One looks for superconductivity at room temperature. But the superconductivity is a purely quantum phenomenon and it is worthwhile to search it in situations in which the quantum nature of substances is most clearly defined. These are, naturally, the low temperature, and besides the state of plasma for comparatively high temperature. I believe that plasma can be a superconductor and the verification of this suggestion is the phenomenon of fireball.

In the work "On the nature of fireball" [13] P.L.Kapitza calculated that the whole energy which can be contained in a ball with the size of a standard fireball (even for the case of full ionization) would irradiated during the time of order 0.01 sec. But the life time of fireballs is several minutes, that has increased the theoretical value ten thousand times. Because of this P.L.Kapitza was forced to propose the hypothesis of constant feeding fireballs by the energy of external magnetic field. But the succeeding experiments of generation such fields by Super-High-Frequency vibrations (SHF) to derive fireballs have been unsuccessful. There were tens of efforts to find a physical mechanism which could explain such a long life time of fireballs. It is generally agreed that all of them collapsed. It seems to me that there is the only way out of the difficulty. To accept that the groundwork of the phenomenon is the superconductivity of plasma.

B.B.Kadomzev in the book "Collective phenomena in plasma" [12] considers oscillations in plasma with negative energy which is indirect evidence for a gap in energy spectrum and hence for the possibility of superconductivity. In the paper of collaborators of Lawrens Livermore National Laboratory "Temperature Measurements of Shock Compressed Liquid Deuterium up to 230 GPa" [17] it was described that at high pressures and temperatures deuterium behaves like a degenerate Fermi-liquid metal: its compressibility and conductivity abruptly increased, it appears a highly reflective state which is characteristic of a liquid metal. Such behaviour also points out to a possibility of superconductivity. An added reason for superconductivity is the Meisner-Ochsenfeld effect. The fact that magnetic fields penetrate into electronic plasma only to the depth of $\lambda = c/\omega$ shows strong diamagnetic properties of plasma. Here $c$ is the light velocity; $\omega$ is the Langmure frequency (the frequency of "electronic sound"); $\omega = \sqrt{4\pi e^2 n/m}$, where $n$ is the electronic density, $m$ is the mass of electron, $e$ is its charge [12]. The value $\lambda = c/\omega$ plays the part of Debye wavelength. In the paper of Arsenjev [1] there is another indirect evidence in favour of superconductivity. He indicates that collisions in plasma leads to very small changes of momentum.

Let us try to describe a qualitative picture of the phenomenon under the conjecture of superconductivity of plasma. Let us imagine ionized plasma that generates a positively charged ball and a spherical cover around this ball generated by negatively charged electronic plasma. Suppose at last that in this cover it is realized superconductivity. Electrons with momentum which tangent to the sphere are deflected by the positive charge of the central part of the ball remaining in the spherical cover. The superconductivity means that the energy of electrons do not looses by collisions. Any continuous vector field on a sphere has singularities. To avoid it, suppose that in our case something like a geodesic flow on the sphere takes place. Preservation of the ionization state means that one deals with the cold plasma. This plasma does not failed due to superconductivity. It explains
such a long life time of fireballs. A ball of plasma is hanged in the air and in this ball takes place the current. It is unusual current since it does not have a fixed direction at any point and at any moment. It is spreaded in all possible tangent directions. Inside the ball there is positive charge which directs and maintains this current. Such phenomenon is possible at the state of superconductivity only — a current that does not have a direction. Electronic liquid flows indeed, but it flows simultaneously in all directions. Hence the fireball is a physical realization of the mental idea of Vlasov (which was used by him as the basis for his equation) [8], the idea of probability distribution of electrons relative to its positions and velocities.

How to organize such current at the low temperature under the usual superconductivity? It would be interesting to take a negatively charged superconductor of the spherical form and to put inside it a sufficiently large positive charge.

Thus the fireball, having the same density as the air and being of no different from it except for its state, can freely float in the air which is observed in reality. Its movement can obey airflows or electromagnetic forces — it is of no importance. What means the explosion of fireballs? Some leakage of energy does exist, especially, by contacts with charged bodies. Balance of energy which is necessary to retain the current is failed, and plasma is extinguished. More exactly, the charge of the kernel becomes unsufficient to maintain electrons, they begin to fall into the kernel and this falling decreases still further its charge. The process becomes avalanche-type and the fireball is quenched instantly. The pressure has no time to level off. The difference in pressure stimulate a shock wave like that of the spark or of the usual lightning. This wave is perceived as an explosion. The known phenomenon — the breakdown of the superconductivity by sufficiently strong magnetic field — explains nonstandard interactions of fireballs with electrical equipment [23].

The fireball is something like the model of the atom: a kernel and electrons spreaded on orbits. Of course, we use the term "geodesic flow" as a figurative expression. In accordance with the quantum theory and with the Heisenberg’s indeterminacy principle, electrons are defined not as moving points but as a probability distribution. Nevertheless the term "geodesic flow" reflects much more adequately the bound state of electrons under the influence of powerful electrostatic field of the positively charged kernel.

At latest works on the creation of fireballs [11] it was generated plasmoids (that was regarded by the authors as fireballs). It has been found experimentally that: "It was formed the negatively charged layer around the polarized kernel".

It may appear that the geodesic flow must lead to perpetual collisions of electrons. But the volume of the cover is macroscopic and it is huge by comparison with the size of an electron. Though any two big circles of $S^2$ intersect but electrons moving along these orbits (no matter how many they will be) can avoid collisions. Moreover, in the approximation when electrons are considered as points, the probability of collisions equals to zero. But the main reason: each separate electron cannot scatter by himself being bounded coherently with other electrons generating joint "plasma-sounded" vibrations. Thus the approximation of the geodesic flow is reasonable.

Let us remark that all attempts to confine plasma was based on ring-shaped regions. But it needs strong magnetic fields. May be it would be better to try electrostatic fields. If there exists superconductivity of plasma, then it must help. Attempts of confinement are very difficult to realise because one endeavour to feed plasma by energy, to clamp it down, to make a pinch. However we may give freedom to plasma, to set it in a situation
of "singing", when plasma will have its own nondamping "sound" vibrations, moreover, vibrations which resist attempts of attenuation. Then it would be a confinement and a source of energy. It is exactly the state of superconductivity of fireballs.

The Kapitza’s conjecture on the nature of fireball was not exactly correct, because his experiments with SHF-vibrations gave a ring-shaped plasma rather than a ball. This was quite natural because of the lack of the kernel. But the idea of feeding is very important. One needs only firstly to create "atom", and after that to feed it by energy.

One may imagine a mechanism of the fireball birth in the following way: suppose, for instant, that the curl of positively charged particles falls on the course of usual lightning. Then the negatively charged plasma may surround this curl, and the fireball being stable begins its life. This idea leaves room for experiments. For instance, to blow a ball of ionized, positively charged gas with the electron beam, or to lay it into the negatively charged medium. If it has been possible to create fireballs artificially then an incorporation in it exact doses of "fuel" and an energy-rejection would give a safe atomic reactor.

So far as in the picture of superconductivity of Bardeen-Cooper-Schrieffer (BCS) the key part play phonons (quasi-particles related to the fundamental frequences of the cristal lattice), our goal is to find eigenfunctions of operator that describes oscillations of spherical cover of plasma into the centrally symmetrical electrostatic field and to prove the existence of a gap in the energy spectrum which provides superconductivity.

Thus we take as a base the geodesic flow on the sphere $S^2$, that is the uniform distribution of electrons on the bundle $T^1_*(S^2)$ of tangent circles of radius 1 to the two-dimensional sphere of radius $R$. The geodesic flow of electrons must be stable relative to small disturbances, since, due to Coulomb interaction, electrons resist compressions. Deviations from this stable movement provide oscillations which play the role of phonons. Interaction of phonons with electrons is the source of superconductivity. One needs to interpret elementary vibrations around the ground-state as independent oscillators and to replace it by operators. As in an atom, the potential of the kernel exceeds the interaction of electrons and it may be thought that electrons move along great circles with the constant speed. That was the reason that we begin with the bundle $T^1_*(S^2)$ of tangent circles to sphere.

The kernel of the fireball consists of positively charged ions. The external cover of the fireball is the negatively charged plasma in which there is a current — the geodesic flow of electrons. That means that the density of electrons is uniformly distributed on the sphere; their velocities are uniformly distributed on the tangent directions to the sphere and its absolute value equal to the speed of circular motion around the kernel in response to its Coulomb attraction. As a result, the potential of the external cover acting upon internal points of the kernel equals to zero. This provide the stability of the kernel. Indeed, if an ion penetrates into the cover from the kernel then the electrostatic potential of the cover draw it back, since the potential of the external (relative to this ion) part of the cover equals to zero, meanwhile the potential of the internal part of the cover draws it to the center. Electrons located outside of the cover can not penetrate to the kernel, since the cover obstructs this penetration. Namely, though the total charge of the fireball is neutral, but in the vicinity of it, the negative charge of the cover is prevailing that repuls closely-spaced negative charges preventing its penetration into the kernel. Thus the positive charge of the kernel remains practically unchanged. So, to explore the kernel one can use the theory of nuclear matter [5]. Electrostatic potential sustains the movement of electrons in the cover which vaguely resembles the skin-effect in superconductors.
The fireball is something like soap-bubble. Soap-bubbles usually slowly fall down in the air, because the pressure inside it (caused by surface tension), and hence the density inside it, is a little more than outside. So the weight of the bubble is a little greater than the weight of the air. Similarly behave the fireballs. The additional complication is that the kernel tends to spread due to Coulomb forces, and this electrostatic pressure is added to the usual pressure in gas. The resulting pressure turns out to be a little greater than atmospheric pressure which bring down the density in the central part of the kernel. Meanwhile in the boundary part of the kernel (at the vicinity of the cover) as well as in the cover itself the density must be greater due to the pressure in the central part. As a result the mean density of the fireball is only slightly greater than the density of the air. Consequently the fireball can freely float in the air with the slight tendency toward falling.

The difference in pressure is responsible for the effect of surface tension as well as for the explosion of fireballs and the arising of shock waves when fireballs are extinguished.

The main question remains to be answered. The question about losses of the energy by scattering. Take as a pattern the BCS-theory of superconductivity. In plasma of the cover there exist oscillations like that of Bloch-waves in cristallic lattice. The difference is that Bloch-waves are linked with the periodic potential of the lattice, while collective oscillations of the cover are caused by the spherical form of configuration space. The attracting field of the kernel and the difference in the pressure inside and outside of the cover make the cover resemble a spherical elastic film in which arise spherical oscillations. Further on, the mechanism of plasma stability becomes analogous to that of usual superconductivity: scattering of electrons excite oscillations of the cover and that in its turn stimulate electrons. This process can be regarded (in the quantum level) as an exchanging interaction of electrons by using the analog of phonons. Our aim is to find the energetic spectrum of the secondary quantized problem and to detect a gap in this spectrum.

Let us recall that the notion of current in our case is different from the usual electrodynamics notion. The movement of electrons is uniformly distributed on all tangent directions. So in place of a current one has only "microcurrents" which superimpose and do not give electromagnetic fields. Consequently, the energy does not release: the fireball does not radiate electromagnetic waves and there are no losses of energy for that process. The fireball only fluoresce as usual plasma.

1. Electronic cover

Consider at first vibrations of the density of the geodesic flow on a sphere. Imagine the cover as a homogeneous flux of a compressible fluid on the bundle $T^1_1(S^2)$ of tangent circles at the two-dimensional sphere of the radius $R$. On the sphere arise circular waves that can be parametrized by its centers — points of emittance of waves. At an initial moment, take the disturbance of the density being the $\delta$-function at a point $O$ with velocities uniformly distributed on a tangent circle. As far as the velocity of superfluous density is always directed outside, the wave bend the sphere passing the equator, focusses to the diametrically opposite point (−0) turning again into the $\delta$-function. We will call such waves elementary. If a wave is not elementary then in place of focal points appear another caustics. Each elementary wave on the bundle $T^1_1(S^2)$ of tangent circles to the sphere is the torus $T^2_0$. Indeed in the initial moment one has the circle $S^1$ at the fiber over the point $O$. Further, it is projected on a circle of the sphere $S^2$ with the unically defined velocities which is orthogonal to the circle. After two focusing, the initial circle is
identically maps to itself. Any two such torus intersect by a pair of anti-directed geodesics, that connect centers of the corresponded waves.

Besides these waves of condensation there exist similar waves of rarefaction. To quantize means to correlate an operator with each elementary wave. Let us find the Hamiltonian for corresponding periodic oscillations.

Note that the potential energy of the flow (let us denote it by $U$) does not change during these oscillations. Indeed, from the point of view of geodesic flow, the disturbances at each moment are condensed at points of a circle which moves without deformation with the constant speed along the torus $\mathbb{T}^2$. The degenerations of the projection to a point in poles does not connect with the geodesic flow as itself, but with its projection on the base. More exactly, $T_1^1(S^2)$ — the bundle of unit tangent vectors to $S^2$ — is diffeomorphic to the real three-dimensional projective space $\mathbb{RP}^3$. Tangent vectors to the geodesic flow are, say, left-invariant vector field on $\mathbb{RP}^3$ or on its two-fold covering $S^3$, that may be realized as the group of unit in absolute value quaternions. Under the stereographic projection from the center of the sphere $S^3$ on $\mathbb{R}^3$, trajectories of this field transfer into a family of Clifford parallel lines in $Ell(3)$ which is the space $\mathbb{R}^3$ equipped by elliptic metric induced from $S^3$ by this projection. Shifting along the trajectories of this family are isometries. To any fiber $S^1 \in T_1^1(S^2)$, i.e. to the unit circle in the tangent plane at a point $l$ of the base $S^2$, correspond quaternions that commute with $l$: the circle in $S^3$ passing through $l$ and the neutral quaternion. It is a greate circle and so it projects onto a line in $Ell(3)$ passing through the image of the neutral element. Under the Clifford shift, these lines run along a toric surface. Hence, the set of trajectories of points of a fiber over some point $x_0 \in S^2$ is two-dimensional torus. As was shown by Clifford, the metric of this torus induced by the ambient elliptic metric is Euclidian. Thus the potential energy of an elementary wave remains constant.

Meanwhile, in contrast with the movement in a flat space, the momentum of the wave changes. This is caused by the fact that the movement is not free: it takes place not in the whole space but along the surface of a sphere. Take the spherical coordinate on the sphere with the pole $O$. Fronts of wave centered in $O$ are circles $\psi = Const$. As an momentum of this wave it should be taken the operator of differentiation by $\psi$. But since this operator must be defined at all points (in particular at poles), it is necessary to multiply it by a function which is equal to zero at poles. The simplest such function is $\cos \psi$. Thus as an momentum we consider the operator

$$p = i \frac{\hbar}{2} \cos \psi \frac{\partial}{\partial \psi}.$$  

We will use the rational system of units in which the light velocity $c$ and the Plank’s constant $\hbar$ equal to 1. So, the Hamiltonian is defined by the operator

$$H f = - \cos \psi \frac{\partial}{\partial \psi} (\cos \psi \frac{\partial}{\partial \psi}) f + U f, \quad (1)$$

where $U$ up to multiplicative constant corresponds to potential energy. We are interesting in solutions depending from $\psi$ only, so $f = f(\psi)$. The equation for oscillations is

$$- \cos^2 \psi \frac{d^2 f}{d\psi^2} + \cos \psi \sin \psi \frac{df}{d\psi} + U f(\psi) = 0,$$  

5
or
\[ L = \frac{d^2f}{d\psi^2} - \tan \psi \frac{df}{d\psi} - \frac{U}{\cos^2 \psi} f = 0. \] (2)

We seek for the spectrum of the operator \( L \) defined by the equation (2). Eigenvalues of \( L \) will be denoted by \( \alpha \). We obtain the equation

\[ (\cos^2 \psi) f'' - (\sin \psi \cos \psi) f' + (\alpha \cos^2 \psi - U) f = 0. \] (3)

Boundary conditions for the equation (3) are

\[ f \left( \frac{\pi}{2} \right) = f \left( -\frac{\pi}{2} \right) = 0, \] (4)

since the function \( f \) has to be independent on \( \phi \) at poles. Let us note that the boundary value problem (3)-(4) defines a selfajoint operator, because the equation (3) can be written in the form

\[ L_1(f) = \frac{d}{d\psi} \left( \cos \psi \frac{df}{d\psi} \right) - \frac{U}{\cos \psi} f = -\alpha \cos \psi f. \]

Eigenfunctions of the selfajoint operator \( L_1 \) with the boundary conditions (4) relative to the positively defined selfajoint operator of multiplication by the function \( \cos \psi \) are mutually orthogonal on the interval \([-\pi/2, \pi/2]\) with the weight \( \cos \psi \).

We found oscillations of density or the "longitudinal" oscillations on the sphere. Let us find its "cross" oscillations, that is the oscillations of the sphere itself from purely phenomenological point of view. With this in mind, let us solve the following problem that has an independent interest. It may be interpreted as a problem of oscillation of the balloon cover.

Consider an elastic spherical film surrounding a volume with the internal pressure greater than that of the outer air. We seek for its eigen oscillations due to surface tension.

On any element of the surface act forces of surface tension which are applied to the boundary points of the element. The resultant of two forces which act on the opposite points of a given direction at a point \( O \) is normal to the surface and equals to the normal curvature of this direction. The resultant of all forces is the normal to the surface which equals to the integral of normal curvatures, hence, equals to the mean curvature at a point \( O \). For the stationary state, these forces are balanced out by the difference in pressure. The difference between resultant and the force caused by the difference in pressure (acting in the same direction) give the force acting on the cover. As far as we know, the problem in this statement was not considered. This model is different from usual statements of problems in elastic theory since usually one considers forces to be proportional to displacements.

Let us formalize the problem obtained. Introduce spherical coordinates on the sphere of radius \( R \).

\[ \{ R \cos \psi \cos \phi, R \cos \psi \sin \phi, R \sin \psi \}. \]

Denote the corresponding point of the unit sphere by

\[ r = \{ \cos \psi \cos \phi, \cos \psi \sin \phi, \sin \psi \}. \]

Denote a displacement of the point \( R, \psi, \phi \) by
\[ R\Pi = \{ R\xi(\psi, \phi), R\eta(\psi, \phi), R\zeta(\psi, \phi) \}. \]

The coordinates of the displaced point are \( X = Rr + R\Pi \). Let us introduce the moving frame on the unit sphere

\[
\begin{align*}
  r &= \{ \cos \psi \cos \phi, \cos \psi \sin \phi, \sin \psi \}, \\
  k &= \{ -\sin \phi, \cos \phi, 0 \} = \frac{r}{\cos \psi}, \\
  l &= \{ -\sin \psi \cos \phi, -\sin \psi \sin \phi, \cos \psi \} = r\psi.
\end{align*}
\]

One has

\[
\begin{align*}
  r_{\phi\phi} &= (\cos \psi)k_{\phi}, \\
  k_{\phi} &= (-\cos \psi)r + (\sin \psi)l, \\
  k_{\psi} &= 0, \\
  l_{\psi} &= -r;
\end{align*}
\]

Its vector products are

\[
[r \times k] = l, \\
[k \times l] = r, \\
[l \times r] = k.
\]

Calculate the mean curvature of the displaced surface, neglecting members of second order relative to \( \Pi \) and its derivatives

\[
X_\phi = (R \cos \psi)k + R\Pi_\phi, \\
X_\psi = Rl + R\Pi_\psi.
\]

The coefficients of the first quadratic form \( ds^2 \) equal

\[
\begin{align*}
  (X_\phi, X_\phi) &= R^2(\cos^2 \psi + 2 \cos \psi (k, \Pi_\phi)), \\
  (X_\phi, X_\psi) &= R^2(\cos \psi (k, \Pi_\psi) + (l, \Pi_\phi)), \\
  (X_\psi, X_\psi) &= R^2(1 + 2(l, \Pi_{\psi\phi})).
\end{align*}
\] (5)

\[
[X_\phi \times X_\psi] = R^2\{ (\cos \psi)r + (\cos \psi)[k \times \Pi_\psi] - [l \times \Pi_\phi] \}. \\
|[X_\phi \times X_\psi]| = R^2\sqrt{\cos^2 \psi + 2 \cos^2 \psi (l, \psi) + 2 \cos \psi (k, \Pi_\phi)}.
\]

\[
\frac{1}{|[X_\phi \times X_\psi]|} = \frac{1}{R^2 \cos \psi} \{ 1 - (l, \Pi_\psi) - \frac{(k, \Pi_\phi)}{\cos \psi} \}.
\]

Normal to the surface is defined by the expression

\[
n = r + [k \times \Pi_\psi] - \frac{[l \times \Pi_\phi]}{\cos \psi} - (l, \Pi_\psi)r - \frac{(k, \Pi_\phi)}{\cos \psi}r.\] (6)

We have

\[
\begin{align*}
  X_{\phi\phi} &= R\{ (-\cos^2 \phi)r + (\cos \phi \sin \phi)l + \Pi_{\phi\phi} \}, \\
  X_{\phi\psi} &= R\{ (-\sin \psi)k + \Pi_{\phi\psi} \}, \\
  X_{\psi\psi} &= R\{ -r + \Pi_{\psi\psi} \}.
\end{align*}
\]

The coefficients of the second quadratic form \( dN^2 \) equal

\[
\begin{align*}
  (X_{\phi\phi}, n) &= R\{ -\cos^2 \psi - \cos \psi \sin \psi (r, \Pi_\psi) \} + (r, \Pi_{\phi\phi}), \\
  (X_{\phi\psi}, n) &= R\{ \tan \psi (r, \Pi_\phi) + (r, \Pi_{\phi\psi}) \}, \\
  (X_{\psi\psi}, n) &= R\{ -1 + (r, \Pi_{\psi\psi}) \}.\] (7)
\]
Denote \((2 \times 2)\)-matrix of the quadratic form \(dN^2 - \lambda ds^2\) by \(A(\lambda)\). Using (5) and (7) find entries of this matrix

\[
A_{11} = R\{-\cos^2 \psi - \cos \psi \sin \psi (r, \Pi_\psi) + (r, \Pi_{\psi \psi})\} - \lambda R^2\{\cos^2 \psi + 2 \cos \psi (k, \Pi_\psi)\},
A_{12} = R\{\tan \psi (r, \Pi_\psi) + (r, \Pi_{\psi \psi})\} - \lambda R^2\{\cos \psi (k, \Pi_\psi) + (l, \Pi_\psi)\},
A_{22} = R\{-1 + (r, \Pi_{\psi \psi})\} - \lambda R^2\{1 + 2(l, \Pi_\psi)\}.
\]

Roots of the equation \(det.A = 0\) define main curvatures. By dividing half of the coefficient of \(\lambda\) in this equation by the first coefficient and neglecting members of the second order relative to \(\Pi\) we find the mean curvature

\[
K = \frac{1}{R} \left\{-1 - \frac{\tan \psi}{2} (r, \Pi_\psi) + \frac{(r, \Pi_{\psi \phi})}{2 \cos^2 \psi} + \frac{(r, \Pi_{\psi \psi})}{2} + (l, \Pi_\psi) + \frac{(k, \Pi_\phi)}{\cos \psi} \right\}.
\]

The force which act on the surface at a point \((\phi, \psi)\) is directed along the normal \(n\) (see (6)). It is proportional to \(K\) with the coefficient of proportionality which we denote by \(\beta\). The inner pressure gives the force which is proportional to \(\frac{1}{R} n\). To have an equilibrium for the case when the displacement of surface equals to zero (sphere), the coefficient of proportionality must be also equal to \(\beta\). The summary force is equal

\[
\beta r \left\{-\frac{\tan \psi}{2} (r, \Pi_\psi) + \frac{(r, \Pi_{\psi \phi})}{2 \cos^2 \psi} + \frac{(r, \Pi_{\psi \psi})}{2} + (l, \Pi_\psi) + \frac{(k, \Pi_\phi)}{\cos \psi} \right\}.
\]

One obtains the equation

\[
R \frac{\partial^2 \Pi}{\partial t^2} = \beta r \left\{-\frac{\tan \psi}{2} (r, \Pi_\psi) + \frac{(r, \Pi_{\psi \phi})}{2 \cos^2 \psi} + \frac{(r, \Pi_{\psi \psi})}{2} + (l, \Pi_\psi) + \frac{(k, \Pi_\phi)}{\cos \psi} \right\}.
\]

Rewrite the equation (8) in the moving frame coordinates: \(u = (r, \Pi), v = (k, \Pi), w = (l, \Pi)\).

\[
\frac{1}{\beta} \frac{\partial^2 u}{\partial t^2} = \frac{1}{R} \left\{-\frac{\tan \psi}{2} u_\psi + \frac{u_{\psi \phi} + (\cos^2 \psi) u_{\psi \psi}}{2 \cos^2 \psi} + u_\psi + \frac{u_\phi}{\cos \psi} \right\},
\]

\[
\frac{1}{\beta} \frac{\partial^2 v}{\partial t^2} = (k, \frac{\partial^2 \Pi}{\partial t^2}) = 0,
\]

\[
\frac{1}{\beta} \frac{\partial^2 w}{\partial t^2} = (l, \frac{\partial^2 \Pi}{\partial t^2}) = 0.
\]

If at the initial moment \(v = 0, \dot{v} = 0\) and \(w = 0, \dot{w} = 0\) then \(v \equiv 0, w \equiv 0\). Thus, the linearization leads to purely radial vibrations. The final equation is

\[
\frac{1}{\beta} \frac{\partial^2 u}{\partial t^2} = \frac{1}{R} \left\{-\frac{\tan \psi}{2} u_\psi + \frac{u_{\psi \phi} + (\cos^2 \psi) u_{\psi \psi}}{2 \cos^2 \psi} \right\}.
\]

As expected, the right hand side of the equation (9) gives the Laplace-Beltrami operator on the sphere.

As soon as the equation (9) is autonomous relative to \(\phi\), we find the solution in the form of traveling wave relative to \(\phi\), that is \(u = u(\phi - \kappa t, \psi)\). We have

\[
\frac{\kappa^2}{\beta} u_{\phi \phi} = \frac{1}{R} \left\{-\frac{\tan \psi}{2} u_\psi + \frac{u_{\phi \phi} + (\cos^2 \psi) u_{\psi \psi}}{2 \cos^2 \psi} \right\}.
\]

Separation of variables \(u = \Phi(\phi)\Psi(\psi)\) gives
\[ \frac{\Phi''}{\Phi} = \frac{(-\sin \psi \cos \psi)\Psi' + (\cos^2 \psi)\Psi''}{2R \kappa^2 \cos^2 \psi - \beta} = C. \]

The period of the function \( \Phi \) must be equal to \( 2\pi \), hence \( C = -n^2 \), \( n \in \mathbb{Z} \). The equation for \( \Psi \) has the form

\[
(\cos^2 \psi)\Psi'' - (\sin \psi \cos \psi)\Psi' + (2n^2R \kappa^2 \cos^2 \psi - n^2 \beta)\Psi = 0. \tag{10}
\]

The boundary conditions for the equation (10) are

\[
\Psi\left(\frac{\pi}{2}\right) = \Psi\left(-\frac{\pi}{2}\right) = 0, \tag{11}
\]

since the function \( \Psi \) does not depend on \( \phi \) at poles. The expression \( \alpha = 2n^2R \kappa^2 \) can be considered as the spectral parameter for the problem (10)-(11). Zonal spherical harmonics are its solutions.

The spectral problem (10)-(11) appears exactly the same as (3)-(4). The striking fact that entirely different approaches and different phenomena lead to the same equation (10)=(3) give confidence to the Hamiltonian (1).

Eigenfunctions \( u_{m,l} \) of the Laplace-Beltrami operator on the two-dimensional sphere are enumerated by pairs of integers \((m, l)\), \(|m| \leq l \) and relate to the eigenvalues \(-l(l+1)\).

In accordance with the established linguistic tradition, we shall call particles which are obtained by quantization of the above-mentioned waves (both longitudinal and cross waves) by spherons.

**Theorem 1** The solution \( u = 0 \) of the equation (9) is stable in Liapunov’s sense.

**Proof**. The equation (9) can be rewritten as the ordinary differential equation in the space \( l^2 \)

\[
\frac{\partial^2 u}{\partial t^2} = Au. \tag{12}
\]

The spectrum of the operator \( A \), which stand at the right hand side of the equation (12), equals

\[
\lambda_{m,l} = -l(l+1), \quad l \geq |m|.
\]

The equation (12) can be written in normal form as a system

\[
\frac{\partial u}{\partial t} = v, \quad \frac{\partial v}{\partial t} = Au. \tag{13}
\]

The spectrum of the matrix

\[
\begin{pmatrix}
0 & I \\
A & 0
\end{pmatrix},
\]

standing at the right hand side of the system (13), is purely imaginary. It is equal
Indeed, if \( f_k, \ k \in \mathbb{N} \) is a basis of eigenvectors of the operator \( A \) responding to eigenvalues \( -\lambda_k^2 \), then
\[
  u = f_k e^{\pm i\lambda_k t}, \quad v = \pm i\lambda_k f_k e^{\pm i\lambda_k t}
\]
is a fundamental system of solutions to the equation (13).

The theorem is proved.

\( \square \)

2. Second quantization

We will quantize the equation obtained in momentum space. The role of the momentum space will play the space of spherical harmonics. Let us note that it has the discrete basis that may be defined by eigenfunctions of the Laplace-Beltrami operator. The technique of quasi-discrete representation, i.e. the preliminary consideration of the problem in finite parallelepiped with periodic boundary conditions and after that the passage to the limit from discrete case to the continuous momentum representation [6] (one of the feeble item in the ground of quantum statistics), is found to be unnecessary due to compactness of the phase state.

Denote the spherical harmonics by \( v_{m,l}(s), \ s \in S^2 \). Expand a solution to the equation
\[
\frac{\partial^2 u}{\partial t^2} = \Delta u,
\]
where \( \Delta \) is the Laplace-Beltrami operator on the sphere \( S^2 \), in the series in terms of spherical harmonics \( v_{m,l}(s) \). Coefficients of this series are defined by the expressions
\[
\hat{u}_{m,l}(t) = \int_{S^2} u(s,t)v_{m,l}(s) \, ds.
\]

Apply the operator \( \frac{\partial^2}{\partial t^2} \) to the (15) taking into account that the Laplace-Beltrami operator is selfajoint and that the functions \( v_{m,l}(s) \) are its eigenfunctions with the eigenvalues \(-l(l+1)\)
\[
\frac{\partial^2 \hat{u}_{m,l}(t)}{\partial t^2} = \int_{S^2} \Delta u(s,t) \, v_{m,l}(s) \, ds = \int_{S^2} u(s,t) \, \Delta v_{m,l}(s) \, ds = -l(l+1)\hat{u}_{m,l}(t).
\]

One obtains a decomposed infinite system of ordinary differential equations
\[
\frac{\partial^2 \hat{u}_{m,l}(t)}{\partial t^2} = -l(l+1)\hat{u}_{m,l}(t). \tag{16}
\]

Each equation of the system (16) is the equation of harmonic oscillator with the unit mass and the frequency \( \sqrt{l(l+1)} \). We shall call it by the oscillator \( (m,l) \). Apply to it the standard procedure of quantization. That leads to second quantization of the oscillation of the geodesic flow of electrons.

The Hamiltonian of the oscillator \( (m,l) \) is
\[
H_{m,l}^0 = \frac{1}{2}(p^2 + l(l + 1)q^2).
\]

We enumerate eigenfunctions of the operator \( H_{m,l}^0 \) by indices \( n \) and denote it by \( \xi_n(m,l) \) (the number of an eigenfunction is denoted by the lower index). It is well known that the corresponding eigenvalues are \( E_n = (n + \frac{1}{2})\sqrt{l(l + 1)} \). Introduce the operators \([7]\) acting on eigenfunction as follows

\[
a^+(m,l)\xi_n(m,l) = \sqrt{n + 1}\xi_{n+1}(m,l), \quad a^-(m,l)\xi_n(m,l) = \sqrt{n}\xi_{n-1}(m,l).
\]  

(17)

The ground-state of the system corresponds to \( n = 0 \), since in this case the next conditions are fullfiled

\[
a^-(m,l)\xi_0(m,l) = 0, \quad \xi_n(m,l) = (a^+(m,l))^{n}\xi_0(m,l).
\]  

(18)

The formulas (17) show that states \( \xi_n(m,l) \) are eigenfunctions of the operators \( a^-(m,l)a^+(m,l) \) and \( a^+(m,l)a^-(m,l) \) with the eigenvalues \( n + 1 \) and \( n \) correspondingly. Consequently, it is natural to call the operator \( a^+(m,l)a^-(m,l) \) by the operator of number of spherons of type \( (m,l) \). Denote this operator by \( \hat{n}^s(m,l) \). The commutation relations take the standard form

\[
[a^-(m,l),a^+(m,l)] = 1, \quad [a^-(m,l),a^-(m,l)] = [a^+(m,l),a^+(m,l)] = 0.
\]

Meanwhile operators that relate to noncoinciding values of indices commute. The Hamiltonian for an oscillator is \( H_0^0(m,l) = \sqrt{l(l + 1)}(a^+(m,l)a^-(m,l) + \frac{1}{2}I^s(m,l)) \), or

\[
H_0^0(m,l) = \sqrt{l(l + 1)}(\hat{n}^s(m,l) + \frac{1}{2}I^s(m,l)),
\]

where \( I^s(m,l) \) is the identity operator on the space of states of the oscillator \( (m,l) \). The full Hamiltonian of the field of the cross waves is represented by the sum

\[
H^0 = \sum_{l,|m|\leq t} H_0^0(m,l) = \sum_{l,|m|\leq t} \sqrt{l(l + 1)}(\hat{n}^s(m,l) + \frac{1}{2}I^s(m,l)).
\]

(19)

One can express the Hamiltonian in the representation of occupation numbers. For brevity sake, let us relabel Fourier coefficients \( (m,l) \) of spherical harmonics by unique index \( h \) and denote the corresponding operators by \( a^+(h) \) and \( a^-(h) \). The state of the system is characterized by a set of occupation numbers \( h_1, h_2, ..., h_N \), i.e. the state

\[
\xi(h_1, ..., h_N) = \prod_{1 \leq k \leq N} \frac{(a^+(k))^{h_k}}{\sqrt{h_k!}}\xi_0(k)
\]

means that there are \( h_i \) particles of the sort \( \xi_i \).

The mechanism of the superconductivity of plasma is somewhat different from that of electron-phonon interaction in BCS-theory. The later is connected with interactions of electrons and vibrations of ion lattice, that is with something external relative to the electron flow itself. Whereas electron-spheron interaction is the interaction of electrons
with vibrations of the flow of electrons. In other words, this is something like the selfconsistency of Hartree [26]. The only difference is that in Hartree theory the movement of an electron is consistent with its own field, whereas in plasma, the movement of an electron which affect the creation of the field of the flow must be consistent with the field of the flow as a whole. In this situation the field of the flow is not the electromagnetic field, but the field of spheron’s oscillations. Roughly speaking, fluctuations in state of an electron as if “damp” the electron flow giving to it quants of energy. In response, the coherent flow slightly “pushes” this electron returning it back into “formation”.

As the components of the field of the flow it may be taken longitudinal and cross waves. The equation (3) describes longitudinal oscillations in the absence of cross ones; the equation (10) describes cross oscillations in the absence of longitudinal ones. But in the general case longitudinal and cross oscillations are mutually connected. Local deformations of the sphere lead to the changes of density; local decreasing of density decrease local forces of surface tension that leads to deformation of the surface. Note that the increasing of the function \( f \) decrease the function \( \Psi \) and vice versa. In other words oscillations of the longitudinal and cross waves are in opposite phase. Taking into account that these oscillations satisfy to the same equation, it will be supposed that \( \Psi = -f \).

So spherons will be described by a scalar field with the Hamiltonian (19). Note further, that an elementary wave of density arrive at the diametrically opposite point being in the same phase as at the initial point, that is the excess of density at the initial point leads to the excess of density after focussing. Thus we will be interested only in even functions relative to the antipodal mapping. Hence, the spectral parameter \( l \) must be an even number. The corresponding system of eigenfunctions are complete in the projective space \( \mathbb{R}P^2 \). The Hamiltonian of the field of spherons takes the form

\[
H^0 = \sum_{l,|m|\leq 2l} \sqrt{2l(2l+1)}(\hat{n}^*(m,2l) + \frac{1}{2}\hat{l}^*(m,2l)).
\] (20)

Our aim is to write the Hamiltonian which consists of
1). the free Hamiltonian of the spherons field (20),
2). the free Hamiltonian of the field of electrons, and
3). the interaction Hamiltonian.

3. The gap

We wish to reduce the situation to the case of the Bose statistics as in usual theory of superconductivity. In that theory it was used quasi-particles — Cooper’s pairs. There is some mystery in the fact that a pair consists of electrons with the opposite momentum and spins. It is rather strange that particles flying in opposite directions are linked. Namely Cooper that proposes this idea calls this connection ”deeply mysterious”. Though, there was some statistical arguments in favour of it (see, for instant, [29]). The main reason is that the combined spin becomes zero which leads to the Bose statistics. So in the state with minimum energy there are plenty of particles, so called coherent condensate, having high-capacity inertia which obstacle small changes. Together with the energetic gap this gives the intuitive explanation of the superconductivity.

In our case the combination in Cooper’s pairs is much more natural than in BCS-theory. We again combine electrons with the opposite spin and momentum. But now electrons of the pair move along the same geodesic in the same direction. It is convenient
to think on it as situated at the diametrically opposite points. More exactly, the support
of the function that describes the joint state of the pair belongs to the antipodal diagonal
of the direct product $S^2 \times S^2$, that is it belongs to the manifold $\{x, -x\} \in S^2 \times S^2$. In
particular this means that if one of the electrons was detected at the point $x$ then its
pair will be at $-x$. Thus if one of the electrons of the pair jumps from the state with
the momentum $p$ into that with the momentum $k$ then the wave of density having the
concentration at diametrically opposite points transfers its pair from the state with the
momentum $-p$ into that with the momentum $-k$.

Let us show that combining electrons in Cooper’s pair provides the gap in energy
spectrum. Denote the width of the electronic cover by $r_0$. The phase space for electronic
cover related to spheres with radii from $R$ to $(R + r_0)$ is the direct product of the man-
ifold of geodesics on a sphere (that is $\mathbb{RP}^2$) by the interval $[R, R + r_0]$. An momentum
$k$ and a spin, taking values $\pm 1/2$, parametry eigenfunctions. The set of eigenfunctions
plays the part of a crystalline lattice being something like a framework maintaining super-
conductivity. Accidental disturbance in density of electrons spreads along the sphere as a
spheron and compensates the related disturbance. It is possible since the phase velocity
of waves is far much than that of electrons. The kinetic energy of an electron is defined
by the radius of the sphere on which it moves. Aside from interchanging processes due
to sphersons, the potential energy is defined by interelectron repulsion which we will often
neglect in what follows. Interchanging processes give the effective attraction, and this attrac-
tion is far-acting since oscillations of electronic cover is a global process acting on the
whole sphere. So the interchanging processes give a negative contribution into potential
energy of the system. That is the physical reason for arising of the energetic gap in the
spectrum. Let us note that the role of Debye wavelength which defines the characteristic
linear length of the system, plays the width of the electronic cover $r_0$.

Denote by $Q$ the charge of the kernel of a fireball, and by $R$ its radius. Let $\rho$ be the
average volume density of ions, and $e$ its charge (in the case of single ionization). One
has $Q = \frac{4}{3} \pi R^3 \rho e$. Let $\mu$ be the average volume density of electrons in the cover. The
volume between spheres of the radii $R$ and $R + r$ equals

$$\frac{4}{3} \pi (R + r)^3 - \frac{4}{3} \pi R^3.$$

The attractive force to the kernel acting on an electron that lie at the distance $R + r$
from the center of the kernel equals $\frac{Qe}{(R+r)^2} = \frac{4\pi \rho R^3 e^2}{3(R+r)^2}$. But this force is shielded by the negative charge of the layer between radii $R$ and $R + r$. Denote the shielding coefficient (depended on $r$ only) by $\kappa(r)$. Then the shielded attractive force of the kernel equals $\kappa(r) \frac{4\pi \rho R^3 e^2}{3(R+r)^2}$. Besides this force, on an electron acts the force of repulsion from electrons of the layer $(R, R + r)$ only, since the resultant force acting from the external layer equals to zero. The resultant force acting from the internal layer equals to that of repulsion from the center induced by the overall charge of the layer $(R, R + r)$. Hence, the full force is equal to

$$F = \kappa(r) \frac{4\pi \rho R^3 e^2}{3(R+r)^2} - \left( \frac{4\pi (R + r)^3}{3(R+r)^2} - \frac{4\pi R^3}{3(R+r)^2} \right) \mu e^2.$$

This force equals to zero at $r = r_0$, consequently, the equation defining $r_0$ is

$$\kappa(r_0) \rho R^3 = ((R + r_0)^3 - R^3) \mu.$$

(21)
The equation (21) is in a sense the condition of neutrality of the fireball. If one denote \( \frac{\mathbf{r}}{R} = \delta \), then the equation for \( \delta \) is

\[
\delta^3 + 3\delta^2 + 3\delta = \frac{\kappa(\delta)\rho}{\mu}.
\]  

(22)

Find the velocity \( v \) of electron moving at the orbit of radius \( R + r \). The equation of movement at the plane \((x, y)\) are

\[
x = (R + r) \cos \frac{v}{R + r}t; \quad y = (R + r) \sin \frac{v}{R + r}t.
\]

By differentiation one obtains the centrifugal acceleration

\[
a = \frac{F}{m} = \frac{v^2}{R + r},
\]

where \( m \) is the mass of the electron. Hence,

\[
v = \sqrt{\frac{F(R + r)}{m}}.
\]

The kinetic energy of an electron is

\[
\varepsilon(R + r) = \frac{mv^2}{2} = \frac{2\pi e^2}{3(R + r)}[\kappa(r)\rho R^3 - \mu((R + r)^3 - R^3)].
\]

It follows that

\[
\varepsilon(R) = \frac{2\pi e^2}{3} R^2.
\]

Besides, in view of (22), \( \varepsilon(R + r_0) = 0 \).

As in [28], imagine all admissible states in which electrons with momentum and spin \(((p, 1/2), (-p, -1/2))\) are combined in pairs. The scattering of a pair \(((p, 1/2), (-p, -1/2))\) on a pair \(((k, 1/2), (-k, -1/2))\) means the transition from the state \( \psi \) in which the cell \(((k, 1/2), (-k, -1/2))\) is free and the cell \(((p, 1/2), (-p, -1/2))\) is occupied, into the state \( \chi \) being in opposite condition. Denote by \( x_p^2 \) the probability that a cell \(((p, 1/2), (-p, -1/2))\) is occupied. Then the probability that a cell \(((k, 1/2), (-k, -1/2))\) is free equals \( y_k^2 = 1 - x_k^2 \). The probability of the state \( \psi \) in which the cell \(((k, 1/2), (-k, -1/2))\) is free and the cell \(((p, 1/2), (-p, -1/2))\) is occupied is the product of probabilities \( P^2_\psi = x_p^2(1 - x_k^2) \). The amplitude of the state \( \psi \) is \( P_\psi = x_p y_k \). Analogously the amplitude of the state \( \chi \) in which the cell \(((k, 1/2), (-k, -1/2))\) is free and the cell \(((p, 1/2), (-p, -1/2))\) is occupied equals \( P_\chi = x_k y_p \). Define the energy of an electron with the momentum \( p \) by \( \varepsilon(p) \), and denote the matrix element of the potential energy of interaction for the process in question by \( W_{p,k} \). The full energy of electronic cover (discarding the Coulomb forces of electronic repulsion) is

\[
E = \sum_p 2\varepsilon(p)x_p^2 + \sum_{p,k} W_{p,k}x_p x_k y_p y_k = \sum_p 2\varepsilon(p)x_p^2 - W(\sum_p x_p y_p)(\sum_k x_k y_k).
\]

(23)

Here \( W_{p,k} \) is changed by \( -W \) for momentum related to tangent bundle to spheres of radii in the interval \((R, R + r_0)\), and \( W_{p,k} = 0 \) for other momentum. The factor 2 of
the first sum arises from the fact that electrons in the states both \((p, 1/2)\) and \((p, -1/2)\) have the energy \(\varepsilon(p)\).

The ground-state gives the minimum to the energy \(E\). Let us find this minimum.

**Theorem 2** *The following relation is fulfilled at the point of minimum of the energy \(E\)*

\[ 1 = \sum_l \frac{2W}{\sqrt{\Delta^2 + \varepsilon^2(l)}}, \]

*where*

\[ \Delta = W \sum_k x_k y_k = W \sum_k \sqrt{x_k^2 - x_k^4}. \]

**Proof.**

Differentiate the expression for \(E\) relative to \(x_l^2\)

\[ \frac{\partial E}{\partial (x_l^2)} = 2\varepsilon(l) - W \left( \sum_k x_k y_k \right) \frac{1 - 2x_l^2}{2x_l y_l} = 0. \]

The notation used in the statement of the theorem:

\[ \Delta = W \sum_k x_k y_k = W \sum_k \sqrt{x_k^2 - x_k^4} \quad (24) \]

gives the possibility to rewrite the preceding equation in the form

\[ 2\varepsilon(l) \sqrt{x_l^2 - x_l^4} = \Delta (1 - 2x_l^2). \quad (25) \]

By squaring we obtain the quadratic equation relative to \(x_l^2\).

\[ x_l^4 - x_l^2 + \frac{\Delta^2}{4(\Delta^2 + \varepsilon^2(l))} = 0. \]

\[ x_l^2 = \frac{1}{2} \left( 1 \pm \sqrt{\frac{\varepsilon^2(l)}{\Delta^2 + \varepsilon^2(l)}} \right). \]

The sign minus should be taken in this expression, since otherwise the right hand side of the equation (25) would be negative. So

\[ y_l^2 = \frac{1}{2} \left( 1 + \sqrt{\frac{\varepsilon^2(l)}{\Delta^2 + \varepsilon^2(l)}} \right). \]

We shall show below that, as in the classical theory of superconductivity, the value \(\Delta\) corresponds to the gap in the energy spectrum of the system. Substituting this expression into the formula (24), we obtain the analog of the famous gap equation:

\[ \Delta = \sum_l \frac{W}{2} \left( 1 - \frac{\varepsilon^2(l)}{\Delta^2 + \varepsilon^2(l)} \right)^{1/2} \]

or

\[ 1 = \sum_l \frac{2W}{\sqrt{\Delta^2 + \varepsilon^2(l)}} \quad (26) \]
The theorem is proved. □

Let us introduce \( \nu(\varepsilon) \) — the density of states of electrons with the energy \( \varepsilon \). Using this notion rewrite the gap equation in an integral form

\[
1 = W \int_{\varepsilon(R)}^{\varepsilon(R+r_0)} \frac{\nu(\varepsilon)d\varepsilon}{2\sqrt{\Delta^2 + \varepsilon^2}}.
\]

The oscillation of the function \( \nu(\varepsilon) \) on the interval \((R, R + r_0)\) is insignificant. We shall denote \( \nu(\varepsilon) \) simply by \( \nu \) and it can be carried out of the sign of the integral. We obtain

\[
\frac{2}{W\nu} = \int_{\varepsilon(R+r_0)}^{\varepsilon(R)} \frac{d\varepsilon}{\sqrt{\Delta^2 + \varepsilon^2}}.
\] (27)

The integral at the right hand side of (27) monotonically decreases from \(+\infty\) to 0 as \( \Delta \) runs from 0 to \(+\infty\). Consequently, a solution \( \Delta \) to the equation (27) exists and is unique. Let us find the explicit expression for \( \Delta \) by calculating the integral (27).

\[
\frac{2}{W\nu} = \text{Arsh} \frac{\varepsilon(R)}{\Delta} - \text{Arsh} \frac{\varepsilon(R + r_0)}{\Delta} = \text{Arsh} \frac{2\pi e^2 \rho R^2}{3\Delta}.
\]

Hence,

\[
\text{sh} \frac{2}{W\nu} = \frac{2\pi e^2 \rho R^2}{3\Delta}.
\] (28)

It follows

\[
\Delta = \frac{2\pi e^2 \rho R^2}{3} \text{sh}^{-1} \frac{1}{2W\nu}. \quad (29)
\]

Let us show that \( \Delta \) is actually the gap in the energy spectrum. Having this in mind, we evaluate the contribution \( \xi(l) \) from the adding of the combined pair of electrons with momentum \( \pm l \) into the energy of the ground-state (when all electrons are combined in pairs). From (23) we have

\[
\xi(l) = 2\varepsilon(l)x_l^2 - 2Wx_l y_l \sum_k x_k y_k.
\]

Indeed, the first summand in this formula is the kinetic energy of the pair \((l, -l)\), and the second is the contribution into the negative part of the energy of the ground-state arising from the possibility of interaction processes of the pair in question with all other pairs \((k, -k)\). The coefficient 2 reflects the fact that the pair \((l, -l)\) in the sum (23) occurs twice: in summing by \( k \), and in summing by \( p \). Introduce the notation \( E_l = \sqrt{\Delta^2 + \varepsilon^2(l)} \).

Taking into account values: \( x_l^2 = \frac{1}{2}(1 - \frac{\varepsilon(l)}{E_l}) \), and denotation \( \Delta = W \sum_k x_k y_k \), we find

\[
\xi(l) = 2\varepsilon(l) \frac{1}{2}(1 - \frac{\varepsilon(l)}{E_l}) - 2\Delta \left( \frac{1}{4}(1 - \frac{\varepsilon^2(l)}{E_l^2}) \right)^{1/2} = \\
\varepsilon(l) - \frac{\varepsilon^2(l)}{E_l} - \Delta \frac{\Delta}{E_l} = \varepsilon(l) - E_l.
\]
If to add to the ground-state merely one (unpaired) electron with the momentum $l$ then the pairs $(l, -l)$ cannot contribute into the energy of the ground-state, and hence the energy of the system after such adding become

$$E - \xi(l) + \varepsilon(l) = E + \sqrt{\Delta^2 + \varepsilon^2(l)}.$$ 

Consequently, the adding of an isolated electron raises the energy of the ground-state at least on $\Delta$. This means that to change the ground-state, i.e. to disjoin a combined pair, it is required the energy non less than $2\Delta$. This is the same as saying that there is a gap in the energy spectrum.

To estimate the shielding effect $\kappa(r)$, we will use the methodology of Tomas-Fermi, developed for computations of heavy metals spectra. The Tomas-Fermi model for centrally symmetric fields is based on the supposition that there are great many electrons in a domain where the potential changes slowly, and in this domain the Fermi statistics is applicable. This supposition is fulfilled in our model of fireballs.

Firstly, let us reminded the deduction of the Tomas-Fermi equation in suitable denotations. Denote by $p_0(r)$ the maximal absolute value of momentum of electrons situated at the distance $r$ from the center. Let us agree that $\frac{p_0^2(r)}{2m}$ is the excess of the energy of an electron over the level of the termal potential. We have

$$V(r) + \frac{p_0^2(r)}{2m} = 0.$$ \hspace{2cm} (30)

Let an electron moves freely in a domain $\Omega$ with the electron density $\mu = \frac{N}{\Omega}$. Then the number of quantum states with the absolute values of momentum from $p$ to $p + dp$ is

$$2 \frac{\Omega}{(2\pi)^3} 4\pi k^2 dk,$$

where $\hbar k = p$, and the factor 2 takes into account two possible orientations of the spin. We integrate from 0 to $k_0$ and equate the result to the full number of electrons $N$:

$$2 \frac{\Omega}{(2\pi)^3} 4\pi k_0^3 = N,$$

or

$$k_0^3 = 3\pi^2 \mu.$$

Let now suppose that a subdomain $O \subset \Omega$ is selected being sufficiently large for correctness of (30) and sufficiently small, such that the potential energy in it changes not too much. Then

$$\mu = \frac{(2m)^{3/2}}{3\pi^2 \hbar^3} (-V^{3/2}).$$ \hspace{2cm} (31)

By substitution (31) into the Poisson equation for electrostatic potential $-V/e$ with the density of charge $-e\mu$:

$$\Delta V = -4\pi e^2 \mu,$$
one obtains
\[ \frac{1}{r} \frac{d^2}{dr^2}(rV(r)) = -\frac{4e^2}{3\pi\hbar^3}(2m)^{3/2}(-V)^{3/2}. \]

Using dimensionless variables
\[ x = \frac{r}{b} = \frac{2^{7/3}}{(3\pi)^{2/3}}r; \quad f = -\frac{rV}{Qe^2}, \]
where \( Q \) is the total charge of the kernel, one obtains the Tomas-Fermi equation
\[ \frac{d^2f}{dx^2} = \frac{f^{3/2}}{\sqrt{x}}. \quad (32) \]

Let us apply the equation (32) to our theory of fireballs. While calculating heavy metal spectra, the following boundary values for the equation (32) is customarily used (cf. [28], [4])
\[ f(0) = 1; \quad \lim_{x \to \infty} f(x) = 0. \quad (33) \]

However, we need for our model another boundary conditions. It must be taken that at the distant \( R \) we have the Coulomb potential, and the cover ends at points where termal fluctuations of electrons majorize the attractive forces. At the boundary points, the potential will be equal to the fixed value \(-\zeta\), where \( \zeta \) corresponds to the termal energy of electrons at the temperature \( T \). The related boundary conditions are
\[ f(R) = 1; \quad f(R + r_0) = \frac{R + r_0}{Qe^2} \zeta. \quad (34) \]

Let us explore the geometrical structure of solutions to the equation (32).

**Theorem 3** The equation (32) admits the group \( G \) of scale symmetries allowing to reduce it to a two-dimensional system which does not depend on the argument \( x \).

**Proof.**
Let us rearrange the equation (32) as the system
\[ \frac{df}{dx} = g, \quad \frac{dg}{dx} = \frac{f^{3/2}}{\sqrt{x}}. \quad (35) \]

The group \( G \) can be found in the form \( f \mapsto \lambda f, \quad g \mapsto \mu g, \quad x \mapsto \nu x \). These transformations transfer the system (35) into the system
\[ \frac{\lambda df}{\nu dx} = \nu g, \quad \frac{\mu dg}{\nu dx} = \sqrt{\frac{\lambda^3}{\nu^3}} \frac{f^{3/2}}{\sqrt{x}}. \]

To hold the initial shape of the system (35) it is necessary that \( \lambda = \mu \nu, \quad \mu^2 = \lambda^3 \nu \)
or, equivalently, \( \lambda = \nu^{-3}, \quad \mu = \nu^{-4} \). Hence, the system (35) turns into itself if \( f \mapsto \nu^{-3} f, \quad g \mapsto \nu^{-4} g, \quad x \mapsto \nu x \). It allows to reduce the dimension of the system. To do this, one needs to take as new variables such combinations of old ones that are invariant under the action of \( G \). The simplest such combinations are \( y = x^3 f, \quad z = x^4 g \). In
order that the system in the new coordinates becomes autonomous, (i.e. does not content
the independent variable) we put \( x = \exp \rho \). In this case shifts of \( \rho \) transfers into
multiplications by a constants \( \nu \). Then the system (35) takes the form
\[
\begin{align*}
\frac{dy}{d\rho} &= 3y + z, \\
\frac{dz}{d\rho} &= 4z + \sqrt{y^2}.
\end{align*}
\] (36)

The theorem is proved.
\( \square \)

Now we can explore the phase portrait of the system (36) on the two-dimensional
phase plane \((y, z)\). Besides the obvious equilibrium at the origin we have the singular
point \( A = (y = 144, z = 3 \cdot 144) \). By returning to the initial variables, the singular
point \( A \) gives the famous partial solution to the equation (32) guessed by Sommerfeld,
\( f = 144x^{-3} \). It is agreed (cf. \[4\]) that this solution gives adequate asymptotics of the
exact solution for atoms as \( r \to +\infty \), but does not satisfy the boundary conditions at the
origin. This fact is understandable if to use the system (36). It is easy to check that the
origin is the unstable knot of the system (36), and the point \( A \) is the saddle. The exact
solution for the atom that satisfies both boundary conditions is the separatrix passing
from the origin and tending to \( A \) as \( \rho \to +\infty \). It is not hard to show the existence of the
separatrix. And for it \( r \to 0 \) as \( \rho \to -\infty \).

However we are interesting in boundary conditions (34). In terms of the system (36)
these boundary conditions transfer into the following ones (let us recall that \( y = x^3f, z = x^4g \).)
\[
y(\rho_0) = \frac{2^7R^3}{(3\pi)^2}, \quad y(\rho_1) = \frac{2^7(R + r_0)^4}{Qe^2(3\pi)^2} \zeta,
\]
where
\[
\rho_0 = \ln \frac{2^{7/3}R}{(3\pi)^{2/3}}, \quad \rho_1 = \ln \frac{2^{7/3}(R + r_0)}{(3\pi)^{2/3}}.
\]

Finally we have
\[
\begin{align*}
y(\ln \frac{2^{7/3}R}{(3\pi)^{2/3}}) &= \frac{2^7R^3}{(3\pi)^2}, \\
y(\ln \frac{2^{7/3}(R + r_0)}{(3\pi)^{2/3}}) &= \frac{2^7(R + r_0)^4}{Qe^2(3\pi)^2} \zeta.
\end{align*}
\]

Thus we need a trajectory of the phase plane \((y, z)\) that starts with the vertical line
\( y = \frac{2^7R^3}{(3\pi)^2} \) and leads to the other vertical line \( y = \frac{2^7(R + r_0)^4}{Qe^2(3\pi)^2} \zeta \) during the time \( \rho_1 - \rho_0 = \ln r_0 + \frac{1}{3} \ln \frac{128}{(3\pi)^2} \) Coordinates of the final point \((y(\rho_1), z(\rho_1))\) of that trajectory give the
derivative of the potential \( \frac{\partial \zeta}{\partial r}(R + r_0) \), and that in its turn is equal to the shielded force
of attraction of the kernel. Because of \( z = gx^4, g = \frac{\partial f}{\partial x}, f = -\frac{rV}{Qe^2} \), we obtain
\[
\begin{align*}
z(\rho_1) &= \frac{(R + r_0)^4}{k^4} \frac{\partial (eV)}{\partial r} \partial r \\
&= \frac{-2Qe^2}{(3\pi)^2} (V + r \frac{\partial (eV)}{\partial r}) = \frac{Qe}{2Qe^2} \left( \zeta + \kappa(r_0) \frac{Qe}{(R + r_0)} \right) \quad \text{(37)}
\end{align*}
\]

Substitute into the equation (37) the value \( \kappa(r_0) \) from the equation (21). We obtain
the relationship for finding the unknown parameter \( r_0 \). Let us remark that it is natural
to take for \( R \) values of fireballs that have been surveyed \( R \approx 10 \text{cm} \) (cf. \[23\]).
The system (37), (21) can be solved by Newton’s method. It is possible to take for initial data nonshielded potential \( V_1(r) = -Qe^2/(R+r) \); \( \kappa_1(r_0) = 1 \), and sufficiently small value \( r_0 \), say, \( (r_0)_1 = R/100 \). Find the corresponding value \( (\rho_0), (\rho_1), y(\rho_0), y(\rho_1) \). Using (36) and reversing the time current, we obtain the trajectory \( y_1(\cdot), z_1(\cdot) \). To calculate the derivative of the first approximation, we have to consider the variational equation of the system (36) for the trajectory \( y_1(\cdot), z_1(\cdot) \). We obtain the derivative of \( \kappa_1 \) at the point \( \kappa_1(r_0) \). Using Newton’s algorithm, find \( \kappa_2 \) and new \( (r_0)_2 \). After that the process is repeated.

4. The field of electrons.

Our aim is to quantize the field of electrons itself that moves at trajectories of the geodesic flow on the sphere. To simplify the construction we will take the tangent bundle of unit tangent vectors \( T^1S^2 \) to the two-dimensional sphere as the phase space. This gives us the possibility disregard the electromagnetic field of the kernel, since its attraction was properly accounted. Objects of quantization are pairs of electrons, i.e. bosons having integral-valued spin.

One of the way is to quantize the manifold \( T^1S^2 \), using the technique of geometrical quantization in terms of Kirilov-Kostant-Souriau [2], [14], [15], [16], [19]. [21], [22], [25], [27]. It would be very interesting to try this approach and to develop the technique of geometrical quantization, considering bundles over the Kepler manifold and the projectivization of its sections as states.

But we will use another approach. As far as electrons are moving at trajectories of the geodesic flow, we take as the main object of the theory (quasi-particles) geodesics itself. Consider the manifold of nonoriented geodesics on the sphere \( S^2 \) [3]. This manifold is \( \mathbb{R}P^2 \). Let us recall that the tangent bundle of unit tangent vectors \( T^1S^2 \) to the two-dimensional sphere is \( \mathbb{R}P^3 \). The stereographic projection of \( \mathbb{R}P^3 \) from the center of hemisphere induces on \( \mathbb{R}^3 \) the metric of the elliptic space \( Ell(3) \). The geodesic flow gives by this projection the set of parallel (in the sense of Clifford) lines. This set constitute the manifold \( \mathbb{R}P^2 \). The distant between Clifford-parallels remains constant in the metric of \( Ell(3) \). It is natural to take it as a distant between geodesics on \( S^2 \). So, our quasi-particles are points of \( \mathbb{R}P^2 \).

Let us find interrelation of quasi-particles. Denote the corresponding potential by \( V(\rho(x,y)) \), where \( \rho(x,y) \) is the distant from \( x \) to \( y \) in \( \mathbb{R}P^2 \). To be uniquely defined on \( \mathbb{R}P^2 \times \mathbb{R}P^2 \), the function \( V(\rho) \) must be zero at infinity, i.e. \( V(\pi/2) = 0 \), and besides, \( V(\rho) \to \infty \) as \( \rho \to 0 \). This defines a dynamical system on a set of points of \( \mathbb{R}P^2 \). For finite sets of points it is the system of ordinary differential equations. Equidistant systems of points on \( \mathbb{R}P^2 \) are stable equilibriums. These equilibriums are not isolated, since the set of points can move along \( \mathbb{R}P^2 \) as a rigid body. If equidistant systems do not exist then we have no equilibriums. But, in view of condition \( V(\rho) \to \infty \) as \( \rho \to 0 \), for any fixed value of energy, distant between points during the motion admits lower estimation by a positive constant.

Thus, the phase space is the projective plane \( \mathbb{R}P^2 \) (recall that spherons are parametrized by points of \( \mathbb{R}P^2 \) also). To find the law of interaction of quasi-particles, let us calculate the interaction of two uniformly charged geodesics on \( S^2 \).
Theorem 4 The potential of interaction of two uniformly charged geodesics on $S^2$ equals

$$\frac{1}{\sin \theta},$$

where $\theta$ is the angle between these geodesics.

Proof.
First of all, find the interaction of a point $A \in S^2$ with the uniformly charged geodesic $\sigma_1 \in S^2$. Introduce in $S^2$ the spherical coordinates $x = \cos \phi \cos \psi$, $y = \sin \phi \cos \psi$, $z = \sin \psi$. Without loss of generality, take $\sigma_1$ as

$$x = \cos t, \quad y = \sin t, \quad z = 0, \quad (0 \leq t \leq 2\pi).$$

Take a point $A$ with the coordinates $(\phi, \psi)$. The vector $l$ that joins $A$ with the moving point $t$ of $\sigma_1$, is

$$l = (\cos \phi \cos \psi - \cos t, \sin \phi \cos \psi - \sin t, \sin \psi),$$

$$|l| = \sqrt{2 - 2 \cos t \cos \phi \cos \psi - 2 \sin t \sin \phi \cos \psi}.$$

The resultant force for all points of the $\sigma_1$ is aligned with the meridian passing through the point $A$ that is orthogonal to $\sigma_1$, i.e. along the tangent to the circle $(x = \cos \phi \cos s$, $y = \sin \phi \cos s$, $z = \sin s$ at the point $s = \psi$. This is the vector $e = (-\cos \phi \sin \psi, -\sin \phi \sin \psi, \cos \psi)$. The angle between $l$ and $e$ will be denoted by $\gamma$.

Then

$$\cos \gamma = (l, e) = \frac{\cos t \cos \phi \sin \psi + \sin t \sin \phi \sin \psi}{\sqrt{2 - 2 \cos t \cos \phi \cos \psi - 2 \sin t \sin \phi \cos \psi}}.$$

The element of the force acting on $A$ from the portion $dt$ of the circle $\sigma_1$, equals

$$dF(t) = \frac{ldt}{|l|^3} = \frac{ldt}{(2 - 2 \cos \psi \cos(t - \phi))^{3/2}}.$$

The projection on the direction of the resultant is

$$(e, dF(t)) = 2 \int_0^\pi \sin \psi \cos(t - \phi) \frac{d\tau}{(2 - 2 \cos \psi \cos(t - \phi))^2}.$$

The force acting to the point $A$ from all circle $\sigma_1$ equals

$$\int_0^{2\pi} \frac{\sin \psi \cos(t - \phi)dt}{(2 - 2 \cos \psi \cos(t - \phi))^2} = 2 \int_0^\pi \frac{\sin \psi \cos(t)dt}{(2 - 2 \cos \psi \cos(t))^2}.$$

Rearrange this integral by the change of variable $\tan \frac{\tau}{2} = u$

$$\frac{\sin \psi}{2 \cos^3 \frac{\tau}{2}} \int_0^\infty \frac{(1-u^2)du}{(u^2+\tan^2 \frac{\tau}{2})^2} = \pi \frac{\cos \psi}{2 \sin^2 \psi}.$$

Find the interaction of the geodesic $\sigma_1$ with that $\sigma_2$ obtained from it by rotation round the axis $Ox$ by an angle $\theta$. 

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\[ \sigma_2 = (\cos \tau, \sin \tau \cos \theta, \sin \tau \sin \theta). \]

The spherical coordinates of a moving point \( \tau \in \sigma_2 \) are

\[ \tan \phi_\tau = \cos \theta \tan \tau; \quad \cos \phi_\tau = \frac{1}{\sqrt{1+\cos^2 \theta \tan^2 \tau}}; \quad \sin \phi_\tau = \frac{\cos \theta \tan \tau}{\sqrt{1+\cos^2 \theta \tan^2 \tau}}; \]
\[ \sin \psi_\tau = \sin \tau \sin \theta; \quad \cos \psi_\tau = \sqrt{1-\sin^2 \tau \sin^2 \theta}. \]

Let us remark that \( \sqrt{1-\sin^2 \tau \sin^2 \theta} = \sqrt{\cos^2 \tau + \sin^2 \tau \cos^2 \theta} \). The force \( F_\tau \) with the absolute value

\[ |F_\tau| = \frac{\pi \cos \psi_\tau}{2 \sin^2 \psi_\tau} = \frac{\pi \sqrt{\cos^2 \tau + \sin^2 \tau \cos^2 \theta}}{2 \sin^2 \tau \sin^2 \theta}. \]

acts (along the meridian) on a point \( \tau \). We need its moment relative to the axis \( O_x \).

Hence, we project \( F_\tau \) on the normal to the plane \( \sigma_2 \), i.e. on the vector

\[ \eta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -\sin \theta \\ \cos \theta \end{pmatrix}. \]

The unit vector \( \xi \) tangent to the meridian passing through the point \( \tau \), equals

\[ \xi = (-\cos \phi_\tau \sin \psi_\tau, -\sin \phi_\tau \sin \psi_\tau, \cos \psi_\tau) = \left( -\frac{\sin \tau \sin \theta \cos \tau}{\sqrt{\cos^2 \tau + \sin^2 \tau \cos^2 \theta}}, -\frac{\sin \tau \sin \theta \cos \tau}{\sqrt{\cos^2 \tau + \sin^2 \tau \cos^2 \theta}}, \frac{\sqrt{\cos^2 \tau + \sin^2 \tau \cos^2 \theta}}{\sqrt{\cos^2 \tau + \sin^2 \tau \cos^2 \theta}} \right). \]

Further on

\[ (\xi, \eta) = -\frac{\sin^2 \tau \sin^2 \theta \cos \tau}{\sqrt{\cos^2 \tau + \sin^2 \tau \cos^2 \theta}} + \cos \tau \sqrt{\cos^2 \tau + \sin^2 \tau \cos^2 \theta} = \frac{\cos \theta}{\sqrt{\cos^2 \tau + \sin^2 \tau \cos^2 \theta}}. \]

This means that we have the projection

\[ |F_\tau|(\xi, \eta) = \frac{\pi \cos \theta}{\sin^2 \tau \sin^2 \theta}. \]

The arm of this force is the distant from the point \( (\cos \tau, \sin \tau \cos \theta, \sin \tau \sin \theta) \) to the axis \( O_x \). It equals \( \sin^2 \tau \). The moment acting on the portion \( d\tau \) of the circle \( \sigma_2 \) equals

\[ M_\tau d\tau = \frac{\pi \cos \theta}{2 \sin^2 \theta} d\tau. \]

The total moment acting on the \( \sigma_2 \) is

\[ \int_0^{2\pi} M_\tau d\tau = \frac{\pi^2 \cos \theta}{\sin^2 \theta}. \]

Thus, points of the projective plane \( \mathbb{RP}^2 \) are repelling (up to a multiplicative constant) with the force

\[ \frac{\cos \theta}{\sin^2 \theta}. \]
where $\theta$ is the angular distant between them. The potential of this force is

$$
\frac{1}{\sin \theta}.
$$

The theorem is proved.

It is reasonable to believe that the kinetic energy of quasi-particles, as well as for spherons, is described by the Laplace-Beltrami operator. In the homogenous coordinats $(x, y, z)$ on the projective space, this operator has the form

$$D = \left( y^2 + z^2 \right) \frac{\partial^2}{\partial x^2} + \left( z^2 + x^2 \right) \frac{\partial^2}{\partial y^2} + \left( x^2 + y^2 \right) \frac{\partial^2}{\partial z^2} - \frac{2xy}{\partial xy} - \frac{2yz}{\partial yz} - \frac{2zx}{\partial zx} - 2x \frac{\partial}{\partial x} - 2y \frac{\partial}{\partial y} - 2z \frac{\partial}{\partial z}. \tag{38}$$

The distant between points of $\mathbb{RP}^2$ is defined by the logarithm of the cross ratio. In view of the preceding theorem, the potential of the pairwise inter relation of points with the coordinates $(x_i, y_i, z_i)$ $(x_j, y_j, z_j)$ is the sine in power $-1$ of the corresponding angle

$$V_{ij} = g \frac{\sqrt{(x_i^2 + y_i^2 + z_i^2)(x_j^2 + y_j^2 + z_j^2)}}{\sqrt{(x_i y_j - x_j y_i)^2 + (y_i z_j - y_j z_i)^2 + (z_i x_j - z_j x_i)^2}},$$

where $g$ is the interaction parameter.

Denote by $D_i$ the Laplace-Beltrami operator in $Ell(\mathbb{R}^2)$ applying to the coordinates $q_i$. The Hamiltonian of the system that consists of $N$ interacting quasi-particles is

$$H^1 = -\gamma \sum_{i=1}^{N} D_i + \sum_{i<j} V_{ij}. \tag{39}$$

As it was expected, the Hamiltonian is homogeneous. We consider it on the space $\mathbb{RP}^2$, i.e. on functions of zero degree homogeneity relative to $x_i, y_i, z_i$ for any $i$. Besides, it is invariant relative to the group of permutations of points. We obtain an analogue of the Calogero-Sutherland-Moser problem (CSM) [24]. But here the part of the circle plays the projective space $\mathbb{RP}^2$. It would be good to find the asymptotics of the spectrum as $N \to \infty$.

We will use the Euler formulas of the first and second orders for functions of zero degree homogeneity.

$$
x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z} = 0;
$$

$$
x^2 \frac{\partial^2 f}{\partial x^2} + y^2 \frac{\partial^2 f}{\partial y^2} + z^2 \frac{\partial^2 f}{\partial z^2} + xy \frac{\partial^2 f}{\partial xy} + yz \frac{\partial^2 f}{\partial yz} + zx \frac{\partial^2 f}{\partial zx} = 0.
$$

By adding to and subtracting from $D$ the value

$$x^2 \frac{\partial^2}{\partial x^2} + y^2 \frac{\partial^2}{\partial y^2} + z^2 \frac{\partial^2}{\partial z^2},$$

and then by using the Euler formulas, we reduce (38) to the form

$$D_i = |q_i|^2 \left( \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + \frac{\partial^2}{\partial z_i^2} \right).$$

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Hence, the operator $H^1$ takes the form

$$H^1 = -\frac{\gamma}{2} \sum_{i=1}^{N} |q_i|^2 \Delta_i + \sum_{i<j} V_{ij}. \quad (40)$$

Enumerate eigenvalues of the Hamiltonian (40) by natural numbers $k$ and denote them by $-\sigma_k$. The corresponding complete orthonormal system of eigenfunctions is denoted by $Y_k(S), S \in (S^2)^N$. Expand a solution of the equation

$$\frac{\partial^2 U}{\partial t^2} = H^1(U) \quad (41)$$
in term of functions $Y_k(S)$. Coefficients of the series are defined by expression

$$\hat{U}_k(t) = \int_{(S^2)^N} U(S,t)Y_k(S) \, dS. \quad (42)$$

Apply the operator $\frac{\partial^2}{\partial t^2}$ to (42). Since the operator $H^1$ is selfajoint and $Y_k(S)$ are its eigenfunctions with the eigenvalues $-\sigma_k$, we obtain

$$\frac{\partial^2 \hat{U}_k(t)}{\partial t^2} = \int_{(S^2)^N} H^1U(S,t)Y_k(S) \, dS = \int_{(S^2)^N} U(S,t) H^1Y_k(S) \, dS = -\sigma_k \hat{U}_k(t).$$

Hence, we get the disjoin system of ordinary differential equations

$$\frac{\partial^2 \hat{U}_k(t)}{\partial t^2} = -\sigma_k \hat{U}_k(t). \quad (43)$$

Each equation of the system (43) is the equation of the harmonic oscillator with the unit mass and the frequency $\sqrt{\sigma_k}$ (we shall call it by the oscillator $k$). The application of the standard quantization procedure leads to the secondary quantization of the quasi-partial oscillations. The Hamiltonian of the oscillator $k$ is

$$H_k^1 = \frac{1}{2}(p^2 + \sigma_k q^2).$$

We enumerate eigenfunctions of the operator $H_k^1$ by indices $n$ and denote by $\Xi_n(k)$. It is known that the corresponding eigenvalues are

$$E_n(k) = (n + \frac{1}{2}) \sqrt{\sigma_k}.$$  

We introduce operators as in (17).

$$A^+ (k) \Xi_n (k) = \sqrt{n + 1} \Xi_{n+1}(k), \quad A^- (k) \Xi_n (k) = \sqrt{n} \Xi_{n-1}(k). \quad (44)$$

The ground-state of the system corresponds to $n = 0$ since

$$A^- (k) \Xi_0 (k) = 0,$$

and

$$\Xi_n (k) = \frac{(A^+ (k))^n}{\sqrt{n!}} \Xi_0 (k). \quad (45)$$
The formulas (44) show that states $\Xi_n(k)$ are eigenfunctions of the operators $A^-(k)A^+(k)$ and $A^+(k)A^-(k)$ with the eigenvalues $n + 1$ and $n$ correspondingly. Consequently, it is natural to call the operator $A^+(k)A^-(k)$ by the operator of number of quasi-particles of type $k$. Denote this operator by $\hat{n}^q(k)$. The commutation relations take the standard form

$$[A^-(k), A^+(k)] = 1, \quad [A^-(k), A^-(k)] = [A^+(k), A^+(k)] = 0.$$  

Meanwhile operators that relate to noncoinciding values of indices commute. The Hamiltonian for an oscillator is

$$H^1(k) = \sqrt{\sigma_k}(A^+(k)A^-(k) + \frac{1}{2}I^q(k)),$$

or

$$H^1(k) = \sqrt{\sigma_k}(\hat{n}^q + \frac{1}{2}I^q(k)),$$

where $I^q(k)$ is the identity operator on the space of states of the oscillator $k$. The full Hamiltonian of the field of the quasi-particles is represented by the sum

$$H^1 = \sum_k H^1(k) = \sum_k \sqrt{\sigma_k}(\hat{n}^q + \frac{1}{2}I^q(k)). \quad (46)$$

Let us find the operator $H^{int}$ of interaction of spherons with quasi-particles. The interaction annihilates a quasi-particle of an energy $E_3$ that corresponds to the oscillator $k_3$, and this excites a wave of an energy $E_1$ that corresponds to the spheron $(m_1, 2l_1)$. This wave create quasi-particle of an energy $E_2$ that corresponds to the spheron $k_2$, and this annihilates a wave of an energy $E_4$ that corresponds to the spheron $(m_4, 2l_4)$. So, we can write the operator of interaction $H^{int}$.

$$H^{int} = -W \sum_{k_j, l_j, \mid m_i \mid \leq 2l_j} (a^+(m_1, 2l_1)A^+(k_2)A^-(k_3)a^-(m_4, 2l_4))\delta(E_1 + E_2 - E_3 - E_4), \quad (47)$$

where $E_i$ are energies of the corresponding spherons and quasi-particles. The $\delta$-function at the right hand side of (47) provides energy conservation.

The full Hamiltonian is

$$H = H^0 + H^1 + H^{int} = \sum_{\mid m \mid \leq 2l} \sqrt{2l(2l + 1)}(\hat{n}^s(m, 2l) + \frac{1}{2}I^s(m, 2l)) + \sqrt{\sigma_k}(\hat{n}^q + \frac{1}{2}I^q(k)) - W \sum_{k_j, l_j, \mid m_i \mid \leq 2l_j} (a^+(m_1, 2l_1)A^+(k_2)A^-(k_3)a^-(m_4, 2l_4))\delta(E_1 + E_2 - E_3 - E_4). \quad (48)$$

The problem is significantly simplified if we neglect terms of Coulomb interaction $V_{ij}$ considering free quasi-particles as a first approximation.

At first, take the special case when $\gamma = 1$. In this case both Hamiltonians $H^0$ and $H^1$ act on isomorphic spaces that we denote by $\mathfrak{x}^s$ and $\mathfrak{x}^q$ correspondingly. Both these operators are diagonal in basis consisted of spherical harmonics. Its spectra coincide with
N-multiple spectrum of the harmonic oscillator. Note that operators $a^\pm$ and $A^\pm$ commute, acting on different objects (different components of a state-vector) The Hamiltonian equals

$$H = \sum_{l,m} \sum_{n,m} \sqrt{2l(2l+1)} \left( \frac{1}{2} I^s(m, 2l) + \hat{n}^s(m, 2l) \right) + \sum_{l,m} \sum_{n,m} \sqrt{2l(2l+1)} \left( \frac{1}{2} I^q(m, 2l) \right) - W \sum_{l,m} \sum_{n,m} \left( a^+(m_1, 2l_1) A^+(m_2, 2l_2) A^-(m_3, 2l_3) a^-(m_4, 2l_4) \right) \delta(l_1 + l_2 - l_3 - l_4).$$

(49)

We shall call the interaction switch-back if $m_1 = m_3$, $l_1 = l_3$ and $m_2 = m_4$, $l_2 = l_4$. Find the spectrum of Hamiltonian under the natural conjecture of switch-back interaction. In this case the interaction behaves as if it were two different acts. The first act is the annihilation of a quasi-particle of type $(m_1, 2l_1)$ with the synchronous excitation of a spheron of the same type. The second one is the annihilation of a spheron of type $(m_2, 2l_2)$ with the birth of a quasi-particle of the same type. Since the Hamiltonian contains the sum of all these processes, one can imagine that the state-vector of quasi-particles turns into the state-vector of spheron and conversely. The interaction Hamiltonian is

$$H = \sum_{l,m} \sum_{n,m} \sqrt{2l(2l+1)} \left( \frac{1}{2} I^s(m, 2l) + \hat{n}^s(m, 2l) \right) + \sum_{l,m} \sum_{n,m} \sqrt{2l(2l+1)} \left( \frac{1}{2} I^q(m, 2l) \right) - W \sum_{l,m} \sum_{n,m} \left( a^+(m_1, 2l_1) A^+(m_2, 2l_2) A^-(m_3, 2l_3) a^-(m_4, 2l_4) \right) \delta(l_1 + l_2 - l_3 - l_4).$$

(50)

Let us rewrite the operator $H$ given by the formula (50), by using the polarization of the state-space $\mathfrak{a} = \mathfrak{a}^s \oplus \mathfrak{a}^q$. Take in both isomorphic subspaces $\mathfrak{a}^s$ and $\mathfrak{a}^q$ the concordant bases. Then the operator $H^{int}$ interchanges vectors of these subspaces and multiply that by $-W$, i.e. $H^{int} : (x, y) \mapsto (-Wy, -Wx)$. Consequently, the operator $H$ has the block form:

$$\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

(51)

where $B_{11} = B_{22} = Diag(b_1, b_2, \ldots)$, $B_{12} = B_{21} = -WI$.

To find the spectrum of the operator $H$, we explore a little more general problem. Consider an operator $A$, acting on the direct sum of two isomorphic separable Hilbert spaces $K \oplus K$, with the block structure

$$A = \begin{pmatrix} B_1 & B_2 \\ B_2 & B_1 \end{pmatrix},$$

(52)

where $B_i$ are commuting selfajoint operators acting on the space $K$. In view of the von-Neumann theorem [18], there exists selfajoint operator $C$ such that both operators $B_i$ are functions of $C$. Take $E_\lambda$ — the spectral decomposition of the space $K$, corresponding to the operator $C$. Then

$$B_1 = \int \alpha(\lambda) dE_\lambda, \quad B_2 = \int \beta(\lambda) dE_\lambda.$$

Note that the diagonal subspaces $K_1 = \{(x, x)\}$ and $K_2 = \{(x, -x)\}$ are invariant relative to the operator $A$. By using the polarization $K \oplus K = K_1 \oplus K_2$, we reduce the operator $A$ to the block-diagonal form.
A = \begin{pmatrix} (B_1 + B_2) & 0 \\ 0 & (B_1 - B_2) \end{pmatrix}. \quad (53)

Denote by $E_\lambda \oplus E_\lambda$ the operators that correspond to the spectral family $E_\lambda$ into the subspace $K_1 = \{(x, x)\}$, and by $E_\lambda \ominus E_\lambda$ the operators that correspond to the spectral family $E_\lambda$ into the subspace $K_2 = \{(x, -x)\}$.

We proved the following theorem

**Theorem 5** The spectral representation of the operator $A$ on the space $K_1$ is

$$A|_{K_1} = \int (\alpha(\lambda) + \beta(\lambda)) d(E_\lambda \oplus E_\lambda).$$

The spectral representation of the operator $A$ on the space $K_2$ is

$$A|_{K_2} = \int (\alpha(\lambda) - \beta(\lambda)) d(E_\lambda \ominus E_\lambda).$$

**Corollary 1** The spectrum of the operator $H$ can be obtained from the spectrum of the operator $B_{11} = B_{22}$ by the shifts (as a whole) both to the right and to the left for a distance of $W$.

Now consider the case when $\gamma = n/m$ any rational number. As before we neglect the interactions defined by terms $V_{ij}$. Let us unify $n$ exciting states of spherons in a cluster that we call pseudo-spheron. In the same way, $m$ exciting states of quasi-particles will be a cluster called pseudo-particle. For sufficiently many quasi-particles and quasi-spherons, the Hamiltonians $H^0$ and $H^1$ will asymptotically coincide. Irrational values of $\gamma$ can be approximated by rational ones preserving the asymptotics of the spectrum. So, the problem can be reduced to the special case when $\gamma = 1$.

Since the spectrum of the operator $B_{11} = B_{22}$ is known, then at the approximation in question we obtain the spectrum of the Hamiltonian $H$. So, we obtain a possibility (in principle) to find mean values of important operators. Besides, our model Hamiltonian provides the basis for application of the perturbation theory for more realistic Hamiltonians.

The question:

Does there exist a coherence of electrons under a line (usual) lightning? To resolve the question on the quantum character of the usual lightning (or any spark discharge) one needs to explore oscillations of cylindrical flow of electrons. Propose some conjectures concerning this question.

5. Conjectures

The inner canal of a lightning is the ionized, positively charged plasma. Electrons are winding round this canal into a cylindrical cover. Stepped leader is, apparently, a
process of transformation into a superconductive state. And the superconductive state itself is connected with macroscopic separation of positive and negative plasma. It seems that Abrikosov’s threads in the theory of usual (low-temperature) superconductivity have the same structure: canal from ions surrounded by a cover of winding electrons (not for nothing these threads are called vortical). It is precisely this separation of plasma which gives rise to superconductivity. The ionized air in the course of thunderstorms behaves like a superconductor of the second kind. Stepped leader is the process of generation of analogs of Abrikosov’s threads — media of plasma superconductivity. These threads can curve, branch, and terminate due to rushes of the wind. The lightning flying through these canals repeats its form. The stepped leader constructs the canal of lightning by steps, as if it makes links of a chain. So, one of the possible ways of forming fireballs is separation of one of the chain from the canal of lightning. Nice recent theory of lightning (Runaway Breakdown) [9] takes into account the leading role of high-energy cosmic rays. It seems that Runaway Breakdown relates rather to the stepped leader then to the lightning itself. This is in agreement with the recent observations [10], where it was shown that steps of the stepped leader were well correlated with the high-energetic microsecond burst of gamma emission.

The question arises: why the lightning extincts so fast if there exists the canal of superconductivity? Perhaps it happens due to the magnetic field. Too strong magnetic field, that is excited by the powerful current in the superconductor, destruct its creator — the state of superconductivity that induces this current. Electrons begin to fall from the cover into the canal, superconductivity disappears, and the lightning extincts. The difference in pressure generates the shock wave — the thunder.

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