An intrinsic characterization of the Kerr metric

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Abstract
We give the necessary and sufficient (local) conditions for a metric tensor to be the Kerr solution. These conditions exclusively involve explicit concomitants of the Riemann tensor.

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1. Introduction

The Kerr solution plays an essential role in relativistic astrophysics to model the exterior gravitational field of a rotating mass. Its prominence among stationary vacuum solutions comes from the black hole uniqueness theorem which states that, under rather general conditions, the Kerr spacetime is the only asymptotically flat, stationary, vacuum black hole.

Nowadays, the study of dynamical black hole spacetimes is based on numerical simulations that usually make use of the 3+1 formalism. Thus, it is important to understand the properties of the Kerr metric on a generic spacelike hypersurface. In a recent work [1], the Cauchy initial data have been analyzed on a generic slice of the Kerr geometry and a characterization of these data is offered. This study is based on the spacetime characterization of the Kerr solution given by Mars [2, 3].

The Kerr initial data characterization given in [1] fails to be completely algorithmic as claimed by the authors. In contrast, in a previous paper [4] the same authors obtained a fully algorithmic characterization of the Schwarzschild initial data. The main reason for these differences is that the study of the initial data made in [4] is based on a local \textit{intrinsic} and \textit{explicit} spacetime characterization of the Schwarzschild solution that we presented ten years ago [5]. However, the characterization of the Kerr geometry by Mars [2, 3] used in [1] does not possess such good qualities. Nevertheless, these two works study the line suggested by Simon [6] in depth, and have also been useful to improve the black hole uniqueness theorem [7].
In this paper we present a local intrinsic spacetime characterization of the Kerr solution that exclusively uses explicit concomitants of the Weyl tensor. Our labeling of the Kerr geometry is algorithmic and must allow an algorithmic characterization of the Kerr initial data. But the relevance of this intrinsic characterization goes further than the above-mentioned application.

Since the beginning of the Riemannian geometry the invariant characterization of metrics is a subject that has been tackled from several points of view. Thus we have the historic theorems that characterize locally flat Riemann spaces \[8\], Riemann spaces with a maximal group of isometries \[9\] and locally conformally flat Riemann spaces \[10–12\]. It is worth remarking that, in all these historic results, the conditions involve explicit concomitants of the curvature tensor (Riemann, Weyl and Cotton tensors). Then, they are certainly intrinsic (depend solely on the metric tensor \(g\)) and, besides, the explicit expression of these concomitants in terms of \(g\) is known.

Cartan \[13\] showed that a Riemannian geometry may be characterized in terms of the Riemann tensor and its covariant derivatives. Brans \[14\] introduced the Cartan invariant scheme in general relativity, and after Karlhede’s work \[15\], this approach became more helpful within the relativistic framework. The Cartan–Karlhede method to study the equivalence of two metric tensors is based on working in an orthonormal (or a null) frame, fixed by the underlying geometry of the Riemann tensor.

Nevertheless, the historic theorems quoted above show that the determination of a Riemannian canonical frame is not necessary in labeling specific families of spacetimes. We find a similar situation in the characterization of the Stephani Universes (conformally flat perfect fluid solutions) or in the different ways that the Friedmann–Lemaître–Robertson–Walker Universes can be intrinsically labeled (conformally flat barotropic perfect fluid solutions; perfect fluid solutions with vorticity-free and shear-free geodesic velocity).

A suitable procedure is to analyze every particular case in order to understand the minimal set of elements of the curvature tensor that are necessary to label these geometries, an approach adapted to each particular geometry we want to characterize. The examples quoted above show that the characterization conditions usually involve tensorial concomitants whereas the Cartan–Karlhede scheme only uses scalar concomitants. Whatever the method, the covariant obtention of the underlying geometry of the Weyl and Ricci tensors is a necessary tool to characterize spacetimes intrinsically.

The algebraic classification of the spacetime symmetric tensors was given by Churchill \[16\], and since the 1960s a lot of work has been devoted to studying the Ricci tensor from an algebraic point of view (see, for example, the Plebański paper \[17\]). Nevertheless, the general covariant method to determine the Ricci eigenvectors and their causal character was presented by Morales in his Ph D thesis (1988) and published in \[18\].

The pioneering papers by Petrov \[19\] and Bel \[20\] studied the algebraic classification of the Weyl tensor. Since then several algorithms have been proposed to determine the Petrov–Bel-type. The general covariant expressions that give the underlying geometry of the Weyl tensor for every Petrov–Bel-type were obtained by Sáez in his Ph D thesis (2001) and published in \[21\].

The Kerr metric is a vacuum solution and, consequently, the Ricci tensor vanishes, \(Ric(g) = 0\). On the other hand, the Weyl tensor \(W = Weyl(g)\) is Petrov–Bel-type D. The first one is an intrinsic and explicit condition in the metric tensor. The second one is intrinsic and we must write it in an explicit form. Thus, in the first step in labeling the Kerr solution we must impose a Weyl tensor of type D, and we must obtain the explicit expressions for its scalar invariants and its underlying geometry. In section 2, we solve these two questions by making use of the results in \[21\].
Walker and Penrose [22] showed that a Killing tensor exists in the charged Kerr black hole, and Hougston and Sommers [23] proved that, with the exception of the generalized charged C-metrics, the other charged counterpart of the type D vacuum solutions also have this property. The type D vacuum solutions with a Killing tensor are called the vacuum Kerr-NUT metrics. In [24], we have studied the symmetries and other invariant properties of the type D vacuum metrics. These results allow us to give in section 3 an intrinsic labeling of the Kerr-NUT vacuum solutions. In this section we also offer an alternative characterization of the vacuum Kerr-NUT metrics more based on the Weyl scalar invariants.

In section 4 we give the coordinate expression of the basic Weyl invariants in a Kerr-NUT vacuum solution, and we also study some restrictions on the Weyl invariants that hold in these spacetimes. We need these results in order to obtain new intrinsic conditions that distinguish the Kerr solution from the other vacuum Kerr-NUT metrics.

The Kerr characterization theorem is presented in section 5. We offer two alternative intrinsic and explicit labeling of the Kerr geometry. The second one is more adapted to the Weyl scalar invariants and it seems suitable to obtain an algorithmic characterization of the Kerr initial data. We also offer the invariant expression of the Kerr mass and angular momentum, as well as, of the ‘stationary’ Killing vector field.

Finally, section 6 is devoted to remark the algorithmic nature of our characterization theorem by presenting a summary of the results as a flow chart.

In this paper we work on an oriented spacetime with a metric tensor $g$ of the signature $\{-, +, +, +\}$. The Ricci and Weyl tensors are defined as given in [25] and denoted, respectively, by $Ric$ and $W$. For the metric product of two vectors we write $(x, y) = g(x, y)$, and we put $x^2 = g(x, x)$. Other basic notation used in this work is summarized in appendix A.

2. Labeling Petrov–Bel-type D spacetimes

The Kerr solution is a Petrov–Bel-type D metric. In this section we give the equations that make explicit this intrinsic condition. First, in the self-dual complex formalism which is well adapted to the study of the Weyl eigenvalue problem, and second, in real formalism.

2.1. Type D metrics in self-dual formalism

In order to analyze the conditions for a spacetime to be a Petrov–Bel-type D solution we now introduce the necessary notation. A self-dual 2-form is a complex 2-form $F$ such that $\ast F = iF$, where $\ast$ denotes the Hodge dual operator. We can associate biunivocally with every real 2-form $F$ the self-dual 2-form $F = \frac{1}{\sqrt{2}}(F - i\ast F)$. Here we refer to a self-dual 2-form as a SD bivector. The endowed metric on the three-dimensional complex space of the SD bivectors is $G = \frac{1}{2}(G - i\eta)$, $\eta$ being the metric volume element of the spacetime and $G$ the metric on the space of 2-forms, $G = \frac{1}{2}g \wedge g$.

Every double 2-form, and in particular the Weyl tensor $W$, can be considered as an endomorphism on the space of the 2-forms. The restriction of the Weyl tensor on the SD bivectors space is the self-dual Weyl tensor and is given by $\mathcal{W} = \frac{1}{2}(W - i\ast W)$. The algebraic classification of the Weyl tensor $W$ can be obtained by studying the traceless linear map defined by the self-dual Weyl tensor $\mathcal{W}$ on the SD bivectors space [19–21]. The characteristic equation reads $x^3 - \frac{a}{4}x - \frac{b}{4} = 0$, where the complex invariants $a$ and $b$ are given by $a \equiv \text{Tr} \mathcal{W}^2$, $b \equiv \text{Tr} \mathcal{W}^3$.

In a Petrov–Bel-type D spacetime the self-dual Weyl tensor satisfies $6b^2 = a^3$ and has a minimal polynomial of degree two [20]. Then, it has a double eigenvalue $w$ and admits the...
canonical expression [21]:
\[ W = 3 w U \otimes U + w G, \quad w = -\frac{b}{a}, \quad \text{with} \quad U \text{ normalized eigenbivector associated with the simple eigenvalue} -2w. \]

The canonical bivector \( U = \frac{1}{\sqrt{2}} (U - i U) \) determines the two principal planes of a type D Weyl tensor. The projector on the timelike (resp., spacelike) principal plane is \( v = U^2 \) (resp., \( h = g - v = -(U^2) \)).

We summarize the results that we need below in two lemmas [21].

**Lemma 1.** A spacetime is Petrov–Bel-type D if, and only if, the self-dual Weyl tensor satisfies
\[ a \neq 0, \quad W^2 - \frac{b}{a} W - \frac{a}{3} G = 0; \quad a = \text{Tr} W^2, \quad b = \text{Tr} W^3. \]

**Lemma 2.** The eigenvalue \( w \) and the SD canonical bivector \( U \) of a Petrov–Bel-type D Weyl tensor can be obtained as
\[ w = -\frac{b}{a}, \quad U = \frac{P(Z)}{\sqrt{-P^2(Z, Z)}}, \quad P = W + \frac{b}{a} G. \]

where \( Z \) is an arbitrary bivector.

2.2. Type D metrics in real formalism

Now, we express the statements of the above lemmas in real formalism. The real Weyl scalar invariants are defined as [20]:
\[ A \equiv \frac{1}{2} \text{Tr} W^2, \quad B \equiv \frac{1}{2} \text{Tr}(W \circ \ast W), \quad D \equiv \frac{1}{2} \text{Tr} W^3, \quad E \equiv \frac{1}{2} \text{Tr}(W^2 \circ \ast W), \]

and they are related with the complex ones by \( a = A - i B, b = D - i E \).

If we denote \( \alpha \) and \( \beta \) the real and imaginary parts of the double Weyl eigenvalue \( w, w = \alpha + i \beta \), we find that, in terms of the real Weyl scalar invariants, they are given by
\[ \alpha = -\frac{AD + BE}{A^2 + B^2}, \quad \beta = \frac{AE - BD}{A^2 + B^2}. \]

Then, from this notation, lemma 1 becomes:

**Lemma 3.** A spacetime is Petrov–Bel-type D if, and only if, the Weyl tensor satisfies
\[ A^2 + B^2 \neq 0, \quad W^2 + \alpha W + \beta \ast W - \frac{1}{3}(AG - B\eta) = 0, \]

where \( A, B, \alpha \) and \( \beta \) are given in (4) and (5).

From (1) we can obtain the canonical expression for the real Weyl tensor \( W \) in terms of the Weyl scalar invariants and the canonical bivector
\[ W = 3\alpha(U \otimes U - \ast U \otimes \ast U) + 3\beta U \otimes \ast U + \alpha G + \beta \eta. \]

On the other hand, the real part \( P \) of the SD double-2-form \( P \) given in (3) projects every bivector \( Z \) on the plane of bivectors generated by \( U \) and \( \ast U \). Then a straightforward calculation leads to

**Lemma 4.** For a Petrov–Bel-type D Weyl tensor, the real and imaginary parts of the Weyl eigenvalue, \( \alpha \) and \( \beta \), are given by (5), and the canonical bivector \( U \) can be obtained as
\[ U = \frac{1}{\sqrt{\chi + f}} ((\chi + f) F + \tilde{f} \ast F); \quad F = P(Z), \]

where \( Z \) is an arbitrary bivector and
\[ P = W - \alpha G - \beta \eta, \quad \chi = \sqrt{f^2 + \tilde{f}^2}, \quad f = \text{tr} F^2, \quad \tilde{f} = \text{tr}(F \cdot \ast F). \]
3. Intrinsic characterization of the Kerr-NUT vacuum solutions

The Kerr metric is a type D vacuum solution. We have an intrinsic and explicit expression for the vacuum condition, $Ric(g) = 0$. And now, after our above analysis, we also have an intrinsic and explicit expression (2) (or, equivalently, (6)) for the type D condition. Moreover, we have obtained the explicit expressions (3) (or, equivalently, (5) and (8), (9)) for the Weyl eigenvalue $w = \alpha + i\beta$ and the canonical bivector $U$. Next, we must use these Weyl concomitants to distinguish the Kerr solution from the other type D vacuum solutions.

In order to carry out this step we remember some features of the type D vacuum solutions and of their charged counterpart, the so-called $D$-metrics. Recently [26], we have obtained an explicit and intrinsic characterization of the $D$-metrics. Our conditions exclusively involve algebraic concomitants of the Ricci and Weyl tensors.

An important property of the $D$-metrics is that the Weyl invariant $\xi = w^{-1}\delta U$ is a complex Killing vector ($\delta$ denotes the exterior co-differential). On the other hand, Walker and Penrose [22] showed that a Killing tensor exists in the charged Kerr black hole, and Hougston and Sommers [23] proved that, with the exception of the generalized charged C-metrics, the other $D$-metrics also have this property. The $D$-metrics with a Killing tensor are called the Kerr-NUT metrics.

In [24] we have studied the symmetries and other invariant properties of the $D$-metrics. Moreover we have shown that they can be intrinsically labeled by adding some differential conditions on the canonical bivector given in (3) to the conditions (2) that require a type D Weyl tensor. These restrictions impose that: (i) the principal null directions determine shear-free geodesic null congruences, (ii) the principal planes define a Maxwellian structure and (iii) the Ricci tensor commutes with $U$. An important fact is that these three conditions identically hold in the vacuum case [27].

3.1. Labeling the Kerr-NUT metrics. (i) Using the canonical bivector

In [24], we have also presented some equivalent conditions that distinguish the charged Kerr-NUT metrics from the other $D$-metrics. The first one is a result by Hougston and Sommers [28]: in a Kerr-NUT metric the complex invariant vector $\xi$ defines a unique Killing direction. The second one states that the Kerr-NUT metrics are the $D$-metrics with a Killing two-form $\nabla \xi$ aligned with the Weyl geometry, $\nabla \xi = \gamma_1 U + \gamma_2 \ast U$. And the third one imposes the canonical bivector to satisfy the following first order differential equation:

$$\delta U \wedge \delta \ast U = 0.$$ (10)

A remark on the second statement: it characterizes the Kerr-NUT metrics and, in the vacuum case, it leads to the Kerr solution under asymptotic flatness behavior. This result by Mars [2, 3] has recently been used to improve the black hole uniqueness theorems [7] and to study the Kerr initial data [1].

On the other hand, note that condition (10) only involves the canonical bivector, i.e., it is intrinsic. Consequently, we have an intrinsic and explicit characterization of the Kerr-NUT vacuum solutions:

**Proposition 1.** The Kerr-NUT vacuum solutions are the spacetimes with vanishing Ricci tensor, $Ric(g) = 0$, and whose Weyl tensor satisfies (6) and (10), where $U$ is the canonical bivector given in (8).
3.2. Labeling the Kerr-NUT metrics. (ii) Without using the canonical bivector

The intrinsic labeling of the Kerr-NUT metrics given in proposition 1 is a direct application of the results in [24]. The condition (10) has a simple geometrical interpretation in terms of the Weyl principal planes. Nevertheless, its strong dependence on the canonical bivector \( U \) could be a handicap for subsequent applications. Indeed, \( U \) is a concomitant of the Weyl tensor, but in its explicit expression (8) appears, necessarily, an arbitrary bivector \( Z \). This fact could make difficult to work with equations, like (10), involving \( U \).

Here we offer an alternative intrinsic labeling of the Kerr-NUT vacuum metrics that uses the gradients of the Weyl scalar invariants. This approach is more suitable to study the Kerr initial data.

We shall start remembering that, for a vacuum metric, the Bianchi identity states

\[
3\delta U = d\ln w. \tag{11}
\]

In order to write this equation in real formalism, we consider the invariants \( \rho \) and \( \theta \) which are essentially the logarithm of the modulus and the argument of the eigenvalue. If we put \( w = e^{\rho+i\theta} \), then (11) becomes

\[
*U(\delta U) - U(\delta U) = \frac{2}{3}d\rho, \quad *U(\delta U) + U(\delta U) = \frac{2}{3}d\theta. \tag{12}
\]

Note that the argument \( \theta \) is not uniquely defined, but its gradient is, and we can compute \( d\rho \) and \( d\theta \) using the fact that \( d(\rho + i\theta) = dw = \frac{dw}{w} \). Then, taking into account that \( w^2 = \frac{a^3}{L^2} \), we have the explicit Weyl invariants

\[
\frac{2}{3}d\rho = \frac{A}{3(A^2 + B^2)} dA + \frac{B}{3(A^2 + B^2)} dB \equiv R, \quad \frac{2}{3}d\theta = \frac{B}{3(A^2 + B^2)} dA - \frac{A}{3(A^2 + B^2)} dB \equiv \Theta. \tag{13}
\]

Then, conditions (12) can be written as

\[
\delta U = -U(R) - *U(\Theta), \quad \delta *U = U(\Theta) - *U(R). \tag{14}
\]

These conditions that relate the first derivatives of both, the Weyl canonical bivector and the Weyl scalar invariants, allow the Kerr-NUT invariant condition (10) to be expressed without an explicit use of the canonical bivector. Indeed, if we exclude the B-metrics where \( \delta U \) identically vanishes, condition (10) states that a scalar \( \lambda \) exists such that

\[
\delta *U = \lambda \delta U. \tag{15}
\]

Under this assumption, (14) is equivalent to

\[
U(\Theta) = -\lambda U(R), \quad *U(R) = \lambda *U(\Theta). \tag{16}
\]

These two vectorial equations can be stated as a tensorial one that is also satisfied by the B-metrics

\[
(U \circ *U)(R, R) + (U \circ *U)(\Theta, \Theta) = 0, \tag{17}
\]

where, for a double 2-form \( P \) and two vectors \( x, y \), \( P(x, y) \) denotes the 2-tensor with components \( P_{\mu\nu} = P_{\mu\nu} = x^\mu y^\nu \). Note that the double 2-form \( U \circ *U \) can be obtained from the canonical expression (7) in terms of the Weyl tensor and the scalar invariants. Moreover, equation (17) is invariant if we modify \( U \circ *U \) by a factor or by the addition of a fully skew-symmetric tensor. Finally, if we use the expressions (13) for \( R \) and \( \Theta \) we arrive to the following labeling of the Kerr-NUT vacuum solutions:
Proposition 2. The Kerr-NUT vacuum solutions are the spacetimes with vanishing Ricci tensor, \( \text{Ric}(g) = 0 \), and whose Weyl tensor satisfies (6) and
\[
Q(A, dA) + Q(B, dB) = 0,
\]
where \( A, B, \alpha \) and \( \beta \) are given in (4) and (5).

It is worth remarking that condition (10) distinguishes the Kerr-NUT metrics from the other \( D \)-metrics. Thus, it also applies in the charged case although here we have only used it for the vacuum (proposition 1). Nevertheless, condition (18) has been obtained by using the Bianchi identities for the vacuum case. Thus, it only distinguishes the vacuum Kerr-NUT metrics from the other type D vacuum solutions (proposition 2).

4. Some remarks on the vacuum Kerr-NUT metrics

Once the Kerr-NUT metrics have been labeled, we need to obtain the complementary restrictions on the Weyl invariants that allow us to characterize the Kerr metric. In order to tackle this point in the following section, here we present the same useful features of a family of vacuum Kerr-NUT metrics which contains the Kerr solution. This family is defined by the regularity conditions
\[
Q(\delta U, \delta U) \neq 0,
\]
where \( Q \) is given in (18) and \( v \) is the projector on the time-like principal plane. The first regularity condition in (19) imposes that the invariant directions defined by the first derivatives of the Weyl tensor have a non-vanishing projection on the two principal planes.

On the other hand, the \( D \)-metrics admit, at least, a commutative two-dimensional group of isometries. From the results in [24], it follows that the Killing vectors plane is null if, and only if, \( v(\delta U) \) is a null direction everywhere. Thus, the second regularity condition in (19) avoids this case.

The scalar \( \lambda \) introduced in (15) plays an important role in the intrinsic study of the Kerr-NUT metrics and, in particular, in the characterization of the Kerr solution. Note that \( \lambda \) is a covariant scalar and, if we exclude the B-metrics, it can be explicitly obtained as
\[
\lambda = \frac{(s, \delta * U)}{(s, \delta U)},
\]
with \( s \) being an arbitrary vector such that \( (s, \delta U) \neq 0 \).

Moreover, the scalar invariant \( \lambda \) is related with the derivatives of the algebraic Weyl scalars as a consequence of restrictions (14) and (16). Indeed, a straightforward calculation leads to
\[
K \lambda^2 - 2\lambda - K = 0,
\]
where \( K \equiv \frac{2(R, \Theta)}{R^2 - \Theta^2} \).

The type D vacuum solutions were first obtained by Kinnersley [29]. The explicit integration of their charged counterpart, the \( D \)-metrics, has been acquired by several authors, and they can be deduced from the Plebański and Demiański [30] line element by means of several limiting procedures (see [25] and references therein and the recent paper by Griffiths and Podolsky [31] for a detailed analysis). The vacuum Kerr-NUT metrics satisfying the regularity conditions (19) admit a coordinate line element that can be found in [25]. Moreover, we can obtain the basic Weyl invariants in this coordinate system.

Lemma 5. In the Kerr-NUT vacuum solutions such that \( Q(\delta U, \delta U) \neq 0 \) and \( v(\delta U, \delta U) \neq 0 \), the metric line element takes the expression
\[
g = -\frac{Y}{y^2 + x^2} (dt + x^2 dz)^2 + \frac{y^2 + x^2}{Y} dy^2 + \frac{X}{y^2 + x^2} (dt - y^2 dz)^2 + \frac{y^2 + x^2}{X} dx^2.
\]
where

\[ Y \equiv \epsilon y^2 - 2\mu y + \gamma, \quad X \equiv -\epsilon x^2 + 2\nu x + \gamma > 0. \tag{23} \]

Moreover, in this coordinate system the double Weyl eigenvalue \( w \) and the principal two-forms \( U \) and \( \ast U \) are

\[ w = -\frac{\mu + i \nu}{(y + i x)^3}, \quad U = -(dt + x^2 \, dz) \wedge dy, \quad \ast U = (dt - y^2 \, dz) \wedge dx. \tag{24} \]

From the coordinate expression (24) of the basic Weyl invariants and the expression (20) we have

\[ \delta U = yu, \quad \delta \ast U = xu; \quad \lambda = \frac{x}{y}. \tag{25} \]

Then, we can obtain the coordinate expression of the invariant

\[ \Omega \equiv \omega (1 + i \lambda)^3 = -\frac{1}{y^3} (\mu + i \nu). \tag{27} \]

The metric line element of the vacuum Kerr-NUT metrics with null orbits can also be found in [25]. From this, we can obtain the Weyl complex scalar invariant \( \Omega \) and we see that it is a purely imaginary scalar. Thus, we have

**Lemma 6.** A Kerr-NUT vacuum solution has null orbits if, and only if, \( v(\delta U, \delta U) = 0 \). Moreover, in this case (27) is a purely imaginary Weyl scalar invariant.

### 5. The characterization theorem for the Kerr metric

The Kerr solution is a vacuum Kerr-NUT metric which satisfies the regularity conditions (19). Thus, its metric line element follows from (22) for particular values of the coordinate parameters. More precisely, we have [25]:

**Lemma 7.** The line element (22) and (23) becomes the Kerr solution if we take \( \nu = 0 \) and \( \epsilon > 0 \). Moreover the Kerr mass and angular momentum are given, respectively, by

\[ m = \frac{|\mu|}{\epsilon \sqrt{\epsilon}}, \quad a = \frac{\sqrt{\nu}}{\epsilon}. \tag{28} \]

Then, if we want to label the Kerr metric we must add to the conditions that characterize the vacuum Kerr-NUT metrics (propositions 1 or 2): (i) regularity conditions (19), (ii) intrinsic and explicit expressions of the coordinate restrictions \( \nu = 0 \) and \( \epsilon > 0 \).

#### 5.1. Labeling the Kerr solution. (i) Using the canonical bivector

From the coordinate expression (27) for the complex scalar \( \Omega \), we obtain that this scalar is real if, and only if, \( \nu = 0 \). Then, if we obtain the imaginary part of the invariant expression of \( \Omega \), we have

\[ \nu = 0 \iff (1 - 3\lambda^2)\beta + \lambda(3 - \lambda^2)\alpha = 0. \tag{29} \]

Moreover, this invariant condition avoids the case of null orbits as a consequence of lemma 6.
In order to write the restriction $\epsilon > 0$ intrinsically, we define the explicit Weyl scalar invariant
\[ \sigma \equiv \frac{2\alpha}{3\lambda^2 - 1} - \frac{1}{4}(\delta U)^2. \]  
(30)
From (24)–(26) we can compute the coordinate expression of $\sigma$ and we obtain
\[ \sigma = \frac{y^2}{(x^2 + y^2)^2} \epsilon. \]  
(31)
Thus, $\epsilon > 0$ if, and only if, $\sigma > 0$. On the other hand, note that we can consider $s = \delta U$ in expression (20) because $\delta U$ is a non-null vector almost everywhere as follows from the regularity conditions (19).

Then, from these considerations and proposition 1, we arrive to the following intrinsic and explicit characterization of the Kerr geometry:

**Theorem 1.** A gravitational field $g$ is the Kerr solution if, and only if, its Ricci tensor vanishes, $\text{Ric}(g) = 0$, and its Weyl tensor satisfies the algebraic conditions:
\[ A^2 + B^2 \neq 0, \quad W^2 + \alpha W + \beta * W - \frac{1}{4}(AG - B\eta) = 0, \]  
(32)
and the first order differential restrictions
\[ \delta U \wedge \delta * U = 0, \quad Q(\delta U, \delta U) \neq 0, \]  
(33)
\[ (1 - 3\lambda^2)\beta + \lambda(3 - \lambda^2)\alpha = 0, \quad \sigma > 0, \]  
(33)
$Q = Q(g)$, $\lambda = \lambda(g)$ and $\sigma = \sigma(g)$ being the Weyl concomitants:

\[ Q \equiv \beta W + \alpha * W, \quad \lambda \equiv \frac{(\delta U, \delta * U)}{(\delta U)^2}, \quad \sigma \equiv \frac{2\alpha}{3\lambda^2 - 1} - \frac{1}{4}(\delta U)^2, \]  
(34)
where $A = A(g)$, $B = B(g)$, $D = D(g)$ and $E = E(g)$, $\alpha = \alpha(g)$ and $\beta = \beta(g)$, and $U = U(g)$ are the explicit Weyl concomitants given in (4), (5) and (8)–(9), respectively, and $2G \equiv g \wedge g$.

On the other hand, we can compute the Killing (invariant) vector $\xi = w^{-1} \delta U$ for the Kerr metric. In this case it is real and coincides with the stationary Killing vector in the outside region of the Kerr geometry.

Moreover, from the coordinate expressions of the Weyl invariants given in section 4 we can obtain the invariant expression of the coordinate parameters, and using (28), the invariant expression of the Kerr mass and angular momentum. Then, we reach

**Proposition 3.** In terms of the Weyl invariants defined in theorem 1, the Kerr mass $m$ and angular momentum $a$, and the stationary Killing vector $\xi$ are given, respectively, by
\[ m = \frac{|\alpha|}{\sigma \sqrt{\alpha(3\lambda^2 - 1)}}, \quad a = \frac{1}{2\sigma \sqrt{1 + \lambda^2}} \left[ h(\delta U, \delta U) + \frac{4\alpha^2 \lambda^2}{1 + \lambda^2} \right]^{1/2}, \]  
(35)
\[ \xi = \left( \frac{1 - 3\lambda^2}{\alpha} \right)^{1/2} \frac{1}{\sqrt{2}} \delta U. \]  
(36)

5.2. Labeling the Kerr solution. (ii) Without using the canonical bivector

Now we offer an alternative characterization of the Kerr geometry which could be useful in subsequent applications because it avoids the explicit calculation of $U$. 


At first, the two first conditions in (33) can be replaced by the equivalent ones $Q(dA, dA) = -Q(dB, dB) \neq 0$. This equality ensures that $\delta U$ and $\delta \* U$ are proportional vectors. We know that the proportionality factor $\lambda$ fulfills (21) and that, for the Kerr metric, it must also satisfy equation (29). If we use (21) to obtain $\lambda^2$ and $\lambda^3$ in terms of $\lambda$ and $K$, and remove them in (29), we obtain $\lambda$ as a function of $K$ and $T \equiv \frac{\lambda}{K}$ that must be satisfied if $\nu = 0$ as happens in the Kerr metric. But this is not a sufficient condition because $-\frac{1}{2}$ also satisfies (21). It can be shown that this case leads to $\mu = 0$ in (22). Consequently, we need another condition to distinguish which of the two solution of (21), $\lambda$ and $-\frac{1}{2}$, satisfies (29). To that end, let us define

$$\Xi = \frac{1}{(1-\lambda^2)/\sqrt{A^2+B^2}}[\Pi(R, R) - \Pi(\Theta, \Theta) + Q(R, \Theta) + Q(\Theta, R)],$$

(37)

where $\Pi \equiv \alpha W - \beta \* W - (\alpha^2 + \beta^2)G$. A straightforward calculation shows that $\Xi$ is a non-negative (respectively, non-positive) quadratic form when $\nu = 0$ (respectively, $\mu = 0$). Then, the Kerr solution follows if the semi-definite quadratic form $\Xi$ is non-negative. In order to impose this condition, it is enough to take the trace with an arbitrary elliptical metric associated with $g$.

On the other hand, from (14) and (16) we can determine $(\delta U)^2$ in terms of $\lambda$, $R$, and $\Theta$, and we can replace them in the expressions (30) of the scalar $\sigma$. Then, we obtain the following labeling of the Kerr geometry:

**Theorem 2.** A gravitational field $g$ is the Kerr solution if, and only if, its Ricci tensor vanishes, $Ricc(g) = 0$, and its Weyl tensor satisfies the algebraic conditions

$$A^2 + B^2 \neq 0, \quad W^2 + \alpha W + \beta \* W - \frac{1}{2} (AG - B\eta) = 0,$$

(38)

and the first order differential restrictions

$$Q(dA, dA) = -Q(dB, dB) \neq 0, \quad 2\Xi(x, x) + \text{tr} \Xi > 0, \quad (1-3\lambda^2)\beta + \lambda(3 - \lambda^2)\alpha = 0, \quad \sigma > 0,$$

(39)

with $x$ being an arbitrary timelike unitary vector, and $\Xi = \Xi(g)$, $Q = Q(g)$, $\sigma = \sigma(g)$ and $\lambda = \lambda(g)$ being the Weyl concomitants

$$\Xi = \frac{1}{(1-\lambda^2)/\sqrt{A^2+B^2}}[\Pi(R, R) - \Pi(\Theta, \Theta) + Q(R, \Theta) + Q(\Theta, R)],$$

(40)

$$Q \equiv \beta W + \alpha \* W, \quad \Pi \equiv \alpha W - \beta \* W - (\alpha^2 + \beta^2)G;$$

(41)

$$\sigma \equiv \frac{2\alpha}{3\lambda^2 - 1} - \frac{R^2 - \Theta^2}{4(\lambda^2 - 1)}; \quad \lambda \equiv \frac{K(\lambda T + 1)}{K^2 - 3KT - 2}; \quad T \equiv \frac{\beta}{\alpha}; \quad K \equiv \frac{2(R, \Theta)}{R^2 - \Theta^2};$$

(42)

where $A = A(g)$, $B = B(g)$, $D = D(g)$ and $E = E(g)$, $\alpha = \alpha(g)$ and $\beta = \beta(g)$, and $R = R(g)$ and $\Theta = \Theta(g)$ are the explicit Weyl concomitants given in (4), (5) and (13), respectively, and $2G \equiv g \wedge g$.

Moreover, we can obtain the invariant vector $\delta U$ that gives the Killing vector $\xi$ in (35) without making use of the expression of the canonical bivector $U$. Indeed, from the expressions (7), (14) and (16), we obtain that, under the conditions of theorem 2, the quadratic form (40) satisfies $2\Xi = \delta U \otimes \delta U$. On the other hand, from (14) and (16) we can determine $h(\delta U, \delta U)$ in terms of $\lambda$, $R$, and $\Theta$, and we can replace them in the expressions (35) of the angular momentum $a$. Then, we can state
Proposition 4. In terms of the Weyl invariants defined in theorem 2, the Kerr mass \( m \) and angular momentum \( a \), and the stationary Killing vector \( \xi \) are given, respectively, by

\[
m = \frac{\|\alpha\|}{\sigma \sqrt{\sigma (3\lambda^2 - 1)}}, \quad a = \frac{1}{2\sigma (1 + \lambda^2)} [\lambda (R, \Theta) + \Theta^2 + 4\sigma \lambda^2]^{1/2};
\]

(43)

\[
\xi = \left( 1 - \frac{3\lambda^2}{\alpha} \right)^{1/2} \frac{\Xi(x)}{\sqrt{\Xi(x, x)}},
\]

(44)

where \( x \) is an arbitrary vector such that \( \Xi(x) \neq 0 \).

6. Summary and ending comments

In this work we present the intrinsic and explicit labeling of the Kerr solution in two different versions (theorems 1 and 2). These results extend the intrinsic labeling of the Schwarzschild geometry given in [5] to the rotating case. The algorithmic nature of this last result has enabled an algorithmic characterization of the Schwarzschild initial data to be given [4]. Similarly, our Kerr characterization theorem must be a basic tool in giving a fully algorithmic characterization of the Kerr initial data which could improve the results presented in [1].

The algorithmic nature of our results is a direct consequence of their intrinsic and explicit presentation. We make this algorithmic nature more evident by summarizing the Kerr characterization in a flow chart (see below). The diagram shows the role played by every condition in the theorem. Note that the algebraic conditions label the type D vacuum solutions. The first differential equation in (33) distinguishes the Kerr-NUT metrics from the C-metrics. Finally, the other three differential conditions in (33) determine the Kerr geometry.

It is worth remarking that theorem 2 labels the Schwarzschild geometry as a limit case. Indeed, we can easily prove that, if we remove in theorem 2 the regularity requirement \( Q(dB, dB) \neq 0 \), the three conditions \( \beta = 0, \lambda = 0, a = 0 \), are equivalent restrictions on the Weyl tensor. Then, the other conditions in theorem 2 become the conditions of the Schwarzschild characterization theorem given in [5]. Moreover, the invariant expressions (43) for \( m \) and (44) for \( \xi \) become those given in [5] for the Schwarzschild mass and the static Killing vector.
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Appendix. Notation

(i) Products and other formulae involving 2-tensors $A$ and $B$:

(a) Composition as endomorphisms: $A \cdot B$,
$$ (A \cdot B)_{\rho \beta} = A_{\mu \rho} B_{\beta \nu}. \tag{A.1} $$

(b) Square and trace as an endomorphism:
$$ A^2 = A \cdot A, \quad \text{tr} A = A_{\mu \mu}. \tag{A.2} $$

(c) Action on a vector $x$, as an endomorphism $A(x)$, and as a quadratic form $A(x, x)$:
$$ A(x)_{\rho \beta} = A_{\mu \rho} x_{\beta \nu}, \quad A(x, x) = A_{\alpha \beta} x^\alpha x^\beta. \tag{A.3} $$

(d) Exterior product as double 1-forms:
$$ (A \wedge B)_{\rho \sigma \mu \nu}, \quad (A \wedge B)_{\rho \beta \mu \nu} = A_{\alpha \mu} B_{\beta \nu} + A_{\beta \nu} B_{\alpha \mu} - A_{\alpha \nu} B_{\beta \mu} - A_{\beta \mu} B_{\alpha \nu}. \tag{A.4} $$

(ii) Products and other formulae involving double 2-forms $P$ and $Q$:

(a) Composition as endomorphisms of the bivectors space: $P \circ Q$,
$$ (P \circ Q)_{\rho \sigma \mu \nu} = \frac{1}{2} P_{\rho \sigma \mu \nu} Q_{\rho \sigma \mu \nu}. \tag{A.5} $$

(b) Square and trace as an endomorphism:
$$ P^2 = P \circ P, \quad \text{Tr} P = \frac{1}{2} P_{\rho \sigma \rho \sigma}, \tag{A.6} $$

(c) Action on a bivector $X$, as an endomorphism $P(X)$, and as a quadratic form $P(X, X)$,
$$ P(X)_{\rho \sigma} = \frac{1}{2} P_{\rho \sigma \mu \nu} X_{\mu \nu}, \quad P(X, X) = \frac{1}{4} P_{\rho \sigma \rho \sigma} X_{\mu \nu} X_{\mu \nu}. \tag{A.7} $$

(d) The Hodge dual operator is defined as the action of the metric volume element $\eta$ on a bivector $F$ and a double 2-form $W$:
$$ *F = \eta(F), \quad *W = \eta \circ W. \tag{A.8} $$

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