Ray-tracing in pseudo-complex General Relativity

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ABSTRACT
Motivated by possible observations of the black hole candidate in the centre of our Galaxy and the galaxy M87, ray-tracing methods are applied to both standard General Relativity (GR) and a recently proposed extension, the pseudo-complex GR (pc-GR). The correction terms due to the investigated pc-GR model lead to slower orbital motions close to massive objects. Also the concept of an innermost stable circular orbit is modified for the pc-GR model, allowing particles to get closer to the central object for most values of the spin parameter $a$ than in GR. Thus, the accretion disc, surrounding a massive object, is brighter in pc-GR than in GR. Iron Kα emission-line profiles are also calculated as those are good observables for regions of strong gravity. Differences between the two theories are pointed out.

Key words: black hole physics – gravitation – celestial mechanics – galaxies: nuclei.

1 INTRODUCTION
Taking a picture of a black hole is not possible as long as an ambient light source is missing. However, we can image a black hole and its strong gravitational effects by following light rays coming from a source near the black hole. A powerful standard technique is called ray-tracing. The basic idea is to follow light rays (on null geodesics) in a curved background space–time from their point of emission, e.g. in an accretion disc, around a massive object.1 In this way, one can create an image of the black holes direct neighbourhood. There are numerous groups using ray-tracing for this purpose, see, e.g., Fanton et al. (1997), Müller & Camenzind (2004), Vincent et al. (2011) and Bambi & Malafarina (2013). Aside from an image of the black hole, it is also possible to calculate emission-line profiles using the same technique, but adding in a second step the evaluation of an integral for the spectral flux. This is of particular interest as the emission profile of, e.g., the iron Kα line is one of the few good observables in regions with strong gravitation.

In the near future, it will be possible to resolve the central massive object Sagittarius A* in the centre of our Galaxy (Eisenhauer et al. 2011; Gillessen et al. 2012) and the one in M87 with the planned Event Horizon Telescope (Doelmann et al. 2009; Falcke et al. 2012). This offers a great opportunity to test General Relativity (GR) and its predictions.

Predicting the expected picture from theory gets even more important, noting that during 2013/2014 a gas cloud approaches close to the centre of our Galaxy (Gillessen et al. 2012) and probably a portion of it may become part of an accretion disc. Once formed, we assume it also may exhibit hot spots, seen as quasi-periodic oscillations (QPOs) (Belloni, Méndez & Homan 2005) with the possibility to measure the iron Kα line. This gives the chance to test a theory, measuring the periodicity of the QPO and simultaneously the redshift.

Recently, in Hess & Greiner (2009) and Caspar et al. (2012), a pseudo-complex extension to GR (pc-GR) was proposed, which adds to the usual coupling of mass to the geometry of space as a new ingredient the presence of a dark energy fluid with negative energy density. The resulting changes to Einstein’s equations could also be obtained by introducing a non-vanishing energy momentum tensor in standard GR but arise more naturally when using a pc-description. By another group, in Visser (1996), the coupling of the mass to the local quantum property of vacuum fluctuations was investigated, applying semi-classical quantum mechanics, where the decline of the energy density is dominated by a $1/r^2$ behaviour. However, in Visser (1996), no recoupling to the metric was considered. In pc-GR, the recoupling to the metric is automatically included and the energy density is modelled to decline as $1/r^3$; however, there is no microscopical description for the dark energy yet. The fall-off of the order of $1/r^5$ can neither be noted yet by Solar system experiments (Will 2006) nor in systems of two orbiting neutron stars (Hulse & Taylor 1975). A model description of the Hulse–Taylor binary, including pc-GR terms, showed that corrections become

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1 Technically, this is not correct. It is computationally less expensive to follow light rays from an observer’s screen back to their point of emission.

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significant 12 orders of magnitude beyond current accuracy. Other models concerning the physics of neutron stars are currently prepared for publication (Rodríguez et al. 2014)

The effects of the dark energy become important near the Schwarzschild radius of a compact object towards smaller radial distances. A parameter $B = bm^3$ is introduced, which defines the coupling of the mass with the vacuum fluctuations. In contrast to the work by Visser (1996), the coupling of the mass to vacuum fluctuations on a macroscopical level, described with the parameter $B$, allows us to include alterations to the metric. A downside is that this modification yet lacks a complete microscopic description; thus, one can see the work by Visser (1996) and pc-GR as complementary. In Caspar et al. (2012), investigations showed that a value of $B > (64/27)m^3$ leads to a metric with no event horizons. Thus, an external observer can in principle still look inside, though a large redshift will make the grey star look like a black hole. In the following, we will use the critical value $B = (64/27)m^3$ if not otherwise stated.

In Schönenbach et al. (2013), several predictions were made, related to the motion of a test particle in a circular orbit around a massive compact object (labelled there as a grey star), which is relevant for the observation of a QPO and the redshift. One distinct feature is that at a certain distance in pc-GR the orbital frequency shows a maximum, from which it decreases again towards lower radii, allowing near the surface of the star a low orbital frequency correlated with a large redshift. This will affect the spectrum as seen by an observer at a large distance. Thus, it is of interest to know how the accretion disc would look like by using pc-GR. In addition to the usual assumptions made for modelling accretion discs, e.g. in Page & Thorne (1974), we assume the coupling of the dark energy to the matter of the disc to be negligible compared to the coupling to the central object. This is justified in the same way as one usually neglects the mass of the disc material in comparison to the central object.

In the following, we will first briefly review the theoretical background on the methods used, where we will also discuss the two models we used to describe accretion discs. After that we will present results obtained with the open-source ray-tracing code gyoto² (Vincent et al. 2011) for the simulation of disc images and emission-line profiles.

2 THEORETICAL BACKGROUND

We will use the Boyer–Lindquist coordinates of the Kerr metric and its pc-equivalent, which we will write in a slightly different form³ than in Caspar et al. (2012) as

$$g_{00} = -\left(1 - \frac{\psi}{\Sigma}\right),$$

$$g_{11} = \Sigma,$$

$$g_{22} = \Delta,$$

$$g_{33} = \left((r^2 + a^2) + \frac{a^2 \psi}{\Sigma}\sin^2 \theta\right)\sin^2 \theta,$$

² gyoto is obtainable at http://gyoto.obspm.fr/.
³ Here, we use the convention $a = \frac{J}{M}$ instead of $a = -\frac{J}{M}$, where $J$ is the angular momentum of the central massive object, and signature $(-,+,+)$ in contrast to previous work in Caspar et al. (2012) and Schönenbach et al. (2013).

$$g_{03} = -a \frac{\psi}{\Sigma}\sin^2 \theta,$$

with

$$\Sigma = r^2 + a^2 \cos^2 \theta,$$

$$\Delta = r^2 + a^2 - \psi,$$

$$\psi = 2mr - \frac{B}{2r}.$$  (2)

Here, $m = \kappa M$ is the gravitational radius of a massive object, $M$ is its mass, $a$ is a measure for the specific angular momentum or spin of this object and $\kappa$ is the gravitational constant. In addition, we set the speed of light $c = 1$. One can easily see that this metric differs from the standard Kerr metric only in the use of the function $\psi$ which reduces to $2mr$ in the limit $B = 0$. Bearing this in mind one can simply follow the derivation of the Lagrange equations given, e.g., in Levin & Perez-Giz (2008) (which are the basis for the implementation in gyoto) and modify the occurrences of the Boyer–Lindquist $\Delta$-function and the newly introduced $\psi$.

To derive the desired equations, we exploit all conserved quantities along geodesics which are the test particle’s mass $m_0$, the energy at infinity $E$, the angular momentum $L_z$ and the Carter constant $Q$ (Carter 1968; Levin & Perez-Giz 2008). The usual way to proceed from this is to follow Carter (1968) and demand separability of Hamilton’s principal function as

$$S = -\frac{1}{2}\dot{\lambda} - \dot{E}t + L_z \dot{\phi} + S_r + S_\theta.$$  (3)

Here, $\lambda$ is an affine parameter, and $S_r$ and $S_\theta$ are functions of $r$ and $\theta$, respectively. Demanding separability now for equation (3) leads Carter (1968) to

$$\left(\frac{dS_r}{dr}\right)^2 = \frac{R}{\Delta^2} \quad \text{and} \quad \left(\frac{dS_\theta}{d\theta}\right)^2 = \Theta,$$  (4)

with the auxiliary functions

$$R(r) := \left[(r^2 + a^2)E - aL_z^2\right]^2 - \Delta \left[Q + (aE - L_z)^2 + m_0^2r^2\right],$$

$$\Theta(\theta) := Q - \left[\frac{L_z^2}{\sin^2 \theta} - a^2E^2 + m_0^2a^2\right] \cos^2 \theta.$$  (5)

Taking these together with

$$\dot{x}^a = g^{\alpha\mu} p_\mu = g^{\alpha\mu} \frac{\partial S}{\partial x^\alpha}$$  (6)

leads to a set of first-order equations of motion in the coordinates

$$\dot{t} = \frac{1}{\Sigma \Delta} \left\{\left[(r^2 + a^2)^2 + a^2 \Delta \sin^2 \theta\right] - aL_z\right\},$$

$$\dot{r} = \pm \frac{\sqrt{R}}{\Sigma},$$

$$\dot{\theta} = \pm \frac{\sqrt{\Theta}}{\Sigma},$$

$$\dot{\phi} = \frac{1}{\Sigma \Delta} \left[\left(\frac{\Delta}{\sin^2 \theta} - a^2\right) L_z + a \psi E\right],$$  (7)

where the dot represents the derivative with respect to the proper time $\tau$.

Levin & Perez-Giz (2008) however argue that those equations contain an ambiguity in the sign for the radial and azimuthal
velocities. In addition to that, using Hamilton’s principle to get the geodesics leads to the integral equation (Carter 1968)

\[
\int_{\gamma_{\text{em}}} \frac{dr}{\sqrt{\mathcal{R}}} = \int_{\gamma_{\text{em}}} \frac{d\theta}{\sqrt{\mathcal{E}}}. \tag{8}
\]

To solve this equation, Fanton et al. (1997) and Müller & Camenzind (2004) make use of the fact that \( R \) is a fourth-order polynomial in \( r \). This is not possible anymore with the pc-correction terms as the order of \( R \) increases.

Luckily, the equations used in GYOTO are based on the use of different equations of motion derived by Levin & Perez-Giz (2008). Therefore, we follow Levin & Perez-Giz (2008) who make use of the Hamiltonian formulation in addition to the separability of Hamilton’s principal function. The canonical four-momentum of a particle is given as

\[
p^\mu := \dot{x}^\mu,
\]

when we write the Lagrangian as

\[
\mathcal{L} = \frac{1}{2} \xi_{\mu} \dot{\xi}^\mu.
\]

Given explicitly in their covariant form, the momenta are (Levin & Perez-Giz 2008)

\[
p_0 = -\left(1 - \frac{\psi}{\Sigma}\right) i - \frac{\psi a \sin^2 \theta}{\Sigma} \phi,
\]

\[
p_1 = \frac{\Sigma}{\Delta} \dot{r},
\]

\[
p_2 = \Sigma \dot{\theta},
\]

\[
p_3 = \sin^2 \theta \left(r^2 + a^2 + \frac{\psi a^2 \sin^2 \theta}{\Sigma} \phi - \psi a \sin^2 \theta r\right).
\]

After a short calculation, the Hamiltonian \( \mathcal{H} = p_\mu \dot{x}^\mu - \mathcal{L} = \frac{1}{2} g^{\mu \nu} p_\mu p_\nu \) can be rewritten as (Levin & Perez-Giz 2008)

\[
\mathcal{H} = \frac{\Delta}{2\Sigma} p_1^2 + \frac{1}{2\Sigma} p_2^2 - \frac{R(r) + \Delta \Theta(\theta)}{2\Delta \Sigma} - m_0 a^2.
\]

The momenta associated with time \( t \) and azimuth \( \phi \) are conserved and can be identified with the energy at infinity \( p_0 = -E \) and the angular momentum \( p_3 = L_\phi \), respectively (Carter 1968).

Now, Hamilton’s equations \( \dot{x}_k = \frac{\partial \mathcal{H}}{\partial p_k} \) and \( \dot{p}_k = -\frac{\partial \mathcal{H}}{\partial x_k} \) yield the wanted equations of motion\(^4\) (Levin & Perez-Giz 2008; Vincent et al. 2011) as

\[
\dot{i} = \frac{1}{2\Delta \Sigma} \frac{\partial}{\partial E} (R + \Delta \Theta),
\]

\[
\dot{r} = \frac{\Delta}{\Sigma} p_1,
\]

\[
\dot{\theta} = \frac{1}{\Sigma} p_2,
\]

\[
\dot{\phi} = -\frac{1}{2\Delta \Sigma} \frac{\partial}{\partial L_\phi} (R + \Delta \Theta),
\]

\[
\dot{p}_0 = 0,
\]

\(^4\) Each occurrence of \( p_0 \) and \( p_3 \) here is already replaced by the constants of motion \(-E\) and \( L_\phi \), respectively.

\[
p_1 = -\left(\frac{\Delta}{2\Sigma}\right) |p_1|^2 - \left(\frac{1}{2\Sigma}\right) |p_2|^2 + \frac{(R + \Delta \Theta)}{2\Delta \Sigma},
\]

\[
p_2 = -\left(\frac{\Delta}{2\Sigma}\right) |p_1|^2 - \left(\frac{1}{2\Sigma}\right) |p_2|^2 + \frac{(R + \Delta \Theta)}{2\Delta \Sigma},
\]

\[
p_3 = 0.
\]

Here, \( |r| \) and \( |\phi| \) stand for the partial derivatives with respect to \( r \) and \( \theta \).

In addition to the modification of the metric and thus the evolution equations, one has to modify the orbital frequency of particles around a compact object. This has been done in Schönénbach et al. (2013) with the resulting frequencies as

\[
\omega_\Lambda = \frac{1}{a + \sqrt{\frac{\Sigma}{\Theta}}},
\]

where \( \omega_\Lambda \) describes prograde motion and \( h(r) = \frac{2m_\Lambda}{r} - \frac{3b}{r^2} \). Equation (14) reduces to the well-known \( \omega_\Lambda \) for \( B = 0 \).

Finally, the concept of an innermost stable circular orbit (ISCO) has to be revised, as the pc-equivalent of the Kerr metric only shows an ISCO for some values of the spin parameter \( a \). For values of \( a \) greater than 0.416 and \( B = \frac{\Sigma}{\Theta} m^3 \), there is no region of unstable orbits anymore (Schönénbach et al. 2013).

After including all those changes due to correction terms of the pc-equivalent of the Kerr metric one can straightforwardly adapt the calculations done in GYOTO. The adapted version will be published online soon.

### 2.1 Obtaining observables

After setting the stage for geodesic evolution used for ray-tracing, we will briefly discuss physical observables used in ray-tracing studies. First of all, let us note that we will focus on ray-tracing of null geodesics and thus our observables are of radiative nature. The first quantity of interest is the intensity of the radiation. The intensity of radiation emitted between a point \( s_0 \) and the position \( s \) in the emitters frame is given by (Rybicki & Lightman 2004; Vincent et al. 2011)

\[
I_r(s) = \int_{s_0}^{s} \exp \left( -\int_{s'}^s \alpha_s(s') \, ds' \right) j_r(s') \, ds'.
\]

Here, \( \alpha_s \) is the absorption coefficient and \( j_r \) the emission coefficient in the comoving frame.

Using the invariant intensity \( I = I_r/v^3 \) (Misner, Thorne & Wheeler 1973), one gets the observed intensity via

\[
I_{\text{obs}} = g^3 I_{\text{em}},
\]

where we introduced the relativistic generalized redshift factor \( g := \frac{\lambda_{\text{em}}}{\lambda_{\text{obs}}} \). The quantity observed however is the flux \( F \) which is given by

\[
dF_{\text{obs}} = I_{\text{obs}} \cos \theta d\Omega,
\]

where \( \theta \) describes the angle between the normal of the observer’s screen and the direction of incidence and \( \Omega \) gives the solid angle in which the observer sees the light source (Vincent et al. 2011).

In the following, we will consider two special cases for the intensity. First, the emission line in an optically thick, geometrically thin accretion disc, which can be modelled by (Fanton et al. 1997; Vincent et al. 2011)

\[
I_r \propto \delta(v_{\text{em}} - v_{\text{line}}) \epsilon(r),
\]
where the radial emissivity $\varepsilon(r)$ is given by a power law

$$\varepsilon(r) \propto r^{-\alpha},$$

with $\alpha$ being the single power-law index.

The second emission model we consider is a geometrically thin, infinite accretion disc first modelled by Page & Thorne (1974). The intensity profile here is strongly dependent on the used metric and thus some modifications have to be done. Fortunately, most of the results of Page & Thorne (1974) can be inherited and only at the end one has to insert the modified metric. Equation 12 in Page & Thorne (1974)

$$f = -\omega_f (E - \omega L_z)^2 \int_{r_{\text{max}}}^{r} (E - \omega L_z) L_{z1}\,dr$$

builds the core for the computation of the flux (Page & Thorne 1974) as

$$F = \frac{M_0}{4\pi\sqrt{-g}} f.$$ (21)

Assuming $M_0 = 1$ as in Vincent et al. (2011) and observing that the determinant of the metric $\sqrt{-g}$ is the same for both GR and pc-GR, we see that the only difference in the flux lies in the function $f$ given by equation (20).

In addition to the assumptions made by Page & Thorne (1974), we have to include the assumption that the stresses inside the disc carry angular momentum and energy from faster to slower rotating parts of the disc. In the case of standard GR, this assumption means that energy and angular momentum get transported outwards. In pc-GR, equation (20) then has to be modified to

$$f = -\omega_f (E - \omega L_z)^2 \int_{r_{\text{max}}}^{r} (E - \omega L_z) L_{z1}\,dr,$$ (22)

where $\omega_{\text{max}}$ describes the orbit where the angular frequency $\omega$ has its maximum (this is the last stable orbit in standard GR).

Equation (22) gives a concise way to write down the flux in the following two regions ($r_{\text{in}}$ describes the inner edge of an accretion disc):

(i) $r_{\text{max}} < r_{\text{in}} \leq r$: this is also the standard GR case, where $\omega_f < 0$ and the flux in equations (20) and (22) is positive.

(ii) $r_{\text{in}} \leq r < r_{\text{max}}$: here, $\omega_f > 0$, but the upper integration limit in equation (22) is smaller than the lower one. Thus, there are overall two sign changes and the flux $f$ is positive again.

Thus, if we consider a disc whose inner radius is below $r_{\text{max}}$, which is the case in the pc-GR model for $a > 0.416\,m$, equation (22) guarantees a positive flux function $f$.

All quantities $E$, $L_z$, $\omega$ in equation (22) were already computed in Schönenbach et al. (2013). The angular frequency $\omega$ is given in equation (14), $E$ and $L_z$ are given as

$$L_z^2 = \frac{(g_{00} + \omega g_{33})^2}{-g_{33}\omega^2 - 2g_{00}\omega - g_{00}},$$

$$E^2 = \frac{(g_{33} + \omega g_{00})^2}{-g_{33}\omega^2 - 2g_{00}\omega - g_{00}}.$$ (23)

Unfortunately, the derivatives of $E$ and $L_z$ become lengthy in pc-GR and the integral in equation (20) has no analytic solution anymore.

However, it can be solved numerically and thus we are able to modify the original disc model by Page & Thorne (1974) to include pc-GR correction terms.

### 3 RESULTS

As shown in Schönenbach et al. (2013), the concept of an ISCO is modified in the pc-GR model. For the following results, we used as the inner radius for the discs in the pc-GR case the values depicted in Table 1. Values of $r_{\text{in}}$ for $a \leq 0.4\,m$ correspond to the modified last stable orbit. The value of $r_{\text{in}}$ for values of $a$ above $0.416\,m$ is chosen slightly above the value $r = (4/3)\,m$. For smaller radii, equation (14) has no real solutions anymore in the case of $B = (64/27)\,m^3$. The same also holds for general (not necessarily geodesic) circular orbits, where the time component $u_0 = \sqrt{-g_{00} - 2g_{03} - g_{33}/2}$ of the particles four-velocity also turns imaginary for radii below $r = (4/3)\,m$ in the case of $B = (64/27)\,m^3$.

We assume that the compact massive object extends up to at least this radius. For all simulations however we did neglect any radiation from the compact object. This is a simplification which will be addressed in future works.

The angular size of the compact object is also modified in the pc-GR case. It is proportional to the radius of the central object (Mueller 2006), which varies in standard GR between 1 and 2 m, leading to angular sizes of approximately 10–20$\,\mu$as for Sagittarius A*.

The size of the central object in pc-GR is fixed at $r = (4/3)\,m$ in the limiting case for $B = (64/27)\,m^3$ thus leading to an angular size of approximately 13$\,\mu$as.

#### 3.1 Images of an accretion disc

In Figs 3 and 4, we show images of infinite geometrically thin accretion discs according to the model of Page & Thorne (1974, see Section 2.1) in certain scenarios. Shown is the bolometric intensity $I(\text{erg cm}^{-2}\text{s}^{-1}\text{sr}^{-1})$ which is given by $I = \frac{1}{2}F$ (Vincent et al. 2011). To make differences comparable, we adjusted the scales for each value of the spin parameter $a$ to match the scale for the pc-GR scenario. The plots of the Schwarzschild object ($a = 0\,m$) and the first Kerr object ($a = 0.3\,m$) use a linear scale whereas the plots for the other Kerr objects ($a = 0.6$ and $0.9\,m$, respectively) use a log scale for the intensity. This is a compromise between comparability between both theories and visibility in each plot. One has to keep in mind that scales remain constant for a given spin parameter $a$ and change between different values for $a$.

The overall behaviour is similar in GR and pc-GR. The most prominent difference is that the pc-GR images are brighter. An explanation for this effect is the amount of energy which is released for particles moving to smaller radii. This energy is then

| Spin parameter $a$ (m) | $r_{\text{in}}$ (m) |
|------------------------|---------------------|
| 0.0                    | 5.243 92            |
| 0.1                    | 4.823 65            |
| 0.2                    | 4.359 76            |
| 0.3                    | 3.815 29            |
| 0.4                    | 2.999 11            |
| 0.5 and above          | 1.334               |
Figure 1. Normalized energy of particles on stable prograde circular orbits. The pc-parameter $B$ is set to the critical value of $(64/27)m^3$. In the pc-GR case, more energy is released as particles move to smaller radii, where the amount of released energy increases significantly in the case where no last stable orbit is present anymore. The lines end at the last stable orbit or at $r = 1.334$ m, respectively.

Figure 2. Shown is the flux function $f$ from equations (20) and (22) for different values of $a$ (and $B$). If not stated otherwise, $B = (64/27)m^3$ is assumed for the pc-GR case. (a) Flux function $f$ for varying spin parameter $a$ and inner edge of the accretion disc. In the standard GR case, the ISCO is taken as inner radius; for the pc-GR case, see Table 1. (b) Dependence of the flux function $f$ on the inner radius of the disc.

transported via stresses to regions with lower angular velocity, thus making the disc overall brighter. In Fig. 1, we show this energy for particles on stable circular orbits.

At first puzzling might be the fact that the fluxes differ significantly for radii above 10 m, although here the differences between the pc-GR and standard GR metric become negligible. However, the flux $f$ in equation (22) at any given radius $r$ depends on an integral overall radii starting from $r_{\text{max}}$ up to $r$. Thus, the flux at relatively large radii is dependent on the behaviour of the energy at smaller radii, which differs significantly from standard GR.

It is important to stress that the difference in the flux between the standard GR and pc-GR scenarios is thus also strongly dependent on the inner radius of the disc. This is due to the fact that the values for the energy too are strongly dependent on the radius, see Fig. 1(b). In Fig. 2(b), we compare the pc-GR and GR case for the same inner radius. There is still a significant difference between both curves but not as strongly as in Fig. 2(a).

The next significant difference to the standard disc model by Page & Thorne (1974) is the occurrence of a dark ring in the case of $a \geq 0.416$. This ring appears in the pc-GR case due to the fact that the angular frequency of particles on stable orbits now has a maximum at $r = r_{\text{max}} \approx 1.72$ m (Schönenbach et al. 2013) and the discs extend up to radii below $r_{\text{max}}$. At this point, the flux function vanishes, see Section 2.1. Going further inside, the flux increases again, which is a new feature of the pc-GR model. This is the reason of the ring-like structure for $a > 0.416$ m. Note that the bright inner ring may be mistaken for second-order effects although these do not appear as the disc extends up to the central object.

In Fig. 2(a), we show the radial dependence of the flux function, see equations (20) and (22). For small values of $a$, we still have an ISCO in the pc-GR case and the flux looks similar to the standard GR flux – it is comparable to standard GR with higher values of $a$. If $a$ increases and we do not have a last stable orbit in the pc-GR case, the flux gets significantly larger and now has a minimum. This minimum can be seen as a dark ring in the accretion discs in Fig. 4.

Another feature is the change of shape of the higher order images. For spin values of $a \geq 0.416$ m, the disc extends up to the central
Figure 3. Infinite, counter-clockwise geometrically thin accretion disc around static and rotating compact objects viewed from an inclination of 70°. The disc model was developed originally by Page & Thorne (1974). Scales change between the images. (a) Standard GR $a = 0.0$ m. (b) Standard GR $a = 0.3$ m. (c) Standard GR $a = 0.6$ m. (d) Standard GR $a = 0.9$ m.

Object in the pc-GR model, as it is the case for (nearly) extreme spinning objects in standard GR. Therefore, no higher order images can be seen in this case. However, in Figs 3(a)–(d), 4(a) and (b), images of higher order occur. The ring-like shapes in Figs 4(c) and (d) are not images of higher order but still parts of the original disc, as described above. They could be mistaken for images of higher order although they differ significantly on the redshifted side of the disc.

Figure 4. Infinite, counter-clockwise geometrically thin accretion disc around static and rotating compact objects viewed from an inclination of 70°. The disc model has been modified to include pc-GR correction terms as described in Section 2.1. Scales change between the images. (a) pc-GR $a = 0.0$ m. (b) pc-GR $a = 0.3$ m. (c) pc-GR $a = 0.6$ m. (d) pc-GR $a = 0.9$ m.
3.2 Emission-line profiles for the iron Kα line

As mentioned earlier, emission-line profiles allow us to investigate regions of strong gravity. All results in this section share the same parameter values for the outer radius of the disc ($r = 100$ m), the inclination angle ($\theta = 40$ deg) and the power-law parameter $\alpha = 3$ (as suggested for discs first modelled by Shakura & Sunyaev 1973), see equation (19). We use this simpler model to simulate emission lines as it is widely used in the literature and thus results are easily comparable. The angle of $\theta = 40$ deg is just an exemplary value and can be adjusted. As rest energy for the iron Kα line, we use 6.4 keV. The inner radius of the discs is determined by the ISCO and thus varies with varying values for $a$. Shown is the flux in arbitrary units. In Figs 5(a) and (b), we compare the influence of the objects spin on the shape of the emission-line profile in GR and pc-GR separately. Both in GR and pc-GR, we observe the characteristic broad and smeared out low-energy tail, which grows with growing spin. It is more prominent in the case of pc-GR. The overall behaviour is the same in both theories. A closer comparison of both theories and their differences is then done in Figs 6 and 7, where we compare the two theories for different values of the spin parameter $a$. For slow-rotating objects (Schwarzschild limit), almost no difference is observable. As the spin grows, we observe an increase of the low-energy tail in the pc-GR scenario compared to the GR one. The blueshifted peak however stays nearly the same. If we compare both theories for different values of the spin parameter $a$, they get almost indistinguishable for certain choices of parameters, see Fig. 8.

To better understand the emission-line profiles, we have a look at the redshift in two ways. The redshift can be written as (Fanton et al. 1997)

$$g = \frac{1}{u_0^\text{em}(1 - \omega \lambda)},$$

(24)

where $u_0^\text{em} = \frac{1}{\sqrt{g_{00} - 2g_{0i}\omega g_{i3} - g_{i3}^2}}$ is the time component of the emitter’s four-velocity, $\omega$ is the angular frequency of the emitter and $\lambda$ is the ratio of the emitted photons energy to angular momentum. Cisneros et al. (2012) derived an expression for photons emitted directly in the direction of the emitters movement as

$$\lambda_{\text{cis}} = \frac{-g_{03} - \sqrt{g_{03}^2 - g_{00}g_{33}}}{g_{00}}.$$  (25)
Figure 7. Comparison between theories. The plots are done for parameter values $r = 100 \text{ m}$ for the outer radius of the disc, $\theta = 40 \text{ deg}$ for the inclination angle and $\alpha = 3$ for the power-law parameter. The inner radius of the discs is determined by the ISCO and thus varies for varying $a$. (a) $a = 0.6 \text{ m}$. (b) $a = 0.9 \text{ m}$.

Figure 8. Comparison between theories for different values for the spin parameter. The plot is done for parameter values $r = 100 \text{ m}$ for the outer radius of the disc, $\theta = 40 \text{ deg}$ for the inclination angle and $\alpha = 3$ for the power-law parameter. The inner radius of the discs is determined by the ISCO. (a) $a = 0.3 \text{ m}$. (b) $a = 0.6 \text{ m}$.

Figure 9. Combined effects of relativistic Doppler blueshift and gravitational redshift as a function of the radius. The inclination is given as $\theta = 40 \text{ deg}$. Values greater than 1 represent a blueshift. The plots start at the inner edge of the disc and are done for photons emitted parallel to the direction of movement of the emitter, i.e. where the highest blueshift occurs. (a) $a = 0.3 \text{ m}$. (b) $a = 0.6 \text{ m}$.

We take this expression and use it to approximate the redshift viewed from an inclination angle $\theta_{\text{obs}}$ as

$$\frac{1}{\nu_{\text{em}}(1 - \omega \lambda_{\text{em}} \sin \theta_{\text{obs}}}).$$

(26)

In Fig. 9, we show plots for different values of the spin parameter for both GR and the pc-GR model for particles moving towards the observer, where we expect the highest blueshift to occur. To obtain the full frequency shift, one needs in general to know the emission angle of the photon at the point of emission, which can be obtained by using ray-tracing techniques. In Fig. 10, we display this redshift obtained with $\text{GYOTO}$ for a thin disc. Several features can now be seen in Figs 9 and 10. First, we see that the maximal blueshift is almost the same in both the GR and pc-GR case. Then, as the discs extend to smaller radii in pc-GR, we observe that there is a region where photons get redshifted, which is not accessible in GR for the same values of the spin parameter $a$. This can explain the excess of flux in the redshifted region seen in Figs 6 and 7 even for low values of the spin parameter. Finally, the similarity of both theories
Ray-tracing in pc-GR

Figure 10. Redshift for a thin accretion disc. The inclination angle is $40^\circ$. The outer radius is set to $r_{\text{out}} = 50\, \text{m}$. The inner radius is set to the ISCO, if exists. For the pc-GR case, see Table 1. The similarity between the pc-GR case for $a = 0.3\, \text{m}$ and standard GR for $a = 0.6\, \text{m}$ can also be seen in Figs 10(b) and (c).

4 CONCLUSION

We have adapted two models, which are implemented in GYOTO (Vincent et al. 2011) – an infinite, geometrically thin and optically thick accretion disc (Page & Thorne 1974) and the iron K$\alpha$ emission-line profile of a geometrically thin and optically thick disc (Fanton et al. 1997) – to incorporate correction terms due to a pc-extension of GR. In both models, we can see differences between standard GR and pc-GR. These differences can be attributed to the modification of the last stable orbit in pc-GR and thus discs which extend further in for a big range of spin parameter values of the massive object. In addition, the gravitational redshift and orbital frequencies of test particles have to be modified. Both the accretion disc images and emission-lines profiles show an increase in the amount of outgoing radiation thus turning the massive objects brighter in pc-GR than in GR, assuming that all other parameters are the same. Although the difference in the emission-line profiles is in principle big enough to be used to discriminate between GR and pc-GR, the effects of the pc-correction terms on the results are not as strong as the modifications presented, e.g., in Bambi & Malafarina (2013). Also an uncertainty in, e.g., the spin parameter $a$ can make it very difficult to discriminate between both theories as we have seen in Fig. 8.

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