Two-Dimensional Dilaton Gravity and Toda - Liouville Integrable Models

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November 27, 2008

Abstract

General properties of a class of two-dimensional dilaton gravity (DG) theories with multi-exponential potentials are studied and a subclass of these theories, in which the equations of motion reduce to Toda and Liouville equations, is treated in detail. A combination of parameters of the equations should satisfy a certain constraint that is identified and solved for the general multi-exponential model. From the constraint it follows that in DG theories the integrable Toda equations, generally, cannot appear without accompanying Liouville equations. We also show how the wave-like solutions of the general Toda-Liouville systems can be simply derived. In the dilaton gravity theory, these solutions describe nonlinear waves coupled to gravity as well as static states and cosmologies. A special attention is paid to making the analytic structure of the solutions of the Toda equations as simple and transparent as possible, with the aim to gain a better understanding of realistic theories reduced to dimensions 1+1 and 1+0 or 0+1.

1 Introduction

The theories of (1 + 1)-dimensional dilaton gravity coupled to scalar matter fields are known to be reliable models for some aspects of higher-dimensional black holes, cosmological models and waves. The connection between higher and lower dimensions was demonstrated in different contexts of gravity and string theory and, in several cases, has allowed finding the general solution or special classes of solutions in high-dimensional theories1. A generic example is the spherically symmetric gravity coupled to Abelian gauge fields and scalar matter fields. It exactly reduces to a (1+1)-dimensional dilaton gravity and can be explicitly solved if the scalar fields are constants independent of the coordinate2. These solutions can describe interesting physical objects – spherical static black holes and simplest cosmologies. However, when the scalar matter fields, which presumably play a significant cosmological role, are nontrivial, not many exact analytical solutions of high-dimensional theories are known3. Correspondingly, the two-dimensional models of DG that nontrivially couple to scalar matter are usually not integrable.

To construct integrable models of this sort one usually must make serious approximations, in other words, deform the original two-dimensional model obtained by direct dimensional reductions of realistic higher-dimensional theories. Nevertheless, the deformed models can quali-

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1See, e.g., [1]-[28] for a more detailed discussion of this connection, references, and solution of some integrable two-dimensional and one-dimensional models of dilaton gravity.

2This is not possible for arbitrary dependence of the potentials on the scalar fields, as will be clear in a moment.

3See, e.g., [8], [11], [12], [17]-[23]; a review and further references can be found in [20], [27] and [29].
tatively describe certain physically interesting solutions of higher-dimensional gravity or supergravity theories related to the low-energy limit of superstring theories. We note that several important four-dimensional space-times with symmetries defined by two commuting Killing vectors may also be described by two-dimensional models of dilaton gravity coupled to scalar matter. For example, cylindrical gravitational waves can be described by a $(1+1)$−dimensional dilaton gravity coupled to one scalar field [29]-[31], [22]. The stationary axially symmetric pure gravity ([32], [11]) is equivalent to a $(0+2)$−dimensional dilaton gravity coupled to one scalar field. Similar but more general dilaton gravity models were also obtained in string theory. Some of them can be solved by using modern mathematical methods developed in the soliton theory (see e.g. [1], [2], [11], [19]). Note also that the theories in dimension 1+0 (cosmologies) and 0+1 (static states, in particular black holes) may be integrable in spite of the fact that their 1+1 dimensional ‘parent’ theory is not integrable without a deformation (see [23] and an example given in this paper).

In our previous work (see, e.g., [20] - [23] and references therein) we constructed and studied some explicitly integrable models based on the Liouville equation. Recently, we attempted to find solutions of some realistic two-dimensional dilaton gravity models (derived from higher-dimensional gravity theories by dimensional reduction) using a generalized separation of variables introduced in [21], [22]. These attempts showed that seemingly natural ansatzes for the structure of the separation, which proved a success in previously studied integrable models, do not give interesting enough solutions (‘zero’ approximation of a perturbation theory) in realistic nonintegrable models. Thus an investigation of more complex dilaton gravity models, which are based on the two dimensional Toda chains, was initiated in [24].

At first sight it seems that it should be not difficult to find a potential in DG theory that will give integrable Toda equations of motion. However in reality it is not as simple as that, and the Toda theory may only emerge in company with a Liouville theory (this was mentioned in footnote in ref. [24]). In fact, even the $N$−Liouville theory satisfies the same constraint. It was known to the authors of [23] and [24] since long time but the meaning of this fact was not clearly understood.

In this paper we first introduce the general multi-exponential DG and present the equations of motion in a form that resembles the Toda equations. In addition to the equations, in the DG theory one should satisfy two extra equations which in General Relativity are called the energy and momentum constraints. In the $N$−Liouville theory these constraints were explicitly solved but in the general case solving the constraints is a difficult problem which we discuss in Section 4.

Section 3 is devoted to the problem of reconstructing the dilaton gravity from the ‘one-exponential’ form of the equation of motion

$$\partial_u \partial_v x_m = g_m \exp \sum_n A_{mn} x_n. \quad (1)$$

This amounts to finding the matrix $\hat{a}$ satisfying the matrix equation

$$\hat{a}^T \hat{e} \hat{a} = \hat{A} \quad (\hat{e} \text{ is a diagonal matrix to be introduced later}).$$

Evidently, this equation may have many solutions for a fixed matrix $\hat{A}$ (e.g., if $\hat{a}$ is a solution, then $\hat{O} \hat{a}$, where $\hat{O}^T \hat{e} \hat{O} = 1$, is also a solution). The important fact is however that the solution is not possible for an arbitrary symmetric matrix $\hat{A}^T = \hat{A}$. In Section 3 we establish the class of ‘solvable’ matrices $\hat{A}$ (satisfying the A-condition) and introduce a recursive procedure in order to find all possible solutions for any matrix satisfying the A-condition.

We show that the Cartan matrices for simple Lie groups do not satisfy the A-condition and thus the generic DG cannot be reduced to the Toda equations. However, adding at least one Liouville equation to the Toda system (Toda - Liouville System, or TL) solves this constraint and in Section 4 we briefly introduce the simplest form of solution of TLS in the

\[\text{We call it the A-equation.}\]
case of the $A_n$ Cartan matrices. We also discuss the problem of the energy and momentum constraints and solve the constraints for a class of Toda-Liouville theories.

Finally, we briefly discuss possible applications of our results to the theory of black holes, cosmological models and waves which, at least in integrable theories, are closely related.

## 2 Multi-exponential model of (1+1)-dimensional dilaton gravity minimally coupled to scalar matter fields.

The effective Lagrangian of the (1+1)-dimensional dilaton gravity coupled to scalar fields $\psi_n$ obtainable by dimensional reductions of a higher-dimensional spherically symmetric (super)gravity can usually be (locally) transformed to the form:

$$\mathcal{L}^{(2)} = \sqrt{-g} \left[ \varphi R(g) + V(\varphi, \psi) + \sum_{m,n} Z_{mn}(\varphi, \psi) g^{ij} \partial_i \psi_m \partial_j \psi_n \right]$$  \hspace{1cm} (2)

(see [20] - [23] for a detailed motivation and examples). In Eq. (2), $g_{ij}(x^0, x^1)$ is the (1+1)-dimensional metric with signature (-1,1), $g \equiv \det(g_{ij})$, $R$ is the Ricci curvature of the two-dimensional space-time with the metric

$$ds^2 = g_{ij} \, dx^i \, dx^j, \quad i, j = 0, 1. \hspace{1cm} (3)$$

The effective potentials $V$ and $Z_{mn}$ depend on the dilaton $\varphi(x^0, x^1)$ and on $N - 2$ scalar fields $\psi_n(x^0, x^1)$ (we note that the matrix $Z_{mn}$ should be negative definite to exclude the so-called ‘phantom’ fields). They may depend on other parameters characterizing the parent higher-dimensional theory (e.g., on charges introduced in solving the equations for the Abelian fields). Here we consider the ‘minimal’ kinetic terms with diagonal and constant $Z$-potentials, $Z_{mn}(\varphi, \psi) = \delta_{mn} Z_n$. This approximation excludes the important class of the sigma-model-like scalar matter discussed, e.g., in [23]; such models can be integrable if $V \equiv 0$ and $Z_{mn}(\varphi, \psi)$ satisfy certain rather stringent conditions. In [2] we also used the Weyl transformation to eliminate the gradient term for the dilaton. To simplify derivations, we write the equations of motion in the light-cone metric, $ds^2 = -4f(u, v) \, du \, dv$. Now, by first varying the Lagrangian in generic coordinates and then passing to the light-cone coordinates we obtain the equations of motion ($Z_n$ are constants!)

$$\partial_u \partial_v \varphi + f \, V(\varphi, \psi) = 0, \hspace{1cm} (4)$$

$$f \partial_i (\partial_i \varphi / f) = \sum_{n} Z_n \left( \partial_i \psi_n \right)^2, \quad i = u, v. \hspace{1cm} (5)$$

$$2Z_n \partial_u \partial_v \psi_n + f \, V_{\psi_n}(\varphi, \psi) = 0, \hspace{1cm} (6)$$

$$\partial_u \partial_v \ln |f| + f \, V_\varphi(\varphi, \psi) = 0, \hspace{1cm} (7)$$

where $V_{\varphi} \equiv \partial_{\varphi} V$, $V_{\psi_n} \equiv \partial_{\psi_n} V$. These equations are not independent. Actually, (7) follows from (4) - (6). Alternatively, if (4), (5), and (7) are satisfied, one of the equations (6) is also satisfied. Note that the equations may have the solution with $\psi_n = \psi_n^0 = \text{const}$ only if $V_{\psi_n}(\varphi, \psi_n^0) \equiv 0$.

The higher-dimensional origin of the Lagrangian (2) suggests that the potential is the sum of exponentials of linear combinations of the scalar fields and of the dilaton $\varphi$. In our previous work [23] we studied the constrained Liouville model, in which the system of equations of motion (4), (5) and (7) is equivalent to the system of independent Liouville equations for the linear combinations of fields $q_n \equiv F + \psi_n^0$, where $F \equiv \ln |f|$. The easily derived solutions of these equations should satisfy the constraints (5), which was the most difficult part of the problem. The solution of the whole problem revealed an interesting structure of the moduli space of

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*Actually, the potential $V$ usually contains terms non exponentially depending on $\varphi$ (e.g., linear in $\varphi$), and then the exponentiation of $\varphi$ is only an approximation, see the discussion in [23].
the solutions that allowed us to easily identify static, cosmological and wave-like solutions and effectively embed these essentially one-dimensional (in a broad sense) solutions into the set of all two-dimensional solutions and study their analytic and asymptotic properties.

Here we propose a natural generalization of the Liouville model to the model in which the fields are described by the Toda equations (or by nonintegrable deformations of them). To demonstrate that the model shares many properties with the Liouville one and to simplify a transition from the integrable models to nonintegrable theories we suggest a different representation of the Toda solutions which is not directly related to their group-theoretical background.

Consider the theory defined by the Lagrangian (2) with the potential \( Z_n = -1 \):

\[
V = \sum_{n=1}^{N} 2g_n \exp q_n^{(0)}, \quad q_n^{(0)} = a_n \varphi + \sum_{m=3}^{N} \psi_m a_{mn}.
\]

In what follows we also use

\[
q_n \equiv F + q_n^{(0)} \equiv \sum_{m=1}^{N} \psi_m a_{mn},
\]

where \( \psi_1 + \psi_2 \equiv \ln |f| \equiv F, \psi_1 - \psi_2 \equiv \varphi \) and hence \( a_{1n} = 1 + a_n, a_{2n} = 1 - a_n \).

Rewriting the equations of motion in terms of \( \psi_n \) we find that Eqs. (4) - (7) are equivalent to \( N \) equations of motion for \( N \) functions \( \psi_n \) (\( \varepsilon \) is the sign of the metric \( f \)),

\[
\partial_\alpha \partial_\beta \psi_n = \varepsilon \sum_{m=1}^{N} \epsilon_n a_{nm} g_m \exp (q_m) \quad (\epsilon_1 = -1, \ \epsilon_n = +1 \text{ if } n \geq 2),
\]

and two constraints,

\[
C_i \equiv \partial_i^2 \varphi + \sum_{n=1}^{N} \epsilon_n (\partial_\psi \psi_n)^2 = 0, \quad i = u, v.
\]

With arbitrary parameters \( a_{nm} \), these equations of motion are not integrable. But as proposed in [16] - [18], [20] [23], Eqs. (10) are integrable and the constraints (11) can be solved if the \( N \)-component vectors \( v_n \equiv (a_{mn}) \) are pseudo-orthogonal.

Now, consider more general nondegenerate matrices \( a_{mn} \) and define the new scalar fields \( x_n \):

\[
x_n \equiv \sum_{m=1}^{N} a^{-1}_{nm} \epsilon_m \psi_m, \quad \psi_n \equiv \sum_{m=1}^{N} \epsilon_n a_{nm} x_m.
\]

In terms of these fields, Eqs. (10) read as

\[
\partial_\alpha \partial_\beta x_m = \varepsilon g_m \exp \left( \sum_{k,n=1}^{N} \epsilon_n a_{nm} a_{nk} x_k \right) \equiv \varepsilon g_m \exp \left( \sum_{k=1}^{N} A_{mk} x_k \right),
\]

and we see that the symmetric matrix

\[
\hat{A} \equiv \hat{a}^T \hat{\epsilon} \hat{a}, \quad \epsilon_{mn} \equiv \epsilon_m \delta_{mn},
\]

defines the main properties of the model.

If \( \hat{A} \) is a diagonal matrix we return to the \( N \)-Liouville model. If \( \hat{A} \) were the Cartan matrix of a simple Lie algebra, the system (13) would coincide with the corresponding Toda system, which is integrable and can be more or less explicitly solved (see, e.g., [33], [34]). However, it can be shown that the Cartan matrices of the simple Lie algebras (symmetrized when necessary) cannot be represented in the form (14). Nevertheless, a very simple extension of the Toda equations obtained by adding one or more Liouville equations can solve this problem. In fact, a symmetric matrix \( A_{mn} \) that is the direct sum of a diagonal \( L \times L \)-matrix \( \gamma_n^{-1} \delta_{mn} \) and of an
arbitrary symmetric matrix $A_{mn}$, can be represented in form \((14)\) if the sum of $\gamma_n^{-1}$ is a certain function of the matrix elements $A_{mn}$. If $A_{mn}$ is a Cartan matrix, the system \((13)\) thus reduces to $L$ independent Liouville (Toda $A_1$) equations and the higher-rank Toda system (TLS).

The solution of TLS can be derived in several ways. The most general one is provided by the group-theoretical construction described in \[33\], \[34\]. Here, in Section 4 we outline an analytical method directly applicable to solving $A_N$ TLS proposed in \[24\]. However, solving the equations of motion is not the whole story. Once the equations are solved, their solutions must be constrained to satisfy the zero energy-momentum conditions \((11)\) that in terms of $x_n$ are:

\[ - C_i = 2 \sum_{n=1}^{N} \partial^2_n x_n - \sum_{n,m=1}^{N} \partial_c x_m A_{mn} \partial_c x_n = 0, \quad i = u, v. \]  

(15)

In the $N$-Liouville model the most difficult problem was to satisfy the constraints \((15)\) but this problem was eventually solved. In the general nonintegrable case of an arbitrary matrix $A$, we do not know even how to approach this problem. The Toda case is discussed below.

To study the general properties of the solutions of equations \((13)\) and of the constraints \((15)\) we first rewrite the general equations in a form that is particularly useful for the Toda-Liouville systems. Introducing notation

\[ X_n \equiv \exp(-\frac{1}{2}A_{mn}x_n), \quad \Delta_2(X) \equiv X \partial_u \partial_v X - \partial_u X \partial_v X, \quad \alpha_{mn} \equiv -2A_{mn}/A_{nn}, \]  

(16)

it is easy to rewrite Eqs.\((13)\) in the form:

\[ \Delta_2(X_n) = -\frac{1}{2} \varepsilon g_n A_{nn} \prod_{m \neq n} X_n^{\alpha_{nm}}. \]  

(17)

The multiplier $|-\frac{1}{2} \varepsilon g_n A_{nn}|$ can be removed by using the transformation $x_n \mapsto x_n + \delta_n$ and the final (standard) form of the equations of motion is

\[ \Delta_2(X_n) = \varepsilon_n \prod_{m \neq n} X_n^{\alpha_{nm}}, \quad \varepsilon_n \equiv \pm 1. \]  

(18)

These equations are in general not integrable. However, when $A_{mn}$ are Toda plus Liouville matrices, they simplify to integrable equations (see \[33\]). The Liouville part is diagonal while the Toda part is non-diagonal. For example, for the Cartan matrix of $A_N$, only the near-diagonal elements of the matrix $\alpha_{mn}$ are nonvanishing, $\alpha_{n+1,n-1} = \alpha_{n-1,n+1} = 1$. This allows one to solve Eq.\((18)\) for any $N$. The parameters $\alpha_{mn}$ are invariant w.r.t. transformations $x_n \mapsto \lambda_n x_n + \delta_n$. This means that the non-symmetric Cartan matrices of $B_N$, $C_N$, $G_2$, and $F_4$ can be symmetrized while not changing the equations. In this sense, $\alpha_{mn}$ are the fundamental parameters of the equations of motion. From this point of view, the characteristic property of the Cartan matrices is the simplicity of Eqs.\((18)\) which allow one to solve them by a generalization of separation of variables. As is well known, when $A_{mn}$ is the Cartan matrix of any simple algebra, this procedure gives the exact general solution (see \[33\]). In Section 4 we show how to construct the exact general solution for the $A_N$ Toda system and write a convenient representation for the general solution that differs from the standard one given in \[33\].

Unfortunately, as we emphasized above, solving equations \((18)\) is not sufficient for finding the solution of the whole problem. We also must solve the constraints \((15)\), and this is a more difficult task. In our previous papers we succeeded in solving the constraints of the $N$-Liouville theory. So, let us try to formulate the problem of the constraints in the Toda-Liouville case as close as possible to the $N$-Liouville case. First, it is not difficult to show that $\partial_v C_u = \partial_u C_v = 0$ and thus $C_u = C_u(u)$, $C_v = C_v(v)$ as in the Liouville case. To prove this one should differentiate \((15)\) and use \((13)\) to get rid of $\partial_u \partial_v x_m$ and $\partial_u \partial_v x_n$. 


Up to now we considered an arbitrary symmetric matrix $\hat{A}$. At this point we should use a more detailed information about $A_{mn}$ and about the structure of the solution. To see whether the constraints can be solved we first rewrite them in terms of $X_n$ and then consider the Toda - Liouville matrices and the explicit solutions of the equations. It is not difficult to see that the constraints (15) can be written in the form ($i = u$ or $i = v$ and the prime denotes $\partial_i$):

$$
\frac{1}{4} C_i = \sum_{n=1}^{N} \frac{1}{A_n} X_n'' + \sum_{m<n}^{N} \frac{2A_{mn}}{A_mA_n} X_m' X_n'.
$$

(19)

The first term looks exactly as in the case of the $N$–Liouville model. However, in the Liouville case we also knew that

$$
\partial_u \left( X_n^{-1} \partial_u^2 X_n \right) = 0, \quad \partial_v \left( X_n^{-1} \partial_v^2 X_n \right) = 0,
$$

(20)

which is not true in the general case. Moreover, the first and the second terms in r.h.s. of Eq.(19) are in general not functions of a single variable (above we have only proved that in general $C_u = C_u(u)$ and $C_v = C_v(v)$).

Nevertheless, let us try to push the analogy with the Liouville case as far as possible, at least in the integrable Toda - Liouville case. Thus, suppose that the first $N_1$ equations are the Toda ones and the remaining $N_2 = N - N_1$ equations are the Liouville ones. This means that $A_{mn} = \hat{A}_{mn} (1 \leq m, n \leq N_1)$, where $\hat{A}_{mn}$ is a Cartan matrix while for $N_1 + 1 \leq m, n \leq N$ we have $A_{mn} = \delta_{mn}\gamma_n^{-1}$. Then the constraints split into the Toda and the Liouville parts:

$$
\frac{1}{4} C_i = \sum_{n=1}^{N_1} \frac{1}{A_n} X_n'' + \sum_{m<n}^{N_1} \frac{2A_{mn}}{A_mA_n} X_m' X_n' + \sum_{n=N_1+1}^{N} \gamma_n \frac{X_n''}{X_n}. \tag{21}
$$

They are significantly different: first, because the Liouville solutions $X_n$ for $n \geq N_1 + 1$ satisfy the second order differential equation while the Toda solutions $X_n$ satisfy higher order ones (see Section 4). In the general $A_N$ Toda case $X_1$ can be written as

$$
X_1 = \sum_{i,j=1}^{N_1+a} a_i(u) b_j(v), \tag{22}
$$

while in the Liouville case the solution is simply the sum of two terms and (see Section 4). Moreover, for the Liouville solution we have

$$
X_n^{-1} \partial_{u}^2 X_n = \frac{a_n''(u)}{a_n(u)}, \quad X_n^{-1} \partial_{v}^2 X_n = \frac{b_n''(v)}{b_n(v)}, \tag{23}
$$

while in the Toda case everything is much more complex.

To understand better this fact we consider the case $N_1 = 2$, $N = 3$ with $A_{mn}(1 \leq m, n \leq 2)$ being the $A_2$– Cartan matrix and $A_{3n} = \delta_{3n}A_3$. Using $A_1 = A_2 = 2$, $A_{12} = A_{21} = -1$, we find

$$
\frac{1}{2} C_i = \left( \frac{X_2'}{X_1} + \frac{X_3'}{X_2} - \frac{X_1'}{X_1} \cdot \frac{X_2'}{X_2} \right) - 4 \frac{X_3'}{X_3} = 0 \tag{24}
$$

where $X_2 = \varepsilon_1 \Delta_2(X_1)$, $\varepsilon_2 = \pm 1$, $X_3$ is the Liouville solution (note that according to the constraint on $A_{ij}$ we have in this case $\gamma_3 = A_3^{-1} = -2$). Although we know that $X_3''/X_3$ and $C_i$ are functions of one variable, we do not have at the moment simple and explicit expressions for $C_i$. Indeed, using (22) it is not difficult to find that

$$
\partial_v (X_1^{-1} \partial_u^2 X_1) = \left( \sum_{j=1}^{3} a_j b_j \right)^{-2} \sum_{i>j} W'[a_i, a_j] W[b_i, b_j] \neq 0. \tag{25}
$$

So, we should first write the explicit expression for $X_2(u, v)$ in terms of $a, b$, and then derive the complete first term in $C_i$. We construct solutions of the $A_2 + A_1$ constraints in Section 4.
3 Solving $\hat{a}^T \hat{e} \hat{a} = \hat{A}$

In this section we show how to solve Eq. (14) for the matrix $\hat{a}$ in the standard DG. This is possible if and only if $\hat{A}$ satisfies certain conditions, which we explicitly derive. First, $\det \hat{A} = - \det \hat{a}^2 < 0$. This restricts the matrices $\hat{A}$ of even order but is not so severe a restriction for the odd order matrices. In fact, we can then change sign of $\hat{A}$ and of all the variables $x_n$ and the only effect will be that all $\varepsilon_n$ in Eq. (15) change sign. If these signs are unimportant and the two systems of equations may be considered as equivalent, the restriction does not work. As the determinants of all (symmetrized) Cartan matrices for simple groups are positive (and their eigenvalues are positive), it follows that the even-order Cartan matrices do not satisfy this restriction. A more severe restriction is related to the special structure of the matrices $a_{mn}$ in (9). In consequence, the matrix $\hat{A}$ must satisfy one equation that we derive and explicitly solve below.

Let us now take the general $N \times N$ matrix $\hat{a}$ of DG, with the only restriction: $a_{1n} = 1 + a_n$ and $a_{2n} = 1 - a_n$. The equations defining $a_{mn}$ in terms of $A_{mn}$ are

$$-2(a_m + a_n) + V_m \cdot V_n = A_{mn}, \quad -4a_n = A_n - V_n^2, \quad m, n = 1, \ldots, N$$

(26)

where we introduced notation $V_n \equiv (a_{3n}, \ldots, a_{Nn})$. As follows from (26), our $N$ vectors $V_i$ in the $(N - 2)$-dimensional space have $N(N - 2)$ components and satisfy $N(N - 1)/2$ equations:

$$(V_m - V_n)^2 = A_m + A_n - 2A_{mn}, \quad m > n, \quad m, n = 1, \ldots, N.$$ \hspace{1cm} (27)

These equations are invariant under $(N - 2)(N - 3)/2$ rotations of the $(N - 2)$-dimensional space and under $N - 2$ translations. It follows that the vectors $V_m$ in fact depend on invariant parameters. The $N(N - 1)/2$ equations should define $(N - 2)(N + 1)/2$ parameters. Thus one can see that the number of equations minus the number of parameters is equal to one, and thus one of the equations will give a relation between the parameters.

It is possible to give a more constructive approach directly utilizing the invariant equations that follow from the equations (27) above. Define $v_k \equiv V_k - V_1$, where $k = 2, \ldots, N$. Then, from (27) we have:

$$v_k^2 \equiv (V_k - V_1)^2 = A_1 + A_k - 2A_{1k} \equiv \tilde{A}_{1k},$$

$$(v_k - v_l)^2 \equiv \tilde{A}_{1k} + \tilde{A}_{1l} - 2v_k \cdot v_l, \quad k > l; \quad k, l = 2, \ldots, N.$$ \hspace{1cm} (28)

Thus the general invariant equations for $v_k$ can be written:

$$v_k \cdot v_l = A_1 - A_{1k} - A_{1l} + A_{kl}, \quad k \geq l.$$ \hspace{1cm} (29)

As these equations are valid also for $l = k$ we have $N(N - 1)/2$ equations for the same number of the invariant parameters $v_k \cdot v_l$, as it should be. But, of course, there is one relation between these parameters because there exist a linear relation between $N - 1$ vectors $v_k$ in the $(N - 2)$-dimensional space. For example, $v_k^2$ can be expressed in terms of the remaining parameters $v_2^2, \ldots, v_{N-1}^2$ and $v_k \cdot v_l$, $k > l$ (their number is $(N - 2)(N + 1)/2$, as above). As the equations for $v_k$ express $v_k \cdot v_l$ in terms of the matrix elements $A_{kl}$, we thus can derive the necessary relation between $A_{kl}$ (e.g. an expression of $A_1 \equiv A_{11}$ in terms of the remaining matrix elements).

Using the vectors $v_k$ we can give an explicit construction of the solutions and derive the constraint on the matrix elements $A_{mn}$. The construction of the solution of the equations for $a_{mn}$ can be given as follows. It is not difficult to understand that we only need to find the unit vectors,

$$\hat{v}_k \equiv v_k/|v_k| = v_k \tilde{A}_{1k}^{-1/2},$$ \hspace{1cm} (29)
in any fixed coordinate system in the \((N - 2)\)-dimensional space. Then we can reconstruct the general solution by applying to \(\hat{v}_k\) rotations and translations (i.e., choosing arbitrary \(a_{n1}, n = 3, ..., N\)). Let us introduce the temporary notation
\[
c_{kl} \equiv \cos \theta_{kl} \equiv \hat{v}_k \cdot \hat{v}_l = (A_1 - A_{1k} - A_{1l} + A_{kl}) (\tilde{A}_{1k} \tilde{A}_{1l})^{-1/2}. \tag{30}
\]
As \(v_k = (a_{3k} - a_{31}, ..., a_{Nk} - a_{N1})\), we denote \(\alpha_{nk} \equiv (a_{nk} - a_{n1})/|v_k|\) and thus \(\hat{v}_k = (\alpha_{3k}, ..., \alpha_{Nk})\). Choosing the coordinate system in which \(\hat{v}_2 = (1, 0, .. 0)\) we see that \(\alpha_{3k} = c_{k2} \equiv \cos \theta_{2k}\) and \(\hat{v}_3\) can be chosen with two nonvanishing components,
\[
\hat{v}_3 = (c_{23}, s_{23}, 0, ..., 0), \tag{31}
\]
where \(s_{23} \equiv \sin \theta_{23}\) and in general \(s_{kl} = \sin \theta_{kl}\). The further invariant parameters \(\alpha_{nk}\) can be derived recursively. The vectors \(\hat{v}_k, ..., \hat{v}_N\) for \(k \geq 4\) are constructed as follows (it is easy to check!). We take \(\alpha_{3k} = c_{2k}, \alpha_{nk} = 0\) if \(k \leq N - 2\) and \(n \geq k + 2\). Thus
\[
\hat{v}_k = (c_{2k}, \alpha_{4k}, \alpha_{5k}, ..., \alpha_{(k+1)k}, 0, 0, ...) \tag{32}
\]
and the parameters \(\alpha_{nk}\) can be recursively derived from the relations \((k \geq 4)\)
\[
\sum_{n=4}^{l+1} \alpha_{nk} \alpha_{nl} = c_{kl} - c_{k2} c_{l2}, \quad k > l, \quad \sum_{n=4}^{k+1} \alpha_{nk}^2 = s_{k2}^2, \quad k \leq N - 1. \tag{33}
\]
The normalization condition for \(\hat{v}_N\) (not included in the above equations),
\[
\sum_{n=4}^{N} \alpha_{nN}^2 = s_{N2}^2, \tag{34}
\]
then gives a relation between the \(c_{kl}\)'s (and thus between the \(A_{ij}\)'s).

Using this solution we can find the expression for \(A_1 \equiv A_{11}\) in terms of \(A_{kl}\). However, this derivation is rather awkward. It can be somewhat simplified if we consider simpler matrices \(A_{kl}\) for which \(A_{1k} = A_{k1} = 0, k \neq 1\). Then one can find that the equation for \(A_1\) is linear and thus has the unique solution. Nevertheless, it is not a good idea to derive the constraint on \(A_{kl}\) in this rather indirect way. The linearity of the constraint in \(A_1\) suggests that there exists a simple and general formula directly expressing \(A_1\) in terms of the other elements \(A_{kl}\).

The simplest way to find \(A_1\) in terms of the other \(A_{ij}\) is the following: one of the vectors \(v_2, v_3, ..., v_N\) must be given by a linear combination of \(N - 2\) other vectors. Suppose that
\[
v_2 = \sum_{p=3}^{N} v_p \hat{z}_p. \tag{35}
\]
Then we can find \(\hat{z}_p\) in terms of \(A_{mn}\) by solving the equations
\[
v_p \cdot v_2 = \sum_{q=3}^{N} (v_p \cdot v_q) \hat{z}_q, \quad p = 3, ..., N. \tag{36}
\]
The solution is given by \(\hat{z}_p = D_p/D\), where \(D\) is the determinant of the \((N - 2) \times (N - 2)\) matrix \((v_p \cdot v_q)\), and the \(D_p\) are the determinants of the same matrix but with the \(p\)-th column replaced by \((v_p \cdot v_2)\).

Now it is clear that the expression of \(v_2^2\) in terms of the solution of \(\text{[36]}\),
\[
v_2^2 = \sum_{q=3}^{N} (v_2 \cdot v_q) \hat{z}_q = \sum_q (v_2 \cdot v_q) \cdot D_q/D, \tag{37}
\]
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gives us the desired constraint on $A_{mn}$. Using (28) we rewrite it in the form

$$(A_1 + A_2 - 2A_{12}) D = \sum_{p=3}^{N} (A_1 + A_{p2} - A_{12} - A_{1p}) D_p,$$

where the determinants $D$ and $D_p$ should be expressed in terms of $A_{mn}$. They evidently depend on $A_1$ linearly and thus Eq. (38) is at most quadratic in $A_1$. In fact, it is just linear. To prove this it is sufficient to show that

$$\frac{dD}{dA_1} = \sum_{p=3}^{N} \frac{dD_p}{dA_1}.$$  

(39)

This is not very difficult but we omit the proof because of the space restrictions.

4 Solution of the $A_N$ Toda system

The equations (18) for the $A_N$-theory are extremely simple,

$$\Delta_2(X_n) = \varepsilon_n X_{n-1} X_{n+1}, \quad X_0 \mapsto 1, \quad X_{N+1} \mapsto 1, \quad n = 1, \ldots, N,$$

(40)

where $\varepsilon_n^2 = 1$. As is well known, their solution can be reduced to solving just one higher-order equation for $X_1$ by using the relation (see [33]):

$$\Delta_2(\Delta_n(X)) = \Delta_{n-1}(X) \Delta_{n+1}(X), \quad \Delta_1(X) \equiv X, \quad n \geq 2.$$  

(41)

Indeed, using Eqs. (40), (41) one can prove that for $n \geq 2$

$$X_n = \Delta_n(X_1) \prod_{k=1}^{[n/2]} \varepsilon_{n+1-2k},$$  

(42)

where the square brackets denote the integer part of $n/2$. Thus the condition $X_{N+1} = 1$ gives the equation for $X_1$,

$$\Delta_{N+1}(X_1) = \prod_{k=1}^{[(N+1)/2]} \varepsilon_{N+2-2k} \equiv \tilde{\varepsilon}_{N+1} = \pm 1.$$  

(43)

This equation looks horrible but it is known to be exactly solubles by a special separation of variables, Eq. (22). We present its solution in a form that is equivalent to the standard one [33] but is more compact and more suitable for constructing effectively one-dimensional solutions, generalizing those studied in [23].

Let us start with the Liouville ($A_1$ Toda) equation $\Delta_2(X) = \tilde{\varepsilon}_2 \equiv \varepsilon_1$ (see [33], [35], [33], [23]). Calculating the derivatives of $\Delta_2(X)$ in the variables $u$ and $v$, it is not difficult to prove Eqs. (20). It follows that there exist some ‘potentials’ $U(u), V(v)$ such that

$$\partial_u^2 X - U(u) X = 0, \quad \partial_v^2 X - V(v) X = 0,$$

(44)

and thus $X$ can be written in the ‘separated’ form given in [22] with $N = 1$ where $a_i(u), b_j(u)$ ($i, j = 1, 2$) are linearly independent solutions of the equations (Eq. (23) follows from this):

$$a_i''(u) - U(u) a_i(u) = 0, \quad b_j''(v) - V(v) b_j(v) = 0.$$  

(45)

For $i = 1$ these equations define the potentials for any choice of $a_1, b_1$, while $a_2, b_2$ then can be derived from the Wronskian first-order equations

$$W[a_1(u), a_2(u)] = w_a, \quad W[b_1(v), b_2(v)] = w_b, \quad w_a w_b = \varepsilon_1.$$  

(46)
We have repeated this well known derivation because it is applicable to the $A_N$ Toda equation \((43)\). By similar derivations it can be shown that $X_1$ satisfies the equations
\[
\partial v^{N+1} X + \sum_{n=0}^{N-1} U_n(u) \partial v^n X = 0, \quad \partial v^{N+1} X + \sum_{n=0}^{N-1} V_n(v) \partial v^n X = 0. \tag{47}
\]
Thus the solution of \((43)\) can be written in the same ‘separated’ form \((22)\), where now $a_i(u), b_i(v)$ ($i = 1, \ldots, N + 1$) satisfy the ordinary linear differential equations corresponding to \((17)\), with the constant Wronskians normalized by the conditions (one can choose any other normalization in which the product of the two Wronskians is the same):
\[
W[a_1(u), \ldots, a_{N+1}(u)] = w_a, \quad W[b_1(v), \ldots, b_{N+1}(v)] = w_b, \quad w_aw_b = \bar{\epsilon}_{N+1}. \tag{48}
\]
The potentials $U_n(u), V_n(v)$ can easily be expressed in terms of the arbitrary functions $a_i(u)$ and $b_i(v)$, $i = 1, \ldots, N$. To find the expressions one should differentiate the determinants \((48)\) to obtain the homogeneous differential equations for $a_{N+1}(u), b_{N+1}(v)$. For example, for $N = 2$:
\[
U_1(u) = -(a_1 a_2'' - a_2' a_2') / W[a_1, a_2], \quad U_0(u) = (a_1' a_2'' - a_1'' a_2') / W[a_1, a_2]. \tag{49}
\]
Let us return to the general solution of Eq.\((43)\). In fact, considering Eqs.\((48)\) as inhomogeneous differential equations for $a_{N+1}(u), b_{N+1}(v)$ with arbitrary chosen functions $a_i(u), b_i(v)$ ($1 \leq i \leq N$), it is easy to write the explicit solution of this problem:
\[
a_{N+1}(u) = \sum_{i=1}^{N} a_i(u) \int_{u_0}^{u} \frac{d\bar{u}}{W_N^{2}(\bar{u})} M_{N,i}(\bar{u}). \tag{50}
\]
Here $W_N \equiv W[a_1(u), \ldots, a_N(u)]$ is the Wronskian of $N$ arbitrary chosen functions $a_i$ and $M_{N,i}$ are the complementary minors of the last row in the Wronskian. (Replacing $a$ by $b$ and $u$ by $v$ we can find the expression for $b_{N+1}(v)$ from the same formula \((50)\)). For the simplest $A_2$-case:
\[
a_3(u) = \sum_{i=1}^{2} a_i(u) \int_{u_0}^{u} \frac{d\bar{u}}{W_{2}^{2}(\bar{u})} M_{2,i}(\bar{u}) \equiv \int_{u_0}^{u} d\bar{u} \frac{a_1(\bar{u})a_2(\bar{u}) - a_1(u)a_2(\bar{u})}{(a_1(\bar{u})a_2'(\bar{u}))^2 - (a_1'(\bar{u})a_2(\bar{u}))^2}. \tag{51}
\]
Thus we have found the expression for the basic solution $X_1$ in terms of $2N$ arbitrary chiral functions $a_i(u)$ and $b_i(v)$. To complete constructing the solution we should derive the expressions for all $X_n$ in terms of $a_i$ and $b_i$. This can be done with simple combinatorics that allows one to express $X_n$ in terms of the $n$-th order minors. For example, it is easy to derive the expressions for $X_2$:
\[
X_2 = \varepsilon_1 \Delta_2(X_1) = \varepsilon_1 \sum_{i<j} W[a_i(u), a_j(u)] W[b_i(v), b_j(v)], \tag{51}
\]
which is valid for any $N \geq 1$ ($i, j = 1, \ldots, N + 1$). Note that expressions for all $X_n$ have a similar separated form with higher-order determinants.

Our simple representation of the $A_N$ Toda solution is completely equivalent to what one can find in \((33)\) but is more convenient for treating some problems. For example, it is useful in discussing asymptotic and analytic properties of the solutions of the original physical problems. It is especially appropriate for constructing wave-like solutions of the Toda system which are similar to the wave solutions of the $N$-Liouville model. In fact, quite like the Liouville model, the Toda equations support the wave-like solutions. To derive them let us first identify the moduli space of the Toda solutions. Recalling the $N$-Liouville case, we may try to identify the moduli space with the space of the potentials $U_n(u), V_n(v)$. Possibly, this is not the best choice and, in fact, in the Liouville case we finally made a more useful choice suggested by the solution of the constraints. For our present purposes the choice of the potentials is as good as any other because
each choice of \( U_0(u) \) and \( V_n(v) \) defines some solution and, vice versa, any solution given by the set of the functions \((a_1(u), \ldots, a_{N+1}(u)), (b_1(v), \ldots, b_{N+1}(v))\) satisfying the Wronskian constraints \([48]\) defines the corresponding set of potentials \((U_0(u), \ldots, U_{N-1}(u)), (V_0(v), \ldots, V_{N-1}(v))\).

Now, as in the Liouville case, we may consider the reduction of the moduli space to the space of constant ‘vectors’ \((U_0, \ldots, U_{N-1}), (V_0, \ldots, V_{N-1})\). The fundamental solutions of the equations \([47]\) with these potentials are exponentials (in the nondegenerate case): \(\exp(\mu_i u), \exp(\nu_i v)\). Then \(X_1\) can be written as (for simplicity we take \(f_i > 0\)):

\[
X_1 = \sum_{i=1}^{N+1} a_i(u)b_i(v) = \sum_{i=1}^{N+1} f_i \exp(\mu_i u) \exp(\nu_i v) \equiv \sum_{i=1}^{N+1} \exp[\mu_i u + u_i] \exp[\nu_i v + v_i],
\]

where the parameters must satisfy the conditions \([48]\). Calculating the determinant \(\Delta_{N+1}(X_1)\) and denoting the standard Vandermonde determinants by

\[
D_\mu = \prod_{i>j}(\mu_i - \mu_j), \quad D_\nu = \prod_{i>j}(\nu_i - \nu_j),
\]

one can easily find that \([48]\) is satisfied if

\[
\sum_{i=1}^{N+1} \mu_i = \sum_{i=1}^{N+1} \nu_i = 0, \quad \prod_{i=1}^{N+1} f_i D_\mu D_\nu = \tilde{\varepsilon}_{N+1}.
\]

By the way, instead of the last condition we could write the equivalent conditions \([48]\):

\[
\prod_{i=1}^{N+1} \exp u_i = w_a, \quad \prod_{i=1}^{N+1} \exp v_i = w_b, \quad w_aw_b = (D_\mu D_\nu)^{-1}\tilde{\varepsilon}_{N+1},
\]

where \(\exp u_i\) and \(\exp v_i\) are not necessary positive (e.g., we can make \(\exp u_i\) negative by supposing that \(u_i\) has the imaginary part \(i\pi\)) but here we mostly consider positive \(f_i\).

In this reduced case we may regard the space of the parameters \((\mu_i, \nu_i, u_i, v_i)\) as the new moduli space, in complete agreement with the Liouville case. Having the basic solution \(X_1\) given by Eqs.\([52]-[53]\) it is not difficult to derive \(X_n\) recursively by using \([10]\). For illustration, consider the simplest TL theory \(A_1 + A_2\). Then \(X_2\) is given by \([51]\) and \([52]-[53]\):

\[
X_2 = \tilde{\varepsilon}_2(D_\mu D_\nu)^{-1}\sum_{i=1}^{3} \exp[-\mu_i u - u_i] \exp[-\nu_i v - v_i](\mu_j - \mu_k)(\nu_j - \nu_k),
\]

where \((ijk)\) is a cyclic permutation of \((123)\). The next step is to consider the constraints \([21]\), where \(X_3\) is the solution of the Liouville equation (in order not to mix it with the \(X_3\) of the \(A_2\)-solution that is equal to 1, we better denote it by \(\tilde{X}\)). Of course, we should suppose that this solution has the form \([22]\) with exponential functions \([52]\). In the Liouville case \(N = 2\) and thus \(\tilde{X''}/\tilde{X}\) is simply \(\tilde{\mu}^2\) or \(\tilde{\nu}^2\) (see \([23]\)).

Now, using Eqs.\([52]-[55]\), one can find that the constraints are equivalent to the following equations:

\[
\sum_{i<j}(\mu_i - \mu_k)(\nu_j - \nu_k)[3\mu_k^2 - C_\mu] = 0, \quad \sum_{i<j}(\mu_i - \mu_k)(\nu_j - \nu_k)[3\nu_k^2 - C_\nu] = 0,
\]

\[
\mu_1^2 + \mu_2^2 + \mu_1\mu_2 = C_\mu, \quad \nu_1^2 + \nu_2^2 + \nu_1\nu_2 = C_\nu,
\]

where \(C_\mu\) and \(C_\nu\) represent contribution of the Liouville term. Computing the sums in Eq.\([56]\) we find that equations \([56]\) are equivalent to the relations

\[
[(\mu_1^2 + \mu_2^2 + \mu_1\mu_2) - C_\mu] \sum \mu_i\nu_i = 0, \quad [(\nu_1^2 + \nu_2^2 + \nu_1\nu_2) - C_\nu] \sum \mu_i\nu_i = 0,
\]

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which are satisfied as soon as Eqs.\((57)\) are satisfied.

It is not difficult to check that the potentials \(U_1(u), V_1(u)\) for the exponential solutions are
\[
U_1(u) = -(\mu_1^2 + \mu_2^2 + \mu_1 \mu_2), \quad V_1(u) = -(\nu_1^2 + \nu_2^2 + \mu_1 \nu_2),
\]
and thus the constraints have extremely simple and natural form:
\[
U_1 + C_\mu = 0, \quad V_1 + C_\nu = 0.
\]

For the (1+1)-dimensional \(A_2\) Toda plus Liouville case we have found that the constraints \((21)\) with any number of Liouville terms are satisfied for the general solution (i.e. if we put into \((60)\) the expression \((49)\)). Note that in case of just one Liouville term this does not help to find an explicit solution of the constraint. However, if the number of the Liouville terms in Eq.\((21)\) is greater than two, and if \(\sum \gamma_n\) for these terms vanishes, one can easily derive the explicit general solution by applying the method described in \([20],[23]\). A detailed account of these results will be published elsewhere.

5 Conclusion

Let us briefly summarize the main results and possible applications. We introduced a simple and compact formulation of the general (1+1)-dimensional dilaton gravity with multi-exponential potentials and derived the conditions allowing to find its explicit solutions in terms of the Toda theory. The simplest class of theories satisfying these conditions is the Toda-Liouville theory.\(^6\)

We proposed a simple approach to solving the equations and constraints in the case of the \(A_N\) Toda part.

Of special interest are simple exponential solutions derived in the last section. They explicitly unify the static (black hole) solutions\(^7\), cosmological models, and waves of the Toda matter coupled to gravity. Some of these solutions can be related to cosmologies with spherical inhomogeneities or to evolving black holes but this requires special studies. Earlier we studied similar but simpler solutions in the \(N\)-Liouville theories in paper \([23]\). The main results of that paper, in particular, the existence of nonsingular exponential solutions, are true also in the Toda-Liouville theory.

Note that one-dimensional Toda-Liouville cosmological models were met long time ago in dimensional reductions of higher-dimensional (super)gravity theories (see, e.g., \([15]\)). Considerations of the two-dimensional TL theories of this paper are equally applicable to the one-dimensional case. A preliminary discussion can be found in \([24]\) and the detailed consideration will be published elsewhere, together with a detailed presentation of the results that were only briefly described here.

Finally, note that here we only give an account of the first part of the report presented at the workshop ‘Quarks-2008’ (see the presentation of our report at the site \(\text{http://inr.ac.ru}\)). In the second part, a brief summary and a new interpretation of A.Einstein’s paper \([37]\) was proposed by one of the present authors (ATF). The proposal is that Einstein’s theory (that he regarded as a unified theory of gravity and electromagnetism) is in fact a first unified model of dark energy (dictated by the geometry cosmological constant) and dark matter (dictated by the geometry neutral massive vector field coupled to gravity only). Unfortunately even one-dimensional spherically symmetric reductions of this theory are not integrable (a preliminary analysis\(^8\) of these solutions can be found in \(\text{http://atfilippov.googlepages.com/ogiev.ppt}\)).

\(^6\) In \([24]\), it was shown that the models with the potential independent of the dilaton \(\varphi\) can be explicitly solved if \(A_{mn}\) is any Cartan matrix. In this case adding the Liouville part is unnecessary.

\(^7\)These solutions normally have two horizons defined by zeroes of the metric, i.e. \(F \to -\infty\). In the Liouville case they were analyzed in \([12],[15],[20],[23]\).

\(^8\)It is interesting that the static solutions may have two horizons, like the Reissner - Nordstrom black holes, although there is no electric charge in the model.
Acknowledgment:
One of the authors appreciates financial support from the Department of Theoretical Physics of the University of Turin and INFN (Turin Section), where this work was completed.
This work was supported in part by the Russian Foundation for Basic Research (Grant No. 06-01-00627-a).

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