A NEW CONSTRUCTION OF THE ASYMPTOTIC ALGEBRA ASSOCIATED TO THE \( q \)-SCHUR ALGEBRA

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ABSTRACT. We denote by \( A \) the ring of Laurent polynomials in the indeterminate \( v \) and by \( K \) its field of fractions. In this paper, we are interested in representation theory of the “generic” \( q \)-Schur algebra \( S_q(n, r) \) over \( A \). We will associate to every non-degenerate symmetrising trace form \( \tau \) on \( K S_q(n, r) \) a subalgebra \( \mathcal{J}_\tau \) of \( K S_q(n, r) \) which is isomorphic to the “asymptotic” algebra \( \mathcal{J}(n, r)_A \) defined by J. Du. As a consequence, we give a new criterion for James’ conjecture.

1. INTRODUCTION

This article is concerned with the representation theory of the “generic” \( q \)-Schur algebra \( S_q(n, r) \) over \( A = \mathbb{Z}[v, v^{-1}] \). The \( q \)-Schur algebra was introduced by Dipper and James in [3] and [4]. There is an interest in studying the representations of this algebra, because they relate informations about the modular representation theory of the finite general linear group \( \text{GL}_n(q) \) and of the quantum groups.

Using a new basis of \( S_q(n, r) \) constructed in [5] (which is analogous to the Kazhdan-Lusztig basis in Iwahori-Hecke algebras), J. Du introduced in [7] the asymptotic algebra \( \mathcal{J}(n, r)_A \) over \( A \) and defined a homomorphism, \( \Phi : S_q(n, r) \to \mathcal{J}(n, r)_A \), the so-called Du-Lusztig homomorphism because its construction is similar to the Lusztig homomorphism for Iwahori-Hecke algebras.

There is a relevant open question in the representation theory of the \( q \)-Schur algebra, the so-called James’ conjecture. A precise formulation of this conjecture is recalled in Section 6. In [9] Meinolf Geck obtained a new formulation of this conjecture. More precisely, for \( k \) any field of characteristic \( \ell \) and for \( R \) any integral domain with quotient field \( k \), if \( q \in R \) is invertible, we can define the corresponding \( q \)-Schur algebra \( S_q(n, r)_R \) over \( R \) and its extension of scalars \( S_q(n, r)_k \). Similarly, we can define \( \mathcal{J}(n, r)_k \).

In [9, 1.2] M. Geck has shown that James’ conjecture holds if and only if, for \( \ell > r \), the rank of the homomorphism \( \Phi_k : S_q(n, r)_k \to \mathcal{J}(n, r)_k \) only depends on the multiplicative order of \( q \) in \( k^\times \), but not on \( \ell \).

Thus, in order to prove James’ conjecture, it is relevant to understand the rank of the Du-Lusztig homomorphism. The motivation of this paper is to develop new methods allowing to study this rank. More precisely, we will give a new construction of the asymptotic algebra. Indeed, thanks to methods developed in [14] by the second author and adapted to our situation, we prove that \( \mathcal{J}(n, r)_A \) is isomorphic to an algebra \( \mathcal{J}_\tau \), which only depends on the choice of a non-degenerate symmetrising trace form \( \tau \) on the semisimple algebra \( K S_q(n, r) \) (here \( K = \mathbb{Q}(v) \)) such that

\[ S_q(n, r) \subseteq \mathcal{J}_\tau \subseteq K S_q(n, r). \]

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Our main tool is to use the structure of the left cell modules of $\mathcal{S}_q(n, r)$ to construct an explicit Wedderburn basis of $K\mathcal{S}_q(n, r)$ (see Theorem 4.11). The main result of this paper is Theorem 5.5.

The article is organized as follows. In Section 2 we recall the definition of the “generic” $q$-Schur algebra and of its analogue of the Kazhdan-Lusztig basis for Iwahori-Hecke algebras. In Section 3 we prove that the $q$-Schur algebra satisfies properties which are very similar to Lusztig’s conjectures $\textbf{P1}, \ldots, \textbf{P15}$ for Iwahori-Hecke algebras. In Section 4 we develop some tools to prove our main result in Section 5. Finally, in Section 6 we state a new criterion for James’ conjecture.

2. The Iwahori-Hecke Algebra of Type A and the $q$-Schur Algebra

Let $v$ be an indeterminate. We set $A = \mathbb{Z}[v, v^{-1}]$ to be the ring of Laurent polynomials in $v$ and $K := \mathbb{Q}(v)$ its field of fractions. In order to introduce the $q$-Schur algebra over $A$, we have to recall some definitions and properties about Iwahori-Hecke algebras. We follow [13].

2.1. Iwahori-Hecke algebras and the Kazhdan-Lusztig basis. Let $(W, S)$ be a Coxeter group (here $S$ is the set of simple reflections). We define the corresponding Iwahori-Hecke algebra $\mathcal{H}$ as the free $A$-module with basis $\{T_w \mid w \in W\}$ satisfying

$$T_w T_{w'} = T_{ww'} \text{ if } l(ww') = l(w) + l(w'),$$

$$(T_s - v)(T_s + v^{-1}) = 0 \text{ for } s \in S,$$

where $l$ is the length function on $W$. In [12] §1 Kazhdan and Lusztig define an $A$-basis $\{C_w \mid w \in W\}$ of $\mathcal{H}$ which satisfies

$$C_w = C_w \text{ and } C_w = \sum_{y \leq w} p_{y, w} T_y \text{ for } w \in W,$$

where $\leq$ is the Bruhat-Chevalley order on $W$, and $\sim : \mathcal{H} \to \mathcal{H}$ is the involutive automorphism of $\mathcal{H}$ defined by $\tau = v \tau$, $\tau$ is a Coxeter group given by $\tau = v \tau$, and $p_{y, w} \in \{v^k \mid k \leq 0\} \mathbb{Z}$ and $p_{w, w} = 1$.

Note that we use the more modern notation from [13], that is, our elements $T_w$ here are the same as in [12] and were denoted by $v^{-l(w)} T_w$ in [12], and our elements $C_w$ here were denoted by $C_w$ in [12] and by $C_w$ in [13].

We denote by $g_{x, y, z}$ the structure constants of $\mathcal{H}$ with respect to the basis $\{C_w \mid w \in W\}$, that is, we have

$$C_x C_y = \sum_{z \in W} g_{x, y, z} C_z \text{ for } x, y \in W.$$

We define a relation $y \preceq_L w$ on $W$ by: either $y = w$ or there is an $s \in S$ such that $g_{s, w, y} \neq 0$. Let $\leq_L$ be the transitive closure of the relation $\preceq_L$ and denote by $\sim_L$ the associated equivalence relation on $W$. The classes for this relation are the so-called left cells. Similarly, we define $\leq_R$ and $\sim_R$, and we call the corresponding equivalence classes right cells. For $y, w \in W$, we write $y \preceq_{L,R} w$ if there is a sequence $y = y_0, y_1, \ldots, y_n = w$ of elements of $W$ such that, for $i \in \{0, 1, \ldots, n - 1\}$, we have $y_i \leq_L y_{i+1}$ or $y_i \leq_R y_{i+1}$.

In [13] §3.6, Lusztig shows that for $z \in W$, there is a unique integer $\alpha(z)$ such that for every $x, y \in W$, we have $g_{x, y, z} v^{\alpha(z)} \in \mathbb{Z}[v^{-1}]$ and $g_{x, y, z} v^{\alpha(z) - 1} \notin \mathbb{Z}[v^{-1}]$. Moreover,
for \( z \in W \), we define \( \Delta(z) = - \deg p_{1,z} \). For \( x, y, z \in W \), we write \( \gamma_{x,y,z} \in \mathbb{Z} \) for the coefficient of \( u^a(z) \) in \( g_{x,y,z} \) and we set
\[
\mathcal{D} = \{ d \in W \mid a(d) = \Delta(d) \},
\]
the set of distinguished involutions. In the case that \( W \) is a finite Weyl group, an affine Weyl group, or a dihedral group, Lusztig proved that the following conjectures hold (see [13, §§15–17]):

1. **P1** For any \( z \in W \) we have \( a(z) \leq \Delta(z) \).
2. **P2** Let \( x, y \in W \); if \( \gamma_{x,y} \neq 0 \) for some \( d \in \mathcal{D} \), then we have \( x = y^{-1} \).
3. **P3** If \( y \in W \), there exists a unique \( d \in \mathcal{D} \) such that \( \gamma_{y^{-1},y,d} \neq 0 \).
4. **P4** If \( x, y \in \mathcal{D} \), then \( a(x) \geq a(y) \).
5. **P5** If \( d \in \mathcal{D} \) and \( y \in W \) are such that \( \gamma_{y^{-1},y,d} = 0 \), then \( \gamma_{y^{-1},y,d} = \pm 1 \).
6. **P6** For \( d \in \mathcal{D} \), we have \( d = d^{-1} \).
7. **P7** For every \( x, y, z \in W \), we have \( \gamma_{x,y,z} = \gamma_{y,z,x} = \gamma_{z,x,y} \).
8. **P8** Let \( x, y, z \in W \) be such that \( \gamma_{x,y,z} \neq 0 \), then \( x \sim_L y^{-1}, y \sim_L z^{-1} \) and \( z \sim_L x^{-1} \).
9. **P9** If \( x \leq y \) and \( a(x) = a(y) \), then \( x \sim_L y \).
10. **P10** If \( x \leq y \) and \( a(x) = a(y) \), then \( x \sim_R y \).
11. **P11** If \( x \leq y \) and \( a(x) = a(y) \), then \( x \sim_R y \).
12. **P13** Every left cell contains a unique \( d \in \mathcal{D} \) and \( \gamma_{y^{-1},y,d} = 0 \) for every \( y \sim_L d \).
13. **P14** For every \( x \in W \), we have \( x \sim_R x^{-1} \).
14. **P15** Let \( v' \) be a second indeterminate and let \( g_{x,y,z}' \in \mathbb{Z}[v', v'-1] \) be obtained from \( g_{x,y,z} \) by the substitution \( v \mapsto v' \). If \( x, y, z \in W \) satisfy \( a(w) = a(y) \), then
\[
\sum_{y'} g_{x,y',y} g_{x,y',y} = \sum_{y'} g_{x,w,y} g_{y',x,y}.\]

Note that in this paper we only consider the case of type A, in which \( W \) is the symmetric group on \( |S| + 1 \) points.

2.2. The \( q \)-Schur algebra \( S_q(n,r) \). In the following, we denote by \( W \) the symmetric group of degree \( r \), and by \( S \) the set of transpositions \( s_i = (i, i + 1) \) for \( 1 \leq i \leq r - 1 \) and \( \mathcal{H} \) is the associated Iwahori-Hecke algebra as in [24]. Let \( n, r \geq 1 \), we denote by \( \Lambda(n) \) the set of compositions of \( r \) into at most \( n \) parts. For \( \lambda, \mu \in \Lambda(n,r) \), we denote by \( W_\lambda \subseteq W \) the corresponding Young subgroup. For \( \lambda, \mu \in \Lambda(n,r) \), we set \( D_{\lambda,\mu} \) to be the set of distinguished double coset representatives of \( W \) with respect to \( W_\lambda \) and \( W_\mu \). We set
\[
M(n,r) = \{ (\lambda, w, \mu) \mid \lambda, \mu \in \Lambda(n,r), w \in D_{\lambda,\mu} \}.
\]
For \( \underline{a} = (\lambda, w, \mu) \in M(n,r) \), we write \( ro(\underline{a}) = \lambda \) and \( co(\underline{a}) = \mu \) and we set \( \underline{a}' = (\mu, w^{-1}, \lambda) \). For \( \lambda, \mu \in \Lambda(n,r) \), we set \( M_{\lambda,\mu} = \{ \underline{a} \in M(n,r) \mid ro(\underline{a}) = \lambda, co(\underline{a}) = \mu \} \). We remark that if \( w \in D_{\lambda,\mu} \), then the double coset \( W_\lambda w W_\mu \) has a unique longest element. To prove this, we can proceed as follows: we denote by \( w_0 \) the longest element of \( W \), then \( w_0 W_\mu = W_\mu \). Here \( \mu = (\mu_1, \ldots, \mu_n) \), where \( \mu = (\mu_1, \ldots, \mu_n) \). Moreover, \( r_{w_0} : W \to W, x \mapsto xw_0 \) induces a bijection from the double coset \( W_\lambda w W_\mu \) to the double coset \( W_\lambda w W_\mu \). Thanks to [13, 11.3], we deduce that \( w_0 \) reverses the Bruhat-order. Since the double coset \( W_\lambda w W_\mu \) has a unique element of minimal length, the result follows. We write \( D_{\lambda,\mu}^+ \) for the set of double coset representatives of maximal length. We denote by \( \ell_{\lambda,\mu} \) the bijection from \( D_{\lambda,\mu} \) to \( D_{\lambda,\mu}^+ \) that associates to the representative of
minimal length $w$ of the double coset $W_\lambda W_\mu$, the representative of maximal length. We remark that if $w \in D_{\lambda,\mu}$, then $w^{-1} \in D_{\mu,\lambda}$. Moreover, we have
\[
\ell_{\lambda,\mu}(w)^{-1} = \ell_{\mu,\lambda}(w^{-1}).
\]
In the following, we set $\sigma(\omega) := \ell_{\lambda,\mu}(w)$ for $\omega = (\lambda, w, \mu)$.

We now recall the definition of the q-Schur algebra $S_q(n, r)$ introduced by Dipper and James in [3]. We set $q = v^2$, then the q-Schur algebra $S_q(n, r)$ of degree $(n, r)$ is the endomorphism algebra
\[
S_q(n, r) = \text{End}_H \left( \bigoplus_{\lambda \in \Lambda(n, r)} x_\lambda H \right),
\]
where $x_\lambda = \sum_{w \in W_\lambda} v^{l(w)} T_w \in H$. In [2, 3.4] Dipper and James prove that $S_q(n, r)$ has a standard basis $\{\phi_{\lambda,\mu}^z \mid (\lambda, w, \mu) \in M(n, r)\}$ indexed by the set $M(n, r)$, which plays the same role as the basis $\{T_w \mid w \in W\}$ for the Iwahori–Hecke algebra $H$. Moreover, in [5] Du proves that $S_q(n, r)$ has another basis $\{\theta_\omega \mid \omega \in M(n, r)\}$ whose construction is analogous to the Kazhdan–Lusztig basis of $H$. We denote by $f_{\alpha,\beta,\epsilon} \in A$ the structure constants with respect to this basis, that is, we have
\[
\theta_\alpha \theta_\beta = \sum_{\epsilon \in M(n, r)} f_{\alpha,\beta,\epsilon} \theta_\epsilon \quad \text{for all } \alpha, \beta, \epsilon \in M(n, r).
\]

We recall the following lemma:

**Lemma 2.3.** We have $f_{\alpha,\beta,\epsilon} \neq 0$ only if $\text{co}(\alpha) = \text{ro}(\beta)$ and $(\text{ro}(\alpha), \text{co}(\beta)) = (\text{ro}(\epsilon), \text{co}(\epsilon))$. In this case, we have
\[
f_{\alpha,\beta,\epsilon} = h_\mu^{-1} g_{\sigma(\alpha),\sigma(\beta),\sigma(\epsilon)}
\]
where $\mu = \text{co}(\alpha) = \text{ro}(\beta)$ and $h_\mu = \sum_{w \in W_\mu} v^{2l(w) - l(w_\mu)}$ (here $w_\mu$ denotes the longest element in $W$) and $g_{\sigma(\alpha),\sigma(\beta),\sigma(\epsilon)}$ is the structure constant of $H$ defined in Section 2.1.

**Proof.** See [5, Prop. 3.4]. We want to explain why we have a further hypothesis here than in [5, Prop. 3.4]: For $\omega = (\lambda, w, \mu) \in M(n, r)$ the element $\theta_\omega$ is by definition a linear combination of basis elements $\phi_{\lambda,\mu}^z$ for $z \in D_{\lambda,\mu}$. Thus, viewed as endomorphism of $\bigoplus_{\lambda \in \Lambda(n, r)} x_\lambda H$ it vanishes on all summands except on $x_\mu H$ and maps into the summand $x_\lambda H$. Thus, if either $\text{co}(\omega) \neq \text{ro}(\beta)$ or $(\text{ro}(\alpha), \text{co}(\beta)) \neq (\text{ro}(\epsilon), \text{co}(\epsilon))$, the structure constant $f_{\alpha,\beta,\epsilon}$ vanishes also. If both equations hold, the proof in [5, Prop. 3.4] works using $g_{\sigma(\alpha),\sigma(\beta),\sigma(\epsilon)}$.

We are not claiming that [5, Prop. 3.4] is wrong as stated there. However, the notation $g_{\alpha,\beta,\epsilon}$ there needs proper interpretation (see [5, Section 3.3]), a problem we avoid here. □

**Remark 2.4.** To further explain the just mentioned change of notation, consider the following: Let $n = r = 3$, $\lambda := (2,1,0)$, $\mu := (1,1,1)$, and $\nu := (2,1,0)$. Then $W$ is the symmetric group on 3 letters, generated by the two Coxeter generators $s_1 = (1,2)$ and $s_2 = (2,3)$. Thus $D_{\lambda,\mu} := \{s_1, s_1 s_2, s_1 s_2 s_1\}$, $D_{\mu,\nu} := \{s_1, s_2 s_1, s_1 s_2 s_1\}$, and $D_{\lambda,\nu} := \{s_1, s_1 s_2 s_1\}$.

By the relations, we have $T_{s_1} \cdot T_{s_2 s_1} = T_{s_1 s_2 s_1}$ and thus $g_{s_1 s_2 s_1, s_1 s_2 s_1} = 1$. We now set $\alpha := (\lambda, id, \mu)$, $\beta := (\mu, s_2, \nu)$ and $\epsilon := (\lambda, s_2, \nu)$. Thus, we get
\[
f_{\alpha,\beta,\epsilon} = 1 \cdot g_{\sigma(\alpha),\sigma(\beta),\sigma(\epsilon)} = g_{s_1, s_2 s_1, s_1 s_2 s_1} = 1,
\]
since $h_\mu = 1$ here.
However, if we set \( a' := (\mu, s, \mu) \), then \( f_{a', b, c} = 0 \), because of \( \text{ro}(a') \neq \text{ro}(c) \) and the arguments in the proof of Lemma 2.3. On the other hand, we have \( \text{ro}(a') = \text{co}(b) \) and \( g_{(s, a, \mu)} = g_{s_1, s_2, s_1, s_2 s_1} = 1 \). This shows, that we indeed need all the hypothesis in Lemma 2.3. The statement in \([5\text{ Prop.} 3.4]\) is true if one interprets \( g_{a', b, c} \) to be zero.

**Definition 2.5** (The \( a \)-function and the distinguished elements). Following \([7\text{ Section} 2]\), we extend the \( a \)-function to \( M(n, r) \) by setting \( a(a) = a(\sigma(a)) \) for every \( a \in M(n, r) \) and we extend the set \( D \) to the set

\[
D(n, r) = \{ d \in M(n, r) \mid \text{co}(d) = \text{ro}(d), \, \sigma(d) \in D \}.
\]

Moreover, for every \( a, b, c \in M(n, r) \), we define

\[
\gamma_{a, b, c} = \begin{cases} 
\gamma_{\sigma(a), \sigma(b), \sigma(c)} = \gamma_{\sigma(a), \sigma(b), \sigma(c)}^{-1} & \text{if } f_{a, b, c} \neq 0, \\
0 & \text{otherwise}.
\end{cases}
\]

**Remark 2.6.** Note that our definition for \( \gamma_{a, b, c} \) differs slightly from the one in \([7\text{ Section} 2.2]\). His \( \gamma_{a, b, c} \) is our \( \gamma_{a', b, c} \). With our definition we follow the setup in \([13]\) more closely and get nicer cyclic symmetries in our formulas.

**Remark 2.7.** In comparison to \([7\text{ Section} 2.1]\) we added the explicit hypothesis for the elements \( d \in D(n, r) \) that \( \text{ro}(d) = \text{co}(d) \). However, this hypothesis is implicit in \([7\text{ 4.1.(a)–(d)}]\) and some others would not be true.

Now, for \( a, b \in M(n, r) \), if there is \( c \in M(n, r) \) such that \( f_{a, b, c} \neq 0 \) then we write \( a \leq_L b \). We define \( \leq_R \) by \( a \leq_R b \) if and only if \( a' \leq_L b' \). Moreover, we define \( \leq_{LR} \) as in the Iwahori-Hecke algebra case. These relations induce corresponding equivalence relations \( \sim_L, \sim_R \) and \( \sim_{LR} \). We call the corresponding equivalence classes the left, right and two-sided cells of \( M(n, r) \) respectively.

Let \( \Gamma \) be a left cell of \( M(n, r) \). We set

\[
S_{\leq \Gamma} = \sum_{a \leq \Gamma} A \theta_a \quad \text{and} \quad S_{< \Gamma} = \sum_{a < \Gamma} A \theta_a,
\]

for some \( a \in \Gamma \), both are clearly left ideals of \( S(n, r) \) by the definition of \( \leq_L \). Then the left cell module \( L \mathcal{C}^{(\Gamma)} \) corresponding to \( \Gamma \) is defined as the quotient \( S_{\leq \Gamma}/S_{< \Gamma} \).

We define the right cell module \( R \mathcal{C}^{(\Gamma)} \) corresponding to a right cell \( \Gamma \) of \( M(n, r) \) similarly. To see that we get right ideals we have to use Lemma 2.3 and \( g_{x, y, z} = g_{y^{-1}, x, z^{-1}} \) for \( x, y, z \in W \) (see \([13\text{ 13.2.(a)}]\)) together with \( \sigma(x') = \sigma(x)^{-1} \). This implies \( f_{a', a, c} = 0 \) if and only if \( f_{a', a, c} = 0 \).

### 3. Lusztig’s conjectures for the \( q \)-Schur algebra

In this section, we prove that the \( q \)-Schur algebra satisfies properties very similar to \( P_1, \ldots, P_{15} \) for the Iwahori-Hecke algebra. First, we give some preliminary results.

**Lemma 3.1.** If \( a \leq_L b \) (resp. \( a \leq_R b \)), then \( \sigma(a) \leq_L \sigma(b) \) (resp. \( \leq_R \)).

**Proof.** Since \( a \leq_L b \), there is \( c \in M(n, r) \) such that \( f_{a, b, c} \neq 0 \). But we have \( f_{a, b, c} = h_{a, c}^{-1} g_{\sigma(a), \sigma(c)} h_{a, c} \) with \( h_{a, c}^{-1} \neq 0 \). Thus \( g_{\sigma(a), \sigma(c)} h_{a, c} \neq 0 \) and \( \sigma(a) \leq_L \sigma(b) \).

**Lemma 3.2.** If \( a \leq_L b \) then \( \text{co}(a) = \text{co}(b) \). If \( a \leq_R b \) then \( \text{ro}(a) = \text{ro}(b) \).

**Proof.** Since \( a \leq_L b \) there is \( c \in M(n, r) \) such that \( f_{a, b, c} \neq 0 \). From Lemma 2.3 follows that \( \text{ro}(a) = \text{ro}(c) \) and the result is proved. \( \Box \)
Lemma 3.3. Let $\lambda$, $\mu$, $\nu \in \Lambda(n,r)$, $x \in D^+_\lambda$ and $y \in D^+_{\mu,\nu}$. If $g_{x,y,z} \neq 0$ for some $z \in W$, then $z \in D^+_\lambda$.

Proof. For $\lambda \in \Lambda(n,r)$ we set $S_\lambda := W_\lambda \cap S$, the set of Coxeter generators of the parabolic subgroup $W_\lambda$. Let $x \in D^+_\lambda$ and $y \in D^+_{\mu,\nu}$ and $g_{x,y,z} \neq 0$. On one hand, this means that $l(sx) < l(x)$ for all $s \in S_\lambda$ and $l(ys) < l(y)$ for all $s \in S_\mu$. On the other hand, we get $z \leq L y$ and $z \leq L x$ and thus $l(zs) < l(z)$ for all $s \in S$ with $l(ys) < l(y)$ and $l(sz) < l(s)$ for all $s \in S$ with $l(sx) < l(x)$ by [13 Lemma 8.6]. Thus we have in particular that $l(zs) < l(z)$ for all $s \in S_\lambda$ and $l(sz) < l(z)$ for all $s \in S_\lambda$. Hence $z$ is the longest element in its $W_\lambda$-$W_\mu$-double coset in $W$.

Lemma 3.4. We have $a \leq_r b$ if and only if there is a $z \in M(n,r)$ with $f_{a,z} \neq 0$.

Proof. By definition, $a \leq_r b$ is equivalent to $a^t \leq_r b^t$. This in turn means that there is a $z \in M(n,r)$ such that $f_{z,a^t} \neq 0$. As mentioned at the end of Section 2.2, we have $f_{z,a^t} = 0$ if and only if $f_{z,a^t} = 0$ which directly implies the statement in the lemma.

Proposition 3.5. The following properties hold for the q-Schur algebra:

Q1 For any $a \in M(n,r)$ we have $a(\sigma(a)) \leq \Delta(a)$.
Q2 If $\gamma_{a,b,d} \neq 0$ for some $d \in D(n,r)$, then we have $b = a^t$.
Q3 For every $a \in M(n,r)$, there is a unique $d \in D(n,r)$ with $\gamma_{a^t,a,d} \neq 0$.
Q4 If $a \leq_l b$ then $a(\sigma(a)) \geq b(\sigma(b))$.
Q5 If $d \in D(n,r)$ and $a \in M(n,r)$ are such that $\gamma_{a^t,a,d} \neq 0$, then $\gamma_{a^t,a,d} = 1$.
Q6 For $d \in D(n,r)$, we have $\sigma(d) = d^t$.
Q7 For every $a, b, c \in M(n,r)$, we have $\gamma_{a,b,c} = \gamma_{b,c,a} = \gamma_{c,a,b}$.
Q8 Let $a, b, c \in M(n,r)$ be such that $\gamma_{a,b,c} \neq 0$, then $a \sim_r b^t, b \sim_r c^t$ and $c \sim_r a^t$.
Q9 If $a \leq_l b$ and $a(\sigma(a)) = b(\sigma(b))$, then $a \sim_l b$.
Q10 If $a \leq_l b$ and $a(\sigma(a)) = b(\sigma(b))$, then $a \sim_{l,n} b$.
Q11 If $a \leq_l b$ and $a(\sigma(a)) = b(\sigma(b))$, then $a \sim_{l,n} b$.
Q12 Every left cell contains a unique element $d \in D(n,r)$ and $\gamma_{a^t,a,d} \neq 0$ for every $a \sim_{l,n} d$.
Q13 For every $a \in M(n,r)$, we have $a \sim_{l,n} a^t$.
Q14 Let $v' \in L$ be a second indeterminate and let $f_{x,y,z} \in \mathbb{Z}[v', v'^{-1}]$ be obtained from $f_{x,y,z}$ by the substitution $v \mapsto v'$. If $a, a', b, c \in W$ satisfy $a(\sigma(a)) = b(\sigma(b))$, then

$$\sum_{\lambda} f_{x,y,z}^{a'} \overline{f}_{a'} b = \sum_{\lambda} f_{x,y,z}^{a} \overline{f}_{a} b.$$
representative of minimal length of the coset $W_\mu d W_\mu$ and we set $\dd := (\mu, \dd, \mu)$. Then $\dd \in D(n, r)$ and $\sigma(\dd) = \dd$. It follows that $\gamma_{\dd} \neq 0$ and thus $Q_3$ holds.

The property $Q_4$ follows from $P_4$ and Lemma 3.1. The property $Q_5$ directly follows from $P_5$, since in our case $W$ is of type $A$ and thus all coefficients of all Kazhdan-Lusztig polynomials are non-negative by [13, 15.1].

Let $\dd = (\lambda, w, \lambda) \in D(n, r)$; we have $\sigma(\dd) \in D$, thus $P_6$ gives $\sigma(\dd)^{-1} = \sigma(\dd)$. Therefore, we have $\ell_{\lambda, \lambda}(w) = \sigma(\dd)^{-1} = \sigma(\dd') = \ell_{\lambda, \lambda}(w^{-1})$, and it follows that $w = w^{-1}$; thus $Q_6$ holds. The property $Q_7$ follows directly from $P_7$.

Suppose that $\gamma_{\dd} \neq 0$ for some $\dd \in D(n, r)$, then $f_{\dd} \neq 0$ and it follows that $\gamma_{\dd} \neq 0$. Moreover, we have $\gamma_{\dd} = \gamma_{\dd'}$, and thus $Q_8$ holds.

Next we prove $Q_9$. Let $a, b \in D(n, r)$ with $a \neq b$ and $a(b) = a(b)$. We denote the unique element of $D(n, r)$ in the left cell of $a$ by $d_n$ (resp. $d_n'$ for $b$). Using $Q_4$ we deduce that $\sigma(d_n) = a(a)$ and $\sigma(d_n') = a(a)$. Moreover, we have $d_n \leq d_n'$. Thus using Lemma 3.1 shows that $\sigma(d_n) \leq \sigma(d_n')$. Hence, using Property $P_9$, we have $\sigma(d_n) \leq \sigma(d_n')$ and $\sigma(d_n') \leq \sigma(d_n)$. However, $\sigma(d_n)$ and $\sigma(d_n')$ lie in $D$. Therefore, using $P_13$ in the Iwahori-Hecke algebra, we deduce that $\sigma(d_n') = \sigma(d_n')$. We now prove that $f_{\dd, d_n, d_n'} \neq 0$. Since $\sigma(d_n) = \sigma(d_n') = \sigma(d_n')$ (thanks to Lemma 3.2), we deduce that $f_{\dd, d_n, d_n'} = h_{\sigma(d_n)} g_{\sigma(d_n), \sigma(d_n')}$.

Using $P_13$, we deduce that $\gamma_{\sigma(d_n)} \neq 0$. Since $h_{\sigma(d_n)} \neq 0$, it follows that $f_{\dd, d_n, d_n'} \neq 0$. Hence $d_n \leq d_n'$, and $Q_9$ follows.

Property $Q_10$ follows from $Q_9$ by transposition since $a(a) = a(a)$ for all $a \in M(n, r)$ (use [13, 13.9 (a)]). Property $Q_11$ follows from $Q_9$ and $Q_10$ and induction.

Let $a \in M(n, r)$ and $\dd \in D(n, r)$ be the unique element such that $a \sim_\dd \dd$ given by $Q_{13}$. Then $a \sim_\dd \dd$ and $Q_{14}$ holds.

Finally, we prove $Q_{15}$. We first remark that $f_{\dd, d_n', d_n'} \neq 0$ if and only if $f_{\dd, d_n', d_n'} \neq 0$. Moreover, if $f_{\dd, d_n', d_n'} \neq 0$, then $f_{\dd, d_n', d_n'} = h_{\sigma(d_n')} g_{\sigma(d_n), \sigma(d_n')}$ and $f_{\dd, d_n', d_n'} = h_{\sigma(d_n')} g_{\sigma(d_n), \sigma(d_n')}$. If $f_{\dd, d_n', d_n'} \neq 0$, then $f_{\dd, d_n', d_n'} = h_{\sigma(d_n')} g_{\sigma(d_n), \sigma(d_n')}$ and $f_{\dd, d_n', d_n'} = h_{\sigma(d_n')} g_{\sigma(d_n), \sigma(d_n')}$. Here $h_{\sigma(d_n')}$ is obtained from $h_{\sigma(d_n)}$ by the substitution $v \mapsto v'$. We note that $h_{\sigma(d_n')} = h_{\sigma(d_n)}$ do not depend on $d_n'$. It follows from $P_{15}$ that

$\sum_{v} f_{\dd, d_n', d_n'} = h_{\sigma(d_n')} h_{\sigma(d_n)} \sum_{v} g_{\sigma(d_n), \sigma(d_n'), \sigma(d_n')} g_{\sigma(d_n), \sigma(d_n'), \sigma(d_n)}$

$= h_{\sigma(d_n')} h_{\sigma(d_n)} \sum_{v} g_{\sigma(d_n), \sigma(d_n'), \sigma(d_n')} f_{\sigma(d_n'), \sigma(d_n')} \sigma(d_n')$

$= \sum_{\dd} f_{\dd, d_n', d_n'}$. 


Proposition 3.6. If \( a \sim_L b \) and \( a \sim_R b \), then \( a = b \).

Proof. Let \( a = (\lambda, w, \mu_a) \) and \( b = (\lambda, w, \mu_b) \) be such that \( a \sim_L b \) and \( a \sim_R b \). We have \( a \leq_L b \) and \( a^* \leq_L b^* \), then using Lemma 3.2 we deduce that \( \mu_a = \mu_b \) and \( \lambda_a = \lambda_b \). Using Lemma 3.1 we deduce that \( \sigma(a) \sim_L \sigma(b) \) and \( \sigma(a) \sim_R \sigma(b) \). Since \( \mathcal{H} \) is of type \( A \), it follows that \( \sigma(a) = \sigma(b) \), that is \( f_{\lambda_a, \mu_a}(w_a) = f_{\lambda_b, \mu_b}(w_b) \). Hence we get \( w_a = w_b \).

4. Irreducible Cell Modules and Dual Basis

In this section we view the extension of scalars \( KS_q(n, r) \) of the \( q \)-Schur algebra \( S_q(n, r) \) as a symmetric algebra. This is possible, since it is semisimple (see [11] (9.8)). We can take as symmetrising trace form any \( K \)-linear form \( \tau : KS_q(n, r) \to K \) that is a \( K \)-linear combination

\[
\tau = \sum_{\chi \in \text{Irr}(K S_q(n, r))} c_\chi \chi
\]

of the irreducible characters where the \( c_\chi \) are non-zero constants, the so-called Schur elements (see [10] 7.1.1 and 7.2.6). Clearly, \( \tau \) is non-degenerate.

Having fixed \( \tau \), we denote for any \( K \)-basis \( (B_{\underline{a}})_{\underline{a} \in M(n, r)} \) of \( KS_q(n, r) \) its dual basis with respect to \( \tau \) by \( (B_{\underline{a}}^\vee)_{\underline{a} \in M(n, r)} \). That is, we have \( \tau(B_{\underline{a}} \cdot B_{\underline{b}}^\vee) = \tau(B_{\underline{a}}^\vee \cdot B_{\underline{b}}) = \delta_{\underline{a}, \underline{b}} \) for all \( \underline{a}, \underline{b} \in M(n, r) \). Note that this immediately implies that we can write every element \( x \in KS_q(n, r) \) in the following form:

\[
x = \sum_{\underline{a} \in M(n, r)} \tau(x \cdot B_{\underline{a}}^\vee)B_{\underline{a}} = \sum_{\underline{a} \in M(n, r)} \tau(x \cdot B_{\underline{a}})B_{\underline{a}}^\vee
\]

(just write \( x \) as a linear combination of the \( B_{\underline{a}} \), multiply by some \( B_{\underline{b}} \) and apply \( \tau \)).

Remark 4.1. We have \( f_{\underline{a}, \underline{b}, \underline{c}} = \tau(\theta_{\underline{a}} \cdot \theta_{\underline{b}} \cdot \theta_{\underline{c}}^\vee) \) for all \( \underline{a}, \underline{b}, \underline{c} \in M(n, r) \). Moreover, we note that formula (4.1) immediately gives us nice formulas for the matrix representations coming from the left cell modules. For a left cell \( \Gamma \) and an element \( h \in S_q(n, r) \) the representing matrix of \( h \) on the left cell module \( LC(\Gamma) \) with respect to the basis \( \{\theta_{\underline{a}} + S_{\leq \Gamma} | \underline{a} \in \Gamma\} \) is \( (\tau(\theta_{\underline{a}} \cdot h \cdot \theta_{\underline{b}}))_{\underline{a}, \underline{b} \in \Gamma} \) since \( h \cdot \theta_{\underline{a}} = \sum_{\underline{b} \in M(n, r)} \tau(\theta_{\underline{b}} \cdot h \cdot \theta_{\underline{a}} \cdot \theta_{\underline{b}}) = 0 \) and it is enough to sum over those \( \underline{b} \) with \( \underline{b} \leq_L \underline{a} \).

Lemma 4.2 (Characterisation of \( \leq_L \) and \( \leq_R \)). We have \( \underline{a} \leq_L \underline{b} \) if and only if \( \theta_{\underline{a}} \theta_{\underline{b}}^\vee \neq 0 \) and \( \underline{a} \leq_R \underline{b} \) if and only if \( \theta_{\underline{a}} \theta_{\underline{b}}^\vee \neq 0 \).

Proof. We only show the version with \( \leq_L \), the other is completely analogous thanks to Lemma 3.4. If \( \underline{a} \leq_L \underline{b} \) there exists \( \underline{c} \in M(n, r) \) with \( f_{\underline{a}, \underline{b}, \underline{c}} = \tau(\theta_{\underline{a}} \theta_{\underline{b}}^\vee) \neq 0 \) which implies \( \theta_{\underline{a}} \theta_{\underline{b}}^\vee \neq 0 \). If we assume the latter, then by the non-degeneracy of \( \tau \) there is some \( \underline{c} \in M(n, r) \) with \( \tau(\theta_{\underline{c}} \theta_{\underline{a}}) \neq 0 \) and \( \underline{a} \leq_L \underline{b} \) follows.

The other major ingredient is the fact that cell modules are simple, more precisely:

Theorem 4.3 (Simple cell modules, see [6] or [7] 4.3). Let \( \Gamma \) be a left cell and recall \( K = \mathbb{Q}(n) \). The extension of scalars \( K LC(\Gamma) \) of the left cell module \( LC(\Gamma) \) for a left cell \( \Gamma \) is a simple \( KS_q(n, r) \)-module.

Proof. See [6] or [7] 4.3.
Remark 4.4. This in particular implies that all simple \(K\mathcal{S}_q(n, r)\)-modules can be realised over the ring \(A\), since their corresponding representing matrices involve only structure constants of \(\mathcal{S}_q(n, r)\).

We now directly obtain useful algebra elements by using the simple cell modules:

**Theorem 4.5** (Basis of an isotypic component). Let \(\Gamma\) be a left cell and \(\chi\) the corresponding irreducible character of the left cell module \(LC(\Gamma)\), then the elements

\[
(c^{-1}_\chi \theta_a \theta_a')_{a, a' \in \Gamma}
\]

are \(K\)-linearly independent and span the isotypic component of \(K\mathcal{S}_q(n, r)\) belonging to the character \(\chi\). Furthermore, we have the relations

\[
(c^{-1}_\chi \theta_a \theta_a') \cdot (c^{-1}_\chi \theta_{a'} \theta_{a'}) = \delta_{a, a'} \cdot c^{-1}_\chi \theta_{a'}
\]

for all \(a, a', b, b' \in \Gamma\). That is, these elements form a matrix unit for the isotypic component of \(K\mathcal{S}_q(n, r)\) corresponding to the simple module \(KL C(\Gamma)\).

**Proof.** By [10, 7.2.7] we get a matrix unit for the isotypic component of \(K\mathcal{S}_q(n, r)\) corresponding to the simple module \(KL C(\Gamma)\) by the elements

\[
\frac{1}{c_\chi} \sum_{a, b \in \Gamma} \tau(\theta_a \theta_a') \cdot \theta_{a'} = \frac{1}{c_\chi} \sum_{a, b \in \Gamma} \tau(\theta_a \cdot \theta_{a'}) \cdot \theta_{a'}
\]

for \(a, b \in \Gamma\). But this is equal to \(c^{-1}_\chi \theta_a \theta_a'\) by Formula (4.1). \(\Box\)

**Corollary 4.6.** Let \(\Gamma\) be a left cell and \(\chi\) the corresponding irreducible character of the left cell module \(LC(\Gamma)\). Then the element

\[
e_{\Gamma} := \frac{1}{c_\chi} \sum_{a \in \Gamma} \theta_a \theta_a'
\]

is the central primitive idempotent of \(K\mathcal{S}_q(n, r)\) corresponding to the irreducible character \(\chi\).

**Proof.** By Theorem 4.5, \(e_{\Gamma}\) lies in the isotypic component corresponding to the character \(\chi\) and is mapped to the identity matrix in the corresponding matrix representation. \(\Box\)

**Lemma 4.7** (Isomorphism of left cell modules and two-sided cells). Let \(\Gamma\) and \(\Gamma'\) be left cells. If \(KL C(\Gamma)\) and \(KL C(\Gamma')\) are isomorphic \(K\mathcal{S}_q(n, r)\)-modules then \(\Gamma\) and \(\Gamma'\) lie in the same two-sided cell.

**Proof.** Let \(\chi\) be the irreducible character of the left cell module \(KL C(\Gamma)\) and \(\chi'\) that of \(KL C(\Gamma')\). The modules \(KL C(\Gamma)\) and \(KL C(\Gamma')\) are isomorphic if and only if \(e_{\Gamma} \cdot e_{\Gamma'} = e_{\Gamma'} \cdot e_{\Gamma} \neq 0\) (and in this case \(e_{\Gamma} = e_{\Gamma'}\)). Now assume this case. Then

\[
0 \neq \frac{1}{c^2_\chi} \sum_{\alpha, \beta \in \Gamma} \sum_{\beta' \in \Gamma'} \theta_{\alpha} \theta_{\beta} \theta_{\alpha} \theta_{\beta'} = \frac{1}{c^2_\chi} \sum_{\alpha, \beta \in \Gamma} \sum_{\beta' \in \Gamma'} \theta_{\beta} \theta_{\beta'} \theta_{\alpha} \theta_{\alpha}'
\]

and thus there is at least one pair \((a, b) \in \Gamma \times \Gamma'\) such that \(\theta_a \theta_{a'} \neq 0\). By Lemma 4.2 this implies \(a \leq_{\Gamma} b\). Since \(e_{\Gamma}\) and \(e_{\Gamma'}\) commute, the same argument shows \(b' \leq_{\Gamma'} a'\) for some \(a' \in \Gamma\) and \(b' \in \Gamma'\). Thus, \(\Gamma\) and \(\Gamma'\) lie in the same two-sided cell in that case. \(\Box\)

For what follows we need the following statement about Iwahori-Hecke-Algebras of type \(A\):
Theorem 4.8 (Equal cell modules in the Iwahori-Hecke algebra). Let \( H \) be a generic Iwahori-Hecke-Algebra of type A as in Section 2. If \( x \sim_L y \) and \( z \sim_L w \) and \( x \sim_R z \) and \( y \sim_R w \), then \( C_x D_{y^{-1}} = C_z D_{w^{-1}} \). In particular, we have

\[
g_{u, x, y} = \tau(C_u C_x D_{y^{-1}}) = \tau(C_u C_z D_{w^{-1}}) = g_{u, z, w}
\]

for all \( u \in W \).

Proof. This statement is already implicitly stated in [12]. Namely, it is shown there in the proof of Theorem 4.1 that the two left cell modules defined by the left cell containing \( x, y \) and the one containing \( z, w \) are isomorphic since all four lie in the same two-sided. The exact statement there is that two \( W \)-graphs are isomorphic, which means in particular that not only the two left cell modules are isomorphic, but that even the matrix representations with respect to the bases \( \{ C_v \mid v \sim_L x \} \) and \( \{ C_w \mid w \sim_L z \} \) are equal. But this exactly means, that

\[
\tau(D_{y^{-1}} C_u C_x) = \tau(D_{w^{-1}} C_u C_z)
\]

for all \( u \in W \) which we claim. \( \square \)

Now we begin to use statements Q1 to Q14:

Theorem 4.9 (Equality of different left cell modules). Let \( \Gamma, \Gamma' \) be left cells such that \( K \mathrm{LC}(\Gamma) \) and \( K \mathrm{LC}(\Gamma') \) are isomorphic \( K S_n(n, r) \)-modules. Let \( \bar{d} \) be the unique element in \( \Gamma' \cap D(n, r) \) (use (13) and \( \bar{c} \sim_L d \) that is \( \bar{c} \in \Gamma' \). Then there are unique \( a, b, \bar{b} \in \Gamma \) with \( a \sim_n \bar{c} \) and \( \bar{b} \sim_R d \) and we have \( \theta_{\bar{b}, \bar{b}} = \theta_{a, a} \).

Proof. Let \( \chi \) be the irreducible character of the left cell module \( \mathrm{LC}(\Gamma') \). We denote by \( c_\chi \) the corresponding Schur element. Since \( \bar{c} \sim_L d \), it follows from Theorem 4.5 that

\[
\theta_{\bar{b}, \bar{b}} = c_\chi \theta_{a, a}.
\]

Therefore we have \( \tau(\theta_{\bar{b}, \bar{b}}) \neq 0 \) and hence \( \theta_{\bar{b}, \bar{b}} \) acts non-trivially on the module \( \mathrm{LC}(\Gamma') \) (see Remark 4.1) and thus also on the isomorphic module \( \mathrm{LC}(\Gamma) \).

This means that there is at least one pair \( (a, b) \in \Gamma \times \Gamma \) such that

\[
\tau(\theta_{a, a} \theta_{\bar{b}, \bar{b}}) = \tau(\theta_{\bar{b}, \bar{b}}) \neq 0.
\]

But then in particular \( \theta_{a, a} \theta_{\bar{b}, \bar{b}} \neq 0 \) and thus \( a \sim_n \bar{c} \) by Lemma 4.2. Since \( \Gamma \) and \( \Gamma' \) lie in the same two-sided cell by Lemma 4.7 we conclude \( a \sim_L \bar{c} \) and thus by Q4 and Q10 \( a \sim_R \bar{c} \). Analogously, we show \( b \sim_R \bar{d} \). By Proposition 2.6 we conclude that there is only one such pair \( (a, b) \) since both are uniquely defined by their membership in a left and a right cell.

We now show that \( f_{\bar{c}, \bar{b}, \bar{b}} = f_{\bar{c}, \bar{b}, \bar{d}} \) for all \( \bar{c} \in M(n, r) \) and thus \( \theta_{\bar{b}, \bar{b}} = \theta_{\bar{b}, \bar{d}} \). We have \( \mathrm{co}(\bar{a}) = \mathrm{co}( \bar{b} ) \) and \( \mathrm{co}(\bar{c}) = \mathrm{co}( \bar{d} ) \) and \( \mathrm{ro}(\bar{a}) = \mathrm{ro}(\bar{b}) \) and \( \mathrm{ro}(\bar{c}) = \mathrm{ro}(\bar{d}) \) by Lemma 3.2 and the fact that \( \bar{d} \in D(n, r) \). Thus, if \( \mathrm{ro}(\bar{a}) \neq \mathrm{ro}(\bar{b}) \) or \( \mathrm{co}(\bar{c}) \neq \mathrm{co}(\bar{d}) \) then both sides are zero by Lemma 2.3. Otherwise, we have

\[
f_{\bar{c}, \bar{b}, \bar{b}} = h_{\mathrm{co}(\bar{c})}^{-1} \cdot g_{\sigma(\bar{c}), \sigma(\bar{a}), \sigma(\bar{b})} \quad \text{and} \quad f_{\bar{c}, \bar{b}, \bar{d}} = h_{\mathrm{co}(\bar{c})}^{-1} \cdot g_{\sigma(\bar{c}), \sigma(\bar{a}), \sigma(\bar{d})}
\]

and thus the equality \( f_{\bar{c}, \bar{b}, \bar{b}} = f_{\bar{c}, \bar{b}, \bar{d}} \) follows from

\[
\sigma(\bar{a}) \sim_L \sigma(\bar{b}) \sim_R \sigma(\bar{d}) \sim_L \sigma(\bar{c}) \sim_R \sigma(\bar{a})
\]

using Lemma 3.1 and Theorem 4.8. The non-degeneracy of \( \tau \) now immediately implies \( \theta_{\bar{b}, \bar{b}} = \theta_{\bar{b}, \bar{d}} \). \( \square \)

With this we get the following result, for which we first need one more piece of notation:
Definition 4.10 (Schur elements of characters of left cell modules). Let $d \in D(n, r)$ and $\Gamma$ the unique left cell with $d \in \Gamma$ (remember Q13). We denote the left cell module $LC^{(\Gamma)}$ by $LC(d)$ and the Schur element corresponding to the irreducible character of $LC(d)$ by $c_d$.

Theorem 4.11 (Wedderburn basis). Let $\tau$ be an arbitrary non-degenerate symmetrising trace form on $KS_q(n, r)$. The set

$$B := \{c_d^{-1} \theta_{\xi}^{\gamma} \mid \xi \in M(n, r), d \in D(n, r), \xi \sim_L d\}$$

is a Wedderburn basis of $KS_q(n, r)$. Two elements $c_d^{-1} \theta_{\xi}^{\gamma}$ and $c_{d'}^{-1} \theta_{\xi'}^{\gamma'}$ lie in the same isotypic component if and only if $LC(d) \cong LC(d')$.

For $c_d^{-1} \theta_{\xi}^{\gamma}$, $c_{d'}^{-1} \theta_{\xi'}^{\gamma'}$, $d, d' \in B$ we have the following equation:

$$c_d^{-1} \theta_{\xi}^{\gamma} \cdot c_{d'}^{-1} \theta_{\xi'}^{\gamma'} = \begin{cases} 0 & \text{if } LC(d) \neq LC(d') \\ 0 & \text{if } LC(d) \cong LC(d') \text{ and } d \neq r c' \\ c_{d'}^{-1} \theta_{\xi'}^{\gamma'} & \text{if } LC(d) \cong LC(d') \text{ and } d \sim r c' \end{cases}$$

Here, $c''$ in the last case is the unique element with $c''_L = d''$ and $c''_L = r c''$ and the statement contains the information that such a $c''$ in fact exists.

Proof. By Theorem 4.3 the elements $c_d^{-1} \theta_{\xi}^{\gamma}$ and $c_{d'}^{-1} \theta_{\xi'}^{\gamma'}$ both lie in an isotypic component. Thus, if $LC(d) \neq LC(d')$ then clearly their product is zero.

Now assume that the left cell modules are isomorphic. Let $\Gamma$ be an arbitrary left cell, such that $KLC^{(\Gamma)}$ is isomorphic to $KLC(d)$ and $KLC^{(\Gamma)}$ and denote the corresponding irreducible character by $\chi$. By Theorem 4.9 there are unique $a, b, a', b' \in \Gamma$ with

$$a \sim_r \xi \quad \text{and} \quad b \sim_r d \quad \text{and} \quad a' \sim_r \xi' \quad \text{and} \quad b' \sim_r d'$$

and we have $\theta_\chi \theta_\chi^{\gamma} = \theta_\chi \theta_\chi^{\gamma'}$ and $\theta_{\xi'} \theta_{\xi'}^{\gamma'} = \theta_{\xi} \theta_{\xi}^{\gamma}$. Thus, Theorem 4.3 implies that the product in the theorem is $0$ if $b \neq a'$ and equal to $c_{d'}^{-1} \theta_{\xi'}^{\gamma'}$ otherwise. We remark that if $d \sim_r c'$, then $a' \sim_r d$ by transitivity. But using Proposition 4.6, $a', b \in \Gamma$ implies $b = a'$. Hence $b = a'$ if and only if $d \sim_r c'$ which proves case two in the equation.

Finally, we assume also $d \sim_r c'$. Then, as $c''_L$ runs through the left cell that contains $d'$, we can apply Theorem 4.9 to each $\theta_{\nu} \theta_{\nu}^{\gamma}$ and the left cell $\Gamma$. Since $d' \in \Gamma$ and $b' \sim_r d'$ we get that

$$\{\theta_{\nu} \theta_{\nu}^{\gamma} \mid \nu \sim_L \Gamma, d'\} = \{\theta_{\nu} \theta_{\nu}^{\gamma} \mid \nu \in \Gamma\}$$

and both sets have cardinality $|\Gamma|$. Thus, there is a unique $c''_L$ with $\theta_{\nu} \theta_{\nu}^{\gamma} = \theta_{\nu} \theta_{\nu}^{\gamma}$ characterised by $a \sim_{r''} c''_L \sim_{r''} d'$. Hence the product is proved.

Corollary 4.12 (Idempotents). The elements $c_d^{-1} \theta_\alpha^{\gamma}$ with $d \in D(n, r)$ are pairwise orthogonal primitive idempotents whose sum is the identity $1 \in S_q(n, r)$. The central primitive idempotent corresponding to an irreducible character $\chi$ of $KS_q(n, r)$ is equal to

$$\sum_{d \in D(n, r)} c_d^{-1} \theta_\alpha^{\gamma}$$

Proof. This follows directly from Theorems 4.11, 4.9, and 4.5.

Corollary 4.13 (Left cell modules as submodules). Let $d \in D(n, r)$. Then the $A$-span

$$L_A := \{\theta_\alpha^{\gamma} \mid \alpha \sim_L d\}$$

is a left $S_q(n, r)$-module by the multiplication in $KS_q(n, r)$ that is isomorphic to the left cell module $LC(d)$. In fact, the representing matrices with respect to the basis $\theta_\alpha^{\gamma}$ are $A$-span
are equal to the representing matrices coming from the left cell module $LC(\overline{d})$ with respect to its standard basis.

Proof. Let $\Gamma$ be the left cell that contains $d$. Then by Formula (4.1) we have for every $h \in S_q(n, r)$:

$$h\theta^\vee_a = \sum_{\theta^\vee_c \in M(n, r)} \tau(\theta^\vee_c \cdot h\theta^\vee_a) \cdot \theta^\vee_c.$$ 

Moreover, for $a \in A$, there is $\alpha_a \in A$ such that

$$h = \sum_{\theta^\vee_c \in M(n, r)} \alpha_a \theta^\vee_c.$$ 

Hence, for $\theta^\vee_c, \theta^\vee_c' \in M(n, r)$, we have $\tau(\theta^\vee_c' \cdot h\theta^\vee_a) \in A$, because $\tau(\theta^\vee_c' \cdot \alpha_a \theta^\vee_c) \in A$ (see Remark 4.1). Multiplying this from the right with $\theta^\vee_c$ we get

$$h\theta^\vee_a \theta^\vee_c = \sum_{\theta^\vee_c' \in M(n, r)} \tau(h\theta^\vee_a \theta^\vee_c') \cdot \theta^\vee_c,$$

where we only have to sum over $\theta^\vee_c' \in \Gamma$, since all the summands are zero unless $d \leq \theta^\vee_c \leq \theta^\vee_c'$ by Lemma 4.2 which is equivalent to $\theta^\vee_c' \in \Gamma$. We then deduce that $Lq_d\theta^\vee_a$ is a left $S_q(n, r)$-module. Moreover, comparing with Remark 4.1 this shows the statement about the representing matrices.

Corollary 4.14. The Schur algebra $S_q(n, r)$ is contained in the $A$-span of the Wedderburn basis $B$:

$$S_q(n, r) \subseteq \langle B \rangle_A$$

Proof. Let $\Gamma_1, \ldots, \Gamma_n$ be left cells, such that the corresponding left cell modules form a system of representatives for the isomorphism types of simple left $KS_q(n, r)$-modules. The mapping that maps $h \in KS_q(n, r)$ to its tuple of representing matrices in the cell-modules $LC(\Gamma_1), \ldots, LC(\Gamma_n)$ with respect to their standard basis is an explicit isomorphism to a direct sum of full matrix rings over $K$. In this isomorphism, the elements of $B$ are mapped to a matrix unit, that is, to tuples of matrices, in which exactly one matrix is non-zero, and this matrix contains exactly one non-zero coefficient equal to 1. The elements of $S_q(n, r)$ are mapped to tuples of matrices with entries in $A$, since their representing matrices on the cell modules have entries in $A$ (see the remark after Theorem 4.3). Therefore, $S_q(n, r)$ lies in the $A$-span of $B$.

Proposition 4.15. Let $\tau$ be a non-degenerate symmetrising trace form on $KS_q(n, r)$. We denote by $B$ the corresponding Wedderburn basis obtained in Theorem 4.11. Then, the dual basis of $B$ relative to $\tau$ is

$$B^\vee = \{ \theta^\vee_{\theta^\vee_c} | \theta^\vee_c \in M(n, r), d \in D(n, r), \theta^\vee_c \sim_L \theta^\vee_d \}.$$ 

Proof. Note first, that since $\tau$ is non-degenerate and $B$ is a basis of $KS_q(n, r)$, there must be at least one element $e_{\theta^\vee_a}^{-1} \theta^\vee_c \theta^\vee_d' \in B$ such that $\tau(e_{\theta^\vee_a}^{-1} \theta^\vee_c \theta^\vee_d') = 0$. Since $c_{\theta^\vee_d} \neq 0$, we have in particular $\tau(e_{\theta^\vee_a}^{-1} \theta^\vee_c \theta^\vee_d' \theta^\vee_{\theta^\vee_c'}) = 0$. We try to find out, which element $\theta^\vee_{\theta^\vee_c'}$ this can be:

By Theorem 4.11 the value $\tau(e_{\theta^\vee_a}^{-1} \theta^\vee_c \theta^\vee_d' \theta^\vee_{\theta^\vee_c'})$ is equal to zero, if $LC(\theta^\vee_c) \neq LC(\theta^\vee_d')$ or $\theta^\vee_c \sim_L \theta^\vee_d'$. If however $LC(\theta^\vee_c) \equiv LC(\theta^\vee_d')$ and $\theta^\vee_c \sim_L \theta^\vee_d'$, then it is equal to $\tau(\theta^\vee_d' \theta^\vee_{\theta^\vee_c'})$ where $\theta^\vee_{\theta^\vee_c'}$ is uniquely defined by $e_{\theta^\vee_a}^{-1} \theta^\vee_c \theta^\vee_d' \sim_L \theta^\vee_{\theta^\vee_c'}$. If $\theta^\vee_{\theta^\vee_c'} \neq \theta^\vee_d'$, then this value is also equal to 0 because of the original definition of $\{ \theta^\vee_{\theta^\vee_c} | \theta^\vee_c \in M(n, r) \}$. If however $\theta^\vee_{\theta^\vee_c'} = \theta^\vee_d'$ we can
show that \( c'_r = c'_t \) using Proposition 3.16. Namely, we have \( c'_r \sim_L d'_r = c''_r \sim \mathcal{L} c_r \) and thus \( c'_r \sim_L c'_t \) by transposition. Further, we have \( c'_t \sim_L d'_t \) and thus again by transposition \( c'_t \sim_L c'_r \). Thus, \( c'_t \) and \( c'_r \) are both left and right equivalent and therefore equal.

Thus, we deduce that

\[
\tau(c_{c'_r}^{-1} \theta \cdot \theta'_{d'_r}) = \delta_{c'_c, c'_t}
\]

for all \( c \in M(n, r) \) and \( d \in \mathcal{D}(n, r) \) with \( c \sim_L d \) and all \( c'_r, c'_t \in M(n, r) \) and \( d'_r, d'_t \in \mathcal{D}(n, r) \) with \( c'_r \sim_L d'_r \).

□

Remark 4.16. Note that as a byproduct we have proved the following result: If \( c \in M(n, r) \) and \( d \in \mathcal{D}(n, r) \) with \( c \sim_L d \), then \( LC(\mathcal{L} c) \cong LC(\mathcal{L} d) \).

We now talk about \( A \)-sublattices of \( KS_q(n, r) \).

Definition/Proposition 4.17 \((A\)-sublattices of \( KS_q(n, r) \) and their duals). By an \( A \)-lattice in \( KS_q(n, r) \) we mean an \( A \)-free \( A \)-submodule that contains a \( K \)-basis of \( KS_q(n, r) \). Let \( L \subseteq KS_q(n, r) \) be an \( A \)-lattice. Then we set

\[
L^\vee := \{ h \in KS_q(n, r) \mid \tau(hx) \in A \text{ for all } x \in L \}
\]

and call it the dual lattice of \( L \). Since \( \tau \) is non-degenerate, \( L^\vee \) is again an \( A \)-lattice in \( KS_q(n, r) \), namely, if \( (b_{a, a})_{a \in M(n, r)} \) is an \( A \)-basis of \( L \), then the dual basis \( (b_{a, a})_{a \in M(n, r)} \) is an \( A \)-basis of \( L^\vee \). Clearly, if \( L \subseteq N \) are two \( A \)-lattices in \( KS_q(n, r) \), then \( N^\vee \subseteq L^\vee \).

Note that we do not require an \( A \)-lattice to be an \( A \)-algebra!

□

Proposition 4.18 (The dual is an \( S_q(n, r) \)-module). We have \( S_q(n, r) : S_q(n, r)^\vee \subseteq S_q(n, r)^\vee \).

Proof. Fix \( h \in S_q(n, r) \) and \( k \in S_q(n, r)^\vee \). We have to show that \( hk \in S_q(n, r)^\vee \). However, for every \( x \in S_q(n, r) \) holds \( \tau(hkx) = \tau(kxh) \). Since \( xh \in S_q(n, r) \) (because \( S_q(n, r) \) is an algebra), and \( k \in S_q(n, r)^\vee \) we get \( \tau(kxh) \in A \).

□

For the rest of this section we let \( \tau = \sum_{\chi \in \text{Irr}(KS_q(n, r))} \chi \), that is, we choose \( \tau \) such that all Schur elements are equal to 1.

Proposition 4.19 (The Wedderburn-basis is self-dual). Let \( \tau = \sum_{\chi \in \text{Irr}(KS_q(n, r))} \chi \). Then

\[
\langle B \rangle_A^\vee = \langle B \rangle_A
\]

for the Wedderburn basis \( B \) from Theorem 4.14.

Proof. Since \( \tau \) is the sum of the irreducible characters, all Schur elements \( c_\chi \) are equal to one. It is then a direct consequence of Proposition 4.15.

□

Corollary 4.20 (The dual of \( S_q(n, r) \)). From Lemma 4.14 and Proposition 4.19 follows

\[
\langle B \rangle_A \subseteq S_q(n, r)^\vee
\]

Proof. Dualising reverses inclusion.

□

5. THE ASYMPTOTIC ALGEBRA AND THE DU-LUSZTIG HOMOMORPHISM

In this section we briefly recall the definition of the asymptotic algebra \( \mathcal{J}(n, r) \) for the \( q \)-Schur algebra \( S_q(n, r) \) and of the Du-Lusztig homomorphism \( \Phi \) from \( S_q(n, r) \) to \( \mathcal{J}(n, r) \). We then show that this algebra is isomorphic to the algebra \( \langle B \rangle_A \) spanned by our Wedderburn basis \( B \) and that the Du-Lusztig homomorphism can be interpreted as the inclusion of \( S_q(n, r) \) into \( \langle B \rangle_A \).
**Definition 5.1** (The asymptotic algebra $J(n,r)$). Let $J(n,r)$ be the free abelian group with basis $\{ t_a | a \in M(n,r) \}$. We define a multiplication on $J(n,r)$ by setting

$$t_at_b = \sum_{c \in M(n,r)} \gamma_{a,b,c} t_c.$$ 

We set $D(n,r)_\lambda := D(n,r) \cap M_{\lambda, \lambda}$. Following Du, we denote the extension of scalars of $J(n,r)$ to $A$ by $J(n,r)_A$.

**Lemma 5.2** (See \cite{7} (2.2.1)). The $\mathbb{Z}$-algebra $J(n,r)$ is associative with the identity element

$$\sum_{d \in D(n,r)} t_d,$$

**Theorem 5.3** (The Du-Lusztig homomorphism $\Phi$, see \cite{7} (2.3)). The $A$-linear map $\Phi: S_q(n,r) \to J(n,r)_\lambda$ defined by

$$\Phi(\theta_a) := \sum_{b \in M(n,r)} \sum_{d \in D(n,r), \lambda - a(b)} f_{a,b,d} t_b,$$

where $\mu = \text{co}(\mu)$ is an algebra homomorphism and becomes an isomorphism $K_S_q(n,r) \to J(n,r)_K$ when tensored with the field of fractions $K$ of $A$.

**Proof.** See \cite{7} 2.3. The latter equation holds, since $f_{a,b,c} = 0$ unless $c \leq \lambda$, $b$, and \textbf{Q9} implies $\sim d \sim b$ in this case. Also we can safely sum over all of $D(n,r)$ neglecting the index $\mu$, since all elements $d \in D(n,r)$ fulfill $\text{ro}(d) = \text{co}(d)$ by definition (see Definition \textbf{2.5} and the remark there) and $f_{a,b,c} = 0$ unless $\text{co}(a) = \text{ro}(d)$ anyway. \hfill $\square$

We can now present our main theorem, which links our Wedderburn basis $B$ to the asymptotic algebra:

**Theorem 5.4** (Preimage of the $t$-basis under the Du-Lusztig homomorphism). Let $\tau$ be an arbitrary non-degenerate symmetrizing trace form. All dual bases in the following are meant with respect to $\tau$.

With the above notation we have

$$\Phi(c^{-1}_a \theta_a \theta_d^\vee) = t_c \quad \text{for all } c \in M(n,r).$$

**Proof.** The rightmost sum in Theorem 5.3 has the advantage that it provides a formula for the image of an arbitrary element $h \in KS_q(n,r)$ under the Du-Lusztig homomorphism, since it is obviously $K$-linear in $\theta_a$:

$$\Phi(h) = \sum_{b \in M(n,r)} \sum_{d \in D(n,r), \lambda - a(b)} \tau(h \cdot \theta_d \theta_d^\vee) \cdot t_b$$

(recall $\tau(\theta_d \theta_d^\vee) = f_{a,b,c}$). But now we can immediately set $h := c^{-1}_a \theta_a \theta_d^\vee$ for some $c \in M(n,r)$ and $d \in D(n,r)$ with $c \sim \lambda d$. The value $\tau(c^{-1}_a \theta_a \theta_d^\vee \cdot \theta_d \theta_d^\vee)$ is zero (see Lemma \textbf{4.2}) unless $b \leq \lambda c \sim \lambda d \leq \lambda d' \sim \lambda b$ and this implies $b \sim \lambda c$ and $d' \sim \lambda d$ using \textbf{Q4} and \textbf{Q10}. But this means $d' = d$ by \textbf{Q13} and the definition of $\sim$, and thus $b = c$ because of Lemma \textbf{3.6}. Thus, in the sum there is only one non-zero summand, which is $\tau(c^{-1}_a \theta_a \theta_d^\vee \cdot \theta_d \theta_d^\vee) t_c$.

Now everything is in a single left cell such that we can use Theorem \textbf{4.5} to get

$$\tau(c^{-1}_a \theta_a \theta_d^\vee \cdot \theta_d \theta_d^\vee) \cdot t_c = \tau(\theta_a \theta_d^\vee) \cdot t_c = t_c.$$
as claimed.

We can summarise our results in the following way:

**Theorem 5.5** (New interpretation of the Du-Lusztig homomorphism). Let $\tau$ be an arbitrary non-degenerate symmetrising trace form on $KS_q(n,r)$. We define the set $B$ as in Theorem 4.11 and we set

$$\mathcal{J}_\tau = \langle B \rangle_A.$$

The following diagram commutes and all unmarked arrows are identities or natural inclusions:

$$
\begin{array}{ccc}
S_q(n,r) & \longrightarrow & J(n,r) \\
\phi & \cong & \phi \\
S_q(n,r) & \phi & \longrightarrow \mathcal{J}(n,r)_A & \longrightarrow & J(n,r)_{K}
\end{array}
$$

Thus, the asymptotic algebra $J(n,r)_A$ is nothing but the $A$-span of our Wedderburn basis and the Du-Lusztig homomorphism $\Phi$ can simply be interpreted as the inclusion of $S_q(n,r)$ into $\langle B \rangle_A$. Furthermore, our results directly and explicitly show that $\langle B \rangle_A$ is isomorphic as an $A$-algebra to a direct sum of full matrix rings over $A$.

6. A CRITERION FOR JAMES’ CONJECTURE

In this section we show how our results provide an equivalent formulation of a conjecture about the representation theory of specialisations of the $q$-Schur algebra. We first recall the conjecture.

The construction of the Iwahori-Hecke algebra of type A and of the $q$-Schur algebra as in Section 2 together with their Kazhdan-Lusztig bases can be carried out over an arbitrary integral domain $R$ with quotient field $k$ and with an arbitrary invertible parameter $q \in R$ having a square root in that domain. We denote the resulting algebra by $S_q(n,r)_R$ and its extension of scalars to $k$ by $S_q(n,r)_k$. This is called a “specialisation”.

It is known, that $S_q(n,r)_k$ is semisimple unless $q$ is an $e$-th root of unity. If $q$ is a root of unity, then there is a decomposition matrix, which records the multiplicities of the simple modules in the so-called “standard modules”. For the case that $k$ has characteristic zero, recent work by Lascoux, Leclerc and Thibon, and Varagnolo and Vasserot yields a complete determination of these decomposition matrices (see [15], [8] and the references there). However, the case of positive characteristic is still open.

James’ conjecture is a statement about this modular case. Roughly speaking, it asserts that if $k$ is a field of characteristic $\ell$ and the multiplicative order $e$ of the parameter $q \in k$ is greater than $r$, then the decomposition matrix of $S_q(n,r)_k$ does not depend on the particular value of $\ell$ but only on $e$.

We now want to make this statement more precise. Both the simple modules and the standard modules have a labelling by the set $\Lambda(n,r)$. Let $V^\lambda\nabla$ denote the standard module and $M_{k,q}^\lambda$ the simple module of $S_q(n,r)_k$ corresponding to $\lambda$ and $\mu$ respectively. Then the decomposition matrix for $S_q(n,r)_k$ consists of the numbers

$$d^k_{\lambda,\mu} := \text{multiplicity of } M_{k,q}^\mu \text{ in } V_{k,q}^\lambda.$$

Conjecture 6.1 (James, see [11] §4 and [8] §3). If $\ell > r$ and $e$ is the multiplicative order of $q \in k$, then $c_{\lambda,\mu}^{k,q} = d_{\lambda,\mu}^{\zeta_{c}}$ for all $\lambda, \mu \in \Lambda(n, r)$, where $\zeta_{c}$ is a complex primitive $e$-th root of unity.

Mieolf Geck has shown in [9, Theorem 1.2] that this statement is equivalent to the fact, that for $\ell > r$, the rank of the Du-Lusztig homomorphism $\Phi : S_{q}(n, r)_{k} \to \mathcal{F}(n, r)_{k}$ with respect to the two bases $(\theta_{a})_{a \in M(n, r)}$ and $(t_{a})_{a \in M(n, r)}$ respectively only depends on the multiplicative order $e$ of $q \in k$ and not on the characteristic $\ell$ of $k$.

In view of our Theorem 5.5 this immediately implies:

Theorem 6.2 (An equivalent formulation of James’ conjecture). Let $\{ \theta_{a} | a \in M(n, r) \}$ be the Du-Kazhdan-Lusztig-basis of $S_{q}(n, r)$ and let $\tau$ be a non degenerate symmetrising trace form for $KS_{q}(n, r)$. Let $\{ \theta'_{a} | a \in M(n, r) \}$ be the dual basis of $\{ \theta_{a} | a \in M(n, r) \}$ with respect to $\tau$. Let $B$ be the basis defined in Theorem 4.4.4. Let $s := |M(n, r)|$ and $M = (m_{a,b})_{a,b \in M(n, r)} \in A^{s \times s}$ be the matrix, for which

$$\theta_{a} = \sum_{\zeta \in \mathbb{Z}(n, r)} m_{a,\zeta} \cdot c_{\zeta}^{-1} \theta_{\zeta}$$

with $c_{\zeta}^{-1} \theta_{\zeta} \theta'_{\zeta} \in B$ holds for all $a \in M(n, r)$.

Let $\ell_{1}, \ell_{2}$ be two primes and $\varphi_{1} : \mathbb{Z}[v, v^{-1}] \to \mathbb{F}_{\ell_{1}}$ and $\varphi_{2} : \mathbb{Z}[v, v^{-1}] \to \mathbb{F}_{\ell_{2}}$ two ring homomorphisms, such that the multiplicative orders of $\varphi_{1}(v^{2})$ and $\varphi_{2}(v^{2})$ are equal. Denote by $\varphi_{i}(M)$ the matrix in $\mathbb{F}_{\ell_{i}}^{s \times s}$ that one gets by applying the ring homomorphism $\varphi_{i}$ to every entry of $M$.

Then James’ conjecture is equivalent to the fact, that for $\ell_{1}, \ell_{2} > r$ the ranks of $\varphi_{1}(M)$ and of $\varphi_{2}(M)$ are equal.

Let $\tau$ be a non-degenerate symmetrising trace form on $KS_{q}(n, r)$. We denote by $\{ \theta_{a} | a \in M(n, r) \}$ the Du-Kazhdan-Lusztig-basis of $S_{q}(n, r)$ and by $\{ \theta'_{a} | a \in M(n, r) \}$ its dual basis relative to $\tau$. As above, we denote by $B$ the Wedderburn basis obtained in Theorem 4.4.4. Moreover, we denote by $M = (m_{a,b})_{a,b \in M(n, r)}$ the change of basis matrix from $\{ \theta_{a} | a \in M(n, r) \}$ to $B$ as above and by $P_{\tau} = (p_{a,b})_{a,b \in M(n, r)}$ the change of basis matrix from $\{ \theta_{a} | a \in M(n, r) \}$ to $\{ \theta'_{a} | a \in M(n, r) \}$, that is:

$$\theta_{a} = \sum_{b \in M(n, r)} p_{a,b} \cdot \theta'_{b}$$

for all $a \in M(n, r)$. Formula (4.1) implies that

$$P_{\tau} = (\tau(\theta_{a} \theta'_{b}))_{a,b \in M(n, r)} \quad \text{and} \quad P_{\tau}^{-1} = (\tau(\theta'_{a} \theta_{b}))_{a,b \in M(n, r)}.$$ 

Lemma 6.3. With the above notation, the matrix

$$D = M^{T} P_{\tau}^{-1} M$$

is monomial and its entries are the Schur elements $c_{a}$ associated to $a \in D(n, r)$ as in Definition 4.10.

Proof. The matrix $M^{T}$ is the change of basis matrix from $B$ to $\{ \theta'_{a} | a \in M(n, r) \}$ and thus the matrix $D$ is the change of basis matrix from $B^{T}$ to $B$, that is:

$$\theta'_{a} = \sum_{c' \in M(n, r)} d_{a,c'} c' c_{a}^{-1} \theta'_{c'}$$

for all $\theta'_{a} \in B^{T}$. Using Proposition 4.15 the result follows. \qed
Proposition 6.4 (A criterion for James’ conjecture). Let $\tau$ be a non-degenerate symmetrising trace form on $K S_q(n, r)$. Let $\varphi_e: A \to \mathbb{Z}[\zeta_{2e}], v \mapsto \zeta_{2e}$ be a specialisation to characteristic 0 where $v^2$ is mapped to a primitive $e$-th root of unity in a cyclotomic field and $\varphi_\ell : A \to \mathbb{F}_\ell$ is a second specialisation to characteristic $\ell$ such that there is a ring homomorphism $\varphi_\ell^e : \mathbb{Z}[\zeta_{2e}] \to \mathbb{F}_\ell$ with $\varphi_\ell = \varphi_\ell^e \circ \varphi_e$. We suppose that $\ell > r$ and the following hypotheses on $\tau$:

- The Schur elements $c_d$ for $d \in D(n, r)$ lie in $A$.
- The coefficients of the matrix $P_\tau^{-1}$ lie in $A$.
- Let $a$ be the number of Schur elements $c_d$ for $d \in D(n, r)$ that do not vanish under $\varphi_e$ and $b$ the number of Schur elements that do not vanish under $\varphi_\ell$. The numbers $a$ and $b$ are both equal to the rank over $\mathbb{Q}(\zeta_{2e})$ of the matrix $\varphi_e(M)$ for $M$ from above.

Note that we denote with the notation $\varphi_e(M)$ the matrix one gets from $M$ by applying the ring homomorphism $\varphi_e$ on every entry.

If $\tau$ can be found fulfilling all these hypotheses, then James’ conjecture holds for all $\ell > r$ for which $\varphi_\ell$ as above exist.

Proof. We denote by $B$ the change of basis matrix from $\{\theta_\underline{a} | \underline{a} \in M(n, r)\}$ to $B$ as above. Then Lemma 6.3 asserts that

$$D = M^T P_\tau^{-1} M.$$ 

Thanks to Theorem 4.11 the coefficients of the matrix $M$ lie in $A$. By hypothesis, the matrix $P_\tau^{-1}$ has coefficients in $A$. By Lemma 6.3 and the first hypothesis the entries of $D$ are also in $A$.

Since the matrices $D$, $M$, $M^T$, and $P_\tau^{-1}$ have coefficients in $A$, the matrices $\varphi_e(D)$, $\varphi_e(M)$, $\varphi_\ell(D)$, $\varphi_\ell(M)$, $\varphi_\ell(M^T)$ and $\varphi_\ell(P_\tau^{-1})$ are well-defined. We then have the following equality

$$\varphi_\ell(D) = \varphi_\ell(M^T) \cdot \varphi_\ell(P_\tau^{-1}) \cdot \varphi_\ell(M),$$ 

implying that $\text{rk}_{\mathbb{Q}(\zeta_{2e})}(\varphi_\ell(D)) \leq \text{rk}_{\mathbb{Q}(\zeta_{2e})}(\varphi_\ell(M))$. Moreover we have $\varphi_\ell(M) = \varphi_\ell^e(\varphi_e(M))$. Since $\varphi_\ell^e$ is a ring homomorphism, we deduce that

$$\text{rk}_{\mathbb{Q}(\zeta_{2e})}(\varphi_\ell(M)) \leq \text{rk}_{\mathbb{Q}(\zeta_{2e})}(\varphi_e(M)).$$ 

Since $D$ is a monomial matrix containing only the Schur elements as non-zero entries, the numbers $a$ and $b$ from the hypotheses are the ranks of $\varphi_e(D)$ and $\varphi_\ell(D)$ respectively. However, if as in the last hypothesis the ranks of $\varphi_e(M)$ and $\varphi_\ell(D)$ are equal, then it follows that $\text{rk}_{\mathbb{Q}(\zeta_{2e})}(\varphi_\ell(M)) \leq \text{rk}_{\mathbb{Q}(\zeta_{2e})}(\varphi_\ell(D))$. We then deduce that

$$\text{rk}_{\mathbb{Q}(\zeta_{2e})}(\varphi_\ell(M)) = \text{rk}_{\mathbb{Q}(\zeta_{2e})}(\varphi_\ell(D)),$$

and the result now follows from Theorem 6.2. \hfill $\square$

Remark 6.5. To prove James’ conjecture it is enough to find a symmetrising trace form $\tau$ on $K S_q(n, r)$ such that the hypotheses of Proposition 6.4 are satisfied. We notice that the assumption on $P_\tau$ in the statement of Proposition 6.4 is “generic” in the sense that this property only depending on the “generic” $q$-Schur algebra, but not on specialisations over finite fields.

Remark 6.6. We can replace the second assumption of Proposition 6.4 by the fact that the matrix $P_\tau^{-1} M$ (or $M^T P_\tau^{-1}$) has its coefficients in $A$. 


Remark 6.7. For the usual trace form \( \tau \) on Hecke algebras of type \( A \), we note that the assumptions of Proposition 6.4 hold. Then using [14], we can prove in a way similar to the one of the proof of Proposition 6.4 that the rank of the Lusztig homomorphism (specialized in a finite field \( \mathbb{F}_\ell \) by \( \phi_\ell: A \to \mathbb{F}_\ell \) mapping \( v^2 \) to an element \( q \in \mathbb{F}_\ell \) with multiplicative order \( e \) as above) does not depend on \( \ell \). However as noted by Geck in [9] an analogue result as Theorem 6.2 in Hecke algebras does not imply the Hecke algebras James’ conjecture.

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