APPROXIMATION IN $K$-THEORY FOR WALDHAUSEN QUASICATEGORIES

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Abstract. We prove that an exact functor of Waldhausen quasicategories induces a stable equivalence of $K$-theory spectra if it induces an equivalence of cofibration homotopy categories. As a special case, if an exact functor reflects cofibrations and induces an equivalence of homotopy categories, then it induces a stable equivalence of $K$-theory spectra.

The main technical result to prove these versions of Waldhausen Approximation is that a functor of quasicategories $F : \mathcal{A} \to \mathcal{B}$ induces a weak homotopy equivalence of maximal Kan complexes if it is essentially surjective, $\mathcal{A}$ admits colimits of diagrams in its maximal Kan complex indexed by finite posets, $F$ preserves them, and $F$ reflects equivalences.

We also prove that $S_n^\infty$ is Waldhausen equivalent to $F_n^\infty - 1$ using the mid anodyne maps known as spine inclusions, and clarify how hypotheses and notions in Waldhausen structures are related in new ways in the context of quasicategories. In the last section, we give conditions for an exact functor to induce an equivalence between (cofibration) homotopy categories of higher $S_n^\infty$-iterates.

Key words: quasicategory, $\infty$-category, $K$-theory, Waldhausen’s Approximation Theorem, Waldhausen quasicategory, Waldhausen $\infty$-category

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1. Introduction

When does a map $G$ induce an equivalence of algebraic $K$-theory spectra? Waldhausen’s Approximation Theorem [27, Theorem 1.6.7] gives just such a criterion: if $G : \mathcal{C} \to \mathcal{D}$ is an exact functor between reasonable Waldhausen categories, $G$ reflects weak equivalences, and any map $G(C) \to D$ factors as (w.e.) $\circ G(\text{cofibration})$, then $G$ induces an equivalence of $K$-theory spectra (even $wS_n G$ is a weak homotopy equivalence).
The main purpose of the present paper is to prove the most general Approximation Theorem presently available in the context of quasicategories: if an exact functor of Waldhausen quasicategories induces an equivalence of cofibration homotopy categories, then it induces a stable equivalence of $K$-theory spectra. Recent general results in this direction worked with localizations of homotopical categories, or assumed that every map is a cofibration (see the literature review later in this introduction).

A few years after Waldhausen’s Approximation Theorem appeared, Thomason–Trobaugh published a proof in [25, 1.9.8] that a functor which induces an equivalence of derived homotopy categories induces an equivalence in $K$-theory, under appropriate hypotheses. In 1.9.9, Thomason remarks “Morally, it says that $K(A)$ essentially depends only on the derived category $w^{-1}A$, and thus that Waldhausen $K$-theory gives essentially a $K$-theory of the derived category.”

Since then, many mathematicians have obtained results in this direction in various contexts: Neeman [18, 19] and Dugger-Shipley [7] for algebraic $K$-theory of rings, Schlichting’s counterexample for Frobenius categories [22], Toën–Vezzosi [26] and Blumberg–Mandell [4] for Dwyer–Kan simplicial localizations of Waldhausen categories, and Cisinski [6] for right exact functors between reasonable Waldhausen categories. Sagave [21] showed how to loosen Waldhausen’s requirement of factorizing $G(C) \to D$ mentioned above to a requirement of factorizing $G(C) \to D$ when $D$ is a “special object”. In Appendix A of [23], Schlichting showed how to replace Waldhausen’s cylinder functor and cylinder axiom in the Approximation Theorem by requiring factorization of any map into a cofibration followed by a weak equivalence.

The main contribution of the present article is to prove several versions of Waldhausen Approximation in the context of the Waldhausen quasicategories of Barwick [1] and Fiore–Lück [8], see Definition 3.1, Theorems 4.10 and 4.12 and Corollaries 4.16 and 4.18. A quasicategory, or $\infty$-category, is a simplicial set in which every inner horn has a filler. For instance Kan complexes and nerves of categories are quasicategories. A Waldhausen quasicategory is equipped with a distinguished zero object and a 1-full subquasicategory of cofibrations containing the equivalences, and is required to have pushouts along cofibrations. The subquasicategory of “weak equivalences” is always the 1-full subquasicategory on the equivalences of the underlying quasicategory, namely the 1-full subquasicategory on the maps that are invertible in its homotopy category.
The main results of this paper begin with Theorem 4.10: if an exact functor \( G \) induces an equivalence of cofibration homotopy categories, then it induces a stable equivalence of \( K \)-theory spectra. The next main result, Theorem 4.12, says that if \( G \) reflects cofibrations and induces an equivalence of homotopy categories, then \( G \) induces a stable equivalence of \( K \)-theory spectra. Proposition 4.5 (without Waldhausen structures) is the main technical step, and states that a functor \( F : \mathcal{A} \to \mathcal{B} \) between quasicategories induces a weak homotopy equivalence of maximal Kan complexes if \( F \) is essentially surjective, \( \mathcal{A} \) admits colimits of diagrams in its maximal Kan complex indexed by finite posets, \( F \) preserves them, and \( F \) reflects equivalences. The idea for the proof of Proposition 4.5 is due to Waldhausen: use Lemma 4.4 and Quillen’s Theorem A. However, we also incorporate a quasicategorical implementation of an idea of Schlichting [23, page 132], see the discussion preceding Proposition 4.5.

Factorization is not required for the main results (indeed, factorization in the quasicategorical context is equivalent to requiring all maps to be cofibrations).

Related results for quasicategories are the following. Barwick proved in [11 Proposition 2.10] in the special case that all maps are cofibrations that an exact functor \( G : \mathcal{A} \to \mathcal{B} \) between Waldhausen quasicategories is an equivalence if and only if it induces an equivalence of homotopy categories \( \text{ho} \mathcal{A} \to \text{ho} \mathcal{B} \). Similarly, Blumberg–Gepner–Tabuada conclude in [2 Corollary 4.10] that a map of stable quasicategories is an equivalence if and only if it induces an equivalence of homotopy categories (again all maps are considered cofibrations). In general, Blumberg–Gepner–Tabuada study \( K \)-theory in [2] as an invariant of stable quasicategories. In the present paper, we do not require stability.

Simplicial categories, on the other hand, are another model for \( \infty \) categories. Toën–Vezzosi [26] observed already in 2004 that the \( K \)-theory of a “good” category with fibrations and weak equivalences is an invariant of the underlying \( \infty \)-category, namely of its Dwyer–Kan hammock localization, despite the fact that the \( K \)-theory cannot be reconstructed from the triangulated homotopy category, as proved by Neeman [17]. Equivalence of localizations is in fact closely correlated to approximation: Blumberg–Mandell [4 Theorem 1.5] prove that equivalence of Dwyer–Kan localizations follows from Waldhausen’s approximation axioms and Cisinski [6] proves that this is actually an “if and

\[ A \] a quasicategory is stable if it admits finite limits and colimits and pushout and pullback squares coincide, see [16].
only if" statement (see Theorems 2.9 and 3.25, Proposition 4.5, and Scholie 4.15, all in [6]). Assuming Dwyer-Kan equivalence of Dwyer-Kan localizations of weak cofibration subcategories, and a few other hypotheses, a consequence of [4] is a stable equivalence of $K$-theory spectra, see Remark 4.13 for details. The present article remains entirely in the world of Waldhausen quasicategories.

**Outline of Paper.** In Section 2 I recall all the prerequisites from the theory of quasicategories as developed by Boardman–Vogt, Joyal, and Lurie, including homotopy, join, slice, and colimits. In Section 3 I recall the notion of Waldhausen quasicategory, discuss some of its consequences, and introduce the variants $S^\infty$ and $F^\infty$ of Waldhausen’s constructions using spines $I[n]$ rather than simplices $\Delta[n]$. Section 4 contains the proofs of various quasicategorical Approximation Theorems and the main technical result Proposition 4.5 and discusses overquasicategories for the quasicategorical version of Quillen’s Theorem A. Section 4 also contains a comparison with other results in the literature. Section 5 asks when an exact functor induces an equivalence of homotopy categories for higher $S^\infty_n$ iterates, and discusses homotopy of natural transformations in terms of lifting conditions.

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2. **Recollections on Quasicategories, Join, Slice, and Colimits**

Boardman and Vogt [5] originally defined the concept of quasicategory under the name weak Kan complex. Joyal [13] and Lurie [15], [16] have extensively developed the theory of quasicategories, and we review some of the fundamentals in this section.
2.1. Quasicategories and Functors. A quasicategory is a simplicial set $X$ in which every inner horn admits a filler. That is, for any $0 < k < n$ and any map $\Lambda^k[n] \to X$, there exists a map $\Delta[n] \to X$ such that the diagram

\[
\begin{array}{ccc}
\Lambda^k[n] & \xrightarrow{} & X \\
\downarrow & & \downarrow \exists \\
\Delta[n] & \xrightarrow{} & \end{array}
\]

commutes. A functor between quasicategories is merely a map of simplicial sets. The quasicategory of functors from a simplicial set $A$ to a quasicategory $X$ is the usual simplicial set $X^A$. The 1-simplices of $X^A$ are the natural transformations, that is, simplicial set maps $\alpha: A \times \Delta[1] \to X$. A functor $F: X \to Y$ between quasicategories is fully faithful if $F(a, b): X(a, b) \to Y(Fa, Fb)$ is a weak homotopy equivalence, while $F$ is essentially surjective if $\tau_1 F$ is essentially surjective. Here $X(a, b)$ is the mapping space recalled below, and $\tau_1: \SSet \to \Cat$ is the left adjoint to the nerve functor $N: \Cat \to \SSet$. A functor between quasicategories is called an equivalence if it is an equivalence in the 2-category $\SSet^{\tau_1}$ which has simplicial sets as its objects and $\SSet^{\tau_1}(A, B) := \tau_1(B^A)$. A functor between quasicategories is an equivalence if and only if it is fully faithful and essentially surjective \cite[Propositions 3.19 and 3.20]{approximation_for_quasicategories}.

A subset $A$ of a quasicategory $X$ is called full or 0-full if any simplex of $X$ is in $A$ if and only if all of its vertices are in $A$. Any 0-full subset of a quasicategory is a quasicategory. A quasicategory $W$ of a quasicategory $X$ is called 1-full if any simplex of $X$ is in $W$ if and only if all of its edges are in $W$.

2.2. Homotopy and Equivalence in a Quasicategory. Boardman and Vogt associated to any quasicategory $X$ its homotopy category $\text{ho}X$. A 1-simplex of a quasicategory $X$ is called a morphism. Two parallel morphisms $f, g: a \to b$ are left homotopic if there exists a 2-simplex $\sigma \in X_2$ with boundary $\partial \sigma = (d_0 \sigma, d_1 \sigma, d_2 \sigma) = (1_b, g, f)$. They are right homotopic if there exists a 2-simplex $\sigma \in X_2$ with boundary $\partial \sigma = (g, f, 1_a)$. They are homotopic if they are in the same path component of the mapping space $X(a, b)$, which is the following pullback.

\[
\begin{array}{ccc}
X(a, b) & \xrightarrow{} & X^\Delta[1] \\
\downarrow \text{pullback} & & \downarrow (s,t) \\
(a,b) & \xrightarrow{} & X \times X
\end{array}
\]
All three notions of homotopy coincide and are an equivalence relation on the morphisms of the quasicategory $X$. The 0-simplices of $X$ together with the homotopy classes of morphisms form the homotopy category $\text{ho}X$ [3]. This category is isomorphic to the fundamental category $\tau_1X$, so the categories $\text{ho}X$ and $\tau_1X$ are identified without further mention. Here $\tau_1$ is the left adjoint to the nerve functor.

A morphism in $X$ is an equivalence if its homotopy class is an isomorphism in $\text{ho}X$, which is the case if and only if the morphism has a “homotopy inverse” in $X$. We denote the 1-full subquasicategory of $X$ on the equivalences by $X_{\text{equiv}}$. This is the maximal Kan subcomplex of $X$ and $(-)_{\text{equiv}}$ is right adjoint to the inclusion $\text{Kan} \hookrightarrow \text{SSet}$ [13, Theorems 4.18 and 4.19]. If $X$ and $Y$ are quasicategories, then a natural transformation $\alpha : X \times \Delta[1] \to Y$ is an equivalence in the quasicategory $Y^X$ if and only if each component $\alpha_x = \alpha(x, -)$ is an equivalence in $Y$, see [13, Theorem 5.14] or [8, Corollary 3.7].

**Example 2.2** (Nerve of a category). The nerve of any category $C$ is a quasicategory. The homotopy class of a morphism $f$ in $NC$ is simply $\{f\}$. The mapping space $(NC)(a,b)$ is $C(a,b)$ viewed as a discrete simplicial set. Consequently, the homotopy category of $NC$ is just $C$, and a morphism in $NC$ is an equivalence if and only if it is an isomorphism in $C$. Both of these consequences also follow from the fact that $\tau_1N \cong \text{Id}_{\text{Cat}}$ (nerve is fully faithful). As expected, we now see as a special case of the above-mentioned [13, Theorem 5.14], the classical lemma that a natural transformation $C \times [1] \to D$ has components isomorphisms if and only if it is invertible in $\text{Cat}(C,D)$.

2.3. **Join and Slice.** Recall that if $A$ and $B$ are categories, then the join $A \star B$ is the category with objects and morphisms

\[
\text{Ob } A \star B := \text{Ob } A \sqcup \text{Ob } B
\]

\[
\text{Mor}(A \star B) := \text{Mor}(A) \sqcup \text{Mor}(B) \sqcup \left( \bigsqcup_{a \in \text{Ob } A, b \in \text{Ob } B} \{f_{a,b} : a \to b\} \right).
\]

For, $a, a' \in \text{Ob } A$ and $b, b' \in \text{Ob } B$, the composite $a' \to a \to b \to b'$ is the unique map $a' \to b'$. All other possible compositions are already defined. For example, the join of any $(m+1)$-element linearly ordered set with any $(n+1)$-element linearly ordered set is an $(m+n+2)$-element linearly ordered set, so $[m] \star [n] = [m+n+1]$. If 1 denotes the terminal category, then $A \star 1$ is $A$ with a terminal object adjoined, while $1 \star B$ is $B$ with an initial object adjoined.
For any simplicial sets $A$ and $B$, the join operation \cite{13} page 243 satisfies

\begin{equation}
(A \star B)_n = A_n \sqcup B_n \sqcup \left( \bigsqcup_{i+1+j=n} A_i \times B_j \right)
\end{equation}

by \cite{13} Proposition 3.2. The face and degeneracy maps can be understood from the case where $A$ and $B$ are nerves of categories. For this paper, we may take (2.3) as a definition. Clearly, $A \subseteq A \star B$ and $B \subseteq A \star B$, and there are two functors

$$A \star (-): \text{SSet} \longrightarrow (A \downarrow \text{SSet})$$

$$(-) \star B: \text{SSet} \longrightarrow (B \downarrow \text{SSet}).$$

These both admit a right adjoint called slice \cite{13} Proposition 3.12, denoted $a \backslash X$ and $X/b$ respectively for $a: A \rightarrow X$ and $b: B \rightarrow X$ in $\text{SSet}$. In particular, there are bi-natural bijections

$$\left( A \downarrow \text{SSet} \right) \begin{pmatrix}
A & A \\
\downarrow & \downarrow \\
A \star B & X
\end{pmatrix} \cong \text{SSet} (B, a \backslash X)$$

$$\left( B \downarrow \text{SSet} \right) \begin{pmatrix}
B & B \\
\downarrow & \downarrow \\
A \star B & X
\end{pmatrix} \cong \text{SSet} (A, X/b).$$

An $n$-simplex $\Delta [n] \rightarrow a \backslash X$ is a map $A \star \Delta [n] \rightarrow X$ which extends $a$ along $A \subseteq A \star \Delta [n]$, while an $n$-simplex $\Delta [n] \rightarrow X/b$ is a map $\Delta [n] \star B \rightarrow X$ which extends $b$ along $B \subseteq \Delta [n] \star B$.

When $X$ is a quasicategory, so are the slices $a \backslash X$ and $X/b$, see \cite{13} Corollary 3.20, page 256 for the case of $X/b$.

Join and slice of simplicial sets is compatible with join and slice of categories. The nerve functor sends the join of two categories to the join of their nerves \cite{13} Corollary 3.3. So for instance, we may prove $\Delta [m] \star \Delta [n] \cong \Delta [m + n + 1]$ by the sequence of isomorphisms

$$\Delta [m] \star \Delta [n] \cong N([m] \star [n]) \cong N([m + n + 1]) = \Delta [m + n + 1].$$

The nerve preserves the slice operations \cite{13} Proposition 3.13.

$$N(a \backslash C) \cong a \backslash N(C) \quad N(C/b) \cong N(C)/b$$
2.4. **Colimits in a Quasicategory.** Joyal defined the notion of colimit in a quasicategory $X$ using join and slice as follows [12, Definition 4.5], see also [13, page 159] and [15, pages 46-49]. An object $i$ in a quasicategory $X$ is *initial* if for every object $x$ in $X$ the map $X(i, x) \to \text{pt}$ is a weak homotopy equivalence. Any two initial objects of $X$ are equivalent, in the sense that there is an equivalence from one to the other. Clearly, this equivalence is even homotopically unique. If $A$ is a simplicial set and $a: A \to X$ is a map of simplicial sets, then a *cocone with base $a$* is a 0-simplex of $a \setminus X$. In other words, a cocone with base $a$ is a map $A \star 1 \to X$ which extends $a: A \to X$ along $A \subseteq A \star 1$. A *colimiting cocone for $a$* is an initial object of $a \setminus X$, in other words a cocone $\eta: A \star 1 \to X$ with base $a$ which is initial. A *colimit of $a$* is the value of a colimiting cocone $A \star 1 \to X$ at the unique vertex of the terminal simplicial set 1.

If $\mathcal{C}$ is a category, then all of these concepts in $\mathcal{N}\mathcal{C}$ coincide with the usual 1-category notions in $\mathcal{C}$ because nerve commutes with join and slice, nerve is fully faithful, and $(\mathcal{N}\mathcal{C})(a, b)$ is $\mathcal{C}(a, b)$ viewed as a discrete simplicial set.

The following proposition seems not to be in the literature. I thank David Gepner for sketching the main idea to me. This proposition will be used in Proposition 3.11, the equivalence of $S^\infty_n \mathcal{C}$ with $\mathcal{F}^\infty_n - 1 \mathcal{C}$.

**Proposition 2.4.** Let $A$ be a simplicial set, $X$ a quasicategory which admits $A$-shaped colimits, and $\text{colim}(X^{A \star 1})$ the subquasicategory of $X^{A \star 1}$ 0-full on the colimiting cocones. Then the restriction map

$$r: \text{colim}(X^{A \star 1}) \longrightarrow X^A$$

is an equivalence of quasicategories.

**Proof.** Since $A \hookrightarrow A \star 1$ is a monomorphism, and $X$ is a quasicategory, the map $X^{A \star 1} \to X^A$ is a pseudo fibration (take $Y = \ast$ in [13 Theorem 5.13]). Its fiber above $a: A \to X$ is isomorphic to the slice $a \setminus X$. Namely a $q$-simplex in the fiber over $a$ is a map $f: (A \star 1) \times \Delta[q] \to X$ such that the composite

$$A \times \Delta[q] \longrightarrow (A \star 1) \times \Delta[q] \xrightarrow{f} X$$

is the same as

$$A \times \Delta[q] \xrightarrow{\text{proj}} A \xrightarrow{a} X.$$
Such an \( f \) is the same as a map \( \overline{f}: A \ast \Delta[q] \to X \) such that

\[
\begin{array}{c}
A \\
\downarrow \downarrow \downarrow \downarrow \downarrow a
\end{array}
\begin{array}{c}
A \ast \Delta[q]
\end{array}
\begin{array}{c}
\overline{f}
\end{array}
\begin{array}{c}
X
\end{array}
\]

commutes. Given \( f \) for instance, \( \overline{f}: A \ast \Delta[q] \to X \) on the copy of \( A \) in (2.3) is \( f|_{A \times \Delta[q]} \), while \( \overline{f} \) on the copy of \( \Delta[q] \) in (2.3) is \( f|_{1 \times \Delta[q]} \). On the \( n \)-simplices in \( A_i \times \Delta([j],[q]) \), the function \( \overline{f}_n \) is defined via \( f_j \).

The restriction of the pseudo fibration \( X^{A*1} \to X^A \) to the 0-full subquasicategory on the colimiting cocones, which we denote \( r: \text{colim}(X^{A*1}) \to X^A \) is also a pseudo fibration by 0-fullness. From the isomorphism above, the fiber of \( r \) over a vertex \( a: A \to X \) in \( X^A \) is the quasicategory of colimiting cocones of \( a \), which is a contractible Kan complex [12, Proposition 4.6].

The map \( r \) is a mid-fibration (since it is a pseudo fibration). In fact, \( r \) is even a Kan fibration. Since the fibers of the Kan fibration \( r \) are contractible, it is a trivial fibration (see for instance [13, Proposition 8.23] for a proof of this classical result). To show \( r \) is a trivial fibration, it would suffice to even have \( r \) a left or a right fibration, since a left or right fibration is a trivial fibration if and only if its fibers are contractible by [13, Proposition 8.27].

But now \( r \) is an equivalence of quasicategories, since a pseudo fibration between quasicategories is a trivial fibration if and only if it is an equivalence of quasicategories [13, Theorem 5.15].

A different way to prove \( r \) is an equivalence is to show for each \( \alpha, \beta \in \text{colim}(X^{A*1}) \) the map of Kan complexes

\[
\text{colim}(X^{A*1})(\alpha, \beta) \longrightarrow X^A(\alpha|_A, \beta|_A)
\]

is a Kan fibration with contractible fibers. \( \square \)

### 3. Waldhausen Quasicategories

We next recall some background on Waldhausen quasicategories from the paper of Fiore–Lück [8], discuss consequences of the definition, and introduce a variant of Waldhausen’s \( F_n \mathcal{C} \). Recall that a subquasicategory \( X \) of a quasicategory \( Y \) is 1-full if any simplex of \( Y \) is in \( X \) if and only if all of its edges are in \( X \).

#### 3.1. Waldhausen Quasicategories and Exact Functors.

**Definition 3.1** (Waldhausen quasicategory, [8]). A **Waldhausen quasicategory** consists of a quasicategory \( \mathcal{C} \) together with a distinguished
zero object \(*\) and a subquasicategory \(coC\), the 1-simplices of which are called cofibrations and denoted \(\Rightarrow\), such that

(i) The subquasicategory \(coC\) is 1-full in \(C\) and contains all equivalences in \(C\),

(ii) For each object \(A\) of \(C\), every morphism \(* \to A\) is a cofibration,

(iii) The pushout of a cofibration along any morphism exists, and every pushout of a cofibration along any morphism is a cofibration.

Barwick's notion of Waldhausen \(\infty\)-category in [1, Definition 2.4] is equivalent to Definition 3.1, though he does not distinguish a zero object.

Though Definition 3.1 looks much like the classical definition, there are some important differences which have far-reaching consequences. Perhaps most prominently, a classical Waldhausen category comes equipped with a class of “weak equivalences” which contains the isomorphisms, whereas a Waldhausen quasicategory has its class of “weak equivalences” pre-selected as the equivalences of the underlying quasicategory. These are exactly the maps which become isomorphisms in the homotopy category.

Consequently, extra hypotheses on the equivalences (typically needed in a discussion of Approximation) are rarely needed because they automatically hold. For instance, the 3-for-2 property for equivalences in a quasicategory \(X\) follows immediately from the 3-for-2 property for isomorphisms in the homotopy category \(\tau_1X\) and the fact that 2-simplices in \(X\) give rise to commutative triangles in \(\tau_1X\). Even more strongly, the equivalences in a quasicategory \(X\) satisfy the 6-for-2 property\(^2\) by a similar argument applied to \(Sk^2\Delta[3] \to X\) using the 6-for-2 property of the isomorphisms in \(\tau_1X\) (the isomorphisms in any category satisfy the 6-for-2 property).

Another consequence of choosing the equivalences as the “weak equivalences” is that any map homotopic to a cofibration is also a cofibration, see [8, Section 4]. From the 1-fullness of \(coC\) in \(C\) in Definition 3.1 it follows that for any cofibrations \(f, g, h\) there is a 2-simplex in \(coC\) with boundary \((g, h, f)\) if and only if there is a 2-simplex in \(C\) with boundary \((g, h, f)\). We see that \(\tau_1(coC)\) is naturally a subcategory of \(\tau_1C\) which contains the isomorphisms of \(\tau_1C\).

\(^2\)A class of maps has the 6-for-2 property if whenever we have any three composable maps \(u \to v \to w\) with \(vu\) and \(uv\) in the class, we can conclude that \(u, v, w,\) and \(wvu\) are also in the class.
The choice of the “weak equivalences” as the equivalences also changes the way that classical hypotheses are related to one another. For instance, the factorization axiom in a Waldhausen quasicategory is equivalent to the requirement that every map is a cofibration. A Waldhausen quasicategory \( C \) is said to admit factorization if for any morphism \( f \) of \( C \) there exists a 2-simplex \( \sigma \) with boundary
\[
\partial \sigma = (d_0 \sigma, d_1 \sigma, d_2 \sigma) = (\text{equivalence}, f, \text{cofibration}).
\]
As a proof, if \( C \) admits factorization, and \( f \) is any morphism in \( C \), then \([f] = [w][c]\) in the homotopy category for some equivalence \( w \) and some cofibration \( c \). But every equivalence is a cofibration, and \( \tau_1(coC) \) is naturally a subcategory of \( \tau_1 C \), so \([f]\) is the homotopy class of a cofibration. But any map homotopic to a cofibration is a cofibration, so \( f \) is a cofibration. The converse is clear.

Every “weak cofibration” in a Waldhausen quasicategory is actually a cofibration, so we make no distinction. More precisely, in [3] and [4], Blumberg–Mandell call a morphism \( f \) in a classical Waldhausen category a weak cofibration if there is a zig-zag of weak equivalences in the arrow category from \( f \) to a cofibration. If we have a commutative square in a Waldhausen quasicategory \( C \) (that is a map \( \Delta[1] \times \Delta[1] \to C \) (3.2)
\[
\begin{array}{ccc}
x & \xrightarrow{r} & y \\
\downarrow^u & & \downarrow^v \\
x' & \xrightarrow{r'} & y'
\end{array}
\]
then in the homotopy category \( \tau_1 C \) we have
\[
[r] = [v^{-1}][r'][u] \quad \text{and} \quad [r'] = [v][r][u^{-1}],
\]
so \([r]\) is in \( \tau_1(coC) \) if and only if \([r']\) is in \( \tau_1(coC) \) (recall that \( \tau_1(coC) \) contains the isomorphisms of \( \tau_1 C \)). But the homotopy class \([r]\) is in \( \tau_1(coC) \) if and only if the morphism \( r \) is in \( coC \), and similarly for \([r']\) and \( r' \). So in any vertical zig-zag of commutative squares like (3.2) in a Waldhausen quasicategory, with all the vertical arrows weak equivalences, if any one of the horizontal arrows is a cofibration, then all of the horizontal arrows are cofibrations, and weak cofibrations are cofibrations.

Similarly, a homotopy cocartesian square in the sense of [3] and [4], but in a Waldhausen quasicategory, is merely a pushout square in which one of the legs is a cofibration (the opposite morphism will then also be a cofibration). This follows from the fact that pushouts in a quasicategory are invariant under equivalence.
**Definition 3.3** (Exact functor). Let \( \mathcal{C} \) and \( \mathcal{D} \) be Waldhausen quasicategories. A functor \( f : \mathcal{C} \to \mathcal{D} \) is called *exact* if \( f \) maps the distinguished zero object to the distinguished zero object, \( f \) maps each cofibration to a cofibration, and \( f \) maps each pushout square along a cofibration to a pushout square along a cofibration.

By 1-fullness of the subquasicategory of cofibrations, we have \( f(\text{co}\mathcal{C}) \subseteq \text{co}\mathcal{D}. \) Every map of quasicategories sends equivalences to equivalences, so we also have \( f(\mathcal{C}_{\text{equiv}}) \subseteq \mathcal{D}_{\text{equiv}} \) by 1-fullness of the maximal sub Kan complex. By the comments preceding the definition, an exact functor between Waldhausen quasicategories preserves weak cofibrations and homotopy cocartesian squares.

### 3.2. The \( S^\infty \) Construction

We next recall the \( S^\infty \) construction. See also [16, 1.2.2.2 and 1.2.2.5] of Lurie, and [2, Section 6.1] where Blumberg–Gepner–Tabuada compare \( S^\infty \) with the \( S' \) construction of [3] and [4] in the case of a simplicial model category which admits all finite homotopy colimits. See also Fiore–Lück [8]. For a fibrational version of \( S^\infty \), see Barwick [1, Section 5].

Let \( \text{Ar}[n] \) be the *category of arrows* in \([n]\). It is the partially ordered set with elements \((i, j)\) such that \(0 \leq i \leq j \leq n\), and with the order \((i, j) \leq (i', j')\) whenever \( i \leq i' \) and \( j \leq j' \). The category \( \text{Ar}[n] \) is an upper-triangular subgrid of an \( n \times n \)-grid of squares.

**Definition 3.4** (*\( S^\infty \) Construction*). Let \( \mathcal{C} \) be a Waldhausen quasicategory. An \([n]\)-*complex* is a map of simplicial sets \( A : N\text{Ar}[n] \to \mathcal{C} \) such that

(i) For each \( i \in [n] \), \( A(i, i) \) is the distinguished zero object of \( \mathcal{C} \),

(ii) For each \( i \leq j \leq k \), the morphism \( A(i, j) \to A(i, k) \) is a cofibration.

---

In [2, Corollary 7.7], Blumberg–Gepner–Tabuada consider a simplicial model category \( \mathcal{A} \) and a small full subcategory \( \mathcal{C} \) which has all finite homotopy colimits. They apply the \( S^\infty \) construction of Lurie to the simplicial nerve of the full sub-simplicial category on the fibrant-cofibrant objects and compare it with the nerve of \( wS^\infty \). Blumberg–Gepner–Tabuada primarily work with Waldhausen quasicategories in which all maps are cofibrations, so they do not usually explicitly mention cofibrations. For instance, every map in a model category is a weak cofibration, and in a Waldhausen quasicategory weak cofibrations are the same as cofibrations, so in [2, Corollary 7.7] there is no need to specify that the horizontal maps in the \( S^\infty \) construction applied to a model category are cofibrations.
(iii) For each $i \leq j \leq k$, the diagram

$$
\begin{array}{c}
A(i, j) \longrightarrow A(i, k) \\
\downarrow \quad \downarrow \\
A(j, j) \longrightarrow A(j, k)
\end{array}
$$

is a pushout square in $\mathcal{C}$.

Let $\text{Gap}([n], \mathcal{C})$ be the full sub-simplicial set of $\text{Map}(N\text{Ar}[n], \mathcal{C})$ with vertices the $[n]$-complexes. The $S^\infty_\bullet$ construction of $\mathcal{C}$ is the simplicial quasicategory $S^\infty_\bullet \mathcal{C}$ defined by

$$
S^\infty_n \mathcal{C} = \text{Gap}([n], \mathcal{C}) \subseteq \text{SSet}(N(\text{Ar}[n]) \times \Delta[-], \mathcal{C})
$$

The objects of $S^\infty_n \mathcal{C}$ are sequences of cofibrations in $\mathcal{C}$

$$
(*) : A_{0,1} \longrightarrow A_{0,2} \longrightarrow \cdots \longrightarrow A_{0,n}
$$

with a choice of quotient $A_{i,j} = A_{0,j}/A_{0,i}$ for each $i \leq j$, and a choice of all composites with simplices that fill them. The face and degeneracy maps of the simplicial object $S^\infty_\bullet \mathcal{C}$ in the category $\text{QCat}$ are induced from $\Delta$ via the category of arrows construction.

Each quasicategory $S^\infty_n \mathcal{C}$ is a Waldhausen quasicategory. A morphism $f : A \rightarrow B$ in $S^\infty_n \mathcal{C}$ is a cofibration in $S^\infty_n \mathcal{C}$ if each $f_{0,j}$ is a cofibration and each pushout morphism

$$
A_{0,j} \cup_{A_{0,j-1}} B_{0,j-1} \rightarrow B_{0,j}
$$

is a cofibration in $\mathcal{C}$ for each $1 \leq j \leq n$. If a morphism is levelwise homotopic to a cofibration in $S^\infty_n \mathcal{C}$, then it is also a cofibration in $S^\infty_n \mathcal{C}$. A natural transformation $f : A \rightarrow B$ is an equivalence in $S^\infty_n \mathcal{C}$ if and only if each component $f_{i,j}$ is an equivalence in $\mathcal{C}$ by [13, Theorem 5.14] or [8, Corollary 3.7]. Because all the squares in an object are pushouts, this is equivalent to requiring every $f_{0,j} : A_{0,j} \rightarrow B_{0,j}$ to be an equivalence in $\mathcal{C}$ (the pushout of an equivalence in a quasicategory along any morphism is also an equivalence, see [8, Lemma 3.23]). See [20, 8.3.15] for the case of Waldhausen categories.

**Definition 3.6.** The $n$-th $K$-theory space is

$$
K(\mathcal{C})_n := \left| (S^\infty_\bullet \cdots S^\infty_\bullet \mathcal{C})_{\text{equiv}} \right|
$$

for $n \geq 0$, where $S^\infty_\bullet$ appears $n$ times.

The $K$-theory spaces form an $\Omega$-spectrum beyond the 0-th term.
3.3. The $S^\infty$ Construction, or the $S^\infty$ Construction without Composites. An important difference between categories and quasi-categories motivates our variant of the classical definition: a sequence of $n$ morphisms in a category already has uniquely determined composites of all subsequences and their composites, whereas in a quasi-category such composites for a sequence exist but are not chosen. For a category $\mathcal{D}$ for instance, if $I[n]$ denotes the spine of the $n$-simplex $\Delta[n]$, namely the 1-dimensional subcomplex of $\Delta[n]$ given by the union of the edges $(i - 1, i)$ for $1 \leq i \leq n$, then we have

$$\text{SSet}(I[n], N\mathcal{D}) \cong \text{Cat}(\tau_1 I[n], \mathcal{D}) \cong \text{Cat}(\tau_1 \Delta[n], \mathcal{D}) \cong \text{SSet}(\Delta[n], N\mathcal{D}).$$

In other words, indicating a sequence of $n$ morphisms in a category is tantamount to indicating all of the composites of subsequences (this is one reason why it is customary in category theory to only draw the $n$ morphisms to indicate the $n$ morphisms along with all of their composites).

To see how such composites of subsequences exist in a quasicategory $X$, notice that a map $I[n] \to X$ is a sequence of $n$ morphisms in $X$, while a map $\Delta[n] \to X$ is a sequence of $n$ morphisms in $X$ together with a choice of composites for all possible subsequences and their composites, together with simplices which make them commute. The spine inclusion $I[n] \hookrightarrow \Delta[n]$ is mid anodyne [13, Proposition 2.13] so that the following lift exists.

Nevertheless, there is little theoretical difference in considering maps $I[n] \to X$ versus maps $\Delta[n] \to X$ because for any quasicategory $X$, the spine inclusion induces an equivalence $X^{\Delta[n]} \to X^{I[n]}$ by [13] Propositions 2.27 and 2.29. Practically, it is easier to work with $I[n]$, and this is what we do now for $S^\infty$ and $F^\infty$ in the next Section.

**Definition 3.7 ($S^\infty$ Construction).** Let $N\text{Ar}[n]$ be the intersection of $N\text{Ar}[n]$ with the 2-dimensional simplicial set $I[n] \times I[n]$. A restricted $[n]$-complex in a Waldhausen quasicategory $\mathcal{C}$ is a map of simplicial sets $A: N\text{Ar}[n] \to \mathcal{C}$ such that

- (i) For each $i \in [n]$, $A(i, i)$ is the distinguished zero object of $\mathcal{C}$,
- (ii) For each $i \leq j$, the morphism $A(i, j) \to A(i, j + 1)$ is a cofibration,
(iii) For each \(i \leq j\), the diagram

\[
\begin{array}{ccc}
A(i, j) & \longrightarrow & A(i, j + 1) \\
\downarrow & & \downarrow \\
A(i + 1, j) & \longrightarrow & A(i + 1, j + 1)
\end{array}
\]

is a pushout square in \(\mathcal{C}\).

Then \(S_n^\infty\mathcal{C}\) is the full sub-simplicial set of \(\text{Map}(\overline{\text{NAr}}[n], \mathcal{C})\) on the restricted \([n]\)-complexes. The cofibrations in \(S_n^\infty\mathcal{C}\) are defined just as in \(S_n\mathcal{C}\).

Note that the \(S_n^\infty\) construction is not a simplicial object.

**Proposition 3.8.** For any Waldhausen quasicategory \(\mathcal{C}\), the forgetful functor \(S_n^\infty\mathcal{C} \to S_n^\infty\mathcal{C}\) induced by the inclusion \(\overline{\text{NAr}}[n] \hookrightarrow \text{NAr}[n]\) is a Waldhausen equivalence, that is, it is an exact functor which admits an exact pseudo inverse (see [8, Section 6] for more on Waldhausen equivalences).

**Proof.** Every mid anodyne map is a weak categorical equivalence [13, Corollary 2.29], and the Cartesian product of two weak categorical equivalences is a weak categorical equivalence [13, Proposition 2.28], so \(I[n] \times I[n] \hookrightarrow \Delta[n] \times \Delta[n]\) is a weak categorical equivalence. Then the induced map \(\mathcal{C}^{\Delta[n] \times \Delta[n]} \to \mathcal{C}^{I[n] \times I[n]}\) is an equivalence of quasicategories by [13, Proposition 2.27]. Restricting to full subquasicategories, we have an equivalence \(\mathcal{C}^{\overline{\text{NAr}}[n]} \to \mathcal{C}^{\text{NAr}[n]}\). Restricting further still, we have the equivalence \(S_n^\infty\mathcal{C} \to S_n^\infty\mathcal{C}\). This equivalence is exact and reflects cofibrations, so it is a Waldhausen equivalence by [8, Proposition 6.9].

For later use, we present the following Lemma, which relies on the use of \(I[n]\) over \(\Delta[n]\).

**Lemma 3.9** (\(\tau_1(X^{I[n]}\]) versus \(nX\)). Let \(X\) be a quasicategory, and let \(nX\) denote the category with objects sequences of \(n\) morphisms of \(X\) and morphisms the diagrams of the form

\[
\begin{array}{cccccc}
A_0 & \longrightarrow & A_1 & \longrightarrow & \cdots & \longrightarrow & A_n \\
\downarrow^{[g_0]} & & \downarrow^{[g_1]} & & \cdots & & \downarrow^{[g_n]} \\
B_0 & \longrightarrow & B_1 & \longrightarrow & \cdots & \longrightarrow & B_n
\end{array}
\]
such that \([g_i]\) is a morphism in \(\tau_1X\) with the indicated source and target, and such that each square commutes in \(\tau_1X\). Consider the identity-on-objects functor

\[\varphi: \tau_1(X^{I[n]}) \longrightarrow nX\]

which sends a morphism \([\alpha]\) to the morphism as in (3.10) consisting of the homotopy classes of its components \(([\alpha_i])_{1 \leq i \leq n}\). Then

(i) The functor \(\varphi\) is full.

(ii) The functor \(\varphi\) is faithful if and only if any two natural transformations \(\alpha: I[n] \times \Delta[1] \rightarrow X\) with homotopic components are homotopic.

Proof. The map \(\varphi\) is well defined, because if \(\alpha, \beta: I[n] \times \Delta[1] \rightarrow X\) are left homotopic via \(\sigma: I[n] \times \Delta[2] \rightarrow X\), then \(\sigma(i, -)\) is a left homotopy for \(\alpha(i, -)\) and \(\beta(i, -)\).

For fullness in claim \(\text{(i)}\), to define \(\alpha: I[n] \times \Delta[1] \rightarrow X\) from a morphism \(([g_i])_{1 \leq i \leq n}\) as in (3.10), it suffices to define \(\alpha\) on the 2-skeleton of \(I[n] \times \Delta[1]\) because \(I[n] \times \Delta[1]\) has dimension 2 (unlike \(\Delta[n] \times \Delta[1]\)). Since square \(i\) of (3.10) commutes in the homotopy category, there exist two 2-simplices of \(X\) as below.

\[
\begin{array}{ccc}
A_{i-1} & \xrightarrow{c_i} & A_i \\
| & \searrow & \searrow \\
B_{i-1} & \xrightarrow{d_i} & B_i
\end{array}
\]

This is the definition of \(\alpha\) on the 2-skeleton \((i, i - 1) \times \Delta[1]\). Gluing these, we have \(\alpha\) on the 2-skeleton of \(I[n] \times \Delta[1]\), which we then extend to all of \(I[n] \times \Delta[1]\).

Claim \(\text{(ii)}\) is clear. \(\square\)

3.4. The \(F_\infty\) Construction. We next define the quasicategory \(F_\infty^\infty C\) of sequences of \(n\)-many cofibrations. Let \(C\) be a Waldhausen quasicategory. The quasicategory \(F_\infty^\infty C\) is the 0-full subsimplicial set of \(C^{I[n]}\) on the functors \(I[n] \rightarrow coC\). Strictly speaking, we should denote this as \(\overline{F_\infty^\infty C}\) to be consistent with \(\overline{S_\infty^\infty C}\), but we refrain from the underline on \(F\) for readability. For a fibrational version of an \(F_\infty^\infty\), see Barwick [1, Section 5].

As in the classical work of Waldhausen [27], pages 324 and 328], the quasicategory \(F_\infty^\infty C\) is a Waldhausen quasicategory, the cofibrations in \(F_\infty^\infty C\) are levelwise cofibrations with the additional property that the map from each pushout to each lower corner is a cofibration in \(C\).
Proposition 3.11. The forgetful functor $S^\infty_n \mathcal{C} \to \mathcal{F}^\infty_n \mathcal{C} \to \mathcal{F}^\infty_{n-1} \mathcal{C}$ is a Waldhausen equivalence of Waldhausen quasicategories.

Proof. This forgetful functor factors as the composite $S^\infty_n \mathcal{C} \to S^\infty_n \mathcal{C} \to \mathcal{F}^\infty_n \mathcal{C} \to \mathcal{F}^\infty_{n-1} \mathcal{C}$, the first map of which is a Waldhausen equivalence by Proposition 3.8. An inductive application of Proposition 2.4, forgetting one row at a time, shows that the second map is also an equivalence. Each of these intermediate functors is exact, reflects cofibrations, and is an equivalence, so is a Waldhausen equivalence by [8, Proposition 6.9]. For instance, to apply Proposition 2.4, we consider row 1 and row 2 of a restricted $[n]$-complex as part of a colimiting cocone consisting of rows 1 and 2, a row of $A_1 = A_1 = \cdots = A_1$, and a row of $* = * = \cdots = *$, so that we have a row of pushout squares. □

Proposition 3.11 and the earlier results it references, provides some of the details to [2, Lemma 7.3] of Blumberg–Gepner–Tabuada.

The equivalence of quasicategories $S^\infty_n \mathcal{C} \to \mathcal{F}^\infty_n \mathcal{C}$ in Proposition 3.11 induces an equivalence of homotopy categories since $\tau_1$ is a 2-functor [13, Proposition 1.27].

4. Approximation Theorems for Waldhausen Quasicategories

We now turn to the proof of various Approximation Theorems for Waldhausen quasicategories, namely Theorems 4.10 and 4.12, Corollaries 4.16 and 4.18. The proof uses a quasicategorical version of Quillen’s Theorem A. The main technical step is Proposition 4.5 which states that a functor of quasicategories $F : \mathcal{A} \to \mathcal{B}$ induces a weak homotopy equivalence of maximal Kan complexes if it is essentially surjective, $\mathcal{A}$ admits colimits of diagrams in its maximal Kan complex indexed by finite posets, $F$ preserves them, and $F$ reflects equivalences. This main technical step does not require Waldhausen structures.

Definition 4.1 (Overquasicategory $(Y \downarrow y)$). Let $y$ be an object of a quasicategory $Y$. An $n$-simplex over $y$ is an $(n+1)$-simplex $z \in Y_{n+1}$ such that $\iota_{n+1}(z) = y$, where $\iota_{n+1} : [0] \to [n+1]$ is $\iota_{n+1}(0) = n+1$. The simplices over $y$ form the overquasicategory $(Y \downarrow y)$. The projection

$$q : (Y \downarrow y) \longrightarrow Y$$

is $z \mapsto d_{n+1}z$.

The overquasicategory $(Y \downarrow y)$ is isomorphic to the slice $Y/y$ in Joyal’s notation since a map $z : \Delta[n] \times \Delta[0] \to Y$ which extends $y : \Delta[0] \to Y$ is the same as a map $z : \Delta[n+1] \to Y$ with $z \circ \Delta[\iota_{n+1}] = y$. The simplicial set $Y \downarrow y$ is a quasicategory when $Y$ is by [13] Corollary 3.20,
page 256]. Joyal denotes \((Y \downarrow y)\) by \(Y/y\) [13 page 249], while Lurie writes \(Y/y\), see [15 pages 42-43].

The nerve preserves the slice operations, so if \(C\) is a category, then \(N(C \downarrow c) = (NC \downarrow c)\), see [13 Proposition 3.13, page 250].

**Definition 4.2** (Over simplicial set \((G \downarrow y)\)). Let \(X\) be a simplicial set, \(y\) an object of a quasicategory \(Y\), and \(G : X \to Y\) a map of simplicial sets. An \(n\)-simplex \(G\)-over \(y\) is a pair \((x \in X_n, z \in (Y \downarrow y)_n)\) such that \(G(x) = q(z)\). The simplices \(G\)-over \(y\) form the over simplicial set \((G \downarrow y)\), it is the pullback

\[
\begin{array}{ccc}
(G \downarrow y) & \longrightarrow & X \\
\downarrow & & \downarrow G \\
(Y \downarrow y) & \longrightarrow & Y,
\end{array}
\]

In the notation of Lurie, \((G \downarrow y)\) would be \(X \times_Y Y/y\), see [15 page 236]. The simplicial set \((G \downarrow y)\) is like a homotopy fiber. The left fiber \(G/(0, y)\) on page 337 of Waldhausen [27], which is the pullback of \(G : X \to Y\) along \(y : \Delta[0] \to Y\), is like a strict fiber.

**Theorem 4.3** (Quasicategorical Quillen Theorem A, [11]). Let \(Y\) be a quasicategory, \(X\) a simplicial set, and \(G : X \to Y\) a map of simplicial sets. If the over simplicial set \((G \downarrow y)\) is weakly contractible for every object \(y\) of \(Y\), then \(G\) is a weak homotopy equivalence.

**Proof.** For a proof of the analogous version for \((y \downarrow G)\), see Proposition 4.1.1.3.(3) and Theorem 4.1.3. of Lurie on pages 223 and 236 of [15]. \(\square\)

The next lemma is a simplicial version of [23 Lemma 14 on page 131] by Schlichting, which he extracted from Waldhausen’s paper [27, pages 354–356]. This proof is an adaptation of Schlichting’s extraction to the situation of \(C\) a simplicial set (instead of a category), with a few more details.

**Lemma 4.4.** Let \(C\) be a nonempty simplicial set. If for every finite poset \(P\), every map \(NP \to C\) can be extended to a map \((NP) \star 1 \to C\), then \(C\) is weakly contractible.

**Proof.** For \(n = 0\), we have \(\pi_0|C| = *\), since if \(a, b \in C_0\) the map \(\{a, b\} \hookrightarrow C\) extends to \(N\{a \to 1 \leftarrow b\} \to C\).

Let \(n \geq 1\), and let \(\text{Ex}^\infty : \text{SSet} \to \text{SSet}\) be the fibrant replacement functor of Kan. Fix a vertex \(c\) of \(C\), and let

\[
[|\alpha|] \in \pi_n(|C|, c) = [S^n, |C|]_{\text{based}} \cong \left[ S^n_{\text{simp}}, \text{Ex}^\infty C \right]_{\text{based}},
\]

where \([S^n_{\text{simp}}, \text{Ex}^\infty C]_{\text{based}}\) is a weak equivalence.
where $S^n_{\text{simp}}$ is any simplicial model of the $n$-sphere with only finitely many non-degenerate simplices. Since $\text{SSet}(S^n_{\text{simp}}, -)$ commutes with directed colimits and $\text{Ex}^\infty C$ is the colimit of the directed diagram of weak equivalences

$$
\mathcal{C} \xrightarrow{\text{Ex}} \mathcal{C} \xrightarrow{\text{Ex}^2} \mathcal{C} \xrightarrow{\text{Ex}^3} \mathcal{C} \xrightarrow{} \cdots
$$

(see [14, Section 4] or [10 page 188]), the map $\alpha: S^n_{\text{simp}} \to \text{Ex}^\infty \mathcal{C}$ factors through some $\text{Ex}^k \mathcal{C}$. More concretely, for each non-degenerate simplex $e$ of $S^n_{\text{simp}}$, pick an index $\ell(e)$ such that $\alpha(e)$ is in the image of $\text{Ex}^{\ell(e)} \mathcal{C}$ in $\text{Ex}^\infty \mathcal{C}$, and then let $k = 2 + \max_e \ell(e)$. The barycentric subdivision $\text{Sd}: \text{SSet} \to \text{SSet}$ is left adjoint to $\text{Ex}$ (see [14, Section 7] or [10 page 183] or [9, 4.6]), so $\alpha$ corresponds to a map $\beta: \text{Sd}^k S^n_{\text{simp}} \to \mathcal{C}$. We have now proved that any homotopy class in $\pi_n(|\mathcal{C}|, c)$ can be represented by the geometric realization of a map $\beta: \text{Sd}^k S^n_{\text{simp}} \to \mathcal{C}$ with $k \geq 2$.

Let $\beta: \text{Sd}^k S^n_{\text{simp}} \to \mathcal{C}$ with $k \geq 2$. Since $k \geq 2$, the finite simplicial set $\text{Sd}^k S^n_{\text{simp}}$ is the nerve of a finite poset (see [24]), and $\beta$ extends to $\overline{\beta}: (\text{Sd}^k S^n_{\text{simp}}) \ast 1 \to \mathcal{C}$ by hypotheses. The space $|\overline{\beta}|(\text{Sd}^k S^n_{\text{simp}}) \ast 1|$ is contractible, as it is the realization of the nerve of a category with a terminal object. So $|\beta|$ is nullhomotopic, as is its restriction $|\beta|$. The space $|\mathcal{C}|$ is path connected, so $|\beta|: S^n \to |\mathcal{C}|$ is homotopic to the constant map $c$ along a basepoint preserving homotopy, and $\pi_n(|\mathcal{C}|, c)$ is trivial. □

We now turn to the main technical result in this paper, which has variants of Waldhausen Approximation among its consequences. The idea goes back to Waldhausen in the classical context of Approximation, namely use Lemma 4.4 and Quillen’s Theorem A. However, the implementation here is different. A key ingredient comes from Schlichting’s appendix [23, page 132], where he proves Approximation for Waldhausen categories, assuming factorizations in place of a cylinder functor. There, Schlichting takes a colimit of a (cofibrant replacement of a) $\mathcal{P}$-shaped diagram, and applies the 3-for-2 property to an appropriate commutative triangle. We proceed in this way (without cofibrant replacement) for a similar triangle in (4.7).

We do not assume factorizations, nor a cylinder functor, nor even a Waldhausen structure for the technical result.

**Proposition 4.5.** Let $F: \mathcal{A} \to \mathcal{B}$ be a functor between quasicategories (no Waldhausen structure is assumed). Suppose:

(i) The functor $F$ is essentially surjective.
(ii) The quasicategory \( \mathcal{A} \) admits all colimits of diagrams in \( \mathcal{A}_{\text{equiv}} \) indexed by finite (not necessarily connected) posets, and \( F \) preserves such colimits. Note that \( \mathcal{A}_{\text{equiv}} \) itself is not assumed to admit such colimits.

(iii) \( F \) reflects equivalences.

Then \( F_{\text{equiv}} : \mathcal{A}_{\text{equiv}} \rightarrow \mathcal{B}_{\text{equiv}} \) is a weak homotopy equivalence.

Proof. Let \( Y \) be the subquasicategory of \( \mathcal{B}_{\text{equiv}} \) which is 0-full on the vertices in the (strict) image of \( F_{\text{equiv}} : \mathcal{A}_{\text{equiv}} \rightarrow \mathcal{B}_{\text{equiv}} \). The inclusion \( Y \hookrightarrow \mathcal{B}_{\text{equiv}} \) is fully faithful, as \( Y(b, b') = \mathcal{B}_{\text{equiv}}(b, b') \). This inclusion is also essentially surjective by hypothesis. Hence \( Y \hookrightarrow \mathcal{B}_{\text{equiv}} \) is a weak homotopy equivalence.

Let \( G : \mathcal{A}_{\text{equiv}} \rightarrow Y \) be the functor \( F_{\text{equiv}} \) with restricted codomain \( Y \). We show that \( G \) is a weak homotopy equivalence using the quasicategorical version of Quillen’s Theorem A recalled above in Theorem 4.3.

Let \( y \in Y_0 \). Then the over simplicial set \( (G \downarrow y) \) (see Definition 4.2) is non-empty by definition of \( Y \). Let \( \mathcal{P} \) be a nonempty finite poset and consider a map of simplicial sets \( \Phi : N\mathcal{P} \rightarrow (G \downarrow y) \). This corresponds to two maps \( p : N\mathcal{P} \rightarrow (Y \downarrow y) \) and \( r : N\mathcal{P} \rightarrow \mathcal{A}_{\text{equiv}} \) such that \( q \circ p = G \circ r \) and the two corresponding triangles commute.

![Diagram](4.6)

We show that \( \Phi \) extends to \( (N\mathcal{P}) \star 1 \) by extending \( p \) and \( r \) to \( \overline{p} \) and \( \overline{r} \) on \((N\mathcal{P}) \star 1\) in such a way that \( G \circ \overline{r} = q \circ \overline{p} \). The method is to extend \( r \) to \( \overline{r} \) with codomain \( \mathcal{A} \), then show that its image actually is in \( \mathcal{A}_{\text{equiv}} \). To do this, we use an equivalence \( \alpha \) between two colimiting cocones of \( F \circ r \), namely between \( F \circ \overline{r} \) and the transpose \( p^! \). This equivalence will induce \( \overline{p} \).

The map \( r \) extends in \( \mathcal{A} \) to a colimiting cocone \( \overline{r} : (N\mathcal{P}) \star 1 \rightarrow \mathcal{A} \), as \( \mathcal{A} \) admits all colimits of diagrams in \( \mathcal{A}_{\text{equiv}} \) indexed by finite posets. Moreover, \( F \circ \overline{r} \) is also a colimiting cocone in \( \mathcal{B} \) for \( F \circ r \), because \( F \) preserves such colimiting cocones.
On the other hand, $F \circ r$ admits another colimiting cocone in $\mathcal{B}$, namely the transpose $p^\dagger : ((N\mathcal{P}) \star 1 \to Y) \subseteq \mathcal{B}_{\text{equiv}}$, which has $p^\dagger(t) = y$. The map $p^\dagger$ is a colimiting cocone of $p^\dagger|_{N\mathcal{P}} = p \circ q = F \circ r$ in $\mathcal{B}$ because all 1-morphisms in the cocone $p^\dagger$ are equivalences. For the slice quasicategory $(F \circ r) \backslash \mathcal{B}$ recalled in Section 2 and its 0-simplices $F \circ \tau$ and $p^\dagger$, the mapping space in equation (2.1) $(F \circ r) \backslash \mathcal{B} (F \circ \tau, p^\dagger)$ is weakly contractible, as $F \circ \tau$ is a colimiting cocone. Thus there is a (homotopically unique) 1-simplex $\alpha : [1] \to (F \circ r) \backslash \mathcal{B}$ with $d_1 \alpha = F \circ \tau$ and $d_0 \alpha = p^\dagger$. Not only that, $\alpha$ must be an equivalence in the quasicategory $(F \circ r) \backslash \mathcal{B}$ by the uniqueness of colimits. By an adjunction in Section 2, the equivalence $\alpha$ corresponds to a map $N\mathcal{P} \star [1] \to \mathcal{B}$ which extends $F \circ r$ along $N\mathcal{P} \subseteq N\mathcal{P} \star [1]$. We denote this map by $\alpha^\dagger : N\mathcal{P} \star [1] \to \mathcal{B}$.

Let $x \in \mathcal{P}$. Then the 2-simplex $\{x\} \star [1] \subseteq N\mathcal{P} \star [1]$ is mapped by $\alpha^\dagger$ to a 2-simplex in $\mathcal{B}$ with boundary

\[
\begin{array}{ccc}
\alpha^\dagger(x) & \xrightarrow{p^\dagger(x < t)} & F \circ \tau(x < t)\\
& \downarrow & \downarrow & \downarrow \\
F \circ \tau(t) & \xrightarrow{\alpha(0 < 1)} & y = p^\dagger(t),
\end{array}
\]

where $\alpha^\dagger(x) = F \circ r(x)$. Classically, this boundary would simply be the commutative triangle arising from the isomorphism of two colimits of a diagram $F \circ r$ arising from the universal property. The right diagonal arrow is an equivalence because $p^\dagger((N\mathcal{P}) \star 1) \subseteq \mathcal{B}_{\text{equiv}}$, while the bottom horizontal arrow is an equivalence because $\alpha$ is. By the 3-for-2 property of equivalences in $\mathcal{B}$, we have that the left vertical arrow $F \circ \tau(x < t)$ is also an equivalence. This holds for all $x \in \mathcal{P}$, and from diagram (4.6) we also have

\[
F \circ r(N\mathcal{P}) \subseteq \mathcal{B}_{\text{equiv}},
\]

so $F \circ \tau$ maps all 1-morphisms of $(N\mathcal{P}) \star 1$ to $\mathcal{B}_{\text{equiv}}$, and we have

\[
F \circ \tau((N\mathcal{P}) \star 1) \subseteq \mathcal{B}_{\text{equiv}}
\]

(recall that $\mathcal{B}_{\text{equiv}}$ is 1-full in $\mathcal{B}$ by [13, Lemma 4.18], so it suffices to check 1-morphisms to determine containment of the image in $\mathcal{B}_{\text{equiv}}$).
By hypothesis, $F$ reflects equivalences, so equation (4.8) implies
\[ \tau\left((NP) \ast 1\right) \subseteq A_{\text{equiv}}, \] again by 1-fullness of $A_{\text{equiv}}$ in $A$. Consequently, $\tau$ goes to $A_{\text{equiv}}$. Also, $\alpha^\dagger$ goes to $B_{\text{equiv}}$.

Writing $t_0$ and $t_1$ for 0 and 1, and making the identification $NP \ast \Delta[1] \cong (NP) \ast \{t_0\} \ast \{t_1\}$, we now have
\[ \alpha^\dagger: (NP) \ast \{t_0\} \ast \{t_1\} \to B_{\text{equiv}} \]
and
\[ \alpha^\dagger|_{(NP) \ast \{t_0\}} = G \circ \tau, \quad \alpha^\dagger|_{(NP) \ast \{t_1\}} = p^\dagger. \]

By $\alpha^\dagger(t_1) = p^\dagger(t_1) = y$ and (4.9), $\alpha^\dagger$ corresponds via adjunction to a map $\rho: (NP) \ast \{t_0\} \to (Y \downarrow y)$ which satisfies $q \circ \rho = G \circ \tau$. We have now constructed the desired extension $\omega: (NP) \ast 1 \to (G \downarrow y)$, so $(G \downarrow y)$ is contractible by Lemma 4.3 and $G$ is a weak homotopy equivalence by the quasicategorical version of Quillen’s Theorem A, recalled above in Theorem 4.3.

We next prove the main Approximation result of the paper. Instead of requiring an equivalence of homotopy categories as in the classical case, it is actually sufficient to require only an equivalence of the cofibration homotopy categories. Other variants of Approximation will follow from this one.

**Theorem 4.10 (Approximation, when $G$ induces an equivalence of cofibration homotopy categories).** Let $G: A \to B$ be an exact functor between Waldhausen quasicategories. If $G$ induces an equivalence of cofibration homotopy categories, i.e. the functor
\[ \tau_1(coG): \tau_1(coA) \to \tau_1(coB) \]
is an equivalence of categories, then $G$ induces an equivalence of $K$-theory spectra.

**Proof.** It suffices to prove that $K(G)_1$ is a weak homotopy equivalence because the $K$-theory spaces recalled in Definition 3.6 form an $\Omega$-spectrum beyond the 0-th term. Namely, if $K(G)_1$ is a weak homotopy equivalence then $\Omega K(G)_1$ is a weak homotopy equivalence (recall $\pi_n \Omega X \cong \pi_{n+1} X$ naturally), so the commutative diagram
\[
\begin{array}{ccc}
\Omega K(A)_1 & \xrightarrow{\text{w.e.}} & K(A)_2 \\
\Omega K(G)_1 \downarrow & & \downarrow K(G)_2 \\
\Omega K(B)_1 & \xrightarrow{\text{w.e.}} & K(B)_2
\end{array}
\]
implies that $K(G)_2$ is a weak homotopy equivalence by the 3-for-2 property of weak homotopy equivalences. Continuing in this way, every $K(G)_n$ is a weak homotopy equivalence for $n \geq 1$, and this is a stable equivalence of spectra.

But to prove that $K(G)_1$ is a weak homotopy equivalence, it suffices (by the Realization Lemma for bisimplicial sets) to prove that every $(S^\infty_n G)_{\text{equiv}}$ is a weak equivalence of simplicial sets. But $S^\infty_n A$ is naturally Waldhausen equivalent to $F^\infty_n A$ by Proposition 3.11, so it suffices to show $(F^\infty_n G)_{\text{equiv}}$ is a weak equivalence of simplicial sets for all $n \geq 0$.

I claim that $(F^\infty_n G)_{\text{equiv}}$ is a weak equivalence of simplicial sets for all $n \geq 0$, and to prove this I apply Proposition 4.5 to $F^\infty_n$. (i) For essential surjectivity of $F^\infty_n$, suppose $b_1: B_0 \rightarrow B_1$ is a cofibration in $B$. Since $\tau_1(coG)$ is essentially surjective by hypothesis, there exist objects $A_0, A_1$ in $A$ and equivalences $w_0$ and $w_{-1}$ in $coB$ as below.

\[
\begin{array}{ccc}
G A_0 & \xrightarrow{w_0} & G A_1 \\
\downarrow & & \uparrow_{w_{-1}} \\
B_0 & \xrightarrow{b_1} & B_1
\end{array}
\]

Since the morphism $[w_{-1}][b_1][w_0]: GA_0 \rightarrow GA_1$ is in $\tau_1(coB)$, there is a homotopy class $[a_1]: A_0 \rightarrow A_1$ in $\tau_1(coA)$ with $G[a_1] = [w_{-1}][b_1][w_0]$, by the hypotheses that $\tau_1(G)$ is fully faithful.

Continuing in this way, we construct for an object $b_*$ of $\tau_1(F^\infty_n B)$ an object $a_*$ of $\tau_1(F^\infty_n A)$ and a morphism

\[ ([w_*]): G(a_*) \rightarrow b_* \]

in $nB$ as in (3.10), which then extends to a homotopy class of a natural transformation $G(a_*) \rightarrow b_*$ by Lemma 3.9(i). Since its components are equivalences, this natural transformation is a natural equivalence by [13 Theorem 5.14] or [8 Corollary 3.7]. (ii) The quasicategory $F^\infty_n A$ is a Waldhausen quasicategory, so $co(F^\infty_n A)$ admits finite colimits (every finite colimit can be built out of an initial object and pushouts). Since $(F^\infty_n A)_{\text{equiv}} \subseteq co(F^\infty_n A)$, every diagram in $(F^\infty_n A)_{\text{equiv}}$ admits a colimit in $co(F^\infty_n A)$, so also in $F^\infty_n A$. The functor $F^\infty_n G$ is exact, and so preserves these.
(iii) Observe that $G$ reflects equivalences because every equivalence is a cofibration, and the equivalence of categories $\tau_1(coG)$ reflects isomorphisms. From this, $\mathcal{F}_n G$ also reflects equivalences because a natural transformation of quasicategories is an equivalence if and only if each component is an equivalence, see [13, Theorem 5.14] or [8, Corollary 3.7].

Although $G_{\text{equiv}}$ may not be an equivalence of quasicategories in the previous proof, we remark that $\tau_1(G_{\text{equiv}})$ is an equivalence of categories because $\tau_1(coG)$ is assumed to be so and $\tau_1((co\mathcal{A})_{\text{equiv}}) = \tau_1(co\mathcal{A})_{\text{iso}}$ and $\tau_1((co\mathcal{B})_{\text{equiv}}) = \tau_1(co\mathcal{B})_{\text{iso}}$.

**Lemma 4.11.** Let $G : \mathcal{A} \to \mathcal{B}$ be an exact functor between Waldhausen quasicategories such that $\tau_1(G)$ is an equivalence of categories. Then the functor $G$ reflects cofibrations if and only if it induces an equivalence of cofibration homotopy categories, that is, $G$ reflects cofibrations if and only if

$$\tau_1(coG) : \tau_1(co\mathcal{A}) \to \tau_1(co\mathcal{B})$$

is an equivalence of categories.

**Proof.** Since $\tau_1G$ is an equivalence of categories, $\tau_1co\mathcal{A}$ is naturally a subcategory of $\tau_1\mathcal{A}$, and equivalences are cofibrations, the exact functor $G$ induces a faithful and essentially surjective functor $\tau_1coG : \tau_1co\mathcal{A} \to \tau_1co\mathcal{B}$. Then $G$ reflects cofibrations if and only if $\tau_1coG$ is full (recall that any map homotopic to a cofibration is a cofibration). Consequently $G$ reflects cofibrations if and only if $\tau_1coG$ is an equivalence of categories. $\square$

**Theorem 4.12** (Approximation, when $G$ reflects cofibrations and induces equivalence of homotopy categories). Let $G : \mathcal{A} \to \mathcal{B}$ be an exact functor between Waldhausen quasicategories. Suppose:

(i) The functor $G$ reflects cofibrations.

(ii) The functor $\tau_1(G) : \tau_1(\mathcal{A}) \to \tau_1(\mathcal{B})$ is an equivalence of categories.

Then $G$ induces an equivalence of $K$-theory spectra.

**Proof.** By Lemma 4.11, the functor $\tau_1(coG)$ is an equivalence of categories, so the result follows from Theorem 4.10. $\square$

**Remark 4.13.** In the classical context, a related result to Theorems 4.10 and 4.12 is the following. Let $\mathcal{A}$ and $\mathcal{B}$ be classical Waldhausen categories both of which have the 3-for-2 property for weak equivalences,
and both of which admit functorial mapping cylinders for weak cofibrations (FMCWC) in the sense of [1] Definition 2.6. If $G: \mathcal{A} \to \mathcal{B}$ is an exact functor which induces an equivalence of homotopy categories and induces a Dwyer-Kan equivalence of DK-localizations of subcategories of weak cofibrations, then it follows from [1] Theorem 2.7] of Blumberg–Mandell that $G$ induces a stable equivalence in $K$-theory.

**Remark 4.14.** In Theorem 4.12 we made no assumption that $G$ satisfies the approximation axiom of Waldhausen, and instead required reflection of cofibrations and an equivalence of homotopy categories. This is reasonable because Blumberg–Mandell prove in [1] Theorems 1.5 and 9.1] that if $\mathcal{C}$ and $\mathcal{D}$ are both classical Waldhausen categories whose weak equivalences have the 3-for-2 property and $\mathcal{C}$ admits factorization, then any exact functor $G: \mathcal{C} \to \mathcal{D}$ that satisfies the approximation axiom (App 2) and reflects weak equivalences induces an equivalences of homotopy categories. Moreover, in [1] Theorem 1.3 (ii) they prove: if $\mathcal{C}$ and $\mathcal{D}$ are both classical Waldhausen categories which admit factorization and whose classes of weak equivalences both have the 3-for-2 property, then any weakly exact functor $G: \mathcal{C} \to \mathcal{D}$ that reflects weak equivalences and induces an equivalence of homotopy categories then induces an equivalence of $K$-theory spectra. Cisinski also proved this latter theorem in [6] for a right exact functor $G: \mathcal{C} \to \mathcal{D}$ for which each Waldhausen category $\mathcal{C}$ and $\mathcal{D}$ is a category of cofibrant objects, has a null object, and satisfies saturation conditions. The present Theorem 4.10 requires only an equivalence of cofibration categories.

**Remark 4.15.** A different variant of Approximation for an exact functor $G: \mathcal{C} \to \mathcal{D}$ between classical Waldhausen categories was proved by Sagave in [21] Theorem 2.8. Instead of requiring Waldhausen’s approximation axiom (App 2) for all maps $G(A) \to B$ in the Waldhausen category $\mathcal{D}$, he requires (App 2) only for maps with codomain $B$ in a full subcategory of special objects. The category $\mathcal{D}$ is equipped with a functorial replacement of any object by a special one. If $\mathcal{C}$ and $\mathcal{D}$ are both classical Waldhausen categories whose weak equivalences have the 3-for-2 property and $\mathcal{C}$ admits factorization, then any exact functor $G: \mathcal{C} \to \mathcal{D}$ that satisfies special approximation and reflects weak equivalences induces an equivalence of algebraic $K$-theory spectra. An example of such a functor arises when $\mathcal{C}$ and $\mathcal{D}$ are full subcategories of pointed model categories, $\mathcal{C}$ and $\mathcal{D}$ both have full subcategories of special objects with functorial replacement, and $\text{HoF}: \text{HoC} \to \text{HoD}$ is an equivalence, see [21] Section 3.2].

**Corollary 4.16** (Approximation, when all maps of domain are cofibrations). Let $G: \mathcal{A} \to \mathcal{B}$ be an exact functor between Waldhausen
quasicategories, and suppose every morphism in \( \mathcal{A} \) is a cofibration. If \( G \) induces an equivalence of homotopy categories, then \( G \) induces an equivalence of \( K \)-theory spectra.

**Remark 4.17.** Barwick proved a quasicategorical version of Approximation in the case that all maps are cofibrations [1, Proposition 2.10]: if \( \mathcal{A} \) and \( \mathcal{B} \) are Waldhausen quasicategories, and all maps are cofibrations, then an exact functor \( G: \mathcal{A} \to \mathcal{B} \) is an equivalence if and only if it induces an equivalence of homotopy categories \( ho\mathcal{A} \to ho\mathcal{B} \). Blumberg–Gepner–Tabuada prove the same result for stable quasicategories in [2, Corollary 4.10].

**Corollary 4.18** (Approximation for factorization in \( \mathcal{B} \)). Let \( G: \mathcal{A} \to \mathcal{B} \) be an exact functor between Waldhausen quasicategories. If \( \mathcal{A} \) admits factorization (see Section 3) and \( G \) induces an equivalence of homotopy categories, then \( G \) induces an equivalence of \( K \)-theory spectra.

**Proof.** This follows from Corollary 4.16 because \( \mathcal{A} \) admits factorization if and only if every morphism in \( \mathcal{A} \) is a cofibration, as discussed in Section 3.

\( \square \)

5. Equivalence of Homotopy Categories of \( S_n \)

In Section 4, we proved Approximation results *without* resorting to proving that each \( \tau_1(S_n^\infty \mathcal{G}) \) is an equivalence of categories. But one may still ask, when is \( \tau_1(S_n^\infty \mathcal{G}) \) an equivalence? In this section we provide some answers to the question: if an exact functor \( G \) reflects cofibrations and \( \tau_1(G) \) (or \( \tau_1(coG) \)) is an equivalence of categories, when is \( \tau_1\left(S_{n_k}^\infty \cdots S_{n_1}^\infty \mathcal{G}\right) \) an equivalence? A key condition in this paper is that any two natural transformations from \( I[n] \) into the domain (or codomain) with homotopic components are homotopic. This is true for any left or right Kan complex for instance.

5.1. Homotopy of Natural Transformations. In Theorems 5.4 and 5.5 on equivalences of (cofibration) homotopy categories of \( S_n^\infty \)-iterates, we assume that any two natural transformations of the form \( I[p] \times \Delta[1] \to \mathcal{A}^{[p]} \) with homotopic components are homotopic. We now find two situations in which that is true, namely when \( \mathcal{A} \) is a left or right Kan complex. Note that those theorems do not require that the homotopy of natural transformations extends the homotopy of components, rather just that a homotopy of natural transformations exists when components are homotopic. As a warmup, the following simple example illustrates that homotopies of natural transformations may
still exist even when none exists which extends the homotopies of the components.

**Example 5.1.** Let $\mathcal{B}$ be the singular simplicial set of the cylinder boundary

$$C = (S^1 \times [0, 1]) \cup (D^2 \times \{0\}) \cup (D^2 \times \{1\}),$$

where we consider $D^2$ and $S^1$ in the complex plane for simplicity. Let $\pi = 1$ so $I[1] = \Delta[1]$. By adjunction, we let the functors $K, L : \mathcal{B} \to \Delta[1]$ be the two positively oriented paths $K, L : |\Delta[1]| \to C$ with velocity 1 given by $\{-i\} \times [0, 1]$ and $\{i\} \times [0, 1]$, and we let $\alpha, \beta : |\Delta[1]| \times |\Delta[1]| \to C$ be the two natural transformations $K \to L$ determined by the two possible semi-circle edges of the lower disc and upper disc, and the two vertical sheets between them. The components of $\alpha$ and $\beta$ are left homotopic via the disks, but these two homotopies cannot be extended to a left homotopy $h : |\Delta[1]| \times |\Delta[2]| \to C$ of $\alpha$ and $\beta$ because $h$ restricted to the boundary of the prism would be a nontrivial element of $\pi_2(C)$, a contradiction.

However, $\alpha$ and $\beta$ are homotopic via a homotopy $h' : |\Delta[1]| \times |\Delta[2]| \to C$ which maps one square of the prism to the vertical segment $K$, maps another square of the prism to the sheet $\alpha$, and maps a third square of the prism to the sheet $\beta$. The bottom triangle maps to the bottom disk of $C$. To say where the interior of the prism and the interior of the top triangle go, we radially project those from the center of the top triangle downward to the boundary part of the prism where the homotopy is already defined, and then apply $h'$ as defined thus far. In particular, the interior of the top triangle does not map to the top disk. This $h'$ is a homotopy between $\alpha$ and $\beta$.

I thank André Joyal for pointing out to me that $h'$ exists here because $\mathcal{B}$ is a Kan complex (it is a singular simplicial set).

**Proposition 5.2 (Homotopy from homotopy of last/first components).**

(i) Suppose $X$ is a quasicategory such that any horn $\Lambda^3[3] \to X$ admits a filler to $\Delta[3]$. If $\alpha, \beta : I[n] \times \Delta[1] \to X$ are natural transformations such that the last components $\alpha_n$ and $\beta_n$ are homotopic as morphisms in $X$, then $\alpha$ and $\beta$ are homotopic as natural transformations.

(ii) Suppose $Y$ is a quasicategory such that any horn $\Lambda^0[3] \to Y$ admits a filler to $\Delta[3]$. If $\alpha, \beta : I[n] \times \Delta[1] \to Y$ are natural transformations such that the first components $\alpha_0$ and $\beta_0$ are homotopic as morphisms in $Y$, then $\alpha$ and $\beta$ are homotopic as natural transformations.
Proof. Consider (i).

We first prove the case $n = 1$. Suppose that $\alpha_1$ is right homotopic to $\beta_1$, so that there is a map $N\{p < q < r\} \to X$ which sends $pr$ to $\alpha_1$, $qr$ to $\beta_1$ and $pq$ to the identity on the object $\text{dom} \alpha_1$. We extend this to a map $h: I[1] \times \Delta[2] \to X$ with the required properties as follows. We label the vertices of the top prism end $\{0\} \times \Delta[2]$ by $a < b < c$ and label the vertices of the bottom prism end $\{1\} \times \Delta[2]$ by $p < q < r$. The two 2-simplices on the square $abpq$ map to the degeneracy of $\text{dom} \alpha_1$. The other two squares of the prism boundary are mapped according to $\alpha$ and $\beta$.

The rest of $h$ is constructed via successive three-dimensional horn filling. The map $h$ is already defined on the inner horn $\Lambda^1[3]$ subcomplex of the bottom 3-simplex $a < p < q < r$, so $h$ extends to the entire bottom 3-simplex. Now we have the map $h$ already defined on the inner horn $\Lambda^2[3]$ subcomplex of the middle 3-simplex $a < b < q < r$, so $h$ extends to the entire middle 3-simplex. Finally, we have the map $h$ already defined on the right horn $\Lambda^3[3]$ subcomplex of the top 3-simplex $a < b < c < r$, so $h$ extends to the top 3-simplex of the prism by hypothesis. Thus we have constructed the desired right homotopy $h: I[1] \times \Delta[2] \to X$ between $\alpha$ and $\beta$.

For general finite $n$, the homotopy $I[n] \times \Delta[2] \to X$ is built from the above case by starting at the bottom and successively filling the prism segments. Note that this gluing works because we are using $I[n]$ in place of $\Delta[n]$.

The proof of (ii) is similar to (i) we just fill from the top down instead. □

Proposition 5.3 (Homotopy in $X^W$ resp. $Y^W$ from homotopy of last resp. first components).

(i) Suppose $X$ is a right Kan complex, that is, suppose any horn $\Lambda^k[n] \to X$ with $0 < k \leq n$ admits a filler. Let $W$ be a simplicial set. If $\alpha, \beta: I[n] \times \Delta[1] \to X^W$ are natural transformations such that the last components $\alpha_n$ and $\beta_n$ are homotopic as morphisms in $X$, then $\alpha$ and $\beta$ are homotopic as natural transformations.

(ii) Suppose $Y$ is a left Kan complex, that is, suppose any horn $\Lambda^k[n] \to Y$ with $0 \leq k < n$ admits a filler. Let $W$ be a simplicial set. If $\alpha, \beta: I[n] \times \Delta[1] \to Y^W$ are natural transformations such that the first components $\alpha_0$ and $\beta_0$ are homotopic as morphisms in $X$, then $\alpha$ and $\beta$ are homotopic as natural transformations.
Proof. We prove (i). Since $X$ is a right Kan complex, so is $X^W$ by [13, Theorem 2.18], in a way completely analogous to [13, Corollary 2.19]. Then Proposition 5.2 applies. □

5.2. Equivalence of Homotopy Categories of Higher $S_n^\infty$ Iterates. We now present situations in which $\tau_1\left(S_n^\infty G\right)$ and $\tau_1\left(co(S_n^\infty G)\right)$ are equivalences of categories.

Theorem 5.4 (Equivalence of homotopy categories of higher $S_n^\infty$ iterates). Let $G: \mathcal{A} \to \mathcal{B}$ be an exact functor between Waldhausen quasicategories. Suppose:

(i) The functor $G$ reflects cofibrations.
(ii) The functor $\tau_1(G): \tau_1(\mathcal{A}) \to \tau_1(\mathcal{B})$ is an equivalence of categories.
(iii) The quasicategory $\mathcal{A}$ has the property that for all $p, k \geq 0$ and all $\overline{n} \in \mathbb{N}^k$, any two natural transformations $I[p] \times \Delta[1] \to \mathcal{A}[\overline{n}]$ with respective homotopic components are homotopic. Here $I[\overline{n}] := I[n_1] \times I[n_2] \times \cdots \times I[n_k]$ and $I[n_j]$ is the spine recalled in Section 3.4.
(iv) The quasicategory $\mathcal{B}$ similarly has the property that for all $p, k \geq 0$ and all $\overline{n} \in \mathbb{N}^k$, any two natural transformations $I[p] \times \Delta[1] \to \mathcal{B}[\overline{n}]$ with respective homotopic components are homotopic.

Then the functor $S_n^\infty G$ reflects cofibrations and the functor

$$\tau_1\left(S_n^\infty G\right): \tau_1\left(S_n^\infty \mathcal{A}\right) \longrightarrow \tau_1\left(S_n^\infty \mathcal{B}\right)$$

is an equivalence of categories, where $S_n^\infty := S_{n_k}^\infty \cdots S_{n_2}^\infty S_{n_1}^\infty$.

Proof. For $\overline{n} = (n_1, \ldots, n_k) \in \mathbb{N}^k$, let $F^\infty_{\overline{n}} := F_{n_k}^\infty \cdots F_{n_2}^\infty F_{n_1}^\infty$. Since $S_j^\infty \mathcal{A}$ and $F_j^\infty \mathcal{A}$ are naturally Waldhausen equivalent Waldhausen quasicategories, it suffices to prove that $F^\infty_{\overline{n}} G$ reflects cofibrations (recall Waldhausen equivalences reflect cofibrations by the Appendix of [8]) and $\tau_1(F^\infty_{\overline{n}} G)$ is an equivalence of categories for all $\overline{n} \in \mathbb{N}^k$.

We proceed to prove by induction on $k$ that $F^\infty_{\overline{n}} G$ reflects cofibrations and $\tau_1(F^\infty_{\overline{n}} G)$ is an equivalence. The conclusion for the case $k = 0$ is hypotheses (i) and (ii). Suppose $F^\infty_{\overline{m}} G$ reflects cofibrations and $\tau_1(F^\infty_{\overline{m}} G)$ is an equivalence for $k - 1$ and all $\overline{m} \in \mathbb{N}^{k-1}$. Let $m_k \in \mathbb{N}$, and let $\mathcal{C} := F^\infty_{\overline{m}} \mathcal{A}$ and $\mathcal{D} := F^\infty_{\overline{m}} \mathcal{B}$ and $H := F^\infty_{\overline{m}} G$, so that $H: \mathcal{C} \to \mathcal{D}$ is $F^\infty_{\overline{n}} G: F^\infty_{\overline{m}} \mathcal{A} \to F^\infty_{\overline{m}} \mathcal{B}$.

(i) The functor $H$ reflects cofibrations by the induction hypothesis. Since $H$ is exact, it preserves pushouts along cofibrations.
Given a commutative square in \( \mathcal{C} \) with one leg a cofibration, any induced map from the pushout to the lower right corner is a cofibration if and only if its image in \( \mathcal{D} \) is a cofibration (recall that the induced map is determined only up to homotopy, but this is fine since any map homotopic to a cofibration is a cofibration). Arguing in this way for each square of a morphism in \( \mathcal{F}^\infty_{m_k} \mathcal{C} \), we see that \( \mathcal{F}^\infty_{m_k} H \) reflects cofibrations.

(ii) The essential surjectivity of \( \tau_1 (\mathcal{F}^\infty_{m_k} H) \) is similar to the proof of essential surjectivity of \( \mathcal{F}^\infty_n G \) in the proof of Theorem 4.10 part (i).

For the faithfulness of \( \tau_1 (\mathcal{F}^\infty_{m_k} H) \), if \([\alpha]\) and \([\beta]\) are parallel morphisms in \( \tau_1 (\mathcal{F}^\infty_{m_k} \mathcal{C}) \) such that \([H \circ \alpha] = [H \circ \beta]\), then the components of the natural transformations \( H \circ \alpha \) and \( H \circ \beta \) are homotopic, so the components of \( \alpha \) and \( \beta \) are homotopic, and \( \alpha \) and \( \beta \) are homotopic by hypothesis (iii).

For the fullness of \( \tau_1 (\mathcal{F}^\infty_{m_k} H) \), if \([\gamma]: H(c) \to H(c')\) in \( \tau_1 (\mathcal{F}^\infty_{m_k} \mathcal{D}) \), then we find a preimage in \( \mathcal{C} \) of the homotopy class of each component by the fully faithfulness of \( H \), then we extend this to the homotopy class of a natural transformation \([\alpha]\) by Lemma 3.9 (i). By hypothesis (iv) \([H \circ \alpha] = [\gamma]\) because the components of \( H \circ \alpha \) and \( \gamma \) are homotopic.

\[\square\]

**Theorem 5.5** (Equivalence of cofibration homotopy categories of higher \( S_n^\infty \) iterates). Let \( G: \mathcal{A} \to \mathcal{B} \) be an exact functor between Waldhausen quasicategories. Suppose:

(i) The functor \( G \) reflects cofibrations.

(ii) The functor \( \tau_1 (\text{co} G): \tau_1 (\text{co} \mathcal{A}) \to \tau_1 (\text{co} \mathcal{B}) \) is an equivalence of categories.

(iii) The quasicategory \( \mathcal{A} \) has the property that for all \( p, k \geq 0 \) and all \( \overline{n} \in \mathbb{N}^k \), any two natural transformations \( I[p] \times \Delta[1] \to \mathcal{A}[^{\overline{n}}] \) with respective homotopic components are homotopic.

(iv) The quasicategory \( \mathcal{B} \) similarly has the property that for all \( p, k \geq 0 \) and all \( \overline{n} \in \mathbb{N}^k \), any two natural transformations \( I[p] \times \Delta[1] \to \mathcal{B}[^{\overline{n}}] \) with respective homotopic components are homotopic.

Then the functor \( S_{n_1}^\infty G \) reflects cofibrations and the functor

\[
\tau_1 \left( \text{co} \left( S_{n_1}^\infty \mathcal{G} \right) \right) \colon \tau_1 \left( \text{co} \left( S_{n_1}^\infty \mathcal{A} \right) \right) \rightarrow \tau_1 \left( \text{co} \left( S_{n_1}^\infty \mathcal{B} \right) \right)
\]

is an equivalence of categories, where \( S_{n_1}^\infty := S_{n_1}^\infty \cdots S_{n_2}^\infty S_{n_1}^\infty \).
Proof. We proceed to prove by induction on $k$ that $\mathcal{F}^\infty_\infty G$ reflects cofibrations and $\tau_1 (co(\mathcal{F}^\infty_\infty G))$ is an equivalence. The conclusion for case $k = 0$ is hypotheses (i) and (ii).

Suppose $\mathcal{F}^\infty_\infty G$ reflects cofibrations and $\tau_1 (co(\mathcal{F}^\infty_\infty G))$ is an equivalence for $k - 1$ and all $\overline{m} \in \mathbb{N}^{k-1}$. Let $m_k \in \mathbb{N}$, and let $C := \mathcal{F}^\infty_\infty A$ and $D := \mathcal{F}^\infty_\infty B$ and $H := \mathcal{F}^\infty_\infty G$, so that $G: C \to D$ is $\mathcal{F}^\infty_\infty F: \mathcal{F}^\infty_\infty A \to \mathcal{F}^\infty_\infty B$.

(i) The functor $\mathcal{F}^\infty_\infty H$ reflects cofibrations just as in Theorem 5.4 (i).

(ii) The proof of the essential surjectivity and faithfulness of $\tau_1 (co(\mathcal{F}^\infty_\infty H))$ is similar to the proof of essential surjectivity and faithfulness of $\tau_1 (\mathcal{F}^\infty_\infty H)$ in Theorem 5.4 part (ii).

For the fullness of $\tau_1 (co(\mathcal{F}^\infty_\infty H))$, if $[\gamma] : H(c_\ast) \to H(c'_\ast)$ is the homotopy class of a cofibration, then we find a preimage in $coC$ of the homotopy class of each component by induction hypothesis (ii), then we extend this to the homotopy class of a natural transformation $[\alpha]$ by Lemma 3.9 (i). By hypothesis (iv) $[H \circ \alpha] = [\gamma]$ because the components of $H \circ \alpha$ and $\gamma$ are homotopic. We have already proved in (i) of Theorem 5.4 that $\mathcal{F}^\infty_\infty H$ reflects cofibrations, so $\alpha$ is also a cofibration.

5.3. Strong lifting conditions. In Theorems 5.4 and 5.5 we assumed that natural transformations of functors $I[p] \to B^{|\overline{m}|}$ with homotopic components are homotopic as natural transformations (same for functors $I[p] \to A^{|\overline{m}|}$). By adjunction, we can equivalently consider homotopy of natural transformations of functors $I[\overline{m}] \times I[p] \to \mathcal{B}$, or simply homotopies of natural transformations of functors $I[\overline{m}] \to \mathcal{B}$, since $\overline{m}$ was general.

We now give some stronger lifting conditions which may replace hypotheses (iii) and (iv) in Theorems 5.4 and 5.5. The replacement condition for (iv), which concerns natural transformations in $\mathcal{B}$, is stronger because it will imply that the homotopies of components extend to homotopies of natural transformations, which our results do not require (recall Example 5.1). The replacement condition for (iii) on natural transformations in $\mathcal{A}$ is in terms of lifting condition on $G$. 

□
Proposition 5.6 (Strong Replacement for hypothesis (iv)). Suppose \( \mathcal{B} \) is a quasicategory in which every diagram of the form
\[
\left[ (\text{Obj} \mathcal{I}) \times \Delta[2] \right] \cup \left[ \mathcal{I} \times (\partial_1 \Delta[2] \cup \partial_2 \Delta[2]) \right] \cup \left[ \mathcal{I} \times \partial_0 \Delta[2] \right] \rightarrow \mathcal{B}
\]
admits a lift. Then any left homotopy of components of natural transformations \( \mathcal{I} \times \Delta[1] \rightarrow \mathcal{B} \) extends to a left homotopy of natural transformations.

Proof. A left homotopy of components of \( \alpha \) and \( \beta \), along with \( \alpha \) and \( \beta \) and the identity extension to \( \mathcal{I} \times \partial_0 \Delta[2] \) is a map of the form in the top arrow. A diagonal map is then an extension to a homotopy between \( \alpha \) and \( \beta \). \( \square \)

Hypothesis (iii) was used only for the faithfulness of \( \tau_1(\mathcal{F}_{\mathcal{H}}^\infty) \). Instead of a lifting condition on \( \mathcal{A} \), we can place a lifting condition on \( \mathcal{G} \).

Proposition 5.7 (Replacement for hypothesis (iii)). Let \( \mathcal{G} : \mathcal{A} \rightarrow \mathcal{B} \) be an exact functor between Waldhausen quasicategories. Suppose that for all \( k \geq 0 \) and all \( \overline{\pi} \in \mathbb{N}^k \), the functor \( \mathcal{G} \) has the right lifting property with respect to
\[
\mathcal{I}[\overline{\pi}] \times \partial \Delta[2] \rightarrow \mathcal{I}[\overline{\pi}] \times \Delta[2].
\]
Then for all \( k \geq 0 \) and all \( \overline{\pi} \in \mathbb{N}^k \), the functor \( \tau_1(\mathcal{F}_{\mathcal{G}}^\infty) \) is fully faithful.

Proof. If \( [\alpha] \) and \( [\beta] \) are parallel morphisms in \( \tau_1(\mathcal{F}_{\mathcal{A}}^\infty) \) such that \( [\mathcal{F}_{\mathcal{G}}^\infty \circ \alpha] = [\mathcal{F}_{\mathcal{G}}^\infty \circ \beta] \), then we have a commutative square
\[
\begin{array}{ccc}
\mathcal{I}[\overline{\pi}] \times \partial \Delta[2] & \xrightarrow{(\beta, \alpha, \text{id})} & \mathcal{A} \\
\downarrow & & \downarrow \mathcal{G} \\
\mathcal{I}[\overline{\pi}] \times \Delta[2] & \xrightarrow{} & \mathcal{B}
\end{array}
\]
expressing that \( \mathcal{F}_{\mathcal{G}}^\infty \circ \alpha \) and \( \mathcal{F}_{\mathcal{G}}^\infty \circ \beta \) are homotopic. There is a lift by hypothesis, so that \( [\alpha] = [\beta] \). \( \square \)

References

[1] Clark Barwick. On the algebraic K-theory of higher categories, I. the universal property of Waldhausen K-theory. [http://arxiv.org/abs/1204.3607](http://arxiv.org/abs/1204.3607)

[2] Andrew Blumberg, David Gepner, and Gonçalo Tabuada. A universal characterization of higher algebraic K-theory. *Geometry and Topology.*
[3] Andrew J. Blumberg and Michael A. Mandell. The localization sequence for the algebraic $K$-theory of topological $K$-theory. *Acta Math.*, 200(2):155–179, 2008.

[4] Andrew J. Blumberg and Michael A. Mandell. Algebraic $K$-theory and abstract homotopy theory. *Adv. Math.*, 226(4):3760–3812, 2011.

[5] J. M. Boardman and R. M. Vogt. *Homotopy invariant algebraic structures on topological spaces*. Lecture Notes in Mathematics, Vol. 347. Springer-Verlag, Berlin, 1973.

[6] Denis-Charles Cisinski. Invariance de la $K$-théorie par équivalences dérivées. *J. K-Theory*, 6(3):505–546, 2010.

[7] Daniel Dugger and Brooke Shipley. $K$-theory and derived equivalences. *Duke Math. J.*, 124(3):587–617, 2004.

[8] Thomas M. Fiore and Wolfgang Lück. Waldhausen Additivity: Classical and Quasicategorical. *Preprint*, 2012. [http://arxiv.org/abs/1207.6613](http://arxiv.org/abs/1207.6613).

[9] Rudolf Fritsch and Renzo A. Piccinini. *Cellular structures in topology*, volume 19 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.

[10] Paul G. Goerss and John F. Jardine. *Simplicial homotopy theory*. Modern Birkhäuser Classics. Birkhäuser Verlag, Basel, 2009. Reprint of the 1999 edition.

[11] André Joyal. *Theory of quasicategories I*. In preparation.

[12] André Joyal. Quasi-categories and Kan complexes. *J. Pure Appl. Algebra*, 175(1-3):207–222, 2002. Special volume celebrating the 70th birthday of Professor Max Kelly.

[13] André Joyal. *The theory of quasi-categories and its applications*. Quadern 45, Vol. II, Centre de Reerca Matemàtica Barcelona. 2008, [http://mat.uab.cat/~kock/crm/hocat/advanced-course/Quadern45-2.pdf](http://mat.uab.cat/~kock/crm/hocat/advanced-course/Quadern45-2.pdf).

[14] Daniel M. Kan. On c. s. s. complexes. *Amer. J. Math.*, 79:449–476, 1957.

[15] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.

[16] Jacob Lurie. Higher algebra. *Preprint*, 2011, [http://www.math.harvard.edu/~lurie/](http://www.math.harvard.edu/~lurie/).

[17] Amnon Neeman. Stable homotopy as a triangulated functor. *Invent. Math.*, 109(1):17–40, 1992.

[18] Amnon Neeman. $K$-theory for triangulated categories. I(A). Homological functors. *Asian J. Math.*, 1(2):330–417, 1997.

[19] Amnon Neeman. $K$-theory for triangulated categories. I(B). Homological functors. *Asian J. Math.*, 1(3):435–529, 1997.

[20] John Rognes. Lecture notes on algebraic $K$-theory. [http://folk.uio.no/rognes/kurs/mat9570v10/akt.pdf](http://folk.uio.no/rognes/kurs/mat9570v10/akt.pdf).

[21] Steffen Sagave. On the algebraic $K$-theory of model categories. *J. Pure Appl. Algebra*, 190(1-3):329–340, 2004.

[22] Marco Schlichting. A note on $K$-theory and triangulated categories. *Invent. Math.*, 150(1):111–116, 2002.

[23] Marco Schlichting. Negative $K$-theory of derived categories. *Math. Z.*, 253(1):97–134, 2006.

[24] R. W. Thomason. Cat as a closed model category. *Cahiers Topologie Géom. Différentielle*, 21(3):305–324, 1980.
[25] R. W. Thomason and Thomas Trobaugh. Higher algebraic $K$-theory of schemes and of derived categories. In The Grothendieck Festschrift, Vol. III, volume 88 of Progr. Math., pages 247–435. Birkhäuser Boston, Boston, MA, 1990.

[26] Bertrand Toën and Gabriele Vezzosi. A remark on $K$-theory and $S$-categories. Topology, 43(4):765–791, 2004.

[27] Friedhelm Waldhausen. Algebraic $K$-theory of spaces. In Algebraic and geometric topology (New Brunswick, N.J., 1983), volume 1126 of Lecture Notes in Math., pages 318–419. Springer, Berlin, 1985.

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