FOLIATIONS MODELING NONRATIONAL SIMPLICIAL TORIC VARIETIES

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Abstract. We establish a correspondence between simplicial fans, not necessarily rational, and certain foliated compact complex manifolds called LVMB-manifolds. In the rational case, Meersseman and Verjovsky have shown that the leaf space is the usual toric variety. We compute the basic Betti numbers of the foliation for shellable fans. When the fan is in particular polytopal, we prove that the basic cohomology of the foliation is generated in degree two. We give evidence that the rich interplay between convex and algebraic geometries embodied by toric varieties carries over to our nonrational construction. In fact, our approach unifies rational and nonrational cases.

1. Introduction

Rational convex polytopes and toric varieties. The correspondence between rational convex polytopes and projective toric varieties is well-known. It pertains to several fields, including combinatorics, convex geometry, symplectic geometry, algebraic geometry.

Within this picture, simple rational polytopes correspond to toric varieties that are rationally smooth (i.e., having at most orbifold singularities). We will only consider this restricted correspondence, which has long been known to provide fruitful links between the fields listed above [14, 22, 29, 44, 42].

On the other hand, simple polytopes come in continuous families (by perturbing the facets’ directions), whereas toric varieties, dubbed “frigid crystals” in [15], do not. The reason is that no toric variety corresponds to a nonrational polytope. A solution to this problem was given by Prato in [40], by introducing a generalization of toric orbifolds which are non Hausdorff when the polytope is nonrational.

Relying on works by Meersseman and Verjovsky [34, 35], and by Prato [40] (and also on [7, 31, 32, 11]), we take a new approach by realizing the toric space corresponding to any simple convex polytope as the leaf space of a smooth foliation. This simultaneously provides an object in the nonrational case and removes all singularities. Even though we are in principle interested in the leaf space, we lift all statements and proofs to the level of the foliation, where everything is smooth and Hausdorff.

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A generalized correspondence. A more accurate formulation of the above correspondence is made in terms of fans: recall that to any convex polytope $P$, we can associate its normal fan $\Delta$, which is complete and polytopal. The map $P \mapsto X$ taking a rational simple convex polytope $P$ to a toric variety $X$ factors out as $P \mapsto \Delta \mapsto X$. The first map is non-injective and the second one is the classical one-to-one correspondence between complete simplicial rational polytopal fans and rationally smooth projective toric varieties (these objects being seen up to isomorphism).

We propose a four-way generalization of the correspondence $\Delta \mapsto X$. The roles of varieties and fans will be played, respectively, by leaf spaces of foliations on so-called LVMB-manifolds and suitable triangulated vector configurations.

Notice that a rational fan $\Delta$ determines implicitly a vector configuration $V$ on a lattice. Namely, $V$ is the set of primitive generators of the fan’s rays. We will stress the importance of $V$ and its generalizations, to which we give the main role on the convex geometric side.

The four generalizations are, in increasing order of importance from our point of view:

1. We allow non-polytopal fans. This translates into using nonregular triangulations (cf. 4.5.1).
2. We allow orbifold multiplicities. This amounts to taking nonprimitive generators on rays.
3. We have more generators than rays: some vectors in the configuration may not correspond to a ray.
4. We do not require rationality of the configuration. This means dropping the closedness condition of the lattice.

Each of these generalizations has already appeared in the literature. We will simultaneously refine, generalize or desingularize several known constructions, thus giving a unified picture of the smooth, orbifold and nonrational cases.

Related works. We give a simplified account of earlier constructions. Each starts from a convex-geometric object, and (using Gale duality) defines an algebraic- or complex-geometric object. From now on, all fans (resp. polytopes) are simplicial (resp. simple).

A rational fan. To each rational simplicial fan in a vector space $L \otimes_{\mathbb{Z}} \mathbb{R}$, with $L$ a lattice, there corresponds a rationally smooth toric variety $X$.

A Delzant polytope (necessarily rational). On the symplectic side Delzant proves the existence of a unique symplectic toric manifold in correspondence to each Delzant polytope, i.e., whose normal fan satisfies suitable integrality conditions [14].

A rational polytope and multiplicities attached to facets (equivalently, a rational polytopal fan with multiplicities attached to rays, and a certain height function).

1. Symplectic orbifolds. Lerman and Tolman generalize Delzant’s theorem to the class of symplectic toric orbifolds, by allowing any rational convex
(2) \textit{Generalized Calabi-Eckmann fibrations I}. Meersseman and Verjovsky prove in \cite{35} that toric varieties and “toric varieties with orbifold multiplicities” can be viewed as leaf spaces of the foliations on \textit{rational LVM-manifolds} \cite{34} (see below).

A \textit{nonrational polytope, a quasilattice, and rays generators}. Prato generalizes the Delzant procedure to any simple convex polytope. A key point is to replace the lattice with a \textit{quasilattice}, i.e., a \(\mathbb{Z}\)-module in a vector space, generated by a finite spanning set. For a given simple convex polytope, different choices of a quasilattice and rays generators contained therein yield a family of symplectic spaces called \textit{quasifolds}. When the polytope is rational, this family strictly contains the cases above. When the quasilattice is not a lattice the corresponding spaces are non Hausdorff: nonrationality forces quotient singularities of finite type to become of “infinite type” \cite{40}. We shall refer to these spaces as \textit{toric quasifolds}.

A \textit{nonrational polytope}. Generalizing the works of López de Medrano and Verjovsky \cite{32}, and of Lœb and Nicolau \cite{31}, Meersseman \cite{34} constructs the so-called LVM-manifolds. It is formed by the compact complex foliated manifolds corresponding to our polytopal case. To each such manifold \(N\), he associates a non necessarily rational polytope \(P\); he establishes a one-to-one correspondence between the combinatorial type of \(P\) and \(N\) up to deformation \cite[Th. 13]{34}.

A \textit{(n implicit) nonrational fan}. Generalizing Meersseman’s construction, Bosio constructs the family of so-called LVMB-manifolds that we consider here (cf. \cite{7} and the interpretation in \cite{11}).

A \textit{stacky fan (equivalently, a rational fan with multiplicities attached to rays)}.

(1) \textit{Generalized Calabi-Eckmann fibrations II}. Tambour constructs and studies certain LVMB-manifolds in \cite{43}. He discusses the relationship with toric varieties.

(2) \textit{Stacks}. Another approach for handling the orbifold structure, and turning orbifolds into smooth objects, is to use stacks. We refer to Iwanari’s article \cite{26} for this point of view, initiated by Borisov-Chen-Smith in \cite{8}.

A \textit{nonrational fan}. Panov and Ustinovsky construct complex structures on moment-angle manifolds and their quotients by real tori in \cite{39}. Under rationality assumptions, they discuss the relationship with toric varieties.

Since we posted this article on the arXiv, several related results have appeared: Ishida \cite{25} discovered an interesting group-theoretic characterization of a class of manifolds strictly containing LVMB-manifolds. Ustinovsky showed in \cite{48} that Ishida’s manifolds coincide with that of \cite{39}. Another recent generalization was described by Battisti and Oeljeklaus \cite{5}. Finally in the note \cite{28} Katzarkov, Lupercio, Meersseman and Verjovsky investigate a different approach to defining simplicial toric varieties in a nonrational setting. Their main technical tool is
an extension of LVM theory to the nonrational case (i.e. they assume fans to be polytopal; cf. our Sect. 4.5), but they consider the leaf space from the noncommutative and diffeological (cf. [24]) viewpoints.

**Summary of our results.** We propose to encode all of the convex-geometric data needed for the construction of a space $X$ —that is, a fan, a choice of a point on each ray and a (quasi)lattice containing each of those points— in a unique and well-studied object (cf. [13]): a triangulated vector configuration $(V, T)$. Notice that a nonrational fan is not sufficient to determine a unique $X$. We develop a framework in which it is possible to obtain nonrational toric varieties by means of LVMB-manifolds. In fact, we construct a well-defined map from $(V, T)$ to a complex-geometric object $X$ (the leaf space of an LVMB-manifold).

We define in Sect. 2.1.1 two integers $a$ and $b$ that are quantitative measures of the nonrationality of the configuration (the integer $a$ was defined in [34]). In correspondence to the configuration, we construct an LVMB-manifold $N$, endowed with a smooth holomorphic foliation $F$ whose topology depends on $a$ and $b$. The leaf space $X$ can be, in increasing generality, a smooth toric variety, a toric orbifold or a toric quasifold. In the latter cases, we see our smoothly foliated manifold $N$ as a desingularisation of the space $X$. In Sect. 4 we include several fully worked out examples.

Beyond the introduction of $(V, T)$ as the main convex-geometric object, our main result is that, in both rational and nonrational cases, the cohomological study can be lifted to the foliation by using basic cohomology. In the case of a shellable fan, we compute the basic Betti numbers of $(N, F)$. In particular, we show that they only depend on the combinatorial type of the fan (Th. 3.1). When the fan is polytopal we prove that the basic cohomology algebra of $(N, F)$ is generated in degree two (Th. 3.5). In Sect. 4.4 we show that our framework handles Stanley’s proof of the necessity part of the $g$-theorem, by applying El Kacimi’s basic version of the hard Lefschetz theorem [16, 3.4.7]. Finally, in Sect. 4.5 we illustrate some specific features of the polytopal case.

Throughout, we try to delineate the combinatorial, topological, and convex geometric aspects, each of which being of independent interest. On the convex-geometric side, we emphasize the relevance of methods such as triangulations, shellings and Gale duality. On the complex-geometric side, we explore and extend toric methods, giving evidence that at least part of the technology available with toric varieties carries over to our foliated model, which makes no distinction between rational and nonrational cases. LVMB-manifolds thus establish a tight link between convex geometry and complex geometry, and may also contribute to a more geometric understanding of the nonsimplicial nonrational case.

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2. Construction

2.1. Triangulated configurations. Let $E$ be an $\mathbb{R}$-vector space of dimension $d$.

2.1.1. Vector configurations. A vector configuration $V = (v_1, \ldots, v_n)$ is a finite, ordered list of vectors, allowing repetitions. We will assume that $\text{Span}_{\mathbb{R}}\{v_1, \ldots, v_n\} = E$.

Consider the space of linear relations among $v_1, \ldots, v_n$

$$\text{Rel}(V) := \{c \in \mathbb{R}^n \mid \sum_{1 \leq j \leq n} c_j v_j = 0\},$$

which has dimension $n - d$. We say that a real subspace of $\mathbb{R}^n$ is rational when it admits a real basis of vectors in $\mathbb{Q}^n$ (equivalently, $\mathbb{Z}^n$). We define $a(V)$ as the dimension of the largest rational space contained in $\text{Rel}(V)$, and $b(V)$ as the dimension of the smallest rational space containing $\text{Rel}(V)$. Then $0 \leq a(V) \leq n - d \leq b(V) \leq n$.

The configuration is called rational when $\text{Rel}(V)$ is rational or, equivalently, $a(V) = n - d$ or $b(V) = n - d$. Otherwise $2 + a(V) \leq b(V)$, and all such values are possible.

2.1.2. Triangulations. Our main reference for triangulations and related concepts is the book [13]. Let $\tau \subset \{1, \ldots, n\}$. The cone over $\tau$ is defined as $\text{cone}(\tau) = \{\sum_{j \in \tau} \mathbb{R}_{\geq 0} v_j\}$. By convention, $\text{cone}(\emptyset) = \{0_E\}$. We say that $\tau$ is a simplex when the vectors indexed by $\tau$ are linearly independent (in particular, pairwise distinct). A simplicial cone is a cone over a simplex.

A triangulation $T$ of a configuration $V$ is a collection of simplices such that:

- If $\tau \in T$ and $\tau' \subset \tau$ then $\tau' \in T$;
- For all $\tau, \tau' \in T$, $\text{cone}(\tau) \cap \text{cone}(\tau') = \text{cone}(\tau \cap \tau')$;
- $\cup_{\tau \in T} \text{cone}(\tau) \supset \text{cone}(V)$.

This definition allows that some vectors among $v_1, \ldots, v_n$ do not belong to any simplex of $T$. We denote by $k \geq 0$ the number of such “ghost vectors”. We will always assume that they are at the end of the list $v_1, \ldots, v_n$. The pair $(V, T)$ is said to be a triangulated configuration.

2.1.3. Relations to other convex-geometric data. Suppose first that a triangulated configuration $(V, T)$ is given.

Where is the fan? The collection of cones on all of the simplices of $T$ is a simplicial fan $\Delta$, of dimension $d$. That is, a collection of simplicial cones such that: each nonempty face of a cone in $\Delta$ is a cone in $\Delta$; the intersection of any two cones in $\Delta$ is a face of each [50]. Notice that the fan $\Delta$ does not keep track of the ghost vectors and of the position of the other vectors on their respective rays. The non-ghost vectors play the role of generators of the rays of $\Delta$, as in [39]; they correspond to the vertices of the star-shaped simplicial sphere considered in [43].

Where is the polytope? In general there is no relevant polytope associated to $(V, T)$. In the important special case of $\Delta$ being polytopal, there are infinitely
many polytopes whose normal fan is $\Delta$, all of the same combinatorial type. Some extra data is needed in order to determine a particular polytope.

Where is the (generalized) lattice? The $\mathbb{Z}$-submodule of $E$ generated by all the vectors $v_1, \ldots, v_n$ is a quasilattice in $E$. By lattice in $E$ we shall mean a quasilattice that is closed, equivalently of rank $d$. The configuration $V$ being rational is equivalent to $Q$ being a lattice. It is well-known that fixing a rational fan but varying the lattice will change the associated toric variety. Analogously, starting from a triangulated configuration and modifying the quasilattice (by adding or deleting ghost vectors) will alter the geometry of the leaf space. This is exemplified in 4.3 which can be understood as a realization theorem, showing a substantial freedom in the construction even in dimension one.

Conversely, assume given Prato’s data of: a nonnecessarily rational simple polytope $P$ with $h$ facets; normal vectors $v_1, \ldots, v_h$; a quasilattice $Q$ containing these vectors. Choose $v_{h+1}, \ldots, v_n$ such that $v_1, \ldots, v_n$ generate $Q$. The vectors $v_1, \ldots, v_h$ generate the rays of the normal fan $\Delta$ of $P$. This fan determines a triangulation $\mathcal{T}$ on $V = (v_1, \ldots, v_n)$ with $v_{h+1}, \ldots, v_n$ as ghost vectors.

Actually some information is lost — $P$ can’t be recovered from $\Delta$—, but this information is not necessary to build the toric quasifold $X$ as a complex quotient [4]. As with toric varieties, the benefits of the symplectic reduction construction are an a priori symplectic/Kähler structure and compactness, whereas the advantages of the complex quotient are: an a priori complex structure; a generalization to the non polytopal case. We will give more details later on how to encode and use that extra piece of information, that can exist only in the polytopal case.

Finally, starting from a stacky fan, we encode it in a similar way: we add ghost vectors to generate the ambient lattice, as in [35].

2.2. Construction of the LVMB-manifold $N$.

2.2.1. Balanced and odd triangulations. Let $(V, \mathcal{T})$ be a triangulated vector configuration satisfying:

(i) $n - d = 2m + 1$ with $m$ a positive integer,
(ii) $\sum v_i = 0$.

By (ii) and our assumption that $V$ spans the ambient space $E$, the vectors of $V$ can not be contained in any half-space, so $\text{cone}(V) = E$. Thus, the third defining property of triangulations (cf. 2.1.2) implies that $\Delta$ is a complete fan. Conditions (i) and (ii) are mild restrictions: starting with the weaker assumption that $(V, \mathcal{T})$ is a triangulated configuration whose associated fan is complete, we easily obtain (i) and (ii) while keeping both the quasilattice and the fan unchanged (this fact is used in [MV]). Namely, we apply the following algorithm:
Step 1. If $\sum v_i \neq 0$, append $-\sum v_i$ as a new ghost vector of the configuration (and increase $n$ by 1);
Step 2. If $n - d$ is even, append 0 as a new ghost vector of the configuration (and increase $n$ by 1);
Step 3. If $n - d = 1$, append 0 and 0 as new ghost vectors of the configuration (and increase $n$ by 2).
2.2.2. Virtual chamber and \( U(T) \). Denote the set of maximal simplices of \( T \) by \( \{E_\alpha\}_\alpha \). Define the virtual chamber \( E := \{E_\alpha \mid \forall j \in E_\alpha, z_j \neq 0\} \). Define \( U(T) := \bigcup_\alpha U_\alpha \).

2.2.3. The dual configuration. Define a matrix \( M \in \mathbb{R}^{n \times (2m+1)} \) by

\[
M = \begin{bmatrix}
1 & a_1^1 & \cdots & a_1^{2m} \\
\vdots & \vdots & \ddots & \vdots \\
1 & a_n^1 & \cdots & a_n^{2m}
\end{bmatrix},
\]

where the columns form a basis of \( \text{Rel}(V) \). Now define a vector configuration \( \hat{\Lambda}^R = (\hat{\Lambda}_1^R, \ldots, \hat{\Lambda}_n^R) \) in \( \mathbb{R}^{2m+1} \), called a Gale dual of \( V = (v_1, \ldots, v_n) \), and a configuration \( \Lambda^R = (\Lambda_1^R, \ldots, \Lambda_n^R) \) in \( \mathbb{R}^{2m} \) by

\[
M = \begin{bmatrix}
-\hat{\Lambda}_1^R \\
\vdots \\
-\hat{\Lambda}_n^R
\end{bmatrix} = \begin{bmatrix}
1 & -\Lambda_1^R \\
\vdots \\
1 & -\Lambda_n^R
\end{bmatrix}.
\]

Notice that \( M \) is only defined up to right multiplication by a matrix of form \( T = \begin{bmatrix} 1 & B \\ 0 & A \end{bmatrix} \) where \( B = (b_1, \ldots, b_{2m}) \in \mathbb{R}^{2m} \) and \( A \in GL(2m, \mathbb{R}) \). Therefore, a Gale dual is not unique, and \( \Lambda^R \) is only defined up to the invertible real affine transformation of the ambient \( \mathbb{R}^{2m} \) given by \( X \mapsto AX + B \). Thus, \( \Lambda^R \) is to be seen as a configuration of points, i.e., affine objects. We refer to Sect. 4 for examples.

The quantitative measures of nonrationality of the dual configurations \( V \) and \( \hat{\Lambda}^R \) are linked by the relations

\[
a(V) + b(\hat{\Lambda}^R) = n \quad \text{and} \quad a(\hat{\Lambda}^R) + b(V) = n,
\]

which follow from \( \text{Rel}(\hat{\Lambda}^R) = \text{Ker} \ M^t = (\text{Im} \ M)^\perp = \text{Rel}(V)^\perp \). We note also that \( a(\hat{\Lambda}^R) \) is denoted \( a \) in [34, Th. 4], where it is shown that the algebraic dimension of \( N \) is at least \( a \), with equality in the absence of ghost vectors.

2.2.4. The \( C^m \)-action and \( N \). Consider the holomorphic \( C^m \)-action on \( U(T) \) defined by

\[
C^m \times U(T) \longrightarrow U(T) \quad (u ; [z_1 : \cdots : z_n]) \longmapsto [e^{2\pi i \Lambda_1(u)} z_1 : \cdots : e^{2\pi i \Lambda_n(u)} z_n],
\]

where

\[
\Lambda_j := \begin{bmatrix}
a_j^1 + ia_j^{n+1} \\
\vdots \\
a_j^m + ia_j^{2m}
\end{bmatrix} \in \mathbb{C}^m
\]

with \( a_1^1, \ldots, a_j^{2m} \) denoting the entries of \( \Lambda_j^R \), and \( \Lambda_j(u) \) denotes the dot product.

Bosio has given in [7] sufficient conditions for this action to be proper and cocompact. We show below that action (1) is free and Bosio’s conditions hold,
thus the quotient of $U(T)$ by this action is a compact complex manifold that we denote $N$. Note that acting on $V$ by a linear automorphism of $E$ is immaterial for the construction we have described, since Rel($V$) is unchanged by such a transformation. We refer to [7, 34] for properties of $N$, and note here that the standard holomorphic $(\mathbb{C}^*)^n$-action on $\mathbb{C}P^{n-1}$ induces a decomposition of $N$ (cf. [21 p. 36]). define $N(\tau) \subset N$ as the image in $N$ of $\{ [z] \in U(T) \mid z_j \neq 0 \text{ iff } j \notin \tau \}$. Then $N$ is the disjoint union of the $(\mathbb{C}^*)^n$-orbits $\coprod_{\tau \in T} N(\tau)$, with a unique open orbit $N(\emptyset)$.

2.2.5. Proof that Bosio’s conditions hold. By properties of Gale duality (see [13] Def. 5.4.3 and the comment below), for each $\alpha, \beta$, $\{ \hat{\Lambda}_j^\alpha \mid j \in E^c_\alpha \}$ is a simplex, i.e., a linear basis of $\mathbb{R}^{2m+1}$. Let $P_\alpha$ denote the convex hull of $\{ \Lambda_j^\alpha \mid j \in E^c_\alpha \} \subset \mathbb{R}^{2m}$. Then $\hat{P}_\alpha \neq \emptyset$, and it follows that action (1) has trivial isotropy at any element of $U_\alpha$, so this action is free on $U(T) = \cup_\alpha U_\alpha$. The statement on the isotropy, found in [7, Rem. 1.1] or Meersseman’s thesis, is proved as follows: let $u \in \mathbb{C}^m$ be in the isotropy at $z \in U_\alpha$ and suppose without loss of generality that $n \in E^c_\alpha$. This implies $\text{Im} [(\Lambda_j - \Lambda_n)(u)] = 0$ for all $j \in E^c_\alpha \setminus \{n\}$. This in turn implies $\text{Im} \left( \frac{u}{\text{Re} \ u} \right) = 0$. Since $\hat{P}_\alpha \neq \emptyset$, the vectors $\Lambda_j^\alpha - \Lambda_n^\alpha$, with $j \in E^c_\alpha \setminus \{n\}$, are a basis of $\mathbb{R}^{2m}$, therefore $u = 0$. The result below belongs to a circle of ideas that appear in the works of Białynicki-Birula and Świącicka. Similar results include also [6] Lemma 3.5 and [13] Prop. 2.3 and Cor. 2.4.

**Proposition 2.1.** Bosio’s conditions hold here, i.e.,

(i) $\hat{P}_\alpha \cap \hat{P}_\beta \neq \emptyset$ for every $\alpha, \beta$;

(ii) for every $E^c_\alpha \in \mathcal{E}$ and every $i \in E_\alpha$,

there exists $k \in E^c_\alpha$ such that $(E^c_\alpha \setminus \{k\}) \cup \{i\} \in \mathcal{E}$.

**Proof.** (i) Pick in $\mathcal{T}$ any two distinct maximal simplices $E_\alpha$ and $E_\beta$, and choose a linear form $\varphi$ that separates the respective cones, in the sense that $\varphi$ is positive on cone($E_\alpha$) and negative on cone($E_\beta$), except on cone($E_\alpha$) $\cap$ cone($E_\beta$), where it is zero. A linear evaluation such as

\[
(\varphi(v_1), \ldots, \varphi(v_n))
\]

corresponds (cf. [13] p. 244) to a linear relation on the Gale dual with coefficients given by $\varphi(v_1), \ldots, \varphi(v_n)$. Here the relation has the form

\[
\sum_{j \in E_\alpha \setminus E_\beta} a_j \hat{\Lambda}_j^\alpha - \sum_{j \in E_\beta \setminus E_\alpha} b_j \hat{\Lambda}_j^\beta + \sum_{j \notin E_\alpha \cup E_\beta} c_j \hat{\Lambda}_j^\beta = 0,
\]

where all $a_j$’s and $b_j$’s are positive. For all $j \notin E_\alpha \cup E_\beta$, we write $c_j$ as the difference of two positive numbers $a_j - b_j$. Then

\[
\sum_{j \in E_\alpha \setminus E_\beta} a_j \hat{\Lambda}_j^\alpha + \sum_{j \notin E_\alpha \cup E_\beta} a_j \hat{\Lambda}_j^\alpha = \sum_{j \in E_\beta \setminus E_\alpha} b_j \hat{\Lambda}_j^\alpha + \sum_{j \notin E_\alpha \cup E_\beta} b_j \hat{\Lambda}_j^\beta, \text{ i.e.,}
\]

\[
\sum_{j \in E_\beta} a_j \hat{\Lambda}_j^\beta = \sum_{j \in E_\alpha} b_j \hat{\Lambda}_j^\alpha.
\]
Thus \[
\sum_{j \in \mathcal{E}_\beta^c} a_j = \sum_{j \in \mathcal{E}_\alpha^c} b_j =: s, \quad \text{and} \quad \frac{1}{s} \sum_{j \in \mathcal{E}_\beta^c} a_j \Lambda_j^\mathbb{R} = \frac{1}{s} \sum_{j \in \mathcal{E}_\alpha^c} b_j \Lambda_j^\mathbb{R}.
\]

The left hand side and right hand side belong to \( \hat{P}_\beta \) and \( \hat{P}_\alpha \) respectively. Therefore the intersection is nonempty.

(ii) Pick \( \mathcal{E}_\alpha^c \in \mathcal{E} \) and \( i \in \mathcal{E}_\alpha \). The facet of cone(\( \mathcal{E}_\alpha \)) determined by omitting \( v_i \) is shared by one and only one maximal cone, say cone(\( \mathcal{E}_\beta \)). Then \( \mathcal{E}_\beta = (\mathcal{E}_\alpha \setminus \{ i \}) \cup \{ k \} \) for some \( k \), and \( k \not\in \mathcal{E}_\alpha \) by convexity of cone(\( \mathcal{E}_\alpha \)). Then \( (\mathcal{E}_\alpha \setminus \{ k \}) \cup \{ i \} = \mathcal{E}_\beta^c \in \mathcal{E} \). \qed

2.3. The foliation \( \mathcal{F} \) on \( N \). Consider on \( U(\mathcal{T}) \) the following holomorphic action by \( \mathbb{C}^{2m} \):

\[
\ell.[z_1 : \cdots : z_n] = [e^{2\pi i \Lambda_j^\mathbb{R}(\ell)} z_1 : \cdots : e^{2\pi i \Lambda_j^\mathbb{R}(\ell)} z_n].
\]

Fix a \( [z] \in U(\mathcal{T}) \). Direct computations show that the isotropy at \( [z] \) is a closed \( \mathbb{Z} \)-module \( L_z \subset \mathbb{R}^{2m} \subset \mathbb{C}^{2m} \) of rank at most \( 2m \).

Action \((2)\) commutes with \((1)\), so it descends to \( N \). The restriction of action \((2)\) to \( \mathbb{C}^m_N := \{ t \in \mathbb{C}^{2m} | \ell \cdot t = \left( \begin{array}{c} u \\ iu \end{array} \right), \; u \in \mathbb{C}^m \} \)
gives action \((1)\). Define \( \mathbb{C}^m_F := \{ t \in \mathbb{C}^{2m} | \ell \cdot t = \left( \begin{array}{c} u \\ 0 \end{array} \right), \; u \in \mathbb{C}^m \} \).

The projection \( \pi : \mathbb{C}^{2m} = \mathbb{C}^m_N \oplus \mathbb{C}^m_F \to \mathbb{C}^m_F \) is given by \( (x, y) \mapsto (x + iy, 0) \). The isotropy of \( [z] \in N \) for the action of \( \mathbb{C}^m_F \) on \( N \) is \( \pi(L_z) \). Therefore this action has discrete isotropy, so it induces on \( \hat{N} \) a smooth foliation \( \mathcal{F} \) of dimension \( m \).

In the polytopal case, this foliation appears in \([31]\) and \([34]\) (cases \( m = 1 \) and \( m \geq 1 \) respectively). The foliation \( \mathcal{F} \) is holomorphic, and in particular transversely orientable. We show below the stronger statement that \( \mathcal{F} \) is homologically orientable (cf. \( [34] \) in Sect. \( 4.4 \)).

The leaf \( \mathcal{F}_z \) through a point \( [z] \in N \) is the image, via an injective immersion, of \( \mathbb{C}^m_F/\pi(L_z) \). By varying the choice of the Gale dual, the \( \mathbb{Z} \)-module \( L_z \) becomes \( A^{-1}L_z \), with \( A \in GL(2m, \mathbb{R}) \) (cf. Sect. \( 2.2.3 \)), so the holomorphic structure on \( \mathcal{F}_z \) varies among all complex abelian groups on a fixed topological type. There is a unique \( \tau \) such that \( \mathcal{F}_z \subset N(\tau) \). Define a subconfiguration of \( \hat{\Lambda}^\mathbb{R} \) by \( \hat{\Lambda}^\mathbb{R}(\tau) := (\hat{\Lambda}_j^\mathbb{R})_{j \not\in \tau} \). By computing rank(\( \pi(L_z) \)) we obtain the topological type of the leaf \( \mathcal{F}_z \approx (S^1)^{B(\tau)-1} \times \mathbb{R}^{2m-B(\tau)+1} \)
where \( B(\tau) = n - \#\tau - b(\hat{\Lambda}^\mathbb{R}(\tau)) \). The topological type of the leaf closure is \( \overline{\mathcal{F}_z} \approx (S^1)^{A(\tau)-1} \).
where $A(\tau) = n - \#\tau - a(\hat{\Lambda}^R(\tau))$. In particular, these topological types depend on $V$ and $\tau$, but not on the choice of the Gale dual.

Generic leaves (i.e. lying in the open orbit $N(\emptyset)$) correspond to $\tau = \emptyset$. Since $a(V) = n - b(\hat{\Lambda}^R) \ (cf. \ 2.2.3)$, they are homeomorphic to $(S^1)^{a(V)-1} \times \mathbb{R}^{2m-a(V)+1}$. If the configuration is rational, that is $a(V) = b(V) = 2m+1$, all leaves are closed (cf. [35]). On the other hand there are nonrational configurations $V$ such that $a(V) = 1$; in these cases the generic leaf is $\mathbb{C}^m$.

2.3.1. The leaf space. Let $\Delta$, $v_1, \ldots, v_h$, and $Q$ be the fan, the rays generators (i.e., non-ghost vectors, so $h = n - k$), and the quasilattice associated to $(V,T)$ (see Sect. 2.1.3). From the Audin-Cox construction and its nonrational complex generalization [4], it is known that to this data there corresponds a geometric quotient $X = U'(\Delta)/G$, where $U'(\Delta)$ is an open subset of $\mathbb{C}^h$ that depends on the combinatorics of $\Delta$, and $G$ is a complex subgroup of $(\mathbb{C}^*)^h$ that depends on $Q$ and on the vectors $v_1, \ldots, v_h$. If the configuration is rational (resp. nonrational), then $X$ is a complex manifold or a complex orbifold (resp. a non Hausdorff complex quasifold) of dimension $d$, acted on holomorphically by the torus (resp. quasitorus) $\mathbb{C}^d/Q$ (cf. [1, 10, 40, 4]; the construction in [4, Thm 2.2] can be adapted to the nonpolytopal case). Quasifolds generalize orbifolds: the local model is a quotient of a manifold by the smooth action of a finite or countable group, non free on a closed subset of topological codimension at least 2 [40]. Let $(N,F)$ be any foliated complex manifold corresponding to $(V,T)$. The complex structure induced by $(N,F)$ on the leaf space depends on the initial data $(V,T)$, but not on the choice of a Gale dual.

Remark 2.2. The action of the group $G$ does induce a holomorphic foliation on $U'(\Delta)$. However, since $G$ is in general, for rational and nonrational configurations, not connected (cf. Ex. 4.2), the leaf space is not $X$. This problem is overcome in our construction by “increasing the dimension”.

2.3.2. The foliation is Riemannian. Consider the $(S^1)^{n-1}$-action on $N$ induced by the $(\mathbb{C}^*)^{n-1}$-action on $\mathbb{C}P^{n-1}$, and construct a Riemannian metric on the compact manifold $N$ such that the compact group $(S^1)^{n-1}$ acts by isometries. Now we observe that $\mathbb{C}^m_F$ acts on $N$ as a subgroup of $(S^1)^{n-1}$: fix $[\bar{z}] \in N$ and $t_1 = \left( \begin{array}{c} u \\ 0 \end{array} \right) \in \mathbb{C}^m_F$. Define $t_2 = \left( \begin{array}{c} u \\ iu \end{array} \right) \in \mathbb{C}^m_N$, with $u = -i\text{Im}(\bar{z})$. Then $t_1 \cdot (t_2 \cdot [\bar{z}]) = t \cdot [\bar{z}]$ with $t = \left( \begin{array}{c} \text{Re}(u) \\ \text{Im}(u) \end{array} \right)$, where the action used here is action (2). Now, the $\mathbb{C}^m_F$-action being locally free implies that the induced foliation $F$ is Riemannian [36, Ex. 2, p.100]. The same argument shows that the foliation is moreover Killing [37].

3. Topological results on the basic cohomology algebra

In this section we show how the combinatorics of a balanced and odd triangulated configuration $(V,T)$ relate to the basic Betti numbers of any foliated manifold
(N, F) built from (V, T). The formulas are the same as the usual Betti numbers of simplicial toric varieties.

For the combinatorial part we refer the reader to [50, Sect. 8.3]; for basic cohomology, see [47]. We recall definitions and results in the form we need for our purposes.

3.1. Shellings and h-vector. Fix a triangulated vector configuration (V, T). In particular, T is an abstract simplicial complex of pure dimension d − 1 (topologically, a sphere). The dimension of a (possibly empty) simplex τ ∈ T is #τ − 1. Recall that the f-vector (f − 1, f0, f1, . . . , fd−1) records the number of simplices in each dimension. The fan ∆ gives a “linear realization” of this simplicial complex, with simplices of dimension l corresponding bijectively to cones of dimension l + 1.

A shelling of T (or of ∆) is a linear ordering of the maximal simplices E1, . . . , Ed−1 such that for all α ≥ 2, cone(Eα) intersects cone(E1) ∪ · · · ∪ cone(Eα−1) along a nonempty union of facets of cone(Eα). The number of such facets, called the index of Eα w.r.t. the shelling, is denoted iα. We take i1 = 0.

Polytopal fans are shellable, i.e., they admit a shelling ([50, Sect. 8.2]). The h-vector (h0, h1, . . . , hd) of T (or ∆) records the number of maximal simplices of each index in a given shelling. It is well-known that the h-vector is completely determined by the f-vector—in particular, it is independent of the choice of a shelling—and conversely it determines the f-vector.

3.2. Basic cohomology. Let M be a smooth manifold with a smooth foliation G. A differential form ω ∈ Ω•(M) is called basic when for all vector fields X tangent to G, ıXω = 0 and ıXdω = 0. When the foliation is given by the orbits of a Lie group G, this means that the form is G-invariant and its kernel contains the tangent space to G. The cohomology of the complex of basic forms is in some sense the de Rham cohomology of the leaf space. The dimensions of these groups are called the basic Betti numbers. An example that gives some intuition for the proofs below is the torus M = S1 × S1 with G given by lines of slope s. When s is rational, the leaf space is a circle and b1G(M) = 1. When s is irrational, the leaf space is not Hausdorff but, again, b1G(M) = 1. Cohomologically, the leaf space is still a circle. Notice however that the basic Betti numbers of a foliated compact manifold can be infinite-dimensional in general, and that basic Betti numbers are not invariant under small deformations of Riemannian foliations [38, Example 7.4].

3.3. Computation of the basic Betti numbers.

Theorem 3.1. Let (V, T) be a shellable, balanced and odd triangulated vector configuration, with dim(V) = d and h-vector (h0, . . . , hd). Let (N, F) be any foliated manifold built from (V, T). Then the basic Betti numbers are

\[ b^{2j+1}_F(N) = 0 \]

and

\[ b^{2j}_F(N) = h_j \]
for $j = 0, \ldots, d$.

Proof. We use a “Morse-theoretic” method due to Khovanskii for simple polytopes. Working dually with simplicial fans, we see that his method extends (from polytopal fans) to shellable fans. Let $E_1, \ldots, E_{f_d-1}$ be a shelling of $\mathcal{T}$. Consider the open subsets $U_{\alpha}$ defined in 2.2.2 and their image, $N_{\alpha}$, in $N$. We consider the $F$-saturated open covering of $N$ defined as follows:

$$W_1 = N_1, \quad W_\alpha = W_{\alpha-1} \cup N_\alpha, \quad \alpha = 2, \ldots, f_d-1.$$  

Therefore

$$N_1 = W_1 \subset W_2 \subset \cdots \subset W_{f_d-1} = N.$$  

We compute inductively the basic cohomology of the foliated manifolds $W_\alpha$ by means of a Mayer-Vietoris sequence. For this we need a basic partition of unity: pick any partition of unity relative to the decomposition $W_\alpha = W_{\alpha-1} \cup N_\alpha$. The natural $(S^1)^d$-action on $\mathbb{C}^d$ descends to $N$. Averaging out the functions over this action will in particular make them constant on the leaves.

From Lemma 3.2 below, case $r = 0$, we know that $b^l_F(W_1 = N_1) = \delta_{0,l}$ now fix $\alpha \geq 2$ and make the induction hypothesis:

$$ \left( H_{\alpha-1} \right) \text{ if } l \text{ is odd then } b^l_F(W_{\alpha-1}) = 0.$$  

We claim that

$$b^l_F(W_\alpha) = \begin{cases} b^l_F(W_{\alpha-1}) & \text{if } l \neq 2i_\alpha, \\ b^l_F(W_{\alpha-1}) + 1 & \text{if } l = 2i_\alpha. \end{cases}$$  

In particular, $\left( H_\alpha \right)$ holds, and the theorem follows by induction.

Now we prove the claim. Using the notation of Lemma 3.2, remark first that $W_{\alpha-1} \cap N_\alpha$ is of the form $N_{\alpha, \tau}$, where $\tau$ is the restriction of $E_\alpha$, defined in [50, 8.3] as

$$\tau = \{ i \in E_\alpha \mid (E_\alpha \setminus i) \subseteq E_\beta \text{ with } \beta < \alpha \}.$$  

Notice that $\# \tau = i_\alpha$.

Then Lemma 3.2 tells us that $W_{\alpha-1} \cap N_\alpha$ has no basic cohomology in positive even dimension. Thus by Mayer-Vietoris, for any odd integer $p$,

$$0 \to H^p_F(W_\alpha) \to H^p_F(W_{\alpha-1}) \oplus H^p_F(N_\alpha) \to H^p_F(W_{\alpha-1} \cap N_\alpha) \to H^{p+1}_F(W_\alpha) \to$$

$$\quad \tag{*} \text{by Lemma 3.2} \quad \text{by Lemma 3.2} \quad \text{by Lemma 3.2}$$

We see that the second term, $H^{p}_F(W_\alpha)$, must be zero. Again by Lemma 3.2 ($*$) is zero unless $p = 2i_\alpha - 1$, in which case it is of dimension one.

\begin{lemma}
Let $(V, \mathcal{T})$ be a balanced and odd triangulated vector configuration in a vector space of dimension $d$. Let $\tau$ be a subset of a maximal simplex $E_\alpha$ of $\mathcal{T}$. We define an $\mathcal{F}$-saturated open subset of $N$, denoted $N_{\alpha, \tau}$, as the image in $N$ of

$$U_{\alpha, \tau} = U_\alpha \setminus \{ [z_1 : \cdots : z_n] \mid \forall j \in \tau, z_j = 0 \}.$$  

\end{lemma}
In particular, $N_{\alpha,\tau} = N_{\alpha}$ when $\tau$ is empty. Then, denoting $r = \#\tau$,

$$\forall l \geq 0, \ H^l_F(N_{\alpha,\tau}) \approx \begin{cases} \mathbb{R} & \text{if } l = 0 \text{ or } l = 2r - 1, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, the leaf space $N_{\alpha,\tau}/F$ is cohomologically a point when $r = 0$, and a $2r - 1$-sphere when $r > 0$.

**Proof.** By definition of $F$ we have $H^l_F(N_{\alpha,\tau}) \approx H^l\left(\Omega_{\mathbb{C}^{2m}}(U_{\alpha,\tau})\right)$, where $\Omega_{\mathbb{C}^{2m}}(U_{\alpha,\tau})$ denotes the complex of forms on $U_{\alpha,\tau}$ that are basic with respect to the foliation induced by the $\mathbb{C}^{2m}$-action $[2]$.

Suppose for now that $r > 0$, and assume for simplicity that $\mathcal{E}_\alpha = \{1, \ldots, d\}$ and $\tau = \{1, \ldots, r\}$. We know that $\left\{\Lambda_j^R - \Lambda_n^R\right\}_{j=d+1 \ldots n-1}$ is an $\mathbb{R}$-basis of $\mathbb{R}^{2m}$, so it is a $\mathbb{C}$-basis of $\mathbb{C}^{2m}$. This implies surjectivity of the map

$$g : (\mathbb{C}^r \setminus \{0\} \times \mathbb{C}^{d-r}) \times \mathbb{C}^{2m} \to U_{\alpha,\tau}$$

$$(\{z_1, \ldots, z_d\}; \ell) \mapsto \ell, [z_1 : \ldots : z_d : \underbrace{\vdots & \vdots & \ldots & \vdots}_{2m+1}].$$

On the other hand, $g(\{z_1, \ldots, z_d\}; \ell) = g(\{w_1, \ldots, w_d\}; \varepsilon)$ is equivalent to

$$\left\{ (w_1, \ldots, w_d) = \left( e^{2\pi i (\Lambda_j^R - \Lambda_n^R)(\ell - \varepsilon) z_1, \ldots, e^{2\pi i (\Lambda_j^R - \Lambda_n^R)(\ell - \varepsilon) z_d} \right) \right\}$$

$$\left( \Lambda_j^R - \Lambda_n^R \right)(\ell - \varepsilon) \in \mathbb{Z}, \ j = d + 1 \ldots n - 1.$$ 

The second condition implies that $\ell - \varepsilon \in \mathbb{Z}_{\mathfrak{y}}$ with $[\mathfrak{y}]$ any point in $N(\mathcal{E}_\alpha)$. Therefore $\mathbb{Z}_{\mathfrak{y}}$ is a lattice in $\mathbb{R}^{2m}$ that we denote $\mathcal{F}$. This shows that the fibers of $g$ are the orbits of the $\mathcal{F}$-action on $(\mathbb{C}^r \setminus \{0\} \times \mathbb{C}^{d-r}) \times \mathbb{C}^{2m}$ defined by

$$(4) \quad \gamma_{\mathfrak{y}}((z_1, \ldots, z_d); \ell) = \left( e^{2\pi i (\Lambda_j^R - \Lambda_n^R)(\gamma z_1, \ldots, e^{2\pi i (\Lambda_j^R - \Lambda_n^R)(\gamma z_d)}; \ell - \gamma \right).$$

Notice that the action of $\Gamma$ on the first factor does not depend on the choice of a Gale dual: changing this choice, $\Lambda_j^R - \Lambda_n^R$ becomes $(\Lambda_j^R - \Lambda_n^R)A$ and $\Gamma$ becomes $A^{-1}\Gamma$ (cf. 2.2.3), thus $(\Lambda_j^R - \Lambda_n^R)A(A^{-1}\gamma) = (\Lambda_j^R - \Lambda_n^R)(\gamma)$. Remark also that the map $g$ induces a foliation-preserving homeomorphism

$$(\mathbb{C}^r \setminus \{0\} \times \mathbb{C}^{d-r}) \times \mathbb{C}^{2m} / \simeq U_{\alpha,\tau}.$$ 

This in turn implies

$$(5) \quad (\mathbb{C}^r \setminus \{0\} \times \mathbb{C}^{d-r}) \times \mathbb{C}^{2m}_F / \Gamma \simeq N_{\alpha,\tau}.$$

Now, $\omega \mapsto g^*(\omega)$ maps isomorphically the complex $\Omega^\bullet_{\mathbb{C}^{2m}}(U_{\alpha,\tau})$ onto the complex $\mathbb{C}^r$ of forms on $(\mathbb{C}^r \setminus \{0\} \times \mathbb{C}^{d-r}) \times \mathbb{C}^{2m}$ that are: (a) $\Gamma$-invariant; (b) basic with respect to the foliation with leaves $\{\ell\} \times \mathbb{C}^{2m}$. But a form satisfies condition (b) if and only if it is the pull-back of a form by the projection $(\mathbb{C}^r \setminus \{0\} \times \mathbb{C}^{d-r}) \times \mathbb{C}^{2m} \to \mathbb{C}^r \setminus \{0\} \times \mathbb{C}^{d-r}$. Therefore $\mathbb{C}^r$ is (isomorphic to) the complex of $\Gamma$-invariant forms $\Omega^\bullet(\mathbb{C}^r \setminus \{0\} \times \mathbb{C}^{d-r})^\Gamma$. By (4), we see that the
action of $\Gamma$ factors through the standard $(S^1)^d$-action on $\mathbb{C}^d$. Therefore we can apply [3] Lemma 2.2 to conclude that for every $l \geq 0$,
\[
H^l\left(\Omega^\bullet(\mathbb{C}^r \setminus \{0\} \times \mathbb{C}^{d-r})^\Gamma\right) \approx H^l\left(\Omega^\bullet(\mathbb{C}^r \setminus \{0\} \times \mathbb{C}^{d-r})\right) \\
\approx H^l(\mathbb{C}^r \setminus \{0\}) \approx H^l(S^{2r-1}).
\]
In the case $r = 0$ the proof is similar: replace every $\mathbb{C}^r \setminus \{0\} \times \mathbb{C}^{d-r}$ with $\mathbb{C}^d$, and omit the last line.

After a first version this paper was completed, we came across the article [20] by O. Goertsches and D. Töben, that contains results in the spirit of our Th. 3.1. We use their techniques in Sect. 3.4.

3.4. The basic cohomology algebra is generated in degree two. Let $(V, \mathcal{T})$ be a balanced and odd triangulated vector configuration, with associated fan denoted $\Delta$.

In this section we assume polytopality, i.e. there exists a (necessarily) simple polytope $P \subset \mathbb{R}^d$ whose normal fan is $\Delta$. There is an inclusion-reversing duality between faces of $P$ and cones of $\Delta$ (or simplices of $\mathcal{T}$). In particular, each vertex of $P$ corresponds to a maximal simplex of $\mathcal{T}$. We fix a shelling of $\mathcal{T}$ in the following way: thinking of the last coordinate of the ambient space as the “height”, we rotate $P$ until no two of its vertices have same height. We order the vertices from lowest to highest. It is easy to check that the corresponding order on the maximal simplices is a shelling of $\mathcal{T}$, that we denote $\mathcal{E}_1, \ldots, \mathcal{E}_{f_{d-1}}$.

Fix a maximal simplex $\mathcal{E}_\alpha$. Denote its restriction, defined in [3], with respect to the shelling by $\tau_\alpha$, and let $\tau_\alpha^- := \mathcal{E}_\alpha \setminus \tau_\alpha$. Now denote by $V_\alpha$, $F_\alpha$ and $F_\alpha^-$ the closed faces of $P$ that are dual to $\mathcal{E}_\alpha$, $\tau_\alpha$ and $\tau_\alpha^-$ respectively. We remark that $V_\alpha$ is the lowest (resp. highest) vertex of $F_\alpha$ (resp. $F_\alpha^-$). Recall that for any simplex $\sigma \in \mathcal{T}$, $N(\sigma)$ is the image in $N$ of the set $\{z \in U(\mathcal{T}) \mid z_l \neq 0 \text{ iff } l \in \sigma^c\}$, and $\overline{N(\sigma)}$ is the disjoint union $\sqcup_{\sigma \subset \sigma'} N(\sigma')$.

**Lemma 3.3.**

(i) For all $\alpha, \beta$ such that $\beta < \alpha$, $\overline{N(\tau_\alpha)}$ and $\overline{N(\tau^-_\beta)}$ are disjoint;

(ii) For all $\alpha$, $\overline{N(\tau_\alpha)}$ and $\overline{N(\tau^-_\alpha)}$ intersect transversally along $N(\mathcal{E}_\alpha)$, which is a compact leaf of $\mathcal{F}$.

**Proof.**

(i) We know that $F_\alpha$ has no point below $V_\alpha$, and $F^-_\beta$ has no point above $V_\beta$.

As $V_\beta$ is lower than $V_\alpha$, the closed faces $F_\alpha$ and $F^-_\beta$ are disjoint, i.e. they have no common face. Dually, this means that no simplex contains both $\tau_\alpha$ and $\tau^-_\alpha$. This implies that the disjoint unions making up $\overline{N(\tau_\alpha)}$ and $\overline{N(\tau^-_\alpha)}$ have no common component, and the result follows.

(ii) On the image in $N$ of $U_\alpha$ (which is of the form $\mathbb{C}^d \times \mathbb{C}^{d/2}_\mathbb{P} / \Gamma$; cf. [5]), the two submanifolds $\overline{N(\tau_\alpha)}$ and $\overline{N(\tau^-_\alpha)}$ become respectively $\{0\} \times \mathbb{C}^{d-r} \times \mathbb{C}^r_\mathbb{P} / \Gamma$.
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and \( C' \times \{ 0 \} \times \mathbb{C}^n / \Gamma \). The intersection is \( \{ 0 \} \times \{ 0 \} \times \mathbb{C}^n / \Gamma \), a compact torus.

\[ \Box \]

From the polytopality assumption, we know that \( N \) is an LVM-manifold (cf. 4.5.1). For each simplex \( \sigma \in T \), the subset \( N(\sigma) \) is defined by the vanishing of certain coordinates. From Property 5 in 4.5.1, it follows that \( N(\sigma) \) is itself an LVM-manifold. In particular it is a smooth, compact, \( \mathcal{F} \)-saturated complex submanifold. Moreover, \( \mathcal{F} \) is holomorphic, so there is a fiber orientation form on the normal bundle \( \nu_{N(\sigma)} = T_{N(\sigma)} \mathbb{M} / T_{N(\sigma)} \mathcal{F} \) that is invariant by the foliation's holonomy. It follows that this normal bundle is oriented as a foliated bundle (cf. paragraph above [46, Cor. 4.8], where the author defines on the normal bundle a foliation \( \mathcal{G} \) induced by the holonomy of \( \mathcal{F} \)). In this situation, it is possible \([46, \text{Sect. 4}]\) to define the integration along the fibers of basic forms with compact vertical support \( \pi^* : \Omega_{G,cv}^{\# \sigma}(\nu_{N(\sigma)}) \to \Omega_{\mathcal{F}}(N(\sigma)) \). This yields a homomorphism in cohomology \( \pi_* : H^*_{G,cv}^{\# \sigma}(\nu_{N(\sigma)}) \to H^*_{\mathcal{F}}(N(\sigma)) \), which is an isomorphism \([46, \text{Sect. 4, Th. 4.6}]\). In particular \( N(\sigma) \) admits a basic Thom class \( [\Phi_{\sigma}] \), of degree \( \# \sigma \), that can be viewed as a basic class on \( N \). Note that the basic Poincaré dual class exists, but the identification with the basic Thom class is not established. Therefore we don’t know the behaviour of basic Poincaré classes under intersections, so we use basic Thom classes instead. We need the following foliation-theoretic result:

**Proposition 3.4.** Let \( \mathcal{F} \) be a transversely oriented, Killing, Riemannian foliation on a compact connected manifold \( M \). If \( L \) is a compact leaf, then its basic Thom class is a generator of the top basic cohomology of \( M \).

**Proof.**

**Step 1 – Preliminaries.** In order to prove this statement we need to use the notion of transverse integration \([11]\), which in turn involves the so-called Molino bundle \( \hat{M} \) and basic manifold \( W \). We briefly recall here the notation and main properties of the Molino construction; for details we refer to \([36, 20]\). Denote by \( q \) the codimension of \( \mathcal{F} \). By assumption there is a metric \( g \) on the normal bundle \( \nu_{\mathcal{F}} \) with respect to which \( \mathcal{F} \) is Riemannian. Since \( \mathcal{F} \) is transversely oriented, we can consider the \( SO(q) \)-principal bundle \( \pi : \hat{M} \to M \) of positively oriented orthonormal frames of \( \nu_{\mathcal{F}} = T_{\hat{M}} / T_{\mathcal{F}} \). Let \( \omega \) be the transverse Levi-Civita connection on \( \hat{M} \) and let \( H = \ker \omega \) be the corresponding horizontal distribution. In particular, at each \( \hat{m} \in \hat{M} \) we have the splitting \( T_{\hat{m}} \hat{M} = H_{\hat{m}} \oplus V_{\hat{m}} \), where \( V_{\hat{m}} \) is tangent to the fibre \( \pi^{-1}(\pi(\hat{m})) \). The horizontal lift of \( \mathcal{F} \) to \( \hat{M} \) is a transversely parallelizable foliation \( \hat{\mathcal{F}} \) of same dimension, Riemannian for a certain metric \( \hat{g} \) on \( \nu_{\hat{\mathcal{F}}} = T_{\hat{M}} / T_{\hat{\mathcal{F}}} \). On each leaf \( \hat{L} \) of \( \hat{\mathcal{F}} \), \( \pi \) is a Galois covering of a corresponding leaf \( L \) in \( N \) (the group acting on \( \hat{L} \) is the holonomy of \( L \)). The commuting sheaf of germs of \( \hat{\mathcal{F}} \)-transverse fields that commute with all global transverse fields of \( (\hat{M}, \hat{\mathcal{F}}) \) is locally constant. In this case the stalk is an abelian Lie algebra \( a \). This gives
rise to two transverse and \( \pi \)-equivariant actions by \( \mathfrak{a} \) on \( M \) and \( \hat{M} \). In both \( M \) and \( \hat{M} \), the \( \mathfrak{a} \)-orbit of any leaf is its closure, which on \( \hat{M} \) has always dimension \( \dim F + \dim \mathfrak{a} \). The space of leaf closures \( \hat{M}/\hat{F} \) is a smooth manifold \( W \) called the basic manifold. The action of \( SO(q) \) on \( M \) induces a smooth action of \( SO(q) \) on \( W \). The projection \( \rho : \hat{M} \to W \) is locally trivial and the orbit space \( W/\hat{SO}(q) \) is homeomorphic to the space of leaf closures \( M/\hat{F} \). Note that if \( F \) is Killing and transversely oriented then \( W \) is orientable \[46\] Sect. 5).

Denote \( l = \dim SO(q) \) and consider a volume element \( \nu \) on \( \mathfrak{so}(q) \). Following \[46\] Sect. 5) (see also \[41\]) we then consider the basic \( l \)-form \( \nu \) on \( \hat{M} \) defined at each \( \hat{m} \in \hat{M} \) by:

\[
\nu_{\hat{m}}(X_1, \ldots, X_l) = \nu(\omega(X_1), \ldots, \omega(X_l)), \quad X_i \in T_{\hat{m}}\hat{M}.
\]

The transverse integration of a given basic \( q \)-form \( \alpha \) on \( M \) is defined by

\[
\int_{\hat{F}} \alpha = \int_{\hat{F}} (\pi^*\alpha) \wedge \nu,
\]

where \( \int_{\hat{F}} \) denotes transverse integration on \( \hat{M} \), which can in turn be defined as follows: let \( \beta \in \Omega^{q+l}(\hat{M}) \) (in other degrees \( \int_{\hat{F}} \) evaluates to zero). Denote by \( \iota_X\beta \) the contraction of \( \beta \) with the fundamental transverse fields \( \hat{X}_1, \ldots, \hat{X}_{\dim \mathfrak{a}} \) of the \( \mathfrak{a} \)-action. Then \( \iota_X\beta \) can be written \( \rho^*(\rho_!\beta) \) for some top form \( \rho_!\beta \) on \( W \). Now define

\[
\int_{\hat{F}} \beta = \int_W \rho_!\beta.
\]

Step 2. Now consider the compact leaf \( L \) in \( M \), and its normal bundle \( \nu L \). We consider on \( \hat{M} \) the transverse horizontal bundle \( \hat{H} = H/T\hat{F} \). The projection \( \pi \) induces isomorphisms \( H_{\hat{m}} \simeq T_{\hat{m}}\hat{M} \) and \( \hat{H}_{\hat{m}} \simeq T_{\hat{m}}\hat{M}/T_{\hat{m}}\hat{F} \simeq \nu L_m \), where \( m = \pi(\hat{m}) \).

According to \[46\] Cor. 4.8, \( \nu L \) is oriented as a foliated bundle. Therefore, we can consider a basic Thom form \( \Phi \in \Omega^q_F(M) \). We claim that \( \pi^*\Phi \in \Omega^q(\hat{M}) \) is \( \hat{F} \)-basic: For a vector \( \hat{X} \) tangent to \( \hat{F} \), \( \iota_{\hat{X}}\pi^*\Phi = \iota_{\pi_*\hat{X}}\Phi \circ \pi_* \), but \( \iota_{\pi_*\hat{X}}\Phi = 0 \) as \( \Phi \) is \( F \)-basic. As \( \pi^*\Phi \) is closed, we conclude that it is basic with respect to \( \hat{F} \).

Now, let \( \eta \) be the \((q+\dim \mathfrak{a})\)-form on \( W \) such that \( \rho^*(\eta) = \iota_X(\pi^*\Phi \wedge \nu) \), i.e. \( \eta = \rho_!(\pi^*\Phi \wedge \nu) \). The form \( \eta \) is a top form on \( W \), and by definition of transverse integration

\[
\int_{\hat{F}} \Phi = \int_W \eta.
\]

We are left to show that the right hand side is nonzero: by \[41\] Sect. 2], \( \Phi \) is then not exact, thus \([\Phi]\) is nonzero in \( H^q_F(M) \), which implies that \( H^q_F(M) \) is one-dimensional and generated by \([\Phi]\) (cf. \[6\] p.\[19\] and comments below). Let \( \hat{m} \in \hat{M} \). Since \( H_{\hat{m}} \simeq T_{\hat{m}}\hat{M}/T_{\hat{m}}\hat{F} \) and \( \nu \) is zero on horizontal vectors, the \( \hat{F} \)-basic top form \( \pi^*\Phi \wedge \nu \) is nonzero at \( \hat{m} \) is and only if \( \pi^*\Phi \) is. Since \( L \) is connected, we can assume that the subset \( \{ \hat{m} \in \hat{M} \mid (\pi^*\Phi)_{\hat{m}} \neq 0 \} \) is open, connected, and \( \hat{F} \)-saturated, since \( \pi^*\Phi \) is \( \hat{F} \)-basic and therefore \( \mathfrak{a} \)-invariant \[20\] Lem. 3.15]. The top form \( \eta \) is nonzero at \( w \in W \) if and only if \( w \) lies in the open, connected subset \( \rho(\{ \hat{m} \in \hat{M} \mid (\pi^*\Phi)_{\hat{m}} \neq 0 \}) \). It follows that \( \int_W \eta \neq 0 \).
Theorem 3.5. Let \((V, \mathcal{T})\) be a polytopal, balanced and odd triangulated vector configuration. Let \((N, \mathcal{F})\) be any foliated manifold built from \((V, \mathcal{T})\). Then the basic cohomology algebra \(H^*_\mathcal{F}(N)\) is generated by classes of degree two.

Proof. Fix \(r\) such that \(0 \leq r \leq d\). By Th. \[3.1\] \(\dim b^r_\mathcal{F} = h_r\). Moreover, from the proof of Th. \[3.1\] there are, in the shelling \(\mathcal{E}_1, \ldots, \mathcal{E}_{d+1}\), exactly \(h_r\) maximal simplices \(\mathcal{E}_{a_1}, \ldots, \mathcal{E}_{a_{h_r}}\) whose restrictions \(\tau_{a_j}\)'s have cardinality equal to \(r\).

For every \(j\), denote the basic Thom classes of \(N(\tau_{a_j})\) and \(N(\tau_{a_j})\) by \([\psi_j]\) and \([\psi_j]\), of degree \(2r\) and \(2d - 2r\) respectively.

We prove below that the classes \([\psi_j]\), \(j = 1, \ldots, h_r\) are linearly independent, and therefore give a basis of the vector space \(H^*_\mathcal{F}(N)\). Consider a linear combination \(\sum_{j=1}^{h_r} a_j[\psi_j] = 0\), with \(a_j \in \mathbb{R}\).

By Lem. \[3.3\] \((i)\), the cup product \([\psi_j] \cup [\psi_j]\) is zero when \(j > 1\) as these forms have disjoint supports, so the cup product of the above equality with \([\psi_1]\) gives \(a_1[\psi_1] \cup [\psi_1] = 0\).

The classical proof \[9\] can be adapted to show that given two transversely intersecting, smooth, compact, saturated, submanifolds, with oriented foliated normal bundles, the basic Thom class of their intersection is the cup product of their basic Thom classes. We know that \(\mathcal{F}\) is Riemannian and Killing (cf. \[2.3.2\]). Therefore, by Lem. \[3.3\] \((ii)\) and Prop. \[3.4\] \([\psi_j] \cup [\psi_1]\) is a nonzero generator of the top basic cohomology of \((M, \mathcal{F})\), which is one-dimensional by \[6\] below. Then \(a_1\) must be zero, and repeating this argument shows that all coefficients vanish.

Now, for each index \(i\) in \(\{1, \ldots, h\}\), the basic Thom class of \(N\{i\}\) is a basic class \([\tau_i]\) of degree 2 on \(N\). Remark that for each \(j\), \(N(\tau_{a_j})\) is the transverse intersection \(\cap_{i \in a_j} N\{i\}\) (an analogous idea is used in the proof of \[12, Prop. 3.10\] (ii)), we conclude that \([\psi_j]\) is a basic generator of \(H^*_\mathcal{F}(N)\). It follows that the algebra \(H^*_\mathcal{F}(N)\) is generated in degree 2.

4. EXAMPLES

4.1. The projective line \(\mathbb{CP}^1\) and variants. Let \(E = \mathbb{R}\). Consider the configuration \(V := (p, -q, q - p, 0)\), where \(p\) and \(q\) are positive reals. We have \(d = 1\) and \(n = 4\), so \(m = 1\). As noticed in \[2.2.3\] we can use an ambient automorphism to normalize \(V\) to \((\frac{p}{q}, -1, 1 - \frac{p}{q}, 0)\). Therefore, there is only one real parameter here, namely the fraction \(\frac{p}{q}\). We will distinguish the following cases

(a) \(\frac{p}{q} \in \mathbb{Q}\) with \(p, q\) coprime integers;
(b) \(\frac{p}{q} \in \mathbb{R} \setminus \mathbb{Q}\).

We triangulate \(V\) by \(\mathcal{T} = \{\mathcal{E}_1 = \{1\}, \mathcal{E}_2 = \{2\}, \emptyset\}\) —in particular \(v_3 = 1 - \frac{p}{q}\) and \(v_4 = 0\) are ghosts. Then the fan and quasilattice associated with \((V, \mathcal{T})\) are respectively: the one dimensional fan whose maximal cones are cone(\(\mathcal{E}_1\)) = \(\text{Span}_{\mathbb{Z}} v_1 = \mathbb{R}_{\geq 0}\) and cone(\(\mathcal{E}_2\)) = \(\text{Span}_{\mathbb{R}_{\geq 0}} v_2 = \mathbb{R}_{\leq 0}\), that is the usual fan of \(\mathbb{CP}^1\); the quasilattice \(Q = \text{Span}_{\mathbb{Z}} \{1, \frac{p}{q}\}\). In case (a) \(Q\) is \(\mathbb{Z}\). In case (b), \(Q\) is dense and has rank two. The virtual chamber is \(\mathcal{E} = \{\mathcal{E}_1 \setminus \{234\}, \mathcal{E}_2 \setminus \{134\}\}\)
and

\[ U(\mathcal{T}) = U_1 \cup U_2 = \left\{ [z] \in \mathbb{C}P^3 \mid z_2 \neq 0, z_3 \neq 0, z_4 \neq 0 \right\} \cup \left\{ [z] \in \mathbb{C}P^3 \mid z_1 \neq 0, z_3 \neq 0, z_4 \neq 0 \right\}. \]

Choose \( M = \begin{bmatrix} 1 & 1 & 0 \\ \frac{p}{q} & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \) so \( \Lambda^R = \begin{bmatrix} 1 & \frac{p}{q} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \) and \( \Lambda = \begin{bmatrix} 1 & \frac{p}{q} & 0 & i \end{bmatrix} \). It is now straightforward to write explicitly action \([1]\) and action \([2]\). The leaf \( \mathcal{F}_1 \) corresponding to the simplex \([1]\) is given by \( \mathbb{C}/\pi(L_1) \approx \mathbb{C}/\text{Span}_Z\{\frac{3}{p}, i\} \), while the leaf \( \mathcal{F}_2 \) corresponding to the simplex \([2]\) is given by \( \mathbb{C}/\pi(L_2) \approx \mathbb{C}/\text{Span}_Z\{1, i\} \). Now let \([z] \) be a generic point, i.e., \([z] \in N(\mathcal{Z}) \). Then \( (t, u) \in L_z \Leftrightarrow qt, pt, u \in \mathbb{Z} \). In case (a), \( L_z = \mathbb{Z}^2 \), so \( \mathcal{F}_z \approx \mathbb{C}/\text{Span}_Z\{1, i\} \).

In case (b), \( \text{Rank}(L_z) = 0 \), the generic leaf is \( \mathbb{C} \) and its closure in \( N \) is an \((S^1)^3\).

From the proof of Lemma 3.2, we know that \( \mathbb{C} \hookrightarrow U_1, z_1 \mapsto [z_1 : 1 : 1 : 1] \) gives a local slice for action \([2]\). This slice is stabilized by \( \frac{2}{p} \mathbb{Z} \times \mathbb{Z} \subset \mathbb{C}^2 \). Hence, \( X_1 := N_1/\mathcal{F} = U_1/\mathbb{C}^2 \) can be identified with \( \mathbb{C} \) modulo \( z_1 \mapsto e^{2\pi i\frac{p}{q} z_1} \). Similarly \( X_2 := N_2/\mathcal{F} = U_2/\mathbb{C}^2 \) can be identified with \( \mathbb{C} \) modulo \( z_2 \mapsto e^{2\pi i\frac{q}{t} z_2} \). The leaf space is then \( X = X_1 \cup X_2 \). In case (a), if \( p = q = 1 \) the leaf space \( X \) is \( \mathbb{C}P^1 \).

For \( p, q \) any coprime integers, the leaf space is a weighted projective space, that is the quotient of \( \mathbb{C}^2 \setminus \{0\} \) by the action of \( \mathbb{C}^* \) with weights \( q \) and \( p \).

In case (b), the local groups at the poles are infinite, of rank one. The leaf space is the toric quasifold described in detail in [10] Ex. 1.13,3,5 and [4] Ex. 2.6.

To describe \( \mathcal{F} \), and see how it desingularizes \( X \), compose the above \( \mathbb{C} \hookrightarrow U_1 \) with the quotient \( U_1 \to N_1 \). We obtain a slice \( \mathbb{C} \hookrightarrow N_1 \) for the action of \( \mathcal{F} \). The leaf passing through \( z_1 = 0 \) is the leaf \( \mathcal{F}_1 \) above; it intersects the slice only once. The leaf through any \( z_1 \neq 0 \) hits the slice \( p \) times in case (a) and infinitely many times in case (b). Hence it wraps around \( \mathcal{F}_1 \) \( p \) times or infinitely many times respectively.

4.2. The non-necessarily reduced orbifold \( \mathbb{C}P^1 \). Now we encode in a vector configuration the case of \( p \) and \( q \) non necessarily coprime, \( Q = \mathbb{Z} \). Choose \( V = (p, -q, 1, q - p - 1) \) and take \( M = \begin{bmatrix} 1 & q & 0 \\ p & 1 & 0 \\ 1 & 0 & q \\ 1 & 0 & 0 \end{bmatrix} \) so \( \Lambda^R = \begin{bmatrix} p & 0 & 0 \\ 0 & q & 0 \end{bmatrix} \) and \( \Lambda = \begin{bmatrix} q & p + i & 0 \\ p + i & q & 0 \end{bmatrix} \).

Action \([1]\) becomes \( \pi_i[z] = [e^{2\pi iqt}z_1 : e^{2\pi i(pt+u)}z_2 : e^{-2\pi iqt}z_3 : z_4] \) and action \([2]\) becomes \( (t, u), [z] = [e^{2\pi iqt}z_1 : e^{2\pi i(pt+u)}z_2 : e^{2\pi iqv}z_3 : z_4] \).

We take the same triangulation \( \mathcal{T} \) as above, so we can use the same slices, which are stabilized respectively by \( \{(t, u) = (\frac{k}{pq}, \frac{l}{q}) \mid k, l \in \mathbb{Z}\} \subset \mathbb{C}^2 \) and \( \{(t, u) = (\frac{k}{q}, \frac{l}{q}) \mid k, l \in \mathbb{Z}\} \subset \mathbb{C}^2 \). These groups act on the slices by \( (k, l), z_1 = e^{2\pi iqt}z_1 = e^{2\pi i\frac{k-l}{p}z_1} \) and \( (k, l), z_2 = e^{2\pi i(pt+u)}z_2 = e^{2\pi i\frac{k-l}{q}}z_2 \). The leaf space is
therefore an orbifold with singularities at the poles of order \( p, q \in \mathbb{Z}_{\geq 1} \). This is similar to [35] Ex. 5.3, although our construction does not involve the choice of a Kähler class (we give an interpretation of this extra piece of information in 4.5.2). In conclusion, in the rational case, one can prescribe at the poles orbifold singularities of arbitrary order \( p, q \in \mathbb{Z}_{\geq 1} \). Referring to Rem. 2.2, consider the case \( \gcd(p,q) = e > 1 \) and let \( a, b \in \mathbb{Z} \) such that \( ap + bq = e \). Then the leaf space is \( X = U'/(\Delta)/G \), where \( U'/(\Delta) = \mathbb{C}^2 \setminus \{0\} \) and \( G = \mathbb{C} \times \mathbb{Z}/e \mathbb{Z} \) acts on \( U'(\Delta) \) by \((t, [n]).(z_1, z_2) = (e^{2\pi i(qt + \frac{z}{n})}z_1, e^{2\pi i(pt - \frac{z}{n})}z_2)\).

4.3. The nonrational \( \mathbb{CP}^1 \). By a suitable choice of a vector configuration— in particular of the ghosts vectors— one can prescribe an arbitrary finitely generated subgroup \( A \) of the circle as local group at both poles of the corresponding toric quasifold. Assume without loss of generality that \( A \) is generated by \( e^{2\pi ir_1}, \ldots, e^{2\pi ir_{2m-1}} \) with \( r_j \in \mathbb{R} \). Let \( V = (1, -1, r_1, \ldots, r_{2m-1}, -r_1 - \cdots - r_{2m-1}) \). Then the quasilattice \( Q = \text{Span}_{\mathbb{Z}}\{1, r_1, \ldots, r_{2m-1}\} \). Keep \( T \) as above, so \( v_3 \ldots v_{2m+2} \) are ghosts. Take 

\[
M = \begin{bmatrix}
1 & 1 & -r_1 & \cdots & \cdots & -r_{2m-1} \\
1 & 0 & \cdots & \cdots & \cdots & 0 \\
1 & 1 & 0 & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & 0 \\
1 & 0 & 0 & \cdots & 0 & 1 \\
1 & 0 & 0 & \cdots & \cdots & 0
\end{bmatrix},
\]

so \( A^\mathbb{R} = \left( \begin{bmatrix} 1 \\ -r_1 \\ \vdots \\ -r_{2m-1} \end{bmatrix}, e_1, \ldots, e_{2m}, 0 \right) \), where \( e_1, \ldots, e_{2m} \) is the canonical basis of \( \mathbb{R}^{2m} \). The slice \( \mathbb{C} \hookrightarrow U_1, z_1 \hookrightarrow [z_1 : 1 : \cdots : 1] \) is stabilized by \( \mathbb{Z}^{2m-1} \subset \mathbb{C}^{2m} \), which acts by \( h(z_1) = e^{2\pi i(r_1 h_2 + \cdots + r_{2m-1} h_{2m})}z_1 \). At the other pole the local group is also \( A \), which acts on the corresponding slice in the same way.

4.4. Stanley’s proof in our setting.
Let \( \Delta \) be a polytopal simplicial fan. Denote by \( (h_0, \ldots, h_d) \) its \( h \)-vector (cf. Sect. 3.1). Define its \( g \)-vector \( (g_1, \ldots, g_b) \), with \( \delta = \left[ \frac{d}{2} \right] \), by \( g_j = h_j - h_{j-1}, j = 1 \ldots \delta \). Choose a triangulated vector configuration \( (V, F) \) whose associated fan is \( \Delta \), and a corresponding \( (N, F) \) (cf. Sect. 2.2).

In close analogy to [42], we show below that the combinatorial properties that characterize the \( g \)-vector of a simple polytope have a direct interpretation, and proof, in terms of the basic cohomology of \( (N, F) \).

We first remark that by Th. 3.1, \( F \) is homologically orientable, i.e.

\[
H^{2d}_F(N) \neq 0,
\]

which is equivalent to either \( H^{2d}_F(N) = 1 \), or to Poincaré duality for the basic cohomology of \( (N, F) \) [17, 23, 41].
4.4.1. **Dehn-Sommerville equations.** By our computation of the basic Betti numbers in Th. 3.1, the Dehn-Sommerville equations
\[ h_{d-j} = h_j \quad \text{for all } j \]
are only a restatement of basic Poincaré duality.
As is well-known, these relations follow more simply by computing the \( h_i \)'s using a shelling and the reverse shelling [50, 8.21].

4.4.2. **Nonnegativity of the \( g \)-vector.** The foliation \( \mathcal{F} \) is transversely Kähler by Loeb-Nicolau (for \( m = 1 \), in [31]) and Meersseman (for \( m \geq 1 \), in [34]). By (6) and [16, 3.4.7], \( H^\bullet_{\mathcal{F}}(N) \) has the hard Lefschetz property. In particular there exists an injective map \( L : H^2_{\mathcal{F}}(N) \rightarrow H^2_{\mathcal{F}}(N) \) for all \( j \leq \delta \). Therefore \( b_{2j}^\mathcal{F}(N) - b_{2j-2}^\mathcal{F}(N) \geq 0 \). By Th. 3.1, \( g_j = h_j - h_{j-1} \geq 0 \) for \( j = 1 \ldots \delta \).

4.4.3. **Bound on growth of \( g_j \).** The usual numerical condition (cf. [19, p.127 II(b)]) is equivalent to the existence of a graded commutative algebra \( R = R_0 \oplus R_1 \oplus \cdots \oplus R_\delta \) over the field \( K = R_0 \), generated by \( R_1 \), and such that \( g_j = \dim R_j \) for \( j = 1 \ldots \delta \). We take \( R_j := H^2_{\mathcal{F}}(N)/L(H^2_{\mathcal{F}}) \). The result follows from the hard Lefschetz property and Th. 3.5.

**Remark 4.1.** This shows that at least part of the technology available to toric varieties survives to the nonrational case. However, in order to prove Stanley’s result it is possible to bypass toric geometry: we refer the reader to [18] for the latest in a series of results initiated by McMullen [33] and continued by Timorin [45] and others.

4.5. **Brief account of the polytopal case.**

4.5.1. **Preliminaries.** An important special case is when \( \mathcal{T} \) is a regular triangulation. Regularity has several characterisations:

1. The triangulation is regular
2. The fan \( \Delta \) is polytopal
3. There exists a height function on \( V \) that induces \( \mathcal{T} \)
4. The virtual chamber defines a nonempty chamber, i.e., \( \bigcap_\alpha \check{\mathcal{P}}_\alpha \neq \emptyset \) (cf. [2.2.5])
5. There exists \( \nu \in \mathbb{R}^m \) s.t. \( \forall \tau \subset \{1 \ldots n\}, \tau \in \mathcal{T} \) if and only if \( \nu \) is in the interior of the convex hull of \( \left\{ \Lambda^R_j \mid j \in \tau^c \right\} \)

The last condition implies that, by definition, the corresponding manifold \( N \) is an LVM-manifold [34]. This in turn implies that the foliation \( \mathcal{F} \) is transversely Kähler by [31] (for \( m = 1 \)) and [34] (for \( m \geq 2 \)).

**Correspondence between regular triangulations and chambers.** Regular triangulations of \( V \) are in one-to-one correspondence with chambers of \( \Lambda^R \), i.e., bounded connected components of \( \mathbb{R}^{2m} - L \), where \( L \) is the union of all affine \( 2m-1 \)-planes determined by \( \Lambda^R_1, \ldots, \Lambda^R_n \). Explicitly: from \( \mathcal{T} \) we obtain the chamber \( \bigcap_\alpha \check{\mathcal{P}}_\alpha \); from a chamber \( C \), we define \( \mathcal{T} \) by \( \tau \in \mathcal{T} \Leftrightarrow \) the convex hull of \( \left\{ \Lambda^R_j \mid j \in \tau^c \right\} \) contains \( C \).
Correspondence between height functions and points in a chamber. A triangulation is regular when there exists a so-called height function $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{R}^n$ such that: there is a (necessarily unique) convex function $\psi_\omega : \mathbb{R}^d \to \mathbb{R}$, restricting to pairwise distinct linear forms on the maximal cones of $\Delta$, such that $\psi_\omega(v_i) = \omega_i$ for each non ghost vector $v_i$ and $\psi_\omega(v_i) < \omega_i$ for each ghost vector $v_i$.

By a result of Carl Lee [13, Lemma 5.4.4], $\omega$ induces a triangulation $T$ if and only if $\nu := \frac{1}{\omega_1} \sum \omega_i \Lambda_i^\mathbb{R}$ belongs to the chamber associated to $T$ as above. Therefore, conversely, starting from a point in a chamber written as a convex linear combination $\sum \omega_i \Lambda_i^\mathbb{R}$, we obtain a height function $\omega$ inducing a regular triangulation.

Conclusion: the map $\omega \mapsto \nu$ gives a quantitative refinement of the qualitative correspondence between regular triangulations and chambers described above.

4.5.2. Example. We choose the same data as in 4.2 (but here $p, q$ can be any positive reals): $V = (p, -q, 1, q - p - 1)$ and $\Lambda^\mathbb{R} = \begin{bmatrix} q & p & 0 & 0 \\ 0 & 1 & q & 0 \end{bmatrix}$.

To induce the triangulation $T = \{E_1 = \{1\}, E_2 = \{2\}, \emptyset\}$, we can choose, for example, $\omega_1 = p$, $\omega_2 = q$ (so $\psi_\omega = |.|$), $\omega_3 > 1$ and $\omega_4 > |q - p - 1|$:

which in turns gives the point $\nu = \frac{1}{\omega_1 + \omega_2 + \omega_3 + \omega_4}(\omega_1 \Lambda_1^\mathbb{R} + \omega_2 \Lambda_2^\mathbb{R} + \omega_3 \Lambda_3^\mathbb{R} + \omega_4 \Lambda_4^\mathbb{R})$ contained in one of the four chambers of the configuration $\Lambda^\mathbb{R}$.
Using \[34\], this point can be used to give a \(C^\infty\) embedding \(N \hookrightarrow \mathbb{CP}^{n-1}\) as

\[
N = \left\{ [z] \in \mathbb{CP}^{n-1} \mid \sum_{j=1}^{n} (\Lambda_j^R - \nu_j) |z_j|^2 = 0 \right\}.
\]

This solves a problem mentioned in \[35\] Rem. 4.11. Pulling-back the Fubini-Study Kähler form by this embedding endows \(N\) with a 2-form \(\phi\) transversely Kähler with respect to \(F\) \[35\]. The form \(\phi\) defines on \(X\) a Kähler form, whose moment polytope is \([- \frac{p+q+\omega_3+\omega_4}{p+q+\omega_3+\omega_4}, \frac{1}{p+q+\omega_3+\omega_4}]\).

**Concluding remarks**

- Using the above notation, notice that we can also interpret the choice of \(\nu\) on the boundary of a chamber: this corresponds to a non simplicial polyhedral decomposition of the vector configuration \(V\) (cf. \[13\]). We intend to use our methods to investigate this case.

- Starting from a rational triangulated vector configuration, we can rescale each vector to obtain a nonrational configuration, which is in a sense "weakly" nonrational (the fan is still rational). It would be interesting to characterize geometrically the class of foliated manifolds obtained from such configurations.

- In order to prove Th. 3.1, we assume that our simplicial fan is shellable. It is an open problem to decide if all simplicial fans are shellable (specialists we consulted lean toward a negative answer). It seems possible to prove Th. 3.1 in general by constructing an ad hoc spectral sequence.

- We expect the basic cohomology ring to have a similar description to the real cohomology ring of simplicial toric varieties.

- We conjecture that the basic Hodge numbers of these foliations are concentrated on the diagonal.

- Using the result of Th. 3.1, we hope to be able to prove the following result (conjectured in \[11\]): an LVMB-manifold is an LVM-manifold if and only if the foliation \(F\) is transversely Kähler.

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