A NOTE ON 4-RANK DENSITIES
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Abstract. For certain real quadratic number fields, we prove density results concerning 4-ranks of tame kernels. We also discuss a relationship between 4-ranks of tame kernels and 4-class ranks of narrow ideal class groups. Additionally, we give a product formula for a local Hilbert symbol.

1. Introduction

Let $F$ be a real quadratic number field and $\mathcal{O}_F$ its ring of integers. In [4], the authors gave an algorithm for computing the 4-rank of the tame kernel $K_2(\mathcal{O}_F)$. The idea of the algorithm is to consider matrices with Hilbert symbols as entries and compute matrix ranks over $\mathbb{F}_2$. Recently, the author used these matrices to obtain “density results” concerning the 4-rank of tame kernels, see [6], [7].

In this note, we consider the 4-rank of $K_2(\mathcal{O})$ for the real quadratic number fields $\mathbb{Q}(\sqrt{p_1p_2p_3})$ for primes $p_1 \equiv p_2 \equiv p_3 \equiv 1 \mod 8$. We will see that

$$4\text{-rank } K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{p_1p_2p_3})}) = 0, 1, 2, \text{ or } 3.$$

For squarefree, odd integers $d$, consider the set $X = \{d : d = p_1p_2p_3, p_i \equiv 1 \mod 8\}$ for distinct primes $p_i$.

Using GP/PARI [1], we computed the following: For $50881 \leq d < 2 \times 10^7$, there are 7257 d’s in $X$. Among them, there are 2121 d’s (29.23%) yielding 4-rank 0, 3977 d’s (54.80%) yielding 4-rank 1, 1086 d’s (14.96%) yielding 4-rank 2, and 73 d’s (1.01%) yielding 4-rank 3. In fact, we prove

**Theorem 1.1.** For the fields $\mathbb{Q}(\sqrt{p_1p_2p_3})$, 4-rank 0, 1, 2, and 3 appear with natural density $\frac{1}{4}, \frac{17}{32}, \frac{13}{64}$, and $\frac{1}{64}$ respectively in $X$.

In the appendix we point out a beautiful result which may not be well known. It is a product formula from [11] for a certain local Hilbert symbol. This product formula both simplifies numerical computations and is a generalization of Propositions 4.6 and 4.4 in [2] and [7], respectively.

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2. Matrices

Hurrelbrink and Kolster \cite{4} generalize Qin’s approach in \cite{8}, \cite{9} and obtain 4-rank results by computing $\mathbb{F}_2$-ranks of certain matrices of local Hilbert symbols. Specifically, let $F = \mathbb{Q}(\sqrt{d})$, $d > 1$ and squarefree. Let $p_1, p_2, \ldots, p_t$ denote the odd primes dividing $d$. Recall $2$ is a norm from $F$ if and only if all $p_i$’s are $\equiv \pm 1 \mod 8$. If so, then $d$ is a norm from $\mathbb{Q}(\sqrt{2})$, thus

$$d = u^2 - 2w^2$$

for $u, w \in \mathbb{Z}$. Now consider the matrix:

$$M_{F/Q} = \begin{pmatrix}
(-d, p_1)_2 & (-d, p_1)_{p_1} & \ldots & (-d, p_1)_{p_t} \\
(-d, p_2)_2 & (-d, p_2)_{p_1} & \ldots & (-d, p_2)_{p_t} \\
\vdots & \vdots & \ddots & \vdots \\
(-d, p_{t-1})_2 & (-d, p_{t-1})_{p_1} & \ldots & (-d, p_{t-1})_{p_t} \\
(-d, v)_2 & (-d, v)_{p_1} & \ldots & (-d, v)_{p_t} \\
(d, -1)_2 & (d, -1)_{p_1} & \ldots & (d, -1)_{p_t}
\end{pmatrix}.$$

If $2$ is not a norm from $F$, set $v = 2$. Otherwise, set $v = u + w$. Replacing the $1$’s by $0$’s and the $-1$’s by $1$’s, we calculate the matrix rank over $\mathbb{F}_2$.

From \cite{4},

**Lemma 2.1.** Let $F = \mathbb{Q}(\sqrt{d})$, $d > 0$ and squarefree. Then

$$4\text{-rank } K_2(O_F) = t - rk (M_{F/Q}) + a' - a$$

where

$$a = \begin{cases}
0 & \text{if } 2 \text{ is a norm from } F \\
1 & \text{otherwise}
\end{cases}$$

and

$$a' = \begin{cases}
0 & \text{if both } -1 \text{ and } 2 \text{ are norms from } F \\
1 & \text{if exactly one of } -1 \text{ or } 2 \text{ is a norm from } F \\
2 & \text{if none of } -1 \text{ or } 2 \text{ are norms from } F.
\end{cases}$$

Recall that our case is $\mathbb{Q}(\sqrt{p_1p_2p_3})$ for primes $p_1 \equiv p_2 \equiv p_3 \equiv 1 \mod 8$. In this case $a = a'$ and we may delete the last row of $M_{F/Q}$ without changing its rank (see discussions preceding Proposition 5.13 and Lemma 5.14 in \cite{4}). Also note that $v$ is an $p_1$-adic unit and hence

$$(-p_1p_2p_3, v)_{p_1} = (p_1, v)_{p_1} = \left(\frac{v}{p_1}\right).$$

Similarly, $(-p_1p_2p_3, v)_{p_2} = \left(\frac{v}{p_2}\right)$ and $(-p_1p_2p_3, v)_{p_3} = \left(\frac{v}{p_3}\right)$. From Lemma 2.1 we have
4-rank $K_2(\mathcal{O}_F) = 3 - \text{rk} (M_{F/Q})$
and the matrix $M_{F/Q}$ is of the form

$$
\begin{pmatrix}
1 & \left( \frac{p_2}{p_1} \right) & \left( \frac{p_3}{p_1} \right) & \left( \frac{p_1}{p_2} \right) & \left( \frac{p_4}{p_3} \right) \\
1 & \left( \frac{p_2}{p_1} \right) & \left( \frac{p_3}{p_2} \right) & \left( \frac{p_4}{p_3} \right) \\
(-d, u + w)_2 & \left( \frac{u}{p_1} \right) & \left( \frac{w}{p_2} \right) & \left( \frac{v}{p_3} \right)
\end{pmatrix}.
$$

Let us now prove Theorem 1.1.

**Proof.** The idea in [6] and [7] is to first consider an appropriate normal extension $N$ of $\mathbb{Q}$ and then relate the splitting of the primes $p_i$ in $N$ to their representation by certain quadratic forms. The next step is classifying 4-rank values in terms of values of the symbols $(-d, v)_2$, $\left( \frac{u}{p_i} \right)$. The values of these symbols are then characterized in terms of $p_i$ satisfying the alluded to quadratic forms. We then associate Artin symbols to the primes $p_i$ and apply the Chebotarev density theorem. In what follows, we classify the 4-rank values in terms of the symbols $(-d, v)_2$, $\left( \frac{u}{p_i} \right)$ and in parenthesis give the relevant densities in $X$ obtained by using the above machinery. Let us consider the following four cases (see Table III in [9]).

Case 1: Suppose $\left( \frac{p_2}{p_1} \right) = \left( \frac{p_4}{p_3} \right) = \left( \frac{p_4}{p_2} \right) = 1$. Then we immediately have that

- 4-rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{p_1p_2p_3})}) = 3 \iff \text{rank} (M_{F/Q}) = 0 \iff (-d, v)_2 = 1$ and $\left( \frac{u}{p_1} \right) = \left( \frac{w}{p_2} \right) = \left( \frac{v}{p_3} \right) = 1 \left( \frac{1}{64} \right)$.

- 4-rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{p_1p_2p_3})}) = 2 \iff \text{rank} (M_{F/Q}) = 1 \iff (-d, v)_2 = -1$ or $(-d, v)_2 = 1$ and $\left( \frac{u}{p_1} \right) = \left( \frac{w}{p_2} \right) = -1$ and $\left( \frac{v}{p_3} \right) = 1$ or $(-d, v)_2 = 1$ and $\left( \frac{u}{p_1} \right) = \left( \frac{w}{p_3} \right) = -1$ and $\left( \frac{v}{p_2} \right) = 1$ or $(-d, v)_2 = 1$ and $\left( \frac{u}{p_3} \right) = -1$ and $\left( \frac{v}{p_1} \right) = 1 \left( \frac{7}{64} \right)$.

Case 2: Suppose $\left( \frac{p_2}{p_1} \right) = \left( \frac{p_3}{p_1} \right) = 1$. Thus

- 4-rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{p_1p_2p_3})}) = 2 \iff \text{rank} (M_{F/Q}) = 1 \iff (-d, v)_2 = 1$ and $\left( \frac{u}{p_1} \right) = \left( \frac{w}{p_2} \right) = 1$ or $(-d, v)_2 = 1$ and $\left( \frac{v}{p_3} \right) = \left( \frac{u}{p_2} \right) = -1 \left( \frac{3}{32} \right)$.

- 4-rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{p_1p_2p_3})}) = 1 \iff \text{rank} (M_{F/Q}) = 2 \iff (-d, v)_2 = -1$ or $(-d, v)_2 = 1$ and $\left( \frac{u}{p_1} \right) = \left( \frac{w}{p_3} \right) = -1$ and $\left( \frac{v}{p_2} \right) = 1$ or
\((-d, v)_2 = 1\) and \(\left(\frac{v}{p_2}\right) = \left(\frac{v}{p_3}\right) = -1\) and \(\left(\frac{v}{p_1}\right) = 1 \left(\frac{3}{32}\right).

Case 3: Suppose \(\left(\frac{p_2}{p_1}\right) = \left(\frac{p_3}{p_1}\right) = -1, \left(\frac{p_3}{p_2}\right) = 1.\) Thus

- 4-rank \(K_2(O_{\sqrt{d}p_1p_2p_3}) = 1 \iff \text{rank} (M_{F/Q}) = 2 \iff (-d, v)_2 = 1 \left(\frac{3}{16}\right).

- 4-rank \(K_2(O_{\sqrt{d}p_1p_2p_3}) = 0 \iff \text{rank} (M_{F/Q}) = 3 \iff (-d, v)_2 = -1 \left(\frac{3}{16}\right).

Case 4: Suppose \(\left(\frac{p_2}{p_1}\right) = \left(\frac{p_3}{p_1}\right) = \left(\frac{p_3}{p_2}\right) = -1.\) Then

- 4-rank \(K_2(O_{\sqrt{d}p_1p_2p_3}) = 1 \iff \text{rank} (M_{F/Q}) = 2 \iff (-d, v)_2 = 1 \left(\frac{3}{16}\right).

- 4-rank \(K_2(O_{\sqrt{d}p_1p_2p_3}) = 0 \iff \text{rank} (M_{F/Q}) = 3 \iff (-d, v)_2 = -1 \left(\frac{3}{16}\right).

Thus 4-rank 0, 1, 2, and 3 occur with natural density \(\frac{1}{16} + \frac{3}{16} = \frac{1}{4}\), \(\frac{1}{16} + \frac{3}{32} + \frac{7}{64} = \frac{13}{64}\), and \(\frac{1}{64}\).

\[\square\]

**Remark 2.2.** The matrices in [4] are related to Rédei matrices which were used in the 1930’s to study the structure of narrow ideal class groups. Namely, for \(\mathbb{Q}(\sqrt{d})\), we considered the case that all odd primes divisors of \(d\) are \(\equiv 1 \mod 8\). Thus 2 is a norm from \(F = \mathbb{Q}(\sqrt{d})\) and we have the representation

\[d = u^2 - 2w^2.\]

Let \(d' = \prod_{i=1}^t p_i\). The matrix \(M_{F/Q}\) has the form:

\[
\begin{pmatrix}
1 \\
1 \\
\vdots \\
1 \\
(-d, v)_2 & (-d, v)_{p_1} & \ldots & (-d, v)_{p_t}
\end{pmatrix}.
\]

The \((t - 1)\) by \(t\) matrix \(\hat{R}_{F/Q}\) can be extended, without changing its rank, to a \(t\) by \(t\) matrix \(R_{F/Q}\) by adding the last row

\[(-d, p_t)_{p_1}, (-d, p_t)_{p_2}, \ldots, (-d, p_t)_{p_t}.
\]

\(R_{F/Q}\) is known as the Rédei matrix of the field \(F' := \mathbb{Q}(\sqrt{d'})\) (see [5] or [10]). Its rank determines the 4-rank of the narrow ideal class group \(C'_{F'}\) of the field \(F'\) by
4-rank $C^t = t - 1 - \text{rank}(R_{F/Q})$.

Combining this information with Lemma 2.1, we have that if $(-d, u + w)_2 = -1$, then 4-rank $K_2(O_F) = 4\text{-rank } C^t_F$. Using Rédei matrices, Gerth in [3] derived an effective algorithm for computing densities of 4-class ranks of narrow ideal class groups of quadratic number fields. It would be interesting to see if density results concerning 4-class ranks of narrow ideal class groups (coupled with the product formula in the appendix) can be used to obtain asymptotic formulas for 4-rank densities of tame kernels.

3. Appendix: A Product Formula

Most of the local Hilbert symbols in the matrix $M_{Q/\mathbb{Q}}$ are calculated directly. Difficulties arise when $d$ is a norm from $Q(\sqrt{2})$. In this case, we need to calculate the Hilbert symbols $(-d, u + w)_l$ and $(-d, u + w)_{p_k}$. The local symbol at 2 is calculated using Lemmas 5.3 and 5.4 in [4]. In this appendix we provide a product formula which allows one to calculate $(-d, u + w)_l$ using 2 factors of $d$ at a time.

Let $d$ be a squarefree integer and assume that all odd prime divisors of $d$ are $\equiv \pm 1 \mod 8$. Then $d$ is a norm from $F = Q(\sqrt{2})$ and we have the representation

$$d = u^2 - 2w^2$$

with $u > 0$. Let $l$ be any odd prime dividing $d$. Note that $l$ does not divide $u + w$ and so

**Remark 3.1.** $(-d, u + w)_l = (l, u + w)_l = \left(\frac{u + w}{l}\right)$.

Recall that any odd prime divisor $l$ of $d$ is $\equiv \pm 1 \mod 8$. We fix $x$ and $y$ according to the representation:

$$(-1)^{\frac{l-1}{2}}l = N_{Q(\sqrt{2})/Q}(x + y\sqrt{2}) = x^2 - 2y^2$$

with $x \equiv 1 \mod 4$, $x, y > 0$. Observe that mod 8, $x$ is odd. Also we can arrange for $x \equiv 1 \mod 4$ by multiplying $x + y\sqrt{2}$ by $(1 + \sqrt{2})^2$.

For $l \equiv 1 \mod 8$, we have $l = x^2 - 2y^2$ and so $\left(\frac{4}{l}\right) = 1$. Thus $\left(\frac{4}{7}\right) = 1$.

For $l \equiv 7 \mod 8$,

$$1 = \left(\frac{-1}{y}\right) = \left(\frac{-1}{y}\right)\left(\frac{4}{7}\right) = (-1)^{\frac{y-1}{2}}(-1)^{\frac{y-1}{2}}\left(\frac{4}{7}\right) = \left(\frac{4}{7}\right).$$

Now let $r$ be an integer not divisible by $l$ which can be represented as a norm from $Q(\sqrt{2})$. Denote by $\pi_r = s + t\sqrt{2}$ an element such that $N_{Q(\sqrt{2})/Q}(\pi_r) = r$ with $s, t > 0$. Now let $u_r$ and $w_r$ be such that

$$u_r + w_r\sqrt{2} = (1 + \sqrt{2})(x + y\sqrt{2})(s + t\sqrt{2}).$$

By the choice of $x, y, s, t$, we have $u_r > 0$. Note that

$$N_{Q(\sqrt{2})/Q}(u_r + w_r\sqrt{2}) = -(-1)^{\frac{r-1}{2}}r.$$
Now fix \( l = \langle x - y\sqrt{2} \rangle \) a prime ideal above \( l \) in \( \mathbb{Q}(\sqrt{2}) \). As \( l \) splits in \( \mathbb{Q}(\sqrt{2}), \mathbb{Z}[\sqrt{2}]/l \cong \mathbb{Z}/l\mathbb{Z} \). This allows us to work mod \( l \) as opposed to mod \( l \). From the above, \( u_r + w_r = 2xs + 3tx + 3sy + 4yt \). Modulo \( l \), we have

\[
\begin{align*}
  u_r + w_r &\equiv 2sy\sqrt{2} + 3t\sqrt{x} + 3sy + 4yt \\
                  &\equiv y(3 + 2\sqrt{2})(s + t\sqrt{2}).
\end{align*}
\]

As \( \left( \frac{y}{l} \right) = 1 \), \( \left( \frac{u_r + w_r}{l} \right) = \left( \frac{y}{l} \right) \left( \frac{u_r}{l} \right) = \left( \frac{u_r}{l} \right) \) where

\[
\left( \frac{u_r}{l} \right) = \begin{cases} 
  1 & \text{if } x^2 \equiv \pi_r \text{ mod } l \text{ is solvable} \\
  -1 & \text{otherwise}.
\end{cases}
\]

In the case \( r = \prod_i^{t-1} p_i \) where \( p_i \equiv \pm 1 \text{ mod } 8 \), we obtain for each \( p_i \) an element \( \pi_i \in \mathbb{Q}(\sqrt{2}) \) of norm \( (-1)^{\frac{p_i-1}{2}} p_i \). Let \( c \) be the number of primes dividing \( r \) which are congruent to 7 modulo 8. Then we have (up to squares of units) \( \pi_r = (1 + \sqrt{2})^c \prod_i^{t-1} \pi_i \). This yields

\[
\left( \frac{u_r + w_r}{l} \right) = \left( \frac{1 + \sqrt{2}^c}{l} \right) \left( \prod_i^{t-1} \left( \frac{\pi_i}{l} \right) \right),
\]

and so

\[
\left( \frac{u_r + w_r}{l} \right) = \left( \frac{u_r + w_r}{l} \right)^c \prod_i^{t-1} \left( \frac{u_r + w_r}{l} \right). 
\]

As \(-1\) and \(2\) are also norms from \( \mathbb{Q}(\sqrt{2}) \), we can include \( r \)'s having factors \(-1\) or \(\pm 2\). Thus for \( r = (-1)^m(2)^n \prod_i^{t-1} p_i \) with \( m, n = 0, 1 \), and each \( p_i \equiv \pm 1 \text{ mod } 8 \) and \( l \neq p_i \) for any \( i \), we have

**Remark 3.2.** \( \left( \frac{u_r + w_r}{l} \right) = \left( \frac{u_r + w_r}{l} \right)^{n+c} \left( \frac{u_r + w_r}{l} \right)^m \prod_i^{t-1} \left( \frac{u_r + w_r}{l} \right). \)

Setting \( r = \frac{d}{l} \), we have \( -(-1)^{\frac{l-1}{2}} d = N_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}(u_r + w_r\sqrt{2}) \). So for any prime \( l \equiv 7 \text{ mod } 8 \), we have \( N_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}(u_r + w_r\sqrt{2}) = d \equiv N_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}(u + w\sqrt{2}). \) Then, up to squares, \( \left( \frac{u_r + w_r}{l} \right) = \left( \frac{u + w}{l} \right) \). For prime divisors \( l \equiv 1 \text{ mod } 8 \), we have \( -d = N_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}(u_r + w_r\sqrt{2}) \) and so we include \( \left( \frac{u_r + w_r}{l} \right) \). To summarize,

**Remark 3.3.** \( (-d, u + w)_l = \begin{cases} 
  \left( \frac{u_r + w_r}{l} \right) & \text{if } l \equiv 7 \text{ mod } 8 \\
  \left( \frac{u_r + w_r}{l} \right)^{n+c} \left( \frac{u_r + w_r}{l} \right)^m \prod_i^{t-1} \left( \frac{u_r + w_r}{l} \right) & \text{if } l \equiv 1 \text{ mod } 8.
\end{cases} \)

We may now reduce to the following \( d = rl \): \( d = -l, d = 2l \), and \( d = pl \), i.e. calculate the symbols \( \left( \frac{u_r + w_r}{l} \right), \left( \frac{u_r + w_r}{l} \right)^{n+c}, \left( \frac{u_r + w_r}{l} \right)^m \prod_i^{t-1} \left( \frac{u_r + w_r}{l} \right). \) The first two symbols can be calculated using the following two elementary lemmas.
Lemma 3.4. \( \left( \frac{u-1+w-1}{l} \right) = 1 \iff (-1)^{\frac{l-1}{2}}l = a^2 - 32b^2 \) for some \( a, b \in \mathbb{Z} \) with \( a \equiv 1 \mod 4 \).

Lemma 3.5. \( \left( \frac{u+u_2}{l} \right) = 1 \iff l \equiv \pm 1 \mod 16 \).

A little care is necessary in computing \( \left( \frac{u^p+w^p}{l} \right) \). If \( \left( \frac{(-1)^{\frac{p-1}{2}}}{l} \right) = 1 \), then the symbol \( \left( \frac{\pi}{l} \right) \) is well defined (see discussion preceding Proposition 3.5 in [2]) and can be computed using

\[
\text{Lemma 3.6. For } K = \mathbb{Q}(\sqrt{(-1)^{\frac{p-1}{2}}2p}) \text{ with } p \equiv \pm 1 \mod 8 \text{ and } h^+(K) \text{ the narrow class number of } K, \text{ we have}
\]
\[
\left( \frac{\pi}{l} \right) = 1 \iff l^{\frac{h^+(K)}{4}} = n^2 - 2pm^2 \text{ for some } n, m \in \mathbb{Z} \text{ with } m \not\equiv 0 \mod l.
\]

For \( K = \mathbb{Q}(\sqrt{-2p}) \) with \( p \equiv 7 \mod 8 \), \( \left( \frac{\pi}{l} \right) = -1 \iff l^{\frac{h^+(K)}{4}} = 2n^2 + pm^2 \) for some \( n, m \in \mathbb{Z} \) with \( m \not\equiv 0 \mod l \).

For \( K = \mathbb{Q}(\sqrt{-2p}) \) with \( p \equiv 1 \mod 8 \), \( \left( \frac{\pi}{l} \right) = -1 \iff l^{\frac{h^+(K)}{4}} = pm^2 - 2m^2 \) for some \( n, m \in \mathbb{Z} \) with \( m \not\equiv 0 \mod l \).

In fact,

\[
\text{Lemma 3.7. If } \left( \frac{(-1)^{\frac{p-1}{2}}}{l} \right) = 1, \text{ then}
\]
\[
\left( \frac{u^p+w^p}{l} \right) = \begin{cases} 
\left( \frac{\pi}{l} \right) & \text{for } p \equiv 1 \mod 8 \\
\left( \frac{u_{-1}+w_{-1}}{l} \right) \left( \frac{\pi}{l} \right) & \text{for } p \equiv 7 \mod 8.
\end{cases}
\]

The case where \( \left( \frac{(-1)^{\frac{p-1}{2}}}{l} \right) = -1 \) can be done by finding \( u_p \) and \( w_p \) from the presentation

\[
N_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}(u_p + w_p \sqrt{2}) = -(1)^{\frac{p-1}{2}}pl.
\]

Combining Remarks 3.1, 3.2, and 3.3, we have

\[
\text{Theorem 3.8. For } d = (-1)^n(2)^m \prod_{i=1}^{l} p_i, \text{ with each } p_i \equiv \pm 1 \mod 8, \text{ we have}
\]
\[
(-d, u + w)_{p_k} = \left( \frac{u_{-1}+w_{-1}}{p_k} \right)^{n+(\frac{p-1}{2})} \left( \frac{u_{+1}+w_{+1}}{p_k} \right)^m \prod_{1 \not\equiv k} \left( \frac{u_{p_i}+w_{p_i}}{p_k} \right).
\]

\[
\text{Example 3.9. Consider the cases } d = \pm pl, \pm 2pl \text{ with } p \equiv 7 \mod 8, l \equiv 1 \mod 8, \text{ and } \left( \frac{l}{p} \right) = 1 \text{ (see Proposition 4.6 in [2]). Note that } \left( \frac{\pi}{l} \right) \text{ is well defined and so Lemma 3.7 is applicable.}
\]
For $d = pl$, we have $n = 0, m = 0$ and so
\[
(-d, u + w)_l = \left(\frac{u_1 + w_{-1}}{l}\right)^{-1} \left(\frac{u_2 + w_2}{l}\right)^0 \left(\frac{u_{-1} + w_{-1}}{l}\right) \left(\frac{\pi}{l}\right)
\]
\[= \left(\frac{\pi}{l}\right).
\]

For $d = 2pl$, we have $n = 0, m = 1$. Thus
\[
(-d, u + w)_l = \left(\frac{u_1 + w_{-1}}{l}\right)^{-1} \left(\frac{u_2 + w_2}{l}\right)^1 \left(\frac{u_{-1} + w_{-1}}{l}\right) \left(\frac{\pi}{l}\right)
\]
\[= \left(\frac{2 + \sqrt{2}}{l}\right) \left(\frac{\pi}{l}\right).
\]

For $d = -pl$, we have $n = 1, m = 0$. This yields
\[
(-d, u + w)_l = \left(\frac{u_1 + w_{-1}}{l}\right)^0 \left(\frac{u_2 + w_2}{l}\right)^0 \left(\frac{u_{-1} + w_{-1}}{l}\right) \left(\frac{\pi}{l}\right)
\]
\[= \left(\frac{1 + \sqrt{2}}{l}\right) \left(\frac{\pi}{l}\right).
\]

Finally, for $d = -2pl$, we have $n = 1, m = 1$. So
\[
(-d, u + w)_l = \left(\frac{u_1 + w_{-1}}{l}\right)^0 \left(\frac{u_2 + w_2}{l}\right)^1 \left(\frac{u_{-1} + w_{-1}}{l}\right) \left(\frac{\pi}{l}\right)
\]
\[= \left(\frac{2 + \sqrt{2}}{l}\right) \left(\frac{1 + \sqrt{2}}{l}\right) \left(\frac{\pi}{l}\right).
\]

**Example 3.10.** Consider the cases $d = \pm pl$ with $p \equiv l \equiv 1 \mod 8$, and \(\left(\frac{l}{p}\right) = 1\) (see Proposition 4.4 in [7]). Again \(\left(\frac{\pi}{l}\right)\) is well defined and so Lemma 3.7 is applicable.

For $d = pl$, we have $n = 0, m = 0$, and so
\[
(-d, u + w)_l = \left(\frac{u_1 + w_{-1}}{l}\right)^{-1} \left(\frac{u_2 + w_2}{l}\right)^0 \left(\frac{u_{-1} + w_{-1}}{l}\right) \left(\frac{u_p + w_p}{l}\right)
\]
\[= \left(\frac{1 + \sqrt{2}}{l}\right) \left(\frac{\pi}{l}\right).
\]

For $d = -pl$, we have $n = 1, m = 0$. Thus
\[
(-d, u + w)_l = \left(\frac{u_1 + w_{-1}}{l}\right)^0 \left(\frac{u_2 + w_2}{l}\right)^0 \left(\frac{u_{-1} + w_{-1}}{l}\right) \left(\frac{u_p + w_p}{l}\right)
\]
\[= \left(\frac{\pi}{l}\right).
\]
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