Financial Market Dynamics

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Abstract

Distributions derived from non-extensive Tsallis statistics are closely connected with dynamics described by a nonlinear Fokker-Planck equation. The combination shows promise in describing stochastic processes with power-law distributions and superdiffusive dynamics. We investigate intra-day price changes in the S&P500 stock index within this framework by direct analysis and by simulation. We find that the power-law tails of the distributions, and the index’s anomalously diffusing dynamics, are very accurately described by this approach. Our results show good agreement between market data, Fokker-Planck dynamics, and simulation. Thus the combination of the Tsallis non-extensive entropy and the nonlinear Fokker-Planck equation unites in a very natural way the power-law tails of the distributions and their superdiffusive dynamics.
I. INTRODUCTION

In anomalously diffusing systems, a mean-square displacement scale $s$ with time according to a power law, $t^\alpha$, with $\alpha > 1$ (superdiffusion) or $\alpha < 1$ (subdiffusion). (The case $\alpha = 1$ corresponds to normal diffusion.) Anomalous diffusion has been observed in systems as widely varied as plasma flow [1], surface growth [2], and financial markets [3]. A general framework for treating superdiffusive systems is provided by the nonlinear Fokker-Planck equation, which is associated with an underlying Ito-Langevin process [4–6]. This in turn has a very interesting connection to the nonextensive entropy proposed by Tsallis [7]: the nonlinear Fokker-Planck equation is solved by time-dependent distributions which maximize the Tsallis entropy [5,8]. This unexpected connection between thermostatics and anomalous diffusion gives an entirely new way to approach the study of dynamics of anomalously diffusing systems. In this paper we use this viewpoint to address the dynamics of financial markets.

Several financial markets’ indices as well as their member stocks are characterized by price changes whose variances have been shown to undergo anomalous (super) diffusion under time evolution [3,9–11]. Moreover, the probability distributions of the price changes have power law tails [3,12]. An open long-term question is how best to describe these distributions and their time evolution. Earlier work has shown that the power-law tails can be described by a distribution which maximizes the Tsallis entropy [13,14]. The connection mentioned above then suggests that the market dynamics might be controlled by a nonlinear Fokker-Planck equation. In this description the power-law tails and the anomalous diffusion arise together quite naturally. Here we show that this approach seems to accurately describe high-frequency intra-day price changes for the S&P500 index. It may also be suitable to describe the superdiffusion and power law behaviors observed in a broad range of markets and exchanges [3].

The description we will use was developed in the general context of anomalously diffusing systems by Tsallis and Bukman [5] and Zanette [8]. Here we briefly summarize their results, which are based on a maximization of entropy subject to certain constraints. The central ingredient is a time-dependent probability distribution $P(x,t)$ of a stochastic variable $x$. The point of departure from normal maximum entropy approaches is the feature that the entropy used is the non-extensive Tsallis entropy,

$$S_q = -\frac{1}{1-q} \left(1 - \int P(x,t)^q \, dx\right). \tag{1}$$

The Tsallis parameter $q$ characterizes the non-extensivity of the entropy. In the limit $q \to 1$ the entropy becomes the usual logarithmic expression $S = -\int P \ln P$.

Associated with the non-extensive entropy is the use of the constraints

$$\int P(x,t) \, dx = 1, \tag{2}$$

$$\langle x - \bar{x}(t) \rangle_q \equiv \int [x - \bar{x}(t)] P(x,t)^q \, dx = 0, \tag{3}$$

$$\langle (x - \bar{x}(t))^2 \rangle_q \equiv \int [x - \bar{x}(t)]^2 P(x,t)^q \, dx = \sigma_q(t)^2. \tag{4}$$

The first of these is simply the normalization of the probability. However, in the latter two equations the probability distribution function is raised to the power $q$. Unless $q = 1$ these
are not the usual constraints leading to the mean and variance, and $\sigma_q^2$ (the ‘$q$-variance’) is not the ordinary variance. Notice that the Tsallis nonextensivity parameter $q$ is independent of time.

Maximizing the Tsallis entropy subject to these constraints, for fixed $q$, yields

$$P(x, t) = \frac{1}{Z(t)} \left\{ 1 + \beta(t)(q - 1)[x - \bar{x}(t)]^2 \right\}^{-\frac{1}{q-1}}. \tag{5}$$

Here parameters $Z$ (a normalization constant) and $\beta$ are Lagrange multipliers associated with the first and third constraints, and are given by

$$Z(t) = B\left(\frac{1}{2}, \frac{1}{q-1} - \frac{1}{2}\right)/\sqrt{(q-1)\beta(t)}, \tag{6}$$

$$\beta(t) = \frac{1}{2\sigma_q(t)^2 Z(t)^{q-1}}, \tag{7}$$

where $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ is Euler’s Beta function.

The ordinary variance of the distribution Eq. (5) is

$$\sigma^2(t) = \langle (x - \bar{x}(t))^2 \rangle_1 = \begin{cases} \frac{1}{(5 - 3q)\beta(t)}, & q < \frac{5}{3}, \\ \infty, & q \geq \frac{5}{3}. \end{cases} \tag{8}$$

Hence for applications to data of finite variance, the Tsallis parameter $q$ must lie within the range

$$1 \leq q < \frac{5}{3}. \tag{9}$$

The Tsallis function Eq. (5) can be viewed as a least biased probability distribution compatible with observed stochastic data with a certain mean $\bar{x}(t)$ and variance $\sigma^2(t)$ (but with a particular choice of $q$).

An important property of the probability distribution function Eq. (5) is that, with appropriate time-dependent parameters, it is the solution of a time evolution equation which leads naturally to anomalous diffusion [5,8]. Consider the nonlinear Fokker-Planck equation

$$\frac{\partial P(x, t)^\mu}{\partial t} = -\frac{\partial}{\partial x} [F(x)P(x, t)^\mu] + \frac{D}{2} \frac{\partial^2 P(x, t)^\nu}{\partial x^2}, \tag{10}$$

where $F(x)$ is a linear drift force, $F(x) = a - bx$. One can show [5,8] that a function of the form Eq. (5) solves this as long as

$$q = 1 + \mu - \nu \tag{11}$$

and the time dependences of the parameters are given by

$$-\frac{\mu}{\mu + \nu} \frac{dZ^{\mu+\nu}(t)}{dt} + 2\nu D\beta(t_0)Z(t_0)^{2\mu} - bZ^{\mu+\nu} = 0, \tag{12}$$

$$\frac{\beta(t)}{\beta(t_0)} = \left(\frac{Z(t_0)}{Z(t)}\right)^{2\mu}, \tag{13}$$

$$\frac{d\bar{x}}{dt} = a - b\bar{x}. \tag{14}$$
By comparing Eqs. (6) and (13) one easily shows that the nonlinear Fokker-Planck equation preserves the norm \( \int P(x,t) \, dx \) only if \( \mu = 1 \). Hereafter we specialize to this case. Then Eqs. (11, 12, 13) give

\[
\beta(t) - 3 - q^2 = \beta(t_0) - 3 - q^2 e^{-b(3-q)(t-t_0)} - 2D b^{-1} (2 - q) \left[ \beta(t_0) Z^2(t_0) \right]^{q+1} \left( e^{-b(3-q)(t-t_0)} - 1 \right). \tag{15}
\]

Z(t) is then obtained from Eq. (9).

II. APPLICATION TO A MARKET INDEX

It is apparent from the results sketched in the previous section that the Tsallis probability distribution function \( P(x,t) \) has properties which make it a good candidate for describing the anomalous diffusion of financial market indices, member stocks, and currency exchanges. First, in the large \( x \) limit \( P(x,t) \) becomes a power law distribution,

\[
P(x,t) \sim x^{-\frac{2}{q-1}}. \tag{16}
\]

This is in keeping with the observed power law tails in market distributions and, indeed, certain market distributions have been well fit by a distribution of the Tsallis form \( \text{(13)} \). In fact this fit to market data has a longer history — the Tsallis function is an extension to continuous values of \( q \) of the student-\( t \) distribution, which has been known for some time to provide a good fit to certain market data \( \text{(14)} \).

What has not been widely recognized in the finance-related applications of statistics is that the Tsallis distribution solves the nonlinear Fokker-Planck equation. Consequently it may provide a framework for understanding the dynamics of certain market data, including anomalous diffusion. It is this possibility that we will now discuss.

For application to market indices the Tsallis parameter \( q \) must lie within the range Eq. (9), ensuring that the regular variance remains finite and evolves for moderately long times as

\[
\sigma^2(t) \sim 1/\beta(t) \sim t^{\frac{2}{3-q}} \tag{17}
\]

[using Eq. (6, 13) in the case \( b \ll 1 \)]. Thus the Fokker-Planck equation can describe superdiffusive processes (with \( 1 < q < 5/3 \)).

Here we investigate one test case, the S&P500 stock market index, using fairly high-frequency 1-minute-interval data collected from July 2000 to January 2001. This data consists of a set of prices (index values) \( p(\tau) \) at discrete times \( \tau (\tau = 1, 2, \ldots, 50000 \text{ trading minutes}) \) obtained from the Terra-Lycos QCharts server \( \text{(10)} \). (The time period chosen has no special significance.) The quantity amenable to a stochastic analysis of the Ito-Langevin type is the price change during various time intervals \( \text{(10, 14, 16)} \). To agree with the notation used in the previous section we define \( x \) to be a price change during a time interval \( t \). For each fixed time interval \( t \) we generate a sequence of non-overlapping price changes \( \{x_1, x_2, \ldots\} \), with \( x_j = p(jt + 1) - p((j-1)t + 1) \). We view the data thus generated for each fixed time interval \( t \) as a sample selected from a population with some distribution \( P_{\text{market}}(x,t) \). Thus in this application the presumably stochastic variable \( x \) is a price change, and the ‘time’ variable \( t \) is really the corresponding time interval.
Anomalous diffusion occurs in random systems with correlations in time. Because the price change during a longer interval is a sum of price changes during shorter intervals, one has

\[ P(x, t_1 + t_2) = \int dx_1 P(x - x_1, t_2|x_1, t_1) P(x_1, t_1). \]  

(18)

where \( P(\cdot | \cdot) \) is a conditional probability. In the absence of correlation, price changes in different intervals are independent, so that \( P(x - x_1, t_2|x_1, t_1) = P(x - x_1, t_2) \). In this case it is easy to see that the variance grows linearly with time. Anomalous diffusion thus means that price changes during successive time intervals are not independent — naturally enough, since traders respond to earlier changes. The observed superdiffusion of financial markets thus indicates correlation, and consequently a nontrivial time dependence. Autocorrelation analyses of the S&P500 market have found strong correlations for times on the order of a few minutes, with weak persistent correlations at longer times. We find the same for our data set. Consequently in this work we will concentrate on the intra-day market dynamics of intervals less than one hour (1 min \( \leq t \leq 60 \) min).

If the dynamics are of Fokker-Planck form, then there must be a consistency between the distribution \( P_{\text{market}}(x, t) \) at a certain time \( t \) and the way the distributions evolve in time. One would have to find that the data at different times can be fit by distributions of Tsallis form, with a Tsallis parameter \( q \) independent of time. If so, then the variance will evolve in time according to Eq. (8,15) with the same \( q \) and appropriate values for \( D, b \).

We investigate the degree to which the S&P500 price changes at different time intervals \( t \) can be described by a Tsallis distribution evolving according to a Fokker-Planck equation. The approach is to bin the data and perform a direct nonlinear \( \chi^2 \) fit of Eq. (5). The ‘initial’ distribution corresponds to the shortest value of \( t \), here \( t_0 = 1 \) minute. The \( t_0 = 1 \) min price changes and a fit of Tsallis form are shown in Fig. 1. The fit parameters are \( q = 1.64 \pm 0.02 \) and \( \beta(t_0) = 4.90 \pm 0.11 \), and consequently [from Eq. (3)] \( Z(t_0) = 1.09 \pm 0.02 \). We then fit the data at different time intervals \( t \), with the Tsallis parameter \( q = 1.64 \) fixed and \( \beta(t) \) determined by the fit. It is important to this analysis that the data at all times continue to be well fit by the Tsallis distribution, and indeed this is what we find. (See Fig. 1.)

The resulting inverse variance \( \beta(t) \) extracted from the fits is shown in Fig. 2. Time evolution controlled by the Fokker-Planck equation predicts the time dependence for \( \beta(t) \) given in Eq. (15). Accordingly we fit this form to the extracted values, finding \( D = 0.217 \pm 0.003 \) and \( b = 0.047 \pm 0.004 \). The Fokker-Planck form for \( \beta(t) \) is also shown in Fig. (2).

It is clear from Fig. 2 that the agreement is quite good, indicating time evolution of nonlinear Fokker-Planck form.

As a check on our analysis we also generated simulated data using a Tsallis distribution Eq. (3) evolving in time as described in Eqs. (12,15). We then performed the same analysis that we performed on the S&P500 data. Briefly, the approach is as follows. We use the parameters \( q, D, b \) and initial values \( \beta(t_0), Z(t_0) \) obtained from the S&P500 data. Parameters \( \beta(t), Z(t) \) at later times are obtained from Eq. (15). At several times \( t \) we generate a set of random numbers \( x \) with the Tsallis distribution Eq. (3). This can be done by a transformation from a uniformly distributed set of random numbers [19], as follows. Consider a uniform distribution \( P_{\text{uniform}}(y) \) which is unity for \( 0 \leq y \leq 1 \) and zero otherwise. By choosing an appropriate function \( y(x) \) we can obtain the desired distribution from the uniform distribution in the form
\[
P(x,t) = P_{\text{uniform}}(y) \left| \frac{dy}{dx} \right|.
\]  
(19)

This can be solved for \( y \) as a hypergeometric function:

\[
y = \int_0^x P(x',t)dx' = x \cdot 2F_1 \left[ \frac{1}{2}, \frac{1}{q-1}, \frac{3}{2}, -(q-1)\beta(t)x^2 \right].
\]  
(20)

For each \( y \) generated uniformly, we numerically invert this to find \( x \). The resulting \( x \)'s are distributed according to the Tsallis form, Eq. (5). Thus the simulated price-change data is guaranteed to have a Tsallis distribution evolving in time according to the Fokker-Planck equation. We then analyzed this simulated data exactly as we did the financial data as a check. Fig. 2 shows that simulated data closely tracks the actual market data.

III. CONCLUSIONS

The nonlinear Fokker-Planck equation for a probability distribution \( P(x,t) \) has a time-dependent solution equivalent to the distribution obtained by maximization of the Tsallis nonextensive entropy. Consequently if a stochastic process has a Tsallis distribution it is natural to attempt to work backwards, and ask whether the underlying dynamics are Fokker-Planck. We have investigated this possibility in one market, analyzing the high-frequency intra-day dynamics of the S&P500 index within this framework. We found that the distribution \( P(x,t) \) is well fit by the Tsallis form at all times, and evolves in time according to the Fokker-Planck equation. Thus the combination of the Tsallis distribution and the Fokker-Planck equation unites in a very natural way the power-law tails of the distributions and their superdiffusive dynamics.

It would be of interest to apply this analysis to longer time windows, inter-day (long term) dynamics, and also to other markets.

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FIGURES

FIG. 1. Distribution of S&P500 index price changes at intervals $t = 1, 10, \text{ and } 60$ minutes, and
a fit of Tsallis form [Eq. (5)]. (Note: horizontal scales in the three plots are different.)

FIG. 2. Parameter $\beta(t)$ (which is proportional to the inverse of the variance) as a function of
time interval $t$. Filled circles: S&P500 index data. Solid curve: Fit of the Fokker-Planck $\beta(t)$ from
Eq. (15). Open diamonds: Simulated data using parameters from the fit. Along most of the curve
the simulated data are indistinguishable from the actual market data.
Price change distributions

Fig. 1 F. Michael
Fig. 2 F. Michael

The graph shows the function $\beta(t)$ as a function of time interval $t$ in minutes. The function decreases rapidly initially and approaches a horizontal asymptote as $t$ increases.