REGULAR BRANEWORLDS WITH BULK FLUIDS

IGNATIOS ANTONIADIS\textsuperscript{1,2,*} SPIROS COTSAKIS\textsuperscript{3,4†} IFIGENIEA KLAoudATOU\textsuperscript{4‡}

\textsuperscript{1} Laboratoire de Physique Théorique et Hautes Energies - LPTHE
Sorbonne Université, CNRS 4 Place Jussieu, 75005 Paris, France
\textsuperscript{2} Institute for Theoretical Physics, KU Leuven
Celestijnenlaan 200D, B-3001 Leuven, Belgium
\textsuperscript{3} Institute of Gravitation and Cosmology, RUDN University
ul. Miklukho-Maklaya 6, Moscow 117198, Russia
\textsuperscript{4} Research Laboratory of Geometry, Dynamical Systems and Cosmology
University of the Aegean, Karlovassi 83200, Samos, Greece

October 29, 2021

\textsuperscript{*} antoniad@lpthe.jussieu.fr
\textsuperscript{†} skot@aegean.gr
\textsuperscript{‡} iklaoud@aegean.gr
Abstract

We review studies on the singularity structure and asymptotic analysis of a 3-brane (flat or curved) embedded in a five-dimensional bulk filled with a ‘perfect fluid’ with an equation of state $p = \gamma \rho$, where $p$ is the ‘pressure’ and $\rho$ is the ‘density’ of the fluid, depending on the 5th space coordinate. Regular solutions satisfying positive energy conditions in the bulk exist only in the cases of a flat brane for $\gamma = -1$ or of AdS branes for $\gamma \in [-1, -1/2)$. More cases can be found by gluing two regular bunches of solutions at the position of the brane. However, only a flat brane for $\gamma = -1$ leads to finite Planck mass on the brane and thus localises gravity. In a more recent work, we showed that a way to rectify the previous findings and obtain a solution for a flat brane and a range of $\gamma$, that is both free from finite-distance singularities and compatible with the physical conditions of energy and finiteness of four-dimensional Planck mass, is by introducing a bulk fluid component that satisfies a non-linear equation of state of the form $p = \gamma \rho^\lambda$ with $\gamma < 0$ and $\lambda > 1$. 
## Contents

1 Introduction .................................................. 4

2 Setup and field equations ................................. 8

3 Energy conditions ........................................... 11

4 Linear fluid ...................................................... 13
   4.1 Flat brane .................................................. 14
      4.1.1 Energy conditions ...................................... 17
      4.1.2 Planck Mass ............................................. 18
   4.2 The special case $\gamma = -1$ for a flat brane ............ 19
   4.3 Curved brane ................................................. 20
      4.3.1 The null energy condition .............................. 26
      4.3.2 Localisation of gravity .................................. 26
   4.4 The special case $\gamma = -1$ for a curved brane .......... 27

5 Non-linear fluid ............................................... 29
   5.1 The case of $\lambda = 3/2$ .................................... 30
   5.2 Solutions for general $\lambda$ ................................. 31

6 Conclusions and open questions ......................... 33

Acknowledgments ................................................. 34

Bibliography .................................................... 34
1 Introduction

In this paper, we review the dynamical evolution and physical properties of a class of higher-dimensional models analysed in \[1, 2, 3, 4, 5, 6, 7\]. Our motivation for studying higher-dimensional models stems from the fact that they propose alternative approaches towards understanding and hopefully improving our view on challenging issues in cosmology and particle physics. In this review specifically, we focus on a class of brane-world models that offers interesting implications on the cosmological constant problem (hereafter abbreviated cc-problem).

The cc-problem arises from the disagreement between predictions from quantum field theories and observations regarding the value of the cosmological constant. In particular, the theoretical quantum corrections to the cosmological constant are naturally some 120 orders of magnitude higher than its observed value. Based on theory, the huge value of the cosmological constant, would automatically imply a huge value of the vacuum energy which would in turn give rise to a highly curved universe, a prediction that is not compatible with observations. To resolve the discrepancy between theory and observations, the bare value of the cosmological constant has to be unreasonably fine tuned.

A viable approach to this puzzling problem, is to re-examine it in the context of higher-dimensional models. A first study towards this way of resolving the cc-problem, was explored in \[8\]. The main idea was that in the framework of higher dimensions (4+2, where 2 are extra compactified a la Kaluza-Klein spatial dimensions), the cosmological constant could only curve the extra dimensions, leaving the four-dimensional observed universe (almost) flat. Following this scenario, a class of solutions with arbitrary (including zero) values of the observable cosmological constant was found in \[8\]. However, the question of why the solutions with a vanishing cosmological constant should be singled out was kept unanswered, while various factors that could favour these solutions were discussed, such as appropriate quantum corrections and/or additional interactions, or, even stability considerations.
Later on, the idea to use extra dimensions to attack the cc-problem, was revisited in \[9, 10, 11, 12, 13, 14, 15\]. The setup of these models is based on the idea that the observed universe is modeled by a four-dimensional hypersurface situated at a fixed position of an extra spatial dimension. The whole spacetime is five dimensional: there are four dimensions of space, out of which only three are spanned by the hypersurface and one dimension of time. Such a hypersurface is called 3-brane while the full higher-dimensional spacetime is called the bulk. Branes play an important role in string theory as they are the designated locations where open strings end and at the same time they are essential in proving duality between different versions of the theory. Additional interest on brane-worlds was sparked from the realization of a novel proposal towards resolving the hierarchy problem \[16, 17, 18, 19, 20, 21\] within their context.

In the brane-world scenario, the brane behaves like a thin surface layer that is embedded in the bulk and it is associated with a surface-energy momentum tensor that acts as a delta-source of matter. The surface-energy momentum tensor creates a jump in the extrinsic curvature and describes all Standard Model fields. Gravity and some non-standard fields on the other hand, experience also the fifth spatial dimension and interact with the fields on the brane, through a coupling function (tension) that depends on the setup of each model. In particular, the specific class of models that we review here, have a warped geometry with a line-element of the form

\[
ds^2 = a^2(Y)g_4 + dY^2
\]  

(1.1)

where \(Y\) is the extra dimension, \(g_4\) is the 4D metric and \(a(Y) > 0\) is the warp factor that describes deformations with respect to the extra dimension. The embedding of the brane in the bulk, introduces a \(Y \rightarrow -Y\) mirror symmetry and a set of Israel junction conditions \[22, 23\] that relate the tension of the brane with the value of the warp factor at the position of the brane (chosen at the origin of \(Y\)). This latter relation plays a key role as we discuss below, since it implicitly connects the tension of the brane with the value of the 4D scalar curvature.

Refs. \[9\] and \[10\] use higher-dimensional models in an effort to ameliorate the cc-
problem. Their proposal is to explore the possibility of a self-tuning mechanism. To understand better this mechanism, we have to keep in mind that the tension of the brane receives quartically divergent quantum corrections from vacuum energy and through the Israel conditions which, as mentioned above, are essentially boundary conditions describing the embedding of the brane in the bulk, transfers these corrections to the curvature of the brane. However, if it is possible to find flat-brane solutions irrespectively from the fluctuations of the brane tension, then we could end-up with a universe that is self-tuned to a vanishing cosmological constant on the brane.

In [9], the bulk matter is modeled by a scalar field that is minimally coupled to gravity and conformally coupled to the fields on the brane. The conformal coupling is carefully chosen to allow only for a flat brane in support to the self-tuning mechanism. In [10] on the other hand, the bulk matter can also contain a 5D cosmological constant and a variety of forms of tension.

A common feature of both models of [9] and [10], is the emergence of singularities that arise within finite distance from the position of the brane. While investigating the puzzling nature of the finite-distance singularities, it was argued initially in [9] that these singularities, on one hand, can be viewed to act like a reservoir through which all the vacuum energy is emptied, while on the other hand, can serve at successfully compactifying the extra dimension. Thus, it was implied that the existence of such singularities serves in achieving both 4D gravity macroscopically and a vanishing cosmological constant. The details of a mechanism that could lead to a smooth transition to 4D dynamics was not further explored.

Later works [11] however, showed that the flat-brane solutions of [10] containing finite-distance singularities fail to meet a consistency condition that would ensure that the field equations are globally satisfied. This result was extended in [12] by considering a variety of vacuum configurations. As it was pointed out in [11, 12], to obtain consistency, the singularities have to be resolved. One way to achieve this, is by introducing extra branes at the positions of the singularities. Unfortunately, embedding more branes
entails defining new boundary conditions which in turn introduces a type of fine tuning in the model. An alternative way to rectify the singularities, is by exploiting the mirror symmetry introduced by the embedding of the brane in the bulk and choosing the parameters of the model appropriately to construct a regular matching solution, by cutting and matching the part of the bulk that does not contain singularities. Again, this leads to more issues, since the matching solution gives an infinite Planck mass, thus failing to localize gravity on the brane.

A way to overcome this was explored in [13], with the use of a bulk scalar field with an unorthodox Lagrangian. It was shown that by choosing the range of parameters appropriately and using the matching mechanism mentioned above, it is possible to avoid finite-distance singularities and construct a flat-brane solution that is both regular and at the same time suitable for localizing gravity on the brane. However, the derived solution faces stability issues.

In [14], the possibility of finding general forms of bulk potentials leading to self-tuned solutions without a finite-distance singularity and being valid for a range of brane tensions compatible with localized gravity on the brane, was explored. It was found that singularities in self-tuned solutions are generic for localizing gravity on the brane. Resolving the singularities essentially re-introduces fine-tuning in accordance with [11, 12]. However, it was suggested that additional fields may help revive the self-tuning mechanism.

In [1, 2, 3, 4, 5, 6, 7], we generalized the work of [9] by modeling the bulk matter with a variety of components such as an analog of a perfect fluid $p = \gamma \rho$, where the ‘pressure’ $p$ and the ‘density’ $\rho$ are functions of the extra dimension only, an interacting mixture, or more recently, a non-linear fluid with equation of state $p = \gamma \rho^\lambda$. In most of these cases our models contained also a curved brane. We find that the non-linear equation of state $p = \gamma \rho^\lambda$, is the most appropriate type of bulk matter for generating regular solutions with fundamental physical properties. Such an equation of state has been studied previously in cosmology for its role in avoiding big-rip singularities during
late time asymptotics \cite{24, 25, 26}, obtaining inflationary models with special properties \cite{27}, unifying models of dark energy and dark matter \cite{28, 29}, but also in the analysis of singularities \cite{30, 31, 32}.

The mathematical tools that we used for performing our analysis in \cite{1, 2, 3, 4, 5, 6, 7}, include the method of asymptotic splittings \cite{33} that detects all possible asymptotic behaviors of solutions around a singularity, combined with a method for tracing envelopes, cf. \cite{4}, Section 2, which are essentially solutions with a smaller number of arbitrary constants but which, nonetheless, play a crucial role in determining the general behavior of solutions. Another tool is the analysis of asymptotic behaviors of Gaussian hypergeometric functions that arise in solutions of curved branes, or, even of flat branes for a non-linear bulk fluid. In this paper, we overview the basic mathematical details and physical implications of the body of work of \cite{1, 2, 3, 4, 5, 6, 7}.

The structure of this paper is as follows. In Section 2, we give the setup and field equations for our brane-worlds. In Section 3, we study the weak, strong and null energy conditions for the bulk fluid. Next, in Section 4, we overview the case of a linear fluid for a flat, or, curved brane and analyse the corresponding energy conditions, as well as, the possibility of localizing gravity on the brane. Then, in Section 5, we study from the same perspective, a non-linear fluid, first, for $\lambda = 3/2$ and then for general $\lambda$. Finally, in Section 6, we present our conclusions and discuss open questions.

\section{Setup and field equations}

We study a class of brane-world models that consist of a flat, or, a curved 3-brane embedded in a five-dimensional bulk. The bulk metric is given by

$$g_5 = a^2(Y)g_4 + dY^2,$$

(2.1)

where $g_4$ is the four-dimensional flat, de Sitter or anti de Sitter metric, \textit{i.e.},

$$g_4 = -dt^2 + f_k^2g_3,$$

(2.2)
with
\[ g_3 = dr^2 + h_k^2 g_2, \] (2.3)
and
\[ g_2 = d\theta^2 + \sin^2 \theta d\phi^2, \] (2.4)
with \( f_k = 1, \cosh(HT)/H, \cos(HT)/H \) \((H^{-1} \text{ is the de Sitter (or AdS) curvature radius)}), \( h_k = r, \sin r, \sinh r \), respectively and \( a(Y) \) is the warp factor \((a(Y) > 0) \) which we simply denote by \( a \).

In our notation, capital Latin indices take the values \( A, B, \cdots = 1, 2, 3, 4, 5 \), while lowercase Greek indices are taken to range as \( \alpha, \beta, \ldots = 1, 2, 3, 4 \), with \( t \) being the timelike coordinate, \((r, \theta, \phi, Y) \) the remaining spacelike ones and the 5th coordinate corresponding to \( Y \). The 5-dimensional Riemann tensor is defined by the formula,
\[ R^A_{
abla BCD} = \partial_C \Gamma^A_{BD} - \partial_D \Gamma^A_{BC} + \Gamma^M_{BD} \Gamma^A_{MC} - \Gamma^M_{BC} \Gamma^A_{MD} \] (2.5)
the Ricci tensor is the contraction,
\[ R_{AB} = R^C_{ACB}, \] (2.6)
and the five-dimensional Einstein equations on the bulk space are given by,
\[ G_{AB} = R_{AB} - \frac{1}{2} g_{AB} R = \kappa_5^2 T_{AB}. \] (2.7)

We assume that the bulk is filled with a fluid analogue with energy-momentum tensor of the form
\[ T_{AB} = (\rho + p) u_A u_B - p g_{AB}, \] (2.8)
where the ‘pressure’ \( p \) the ‘density’ \( \rho \) are functions only of the fifth dimension, \( Y \), and the velocity vector field is \( u_A = (0, 0, 0, 0, 1) \), that is \( u_A = \partial/\partial Y \), parallel to the \( Y \)-dimension. The five-dimensional Einstein equations can then be written as
\[ \frac{a'^2}{a^2} = \frac{\kappa_5^2}{6} \rho + \frac{kH^2}{a^2}, \] (2.9)
\[ \frac{a''}{a} = -\frac{\kappa_5^2}{6} (\rho + 2p), \] (2.10)

9
where \( k = \pm 1 \), and the prime (‘) denotes differentiation with respect to \( Y \). On the other hand, the equation of conservation,

\[
\nabla_B T^{AB} = 0,
\]

gives

\[
\rho' + 4\frac{a'}{a}(\rho + p) = 0. \tag{2.11}
\]

We assume that the density and pressure of the fluid are related according to the general equation of state

\[
p = \gamma \rho^\lambda, \tag{2.12}
\]

where \( \gamma \) and \( \lambda \) are constants. Inputting (2.12) in (2.9)-(2.11), we find

\[
\frac{a'^2}{a^2} = \frac{\kappa_5^2}{6} \rho + \frac{kH^2}{a^2}, \tag{2.13}
\]

\[
\frac{a''}{a} = -\frac{\kappa_5^2}{6}(\rho + 2\gamma \rho^\lambda), \tag{2.14}
\]

\[
\rho' + 4\frac{a'}{a}(\rho + \gamma \rho^\lambda) = 0. \tag{2.15}
\]

Before solving the system of Eqs. (2.13)-(2.15) and studying its asymptotic behaviors, we find it useful to outline below, the possible types of singularity that we will encounter in the next Sections. Denoting with \( Y_s \), the finite value of \( Y \) labeling the position of the singularity, we say that a finite-distance singularity is a

- **collapse** singularity, if \( a \to 0^+ \), as \( Y \to Y_s \),

- **big-rip** singularity, if \( a \to \infty \), as \( Y \to Y_s \).

Depending on the values of \( k, \gamma \) and \( \lambda \), the above behaviors may be accompanied by a divergence in the density, or, even in the pressure of the fluid. We emphasize that, these singularities are not related to geodesic incompleteness as in standard cosmology, but rather on a pathological behavior of the warp factor. In the absence of finite-distance singularities, we call the solutions *regular* and include in this category the behaviors
of the warp factor given above, provided that these occur *only* at infinite distance, i.e. $Y \to \pm \infty$.

Our study will be completed once we find a solution that has the following fundamental properties:

- it is regular (no finite-distance singularities)
- it satisfies physical conditions, such as energy conditions
- it leads to a finite Planck mass, thus, it localizes gravity on the brane.

Since the behaviors of solutions depend strongly on the curvature of the brane, as well as on the linearity/non-linearity of the equation of state, we present the various possibilities in separate Sections. Also, in the next Section, we explain briefly a way to formulate the weak, strong and null energy condition for our type of bulk matter. Detailed proofs of our results can be found in [4, 6].

## 3 Energy conditions

The weak, strong and null energy condition are physical requirements that we wish our solutions to fulfill. They can help us single out those solutions of Eqs. \((2.13)-(2.15)\) as more plausible from a physics point of view. Classically, it is convenient to translate the energy conditions to restrictions imposed on $p$ and $\rho$.

To work out the energy conditions for our type of fluid, we start by noting that in the formulation of the field equations, both the metric given by \((2.1)\) and the bulk fluid described by \((2.8)\) and \((2.12)\), appear as static with respect to the time coordinate $t$, because the evolution is taken with respect to the fifth spatial coordinate $Y$. Using this fact, we can reinterpret our fluid analogue as a real anisotropic fluid having the following energy momentum tensor,

$$
T_{AB} = (\rho^0 + p^0)u^0_A u^0_B - p^0 g_{\alpha\beta} \delta^\alpha_A \delta^\beta_B - p_Y g_{55} \delta^5_A \delta^5_B.
$$

(3.1)
where \( u^0_A = (a(Y), 0, 0, 0) \), \( A, B = 1, 2, 3, 4, 5 \) and \( \alpha, \beta = 1, 2, 3, 4 \). When we combine (2.8) with (3.1), we get the following set of relations,

\[
\begin{align*}
    p_Y &= -\rho \quad (3.2) \\
    \rho^0 &= -p \quad (3.3) \\
    p^0 &= p. \quad (3.4)
\end{align*}
\]

Note that, the last two relations imply that

\[
    p^0 = -\rho^0, \quad (3.5)
\]

which means that this type of matter satisfies a cosmological constant-like equation of state. Substituting (3.2)-(3.5) in (3.1), we find that

\[
    T_{AB} = -pg_{\alpha\beta}\delta^\alpha_A\delta^\beta_B + \rho g_{55}\delta^5_A\delta^5_B. \quad (3.6)
\]

At this point, we can start to formulate the energy conditions. First, we study the weak energy condition according to which, every future-directed timelike vector \( v^A \) should satisfy

\[
    T_{AB}v^Av^B \geq 0. \quad (3.7)
\]

This condition implies that the energy density should be non negative for all forms of physical matter [34]. Here we find that it translates to

\[
    p \geq 0, \quad (3.8)
\]

and

\[
    p + \rho \geq 0. \quad (3.9)
\]

Second, we work out the strong energy condition which states that

\[
    \left( T_{AB} - \frac{1}{3} T g_{AB} \right) v^Av^B \geq 0, \quad (3.10)
\]

for every future-directed unit timelike vector \( v^A \). In our case,

\[
    -p + \rho \geq 0, \quad (3.11)
\]
and
\[ p + \rho \geq 0. \] (3.12)

Finally, we study the null energy condition according to which, every future-directed null vector \( k^{A} \) should satisfy [35]
\[ T_{AB}k^{A}k^{B} \geq 0. \] (3.13)

Here we find that it translates to
\[ p + \rho \geq 0. \] (3.14)

In later Sections we are going to express these conditions with respect to the values of the parameters \( \gamma \) and \( \lambda \) of the equation of state, as this will enable us to automatically recognize those solutions that are compatible with the energy conditions.

### 4 Linear fluid

In this section, we review the behaviors of solutions for a linear fluid, which can be also viewed as an analogue of a ‘perfect’ fluid.

Inputting \( \lambda = 1 \) in the field equations (2.13)-(2.14), we find
\[
\frac{a'^2}{a^2} = \frac{k^2}{6} \rho + \frac{kH^2}{a^2}, \quad (4.1)
\]
\[
\frac{a''}{a} = -\frac{k^2}{6} (1 + 2\gamma) \rho, \quad (4.2)
\]
while (2.15) gives
\[ \rho' + 4(1 + \gamma) \frac{a'}{a} \rho = 0. \] (4.3)

Naturally, the forms of solutions of Eqs. (4.1)-(4.3), depend strongly on the values of \( k \) and \( \gamma \). We classify the types of solutions and examine each class, separately, in the following subsections.
4.1 Flat brane

To study the case of a flat brane, we first substitute $k = 0$ in (4.1) and find

$$\frac{a'^2}{a^2} = \frac{\kappa_5^2}{6} \rho. \quad (4.4)$$

Next, we integrate (4.3) to obtain the relation between $\rho$ and $a$ which reads

$$\rho = c_1 a^{-4(\gamma+1)}, \quad (4.5)$$

with $c_1$ an arbitrary constant.

We can now substitute (4.5) in (4.4) and integrate to derive the form of the warp factor, $a$,

$$a = \left(2(\gamma + 1) \left(\pm \sqrt{\frac{2A c_1}{3} Y + c_2}\right)^{1/(2(\gamma+1))}\right)^{1/(2(\gamma+1))}, \quad \gamma \neq -1, \quad (4.6)$$

where $A = \kappa_5^2/4$. Finally, we input (4.6) in (4.5) to find $\rho$:

$$\rho = c_1 \left(2(\gamma + 1) \left(\pm \sqrt{\frac{2A c_1}{3} Y + c_2}\right)^{1/(2(\gamma+1))}\right)^{-2}. \quad (4.7)$$

Substitution of our solution for $k = 0$ of $a$ and $\rho$ in (4.2) shows that the latter equation is satisfied.

Our solution (4.6) and (4.7) holds for all values of $\gamma$ except from $\gamma = -1$. The case of $\gamma = -1$ is a special one and is studied separately in a following subsection. For all other values of $\gamma$, we see that for a flat brane and a linear equation of state, there is always a finite-distance singularity located at $Y_s = \mp c_2 \sqrt{3/(2Ac_1)}$. The nature of the singularity depends on whether $\gamma$ is less, or, greater than $-1$, and can be classified into a collapse type of singularity with

$$a \rightarrow 0, \quad \rho \rightarrow \infty, \quad Y \rightarrow \mp c_2 \sqrt{3/(2Ac_1)}, \quad \gamma > -1, \quad (4.8)$$

or, big-rip type with

$$a \rightarrow \infty, \quad \rho \rightarrow \infty, \quad Y \rightarrow \mp c_2 \sqrt{3/(2Ac_1)}, \quad \gamma < -1. \quad (4.9)$$
Combining this outcome with similar results found in [9] for a massless scalar field, which can be also viewed as a fluid with $\gamma = 1$, we realize that the emergence of finite-distance singularities persists even for this more general type of bulk matter. A next step should therefore be to look for possible ways to rectify these singularities.

In [4], we explored the possibility of avoiding the singularities by constructing a regular matching solution from (4.6) and (4.7). The procedure we followed there, is similar to the one used in [13]: we cut and match the part of the bulk that is free from finite-distance singularities. This is indeed possible for an appropriate choice of the range of parameters. In particular, we have examined the following two choices:

I) $\gamma < -1$, $c_2 \leq 0$, with the $+$ sign for $Y < 0$ and the $-$ sign for $Y > 0$.

The matching solution then reads

$$a = \left(2(\gamma + 1) \left(-\sqrt{\frac{2A c_1}{3}}|Y| + c_2\right)\right)^{1/(2(\gamma+1))}, \quad (4.10)$$

and

$$\rho = c_1 \left(2(\gamma + 1) \left(-\sqrt{\frac{2A c_1}{3}}|Y| + c_2\right)\right)^{-2}, \quad (4.11)$$

with the brane placed at the origin $Y = 0$. Clearly then, both $a$ and $\rho$ are non-singular since the term

$$\left(-\sqrt{2A c_1/3}|Y| + c_2\right)$$

is always negative.

II) $\gamma > -1$, $c_2 \geq 0$, with the $+$ sign for $Y > 0$ and the $-$ sign for $Y < 0$. Then the matching solution is

$$a = \left(2(\gamma + 1) \left(\sqrt{\frac{2A c_1}{3}}|Y| + c_2\right)\right)^{1/(2(\gamma+1))}, \quad (4.12)$$

and

$$\rho = c_1 \left(2(\gamma + 1) \left(\sqrt{\frac{2A c_1}{3}}|Y| + c_2\right)\right)^{-2}. \quad (4.13)$$
Again, $a$ and $\rho$ are non-singular, since the term
\[
(\sqrt{2Ac_1/3}|Y| + c_2)
\]
is always positive.

We are going to focus only on the solution described by (4.10)-(4.11), which corresponds to $\gamma < -1$, since it is the only one of the two possibilities that leads to a finite four-dimensional Planck mass.

To examine further the adequacy of this solution, we ought to check the boundary conditions that describe the embedding of the brane in the bulk. A natural condition to impose is, that the warp factor and energy density are continuous functions. Note that in what follows, by writing $c_i^+$ ($c_i^-$) we refer to the value of an arbitrary constant $c_i$ at $Y > 0$ ($Y < 0$). The continuity of the warp factor at $Y = 0$ leads to the condition
\[
(2(\gamma + 1)c_2^+)^{1/(2(\gamma+1))} = (2(\gamma + 1)c_2^-)^{1/(2(\gamma+1))},
\]
or, since $c_2^+ \text{ and } c_2^-$ are real numbers, we have
\[
c_2^+ = \pm c_2^-,
\]
depending on the value of $\gamma$. Similarly, continuity of the density gives
\[
\frac{c_1^+}{(c_2^+)^2} = \frac{c_1^-}{(c_2^-)^2},
\]
and using (4.15) we find
\[
c_1^+ = c_1^-.
\]
On the other hand, the jump of the extrinsic curvature $K_{\alpha\beta} = 1/2(\partial g_{\alpha\beta}/\partial Y)$ ($\alpha, \beta = 1, 2, 3, 4$), is given by
\[
K_{\alpha\beta}^+ - K_{\alpha\beta}^- = -\kappa_5^2 \left( S_{\alpha\beta} - \frac{1}{3}g_{\alpha\beta}S \right),
\]
where the surface energy-momentum tensor $S_{\alpha\beta}$ (defined only on the brane and vanishing off the brane) is taken to be
\[
S_{\alpha\beta} = -g_{\alpha\beta} f(\rho),
\]
with \( f(\rho) \) denoting the brane tension and \( S = g^{\alpha\beta}S_{\alpha\beta} \) the trace of \( S_{\alpha\beta} \). For our type of geometry, (4.18) becomes
\[
a'(0^+) - a'(0^-) = -\frac{\kappa_5^2}{3} f(\rho(0)) a(0). \tag{4.20}
\]
Substitution of (4.10) and (4.19) in (4.20), leads to a junction condition for the arbitrary constants
\[
\sqrt{c_1} \left( \frac{1}{c_2^+} + \frac{1}{c_2^-} \right) = 4 \sqrt{\frac{2A}{3}} (\gamma + 1) f(\rho(0)), \tag{4.21}
\]
from which we see that we have to choose the plus sign in (4.15) and then (4.21) becomes
\[
\frac{\sqrt{c_1}}{c_2} = 2 \sqrt{\frac{2A}{3}} (\gamma + 1) f(\rho(0)). \tag{4.22}
\]

4.1.1 Energy conditions

So far, we were able to construct a regular matching solution for \( \gamma < -1 \), however, we still have to check if this range of \( \gamma \) is compatible with the energy conditions.

We begin with the weak energy condition: Keeping (3.8) as it is and inputting \( p = \gamma \rho \) in (3.9) gives
\[
p \geq 0 \tag{4.23}
\]
and
\[
(\gamma + 1) \rho \geq 0 \tag{4.24}
\]
so that either
\[
\gamma \geq -1 \text{ and, } \rho \geq 0, \text{ or, } \gamma \leq -1 \text{ and } \rho \leq 0 \tag{4.25}
\]
For a flat brane, \( \rho \geq 0 \) because of (4.4), so we end up with \( \gamma > -1 \) which is further refined to \( \gamma > 0 \) upon the requirement of (3.8). We end up with the condition
\[
p \geq 0 \text{ and } \gamma > 0. \tag{4.26}
\]

For the strong energy condition, on the other hand, we input \( p = \gamma \rho \) in (3.11) and (3.12). We find that
\[
(-\gamma + 1) \rho \geq 0 \text{ and } (\gamma + 1) \rho \geq 0 \tag{4.27}
\]
so

\[-1 \leq \gamma \leq 1,\]

from which we find that it is possible to have either \(p = 0\), or,

\[p < 0 \quad \text{and} \quad -1 \leq \gamma < 0,\]  \hspace{1cm} (4.28)

or,

\[p > 0 \quad \text{and} \quad 0 < \gamma \leq 1.\]  \hspace{1cm} (4.29)

Finally the null energy condition (3.14) for \(p = \gamma \rho\) gives (4.24), which again implies that

\[\gamma \geq -1.\]  \hspace{1cm} (4.30)

The inequalities (4.26), (4.28), (4.29) and (4.30) show that the energy conditions restrict \(\gamma\) to be at least greater than or equal to \(-1\), which means that the regular solution for \(\gamma < -1\), cannot satisfy the energy conditions.

### 4.1.2 Planck Mass

In this Section, we will show that the solution (4.10), provides the appropriate range of \(\gamma\) to obtain a finite four-dimensional Planck mass.

The value of the four-dimensional Planck mass, \(M^2_p = 8\pi/\kappa\), is determined by the following integral [13]

\[\frac{\kappa^2}{\kappa^2} = \int_{-Y_c}^{Y_c} a^2(Y) dY.\]  \hspace{1cm} (4.31)

For our solution, Eq. (4.10), the above integral becomes [4],

\[\int_{-Y_c}^{Y_c} \left(2(\gamma + 1) \left(-\sqrt{\frac{2Ac_1}{3}}|Y| + c_2\right)\right)^{1/(\gamma+1)} dY =\]  \hspace{1cm} (4.32)

\[= \frac{1}{2(\gamma + 2)} \sqrt{\frac{3}{2Ac_1}} \left(2(\gamma + 1) \left(\sqrt{\frac{2Ac_1}{3}}Y + c_2\right)^{(\gamma+2)/(\gamma+1)} \right.\]  \hspace{1cm} (4.33)

\[\left. - \frac{1}{2(\gamma + 2)} \sqrt{\frac{3}{2Ac_1}} \left(2(\gamma + 1) \left(-\sqrt{\frac{2Ac_1}{3}}Y + c_2\right)^{(\gamma+2)/(\gamma+1)} \right) \right|_{0}^{Y_c},\]  \hspace{1cm} (4.34)
In the limit $Y_c \rightarrow \infty$, we see that the Planck mass remains finite only for

$$-2 < \gamma < -1,$$

(4.35)

and takes the form

$$\frac{\kappa_5^2}{\kappa} = \sqrt{\frac{3}{2A c_1} \frac{(2(\gamma + 1)c_2)^{\frac{\gamma+2}{\gamma+1}}}{\gamma+2}},$$

(4.36)

This means that the interval $(-\infty, -1)$ for which the solution (4.10) is defined, has to be refined to $(-2, -1)$, after taking into account the requirement of a finite Planck mass. Combining this fact with the results of the previous subsection, we conclude that for a flat brane and a linear fluid with $\gamma \neq -1$, it is not feasible to construct a regular solution that satisfies both the requirement for a finite Planck mass given by (4.35) and the energy conditions.

### 4.2 The special case $\gamma = -1$ for a flat brane

Putting $\gamma = -1$ in (4.3), we find

$$\rho = c_1,$$

(4.37)

where $c_1$ is an integration constant. Since $\rho \geq 0$ from (4.4), we see that $c_1$ has to be non-negative. Substituting (4.37) in (4.4) we find

$$a(Y) = e^{\pm \sqrt{\left(\kappa_5^2/6\right)c_1} Y + c_2},$$

(4.38)

where $c_2$ is an integration constant. We note that this solution has no finite distance singularities and satisfies trivially the null energy condition ($\gamma = -1$). For a finite four-dimensional Planck mass, we can make the following choice: we can choose the $+$ sign for $Y < 0$ and the $-$ sign for $Y > 0$ and place the brane at $Y = 0$. Then the matching solution reads

$$a(Y) = e^{-\sqrt{\left(\kappa_5^2/6\right)c_1} Y + c_2},$$

(4.39)

This reduces to the Randall-Sundrum solution of [20], by setting $c_2 = 0$ and $c_1 = -\Lambda$, where $\Lambda < 0$ is the bulk cosmological constant in that model.
Continuity of the warp factor and density at the position of the brane give

\[ c_2^+ = c_2^-, \quad \text{and} \quad c_1^+ = c_1^- . \]  

(4.40)

For simplicity, we can also set \( c_2 = 0 \). Then using the junction condition (4.20), we can find the form of the brane tension which reads

\[ f(\rho(0)) = \frac{\sqrt{6c_1}}{\kappa_5} , \]  

(4.41)

and we note that the tension is positive.

Finally, the four-dimensional Planck mass is determined by (4.31). Here we have

\[ a^2(Y) = e^{-2\sqrt{(\kappa_5^2/6)c_1}|Y|} , \]  

(4.42)

and using the symmetry of the solution we find

\[ \frac{\kappa_5^2}{\kappa} = \int_{-Y_c}^{Y_c} a^2(Y) dY = 2 \int_{0}^{Y_c} e^{-2\sqrt{(\kappa_5^2/6)c_1}Y} dY = -\frac{\sqrt{6}}{\kappa_5\sqrt{c_1}} e^{-2\sqrt{(\kappa_5^2/6)c_1}Y_c} \bigg|_{0}^{Y_c} . \]  

(4.43)

Taking \( Y_c \to \infty \), we see that the four-dimensional Planck mass remains finite and is proportional to

\[ \frac{\sqrt{6}}{\kappa_5\sqrt{c_1}} . \]  

(4.44)

### 4.3 Curved brane

The impossibility of finding for a range of \( \gamma \) a flat-brane solution bearing the required physical properties mentioned in the previous subsections, led us to further research the question of whether the situation could be resolved by allowing for a nonzero brane curvature. Of course, this would inevitably bring back the cc-problem, as mentioned in the introduction. Still, it is worth exploring the impact of the curvature, as this would offer a deeper understanding of the factors that monitor the dynamics and evolution of the brane-worlds under investigation.

Assuming \( k \neq 0 \) in Eqs. (4.1) and (4.2) and substituting (4.5) in (4.1), we find

\[ a^2 = \frac{2}{3} Ac_1 a^{-2(2\gamma+1)} + kH^2 . \]  

(4.45)
For simplicity, we can set $C = 2/3A_c$ and keep in mind that the sign of $C$ follows the sign of $\rho$. Then (4.45) can be written as

$$a^2 = Ca^{-2(2\gamma + 1)} + kH^2. \quad (4.46)$$

It automatically follows from (4.46) that the LHS of this equation restricts the acceptable combinations of the signs of $C$ and $k$ and the possible asymptotic behaviors of $a$. For example, the case $C < 0$ and $k < 0$ becomes an impossible combination. Also, the case $C < 0$, $k > 0$ becomes possible only for,

$$\text{dS brane-world: } 0 < a^{-2(2\gamma + 1)} < -\frac{kH^2}{C}, \quad (4.47)$$

while, the case $C > 0$, $k < 0$ is allowed only for

$$\text{AdS brane-world: } a^{-2(2\gamma + 1)} > -\frac{kH^2}{C} > 0. \quad (4.48)$$

Clearly, the above inequalities show that both cases are likely to give rise to regular solutions. In particular, for the case $C < 0$ and a dS brane with $\gamma > -1/2$, (4.47) implies that $a^{2(2\gamma + 1)} > -C/(kH^2) > 0$, so that the warp factor, $a$, is bounded away from zero, which prevents collapse singularities from happening. The only way that this case may introduce a finite-distance singularity, is by allowing the warp factor to become divergent within a finite distance, thus signalling a big-rip singularity. However, further calculations presented in [6], showed that this singular behaviour is also excluded, which means that for $C < 0$ and a dS brane there is indeed a regular solution.

On the other hand, for the case $C > 0$ and an AdS brane with $\gamma < -1/2$, (4.48) implies that the warp factor is again bounded away from zero, thus excluding the existence of collapse singularities here, as well. However, for this latter case, we have to further restrict $\gamma$ on the interval $(-1, -1/2)$, in order to avoid a finite-distance big-rip singularity [6].

To derive solutions for a curved brane, we can proceed by writing Eq. (4.46) in the form,

$$\int \frac{da}{\sqrt{C^2a^{-2(2\gamma + 1)} + kH^2}} = \pm \int dY. \quad (4.49)$$
Naturally, the integration is more complicated in this case and cannot be performed directly. We can nevertheless, express our solutions implicitly by the Gaussian hypergeometric function $2F_1(\alpha, b, c; z)$, defined by

$$2F_1(\alpha, b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-\alpha} dt$$

and convergent for $|z| < 1$, where $\Gamma$ is the Gamma function. For this purpose, we use standard substitution formulas and bring the integral on the LHS of Eq. (4.49) in the form of an integral representation of a hypergeometric function $2F_1(\alpha, b, c; z)$ given by

$$2F_1(\alpha, b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!}$$

and provided that

$$0 < Re(b) < Re(c).$$

In our solutions, the corresponding parameters $b$ and $c$ are, either, constants or, functions of $\gamma$. This means that condition (4.52) above, determines the range of $\gamma$, for which the representations of solutions in terms of hypergeometric functions are valid.

We give a full list of solutions derived, in this way, below.

$I$) For a dS brane, $C > 0$ and

$II$) For dS brane, $C < 0$ and
IIa) \( \gamma < -1/2 \), the solution is given by (4.53).

IIb) \( \gamma > -1/2 \), the solution is,

\[
\pm Y + C_2 = \frac{(-C)^{\frac{1}{1+2\gamma}} (k H^2)^{\frac{1}{1+2\gamma}}}{2(2\gamma + 1)} \sqrt{a^{2(2\gamma + 1)} + \frac{C}{k H^2}} \times \\
\times {}_2F_1\left(\frac{1}{2}, \frac{\gamma}{2\gamma + 1}, \frac{3}{2}; 1 + \frac{k H^2}{C} a^{2(2\gamma + 1)}\right). \tag{4.55}
\]

III) For AdS, \( C > 0 \) and

IIIa) \( \gamma > -1/2 \), or, \( \gamma < -1 \) the solution is given by Eq. (4.54).

IIIb) \(-1 < \gamma < -1/2 \) the solution is,

\[
\pm Y + C_2 = -\frac{C^{\frac{1}{1+2\gamma}}}{2(2\gamma + 1)(-k H^2)^{\frac{2(2\gamma + 1)}{1+2\gamma}}} \sqrt{a^{-2(2\gamma + 1)} + \frac{k H^2}{C}} \times \\
\times {}_2F_1\left(\frac{4\gamma + 3}{2(2\gamma + 1)}, \frac{1}{2}, \frac{3}{2}; 1 + \frac{C}{k H^2} a^{-2(2\gamma + 1)}\right). \tag{4.56}
\]

We can deduce the asymptotic behaviors of the warp factor either directly, or, indirectly from the above implicit solutions. For example, take the case Ia) that is valid for a dS brane with \( C > 0 \) and \( \gamma < -1/2 \): the argument of the hypergeometric function is

\[
z = -\frac{C}{k H^2} a^{-2(2\gamma + 1)}.
\]

Note that, the power of \( a \) is positive and so, letting \( a \to 0 \), makes \( z \to 0 \) which means that the hypergeometric function is convergent and that it, actually, approaches one. Then from the LHS of (4.53), we see that \( Y \) will approach the finite value \( \pm C_2 \), which shows that this solution has a finite-distance collapse singularity.

A more indirect case is the one described by IIIb). As it follows from (4.48), the warp factor is bounded away from zero so that no collapse singularities exist in this case. We should, however, check if it is possible to have a divergent warp factor within finite distance. Since the power of \( a \) in the argument

\[
z = 1 + \frac{C}{k H^2} a^{-2(2\gamma + 1)}
\]
of the hypergeometric function is positive, letting \( a \to \infty \) leads us outside the disc of convergence of \( _2F_1 \). In order to proceed and find the behavior of the solution as \( a \to \infty \), we use the following rule \[36\]

\[
_2F_1(\alpha, b, c; z) = A(-z)^{-\alpha} _2F_1(\alpha, \alpha - c + 1, \alpha - b + 1; 1/z) + B(-z)^{-b} _2F_1(b, b - c + 1, b - \alpha + 1; 1/z), \tag{4.57}
\]

where the constants \( A \) and \( B \) are given by

\[
A = \frac{\Gamma(c)\Gamma(b - \alpha)}{\Gamma(c - \alpha)\Gamma(b)} \tag{4.58}
\]

and

\[
B = \frac{\Gamma(c)\Gamma(\alpha - b)}{\Gamma(c - b)\Gamma(\alpha)}. \tag{4.59}
\]

Using \(\text{(4.57)}\), the hypergeometric function of solution \( IIIb \) transforms to

\[
_2F_1 \left( \frac{4\gamma + 3}{2(2\gamma + 1)}, \frac{1}{2}; \frac{3}{2}, \frac{1 + C}{kH^2}a^{-2(2\gamma+1)} \right) = \]

\[
A_1 \left( -1 - \frac{C}{kH^2}a^{-2(2\gamma+1)} \right)^{-\frac{4(\gamma+3)}{4\gamma+2}} \times \]

\[
\times _2F_1 \left( \frac{4\gamma + 3}{2(2\gamma + 1)}, \frac{\gamma + 1}{2\gamma + 1}, \frac{3\gamma + 2}{2\gamma + 1}; \left( 1 + \frac{C}{kH^2}a^{-2(2\gamma+1)} \right)^{-1} \right) + \]

\[
+ B_1 \left( -1 - \frac{C}{kH^2}a^{-2(2\gamma+1)} \right)^{-1/2} \times \]

\[
\times _2F_1 \left( \frac{1}{2}, 0, \frac{\gamma}{2\gamma + 1}; \left( 1 + \frac{C}{kH^2}a^{-2(2\gamma+1)} \right)^{-1} \right), \tag{4.60}
\]

where \( A_1 \) and \( B_1 \) are constants given by,

\[
A_1 = \frac{\Gamma(-\gamma + 1)/(2\gamma + 1)}{2\Gamma(\gamma/(2\gamma + 1))}, \tag{4.61}
\]

and

\[
B_1 = \frac{\sqrt{\pi} \Gamma((\gamma + 1)/(2\gamma + 1))}{2\Gamma((4\gamma + 3)/(2(2\gamma + 1)))}. \tag{4.62}
\]
Substituting (4.60) in (4.56), we deduce that \( a \) behaves according to (we denote this below, with the symbol \( \sim \)) \[6\]

\[
a^{2(\gamma+1)} \sim \pm Y + C_2,
\]

and so, letting \( a \to \infty \), gives \( Y \to \pm \infty \). This means that the solution (4.56) is indeed regular.

We can use the same procedure for determining the singular, or, regular nature of solutions. For a convenient overall view of the most important behaviors of the solutions of cases Ia)-IIIb), we use the table below to illustrate them. Each behavior is accompanied with information regarding the corresponding range of \( \gamma \) and the solution from which it arises. For brevity, regular behaviors like \( a \to \infty \) as \( Y \to \infty \), that coexist with finite-distance singularities are not depicted in the table but can be found in \[6\].

**Table 1: Asymptotic behaviors for a curved brane and a linear fluid**

| \( \gamma \) | \((-\infty,-1)\) | \((-1,-1/2)\) | \((-1/2,1/2)\) | \((1/2,\infty)\) |
|--------------|-----------------|---------------|---------------|----------------|
| dS, \( C > 0 \) | collapse sing., Ia | collapse sing., Ia | collapse sing., Ib | collapse sing., Ib |
|               | big-rip sing., Ia |               |               |                |
| dS, \( C < 0 \) | collapse sing., IIa | collapse sing., IIa | \( \text{regular, IIIb} \) | \( \text{regular, IIIb} \) |
| AdS, \( C > 0 \) | big-rip sing., IIIa | \( \text{regular, IIIb} \) | collapse sing., IIIa | collapse sing., IIIa |

Summarizing, for a curved brane we obtain regular solutions from the following two cases:

- **IIb)**, referring to a dS brane with negative density and \( \gamma > -1/2 \)
- **IIIb)**, referring to an AdS brane with positive density and \(-1 < \gamma < -1/2\).

We note here that the special case of \( \gamma = -1 \), is studied separately in a subsequent subsection.
4.3.1 The null energy condition

The case of curved branes, has already proved successful in providing regular solutions, which was an impossible outcome for a flat brane. We still have to check whether the regular solutions can fulfill energy conditions as well as the requirement of a finite Planck mass.

For a linear fluid, we have already seen that the null energy condition is given by (4.25). For a curved brane it translates to having

$$\rho \geq 0 \quad \text{and} \quad \gamma \geq -1, \quad \text{or,} \quad \rho \leq 0 \quad \text{and} \quad \gamma \leq -1. \quad (4.64)$$

These two conditions may be written equivalently with respect to $C$ instead of $\rho$ as,

$$C \geq 0 \quad \text{and} \quad \gamma \geq -1, \quad (4.65)$$

and

$$C \leq 0 \quad \text{and} \quad \gamma \leq -1. \quad (4.66)$$

Combining conditions (4.65) and (4.66) with the ranges of $\gamma$ and $C$ for which the regular solutions of cases IIb) and IIIb) are defined, we see that only the regular solution of IIIb) is compatible with the null energy condition.

4.3.2 Localisation of gravity

To complete our study for a curved brane, we examine in this subsection, the requirement of a finite Planck mass.

We continue, for illustration purposes, to focus on the regular solution of case IIIb). The 4D-Planck mass is given by the integral of Eq. (4.31). The behavior of $a^2$ that we need to substitute in Eq. (4.31), can be deduced from (4.63)

$$a^2 \sim (|Y| + c_2)^{\frac{1}{\gamma+1}}, \quad (4.67)$$

after setting $\pm Y = |Y|$ and positioning the brane at $Y = 0$ [6]. It is straightforward to see that integration of $a^2$ gives an expression with $Y$ raised to the exponent,

$$\frac{\gamma + 2}{\gamma + 1}, \quad (4.68)$$
which is positive, since $-1 < \gamma < -1/2$ for the case IIIb). Therefore, the Planck mass is infinite in this case. We note that for a finite Planck mass, we need to have

$$-2 < \gamma < -1.$$  

As shown in [6], also the second regular solution of case IIb) fails to give a finite Planck mass, as well. The problem of localizing gravity on the brane, persists further in the case of regular matching solutions that can be constructed out of the singular solutions Ia) and Ib).

Summarizing, for the case of a curved brane and a linear fluid with $\gamma \neq -1$, there exist regular solutions for ranges of $\gamma$, however these ranges are inconsistent, with the requirement of a finite Planck mass (AdS brane with positive density), or, with both the null energy condition and the requirement of a finite Planck mass (dS brane with negative density). On the other hand, regular matching solutions that can be constructed by cutting the bulk and gluing the parts that are free from singularities, satisfy the null energy conditions but fail to localize gravity on the brane [6].

### 4.4 The special case $\gamma = -1$ for a curved brane

For $\gamma = -1$ and a curved brane, we find from (4.3), that

$$\rho = c_3,$$

where $c_3$ is an integration constant. Substituting (4.69) and $\gamma = -1$ in (4.2) we find

$$a'' - \kappa_5^2 \frac{c_3}{6} a = 0.$$  

For $c_3 > 0$ the above equation has the general solution

$$a(Y) = c_1 e^{\kappa_5 \sqrt{c_3/6} Y} + c_2 e^{-\kappa_5 \sqrt{c_3/6} Y},$$  

where $c_1$ and $c_2$ are arbitrary constants. Substitution of (4.71) in (4.1), determines the arbitrary constant $c_3$ in terms of $c_1$ and $c_2$. Here we find

$$c_3 = -\frac{3kH^2}{2c_1 c_2 \kappa_5^2}.$$  

27
Since \( c_3 > 0 \) we need to have the following restrictions on the signs of \( c_1, c_2 \) and \( k \)

\[
\text{either } c_1 c_2 < 0 \quad \text{and} \quad k > 0, \quad \text{or,} \quad c_1 c_2 > 0 \quad \text{and} \quad k < 0. \tag{4.73}
\]

For \( c_1 c_2 < 0 \) and \( k > 0 \), there is a finite-distance singularity at

\[
Y_s = \frac{\sqrt{6}}{2\kappa_5 \sqrt{c_3}} \ln \left( -\frac{c_2}{c_1} \right). \tag{4.74}
\]

For \( c_1 c_2 > 0 \) and \( k < 0 \), on the other hand, the above singularities are excluded. Also, from (4.71), it follows that both \( c_1 \) and \( c_2 \) have to be positive. This is exactly why the warp factor cannot approach zero within finite distance. Since divergence of the warp factor within finite distance is also impossible, we conclude that this solution is regular. In addition, we note that this solution satisfies the null energy condition trivially \((\gamma = -1)\). Next we introduce a brane, examine the boundary conditions at the position of the brane and comment on the resulting four-dimensional Planck mass.

As before, we can place the brane at \( Y = 0 \) and construct the matching solution

\[
a(Y) = c_1 e^{\sqrt{H^2/(4c_1 c_2)|Y|}} + c_2 e^{-\sqrt{H^2/(4c_1 c_2)|Y|}}. \tag{4.75}
\]

Then we can express the conditions imposed by the continuity of the warp factor and density at the position of the brane, in terms of the arbitrary constants \( c_1 \) and \( c_2 \), we find

\[
c_1^+ + c_2^- = c_1^- + c_2^+ \quad \text{and} \quad c_1^+ c_2^+ = c_1^- c_2^- \tag{4.76}
\]

which implies that \( c_1^+ = c_1^- = c_1 \) and \( c_2^+ = c_2^- = c_2 \). Using the junction condition (4.20), we can find the form of the brane tension, it reads

\[
f(\rho(0)) = \frac{3}{\kappa_5^2} \sqrt{\frac{H^2 c_2 - c_1}{c_1 c_2 c_1 + c_2}}. \tag{4.77}
\]

We note that the sign of the tension depends on the ordering between \( c_1 \) and \( c_2 \): for \( c_2 > c_1 \) the tension is positive, while for \( c_2 < c_1 \) the tension is negative.

Finally, we see that for the solution (4.73), the integral in (4.31) that determines the four-dimensional Planck mass, diverges. The only way of ending up with a finite Planck mass from this solution is by compactifying \( Y \) with a second brane.
5 Non-linear fluid

The problems we faced with the solutions of flat, or, curved branes analysed above, can be resolved by allowing \( \lambda \neq 1 \), in the equation of state (2.12). We are going to briefly review here, the main features of solutions for a non-linear equation of state and a flat brane, while full details can be found in [7].

We start by substituting \( k = 0 \) in the system (2.13)-(2.15), leading to

\[
\frac{a'^2}{a^2} = \frac{\kappa^2_5}{6} \rho,
\]

(5.1)

\[
\frac{a''}{a} = -\frac{\kappa^2_5}{6} (2 \gamma \rho^\lambda + \rho),
\]

(5.2)

and

\[
\rho' + 4(\gamma \rho^\lambda + \rho) \frac{a'}{a} = 0.
\]

(5.3)

Before solving the system of equations above, we can first check the restrictions that the null energy condition imposes on the parameters of a non-linear fluid. Inputting \( p = \gamma \rho^\lambda \) in the null energy condition (3.14), we obtain

\[
\gamma \rho^\lambda + \rho \geq 0,
\]

(5.4)

or, equivalently,

\[
\rho^\lambda (\gamma + \rho^{1-\lambda}) \geq 0.
\]

(5.5)

Since \( \rho \geq 0 \) from Eq. (5.1), we see that the null energy condition can be written as

\[
\gamma + \rho^{1-\lambda} \geq 0.
\]

(5.6)

To derive a solution of the system of Eqs. (5.1)-(5.3), we integrate the continuity equation (5.3) to find the relation between the warp factor and the density. In the integration process we arrive at a logarithmic term of the form \( \ln |\gamma + \rho^{1-\lambda}| \). To incorporate from the beginning the null energy condition (5.6), we choose to ignore the absolute value and simply put this term equal to \( \ln(\gamma + \rho^{1-\lambda}) \). The resulting relation between \( \rho \) and \( a \) is

\[
\rho = (-\gamma + c_1 a^{4(\lambda-1)})^{1/(1-\lambda)},
\]

(5.7)
where
\[ c_1 = \frac{\gamma + \rho_0^{1-\lambda}}{a_0^{4(\lambda-1)}} \]  
(5.8)
with \( \rho_0 = \rho(Y_0) \), \( a_0 = a(Y_0) \) being the initial conditions. According to (5.6) this translates to \( c_1 \geq 0 \).

To avoid the singularity in the density with \( \rho \to \infty \) for \( \lambda > 1 \) and
\[ a^{4(\lambda-1)} = \frac{\gamma}{c_1} \]  
(5.9)
we take, in what follows, \( \gamma < 0 \).

Next, we substitute (5.7) in (5.1) and integrate. We find
\[ \int \frac{a}{(c_1 - \gamma a^{4(1-\lambda)})^{1/(2(1-\lambda))}} \, da = \pm \frac{\kappa_5}{\sqrt{6}} \int dY. \]  
(5.10)
We note that we can calculate directly the above integral for values of \( \lambda \) that make \( 1/(2(1-\lambda)) \) a negative integer. These are: \( \lambda = (n+1)/n \), with \( n = 2k \) and \( k \) a positive integer. We study a characteristic example of such choice in the next paragraph.

5.1 The case of \( \lambda = 3/2 \)

Let us focus first on the simplest case, \( n = 2 \) corresponding to \( \lambda = 3/2 \). This makes the exponent \( 1/(2(1-\lambda)) \) in the integral on the LHS of Eq. (5.10), equal to \(-1\). It is then straightforward to integrate. We arrive at the following implicit solution
\[ \pm Y + C_2 = \frac{\sqrt{6}}{\kappa_5} \left( \frac{c_1}{2} a^2 - \gamma \ln a \right), \]  
(5.11)
where \( C_2 \) is an integration constant. Looking at solution (5.11), we see that we can have the following asymptotic behaviors
\[ a \to \infty, \quad \rho \to 0, \quad p \to 0, \quad \text{as} \; Y \to \pm \infty \]  
(5.12)
\[ a \to 0^+, \quad \rho \to 1/\gamma^2, \quad p \to -1/\gamma^2, \quad \text{as} \; Y \to \pm \infty, \]  
(5.13)
which show that all pathological behaviors of \( a \), become possible only at infinite distance, and therefore this solution is regular.
In addition to its good features of regularity and compatibility with the null energy condition, the solution (5.11) also offers the possibility to construct a matching solution that leads to a finite 4D-Planck mass; hence, it embodies all the required physical properties. The matching solution reads

\[ |Y| = \frac{\sqrt{6}}{\kappa_5} \left( -\frac{c_1}{2} a^2 + \gamma \ln a - \frac{\gamma}{2} - \gamma \ln \sqrt{-\frac{\gamma}{c_1}} \right), \quad 0 < a \leq \sqrt{-\frac{\gamma}{c_1}}, \]  

(5.14)

with the brane positioned at \( Y = 0 \).

To calculate the 4D-Planck mass, we first figure out from (5.14) the behaviour of \( a^2 \) as \( Y \to -\infty \), which reads

\[ a^2 \sim e^{-\left(\frac{\sqrt{6} \kappa_5}{(3\gamma)}\right)Y}. \]  

(5.15)

Then by using the symmetry of (5.14), we write the integral (4.31) in the following form

\[ \int_{-Y_c}^{Y_c} a^2(Y) dY = 2 \int_{-Y_c}^{0} a^2(Y) dY \sim 2 \int_{-Y_c}^{0} e^{-\left(\frac{\sqrt{6} \kappa_5}{(3\gamma)}\right)Y} dY = -\frac{\sqrt{6}}{\kappa_5} \frac{\gamma}{(1 - e^{\left(\frac{\sqrt{6} \kappa_5}{(3\gamma)}\right)Y_c})}. \]  

(5.16)

Taking \( Y_c \to \infty \) and keeping in mind that we consider only negative values of \( \gamma \), we see that the Planck mass remains finite and is proportional to

\[ -\frac{\sqrt{6}}{\kappa_5} \frac{\gamma}{}. \]

### 5.2 Solutions for general \( \lambda \)

For general \( \lambda \), solving the system (5.1)-(5.3) becomes much more complicated. For a convenient overview, we outline below, the types of new solutions that we obtain and comment on their asymptotic behaviors. Full details can be found in [7].

For all values of \( \lambda > 1 \), the solutions share all the fine qualities, previously, encountered in the solution for \( \lambda = 3/2 \). In particular, for \( \lambda = 1 + 1/(2k) \), with \( k \) a positive integer, we find the following form of solution

\[ \pm Y + c_2 = \frac{\sqrt{6}}{\kappa_5} \left( \sum_{s=0}^{k-1} \frac{k!}{(k-s)!s!} \frac{c_1^{k-s}}{2 - 2s/k} a^{2s/k} (-\gamma)^s + (-\gamma)^k \ln a \right). \]  

(5.17)
Furthermore, for \( \lambda > 3/2 \) we have the solution
\[
\pm Y + c_2 = \sqrt{6} \frac{a^2}{\kappa_5} \left( \frac{a^2}{2} (c_1 - \gamma a^{4(1-\lambda)})^{1/(2(\lambda-1))} - \frac{\gamma c_1^{(3-2\lambda)/(2(\lambda-1))}}{2(3-2\lambda)} a^{2(3-2\lambda)} \times 2F_1 \left( \frac{3 - 2\lambda}{2(1 - \lambda)}, \frac{3 - 2\lambda}{2(1 - \lambda)}, \frac{3 - 2\lambda}{2(1 - \lambda)} + 1; \frac{\gamma a^{4(1-\lambda)}}{c_1} \right) \right),
\] (5.18)
For \( 1 < \lambda < 3/2 \), on the other hand, we have solutions valid inside intervals of the form \((1 + 1/(2k), 1 + 1/2(k - 1))\) with \(k\) a positive integer such that \(k \geq 2\), given by
\[
\pm Y + c_2 = \sqrt{6} \frac{n}{\kappa_5} \left( \sum_{s=0}^{n/2-1} \frac{(-\gamma)^s}{2(1 - 2s(\lambda - 1))} (c_1 a^{4(\lambda-1)} - \gamma)^{1/(2(\lambda-1)) - s} \right. \\
+ \left. \frac{(-\gamma)^{n/2}(c_1)^{(n+1)-n\lambda}/(2(\lambda-1))}{2((n+1) - n\lambda)} a^{2((n+1)-n\lambda)} \times 2F_1 \left( \frac{(n+1) - n\lambda}{2(1 - \lambda)}, \frac{(n+1) - n\lambda}{2(1 - \lambda)} + 1; \frac{\gamma a^{4(1-\lambda)}}{c_1} \right) \right).\] (5.19)
All solutions for \( \lambda > 1 \) are free from finite-distance singularities, and follow the asymptotic behaviors
\[
a \rightarrow \infty, \quad \rho \rightarrow 0, \quad p \rightarrow 0, \quad \text{as} \quad Y \rightarrow \pm \infty \quad \text{(5.20)}
\]
\[
a \rightarrow 0^+, \quad \rho \rightarrow (-\gamma)^{1/(1-\lambda)}, \quad p \rightarrow -(-\gamma)^{1/(1-\lambda)}, \quad \text{as} \quad Y \rightarrow \pm \infty. \quad \text{(5.21)}
\]
We can also construct a matching solution with a finite Planck mass from every solution with \( \lambda > 1 \), by applying the method presented in the previous Sections [7].

Finally, for \( \lambda < 1 \), the situation changes drastically, because of the emergence of finite-distance singularities of the collapse type. In particular, the solution for \( \lambda < 1 \) reads
\[
\pm Y + c_2 = \sqrt{6} \frac{1}{\kappa_5} c_1^{1/(2(\lambda-1))} a^2 2F_1 \left( \frac{1}{2(1 - \lambda)}, \frac{1}{2(1 - \lambda)}; \frac{1}{2(1 - \lambda)} + 1; \frac{\gamma a^{4(1-\lambda)}}{c_1} \right), \quad \text{(5.22)}
\]
As shown in [7], this solution has a finite-distance singularity at \( Y \rightarrow \pm c_2 \), with
\[
a \rightarrow 0^+, \quad \rho \rightarrow \infty, \quad p \rightarrow 0, \quad \text{if} \quad \lambda < 0 \quad \text{(5.23)}
\]
\[
a \rightarrow 0^+, \quad \rho \rightarrow \infty, \quad p \rightarrow \infty, \quad \text{if} \quad 0 < \lambda < 1. \quad \text{(5.24)}
\]
The behaviors of \( p \) and \( \rho \) above, have been deduced from Eqs. (2.12) and (5.7).
6 Conclusions and open questions

We have reviewed the effect of the curvature and equation of state of the bulk fluid in the behavior of solutions of brane-worlds, consisting of a 3-brane embedded in a five-dimensional bulk.

For a linear equation of state with $\gamma \neq -1$ and a flat brane, there is always a finite-distance singularity. The types of singularity that arise in this case are, the collapse type which is determined by a vanishing warp factor and a divergent density and pressure (for $\gamma \neq 0$), or, a big-rip type that is signaturred by a divergent warp factor and also density and pressure. The avoidance of such singularities becomes possible only after cutting and matching the part of the bulk that is free from singularities. Still, the solutions we obtain in this way, cannot satisfy, simultaneously, requirements set by energy conditions and localization of gravity on the brane. An exception to this result, is the case of $\gamma = -1$ which gives a solution that can be translated to the one arising within the scenario of [20]. For this particular value of $\gamma$, it is possible to have a regular solution (which is half of AdS$_5$) that trivially satisfies the null energy condition and at the same time gives a finite four-dimensional Planck mass.

For a curved brane and a linear fluid, on the other hand, the situation improves in the sense that, regular solutions now become possible for a range of $\gamma$. Some of the regular solutions can even satisfy the null energy condition; they correspond to AdS branes for $\gamma$ in the region $[-1, 1/2)$. However, the problem of localizing gravity on the brane met, previously, in the case of a flat brane, continues to arise and the only way to overcome it, is by compactifying the bulk with a second brane as in [19].

Finally, for a flat brane and a non-linear equation of state, the situation is resolved: we can construct a regular solution consistent with the null energy condition which also localizes successfully gravity on the brane. It is an interesting open question whether such non-linear equation of state satisfying the null energy condition can be realised using an underlying microscopic description.

In this paper, we have also presented results which establish that there is a close
connection between 3-brane setups in a five-dimensional bulk and cosmological solutions having a 4-dimensional spatial slice and evolving in proper time. This is accomplished by transforming the $Y \equiv Y_{OLD}$-dimension into proper time, $Y_{OLD} \to it_{NEW}$, and the timelike dimension on the brane into a fifth spatial coordinate, $t_{OLD} \to -iY_{NEW}$, leaving the remaining three spatial coordinates intact. Therefore we can map from a braneworld, where everything depends on the transverse bulk space coordinate $Y_{OLD}$, to a cosmological spacetime in 4+1-dimensions evolving in time. This establishes a possible connection between cosmological phenomena and braneworld properties. In particular, the interpretation of the singular (in the transverse bulk coordinate) brane solutions, or those with some regularity (as discussed here), may correspond to cosmological solutions with special properties. For instance, the entropy of black holes in the braneworld may correspond to the cosmological entropy of standard (3+1)-spacetime. We expect to analyse this point in a future publication.

Acknowledgments

Work partially performed by I.A. as International professor of the Francqui Foundation, Belgium.

References

[1] Antoniadis I, Cotsakis S, Klaoudatou I. 2007. Braneworld cosmological singularities. Proceedings of MG11 meeting on General Relativity World Scientific 3 2054-2056 [arXiv:0701033 [gr-qc]].

[2] Antoniadis I, Cotsakis S, Klaoudatou I. 2010. Brane singularities and their avoidance. Class. Quant. Grav. 27 235018 [arXiv:1010.6175 [gr-qc]].

[3] Antoniadis I, Cotsakis S, Klaoudatou I. 2013. Brane singularities with mixtures in the bulk. Fortschr. Phys. 61 20-49 [arXiv:1206.0090 [hep-th]].
[4] Antoniadis I, Cotsakis S, Klaoudatou I. 2014. Enveloping branes and braneworld singularities. *Eur. Phys. J. C* 74 3192 [arXiv:1406.0611v2 [hep-th]].

[5] Antoniadis I, Cotsakis S, Klaoudatou I. 2015. Dynamics and asymptotics of braneworlds. Proceedings of the 13th Marcel Grossmann Meeting on General Relativity. *World Scientific* 1859-1861, doi: 10.1142/9789814623995-0299.

[6] Antoniadis I, Cotsakis S, Klaoudatou I. 2016. Curved branes with regular support. *Eur. Phys. J. C* 76 511 [arXiv:1606.09453 [hep-th]].

[7] Antoniadis I, Cotsakis S, Klaoudatou I. 2021. Regular braneworlds with nonlinear bulk-fluids. *Eur. Phys. J. C* 81 8, 771 [arXiv:2106.15669v2 [hep-th]].

[8] Rubakov VA, Shaposhnikov ME. 1983. Extra space-time dimensions: Towards a solution to the cosmological constant problem. *Phys. Lett. B* 125 139.

[9] Arkani-Hamed N, Dimopoulos S, Kaloper N, Sundrum R. 2000. A small cosmological constant from a large extra dimension. *Phys. Lett. B* 480 193-199 [arXiv:0001197v2 [hep-th]].

[10] Kachru S, Schulz M, Silverstein E. 2000. Bounds on curved domain walls in 5d gravity. *Phys. Rev. D* 62 085003 [arXiv:0002121 [hep-th]].

[11] Forste S, Lalak Z, Lavignac S, Nilles H P. 2000. A Comment on Self-Tuning and Vanishing Cosmological Constant in the Brane World *Phys. Lett. B* 481 360-364 [arXiv:0002164v2 [hep-th]].

[12] Forste S, Lalak Z, Lavignac S, Nilles H P. 2000. The Cosmological Constant Problem from a Brane-World Perspective. *JHEP* 09 034 [arXiv:0006139v2 [hep-th]].

[13] Forste S, Nilles H P, Zavala I. 2011. Nontrivial Cosmological Constant in Brane Worlds with Unorthodox Lagrangians. *JCAP* 07 007 [arXiv:1104.2570 [hep-th]].
[14] Csaki C, Erlich J, Grojean C, Timothy Hollowood T. 2000. General Properties of the Self-tuning Domain Wall Approach to the Cosmological Constant Problem. *Nucl. Phys. B* 584 359-386 [arXiv:0004133v2 hep-th]].

[15] Gubser SS. 2000. Curvature singularities: The good, the bad, and the naked. *Adv. Theor. Math. Phys.* 4 679-745 [arXiv:0002160 [hep-th]].

[16] Antoniadis I, A possible new dimension at a few TeV. 1990. *Phys. Lett. B* 246 377-384.

[17] Arkani-Hamed N, Dimopoulos S, Dvali G. 1998. The hierarchy problem and new dimensions at a millimeter. *Phys. Lett. B* 429 263-272 [arXiv:9803315 [hep-ph]].

[18] Antoniadis I, Arkani-Hamed N, Dimopoulos D, Dvali G 1998. New dimensions at a millimeter to a Fermi and superstrings at a TeV. *Phys. Lett. B* 436 257-263 [arXiv:hep-ph/9804398 [hep-ph]].

[19] Randall L, Sundrum R. 1999. A large mass hierarchy from a small extra dimension, *Phys. Rev. Lett.* 83 3370-3373 [arXiv:9905221 [hep-ph]].

[20] Randall L, Sundrum R. 1999. An alternative to compactification. *Phys. Rev. Lett.* 83 4690-4693 [arXiv:9906064 hep-th]].

[21] Antoniadis I. 2006. Physics of extra dimensions. *J. Phys. Conf. Ser.* 33 170-181.

[22] Israel W. 1966. Singular hypersurfaces and thin shells in general relativity. *Nuovo Cimento B* 44 1–14.

[23] Israel W. 1967. Singular hypersurfaces and thin shells in general relativity. *Nuovo Cimento B* 48 463.

[24] Srivastava SK. 2005. Future universe with $w < -1$ without big smash. *Phys. Lett. B* 619 1-4 [arXiv:0407048v4 [astro-ph]].
[25] Gonzalez-Diaz PF. 2003. You need not be afraid of phantom energy. *Phys. Rev. D* **68** 021303 [arXiv:0305559 [astro-ph]].

[26] Astashenok AV, Nojiri S, Odintsov SD, Yurov AV. 2012. Phantom Cosmology without Big Rip Singularity. *Phys. Lett. B* **709** 396-403 [arXiv:1201.4056 [gr-qc]].

[27] Barrow JD. 1990. Graduated inflationary universes. *Phys. Lett. B* **235** 40-43.

[28] Kamenshchik AY, Moschella U, Pasquier V. 2001. An alternative to quintessence. *Phys. Lett. B* **511** 265-268 [arXiv:0103004 [gr-qc]].

[29] Bento MC, Bertolami O, Sen AA. 2002. Generalized Chaplygin gas, accelerated expansion and dark energy-matter unification. *Phys. Rev. D* **66** 043507 [arXiv:0202064 [gr-qc]].

[30] Cotsakis S, Klaoudatou I. 2005. Future Singularities of Isotropic Cosmologies. *J. Geom. Phys.* **55** 306-315 [arXiv:0409022 [gr-qc]].

[31] Cotsakis S, Klaoudatou I. 2007. Cosmological Singularities and Bel-Robinson Energy. *J. Geom. Phys.* **57** 1303-1312 [arXiv:0604029 [gr-qc]].

[32] Nojiri S, Odintsov SD, Tsujikawa S. 2005. Properties of singularities in (phantom) dark energy universe. *Phys. Rev. D* **71** 063004 [arXiv:0501025 [hep-th]].

[33] S. Cotsakis and J. D. Barrow, The dominant balance at cosmological singularities, *J. Phys. Conf. Ser.* **68** (2007) 012004.

[34] Wald RM. 1984 *General Relativity*. University of Chicago Press.

[35] Poisson E. 2004 *A Relativist’s Toolkit*. Cambridge University Press.

[36] Wang ZX, Guo DR. 1989 *Special Functions*. World Scientific.
This figure "Capture01.PNG" is available in "PNG" format from:

http://arxiv.org/ps/2110.15079v1
This figure "Capture02.PNG" is available in "PNG" format from:

http://arxiv.org/ps/2110.15079v1
This figure "Capture3.PNG" is available in "PNG" format from:

http://arxiv.org/ps/2110.15079v1
This figure "Capture4.PNG" is available in "PNG" format from:

http://arxiv.org/ps/2110.15079v1
This figure "Capture5.PNG" is available in "PNG" format from:

http://arxiv.org/ps/2110.15079v1
This figure "Capture6.PNG" is available in "PNG" format from:

http://arxiv.org/ps/2110.15079v1