \) admits a unitary flat connection for all \( 1 \leq i \leq \ell \) (see \([\text{DPS}, \text{p. 311, Theorem 1.18}]\)). This immediately implies the following:

1. the Chern class \( c_i(f^*E) \in H^{2i}(Y, \mathbb{R}) \) vanishes for every \( i \geq 1 \) (see \([\text{DPS}, \text{p. 311, Corollary 1.19}]\)), and
2. the vector bundle \( f^*E \) is pseudostable (see \([\text{BG}, \text{p. 23, Definition 2.1}]\) for pseudostable bundles; set the Higgs field \( \theta \) in \([\text{BG}, \text{Definition 2.1}]\) to be the zero section).

Consequently, \( f^*E \) admits a flat holomorphic connection \( \widetilde{\nabla} \) such that

1. the connection \( \widetilde{\nabla} \) preserves the subbundle \( V_i \) in (2.1) for every \( 1 \leq i \leq \ell \), and
2. the holomorphic connection on \( V_i/V_{i-1} \) induced by \( \widetilde{\nabla} \) is unitary flat for every \( 1 \leq i \leq \ell \).

(See \([\text{BG}, \text{p. 20, Theorem 1.1}]\); set the Higgs field in \([\text{BG}, \text{Theorem 1.1}]\) to be the zero section.)

For any \( 1 \leq i \leq \ell \), let \( \widetilde{\nabla}^i \) be the connection on \( V_i \) induced by \( \widetilde{\nabla} \).
Since the map $f$ is a bimeromorphism, the induced homomorphism of fundamental groups
$$f_* : \pi_1(Y) \longrightarrow \pi_1(M)$$
is actually an isomorphism. Therefore, we conclude the following:

(1) the flat connection $\tilde{\nabla}$ on $f^*E$ is the pull-back of a flat connection $\nabla$ on $E$, and
(2) the flat subbundle $(V_i, \tilde{\nabla}^i)$ of $(f^*E, \tilde{\nabla})$ descends to a flat subbundle $(E_i, \nabla^i)$ of $(E, \nabla)$ for all $1 \leq i \leq \ell$. In other words, $(f^*E, \tilde{\nabla})$ is the pullback of the flat subbundle $(E_i, \nabla^i)$ of $(E, \nabla)$.

Furthermore, the connection on $E_i/E_{i-1}$ induced by $\nabla^i$ is unitary flat, because the connection on $V_i/V_{i-1}$ induced by $\tilde{\nabla}$ is unitary flat. The vector bundle $E_i/E_{i-1}$ is numerically flat because it admits a unitary flat connection. Since the extension of a nef vector bundle by a nef vector bundle is again nef [DPS, p. 308, Proposition 1.15(ii)], it follows immediately that the extension of a numerically flat vector bundle by a numerically flat vector bundle is again numerically flat. Consequently, the vector bundle $E$ is numerically flat. \hfill \Box

The following is a consequence of the proof of Proposition 2.1.

Corollary 2.2. Let $E$ be a holomorphic vector bundle on a compact connected Fujiki manifold $M$. Then $E$ is numerically flat if and only if there is a filtration of holomorphic subbundles of $E$
$$0 = E_0 \subset E_1 \subset \cdots \subset E_{\ell-1} \subset E_{\ell} = E$$
and a flat holomorphic $\nabla$ on $E$ such that

1. $\nabla$ preserves $E_i$ for all $1 \leq i \leq \ell$, and
2. the connection on $E_i/E_{i-1}$ induced by $\nabla$ is unitary for all $1 \leq i \leq \ell$.

Proof. If there is a filtration as above and a flat holomorphic $\nabla$ on $E$ satisfying the above two conditions, then $E_i/E_{i-1}$ is numerically flat for all $1 \leq i \leq \ell$. Therefore, using [DPS, p. 308, Proposition 1.15(ii)] it is deduced that $E$ is numerically flat.

To prove the converse assume that $E$ is numerically flat. Take any surjective bimeromorphic map
$$f : Y \longrightarrow X$$
as in Proposition 2.1 with $Y$ Kähler. Then $f^*E$ is numerically flat by Proposition 2.1. As we saw in the proof of Proposition 2.1, this implies that there is a filtration as in the statement of the corollary and a flat holomorphic $\nabla$ on $E$ satisfying the two conditions in the statement of the corollary. \hfill \Box

3. Principal bundles on Fujiki manifolds

Let $G$ be a connected complex semisimple affine algebraic group with one simple factor. Fix a parabolic subgroup
$$P \subset G.$$
The Lie algebras of $G$ and $P$ will be denoted by $\mathfrak{g}$ and $\mathfrak{p}$ respectively.

As before, $M$ is a compact connected Fujiki manifold. Let $E_G$ be a holomorphic principal $G$–bundle on $M$. Let

$$\text{ad}(E_G) := E_G \times^G \mathfrak{g} \longrightarrow M$$

be the adjoint bundle for $E_G$; its fibers are Lie algebras identified with $\mathfrak{g}$ uniquely up to conjugations.

Let $p : E_G \rightarrow E_G/P$ and

$$\varphi : E_G/P \rightarrow M$$

be the natural projections. Consider the projection

$$\varphi^*E_G := (E_G/P) \times_M E_G \xrightarrow{\text{Id} \times p} (E_G/P) \times_M (E_G/P).$$

Let

$$\mathcal{E}_P \subset \varphi^*E_G$$

be the inverse image of the diagonal $E_G/P \subset (E_G/P) \times_M (E_G/P)$ under this projection. It is straight-forward to check that $\mathcal{E}_P$ is a holomorphic reduction of structure group of the principal $G$–bundle $\varphi^*E_G$ to the subgroup $P \subset G$.

**Theorem 3.1.** Let $E_G$ be a holomorphic principal $G$–bundle on a compact connected Fujiki manifold $M$. Then the following two statements are equivalent:

1. The holomorphic vector bundle $\text{ad}(E_G)$ in (3.1) is numerically flat.
2. The holomorphic line bundle $\bigwedge^{\text{top}} \text{ad}(\mathcal{E}_P)^* \rightarrow E_G/P$ is nef, where $\mathcal{E}_P$ in the holomorphic principal $P$–bundle constructed in (3.3).

**Proof.** First assume that $\text{ad}(E_G)$ is numerically flat. We will show that $\bigwedge^{\text{top}} \text{ad}(\mathcal{E}_P)^*$ is nef. For that the following lemma will be used.

**Lemma 3.2.** The direct image $\varphi_* \bigwedge^{\text{top}} \text{ad}(\mathcal{E}_P)^* \rightarrow M$, where $\varphi$ is the projection in (3.2), is a vector bundle of positive rank. If this vector bundle $\varphi_* \bigwedge^{\text{top}} \text{ad}(\mathcal{E}_P)^*$ is nef, then $\bigwedge^{\text{top}} \text{ad}(\mathcal{E}_P)^*$ is also nef.

**Proof of Lemma 3.2.** The line bundle

$$\bigwedge^{\text{top}} \text{ad}(\mathcal{E}_P)^* \rightarrow E_G/P$$

is the relative anti-canonical line bundle for the projection $\varphi$. So $\bigwedge^{\text{top}} \text{ad}(\mathcal{E}_P)^*$ is relatively ample (the anti-canonical line bundle of $G/P$ is ample). Therefore, $\bigwedge^{\text{top}} \text{ad}(\mathcal{E}_P)^*$ is relatively very ample (an ample line bundle on $G/P$ is very ample; see [Sn, Theorem 6.5(2)], [Se]). Also the higher direct images of $\bigwedge^{\text{top}} \text{ad}(\mathcal{E}_P)^*$ vanish by the Kodaira vanishing theorem. These imply that $\varphi_* \bigwedge^{\text{top}} \text{ad}(\mathcal{E}_P)^* \rightarrow M$ is a holomorphic vector bundle of positive rank.

Since $\bigwedge^{\text{top}} \text{ad}(\mathcal{E}_P)^*$ is relatively very ample, we get an embedding

$$\eta : E_G/P \rightarrow \mathbb{P} \left( \varphi_* \bigwedge^{\text{top}} \text{ad}(\mathcal{E}_P)^* \right).$$
The pulled back line bundle $\eta^*\mathcal{O}_{P(\varphi^* \Lambda_{\text{top}}^\text{ad}(\mathcal{E}_P)^*)}(1)$ is identified with $\Lambda_{\text{top}}^\text{ad}(\mathcal{E}_P)^*$. Consequently, $\Lambda_{\text{top}}^\text{ad}(\mathcal{E}_P)^*$ is nef if $\varphi^* \Lambda_{\text{top}}^\text{ad}(\mathcal{E}_P)^*$ is nef, because $\mathcal{O}_{P(\varphi^* \Lambda_{\text{top}}^\text{ad}(\mathcal{E}_P)^*)}(1)$ is nef in that case.

Let $Z_G \subset G$ be the center of $G$. We note that the left-translation action of $G$ on $G/P$ produces an action of $G$ on $H^0(G/P, K_{G/P}^{-1})$, where $K_{G/P}^{-1}$ is the anti-canonical line bundle. The action of $Z_G$ on $G/P$ is trivial because $Z_G \subset P$. Also, the action of $Z_G$ on $H^0(G/P, K_{G/P}^{-1})$ is trivial, because the adjoint action of $Z_G$ is trivial. So we get an action of $G/Z_G$ on $H^0(G/P, K_{G/P}^{-1})$.

Since $G$ is semisimple, the adjoint action of $G/Z_G$ on $\mathfrak{g}$ is faithful. Also, the $G/Z_G$–module $\mathfrak{g}$ is isomorphic to $\mathfrak{g}^*$ using the Killing form on $\mathfrak{g}$. Therefore, there are nonnegative integers $t_1, \cdots, t_n$ such that the $G/Z_G$–module $H^0(G/P, K_{G/P}^{-1})$ is a direct summand of the $G/Z_G$–module

$$\bigoplus_{j=1}^n \mathfrak{g}^{\otimes t_j}$$

(see [DM, p. 40, Proposition 3.1(a)]). This implies that the holomorphic vector bundle $\varphi^* \Lambda_{\text{top}}^\text{ad}(\mathcal{E}_P)^*$ is a direct summand of the holomorphic vector bundle

$$\bigoplus_{j=1}^n \text{ad}(\mathcal{E}_G)^{\otimes t_j}.$$ 

Indeed, if $\bigoplus_{j=1}^n \mathfrak{g}^{\otimes t_j} = H^0(G/P, K_{G/P}^{-1}) \oplus A$, where $A$ is a $G/Z_G$–module, then

$$\bigoplus_{j=1}^n \text{ad}(\mathcal{E}_G)^{\otimes t_j} = (\varphi^* \Lambda_{\text{top}}^\text{ad}(\mathcal{E}_P)^*) \oplus \tilde{A},$$

where $\tilde{A} \rightarrow M$ is the holomorphic vector bundle associated to the principal $G$–bundle $E_G$ for the $G$–module $A$ (any $G/Z_G$–module is also a $G$–module).

Since $\text{ad}(\mathcal{E}_G)$ is nef, the vector bundle $\text{ad}(\mathcal{E}_G)^{\otimes t_j}$ is nef [DPS, p. 307, Proposition 1.14(i)], hence $\bigoplus_{j=1}^n \text{ad}(\mathcal{E}_G)^{\otimes t_j}$ is nef [DPS, p. 308, Proposition 1.15(ii)], and therefore, its direct summand $\varphi^* \Lambda_{\text{top}}^\text{ad}(\mathcal{E}_P)^*$ is nef [DPS, p. 308, Proposition 1.15(i)]. Now Lemma 3.2 implies that $\Lambda_{\text{top}}^\text{ad}(\mathcal{E}_P)^*$ is nef.

To prove the converse, assume that $\Lambda_{\text{top}}^\text{ad}(\mathcal{E}_P)^*$ is nef. We will prove that $\text{ad}(\mathcal{E}_G)$ is numerically flat.

Since the holomorphic line bundle

$$\Lambda_{\text{top}}^\text{ad}(\mathcal{E}_P)^* = (\Lambda_{\text{top}}^\text{ad}(\mathcal{E}_P)^*)^{\otimes 2} \otimes \Lambda_{\text{top}}^\text{ad}(\mathcal{E}_P) = (\Lambda_{\text{top}}^\text{ad}(\mathcal{E}_P)^*)^{\otimes 2} \otimes K_{\varphi}$$

is nef, where $K_{\varphi}$ is the relative canonical bundle for the projection $\varphi$ in (3.2), we conclude that the direct image $\varphi_*(\Lambda_{\text{top}}^\text{ad}(\mathcal{E}_P)^*)^{\otimes 2}$ is a nef vector bundle [Mo, p. 895, Théorème 2].
On the other hand, \( \varphi_*(\bigwedge^{\text{top}} \text{ad}(\mathcal{E}_P)^*) \otimes^2 \) coincides with the holomorphic vector bundle on \( M \) associated to the principal \( G \)-bundle \( E_G \) for the \( G \)-module \( H^0(G/P, (K_{G/P}^{-1})^2) \). Since the group \( G \) is semisimple, it does not have any nontrivial character, in particular, \( \bigwedge^{\text{top}} H^0(G/P, (K_{G/P}^{-1})^2) \) is the trivial \( G \)-module. This implies that the associated holomorphic line bundle \( \bigwedge^{\text{top}} \big( \varphi_*(\bigwedge^{\text{top}} \text{ad}(\mathcal{E}_P)^*) \big)^2 \) is trivial. Since \( \varphi_*(\bigwedge^{\text{top}} \text{ad}(\mathcal{E}_P)^*)^2 \) is nef, this implies that \( \varphi_*(\bigwedge^{\text{top}} \text{ad}(\mathcal{E}_P)^*)^2 \) is numerically flat (see [DPS, p. 311, Definition 1.17]). Since the tensor product of two nef bundles is nef [DPS, p. 307, Proposition 1.14(i)], and both \( \varphi_*(\bigwedge^{\text{top}} \text{ad}(\mathcal{E}_P)^*)^2 \) and \( \varphi_*(\bigwedge^{\text{top}} \text{ad}(\mathcal{E}_P)^*)^2 \) are nef, we conclude that the vector bundle
\[
\text{End}(\varphi_*(\bigwedge^{\text{top}} \text{ad}(\mathcal{E}_P)^*)^2) = (\varphi_*(\bigwedge^{\text{top}} \text{ad}(\mathcal{E}_P)^*)^2) \otimes (\varphi_*(\bigwedge^{\text{top}} \text{ad}(\mathcal{E}_P)^*)^2)^*
\]
is nef. As \( \bigwedge^{\text{top}} \text{End}(\varphi_*(\bigwedge^{\text{top}} \text{ad}(\mathcal{E}_P)^*)^2) \) is the trivial line bundle, it now follows that \( \text{End}(\varphi_*(\bigwedge^{\text{top}} \text{ad}(\mathcal{E}_P)^*)^2) \) is numerically flat (see [DPS, p. 311, Definition 1.17]).

Consider the \( G/Z_G \)-module \( H^0(G/P, (K_{G/P}^{-1})^2) \) (the center \( Z_G \) acts trivially on it). Let
\[
\rho : G/Z_G \rightarrow \text{GL}(H^0(G/P, (K_{G/P}^{-1})^2))
\]
be the corresponding homomorphism. We note that \( G/Z_G \) is simple without center because \( G \) has only one simple factor. Hence the above homomorphism \( \rho \) is injective. Therefore, the \( G \)-module \( \text{Lie}(G/Z_G) = \mathfrak{g} \) is a direct summand of the \( G \)-module
\[
\text{Lie}(\text{GL}(H^0(G/P, (K_{G/P}^{-1})^2))) = \text{End}(H^0(G/P, (K_{G/P}^{-1})^2)).
\]

On the other hand, the holomorphic vector bundle on \( M \) associated to the principal \( G \)-bundle \( E_G \) for the \( G \)-module \( \text{End}(H^0(G/P, (K_{G/P}^{-1})^2)) \) coincides with the vector bundle in (3.5). Since the \( G \)-module \( \mathfrak{g} \) is a direct summand of \( \text{End}(H^0(G/P, (K_{G/P}^{-1})^2)) \), we conclude that \( \text{ad}(E_G) \) is a direct summand of the vector bundle in (3.5). We saw that the vector bundle in (3.5) is numerically flat. So its direct summand \( \text{ad}(E_G) \) is also numerically flat [DPS, p. 308, Proposition 1.15(i)]. This completes the proof of the theorem. \( \square \)

4. Pullback to Kähler manifolds

Take \( M \) and \( E_G \) as before. Let
\[
(Z, \omega)
\]
be a reduced irreducible compact complex analytic space \( Z \) with a Kähler form \( \omega \), and let
\[
\gamma : Z \rightarrow M
\]
be a holomorphic map. Consider the holomorphic principal \( G \)-bundle \( \gamma^*E_G \) on \( Z \). Giving a holomorphic reduction of structure group
\[
E_P \subset \gamma^*E_G
\]
of the principal \( G \)-bundle \( \gamma^*E_G \) to \( P \subset G \) is equivalent to giving a holomorphic section
\[
\sigma : Z \rightarrow (\gamma^*E_G)/P = \gamma^*(E_G/P)
\]
of the natural projection $(\gamma^*E_G)/P \to Z$. Indeed, the inverse image of $\sigma(Z) \subset (\gamma^*E_G)/P$ for the quotient map $\gamma^*E_G \to (\gamma^*E_G)/P$ is a holomorphic reduction of structure group of $\gamma^*E_G$ to $P$.

Take any holomorphic reduction of structure group $E_P \subset \gamma^*E_G$ of $\gamma^*E_G$ to $P$. Let

$$\text{ad}(E_P) := E_P \times^P p \to Z$$

(4.3)

be the adjoint bundle of $E_P$; the inclusion map of $p$ in $g$ produces a map

$$\text{ad}(E_P) \to \text{ad}(\gamma^*E_G) = \gamma^*\text{ad}(E_G),$$

so $\text{ad}(E_P)$ is a subbundle of $\gamma^*\text{ad}(E_G)$.

For a holomorphic vector bundle $W$ on $Z$, define

$$\text{degree}(W) := \int_Z c_1(W) \wedge \omega^{d-1} \in \mathbb{R},$$

where $d = \dim \mathbb{C} Z$; see [Ko] p. 168, (7.1).

**Theorem 4.1.** Let $E_G$ be a holomorphic principal $G$–bundle on a compact connected Fujiki manifold $M$. Then the following two statements are equivalent:

1. The holomorphic vector bundle $\text{ad}(E_G)$ in (3.1) is numerically flat.
2. For every triple $(Z, \omega, \gamma)$ as in (4.1) and (4.2), and every holomorphic reduction of structure group $E_P \subset \gamma^*E_G$ of $\gamma^*E_G$ to $P$, the inequality

$$\text{degree}(\text{ad}(E_P)) \leq 0$$

holds, where $\text{ad}(E_P)$ is the adjoint bundle in (4.3).

**Proof.** First assume that $\text{ad}(E_G)$ is numerically flat. Take any $(Z, \omega, \gamma)$ as in (4.1) and (4.2). Since $\text{ad}(E_G)$ is numerically flat, it follows that $\gamma^*\text{ad}(E_G) = \text{ad}(\gamma^*E_G)$ is also numerically flat [DPS] p. 305, Proposition 1.10].

Let

$$\phi : (\gamma^*E_G)/P \to Z$$

(4.4)

be the natural projection. The holomorphic principal $P$–bundle

$$\gamma^*E_G \to (\gamma^*E_G)/P = \gamma^*(E_G/P)$$

over $(\gamma^*E_G)/P$ will be denoted by $\mathcal{F}_P$; it is a holomorphic reduction of structure group of the principal $G$–bundle $\phi^*\gamma^*E_G$ to $P \subset G$. Since $\gamma^*\text{ad}(E_G)$ is numerically flat, from Theorem 3.1 we know that the line bundle $\bigwedge^{\text{top}}\text{ad}({\mathcal{F}}_P)^*$ is nef.

Let $E_P \subset \gamma^*E_G$ be a holomorphic reduction of structure group of the principal $G$–bundle $\gamma^*E_G$ to the subgroup $P$. It corresponds to a section

$$\beta : Z \to (\gamma^*E_G)/P$$

of the projection $\phi$ in (4.4); the holomorphic principal $P$–bundle $E_P$ is the pullback $\beta^*\mathcal{F}_P$, where $\mathcal{F}_P$ is the principal $P$–bundle defined above. Therefore, we have

$$\bigwedge^{\text{top}}\text{ad}(E_P) = \beta^*\bigwedge^{\text{top}}\text{ad}({\mathcal{F}}_P).$$
This, and the above observation that $\wedge^{\text{top}} \text{ad}(F_P)^*$ is nef, together imply that $\wedge^{\text{top}} \text{ad}(E_P)^*$ is nef. This immediately implies that

$$\text{deg}(\text{ad}(E_P)) \leq 0.$$ 

To prove that converse, assume that

$$\text{deg}(\text{ad}(E_P)) \leq 0$$ 

for every triple $(Z, \omega, \gamma)$ as in (4.1) and (4.2), and every holomorphic reduction of structure group $E_P \subset \gamma^* E_G$ of $\gamma^* E_G$ to $P$. We will prove that $\text{ad}(E_G)$ is numerically flat.

Take a surjective bimeromorphic map

$$f : Y \to M,$$  \hspace{1cm} (4.5)

where $Y$ is a compact connected Kähler manifold. From Proposition 2.1 we know that $\text{ad}(E_G)$ is numerically flat if $f^* \text{ad}(E_G) = \text{ad}(f^* E_G)$ is numerically flat.

Let

$$\psi : (f^* E_G)/P = f^*(E_G/P) \to Y$$ \hspace{1cm} (4.6)

be the natural projection. The holomorphic principal $P$–bundle

$$f^* E_G \to (f^* E_G)/P = f^*(E_G/P)$$ \hspace{1cm} (4.7)

will be denoted by $\tilde{F}_P$. Note that $\tilde{F}_P$ is a holomorphic reduction of structure group of the principal $G$–bundle $(f \circ \psi)^* E_G = \psi^* f^* E_G$ to $P \subset G$. To show that $f^* \text{ad}(E_G)$ is numerically flat, first note that Theorem 3.1 says that it suffices to prove that $\wedge^{\text{top}} \text{ad}(\tilde{F}_P)^*$ is nef. Now, to prove that $\wedge^{\text{top}} \text{ad}(\tilde{F}_P)^*$ is nef, we will use the following criterion of Demailly and Paun, [DP], for nefness.

A holomorphic line bundle $L$ on a compact Kähler manifold $N$ is nef if and only if for every Kähler form $\omega_N$ on $N$, and every irreducible closed connected analytic subspace $S \subset N$, the inequality

$$\int_S c_1(L) \wedge \omega_N^{s-1} \geq 0$$ \hspace{1cm} (4.8)

holds, where $s = \dim \mathbb{C} S$ [DP] p. 1248, Corollary 0.4].

Set

$$(N, L) = ((f^* E_G)/P, \wedge^{\text{top}} \text{ad}(\tilde{F}_P)^*).$$

Take any $(\omega_N, S)$ as above, so

$$S \subset (f^* E_G)/P.$$ 

Set $Z$ in the statement of the theorem to be $S$, and set $\gamma$ in the statement of the theorem to be the composition of maps

$$S \hookrightarrow (f^* E_G)/P \xrightarrow{\psi} Y \xrightarrow{f} M,$$

where $\psi$ and $f$ are the maps in (4.6) and (4.5) respectively. Set the reduction $E_P \subset \gamma^* E_G$ to be the restriction of the reduction

$$\tilde{F}_P \subset (f \circ \psi)^* E_G.$$
(see (4.7)) to $S \subset (f^*E_G)/P$. So we have

$$E_P = (\tilde{F}_P)|_S.$$ 

This implies that

$$\bigwedge^\top \text{ad}(E_P) = \bigwedge^\top \text{ad}(\tilde{F}_P).$$

Therefore, the given condition that

$$\text{degree}(\text{ad}(E_P)) \leq 0$$

implies that the inequality in (4.8) holds. Now the above mentioned criterion of [DP] for nefness implies that $\bigwedge^\top \text{ad}(\tilde{F}_P)^*$ is nef. 

\[\square\]

REFERENCES

[BG] I. Biswas and T. L. Gómez, Connections and Higgs fields on a principal bundle, *Ann. Global Anal. Geom.* 33 (2008), 19–46.

[BB] I. Biswas and U. Bruzzo, On semistable principal bundles over a complex projective manifold, *Int. Math. Res. Not.* (2008), no. 12, Art. ID rnn035.

[BH] U. Bruzzo and D. Hernández Ruipérez, Semistability vs. nefness for (Higgs) vector bundles, *Differential Geom. Appl.* 24 (2006), 403–416.

[DM] P. Deligne and J. S. Milne, Tannakian Categories, *Hodge cycles, motives, and Shimura varieties*, by P. Deligne, J. S. Milne, A. Ogus and K.-Y. Shih, pp. 101–228, Lecture Notes in Mathematics, 900, Springer-Verlag, Berlin-Heidelberg-New York, 1982.

[DP] J.-P. Demailly and M. Paun, Numerical characterization of the Kähler cone of a compact Kähler manifold, *Ann. of Math.* 159 (2004), 1247–1274.

[DPS] J.-P. Demailly, T. Peternell and M. Schneider, Compact complex manifolds with numerically effective tangent bundles, *J. Algebraic Geom.* 3 (1994), 295–346.

[Fu1] A. Fujiki, Closedness of the Douady spaces of compact Kähler spaces, *Publ. Res. Inst. Math. Sci.* 14 (1978), 1–52.

[Fu2] A. Fujiki, On automorphism groups of compact Kähler manifolds, *Invent. Math.* 44 (1978), 225–258.

[Ko] S. Kobayashi, *Differential geometry of complex vector bundles*, Publications of the Mathematical Society of Japan, 15. Kano Memorial Lectures, 5, Princeton University Press, Princeton, NJ; Princeton University Press, Princeton, NJ, 1987.

[Mi] Y. Miyaoka, The Chern classes and Kodaira dimension of a minimal variety. *Algebraic geometry*, Sendai, 1985, 449–476, Adv. Stud. Pure Math., 10, North-Holland, Amsterdam, 1987.

[Mo] C. Mourougane, Images directes de fibrés en droites adjoints, *Publ. Res. Inst. Math. Sci.* 33 (1997), 893–916.

[Se] J.-P. Serre, Représentations linéaires et espaces homogènes kählériens des groupes de Lie compacts (d’après Armand Borel et André Weil), Séminaire Bourbaki, Vol. 2, Exp. No. 100, 447–454, Soc. Math. France, Paris, 1995.

[Sn] D. M. Snow, Homogeneous vector bundles, https://www3.nd.edu/~snow/Papers/HomogVB.pdf.

[Va] J. Varouchas, Kähler spaces and proper open morphisms, *Math. Ann.* 283 (1989), 13–52.

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