Structure relations for the symmetry algebras of quantum superintegrable systems

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Abstract. A quantum superintegrable system is an integrable n-dimensional Hamiltonian system with potential $H = \Delta_n + V$ that admits $2n - 1$ algebraically independent partial differential operators commuting with the Hamiltonian, the maximum number possible. Here, $n \geq 2$. The system is of order $\ell$ if the maximum order of the symmetry operators other than the Hamiltonian is $\ell$. Typically, the algebra generated by the symmetry operators has been shown to close. There is an analogous definition for classical superintegrable systems with the operator commutator replaced by the Poisson bracket. Superintegrability captures what it means for a Hamiltonian system to be explicitly algebraically and analytically solvable, not just solvable numerically. Until recently there were very few examples of superintegrable systems of order $\ell$ with $\ell > 3$ and virtually no structure results. The situation has changed dramatically in the last two years with the discovery of families of systems depending on a rational parameter $k = p/q$ that are superintegrable for all $k$ and of arbitrarily high order, such as $\ell = p + q + 1$. We review a method, based on recurrence formulas for special functions, that proves superintegrability of these higher order quantum systems, and allows us to determine the structure of the symmetry algebra. Just a few months ago, these constructions seemed out of reach.

1. Introduction and definitions

Most classical and quantum Hamiltonian systems that appear in physical applications can only be solved numerically. A few, called integrable, can be solved by quadratures. However, a much smaller group, called superintegrable, can be solved explicitly (algebraically and analytically). This small subset is of immense historical and practical importance. Most of the examples that students learn from textbooks are superintegrable, with adjustable parameters. These exactly solvable systems also serve as the foundation for perturbation expansions. For example the Kepler 2-body problem is superintegrable, with conic section trajectories that can be determined algebraically. These trajectories serve as starting points for computing many-body trajectories via perturbation. The quantum hydrogen atom system is superintegrable and it serves as the foundation of the periodic table of the elements via perturbation analysis. The quantum isotropic oscillator is superintegrable and it is the foundation for the bosonic calculus employed widely in quantum field theory. The concept of superintegrability is the most successful yet in capturing what it means for a Hamiltonian system to be exactly solvable. Once such a system is discovered it can be employed as a model for physical or mathematical application. Classifying all such systems and exploiting them for applications is a valuable endeavor, [1, 2].

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What is it about these systems that enables their solvability? Not unexpectedly, it is their symmetry. However, this symmetry is not normally just group symmetry, but so-called “hidden symmetry” that goes beyond group theory. This talk surveys some basic facts about these systems and introduces some very recent results that have excited specialists in superintegrability.

Consider a quantum system with Schrödinger operator

\[ H = \Delta_n + V(x) \]

where \( \Delta = \frac{1}{\sqrt{g}} \sum_{i,j} \partial_{x_i} (\sqrt{g} g^{ij}) \partial_{x_j} \) is the Laplace-Beltrami operator on an \( n \)-dimensional Riemannian manifold, in local coordinates \( x_j \). We say that the system is (maximally) superintegrable if it admits \( 2n - 1 \) algebraically independent globally defined differential symmetry operators

\[ L_j, \ j = 1, \cdots, 2n - 1, \ n \geq 2, \]

with \( L_1 = H \) and \([H, L_j] \equiv HL_j - L_jH = 0\). This is a very restrictive condition and \( 2n - 1 \) is the maximum number of such symmetries. A superintegrable system is of order \( \ell \) if \( \ell \) is the maximum order of the generating symmetries other than the Hamiltonian as a differential operator for the quantum system. Historically, systems with \( \ell = 1, 2 \) have been the most tractable due to the association with Lie algebras and separation of variables.

Classically a system with Hamiltonian

\[ H = \sum g^{jk} p_j p_k + V(x) \]

on phase space with local coordinates \( x_j, p_j \), where \( ds^2 = \sum_{j,k=1}^n g_{jk} dx_j \ dx_k \) is maximally superintegrable if there are \( 2n - 1 \) functionally independent constants of the motion, i.e., functions

\[ S_j(p,x), \ j = 1, \cdots, 2n - 1 \]

with \( \mathcal{L}_1 = \mathcal{H} \), globally defined and polynomial in the momenta, such that \( \{ \mathcal{L}_j, \mathcal{H} \} = 0 \), where

\[ \{G, F\} = \sum_{j=1}^n \left( \frac{\partial F}{\partial x_j} \frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j} \frac{\partial G}{\partial x_j} \right) \]

is the Poisson bracket. A classical superintegrable system is of order \( \ell \) if \( \ell \) is the maximum order of the generating symmetries other than the Hamiltonian as polynomials in the momenta. Although there are many parallels in the theory of classical and quantum systems and the quantization problem is very interesting, I shall focus on quantum systems in this talk.

It is important to distinguish integrability and superintegrability. An integrable quantum system has \( n \) algebraically independent COMMUTING symmetry operators. However, a superintegrable system has \( 2n - 1 \) algebraically independent symmetry operators and it is impossible for them all to commute. (For many researchers the definition of superintegrability also requires integrability, but it isn’t yet settled if this is automatically satisfied.) The symmetry algebra generated by the symmetries of a merely integrable system is abelian. However, the algebra corresponding to a superintegrable system is nonabelian and this structure can be used to deduce additional properties of the system.

Among the familiar and important superintegrable systems we mention the following:

- The isotropic harmonic oscillator is classically and quantum 2nd order superintegrable.
- Celestial navigation including sending satellites into lunar and earth orbits depends on the 2nd order superintegrability of the Kepler system (Hohmann transfer) \[3, 4\].
The quantum Coulomb or hydrogen atom problem is second order superintegrable. Thus the spectrum can be deduced algebraically. An important reason that the Balmer series was discovered far in advance of quantum mechanics.\[5\]

Also there are deep connections with other important fields in mathematical physics, such as

- Quasi-exactly solvable quantum systems, [6, 7, 8].
- Supersymmetry, [9, 10].
- PT-symmetric systems, [11, 12].
- Special functions and orthogonal polynomials. These occur in two distinct ways: 1) Solutions of the Schrödinger eigenvalue equation, [13, 14]. 2) Irreducible representations of the symmetry algebras, [15, 16]

2. Some examples and structure theorems

**Example 1** A simple 2nd order flat space superintegrable system is an extension of an anharmonic oscillator. Here \( n = 2 \), so \( 2n - 1 = 3 \). The Schrödinger operator is

\[
H = \partial_x^2 + \partial_y^2 - \alpha^2(4x^2 + y^2) + bx + \frac{1}{y^2} - \frac{c^2}{y^2}.
\]

The generating symmetry operators are \( H, L_2, L_3 \) where

\[
L_2 = \partial_x^2 - 4\alpha^2 x^2 + bx
\]

\[
L_3 = \frac{1}{2} \{ M, \partial_y \} + y^2 \left( \frac{b}{4} - x\alpha^2 \right) - \left( \frac{1}{4} - c^2 \right) \frac{x}{y^2}.
\]

Here \( M = x\partial_y - y\partial_x \) and \( \{ A, B \} = AB + BA \). Add the commutator \( R = [L_2, L_3] \) to the symmetry algebra. Then

\[
[L_2, R] = 2bH + 16\alpha^2 L_3 - 2bL_2
\]

\[
[L_3, R] = 8L_2 H - 6L_2^2 - 2H^2 + 2bL_3 - 8\alpha^2(1 - c^2).
\]

\[
R^2 + 4L_3^2 + 4L_2 H^2 - 8L_2^2 H + 16\alpha^2 L_3^2 + 4bL_3 H - 2b\{L_2, L_3\}
\]

\[
+16\alpha^2(3 - c^2)L_2 - 32\alpha^2 H - b^2(1 - c^2) = 0
\]

We have a second order superintegrable system whose symmetry algebra closes at order 6 in the derivatives.

Note that if \( \Psi \) is an eigenvector of \( H \) with eigenvalue \( E \), i.e., \( H\Psi = E\Psi \), and \( L \) is a symmetry operator then also \( H(L\Psi) = E(L\Psi) \), so the symmetry algebra preserves eigenspaces of \( H \). Thus we can use the irreducible representations of the symmetry algebra to explain the “accidental” degeneracies of the eigenspaces of \( H \). By classifying the finite dimensional irreducible representations of the symmetry algebra we can show that the possible bound state energy levels for this system must take the form

\[
E_m = 4\alpha(m - 2c) + \frac{b^2}{16\alpha^2}
\]

where \( m = 0, 1, \cdots \) and the multiplicity of the energy level \( E_m \) is \( m + 1 \),[17, 18].

For all 2nd order superintegrable systems in 2D and for 2nd order systems in 3D with nondegenerate (4 parameter potentials) there is a detailed structure and classification theory, [19, 20, 21, 22, 23, 24, 25, 26, 27]. For example, if \( \mathcal{H} \) is the Hamiltonian of a 2D second order superintegrable system with nondegenerate (3 parameter) potential then
• The space of 2nd order symmetries of this Hamiltonian is 3 dimensional.
• The space of 3rd order symmetries is 1 dimensional.
• The space of 4th order symmetries is 6 dimensional and spanned by symmetric polynomials in the 2nd order symmetries.
• The space of 6th order symmetries is 10 dimensional and spanned by symmetric polynomials in the 2nd order symmetries.

This result shows that the structure algebra must close on dimensional grounds. Closure is not a happy accident. There is also a theory describing the structure preserving transformations between distinct superintegrable systems, [28, 29, 30, 31, 32, 33].

Example 2 A quantum system on the 2-sphere, a spherical analog of the hydrogen atom, is determined by

\[ H = J_1^2 + J_2^2 + J_3^2 + \frac{\alpha s_3}{\sqrt{s_1^2 + s_2^2}}, \]

where, \( J_1, J_2, J_3 \) are the angular momentum operators, \( s_1^2 + s_2^2 + s_3^2 = 1 \) and \( H \Psi = E \Psi \) Here \( \Psi(s_1, s_2, s_3) \) is a function on the unit sphere, square integrable with respect to the area measure on the sphere. The quantum basis of generators for the symmetry operators is

\[ L_1 = J_1 J_3 + J_3 J_1 - \frac{\alpha s_1}{\sqrt{s_1^2 + s_2^2}}, \quad L_2 = J_2 J_3 + J_3 J_2 - \frac{\alpha s_2}{\sqrt{s_1^2 + s_2^2}}, \quad X = J_3. \]

The structure relations are

\[ [H, L_1] = [H, L_2] = [H, X] = 0, \quad [X, L_1] = -L_2, \quad [X, L_2] = L_1, \]

\[ [L_1, L_2] = 4HX - 8X^3 + X. \]

The Casimir relation is

\[ L_1^2 + L_2^2 + 4X^4 - 4HX^2 + H - 5X^2 = \alpha^2. \] (1)

Note that the algebra closes, but also that there are 4 linearly independent generators whereas \( 2n - 1 = 3 \) is the maximum number of algebraically independent generators possible. Hence there must exist an algebraic relation between the 4 generators. That is the Casimir relation. Here the lefthand side of (1) commutes with all of the symmetry algebra generators, the analog of the Casimir operator for simple Lie algebras.

The structure of the symmetry algebra gives vital information about the bound state spectra for \( H \), information that can be obtained algebraically:

• The symmetry operators leave each eigenspace of \( H \) invariant. In each eigenspace \( E \) is a constant.
• Thus the degeneracies can be explained in terms of dimensions of the irreducible representations of the symmetry algebra.
• A standard highest weight vector argument (analogous to that used for the representation theory of simple Lie algebras) can be employed to find these finite dimensional representations. [15, 18, 16]
• The quantum spectrum follows from irreducible representations of the symmetry algebra. The energy levels turn out to be

\[ E_m = -\frac{1}{4}(2m + 1)^2 + \frac{1}{4} + \frac{\alpha^2}{(2m + 1)^2}, \]

with multiplicity \( m + 1 \).
• From the structure equations there is a second solution for the energy levels, but they are complex. However, if we choose the charge on the electron to be pure imaginary then the energy levels are real:

\[ E_n = -\frac{1}{4}(2\mu_n + 1)^2 + \frac{1}{4} - \frac{\alpha^2}{(2\mu_n + 1)^2}, \]

where \(2\mu_n = n + \sqrt{(n+1)^2 + 2\alpha} \) and the charge \(\alpha = ia\) is imaginary. This is an example of PT-symmetry a topic of great current interest in mathematical physics, \([11, 12]\). The potential is complex but the spectrum is real. Superintegrability theory provides a powerful method for finding PT-symmetric systems.

The classical analog of this system, the Kepler system on the 2-sphere is also 2nd order superintegrable. Analogs of Kepler’s 3 laws hold, due to superintegrability. Also the Hohmann transfer procedure for orbital maneuvers in a spherical universe holds.

**Example 3** The generic potential on the \(n\)-sphere: For \(n = 2\) we define the generic sphere system by embedding the unit 2-sphere \(x_1^2 + x_2^2 + x_3^2 = 1\) in three dimensional flat space. The Hamiltonian operator is

\[ H = \sum_{1 \leq i < j \leq 3} (x_i \partial_j - x_j \partial_i)^2 + \sum_{k=1}^3 \frac{a_k}{x_k}, \quad \partial_i \equiv \partial_{x_i}. \]

The 3 operators that generate the symmetries are \(L_1 = L_{12}, L_2 = L_{13}, L_3 = L_{23}\) where

\[ L_{ij} \equiv L_{ji} = (x_i \partial_j - x_j \partial_i)^2 + \frac{a_i}{x_i^2} + \frac{a_j}{x_j^2}, \]

for \(1 \leq i < j \leq 3\). Here,

\[ H = \sum_{1 \leq i < j \leq 3} L_{ij} + \sum_{k=1}^3 a_k = H_0 + V, \quad V = \frac{a_1}{x_1^2} + \frac{a_2}{x_2^2} + \frac{a_3}{x_3^2}. \]

The system is 2nd order superintegrable, \([17, 8]\). The structure equations are:

(i) \(R = \{L_{23}, L_{13}\}\),

(ii) \(\epsilon_{ijk}[L_{jk}, R] = 4\{L_{jk}, L_{ij}\} - 4\{L_{jk}, L_{ik}\} - (8 + 16a_3)L_{ik} + (8 + 16a_k)L_{ij} + 8(a_j - a_k)\),

(iii) \(R^2 = \frac{6}{5}\{L_{23}, L_{13}, L_{12}\} - (16a_1 + 12)L_{23}^2 - (16a_2 + 12)L_{13}^2 - (16a_3 + 12)L_{12}^2 + \frac{52}{45}\{\{L_{23}, L_{13}\} + \{L_{13}, L_{12}\} + \{L_{12}, L_{23}\}\} + \frac{1}{5}(16 + 176a_1)L_{23} + \frac{6}{5}(16 + 176a_2)L_{13} + \frac{6}{5}(16 + 176a_3)L_{12} + \frac{82}{3}(a_1 + a_2 + a_3) + 48(a_1a_2 + a_2a_3 + a_3a_1) + 64a_1a_2a_3\).

(iv) Here \(\epsilon_{ijk}\) is the skew-symmetric tensor and \(\{A, B, C\}\) is the symmetrizer.

This algebra is the structure algebra of the Wilson polynomials. These are most general polynomials that satisfy difference or differential equations: All of the familiar classical orthogonal polynomials are limiting cases of Wilson polynomials and their finite version, the Racah polynomials, \([34]\). The Wilson polynomials \(\Phi_n^{(\alpha,\beta,\gamma,\delta)}(t^2)\) can be expressed as generalized hypergeometric functions of unit argument:

\[ _4F_3 \left( \begin{array}{c} -n, \alpha + \beta + \gamma + \delta + n - 1, \alpha - t, \alpha + t \\ \alpha + \beta, \alpha + \gamma, \alpha + \delta \end{array} ; 1 \right) = (\alpha + \beta)_n(\alpha + \gamma)_n(\alpha + \delta)_n\Phi_n^{(\alpha,\beta,\gamma,\delta)}(1^2). \]
They are eigenfunctions of a divided difference operator
\[ \tau^* \tau \Phi_n = n(n + \alpha + \beta + \gamma + \delta - 1) \Phi_n, \]
where
\[ E_A^1 F(t) = F(t + A), \quad \tau = \frac{1}{2t}(E^{1/2} - E^{-1/2}), \]
\[ \tau^* = \frac{1}{2t}[(\alpha + t)(\beta + t)(\gamma + t)(\delta + t)E^{1/2} - (\alpha - t)(\beta - t)(\gamma - t)(\delta - t)E^{-1/2}]. \]

They appear when we determine the finite and infinite dimensional bounded below irreducible representations of the symmetry algebra for the generic potential system on the 2-sphere and construct a function space model of the algebra action where the symmetry operators are divided difference operators, [34]. Indeed, determining the action of \( L_{12} \) on an \( L_{23} \) eigenbasis we find a 3-term recurrence relations which is identical to the corresponding relation for the Wilson polynomials with the identifications:
\[ 4a_j = \frac{1}{4} - b_j^2, \quad \alpha = -\frac{b_1 + b_2 + 1}{2} - \mu, \quad \beta = \frac{b_1 + b_2 + 1}{2}, \]
\[ \gamma = \frac{b_2 - b_1 + 1}{2}, \quad 4\delta = \frac{b_1 + b_2 - 1}{2} + b + \mu + 2, \]
The algebra is realized by \( H = E \) and
\[ L_{12} = -4t^2 + b_1^2 + b_2^2 \]
\[ L_{23} = -4\tau^* \tau - 2(b_2 + 1)(b_3 + 1) + \frac{1}{2}, \]
\[ -E = \frac{4\mu + 2(b_1 + b_2 + b_3) + 5)(4\mu + 2(b_1 + b_2 + b_3) + 3}{4} + \frac{3}{2} - b_1^2 - b_2^2 - b_3^2. \]

Similarly, the symmetry algebra for superintegrable system of the generic potential on the 3-sphere is the structure algebra for 2-variable Wilson polynomials, a very recent result, [35]. This illustrates the intimate relation between special functions and superintegrable systems. Indeed one can make the case that most special functions are “special” due to their connection with analytically solvable problems, i.e., due to their connection with superintegrable systems.

**Example 4** Another illustration of the the close connection with special functions, is
\[ H = \partial_x^2 + \partial_y^2 + \hbar^2 \omega_1^2 P_1(\omega_1 x) + \hbar^2 \omega_2^2 P_1(\omega_2 y). \]
Here, \( P_1 \) is the first Painlevé transcendent. This system is 3rd order quantum superintegrable but not classically superintegrable. There are other examples with the transcendent \( P_2, P_3, P_4 \), [36, 10].

**3. The problems and the breakthroughs**

For possible superintegrable systems of order \( > 2 \) it is much harder to verify superintegrability and to compute the symmetry algebra. Until about 2 years ago, few examples were known for \( n > 3 \) and there was almost no structure theory. There were two basic problems:

- How to construct numerous examples of superintegrable systems of higher order and in many dimensions, in order to develop enough insight to produce a classification theory.
How to compute efficiently the commutators and products of symmetry operators of arbitrarily high order, to verify superintegrability and to determine the structure equations for the symmetry algebra.

A breakthrough for the first problem came with the publication of two papers by Trembley, Turbiner and Winternitz, [37, 38], the first in 2009. They started with the 2nd order superintegrable system

\[ H = \partial_x^2 + \partial_y^2 - \omega^2(x^2 + y^2) + \frac{\alpha}{x^2} + \frac{\beta}{y^2}, \]

well known to be 2nd order superintegrable. Then they wrote the Schrödinger equation in polar coordinates \( r, \theta \), separable for this system, and replaced \( \theta \) by \( k\theta \) where \( k \) is any rational number. Then after renormalization, the system, which we call the TTW system, became

\[ H = \partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial^2_{\theta} - \omega^2 r^2 + \frac{1}{r^2} \left( \frac{\alpha}{\sin^2(k\theta)} + \frac{\beta}{\cos^2(k\theta)} \right). \]

Note that the TTW eigenvalue equation \( H\Phi = E\Phi \) is still separable in \( r, \theta \) coordinates so it admits \( H \) and the 2nd order operator \( L_2 \) responsible for the separation as symmetries. The TTW system is superintegrable if there exists a third symmetry operator, which may be of arbitrarily high order, dependent on \( k \). They noted that for small values of \( p, q \) these systems were already known to be superintegrable (but usually represented in Cartesian coordinates). For \( k = 1 \) this is the caged isotropic oscillator, for \( k = 1/2 \) it is [E16] in the constant curvature system list in [24], for \( k = 2 \) it is a Calogero system on the line, [39, 40], etc. They conjectured and gave strong evidence that the TTW systems were classically and quantum superintegrable for ALL rational \( k \). This conjecture had broad influence in the community and led to a flurry of activity to prove the conjectures. (As a result of its impact, the first TTW paper won the Journal of Physics A Best Paper Prize for 2011.) This simple approach showed how an infinite number of higher order superintegrable systems could be generated from a single 2nd order system. It was immediately applied to generate numerous other families of higher order superintegrable systems and in \( n > 2 \) variables, [41, 42].

The basic problem for proof of the conjectures was in dealing with symmetries of arbitrarily high order. Already for choices such as \( k = 6 \) the expression for the third symmetry operator required several pages The problem was solved for the classical TTW system in the papers [43, 42, 41]. The quantum problem was solved first in [44] by a very general approach that verified superintegrability but didn’t give information about the structure equations, see also [45, 46, 47]. Then in [48] an explicit method was introduced that enabled superintegrability to be proved and explicit structure equations to be calculated. This is the recurrence approach.

Here is the basic idea for \( n = 2 \):

- We require that the system admit a 2nd order symmetry that determines a separation of variables.
- We note that the formal eigenspaces of the Hamiltonian are invariant under action of any symmetry operator, so the operator must induce recurrence relations for the basis of separated eigenfunctions.
- Suppose that the separated eigenfunctions are of hypergeometric type. Use the known recurrence relations for hypergeometric functions to reverse this process and determine a symmetry operator from the recurrence relations.
- We can compute the symmetry operators and structure equations by restricting ourselves to a formal “basis” of separated eigenfunctions.
- Then we appeal to our theory of canonical forms for symmetry operators to show that results obtained on restriction to a formal eigenbasis actually hold as true identities for purely differential operators defined independent of “basis” functions.
We sketch how this method works for the TTW system, $H\Psi = E\Psi$, $H = L_1$. A formal eigenbasis takes the form

$$\Psi = e^{-\frac{2}{r^2} r^k(2n+a+b+1)} L^k(2n+a+b+1) (\omega r^2) (\sin(k\theta))^{a+\frac{1}{2}} (\cos(k\theta))^{b+\frac{1}{2}} P_n^a,b(\cos(2k\theta)).$$

where $\alpha = k^2(\frac{1}{4} - a^2)$ and $\beta = k^2(\frac{1}{4} - b^2)$. The $L_n^k(z)$ are associated Laguerre functions and the $P_n^a,b(w)$ are Jacobi functions, [34]. We use a gauge transformation to get eigenfunctions

$$\Pi = e^{-\frac{2}{r^2} r^k(2n+a+b+1)} L^k(2n+a+b+1) (\omega r^2) P_n^a,b(x)$$

where $x = \cos(2k\theta)$, $\mu = 2n + a + b + 1$. The energy eigenvalue is

Energy : $E = -2\omega [2(m + nk) + 1 + (a + b + 1)k]$ and the 2nd order symmetry operator responsible for the separation of variables is

Separation : $L_2\Psi = (\partial_\theta^2 + \frac{\alpha}{\sin^2(k\theta)} + \frac{\beta}{\cos^2(k\theta)})\Psi = -k^2\mu^2\Psi$

where $\mu = 2n + a + b + 1$, $k = p/q$. Recall that the separation constant is just the eigenvalue of the symmetry operator $L_2$.

Our strategy is to look for recurrences that change $m$, $n$ but fix $u = m + nk$. This will map eigenfunctions of $H$ corresponding to eigenvalue $E$ to eigenfunctions that correspond to the same eigenvalue $E$. The two transformations

(1) $n \rightarrow n + q$, $m \rightarrow m - p$,
(2) $n \rightarrow n - q$, $m \rightarrow m + p$

each achieve this.

First, consider $X_n^{a,b}(x)$. Raise or lower $n$ with

(1) : $J_n^+ X_n^{a,b} = (2n + a + b + 2)(1 - x^2) \partial_x X_n^{a,b}$

$$+(n + a + b + 1)(-2n + a + b + 2)x - (a - b))X_n^{a,b} = 2(n + 1)(n + a + b + 1)X_{n+1}^{a,b},$$

(2) : $J_n^- X_n^{a,b} = -2n + a + b)(1 - x^2) \partial_x X_n^{a,b}$

$$-n((2n + a + b)x - (a - b))X_n^{a,b} = 2(n + a)(n + b)X_{n-1}^{a,b}.$$ For $Y_{m}^{k,\mu}(R) = \omega^{k/2} Y_{m}^{k,\mu}(r)$ where $R = r^2$ change $m$ by

$$K_{k\mu,m}^{+} Y_{m}^{k,\mu} = \left\{ (k\mu + 1) \partial_R - \frac{E}{4} - \frac{1}{2R} k\mu(k\mu + 1) \right\} Y_{m}^{k,\mu}$$

$$= -\omega Y_{m-1}^{k,\mu+2},$$

$$K_{k\mu,m}^{-} Y_{m}^{k,\mu} = \left\{ (-k\mu + 1) \partial_R - \frac{E}{4} + \frac{1}{2R} k\mu(1 - k\mu) \right\} Y_{m}^{k,\mu}$$

$$= -\omega(m + 1)(m + k\mu) Y_{m+1}^{k,\mu-2}.$$
From these recurrences we construct the two operators

\[ \Xi_+ = K_{k\mu+2(p-1),m-(p-1)}^+ \cdots K_{k\mu,m}^+ J_{n+q-1}^+ \cdots J_n^+ \]

\[ \Xi_- = K_{k\mu-2(p-1),m+p-1}^- \cdots K_{k\mu,m}^- J_{n-q+1}^- \cdots J_n^- . \]

For fixed \( u = m + kn \), we have

\[ \Xi_+ \Psi_n = 2^q (-1)^p \omega^p (n+1) \eta(n + a + b + 1) \eta \Psi_{n+q}, \]

\[ \Xi_- \Psi_n = 2^q \omega^p (-n-a) \eta(-n-b) \eta(u - kn + 1) \eta(-u - k[n + a + b + 1]) \eta \Psi_{n-q}. \]

These are basis dependent operators. However, under the transformation \( n \rightarrow -n - a - b - 1 \), i.e., \( \mu \rightarrow -\mu \), we have \( \Xi_+ \rightarrow \Xi_- \) and \( \Xi_- \rightarrow \Xi_+ \). Thus \( \Xi = \Xi_+ + \Xi_- \) is a polynomial in \( \mu^2 \) and \( u \) is unchanged under this transformation.

Therefore we can replace \((2n + a + b + 1)^2 = \mu^2 \) by \( L_2/k^2 \) and \( E \) by \( H \) everywhere they occur, and express \( \Xi \) as a pure differential symmetry operator.

Note also that under the transformation \( n \rightarrow -n - a - b - 1 \), i.e., \( \mu \rightarrow -\mu \) the operator \( \Xi_+ - \Xi_- \) changes sign, hence

\[ \tilde{\Xi} = \frac{\Xi_+ - \Xi_-}{\mu} \]

is unchanged under this transformation. This defines \( \tilde{\Xi} \) as a symmetry operator. We set

\[ L_3 = \Xi, \quad L_4 = \tilde{\Xi}. \]

- We have shown that \( L_3, L_4 \) commute with \( H \) on any formal eigenbasis.
- In fact, we have constructed pure differential operators which commute with \( H \), independent of basis. This takes some proof and we have verified this in general using a canonical form for higher order differential operators, [44].
- Indeed, we can prove that operator relations verified on formal eigenbases of separated solutions must actually hold identically.

Applying a product of raising and lowering operators to a basis function we obtain

\[ \Xi_- \Xi_+ \Psi_n = (-1)^p 2^q \omega^p (n+1) \eta(n + a + 1) \eta(n + b + 1) \eta \]

\[ \times (n + a + b + 1) \eta(-u + kn) \eta(u + k[n + a + b + 1] + 1) \eta \Psi_n \]

\[ = \xi_n \eta_n \Psi_n . \]

\[ \Xi_+ \Xi_- \Psi_n = (-1)^p 2^q \omega^p (-n-a) \eta(-n-b) \eta \]

\[ \times (-n-a-b) \eta(u - kn + 1) \eta(-u - k[n + a + b + 1]) \eta \Psi_n \]

\[ = \eta_n \xi_n \Psi_n . \]

Thus \( \Xi^{(+)} = \Xi_- \Xi_+ + \Xi_+ \Xi_- \) multiplies any basis function by \( \xi_n + \eta_n \). However, the transformation \( n \rightarrow -n - a - b - 1 \), i.e. \( \mu \rightarrow -\mu \) maps \( \Xi_- \Xi_+ \leftrightarrow \Xi_+ \Xi_- \) and \( \xi_n \leftrightarrow \eta_n \).

Thus \( \Xi^{(+)} \) is an even polynomial operator in \( \mu \), polynomial in \( u \), and \( \xi_n + \eta_n \) is an even polynomial function in \( \mu \), polynomial in \( u \). Furthermore, each of \( \Xi_- \Xi_+ \) and \( \Xi_+ \Xi_- \) is unchanged under \( u \rightarrow -u - (a + b + 1) - 1 \), hence a polynomial of order \( p \) in \([2u + (a + b + 1)k + 1]^2 \).

We conclude that

\[ \Xi^{(+)} = P^{(+)}(H^2, L_2, \omega^2, a, b). \]

Similarly

\[ \Xi^{(-)} = (\Xi_- \Xi_+ - \Xi_+ \Xi_-)/\mu = P^{(-)}(H^2, L_2, \omega^2, a, b). \]
4. Structure equations

Setting \( R = -4k^2qL_3 - 4k^2q^2L_4 \), we find the structure equations

\[
[L_2, L_4] = R, \quad [L_2, R] = -8k^2q^2\{L_2, L_4\} - 16k^4q^4L_4,
\]
\[
[L_4, R] = 8k^2q^2L_4^2 - 8k^2qP(-)(H^2, L_2, \omega^2, a, b),
\]
\[
\frac{3}{8k^2q^2}R^2 = -22k^2q^2L_4^2 + \{L_2, L_4, L_4\} + 4k^2qP(-)(H^2, L_2, \omega^2, a, b) + 12k^2P(+)(H^2, L_2, \omega^2, a, b).
\]

Thus \( H, L_2, L_4 \) generate a closed algebra. Note that the symmetries \( P(-), P(+) \) can be of arbitrarily high order, depending on \( p \) and \( q \), but we have explicit compact expressions for these symmetries.

There is a complication: Examples show that \( L_4 \) is not the lowest order generator. To find the lowest order generator we look for a symmetry operator \( L_5 \) such that \( [L_2, L_5] = L_4 \). Applying this condition to a formal eigenbasis of functions \( \Psi_n \) we obtain the general solution

\[
L_5 = -\frac{1}{4qk^2}\left(\frac{\Xi_+}{(\mu + q)\mu} + \frac{\Xi_-}{(\mu - q)\mu}\right) + \frac{\beta_n}{\mu^2 - q^2},
\]

where \( \beta_n \) is a polynomial function of \( H \) to be determined.

Simple algebra gives

\[
L_5(-\mu^2 + q^2) \equiv L_5(\frac{L_2}{k^2} + q^2) = \frac{1}{4qk^2}(L_4 - qL_3) - \beta_n.
\]

To find \( \beta_n \) we take the limit as \( \mu \to -q \). There are 3 cases, depending on the relative parities of \( p \) and \( q \). For \( p \), \( q \) both odd we have

\[
\beta_n \equiv Q(H) = -\frac{H(a^2 - b^2)}{4}\Pi^{(p-1)/2}_{\ell=1}\left[(-\omega^2)(\frac{H}{4\omega} - \ell)(\frac{H}{4\omega} + \ell)\right] \times \Pi^{(q-1)/2}_{s=1}\left[\frac{1}{4}(-a - b + 2s)(a + b + 2s)(a - b + 2s)(-a + b + 2s)\right]
\]

The other cases are similar.

Note that the raising and lowering operators \( \Xi_\pm \) are key to the success of this method. See also [46]. They themselves are not symmetry operators, indeed they are not even differential operators. However, all of the true symmetry operators can be constructed from them and the structure equations computed explicitly with their help.

These results can be extended to systems in \( n \geq 3 \) variables, and there are analogs of the raising and lowering operators for classical superintegrable systems, [49].

5. Final comments

- The most interesting aspect of superintegrability theory is the symmetry algebra, its irreducible representations and their realization on function spaces. Many fascinating particular results are known, but there is no general representation theory for these algebras. It should be analogous to the representation theory of simple Lie algebras.
- There are quantum superintegrable systems in which the separated eigenfunctions are not of hypergeometric type. How do we treat these systems?
- In many cases in dimensions \( n \geq 3 \) the symmetry algebra closes rationally but not polynomially. To our knowledge, the representation theory of such rational algebras has not been studied.
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