SYMMETRIC RING SPECTRA AND TOPOLOGICAL
HOCHSCHILD HOMOLOGY

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1. Introduction

The category of symmetric spectra introduced by Jeff Smith is a closed symmetric monoidal category whose associated homotopy category is equivalent to the traditional stable homotopy category, see [HSS]. In this paper, we study symmetric ring spectra, i.e., the monoids in the category of symmetric spectra. The category of symmetric ring spectra is closely related to the category of “functors with smash product defined on spheres” defined for instance in [HM, 2.7]. Actually, the category of symmetric ring spectra is equivalent to the category of FSPs defined on spheres if the usual connectivity and convergence conditions on FSPs are removed. The choices of equivalences must also be changed when considering symmetric ring spectra instead of FSPs on spheres. A map of FSPs on spheres is a weak equivalence when the map is a $\pi_*$-isomorphism, i.e., when it induces an isomorphism in the associated stable homotopy groups. As with symmetric spectra, one must consider a broader class of equivalences called stable equivalences when working with symmetric ring spectra, see 2.1.9. In section 2.2, the model category structure on symmetric ring spectra is defined with these stable equivalences. In [MMSS], we show that the associated homotopy category is equivalent to the traditional category of $A_\infty$-ring spectra. Because there are more stable equivalences than $\pi_*$-isomorphisms, the classically defined stable homotopy groups are not invariants of the homotopy types of symmetric spectra or symmetric ring spectra. Hence stable equivalences can be hard to identify. To remedy this we consider a detection functor, $D$, which turns stable equivalences into $\pi_*$-isomorphisms. Theorem 3.1.2 shows that $X \to Y$ is a stable equivalence if and only if $DX \to DY$ is a $\pi_*$-isomorphism. Thus, the classical stable homotopy groups of $DX$ are invariants of the stable homotopy type of $X$. There is also a spectral sequence for calculating the classical stable homotopy groups of $DX$, see Proposition 2.3.4.

The category of FSPs was defined in [B] in order to define the topological Hochschild homology for an associative ring spectrum $R$. In section 4, three different definitions of topological Hochschild homology for a symmetric ring spectrum are considered: Bökstedt’s original definition restated for symmetric ring spectra, derived smash product definition, and a definition which mimics the standard Hochschild complex from algebra. Theorems 4.1.10 and 4.2.8 show that under certain cofibrancy conditions these definitions all agree.

Perhaps the most surprising of these results is the agreement of Bökstedt’s definition with the others without any connectivity or convergence conditions. Some
conditions are indeed necessary to apply Bökstedt’s approximation theorem, [B, 1.6], though the usual connectivity and convergence conditions can be weakened, see Corollary [1.17]. For spectra which do not satisfy these hypotheses the model category structure on symmetric ring spectra is used instead to prove comparison results such as Theorem [3.1.2]. Also, without any extra conditions, Bökstedt’s original definition of THH takes stable equivalences of symmetric ring spectra to \( \pi_*\)-isomorphisms, see Corollary [4.2.10]. See also Remark [2.2.2].

Outline. In the first section we recall various definitions from [HSS], define symmetric ring spectra, and discuss homotopy colimits. Section 2.2 uses the results of [SS] and [HSS] to establish model category structures for symmetric ring spectra, for \( R\)-modules over any symmetric ring spectrum \( R\), and for \( R\)-algebras over any commutative symmetric ring spectrum \( R\). In section 2.3 we define the homotopy colimit of diagrams of symmetric spectra and state several comparison results for homotopy colimits which are used in sections 3 and 4. A functor, \( D\), which detects stable equivalences is defined in section 3. In section 4 three different definitions of topological Hochschild homology are defined and compared. The detection functor from section 3 is used in one of the comparisons in section 4.

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2. Basic definitions

In this section we state the basic definitions which are needed in sections 3 and 4. Most of the definitions in the first subsection come from [HSS]. The second subsection, 2.2, considers symmetric ring spectra and model categories for \( R\)-modules and \( R\)-algebras. In the last subsection, 2.3, we consider the properties of the homotopy colimit needed for sections 3 and 4.

2.1. Symmetric spectra. We first define the symmetric monoidal category of symmetric spectra. Next we define certain model category structures on symmetric spectra. Then we consider a subcategory of symmetric spectra, the semistable spectra, between which stable equivalences are exactly the \( \pi_*\)-isomorphisms. Throughout this paper “space” means simplicial set, except in Remark 3.1.4.

Definition 2.1.1. Let \( \Sigma \) be the skeleton of the category of finite sets and bijections with objects \( n = \{1, \cdots, n\} \). The category of symmetric sequences \( \Sigma^\ast \) is the category of functors from \( \Sigma \) to \( \Sigma_n^\ast \), the category of pointed simplicial sets. Thus, a symmetric sequence is a sequence \( X_n \) of spaces, where \( X_n \) is equipped with a basepoint preserving action of the symmetric group \( \Sigma_n \).

This category of symmetric sequences is a symmetric monoidal category with the following definition of the tensor product of two symmetric sequences.

Definition 2.1.2 (HSS). Given two symmetric sequences \( X \) and \( Y \), we define their tensor product, \( X \otimes Y \),

\[
(X \otimes Y)_n = \bigvee_{p+q=n} \Sigma_n^+ \wedge_{\Sigma_p \times \Sigma_q} (X_p \wedge Y_q).
\]

Note here, as elsewhere in the paper, we sometimes denote an extra basepoint as \( X^+ \) to make the notation more readable.
Let $S^1 = \Delta[1]/\Delta[1]$ and $S^n = (S^1)^{\wedge n}$ for $n > 1$. Then $S = (S^0, S^1, \ldots, S^n, \ldots)$ is a symmetric sequence. In fact, $S$ is a commutative monoid.

**Definition 2.1.3.** The category of symmetric spectra, $Sp^\Sigma$, is the category of left $S$-modules in the category of symmetric sequences.

A symmetric spectrum is then a sequence of pointed spaces with a left, pointed $\Sigma_p$ action on $X_p$ and associative, unital, $\Sigma_p \times \Sigma_q$-equivariant maps $Sp^\Sigma X_q \to X_{p+q}$.

The category of symmetric spectra is a symmetric monoidal category with the following definition of the smash product of two symmetric spectra, see [HSS].

**Definition 2.1.4.** Given two symmetric spectra $X$ and $Y$ we define their smash product $X \wedge_S Y$ as the coequalizer of the two maps

$$X \otimes S \otimes Y \rightrightarrows X \otimes Y.$$

We now describe certain symmetric spectra which play an important role in the model category structures and in the later sections of this paper. Let $I$ be the skeleton of the category of finite sets and injections with objects $n$. Note that $\text{hom}_I(n, m) \cong \Sigma_m/\Sigma_{m-n}$ as $\Sigma_m$ sets.

**Definition 2.1.5.** Define $F_n: S_* \to Sp^\Sigma$ by $F_n K = S \otimes G_n K$ where $G_n K$ is the symmetric sequence with $\text{hom}_I(n, m)_+ \wedge K$ in degree $n$ and the basepoint elsewhere. So $(F_n K)_m = \Sigma_m^+ \wedge_{\Sigma_m} S^{m-n} \wedge K \cong \text{hom}_I(n, m)_+ \wedge S^{m-n} \wedge K$ where $S^0 = *$ for $n < 0$.

$F_n$ is left adjoint to the $n$th evaluation functor $Ev_n: Sp^\Sigma \to S_*$ where $Ev_n(X) = X$. There is a natural isomorphism $F_n K \wedge_S F_m L \to F_{n+m} (K \wedge L)$.

*Model category structures.* There are two model category structures on symmetric spectra which we consider; the injective model category and the stable model category. The injective model category is a stepping stone for defining the stable model category. The homotopy category associated to the stable model category is equivalent to the stable homotopy category of spectra, see [HSS]. Hence, the stable model category is the model category which we refer to most often. See [Q] or [DS] for the basic definitions for model categories.

**Definition 2.1.6.** Let $f: X \to Y$ be a map in $Sp^\Sigma$. The map $f$ is a level equivalence if each $f_n: X_n \to Y_n$ is a weak equivalence of spaces, ignoring the $\Sigma_n$ action. It is a level cofibration if each $f_n$ is a cofibration of spaces.

With level equivalences, level cofibrations, and fibrations the maps with the right lifting property with respect to all maps which are trivial cofibrations, $Sp^\Sigma$ forms a simplicial model category referred to as the injective model category. A cofibration here is just a monomorphism. A fibrant object here is called an injective spectrum. An injective spectrum is a spectrum with the extension property with respect to every monomorphism that is a level equivalence.

To define the equivalences for the stable model category we first need the following definition.

**Definition 2.1.7.** A spectrum $X$ in $Sp^\Sigma$ is an $\Omega$-spectrum if $X$ is fibrant on each level and the adjoint to the structure map $S^1 \wedge X_n \to X_{n+1}$ is a weak equivalence of spaces for each $n$.

Define shifting down functors, $\text{sh}_n: Sp^\Sigma \to Sp^\Sigma$, by $\text{sh}_n (X)_k = X_{n+k}$. Then the adjoint of the structure maps give a map $i_X: X \to \Omega \text{sh}_1 X$. 

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Lemma 2.1.8. \( X \) is an \( \Omega \)-spectrum if and only if \( X \) is level fibrant and \( X \to \Omega sh_1 X \) is a level equivalence.

Definition 2.1.9. Let \( f : X \to Y \) be a map in \( Sp^\Sigma \). The map \( f : X \to Y \) is a stable equivalence if
\[
\pi_0 \text{map}(Y, Z) \to \pi_0 \text{map}(X, Z)
\]
is an isomorphism for all injective \( \Omega \)-spectra \( Z \). The map \( f \) is a stable cofibration if it has the left lifting property with respect to each level trivial fibration, i.e., a map that is a trivial fibration on each level. The map \( f \) is a stable fibration if it has the right lifting property with respect to each map which is both a stable cofibration and a stable equivalence.

Theorem 2.1.10 (HSS). With these definitions of stable equivalences, stable cofibrations, and stable fibrations, \( Sp^\Sigma \) forms a model category referred to as the stable model category. A map is a stable trivial fibration if and only if it is a level trivial fibration. Moreover, the fibrant objects are the \( \Omega \)-spectra and a map between \( \Omega \)-spectra is a stable equivalence if and only if it is a level equivalence.

As shown in [HSS], the stable model category is in fact a cofibrantly generated model category. In particular, this means that a transfinite version of Quillen’s small object argument, [Q, II 3.4], exists. This argument is central to the proofs in sections 3 and 4.

Proposition 2.1.11 (HSS). There is a set of maps \( J \) in \( Sp^\Sigma \) such that any stable trivial cofibration is a retract of a directed colimit of pushouts of maps in \( J \). Similarly, there is a set of maps \( I \) which generate the stable cofibrations. These maps are called the generating stable (trivial) cofibrations.

Let \( I = \{ F_n(\Delta[k]_+) \to F_n(\Delta[k]_+) \} \) and \( J' = \{ F_n(\Delta^1[k]_+) \to F_n(\Delta[k]_+) \} \). \( I \) is the set mentioned in the proposition which generates the stable cofibrations. The set \( J \) in the proposition is the union of \( J' \) with a set \( K \) which we now describe.

First, consider the map \( \sigma : F_1(S^1) \to F_0(S^0) \) which is adjoint to the identity map \( S^1 \to S^1 = Ev_1(F_0S^0) \). There is a factorization of \( \sigma \) as a level cofibration \( c : F_1(S^1) \to C \) followed by a level trivial fibration \( r : C \to F_0(S^0) \). \( C \) is defined as the pushout in the following square.

\[
\begin{array}{ccc}
F_1S^1 \wedge_S F_0(\Delta[0]_+) & \xrightarrow{\sigma} & F_0S^0 \\
\downarrow^{1 \wedge_S i_0} & & \downarrow^i \\
F_1S^1 \wedge_S F_0(\Delta[1]_+) & \xrightarrow{b} & C
\end{array}
\]

Define \( c : F_1S^1 \to C \) as the composite \( b \circ (1 \wedge_S i_1) \). In [HSS], \( c \) is shown to be a stable trivial cofibration. The left inverse, \( r_0 \), to \( i_0 \) induces a map \( \sigma \circ (1 \wedge_S r_0) : F_1S^1 \wedge_S F_0(\Delta[1]_+) \to F_0S^0 \). Use this map and the identity map on \( F_0S^0 \) to induce \( r : C \to F_0S^0 \) using the property of the pushout.

The generating stable trivial cofibrations are built from the map \( c : F_1S^1 \to C \) as follows. Let \( P_{m,r} \) be the pushout of the following square.

\[
\begin{array}{ccc}
F_1S^1 \wedge_S F_m(\Delta[r]_+) & \xrightarrow{c \wedge_S 1} & C \wedge_S F_m(\Delta[r]_+) \\
\downarrow^{1 \wedge_S F_m r} & & \downarrow \\
F_1S^1 \wedge_S F_m(\Delta[r]_+) & \xrightarrow{P_{m,r}} & P_{m,r}
\end{array}
\]
The map $P(c, F_{m}g_{r}) : P_{m, r} \to C \wedge S F_{m} (\Delta [r]_{+})$ is induced by the property of the pushout by the maps $c \wedge S 1 F_{m} (\Delta [r]_{+})$ and $1 C \wedge S F_{m}g_{r}$. Let $J = J' \cup K$ where $K = \{ P(c, F_{m}g_{r}), m, r \geq 0 \}$ and $J'$ is the set of generating level trivial cofibrations defined above.

Quillen’s small object argument [Q, p. II 3.4] has an analogue which allows one to functorially factor maps whenever the model category is cofibrantly generated, see [HSS].

**Definition 2.1.12.** Let $L$ be the functorial stable fibrant replacement functor defined by functorially factoring the map $X \to \ast$ into a stable trivial cofibration, $X \to LX$ and a stable fibration $LX \to \ast$. This is the factorization one defines using the small object argument applied to the set of maps $J$. Using $J'$ instead, one defines $L'$ as the functorial level fibrant replacement with $X \to L' X$ a level trivial cofibration and $L' X \to \ast$ a level fibration.

This analogue of the small object argument provides a general method for proving that the class of cofibrations or trivial cofibrations has some property. One only needs to show that the generating maps have some property and that the property is preserved under pushouts, colimits, and retracts. This general method is used in sections 3 and 4.

We also need to know that the symmetric monoidal structure and the model category structure fit together to give a monoidal model category structure, see [HSS].

**Proposition 2.1.13** (HSS). The stable model category is a monoidal model category. That is, the symmetric monoidal structure satisfies the following pushout product axiom. Let $f : A \to B$ and $g : C \to D$ be two stable cofibrations. Then $Q(f, g) : (A \wedge D) \cup_{A \wedge C} (B \wedge C) \to B \wedge D$ is a stable cofibration. If one of $f$ or $g$ is also a stable equivalence, then so is $Q(f, g)$.

This structure can also be restricted to a simplicial structure via the functor $F_{0} : S_{+} \to Sp^{\Sigma}$.

**Semistable objects and $\pi_{\ast}$-isomorphisms.** It is often useful to compare symmetric spectra to the model category of spectra, $Sp^{N}$, defined in [BF]. There is a forgetful functor $U : Sp^{\Sigma} \to Sp^{N}$ which forgets the action of the symmetric groups and uses the structure maps $S^{1} \wedge X_{n} \to X_{1+n}$.

**Definition 2.1.14.** Let $\pi_{k}(X) = \pi_{k}(UX) = \text{colim}_{i} \pi_{k+i} X_{i}$. A map $f$ of symmetric spectra is a $\pi_{\ast}$-isomorphism if it induces an isomorphism on these classical stable homotopy groups.

As seen in [HSS] these classical stable homotopy groups are NOT the maps in the homotopy category of symmetric spectra of the sphere into $X$. For example $\sigma : F_{1} S^{1} \to F_{0} S^{0}$ is a stable equivalence but it is not a $\pi_{\ast}$-isomorphism. As shown in [HSS], though, a $\pi_{\ast}$-isomorphism is a particular example of a stable equivalence. Hence, to avoid confusion, we use the term $\pi_{\ast}$-isomorphism instead of stable homotopy isomorphism and call these the classical stable homotopy groups instead of just stable homotopy groups. In section 3 we construct a functor, $D$, which converts any stable equivalence into a $\pi_{\ast}$-isomorphism between semistable spectra, see Definition 2.1.16 below.

As in [BF], we define a functor $Q$ for symmetric spectra.
Definition 2.1.15. Define $QX = \text{colim}_n \Omega^n L' \text{sh}_n X$.

This functor does not have the same properties as in [BF]. For instance $QX$ is not always an $\Omega$-spectrum and $X \to QX$ is not always a $\pi_*$-isomorphism. One property that does continue to hold, however, is that a map $f$ is a $\pi_*$-isomorphism if and only if $Qf$ is a level equivalence. Also, $QX$ is always level fibrant.

Definition 2.1.16. A semistable symmetric spectrum is one for which the stable fibrant replacement map, $X \to LX$, is a $\pi_*$-isomorphism.

Of course $X \to LX$ is always a stable equivalence, but not all spectra are semistable. For instance $F_1 S^1$ is not semistable. Any stably fibrant spectrum, i.e., an $\Omega$-spectrum, is semistable though. The following proposition shows that on semistable spectra $Q$ has the same properties as in [BF] on $Sp^N$.

Proposition 2.1.17. The following are equivalent.

1. The symmetric spectrum $X$ is semistable.
2. The map $X \to \Omega L' \text{sh}_1 X$ is a $\pi_*$-isomorphism.
3. $X \to QX$ is a $\pi_*$-isomorphism.
4. $QX$ is an $\Omega$-spectrum.

Before proving this proposition we need the following lemma.

Lemma 2.1.18. Let $X \in Sp^N$. Then $\pi_k(QX)_n$ and $\pi_{k+1}(QX)_{n+1}$ are isomorphic groups, and $i_{QX} : (QX)_n \to \Omega(Q \text{sh}_1 X)_{n+1}$ induces a monomorphism $\pi_k(QX)_n \to \pi_{k+1}(QX)_{n+1}$.

Proof. We can assume $X$ is level fibrant. Both $\pi_k(QX)_n$ and $\pi_{k+1}(QX)_{n+1}$ are isomorphic to the $(k-n)$th classical stable homotopy group of $X$. However, the map $\pi_k i_{QX}$ need not be an isomorphism. Indeed, $\pi_k i_{QX}$ is the map induced on the colimit by the vertical maps in the diagram

$$
\begin{array}{cccc}
\pi_k X_n & \to & \pi_{k+1} X_{n+1} & \to & \pi_{k+2} X_{n+2} & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\pi_{k+1} X_{n+1} & \to & \pi_{k+2} X_{n+2} & \to & \pi_{k+3} X_{n+3} & \to & \cdots \\
\end{array}
$$

where the vertical maps are not the same as the horizontal maps, but differ from them by isomorphisms. The induced map on the colimit is injective in such a situation, though not necessarily surjective. \[Q\]

Proof of Proposition 2.1.17. First we show that (1) implies (2) by using the following diagram.

$$
\begin{array}{ccc}
X & \to & \Omega L' \text{sh}_1 X \\
\downarrow & & \downarrow \\
LX & \to & \Omega L' \text{sh}_1 LX
\end{array}
$$

Since $\Omega L' \text{sh}$ preserves $\pi_*$-isomorphisms both vertical arrows are $\pi_*$-isomorphisms. The bottom map is a level equivalence since $LX$ is an $\Omega$-spectrum. Hence the top map is also a $\pi_*$-isomorphism.

Also, (2) easily implies (3). Since $\Omega$ and $\text{sh}$ commute and both preserve $\pi_*$-isomorphisms, $X \to QX$ is a colimit of $\pi_*$-isomorphisms provided $X \to \Omega L' \text{sh}_1 X$ is a $\pi_*$-isomorphism.
Next we show that (3) and (4) are equivalent. The map \( \pi_* X \to \pi_* QX \) factors as \( \pi_* X \to \pi_* (QX)_0 \to \pi_* QX \) where the first map here is an isomorphism by definition. Then by Lemma 2.1.18 we see that \( \pi_* (QX)_0 \to \pi_* QX \) is an isomorphism if and only if \( \pi_* (QX)_n \to \pi_* (QX)_{n+1} \) is an isomorphism for each \( n \).

To see that (3) implies (1), consider the following diagram.

\[
\begin{array}{ccc}
X & \longrightarrow & LX \\
\downarrow & & \downarrow \\
QX & \longrightarrow & QLX
\end{array}
\]

By (3) and (4) the left arrow is a \( \pi_* \)-isomorphism to an \( \Omega \)-spectrum, \( QX \). Since \( LX \) is an \( \Omega \)-spectrum the right arrow is a level equivalence. Since \( X \to LX \) is a stable equivalence, the bottom map must also be a stable equivalence. But a stable equivalence between \( \Omega \)-spectra is a level equivalence, so the bottom map is a level equivalence. Hence the top map is a \( \pi_* \)-isomorphism.

Two classes of semistable spectra are described in the following proposition. The second class includes the connective and convergent spectra.

**Proposition 2.1.19.**

1. If the classical stable homotopy groups of \( X \) are all finite then \( X \) is semistable.
2. Suppose that \( X \) is a level fibrant symmetric spectrum and there exists some \( \alpha > 1 \) such that \( X_n \to \Omega X_{n+1} \) induces an isomorphism \( \pi_k X_n \to \pi_{k+1} X_{n+1} \) for all \( k \leq \alpha n \) for sufficiently large \( n \). Then \( X \) is semistable.

**Proof.** By Lemma 2.1.18, \( \pi_k (QX)_n \to \pi_{k+1} (QX)_{n+1} \) is a monomorphism between two groups which are isomorphic. In the first case these groups are finite, so this map must be an isomorphism. Hence \( QX \) is an \( \Omega \)-spectrum, so \( X \) is semistable.

For the second part we also show that \( QX \) is an \( \Omega \)-spectrum. Since for fixed \( k \) the maps \( \pi_{k+i} X_{n+i} \to \pi_{k+1+i} X_{n+1+i} \) are isomorphisms for large \( i \), \( \pi_k (QX)_n \to \pi_{k+1} (QX)_{n+1} \) is an isomorphism for each \( k \) and \( n \).

The next proposition shows that stable equivalences between semistable spectra are particularly easy to understand.

**Proposition 2.1.20.** Let \( f : X \to Y \) be a map between two semistable symmetric spectra. Then \( f \) is a stable equivalence if and only if it is a \( \pi_* \)-isomorphism.

**Proof.** Since \( X \to LX \) and \( Y \to LY \) are \( \pi_* \)-isomorphisms, \( Lf \) is a \( \pi_* \)-isomorphism if an only if \( f \) is a \( \pi_* \)-isomorphism. But \( Lf \) is a map between \( \Omega \)-spectra so it is a \( \pi_* \)-isomorphism if an only if it is a level equivalence. But, in general, \( f \) is a stable equivalence if and only if \( Lf \) is a level equivalence.

Finally, we show that any spectrum \( \pi_* \)-isomorphic to a semistable spectrum is itself semistable.

**Proposition 2.1.21.** If \( f : X \to Y \) is a \( \pi_* \)-isomorphism and \( Y \) is semistable then \( X \) is semistable.

**Proof.** Since \( Lf \) and \( Y \to LY \) are \( \pi_* \)-isomorphisms, \( X \to LX \) is also a \( \pi_* \)-isomorphism.
2.2. **Symmetric ring spectra.** In this section, rings, modules, and algebras are defined for symmetric spectra. We also discuss the model category structures on these categories.

**Definition 2.2.1.** A *symmetric ring spectrum* is a monoid in the category of symmetric spectra. In other words, a symmetric ring spectrum is a symmetric spectrum, \( R \), with maps \( \mu : R \wedge_S R \to R \) and \( \eta : S \to R \) such that they are associative and unital, i.e., \( \mu \circ (\mu \wedge_S \text{id}) = \mu \circ (\text{id} \wedge_S \mu) \) and \( \mu \circ (\eta \wedge_S \text{id}) \cong \text{id} \cong \mu \circ (\text{id} \wedge_S \eta) \).\( R \) is called *commutative* if \( \mu \circ \text{tw} = \mu \) where \( \text{tw}: R \wedge_S R \to R \wedge_S R \) is the twist isomorphism.

Since symmetric ring spectra are the only type of ring spectra in this paper we also refer to them as simply *ring spectra*. Using formal properties of symmetric monoidal categories, one can show that a monoid in the category of \( S \)-modules is the same as a monoid in the category of symmetric sequences with a monoid map \( \eta : S \to R \) which is central in the sense that \( \mu \circ (\text{id} \wedge_S \eta) \circ \text{tw} = \mu \circ (\eta \wedge_S \text{id}) \).

**Remark 2.2.2.** This description of a symmetric ring spectrum agrees with the definition of a functor with smash product defined on spheres as in \([HM, 2.7]\). The centrality condition mentioned above, and in \([HM, 2.7.ii]\), is necessary but was not included in some earlier definitions of FSPs defined on spheres. Note, however, that there are no connectivity (e.g. \( F(S^{n+1}) \) is \( n \)-connected) or convergence conditions (e.g. the limit is attained at a finite stage in the colimit defining \( \pi_n \) for each \( n \)) placed on symmetric ring spectra. These conditions are usually assumed although not always explicitly stated when using FSPs. In particular, these conditions are necessary for applying Bökstedt’s approximation theorem, \([B, 1.6]\). Corollary 3.1.7 shows that a special case of this approximation theorem holds for any semistable spectrum. To consider non-convergent spectra we use Theorem 3.1.2 in place of the approximation theorem. This theorem does not require any connectivity or convergence conditions.

Proposition 2.1.19 shows that the connectivity and convergence conditions on an FSP ensure that the associated underlying symmetric spectrum is semistable. Proposition 2.1.20 shows that stable equivalences between such FSPs are exactly the \( \pi_* \)-isomorphisms. As with the category of symmetric spectra, inverting the \( \pi_* \)-isomorphisms is not enough to ensure that the homotopy category of symmetric ring spectra is equivalent to the homotopy category of \( A_\infty \)-ring spectra. So once the connectivity and convergence conditions are removed one must consider stable equivalences instead of just \( \pi_* \)-isomorphisms.

We also need the following definitions of \( R \)-modules and \( R \)-algebras in later sections.

**Definition 2.2.3.** Let \( R \) be a symmetric ring spectrum. A (left) \( R \)-module is a symmetric spectrum \( M \) with a map \( \alpha : R \wedge_S M \to M \) that is associative and unital.

**Definition 2.2.4.** Let \( R \) be a commutative ring spectrum. An \( R \)-algebra is a monoid in the category of \( R \)-modules. That is, an \( R \)-algebra is a symmetric spectrum \( A \) with \( R \)-module maps \( \mu : A \wedge_R A \to A \) and \( R \to A \) that satisfy the usual associativity and unity diagrams.

Note that symmetric ring spectra are exactly the \( S \)-algebras.

*Model category structures for rings and modules* We developed techniques in [SS] to form model category structures for algebras and modules over a cofibrantly
generated, monoidal model category. To apply the results from [SS] the smash product is required to satisfy the monoid axiom, which is verified in [HSS]. Hence, the category of modules over a given symmetric ring spectrum $R$ and the category of $R$-algebras for any given commutative ring spectrum $R$ are model categories, see Theorems 2.2.3 and 2.2.7. Each of these model category structures uses the underlying stable equivalences and underlying stable fibrations as the new weak equivalences and fibrations. A map is then a cofibration if it has the left lifting property with respect to the underlying stable trivial fibrations. These model category structures are used in section 4.

After establishing these model category structures we state certain comparison theorems which show that a weak equivalence of ring spectra induces an equivalence of the homotopy theories of the respective modules and algebras. These comparison theorems show that the cofibrancy condition that appears in certain theorems in section 4 is not too restrictive.

In [HSS] $Sp^S$ is shown to be a cofibrantly generated, monoidal model category which satisfies the monoid axiom. Hence Theorem 4.1 in [SS] applies to give the following results in the category of symmetric spectra.

**Theorem 2.2.5.** Let $R$ be a symmetric ring spectrum. Then the category of $R$-modules is a cofibrantly generated model category with weak equivalences and fibrations given by the underlying stable model category structure of $Sp^S$. The generating cofibrations and trivial cofibrations are given by applying $R \wedge_S -$ to the generating maps in $Sp^S$.

*Proof.* This follows from Theorem 4.1 (1) in [SS], once we note that the domains of the generating cofibrations and trivial cofibrations in $Sp^S$ are small with respect to the whole category, see [HSS].

The next lemma is used in proving Theorem 2.2.7. It shows, for any commutative monoid $R$, that the category of $R$-modules has properties similar to the underlying category of $S$-modules.

**Lemma 2.2.6.** Let $R$ be a commutative ring in $Sp^S$. Then the model category structure on $R$-modules given above is a monoidal model category which satisfies the monoid axiom.

*Proof.* This follows from [SS, 4.1 (2)].

**Theorem 2.2.7.** Let $R$ be a commutative monoid in $Sp^S$. Then the category of $R$-algebras is a cofibrantly generated model category with weak equivalences and fibrations given by the underlying stable model category structure of $Sp^S$. The generating cofibrations and trivial cofibrations are given by applying the free monoid functor to the generating (trivial) cofibrations of $R$-modules. Moreover, if $f : A \to B$ is a cofibration of $R$-algebras with $A$ cofibrant as an $R$-module then $f$ is also a cofibration in the underlying category of $R$-modules. In particular, this shows that any cofibrant $R$-algebra is also cofibrant as an $R$-module.

*Proof.* This follows from Theorem 4.1 (3) in [SS]. The facts about cofibrant objects and cofibrations of $R$-algebras follow from [SS, 4.1 (3)] because $S$ is cofibrant in $Sp^S$.

Since symmetric ring spectra are exactly the $S$-algebras, the following is a corollary of Theorem 2.2.7.
**Corollary 2.2.8.** The category of symmetric ring spectra, $\text{S-alg}$, is a cofibrantly generated model category with weak equivalences and fibrations the underlying stable equivalences and stable fibrations of $\text{Sp}^\Sigma$.

The following lemma is needed to apply the comparison theorems of [SS, 4.3-4] which show that a weak equivalence of symmetric ring spectra induces an equivalence on the respective homotopy theories of modules and algebras. This lemma is also needed for section 4.

**Lemma 2.2.9 (HSS).** Let $R$ be a symmetric ring spectrum and $M$ a cofibrant $R$-module. Then $M \wedge_R -$ takes level equivalences of $R$-modules to level equivalences in $\text{Sp}^\Sigma$ and it takes stable equivalences of $R$-modules to stable equivalences in $\text{Sp}^\Sigma$.

The following two comparison theorems follow from Lemma 2.2.9, [SS, 4.3-4], and the fact that $S$ is cofibrant in $\text{Sp}^\Sigma$.

**Theorem 2.2.10.** If $A \sim - \rightarrow B$ is a map of symmetric ring spectra which is an underlying stable equivalence, then the total derived functors of restriction and extension of scalars induce equivalences of homotopy theories

$$\text{Ho}(A\text{-mod}) \cong \text{Ho}(B\text{-mod}).$$

**Theorem 2.2.11.** If $A \sim - \rightarrow B$ is a map of commutative ring spectra which is an underlying stable equivalence, then the total derived functors of restriction and extension of scalars induce equivalences of homotopy theories

$$\text{Ho}(A\text{-alg}) \cong \text{Ho}(B\text{-alg}).$$

The following two lemmas from [HSS] are used to verify some of the properties of the smash product that we have mentioned here. They are also needed in section 4.

**Lemma 2.2.12.** Let $X \rightarrow Y$ be a level equivalence. Then $A \wedge_S X \rightarrow A \wedge_S Y$ is a level equivalence for any cofibrant spectrum $A$.

**Lemma 2.2.13.** Let $X \rightarrow Y$ be a stable equivalence. Then $A \wedge_S X \rightarrow A \wedge_S Y$ is a stable equivalence for any cofibrant spectrum $A$.

### 2.3. Homotopy colimits

In this section we list some of the properties of the homotopy colimit functor for symmetric spectra which are used in the latter parts of this paper. The most important property is that the homotopy colimit in symmetric spectra can be defined by using the homotopy colimit of spaces at each level, see Definition 2.3.1. This is useful not only because the homotopy colimit of spaces is well understood, but also to show that the homotopy colimit preserves level equivalences of symmetric spectra, see Proposition 2.3.2. We use the basic construction of the homotopy colimit for spaces from [BK].

**Definition 2.3.1.** Let $B$ be a small category and $F : B \rightarrow \text{Sp}^\Sigma$ a diagram of symmetric spectra. Let $F_i$ denote the diagram of spaces at level $l$. Then

$$(\text{hocolim}^B_{\text{Sp}^\Sigma} F)_l = \text{hocolim}^B_{\text{Sp}_l} F_i.$$  

This definition makes sense because any stable cofibration is a level cofibration and colimits in $\text{Sp}^\Sigma$ are created on each level. Also, we show that this homotopy colimit has the usual properties of a homotopy colimit. Namely, a map between diagrams which is objectwise a level equivalence, a $\pi_*$-isomorphism, or a stable
equivalence induces the same type of equivalence on the homotopy colimit. The next two propositions consider the first two cases. The case of stable cofibrations could be proved by generalizing [BK, XII 4.2] to arbitrary model categories. Instead, here we use the detection functor developed in section 3 to verify this property in Lemma 4.1.5.

**Proposition 2.3.2.** Let $F, G : B \to Sp^\Sigma$ be two diagrams of symmetric spectra with a natural transformation $\eta : F \to G$ between them. If $\eta(b) : F(b) \to G(b)$ is a level equivalence at each object $b \in B$, then $\text{hocolim}^B F \to \text{hocolim}^B G$ is a level equivalence.

**Proof.** Using Proposition 2.3.1 this statement reduces to asking that the homotopy colimit preserve objectwise weak equivalences of simplicial sets. This is the dual of [BK, XI 5.6]. Cofibrancy conditions are not required here since any space (i.e., simplicial set) is cofibrant. \hfill \Box

We also need to know that the homotopy colimit of an objectwise $\pi_*^s$-isomorphism is a $\pi_*^s$-isomorphism. For this we form a spectral sequence for calculating the classical stable homotopy groups of the homotopy colimit. Following [BK, XII 5] one can form a spectral sequence for calculating any homology theory applied to the homotopy colimit of spaces.

Applying homotopy and taking colimits in the two different directions finishes the proof. In one direction, one gets colim $\pi_*^s \Omega^j L^j \Sigma^j X_i$, but each of these terms and hence the colimit is isomorphic to $\pi_*^s X_i$. In the other direction, one has the colim $\Omega^i \pi_*^s L^i \Sigma^i X_i$.

So applying the homology theory $\pi_*^s$, the above spectral sequence calculates $\pi_*^s$ of each level of the homotopy colimit. Since $\pi_*^s X = \text{colim}_n \pi_*^s X_n$ and a sequential colimit of spectral sequences is a spectral sequence, taking the colimit of these level spectral sequences produces a spectral sequence.

**Proposition 2.3.4.** For $F : B \to Sp^\Sigma$, there is a spectral sequence converging to $\pi_* \text{hocolim}^B_{Sp^\Sigma} F$ with $E^2$-term
\[ E^2_{s,t} = \text{colim}^h_{h,t}(h,F). \]

Using this spectral sequence we can show that homotopy colimits preserve objectwise $\pi_*^s$-isomorphisms.

**Proposition 2.3.5.** Let $F, G : B \to Sp^\Sigma$ be two diagrams of symmetric spectra with a natural transformation $\eta : F \to G$ between them. If $\eta(b) : F(b) \to G(b)$ induces a $\pi_*^s$-isomorphism at each object $b \in B$ then $\text{hocolim}^B F \to \text{hocolim}^B G$ induces a $\pi_*^s$-isomorphism.
Proof. Since \( \eta \) induces a \( \pi_* \)-isomorphism between the two diagrams in question, it induces an \( E^2 \)-isomorphism. Thus it induces an isomorphism on the \( E^n \)-term and hence a \( \pi_* \)-isomorphism on the homotopy colimits.

For section \( 3 \) we also need the following two cofinality results which are from [BK, XI 9.2].

Given a functor between two small categories \( f : A \to B \) one has a natural map \( \text{hocolim}^A f_*F \to \text{hocolim}^B F \). A functor \( f : A \to B \) is called terminal or right cofinal, if for every object \( b \in B \) the under category \( (b \downarrow f) \) is contractible see [BK, XI 9].

**Proposition 2.3.6.** Let \( f : A \to B \) be a functor which is terminal. Then for any functor \( F : B \to \text{Sp}^\Sigma \), \( \text{hocolim}^A f_*F \to \text{hocolim}^B F \) is a level equivalence.

**Proof.** The dual of [BK, XI 9.2] states this property for objectwise cofibrant diagrams of spaces. Applying this on each level and using the fact that spaces are cofibrant proves this statement.

In section \( 3 \) we consider diagrams over the skeleton of the category of finite sets and injections, \( I \), with objects \( n \). Let \( I_m \) denote the full subcategory of \( I \) whose objects are \( n \) where \( n \) is greater than or equal to \( m \). The following lemma states the cofinality information relating these categories.

**Lemma 2.3.7.** Let \( F : I \to \text{Sp}^\Sigma \) be a diagram of spectra. The inclusion \( u_m : I_m \to I \) is terminal, hence \( \text{hocolim}^{I_m} u_m^*F \to \text{hocolim}^I F \) is a level equivalence.

**Proof.** Consider the functor \( +m : I \to I_m \) which induces a functor on the under categories. There is a natural transformation from the identity functor to both \( u_m \circ (+m) \) and \( (-+m) \circ u_m \). Hence the under categories are each homotopy equivalent to \( (i \downarrow I) \). But \( (i \downarrow I) \) is contractible because it has an initial object \( 1 : i \to i \). The homotopy colimit statements follow from Proposition 2.3.6.

Using these cofinality results we can prove the following proposition.

**Proposition 2.3.8.** Let \( F,G : I \to S_* \) be two diagrams of spaces with a natural transformation \( \eta : F \to G \) between them. Assume that \( \eta (n) : F(n) \to G(n) \) is a \( \lambda(n) \) connected map, where \( \lambda(n) \leq \lambda(n+1) \) and \( \lim_n \lambda(n) = \infty \). Then \( \text{hocolim}^I F \to \text{hocolim}^I G \) is a weak equivalence.

**Proof.** We show that the map is an \( N \)-equivalence for every \( N > 0 \). Choose an \( n \) such that \( \lambda(n) > N \). Then for every object \( m \) in \( I_n' \) the map \( \eta : F(m) \to G(n) \) is an \( N \)-equivalence, and so we can conclude that \( \eta : \text{hocolim}^{I_n} u_n^*F \to \text{hocolim}^{I_n} u_n^*G \) is an \( N \)-equivalence. The proposition follows by Lemma 2.3.7.

We also need the following proposition which shows that the homotopy colimit of a diagram of level equivalences over \( I \) is level equivalent to its value on \( 0 \).

**Proposition 2.3.9.** Let \( F : I \to \text{Sp}^\Sigma \) be a diagram of spectra. Assume that for each morphism \( f \) in \( I \), \( F(f) \) is a level equivalence. Then the inclusion \( F(0) \to \text{hocolim}^I F \) is a level equivalence.

**Proof.** Consider the constant functor \( C : I \to \text{Sp}^\Sigma \) with constant value \( F(0) \). Then at each object the map \( C(n) = F(0) \to F(n) \) induced by the unique map \( 0 \to n \) is a level equivalence. Hence, by Proposition 2.3.2 it induces a level equivalence on the homotopy colimits, \( F(0) \to \text{hocolim}^I F \).
Finally, we need the following proposition due to Jeff Smith, [S]. Let \( T \) be the category with objects \( n = \{1, \ldots, n\} \) and morphisms the standard inclusions. Homotopy colimits over \( T \) are weakly equivalent to telescopes. Let \( \omega \) be the ordered set of natural numbers and \( I_\omega \) be the category whose objects are the finite sets \( n \) and the set \( \omega \) and whose morphisms are inclusions. Let \( L_h F : I_\omega \to S_* \) be the left homotopy Kan extension of \( F : I \to S_* \) along the inclusion of categories \( i : I \to I_\omega \).

**Proposition 2.3.10.** Let \( M \) be the monoid of injective maps \( i : \omega \to \omega \) under composition. Given any functor \( F : I \to S_* \), then

1. \( \text{hocolim}^I F \) is weakly equivalent to \( (L_h F(\omega))_{hM} \) where \( (-)_{hM} \) is the homotopy orbits with respect to the action of \( M \), and
2. \( L_h F(\omega) \) is weakly equivalent to \( \text{hocolim}^T F \).

**Proof.** For the convenience of the reader we sketch Smith’s proof of this proposition. Since \( L_h F \) is the homotopy Kan extension, \( \text{hocolim}^I F \simeq \text{hocolim}^I_{h}\omega \text{L}F \). Next, consider the full subcategory, \( A \) of \( I_\omega \) with just one object, \( \omega \). Since the inclusion of \( A \) in \( I_\omega \) is terminal, \( \text{hocolim}^I_{h}\omega \text{L}F \) is weakly equivalent to \( \text{hocolim}^A \omega \text{L}F \). Since \( \text{Hom}_A(\omega, \omega) = M \), \( \text{hocolim}^A \omega \text{L}F \) is the homotopy orbit space \( (\omega \text{L}F(\omega))_{hM} \).

For the second statement, \( L_h F(\omega) = \text{hocolim}^i_{i(\omega) \rightarrow \omega \in (i(\omega))} F(n) \). Here \( i \) is the inclusion \( I \to I_\omega \). The category \( T \) described above is equivalent to the category \( (i \circ \alpha \downarrow \omega) \) for the inclusion \( \alpha : T \to I \). This category \( (i \circ \alpha \downarrow \omega) \) is terminal in \( (i \downarrow \omega) \), because every under category has an initial object. So by Proposition 2.3.6, \( L_h F(\omega) \) is weakly equivalent to \( \text{hocolim}^T F \). \( \square \)

### 3. Detecting stable equivalences

In this section we introduce a functor, \( D \), which detects stable equivalences in the sense that a map \( X \to Y \) is a stable equivalence if and only if \( DX \to DY \) is a \( \pi_* \)-isomorphism. Of course the stable fibrant replacement functor, \( L \), also has this property. It even turns stable equivalences into level equivalences. The drawback of \( L \) is that its only description is via the small object argument. Hence it is difficult to say much about \( L \) apart from its abstract properties. The advantage of the functor \( D \) is that it has a more explicit definition. In particular, there is a spectral sequence for calculating the classical stable homotopy groups of \( DX \), see Proposition 2.3.4. Moreover, these groups are invariants of the stable equivalence type of \( X \) because \( D \) takes stable equivalences to \( \pi_* \)-isomorphisms.

In Section 4 we see that \( D \) fits into a sequence of functors used to define THH in [B]. We use the notation \( D \) instead of \( \text{THH}_0 \) because \( D \) is defined on any symmetric spectrum, not just on ring spectra.

**3.1. Main statements and proofs.** The detection functor \( D \) is defined as a homotopy colimit over the diagram category of the skeleton of finite sets and injections, \( I \). Given a symmetric spectrum \( X \), define a functor \( \mathbb{D}_X : I \to Sp^\Sigma \) whose value on the object \( n \) is \( \Omega^n L' F_0 X_n \). Recall \( L' \) is just a level fibrant replacement functor. For a standard inclusion of a subset \( \alpha : n \subset m \) the map \( \mathbb{D}_X(\alpha) \) is just \( \Omega^n L' \) applied to the composition of maps \( F_0 X_n \to F_0 \Omega^m X_m \to \Omega^{m-n} F_0 X_m \) induced by the structure maps of \( X \). For an isomorphism, the action is given by the conjugation action on the loop coordinates and on \( X_n \). All morphisms in \( I \) are compositions of isomorphisms and these standard inclusions.
**Definition 3.1.1.** The detection functor $D : Sp^\Sigma \to Sp^\Sigma$ is defined by

$$DX = \text{hocolim}_{Sp^\Sigma}^I D_X.$$ 

As defined in definition 2.3.1, the homotopy colimit of symmetric spectra is given by a level homotopy colimit of spaces. Hence

$$(DX)_n = \text{hocolim}_{k \in I}^{k \in \mathbb{N}} \Omega^k L^\Sigma^n X_k.$$ 

As mentioned above, the main reason for considering $D$ is that it detects stable equivalences. This is stated in the next theorem.

**Theorem 3.1.2.** The following are equivalent.

1. $X \to Y$ is a stable equivalence.
2. $DX \to DY$ induces a $\pi_\ast$-isomorphism.
3. $D^2 X \to D^2 Y$ is a level equivalence.
4. $QDX \to QDY$ is a level equivalence.

**Remark 3.1.3.** Notice that one can apply the forgetful functor $U : Sp^\Sigma \to Sp^N$ after applying $D$. In that case, this theorem says that although the usual forgetful functor does not detect and preserve stable equivalences, the composition of this detection functor with the forgetful functor does detect and preserve weak equivalences. Note also that although the classical stable homotopy groups are not invariants of stable equivalence types in symmetric spectra this theorem shows that after applying $D$ the classical stable homotopy groups are invariants.

**Remark 3.1.4.** We could also consider symmetric spectra over topological spaces instead of simplicial sets here, see [HSS]. In that case, Theorem 3.1.2 and all of the statements leading up to it in this section and in section 2.3 which do not involve the functor $Q$ hold when the objects involved are levelwise non-degenerately based spaces. Hence, $D$ also detects stable equivalences between symmetric spectra based on topological spaces. More precisely, let $c$ be a cofibrant replacement functor of spaces applied levelwise, then $X \to Y$ is a stable equivalence if and only if $DeX \to DeY$ is a $\pi_\ast$-isomorphism.

The only fact that is needed to modify all of these statements for topological spaces is that homotopy colimits of non-degenerately based spaces are invariant under weak homotopy equivalences. For the statements involving $Q$ one needs to consider stably cofibrant symmetric spectra because these statements require that homotopy groups commute with directed colimits. But these statements are separate from the statements involving $D$.

**Theorem 3.1.5.** Consider the properties of $D$ with respect to morphisms. The following theorem considers the properties of $D$ on objects.

**Theorem 3.1.5.** Let $X$ be a symmetric spectrum.

1. $DX$ is semistable.
2. If $X$ is semistable, then the level fibrant replacement of $DX$, $L' DX$, is an $\Omega$-spectrum.

Since stable equivalences between semistable spectra are $\pi_\ast$-isomorphisms and between $\Omega$-spectra are level equivalences, Theorem 3.1.3 shows that the second and third statements of Theorem 3.1.2 really just say that $D$ and $D^2$ preserve and detect stable equivalences.
Theorem 3.1.2 shows that the classical stable homotopy groups of $DX$ are a stable equivalence invariant. In the next theorem we show that they are in fact the derived classical stable homotopy groups, i.e., they are isomorphic to $\pi_* LX$.

**Theorem 3.1.6.** Let $X$ be a symmetric spectrum.

1. There is a natural zig-zag of functors inducing $\pi_*$-isomorphisms between $LX$ and $DX$.
2. There are natural zig-zags of functors inducing level equivalences between $LX$, $D^2 X$, and $QDX$.

This theorem shows that the fibrant replacement functor is determined up to $\pi_*$-isomorphism by $D$ or up to level equivalence by $D^2$ or $QD$. The spectral sequence for calculating the classical stable homotopy groups of $DX$, Proposition 2.3.4, calculates the derived stable homotopy groups $\pi_* DX \cong \pi_* LX$.

**Corollary 3.1.7.** For $X$ any semistable spectrum, $X$ and $DX$ are $\pi_*$-isomorphic.

**Remark 3.1.8.** This corollary is a special case of [B, 1.6] where the convergence and connectivity conditions are replaced by the semistable condition. By Proposition 2.1.19 we recover a statement with convergence conditions but no connectivity conditions.

The proofs of Theorems 3.1.2 and 3.1.5 use the following properties of the functor $D$.

**Proposition 3.1.9.** Let $f: X \rightarrow Y$ be a map of symmetric spectra.

1. If $f$ is a stable equivalence then $Df$ is a $\pi_*$-isomorphism.
2. If $f$ is a $\pi_*$-isomorphism then $Df$ is a level equivalence.
3. For any semistable spectrum $X$, there is a natural zig-zag of functors inducing level equivalences between $LX$ and $DX$.

We assume Proposition 3.1.9 to prove Theorems 3.1.2, 3.1.5, and 3.1.6. The proof of Proposition 3.1.9 is technical, so it is delayed until the next subsection.

**Proof of Theorem 3.1.6.** By Proposition 3.1.9 (3) applied to $LX$ there is a zig-zag of level equivalences between $LLX$ and $DLX$. By Proposition 3.1.9 (1) since $X \rightarrow LX$ is a stable equivalence $DX \rightarrow DLX$ is a $\pi_*$-isomorphism. Hence, putting these equivalences together with the fact that $LLX$ is level equivalent to $LX$, we get a zig-zag of $\pi_*$-isomorphisms between $LX$ and $DX$.

Applying $D$ to the zig-zag of $\pi_*$-isomorphisms between $LX$ and $DX$ shows that $DLX$ and $D^2 X$ are level equivalent by Proposition 3.1.9 (2). Combining this with the zig-zag of level equivalences between $LX$ and $DLX$ produces the level equivalence of $LX$ and $D^2 X$. The equivalences for $QDX$ are similar.

**Proof of Theorem 3.1.5.** By Theorem 3.1.6 $DX$ is $\pi_*$-isomorphic to $LX$. $LX$ is an $\Omega$-spectrum, hence it is semistable. So by Proposition 2.1.21, $DX$ is semistable.

For $X$ semistable, Proposition 3.1.9 shows that $DX$ is level equivalent to $LX$, an $\Omega$-spectrum. Hence $L'DX$ is an $\Omega$-spectrum.

**Proof of Theorem 3.1.4.** Proposition 3.1.6 shows that (1) implies (2) and (2) implies (3). A map $f$ is a $\pi_*$-isomorphism if and only if $Qf$ is a level equivalence. Hence the second and fourth statements are also equivalent.
By Theorem 3.1.6 part 2, $LX$ and $D^2X$ are naturally level equivalent. Hence if $D^2X \to D^2Y$ is a level equivalence then so is $LX \to LY$. But this is equivalent to $X \to Y$ being a stable equivalence.

3.2. Proof of Proposition 3.1.9. As mentioned above the proof of Proposition 3.1.9 is more technical. In this subsection we first prove the second part of Proposition 3.1.9. Using this we prove the third part. Then, for the first part of Proposition 3.1.9 we state and prove several lemmas which together finish the proof. Throughout this section we use several of the properties of the homotopy colimit developed in section 2.3.

For the proof of the second part of Proposition 3.1.9 we use Proposition 2.3.10, due to Jeff Smith.

Proof of Proposition 3.1.9 Part 2. We apply Lemma 2.3.10 to each level of $D$. Consider the 0th level first. If $f$ is a $\pi_*$-isomorphism then hocolim$^T \Omega^n L' f_n$ is a weak equivalence, since $\pi_* X = \pi_* \text{hocolim}^T \Omega^n L' X_n$. Since taking homotopy orbits preserves weak equivalences this shows that the 0th level of $DX \to DY$ is a weak equivalence, i.e., hocolim$^T \Omega^n L' f_n$ is a weak equivalence.

The $k$th level of $DX$ is the 0th level of $D\Sigma^k X$. Since $\Sigma^k f$ is a $\pi_*$-isomorphism if $f$ is, this shows that each level is a weak equivalence.

For the third part of Proposition 3.1.9 we need the following functor. Recall that $(sh^n X)_k = X_{n+k}$.

Definition 3.2.1. Define $MX = \text{hocolim}^I \Omega^n L' sh_n X$.

Proof of Proposition 3.1.9 Part 3. First we develop the transformations which play a part in the zig-zag mentioned in the proposition. The inclusion of the object $0$ in $I$ induces a natural map $X \to MX$. There is also a natural transformation of functors $D \to M$. The structure maps on $X$ induce a natural map of symmetric spectra $F_0 X_n \to sh_n X$. Applying $\Omega^n L'$ this map induces a map of diagrams over $I$, and hence a natural map of homotopy colimits. So there is a natural zig-zag $X \to MX \leftarrow DX$. The zig-zag mentioned in the proposition is this zig-zag applied to $LX$ along with the natural map $DX \to DLX$.

For semistable $X$, the map $X \to LX$ is a $\pi_*$-isomorphism. So $DX \to DLX$ is a level equivalence by Proposition 3.1.9 part 2. So we only need to show that if $X$ is an $\Omega$-spectrum, then both of the maps $X \to MX \leftarrow DX$ are level equivalences.

First we show that the map $X \to MX$ is a level equivalence for any $\Omega$-spectrum $X$. By definition an $\Omega$-spectrum is a level fibrant spectrum such that $X \to \Omega sh_1 X$ is a level equivalence. Using this and the fact that both shift and $\Omega$ preserve level equivalences (on level fibrant spectra), one can show that each of the maps in the diagram over $I$ used to define $MX$ is a level equivalence. By Proposition 2.3.9 this implies that $X \to MX$ is a level equivalence.

To show that $DX \to MX$ is a level equivalence for any $\Omega$-spectrum $X$, we need to consider connective covers. Given a level fibrant spectrum $X$ define its $k$th connective cover, $C_k X$, as the homotopy fibre of the map from $X$ to its $k$th Postnikov stage $P_k X$. The $k$th Postnikov functor is the localization functor given by localizing with respect to the set of maps $\{ F_n \partial \Delta [m+n+k+2] \to F_n \Delta [m+n+k+2]; \ m, n \geq 0 \}$. At level $n$, this functor is weakly equivalent to the $(n+k)$th Postnikov functor on spaces which is given by localization with respect to the set of maps $\{ \partial \Delta [m+n+k+2] \to \Delta [m+n+k+2]; \ m \geq 0 \}$. See also [F2]. Then $(C_k T)_n$
is $n + k$ connected and $\pi_i(C_k T)_n \rightarrow \pi_i T_n$ is an isomorphism for $i > n + k$. Note that any level fibrant spectrum is level equivalent to the homotopy colimit over its connective covers. As $-k$ decreases, the homotopy type of each level of $C_{-k} X$ eventually becomes constant. So hocolim$_k C_{-k} X \rightarrow X$ is a level equivalence.

Because $\Omega^m L'$ and $F_0$ commute up to level equivalence with directed homotopy colimits and homotopy colimits commute, hocolim$_n DC_{-n} X$ is level equivalent to $DX$. The shift functor also commutes with homotopy colimits so hocolim$_n MC_{-n} X$ is level equivalent to $MX$. So we first show that for each $n$, $DC_n X$ and $MC_n X$ are level equivalent.

In the diagrams creating these homotopy colimits, consider level $l$ at the object $m \in I$. The map in question is $\Omega^m L'(S^l \land C_n X_m) \rightarrow \Omega^m L'C_n X_{m+l}$. In general the map $\Omega^m L'(S^l \land Y) \rightarrow \Omega^m L' Y$ is $2N - l - m + 1$ connected when $Y$ is $N$ connected. Hence for $Y = C_n X_{m+l}$ the map in question is $2n + m + l + 1$ connected.

Using Proposition 2.3.8 we see that this connectivity implies that $(DC_n X)_k \rightarrow (MC_n X)_k$ is a weak equivalence. Homotopy commutes with directed colimits, so taking colimits over $n$ on both sides we get a weak equivalence $DX_k \rightarrow MX_k$. So $DX \rightarrow MX$ is a level equivalence. This is what we needed to finish the third part of Proposition 3.1.9. Note also that since $MX$ is an $\Omega$-spectrum this shows that the level fibrant replacement of $DX$ is also an $\Omega$-spectrum.

The proof of Proposition 3.1.9 part 1 breaks up into several parts. For the case of stable trivial cofibrations we split the problem into showing that $D$ of any generating stable trivial cofibration is a $\pi_*$-isomorphism and that $D$ behaves well with respect to push outs, i.e., that the following two lemmas hold.

**Lemma 3.2.2.** Let $j : A \rightarrow B$ be a generating stable trivial cofibration. Then $D j : DA \rightarrow DB$ is a $\pi_*$-isomorphism.

**Lemma 3.2.3.** If

\[
\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow & & \downarrow \\
B & \longrightarrow & Y
\end{array}
\]

is a pushout square with $A \rightarrow B$ a cofibration, then

\[
\begin{array}{ccc}
DA & \longrightarrow & DX \\
\downarrow & & \downarrow \\
DB & \longrightarrow & DY
\end{array}
\]

is a homotopy pushout square. I.e., if $P$ is the homotopy colimit of $DB \leftarrow DA \rightarrow DX$, then $P \rightarrow DY$ is a stable equivalence. In fact, $P \rightarrow DY$ is a $\pi_*$-isomorphism.

Combining this lemma with the next shows that if $DA \rightarrow DB$ is a $\pi_*$-isomorphism then $DX \rightarrow DY$ is also a $\pi_*$-isomorphism.

**Lemma 3.2.4.** Let

\[
\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow & & \downarrow \\
B & \longrightarrow & Y
\end{array}
\]
be a square in $\mathcal{Sp}^\Sigma$ with $Y$ $\pi_*$-isomorphic to the homotopy pushout. Assume $A \to B$ is a $\pi_*$-isomorphism. Then $X \to Y$ is a $\pi_*$-isomorphism.

For a proper model category this is a standard fact, that the homotopy pushout of a weak equivalence is a weak equivalence. But no model category on symmetric spectra has been written down with weak equivalences the $\pi_*$-isomorphisms.

Proof of Proposition 3.1.9 Part 1. Assuming Lemmas 3.2.2, 3.2.3, and 3.2.4, we can finish this proof. First note that since any stable equivalence can be factored as a stable trivial cofibration followed by a level trivial fibration, we only need to show that $D$ takes both stable trivial cofibrations and level equivalences to $\pi_*$-isomorphisms.

A level equivalence induces a level equivalence at each object in the diagram for defining $D$ and homotopy colimits preserve level equivalences, by Proposition 2.3.2. Hence, $D$ of a level equivalence is a level equivalence, and thus a $\pi_*$-isomorphism.

Any stable trivial cofibration is a retract of a directed colimit of pushouts of maps in $J$. Since retracts and directed colimits preserve $\pi_*$-isomorphisms we only need to consider pushouts of generating stable trivial cofibrations. By Lemma 3.2.3, $D$ of a generating trivial cofibration is a $\pi_*$-isomorphism. Hence, by Lemmas 3.2.3 and 3.2.4, $D$ of any map formed by a pushout of a generating stable trivial cofibration is a $\pi_*$-isomorphism.

Proof of Lemma 3.2.4. Factor the map $A \to X$ as a stable cofibration followed by a level trivial fibration $A \to Z \to X$. Then form the pushout square as follows,

\[
\begin{array}{ccc}
A & \longrightarrow & Z \\
\downarrow & & \downarrow \\
B & \longrightarrow & P.
\end{array}
\]

Since the top map is a level cofibration, $P$ is the homotopy pushout of this square. Since $A \to B$ is a $\pi_*$-isomorphism, $Z \to P$ is a $\pi_*$-isomorphism because $\pi_*$ is a homology theory. Since $Z \to X$ is a level equivalence, to see that $X \to Y$ is a $\pi_*$-isomorphism it is enough to know that $P \to Y$ is a $\pi_*$-isomorphism. But this is assumed as part of the hypotheses.

Now we proceed with the proof of Lemma 3.2.3.

Proof of Lemma 3.2.3. To see that $P \to DY$ is a $\pi_*$-isomorphism, we use the fact that homotopy colimits commute. $P$ is the homotopy colimit of $DB \leftarrow DA \to DX$, so $P$ is level equivalent to the homotopy colimit over $I$ of the homotopy pushout of the squares at each object in $I$. In other words, let $P^n$ be the homotopy pushout at the object $n \in I$. Then $P$ is level equivalent to $\text{hocolim}^I P^n$.

Proposition 2.3.2 shows that a map of diagrams which is a $\pi_*$-isomorphism at each object induces a $\pi_*$-isomorphism on the homotopy colimits. Hence, it is enough to show that $P^n \to \Omega^n L'F_0(Y_n)$ is a $\pi_*$-isomorphism for each $n$.

Since cofibrations induce level cofibrations and $F_0$ preserves cofibrations and pushouts, $F_0$ applied to each level of the pushout square in the lemma is a homotopy pushout square. Since $X \to L' X$ is a level equivalence it preserves homotopy pushout squares up to level equivalence. Since $\Omega^n$ only shifts $\pi_*$ by $n$, it preserves homotopy pushouts up to $\pi_*$-isomorphism. Hence $P^n \to \Omega^n L'F_0(Y_n)$ is a $\pi_*$-isomorphism.
We are left with proving Lemma 3.2.4. First we prove the following lemma which identifies the stable homotopy type of \(DF_m(K)\).

**Lemma 3.2.5.** There is a \(2l-m-1\) connected map \(\psi_l \colon \Omega^mL'(S^l \wedge K) \to (DF_mK)_l\). These maps fit together to give a map of symmetric spectra \(\psi : \Omega^mL'F_0K \to DF_mK\) which is a \(\pi_*\)-isomorphism.

To prove this lemma we define another functor on the category \(I\).

**Definition 3.2.6.** Define \(\mathcal{F}_mK : I \to S_*\) by \((\mathcal{F}_mK)(n) = \text{hom}_I(m, n)_+ \wedge K\).

\(\mathcal{F}_m(-)\) is left adjoint to the functor from \(I\)-diagrams over \(S_*\) to \(S_*\) which evaluates the diagram at \(m \in I\). Hence a natural transformation from \(\mathcal{F}_mK\) into any diagram over \(I\) is determined by a map from \(K\) to the diagram evaluated at \(m\).

**Proof of Lemma 3.2.4.** Let \(\mathcal{D}_{F_mK}^l : I \to S_*\) be the functor given by the \(l\)th level of the functor \(\mathcal{D}_{F_mK}\). Then there is a map \(\phi_l : \mathcal{F}_m\Omega^mL'(S^l \wedge K) \to \mathcal{D}_{F_mK}^l\) determined by the inclusion of the wedge summand corresponding to the identity map, \(\Omega^mL'(S^l \wedge K) \to \Omega^mL'(S^l \wedge \text{hom}_I(m, m)_+ \wedge K)\). Because the homotopy colimit of a free diagram is weakly equivalent to the colimit, see [F1], the homotopy colimit of this map is the map \(\psi_l : \Omega^mL'(S^l \wedge K) \to (DF_mK)_l\) mentioned in the lemma.

We show that the map of diagrams is \(2l-m-1\) connected at each spot. At each \(n \in I\), \(\phi_l(n)\), factors into two maps as follows,

\[
\text{hom}_I(m, n)_+ \wedge \Omega^mL'(S^l \wedge K) \to \Omega^mL'(\text{hom}_I(m, n)_+ \wedge S^l \wedge K)
\]

\[
\to \Omega^m\Omega^{n-m}L'\Sigma^{n-m}(\text{hom}_I(m, n)_+ \wedge S^l \wedge K).
\]

The first map is \(2l-m-1\) connected by the application of the Blakers-Massey theorem which shows that a wedge of loop spaces, \(\Omega X \vee \Omega Y\), is equivalent in the stable range to the loop of the wedge, \(\Omega(X \vee Y)\). The second map is \(2l-m-1\) connected by the Freudenthal suspension theorem, which for simplicial sets concerns the map \(X \to \Omega L' \Sigma X\). Hence the map at each spot in the diagram, \(\phi_l(n)\) and thus the map of homotopy colimits, \(\psi_l\), is \(2l-m-1\) connected.

To see that these levels fit together, note that we can prolong \(\mathcal{F}_m\) to a functor from symmetric spectra to \(I\)-diagrams of symmetric spectra, then there is a map \(\phi : \mathcal{F}_m(\Omega^mL'F_0K) \to \mathcal{D}_{F_mK}\) which on level \(I\) is given by the map, \(\phi_l\), above. Hence, taking homotopy colimits, this induces a map \(\psi : \Omega^mL'F_0K \to DF_mK\) which is a \(\pi_*\)-isomorphism.

**Proof of Lemma 3.2.5.** We must show that \(D\) of a generating trivial cofibration is a \(\pi_*\)-isomorphism. First recall that the generating trivial cofibrations are the maps \(P(c, F_mg_r) : P_{m,r} \to C \wedge S F_m(\Delta[r]_+)\) where \(P_{m,r}\) is the pushout below.

\[
\begin{array}{ccc}
F_1S^1 \wedge S F_m(\Delta[r]_+) & \longrightarrow & F_1S^1 \wedge S F_m(\Delta[r]_+) \\
\downarrow & & \downarrow \\
C \wedge S F_m(\Delta[r]_+) & \longrightarrow & P_{m,r}
\end{array}
\]

To show that \(D\) of \(P(c, F_mg_r)\) is a \(\pi_*\)-isomorphism it is only necessary to show that \(D\) of \(cK : F_1S^1 \wedge S F_mK \to C \wedge S F_mK\) is a \(\pi_*\)-isomorphism for \(K = \Delta[r]_+\) or \(\Delta[r]_+\). This is enough because Lemma 3.2.4 shows that if \(D(F_1S^1 \wedge S F_m(\Delta[r]_+)) \to D(C \wedge S F_m(\Delta[r]_+))\) is a \(\pi_*\)-isomorphism then the pushout \(D(F_1S^1 \wedge S F_m(\Delta[r]_+)) \to D(P_{m,r})\) is also a \(\pi_*\)-isomorphism. If \(D(F_1S^1 \wedge S F_m(\Delta[r]_+)) \to D(C \wedge S F_m(\Delta[r]_+))\)
is also a $\pi_*$-isomorphism, this implies that $D(P_{m,r}) \to D(C \wedge_S F_m(\Delta[v]_+))$ is a $\pi_*$-isomorphism.

Since $F_mK$ is cofibrant and $C \to F_0S^0$ is a level equivalence, the map $C \wedge F_mK \to F_0S^0 \wedge F_mK$ is a level equivalence, by Lemma 2.2.12. As already noticed, $D$ takes level equivalences to $\pi_*$-isomorphisms so we can assume that $C$ is replaced by $F_0S^0$ in $c_K$ for both values of $K$.

Thus we only need to show that $Dc_K : DF_{m+1}(S^1 \wedge K) \to DF_mK$ is a $\pi_*$-isomorphism. To see that this map induces an isomorphism on the stable homotopy groups, note that $F_{m+1}(S^1 \wedge K) \to F_mK$ is induced by $\text{hom}_I(m+1,n) \to \text{hom}_I(m,n)$ which in turn is induced by the inclusion of $m$ in $m+1$. Now consider homotopy applied to the map of diagrams, $Dc_K$. Using the $\pi_*$-isomorphisms from Lemma 3.2.3 above, this map is a map of free diagrams, $\text{hom}_I(m+1,-) \otimes \pi_*^{m+1}S^1 \wedge K \to \text{hom}_I(m,-) \otimes \pi_*^mK$. This map induces an isomorphism on the colimits and all of the higher colim vanish. Hence, using the spectral sequence for calculating the homotopy of homotopy colimits, see section 2.3, $Dc_K$ is a $\pi_*$-isomorphism. One can also see this by considering the associated map of free diagrams directly.

### 4. Topological Hochschild Homology

Let $k$ be a commutative symmetric ring spectrum. Let $R$ be a $k$-algebra. Define $R^e = R \wedge_k R^{op}$. Let $M$ be a $k$-symmetric $R$-bimodule, i.e., an $R^e$-module. With this set up we have two different definitions of topological Hochschild homology, one using a derived tensor product definition, the other mimicking the usual Hochschild complex. In Theorem 4.1.10 we see that these definitions construct stably equivalent $k$-modules. Of course, since the smash product is only stably invariant for cofibrant spectra, the case where $R$ is a cofibrant $k$-module is the only one of interest.

The idea to define topological Hochschild homology by mimicking algebra in this way is due to Goodwillie. But because a symmetric monoidal category of spectra was not available until recently, one could not simply implement this idea. Bökstedt was the first one to define topological Hochschild homology by modifying this idea to work with certain rings up to homotopy. This original definition of topological Hochschild homology concerns the case when $k = S$. We restate the definition of the simplicial spectrum $\text{THH}(R)$ and its realization, $\text{THH}(R)$, from [B] for a symmetric ring spectrum. See Definition 4.2.4. In Theorem 4.2.8 we show that for $k = S$ our new definitions are stably equivalent to the original definition when $R$ is a cofibrant symmetric ring spectrum. As a corollary to this comparison theorem we see that Bökstedt’s definition of $\text{THH}$ takes stable equivalences of $S$-algebras to $\pi_*$-isomorphisms. Hence it always determines the right homotopy type, even on non-connective and non-convergent ring spectra, whereas the other two definitions give the right homotopy type only on cofibrant symmetric ring spectra.

#### 4.1. Two definitions of relative topological Hochschild homology.

The first definition corresponds to the derived tensor product notion of algebraic Hochschild homology. The second definition mimics the Hochschild complex from algebraic Hochschild homology. As we see in Theorem 4.1.10 these notions are stably equivalent when $M$ is a cofibrant $R^e$-module.

**Definition 4.1.1.** Define $\text{thh}^k(R;M)$ by $M \wedge_{R^e} R$. 

Lemma 4.1.5. is a stable equivalence at each simplicial level is a stable equivalence.

Proposition 4.1.4. [BK, XII, 4.3], proves the following proposition. A bisimplicial set is weakly equivalent to the diagonal simplicial set, see [BK, XII, 4.3] by the homotopy colimit of each level, the fact that the homotopy colimit of a structure maps of $R$ as in [CE].

Definition 4.1.3. Define the diagonal to define the realization of this simplicial symmetric spectrum.

Definition 4.1.2. $\text{tHH}^k(R; M)$ is the simplicial $k$-module with $s$-simplices $M \wedge_k R^s$. The simplicial face and degeneracy maps are given by

$$d_i = \begin{cases} 
\phi_r \wedge (id_R)^{s-1} & \text{if } i = 0 \\
(id_M) \wedge (id_R)^{i-1} \wedge \mu \wedge (id_R)^{s-i-1} & \text{if } 1 \leq i < s \\
(\phi_l \wedge (id_R)^{s-1}) \circ \tau & \text{if } i = s 
\end{cases}$$

and $s_i = id_M \wedge (id_R)^i \wedge \eta \wedge (id_R)^{s-1}$.

Each level of this simplicial symmetric spectrum is a bisimplicial set. Since the realization of bisimplicial sets is equivalent to taking the diagonal, we use the diagonal to define the realization of this simplicial symmetric spectrum.

Definition 4.1.1. Define the $k$-module $\text{tHH}^k(R; M)$ as the diagonal of the bisimplicial set at each level of this simplicial $k$-module. For the special cases of $k = S$ or $M = R$ we delete them from the notation.

Since the homotopy colimit of a diagram of symmetric spectra is determined by the homotopy colimit of each level, the fact that the homotopy colimit of a bisimplicial set is weakly equivalent to the diagonal simplicial set, see [BK, XII 4.3], proves the following proposition.

Proposition 4.1.4. The map $\text{hocolim}^\Delta_{Sp^E} \text{tHH}^k(R; M) \to \text{tHH}^k(R; M)$ is a level equivalence.

Using $D$ and this Proposition we can show that the realization of a map which is a stable equivalence at each simplicial level is a stable equivalence.

Lemma 4.1.5. Let $F, G : B \to Sp^E$ be two diagrams of symmetric spectra with a natural transformation $\eta : F \to G$ between them. If $\eta(b) : F(b) \to G(b)$ is a stable equivalence for each object $b$ in $B$ then $\text{hocolim}^B F \to \text{hocolim}^B G$ is a stable equivalence.

Proof. Consider $D\eta : DF \to DG$. By Theorem 3.1.2 this is a $\pi_\ast$-isomorphism at each object, so by Proposition 2.3.5 the homotopy colimits are $\pi_\ast$-isomorphic. Since $L', F_0$ and homotopy colimits commute with homotopy colimits and $\Omega^n$ commutes with homotopy colimits up to $\pi_\ast$-isomorphism, $\text{hocolim}^B DF$ is $\pi_\ast$-isomorphic to $D \text{hocolim}^B F$. Hence, $D \text{hocolim}^B F \to D \text{hocolim}^B G$ is a $\pi_\ast$-isomorphism. Thus, by Theorem 5.1.2, $\text{hocolim}^B F \to \text{hocolim}^B G$ is a stable equivalence.

Corollary 4.1.6. A map between simplicial symmetric spectra which is a stable equivalence on each level induces a stable equivalence on the realizations.

Proof. This just combines Lemma 4.1.5 and [BK, XII, 4.3].

Proposition 4.1.7. Let $R \to R'$ be a stable equivalence between $k$-algebras which are cofibrant as $k$-modules, $M$ a $R'$-module, $N$ a $(R')^e$-module, and $M \to N$ a stable equivalence of $R'$-modules. Then $\text{tHH}^k(R; M) \to \text{tHH}^k(R'; N)$ is a stable
equivalence. In particular, $\text{tHH}^k(R) \to \text{tHH}^k(R')$, $\text{tHH}^k(R; M) \to \text{tHH}^k(R; N)$, and $\text{tHH}^k(R; N) \to \text{tHH}^k(R'; N)$ are stable equivalences.

First note that a cofibrant $k$-algebra is also cofibrant as a $k$-module by Theorem 2.2.7, so there are many examples of $k$-algebras which are cofibrant as $k$-modules.

**Proof.** Lemma 2.2.9 applied to $k$ shows that $P \wedge_k -$ preserves stable equivalences of $k$-modules if $P$ is a cofibrant $k$-module. Hence, $R^s \to R'^s$ is a stable equivalence between cofibrant $k$-modules. So both $M \wedge_k R^s \to N \wedge_k R^s$ and $N \wedge_k R^s \to N \wedge_k R'^s$ are also stable equivalences. Thus each simplicial level is a stable equivalence. Then Corollary 4.1.4 shows that this map induces a stable equivalence on $\text{tHH}^k$.

To compare these two definitions of topological Hochschild homology we first define certain bar constructions. Let $M$ be a right $R$-module, with $\phi_M : M \wedge_k R \to M$. We define the topological bar construction $B^k(M, R, N)$ by mimicking algebra.

**Definition 4.1.8.** The bar construction $B^k(M, R, N)$ is the simplicial $k$-module with $s$-simplices $M \wedge_k R^s \wedge_k N$. The face and degeneracy maps are given by

$$d_i = \begin{cases} 
\phi_M \wedge (id_R)^{s-1} \wedge id_N & \text{if } i = 0 \\
id_M \wedge (id_R)^{1-1} \wedge \mu \wedge (id_R)^{s-i-1} \wedge id_N & \text{if } 1 \leq i < s \\
id_M \wedge (id_R)^{s-1} \wedge \phi_N & \text{if } i = s
\end{cases}$$

Let $B^k(M, R, N)$ be the realization of this simplicial $k$-module.

Let $c.(X)$ be the constant simplicial object with $X$ in each simplicial degree. Using the identification $M \wedge_R R \cong M$, the map $\eta : k \to R$ induces a simplicial $k$-module map $B^k(M, R, N) \to c.(M \wedge_R N)$.

**Lemma 4.1.9.** For $M$ a cofibrant $R$-module, the simplicial map of $k$-modules, $B^k(M, R, N) \to c.(M \wedge_R N)$, induces a stable equivalence of $B^k(M, R, N) \to M \wedge_R N$.

**Proof.** Note that $B^k(M, R, N) \cong c.M \wedge_R B^k(R, R, N)$. Since realization commutes with smash products, $B^k(M, R, N) \cong M \wedge_R B^k(R, R, N)$. So using Lemma 2.2.9 it is enough to show that $B^k(R, R, N) \to N$ is a stable equivalence. The map $N \cong k \wedge_k N \to R \wedge_k N$ provides a simplicial retraction for $B^k(R, R, N)$. Hence the spectral sequence for computing the classical stable homotopy groups of the homotopy colimit of this simplicial $k$-module collapses. So the map $B^k(R, R, N) \to c.N$ induces a $\pi_*$-isomorphism on the realizations.

Using the bar construction we now show that the two definitions of topological Hochschild homology are stably equivalent when $M$ is a cofibrant $R^c$-module.

**Theorem 4.1.10.** There is a natural map of $k$-modules $\text{tHH}^k(R; M) \to \text{thh}^k(R; M)$ which is a stable equivalence for $M$ a cofibrant $R^c$-module.

**Proof.** We show that $\text{tHH}^k(R; M)$ is naturally isomorphic to $M \wedge_{R^c} B^k(R, R, R)$ below. Then the map $\text{tHH}^k(R; M) \to \text{thh}^k(R; M)$ is given by $M \wedge_{R^c} \phi$ for $\phi : B^k(R, R, R) \to R$. $R$ is always a cofibrant $R$-module, hence $\phi$ is a stable equivalence by Lemma 4.1.1. Then Proposition 2.2.9 shows that $M \wedge_{R^c} \phi$ is a stable equivalence since $M$ is a cofibrant $R^c$-module.
To see that $\text{tHH}^k(R; M)$ is naturally isomorphic to $M \wedge_{R^e} B_k^k(R, R, R)$ we show that $\text{tHH}^k(R; M)$ is naturally isomorphic to $c_\ast(M) \wedge_{R^e} B_k^k(R, R, R)$. On each simplicial level there are natural isomorphisms

$$M \wedge_k R^e \cong M \wedge_{R^e} (R^e \wedge_k R^e) \cong M \wedge_{R^e} (R \wedge_k R^e \wedge_k R) = M \wedge_{R^e} B_k^k(R, R, R).$$

These isomorphisms commute with the simplicial structure. Hence the simplicial $k$-modules are naturally isomorphic, so their realizations are also naturally isomorphic.

4.2. Bökstedt’s definition of topological Hochschild homology. We now define the simplicial spectrum $\text{THH}(R; M)$ and its realization $\text{THH}(R; M)$ following Bökstedt’s original definitions. Each of the levels of the simplicial spectrum $\text{THH}$ can be defined for a general symmetric spectrum $X$. A ring structure is only necessary for defining the simplicial structure. In fact, each of the levels of $\text{THH}$ can be thought of as a functor which gives the correct $\pi$-isomorphism type for the smash product of symmetric spectra. We start by considering each of these levels as a functor of several variables.

Let $X$ denote a sequence of $j + 1$ spectra, $X^0, \ldots, X^j$. Define a functor $D^j_X$ from $I^{j+1}$ to $Sp$ which at $n = (n_0, \ldots, n_j)$ takes the value,

$$D^j_X(n) = \Omega^n L^i F_0(X^0_{n_0} \wedge \ldots \wedge X^j_{n_j})$$

where $L^i$ is a level fibrant replacement functor and $n = \sum n_i$, the sum of the $n_i$. Note that $D^0(X)$ is $D_X$, the functor defined at the beginning of section 3. To see that $D^j_X$ is defined over $I^{j+1}$ one uses maps similar to those described for $D_X$.

**Definition 4.2.1.** Let $X^0, \ldots, X^j$ be symmetric spectra. Define

$$T_j X = \text{hocolim}_{I^{j+1}} D^j_X.$$

We now define a natural transformation $\phi_j X: T_j X \to D(X^0 \wedge_S \ldots \wedge_S X^j)$. Let $\mu : I^{j+1} \to I$ be the functor induced by concatenation of all of the factors in $I^{j+1}$. Then there is a natural transformation from $D^j_X$ to $\mu^* D^0(X^0 \wedge_S \ldots \wedge_S X^j)$. This natural transformation is induced by the map from $X^0_{n_0} \wedge \ldots \wedge X^j_{n_j}$ to the $n$th level of $X^0 \wedge_S \ldots \wedge_S X^j$. This map is $\Sigma n_0 \times \ldots \times \Sigma n_j$ equivariant, which is exactly what is necessary over $I^{j+1}$. Hence, on homotopy colimits there is a natural map $\text{hocolim}_{I^{j+1}} D^j_X \to \text{hocolim}_{I^{j+1}} \mu^* D^0(X^0 \wedge_S \ldots \wedge_S X^j)$.

**Definition 4.2.2.** There is a natural transformation $\phi_j X: T_j X \to D(X^0 \wedge_S \ldots \wedge_S X^j)$. It is given by the composition

$$\text{hocolim}_{I^{j+1}} D^j_X \to \text{hocolim}_{I^{j+1}} \mu^* D^0(X^0 \wedge_S \ldots \wedge_S X^j) \to \text{hocolim}_{I^j} D^0(X^0 \wedge_S \ldots \wedge_S X^j).$$

**Proposition 4.2.3.** For any cofibrant symmetric spectra, $X^0, \ldots, X^j$, the map $\phi_j X$ is a $\pi_\ast$-isomorphism.

This proposition is proved in subsection 4.3. It is used in proving the comparison theorem between Bökstedt’s definition of $\text{THH}$ and our previous definition of $\text{tHH}$. As a corollary of this proposition, $T_j$ gives the correct $\pi_\ast$-isomorphism type for the derived smash product of $j + 1$ symmetric spectra. Recall that the smash product is only homotopy invariant on cofibrant spectra, so the derived smash product is the smash product of the cofibrant replacements. In the stable model category of
Corollary 4.2.4. \( \pi_\ast T_j X \) is isomorphic to \( \pi_\ast L(CX^0 \wedge S \ldots \wedge S CX^j) \), the derived homotopy of the derived smash product of \( X^0, \ldots, X^j \).

Proof. Since \( C \) is a cofibrant replacement functor, \( CX \to X \) is a level equivalence. Hence \( T_j(CX^0, \ldots, CX^j) \to T_j(X^0, \ldots, X^j) \) is a level equivalence by Proposition 2.3.2 because the map is a level equivalence at each object in the diagram defining \( T_j \). So this corollary follows from Proposition 4.2.3 since \( \pi_\ast D(CX^0 \wedge S \ldots \wedge S CX^j) \) is isomorphic to \( \pi_\ast L(CX^0 \wedge S \ldots \wedge S CX^j) \) by Theorem 3.1.6.

We now use this functor \( T_j \) to define \( \text{THH} \) following Bökstedt’s definition in [B].

Definition 4.2.5. Let \( R \) be a symmetric ring spectrum with \( M \) an \( R^e \)-module. Define \( \text{THH}_j(R; M) = T_j(M, R, \ldots, R) \).

The functors \( \text{THH}_j(R; M) \) fit together to form a simplicial symmetric spectrum \( \text{THH}(R; M) \). Although the definition of \( \text{THH}_j(R; M) \) does not use the ring structure of \( R \) or the module structure of \( M \), the simplicial structure of \( \text{THH}(R; M) \) does use both the multiplication and unit maps. The \( i \)th face map uses the functor \( \delta_i : P^{i+1} \to P^i \) defined by concatenation of the sets in factors \( i \) and \( i + 1 \). The last face map uses the cyclic permutation of \( P^{i+1} \) followed by concatenation of the first two factors. For ease of notation let \( \mathcal{D}^j(R; M) = \mathcal{D}^j(M, R, \ldots, R) \). The multiplication of \( R \) and \( M \) defines a natural transformation of functors from \( \mathcal{D}^j(R; M) \) to \( \delta^*_j \mathcal{D}^{j-1}(R; M) \). So \( d_i \) is the composition

\[
d_i : \text{hocolim}^{j+1} \mathcal{D}^j(R; M) \to \text{hocolim}^{j+1} \delta^*_j \mathcal{D}^{j-1}(R; M) \to \text{hocolim}^j \mathcal{D}^{j-1}(R; M).
\]

The degeneracy maps are similar.

Definition 4.2.6. Define \( \text{THH}(R; M) \) as the diagonal of the bisimplicial set at each level of the simplicial symmetric spectrum \( \text{THH}(R; M) \).

One can check that each level in this spectrum agrees with the definition in [B] when \( M = R \).

As in Proposition 4.1.4, we have the following equivalence.

Proposition 4.2.7. The map \( \text{hocolim}_{\triangle}^{\mathcal{D}^j} \text{THH}(R; M) \to \text{THH}(R; M) \) is a level equivalence.

The next theorem shows that the definition of topological Hochschild homology which mimics the Hochschild complex is stably equivalent to the original definition of topological Hochschild homology.

Theorem 4.2.8. Let \( R \) be a cofibrant ring spectrum. Then there is a natural zig-zag of stable equivalences between \( \text{tHH}(R; M) \) and \( \text{THH}(R; M) \).

Proof. The zig-zag of functors between \( \text{tHH} \) and \( \text{THH} \) is induced by a zig-zag of maps between the simplicial complexes defining \( \text{tHH} \) and \( \text{THH} \). First one applies the zig-zag of functors \( 1 \xrightarrow{\psi} L \to ML \leftarrow DL \xleftarrow{D\psi} D \) to each simplicial level of the Hochschild complex defining \( \text{tHH} \). Here, \( L \) is the fibrant replacement functor, \( M \), \( D \), and the natural transformations are defined in section 3.3, see [2.1.3, 2.2.3], and the proof of [3.1.9] part 3. Then there is a natural map \( \phi_j : \text{THH}_j(R; M) \to \text{D}(M \wedge S R^j) \).

To see that the \( \phi_j \) maps commute with the simplicial maps, one needs to note that
the multiplication maps commute with the first map in the composite defining \( \phi_j \).
This follows since the map \( R_n \wedge R_m \to R_{n+m} \) is the map on the appropriate wedge summand of the map \( R \otimes R \to R \) which induces the map \( R \wedge_S R \to R \). The maps involving \( M \) are similar. Putting these simplicial levels together one gets a zig-zag of natural transformations from \( \text{tHH}(\cdot, (-)) \) to \( \text{THH}(\cdot, (-)). \) Note that by composing maps this zig-zag is only of length 2.

The zig-zag of functors between \( I \) and \( D \) was investigated in section 3. \( \psi \) induces a stable equivalence on any object by definition of the fibrant replacement functor \( L \). So Corollary 4.1.6 applies to show that \( \psi \) induces a stable equivalence on the realization of these simplicial \( S \)-modules.

Since \( \psi \) is always a stable equivalence, \( D\psi \) is a \( \pi_* \)-isomorphism on each simplicial level by Theorem 3.1.2. The two natural transformations of middle functors \( L \to ML \leftarrow DL \) induce level equivalences by Proposition 3.1.9. Propositions 2.3.2 and 2.3.3 apply to these natural transformations to show that they also induce level equivalences and \( \pi_* \)-isomorphisms on the realizations.

So the only part of the zig-zag between \( \text{tHH}(R; M) \) and \( \text{THH}(R; M) \) that is left is \( \text{THH}_j(R; M) \to D(\text{tHH}_j(R; M)) \). Let \( CM \to M \) be a cofibrant replacement of \( M \) as an \( R^c \)-module. Then by Proposition 4.1.7, \( \text{tHH}_j(R; CM) \to \text{tHH}_j(R; M) \) is a stable equivalence. Similarly, \( \text{tHH}_j(R; CM) \to \text{THH}_j(R; M) \) is a stable equivalence since \( CM \to M \) is a level equivalence and hence induces a level equivalence on the homotopy colimits used to define \( \text{THH}_j \). So it is enough to prove that \( \text{THH}_j(R; M) \to D(\text{tHH}_j(R; M)) \) is a stable equivalence in the case when \( M \) is cofibrant.

Since \( R \) is cofibrant as an \( S \)-algebra, it is also cofibrant as an \( S \)-module. Since \( M \) is cofibrant as an \( R^c \)-module and \( R^c \) is cofibrant, \( M \) is also cofibrant as an \( S \)-module. Proposition 4.2.3 shows that if \( R \) and \( M \) are any cofibrant \( S \)-modules then \( \text{THH}_j(R; M) \to D(M \wedge_S R^c) \) is a \( \pi_* \)-isomorphism. Then Proposition 2.3.3 shows that this is enough to ensure that the map on the realizations is a \( \pi_* \)-isomorphism. Hence, assuming Proposition 4.2.3 this finishes the proof of Theorem 4.2.8.

**Corollary 4.2.9.** The derived stable homotopy groups of \( \text{tHH}(R; M) \) are isomorphic to \( \pi_* \text{THH}(R; M) \).

Since \( \text{tHH}(R; M) \) and \( \text{THH}(R; M) \) are stably homotopic their derived stable homotopy groups must be isomorphic. So this corollary says that the derived stable homotopy groups of \( \text{tHH}(R; M) \) are isomorphic to the classical stable homotopy groups of \( \text{THH}(R; M) \).

**Proof.** The proof of Theorem 4.2.8 shows that the map from \( \text{THH}(R; M) \) to the realization of \( D \text{tHH}((R; M) \) is a \( \pi_* \)-isomorphism. But this realization is \( \pi_* \)-isomorphic to \( D \text{tHH}(R; M) \) as shown in the proof of Proposition 4.1.3. So the derived stable homotopy groups of \( \text{tHH}(R; M) \), i.e., \( \pi_* \text{tHH}(R; M) \), are isomorphic to the classical stable homotopy groups of \( \text{THH}(R; M) \).

Using this comparison we can show that Bökstedt’s original definition of \( \text{THH} \) takes stable equivalences of ring spectra to \( \pi_* \)-isomorphisms. This is a stronger result than for \( \text{tHH} \) because no cofibrancy condition is needed here and the map is a \( \pi_* \)-isomorphism, not just a stable equivalence.
Corollary 4.2.10. Let \( R \rightarrow R' \) be a stable equivalence of ring spectra, \( M \) a \( R' \)-module, \( N \) a \( (R')^e \)-module, and \( M \rightarrow N \) a stable equivalence of \( R' \)-modules. Then \( \text{THH}(R; M) \rightarrow \text{THH}(R'; N) \) is a \( \pi_* \)-isomorphism.

Remark 4.2.11. This corollary could also be proved without using these comparison results. One can show that each \( \text{THH}_j \) takes stable equivalences to \( \pi_* \)-isomorphisms following arguments similar to those for \( \text{THH}_0 = D \) in section 3. Then Proposition 2.3.3 shows that the realization, \( \text{THH} \), also takes stable equivalences to \( \pi_* \)-isomorphisms.

Proof. In the category of symmetric ring spectra, define a functorial cofibrant replacement functor, \( C \). Applying this functor we have the following square.

\[
\begin{array}{ccc}
CR & \longrightarrow & CR' \\
\downarrow & & \downarrow \\
R & \longrightarrow & R'
\end{array}
\]

Each of the vertical maps is a level trivial fibration and hence a level equivalence. The bottom map is a stable equivalence by assumption. Hence the top map is also a stable equivalence. To show that \( \text{THH} \) applied to the bottom map is a \( \pi_* \)-isomorphism we show that \( \text{THH} \) applied to the other three maps in this square are \( \pi_* \)-isomorphisms.

We also need to consider cofibrant replacements of the modules in question. \( M \) is a \( (CR)^e \)-module and \( N \) is a \( (CR')^e \)-module. We replace them by modules which are cofibrant as underlying \( S \)-modules. Since \( CR \) is a cofibrant \( S \)-algebra it is a cofibrant \( S \)-module. Thus \( (CR)^e \) is also cofibrant as an \( S \)-module by Proposition 2.1.13. Hence the cofibrations in the category of \( (CR)^e \)-modules are also underlying cofibrations. So let \( CM \rightarrow M \) be the cofibrant replacement of \( M \) in the category of \( (CR)^e \)-modules. Similarly, \( R' \) let \( CN \rightarrow N \) be the cofibrant replacement of \( N \) as a \( (CR')^e \)-module. Then both \( CM \) and \( CN \) are cofibrant as \( S \)-modules. Also, in the category of \( (CR)^e \)-modules by the lifting property in the model category of \( (CR)^e \)-modules we have a map \( CM \rightarrow CN \) because \( CN \rightarrow N \) is a level trivial fibration. This map \( CM \rightarrow CN \) is a stable equivalence by the two out of three property.

The level equivalences \( CR \rightarrow R \) and \( CM \rightarrow M \) induce a level equivalence on each object of the diagram defining \( \text{THH}_j \). So by applying Proposition 2.3.2 and Lemma 4.2.7 this shows that \( \text{THH}(CR; CM) \rightarrow \text{THH}(R; M) \) is a level equivalence. Similarly \( \text{THH}(CR'; CN) \rightarrow \text{THH}(R'; N) \) is a level equivalence.

For the top map, first consider applying \( t\text{HH} \). Proposition 4.1.7 implies that \( t\text{HH}(CR; CM) \rightarrow t\text{HH}(CR'; CN) \) is a stable equivalence. Hence by Theorem 3.1.2, \( D t\text{HH}(CR; CM) \rightarrow D t\text{HH}(CR'; CN) \) is a \( \pi_* \)-isomorphism. But in the proof of Theorem 4.2.8 we showed that \( \text{THH} \rightarrow D t\text{HH} \) induces a \( \pi_* \)-isomorphism if the ring and module are cofibrant as \( S \)-modules. So \( \text{THH}(CR; CM) \rightarrow \text{THH}(CR'; CN) \) is a \( \pi_* \)-isomorphism. Stringing these equivalences together finishes the proof of this corollary. Because Proposition 4.2.3 applies to each level, we have actually shown that each \( \text{THH}_j(R; M) \rightarrow \text{THH}_j(R'; N) \) is also a \( \pi_* \)-isomorphism.

4.3. Proof of Proposition 4.2.3. To prove Proposition 4.2.3 we follow an outline similar to the proof that \( D \) takes stable trivial cofibrations to \( \pi_* \)-isomorphisms. We show that \( \phi_j \) is a \( \pi_* \)-isomorphism when it is evaluated only on free symmetric
spectra, \textit{i.e.}, some $F_n K$. Then we prove an induction step lemma which deals with pushouts over generating stable trivial cofibrations. Using these lemmas we show that $\phi_j$ is a $\pi_\ast$-isomorphism on any collection of cofibrant spectra.

**Lemma 4.3.1.** $\phi_j(F_{n_0} K_0, \ldots, F_{n_j} K_j)$ is a $\pi_\ast$-isomorphism.

**Lemma 4.3.2.** Let $A \to B$ be a stable cofibration and $X^0, \ldots, X^j$ be cofibrant $S$-modules. Consider the following pushout square.

\[
\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow & & \downarrow \\
B & \longrightarrow & Y
\end{array}
\]

Assume that $\text{THH}_{j+1}(X^0, \ldots, Z, \ldots, X^j) \to D(X^0 \wedge_S \ldots \wedge_S Z \wedge_S \ldots \wedge_S X^j)$ is a $\pi_\ast$-isomorphism for $Z = A, B,$ or $X$ where $Z$ is inserted between the $i$th and $i + 1$st spots. Then $\text{THH}_{j+1}(X^0, \ldots, Y, \ldots, X^j) \to D(X^0 \wedge_S \ldots \wedge_S Y \wedge_S \ldots \wedge_S X^j)$ is a $\pi_\ast$-isomorphism.

Using these two lemmas we can now prove Proposition 4.2.3.

**Proof of Proposition 4.2.3.** We prove this by induction on $\pi$ with the induction assumption that $\phi_j$ is a $\pi_\ast$-isomorphism when $j - i$ variables are free spectra and the other variables are cofibrant. Lemma 4.3.1 verifies this for $i = 0$. For the induction step, in one variable we build up a cofibrant spectrum from the initial spectrum by retracts, colimits, and pushouts over generating cofibrations. Since retracts of $\pi_\ast$-isomorphisms are $\pi_\ast$-isomorphisms and $\phi_j$ of a retract is a retract we only need to consider colimits and pushouts.

Because $F_0$, smash products, $L'$, $\Omega^n$, and homotopy colimits commute with filtered colimits, $T_j$ of a colimit in one of the variables is a colimit. This is also true of $D$. Since a filtered colimit of $\pi_\ast$-isomorphisms is a $\pi_\ast$-isomorphism this means $\phi_j$ of a colimit in one variable is a $\pi_\ast$-isomorphism if it is a $\pi_\ast$-isomorphism at each spot in the sequence. Hence we are only left with pushouts.

Since $\phi_j$ is a level equivalence between trivial spectra if one of the variables is the initial spectrum, $\ast$, one can proceed by induction to verify the pushout property. By induction the two corners in the pushout corresponding to the generating cofibration are $\pi_\ast$-isomorphisms. This is because generating cofibrations are of the form $F_n K \to F_n L$, so these two corners have one extra variable a free spectrum and hence fall into the case covered by the previous induction step. The third corner is assumed to be a $\pi_\ast$-isomorphism by induction, hence $\phi_j$ is a $\pi_\ast$-isomorphism on the pushout corner by Lemma 4.3.2.

**Proof of Lemma 4.3.2.** To show that $\phi_j(F_{n_0} K_0, \ldots, F_{n_j} K_j)$ is a $\pi_\ast$-isomorphism we first establish the stable homotopy type of $T_j(F_{n_0} K_0, \ldots, F_{n_j} K_j)$. There is a functor $T_{(n_0, \ldots, n_j)} X : I^{j+1} \to Sp^S$ defined by

$T_{(n_0, \ldots, n_j)} X(m_0, \ldots, m_j) = \text{hom}_{I^{j+1}}((n_0, \ldots, n_j), (m_0, \ldots, m_j)) \wedge X$.

Then $T_{(n_0, \ldots, n_j)}(-)$ is left adjoint to the functor from $I$-diagrams over $Sp^S$ to $Sp^S$ which evaluates the diagram at $(n_0, \ldots, n_j) \in I^{j+1}$. There is a map of diagrams $T_{(n_0, \ldots, n_j)}(\Omega^j L'/F_0(K_0 \wedge_S \ldots \wedge_S K_j)) \to D_j F_n K_0, \ldots, F_{n_j} K_j$ where $n = \Sigma n_i$. Each spot in this diagram is a $\pi_\ast$-isomorphism. This is similar to the proof of Lemma
on each level the map is an equivalence in the stable range by the Blakers-Massey and the Freudenthal suspension theorems. Hence the map on homotopy colimits is also a \( \pi_* \)-isomorphism, \( \Omega^n L' F_0(K_0 \wedge \ldots \wedge K_j) \rightarrow T_j(F_{n_0} K_0, \ldots, F_{n_j} K_j) \).

By Lemma 3.2.3, \( \Omega^n L' F_0(K_0 \wedge \ldots \wedge K_j) \rightarrow D(F_n(K_0 \wedge \ldots \wedge K_j)) \) is also a \( \pi_* \)-isomorphism. To see that \( \phi_j \) induces a \( \pi_* \)-isomorphism, note that on the free diagrams there are similar maps \( \text{hocolim}_j \mathcal{F}_n \mu^* \mathcal{F}_n \Omega^n L' F_0(K_0 \wedge \ldots \wedge K_j) \rightarrow \text{hocolim}_j \mathcal{F}_n \Omega^n L' F_0(K_0 \wedge \ldots \wedge K_j) \) which induce level equivalences on the homotopy colimits.

To prove Lemma 4.3.2 we first need to show that \( T_j \) of a homotopy pushout in one variable is a homotopy pushout.

**Lemma 4.3.3.** Let \( X^0, \ldots, X^j \) be cofibrant spectra. If

\[
\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow & & \downarrow \\
B & \longrightarrow & Y
\end{array}
\]

is a pushout square with \( A \rightarrow B \) a cofibration, then

\[
\begin{array}{ccc}
T_{j+1}(X^0, \ldots, A, \ldots X^j) & \longrightarrow & T_{j+1}(X^0, \ldots, X^j) \\
\downarrow & & \downarrow \\
T_{j+1}(X^0, \ldots, B, \ldots X^j) & \longrightarrow & T_{j+1}(X^0, \ldots, Y, \ldots X^j)
\end{array}
\]

is a homotopy pushout square. I.e., if \( P \) is the homotopy pushout of the second square then \( P \rightarrow T_{j+1}(X^0, \ldots, Y, \ldots X^j) \) is a stable equivalence. In fact, \( P \rightarrow T_{j+1}(X^0, \ldots, Y, \ldots X^j) \) is a \( \pi_* \)-isomorphism.

**Proof.** This proof is similar to the proof of Lemma 3.2.3. As with Lemma 3.2.3, it is enough to consider each object in \( P^{j+1} \) since homotopy colimits commute.

The following square is a pushout square with the left map a cofibration since each \( X^i \) is cofibrant.

\[
\begin{array}{ccc}
X^0_{n_0} \wedge \ldots \wedge A_{n_i} \wedge \ldots X^j_{n_j} & \longrightarrow & X^0_{n_0} \wedge \ldots \wedge X_{n_i} \wedge \ldots X^j_{n_j} \\
\downarrow & & \downarrow \\
X^0_{n_0} \wedge \ldots \wedge B_{n_i} \wedge \ldots X^j_{n_j} & \longrightarrow & X^0_{n_0} \wedge \ldots \wedge Y_{n_i} \wedge \ldots X^j_{n_j}
\end{array}
\]

The first step in constructing \( T_j \) is just applying \( F_0 \) to this square. \( F_0 \) preserves cofibrations and pushouts, hence \( F_0 \) applied to this square is a homotopy pushout. \( L' \) preserves homotopy pushout squares up to level equivalence and \( \Omega^{\Sigma n_i} \) preserves homotopy pushout squares up to \( \pi_* \)-isomorphism. Hence the map from the homotopy pushout to the bottom right corner is a \( \pi_* \)-isomorphism. Since the homotopy colimit of \( \pi_* \)-isomorphisms is a \( \pi_* \)-isomorphism, this finishes the proof.

**Proof of Lemma 4.3.3.** Both \( T_j \) and \( D \) take homotopy pushouts in one variable to homotopy pushouts where the map from the pushout to the bottom right corner is a \( \pi_* \)-isomorphism by Lemmas 4.3.3 and 3.2.3. Hence, this lemma follows from the fact that homotopy colimits preserve \( \pi_* \)-isomorphisms, Lemma 2.3.3. \( \square \)
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