Supersymmetric Gauged $O(3)$ Sigma Model and 
Self-dual Born-Infeld Theory

Prasanta K. Tripathy

Institute of Physics, Bhubaneswar 751005, India

Abstract

We study the supersymmetric extension of the gauged $O(3)$ sigma model in 2 + 1 dimensions and find the supersymmetry algebra. We also discuss soliton solutions in case the Maxwell term is replaced by the Born-Infeld term. We show that by appropriate choice of the potential, the self-dual equations in the Born-Infeld case coincide with those of the Maxwell’s case.

I. INTRODUCTION

The $O(3)$ sigma model in 2 + 1 dimensions is well studied [1], [2]. It is well known that this model is integrable and has Bogomol’nyi type bound on energy [3]. All finite energy static solutions of the Bogomol’nyi equations are well known and can be expressed in terms of rational functions [1], [2]. The model is scale invariant and hence the solitons are of arbitrary size. Bogomol’nyi type bound has also been obtained by adding a suitable potential to the model so as to break the scale invariance [4]. Some time ago, gauged $O(3)$ sigma model was also well studied when the gauge field dynamics is governed by a Maxwell term [5] and Bogomol’nyi bound was obtained. It was shown that these solitons are topologically stable even though they carry arbitrary value of magnetic flux. Around the same time, people also

*E-mail: prasanta@iopb.res.in
considered gauged $O(3)$ sigma model when the dynamics is governed by the Chern-Simons term alone and all static solutions were obtained for both abelian as well as non-abelian gauge fields [5], [7]. Gauged $O(3)$ sigma models in the presence of both the Maxwell and the Chern-Simons term also admit static self-dual solutions when anomalous magnetic moment interaction is incorporated [8].

It is well known that self-duality is closely related to $N = 2$ supersymmetry [9]. Supersymmetric pure $O(3)$ sigma model (without gauge field) was indeed constructed [10] long time ago. Recently supersymmetric version of the gauged sigma model has also been constructed in case the gauge field dynamics is governed by the abelian Chern-Simons term [11]. However, to the best of our knowledge no supersymmetric model has been constructed for gauged sigma model with Maxwell term. The aim of this paper is to carry out this task.

Recently Born-Infeld theory [12], [13] has received wide publicity, specially in the context of string theory [14]. Different models containing Born-Infeld Lagrangian are found to possess domain wall, vortex and monopole solutions [15]. Supersymmetry and BPS saturated solutions in connection with D-brane dynamics [16] (which contains Born-Infeld action) have also been investigated. Recently self-dual solutions have been obtained in abelian Higgs model in $2 + 1$ dimensions in case the Maxwell term is replaced by the abelian Born-Infeld term [17]. In particular, it has been shown that by suitably modifying the Higgs potential, the self-dual equations for the Born-Infeld abelian Higgs model are identical to that of the corresponding Maxwell abelian Higgs model. It is then worthwhile enquiring if one can also obtain self-dual solutions in the gauged $O(3)$ sigma model in case the Maxwell term is replaced by the Born-Infeld action. In this paper we show that the answer to the question is yes. In particular, we show that the self-dual equations are identical to the gauged $O(3)$ sigma model with Maxwell term provided the potential is adjusted suitably.

The plan of the paper is the following. In section II we construct the supersymmetric version of the gauged $O(3)$ sigma model with Maxwell term and find the corresponding supersymmetry algebra. In section III we consider the gauged $O(3)$ sigma model with the Born-Infeld action and obtain the Bogomol’nyi bound. Finally we conclude the results in
section IV and point out some of the open problems.

II. SUPERSYMMETRIC GAUGED SIGMA MODEL

We consider three component real superfield $\Phi^a$ containing scalar field $\phi^a$, Majorana spinor $\psi^a$ and auxiliary field $F^a$ in 2 + 1 dimensions. The real spinor gauge superfield $W^\alpha$ contains a gauge field $A_{\mu}$ and a Majorana spinor $\lambda^\alpha$. A real scalar superfield $S$ consists of a real scalar $M$, a Majorana fermion $\chi$ and a real auxiliary field $D$. The superfield $\Phi^a$ is constrained to satisfy the relation $\Phi^a \Phi^a = 1$, $a = 1,...,3$ which yields the following three constraints

\[ \phi^a \phi^a = 1 \]
\[ \phi^a \psi^a = 0 \]
\[ \phi^a F^a + \frac{1}{2} \bar{\psi}^a \psi^a = 0. \]  (1)

The action in terms of component fields can be written as

\[ S = \int d^3x \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \partial_\mu M \partial^\mu M + \frac{1}{2} D_\mu \phi^a D^\mu \phi^a \right. \\
+ \frac{1}{2} F^a F^a + \frac{1}{2} D^2(x) + D(x)(1 - n^a \phi^a) \\
+ \frac{i}{2} \bar{\lambda} \gamma_\mu \partial^\mu \lambda + \frac{i}{2} \bar{\chi} \gamma_\mu \partial^\mu \chi + \frac{i}{2} \bar{\psi}^a \gamma_\mu \partial^\mu \psi^a \\
- \bar{\lambda} \psi^a (\hat{n} \times \phi)^a - M n^a F^a - n^a \bar{\psi}^a \chi \right\} \]  (2)

where we have

\[ D_\mu \phi^a = \partial_\mu \phi^a + \epsilon^{abc} A_\mu n^b \phi^c. \]

The auxiliary fields $F^a$ and $D$ can be removed from the action by using their equations of motion

\[ D = -(1 - n^a \phi^a) \]
\[ F^a = (n^a - \phi^a n^b \phi^b)M - \frac{1}{2} \phi^a \bar{\psi}^b \psi^b \]  (3)
Note that the constraints in Eq.(1) have been used to obtain the second of the above equations. Using Eq.(3) the action (2) becomes

\[ S = \int d^3x \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \partial_\mu M \partial^\mu M + \frac{1}{2} D_\mu \phi^a D^\mu \phi^a \\
+ \frac{i}{2} \bar{\lambda} \gamma_\mu \partial^\mu \lambda + \frac{i}{2} \bar{\chi} \gamma_\mu \partial^\mu \chi + \frac{i}{2} \bar{\psi} \gamma_\mu D^\mu \psi \\
- \frac{1}{2} (1 - n^a \phi^a)^2 - \frac{1}{2} M^2 \left[ 1 - (n^b \phi^b)^2 \right] \\
- \lambda \psi^a (\hat{n} \times \phi)^a - n^a \bar{\psi}^a \chi \\
+ \frac{1}{2} M n^b \phi^b \bar{\psi}^c + \frac{1}{4} (\bar{\psi}^b \psi^b)^2 \right\} \]  

Equation (4)

The action (4) is invariant under the following $N = 2$ supersymmetry transformations

\[ \delta \phi^a = \bar{\epsilon} \psi^a \]
\[ \delta A_\mu = i \bar{\epsilon} \gamma_\mu \lambda \]
\[ \delta M = \bar{\epsilon} \chi \]
\[ \delta \lambda = \frac{i}{2} \epsilon^{\mu\nu\rho} F_{\mu\nu} \gamma^\rho \epsilon \]
\[ \delta \chi = -i \gamma_\mu \partial^\mu M \epsilon + (1 - n^a \phi^a) \epsilon \]
\[ \delta \psi^a = -i \gamma_\mu D^\mu \phi^a \epsilon - \left[ (n^a - \phi^a n^b \phi^b) M - \frac{1}{2} \phi^a \bar{\psi}^b \psi^b \right] \epsilon \]

Equation (5)

The corresponding supercharges are

\[ Q^1 = \int d^2x \left[ \gamma^\mu \gamma^0 \psi^a D_\mu \phi^a + i \gamma^0 \psi^a n^a M \\
- \epsilon_\rho_\mu \gamma^0 \gamma^0 \lambda \partial^\mu A^\nu + \gamma^\mu \gamma^0 \chi \partial_\mu M + i \gamma^0 \chi (1 - n^a \phi^a) \right] \]

Equation (6)

and

\[ Q^2 = \int d^2x \left[ (\phi \times D_\mu \phi)^a \gamma^\mu \gamma^0 \psi^a - i (n \times \phi)^a \gamma^0 \psi^a M \\
- \epsilon_\rho_\mu \gamma^0 \gamma^0 \lambda \partial^\mu A^\nu + \gamma^\mu \gamma^0 \lambda \partial_\mu M + i \gamma^0 \lambda (1 - n^a \phi^a) \right] \]

Equation (7)

which satisfy the following anticommutation relations

\[ \frac{1}{2} \left\{ Q^A_\alpha, Q^B_\beta \right\} = (\gamma_0)_{\alpha\beta} P_0 \delta^{AB} + T \epsilon_{\alpha\beta} \epsilon^{AB} \]

Equation (8)
where \( P_0 \), the static energy is given by
\[
\int d^2x \left[ \frac{1}{4} F_{ij} F^{ij} + \frac{1}{2} | D_i \phi |^2 + \frac{1}{2} (1 - n^a \phi^a)^2 \right]
\]  
(9)
and the central charge
\[
T = \int d^2x \left[ \epsilon^{abc} \phi^a D_1 \phi^b D_2 \phi^c + F_{12}(1 - n^a \phi^a) \right]
\]  
(10)
Note that after calculating the superalgebra we restrict the model to its bosonic sector by putting the fermion fields to zero. As expected, the above expression for static energy coincides with that in ref. [5] and the central charge is identical to the topological charge derived in that paper. Note that the anticommutator in Eq.(8) is hermitian and hence the trace of its square is positive semi-definite.
\[
\sum_{AB} \{ Q^{(A)} \alpha, Q^{(B)} \beta \} \{ Q^{(A)} \alpha, Q^{(B)} \beta \} \geq 0
\]  
(11)
\[
\Rightarrow P_0 \geq | T |
\]  
(12)
which is the Bogomol’nyi bound on the energy.

III. THE BORN-INFELD GAUGED \( O(3) \) MODEL

Now we construct the action for the gauged \( O(3) \) sigma model with the Born-Infeld term. Taking a clue from the abelian-Higgs model with the Born-Infeld term [17], here we introduce a potential which appears as a multiplicative factor inside the square root of the Born-Infeld term. We start with the following action for scalar fields with a Born-Infeld term
\[
S = \int d^3x \left[ \frac{1}{2} D_\mu \phi^a D^\mu \phi^a + \beta^2 - \beta^2 \sqrt{\left(1 + \frac{1}{2 \beta^2} F_{\mu\nu} F^{\mu\nu}\right)} V(\phi) \right]
\]  
(13)
where \( V(\phi) \) is given as
\[
V(\phi) = 1 + \frac{1}{\beta^2} (1 - n^a \phi^a)^2
\]  
(14)
As before we restrict the field $\phi^a$ be $\phi^a \phi^a = 1$. Note that in the limit $\beta \to \infty$ the above action reduces to

$$S = \int d^3 x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} D_\mu \phi^a D^\mu \phi^a - \frac{1}{2} (1 - n^a \phi^a)^2 \right]$$

which is the action for the gauged $O(3)$ sigma model with the Maxwell term.

The equations of motion are

$$D_\mu D^\mu \phi^a - (\phi^b D_\mu D^\mu \phi^b) \phi^a + (n^a - \phi^a n^b \phi^b)(1 - n^c \phi^c) \sqrt{\frac{1}{1 + \frac{1}{2 \beta^2} F_{\mu\nu} F^{\mu\nu}}} = 0$$

and

$$\partial_\mu \left( F^{\mu\nu} \sqrt{\frac{V(\phi)}{1 + \frac{1}{2 \beta^2} F_{\mu\nu} F^{\mu\nu}}} \right) + \epsilon^{abc} D_\nu \phi^a n^b \phi^c = 0.$$

The static energy is found to be

$$E = \int d^2 x \left( \frac{1}{2} (D_i \phi^a)^2 + \beta^2 \left[ \left( 1 + \frac{F_{12}^2}{\beta^2} \right) \left( 1 + \frac{1}{\beta^2} (1 - n^a \phi^a)^2 \right) - 1 \right] \right)$$

The energy can be rearranged as

$$E = \int d^2 x \left( \frac{1}{2} (D_1 \phi^a + \epsilon^{abc} \phi^b D_2 \phi^c)^2 + \epsilon^{abc} \phi^a D_1 \phi^b D_2 \phi^c \\
+ \beta^2 \left[ \left( \frac{1}{\beta^2} F_{12} - (1 - n^a \phi^a) \right)^2 + \left( 1 + \frac{1}{\beta^2} F_{12} (1 - n^a \phi^a) \right)^2 - 1 \right] \right)$$

From the above expression we get the following bound on the energy

$$E \geq \int d^2 x \left( \epsilon^{abc} \phi^a D_1 \phi^b D_2 \phi^c + F_{12} (1 - n^a \phi^a) \right)$$

which is same as that for the Maxwell case. The bound is saturated when the fields satisfy the Bogomoln’yi equations

$$D_1 \phi^a + \epsilon^{abc} \phi^b D_2 \phi^c = 0$$

$$F_{12} - (1 - n^a \phi^a) = 0$$
which are identical to the Bogomol’nyi equations for the Maxwell’s case \[3\]. Note that
these are also consistent with the second order field equations Eqs. (16-17). Hence all the
properties like flux and charge for this model will be same as those for the Maxwell’s case.

It is well known that results similar to $O(3)$ sigma model also holds in $CP^1$ model. As a
confirmation of the fact, we now obtain a similar bound on energy in the case of gauged $CP^1$
model with Born-Infeld action. Let $\xi$ be the $CP^1$ field such that $\xi^\dagger \xi = 1$. Then consider
the following action

$$S = \int d^3x \left[ \left| \nabla_\mu \xi \right|^2 + \beta^2 - \beta^2 \sqrt{\left( 1 + \frac{1}{2\beta^2} F_{\mu\nu} F^{\mu\nu} \right)} V(\xi) \right]$$

(22)

where

$$\nabla_\mu \xi = D_\mu \xi - (\xi^\dagger D_\mu \xi) \xi$$

(23)

$$V(\xi) = 1 + \frac{1}{\beta^2} (1 - \xi^\dagger \sigma_3 \xi)^2$$

(24)

and

$$(D_\mu \xi)^T = ([\partial_\mu - iA_\mu] \xi_1, \partial_\mu \xi_2)$$

(25)

The static energy is

$$E = \int d^2x \left( \left| \nabla_i \xi \right|^2 + \beta^2 \left[ \sqrt{\left( 1 + \frac{F_{12}^2}{\beta^2} \right)} \left( 1 + \frac{1}{\beta^2} (1 - \xi^\dagger \sigma_3 \xi)^2 \right) - 1 \right] \right)$$

(26)

Using the identity

$$\left| \nabla_i \xi \right|^2 = \left| (\nabla_1 + i\nabla_2) \xi \right|^2 + i\epsilon_{jk} (\nabla_k \xi)^\dagger (\nabla_j \xi)$$

(27)

the energy can be rewritten as,

$$E = \int d^2x \left( \left| (\nabla_1 + i\nabla_2) \xi \right|^2 + i\epsilon_{jk} (\nabla_k \xi)^\dagger (\nabla_j \xi) \right.$$

$$\left. + \beta^2 \left[ \sqrt{\frac{1}{\beta^2} \left( F_{12} - (1 - \xi^\dagger \sigma_3 \xi) \right)^2 + \left( 1 + \frac{1}{\beta^2} F_{12}(1 - \xi^\dagger \sigma_3 \xi) \right)^2 - 1 \right] \right)$$

(28)
and hence we have the inequality

\[ E \geq \int d^2x \left( i \epsilon_{jk}(\nabla_k \xi)^\dagger(\nabla_j \xi) + F_{12}(1 - \xi^\dagger \sigma^3 \xi) \right) \]  

(29)

The above bound is saturated if the following Bogomol’nyi equations hold.

\[ \nabla_1 \xi + i \nabla_2 \xi = 0 \]  

(30)

\[ F_{12} - (1 - \xi^\dagger \sigma_3 \xi) = 0 \]  

(31)

which are the self-dual equations for the gauged \( CP^1 \) model with Maxwell’s term.

IV. CONCLUSION

In this paper we have constructed a supersymmetric extension of the gauged \( O(3) \) sigma model with the Maxwell term. Replacing the Maxwell term by the Born-Infeld term in the above model we have obtained the self-dual equations and have shown that by changing the potential appropriately, these self-dual equations are same as in the Maxwell’s case. This work raises several problems which deserves further study. Some of these are

(i) can one find supersymmetric extension of the gauged \( O(3) \) sigma model with both the Maxwell and the Chern-Simons terms?

(ii) In case of abelian Higgs model the maximal supersymmetry \( [18] \) is \( N = 3 \). Thus it is natural to enquire about the maximal supersymmetry in the case of sigma model with Maxwell term.

(iii) Can one obtain self-dual solutions in the gauged sigma model with both Chern-Simons and Born-Infeld action?

(iv) Further can one obtain the supersymmetric version of the above?

We hope to report on some of these issues in the near future.

V. ACKNOWLEDGMENTS

I am indebted to Avinash Khare for helpful discussions as well as for careful reading of the manuscript.
REFERENCES

[1] R. Rajaraman, Solitons and Instantons (North-Holland, Amsterdam)

[2] A. A. Belavin and A. M. Ployakov, JETP Lett. 22, 245 (1975).

[3] E. Bogomol’nyi, Sov. J. Nucl. Phys. 24, 449 (1976).

[4] R. A. Leese, Nucl. Phys. B344, 33 (1990).

[5] B. J. Schroers, Phys. Lett. B356, 291 (1995).

[6] G. Nardelli, Phys. Rev. Lett. 73, 2524 (1994); Phys. Rev. D52, 5944 (1995).

[7] Y. M. Cho and K. Kimm, Phys. Rev. D52, 7325 (1995).

[8] P. K. Ghosh, Phys. Lett. B381, 237 (1996).

[9] C. Lee, K. Lee and E. J. Weinberg, Phys. Lett. B234, 105 (1990); E. Ivano, Phys. Lett. B268, 203 (1991); S. J. Gates and N. Nishino, Phys. Lett. B281, 72 (1992); Z. Housek and D. Spector, Nucl. Phys. B397, 173 (1993).

[10] E. Witten, Phys. Rev. D16, 2991 (1977); P. Di Vecchia and S. Ferrara, Nucl. Phys. B130, 93 (1977).

[11] K. Kimm and B. Sul, hep-th/9703185.

[12] M. Born, Proc. R. Sco. London A 143, 410 (1934).

[13] M. Born and M. Infeld, Proc. R. Sco. London A 144, 425 (1943).

[14] E. S. Fradkin and A. A. Tseytlin, Phys. Lett. B 163, 123 (1985); A. A. Tseytlin, Nucl. Phys. B 276, 391 (1986); R. G. Leigh, Mod. Phys. Lett. A 4, 2073, 2767 (1989).

[15] A. Nakamura and K. Shiraishi, Int. Jour. of Mod. Phys. A 6, 2635 (1991); Hadronic Jour. 14, 369 (1991); G. Gibbons, hep-th/9709027.

[16] C. G. Callan and J. M. Maldacena, hep-th/9708147; J. P. Gauntlett, J. Gomis and P.
K. Townsend, hep-th/9711205.

[17] S. Gonorazky, C. Nunez, F. A. Schaposnik and G. Silva, hep-th/9805054 (To be published in Nucl. Phys. B).

[18] H.-C. Kao, Phys. Rev. D50, 2881 (1994).