An Improved Model for Air Damping of Perforated Structures

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Abstract: the prediction of air damping of micromachined mechanical resonant structures is significant in the design of high quality factor devices. In rarefied air, based on Bao’s molecule model, Li gives an analytical model for air damping of perforated structures. By studying the action of molecules going through holes and reflected by the fixed plate, this paper gives a probability of molecules through holes going into the gap between the moving plate and the fixed one. Comparison with Li’s model, the new model can play a better performance of air damping for perforated structures, at a wide range of size of holes.

1. Introduction

Micromachined mechanical resonant structures have been applied extensively due to their high sensitivity and low fabrication cost[1]. A common approach to evaluate resonant structures is to measure the quality factor of the structure defined as the ratio of total input energy to dissipated energy. However, even in rarefied air, the squeeze film damping cannot be neglected[2]. The character of evaluation of air rarefaction effect is Kn number which is the ratio of the molecule mean free path and the characteristic length of the structures. According to the value of Kn, the different air regimes can be briefly separated into 4 kinds[3]: continuum regime $Kn \leq 10^{-3}$; Slip regime $10^{-3} \leq Kn \leq 10^{-1}$; transition regime $10^{-1} \leq Kn \leq 10$; free molecule regime $Kn > 10$.

For the prediction of the quality factor of air damping in free molecule regime, owing to several reasonable assumptions, Bao et al[4] calculate the change of kinetic energy of molecules in the gap and give a simple but widely used expression of quality factor:

$$Q_{bao} = (2\pi)^{3} \rho H \frac{d}{L} \left( \frac{RT}{M_{m}} \right)^{\frac{1}{2}} \frac{1}{P}$$

(1)

where $\rho$, $H$, $L$ is the specific mass, thickness and peripheral length of the moving plate, $\omega$ is the vibrating frequency, $d$ is the initial gap between the moving plate and the fixed one, $R$ the universal gas constant, $T$ is the absolute temperature, $M_{m}$ and $P$ are the molar weight and pressure. Based on Bao’s model, Li et al[5] devised a modification for perforated structures, the total quality factor expression for perforated structures is presented:

$$Q_{Li} = r \left[ f(c_{1}) + f(c_{2}) N_{x} N_{y} L_{2} (2L)^{-1} \right]^{-1} Q_{bao}$$

(2)

where $L_{2}$ is the peripheral length of the square hole, $N_{x} N_{y}$ is the number of holes along $x$ direction and $y$ direction. However, a limitation exists in Li’s model that the model is valid only for $g > 1$ or more, where $g$ is ratio of $l_{hole}$ and $d$, $l_{hole}$ is the size of the square hole.

This paper is an extension of Li’s model[5] for air damping of perforated structures. In section 2, the brief description of Li’s model is given. In section 3, a detailed description of molecules in gap between plates, and the probability of molecules through holes into the gap are found. In section 4, the
improved model is found and the validation is proved compared with Li’s model.

2. A description of Li’s model

According to Li’s model, there are two parts of molecules going into the gap. The first part is the molecules going through four side edges of the moving plate into the gap, \( N_f \) is the molecule number; the second part is the molecules entering the gap through the holes, \( N_2 \) is the molecule number, so the total number of molecules entering the gap is \( N = N_f + N_2 \).

To the first part of molecules going into the gap, \( N_f = \frac{1}{4}nVd \), where \( n \) is the concentration of the molecules and \( V = [8\pi T(4\pi M_m)^{1/2}] \) is the average velocity of the molecules, \( k \) is the Boltzmann constant.

The quality factor of the first part of molecules is given:

\[
Q_f = 2\pi \frac{E_m}{\Delta E_{cycle}} = \frac{r}{f(c_1)} Q_{bars}
\]

where \( E_m = 1/2M_{vio2}(4\pi c2)_2 \) is the total mechanical energy of the moving plate, \( M_{vio2} = \rho hL_cH \), \( L_c, L_y \) and \( H \) are the length, width and thickness of the plate, \( r = (L_r - L_r - L_rN_r) \theta / (L_x L_y) \), \( a_1 = L_r - r, \theta = \theta_{bars} \), \( \theta_{bars} \) is the size of the hole, \( c_1 = [L_c L_y / (4\pi c2)]^{1/2} \) is the average number of collisions with the moving plate for the first part molecules leaving from the four edges of the plate.

To the second part of molecules entering the gap from the holes, Li [5] treated \( N_2 = (1/4nVd)N_rN_r, L_x = 4l_h \). The quality factor of the second part of molecules going through the holes is found:

\[
Q_2 = 2\pi \frac{E_m}{\Delta E_{cycle}} = \frac{2L}{f(c_2)} \frac{N_r N_r L_y}{N_r N_r L_y} Q_{bars}
\]

where \( c_2 = [L_h / (4\pi c_2)]^{1/2} \) is the average number of collisions with the moving plate for the second part molecules and leaving through four edges.

After above calculations, the total quality factor of Li’s model [5] is found:

\[
Q_{Li} = (Q_f^{-1} + Q_2^{-1})^{-1} = r f(c_1) + f(c_2)N_r L_y (2L)^{-1}] Q_{bars}
\]

However, there is a limitation in Li’s model, that the model is reasonable only for a high value of \( g \). For a low value of \( g \), equation (4) overestimates the number of molecules that enter the gap from the holes. A discussion in detail will be given in section 5.

3. An improved model

(a) hole at large \( g \); (b) hole at small \( g \)

**Figure 1.** Two holes when \( g \) is large or small

**Figure 2.** The motion of a molecule in a hole

Based on the above analysis, it is critical to find the exact number of molecules entering the gap through holes. When molecules enter the hole of structures, some of them can go straight into the gap; others collide with the fixed plate and go back to the space through the hole, only the molecules enter the gap that could count for the energy change with the moving plate. This paper gives two holes that \( g \) is extremely large and small in figure 1.
The total number of molecules entering a hole is \( N_{\text{hole}} = 1/4nvL_d \). When \( g \) is extremely large in figure 1(a), molecules entering the hole can easily be reflected through the hole by the fixed plate. So the exact number of molecules that can go into the gap is \( 1/4nvL_d \). Use the probability \( P \) to estimate molecules go into the gap from the hole, \( P_{L-g} \approx (1/4nvL_d)/N_{\text{hole}} = 4/g \), the subscript \( L-g \) means \( P \) at large \( g \). When \( g \) is extremely small in figure 1(b), molecules entering the hole are hardly reflected by the fixed plate and back into the space. As a result, most of molecules from the hole can easily reach the gap and \( P_{S-g} \approx (1/4nvA_{\text{hole}})/N_{\text{hole}} = 1 \). According to the analysis above, using \( P_{L-g} \) or \( P_{S-g} \) to calculate the number of molecules that enters the gap from holes isn’t reasonable theoretically. To get an exact value of \( P \), a particular motion of a molecule in a hole is shown in figure 2. When a molecule goes through the hole, its velocity \( V \) can be divided into two parts: \( V_x \) in \( x-z \) plane and \( V_z \) in \( y-z \) plane. The location of a molecule and velocity \( V_x, V_z \) decide whether the molecule goes into the gap or not. The depth of the hole in figure 3(b) is \( 2d \). When a molecule goes back to the space through the hole after the collision with the fixed plate in figure 3(a), it is equal that a molecule can reach plate C from the hole. The probability of molecules to go into the gap is equal to that of molecules which cannot reach plate C from the hole.

Considering \( x-z \) plane in figure 4, the velocity direction and locations of molecules are both uniformly distributed. To a special location \( x \), the injection angle of a molecule is \( Q \), and the small interval of length along \( x \) direction is \( dx \). the probability of molecules from \( dx \) reaching C plate is:

\[
P_x = \frac{1}{\pi} [\arctan(\frac{x}{2d}) + \arctan(\frac{l_{\text{hole}}-x}{2d})] \quad (6)
\]

\[
P_{x\text{plane}} = \frac{1}{l_{\text{hole}}} \int_{0}^{l_{\text{hole}}} P_x \cdot dx = \frac{8d}{\pi a} \frac{l_{\text{hole}}}{2d} \arctan(\frac{l_{\text{hole}}}{2d}) + \frac{1}{2} \ln(\cos(\arctan(\frac{g}{2d}))) \quad (7)
\]

\[
P = 1 - (P_{x\text{plane}})^2 = 1 - \left[ -\frac{4}{\pi g} \frac{g}{2} \arctan(\frac{g}{2}) + \ln(\cos(\arctan(\frac{g}{2}))) \right]^2 \quad (8)
\]

where \( P_{x\text{plane}} \) is the probability of molecules reaching C plate from the hole in \( x-z \) plane, due to the square hole, \( P \) is the exact probability of molecules from the hole going to the gap.

**4. Validation of the model of \( P \)**
To validate the probability of molecules reflected through a hole, $P_{\text{com}}$ calculated by computer is used, then $P_{L-g}=4/g$, $P_{S-g}=1$, $P$ and $P_{\text{com}}$ are discussed in figure 5. In figure 5(a), both values of $P_{\text{com}}$ and $P$ decrease as $g$ decreases. During the whole range of small $g$, values of $P$ can match those of $P_{\text{com}}$ very well with the largest description is less than 0.1%. Figure 5(b) shows that both values of $P_{\text{com}}$ and $P$ decrease as $g$ decreases. Although the description of $P_{\text{com}}$ and $P$ is growing slowly as $g$ is growing, the largest description is less than 9%. In figure 5(c), when $g$ is extremely large, the values of $P$ are still close to those of $P_{\text{com}}$. As the above analysis, all the values of $P$ can match $P_{\text{com}}$ well.

Combining equation(8) and equation(5), a new expression of quality factor of a rigid rectangular structures perforated with square holes is obtained:

$$Q_{N} = (Q_{1}^{-1} + Q_{2}^{-1})^{-1} = r[f(c_{1}) + f(c_{2})N_{N}N_{y}A_{\text{hole}}(1 - P_{N}^{2})(2Ld)^{-1}]^{-1}Q_{Bao}$$

(9)

5. Validation of the improved model

To check the accuracy of the improved model, we use the improved model to predict the air damping constant of Kwork’s gyroscopies[6]. Obviously, Bao’s model[4] can only give the prediction of air damping of non-perforated structure. It is well known that with the same overall sizes, the air damping of structures perforated with holes is much small than that of non-perforated structures. The comparison of these models at different values of $g$ is shown in figure 6. At $g=0-1.25$, while Li’s model[5] fails to give an acceptable prediction of air damping, the improved model can work well. The biggest description of Li’s model and the improved model that occurs at $g=0.9$ is 7%, as $g$ is growing; the description becomes small and small. Consequently, the improved model can remove the limitation of Li’s model, and it can give a better prediction of air damping of perforated structures.

6. Summary and conclusions

For perforated structures in rarefied air, Li’s model[5] is valid only for large values of $g$. By researching on molecules going through holes, a better prediction model of the probability molecules entering the gap though the hole is found. Compared with the computer numerical simulation, the
improved model is validated at a wide range of $g$. The improved model can make a better performance for predicting the air damping of perforated structures than Li’s model.

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