A DECOMPOSITION OF THE CURVATURE TENSOR
ON $SU(3)/T(k,l)$ WITH A $SU(3)$-INARIANT METRIC

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Abstract. In this paper, we decompose the curvature tensor (field) on the homogeneous Riemannian manifold $SU(3)/T(k,l)$ with an arbitrarily given $SU(3)$-invariant Riemannian metric into three curvature-like tensor fields, and investigate geometric properties.

1. Introduction

Let $(V, < , >)$ be an $n$-dimensional real inner product space. In this paper, we use the notion of a curvature-like tensor of type $(1,3)$ on $(V, < , >)$ (cf. (2.1)). We put

$L(V) := \{ L \mid L \text{ is a curvature-like tensor on } (V, < , >) \}$,

$L_1(V) := \{ L \in L(V) \mid L(u,v) = c \ u \wedge v \text{ for } u,v \in V \text{ and some } c \in \mathbb{R} \}$,

$L_\omega(V) := \{ L \in L(V) \mid \text{the Ricci tensor } Ric_L \text{ of } L \text{ is zero} \}$,

$L_2(V) := \{ L \in L_1(V) \mid < L, L' > = 0 \text{ for all } L' \in L_\omega(V) \}$.

Then $L(V)$ is decomposed into the orthogonal direct sum $L_1(V) \oplus L_\omega(V) \oplus L_2(V)$. Let $L = L_1 + L_\omega + L_2$ ($L \in L(V)$) be the decomposition corresponding to $L_1(V) \oplus L_\omega(V) \oplus L_2(V)$. The component $L_\omega$ of $L \in L(V)$ is said to be the Weyl tensor of $L$. The curvature-like tensors $L_1, L_\omega, L_2$ of $L = L_1 + L_\omega + L_2 \in L(V)$ are given in terms of the Ricci tensor $Ric_L$ and the scalar curvature $S_L$ of $L$ (cf. Lemma 2.1).

In this paper, using Lemma 2.1 we decompose the curvature tensor on the homogeneous Riemannian manifold $(SU(3)/T(k,l), g_{(\lambda_1,\lambda_2,\lambda_3)})$ into three curvature-like tensor fields. On the manifold $SU(3)/T(k,l)$, we deal with an arbitrary $SU(3)$-invariant Riemannian metric $g = g_{(\lambda_1,\lambda_2,\lambda_3)}$.

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Geometric properties on $SU(3)/T(k, l)$ have been studied by many mathematicians (cf. [1, 6, 9, 10]).

Now, let $R$ be the curvature tensor (field) on the homogeneous manifold $(SU(3)/T(k, l), g(\lambda_1, \lambda_2, \lambda_3))$, and $R = R^{(1)} + R^\omega + R^{(2)}$ the orthogonal decomposition of the curvature tensor $R$ corresponding to

$$\mathfrak{L}(T_o(G/H)) = \mathfrak{L}_1(T_o(G/H)) \oplus \mathfrak{L}_\omega(T_o(G/H)) \oplus \mathfrak{L}_2(T_o(G/H))$$

(cf. Lemma 2.1), where $G := SU(3)$, $H := T(k, l)$ and $O := \{T(k, l)\}$.

Let $m$ be the subspace of $su(3)$ such that $B(m, t(k, l)) = 0$ and $\text{Ad}(h)m \subset m$ $(h \in T(k, l))$, where $su(3)$ is the Lie algebra of $SU(3)$, $B$ is the negative of the Killing form of $su(3)$, $t(k, l)$ is the Lie algebra of $T(k, l)$, and $\text{Ad}$ is the adjoint representation of $SU(3)$ on $su(3)$.

In this paper, we represent the curvature-like tensors $R^{(1)}$, $R^\omega$ and $R^{(2)}$ in the orthogonal decomposition $R = R^{(1)} + R^\omega + R^{(2)}$ $(\in \mathfrak{L}_1(V) \oplus \mathfrak{L}_\omega(V) \oplus \mathfrak{L}_2(V))$ of the curvature tensor $R$ on $(SU(3)/T(k, l), g(\lambda_1, \lambda_2, \lambda_3))$ for $(k, l) \in D$, where

$$D := \mathbb{Z}^2 \setminus \{(0, t), (t, 0), (t, t), (t, -t), (t, -2t), (2t, -t) \mid t \in \mathbb{R}\}$$

(cf. Theorem 4.3). And then, under the condition $(k, l) \in D \subset \mathbb{Z}^2$, we obtain the Ricci tensor $Ric^{(2)}$ of the component $R^{(2)}$ of the curvature $R = R^{(1)} + R^\omega + R^{(2)}$ on the homogeneous space $(SU(3)/T(k, l), g(\lambda_1, \lambda_2, \lambda_3))$ (cf. Corollary 4.4). Furthermore, we estimate the Ricci curvature $r^{(2)}$ of the curvature-like tensor $R^{(2)}$ (cf. Proposition 4.5).

2. Preliminaries

Let $(V, <, >)$ be an $n$-dimensional real inner product space and $\mathfrak{gl}(V)$ the vector space of all endomorphisms of $V$. We denote by $\mathfrak{L}(V)$ the vector space of all tensors of type $(1, 3)$ on $V$ which satisfy the following properties:

$$L : V \times V \rightarrow \mathfrak{gl}(V)$$

is an $\mathbb{R}$-bilinear map such that, for all $v_1, v_2, v_3, v_4 \in V$,

$$(2.1) \quad < L(v_1, v_2)v_3, v_4 > - < L(v_2, v_1)v_3, v_4 >= - < L(v_1, v_2)v_4, v_3 >,
\quad < L(v_1, v_2)v_3, v_4 > + < L(v_2, v_3)v_1, v_4 > + < L(v_3, v_1)v_2, v_4 >= 0.$$
A tensor $L \in \mathcal{L}(V)$ (of type $(1,3)$ on $(V,\langle , \rangle)$ which satisfies the condition (2.1)) is called a curvature-like tensor (cf. [3, 4]). If $L \in \mathcal{L}(V)$, then we get from (2.1)

\[(2.2) \quad \langle L(v_1, v_2)v_3, v_4 \rangle = \langle L(v_3, v_4)v_1, v_2 \rangle \quad (v_1, v_2, v_3, v_4 \in V).\]

From now on, let $\{e_i\}_{i=1}^n$ be an orthonormal basis of $(V,\langle , \rangle)$. The Ricci tensor $\text{Ric}_L$ of type $(0,2)$ with respect to a curvature-like tensor $L$ on $V$ is defined by

\[(2.3) \quad \text{Ric}_L(v, w) := \sum_{i=1}^n \langle L(e_i, v)w, e_i \rangle \quad (v, w \in V).\]

The Ricci tensor $\text{Ric}_L$ of type $(1,1)$ with respect to $L \in \mathcal{L}(V)$ is defined by

\[(2.4) \quad \langle \text{Ric}_L(v), w \rangle := \text{Ric}_L(v, w) \quad (v, w \in V).\]

For $L \in \mathcal{L}(V)$, we obtain from (2.1) $\sim$ (2.4)

\[
\text{Ric}_L(v, w) = \langle \text{Ric}_L(v), w \rangle = \text{Ric}_L(w, v) = \langle \text{Ric}_L(w), v \rangle
\]

for $v, w \in V$.

The trace of $\text{Ric}_L$ for $L \in \mathcal{L}(V)$

\[(2.5) \quad S_L := \sum_{i=1}^n \langle \text{Ric}_L(e_i), e_i \rangle = \sum_{i,j=1}^n \langle L(e_j, e_i)e_i, e_j \rangle
\]

is called the scalar curvature with respect to $L \in \mathcal{L}(V)$. The sectional curvature $K_L(\sigma)$ ($L \in \mathcal{L}(V)$) for each plane $\sigma = \{v, w\}_{\mathbb{R}}(\subset V)$ is defined by

\[
K_L(\sigma) = \frac{\langle L(v, w)w, v \rangle}{\langle v, v \rangle \langle w, w \rangle - \langle v, w \rangle^2}.
\]

In general, the inner product $\langle , \rangle$ on $\mathcal{L}(V)$ is defined by

\[(2.6) \quad \langle L, L' \rangle = \sum_{i,j,k,l=1}^n L_{ijk}^l \cdot L'_{ij}^l,
\]

where $L_{ijk}^l = \langle L(e_i, e_j)e_k, e_l \rangle$.

Let $\mathfrak{L}_1(V)$ be the subspace of $\mathcal{L}(V)$ which consists of all elements $L \in \mathcal{L}(V)$ such that

$L(v, w) = cv \wedge w$ for $v, w \in V$ and some $c \in \mathbb{R}$.

Here $v \wedge w$ is an element of $\mathfrak{gl}(V)$ which is defined by

\[v \wedge w : V \ni z \mapsto (v \wedge w)(z) = \langle w, z \rangle - \langle v, z \rangle \wedge w \in V.\]
We put
\[ L_{1}(V) := \{ L \in \mathcal{L}(V) \mid <L, L'> = 0 \text{ for all } L' \in L_{1}(V) \}. \]
Then \( L_{1}(V) \perp = \{ L \in \mathcal{L}(V) \mid S_L = 0 \}. \) In fact, for \( L \in \mathcal{L}(V) \) and \( L' \in L_{1}(V) \), we get from (2.5) and (2.6), and the definition of \( L_{1}(V) \)
\begin{equation}
< L, L' > = 2c S_L,
\end{equation}
where \( L'(v, w) = cv \wedge w \) for some \( c \in \mathbb{R} \). From (2.7), we obtain the following:
\[ < L, L' > = 0 \text{ for all } L' \in L_{1}(V) \iff 2c S_L = 0 \text{ for all } c \in \mathbb{R} \iff S_L = 0. \]
Putting
\[ \{ L \in L_{1}(V) \mid \text{Ric}_L = 0 \} =: L_{\omega}(V) \]
and
\[ \{ L \in L_{1}(V) \mid < L, L' > = 0 \text{ for all } L' \in L_{\omega}(V) \} =: L_{2}(V), \]
we get the orthogonal direct sum decomposition of \( \mathcal{L}(V) \) as follows:
\[ \mathcal{L}(V) = L_{1}(V) \oplus L_{\omega}(V) \oplus L_{2}(V). \]
Putting together the results above, we obtain the following (cf. [5, Chapter 5])

**Lemma 2.1.** Let \( V \) be an \( n(\geq 3) \)-dimensional real inner product space and \( L \in \mathcal{L}(V) \). Then components \( L_1 \in L_{1}(V) \), \( L_\omega \in L_{\omega}(V) \) and \( L_2 \in L_{2}(V) \) of \( L(= L_{1} + L_{\omega} + L_{2}) \) are given as follows:
\begin{equation}
L_{1}(u, v) = \frac{S_L}{n(n-1)} u \wedge v,
L_{2}(u, v) = \frac{1}{n-2} \left( \text{Ric}_L(u) \wedge v + u \wedge \text{Ric}_L(v) - \frac{2S_L}{n} u \wedge v \right),
L_{\omega}(u, v) = L(u, v) - \frac{1}{n-2} \left( \text{Ric}_L(u) \wedge v + u \wedge \text{Ric}_L(v) \right),
\end{equation}
\begin{equation}
= \frac{S_L}{(n-1)(n-2)} u \wedge v.
\end{equation}

**Proof.** The fact that \( L_1, L_2, L_\omega \) appeared in (2.8) belong to \( \mathcal{L}(V) \) is easily verified. And, \( L = L_{1} + L_{\omega} + L_{2} \). Moreover from straightforward computations we get
\[ S_{L_{2}} = 0, \quad \text{Ric}_{L_{\omega}} = 0, \quad < L_{2}, L_{\omega} > = 0. \]
Thus the proof of Lemma 2.1 is completed. \( \square \)
3. Inequivalent isotropy irreducible representations in
$SU(3)/T(k,l)$

3.1. Isotropy irreducible representations

Let $G$ be a compact connected semisimple Lie group and $H$ a closed subgroup of $G$. The homogeneous space $G/H$ is reductive, that is, in the Lie algebra $g$ of $G$ there exists a subspace $m$ such that $g = h + m$ (direct sum of vector subspaces) and $\text{Ad}(h) m \subset m$ for all $h \in H$, where $h$ is the subalgebra of $g$ corresponding to the identity component $H_o$ of $H$ and $\text{Ad}(h)$ denotes the adjoint representation of $H$ in $m$.

Let $\tau_x (x \in G)$ be the transformation of $G/H$ which is induced by $x$. Taking differentials of $\tau_x$ at $p_o := \{H\} \in G/H$, we obtain the fact that the tangent space $T_{p_o}(G/H) = m$ is $\text{Ad}(H)$-invariant. The homogeneous space $G/H$ is said to be isotropy irreducible if $(T_{p_o}(G/H), \text{Ad}(H))$ is an irreducible representation.

3.2. Inequivalent isotropy irreducible summands in
$SU(3)/T(k,l)$

Here and from now on, without further specification, we use the following notations:

$G = SU(3), \ g :$ the Lie algebra of $SU(3), \ i = \sqrt{-1},$

$T = T(k,l) = \{\text{diag}[e^{2\pi ik\theta}, e^{2\pi il\theta}, e^{-2\pi i(k+l)\theta} | \theta \in \mathbb{R}] \} \text{ for } (k,l) \in \mathbb{Z}^2$
and $|k| + |l| \neq 0,$

$t(k,l) :$ the Lie algebra of $T(k,l), \ \gamma = k^2 + kl + l^2,$

$(X,Y)_0 = B(X,Y) = -6 \text{Trace}(XY), \ X,Y \in g :$ the negative of the Killing form of $g.$

Let $E_{ij}$ be a real $3 \times 3$ matrix with 1 on entry $(i,j)$ and 0 elsewhere. And we put

$$X_1 = \frac{1}{\sqrt{12}}(E_{12} - E_{21}), \quad X_2 = \frac{i}{\sqrt{12}}(E_{12} + E_{21}),$$
$$X_3 = \frac{1}{\sqrt{12}}(E_{13} - E_{31}), \quad X_4 = \frac{i}{\sqrt{12}}(E_{13} + E_{31}),$$
$$X_5 = \frac{1}{\sqrt{12}}(E_{23} - E_{32}), \quad X_6 = \frac{i}{\sqrt{12}}(E_{23} + E_{32}),$$

(3.1)
\[ X_7 = \frac{i}{\sqrt{36\gamma}} \text{diag}[(k + 2l), -(2k + l), (k - l)], \]
\[ X_8 = \frac{i}{\sqrt{12\gamma}} \text{diag}[k, l, -(k + l)]. \]

Then \( \{X_1, \cdots, X_7\} \) (resp. \( \{X_8\} \)) is an orthonormal basis of \( m \) (resp. \( t(k, l) \)) with respect to \((\cdot, \cdot)_0\) such that
\[ g = m + t(k, l) \text{ and } (m, t(k, l))_0 = 0. \]

If we put \( \{X_1, X_2\}_R = m_1, \{X_3, X_4\}_R = m_2, \{X_5, X_6\}_R = m_3, \) and \( \{X_7\}_R = m_4, \) then \( m_i \) are irreducible \( \text{Ad}(T) \)-representation spaces.

In general, two representations \((\mu_1, V_1)\) and \((\mu_2, V_2)\) of a Lie group \( G \) are called equivalent if there exists a linear isomorphism \( \rho \) of \( V_1 \) onto \( V_2 \) such that \( \rho \circ \mu_1(x) = \mu_2(x) \circ \rho \) for all \( x \in G \).

Park obtained the following

**Theorem 3.1.** ([9]) Assume that \(|k| + |l| \neq 0 \) \((k, l \in \mathbb{Z})\). Then a necessary and sufficient condition for \((m_i, \text{Ad}(T(k, l)))\) \((i = 1, 2, 3, 4)\) to be mutually inequivalent is
\[ k \neq 0, \ l \neq 0, \ k \neq \pm l, \ k \neq -2l \text{ and } l \neq -2k. \]

4. A decomposition of the curvature tensor on \( SU(3)/T(k, l) \) with an arbitrarily given \( SU(3) \)-invariant Riemannian metric

4.1. The curvature tensor field on a homogeneous Riemannian space

Let \( G \) be a compact connected semisimple Lie group and \( H \) a closed subgroup of \( G \). We denote by \( g \) and \( h \) the corresponding Lie algebras of \( G \) and \( H \), respectively. Let \( B \) be the negative of the Killing form of \( g \). We consider the \( \text{Ad}(H) \)-invariant decomposition \( g = h + m \) with \( B(h, m) = 0 \). Then the set of \( G \)-invariant symmetric covariant 2-tensor fields on \( G/H \) can be identified with the set of \( \text{Ad}(H) \)-invariant symmetric bilinear forms on \( m \). In particular, the set of \( G \)-invariant Riemannian metrics on \( G/H \) is identified with the set of \( \text{Ad}(H) \)-invariant inner products on \( m \) (cf. [2, 5, 8, 9]).

Let \( < , > \) be an inner product which is invariant with respect to \( \text{Ad}(H) \) on \( m \), where \( \text{Ad} \) denotes the adjoint representation of \( H \) in \( g \).
This inner product $< , >$ determines a $G$-invariant Riemannian metric $g_{<,>}$ on $G/H$.

For the sake of the calculus, we take a neighborhood $V$ of the identity element $e$ in $G$ and a subset $N$ (resp. $N_H$) of $G$ (resp. $H$) in such a way that

(i) $N = V \cap \exp(m)$, $N_H = V \cap \exp(h)$,
(ii) the map $N \times N_H \ni (c,h) \mapsto ch \in N \cdot N_H$ is a diffeomorphism,
(iii) the projection $\pi$ of $G$ onto $G/H$ is a diffeomorphism of $N$ onto a neighborhood $\pi(N)$ of the origin $\{H\}$ in $G/H$. Here, $\{\exp(tX) \mid t \in \mathbb{R}\}$ for $X \in g$ is a 1-parameter subgroup of $G$.

Now for an element $X \in m$, we define a vector field $X^*$ on the neighborhood $\pi(N)$ of $\{H\}$ in $G/H$ by

$$X^*_\pi(c) := (\tau_c)_*X_{\pi(c)} \in T_{\pi(c)}G/H \quad (c \in N),$$

where $\tau_c$ denotes the transformation of $G/H$ which is induced by $c$. Let $\{X_i\}_i$ be an orthonormal basis of the inner product space $(m, < , >)$. Then $\{X_i\}_i$ is an orthonormal frame on $\pi(N) \subset G/H$.

On the other hand, the connection function $\alpha$ (cf. [7, p.43]) on $m \times m$ corresponding to the invariant Riemannian connection of $(G/H, g_{< , >})$ is given as follows (cf. [7, p.52]):

$$\alpha(X,Y) = \frac{1}{2}\{[X,Y]_m + U(X,Y)\} \quad (X,Y \in m),$$

where $U(X,Y)$ is determined by

$$2 < U(X,Y), Z > = < [Z, X]_m, Y > + < X, [Z, Y]_m >$$

for $X, Y, Z \in m$, and $X_m$ denotes the $m$-component of an element $X \in g = h + m$. Let $\nabla$ be the Levi-Civita connection on the Riemannian manifold $(G/H, g_{< , >})$. Then on $\pi(N)$ $(\nabla X \cdot Y^*)_\pi(N) = \alpha(X,Y) (X,Y \in m)$. Moreover, the expression for the value at $p_o := \{H\} \in G/H$ of the curvature tensor field is as follows (cf. [7, p.47]):

$$R(X,Y)Z = \alpha(X,\alpha(Y,Z)) - \alpha(Y,\alpha(X,Z))$$

$$- \alpha([X,Y]_m, Z) - [[X,Y]_h, Z] \quad (X,Y,Z \in m),$$

where $X_m$ (resp. $X_h$) denotes the $m$ - component (resp. $h$ -component) of an element $X \in g = h + m$.

In general, the Ricci tensor field $Ric$ of type $(0,2)$ on a Riemannian manifold $(M, g)$ is defined by

$$Ric(Y,Z) = Trace \{X \mapsto R(X,Y)Z\} \quad (X,Y,Z \in \mathfrak{X}(M)).$$
Let \( \{ Y_j \}_j \) be an orthonormal basis of the inner product \((\mathfrak{m}, \langle , \rangle)\). Since the group \( G \) is unimodular, we obtain the fact (cf. [2, p.184]) that
\[
\sum_j U(Y_j, Y_j) = 0.
\]
Using (4.1), (4.2) and (4.3), we obtain the following expression (cf. [2, p.184-185]) for the value at \( p_o \) of the Ricci tensor field \( \text{Ric} \) on \((G/H, g_{< , >})\):
\[
\text{Ric}(Y,Y) = -\frac{1}{2} \sum_j \langle [Y,Y_j]_{\mathfrak{m}}, [Y,Y_j]_{\mathfrak{m}} \rangle + \frac{1}{2} B(Y,Y) + \frac{1}{4} \sum_{i,j} \langle [Y_i,Y_j]_{\mathfrak{m}}, Y \rangle^2
\]
for \( Y \in \mathfrak{m} \), where \( B \) is the negative of the Killing form of the Lie algebra \( \mathfrak{g} \).

4.2. Ricci tensor fields on inequivalent isotropy irreducible homogeneous spaces

We retain the notation as in Section 4.1. The set of \( G \)-invariant symmetric tensor fields of type \((0,2)\) on \( G/H \) can be identified with the set of \( \text{Ad}(H) \)-invariant symmetric bilinear forms on \( \mathfrak{m} \). In particular, the set of \( G \)-invariant metrics on \( G/H \) is identified with the set of \( \text{Ad}(H) \)-invariant inner products on \( \mathfrak{m} \).

Let \( \langle , \rangle_o \) be an \( \text{Ad}(G) \)-invariant inner product on \( \mathfrak{g} \) such that \( (\mathfrak{m}, \mathfrak{h})_o = 0 \). For the sake of simplicity, we put \( \langle , \rangle_o =: B \). Let \( \mathfrak{m} = \mathfrak{m}_1 + \cdots + \mathfrak{m}_q \) be an orthogonal \( \text{Ad}(H) \)-invariant decomposition of the space \((\mathfrak{m}, B)\) such that \( \text{Ad}(H)_{\mathfrak{m}_i} \) is irreducible for \( i = 1, \ldots, q \), and assume that \( (\mathfrak{m}_i, \text{Ad}(H)) \) are mutually inequivalent irreducible representations. Then, the space of \( G \)-invariant symmetric tensor fields of type \((0,2)\) on \( G/H \) is given by
\[
\{ \lambda_1 B|_{\mathfrak{m}_1} + \cdots + \lambda_q B|_{\mathfrak{m}_q} \mid \lambda_1, \ldots, \lambda_q \in \mathbb{R} \}
\]
and the space of \( G \)-invariant Riemannian metrics on \( G/H \) is given by
\[
\{ \lambda_1 B|_{\mathfrak{m}_1} + \cdots + \lambda_q B|_{\mathfrak{m}_q} \mid \lambda_1 > 0, \ldots, \lambda_q > 0 \}
\]
In fact, for an arbitrarily given \( \text{Ad}(H) \)-invariant inner product \( < , > \) on \( \mathfrak{m} \), we have \( < , > |_{\mathfrak{m}_i} = \lambda_i B|_{\mathfrak{m}_i} \) on each \( \mathfrak{m}_i \) by the help of Shur’s lemma ([cf. [12, 13]]), and \( < \mathfrak{m}_i, \mathfrak{m}_j > = 0 \) for \( i, j \ (i \neq j) \) since \( (\mathfrak{m}_i, \text{Ad}(H)) \) are mutually inequivalent (cf. [8, 9, 11]).

Note that the Ricci tensor field \( \text{Ric} \) of a \( G \)-invariant Riemannian metric on \( G/H \) is a \( G \)-invariant symmetric tensor field of type \((0,2)\) on
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$G/H$, and we identify $\text{Ric}$ with an $\text{Ad}(H)$-invariant symmetric bilinear form on $\mathfrak{m}$. Thus, if $(m_i, \text{Ad}(H))$ are mutually inequivalent irreducible representations, then $\text{Ric}$ is written as

$$\text{Ric} = y_1B\mid_{m_1} + \cdots + y_qB\mid_{m_q}$$

for some $y_1, \ldots, y_q \in \mathbb{R}$.

4.3. The Ricci tensor field and the scalar curvature on $SU(3)/T(k,l)$ with an arbitrarily given $SU(3)$-invariant metric

We retain the notation as in Section 4.2. In this section, we assume that the isotropy irreducible representations $(m_i, \text{Ad}(T(k,l)))$ ($i = 1, 2, 3, 4; k, l \in \mathbb{Z}$) are mutually inequivalent. For the sake of simplicity, we put

$$D := \mathbb{Z}^2 \setminus \{(0,t), (t,0), (t,t), (t,-t), (t,-2t), (2t,-t) \mid t \in \mathbb{Z}\}.$$ 

Let $(\ , \ )_0$ be the negative of the Killing form of $\mathfrak{su}(3)$, and $< \ , \ >$ an arbitrarily given $\text{Ad}(T(k,l))$-invariant inner product on $\mathfrak{m}$. By Theorem 3.1, we obtain the fact that the isotropy irreducible representations $(m_i, \text{Ad}(T(k,l)))$ $(i = 1, 2, 3, 4; k, l \in \mathbb{Z})$ are mutually inequivalent if and only if $(k,l)$ in $T(k,l)$ belongs to $D$. Since $(m_i, \text{Ad}(T(k,l)))$ are mutually inequivalent, for the inner product $< \ , \ >$ on $\mathfrak{m}$ there are corresponding positive numbers $\lambda_1, \lambda_2, \lambda_3$ and $\lambda_4$ such that

$$\begin{align*}
X_1/\sqrt{\lambda_1} &=: Y_1, & X_2/\sqrt{\lambda_1} &=: Y_2, & X_3/\sqrt{\lambda_2} &=: Y_3, \\
X_3/\sqrt{\lambda_2} &=: Y_4, & X_5/\sqrt{\lambda_3} &=: Y_5, & X_6/\sqrt{\lambda_3} &=: Y_6, \\
X_7/\sqrt{\lambda_4} &=: Y_7
\end{align*}$$

is an orthonormal basis of $\mathfrak{m}$ with respect to the inner product $< \ , \ >$, by virtue of (3.1), Theorem 3.1 and (4.5). This inner product $< \ , \ >$ determines a $SU(3)$-invariant Riemannian metric $g_{(\lambda_1,\lambda_2,\lambda_3,\lambda_4)}$ on $SU(3)/T(k,l)$.

From now on, we normalize $SU(3)$-invariant Riemannian metrics on $SU(3)/T(k,l)$ by putting $\lambda_4 = 1$, and denote by $g_{(\lambda_1,\lambda_2,\lambda_3)}$ the metric defined by

$$\lambda_1B\mid_{m_1} + \lambda_2B\mid_{m_2} + \lambda_3B\mid_{m_3} + B\mid_{m_4}.$$

By virtue of (3.1), (4.4), (4.6) and (4.7), we obtain the following result.

**Lemma 4.1.** ([9]) Assume that $(k,l) \in D$. Then the Ricci tensor $\text{Ric}$ on the Riemannian homogeneous space $(SU(3)/T(k,l), g_{(\lambda_1,\lambda_2,\lambda_3)})$...
is given as follows:

\[
\begin{align*}
Ric(Y_i, Y_j) &= 0 \ (i \neq j), \\
Ric(Y_1, Y_1) &= Ric(Y_2, Y_2) = \frac{\lambda_1^2 - \lambda_2^2 - \lambda_3^2 + 6\lambda_2\lambda_3}{12\lambda_1\lambda_2\lambda_3} - \frac{(k + l)^2}{8\gamma\lambda_1^2}, \\
Ric(Y_3, Y_3) &= Ric(Y_4, Y_4) = \frac{\lambda_2^2 - \lambda_3^2 - \lambda_1^2 + 6\lambda_3\lambda_1}{12\lambda_1\lambda_2\lambda_3} - \frac{l^2}{8\gamma\lambda_2^2}, \\
Ric(Y_5, Y_5) &= Ric(Y_6, Y_6) = \frac{\lambda_3^2 - \lambda_1^2 - \lambda_2^2 + 6\lambda_1\lambda_2}{12\lambda_1\lambda_2\lambda_3} - \frac{k^2}{8\gamma\lambda_3^2}, \\
Ric(Y_7, Y_7) &= \frac{1}{8\gamma} \left\{ \frac{(k + l)^2}{\lambda_1^2} + \frac{l^2}{\lambda_2^2} + \frac{k^2}{\lambda_3^2} \right\},
\end{align*}
\]

where \(\gamma := k^2 + kl + l^2\).

The trace of the Ricci tensor \(Ric\) of a Riemannian manifold \((M, g)\), (i.e., \(\sum_j Ric(e_j, e_j)\), where \(\{e_j\}_j\) is a (locally defined) orthonormal frame on \((M, g)\)), is called the scalar curvature of \((M, g)\).

By virtue of Lemma 4.1, we get

**Lemma 4.2.** ([9]) The scalar curvature \(S_{(\lambda_1, \lambda_2, \lambda_3)}\) of the Riemannian homogeneous space \((SU(3)/T(k, l), g_{(\lambda_1, \lambda_2, \lambda_3)})\), \((k, l) \in D\), is given as follows:

\[
S_{(\lambda_1, \lambda_2, \lambda_3)} = \frac{-(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) + 6(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1)}{6\lambda_1\lambda_2\lambda_3} - \frac{1}{8\gamma} \left\{ \frac{(k + l)^2}{\lambda_1^2} + \frac{l^2}{\lambda_2^2} + \frac{k^2}{\lambda_3^2} \right\},
\]

where \(\gamma := k^2 + kl + l^2\).

**4.4. A decomposition of the curvature tensor field on**

\((SU(3)/T(k, l), g_{(\lambda_1, \lambda_2, \lambda_3)})\)

We retain the notation as in Section 4.3. Let \(\nabla\) be the Levi-Civita connection on the homogeneous space \((SU(3)/T(k, l), g_{(\lambda_1, \lambda_2, \lambda_3)})\) and \(\nabla R\) the curvature tensor field with respect to \(\nabla\).

For the sake of convenience, we use the following notations:
A decomposition of the curvature tensor on $SU(3)/T(k,l)$

$$V := \mathcal{T}(SU(3)/T(k,l)),$$

$$(V, g_{(\lambda_1, \lambda_2, \lambda_3)}|V) = (V, \langle , \rangle), \quad \nabla R =: R,$$

$${\mathcal{L}}(V) := \{L | L is a curvature-like tensor on V\},$$

$${\mathcal{L}}_1(V) := \{L \in {\mathcal{L}}(V) | L(X, Y) = c X \wedge Y \text{ for } X, Y \in V$$

and some $c \in \mathbb{R}\},$$

$${\mathcal{L}}_\omega(V) := \{L \in {\mathcal{L}}(V) | \text{the Ricci tensor of } L \text{ is zero}\},$$

$${\mathcal{L}}_2(V) := \{L \in {\mathcal{L}}_1(V) | \langle L, L' \rangle = 0 \text{ for all } L' \in {\mathcal{L}}_\omega(V)\}.$$ Then, we get the orthogonal direct sum decomposition of $\mathcal{L}(V)$ as follows:

$${\mathcal{L}}(V) = {\mathcal{L}}_1(V) \oplus {\mathcal{L}}_\omega(V) \oplus {\mathcal{L}}_2(V).$$

So, the curvature tensor $R$ at $p_0(=\{T(k,l)\})$ of the homogeneous space $(SU(3)/T(k,l), g_{(\lambda_1, \lambda_2, \lambda_3)})$ is uniquely decomposed as

$$R = R^{(1)} + R^{\omega} + R^{(2)}$$

$$(R^{(1)} \in {\mathcal{L}}_1(V), \quad R^{\omega} \in {\mathcal{L}}_\omega(V), \quad R^{(2)} \in {\mathcal{L}}_2(V)).$$

The curvature-like tensor $R^{\omega}$ appeared in (4.8) is said to be the Weyl tensor (field) of the curvature tensor field $R$ on $(SU(3)/T(k,l), g_{(\lambda_1, \lambda_2, \lambda_3)})$.

Then, by virtue of (2.8), Lemmas 4.1 and 4.2, we obtain

\textbf{Theorem 4.3. Let $R^{(1)}$, $R^{\omega}$ and $R^{(2)}$ be the curvature-like tensors appeared in the curvature tensor $R = R^{(1)} + R^{\omega} + R^{(2)} \in {\mathcal{L}}_1(V) \oplus {\mathcal{L}}_\omega(V) \oplus {\mathcal{L}}_2(V))$ on $(SU(3)/T(k,l), g_{(\lambda_1, \lambda_2, \lambda_3)})$. Assume that $(k,l)$ belongs to $D$. Then}

$$R^{(1)}(Y_i, Y_j) = \frac{1}{42} S_{(\lambda_1, \lambda_2, \lambda_3)} Y_i \wedge Y_j,$$

$$R^{(2)}(Y_i, Y_j) = \frac{1}{5} \{\text{Ric}(Y_i) \wedge Y_j + Y_i \wedge \text{Ric}(Y_j)\} - \frac{2}{35} S_{(\lambda_1, \lambda_2, \lambda_3)} Y_i \wedge Y_j,$$

$$R^{\omega}(Y_i, Y_j) = R(Y_i, Y_j) - \frac{1}{5} \{\text{Ric}(Y_i) \wedge Y_j + Y_i \wedge \text{Ric}(Y_j)\}$$

$$+ \frac{1}{30} S_{(\lambda_1, \lambda_2, \lambda_3)} Y_i \wedge Y_j,$$

where $\{Y_i\}_{i=1}^7$ is an orthonormal basis on $(m, \langle , \rangle)$ and $S_{(\lambda_1, \lambda_2, \lambda_3)}$ is the scalar curvature of $(SU(3)/T(k,l), g_{(\lambda_1, \lambda_2, \lambda_3)}).$
In general, the Ricci curvature $r$ of a Riemannian manifold $(M, g)$ with respect to a nonzero vector $v \in TM$ is defined by

$$r(v) = \frac{\text{Ric}(v, v)}{||v||^2_g}.$$ 

From Theorem 4.3, we get

**Corollary 4.4.** Let $R^{(2)}$ be the curvature-like tensor appeared in the curvature tensor $R = R^{(1)} + R^\omega + R^{(2)}$ on $(SU(3)/T(k, l), g(\lambda_1, \lambda_2, \lambda_3))$, where $(k, l) \in D$. Then the Ricci tensor of $R^{(2)}$ is given as follows:

$$\text{Ric}^{(2)}(Y_i, Y_j) = -\frac{1}{7} S(\lambda_1, \lambda_2, \lambda_3) \delta_{ij} + \text{Ric}(Y_i, Y_j).$$

By the help of Lemma 4.1 and Corollary 4.4, we obtain

**Proposition 4.5.** Assume that $(k, l) \in D, k > l > 0,$ and

$$\lambda \geq \frac{3l^2}{10(k^2 + kl + l^2)}$$

in $(SU(3)/T(k, l), g(\lambda, \lambda, \lambda_3))$, $\lambda > 0$. Then the Ricci curvature $r^{(2)}$ of the curvature-like tensor $R^{(2)}$ in the curvature tensor $R = R^{(1)} + R^\omega + R^{(2)}$ on $(SU(3)/T(k, l), g(\lambda, \lambda, \lambda_3))$ is estimated as follows:

$$r^{(2)}(Y_i) = r^{(2)}(Y_2) \leq r^{(2)}(Y_7),$$

where $r^{(2)}(Y_i) = \text{Ric}^{(2)}(Y_i, Y_i)$ for $i = 1, 2, \ldots, 7$.

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