Extremal limit for charged and rotating 2+1–dimensional black holes and Bertotti-Robinson geometry

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We consider 2+1–dimensional analogues of the Bertotti-Robinson (BR) spacetimes in the sense that the coefficient at the angular part is a constant. We show that such BR-like solutions are either pure static or uncharged rotating. We trace the origin of the inconsistency between a charge and rotation, considering the BR-like spacetime as a result of the limiting transition of a non-extremal black hole to the extremal state. We also find that the quasilocal energy and angular momentum of such BR-like spacetimes calculated within the boundary $l = \text{const}$ ($l$ is the proper distance) are constants independent of the position of the boundary.

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I. INTRODUCTION

In recent years, interest to $AdS_2 \times S_2$ geometries increased in the context of string theory and AdS/CFT correspondence [1,2]. Apart from this, the geometries of this type appear naturally in black hole physics. If one considers a charged black hole and makes limiting transition $T_H \rightarrow 0$ ($T_H$ is the Hawking temperature) from the non-extremal black hole geometry to the extremal state, such that the canonical gravitating thermal ensemble remains well-defined [3,4], the Reissner-Nordström (RN) metric turns into the Bertotti-Robinson (BR)
spacetime [5], [6] with the black hole horizon turning into the acceleration one. The similar geometries are also relevant for non-linear electrodynamics [7], string dust sources [8], higher-dimensional spacetimes [9,10], and quadratic gravity [11]. The thermodynamic properties of acceleration horizons are considered from a general viewpoint in [12]. Moreover, it should be emphasised that the limiting procedure is defined not only for spherically-symmetrical spacetimes but also for generic static black hole configurations [13], and, in particular, can be applied to different versions of C-metric [14]. In a similar way, the rotating analogs of BR spacetimes are obtained from the Kerr solutions in Ref. [13] for the non-extremal horizons and in [15] for the extremal ones. Such solution appear naturally in the context of the dilaton-axion gravity [16].

For 2+1 rotating uncharged black holes the limiting procedure under consideration has been carried out in [4]. The resulting solution coincides with that found in [17], [18]. For the charged unrotating 2+1 black holes the general procedure of [4] applies as well and gives solutions found in [19] and discussed also in [20].

The aim of this paper is to elucidate, whether such BR-like 2+1-dimensional solutions exist when both rotation and charge are present. (By BR-like we mean the $AdS_2 \times S_1$ structure, the coefficient at the angular part being a constant.) In this case the original black hole solutions, to which the limiting procedure should apply, become rather complicated [19], [21], [25]. Meanwhile, the advantage of the general limiting procedure elaborated in [13] consists just in the fact that it enables to guess the general form of the metric. Therefore, instead of analysing the original solutions with the subsequent applying the limiting transition, we start from the Einstein equations directly in which we substitute the anticipated form of the metric. As we will see below, such an approach enables us to find at once a quite unexpected result: when the charge and rotation are both present, BR-like geometries are impossible.

II. FIELD EQUATIONS

Consider stationary 2+1–dimensional geometry described by a line element of the form

$$ds^2 = -N^2(\rho) f^2(\rho) dt^2 + f(\rho)^{-2} d\rho^2 + r^2(\rho) \left( d\phi + N^I(\rho) dt \right)^2.$$  (1)

Of matter source we assume that it is purely electromagnetic, with the stress-energy tensor given by

$$8\pi T_\mu^\nu \equiv \theta_\mu^\nu = 2 F^{\mu\alpha} F_{\nu\alpha} - \frac{1}{2} \delta_\mu^\nu F_{\alpha\beta} F^{\alpha\beta},$$  (2)

where $F_{\alpha\beta}$ is the electromagnetic tensor. We adopt the notations $t = x^0$, $\rho = x^1$, $\phi = x^2$.  

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For the electromagnetic tensor compatible with the assumed symmetries, it follows from the Maxwell equations that $F^{01} = -\frac{Q}{N^2}$, $F^{21} = \frac{P}{N^2}$, where the constants $Q$ and $P$ have the meaning of an electric and magnetic charge, respectively.

The metric (1) preserves its form under the coordinate transformation $\phi = \phi' + \Omega t$, where $\Omega$ is a constant. In doing so, $(N^\phi)' = N^\phi + \Omega$ and $(F^{21})' = F^{21} - F^{01}\Omega$. As both $F^{21}$ and $F^{01}$ have the same coordinate dependence, one can always achieve $F^{12} = 0$, choosing $\Omega = -\frac{P}{Q}$. In what follows we assume that this condition is satisfied. Then, the only nonvanishing components of $\theta^\mu_\nu$ are simply $\theta^0_0 = Q^2\left[\frac{(N^\phi)^2}{N^2} - \frac{1}{r^2}\right] = \theta^1_1 = -\theta^\phi_\phi$, and $\theta^0_\phi = \frac{2Q^2N^\phi}{N^2f^2}$. It is worth noting that the regularity of the stress-energy tensor on the event horizon requires $N^\phi \sim f^2 \to 0$. Thus, our frame turns out to be corotating with the horizon automatically.

For the line element (1) it is convenient to use the following combinations of Einstein equations (where the cosmological constant $\Lambda = -|\Lambda| < 0$):

\[
G^t_t - N^\phi G^t_\phi = \theta^t_t - N^\phi \theta^t_\phi - \Lambda,
\]

\[
NG^t_\phi = N\theta^t_\phi,
\]

\[
G^t_\phi N^\phi - G^t_t + G^\rho_\rho = \theta^t_\phi N^\phi - \theta^t_t + \theta^\rho_\rho,
\]

\[
G^t_t + G^\phi_\phi = \theta^t_t + \theta^\phi_\phi - 2\Lambda.
\]

Henceforth, we shall use the proper length $l$ as a radial coordinate, i.e., we will work in the gauge

\[
\frac{2}{M^2} \frac{d^2r}{dl^2} + 2\Lambda r + \frac{1}{2} \frac{r^3}{M^2} \left( \frac{dN^\phi}{dl} \right)^2 + 2 \frac{Q^2}{r} + \frac{2\left(N^\phi Q\right)^2 r}{M^2} = 0,
\]

\[
\frac{d}{dl} \left( \frac{r^3}{M} \frac{dN^\phi}{dl} \right) = 2\frac{Q^2 N^\phi r}{M},
\]

\[
\frac{1}{M} \frac{dM}{dl} \frac{dr}{dl} - \frac{d^2r}{dl^2} = 2\frac{r \left(Q N^\phi\right)^2}{M^2},
\]

\[
\frac{1}{M} \frac{d^2M}{dl^2} + \frac{1}{r} \frac{d^2r}{dl^2} - \frac{1}{2} \frac{r^2}{M^2} \left( \frac{dN^\phi}{dl} \right)^2 + 2\Lambda = 0.
\]
III. $r(l) = \text{const} = r_0$

Eqs. (8) - (10) describe two qualitatively different situations. If $r(l)$ is not constant identically, Eqs. (8) - (10) comprise the full set of three independent equations for three unknown functions $r(l)$, $N(l)$ and $N^\phi(l)$. Remaining equations, as for instance, $(\phi^\mu)_{\phi}$ equation, can be easily obtained from them with the help of Bianchi identities. However, if $r(l) = \text{const} \equiv r_0$, only two equations of (8) - (10) are independent. Indeed, it could be easily demonstrated that the left hand side of Eq. (10) identically vanishes, and, consequently,

$$QN^\phi = 0.$$  \hspace{1cm} (12)

As the equation for determining $M$ is now missing, Eqs. (8) - (10) are insufficient for the case $r = r_0$ and should be supplemented by the additional equation, as for example Eq. (6) or, equivalently, (11). It is worth stressing that, being the consequence of Eqs. (8) - (10) in the case $r(l) \neq \text{const}$, now it represents a new independent equation, which in the present context gives

$$\frac{1}{M} \frac{d^2M}{dl^2} = \alpha^2, \quad \alpha \equiv \frac{r_0}{M} \frac{dN^\phi}{dl} = \frac{2}{L}. \hspace{1cm} (13)$$

It follows from Eq. (12) that the general case is splitted to two subcases. Although, as is mentioned in Introduction, the solutions for each of them are known, for completeness we rederive them below in a rather straightforward manner.

1) $Q = 0, N^\phi \neq 0$.

Then

$$M = \frac{L}{2} m(\alpha l), \quad N^\phi = \frac{L}{2r_0} n(\alpha l), \hspace{1cm} (14)$$

where we choose the normalization of time (i.e. the coefficient at $M$) in accordance with the limiting transition for 2+1 black holes to the extremal state (see below). There are three physically different solutions ($x \equiv \alpha l$): (1a) $m(x) = \sinh x$, $n(x) = 2 \sinh^2 \frac{x}{2}$. (1b) $m = \exp(x)$, $n = \exp(x)$, (1c) $m = \cosh x$, $n = \sinh x$. The cases (1a) - (1c) correspond to the solutions found in [17], [18]. The case (1a) can be also obtained by taking the extremal limit of the non-extremal 2+1 black hole [4].

2) $Q \neq 0, N^\phi = 0$.

It follows from Eq. (8)

$$r_0^2 = Q^2 L^2. \hspace{1cm} (15)$$
Eq. (13) with $\alpha = 0$ gives us again three possibilities (2a) $M = \frac{\sinh a}{a}$, (2b) $M = \cosh a$, (2c) $M = \exp(al)$, where $a^2 = \frac{\Lambda}{L^2}$. In the limit $l \to \infty$ ($\Lambda \to 0$), $Q \to 0$, $r_0 = \text{const}$ we obtain the flat spacetime. The solutions (2a) - (2c) correspond to those found in [19].

All solutions (1a–1c) and (2a–2c) share a rather peculiar property which was not noticed before. One can calculate the quasilocal energy density $\varepsilon = \frac{k}{8\pi} + \varepsilon_0$, where $\varepsilon_0$ comes from the contribution of the reference background, $k$ is the extrinsic curvature of the boundary embedded into the three-dimensional spacelike surface [26]. Choosing the foliation $t = \text{const}$ and the boundary at fixed $l$, we find that the only non-vanishing component of the unit normal to the boundary is $n^1 = 1$ and $k = 0$. Thus, the energy density $\varepsilon = \varepsilon_0(r_0) = \text{const}$ since $r_0 = \text{const}$. This is the typical feature of acceleration horizons [12].

In a similar way one can calculate the quasilocal angular momentum [29]

$$J_B = \frac{1}{\pi} \int d\sigma K_{ij} \xi^i n^j,$$

(16)

d$\sigma$ is the proper element of the boundary, $\xi^i$ is the axial Killing vector, the extrinsic curvature tensor

$$K_{ij} = \frac{1}{2N} (N_{ij} + N_{ji}),$$

(17)

$N_{ij}$ is the covariant derivative of the shift vector with respect to the slice $t = \text{const}$. Then, direct calculation gives us for the boundary with an arbitrary fixed $l$ that

$$J_B = 2r_0^2/L.$$  

(18)

Thus, both the energy and angular momentum turned out to be constant.

**IV. LIMITING TRANSITION**

As we saw in the preceding sections, BR-like configurations with simultaneously nonzero $N^\phi$ and $Q$ are impossible. To understand better this fact, we shall employ the approach based on the limiting transition [4] relying only on the structure of field equations. For the metric (1) one can always achieve $N^\phi = 0$ on the horizon passing to the frame corotating with a black hole. Using in Eq. (1) the gauge $r = \rho$ and assuming for simplicity $N = 1$, we can write the asymptotic expansion of metric potentials near the extremal state in the form

$$f^2 = 4\pi T_H(\rho - \rho_+) + b(\rho - \rho_+)^2, \quad N^\phi = c(\rho - \rho_+)$$

(19)

and

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\[
\rho - \rho_+ = 4\pi T_H b^{-1} \sinh^2 \frac{x}{2}, \quad t = \frac{\tilde{t}}{2\pi T_H}.
\] (20)

In the limit \( T_H \to 0 \) we obtain
\[
ds^2 = b^{-1} \left( -\sinh^2 x d\tilde{t}^2 + dx^2 \right) + \rho_+^2 (d\phi + d\sinh^2 \frac{x}{2} d\tilde{t})^2,
\] (21)
where \( d = 2cb^{-1} \).

In particular, for 2+1 black holes [22], [23]
\[
f^2 = \frac{\rho^2}{L^2} - M + \frac{J^2}{4\rho^2}, \quad b^{-1} = \frac{L^2}{4}
\] (22)
and
\[
N^\phi = \frac{J}{2} \left( \frac{1}{\rho^2_+} - \frac{1}{\rho^2} \right), \quad (23)
\]
where the constant \( J \) represents the angular momentum. We see that \( c = \frac{J}{\rho_+} \). In the extremal limit
\[
J = \frac{2\rho_+^2}{L}, \quad c = \frac{2\rho_+}{L}, \quad d = \frac{L}{\rho_+}.
\] (24)

For uncharged black holes, the metric (21) corresponds to the case (1a) and agrees with Eq. (22) of Ref. [4] (with typographical errors corrected). However, if \( Q \neq 0 \), the quantity \( N \neq 1 \) [25]. It follows from eqs. (17), (19) of [25] or directly from eq. (5) with \( r(\rho) = \rho \) that
\[
\frac{d}{d\rho} N = \frac{(Q N^\phi)^2 \rho}{2N f^4}.
\] (25)

Therefore, we see from (25) that near the horizon \( N^\phi \sim f^2 \sim T_H (\rho - \rho_+) \). Thus, the coefficient \( c \) is proportional to \( T_H \) and vanishes as \( T_H \to 0 \). As in the process of the limiting transition the radial coordinate of all points of the manifold approaches the horizon value \( \rho = \rho_+ \), it turns out that the coefficient \( d \to 0 \), so the system becomes static. Thus, the interpretation of the metrics under discussion as extremal limits of corresponding non-extremal configuration explains why the limiting configuration cannot be simultaneously rotating and charged.

It is worth noting that for 2+1 black holes the angular momentum \( J_B = \text{const} = J \) [27], [28]. Therefore, the fact that \( J = \text{const} \) also for the limiting form of the metric (established in the previous Section) is not surprising. In doing so, eq.(24) with \( \rho_+ = r_0 \) agrees with (18). However, for 2+1 black holes the energy is not a constant [27], [28]. Nevertheless, after the limiting transition under discussion it becomes constant. Indeed, calculating the quasilocal
energy for the boundary $\rho = r_0(l_0)$, one obtains $\varepsilon = -\frac{\omega}{4\pi r_0} f(r_0) + \varepsilon_0(r_0)$, where $\varepsilon_0(r_0)$ is the subtraction term. After the limiting transition $r_0 = \rho_+ = \text{const}$, so that $f(r_0) = 0$ and we obtain that $\varepsilon = \varepsilon_0$ does not depend on the distance from the horizon. Thus, for the solution with an acceleration non-extremal horizon (1a) and (2a) the limiting procedure under discussion explains also the property of the energy indicated at the end of Section IV.

V. SUMMARY

In this note we established that 2+1 BR-like geometries ($r(\rho) = r_0 = \text{const}$) can exist only for the rotating or charged case separately but are forbidden when rotation and charge are present simultaneously. (Throughout the paper we assumed that our frame is corotating with the horizon, so one can make coordinate transformations to the new system rotating with a constant angular velocity but such a transition is a trivial and gives no new solutions.) This fact is in a sharp contrast with the 3+1 case when generalizations of BR-like solutions do exist and can be obtained form the extremal Kerr-Newman geometries by the suitable limiting transition [13], [15]. As the corresponding metrics with extremal horizons describe the near-horizon region of the extremal black holes, the property under discussion may have important consequences for the late stage of 2+1 collapse to the extremal state, if rotation is supplemented by the presence of a charge, however small it be.

It is also worth noting that the BR-like configurations under discussion turned out to be “poor” in the dynamic sense since both the angular momentum (for $Q = 0$, $N^\phi \neq 0$) and the energy calculated within the boundary $l = \text{const} = l_B$ do not depend on $l_B$. This is in agreement with the symmetry of the system according to which the geometrical properties of sections with different $l_B$ including the horizon are equivalent.

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[1] J. Maldacena, Adv. Theor. Math. Phys. 2 (1998) 231.

[2] E. Witten, Adv. Theor. Math. Phys. 2 (1998) 253.
[3] O. B. Zaslavskii, Phys. Rev. Lett. 76 (1996) 2211.

[4] O. B. Zaslavskii, Phys. Rev. D 56 (1997) 2188. Erratum-ibid. D59 (1999) 069901.

[5] I. Robinson, Bull. Acad. Pol. Sci. 7 (1959) 351.

[6] B. Bertotti, Phys. Rev. 116 (1959) 1331.

[7] J. Matyjasek, Extremal limit of the regular charged black holes in nonlinear electrodynamics, gr-qc/0403109.

[8] N. Dadhich, On product spacetime with 2-sphere of constant curvature, gr-qc/0003026.

[9] M. Cardarelli, L. Vanzo and Z. Zerbini, in Geometrical Aspects of Quantum Fields, edited by A. A. Bytsenko, A. E. Goncalves and B. M. Pimentel (World Scientific, Singapore, 2001); hep-th/0008136.

[10] V. Cardoso, O. J.C. Dias, J. P.S. Lemos, Nariai, Bertotti-Robinson and anti-Nariai solutions in higher dimensions; hep-th/0401192.

[11] J. Matyjasek and D. Tryniecki, Phys. Rev. D 69 (2004) 124016.

[12] O. B. Zaslavskii, Class. Quant. Grav. 17 (2000) 497.

[13] O. B. Zaslavskii, Class. Quant. Grav. 15 (1998) 3251.

[14] O. J. C. Dias and J. P. S. Lemos, Phys. Rev. D 68 (2003) 104010.

[15] J. Bardeen and G. T. Horowitz, Phys. Rev. D60 (1999) 104030.

[16] G. Clément and D. Gal’tsov, Phys. Rev. D 63 (2001) 124011.

[17] G. Clément, Phys. Rev. D49 (1994) 5131.

[18] O. Coussaert and M. Henneaux, in the Black hole, 25 years later, edited by C. Teitelboim and J. Zanelli (World Scientific, Singapore 1998), hep-th/9407181.

[19] G. Clément, Class. Quant. Grav. 10 (1993) L49-L54.
[20] G. Clément and A. Fabbri, Class. Quant. Grav. 17 (2000) 2537.

[21] G. Clément, Phys. Lett. B367 (1996) 70.

[22] M. Bañados, C. Teitelboim and J. Zanelli, Phys. Rev. Lett. 69 (1992) 1849.

[23] M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli, Phys. Rev. D 48 (1993) 1506.

[24] E. Ayon-Beato, C. Martinez, and J. Zanelli, Birkhoff’s theorem for three-dimensional AdS gravity; hep-th/0403227.

[25] C. Martinez, C. Teitelboim and J. Zanelli, Phys. Rev. D61, 104013 (2000).

[26] J. D. Brown and J. W. York, Jr., Phys. Rev. D 47 (1993) 1407.

[27] O. B. Zaslavskii, Class. Quant. Grav. 11 (1994) L33-L38.

[28] J.D. Brown, J. Creighton, R.B. Mann, Phys.Rev. D50 (1994) 6394.

[29] J.D. Brown, E. A. Martinez and J. W. York, Jr., Phys. Rev. Lett. 66 (1991) 2281.