BOOTSTRAPPING HIGH DIMENSIONAL TIME SERIES

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This article studies bootstrap inference for high dimensional weakly dependent time series in a general framework of approximately linear statistics. The following high dimensional applications are covered: (i) uniform confidence band for mean vector; (ii) specification testing on the second order property of time series such as white noise testing and bandedness testing of covariance matrix; (iii) specification testing on the spectral property of time series. In theory, we first derive a Gaussian approximation result for the maximum of a sum of weakly dependent vectors, where the dimension of the vectors is allowed to be exponentially larger than the sample size. In particular, we illustrate an interesting interplay between dependence and dimensionality, and also discuss one type of “dimension free” dependence structure. We further propose a blockwise multiplier (wild) bootstrap that works for time series with unknown autocovariance structure. These distributional approximation errors, which are finite sample valid, decrease polynomially in sample size. A non-overlapping block bootstrap is also studied as a more flexible alternative. The above results are established under the general physical/functional dependence framework proposed in Wu (2005). Our work can be viewed as a substantive extension of Chernozhukov et al. (2013) to time series based on a variant of Stein’s method developed therein.

1. Introduction. High-dimensional data are increasingly encountered in many applications of statistics such as bioinformatics, information technology, medical imaging, astronomy and financial studies. In recent years, there is a growing body of literature concerning inference on the first and second order properties of high dimensional data; see [7–9, 13, 14, 26, 33] among others. The validity of these procedures is generally established under independence amongst the data vectors, which can be quite restrictive for situations that involve temporally observed data. Examples include spatial-temporal modeling [39] and financial study of a large number of asset returns [37].

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Although high dimensional statistics has witnessed unprecedented development, statistical inference for high dimensional time series remains largely untouched so far. In the conventional low dimensional setting, inference for time series data typically involves the direct estimation of the asymptotic covariance matrix, which is known to be difficult in the presence of heteroscedasticity and autocorrelation of unknown forms [1]. In the high dimensional setting, where the dimension is comparable or even larger than sample size, the classical inferential procedures designed for the low dimensional case are no longer applicable, e.g., the asymptotic covariance matrix is singular. Along a different line, alternative nonparametric procedures including block bootstrap, subsampling and blockwise empirical likelihood [10, 23, 24, 28, 32] have been proposed to avoid the direct estimation of covariance matrices. However, the extension of these procedures (coupled with suitable testing procedures) to the high dimensional setting remains unclear. One relevant high dimensional work ([12]) we are aware is on the estimation rates of the covariance/precision matrices of time series.

In this paper, we establish a general framework of conducting bootstrap inference for high dimensional stationary time series under weak dependence. We start from three motivating examples that are mainly concerned with first or second order property of time series: (i) uniform confidence band for mean vector; (ii) testing for serial correlation; (iii) testing on the bandedness of covariance matrix. The proposed bootstrap procedures are rather simple to implement and supported by simulation results. We want to emphasize that neither Gaussian assumption nor strong restrictions on the covariance structure are imposed in these applications. An important by-product of Examples (ii) and (iii) is the covariance structure testing for high dimensional time series that even does not rely on the existence of the null limit distribution. This new result is in sharp contrast with the existing literature for i.i.d data such as [8, 13, 14]. We also remark that the maximum-type testing procedure considered in these examples is expected to be particularly powerful for detecting sparse alternatives (see [8]). A comprehensive investigation along this line is left as our future topic.

The underlying theory in supporting these high dimensional applications is a general Gaussian approximation theory and its bootstrap version. The Gaussian approximation theory quantifies the Kolmogorov distance between the largest element of a sum of weakly dependent vectors and its Gaussian analog that shares the same autocovariance structure. We develop our theory in the general framework of dependency graph, which leads to delicate bounds on the Kolmogorov distance for various types of time series. The approximation error, which is finite sample valid, decreases polynomially in sample size even when the data dimension is exponentially high. Moreover, we study two important dependence structures in more details: $M$-dependent time series and weakly dependent time series. Although the sharpness of Kolmogorov distance is not established in this paper, our theoretical results (also see Figure 1) strongly indicate an interesting interplay between dependence and dimensionality: the less dependent of the data vectors, the faster diverging rate of the dimension is allowed for obtaining an accurate Gaussian approximation. We also propose an interesting “dimension free” dependence structure that allows the dimension to diverge at the
rate as if the data were independent. However, in practice, the intrinsic dependence structure of time series is usually unknown. This motivates us to develop a bootstrap version of the Gaussian approximation theory that does not require such knowledge. Specifically, we propose a blockwise multiplier bootstrap that is able to capture the dependence amongst and within the data vectors. Moreover, it inherits the high quality approximation without relying on the autocovariance information. We also introduce a non-overlapping block bootstrap as a more flexible alternative. The above theoretical results are major building blocks of a general framework of conducting bootstrap inference for high dimensional time series. This general framework assumes that the quantity of interest admits an approximately linear expansion, and thus covers the three examples mentioned above. This quantity of interest can be expressed as a functional of the distribution of the time series with finite or infinite length. Hence, our result is also useful in making inference for the spectrum of time series.

Our general Gaussian approximation theory and its block bootstrap version substantially relax the independence assumption in [2, 16], and is established using several techniques including the Slepian interpolation [35], leave-one-block-out argument (modification of Stein’s leave-one-out argument [36]), self-normalization [17], weak dependence measure [40], and $M$-dependent approximation [29]. It is worth pointing out that our results are established under the physical/functional dependence measure proposed in [40]. This framework (or its variants) is known to be very general and easy to verify for linear and nonlinear data-generating mechanisms, and it also provides a convenient way for establishing large-sample theories for stationary causal processes [12, 40, 41]. In particular, our work is largely inspired by a recent breakthrough in Gaussian approximation for i.i.d data ([16]) that obtained an astounding improvement over the previous results in [3] by allowing the dimension of the data vectors to be exponentially larger than the sample size.

The rest is organized as follows. In Section 2, we describe three concrete bootstrap inference procedures mentioned above in details. Section 3 gives the Gaussian approximation result that works even when the dimension is exponentially larger than sample size, and Section 4 proposes the blockwise multiplier (wild) bootstrap and also the non-overlapping block bootstrap that do not depend on the autocovariance structure of time series. Building on the results in Sections 3 and 4, a general framework of conducting bootstrap inference based on approximately linear statistics is established in Section 5. Three examples considered in 2 and one spectral testing example are covered by this framework. All the proofs are gathered in the supplementary material.

2. High Dimensional Inference. To motivate our general theory, we consider three concrete bootstrap inference procedures for high dimensional time series: uniform confidence band; white noise testing; and bandedness testing for covariance matrix. These procedures are rather straightforward to implement. The main focus of this section is mostly on the methodological side, and the general theoretical results are deferred to Section 5. An ad-hoc way of choosing block size in
2.1. **Uniform confidence band.** Consider \( n \) observations from a sequence of weakly dependent \( p \)-dimensional time series \( \{x_i\} \) with \( x_i = (x_{i1}, \ldots, x_{ip})' \). We are interested in constructing a \( 100(1 - \alpha) \)th uniform confidence band for the mean vector \( \mu_0 = (\mu_{01}, \mu_{02}, \ldots, \mu_{0p})' \) in the form of

\[
\left\{ \mu = (\mu_1, \ldots, \mu_p)' \in \mathbb{R}^p : \sqrt{n} \max_{1 \leq j \leq p} |\mu_j - \bar{x}_{nj}| \leq c(\alpha) \right\},
\]

where \( \bar{x}_n = (\bar{x}_{n1}, \ldots, \bar{x}_{np})' = \sum_{i=1}^{n} x_i/n \). In the traditional low dimensional regime, confidence region for the mean of a multivariate time series is typically constructed by inverting a suitable Wald type test. A common choice is the Wald type test which is of the form

\[
\frac{\hat{\mu} - \mu_0}{\sqrt{\hat{\Sigma}}} \sim \mathcal{N}(0, \mathbb{I})
\]

where \( \hat{\Sigma} = N \sigma^2 \) is a consistent estimator of the variance matrix \( \Sigma \). However, obtaining a consistent \( \hat{\Sigma} \) could be difficult in practice due to the unknown dependence structure. To avoid this hassle, several appealing nonparametric alternatives, e.g., moving block bootstrap method [10, 24, 28], subsampling approach [32] and block-wise empirical likelihood [23], have been proposed. In the high dimensional regime, where the dimension of the time series is comparable with or even much larger than the sample size, inverting the Wald type test is no longer applicable because the long run variance estimator \( \hat{\Sigma} \) is singular for \( p > n \). Moreover, the direct application of the nonparametric approaches described above to the high dimensional setting is unclear yet.

In this subsection, we propose a bootstrap-assisted method to obtain the critical value \( c(\alpha) \) in (1), whose theoretical validity will be justified in Section 5.1. Specifically, we introduce the following blockwise multiplier (wild) bootstrap. For simplicity, suppose \( n = b_n l_n \) with \( b_n, l_n \in \mathbb{Z} \). Define the non-overlapping block sums,

\[
\hat{A}_{ij} = \sum_{l=(i-1)b_n+1}^{ib_n} (x_{lj} - \bar{x}_{nj}), \quad i = 1, 2, \ldots, l_n,
\]

and the bootstrap statistic,

\[
T_A = \max_{1 \leq j \leq p} \frac{1}{\sqrt{n}} \left| \sum_{i=1}^{l_n} \hat{A}_{ij} e_i \right|,
\]

where \( \{e_i\} \) is a sequence of i.i.d. \( N(0, 1) \) random variables independent of \( \{x_i\} \). The bootstrap critical value is defined as \( c(\alpha) := \inf \{t \in \mathbb{R} : P(T_A \leq t|\{x_i\}_i=1^n) \geq 1 - \alpha \} \).

We next conduct a small simulation study to assess the finite sample coverage probability of the uniform confidence band. Consider a \( p \)-dimensional VAR(1) (vector autoregressive) process,

\[
x_t = \rho x_{t-1} + \sqrt{1 - \rho^2} \epsilon_t,
\]

where \( \epsilon_t = (\epsilon_{t1}, \ldots, \epsilon_{tp})' \). For the error process \( \{\epsilon_t\} \), we consider three cases: (i) \( \epsilon_{tj} = (\epsilon_{tj} + \epsilon_{t0})/\sqrt{2} \), where \( (\epsilon_{t0}, \epsilon_{t1}, \ldots, \epsilon_{tp})' \sim i.i.d. N(0, I_{p+1}); \) (ii) \( \epsilon_{tj} = \rho_1 \zeta_{tj} + \rho_2 \zeta_{t(j+1)} + \cdots + \rho_p \zeta_{t(j+p-1)} \), where \( \{\rho_j\}_j=1^p \)
are generated independently from Unif(2,3) (uniform distribution on \([2,3]\)), and \(\{\zeta_{tj}\}\) are i.i.d \(N(0,1)\) random variables; (iii) \(\epsilon_{tj}\) is generated from the moving average model in (ii) with \(\{\zeta_{tj}\}\) being i.i.d centralized Gamma(4,1) random variables. Set \(n = 120, p = 500, 1000, \) and \(\rho = 0.2\) or \(0.5\) in (2). To implement the blockwise multiplier bootstrap, we choose \(b_n = 4, 6, 8, 10, 12, 15, 20\).

Table 1 reports the coverage probabilities at 90% and 95% nominal levels based on 5000 simulations and 499 bootstrap resamples. We note that the coverage probabilities appear to be low for relatively small block size. When \(\rho\) increases, a larger block size is generally required to capture the dependence. Although the coverage probability is generally sensitive to the choice of the block size, with a proper block size, the coverage probability can be reasonably close to the nominal level. For univariate time series, there are two major approaches for selecting the optimal block size: the non-parametric plug-in method (e.g. [6]) and the empirical criteria-based method [19]. However, these selection procedures are deduced based on the bias-variance tradeoff, which are not intended to guarantee the best coverage of confidence interval. Moreover, it is still unclear how these selection rules can be extended to the high dimensional context.

Hence, we provide an ad-hoc way for choosing the block size below. Given a set of realizations \(\{x_t\}_{t=1}^n\), we pick an initial block size \(b_{int}\) such that \(n = b_{int}l_{int}\) where \(b_{int}, l_{int} \in \mathbb{Z}\). Conditional on the sample \(\{x_t\}_{t=1}^n\), we let \(s_1, \ldots, s_{l_{int}}\) be i.i.d uniform random variables on \(\{0, \ldots, l_{int} - 1\}\) and define \(x^*_t = x_{s_jb_{int} + i}\) with \(1 \leq j \leq l_{int}\) and \(1 \leq i \leq b_{int}\). In other words, \(\{x^*_t\}_{t=1}^n\) is a non-overlapping block bootstrap sample with block size \(b_{int}\). For each \(b_n\) (block size for the original sample), we can compute the times that the sample mean \(\bar{x}_n\) is contained in the uniform confidence band constructed based on the bootstrap sample \(\{x^*_t\}_{t=1}^n\) and then compute the empirical coverage probabilities based on \(B\) bootstrap samples. This is based on the notion that \(\bar{x}_n\) is the true mean for the bootstrap sample conditional on \(\{x_t\}_{t=1}^n\). In this case, the block size, which delivers the most accurate coverage for \(\bar{x}_n\), can be viewed as an estimate of the optimal \(b_n\) for the original series. We employ the above procedure with \(b_{int} = 6\) and \(B = 500\) to choose the optimal block size. Based on 200 realizations from the original data generating process, the coverage probabilities (given the selected block size) in different simulation setup are summarized in Table 2. We observe that the coverage probability based on the optimal block size is close to the best coverage presented in Table 1. Finally we point out that it might be possible to iterate the above procedure to further improve the empirical performance.

2.2. Testing for serial correlation. Covariance matrix plays a crucial role in many areas of statistical inference. For independent vectors, many methods have been developed for testing specific structures of covariance matrices (see e.g. [9, 14, 26, 33] for some recent developments). In this subsection, we examine the serial correlation of a sequence of time series data by testing its autocovariance matrix (a more general measure than covariance matrix).

To illustrate the idea, let \(\gamma(l) = (\gamma_{jk}(l))_{j,k=1}^p = \mathbb{E}x_ix'_{i+l} \in \mathbb{R}^{p \times p}\) be the autocovariance matrix of
The cardinality of \( \Lambda \) is allowed to grow with the dimension \( p \). To obtain the critical value for the \( p \)-dimensional setting, i.e., \( \gamma \), we employ the blockwise multiplier bootstrap. A diagnostic procedure in time series analysis, e.g., Table 1, fills in this gap. Again, we employ the blockwise multiplier bootstrap to obtain the critical value \( c(\alpha) \). To proceed, we let

\[
\sqrt{n} \max_{l \in \Lambda} \max_{1 \leq j, k \leq p} |\tilde{\gamma}_{jk}(l) - \gamma_{jk}(l)| > c(\alpha),
\]

where \( c(\alpha) \) denotes the bootstrap critical value at level \( \alpha \). This framework includes several important applications such as white noise testing (i.e., testing for serial correlation) and covariance testing.

In the white noise testing, we consider \( H_0 : \gamma(l) = 0_{p \times p} \) for any \( 1 \leq l \leq L \) v.s. \( H_a : \gamma(l) \neq 0_{p \times p} \) for some \( 1 \leq l \leq L \), where \( 0_{p \times p} \) denotes a \( p \times p \) matrix of all zeros. This is a standard diagnostic procedure in time series analysis, e.g., [4, 18, 22, 34] among others. However, in the high dimensional setting, i.e., \( p^2 \gg n \), there seems no systematic method available to test the white noise assumption. The proposed test statistic \( \sqrt{n} \max_{1 \leq l \leq L} \max_{1 \leq j, k \leq p} |\tilde{\gamma}_{jk}(l)| \) fills in this gap. Again, we employ the blockwise multiplier bootstrap to obtain the critical value \( c(\alpha) \). To proceed, we let

| \( \rho = 0.2 \) | \( \rho = 0.2, (i) \) | \( \rho = 0.2, (ii) \) | \( \rho = 0.5, (i) \) | \( \rho = 0.5, (ii) \) | \( \rho = 0.5, (iii) \) |
|---|---|---|---|---|---|
| \( p = 500 \) | 90% | 95% | 90% | 95% | 90% | 95% | 90% | 95% |
| \( b_n = 4 \) | 85.0 | 92.2 | 85.6 | 92.6 | 85.5 | 91.7 | 86.0 | 92.8 |
| \( b_n = 6 \) | 87.8 | 93.8 | 85.4 | 93.7 | 86.0 | 92.4 | 87.7 | 94.5 |
| \( b_n = 8 \) | 89.1 | 95.5 | 85.7 | 92.3 | 86.4 | 93.1 | 89.2 | 95.1 |
| \( b_n = 10 \) | 89.5 | 95.7 | 85.7 | 92.3 | 85.2 | 92.1 | 90.7 | 96.0 |
| \( b_n = 12 \) | 89.2 | 95.3 | 85.4 | 91.8 | 85.4 | 92.5 | 90.4 | 96.5 |
| \( b_n = 15 \) | 90.3 | 96.0 | 84.6 | 91.8 | 85.2 | 92.3 | 90.2 | 96.4 |
| \( b_n = 20 \) | 90.2 | 96.5 | 83.0 | 90.7 | 83.2 | 90.8 | 91.2 | 96.9 |

| \( p = 500, (ii) \) | \( p = 500, (iii) \) | \( p = 1000, (i) \) | \( p = 1000, (ii) \) | \( p = 1000, (iii) \) |
|---|---|---|---|---|
| \( b_n = 4 \) | 85.0 | 92.2 | 85.6 | 92.6 | 85.5 | 91.7 | 86.0 | 92.8 |
| \( b_n = 6 \) | 87.8 | 93.8 | 85.4 | 93.7 | 86.0 | 92.4 | 87.7 | 94.5 |
| \( b_n = 8 \) | 89.1 | 95.5 | 85.7 | 92.3 | 86.4 | 93.1 | 89.2 | 95.1 |
| \( b_n = 10 \) | 89.5 | 95.7 | 85.7 | 92.3 | 85.2 | 92.1 | 90.7 | 96.0 |
| \( b_n = 12 \) | 89.2 | 95.3 | 85.4 | 91.8 | 85.4 | 92.5 | 90.4 | 96.5 |
| \( b_n = 15 \) | 90.3 | 96.0 | 84.6 | 91.8 | 85.2 | 92.3 | 90.2 | 96.4 |
| \( b_n = 20 \) | 90.2 | 96.5 | 83.0 | 90.7 | 83.2 | 90.8 | 91.2 | 96.9 |

| \( p = 500, (i) \) | \( p = 500, (ii) \) | \( p = 500, (iii) \) | \( p = 1000, (i) \) | \( p = 1000, (ii) \) | \( p = 1000, (iii) \) |
|---|---|---|---|---|---|
| \( b_n = 4 \) | 85.0 | 92.2 | 85.6 | 92.6 | 85.5 | 91.7 | 86.0 | 92.8 |
| \( b_n = 6 \) | 87.8 | 93.8 | 85.4 | 93.7 | 86.0 | 92.4 | 87.7 | 94.5 |
| \( b_n = 8 \) | 89.1 | 95.5 | 85.7 | 92.3 | 86.4 | 93.1 | 89.2 | 95.1 |
| \( b_n = 10 \) | 89.5 | 95.7 | 85.7 | 92.3 | 85.2 | 92.1 | 90.7 | 96.0 |
| \( b_n = 12 \) | 89.2 | 95.3 | 85.4 | 91.8 | 85.4 | 92.5 | 90.4 | 96.5 |
| \( b_n = 15 \) | 90.3 | 96.0 | 84.6 | 91.8 | 85.2 | 92.3 | 90.2 | 96.4 |
| \( b_n = 20 \) | 90.2 | 96.5 | 83.0 | 90.7 | 83.2 | 90.8 | 91.2 | 96.9 |

Table 1: Coverage probabilities of the uniform confidence band for the mean, where the block size \( b_n = 4, 6, 8, 10, 12, 15, 20, \) and \( n = 120 \).

Table 2: Coverage probabilities of the uniform confidence band for the mean, where the block size is chosen automatically, \( p = 500, n = 120 \), and the nominal level is 95%.
\[ \nu_i = (\nu_{i,1}, \ldots, \nu_{i,p^2L}) = (\text{vec}(x_{i,x_{i+1}}'), \ldots, \text{vec}(x_{i,x_{i+L}}'))' \in \mathbb{R}^{p^2L} \text{ for } i = 1, \ldots, N := n - L, \]

where vec denotes the operator that stacks the columns of a \( p \times p \) matrix as a vector with \( p^2 \) components. Suppose \( N = b_n l_n \) for \( b_n, l_n \in \mathbb{Z} \). Define

\[
\tilde{T}_A = \max_{1 \leq j \leq p^2L} \frac{1}{\sqrt{n}} \left| \sum_{i=1}^{l_n} \hat{A}_{ij} e_i \right| , \quad \hat{A}_{ij} = \sum_{l=(i-1)b_n+1}^{ib_n} (\nu_{ij} - \bar{\nu}_{nj}),
\]

where \( \{e_i\} \) is a sequence of i.i.d standard normal independent of \( \{x_i\} \), and \( \bar{\nu}_{nj} = \sum_{i=1}^{N} \nu_{ij} / N \). The bootstrap critical value is then given by \( c(\alpha) := \inf \{t \in \mathbb{R} : P(\tilde{T}_A \leq t | \{x_i\}_{i=1}^{n}) \geq 1 - \alpha \} \). The above procedure can be easily modified to get the critical value for the general test described in (3).

When assuming \( \Lambda = \{0\} \), we obtain an important by-product: covariance structure testing for high dimensional vector. In this case, our test reduces to \( \sqrt{n} \max_{1 \leq j \leq k \leq p} |\tilde{\gamma}_{jk}(0) - \bar{\gamma}_{jk}(0)| > c(\alpha) \). Compared to the existing work in the independence case, e.g., [8], our test enjoys three appealing features: (i) it allows dependence amongst data vectors and relaxes the Gaussian assumption; (ii) it does not require the existence of a null limit distribution such as the extreme distribution of Type I in [9]. Hence, we can avoid the slow convergence issue of the extreme value distribution (see [30]), which causes an inaccurate critical value. Rather, a blockwise multiplier bootstrap is employed to provide high quality approximation; (iii) it does not impose strong restrictions on the covariance structure such as sparsity on the precision matrix [8] or pseudo-independence among its components [13, 14].

To evaluate the finite sample performance of the white noise testing procedure, we consider the following data generating processes: (i) independent normal random vectors whose covariance structure is determined by a moving average model \( x_{ij} = \rho_1 \xi_{ij} + \rho_2 \xi_{i(j+1)} + \cdots + \rho_p \xi_{i(j+p-1)} \), where \( \{\rho_j\}_{j=1}^{p} \) are generated independently from Unif(2, 3), and \( \{\xi_{ij}\} \) are i.i.d \( N(0,1) \) random variables; (ii) multivariate ARCH model defined as \( x_i = \Sigma_i^{1/2} e_i \) with \( e_i \sim N(0, I_p) \) and \( \Sigma_i = 0.1I_p + 0.9x_{i-1}x_{i-1}' \), where \( \Sigma_i^{1/2} \) is a lower triangular matrix based on the Cholesky decomposition of \( \Sigma_i \); (iii) VAR(1) model \( x_i = \rho x_{i-1} + \sqrt{1-\rho^2} e_i \), where \( \rho = 0.3 \) and the errors \( \{e_i\} \) are generated according to (i). We consider \( n = 60 \) and \( p = 30 \) or 50. Notice that the actual number of parameters in consideration is \( p^2 \times L \), where \( L \) is the number of lags specified in the hypothesis. Table 3 summarizes the rejection probabilities at 10% and 5% nominal levels based on 5000 simulations and 499 bootstrap resamples. In general, the proposed method delivers reasonable size and power, although we still observe some downward size distortion and power loss especially for \( L = 3 \). The power loss here is presumably due to the correlation structure of the VAR(1) model. It is also worth noting that the choice of \( b_n = 1 \) generally performs well for the martingale difference sequences considered under the null.

**Remark 2.1.** The simulation results demonstrate the usefulness of the proposed method but they also leave room for improvement. Here we point out two possibilities: (i) it is of interest
Table 3

Rejection percentages for testing the uncorrelatedness, where the block size $b_n = 1, 2, 3, 4, 5, 6$, and $n = 60$. Cases (i) and (ii) are under null, while case (iii) is under alternative.

| $p$  | 10% | 5% | 10% | 5% | 10% | 5% | 10% | 5% | 10% | 5% | 10% | 5% |
|------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $L = 1$ |     |     |     |     |     |     |     |     |     |     |     |     |
| $b_n = 1$ | 8.1 | 3.5 | 9.2 | 3.2 | 73.1 | 59.1 | 8.3 | 3.9 | 8.9 | 2.8 | 72.4 | 58.9 |
| $b_n = 2$ | 9.2 | 3.6 | 7.0 | 2.4 | 67.0 | 49.6 | 8.7 | 3.2 | 6.7 | 2.2 | 68.2 | 49.4 |
| $b_n = 3$ | 10.8 | 4.6 | 6.8 | 2.5 | 66.0 | 46.4 | 9.9 | 4.1 | 6.9 | 2.6 | 66.4 | 46.7 |
| $b_n = 4$ | 10.9 | 4.6 | 6.7 | 3.0 | 67.0 | 46.8 | 11.0 | 4.1 | 6.9 | 2.8 | 67.4 | 46.3 |
| $b_n = 5$ | 11.4 | 4.5 | 7.8 | 3.7 | 69.2 | 47.5 | 11.6 | 4.5 | 7.8 | 3.7 | 67.6 | 46.6 |
| $b_n = 6$ | 12.7 | 5.2 | 9.2 | 4.7 | 67.7 | 47.7 | 12.3 | 5.1 | 8.3 | 4.4 | 68.2 | 48.2 |
| $L = 3$ |     |     |     |     |     |     |     |     |     |     |     |     |
| $b_n = 1$ | 7.2 | 2.4 | 8.5 | 3.3 | 58.3 | 43.8 | 6.7 | 2.5 | 8.6 | 3.2 | 58.7 | 43.4 |
| $b_n = 2$ | 7.6 | 2.7 | 5.4 | 2.1 | 51.3 | 33.0 | 7.9 | 3.0 | 5.3 | 2.4 | 51.3 | 32.4 |
| $b_n = 3$ | 6.9 | 2.3 | 3.9 | 1.5 | 46.4 | 28.1 | 6.4 | 2.0 | 3.7 | 1.6 | 46.9 | 27.7 |
| $b_n = 4$ | 7.0 | 2.3 | 3.8 | 2.0 | 47.0 | 27.4 | 6.6 | 2.0 | 4.2 | 2.2 | 47.5 | 28.2 |
| $b_n = 5$ | 7.8 | 2.4 | 5.1 | 2.4 | 48.6 | 28.2 | 7.4 | 2.2 | 4.6 | 2.5 | 47.7 | 27.5 |
| $b_n = 6$ | 7.9 | 2.5 | 6.4 | 3.8 | 49.3 | 28.1 | 8.7 | 2.7 | 5.9 | 3.2 | 49.1 | 28.1 |

2.3. Bandedness testing of covariance matrix. In this subsection, we consider testing the bandedness of covariance matrix $\gamma(0)$. This problem arises, for example, in econometrics when testing certain economic theories; see [1, 27] and reference therein. Also see [9, 33] for independent case. For any integer $\iota \geq 1$ (which possibly depends on $n$ or $p$), we want to test

$$H_0: \gamma_{jk}(0) = 0, \quad |j - k| \geq \iota.$$  

Our setting significantly generalizes the one considered in [9] which focuses on independent Gaussian vectors. Here, we shall allow non-Gaussian and dependent random vectors.

We define the test statistic as

$$T_{band} = \sqrt{n} \max_{|j-k| \geq \iota} \left| \frac{\hat{\gamma}_{jk}(0)}{\sqrt{\hat{\gamma}_{jj}(0)\hat{\gamma}_{kk}(0)}} \right| = \max_{|j-k| \geq \iota} \frac{1}{\sqrt{n}} \left| \sum_{i=1}^{n} \frac{x_{ij}x_{ik}}{\sqrt{\hat{\gamma}_{jj}(0)\hat{\gamma}_{kk}(0)}} \right|. $$

For $n = b_nl_n$ with $b_n, l_n \in \mathbb{Z}$, we define the block sums

$$\hat{A}_{i,jk} = \sum_{l=(i-1)b_n+1}^{ib_n} \frac{x_{ij}x_{ik} - \hat{\gamma}_{jk}(0)}{\sqrt{\hat{\gamma}_{jj}(0)\hat{\gamma}_{kk}(0)}}, \quad i = 1, 2, \ldots, l_n,$$
and the bootstrap statistic

\[ T_{\text{band}, \hat{A}} = \max_{|j-k| \geq 1} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{t_n} \hat{A}_{i,jk} e_i \right|, \]

where \( \{e_i\} \) is a sequence of i.i.d \( N(0, 1) \) independent of \( \{x_i\} \). We reject the null \( H_0 \) if \( T_{\text{band}, \hat{A}} > c_{\text{band}}(\alpha) \), where \( c_{\text{band}}(\alpha) := \inf \{ t \in \mathbb{R} : P(T_{\text{band}, \hat{A}} \leq t | \{x_i\}_{i=1}^{n}) \geq 1 - \alpha \} \). Alternatively, one can employ the non-overlapping block bootstrap (to be presented in Sections 4.2) to obtain the critical value.

### 3. Gaussian Approximation Theory

In this section, we derive a Gaussian approximation theory that serves as the first step in studying high dimensional inference procedures in Section 2. Consider a sequence of p-dimensional dependent random vectors \( \{x_i\}_{i=1}^{n} \) with \( x_i = (x_{i1}, \ldots, x_{ip})' \). Suppose \( \mathbb{E}x_i = 0 \) and \( \Sigma_{ij} := \text{cov}(x_i, x_j) \in \mathbb{R}^{p \times p} \). The Gaussian counterpart is defined as a sequence of Gaussian random variables \( \{y_i\}_{i=1}^{n} \) independent of \( \{x_i\}_{i=1}^{n} \). In addition, \( \{y_i\}_{i=1}^{n} \) preserves the autocovariance structure of \( \{x_i\} \) in the sense that \( \mathbb{E}y_i = 0 \) and \( \text{cov}(y_i, y_j) = \Sigma_{ij} \). (note that this assumption can be weakened, see Remark 3.1). Gaussian approximation theory quantifies the Kolmogorov distance defined as

\[ \rho_n := \sup_{t \in \mathbb{R}} |P(T_X \leq t) - P(T_Y \leq t)|, \]

where \( T_X = \max_{1 \leq j \leq p} X_j \), \( T_Y = \max_{1 \leq j \leq p} Y_j \), and

\[ X = (X_1, \ldots, X_p)' = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i, \quad Y = (Y_1, \ldots, Y_p)' = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} y_i. \]

Chernozhukov et al (2013) recently showed that for independent data vectors, \( \rho_n \) decays to zero polynomially in the sample size. In Section 3.1, we substantially relax their independence assumption by first establishing a general proposition, i.e., Proposition 3.1, in the framework of dependency graph. This general result leads to delicate bounds on the Kolmogorov distance for various types of weakly dependent time series even when their dimension is exponentially high, i.e., Sections 3.2 – 3.3.

#### 3.1. General framework: dependency graph

In this subsection, we introduce a flexible framework in modelling the dependence among a sequence of \( p \)-dimensional dependent (unnecessarily identical) random vectors \( \{x_i\}_{i=1}^{n} \). We call it as dependency graph \( G_n = (V_n, E_n) \), where \( V_n = \{1, 2, \ldots, n\} \) is a set of vertices and \( E_n \) is the corresponding set of undirected edges. For any two disjoint subsets of vertices \( S, T \subseteq V_n \), if there is no edge from any vertex in \( S \) to any vertex in \( T \), the collections \( \{x_i\}_{i \in S} \) and \( \{x_i\}_{i \in T} \) are independent. Let \( D_{\max, n} = \max_{1 \leq i \leq n} \sum_{j=1}^{n} I\{\{i, j\} \in E_n\} \) be the maximum degree of \( G_n \) and denote \( D_n = 1 + D_{\max, n} \). Throughout the paper, we allow \( D_n \) to
grow with the sample size $n$. For example, if an array $\{x_{i,n}\}_{i=1}^n$ is a $M := M_n$ dependent sequence (that is $x_{i,n}$ and $x_{j,n}$ are independent if $|i - j| > M$), then we have $D_n = 2M + 1$.

Within this general framework, we want to understand the largest possible diverging rate of $p$ (w.r.t. $n$) under which the Kolmogorov distance between the distributions of $T_X$ and $T_Y$, i.e., $\rho_n$ defined in (6), converges to zero. Recall that $T_X = \max_{1 \leq j \leq p} X_j$, $T_Y = \max_{1 \leq j \leq p} Y_j$. The problem of comparing distributions of maxima is nontrivial since the maximum function $z = (z_1, \ldots, z_p)' \to \max_{1 \leq j \leq p} z_j$ is non-differentiable. To overcome this difficulty, we consider a smooth approximation of the maximum function,

$$F_\beta(z) := \beta^{-1} \log \left( \sum_{j=1}^p \exp(\beta z_j) \right), \quad z = (z_1, \ldots, z_p)' ,$$

where $\beta > 0$ is the smoothing parameter that controls the level of approximation. Simple algebra yields that (see [11]),

$$0 \leq F_\beta(z) - \max_{1 \leq j \leq p} z_j \leq \beta^{-1} \log p .$$

Denote by $C^k(\mathbb{R})$ the class of $k$ times continuously differentiable functions from $\mathbb{R}$ to itself, and denote by $C_b^k(\mathbb{R})$ the class of functions $f \in C^k(\mathbb{R})$ such that $\sup_{z \in \mathbb{R}} |\partial^j f(z)/\partial z^j| < \infty$ for $j = 0, 1, \ldots, k$. Set $m = g \circ F_\beta$ with $g \in C_b^3(\mathbb{R})$. In Proposition 3.1 below, we derive a non-asymptotic upper bound for the quantity $[E[m(X) - m(Y)]]$ by employing the Slepian interpolation [35], and modifying Stein’s leave-one-out argument [36] to the leave-one-block-out argument for capturing the local dependence of the data.

Denote the truncated variables $\widetilde{x}_{ij} = (x_{ij} \wedge M_x) \vee (-M_x) - \mathbb{E}[(x_{ij} \wedge M_x) \vee (-M_x)]$ and $\widetilde{y}_{ij} = (y_{ij} \wedge M_y) \vee (-M_y)$ for some $M_x, M_y > 0$. Let $\bar{x}_i = (\bar{x}_{i1}, \ldots, \bar{x}_{ip})'$ and $\bar{y}_i = (\bar{y}_{i1}, \ldots, \bar{y}_{ip})'$. For $1 \leq i \leq n$, let $N_i = \{ j : \{i, j\} \in E_n \}$ be the set of neighbors of $i$, and $\tilde{N}_i = \{i\} \cup N_i$. Let $\phi(M_x)$ be a constant depending on the threshold parameter $M_x$ such that

$$\max_{1 \leq j, k \leq p} \frac{1}{n} \sum_{i=1}^n \left| \sum_{l \in \tilde{N}_i} (\mathbb{E}x_{ij}x_{lk} - \mathbb{E}\tilde{x}_{ij}\tilde{x}_{lk}) \right| \leq \phi(M_x) .$$

Analogous quantity $\phi(M_y)$ can be defined for $\{y_i\}$. Set $\phi(M_x, M_y) = \phi(M_x) + \phi(M_y)$. Define

$$m_{x,k} = (\mathbb{E} \max_{1 \leq j \leq p} |x_{ij}|^k)^{1/k}, \quad m_{y,k} = (\mathbb{E} \max_{1 \leq j \leq p} |y_{ij}|^k)^{1/k} ,$$

$$\bar{m}_{x,k} = \max_{1 \leq j \leq p} (\mathbb{E}|x_{ij}|^k)^{1/k}, \quad \bar{m}_{y,k} = \max_{1 \leq j \leq p} (\mathbb{E}|y_{ij}|^k)^{1/k} ,$$

where $\mathbb{E}[z_i] = \sum_{i=1}^n \mathbb{E}z_i/n$ for a sequence of random variables $\{z_i\}_{i=1}^n$. Note that $\bar{m}_{x,k} \leq m_{x,k}$ and $\bar{m}_{y,k} \leq m_{y,k}$. Further define an indicator function,

$$I := I_\Delta = 1 \left\{ \max_{1 \leq j \leq p} |X_j - \bar{X}_j| \leq \Delta, \max_{1 \leq j \leq p} |Y_j - \bar{Y}_j| \leq \Delta \right\} ,$$

where $\bar{X} = (\bar{X}_1, \ldots, \bar{X}_p)' = \frac{1}{\sqrt{n}} \sum_{i=1}^n \bar{x}_i$ and $\bar{Y} = (\bar{Y}_1, \ldots, \bar{Y}_p)' = \frac{1}{\sqrt{n}} \sum_{i=1}^n \bar{y}_i$. 


Proposition 3.1. Assume that \(2\sqrt{5} \beta D^2_n M_{xy} / \sqrt{n} \leq 1\) with \(M_{xy} = \max\{M_x, M_y\}\). Then we have for any \(\Delta > 0\),

\[
|\mathbb{E}[m(X) - m(Y)]| \lesssim (G_2 + G_1 \beta)\phi(M_x, M_y) + (G_3 + G_2 \beta + G_1 \beta^2) \frac{D^2_n}{\sqrt{n}} (\bar{m}_{x,3}^3 + \bar{m}_{y,3}^3)
+ (G_3 + G_2 \beta + G_1 \beta^2) \frac{D^3_n}{\sqrt{n}} (m_{x,3}^3 + m_{y,3}^3) + G_1 \Delta + G_0 \mathbb{E}[1 - I],
\]

where \(G_k = \sup_{z \in \mathbb{R}} |\partial^k g(z) / \partial z^k| \) for \(k \geq 0\). In addition, if \(2\sqrt{5} \beta D^3_n M_{xy} / \sqrt{n} \leq 1\), we can replace \(m_{x,3}^3 + m_{y,3}^3\) by \(\bar{m}_{x,3}^3 + \bar{m}_{y,3}^3\) in the above expression.

The proof of Proposition 3.1 is adapted from that of Theorem 2.1 in [16] for i.i.d case.

By approximating the indicator function \(I\{a \leq t\}\) with a suitable smooth function \(g(\cdot)\), Proposition 3.1 leads to an upper bound on the Kolmogorov distance, i.e., \(\rho_n\) defined in (6). In fact, the upper bound in (9) can be further simplified using the self-normalization technique (see Lemma 3.1) and certain arguments under weak dependence assumption. Finally, by optimizing the simplified upper bound (see Theorem 3.1), we obtain various convergence rates for \(\rho_n\) in Sections 3.2 – 3.3.

Remark 3.1. In view of the proof of Proposition 3.1 (see e.g. (S.4)), the assumption that \(\{y_i\}\) preserves the autocovariance structure of \(\{x_i\}\) can be weakened by assuming that for all \(i\),

\[
\sum_{k \in \hat{N}_i} \mathbb{E} x_i x'_k = \sum_{k \in \hat{N}_i} \mathbb{E} y_i y'_k.
\]

Thus \(\{y_i\}\) is allowed to be a sequence of independent (mean-zero) \(p\)-dimensional Gaussian random variables such that \(\text{cov}(y_i) = \sum_{k \in \hat{N}_i} \mathbb{E} x_i x'_k\) (provided that \(\sum_{k \in \hat{N}_i} \mathbb{E} x_i x'_k\) is positive-definite).

Remark 3.2. The arguments in the proof of Proposition 3.1 allow us to derive a non-asymptotic upper bound on \(\mathbb{E}[m^*(X) - m^*(Y)]\) for a more general function \(m^*(\cdot)\) on the high dimensional vector sum (after some suitable componentwise transformation); see Section S.5. Such general results are potentially useful in studying higher criticism test ([43]); see Example S.1 and Remark S.1.

3.2. Dependence structure I: \(M\)-dependent time series. This subsection is devoted to the analysis of \(M\)-dependent time series, which fits in the framework of dependency graph. Here, we allow \(M\) to grow slowly with the sample size \(n\). Using the arguments in the proof of Proposition 3.1, we obtain the following result for \(M\)-dependent (unnecessarily stationary) sequence.

Corollary 3.1. When \(\{x_i\}\) is a \(M\)-dependent sequence, under the assumption that \(2\sqrt{5} \beta (6M + 1) M_{xy} / \sqrt{n} \leq 1\), we have

\[
|\mathbb{E}[m(X) - m(Y)]| \lesssim (G_3 + G_2 \beta + G_1 \beta^2) \frac{(2M + 1)^2}{\sqrt{n}} (\bar{m}_{x,3}^3 + \bar{m}_{y,3}^3)
+ (G_2 + G_1 \beta)\phi(M_x, M_y) + G_1 \Delta + G_0 \mathbb{E}[1 - I].
\]
Let $n = (N + M)r$, where $N \geq M$ and $N, M, r \to +\infty$ as $n \to +\infty$. Define the block sums

$$A_{ij} = \sum_{l=0}^{iN+(i-1)M} x_{lj}, \quad B_{ij} = \sum_{l=iN+(i-1)M-N+1}^{i(N+M)} x_{lj}.$$  

(11)

It is not hard to see that $\{A_{ij}\}_{i,j=1}^{r}$ and $\{B_{ij}\}_{i,j=1}^{r}$ with $1 \leq j \leq p$ are two sequences of i.i.d random variables. Let $V_{nj} = \sqrt{V_{nj}^2 + V_{2nj}^2}$ with $V_{nj}^2 = \sum_{i=1}^{r} A_{ij}^2$ and $V_{2nj}^2 = \sum_{i=1}^{r} B_{ij}^2$. By generalizing Theorem 2.16 of de la Peña et al (2009), we obtain the following lemma.

**Lemma 3.1.** Suppose $\{x_i\}$ is a $p$-dimensional $M$-dependent sequence. Assume that there exist $a_j, b_j > 0$ such that

$$P\left(\sum_{i=1}^{n} x_{ij} > a_j\right) \leq 1/4, \quad P(V_{nj}^2 > b_j^2) \leq 1/4.$$

Then we have

$$P\left(\sum_{i=1}^{n} x_{ij} \geq x(a_j + b_j + V_{nj})\right) \leq 8 \exp(-x^2/8),$$

(12)

for any $1 \leq j \leq p$. In particular, we can choose $b_j^2 = 4E V_{nj}^2$ and $a_j^2 = 2b_j^2 = 8 EV_{nj}^2$.

It is worth noting that Lemma 3.1 holds without the stationarity assumption. This lemma is particularly useful in controlling the last two terms in (10).

Throughout the rest of this subsection, we consider the case where $\{x_i\}$ is a $M$-dependent stationary time series. Define $\gamma_{x,jk}(l) = \mathbb{E} x_{lj} x_{(l+i)k}$ for $l \geq 0$ and $\gamma_{x,jk}(l) = \gamma_{x,kj}(-l)$ for $l < 0$, where $1 \leq j, k \leq p$. Let

$$\sigma_{x,jk}(n) := \sigma_{x,jk}(M) = \sum_{l=1-n}^{n-1} (n-|l|) \gamma_{x,jk}(l)/n, \quad \sigma_{x,kj} := \sigma_{x,kj}(M) = \sum_{l=-\infty}^{\infty} \gamma_{x,jk}(l)$$

and $\sigma_{x,j}^2 = \sigma_{x,jk}(M) = \sum_{l=-\infty}^{\infty} |\gamma_{x,jk}(l)|$. Let $\varphi(M_x) := \varphi_{N,M}(M_x)$ be the smallest finite constant which that uniformly for $j$,

$$E(A_{ij} - \tilde{A}_{ij})^2 \leq N \varphi^2(M_x) \sigma_{x,j}^2, \quad E(B_{ij} - \tilde{B}_{ij})^2 \leq M \varphi^2(M_x) \sigma_{x,j}^2,$$

(13)

where $\tilde{A}_{ij}$ and $\tilde{B}_{ij}$ are the truncated versions of $A_{ij}$ and $B_{ij}$ defined as follows:

$$\tilde{A}_{ij} = \sum_{l=iN+(i-1)M-N+1}^{iN+(i-1)M} (x_{lj} \wedge M_x) \vee (-M_x),$$

$$\tilde{B}_{ij} = \sum_{l=i(N+M)-M+1}^{i(N+M)} (x_{lj} \wedge M_x) \vee (-M_x).$$

Similarly, we can define the quantity $\varphi(M_y)$ for the Gaussian sequence $\{y_i\}$. Set $\varphi(M_x, M_y) = \varphi(M_x) \vee \varphi(M_y)$. Further let $u_x(\gamma)$ and $u_y(\gamma)$ be the smallest quantities such that

$$P\left(\max_{1 \leq i \leq n} \max_{1 \leq j \leq p} |x_{ij}| \leq u_x(\gamma)\right) \geq 1 - \gamma, \quad P\left(\max_{1 \leq i \leq n} \max_{1 \leq j \leq p} |y_{ij}| \leq u_y(\gamma)\right) \geq 1 - \gamma.$$
Building on the above results, we are ready to derive an upper bound for $\rho_n$. To this end, consider a “smooth” indicator function $g_0 \in C^3(\mathbb{R}): \mathbb{R} \to [0, 1]$ such that $g_0(s) = 1$ for $s \leq 0$ and $g_0(s) = 0$ for $s \geq 1$. Fix any $t \in \mathbb{R}$ and define $g(s) = g_0(\psi(s - t - \epsilon_\beta))$ with $\epsilon_\beta = \beta^{-1} \log p$. For this function $g$, $G_0 = 1$, $G_1 \lesssim \psi$, $G_2 \lesssim \psi^2$ and $G_3 \lesssim \psi^3$. Here, $\psi$ is a smoothing parameter we will choose carefully in the proof. Corollary 3.1 and Lemma 3.1 imply the following result.

**Theorem 3.1.** Consider a $M$-dependent stationary time series $\{x_i\}$. Suppose $2\sqrt{5}\beta (6M+1) Mxy/\sqrt{n} \leq 1$ with $M_{xy} = \max\{M_x, M_y\}$, and $M_x > u_x(\gamma)$ and $M_y > u_y(\gamma)$ for some $\gamma \in (0, 1)$. Further suppose that there exist constants $0 < c_1 < c_2$ such that $c_1 < \min_{1 \leq j \leq p} \sigma_j^{(n)} \leq \max_{1 \leq j \leq p} \sigma_j^{(n)} < c_2$ uniformly holds for all large enough $n$, $M$ and $p$. Then for any $\psi > 0$,

$$\rho_n = \sup_{t \in \mathbb{R}} |P(T_X \leq t) - P(T_Y \leq t)|$$

$$\lesssim (\psi^2 + \psi \beta) \phi(M_x, M_y) + (\psi^3 + \psi^2 \beta + \psi \beta^2) \frac{(2M + 1)^2}{\sqrt{n}} (\tilde{m}_{x, \beta}^3 + \tilde{m}_{y, \beta}^3)$$

$$+ \psi \phi(M_x, M_y) \sigma_j \sqrt{8 \log(p/\gamma) + \gamma + (\epsilon_\beta + \psi^{-1}) \sqrt{1 + \log(p\psi)}}.$$ 

We point out that the stationarity assumption is non-essential in the proof of Theorem 3.1.

To characterize the dependence of $M$-dependent time series, we adopt the idea of viewing the weakly dependent time series as outputs on inputs in physical systems [40]. This framework is very general and easy to verify for specific (linear or nonlinear) data-generating mechanism; see [41]. With some abuse of notation, let $\epsilon_i$ be a sequence of mean-zero i.i.d random variables. Consider a physical system $\mathcal{G}(\ldots, \epsilon_{i-1}, \epsilon_i)$, where $\{\epsilon_i\}$ are the inputs and $\mathcal{G} = (\mathcal{G}_1, \ldots, \mathcal{G}_p)'$ is a ($p$-dimensional) measurable function such that its output is well defined. Define the sigma field $\mathcal{F}_M(i) = \sigma(\epsilon_{i-M}, \epsilon_{i-M+1}, \ldots, \epsilon_i)$ with $M \geq 0$. We suppose the $M$-dependent sequence $\{x_i\}$ has the following representation (also see the discussions in the next subsection),

$$x_i := x_i^{(M)} = \mathbb{E}[\mathcal{G}(\ldots, \epsilon_{i-1}, \epsilon_i) | \mathcal{F}_M(i)] := G^{(M)}(\epsilon_{i-M}, \epsilon_{i-M+1}, \ldots, \epsilon_i).$$

For any $l \in \mathbb{N}$, let $x_i^{(l-1)} = \mathbb{E}[x_i | \epsilon_{i-1-l}, \ldots, \epsilon_i] = \mathbb{E}[\mathcal{G}(\ldots, \epsilon_{i-1}, \epsilon_i) | \mathcal{F}_{l-1}(i)]$ for $l \leq M$, and $x_i^{(l-1)} = x_i$ for $l > M$. By construction, $x_{1j}$ and $x_{(1+l)k}$ are independent for any $1 \leq j, k \leq p$.

Let $h : [0, +\infty) \to [0, +\infty)$ be a convex and strictly increasing function with $h(0) = 0$. Denote by $h^{-1}(\cdot)$ the inverse function of $h(\cdot)$. Let $l_n := l_n(p, \gamma) = (\log(pn/\gamma))^{\gamma}$. 

**Assumption 3.1.** Suppose one of the following two conditions holds: (i) $\mathbb{E}h(\max_{1 \leq j \leq p} |x_{ij}| / D_n) \leq 1$ with $D_n > 0$, and

$$n^{3/8} M^{-1/2} l^{-5/8}_n \geq C_1 \max\{D_n h^{-1}(n/\gamma), l^{1/2}_n\}, \quad n^{7/4} M^{-1} l^{-9/4}_n \geq C_2 N,$$

for some constants $C_1, C_2 > 0$; (ii) $\max_{1 \leq j \leq p} \mathbb{E} \exp(|x_{ij}| / D_n) \leq 1$ with $D_n > 0$, and

$$n^{3/8} M^{-1/2} l^{-5/8}_n \geq C_3 \max\{D_n l_n, l^{1/2}_n\}, \quad n^{7/4} M^{-1} l^{-9/4}_n \geq C_4 N,$$
for some constants $C_3, C_4 > 0$.

**Theorem 3.2.** Assume that there exist constants $c_1, c_2, c_3 > 0$ such that

$$c_1 \leq \min_{1 \leq j \leq p} \sigma_{j, j}^{(n)} \leq \max_{1 \leq j \leq p} \sigma_{j, j}^{(n)} < c_2, \quad \max_{1 \leq j \leq p} \sigma_j^2 < c_3,$$

uniformly for all large enough $M, p$, and

$$\limsup_{p} \max_{1 \leq k \leq p} \mathbb{E}[G_k(\ldots, \epsilon_{i-1}, \epsilon_i)]^4 < \infty,$$

$$\limsup_{M, p} \max_{1 \leq k \leq p} \sum_{i=1}^M (\mathbb{E}[x_{(1+4)k} - x_{(1+4)k}]^3)^{1/3} < \infty.$$

Condition (18) also holds for $\{y_i\}$. Then under Assumption 3.1, we have

$$\rho_n = \sup_{t \in \mathbb{R}} \left| P(T_X \leq t) - P(T_Y \leq t) \right| \lesssim n^{-1/8} M^{1/2} T_n^{7/8} + \gamma.$$

Suppose $\mathbb{E}(\max_{1 \leq j \leq p} |x_{ij}|/\mathcal{D}_n)^4 \leq 1$. Then with $p \lesssim \exp(n^b)$, $M \times N \lesssim n^{b'}$, $\gamma \propto n^{-(1-4b'-7b)/8} = o(1)$, and $\mathcal{D}_n \lesssim n^{(3-12b'-13b)/32}$, we have Condition (i) in Assumption 3.1 holds with $h(x) = x^4$, and

$$\rho_n \lesssim n^{-(1-4b'-7b)/8}.$$

If Condition (ii) in Assumption 3.1 holds, we can still have (20) when $\max_{1 \leq j \leq p} \mathbb{E} \exp(|x_{ij}|/\mathcal{D}_n) \leq 1$, $p \lesssim \exp(n^b)$, $M \times N \lesssim n^{b'}$, $\gamma \propto n^{-(1-4b'-7b)/8} = o(1)$ and $\mathcal{D}_n \lesssim n^{(3-12b'-13b)/8}$.

When $b' = 0$ (i.e. $M = O(1)$), our result allows $p = O(\exp(n^b))$ with $b < 1/7$, which is consistent with Corollary 2.1 in [16] for i.i.d random vectors (assuming that $B_n = O(1)$ therein).

**Remark 3.3.** The sharpness of $\rho_n$ is not established in Theorem 3.2. However, the upper bound of $\rho_n$ given in (20) leads to two conjectures: (i) Gaussian approximation becomes less accurate when the data vectors are more dependent or the data dimension diverges at a faster rate; (ii) the less dependent of the data vectors, the faster diverging rate of the dimension is allowed for obtaining an accurate Gaussian approximation. The above phenomena will also be observed for the weakly dependent data in Section 3.3. Interestingly, we will show some empirical evidence of both conjectures in that section.

**Remark 3.4.** Assumption 3.1 and (17) impose tail restrictions on $\{x_i\}$. Condition (18) requires $\{x_i\}$ to be weakly dependent uniformly as $M$ grows, and, in particular, (18) allows us to quantify $\phi(M_x, M_y)$ and $\varphi(M_x, M_y)$; see \ref{S.10}.
3.3. Dependence structure II: Weakly dependent time series. In this subsection, we extend the results in Section 3.2 to the weakly dependent case, i.e., $D_n = n + 1$. The key idea here is to approximate the weakly dependent time series by a $M$-dependent time series, see the approximation error (24) below.

With slightly abuse of notation, suppose the sequence $\{x_i\}$ has the following causal representation,

$$x_i := x_i^{(\infty)} = G(\ldots, \epsilon_{i-1}, \epsilon_i),$$

where $G = (G_1, \ldots, G_p)'$ is a $p$-dimensional measurable function such that $x_i$ is well defined. To measure the strength of dependence, we let $\{\epsilon'_i\}$ be an i.i.d copy of $\{\epsilon_i\}$ and $x^*_i = G(\ldots, \epsilon'_0, \epsilon_1, \ldots, \epsilon_i)$, and define

$$\theta_{i,j,q}(x) = (\mathbb{E}|x_{ij} - x^*_{ij}|^q)^{1/q}, \quad \Theta_{i,j,q}(x) = \sum_{l=i}^{+\infty} \theta_{l,j,q}(x).$$

In the subsequent discussions, we assume that the dependence measure $\sup_{1 \leq j \leq p} \Theta_{i,j,q}(x) < \infty$ for some $q > 0$. Analogous quantity $\theta_{i,j,q}(y)$ can be defined for the Gaussian sequence $\{y_i\}$.

Let $x^{(M)}_i = (x^{(M)}_{i1}, \ldots, x^{(M)}_{ip})'$ be the $M$-dependent approximation sequence for $\{x_i\}$. Define $X^{(M)}$ in the same way as $X$ by replacing $x_i$ with $x^{(M)}_i$. Because $|m(x) - m(y)| \leq 2G_0$ and $|m(x) - m(y)| \leq G_1 \max_{1 \leq j \leq p} |x_j - y_j|$ (by the Lipschitz property of $F_\beta$), we have

$$||E[m(X) - m(X^{(M)})]| \leq ||E[(m(X) - m(X^{(M)}))I_M]|| + ||E[(m(X) - m(X^{(M)}))(1 - I_M)]|| \lesssim G_1 \Delta_M + G_0 E[1 - I_M],$$

where $I_M := I_{\Delta_M} = 1\{\max_{1 \leq j \leq p} |X_j - X^{(M)}_j| \leq \Delta_M\}$ for some $\Delta_M > 0$ depending on $M$. Suppose $\max_{1 \leq j \leq p} E|x_{ij}|^q < \infty$ for some $q > 0$. By Lemma A.1 of [29], we have

$$\mathbb{E}|X_j - X^{(M)}_j|^q \lesssim C_q n^{1-q'/2} q' \Theta_{M,j,q}(x),$$

where $q' = \min(2, q)$ and $C_q$ is a positive constant depending on $q$. For any $q \geq 2$, we obtain

$$\mathbb{E}[1 - I_M] \lesssim \sum_{j=1}^{p} P(|X_j - X^{(M)}_j| \geq \Delta_M) \leq \sum_{j=1}^{p} \frac{1}{\Delta_M^q} \mathbb{E}|X_j - X^{(M)}_j|^q \lesssim \sum_{j=1}^{p} \frac{C_q^2 \Theta_{M,j,q}(x)}{\Delta_M^q} = \sum_{j=1}^{p} \frac{C_q^2}{\Delta_M^q} \left( \sum_{l=M}^{+\infty} \theta_{l,j,q}(x) \right)^q.$$

Optimizing the bound with respect to $\Delta_M$ in (23), we deduce that

$$||E[m(X) - m(X^{(M)})]| \lesssim (G_0 C_1)^{1/(1+q)} \left( \sum_{j=1}^{p} \Theta_{M,j,q}(x) \right)^{1/(1+q)}.$$
which along with (8) implies that
\[
|\mathbb{E}[g(T_X) - g(T_{X^{(M)})}]| \lesssim (G_0 G_1^q)^{1/(1+q)} \left( \sum_{j=1}^p \Theta_{M,j,q}^q(x) \right)^{1/(1+q)} + \beta^{-1} G_1 \log p,
\]
with \( T_{X^{(M)}} = \max_{1 \leq j \leq p} \sum_{i=1}^n x_{ij}^{(M)} / \sqrt{n}. \)

We give an explicit expression of the approximation error (24) in the following two examples.

**Example 3.1.** Consider a stationary linear process,
\[
x_{ij} = \sum_{l=0}^{+\infty} b_{lj} \epsilon_{(i-1)j}, \quad 1 \leq j \leq p,
\]
where \( \sum_{l=0}^{+\infty} |b_{lj}| < \infty \) and \( \epsilon_i = (\epsilon_1, \ldots, \epsilon_p)' \) is a sequence of i.i.d random variables. Simple calculation yields that \( X_j - X_j^{(M)} = \frac{1}{\sqrt{n}} \sum_{i=1}^n \sum_{l=0}^{+\infty} b_{lj} \epsilon_{(i-1)j} \) and \( \theta_{i,j,q}(x) = |b_{lj} (|\mathbb{E}|\epsilon_{0j} - \epsilon'_{0j}|)q^{1/4}. \) For \( q \geq 2, \) we have
\[
|\mathbb{E}[m(X) - m(X^{(M)})]| \lesssim (G_0 G_1^q)^{1/(1+q)} \max_{1 \leq j \leq p} (\mathbb{E}|\epsilon_{0j} - \epsilon'_{0j}|q^{1/(q+1)} \left( \sum_{j=1}^p \left( \sum_{l=M}^{+\infty} |b_{lj}| \right)^q \right)^{1/(q+1)}.
\]
Under the assumption that \( \limsup_p \max_{1 \leq j \leq p}(\mathbb{E}|\epsilon_{0j}|q^{1/q}) < \infty \) and \( b_{lj} = \rho^l \) with \( \rho < 1, \) we get
\[
|\mathbb{E}[m(X) - m(X^{(M)})]| \lesssim (G_0 G_1^q)^{1/(1+q)} p^{1/(1+q)} \rho^{(q^{(q+1)})/(1+q)}.
\]

**Example 3.2.** Consider a stationary Markov chain defined by an iterated random function
\[
x_i = H(x_{i-1}, e_i).
\]
Here \( e_i \)'s are i.i.d. innovations, and \( H(\cdot, \cdot) \) is an \( \mathbb{R}^p \)-valued and jointly measurable function, which satisfies the following two conditions: (i) there exists some \( x_0 \) such that \( \mathbb{E}|H(x_0, e_0)|^{2q} < \infty \) and (ii)
\[
\rho := \sup_{x \neq x'} \frac{(\mathbb{E}|H(x, e_0) - H(x', e_0)|^{2q})^{1/(2q)}}{|x - x'|} < 1,
\]
where \( |\cdot| \) denotes the Euclidean norm for a \( p \)-dimensional vector. Then it can be shown that \( \{x_i\} \) has the geometric moment contraction (GMC) condition property [42] and \( \max_{1 \leq j \leq p} \Theta_{m,j,2q}(x) = O(\rho^{-m}) \) (see Example 2.1 in [42]). Hence
\[
|\mathbb{E}[m(X) - m(X^{(M)})]| \lesssim (G_0 G_1^q)^{1/(1+q)} p^{1/(1+q)} \rho^{(q^{(q+1)})/(1+q)}.
\]
We are now ready to present the main result. Recall that \( h(\cdot) \) and \( l_n \) are defined in Section 3.2.
Theorem 3.3. Suppose \( \{x_i\} \) is a stationary time series which admits the representation (21). Assume that \( \max_{1 \leq j \leq p} \mathbb{E} x_{ij}^4 < C_1 \), and

\[
(25) \quad c_1 < \min_{1 \leq j \leq p} \sigma_{j,j}^{(n)} \leq \max_{1 \leq j \leq p} \sigma_{j,j}^{(n)} < c_2, \quad \max_{1 \leq j \leq p} \sigma_j^2 < c_3,
\]

\[
(26) \quad \max_{1 \leq k \leq p} \{ \theta_{j,k,3}(x) \vee \theta_{j,k,3}(y) \} \leq \ell_j \text{ with } \sum_{j=1}^{+\infty} \ell_j < \infty,
\]

for some constants \( C_1, c_3 > 0 \) and \( 0 < c_1 < c_2 \). Suppose that there exist \( N \) and \( M \) such that \( N \geq M \) and Assumption 3.1 is fulfilled. Then for \( q \geq 2 \), we have

\[
(27) \quad \rho_n \lesssim n^{-1/8}M^{1/2}l_n^{7/8} + \gamma + (n^{1/8}M^{-1/2}l_n^{-3/8}q/(1+q)) \left( \sum_{j=1}^{p} \Theta_{M,j,q}^{(j)} \right)^{1/(1+q)},
\]

where \( \Theta_{i,j,q} = \Theta_{i,j,q}(x) \vee \Theta_{i,j,q}(y) \).

The approximation parameter \( M \) will be chosen appropriately to optimize the bound (27). The Gaussian sequence \( \{y_t\} \) can be constructed as a causal linear process (e.g. based on the Wold representation theorem) to capture the second order property of \( \{x_i\} \).

We note that the conditions in Theorem 3.3 can be categorized into two types: tail restriction and weak dependence assumption. Assumption 3.1 and the condition that \( \max_{1 \leq j \leq p} \mathbb{E} x_{ij}^4 < C_1 \) impose restrictions on the tails of \( \{x_{ij}\}_{j=1}^{p} \) uniformly across \( j \), while conditions (25)-(26) essentially require weak dependence uniformly across all the components of \( \{x_i\} \). When \( \max_{1 \leq j \leq p} \Theta_{M,j,q} = O(p^M) \) for \( p < 1 \), we have

\[
(28) \quad \rho_n \lesssim n^{-1/8}M^{1/2}l_n^{7/8} + \gamma + (n^{1/8}M^{-1/2}l_n^{-3/8}q/(1+q)) p^{1/(1+q)} \rho^{(qM)/(1+q)},
\]

Suppose \( p \lesssim \exp(n^b) \) for some \( 0 \leq b < 1/11 \), and \( \mathbb{E}(\max_{1 \leq j \leq p} |x_{ij}|^4 / \mathcal{D}_n) \lesssim 1 \). Then by choosing \( M \asymp N \lesssim n^{\beta} \) with \( 4\beta + 7b < 1 \) and \( 1 > \beta' > b \), \( \gamma \asymp n^{-(4\beta'-7b)/8} = o(1) \) and assuming that \( \mathcal{D}_n \lesssim n^{(3-12\beta'-13b)/32} \), Condition (i) in Assumption 3.1 holds with \( h(x) = x^4 \), and

\[
\rho_n \lesssim n^{-(4\beta'-7b)/8}.
\]

The same conclusion holds under condition (ii) in Assumption 3.1 provided that \( p \lesssim \exp(n^b) \), \( M \asymp N \lesssim n^{\beta} \), \( \gamma \asymp n^{-(1-4\beta'-7b)/8} = o(1) \) and \( \mathcal{D}_n \lesssim n^{(3-4\beta'-13b)/8} \) with \( 1 > \beta' > b \).

Below we provide some empirical evidence for two conjectures proposed in Remark 3.3, in particular the interplay between dependence and dimensionality. To this end, we generate \( \{x_i\} \) from a multivariate ARCH model \( x_i = \Sigma_i^{1/2} \epsilon_i \), where \( \epsilon_i = (\epsilon_{i1}, \ldots, \epsilon_{ip})' \) with \( \sqrt{2} \epsilon_{ij} \) being a sequence of i.i.d. \( t(4) \) random variables, and \( \Sigma_i = (1 - \beta_0)D_p + \beta_0 x_{i-1}x_{i-1}' \) with \( \Sigma_i^{1/2} \) being a lower triangular matrix based on the Cholesky decomposition of \( \Sigma_i \). Here \( D_p = (d_{ij})_{i,j=1}^{p} \) with \( d_{jj} = 1 \) and \( d_{ij} = 0.5 \) for \( i \neq j \). Notice that \( \{x_i\} \) are uncorrelated and \( \text{cov}(x_i) = D_p \). To capture the second order property
of \{x_i\}, we generate independent Gaussian vectors \{y_i\} from \(N(0, D_p)\). Figure 1 illustrates the interplay between dependence and dimensionality using the P-P plots for \(n = 60, p = 100, 300, 500\), and \(\beta_0 = 0, 0.2, 0.5\). For moderate \(p\) and \(\beta_0\), the Gaussian approximation is reasonably good, which is consistent with our theory. Moreover, we also observe the following phenomena. On one hand, as \(p\) increases, the approximation deteriorates for the same \(\beta_0\) which controls the strength of dependence; on the other hand, for fixed \(p\), the approximation becomes worse in the right tail which is most relevant for practical applications, as \(\beta_0\) increases. Note that our theoretical results are finite sample valid, and thus the sample size supposed not to play any role here. Hence, we believe that the less dependent of the data vectors, the faster diverging rate of the dimension is allowed for obtaining an accurate Gaussian approximation.

![Interplay between dependence and dimensionality: P-P plots comparing distributions of \(T_X\) and \(T_Y\).](image)

In the end, we discuss an intriguing question: is there any so-called “dimension free dependence structure”? In other words, what kind of dependence assumption will not affect the dimension increase rate (as compared to the independence case in [16])? To address this question, we consider one possibility: the original \(p\)-dimensional vector can be decomposed into two components namely one times series component and one independence component, where the former component is asymptotically ignorable comparing to the latter as \(n\) grows. Our contribution here is to precisely characterize such a “dimension free” dependence structure.

**Proposition 3.2.** Consider a \(p\)-dimensional time series \(\{x_i\}\). Suppose there exists a permutation \(\pi(\cdot)\) such that \((x_{i\pi(1)}, \ldots, x_{i\pi(p)}) = (z'_i, z''_i)'\), where \(\{z_i\} = \{z'_i\} = \{z''_i\}\) is a \(q\)-dimensional (possibly nonstationary) time series and \(\{z''_i\}\) is a \(p - q\) dimensional sequence of independent variables. Suppose \(\{z'_i\}\) and \(\{z''_i\}\) are independent. When \(\{z''_i\}\) satisfies the assumptions in Corollary 2.1 of [16], we have

\[
\sup_{z \in \mathbb{R}} \left| P \left( \max_{q+1 \leq j \leq p} X_{\pi(j)} \leq z \right) - P \left( \max_{q+1 \leq j \leq p} Y_{\pi(j)} \leq z \right) \right| \lesssim n^{-c}, \quad c > 0. \tag{29}
\]
Recall that $X_{\pi(j)}$ is $\sum_{i=1}^n x_{i\pi(j)}/\sqrt{n}$ and $Y_{\pi(j)}$ is defined in a similar manner. Then under the additional assumption that

$$qn^{-c} + q/E \max_{q+1 \leq j \leq p} Y_j = O(n^{-c'}) \quad c' > 0,$$

and $\max_{1 \leq j \leq p} E|X_j|^2 < +\infty$, we have

$$\sup_{z \in \mathbb{R}} \left| P \left( \max_{1 \leq i \leq p} X_i \leq z \right) - P \left( \max_{1 \leq i \leq p} Y_i \leq z \right) \right| \lesssim n^{-c''}, \quad c'' > 0.$$  

The additional assumption (30) implies that $q$ is of a polynomial order w.r.t. $n$ while $(p - q)$ achieves the exponential order as specified in Corollary 2.1 of [16]. Therefore, the largest possible diverging rate of $p$ allowed in Proposition 3.2 remains the same as that in the independence case ([16]). The independence assumption between $\{z_{i1}\}$ and $\{z_{i2}\}$ might be relaxed. Here, we assume it mainly for technical simplicity so that only one single dependence assumption $\max_{1 \leq j \leq q} E|X_{\pi(j)}|^2 < \infty$ needs to be imposed on $\{z_{i1}\}$.

4. Bootstrap Inference. In practice, the intrinsic dependence structure of time series data is usually unknown. Hence, the Gaussian approximation theory becomes too restrictive to use. However, this general theory provides a foundation in developing the bootstrap inference theory that do not require such knowledge. In this section, we consider two types of bootstrap procedures: (i) blockwise multiplier bootstrap; and (ii) non-overlapping block bootstrap. The former is employed in Section 2, while the latter is a more flexible alternative.

4.1. Blockwise multiplier bootstrap. To approximate the quantiles of $T_X$, we introduce a blockwise multiplier bootstrap procedure for $M$-dependent and weakly dependent time series considered in Sections 3.2 and 3.3. Suppose $n = (N + M)r$, where $N \geq M$ and $N, M, r \to +\infty$ as $n \to +\infty$. Let $\{(e_i, \tilde{e}_i)\}$ be a sequence of i.i.d $N(0, I_2)$ variables that are independent of $\{x_i\}$. Define

$$T_D = \max_{1 \leq j \leq p} \frac{1}{\sqrt{n}} \sum_{i=1}^r D_{ij}, \quad D_{ij} = A_{ij}e_i + B_{ij}\tilde{e}_i.$$  

Recall the definitions of $A_{ij}$ and $B_{ij}$ in (11). Conditional on $\{x_i\}$, $D_{ij}$ are mean-zero Gaussian random variables such that

$$\text{cov}(D_{ij}, D_{i'k}) = \delta_{ii'}(A_{ij}A_{i'k} + B_{ij}B_{i'k}), \quad \delta_{ii'} = 1\{i = i'\}.$$  

Thus we have

$$\text{cov} \left( \sum_{i=1}^r D_{ij}/\sqrt{n}, \sum_{i=1}^r D_{ik}/\sqrt{n} \right) = \frac{1}{n} \sum_{i=1}^r (A_{ij}A_{ik} + B_{ij}B_{ik}).$$

Conditional on the sample $\{x_i\}_{i=1}^n$, define the $\alpha$-quantile of $T_D$ as

$$c_{T_D}(\alpha) := \inf \{ t \in \mathbb{R} : P(T_D \leq t|\{x_i\}_{i=1}^n) \geq \alpha \}.$$
Our goal below is to quantify

\[(36) \quad \tilde{\rho}_n := \sup_{\alpha \in (0,1)} |P(T_X \leq c_{T_\alpha}(\alpha)) - \alpha|.\]

To this end, consider the estimation errors

\[(37) \quad E_A := \max_{1 \leq j,k \leq p} \left| \frac{1}{n} \sum_{i=1}^r A_{ij} A_{ik} / N - \sigma_{j,k}^{(n)} \right|,
\quad E_B := \max_{1 \leq j,k \leq p} \left| \frac{1}{n} \sum_{i=1}^r B_{ij} B_{ik} / M - \sigma_{j,k}^{(n)} \right|,
\quad E_{AB} := \max_{1 \leq j,k \leq p} \left| \frac{1}{n} \sum_{i=1}^r (A_{ij} A_{ik} + B_{ij} B_{ik}) - \sigma_{j,k}^{(n)} \right|,
\]

where \(\sigma_{j,k}^{(n)} = \frac{1}{n} \sum_{l=1-n}^{n-1} (n - |l|) \gamma_{x,jk}(l)\). Recall that \(h(\cdot)\) is a nondecreasing convex function with \(h(0) = 0\). Define the Orlicz norm as

\[||X||_h = \inf \left\{ B > 0 : \mathbb{E} h \left( \frac{|X|}{B} \right) \leq 1 \right\}.\]

We first consider \(M\)-dependent stationary sequence where \(M\) is allowed to grow with the sample size \(n\). Define the following quantities which characterize the higher order properties of the time series (e.g., \(\tilde{\sigma}_{x,N}^2\) and \(\varsigma_{x,N}\) below characterize the fourth order property of \(\{x_t\}\)),

\[\tilde{\sigma}_{x,N}^2 = \max_{1 \leq j \leq p} \left\{ \frac{1}{N} \sum_{i=1}^{+\infty} |\text{cum}(x_{ij}, x_{i2j}, x_{i3j}, x_{0j})| + \sigma_j^4 \right\},\]
\[\varsigma_{x,N} = \left( \mathbb{E} \max_{1 \leq j \leq p} \left| \sum_{i=1}^N x_{ij} / \sqrt{N} \right|^4 \right)^{1/4},\]
\[\varsigma_{x,h,N} = \max_{1 \leq j \leq p} \left| \sum_{i=1}^N x_{ij} / \sqrt{N} \right|_h, \quad \varsigma_x = \max_{1 \leq j \leq p} \sum_{l=-\infty}^{+\infty} |l| |\mathbb{E} x_{i,j} x_{i+l,k}|,
\]

where \(\text{cum}\) denotes the cumulant (see e.g. [5]) and \(\sigma_j^2 = \sum_{l=-\infty}^{+\infty} |\gamma_{x,jj}(l)|\).

The following lemma plays an important role in the subsequent derivations.

**Lemma 4.1.** Suppose \(\{x_t\}\) is a \(M\)-dependent stationary sequence. Then with \(h(x) = \exp(x) - 1\),

\[\mathbb{E} E_A \lesssim \tilde{\sigma}_{x,N} \sqrt{\log p / r} + \log p \{\log(rp)\}^2 \varsigma_{x,h,N} / r + \varsigma_x / N,
\quad \mathbb{E} E_B \lesssim \tilde{\sigma}_{x,M} \sqrt{\log p / r} + \log p \{\log(rp)\}^2 \varsigma_{x,h,M} / r + \varsigma_x / M.\]

Alternatively, we have

\[\mathbb{E} E_A \lesssim \tilde{\sigma}_{x,N} \sqrt{\log p / r} + \log p \varsigma_{x,N}^2 / \sqrt{r} + \varsigma_x / N,
\quad \mathbb{E} E_B \lesssim \tilde{\sigma}_{x,M} \sqrt{\log p / r} + \log p \varsigma_{x,M}^2 / \sqrt{r} + \varsigma_x / M.\]
Let \( c_{TV}(\alpha) = \inf\{t \in \mathbb{R} : P(T_Y \leq t) \geq \alpha\} \). In the spirit of Lemma 3.2 in [16], we can show that when \( c_1 < \min_{1 \leq j \leq p} \sigma_{j,j}^{(n)} \leq \max_{1 \leq j \leq p} \sigma_{j,j}^{(n)} < c_2 \) for some \( 0 < c_1 < c_2 \),

\[
P(c_{TV}(\alpha) \leq c_{TV}(\alpha + \pi(\nu))) \geq 1 - P(E_{AB} > \nu),
\]

\[
P(c_{TV}(\alpha) \leq c_{TV}(\alpha + \pi(\nu))) \geq 1 - P(E_{AB} > \nu),
\]

where \( \pi(\nu) = C\nu^{1/3}(1 \lor \log(p/\nu))^{2/3} \) for some constant \( C > 0 \) depending on \( c_1, c_2 \). Using the arguments in Theorem 3.1 of [16], it is not hard to show that

\[
\sup_{\alpha \in (0,1)} |P(T_X \leq c_{TD}(\alpha)) - \alpha| \leq \rho_n + \pi(\nu) + P(E_{AB} > \nu).
\]

Because \( \mathbb{E}E_{AB} \leq \mathbb{E}E_A + \mathbb{E}E_B \), we deduce that

\[
\tilde{\rho}_n := \sup_{\alpha \in (0,1)} |P(T_X \leq c_{TD}(\alpha)) - \alpha| \leq \rho_n + \nu^{1/3}(1 \lor \log(p/\nu))^{2/3} + \mathbb{E}E_A/\nu + \mathbb{E}E_B/\nu.
\]

**Assumption 4.1.** Suppose \( p \leq \exp(n^b) \) with \( 0 \leq b < 1/15 \). Set \( M \leq n^{b'} \) and \( N \leq n^{b''} \) with \( 1 > b'' \geq b', 4b' + 7b < 1 \) and \( b' > 2b \). Assume that \( \mathcal{D}_n \leq n^{(3 - 12b' - 13b)/32} \) under Condition (i) in Assumption 3.1 with \( h(x) = x^4 \) or \( \mathcal{D}_n \leq n^{(3 - 4b' - 13b)/8} \) under Condition (ii) in Assumption 3.1. Further assume that one of the following two conditions holds.

**Condition 1:** \( \bar{s}_x,M \vee \bar{s}_x,N \leq n^{s_1}, \zeta_x,h,M \vee \zeta_x,h,N \leq n^{s_2}/2, \bar{\omega}_x \leq n^{s_3}, \) where \( h(x) = \exp(x) - 1 \) and \( s_1, s_2, s_3 \) satisfy that

\[
s_b := ((1 - 5b - b'')/2 - s_1) \land (1 - 5b - b'' - s_2) \land (b' - 2b - s_3) > 0.
\]

**Condition 2:** \( \bar{s}_x,M \vee \bar{s}_x,N \leq n^{s_1}, \zeta_x,M \vee \zeta_x,N \leq n^{s_2}/2, \bar{\omega}_x \leq n^{s_3}, \) and \( s_1, s_2, s_3 \) satisfy that

\[
s'_b := ((1 - 5b - b'')/2 - s_1) \land ((1 - 6b - b'')/2 - s'_2) \land (b' - 2b - s_3) > 0.
\]

We are now in position to present the first main result in this section.

**Theorem 4.1.** Consider a \( M \)-dependent stationary time series \( \{x_i\} \). Under the assumptions in Theorem 3.2 and Assumption 4.1,

\[
\sup_{\alpha \in (0,1)} |P(T_X \leq c_{TD}(\alpha)) - \alpha| \leq \begin{cases} n^{-c}, & c = \min\{s_b/4, (1 - 4b' - 7b)/8\}, \\ n^{-c'}, & c' = \min\{s'_b/4, (1 - 4b' - 7b)/8\}, \end{cases}
\]

under Condition 1,

under Condition 2.

Our next theorem extends the above result to weakly dependent stationary time series.
Theorem 4.2. Consider a weakly dependent stationary time series \( \{x_i\} \). Suppose \( \max_{1 \leq j \leq p} \Theta_{M,j,q} = O(\rho^M) \) for \( \rho < 1 \) and some \( q \geq 4 \). Then under the assumptions in Theorem 3.3 and Assumption 4.1,

\[
(41) \quad \sup_{\alpha \in (0, 1)} |P(T_X \leq c_{TD}(\alpha)) - \alpha| \lesssim \begin{cases} n^{-c}, & c = \min\{s_b/4, (1 - 4b' - 7b)/8\}, \\ n^{-c'}, & c' = \min\{s_b'/4, (1 - 4b' - 7b)/8\}, \\ \end{cases}
\]
under Condition 1,

\[
\quad n^{-c'}, & c' = \min\{s_b'/4, (1 - 4b' - 7b)/8\},
\]
under Condition 2.

Remark that the results of Theorems 4.1 and 4.2 are still valid even when \( p \) is fixed or \( p \) grows slower than the exponential rate required in Assumption 4.1.

Remark 4.1. When \( \{x_i\} \) has the so-called geometric moment contraction (GMC) property (uniformly across its components), we have \( \bar{\sigma} _{x,M} \vee \bar{\sigma} _{x,N} \lesssim 1 \) (i.e., \( s_1 = 0 \)) by Proposition 2 of [42] and the assumption that \( \max_j \sigma_j < \infty \).

Remark 4.2. It is known that in the low dimensional setting, the tapered block bootstrap method yields an improvement over the block bootstrap in terms of the bias for variance estimation, and thus provides a better MSE rate; see [31]. Hence, we may also want to combine the blockwise multiplier bootstrap method proposed here with the data tapering scheme. For example, let \( K: \mathbb{R} \to \mathbb{R} \) be a data taper with \( K(x) = 0 \) for \( x \notin [0, 1) \). One can consider the following modification,

\[
T_{K,D} = \max_{1 \leq j \leq p} \frac{1}{\sqrt{n}} \sum_{i=1}^r D_{K,ij}, \quad D_{K,ij} = A_{K,ij}e_i + B_{K,ij}\tilde{e}_i,
\]

\[
A_{K,ij} = \sum_{l=iN+(i-1)M+1}^{iN+(i-1)M} K\left(\frac{l - (i - 1)(N + M)}{N}\right) x_{lj},
\]

\[
B_{K,ij} = \sum_{l=iN+(i-1)M+1}^{iN+M} K\left(\frac{l - iN + (i - 1)M}{M}\right) x_{lj}.
\]

More detailed investigation along this direction is left for future study.

4.2. Non-Overlapping Block bootstrap. In this subsection, we propose an alternative bootstrap procedure in the high dimensional setting: non-overlapping block bootstrap ([10]). In general, this bootstrap procedure may avoid estimating the influence function (defined in Section 5) in contrast with blockwise multiplier bootstrap. We provide theoretical justifications for this procedure through establishing its equivalence with multiplier bootstrap; see (42).

Assume for simplicity that \( n = b_nl_n \), where \( b_n, l_n \in \mathbb{Z} \). Conditional on the sample \( \{x_i\}_{i=1}^n \), we let \( \varphi_1, \ldots, \varphi_n \) be i.i.d uniform random variables on \( \{0, \ldots, l_n - 1\} \) and define \( x^*_{(j-1)b_n + i} = x_{\varphi_jb_n + i} \) with
\begin{align*}
1 \leq j \leq l_n \text{ and } 1 \leq i \leq b_n. \text{ In other words, } \{x_i^n\}_{i=1}^n \text{ is a non-overlapping block bootstrap sample with block size } b_n. \text{ Define}
\begin{align*}
T_{X^*} &= \max_{1 \leq j \leq p} \frac{1}{\sqrt{n}} \sum_{i=1}^n (x_{ij}^* - \bar{x}_{nj}) = \max_{1 \leq j \leq p} \frac{1}{\sqrt{n}} \sum_{i=1}^{l_n} (A_{ij}^* - \bar{A}_{nj}),
\end{align*}
where \( \bar{x}_{nj} = \sum_{i=1}^n x_{ij}/n, \) \( \bar{A}_{nj} = \sum_{i=1}^{l_n} A_{ij}/l_n, \) and \( A_{ij} = \sum_{t=(i-1)b_n+1}^{ib_n} x_{tj}, \) and \( A_{ij}^*, \ldots, A_{ln}^* \) are i.i.d draws from the empirical distribution of \( A_{1j}, \ldots, A_{l_nj}. \) Also define
\begin{align*}
T_{\bar{X}} &= \max_{1 \leq j \leq p} \frac{1}{\sqrt{n}} \sum_{i=1}^{l_n} A_{ij} e_i,
\end{align*}
where \( \{e_i\}_{i=1}^{l_n} \) is a sequence of i.i.d \( N(0,1). \) Throughout the following discussions, we suppose that \( b_n \geq M. \) The theoretical validity of the multiplier bootstrap based on \( T_{\bar{X}} \) can be justified using similar arguments in the previous subsection because the same arguments go through when \( A_{ij} \) and \( B_{ij} \) are replaced by \( A_{ij} \) (provided that \( b_n \geq M). \) By showing that with probability \( 1 - Cn^{-c}, \)
\begin{align*}
\sup_{t \in \mathbb{R}} |P(T_{X^*} \leq t | \{x_i^n\}_{i=1}^n) - P(T_{\bar{X}} \leq t | \{x_i^n\}_{i=1}^n)| \lesssim n^{-c'}, \quad c' > 0,
\end{align*}
we establish the validity of non-overlapping block bootstrap in Theorem 4.3.

**Assumption 4.2.** Assume that \( \bar{\sigma}_{x,b_n} \sqrt{\log p} / l_n \lesssim n^{-c_0} \) and \( \zeta_{x,h,b_n}^2 \{\log(l_n)\}^9 / l_n \lesssim n^{-c_0'} \) with \( h(x) = \exp(x) - 1, \) where \( c_0, c_0' > 0. \)

**Theorem 4.3.** Suppose that \( c_1 < \min_{1 \leq j \leq p} \sigma_{j,j}^{(b_n)} \leq \max_{1 \leq j \leq p} \sigma_{j,j}^{(b_n)} < c_2 \) and \( \max_{1 \leq j \leq p} \sigma_{j,j}^{(b_n)} \leq c_3 \) for some constants \( 0 < c_1 < c_2 < \infty \) and \( c_3 > 0, \) where \( \sigma_{j,j}^{(b_n)} = \sum_{t=1}^{b_n-1} (b_n - |t|) \gamma_{x,j,j}^{(b_n)}(|t|)/b_n. \) Further assume that the assumptions in Theorem 4.1 or Theorem 4.2 hold with \( M = N = b_n \) and \( r = n/(2b_n). \) Then (42) holds with probability \( 1 - Cn^{-c} \) for some \( c, C > 0. \) Moreover, we have
\begin{align*}
\sup_{\alpha \in (0,1)} \left| P(T_{X^*} \leq c \alpha) - \alpha \right| \lesssim n^{-c''}, \quad c'' > 0.
\end{align*}
where \( c_{T_{X^*}}(\alpha) = \inf\{t \in \mathbb{R} : P(T_{X^*} \leq t | \{x_i^n\}_{i=1}^n) \geq \alpha \} \) and \( c'' > 0. \)

**5. General Inferential Theory.** In this section, we establish a general framework of conducting bootstrap inference for high dimensional time series based on the theoretical results in Section 4. This general framework assumes that the \( q \)-dimensional quantity of interest, denoted as \( \Theta_{0q}, \) admits an approximately linear expansion, and thus covers three examples considered in Section 2. In particular, \( \Theta_{0q} \) is expressed as a functional of the distribution of a \( p \)-dimensional weakly dependent stationary time series \( \{u_i\}. \)

\footnote{Note that \( p \) here is different from the dimension of \( x_i \) discussed in previous sections.}
5.1. Approximately linear statistics. In this subsection, we consider the quantities that can be expressed as functionals of the marginal distribution of a block time series with length \(d_0\): \(\{v_i\}_{i=1}^{N_0}\), where \(v_i := (u_i, \ldots, u_{i+d_0-1})'\) and \(N_0 = n - d_0 + 1\). Here, we allow the integer \(d_0\) to grow with \(n\). Define \(F_{d_0}^{N_0} = \frac{1}{N_0} \sum_{i=1}^{N_0} \delta_{v_i/N_0}\) as the empirical distribution for \(\{v_i\}_{i=1}^{N_0}\). The distribution function of \(v_i\) is denoted as \(F_{d_0}\). We are interested in testing the parameter \(\Theta_{q_0} = (\theta_0, \ldots, \theta_{q_0})' = T(F_{d_0})\) for some functional \(T := T_{q_0,d_0}\). The parameter dimension \(q_0\) depends on either \(p\) or \(d_0\), e.g., \(q_0 = p, p^2\) or \(d_0 p^2\). A natural estimator for \(\Theta_{q_0}\) is then given by \(\hat{\Theta}_{q_0} = (\hat{\theta}_1, \ldots, \hat{\theta}_{q_0})' = T(F_{d_0}^{N_0})\).

Assume \(\hat{\Theta}_{q_0}\) admits the following approximately linear expansion in a neighborhood of \(F_{d_0}\):

\[
\hat{\Theta}_{q_0} = \Theta_{q_0} + \frac{1}{N_0} \sum_{i=1}^{N_0} IF(v_i, F_{d_0}) + R_{N_0},
\]

where \(IF(v_i, F_{d_0}) = (IF_1(v_i, F_{d_0}), \ldots, IF_{q_0}(v_i, F_{d_0}))'\) is called “influence function” (see e.g. [21]) and \(R_{N_0} := R_{N_0}(v_1, \ldots, v_{N_0}) = (R_{1,N_0}, \ldots, R_{q_0,N_0})'\) is a remainder term. Examples of approximately linear statistics include various location and scale estimators for the marginal distribution of \(\{u_i\}\), von Mises statistics and \(M\)-estimators of time series models (see [24]).

We are interested in testing the null hypothesis \(H_0 : \Theta_{q_0} = \bar{\Theta}_{q_0}\) versus the alternative \(H_a : \Theta_{q_0} \neq \bar{\Theta}_{q_0}\), where \(\bar{\Theta}_{q_0} = (\bar{\theta}_1, \ldots, \bar{\theta}_{q_0})'\). The test is proposed as

\[
\phi(\hat{\Theta}_{q_0}; c(\alpha)) = \begin{cases} 
1, & \max_{1 \leq j \leq q_0} \sqrt{N_0} |\hat{\theta}_j - \bar{\theta}_j| \geq c(\alpha), \\
0, & \text{otherwise.}
\end{cases}
\]

We next apply the bootstrap theory in Section 4 to obtain the critical value \(c(\alpha)\). Specifically, we define \(x_i = (IF(v_i, F_{d_0})', -IF(v_i, F_{d_0})')'\) and \(\bar{x}_i = (\hat{IF}(v_i, F_{d_0}^{N_0})', -\hat{IF}(v_i, F_{d_0}^{N_0})')'\), where \(\hat{IF}(v_i, F_{d_0}^{N_0})\) is some estimate of \(IF(v_i, F_{d_0})\). Suppose \(N_0 = (N_1 + M_1) r_1\), where \(N_1 \geq M_1\) and \(N_1, M_1, r_1 \to +\infty\) as \(N_0 \to +\infty\). Define the estimated block sums

\[
\hat{A}_{ij} = \sum_{l = (i-1)(N_1+M_1)+1}^{iN_1+(i-1)M_1} \hat{x}_{lj}, \quad \hat{B}_{ij} = \sum_{l = iN_1+(i-1)M_1+1}^{i(N_1+M_1)} \hat{x}_{lj},
\]

where \(1 \leq i \leq r_1\) and \(1 \leq j \leq 2q_0\). Let

\[
T_D = \max_{1 \leq j \leq 2q_0} \frac{1}{\sqrt{n}} \sum_{i=1}^{r_1} \hat{D}_{ij},
\]

where \(\hat{D}_{ij} = \hat{A}_{ij} - \hat{A}_{ij} \hat{e}_i + \hat{B}_{ij} \hat{e}_i\) with \(\{(e_i, \hat{e}_i)\}\) being a sequence of i.i.d \(N(0, I_2)\) independent of \(\{u_i\}\). The bootstrap critical value is given by

\[
c_1(\alpha) := \inf \{t \in \mathbb{R} : P(T_D \geq t | \{x_i\}_{i=1}^n) \geq 1 - \alpha\}.
\]

We next justify the validity of the test in (45) with \(c(\alpha) = c_1(\alpha)\) in Theorems 5.1 and 5.2.
ASSUMPTION 5.1. Assume that $P(\max_{1 \leq j \leq \varrho_0} \sqrt{N_0} |R_{ji}N_0| > C_1 n^{-c_1}/\sqrt{\log(2\varrho_0)}) < C_1 n^{-c_1}$ and $P(\mathcal{E}_{AB}\{\log(2\varrho_0)\})^2 > C_2 n^{-c_2}$, where $c_1, C_1, c_2, C_2 > 0$, and

$$
\mathcal{E}_{AB} = \max_{1 \leq j \leq 2\varrho_0} \left| \frac{1}{n} \sum_{i=1}^{r} \{(A_{ij} - \hat{A}_{ij})^2 + (B_{ij} - \hat{B}_{ij})^2\} \right|,
$$

with $A_{ij} = \sum_{t=(i-1)(N_1+M_1)+1}^{iN_1+(i-1)M_1+1} x_{ij}$ and $B_{ij} = \sum_{t=(i-1)(N_1+M_1)+1}^{iN_1+(i-1)M_1+1} x_{lj}$.

THEOREM 5.1. Suppose the assumptions in Theorem 4.1 or Theorem 4.2 hold for $\{x_i\}$, where $p$ is replaced by $2\varrho_0$. Then under Assumption 5.1 and $H_0$, we have

$$
(48) \quad \sup_{\alpha \in (0,1)} \left| P \left( \max_{1 \leq j \leq \varrho_0} \sqrt{N_0} |\theta_j - \hat{\theta}_j| \geq c_1(\alpha) \right) - \alpha \right| \lesssim n^{-c}, \quad c > 0.
$$

Theorem 5.1 applies directly to the methods described in Sections 2.1-2.2 for both M-dependent and weakly dependent stationary time series. For example, consider the white noise testing problem in Section 2.2. Suppose $\mathbb{E}u_i = 0$. In this example, $\Theta_{\varrho_0} = (\text{vec}(\gamma_u(1))', \ldots, \text{vec}(\gamma_u(L))')'$ with $\gamma_u(h) = \mathbb{E}u_i u_i', h$ and $q_0 = Lp^2$. Then we have $IF(v_i, F_d) = \nu_i - \Theta_{\varrho_0}$ and $\hat{IF}(v_i, F_d N_0) = \nu_i - \sum_{i=1}^{N_0} \nu_i/n$ with $\nu_i = (\text{vec}(u_i u_i')', \ldots, \text{vec}(u_i u_i'+L)')'$ and $N_0 = n - L$. Note that the bootstrap procedures considered in Section 2 are in fact simplified versions of the blockwise multiplier bootstrap in Section 4 with $N = M = b_n$ and $r = l_n/2$.

Our next theorem covers the problem of testing the bandedness of covariance matrix in Section 2.3. Recall that

$$
T_{\text{band}} = \max_{|j - k| \geq 1} \frac{1}{n} \left| \sum_{i=1}^{n} (u_{ij} u_{ik})/\sqrt{\hat{\gamma}_{u,ij}(0)\hat{\gamma}_{u,ik}(0)} \right|,
$$

where $\hat{\gamma}_{u,jk}(0) = \sum_{i=1}^{n} u_{ij} u_{ik}/n$. With some abuse of notation, let $x_i = (\tilde{u}_{i1}, \tilde{u}_{i1}, \ldots, \tilde{u}_{i1}, \tilde{u}_{ip}, \ldots, \tilde{u}_{ip}, \tilde{u}_{i1}, \ldots, \tilde{u}_{ip}, \tilde{u}_{ip})$ with $\tilde{u}_{ij} = u_{ij}/\sqrt{\gamma_{u,ij}(0)}$.

THEOREM 5.2. Suppose the assumptions in Theorem 4.1 or Theorem 4.2 hold for $\{x_i\}$, where $p$ is replaced by the cardinality of the set $\{1 \leq j, k \leq p : |j - k| \geq \iota\}$. Then under Assumption S.1 in the supplementary material and $H_0$, we have

$$
(49) \quad \sup_{\alpha \in (0,1)} \left| P \left( T_{\text{band}} \geq c_{\text{band}}(\alpha) \right) - \alpha \right| \lesssim n^{-c}, \quad c > 0,
$$

where $c_{\text{band}}(\alpha)$ is given in Section 2.3.

The proof of Theorem 5.2 is similar as that of Theorem 5.1, and thus skipped. In Section S.4, we show that Assumption S.1 can be verified under suitable primitive conditions.

To avoid direct estimation of the influence function, we may alternatively apply the non-overlapping block bootstrap procedure in Section 4.2. Assume for simplicity that $N_0 = b_n l_n$, where $b_n, l_n \in \mathbb{Z}$. Let $\varrho_1, \ldots, \varrho_n$ be i.i.d uniform random variables on $\{0, \ldots, l_n - 1\}$ and define $v_{(j-1)b_n+i} = v_{\varrho_i b_n+i}$.
with $1 \leq j \leq l_n$ and $1 \leq i \leq b_n$. Compute the block bootstrap estimate $\hat{\Theta}_{q_0}^*$ based on the bootstrap sample $\{v^*_i\}_{i=1}^{N_0}$. Let $c_2(\alpha)$ be the $100(1-\alpha)$th quantile of the distribution of $\max_{1 \leq j \leq q_0} \sqrt{N_0} |\tilde{\theta}_j - \tilde{\theta}_j|$ conditional on the sample $\{u_i\}$. In what follows, we further justify the validity of the non-overlapping block bootstrap in the same framework.

**Assumption 5.2.** Assume that

$$P \left( P \left( \sqrt{N_0} \max_{1 \leq j \leq q_0} |R_{jN_0}^* - R_{jN_0}| > C_3 n^{-c_3} / \sqrt{\log(2q_0)} \right\} \right) > C_4 n^{-c_4},$$

where $R_{N_0}^* = (R_{1N_0}^*, \ldots, R_{q_0N_0}^*) = R_{N_0}(v_1^*, \ldots, v_N^*)$, and $c_3, C_3, c_4, C_4 > 0$.

**Theorem 5.3.** Suppose the assumptions in Theorem 4.3 hold for $\{x_i\}$, where $p$ is replaced by $2q_0$. Then under Assumptions 5.1-5.2, we have

$$P \left( \max_{1 \leq j \leq q_0} \sqrt{N_0} |\hat{\theta}_j - \tilde{\theta}_j| \geq c_2(\alpha) \right) - \alpha \lesssim n^{-c}, \quad c > 0. \tag{50}$$

**Remark 5.1.** An alternative way to construct the uniform confidence band or perform hypothesis testing is based on the studentized statistic. For example, let $\hat{\sigma}_j^2$ be a consistent estimator of $\lim_{n \to \infty} N_0 \text{var}(\hat{\theta}_j)$. Then the uniform confidence band can be constructed as

$$\left\{ \Theta_{q_0} = (\theta_1, \ldots, \theta_{q_0})' \in \mathbb{R}^{q_0} : \max_{1 \leq j \leq q_0} \sqrt{N_0} |\hat{\theta}_j - \tilde{\theta}_j| / \hat{\sigma}_j \leq \hat{c}(\alpha) \right\}. $$

The blockwise multiplier bootstrap or non-overlapping block bootstrap can be modified accordingly to obtain the critical value $\hat{c}(\alpha)$.

5.2. Extension to infinite dimensional parameters. To broaden the applicability of our method, we extend the above results to cover infinite dimensional parameters that are functionals of the joint distribution of $\{u_i\}_{i \in \mathbb{Z}}$, denoted as $F_\infty$. A typical example is the spectral quantities that depend on the distribution of the whole time series rather than any finite dimensional distribution; see Example 5.1. Hence, the extension in this section is useful in conducting inference for the spectrum of high dimensional time series.

Suppose $\Theta_{q_0} = (\theta_1, \ldots, \theta_{q_0})' = \mathcal{T}_\infty(F_\infty)$ and its estimator is $\hat{\Theta}_{q_0} := \hat{\Theta}_{q_0}(u_1, \ldots, u_n) = (\hat{\theta}_1, \ldots, \hat{\theta}_{q_0})'$. Again, $q_0$ is allowed to grow with $n$ or $p$. Assume that there exists a sequence of approximating statistics for $\hat{\Theta}_{q_0}$ that is a functional of $\theta_n$-dimensional empirical distribution, and a sequence of approximating (non-random) quantities $\tilde{\Theta}_{q_0} = (\tilde{\theta}_1, \ldots, \tilde{\theta}_{q_0})'$ for $\Theta_{q_0}$. Then our bootstrap method as proposed in Section 5.1 still works provided that these two approximation errors can be well controlled and similar regularity conditions hold for the expansion of the approximating statistics around $\tilde{\Theta}_{q_0}$, i.e., (51). To be more precise, we impose the following assumption.
Assumption 5.3. For a sequence of positive integers $\vartheta_n$ that grow with $n$, let $v_i,\vartheta_n = (u_i, \ldots, u_i + \vartheta_n - 1)$ with $i = 1, 2, \ldots, N_0, \vartheta_n := n - \vartheta_n + 1$. Assume the expansion,

$$\mathcal{T}_{\vartheta_n}(F_{\vartheta_n}^{N_0, \vartheta_n}) := (\mathcal{T}_{1,\vartheta_n}(F_{\vartheta_n}^{N_0, \vartheta_n}), \ldots, \mathcal{T}_{q_0,\vartheta_n}(F_{\vartheta_n}^{N_0, \vartheta_n}))'$$

$$= \tilde{\Theta}_0 + \frac{1}{n} \sum_{i=1}^{N_0, \vartheta_n} IF(v_i, \vartheta_n, F_{\vartheta_n}) + R_{N_0, \vartheta_n},$$

where $R_{N_0, \vartheta_n} = (R_{1,N_0, \vartheta_n}, \ldots, R_{q_0,N_0, \vartheta_n})'$ is a remainder term. Denote $\Upsilon_{j,\vartheta_n} = |\mathcal{R}_{j,N_0, \vartheta_n}| + |\tilde{\theta}_j - \mathcal{T}_{j,\vartheta_n}(F_{\vartheta_n}^{N_0, \vartheta_n})|$. Suppose that

$$P\left(\max_{1 \leq j \leq q_0} \sqrt{N_0} \Upsilon_{j,\vartheta_n} > C_1 n^{-c_1}/\sqrt{\log(2q_0)}\right) < C_1 n^{-c_1},$$

and $n^{c_1} \sqrt{\log(2q_0)} \max_{1 \leq j \leq q_0} \sqrt{N_0} |\tilde{\theta}_j - \theta_j| = o(1)$ for some $c_1, C_1 > 0$.

We next illustrate the validity of expansion (51) using a spectral mean example.

Example 5.1. Consider the spectral mean $G(F_u, \phi) = \int_{-\pi}^{\pi} \text{tr}(\phi(\lambda)F_u(\lambda))d\lambda$, where $\text{tr}$ denotes the trace of a square matrix, $F_u(\cdot)$ is the spectral density of $\{u_t\}$ and $\phi(\cdot) : [-\pi, \pi] \to \mathbb{R}^{p \times p}$. For simplicity, assume that $\mathbb{E}u_t = 0$. Suppose the quantity of interest is $\Theta_0 = (G(F_u, \phi_1), \ldots, G(F_u, \phi_{q_0}))'$ with $\phi_k(\cdot) : [-\pi, \pi] \to \mathbb{R}^{p \times p}$ for $1 \leq k \leq q_0$. Here $\Theta_0$ can be interpreted as the projection of the spectral density matrix onto $q_0$ directions defined by $\phi_k(\cdot)$ with $1 \leq k \leq q_0$. A sample analogue of $F_u(\lambda)$ is the periodogram $\mathcal{I}_{n,u}(\lambda) = (2\pi n)^{-1} \sum_{i=1}^{n} u_i u_i' \exp(\text{i}(i-j)\lambda)$ with $\text{i} = \sqrt{-1}$. Then a plug-in estimator for $\Theta_0$ is given by $\tilde{\Theta}_0 = (G(\mathcal{I}_{n,u}, \phi_1), \ldots, G(\mathcal{I}_{n,u}, \phi_{q_0}))'$. Letting $\hat{\Gamma}_{n,h} = \sum_{k=-h}^{h} u_{j+h} u_{j}'/n$, then $G(\mathcal{I}_{n,u}, \phi_k) = \sum_{h=1-h}^{h} \text{tr}(\hat{\phi}_{hk} \hat{\Gamma}_{n,h})$ with $\hat{\phi}_{hk} = \int_{-\pi}^{\pi} \phi_k(\lambda) \exp(\text{i}h\lambda)d\lambda/(2\pi)$. Consider the approximating quantity $\hat{\theta}_j = \sum_{h=1-h}^{h} \text{tr}(\hat{\phi}_{hk} \hat{\Gamma}_{h})$ with $\hat{\Gamma}_{h} = \mathbb{E}u_{j+h} u_{j}'$. It is then straightforward to see that

$$\mathcal{T}_{\vartheta_n}(F_{\vartheta_n}^{N_0, \vartheta_n}) := \sum_{h=1-h}^{h} \text{tr}(\hat{\phi}_{hk} \hat{\Gamma}_{h}) = \tilde{\theta}_j + \frac{1}{n} \sum_{i=1}^{N_0, \vartheta_n} IF(v_i, \vartheta_n, F_{\vartheta_n}) + R_{j,N_0, \vartheta_n},$$

where $IF(v_i, \vartheta_n, F_{\vartheta_n}) = \sum_{h=1-h}^{h} \text{tr}(\hat{\phi}_{hk}(u_{i+h} u_{i}' - \Gamma_{h}))$ and $R_{j,N_0, \vartheta_n}$ is the corresponding remainder term.

Recall that $\Theta_{q_0} = \mathcal{T}_{\infty}(F_\infty)$ with $F_\infty$ being the joint distribution of $\{u_t\}_{t \in \mathbb{Z}}$. The statistic for testing the null hypothesis $H_0 : \Theta_{q_0} = \tilde{\Theta}_{q_0}$ versus the alternative $H_0 : \Theta_{q_0} \neq \tilde{\Theta}_{q_0}$, where $\tilde{\Theta}_{q_0} = (\tilde{\theta}_1, \ldots, \tilde{\theta}_{q_0})'$, is given by

$$\max_{1 \leq j \leq q_0} \sqrt{N_0} |\tilde{\theta}_j - \tilde{\theta}_j| \geq c(\alpha).$$

With some abuse of notation, we now define $x_i := x_{i,n} = (IF(v_i, \vartheta_n, F_{\vartheta_n})', -IF(v_i, \vartheta_n, F_{\vartheta_n})')'$ and $\widehat{x}_i = (IF(v_i, \vartheta_n, F_{\vartheta_n}^{N_0, \vartheta_n})', -IF(v_i, \vartheta_n, F_{\vartheta_n}^{N_0, \vartheta_n})')'$ with $IF(v_i, \vartheta_n, F_{\vartheta_n}^{N_0, \vartheta_n})'$ being some estimate of
IF\left(v_{i,\vartheta_n},F_{\vartheta_n}\right)$ (note that in this case $\{x_{in}\}_{i=1}^{N_0,\vartheta_n}$ is an array). Suppose $N_0,\vartheta_n = (N_1,\vartheta_n + M_1,\vartheta_n)_{r_1,\vartheta_n}$. We can define $\hat{A}_{ij}$ and $\hat{B}_{ij}$ in a similar way as before (see (46)), where $1 \leq i \leq r_1,\vartheta_n$ and $1 \leq j \leq 2q_0$. Let

$$T_{\hat{D}} = \max_{1 \leq j \leq 2q_0} \frac{1}{\sqrt{n}} \sum_{i=1}^{r_1,\vartheta_n} \hat{D}_{ij},$$

where $\hat{D}_{ij} = \hat{A}_{ij}c_i + \hat{B}_{ij}\bar{c}_i$ with $\{(c_i,\bar{c}_i)\}$ being a sequence of i.i.d $N(0, I_2)$ independent of $\{u_i\}$. The bootstrap critical value is then given by

$$(54) \quad c_1(\alpha) := \inf\{t \in \mathbb{R} : P(T_{\hat{D}} \leq t|\{x_{i,\vartheta_n}\}_{i=1}^{\vartheta_n}) \geq 1 - \alpha\}.$$ 

Following the arguments in the proof of Theorem 5.1, we obtain the following result.

**Theorem 5.4.** Suppose Assumption 5.3 holds and the assumptions in Theorem 4.1 or Theorem 4.2 are satisfied for $\{x_i\}$, where $p$ is replaced by $2q_0$. Assume in addition that $P(\mathcal{E}_{AB}|\log(2q_0)|^2 > C_2n^{-c_2}) \leq C_2n^{-c_2}$, where $c_2, C_2 > 0$, and

$$\mathcal{E}_{AB} = \max_{1 \leq j \leq 2q_0} \left\{ \frac{1}{n} \sum_{i=1}^{r} (A_{ij} - \hat{A}_{ij})^2 + (B_{ij} - \hat{B}_{ij})^2 \right\},$$

with $A_{ij} = \sum_{l=iN_1,\vartheta_n+(i-1)M_1,\vartheta_n}^{iN_1,\vartheta_n+M_1,\vartheta_n} x_{lj}$ and $B_{ij} = \sum_{l=i(N_1,\vartheta_n+M_1,\vartheta_n)-(M_1,\vartheta_n)+1}^{i(N_1,\vartheta_n+M_1,\vartheta_n)} x_{lj}$. Then we have for some $c > 0$,

$$(55) \quad \sup_{\alpha \in (0,1)} P\left( \max_{1 \leq j \leq 2q_0} \sqrt{N_0}\hat{\theta}_j - \bar{\theta}_j \geq c_1(\alpha) \right) - \alpha \leq n^{-c}.$$ 

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Supplementary Material

Throughout the supplementary material, define the generic constants $C$ and $C'$ that are independent of $n$ and $p$. For a set $\mathcal{A}$, denote by $|\mathcal{A}|$ its cardinality.

S.1. Proofs of the main results in Section 3.

**Proof of Proposition 3.1.** Define $Z(t) = \sum_{i=1}^n Z_i(t)$ with the Slepian interpolation $Z_i(t) = (\sqrt{t}x_i + \sqrt{1-t}y_i)/\sqrt{n}$ and $0 \leq t \leq 1$. Let $\Psi(t) = \mathbb{E}m(Z(t))$. Define $V^{(i)}(t) = \sum_{j \in \tilde{N}_i} Z_j(t)$ and $Z^{(i)}(t) = Z(t) - V^{(i)}(t)$. Write $\partial_j m(x) = \partial m(x)/\partial x_j$, $\partial_{jk} m(x) = \partial^2 m(x)/\partial x_j \partial x_k$ and $\partial_{jk}m(x) = \partial^3 m(x)/\partial x_j \partial x_k \partial x_l$ for $j,k,l = 1,2,\ldots,p$, where $x = (x_1, x_2, \ldots, x_p)'$. Note that

$$
\mathbb{E}m(\tilde{X}) - \mathbb{E}m(\tilde{Y}) = \Psi(1) - \Psi(0) = \int_0^1 \Psi'(t) dt = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^p \int_0^1 \mathbb{E}[\partial_j m(Z(t))\tilde{Z}_{ij}(t)] dt \tag{S.1}
$$

$$
= \frac{1}{2}(I_1 + I_2 + I_3),
$$

where $\tilde{Z}_{ij}(t) = \{\tilde{x}_{ij}/\sqrt{t} - \tilde{y}_{ij}/\sqrt{1-t}\}/\sqrt{n}$, and

$$
I_1 = \sum_{i=1}^n \sum_{j=1}^p \int_0^1 \mathbb{E}[\partial_j m(Z^{(i)}(t))\tilde{Z}_{ij}(t)] dt,
$$

$$
I_2 = \sum_{i=1}^n \sum_{k,j=1}^p \int_0^1 \mathbb{E}[\partial_k \partial_j m(Z^{(i)}(t))\tilde{Z}_{ij}(t)V^{(i)}(t)] dt,
$$

$$
I_3 = \sum_{i=1}^n \sum_{k,l,j=1}^p \int_0^1 \int_0^1 (1 - \tau) \mathbb{E}[\partial_k \partial_l m(Z^{(i)}(t)) + \tau V^{(i)}(t))\tilde{Z}_{ij}(t)V^{(i)}(t)V^{(i)}(t)] dt d\tau.
$$

Using the fact that $Z^{(i)}(t)$ and $\tilde{Z}_{ij}(t)$ are independent, and $\mathbb{E}\tilde{Z}_{ij}(t) = 0$, we have $I_1 = 0$. To bound the second term, define the expanded neighborhood around $N_i$,

$$
N_i = \{j : \{j,k\} \in E_n \text{ for some } k \in N_i\},
$$

and $Z^{(i)}(t) = Z(t) - \sum_{t \in N_i \cup \tilde{N}_i} Z_t(t) = Z^{(i)}(t) - V^{(i)}(t)$, where $V^{(i)}(t) = \sum_{t \in N_i \setminus \tilde{N}_i} Z_t(t)$ with $N_i \setminus \tilde{N}_i =$
where $U$ with $X. ZHANG AND G. CHENG$

\{k \in \mathcal{N}_i : k \notin \tilde{\mathcal{N}}_i\}$. By Taylor expansion, we have

\[
I_2 = \sum_{i=1}^{n} \sum_{k,j=1}^{p} \int_0^1 \mathbb{E}[\partial_k \partial_j m(\mathcal{Z}^{(i)}(t)) \dot{Z}_{ij}(t) V_k^{(i)}(t)] dt \\
+ \sum_{i=1}^{n} \sum_{k,j,l=1}^{p} \int_0^1 \int_0^1 \mathbb{E}[\partial_k \partial_j \partial_l m(\mathcal{Z}^{(i)}(t) + \tau \mathcal{V}^{(i)}(t)) \dot{Z}_{ij}(t) V_k^{(i)}(t) V_l^{(i)}(t)] dt \, d\tau \\
= \sum_{i=1}^{n} \sum_{k,j=1}^{p} \int_0^1 \mathbb{E}[\partial_k \partial_j m(\mathcal{Z}^{(i)}(t)) \mathbb{E}[\dot{Z}_{ij}(t) V_k^{(i)}(t)] dt \\
+ \sum_{i=1}^{n} \sum_{k,j,l=1}^{p} \int_0^1 \int_0^1 \mathbb{E}[\partial_k \partial_j \partial_l m(\mathcal{Z}^{(i)}(t) + \tau \mathcal{V}^{(i)}(t)) \dot{Z}_{ij}(t) V_k^{(i)}(t) V_l^{(i)}(t)] dt \, d\tau \\
= I_{21} + I_{22},
\]

where we have used the fact that $\dot{Z}_{ij}(t) V_k^{(i)}(t)$ and $\mathcal{Z}^{(i)}(t)$ are independent.

Let $M_{xy} = \max\{M_x, M_y\}$. By the assumption that $2\sqrt{5} \beta D_n^2 M_{xy} / \sqrt{n} \leq 1$,

\[
\max_{1 \leq j \leq p} \left| \sum_{l \in \mathcal{N}_i \cup \tilde{\mathcal{N}}_i} Z_{lj}(t) \right| \leq \max_{1 \leq j \leq p} \sum_{l \in \mathcal{N}_i \cup \tilde{\mathcal{N}}_i} |Z_{lj}(t)| \leq D_n^2 \sup_{t \in [0,1]} (2\sqrt{t} + \sqrt{1-t}) M_{xy} / \sqrt{n} \\
\leq \sqrt{5} D_n^2 M_{xy} / \sqrt{n} \leq \beta^{-1} / 2 \leq \beta^{-1},
\]

where the second inequality comes from the facts that $|\tilde{x}_{ij}| \leq 2M_{xy}$, $|\tilde{y}_{ij}| \leq M_{xy}$ and $|\mathcal{N}_i \cup \tilde{\mathcal{N}}_i| \leq D_n^2$. By Lemma A.5 in [16], we have for every $1 \leq j, k, l \leq p$,

\[
|\partial_j \partial_k m(z)| \leq U_{jk}(z), \quad |\partial_j \partial_k \partial_l m(z)| \leq U_{jk l}(z),
\]

where $U_{jk}(z)$ and $U_{jk l}(z)$ satisfy that

\[
\sum_{j,k=1}^{p} U_{jk}(z) \leq (G_2 + 2G_1 \beta), \quad \sum_{j,k,l=1}^{p} U_{jk l}(z) \leq (G_3 + 6G_2 \beta + 6G_1 \beta^2),
\]

with $G_k = \sup_{z \in \mathbb{R}} |\partial^k g(z) / \partial z^k|$ for $k \geq 0$. Along with Lemma A.6 in [16], we obtain

\[
|I_{21}| \leq \sum_{i=1}^{n} \sum_{k,j=1}^{p} \int_0^1 \mathbb{E}[U_{jk}(\mathcal{Z}^{(i)}(t))] \mathbb{E}[\dot{Z}_{ij}(t) V_k^{(i)}(t)] dt \\
\lesssim \sum_{i=1}^{n} \sum_{k,j=1}^{p} \int_0^1 \mathbb{E}[U_{jk}(\mathcal{Z}(t))] \mathbb{E}[\dot{Z}_{ij}(t) V_k^{(i)}(t)] dt \\
\lesssim (G_2 + G_1 \beta) \int_0^1 \max_{1 \leq j \leq p} \sum_{i=1}^{n} \mathbb{E}[\dot{Z}_{ij}(t) V_k^{(i)}(t)] dt.
\]
Since $2\sqrt{5}D_n^2 M_{xy}/\sqrt{n} \leq 1$, we have

$$|I_{22}| \leq \sum_{i=1}^{n} \sum_{k,j,l=1}^{p} \int_{0}^{1} \int_{0}^{1} \mathbb{E}[|\partial_k \partial_j \partial_l m(Z^{(i)}(t) + \tau V^{(i)}(t)) \cdot \hat{Z}_{ij}(t)V^{(i)}_k(t)V^{(i)}_l(t)|]dtd\tau$$

$$\leq \sum_{i=1}^{n} \sum_{k,j,l=1}^{p} \int_{0}^{1} \int_{0}^{1} \mathbb{E}[U_{kjl}(Z(t))|\hat{Z}_{ij}(t)V^{(i)}_k(t)V^{(i)}_l(t)||dtd\tau$$

$$\leq \sum_{i=1}^{n} \sum_{k,j,l=1}^{p} \int_{0}^{1} \mathbb{E}\left[\sum_{1 \leq k,j,l \leq p} U_{kjl}(Z(t)) \max_{1 \leq k,j,l \leq p} \sum_{i=1}^{n} |\hat{Z}_{ij}(t)V^{(i)}_k(t)V^{(i)}_l(t)|\right]dtd\tau$$

(S.3) $\lesssim (G_3 + G_2\beta + G_1\beta^2) \int_{0}^{1} \mathbb{E} \max_{1 \leq k,j,l \leq p} \sum_{i=1}^{n} |\hat{Z}_{ij}(t)V^{(i)}_k(t)V^{(i)}_l(t)||dtd\tau$.

To bound the integration on (S.3), we let $w(t) = 1/(\sqrt{t} \wedge \sqrt{1-t})$ and note that

$$\int_{0}^{1} \mathbb{E} \max_{1 \leq k,j,l \leq p} \sum_{i=1}^{n} |\hat{Z}_{ij}(t)V^{(i)}_k(t)V^{(i)}_l(t)|dt$$

$$\leq \int_{0}^{1} \mathbb{E} \max_{1 \leq k,j,l \leq p} \left( \sum_{i=1}^{n} |\hat{Z}_{ij}(t)|^3 \right)^{1/3} \left( \sum_{i=1}^{n} |V^{(i)}_k(t)|^3 \right)^{1/3} \left( \sum_{i=1}^{n} |V^{(i)}_l(t)|^3 \right)^{1/3} dt$$

$$\leq \int_{0}^{1} w(t) \left( \mathbb{E} \max_{1 \leq k,j,l \leq p} \sum_{i=1}^{n} |\hat{Z}_{ij}(t)/w(t)|^3 \mathbb{E} \max_{1 \leq k,j,l \leq p} \sum_{i=1}^{n} |V^{(i)}_k(t)|^3 \mathbb{E} \max_{1 \leq l \leq p} \sum_{i=1}^{n} |V^{(i)}_l(t)|^3 \right)^{1/3} dt.$$
We first consider the term \( \mathbb{E} \max_{1 \leq j \leq p} \sum_{i=1}^{n} |\hat{Z}_{ij}(t)/w(t)|^3 \). Using the fact that \( |\hat{Z}_{ij}(t)/w(t)| \leq (|\bar{x}_{ij}| + |\bar{y}_{ij}|)/\sqrt{n} \), we get

\[
\mathbb{E} \max_{1 \leq j \leq p} \sum_{i=1}^{n} |\hat{Z}_{ij}(t)/w(t)|^3 \lesssim \frac{1}{n^{3/2}} \mathbb{E} \max_{1 \leq j \leq p} \sum_{i=1}^{n} (|\bar{x}_{ij}| + |\bar{y}_{ij}|) \lesssim \frac{1}{\sqrt{n}} (m_{x,3}^3 + m_{y,3}^3).
\]

On the other hand, notice that

\[
\mathbb{E} \max_{1 \leq k \leq p} \sum_{i=1}^{n} |V^{(i)}_k(t)|^3 \leq D_n^2 \mathbb{E} \max_{1 \leq k \leq p} \sum_{i=1}^{n} \sum_{j \in N_i} |Z_{jk}(t)|^3 \lesssim \frac{D_n^2}{n^{3/2}} \mathbb{E} \max_{1 \leq k \leq p} \sum_{i=1}^{n} \sum_{j \in N_i} (|\bar{x}_{ij}|^3 + |\bar{y}_{ij}|^3)
\]

\[
\lesssim \frac{D_n^2}{\sqrt{n}} (m_{x,3}^3 + m_{y,3}^3).
\]

Similarly, we have

\[
\mathbb{E} \max_{1 \leq l \leq p} \sum_{i=1}^{n} |V^{(i)}_l(t)|^3 \leq D_n^4 \mathbb{E} \max_{1 \leq l \leq p} \sum_{i=1}^{n} \sum_{j \in N_i} |Z_{jl}(t)|^3 \leq \frac{D_n^4}{n^{3/2}} \mathbb{E} \max_{1 \leq l \leq p} \sum_{i=1}^{n} \sum_{j \in N_i} (|\bar{x}_{jl}|^3 + |\bar{y}_{jl}|^3)
\]

\[
\lesssim \frac{D_n^4}{\sqrt{n}} (m_{x,3}^3 + m_{y,3}^3).
\]

Note that \( \int_0^1 w(t) dt \lesssim 1 \). Summarizing the above results, we have

\[
I_2 \lesssim (G_2 + G_1 \beta) \phi(M_x, M_y) + (G_3 + G_2 \beta + G_1 \beta^2) \frac{D_n^3}{\sqrt{n}} (m_{x,3}^3 + m_{y,3}^3),
\]

\[
I_3 \lesssim (G_3 + G_2 \beta + G_1 \beta^2) \frac{D_n^2}{\sqrt{n}} (m_{x,3}^3 + m_{y,3}^3).
\]

Alternatively, we can bound \( I_3 \) in the following way. By Lemmas A.5 and A.6 in [16], we have

\[
|I_3| = \sum_{i=1}^{n} \sum_{k,j,l=1}^{p} \int_0^1 \int_0^1 (1 - \tau) \mathbb{E} [\partial_t \partial_k \partial_j m(Z^{(i)}(t) + \tau V^{(i)}(t)) \hat{Z}_{ij}(t) V^{(i)}_k(t) V^{(i)}_l(t)] dt d\tau
\]

\[
\lesssim \sum_{i=1}^{n} \sum_{k,j,l=1}^{p} \int_0^1 \mathbb{E} [U_{kjl}(Z^{(i)}(t))] \mathbb{E} [\hat{Z}_{ij}(t) V^{(i)}_k(t) V^{(i)}_l(t)] dt
\]

\[
\lesssim \sum_{i=1}^{n} \sum_{k,j,l=1}^{p} \int_0^1 \mathbb{E} [U_{kjl}(Z(t))] \mathbb{E} [\hat{Z}_{ij}(t) V^{(i)}_k(t) V^{(i)}_l(t)] dt
\]

\[
\leq n(G_3 + G_2 \beta + G_1 \beta^2) \int_0^1 w(t) \max_{1 \leq k,j,l \leq p} (\mathbb{E} |\hat{Z}_{ij}(t)/w(t)|^3)^{1/3} (\mathbb{E} |V^{(i)}_k(t)|^3)^{1/3} (\mathbb{E} |V^{(i)}_l(t)|^3)^{1/3} dt.
\]

Notice that

\[
\max_{1 \leq j \leq p} \mathbb{E} |\hat{Z}_{ij}(t)/w(t)|^3 \leq \frac{1}{n^{3/2}} \max_{1 \leq j \leq p} \mathbb{E} (|\bar{x}_{ij}| + |\bar{y}_{ij}|)^3 \lesssim \frac{1}{n^{3/2}} (m_{x,3}^3 + m_{y,3}^3).
\]

It is not hard to see that

\[
\max_{1 \leq k \leq p} \mathbb{E} |V^{(i)}_k(t)|^3 \leq D_n^2 \max_{1 \leq k \leq p} \mathbb{E} \sum_{j \in N_i} |Z_{jk}(t)|^3 \lesssim \frac{D_n^3}{n^{3/2}} (m_{x,3}^3 + m_{y,3}^3).
\]
Thus we derive that

\[ I_3 \lesssim (G_3 + G_2\beta + G_1\beta^2) \frac{D_n^2}{\sqrt{n}} (\tilde{m}_{x,3} + \tilde{m}_{y,3}). \]

Therefore, we obtain

\[
|\mathbb{E}[m(\tilde{X}) - m(\tilde{Y})]| \lesssim (G_2 + G_1\beta)\phi(M_x, M_y) + (G_3 + G_2\beta + G_1\beta^2) \frac{D_n^3}{\sqrt{n}} (m_{x,3} + m_{y,3}) \\
+ (G_3 + G_2\beta + G_1\beta^2) \frac{D_n^2}{\sqrt{n}} (\tilde{m}_{x,3} + \tilde{m}_{y,3}).
\]

Using the above arguments, we can show that

\[ I_{22} \lesssim (G_3 + G_2\beta + G_1\beta^2) \frac{D_n^3}{\sqrt{n}} (\tilde{m}_{x,3} + \tilde{m}_{y,3}) \]

provided that \(2\sqrt{5}\beta D_n^3 M_{xy}/\sqrt{n} \leq 1\). This proves the last statement of Proposition 3.1.

Note that \(|m(x) - m(y)| \leq 2G_0\) and \(|m(x) - m(y)| \leq G_1\max_{1 \leq j \leq p} |x_j - y_j|\) with \(x = (x_1, \ldots, x_p)'\) and \(y = (y_1, \ldots, y_p)'\). So

\[
|\mathbb{E}[m(X) - m(\tilde{X})]| \leq |\mathbb{E}[(m(X) - m(\tilde{X}))(1 - I)]| + |\mathbb{E}[(m(X) - m(\tilde{X}))(1 - I)]|
\]

\[
\lesssim G_1\Delta + G_0\mathbb{E}[1 - I], \\
\mathbb{E}[m(Y) - m(\tilde{Y})] \lesssim G_1\Delta + G_0\mathbb{E}[1 - I].
\]

The conclusion follows by combining (S.5), (S.6) and (S.7). \(\diamondsuit\)

**Proof of Corollary 3.1.** Notice that \(D_n = 2M + 1\), \(|\mathcal{N}_i| \leq 2M + 1\) and \(|\mathcal{N}_i \cup \mathcal{N}_j| \leq 4M + 1\). Define the \(\mathcal{N}_i = \{j : (j, k) \in E_n\ \text{for some} \ k \in \mathcal{N}_j\}\). Then \(|\mathcal{N}_i \cup \mathcal{N}_j \cup \mathcal{N}_k| \leq 6M + 1\). Following the arguments in the proof of Proposition 3.1, we can show that

\[
\max_{1 \leq i \leq p} \mathbb{E}[\mathcal{V}^{(t)}_i (t)] \lesssim \frac{D_n^3}{n^{3/2}} (\tilde{m}_{x,3} + \tilde{m}_{y,3}),
\]

which implies that

\[ I_{22} \lesssim (G_3 + G_2\beta + G_1\beta^2) \frac{D_n^2}{\sqrt{n}} (\tilde{m}_{x,3} + \tilde{m}_{y,3}). \]

The conclusion follows from the proof of Proposition 3.1. \(\diamondsuit\)

**Proof of Lemma 3.1.** We only need to prove the result for \(x > 1\) as the inequality holds trivially for \(x < 1\). Suppose that the distributions of \(A_i\) and \(B_i\) are both symmetric, then we have

\[
P\left( \sum_{i=1}^n x_{ij} > xV_{nj} \right) \leq P\left( \sum_{i=1}^r (A_{ij} + B_{ij}) > xV_{nj} \right)
\]

\[
\leq P\left( \sum_{i=1}^r A_{ij} > xV_{nj}/2 \right) + P\left( \sum_{i=1}^r B_{ij} > xV_{nj}/2 \right)
\]

\[
\leq P\left( \sum_{i=1}^r A_{ij} > xV_{nj}/2 \right) + P\left( \sum_{i=1}^r B_{ij} > xV_{nj}/2 \right)
\]

\[
\leq 2 \exp(-x^2/8),
\]
where we have used Theorem 2.15 in [17].

Let \( \{ \xi_{ij} \}_{i=1}^{n} \) be an independent copy of \( \{ x_{ij} \}_{i=1}^{n} \) in the sense that \( \{ \xi_{ij} \}_{i=1}^{n} \) have the same joint distribution as that for \( \{ x_{ij} \}_{i=1}^{n} \), and define \( V_{n_j}^r \) \((A'_{ij} \text{ and } B'_{ij})\) in the same way as \( V_{n_j} \) \((A_{ij} \text{ and } B_{ij})\) by replacing \( \{ x_{ij} \}_{i=1}^{n} \) with \( \{ \xi_{ij} \}_{i=1}^{n} \). Following the arguments in the proof of Theorem 2.16 in [17], we deduce that for \( x > 1 \),

\[
\begin{align*}
\left\{ \sum_{i=1}^{n} x_{ij} &> x(a_j + b_j + V_{n_j}), \sum_{i=1}^{n} \xi_{ij} \leq a_j, V_{n_j}' \leq b_j \right\} \\
\subset & \left\{ \sum_{i=1}^{n} (x_{ij} - \xi_{ij}) \geq x(a_j + b_j + V_{n_j}) - a_j, V_{n_j}' \leq b_j \right\} \\
\subset & \left\{ \sum_{i=1}^{n} (x_{ij} - \xi_{ij}) \geq x(a_j + b_j + V_{n_j}^r - V_{n_j}' - a_j, V_{n_j}' \leq b_j \right\} \\
\subset & \left\{ \sum_{i=1}^{n} (x_{ij} - \xi_{ij}) \geq xV_{n_j}^r \right\},
\end{align*}
\]

where we have used the fact that

\[
V_{n_j}^r = \left[ \sum_{i=1}^{r} (A_{ij} - A'_{ij})^2 + \sum_{i=1}^{r} (B_{ij} - B'_{ij})^2 \right] \leq V_{n_j} + V_{n_j}'.
\]

We note that \( A_{ij} - A'_{ij} \) and \( B_{ij} - B'_{ij} \) are symmetric, and

\[
P \left( \sum_{i=1}^{n} \xi_{ij} \leq a_j, V_{n_j}' \leq b_j \right) \geq 1/2.
\]

Thus we obtain

\[
P \left( \sum_{i=1}^{n} x_{ij} \geq x(a_j + b_j + V_{n_j}) \right) = \frac{P(\sum_{i=1}^{n} x_{ij} \geq x(a_j + b_j + V_{n_j}), \sum_{i=1}^{n} \xi_{ij} \leq a_j, V_{n_j}' \leq b_j)}{P(\sum_{i=1}^{n} \xi_{ij} \leq a_j, V_{n_j}' \leq b_j)}
\]

\[
\leq 2P \left( \sum_{i=1}^{n} x_{ij} \geq x(a_j + b_j + V_{n_j}), \sum_{i=1}^{n} \xi_{ij} \leq a_j, V_{n_j}' \leq b_j \right)
\]

\[
\leq 2P \left( \sum_{i=1}^{n} (x_{ij} - \xi_{ij}) \geq xV_{n_j}^r \right)
\]

\[
\leq 4 \exp(-x^2/8).
\]

Hence we get

\[
P \left( \left| \sum_{i=1}^{n} x_{ij} \right| \geq x(a_j + b_j + V_{n_j}) \right) \leq 8 \exp(-x^2/8).
\]

In particular, we can choose \( b_j^2 = 4E V_{n_j}^2 \) and \( a_j^2 = 2b_j^2 = 8E V_{n_j}^2 \) because \( 4E(\sum_{i=1}^{n} x_{ij})^2 \leq 8E(\sum_{j=1}^{r} A_j)^2 + 8E(\sum_{j=1}^{r} B_j)^2 = 8E V_{n_j}^2 \). \( \diamond \)
Proof of Theorem 3.1. Note that

\[
\mathbb{E}[1 - I] \leq P\left( \max_{1 \leq j \leq p} |X_j - \tilde{X}_j| > \Delta \right) + P\left( \max_{1 \leq j \leq p} |Y_j - \bar{Y}_j| > \Delta \right)
\]

\[
\leq \sum_{j=1}^{p} \left\{ P(|X_j - \tilde{X}_j| > \Delta) + P(|Y_j - \bar{Y}_j| > \Delta) \right\}.
\]

Let

\[
\Lambda_j \equiv (2 + 2\sqrt{2}) \sqrt{\frac{r}{n} \sum_{i=1}^{r} \mathbb{E}(A_{ij} - \tilde{A}_{ij})^2 + \frac{r}{n} \sum_{j=1}^{r} \mathbb{E}(B_{ij} - \tilde{B}_{ij})^2}
\]

\[
+ \sqrt{\sum_{i=1}^{r} (A_{ij} - \tilde{A}_{ij})^2/n + \sum_{j=1}^{r} (B_{ij} - \tilde{B}_{ij})^2/n} = \Lambda_{1j} + \Lambda_{2j},
\]

where

\[
\tilde{A}_{ij} = \sum_{l=(i-1)(N+M)+1}^{iN+(i-1)M} \tilde{x}_{lj}, \quad \tilde{B}_{ij} = \sum_{l=iN+(i-1)M+1}^{i(N+M)} \tilde{x}_{lj}.
\]

Applying Lemma 3.1 and using the union bound, we have with probability at least \(1 - 8\gamma\),

\[
|X_j - \tilde{X}_j| \leq \Lambda_j \sqrt{8 \log(p/\gamma)}, \quad 1 \leq j \leq p.
\]

By the assumption,

\[
P(\max_{1 \leq i \leq 1 \leq j \leq p} |x_{ij}| \leq M_x) \geq 1 - \gamma, \quad P(\max_{1 \leq i \leq 1 \leq j \leq p} |y_{ij}| \leq M_y) \geq 1 - \gamma.
\]

Therefore with probability at least \(1 - \gamma\),

\[
\Lambda_j \leq (2 + 2\sqrt{2}) \sqrt{\sum_{i=1}^{r} \mathbb{E}(A_{ij} - \tilde{A}_{ij})^2/n + \frac{r}{n} \sum_{j=1}^{r} \mathbb{E}(B_{ij} - \tilde{B}_{ij})^2/n}
\]

\[
+ \sqrt{\sum_{i=1}^{r} (\mathbb{E}\tilde{A}_{ij})^2/n + \sum_{j=1}^{r} (\mathbb{E}\tilde{B}_{ij})^2/n},
\]

\[
\leq (3 + 2\sqrt{2}) \varphi(M_x) \sqrt{N r \sigma^2_j/n + M r \sigma^2_j/n} \lesssim \varphi(M_x) \sigma_j,
\]

where we have used the fact that \(\mathbb{E}A_{ij} = \mathbb{E}B_{ij} = 0\) and the Cauchy-Schwarz inequality. The same argument applies to the Gaussian sequence \(\{y_i\}\).

Summarizing the above results and along with (10), we deduce that

\[
|\mathbb{E}[m(X) - m(Y)]| \lesssim (G_2 + G_1 \beta) \varphi(M_x, M_y) + (G_3 + G_2 \beta + G_1 \beta^2) (2M + 1)^2 \sqrt{\frac{m_3^2 + m_3^2}{n}} (\tilde{m}_x, 3 + \tilde{m}_y, 3)
\]

\[
+ G_1 \varphi(M_x, M_y) \sigma_j \sqrt{8 \log(p/\gamma)} + G_0 \gamma,
\]

(S.8)
which also implies that

\[
|E[g(T_X) - g(T_Y)]| \leq (G_2 + G_1 \beta \phi(M_x, M_y) + (G_3 + G_2 \beta + G_1 \beta^2) \frac{(2M + 1)^2}{\sqrt{n}} (\bar{m}_{x,3} + \bar{m}_{y,3}) + G_1 \varphi(M_{xy}) \sigma_j \sqrt{8 \log(p/\gamma)} + G_0 \gamma + \beta^{-1} G_1 \log p,
\]

for \( M \)-dependent sequence, provided that \( 2 \sqrt{5} \beta (6M + 1) M_{xy}/\sqrt{n} < 1 \). Consider a “smooth” indicator function \( g_0 \in C^3(\mathbb{R}) : \mathbb{R} \to [0,1] \) such that \( g_0(s) = 1 \) for \( s \leq 0 \) and \( g_0(s) = 0 \) for \( s \geq 1 \). Fix any \( t \in \mathbb{R} \) and define \( g(s) = g_0(\psi(s - t - e\beta)) \) with \( e\beta = \beta^{-1} \log p \). The conclusion follows from the proof of Corollary F.1 in [16] and Lemma 2.1 in [15] regarding the anti-concentration property for Gaussian distribution. We omit the details to conserve the space. \( \diamond \)

**Proof of Theorem 3.2.** Let \( \hat{x}_{ij} = x_{ij} - \bar{x}_{ij} \). Define \( \chi_{(l+1)k} = (x_{(l+1)k} \wedge M_x) \vee (-M_x) \) and \( \chi_{(l+1)k} = (x_{(l+1)k} \wedge M_x) \vee (-M_x) \). Using the fact that \( x_{ij} \) and \( x_{(l+1)k} \) are independent for any \( 1 \leq j, k \leq p \) and \( E_{x_{ij}} = E \hat{x}_{ij} = 0 \), we obtain for \( l > 0 \),

\[
|E_{x_{(l+1)k}} \hat{x}_{(l+1)k}| = |E x_{(l+1)k} - x_{(l+1)k}^{(l-1)}}| \\
\leq (E x_{(l+1)k}^{2})^{1/2} (E (x_{(l+1)k} - x_{(l+1)k}^{(l-1)}/2) \leq (E x_{(l+1)k}^{2})^{1/2} (E (x_{(l+1)k} - x_{(l+1)k}^{(l-1)})^{2} / M_x.
\]

Using the fact that the map \( x \to (x \wedge M_x) \vee (-M_x) \) is lipschitz continuous, we deduce that

\[
|Ex_{(l+1)k} \hat{x}_{(l+1)k}| = |Ex_{l+1k} \{ \hat{x}_{(l+1)k} - \hat{x}_{(l+1)k}^{(l-1)} - E(\chi_{(l+1)k} - \chi_{(l+1)k}^{(l-1)}) \} I{|x_{(l+1)k} > M_x} or |x_{(l+1)}^{(l-1)} > M_x} \\
\leq (E x_{(l+1)k}^{2})^{1/3} (E (\hat{x}_{(l+1)k} - \hat{x}_{(l+1)k}^{(l-1)})^{3} + E |\chi_{(l+1)k} - \chi_{(l+1)k}^{(l-1)}|^{3/3} \\
(\hat{x}_{(l+1)k} > M_x) + P(|x_{(l+1)k}^{(l-1)} > M_x) / M_x) \\
\leq (E x_{(l+1)k}^{2})^{1/3} (E (x_{(l+1)k} - x_{(l+1)k}^{(l-1)})^{3})^{1/3} (E |x_{(l+1)k}|^{3} + E |x_{(l+1)k}^{(l-1)}|^{3})^{1/3} / M_x.
\]

Note for \( l = 0 \), \( |Ex_{(l+1)k} \hat{x}_{(l+1)k}| \leq (E x_{(l+1)k}^{2})^{1/2} (E x_{(l+1)k}^{2})^{1/2} / M_x. \) It is not hard to show that the above result holds if \( x_{ij} \) (or \( x_{(l+1)k} \)) is replaced by its \( \hat{z}_{1j} \) (or \( \hat{z}_{(l+1)k} \)). Therefore by (S.4) and the assumptions, we have

\[
\max_{1 \leq i, k \leq p} \sum_{j=1}^{n} |E \hat{Z}_{ij}(t)V_{(j)}^{(i)}(t)| \leq (M_x + M_y).
\]

Thus we may set \( \phi(M, M_y) = C(1/M_x + 1/M_y) \) for some constant \( C > 0 \).
Next we consider $\varphi(M_x, M_y)$. By the stationarity, we have

$$
\sum_{t=1}^{N-1} \left| \mathbb{E}x_{1k} \bar{x}_{(1+t)k} \right| = 2 \sum_{t=1}^{N-1} \left| \mathbb{E}x_{1k} \left\{ \bar{x}_{(1+t)k} - \bar{x}_{\left(1+(t-1)\right)k} - \mathbb{E}(\chi_{(1+t)k}^{(t-1)}) \right\} \right|
$$

$$
\leq 2 \sum_{t=1}^{N-1} \left( \mathbb{E}x_{1k}^2 \right)^{1/2} \left( \mathbb{E}(\chi_{(1+t)k}^{(t-1)})^3 + \mathbb{E}(\chi_{(1+t)k}^{(t-1)})^3 \right)^{1/3}
$$

(S.10)

Also note that $(\mathbb{E}A_{ij})^2/N = N(\mathbb{E}(\chi_{1j} - x_{1j}))^2 \leq N(\mathbb{E}x_{1j}^2/M_x^2)^2$ and $(\mathbb{E}B_{ij})^2/M \leq M(\mathbb{E}x_{ij}^4/M_x^2)^2$. Because $E(A_{ij} - \bar{A}_{ij})^2/N \leq 1/M_x^{3/2}$ and $E(B_{ij} - \bar{B}_{ij})^2/M \leq 1/M_x^{3/2}$ by (S.10), we can choose $\varphi(M_x) = C'(1/M_x^{5/6} + \sqrt{N/M_y^2})$ for some constant $C' > 0$. By the assumption that $\max(1 \leq k \leq p) \mathbb{E}G_k(\ldots, e_{i-1}, e_i)^4 < \infty$ and the fact that $\mathbb{E}y_{ij}^2 = \mathbb{E}x_{ij}^2$, we have $E|x_{ij}|^3 \leq (\mathbb{E}G_j(\ldots, e_{i-1}, e_i)^4)^{3/4}$.

$$
\mathbb{E}|y_{ij}|^3 \leq (\mathbb{E}|y_{ij}|^3)^{3/4} \leq (\mathbb{E}|y_{ij}|^2)^{3/2} \leq (\mathbb{E}|x_{ij}|^2)^{3/2} \leq |x_{ij}|^3 < \infty,
$$

and

$$
E|y_{ij}|^4 \leq (E|y_{ij}|^2)^2 = (E|x_{ij}|^2)^2 \leq |x_{ij}|^4 < \infty.
$$

Using similar arguments, we can show that $\varphi(M_y) = C''(1/M_y^{3/4} + \sqrt{N/M_y^2})$ for some constant $C'' > 0$. The above argument also implies that $\tilde{m}_{x,3}^3 + \tilde{m}_{y,3}^3 < \infty$. Thus we ignore the constants and set $\psi = O(n^{1/8}M^{-1/2}l_n^{-3/8})$ and $M_x = M_y = u = O(n^{3/8}M^{-1/2}l_n^{-5/8})$.

Let $2\sqrt{5}3(6M + 1)M_{xy}/\sqrt{n} = 1$, that is $\beta = O(\sqrt{n}/(uM))$. It is straightforward to check the following:

$$
(\psi^2 + \psi^3)\phi(M_x, M_y) \leq \psi^2/u + \psi\sqrt{n}/(u^2M) \lesssim n^{-1/8}M^{1/2}l_n^{7/8},
$$

$$
(\psi^2 + \psi^3 + \psi^2)\phi(2M + 1)^{1/2} / \sqrt{n} \leq \psi^2M^2 / \sqrt{n} + \psi^2M / u + \psi\sqrt{n} / u^2 \lesssim n^{-1/8}M^{1/2}l_n^{7/8},
$$

$$
\psi\phi(M_x, M_y)\sigma_j\sqrt{\log(p/\gamma)} \leq \psi\log^{1/2} / u^{5/6} + \sqrt{\gamma} \psi\log^{1/2} / u^3 \lesssim n^{-1/8}M^{1/2}l_n^{7/8},
$$

$$
(e_\beta + \psi^{-1})\sqrt{\log(p\psi)} \leq \sqrt{n} / u + \psi^{-1} / n^2 \leq n^{-1/8}M^{1/2}l_n^{7/8}.
$$

Therefore we get

(S.11) 

$$
\rho_n := \sup_{t \in \mathbb{R}} |P(TX \leq t) - P(TY \leq t)| \lesssim n^{-1/8}M^{1/2}l_n^{7/8} + \gamma.
$$
Using similar arguments, we can prove the result under Condition (ii) in Assumption 3.1. Therefore, we obtain

\[ \mathbb{E} h(\max_{1 \leq j \leq p} |x_{ij}| / \mathcal{D}_n) \leq 1. \]

By Lemma 2.2 in [16], we have \( u_x(\gamma) \lesssim \max\{ \mathcal{D}_n h^{-1}(n/\gamma), l_{n/2}^1 \} \) and \( u_y(\gamma) \lesssim l_{n/2}^1 \). Because \( n^{3/8} M^{1/2} l_{n/2}^{-5/8} \geq C \max\{ \mathcal{D}_n h^{-1}(n/\gamma), l_{n/2}^1 \} \), we can always choose \( u = O(n^{3/8} M^{-1/2} l_{n/2}^{-5/8}) \) such that

\[ P(\max_{1 \leq i \leq n} |x_{ij}| \leq u) \geq 1 - \gamma, \quad P(\max_{1 \leq i \leq n} |y_{ij}| \leq u) \geq 1 - \gamma. \]

Using similar arguments, we can prove the result under Condition (ii) in Assumption 3.1. The proof is thus completed.

The following lemma verifies condition (18).

**Lemma S.1.** Assume that \( \max_{1 \leq k \leq p} \sup_{l=1}^{+\infty} j \theta_{j,k,3}(x) < \infty \). Then

\[ \sup_{M} \max_{1 \leq k \leq p} \sum_{l=1}^{M} (\mathbb{E} |x^{(M)}_{(1+l)k} - x^{(l-1)}_{(1+l)k}|^3)^{1/3} \leq \max_{1 \leq k \leq p} \sum_{j=1}^{+\infty} j \theta_{j,k,3}(x) < \infty. \]

**Proof of Lemma S.1.** Define the projection \( \mathcal{P}_j x_{ik} = \mathbb{E}[x_{ik} | \mathcal{F}_j(i)] - \mathbb{E}[x_{ik} | \mathcal{F}_{j-1}(i)] \). Then we have

\[ x^{(M)}_{(1+l)k} - x^{(l-1)}_{(1+l)k} = \mathbb{E}[\mathcal{G}_k(\ldots, \epsilon_l, \epsilon_{l+1}) | \mathcal{F}_M(l+1)] - \mathbb{E}[\mathcal{G}_k(\ldots, \epsilon_l, \epsilon_{l+1}) | \mathcal{F}_{l-1}(l+1)] = \sum_{j=l}^{M} \mathcal{P}_j x_{(l+1)k}. \]

Note that

\[ \mathcal{P}_j x_{ik} = \mathbb{E}[x_{ik} | \mathcal{F}_j(i)] - \mathbb{E}[x_{ik} | \mathcal{F}_{j-1}(i)] = \mathbb{E}[\mathcal{G}_k(\ldots, \epsilon_{i-1}, \epsilon_i) - \mathcal{G}_k(\ldots, \epsilon_{i-1}, \epsilon_{i+1}, \ldots, \epsilon_i, \epsilon_{i+1}) | \mathcal{F}_j(i)] = \mathbb{E}[\mathcal{G}_k(\ldots, \epsilon_{i-1}, \epsilon_i) - \mathcal{G}_k(\ldots, \epsilon_{i+1}, \ldots, \epsilon_{i-1}, \epsilon_i) | \mathcal{F}_j(i)]. \]

Jensen’s inequality yields that \( (\mathbb{E} |\mathcal{P}_j x_{ik}|^q)^{1/q} \leq \theta_{j,k,q}(x) \) which implies that

\[ (\mathbb{E} |x^{(M)}_{(1+l)k} - x^{(l-1)}_{(1+l)k}|^3)^{1/3} \leq \sum_{j=l}^{M} (\mathbb{E} |\mathcal{P}_j x_{(l+1)k}|^3)^{1/3} \leq \sum_{j=l}^{M} \theta_{j,k,3}(x). \]

Therefore, we obtain

\[ \sup_{M} \max_{1 \leq k \leq p} \sum_{l=1}^{M} (\mathbb{E} |x^{(M)}_{(1+l)k} - x^{(l-1)}_{(1+l)k}|^3)^{1/3} \leq \sup_{M} \max_{1 \leq k \leq p} \sum_{l=1}^{M} \sum_{j=l}^{M} \theta_{j,k,3}(x) \leq \max_{1 \leq k \leq p} \sum_{j=1}^{+\infty} j \theta_{j,k,3}(x) < \infty. \]

**Proof of Theorem 3.3.** We need to verify that the \( M \)-dependent approximation \( \{x^{(M)}_i\} \) satisfies the assumptions in Theorem 3.2. Using the convexity of \( h(\cdot) \) and Jensen’s inequality we have

\[ \mathbb{E} h(\max_{1 \leq j \leq p} |x^{(M)}_{ij}| / \mathcal{D}_n) \leq \mathbb{E} h(\max_{1 \leq j \leq p} |x_{ij}| / \mathcal{D}_n) \leq 1, \]
under Condition (i) in Assumption 3.1, and
\[ \max_{1 \leq j \leq p} \mathbb{E} \exp(|x_{ij}^{(M)}|/D_n) \leq \max_{1 \leq j \leq p} \mathbb{E} \exp(|x_{ij}|/D_n) \leq 1, \]
under Condition (ii) in Assumption 3.1.

We claim that as \( M \to +\infty \),

\[
\sup_p \max_{1 \leq j \leq p} \sum_{h=-\infty}^{+\infty} |\mathbb{E} x_{ij}^{(M)} x_{(i+h)j} - \mathbb{E} x_{ij} x_{(i+h)j}| \to 0,
\]
which implies that \( \max_{1 \leq j \leq p} |\mathcal{Q}_{j}^{(M,n)} - \mathcal{Q}_{j}^{(n)}| \to 0 \) and \( \max_{1 \leq j \leq p} |(\mathcal{Q}_{j}^{(M)})^2 - \mathcal{Q}_{j}^{2}| \to 0 \) with \( \mathcal{Q}_{j}^{(M,n)} = \sum_{h=1-n}^{n-1} (n-h)\mathbb{E} x_{ij}^{(M)} x_{(i+h)j}/n \) and \( (\mathcal{Q}_{j}^{(M)})^2 = \sum_{h=-\infty}^{+\infty} |\mathbb{E} x_{ij}^{(M)} x_{(i+h)j}| \). Thus under the assumptions in Theorem 3.3, we have \( c_1/2 < \min_{1 \leq j \leq p} \mathcal{Q}_{j}^{(M,n)} \leq \max_{1 \leq j \leq p} \mathcal{Q}_{j}^{(M,n)} < 2c_2 \) for some constants \( 0 < c_1 < c_2 \) uniformly for all large enough \( M \).

To show (S.13), we note that

\[
\sum_{h=-\infty}^{+\infty} |\mathbb{E} x_{ij}^{(M)} x_{(i+h)j} - \mathbb{E} x_{ij} x_{(i+h)j}| = \sum_{h=-M}^{M} |\mathbb{E} x_{ij}^{(M)} x_{(i+h)j} - \mathbb{E} x_{ij} x_{(i+h)j}| + \sum_{|h|>M} |\mathbb{E} x_{ij} x_{(i+h)j}| = I_{1j}(M) + I_{2j}(M).
\]

For the first term, we have

\[
I_{1j}(M) \leq \sum_{h=-M}^{M} |\mathbb{E} x_{ij}^{(M)} x_{(i+h)j} - \mathbb{E} x_{ij} x_{(i+h)j}| = \sum_{h=-M}^{M} |\mathbb{E} (x_{ij}^{(M)} - x_{ij}) x_{(i+h)j}|
\leq \sum_{h=-M}^{M} \left\{ \mathbb{E} (x_{ij}^{(M)})^2 \right\}^{1/2} \left\{ \mathbb{E} (x_{ij}^{(M)} - x_{ij})^2 \right\}^{1/2} + \sum_{h=-M}^{M} \left\{ \mathbb{E} (x_{ij}^{(M)} - x_{ij})^2 \right\}^{1/2} \left\{ \mathbb{E} (x_{(i+h)j})^2 \right\}^{1/2}
\leq M \left\{ \mathbb{E} (x_{ij}^{(M)})^2 \right\}^{1/2} \left\{ \mathbb{E} (x_{ij}^{(M)} - x_{ij})^2 \right\}^{1/2} \leq M \left\{ \mathbb{E} (x_{ij}^2) \right\}^{1/2} \left\{ \mathbb{E} (x_{ij})^2 \right\}^{1/2} \sum_{l=M+1}^{+\infty} \left\{ \mathbb{E} |\mathcal{P}_l x_{ij}| \right\}^{2/2} \sum_{l=M+1}^{+\infty} \left\{ l \theta_{l,j,2}(x) \right\}^{2/2} \sum_{l=M+1}^{+\infty} \left\{ l \theta_{l,j,3}(x) \right\}
\]

where we have used the fact that \( x_{ij} - x_{ij}^{(M)} = \sum_{l=M+1}^{+\infty} \mathcal{P}_l x_{ij} \) and \( (\mathbb{E} |\mathcal{P}_l x_{ij}|^q)^{1/q} \leq \theta_{l,j,q}(x) \). Under the assumption that \( \sum_{j=1}^{+\infty} \max_{1 \leq k \leq p} j \theta_{j,k,3}(x) \leq \sum_{j=1}^{+\infty} \ell_j < \infty \), we have \( \max_{1 \leq j \leq p} I_{1j}(M) \to 0 \) as \( M \to +\infty \). On the other hand, note that for \( h > M \)

\[
\mathbb{E} x_{ij} x_{(i+h)j} = \mathbb{E} x_{ij} (x_{(i+h)j} - x_{(i+h)j}^{(h-1)}) \leq \left\{ \mathbb{E} x_{ij}^2 \right\}^{1/2} \left\{ \mathbb{E} (x_{(i+h)j} - x_{(i+h)j}^{(h-1)})^2 \right\}^{1/2}.
\]

Thus we have

\[
\max_{1 \leq j \leq p} I_{2j}(M) \leq \max_{1 \leq j \leq p} \sum_{h>M} \sum_{l \geq h} \theta_{l,j,3}(x) \leq \max_{1 \leq j \leq p} \sum_{l=M+1}^{+\infty} \theta_{l,j,3}(x) \leq \sum_{l=M+1}^{+\infty} \ell_l.
\]
which implies that \( \max_{1 \leq j \leq p} I_{2j}(M) \to 0 \) as \( M \to +\infty \).

Lemma S.1 verifies the first condition in (17). The same arguments apply to \( \{y_i\} \). The triangle inequality and (24) imply that

\[
|\mathbb{E}[m(X) - m(Y)]| \lesssim |\mathbb{E}[m(X^{(M)}) - m(Y^{(M)})]| + (G_0\varphi_{\epsilon})^{1/(1+q)} \left( \sum_{j=1}^{p} \Theta_{M,j,q}^q \right)^{1/(1+q)},
\]

where \( Y^{(M)} = \sum_{i=1}^{n} y_i^{(M)}/\sqrt{n} \) with \( y_i^{(M)} \) being the \( M \)-dependent approximation for \( \{y_i\} \). The conclusion thus follows from Theorem 3.1 and Theorem 3.2.

\[
\Box
\]

**Proof of Proposition 3.2.** Without loss of generality, we assume that \( \pi(i) = i \). Define two events \( D_x = \{ \max_{1 \leq j \leq q} X_j > \max_{q+1 \leq j \leq p} X_j \} \) and \( D_y = \{ \max_{1 \leq j \leq q} Y_j > \max_{q+1 \leq j \leq p} Y_j \} \). Simple algebra yields that uniformly for all \( z \in \mathbb{R} \),

\[
\left| P\left( \max_{1 \leq j \leq p} X_j \leq z \right) - P\left( \max_{q+1 \leq j \leq p} X_j \leq z \right) \right| \\
\leq P\left( \max_{1 \leq j \leq p} X_j \leq z, D_x \right) + P\left( \max_{1 \leq j \leq p} X_j \leq z, D_x^c \right) - P\left( \max_{q+1 \leq j \leq p} X_j \leq z \right) \\
\leq P\left( \max_{1 \leq j \leq q} X_j \leq z, D_x \right) + P\left( \max_{1 \leq j \leq q} X_j \leq z, D_x^c \right) - P\left( \max_{q+1 \leq j \leq p} X_j \leq z \right) \\
\leq P\left( \max_{1 \leq j \leq q} X_j \leq z, D_x \right) - P\left( \max_{q+1 \leq j \leq p} X_j \leq z, D_x \right) \leq 2P(D_x).
\]

Next we analyze \( P(D_x) \) and \( P(D_y) \). Under the assumptions in Corollary 2.1 of [16], we have

\[
(S.14) \quad \sup_{z \in \mathbb{R}} \left| P\left( \max_{q+1 \leq i \leq p} X_i \leq z \right) - P\left( \max_{q+1 \leq i \leq p} Y_i \leq z \right) \right| \lesssim n^{-c}, \quad c > 0.
\]

Notice that in this case, we allow \( p = O(\exp(n^b)) \) with \( b < 1/7 \) (assuming that \( B_n = O(1) \) in Corollary 2.1 of [16]). By (S.14), and the independence between \( \{z_{i1}\} \) and \( \{z_{i2}\} \), we obtain

\[
P(D_x) \leq \sum_{j=1}^{q} \mathbb{E} \left[ P\left( \max_{q+1 \leq i \leq p} X_i < X_j \mid X_j \right) \right] \lesssim \sum_{j=1}^{q} \mathbb{E} \left[ P_y \left( \max_{q+1 \leq i \leq p} Y_i < X_j \right) \right] + qn^{-c},
\]

where \( P_y \) denotes the probability measure with respect to \( (Y_{q+1}, \ldots, Y_p) \).

Let \( \sigma = \max_{1 \leq j \leq p} \sigma_{j,j} \). Using the concentration inequality (see e.g. (7.3) of [25] and Theorem A.2.1 of [38]),

\[
P\left( \max_{q+1 \leq i \leq p} Y_i \leq \mathbb{E}\max_{q+1 \leq i \leq p} Y_i - r \right) \leq e^{-r^2/(2\sigma)}
\]

for \( r > 0 \), we have

\[
P\left( \max_{q+1 \leq i \leq p} Y_i < x \right) \leq \exp\left( -\frac{1}{2\sigma} \left( \mathbb{E}\max_{q+1 \leq i \leq p} Y_i - x \right)^2 \right),
\]
where $x_+ = x I\{x \geq 0\}$. Under the assumption that $q/E \max_{q+1 \leq i \leq p} Y_i \to 0$, we can choose $\tilde{q} \to +\infty$ such that $\tilde{q}/E \max_{q+1 \leq i \leq p} Y_i \to 0$ and $q/\tilde{q} \to 0$. Then we have

$$
\sum_{j=1}^q E \left[ P_y \left( \max_{q+1 \leq i \leq p} Y_i < X_j \right) \right] \leq \sum_{j=1}^q E \exp \left( -\frac{1}{2\sigma} \left( E \max_{q+1 \leq i \leq p} Y_i - X_j \right)^2 \right) + \sum_{j=1}^q E \{ X_j \leq \tilde{q} \} + q \max_{1 \leq j \leq q} E|X_j|/\tilde{q} = o(1)
$$

Moreover, if $q/E \max_{q+1 \leq j \leq p} Y_j = O(n^{-c'})$ for $c' > 0$, we can replace $o(1)$ by $O(n^{-c''})$ for some $c'' > 0$. Thus we get

$$
\sup_{z \in \mathbb{R}} \left| P \left( \max_{1 \leq j \leq p} X_j \leq z \right) - P \left( \max_{q+1 \leq j \leq p} X_j \leq z \right) \right| \leq 2P(D_x)
$$

Similar argument applies to $\{Y_i\}$ and the conclusion follows from (S.14). \hfill \Box

### S.2. Proofs of the main results in Section 4.

#### Proof of Lemma 4.1. By the triangle inequality and the stationarity, we have

$$
E_A \leq \max_{1 \leq j,k \leq p} \left| \frac{1}{N^r} \sum_{i=1}^r (A_{ij} A_{ik} - E A_{ij} A_{ik}) \right| + \max_{1 \leq j,k \leq p} \left| \frac{1}{N^r} \sum_{i=1}^r E A_{ij} A_{ik} - \sigma_{j,k} \right| + \max_{1 \leq j,k \leq p} |\sigma_{j,k} - \sigma_{j,k}^{(n)}|
$$

$$
\leq \max_{1 \leq j,k \leq p} \left| \frac{1}{N^r} \sum_{i=1}^r (A_{ij} A_{ik} - E A_{ij} A_{ik}) \right| + \max_{1 \leq j,k \leq p} \sum_{|l| \geq N} |E x_{i+l,j} x_{i,k}| + \frac{1}{N} \sum_{l=1-N}^{N-1} |l||E x_{i+l,j} x_{i,k}| + \max_{1 \leq j,k \leq p} |\sigma_{j,k} - \sigma_{j,k}^{(n)}|
$$

Note that for any $1 \leq j,k \leq p$, $\{A_{ij} A_{ik}\}$ is a sequence of i.i.d random variables. Let $\sigma_{A,N}^2 = \max_{1 \leq j,k \leq p} E (A_{ij} A_{ik})^2 / N^2$ and $M_{A,N} = \max_{1 \leq i \leq r} \max_{1 \leq j,k \leq p} |A_{ij}/\sqrt{N}|^4$. Then by Lemma A.1 in [16], we have

$$
E \max_{1 \leq j,k \leq p} \left| \frac{1}{N^r} \sum_{i=1}^r (A_{ij} A_{ik} - E A_{ij} A_{ik}) \right| \leq \sigma_{A,N} \sqrt{2 \log p/r + 2 \log p \sqrt{E M_{A,N}/r}}.
$$
Cauchy-Schwarz inequality yields that

\[
\sigma^2_{A,N} \leq \frac{1}{N^2} \max_{1 \leq j \leq p} \mathbb{E}(A_{ij})^4 \leq \frac{1}{N^2} \max_{1 \leq j \leq p} \mathbb{E} \sum_{i_1, i_2, i_3, i_4 = 1}^{N} x_{i_1 j} x_{i_2 j} x_{i_3 j} x_{i_4 j} ~ \leq \frac{1}{N^2} \max_{1 \leq j \leq p} \sum_{i_1, i_2, i_3, i_4 = 1}^{N} \left\{ \text{cum}(x_{i_1 j}, x_{i_2 j}, x_{i_3 j}, x_{i_4 j}) + \gamma_{x,jj}(i_1 - i_3)\gamma_{x,jj}(i_2 - i_4) + \gamma_{x,jj}(i_1 - i_4)\gamma_{x,jj}(i_2 - i_3) \right\} ~ \leq \max_{1 \leq j \leq p} \left\{ \frac{1}{N} \sum_{i_1, i_2, i_3 = -\infty}^{+\infty} |\text{cum}(x_{i_1 j}, x_{i_2 j}, x_{i_3 j}, x_{0 j})| + 3 \left( \sum_{h = -\infty}^{+\infty} |\gamma_{x,jj}(h)| \right)^2 \right\} \lesssim \sigma^2_{x,N}.
\]

On the other hand, with \( h(x) = \exp(x) - 1 \), we have

\[
(\mathbb{E} \max_{1 \leq i \leq r} \max_{1 \leq j \leq p} |A_{ij}/\sqrt{N}|^4)^{1/4} \lesssim \left( \max_{1 \leq i \leq r} \max_{1 \leq j \leq p} |A_{ij}/\sqrt{N}| \right)_{h} \lesssim \log(rp) \max_{1 \leq i \leq r} \max_{1 \leq j \leq p} ||A_{ij}/\sqrt{N}||_{h} = \log(rp) \max_{1 \leq j \leq p} ||A_{1j}/\sqrt{N}||_{h},
\]

where we have used Lemma 2.2.2 in [38]. It implies that

\[
\sqrt{\mathbb{E}M_{A,N}} \leq (\log(rp))^2 \max_{1 \leq j \leq p} \left( \sum_{i=1}^{N} x_{ij}/\sqrt{N} \right)_{h}^2.
\]

Combining the above arguments, we deduce that

\[
\mathbb{E}A \lesssim \sigma_{x,N} \sqrt{\log p/r + \log p \{\log(rp)\}^2} \zeta_{x,h,N}/r + \omega_x/N,
\]

\[
\mathbb{E}B \lesssim \sigma_{x,M} \sqrt{\log p/r + \log p \{\log(rp)\}^2} \zeta_{x,h,M}/r + \omega_x/M.
\]

Alternatively, note that \( (\mathbb{E} \max_{1 \leq i \leq r} \max_{1 \leq j \leq p} |A_{ij}/\sqrt{N}|^4)^{1/4} \leq r^{1/4} \zeta_{x,N} \). The conclusion follows from the above arguments.

**Proof of Theorem 4.1.** By Theorem 3.2, \( \rho_n \lesssim n^{-1/8} M^{1/2} l_{\omega}^{7/8} + \gamma \). Choosing \( \gamma = O(n^{-c'}) \) for some \( c' > (1 - 4b' - 7b)/8 \), we have \( \rho_n = O(n^{(-1 - 4b' - 7b)/8}) \). Pick \( \nu = O(n^{-v}) \) with

\[
v = 3((1 - 5b - b'')/2 - s_1) \wedge (1 - 5b - b'' - s_2) \wedge (b' - 2b - s_3)/4 + 2b.
\]

Then it is easy to verify that the terms \( \nu^{1/3}(1 \vee \log(p/\nu)^{2/3} \) and \( \mathbb{E}A/\nu + \mathbb{E}B/\nu \) are both of order \( O(n^{-c''}) \) with \( c'' = s_b/4 \). Finally by (39), we have

\[
\sup_{\alpha \in (0,1)} |P(T_X \leq c_{T_0}(\alpha)) - \alpha| \lesssim n^{-c}, \quad c = \min\{s_b/4, (1 - 4b' - 7b)/8\}.
\]

The result under Condition 2 can be proved in a similar manner.
Proof of Theorem 4.2. Let \( x_i^{(M)} \) be the \( M \)-dependent approximation sequence for \( x_i \). Define \( A_{ij}^{(M)}, B_{ij}^{(M)}, E_A^{(M)} \) and \( E_B^{(M)} \) in a similar way as \( A_{ij}, B_{ij}, E_A \) and \( E_B \) by replacing \( x_i \) with \( x_i^{(M)} \). Notice that

\[
E \max_{1 \leq j, k \leq p} \left| \frac{1}{r} \sum_{i=1}^{r} (A_{ij}^{(M)} A_{ik}^{(M)} - A_{ij} A_{ik}) / N \right| \\
\leq \frac{1}{rN} \sum_{1 \leq j, k \leq p} E \left| \sum_{i=1}^{r} (A_{ij}^{(M)} A_{ik}^{(M)} - A_{ij} A_{ik}) + A_{ij} A_{ik}^{(M)} - A_{ij} A_{ik} \right| \\
\leq \frac{1}{N} \sum_{1 \leq j, k \leq p} \left( E \left| A_{ij}^{(M)} A_{ik}^{(M)} - A_{ij} A_{ik} \right| + E \left| A_{ij} A_{ik}^{(M)} - A_{ij} A_{ik} \right| \right) \\
\leq \frac{1}{N} \sum_{1 \leq j, k \leq p} \left\{ \left( E \left| A_{ij}^{(M)} - A_{ij} \right|^2 \right)^{1/2} \left( E \left| A_{ik}^{(M)} \right|^2 \right)^{1/2} + \left( E \left| A_{ik}^{(M)} - A_{ik} \right|^2 \right)^{1/2} \left( E \left| A_{ij} \right|^2 \right)^{1/2} \right\} .
\]

By Lemma A.1 of [29], we have \( (E \left| A_{ij}^{(M)} - A_{ij} \right|^2)^{1/2} \sqrt{N} \leq C_q \Theta_{M,j,q}(x) \) for some \( q \geq 2 \). It follows that

\[
E \max_{1 \leq j, k \leq p} \left| \frac{1}{r} \sum_{i=1}^{r} (A_{ij}^{(M)} A_{ik}^{(M)} - A_{ij} A_{ik}) / N \right| \lesssim p^2 \rho^M.
\]

Similarly we have

\[
E \max_{1 \leq j, k \leq p} \left| \frac{1}{r} \sum_{i=1}^{r} (B_{ij}^{(M)} B_{ik}^{(M)} - B_{ij} B_{ik}) / M \right| \lesssim p^2 \rho^M.
\]

Using similar arguments in the proof of Theorem 3.3, we have

\[
\max_{1 \leq j, k \leq p} \sum_{h=1-n}^{n-1} \left| E^{(M)}_{x_{ij}^{(M)} x_{(i+h)k}^{(M)}} - E_{x_{ij} x_{(i+h)k}} \right| \lesssim M \rho^M.
\]

Thus by (39), we have

\[
\sup_{\alpha \in (0,1)} \left| P(T_X \leq cT_0(\alpha)) - \alpha \right| \lesssim \rho_n + \nu^{1/3} (1 \vee \log(p/n))^{2/3} + \sqrt{E \left| A_{ij}^{(M)} \right|^2 / \nu} + \sqrt{E \left| B_{ij}^{(M)} \right|^2 / \nu} + (p^2 + M) \rho^M / \nu.
\]

Then by Lemma A.1 in [16], we have

\[
E \max_{1 \leq j, k \leq p} \left| \frac{1}{N^r} \sum_{i=1}^{r} (A_{ij}^{(M)} A_{ik}^{(M)} - E A_{ij}^{(M)} A_{ik}^{(M)}) \right| \\
\lesssim \frac{1}{N} \max_{1 \leq j \leq p} \left\{ E(A_{ij}^{(M)})^4 \right\}^{1/2} 2 \log p / r + 2 \log p \sqrt{E \max_{1 \leq i \leq r \leq j \leq p} \left| A_{ij}^{(M)} / \sqrt{N} \right|^{4/r}} \\
\lesssim \frac{1}{N} \max_{1 \leq j \leq p} \left\{ E(A_{ij})^4 \right\}^{1/2} 2 \log p / r + 2 \log p \sqrt{E \max_{1 \leq i \leq r \leq j \leq p} \left| A_{ij} / \sqrt{N} \right|^{4/r}} \\
+ \sqrt{2 \log p / r} \max_{1 \leq j \leq p} \Theta_{M,j,4}^2(x) + 2 \log p \sqrt{rp \max_{1 \leq j \leq p} \Theta_{M,j,4}^2(x) / r} ,
\]

where \( \Theta_{M,j,4}(x) \) is the be the \( x \)-dependent approximation sequence for \( x \).
where the first two terms can be bounded using similar arguments in the proof of Lemma 4.1, and the last two terms decay exponentially. The same arguments apply to the terms associated with $B_{ij}$.

By Theorem 3.3, we have

$$\rho_n \leq n^{-1/8} M^{1/2} l_n^{-7/8} + \gamma + (n^{1/8} M^{-1/2} l_n^{-3/8})^{q/(1+q)} \left( \sum_{j=1}^{p} \Theta_{M,j,q}^q \right)^{1/(1+q)}.$$  

The assumption that $\max_{1 \leq j \leq p} \Theta_{M,j,q} = O(p^M)$ for $p < 1$, and $M = O(n^{b'})$ with $b' > 2b$ implies that $\left( \sum_{j=1}^{p} \Theta_{M,j,q}^q \right)^{1/(1+q)}$ decays exponentially. The rest of the proof is similar to those in the proof of Theorem 4.1.

\[ \Box \]

**Proof of Theorem 4.3.** Our arguments below apply to $M$-dependent time series, and can be easily extended to weakly dependent time series by employing the $M$-approximation techniques (that incurs only an asymptotically ignorable error).

Let $c$, $c^*$, and $C$ be some generic constants which can be different from line to line. Define

$$\tilde{T}_X = \max_{1 \leq j \leq p} \frac{1}{\sqrt{n}} \sum_{i=1}^{l_n} (A_{ij} - \bar{A}_j) e_i.$$  

Following the arguments in the proof of Lemma 4.1, we have

$$\mathbb{E} \max_{1 \leq j \leq p} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{l_n} (A_{ij} / \sqrt{b_n})^2 - \sigma_{j,j}^{(b_n)} \right| \leq \tilde{\sigma}_{x,b,n} \sqrt{\log(p/l_n) + \log p(\log(l_n p))^2} \zeta_{x,h,b,n}^2 / l_n \leq C n^{-c}.$$  

Similarly we can show that

$$\mathbb{E} \max_{1 \leq j \leq p} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{l_n} A_{ij} / \sqrt{b_n} \right| \leq \max_{1 \leq j \leq p} \sigma_{j} \sqrt{\log(p/l_n) + \log p(\log(l_n p))^2} \zeta_{x,h,b,n}^2 / l_n \leq C n^{-c}$$

where we have used the fact that

(S.16) \[ \mathbb{E} \max_{1 \leq j \leq p} \max_{1 \leq l_n} |A_{ij} / \sqrt{b_n}| \leq \log(l_n p) \zeta_{x,h,b,n}. \]

By Markov’s inequality, we have with probability $1 - C n^{-c}$,

$$\left| \frac{1}{\sqrt{n}} \sum_{i=1}^{l_n} (A_{ij} - \bar{A}_j)^2 - \sigma_{j,j}^{(b_n)} \right| \leq (c_1/2) \wedge c_2,$$

uniformly for $1 \leq j \leq p$. It implies that with probability $1 - C n^{-c}$, $c_1/2 \leq \frac{1}{\sqrt{n}} \sum_{i=1}^{l_n} (A_{ij} - \bar{A}_j)^2 \leq 2c_2$. By (S.16), we have with probability with $1 - C n^{-c}$, $\max_{1 \leq i \leq l_n} \max_{1 \leq j \leq p} |A_{ij} / \sqrt{b_n}| \leq n^{c'} \log(l_n p) \zeta_{x,h,b,n}$ for some small $c' > 0$. Because $\zeta_{x,h,b,n}^2 \log(l_n p) / l_n \leq n^{-c'}$, we can apply Corollary 2.1 in [16] to conclude that with probability $1 - C n^{-c}$,

(S.17) \[ \sup_{t \in \mathbb{R}} |P(T_X \leq t | \{x_i\}_{i=1}^{n}) - P(\tilde{T}_X \leq t | \{x_i\}_{i=1}^{n})| \leq n^{-c'}, \quad c' > 0. \]
Next, notice that
\[
|\hat{T}_X - T_X| \leq \max_{1 \leq j \leq p} |\hat{A}_j / \sqrt{b_n}| \left| \frac{1}{\sqrt{t_n}} \sum_{i=1}^t e_i \right|.
\]
With probability $1 - Cn^{-c}$, we have $|\hat{T}_X - T_X| \leq n^{c'} \sqrt{\log p / t_n} \left| \frac{1}{\sqrt{t_n}} \sum_{i=1}^t e_i \right|$. Using the tail property of standard normal distribution, we can choose $\zeta = n^{2c'} \sqrt{\log p / t_n}$ such that with probability $1 - o(1)$,
\[
P(|\hat{T}_X - T_X| > \zeta | \{x_i\}_{i=1}^n) \lesssim n^{-c'},
\]
and $\sqrt{\log p} \zeta \lesssim n^{-c'}$ for some properly chosen $c'$ and $c$. Therefore by Lemma 2.1 in [16], we obtain that with probability $1 - Cn^{-c},$
\[
(S.18) \quad \sup_{t \in \mathbb{R}} |P(T_X \leq t | \{x_i\}_{i=1}^n) - P(\hat{T}_X \leq t | \{x_i\}_{i=1}^n)| \lesssim n^{-c'}.
\]
By (S.17) and (S.18), (42) holds with probability $1 - Cn^{-c}$. The second part of the theorem follows from Theorem 4.1 and Theorem 4.2.

\[\Box\]

S.3. Proofs of the main results in Section 5.

**Proof of Theorem 5.1.** Define $T_D = \max_{1 \leq j \leq 2q_0} \frac{1}{\sqrt{n}} \sum_{i=1}^{r_1} D_{ij}$, where $D_{ij} = A_{ij} \tilde{e}_i + B_{ij} \tilde{e}_j$ and
\[
A_{ij} = \sum_{l=(i-1)(N_1+M_1)+1}^{iN_1+M_1} x_{ij}, \quad B_{ij} = \sum_{l=(i-1)(N_1+M_1)+1}^{iN_1+M_1} x_{ij}, \quad 1 \leq i \leq r_1, 1 \leq j \leq 2q_0.
\]
Since $\max_{1 \leq j \leq 2q_0} \left| \frac{1}{\sqrt{N_0}} \sum_{i=1}^{N_0} \mathcal{F}(v_i, F_{d_0}) \right| / \sqrt{N_0} = \max_{1 \leq j \leq 2q_0} \sum_{i=1}^{N_0} x_i / \sqrt{N_0}$, we have
\[
\max_{1 \leq j \leq 2q_0} \sqrt{N_0} |\hat{\theta}_j - \tilde{\theta}_j| - \max_{1 \leq j \leq 2q_0} \sum_{i=1}^{N_0} x_i / \sqrt{N_0} \leq \max_{1 \leq j \leq 2q_0} \sqrt{N_0} |R_{jN_0}|.
\]

Let $\zeta_1 = Cn^{-c} / \sqrt{\log(2q_0)}$ and $\zeta_2 = Cn^{-c}$ for some large enough $C$ and small enough $c$ (e.g., $c < c_1$) such that
\[
P(\max_{1 \leq j \leq 2q_0} \sqrt{N_0} |R_{jN_0}| > \zeta_1) < \zeta_2.
\]
We show that $P(|T_D - \tilde{T}_D| > \zeta_1 | \{x_i\}_{i=1}^n) \geq \zeta_2$. Because $|T_D - \tilde{T}_D| \leq \max_{1 \leq j \leq 2q_0} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{r} (D_{ij} - \tilde{D}_{ij}) \right|$ and
\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{r} (D_{ij} - \tilde{D}_{ij}) \sim N \left( 0, \frac{1}{n} \sum_{i=1}^{r} \{(A_{ij} - \hat{A}_{ij})^2 + (B_{ij} - \hat{B}_{ij})^2 \} \right)
\]
conditional on $\{x_i\}_{i=1}^{2q_0}$, we have $\mathbb{E}[|T_D - \tilde{T}_D| | \{x_i\}_{i=1}^n] \leq C' \sqrt{\mathbb{E}_{AB} \log(2q_0)}$ for some large enough constant $C'$. It thus implies that
\[
P(|T_D - \tilde{T}_D| > \zeta_1 | \{x_i\}_{i=1}^n) \geq \zeta_2 \leq P(\mathbb{E}[|T_D - \tilde{T}_D| | \{x_i\}_{i=1}^n] > \zeta_1 \zeta_2)
\]
\[
\leq P(\mathbb{E}_{AB} \{C' \log(2q_0) \}^2 > C^4 n^{-4c}) \leq Cn^{-c},
\]
for large enough $C$ and sufficiently small $c$ (e.g. $c < c_2/4$). By Theorem 4.2, and Lemma 3.3 and the arguments in the proof of Theorem 3.2 in [16], we derive that under $H_0$,

$$
\sup_{\alpha \in (0,1)} |P(\max_{1 \leq j \leq q_0} \sqrt{N_0} |\hat{\theta}_j - \tilde{\theta}_j| > c_1(\alpha)) - \alpha| \lesssim n^{-c} + \zeta_1 \sqrt{1 / \log(2q_0 / \zeta_1)} + \zeta_2 \lesssim n^{-c''},
$$

for $c'' > 0$, where $\bar{c} = c$ or $c'$, which are defined in Theorem 4.2.

\[\Box\]

**Proof of Theorem 5.3.** Note that

$$
\sqrt{N_0}(\hat{\Theta}^* - \hat{\Theta}) = \frac{1}{\sqrt{N_0}} \sum_{i=1}^{N_0} \left\{ IF(u_i^*, F_{d_0}) - \frac{1}{N_0} \sum_{i=1}^{N_0} IF(u_i, F_{d_0}) \right\} + \sqrt{N_0}(R_{N_0} - R_{N_0}),
$$

which implies that

$$
J \equiv \max_{1 \leq j \leq q_0} \sqrt{N_0} |\hat{\theta}_j^* - \tilde{\theta}_j| - \max_{1 \leq j \leq q_0} \frac{1}{\sqrt{N_0}} \sum_{i=1}^{N_0} (x_{ij}^* - \bar{x}_j) \leq \max_{1 \leq j \leq q_0} \sqrt{N_0} |R_{N_0} - R_{N_0}|,
$$

where $\bar{x}_j = \sum_{i=1}^{N_0} x_{ij} / N_0$.

Denote by $\tilde{c}_2(\alpha)$ the $(1 - \alpha)$ quantile of the distribution of $\max_{1 \leq j \leq q_0} \{ \sum_{i=1}^{N_0} (x_{ij}^* - \bar{x}_j) / \sqrt{N_0} \}$ conditional on the sample $\{u_i\}$. Let $\zeta_1 = C n^{-c} / \sqrt{\log(2q_0)}$ and $\zeta_2 = C n^{-c}$ for $C > C_i$ and $c < c_i$ with $i = 1, 2, 3, 4$. Assumption 5.2 and Lemma 3.3 of [16] imply that

$$
P(P(J > \zeta_1 | \{u_i\}_{i=1}^n) > \zeta_2) \leq \zeta_2,
$$

and thus

$$
P(\tilde{c}_2(\alpha) \leq c_2(\alpha + \zeta_2) + \zeta_1) \geq 1 - \zeta_2,
$$

$$
P(c_2(\alpha) \leq \tilde{c}_2(\alpha + \zeta_2) + \zeta_1) \geq 1 - \zeta_2.
$$

Then on the event $\{c_2(\alpha) \leq \tilde{c}_2(\alpha + \zeta_2) + \zeta_1 \} \cup \{\tilde{c}_2(\alpha - \zeta_2) \leq c_2(\alpha) + \zeta_1 \} \cup \{\max_{1 \leq j \leq q_0} \sqrt{N_0} |R_{N_0}| \leq \zeta_1 \}$, we have

$$
P \left( \max_{1 \leq j \leq q_0} \frac{1}{\sqrt{N_0}} \sum_{i=1}^{N_0} x_{ij} / \sqrt{N_0} \leq \tilde{c}_2(\alpha) \right) - P \left( \max_{1 \leq j \leq q_0} \sqrt{N_0} |\hat{\theta}_j - \tilde{\theta}_j| \leq c_2(\alpha) \right) \leq P \left( \tilde{c}_2(\alpha - \zeta_2) - 2\zeta_1 \leq \max_{1 \leq j \leq q_0} \frac{1}{\sqrt{N_0}} \sum_{i=1}^{N_0} x_{ij} / \sqrt{N_0} \leq \tilde{c}_2(\alpha) \right) + P \left( \tilde{c}_2(\alpha) \leq \max_{1 \leq j \leq q_0} \frac{1}{\sqrt{N_0}} \sum_{i=1}^{N_0} x_{ij} / \sqrt{N_0} \leq \tilde{c}_2(\alpha + \zeta_2) + 2\zeta_1 \right).
$$

The conclusion follows from similar arguments in the proof of Theorem 4.3. \[\Box\]
S.4. **Technical details for Section 2.3.** To justify the validity of the procedure in Section 2.3, we impose the following assumptions which are parallel to those in Assumption 5.1.

**Assumption S.1.** Assume that under $H_0$,

$$
P\left(\max_{|l-k|\geq l} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\sqrt{\gamma_{u,jj}(0)\gamma_{u,kk}(0)} - \sqrt{\gamma_{u,jj}(0)\gamma_{u,kk}(0)} u_{ij} u_{ik}}{\sqrt{\gamma_{u,jj}(0)\gamma_{u,kk}(0)\gamma_{u,jj}(0)\gamma_{u,kk}(0)}} \right| > C_1 n^{-c_1}/\sqrt{\log(p)} \right) < C_1 n^{-c_1}
$$

and $P(\mathcal{E}_{AB}\{\log(p)\})^2 > C_2 n^{-c_2}$, where $c_1, C_1, c_2, C_2 > 0$, and

$$
\mathcal{E}_{AB} = \max_{|l-k|\geq l} \left| \frac{1}{n} \sum_{i=1}^{r} \left( (\bar{A}_{i,jk} - \hat{A}_{i,jk})^2 + (\bar{B}_{i,jk} - \hat{B}_{i,jk})^2 \right) \right|,
$$

with $\bar{A}_{i,jk} = \sum_{l=iN+(i-1)M-N+1}^{iN+(i-1)M-N+1} \bar{u}_{lj} \bar{u}_{lk}$ and $\bar{B}_{i,jk} = \sum_{l=i(N+M)-M+1}^{i(N+M)-M+1} \bar{u}_{lj} \bar{u}_{lk}$.

Below we provide some primitive conditions under which Assumption S.1 holds. To this end, we consider a $M$-dependent stationary sequence $\{x_i\}$, where $M$ is allowed to grow with the sample size.

**Lemma S.2.** Assumption S.1 holds under the following conditions,

$$
c_0 < \min_{j} \gamma_{u,jj}(0) \leq \max_{j} \gamma_{u,jj}(0) < C_0, \quad c_0, C_0 > 0,
$$

$$
\max_{1 \leq j, k \leq p} \{ \mathbb{E}(A_{1,jk}/\sqrt{N})^2 \}^{1/2} \vee \{ \mathbb{E}(B_{1,jk}/\sqrt{M})^2 \}^{1/2} \lesssim n^{s_1},
$$

$$
\max_{1 \leq j, k \leq p} ||A_{1,jk}/\sqrt{N}||_h \vee ||B_{1,jk}/\sqrt{M}||_h \lesssim n^{s_2},
$$

$$
n^{s_1} \sqrt{\log(p)/(rM)} + n^{s_2} \log(rp) \log(p)/(r\sqrt{M}) \lesssim n^{-c},
$$

$$
N n^{-c}(\log(p))^2 \lesssim n^{-c'}, \quad \frac{\sqrt{n \log p}}{n^{3c'/2}} \lesssim n^{-2c''},
$$

$$
n^{-c} \sqrt{N(\log(p))^2 \log(rp)} n^{s_1} \lesssim n^{-c''}.
$$

**Proof of Lemma S.2.** Define the block sums $A_{i,jk} = \sum_{l=iN+(i-1)M-N+1}^{iN+(i-1)M-N+1} u_{lj} u_{lk}$ and $B_{i,jk} = \sum_{l=i(N+M)-M+1}^{i(N+M)-M+1} u_{lj} u_{lk}$. Note that

$$
P\left( \max_{1 \leq j, k \leq p} |\gamma_{u,jj}(0) - \hat{\gamma}_{u,jj}(0)| > j \right) \leq \mathbb{E} \max_{1 \leq j, k \leq p} \left| \gamma_{u,jj}(0) - \hat{\gamma}_{u,jj}(0) \right| / j
$$

$$
\leq \frac{1}{j} \mathbb{E} \max_{1 \leq j, k \leq p} \left| \sum_{i=1}^{r} (A_{i,jk} - \mathbb{E}A_{i,jk})/(Nr) \right| + \frac{1}{j} \mathbb{E} \max_{1 \leq j, k \leq p} \left| \sum_{i=1}^{r} (B_{i,jk} - \mathbb{E}B_{i,jk})/(Mr) \right| .
$$
By Lemma A.1 in [16] and the assumptions,
\[
\mathbb{E} \max_{1 \leq j, k \leq p} \left| \sum_{i=1}^{r} (A_{i,jk} - \mathbb{E} A_{i,jk}) / (Nr) \right| 
\lesssim \max_{1 \leq j, k \leq p} \{ \mathbb{E}(A_{1,jk}/N)^2 \}^{1/2} \sqrt{\log(p)/r} + \max_{1 \leq i \leq r} \max_{1 \leq j, k \leq p} |A_{i,jk}/N|^2 \log(p)/r 
\lesssim \max_{1 \leq j, k \leq p} \{ \mathbb{E}(A_{1,jk}/\sqrt{N})^2 \}^{1/2} \sqrt{\log(p)/(rN)} + \max_{1 \leq j, k \leq p} |A_{1,jk}/\sqrt{N}| h \log(rp) \log(p)/(r\sqrt{N}) 
\lesssim n^{s_1} \sqrt{\log(p)/(rN)} + n^{s_2} \log(rp) \log(p)/(r\sqrt{N}) \lesssim n^{-c}.
\]

With \( j = n^{-c/2} \), we have
\[
P \left( \max_{1 \leq j, k \leq p} |\gamma_{u,jk}(0) - \hat{\gamma}_{u,jk}(0)| > n^{-c/2} \right) \lesssim n^{-c/2}.
\]

On the event \( \max_{1 \leq j, k \leq p} |\gamma_{u,jk}(0) - \hat{\gamma}_{u,jk}(0)| \leq n^{-c/2} \), we have \( c_0/2 \leq \hat{\gamma}_{u,jj}(0) \leq 2C_0 \) uniformly for \( 1 \leq j \leq p \) and some \( c_0, C_0 > 0 \). Hence we get
\[
\frac{\sqrt{\gamma_{u,jj}(0)\gamma_{u,kk}(0) - \hat{\gamma}_{u,jj}(0)\hat{\gamma}_{u,kk}(0)}}{\sqrt{\gamma_{u,jj}(0)\gamma_{u,kk}(0)\gamma_{u,jj}(0)\gamma_{u,kk}(0)}} \lesssim \sqrt{\gamma_{u,jj}(0)\gamma_{u,kk}(0) - \hat{\gamma}_{u,jj}(0)\hat{\gamma}_{u,kk}(0)} 
\lesssim \max_{1 \leq j, k \leq p} |\gamma_{u,jj}(0) - \hat{\gamma}_{u,jj}(0)| \lesssim \max_{1 \leq j, k \leq p} |\gamma_{u,jj}(0) - \hat{\gamma}_{u,jj}(0)|.
\]

On the other hand, using similar arguments above, we have
\[
I = P \left( \max_{|j-k| \geq t} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} u_{ij}u_{ik} \right| > 1 \right) + n^{-c/2} \leq \frac{1}{f} \mathbb{E} \max_{|j-k| \geq t} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} u_{ij}u_{ik} \right| + n^{-c/2}
\leq \frac{\sqrt{n}}{f} \mathbb{E} \max_{|j-k| \geq t} \left| \sum_{i=1}^{r} A_{i,jk}/(Nr) \right| + \frac{\sqrt{n}}{f} \mathbb{E} \max_{|j-k| \geq t} \left| \sum_{i=1}^{r} B_{i,jk}/(Mr) \right| + n^{-c/2},
\]

where \( f = Cn^{c'/2-c_1}/\sqrt{\log p} \).

Again by Lemma A.1 in [16],
\[
\mathbb{E} \max_{|j-k| \geq t} \left| \sum_{i=1}^{r} A_{i,jk}/(Nr) \right| \lesssim \max_{|j-k| \geq t} \{ \mathbb{E}(A_{1,jk}/N)^2 \}^{1/2} \sqrt{\log(p)/r}
\lesssim \max_{1 \leq i \leq r} \mathbb{E} \max_{|j-k| \geq t} |A_{i,jk}/N|^2 \log(p)/r 
\lesssim n^{s_1} \sqrt{\log(p)/(rN)} + n^{s_2} \log(rp) \log(p)/(r\sqrt{N}) \lesssim n^{-c},
\]

which implies that
\[
I \lesssim P \left( \max_{|j-k| \geq t} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} u_{ij}u_{ik} \right| > f \right) + n^{-c/2} \lesssim \frac{\sqrt{n} \log p}{n^{3c/2-c_1}} + n^{-c/2} \lesssim n^{-c_1},
\]
for properly chosen $c_1$. Next we show that $P(\mathcal{E}_A \log(p))^2 > C_2 n^{-c_2}) \leq C_2 n^{-c_2}$.

Let

$$\mathcal{E}_A = \max_{|j-k| \geq r} \left| \frac{1}{n} \sum_{i=1}^{r} (\hat{A}_{i,jk} - \hat{A}_{i,jk})^2 \right|, \quad \mathcal{E}_B = \max_{|j-k| \geq r} \left| \frac{1}{n} \sum_{i=1}^{r} (\hat{B}_{i,jk} - \hat{B}_{i,jk})^2 \right|.$$ 

We shall show that

$$P(\mathcal{E}_A \log(p))^2 > C_2 n^{-c_2}) \leq C_2 n^{-c_2}.$$ 

Similar arguments apply to $\mathcal{E}_B$. Note that

$$P\left( \max_{1 \leq i \leq r} \max_{|j-k| \geq r} |A_{i,jk}/\sqrt{N}| > a \right) \leq \frac{1}{a} E \max_{1 \leq i \leq r} \max_{|j-k| \geq r} |A_{i,jk}/\sqrt{N}| \leq \frac{n^{s_1} \log(rp)}{a},$$

where the value of $a$ will be determined later. On the events $\max_{1 \leq i \leq r} \max_{|j-k| \geq r} |A_{i,jk}/\sqrt{N}| < a$ and $\max_{1 \leq i \leq r} \max_{|j-k| \geq r} |\hat{A}_{i,jk}(0) - \hat{A}_{i,jk}(0)| \leq n^{-c/2}$, we have

$$\frac{1}{n} \sum_{i=1}^{r} A_{i,jk}^2 \leq \frac{1}{n} \sum_{i=1}^{r} \sqrt{N} a |A_{i,jk}| \leq \frac{1}{n} \sum_{i=1}^{r} \sqrt{N} \frac{a |\hat{A}_{i,jk}(0)|}{\sqrt{\hat{A}_{i,jk}(0) \hat{A}_{i,jk}(0)}} \leq \sqrt{Na}.$$ 

We also note that

$$\mathcal{E}_A \lesssim \max_{|j-k| \geq r} \frac{1}{n} \sum_{i=1}^{r} \left\{ \frac{A_{i,jk}(\sqrt{\gamma_{u,j}(0) \gamma_{u,k}(0)} - \sqrt{\gamma_{u,j}(0) \gamma_{u,k}(0)})}{\sqrt{\gamma_{u,j}(0) \gamma_{u,k}(0) \gamma_{u,j}(0) \gamma_{u,k}(0)}} \right\}^2 + N^2 r \max_{|j-k| \geq r} |\hat{\gamma}_{jk}|^2/n.$$ 

$$\lesssim n^{-c} \max_{|j-k| \geq r} \frac{1}{n} \sum_{i=1}^{r} \frac{A_{i,jk}^2}{\gamma_{u,j}(0) \gamma_{u,k}(0)} + N n^{-c} \lesssim n^{-c} \sqrt{Na} + N n^{-c}.$$ 

The conclusion therefore follows provided that $a = n^{s_1} \log(rp)n^{c''}/2$, $N n^{-c}(\log p)^2 \lesssim n^{-c'}$ and

$n^{-c} \sqrt{N}(\log p)^2 \log(rp)n^{s_1} \lesssim n^{-c''}$ for some $c', c'' > 0$.

\[ \diamond \]

S.5. General functions on vector sum. In this section, we extend the results in Section 3.1 to general smooth functions $\mathcal{L} : \mathbb{R}^p \to \mathbb{R}$ on the high-dimensional vector sum. We impose the following assumption S.2 regarding the smoothness of $\mathcal{L}$. Write $\partial_j \mathcal{L}(x) = \partial \mathcal{L}(x)/\partial x_j$, $\partial_{jk} \mathcal{L}(x) = \partial^2 \mathcal{L}(x)/\partial x_j \partial x_k$ and $\partial_{jkl} \mathcal{L}(x) = \partial^3 \mathcal{L}(x)/\partial x_j \partial x_k \partial x_l$ for $j, k, l = 1, 2, \ldots, p$, where $x = (x_1, x_2, \ldots, x_p)'$.

Assumption S.2. Suppose that

(S.19) \[ \sum_{j=1}^{p} |\partial_j \mathcal{L}(x)| \lesssim L_1(p), \quad \sum_{j,k=1}^{p} |\partial_{jk} \mathcal{L}(x)| \lesssim L_2(p), \quad \sum_{j,k,l=1}^{p} |\partial_{jkl} \mathcal{L}(x)| \lesssim L_3(p), \]

where the constants $L_1(p)$, $L_2(p)$ and $L_3(p)$ do not depend on $x$. Further assume that for any $\omega = (\omega_1, \ldots, \omega_p)' \in \mathbb{R}^p$ with $\max_{1 \leq j \leq p} |\omega_j| \in B_p$ for some set $B_p \subset \mathbb{R}$,

$$\partial_j \mathcal{L}(x) \lesssim \partial_j \mathcal{L}(x + \omega) \lesssim \partial_j \mathcal{L}(x),$$

$$\partial_{jk} \mathcal{L}(x) \lesssim \partial_{jk} \mathcal{L}(x + \omega) \lesssim \partial_{jk} \mathcal{L}(x),$$

$$\partial_{jkl} \mathcal{L}(x) \lesssim \partial_{jkl} \mathcal{L}(x + \omega) \lesssim \partial_{jkl} \mathcal{L}(x),$$

where the constants $L_1(p)$, $L_2(p)$ and $L_3(p)$ do not depend on $x$. Further assume that for any $\omega = (\omega_1, \ldots, \omega_p)' \in \mathbb{R}^p$ with $\max_{1 \leq j \leq p} |\omega_j| \in B_p$ for some set $B_p \subset \mathbb{R}$,
where $1 \leq j, k, l \leq p$. Here, "\( \lesssim \)" means \( \leq \) up to a universal constant.

**Example S.1.** Consider \( \mathcal{L}_\lambda(x) = \sum_{j=1}^{p} g_{j,\lambda}(x_j)/p \), where \( x = (x_1, \ldots, x_p) \) and \( \lambda \) is a thresholding parameter. Here we assume that \( g_{j,\lambda}(x) = 0 \) for \( |x| < \lambda \) and \( g_{j,\lambda} \) satisfies that \( \sum_{j=1}^{p} |\partial g_{j,\lambda}(x)/\partial x_j|/p \leq C \) for some constant \( C > 0 \). It is straightforward to verify that \( \sum_{j=1}^{p} |\partial \mathcal{L}_\lambda(x)|/p \leq C \), \( \partial \mathcal{L}_\lambda(x) = 0 \) and \( \partial_j \partial_k \partial_L \mathcal{L}_\lambda(x) = 0 \) for \( 1 \leq j, k, l \leq p \). Note that with proper choice of \( g_{j,\lambda} \), \( \mathcal{L}_\lambda(x) \) provides a smooth approximation to the function \( \sum_{j=1}^{p} |x_j| \{ |x_j| > \lambda \} \) which serves as a building block for the higher criticism test in [43].

Assumption S.2 generalizes the results in Lemmas A.5 and A.6 of [16]. Consider the dependency graph in Section 3.1. Parallel to Proposition 3.1, we have the following result. With slightly abuse of notation, set \( m = g \circ \mathcal{L} \) with \( g \in C_0^0(\mathbb{R}) \).

**Proposition S.1.** Assume that \( 2\sqrt{5}D_n^2 M_{xy}/\sqrt{n} \in \mathcal{B}_p \) with \( M_{xy} = \max\{M_x, M_y\} \). Then under Assumption S.2, we have for any \( \Delta > 0 \),

\[
|\mathbb{E}[m(X) - m(Y)]| \lesssim \{G_2L_1^2(p) + G_1L_2(p)\} \phi(M_x, M_y) \\
+ \{G_3L_1^3(p) + 3G_2L_1(p)L_2(p) + G_1L_3(p)\} \frac{D_n^2}{\sqrt{n}} (m_{x,3}^3 + m_{y,3}^3) \\
+ \{G_3L_1^3(p) + 3G_2L_1(p)L_2(p) + G_1L_3(p)\} \frac{D_n^3}{\sqrt{n}} (m_{x,3}^3 + m_{y,3}^3) + G_1\Delta + G_0\mathbb{E}[1 - \mathcal{I}],
\]

where \( G_k = \sup_{z \in \mathbb{R}} |\partial^k g(z)/\partial z^k| \) for \( k \geq 0 \). In addition, if \( 2\sqrt{5}D_n^2 M_{xy}/\sqrt{n} \in \mathcal{B}_p \), we can replace \( m_{x,3}^3 + m_{y,3}^3 \) by \( m_{x,3}^3 + m_{y,3}^3 \) in the above upper bound.

With the aid of Assumption S.2, Proposition S.1 follows from similar arguments in the proof of Proposition 3.1 (the technical details are omitted to conserve space). When specialized to stationary \( M \)-dependent time series, we have the following result.

**Theorem S.1.** Suppose \( 2\sqrt{5}(6M+1)M_{xy}/\sqrt{n} \in \mathcal{B}_p \) with \( M_{xy} = \max\{M_x, M_y\} \), and \( M_x > u_x(\gamma) \) and \( M_y > u_y(\gamma) \) for some \( \gamma \in (0, 1) \). Then

\[
|\mathbb{E}[m(X) - m(Y)]| \lesssim \{G_2L_1^2(p) + G_1L_2(p)\} \phi(M_x, M_y) \\
+ \{G_3L_1^3(p) + 3G_2L_1(p)L_2(p) + G_1L_3(p)\} \frac{(2M + 1)^2}{\sqrt{n}} (m_{x,3}^3 + m_{y,3}^3) \\
+ G_1\varphi(M_x, M_y)\sigma_j \sqrt{8\log(\gamma/p)} + G_0\gamma.
\]

Under Condition (18), we may set \( \phi(M_x, M_y) = C(1/M_x + 1/M_y) \) and \( \varphi(M_x, M_y) = C'(1/M_x^{5/6} + 1/M_y^{5/6}) \) for some constants \( C, C' > 0 \) in (S.21).
Remark S.1. Consider $L_\lambda(x) = \sum_{j=1}^p g_{j,\lambda}(x_j)/p$ in Example S.1. When $M_x > u_x(\gamma)$ and $M_y > u_y(\gamma)$ for some $\gamma \in (0, 1)$, we have
\[
|E[m_\lambda(X) - m_\lambda(Y)]| \lesssim G_2 \phi(M_x, M_y) + G_3 \frac{(2M + 1)^2}{\sqrt{n}} (\bar{m}_{x,3}^3 + \bar{m}_{y,3}^3) + G_1 \varphi(M_x, M_y) \sigma_j \sqrt{8 \log(p/\gamma)} + G_0 \gamma,
\]
where $m_\lambda = g \circ L_\lambda$. Under Condition (18),
\[
|E[m_\lambda(X) - m_\lambda(Y)]| \lesssim G_2 (1/M_x + 1/M_y) + G_3 \frac{(2M + 1)^2}{\sqrt{n}} (\bar{m}_{x,3}^3 + \bar{m}_{y,3}^3) + G_1 (1/M_x^{5/6} + 1/M_y^{5/6}) \sigma_j \sqrt{8 \log(p/\gamma)} + G_0 \gamma.
\]
By letting $M_x \to +\infty$, $M_y \to +\infty$, and $\gamma = M^2/\sqrt{n}$, we deduce that $|E[m_\lambda(X) - m_\lambda(Y)]| \lesssim M^2(\bar{m}_{x,3}^3 + \bar{m}_{y,3}^3)/\sqrt{n}$. Note that in this case, $p$ is allowed to grow arbitrarily.

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