Train Track Maps and CTs on Graphs of Groups

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Abstract

In this paper we develop the theory of train track maps on graphs of groups. We prove the existence of CTs representing outer automorphisms of free products. We generalize an index inequality due to Feighn–Handel to the graph of groups setting, sharpening a result of Martino.

A homotopy equivalence $f: G \to G$ of a graph $G$ is a train track map when the restriction of any iterate of $f$ to an edge of $G$ yields an immersion. (Relative) train track maps were introduced in [BH92]; they are perhaps the main tool for studying outer automorphisms of free groups. Choosing a basepoint $\star$ in $G$ and a path from $\star$ to $f(\star)$ determines an automorphism $f^\#: \pi_1(G, \star) \to \pi_1(G, \star)$ and a lift of $f$ to the universal covering tree $\Gamma$ of $G$. The map $\tilde{f}: \Gamma \to \Gamma$ is equivariant in the sense that for $g \in \pi_1(G, \star)$ and $x \in \Gamma$, we have $\tilde{f}(g.x) = f^\#(g).\tilde{f}(x)$. The lift $\tilde{f}: \Gamma \to \Gamma$ also satisfies the definition of a train track map. This formulation of train track maps as equivariant maps of trees can be adapted in a straightforward way to automorphisms of groups acting on trees. However, one is primarily interested in using train track maps to study outer automorphisms, for which it would be more convenient to be able to work directly in the quotient graph of groups. This is one purpose of this paper.

**Theorem A.** Let $\mathcal{G}$ be a graph of groups. If an outer automorphism $\varphi$ of $\pi_1(\mathcal{G})$ admits a topological representative, then there exists a relative train track map $f: \mathcal{G}' \to \mathcal{G}'$ representing $\varphi$. If $\varphi$ is irreducible, then the relative train track map is a train track map.

This is not the first construction of relative train track maps on graphs of groups, see [CT94], [FM15] and [Syk04], but it is the most general, allowing in particular for infinite edge groups.

One class of groups acting on trees of particular interest are free products, which act with trivial edge stabilizers. This greater flexibility allows for stronger results. The other purpose of this paper is to extend Feighn–Handel’s completely split relative train track maps, CTs [FH11], to outer automorphisms of free products.

Let $F = A_1 * \cdots * A_n * F_k$ be the free product of the groups $A_i$ with a free group of rank $k$ represented as the fundamental group of a graph of groups $\mathcal{G}$, where the $A_i$ are the nontrivial vertex groups of $\mathcal{G}$, the edge groups of $\mathcal{G}$ are trivial, and the ordinary fundamental group of the (underlying graph of) $\mathcal{G}$ is free of rank $k$.

**Theorem B.** Every $\varphi \in \Out(F)$ which is represented by a rotationless relative train track map $f: \mathcal{G} \to \mathcal{G}$ on a $\mathcal{G}$-marked graph of groups $\mathcal{G}$ is represented by a CT (see Section 6 for the definition) $f': \mathcal{G}' \to \mathcal{G}'$ on a $\mathcal{G}'$-marked graph of groups.

In [FH18], Feighn and Handel define an index $j(\varphi)$ for an outer automorphism $\varphi \in \Out(F_n)$ (see Section 7 for a definition) and prove it satisfies an inequality strengthening an inequality due to Gaboriau, Jaeger, Levitt and Lutsig [GJLL98]. Our final result extends this definition and inequality to the free product setting, strengthening a result of Martino [Mar99].
Theorem C. If \( \varphi \in \text{Out}(F) \) is represented on a \( \mathbb{G} \)-marked graph of groups, then
\[
j(\varphi) \leq n + k - 1.
\]

Part of this work originally appeared in the author’s thesis [Lym20]. The author would like to thank Lee Mosher for many helpful conversations.

The strategy of the proof is to find the correct equivariant perspective so that the original arguments in [BH92] and [FH11], as well as their algorithmic counterparts in [FH18], can be adapted without too much extra effort. We build up this perspective in Section 4. This done, the proof of Theorem A follows the outline in [BH92]; we give a brisk treatment for completeness in Section 2 and Section 3. Attracting laminations for free products are considered in Section 4. We then turn to the analogue of [FH11, Theorem 2.19] in Section 5 and the construction of CTs for outer automorphisms of free products in Section 6. Theorem C is proven in Section 7.

1 Maps of Graphs of Groups

The purpose of this section is to develop tools for working with maps of graphs of groups. Let us take up the discussion from the introduction: let \( F \) be a group acting on a (simplicial) tree \( T \), let \( \Phi: F \rightarrow F \) be an automorphism, and suppose there is a map \( \bar{f}: T \rightarrow T \) which is equivariant in the sense that we have for all \( g \in F \) and \( x \in T \)
\[
\bar{f}(g.x) = \Phi(g).\bar{f}(x).
\]
The map \( \bar{f} \) is equivariantly homotopic to a map \( \bar{f}' \) which sends vertices to vertices and either collapses edges to vertices or linearly expands edges to edge paths. It is maps of this kind and homotopies between them that we would like to represent on the quotient graph of groups \( \mathcal{G} \). Call a map of graphs a morphism when it maps vertices to vertices and either maps edges to edges or collapses them to vertices. The kind of map of trees we are interested in becomes a morphism after subdividing edges of the domain tree into finitely many edges.

In [Bas93], a notion of morphism of a graph of groups is defined and a correspondence is established between equivariant morphisms of Bass–Serre trees in the sense above and morphisms of the quotient graph of groups. To anticipate the definition, which we explain below, recall that an identification of the universal cover of \( \mathcal{G} \) with \( T \), unlike the case of the universal cover of a graph, depends additionally on a choice of fundamental domain for the action of \( F = \pi_1(\mathcal{G}) \) on \( T \). Given a morphism \( \bar{f}: T \rightarrow T \), the corresponding morphism \( f: \mathcal{G} \rightarrow \mathcal{G} \) needs to account for the failure of \( f \) to take the fundamental domain in the source to the fundamental domain in the target.

Morphisms of Graphs of Groups. Here is the definition; it depends on a slightly unusual description of what a graph of groups is. A curious reader is referred to [Bas93] or [Lym20, Chapter 1] for more details. A graph \( \mathcal{G} \) determines a small category \( G \) with objects the vertices of the first barycentric subdivision of \( \mathcal{G} \) and nonidentity arrows from barycenters of edges to the incident vertices. A graph of groups \( \mathcal{G} \) is a graph \( G \) and a functor \( \mathcal{G}: G \rightarrow \text{Group}^{\text{mono}} \) from the small category \( G \) to the category of groups with arrows injective homomorphisms.

Thus as usual there are vertex groups \( \mathcal{G}_v \) and edge groups \( \mathcal{G}_e \). If \( \tau(e) \) denotes the terminal vertex of the edge \( e \) in some choice of orientation and \( \bar{e} \) denotes \( e \) with its orientation reversed, we have injective homomorphisms \( I_e: \mathcal{G}_e \rightarrow \mathcal{G}_{\tau(e)} \) and \( I_{\bar{e}}: \mathcal{G}_e \rightarrow \mathcal{G}_{\bar{\tau}(e)} \). We have \( \mathcal{G}_e = \mathcal{G}_{\bar{e}} \).

A morphism of graphs of groups \( f: \mathcal{G} \rightarrow \mathcal{G}' \) is a morphism \( f: G \rightarrow G' \) of underly graphs together with a pseudonatural transformation of functors \( f: \mathcal{G} \Rightarrow \mathcal{G}'.f \). This comprises two kinds of data: for \( v \) a vertex and \( e \) an edge of \( G \) there are homomorphisms \( f_v: \mathcal{G}_v \rightarrow \mathcal{G}'_{f(v)} \) and \( f_e: \mathcal{G}_e \rightarrow \mathcal{G}'_{f(e)} \), and for each oriented edge \( e \) there is an element
\( \delta_e \in \mathcal{G}_{\tau(f(e))} \) such that the following diagram commutes

\[
\begin{align*}
\mathcal{G}_e & \xrightarrow{f_e} \mathcal{G}_{f(e)}^\prime \\
\downarrow \tau_e & \downarrow \text{ad}(\delta_e)_{f(e)} \\
\mathcal{G}_{\tau(e)} & \xrightarrow{f_{\tau(e)}} \mathcal{G}_{f(\tau(e))}^\prime \\
\end{align*}
\]

\( g \in \mathcal{G}_e \quad \xrightarrow{\tau(g)} \quad f_e(g) \)

Here \( \text{ad}(\delta_e) \) is the inner automorphism \( g \mapsto \delta_e g \delta_e^{-1} \). If the edge \( e \) is collapsed to a vertex by the morphism \( f \), then \( \tau(f(e)) \) should be replaced with the identity of \( \mathcal{G}_{f(e)} \).

An edge path \( \gamma \) in a graph of groups \( \mathcal{G} \) is a finite sequence

\[
\gamma = e'_1 g_1 e_2 g_2 \cdots g_k = e_{k-1} e_k,
\]

where \( e_2, \ldots, e_{k-1} \) are edges of \( \Gamma \), \( e'_1 \) and \( e'_k \) are terminal and initial segments of edges \( e_1 \) and \( e_k \), respectively, where \( g_i \in \mathcal{G}_{v_i} \), and where \( v_i = \tau(e_i) = \tau(e_{i-1}) \). We allow the case where \( e'_1 \) and \( e'_k \) are empty, in which case they will be dropped from the notation. A path is nontrivial if it contains (a segment of) an edge. There is a notion of homotopy (rel endpoints) for edge paths: it is generated by replacing a segment of the form \( e_t(h) \) with \( e_t(h)e \), where \( e \) is an edge and \( h \in \mathcal{G}_e \), and by adding or removing segments of the form \( e\tilde{e} \) for an edge \( e \). An edge path \( \gamma \) is tight if the number of edges in \( \gamma \) cannot be shortened by a homotopy.

Let \( p \) be a point of \( G \). The fundamental group \( \pi_1(G, p) \) is the group of homotopy classes of edge paths that form loops based at \( p \). The fundamental theorem of Bass–Serre theory asserts the existence of a tree \( \Gamma \) and an action of \( \pi_1(G, p) \) on \( \Gamma \) such that the quotient graph of groups is naturally identified with \( \mathcal{G} \).

Suppose

\[
\gamma = e'_1 g_1 e_2 g_2 \cdots g_k = e_{k-1} e_k,
\]

is an edge path in \( \mathcal{G} \), where \( g_i \in \mathcal{G}_{v_i} \). A morphism \( f : \mathcal{G} \to \mathcal{G}' \) acts on \( \gamma \), sending it to

\[
f(\gamma) = f(e'_1) \delta_{e_1}^{-1} f_{v_1}(g_1) \delta_{e_2} f(e_2) \delta_{e_2}^{-1} f_{v_2}(g_2) \cdots f(e_{k-1}) \delta_{e_{k-1}}^{-1} f_{v_{k-1}}(g_{k-1}) \delta_{e_k} f(e_k).
\]

The pseudonaturality condition ensures that \( f \) preserves homotopy classes of paths, and thus induces a homomorphism \( f_* : \pi_1(\mathcal{G}, p) \to \pi_1(\mathcal{G}', f(p)) \).

**Example 1.1.** The most important example of a morphism between graphs of groups is the natural projection \( \pi : \Gamma \to \mathcal{G} \), where \( \Gamma \) is the Bass–Serre tree for \( \mathcal{G} \). The definition depends on a choice of basepoint \( p \in \mathcal{G} \), a lift \( \tilde{p} \in \Gamma \), and a fundamental domain containing \( \tilde{p} \) — thus fixing an action of \( \pi_1(\mathcal{G}, p) \) on \( \Gamma \). Since the vertex and edge groups of \( \Gamma \) are trivial, for \( \pi \) to define a morphism of graphs of groups, we need to define \( \delta_e \) for each oriented edge \( \tilde{e} \) of \( \Gamma \).

Recall [Bass93, 1.12] that given a choice, for each oriented edge \( e \), a set \( S_e \) of left coset representatives for \( \mathcal{G}_{\tau(e)}/\tau_e(\mathcal{G}_e) \) containing \( 1 \in \mathcal{G}_{\tau(e)} \), there is a normal form for edge paths in \( \mathcal{G} \). For each vertex \( \tilde{v} \in \Gamma \), there is a unique path \( \gamma \) starting at \( \tilde{p} \) in \( \mathcal{G} \) in the normal form

\[
\gamma = e'_1 s_1 e_2 \cdots s_{k-1} e_k 1,
\]

where \( s_i \in S_{\tilde{v}_i} \), for each \( i \), which lifts to a tight path \( \tilde{\gamma} \) from \( \tilde{p} \) to \( \tilde{v} \). If \( \tilde{e} \) is an oriented edge of \( \Gamma \), let \( \tilde{\gamma} \) be the unique tight path in \( \Gamma \) connecting \( \tilde{p} \) to \( \tau(\tilde{e}) \). The path \( \tilde{\gamma} \) corresponds to a unique path \( \gamma \) in \( \mathcal{G} \) of the above form. If \( \pi(\tilde{e}) = e_k \) is the last edge appearing in \( \gamma \), define \( \delta_{\gamma} = s_{k-1} \). Otherwise define \( \delta_{\gamma} = 1 \). Observe that under this definition, we have \( \pi(\tilde{\gamma}) = \gamma \) as a path in \( \mathcal{G} \).

Given a vertex \( v \) in \( \mathcal{G} \), let \( st(v) \) denote the set of oriented edges \( e \) of \( \mathcal{G} \) with terminal vertex \( \tau(e) = v \). Recall that for each lift \( \tilde{v} \in \pi^{-1}(v) \) there is a \( \mathcal{G}_v \)-equivariant bijection

\[
st(\tilde{v}) \cong \bigcup_{e \in st(\tilde{v})} \mathcal{G}_e/\tau_e(\mathcal{G}_e) \times \{ e \}.
\]

In fact, the morphism \( \pi \) induces this bijection: if \( \pi(\tilde{e}) = \delta_{\gamma} e_{\delta^{-1}} \), the above map is \( \tilde{e} \mapsto ([\delta_{\gamma}], e) \), where \([\delta_{\gamma}]\) denotes the left coset of \( \tau_e(\mathcal{G}_e) \) in \( \mathcal{G}_e \) represented by \( \delta_{\gamma} \).
Proposition 1.2. Let \( f : G \to G' \) be a morphism, \( p \in G \) and \( q = f(p) \). Write \( \Gamma \) and \( \Gamma' \) for the Bass–Serre trees of \( G \) and \( G' \), respectively. There is a unique morphism \( \tilde{f} : \Gamma \to \Gamma' \) such that the following diagram commutes

\[
\begin{array}{ccc}
(\Gamma, \bar{p}) & \xrightarrow{\tilde{f}} & (\Gamma', \bar{q}) \\
\downarrow_{\pi_\Gamma} & & \downarrow_{\pi_{\Gamma'}} \\
(\bar{G}, p) & \xrightarrow{f} & (\bar{G}', q)
\end{array}
\]

where \( \pi_\Gamma \) and \( \pi_{\Gamma'} \) are the natural projections, and \( \bar{p} \) and \( \bar{q} \) are the distinguished lifts of \( p \) and \( q \), respectively.

Proof. If \( \tilde{x} \) is a point of \( \Gamma \), there is a unique tight path \( \gamma \) from \( \bar{p} \) to \( x \). The path \( f \pi_\Gamma(\gamma) \) has a (unique) lift \( \tilde{\gamma} \) to \( \Gamma' \) beginning at \( \bar{q} \) such that \( \pi_{\Gamma'}(\tilde{\gamma}) \) differs from \( f \pi_\Gamma(\gamma) \) only possibly by an element of \( G'_{f \pi_\Gamma(\tilde{x})} \) when \( \tilde{x} \) is a vertex. In particular, the terminal endpoint \( \tilde{\gamma}(1) \) is well-defined, and depends only on the homotopy class rel endpoints of \( f \pi_\Gamma(\gamma) \). Define \( \tilde{f}(\tilde{x}) = \tilde{\gamma}(1) \). It is easy to see that \( \tilde{f} : \Gamma \to \Gamma' \) defines a morphism, and that the diagram commutes. For uniqueness, observe that any morphism \( \tilde{f}' \) making the above diagram commute must satisfy

\[ \pi_{\Gamma'} \tilde{f}'(\bar{\eta}) = f \pi_\Gamma(\bar{\eta}) \]

for any path \( \bar{\eta} \) in \( \Gamma \), so specializing to paths connecting \( \bar{p} \) to \( \tilde{x} \), we see that \( \tilde{f}'(\tilde{x}) = \tilde{f}(\tilde{x}) \) for all \( \tilde{x} \in \Gamma \). \( \Box \)

The map \( \tilde{f} \) is \( f_\Gamma \)-equivariant, again in the sense that for \( g \in \pi_1(\bar{G}, p) \) and \( x \in \Gamma \) we have

\[ \tilde{f}(g.x) = f_\Gamma(g).\tilde{f}(x). \]

Proposition 1.3 (cf. 4.1–4.5 of [Bas93]). Let \( f_\pi : \pi_1(\bar{G}, p) \to \pi_1(\bar{G}', q) \) be a homomorphism, and let \( \tilde{f} : (\bar{G}, \bar{p}) \to (\bar{G}', \bar{q}) \) be an \( f_\pi \)-equivariant morphism of Bass–Serre trees in the sense above. There is a morphism \( f : (\bar{G}, p) \to (\bar{G}', q) \) which induces \( f_\pi \) and \( \tilde{f} \) and such that the following diagram commutes

\[
\begin{array}{ccc}
(\bar{G}, p) & \xrightarrow{f} & (\bar{G}', q) \\
\downarrow_{\pi_\Gamma} & & \downarrow_{\pi_{\Gamma'}} \\
(\bar{G}, p) & \xrightarrow{f_\pi} & (\bar{G}', q)
\end{array}
\]

Proof. As a morphism of graphs \( f \) is easy to describe. By \( f_\Gamma \)-equivariance, the map \( \pi_{\Gamma'} \tilde{f} \) yields a well-defined map on \( \pi_1(\bar{G}, p) \)-orbits; this is the map \( f : \Gamma \to \Gamma' \) as a morphism of graphs.

Identify the graph of groups structures \( G \) and \( G' \) with those induced by the actions of the respective fundamental groups on \( \Gamma \) and \( \Gamma' \). For \( \Gamma \) this involves a choice of fundamental domain \( T \subset \Gamma \) containing \( \bar{p} \). Each edge \( e \in G \) has a single preimage \( \bar{e} \in T \). If \( e \) is not collapsed by \( f \), define \( \delta_e \) for the morphism \( \tilde{f} \) as \( \delta_{f(\bar{e})} \) for the morphism \( \pi_{\Gamma'} \). If \( e \) is collapsed, define \( \delta_e = 1 \). Thus for \( \tilde{\gamma} \) a path in \( T \), \( \pi_{\Gamma'} \tilde{f}(\tilde{\gamma}) = f \pi_\Gamma(\tilde{\gamma}) \).

Let \( v \) be a vertex of \( G \) and write \( w = f(v) \). To define \( f_\pi : G_v \to G'_w \), recall that under the identification of graph of groups structures, \( G_v \) is the stabilizer of a vertex \( \bar{v} \in \pi_1^{-1}(v) \cap T \). Similarly, \( G'_w \) is identified with the stabilizer of some vertex \( \bar{w} \in \Gamma' \). The stabilizers of \( f(\bar{v}) \) and \( \bar{w} \) are conjugate in \( \pi_1(\bar{G}', q) \) via some element \( g_w \) such that \( g_w.f(\bar{v}) = \bar{w} \). Furthermore recall that there is a preferred translate for the fundamental domain for the action of \( \pi_1(\bar{G}', q) \) containing \( f(\bar{v}) \) and another preferred translate containing \( \bar{w} \). Namely, the translate containing the edges corresponding to those in

\[ \{([1], e) : e \in st(w) \} \]
under the correspondences between $\text{st}(\tilde{f}(\tilde{v}))$ and $\text{st}(\tilde{w})$ with

$$\prod_{e \in \text{st}(w)} \mathcal{G}_w/\iota_e(\mathcal{G}_e) \times \{e\}.$$ 

We require $g_w$ to take the preferred translate for $\tilde{f}(\tilde{v})$ to the preferred translate for $\tilde{w}$. The restriction of

$$h \mapsto g_w f_\tilde{v}(h) g_w^{-1}$$

to the stabilizer of $\tilde{v}$ defines a homomorphism $f_\tilde{v} : \mathcal{G}_v \to \mathcal{G}_w'$.

Now let $e$ be an edge of $G$ and write $a = f(e)$. The story is similar: the stabilizer of the preimage $\tilde{e}$ in $T$ is identified with $\mathcal{G}_e$, and some element $g_a \in \pi_1(\mathcal{G}', q)$ takes $\tilde{f}(\tilde{e})$ to the preferred preimage $\tilde{a}$. If $a$ is a vertex, we again require $g_a$ to match up preferred fundamental domains. The homomorphism $f_a : \mathcal{G}_e \to \mathcal{G}_a'$ is

$$h \mapsto g_a f_\tilde{e}(h) g_a^{-1}.$$ 

Whether $e$ is an edge of $G$ or $G'$, the monomorphism $\iota_e$ has a uniform description. If $v = \tau(e)$ and $\tau(\tilde{e}) = \tilde{v}$, then the monomorphism $\iota_e$ is the inclusion of the stabilizer of $\tilde{e}$ into the stabilizer of $\tilde{v}$. If not, there is some element $t_e$ with $t_e \tau(\tilde{e}) = \tilde{v}$ matching up preferred fundamental domains. In this latter case, $\iota_e$ is the map

$$h \mapsto t_e h t_e^{-1}.$$ 

In the former case, write $t_e = 1$. In the case where $f$ collapses $e$ to a vertex $a$, let $t_a = 1_{\mathcal{G}_a}$ and $t_a = 1$. Tracing an element $h \in \mathcal{G}_e$ around the pseudonaturality square, we have

$$\begin{array}{ccc}
\mathcal{G}_e & \xrightarrow{f_e} & g_a f_\tilde{e}(h) g_a^{-1} \\
\iota_e & \downarrow & \iota_e \downarrow \text{ad}(\delta_e) \iota_e & \iota_e \downarrow \text{ad}(\delta_e) \iota_e \\
t_e h t_e^{-1} & \xrightarrow{f_v} & g_w f_\tilde{v}(t_e h t_e^{-1}) g_w^{-1} = \delta_e t_a g_a f_\tilde{v}(h) g_a^{-1} t_a^{-1} \delta_e^{-1}.
\end{array}$$

Equality holds: to see this, we only need to check the case where $e$ is not collapsed. In this case, note that $\delta_e \in \mathcal{G}_w'$ was defined to take $(t_a g_a)$, $\tilde{f}(\tilde{e})$, whose image under the correspondence

$$\text{st}(\tilde{w}) \cong \prod_{e \in \text{st}(w)} \mathcal{G}_w'/\iota_e(\mathcal{G}_e') \times \{e\}$$

is $([1], e)$ to $(g_w f_\tilde{v}(t_e)). \tilde{f}(\tilde{e})$. This completes the definition of $f$ as a morphism of graphs of groups. Checking that $f$ induces $\tilde{f}$ and $f_\tilde{v}$ is straightforward; we leave it to the reader. □

The correspondence established above is not quite perfect: the following two operations on $f : (\mathcal{G}, p) \to (\mathcal{G}', q)$ do not change $f_\tilde{v}$ nor $\tilde{f}$.

1. Let $v \neq p$ be a vertex of $G$ and $g \in \mathcal{G}'_{f(v)}$. Replace $f_v$ with $\text{ad}(g) \circ f_v$ and replace $\delta_e$ with $g \delta_e$ for each oriented edge $e$ with $\tau(e) = v$.

2. Let $e$ be an edge of $G$ not containing $p$ and $g \in \mathcal{G}'_{f(e)}$. Replace $\delta_e$ and $\delta_{\tilde{e}}$ with $\delta_e t_{f(e)}(g)$ and $\delta_{\tilde{e}} t_{f(e)}(g)$, respectively.

We shall consider two morphisms $f$ and $f'$ equivalent fixing $p$ if $f$ can be transformed into $f'$ by a finite sequence of the above operations. If one ignores the stipulations around the basepoint, we say the two morphisms are equivalent. We are only interested in morphisms up to equivalence.
The previous propositions give us a method for transferring between morphisms \( f: G \to G' \) and equivariant pairs \( \tilde{f}: \Gamma \to \Gamma' \) and \( f^\#: \pi_1(G, p) \to \pi_1(G', q) \), and more generally, those maps which become morphisms after subdividing each edge in the domain into finitely many edges. A map \( f: G \to G' \) is a homotopy equivalence if there exists a map \( g: G' \to G \) such that each double composition \( gf \) and \( fg \) is homotopic to the identity—one can see this homotopy as maps of graphs of groups or by lifting to maps of Bass–Serre trees which are equivariantly homotopic to the identity. A homotopy equivalence \( f: G \to G' \) is a topological representative if it maps vertices to vertices and edges to nontrivial tight edge paths. In the following sections, we prove Theorem A and Theorem B by performing a number of operations on topological representatives.

Given a graph of groups \( G \), the collection of outer automorphisms of \( F = \pi_1(G) \) that admit a topological representative \( f: G \to G \) forms a subgroup of \( \text{Out}(F) \). We suggest the name modular group or mapping class group of \( G \) for this group. In some cases this subgroup is all of \( \text{Out}(F) \). One case where this happens is when \( F \) is virtually free and vertex groups of \( G \) are finite. An outer automorphism \( \varphi \in \text{Out}(F) \) belongs to this subgroup if for any conjugacy class \([g]\) in \( F \), the conjugacy class \( \varphi([g]) \) is elliptic in the Bass–Serre tree for \( G \) if and only if \([g]\) is elliptic.

Another case where this “modular group” is all of \( \text{Out}(G) \) is when \( F \) is a free product \( A_1 \ast \cdots \ast A_n \ast F_k \) where \( F_k \) is a free group and \( A_1, \ldots, A_n \) are freely indecomposable and not infinite cyclic. In this case one may take the graph of groups \( G \) to be the thistle with \( n \) prickles and \( k \) petals. This is a graph of groups with one vertex \( \star \) with trivial vertex group, \( n \) vertices with vertex group each of the \( A_i \), and \( n + k \) edges. The first \( n \) edges connect vertices with nontrivial vertex group to \( \star \), and the remaining \( k \) edges form loops based at \( \star \).

**Example 1.4.** Consider

\[
F = C_2 \ast C_2 \ast C_2 \ast C_2 = \langle a, b, c, d \mid a^2 = b^2 = c^2 = d^2 = 1 \rangle
\]

the free product of four copies of the cyclic group of order two. Let \( \Phi: F \to F \) be the automorphism

\[
\Phi \begin{cases}
  a \mapsto b \\
  b \mapsto c \\
  c \mapsto d \\
  d \mapsto cbdadbc,
\end{cases}
\]

(notice that, e.g. \( c^{-1} = c \)). A topological representative \( f: G \to G \) of \( \Phi \) on the thistle with four prickles is depicted in Figure 1.

![Figure 1: The topological realization \( f: G \to G \).](image)

## 2 Train Track Maps

The purpose of this section is to prove the irreducible case of Theorem A. The strategy is a straightforward adaptation of the arguments of [BH92] Section 1 to graphs of groups.
At the end of the section we prove a proposition characterizing irreducibility for outer automorphisms of free products.

Fix once and for all a graph of groups $G$. A marked graph of groups is a graph of groups $G$ together with a homotopy equivalence $\sigma : G \to \mathcal{G}$.

Given a topological representative $f : \mathcal{G} \to \mathcal{G}$ and an ordering $e_1, \ldots, e_m$ of the edges of $G$, there is an associated $m \times m$ transition matrix $M$ with $ij$th entry counting the number of times the $f$-image of the $j$th edge crosses the $i$th edge in either direction. The map $f$ is irreducible if the matrix $M$ is irreducible. Associated to every irreducible matrix is its Perron–Frobenius eigenvalue $\lambda \geq 1$. An irreducible matrix with Perron–Frobenius eigenvalue $\lambda = 1$ is a transitive permutation matrix. The transition matrix of Example 1.4 is

$$
\begin{pmatrix}
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 2 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 2
\end{pmatrix},
$$

which is irreducible and for which the Perron–Frobenius eigenvalue is the largest real root of the polynomial $x^4 - 2x^3 - 2x^2 - 2x - 1$ and satisfies $\lambda \approx 2.948$.

Call a vertex $v$ of $G$ inessential if for some oriented edge $e$ with $\tau(e) = v$, the homomorphism $\iota_e : G_v \to G_v$ is surjective.

A subgraph $G_0$ of $G$ is invariant with respect to the topological representative $f : \mathcal{G} \to \mathcal{G}$ if $f(G_0) \subset G_0$. It is a forest if each component $C$ of $G_0$ is a tree and in the induced graph of groups structure we have that $\pi_1(G|C)$ acts with global fixed point on its Bass–Serre tree. In $C$, this means there is a choice of vertex $v$ in $C$ and an orientation of each edge $e$ of $C$ toward this vertex such that each homomorphism $\iota_e : G_v \to G_{\tau(e)}$ is surjective. A forest is nontrivial if it contains at least one edge. An outer automorphism $\varphi \in \text{Out}(\pi_1(G))$ is irreducible if it admits a topological representative $f : \mathcal{G} \to \mathcal{G}$ and if whenever $G$ has no inessential valence-one vertices and no nontrivial invariant forests, then $f$ is irreducible.

A homotopy equivalence $f : \mathcal{G} \to \mathcal{G}$ taking vertices to vertices is tight if for each edge $e$, either $f(e)$ is a tight edge path, or $f(e)$ is a vertex. A homotopy equivalence may be tightened to a tight homotopy equivalence by a homotopy relative to the vertices of $G$.

**Lemma 2.1** ([BH92] p. 7). If $f : \mathcal{G} \to \mathcal{G}$ is a tight homotopy equivalence, collapsing a maximal pretrivial forest in $G$ produces a topological representative $f' : \mathcal{G}' \to \mathcal{G}'$. If instead $f : \mathcal{G} \to \mathcal{G}$ is a topological representative of an irreducible outer automorphism and $G$ has no inessential valence-one vertices, collapsing a maximal invariant forest yields an irreducible topological representative $f' : \mathcal{G}' \to \mathcal{G}'$.

**Proof.** A forest in $G$ is pretrivial with respect to a homotopy equivalence $f : \mathcal{G} \to \mathcal{G}$ if each edge in the forest is eventually mapped to a point. Maximal pretrivial forests are in particular invariant. We describe how to collapse invariant forests.

If $f : \mathcal{G} \to \mathcal{G}$ is a tight homotopy equivalence and $G_0 \subset G$ is an invariant forest, define $G_1 = \mathcal{G}/G_0$ to be the quotient graph of groups obtained by collapsing each component $C$ of $G_0$ to a vertex. The vertex group of the vertex determined by $C$ is $\pi_1(G|C)$. Since $G_0$ is a forest, this fundamental group is equal to some vertex group in $C$. Let $\pi : \mathcal{G} \to \mathcal{G}_1$ be the quotient map, and define $f_1 = \pi f \pi^{-1} : \mathcal{G}_1 \to \mathcal{G}_1$. Since $G_0$ was $f$-invariant, this is well-defined. If $e \subset G$ is an edge not in $G_0$, then the edge path for $f_1(e)$ is obtained from $f(e)$ by deleting all occurrences of edges in $G_0$. Since $f$ was tight, if $e\sigma\bar{e}$ is a subpath of the $f$-image of some edge $e'$ not in $G_0$, where $\sigma$ is a nontrivial path in $G_0$, then $\sigma$ must be of the form $\sigma'\gamma\sigma''$ for some path $\sigma'$ in $G_0$ and $\gamma$ an element of some vertex group. In $f_1(e')$, the path $e\sigma\bar{e}$ is replaced by $eg\bar{e}$. This implies that $f_1 : \mathcal{G}_1 \to \mathcal{G}_1$ is tight. The transition matrix for $f_1 : \mathcal{G}_1 \to \mathcal{G}_1$ is obtained from the transition matrix for $f : \mathcal{G} \to \mathcal{G}$ by deleting the rows and columns associated to the edges of $G_0$. 

\[ \square \]
Recall we write \( \text{st}(v) \) for the set of oriented edges \( e \) with terminal vertex \( \tau(e) = v \). A turn at \( v \) is a pair of elements of the set

\[
\prod_{e \in \text{st}(v)} G_v / \langle \delta_e \rangle \times \{ e \}.
\]

If \( f : \mathcal{G} \to \mathcal{G} \) is a morphism of graphs of groups, \( f \) determines a map \( Df \) sending a turn based at \( v \) to a turn based at \( f(v) \) via the rule

\[
\{(g_1, e_1), (g_2, e_2)\} \mapsto \{(f_\circ g_1 \delta_{e_1}, f(e_1)), (f_\circ g_2 \delta_{e_2}, f(e_2))\}.
\]

The pseudonaturality condition ensures that this map is well-defined; we have

\[
f_\circ (g_\circ e(h)) \delta_e = f_\circ (g(h)) f_\circ (e(h)) \delta_e = f_\circ (g(h)) \delta_{f(e)(h)}(f_\circ (e(h))).
\]

If \( f \) is a topological representative instead of a morphism, there is nonetheless a well-defined map \( Df \) of turns given by first subdividing so that \( f \) becomes a morphism and then acting via the above rule. In Example 1.4, the vertex \( * \) is mapped to itself by \( f \); the restriction of \( Df \) to \( * \) is determined by the dynamical system \( e_1 \mapsto e_2 \mapsto e_3 \leftrightarrow e_4 \).

A turn is degenerate if it consists of a pair of identical elements and is nondegenerate otherwise. A turn is illegal with respect to a topological representative \( f : \mathcal{G} \to \mathcal{G} \) if its image under some iterate of \( Df \) is degenerate and is legal otherwise. In Example 1.4 a turn \( \{ e_i, e_j \} \) based at \( * \) is illegal if \( i \) and \( j \) are equal mod 2, and is legal otherwise.

Consider the edge path

\[
\gamma = g_1 e_1 g_2 e_2 \cdots e_k g_{k+1}.
\]

We say \( \gamma \) takes the turns \( \{(1), e_1\}, \{(g_{i+1}), e_{i+1}\} \). The path \( \gamma \) is legal if it takes only legal turns.

A topological representative \( f : \mathcal{G} \to \mathcal{G} \) is a train track map if for each edge \( e \) of \( \Gamma \). Equivalent, \( f \) is a train track map if for each \( k \geq 1 \) and each edge \( e \) of \( \Gamma \), we have that \( f^k(e) \) is a tight edge path. In Example 1.4 \( f \) is not a train track map because the image of \( e_4 \) takes the illegal turn \( \{ e_4, e_2 \} \).

**Example 1.4 Continued.** Let us fold \( f \) at the illegal turn \( \{ e_2, e_4 \} \). To do this, first subdivide \( e_4 \) at the preimage of the vertex with vertex group \( \langle c \rangle \) so \( e_4 \) becomes the edge path \( c e_4 c \) and identify \( e'_2 \) with \( e_2 \). The action of the resulting map \( f' : \mathcal{G}_1 \to \mathcal{G}_1 \) is obtained from \( f \) by replacing instances of \( e_4 \) with \( e'_2 e_2 \). Thus we have

\[
f'(e_4) = e_1 e_2 e'_2 e_2 e_2 e_2 e_2 e_2 e_2 c.
\]

Tighten \( f' \) by a homotopy with support on \( e'_1 \) to remove \( e_2 e_2 \), yielding an irreducible topological representative \( f_1 : \mathcal{G}_1 \to \mathcal{G}_1 \). See Figure 2.

The Perron–Frobenius eigenvalue \( \lambda_1 \) for \( f_1 : \mathcal{G}_1 \to \mathcal{G}_1 \) is the largest real root of the polynomial \( x^4 - 2x^3 - 2x^2 + x - 1 \) and satisfies \( \lambda_1 \approx 2.663 \); thus \( \lambda_1 < \lambda \). However, \( f_1 \) is still not a train track map: \( Df_1 \) sends the turn \( \{ (1), e'_4 \}, \{(b, c)\} \), which is crossed by \( f_1(e'_4) \) to \( \{(c, c), (c), c\} \); thus this turn is illegal. We cannot quite fold \( e_2 \) and the end of \( e'_4 \) because the \( f_1 \)-image of the latter ends with \( e_3 c \). Lifting to the universal cover \( \tilde{f}_1 : \Gamma_1 \to \Gamma_1 \), it is not the edge \( \tilde{e}_4 \) which is folded with \( \tilde{e}_2 \) but \( b \tilde{e}_4 \). We may remedy the situation by changing the fundamental domain in \( \Gamma_1 \), or equivalently by changing the marking on \( \mathcal{G}_1 \) by twisting the edge \( e'_4 \) by \( b^{-1} = b \). This replaces \( (d) \) with \( \langle d b d \rangle \), replaces \( f_1(e_3) \) with \( e'_3 b e_2 \) and replaces \( f_1(e'_4) \) with \( c e_2 e'_2 e'_4 e_2 e_2 \). Then we fold \( e'_4 \) and \( e_2 \). The resulting graph of groups \( \mathcal{G}_2 \) is abstractly isomorphic to our original graph of groups \( \mathcal{G} \), but the marking differs. The action of the resulting map \( f'' : \mathcal{G}_2 \to \mathcal{G}_2 \) on edges is obtained by replacing instances of \( e'_4 \) with \( e''_4 e_2 \). Thus we have

\[
f''(e''_4) = e_1 e_2 b e_2 e'_4 b d e''_4 e_2 e_2.
\]
and we may tighten to produce an irreducible topological representative $f_2: \mathcal{G}_2 \to \mathcal{G}_2$. See Figure 3. The Perron–Frobenius eigenvalue $\lambda_2$ is the largest real root of $x^4 - 2x^3 - 2x^2 + 2x - 1$ and satisfies $\lambda_2 \approx 2.539$; thus $\lambda_2 < \lambda_1$. The restriction of $Df_2$ to turns incident to $\star$ is determined by the dynamical system $e_1 \mapsto e_2 \leftrightarrow e_3$, $e_4 \mapsto e_4$. The only illegal turn in $\mathcal{G}_2$ is $\{e_1, e_3\}$, which is not crossed by the $f_2$-image of any edge, so $f_2: \mathcal{G}_2 \to \mathcal{G}_2$ is a train track map.

The main result of this section is the following theorem.

**Theorem 2.2.** Suppose $\varphi \in \text{Out}(\pi_1(\mathcal{G}))$ is irreducible. Then there exists a train track map representing $\varphi$.

The broad-strokes outline of the proof of Theorem 2.2 is much the same as the previous example. By folding at illegal turns, we often produce nontrivial tightening, which decreases the Perron–Frobenius eigenvalue. By controlling the presence of valence-one and valence-two vertices, we may argue that the transition matrix lies in a finite set of matrices, thus the Perron–Frobenius eigenvalue may only be decreased finitely many times. In the remainder of this section, we make this precise by recalling Bestvina and Handel’s original analysis. The proofs are identical to the original, so we omit them.

**Subdivision.** Given a topological representative $f: \mathcal{G} \to \mathcal{G}$, if $p$ is a point in the interior of an edge $e$ such that $f(p)$ is a vertex, we may give $\mathcal{G}$ a new graph of groups structure by declaring $p$ to be a vertex, with vertex group equal to $\mathcal{G}_e$.

**Lemma 2.3** (Lemma 1.10 of [BH92]). If $f: \mathcal{G} \to \mathcal{G}$ is a topological representative and $f_1: \mathcal{G}_1 \to \mathcal{G}_1$ is obtained by subdivision, then $f_1$ is a topological representative. If $f$ is irreducible, then $f_1$ is too, and the associated Perron–Frobenius eigenvalues are equal. \(\square\)
Valence-One Homotopy. Recall that a valence-one vertex \( v \) with incident edge \( e \) is inessential if the monomorphism \( \iota_e \colon G_e \to G_v \) is an isomorphism.

If \( v \) is an inessential valence-one vertex with incident edge \( e \), let \( G_1 \) denote the subgraph of groups determined by \( G \setminus \{ e, v \} \), and let \( \pi : G \to G_1 \) be the map collapsing \( e \). Let \( f_1 : G_1 \to G_1 \) be the topological representative obtained from \( \pi f \vert_{G_1} \) by tightening and collapsing a maximal pretrivial forest. We say that \( f_1 : G_1 \to G_1 \) is obtained from \( f : G \to G \) by a valence-one homotopy.

Lemma 2.4 (Lemma 1.11 of [BH92]). If \( f : G \to G \) is an irreducible topological representative with Perron–Frobenius eigenvalue \( \lambda \) and \( f_1 : G_1 \to G_1 \) is obtained from \( f : G \to G \) by performing valence-one homotopies on all inessential valence-one vertices of \( G \) followed by the collapse of a maximal invariant forest, then \( f_1 : G_1 \to G_1 \) is irreducible, and the associated Perron–Frobenius eigenvalue \( \lambda_1 \) satisfies \( \lambda_1 < \lambda \).

Valence-Two Homotopy. We likewise distinguish two kinds of valence-two vertices. A valence-two vertex \( v \) with incident edges \( e_i \) and \( e_j \) is inessential if at least one of the monomorphisms \( \iota_{e_i} : G_{e_i} \to G_v \) and \( \iota_{e_j} : G_{e_j} \to G_v \) is an isomorphism, say \( \iota_{e_j} : G_{e_j} \to G_v \). Let \( \pi \) be the map that collapses \( e_j \) to a point and expands \( e_i \) over \( e_j \). Define a map \( f' : G \to G \) by tightening \( \pi f \). Observe that no vertex of \( G \) is mapped to \( v \). Thus we may define a new graph of groups structure \( G' \) by removing \( v \) from the set of vertices. Thus the edge path \( e_i e_j \) is now an edge, which we will call \( e_i \), with edge group \( G_{e_i} \). Let \( f'' : G' \to G' \) be the map obtained by tightening \( f''(e_i) = f'(e_i e_j) \). Finally, let \( f_1 : G_1 \to G_1 \) be the topological realization obtained by collapsing a maximal pretrivial forest. We say that \( f_1 : G_1 \to G_1 \) is obtained by a valence-two homotopy of \( v \) across \( e_j \).

Lemma 2.5 (Lemma 1.12 of [BH92]). Let \( f : G \to G \) be an irreducible topological representative, and suppose \( G \) has no inessential valence-one vertices. Suppose \( f_2 : G_2 \to G_2 \) is the irreducible topological representative obtained by performing a valence-two homotopy of \( v \) across \( e_j \) followed by the collapse of a maximal invariant forest. Let \( M \) be the transition matrix of \( f \) and choose a positive eigenvector \( \bar{w} \) with \( M \bar{w} = \lambda \bar{w} \). If \( w_i < w_j \), then \( \lambda_2 \leq \lambda \); if \( w_i < w_j \), then \( \lambda_2 < \lambda \).

Remark 2.6. The statement of the lemma hides a problem: if we cannot freely choose which edge incident to an inessential valence-two vertex to collapse via a valence-two homotopy, we may be forced to increase \( \lambda \). Fortunately, such valence-two vertices are not produced in the proof of Theorem 2.2.

Folding. Suppose some pair of edges \( e_1, e_2 \) in \( G \) have the same \( f \)-image. Define a new graph of groups \( G'_1 \) by identifying \( e_1 \) and \( e_2 \) to a single edge \( e \). The map \( f : G \to G \) descends to a well-defined homotopy equivalence \( f_1 : G_1 \to G'_1 \). This is an elementary fold. More generally if \( e'_1 \) and \( e'_2 \) are maximal initial segments of \( e_1 \) and \( e_2 \) with equal \( f \)-images and endpoints sent to a vertex by \( f \), we first subdivide at the endpoints of \( e'_1 \) and \( e'_2 \) if they are not already vertices and then perform an elementary fold on the resulting edges.

Lemma 2.7 (Lemma 1.15 of [BH92]). Suppose \( f : G \to G \) is an irreducible topological representative and that \( f_1 : G_1 \to G'_1 \) is obtained by folding a pair of edges. If \( f_1 \) is a topological representative, then it is irreducible, and the associated Perron–Frobenius eigenvalues satisfy \( \lambda_1 = \lambda \). Otherwise, let \( f_2 : G_2 \to G_2 \) be the irreducible topological representative obtained by tightening, collapsing a maximal pretrivial forest, and collapsing a maximal invariant forest. Then the associated Perron–Frobenius eigenvalues satisfy \( \lambda_2 < \lambda \).

Proof of Theorem 2.2. Let \( f : G \to G \) be an irreducible topological representative of \( \varphi \). Suppose the Perron–Frobenius eigenvalue \( \lambda \) satisfies \( \lambda = 1 \). Then \( f \) transitively permutes the edges of \( G \) and is thus a train track map.

So assume \( \lambda > 1 \). By performing valence-one and valence-two homotopies, we may assume that \( G \) has no inessential valence-one or valence-two vertices except in the corner.
Call a vertex $v$ of $G$ essential if for all oriented edges $e \in \text{st}(v)$, the monomorphism $\nu_{e}: G_{e} \to G_{v}$ is not surjective. Let $\eta(G)$ be the number of essential vertices of $G$, and let $\beta(G)$ be the first Betti number of $G$. A homotopy equivalence of graphs of groups sends essential vertices to essential vertices, so the quantities $\eta(G)$ and $\beta(G)$ are equal for any marked graph of groups obtained from $G$ by any of the operations described in this section. We claim that $G$ has at most $2\eta(G) + 3\beta(G) - 3$ edges. To see this, form a new graph $G'$ from $G$ by cyclically ordering the essential vertices of $G$ and attaching an edge from each essential vertex to its neighbors in the cyclic ordering. The graph $G'$ has no valence-one or valence-two vertices and first Betti number $\eta(G) + \beta(G)$. An Euler characteristic argument reveals that $G'$ has at most $3(\eta(G) + \beta(G)) - 3$ edges, from which the stated bound for $G$ follows.

We will show that if $f: G \to G$ is not a train track map, then there is an irreducible topological representative $f_{1}: G_{1} \to G_{1}$ without inessential valence-one or valence-two vertices such that the associated Perron–Frobenius eigenvalues satisfy $\lambda_{1} < \lambda$. The argument in the previous paragraph shows that the size of the transition matrix of $f_{1}$ is uniformly bounded. Furthermore, if $M$ is an irreducible matrix, its Perron–Frobenius eigenvalue $\lambda$ is bounded below by the minimum sum of the entries of a row of $M$. To see this, let $\vec{w}$ be a positive eigenvector. If $w_{j}$ is the smallest entry of $\vec{w}$, $\lambda w_{j} = (M\vec{w})_{j}$ is greater than $w_{j}$ times the sum of the entries of the $j$th row of $M$. Thus if we iterate this argument reducing the Perron–Frobenius eigenvalue, there are only finitely many irreducible transition matrices that can occur, so at some finite stage the Perron–Frobenius eigenvalue will reach a minimum. At this point, we must have a train track map.

To complete the proof, we turn to the question of decreasing $\lambda$. Suppose $f: G \to G$ is not a train track map. Then there exists a point $p$ in the interior of an edge such that $f(p)$ is a vertex, and $f^{k}$ is not locally injective (as a map of graph of groups) at $p$ for some $k > 1$. We assume that topological representatives act linearly on edges with respect to some metric on $G$. Since $\lambda > 1$, this means the set of points of $G$ eventually mapped to a vertex is dense. Thus we can choose a neighborhood $U$ of $p$ so small that it satisfies the following conditions.

1. The boundary $\partial U$ is a two-point set $\{s, t\}$, where $f^{i}(s)$ and $f^{i}(t)$ are vertices for some $\ell \geq 1$.
2. $f^{i}|_{U}$ is injective for $1 \leq i \leq k - 1$.
3. $f^{k}$ is two-to-one on $U \setminus \{p\}$, and $f^{k}(U)$ is contained within a single edge.
4. $p \notin f^{i}(U)$, for $1 \leq i \leq k$.

First we subdivide at $p$. Then we subdivide at $f^{i}(s)$ and $f^{i}(t)$ for $1 \leq i \leq \ell - 1$ (in reverse order so that subdivision is allowed). The vertex $p$ has valence two; denote the incident edges by $e$ and $e'$. Observe that $f^{i-1}(e)$ and $f^{i-1}(e')$ are single edges that are identified by $f$. Thus we may fold. The resulting map $f': G' \to G'$ may be a topological realization, in which case the Perron–Frobenius eigenvalue $\lambda'$ satisfies $\lambda' = \lambda$. In this case $f^{k-2}(e)$ and $f^{k-2}(e')$ are single edges that are identified by $f$. In the contrary case, nontrivial tightening occurs. After collapsing a maximal pretrivial forest and a maximal invariant forest, the resulting irreducible topological representative $f'': G'' \to G''$ has Perron–Frobenius eigenvalue $\lambda''$ satisfies $\lambda'' < \lambda$.

Repeating this dichotomy $k$ times if necessary, we have either decreased $\lambda$, or we have folded $e$ and $e'$ so that $p$ is now an inessential valence-one vertex.

We remove inessential valence-one and valence-two vertices by the appropriate homotopies. Since valence-one homotopy always decreases the Perron–Frobenius eigenvalue, the resulting irreducible topological representative $f_{1}: G_{1} \to G_{1}$ has Perron–Frobenius eigenvalue $\lambda_{1}$ satisfying $\lambda_{1} < \lambda$. \qed
Remark 2.8. As in the original, the proof of Theorem 2.2 provides in outline an algorithm that takes as input a topological representative of an irreducible outer automorphism and returns a train track map.

A reduction for an outer automorphism \( \varphi \in \text{Out}(\pi_1(\mathcal{G})) \) is a topological representative \( f : \mathcal{G} \to \mathcal{G} \) which has no inessential valence-one vertices and no invariant forests but has a nontrivial invariant subgraph. If \( \varphi \) has a reduction, then it is reducible—i.e., not irreducible. Let \( F = A_1 \ast \cdots \ast A_k \ast F_k \) be a free product, represented as the fundamental group of a graph of groups \( \mathcal{G} \) with trivial edge groups, where the \( A_i \) are vertex groups of \( \mathcal{G} \) and the ordinary fundamental group of \( \mathcal{G} \) is free of rank \( k \). For example the \( A_i \) might be freely indecomposable and not infinite cyclic, in which case \( \mathcal{G} \) is a Grushko splitting of \( F \). Define the complexity of \( F \) relative to \( \mathcal{G} \) to be the quantity \( n + 2k - 1 \). If \( F \) is a free factor of \( F \) relative to this free product decomposition, we may define the complexity of \( F \) relative to \( \mathcal{G} \) analogously. The final result of this section is the following characterization of reducibility for outer automorphisms \( \varphi \in \text{Out}(F) \) represented on \( \mathcal{G} \)-marked graphs of groups.

**Proposition 2.9.** Let \( F \) be a free product. An outer automorphism \( \varphi \in \text{Out}(F) \) is reducible relative to \( \mathcal{G} \) if and only if there are free factors \( F^1, \ldots, F^m \) of \( F \) with positive complexity such that \( F^1 \ast \cdots \ast F^m \) is a free factor of \( F \) and \( \varphi \) cyclically permutes the conjugacy classes of the \( F^i \).

**Proof.** Suppose first that \( \varphi \) is reducible relative to \( \mathcal{G} \); let \( f : \mathcal{G} \to \mathcal{G} \) be a reduction and let \( G_i = f^i(\mathcal{G}_i), 0 \leq i \leq m - 1 \) denote distinct noncontractible components of an \( f \)-invariant subgraph. Then each \( \pi_1(\mathcal{G}_i) \) determines a free factor \( F^i \) with positive complexity such that \( F^1 \ast \cdots \ast F^m \) is a free factor of \( F \) and such that \( \varphi \) cyclically permutes the conjugacy classes of the \( F^i \).

Conversely, suppose \( F^1, \ldots, F^m \) are free factors with positive complexity as in the statement of the proposition. Take \( F^{m+1} \) a free factor so that \( F = F^1 \ast \cdots \ast F^m \ast F^{m+1} \). Suppose that \( n_i \) and \( k_i \) are the data determining the complexity of \( F^i \) for \( 1 \leq i \leq m + 1 \). Let \( G_i \) be the thistle with \( n_i \) pricks and \( k_i \) petals (if \( n_m + 1 = k_{m+1} = 0 \), then \( G_{m+1} \) is a vertex) and distinguished vertex \( \star_i \). For each \( i \) satisfying \( 1 \leq i \leq m \) choose automorphisms \( \Phi_i : F \to F \) representing \( \varphi \) such that \( \Phi(F^i) = F^{i+1} \), with indices taken mod \( m \), and let \( f_i : G_i \to G_{i+1} \) be the corresponding topological representatives taking \( \star_i \) to \( \star_{i+1} \). Define \( \mathcal{G} \) to be the union of the \( G_i \) for \( 1 \leq i \leq m + 1 \) together with, for \( 1 \leq i \leq m \), an oriented edge \( E_i \) connecting \( \star_i \) to \( \star_{i+1} \).

Collapsing the \( E_i \) to a point yields a homotopy equivalence \( \mathcal{G} \to \mathcal{G} \), where \( \mathcal{G} \) is the thistle with \( n \) pricks and \( k \) petals. Identifying the image of \( \pi_1(\mathcal{G}_i, \star_i) \) with \( G^i \) will serve as (the inverse of) a marking. We will use \( \Phi_1 \) to create a topological representative \( f : \mathcal{G} \to \mathcal{G} \) for \( \varphi \). Define \( f(G^i) = f_i(G^i) \) for \( 1 \leq i \leq m \). By assumption there exist \( c_i \in F \) such that \( \Phi_1(x) = c_i \Phi_1(x) c_i^{-1} \). Choose \( \gamma_i \) a closed tight edge path based at \( \star_{m+1} \) representing \( c_i \) (so \( \gamma_1 \) is the trivial path) and define \( f(E_i) = \gamma_i E_{i+1} \) with indices taken mod \( m \). Finally define \( f(G^{m+1}) \) by \( \Phi_1 \) and the marking on \( (\Gamma, \mathcal{G}) \).

The topological representative \( f : \mathcal{G} \to \mathcal{G} \) is a reduction for \( \varphi \) unless \( \mathcal{G} \) has an invariant contractible forest. Since thistles have contractible subgraphs, there are a few possibilities. If there is a family of edges \( e_1, \ldots, e_m \) with \( e_i \in G^i \) and \( f(e_i) = e_{i+1} \) with indices mod \( m \), we may collapse each of these edges. Likewise if some edge of \( G^{m+1} \) is sent to itself, we may collapse it. If each \( c_i = 1 \) in \( F \), then the \( E_i \) also form an invariant forest that is contractible if the subgraph they span contains at most one vertex with vertex group some \( A_i \). After all these forest collapsings, the only worry is that \( F^{m+1} \) has complexity zero and the \( E_i \) would be collapsed, leaving \( G \) as the only \( f \)-invariant subgraph. In this case, choose \( A \) an edge of \( G^1 \) sharing an initial vertex with \( E_1 \), and change \( f \) via a homotopy with support in \( E_1 \) so that \( f(E_1) = f(A)f(A)E_2 \) and then fold the initial segment of \( E_1 \) mapping to \( f(A) \) with all of \( A \). The resulting graph is combinatorially identical to \( \mathcal{G} \) but the markings differ. Now \( f(E_1) = f(A)E_2 \) and \( f(E_k) = AE_1 \) so the \( E_i \) no longer form an invariant forest. \( \square \)
3 Relative Train Track Maps

The purpose of this section is to prove the general case of Theorem A. The strategy is to adapt arguments in [BH92, Section 5] and [FH18, Section 2]. At the end of the section we prove that given an outer automorphism \( \varphi \) of a free product and a nested sequence of \( \varphi \)-invariant free factor systems, there is a relative train track map such that the free factor systems are realized by filtration elements.

**Filtrations.** A *filtration* on a marked graph of groups \( \mathcal{G} \) with respect to a topological representative \( f: \mathcal{G} \to \mathcal{G} \) is an increasing sequence \( \emptyset = G_0 \subset G_1 \subset \cdots \subset G_m = \mathcal{G} \) of \( f \)-invariant subgraphs. The subgraphs are not required to be connected.

**Strata.** The \( r \)th *stratum* of \( \mathcal{G} \) is the subgraph \( H_r \) containing those edges of \( G_r \) not contained in \( G_{r-1} \). An edge path has height \( r \) if it is contained in \( G_r \) and meets the interior of \( H_r \). If both edges of a turn \( T \) are contained in a stratum \( H_r \), then \( T \) is a *turn in \( H_r \). If a path has height \( r \) and contains no illegal turns in \( H_r \) then it is \( r \)-legal.

**Transition Submatrices.** Relabeling the edges of \( G \) and thus permuting the rows and columns of the transition matrix \( M \) so that the edges of \( H_i \) precede those of \( H_{i+1} \), \( M \) becomes block upper-triangular, with the \( i \)th block \( M_i \) equal to the square submatrix of \( M \) containing those rows and columns corresponding to edges in \( H_i \).

A filtration is *maximal* when each \( M_i \) is either irreducible or the zero matrix. If \( M_i \) is irreducible, call \( H_i \) an *irreducible stratum* and a zero stratum otherwise. If \( H_i \) is irreducible, \( M_i \) has an associated Perron–Frobenius eigenvalue \( \lambda_i \geq 1 \). If \( \lambda_i > 1 \), then \( H_i \) is an *exponentially-growing stratum*. Otherwise \( \lambda_i = 1 \), we say \( H_i \) is *non-exponentially-growing* and \( M_i \) is a transitive permutation matrix.

**Eigenvalues.** Let \( H_{r_1}, \ldots, H_{r_k} \) be the exponentially-growing strata for \( f: \mathcal{G} \to \mathcal{G} \). We define \( \text{PF}(f) \) to be the sequence of associated Perron–Frobenius eigenvalues \( \lambda_{r_1}, \ldots, \lambda_{r_k} \) in nonincreasing order. We order the set

\[
\{ \text{PF}(f) \mid f: \mathcal{G} \to \mathcal{G} \text{ is a topological representative} \}
\]

lexicographically; thus if \( \text{PF}(f) = \lambda_1, \ldots, \lambda_k \) and \( \text{PF}(f') = \lambda'_1, \ldots, \lambda'_k \), then \( \text{PF}(f) < \text{PF}(f') \) if there is some \( j \) with \( \lambda_j < \lambda'_j \) and \( \lambda_i = \lambda'_i \) for \( i \) satisfying \( 1 \leq i < j \), or if \( k < \ell \) and \( \lambda_i = \lambda'_i \) for \( i \) satisfying \( 1 \leq i \leq k \).

**Relative Train Track Maps.** Throughout the paper, we will assume our filtrations are maximal unless otherwise specified. Given \( \sigma \) a path in \( \mathcal{G} \), let \( f_2(\sigma) \) denote the tight path homotopic rel endpoints to \( f(\sigma) \). We will denote the filtration associated to \( f: \mathcal{G} \to \mathcal{G} \) as \( \emptyset = G_0 \subset \cdots \subset G_m = \mathcal{G} \). A topological representative \( f: \mathcal{G} \to \mathcal{G} \) is a *relative train track map* if for every exponentially-growing stratum \( H_r \), we have

(EG-i) Turns in \( H_r \) are mapped to turns in \( H_r \) by \( Df \); every turn with one edge in \( H_r \) and the other in \( G_{r-1} \) is legal.

(EG-ii) If \( \sigma \subset G_{r-1} \) is a nontrivial path with endpoints in \( H_r \cap G_{r-1} \), then \( f_2(\sigma) \) is as well.

(EG-iii) If \( \sigma \subset \Gamma_r \) is a legal path, then \( f(\sigma) \) is a (tight) \( r \)-legal path.

The main result of this section is

**Theorem 3.1.** There is an algorithm that takes as input a topological representative \( f: \mathcal{G} \to \mathcal{G} \) and improves it to a relative train track map \( f': \mathcal{G}' \to \mathcal{G}' \).
We sketch the outline of the proof: we begin with a topological representative that is \textit{bounded}, a term which will be defined below. We use two new operations, described in Lemma 3.3 and Lemma 3.4 so that the resulting topological representative satisfies (\textit{EG-i}) and (\textit{EG-ii}). If (\textit{EG-iii}) is not satisfied, as in \cite{BH92} and \cite{FH18}, we modify the algorithm in the proof of Theorem 2.2 to reduce \(\text{PF}(f)\), the set of Perron–Frobenius eigenvalues for the exponentially-growing strata of \(f : \mathcal{G} \to \mathcal{G}\), while remaining bounded. The boundedness assumption ensures that we will hit a minimum value after a finite number of moves, at which point (\textit{EG-iii}) will be satisfied.

**Bounded Representatives.** As we observed in the proof of Theorem 2.2 if \(\mathcal{G}\) is a marked graph of groups without inessential valence-one or valence-two vertices, then \(\mathcal{G}\) has at most \(2\eta(\mathcal{G}) + 3\beta(\mathcal{G}) - 3\) edges, and this bound applies to any marked graph of groups homotopy equivalent to \(\mathcal{G}\). Our assumption that \(\varphi\) was irreducible allowed us to remove valence-two vertices, but we cannot always do this in the general case. Instead, call a topological representative \(f : \mathcal{G} \to \mathcal{G}\) \textit{bounded} if there are at most \(2\eta(\mathcal{G}) + 3\beta(\mathcal{G}) - 3\) exponentially-growing strata, and if, for each exponentially-growing stratum \(H_r\), the associated Perron–Frobenius eigenvalue \(\lambda_r\) is also the Perron–Frobenius eigenvalue of a matrix with at most \(2\eta(\mathcal{G}) + 3\beta(\mathcal{G}) - 3\) rows and columns. As in the proof of Theorem 2.2 if \(f : \mathcal{G} \to \mathcal{G}\) is bounded, the set of \(\text{PF}(f')\) for \(f' : \mathcal{G}' \to \mathcal{G}'\) a bounded representative of \(\varphi\) satisfying \(\text{PF}(f') \leq \text{PF}(f)\) is finite, so operations decreasing \(\text{PF}(f)\) will eventually reach a minimum, which we will denote \(\text{PF}_{\text{min}}\).

**Elementary Moves Revisited.** In \cite{BH92} Lemmas 5.1–5.4, Bestvina and Handel revisit the four elementary moves \textit{subdivision}, \textit{valence-one homotopy}, \textit{valence-two homotopy} and \textit{folding} to analyze their impact on \(\text{PF}(f)\). All of these moves except valence-two homotopy produce a topological representative \(f' : \mathcal{G}' \to \mathcal{G}'\) such that the associated Perron–Frobenius eigenvalues satisfy \(\text{PF}(f') \leq \text{PF}(f)\). In the case of valence-two homotopy within a single exponentially-growing stratum \(H_r\), it may happen that \(\lambda_r\) is replaced by some number of eigenvalues \(\lambda'\) that all satisfy \(\lambda' \leq \lambda_r\), so it is possible that \(\text{PF}(f') > \text{PF}(f)\). Nonetheless, we have the following result. Call an elementary move \textit{safe} if performing it on a topological representative \(f : \mathcal{G} \to \mathcal{G}\) yields a new topological representative \(f' : \mathcal{G}' \to \mathcal{G}'\) with \(\text{PF}(f') \leq \text{PF}(f)\).

**Lemma 3.2** (\cite{BH92} Lemma 5.5). \textit{If} \(f : \mathcal{G} \to \mathcal{G}\) \textit{is a bounded topological representative and} \(f' : \mathcal{G}' \to \mathcal{G}'\) \textit{is obtained from} \(f\) \textit{by a sequence of safe moves with} \(\text{PF}(f') < \text{PF}(f)\), \textit{then there is a bounded topological representative} \(f'' : \mathcal{G}'' \to \mathcal{G}''\) \textit{with} \(\text{PF}(f'') < \text{PF}(f)\).

The idea of the proof is the following: given \(f' : \mathcal{G}' \to \mathcal{G}'\), perform all valence-one homotopies and safe valence-two homotopies. Then perform valence-two homotopies until the resulting topological representative \(f'' : \mathcal{G}'' \to \mathcal{G}''\) is bounded. We have \(\text{PF}(f') \leq \text{PF}(f'') < \text{PF}(f)\).

**Invariant Core Subdivision.** We recall the construction of the \textit{invariant core subdivision} of an exponentially-growing stratum \(H_r\). Assume that a topological representative \(f : \mathcal{G} \to \mathcal{G}\) linearly expands edges over edge paths with respect to some metric on \(G\). If \(f(H_r)\) is not entirely contained in \(H_r\), then the set

\[I_r := \{ x \in H_r \mid f^k(x) \in H_r \text{ for all } k > 0 \}\]

is an \(f\)-invariant Cantor set. The \textit{invariant core} of an edge \(e\) in \(H_r\) is the smallest closed subinterval of \(e\) containing the intersection of \(I_r\) with the interior of \(e\). The endpoints of invariant cores of edges in \(H_r\) form a finite set which \(f\) sends into itself. Declaring elements of this finite set to be vertices is called \textit{invariant core subdivision}.

The following lemma says that invariant core subdivision can be used to create topological representatives whose exponentially-growing strata satisfy (\textit{EG-i}).
Lemma 3.3 ([BH92] Lemma 5.13). If $f' : \mathcal{G}' \to \mathcal{G}'$ is obtained from $f : \mathcal{G} \to \mathcal{G}$ by an invariant core subdivision of an exponentially-growing stratum $H_r$, then $\text{PF}(f') = \text{PF}(f)$, and the map $Df'$ maps turns in the resulting exponentially-growing stratum $H_r'$ to itself, so $H_r'$ satisfies $(\text{EG-i})$. Suppose $H_j$ is another exponentially-growing stratum for $f : \mathcal{G} \to \mathcal{G}$ that satisfies $(\text{EG-i})$ or $(\text{EG-ii})$, then the resulting exponentially-growing stratum $H_j'$ for $f' : \mathcal{G}' \to \mathcal{G}'$ still satisfies those properties.

In fact, invariant core subdivision affects only edges in $H_r$. If new vertices are created, then one or more non-exponentially-growing strata are added to the filtration below $H_r$.

Collapsing Inessential Connecting Paths. The following lemma says that an application of operations already defined may be used to construct topological representatives whose exponentially-growing strata satisfy $(\text{EG-ii})$.

Lemma 3.4 ([BH92] Lemma 5.14). Let $f : \mathcal{G} \to \mathcal{G}$ be a topological representative with exponentially-growing stratum $H_r$. If $\alpha \subset G_{r-1}$ is a path with endpoints in $H_r \cap G_{r-1}$ such that $f_\alpha(\alpha)$ is trivial, we construct a new topological representative $f' : \mathcal{G} \to \mathcal{G}'$ such that if $H_r'$ is the stratum of $\mathcal{G}'$ determined by $H_r$, then $H_r' \cap G_{r-1}$ has fewer points than $H_r \cap G_{r-1}$.

The new topological representative $f' : \mathcal{G}' \to \mathcal{G}'$ is constructed by subdividing at the preimages of vertices in $\alpha$ and repeatedly folding, followed by tightening and collapsing a pretrivial forest. As such, $\text{PF}(f') \leq \text{PF}(f)$. If $H_j$ is an exponentially-growing stratum for $f : \mathcal{G} \to \mathcal{G}$ with $j > r$ that satisfies $(\text{EG-i})$ or $(\text{EG-ii})$, the resulting exponentially-growing stratum $H_j'$ still satisfies these properties. If $H_r$ satisfies $(\text{EG-i})$ then so does $H_r'$.

Lemma 3.5 ([FH18] Lemma 2.4). There is an algorithm that checks whether a topological representative $f : \mathcal{G} \to \mathcal{G}$ is a relative train track map.

Proof. Since $(\text{EG-i})$ is a finite property, we may assume that each exponentially-growing stratum satisfies $(\text{EG-i})$. Suppose $H_j$ is an exponentially-growing stratum. A connecting path for $H_j$ is a tight path $\alpha$ in $G_{r-1}$ with endpoints in $H_r \cap G_{r-1}$. Let $C$ be a component of $G_{r-1}$. If $C$ is contractible, then there are only finitely many tight paths in $C$ with endpoints at vertices, so checking $(\text{EG-ii})$ for connecting paths in $C$ is a finite property.

So suppose $C$ is noncontractible. We claim that $(\text{EG-ii})$ for $C$ is equivalent to the property that each vertex in $H_j \cap C$ is periodic. If this latter property fails, there is some $k > 0$ and points $v$ and $w$ in $H_j \cap C$ with $f^k(v) = f^k(w)$. In this case, there is some path connecting $f^{k-1}(v)$ and $f^{k-1}(w)$ whose $f_\alpha$-image is trivial (consider what a homotopy inverse for $f$ does to $f^{k}(v)$), so $(\text{EG-ii})$ fails. If each vertex in $H_j \cap C$ is periodic, each connecting path $\alpha$ in $C$ either has distinct endpoints or determines a nontrivial loop in the fundamental group of $\mathcal{G}$ based at its endpoint. Both of these properties are preserved by $f_\alpha$, so $(\text{EG-ii})$ holds.

Finally, $(\text{EG-iii})$ for $H_j$ is equivalent to checking that $f(e)$ is $j$-legal for each edge $e \in H_j$, so is a finite-property.

Proof of Theorem 3.7. We begin with a topological representative and applying valence-one and valence-two homotopies until the resulting topological representative $f : \mathcal{G} \to \mathcal{G}$ is bounded. Consider the highest exponentially-growing stratum $H_r$ of $\mathcal{G}$. We check whether $H_r$ satisfies $(\text{EG-i})$ and $(\text{EG-ii})$ using Lemma 3.5. If not, apply Lemma 3.3 and Lemma 3.4 to create a new topological representative, still called $f : \mathcal{G} \to \mathcal{G}$ such that the resulting exponentially-growing stratum $H_r$ satisfies $(\text{EG-i})$ and $(\text{EG-ii})$. Repeat with the next highest exponentially-growing stratum until all exponentially-growing strata satisfy these properties.

Check whether the resulting topological representative, which we still call $f : \mathcal{G} \to \mathcal{G}$, satisfies $(\text{EG-iii})$. If it does, we are done.

If not, then there is some edge $e$ in an exponentially-growing stratum $H_r$ such that $f(e)$ is not $r$-legal. We apply the algorithm in the proof of Theorem 2.2 there is a point $P$ in $H_r$ where $f^k$ is not injective at $P$ for some $k > 1$. We subdivide and then repeatedly fold.
Either we have reduced the eigenvalue for $H_r$ or produced a valence-one vertex. We remove all valence-one vertices via homotopies and perform all valence-two homotopies which do not increase $\text{PF}(f)$. At this point we have created a new topological representative $f: G \to G'$ with $\text{PF}(f') < \text{PF}(f)$, but $f'$ may not be bounded. Apply Lemma 3.2 to produce a new bounded topological representative $f'': G'' \to G''$ with $\text{PF}(f'') < \text{PF}(f)$. If $[\text{EG-i}]$ and $[\text{EG-ii}]$ are not satisfied by $f''$, we may restore these properties without increasing $\text{PF}(f'')$. Because $\text{PF}(f)$ can only be decreased finitely many times before reaching $\text{PF}_{\text{min}}$, eventually this process terminates, yielding a relative train track map.

**Corollary 3.6.** If $f: G \to G$ is a bounded topological representative satisfying $[\text{EG-i}]$ and with $\text{PF}(f) = \text{PF}_{\text{min}}$, then the exponentially-growing strata of $f$ satisfy $[\text{EG-iii}]$.

**Free Factor Systems.** Let $F = A_1 \ast \cdots \ast A_n \ast F_k$ be a free product, represented as the fundamental group of a graph of groups $G$ with trivial edge groups, where the $A_i$ are vertex groups of $G$, and the ordinary fundamental group of $G$ is free of rank $k$. In Section 2 we showed that if $\varphi \in \text{Out}(F)$ is represented on a $G$-marked graph of groups, then $\varphi$ is irreducible relative to $G$ if it preserved the conjugacy class of no free factor relative to $G$ of positive complexity. If $F^1$ is a free factor of $F$ relative to $G$, let $[[F^1]]$ denote its conjugacy class. If $F^1, \ldots, F^m$ are free factors of $F$ relative to $G$ with positive complexity and $F^1 \ast \cdots \ast F^m$ is a free factor of $F$, the collection $\{[[F^1]], \ldots, [[F^m]]\}$ is a free factor system.

**Example 3.7.** If $f: G \to G$ is a topological representative where $G$ is a $G$-marked graph of groups and $G_r \subset G$ is an $f$-invariant subgraph with noncontractible connected components $C_1, \ldots, C_k$, then the conjugacy classes $[[\pi_1(G|C_1)]]$ of the fundamental groups of the $C_i$ are well-defined. We define

$$\mathcal{F}(G_r) := \{[[\pi_1(G|C_1)]], \ldots, [[\pi_1(G|C_k)]]\}.$$ 

Notice that each $[[\pi_1(G|C_i)]]$ has positive complexity. We say that $G_r$ realizes $\mathcal{F}(G_r)$.

The subgroup of $\text{Out}(F)$ representable on a $G$-marked graph of groups acts on the set of conjugacy classes of free factors of $F$ relative to $G$. If $\varphi$ is such an outer automorphism and $F'$ a free factor, $[[F']]$ is $\varphi$-invariant if $\varphi([[F']]) = [[F']]$. In this case there is some automorphism $\Phi: F \to F$ representing $\varphi$ such that $\Phi(F') = F'$ and $\Phi|_{F'}$ is well-defined up to an inner automorphism of $F'$, so it induces an outer automorphism $\varphi|_{F'} \in \text{Out}(F')$, which we will call the restriction of $\varphi$ to $F'$.

There is a partial order $\sqsubseteq$ on free factor systems: we say $[[F^1]] \sqsubseteq [[F^2]]$ if $F^1$ is a free factor of $F^2$. We say that $\mathcal{F}_1 \sqsubseteq \mathcal{F}_2$ for free factor systems $\mathcal{F}_1$ and $\mathcal{F}_2$ if for each $[[F^i]] \in \mathcal{F}_1$, there exists $[[F^j]] \in \mathcal{F}_2$ such that $[[F^j]] \sqsubseteq [[F^i]]$.

**Proposition 3.8.** Suppose $\varphi \in \text{Out}(F)$ is representable on a $G$-marked graph of groups. If $\mathcal{F}_1 \sqsubseteq \cdots \sqsubseteq \mathcal{F}_d$ is a nested sequence of $\varphi$-invariant free factor systems relative to $G$, then there is a relative train track map $f: G \to G$ representing $\varphi$ such that each free factor system is realized by some element of the filtration $\mathcal{F} = G_0 \subset \cdots \subset G_m = G$.

**Proof.** The first step is to construct a topological representative $f: G \to G$ and an associated filtration such that each $\mathcal{F}_i$ is realized by a filtration element. We proceed by induction on $d$, the length of the nested sequence of $\varphi$-invariant free factor systems. The case $d = 1$ is accomplished in Proposition 2.9. Let $\mathcal{F}_d = \{[[F^1]], \ldots, [[F^d]]\}$. For each $j$ with $1 \leq j \leq d - 1$ and each $i$ satisfying $1 \leq i \leq \ell$, write $\mathcal{F}_j^i$ for the set of conjugacy classes of free factors in $\mathcal{F}_j$ that are conjugate into $F^i$. Then for each $i$,

$$\mathcal{F}_1^i \subset \cdots \subset \mathcal{F}_d^i$$

is a nested sequence of $\varphi$-invariant free factor systems. By induction, there are graphs of groups $G^i$ and topological representatives $f_i: G^i \to G^i$ representing the restriction $\varphi|_{F^i}$ of
φ to \( F^i \) together with associated (not necessarily maximal!) filtrations \( G_i^1 \subset \cdots \subset G_i^d \) such that \( G_i^j \) realizes \( F_i^j \). Inductively, we may assume that \( f_i \) fixes some vertex \( v_i \) of \( G_i \), and that \( G_i \) has no inessential valence-one or valence-two vertices.

As in the proof of Proposition \ref{prop:realizing_free_factors} take a complementary free factor \( F^{i+1} \) so that \( F = F^1 \ast \cdots \ast F^k \ast F^{i+1} \), and an associated thistle \( G^{i+1} \). The graph of groups \( G \) is constructed as follows: begin with the disjoint union of the \( G_i \). Glue \( G^{i+1} \) to \( G^i \) by identifying the vertex \( * \) of \( G^{i+1} \) with the fixed point \( v_1 \). Then attach an edge connecting \( v_1 \) to \( v_i \) for \( 2 \leq i \leq k \). The resulting graph of groups has no inessential valence-one or valence-two vertices. Define \( f : G \rightarrow G \) from the \( f_i \) as in the proof of Proposition \ref{prop:realizing_free_factors} and observe that \( f \) is bounded. For \( 1 \leq j \leq d - 1 \), define \( G_j = \bigcup_{i=1}^d G_j^i \), and define \( G_d = \bigcup_{i=1}^d G_i^i \). Then \( \emptyset = G_0 \subset \cdots \subset G_d \subset G_{d+1} = G \) is an \( f \)-invariant filtration, and each \( F_i \) for \( i \) satisfying \( 1 \leq i \leq d \) is realized by \( G_i \). Complete this filtration to a maximal filtration; this completes the first step.

The next step is to promote our topological representative to a relative train track map. Note that (cf. the proof of \cite[Lemma 2.6.7]{BFH00}) that the moves described in Section 2 and this section all preserve the property of realizing free factors. More precisely, suppose \( C_1 \) and \( C_2 \) are disjoint noncontractible components of some filtration element \( G_i \), and that \( f' : G' \rightarrow G' \) is obtained from \( f : G \rightarrow G \) by collapsing a pretrivial forest, folding, subdivision, invariant core subdivision, valence-one homotopy or (properly restricted) valence-two homotopy. If \( p : G \rightarrow G' \) is the identifying homotopy equivalence, then \( p(C_1) \) and \( p(C_2) \) are disjoint, noncontractible subgraphs of \( G' \). Thus we may work freely and apply the proof of Theorem \ref{thm:main} to produce a relative train track map.

4 Attracting Laminations for Free Products

The purpose of this section is to construct the attracting laminations associated to an outer automorphism of a free product. We follow \cite[Section 3]{BFH00} in our treatment, and we defer to that paper for proofs and additional details.

Let \( F = A_1 \ast \cdots \ast A_n \ast F_k \) be a free product represented by a graph of groups \( G \) with trivial edge groups, where the \( A_i \) are vertex groups of \( G \), and the ordinary fundamental group of \( G \) is free of rank \( k \). Let \( \Gamma \) be the Bass–Serre tree for \( G \). A line in \( \Gamma \) is a proper, linear embedding \( \sigma : \mathbb{R} \rightarrow \Gamma \). We are interested in the unoriented image of \( \sigma \) more than the map (cf. \cite[2.2]{BFH00}).

**The Space of Lines.** There is a space of lines in \( \Gamma \), denoted \( \mathcal{B}(\Gamma) \). It is equipped with a “compact–open” topology: if \( \gamma \subset \Gamma \) is an edge path (say with endpoints at vertices), define \( N(\gamma) \subset \mathcal{B}(\Gamma) \) to be the set of lines that contain \( \gamma \) as a subpath. The sets \( N(\gamma) \) form a basis for the topology on \( \mathcal{B}(\Gamma) \). Let \( \pi : \Gamma \rightarrow G \) be the natural projection. By declaring preimages of vertices of \( \Gamma \) to be vertices, we may give \( \mathbb{R} \) a graph structure and consider \( \pi \circ \sigma : \mathbb{R} \rightarrow G \) as a morphism of graphs of groups (see Example \ref{ex:graph_morphism}). This is a line in \( G \).

There is a space of lines in \( G \), denoted \( \mathcal{B}(G) \) and a projection \( \mathcal{B}(\Gamma) \rightarrow \mathcal{B}(G) \). We give \( \mathcal{B}(G) \) the quotient topology. Given an edge path \( \gamma \) in \( G \), define \( N(\gamma) \) to be the set of those lines in \( \mathcal{B}(G) \) that contain \( \gamma \) as a subpath. The sets \( N(\gamma) \) form a basis for the topology on \( \mathcal{B}(G) \).

**Abstract Lines.** Bestvina–Feighn–Handel define spaces of abstract lines, \( \mathcal{B} \) and \( B \), using the Gromov boundary of the free group. In the case of a free product, the Gromov boundary of \( G \) is not compact in general. If one wanted a compact space, one could use the boundary Guirardel and Horbez define in \cite{GH19}. Let \( \partial \Gamma \) denote the Gromov boundary of \( \Gamma \). If \( f : \Gamma \rightarrow \Gamma \) is a homotopy equivalence and \( f : \Gamma \rightarrow \Gamma' \) is any lift to the Bass–Serre trees, that \( f \) induces a homeomorphism \( \partial f : \partial \Gamma \rightarrow \partial \Gamma' \). A line \( \sigma \) in \( \Gamma \) is determined by its endpoints, an unordered pair of distinct points \( \{\alpha, \omega\} \) in \( \partial \Gamma \). Conversely, given a pair of distinct points \( \{\alpha, \omega\} \) in \( \partial \Gamma \), there is a unique line \( \sigma \) in \( \Gamma \) connecting them. Thus we have a homeomorphism between \( \mathcal{B}(\Gamma) \) and \( B = (\partial \Gamma \times \partial \Gamma \setminus \Delta)/\mathbb{Z}/2\mathbb{Z} \), where \( \Delta \) denotes the diagonal and \( \mathbb{Z}/2\mathbb{Z} \) acts
by interchanging the factors. This homeomorphism is equivariant, taking the action of $F$ on lines in $\Gamma$ to the diagonal action of $F$ on $\partial \Gamma \times \partial \Gamma$, thus it yields a homeomorphism between the quotient of $\mathcal{B}$ by the action of $F$, which we denote $\mathcal{B}$, and $\mathcal{B}(G)$. If $f : G \to G$ is a homotopy equivalence, the homeomorphism $\partial f : \partial \Gamma \to \partial \Gamma'$ yields a homeomorphism $\partial f : \mathcal{B}(\Gamma) \to \mathcal{B}(\Gamma')$ and a homeomorphism $f_{\sharp} : \mathcal{B}(G) \to \mathcal{B}(G)$. If $\beta \in \mathcal{B}$ corresponds to $\gamma \in \mathcal{B}(G)$, we say $\gamma$ realizes $\beta$ in $G$.

An element of $F$ is peripheral in a $G$-marked graph of groups $G$ if it is conjugate into some vertex group. The conjugacy class of a nonperipheral element determines a periodic birecurrent, the homeomorphism $f_{\sharp}$ occurs infinitely often as a subpath of each end of $\infty$ in the closure of the periodic lines in $G$ determined by the conjugacy classes of nonperipheral elements of $F^i$. If $G$ is a marked graph of groups and $K \subset G$ is a connected subgraph such that $[[\pi_1(G|K)]] = [[F^i]]$, then $\beta$ is carried by $[[F^i]]$ if and only if the realization of $\beta$ in $G$ is contained in $K$.

Let $\varphi \in \text{Out}(F)$ be representable on $G$. We say $\beta' \in \mathcal{B}$ is weakly attracted to $\beta \in \mathcal{B}$ under the action of $\varphi$ if $\varphi_k^i(\beta') \to \beta$ (note that $\mathcal{B}$ is not Hausdorff). A subset $U \subset \mathcal{B}$ is an attracting neighborhood of $\beta \in \mathcal{B}$ for the action of $\varphi$ if $\varphi_k(U) \subset U$ and if $\{\varphi_k(U) : k \geq 0\}$ is a neighborhood basis for $\beta \in \mathcal{B}$. A line $\sigma : \mathbb{R} \to G$ in $G$ is birecurrent if every subpath of $\sigma$ occurs infinitely often as a subpath of each end of $\sigma$.

**Lemma 4.1** (BFH00 Lemma 3.1.4). *If some realization of $\beta \in \mathcal{B}$ in a $G$-marked graph of groups is birecurrent, then every realization of $\beta$ in a marked graph of groups is birecurrent. If $\beta$ is birecurrent, then $\varphi_k(\beta)$ is birecurrent for every $\varphi \in \text{Out}(F)$ that is representable on $G$.***

A closed subset $\Lambda_+ \subset \mathcal{B}$ is an attracting lamination for $\varphi$ if it is the closure of a single point $\beta$ such that:

1. The line $\beta$ is birecurrent.
2. The line $\beta$ has an attracting neighborhood for the action of some iterate of $\varphi$.
3. The line $\beta$ is not carried by any $F_i$ or $\mathbb{Z}/2\mathbb{Z} \ast \mathbb{Z}/2\mathbb{Z}$ free factor.

The line $\beta$ is said to be generic for $\Lambda_+$. The set of attracting laminations for $\varphi$ is denoted $\mathcal{L}(\varphi)$.

**Lemma 4.2** (BFH00 Lemma 3.1.6). $\mathcal{L}(\varphi)$ is $\varphi$-invariant.

A nonnegative integral matrix $M$ is aperiodic if there is some power $k$ such that $M^k$ has all positive entries. Aperiodic matrices are irreducible. Suppose $f : G \to G$ is a relative train track map with filtration $\emptyset = G_0 \subset G_1 \subset \cdots \subset G_m = G$ and that $H_r$ is an exponentially-growing stratum. We say that $H_r$ is aperiodic if its transition matrix is aperiodic, and that $f : G \to G$ is eg-aperiodic if each exponentially-growing stratum is aperiodic.

**Lemma 4.3** (BFH00 Lemma 3.1.9). *Suppose that $f : G \to G$ is a relative train track map and that $H_r$ is an aperiodic exponentially-growing stratum. There is an attracting lamination $\Lambda_+$ with generic leaf $\beta$ such that $H_r$ is the highest stratum crossed by the realization $\lambda$ of $\beta$ in $G$.***

**Proof.** The strategy of Bestvina–Feighn–Handel’s proof is to look at weak limits of $f^k(E)$ for $E$ an edge of $H_r$. They show that $\lambda$ is not a periodic line, thus it cannot be carried by any $F_i$ nor $\mathbb{Z}/2\mathbb{Z} \ast \mathbb{Z}/2\mathbb{Z}$ free factor.

**Lemma 4.4** (BFH00 Lemma 3.1.10). *Assume that $\beta \in \mathcal{B}$ is a generic line of some $\Lambda_+ \in \mathcal{L}(\varphi)$, that $f : G \to G$ and $\emptyset = G_0 \subset G_1 \subset \cdots \subset G_m = G$ are a relative train track map and filtration representing $\varphi$ and that $\lambda$ is the realization of $\beta$ in $G$. Then the highest stratum $H_r$ crossed by $\lambda$ is exponentially-growing and $\lambda$ is $r$-legal.*
Proof. We argue by induction on the structure of $F$. The base cases of $F = A_1, F_1$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ are trivial, so we assume the result holds for outer automorphisms of proper free factors of $F$. The rest of the proof follows from Bestvina–Feighn–Handel’s arguments. \qed

**Corollary 4.5** ([BFH00] Corollary 3.1.11). Assume that $f : G \to G$ and $\varnothing = G_0 \subset G_1 \subset \cdots \subset G_m = G$ are a relative train track map and filtration representing $\varphi$, and that $H_\tau$ is an aperiodic exponentially-growing stratum. Assume further that $\beta \in B$ is $\Lambda^+$-generic for some $\Lambda^+ \in \mathcal{L}(\varphi)$ and that $H_\tau$ is the highest stratum crossed by the realization of $\beta$ in $G$. All such $\beta$ have the same closure.

**Lemma 4.6** ([BFH00] Lemma 3.1.13). $\mathcal{L}(\varphi)$ is finite.

**Lemma 4.7** ([BFH00] Lemma 3.1.14). The following are equivalent:

1. Each element of $\mathcal{L}(\varphi)$ is $\varphi$-invariant.
2. Each element of $\mathcal{L}(\varphi)$ has an attracting neighborhood for $\varphi$.
3. Every relative train track map $f : G \to G$ representing $\varphi$ is eg-aperiodic.
4. Some relative train track map $f : G \to G$ representing $\varphi$ is eg-aperiodic.

## 5 Improving Relative Train Track Maps

As a step towards the existence of CTs, Feighn and Handel construct in [FH11] Theorem 2.19 relative train track maps that satisfy a number of extra properties. The goal of this section is to prove the existence of such relative train track maps for outer automorphisms of free products. To state the theorem we require some more terminology.

**Nielsen Paths.** A path $\sigma$ is a periodic Nielsen path with respect to a topological representative $f : G \to G$ if $\sigma$ is nontrivial and $f^k(\sigma) = \sigma$ for some $k \geq 1$. The minimal such $k$ is the period of $\sigma$, and $\sigma$ is a Nielsen path if it has period 1. A periodic Nielsen path is indivisible if it cannot be written as a concatenation of nontrivial periodic Nielsen paths.

**Non-Exponentially-Growing Strata.** If $H_r$ is a non-exponentially growing but not periodic stratum for a topological representative $f : G \to G$, each edge $e$ has a subinterval which is eventually mapped back over $e$, so the subinterval contains a periodic point. After declaring all of these periodic points to be vertices, reordering, reorienting, and possibly replacing $H_r$ with two non-exponentially-growing strata, we may assume the edges $E_1, \ldots, E_k$ of $H_r$ satisfy $f(E_i) = E_{i+1}u_i$, where indices are taken mod $k$ and $u_i$ is a path in $G_{r-1}$. Henceforth we always adopt this convention. Note that in the case where $g$ is an element of $\tau(E_{i+1})$ and $E_i$ satisfies $f(E_i) = gE_{i+1}u_i$, we still perform a subdivision.

If $H_r$ is a non-exponentially-growing stratum with edges $E_1, \ldots, E_k$ such that $f(E_i) = E_{i+1}g_i$, for some element $g \in G_{\tau(E_{i+1})}$ with indices taken mod $k$, we say that $H_r$ is an almost periodic stratum and each $E_i$ is an almost periodic edge. If $H_r$ is an almost periodic stratum consisting of a single edge $E_r$, the stratum and the edge are almost fixed.

For a topological representative $f : G \to G$, let $\text{Per}(f)$ denote the set of $f$-periodic points in $G$. The subset of points with period 1 is $\text{Fix}(f)$. A subgraph $C \subset G$ is wandering if $f^k(C) \subset G \setminus C$ for all $k \geq 1$ and is non-wandering otherwise.

The core of a subgraph $C \subset G$ is the minimal subgraph (of groups) $K$ of $C$ such that the inclusion is a homotopy equivalence.
Enveloped Zero Strata. Suppose that $f : \mathcal{G} \to \mathcal{G}$ is a topological representative, that $u < r$ and that the following hold.

1. The stratum $H_u$ is irreducible.
2. The stratum $H_r$ is exponentially-growing and all components of $G_r$ are noncontractible.
3. For each $i$ satisfying $u < i < r$, the stratum $H_i$ is a zero stratum that is a component of $G_{r-1}$, and each vertex of $H_i$ has valence at least two in $G_r$.

Then we say that each $H_i$ is enveloped by $H_r$, and write $H^*_r = \bigcup_{k=u+1}^{r} H_k$.

Finally, recall from the end of Section 3 our discussion on free factor systems, the partial order $\sqsubseteq$ on free factor systems, and the free factor system $\mathcal{F}(C)$ realized by a subgraph of groups $C$.

Let $F = A_1 \ast \cdots \ast A_n \ast F_k$ be a free product represented as $\pi_1(\mathcal{G})$, where $\mathcal{G}$ is a graph of groups with trivial edge groups, vertex groups the $A_i$, and ordinary fundamental group free of rank $k$.

**Theorem 5.1** ([FH11] Theorem 2.19). Given an outer automorphism $\varphi \in \text{Out}(F)$ representable on a $\mathcal{G}$-marked graph of groups, there is a relative train track map $f : \mathcal{G} \to \mathcal{G}$ on a $\mathcal{G}$-marked graph of groups and filtration $\mathcal{F} = G_0 \subset G_1 \subset \cdots \subset G_m = G$ representing $\varphi$ satisfying the following properties:

(V) The endpoints of all indivisible periodic Nielsen paths are vertices.

(P) If a periodic or almost periodic stratum $H_m$ is a forest, then there exists a filtration element $G_j$ such that $\mathcal{F}(G_j) \neq \mathcal{F}(G_i \cup H_m)$ for any filtration element $G_i$.

(Z) Each zero stratum $H_i$ is enveloped by an exponentially-growing stratum $H_r$. Each vertex of $H_i$ is contained in $H_r$ and meets only edges in $H_i \cup H_r$.

(NEG) The terminal endpoint of an edge in a non-periodic, non-exponentially-growing stratum $H_i$ is periodic, and is contained in a filtration element $G_j$ with $j < i$ that is its own core.

(F) The core of a filtration element is a filtration element.

Moreover, if $\mathcal{F}_1 \sqsubseteq \cdots \sqsubseteq \mathcal{F}_d$ is a nested sequence of $\varphi$-invariant free factor systems, we may assume that each free factor system is realized by some filtration element.

**Proof.** We adapt the proof of [FH11] Theorem 2.19, pp. 56–62. Begin with a relative train track map $f : \mathcal{G} \to \mathcal{G}$ which has no inessential valence-one vertices.

**Property (V).** To prove (V) we need to collect more information about Nielsen paths.

**Lemma 5.2** (cf. [FH11] Lemma 2.11). Suppose $f : \mathcal{G} \to \mathcal{G}$ is a relative train track map and $H_r$ is an exponentially-growing stratum.

1. There are only finitely many indivisible periodic Nielsen paths of height $r$.
2. If $\sigma$ is an indivisible periodic Nielsen path of height $r$, then $\sigma$ contains exactly one illegal turn in $H_r$.

Feighn and Handel use the foregoing to show that in fact,

**Lemma 5.3** ([FH11] Lemma 2.12). If $f : \mathcal{G} \to \mathcal{G}$ is a relative train track map, there are only finitely many points in $G$ that are the endpoints of an indivisible periodic Nielsen path. If these points are not already vertices, they lie in the interior of exponentially-growing strata.
Note that if $E$ is a periodic edge, then it is a periodic Nielsen path, but it is not indivisible. This latter lemma implies that $[V]$ can be accomplished by declaring the periodic points in the statement to be vertices. The resulting topological representative is still a relative train track map.

**Sliding.** The move *sliding* was introduced in [BFH00, Section 5.4, p. 579]. Suppose $H_i$ is a non-periodic, non-exponentially-growing stratum that satisfies our convention: the edges $E_1, \ldots, E_k$ of $H_i$ satisfy $f(E_j) = E_{j+1}u_j$ where indices are taken mod $k$ and $u_j$ is a path in $G_{i-1}$. We will call the edge of $H_i$ we focus on $E_1$. Let $\alpha$ be a path in $G_i$ from the terminal vertex of $E_1$ to some vertex of $G_{i-1}$. Define a new graph of groups $G'$ by removing $E_1$ from $G$ and gluing in a new edge $E'_1$ with initial vertex equal to the initial vertex of $E_1$ and terminal vertex the terminal vertex of the path $\alpha$. See Figure 4. Define homotopy equivalences $p: G \rightarrow G'$ and $p': G' \rightarrow G$ by sending each edge other than $E_1$ and $E'_1$ to itself, and defining $p(E_1) = E'_1\bar{\alpha}$ and $p'(E'_1) = E_1\alpha$. Define $f': G' \rightarrow G'$ by tightening $pfp': G \rightarrow G'$. If $G_r$ is a filtration element of $G$, define $G'_r = p(G_r)$. The $G'_r$ form the filtration for $f': G' \rightarrow G'$.

**Lemma 5.4 ([FH11] Lemma 2.17).** Suppose $f': G' \rightarrow G'$ is obtained from $f: G \rightarrow G$ by sliding $E_1$ along $\alpha$ as described above. Let $H_i$ be the non-exponentially-growing stratum of $G$ containing $E_1$, and let $k$ be the number of edges of $H_i$.

1. $f': G' \rightarrow G'$ is a relative train track map if $f: G \rightarrow G$ was.
2. $f'|_{G_{r-1}} = f|_{G_{r-1}}$.
3. If $k = 1$, then $f'(E'_1) = E'_1[\bar{\alpha}u_1f(\alpha)]$, where $[\gamma]$ denotes the path obtained from $\gamma$ by tightening.
4. If $k \neq 1$, then $f'(E_k) = E'_1[\bar{\alpha}u_k]$, $f'(E'_1) = E_2[u_1f(\alpha)]$ and $f'(E_j) = E_{j+1}u_j$ for $2 \leq j \leq k-1$.
5. For each exponentially-growing stratum $H_r$, $p_r$ defines a bijection between the set of indivisible periodic Nielsen paths in $G$ of height $r$ and the indivisible Nielsen paths in $G'$ of height $r'$.

**Lemma 5.5 ([FH11] Lemma 2.10).** Let $H_r$ be an exponentially-growing stratum of a relative train track map $f: G \rightarrow G$ and $v$ a vertex of $H_r$. Then there is a legal turn in $G_r$ based at $v$.

In particular, the lemma implies that if $v$ has valence one in $G_r$, then $v$ has nontrivial vertex group.
Property (NEG) Part One. We will first show that the terminal vertex of an edge in a non-periodic, non-exponentially-growing stratum \( H_i \) is either periodic or has valence at least three. Let \( E_1, \ldots, E_k \) be the edges of \( H_i \). As usual, assume \( f(E_j) = E_{j+1}u_j \), where indices are taken mod \( k \) and \( u_j \) is a path in \( G_{i-1} \). Suppose the terminal vertex \( v_1 \) of \( E_1 \) is not periodic and has valence two. Then \( v_1 \) has trivial vertex group. If \( E \) is the other edge incident to \( v_1 \), then \( E \) does not belong to an exponentially-growing stratum by the previous lemma.

We perform a valence-two homotopy of \( \bar{E}_1 \) over \( E \). If \( v \) is a vertex of an exponentially-growing stratum, \( f(v) \neq v_1 \), so before collapsing the pretrivial forest, properties \([\text{EG-i}]\) through \([\text{EG-iii}]\) are preserved. The pretrivial forest is inductively constructed as follows: any edge which was mapped to \( E \) is added, then any edge which is mapped into the pretrivial forest is added. Thus the argument above shows that no vertex of an exponentially-growing stratum is incident to any edge in the pretrivial forest, so the property of being a relative train track map is preserved. After repeating this process finitely many times, the terminal vertex of \( E_1 \) is either periodic or has valence at least three in \( \mathcal{G} \).

Finally, we arrange that \( v_1 \) is periodic: the component of \( G_{i-1} \) containing \( v_1 \) is non-wandering (because \( f^{k-1}(u_1) \) is contained in it), so contains a periodic vertex \( u_1 \). Choose a path \( \alpha \) in \( G_{i-1} \) from \( v_1 \) to \( w_1 \) and slide \( E_1 \) along \( \alpha \). No valence-one vertices are created because \( v_1 \) was assumed to have valence at least three. Repeating this process for each edge in a non-periodic, non-exponentially-growing stratum, we establish the first part of \([\text{NEG}]\), namely the following.

\((\text{NEG}^*)\) The terminal vertex of each edge in a non-periodic, non-exponentially-growing stratum is periodic.

Property (Z) Part One. Property \([\text{Z}]\) has several parts. Let \( H_i \) be a zero stratum. For \( H_i \) to be enveloped, let \( H_u \), the first irreducible stratum below \( H_i \), and \( H_r \), the first irreducible stratum above. One condition we need is that no component of \( G_r \) is contractible. Following \cite[p. 58]{FH}, we postpone that step and merely show here that each component is non-wandering.

First we arrange that if a filtration element \( G_i \) has a wandering component, then \( H_i \) is a wandering component. Suppose that \( G_i \) has wandering components. Call their union \( W \) and their complement \( N \). Since \( N \) is \( f \)-invariant, it is contained in a union of strata. If \( N \) is not precisely equal to a union of strata, the difference is that \( N \) contains part but not all of a zero stratum, so we may divide this zero stratum to arrange so that \( N \) is a union of strata. Thus \( W \) is a union of zero strata. Since \( N \) is \( f \)-invariant, we may push all strata in \( W \) higher than all strata in \( N \). We define a new filtration. Strata in \( N \) and higher than \( G_i \) remain unchanged. The strata that make up \( W \) will be the components of \( W \). If \( C \) and \( C' \) are such components, \( C' \) will be higher than \( C \) if \( C' \cap f^k(C) = \emptyset \) for all \( k \geq 0 \). We complete this to an ordering on the components of \( W \), yielding the new filtration.

Now we work toward showing that zero strata are enveloped by exponentially-growing strata. Suppose that \( K \) is a component of the union of all zero strata in \( \mathcal{G} \), that \( H_i \) is the highest stratum that contains an edge of \( K \) and that \( H_u \) is the highest irreducible stratum below \( H_i \). We aim to show that \( K \cap G_u = \emptyset \). So assume \( K \cap G_u \neq \emptyset \). By the previous paragraph, because \( H_u \) is irreducible, each component of \( G_u \) is non-wandering, so \( K \) meets \( G_u \) in a unique component \( C \) of \( G_u \). If each vertex of \( K \) has valence at least two in \( C \cup K \), then each edge of \( K \) belongs to a tight path in \( K \) with endpoints in \( C \), and we may close this path up in \( C \) to form a tight loop in \( K \cup C \). But some iterate of \( f: \mathcal{G} \to \mathcal{G} \) maps \( K \cup C \) into \( C \), so this is a contradiction. Therefore some vertex \( v \in K \) has valence one in \( K \cup C \).

This valence-one vertex \( v \) is not periodic, so by \([\text{NEG}^*]\) \( v \) is not an endpoint of an edge in a non-exponentially-growing stratum. We saw in Section \[3\] that \([\text{EG-ii}]\) for an exponentially-growing stratum \( H_r \) is equivalent to the condition that vertices of \( H_r \cap G_{r-1} \) contained in in non-wandering components of \( G_{r-1} \) are periodic. Thus \( v \) is not the endpoint of an edge in an exponentially-growing stratum above \( H_i \). By construction, \( v \) is also not
the endpoint of an edge of another zero stratum. Thus \( v \) has valence one in \( G \), but we have not produced any inessential valence-one vertices so far. This contradiction shows that \( G_u \cap K = \emptyset \).

The lowest edge in \( K \) is mapped either to another zero stratum or into \( G_u \). In any case, by connectivity, \( K \) is wandering, so we can reorganize zero strata so that \( K = H_i \). Repeating this for each component of the union of all zero strata, we have arranged that if \( H_i \) is a zero stratum and \( H_r \) is the first irreducible stratum above \( H_i \), then \( H_i \) is a component of \( G_{r-1} \).

Let \( H_i \) be the first irreducible stratum above \( H_r \). Because \( H_r \) is irreducible, no component of \( G_r \) is periodic, so \((\text{NEG}^*)\) implies \( H_r \) is exponentially-growing, and the argument above shows that vertices of \( H_i \) meet only edges of \( H_i \) and \( H_r \) and every edge of \( H_i \) has valence at least two in \( G_r \). This satisfies every part of the definition of the zero stratum \( H_i \), being enveloped by \( H_r \), except that we have not shown that all components of \( G_r \) are non-contractible, only that they are non-wandering.

Note also that \( H_i \) is contained in the core of \( G_r \): one obtains the core of \( G_r \) by repeatedly removing from \( G_r \) any edge incident to a valence-one vertex with trivial vertex group. Each vertex of \( H_i \) has valence at least two in \( G_r \), and Lemma 5.5 says that each valence-one vertex of \( H_r \) has nontrivial vertex group.

Tree Replacement. The final part of property \([Z]\) we will establish now is that every vertex of \( H_i \) is contained in \( H_r \). We do so by Feighn and Handel’s method of tree replacement. Replace \( H_i \) with a tree \( H_i' \) whose vertex set is exactly \( H_i \cap H_r \). We may do so for every zero stratum at once, (with \textit{a priori} different exponentially-growing strata \( H_r \), of course) and call the resulting graph of groups \( G' \). Let \( X \) denote the union of all irreducible strata. There is a homotopy equivalence \( p' : G' \to G \) that is the identity on edges in \( X \) and sends each edge in a zero stratum \( H_i' \) to the unique path in \( H_i \) with the same endpoints. Choose a homotopy inverse \( p : G \to G' \) that also restricts to the identity on edges in \( X \) and maps each zero stratum \( H_i \) to the corresponding tree \( H_i' \). Define \( f : G' \to G' \) by tightening \( pfp' : G' \to G' \). The map \( f \) still satisfies \((\text{EG-i})\). Because \( p \) and \( p' \) send nontrivial paths with endpoints in \( X \) to nontrivial paths with endpoints in \( X \), \((\text{EG-ii})\) is preserved as well. Because \( \text{PF}(f) = \text{PF}(f') \), Corollary 3.6 implies \( f' \) satisfies \((\text{EG-iii})\) as well. Nothing we have done so far changes the realization of free factor systems by filtration elements as well. We replace \( f : G \to G' \) by \( f' : G' \to G' \) in what follows.

Property \((P)\). Let \( F_1 \sqsubset \cdots \sqsubset F_d \) be our chosen nested sequence of \( \varphi \)-invariant free factor systems. If none was specified, instead define this sequence to be the sequence determined by \( F(G_r) \) as \( G_r \) varies among the filtration elements of \( G \). We will show that if \( H_m \) is a periodic or almost periodic forest, then there is some \( F_i \) that is not realized by \( H_m \cup G_i \) for any filtration element \( G_i \). Assume that this does not hold for some almost periodic forest \( H_m \). Then for all \( i \) satisfying \( 1 \leq i \leq d \), there is some filtration element \( G_i \) such that \( F(H_m \cup G_i) = F_i \). In this case we will collapse an invariant forest containing \( H_m \), reducing the number of non-exponentially-growing strata. Iterating this process establishes \([P]\).

Let \( Y \) be the set of all edges in \( G \setminus H_m \) eventually mapped into \( H_m \) by some iterate of \( f \). Each edge of \( Y \) is thus contained in a zero stratum. We want to arrange that if \( \alpha \) is a tight path in a zero stratum with endpoints at vertices that is not contained in \( Y \), then \( f_2(\alpha) \) is not contained in \( Y \cup H_m \). If there is such a path \( \alpha \), let \( E_1 \) be an edge crossed by \( \alpha \) and not contained in \( Y \). Perform a tree replacement as above, removing \( E_1 \) and adding in an edge with endpoints at the endpoints of \( \alpha \). By our preliminary form of property \([Z]\) if a vertex incident to \( E_1 \) has valence two, then it is an endpoint of \( \alpha \), so this process does not create inessential valence-one vertices. The image of the new edge is contained in \( Y \cup H_m \), so we add it to \( Y \). Because there are only finitely many paths in zero strata with endpoints at vertices, we need only repeat this process finitely many times if necessary.

Let \( G' \) be the graph of groups obtained by collapsing each component of \( H_m \cup Y \) to a point, and let \( p : G \to G' \) be the quotient map. Identify the edges of \( G \setminus (Y \cup H_m) \) with the
edges of $G'$ and define $f: G' \to G'$ on each edge $E$ of the complement by tightening $pf(E)$. By construction, $f': G' \to G'$ is a topological representative of $\varphi$, and $f'(E)$ is obtained from $f(E)$ by removing all occurrences of edges in $Y \cup H_m$. If $p(H_r)$ is not collapsed to a point, the stratum $H_r$ and $p(H_r)$ are thus of the same type, and we see that $f'$ has one less non-exponentially-growing stratum and possibly fewer zero strata. The previous properties, $(\text{NEG}^*)$ and our preliminary form of $(\text{Z})$ are still satisfied.

Let $H_r$ be an exponentially-growing stratum. As remarked above, checking $(\text{EG-ii})$ for $p(H_r)$ is equivalent to checking that each vertex of $p(H_r) \cap C$ is periodic for each non-wandering component $C$ of $p(G_{r-1})$. Let $v'$ be such a vertex. By assumption, there is a vertex $v \in H_r$ such that $p(v) = v'$. If $v$ is periodic, we are done. If not, the component of $G_{r-1}$ containing $v$ is wandering, contradicting the assumption that $C$ is non-wandering.

This verifies $(\text{EG-ii})$; it is easy to see that $(\text{EG-i})$ is still satisfied, and that $\text{PF}(f) = \text{PF}(f')$, so Corollary 3.6 implies $(\text{EG-iii})$ is still satisfied.

It remains to check that our family of free factor systems is still realized. Let $F_j$ be such a free factor system. By assumption on $H_m$, there is $G_j$ such that $\mathcal{F}(G_j) = \mathcal{F}_j$. Each non-contractible component of $G_j \cup H_m$ is mapped into itself by some iterate of $f$.

Since $Y$ is eventually mapped into $H_m$, some iterate of $f$ induces a bijection between the non-contractible components of $G_j \cup H_m \cup Y$ and those of $G_j \cup H_m$, so $p(G_j)$ realizes $\mathcal{F}_j$. Repeating this process decreases the number of non-exponentially-growing strata, so eventually property $(\text{P})$ is established.

**Property (Z).** Suppose that $C$ is a non-wandering component of some filtration element. We will show that $C$ is non-contractible. The lowest stratum $H_i$ containing an edge of $C$ is either exponentially-growing or periodic. If $H_i$ is exponentially-growing, Lemma 5.5 shows that each vertex of $H_i$ has valence at least two in $H_i$ or has nontrivial vertex group, showing that $C$ is non-contractible. If instead $H_i$ is periodic, we show that $(\text{P})$ implies $H_i$ is not a forest. If it were, $(\text{P})$ says in particular that there is some filtration element $G_j$ such that $\mathcal{F}(G_j) \neq \mathcal{F}(G_j \cup H_i)$. Thus $j < i$. This is only possible if $H_i$ is not disjoint from $G_j$, but it is by assumption. Therefore $H_i$ must not be a forest, and in particular $C$ is not contractible. This proves that $(\text{Z})$ follows from the form of $(\text{Z})$ we have already established.

**Property (NEG).** Let $E$ be an edge in a non-periodic, non-exponentially-growing stratum $H_i$. Let $C$ be the component of $G_{i-1}$ containing the terminal vertex $v$ of $E$; it is non-wandering by our work proving $(\text{NEG}^*)$. By the argument in the previous paragraph, if $H_i$ is the lowest stratum containing an edge of $C$, then $H_i$ is either periodic or exponentially-growing, and the argument in the previous paragraph shows that $H_i$ is non-contractible. In fact, $H_i$ is its own core. In the exponentially-growing case this follows since the argument above shows in either case that every valence-one vertex of $H_i$ has nontrivial vertex group. To see this in the periodic case, observe that if some edge of $H_i$ is incident to a vertex with trivial vertex group and valence one in $H_i$, then every edge has this property and $H_i$ is a forest, in contradiction to $(\text{P})$.

Choose a periodic vertex $w$ in $H_i$ and a path $\gamma$ from $v$ to $w$. Slide $E$ along $\gamma$. The result is a relative train track map which still realizes our sequence of free factor systems and still satisfies $(\text{Z})$. Working up through the filtration repeating this process establishes $(\text{NEG})$. This time sliding may have introduced inessential valence-one vertices, but $(\text{NEG})$, $(\text{Z})$ and Lemma 5.5 imply that only valence-one vertices are mapped to the valence-one vertices created. We perform valence-one homotopies to remove each of these vertices. If property $(\text{P})$ is not satisfied, restore it using the process above. Since the number of non-exponentially-growing strata decreases, this process terminates.

**Property (F).** We want to show that the core of each filtration element is a filtration element. If $H_i$ is a zero stratum, then $\mathcal{F}(G_i) = \mathcal{F}(G_{i-1})$, so assume that $H_i$ is irreducible, and thus $G_i$ has no contractible components. If a vertex $v$ with trivial vertex group has
valence one in $G_i$, then Lemma 5.5 and (Z) imply the incident edge $E$ belongs to a non-exponentially-growing stratum $H_i$. If $H_i$ were periodic, it would be a forest, because every edge would be incident to a vertex of valence one in $H_i$, so (P) implies that some and hence every valence-one vertex of $H_i$ is contained in some lower filtration element. This exhausts the possibilities: $v$ must be the initial endpoint of a non-periodic non-exponentially-growing edge. All edges in such a stratum have initial vertex a valence-one vertex of $G_i$, and no vertex of valence at least two in $G_i$ maps to them. Thus we may push all such non-exponentially-growing strata $H_i$ above $G_i \setminus H_i$. After repeating this process finitely many times, $\mathcal{F}(G_i)$ is realized by a filtration element that is its own core. Working upwards through the strata, (F) is satisfied.

6 CTs for Free Products

In this section we prove the existence of CTs for free products.

Let $G$ be a graph of groups with trivial edge groups and $\Gamma$ its Bass–Serre tree. Let $x \in G$ be a point and $\tilde{x} \in \Gamma$ a lift of $x$. A direction at $x$ is a component of the complement $\Gamma \setminus \tilde{x}$. If $x$ is in the interior of an edge, there are two directions at $x$, and if $x$ is a vertex, the set of directions at $x$ is in one-to-one correspondence with the set

$$\prod_{e \in \text{st}(x)} G_e \times \{e\}.$$ 

Just as a topological representative $f : G \to G$ acts on the set of turns based at a vertex $v$, there is an action of the map $f$ sending the set of directions at $x$ to the set of directions at $f(x)$.

Let $f : G \to G$ be a relative train track map. A periodic point $x \in \text{Per}(f)$ is principal (cf. [FH18, Definition 3.5]) if none of the following conditions hold.

1. $x$ is not an endpoint of a nontrivial periodic Nielsen path and there are exactly two periodic directions at $x$, both of which are contained in the same exponentially-growing stratum.

2. $x$ is contained in a component $C$ of $\text{Per}(f)$ that is either a topological circle or an interval with each endpoint a vertex with nontrivial vertex group and each point of $C$ has exactly two periodic directions.

3. $x$ has infinite vertex group $G_x$ and there are no periodic directions at $x$.

If each principal periodic vertex is fixed and if each periodic direction based at a principal periodic vertex is fixed, we say $f$ is rotationless.

As before, let $F = A_1 \ast \cdots \ast A_n \ast F_k$ be a free product represented as $\pi_1(G)$, where $G$ is a graph of groups with trivial edge groups, vertex groups the $A_i$, and ordinary fundamental group free of rank $k$. Given $a \in F$, let $[a]_u$ be the unoriented conjugacy class of $a$, i.e. $[a]_u = [b]_u$ if and only if $b$ is conjugate to $a$ or $a^{-1}$. If $\sigma$ is a closed path, we let $[\sigma]_u$ be the unoriented conjugacy class determined by $\sigma$.

**Non-Exponentially-Growing Strata.** Suppose that $f : G \to G$ is a rotationless relative train track map satisfying the conclusions of Theorem 5.1. The initial endpoint of an edge in a non-periodic non-exponentially-growing stratum $H_i$ is principal, so each such stratum consists of a single edge $E_i$ satisfying $f(E_i) = E_i u_i$ for some closed path $u_i \subset G_{i-1}$. If $u_i$ is a Nielsen path, $E_i$ is a linear edge, and we define the axis for $E_i$ to be $[w_i]_u$, where $w_i$ is root-free and $u_i = w_i^d_i$ for some $d_i \neq 0$.

If $E_i$ and $E_j$ are linear edges and there exist $m_i, m_j > 0$ and a closed root-free Nielsen path $w$ such that $u_i = w^{m_i}$ and $u_j = w^{m_j}$, then a path of the form $E_i w^p E_j$ for $p \in \mathbb{Z}$ is called an exceptional path. The map $f_2$ sends exceptional paths to exceptional paths.
Reduced Filtrations. A filtration \( \varnothing = G_0 \subset G_1 \subset \cdots \subset G_m = G \) is reduced with respect to \( \varphi \in \text{Out}(F) \) if it satisfies the following property: if \( \mathcal{F} \) is a free factor system is \( \varphi^k \)-invariant for some \( k > 0 \) and if \( \mathcal{F}(G_{r-1}) \subset \mathcal{F} \subset \mathcal{F}(G_r) \), then either \( \mathcal{F} = \mathcal{F}(G_{r-1}) \) or \( \mathcal{F} = \mathcal{F}(G_r) \).

Taken Paths, Splittings. If \( \sigma \) is a maximal subpath of \( f^k(E) \) in a zero stratum, where \( k > 0 \) and \( E \) is an edge of an irreducible stratum \( H_r \), we say \( \sigma \) is \( (r-)\)taken. A decomposition of a path \( \sigma \) into subpaths \( \sigma = \sigma_1 \cdots \sigma_n \) is a splitting if \( f^k(\sigma) = f^k(\sigma_1) \cdots f^k(\sigma_n) \): that is, \( f^k(\sigma) \) is obtained from \( f(\sigma) \) by tightening each \( f(\sigma_i) \) and then concatenating. For our purposes, merely performing multiplication in a vertex group does not count as tightening. A nontrivial path is completely split if it has a splitting, called a complete splitting into subpaths, each of which is either a single edge in an irreducible stratum (possibly together with a vertex group element), an indivisible Nielsen path, an exceptional path, or a path in a zero stratum \( H_i \) with endpoints in an exponentially-growing stratum that is both maximal (i.e. not contained in a larger such subpath of \( \sigma \)) and taken.

A relative train track map \( f: \mathcal{G} \to \mathcal{G} \) is completely split if

1. \( f(E) \) is completely split for each edge \( E \) of an irreducible stratum.
2. if \( \sigma \) is a taken path in a zero stratum with endpoints in an exponentially-growing stratum, then \( f^k(\sigma) \) is completely split.

CTs. A relative train track map \( f: \mathcal{G} \to \mathcal{G} \) and filtration \( \varnothing = G_0 \subset G_1 \subset \cdots \subset G_m = G \) is a CT if it satisfies the following properties (cf. \cite[Definition 4.7]{FH11}).

1. (Rotationless) The map \( f: \mathcal{G} \to \mathcal{G} \) is rotationless.
2. (Completely Split) The map \( f: \mathcal{G} \to \mathcal{G} \) is completely split.
3. (Filtration) The filtration is reduced. The core of a filtration element is a filtration element.
4. (Vertices) The endpoints of all indivisible periodic (hence fixed by \cite[Lemma 3.28]{FH11}) Nielsen paths are principal vertices. The terminal endpoint of each non-fixed non-exponentially-growing edge which is not almost fixed (see item 7) is principal and hence fixed.
5. (Periodic Edges) Each periodic edge is fixed and each endpoint of a fixed edge is principal. If a fixed stratum \( H_r \) is disjoint from \( G_{r-1} \), it is noncontractible. If not, then \( G_{r-1} \) is a core graph and any endpoint of the unique edge \( E_r \) not contained in \( G_{r-1} \) has nontrivial vertex group.
6. (Almost Periodic Edges) The initial endpoint of an almost periodic edge is principal, so an almost periodic edge is almost fixed. If an almost fixed stratum \( H_r \) is disjoint from \( G_{r-1} \), it is noncontractible. If not, then \( G_{r-1} \) is a core graph and any endpoint of the unique edge \( E_r \) not contained in \( G_{r-1} \) has nontrivial vertex group.
7. (Zero Strata) If \( H_i \) is a zero stratum, then \( H_i \) is enveloped by an exponentially-growing stratum \( H_r \), each edge in \( H_i \) is \( r \)-taken and each vertex in \( H_i \) is contained in \( H_r \) and meets only edges in \( H_i \cup H_r \).
8. (Linear Edges) For each linear edge \( E_i \) there is a closed root-free Nielsen path \( w_i \) such that \( f(E_i) = E_i w_i^{d_i} \) for some \( d_i \neq 0 \). If \( E_i \) and \( E_j \) are distinct linear edges with the same axis then \( w_i = w_j \) and \( d_i \neq d_j \).
9. (NEG Nielsen Paths) If the highest edges in an indivisible Nielsen path \( \sigma \) belong to a non-exponentially-growing stratum then either
   
   (a) there is a linear edge \( E_i \) with \( w_i \) as in (Linear Edges) and there exists \( k \neq 0 \) such that \( \sigma = E_i w_i^k E_i \), or
Let us remark that in the corner cases of $F$ entirely from Section 3 of [FH11], so we omit it, directing the interested reader to [FH11]. Guirardel–Horbez and discussed in Section 4. The development of this theory follows in the case of a free product, a parallel theory is possible using the boundary proof of Lemma 5.2.4, if $\sigma$. The lemma follows from the proofs of Lemmas 5.2.3 and 5.2.4 of [BFH00]. In the Proof.

In the fold at the illegal turn of $\sigma$ a relative train track map

First a word about terminology: every indivisible Nielsen path

Property (EG Nielsen Paths). We will argue that if $f \cdot \sigma$ is not satisfied, then there is a relative train track map $f': C' \to C'$ representing $\varphi$ with $N(f') < N(f)$, where $N(f)$ denotes the number of indivisible Nielsen paths (with respect to $f: C \to C$) of exponentially-growing height.

First a word about terminology: every indivisible Nielsen path $\sigma$ of exponentially-growing height may be folded. The fold at the illegal turn of $\sigma$ may be full or partial, and a full fold may be proper or improper. Our first lemma says that if the fold at $\sigma$ is partial, then $N(f)$ may be decreased.

Lemma 6.2 ([FH11] Lemma 4.29). Suppose that $H$ is an exponentially-growing stratum of a relative train track map $f: C \to C$ and that $\sigma$ is an indivisible Nielsen path of height $r$. If the fold at the illegal turn of $\sigma$ is partial then there is a relative train track map $f': C' \to C'$ satisfying $N(f') < N(f)$.

Proof. The lemma follows from the proofs of Lemmas 5.2.3 and 5.2.4 of [BFH00]. In the proof of Lemma 5.2.4, if $\sigma = \alpha \beta$ is the decomposition of $\sigma$ into maximal $r$-legal subpaths, it is argued that if $\alpha$ and $\beta$ are single edges, then $N(f)$ may be reduced by folding them. In order to perform this fold, we must know that the initial vertices of $\alpha$ and $\beta$ do not both
have nontrivial vertex group. If they did, the free product of the two vertex groups would
determine a free factor of $F$ invariant under $\varphi$. This contradicts our assumption that the
filtration is reduced unless $\alpha$ and $\beta$ are the only edges in $H_r$ and the terminal vertex $v$ of $\alpha$
has trivial vertex group. But in this case there is only one turn in $H_r$ based at $v$, and this
turn is illegal. This contradicts Lemma 5.5. Therefore we may indeed fold $\alpha$ and $\beta$.

Lemma 6.3 (FH11 Lemma 4.30). Suppose that $H_r$ is an exponentially-growing stratum
of a relative train track map $f : \mathcal{G} \to \mathcal{G}$ and that $\sigma$ is an indivisible Nielsen path of height $r$. Suppose that the fold at the illegal turn of $\sigma$ is proper, and let $f' : \mathcal{G}' \to \mathcal{G}'$ be the relative train track map obtained from $f : \mathcal{G} \to \mathcal{G}$ by folding $\sigma$. Then $N(f') = N(f)$ and there is a
bijection $H_s \to H'_s$ between the exponentially-growing strata of $f$ and those of $f'$ such that
$H_s$ and $H'_s$ have the same number of edges for all $s$.

Lemma 6.4 (FH11 Lemma 4.31). Suppose that $H_r$ is an exponentially-growing stratum
of a relative train track map $f : \mathcal{G} \to \mathcal{G}$ and that $\sigma$ is an indivisible Nielsen path of height $r$. Suppose that the fold at the illegal turn of $\sigma$ is improper. Then there exists a relative train
track map $f' : \mathcal{G}' \to \mathcal{G}'$ and a bijection $H_s \to H'_s$ between the exponentially-growing strata of $f$ and those of $f'$ with the following properties.

1. $N(f) = N(f')$.
2. $H'_s$ has fewer edges than $H_r$.
3. If $s > r$ then $H'_s$ and $H_s$ have the same number of edges.

Lemma 6.5 (FH11 Corollary 4.33). Suppose that $H_r$ is an exponentially-growing stratum
of a relative train track map $f : \mathcal{G} \to \mathcal{G}$ and that $\sigma$ is an indivisible Nielsen path of height $r$. Then the fold at the illegal turn of each indivisible Nielsen path obtained by iteratively
folding $\sigma$ is proper if and only if $H_r$ satisfies (EG Nielsen Paths).

If some exponentially-growing stratum of $f : \mathcal{G} \to \mathcal{G}$ does not satisfy (EG Nielsen Paths),
let $H_r$ be the highest such stratum. By the previous lemma there is a sequence of proper
folds leading to a relative train track map and an indivisible Nielsen path with either a
partial or improper fold. We may apply Lemma 6.2 or Lemma 6.4, respectively. In the
former situation $N(f)$ decreases, while in the latter the number of edges of $H_r$ decreases.
Since both quantities are finite, eventually all further folds are proper, at which point the
previous lemma implies that $H_r$ satisfies (EG Nielsen Paths).

In the following steps, the number of edges in each exponentially-growing strata and the
number of indivisible Nielsen paths of exponentially-growing height are not increased. If
after some step (EG Nielsen Paths) fails, then we may return to this step and restore this
property. By the above argument this process terminates.

Applying Theorem 5.1. Apply the proof of Theorem 5.1 to produce a new relative train
track map $f : \mathcal{G} \to \mathcal{G}$ satisfying the conclusions of that theorem. Note that in the process
of the proof, the number of edges in each exponentially-growing stratum and the number
of indivisible Nielsen paths of exponentially-growing height is unchanged. As noted in the
previous paragraph, we may assume that (EG Nielsen Paths) remains satisfied.

Properties (Rotationless), (Filtration) and (Zero Strata). Properties (Rotationless)
and (Filtration) follow from FH11 Proposition 3.29 and property (F) of Theorem 5.1.
By property (Z) of Theorem 5.1 to prove property (Zero Strata) it suffices to arrange that
every edge in a zero stratum $H_i$ is $r$-taken. Each edge $E$ in $H_i$ is contained in an $r$-taken
path $\sigma \subset H_i$. If $E$ itself is not $r$-taken, perform a tree replacement, replacing $E$ by a path
that has the same endpoints as $\sigma$ and is marked by $\sigma$.
**Property (Periodic Edges) and (Almost Periodic Edges).** First suppose that no component \( C \) of \( \text{Per}(f) \) is topologically a circle or an interval with both endpoints vertices with nontrivial vertex group, with each point in \( C \) having exactly two periodic directions. Then the endpoints of any periodic edge are principal, each periodic edge is fixed and each periodic stratum \( H_r \) is a single edge \( E_r \). Similarly, the initial endpoint of an almost periodic edge is principal, so each almost periodic edge is almost fixed and each almost periodic stratum \( H_r \) is a single edge \( E_r \). If the conclusions of [Periodic Edges] or [Almost Periodic Edges] do not hold for the edge \( E_r \), then one could collapse \( E_r \) without changing the free factor systems realized by the filtration elements. This would violate [P], so [Periodic Edges] and [Almost Periodic Edges] are satisfied in this case.

In the general case we will reduce the number of components of \( \text{Per}(f) \) that are circles or intervals with both endpoints vertices with nontrivial vertex group, with every point having exactly two periodic directions until we are in the former case.

By [FH11, Lemma 3.30(1)], if \( C \) is such a component, then \( C \) is \( f \)-invariant. If \( C \) is an interval, then \( C \) is fixed. If \( C \) is a circle, then \( g = f|_C \) is orientation-preserving. In the case where \( C \) is a circle there are two steps: first arranging that \( C \subset \text{Fix}(f) \), and then “untwisting” near \( C \) to add another edge to \( \text{Fix}(f) \). In the case where \( C \) is an interval only the latter step is necessary.

By [Zero Strata] and the fact that there are no periodic directions based in \( C \) and pointing out of \( C \), every edge not in \( C \) that has an endpoint in \( C \) is non-periodic and non-exponentially-growing and intersects \( C \) in its terminal endpoint. Since all non-periodic vertices are contained in exponentially-growing strata, no vertex in the complement of \( C \) maps into \( C \). By [NEG] \( C \) is a component of some filtration element.

For the first step, suppose \( C \) is a circle. Extend the rotation \( g^{-1} : C \rightarrow C \) to a map \( h : \mathcal{G} \rightarrow \mathcal{G} \) that has support on a small neighborhood of \( C \), that is homotopic to the identity and such that \( h(E_j) \subset E_j \cup C \) for each non-periodic non-exponentially-growing edge \( E_j \) that has terminal endpoint in \( C \). Redefine \( f \) on each edge \( E_j \) to be \( h(E_j) f_1(E_j) \). Edges in \( C \) are now fixed. If \( f(E_j) = E_j u_j \), then the new path \( u_j \) and the old \( u_j \) differ only by possible initial and terminal segments in \( C \); the \( f \)-image of all other edges is unchanged. The map \( f : \mathcal{G} \rightarrow \mathcal{G} \) is a relative train track map that has all the properties we have already established, with the possible exception of [P] which fails if one or more edge with terminal vertex in \( C \) is now a fixed edge that should be collapsed as in the proof of Theorem 5.1.

Let \( E_m \) be the first non-periodic non-exponentially-growing edge that has terminal endpoint in \( C \). Then \( f(E_m) = E_m u \), where \( u \) is a closed tight path that wraps around \( C \)—which may now be either a circle or an interval with endpoints vertices with nontrivial vertex group—some number of times. We will alter \( f \) so that \( E_m \) becomes a fixed edge. Choose \( h' : \mathcal{G} \rightarrow \mathcal{G} \) that is the identity on \( C \), that satisfies \( h'(E_j) = E_j u \) for each \( E_j \) with terminal vertex in \( C \) and is the identity otherwise. The map \( h' \) is homotopic to the identity. Redefine \( f \) on each edge to be \( h'_j f(E_j) \). Now \( E_m \) is fixed, so \( C \) no longer forms a problematic component. If necessary to restore [P] collapse fixed edges with endpoint in \( C \) and repeat this step.

**Induction: The NEG Case.** We establish the rest of the properties by induction up the filtration. Let \( M \) be the number of irreducible strata in the filtration, and for \( 0 \leq m \leq M \) let \( G_{i(m)} \) be the smallest filtration element containing the first \( m \) irreducible strata. We prove by induction on \( m \) that one can modify \( f \) so that \( f|_{G_{i(m)}} \) is a CT. The base case \( m = 0 \) is trivial. So assuming for \( \tau = i(m) \) that \( f|_{G_{\tau}} \) is a CT, we will arrange that \( f|_{G_{\tau+1}} \) is a CT for \( s = i(m+1) \). In this step we assume that \( H_s \) is non-exponentially-growing and hence is a single edge \( E_s \) satisfying \( f(E_s) = E_s u_s \) for some path \( u_s \subset G_{s-1} \). By [Zero Strata] \( \tau = s - 1 \). By choosing a path \( \tau \in G_{s-1} \), we may slide \( E_s \) along \( \tau \) so that after sliding we have \( f(E_s) = E_s [\tau u_s f(\tau)] \), where \([\gamma]\) denotes the path obtained from \( \gamma \) by tightening. As remarked in the [EG Nielsen Paths] step, we may assume sliding preserves [EG Nielsen Paths].

Suppose first that there is some path \( \tau \) so that after sliding \( E_s \) along \( \tau \) we have \( E_s \subset
Fix($f$). This is equivalent to $[τu_s f(τ)]$ being trivial, and hence to $f_1(E_sτ) = E_s[u_s f(τ)] = E_sτ$; that is, to $E_sτ$ being a Nielsen path. In this case if the initial vertex of $E_s$ has nontrivial vertex group or is contained in $G_{s-1}$, then (Periodic Edges) is satisfied, as are all the previous properties; the remaining properties of a CT follow by induction.

If on the other hand the initial vertex of $E_s$ is not contained in $G_{s-1}$ and has trivial vertex group, then collapse $E_s$ to a point as in the (Periodic Edges) step. None of our previously achieved properties are lost, and the remaining properties of a CT follow by induction.

Now suppose that there is no choice of $τ$ such that $E_sτ$ is a Nielsen path. If $f(E_s) = E_sg_s$ for some $g_s ∈ G(τ(E_s))$, i.e. $E_i$ is almost fixed, then $f|_{G_s}$ satisfies (Almost Fixed Edges) and the remaining properties of a CT, follow by induction, so we are done with the inductive step. So suppose $E_i$ is not almost fixed.

**Proposition 6.6** ([FH11] Proposition 4.35). Suppose that

1. $f: G → G$ is a relative train track map that satisfies (EG Nielsen Paths).
2. $f|_{G_{s-1}}$ is a CT, and
3. $H_s$ is a non-exponentially-growing stratum with single edge $E_s$ for which there does not exist $μ ∈ G_{s-1}$ such that $E_sμ$ is a Nielsen path.

Then there exists a path $τ ⊂ G_{s-1}$ along which we may slide $E_s$ such that after sliding, the following conditions hold.

1. $f(E_s) = E_s · u_s$ is a nontrivial splitting.
2. If $σ$ is a circuit or path with endpoints at vertices and if $σ$ has height $s$, then there exists $k ≥ 0$ such that $f^k|_σ$ splits into subpaths of the following type.
   - (a) $E_s$ or $E_s$.
   - (b) an exceptional path of height $s$.
   - (c) a subpath of $G_{s-1}$.
3. $u_s$ is completely split and its initial vertex is principal.
4. $f|_{G_s}$ satisfies (Linear Edges).

After performing the slide described in the proposition, since $u_s$ is nontrivial, $f|_{G_s}$ satisfies (Periodic Edges) and all the previously established properties. Properties (Completely Split), (Vertices), (Almost Fixed Edges), (NEG Nielsen Paths), and (Linear Edges) for $f|_{G_s}$ follow from the proposition and the inductive hypothesis. This completes the inductive step in the case that $H_s$ is non-exponentially-growing.

**Induction: The EG Case.** Suppose now that $H_s$ is exponentially-growing. Properties (Vertices), (Almost Fixed Edges), (NEG Nielsen Paths), and (Linear Edges) for $f|_{G_s}$ follow from these properties for $f|_{G_{s-1}}$. Thus it suffices to establish (Completely Split).

For each edge $E ⊂ H_s$, there is a decomposition $f(E) = μ_1 · ν_1 · μ_2 · ν_2 · · · μ_t · ν_t$, where the $ν_i$ are maximal subpaths in $G_r$. Let $\{ν_i\}$ be the collection of all such paths that occur as $E$ varies over the edges of $H_s$. By (Zero Strata) and (EG-ii), we have that $f^k|_{ν_i}$ is nontrivial for each $k$ and $i$. By [FH11] Lemma 4.20, since $f|_{G_{s-1}}$ is a CT, we may choose $k$ so large that each $f^k|_{ν_i}$ is completely split. We may also assume that the endpoints of $f^k|_{ν_i}$ are periodic and hence principal. There are finitely many paths $σ$ with endpoints in $H_s ∩ G_{s-1}$ contained in the strata between $G_s$ and $H_s$. Each $f(σ)$ is either a homotopically nontrivial path with endpoints in $H_s ∩ G_{s-1}$ or a path in $G_s$ with fixed endpoints. We therefore may assume that $f^k|_{ν_i}$ is completely split for each such $σ$. After applying the following move $k$ times with $j = r$, we have that $f|_{G_s}$ is completely split, completing the induction step and the proof of the theorem.
Changing the Marking. Suppose that $f: \mathcal{G} \to \mathcal{G}$ is a rotationless relative train track map satisfying the conclusions of Theorem 5.1 with respect to the filtration $\mathcal{G} = G_0 \subset G_1 \subset \cdots \subset G_m = \mathcal{G}$, that $1 \leq j \leq m$, that every component of $G_j$ is noncontractible and that $f$ fixes every vertex in $G_j$, that meets edges of $G \setminus G_j$. Define a homotopy equivalence $g: \mathcal{G} \to \mathcal{G}$ by $g|_{G_j} = f|_{G_j}$ and $g|_{(G \setminus G_j)}$ equal to the identity.

Define a new marked graph of groups $\mathcal{G}'$ from $\mathcal{G}$ by changing the marking via $g$. That is, as a graph of groups we have $\mathcal{G} = \mathcal{G}'$, and if $r: \mathcal{G} \to \mathcal{G}$ is the marking for $\mathcal{G}$, then $gr: \mathcal{G} \to \mathcal{G}'$ is the marking for $\mathcal{G}'$.

Define $f': \mathcal{G}' \to \mathcal{G}'$ by $f'|_{G_j} = f|_{G_j}$ and $f'(E) = (gf)_E(E)$ for all edges $E$ higher than $G_j$. We say that $f': \mathcal{G}' \to \mathcal{G}'$ is obtained from $f: \mathcal{G} \to \mathcal{G}$ by changing the marking on $G_j$ via $f$.

Lemma 6.7 ([FH11] Lemma 4.27). Suppose that $f': \mathcal{G}' \to \mathcal{G}'$ is obtained from $f: \mathcal{G} \to \mathcal{G}$ by changing the marking on $G_j$ via $f$. The following hold.

1. $f'|_{G_j} = f|_{G_j}$;
2. for every path $\sigma \subset G$ with endpoints at vertices and every $k > 0$, we have $g_2f^k_2(\sigma) = (f')^k_2g(\sigma)$;
3. $f': \mathcal{G}' \to \mathcal{G}'$ is a homotopy equivalence that represents the same element of Out$(F)$ as $f: \mathcal{G} \to \mathcal{G}$;
4. there is a one-to-one correspondence between Nielsen paths for $f$ and Nielsen paths for $f'$; and
5. $f': \mathcal{G}' \to \mathcal{G}'$ is also a rotationless relative train track map satisfying the conclusions of Theorem 5.1 with respect to the same filtration as $f$.

\[\square\]

7 An Index Inequality

Gaboriau, Jaeger, Levitt and Lustig define in [GJLL98, Theorem 4] an index for (outer) automorphisms of free groups and show that it satisfies an inequality that reproves and sharpens Bestvina–Handel’s proof of the Scott conjecture. Martino [Mar99] proved the analogous result for free products. We remark that Martino’s result is stated for the Grushko decomposition of the free product, but that the proof does not use this assumption in any essential way. More recently Feighn–Handel sharpen this inequality [FH11, Proposition 15.14]. The purpose of this section is to extend this latter inequality to the setting of free products.

As usual, let $F = A_1 \ast \cdots \ast A_n \ast F_k$ be represented by the graph of groups $\mathcal{G}$ with vertex groups the $A_i$, trivial edge groups, and ordinary fundamental group free of rank $k$. Suppose that $\Phi \in \text{Aut}(F)$ admits a topological representative on a $\mathcal{G}$-marked graph of groups. Recall from Section 4 that if $\Gamma$ is the Bass–Serre tree of $\mathcal{G}$, then $\Phi$ acts on the Gromov boundary $\partial \Gamma$. The subgroup Fix$(\Phi)$ inherits a splitting as a free product $B_1 \ast \cdots \ast B_m \ast F_\ell$ from $\mathcal{G}$. Define the rank (or $G$-rank) of Fix$(\Phi)$ to be the quantity $m + \ell$. Let $a(\Phi)$ denote the number of Fix$(\Phi)$-orbits of attracting fixed points of $\Phi$ in its action on $\partial \Gamma$, and let $b(\Phi)$ denote the number of Fix$(\Phi)$-orbits of attracting fixed points of $\Phi$ associated to NEG rays (see below for a definition). Define the quantity $j(\Phi)$ as

$$j(\Phi) = \max \left\{ 0, \text{rank}(\text{Fix}(\Phi)) + \frac{1}{2}a(\Phi) + \frac{1}{2}b(\Phi) - 1 \right\}.$$  

Two automorphisms $\Phi$ and $\Phi'$ are isogredient if there exists $w \in F$ such that $\Phi' = \text{ad}(w)\Phi\text{ad}(w)^{-1}$, where $\text{ad}(w)$ is the inner automorphism $g \mapsto wgw^{-1}$. Note that isogredience is an equivalence relation and that if $\Phi$ and $\Phi'$ are isogredient, then $j(\Phi) = j(\Phi').$
Given $\varphi \in \text{Out}(F)$ an automorphism that admits a topological representative on a $G$-marked graph of groups, define the quantity $j(\varphi)$ as
\[
j(\varphi) = \sum j(\Phi),\]
where the sum is taken over representatives of the isogredience classes of automorphisms $\Phi$ representing $\varphi$. If $j(\Phi)$ is positive, then $\Phi$ is a principal automorphism (see [PH11, Definition 3.1]). By [PH11] Remark 3.9, there are only finitely many isogredience classes of principal automorphisms, so this sum has only finitely many nonzero terms.

The main result of this section is the following theorem.

**Theorem 7.1.** If $\varphi \in \text{Out}(F)$ admits a topological representative on a $G$-marked graph of groups, then
\[
j(\varphi) \leq n + k - 1.
\]

The strategy of the proof of Theorem 7.1 is to firstly show that if $\psi = \varphi^K$ is a rotationless iterate of $\varphi$, then $j(\varphi) \leq j(\psi)$. Then, using a CT $f: G \to G$ representing $\psi$, to construct a graph of groups $S_N(f)$, invariants of which calculate $j(\psi)$, and argue by induction up through the filtration that $j(\psi) \leq n + k - 1$.

**Rays and Attracting Fixed Points** We recall [FH18, Definition 3.9]: let $f: G \to G$ be a CT representing $\varphi \in \text{Out}(F)$, and let $E$ denote the set of oriented, non-fixed, non-linear edges whose initial vertex is principal and which support a fixed direction at that vertex. If $E$ is an edge of $\mathcal{E}$, such that the direction $gE$ is fixed, there is a path $u$ such that $f^k(gE)$ has a splitting of the form $gE \cdot u \cdot f_1(u) \cdot \cdots \cdot f_k(u)$ for all $k \geq 1$ and such that the length $|f^k(u)| \to \infty$ as $k$ tends to infinity. The limit of the increasing sequence $gE \subset f(gE) \subset f^2(gE) \subset \cdots$

is a ray $R_E$ in $\mathcal{G}$.

If $\Gamma$ is the Bass-Serre tree for $G$, each lift of $R_E$ to $\Gamma$ has a well-defined endpoint in $\partial \Gamma$, so $R_E$ determines an $F$-orbit of points in $\partial \Gamma$. If $E$ is a lift of $E$ to $\Gamma$ and $\tilde{f}: \Gamma \to \Gamma$ is the lift of $f$ sending $E$ to $\tilde{E}u$, then the lift $\tilde{R}_E$ of $R_E$ with initial edge $\tilde{E}$ determines an attracting fixed point for $\tilde{f}$. The lift $\tilde{f}$ is a principal lift (see [PH11, Definition 3.1]). By [FH18, Lemma 3.10], every $F$-orbit of attracting fixed points for every principal lift of $f$ is represented by some $R_E$ for $E \in E$.

If $P$ is an attracting fixed point for the automorphism $\Phi$ determined by the principal lift $\tilde{f}$ which is represented by the lifted ray $\tilde{R}_E$, we say $P$ is an NEG ray for $\Phi$ if the stratum of $f: G \to G$ containing $E$ is non-exponentially growing. By [FH18, Lemma 15.4] and [HM20] Part II, Definitions 2.9 and 2.10 and Lemma 2.11, this definition is independent of the choice of CT representing $\varphi \in \text{Out}(F)$. If $\varphi$ is not rotationless (and so not represented by a CT), then we say $P$ is an NEG ray for $\Phi$ if it is an NEG ray for $\Phi^K$, where $\varphi^K$ is a rotationless iterate of $\varphi$. This definition is independent of the choice of iterate by [FH18, Remark 15.6].

**Lemma 7.2.** If $\psi = \varphi^K$ is a rotationless iterate of $\varphi$, then $j(\varphi) \leq j(\psi)$.

**Proof.** Let $\Phi_1, \ldots, \Phi_N$ be a set of representatives of isogredience classes of principal automorphisms representing $\varphi$. Feighn and Handel prove in [FH18, Lemma 15.8] that the sum $\sum_{i=1}^N a(\Phi_i)$ (respectively, $\sum_{i=1}^N b(\Phi_i)$) is equal to the number of $F$-orbits of attracting fixed points of $\Phi$ (respectively, $F$-orbits of NEG rays for $\Phi$) as $\Phi$ varies over all principal automorphisms representing $\varphi$. Then they observe in [FH18, Lemma 15.9] that by definition if $P$ is an attracting fixed point (respectively an NEG ray) for the principal automorphism $\Phi$, then $P$ is also an attracting fixed point (respectively an NEG ray) for the principal automorphism $\Phi^K$ representing $\psi$. Therefore if we write $a(\varphi) = \sum a(\Phi_i)$ and $b(\varphi) = \sum b(\Phi_i)$...
where the sum is over representatives of isogredience classes of principal automorphisms \( \Phi \) representing \( \varphi \), we see that \( a(\varphi) \leq a(\psi) \) and \( b(\varphi) \leq b(\psi) \).

Recall that for \( \Phi \) an automorphism representing \( \varphi \), \( \text{Fix}(\Phi) \) inherits a splitting as a free product from \( \mathbb{G} \). By the main result of [CT94], the free product is of the form

\[
\text{Fix}(\Phi) = B_1 \ast \cdots \ast B_m \ast F_\ell
\]

(i.e. there are finitely many factors). Define \( \hat{r}(\Phi) = \max\{0, m + \ell - 1\} \), and define \( \hat{r}(\varphi) = \sum \hat{r}(\Phi) \), where the sum is over representatives of isogredience classes of principal automorphisms \( \Phi \) representing \( \varphi \). To finish the proof, we show that \( \hat{r}(\varphi) \leq \hat{r}(\psi) \).

So suppose \( \hat{r}(\varphi) > 0 \); we argue as in [FH18 Lemma 15.11]. Let \( \Phi_1, \ldots, \Phi_s \) be the representatives of isogredience classes of principal automorphisms for which \( \hat{r}(\Phi_i) > 0 \). For each \( i \) satisfying \( 1 \leq i \leq s \), there exists a principal automorphism \( \Psi_j \) representing \( \psi \) such that \( \Phi_i^k \) is isogredient to \( \Psi_j^k \). By replacing \( \Phi_i \) within its isogredience class we may assume that \( \Phi_i^k = \Psi_j^k \). Let \( \Psi_1, \ldots, \Psi_t \) denote the representatives of isogredience classes of principal automorphisms for which there exists some \( \Phi_i \) with \( \Phi_i^k = \Psi_j^k \). The assignment \( i \mapsto j \) defines a function \( p: \{1, \ldots, s\} \rightarrow \{1, \ldots, t\} \). It suffices to show that

\[
\sum_{i=1}^s \hat{r}(\Phi_i) \leq \sum_{j=1}^t \hat{r}(\Psi_j),
\]

which will hold if we can show that for each \( j \) satisfying \( 1 \leq j \leq t \), we have

\[
\sum_{i \in p^{-1}(j)} \hat{r}(\Phi_i) \leq \hat{r}(\Psi_j).
\]

Fix \( j \), write \( F = \text{Fix}(\Psi_j) \) and take \( i \in p^{-1}(j) \). We have \( \Phi_i^k = \Psi_j^k \), so \( F = \text{Fix}(\Phi_i^k) \). Since \( \psi \) is rotationless (see [FH11 Definition 3.13]), we have that \( \Phi_i(F) = F \), that \( \text{Fix}(\Phi_i) \subset F \), and that \( \Phi_i|_F \) is a finite order automorphism.

We claim that if \( i' \in p^{-1}(j) \), then \( \text{Fix}(\Phi_i) \) and \( \text{Fix}(\Phi_{i'}) \) are not conjugate in \( F \). For if \( \text{Fix}(\Phi_i) = h \text{Fix}(\Phi_{i'}) h^{-1} \), then we have \( \text{Fix}(\Phi_i) = \text{Fix}((\text{ad}(h)\Phi_{i'}\text{ad}(h^{-1}))) \), where \( \text{ad}(h) \) is the inner automorphism \( g \mapsto hgh^{-1} \). These automorphisms differ by an inner automorphism \( \text{ad}(k) \), where \( k \) centralizes \( \text{Fix}(\Phi_i) \), hence \( k = 1 \), so they are equal, in contradiction to the assumption that \( \Phi_i \) and \( \Phi_{i'} \) are not isogredient.

There are two cases: either \( F \) is infinite dihedral or it is not. If \( F \) is infinite dihedral, the only possibility for the finite order automorphism \( \Phi_i|_F \) is the identity, and by the argument in the previous paragraph we conclude that \( p^{-1}(j) = \{i\} \), and the displayed inequality for \( \hat{r}(\Psi_j) \) holds. If \( F \) is not infinite dihedral, by [CT94], it inherits a free product decomposition as \( F = B_1 \ast \cdots \ast B_m \ast F_\ell \) from \( \mathbb{G} \) (i.e. there are finitely many factors). The argument in the proof of [FH18 Lemma 3.12] applies to show that \( F \) is self-normalizing, so the restriction \( \varphi|_F \) is well-defined and has finite order.

Martino [Mar99] defines an index \( i(\Phi) \) for \( \Phi \) an automorphism of a free product \( F \) representing the outer automorphism \( \varphi \) as

\[
i(\Phi) = \max \left\{ 0, \text{rank}(\text{Fix}(\Phi)) + \frac{1}{2} a(\Phi) - 1 \right\},
\]

and defines \( i(\varphi) = \sum i(\Phi) \) where the sum is over representatives of isogredience classes of automorphisms \( \Phi \) representing \( \varphi \). If \( F = A_1 \ast \cdots \ast A_n \ast F_k \) is represented as a graph of groups \( \mathbb{G} \) with nontrivial vertex groups the \( A_i \), trivial edge groups, and ordinary fundamental group free of rank \( k \) and \( \varphi \) may be represented on a \( \mathbb{G} \)-marked graph of groups, he shows that \( i(\varphi) \leq n + k - 1 \). (In fact, as remarked above, Martino’s result is stated for the Grushko decomposition of \( F \), but this is not necessary in the proof.) If \( \Phi: F \rightarrow F \) represents \( \varphi|_F \), then \( i(\Phi) = \hat{r}(\Phi) \), since there are no attracting fixed points. Thus Martino’s result applied to \( \varphi|_F \) shows that

\[
\sum_{i \in p^{-1}(j)} \hat{r}(\Phi_i) \leq i(\varphi|_F) \leq m + \ell - 1 = \hat{r}(\Psi_j).
\]
The change in negative Euler characteristic is $\Delta \gamma_\psi$ that $H \rightarrow G$ is a graph of groups equipped with an immersion $f$ defined by including only those elements that are their own core. By [Periodic Edges] and [Almost Fixed Edges] we have $\ell_1 = 1$. The $i$th stratum of the core filtration, $H_{\ell_i}^c$, is

$$H_{\ell_i}^c = \bigcup_{j=\ell_{i-1}+1}^{\ell_i} H_j.$$ 

The change in negative Euler characteristic is $\Delta \gamma^-(G) = \chi^-(G_{\ell_i}) - \chi^-(G_{\ell_{i-1}})$. 

The Index of a Finite-Type Graph of Groups. Let $f : G \rightarrow G$ be a CT representing $\psi \in \text{Out}(F)$. In the proof of Theorem 7.1, we build a finite-type graph of groups $S_N(f)$ considered by Feighn–Handel in [FH18, Section 12] equipped with an immersion $S_N(f) \rightarrow G$ such that $j(S_N(f)) = j(\psi)$, by [FH18, Lemma 12.4]. Each Nielsen path (containing an edge) and each ray $R_E$ for $E \in \mathcal{E}$ lifts to $S_N(f)$. In the proof we build up $S_N(f)$ in stages by induction up through the core filtration. Here we describe its construction without reference to the filtration.

**Nielsen Paths in CTs.** Let $f : G \rightarrow G$ be a CT representing $\psi \in \text{Out}(F)$. Each Nielsen path with endpoints at vertices is a composition of fixed edges and indivisible Nielsen paths. There are three kinds of indivisible Nielsen paths. The first possibility is that $E$ is an almost fixed edge, that is, if $f(E) = Eg$ for some group element $g \in G_{r(E)}$, then for any nontrivial group element $h \in G_{r(E)}$ such that $f_{r(E)}(h) = g^{-1}hg$, $EhE$ is an indivisible Nielsen path. The second is that $E$ is a linear edge, so there exists a closed root-free Nielsen path $w_E$ such that $f(E) = Ew_kE$ for some $d \neq 0$. Then $Ew_kE$ is an indivisible Nielsen path for any $k \neq 0$. By [NEG Nielsen Paths] these are the only possibilities for indivisible Nielsen paths of non-exponentially-growing height. The final possibility is an indivisible Nielsen path $\mu$ of exponentially-growing height $r$. By [FH11, Lemma 4.19], $\mu$ and $\bar{\mu}$ are the only indivisible Nielsen paths of height $r$, and the initial edges of $\mu$ and $\bar{\mu}$ are distinct.

**An Euler Characteristic for Graphs of Groups.** If $G$ is a finite graph of groups with trivial edge groups, we define the (negative) Euler characteristic $\chi^-(G)$ of $G$ to be the number of edges of $G$ minus the number of vertices with trivial vertex group. Note that $G$ defines a splitting of its fundamental group as a free product $B_1 \ast \cdots \ast B_m \ast F_\ell$, where the $B_i$ are the nontrivial vertex groups of $G$ and the ordinary fundamental group of $G$ is free of rank $\ell$. The ordinary negative Euler characteristic of $G$ calculates the quantity $\ell - 1$. In our definition, the $m$ vertices with nontrivial vertex group are not counted, so we see that $\chi^-(G) = m + \ell - 1$.

**The Core Filtration.** If $f : G \rightarrow G$ is a CT, then the core of a filtration element is a filtration element. The core filtration

$$\emptyset = G_0 = G_{\ell_0} \subset G_{\ell_1} \subset \cdots \subset G_{\ell_m} = G$$

is the coarsening of the filtration for $f : G \rightarrow G$ defined by including only those elements that are their own core. By [Periodic Edges] and [Almost Fixed Edges] we have $\ell_1 = 1$. The $i$th stratum of the core filtration, $H_{c \ell_i}^c$, is

$$H_{c \ell_i}^c = \bigcup_{j=\ell_{i-1}+1}^{\ell_i} H_j.$$
Begin with the subgraph of groups of \( G \) consisting of all principal vertices and all fixed edges. For each vertex \( v \), set its vertex group in \( S_N(f) \) to be \( \text{Fix}(f_v) \). We add a piece to \( S_N(f) \) for indivisible Nielsen paths and edges in \( E \) in the following way.

Suppose \( E \) is an almost fixed edge, say \( f(E) = Eg \), such that there exists an indivisible Nielsen path of the form \( EhE \) for \( h \in \mathcal{G}_{\tau(E)} \). To the principal initial vertex of \( E \) in \( S_N(f) \) we attach an edge mapping to \( E \). Although \( E \) may form a loop in \( \mathcal{G} \), the edge we attach does not; we set the vertex group of its terminal vertex to be the subgroup of \( \mathcal{G}_{\tau(E)} \) consisting of those \( h \) such that \( f_v(h) = g^{-1}hg \). We call the newly attached subgraph of groups the pin associated to \( E \).

Suppose now that \( E \) is a linear edge, say \( f(E) = Ew \) for some closed, root-free Nielsen path \( w \). To the principal initial vertex of \( E \) in \( S_N(f) \) we attach a lollipop: a graph consisting of two edges sharing a vertex, one of which forms a loop. The loop maps to \( w \), and the other edge to \( E \).

Suppose now that \( \mu \) is an indivisible Nielsen path of exponentially-growing height. Attach an edge mapping to \( \mu \) to the principal endpoints of \( \mu \) in \( S_N(f) \).

Suppose finally that \( E \in E \). If \( E \) belongs to a non-exponentially-growing stratum or \( E \) belongs to an exponentially-growing stratum and is not the initial edge of an indivisible Nielsen path, attach to its principal initial vertex an infinite ray mapping to \( E \). If \( E \) is the initial edge of an indivisible Nielsen path \( \mu \) and \( E' \) is the initial edge of \( \mu_1 \), then \( R_{E} \) and \( R_{E'} \) have a common terminal subray \( R_{E,E'} \) (see \cite{FH18} Section 12 for more details). Recall that we may write \( \mu = \alpha \beta \), where \( \alpha \) and \( \beta \) are the maximal legal segments of \( \mu \). In this case subdivide the edge mapping to \( \mu \) into two edges, one mapping to \( \alpha \) and the other to \( \beta \), and attach \( R_{E,E'} \) at the newly added vertex. This completes the construction of \( S_N(f) \).

**Proof of Theorem 7.7** We follow the outline of \cite{FH18} Proposition 15.14. Let \( f: \mathcal{G} \to \mathcal{G} \) be a CT representing \( \psi \in \text{Out}(F) \). We construct \( S_N(f) \) by inducting up through the core filtration of \( \mathcal{G} \). For \( i \) satisfying \( 0 \leq i \leq M' \), let \( S_N(i) \) denote the subgraph of groups of \( S_N(f) \) constructed by the \( i \)th term of the core filtration \( G_{\ell_i} \). We shall prove that

\[
j(S_N(i)) \leq \chi^-(G_{\ell_i}),
\]

from which the theorem follows. We prove this inequality by induction. The case \( i = 0 \) holds trivially, since \( S_N(0) \) and \( G_0 \) are empty. Let \( \Delta_k = j(S_N(k)) - j(S_N(k-1)) \). Suppose that the inequality holds for \( k - 1 \). We prove that \( \Delta_k \leq \Delta_k \chi^- \), so that the inequality holds for \( k \) as well. The proof has two cases.

**Case 1.** Suppose that \( H_{\ell_k}^c \) contains no exponentially-growing strata. We claim that one of the following occurs.

(a) We have \( \ell_k = \ell_{k-1} + 1 \). The unique edge in \( H_{\ell_k}^c \) is disjoint from \( G_{\ell_k-1} \) and is fixed or almost fixed. Either it forms a loop, or both incident vertices have nontrivial vertex group.

(b) We have \( \ell_k = \ell_{k-1} + 1 \). The unique edge in \( H_{\ell_k}^c \) either has both endpoints contained in \( G_{\ell_k-1} \) or one endpoint is contained in \( G_{\ell_k-1} \) and the other has nontrivial vertex group.

(c) We have \( \ell_k = \ell_{k-1} + 2 \). The two edges in \( H_{\ell_k}^c \) share an initial endpoint, are not fixed or almost fixed, and have terminal endpoints in \( G_{\ell_k-1} \). (Note that the initial endpoint has trivial vertex group, otherwise we would be in case (b).)

The proof is similar to \cite{FH09} Lemma 8.3. Because \( H_{\ell_k}^c \) contains no exponentially-growing strata, it contains no zero strata, so each \( H_j \) is a single edge \( E_j \) for each \( j \) satisfying \( \ell_{k-1} + 1 \leq j \leq \ell_k \). If some \( E_j \) is fixed or almost fixed, then either (a) or (b) holds by \cite[Periodic Edges]{FH09} or \cite[Almost Periodic Edges]{FH09}, respectively. If each \( E_j \) is not fixed or almost fixed, and (b) does not hold, then \( E_j \) adds an inessential valence-one vertex to \( G_{\ell_k-1} \), and
Case 2. The argument again is similar to \cite[Lemma 8.3]{FH11}. Suppose that $G_H$ from (Periodic Edges), (Almost Periodic Edges) and \cite[Lemma 4.22]{FH11}. We analyze each subcase.

(a) Suppose first that the edge $E_{\ell_k}$ is fixed. To $S_N(k)$ we add a new component containing the subgraph spanned by $E_{\ell_k}$. If $E_{\ell_k}$ forms a loop with trivial incident vertex group, we see that $\Delta_{kj} = \Delta_k \chi = 0$. Otherwise $\Delta_{kj} \leq \Delta_k \chi = 1$.

If instead $E_{\ell_k}$ is almost fixed, then its initial vertex $v$ is principal and we add it to $S_N(k)$. If we add a pin corresponding to $E_{\ell_k}$, then $\Delta_{kj} = \Delta_k \chi = 1$, otherwise $\Delta_{kj} = 0$ and $\Delta_k \chi = 1$.

(b) In this case we always have $\Delta_k \chi = 1$. If $E_{\ell_k}$ is fixed, we add it to $S_N(k)$ as in item (a). If both endpoints are contained in $G_{\ell_{k-1}}$ or the valence-one vertex $v$ has Fix($f_{v}$) nontrivial, then $\Delta_{kj} = 1$. Otherwise $\Delta_{kj} = 0$.

If instead $E_{\ell_k}$ is almost fixed and we add a pin, then $\Delta_{kj} = 1$, otherwise $\Delta_{kj} = 0$. If the initial vertex of $E_{\ell_k}$ is not contained in $G_{\ell_{k-1}}$, then $E_{\ell_k}$ is added, if at all, as a new component to $S_N(k)$. In this case we have $\Delta_{kj} = 1$ if the pin is added and Fix($f_{v}$) is nontrivial for $v$ the initial vertex of $E_{\ell_k}$; otherwise $\Delta_{kj} = 0$.

Suppose then that $E_{\ell_k}$ is linear; to $S_N(k)$ we add a lollipop. If the initial vertex of $E_{\ell_k}$ is contained in $G_{\ell_{k-1}}$, then $\Delta_{kj} = 1$. If not, then the lollipop is added as a new component to $S_N(k)$; we have $\Delta_{kj} = 1$ if Fix($f_{v}$) is nontrivial for $v$ the initial vertex of $E_{\ell_k}$, and $\Delta_{kj} = 0$ if not.

Finally suppose $E_{\ell_k}$ is nonlinear. Then to the initial vertex $v$ of $E_{\ell_k}$ in $S_N(k)$ we attach the NEG ray $R_{E_{\ell_k}}$. We have $\Delta_{kj} = 1$ if $v \in G_{\ell_{k-1}}$ or Fix($f_{v}$) $\neq 1$, and $\Delta_{kj} = 0$ otherwise.

(c) In this case we have $\Delta_k \chi = 1$. The vertex in $G_{\ell_k}$ not contained in $G_{\ell_{k-1}}$ is principal, so determines a new component of $S_N(k)$. To this new vertex we attach a lollipop for each linear edge, and an NEG ray for each nonlinear edge, for a total of three possible combinations. In all cases $\Delta_{kj} = 1$.

Case 2. The argument again is similar to \cite[Lemma 8.3]{FH09}. Suppose that $H_{\ell_k}$ contains an exponentially-growing stratum $H_{r}$. Since the core of a filtration element is a filtration element and $G_{r-1}$ does not carry the attracting lamination associated to $H_{r}$, we see that $G_{r}$ is its own core. Thus $H_{\ell_k}$ is the unique exponentially-growing stratum in $H_{\ell_{k}}$. We claim that there exists some $u_k$ satisfying $\ell_{k-1} \leq u_k < \ell_k$ such that both of the following hold.

(a) For $j$ satisfying $\ell_{k-1} < j \leq u_k$, the stratum $H_j$ is a single edge which is not fixed nor almost fixed whose terminal vertex is in $G_{\ell_{k-1}}$ and whose initial vertex is an inessential valence-one vertex in $G_{u_k}$.

(b) For $j$ satisfying $u_k < j < \ell_k$, the stratum $H_j$ is a zero stratum.

The existence of $u_k$ and that (b) holds follows from \cite[Zero Strata]{Periodic Edges}. That (a) holds follows from \cite[Almost Periodic Edges]{Periodic Edges} and \cite[Lemma 4.22]{FH11}.

To calculate $\chi^-$, note that $\chi^-(G)$ is equal to the sum over the vertices $v$ of $G$ of $\frac{1}{2}$ valence($v$) $-$ $1$ if $v$ has trivial vertex group and $\frac{1}{2}$ valence($v$) if not. Note that each edge contributes valence to two vertices.

For $j$ as in item (a) above, the contribution to $S_N(k)$ is the addition of a new component, with either a lollipop or an NEG ray. Thus in each case $\Delta_{kj} = 0$, while $\Delta_k \chi = 0$. This completes the analysis up to $u_k$.

For each vertex $v \in H_{\ell_k}$, let $\Delta_{kj}(v)$ and $\Delta_k \chi^-(v)$ be the contributions to $\Delta_{kj}$ and $\Delta_k \chi$ from $v$ not already considered. If $v$ is principal, let $\kappa(v)$ denote the number of oriented edges of $H_{\ell_k}$ incident to $v$ that do not support a fixed direction at $v$. If $v \in H_{\ell_k}$ is not principal, let $\kappa(v) = \text{valence}(v) - 2$ if $v$ has trivial vertex group and $\kappa(v) = \text{valence}(v)$ otherwise.
Suppose first that there are no indivisible Nielsen paths of height $\ell_k$. If the vertex $v$ is not principal, then $\Delta_k j(v) = 0$. By [FH18, Lemma 3.8], either $v$ is a new vertex or it has infinite vertex group. In either case, $\Delta_k \chi^-(v) = \frac{1}{2} \kappa(v)$ and we have $\Delta_k \chi^-(v) - \Delta_k j(v) = \frac{1}{2} \kappa(v) \geq 0$.

If instead the vertex $v$ is principal, let $L(v)$ denote the number of oriented edges based at $v$ in $E \cap H_{\ell_k}$. If $v$ has already been added to $S_N(k)$, then $\Delta_k j(v) = \frac{1}{2} L(v)$ and $\Delta_k \chi^- = \frac{1}{2}(L(v) + \kappa(v))$. If $v$ has nontrivial vertex group, then $\Delta_k \chi^-(v) = \frac{1}{2}(L(v) + \kappa(v))$ and $\Delta_k j(v) = \frac{1}{2} L(v)$ if $\text{Fix}(f_v) \neq 1$ and $\frac{1}{2} L(v) - 1$ otherwise. Finally if $v$ is new and has trivial vertex group, then $\Delta_k j(v) = \frac{1}{2} L(v) - 1$ and $\Delta_k \chi^-(v) = \frac{1}{2}(L(v) + \kappa(v)) - 1$. In all cases we see that $\Delta_k \chi^-(v) - \Delta_k j(v) \geq \frac{1}{2} \kappa(v) \geq 0$.

Now suppose finally that there is an indivisible Nielsen path $\mu$ of height $\ell_k$. By [FH18, Lemma 4.24], we have $\ell_k = n + 1$. Let $w$ and $w'$ be the endpoints of $\mu$. By [BFH00, Lemma 5.1.7], if $w \neq w'$, then at least one of $w$ and $w'$ is a new vertex. By [FH11, Corollary 4.19] if $w = w'$, then $w$ is new. Let $e$ and $e'$ be the initial edges of $\mu$ and $\bar{\mu}$ respectively.

Let $V$ be the set of vertices of $H_{\ell_k}$ that are not endpoints of $\mu$. Each vertex of $V$ may be handled as in the case without indivisible Nielsen paths, so we conclude

$$\sum_{v \in V} \Delta_k \chi^-(v) - \sum_{v \in V} \Delta_k j(v) \geq \sum_{v \in V} \frac{1}{2} \kappa(v) \geq 0.$$

Let $\Delta_k j(\mu)$ and $\Delta_k \chi^-(\mu)$ be the contributions to $\Delta_k j$ and $\Delta_k \chi^-$ coming from the endpoints of $\mu$ and not already considered. There are three subcases to consider according to whether $w = w'$ and whether $w$ and $w'$ are new.

1. Suppose $\mu$ is a closed path based at the new vertex $w$. In $S_N(k)$, we add a new vertex $w$, a loop at $w$ mapping to $\mu$, a ray for each $E \in E$ incident to $w$ other than $e$ and $e'$, and then one ray corresponding to $e$ and $e'$. Therefore,

$$\Delta_k j(\mu) = 1 + \frac{1}{2}(L(w) - 1) - \frac{1}{2} = \frac{1}{2}L(w) - \frac{1}{2}$$

if $w$ has trivial vertex group or $\text{Fix}(f_w) = 1$ and $\Delta_k j(\mu) = \frac{1}{2}L(w) + \frac{1}{2}$ otherwise. Similarly we have

$$\Delta_k \chi^-(\mu) = \frac{1}{2}(L(w) + \kappa(w)) - 1$$

if $w$ has trivial vertex group and $\Delta_k \chi^-(\mu) = \frac{1}{2}(L(w) + \kappa(w))$ otherwise. Thus

$$\Delta_k \chi^-(\mu) - \Delta_k j(\mu) \geq \frac{1}{2} \kappa(w) - \frac{1}{2}.$$

Since there is always at least one illegal turn in $H_{\ell_k}$—for instance the one in $\mu$—and since illegal but nondegenerate turns are determined by distinct edges, there must be at least one vertex of $H_{\ell_k}$ with $\kappa(v) \neq 0$, so we conclude that $\Delta_k j \leq \Delta_k \chi^-$ as desired.

2. Suppose that $w$ is new and $w'$ is old. In $S_N(k)$ we add the new vertex $w$, an edge connecting $w$ to $w'$, one ray for each $E \in E$ based at $w$ or $w'$ other than $e$ and $e'$ and one ray corresponding to $e$ and $e'$. Thus we have

$$\Delta_k j(\mu) = \frac{1}{2}(L(w) + L(w')) - \frac{1}{2}$$

if $w$ has trivial vertex group or $\text{Fix}(f_w) = 1$ and $\Delta_k j(\mu) = \frac{1}{2}(L(w) + L(w')) + \frac{1}{2}$ otherwise. Similarly,

$$\Delta_k \chi^-(\mu) = \frac{1}{2}(L(w) + L(w') + \kappa(w) + \kappa(w')) - 1$$

if $w$ has trivial vertex group, and $\Delta_k \chi^-(\mu) = \frac{1}{2}(L(w) + L(w') + \kappa(w) + \kappa(w'))$ otherwise. In each case, the argument concludes as in the previous subcase.
3. Suppose \( w \) and \( w' \) are distinct and both new. The change in \( S_N(k) \) is the addition of two new vertices, an edge connecting them, one ray for each \( E \in \mathcal{E} \) based at \( w \) or \( w' \) other than \( e \) and \( e' \) and one ray corresponding to \( e \) and \( e' \). The calculation has several possibilities depending on whether \( w \) or \( w' \) have trivial or nontrivial vertex group, but as in the previous two cases, we see that

\[
\Delta_k \chi^-(\mu) - \Delta_k j(\mu) \geq \frac{1}{2} (\kappa(w) + \kappa(w')) - \frac{1}{2},
\]

and we conclude the argument as before.

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