GLOBAL WELL-POSEDNESS OF THE MHD EQUATIONS IN A HOMOGENEOUS MAGNETIC FIELD

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ABSTRACT. In this paper, we study the MHD equations with small viscosity and resistivity coefficients, which may be different. This is a typical setting in high temperature plasmas. It was proved that the MHD equations are globally well-posed if the initial velocity is close to 0 and the initial magnetic field is close to a homogeneous magnetic field in the weighted Hölder space, where the closeness is independent of the dissipation coefficients.

1. INTRODUCTION

In this paper, we consider the incompressible magneto-hydrodynamics (MHD) equations in \([0, T) \times \Omega, \Omega \subseteq \mathbb{R}^d\):

\[
\begin{align*}
\partial_t v - \nu \Delta v + v \cdot \nabla v + \nabla p &= b \cdot \nabla b, \\
\partial_t b - \mu \Delta b + v \cdot \nabla b &= b \cdot \nabla v, \\
\text{div } v &= \text{div } b = 0,
\end{align*}
\]

where \(v\) denotes the velocity field and \(b\) denotes the magnetic field, and \(\nu \geq 0\) is the viscosity coefficient, \(\mu \geq 0\) is the resistivity coefficient. If \(\nu = \mu = 0\), (1.1) is the so called ideal MHD equations; If \(\nu > 0\) and \(b = 0\), (1.1) is reduced to the Navier-Stokes equations. We refer to \([18]\) for the mathematical introduction to the MHD equations.

It is well-known that the 2-D MHD equations with full viscosities(i.e., \(\nu > 0\) and \(\mu > 0\)) have global smooth solution. In general case, the question of whether smooth solution of the MHD equations develops singularity in finite time is basically open \([18, 9]\). Recently, Cao and Wu \([6]\) studied the global regularity of the 2-D MHD equations with partial dissipation and magnetic diffusion. We refer to \([7, 8, 10, 11, 12, 13]\) and references therein for more relevant results.

In this paper, we are concerned with the global well-posedness of the MHD equations in a homogeneous magnetic field \(B_0\). Recently, there are a lot of works \([1, 14, 16, 17, 19]\) devoted to the case without resistivity (i.e., \(\nu > 0\) and \(\mu = 0\)). Roughly speaking, it was proved that the MHD equations are globally well-posed and the solution decays in time if the initial velocity field is close to 0 and the initial magnetic field is close to \(B_0\). These results especially justify the numerical observation \([5]\): the energy of the MHD equations is dissipated at a rate independent of the ohmic resistivity.

In high temperature plasmas, both the viscosity coefficient \(\nu\) and resistivity coefficient \(\mu\) are usually very small \([5]\). Up to now, the heating mechanism of the solar corona is still an unsolved problem in physics \([15]\). So, it is very interesting to investigate the long-time dynamics of the MHD equations in the case when the dissipation coefficients are very small.

For the simplicity, let us first look at the ideal case. Following \([2]\), we rewrite the system (1.1) in terms of the Elsässer variables

\[Z_+ = v + b, \quad Z_- = v - b.\]
Then the ideal MHD equations (1.1) can be written as

\[
\begin{aligned}
\partial_t Z_+ + Z_- \cdot \nabla Z_+ &= -\nabla p, \\
\partial_t Z_- + Z_+ \cdot \nabla Z_- &= -\nabla p, \\
\text{div} Z_+ = \text{div} Z_- &= 0.
\end{aligned}
\]  

We introduce the fluctuations

\[
z_+ = Z_+ - B_0, \quad z_- = Z_- + B_0.
\]

Then the system (1.2) can be reformulated as

\[
\begin{aligned}
\partial_t z_+ + Z_- \cdot \nabla z_+ &= -\nabla p, \\
\partial_t z_- + Z_+ \cdot \nabla z_- &= -\nabla p, \\
\text{div} z_+ = \text{div} z_- &= 0.
\end{aligned}
\]  

In the case of \( \Omega = \mathbb{R}^d \), Bardos-Sulem-Sulem [2] proved that for large time, the solution \( z_\pm \) of (1.3) tends to linear Alfvén waves:

\[
\partial_t w_\pm \pm \pm B_0 \cdot \nabla w_\pm = 0.
\]

In two recent works [3] and [4], Cai-Lei and He-Xu-Yu studied the global well-posedness of (1.1) for any \( \nu = \mu \geq 0 \) and \( \Omega = \mathbb{R}^3 \). The result in [3] also includes the case of \( \Omega = \mathbb{R}^2 \).

From the physical point of view, it is more natural to consider the MHD equations in a domain with the boundary. One frequently used domain in physics is a slab bounded by two hyperplanes, i.e., \( \Omega = \mathbb{R}^{d-1} \times [0,1] \). More importantly, although both \( \nu \) and \( \mu \) are very small, they should be different in the real case. However, the proof in [3, 4] strongly relies on the facts that \( \Omega \) is a whole space and \( \nu = \mu \).

The main goal of this paper is to prove the global well-posedness of (1.1) in the physical case when \( \Omega \) is a slab and \( \nu \neq \mu \). In this case, we need to impose the suitable boundary conditions on \( z_\pm \). Let \( z_\pm \) be a function of \((t,x,y),(x,y) \in \Omega\). In the case when \( \nu = \mu = 0 \), we impose the nonpenetrating boundary condition

\[
z_\pm^d = 0 \quad \text{on} \quad y = 0,1.
\]  

In the case when \( \nu > 0 \) and \( \mu > 0 \), we impose the Navier-slip boundary condition

\[
z_\pm^d = 0, \quad \partial_i z_\pm^i = 0 \quad i = 1, \cdots, d-1 \quad \text{on} \quad y = 0,1.
\]  

To deal with the boundary case, our idea is to use the symmetric extension and solve the MHD equations in the framework of Hölder spaces \( C^{1,\alpha} \) for \( 0 < \alpha < 1 \). In the ideal case, we give a representation formula of the pressure by using the symmetric extension. Although the extended solution has not the same regularity as the origin one under the nonpenetrating boundary condition, we find that \( \nabla p \) still lies in \( C^{1,\alpha} \) based on the representation formula. In the viscous case, we can reduce the slab domain to \( \Omega = \mathbb{R}^{d-1} \times \mathbb{T} \) by using the symmetric extension, because the extended solution still keeps the \( C^{1,\alpha} \) regularity under the Navier-slip boundary condition.

The most challenging task comes from the case of \( \nu \neq \mu \). In the case of \( \nu = \mu \), the following formulation plays a crucial role in the proof of [3, 4]:

\[
\begin{aligned}
\partial_t z_+ + Z_- \cdot \nabla z_+ &= \nu \Delta z_+ - \nabla p, \\
\partial_t z_- + Z_+ \cdot \nabla z_- &= \nu \Delta z_- - \nabla p.
\end{aligned}
\]

Indeed, the viscosity leads to more technical troubles compared with the ideal case. To handle the case of \( \nu \neq \mu \), we need to introduce many new ideas. First of all, we introduce a key
decomposition: let $\mu_1 = \frac{\nu + \mu}{2}$, $\mu_2 = \frac{\nu - \mu}{2}$, and we decompose $z_+ = z_+^{(1)} + z_+^{(2)}$, $z_- = z_-^{(1)} + z_-^{(2)}$ so that

$$
\begin{align*}
\partial_t z_+^{(1)} + Z_- \cdot \nabla z_+^{(1)} &= \mu_1 \Delta z_+^{(1)} - \nabla p_+^{(1)}, \\
\partial_t z_-^{(1)} + Z_+ \cdot \nabla z_-^{(1)} &= \mu_1 \Delta z_-^{(1)} - \nabla p_-^{(1)}, \\
\partial_t z_+^{(2)} + Z_- \cdot \nabla z_+^{(2)} &= \mu_2 \Delta z_+^{(2)} + \mu_2 \Delta z_- - \nabla p_+^{(2)}, \\
\partial_t z_-^{(2)} + Z_+ \cdot \nabla z_-^{(2)} &= \mu_2 \Delta z_-^{(2)} + \mu_2 \Delta z_+ - \nabla p_-^{(2)}.
\end{align*}
$$

The next task is to establish a closed uniform estimate for the solution $z_+^{(1)}$ and $z_-^{(2)}$ with respect to $\mu_1$ and $t$. For this end, we need the following key ingredients:

- The construction of the weighted Hölder spaces for the solution. Due to the appearance of the extra trouble term $\Delta z_\pm$, we have to work in the spaces with different regularity and weight for the solution $z_+^{(1)}$ and $z_-^{(2)}$. Such incompatibility gives rise to the essential difficulties. Especially, the choice of the weight is very crucial.

- Uniform estimates of the transport equation in the weighted Hölder spaces, which are used to control the growth of Lagrangian map.

- Uniform estimates for the parabolic equation with variable coefficients. The proof is based on the uniform estimates of heat operator in the weighted Hölder spaces.

- Boundedness of Riesz transform and its commutator in the weighted Hölder spaces, which is essentially used to handle the nonlocal pressure. To our knowledge, these results are new, and the proof is highly nontrivial.

In this paper, we consider the MHD equations in a homogeneous magnetic field. In the real case (for example, solar corona), it is more natural to consider the MHD equations in an inhomogeneous magnetic field. An important question is to consider the decay of Alfvén waves in an inhomogeneous magnetic field $B_0(y) = (b_1(y), b_2(y), 0)$. This is similar to the situation of Landau damping.

2. The weighted Hölder spaces and symmetric extension

2.1. Weighted Hölder spaces. Let $\Omega \subseteq \mathbb{R}^d$ be a domain and $\alpha \in (0, 1]$. We denote by $C^{k,\alpha}(\Omega)$, $(k = 0, 1)$ the Hölder space equipped with the norm

$$
|u|_{0,\alpha;\Omega} \overset{\text{def}}{=} |u|_{0;\Omega} + |u|_{0,\alpha;\Omega}, \quad |u|_{1,\alpha;\Omega} \overset{\text{def}}{=} |u|_{0;\Omega} + \|\nabla u\|_{0,\alpha;\Omega},
$$

where

$$
|u|_{0;\Omega} = \sup_{X \in \Omega} |u(X)|, \quad |u|_{0,\alpha;\Omega} = \sup_{X,Y \in \Omega} \frac{|u(X) - u(Y)|}{|X - Y|^\alpha}.
$$

Let $h(X) \in C(\mathbb{R}^d)$ be a positive bounded function. We introduce the following weighted $C^{k,\alpha}$ norms

$$
|u|_{0,\alpha;h,\Omega} \overset{\text{def}}{=} |u|_{0;h,\Omega} + |u|_{0,\alpha;h,\Omega}, \quad |u|_{1,\alpha;h,\Omega} \overset{\text{def}}{=} |u|_{0;h,\Omega} + \|\nabla u\|_{0,\alpha;h,\Omega},
$$

where

$$
|u|_{0;h,\Omega} = \frac{|u|}{h}, \quad |u|_{0,\alpha;h,\Omega} = \sup_{X,Y \in \Omega} \frac{|u(X) - u(Y)|}{(h(X) + h(Y))|X - Y|^\alpha}.
$$

We say that $u \in C^{k,\alpha}_h(\Omega)$ if $|u|_{k,\alpha;h,\Omega} < +\infty$. We also introduce

$$
|u|_{k,\alpha;h,\Omega,T} \overset{\text{def}}{=} \sup_{0 \leq t \leq T} |u(t)|_{k,\alpha;h(t),\Omega}.
$$
When $\Omega = \mathbb{R}^d$, we will omit the subscript $\Omega$ in the norm of Hölder spaces.

The following two lemmas can be proved by using the definition of Hölder norm.

**Lemma 2.1.** Let $h, h_1, h_2$ be the weight functions so that there exists a constant $c_0$ so that

\[
0 < c_0 h(X) \leq h(Y) \quad \text{for any } X, Y \in \mathbb{R}^d, \ |X − Y| \leq 2.
\]

Then there exists a constant $C$ depending only on $c_0$ so that for $k = 0, 1$,

\[
\begin{align*}
|u|_{0, α; h, Ω} &\leq C(|u|_{0, h, Ω} + |∇u|_{0, h, Ω}), \\
|uw|_{k, α; h_1 h_2, Ω} &\leq C|u|_{k, α; h_1, Ω}|w|_{k, α; h_2, Ω}, \\
\int_t^s u(r)dr &\leq \sup_{t ≤ r ≤ s} |u(r)|_{k, α; h(r), Ω}.
\end{align*}
\]

**Lemma 2.2.** Let $Φ$ be a map from $Ω$ to $Ω$ with $∇Φ ∈ C^{0, α}(Ω)$. It holds that

\[
\begin{align*}
|u ∘ Φ|_{0, α; h_0 Φ, Ω} &\leq |u|_{0, α; h, Ω} \max \{|∇Φ|^0_{0, Ω} , 1\}, \\
|u ∘ Φ|_{1, α; h_0 Φ, Ω} &\leq |u|_{1, α; h, Ω} \max \{|∇Φ|^1_{0, Ω} , 1\} \max \{|∇Φ|_{0, Ω} , 1\}.
\end{align*}
\]

Here and in what follows, $|∇Φ|$ denotes the matrix norm defined by

\[
|A| \overset{\text{def}}{=} \sup_{|X| = 1} |AX|.
\]

To deal with the viscous case, we introduce the following scaled weighted Hölder space. Let $α ∈ (0, 1), R ≥ 0$ and define

\[
\begin{align*}
|u|_{0, α; h, R} &\overset{\text{def}}{=} |u|_{0; h} + R^α |u|_{0, h}, \\
|u|_{1, α; h, R} &\overset{\text{def}}{=} |u|_{0, α; h} + \max(R, R^{1-α}) |∇u|_{0, α; h, R}.
\end{align*}
\]

For this kind of weight spaces, we have the analogous of Lemma 2.1 and Lemma 2.2. For example, if $h(X)$ satisfies

\[
0 < c_0 h(X) \leq h(Y) \quad \text{for any } X, Y \in \mathbb{R}^d, \ |X − Y| ≤ 2R.
\]

Then for $R ≥ 1$, we have

\[
|u|_{0, h} + R|∇u|_{0, α; h, R} ≤ |u|_{1, α; h, R} ≤ |u|_{0, α; h, R} + R|∇u|_{0, α; h, R} ≤ C(|u|_{0; h} + R|∇u|_{0, α; h, R}).
\]

Here $C$ is a constant depending only on $c_0$. In the following, we will fix $α ∈ (0, 1)$.

**Lemma 2.3.** Let $γ > 0$ and $h(X) > 0$. Then there exists a constant $C$ independent of $h, γ, t$ so that

\[
\begin{align*}
\int_0^t u(s)ds &\overset{0 < s < t}{\leq} Cγ^{-1} \sup_{0 < s < t} \left( (γs)^{\frac{1}{2}}(γ(t − s))^{\frac{1}{2}} |u(s)|_{0, α; h} + \varphi_α(\sqrt{k + γs})(γ(t − s))^{\frac{3−α}{2}} |∇u(s)|_{1; h}\right),
\end{align*}
\]

where $\varphi_α(R) = \max(R, R^{1+α})$.

**Proof.** We denote by $Cγ^{-1}A$ the right hand side of the inequality. Then we have

\[
\begin{align*}
\int_0^t u(s)ds &\overset{0 < s < t}{\leq} \int_0^t |u(s)|_{0, α; h}ds ≤ \int_0^t (γs)^{-\frac{1}{2}}(γ(t − s))^{-\frac{1}{2}} dsA ≤ Cγ^{-1}A, \\
|∇u|_{0, h} &\overset{0 < s < t}{\leq} \int_0^t |∇u(s)|_{0, h}ds ≤ \int_0^t \varphi_α(\sqrt{k + γs})^{-1}(γ(t − s))^{-1+\frac{α}{2}} dsA.
\end{align*}
\]
Lemma 2.5. It holds that

\[
(\gamma + t)^{-\frac{1}{2}}, (k + \gamma t)^{-\frac{1}{2}} A.
\]

For any \(X, Y \in \mathbb{R}^d\), we have

\[
|\nabla u(s, X) - \nabla u(s, Y)| \leq |X - Y|(h(X) + h(Y))|\nabla u(s)|_{1,h},
\]
\[
|\nabla u(s, X) - \nabla u(s, Y)| \leq |\nabla u(s, X)| + |\nabla u(s, Y)| \leq (h(X) + h(Y))|\nabla u(s)|_{0,h}.
\]

This gives

\[
|\nabla u(s, X) - \nabla u(s, Y)|
\]
\[
\leq \min((\gamma + t)^{\frac{1}{2}}, |X - Y|)(h(X) + h(Y))(|\nabla u(s)|_{1,h} + (\gamma + t)^{-\frac{1}{2}}|\nabla u(s)|_{0,h})
\]
\[
\leq \min((\gamma + t)^{\frac{1}{2}}, |X - Y|)(h(X) + h(Y))^\alpha (\sqrt{k + \gamma s} - 1)(\gamma + t)^{-\frac{1}{2}} A,
\]

therefore,

\[
|\nabla \int_0^t u(s)ds(X) - \nabla \int_0^t u(s)ds(Y)| \leq \int_0^t |\nabla u(s, X) - \nabla u(s, Y)|ds
\]
\[
\leq \int_0^t \min((\gamma + t)^{\frac{1}{2}}, |X - Y|)(h(X) + h(Y))^\alpha (\sqrt{k + \gamma s} - 1)(\gamma + t)^{-\frac{1}{2}} Ads
\]
\[
\leq C(h(X) + h(Y))A\left(\min((\gamma t)^{\frac{1}{2}}, |X - Y|) \int_0^t \phi(\sqrt{k + \gamma s})^{-\frac{1}{2}} ds(t)^{-\frac{1}{2}} A
\]
\[
+ \int_0^t \min((\gamma + t)^{\frac{1}{2}}, |X - Y|)(\gamma + t)^{-\frac{1}{2}} ds \phi(\sqrt{k + \gamma t} - 1)
\]
\[
\leq C(h(X) + h(Y))A\left((\gamma t)^{\frac{1}{2}} |X - Y|^\alpha \phi(\sqrt{k + \gamma t} - 1)(\gamma t)^{-\frac{1}{2}} A
\]
\[
+ \gamma^{-1}|X - Y|^\alpha \phi(\sqrt{k + \gamma t} - 1)
\]
\[
\leq C\gamma^{-1}(h(X) + h(Y))A|X - Y|^\alpha \phi(\sqrt{k + \gamma t} - 1).
\]

Summing up, we deduce our result. \(\square\)

Lemma 2.4. Let \(\Phi\) be a map from \(\mathbb{R}^d\) to \(\mathbb{R}^d\) with \(\nabla \Phi \in C^{0,\alpha}(\mathbb{R}^d)\). It holds that

\[
|u \circ \Phi|_{0,\alpha;h \circ \Phi,R} \leq |u|_{0,\alpha;h,R} \max \left(|\nabla \Phi|_{0,\alpha}^\alpha, 1\right),
\]
\[
|u \circ \Phi|_{1,\alpha;h \circ \Phi,R} \leq |u|_{1,\alpha;h,R} \max \left(|\nabla \Phi|_{1,\alpha}^\alpha, 1\right) \max \left(|\nabla \Phi|_{0,\alpha;1,R, 1}\right).
\]

2.2. Symmetric extension. Let \(\Omega = \mathbb{R}^{d-1} \times [0,1]\) be a strip and \(X = (x,y), x \in \mathbb{R}^{d-1}, y \in [0,1]\) be a point in \(\Omega\).

Let \(T_e\) be an even extension from \(C(\Omega)\) to \(C(\mathbb{R}^d)\) defined by

\[
T_e f(x,2n+y) = T_e f(x,2n-y) = f(x,y)
\]

for \(x \in \mathbb{R}^{d-1}, y \in [0,1], n \in \mathbb{Z}\). Let \(T_o\) be an odd extension from \(C_0(\Omega) = \{ u \in C(\Omega), u = 0\text{ on } \partial \Omega \}\) to \(C(\mathbb{R}^d)\) defined by

\[
T_o f(x,2n-y) = -f(x,y), \quad T_o f(x,2n+y) = f(x,y)
\]

for \(x \in \mathbb{R}^{d-1}, y \in [0,1], n \in \mathbb{Z}\).

Lemma 2.5. It holds that

\[
|T_e f|_{0,\alpha} = |f|_{0,\alpha, \Omega},
\]
\[
|f|_{0,\alpha; \Omega} \leq |T_o f|_{0,\alpha} \leq 2|f|_{0,\alpha, \Omega}.
\]
The same result holds for the weighted Hölder norm $| \cdot |_{0, \alpha; h}$ if the weight function $h(X)$ depends only on $x$.

Proof. First of all, it is obvious that

$$|f|_{0, \alpha; \Omega} \leq |T_{f}f|_{0, \alpha}, \quad |f|_{0, \alpha; \Omega} \leq |T_{0}f|_{0, \alpha},$$

and the same is true for the weighted Hölder norm $| \cdot |_{0, \alpha; h}$. We denote

$$\rho_{0}(y) = \inf_{n \in \mathbb{Z}} |y - 2n| \in [0, 1] \quad \text{for} \ y \in \mathbb{R},$$

$$\rho(X) = (x, \rho_{0}(y)) \in \Omega \quad \text{for} \ X = (x, y) \in \mathbb{R}^{d},$$

and let

$$\Omega_{+} = \bigcup_{n \in \mathbb{Z}} \mathbb{R}^{d-1} \times [2n, 2n + 1], \quad \Omega_{-} = \bigcup_{n \in \mathbb{Z}} \mathbb{R}^{d-1} \times [2n - 1, 2n].$$

Then it is easy to see that

$$T_{f}f = f \circ \rho, \quad T_{0}f = f \circ \rho \quad \text{in} \ \Omega_{+}, \quad T_{0}f = -f \circ \rho \quad \text{in} \ \Omega_{-},$$

$$|\rho_{0}(y) - \rho_{0}(y')| \leq |y - y'|, \quad |\rho(X) - \rho(Y)| \leq |X - Y|,$$

from which, it follows that

$$|T_{f}f|_{0, \alpha} \leq |f|_{0, \alpha; \Omega}, \quad |T_{f}f|_{0, \alpha; h} \leq |f|_{0, \alpha; h, \Omega},$$

$$|T_{0}f|_{0} \leq |f|_{0, \Omega}, \quad |T_{0}f|_{0, h} \leq |f|_{0, h, \Omega}.$$

Given $X = (x, y), \ Y = (x', y') \in \mathbb{R}^{d}$ with $y \leq y'$, if $X, Y \in \Omega_{+}$ or $X, Y \in \Omega_{-}$, then

$$|T_{0}f(X) - T_{0}f(Y)| = |f \circ \rho(X) - f \circ \rho(Y)|$$

$$\leq |f|_{0, \alpha; h, \Omega}(h \circ \rho(X) + h \circ \rho(Y))|\rho(X) - \rho(Y)|^{\alpha}$$

$$\leq |f|_{0, \alpha; h, \Omega}(h(X) + h(Y))|X - Y|^{\alpha}.$$

Here we used $h \circ \rho(X) = h(X)$. Otherwise, there exists $y_{1}, y_{2} \in \mathbb{Z}$ so that $y_{1} - 1 \leq y \leq y_{1} \leq y_{2} \leq y' \leq y_{2} + 1$. Let $X' = (x, y_{1}), \ Y' = (x', y_{2})$. Then for $f \in C_{0}(\Omega)$, we have

$$|T_{0}f(X)| = |f \circ \rho(X)| = |f \circ \rho(X) - f \circ \rho(X')|$$

$$\leq |f|_{0, \alpha; h, \Omega}(h \circ \rho(X) + h \circ \rho(X'))|\rho(X) - \rho(X')|^{\alpha}$$

$$\leq 2|f|_{0, \alpha; h, \Omega} h(X)|X - X'|^{\alpha}.$$

Similarly, we have

$$|T_{0}f(Y)| \leq 2|f|_{0, \alpha; h, \Omega} h(Y)|Y - Y'|^{\alpha}.$$

Then, using $|X - X'| + |Y - Y'| \leq |X - Y|$, we get

$$|T_{0}f(X) - T_{0}f(Y)| \leq 2|f|_{0, \alpha; h, \Omega}(h(X) + h(Y))|X - Y|^{\alpha}.$$

This shows $|T_{0}f|_{\alpha; h} \leq 2|f|_{\alpha; h, \Omega}$. Similarly, $|T_{0}f|_{\alpha} \leq 2|f|_{\alpha; \Omega}$. \qed
3. Global well-posedness for the ideal MHD equations

This section is devoted to the proof of global well-posedness of the ideal MHD equations in $\mathbb{R}^{d-1} \times [0, 1]$ with the boundary condition (1.4). Recall that in terms of the Elsasser variables $z_\pm = Z_\pm \pm B_0$, the ideal MHD equations take as follows

\[
\begin{aligned}
\partial_t z_+ + Z_+ \cdot \nabla z_+ &= -\nabla p, \\
\partial_t z_- + Z_- \cdot \nabla z_- &= -\nabla p, \\
\text{div} z_+ &= \text{div} z_- = 0, \\
z_\pm^+(t, x, y) &= 0 \quad \text{on} \quad y = 0, 1.
\end{aligned}
\]  

(3.1)

Without loss of generality, we take the background magnetic field $B_0 = (1, 0, \cdots, 0)$.

3.1. Main result. Let $f(x, y) = f_0(x_1)$, where $f_0 \in C^1(\mathbb{R})$ is chosen so that $|f_0'| < f_0 < 1$ and for some $C_1 > 0$,

\[
\delta(T) \triangleq \sup_{Y \in \mathbb{R}^d} \int_{-T}^{T} f(Y + 2B_0 t) dt \leq C_1 \quad \text{for any} \quad T > 0,
\]

(3.2)

\[
\int_{\mathbb{R}^d} f(Y) \frac{f(Y)}{1 + |X - Y|^{d+1}} dY \leq C_1 f(X) \quad \text{for any} \quad X \in \mathbb{R}^d,
\]

\[
f(X) \leq 2 f(Y) \quad \text{for any} \quad |X - Y| \leq 2.
\]

In fact, $f_0(r) = (C_0 + r^2)^{-\frac{d+1}{2}}$ satisfies the above conditions for some $C_0 > 1$ and $0 < \delta < 1$.

Now we introduce the weight function $f_{\pm}(t, X)$ given by

\[
f_{\pm}(t, X) \triangleq f(X \pm B_0 t),
\]

which satisfies (2.1) with a uniform constant $c_0$ independent of $t$. Let

\[
M_{\pm}(t) \triangleq \sup_{|s| \leq t} |z_{\pm}(s)|_{1, \alpha; f_{\pm}(s), \Omega}.
\]

Main result of this section is stated as follows.

**Theorem 3.1.** Let $\alpha \in (0, 1)$. There exists $\epsilon > 0$ so that if $M_{\pm}(0) \leq \epsilon$, then there exists a global in time unique solution $(z_+, z_-) \in L^\infty(0, +\infty; C^1, \alpha(\Omega))$ with the pressure $p$ determined by (3.1) to the ideal MHD equations (3.1), which satisfies

\[
M_{\pm}(t) \leq C \epsilon \quad \text{for any} \quad t \in [0, +\infty).
\]

**Remark 3.2.** Thanks to $M_{\pm}(0) \sim |z_\pm(0)|_{(\Omega)}^{1+\delta}$, if $f_0(r) = (C_0 + r^2)^{-\frac{d+1}{2}}$, this means that the initial data decays at infinity only in one direction. This is a key point for the global well-posedness in the strip domain, especially in $\mathbb{R} \times [0, 1]$.

We conclude this subsection by introducing some properties of weight functions. Let

\[
g(t, X) \triangleq \int_{\mathbb{R}^d} f(Y + B_0 t) f(Y - B_0 t) \frac{f(Y) f(Y - B_0 t)}{1 + |X - Y|^{d+1}} dY.
\]

We have the following important facts.

**Lemma 3.3.** There exists a constant $C > 0$ so that for any $X \in \mathbb{R}^d, t \in \mathbb{R}$,

\[
\begin{aligned}
f(X + B_0 t) f(X - B_0 t) &\leq C g(t, X), \\
g(t, X) &\leq C (1 + |X - Y|) \frac{d+1}{2} g(t, Y), \\
\int_{-T}^{T} g(t, X \pm B_0 t) dt &\leq C \delta(T) f(X).
\end{aligned}
\]
Proof. Thanks to \( f(Y) \geq f(X)/2 \) for \( |X - Y| < 2 \), we get
\[
g(t, X) = \int_{B(X, 2)} \frac{f(Y + B_0t) f(Y - B_0t)}{1 + |X - Y|^{d+1}} dY
\]
\[
\geq \frac{1}{4} \int_{B(X, 2)} \frac{f(X + B_0t) f(X - B_0t)}{1 + |X - Y|^{d+1}} dY
\]
\[
\geq C^{-1} f(X + B_0t) f(X - B_0t),
\]
which gives the first inequality.

Using the inequality
\[
\frac{1}{1 + |X - Z|^{d+1}} \leq C \left( 1 + |X - Y|^{d+1} \right)^{\frac{d+1}{d+1}},
\]
we infer
\[
g(t, X) = \int_{\mathbb{R}^d} \frac{f(Z + B_0t) f(Z - B_0t)}{1 + |X - Z|^{d+1}} dZ
\]
\[
\leq C \int_{\mathbb{R}^d} \frac{f(Z + B_0t) f(Z - B_0t)}{1 + |Y - Z|^{d+1}} (1 + |X - Y|^{d+1}) dY
\]
\[
= C \left( 1 + |X - Y|^{d+1} \right) g(t, Y),
\]
which gives the second inequality.

Make a change of variable
\[
g(t, X + B_0t) = \int_{\mathbb{R}^d} \frac{f(Y + B_0t) f(Y - B_0t)}{1 + |X + B_0t - Y|^{d+1}} dY = \int_{\mathbb{R}^d} \frac{f(Y + 2B_0t) f(Y)}{1 + |X - Y|^{d+1}} dY,
\]
which along with (3.2) gives
\[
\int_{-T}^{T} g(t, X + B_0t) = \int_{\mathbb{R}^d} \frac{\int_{-T}^{T} f(Y + 2B_0t) f(Y) dt}{1 + |X - Y|^{d+1}} dY
\]
\[
\leq C \int_{\mathbb{R}^d} \frac{\delta(T) f(Y)}{1 + |X - Y|^{d+1}} dY \leq C \delta(T) f(X).
\]

Similarly, we have
\[
\int_{-T}^{T} g(t, X - B_0t) \leq C \delta(T) f(X).
\]

This completes the proof of the lemma.

3.2. \textbf{Weighted } \( C^{1, \alpha} \) \textbf{estimate for the transport equation.} Let \( Z \in C^1([0, T] \times \Omega) \) be a vector field with \( Z^d = 0 \) on \( \partial \Omega \). We introduce the characteristic associated with \( Z \):

\begin{equation}
\frac{d}{dt} \Phi(s, t, X) = Z(t, \Phi(s, t, X)), \quad \Phi(s, s, X) = X.
\end{equation}

Then \( \Phi(s, t, X) \in C^1([0, T] \times [0, T] \times \Omega) \) is a diffeomorphism from \( \Omega \) to \( \Omega \) and \( \partial \Omega \) to \( \partial \Omega \) having the property
\[
\Phi(r, t) \circ \Phi(s, r) = \Phi(s, t), \quad \Phi(s, s) = Id.
\]

\textbf{Lemma 3.4.} If \( Z(t, X) \) satisfies the extra condition

\begin{equation}
|\nabla Z|_{0, \alpha; h, \Omega, T} \int_{t_0}^{T} h(t, \Phi(T, t, X)) dt \leq A_0 \quad \text{for any } \quad X \in \Omega,
\end{equation}

then it holds that for \( 0 \leq t_0 \leq t \leq s < T \)
\[
|\nabla \Phi(s, t) - Id|_{0; \Omega} \leq e^{A_0} - 1,
\]
Thus, we conclude that
\[
|\nabla \Phi(s,t)|_{0,\Omega} \leq e^{A_0}, \\
|\nabla \Phi(s,t)|_{\alpha,\Omega} \leq 2A_0 e^{(2+\alpha)A_0}.
\]

**Proof.** Thanks to the definition of \( \Phi(s,t) \), we have
\[
\partial_t \nabla \Phi(s,t) = \nabla \Phi(s,t)((\nabla Z(t)) \circ \Phi(s,t)),
\]
\[
\Phi(s,s) = Id, \quad \nabla \Phi(s,s) = Id,
\]
\[
|\nabla \Phi(s,t)| \leq |\nabla \Phi(s,t) - Id| + 1.
\]
Here \( |\nabla \Phi(s,t)| \) is the matrix norm defined by (2.2). Therefore,
\[
|\nabla \Phi(s,t) - Id| \leq \int_t^s |\partial_t \nabla \Phi(s,r)|dr
\leq \int_t^s |\nabla \Phi(s,r)||\nabla Z(r)| \circ \Phi(s,r)|dr
\leq \int_t^s |(\nabla Z(r)) \circ \Phi(s,r)|dr + \int_t^s |\nabla \Phi(s,r) - Id||\nabla Z(r)| \circ \Phi(s,r)|dr,
\]
which implies
\[
|\nabla \Phi(s,t) - Id| \leq \exp\left(\int_t^s |(\nabla Z(r)) \circ \Phi(s,r)|dr\right) - 1.
\]
Thanks to
\[
|(\nabla Z(r)) \circ \Phi(s,r)| \leq |\nabla Z|_{0,\alpha;h,\Omega,T} h(r) \circ \Phi(s,r),
\]
we get by (3.4) that
\[
\int_t^s |(\nabla Z(r)) \circ \Phi(s,r)(X)|dr \leq |\nabla Z|_{0,\alpha;h,\Omega,T} \int_t^s h(r) \circ \Phi(s,r)(X)dr = |\nabla Z|_{0,\alpha;h,\Omega,T} \int_t^s h(r, \Phi(T, r, \Phi(s, T)(X)))dr 
\leq A_0.
\]
Thus, we conclude that
\[
|\nabla \Phi(s,t) - Id|_{0,\Omega} \leq e^{A_0} - 1, \\
|\nabla \Phi(s,t)|_{0,\Omega} \leq e^{A_0}, \\
|\Phi(s,t,X) - \Phi(s,t,Y)| \leq |\nabla \Phi(s,t)|_{0,\Omega}|X - Y| \leq e^{A_0} |X - Y|.
\]
Notice that
\[
|\nabla \Phi(s,t,X) - \nabla \Phi(s,t,Y)| \leq \int_t^s |\nabla \Phi(s,r,X) - \nabla \Phi(s,r,Y)||\nabla Z(r)| \circ \Phi(s,r,X)|dr
\leq \int_t^s |\nabla \Phi(s,r,Y)||\nabla Z|_{0,\alpha;h,\Omega,T}(h(r, \Phi(s,r,X)) + h(r, \Phi(s,r,Y)))|\Phi(s,r,X) - \Phi(s,r,Y)|dr,
\]
from which and Gronwall’s inequality, we infer
\[
|\nabla \Phi(s,t,X) - \nabla \Phi(s,t,Y)|
\leq \int_t^s |\nabla \Phi(s,r,Y)||\nabla Z|_{0,\alpha;h,\Omega,T}(h(r, \Phi(s,r,X)) + h(r, \Phi(s,r,Y)))|\Phi(s,r,X) - \Phi(s,r,Y)|^\alpha dr e^{A_0}
\]
\[
\leq \int_t^s e^{\lambda t} |\nabla Z|_{0, \alpha, h, \Omega, T} (h(r, \Phi(s, r, X)) + h(r, \Phi(s, r, Y))) e^{\lambda \alpha} |X - Y|^\alpha dr A_0
\]
\[
= e^{(2+\alpha) \lambda t} |X - Y|^\alpha \int_t^s (h(r, \Phi(s, r, X)) + h(r, \Phi(s, r, Y))) dr
\]
\[
\leq 2A_0 e^{(2+\alpha) \lambda t} |X - Y|^\alpha,
\]
which shows the last inequality of the lemma. \qed

Next we consider the transport equation
\[
(3.5) \quad \partial_t u + Z \cdot \nabla u = F, \quad u(0, X) = u_0(X).
\]
Using the characteristic, the solution \(u(t, X)\) is given by
\[
(3.6) \quad u(t, X) = u_0(\Phi(t, 0, X)) + \int_0^t F(s, \Phi(s, t, X)) ds.
\]

**Lemma 3.5.** If \(Z\) satisfies \((3.4)\), then we have
\[
|u(t)|_{0, \alpha, \Omega} \leq e^{\lambda \alpha} \left( |u_0|_{0, \alpha, \Omega} + \int_0^t |F(s)|_{0, \alpha, \Omega} ds \right),
\]
\[
|\text{div} u(t)|_{0, \Omega} \leq |\text{div} u_0|_{0, \Omega} + \int_0^t |(\text{tr} \nabla Z \nabla u) - \text{div} F(s)|_{0, \Omega} ds.
\]

**Proof.** Using \((3.6)\), Lemma 2.2 and Lemma 3.4, we get
\[
|u(t)|_{0, \alpha, \Omega} \leq |u_0 \circ \Phi(t, 0)|_{0, \alpha, \Omega} + \int_0^t |F(s) \circ \Phi(t, s)|_{0, \alpha, \Omega} ds
\]
\[
\leq |u_0|_{0, \alpha, \Omega} \max \{ |\nabla \Phi(t, 0)|_{0, \Omega}^{\alpha}, 1 \} + \int_0^t |F(s)|_{0, \alpha, \Omega} \max \{ |\nabla \Phi(t, s)|_{0, \Omega}^{\alpha}, 1 \} ds
\]
\[
\leq e^{\lambda \alpha} \left( |u_0|_{0, \alpha, \Omega} + \int_0^t |F(s)|_{0, \alpha, \Omega} ds \right).
\]

Taking divergence to \((3.5)\), we obtain
\[
\partial_t \text{div} u + Z \cdot \nabla \text{div} u + \text{tr} (\nabla Z \nabla u) = \text{div} F, \quad u(0, X) = u_0(X).
\]
So, we have
\[
\text{div} u(t) = \text{div} u_0 \circ \Phi(t, 0) + \int_0^t (\text{div} F - \text{tr} (\nabla Z \nabla u))(s) \circ \Phi(t, s) ds,
\]
then the second inequality follows easily. \qed

**Proposition 3.6.** If \(|Z + B_0|_{1, \alpha, f, -, \Omega, T} \delta(T) < 1\), then we have
\[
|u|_{1, \alpha, f, +, \Omega, T} \leq C \left( |u_0|_{1, \alpha, f, \Omega} + \delta(T) |F|_{1, \alpha, g, \Omega, T} \right).
\]
If \(|Z - B_0|_{1, \alpha, f, +, \Omega, T} \delta(T) < 1\), then we have
\[
|u|_{1, \alpha, f, -, \Omega, T} \leq C \left( |u_0|_{1, \alpha, f, \Omega} + \delta(T) |F|_{1, \alpha, g, \Omega, T} \right).
\]
Here \(C\) is a constant independent of \(T\).
Proof. We only prove the first inequality, the proof of the second one is similar. Let us claim

\[(3.7)\quad |\Phi(s, t, X) + B_0(t - s) - X| < 2 \quad \text{for} \quad 0 \leq t \leq s \leq T.\]

Otherwise, there exists \( t \in [0, s] \) such that \(|\Phi(s, t, X) + B_0(t - s) - X| = 2\) and \(|\Phi(s, r, X) + B_0(r - s) - X| \leq 2\) for \( r \in [t, s] \). Thus,

\[
|\Phi(s, t, X) + B_0(t - s) - X| \leq \int_t^s |\partial_r \Phi(s, r, X) + B_0| dr
\]

\[= \int_t^s |Z(r, \Phi(s, r, X)) + B_0| dr
\]

\[\leq \int_t^s |Z + B_0|_{1, \alpha; \Omega} \int_t^s ((\Phi(s, r, X))|dr
\]

\[= |Z + B_0|_{1, \alpha; \Omega} \int_t^s f(\Phi(s, r, X) - B_0r) dr,
\]

while, by (3.2),

\[\int_t^s f(\Phi(s, r, X) - B_0r) dr \leq 2 \int_t^s f(X - B_0(r - s) - B_0r) dr \leq 2\delta(T).
\]

This shows

\[|\Phi(s, t, X) + B_0(t - s) - X| \leq 2|Z + B_0|_{1, \alpha; \Omega} \delta(T) < 2,
\]

which is a contradiction, hence (3.7) is true.

Now we verify (3.4) for \( h = f_-, A_0 = 2 \). Indeed, by (3.2) and (3.7),

\[\int_0^T f_-(t, \Phi(T, t, X)) dt = \int_0^T f(\Phi(T, t, X) - B_0t) dt
\]

\[\leq 2 \int_0^T f(X - B_0(t - T) - B_0t) dt \leq 2\delta(T),
\]

which implies (3.4). Then we infer from Lemma 3.3 that

\[(3.8)\quad |\nabla \Phi(t, s)|_{0, \alpha; \Omega} \leq C.
\]

It follows from Lemma 3.3 and (3.7) that

\[\int_0^t g(r, \Phi(t, r, X)) dr \leq C \int_0^t g(r, X - B_0(r - t)) dr \leq C\delta(T) f(X + B_0t),
\]

which implies

\[|u(t)|_{1, \alpha; f_+(t), \Omega} \leq |u_0 \circ \Phi(t, 0)|_{1, \alpha; f_+(t), \Omega} + C\delta(T) \sup_{0 \leq s \leq t} |F(s) \circ \Phi(t, s)|_{0, \alpha; \Omega}.
\]

Using the fact \( f(\Phi(t, 0, X)) \leq 2f(X - B_0(0 - t)) = 2f_+(t, X) \), we get

\[|u_0 \circ \Phi(t, 0)|_{1, \alpha; f_+(t), \Omega} \leq 2|u_0 \circ \Phi(t, 0)|_{1, \alpha; f_0 \Phi(t, 0), \Omega}.
\]

Then by Lemma 2.2 and (3.3), we obtain

\[|u(t)|_{1, \alpha; f_+(t), \Omega} \leq C (|u_0 \circ \Phi(t, 0)|_{1, \alpha; f_+(t), \Omega} + \delta(T) \sup_{0 \leq s \leq t} |F(s)|_{1, \alpha; g(s), \Omega})
\]

\[\times \max \{|\nabla \Phi(t, s)|_{0, \Omega}, 1\} \max \{|\nabla \Phi(t, s)|_{0, \alpha; \Omega}, 1\}
\]

\[\leq C|u_0|_{1, \alpha; f_+, \Omega} + C\delta(T) \sup_{0 \leq s \leq t} |F(s)|_{1, \alpha; g(s), \Omega}.
\]

This shows the first inequality of the lemma. \( \square \)
3.3. Representation formula of the pressure. In this subsection, we give a representation formula of the pressure by using the symmetric extension.

Let \((v, b, p)\) be a smooth solution of \((1.1)\) in \([0, T] \times \Omega\) with the boundary condition \((1.4)\). We make the following symmetric extension for the solution:

\[
\overline{v} = T v \overset{\text{def}}{=} (T_e v^1, \cdots, T_e v^{d-1}, T_o v^d), \quad \overline{p} = T_e p.
\]

Then \((\overline{v}, \overline{b}, \overline{p})\) satisfies \((1.1)\) in \([0, T] \times \mathbb{R}^d\) in the weak sense. Although the solution after symmetric extension has not the same smoothness as the origin one, we have the following important observation.

**Lemma 3.7.** Let \(h\) be a weight satisfying \((2.1)\). Let \(u = (u^1, \cdots, u^d), w = (w^1, \cdots, w^d) \in C^{1, \alpha}_h(\Omega)\) be two vector fields with \(u^d = w^d = 0\) on \(\partial \Omega\). Let \(\overline{v} = Tu, \overline{w} = Tw\). Then it holds that for \(i, j = 1, \cdots, d\),

\[
\begin{align*}
|\partial_i \overline{v}^j \partial_j \overline{w}^i|_{0, \alpha; h} + |\partial_i \overline{v}^j \partial_j \overline{w}^i|_{0, \alpha; h} & \leq C|\nabla u|_{0, \alpha; h, \Omega} |\nabla w|_{0, \alpha; h, \Omega}, \\
|\overline{v}^i \partial_j \overline{w}^i|_{0, \alpha; h} + |\overline{w}^i \partial_j \overline{w}^i|_{0, \alpha; h} & \leq C|u|_{0, \alpha; h, \Omega} |\nabla w|_{0, \alpha; h, \Omega}.
\end{align*}
\]

**Proof.** It is easy to verify that

\[
\begin{align*}
\partial_i \overline{v}^j \partial_j \overline{w}^i &= T_e (\partial_i u^j \partial_j w^i), \\
\partial_i \overline{w}^j \partial_j \overline{w}^i &= T_e (\partial_i w^j \partial_j w^i), \\
|\overline{v}^i \partial_j \overline{w}^i|_{0, \alpha; h} & \leq C |\nabla u|_{0, \alpha; h, \Omega} |\nabla w|_{0, \alpha; h, \Omega}, \\
|\overline{w}^i \partial_j \overline{w}^i|_{0, \alpha; h} & \leq C |u|_{0, \alpha; h, \Omega} |\nabla w|_{0, \alpha; h, \Omega}.
\end{align*}
\]

Then the lemma follows easily from Lemma 2.5. \(\square\)

Taking the divergence to the first equation of \((1.1)\), we get

\[
-\Delta \overline{p} = \partial_i (\overline{v}^i \partial_j \overline{v}^j - \overline{b}^j \partial_j \overline{b}^i).
\]

Formally, we have

\[
\nabla \overline{p}(t, X) = \nabla \int_{\mathbb{R}^d} N(X - Y) \partial_i (\overline{v}^i \partial_j \overline{v}^j - \overline{b}^j \partial_j \overline{b}^i)(t, Y) dY,
\]

where \(N(X)\) is the Newton potential. In terms of the Els"asser variables \(\overline{z}_{\pm}(t, X)\), we have

\[
\nabla \overline{p}(t, X) = \nabla \int_{\mathbb{R}^d} N(X - Y) \partial_i (\overline{z}^i_+ \partial_j \overline{z}^j_-)(t, Y) dY.
\]

However, this integral does not make sense for \(\partial_i (\overline{z}^i_+ \partial_j \overline{z}^j_-) \in C^{0, \alpha}_h\). To overcome this trouble, we introduce a smooth cut-off function \(\theta(r)\) so that

\[
\theta(r) = \begin{cases} 1 & \text{for } |r| \leq 1, \\
0 & \text{for } |r| \geq 2.
\end{cases}
\]

Integration by parts, we can split \(\nabla \overline{p}(t, X)\) as

\[
-\nabla \overline{p}(t, X) = \int_{\mathbb{R}^d} \nabla N(X - Y)(\partial_i \overline{z}^i_+ \partial_j \overline{z}^j_-)(t, Y) dY + \int_{\mathbb{R}^d} \partial_i \partial_j \left( \nabla N(X - Y)(1 - \theta(|X - Y|)) \right)(\overline{z}^i_+ \overline{z}^j_-)(t, Y) dY.
\]

It is easy to check that this representation makes sense for \(\overline{z}_{\pm} \in W^{1, \infty}(\mathbb{R}^d)\).

We denote

\[
T_1 u \overset{\text{def}}{=} \int_{\mathbb{R}^d} \nabla N(X - Y) \theta(|X - Y|) u(Y) dY.
\]
Let \( u, w \in C^{1,\alpha}(\Omega) \) be two vector fields with \( u^d = w^d = 0 \) on \( \partial \Omega \). Let \( \overline{u} = Tu, \overline{w} = Tw \) be the symmetric extension. We denote
\[
I(u, w) \triangleq T_1(\partial_i \overline{w}^i \partial_j \overline{w}^j - \partial_j \overline{w}^j \partial_i \overline{w}^i) + T_{ij}(\overline{w}^i \overline{w}^j).
\]
Here and in what follows, the repeated index denotes the summation. Thanks to
\[
\partial_i \overline{w}^i \partial_j \overline{w}^j - \partial_j \overline{w}^j \partial_i \overline{w}^i = \partial_i(\overline{w}^i \partial_j \overline{w}^j - \overline{w}^i \partial_j \overline{w}^j),
\]
we infer from Lemma 5.1 and Lemma 3.7 that
\[
|I(u, w)|_{0, \alpha, \Omega} \leq C|u|_{0, \alpha, \Omega}|w|_{1, \alpha, \Omega}.
\]
Using Lemma 5.2 and (3.13), let us calculate
\[
\text{div} I(u, w) + (\partial_i u^j \partial_j w^i - \partial_i w^j \partial_j u^i)
= \int_{\mathbb{R}^d} \nabla N(X - Y) \cdot \nabla\theta(|X - Y|) \left( \partial_i \overline{w}^i \partial_j \overline{w}^j - \partial_i \overline{w}^j \partial_j \overline{w}^i \right) (Y) dY
- \int_{\mathbb{R}^d} \partial_i \partial_j \left( \nabla N(X - Y) \cdot \nabla\theta(|X - Y|) \right) (\overline{w}^i \overline{w}^j) (Y) dY
= \int_{\mathbb{R}^d} \partial_i \left( \nabla N(X - Y) \cdot \nabla\theta(|X - Y|) \right) \left(- \overline{w}^i \partial_j \overline{w}^j + \overline{w}^i \partial_j \overline{w}^j + \partial_j (\overline{w}^i \overline{w}^j) \right) (Y) dY
= \int_{\mathbb{R}^d} \partial_i \left( \nabla N(X - Y) \cdot \nabla\theta(|X - Y|) \right) (\overline{w}^i \text{div} \overline{w} + \overline{w}^i \text{div} \overline{w}) (Y) dY,
\]
which implies
\[
|\text{div} I(u, w) - (\partial_i u^j \partial_j w^i - \partial_i w^j \partial_j u^i)|_{0, \Omega} \leq C|u|_{0, \Omega}|\text{div} w|_{0, \Omega} + |w|_{0, \Omega}|\text{div} u|_{0, \Omega}.
\]
In the case of \( \mathbb{R}^d \), the pressure \( p(t, X) \) can also be expressed as
\[
- \nabla p(t, X) = I(z_+, z_-),
\]
where
\[
I(u, w) \triangleq \int_{\mathbb{R}^d} \nabla N(X - Y) \theta(|X - Y|) (\partial_i u^j \partial_j v^i) (Y) dY
+ \int_{\mathbb{R}^d} \partial_i \partial_j \left( \nabla N(X - Y) \left(1 - \theta(|X - Y|) \right) \right) (u^j v^i) (Y) dY.
\]
Notice that the representation formula (3.16) is independent of the choice of \( \theta \) in \( I(u, w) \).

### 3.4. Proof of Theorem 3.1
Since we can not find a well-posedness theory for the ideal MHD equations in the weighted Hölder spaces, we will present a complete proof of Theorem 3.1. In fact, we find that the proof of the existence part is very nontrivial.

Using the representation of the pressure (3.10), we rewrite the system (3.1) as
\[
\begin{align*}
\partial_t z_+ + Z_+ \cdot \nabla z_+ &= -I(z_+, z_-), \\
\partial_t z_- + Z_+ \cdot \nabla z_- &= -I(z_+, z_-), \\
z_+(0, X) &= z_{+0}(X), \quad z_-(0, X) = z_{-0}(X).
\end{align*}
\]
Let \( T > 0 \) be determined later and \( A_1 = |z_{+0}|_{1, \alpha, f, \Omega} + |z_{-0}|_{1, \alpha, f, \Omega} \). When \( A_1 \) is sufficiently small, \( T \) can be taken \( +\infty \). The system (3.18) is solved by the following iteration scheme:
\[
z_+^{(0)} = z_-^{(0)} = 0, \quad Z_+^{(n)} = z_+^{(n)} + B_0, \quad Z_-^{(n)} = z_-^{(n)} - B_0.
\]
Let us inductively assume that $z^{(n)}_{\pm}$ satisfies
\[
|z^{(n)}_+|_{1, \alpha; f_+, \Omega, T} \leq 2C_1A_1, \quad |z^{(n)}_-|_{1, \alpha; f_-, \Omega, T} \leq 2C_1A_1.
\]
Take $T > 0$ so that $4C_1A_1\delta(T) < 1$. Then we have
\[
(3.19) \quad |z^{(n)}_+|_{1, \alpha; f_+, \Omega, T} \delta(T) \leq \frac{1}{2}, \quad |z^{(n)}_-|_{1, \alpha; f_-, \Omega, T} \delta(T) \leq \frac{1}{2}.
\]
Now, the solution $z^{(n+1)}_+, z^{(n+1)}_-$ are determined by
\[
\begin{aligned}
&\partial_t z^{(n+1)}_+ + Z^{(n)}_- \cdot \nabla z^{(n+1)}_+ = -I(z^{(n)}_+, z^{(n)}_-), \\
&\partial_t z^{(n+1)}_- + Z^{(n)}_+ \cdot \nabla z^{(n+1)}_- = -I(z^{(n)}_+, z^{(n)}_-), \\
&z^{(n+1)}_+(0, X) = z_{+0}(X), \quad z^{(n+1)}_-(0, X) = z_{-0}(X).
\end{aligned}
\]
It follows from Proposition 3.6 that
\[
|z^{(n+1)}_+|_{1, \alpha; f_+, \Omega, T} \leq C_1 \left( |z_{+0}|_{1, \alpha; f_+, \Omega} + \delta(T) |z^{(n)}_+|_{1, \alpha; f_+, \Omega, T} \right), \\
|z^{(n+1)}_-|_{1, \alpha; f_-, \Omega, T} \leq C_1 \left( |z_{-0}|_{1, \alpha; f_-, \Omega} + \delta(T) |z^{(n)}_-|_{1, \alpha; f_-, \Omega, T} \right).
\]
Here we used
\[
|I(u, w)|_{1, \alpha; g, \Omega} \leq C |\partial_x u \partial_x w|_{0, \alpha, h} + C |u w|_{0, h} + C |\nabla u|_{0, \alpha, \Omega},
\]
which follows from Lemma 5.1 with $h(t, X) = f_+ f_- (t, X)$ and Lemma 3.7.
Due to (3.19), we obtain
\[
|z^{(n+1)}_+|_{1, \alpha; f_+, \Omega, T} \leq 2C_1A_1, \quad |z^{(n+1)}_-|_{1, \alpha; f_-, \Omega, T} \leq 2C_1A_1.
\]
In particular, we show that for any $n$,
\[
|z^{(n)}_+|_{1, \alpha; f_+, \Omega, T} \leq C, \quad |z^{(n)}_-|_{1, \alpha; f_-, \Omega, T} \leq C.
\]
Next, we show that $\{z^{(n)}_{\pm}\}_{n \geq 0}$ is a Cauchy sequence in $C^{0, \alpha}(\Omega)$. Indeed, we have
\[
\begin{aligned}
&\partial_t (z^{(n+1)}_+ - z^{(n)}_+) + Z^{(n)}_- \cdot \nabla (z^{(n+1)}_+ - z^{(n)}_+) + (z^{(n)}_+ - z^{(n-1)}_+) \cdot \nabla z^{(n)}_+ \\
&+ I(z^{(n)}_+ - z^{(n-1)}_+, z^{(n)}_-) + I(z^{(n-1)}_+, z^{(n)}_- - z^{(n-1)}_-) = 0, \\
&\partial_t (z^{(n+1)}_- - z^{(n)}_-) + Z^{(n)}_+ \cdot \nabla (z^{(n+1)}_- - z^{(n)}_-) + (z^{(n)}_+ - z^{(n-1)}_+) \cdot \nabla z^{(n)}_- \\
&+ I(z^{(n)}_+ - z^{(n-1)}_+, z^{(n)}_- + I(z^{(n-1)}_+, z^{(n)}_- - z^{(n-1)}_-) = 0, \\
&(z^{(n+1)}_+ - z^{(n)}_+)(0, X) = 0, \quad (z^{(n+1)}_- - z^{(n)}_-)(0, X) = 0.
\end{aligned}
\]
Then it follows from Lemma 3.5 and (3.14) that
\[
\begin{aligned}
&|(z^{(n+1)}_+ - z^{(n)}_+)(t)|_{0, \alpha; \Omega} \leq C \int_0^t \left| (z^{(n)}_+ - z^{(n-1)}_+)(s) \right|_{0, \alpha; \Omega} |\nabla z^{(n)}_+(s)|_{0, \alpha; \Omega} ds \\
&\quad + C \int_0^t \left| (z^{(n)}_+ - z^{(n-1)}_+)(s) \right|_{0, \alpha; \Omega} |z^{(n)}_+(s)|_{1, \alpha; \Omega} ds \\
&\quad + C \int_0^t \left| (z^{(n)}_- - z^{(n-1)}_-)(s) \right|_{0, \alpha; \Omega} |z^{(n-1)}_+(s)|_{1, \alpha; \Omega} ds \\
&\quad \leq C_2 \int_0^t \left| (z^{(n)}_+ - z^{(n-1)}_+)(s) \right|_{0, \alpha; \Omega} + \left| (z^{(n)}_- - z^{(n-1)}_-)(s) \right|_{0, \alpha; \Omega} ds.
\end{aligned}
\]
Similarly, we have
\[|(z^{(n+1)}_+ - z^{(n)}_-(t))|_{0,\alpha,\Omega} \leq C_2 \int_0^t \left(|(z^{(n)}_+ - z^{(n-1)}_-(s))|_{0,\alpha,\Omega} + |(z^{(n)}_- - z^{(n-1)}_-(s))|_{0,\alpha,\Omega}\right)ds.\]
This implies that
\[|(z^{(n+1)}_+ - z^{(n)}_-(t))|_{0,\alpha,\Omega} + |(z^{(n+1)}_- - z^{(n)}_-(t))|_{0,\alpha,\Omega} \leq C(2C_2t)^n/n!\]
Therefore, \(z^{(n)}_+, z^{(n)}_-\) converge to some \(z_+, z_-\) uniformly in \([0, t] \times \Omega\) for any \(0 < t < T\). As \(z^{(n)}_+, z^{(n)}_-\) are uniformly bounded in \(C^{1,\alpha}\), we have \(z_+, z_- \in C^{1,\alpha}\). Then \(\nabla z^{(n)}_+, \nabla z^{(n)}_-\) converge to \(\nabla z_+, \nabla z_-\) uniformly in \([0, t] \times \Omega\) for any \(0 < t < T\). Using the equations of \(z^{(n+1)}_+, z^{(n-1)}_-, \partial_t z^{(n)}_+, \partial_t z^{(n)}_-\) also converge uniformly in \([0, t] \times \Omega\) for any \(0 < t < T\). Thus, \(z_+, z_- \in C^{1}([0, t] \times \Omega)\) satisfies (3.18) and \(z^d_+ = z^d_- = 0\) on \(\partial \Omega\).

Finally, it remains to prove that if \(\text{div} z_+ = \text{div} z_- = 0\), then \(\text{div} z_+ = \text{div} z_- = 0\). It follows from Lemma 3.5 and (3.15) that
\[|\text{div} z_+(t)|_{0,\Omega} \leq \int_0^t |(\partial_i z^j_+ \partial_j z^i_+ - \text{div} I(z_+, z_-))(s)|_{0,\Omega} ds \leq C \int_0^t (|\text{div} z_+(s)|_{0,\Omega}|\text{div} z_-|_{0,\Omega}) ds + |\text{div} z_+(s)|_{0,\Omega}|\text{div} z_-(s)|_{0,\Omega} ds \leq C \int_0^t (|\text{div} z_+(s)|_{0,\Omega} + |\text{div} z_-(s)|_{0,\Omega}) ds\]
similarly,
\[|\text{div} z_-(t)|_{0,\Omega} \leq C \int_0^t (|\text{div} z_+(s)|_{0,\Omega} + |\text{div} z_-(s)|_{0,\Omega}) ds.\]
This implies that \(\text{div} z_+ = \text{div} z_- = 0\).

Let us remark that \(I(z_+, z_-)\) can be expressed as \(\nabla p\). Indeed, we can find \(\theta_1, \theta_2 \in C^\infty(0, +\infty)\) such that \(\theta'_1(r) = -\theta(r)N(r), \theta'_2(r) = (\theta(r) - 1)N(r)\). Let \(\theta_{ij}(X) = \partial_i \partial_j \theta_2(|X|)\) and
\[I_*(u, w)(x) = \int_{\mathbb{R}^d} \theta_1(|X - Y|)(\partial_i u^i \partial_j w^j - \partial_i w^i \partial_j u^j)(Y) dY + \int_{\mathbb{R}^d} (\theta_{ij}(X - Y) - \delta_{ij}(-Y))(u^i w^j)(Y) dY.\]
Then we have \(\nabla I_*(u, v) = I(u, v)\). Therefore, we can take \(p = I_*(\bar{x}_+, \bar{z}_-)\), which satisfies \(|p| \leq C \ln(2 + |x|)\). This completes the proof of Theorem 3.1.

4. Global well-posedness for the viscous MHD equations

In this section, we study the global well-posedness for the viscous MHD equations in the slab domain \(\Omega = \mathbb{R}^{d-1} \times [0, 1]\) with the Navier-slip boundary condition. Because we can reduce the slab domain \(\Omega = \mathbb{R}^{d-1} \times [0, 1]\) to \(\mathbb{R}^{d-1} \times \mathbb{T}\) by using the symmetric extension, we will consider more general domain \(\Omega = \mathbb{R}^k \times \mathbb{T}^{d-k}\) for \(2 \leq k \leq d\). The case of \(k = 1\) is more difficult and will be dealt in the future work.

In fact, \(\Omega = \mathbb{R}^k \times \mathbb{T}^{d-k}\) is a special case of \(\mathbb{R}^d\) periodic in \(d - k\) directions \(e_1, \cdots, e_{d-k}\). We will assume that \(e_1, \cdots, e_{d-k}, B_0\) are linearly independent.
4.1. New formulation. Let $\mu_1 = \frac{\nu + \mu}{2}$ and $\mu_2 = \frac{\nu - \mu}{2}$. In terms of the Elsässer variables $Z_\pm = v \pm b$, the MHD equations (1.1) read

$$\begin{cases}
\partial_t z_+ + Z_- \cdot \nabla z_+ = \mu_1 \triangle z_+ + \mu_2 \triangle z_- - \nabla p, \\
\partial_t z_- + Z_+ \cdot \nabla z_- = \mu_1 \triangle z_- + \mu_2 \triangle z_+ - \nabla p, \\
\text{div} z_+ = \text{div} z_- = 0,
\end{cases}$$

(4.1)

where $z_\pm = Z_\pm \pm B_0$. In the case of $\nu = \mu$ (thus, $\mu_2 = 0$), the formulation (4.1) plays a crucial role in the proof of [3, 4]. To deal with the case of $\nu \neq \mu$, we need to introduce the following key decomposition

$$z_+ = z_+^{(1)} + z_+^{(2)}, \quad z_- = z_-^{(1)} + z_-^{(2)},$$

where $z_+^{(1)}$ and $z_-^{(1)}$ are determined by

$$\begin{cases}
\partial_t z_+^{(1)} + Z_- \cdot \nabla z_+^{(1)} = \mu_1 \triangle z_+^{(1)} - \nabla p_+^{(1)}, \\
\partial_t z_-^{(1)} + Z_+ \cdot \nabla z_-^{(1)} = \mu_1 \triangle z_-^{(1)} - \nabla p_-^{(1)}, \\
\text{div} z_+^{(1)} = \text{div} z_-^{(1)} = 0, \\
z_+^{(1)}(0) = z_+(0), \quad z_-^{(1)}(0) = z_-(0),
\end{cases}$$

(4.2)

and

$$\begin{cases}
\partial_t z_+^{(2)} + Z_- \cdot \nabla z_+^{(2)} = \mu_1 \triangle z_+^{(2)} + \mu_2 \triangle z_- - \nabla p_+^{(2)}, \\
\partial_t z_-^{(2)} + Z_+ \cdot \nabla z_-^{(2)} = \mu_1 \triangle z_-^{(2)} + \mu_2 \triangle z_+ - \nabla p_-^{(2)}, \\
\text{div} z_+^{(2)} = \text{div} z_-^{(2)} = 0, \\
z_+^{(2)}(0) = z_+(0), \quad z_-^{(2)}(0) = z_-(0),
\end{cases}$$

(4.3)

To estimate $z_+^{(1)}$, we rewrite (4.2) as

$$\begin{cases}
\partial_t z_+^{(1)} + Z_- \cdot \nabla z_+^{(1)} = \mu_1 \triangle z_+^{(1)} - z_+^{(2)} \cdot \nabla z_+^{(1)} - I(z_-^{(2)} - z_+^{(1)}) - I(z_-^{(2)} - z_-^{(1)}), \\
\partial_t z_-^{(1)} + Z_+ \cdot \nabla z_-^{(1)} = \mu_1 \triangle z_-^{(1)} - z_-^{(2)} \cdot \nabla z_-^{(1)} - I(z_+^{(2)} - z_+^{(1)}) - I(z_+^{(2)} - z_-^{(1)}),
\end{cases}$$

(4.4)

where $I(u, w)$ is defined by (3.17). And we also need to use the equation of $J_\pm^{(1)} = \text{curl} z_\pm^{(1)}$, which is given by

$$\begin{cases}
\partial_t J_+^{(1)} + Z_- \cdot \nabla J_+^{(1)} + \nabla z_+^{(1)} \cdot \nabla J_+^{(1)} + \text{curl}(z_-^{(2)} \cdot \nabla z_+^{(1)}) = \mu_1 \triangle J_+^{(1)}, \\
\partial_t J_-^{(1)} + Z_+ \cdot \nabla J_-^{(1)} + \nabla z_-^{(1)} \cdot \nabla J_-^{(1)} + \text{curl}(z_+^{(2)} \cdot \nabla z_-^{(1)}) = \mu_1 \triangle J_-^{(1)}.
\end{cases}$$

(4.5)

Here $A \wedge B = (AB) - (AB)^T$ is understood as matrix multiplication.

To estimate $z_-^{(2)}$, we need to introduce another formulation in terms of the stream function $\psi_-^{(2)} = \triangle^{-1} \text{curl} z_-^{(2)}$, which satisfies

$$\begin{cases}
\partial_t \psi_-^{(2)} + \triangle^{-1} \text{curl}(Z_- \cdot \nabla z_-^{(2)}) = \mu_1 \triangle \psi_-^{(2)} + \mu_2 J_-, \\
\partial_t \psi_+^{(2)} + \triangle^{-1} \text{curl}(Z_+ \cdot \nabla z_+^{(2)}) = \mu_1 \triangle \psi_+^{(2)} + \mu_2 J_+,
\end{cases}$$

(4.6)

where

$$J_\pm = \text{curl} z_\pm = J_\pm^{(1)} + \text{curl} z_\pm^{(2)}.$$

We introduce

$$\Pi_1(u, w) \triangleq \triangle^{-1} \text{curl} \text{div}(u \otimes w),$$

$$\Pi_2(u, w) \triangleq \triangle^{-1} \text{curl}(u \cdot \nabla w) - u \cdot \nabla \triangle^{-1} \text{curl} w.$$
So, we get
\[ \triangle^{-1} \text{curl}(Z_\gamma \cdot \nabla z_\gamma) = Z_\gamma^{(1)} \cdot \nabla \psi_\gamma^{(2)} + \Pi_1(z_\gamma^{(2)}, z_\gamma^{(2)}) + \Pi_2(z_\gamma^{(2)}, z_\gamma^{(2)}). \]

Then we deduce that
\[
\begin{align*}
(4.7) & \quad \left\{ \begin{array}{l}
\partial_t \psi_\gamma^{(2)} + Z_\gamma^{(1)} \cdot \nabla \psi_\gamma^{(2)} + \Pi_2(z_\gamma^{(2)}, z_\gamma^{(2)}) + \Pi_1(z_\gamma^{(2)}, z_\gamma^{(2)}) = \mu_1 \Delta \psi_\gamma^{(2)} + \mu_2 J_\gamma, \\
\partial_t \psi_\gamma^{(2)} + Z_\gamma^{(1)} \cdot \nabla \psi_\gamma^{(2)} + \Pi_2(z_\gamma^{(2)}, z_\gamma^{(2)}) + \Pi_1(z_\gamma^{(2)}, z_\gamma^{(2)}) = \mu_1 \Delta \psi_\gamma^{(2)} + \mu_2 J_\gamma.
\end{array} \right.
\end{align*}
\]

A direct calculation shows
\[ - (\triangle^{-1} \text{curl}(u \cdot \nabla w))^j_k = \triangle^{-1} (\partial_k \partial_l (u^j w^l) - \partial_j \partial_k (u^i w^k)) = - R_k R_l (u^i w^l) + R_j (u^i w^k), \]
\[ - (u \cdot \nabla (\triangle^{-1} \text{curl } w))^j_k = u^i \partial_l \triangle^{-1} (\partial_k w^l - \partial_j w^k) = u^i (- R_i R_k w^j + R_j R_i w^k), \]

where \( R_i \) is the Riesz transform defined by \( R_i = \partial_i (-\Delta)^{-\frac{1}{2}} \). This gives
\[ \Pi_2(u, w)^j_k = [u^i, R_i R_j] w^k - [u^i, R_i R_k] w^j. \]

4.2. Weighted \( C^{1, \alpha} \) estimates for the parabolic equation. We consider the parabolic equation with variable coefficients
\[ \partial_t u - \gamma \partial_t (a_{ij}(t, X) u) + F_1 + F_2 + \partial_t G^i = 0, \]
where \( \gamma > 0 \) and the coefficients \( a_{ij}(t, X) \) satisfies
\[ \sup_{t \in [0, T]} \left( |a_{ij}(t) - \delta_{ij}|_0 + (1 + \gamma t)^{\alpha/2} |a_{ij}(t)|_\alpha \right) \leq \varepsilon_0, \]
for some \( \alpha \in (0, 1) \), \( \varepsilon_0 > 0 \) and \( T > 0 \).

Let \( f(t, X), h(t, X) \) be two weight functions satisfying (2.1) with a uniform constant \( c_0 \) independent of \( t \) and
\[ \int_0^T H(2\gamma(t-s)) h(s, X) ds \leq c_0^{-1} f(t, X), \quad H(2\gamma(t-s)) f(s, X) \leq c_0^{-1} f(t, X) \]
for all \( 0 \leq s \leq t \leq T, X \in \mathbb{R}^d \), where
\[ H(t) \phi(X) = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|X-Y|^2}{4t}} \phi(Y) dY. \]

Let \( \delta > 0 \). We introduce
\[ \Lambda_1(T, F_1, F_2, G, f, h) \triangleq \sup_{0 < t \leq T} \left( |F_1(t)|_{1, \alpha; h(t), (1+\gamma t)^{1/2}} + \gamma^{-1} ( (\gamma t)^{\frac{1}{2}} + (\gamma t)^{1+\frac{\alpha}{2}} ) |F_2(t)|_{0, \alpha; f(t)} + \gamma^{-1} ( (\gamma t)^{\frac{1}{2}} + (\gamma t)^{1+\frac{\alpha}{2}} ) |G(t)|_{0, \alpha; f(t), (1+\gamma t)^{1/2}} \right), \]
and
\[ \Lambda_0(T, F_1, F_2, G, f, h) \triangleq \sup_{0 < t \leq T} \left( |F_1(t)|_{1, \alpha; h(t), (\gamma t)^{1/2}} + \gamma^{-1} ( (\gamma t)^{\frac{1}{2}} + (\gamma t)^{1+\frac{\alpha}{2}} ) |F_2(t)|_{0, \alpha; f(t)} + \gamma^{-1} ( (\gamma t)^{\frac{1}{2}} + (\gamma t)^{1+\frac{\alpha}{2}} ) |G(t)|_{0, \alpha; f(t), (\gamma t)^{1/2}} \right). \]

Proposition 4.1. There exist \( \varepsilon_0 > 0 \) and \( C > 0 \) independent of \( \gamma \) and \( T \) such that
\[ \sup_{0 < t \leq T} |u(t)|_{1, \alpha; f(t), (1+\gamma t)^{1/2}} \leq C (|u(0)|_{1, \alpha; f(0)} + \Lambda_1(T, F_1, F_2, G, f, h)), \]
\[ \sup_{0 < t \leq T} |u(t)|_{1, \alpha; f(t), (\gamma t)^{1/2}} \leq C (|u(0)|_{0, \alpha; f(0)} + \Lambda_0(T, F_1, F_2, G, f, h)). \]
Proof. Let us first consider the case of $a_{ij} = \delta_{ij}$. Then we get

$$u(t) = H(\gamma t)u(0) + \int_0^t (H(\gamma(t-s))F_1(s) + F_2(s)) + \partial_i H(\gamma(t-s))G^i(s)\,ds.$$  

Using $H(2\gamma t)f(0) \leq c_0^{-1} f(0, X)$, we get by Lemma 5.4 that

$$|H(\gamma t)u(0)|_{1, \alpha; f(t), (1+\gamma t)^{1/2}} \leq C|H(\gamma t)u(0)|_{1, \alpha; H(2\gamma t)f(0), (1+\gamma t)^{1/2}} \leq C|u(0)|_{1, \alpha; f(0), 1},$$

and

$$|H(\gamma t)u(0)|_{1, \alpha; f(t), (1+\gamma t)^{1/2}} \leq C|H(\gamma t)u(0)|_{1, \alpha; H(2\gamma t)f(0), (\gamma t)^{1/2}} \leq C|u(0)|_{0, \alpha; f(0)}.$$

By (4.11) and Lemma 5.4 we have

$$\left|\int_0^t H(\gamma(t-s))F_1(s)\,ds\right|_{1, \alpha; f(t), (k+\gamma t)^{1/2}} \leq C \sup_{0<s<t} |H(\gamma(t-s))F_1(s)|_{1, \alpha; H(2\gamma(t-s))h(s), (k+\gamma t)^{1/2}} = C \sup_{0<s<t} |H(\gamma(t-s))F_1(s)|_{1, \alpha; H(2\gamma(t-s))h(s), (k+\gamma s+\gamma(t-s))^{1/2}} \leq C \sup_{0<s<t} |F_1(s)|_{1, \alpha; h(s), (k+\gamma s)^{1/2}},$$

and by Lemma 5.4,

$$\left|\int_0^t H(\gamma(t-s))F_2(s)\,ds\right|_{1, \alpha; f(t), (k+\gamma t)^{1/2}} \leq C \int_0^t |H(\gamma(t-s))F_2(s)|_{1, \alpha; H(2\gamma(t-s))f(s), (k+\gamma t)^{1/2}}\,ds \leq C \int_0^t \varphi_\alpha(\sqrt{k+\gamma t})/\varphi_\alpha(\sqrt{\gamma(t-s)})|F_2(s)|_{0, \alpha; f(s)}\,ds$$

for $k = 0, 1$. Recall that $\varphi_\alpha(R) = \max(R, R^{1+\alpha})$, for $k = 0, 1$,

$$\int_0^t \varphi_\alpha(\sqrt{\gamma(t-s)})^{-1} \min((\gamma s)^{-1+\frac{\alpha+k(1-\alpha)}{2}}, (\gamma s)^{-1-\frac{\delta}{2}})\,ds \leq \int_0^t (\gamma(t-s))^{-\frac{1}{2}} (\gamma s)^{-1+\frac{\alpha+k(1-\alpha)}{2}}\,ds \leq C\gamma^{-1}(\gamma t)^{-\frac{(1-k)(1-\alpha)}{2}},$$

and

$$\int_0^t \varphi_\alpha(\sqrt{\gamma(t-s)})^{-1} \min((\gamma s)^{-1+\frac{\alpha+k(1-\alpha)}{2}}, (\gamma s)^{-1-\frac{\delta}{2}})\,ds \leq \int_0^t (\gamma(t-s))^{-\frac{1+\alpha}{2}} \min((\gamma s)^{-1+\frac{\alpha+k(1-\alpha)}{2}}, (\gamma s)^{-1-\frac{\delta}{2}})\,ds \leq C \int_0^t (\gamma(t-s))^{-\frac{1+\alpha}{2}} \min((\gamma s)^{-1+\frac{\alpha+k(1-\alpha)}{2}}, (\gamma s)^{-1-\frac{\delta}{2}})\,ds + \int_0^t (\gamma(t-s))^{-\frac{1+\alpha}{2}} (\gamma t)^{-1}\,ds \leq C\gamma^{-1}(\gamma t)^{-\frac{1+\alpha}{2}}.$$

Thus, we have

$$\int_0^t \varphi_\alpha(\sqrt{k+\gamma t})/\varphi_\alpha(\sqrt{\gamma(t-s)}) \min((\gamma s)^{-1+\frac{\alpha+k(1-\alpha)}{2}}, (\gamma s)^{-1-\frac{\delta}{2}})\,ds$$
\[ \leq C\gamma^{-1} \max \left( (k + \gamma t)^{\frac{1}{2}}, (k + \gamma t)^{\frac{1-k}{2}} \right) \min \left( (\gamma t)^{-\frac{(1-k)(1-\alpha)}{2}}, (\gamma t)^{-\frac{1-\alpha}{2}} \right) \]
\[ \leq C\gamma^{-1}. \]
Therefore, we deduce that for \( k = 0, 1 \) and \( j = 1, 2, \)
\[ \left| \int_0^t H(\gamma(t-s))F_j(s) ds \right|_{1,\alpha,f(t),(k+\gamma t)^{1/2}} \leq CA_k(T, F_1, F_2, G, f, h). \]

It follows from Lemma 2.3 and Lemma 5.3 that for \( k = 0, 1, \)
\[ \left| \int_0^t \partial_i H(\gamma(t-s))G^i(s) ds \right|_{(1,\alpha,f(t),(k+\gamma t)^{1/2}} \]
\[ \leq C\gamma^{-1} \sup_{0<s<t} \left( (\gamma s)^{\frac{1}{2}} (\gamma(t-s))^{\frac{1}{2}} \right) \left| \partial_i H(\gamma(t-s))G^i(s) \right|_{0,\alpha,f(t)} \]
\[ + \varphi_\alpha(\sqrt{k + \gamma s}) (\gamma(t-s))^{\frac{1-\alpha}{2}} |\nabla \partial_i H(\gamma(t-s))G^i(s)|_{0,\alpha,f(t)} \]
\[ \leq C\gamma^{-1} \sup_{0<s<t} \left( (\gamma s)^{\frac{1}{2}} |G(s)|_{0,\alpha,f(s)} + \varphi_\alpha(\sqrt{k + \gamma s}) |G(s)|_{0,\alpha,f(s)} \right) \]
\[ \leq C\gamma^{-1} \sup_{0<s<t} \left( (k + \gamma) s^{\frac{1}{2}} + (k + \gamma) s^{\frac{1-\alpha}{2}} \right) |G(s)|_{0,\alpha,f(s),(k+\gamma s)^{\frac{1}{2}}} \]
\[ \leq CA_k(T, F_1, F_2, G, f, h). \]

Summing up, we conclude the proof for the case \( a_{ij} = \delta_{ij}. \)
To deal with the general case, we rewrite (4.9) as
\[ \partial_i u - \gamma \Delta u + F_1 + F_2 + \partial_i \hat{G}^i = 0, \]
where \( \hat{G}^i = G^i - \gamma (a_{ij} - \delta_{ij}) \partial_j u. \) Thus, we have
\[ \sup_{0<t \leq T} |u(t)|_{1,\alpha,f(t),(k+\gamma t)^{1/2}} \leq C \left( |u(0)|_{1,\alpha,f(0)} + \Lambda_k(T, F_1, F_2, \hat{G}, f, h) \right), \]
for \( k = 0, 1, \) where
\[ \Lambda_k(T, F_1, F_2, \hat{G}, f, h) \leq \Lambda_k(T, F_1, F_2, G, f, h) \]
\[ + \sup_{0 \leq t \leq T} \sup_{1 \leq i \leq 2} \left( (k + \gamma t)^{\frac{1}{2}} + (\gamma t)^{\frac{1-\alpha}{2}} \right) |(a_{ij} - \delta_{ij}) \partial_j u(t)|_{0,\alpha,f(t),(k+\gamma t)^{1/2}}, \]
and by (4.10),
\[ \left| (a_{ij} - \delta_{ij}) \partial_j u(t) \right|_{0,\alpha,f(t),(k+\gamma t)^{1/2}} \leq C |a_{ij} (t) - \delta_{ij}|_{0,\alpha,1,(k+\gamma t)^{1/2}} \left| \partial_j u(t) \right|_{0,\alpha,f(t),(k+\gamma t)^{1/2}} \]
\[ \leq C \varepsilon_0 |\nabla u(t)|_{0,\alpha,f(t),(k+\gamma t)^{1/2}} \]
\[ \leq C \varepsilon_0 \min \left( (k + \gamma t)^{-\frac{1}{2}}, (k + \gamma t)^{-\frac{1-\alpha}{2}} \right) |u(t)|_{1,\alpha,f(t),(k+\gamma t)^{1/2}}. \]
This shows that
\[ \sup_{0 < t \leq T} |u(t)|_{1,\alpha,f(t),(k+\gamma t)^{1/2}} \leq C \left( |u(0)|_{1,\alpha,f(0)} + \Lambda_k(T, F_1, F_2, G, f, h) \right) \]
\[ + \varepsilon_0 \sup_{0 \leq t \leq T} |u(t)|_{1,\alpha,f(t),(k+\gamma t)^{1/2}}, \]
which gives the desired result by taking \( \varepsilon_0 \) so that \( C \varepsilon_0 \leq \frac{1}{2}. \) \( \square \)
4.3. Weighted $C^{1,\alpha}$ estimates for the transport-diffusion equation. We consider the transport-diffusion equation with general form

\begin{equation}
\partial_t u + Z \cdot \nabla u - \gamma \Delta u + F_1 + F_2 + \partial_t G^i = 0, \quad u(0, X) = u_0(X).
\end{equation}

Given the divergence free vector field $Z(t, X) \in C^1([0, T] \times \mathbb{R}^d)$ and $s \in [0, T]$, we define

\begin{equation}
\frac{d}{dt} \Phi(s, t, X) = Z(t, \Phi(s, t, X)), \quad \Phi(s, s, X) = X.
\end{equation}

We denote by $D\Phi$ and $\nabla \Phi$ the matrix with the convention

\[
(D\Phi)_{ij} = \partial_j \Phi^i, \quad (\nabla \Phi)_{ij} = \partial_i \Phi^j.
\]

That is, $(D\Phi)^T = (\nabla \Phi)^T$. We introduce

\[
b = (D\Phi)^{-1}, \quad a = (D\Phi)^{-1}(\nabla \Phi)^{-1}, \quad a_{ij} = b_{ik}b_{kj}.
\]

For $v(t, X)$ defined in $[0, T] \times \mathbb{R}^d$, we denote

\[
v^* (t, X) \triangleq v(t, \Phi(s, t, X)).
\]

Using the formulas

\[
(\text{div} G) \circ \Phi = \text{div}((D\Phi)^{-1} G \circ \Phi), \quad (\Delta u) \circ \Phi = \text{div}((D\Phi)^{-1}(\nabla \Phi)^{-1} \nabla u \circ \Phi),
\]

we can transform (4.12) into the following form

\begin{equation}
\partial_t u^*(t) - \gamma \partial_i (a_{ij} \partial_j u^*(t)) + F_{1}^* + F_{2}^* + \partial_i G_s^i = 0,
\end{equation}

where $G_s^i = b_{ij}(G^*)^j$.

We introduce the weight function $f(t, X)$, $\hat{f}(t, X)$, $h(t, X)$, which satisfy (2.1) with a uniform constant $c_0$ and

\begin{equation}
\int_0^t H(2\gamma(t-s))h_+(s, X)ds \leq c_0^{-1} \hat{f}(t, X) \quad \text{for all } 0 \leq t \leq T, \quad X \in \mathbb{R}^d,
\end{equation}

(4.14) \[ \int_0^T f_{\pm}(t, X \pm B_0t)dt = \int_0^T f(t, X \pm 2B_0t)dt \leq c_0^{-1}, \]

\[ H(2\gamma(t-s))\hat{f}(s, X) \leq c_0^{-1} \hat{f}(t, X) \quad \text{for all } 0 \leq s \leq t \leq T, \quad X \in \mathbb{R}^d, \]

where we denote

\[
f_{\pm}(t, X) = U(\pm t)f(t, X), \quad U(t)f(s, X) = f(s, X + B_0t).
\]

**Proposition 4.2.** There exists $\varepsilon_1 > 0$ and $C > 0$ independent of $\gamma$ and $T$ such that if

\[ |Z(t) + B_0|_{1, \alpha; f_{\pm}(t), (1 + \gamma)t)^{1/2}} < \varepsilon_1, \]

and (4.14) holds for the minus sign, then it holds that for $k = 0, 1$,

\[ \sup_{0 \leq t \leq T} |u(t)|_{1, \alpha; \hat{f}_{\pm}(t), (k + \gamma)t)^{1/2}} \leq C(|u_0|_{1, \alpha; \hat{f}(0), k} + \Lambda_k(T, F_1, F_2, G, \hat{f}_{\pm}, h)). \]

Similarly, if

\[ |Z(t) - B_0|_{1, \alpha; f_{\pm}(t), (1 + \gamma)t)^{1/2}} < \varepsilon_1, \]

and (4.14) holds for the plus sign, then it holds that for $k = 0, 1$,

\[ \sup_{0 \leq t \leq T} |u(t)|_{1, \alpha; \hat{f}_{\pm}(t), (k + \gamma)t)^{1/2}} \leq C(|u_0|_{1, \alpha; \hat{f}(0), k} + \Lambda_k(T, F_1, F_2, G, \hat{f}_{\pm}, h)). \]
Proof. We only consider the case of \(|Z(t) + B_0|_{1,\alpha,F^-(t),(1+\gamma)t)^{1/2}} < \varepsilon_1\). In this case, similar to (3.7), we have

\[
|\Phi(s,t,X) + B_0(t-s) - X| < 2 \quad \text{for} \quad 0 \leq t \leq s \leq T.
\]

Then we get by (2.1) and (4.14) that

\[
\sup_{t \leq s \leq T} |\nabla Z(s)|_{0,\alpha,F^-(s)} \int_0^T f_-(s,\Phi(T,s,X))ds \\
\leq \varepsilon_1(1+\gamma t)^{-1/2}c_0^{-1} \int_0^T f_-(s, X-B_0(s-T))ds \leq \varepsilon_1(1+\gamma t)^{-1/2}c_0^{-1},
\]

and by (2.1),

\[
(4.15)
\]

\[
(4.16)
\]

\[
(4.17)
\]

Now we fix \(s \geq 0\) and assume \(0 \leq t \leq s \leq T\). With (4.15), we infer from Lemma 3.4 that

\[
(4.18)
\]

This implies that

\[
|a_{ij}(t) - \delta_{ij}|_{0,\alpha} \leq C\varepsilon_1(1+\gamma t)^{-1/2}; \quad |b_{ij}(t)|_{0,\alpha,1,(1+\gamma t)^{1/2}} \leq C.
\]

Using (4.14), it is easy to verify that

\[
H(2\gamma(t-\tau))U(s)\hat{f}(\tau,X) = U(s)H(2\gamma(t-\tau))\hat{f}(\tau,X) \leq c_0^{-1}U(s)\hat{f}(t),
\]

and

\[
\int_0^t H(2\gamma(t-\tau))U(s-\tau)h(\tau,X)d\tau = \int_0^t H(2\gamma(t-\tau))U(s)h_-(\tau,X)d\tau \\
\leq c_0^{-1}U(s)\hat{f}(t).
\]

Therefore, if we take \(\varepsilon_1 > 0\) so that \(C\varepsilon_1 \leq \varepsilon_0\), then we can apply Proposition 4.1 to obtain

\[
\sup_{0 < t \leq s} |u^*(t)|_{1,\alpha;U(s)\hat{f}(t),(k+\gamma t)^{1/2}} \\
\leq C(|u_0 \circ \Phi(s,0)|_{1,\alpha;U(s)\hat{f}(0),k} + \Lambda_k(s,F_1^*,F_2^*,G_s,U(s)\hat{f},U(s-\cdot)h)).
\]

Thanks to (4.18), we get by Lemma 2.4 (4.16) and (4.17) that

\[
|u_0 \circ \Phi(s,0)|_{1,\alpha;U(s)\hat{f}(0),k} \leq C|u_0 \circ \Phi(s,0)|_{1,\alpha;\hat{f}(0),\Phi(s,0),k} \leq C|u_0|_{1,\alpha;\hat{f}(0),k},
\]

\[
|F_2^s(t)|_{0,\alpha;U(s)\hat{f}(t)} \leq C|F_2^s(t)|_{0,\alpha;\hat{f}^+(t)\circ \Phi(s,t)} \leq C|F_2^s(t)|_{0,\alpha;\hat{f}^+(t)},
\]

\[
|F_1^s(t)|_{1,\alpha;U(s-t)h(t),(k+\gamma t)^{1/2}} \leq C|F_1^s(t)|_{1,\alpha;h(t)\circ \Phi(s,t),(k+\gamma t)^{1/2}} \leq C|F_1^s(t)|_{1,\alpha;h(t),(k+\gamma t)^{1/2}},
\]

and

\[
|G_s(t)|_{0,\alpha;U(s)\hat{f}(t),(k+\gamma t)^{1/2}} \leq C|G_s(t)|_{0,\alpha;\hat{f}^+(t)\circ \Phi(s,t),(k+\gamma t)^{1/2}} \\
\leq C|b(t)|_{0,\alpha,1,(1+\gamma t)^{1/2}}|G(t) \circ \Phi(s,t)|_{0,\alpha;\hat{f}^+(t)\circ \Phi(s,t),(k+\gamma t)^{1/2}} \\
\leq C|G(t)|_{0,\alpha;\hat{f}^+(t),(k+\gamma t)^{1/2}}.
\]

This proves

\[
\Lambda_k(s,F_1^*,F_2^*,G_s,U(s)\hat{f},U(s-t)h) \leq C\Lambda_k(s,F_1,F_2,G,\hat{f}^+,h).
\]
Therefore, we conclude
\[
\sup_{0 < t \leq s} |u^*(t)|_{1, \alpha; U(s)} \leq C \left( |u_0|_{1, \alpha; \hat{U}(0), k} + \Lambda_k(s, F_1, F_2, G, \hat{f}_+, h) \right).
\]

Thanks to \( u^*(s) = u(s) \) and \( U(s) \hat{f}(s) = \hat{f}_+(s) \), we have
\[
|u(s)|_{1, \alpha; \hat{f}_+(s), (k+\gamma t)^{1/2}} \leq C \left( |u_0|_{1, \alpha; \hat{f}(0), k} + \Lambda_k(s, F_1, F_2, G, \hat{f}_+, h) \right)
\]
for all \( 0 < s \leq T \). The case of \( s = 0 \) is trivial. This completes the proof. 

4.4. Main result. Let us first introduce the weight functions
\[
f(t) = H(1 + 2\mu_1 t)\phi_1, \quad f_1(t) = H(1 + 2\mu_1 t)\phi_0,
\]
where if \( B_0 = (1, 0, \cdots, 0) \), we may take
\[
\phi_1(X) = |x_1^2 + x_2^2|^{-\frac{1+\delta}{2}}, \quad \phi_0(X) = |x_2|^{-\delta}
\]
for some \( 0 < \delta < \frac{1}{2} \). Let
\[
g(t, X) \triangleq \int_{\mathbb{R}^d} \frac{f_+(t, Y)f_-(t, Y)}{1 + |X - Y|^{d+1}} dY.
\]

We introduce
\[
M_\pm(t) \triangleq \sup_{0 \leq \tau \leq t} \left( |z_\pm^{(1)}(\tau)|_{1, \alpha; f_\pm(\tau), (1+\mu_1 \tau)^{1/2}} + |J_\pm^{(1)}(\tau)|_{1, \alpha; f_\pm(\tau), (\mu_1 \tau)^{1/2}} + \mu_1^{-1} |\phi_\pm^{(2)}(\tau)|_{1, \alpha; f_1(\tau), (\mu_1 \tau)^{1/2}} \right).
\]

Main result of this section is stated as follows.

**Theorem 4.3.** Let \( \alpha \in (0, 1) \). There exists \( \varepsilon_2 > 0 \) so that if \( M_\pm(0) + \mu_2 / \mu_1 \leq \varepsilon \leq \varepsilon_2 \), then there exists a global in time unique solution \((z_+, z_-) \in L^\infty((0, +\infty) \times \mathbb{R}^d)\) with the pressure \( p \) determined by (4.16) to the viscous MHD equations (4.1) satisfying
\[
M_\pm(t) \leq C\varepsilon \quad \text{for any} \quad t \in [0, +\infty).
\]

**Remark 4.4.** Thanks to \( M_\pm(0) \sim |z_\pm(0)|_{((x_1, x_2))^{1+\delta}} |_{1, \alpha}, \) this means that the initial data decays at infinity only in two directions. This is a key point for the global well-posedness in the strip domain, especially in \( \mathbb{R}^2 \) and \( \mathbb{R}^2 \times [0, 1] \).

To proceed, we need to verify that the weight functions introduced here satisfy some key properties (2.3) and (4.17).

With the choice of (4.19), it is easy to check that for \( k = 0, 1 \),
\[
C^{-1} R^d \min(\phi_k(X), R^{-k-\delta}) \leq \int_{B(X, R)} \phi_k(Y) dY \leq CR^d \min(\phi_k(X), R^{-k-\delta}),
\]
\[
\int_{\mathbb{R}} \phi_1(X + B_0 t) dt \leq C\phi_0(X),
\]
which imply
\[
C^{-1} \min(\phi_1(X), (1 + \mu_1 t)^{-\frac{1+\delta}{2}}) \leq f(t, X) \leq C \min(\phi_1(X), (1 + \mu_1 t)^{-\frac{1+\delta}{2}}),
\]
\[
C^{-1} \min(\phi_0(X), (1 + \mu_1 t)^{-\frac{1}{2}}) \leq f_1(t, X) \leq C \min(\phi_0(X), (1 + \mu_1 t)^{-\frac{1}{2}}),
\]
\[
\int_{\mathbb{R}} f(t, X + B_0 s) ds \leq C f_1(t, X).
\]

Therefore,
\[
\int_{B(X, R)} f_1(t, Y) dY \leq CR^d \min(R^{-\delta}, (1 + \mu_1 t)^{-\frac{1+\delta}{2}}),
\]
and
\begin{equation}
\int_{B(X,R)} h(Y) dY \leq C h(X),
\end{equation}
which is true for \( h = 1, f(t), f_1(t), \) and \( f_\pm(t) \) by translation. Thus,
\begin{equation}
\int_{\mathbb{R}^d} \frac{f_\pm(t,Y) dY}{R^{d+1} + |X-Y|^{d+1}} \leq C R^{-1} f_\pm(t, X).
\end{equation}

Lemma 4.5. (1) The weight functions \( f(t,X), f_1(t,X), g(t,X) \) satisfies \((2.3)\) with \( R = (1 + \mu_1 t)^{\frac{1}{2}} \) and a uniform constant \( c_0 \) independent of \( t \).

(2) Property \((4.14)\) with \( \gamma = \mu_1 \) holds true for \((\tilde{f}, h) = (f,g) \) or \((\tilde{f}, h) = (f_1,f_-) \) for the minus sign or \((f,h) = (f_1,f_+) \) for the plus sign.

Proof. We deduce from \((4.20)\) and \((4.21)\) that
\begin{equation}
\begin{aligned}
H(2\mu_1(t-s)) f(s) & = f(t), \\
H(2\mu_1(t-s)) f_1(s) & = f_1(t)
\end{aligned}
\end{equation}
which give the third inequality of \((4.14)\).

Thanks to
\begin{equation}
\int_0^T f_\pm(t, X \pm B_0 t) dt = \int_0^T f(t, X \pm 2 B_0 t) dt \leq C f_1(t, X) \leq C,
\end{equation}
which gives the second inequality of \((4.14)\).

Thanks to
\begin{equation}
\begin{aligned}
\int_0^t H(2\mu_1(t-s)) f_-(s,X) ds & = \int_0^t H(2\mu_1(t-s)) U(-2s) f(s,X) ds \\
& = \int_0^t f(t, X - 2 B_0 s) ds \leq C f_1(t, X),
\end{aligned}
\end{equation}
which gives the first inequality of \((4.14)\) with minus sign for \((\tilde{f}, h) = (f_1,f_-) \). Similarly, the first inequality of \((4.14)\) with plus sign for \((\tilde{f}, h) = (f_1,f_+) \) is true.

Notice that
\begin{equation}
\begin{aligned}
H(2\mu_1(t-s)) g_\pm(s,X) & = \int_{\mathbb{R}^d} \frac{H(2\mu_1(t-s))(f_\pm(s)f_\pm(s))(Y \pm B_0 s) dY}{1 + |X-Y|^{d+1}} \\
& = \int_{\mathbb{R}^d} \frac{H(2\mu_1(t-s))(f(s)U(\pm 2s) f(s))(Y) dY}{1 + |X-Y|^{d+1}}.
\end{aligned}
\end{equation}
By \((1.20)\), we have
\begin{equation}
f(t,X) \leq C(1 + |Y-X|/\sqrt{1 + \mu_1 t})^{1+\delta} f(t,Y),
\end{equation}
which gives
\begin{equation}
f(s) U(\pm 2s) f(s)(X) \leq C(1 + |Y-X|/\sqrt{1 + \mu_1 s})^{2+2\delta} f(s) U(\pm 2s) f(s)(Y).
\end{equation}
Therefore, for \( t/2 \leq s < t \),
\begin{equation}
H(2\mu_1(t-s))(f(s) U(\pm 2s) f(s))(Y)
\end{equation}
Proof of Theorem 4.3. which gives the first inequality of (4.14) for (\( f_h \)). This shows that

\[ H(2\mu_1(t-s))(f(s)U(\pm2s)f(s)) \leq CH(2\mu_1t)(f(s)U(\pm2s)f(s)) \leq CH(2\mu_1t)(f(0)U(\pm2s)f(0)), \]

therefore,

\[
\int_0^t H(2\mu_1(t-s))(f(s)U(\pm2s)f(s))ds \\
\leq C \int_0^{\frac{t}{2}} H(2\mu_1t)(f(0)U(\pm2s)f(0))ds + C \int_{\frac{t}{2}}^t f(t)U(\pm2s)f(0)ds \\
\leq CH(2\mu_1t)(f(0)f_1(0)) + Cf(t)f_1(0) \leq Cf(t).
\]

This shows that

\[ \int_0^t H(2\mu_1(t-s))g_\pm(s,X)ds \leq C \int_{\mathbb{R}^d} \frac{f(t,Y)}{1 + |X-Y|^{d+1}}dY \leq Cf(t, X), \]

which gives the first inequality of (4.14) for (\( \hat{f}, h \)) = (\( f, g \)).

4.5. Proof of Theorem 4.3. The following lemma gives the relation between the Hölder norms of \( z_\pm^{(i)}(t) \), \( i = 1, 2 \) and \( M_\pm(t) \).

**Lemma 4.6.** It holds that

\[
|z_\pm^{(2)}(t)|_{0,\alpha;f_1(t),(\mu_1t)^{1/2}} \leq C\mu_1 \min((\mu_1t)^{-\frac{1}{2}}, (\mu_1t)^{-\frac{1+\alpha}{2}})M_\pm(t), \\
|z_\pm^{(2)}(t)|_{0,\alpha;1,(1+\mu_1t)^{1/2}} \leq C\mu_1 \min((\mu_1t)^{-\frac{1}{2}}, (\mu_1t)^{-\frac{1+\alpha}{2}})M_\pm(t), \\
|\nabla z_\pm^{(1)}|_{1,\alpha;f_\pm(t),(\mu_1t)^{1/2}} \leq CM_\pm(t).
\]

**Proof.** As \( z_\pm^{(2)} = \text{div} \psi_\pm^{(2)} \), we have

\[
|z_\pm^{(2)}(t)|_{0,\alpha;f_1(t),(\mu_1t)^{1/2}} \leq C|\nabla \psi_\pm^{(2)}(t)|_{0,\alpha;f_1(t),(\mu_1t)^{1/2}} \\
\leq C|\psi_\pm^{(2)}(t)|_{1,\alpha;f_1(t),(\mu_1t)^{1/2}} \min((\mu_1t)^{-\frac{1}{2}}, (\mu_1t)^{-\frac{1+\alpha}{2}}) \\
\leq C\mu_1 \min((\mu_1t)^{-\frac{1}{2}}, (\mu_1t)^{-\frac{1+\alpha}{2}})M_\pm(t),
\]

which along with (4.21) gives

\[
|z_\pm^{(2)}(t)|_{0,\alpha;1,(1+\mu_1t)^{1/2}} \leq |z_\pm^{(2)}(t)|_{0,\alpha;f_1(t),(\mu_1t)^{1/2}} \left( 1 + \frac{1}{\mu_1t} \right)^{\frac{1}{2}} |f_1(t)|_0 \\
\leq C\mu_1 \min((\mu_1t)^{-\frac{1}{2}}, (\mu_1t)^{-\frac{1+\alpha}{2}})M_\pm(t) \left( 1 + \frac{1}{\mu_1t} \right)^{\frac{1}{2}} (1 + \mu_1t)^{-\frac{1}{2}} \\
\leq C\mu_1 \min((\mu_1t)^{-\frac{1}{2}}, (\mu_1t)^{-\frac{1+\alpha}{2}})M_\pm(t).
\]
Obviously, we have

\[ |\nabla z_\pm^{(1)}|_{0, \alpha; f_\pm(t)} \leq |z_\pm^{(1)}|_{1, \alpha; f_\pm(t), (1+\mu_1 t)^{1/2}} \leq M_\pm(s). \]

Thanks to \( \Delta z_\pm^{(1)} = \text{div} J_\pm^{(1)} \), we have

\[ |\Delta z_\pm^{(1)}|_{0, \alpha; f_\pm(t), (\mu_1 t)^{1/2}} \leq C |J_\pm^{(1)}|_{1, \alpha; f_\pm(t), (\mu_1 t)^{1/2}} \min((\mu_1 t)^{-\frac{1}{2}}, (\mu_1 t)^{-\frac{1-\alpha}{2}}) \]

\[ \leq CM_\pm(s) \min((\mu_1 t)^{-\frac{1}{2}}, (\mu_1 t)^{-\frac{1-\alpha}{2}}). \]

Notice that by Lemma 4.5, \( f_\pm(t, X) \leq Cf_\pm(t, Y) \) if \( |X - Y| \leq (1 + \mu_1 t)^{\frac{1}{2}}. \)

Then we infer from Lemma 5.10 that

\[ |\nabla^2 z_\pm^{(1)}|_{0, \alpha; f_\pm(t), (\mu_1 t)^{1/2}} \leq C \left( |\nabla z_\pm^{(1)}|_{0, \alpha; f_\pm(t)} \min((\mu_1 t)^{-\frac{1}{2}}, (\mu_1 t)^{-\frac{1-\alpha}{2}}) + |\Delta z_\pm^{(1)}|_{0, \alpha; f_\pm(t), (\mu_1 t)^{1/2}} \right) \]

\[ \leq CM_\pm(s) \min((\mu_1 t)^{-\frac{1}{2}}, (\mu_1 t)^{-\frac{1-\alpha}{2}}). \]

This proves the third inequality. \( \square \)

Now let's begin with the proof of Theorem 4.3.

**Proof.** For fixed \( \nu > 0 \) and \( \mu > 0 \), the local well-posedness of the MHD equations in the weighted Hölder space can be proved by using the semigroup method and the estimates of heat operator in the weighted Hölder space (see section 6.3). Here we omit the details. The local well-posedness of the linear equations (4.2)-(4.7) in the weighted Hölder space is also true.

The proof of global well-posedness is based on the continuity argument. Let us first assume

(4.26) \( M_\pm(s) < \varepsilon_1. \)

for \( \varepsilon_1 > 0 \) given by Proposition 4.2. This in particular gives

\[ |Z_\pm^{(1)}(t)| \leq B_0|_{1, \alpha; f_\pm(t), (1+\mu_1 t)^{1/2}} < \varepsilon_1. \]

Our next goal is to show that

(4.27) \( M_+(s) \leq C \left( M_+(0) + (M_+(s) + \mu_2/\mu_1)M_-(s) \right), \)

(4.28) \( M_-(s) \leq C \left( M_-(0) + (M_-(s) + \mu_2/\mu_1)M_+(s) \right). \)

With the above estimates, we can deduce our result if \( \varepsilon_2 \) is taken small enough so that

\[ CM_\pm(0) \leq C\varepsilon_2 < \varepsilon_1/2, \quad C^2\varepsilon_2 < 1/2. \]

This condition on \( \varepsilon_2 \) implies that if \( M_\pm(s) < \varepsilon_1 \) then \( M_\pm(s) < 2CM_\pm(0) < \varepsilon_1. \)

The proof of (4.27) and (4.28) is split into three steps.

**Step 1.** \( C^{1, \alpha} \) estimate for \( z_\pm^{(1)} \)

For the system (4.1), we apply Proposition 4.2 to obtain

\[ \sup_{0 \leq t \leq s} |z_\pm^{(1)}(t)|_{1, \alpha; f_\pm(t), (1+\mu_1 t)^{1/2}} \]

\[ \leq C \left( |z_\pm(0)|_{1, \alpha; f_\pm(t)} + \Lambda_1 (s, I(z_\pm^{(1)}), \nabla z_\pm^{(1)} + I(z_\pm^{(2)}), 0, f_\pm, g) \right). \]

By (5.5), we have

\[ |I(z_\pm^{(1)}(t), z_\pm^{(1)}(t))|_{1, \alpha; g(t), (1+\mu_1 t)^{1/2}} \leq CM_\pm(s)M_\pm(s), \]

\[ \leq CM_\pm(s)M_\pm(s). \]
and by (5.6) and Lemma 4.6
\[ |z_+^{(2)} \cdot \nabla z_+^{(1)}(t) + I(z_+^{(2)}(t), z_+^{(1)}(t))|_{0, \alpha; f_+(t)} \leq C|z_+^{(2)}(t)|_{0, \alpha; 1, (1 + \mu_1 t)^{1/2}} |z_+^{(1)}(t)|_{1, \alpha; f_+(t), (1 + \mu_1 t)^{1/2}} (1 + \mu_1 t)^{-\frac{\alpha}{2}} \]
\[ \leq C \mu_1 M_-(s) \min((\mu_1 t)^{-\frac{1}{2}}, (\mu_1 t)^{-\frac{1+\alpha}{2}}) M_+(s) (1 + \mu_1 t)^{-\frac{\alpha}{2}} \]
\[ \leq C \mu_1 M_+(s) M_-(s) \min((\mu_1 t)^{-\frac{1}{2}}, (\mu_1 t)^{-\frac{1+\alpha}{2}}), \]
and obviously,
\[ |z_+(0)|_{1, \alpha; f(0)} \leq M_+(0). \]
Therefore, we obtain
\[ \sup_{0 \leq t \leq s} |z_+^{(1)}(t)|_{1, \alpha; f_+(t), (1 + \mu_1 t)^{1/2}} \leq C \left( M_+(0) + M_+(s) M_-(s) \right). \]
Similarly, we have
\[ \sup_{0 \leq t \leq s} |z_-^{(1)}(t)|_{1, \alpha; f_-(t), (1 + \mu_1 t)^{1/2}} \leq C \left( M_-(0) + M_+(s) M_-(s) \right). \]

**Step 2.** $C^{1, \alpha}$ estimate for $J_+^{(1)}$
For the system (1.3), we apply Proposition 4.2 to obtain
\[ \sup_{0 \leq t \leq s} |J_+^{(1)}(t)|_{1, \alpha; f_+(t), (\mu_1 t)^{1/2}} \leq C \left( |J_+(0)|_{0, \alpha; f(0)} + \Delta_0(s, \nabla z_-^{(1)} \wedge \nabla z_+^{(1)}, 0, z_-^{(2)} \cdot \nabla z_+^{(1)}, f_+, g) \right). \]
Thanks to the choice of weight functions, we have
\[ f_-(t, X) f_+(t, X) \leq C g(t, X). \]
Then by Lemma 4.6 and analogous of Lemma 2.1, we have
\[ |\nabla z_-^{(1)} \wedge \nabla z_+^{(1)}(t)|_{1, \alpha; g(t), (\mu_1 t)^{1/2}} \leq C |\nabla z_-^{(1)}|_{1, \alpha; f_-(t), (\mu_1 t)^{1/2}} |\nabla z_+^{(1)}(t)|_{1, \alpha; f_+(t), (\mu_1 t)^{1/2}} \]
\[ \leq CM_+(s) M_-(s), \]
\[ |z_-^{(2)} \cdot \nabla z_+^{(1)}(t)|_{0, \alpha; f_+(t), (\mu_1 t)^{1/2}} \leq C |z_-^{(2)}(t)|_{0, \alpha; 1, (\mu_1 t)^{1/2}} |\nabla z_+^{(1)}(t)|_{0, \alpha; f_+(t), (\mu_1 t)^{1/2}} \]
\[ \leq C \mu_1 \min((\mu_1 t)^{-\frac{1+\alpha}{2}}, (\mu_1 t)^{-\frac{\alpha}{2}}) M_-(s) M_+(s), \]
and $|j_+(0)|_{0, \alpha; f(0)} \leq M_+(0)$. Therefore, we obtain
\[ \sup_{0 \leq t \leq s} |J_+^{(1)}(t)|_{1, \alpha; f_+(t), (\mu_1 t)^{1/2}} \leq C \left( M_+(0) + M_+(s) M_-(s) \right). \]
Similarly, we have
\[ \sup_{0 \leq t \leq s} |J_-^{(1)}(t)|_{1, \alpha; f_-(t), (\mu_1 t)^{1/2}} \leq C \left( M_-(0) + M_+(s) M_-(s) \right). \]

**Step 3.** $C^{1, \alpha}$ estimate for $\psi_+^{(2)}$
For the system (1.7), we apply Proposition 4.2 to obtain
\[ \sup_{0 \leq t \leq s} |\psi_+^{(2)}(t)|_{1, \alpha; f_1(t), (\mu_1 t)^{1/2}} \leq C \Delta_0(s, \Pi_2(z_-^{(1)}, z_+^{(2)}), -\mu_2 j_-^{(1)}, \Pi_1(z_-^{(2)}, z_+^{(2)}), \mu_2 z_-, f_1, f_-), \]
here we used the fact that $\psi_+^{(2)}(0) = 0, f_1 = f_1$, and the decomposition of $J_\pm$ in (4.6). We get by Proposition 5.6 and Lemma 4.6 that
\[ |\Pi_2(z_-^{(1)}(t), z_+^{(2)}(t)) - \mu_2 j_-^{(1)}(t)|_{1, \alpha; f_-(t), (\mu_1 t)^{1/2}} \leq C |z_-^{(1)}(t)|_{1, \alpha; f_-(t), (\mu_1 t)^{1/2}} |\nabla \psi_+^{(2)}(t)|_{1, \alpha; f_1(t), (\mu_1 t)^{1/2}} + \mu_2 j_-^{(1)}(t)|_{1, \alpha; f(t), (\mu_1 t)^{1/2}} \]
\[ \leq C \mu_1 M_+(s) M_-(s) + \mu_2 M_-(s), \]

and

\[
|\Pi_1(z^{(2)}_-(t), z^{(2)}_+(t))|_{0, \alpha; f_1(t)} \leq C|z^{(2)}_-(t)|_{0, \alpha; f_1(t), (\mu_1 t)^{1/2}} |z^{(2)}_+(t)|_{0, \alpha; f_1(t), (\mu_1 t)^{1/2}} (1 + \mu_1 t)^{-\frac{\tilde{\alpha}}{2}} (1 + (\mu_1 t)^{-\frac{\tilde{\alpha}}{2}})
\]

\[
\leq C \mu_1^2 \min((\mu_1 t)^{-1+\frac{s}{2}}, (\mu_1 t)^{-1+\frac{1-\tilde{\alpha}}{2}}) M_+(s) M_-(s),
\]

and

\[
|\mu_2 z^{(2)}_-(t)|_{0, \alpha; f_1(t), (\gamma t)^{1/2}} \leq C \mu_1 \mu_2 \min((\mu_1 t)^{-\frac{1}{2}}, (\mu_1 t)^{-\frac{1-\alpha}{2}}) M_-(s).
\]

This shows that

\[
\sup_{0 \leq t \leq s} |\psi_2^{(2)}(t)|_{1, \alpha; f_1(t), (\mu_1 t)^{1/2}} \leq C (\mu_1 M_+(s) + \mu_2) M_-(s).
\]

Similarly, we have

\[
\sup_{0 \leq t \leq s} |\psi_2^{(2)}(t)|_{1, \alpha; f_1(t), (\mu_1 t)^{1/2}} \leq C (\mu_1 M_-(s) + \mu_2) M_+(s).
\]

Summing up the estimates in Step 1-Step 3, we conclude (4.27) and (4.28). \(\square\)

5. **Appendix**

5.1. **Weighted \(C^{1, \alpha}\) estimate for the integral operator.** Recall that

\[
T_1 u \triangleq \int_{\mathbb{R}^d} \nabla N(X - Y) \theta(|X - Y|) u(Y) dY
\]

\[
T_{ij} w \triangleq \int_{\mathbb{R}^d} \partial_i \partial_j \left( \nabla N(X - Y)(1 - \theta(|X - Y|)) \right) w(Y) dY,
\]

where the cut-off function \(\theta\) is given by (3.9).

**Lemma 5.1.** Let \(u, w \in C^{0, \alpha}_h(\mathbb{R}^d)\) with the weight \(h\) satisfying (2.1). Then there exists a constant \(C > 0\) depending only on \(c_0\) so that

\[
|T_1 u|_{1, \alpha; h} \leq C |u|_{0, \alpha; h},
\]

\[
|T_{ij} w|_{1, \alpha; g} \leq C |w|_{0, h},
\]

where \(g(X) = \int_{\mathbb{R}^d} \frac{h(Y)}{1 + |X - Y|^q} dY\). In particular, we have

\[
|T_1 u + T_{ij} w|_{1, \alpha; g} \leq C \left( |u|_{0, \alpha; h} + |w|_{0, h} \right).
\]

**Proof.** Thanks to

\[
\left| \nabla^k \partial_i \partial_j \left( \nabla N(X - Y) \cdot (1 - \theta(|x - y|)) \right) \right| \leq \frac{C}{1 + |x - y|^{q+1}}, \quad k = 0, 1, 2,
\]

and \(h(X) \leq C g(X)\), we get

\[
|\nabla^k T_{ij} w(X)| \leq C g(X) |w/h|_0,
\]

which in particular implies

\[
(5.1) \quad |T_{ij} w|_{1, \alpha; g} \leq C |w/h|_0.
\]

To deal with \(T_1 u\), we decompose it as follows

\[
T_1 u = \sum_{k=0}^{+\infty} B_k(u),
\]
where
\[ B_k(u) = \int_{\mathbb{R}^d} \varphi_k(X-Y)u(Y)dY, \quad \varphi_k(X) = \nabla N(X) \cdot (\theta(2^k|X|) - \theta(2^{k+1}|X|)). \]

To proceed, we need to use the following simple facts:
\[
\int_{\mathbb{R}^d} |\varphi_k(X)|dX \leq C2^{-k},
\int_{\mathbb{R}^d} |\nabla \varphi_k(X)||X|^\alpha dX \leq C2^{-k\alpha},
\int_{\mathbb{R}^d} |\nabla^2 \varphi_k(X)||X|^\alpha dX \leq C2^{k(1-\alpha)},
\varphi_k(X) = 0 \quad \text{for} \quad |X| > 2, \ k \geq 0.
\]

Then we have
\[
|B_k(u)(X)| \leq \int_{\mathbb{R}^d} |\varphi_k(X-Y)||h(Y)||u/h|_0 \leq C2^{-k}h(X)|u/h|_0. \tag{5.2}
\]

Notice that
\[
\nabla B_k(u)(X) = \int_{\mathbb{R}^d} \nabla \varphi_k(X-Y)(u(Y) - u(X))dY,
\]
from which, we deduce
\[
|\nabla B_k(u)(X)| \leq \int_{\mathbb{R}^d} |\nabla \varphi_k(X-Y)||X-Y|^\alpha (h(X) + h(Y))dY|u|_{0,\alpha;h} \leq C2^{-k\alpha}h(X)|u|_{0,\alpha;h}. \tag{5.3}
\]

Similarly, we have
\[
|\nabla^2 B_k(u)(X)| \leq C2^{k(1-\alpha)}h(X)|u|_{0,\alpha;h}. \tag{5.4}
\]

It follows from (5.2) and (5.3) that
\[
\sum_{k=0}^{+\infty} |B_k(u)(X)| \leq \sum_{k=0}^{+\infty} C2^{-k}h(X)|u/h|_0 \leq Ch(X)|u/h|_0,
\]
\[
\sum_{k=0}^{+\infty} |\nabla B_k(u)(X)| \leq \sum_{k=0}^{+\infty} C2^{-k\alpha}h(X)|u|_{0,\alpha;h} \leq Ch(X)|u|_{0,\alpha;h}.
\]

It follows from (5.3) and (5.4) that
\[
|\nabla B_k(u)(X) - \nabla B_k(u)(Y)| \leq C2^{-k\alpha}(h(X) + h(Y))|u|_{0,\alpha;h} \min \{1, 2^k|X-Y|\},
\]
which gives
\[
\left| \sum_{k=0}^{+\infty} \nabla \left( B_k(u)(X) - B_k(u)(Y) \right) \right| \leq C(h(X) + h(Y))|u|_{0,\alpha;h} \sum_{k=0}^{+\infty} 2^{-k\alpha} \min \{1, 2^k|X-Y|\}
\]
\[
\leq C\left( h(X) + h(Y) \right)|u|_{0,\alpha;h}|X-Y|^\alpha.
\]

Now we can conclude that
\[
|T_1 u|_{1,\alpha;h} \leq \left| \sum_{k=0}^{+\infty} B_k(u) \right|_{1,\alpha;h} \leq C|u|_{0,\alpha;h}.
\]
This finishes the proof of the lemma. \[\square\]
Lemma 5.2. It holds that
\[
\text{div}(T_1 u + T_{ij}w^{ij}) + u = \int_{\mathbb{R}^d} \nabla N(X - Y) \cdot \nabla \theta(|X - Y|) u(Y) dY \\
- \int_{\mathbb{R}^d} \partial_i \partial_j \left( \nabla N(X - Y) \cdot \nabla \theta(|X - Y|) \right) w^{ij}(Y) dY.
\]

Proof. With the notations in Lemma 5.1 a direct calculation gives
\[
\text{div} T_{ij}(w^{ij}) = - \int_{\mathbb{R}^d} \partial_i \partial_j \left( \nabla N(X - Y) \cdot \nabla \theta(|X - Y|) \right) w^{ij}(Y) dY,
\]
\[
\text{div} B_k(u) = \int_{\mathbb{R}^d} \text{div} \varphi_k(X) u(Y) dY,
\]
where
\[
\text{div} \varphi_k(X) = \nabla N(X) \cdot \nabla \theta(2^k|X|) - \theta(2^{k+1}|X|) = \varphi^*_k(X) - \varphi^*_{k+1}(X),
\]
\[
\varphi^*_k(X) = \nabla N(X) \cdot \nabla \theta(2^k|X|) = -c_d \frac{2^k \theta'(2^k |X|)}{|X|^{d-1}} \geq 0.
\]
Therefore,
\[
\text{div} \sum_{k=0}^{N} B_k(u) + u
= \int_{\mathbb{R}^d} (\varphi^*_0(X - Y) - \varphi^*_{N+1}(X - Y)) u(Y) dY + u(X)
= \int_{\mathbb{R}^d} \varphi^*_0(|X - Y|) u(Y) dY - \int_{\mathbb{R}^d} \varphi^*_{N+1}(X - Y)(u(Y) - u(X)) dY
\triangleq I_0^* - I_{N+1}^*.
\]
Here we used \( \int_{\mathbb{R}^d} \varphi^*_k(X) dX = 1 \). Now,
\[
|I_{N+1}^*| \leq [u]_\alpha \int_{\mathbb{R}^d} \varphi^*_{N+1}(X - Y)|X - Y|^\alpha dY = C[u]_\alpha 2^{-N\alpha} \rightarrow 0,
\]
as \( N \rightarrow +\infty \). This proves the lemma. \( \square \)

We also introduce
\[
T_1(u, R) \triangleq \int_{\mathbb{R}^d} \nabla N(X - Y) \theta(|X - Y|/R) u(Y) dY,
\]
\[
T_{ij}(w, R) \triangleq \int_{\mathbb{R}^d} \partial_i \partial_j \left( \nabla N(X - Y)(1 - \theta(|X - Y|/R)) \right) w(Y) dY,
\]
where \( N(X) \) is the Newton potential. Let \( R \geq 1 \). If \( h(X) \leq C_0 h(Y) \) for \(|X - Y| \leq 2R\), then we can deduce by following the proof of Lemma 5.1 that
\[
|T_1(u, R)|_{1, \alpha; g, R} + |T_{ij}(w, R)|_{1, \alpha; g, R} \leq C \left( R^2 |u|_{0, \alpha; h, R} + |w|_{0; h} \right),
\]
where \( g(X) = \int_{\mathbb{R}^d} \frac{h(Y)}{R^{d+1} |X - Y|^{d+1}} dY \). Due to (4.23), we also have
\[
R^{-1}|T_1(u, R)|_{1, \alpha; f_\pm(t), R} + |T_{ij}(w, R)|_{0, \alpha; f_\pm(t), R} \leq C \left( |u|_{0, \alpha; f_\pm(t), R} + R^{-1} |w|_{0; f_\pm(t)} \right)
\]
for \( R = \sqrt{1 + \mu_1 t} \).

In particular, we have
\[
|I(u, w)|_{1, \alpha; g(t), (1 + \mu_1 t)^{1/2}}
\]
where \(g, f_{\pm}\) are defined as in section 4.4.

For \(\text{div} u = \text{div} w = 0\), we have

\[
I(u, w) = T_1(\partial_t u^j \partial_j w^i, R) + T_{ij}(u^j w^i, R) = \partial_t T_1(u^j \partial_j w^i, R) + T_{ij}(u^j w^i, R).
\]

Therefore, we deduce

\[
|I(u, w)|_{0; \alpha; f_{\pm}}(t) \leq C |u|_{0; \alpha; 1; (1 + \gamma) t}^{1/2} |w|_{1; \alpha; f_{\pm}}(t)(1 + \gamma t)^{-\frac{d}{2}}. \tag{5.6}
\]

5.2. **Weighted Hölder estimates for the heat operator.** Let \(H(t)\) be the heat operator given by

\[
H(t)f(X) \overset{\text{def}}{=} \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|X - Y|^2}{4t}} f(Y) dY = \int_{\mathbb{R}^d} K(t, X - Y) f(Y) dY,
\]

where \(K(t, X) = (4\pi t)^{-d/2} e^{-\frac{|X|^2}{4t}}\). Let \(\alpha \geq 0\) and \(k \in \mathbb{N}\). It is easy to verify the following properties

\[
|\nabla^k K(t, X)| \leq Ct^{-\frac{k}{2}} K(2t, X),
\]

\[
|\nabla^k K(t, X)||X'||^\alpha \leq Ct^{-\frac{k+\alpha}{2}} K(2t, X), \tag{5.7}
\]

\[
|\nabla^k K(t, X) - \nabla^k K(t, Y)| \leq Ct^{-\frac{k+1}{2}} K(2t, X)|X - Y|,
\]

\[
|\nabla^k K(t, X) - \nabla^k K(t, Y)||X'||^\alpha \leq C t^{-\frac{k+1}{2} - \alpha} K(2t, X)|X - Y|,
\]

for any \(X, Y \in B(X, \sqrt{t})\). Here \(C\) is a constant independent of \(t\).

We introduce the following seminorm

\[
[u]_{1; h} \overset{\text{def}}{=} \sup_{X, Y \in \mathbb{R}^d} \frac{|u(X) - u(Y)|}{(h(X) + h(Y))|X - Y|},
\]

Then it is easy to check that

\[
[u]_{\alpha; h} \leq [u]_{0; h}^{\alpha} [u]_{0; h}^{1-\alpha}, \quad |\nabla u|_{0; h} \leq 2[u]_{1; h}. \tag{5.8}
\]

**Lemma 5.3.** Let \(u \in C_h^{0, \alpha}(\mathbb{R}^d)\) with \(0 < h < C_0\) and \(\alpha \in (0, 1)\). Then there exists a constant \(C > 0\) depending only on \(d, \alpha, k\) such that for \(k \in \mathbb{N}\),

\[
|\nabla^k H(t)u|_{0; H(2t)h} \leq Ct^{-\frac{k}{2}} [u]_{0; h},
\]

\[
|\nabla^k H(t)u|_{1; H(2t)h} \leq Ct^{-\frac{k+1}{2}} [u]_{0; h},
\]

\[
|\nabla^k H(t)u|_{\alpha; H(2t)h} \leq Ct^{-\frac{k}{2}} [u]_{\alpha; h},
\]

\[
|\nabla^k H(t)u|_{1; H(2t)h} \leq Ct^{-\frac{k+1}{2} - \alpha} [u]_{\alpha; h}.
\]

**Proof.** Thanks to (5.7), we have

\[
|\nabla^k H(t)u(X)| = \int_{\mathbb{R}^d} |\nabla^k K(t, X - Y)u(Y)| dY
\]

\[
\leq \int_{\mathbb{R}^d} |\nabla^k K(t, X - Y)||u(Y)| dY
\]

\[
\leq Ct^{-\frac{k}{2}} \int_{\mathbb{R}^d} K(2t, X - Y)h(Y)|u|_{0; h} dY.
\]
which gives the first inequality.

If $|X - Y| < \sqrt{t}$, then we get by (5.7) that

$$|\nabla^k H(t)u(X) - \nabla^k H(t)u(Y)| = \left| \int_{\mathbb{R}^d} (\nabla^k K(t, X - X') - \nabla^k K(t, Y - X'))u(X')dX' \right|$$

$$\leq \int_{\mathbb{R}^d} |\nabla^k K(t, X - X') - \nabla^k K(t, Y - X')||u(X')|dX'$$

$$\leq Ct^{-\frac{k}{2}}|X - Y| \int_{\mathbb{R}^d} K(2t, X - X')h(X')dX' |u|_{0,h}$$

$$\leq Ct^{-\frac{k+\alpha}{2}}|X - Y||H(t)h(X)| |u|_{\alpha,h},$$

and if $|X - Y| \geq \sqrt{t}$, then

$$|\nabla^k H(t)u(X) - \nabla^k H(t)u(Y)| \leq |\nabla^k H(t)u(X)| + |\nabla^k H(t)u(Y)|$$

$$\leq Ct^{-\frac{k}{2}}H(2t)h(X)|u|_{0,h} + Ct^{-\frac{k}{2}}H(2t)h(Y)|u|_{0,h}$$

$$\leq Ct^{-\frac{k+\alpha}{2}}|X - Y||(H(2t)h(X) + H(2t)h(Y))|u|_{\alpha,h},$$

which imply the second inequality.

For any $X, Y \in \mathbb{R}^d$, we have

$$|\nabla^k H(t)u(X) - \nabla^k H(t)u(Y)| = \left| \int_{\mathbb{R}^d} \nabla^k K(t, X')u(X' - X)dX' - \int_{\mathbb{R}^d} \nabla^k K(t, X')u(Y' - X)dX' \right|$$

$$\leq \int_{\mathbb{R}^d} |\nabla^k K(t, X')||u(X' - X) - u(Y' - X')|dX'$$

$$\leq Ct^{-\frac{k}{2}} \int_{\mathbb{R}^d} K(2t, X')(h(X - X') + h(Y - X'))dX' |X - Y'|^\alpha |u|_{\alpha,h}$$

$$\leq Ct^{-\frac{k+\alpha}{2}}(H(2t)h(X) + H(2t)h(Y)) |X - Y'|^\alpha |u|_{\alpha,h},$$

which gives the third inequality.

For any $X, Y \in \mathbb{R}^d$, if $|X - Y| < \sqrt{t}$, we take $Y' \in B(X, \sqrt{t})$ so that

$$h(Y') \int_{B(X, \sqrt{t})} K(2t, X - X')dX' \leq \int_{B(X, \sqrt{t})} K(2t, X - X')h(X')dX' \leq H(2t)h(X),$$

which gives $h(Y') \leq CH(2t)h(X)$. Then we deduce for $|X - Y| < \sqrt{t}$,

$$|\nabla^k H(t)u(X) - \nabla^k H(t)u(Y)|$$

$$= \left| \int_{\mathbb{R}^d} (\nabla^k K(t, X - X') - \nabla^k K(t, Y - X'))(u(X') - u(Y'))dX' \right|$$

$$\leq \int_{\mathbb{R}^d} |\nabla^k K(t, X - X') - \nabla^k K(t, Y - X')||u(X') - u(Y')|dX'$$

$$\leq \int_{\mathbb{R}^d} |\nabla^k K(t, X - X') - \nabla^k K(t, Y - X')||X' - Y'||\alpha (h(X') + h(Y'))dX'[u]_{\alpha,h}$$

$$\leq Ct^{-\frac{k+\alpha}{2}}|X - Y| \int_{\mathbb{R}^d} K(2t, X - X')(h(X') + h(Y'))dX'[u]_{\alpha,h}$$

$$\leq Ct^{-\frac{k+\alpha}{2}}|X - Y||H(2t)h(X) + h(Y'))|u|_{\alpha,h}$$
\[ \leq Ct^{-\frac{k+1+\alpha}{2}}|X - Y|H(2t)h(X)[u]_{\alpha,h}. \]

While, if \(|X - Y| \geq \sqrt{t}\), then
\[ |\nabla^k H(t)u(X) - \nabla^k H(t)u(Y)| \leq Ct^{-\frac{k}{2}}(H(2t)h(X) + H(2t)h(Y))|X - Y|^{\alpha}[u]_{\alpha,h} \]
\[ \leq Ct^{-\frac{k+1+\alpha}{2}}(H(2t)h(X) + H(2t)h(Y))|X - Y|[u]_{\alpha,h}. \]

This proves the fourth inequality. \(\square\)

**Lemma 5.4.** Let \(\gamma > 0, k \geq 0\), and \(u \in C^{0,\alpha}([0,T] \times \mathbb{R}^d, H, u)\) with \(0 < h < C_0\). Let \(R \geq \sqrt{t} > 0\). Then there exists a constant \(C > 0\) depending only on \(d, \alpha\) so that
\[ |H(t)u|_{1,\alpha;H(2t)h,\sqrt{k+t}} \leq C|u|_{1,\alpha;h,\sqrt{k}}, \]
\[ |H(t)u|_{1,\alpha;H(2t)h,R} \leq C\varphi_\alpha(R)/\varphi_\alpha(\sqrt{t})|u|_{0,\alpha,h}, \]
where \(\varphi_\alpha(R) = \max(R, R^{1+\alpha})\).

**Proof.** By Lemma 5.3 and (5.8), we have
\[ |H(t)u|_{0,\alpha;H(2t)h} \leq C|u|_{0,h}, \quad |H(t)u|_{1,\alpha;H(2t)h} \leq C|u|_{1,\alpha,h}, \quad |H(t)u|_{0,\alpha;H(2t)h} \leq C|u|_{0,\alpha,h}, \]
\[ |\nabla H(t)u|_{0,\alpha;H(2t)h} \leq \min(Ct^{-\frac{k}{2}}|u|_{0,h}, Ct^{-\frac{k}{2}}|u|_{1,\alpha,h}) \leq C\min(t^{-\frac{k}{2}}, t^{-\frac{k+\alpha}{2}})|u|_{0,\alpha,h}, \]
\[ |\nabla H(t)u|_{1,\alpha;H(2t)h} \leq \min(Ct^{-\frac{k+\alpha}{2}}|u|_{0,h}, Ct^{-\frac{k}{2}}|u|_{1,\alpha,h}) \leq C\min(t^{-\frac{k+\alpha}{2}}, t^{-\frac{k}{2}})|u|_{0,\alpha,h}. \]

Due to \(\nabla H(t)u = H(t)\nabla u\), we have
\[ |\nabla H(t)u|_{0,\alpha;H(2t)h} \leq C|\nabla u|_{0,h}, \quad |\nabla H(t)u|_{1,\alpha;H(2t)h} \leq C|\nabla u|_{1,\alpha,h}. \]

Therefore,
\[ |H(t)u|_{1,\alpha;H(2t)h,\sqrt{k+t}} = |H(t)u|_{1,\alpha;H(2t)h} + \max(2t^{-\frac{k}{2}}, (k + t)^{\frac{k}{2}})|\nabla H(t)u|_{1,\alpha;H(2t)h} \]
\[ \leq C|u|_{0,\alpha,h} + \max(k^{\frac{k}{2}}, (k + t)^{\frac{k}{2}})|\nabla H(t)u|_{0,\alpha;H(2t)h} + \max(t^{\frac{k}{2}}, t^{\frac{k+\alpha}{2}})|\nabla H(t)u|_{0,\alpha;H(2t)h} \]
\[ \leq C|u|_{0,\alpha,h} + \max(k^{\frac{k}{2}}, (k + t)^{\frac{k}{2}})|\nabla u|_{0,h} + C|u|_{0,\alpha,h} \]
\[ + C \max(k^{\frac{k}{2}}, (k + t)^{\frac{k}{2}})|\nabla u|_{0,h} + C|u|_{0,\alpha,h} \leq C|u|_{1,\alpha,h,\sqrt{k}}, \]
which gives the first inequality. Also,
\[ |H(t)u|_{1,\alpha;H(2t)h,R} = |H(t)u|_{0,\alpha;H(2t)h} + \max(R^{1+\alpha}, R)(|\nabla H(t)u|_{0,\alpha;H(2t)h} + R^\alpha|\nabla H(t)u|_{1,\alpha;H(2t)h}) \]
\[ \leq C|u|_{0,\alpha,h} + \max(R, R^{1+\alpha})(t^{-\frac{k}{2}})|\nabla H(t)u|_{0,\alpha;H(2t)h} + |\nabla H(t)u|_{1,\alpha;H(2t)h} \]
\[ \leq C|u|_{0,\alpha,h} + C\varphi_\alpha(R)\min(t^{-\frac{k+\alpha}{2}}, t^{-\frac{k}{2}})|u|_{0,\alpha,h} \]
\[ \leq C\varphi_\alpha(R)/\varphi_\alpha(\sqrt{t})|u|_{0,\alpha,h}, \]
which gives the second inequality. \(\square\)
5.3. Riesz transform in the weighted Hölder spaces. Throughout this subsection, we take \( f, f_1, f_\pm \) be as in section 4.4. We need the following property for the weight functions.

**Lemma 5.5.** For \( h = 1, f_1(t), f(t), f_\pm(t) \), we have

\[
R^{-d} \int_{B(X,R)} h(Y) f_1(t,Y) dY \leq Ch(X) \min(R^{-\delta}, (1 + \mu_1 t)^{-\frac{1}{2}}).
\]

**Proof.** The case of \( h = 1 \) follows from (4.23). We denote

\[
\rho_1(X) = |x_2|, \quad \rho_2(X) = |(x_1, x_2)| \quad \text{for} \quad X = (x_1, \ldots, x_d) \in \mathbb{R}^d.
\]

Then by (4.21), for \( h = f_1(t) \) if \( \rho_1(X) \geq 2R \) or \( \rho_1(X) \leq 2\sqrt{1 + \mu_1 t} \), we have

\[
h(Y) \leq Ch(X) \quad \text{for} \quad |Y - X| \leq R,
\]

which gives,

\[
R^{-d} \int_{B(X,R)} h(Y) f_1(t,Y) dY \leq CR^{-d} \int_{B(X,R)} h(X) f_1(t,Y) dY
\]

\[
\leq Ch(X) \min(R^{-\delta}, (1 + \mu_1 t)^{-\frac{1}{2}}).
\]

Using (4.20), the above inequality holds for \( h = f(t) \) if \( \rho_2(X) \geq 2R \) or \( \rho_2(X) \leq 2\sqrt{1 + \mu_1 t} \). For the case of \( h = f_1(t) \), if \( 2\sqrt{1 + \mu_1 t} \leq \rho_1(X) \leq 2R \), then by (4.21),

\[
h(X) \geq C^{-1}\phi_1(X) \geq C^{-1}R^{-\delta},
\]

\[
h(Y) f_1(t,Y) \leq C\phi_1(Y)^2 = C\rho_1(Y)^{-2\delta},
\]

which imply

\[
R^{-d} \int_{B(X,R)} h(Y) f_1(t,Y) dY \leq CR^{-d} \int_{B(X,R)} \rho_1(Y)^{-2\delta} dY \leq CR^{-2\delta} \leq Ch(X)R^{-\delta}.
\]

For the case of \( h = f(t) \), if \( 2\sqrt{1 + \mu_1 t} \leq \rho_2(X) \leq 2R \), then by (4.20),

\[
h(X) \geq C^{-1}\phi_2(X) \geq C^{-1}R^{-1-\delta},
\]

\[
h(Y) f_1(t,Y) \leq C\phi_1(Y)\phi_2(Y) = C|y_1|^{-\frac{1}{2}-\delta}|y_2|^{-\frac{1}{2}-\delta},
\]

which imply

\[
R^{-d} \int_{B(X,R)} h(Y) f_1(t,Y) dY \leq CR^{-1-2\delta} \leq Ch(X)R^{-\delta}.
\]

Therefore, (5.9) is true for \( h = f_1(t), f(t) \). The case of \( h = f_\pm(t) \) follows from the case of \( h = f(t) \) by translation. \( \square \)

**Proposition 5.6.** It holds that

\[
|\langle u, R_i R_j \partial_k w\rangle|_{1, \alpha; f_\pm(t), (\mu_1 t)^{1/2}} \leq C|u|_{1, \alpha; f_\pm(t), (1+\mu_1 t)^{1/2}}|w|_{1, \alpha; f_1(t), (\mu_1 t)^{1/2}},
\]

\[
|R_i R_j(uw)|_{0, \alpha; f_1(t)} \leq C(1 + \mu_1 t)^{-\frac{1}{2}}(1 + (\mu_1 t)^{-\frac{1}{2}})|u|_{0, \alpha, f_1(t), (\mu_1 t)^{1/2}}|w|_{0, \alpha, f_1(t), (\mu_1 t)^{1/2}}.
\]

The proof of the proposition is very complicated. Let us begin with some reductions. For fixed \( i, j \), we have

\[
R_i R_j w(X) + \delta_{ij} \frac{d}{d} w(X) = -p.v. \int_{\mathbb{R}^d} \partial_i \partial_j N(X - Y) w(Y) dY \triangleq \sum_{n=-\infty}^{\infty} R_{ij}^n(w),
\]
where
\[ R_{i,j}^n(u) = - \int_{\mathbb{R}^d} \varphi_n(X - Y) u(Y) dY \]
with \( \varphi_n(X) = \partial_i \partial_j N(X)(\theta(2^n|X|) - \theta(2^{n+1}|X|)) \). Therefore,
\begin{equation}
[u, R_i R_j] \partial_k w = \sum_{n=-\infty}^{\infty} [u, R_{i,j}^n] \partial_k w.
\end{equation}

**Lemma 5.7.** For \( h = 1, f_1(t), f(t), f_x(t) \), it holds that
\[
|R_i R_j(u)|_{0, \alpha; h, (1+\mu_1)1/2} \leq C(1+\mu_1)^{-\frac{\delta}{2}} |u|_{0, \alpha; h f_1(t), (1+\mu_1)1/2}.
\]

**Proof.** Notice that
\[
\int_{\mathbb{R}^d} \varphi_n(X) dX = 0, \quad \text{supp} \varphi_n \subset B(0, 2^{1-n}) \backslash B(0, 2^{-1-n}), \quad |\nabla^l \varphi_n| \leq C 2^{n(d+l)}, \quad l = 0, 1, 2,
\]
we deduce from Lemma 5.5 that
\[
|R_{i,j}^n(u)(X)| \leq \int_{\mathbb{R}^d} |\varphi_n(X - Y)| h(Y) f_1(t, Y) dY |u|_{0, h f_1(t)}
\leq C 2^{n(d)} \int_{B(X, 2^{1-n})} h(Y) f_1(t, Y) dY |u|_{0, h f_1(t)}
\leq C 2^{n(d)} h(X) |u|_{0, h f_1(t)}.
\]
For \( X \in \mathbb{R}^d \), we have
\[
R_{i,j}^n(u)(X) = - \int_{\mathbb{R}^d} \varphi_n(X - Y)(u(Y) - u(X)) dY,
\]
which along with (4.24) gives
\[
|R_{i,j}^n(u)(X)| \leq \int_{\mathbb{R}^d} |\varphi_n(X - Y)| (h(X) + h(Y)) |X - Y| dY |u|_{0, h}
\leq C 2^{n(d-a)} \int_{B(X, 2^{1-n})} (h(X) + h(Y)) dY |u|_{0, h}
\leq C 2^{-na} h(X) |u|_{0, h}.
\]
By (4.21), we have
\[
[u]_{\alpha; h} \leq C(1+\mu_1)^{-\frac{\delta}{2}} |u|_{0, \alpha; h f_1(t)} \leq C(1+\mu_1)^{-\frac{\alpha+\delta}{2}} |u|_{0, \alpha; h f_1(t), (1+\mu_1)1/2}.
\]
Thus, we can conclude
\[
|R_i R_j(u)(X)| \leq \sum_{n=-\infty}^{\infty} |R_{i,j}^n(u)(X)|
\leq \sum_{n=-\infty}^{\infty} C \min(2^{n\delta}, 2^{-na}(1+\mu_1)^{-\frac{\alpha+\delta}{2}}) h(X) |u|_{0, \alpha; h f_1(t), (1+\mu_1)1/2}
\leq C(1+\mu_1)^{-\frac{\delta}{2}} h(X) |u|_{0, \alpha; h f_1(t), (1+\mu_1)1/2}.
\]
For any \( X, X' \in \mathbb{R}^d, \ |X - X'| \leq 2^{-n}, \)
\[
|R_{i,j}^n(u)(X) - R_{i,j}^n(u)(X')| \leq \int_{\mathbb{R}^d} |\varphi_n(X - Y) - \varphi_n(X' - Y)| (h(X) + h(Y)) |X - Y| dY |u|_{0, \alpha; h}.
\]
\[\leq C2^{n(d+1-\alpha)}|X - X'| \int_{B(X,2^{-n})} (h(X) + h(Y))dY[u]_{\alpha,h}\]
\[\leq C2^{n(1-\alpha)}|X - X'|h(X)[u]_{\alpha,h},\]
which gives for any \(X, X' \in \mathbb{R}^d\),
\[|R^n_{ij}(u)(X) - R^n_{ij}(u)(X')| \leq C2^{-n\alpha}\min(1,2^n|X - X'|)(h(X) + h(X'))[u]_{\alpha,h}.\]

Then we have
\[|R_iR_j(u)(X) - R_iR_j(u)(X')| \leq \sum_{n=-\infty}^{\infty} |R^n_{ij}(u)(X) - R^n_{ij}(u)(X')|\]
\[\leq \sum_{n=-\infty}^{\infty} C2^{-n\alpha}\min(1,2^n|X - X'|)(h(X) + h(X'))[u]_{\alpha,h}\]
\[\leq C|X - X'|^\alpha(h(X) + h(X'))[u]_{\alpha,h},\]
which implies \([R_iR_j]_{\alpha,h} \leq C[u]_{\alpha,h}.\] Thus,
\[|R_iR_j(u)|_{0,\alpha,h,(1+\mu_1)^{1/2}} = |R_iR_j(u)|_{0,h} + (1 + \mu_1t)^{\alpha/2}|R_iR_j(u)|_{\alpha,h}\]
\[\leq C(1 + \mu_1t)^{-\frac{\alpha}{2}}|u|_{0,\alpha,h,f_1(t),(1+\mu_1)^{1/2}} + (1 + \mu_1t)^{\alpha/2}[u]_{\alpha,h}\]
\[\leq C(1 + \mu_1t)^{-\frac{\alpha}{2}}|u|_{0,\alpha,h,f_1(t),(1+\mu_1)^{1/2}},\]
which gives our result.

\[\square\]

Lemma 5.8. For \(l = 0, 1\), it holds that
\[|\nabla^l[u, R^n_{ij}]\partial_kw(X)| \leq C2^{n(l-\alpha)}|\nabla u|_{0;B(X,2^{-n})}[w]_{\alpha}.\]

Proof. Thanks to
\[\nabla^l[u, R^n_{ij}]\partial_kw(X) = \int_{\mathbb{R}^d} \varphi_n(X - Y)(u(Y) - u(X))\partial_kw(Y)dY\]
(5.11)
\[= \int_{\mathbb{R}^d} \partial_k\varphi_n(X - Y)(u(Y) - u(X))w(Y)dY - \int_{\mathbb{R}^d} \varphi_n(X - Y)\partial_kw(Y)w(Y)dY,\]
we deduce that
\[\left|\left[u, R^n_{ij}\right]\partial_kw(X)\right| \leq \int_{\mathbb{R}^d} |\partial_k\varphi_n(X - Y)||X - Y|dY|\nabla u|_{0;B(X,2^{-n})}[w|_{0;B(X,2^{-n})}\]
\[+ \int_{\mathbb{R}^d} \varphi_n(X - Y)|dY|\nabla u|_{0;B(X,2^{-n})}[w|_{0;B(X,2^{-n})}\]
\[\leq C|\nabla u|_{0;B(X,2^{-n})}[w|_{0;B(X,2^{-n})},\]

Thanks to
\[\nabla[u, R^n_{ij}]\partial_kw(X) = \int_{\mathbb{R}^d} \nabla\partial_k\varphi_n(X - Y)(u(Y) - u(X))w(Y)dY\]
(5.12)
\[-\nabla u(X) \int_{\mathbb{R}^d} \partial_k\varphi_n(X - Y)w(Y)dY - \int_{\mathbb{R}^d} \nabla\varphi_n(X - Y)\partial_kw(Y)dY,\]
we can similarly deduce that
\[\left|\nabla[u, R^n_{ij}]\partial_kw(X)\right| \leq C2^n|\nabla u|_{0;B(X,2^{-n})}[w|_{0;B(X,2^{-n})},\]
As \( [u, R^n_{ij}] \partial_k w = [u, R^n_{ij}] \partial_k (w - w(X)) \), we have for \( l = 0, 1 \),
\[
|\nabla^l [u, R^n_{ij}] \partial_k w(X) | \leq C2^{n(l-\alpha)} |\nabla u|_{0, B(X, 2^{1-n})} |w - w(X)|_{0, B(X, 2^{1-n})} \\
\leq C2^{n(l-\alpha)} |\nabla u|_{0, B(X, 2^{1-n})} |w|_\alpha.
\]
This completes the proof. \(\Box\)

**Lemma 5.9.** If \( |u|_{1, \alpha, h, (1+\mu t)^{1/2}} = |w|_{1, \alpha, f_1(t), (\mu t)^{1/2}} = 1 \) for \( h = 1, f_1(t), f(t), f_\pm(t) \), then we have
\[
|u, R^n_{ij}] \partial_k w(X) | \leq Ch(X) \min(2^{n \delta}, (1 + \mu t)^{-\frac{\delta}{2}}, 2^{-\alpha(1 + \mu t)^{-\frac{\delta}{2}}}),
\]
\[
|\partial_1[u, R^n_{ij}] \partial_k w(X) | \leq Ch(X) (1 + \mu t)^{-\frac{1+\delta+\alpha}{2}} \min(2^{n(1-\alpha)}, 2^{-\alpha(\mu t)^{-\frac{1}{2}}}.
\]

**Proof.** As \( |u|_{1, \alpha, h, (1+\mu t)^{1/2}} = |w|_{1, \alpha, f_1(t), (\mu t)^{1/2}} = 1 \), we have
\[
|u(X)| \leq h(X), \ |\nabla u(X) | \leq h(X)(1 + \mu t)^{-\frac{\delta}{2}}, \ |w(X)| \leq f_1(t, X).
\]
Using \( f_1(t, X) \leq C(1 + \mu t)^{-\frac{\delta}{2}} \), we also have
\[
|w|_0 \leq C(1 + \mu t)^{-\frac{\delta}{2}}, \ \ |w|_\alpha \leq C(1 + \mu t)^{-\frac{\delta}{2}},
\]
\[
|\nabla w|_0 \leq C(1 + \mu t)^{-\frac{\delta}{2}} (\mu t)^{-\frac{1}{2}},
\]
\[
|\nabla w|_\alpha \leq C(1 + \mu t)^{-\frac{\delta}{2}} \min((\mu t)^{-\frac{\delta}{2}}, (\mu t)^{-\frac{1+\delta}{2}}) \leq C(1 + \mu t)^{-\frac{\delta+\alpha}{2}} (\mu t)^{-\frac{1}{2}},
\]
and
\[
|w|_\alpha \leq C|w|^{1-\alpha}_0 |\nabla w|^{\alpha}_0 \leq C(1 + \mu t)^{-\frac{\delta}{2}} (\mu t)^{-\frac{\alpha}{2}}.
\]
Therefore
\[
|w|_\alpha \leq C(1 + \mu t)^{-\frac{\delta}{2}} \min(1, (\mu t)^{-\frac{\delta}{2}}) \leq C(1 + \mu t)^{-\frac{\delta+\alpha}{2}}.
\]
Then we deduce from (5.11) and Lemma 5.5 that for \( 2^{-n} \geq \sqrt{1 + \mu t} \),
\[
[u, R^n_{ij}] \partial_k w(X) \leq \int_{\mathbb{R}^d} |\partial_1 \varphi_n(X - Y)| (h(X) + h(Y)) f_1(t, Y) dY
\]
\[
+ (1 + \mu t)^{-\frac{\delta}{2}} \int_{\mathbb{R}^d} |\varphi_n(X - Y)| h(Y) \alpha_1(t, Y) dY
\]
\[
\leq C2^{n(d+1)} \int_{B(X, 2^{-n})} (h(X) + h(Y)) f_1(t, Y) dY
\]
\[
+ C(1 + \mu t)^{-\frac{\delta}{2}} 2^{nd} \int_{B(X, 2^{-n})} h(Y) \alpha_1(t, Y) dY
\]
\[
\leq C2^{n(1+\delta)} h(X) + C(1 + \mu t)^{-\frac{\delta}{2}} 2^{nd} h(X)
\]
\[
\leq C(1 + \mu t)^{-\frac{\delta}{2}} 2^{nd} h(X).
\]
For \( 2^{-n} \leq \sqrt{1 + \mu t} \), we have
\[
|\nabla u|_{0, B(X, 2^{-n})} \leq |\nabla u|_{0, h, B(X, 2^{-n})} |h|_{0, B(X, 2^{-n})} \leq Ch(X)(1 + \mu t)^{-\frac{\delta}{2}},
\]
where we used the fact that \( h \) satisfies (2.3) with \( R = \sqrt{1 + \mu t} \). Similarly, we have
\[
[\nabla u|_{0, B(X, 2^{-n})} \leq [\nabla u|_{0, h, B(X, 2^{-n})} |h|_{0, B(X, 2^{-n})} \leq Ch(X)(1 + \mu t)^{-\frac{1+\alpha}{2}}.
\]
Then we get by Lemma 5.5 that
\[
[u, R^n_{ij}] \partial_k w(X) \leq C2^{n\alpha} h(X) |\nabla u|_{0, B(X, 2^{-n})} |w|_\alpha
\]
\[
\leq C2^{n\alpha} h(X)(1 + \mu t)^{-\frac{1+\delta+\alpha}{2}}.
\]
which gives the first inequality of the lemma.

Similarly, by (5.12), Lemma 5.5 and Lemma 5.8 we can deduce
\[ |\partial_t[u, R^u_{ij}] \partial_k w(X)| \leq C h(X) 2^n \min(2^{n \delta} (1 + \mu t)^{-\frac{1}{2}}, 2^{-n \alpha} (1 + \mu t)^{-\frac{1+4+\alpha}{2}}) \]
\[ \leq C h(X) 2^{n(1-\alpha)} (1 + \mu t)^{-\frac{1+4+\alpha}{2}}. \]

On the other hand,
\[ \partial_t[u, R^u_{ij}] \partial_k w(X) = \int_{\mathbb{R}^d} \partial_t \varphi_n(X - Y)(u(Y) - u(X)) \partial_k w(Y) dY \]
\[ - \partial_t u(X) \int_{\mathbb{R}^d} \varphi_n(X - Y) \partial_k w(Y) dY \]
\[ \triangleq [u, \partial_t R^u_{ij}] \partial_k w(X) + \partial_t u(X) R^u_{ij} \partial_k w(X). \]

From the proof of Lemma 5.7 we can see that
\[ |R^u_{ij} \partial_k w(X)| \leq C 2^{-n \alpha} |\partial_k w| \alpha \leq C 2^{-n \alpha} (1 + \mu t)^{-\frac{1+4+\alpha}{2}} (\mu t)^{-\frac{1}{2}}. \]

By (4.24), we deduce that for \( 2^{-n} \geq \sqrt{1 + \mu t} \),
\[ ||[u, \partial_t R^u_{ij}] \partial_k w(X)|| \leq \int_{\mathbb{R}^d} |\partial_t \varphi_n(X - Y)||(u(Y)| + |u(X)|)| \partial_k w(Y)| dY \]
\[ \leq C 2^{n(d+1)} (1 + \mu t)^{-\frac{1}{2} + \frac{1}{2}} \int_{B(X, 2^{1-n})} (h(Y) + h(X)) dY \]
\[ \leq C 2^n (1 + \mu t)^{-\frac{1}{2}} h(X) \]
\[ \leq C 2^{-na} (1 + \mu t)^{-\frac{1+4+\alpha}{2}} (\mu t)^{-\frac{1}{2}} h(X). \]

For \( 2^{-n} \leq \sqrt{1 + \mu t} \), using the formula
\[ [u, \partial_t R^u_{ij}] \partial_k w(X) = \int_{\mathbb{R}^d} \partial_t \varphi_n(X - Y)(u(Y) - u(X)) (\partial_k w(Y) - \partial_k w(X)) dY \]
\[ + \partial_k w(X) \int_{\mathbb{R}^d} \varphi_n(X - Y)(\partial_t u(Y) - \partial_t u(X)) dY, \]
we deduce that
\[ ||[u, \partial_t R^u_{ij}] \partial_k w(X)|| \leq \int_{\mathbb{R}^d} |\partial_t \varphi_n(X - Y)||X - Y|^{1+\alpha} dY |\nabla u|_{0; B(X, 2^{1-n})}| \partial_k w| \alpha \]
\[ + |\partial_k w|| \partial_t \varphi_n(X - Y)||X - Y|^{\alpha} dY |\nabla u|_{\alpha; B(X, 2^{1-n})} \]
\[ \leq C 2^{-n \alpha} h(X)(1 + \mu t)^{-\frac{1}{2}} (1 + \mu t)^{-\frac{1+4+\alpha}{2}} \mu t)^{-\frac{1}{2}} \]
\[ + C (1 + \mu t)^{-\frac{1}{2}} (\mu t)^{-\frac{1+4+\alpha}{2}} 2^{-n \alpha} h(X)(1 + \mu t)^{-\frac{1+4+\alpha}{2}} \mu t)^{-\frac{1}{2}} \]
\[ \leq C 2^{-n \alpha} h(X)(1 + \mu t)^{-\frac{1+4+\alpha}{2}} (\mu t)^{-\frac{1}{2}}. \]

This shows that
\[ |\partial_t[u, R^u_{ij}] \partial_k w(X)| \leq [u, \partial_t R^u_{ij}] \partial_k w(X) + |\partial_t u(X) R^u_{ij} \partial_k w(X)| \]
\[ \leq C 2^{-n \alpha} h(X)(1 + \mu t)^{-\frac{1+4+\alpha}{2}} (\mu t)^{-\frac{1}{2}} \]
\[ + C h(X)(1 + \mu t)^{-\frac{1}{2}} 2^{-n \alpha} (1 + \mu t)^{-\frac{1+4+\alpha}{2}} (\mu t)^{-\frac{1}{2}} \]
\[ \leq C 2^{-n \alpha} h(X)(1 + \mu t)^{-\frac{1+4+\alpha}{2}} (\mu t)^{-\frac{1}{2}}, \]
which gives the second inequality of the lemma.

Using the formula
\[
\partial_m[u, \partial_t R^n_{ij}]\partial_k w(X) = \int_{\mathbb{R}^d} \partial_m \partial_t \varphi_n(X-Y)(u(Y) - u(X))\partial_k w(Y) dY \\
- \partial_m u(X) \int_{\mathbb{R}^d} \partial_t \varphi_n(X-Y)\partial_k w(Y) dY,
\]
we can also deduce that
\[
(5.13) \quad |\partial_m[u, \partial_t R^n_{ij}]\partial_k w(X)| \leq C2^{n(1-\alpha)}h(X)(1 + \mu t)^{-\frac{1+\delta+\alpha}{2}} (\mu t)^{-\frac{1}{2}}.
\]

Now we are in position to prove Proposition 5.6.

**Proof.** We get by Lemma 5.7 with \( h = f_1(t) \) that
\[
|R_i R_j (uw)| \leq C(1 + \mu t)^{-\frac{\delta}{2}}|uw|_{0, \alpha; f_1(t)^2, (1 + \mu t)^{1/2}} \\
\leq C(1 + \mu t)^{-\frac{\delta}{2}}(1 + (\mu t)^{-\frac{\delta}{2}})|uw|_{0, \alpha; f_1(t)^2, (\mu t)^{1/2}} \\
\leq C(1 + \mu t)^{-\frac{\delta}{2}}(1 + (\mu t)^{-\frac{\delta}{2}})|w|_{0, \alpha; f_1(t), (\mu t)^{1/2}},
\]
which gives the second inequality of the proposition.

For the first inequality, without lose of generality, we can assume
\[
|u|_{1, \alpha; h, (1 + \mu t)^{1/2}} = |w|_{1, \alpha; f_1(t), (\mu t)^{1/2}} = 1,
\]
here \( h = f_{\pm}(t) \).

First of all, by Lemma 5.9 we have
\[
[[u, R_i R_j]\partial_k w(X)] \leq \sum_{n=-\infty}^{\infty} |[u, R^n_{ij}]\partial_k w(X)| \\
\leq C \sum_{n=-\infty}^{\infty} h(X) \min(2^n\delta(1 + \mu t)^{-\frac{1}{2}}, 2^{-\alpha}(1 + \mu t)^{-\frac{1+\delta+\alpha}{2}}) \\
\leq C h(X)(1 + \mu t)^{-\frac{1+\delta}{4}},
\]
and
\[
|\partial_t [u, R_i R_j]\partial_k w(X)| \leq \sum_{n=-\infty}^{\infty} |\partial_t [u, R^n_{ij}]\partial_k w(X)| \\
\leq C \sum_{n=-\infty}^{\infty} h(X)(1 + \mu t)^{-\frac{1+\delta+\alpha}{2}} \min(2^n(1-\alpha), 2^{-\alpha}(1 + \mu t)^{-\frac{1}{2}}) \\
\leq C h(X)(1 + \mu t)^{-\frac{1+\delta+\alpha}{2}} (\mu t)^{-\frac{1-\alpha}{2}}.
\]

Now we consider \( X, Y \in \mathbb{R}^d, |X-Y| \leq \sqrt{1+\mu t} \). It follows from Lemma 5.9 that
\[
|[u, R^n_{ij}]\partial_k w(X) - [u, R^n_{ij}]\partial_k w(Y)| \leq C h(X) 2^{-\alpha}(1 + \mu t)^{-\frac{1+\delta+\alpha}{2}} \min(1, 2^n|X-Y|),
\]
here we used the fact that \( h \) satisfies (2.3) with \( R = \sqrt{1+\mu t} \). Therefore,
\[
|[u, R_i R_j]\partial_k w(X) - [u, R_i R_j]\partial_k w(Y)| \leq \sum_{n=-\infty}^{\infty} |[u, R^n_{ij}]\partial_k w(X) - [u, R^n_{ij}]\partial_k w(Y)|
\]
\[
\leq C \sum_{n=-\infty}^{\infty} h(X)2^{-n\alpha}(1 + \mu t)^{-\frac{1+\delta+\alpha}{2}} \min(1, 2^n|X - Y|) \\
\leq C h(X)(1 + \mu t)^{-\frac{1+\delta+\alpha}{2}}|X - Y|^\alpha.
\]

We write

\[
\partial_t[u, R_iR_j]\partial_k w = [u, \partial_t R_iR_j]\partial_k w + \partial_t u\ R_iR_j\partial_k w,
\]

where

\[
[u, \partial_t R_iR_j]\partial_k w = \sum_{n=-\infty}^{\infty} [u, \partial_t R_{ij}]^n\partial_k w.
\]

We get by Lemma 5.9 and (5.13) that

\[
||[u, \partial_t R_{ij}]^n\partial_k w(X) - [u, \partial_t R_{ij}]^n\partial_k w(Y)|| \leq C h(X)2^{-n\alpha}(1 + \mu t)^{-\frac{1+\delta+\alpha}{2}}(\mu t)^{-\frac{1}{2}} \min(1, 2^n|X - Y|),
\]

which gives

\[
||[u, \partial_t R_iR_j]\partial_k w(X) - [u, \partial_t R_iR_j]\partial_k w(Y)|| \leq \sum_{n=-\infty}^{\infty} ||[u, \partial_t R_{ij}]^n\partial_k w(X) - [u, \partial_t R_{ij}]^n\partial_k w(Y)|| \\
\leq C \sum_{n=-\infty}^{\infty} h(X)2^{-n\alpha}(1 + \mu t)^{-\frac{1+\delta+\alpha}{2}}(\mu t)^{-\frac{1}{2}} \min(1, 2^n|X - Y|) \\
\leq C h(X)(1 + \mu t)^{-\frac{1+\delta+\alpha}{2}}(\mu t)^{-\frac{1}{2}}|X - Y|^\alpha,
\]

Thanks to

\[
|\partial_k w|_{0, \alpha;f_{t1},(1+\mu t)^{1/2}} \leq (1 + (\mu t)^{-\frac{1}{2}})|\partial_k w|_{0, \alpha;f_{t1},(1+\mu t)^{1/2}} \\
\leq (1 + (\mu t)^{-\frac{1}{2}}) \min((\mu t)^{-\frac{1-\alpha}{2}}, (\mu t)^{-\frac{1}{2}}) \leq 2(\mu t)^{-\frac{1}{2}},
\]

we infer from Lemma 5.7 that

\[
|\partial_t u\ R_iR_j\partial_k w|_{0, \alpha;h,(1+\mu t)^{1/2}} \leq C|\partial_t u|_{0, \alpha;h,(1+\mu t)^{1/2}}|R_iR_j\partial_k w|_{0, \alpha;1,(1+\mu t)^{1/2}} \\
\leq C(1 + \mu t)^{-\frac{1}{2}}(1 + \mu t)^{-\frac{3}{2}}|\partial_k w|_{0, \alpha;h,(1+\mu t)^{1/2}} \\
\leq C(1 + \mu t)^{-\frac{1+\delta+\alpha}{2}}(\mu t)^{-\frac{1}{2}},
\]

and

\[
|\partial_t u\ R_iR_j\partial_k w|_{\alpha;h} \leq C(1 + \mu t)^{-\frac{1+\delta+\alpha}{2}}(\mu t)^{-\frac{1}{2}}.
\]

This shows that

\[
|\partial_t[u, R_iR_j]\partial_k w(X) - \partial_t[u, R_iR_j]\partial_k w(Y)|| \leq C h(X)(1 + \mu t)^{-\frac{1+\delta+\alpha}{2}}(\mu t)^{-\frac{1}{2}}|X - Y|^\alpha.
\]

For the case of $X, Y \in \mathbb{R}^d$, $|X - Y| \geq \sqrt{1+\mu t}$, we have

\[
||[u, R_iR_j]\partial_k w(X) - [u, R_iR_j]\partial_k w(Y)|| \leq C(h(X) + h(Y))(1 + \mu t)^{-\frac{1+\delta}{2}}|X - Y|^\alpha \\
\leq C(h(X) + h(Y))(1 + \mu t)^{-\frac{1+\delta+\alpha}{2}}|X - Y|^\alpha,
\]

and

\[
|\partial_t[u, R_iR_j]\partial_k w(X) - \partial_t[u, R_iR_j]\partial_k w(Y)|| \leq C(h(X) + h(Y))(1 + \mu t)^{-\frac{1+\delta+\alpha}{2}}(\mu t)^{-\frac{1-\alpha}{2}} \\
\leq C(h(X) + h(Y))(1 + \mu t)^{-\frac{1+\delta+\alpha}{2}}(\mu t)^{-\frac{1}{2}}|X - Y|^\alpha.
\]
In summary, we conclude
\[
||u, R_t R_j|\partial_k w|_{1, \alpha; (\mu t)}^{1/2} = ||u, R_t R_j|\partial_k w|_{0; h} + ||u, R_t R_j|\partial_k w|_{\alpha; h}
+ \max((\mu t)^{-\frac{1}{2}}, (\mu t)^\frac{1}{2})\left(||u, R_t R_j|\partial_k w|_{0; h} + (\mu t)^\frac{1}{2}||u, R_t R_j|\partial_k w|_{\alpha; h}\right)
\leq C(1 + \mu t)^{-\frac{1}{2}} + C(1 + \mu t)^{-\frac{1}{2}} + C \max((\mu t)^{-\frac{1}{2}}, (\mu t)^\frac{1}{2})\times
\left((1 + \mu t)^{-\frac{1}{2}}(\mu t)^{-\frac{1}{2}} + (\mu t)^\frac{1}{2}(1 + \mu t)^{-\frac{1}{2}}(\mu t)^\frac{1}{2}\right) \leq C,
\]
which gives the first inequality of the proposition. □

5.4. **Weighted Schauder estimate.** Let \(h(X)\) be a positive bounded weight satisfying
\[h(X) \leq C_0 h(Y)\] for \(|X - Y| \leq 2R, \quad R > 0\).

**Lemma 5.10.** Let \(u \in C_h^2(R^d)\). Then we have
\[
|\nabla^2 u|_{0, \alpha; h, R} \leq C \left(|\nabla u|_{0, \alpha; h} \min(R^{-1+\alpha}, R^{-1}) + |\triangle u|_{0, \alpha; h, R}\right).
\]
Here \(C\) is a constant depending only on \(C_0\).

**Proof.** Fix \(X \in R^d\) and consider the function \(w(Y) = u(Y) - u(X) - (Y - X) \cdot \nabla u(X)\). So,
\[
\nabla^2 w = \nabla^2 u, \quad \triangle w = \triangle u, \quad |\triangle u|_{0, \alpha; B(2R), R} \leq 2C_0 h(X)|\triangle u|_{0, \alpha; h, R},
\]
where
\[
|u|_{0, \alpha; B(2R), R} \triangleq |u|_{0; B(2R)} + R^\alpha |u|_{\alpha; B(2R)}.
\]
As \(\nabla w(Y) = \nabla u(Y) - \nabla u(X)\), we have for \(|X - Y| \leq 2R,
|\nabla w(Y)| = |\nabla u(Y) - \nabla u(X)| \leq (h(X) + h(Y))|X - Y|\alpha|\nabla u|_{0, \alpha; h} \leq 4C_0 h(X)R^\alpha |\nabla u|_{0, \alpha; h},
|\nabla w(Y)| \leq |\nabla u(Y)| + |\nabla u(X)| \leq (h(X) + h(Y))|\nabla u|_{0, \alpha; h} \leq 2C_0 h(X)|\nabla u|_{0, \alpha; h}.
\]
This shows that
\[
|\nabla w|_{0; B(2R), R} \leq 4C_0 h(X) \min(R^\alpha, 1)|\nabla u|_{0, \alpha; h},
\]
from which and \(w(X) = 0\), we infer
\[
|w|_{0; B(2R), R} \leq 2R|\nabla w|_{0; B(2R), R} \leq 8C_0 h(X) \min(R^{1+\alpha}, R)|\nabla u|_{0; \alpha; h}.
\]
Then by the (scaled) Schauder estimate, we obtain
\[
|\nabla^2 w|_{0, \alpha; B(X, R), R} \leq C(R^{-2}|w|_{0; B(2R), R} + |\triangle w|_{0, \alpha; B(2R), R})
\leq C h(X) \min(R^{-1+\alpha}, R^{-1})|\nabla u|_{0, \alpha; h} + |\triangle u|_{0, \alpha; h, R} \triangleq C h(X)A,
\]
which in particular shows
\[
|\nabla^2 u(X)| = |\nabla^2 w(X)| \leq |\nabla^2 w|_{0, \alpha; B(X, R), R} \leq C h(X)A.
\]
On the other hand, if \(|Y - X| < R\), then
\[
|\nabla^2 u(X) - \nabla^2 u(Y)| \leq |X - Y|\alpha R^{-\alpha}|\nabla^2 w|_{0, \alpha; B(X, R), R} \leq C h(X)A|X - Y|\alpha R^{-\alpha},
\]
and if \(|Y - X| \geq R\), then
\[
|\nabla^2 u(X) - \nabla^2 u(Y)| \leq |\nabla^2 u(X)| + |\nabla^2 u(Y)|
\leq C h(X)A + C h(Y)A
\leq C h(X) + h(Y)|X - Y|\alpha R^{-\alpha}.
\]
This gives
\[ |\nabla^2 u|_{0;\alpha;1} = |\nabla^2 u|_{0;\alpha} + R^\alpha |\nabla^2 u|_{\alpha;1} \leq C.A. \]

The proof is finished. □

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