Galois Theory – a first course

Brent Everitt*

Contents

0. What is Galois Theory? ................................................................. 3
1. Rings I: Polynomials ................................................................. 8
2. Roots and Irreducibility ............................................................. 14
3. Fields I: Basics, Extensions and Concrete Examples .................... 22
4. Rings II: Quotients ................................................................ 28
5. Fields II: Constructions and More Examples ............................ 33
6. Ruler and Compass Constructions I ......................................... 39
7. Vector Spaces I: Dimensions .................................................... 46
8. Fields III: Splitting Fields and Finite Fields ............................. 53
9. Ruler and Compass Constructions II ....................................... 57
10. Groups I: Soluble Groups and Simple Groups .......................... 63
11. Groups II: Symmetries of Fields ............................................. 70
12. Vector Spaces II: Solving Equations ....................................... 79
13. The Fundamental Theorem of Galois Theory ......................... 82
14. Applications of the Galois Correspondence ............................. 88
15. (Not) Solving Equations .......................................................... 91

Introductory Note

These notes are a self-contained introduction to Galois theory, designed for the student who has done a first course in abstract algebra.

To not clutter up the theorems too much, I have made some restrictions in generality. For example, all rings are with 1; all ideals are principal; all fields are perfect – in fact, extensions

Brent Everitt: Department of Mathematics, University of York, York YO10 5DD, United Kingdom. e-mail: brent.everitt@york.ac.uk.

* version April 16, 2018.
of \( \mathbb{Q} \) or of finite fields; consequently all field extensions are separable; and so on. This won’t be to everyone’s taste.

The following prerequisites are assumed, although there are reminders: the basics of linear algebra, particularly the span and independence of a set of vectors; the idea of a basis and hence the dimension of a vector space. In group theory the fundamentals up to Lagrange’s theorem and the first isomorphism theorem. In ring and field theory the definitions and some examples, but probably not much else.

There are many books on linear algebra and group theory for beginners. My personal favourite is:

[Arm88] M. A. Armstrong, *Groups and symmetry*, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1988. MR965514

Most of the results and proofs are standard and can be found in any book on Galois theory, but I am particularly indebted to the book of Joseph Rotman:

[Rot90] Joseph Rotman, *Galois theory*, Universitext, Springer-Verlag, New York, 1990. MR1064318

In particular the proofs I give of Theorems C and E, the Fundamental Theorem of Algebra and the Theorem of Abel’s-Ruffini are Rotman’s proofs with some elaboration added. The statements (although not the proofs) of Theorems F and G are also his.

The figure depicting the \((a, b)\)-plane at the end of Section 15 is redrawn from the Mathematica poster *Solving the Quintic*.

**The Cover**

The cover shows a Cayley graph for the smallest non-Abelian simple group – the alternating group \( A_5 \). We will see that the simplicity of this group means there is no formula for the roots of the polynomial \( x^5 - 4x + 2 \), using only the ingredients

\[
\frac{a}{b} \in \mathbb{Q}, +, -, \times, \div, \sqrt{2}, \sqrt{3}, \sqrt{5}, \ldots
\]

Therefore, there can be no formula for the solutions of a quintic equation

\[
ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0
\]

that works for all possible \( a, b, c, d, e, f \in \mathbb{C} \).

A Cayley graph is a picture of the multiplication in the group. Let \( \sigma = (1, 2, 3, 4, 5) \). Each blue pentagonal face can be oriented anti-clockwise when you look at it from the outside of the ball. Crossing a blue edge anti-clockwise corresponds to \( \sigma \) and crossing in the reverse direction (clockwise) corresponds to \( \sigma^{-1} \). Crossing a black edge in either direction corresponds to the element \( \tau = (1, 2)(3, 4) \).

The vertices correspond to the 60 elements of \( A_5 \) – the front ones are marked, with the identity element in the center. If a path \( \gamma \) starts at the vertex corresponding to \( \mu_1 \in A_5 \) and finishes at \( \mu_2 \in A_5 \), then reading the \( \sigma \) and \( \tau \) labels off \( \gamma \) as you travel along it gives \( \mu_1 \gamma = \mu_2 \). For example, the red path gives \((1, 2, 3, 4, 5) \cdot \sigma \sigma^2 \sigma^{-2} \tau \sigma^2 = (2, 5)(3, 4) \).

It is a curious coincidence that the smallest non-Abelian simple group has Cayley graph the simplest known pure form of Carbon – Buckminsterfullerine \( C_{60} \).
0. What is Galois Theory?

A quadratic equation $ax^2 + bx + c = 0$ has exactly two – possibly repeated – solutions in the complex numbers. There is a formula for them, that appears in the ninth century Persian book *Hisab al-jabr w’al-muqabala* by Abu Abd-Allah ibn Musa al’Khwarizmi. In modern notation it says:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Less familiar maybe, $ax^3 + bx^2 + cx + d = 0$ has three $\mathbb{C}$-solutions, and they too can be expressed algebraically using Cardano’s formula. One solution turns out to be

$$x = \frac{-b}{3a} + \sqrt[3]{-\frac{1}{2} \left( \frac{2b^3}{27a^3} - \frac{bc}{a^2} + \frac{d}{a} \right)} - \sqrt[3]{\frac{1}{4} \left( \frac{2b^3}{27a^3} - \frac{bc}{a^2} + \frac{d}{a} \right)^2 + \frac{1}{27} \left( \frac{c}{a} - \frac{b^2}{3a^2} \right)^3},$$

and the other two have similar expressions. There is an even more complicated formula, attributed to Descartes, for the roots of a quartic polynomial equation.

---

1 *al-jabr*, hence “algebra.”
What is kind of miraculous is not that the solutions exist, but they can always be expressed in terms of the coefficients and the basic algebraic operations,

\[ +, -, \times, \div, \sqrt[n]{\cdot}, \sqrt{\cdot}, \sqrt[n]{\cdot} \ldots \]

By the turn of the 19th century, no equivalent formula for the solutions to a quintic (degree five) polynomial equation had materialised, and it was Abels who had the crucial realisation: *no such formula exists*.

Such a statement can be interpreted in a number of ways. Does it mean that there are always algebraic expressions for the roots of quintic polynomials, but their form is too complex for one single formula to describe all the possibilities? It would therefore be necessary to have a number, maybe even infinitely many, formulas. The reality turns out to be far worse: there are specific polynomials, such as \( x^5 - 4x + 2 \), whose solutions cannot be expressed algebraically in any way whatsoever.

A few decades later, Evaristé Galois started thinking about the deeper problem: why don’t these formulae exist? Thus, Galois theory was originally motivated by the desire to understand, in a much more precise way, the solutions to polynomial equations.

Galois’ idea was this: study the solutions by studying their “symmetries”. Nowadays, when we hear the word symmetry, we normally think of group theory. To reach his conclusions, Galois kind of invented group theory along the way. In studying the symmetries of the solutions to a polynomial, Galois theory establishes a link between these two areas of mathematics. We illustrate the idea, in a somewhat loose manner, with an example.

### 0.1. The symmetries of the solutions to \( x^3 - 2 = 0 \).

\((0.1)\). We work in \( \mathbb{C} \). Let \( \alpha \) be the real cube root of 2, ie: \( \alpha = \sqrt[3]{2} \in \mathbb{R} \) and, \( \omega = -\frac{1}{2} + \frac{\sqrt[3]{3}i}{2} \). Note that \( \omega \) is a cube root of 1, and so \( \omega^3 = 1 \).

The three solutions to \( x^3 - 2 = 0 \) (or roots of \( x^3 - 2 \)) are the complex numbers \( \alpha, \alpha \omega \) and \( \alpha \omega^2 \), forming the vertices of the equilateral triangle shown. The triangle has what we might call “geometric symmetries”: three reflections, a counter-clockwise rotation through \( \frac{1}{3} \) of a turn, a counter-clockwise rotation through \( \frac{2}{3} \) of a turn and a counter-clockwise rotation through \( \frac{3}{3} \) of a turn = the identity symmetry. Notice for now that if \( s \) and \( t \) are the reflections in the lines shown, the geometrical symmetries are \( s, t, tst, ts \), \( (ts)^2 \) and \( (ts)^3 = id \) (read these expressions from right to left).

The symmetries referred to in the preamble are not so much geometric as “number theoretic”. It will take a little explaining before we see what this means.

#### Definition 0.1 (field – version 1).
A field is a set \( F \) with two operations, called, purely for convenience, + and \( \times \), such that for any \( a, b, c \in F \),

1. \( a + b \) and \( a \times b \) (= \( ab \) from now on) are uniquely defined elements of \( F \),
2. \( a + (b + c) = (a + b) + c \),
3. \( a + b = b + a \),
4. there is an element \( 0 \in F \) such that \( 0 + a = a \),
5. for any \( a \in F \) there is an element \( -a \in F \) with \( (-a) + a = 0 \),
6. \( a(bc) = (ab)c \),
7. \( ab = ba \),
8. there is an element \( 1 \in F \setminus \{0\} \) with \( 1 \times a = a \),
9. for any \( a \neq 0 \in F \) there is an \( a^{-1} \in F \) with \( aa^{-1} = 1 \),
10. \( a(b + c) = ab + ac \).
A field is just a set of things that you can add, subtract, multiply and divide so that the “usual” rules of algebra are satisfied. Familiar examples of fields are $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$; familiar non-examples of fields are $\mathbb{Z}$, polynomials and matrices (you cannot in general divide integers, polynomials and matrices to get integers, polynomials or matrices).

(0.2). A subfield of a field $F$ is a subset that also forms a field under the same $+$ and $\times$. Thus, $\mathbb{Q}$ is a subfield of $\mathbb{R}$ which is in turn a subfield of $\mathbb{C}$, and so on. On the other hand, $\mathbb{Q} \cup \{\sqrt{2}\}$ is not a subfield of $\mathbb{R}$: it is a subset but axiom 1 fails, as both 1 and $\sqrt{2}$ are elements but $1 + \sqrt{2}$ is not.

**Definition 0.2.** If $F$ is a subfield of the complex numbers $\mathbb{C}$ and $\beta \in \mathbb{C}$, then $F(\beta)$ is the smallest subfield of $\mathbb{C}$ that contains both $F$ and the number $\beta$.

What do we mean by smallest? That there is no other field $F'$ having the same properties as $F(\beta)$ which is smaller, i.e. no $F'$ with $F \subseteq F'$ and $\beta \in F'$ too, but $F'$ properly $\subset F(\beta)$. It is usually more useful to say it the other way around:

If $F'$ is a subfield that also contains $F$ and $\beta$, then $F'$ contains $F(\beta)$ too. (*)

Loosely speaking, $F(\beta)$ is all the complex numbers we get by adding, subtracting, multiplying and dividing the elements of $F$ and $\beta$ together in all possible ways.

The construction of Definition 0.2 can be continued: write $F(\beta, \gamma)$ for the smallest subfield of $\mathbb{C}$ containing $F$ and the numbers $\beta$ and $\gamma$, and so on.

(0.3). To illustrate with some trivial examples, $\mathbb{R}(i)$ can be shown to be all of $\mathbb{C}$: it must contain all expressions of the form $bi$ for $b \in \mathbb{R}$, and hence all expressions of the form $a + bi$ with $a, b \in \mathbb{R}$, and this accounts for all the complex numbers; $\mathbb{Q}(2)$ is equally clearly just $\mathbb{Q}$ back again.

Slightly less trivially, $\mathbb{Q}(\sqrt{2})$, the smallest subfield of $\mathbb{C}$ containing all the rational numbers and $\sqrt{2}$, is a field that is strictly bigger than $\mathbb{Q}$ (eg: it contains $\sqrt{2}$) but is much, much smaller than all of $\mathbb{R}$.

**Exercise 0.1.** Show that $\sqrt{3} \notin \mathbb{Q}(\sqrt{2})$.

(0.4). Returning to the symmetries of the solutions to $x^3 - 2 = 0$, we look at the field $\mathbb{Q}(\alpha, \omega)$, where $\alpha = \sqrt[3]{2} \in \mathbb{R}$ and $\omega = \frac{1}{2} + \frac{\sqrt{3}}{2}i$, as before. Since $\mathbb{Q}(\alpha, \omega)$ is by definition a field, and fields are closed under $+$ and $\times$, we have

$$\alpha \in \mathbb{Q}(\alpha, \omega) \text{ and } \omega \in \mathbb{Q}(\alpha, \omega) \Rightarrow \alpha \times \omega = \alpha \omega, \alpha \times \omega \times \omega = \alpha \omega^2 \in \mathbb{Q}(\alpha, \omega) \text{ too}.$$ 

So, $\mathbb{Q}(\alpha, \omega)$ contains all the solutions to the equation $x^3 - 2 = 0$. On the other hand:

**Exercise 0.2.** Show that $\mathbb{Q}(\alpha, \omega)$ has “just enough” numbers to solve the equation $x^3 - 2 = 0$. More precisely, $\mathbb{Q}(\alpha, \omega)$ is the **smallest** subfield of $\mathbb{C}$ that contains all the solutions to this equation. (hint: you may find it useful to do Exercise 0.5 first).

(0.5). A very loose definition of a symmetry of the solutions of $x^3 - 2 = 0$ is that it is a “rearrangement” of $\mathbb{Q}(\alpha, \omega)$ that does not disturb (or is compatible with) the $+$ and $\times$.

To see an example, consider the two fields $\mathbb{Q}(\alpha, \omega)$ and $\mathbb{Q}(\alpha, \omega^2)$. Despite first appearances, they are actually the same: certainly

$$\alpha, \omega \in \mathbb{Q}(\alpha, \omega) \Rightarrow \alpha, \omega^2 \in \mathbb{Q}(\alpha, \omega).$$

But $\mathbb{Q}(\alpha, \omega^2)$ is the smallest field containing $\mathbb{Q}$, $\alpha$ and $\omega^2$, so by (*),

$$\mathbb{Q}(\alpha, \omega^2) \subseteq \mathbb{Q}(\alpha, \omega).$$
Conversely,  
\[ \alpha, \omega^2 \times \omega^2 = \omega^4 = \omega \in \mathbb{Q}(\alpha, \omega^2) \Rightarrow \mathbb{Q}(\alpha, \omega) \subseteq \mathbb{Q}(\alpha, \omega^2). \]
Remember that \( \omega^3 = 1 \) so \( \omega^4 = \omega \). Thus \( \mathbb{Q}(\alpha, \omega) \) and \( \mathbb{Q}(\alpha, \omega^2) \) are indeed the same. In fact, we should think of \( \mathbb{Q}(\alpha, \omega) \) and \( \mathbb{Q}(\alpha, \omega^2) \) as two different ways of looking at the same field, or more suggestively, the same field viewed from two different angles.

When we hear the phrase, “the same field viewed from two different angles”, it suggests that there is a symmetry that moves the field from one point of view to the other. In the case above, there should be a symmetry of the field \( \mathbb{Q}(\alpha, \omega) \) that puts it into the form \( \mathbb{Q}(\alpha, \omega^2) \). Surely this symmetry should send \( \alpha \mapsto \alpha, \) and \( \omega \mapsto \omega^2 \).

We haven’t yet defined what we mean by, “is compatible with the + and \( \times \)”. It will turn out to mean that if \( \alpha \) and \( \omega \) are sent to \( \alpha \) and \( \omega^2 \) respectively, then \( \alpha \times \omega \) should go to \( \alpha \times \omega^2 \); similarly \( \alpha \times \omega \times \omega \) should go to \( \alpha \times \omega^2 \times \omega^2 = \alpha \omega^4 = \alpha \omega \), and so on. The symmetry thus moves the vertices of the equilateral triangle determined by the roots in the same way that the reflection \( s \) of the triangle does (see Figure 0.1).

(This compatibility also means that it would have made no sense to have the symmetry send \( \alpha \mapsto \omega^2 \) and \( \omega \mapsto \alpha \). A symmetry should not fundamentally change the algebra of the field, so that if an element like \( \omega \) cubes to give 1, then its image under the symmetry should too: but \( \alpha \) doesn’t cube to give 1.)

(0.6) In exactly the same way, we can consider the fields \( \mathbb{Q}(\alpha \omega, \omega^2) \) and \( \mathbb{Q}(\alpha, \omega) \). We have
\[ \alpha, \omega \in \mathbb{Q}(\alpha, \omega) \Rightarrow \omega^2, \alpha \omega \in \mathbb{Q}(\alpha, \omega) \Rightarrow \mathbb{Q}(\alpha \omega, \omega^2) \subseteq \mathbb{Q}(\alpha, \omega); \]
and conversely, \( \alpha \omega, \omega^2 \in \mathbb{Q}(\alpha \omega, \omega^2) \Rightarrow \alpha \omega \omega^2 = \alpha \omega^3 = \alpha \in \mathbb{Q}(\alpha \omega, \omega^2) \), and hence also
\[ \alpha^{-1} \alpha \omega = \omega \in \mathbb{Q}(\alpha \omega, \omega^2) \Rightarrow \mathbb{Q}(\alpha, \omega) \subseteq \mathbb{Q}(\alpha \omega, \omega^2). \]

Thus, \( \mathbb{Q}(\alpha, \omega) \) and \( \mathbb{Q}(\alpha \omega, \omega^2) \) are the same field, and we can define another symmetry that sends \( \alpha \mapsto \alpha \omega, \) and \( \omega \mapsto \omega^2 \).

To be compatible with the + and \( \times \),
\[ \alpha \times \omega \mapsto \alpha \omega \times \omega^2 = \alpha \omega^3 = \alpha, \text{ and } \alpha \times \omega \times \omega \mapsto \alpha \omega \times \omega^2 \times \omega^2 = \alpha \omega^4 = \alpha \omega^2. \]
So the symmetry is like the reflection \( t \) of the triangle (see Figure 0.1).

Finally, if we have two symmetries of the solutions to some equation, we would like their composition to be a symmetry too. So if the symmetries \( s \) and \( t \) of the original triangle are to be considered, so should \( t st, st, (st)^2 \) and \( (st)^3 = 1 \).
(0.7). The symmetries of the solutions to \( x^3 - 2 = 0 \) include all the geometrical symmetries of the equilateral triangle. We will see later that any symmetry of the solutions is uniquely determined as a permutation of the solutions. Since there are \( 3! = 6 \) of these, we have accounted for all of them. So the solutions to \( x^3 - 2 = 0 \) have symmetry precisely the geometrical symmetries of the equilateral triangle.

(0.8). If this was always the case, things would be a little disappointing: Galois theory would just be the study of the “shapes” formed by the roots of polynomials, and the symmetries of those shapes. It would be a branch of planar geometry.

Fortunately, if we look at the solutions to \( x^5 - 2 = 0 \), given in Figure 0.2, then something quite different happens. Exercise 0.4 shows you how to find these expressions for the roots.

A pentagon has 10 geometric symmetries, and you can check that all arise as symmetries of the roots of \( x^5 - 2 \) using the same reasoning as in the previous example. But this reasoning also gives a symmetry that moves the vertices of the pentagon according to:

\[
\alpha = \sqrt{2}, \quad \omega = \frac{\sqrt{5} - 1}{4} + \frac{\sqrt{5} \sqrt{5 + \sqrt{5}}}{4}i
\]

This is not a geometrical symmetry – if it was, it would be pretty disastrous for the poor pentagon. Later we will see that for \( p > 2 \) a prime number, the solutions to \( x^p - 2 = 0 \) have \( p(p-1) \) symmetries. While agreeing with the six obtained for \( x^3 - 2 = 0 \), it gives twenty for \( x^5 - 2 = 0 \). In fact, it was a bit of a fluke that all the number theoretic symmetries were also geometric ones for \( x^3 - 2 = 0 \). A \( p \)-gon has \( 2p \) geometrical symmetries and \( 2p \leq p(p-1) \) with equality only when \( p = 3 \).

Further Exercises for Section 0

Exercise 0.3. Show that the picture on the left of Figure 0.3 depicts a symmetry of the solutions to \( x^3 - 1 = 0 \), but the one on the right does not.

Exercise 0.4. You already know that the 3-rd roots of 1 are 1 and \( \frac{-1}{2} \pm \frac{\sqrt{3}}{2}i \). What about the \( p \)-th roots for higher primes?

1. If \( \omega \neq 1 \) is a 5-th root it satisfies \( \omega^4 + \omega^3 + \omega^2 + \omega + 1 = 0 \). Let \( u = \omega + \omega^{-1} \). Find a quadratic polynomial satisfied by \( u \), and solve it to obtain \( u \).
2. Find another quadratic satisfied this time by \( \omega \), with *coefficients involving* \( u \), and solve it to find explicit expressions for the four primitive 5-th roots of 1.

3. Repeat the process with the 7-th roots of 1.

*factoid:* the *n*-th roots of 1 can be expressed in terms of field operations and extraction of pure roots of rationals for any *n*. The details – which are a little complicated – were completed by the work of Gauss and Galois.

**Exercise 0.5.** Let \( F \) be a field such that the element
\[
\underbrace{1 + 1 + \cdots + 1}_n \neq 0,
\]
for any \( n > 0 \). Arguing intuitively, show that \( F \) contains a copy of the rational numbers \( \mathbb{Q} \) (see also Section 3).

**Exercise 0.6.** Let \( \alpha = \sqrt[5]{2} \in \mathbb{R} \) and \( \omega = \frac{1}{2} + \frac{\sqrt{3}}{2}i \). Show that \( \mathbb{Q}(\alpha, \omega) \), \( \mathbb{Q}(\alpha\omega^2, \omega^5) \) and \( \mathbb{Q}(\alpha\omega^4, \omega^5) \) are all the same field.

**Exercise 0.7.**

1. Show that there is a symmetry of the solutions to \( x^5 - 2 = 0 \) that moves the vertices of the pentagon according to:

\[
\alpha \quad \alpha \omega \quad \alpha \omega^3 \quad \alpha \omega^4
\]

where \( \alpha = \sqrt[5]{2} \), and \( \omega^5 = 1, \omega \in \mathbb{C} \).

2. Show that the solutions in \( \mathbb{C} \) to the equation \( x^6 - 5 = 0 \) have 12 symmetries.

1. **Rings I: Polynomials**

(1.1). There are a number of basic facts about polynomials that we will need. Suppose \( F \) is a field (\( \mathbb{Q}, \mathbb{R} \) or \( \mathbb{C} \) will do for now). A polynomial *over* \( F \) is an expression of the form
\[
f = a_0 + a_1x + \cdots + a_nx^n,
\]
where the \( a_i \in F \) and \( x \) is a “formal symbol” (sometimes called an indeterminate). We don’t tend to think of \( x \) as a variable – it is purely an object on which to perform algebraic manipulations. Denote the set of all polynomials over \( F \) by \( F[x] \). If \( a_n \neq 0 \), then \( n \) is called the \textit{degree} of \( f \), written \( \deg(f) \). If the leading coefficient \( a_n = 1 \), then \( f \) is \textit{monic}.

(The degree of a non-zero constant polynomial is thus 0, but to streamline some statements define \( \deg(0) = -\infty \), where \( -\infty < n \) for all \( n \in \mathbb{Z} \). The arithmetic of degrees is just the arithmetic of non-negative integers, except we decree that \( -\infty + n = -\infty \). A polynomial \( f \) is \textit{constant} if \( \deg f \leq 0 \), and \textit{non-constant} otherwise.

(1.2). We can add and multiply elements of \( F[x] \) in the usual way:

if \( f = \sum_{i=0}^{n} a_i x^i \) and \( g = \sum_{i=0}^{m} b_i x^i \),

then,

\[
f + g = \sum_{i=0}^{\max(m,n)} (a_i + b_i)x^i \quad \text{and} \quad fg = \sum_{k=0}^{m+n} c_k x^k \quad \text{where} \quad c_k = \sum_{i+j=k} a_ib_j.
\]

that is, \( c_k = a_0b_k + a_1b_{k-1} + \cdots + a_kb_0 \). The arithmetic of the coefficients (ie: how to work out \( a_i + b_i, a_ib_j \) and so on) is just that of the field \( F \).

\textit{Exercise 1.1.} Convince yourself that this multiplication is really just the “expanding brackets” multiplication of polynomials that you know so well.

(1.3). The polynomials \( F[x] \) together with this addition form an example of an Abelian group:

\textbf{Definition 1.1 (Abelian group).} An \textit{Abelian group} is a set \( G \) endowed with an operation \( (f,g) \mapsto f + g \) such that for all \( f,g,h \in G \):

1. \( f + g \) is a uniquely defined element of \( G \) (closure);
2. \( f + (g + h) = (f + g) + h \) (associativity);
3. there is an \( 0 \in G \) such that \( 0 + f = f = f + 0 \) (identity);
4. for any \( f \in G \) there is an element \(-f \in G \) with \( f + (-f) = 0 = (-f) + f \) (inverses).
5. \( f + g = g + f \) (commutativity).

We will see more general kinds of groups in Section [10] where we will write the operation as juxtaposition. In an Abelian group however, it is customary to write the operation as addition, as we have done above. In \( F[x] \) the identity \( 0 \) is the zero polynomial, and the inverse of \( f \) is

\[
-\left( \sum_{i=0}^{n} a_i x^i \right) = \sum_{i=0}^{n} (-a_i) x^i.
\]

(To see that \( F[x] \) forms an abelian group, we have \( f + g = g + f \) exactly when \( a_i + b_i = b_i + a_i \) for all \( i \). But the coefficients of our polynomials come from the field \( F \), and addition is always commutative in a field.)

(1.4). If we want to include the multiplication, we need the formal concept of a ring:

\textbf{Definition 1.2 (ring).} A ring is a set \( R \) endowed with two operations \( (a,b) \mapsto a + b \) and \( a \times b \) such that for all \( a,b \in R \):

1. \( R \) is an Abelian group under \(+\);
2. for any \( a,b \in R \), \( a \times b \) is a uniquely determined element of \( R \) (closure of \( \times \));
3. \( a \times (b \times c) = (a \times b) \times c \) (associativity of \( \times \));
4. there is an $1 \in R$ such that $1 \times a = a = a \times 1$ (identity of $\times$);
5. $a \times (b + c) = (a \times b) + (a \times c)$ and $(b + c) \times a = (b \times a) + (c \times a)$ (the distributive law).

Loosely, a ring is a set on which you can add (+), subtract (the inverse of + in the Abelian group) and multiply ($\times$), but not necessarily divide (there is no inverse axiom for $\times$).

Here are some well known examples of rings:

$\mathbb{Z}$, $F[x]$ for $F$ a field, $\mathbb{Z}_n$ and $M_n(F),$

where $\mathbb{Z}_n$ is addition and multiplication of integers modulo $n$ and $M_n(F)$ are the $n \times n$ matrices, with entries from $F$, together with the usual addition and multiplication of matrices.

A ring is commutative if the second operation $\times$ is commutative: $a \times b = b \times a$ for all $a, b$.

Exercise 1.2.
1. Show that $fg = gf$ for polynomials $f, g \in F[x]$, hence $F[x]$ is a commutative ring.
2. Show that $\mathbb{Z}$ and $\mathbb{Z}_n$ are commutative rings, but $M_n(F)$ is not for any field $F$ if $n > 2$.

(1.5). The observation that $\mathbb{Z}$ and $F[x]$ are both commutative rings is not just some vacuous formalism. A concrete way of putting it is this: at a very fundamental level, integers and polynomials share the same algebraic properties.

When we work with polynomials, we need to be able to add and multiply the coefficients of the polynomials in a way that doesn’t produce any nasty surprises—in other words, the coefficients have to satisfy the basic rules of algebra that we all know and love. But these basic rules of algebra can be found among the axioms of a ring. Thus, to work with polynomials successfully, all we need is that the coefficients come from a ring.

This observation means that for a ring $R$, we can form the set of all polynomials with coefficients from $R$ and add and multiply them together as we did above. In fact, we are just repeating what we did above, but are replacing the field $F$ with a ring $R$. In practice, rather than allowing our coefficients to some from an arbitrary ring, we take $R$ to be commutative. This leads to,

Definition 1.3. Let $R[x]$ be the set of all polynomials with coefficients from some commutative ring $R$, together with the $+$ and $\times$ defined at (1.7).

Exercise 1.3.
1. Show that $R[x]$ forms a ring.
2. Since $R[x]$ forms a ring, we can consider polynomials with coefficients from $R[x]$; take a new variable, say $y$, and consider $R[x][y]$. Show that this is just the set of polynomials in two variables $x$ and $y$ together with the ‘obvious’ $+$ and $\times$.

(1.6). A commutative ring $R$ is called an integral domain iff for any $a, b \in R$ with $a \times b = 0$, we have $a = 0$, or $b = 0$ or both. Clearly $\mathbb{Z}$ is an integral domain.

Exercise 1.4.
1. Show that any field $F$ is an integral domain.
2. For what values of $n$ is $\mathbb{Z}_n$ an integral domain?

Lemma 1.1. Let $f, g \in R[x]$ for $R$ an integral domain. Then
1. deg$(fg) = \text{deg}(f) + \text{deg}(g)$.
2. $R[x]$ is an integral domain.

The second part means that given polynomials $f$ and $g$ (with coefficients from an integral domain), we have $fg = 0 \Rightarrow f = 0$ or $g = 0$. You have been implicitly using this fact when you solve polynomial equations by factorising them.
Proof. We have
\[ fg = \sum_{k=0}^{m+n} c_k x^k \] where \( c_k = \sum_{i+j=k} a_i b_j \), so in particular \( c_{m+n} = a_m b_n \neq 0 \) as \( R \) is an integral domain. Thus \( \deg(fg) \geq m + n \) and since the reverse inequality is obvious, we have part (1) of the Lemma. Part (2) now follows immediately since \( fg = 0 \) \( \Rightarrow \) \( \deg(fg) = -\infty \) \( \Rightarrow \) \( \deg f + \deg g = -\infty \), which can only happen if at least one of \( f \) or \( g \) has degree \( = -\infty \) (see the footnote at the bottom of the first page). \( \square \)

All your life you have been happily adding the degrees of polynomials when you multiply them. But as Lemma [11] shows, this is only possible when the coefficients of the polynomial come from an integral domain. For example, \( \mathbb{Z}_6 \), the integers under addition and multiplication modulo 6, is a ring that is not an integral domain (as \( 2 \times 3 = 0 \) for example), and sure enough, \[(3x + 1)(2x + 1) = 5x + 1, \]
where all of this is happening in \( \mathbb{Z}_6[x] \).

(1.7). Although we cannot necessarily divide two polynomials and get another polynomial, we can divide up to a possible “error term”, or, as it is more commonly called, a remainder.

**Theorem A (The division algorithm).** Suppose \( f \) and \( g \) are elements of \( R[x] \) where the leading coefficient of \( g \) has a multiplicative inverse in the ring \( R \). Then there exist \( q \) and \( r \) in \( R[x] \) (quotient and remainder) such that
\[ f = qg + r, \]
where the degree of \( r \) is \(<\) the degree of \( g \).

When \( R \) is a field (where you may be more used to doing long division) all the non-zero coefficients of a polynomial have multiplicative inverses (as they lie in a field) so the condition on \( g \) becomes \( g \neq 0 \).

**Proof.** For all \( q \in R[x] \), consider those polynomials of the form \( f - gq \) and choose one, say \( r \), of smallest degree. Let \( d = \deg r \) and \( m = \deg g \). We claim that \( d < m \). This will give the result, as the \( r \) chosen has he form \( r = f - gq \) for some \( q \), giving \( f = gq + r \). Suppose that \( d \geq m \) and consider \( \tilde{r} = (r_d)(g_m^{-1})x^{d-m}g \), a polynomial since \( d - m \geq 0 \). Notice also that we have used the fact that the leading coefficient of \( g \) has a multiplicative inverse. The leading term of \( \tilde{r} \) is \( r_d x^d \), which is also the leading term of \( r \). Thus, \( r - \tilde{r} \) has degree \( < d \). But \( r - \tilde{r} = f - gq - r_d g_m^{-1} x^{d-m} g \) by definition, which equals \( f - g(q - r_d g_m^{-1} x^{d-m}) = f - g\tilde{q} \), say. Thus \( r - \tilde{r} \) has the form \( f - g\tilde{q} \) too, but with smaller degree than \( r \), which was of minimal degree amongst all polynomials of this form—this is our desired contradiction. \( \square \)

**Exercise 1.5.**
1. If \( R \) is an integral domain, show that the quotient and remainder are unique.
2. Show that the quotient and remainder are not unique when you divide polynomials in \( \mathbb{Z}_6[x] \).
Other familiar concepts from the integers are those of divisors, common divisors and greatest common divisors. Since we need no more algebra to define these notions than given by the axioms for a ring, these concepts carry pretty much straight over to polynomial rings. We will state these in the setting of polynomials from $F[x]$ for $F$ a field.

**Definition 1.4.** For $f, g \in F[x]$, we say that $f$ divides $g$ iff $g = fh$ for some $h \in F[x]$. Write $f \mid g$.

**Definition 1.5.** Let $f, g \in F[x]$. Suppose that $d$ is a polynomial satisfying

1. $d$ is a common divisor of $f$ and $g$, ie: $d \mid f$ and $d \mid g$;
2. if $c$ is a polynomial with $c \mid f$ and $c \mid g$ then $c \mid d$;
3. $d$ is monic.

Then $d$ is called (the) greatest common divisor of $f$ and $g$.

As with the division algorithm, we have tweaked the definition from the integers to make it work in $F[x]$. The reason is that we want the gcd to be unique. In the integers, you ensure this by insisting that all gcd's are positive; in $F[x]$ we insist they are monic.

**Theorem 1.1.** Any two $f, g \in F[x]$ have a greatest common divisor $d$. Moreover, there are $a_0, b_0 \in F[x]$ such that

$$d = a_0f + b_0g.$$  

Compare this with the integers! You can replace $F[x]$ by the integers in the following proof to get the corresponding fact for the integers.

**Proof.** Consider the set $I = \{af + bg \mid a, b \in F[x]\}$. Let $d \in I$ be a monic polynomial with minimal degree. Then $d \in I$ gives that $d = a_0f + b_0g$ for some $a_0, b_0 \in F[x]$. We claim that $d$ is the gcd of $f$ and $g$. The following two basic facts are easy to verify:

1. The set $I$ is a subgroup of the Abelian group $F[x]$–exercise.
2. If $u \in I$ and $w \in F[x]$ then $uw \in I$, since $wu = w(af + bg) = (wa)f + (wb)g \in I$.

Consider now the set $P = \{hd \mid h \in F[x]\}$. Since $d \in I$ and by the second observation above, $hd \in I$, and we have $P \subseteq I$. Conversely, if $u \in I$ then by the division algorithm, $u = qd + r$ where $r = 0$ or $\deg(r) < \deg(d)$. Now, $r = u - qd$ and $d \in I$, so $qd \in I$ by (2). But $u \in I$ and $qd \in I$ so $u - dq = r \in I$ by (1) above. Thus, if $\deg(r) < \deg(d)$ we would have a contradiction to the degree of $d$ being minimal, and so we must have $r = 0$, giving $u = qd$. This means that any element of $I$ is a multiple of $d$, so $I \subseteq P$.

Now that we know that $I$ is just the set of all multiples of $d$, and since letting $a = 1, b = 0$ or $a = 0, b = 1$ gives that $f, g \in I$, we have that $d$ is a common divisor of $f$ and $g$. Finally, if $d'$ is another common divisor, then $f = u_1d'$ and $g = u_2d'$, and since $d = a_0f + b_0g$, we have $d = a_0u_1d' + b_0u_2d' = d'(a_0u_1 + b_0u_2)$ giving $d' \mid d$. Thus $d$ is indeed the greatest common divisor. □
Definition 1.6 (Ring homomorphism). Let $R$ and $S$ be rings. A mapping $\varphi : R \rightarrow S$ is called a ring homomorphism if and only if for all $a, b \in R$,

1. $\varphi(a + b) = \varphi(a) + \varphi(b)$;
2. $\varphi(ab) = \varphi(a)\varphi(b)$;
3. $\varphi(1_R) = 1_S$ (where $1_R$ is the multiplicative identity in $R$ and $1_S$ the multiplicative identity in $S$).

The reason we need the last item but not $\varphi(0) = 0$ is because $\varphi(0) = \varphi(0 + 0) = \varphi(0) + \varphi(0)$, and since $S$ is an group under addition, we can cancel (using the existence of inverses under addition!) to get $\varphi(0) = 0$. We can’t do this to get $\varphi(1) = 1$ as we don’t have inverses under multiplication.

You should think of a homomorphism as being like an “algebraic analogy”, or a way of transferring algebraic properties; the algebra in the image of $\varphi$ is analogous to the algebra of $R$.

Exercise 1.7. Let $R$ be a commutative ring and define $\sigma : R \rightarrow S$ by $\sigma(a) = a + b$ for $a, b \in R$. Show that $\sigma$ is a ring homomorphism.

Exercise 1.8. Let $R$ be a commutative ring and define $\partial : R[x] \rightarrow R[x]$ by $\partial(f) = df/dx$ for $f \in R[x]$. Show that $\partial$ is a ring homomorphism.

Exercise 1.9. Let $p$ be a fixed polynomial in the ring $F[x]$ and consider the map $\epsilon_p : F[x] \rightarrow F[x]$ given by $f(x) \mapsto f(p(x))$. Show that $\epsilon_p$ is a homomorphism. (The homomorphism $\epsilon_p$ is a generalisation of the evaluation at $\lambda$ homomorphism $\epsilon_{\lambda}$.)
2. Roots and Irreducibility

(2.1). The early material in this section is familiar for polynomials with real coefficients. The point is that these results are still true for polynomials with coefficients coming from an arbitrary field $F$, and quite often, for polynomials with coefficients from a ring $R$.

Let

$$f = a_0 + a_1x + \cdots + a_nx^n$$

be a polynomial in $R[x]$ for $R$ a ring. We say that $c \in R$ is a root of $f$ if

$$f(c) = a_0 + a_1c + \cdots + a_nc^n = 0 \text{ in } R.$$ 

As a trivial example, the polynomial $x^2 + 1$ is in all three rings $\mathbb{Q}[x], \mathbb{R}[x]$ and $\mathbb{C}[x]$. It has no roots in either $\mathbb{Q}$ or $\mathbb{R}$, but two in $\mathbb{C}$.

(2.2). We start with a familiar result:

**The Factor Theorem.** An element $c \in R$ is a root of $f$ if and only if $f = (x - c)g$ for some $g \in R[x]$.

In English, $c$ is a root precisely when $x - c$ is a factor.

**Proof.** This is an illustration of the power of the division algorithm, Theorem A. Suppose that $f$ has the form $(x - c)g$ for some $g \in R[x]$. Then

$$f(c) = (c - c)g(c) = 0, g(c) = 0,$$

so that $c$ is indeed a root (notice we used that $\epsilon_c$ is a homomorphism, ie: that $\epsilon_c(f) = \epsilon_c(x - c)\epsilon_c(g)$). On the other hand, by the division algorithm, we can divide $f$ by the polynomial $x - c$ to get,

$$f = (x - c)g + a,$$

where $a \in R$ (we can use the division algorithm, as the leading coefficient of $x - c$, being 1, has an inverse in $R$). Since $f(c) = 0$, we must also have $(c - c)g + a = 0$, hence $a = 0$. Thus $f = (x - c)g$ as required. \qed

(2.3). Here is another familiar result that is reassuringly true for polynomials over (almost) any ring.

**Theorem 2.1.** Let $f \in R[x]$ be a non-zero polynomial with coefficients from the integral domain $R$. Then $f$ has at most $\deg(f)$ roots in $R$.

**Proof.** We use induction on the degree, which is $\geq 0$ since $f$ is non-zero. If $\deg(f) = 0$ then $f = \mu$ a nonzero constant in $R$, which clearly has no roots, so the result holds. Assume $\deg(f) \geq 1$ and that the result is true for any polynomial of degree $< \deg(f)$. If $f$ has no roots in $R$ then we are done. Otherwise, $f$ has a root $c \in R$ and

$$f = (x - c)g,$$

for some $g \in R[x]$ by the Factor Theorem. Moreover, as $R$ is an integral domain, $f(a) = 0$ iff either $a - c = 0$ or $g(a) = 0$, so the roots of $f$ are $c$, together with the roots of $g$. Since the degree of $g$ must be $\deg(f) - 1$ (by Lemma 1.1, again using the fact that $R$ is an integral domain), it has at most $\deg(f) - 1$ roots by the inductive hypothesis, and these combined with $c$ give at most $\deg(f)$ roots for $f$. \qed
A polynomial like $x^2 + 2x + 1 = (x + 1)^2$ has 1 as a repeated root. It’s derivative, in the sense of calculus, is $2(x + 1)$, which also has 1 as a root. In general, and in light of the Factor Theorem, call $c \in F$ a repeated root of $f$ if and only if $c$ is a root of $f$. 

1. Using the formal derivative $\partial$ (see Exercise 1.3), show that $c$ is a repeated root of $f$ if and only if $c$ is a root of $\partial(f)$.
2. Show that the roots of $f$ are distinct if and only if $\gcd(f, \partial(f)) = 1$.

For reasons that will become clearer later, a very important role is played by polynomials that cannot be “factorised”.

**Definition 2.1 (irreducible polynomial over $F$).** Let $F$ be a field and $f \in F[x]$ a non-constant polynomial. A non-trivial factorisation of $f$ is an expression of the form $f = gh$, where $g, h \in F[x]$ and $\deg g, \deg h \geq 1$ (equivalently, $\deg g, \deg h < \deg f$). Call $f$ reducible over $F$ iff it has a non-trivial factorisation, and irreducible over $F$ otherwise.

Thus, a polynomial over a field $F$ is irreducible precisely when it cannot be written as a product of non-constant polynomials. Put another way, $f \in F[x]$ is irreducible precisely when it is divisible only by a constant $c \in F$, or $c f$.

*Aside.* For polynomials over a ring the definition is slightly more complicated: let $f \in R[x]$ a non-constant polynomial with coefficients from the ring $R$. A non-trivial factorisation of $f$ is an expression of the form $f = gh$, where $g, h \in R[x]$ and either,

1. $\deg g, \deg h \geq 1$, or
2. if either $g$ or $h$ is a constant $\lambda \in R$, then $\lambda$ has no multiplicative inverse in $R$.

Say $f$ is reducible over $R$ iff it has a non-trivial factorisation, and irreducible over $R$ otherwise. If $R = F$ a field, then the second possibility never arises, as every non-zero element of $F$ has a multiplicative inverse. As an example, $3x + 3 = 3(x + 1)$ is a non-trivial factorisation in $\mathbb{Z}[x]$ but a trivial one in $\mathbb{Q}[x]$.

The “over $F$” that follows reducible or irreducible is crucial; polynomials are never absolutely reducible or irreducible. For example $x^2 + 1$ is irreducible over $\mathbb{R}$ but reducible over $\mathbb{C}$.

There is one exception to the previous sentence: a linear polynomial $f = ax + b \in F[x]$ is irreducible over any field $F$. If $f = gh$ then since $\deg f = 1$, we cannot have both $\deg(g), \deg(h) \geq 1$, for then $\deg(gh) = \deg(g) + \deg(h) \geq 1 + 1 = 2$, a contradiction. Thus, one of $g$ or $h$ must be a constant with $f$ thus irreducible over $F$.

**Exercise 2.2.**
1. Let $F$ be a field and $a \in F$. Show that $f$ is an irreducible polynomial over $F$ if and only if $af$ is irreducible over $F$ for any $a \neq 0$.
2. Show that if $f(x + a)$ is irreducible over $F$ then $f(x)$ is too.

There is the famous:

**Fundamental Theorem of Algebra.** Any non-constant $f \in \mathbb{C}[x]$ has a root in $\mathbb{C}$.

So if $f \in \mathbb{C}[x]$ has $\deg f \geq 2$, then $f$ has a root in $\mathbb{C}$, hence a linear factor over $\mathbb{C}$, hence is irreducible over $\mathbb{C}$. Thus, the only irreducible polynomials over $\mathbb{C}$ are the linear ones.

**Exercise 2.3.** Show that if $f$ is irreducible over $\mathbb{R}$ then $f$ is either linear or quadratic.
(2.8). A common mistake is to equate having no roots in \( F \) with being irreducible over \( F \). But:

1. A polynomial can be irreducible over \( F \) and still have roots in \( F \): we saw above that a linear polynomial \( ax + b \) is always irreducible, and yet has a root in \( F \), namely \(-b/a\). It is true though that if a polynomial \( f \) has degree \( \geq 2 \) and had a root in \( F \), then by the factor theorem it would have a linear factor so would be reducible. Thus, if \( \deg(f) \geq 2 \) and \( f \) is irreducible over \( F \), then \( f \) has no roots in \( F \).

2. A polynomial can have no roots in \( F \) but not be irreducible over \( F \): the polynomial \( x^4 + 2x^2 + 1 = (x^2 + 1)^2 \) is reducible over \( \mathbb{Q} \), but with roots \( \pm i \notin \mathbb{Q} \).

(2.9). There is no general method for deciding if a polynomial over an arbitrary field \( F \) is irreducible. The best we can hope for is an ever expanding list of techniques, of which the first is:

**Proposition 2.1.** Let \( F \) be a field and \( f \in F[x] \) be a polynomial of degree \( \leq 3 \). If \( f \) has no roots in \( F \) then it is irreducible over \( F \).

**Proof.** Arguing by the contrapositive, if \( f \) is reducible then \( f = gh \) with \( \deg g, \deg h \geq 1 \). Since \( \deg g + \deg h = \deg f \leq 3 \), we must have for \( g \) say, that \( \deg g = 1 \). Thus \( f = (ax + b)h \) and \( f \) has the root \(-b/a\). \( \square \)

(2.10). For another, possibly familiar, example of a field: let \( p \) be a prime and \( \mathbb{F}_p \) the set \( \{0, 1, \ldots, p - 1\} \). Define addition and multiplication on this set to be addition and multiplication of integers modulo \( p \). You can verify that \( \mathbb{F}_p \) is a field by directly checking the axioms. The only tricky one is the existence of inverses under multiplication: to show this use the gcd theorem from Section[1] but for \( \mathbb{Z} \) rather than polynomials.

**Exercise 2.4.** Show that a field \( F \) is an integral domain. Hence show that if \( n \) is not prime, then the addition and multiplication of integers modulo \( n \) is not a field.

Arithmetic modulo \( n \), for the various \( n \), thus gives the sequence

\[ \mathbb{F}_2, \mathbb{F}_3, \mathbb{Z}_4, \mathbb{F}_5, \mathbb{Z}_6, \mathbb{F}_7, \mathbb{Z}_8, \mathbb{Z}_9, \mathbb{Z}_{10}, \mathbb{F}_{11}, \ldots \]

of fields \( \mathbb{F}_p \) for \( p \) a prime, and rings \( \mathbb{Z}_n \) for \( n \) composite. In Section[5] we will see that there are fields \( \mathbb{F}_4, \mathbb{F}_8 \) and \( \mathbb{F}_9 \) of orders 4, 8 and 9, but these fields are not \( \mathbb{Z}_4, \mathbb{Z}_8 \) or \( \mathbb{Z}_9 \). They are something quite different.

(2.11). Consider polynomials with coefficients from \( \mathbb{F}_2 \) ie: the ring \( \mathbb{F}_2[x] \), and in particular, the polynomial

\[ f = x^4 + x + 1 \in \mathbb{F}_2[x]. \]

Now \( 0^4 + 0 + 1 \neq 0 \neq 1^4 + 1 + 1 \), so \( f \) has no roots in \( \mathbb{F}_2 \). This doesn’t mean that \( f \) is irreducible over \( \mathbb{F}_2 \), but certainly any factorisation of \( f \) over \( \mathbb{F}_2 \), if there is one, must be as a product of two quadratics. Moreover, these quadratics must themselves be irreducible over \( \mathbb{F}_2 \), for if not, they would factor into linear factors and the factor theorem would then give roots of \( f \).

There are only four quadratics over \( \mathbb{F}_2 \):

\[ x^2, x^2 + 1, x^2 + x \text{ and } x^2 + x + 1 \]

with \( x^2 = xx, x^2 + 1 = (x + 1)^2 \) and \( x^2 + x = x(x + 1) \). You might have to stare at the second of these factorisations for a second. By Proposition[2.1] \( x^2 + x + 1 \) is irreducible. Thus, any factorisation of \( f \) into irreducible quadratics must be of the form,

\[ (x^2 + x + 1)(x^2 + x + 1). \]

But, \( f \) doesn’t factorise this way – just expand the brackets. Thus \( f \) is irreducible over \( \mathbb{F}_2 \).
(2.12). The most important field for the Galois theory of these notes is the rationals $\mathbb{Q}$. Consequently, determining the irreducibility of polynomials over $\mathbb{Q}$ will be of great importance to us. The first useful test for irreducibility over $\mathbb{Q}$ has the following main ingredient: to see if a polynomial can be factorised over $\mathbb{Q}$ it suffices to see whether it can be factorised over $\mathbb{Z}$.

First we recall Exercise [1.7] which is used a number of times in these notes so is worth placing in a,

**Lemma 2.1.** Let $\sigma : R \to S$ be a homomorphism of rings. Define $\sigma^* : R[x] \to S[x]$ by

$$
\sigma^* : \sum_i a_i x^i \mapsto \sum_i \sigma(a_i) x^i.
$$

Then $\sigma^*$ is a homomorphism.

**Lemma 2.2 (Gauss).** Let $f$ be a polynomial with integer coefficients. Then $f$ can be factorised non-trivially as a product of polynomials with integer coefficients if and only if it can be factorised non-trivially as a product of polynomials with rational coefficients.

**Proof.** If the polynomial can be written as a product of $\mathbb{Z}$-polynomials then it clearly can as a product of $\mathbb{Q}$-polynomials as integers are rational. Suppose on the other hand that $f = gh$ in $\mathbb{Q}[x]$ is a non-trivial factorisation. By multiplying through by a multiple of the denominators of the coefficients of $g$ we get a polynomial $g_1 = mg$ with $\mathbb{Z}$-coefficients. Similarly we have $h_1 = nh \in \mathbb{Z}[x]$ and so

$$
mnf = g_1h_1 \in \mathbb{Z}[x].
$$

(2.1)

Now let $p$ be a prime dividing $mn$, and consider the homomorphism $\sigma : \mathbb{Z} \to \mathbb{F}_p$ given by $\sigma(k) = k \mod p$. Then by the lemma above, the map $\sigma^* : \mathbb{Z}[x] \to \mathbb{F}_p[x]$ given by

$$
\sigma^* : \sum_i a_i x^i \mapsto \sum_i \sigma(a_i) x^i,
$$

is a homomorphism. Applying the homomorphism to (2.1) gives $0 = \sigma^*(g_1)\sigma^*(h_1)$ in $\mathbb{F}_p[x]$, as $mn \equiv 0 \mod p$. As the ring $\mathbb{F}_p[x]$ is an integral domain the only way that this can happen is if one of the polynomials is equal to the zero polynomial in $\mathbb{F}_p[x]$, ie: one of the original polynomials, say $g_1$, has all of its coefficients divisible by $p$. Thus we have $g_1 = pg_2$ with $g_2 \in \mathbb{Z}[x]$, and (2.1) becomes

$$
\frac{mn}{p}f = g_2h_1.
$$

Working our way through all the prime factors of $mn$ in this way, we can remove the factor of $mn$ from (2.1) and obtain a factorisation of $f$ into polynomials with $\mathbb{Z}$-coefficients.

So to determine whether a polynomial with $\mathbb{Z}$-coefficients is irreducible over $\mathbb{Q}$, you need only check that it has no non-trivial factorisations with all the coefficients integers.

**Eisenstein Irreducibility Theorem.** Let

$$
f = c_n x^n + \cdots + c_1 x + c_0,
$$

be a polynomial with integer coefficients. If there is a prime $p$ that divides all the $c_i$ for $i < n$, does not divide $c_n$, and such that $p^2$ does not divide $c_0$, then $f$ is irreducible over $\mathbb{Q}$.

**Proof.** By virtue of the previous discussion, we need only show that under the conditions stated, there is no factorisation of $f$ using integer coefficients. Suppose otherwise, ie: $f = gh$ with

$$
g = a_rx^r + \cdots + a_0 \text{ and } h = b_sx^s + \cdots + b_0,
$$

be a factorisation of $f$ with $a_r \neq 0$ and $b_s \neq 0$. Then $p$ must divide $a_r$ and $b_s$. But then $p$ divides all the coefficients of $f$, a contradiction.

$\square$
and the \( a_i, b_i \in \mathbb{Z} \). Expanding \( gh \) and equating coefficients,

\[
\begin{align*}
c_0 &= a_0b_0 \\
c_1 &= a_0b_1 + a_1b_0 \\
\vdots \\
c_i &= a_0b_i + a_1b_{i-1} + \cdots + a_ib_0 \\
\vdots \\
c_n &= a_rb_s.
\end{align*}
\]

By hypothesis, \( p \mid c_0 \). Write both \( a_0 \) and \( b_0 \) as a product of primes, so if \( p \mid c_0 \), ie: \( p \mid a_0b_0 \), then \( p \) must be one of the primes in this factorisation, hence divides one of \( a_0 \) or \( b_0 \). Thus, either \( p \mid a_0 \) or \( p \mid b_0 \), but not both (for then \( p^2 \) would divide \( c_0 \)). Assume that it is \( p \mid a_0 \) that we have. Next, \( p \mid c_1 \), and this coupled with \( p \mid a_0 \) gives \( p \mid c_1 - a_0b_1 = a_1b_0 \) (If we had assumed \( p \mid b_0 \), we would still reach this conclusion). Again, \( p \) must divide one of the these last two factors, and since we’ve already decided that it doesn’t divide \( b_0 \), it must be \( a_1 \) that it divides. Continuing in this manner, we get that \( p \) divides all the coefficients of \( g \), and in particular, \( a_r \). But then \( p \) divides \( a_rb_s = c_n \), the contradiction we were after. \( \square \)

The proof above is a good example of the way mathematics is sometimes created. You start with as few assumptions as possible (in this case that \( p \) divides some of the coefficients of \( f \)) and proceed towards some sort of conclusion, imposing extra conditions as and when you need them. In this way the statement of the theorem writes itself.

(2.13). For example

\[ x^5 + 5x^4 - 5x^3 + 10x^2 + 25x - 35, \]

is irreducible over \( \mathbb{Q} \). Even less obviously

\[ x^n - p, \]

is irreducible over \( \mathbb{Q} \) for any prime \( p \). Thus, we can find polynomials over \( \mathbb{Q} \) of arbitrary large degree that are irreducible, in contrast to the situation for polynomials over \( \mathbb{R} \) or \( \mathbb{C} \).

(2.14). Another useful tool arises with polynomials having coefficients from a ring \( R \) and there is a homomorphism from \( R \) to some field \( F \). If the homomorphism is applied to all the coefficients of the polynomial (turning it from a polynomial with \( R \)-coefficients into a polynomial with \( F \)-coefficients) then a reducible polynomial cannot turn into an irreducible one:

**The Reduction Test.** Let \( R \) be an integral domain, \( F \) a field and \( \sigma : R \to F \) a ring homomorphism. Let \( \sigma^* : R[x] \to F[x] \) be the homomorphism of Lemma [27]. Moreover, let \( f \in R[x] \) be such that

1. \( \text{deg } \sigma^*(f) = \text{deg}(f) \), and
2. \( \sigma^*(f) \) is irreducible over \( F \).

Then \( f \) cannot be written as a product \( f = gh \) with \( g, h \in R[x] \) and \( \text{deg } g, \text{deg } h < \text{deg } f \).

Although it is stated in some generality, the reduction test is very useful for determining the irreducibility of polynomials over \( \mathbb{Q} \). As an example, take \( R = \mathbb{Z}; F = \mathbb{F}_5 \) and \( f = 8x^3 - 6x - 1 \in \mathbb{Z}[x] \). For \( \sigma \), take reduction modulo 5, ie: \( \sigma(n) = n \mod 5 \). It is not hard to show that \( \sigma \) is a homomorphism. Since \( \sigma(8) \equiv 3 \mod 5 \), and so on, we get

\[ \sigma^*(f) = 3x^3 + 4x + 4 \in \mathbb{F}_5[x]. \]
The degree has not changed, and by substituting the five elements of \( \mathbb{F}_5 \) into \( \sigma^*(f) \), one can see that it has no roots in \( \mathbb{F}_5 \). Since the polynomial is a cubic, it must therefore be irreducible over \( \mathbb{F}_5 \). Thus, by the reduction test, \( 8x^3 - 6x - 1 \) cannot be written as a product of smaller degree polynomials with \( \mathbb{Z} \)-coefficients. But by Gauss’ lemma, this gives that this polynomial is irreducible over \( \mathbb{Q} \).

\( \mathbb{F}_5 \) was chosen because with \( \mathbb{F}_2 \) condition (i) fails; with \( \mathbb{F}_3 \) condition (ii) fails.

Proof. Suppose on the contrary that \( f = gh \) with \( \deg g, \deg h < \deg f \). Then \( \sigma^*(f) = \sigma^*(gh) = \sigma^*(g)\sigma^*(h) \), the last part because \( \sigma^* \) is a homomorphism. Now \( \sigma^*(f) \) is irreducible, so the only way it can factorise like this is if one of the factors, \( \sigma^*(g) \) say, is a constant, hence \( \deg \sigma^*(g) = 0 \).

Then

\[
\deg f = \deg \sigma^*(f) = \deg \sigma^*(g)\sigma^*(h) = \deg \sigma^*(g) + \deg \sigma^*(h) = \deg \sigma^*(h) \leq \deg h < \deg f,
\]

a contradiction. (That \( \deg \sigma^*(h) \leq \deg h \) rather than equality necessarily, is because the homomorphism \( \sigma \) may send some of the coefficients of \( h \) – including the leading one – to 0 \( \in \mathbb{F} \).) \( \square \)

(2.15) We’ve already observed the similarity between polynomials and integers. One thing we know about integers is that they can be written uniquely as products of primes. We might hope that something similar is true for polynomials, and it is in certain situations. For the next few results, we deal only with polynomials \( f \in \mathbb{F}[x] \) for \( F \) a field (although they are true in more generality).

**Lemma 2.3.** 1. If \( \gcd(f, g) = 1 \) and \( f \mid gh \) then \( f \mid h \).
2. If \( f \) is irreducible and monic, then for any \( g \) monic with \( g \mid f \) we have either \( g = 1 \) or \( g = f \).
3. If \( g \) is irreducible and monic and \( g \) does not divide \( f \), then \( \gcd(g, f) = 1 \).
4. If \( g \) is irreducible and monic and \( g \mid f_1f_2\ldots f_n \) then \( g \mid f_i \) for some \( i \).

Proof. 1. Since \( \gcd(f, g) = 1 \) there are \( a, b \in \mathbb{F}[x] \) such that \( 1 = af + bg \), hence \( h = afh + bhg \).

We have that \( f \mid bhg \) by assumption, and it clearly divides \( afh \), hence it divides \( afh + bhg = h \) also.

2. If \( g \) divides \( f \) and \( f \) is irreducible, then by definition \( g \) must be either a constant or a constant multiple of \( f \). But \( f \) is monic, so \( g = 1 \) or \( g = f \) are the only possibilities.

3. The \( \gcd \) of \( f \) and \( g \) is certainly a divisor of \( g \), and hence by irreducibility must be either a constant, or a constant times \( g \). As \( g \) is also monic, the \( \gcd \) must in fact be either 1 or \( g \) itself, and since \( g \) does not divide \( f \) it cannot be \( g \), so must be 1.

4. Proceed by induction, with the first step for \( n = 1 \) being immediate. Since \( g \mid f_1f_2\ldots f_n = (f_1f_2\ldots f_{n-1})f_n \), we either have \( g \mid f_n \), in which case we are finished, or not, in which case \( \gcd(g, f_n) = 1 \) by part (3). But then part (1) gives that \( g \mid f_1f_2\ldots f_{n-1} \), and the inductive hypothesis kicks in. \( \square \)

The best way of summarising the lemma is this: monic irreducible polynomials are like the “prime numbers” of \( F[x] \).

(2.16) Just as any integer can be decomposed uniquely as a product of primes, so too can any polynomial as a product of irreducible polynomials:

**Unique factorisation in \( F[x] \).** Every polynomial in \( F[x] \) can be written in the form

\[
cp_1p_2\ldots p_r,
\]

where \( c \) is a constant and the \( p_i \) are monic and irreducible \( \in \mathbb{F}[x] \). Moreover, if \( aq_1q_2\ldots q_s \) is another factorisation with the \( q_j \) monic and irreducible, then \( r = s \), \( c = a \) and the \( q_j \) are just a rearrangement of the \( p_i \).
The last part says that the factorisation is unique, except for the order you write down the factors.

**Proof.** To get the factorisation just keep factorising reducible polynomials until they become irreducible. At the end, pull out the coefficient of the leading term in each factor, and place them all at the front.

For uniqueness, suppose that

\[ cp_1p_2\ldots p_r = aq_1q_2\ldots q_s. \]

Then \( p_r \) divides \( aq_1q_2\ldots q_s \) which by Lemma 2.3 part (4) means that \( p_r | q_i \) for some \( i \). Reorder the \( q \)'s so that it is \( p_r | q_s \) that in fact we have. Since both \( p_r \) and \( q_s \) are monic, irreducible, and hence non-constant, \( p_r = q_s \), which leaves us with

\[ cp_1p_2\ldots p_{r-1} = aq_1q_2\ldots q_{s-1}. \]

This gives \( r = s \) straight away: if say \( s > r \), then repetition of the above leads to \( c = aq_1q_2\ldots q_{s-r} \), which is absurd, as consideration of degrees gives different answers for each side. Similarly if \( r > s \). But then we also have that the \( p \)'s are just a rearrangement of the \( q \)'s, and canceling down to \( cp_1 = aq_1 \), that \( c = a \). \( \Box \)

**Exercise 2.5.** Formulate the example above into a general Theorem.

**Further Exercises for Section 2**

**Exercise 2.6.** Prove that if a polynomial equation has all its coefficients in \( \mathbb{C} \) then it must have all its roots in \( \mathbb{C} \).

**Exercise 2.7.**

1. Let \( f = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \) be a polynomial in \( \mathbb{R}[x] \), that is, all the \( a_i \in \mathbb{R} \). Show that complex roots of \( f \) occur in conjugate pairs, ie: \( \zeta \in \mathbb{C} \) is a root of \( f \) if and only if \( \overline{\zeta} \) is.

2. Find an example of a polynomial in \( \mathbb{C}[x] \) for which part (a) is not true.
Exercise 2.8.

1. Let $m, n$ and $k$ be integers with $m$ and $n$ relatively prime (ie: $\gcd(m, n) = 1$). Show that if $m$ divides $nk$ then $m$ must divide $k$ (hint: there are two methods here. One is to use Lemma 2.3 but in $\mathbb{Z}$. The other is to use the fact that any integer can be written uniquely as a product of primes. Do this for $m$ and $n$, and ask yourself what it means for this factorisation that $m$ and $n$ are relatively prime).
2. Show that if $m/n$ is a root of $a_0 + a_1 x + \ldots + a_n x^n$, $a_i \in \mathbb{Z}$, where $m$ and $n$ are relatively prime integers, then $m|a_0$ and $n|a_r$.
3. Deduce that if $a_r = 1$ then $m/n$ is in fact an integer.

**moral:** If a monic polynomial with integer coefficients has a rational root $m/n$, then this rational number is in fact an integer.

Exercise 2.9. If $m \in \mathbb{Z}$ is not a perfect square, show that $x^2 - m$ is irreducible over $\mathbb{Q}$ (note: it is not enough to merely assume that under the conditions stated $\sqrt{m}$ is not a rational number).

Exercise 2.10. Find the greatest common divisor of $f(x) = x^3 - 6x^2 + x + 4$ and $g(x) = x^3 - 6x + 1$ (hint: look at linear factors of $f(x)$).

Exercise 2.11. Determine which of the following polynomials are irreducible over the stated field:

1. $1 + x^8$ over $\mathbb{R}$;
2. $1 + x^2 + x^4 + x^6 + x^8 + x^{10}$ over $\mathbb{Q}$ (hint: Let $y = x^2$ and factorise $y^n - 1$);
3. $x^4 + 15x^3 + 7$ over $\mathbb{R}$ (hint: use the intermediate value theorem from analysis);
4. $x^{n+1} + (n + 2)! x^n + \cdots + (i + 2)! x^i + \cdots + 3! x + 2!$ over $\mathbb{Q}$.
5. $x^2 + 1$ over $\mathbb{F}_7$.
6. Let $\mathbb{F}$ be the field of order 8 from Section 3 and let $\mathbb{F}[X]$ be polynomials with coefficients from $\mathbb{F}$ and indeterminate $X$. Is $X^3 + (a^2 + a)X + (a^2 + a + 1)$ irreducible over $\mathbb{F}$?
7. $a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$ over $\mathbb{Q}$ where the $a_i \in \mathbb{Z}$; $a_3, a_2$ are even and $a_4, a_1, a_0$ are odd.

Exercise 2.12. If $p$ is prime, show that $p$ divides $\binom{p}{i}$ for $0 < i < p$. Show that $p$ divides $\binom{p^n}{i}$ for $n \geq 1$ and $0 < i < p$.

Exercise 2.13. Show that

$$x^{p-1} + px^{p-2} + \ldots + \binom{p}{i} x^{p-i-1} + \cdots + p,$$

is irreducible over $\mathbb{Q}$.

Exercise 2.14. A complex number $\omega$ is an $n$-th root of unity if $\omega^n = 1$. It is a primitive $n$-th root of unity if $\omega^n = 1$, but $\omega^r \neq 1$ for any $0 < r < n$. So for example, $\pm 1, \pm i$ are the 4-th roots of 1, but only $\pm i$ are primitive 4-th roots.

Convince yourself that for any $n$,

$$\omega = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$$

is an $n$-th root of 1. In fact, the other $n$-th roots are $\omega^2, \ldots, \omega^{n-1}$.

1. Show that if $\omega$ is a primitive $n$-th root of 1 then $\omega$ is a root of the polynomial $x^{n-1} + x^{n-2} + \cdots + x + 1$.

2. Show that for (2.2) to be irreducible over $\mathbb{Q}$, $n$ cannot be even.
3. Show that a polynomial \( f(x) \) is irreducible over a field \( F \) if \( f(x + 1) \) is irreducible over \( F \).

4. Finally, if

\[
\Phi_p(x) = x^{p-1} + x^{p-2} + \cdots + x + 1
\]

for \( p \) a prime number, show that \( \Phi_p(x + 1) \) is irreducible over \( \mathbb{Q} \), and hence \( \Phi_p(x) \) is too (hint: consider \( x^p - 1 \) and use the binomial theorem, Exercise 2.12, and Eisenstein).

The polynomial \( \Phi_p(x) \) is called the \( p \)-th cyclotomic polynomial.

3. Fields I: Basics, Extensions and Concrete Examples

This course studies the solutions to polynomial equations. Questions about these solutions can be restated as questions about fields. It is to these that we now turn.

(3.1). We remembered the definition of a field in Section \([\text{0}]\); we can restate it as:

**Definition 3.1 (field – version 2).** A field is a set \( F \) with two operations, \(+\) and \( \times \), such that for any \( a, b, c \in F \),

1. \( F \) is an Abelian group under \(+\);
2. \( F \setminus \{0\} \) is an Abelian group under \( \times \);
3. the two operations are linked by the distributive law.

The two groups are called the additive and multiplicative groups of the field. In particular, we will write \( F^* \) to denote the multiplicative group (ie: \( F^* \) is the group with elements \( F \setminus \{0\} \) and operation the multiplication from the field). Even more succinctly,

**Definition 3.2 (field – version 3).** A field is a set \( F \) with two operations, \(+\) and \( \times \), such that for any \( a, b, c \in F \),

1. \( F \) is a commutative ring under \(+\) and \( \times \);
2. for any \( a \in F \setminus \{0\} \) there is an \( a^{-1} \in F \) with \( a \times a^{-1} = 1 = a^{-1} \times a \),

In particular a field is a special kind of ring.

(3.2). More concepts from the first lecture that can now be properly defined are:

**Definition 3.3 (extensions of fields).** Let \( F \) and \( E \) be fields with \( F \) a subfield of \( E \). We call \( E \) an extension of \( F \). If \( \beta \in E \), we write \( F(\beta) \), as in Section \([\text{0}]\) for the smallest subfield of \( E \) containing both \( F \) and \( \beta \) (so in particular \( F(\beta) \) is an extension of \( F \)). In general, if \( \beta_1, \ldots, \beta_k \in E \), define \( F(\beta_1, \ldots, \beta_k) = F(\beta_1, \ldots, \beta_{k-1})(\beta_k) \).

The standard notation for an extension is to write \( E/F \), but in these notes we will use the more concrete \( F \subseteq E \), being mindful that this means \( F \) is a subfield of \( E \), and not just a subset.

We say that \( \beta \) is adjoined to \( F \) to obtain \( F(\beta) \). The last bit of the definition says that to adjoin several elements to a field you adjoin them one at a time. The notation seems to adjoin them in a particular order, but the order doesn’t matter. If we have an extension \( F \subseteq E \) and there is a \( \beta \in E \) such that \( E = F(\beta) \), then we call \( E \) a simple extension of \( F \).

(3.3). \( \mathbb{R} \) is an extension of \( \mathbb{Q} \); \( \mathbb{C} \) is an extension of \( \mathbb{R} \), and so on. Any field is an extension of itself!
(3.4). Let $\mathbb{F}_2$ be the field of integers modulo 2 arithmetic. Let $\alpha$ be an “abstract symbol” that can be multiplied so that it has the following property: $\alpha \times \alpha \times \alpha = \alpha^3 = \alpha + 1$ (a bit like decreeing that the imaginary $i$ squares to give $-1$). Let

$$\mathbb{F} = \{a + b\alpha + ca^2 \mid a, b, c \in \mathbb{F}_2\}.$$ 

Define addition on $\mathbb{F}$ by: $(a_1 + b_1\alpha + c_1a^2) + (a_2 + b_2\alpha + c_2a^2) = (a_1 + a_2) + (b_1 + b_2)\alpha + (c_1 + c_2)a^2$, where the addition of coefficients happens in $\mathbb{F}_2$. For multiplication, “expand” the expression $(a_1 + b_1\alpha + c_1a^2)(a_2 + b_2\alpha + c_2a^2)$ like you would a polynomial with $\alpha$ the indeterminate, so that $a_1a_2\alpha = \alpha^3$, the coefficients are dealt with using the arithmetic from $\mathbb{F}_2$, and so on. Replace any $\alpha^3$ that result using the rule $\alpha^3 = \alpha + 1$.

For example,

$$(1 + \alpha + \alpha^2) + (\alpha + \alpha^2) = 1 \text{ and } (1 + \alpha + \alpha^2)(\alpha + \alpha^2) = \alpha + \alpha^4 = \alpha + \alpha(\alpha + 1) = \alpha^2.$$ 

It turns out that $\mathbb{F}$ forms a field with this addition and multiplication – see Exercise 3.10. Taking those elements of $\mathbb{F}$ with $b = c = 0$ we obtain (an isomorphic) copy of $\mathbb{F}_2$ inside of $\mathbb{F}$, and so we have an extension of $\mathbb{F}_2$ that contains 8 elements.

(3.5). $\mathbb{Q}(\sqrt{2})$ is a simple extension of $\mathbb{Q}$ while $\mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{3})$ would appear not to be. But consider $\mathbb{Q}(\sqrt[3]{2} + \sqrt[3]{3})$: certainly $\sqrt[3]{2} + \sqrt[3]{3} \in \mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{3})$, and so $\mathbb{Q}(\sqrt[3]{2} + \sqrt[3]{3}) \subseteq \mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{3})$. On the other hand,

$$(\sqrt[3]{2} + \sqrt[3]{3})^3 = 11\sqrt[3]{2} + 9\sqrt[3]{3},$$

as is readily checked using the Binomial Theorem. Since $(\sqrt[3]{2} + \sqrt[3]{3})^3 \in \mathbb{Q}(\sqrt[3]{2} + \sqrt[3]{3})$, we get

$$(11\sqrt[3]{2} + 9\sqrt[3]{3}) - 9(\sqrt[3]{2} + \sqrt[3]{3}) \in \mathbb{Q}(\sqrt[3]{2} + \sqrt[3]{3}) \Rightarrow 2\sqrt[3]{2} \in \mathbb{Q}(\sqrt[3]{2} + \sqrt[3]{3}).$$

And so $\sqrt[3]{2} \in \mathbb{Q}(\sqrt[3]{2} + \sqrt[3]{3})$ as $\frac{1}{2}$ is there too. Similarly it can be shown that $\sqrt[3]{3} \in \mathbb{Q}(\sqrt[3]{2} + \sqrt[3]{3})$ and hence $\mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{3}) \subseteq \mathbb{Q}(\sqrt[3]{2} + \sqrt[3]{3})$. So

$$\mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{3}) = \mathbb{Q}(\sqrt[3]{2} + \sqrt[3]{3})$$

is a simple extension!

(3.6). What do the elements of $\mathbb{Q}(\sqrt{2})$ actually look like? Later we will be answer this question in general, but for now we give an ad-hoc answer.

Firstly $\sqrt{2}$ and any $b \in \mathbb{Q}$ are in $\mathbb{Q}(\sqrt{2})$ by definition. Since fields are closed under $\times$, any number of the form $b\sqrt{2} \in \mathbb{Q}(\sqrt{2})$. Similarly, fields are closed under $+$, so any $a + b\sqrt{2} \in \mathbb{Q}(\sqrt{2})$ for $a \in \mathbb{Q}$. Thus, the set

$$\mathbb{F} = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\} \subseteq \mathbb{Q}(\sqrt{2}).$$

But $\mathbb{F}$ is a field in its own right using the usual addition and multiplication of complex numbers. This is easily checked from the axioms; for instance, the inverse of $a + b\sqrt{2}$ can be calculated:

$$\frac{1}{a + b\sqrt{2}} \times \frac{a - b\sqrt{2}}{a - b\sqrt{2}} = \frac{a - b\sqrt{2}}{a^2 - 2b^2} = \frac{a}{a^2 - 2b^2} - \frac{b}{a^2 - 2b^2}\sqrt{2} \in \mathbb{F},$$

and you can check the other axioms for yourself. We also have $\mathbb{Q} \subseteq \mathbb{F}$ (letting $b = 0$) and $\sqrt{2} \in \mathbb{F}$ (letting $a = 0, b = 1$). Since $\mathbb{Q}(\sqrt{2})$ is the smallest field having these two properties, we have $\mathbb{Q}(\sqrt{2}) \subseteq \mathbb{F}$. Thus,

$$\mathbb{Q}(\sqrt{2}) = \mathbb{F} = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}.$$
Exercise 3.1. Let \( \alpha \) be a complex number such that \( \alpha^3 = 1 \) and consider the set 

\[ \mathbb{F} = \{ a_0 + a_1 \alpha + a_2 \alpha^2 \mid a_i \in \mathbb{Q} \} \]

1. By row reducing the matrix, 

\[
\begin{pmatrix}
  a_0 & 2a_2 & 2a_1 & 1 \\
  a_1 & a_0 & 2a_2 & 0 \\
  a_2 & a_1 & a_0 & 0
\end{pmatrix}
\]

find an element of \( \mathbb{F} \) that is the inverse under multiplication of \( a_0 + a_1 \alpha + a_2 \alpha^2 \).

2. Show that \( \mathbb{F} \) is a field, hence \( \mathbb{Q}(\alpha) = \mathbb{F} \).

(3.7). The previous exercise shows that the following two fields have the form, 

\[ \mathbb{Q}(\sqrt{2}) = \{ a + b \sqrt{2} + c \sqrt{2}^2 \mid a, b, c \in \mathbb{Q} \} \text{ and } \mathbb{Q}(\beta) = \{ a + b\beta + c\beta^2 \mid a, b, c \in \mathbb{Q} \}, \]

where 

\[ \beta = \sqrt{2}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \in \mathbb{C}. \]

These two fields are different: the first is completely contained in \( \mathbb{R} \), but the second contains \( \beta \), which is obviously complex but not real. Hold that thought.

Definition 3.4 (ring isomorphism). A bijective homomorphism of rings \( \varphi : R \to S \) is called an isomorphism.

(3.8). A silly but instructive example is given by the Roman ring, whose elements are 

\[ \{ \ldots, V, -IV, -III, -II, -I, 0, I, II, III, IV, V, \ldots \}, \]

and with addition and multiplication \( IX + IV = XIII \) and \( IX \times VI = LIV \), etc... Obviously the ring is isomorphic to \( \mathbb{Z} \), and it is this idea of a trivial relabeling that is captured by an isomorphism – two rings are isomorphic if they are really the same, just written in different languages.

But we place a huge emphasis on the way things are labelled. The two fields of the previous paragraph are a good example, for, 

\[ \mathbb{Q}(\sqrt{2}) \text{ and } \mathbb{Q}\left(\sqrt{2}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right) \]

are isomorphic (we will see why in Section 5). To illustrate how we might now come unstuck, suppose we were to formulate the following, 

"Definition". A subfield of \( \mathbb{C} \) is called real if and only if it is contained in \( \mathbb{R} \).

So \( \mathbb{Q}(\sqrt{2}) \) is a real field, but \( \mathbb{Q}\left(\sqrt{2}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right) \) is not. But they are the same field! A definition should not depend on the way the elements are labelled. We will resolve this problem in Section 5 by thinking about fields in a more abstract way.
(3.9) In the remainder of this section we introduce a few more concepts associated with fields.

It is well known that $\sqrt{2}$ and $\pi$ are both irrational real numbers. Nevertheless, from an algebraic point of view, $\sqrt{2}$ is slightly more tractable than $\pi$, as it is a root of a very simple equation $x^2 - 2$, whereas there is no polynomial with integer coefficients having $\pi$ as a root (this is not obvious).

**Definition 3.5 (algebraic element).** Let $F \subseteq E$ be an extension of fields and $\alpha \in E$. Call $\alpha$ algebraic over $F$ if and only if

$$a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_n\alpha^n = 0,$$

for some $a_0, a_1, \ldots, a_n \in F$.

In other words, $\alpha$ is a root of the polynomial $f = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ in $F[x]$. If $\alpha$ is not algebraic, i.e., not the root of any polynomial with $F$-coefficients, then we say that it is transcendental over $F$.

(3.10) Some simple examples:

$$\sqrt{2}, \frac{1 + \sqrt{3}}{2}$$

are algebraic over $\mathbb{Q}$, whereas $\pi$ and $e$ are transcendental over $\mathbb{Q}$; $\pi$ is algebraic over $\mathbb{Q}(\pi)$.

(3.11). A field can contain many subfields: $\mathbb{C}$ contains $\mathbb{Q}(\sqrt{2}), \mathbb{R}, \ldots$. It also contains $\mathbb{Q}$, but no subfields that are smaller than this. Indeed, any subfield of $\mathbb{C}$ contains $\mathbb{Q}$, so the rationals are the smallest subfield of the complex numbers.

**Definition 3.6 (prime subfield).** The prime subfield of a field $F$ is the intersection of all the subfields of $F$.

In particular the prime subfield is contained in every subfield of $F$.

**Exercise 3.2.** Consider the field of rational numbers $\mathbb{Q}$ or the finite field $\mathbb{F}_p$ having $p$ elements. Show that neither of these fields contain a proper subfield (hint: for $\mathbb{F}_p$, consider the additive group and use Lagrange’s Theorem from Section 10). For $\mathbb{Q}$, any subfield must contain 1, and show that it must then be all of $\mathbb{Q}$).

The prime subfield must contain 1, hence any expression of the form $1 + 1 + \cdots + 1$ for any number of summands. If no such expression equals 0 then we have infinitely many distinct such elements, and their inverses under addition, hence a copy of $\mathbb{Z}$ in $F$. Otherwise, if $n$ is the smallest number of summands for which such an expression equals 0, then the elements

$$1, 1 + 1, 1 + 1 + 1, \ldots, \underbrace{1 + \cdots + 1}_{n \text{ times}} = 0,$$

forms a copy of $\mathbb{Z}_n$ inside of $F$. These comments can be made precise as in the following exercise. It looks ahead a little, requiring the first isomorphism theorem for rings in Section 4.

**Exercise 3.3.** Let $F$ be a field and define a map $\mathbb{Z} \to F$ by

$$n \mapsto \begin{cases} 0, & \text{if } n = 0, \\ 1 + \cdots + 1, (n \text{ times}), & \text{if } n > 0 \\ -1 - \cdots - 1, (n \text{ times}), & \text{if } n < 0. \end{cases}$$

Show that the map is a ring homomorphism. If the kernel consists of just $\{0\}$, then show that $F$ contains $\mathbb{Z}$ as a subring. Otherwise, let $n$ be the smallest positive integer contained in the kernel, and show that $F$ contains $\mathbb{Z}_n$ as a subring. As $F$ is a field, hence an integral domain, show that we must have $n = p$ a prime in this situation.
Thus any field contains a subring isomorphic to \( \mathbb{Z} \) or to \( \mathbb{Z}_p \) for some prime \( p \). But the ring \( \mathbb{Z}_p \) is the field \( \mathbb{F}_p \), and we saw in Exercise 3.2 that \( \mathbb{F}_p \) contains no subfields. The conclusion is that in the second case the prime subfield is \( \mathbb{Z} \). In the first case, \( \mathbb{Z} \) is not a field, but each \( m \) in this copy of \( \mathbb{Z} \) has an inverse \( 1/m \) in \( F \), and the product of this with any other \( n \) gives an element \( m/n \in F \). The set of all such elements obtained is a copy of \( \mathbb{Q} \) inside \( F \).

Exercise 3.4. Make these loose statements precise: let \( F \) be a field and \( R \) a subring of \( F \) with \( \varphi : \mathbb{Z} \to R \) an isomorphism of rings (this is what we mean when we say that \( F \) contains a copy of \( \mathbb{Z} \)). Show that this can be extended to an isomorphism \( \tilde{\varphi} : \mathbb{Q} \to F \) with \( \tilde{\varphi}|_{\mathbb{Z}} = \varphi \).

Exercise 3.5. Show that a field \( F \) has characteristic \( p > 0 \) if and only if \( p \) is the smallest number of summands such that the expression \( 1 + 1 + \cdots + 1 \) is equal to 0. Show that \( F \) has characteristic 0 if and only if no such expression is equal to 0.

Thus, all fields of characteristic 0 are infinite, and the only examples we know of fields of characteristic \( p > 0 \) are finite. It is not true though that a field of characteristic \( p > 0 \) must be finite. We give some examples of infinite fields of characteristic \( p > 0 \) below.

Exercise 3.6. Suppose that \( f \) is an irreducible polynomial over a field \( F \) of characteristic 0. Recalling Exercise 2.1 show that the roots of \( f \) in any extension \( E \) of \( F \) are distinct.

Exercise 3.7.

1. Show that these definitions are well-defined, i.e: if \( (a, b) = (a', b') \) and \( (c, d) = (c', d') \), then \( (a, b) + (c, d) = (a', b') + (c', d') \) and \( (a, b)(c, d) = (a', b')(c', d') \).

2. Show that \( F \) is a field.

3. Define a map \( \varphi : F \to \mathbb{Q} \) by \( \varphi(a, b) = a/b \). Show that the map is well defined (i.e: if \( (a, b) = (a', b') \) then \( \varphi(a, b) = \varphi(a', b') \)) and that \( \varphi \) is an isomorphism.

This construction can be generalised as the following Exercise shows:

Exercise 3.8. Repeat the construction above with \( \mathbb{Z} \) replaced by an arbitrary integral domain \( R \).
The resulting field is called the field of fractions of $R$. The field of fractions construction provides some interesting examples of fields, possibly new in the reader’s experience. Let $F[x]$ be the ring of polynomials with $F$-coefficients where $F$ is any field. The field of fractions of this integral domain has elements of the form $f(x)/g(x)$ for $f$ and $g$ polynomials, in other words, rational functions with $F$-coefficients. The field is denoted $F(x)$ and is called the field of rational functions over $F$.

- An infinite field of characteristic $p$: if $\mathbb{F}_p$ is a finite field of order $p$, then the field of rational functions $\mathbb{F}_p(x)$ is infinite as it contains all the polynomials over $\mathbb{F}_p$. But the rational function $1$ still adds to itself only $p$ times to give 0, hence the field has characteristic $p$.
- A field properly containing the complex numbers: $\mathbb{C}$ is properly contained in the field of rational functions $\mathbb{C}(x)$.

Further Exercises for Section 3

Exercise 3.9. Let $\mathbb{F}$ be the set of all matrices of the form $\begin{bmatrix} a & b \\ 2b & a \end{bmatrix}$ where $a, b$ are in the field $\mathbb{F}_3$. Define addition and multiplication to be the usual addition and multiplication of matrices (and also the addition and multiplication in $\mathbb{F}_3$). Show that $\mathbb{F}$ is a field. How many elements does it have?

Exercise 3.10. Let $\mathbb{F}_2$ be the field of integers modulo 2, and $\alpha$ be an “abstract symbol” that can be multiplied so that it has the following property: $\alpha \times \alpha \times \alpha = \alpha^3 = \alpha + 1$ (a bit like decreeing that the imaginary $i$ squares to give $-1$). Let

$$\mathbb{F} = \{a + b\alpha + ca^2 | a, b, c \in \mathbb{F}_2\},$$

Define addition on $\mathbb{F}$ by: $(a_1 + b_1\alpha + c_1\alpha^2) + (a_2 + b_2\alpha + c_2\alpha^2) = (a_1 + a_2) + (b_1 + b_2)\alpha + (c_1 + c_2)\alpha^2,$

where the addition of coefficients happens in $\mathbb{F}_2$. For multiplication, “expand” the expression $(a_1 + b_1\alpha + c_1\alpha^2)(a_2 + b_2\alpha + c_2\alpha^2)$ like you would a polynomial with $\alpha$ the indeterminate, the coefficients are dealt with using the arithmetic from $\mathbb{F}_2$, and so on. Replace any $\alpha^3$ that result using the rule above.

1. Write down all the elements of $\mathbb{F}$.
2. Write out the addition and multiplication tables for $\mathbb{F}$ (ie: the tables with rows and columns indexed by the elements of $\mathbb{F}$, with the entry in the $i$-th row and $j$-th column the sum/product of the $i$-th and $j$-th elements of the field). Hence show that $\mathbb{F}$ is a field (you can assume that the addition and multiplication are associative as well as the distributive law, as these are a bit tedious to verify!) Using your tables, find the inverses (under multiplication) of the elements $1 + \alpha$ and $1 + \alpha + \alpha^2$, ie: find

$$\frac{1}{1 + \alpha} \quad \text{and} \quad \frac{1}{1 + \alpha + \alpha^2} \quad \text{in} \quad \mathbb{F}.$$

3. Is the extension $\mathbb{F}_2 \subset \mathbb{F}$ a simple one?

Exercise 3.11. Take the set $\mathbb{F}$ of the previous exercise, and define addition/multiplication in the same way except that the rule for simplification is now $\alpha^3 = \alpha^2 + \alpha + 1$. Show that in this case you don’t get a field.

Exercise 3.12. Verify the claim in lectures that the set $\mathbb{F} = \{a + b\sqrt{2} | a, b \in \mathbb{Q}\}$ is a subfield of $\mathbb{C}$.

Exercise 3.13. Verify the claim in lectures that $\mathbb{Q}(\sqrt{2}) = \{a + b(\sqrt{2}) + c(\sqrt{2})^2 | a, b, c \in \mathbb{Q}\}$. 

Exercise 3.14. Find a complex number \( \alpha \) such that \( \mathbb{Q}(\sqrt{2}, i) = \mathbb{Q}(\alpha) \).

Exercise 3.15. Is \( \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{7}) \) a simple extension of \( \mathbb{Q}(\sqrt{2}, \sqrt{3}), \mathbb{Q}(\sqrt{2}) \) or even of \( \mathbb{Q} \)?

Exercise 3.16. Let \( \nabla \) be an “abstract symbol” that has the following property: \( \nabla^2 = -\nabla - 1 \) (a bit like \( i \) squaring to give \(-1\)). Let

\[ F = \{a + b\nabla \mid a, b \in \mathbb{R}\}, \]

and define an addition on \( F \) by: \( (a_1 + b_1\nabla) + (a_2 + b_2\nabla) = (a_1 + a_2) + (b_1 + b_2)\nabla \). For multiplication, expand the expression \( (a_1 + b_1\nabla)(a_2 + b_2\nabla) \) normally (treating \( \nabla \) like an indeterminate, so that \( \nabla^2 = \nabla^2 \), and so on), and replace the resulting \( \nabla^2 \) using the rule above. Show that \( F \) is a field, and is just the complex numbers \( \mathbb{C} \). Do exactly the same thing, but with symbol \( \triangle \) satisfying \( \triangle^2 = \sqrt{2}\triangle - \sqrt{5} \). Show that you still get the complex numbers.

4. Rings II: Quotients

In the last section we saw the need to think about fields more abstractly. This section introduces the machinery we need to do this.

(4.1). A subset \( I \) of a ring \( R \) is an ideal if and only if \( I \) is a subgroup of the abelian group \( (R, +) \) and for any \( s \in R \) we have \( sl = \{sa \mid a \in I\} = Is \subseteq I \).

In the rings that most interest us, ideals turn out to have a very simple form:

**Proposition 4.1.** Let \( I \) be an ideal in \( F[x] \) for \( F \) a field. Then there is a polynomial \( f \in F[x] \) such that \( I = \{fg \mid g \in F[x]\} \).

An ideal in a ring of polynomials over a field thus consists of all the multiples of some fixed polynomial. For \( f \) the polynomial given in the Proposition, write \( \langle f \rangle \) for the ideal that it gives, i.e. \( \langle f \rangle = \{fg \mid g \in F[x]\} \), and call \( f \) a generator of the ideal.

**Proof.** If \( I = \{0\} \) (which is an ideal!) then we have \( I = \langle 0 \rangle \), and so the result holds. Otherwise, \( I \) contains non-zero polynomials. Choose \( f \) to be one of minimal degree \( \geq 0 \). Then \( Ig \subseteq I \) for all \( g \) gives \( \langle f \rangle \subseteq I \). Conversely, if \( h \in I \) then dividing \( h \) by \( f \) gives \( h = qf + r \). As \( qI \subseteq I \) we have \( qf \in I \), hence \( h - qf \in I \), as \( I \) is a subgroup under +. Thus \( r \in I \), and as \( deg \ r < deg \ f \) we are only saved from a contradiction if \( deg \ r < 0 \); that is, if \( r = 0 \). Thus \( h = qf \in \langle f \rangle \) and so \( I \subseteq \langle f \rangle \).

To emphasise that from now on, all our ideals will have this special form, we restate the definition:

**Definition 4.1 (ideals of polynomial rings over a field).** An ideal in \( F[x] \) is a set of the form

\[ \langle f \rangle = \{fg \mid g \in F[x]\} \]

for \( f \) some fixed polynomial.

**Exercise 4.1.** 1. Show that \( \langle f \rangle = \langle h \rangle \) if and only if \( h = cf \) for some constant \( c \in F \). Similarly, \( \langle h \rangle = F[x] \) if and only if \( h = c \) some constant. **Moral:** generators are not unique.

2. Let \( I \subset \mathbb{Z}[x] \) consist of those polynomials having even constant term. Show that \( I \) is an ideal but \( I \neq \langle f \rangle \) for any \( f \in \mathbb{Z}[x] \). **Moral:** ideals in \( R[x] \) for \( R \) a commutative ring need not have the special form of Proposition 4.1.
(4.2). In any ring there are the trivial ideals \( \langle 0 \rangle = \{0\} \) (which we have met already in the proof of Proposition 4.1) and \( \langle 1 \rangle = R \).

Exercise 4.2.

1. Show that the only ideals in a field \( F \) are the two trivial ones (hint: use the property of ideals mentioned at the end of the last paragraph).
2. If \( R \) is a commutative ring whose only ideals are \( \{0\} \) and \( R \), then show that \( R \) is a field.
3. Show that in the non-commutative ring \( M_n(F) \) of \( n \times n \) matrices with entries from the field \( F \) there are only the two trivial ideals, but that \( M_n(F) \) is not a field.

(4.3). For another example of an ideal, consider the ring \( \mathbb{Q}[x] \), the number \( \sqrt{2} \in \mathbb{R} \), and the evaluation homomorphism \( \varepsilon_{\sqrt{2}} : \mathbb{Q}[x] \rightarrow \mathbb{R} \) given by

\[
\varepsilon_{\sqrt{2}}(a_nx^n + \cdots + a_0) = a_n(\sqrt{2})^n + \cdots + a_0.
\]

(see Section [1]). Let \( I \) be the set of all polynomials in \( \mathbb{Q}[x] \) that are sent to \( 0 \in \mathbb{R} \) by this map. Certainly \( x^2 - 2 \in I \) (as \( \sqrt{2} - 2 = 0 \)). If \( f = (x^2 - 2)g \in \mathbb{Q}[x] \), then

\[
\varepsilon_{\sqrt{2}}(f) = \varepsilon_{\sqrt{2}}(x^2 - 2)\varepsilon_{\sqrt{2}}(g) = 0 \times \varepsilon_{\sqrt{2}}(g) = 0,
\]

using the fact that \( \varepsilon_{\sqrt{2}} \) is a homomorphism. Thus, \( f \in I \), and so the ideal \( \langle x^2 - 2 \rangle \) is \( \subseteq I \).

Conversely, if \( h \) is sent to \( 0 \) by \( \varepsilon_{\sqrt{2}} \), ie: \( h \in I \), we can divide it by \( x^2 - 2 \) using the division algorithm,

\[
h = (x^2 - 2)q + r,
\]

where \( \deg r < 2 \), so that \( r = ax + b \) for some \( a, b \in \mathbb{Q} \). But since \( \varepsilon_{\sqrt{2}}(h) = 0 \) we have

\[
(x^2 - 2)q(\sqrt{2}) + r(\sqrt{2}) = 0 \Rightarrow r(\sqrt{2}) = 0 \Rightarrow a\sqrt{2} + b = 0.
\]

If \( a \neq 0 \), then \( \sqrt{2} \in \mathbb{Q} \) as \( a, b \in \mathbb{Q} \), which is plainly nonsense. Thus \( a = 0 \), hence \( b = 0 \) too, so that \( r = 0 \), and hence \( h = (x^2 - 2)q \in \langle x^2 - 2 \rangle \), and we get that \( I \subseteq \langle x^2 - 2 \rangle \).

The conclusion is that the set of polynomials in \( \mathbb{Q}[x] \) sent to zero by the evaluation homomorphism \( \varepsilon_{\sqrt{2}} \) is an ideal.

(4.4). This always happens: if \( R, S \) are rings and \( \varphi : R \rightarrow S \) a ring homomorphism, then the kernel of \( \varphi \), denoted \( \ker \varphi \), is the set of all elements of \( R \) sent to \( 0 \in S \) by \( \varphi \), ie:

\[
\ker \varphi = \{ r \in R \mid \varphi(r) = 0 \in S \}.
\]

Proposition 4.2. If \( F \) is a field and \( S \) a ring then the kernel of a homomorphism \( \varphi : F[x] \rightarrow S \) is an ideal.

Proof. Is very similar to the previous example. To get a polynomial that plays the role of \( x^2 - 2 \), choose \( g \in \ker \varphi \), non-zero, of smallest degree. We claim that \( \ker \varphi = \langle g \rangle \), for which we need to show that these two sets are mutually contained within each other. On the one hand, if \( pg \in \langle g \rangle \) then

\[
\varphi(pg) = \varphi(p)\varphi(g) = \varphi(p) \times 0 = 0,
\]

since \( g \in \ker \varphi \). Thus, \( \langle g \rangle \subseteq \ker \varphi \). On the other hand, let \( f \in \ker \varphi \) and use the division algorithm to divide it by \( g \),

\[
f = qg + r,
\]

where \( \deg r < \deg g \). Now, \( r = f - qg \Rightarrow \varphi(r) = \varphi(f - qg) = \varphi(f) - \varphi(q)\varphi(g) = 0 - \varphi(g).0 = 0 \), since both \( f, g \in \ker \varphi \). Thus, \( r \) is also in the kernel of \( \varphi \). If \( r \) was a non-zero polynomial, then we would have a contradiction because \( \deg r < \deg g \), but \( g \) was chosen from \( \ker \varphi \) to have smallest degree. Thus we must have that \( r = 0 \), hence \( f = qg \in \langle g \rangle \), ie: \( \ker \varphi \subseteq \langle g \rangle \). \( \square \)
(4.5). Let \( \langle f \rangle \subset F[x] \) be an ideal and \( g \in F[x] \) any polynomial. The set

\[ g + \langle f \rangle = \{ g + h \mid h \in \langle f \rangle \}, \]

is called the \textit{coset of} \( \langle f \rangle \) \textit{with representative} \( g \) (or the \textit{coset of} \( \langle f \rangle \) \textit{determined} by \( g \)).

(4.6). As an example, consider the ideal \( \langle x \rangle \) in \( F_2[x] \). Thus \( \langle x \rangle \) is the set of all multiples of \( x \), which is the same as the polynomials in \( F_2[x] \) that have no constant term. What are the cosets of \( \langle x \rangle \)? Let \( g \) be any polynomial and consider the coset \( g + \langle x \rangle \). The only possibilities are that \( g \) has no constant term, or it does, in which case this term is 1 (we are in \( F_2[x] \)).

If \( g \) has no constant term, then

\[ g + \langle x \rangle = \langle x \rangle. \]

For, \( g \) added to a polynomial with no constant term is another polynomial with no constant term, ie: \( g + \langle x \rangle \subseteq \langle x \rangle \). On the other hand, if \( f \in \langle x \rangle \) is any polynomial with no constant term, then \( f - g \in \langle x \rangle \) so \( f = g + (f - g) \in g + \langle x \rangle \), ie: \( \langle x \rangle \subseteq g + \langle x \rangle \).

If \( g \) does have a constant term, you can convince yourself in exactly the same way that,

\[ g + \langle x \rangle = 1 + \langle x \rangle. \]

Thus, there are only two cosets of \( \langle x \rangle \) in \( F_2[x] \), namely the ideal \( \langle x \rangle \) itself and \( 1 + \langle x \rangle \).

Notice that these two cosets are completely disjoint, but every polynomial is in one of them.

(4.7). Here are some basic properties of cosets:

- 
  Every polynomial \( g \) is in some coset of \( \langle f \rangle \): for \( g = g + 0 \times f \in g + \langle f \rangle \).
- For any \( q \), we have \( qf + \langle f \rangle = \langle f \rangle \): so multiples of \( f \) get “absorbed” into the ideal \( \langle f \rangle \).
- The following three things are equivalent: (i) \( g_1 \) and \( g_2 \) lie in the same coset of \( \langle f \rangle \); (ii) \( g_1 + \langle f \rangle = g_2 + \langle f \rangle \); (iii) \( g_1 \) and \( g_2 \) differ by a multiple of \( f \): to see this: (iii) \( \Rightarrow \) (ii) If \( g_1 - g_2 = pf \) then \( g_1 = g_2 + pf \) so that \( g_1 + \langle f \rangle = g_2 + pf + \langle f \rangle = g_2 + \langle f \rangle \); (ii) \( \Rightarrow \) (i) Since \( g_1 \in g_1 + \langle f \rangle \) and \( g_2 \in g_2 + \langle f \rangle \), and these cosets are equal we have that \( g_1, g_2 \) lie in the same coset; (i) \( \Rightarrow \) (iii) If \( g_1 \) and \( g_2 \) lie in the same coset, ie: \( g_1, g_2 \in h + \langle f \rangle \), then each \( g_1 = h + p_1f \Rightarrow g_1 - g_2 = (p_1 - p_2)f \).

It can be summarised by saying that \( g_1 \) and \( g_2 \) lie in the same coset if and only if this coset has the two different names, \( g_1 + \langle f \rangle \) and \( g_2 + \langle f \rangle \), as in the left of Figure 4.1.

- The situation on the right of Figure 4.1 opposite never happens, where distinct cosets have non-empty intersection: if the two cosets pictured are called respectively \( g_1 + \langle f \rangle \) and \( g_2 + \langle f \rangle \), then \( h \) is in both and so differs from \( g_1 \) and \( g_2 \) by multiples of \( f \), ie: \( g_1 - h = p_1f \) and \( h - g_2 = p_2f \), so that \( g_1 - g_2 = (p_1 + p_2)f \). Since \( g_1 \) and \( g_2 \) differ by a multiple of \( f \), we have \( g_1 + \langle f \rangle = g_2 + \langle f \rangle \).

Thus, the cosets of an ideal partition the ring.
(4.8). As an example let $x^2 - 2 \in \mathbb{Q}[x]$ and consider the ideal

$$\langle x^2 - 2 \rangle = \{ p(x^2 - 2) | p \in \mathbb{Q}[x] \}.$$ 

$(x^3 - 2x + 15) + \langle x^2 - 2 \rangle$ is then a coset, but it is not written in the nicest possible form. If we divide $x^3 - 2x + 15$ by $x^2 - 2$:

$$x^3 - 2x + 15 = x(x^2 - 2) + 15,$$

we have $x^3 - 2x + 15$ and 15 differ by a multiple of $x^2 - 2$, so that

$$(x^3 - 2x + 15) + \langle x^2 - 2 \rangle = 15 + \langle x^2 - 2 \rangle.$$ 

(4.9). If we look again at the ideal $\langle x \rangle$ in $\mathbb{F}_2[x]$, there were only two cosets,

$$\langle x \rangle = 0 + \langle x \rangle \text{ and } 1 + \langle x \rangle,$$

that corresponded to the polynomials with constant term 0 and the polynomials with constant term 1. We could try “adding” and “multiplying” these two cosets together according to,

$$(0 + \langle x \rangle) + (0 + \langle x \rangle) = 0 + \langle x \rangle, (1 + \langle x \rangle) + (0 + \langle x \rangle) = 1 + \langle x \rangle, (1 + \langle x \rangle) + (1 + \langle x \rangle) = 0 + \langle x \rangle,$$

and so on, where all we have done is to add the representatives of the cosets together using the addition from $\mathbb{F}_2$. Similarly for multiplying the cosets. This looks like $\mathbb{F}_2$, but with $0 + \langle x \rangle$ and $1 + \langle x \rangle$ replacing 0 and 1.

(4.10). Again this always happens. Let $\langle f \rangle$ be an ideal in $F[x]$, and define an addition and multiplication of cosets of $\langle f \rangle$ by,

$$(g_1 + \langle f \rangle) + (g_2 + \langle f \rangle) = (g_1 + g_2) + \langle f \rangle \text{ and } (g_1 + \langle f \rangle)(g_2 + \langle f \rangle) = (g_1g_2) + \langle f \rangle,$$

where the addition and multiplication of the $g_i$’s is happening in $F[x]$.

**Theorem 4.1.** The set of cosets $F[x]/\langle f \rangle$ together with the + and $\times$ above is a ring.

Call this the quotient ring of $F[x]$ by the ideal $\langle f \rangle$. All our rings have a “zero”, a “one”, and so on, and for the quotient ring these are,

| element of a ring | corresponding element in $F[x]/\langle f \rangle$ |
|-------------------|-----------------------------------------------|
| $a$               | $g + \langle f \rangle$                      |
| $-a$              | $(-g) + \langle f \rangle$                   |
| $0$               | $0 + \langle f \rangle = \langle f \rangle$  |
| $1$               | $1 + \langle f \rangle$                      |

**Exercise 4.3.** To prove this theorem:

1. Show that the addition of cosets is well defined, ie: if $g_i' + \langle f \rangle = g_i + \langle f \rangle$, then

$$(g_1' + g_2') + \langle f \rangle = (g_1 + g_2) + \langle f \rangle.$$ 

2. Similarly, show that the multiplication is well defined.
3. Now verify the axioms for a ring.
(4.11). Let \( x^2 + 1 \in \mathbb{R}[x] \), and consider the ideal \( \langle x^2 + 1 \rangle \). We want to see what the quotient \( \mathbb{R}[x]/\langle x^2 + 1 \rangle \) looks like. First, any coset can be put into a nice form: for example,
\[
x^4 + x^2 + x + 1 + \langle x^2 + 1 \rangle = x^2(x^2 + 1) + (x + 1) + \langle x^2 + 1 \rangle,
\]
where we have divided \( x^4 + x^2 + x + 1 \) by \( x^2 + 1 \) using the division algorithm. But
\[
x^2(x^2 + 1) + (x + 1) + \langle x^2 + 1 \rangle = x + 1 + \langle x^2 + 1 \rangle,
\]
as the multiple of \( x^2 + 1 \) gets absorbed into the ideal. In fact, for any \( g \in \mathbb{R}[x] \) we can make this argument:
\[
g + \langle x^2 + 1 \rangle = g(x^2 + 1) + (ax + b) + \langle x^2 + 1 \rangle = ax + b + \langle x^2 + 1 \rangle,
\]
for some \( a, b \in \mathbb{R} \), so the set of cosets can be written as
\[
\mathbb{R}[x]/\langle x^2 + 1 \rangle = \{ ax + b + \langle x^2 + 1 \rangle | a, b \in \mathbb{R} \}.
\]
Now take two elements of the quotient, say \( (x + 1) + \langle x^2 + 1 \rangle \) and \( (2x - 3) + \langle x^2 + 1 \rangle \), and add/multiply them together:
\[
\left((x + 1) + \langle x^2 + 1 \rangle\right) + \left((2x - 3) + \langle x^2 + 1 \rangle\right) = 3x - 2 + \langle x^2 + 1 \rangle,
\]
and
\[
\left((x + 1) + \langle x^2 + 1 \rangle\right) \times \left((2x - 3) + \langle x^2 + 1 \rangle\right) = (2x^2 - x - 5) + \langle x^2 + 1 \rangle
\]
\[
= -x - 5 + \langle x^2 + 1 \rangle.
\]
Now “squint your eyes”, so that \( ax + b + \langle x^2 + 1 \rangle \) becomes the complex number \( ai + b \in \mathbb{C} \). Then
\[
(i + 1) + (2i - 3) = 3i - 2 \quad \text{and} \quad (i + 1)(2i - 3) = -i - 5.
\]
The addition and multiplication of cosets in \( \mathbb{R}[x]/\langle x^2 + 1 \rangle \) looks exactly like the addition and multiplication of complex numbers!

(4.12). To see what quotient rings look like we use:

**First Isomorphism Theorem.** Let \( \varphi : F[x] \to S \) be a ring homomorphism with kernel \( \langle f \rangle \). Then the map \( g + \langle f \rangle \mapsto \varphi(g) \) is an isomorphism
\[
F[x]/\langle f \rangle \to \text{Im } \varphi \subset S.
\]

(4.13). In the example above let \( R = \mathbb{R}[x] \) and \( S = \mathbb{C} \). Let the homomorphism \( \varphi \) be the evaluation at \( i \) homomorphism,
\[
\varepsilon_i : \left( \sum a_k x^k \right) \mapsto \sum a_k (i)^k.
\]
In exactly the same way as an earlier example, you can show that
\[
\ker \varepsilon_i = \langle x^2 + 1 \rangle.
\]
On the other hand, if \( ai + b \in \mathbb{C} \), then \( ai + b = \varepsilon_i(ax + b) \), so the image of the homomorphism \( \varepsilon_i \) is all of \( \mathbb{C} \). Feeding this into the first homomorphism theorem gives,
\[
\mathbb{R}[x]/\langle x^2 + 1 \rangle \cong \mathbb{C}.
\]
Further Exercises for Section 4

Exercise 4.4. Let \( \phi = (1 + \sqrt{5})/2 \) (in fact the Golden Number).

1. Show that the kernel of the evaluation map \( \epsilon_\phi : \mathbb{Q}[x] \to \mathbb{C} \) (given by \( \epsilon_\phi(f) = f(\phi) \)) is the ideal \( \langle x^2 - x - 1 \rangle \).
2. Show that \( \mathbb{Q}(\phi) = \{ a + b\phi \mid a, b \in \mathbb{Q} \} \).
3. Show that \( \mathbb{Q}(\phi) \) is the image in \( \mathbb{C} \) of the map \( \epsilon_\phi \).

Exercise 4.5. Going back to the general case of an ideal \( I \) in a ring \( R \), consider the map \( \eta : R \to R/I \) given by,

\[ \eta(r) = r + I, \]

sending an element of \( R \) to the coset of \( I \) determined by it.

1. Show that \( \eta \) is a homomorphism.
2. Show that if \( J \) is an ideal in \( R \) containing \( I \) then \( \eta(J) \) is an ideal of \( R/I \).
3. Show that if \( J' \) is an ideal of \( R/I \) then there is an ideal \( J \) of \( R \), containing \( I \), such that \( \eta(J) = J' \).
4. Show that in this way, \( \eta \) is a bijection between the ideals of \( R \) containing \( I \) and the ideals of \( R/I \).

5. Fields II: Constructions and More Examples

(5.1). A proper ideal \( \langle f \rangle \) of \( F[x] \) is maximal if and only if the only ideals of \( F[x] \) containing \( \langle f \rangle \) are \( \langle f \rangle \) itself and the whole ring \( F[x] \), ie:

\[ \langle f \rangle \subseteq I \subseteq F[x], \]

with \( I \) an ideal implies that \( I = \langle f \rangle \) or \( I = F[x] \).

(5.2). The main result of this section is,

Theorem B (Constructing Fields). The quotient ring \( F[x]/\langle f \rangle \) is a field if and only if \( \langle f \rangle \) is a maximal ideal.

Proof. By Exercise 4.2 a commutative ring \( R \) is a field if and only if the only ideals of \( R \) are the trivial one \( \{0\} \) and the whole ring \( R \). Thus the quotient \( F[x]/\langle f \rangle \) is a field if and only if its only ideals are the trivial one \( \langle f \rangle \) and the whole ring \( F[x]/\langle f \rangle \). By Exercise 4.5 there is a one to one correspondence between the ideals of the quotient \( F[x]/\langle f \rangle \) and the ideals of \( F[x] \) that contain \( \langle f \rangle \). Thus \( F[x]/\langle f \rangle \) has only the two trivial ideals precisely when there are only two ideals of \( F[x] \) containing \( \langle f \rangle \), namely \( \langle f \rangle \) and \( F[x] = \langle 1 \rangle \), which is the same as saying that \( \langle f \rangle \) is maximal. \( \Box \)

(5.3). Suppose now that \( f \) is an irreducible polynomial over \( F \), and let \( \langle f \rangle \subseteq I \subseteq F[x] \) with \( I \) an ideal. Then \( I = \langle h \rangle \) hence \( \langle f \rangle \subseteq \langle h \rangle \), and so \( h \) divides \( f \). Since \( f \) is irreducible this means that \( h \) must be either a constant \( c \in F \) or \( cf \), so that the ideal \( I \) is either \( \langle c \rangle \) or \( \langle cf \rangle \). But \( \langle cf \rangle \) is just the same as the ideal \( \langle f \rangle \). On the other hand, any polynomial \( g \) can be written as a multiple of \( c \), just by setting \( g = c(c^{-1}g) \), and so \( \langle c \rangle = F[x] \). Thus if \( f \) is an irreducible polynomial then the ideal \( \langle f \rangle \) is maximal.

Conversely, if \( \langle f \rangle \) is maximal and \( h \) divides \( f \), then \( \langle f \rangle \subseteq \langle h \rangle \), so that by maximality \( \langle h \rangle = \langle f \rangle \) or \( \langle h \rangle = F[x] \). By Exercise 4.1 we have \( h = c \) a constant, or \( h = cf \), and so \( f \) is irreducible over \( F \).

Thus, the ideal \( \langle f \rangle \) is maximal precisely when \( f \) is irreducible.

Corollary 5.1. \( F[x]/\langle f \rangle \) is a field if and only if \( f \) is an irreducible polynomial over \( F \).
(5.4). The polynomial $x^2 + 1$ is irreducible over the reals $\mathbb{R}$, so the quotient ring $\mathbb{R}[x]/(x^2 + 1)$ is a field.

(5.5). The polynomial $x^2 - 2x + 2$ has roots $1 \pm i$, hence is irreducible over $\mathbb{R}$, giving the field, $\mathbb{R}[x]/(x^2 - 2x + 2)$.

Consider the evaluation map $\varepsilon_{1+i} : \mathbb{R}[x] \to \mathbb{C}$ given as usual by $\varepsilon_{1+i}(f) = f(1 + i)$. In exactly the same way as the example for $\varepsilon_q$ in Section 4, one can show that $\ker \varepsilon_{1+i} = \langle x^2 - 2x + 2 \rangle$. Moreover, $a + bi = \varepsilon_{1+i}(a - b + bx)$ so that the evaluation map is onto $\mathbb{C}$. Thus, by the first isomorphism theorem we get that,

$$\mathbb{R}[x]/(x^2 - 2x + 2) \cong \mathbb{C}.$$ 

What this means is that we can construct the complex numbers in the following (slightly non-standard) way: start with the reals $\mathbb{R}$, and define a new symbol, $\nabla$ say, which satisfies the algebraic property,

$$\nabla^2 = 2\nabla - 2.$$

Now consider all expressions of the form $c + d\nabla$ for $c, d \in \mathbb{R}$. Add and multiply two such expressions together as follows:

$$(c_1 + d_1\nabla) + (c_2 + d_2\nabla) = (c_1 + c_2) + (d_1 + d_2)\nabla$$
$$(c_1 + d_1\nabla)(c_2 + d_2\nabla) = c_1c_2 + (c_1d_2 + d_1c_2)\nabla + d_1d_2\nabla^2 = c_1c_2 + (c_1d_2 + d_1c_2)\nabla + d_1d_2(2\nabla - 2) = (c_1c_2 - 2d_1d_2) + (c_1d_2 + d_1c_2 + 2d_1d_2)\nabla.$$

Exercise 5.1. By solving the equations $cx - 2dy = 1$ and $cy + dx + 2dy = 0$ for $x$ and $y$ in terms of $c$ and $d$, find the inverse of the element $c + d\nabla$.

Exercise 5.2. According to Exercise 2.3 if $f$ is irreducible over $\mathbb{R}$ then $f$ must be either quadratic or linear. Suppose that $f = ax^2 + bx + c$ is an irreducible quadratic over $\mathbb{R}$. Show that the field $\mathbb{R}[x]/(ax^2 + bx + c) \cong \mathbb{C}$.

(5.6). The next few paragraphs illustrate the construction for finite fields, using a field of order four as a running example.

In the process of doing the example in (2.11), we saw that the only irreducible quadratic over the field $\mathbb{F}_2$ is $x^2 + x + 1$. Thus the quotient

$$\mathbb{F}_2[x]/(x^2 + x + 1),$$

is a field. Each of its elements is a coset of the form $g + \langle x^2 + x + 1 \rangle$. Use the division algorithm, dividing $g$ by $x^2 + x + 1$, to get

$$g + \langle x^2 + x + 1 \rangle = q(x^2 + x + 1) + r + \langle x^2 + x + 1 \rangle = r + \langle x^2 + x + 1 \rangle,$$

where the remainder $r$ is of the form $ax + b$, for $a, b \in \mathbb{F}_2$. Thus every element of the field has the form $ax + b + \langle x^2 + x + 1 \rangle$, of which there are at most 4 possibilities (2 choices for $a$ and 2 choices for $b$).

Indeed these 4 are distinct, for if

$$a_1x + b_1 + \langle x^2 + x + 1 \rangle = a_2x + b_2 + \langle x^2 + x + 1 \rangle$$

then,

$$(a_1 - a_2)x + (b_1 - b_2) + \langle x^2 + x + 1 \rangle = \langle x^2 + x + 1 \rangle \iff (a_1 - a_2)x + (b_1 - b_2) \in \langle x^2 + x + 1 \rangle.$$
Since the non-zero elements of the ideal are multiples of a degree two polynomial, they have degrees that are at least two. Thus the only way the linear polynomial can be an element is if it is the zero polynomial. In particular, \(a_1 - a_2 = b_1 - b_2 = 0\), so the two cosets are the same. The quotient ring is thus a field having the four elements:

\[
\mathbb{F}_4 = \{ax + b + \langle x^2 + x + 1 \rangle | a, b \in \mathbb{F}_2\}
\]

**(5.7).** Generalising the example of the field of order 4 above, if \(\mathbb{F}_p\) is the finite field with \(p\) elements and \(f \in \mathbb{F}_p[x]\) is an irreducible polynomial of degree \(d\), then the quotient \(\mathbb{F}_p[x]/\langle f \rangle\) is a field containing elements of the form,

\[
a_{d-1}x^{d-1} + \cdots + a_0 + \langle f \rangle,
\]

where the \(a_i \in \mathbb{F}_p\). Any two such are distinct by exactly the same argument as above, so we have a field \(\mathbb{F}_q\) with exactly \(q = p^d\) elements.

**(5.8).** Returning to the general situation of a quotient \(\mathbb{F}[x]/\langle f \rangle\) by an irreducible polynomial \(f\), the resulting field contains a copy of the original field \(\mathbb{F}\), obtained by considering the cosets \(a + \langle f \rangle\) for \(a \in \mathbb{F}\).

*Exercise 5.3.* Show that the map \(a \mapsto a + \langle f \rangle\) is an injective homomorphism \(\mathbb{F} \to \mathbb{F}[x]/\langle f \rangle\), and so \(\mathbb{F}\) is isomorphic to its image in \(\mathbb{F}[x]/\langle f \rangle\).

Blurring the distinction between the original \(\mathbb{F}\) and this copy inside \(\mathbb{F}[x]/\langle f \rangle\), we get that \(\mathbb{F} \subset \mathbb{F}[x]/\langle f \rangle\) is an extension of fields.

**(5.9).** Back to the field \(\mathbb{F}_4\) of order 4 and a more convenient notation. Let

\[
\alpha = x + \langle x^2 + x + 1 \rangle
\]

and write \(a \in \mathbb{F}_2\) for the coset \(a + \langle x^2 + x + 1 \rangle\) as in the previous paragraph. Addition and multiplication of cosets gives:

\[
ax + b + \langle x^2 + x + 1 \rangle = (a + \langle x^2 + x + 1 \rangle)(x + \langle x^2 + x + 1 \rangle) + (b + \langle x^2 + x + 1 \rangle) = a\alpha + b.
\]

So we now have that \(\mathbb{F}_4 = \{a\alpha + b | a, b \in \mathbb{F}_2\} = \{0, 1, \alpha, \alpha + 1\}\). But we also have the coset property \(f + \langle f \rangle = \langle f \rangle\), which for \(f = x^2 + x + 1\) translates into

\[
(x + \langle x^2 + x + 1 \rangle)^2 + (x + \langle x^2 + x + 1 \rangle) + (1 + \langle x^2 + x + 1 \rangle) = \langle x^2 + x + 1 \rangle,
\]

or, \(\alpha^2 + \alpha + 1 = 0\). Our field is now \(\mathbb{F}_4 = \{0, 1, \alpha, \alpha + 1\}\), together with the “rule” \(\alpha^2 = \alpha + 1\).

At the risk of labouring the point, here are the multiplication tables for the field \(\mathbb{F}_4\) and the ring \(\mathbb{Z}_4\):

\[
\begin{array}{c|cccc}
\mathbb{F}_4 & 0 & 1 & \alpha & \alpha + 1 \\
\hline
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & \alpha & \alpha + 1 \\
\alpha & 0 & \alpha & \alpha + 1 & 1 \\
\alpha + 1 & 0 & \alpha + 1 & 1 & \alpha \\
\end{array}
\]

\[
\begin{array}{c|cccc}
\mathbb{Z}_4 & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 2 & 3 \\
\alpha & 0 & \alpha & \alpha + 1 & 1 \\
\alpha + 1 & 0 & \alpha + 1 & 1 & \alpha \\
\end{array}
\]

1 appears in every non-zero row of the \(\mathbb{F}_4\) table – so every non-zero element has an inverse – but does not appear in every non-zero row of \(\mathbb{Z}_4\).
(5.10). In general, when \( f \in \mathbb{F}_p[x] \) is irreducible of degree \( d \), we let \( \alpha = x + \langle f \rangle \) and replace \( \mathbb{F}_p \) by its copy in \( \mathbb{F}_p[x]/\langle f \rangle \) (ie: identify \( a \in \mathbb{F}_p \) with \( a + \langle f \rangle \in \mathbb{F}_p[x]/\langle f \rangle \)). This gives, 
\[
\mathbb{F}_p[x]/\langle f \rangle = \{a_{d-1}\alpha^{d+1} + \cdots + a_0 | a_i \in \mathbb{F}_p\},
\]
where two such expressions are added and multiplied like “polynomials” in \( \alpha \). If \( f = b_2x^2 + \cdots + b_1x + b_0 \), and since \( f + \langle f \rangle = \langle f \rangle \), we have the “rule” \( b_0\alpha^d + \cdots + b_1\alpha + b_0 = 0 \), which allows us to remove any powers of \( \alpha \) bigger than \( d \) that occur in such expressions. The element \( \alpha \) is called a generator for the field.

(5.11). The polynomial \( x^3 + x + 1 \) is irreducible over the field \( \mathbb{F}_2 \) (it is a cubic and has no roots) so that 
\[
\mathbb{F}_2[x]/\langle x^3 + x + 1 \rangle,
\]
is a field with \( 2^3 = 8 \) elements of the form \( \mathbb{F} = \{a + bx + cx^2 | a, b, c \in \mathbb{F}_2\} \) subject to the rule \( \alpha^3 + \alpha + 1 = 0 \), ie: \( \alpha^3 = \alpha + 1 \). This is the field \( \mathbb{F} \) of order 8 from Section [3].

Exercise 5.4. Explicitly construct fields with exactly:
1. 125 elements
2. 49 elements
3. 81 elements
4. 243 elements
(By explicit I mean give a general description of the elements and any algebraic rules that are needed for adding and multiplying them together.)

(5.12). To explicitly construct a field of order \( p^d \) with \( d > 3 \) is harder – finding irreducible polynomials of degree bigger than a cubic is not straightforward, as the example in (2.11) shows. One solution is to create the field in a series of steps (or extensions), each of which only involves quadratics or cubics.

We do this for a field of order \( 729 = 3^6 \). As \( 3^6 = (3^2)^3 \), we first create a field of order \( 3^2 \), and then extend this using a cubic.

Consider the polynomial \( f = x^2 + x + 2 \in \mathbb{F}_3[x] \). Substituting the three elements of \( \mathbb{F}_3 \) into \( f \) gives 
\[
0^2 + 0 + 2 = 2, 1^2 + 1 + 2 = 1 \text{ and } 2^2 + 2 + 2 = 2,
\]
so that \( f \) has no roots in \( \mathbb{F}_3 \). As \( f \) is quadratic it is irreducible over the field \( \mathbb{F}_3 \), and so \( \mathbb{F}_9 = \mathbb{F}_3[x]/\langle x^2 + x + 2 \rangle \) is a field of order \( 3^2 \). Let \( \alpha = x + \langle x^2 + x + 2 \rangle \) in \( \mathbb{F}_9 \) be a generator so that the elements have the form \( a + b\alpha \) with \( a, b \in \mathbb{F}_3 \) and multiplication satisfying the rule \( \alpha^2 + \alpha + 2 = 0 \), or equivalently \( \alpha^2 = 2\alpha + 1 \). (\( -1 = 2 \) and \( -2 = 1 \) in \( \mathbb{F}_3 \)).

Now let \( X \) be a new variable, and consider the polynomials \( \mathbb{F}_9[X] \) over \( \mathbb{F}_9 \) in this new variable. In particular the polynomial:
\[
g = X^3 + (2\alpha + 1)X + 1.
\]
(5.1)

As \( g \) is a cubic, it will be irreducible over \( \mathbb{F}_9 \) precisely when it has no roots in this field, which can be verified as usual by straight substitution of the nine elements of \( \mathbb{F}_9 \). For example:
\[
g(2\alpha + 1) = (2\alpha + 1)^3 + (2\alpha + 1)(2\alpha + 1) + 1 = 2\alpha^3 + 1 + \alpha^2 + \alpha + 1 + 1
\]
\[
= 2\alpha(2\alpha + 1) + \alpha^2 + \alpha
\]
\[
= \alpha^2 + 2\alpha + \alpha^2 + \alpha = \alpha + 2
\]
and the others are similar. We have used an energy saving device in these computations:

Exercise 5.5. If \( a, b \in F \), a field of characteristic \( p > 0 \), then \( (a + b)^p = a^p + b^p \) (hint: Exercise 2.12).
Thus the polynomial \( g \) in (5.11) is irreducible over \( \mathbb{F}_2 \), and we have a field:
\[
\mathbb{F}_9[X]/(X^3 + (2\alpha + 1)X + 1)
\]
of order \( 9^3 = 3^6 = 729 \), called \( \mathbb{F}_{729} \). The elements have the form,
\[
A_0 + A_1\beta + A_2\beta^2,
\]
where the \( A_i \in \mathbb{F}_9 \) and \( \beta = X + \langle g \rangle \) is a generator. Multiplication is given by the rule \( \beta^3 = (\alpha + 2)\beta + 2 \). Replacing the \( A_i \) by the earlier description of \( \mathbb{F}_9 \) in terms of the generator \( \alpha \) gives elements:
\[
a_0 + a_1\alpha + a_2\alpha^2 + a_3\alpha + a_4\alpha^2 + a_5\alpha^2,
\]
with the \( a_i \in \mathbb{F}_3 \), subject to the rules \( \alpha^2 = 2\alpha + 1 \) and \( \beta^3 = (\alpha + 2)\beta + \alpha \).

**Exercise 5.6.**
1. Construct a field \( \mathbb{F}_8 \) with 8 elements by showing that \( x^3 + x + 1 \) is irreducible over \( \mathbb{F}_2 \).
2. Find a cubic polynomial that is irreducible in \( \mathbb{F}_8[x] \) (hint: refer to Exercise 2.11).
3. Hence, or otherwise, construct a field with \( 2^9 = 512 \) elements.

**Exercise 5.7.** Explicitly construct fields with exactly:

1. 64 elements
2. challenge: 4096 elements

(5.13). Theorem B and its Corollary solves the problem that we encountered in Section 3 where the fields

\[
\mathbb{Q}(\sqrt{2}) \text{ and } \mathbb{Q}\left(-\frac{\sqrt{2} + \sqrt{3}i}{2}\right) = \mathbb{Q}(\beta)
\]

were different but isomorphic. The polynomial \( x^3 - 2 \) is irreducible over \( \mathbb{Q} \), either by Eisenstein, or by observing that its roots do not lie in \( \mathbb{Q} \). Thus

\[
\mathbb{Q}[x]/\langle x^3 - 2 \rangle,
\]

is an extension field of \( \mathbb{Q} \). Consider the two evaluation homomorphisms \( \varepsilon_{\sqrt{2}} : \mathbb{Q}[x] \to \mathbb{C} \) and \( \varepsilon_\beta : \mathbb{Q}[x] \to \mathbb{C} \). Since, and this is the key bit,

\[
\sqrt{2} \text{ and } \beta = \frac{-\sqrt{2} + \sqrt{3}i}{2}
\]

are both roots of the polynomial \( x^3 - 2 \), we can show in a similar manner to examples at the end of Section 4 that \( \ker \varepsilon_{\sqrt{2}} \equiv \langle x^3 - 2 \rangle \equiv \ker \varepsilon_\beta \). Thus,

\[
\begin{array}{ccc}
\mathbb{Q}[x]/\ker \varepsilon_{\sqrt{2}} & = & \mathbb{Q}[x]/\langle x^3 - 2 \rangle \\
\cong & & \mathbb{Q}[x]/\ker \varepsilon_\beta \\
\text{Im } \varepsilon_{\sqrt{2}} & \cong & \text{1st Isomorphism Theorem} & \cong & \text{Im } \varepsilon_\beta
\end{array}
\]

\((*)\)

To find the image of \( \varepsilon_{\sqrt{2}} \) write a \( g \in \mathbb{Q}[x] \) as \( g = q(x^3 - 2) + (a + bx + cx^2) \) so that

\[
\varepsilon_{\sqrt{2}}(g) = \varepsilon_{\sqrt{2}}(q(x^3 - 2) + (a + bx + cx^2))
\]

\[
= \varepsilon_{\sqrt{2}}(q)\varepsilon_{\sqrt{2}}(x^3 - 2) + \varepsilon_{\sqrt{2}}(a + bx + cx^2)
\]

\[
= \varepsilon_{\sqrt{2}}(q) \cdot 0 + \varepsilon_{\sqrt{2}}(a + bx + cx^2) = a + b\sqrt{2} + c(\sqrt{2})^2.
\]
Hence \( \text{Im} \varepsilon \mathbb{Q}(\sqrt{2}) \subseteq \{a + b \sqrt{2} + c(\sqrt{2})^2 \in \mathbb{C} \mid a, b, c \in \mathbb{Q} \} = \mathbb{Q}(\sqrt{2}) \).

On the other hand \( a + b \sqrt{2} + c(\sqrt{2})^2 \) is the image of \( a + bx + cx^2 \) and so \( \text{Im} \varepsilon \mathbb{Q}(\sqrt{2}) = \mathbb{Q}(\sqrt{2}) \).

Similarly \( \text{Im} \varepsilon_\beta = \mathbb{Q}(\beta) \). Filling this information into the diagram (\( * \)) above gives the claimed isomorphism between \( \mathbb{Q}(\sqrt{2}) \) and \( \mathbb{Q}(\beta) \):

\[
\begin{array}{ccc}
\mathbb{Q}[x]/\ker \varepsilon \mathbb{Q}(\sqrt{2}) & \cong & \mathbb{Q}[x]/(x^3 - 2) \\
\cong & \text{abstract field} & \cong \\
\mathbb{Q}(\sqrt{2}) & \text{concrete versions in } \mathbb{C} & \mathbb{Q}(\beta)
\end{array}
\]

(5.14). In algebraic number theory a field \( \mathbb{Q}[x]/(f) \), for \( f \) an irreducible polynomial over \( \mathbb{Q} \), is called a number field. If \( \{\beta_1, \ldots, \beta_n\} \) are the roots of \( f \), then we have \( n \) mutually isomorphic fields \( \mathbb{Q}(\beta_1), \ldots, \mathbb{Q}(\beta_n) \) inside \( \mathbb{C} \). The isomorphisms from \( \mathbb{Q}[x]/(f) \) to each of these are called the Galois monomorphisms of the number field.

(5.15). Returning to a general field:

**Kronecker’s Theorem.** Let \( f \) be a polynomial in \( F[x] \). Then there is an extension field of \( F \) containing a root of \( f \).

**Proof.** If \( f \) is not irreducible over \( F \), then factorise as \( f = gh \) with \( g \) irreducible over \( F \) and proceed as below but with \( g \) instead of \( f \). The result will be an extension field containing a root of \( g \), and hence of \( f \). Thus we may suppose that \( f \) is irreducible over \( F \) and \( f = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \) with the \( a_i \in F \). Replace \( F \) by its isomorphic copy in the quotient \( F[x]/(f) \), so that instead of \( a_i \), we write \( a_i + \langle f \rangle \), i.e.,

\[
f = (a_n + \langle f \rangle)x^n + (a_{n-1} + \langle f \rangle)x^{n-1} + \cdots + (a_1 + \langle f \rangle)x + (a_0 + \langle f \rangle).
\]

Consider the field \( E = F[x]/\langle f \rangle \) which is an extension of \( F \) and the element \( \mu = x + \langle f \rangle \in E \). If we substitute \( \mu \) into the polynomial then we perform all our arithmetic in \( E \), i.e: we perform the arithmetic of cosets, and the zero of this field is the coset \( \langle f \rangle \):

\[
f(\mu) = f(x + \langle f \rangle)
\]

\[
= (a_n + \langle f \rangle)(x + \langle f \rangle)^n + (a_{n-1} + \langle f \rangle)(x + \langle f \rangle)^{n-1} + \cdots + (a_1 + \langle f \rangle)(x + \langle f \rangle) + (a_0 + \langle f \rangle)
\]

\[
= (a_n x^n + \langle f \rangle) + (a_{n-1} x^{n-1} + \langle f \rangle) + \cdots + (a_1 x + \langle f \rangle) + (a_0 + \langle f \rangle)
\]

\[
= (a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0) + \langle f \rangle = f + \langle f \rangle = \langle f \rangle = 0.
\]

i.e. for \( \mu = x + \langle f \rangle \in E \) we have \( f(\mu) = 0 \). \( \square \)

**Corollary 5.2.** Let \( f \) be a polynomial in \( F[x] \). Then there is an extension field of \( F \) that contains all the roots of \( f \).

**Proof.** Repeat the process described in the proof of Kronecker’s Theorem at most \( \deg f \) number of times, until the desired field is obtained. \( \square \)

**Further Exercises for Section 5**

**Exercise 5.8.** Show that \( x^4 + x^3 + x^2 + x + 1 \) is irreducible over \( F_3 \). How many elements does the resulting extension of \( F_3 \) have?
Exercise 5.9. As linear polynomials are always irreducible, show that the field $F[x]/\langle ax + b \rangle$ is isomorphic to $F$.

Exercise 5.10.
1. Show that $1 + 2x + x^3 \in F_3[x]$ is irreducible and hence that $F = F_3[x]/\langle 1 + 2x + x^3 \rangle$ is a field.
2. Show that every coset can be written uniquely in the form $(a + bx + cx^2) + \langle 1 + 2x + x^3 \rangle$ with $a, b, c \in F_3$.
3. Deduce that the field $F$ has exactly 27 elements.

Exercise 5.11. Find an irreducible polynomial $f(x)$ in $F_5[x]$ of degree 2. Show that $F_5[x]/\langle f(x) \rangle$ is a field with 25 elements.

Exercise 5.12.
1. Show that the polynomial $x^3 - 3x + 6$ is irreducible over $\mathbb{Q}$.
2. Hence, or otherwise, if

$$
\alpha = \sqrt[3]{-2\sqrt{2} - 3}, \beta = -\sqrt[3]{2\sqrt{2} + 3} \quad \text{and} \quad \omega = \frac{1}{2} + \frac{\sqrt{3}}{2}i,
$$

prove that

(a) the fields $\mathbb{Q}(\alpha + \beta)$ and $\mathbb{Q}(\omega \alpha + \omega \beta)$ are distinct (that is, their elements are different), but,
(b) $\mathbb{Q}(\alpha + \beta)$ and $\mathbb{Q}(\omega \alpha + \omega \beta)$ are isomorphic (You can assume that $\omega \alpha + \omega \beta$ is not a real number.)

6. Ruler and Compass Constructions I

If you are a farmer in Babylon around 2500 BC, how do you subdivide your land into plots? You survey it of course. The most basic surveying instruments are wooden pegs and rope, with which you can do two very basic things: two pegs can be set a distance apart and the rope stretched taut between them; also, one of the pegs can be kept stationary and you can take the path traced by the other as you walk around keeping the rope stretched tight. In other words, you can draw a line through two points or you can draw a circle centered at one point and passing through another.

(6.1). Instead of the Euphrates river valley, we work in the complex plane $\mathbb{C}$. We are thus able, given two numbers $z, w \in \mathbb{C}$, to draw a line through them using a straight edge, or to place one end of a compass at $z$, and draw the circle passing through $w$:

Neither of these operations involves any “measuring”. There are no units on the ruler and we don’t know the radius of the circle.
(6.2). With these two constructions we call a complex number $z$ constructible iff there is a sequence of numbers

$$0, 1, i = z_1, z_2, \ldots, z_n = z,$$

with $z_j$ obtained from earlier numbers in the sequence in one of the following three ways:

\begin{itemize}
  \item[(i)] $z_j z_p z_q z_s$
  \item[(ii)] $z_j z_p z_q z_r$
  \item[(iii)] $z_j z_s z_p z_q$
\end{itemize}

In these pictures, $p, q, r$ and $s$ are all $< j$. We are given $0, 1, i$ “for free”, so they are indisputably constructible. The reasoning is this: if you stand in a plane (without coordinates) then your position can be taken as 0; declare a direction to be the real axis and a distance along it to be length 1; construct the perpendicular bisector of the segment from $-1$ to 1 (as in the next paragraph) and measure a unit distance along this new axis (in either direction) to get $i$.

(6.3). In addition to the two basic moves there are others that follow immediately from them. For example, we can construct the perpendicular bisector of a segment $AB$ as in Figure 6.1.

To explain these pictures (and the rest): a ray, centered at some point and tracing out a dotted circle is the compass. If the ray is marked $r$ – as in the first two pictures above – this means that in passing from the first picture to the second, the setting on the compass is kept the same. It does not mean that we know the setting.

The construction works for the following reason: let $S$ be the set of points in $\mathbb{C}$ that are an equal distance from both $A$ and $B$. After a moments thought, you see that this must be the perpendicular bisector of the line segment $AB$ that we are constructing. Lines are determined
by any two of their points, so if we can find two points equidistant from $A$ and $B$, and we draw a line through them, this must be the set $S$ that we want (and hence the perpendicular bisector). But the intersections of the two circular arcs are clearly equidistant from $A$ and $B$, so we are done.

(6.4). As well as bisecting segments, we can bisect angles, ie: if two lines meet in some angle we can construct a third line meeting these in angles that are each half the original one – see Figure 6.2. Remember: none of the angles in this picture can be measured. Nevertheless, the two new angles are half the old one.

(6.5). Given a line and a point $P$ not on it, we can construct a new line passing through $P$ and perpendicular to the line, as in Figure 6.3. This is called “dropping a perpendicular from a point to a line”.

(6.6). Given a line $\ell$ and a point $P$ not on it we can construct a new line through $P$ parallel to $\ell$ – see Figure 6.4. Some explanation for this one: the first step is to drop a perpendicular from $P$ to the line $\ell$, meeting it at the new point $Q$. Next, set your compass to the distance from $P$ to $Q$, and transfer this circular distance along the line to some point, drawing a semicircle that meets $\ell$ at the points $A$ and $B$. Construct the perpendicular bisector of the segment from $A$ to $B$, which meets the semicircle at the new point $R$. Finally, draw a line through the points $P$ and $R$.

(6.7). Figure 6.5 shows some basic examples of constructible numbers. It is less clear how to construct $\frac{\sqrt{129}}{2}$, or the golden ratio:

$$\phi = \frac{1 + \sqrt{5}}{2}.$$
But these numbers are constructible, and the reason is the first non-trivial fact about constructible numbers: they can be added, subtracted, multiplied and divided. Defining $\mathcal{C}$ to be the set of constructible numbers in $\mathbb{C}$, we have,

**Theorem C (Constructible Numbers).** $\mathcal{C}$ is a subfield of $\mathbb{C}$.

**Proof.** We show first that the real constructible numbers form a subfield of the reals, i.e. that $\mathcal{C} \cap \mathbb{R}$ is a subfield of $\mathbb{R}$, for which we need to show that if $a, b \in \mathcal{C} \cap \mathbb{R}$ then so too are $a + b$, $-a$, $ab$ and $1/a$.

1. $\mathcal{C} \cap \mathbb{R}$ is closed under $+$ and $-$: The picture on the left of Figure 6.6 shows that if $a \in \mathcal{C} \cap \mathbb{R}$ then so is $-a$. Similarly, the two on the right of Figure 6.6 give $a, b \in \mathcal{C} \cap \mathbb{R} \Rightarrow a + b \in \mathcal{C} \cap \mathbb{R}$. (In these pictures $a$ and $b$ are $> 0$. You can draw the other cases yourself).

2. $\mathcal{C} \cap \mathbb{R}$ is closed under $\times$: as can be seen by following through the steps in Figure 6.7. Seeing that the construction works involves studying the pair of similar triangles shown in red.

3. $\mathcal{C} \cap \mathbb{R}$ is closed under $\div$: is just the previous construction backwards – see Figure 6.8.

Now to the complex constructible numbers. Observe that $z \in \mathcal{C}$ precisely when $\text{Re} \, z$ and $\text{Im} \, z$ are in $\mathcal{C} \cap \mathbb{R}$. For, if $z \in \mathcal{C}$ then dropping perpendiculars to the real and imaginary axes give the numbers $\text{Re} \, z$ and $\text{Im} \, z \cdot i$, the second of which can be transferred to the real axis by drawing the circle centered at 0 passing through $\text{Im} \, z \cdot i$. On the other hand, if we have $\text{Re} \, z$ and $\text{Im} \, z$ on the real axis, then we have $\text{Im} \, z \cdot i$ too, and constructing a line through $\text{Re} \, z$ parallel to the imaginary axis and a line through $\text{Im} \, z \cdot i$ parallel to the real axis gives $z$.

---

2 In principle you can now throw away your calculator and instead use ruler and compass! To compute $\cos x$ of a constructible number $x$ for example, construct as many terms of the Taylor series,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

as you need (your calculator only ever gives you approximations anyway).
Suppose then that $z, w \in \mathbb{C}$ are constructible complex numbers: we show that $z + w, -z, zw$ and $1/z$ are also constructible. We have:

$$z + w = (\text{Re } z + \text{Re } w) + (\text{Im } z + \text{Im } w)i$$

$$-z = -\text{Re } z - \text{Im } z \cdot i$$

$$zw = (\text{Re } z \text{Re } w - \text{Im } z \text{Im } w) + (\text{Re } z \text{Im } w + \text{Im } w \text{Re } z)i$$

$$\frac{1}{z} = \frac{\text{Re } z}{\text{Re } z^2 + \text{Im } z^2} - \frac{\text{Im } z}{\text{Re } z^2 + \text{Im } z^2}i,$$

so that for example, $z, w \in \mathbb{C} \Rightarrow \text{Re } z, \text{Im } z, \text{Re } w, \text{Im } w \in \mathbb{C} \cap \mathbb{R} \Rightarrow \text{Re } z + \text{Re } w, \text{Im } z + \text{Im } w \in \mathbb{C} \cap \mathbb{R} \Rightarrow \text{Re } (z + w), \text{Im } (z + w) \in \mathbb{C} \cap \mathbb{R} \Rightarrow z + w \in \mathbb{C}$, and the others are similar.

**Corollary 6.1.** Any rational number is constructible.

**Proofs.** *Brute force:* use the example of the construction of 3 to show that $\mathbb{Z} \subset \mathbb{C}$; that $\mathbb{C} \cap \mathbb{R}$ is closed under $\times$ and $\div$ then gives $\mathbb{Q} \subset \mathbb{C}$.
Slightly slicker: by Exercise 0.5, any subfield of $\mathbb{C}$ contains $\mathbb{Q}$.

(6.8). Not only can we perform the four basic arithmetic operations with constructible numbers, but we can construct square roots too:

**Theorem 6.1.** If $z \in \mathbb{C}$ then $\sqrt{z} \in \mathbb{C}$.

**Proof.** We can construct the square root of any positive real number $a \in \mathbb{R}$ as in Figure 6.9. As an Exercise, show that in the red picture in Figure 6.9 the length $x = \sqrt{a}$. Next, the square root of any complex number can be constructed as in Figure 6.10, where we have used the construction of real square roots in the second step. 

6.1. **Constructing angles and polygons**

(6.1). We say that an angle can be constructed when we can construct two lines intersecting in that angle.

**Exercise 6.1.**

1. Show that we can always assume that one of the lines giving an angle is the positive real axis.
\[ \cos \frac{\pi}{17} = \frac{1}{8} \sqrt{2 \left( 2 \sqrt{\frac{17(17 - \sqrt{17})}{2}} - \sqrt{\frac{17 - \sqrt{17}}{2}} - 4 \sqrt{34 + 2 \sqrt{17}} + 3 \sqrt{17 + 17} + \sqrt{34 + 2 \sqrt{17} + \sqrt{17 + 15}} \right) } \]

**Fig. 6.11.** A proof that the 17-gon is constructible.

2. Show that an angle \( \theta \) can be constructed if and only if the number \( \cos \theta \) can be constructed. Do the same for \( \sin \theta \) and \( \tan \theta \).

**Exercise 6.2.** Show that if \( \varphi, \theta \) are constructible angles then so are \( \varphi + \theta \) and \( \varphi - \theta \).

**Exercise 6.3.** Show that a regular \( n \)-gon can be constructed centered at \( 0 \in \mathbb{C} \) if and only if the angle \( \frac{2\pi}{n} \) can be constructed. Show that a regular \( n \)-gon can be constructed centered at \( 0 \in \mathbb{C} \) if and only if the complex number

\[ z = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}, \]

can be constructed.

**Exercise 6.4.** Show that if an \( n \)-gon and an \( m \)-gon can be constructed for \( n \) and \( m \) relatively prime, then so can a \( mn \)-gon (hint: use the \( \mathbb{Z} \)-version of Theorem 1.1).

(6.3) For what \( n \) can you construct a regular \( n \)-gon? It makes sense to consider first the \( p \)-gons for \( p \) a prime. The complete answer even to this question will not be revealed until Section 14. It turns out that the \( p \)-gons that can be constructed are extremely rare. Nevertheless, the first two (odd) primes do work:

**Exercise 6.5.** Show that a regular 3-gon, ie: an equilateral triangle, can be constructed with any side length. Using Exercises 6.4 and 6.3 show that a regular 5-gon can also be constructed.

(6.4) Here is a proof that a regular 17-gon is constructible. Gauss proved the remarkable identity of Figure 6.11 which is still found in trigonometric tables. Thus the number \( \cos \pi/17 \) can be constructed as this expression involves only integers, the four field operations and square roots, all of which are operations we can perform with a ruler and compass. Hence, by Exercise 6.1(2) the angle \( \pi/17 \) can be constructed and so adding it to itself (Exercise 6.2) gives the angle \( 2\pi/17 \). Now apply Exercise 6.3 to get the 17-gon.

**Further Exercises for Section 6**

**Exercise 6.6.** Using the fact that the constructible numbers include \( \mathbb{Q} \), show that any given line segment can be trisected in length.

**Exercise 6.7.** Show that if you can construct a regular \( n \)-sided polygon, then you can also construct a regular \( 2^k n \)-sided polygon for any \( k \geq 1 \).

**Exercise 6.8.** Show that \( \cos \theta \) is constructible if and only if \( \sin \theta \) is.

**Exercise 6.9.** If \( a, b \) and \( c \) are constructible numbers (ie: in \( \mathbb{C} \)), show that the roots of the quadratic equation \( ax^2 + bx + c \) are also constructible.
7. Vector Spaces I: Dimensions

Having met rings and fields we introduce our third algebraic object: vector spaces.

**Definition 7.1 (vector space).** A vector space over a field $F$ is a set $V$, whose elements are called vectors, together with two operations: addition $u, v \mapsto u + v$ of vectors and scalar multiplication $\lambda, v \mapsto \lambda v$ of a vector by an element (or scalar) $\lambda$ of the field $F$, such that:

1. $(u + v) + w = u + (v + w)$, for all $u, v, w \in V$.
2. There exists a zero vector $0 \in V$ such that $v + 0 = 0 + v$ for all $v \in V$.
3. Every $v \in V$ has a negative $-v$ such that $v + (-v) = 0 = -v + v$, for all $v \in V$.
4. $u + v = v + u$, for all $u, v \in V$.
5. $\lambda(u + v) = \lambda u + \lambda v$, for all $u, v$ and $\lambda \in F$.
6. $(\lambda + \mu)v = \lambda v + \mu v$, for all $\lambda, \mu \in F$ and $v \in V$.
7. $\lambda(\mu v) = (\lambda \mu)v$, for all $\lambda, \mu \in F$ and $v \in V$.
8. $1v = v$ for all $v \in V$.

**Aside.** Alternatively, $V$ forms an Abelian group under $+$ (these are the first four axioms) together with a scalar multiplication that satisfies the last four axioms.

(7.1). A homomorphism of vector spaces is a map $\varphi : V_1 \rightarrow V_2$ such that $\varphi(u + v) = \varphi(u) + \varphi(v)$ and $\varphi(\lambda v) = \lambda \varphi(v)$ for all $u, v \in V$ and $\lambda \in F$. (Homomorphisms of vector are more commonly called linear maps.) A bijective homomorphism is an isomorphism.

(7.2). The set $\mathbb{R}^2$ of $2 \times 1$ column vectors is the motivating example of a vector space over $\mathbb{R}$ under the normal addition and scalar multiplication of vectors. Alternatively, the complex numbers $\mathbb{C}$ form a vector space over $\mathbb{R}$, and these two spaces are isomorphic via the map:

$$\varphi : \begin{bmatrix} a \\ b \end{bmatrix} \mapsto a + bi.$$ 

(7.3). The complex numbers are a vector space over themselves: addition of complex numbers gives an Abelian group and now we can scalar multiply a complex number by another one, using the usual multiplication of complex numbers.

(7.4). A vector spaces over a finite field: consider the set of 3-tuples with coordinates from the field $\mathbb{F}_2$ (so are either 0 or 1) and add two such coordinate-wise, using the addition from $\mathbb{F}_2$. Scalar multiply a tuple coordinate-wise using the multiplication from $\mathbb{F}_2$. As there are only two possibilities for each coordinate and three coordinates in total, we get a total of $2^3 = 8$ vectors in this space. They can be arranged around the vertices of a cube as shown, where $abc$ is the vector with the three coordinates $a, b, c \in \mathbb{F}_2$.

(7.5). We saw in Section 3 that the field $\mathbb{Q}(\sqrt{2})$ has elements the $a + b\sqrt{2}$ with $a, b \in \mathbb{Q}$. The identification,

$$a + b\sqrt{2} \leftrightarrow \begin{bmatrix} a \\ b \end{bmatrix}$$

coordinate in “1 direction”

coordinate in “$\sqrt{2}$ direction”

is an isomorphism with the vector space $\mathbb{Q}^2$ of $2 \times 1$ $\mathbb{Q}$-column vectors with the addition $(a + b\sqrt{2}) + (c + d\sqrt{2}) = (a + c) + (b + d)\sqrt{2}$ corresponding to,

$$\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a + c \\ b + d \end{bmatrix}.$$
and scalar multiplication \( c(a + b\sqrt{2}) = ac + bc\sqrt{2} \) corresponding to:

\[
\begin{bmatrix}
  a \\
  b
\end{bmatrix}
\begin{bmatrix}
  c
\end{bmatrix}
= 
\begin{bmatrix}
  ac \\
  bc
\end{bmatrix}.
\]

(7.6). The polynomial \( x^3 - 2 \) is irreducible over \( \mathbb{Q} \) so the quotient ring \( \mathbb{Q}[x]/\langle x^3 - 2 \rangle \) is a field with elements the \((a + bx + cx^2) + \langle x^3 - 2 \rangle \) for \( a, b \in \mathbb{Q} \). It is a \( \mathbb{Q} \)-vector space, isomorphic to \( \mathbb{Q}^3 \) via

\[
(a + bx + cx^2) + \langle x^3 - 2 \rangle \leftrightarrow \begin{bmatrix}
  a \\
  b \\
  c
\end{bmatrix}
\]

coordinate in “1 + \langle x^3 - 2 \rangle” direction”

coordinate in “\( x + \langle x^3 - 2 \rangle \)” direction”

coordinate in “\( x^2 + \langle x^3 - 2 \rangle \)” direction”

(Check for yourself that the addition and scalar multiplications match up).

(7.7). The previous two examples are special cases of the following: if \( F \subseteq E \) is an extension of fields then \( E \) is a vector space over \( F \). The “vectors” are the elements of \( E \) and the “scalars” are the elements of \( F \). Addition of vectors is just the addition of elements in \( E \), and to scalar multiply a \( v \in E \) by a \( \lambda \in F \), multiply \( \lambda v \) using the multiplication of the field \( E \). The first four axioms for a vector space hold because of the addition of the field \( E \), and the second four from the multiplication.

**Definition 7.2 (span and independence).** If \( v_1, \ldots, v_n \in V \) are vectors in a vector space \( V \), then a vector of the form

\[
\alpha_1 v_1 + \ldots + \alpha_n v_n,
\]

for \( \alpha_1, \ldots, \alpha_n \in F \), is called a linear combination of the \( v_1, \ldots, v_n \). The linear span of \( \{v_j : j \in J\} \), where \( J \) is not necessarily finite, is the set of all linear combinations of vectors from the set:

\[
\text{span}\{v_j : j \in J\} = \{\alpha_1 v_{j_1} + \ldots + \alpha_k v_{j_k} : \alpha_j \in F\}.
\]

Say \( \{v_j : j \in J\} \) span \( V \) when \( V = \text{span}\{v_j : j \in J\} \).

A set of vectors \( v_1, \ldots, v_n \in V \) is linearly dependent if and only if there exist scalars \( \alpha_1, \ldots, \alpha_n \), not all zero, such that

\[
\alpha_1 v_1 + \ldots + \alpha_n v_n = 0,
\]

and linearly independent otherwise, ie: \( \alpha_1 v_1 + \ldots + \alpha_n v_n = 0 \) implies that the \( \alpha_i \) are all 0.

(7.8). In the examples above, the complex numbers \( \mathbb{C} \) are spanned, as a vector space over \( \mathbb{R} \), by \{1, i\}, and indeed by any two non-zero complex numbers that are not scalar multiples of each other. As a vector space over \( \mathbb{C} \), the complex numbers are spanned by one element: any \( \zeta \in \mathbb{C} \) can be written as \( \zeta \times 1 \) for example, so every element is a complex scalar multiple of 1. Indeed, \( \mathbb{C} \) is spanned as a complex vector space by any single one of its non-zero elements.

**Definition 7.3 (basis).** A basis for \( V \) is a set of vectors \( \{v_j : j \in J\} \), with \( J \) a not necessarily finite index set, that span \( V \), and such that every finite set of \( v_j \)'s are linearly independent.

It can be proved that there is a 1-1 correspondence between the elements of any two bases for a vector space \( V \). When \( V \) has a finite basis the dimension of \( V \) is defined to be the number of elements in a basis; otherwise \( V \) is infinite dimensional.
(7.9). Thus $\mathbb{C}$ is 2-dimensional as a vector space over $\mathbb{R}$ but 1-dimensional as a vector space over $\mathbb{C}$. We will see later in this section that $\mathbb{C}$ is infinite dimensional as a vector space over $\mathbb{Q}$.

With the other examples above, $\mathbb{Q}(\sqrt{2})$ is 2-dimensional over $\mathbb{Q}$ with basis $\{1, \sqrt{2}\}$ and $\mathbb{Q}[x]/(x^3 - 2)$ is 3-dimensional over $\mathbb{Q}$ with basis the cosets

$$1 + \langle x^3 - 2 \rangle, x + \langle x^3 - 2 \rangle \text{ and } x^2 + \langle x^3 - 2 \rangle.$$ 

In Exercise [13.1] in Section [13] we will see that if $\alpha = \sqrt[4]{2}$, then $\mathbb{Q}(\alpha)$ is a 2-dimensional space over $\mathbb{Q}(\alpha)$ or $\mathbb{Q}(\alpha i)$ or even $\mathbb{Q}((1 + i)\alpha)$; a 4-dimensional space over $\mathbb{Q}(i)$ or $\mathbb{Q}(i\alpha^2)$, and an 8-dimensional space over $\mathbb{Q}$ (and these are almost, but not quite, all the possibilities; see the exercise for the full story).

**Definition 7.4 (degree of an extension).** Let $F \subseteq E$ be an extension of fields. Consider $E$ as a vector space over $F$, and define the degree of the extension to be the dimension of this vector space, denoted $[E : F]$. Call $F \subseteq E$ a finite extension if the degree is finite.

(7.10). The extensions $\mathbb{Q} \subset \mathbb{Q}(\sqrt{2})$ and $\mathbb{Q} \subset \mathbb{Q}[x]/(x^3 - 2)$ have degrees 2 and 3.

(7.11). It is no coincidence that the degree of extensions of the form $F \subseteq F[x]/(f)$ turn out to be the same as the degree of the polynomial $f$:

**Theorem 7.1.** Let $f$ be an irreducible polynomial in $F[x]$ of degree $d$. Then the extension, 

$$F \subseteq F[x]/(f),$$

has degree $d$.

Hence the name degree!

**Proof.** Replace, as usual, the field $F$ by its copy in $F[x]/(f)$, so that $\lambda \in F$ becomes $\lambda + \langle f \rangle \in F[x]/(f)$. Consider the set of cosets,

$$B = \{1 + \langle f \rangle, x + \langle f \rangle, x^2 + \langle f \rangle, \ldots, x^{d-1} + \langle f \rangle\}.$$ 

Then we claim that $B$ is a basis for $F[x]/(f)$ over $F$, for which we have to show that it spans the vector space and is linearly independent. To see that it spans, consider a typical element, which has the form,

$$g + \langle f \rangle = (gf + r)/\langle f \rangle = r + \langle f \rangle = (a_0 + a_1 x + \cdots + a_{d-1} x^{d-1}) + \langle f \rangle.$$ 

Using the division algorithm and basic properties of cosets. This is turn gives,

$$(a_0 + a_1 x + \cdots + a_{d-1} x^{d-1}) + \langle f \rangle = (a_0 + \langle f \rangle)(1 + \langle f \rangle) + (a_1 + \langle f \rangle)(x + \langle f \rangle) + \cdots + (a_{d-1} + \langle f \rangle)(x^{d-1} + \langle f \rangle),$$

where the last is an $F$-linear combination of the elements of $B$. Thus this sets spans the space.

For linear independence, suppose we have an $F$-linear combination of the elements of $B$ giving zero, ie:

$$(b_0 + \langle f \rangle)(1 + \langle f \rangle) + (b_1 + \langle f \rangle)(x + \langle f \rangle) + \cdots + (b_{d-1} + \langle f \rangle)(x^{d-1} + \langle f \rangle) = \langle f \rangle,$$

remembering that the zero of the field $F[x]/(f)$ is the coset $0 + \langle f \rangle = \langle f \rangle$. Multiplying and adding all the cosets on the left hand side gives,

$$(b_0 + b_1 x + \cdots + b_{d-1} x^{d-1}) + \langle f \rangle = \langle f \rangle,$$

so that $b_0 + b_1 x + \cdots + b_{d-1} x^{d-1} \in \langle f \rangle$ (using another basic property of cosets). The elements of $\langle f \rangle$, being multiples of $f$, must have degree at least $d$, except for the zero polynomial. On the other hand $b_0 + b_1 x + \cdots + b_{d-1} x^{d-1}$ has degree $\leq d - 1$. Thus it must be the zero polynomial, giving that all the $b_i$ are zero, hence all the $b_i + \langle f \rangle$ are 0, and that the set $B$ is linearly independent over $F$ as claimed. \[\square\]
(7.12). What is the degree of the extension \( \mathbb{Q} \subset \mathbb{Q}(\pi) \)? If it was finite, say \([\mathbb{Q}(\pi) : \mathbb{Q}] = d\), then any collection of more than \( d \) elements would be linearly dependent. In particular, the \( d + 1 \) elements,

\[
1, \pi, \pi^2, \ldots, \pi^d,
\]

would be dependent, so that \( a_0 + a_1 \pi + a_2 \pi^2 + \ldots + a_d \pi^d = 0 \) for some \( a_0, a_1, \ldots, a_d \in \mathbb{Q} \), not all zero, hence \( \pi \) would be a root of the polynomial \( a_0 + a_1 x + a_2 x^2 + \ldots + a_d x^d \). But this contradicts the fact that \( \pi \) is transcendental over \( \mathbb{Q} \). Thus, the degree of the extension is infinite.

(7.13). In fact this is always true:

**Proposition 7.1.** Let \( F \subseteq E \) and \( \alpha \in E \). If the degree of the extension \( F \subseteq F(\alpha) \) is finite, then \( \alpha \) is algebraic over \( F \).

**Proof.** The proof is very similar to the example above. Suppose that the extension \( F \subseteq F(\alpha) \) has degree \( n \), so that any collection of \( n + 1 \) elements of \( F(\alpha) \) must be linearly dependent. In particular the \( n + 1 \) elements

\[
1, \alpha, \alpha^2, \ldots, \alpha^n
\]

are dependent over \( F \), so that there are \( a_0, a_1, \ldots, a_n \in F \) with

\[
a_0 + a_1 \alpha + \cdots + a_n \alpha^n = 0,
\]

and hence \( \alpha \) is algebraic over \( F \) as claimed. \( \square \)

Thus, any field \( E \) that contains transcendentals over \( F \) will be infinite dimensional as vector spaces over \( F \). In particular, \( \mathbb{R} \) and \( \mathbb{C} \) are infinite dimensional over \( \mathbb{Q} \).

(7.14). The converse to Proposition 7.1 is partly true, as we summarise now in an important result:

**Theorem D (Simple Extensions).** Let \( F \subseteq E \) and \( \alpha \in E \) be algebraic over \( F \). Then,

1. There is a unique polynomial \( f \in F[x] \) that is monic, irreducible over \( F \), and has \( \alpha \) as a root.
2. The field \( F(\alpha) \) is isomorphic to the quotient \( F[x]/(f) \).
3. If \( \deg f = d \), then the extension \( F \subseteq F(\alpha) \) has degree \( d \) with basis \( \{1, \alpha, \alpha^2, \ldots, \alpha^{d-1}\} \), and so,

\[
F(\alpha) = \{a_0 + a_1 \alpha + a_2 \alpha^2 + \cdots + a_{d-1} \alpha^{d-1} \mid a_0, \ldots, a_{d-1} \in F\}.
\]

**Proof.** Hopefully most of the proof will be recognisable from the specific examples we have discussed already. As \( \alpha \) is algebraic over \( F \) there is at least one \( F \)-polynomial having \( \alpha \) as a root. Choose \( f' \) to be a non-zero one having smallest degree. This polynomial must then be irreducible over \( F \), for if not, we have \( f' = gh \) with \( \deg(g), \deg(h) < \deg(f') \), and \( \alpha \) must be a root of one of \( g \) or \( h \), contradicting the original choice of \( f' \). Divide through by the leading coefficient of \( f' \), to get \( f \), a monic, irreducible (by Exercise 2.2) \( F \)-polynomial, having \( \alpha \) as a root. If \( f_1, f_2 \) are polynomials with these properties then \( f_1 - f_2 \) has degree strictly less than either \( f_1 \) or \( f_2 \) and still has \( \alpha \) as a root, so the only possibility is that \( f_1 - f_2 \) is zero, hence \( f \) is unique.

Consider the evaluation homomorphism \( e_\alpha : F[x] \rightarrow E \) defined as usual by \( e_\alpha(g) = g(\alpha) \). To show that the kernel of this homomorphism is the ideal \( \langle f \rangle \) is completely analogous to the example at the beginning of Section 4. Clearly \( \langle f \rangle \) is contained in the kernel, as any multiple of \( f \) must evaluate to zero when \( \alpha \) is substituted into it. On the other hand, if \( h \) is in the kernel of \( e_\alpha \), then by division algorithm,

\[
h = qf + r,
\]
with \(\deg(r) < \deg(f)\). Taking the \(\varepsilon_\alpha\) image of both sides gives

\[
0 = \varepsilon_\alpha(h) = \varepsilon_\alpha(qf) + \varepsilon_\alpha(r) = \varepsilon_\alpha(r),
\]

so that \(r\) has \(\alpha\) as a root. As \(f\) is minimal with this property, we must have that \(r = 0\), so that \(h = qf\), ie: \(h\) is in the ideal \((f)\), and so the kernel is contained in this ideal. Thus, \(\ker \varepsilon_\alpha = (f)\).

In particular we have an isomorphism \(\overline{\varepsilon_\alpha} : F[x]/(f) \to \text{Im} \varepsilon_\alpha \subset E\), given by,

\[
\overline{\varepsilon_\alpha}(g + (f)) = \varepsilon_\alpha(g) = g(\alpha),
\]

with \(F[x]/(f)\) a field as \(f\) is irreducible over \(F\). Thus, \(\text{Im} \varepsilon_\alpha\) is a subfield of \(E\). Clearly, both the element \(\alpha\) (\(\varepsilon_\alpha(x) = \alpha\)) and the field \(F\) (\(\varepsilon_\alpha(c) = c\)) are contained in \(\text{Im} \varepsilon_\alpha\), hence \(F(\alpha)\) is too as \(\text{Im} \varepsilon_\alpha\) is subfield of \(E\), and \(F(\alpha)\) is the smallest one enjoying these two properties. Conversely, if \(g = \sum a_i x^i \in F[x]\) then \(\varepsilon_\alpha(g) = \sum a_i \alpha^i\), which is an element of \(F(\alpha)\) as fields are closed under sums and products. Hence \(\text{Im} \varepsilon_\alpha \subseteq F(\alpha)\) and so these two are the same. Thus \(\overline{\varepsilon_\alpha}\) is an isomorphism between \(F[x]/(f)\) and \(F(\alpha)\).

The final part follows immediately from Theorem [7.1] where we showed that the set of cosets

\[
\{1 + (f), x + (f), x^2 + (f), \ldots, x^{d-1} + (f)\},
\]

formed a basis for \(F[x]/(f)\) over \(F\). Their images under \(\overline{\varepsilon_\alpha}\), namely \([1, \alpha, \alpha^2, \ldots, \alpha^{d-1}]\), must then form a basis for \(F(\alpha)\) over \(F\).

The proof of Theorem D shows that the polynomial \(f\) has the smallest degree of any polynomial having \(\alpha\) as a root.

**Definition 7.5 (minimum polynomial).** The polynomial \(f\) of Theorem D is called the minimum polynomial of \(\alpha\) over \(F\).

(7.15). An important property of the minimum polynomial is that it divides any other \(F\)-polynomial that has \(\alpha\) as a root: for suppose that \(g\) is such an \(F\)-polynomial. By unique factorisation in \(F[x]\), we can decompose \(g\) as

\[
g = \lambda f_1 f_2 \cdots f_k,
\]

where the \(f_i\) are monic and irreducible over \(F\). Being a root of \(g\), the element \(\alpha\) must be a root of one of the \(f_i\). By uniqueness, this \(f_i\) must be the minimum polynomial of \(\alpha\) over \(F\).

(7.16). The last part of Theorem D tells us that to find the degree of a simple extension \(F \subseteq F(\alpha)\), you find the degree of the minimum polynomial over \(F\) of \(\alpha\).

How do you find this polynomial? Its simple: guess! A sensible first guess is a monic polynomial with \(F\)-coefficients that has \(\alpha\) as root. If your guess is also irreducible, then you have guessed right (uniqueness).

The only thing that can go wrong is if your guess is not irreducible. Your next guess should then be a factor of your first guess. In this way, the search for minimum polynomials is “no harder” than determining irreducibility.

(7.17). As an example consider the minimum polynomial over \(\mathbb{Q}\) of the \(p\)-th root of 1,

\[
\cos \frac{2\pi}{p} + i \sin \frac{2\pi}{p},
\]

for \(p\) a prime. Your first guess is \(x^p - 1\) which satisfies all the criteria bar irreducibility as \(x - 1\) is a factor. Factorising gives:

\[
x^p - 1 = (x - 1)\Phi_p(x),
\]

for \(\Phi_p\) the \(p\)-th cyclotomic polynomial, and this was shown to be irreducible over \(\mathbb{Q}\) in Exercise 2.14.
(7.18). How does one find the degree of extensions \( F \subseteq F(a_1, \ldots, a_k) \) that are not necessarily simple? Such extensions are a sequence of simple extensions. If we can find the degrees of each of these simple extensions, all we need is a way to patch the answers together:

**The Tower Law.** Let \( F \subseteq E \subseteq L \) be a sequence or “tower” of extensions. If both of the intermediate extensions \( F \subseteq E \) and \( E \subseteq L \) are of finite degree, then \( F \subseteq L \) is too, with

\[ [L : F] = [L : E][E : F]. \]

(7.19). Before the proof we consider the example \( \mathbb{Q} \subset \mathbb{Q}(\sqrt{2}, i) \), which a sequence of two simple extensions:

\( \mathbb{Q} \subset \mathbb{Q}(\sqrt{2}) \subset \mathbb{Q}(\sqrt{2}, i) \).

We can use Theorem D to find the degrees of each individual simple extension. Firstly, the minimum polynomial over \( \mathbb{Q} \) of \( \sqrt{2} \) is \( x^2 - 2 \), for this polynomial is monic in \( \mathbb{Q}[x] \) with \( \sqrt{2} \) as a root and irreducible over \( \mathbb{Q} \) by Eisenstein (using \( p = 2 \)). Thus the extension \( \mathbb{Q} \subset \mathbb{Q}(\sqrt{2}) \) has degree \( \text{deg}(x^2 - 2) = 2 \) and \( \{1, \sqrt{2}, (\sqrt{2})^2\} \) is a basis for \( \mathbb{Q}(\sqrt{2}) \) over \( \mathbb{Q} \).

Now let \( \mathbb{F} = \mathbb{Q}(\sqrt{2}) \) so that the second extension is \( \mathbb{F} \subset \mathbb{F}(i) \) and where the minimum polynomial of \( i \) over \( \mathbb{F} \) is \( x^2 + 1 \); it is monic in \( \mathbb{F}[x] \) with \( i \) as a root, and irreducible over \( \mathbb{F} \) as its two roots \( \pm i \) are not in \( \mathbb{F} \) (as \( \mathbb{F} \subset \mathbb{R} \)). Thus Theorem D again gives that \( \mathbb{F} \subset \mathbb{F}(i) \) has degree 2 with \( \{1, i\} \) a basis for \( \mathbb{F}(i) \) over \( \mathbb{F} \).

Now consider the elements,

\[ \{1, \sqrt{2}, (\sqrt{2})^2, i, \sqrt{2}i, (\sqrt{2})^2i\}, \]

obtained by multiplying the two bases together. The claim is that they form a basis for \( \mathbb{Q}(\sqrt{2}, i) = \mathbb{F}(i) \) over \( \mathbb{Q} \); we need to show that the \( \mathbb{Q} \)-span of these six gives every element of \( \mathbb{Q}(\sqrt{2}, i) \) and that they are linearly independent over \( \mathbb{Q} \). For the first, let \( x \) be an arbitrary element of \( \mathbb{Q}(\sqrt{2}, i) = \mathbb{F}(i) \). As \( \{1, i\} \) is a basis for \( \mathbb{F}(i) \) over \( \mathbb{F} \), we can express \( x \) as an \( \mathbb{F} \)-linear combination,

\[ x = a + bi, a, b \in \mathbb{F}. \]

As \( \{1, \sqrt{2}, (\sqrt{2})^2\} \) is a basis for \( \mathbb{F} \) over \( \mathbb{Q} \), both \( a \) and \( b \) can be expressed as \( \mathbb{Q} \)-linear combinations,

\[ a = a_0 + a_1 \sqrt{2} + a_2 (\sqrt{2})^2, b = b_0 + b_1 \sqrt{2} + b_2 (\sqrt{2})^2, \]

with the \( a_i, b_i \in \mathbb{Q} \). This gives,

\[ x = a_0 + a_1 \sqrt{2} + a_2 (\sqrt{2})^2 + b_0 i + b_1 \sqrt{2}i + b_2 (\sqrt{2})^2i, \]

a \( \mathbb{Q} \)-linear combination for \( x \) as required.

Suppose now:

\[ a_0 + a_1 \sqrt{2} + a_2 (\sqrt{2})^2 + b_0 i + b_1 \sqrt{2}i + b_2 (\sqrt{2})^2i = 0, \]

with the \( a_i, b_i \in \mathbb{Q} \). Gathering together real and imaginary parts:

\[ (a_0 + a_1 \sqrt{2} + a_2 (\sqrt{2})^2) + (b_0 + b_1 \sqrt{2} + b_2 (\sqrt{2})^2)i = a + bi = 0, \]

for \( a \) and \( b \) now elements of \( \mathbb{F} \). As \( \{1, i\} \) are independent over \( \mathbb{F} \) the coefficients in this last expression are zero, ie: \( a = b = 0 \). This gives:

\[ a_0 + a_1 \sqrt{2} + a_2 (\sqrt{2})^2 = 0 = b_0 + b_1 \sqrt{2} + b_2 (\sqrt{2})^2, \]

and as \( \{1, \sqrt{2}, (\sqrt{2})^2\} \) are independent over \( \mathbb{Q} \) the coefficients in these two expressions are also zero, ie: \( a_0 = a_1 = a_2 = b_0 = b_1 = b_2 = 0 \). The six elements are thus independent and form a basis as claimed.
(7.20). The proof of the tower law is completely analogous to the example above:

**Proof of the Tower Law.** Let \( \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \) be a basis for \( E \) as an \( F \)-vector space and \( \{\beta_1, \beta_2, \ldots, \beta_m\} \) a basis for \( L \) as an \( E \)-vector space, both containing a finite number of elements as these extensions are finite by assumption. We show that the \( mn = [L : E][E : F] \) elements

\[
\{\alpha_i \beta_j\}, 1 \leq i \leq n, 1 \leq j \leq m,
\]

form a basis for the \( F \)-vector space \( L \), thus giving the result. Working “backwards” as in the example above, if \( x \) is an element of \( L \) we can express it as an \( E \)-linear combination of the \( \{\beta_1, \ldots, \beta_m\} \):

\[
x = \sum_{i=1}^{m} a_i \beta_i,
\]

where, as they are elements of \( E \), each of the \( a_i \) can be expressed as \( F \)-linear combinations of the \( \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \):

\[
a_i = \sum_{j=1}^{n} b_{ij} \alpha_j \Rightarrow x = \sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij} \alpha_j \beta_i.
\]

Thus the elements \( \{\alpha_i \beta_j\} \) span the field \( L \). If we have

\[
\sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij} \alpha_j \beta_i = 0,
\]

with the \( b_{ij} \in F \), we can collect together all the \( \beta_1 \) terms, all the \( \beta_2 \) terms, and so on (much as we took real and imaginary parts in the example), to obtain an \( E \)-linear combination

\[
(\sum_{j=1}^{n} b_{1j} \alpha_j) \beta_1 + (\sum_{j=1}^{n} b_{2j} \alpha_j) \beta_2 + \cdots + (\sum_{j=1}^{n} b_{mj} \alpha_j) \beta_m = 0.
\]

The independence of the \( \beta_i \) over \( E \) forces all the coefficients to be zero:

\[
(\sum_{j=1}^{n} b_{1j} \alpha_j) = \cdots = (\sum_{j=1}^{n} b_{mj} \alpha_j) = 0,
\]

and the independence of the \( \alpha_j \) over \( F \) forces all the coefficients in each of these to be zero too, i.e: \( b_{ij} = 0 \) for all \( i, j \). The \( \{\alpha_i \beta_j\} \) are thus independent. \( \square \)

(7.21). We find the minimum polynomial over \( \mathbb{Q} \) of \( \alpha + \omega \), where \( \alpha = \sqrt{2} \) and \( \omega = \frac{1}{2} + \frac{\sqrt{5}}{2}i \). Following the recipe in the proof of Theorem 8.2 (or just brute force) gives \( \mathbb{Q}(\alpha, \omega) = \mathbb{Q}(\alpha + \omega) \) with \( [\mathbb{Q}(\alpha + \omega) : \mathbb{Q}] = [\mathbb{Q}(\alpha, \omega) : \mathbb{Q}] = 6 \) by the Tower law. So we are after a degree 6 polynomial. Indeed, it suffices to find a monic degree 6 polynomial \( g \) over \( \mathbb{Q} \) having \( \alpha + \omega \) as a root, since the minimum polynomial must then divide \( g \), hence be \( g \).

Writing \( \beta = \alpha + \omega \) we thus require \( a, b, c, d, e, f \in \mathbb{Q} \) such that

\[
\beta^6 + a\beta^5 + b\beta^4 + c\beta^3 + d\beta^2 + e\beta + f = 0
\]

(7.1)

Now compute the powers of \( \beta \) and write the answers in terms on the basis \( \{1, \alpha, \alpha^2, \omega, \alpha\omega, \alpha^2\omega\} \) for \( \mathbb{Q}(\alpha, \omega) \) over \( \mathbb{Q} \) given by the tower law. For example,

\[
\beta^3 = \alpha^3 + 3\alpha^2\omega + 3\alpha\omega^2 + \omega^3 = 3\alpha^2\omega - 3\alpha\omega - 3\alpha^2 + 3,
\]

and the others are similar using the facts \( \alpha^3 = 2, \omega^3 = 1 \) and \( \omega^2 = -\omega - 1 \). Substituting the results into (7.1) and collecting terms gives a linear combination of the basis vectors
equal to 0. Independence means the coefficients must be zero, so we get a linear system of equations in the variables $a, \ldots, f$. Solving these gives $a = 3, b = 6, c = 3, d = 0, e = f = 9$ and hence the minimum polynomial
\[ x^6 + 3x^5 + 6x^4 + 3x^3 + 9x + 9. \]

Further Exercises for Section 7

Exercise 7.1.
1. Show that if $F \subseteq L$ are fields with $[L : F] = 1$ then $L = F$.
2. Let $F \subseteq L \subseteq E$ be fields with $[E : F] = [L : F]$. Show that $E = L$.

Exercise 7.2. Let $F = \mathbb{Q}(a)$, where $a^3 = 2$. Express $(1 + a)^{-1}$ and $(a^4 + 1)(a^2 + 1)^{-1}$ in the form $ba^2 + ca + d$, where $b, d, c$ are in $\mathbb{Q}$.

Exercise 7.3. Let $\alpha = \sqrt[3]{5}$. Express the following elements of $\mathbb{Q}(\alpha)$ as polynomials of degree at most 2 in $\alpha$ (with coefficients in $\mathbb{Q}$):

1. $1/\alpha$
2. $\alpha^5 - \alpha^6$
3. $\alpha/(\alpha^2 + 1)$

Exercise 7.4. Find the minimum polynomial over $\mathbb{Q}$ of $\alpha = \sqrt{2} + \sqrt{-2}$. Show that the following are elements of the field $\mathbb{Q}(\alpha)$ and express them as polynomials in $\alpha$ (with coefficients in $\mathbb{Q}$) of degree at most 3:

1. $\sqrt{2}$
2. $-\sqrt{2}$
3. $i$
4. $\alpha^5 + 4\alpha + 3$
5. $1/\alpha$
6. $(2\alpha + 3)/(\alpha^2 + 2\alpha + 2)$

Exercise 7.5. Find the minimum polynomials over $\mathbb{Q}$ of the following numbers:

1. $1 + i$
2. $\sqrt{7}$
3. $\sqrt[3]{5}$
4. $\sqrt{2} + i$
5. $\sqrt{2} + \sqrt[3]{3}$

Exercise 7.6. Find the minimum polynomial over $\mathbb{Q}$ of the following:

1. $\sqrt{7}$
2. $(\sqrt{11} + 3)/2$
3. $(i\sqrt{3} - 1)/2$

Exercise 7.7. For each of the following fields $L$ and $F$, find $[L : F]$ and compute a basis for $L$ over $F$.

1. $L = \mathbb{Q}(\sqrt{2}, \sqrt{3}), F = \mathbb{Q}$;
2. $L = \mathbb{Q}(\sqrt{3}, i), F = \mathbb{Q}(i)$;
3. $L = \mathbb{Q}(\xi), F = \mathbb{Q}$, where $\xi$ is a primitive complex 7th root of unity;
4. $L = \mathbb{Q}(i, \sqrt[3]{3}, \omega), F = \mathbb{Q}$, where $\omega$ is a primitive complex cube root of unity.

Exercise 7.8. Let $a = e^{\pi i/4}$. Find $[F(a) : F]$ when $F = \mathbb{R}$ and when $F = \mathbb{Q}$.

8. Fields III: Splitting Fields and Finite Fields

8.1. Splitting Fields

(8.1). In Section 0 we encountered fields containing “just enough” numbers to solve some polynomial equation. We now make this more precise.

Let $f$ be a polynomial with $F$-coefficients. We say that $f$ splits in an extension $F \subseteq E$ when we can factorise
\[ f = \prod_{i=1}^{\deg f} (x - a_i), \]
in the polynomial ring \( E[x] \). Thus \( f \) splits in \( E \) precisely when \( E \) contains all the roots \( \{\alpha_1, \alpha_2, \ldots, \alpha_{\deg f}\} \) of \( f \).

There will in general be many such extension fields – we are after the smallest one. By Kronecker’s theorem (more accurately, Corollary [5.2]) there is an extension \( F \subseteq K \) such that \( K \) contains all the roots of \( f \). If these roots are \( \alpha_1, \alpha_2, \ldots, \alpha_d \in K \), then let \( E = F(\alpha_1, \alpha_2, \ldots, \alpha_d) \).

**Definition 8.1 (splitting field of a polynomial).** The field extension \( F \subseteq E \) constructed in this way is called a splitting field of \( f \) over \( F \).

**Exercise 8.1.** Show that \( E \) is a splitting field of the polynomial \( f \) over \( F \) if and only if \( f \) splits in \( E \) but not in any subfield of \( E \) containing \( F \) (so in this sense, \( E \) is the smallest field containing \( F \) and all the roots).

(8.2). The splitting field of \( x^2 + 1 \) over \( \mathbb{Q} \) is \( \mathbb{Q}(i) \). The splitting field of \( x^2 + 1 \) over \( \mathbb{R} \) is \( \mathbb{C} \).

(8.3). Our example from Section 0 again: the polynomial \( x^3 - 2 \) has roots \( \alpha, \alpha\omega, \alpha\omega^2 \) where \( \alpha = \sqrt[3]{2} \in \mathbb{R} \) and

\[
\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i.
\]

Thus a splitting field for \( f \) over \( \mathbb{Q} \) is given by \( \mathbb{Q}(\alpha, \alpha\omega, \alpha\omega^2) \), which is the same thing as \( \mathbb{Q}(\alpha, \omega) \).

**Aside.** In Section [11] we will prove (Theorem G) that an isomorphism of a field to itself \( \sigma : F \to F \) can always be extended to an isomorphism \( \tilde{\sigma} : E_1 \to E_2 \) where \( E_1 \) is a splitting field of some polynomial \( f \) over \( F \) and \( E_2 \) is another splitting field of this polynomial. Thus, any two splitting fields of a polynomial over \( F \) are isomorphic.

**Exercise 8.2.**
1. Let \( f = ax^2 + bx + c \in \mathbb{Q}[x] \) and \( \Delta = b^2 - 4ac \). Show that the splitting field of \( f \) over \( \mathbb{Q} \) is \( \mathbb{Q}(\sqrt{\Delta}) \).
2. Let \( f = (x - \alpha)(x - \beta) \in \mathbb{Q}[x] \) and \( D = (\alpha - \beta)^2 \). Show that the splitting field of \( f \) over \( \mathbb{Q} \) is \( \mathbb{Q}(\sqrt{D}) \). Show that the splitting is \( F(\alpha) = F(\beta) \).

8.2. Finite Fields

The construction of Section 5 produced explicit examples of fields having order \( p^d \) for \( p \) a prime. We now show that any finite field must have order \( p^d \) for some prime \( p \) and \( d > 0 \), and there exists a unique such field.

(8.1). Recall from Definition [3.6] that the prime subfield of a field \( F \) is the intersection of all the subfields of \( F \). It is isomorphic to \( \mathbb{F}_p \) for some \( p \) or to \( \mathbb{Q} \). In particular, the prime subfield of a finite field \( F \) must be isomorphic to \( \mathbb{F}_p \).

Using the ideas from Section 7 we have an extension of fields \( \mathbb{F}_p \subseteq F \) and hence the finite field \( F \) forms a vector space over the field \( \mathbb{F}_p \). This space must be finite dimensional (for \( F \) to be finite), so each element of \( F \) can be written uniquely as a linear combination,

\[
a_1\alpha_1 + a_2\alpha_2 + \cdots + a_d\alpha_d,
\]

of some basis vectors \( \alpha_1, \alpha_2, \ldots, \alpha_d \) with the \( a_i \in \mathbb{F}_p \). In particular there are \( p \) choices for each \( a_i \), and the choices are independent, giving \( p^d \) elements of \( F \) in total.

Thus a finite field has \( p^d \) elements for some prime \( p \).
(8.2). Here is an extended example that shows the converse, ie: constructs a field with \( q = p^d \) elements for any prime \( p \) and positive integer \( d > 0 \).

Consider the polynomial \( x^d - x \) over the field \( \mathbb{F}_p \) of \( p \) elements. Let \( L \) be an extension of the field \( \mathbb{F}_p \) containing all the roots of the polynomial, as guaranteed us by the Corollary to Krones’s Theorem. In Exercise 2.11 we used the formal derivative to see whether a polynomial has distinct roots. We have \( \partial(x^d - x) = qx^{d-1} - 1 = p^a x^{p^a-1} - 1 = -1 \) as \( p^a = 0 \) in \( \mathbb{F}_p \). The constant polynomial \(-1\) has no roots in \( L \), and so the original polynomial \( x^d - x \) has no repeated roots in \( L \) by Exercise 2.11.

In fact, the \( p^d \) distinct roots of \( x^d - x \) form a subfield of \( L \), and this is the field of order \( p^d \) that we seek. To show this, let \( a, c \) be roots (so that \( ad = a \) and \( c^d = c \)). We show that \(-a, a + c, ac \) and \( a^{-1} \) are also roots.

Firstly, \((-a)^d - (-a) = (-1)^d a^d + a \). If \( p = 2 \), then \(-1 = 1 \) in \( \mathbb{F}_2 \), so that \((-1)^d a^d + a = a^d + a = a + a = 2a = 0 \). Otherwise \( p \) is odd so that \((-1)^d a^d + a = -a^d + a = -a + a = 0 \). In either case, \(-a \) is a root of the polynomial \( x^d - x \).

Next,
\[
(a + c)^d = \sum_{i=0}^{d} \binom{d}{i} a^i c^{d-i} = a^d + c^d + p(\text{other terms}),
\]
as \( p \) divides the binomial coefficient when \( 0 < i < q \) by Exercise 2.12. Thus \((a + c)^d = a^d + c^d \).

(Compare this with Exercise 5.5.) Substituting \( a + c \) into \( x^d - x \) gives
\[
(a + c)^d - (a + c) = a^d + c^d - a - c = 0,
\]
using \( a^d = a \) and \( c^d = c \). Thus \( a + c \) is also a root of the polynomial.

The product \((ac)^d - ac = a^d c^d - ac = ac - ac = 0 \). Finally, \((a^{-1})^d - (a^{-1}) = (a^{-1})^{d-1} - (a^{-1}) = a^{-1} - a^{-1} = 0 \). In both cases we have used \( a^d = a \).

Thus the \( q = p^d \) roots of the polynomial form a subfield of \( L \) as claimed, and we have constructed a field with this many elements.

(8.3). Looking back at this example, \( L \) was an extension of \( \mathbb{F}_p \) containing the roots of the polynomial \( x^d - x \). In particular, if these roots are \( \{a_1, \ldots, a_q\} \), then \( \mathbb{F}_p(a_1, \ldots, a_q) \) is the splitting field over \( \mathbb{F}_p \) of the polynomial. In the example we constructed the subfield \( \mathbb{F} \) of \( L \) consisting of the roots of \( x^d - x \). As any subfield contains \( \mathbb{F}_p \), we have \( \mathbb{F}_p(a_1, \ldots, a_q) \subseteq \mathbb{F} \), whereas \( \mathbb{F} = \{a_1, \ldots, a_q\} \) so that \( \mathbb{F} \subseteq \mathbb{F}_p(a_1, \ldots, a_q) \). Hence the field we constructed in the example was the splitting field over \( \mathbb{F}_p \) of the polynomial \( x^d - q \).

If \( F \) is now an arbitrary field with \( q \) elements, then it has prime subfield \( \mathbb{F}_p \). Moreover, as the multiplicative group of \( F \) has order \( q - 1 \), by Lagrange’s Theorem (see Section 10), every element of \( F \) satisfies \( x^{q-1} = 1 \), hence is a root of the \( \mathbb{F}_p \)-polynomial \( x^d - x = 0 \). Thus, a finite field of order \( q \) is the splitting field over \( \mathbb{F}_p \) of the polynomial \( x^d - x \), and by the uniqueness of such, any two fields of order \( q \) are isomorphic.

(8.4). We finish with a fact about finite fields that will prove useful later on. Remember that a field is, among other things, two groups spliced together in a compatible way: the elements form a group under addition (the additive group) and the non-zero elements form a group under multiplication (the multiplicative group).

Looking at the complex numbers as an example, we can find a number of finite subgroups of the multiplicative group \( \mathbb{C}^* \) of \( \mathbb{C} \) by considering roots of 1. For any \( n \), the powers of the \( n\)-th root of 1,
\[
\omega = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n},
\]
form a subgroup of \( \mathbb{C}^* \) of order \( n \). Moreover, this subgroup is cyclic.
**Proposition 8.1.** Let $F$ be any field and $G$ a finite subgroup of the multiplicative group $F^*$ of $F$. Then $G$ is a cyclic group.

In particular, if $F$ is a finite field, then the multiplicative group $F^*$ of $F$ is finite, hence cyclic.

*Proof.* By Exercise [10.3] there is an element $g \in G$ whose order $m$ is the least common multiple of all the orders of elements of $G$. Thus, any element $h \in G$ satisfies $h^m = 1$. Hence every element of the group is a root of $x^m - 1$, and since this polynomial has at most $m$ roots in $F$, the order of $G$ must be $\leq m$. As $g \in G$ has order $m$ its powers must exhaust the whole group, hence $G$ is cyclic.

\[\square\]

### 8.3. Algebraically closed fields

(8.1). In the first part of this section we dealt with fields in which a particular polynomial of interest split into linear factors. There are fields like the complex numbers in which any polynomial splits.

A field $F$ is said to be *algebraically closed* if and only if every (non-constant) polynomial over $F$ splits in $F$.

(8.2). If $F$ is algebraically closed and $\alpha$ is algebraic over $F$ then there is a polynomial with $F$-coefficients having $\alpha$ as a root. As $F$ is algebraically closed, this polynomial splits in $F$, so that in particular $\alpha$ is in $F$. This explains the terminology: an algebraically closed field is *closed* with respect to the taking of algebraic elements. Contrast this with fields like $\mathbb{Q}$, over which there are algebraic elements like $\sqrt{2}$ that are not contained in $\mathbb{Q}$.

**Exercise 8.3.** Show that the following are equivalent:

1. $F$ is algebraically closed;
2. every non-constant polynomial over $F$ has a root in $F$;
3. the irreducible polynomials over $F$ are precisely the linear ones;
4. if $F \subseteq E$ is a finite extension then $E = F$.

**Theorem 8.1.** Every field $F$ is contained in an algebraically closed one.

*Sketch proof.* The full proof is beyond the scope of these notes, although the technical difficulties are not algebraic (or even number theoretical) but set theoretical. If the field $F$ is finite or countably infinite, the proof sort of goes as follows: there are countably many polynomials over a countable field, so take the union of all the splitting fields of these polynomials. Note that for a finite field, this is an infinite union, so an algebraically closed field containing even a finite field is very large.

\[\square\]

### 8.4. Simple extensions

(8.1). We saw in Section [3] that the extension $\mathbb{Q} \subset \mathbb{Q}(\sqrt{2}, \sqrt{3})$ is, despite appearances, simple. The fact that the extension is finite turns out to be enough to see that it is simple:

**Theorem 8.2.** Let $F \subset E$ be a finite extension such that the roots of any irreducible polynomial over $E$ are distinct. Then $E$ is a simple extension of $F$, ie: $E = F(\alpha)$ for some $\alpha \in E$.

The following proof is for the case that $F$ is infinite.
Proof. Let \((\alpha_1, \alpha_2, \ldots, \alpha_k)\) be a basis for \(E\) over \(F\) and consider the field \(F_1 = F(\alpha_3, \ldots, \alpha_k)\), so that \(E = F_1(\alpha_1, \alpha_2)\). We will show that \(F_1(\alpha_1, \alpha_2)\) is a simple extension of \(F_1\), i.e: that \(F_1(\alpha_1, \alpha_2) = F_1(\theta)\) for some \(\theta \in E\). Thus \(E = F(\alpha_1, \alpha_2, \ldots, \alpha_k) = F(\theta, \alpha_3, \ldots, \alpha_k)\), and so by repeatedly applying this procedure, \(E\) is a simple extension of \(F\).

Let \(f_1, f_2\) be the minimum polynomials over \(F_1\) of \(\alpha_1\) and \(\alpha_2\), and let \(L\) be an algebraically closed field containing of the field \(F\). As the \(\alpha_i\) are algebraic over \(F\), we have that the fields \(F_1\) and \(E\) are contained in \(L\) too. In particular the polynomials \(f_1\) and \(f_2\) split in \(L\),

\[
f_1 = \prod_{i=1}^{\deg f_1} (x - \beta_i), \quad f_2 = \prod_{i=1}^{\deg f_2} (x - \delta_i),
\]

with \(\beta_1 = \alpha_1\) and \(\delta_1 = \alpha_2\). As the roots of these polynomials are distinct we have that \(\beta_i \neq \beta_j\) and \(\delta_i \neq \delta_j\) for all \(i \neq j\). For any \(i\) and any \(j \neq 1\), the equation \(\beta_i + x\delta_j = \beta_1 + x\delta_1\) has precisely one solution in \(F_1\), namely

\[
x = \frac{\beta_i - \beta_1}{\delta_1 - \delta_j}.
\]

As there only finitely many such equations and infinitely many elements of \(F_1\), there must be an \(e \in F_1\) which is a solution to none of them, i.e: such that,

\[
\beta_i + e\delta_j \neq \beta_1 + e\delta_1
\]

for any \(i\) and any \(j \neq 1\). Let \(\theta = \beta_1 + e\delta_1 = \alpha_1 + ca_2\). We show that \(F_1(\alpha_1, \alpha_2) = F_1(\theta) = F_1(\alpha_1 + ca_2)\).

Clearly \(\alpha_1 + ca_2 \in F_1(\alpha_1, \alpha_2)\) so that \(F_1(\alpha_1 + ca_2) \subseteq F_1(\alpha_1, \alpha_2)\). We will show that \(\alpha_2 \in F_1(\alpha_1 + ca_2) = F_1(\theta)\), for then \(\alpha_1 + ca_2 - ca_2 = \alpha_1 \in F_1(\alpha_1 + ca_2)\), and so \(F_1(\alpha_1, \alpha_2) \subseteq F_1(\alpha_1 + ca_2)\).

We have \(0 = f_1(\alpha_1) = f_1(\theta - ca_2)\), so if we let \(r(t) \in F_1(\theta)[t]\) be given by \(r(t) = f_1(\theta - ct)\), then we have that \(\alpha_2\) is a root of both \(r(t)\) and \(f_2(x)\). If \(\gamma\) is another common root of \(r\) and \(f_2\), then \(\gamma\) is one of the \(\delta_j\), and \(\theta - c\gamma\) (being a root of \(f_1\)) is one of the \(\beta_i\), so that,

\[
\gamma = \delta_j \text{ and } \theta - c\gamma = \beta_i \Rightarrow \beta_i + c\delta_j = \beta_1 + c\delta_1,
\]

a contradiction. Thus \(r\) and \(f_2\) have just the single common root \(\alpha_2\). Let \(h\) be the minimum polynomial of \(\alpha_2\) over \(F_1(\theta)\), so that \(h\) divides both \(r\) and \(f_2\) (recall that the minimum polynomial divides any other polynomial having \(\alpha_2\) as a root). This means that \(h\) must have degree one, for a higher degree would give more than one common root for \(r\) and \(f_2\). Thus \(h = t + b\) for some \(b \in F_1(\theta)\). As \(h(\alpha_2) = 0\) we thus get that \(\alpha_2 = -b\) and so \(\alpha_2 \in F_1(\theta)\) as required.

The theorem is true for finite extensions of finite fields – even without the condition on the roots of the polynomials – but we omit the proof here. We saw in Exercise 3.6 that irreducible polynomials over fields of characteristic 0 have distinct roots. Thus any finite extension of a field of characteristic zero 0 is simple. For example, if \(\alpha_1, \ldots, \alpha_k\) are algebraic over \(Q\), then \(Q(\alpha_1, \ldots, \alpha_k) = Q(\theta)\) for some \(\theta \in C\).

9. Ruler and Compass Constructions II

(9.1). We can completely describe the complex numbers that are constructible:

Theorem E. The number \(z \in C\) is constructible if and only if there exists a sequence of field extensions,

\[
Q = K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n,
\]

such that \(Q(z)\) is a subfield of \(K_n\), and each \(K_i\) is an extension of \(K_{i-1}\) of degree at most 2.
Suppose that \( z \) the lines is
\[
\underbrace{y \in \mathbb{R}}_{\text{under conjugation, Exercise 9.1 gives the real and imaginary parts}}
\]
\[
\text{hence}
\]
The first of these is a technical convenience, the main point of which is illustrated by Exercise 9.1 following the proof.

We will simultaneously show the following two things by induction:

- Each of the fields \( K_i \) is closed under conjugation, i.e.: if \( z \in K_i \) then \( \bar{z} \in K_i \), and
- the degree of each extension \( K_{i-1} \subseteq K_i \) is at most two.

The first of these is a technical convenience, the main point of which is illustrated by Exercise 9.1 following the proof.

Firstly, \( K_1 = \mathbb{Q}(i) = \{a + bi : a, b \in \mathbb{Q}\} \) is certainly closed under conjugation and \( [K_1 : \mathbb{Q}] = \left[\mathbb{Q}(i) : \mathbb{Q}\right] = 2 \) as the minimum polynomial of \( i \) over \( \mathbb{Q} \) is \( x^2 + 1 \). Now fix \( i \) and suppose that \( K_{i-1} \) is closed under conjugation with \( K_i = K_{i-1}(z_i) \).

(i). Suppose that \( z_i \) is obtained as in case (i) of Figure 9.1. The Cartesian equation for one of the lines is \( y = m_1x + c_1 \), passing through the points \( z_p, z_q, \) with \( z_p, z_q \in K_{i-1} \). As \( K_{i-1} \) is closed under conjugation, Exercise 9.1 gives the real and imaginary parts of \( z_p \) and \( z_q \) are in \( K_{i-1} \). Thus,

\[
\begin{align*}
\text{Im } z_q &= m_1 \text{Re } z_q + c_1 \\
\text{Im } z_p &= m_1 \text{Re } z_p + c_1
\end{align*}
\]

so that \( m_1, c_1 \in K_{i-1} \). (If the line is vertical with equation \( x = c_1 \) we get \( c_1 = \text{Re } z_p \in K_{i-1} \)). If the equation of the other line is \( y = m_2x + c_2 \), we similarly get \( m_2, c_2 \in K_{i-1} \). As \( z_i \) lies on both these lines we have

\[
\begin{align*}
\text{Im } z_i &= m_2 \text{Re } z_i + c_2 \\
\text{Im } z_i &= m_1 \text{Re } z_i + c_1
\end{align*}
\]

with \( m_1, m_2, c_1, c_2 \in K_{i-1} \)

hence

\[
\text{Re } z_i = \frac{c_2 - c_1}{m_1 - m_2} \quad \text{and } \quad \text{Im } z_i = \frac{m_1(c_2 - c_1)}{m_1 - m_2} + c_1
\]
must lie in $K_{i-1}$ too. As $K_{i-1}$ is closed under conjugation we get $z_i \in K_{i-1}$ too, so in fact $K_i = K_{i-1}(z_i) = K_{i-1}$. Thus the degree of the extension $K_{i-1} \subseteq K_i$ (being 1) is certainly $\leq 2$. Moreover, $K_i = K_{i-1}$ is closed under conjugation as $K_{i-1}$ is.

(ii). Suppose $z_i$ arises as in case (ii) with the line having equation $y = mx + c$ and the circle having equation $(x - \text{Re } z_i)^2 + (y - \text{Im } z_i)^2 = r^2$, where $r^2 = (\text{Re } z_i - \text{Re } z_{i+1})^2 + (\text{Im } z_i - \text{Im } z_{i+1})^2$. As before, $m, c \in K_{i-1}$; moreover, $z_i, z_{i+1} \in K_{i-1}$, hence $r^2 \in K_{i-1}$. As $z_i$ lies on the line we have $\text{Im } z_i = m\text{Re } z_i + c$, and as it lies on the circle we have

$$(\text{Re } z_i - \text{Re } z_{i+1})^2 + (m\text{Re } z_i + c - \text{Im } z_i)^2 = r^2.$$ 

Thus the polynomial $(x - \text{Re } z_i)^2 + (mx + c - \text{Im } z_i)^2 = r^2$ is a quadratic with $K_{i-1}$ coefficients and having $\text{Re } z_i$ as a root. The minimum polynomial of $\text{Re } z_i$ over $K_{i-1}$ thus has degree at most 2, giving

$$[K_{i-1}(\text{Re } z_i) : K_{i-1}] \leq 2$$

by Theorem D. In fact, $\text{Im } z_i \in K_{i-1}(\text{Re } z_i)$ as well, since $\text{Im } z_i = m\text{Re } z_i + c$. Thus $z_i$ itself is in $K_{i-1}(\text{Re } z_i)$, as $i$ also is, and we have the sequence,

$$K_{i-1} \subseteq K_i = K_{i-1}(z_i) \subseteq K_{i-1}(\text{Re } z_i),$$

giving that the degree of the extension $K_{i-1} \subseteq K_i$ is also $\leq 2$ by the Tower Law.

Finally, we show that the field $K_i$ is closed under conjugation, for which we can assume that $[K_i : K_{i-1}] = 2$ – it is trivially the case if the degree is one. Now, $K_i = K_{i-1}(z_i) = K_{i-1}(\text{Re } z_i)$, so in particular $z_i$ and $\text{Re } z_i$ are in $K_i$, hence

$$\text{Im } z_i = \frac{z_i - \text{Re } z_i}{i}$$

is too. The result is that $\text{Re } z_i - \text{Im } z_i \cdot i = \bar{z}_i$ is in $K_i$ too. A general element of $K_i$ has the form $a + bz_i$ with $a, b \in K_{i-1}$, whose conjugate $\bar{a} + \overline{b}z_i$ is thus also in $K_i$.

(iii). If $z$ arises as in case (iii), then as $z_i$ lies on both circles we have

$$(\text{Re } z_i - \text{Re } z_{i+1})^2 + (\text{Im } z_i - \text{Im } z_{i+1})^2 = r^2$$
and

$$(\text{Re } z_i - \text{Re } z_{i+1})^2 + (\text{Im } z_i - \text{Im } z_{i+1})^2 = s^2,$$

with both $r^2$ and $s^2$ in $K_{i-1}$ for the same reason as in case (ii). Expanding both expressions gives terms of the form $\text{Re } z_i^2 + \text{Im } z_i^2$, and equating leads to,

$$\text{Im } z_i = \frac{\beta_1}{\alpha}\text{Re } z_i + \frac{\beta_2}{\alpha}, \text{ where } \alpha = 2(\text{Im } z_i - \text{Im } z_{i+1}), \beta_1 = 2(\text{Re } z_i - \text{Re } z_{i+1})$$

and

$$\beta_2 = \text{Re } z_i^2 + \text{Im } z_i^2 - (\text{Re } z_{i+1}^2 + \text{Im } z_{i+1}^2) + s^2 - r^2.$$ Combining this $K_{i-1}$-expression for $\text{Im } z_i$ with the first of the two circle equations above puts us into a similar situation as case (ii), from which the result follows in the same way.

Now for the “if” part, which is mercifully shorter. Suppose we have a tower of fields $\mathbb{Q} = K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n$, with $\mathbb{Q}(z)$ in $K_n$, hence $z \in K_n$. We can assume that $z \notin K_{n-1}$ (otherwise stop one step earlier!) and so we have

$$K_{n-1} \subseteq K_{n-1}(z) \subseteq K_n$$

where $z \notin K_{n-1}$ gives $[K_{n-1}(z) : K_{n-1}] \geq 2$. On the other hand $[K_n : K_{n-1}] \leq 2$ so by the tower law we have $[K_{n-1}(z) : K_{n-1}] = [K_n : K_{n-1}]$ and hence $K_n = K_{n-1}(z)$ with $[K_{n-1}(z) : K_{n-1}] = 2$. 

The minimum polynomial of \(z\) over \(K_{n-1}\) thus has the form \(x^2 + bx + c\), with \(b, c \in K_{n-1}\), so that \(z\) is one of,
\[
-1 \pm \sqrt{-4c}
\]
either of which can be constructed from \(1, 2, 4, b, c \in K_{n-1}\), using the arithmetical and square root constructions of Section \([6]\) But in the same way \(b, c\) can be constructed from elements of \(K_{n-2}\), and so on, giving that \(z\) is indeed constructible. \(\square\)

**Exercise 9.1.** Let \(K\) be a field such that \(\mathbb{Q}(i) \subseteq K \subseteq \mathbb{C}\), and suppose that \(K\) is closed under conjugation. Show that \(z \in K\) if and only if the real and imaginary parts of \(z\) are in \(K\).

\((9.2)\). It is much easier to use the “only if” part of the Theorem, which shows when numbers cannot be constructed, so we restate this part as a separate,

**Corollary 9.1.** If \(z \in \mathbb{C}\) is constructible then the degree of the extension \(\mathbb{Q} \subseteq \mathbb{Q}(z)\) must be a power of two.

**Proof.** If \(z\) is constructible then we have the tower of extensions as given in Theorem E, with \(z \in K_n\). Thus we have the sequence of extensions \(\mathbb{Q} \subseteq \mathbb{Q}(z) \subseteq K_n\), which by the tower law gives,
\[
[K_n : \mathbb{Q}] = [K_n : \mathbb{Q}(z)][\mathbb{Q}(z) : \mathbb{Q}].
\]
Thus \([\mathbb{Q}(z) : \mathbb{Q}]\) divides \([K_n : \mathbb{Q}]\), which is a power of two, so \([\mathbb{Q}(z) : \mathbb{Q}]\) must also be a power of two. \(\square\)

To use the “if” part to show that numbers can be constructed by finding a tower of fields as in Theorem E, is a little harder. We will need to know more about the fields sandwiched between \(\mathbb{Q}\) and \(\mathbb{Q}(z)\) before we can do this. The Galois Correspondence in Section \([13]\) will give us the control we need.

\((9.3)\). The Corollary is only stated in one direction. The converse is not true.

\((9.4)\). A regular \(p\)-gon, for \(p\) a prime, can be constructed, by Exercise \([6,3]\) precisely when the complex number \(z = \cos(2\pi/p) + i\sin(2\pi/p)\) can be constructed. By Exercise \([2,14]\) the minimum polynomial of \(z\) over \(\mathbb{Q}\) is the \(p\)-th cyclotomic polynomial,
\[
\Phi_p(x) = x^{p-1} + x^{p-2} + \cdots + x + 1.
\]

The degree of the extension \(\mathbb{Q} \subseteq \mathbb{Q}(z)\) is thus \(p - 1\), so \(p - 1\) must be a power of two if the \(p\)-gon is to be constructed, i.e.
\[
p = 2^m + 1.
\]

Actually, even more can be said. If \(m\) is odd, the polynomial \(x^m + 1\) has \(-1\) as a root, and so can be factorised as \(x^m + 1 = (x + 1)(x^{m-1} - x^{m-2} + x^{m-3} - \cdots - x + 1)\). Thus if \(n = mk\) for \(m\) odd, we have
\[
2^n + 1 = (2^k)^m + 1 = (2^k + 1)((2^k)^{m-1} - (2^k)^{m-2} + (2^k)^{m-3} - \cdots - (2^k) + 1),
\]
giving that \(2^n + 1\) cannot be prime unless \(n\) has no odd divisors; i.e. \(2^n + 1\) can only be prime if \(n\) itself is a power of two.

Thus for a \(p\)-gon to be constructible, we must have that \(p\) is a prime number of the form
\[
p = 2^m + 1,
\]
a so-called Fermat prime. Such primes are extremely rare: the only ones < \(10^{900}\) are
\[
3, 5, 17, 257 \text{ and } 65537.
\]

We will see in Section \([14]\) that the converse is true: if \(p\) is a Fermat prime, then a regular \(p\)-gon can be constructed.
(9.5). A square plot of land can always be doubled in area using a ruler and compass:

Set the compass to the side length $t$ of the plot. As $\sqrt{2}$ is a constructible number, we can construct the point with coordinates $(\sqrt{2}t, \sqrt{2}t)$, hence doubling the area.

(9.6). Is there a similar procedure for a cube? Suppose the original cube has side length 1, so that the task is to produce a new cube of volume 2. If this could be accomplished via a ruler and compass construction, then by setting the compass to the side length of the new cube, we would have constructed $\sqrt[3]{2}$. But the minimum polynomial over $\mathbb{Q}$ of $\sqrt[3]{2}$ is clearly $x^3 - 2$, with the extension $\mathbb{Q} \subset \mathbb{Q}(\sqrt[3]{2})$ thus having degree three. Such a construction cannot therefore be possible.

(9.7). The subset $\Box^n$ of $\mathbb{R}^n$ given by

$$\Box^n = \{x \in \mathbb{R}^n | |x_i| \leq \frac{t}{2} \text{ for all } i\}$$

is an $n$-dimensional cube of side length $t$ having volume $t^n$. In particular, in 4-dimensions we have the hypercube:

The vertices can be placed on the 3-sphere $S^3$ in $\mathbb{R}^4$. Stereographically projecting $S^3$ to $\mathbb{R}^3$ gives the picture above. This object can be doubled in volume with ruler and compass because the point with coordinates $(\sqrt{2}t, \sqrt{2}t, \sqrt{2}t, \sqrt{2}t)$ can be constructed.

(9.8). One of our fundamental constructions was the bisection of an angle. It is natural to ask if there is a construction that trisects an angle. Certainly there are particular angles that can be trisected: if the angle $\phi$ is constructible for example, then the angle $3\phi$ can be trisected.

The angle $\pi/3$ however cannot be trisected. We will see this by showing that the angle $\pi/9$ cannot be constructed.

Exercise 9.2. Evaluate the complex number $(\cos \phi + i \sin \phi)^3$ in two different ways: using the binomial theorem and De Moivre’s theorem. By equating real parts, deduce that

$$\cos 3\phi = 4 \cos^3 \phi - 3 \cos \phi.$$
Derive similar expressions for \( \cos 5\phi \) and \( \cos 7\phi \).

Exercise 6.3 gives that the angle \( \pi/9 \) is constructible precisely when the complex number \( \cos \pi/9 \) can be constructed, for which it is necessary in turn that the degree of the extension \( \mathbb{Q} \subseteq \mathbb{Q}(\cos \pi/9) \) be a power of two. Exercise 9.2 with \( \phi = \pi/9 \) gives

\[
\cos \frac{\pi}{3} = 4 \cos^3 \frac{\pi}{9} - 3 \cos \frac{\pi}{9}, \text{ hence, } 1 = 8 \cos^3 \frac{\pi}{9} - 6 \cos \frac{\pi}{9}.
\]

Thus, if \( u = 2 \cos(\pi/9) \), then \( u^3 - 3u - 1 = 0 \). This polynomial is irreducible over \( \mathbb{Q} \) by the reduction test (with \( p = 2 \)) so it is the minimum polynomial over \( \mathbb{Q} \) of \( 2 \cos(\pi/9) \). The extension \( \mathbb{Q} \subseteq \mathbb{Q}(2 \cos(\pi/9)) = \mathbb{Q}(\cos(\pi/9)) \) thus has degree three, and so the angle \( \pi/9 \) cannot be constructed.

We will be able to say more about which angles of the form \( \pi/n \) can be constructed in Section 14.

Exercise 9.3.

1. Can an angle of 40° be constructed?
2. Assuming 72° is constructible, what about 24° and 8°?
3. Can 72° be constructed? (hint: Section 0)

Further Exercises for Section 9

Exercise 9.4. The octahedron, dodecahedron and icosahedron are three of the five Platonic solids (the other two are the tetrahedron and the cube). See Figure 9.2. The volume of each is given by the formula, where \( x \) is the length of any edge. Show that in each case, there is no general method, using a ruler and compass, to construct a new solid from a given one, and having twice the volume.

Exercise 9.5. Let \( S_O, S_D \) and \( S_I \) be the surface areas of the three Platonic solids of Exercise 9.4. If,

\[
S_O = 2x^2\sqrt{3}, S_D = 3x^2\sqrt{5(5 + 2\sqrt{5})} \text{ and } S_I = 5x^2\sqrt{3},
\]

determine whether or not a solid can be constructed from a given one with twice the surface area.
Exercise 9.6. 1. Using the identity \( \cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta \). Show that is is impossible, using a ruler and compass, to \textit{quinsect} (that is, divide into 5 equal parts) any angle \( \psi \) that satisfies,

\[
\cos \psi = \frac{5}{6}
\]

2. Using the identity, \( \cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta \) show that it is impossible, using ruler and compass, to \textit{septsect} (that is, divide into seven equal parts) any angle \( \varphi \) such that

\[
\cos \varphi = \frac{7}{8}
\]

10. Groups I: Soluble Groups and Simple Groups

This section contains miscellaneous but important reminders from group theory. Not all our groups will be Abelian, so we return to writing the group operation as juxtaposition and writing “id” for the group identity.

(10.1). A permutation of a set \( X \) is a bijection \( X \rightarrow X \). Usually we are interested in the case where \( X \) is finite, say \( X = \{1, 2, \ldots, n\} \), so a permutation is just a rearrangement of these numbers. Permutations are most compactly written using cycle notation

\[
(a_{11}, a_{12}, \ldots, a_{1n})(a_{21}, a_{22}, \ldots, a_{2n}) \ldots (a_{k1}, a_{k2}, \ldots, a_{kn})
\]

where the \( a_{ij} \) are elements of \( \{1, 2, \ldots, n\} \). Each \( (b_1, b_2, \ldots, b_k) \) means that the \( b_i \) are permuted in a cycle:

Cycles are composed from right to left, eg: \( (1, 2)(1, 2, 4, 3)(1, 3)(2, 4) = (1, 2, 3) \). In this way a permutation can be written as a product of disjoint cycles. The set of all permutations of \( X \) forms a group under composition of bijections called the \textit{symmetric group} \( S_X \), or \( S_n \) if \( X = \{1, 2, \ldots, n\} \).

(10.2). A permutation where just two things are interchanged, and everything else is left fixed, is called a \textit{transposition} or \textit{swap} \( (a, b) \). Any permutation can be written as a composition of transpositions, for example:

\[
(1, 2, 3) = (1, 3)(1, 2) = (1, 2)(2, 3) \text{ and } (a_1, a_2, \ldots, a_k) = (a_1, a_k)(a_1, a_{k-1}) \ldots (a_1, a_3)(a_1, a_2).
\]

There will be many such expressions, but they all involve an even number of transpositions or all involve an odd number of them.

We can thus call a permutation \textit{even} if it can be decomposed into an even number of transpositions, and \textit{odd} otherwise. The even permutations in \( S_n \) form a subgroup called the \textit{Alternating group} \( A_n \).
Exercise 10.1. Show that $A_n$ is indeed a group comprising exactly half of the elements of $S_n$. Show that the odd elements in $S_n$ do not form a subgroup.

Exercise 10.2. Recall that the order of an element $g$ of a group $G$ is the least $n$ such that $g^n = \text{id}$. Show that if $g, h$ are elements such that $gh = hg$ then $(gh)^n = g^n h^n$. If in addition the order of $g$ is $n$ and the order of $h$ is $m$ with $\gcd(n, m) = 1$, then the order of $gh$ is the lowest common multiple of $n$ and $m$.

Exercise 10.3. Let $G$ be a finite Abelian group, and let $1 = m_1, m_2, \ldots, m_\ell$ be a list of all the possible orders of elements of $G$. Show that there exists an element whose order is the lowest common multiple of the $m_i$ [hint: let $g_i$ be an element of order $m_i$ and use Exercise 10.2 to show that there are $k_1, \ldots, k_\ell$ with $g_{k_1} \cdots g_{k_\ell}$ the element we seek].

(10.3). If $G$ is a group and $\{g_1, g_2, \ldots, g_n\}$ are elements of $G$, then we say that the $g_i$ generate $G$ when every element $g \in G$ can be obtained as a product
$$g = g_{i_1}^{\pm 1} g_{i_2}^{\pm 1} \cdots g_{i_k}^{\pm 1},$$
of the $g_i$ and their inverses. Write $G = \langle g_1, g_2, \ldots, g_n \rangle$.

(10.4). We find generators for the symmetric and alternating groups. We have already seen that the transpositions $(a, b)$ generate $S_n$, for any permutation can be written as a product
$$(a_1, a_2, \ldots, a_k) = (a_1, a_k)(a_1, a_{k-1}) \cdots (a_1, a_3)(a_1, a_2).$$The transpositions $(a, b)$ can in turn be expressed in terms of just some of them: when $a < b$ we have
$$(a, b) = (a, a + 1)(a + 1, a + 2) \cdots (b - 2, b - 1)(b - 1, b) \cdots (a + 1, a + 2)(a, a + 1)$$as can be seen by considering the picture:

and doing the swaps in the order indicated. Any number strictly in between $a$ and $b$ moves one place to the right and then one place to the left, with net effect that it remains stationary. The number $a$ is moved to $b$ by the top swaps, but then stays there. Similarly $b$ stays put for all but the last of the top swaps and then is moved to $a$ by the bottom swaps.

Any permutation can thus be written as a product of swaps of the form $(a, a + 1)$. Even these transpositions can be further reduced, by transferring $a$ and $a + 1$ to the points 1 and 2, swapping 1 and 2 and transferring the answer back to $a$ and $a + 1$. Indeed, if $\tau = (1, 2, \ldots, n)$ then doing the permutations in the order indicated in the picture:
shows that \((a, a + 1) = \tau a^{-1}(1, 2)\tau^{-1}a\). The conclusion is that \(S_n\) is generated by just two permutations, namely \((1, 2)\) and \((1, 2, \ldots, n)\).

**Exercise 10.4.** Show that the Alternating group is generated by the permutations of the form 
\((a, b, c)\). Show that just the 3-cycles of the form \((1, 2, a)\) will suffice.

(10.5) Lagrange’s theorem says that if \(G\) is a finite group and \(H\) a subgroup of \(G\), then the order \(|H|\) of \(H\) divides the order \(|G|\) of \(G\). The converse, that if a subset of a group has size dividing the order of the group then it is a subgroup, is false.

**Exercise 10.5.** By considering the Alternating group \(A_4\), justify this statement.

**Exercise 10.6.** Show that if \(G\) is a cyclic group, then the converse to Lagrange’s theorem is true, i.e: if \(G\) has order \(n\) and \(k\) divides \(n\) then \(G\) has a subgroup of order \(k\).

**Exercise 10.7.** Use Lagrange’s Theorem to show that if a group \(G\) has order a prime number \(p\), then \(G\) is isomorphic to a cyclic group. Thus any two groups of order \(p\) are isomorphic.

There are partial converses to Lagrange’s Theorem:

**Theorem 10.1 (Cauchy).** Let \(G\) be a finite group and \(p\) a prime dividing the order of \(G\). Then \(G\) has a subgroup of order \(p\).

Indeed, one can show that \(G\) contains an element \(g\) of order \(p\), with the subgroup being the elements \(\{g, g^2, \ldots, g^p = \text{id}\}\).

**Theorem 10.2 (Sylow’s 1st).** Let \(G\) be a finite group of order \(p^k m\), where \(p\) does not divide \(m\). Then \(G\) has a subgroup of order \(p^k\).

(10.6) It will be useful to consider all the subgroups of a group at once, rather than just one at a time.

**Definition 10.1 (lattice of subgroups).** The subgroup lattice is a diagram depicting all the subgroups of \(G\) and the inclusions between them. If \(H_1, H_2\) are subgroups of \(G\) with \(H_1 \subseteq H_2\) they appear in the diagram like so:

```
    H_2
     |
    H_1
```

At the very base of the diagram is the trivial subgroup \(\{\text{id}\}\) and at the apex is the other trivial subgroup, namely \(G\) itself. Denote the lattice by \(\mathcal{L}(G)\).

For example, the group of symmetries of an equilateral triangle has elements

\[\{\text{id}, r, r^2, s, rs, r^2s\}\]

where \(r\) is a rotation counter-clockwise through \(\frac{\pi}{3}\) of a turn (we called it \(ts\) in Section [10.1]) and \(s\) is the reflection in the horizontal axis.

The subgroup lattice \(\mathcal{L}(G)\) is on the left in Figure [10.1]. I’ll leave you to see that they are all subgroups, so it remains to see that we have all of them. Suppose first that \(H\) is a subgroup containing \(r\). Then it must contain all the powers \(\{\text{id}, r, r^2\}\) of \(r\), and so \(3 \leq |H| \leq 6\). By Lagrange’s Theorem \(|H|\) divides 6, so we have \(|H| = 3\) or 6, giving that \(H\) must be \(\{\text{id}, r, r^2\}\) or all of \(G\). This describes all the
subgroups that contain \( r \), and the same argument – and conclusion – applies to the subgroups containing \( r^2 \).

This leaves the subgroups containing one of the reflections \( s, rs, r^2s \) but not \( r \) or \( r^2 \). If \( H \) is a subgroup containing \( s \), then as it also contains \( \text{id} \), and by Lagrange, it must have order 2, 3 or 6. The first possibility gives \( H = \{ \text{id}, s \} \) and the last gives \( H = G \). On the other hand, to have order 3, the subgroup \( H \) must also contain one of \( rs \) or \( r^2s \). In the first case it also contains \( rss = r \), a contradiction. Similarly \( H \) cannot contain \( r^2s \), so there is no subgroup \( H \) containing \( s \) apart from \( \{ \text{id}, s \} \) and \( G \) itself. Similarly for subgroups containing \( rs \) or \( r^2s \). Thus the lattice \( \mathcal{L}(G) \) is indeed as shown in Figure 10.1.

The right part of Figure 10.1 gives the subgroup lattice of the symmetry group of a square. I’ll leave the details to you.

\[(10.7)\] If \( G \) is a finite group and
\[\{ \text{id} \} \triangleleft H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_{n-1} \triangleleft H_n = G,\]
is a nested sequence of subgroups with each \( H_i \) normal in \( H_{i+1} \) and the quotients
\[H_1/H_0, H_2/H_1, \ldots, H_n/H_{n-1}\]
Abelian, then \( G \) is said to be soluble.

\[(10.8)\] If \( G \) is an Abelian group, then we have the sequence
\[\{ \text{id} \} \triangleleft G,\]
with the single quotient \( G/\{ \text{id} \} \cong G \), an Abelian group. Thus Abelian groups are soluble.

\[(10.9)\] For another example let \( G \) be the symmetries, both rotations and reflections, of a regular \( n \)-gon in the plane. In the sequence:
\[\{ \text{id} \} \triangleleft \{ \text{rotations} \} \triangleleft G\]
the normality of the subgroup of rotations in \( G \) follows from the fact that the rotations comprise half of all the symmetries and Exercise [10.14]. Moreover, the rotations are isomorphic to the cyclic group \( \mathbb{Z}_n \), and so the quotients in this sequence are
\[\{ \text{rotations} \}/\{ \text{id} \} \cong \mathbb{Z}_n \text{ and } G/\{ \text{rotations} \} \cong \mathbb{Z}_2,\]
both Abelian groups.
Exercise 10.8. It turns out, although for slightly technical reasons, that a subgroup of a soluble group is also soluble. This exercise and the next demonstrate why. Let $G$ be a group, $H$ a subgroup and $N$ a normal subgroup. Let

$$NH = \{nh \mid n \in N, h \in H\}.$$  

1. Define a map $\varphi : H \to NH/N$ by $\varphi(h) = Nh$. Show that $\varphi$ is an onto homomorphism with kernel $N \cap H$.

2. Use the first isomorphism theorem for groups to deduce that $H/H \cap N$ is isomorphic to $NH/H$.

(This is called the second isomorphism or diamond isomorphism theorem. Why diamond? Draw a picture of all the subgroups–the theorem says that two “sides” of a diamond are isomorphic).

Exercise 10.9. Let $G$ be a soluble group via the series,

$$\langle \text{id} \rangle = H_0 \trianglelefteq H_1 \triangleleft \cdots \triangleleft H_{n-1} \triangleleft H_n = G,$$

and let $K$ be a subgroup of $G$. Show that

$$\langle \text{id} \rangle = H_0 \cap K \triangleleft H_1 \cap K \triangleleft \cdots \triangleleft H_{n-1} \cap K \triangleleft H_n \cap K = K,$$

is a series with Abelian quotients for $K$, and hence $K$ is also a soluble group.

(10.10). The antithesis of the soluble groups are the simple ones: groups $G$ whose only normal subgroups are the trivial subgroup $\langle \text{id} \rangle$ and the whole group $G$.

Whenever we have a normal subgroup we can form a quotient. A group is thus simple when its only quotients are itself $G/\langle \text{id} \rangle \cong G$ and the trivial group $G/G \cong \langle \text{id} \rangle$. Thus simple groups are analogous to prime numbers: integers whose only quotients are themselves $p/1 = p$ and $p/p = 1$.

If $G$ is non-Abelian and simple, then $G$ cannot be soluble. For, the only sequence of normal subgroups that $G$ can have is

$$\langle \text{id} \rangle \triangleleft G,$$

and as $G$ is non-Abelian the quotients of this sequence are non-Abelian. Thus, non-Abelian simple groups provide a ready source of non-soluble groups.

(10.11). Amazingly, there is a complete list of the finite simple groups, compiled over approximately 150 years. The list is contained in Tables 10.1–10.3.

Exercise 10.10. Show that if $p$ is a prime number then the cyclic group $\mathbb{Z}_p$ has no non-trivial subgroups whatsoever, and so is a simple group.
Table 10.2. The simple groups of Lie type

(10.12). In Table 10.1 we see that the Alternating groups $A_n$ are simple for $n \neq 1, 2$ or 4. In particular these Alternating groups are not soluble, and as any subgroup of a soluble group is soluble, any group containing the Alternating group will also not be soluble. Thus, the symmetric groups $S_n$ are not soluble if $n \neq 1, 2$ or 4.

(10.13). Tables 10.2 and 10.3 list the really interesting simple groups. The groups of Lie type are roughly speaking groups of matrices whose entries come from finite fields. We have already seen that if $q = p^n$ a prime power, then there is a field $\mathbb{F}_q$ with $q = p^n$ elements. The group $\text{SL}_n \mathbb{F}_q$ consists of the $n \times n$ matrices having determinant 1 and with entries from this field and the usual matrix multiplication. This group is not simple as

$$N = \{ \lambda I_n | \lambda \in \mathbb{F}_q \},$$

is a normal subgroup. But it turns out that the quotient group,

$$\text{SL}_n \mathbb{F}_q / N,$$

is a simple group. It is denoted $\text{PSL}_n \mathbb{F}_q$, and called the $n$-dimensional projective special linear group over $\mathbb{F}_q$. The remaining groups in Table 10.2 come from more complicated constructions.

Table 10.3 lists groups that don’t fall into any of the other categories. For this reason they are called the “sporadic” simple groups. They arise from various – often quite complicated – constructions that are beyond the reach of these notes. The most interesting of them is the largest one – the Monster simple group (which actually contains quite a few of the others as subgroups).

In any case, the simple groups in Tables 10.2 and 10.3 are all non-Abelian, hence provide more examples of non-soluble groups.

Further Exercises for Section 10

Exercise 10.11. Show that any subgroup of an abelian group is normal.
Table 10.3. The sporadic simple groups

Exercise 10.12. Let $n$ be a positive integer that is not prime. Show that the cyclic group $\mathbb{Z}_n$ is not simple.

Exercise 10.13. Show that $A_2$ and $A_4$ are not simple groups, but $A_3$ is.

Exercise 10.14. Let $G$ be a group and $H$ a subgroup such that $H$ has exactly two cosets in $G$. Let $C_2$ be the group with elements $\{-1, 1\}$ and operation the usual multiplication. Define a map $f : G \rightarrow C_2$ by

$$f(g) = \begin{cases} 1 & g \in H \\ -1 & g \notin H \end{cases}$$

Show that $f$ is a homomorphism. Deduce that $H$ is a normal subgroup.

Exercise 10.15. Consider the group of symmetries (rotations and reflections) of a regular $n$-sided polygon for $n \geq 3$. Show that this is not a simple group.

Exercise 10.16. Show that $S_2$ is simple but $S_n$ is not for $n \geq 3$. Show that $A_n$ has no subgroups of index 2 for $n \geq 5$.

Exercise 10.17. Show that if $G$ is abelian and simple then it is cyclic. Deduce that if $G$ is simple and not isomorphic to $\mathbb{Z}_p$ then $G$ is non-Abelian.

Exercise 10.18. For each of the following groups $G$, draw the subgroup lattice $\mathcal{L}(G)$:
1. $G$ = the group of symmetries of a pentagon or hexagon.
2. $G$ = the cyclic group $\{1, g, g^2, \ldots, g^{n-1}\}$ where $g^n = 1$.

11. Groups II: Symmetries of Fields

We are finally able to bring symmetry into the solutions of polynomial equations.

**Definition 11.1 (automorphism or symmetry of a field).** An automorphism of a field $F$ is an isomorphism $\sigma : F \rightarrow F$, i.e., a bijective map from $F$ to $F$ such that $\sigma(a + b) = \sigma(a) + \sigma(b)$ and $\sigma(ab) = \sigma(a)\sigma(b)$ for all $a, b \in F$.

We remarked in Section 3 that an automorphism is a relabeling of the elements using different symbols but keeping the algebra the same. So it is a way of picking the field up and placing it back down without changing the way it essentially looks.

**Exercise 11.1.** Show that if $\sigma$ is an automorphism of the field $F$ then $\sigma(0) = 0$ and $\sigma(1) = 1$.

(11.1). A familiar example is complex conjugation: $\sigma : z \mapsto \overline{z}$ is an automorphism of $\mathbb{C}$, since

$$\overline{z + w} = \overline{z} + \overline{w} \text{ and } \overline{\overline{z}} = \overline{z},$$

with conjugation a bijection $\mathbb{C} \rightarrow \mathbb{C}$. This symmetry captures the idea that from an algebraic point of view, we could have just as easily adjoined $-i$ to $\mathbb{R}$, rather than $i$, to obtain the complex numbers – they look the same upside down as right side up!

We will see at the end of this section that if a non-trivial automorphism of $\mathbb{C}$ fixes pointwise the real numbers, then it must be complex conjugation. If we drop the requirement that $\mathbb{R}$ be fixed then there may be more possibilities: if we only insist that $\sigma$ fix $\mathbb{Q}$ pointwise then there are infinitely many possibilities.

**Exercise 11.2.** Let $f \in \mathbb{Q}[x]$ with roots $\{\alpha_1, \ldots, \alpha_d\} \in \mathbb{C}$. Show that complex conjugation $z \mapsto \overline{z}$ is an automorphism of the splitting field $\mathbb{Q}(\alpha_1, \ldots, \alpha_d)$. Is it always non-trivial?

**Exercise 11.3.** Show that $a + bi \mapsto -a + bi$ is not an automorphism of $\mathbb{C}$. Show that if $\ell$ is a line through 0 in $\mathbb{C}$, then reflecting in $\ell$ is an automorphism only when $\ell$ is the real axis.

(11.2). We saw in Section 3 that every field $F$ has a prime subfield isomorphic to either $\mathbb{F}_p$ or $\mathbb{Q}$. The elements have the form:

$$\frac{1 + 1 + \cdots + 1}{1 + 1 + \cdots + 1}$$

If $\sigma : F \rightarrow F$ is an automorphism of $F$ then

$$\sigma\left(\frac{1 + 1 + \cdots + 1}{1 + 1 + \cdots + 1}\right) = \sigma\left(\frac{1}{1 + 1 + \cdots + 1}\right) \sigma\left(\frac{1 + 1 + \cdots + 1}{1 + 1 + \cdots + 1}\right)$$

$$= \frac{m}{n} \sigma(1) + \sigma(1) + \cdots + \sigma(1) \left(\frac{1}{\sigma(1) + \sigma(1) + \cdots + \sigma(1)}\right) = \frac{1 + 1 + \cdots + 1}{1 + 1 + \cdots + 1}$$

The elements of the prime subfield are thus fixed pointwise by the automorphism $\sigma$. 
(11.3). This example suggests that we should think about symmetries in a relative way. As symmetries normally arrange themselves into groups we define:

Definition 11.2 (Galois group of an extension). Let $F \subseteq E$ be an extension of fields. The automorphisms of the field $E$ that fix pointwise the elements of $F$ form a group under composition, called the Galois group of $E$ over $F$, and denoted $Gal(E/F)$.

An element $\sigma$ of $Gal(E/F)$ thus has the property that $\sigma(a) = a$ for all $a \in F$.

Exercise 11.4. For $F \subset E$ fields, show that the set of automorphisms $Gal(E/F)$ of $E$ that fix $F$ pointwise do indeed form a group under composition.

(11.4). Consider the field $\mathbb{Q}(\sqrt{2}, i)$. The tower law gives basis $\{1, \sqrt{2}, i, \sqrt{2}i\}$ over $\mathbb{Q}$, so the elements are

$$\mathbb{Q}(\sqrt{2}, i) = \{a + b\sqrt{2} + ci + d\sqrt{2}i \mid a, b, c, d \in \mathbb{Q}\}.$$ 

If $\sigma \in Gal(\mathbb{Q}(\sqrt{2}, i)/\mathbb{Q})$ then

$$\sigma(a + b\sqrt{2} + ci + d\sqrt{2}i) = \sigma(a) + \sigma(b)\sigma(\sqrt{2}) + \sigma(c)\sigma(i) + \sigma(d)\sigma(\sqrt{2}i)$$

$$= a + b\sigma(\sqrt{2}) + c\sigma(i) + d\sigma(\sqrt{2}i)$$

as an element of $Gal(\mathbb{Q}(\sqrt{2}, i)/\mathbb{Q})$ then fixes rational numbers by definition. Thus $\sigma$ is completely determined by its effect on the basis $\{1, \sqrt{2}, i, \sqrt{2}i\}$: once their images are known, then $\sigma$ is known.

(This is no surprise. If $F \subseteq E$ is an extension then, among other things, $E$ is a vector space over $F$ and $\sigma \in Gal(E/F)$ is, among other things, a linear map of vector spaces $E \to E$, hence completely determined by its effect on a basis.)

We can say more: we have $\sigma(1) = 1$ and $\sigma(\sqrt{2}i) = \sigma(\sqrt{2})\sigma(i)$. Thus $\sigma$ is completely determined by its effect on $\sqrt{2}$ and $i$, the elements adjoined to obtain $\mathbb{Q}(\sqrt{2}, i)$.

(11.5). This is a general fact: if $F \subseteq F(a_1, a_2, \ldots, a_k) = E$ and $\sigma \in Gal(E/F)$, then $\sigma$ is completely determined by its effect on $a_1, \ldots, a_k$. For, if $\{\beta_1, \ldots, \beta_n\}$ is a basis for $E$ over $F$, then $\sigma$ is completely determined by its effect on the $\beta_i$. The proof of the tower law gives

$$\beta_i = a_1^{i_1}a_2^{i_2} \cdots a_k^{i_k},$$

a product of the $a_j$’s, so that $\sigma(\beta_i) = \sigma(a_1)^{i_1}\sigma(a_2)^{i_2} \cdots \sigma(a_k)^{i_k}$ is in turn determined by the $\sigma(a_j)$’s.

(11.6). The structure of Galois groups can sometimes be determined via ad-hoc arguments, at least in very simple cases. For example, let $\omega$ be the primitive cube root of 1,

$$\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i,$$

and consider the extension $\mathbb{Q} \subseteq \mathbb{Q}(\omega)$.

Although $\omega$ is a root of $x^3 - 1$, this is reducible over $\mathbb{Q}$ (1 is also a root) and the minimum polynomial of $\omega$ over $\mathbb{Q}$ is in fact $x^2 + x + 1$ by Exercise 2.13. By Theorem D, the field $\mathbb{Q}(\omega) = \{a + b\omega \mid a, b \in \mathbb{Q}\}$, so that $\mathbb{Q}(\omega)$ is 2-dimensional over $\mathbb{Q}$ with basis $\{1, \omega\}$. Let $\sigma \in Gal(\mathbb{Q}(\omega)/\mathbb{Q})$, whose effect is completely determined by where it sends $\omega$. Suppose $\sigma(\omega) = a + b\omega$ for some $a, b \in \mathbb{Q}$ to be determined. We have $\sigma(\omega^3) = \sigma(1) = 1$, but also

$$\sigma(\omega^3) = \sigma(\omega)^3 = (a + b\omega)^3 = (a^3 + b^3 - 3ab^2) + (3a^2b - 3ab^2)\omega$$
with the last bit using $\omega^2 = -\omega - 1$.

As $\{1, \omega\}$ are independent over $\mathbb{Q}$, the elements of $\mathbb{Q}(\omega)$ have unique expressions as linear combinations of these two basis elements. We can therefore “equate the 1 and $\omega$ parts” in these two expressions for $\sigma(\omega^3)$:

$$1 = \sigma(\omega^3) = (a^3 + b^3 - 3ab^2) + (3a^2b - 3ab^2)\omega,$$

so that $a^3 + b^3 - 3ab^2 = 1$ and $3a^2b - 3ab^2 = 0$.

Solving these equations (in $\mathbb{Q}$!) gives three solutions $a = 0, b = 1$ and $a = 1, b = 0$ and $a = -1, b = -1$, corresponding to $\sigma(\omega) = \omega$ and $\sigma(\omega) = 1$ and $\sigma(\omega) = -1 - \omega = \omega^2$. The second one is impossible as $\sigma$ is a bijection and we already have $\sigma(1) = 1$. The first one is the identity map and the third $\sigma(\omega) = \omega^2 = \overline{\omega}$ is complex conjugation (and shown in the figure above), giving $\text{Gal}(\mathbb{Q}(\omega)/\mathbb{Q}) = \{\text{id}, \sigma : z \mapsto \overline{z}\}$ a group of order two. (Now revisit Exercise 0.3).

Exercise 11.5. $\mathbb{Q}(\omega)$ is also spanned, as a vector space, by $\{1, \omega, \omega^2\}$, so that every element has an expression of the form $a + b\omega + c\omega^2$ for some $a, b, c \in \mathbb{Q}$. In particular $\overline{\omega}$ can be written as both $\omega^2$ and as $-1 - \omega$. “Equating the 1 and the $\omega$ and the $\omega^2$ parts” gives $0 = -1$ and $1 = 0$. What has gone wrong?

(11.7). Our first tool for unpicking the structure of Galois groups is:

**Theorem F (The Extension Theorem).** Let $F, K$ be fields, $\tau : F \to K$ an isomorphism and $\tau^* : F[x] \to K[x]$ the ring homomorphism given by $\tau^* : \sum a_ix^i \mapsto \sum \tau(a_i)x^i$. If $a$ is algebraic over $F$, then $\tau$ extends to an isomorphism $\sigma : F(a) \to K(\beta)$ with $\sigma(a) = \beta$ if and only if $\beta$ is a root of $\tau^* f$, where $f$ is the minimum polynomial of $a$ over $F$.

The elements $\alpha$ and $\beta$ are assumed to lie in some extensions $F \subseteq E_1, K \subseteq E_2$; when we say that $\tau$ extends to $\sigma$ we mean that the restriction of $\sigma$ to $F$ is $\tau$.

The theorem seems technical, but has an intuitive meaning. Suppose we have $F = K$ and $\tau$ is the identity isomorphism, hence $\tau^*$ is also the identity. Then we have an extension $\sigma : F(a) \to F(\beta)$ precisely when $\beta$ is a root of the minimum polynomial $f$ of $a$ over $F$.

We can say even more: if $\beta$ is an element of $F(\alpha)$, then $F(\beta) \subseteq F(\alpha)$; as an $F$-vector space $F(\beta)$ is $(\deg f)$-dimensional over $F$ as $\alpha$ and $\beta$ have the same minimum polynomial over $F$. As $F(\alpha)$ has the same dimension we get $F(\beta) = F(\alpha)$. Thus $\sigma$ is an isomorphism of $F(\alpha) \to F(\alpha)$ fixing $F$ pointwise, and so an element of the Galois group $\text{Gal}(F(\alpha)/F)$.

Here is everything we know about Galois groups so far:

**Corollary 11.1.** Let $\alpha$ be algebraic over $F$ with minimum polynomial $f$ over $F$. Then $\sigma : F(\alpha) \to F(\alpha)$ is an element of the Galois group $\text{Gal}(F(\alpha)/F)$ if and only if $\sigma(\alpha) = \beta$ where $\beta$ is a root of $f$ that is contained in $F(\alpha)$.

The elements of the Galois group thus permute those roots of the minimum polynomial that are contained in $F(\alpha)$.

There are slick proofs of the Extension theorem; ours is not going to be one of them. But it does make things nice and concrete. The elements of $F(\alpha)$ are polynomials in $\alpha$, so the simplest way to define $\sigma$ is

$$\sigma : a_m\alpha^m + \cdots + a_1\alpha + a_0 \mapsto \tau(a_m)\beta^m + \cdots + \tau(a_1)\beta + \tau(a_0). \quad (11.1)$$

The complication is that the same element will have many such polynomial expressions; for example $\overline{\omega} \in \mathbb{Q}(\omega)$ can be written both as $\omega^2$ and $-1 - \omega$ (see Exercise 11.5 above) making it unclear if (11.1) is well-defined. The solution is that $\beta$ is a root of $\tau^* f$, the “$K[x]$ version” of $f$. 

Proof of the Extension Theorem. For the “only if” part let \( f = \sum a_i x^i \) with \( f(\alpha) = 0 \). Then \( \sum a_i \alpha^i = 0 \in E_1 \) and \( \sigma(0) = 0 \in E_2 \) gives:

\[
\sigma\left(\sum a_i \alpha^i\right) = 0 \Rightarrow \sum \sigma(a_i)\sigma(\alpha)^i = 0 \Rightarrow \sum \tau(a_i) \beta^i = 0 \Rightarrow \tau^* f(\beta) = 0.
\]

(Compare this argument with the one that shows the roots of a polynomial with real coefficients occur in complex conjugate pairs).

For the “if” part, we need to build an isomorphism \( F(\alpha) \to K(\beta) \) with the desired properties. Define \( \sigma \) by the formula (11.1); in particular \( \sigma(\alpha) = \tau(\alpha) \) for all \( \alpha \in F \) and \( \sigma(\alpha) = \beta \).

(i). \( \sigma \) is well-defined and 1-1: Let

\[
\sum a_i \alpha^i = \sum b_j \alpha^j,
\]

be two expressions for some element of \( F(\alpha) \). Then \( \sum (a_i - b_i)\alpha^i = 0 \) and so \( \alpha \) is a root of the polynomial \( g = \sum (a_i - b_i) x^i \in F[x] \). As \( g \) is the minimum polynomial of \( \alpha \) over \( F \) it is a factor of \( g \), so that \( g = fh \), hence \( \tau^*(g) = \tau^*(f) \tau^*(h) = \tau^*(f) \tau^*(g) \) and \( \tau^*(f) \) is a factor of \( \tau^*(g) \).

\[
\tau^*(g)(\beta) = 0 \iff \sum \tau(a_i - b_i) \beta^i = 0 \iff \sum \tau(a_i) \beta^i = \sum \tau(b_i) \beta^i \iff \sigma(\sum a_i \alpha^i) = \sigma(\sum b_j \alpha^j).
\]

The conclusion is that \( \sum a_i \alpha^i = \sum b_j \alpha^j \) in \( F(\alpha) \) if and only if \( \sigma(\sum a_i \alpha^i) = \sigma(\sum b_j \alpha^j) \) in \( K(\beta) \), hence \( \sigma \) is both well-defined (\( \Rightarrow \)) and 1-1 (\( \Leftarrow \)).

(ii). \( \sigma \) is a homomorphism: Let

\[
\lambda = \sum a_i \alpha^i \text{ and } \mu = \sum b_j \alpha^j,
\]

be two elements of \( F(\alpha) \). Then

\[
\sigma(\lambda + \mu) = \sigma\left(\sum (a_i + b_j) \alpha^i\right) = \sum \tau(a_i + b_j) \beta^i = \sum \tau(a_i) \beta^i + \sum \tau(b_j) \beta^i = \sigma(\lambda) + \sigma(\mu).
\]

Similarly,

\[
\sigma(\lambda \mu) = \sigma\left(\sum_{i+j=k} a_i b_j \alpha^i\right) = \sum_{i+j=k} \tau\left(\sum_{i+j=k} a_i b_j\right) \beta^k = \sum_{i+j=k} \tau(a_i) \tau(b_j) \beta^k = \left(\sum \tau(a_i) \beta^i\right)\left(\sum \tau(b_j) \beta^j\right) = \sigma(\lambda)\sigma(\mu).
\]

(ii). \( \sigma \) is onto: \( \sigma(F(\alpha)) \) is contained in \( K(\beta) \) by (11.1). On the other hand, any \( b \in K \) is the image \( b = \tau(\alpha) \) of some \( \alpha \in F \), as \( \tau \) is onto, and \( \beta = \sigma(\alpha) \) by definition. Thus both \( \beta \) and \( K \) are in \( \sigma(F(\alpha)) \), hence \( K(\beta) \subseteq \sigma(F(\alpha)) \).

(11.8). To compute the Galois group of the extension \( \mathbb{Q} \subseteq \mathbb{Q}(\alpha) \), where \( \alpha = \sqrt[3]{2} \), any automorphism is completely determined by where it sends \( \alpha \). And we are free to send \( \alpha \) to those roots of its minimum polynomial over \( \mathbb{Q} \) that are also contained in \( \mathbb{Q}(\alpha) \). The minimum polynomial is \( x^3 - 2 \), which has roots \( \alpha, \alpha \omega \) and \( \alpha \omega^2 \) where

\[
\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i.
\]

But the roots \( \alpha \omega \) and \( \alpha \omega^2 \) are not contained in \( \mathbb{Q}(\alpha) \) as this field contains only real numbers – whereas \( \alpha \omega \) and \( \alpha \omega^2 \) are clearly non-real. Thus the only possible image for \( \alpha \) under an automorphism is \( \alpha \) itself, and Gal(\( \mathbb{Q}(\alpha)/\mathbb{Q} \)) is the trivial group \{id\}. 


Returning to the example immediately before the Extension theorem, any automorphism of \( \mathbb{Q}(\omega) \) that fixes \( \mathbb{Q} \) pointwise is determined by where it sends \( \omega \), and this must be to a root of the minimum polynomial over \( \mathbb{Q} \) of \( \omega \). As this polynomial is \( 1 + x + x^2 \) with roots \( \omega \) and \( \omega^2 \), we have automorphisms that sends \( \omega \) to itself or sends \( \omega \) to \( \omega^2 = \overline{\omega} \), i.e:

\[
\text{Gal}(\mathbb{Q}(\omega)/\mathbb{Q}) = \{ \text{id}, \sigma : z \mapsto \overline{z} \}.
\]

In particular the figure below left is an automorphism but below right is not:

(11.10). The “only if” part of the Extension Theorem is worth stating separately:

**Corollary 11.2.** Let \( F \subseteq E \) be an extension and \( g \in F[x] \) having root \( a \in E \). Then for any \( \sigma \in \text{Gal}(E/F) \), the image \( \sigma(a) \) is also a root of \( g \).

An immediate and important consequence is:

**Corollary 11.3.** If \( F \subseteq E \) is a finite extension then the Galois group \( \text{Gal}(E/F) \) is finite.

**Proof.** If \( \{ \alpha_1, \alpha_2, \ldots, \alpha_k \} \) is a basis for \( E \) over \( F \), then \( E = F(\alpha_1, \alpha_2, \ldots, \alpha_k) \), with \( \alpha_i \) algebraic over \( F \) (by Proposition 7.1) having minimum polynomial \( f_i \in F[x] \). If \( \sigma \in \text{Gal}(E/F) \) then \( \sigma \) is completely determined by the finitely many \( \sigma(\alpha_i) \), which in turn must be one of the finitely many roots of \( f_i \).

(11.11). Let \( p \) be a prime and

\[
\omega = \cos \frac{2\pi}{p} + i \sin \frac{2\pi}{p},
\]

be a root of 1.

By Corollary 11.1, \( \sigma \in \text{Gal}(\mathbb{Q}(\omega)/\mathbb{Q}) \) precisely when it sends \( \omega \) to a root, contained in \( \mathbb{Q}(\omega) \), of its minimum polynomial over \( \mathbb{Q} \). The minimum polynomial is

\[
\Phi_p = 1 + x + x^2 + \cdots + x^{p-1},
\]

(Exercise 2.14) with roots \( \omega, \omega^2, \ldots, \omega^{p-1} \). All these roots are contained in \( \mathbb{Q}(\omega) \), and so we are free to send \( \omega \) to any one of them. The Galois group thus has order \( p - 1 \), with elements

\[
\{ \sigma_1 = \text{id} : \omega \mapsto \omega, \sigma_2 : \omega \mapsto \omega^2, \ldots, \sigma_{p-1} : \omega \mapsto \omega^{p-1} \}.
\]

If \( \sigma(\omega) = \omega^k \) then \( \sigma^j(\omega) = \omega^{kj} \) (keeping \( \omega^p = 1 \) in mind).

We saw in Section 8 that the multiplicative group of the finite field \( \mathbb{F}_p \) is cyclic: there is a \( k \) with \( 1 < k < p \), such that the powers \( k^j \) of \( k \) exhaust all of the non-zero elements of \( \mathbb{F}_p^\times \), i.e: the powers \( k^j \) run through \( \{1, 2, \ldots, p - 1\} \mod p \) (or \( k \) generates \( \mathbb{F}_p^\times \)).

Putting the previous two paragraphs together, let \( \sigma \in \text{Gal}(\mathbb{Q}(\omega)/\mathbb{Q}) \) be such that \( \sigma(\omega) = \omega^k \) for \( k \) a generator of \( \mathbb{F}_p^\times \). Then the elements

\[
\{ \sigma(\omega), \sigma^2(\omega), \ldots, \sigma^{p-1}(\omega) \} = \{ \omega, \omega^2, \ldots, \omega^{p-1} \}
\]

and so the powers \( \sigma, \sigma^2, \ldots, \sigma^{p-1} \) exhaust the Galois group. \( \text{Gal}(\mathbb{Q}(\omega)/\mathbb{Q}) \) is thus a cyclic group of order \( p - 1 \).
Thus $\tau$ divides $\alpha$ polynomial of $\sigma$ isomorphism

**Proof.** Let $\tau$. Assume also that the roots of $\tau^* f$ in $E_2$ are distinct. Then the number of extensions of $\tau$ to an isomorphism $\sigma : E_1 \to E_2$ is equal to the degree of the extension $K \subseteq E_2$.

As the roots of $f$ we have $f = ph$ in $F[x]$, so $\tau^* f = (\tau^* p)(\tau^* h)$ in $K[x]$. As the roots of $\tau^* f$ are distinct, those of $\tau^* p$ must be too.

The number of possible $\sigma$ then, which is equal to the number of distinct roots of $\tau^* p$, must in fact be equal to the degree of $\tau^* p$. This in turn equals the degree $[K(\beta) : K] > 1$.

We now proceed by induction on the degree $[E_2 : K]$. If $[E_2 : K] = 1$ then $E_2 = K$. An isomorphism $\sigma : E_1 \to E_2$ extending $\tau$ gives $[E_1 : F] = 1$, hence $E_1 = F$. There can then be only one such $\sigma$, namely $\tau$ itself. By the tower law, $[E_2 : K] = [E_2 : K(\beta)][K(\beta) : K]$ where $[E_2 : K(\beta)] < [E_2 : K]$ since $[K(\beta) : K] > 1$. By induction, any isomorphism $\sigma : F(\alpha) \to K(\beta)$ will thus have

$$[E_2 : K(\beta)] = \frac{[E_2 : K]}{[K(\beta) : K]},$$

extensions to an isomorphism $E_1 \to E_2$. Starting from the bottom of the diagram, $\tau$ extends to $[K(\beta) : K]$ possible $\sigma$’s, and extending each in turn gives,

$$[K(\beta) : K]\frac{[E_2 : K]}{[K(\beta) : K]} = [E_2 : K],$$

extensions in total.

The condition that the roots of $\tau^* f$ are distinct is not essential to the theory, but makes the accounting easier: we can relate the number of automorphisms to the degrees of extensions by passing through the midway house of the roots of polynomials.

![Fig. 11.1. The Galois group $\text{Gal}(\mathbb{Q}(\omega)/\mathbb{Q})$ is cyclic for $\omega$ a primitive $p$-th root of 1.](image-url)
(11.13). Theorem D gives a connection between minimum polynomials and the degrees of field extensions, while Theorem [11.1] connects the degrees of extensions with the number of automorphisms of a field. Bolting these together:

**Corollary G.** Let $f$ be a polynomial over $F$ having distinct roots and let $E$ be its splitting field over $F$. Then

$$[\text{Gal}(E/F)] = [E : F].$$  \hfill (11.2)

The polynomial $f$ is over the field $F$, or is contained in the ring $F[x]$, with $E$ a vector space over $F$ and $\text{Gal}(E/F)$ its group of automorphisms. The formula (11.2) thus contains the main objects of undergraduate algebra.

**Proof.** By Theorem [11.1] there are $[E : F]$ extensions of the identity automorphism $F \to F$ to an automorphism of $E$. Conversely any automorphism of $E$ fixing $F$ pointwise is an extension of the identity automorphism on $F$, so we obtain the whole Galois group this way. \hfill \Box

(11.14). That $E$ be a splitting field is important in Corollary G. Consider the extension $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2})$, where $\mathbb{Q}(\sqrt{2})$ is not the splitting field over $\mathbb{Q}$ of $x^3 - 2$, or indeed any polynomial. $\sigma$ is an element of the Galois group $\text{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$ precisely when it sends $\sqrt{2}$ to a root, contained in $\mathbb{Q}(\sqrt{2})$, of its minimum polynomial over $\mathbb{Q}$. These roots are $\sqrt{2}$ itself, with the other two complex, whereas $\mathbb{Q}(\sqrt{2})$ is completely contained in $\mathbb{R}$. The only possibility for $\sigma$ is that it sends $\sqrt{2}$ to itself, ie: $\sigma = \text{id}$.

The Galois group thus has order 1, but the degree of the extension is 3.

(11.15). The following proposition returns to the kind of examples we saw in Section 0:

**Proposition 11.1.** Let $E$ be the splitting field over $F$ of a polynomial with distinct roots. Suppose also that $E = F(\alpha_1, \ldots, \alpha_m)$ for some $\alpha_1, \ldots, \alpha_m \in E$ such that

$$[E : F] = \prod_i [F(\alpha_i) : F].$$  \hfill (11.3)

Then there is a $\sigma \in \text{Gal}(E/F)$ with $\sigma(\alpha_i) = \beta_i$, if and only if $\beta_i$ is a root of the minimum polynomial of $\alpha_i$ over $F$.

**Proof.** Any $\sigma$ in the Galois group must send each $\alpha_i$ to a root of the minimum polynomial $f_i$ of $\alpha_i$ over $F$. Conversely, $\sigma$ is determined by where it sends the $\alpha_i$’s, and there are at most $\deg(f_i)$ possibilities for these images, namely the $\deg(f_i)$ roots of $f_i$. As

$$[\text{Gal}(E/F)] = [E : F] = \prod_i [F(\alpha_i) : F] = \prod_i \deg(f_i),$$

all these possibilities must arise. For any $\beta_i$ a root of $f_i$ there must then be a $\sigma \in \text{Gal}(E/F)$ with $\sigma(\alpha_i) = \beta_i$. \hfill \Box

(11.16). In Section 0 we computed, in an ad-hoc way, the automorphisms of $\mathbb{Q}(\alpha, \omega)$ where

$$\alpha = \sqrt{2} \in \mathbb{R} \text{ and } \omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i.$$ 

The minimum polynomial of $\alpha$ over $\mathbb{Q}$ is $x^3 - 2$ with roots $\alpha, \alpha \omega, \alpha \omega^2$ and the minimum polynomial of $\omega$ over $\mathbb{Q} - \alpha$ and over $\mathbb{Q}(\omega) - \alpha$ is $1 + x + x^2$ with roots $\omega, \omega^2$. By the Tower law:

$$[\mathbb{Q}(\alpha, \omega) : \mathbb{Q}] = [\mathbb{Q}(\alpha, \omega) : \mathbb{Q}(\alpha)] [\mathbb{Q}(\alpha) : \mathbb{Q}] = [\mathbb{Q}(\alpha) : \mathbb{Q}] [\mathbb{Q}(\alpha) : \mathbb{Q}].$$
By Proposition [11.1] we can send $\alpha$ to any of $\alpha, \alpha \omega, \alpha \omega^2$ and $\omega$ to any of $\omega, \omega^2$, and get an automorphism. Following this through with the vertices of the triangle gives three automorphisms with $\omega$ mapped to itself – the top three in Figure [13.1] – and another three with $\omega$ mapped to $\omega^2$ – as in the bottom three.

**Exercise 11.6.** Let $\alpha = \sqrt[5]{2}$ and $\omega = \cos(2\pi/5) + i \sin(2\pi/5)$, so that $\alpha^5 = 2$ and $\omega^5 = 1$. Let $\beta = \alpha + \omega$ and eliminate radicals by considering $(\beta - \omega)^5 = 2$ to find a polynomial of degree 20 having $\beta$ as a root. Show that this polynomial is irreducible over $\mathbb{Q}$ and hence that

$$[\mathbb{Q}(\alpha + \omega) : \mathbb{Q}] = [\mathbb{Q}(\alpha) : \mathbb{Q}][\mathbb{Q}(\omega) : \mathbb{Q}].$$

Show that $\mathbb{Q}(\alpha + \omega) = \mathbb{Q}(\alpha, \omega)$.

[(11.17)] For $\alpha = \sqrt[5]{2}$ and $\omega$ given by the expression below, the extension $\mathbb{Q} \subset \mathbb{Q}(\alpha, \omega)$ satisfies (11.3) by Exercise 11.6. An automorphism is thus free to send $\alpha$ to any root of $x^5 - 2$ and $\omega$ to any root of $1 + x + x^2 + x^3 + x^4$. This gives twenty elements of the Galois group in total; in particular there is an automorphism sending $\alpha$ to itself and $\omega$ to $\omega^3$:

$$\alpha \omega^2 \alpha \omega^3 \alpha \omega^4$$

[(11.18)]. We can get closer to the spirit of Section [0] by defining:

**Definition 11.3 (Galois group of a polynomial).** The Galois group over $F$ of the polynomial $f \in F[x]$ is the group $\text{Gal}(E/F)$ where $E$ is the splitting field of $f$ over $F$.

**Proposition 11.2.** The Galois group of a polynomial of degree $d$ is isomorphic to a subgroup of the symmetric group $S_d$. 

![Fig. 11.2. the elements of Gal(Q(α,ω)/Q) where α = $\sqrt[5]{2}$ and β = $-\frac{1}{2} + \frac{\sqrt[5]{5}}{4}i$.](image)
The possible Galois groups over $\mathbb{Q}$ of $(x-\alpha)(x-\beta)(x-\gamma)$: the subgroup lattice of the group of permutations of $\{\alpha,\beta,\gamma\}$ (aka the symmetric group $S_3$) (left) and example polynomials having Galois group these subgroups (right).

Proof. Let $\{\alpha_1, \ldots, \alpha_d\}$ be the roots of $f$ and write $\{\alpha_1, \ldots, \alpha_d\} = \{\beta_1, \ldots, \beta_k\}$ where the $\beta$'s are distinct (and $k \leq d$). An element $\sigma \in \text{Gal}(E/F)$, for $E = F(\alpha_1, \ldots, \alpha_d) = F(\beta_1, \ldots, \beta_k)$, is determined by where it sends the $\beta_i$'s, and each $\sigma(\beta_i)$ must be a root of (any) polynomial over $F$ having $\beta_i$ as a root. But $f$ is such a polynomial, hence the effect of $\sigma$ on the $\beta_i$ is to permute them among themselves ($\sigma$ is a bijection). Define a map $\text{Gal}(E/F) \to S_k$ that sends $\sigma$ to the permutation of the $\beta_i$ that it realizes. As the group laws in both the Galois group and the symmetric group are composition, this map is a homomorphism, and is injective as each $\sigma$ is determined by its effect on the roots. Thus the Galois group is isomorphic to a subgroup of $S_k$, which in turn is isomorphic to a subgroup of $S_d$ by taking those permutations of $\{1, \ldots, d\}$ that permute only the first $k$ numbers. \hfill \Box

(11.19). Let $f = (x-\alpha)(x-\beta)$ be a quadratic polynomial in $\mathbb{Q}[x]$ with distinct roots $\alpha \neq \beta \in \mathbb{C}$. Then $f$ has splitting field $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$, since $\alpha + \beta$ and $\alpha \beta$ are rational numbers. If $\alpha \in \mathbb{Q}$ (hence $\beta \in \mathbb{Q}$) then the Galois group of $f$ over $\mathbb{Q}$ is the trivial group $\{\text{id}\}$. Otherwise both $\alpha, \beta \notin \mathbb{Q}$ and $f$, being irreducible over $\mathbb{Q}$, is the minimum polynomial of $\alpha$ over $\mathbb{Q}$. There is an element of the Galois group sending $\alpha$ to $\beta$, and this must be the permutation $(\alpha, \beta)$, as it is the only element of $S_2$ that does the job. The Galois group is thus $\{\text{id}, (\alpha, \beta)\}$ when $\alpha \notin \mathbb{Q}$.

(11.20). Similarly if $f = (x-\alpha)(x-\beta)(x-\gamma)$ is a cubic in $\mathbb{Q}[x]$ with distinct roots $\alpha, \beta, \gamma \in \mathbb{C}$. By Proposition\[11.2\], the Galois group of $f$ is a subgroup of the symmetric group $S_3$, the subgroup lattice of which is shown in Figure\[11.3\] (You can come up with this picture either by brute force, or by taking the symmetry group of the equilateral triangle in Figure\[10.1\] labelling the vertices of the triangle $\alpha, \beta, \gamma$, and taking the permutations of these effected by the symmetries). We can find polynomials having each of these subgroups as Galois group.

If $\alpha, \beta, \gamma \in \mathbb{Q}$ then $f$ has splitting field $\mathbb{Q}$, and the Galois group is $\{\text{id}\}$. If $\alpha, \beta \in \mathbb{Q}$ then, as $\alpha + \beta + \gamma \in \mathbb{Q}$, we get $\gamma \in \mathbb{Q}$ too. The next case then is $\alpha \in \mathbb{Q}$ and $\beta, \gamma \notin \mathbb{Q}$, so that $(x-\beta)(x-\gamma)$ is a rational polynomial. As in [11.19], the splitting field of $f$ is $\mathbb{Q}(\beta)$ and the Galois group is $\{\text{id}, (\beta, \gamma)\}$. The other two subgroups of order two in Figure\[11.3\] come about in a similar way.

That leaves the case $\alpha, \beta, \gamma \notin \mathbb{Q}$, and where the key player is the discriminant:

$$D = (\alpha - \beta)^2(\alpha - \gamma)^2(\beta - \gamma)^2$$
or in fact, its square root. The polynomial $f$ is irreducible over $\mathbb{Q}$, hence the minimum polynomial over $\mathbb{Q}$ of $\alpha$. As the roots $\alpha, \beta, \gamma$ are distinct there are distinct elements of the Galois group sending $\alpha$ to each of $\alpha, \beta$ and $\gamma$, and so the Galois group has order 3 or 6.

Suppose that $\sqrt{D} \in \mathbb{Q}$. Then $\sqrt{D}$, like all rational numbers, is fixed by the elements of the Galois group. The permutation $(\alpha, \beta)$ however sends $\sqrt{D} \mapsto -\sqrt{D}$, and so do $(\alpha, \gamma)$ and $(\beta, \gamma)$. None of these can therefore be in the Galois group, which is thus $\{\text{id}, (\alpha, \beta, \gamma), (\alpha, \gamma, \beta)\}$.

We illustrate the final case $\sqrt{D} \notin \mathbb{Q}$ by example. Suppose that $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $\beta, \gamma \in \mathbb{C} \setminus \mathbb{R}$ – in which case $\beta, \gamma$ are complex conjugates. Then complex conjugation is a non-trivial element of the Galois group (see Exercise 11.2) having effect the permutation $(\beta, \gamma)$. The Galois group must then be all of $S_3$. (Incidentally, this and the previous paragraph show that if $\sqrt{D} \in \mathbb{Q}$ then $\alpha, \beta, \gamma \in \mathbb{R}$.)

(11.21). Finding a rational polynomial of degree $d$ that has Galois group a given subgroup of $S_d$ is possible for small values of $d$ like the cases $d = 2, 3$ above. For general $d$ it is an open problem – called the Inverse Galois problem.

Further Exercises for Section 11

Exercise 11.7. Show that the following Galois groups have the given orders:

1. $|\text{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})| = 2$.
2. $|\text{Gal}(\mathbb{Q}(\sqrt{3})/\mathbb{Q})| = 1$.
3. $|\text{Gal}(\mathbb{Q}(-\frac{1}{2} + \frac{\sqrt{3}}{2}i)/\mathbb{Q})| = 2$.
4. $|\text{Gal}(\mathbb{Q}(\sqrt{5}, -\frac{1}{2} + \frac{\sqrt{5}}{2}i)/\mathbb{Q})| = 6$.

Exercise 11.8. Find the orders of the Galois groups $\text{Gal}(L/\mathbb{Q})$ where $L$ is the splitting field of the polynomial:

1. $x - 2$  
2. $x^2 - 2$  
3. $x^5 - 2$

Exercise 11.9. Find the orders of the Galois groups $\text{Gal}(L/\mathbb{Q})$ where $L$ is the splitting field of the polynomial:

1. $1 + x + x^2 + x^3 + x^4$  
2. $1 + x^2 + x^4$  

(hint for the second one: $(x^2 - 1)(1 + x^2 + x^4) = x^6 - 1$).

Exercise 11.10. Let $p > 2$ be a prime number. Show that

1. $|\text{Gal}(\mathbb{Q}(\cos \frac{2\pi}{p} + i \sin \frac{2\pi}{p})/\mathbb{Q})| = p - 1$.
2. $|\text{Gal}(L/\mathbb{Q})| = p(p - 1)$, where $L$ is the splitting field of the polynomial $x^p - 2$. Compare the answer when $p = 3$ and 5 to Section 0.

12. Vector Spaces II: Solving Equations

This short section contains some auxiliary technical results on the solutions of homogeneous linear equations that are needed for the proof of the Galois correspondence in Section 13.
Let $V$ be a $n$-dimensional vector space over the field $F$ with fixed basis \( \{ \alpha_1, \alpha_2, \ldots, \alpha_n \} \). A homogenous linear equation over $F$ is an equation of the form,

\[
a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0,
\]

with the $a_i$ in $F$. A vector $u = \sum_{i=1}^n t_i \alpha_i \in V$ is a solution when

\[
a_1 t_1 + a_2 t_2 + \cdots + a_n t_n = 0.
\]

A system of homogeneous linear equations,

\[
\begin{align*}
  a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n &= 0, \\
  a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n &= 0, \\
  &\vdots \\
  a_{k1} x_1 + a_{k2} x_2 + \cdots + a_{kn} x_n &= 0,
\end{align*}
\]

is independent over $F$ when the vectors,

\[
v_1 = \sum a_1 j \alpha_j, v_2 = \sum a_2 j \alpha_j, \ldots, v_k = \sum a_k j \alpha_j,
\]

are independent. In other words, if $A$ is the matrix of coefficients of the system of equations, then the rows of $A$ are independent.

Here is the key property of independent systems of equations:

**Proposition 12.1.** Let $S$ be an independent system of equations over $F$ and let $S' \subset S$ be a proper subset of the equations. Then the space of solutions in $V$ to $S$ is a proper subspace of the space of solutions in $V$ to $S'$.

**Exercise 12.1.** Prove Proposition 12.1.

**Exercise 12.2.** Let $F \subseteq E$ be an extension of fields and $B$ a finite set. Let $V_F$ be the $F$-vector space with basis $B$, ie: the elements of $V_F$ are the formal sums

\[
\sum \lambda_i b_i,
\]

with the $\lambda_i \in F$ and the $b_i \in B$. Formal sums are added together and multiplied by scalars in the obvious way. Similarly let $V_E$ be the $E$-vector space with basis $B$, and identify $V_F$ with a subset (it is not a subspace) of $V_E$ in the obvious way. Now let $S' \subset S$ be independent systems of equations over $E$. Show that the space of solutions in $V_F$ to $S$ is a proper subspace of the space of solutions in $V_E$ to $S'$.

**Exercise 12.3.** Let $F$ be a field and $\alpha_1, \ldots, \alpha_{n+1} \in F$ distinct elements. Show that the matrix

\[
\begin{pmatrix}
  \alpha_1^n & \cdots & \alpha_1 & 1 \\
  \vdots & \ddots & \vdots & \vdots \\
  \alpha_{n+1}^n & \cdots & \alpha_{n+1} & 1
\end{pmatrix}
\]

has non-zero determinant (hint: suppose otherwise, and find a polynomial of degree $n$ with $n+1$ distinct roots in $F$, contradicting Theorem 2.1).

**Lemma 12.1.** Let $F$ be a field and $f, g \in F[x]$ polynomials of degree $n$ over $F$. Suppose that there exist distinct $\alpha_1, \ldots, \alpha_{n+1} \in F$ such that $f(\alpha_i) = g(\alpha_i)$ for all $i$. Then $f = g$. 

Proof. Letting \( f(x) = \sum a_i x^i \) and \( g(x) = \sum b_i x^i \) gives \( n + 1 \) expressions \( \sum a_i \alpha^j = \sum b_i \alpha^j \), hence the system of equations
\[
\sum a_j y_i = 0,
\]
where \( y_i = a_i - b_i \). The matrix of coefficients of these \( n + 1 \) equations is
\[
\begin{pmatrix}
\alpha_1 & \cdots & \alpha_1 & 1 \\
\vdots & & \vdots & \vdots \\
\alpha_{n+1} & \cdots & \alpha_{n+1} & 1
\end{pmatrix}
\]
with non-zero determinant by Exercise 12.3. The system (12.1) thus has the unique solution
\( y_i = 0 \) for all \( i \), so that \( f = g \).

(12.2) Here is the main result of the section.

**Theorem 12.1.** Let \( F \subseteq E = F(\alpha) \) be a simple extension of fields with the minimum polynomial of \( \alpha \) over \( F \) having distinct roots. Let \( \{\sigma_1, \sigma_2, \ldots, \sigma_k\} \) be distinct non-identity elements of the Galois group \( \text{Gal}(E/F) \). Then
\[
\sigma_1(x) = \sigma_2(x) = \cdots = \sigma_k(x) = x,
\]
is a system of independent linear equations over \( E \).

**Proof.** By Theorem D we have a basis \( \{1, \alpha, \alpha^2, \ldots, \alpha^d\} \) for \( E \) over \( F \) where the minimum polynomial \( f \) of \( \alpha \) over \( f \) has degree \( d + 1 \). Any \( x \in E \) thus has the form
\[
x = x_0 + x_1 \alpha + x_2 \alpha^2 + \cdots + x_d \alpha^d,
\]
for some \( x_i \in F \). By the Extension Theorem, the elements of the Galois group send \( \alpha \) to roots of \( f \). Suppose these roots are \( \{\alpha = \alpha_0, \alpha_1, \ldots, \alpha_d\} \) where \( \sigma_i(\alpha) = \alpha_i \). Then \( x \) satisfies \( \sigma_j(x) = x \) if and only if,
\[
(\alpha_0 - \alpha_j)x_1 + (\alpha_0^2 - \alpha_2^2)x_2 + \cdots + (\alpha_0^d - \alpha_d^d)x_d = 0.
\]
Thus we have a system of equations \( Ax = 0 \) where the matrix of coefficients \( A \) is made up of rows from the larger \( d \times d \) matrix \( \widetilde{A} \) given by,
\[
\widetilde{A} = \begin{pmatrix}
\alpha_0 - \alpha_1 & \alpha_0^2 - \alpha_1^2 & \cdots & \alpha_0^d - \alpha_1^d \\
\alpha_0 - \alpha_2 & \alpha_0^2 - \alpha_2^2 & \cdots & \alpha_0^d - \alpha_2^d \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_0 - \alpha_d & \alpha_0^2 - \alpha_d^2 & \cdots & \alpha_0^d - \alpha_d^d
\end{pmatrix}
\]
Let \( \widetilde{A}b = 0 \) for some vector \( b \in E^n \), so that
\[
b_0\alpha_0 + b_1\alpha_0^2 + \cdots + b_d\alpha_0^d = b_0\alpha_1 + b_1\alpha_1^2 + \cdots + b_d\alpha_1^d,
\]
for each \( 1 \leq i \leq d \). Thus if \( g = b_0x + b_1x^2 + \cdots + b_d x^d \), then we have \( g(\alpha_0) = g(\alpha_1) = g(\alpha_2) = \cdots = g(\alpha_d) = a \), say. The degree \( d \) polynomial \( g - a \) thus agrees with the zero polynomial at \( d + 1 \) distinct values, hence by Lemma 12.1 must be the zero polynomial, and so all the \( b_i \) are zero. The columns of \( \widetilde{A} \) are thus independent, hence so are the rows, and thus also the rows of \( A \).
13. The Fundamental Theorem of Galois Theory

According to Theorem E, a $z \in \mathbb{C}$ is constructible when there is a sequence of extensions:

$$\mathbb{Q} = K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n,$$

with each $[K_{i+1} : K_i] \leq 2$ and $\mathbb{Q}(z) \subset K_n$. To show that $z$ can actually be constructed, we need to find these $K_i$, and so we need to understand the fields sandwiched between $\mathbb{Q}$ and $\mathbb{Q}(z)$. In this section we prove the theorem that gives us that knowledge.

(13.1). We will need a picture of the fields sandwiched in an extension, analogous to the picture of the subgroups of a group in Section 10.

**Definition 13.1 (intermediate fields and their lattice).** Let $F \subseteq E$ be an extension. Then $K$ is an intermediate field when $K$ is an extension of $F$ and $E$ is an extension of $K$: i.e. $F \subseteq K \subseteq E$. The lattice of intermediate fields is a diagram depicting them and the inclusions between them. If $F \subseteq K_1 \subseteq K_2 \subseteq E$ they appear in the diagram like so:

$$
\begin{array}{c}
K_2 \\
| \\
K_1
\end{array}
$$

At the very base of the diagram is $F$ and at the apex is $E$. Denote the lattice by $\mathcal{L}(E/F)$.

(13.2). From now on we will work in the following situation: $F \subseteq E$ is a finite extension such that:

(†) Every irreducible polynomial over $F$ that has a root in $E$ has all its roots in $E$, and these roots are distinct.

We saw in Exercise 3.6 that if $F$ has characteristic 0 then any irreducible polynomial over $F$ has distinct roots. This is also true if $F$ is a finite field, although we omit the proof here.

**The Galois Correspondence (part 1).** Let $F \subseteq E$ be a finite extension satisfying (†) and $G = \text{Gal}(E/F)$ its Galois group. Let $\mathcal{L}(G)$ and $\mathcal{L}(E/F)$ be the subgroup and intermediate field lattices.

1. For any subgroup $H$ of $G$, let

$$E^H = \{ \lambda \in E | \sigma(\lambda) = \lambda \text{ for all } \sigma \in H \}.$$

Then $E^H$ is an intermediate field, called the fixed field of $H$.

2. For any intermediate field $K$, the group $\text{Gal}(E/K)$ is a subgroup of $G$.

3. The maps $\Psi : H \mapsto E^H$ and $\Phi : K \mapsto \text{Gal}(E/K)$ are mutual inverses, hence bijections

$$\Psi : \mathcal{L}(G) \leftrightarrow \mathcal{L}(E/F) : \Phi$$

that reverse order:

$$
H_1 \subset H_2 \quad \Psi \quad E^{H_2} \subset E^{H_1} \\
\Phi \quad K_2 \subset K_1 \quad \text{Gal}(E/K_1) \subset \text{Gal}(E/K_2)
$$

4. The degree of the extension $E^H \subseteq E$ is equal to the order $|H|$ of the subgroup $H$. Equivalently, the degree of the extension $F \subseteq E^H$ is equal to the index $[G : H]$. 
The correspondence in one sentence: turning the lattice of subgroups upside down gives the lattice of intermediate fields, and vice-versa. See Figure [13.1]

The upside down nature of the correspondence may seem puzzling, but it is just the nature of imposing conditions. If $H$ is a subgroup, the fixed field $E^H$ is the set of solutions in $E$ to the system of equations

$$\sigma(x) = x, \text{ for } \sigma \in H.$$  \hspace{1cm} (13.1)

The more equations, the greater the number of conditions being imposed on $x$, hence the smaller the number of solutions. Thus, larger subgroups $H$ should correspond to smaller intermediate fields $E^H$ and vice-versa. That the correspondence is exact – increasing the size of $H$ decreases the size of $E^H$ – will follow from Section 12 and the fact that the equations (13.1) are independent.

**Proof.** In the situation described in the Theorem the extension is of the form $F \subseteq F(\alpha)$ for some $\alpha \in E$ algebraic over $F$. The minimum polynomial $f$ of $\alpha$ over $F$ splits in $E$ by (†). On the other hand any field containing the roots of $f$ contains $F(\alpha) = E$. Thus $E$ is the splitting field of $f$.

1. **$E^H$ is an intermediate field:** we have $E^H \subseteq E$ by definition, and $F \subseteq E^H$ as every element of $G$ – so in particular every element of $H$ – fixes $F$. If $\lambda, \mu \in E^H$ then $\sigma(\lambda + \mu) = \sigma(\lambda) + \sigma(\mu) = \lambda + \mu$, so that $\lambda + \mu \in E^H$, and similarly $\lambda \mu, 1/\lambda \in E^H$.

2. **$\text{Gal}(E/K)$ is a subgroup:** if an automorphism of $E$ fixes the intermediate field $K$ pointwise, then it fixes the field $F$ pointwise, and thus $\text{Gal}(E/K) \subseteq \text{Gal}(E/F)$. If $\sigma, \tau$ are automorphisms fixing $K$ then so is $\sigma \tau^{-1}$. We thus have a subgroup.

3. **$\Phi$ and $\Psi$ reverse order:** if $\lambda$ is fixed by every automorphism in $H_2$, then it is fixed by every automorphism in $H_1$, so that $E^{H_2} \subseteq E^{H_1}$. If $\sigma$ fixes every element of $K_1$ pointwise then it fixes every element of $K_2$ pointwise, so that $\text{Gal}(E/K_1) \subseteq \text{Gal}(E/K_2)$.

4. **The composition $\Phi \Psi : H \to E^H \to \text{Gal}(E/E^H)$ is the identity:** by definition every element of $H$ fixes $E^H$ pointwise, and since $\text{Gal}(E/E^H)$ consists of all the automorphisms of $E$ that fix $E^H$ pointwise, we have $H \subseteq \text{Gal}(E/E^H)$. In fact, both $H$ and $\text{Gal}(E/E^H)$ have the same fixed field, ie: $E^{\text{Gal}(E/E^H)} = E^H$. To see this, any $\sigma \in \text{Gal}(E/E^H)$ fixes $E^H$ pointwise by definition, so $E^H \subseteq E^{\text{Gal}(E/E^H)}$. On the other hand $H \subseteq \text{Gal}(E/E^H)$ and $\Psi$ reverses order, so $E^{\text{Gal}(E/E^H)} \subseteq E^H$.

By the results of Section 12 the elements of the fixed field $E^{\text{Gal}(E/E^H)}$ are obtained by solving the system of linear equations $\sigma(x) = x$ for all $\sigma \in \text{Gal}(E/E^H)$, and these equations are independent. In particular, a proper subset of these equations has a proper superset of solutions. We already have that $H \subseteq \text{Gal}(E/E^H)$. Suppose $H$ is a proper subgroup of $\text{Gal}(E/E^H)$. The fixed field $E^H$ would then properly contain the fixed field $E^{\text{Gal}(E/E^H)}$. As this contradicts the previous paragraph, we have $H = \text{Gal}(E/E^H)$.
5. The composition $\Psi\Phi : K \to \text{Gal}(E/K) \to E^{\text{Gal}(E/K)}$ is the identity: let $E = K(\beta)$ and suppose the minimum polynomial $g$ of $\beta$ over $K$ has degree $d + 1$ with roots $\{\beta = \beta_0, \ldots, \beta_d\}$. $E$ thus has basis $\{1, \beta, \ldots, \beta^d\}$ over $K$ and $G = \text{Gal}(E/K)$ has elements $\{id = \sigma_0, \ldots, \sigma_d\}$ by Theorem G, labelled so that $\sigma_i(\beta) = \beta_i$. An element $x \in E$ has the form

$$x = x_0 + x_1\beta + \cdots + x_d\beta^d$$

with $x \in E^G$ exactly when $\sigma_i(x) = x$ for all $i$, i.e. when

$$x_1(\beta - \beta_1) + \cdots + x_d(\beta^d - \beta^d_i) = 0,$$

a homogenous system of $d$ equations in $d$ unknowns. The system has coefficients given by the matrix $A$ of Theorem [12.1] (but with $\beta$'s instead of $\alpha$'s) and hence, by the argument given there, has the unique solution $x_1 = \cdots = x_d = 0$. Thus $x = x_0 \in K$ and so $E^{\text{Gal}(E/K)} = K$.

6. As $E$ is a splitting field we can apply Theorem G to get $|\text{Gal}(E/E^H)| = [E : E^H]$, where $\text{Gal}(E/E^H) = H$ gives $|H| = [E : E^H]$. $\square$

(13.3) Before an example, a little house-keeping: the condition (†) in (13.2) can be replaced by an easier one to verify:

Proposition 13.1. Let $F \subset E$ be a finite extension such that every irreducible polynomial over $F$ has distinct roots. Then the following are equivalent:

1. Every irreducible polynomial over $F$ that has a root in $E$ has all its roots in $E$.
2. $E = F(\alpha)$ and the minimum polynomial of $\alpha$ over $F$ splits in $E$.

Proof. (1) $\Rightarrow$ (2): the minimum polynomial is irreducible over $F$ with root $\alpha \in F(\alpha) = E$, hence splits by (1).

(2) $\Rightarrow$ (1): apply the argument of part 5 of the proof of the Galois correspondence to $K = F$ to get $E^G = F$ for $G = \text{Gal}(E/F)$. Suppose that $p \in F[x]$ is irreducible over $F$ and has a root $\alpha \in E$ and let $\{\alpha = \alpha_1, \ldots, \alpha_n\}$ be the distinct elements of the set $\{\sigma(\alpha) : \sigma \in G\}$. The polynomial $g = \prod (x - \alpha_i)$ has roots permuted by the $\sigma \in G$, hence its coefficients are fixed by the $\sigma \in G$, i.e. $g$ is a polynomial over $E^G = F$. Both $p$ and $g$ have factor $x - \alpha$, hence their gcd is not 1. As $p$ is irreducible it must then divide $g$, hence all it roots lie in $E$. $\square$

(13.4) Now to our first example. In Section [11] we revisited the example of Section [9] where for $\alpha = \sqrt{2}$ and $\omega = \frac{1}{2} + \frac{\sqrt{3}}{2}$ we had

$$G = \text{Gal}(\mathbb{Q}(\alpha, \omega)/\mathbb{Q}) = \{id, \sigma, \sigma^2, \tau, \sigma\tau, \sigma^2\tau\},$$

with $\sigma(\alpha) = \alpha\omega, \sigma(\omega) = \omega$ and $\tau(\alpha) = \alpha, \tau(\omega) = \omega^2$.

In [7.21] we showed that $\mathbb{Q}(\alpha, \omega) = \mathbb{Q}(\alpha + \omega)$ with the minimum polynomial of $\alpha + \omega$ over $\mathbb{Q}$ having all its roots in $\mathbb{Q}(\alpha, \omega)$. Condition (†) thus holds. The subgroup lattice $L(G)$ is shown on the left in Figure [13.2] - adapted from Figure [11.3]. Applying the Galois Correspondence then gives the lattice $L(E/F)$ of intermediate fields on the right of Figure [13.2] with $F_4$ the fixed field of $\{id, \sigma, \sigma^2\}$ and the others the fixed fields (in no particular order) of the three order two subgroups. By part (4) of the Galois correspondence, each of the extensions $F_i \subset \mathbb{Q}(\alpha, \omega)$ has degree the order of the corresponding subgroup, so that $\mathbb{Q}(\alpha, \omega)$ is a degree three extension of $F_4$, and a degree two extension of the other intermediate fields.

Let $F_1$ be the fixed field of the subgroup $\{id, \tau\}$; we will explicitly describe its elements. The Tower law gives basis for $\mathbb{Q}(\alpha, \omega)$ over $\mathbb{Q}$ the set

$$\{1, \alpha, \alpha^2, \omega, \alpha\omega, \alpha^2\omega\},$$
Proposition 13.2. \( \text{Gal}(Q(\alpha, \omega)/Q) \) with \( \alpha = \sqrt{2} \) and \( \omega = \frac{1}{2} + \frac{\sqrt{3}i}{2} \). If \( \sigma \) fixes \( x \), then:

\[
\sigma(x) = a_0 + a_1\alpha + a_2\alpha^2 + a_3\omega + a_4\alpha\omega + a_5\alpha^2\omega,
\]

with the \( a_i \in \mathbb{Q} \). The element \( x \) is in \( F_1 \) if and only if \( \tau(x) = x \) where,

\[
\tau(x) = a_0 + a_1\alpha + a_2\alpha^2 + a_3\omega + a_4\alpha\omega + a_5\alpha^2\omega^2 = a_0 + a_1\alpha + a_2\alpha^2 + a_3(-1 - \omega) + a_4(1 - \omega) + a_5\alpha^2(1 - \omega) = (a_0 - a_3) + (a_1 - a_4)\alpha + (a_2 - a_5)\alpha^2 - a_3\omega - a_4\alpha\omega - a_5\alpha^2\omega.
\]

Equate coefficients (we are using a basis) to get:

\[
a_0 - a_3 = a_0, a_1 - a_4 = a_1, a_2 - a_5 = a_2, -a_3 = a_3, -a_4 = a_4 \text{ and } -a_5 = a_5.
\]

Thus, \( a_3 = a_4 = a_5 = 0 \) and \( a_0, a_1, a_2 \) are arbitrary. Hence

\[
x = a_0 + a_1\alpha + a_2\alpha^2
\]

so is an element of \( Q(\alpha) \). This gives \( F_1 \subseteq Q(\alpha) \). On the other hand, \( \tau \) fixes \( Q \) pointwise and fixes \( \alpha \), hence fixes \( Q(\alpha) \) pointwise, giving \( Q(\alpha) \subseteq F_1 \) and so \( F_1 = Q(\alpha) \).

The rest of the picture is described in Exercise 13.3.

(13.5). Recall that a subgroup \( N \) of a group \( G \) is normal when \( gNg^{-1} = N \) for all \( g \in G \). This extra property possessed by normal subgroups means they correspond to slightly special intermediate fields.

Let \( F \subseteq E \) be an extension with Galois group \( \text{Gal}(E/F) \). Let \( F \subseteq K \subseteq E \) be an intermediate field and \( \sigma' \in \text{Gal}(E/F) \). The image of \( K \) by \( \sigma' \) is another intermediate field, as on the left of Figure 13.3. Applying the Galois correspondence gives subgroups \( \text{Gal}(E/K) \) and \( \text{Gal}(E/\sigma(K)) \) as on the right. Then:

Proposition 13.2. \( \text{Gal}(E/\sigma(K)) = \sigma\text{Gal}(E/K)\sigma^{-1} \)

Proof: If \( x \in \sigma(K) \), then \( x = \sigma(y) \) for some \( y \in K \). If \( \tau \in \text{Gal}(E/K) \), then \( \sigma\tau\sigma^{-1}(x) = \sigma\tau(y) = \sigma(y) = x \), so that \( \sigma\tau\sigma^{-1} \in \text{Gal}(E/\sigma(K)) \), giving \( \sigma\text{Gal}(E/K)\sigma^{-1} \subseteq \text{Gal}(E/\sigma(K)) \). Replace \( \sigma \) by \( \sigma^{-1} \) to get the reverse inclusion. \( \square \)
The Galois Correspondence (part 2). Suppose we have the assumptions of the first part of the Galois correspondence. If \( K \) is an intermediate field then \( \sigma(K) = K \), for all \( \sigma \in \text{Gal}(E/F) \), if and only if \( \text{Gal}(E/K) \) is a normal subgroup of \( \text{Gal}(E/F) \). In this case,

\[
\text{Gal}(E/F)/\text{Gal}(E/K) \cong \text{Gal}(K/F).
\]

**Proof.** If \( \sigma(K) = K \) for all \( \sigma \) then by Proposition 13.2 \( \sigma \text{Gal}(E/K)\sigma^{-1} = \text{Gal}(E/\sigma(K)) = \text{Gal}(E/K) \) for all \( \sigma \), and so \( \text{Gal}(E/K) \) is normal. Conversely, if \( \text{Gal}(E/K) \) is normal then Proposition 13.2 gives \( \text{Gal}((E/\sigma(K)) = \text{Gal}(E/K) \) for all \( \sigma \), where \( X \mapsto \text{Gal}(E/X) \) is a 1-1 map by the first part of the Galois correspondence. We thus have \( \sigma(K) = K \) for all \( \sigma \).

Define a map \( \text{Gal}(E/F) \to \text{Gal}(K/F) \) by taking an automorphism \( \sigma \) of \( E \) fixing \( F \) pointwise and restricting it to \( K \). We get an automorphism of \( K \) as \( \sigma(K) = K \). The map is a homomorphism as the operation is composition in both groups. A \( \sigma \) is in the kernel if and only if it restricts to the identity map on \( K \) – that is, fixes \( K \) pointwise – when restricted, which happens if and only if \( \sigma \) is in \( \text{Gal}(E/K) \). If \( \sigma \) is an automorphism of \( K \) fixing \( F \) pointwise then by Theorem F, it can be extended to an automorphism of \( E \) fixing \( F \) pointwise. Thus any element of the Galois group \( \text{Gal}(K/F) \) can be obtained by restricting an element of \( \text{Gal}(E/F) \) and the homomorphism is onto. The isomorphism follows by the first isomorphism theorem. \( \square \)

(13.6). Here is a simple application:

**Proposition 13.3.** Let \( F \subseteq E \) be an extension satisfying the conditions of the Galois correspondence. If \( F \subseteq K \subseteq E \) with \( F \subseteq K \) an extension of degree two, then any \( \sigma \in \text{Gal}(E/F) \) sends \( K \) to itself.

Applying the Galois correspondence (part 1), the subgroup \( \text{Gal}(E/K) \) has index two in \( \text{Gal}(E/F) \), hence is normal by Exercise 10.14 Now apply the Galois correspondence (part 2).

**Further Exercises for Section 13**

*In all these exercises, you can assume that the condition (\( \dagger \)) of (13.2) holds.*

**Exercise 13.1.** Let \( \alpha = \sqrt{2} \in \mathbb{R} \) and \( i \in \mathbb{C} \), and consider the field \( \mathbb{Q}(\alpha, i) \subseteq \mathbb{C} \).

1. Show that there are automorphisms \( \sigma, \tau \) of \( \mathbb{Q}(\alpha, i) \) such that

\[
\sigma(i) = i, \sigma(\alpha) = \alpha i, \tau(i) = -i, \text{ and } \tau(\alpha) = \alpha.
\]

Show that

\[
G = \{ \text{id}, \sigma, \sigma^2, \sigma^3, \tau, \sigma \tau, \sigma^2 \tau, \sigma^3 \tau \},
\]

are then distinct automorphisms of \( \mathbb{Q}(\alpha, i) \). Show that \( \tau \sigma = \sigma^3 \tau \).

2. Show that \( \text{Gal}(\mathbb{Q}(\alpha, i)/\mathbb{Q}) = G \) and that the lattice \( \mathcal{L}(G) \) is as on the left of Figure 13.4

3. Find the subgroups \( H_1, H_2 \) and \( H_3 \) of \( G \). If the corresponding lattice of subfields is as shown on the right, then express the fields \( F_1 \) and \( F_2 \) in the form \( \mathbb{Q}(\beta_1, \ldots, \beta_n) \) for \( \beta_1, \ldots, \beta_n \in \mathbb{C} \).
Exercise 13.2. Let \( \omega = \cos \frac{2\pi}{7} + i \sin \frac{2\pi}{7} \in \mathbb{C} \).

1. Show that \( \mathbb{Q}(\omega) \) is the splitting field of the polynomial
   \[ 1 + x + x^2 + x^3 + x^4 + x^5 + x^6. \]
   and deduce that \( |\text{Gal}(\mathbb{Q}(\omega)/\mathbb{Q})| = 6 \). Let \( \sigma \in \text{Gal}(\mathbb{Q}(\omega)/\mathbb{Q}) \) be such that \( \sigma(\omega) = \omega^3 \). Show that,
   \[ \text{Gal}(\mathbb{Q}(\omega)/\mathbb{Q}) = \{\text{id}, \sigma, \sigma^2, \sigma^3, \sigma^4, \sigma^5\}. \]

2. Using the Galois correspondence, show that the lattice of intermediate fields is as shown on the right, where \( F_1 \) is a degree 2 extension of \( \mathbb{Q} \) and \( F_2 \) a degree 3 extension. Find complex numbers \( \beta_1, \ldots, \beta_n \) such that \( F_2 = \mathbb{Q}(\beta_1, \ldots, \beta_n) \).

Exercise 13.3. Complete the lattice of intermediate fields from the example in (13.4).

Exercise 13.4. Let \( \alpha = \sqrt[4]{2} \) and \( \omega = \frac{1}{2} + \frac{\sqrt{3}}{2} i \) and consider the field extension \( \mathbb{Q} \subset \mathbb{Q}(\alpha, \omega) \).

1. Find a basis for \( \mathbb{Q}(\alpha, \omega) \) over \( \mathbb{Q} \) and show that \( |\text{Gal}(\mathbb{Q}(\alpha, \omega)/\mathbb{Q})| = 24 \).
2. Let $\sigma, \tau \in \text{Gal}(\mathbb{Q}(\alpha, \omega)/\mathbb{Q})$ be such that $\tau(\alpha) = \alpha, \tau(\omega) = \omega^5$ and $\sigma(\alpha) = \alpha \omega, \sigma(\omega) = \omega$. Show that $H = \{\text{id}, \sigma, \sigma^2, \sigma^3, \sigma^4, \sigma^5, \tau, \tau \sigma, \tau \sigma^2, \tau \sigma^3, \tau \sigma^4, \tau \sigma^5\}$, are then distinct elements in $\text{Gal}(\mathbb{Q}(\alpha, \omega)/\mathbb{Q})$.

3. Part of the subgroup lattice $L(G)$ is shown on the left of Figure 13.6. Fill in the corresponding part of the lattice of intermediate fields on the right.

Exercise 13.5. Let $\omega = \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}$.

1. Show that $\mathbb{Q}(\omega)$ is the splitting field of the polynomial $1 + x + x^2 + x^3 + x^4$ and deduce that $|\text{Gal}(\mathbb{Q}(\omega)/\mathbb{Q})| = 4$.

2. Let $\sigma \in \text{Gal}(\mathbb{Q}(\omega)/\mathbb{Q})$ be such that $\sigma(\omega) = \omega^2$. Show that $\text{Gal}(\mathbb{Q}(\omega)/\mathbb{Q}) = \{\text{id}, \sigma, \sigma^2, \sigma^3\}$. Find the subgroup lattice $L(G)$ for $G = \text{Gal}(\mathbb{Q}(\omega)/\mathbb{Q})$.

3. Using the Galois correspondence, deduce that the lattice of intermediate fields is as shown on the right. Find a complex number $\beta$ such that $F = \mathbb{Q}(\beta)$.

Exercise 13.6. Consider the polynomial $f(x) = (x^2 - 2)(x^2 - 5) \in \mathbb{Q}[x]$.

1. Show that $\mathbb{Q}(\sqrt{2}, \sqrt{5})$ is the splitting field of $f$ over $\mathbb{Q}$ and that the Galois group $\text{Gal}(\mathbb{Q}(\sqrt{2}, \sqrt{5})/\mathbb{Q})$ has order four. (You can assume that if $a, b, c \in \mathbb{Q}$ satisfy $a\sqrt{2} + b\sqrt{5} + c = 0$ then $a = b = c = 0$.)

2. Show that there are automorphisms $\sigma, \tau$ of $\mathbb{Q}(\sqrt{2}, \sqrt{5})$ defined by $\sigma(\sqrt{2}) = -\sqrt{2}, \sigma(\sqrt{5}) = \sqrt{5}$ and $\tau(\sqrt{2}) = \sqrt{2}, \tau(\sqrt{5}) = -\sqrt{5}$. List the elements of the Galois group $\text{Gal}(\mathbb{Q}(\sqrt{2}, \sqrt{5})/\mathbb{Q})$.

3. Complete the subgroup lattice on the left of Figure 14.1 by listing the elements of $H$, and use your answer to write the field $F$ in the form $\mathbb{Q}(\theta)$ for some $\theta \in \mathbb{C}$.

14. Applications of the Galois Correspondence

14.1. Constructing polygons

If $p$ is a prime number, then a regular $p$-gon can be constructed only if $p$ is a Fermat prime of the form $2^{2^l} + 1$. 

![Fig. 13.6. Exercise 13.4. $\alpha = \sqrt{2}$ and $\omega = \frac{1}{2} + \frac{i}{2}$]
This negative result was proved in Section 9 and required only the degrees of extensions. We didn’t need any symmetries of fields.

Galois theory proper — the interplay between fields and their Galois groups — allows us to prove positive results:

**Theorem 14.1.** If \( p = 2^r+1 \) is a Fermat prime then a regular \( p \)-gon can be constructed.

*Proof.* By Theorem E we need a tower of fields,

\[
\mathbb{Q} \subset K_1 \subset \cdots \subset K_n = \mathbb{Q}(\zeta),
\]

where \( \zeta = \cos(2\pi/p) + i\sin(2\pi/p) \) and \( [K_i : K_{i-1}] = 2 \). We will get the tower by analysing the Galois group \( \text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}) \) and applying the Galois correspondence. As \( \mathbb{Q}(\zeta) \) is the splitting field over \( \mathbb{Q} \) of the \( p \)-th cyclotomic polynomial

\[
\Phi_p(x) = x^{p-1} + x^{p-2} + \cdots + x + 1,
\]

we have by Theorem G:

\[
|\text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})| = [\mathbb{Q}(\zeta) : \mathbb{Q}] = \deg \Phi_p = p - 1 = 2^r = 2^n.
\]

The roots of \( \Phi \) are the powers \( \zeta^k \), and these all lie in \( \mathbb{Q}(\zeta) \). We can thus apply the Galois correspondence by Proposition 13.1. In Section 11 we showed that \( \text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}) \) is a cyclic group, and so by Exercise 10.6, there is a chain of subgroups

\[
\{\text{id}\} = H_0 \subset H_1 \subset \cdots \subset H_n = \text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}),
\]

where the subgroup \( H_i \) has order \( 2^i \). Explicitly, if \( \text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}) = \{g, g^2, \ldots, g^{2^{n-1}}, g^{2^n} = \text{id}\} \) then

\[
\{\text{id}\} \subset \{h_1, h_1^2 = \text{id}\} \subset \{h_2, h_2^2, h_2^3, h_2^4 = \text{id}\} \subset \cdots \subset \{h_{n-1}, h_{n-1}^2, \ldots, h_{n-1}^{2^{n-1}} = \text{id}\} \subset \text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})
\]

where \( h_i = g^{2^{n-i}} \) and \( H_i \) is the subgroup generated by \( h_i \). The Galois correspondence thus gives a chain of fields,

\[
\mathbb{Q} = K_0 \subset K_1 \subset \cdots \subset K_n = \mathbb{Q}(\zeta),
\]

where \( K_{n-i} \) is the fixed field \( E^{H_i} \) of the subgroup \( H_i \). We have \( 2^i = [G : H_{n-i}] = [K_i : \mathbb{Q}] \), so by the tower law

\[
2^i = [K_i : \mathbb{Q}] = [K_i : K_{i-1}][K_{i-1} : \mathbb{Q}] = [K_i : K_{i-1}]2^{i-1}
\]

and hence \([K_i : K_{i-1}] = 2\) as desired. \( \square \)

Theorem 14.1 and (9.4) then give:
Corollary 14.1. If \( p \) is a prime then a \( p \)-gon can be constructed if and only if \( p = 2^{2^t} + 1 \) is a Fermat prime.

Corollary 14.2. If \( n = 2^k p_1 p_2 \ldots p_m \) with the \( p_i \) Fermat primes, then a regular \( n \)-gon can be constructed.

**Proof.** A \( 2^k \)-gon can be constructed by repeatedly bisecting angles, and thus an \( n \)-gon, where \( n \) has the form given, by Exercise 6.4. □

A little more Galois Theory, which we omit, gives the following complete answer to what \( n \)-gons can be constructed:

Theorem 14.2. An \( n \)-gon can be constructed if and only if \( n = 2^k p_1 p_2 \ldots p_m \) with the \( p_i \) Fermat primes.

(14.1). The angle \( \pi/n \) can be constructed precisely when the angle \( 2\pi/n \) can be constructed which in turns happens precisely when the regular \( n \)-gon can be constructed. Thus, the list of submultiples of \( \pi \) that are constructible runs as,

\[
\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{5}, \frac{\pi}{6}, \frac{\pi}{8}, \frac{\pi}{10}, \frac{\pi}{12}, \frac{\pi}{15}, \ldots
\]

**Exercise 14.1.** Give direct proofs of the non-constructability of the angles, \( \frac{\pi}{7}, \frac{\pi}{9}, \frac{\pi}{11} \) and \( \frac{\pi}{13} \).

### 14.2. The Fundamental Theorem of Algebra

We saw this in Section [2]. We now prove it using the Galois correspondence, starting with two observations:

(i). There are no extensions of \( \mathbb{R} \) of odd degree \( > 1 \). Any polynomial in \( \mathbb{R}[x] \) has roots that are either real or occur in complex conjugate pairs, hence a real polynomial with odd degree \( > 1 \) has a real root and is reducible over \( \mathbb{R} \). Thus, the minimum polynomial over \( \mathbb{R} \) of any \( \alpha \notin \mathbb{R} \) must have even degree so that the degree \([\mathbb{R}(\alpha) : \mathbb{R}]\) is even. If \( \mathbb{R} \subset L \) is an extension, then for \( \alpha \in L \setminus \mathbb{R} \), we have

\[
[L : \mathbb{R}] = [L : \mathbb{R}(\alpha)][\mathbb{R}(\alpha) : \mathbb{R}],
\]

is also even.

(ii). There is no extension of \( \mathbb{C} \) of degree two. For if \( \mathbb{C} \subset L \) with \([L : \mathbb{C}] = 2\) then an \( \alpha \in L \setminus \mathbb{C} \) gives the intermediate \( \mathbb{C} \subset \mathbb{C}(\alpha) \subset L \) with \([\mathbb{C}(\alpha) : \mathbb{C}] = 1 \) or 2 by the Tower law. If this degree equals 1 then \( \alpha \in \mathbb{C} \); thus \([\mathbb{C}(\alpha) : \mathbb{C}] = 2\), and hence \( L = \mathbb{C}(\alpha) \). If \( f \) is the minimum polynomial of \( \alpha \) over \( \mathbb{C} \) then

\[
f = x^2 + bx + c \quad \text{for} \quad b, c \in \mathbb{C} \quad \text{with} \quad \alpha \quad \text{one of the two roots}
\]

\[
\frac{-b \pm \sqrt{b^2 - 4c}}{2}.
\]

But these are both in \( \mathbb{C} \), contradicting the choice of \( \alpha \).

**Fundamental Theorem of Algebra.** Any non-constant \( f \in \mathbb{C}[x] \) has a root in \( \mathbb{C} \).
Proof. The proof toggles back and forth between intermediate fields and subgroups of Galois groups using the Galois correspondence. All the fields and groups appear in Figure [14.2]. If \( f = pq \) is reducible over \( \mathbb{R} \), then replace \( f \) in what follows by \( p \). Thus we may assume that \( f \) is irreducible over \( \mathbb{R} \) and let \( E \) be the splitting field over \( \mathbb{R} \), not of \( f \), but of \((x^2 + 1)f). \) We have \( \mathbb{R} \) and \( \pm i \) are in \( E \), hence \( \mathbb{C} \) is too, giving the series of extensions \( \mathbb{R} \subset \mathbb{C} \subseteq E \).

Since \( G = \text{Gal}(E/\mathbb{R}) \) is a finite group, we can factor from its order all the powers of 2, writing \( |G| = 2^km \), where \( m \geq 1 \) is odd. Sylow’s Theorem then gives a subgroup \( H \) of \( G \) of order \( 2^k \), and the Galois correspondence gives the intermediate field \( F = E^H \) with the extension \( F \subset E \) of degree \( 2^k \). As \([E : \mathbb{R}] = [E : F][F : \mathbb{R}] \) with \([E : \mathbb{R}] = |G| = 2^km \), we have that \( F \) is a degree \( m \) extension of \( \mathbb{R} \). As \( m \) is odd and no such extensions exist if \( m > 1 \), we must have \( m = 1 \), so that \( |G| = 2^k \).

Using the Galois correspondence in the reverse direction, the subgroup \( \text{Gal}(E/\mathbb{C}) \) has order dividing \( |G| = 2^k \), hence order \( 2^s \) for some \( 0 \leq s \leq k \). If \( s > 0 \) then there is a non-trivial subgroup \( K \) of \( \text{Gal}(E/\mathbb{C}) \) of order \( 2^{s-1} \), with \( 2^{s-1}[E^H : \mathbb{C}] = [E : \mathbb{C}] = |\text{Gal}(E/\mathbb{C})| = 2^s \). Thus, \( E^H \) is a degree 2 extension of \( \mathbb{C} \), a contradiction to the second observation above. We thus have \( s = 0 \), hence \( |\text{Gal}(E/\mathbb{C})| = 1 \). We now have two fields, \( E \) and \( \mathbb{C} \), that map via the 1-1 map \( X \mapsto \text{Gal}(E/X) \) to the trivial group. The conclusion is that \( E = \mathbb{C} \). As \( E \) is the splitting field of the polynomial \((x^2 + 1)f \), we get that \( f \) has a root (indeed all its roots) in \( \mathbb{C} \).

15. (Not) Solving Equations

We can finally return to the theme of Section [0] finding algebraic expressions for the roots of polynomials.

(15.1). The formulae for the roots of quadratics, cubics and quartics express the roots in terms of the coefficients, the four field operations \(+, -, \times, \div\) and \( n \)-th roots \( \sqrt[n]{\cdot} \). These roots thus lie in an extension of \( \mathbb{Q} \) obtained by adjoining certain \( n \)-th roots.

**Definition 15.1** (radical extension of \( \mathbb{Q} \)). An extension \( \mathbb{Q} \subset E \) is radical when there is a sequence of simple extensions,

\[
\mathbb{Q} \subset \mathbb{Q}(\alpha_1) \subset \mathbb{Q}(\alpha_1, \alpha_2) \subset \cdots \subset \mathbb{Q}(\alpha_1, \alpha_2, \ldots, \alpha_k) = E,
\]

with some power \( \alpha_i^{m_i} \) of \( \alpha_i \) contained in \( \mathbb{Q}(\alpha_1, \alpha_2, \ldots, \alpha_{i-1}) \) for each \( i \).
Each extension in the sequence is thus obtained by adjoining to the previous field in the sequence, the $m_i$-th root of some element.

A simple example:

$$
\mathbb{Q} \subset \mathbb{Q}(\sqrt{2}) \subset \mathbb{Q}(\sqrt{2}, \sqrt{5}) \subset \mathbb{Q}(\sqrt{2}, \sqrt{5}, \sqrt{\sqrt{2} - 7\sqrt{5}}).
$$

By repeatedly applying Theorem D, the elements of a radical extension are seen to have expressions in terms of rational numbers, $+, -, \times, \div$ and \sqrt[n]{} for various $n$.

**Definition 15.2 (polynomial solvable by radicals).** A polynomial $f \in \mathbb{Q}[x]$ is solvable by radicals when its splitting field over $\mathbb{Q}$ is contained in some radical extension.

Notice that we are dealing with a fixed specific polynomial, and not an arbitrary one. The radical extension containing the splitting field will depend on the polynomial.

**15.2.** Any quadratic polynomial $ax^2 + bx + c$ is solvable by radicals, with splitting field in the radical extension

$$
\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{b^2 - 4ac}).
$$

Similarly, the formulae for the roots of cubics and quartics give for any specific such polynomial, radical extensions containing their splitting fields.

**15.3.** Recalling the definition of soluble group given in Section [10], Theorem H (Galois). A polynomial $f \in \mathbb{Q}[x]$ is solvable by radicals if and only if its Galois group over $\mathbb{Q}$ is soluble.

The proof, which we omit, uses the full power of the Galois correspondence, with the sequence of extensions in a radical extension corresponding to the sequence of subgroups

$$
\{1\} = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_{n-1} \triangleleft H_n = G,
$$

in a soluble group.

**15.4.** As a small reality check of Theorem H, we saw in Section [11] that the Galois group over $\mathbb{Q}$ of a quadratic polynomial is either the trivial group $\{\text{id}\}$ or the (Abelian) permutation group $\{\text{id}, (\alpha, \beta)\}$ where $\alpha, \beta \in \mathbb{C}$ are the roots. Abelian groups are soluble – see [10.8] – and this syncs with quadratics being solvable by radicals via the quadratic formula.

Similarly, the possible Galois groups of cubic polynomials are shown in Figure [11.3]. Apart from $S_3$, these are also Abelian. But $S_3$ is the symmetry group of an equilateral triangle lying in the plane – soluble by [10.9].

**15.5.** Somewhat out of chronological order, we have:

**Theorem 15.1 (Abels-Fubini).** The polynomial $f = x^5 - 4x + 2$ is not solvable by radicals.

The roots of $x^5 - 4x + 2$ are algebraic numbers, yet there is no algebraic expression for them.

**Proof.** We show that the Galois group of $f$ over $\mathbb{Q}$ is insoluble. Indeed, we show that the Galois group is the symmetric group $S_5$, which contains the non-Abelian, finite simple group $A_5$. Thus $S_5$ contains an insoluble subgroup, hence is insoluble, as any subgroup of a soluble group is soluble by Exercises [10.8] and [10.9].

If $E$ is the splitting field over $\mathbb{Q}$ of $f$, then

$$
E = \mathbb{Q}(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5),
$$

Brent Everitt
The Galois groups of the quintic polynomials $x^5 + ax + b$ for $-40 \leq a, b \leq 40$ (re-drawn from the *Mathematica* poster, “Solving the Quintic”).

where the $\alpha_i \in \mathbb{C}$ are the roots of $f$ and the Galois group is $\text{Gal}(E/\mathbb{Q})$, itself a subgroup of the group of permutations of $\{\alpha_1, \ldots, \alpha_5\}$ – which is $\cong S_5$.

The polynomial $f$ is irreducible over $\mathbb{Q}$ by Eisenstein, hence is the minimum polynomial of $\alpha_1$ over $\mathbb{Q}$. The extension $\mathbb{Q} \subset \mathbb{Q}(\alpha_1)$ thus has degree five, and the Tower law gives

$$[E : \mathbb{Q}] = [E : \mathbb{Q}(\alpha_1)][\mathbb{Q}(\alpha_1) : \mathbb{Q}].$$

The degree of the extension $\mathbb{Q} \subset E$ is therefore divisible by the degree of the extension $\mathbb{Q} \subset \mathbb{Q}(\alpha_1)$, i.e.: divisible by five. Moreover, by Theorem G, the group $\text{Gal}(E/\mathbb{Q})$ has order the degree $[E : \mathbb{Q}]$, and so the group has order divisible by five. By Cauchy’s Theorem, the Galois group contains an element $\sigma$ of order 5, and a subgroup

$$\{\text{id}, \sigma, \sigma^2, \sigma^3, \sigma^4\},$$

where the permutation $\sigma$ is a 5-cycle $\sigma = (a, b, c, d, e)$ when considered as a permutation of the roots. The graph of $f$ on the right shows that three of the roots are real, and the other two are thus complex conjugates. By Exercise 11.2, complex conjugation is an element of the Galois group having effect the permutation

$$\tau = (b_1, b_2),$$

where $b_1, b_2$ are the two complex roots. But in Section 10 we saw that $S_n$ is generated by a $n$-cycle and a transposition, hence the Galois group is $S_5$ as claimed. \qed
There is nothing particularly special about the polynomial $x^5 - 4x + 2$; among the polynomials having degree $\geq 5$, those that are not solvable by radicals are generic. We illustrate what we mean with some experimental evidence: consider the quintic polynomials

$$x^5 + ax + b,$$

for $a, b \in \mathbb{Z}$ with $-40 \leq a, b \leq 40$.

Figure 15.1 (which is re-drawn from the Mathematica poster, “Solving the Quintic”) shows the $(a, b)$ plane for $a$ and $b$ in this range. The vertical line through $(0, 0)$ corresponds to $f$ with Galois group the soluble dihedral group $D_{10}$ of order 10. The horizontal line through $(0, 0)$ and the two sets of crossing diagonal lines correspond to reducible $f$, as do a few other isolated points. The (insoluble) alternating group $A_5$ arises in a few sporadic places, as does another soluble subgroup of $S_5$. The vast majority of $f$ however, forming the light background, have Galois group the symmetric group $S_5$, and so have roots that are algebraic, but cannot be expressed algebraically.