Effect of gravitational field on collective motion of fish

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Abstract – Fish exhibit various patterns of collective motion, in which individual fish sense the gravitational field and tend to move horizontally. We study the effect of gravity on the collective patterns by incorporating suppression of vertical motion in an agent-based model. The gravitational factor induces a rotating cylinder which is a vertically and highly elongated form of a torus. The vortex axis becomes almost identical to the vertical axis even when the gravitational factor is weak compared to the interaction between fish. We also obtained a vertically elongated polarized school with high frontal density. Our results clarify the effect of gravity on the shape of clusters, individual-level motion, and mobility of the entire cluster.

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Introduction. – Collective motion is ubiquitously found in life of various organisms such as birds, fish, insects and mammals [1]. Patterns of highly coherent motion include polarized schools and rotating clusters. Fish exhibit a wide variety of rotating clusters such as tori, rings, balls, and cylinders [2,3]. They are especially interesting and non-trivial because the rotational symmetry is spontaneously broken by interaction between individual fish [4]. Theoretically, agent-based models [5–10] have been proposed and reproduced a torus and a ring. In addition, a ball-shaped rotating cluster (resembling a “bait-ball” [3]) is reproduced in a recent model by the present authors [11]. The model uses topological interaction [12] and some experimentally observed behaviors of fish, which facilitate formation of a giant rotating cluster.

In Nature, the vortex axis of a rotating cluster of fish is oriented almost vertically [13], presumably due to gravity. Actually, the otolith (“the ear stone”) allows the fish to perceive gravity and swim almost perpendicular to the water surface [14]. In the previous models [5,7,9,11], however, the direction of the vortex axis is not constrained in any particular direction. A systematic study of the effect of gravity on collective motion of fish has been lacking, although a previous study [15] introduced it as a fixed parameter and yielded a polarized school with high frontal density. Moreover, there are no previous models showing a rotating cylinder which is a vertically and highly elongated form of a torus [2]. We consider that constraints on the vertical motion of fish are important for the occurrence of a rotating cylinder.

In this letter, we introduce a gravitational factor into our previous model [11] and study its effect on collective motion. We find emergence of a giant rotating cylinder and a polarized school flattened along the direction of movement. We also study the effect of gravity on the individual-level motion and the mobility of the entire cluster.

Model. – We introduce the equation of motion of fish as

\[ \tau_0 \frac{dv_i}{dt} = (v_0 - |v_i|) \frac{v_i}{|v_i|} + \frac{1}{|L_i|} \sum_{j \in L_i} g(|r_{ij}|)(v_j - v_i) \]

\[ + \frac{1}{|A_i|} \sum_{j \in A_i} \left( v_o \frac{r_{ji}}{|r_{ji}|} - v_i \right) - \zeta v_z e_z, \]

where \( r_i \) and \( v_i = dr_i/dt \) (i = 1, 2, ..., N) are the position and velocity of the i-th agent, respectively, with \( r_{ij} = r_i - r_j \), \( v_z \) being the z-component of \( v_i \), and \( e_z \) is the unit vector along the vertical axis. The speed is set to relax to \( v_0 \) for an isolated agent, and the z-component of the velocity relaxes to zero with the time-scale \( \tau_0/\zeta \): the gravitational factor \( \zeta \) describes the sensitivity of fish to the gravitational field [15].

As for the interactions, there are reorientation, repulsion, and attraction with the neighbor agents, which are expressed by the second, third, and fourth term on the right-hand side of eq. (1), respectively. Firstly, the orientational and repulsive interactions act with up to \( N_u \)-th nearest neighbors in the radius \( r_e \), at which attraction
is balanced with repulsion [16]. \( N_u \) is called the interaction capacity. This reflects “topological interaction” which fixes the number of neighbors to interact with [12], and we estimate \( N_u \) to be a few from typical distributions of fish [17]. The set of agents that can interact with the \( i \)-th agent by these interactions is denoted by \( A_i \), and the number of them by \( |A_i| \). \( v_r \) is the speed of collision avoidance, and the function

\[
g(|r|) = \begin{cases} \frac{r_b}{|r|}, & |r| \leq r_b, \\ 1, & |r| > r_b \end{cases}
\]  

expresses the excluded volume effect (where \( r_b \) is the body length). Note that an experiment shows that fish also have orientational interaction [18].

The attractive interaction operates at distances between \( r_c \) and \( r_e \), and the set of agents that can attract the \( i \)-th agent is denoted by \( A_i, = \{j| r_c < |r_{ij}| \leq r_e \} \) with its size \( |A_i| \). \( v_a \) is the maximum speed reached by fast-start, which is an acceleration mode of fish to escape from predators. The time-dependent strength of attraction \( \lambda_i(t) \) is introduced to model the screening of attraction above \( r_c \) in a cluster [16] and the duration of attraction by fast-start [19]. In other words, we model this by turning on attraction when a fish is located outside a cluster and has only a small number of neighbors within \( r_c \). The attraction lasts for duration \( \tau \) and helps the fish unite with the cluster, which we regard equivalent to escaping from a (hypothetical) predator. The attraction is turned on \( \lambda_i(t) = \lambda \) at the moment when \( |L_i(t)| \) becomes smaller than \( N_u \), and lasts for the duration \( \tau \). If \( |L_i(t)| = N_u \) after the period \( \tau \), the attraction is switched off \( \lambda_i(t) = 0 \). Otherwise, the attraction is maintained for another period \( \tau \), and this will be repeated until we finally get \( |L_i(t)| = N_u \).

In our simulation, all lengths and time are rescaled by the radius of equilibrium \( r_e = 1.5 \) BL (body length) and the characteristic timescale \( \tau_0 = 1 \) s, respectively. We estimated \( \tau_0 \) from the ratio of the typical acceleration and velocity of some species of fish [20,21]. We fixed the parameter values \( N = 3000, r_c = 1, \tau_0 = 1, r_b = 2/3, r_a = 5, v_0 = 1, v_r = 1, v_a = 5, \tau = 0.1 \), and regarded \( N_u, \lambda, \zeta \) as the control parameters. We focus on the parameter region \( 1 \leq N_u \leq 3 \) and \( 5 \leq \lambda \leq 9.0 \) where rotating clusters of various shapes emerge in the absence of gravity [11]. (See ref. [11] for further details of the model and the choice of parameter values.)

We used the Runge–Kutta method for numerical integration of the equation of motion with the time step \( dt = 0.005 \). The simulation box is a cuboid whose dimensions are \( L = 40 \) in the \( x \) and \( y \) directions and \( L_z = 50 \) in the \( z \)-direction with the periodic boundary condition. Unless otherwise stated, we use the initial condition in which the agents are randomly distributed in a single spherical cluster aligned in the same \( (x-) \) direction and speed \( v_0 \). The radius of the sphere was chosen for each set of \( (N_u, \lambda) \) so that a single cluster is obtained and maintained for \( \zeta = 0 \). The same initial radius was used also for \( \zeta > 0 \), for which a single-cluster state is maintained except for rare cases. This initial condition helps reducing the computational cost, while it was checked that the final (dynamical steady) states do not depend on the initial conditions [11].

**Results:** cluster shapes. – Figure 1 shows typical snapshots at the end of the simulations \( (t = 3000) \) when the clusters reach dynamically steady states. Figures 1(a)–(d) demonstrate that rotating clusters are obtained for \( N_u = 1 \) and become elongated as \( \zeta \) increases; a ball-shaped cluster changes to a rotating cylinder. On the other hand, when \( N_u = 3 \), a torus-shaped rotating cluster is transformed into a polarized school in which the agents have almost the same direction, and then the cluster is elongated as \( \zeta \) increases. The time evolution of the clusters are shown in the Supplementary Movie Movie1.gif \((N_u = 1, \lambda = 9.0, \zeta = 2.0)\) and the Supplementary Movie Movie2.gif \((N_u = 3, \lambda = 9.0, \zeta = 2.0)\). We checked that the same patterns emerge with the initial condition in which the agents have randomly distributed positions and directions: see Supplementary Movies Movie3.gif, Movie4.gif.

In order to distinguish between a rotating cluster and a polarized school, we use the polar order parameter \( P \) and the rotational order parameter \( M \). They are defined by

\[
P(t) = \frac{1}{N} \sum_{i=1}^{N} \frac{v_i(t)}{|v_i(t)|}, \quad M(t) = \frac{1}{N} \sum_{i=1}^{N} \frac{c_i(t) \times v_i(t)}{|c_i(t)||v_i(t)|},
\]

where \( c_i = r_i - r_G \) with \( r_G \) being the center of mass position of the cluster. The magnitude of the polar order parameter \( P(t) \) becomes unity for a completely aligned polarized school, and is (almost) zero for random swarming states and rotating clusters with a cylindrical symmetry.

On the other hand, the magnitude of the rotational order parameter \( M(t) \) becomes unity for a ring-shaped rotating cluster, and is zero for a completely ordered polarized school with central symmetry to center of mass. For a cylindrical rotating cluster, \( M \) is a decreasing function of the aspect ratio, and is calculated in appendix A. We define a rotating cluster by the criterion \( \bar{M} > \bar{P} \). Here and hereafter, \( \bar{X} \) is the time-average of any quantity \( X(t) \) over the time interval \([2500, 3000]\). The probability of emergence of a rotating cluster \( p_m \) is defined by the number of cases where \( \bar{M} > \bar{P} \) divided by the number of samples \( (n = 100) \) in which a single cluster is maintained until the end of a simulation \( (t = 3000) \). We plot \( p_m \) as a function of \( \zeta \) in fig. 2. When \( N_u = 2 \) or 3, \( p_m \) decreases around \( \zeta \sim 0.1 \) (fig. 2 inset), and becomes zero for larger \( \zeta \) (i.e., the cluster changes to a polarized school from a ring-, a torus-, and a ball-shaped rotating cluster) as shown in fig. 2. Dominance of rotating clusters \( (p_m = 1) \) is maintained up to larger values of \( \zeta \) as \( N_u \) is smaller or \( \lambda \) is larger. On the other hand, when \( N_u = 1, p_m \) tends to decrease once, but then increases again and reaches 1 as...
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Fig. 1: Snapshots of clusters at the end of simulations ($t = 3000$). Agents are represented by arrows of length $2/3 = 1$ BL, and the color corresponds to the moving direction of the agent $\mathbf{v}/|\mathbf{v}| = (v_x/|\mathbf{v}|, v_y/|\mathbf{v}|, v_z/|\mathbf{v}|)$ according to the color sphere (h). Only a part of the simulation box that contains the agents is shown and the origin is shifted for visibility. A top view is shown in the upper right corner of each snapshot. (a)–(d) Rotating clusters for $N_u = 1, \lambda = 9.0$ with $\zeta = 0.0, 0.5, 1.0, 2.0$. (e)–(g) A rotating cluster and polarized schools for $N_u = 3, \lambda = 9.0$.

Fig. 2: The probability of emergence of rotating clusters $p_m$ as a function of $\zeta$. See the inset for the narrow range $\zeta \in [0.0, 0.2]$ and $N_u = 2, 3$.

$\zeta$ is increased (fig. 2). For $\lambda = 9.0$, $p_m = 1$ is maintained for any $\zeta$.

Next, we define some quantities that characterize the size and shape of the clusters. The outer radius of the cluster is defined as

$$R_o(t) = \max_{i=1, \ldots, N} |c_i^\perp(t)|,$$  \hspace{1cm} (4)

where the radial position $c_i^\perp$ is the projection of $c_i$ onto a vector which is perpendicular to $M(t)$ and $c_i \times M(t)$. For a rotating cluster, we define the spatial distribution of the agents $\rho$ and occupancy ratio $\eta$ as functions of the radial distance $c^\perp$ as

$$\rho(c^\perp, t) = \sum_{i=1}^{N} \delta(c_i^\perp(t) - c^\perp),$$  \hspace{1cm} (5)

$$\eta(c^\perp, t) = \sum_{i=1}^{N} \eta_i(t) \delta(c_i^\perp(t) - c^\perp),$$  \hspace{1cm} (6)

where $c_i^\perp = |c_i^\perp|$ and $\eta_i(t) = |\mathcal{L}_i(t)|/N_u$. The cluster shape is characterized by the moment of inertia tensor $I_{\mu\nu}(t) = \sum_{i=1}^{N} [c_i^2(t) \delta_{\mu\nu} - c_i\mu(t)c_i\nu(t)]$, where $\mu, \nu = x, y, z$ and $\delta_{\mu\nu}$ is Kronecker delta. Diagonalizing it and using its eigenvectors $u_\alpha(t)$ ($\alpha = 1, 2, 3$), with the principal moments of inertia satisfying $I_1(t) \geq I_2(t) \geq I_3(t)$, we define the cluster sizes by

$$S_\alpha(t) = \max_{i=1, \ldots, N} |c_i(t) \cdot u_\alpha(t)|.$$  \hspace{1cm} (7)
time-averaged cluster sizes \( N = \hat{m} \) as functions of \( \zeta \), for (a) \( N_u = 1, \lambda = 9.0 \) and (b) \( N_u = 3, \lambda = 9.0 \). An ensemble average over 100 simulations is taken for each point, and the error bars represent the standard deviation. In (a), the grey lines with the squares and circles show the calculated values for a cylindrical rotating cluster. (See appendix A for the definitions of \( \hat{m} \) and \( \hat{n} \).) These results quantitatively show that the cluster has a giant rotating cylinder for \( N_u = 1, \lambda = 9.0 \), and \( \zeta = 0.1 \).

Note that \( S_1 \) and \( S_3 \) correspond to the semi-minor axis and the semi-major axis for a rod-shaped cluster, and to the half height and the outer radius for a torus-shaped cluster, respectively.

Figure 3(a) shows that, for \( N_u = 1, \lambda = 9.0 \) (which correspond to fig. 1(a)–(d)), \( \hat{m} \) monotonically decreases while \( S_3 \) starts to increase at \( \zeta \sim 0.3 \) and is deviated from \( S_1 \). The height of the cluster (= \( 2S_3 \)) reaches 28 at the maximum, which is much larger than the interaction range \( r_u = 5 \). The order parameter \( \bar{\rho} \) takes very small values (< 0.1) for any \( \zeta \), while \( \hat{m} \) starts at a larger value \( \sim 0.6 \) and decreases from \( \zeta \sim 0.3 \). The decrease in \( \hat{m} \) is well described by the calculated value \( \hat{m} = w(v_x, v_y) \Psi(S_1/S_3) \) for a cylindrical rotating cluster. (See appendix A for the definitions of \( w \) and \( \Psi \).) These results quantitatively show that the cluster has a giant rotating cylinder for \( N_u = 1, \lambda = 9.0 \), and \( \zeta = 2.0 \).

Figure 3(b) shows that, when \( N_u = 3, \lambda = 9.0 \) (which correspond to polarized schools in fig. 1(f)–(g)), \( \bar{\rho} \) keeps a large constant value \( \sim 0.9 \). The aspect ratio \( S_1/S_2 \) increases sharply with \( \zeta \) and is close to two at \( \zeta = 2 \). The inequality \( S_1 > S_2 > S_3 \) holds for any \( \zeta \) showing that the cluster shape is biaxially distorted.

In fig. 3(c), we show the occupancy ratio \( \hat{n}(c^+) \) and normalized distribution \( \tilde{\rho}(c^+) \), where \( \sim \) means time-averaging over the interval \( t \in [1500, 21500] \). For rotating clusters and for any \( N_u, \hat{n}(c^+) \) is less than 1 only near the outer surface of the cluster, which means that only the fish near the surface get attraction. The density \( \tilde{\rho} \) is low near the surface region. Shown in the inset of fig. 3(c) is the lateral area of the cluster \( A = 2\pi R_o \times S_3 \) for \( N_u = 1 \). It is an increasing function of \( \zeta \), which is due to vertical elongation.
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For polarized schools, we also measured the spatial distribution of the agents $f(\rho, t)$ on a vertical cross-section of the cluster. Here, $\rho = (x_P, z)$ is the position vector in the plane containing $P(t)$ and the vertical axis. See appendix B for details of the definition of $f(\rho, t)$. We plot the time-averaged and normalized distribution $\tilde{f}(\rho)$ in fig. 4. For the standard parameter set ($N_u = 3$, $\lambda = 9.0$), the cluster is elongated vertically and then becomes wing-shaped as $\zeta$ increases; see fig. 4(a) and (b). Moreover, the density is larger in the frontal part of the cluster in these cases. On the other hand, when the attractive interaction is weak ($\lambda = 0.5$), we obtained a horizontally elongated cluster with uniform density (fig. 4(c)). From fig. 4(d), we find that the vertical alignment (purple to green) of the particles persists for a long time, but the horizontal alignment (dark to light color) is mixed.

Results: diffusion in the vertical direction. – To clarify the effect of gravity on the individual-level motion in a rotating cylinder, we study the mean square displacement (MSD) in the $z$-direction,

$$\Delta_z(\delta t) = \frac{1}{N} \sum_{i=1}^{N} (c_{z,i}(t_0 + \delta t) - c_{z,i}(t_0))^2, \quad (8)$$

where $\delta t > 0$. Figure 5(a) shows that the MSD increases in proportion to $\delta t^2$ until $\delta t \sim 1$, and then increases in proportion to $\delta t$ until $\delta t = 10^3 - 10^4$, and finally takes a constant value. In other words, an agent performs a ballistic motion on short timescale $\delta t \sim \tau_0 (= 1)$ and Brownian motion on medium timescale $\delta t \gtrsim 10^3 - 10^4$, before reaching the end of the cluster.

We define the vertical diffusion coefficient $D_z$ and the exponent $\gamma$ to quantify the Brownian motion: $\Delta_z(\delta t) = 2 \gamma D_z \delta t^\gamma$. As shown in the inset of fig. 5(a), agents perform almost normal diffusion ($\gamma = 1$) over a wide range of $\zeta$, except the weak superdiffusion ($\gamma \simeq 1.1$) for $\zeta = 0.1$. For $\zeta \gtrsim 0.5$, $\gamma$ slightly and monotonically increases with $\zeta$. This is related with a positive correlation between $\gamma$ and the side area $A$ (data not shown.). The diffusion coefficient $D_z$ decreases as $\zeta$ increases as seen in fig. 5(c). In addition, $D_z$ increases with the strength of attraction $\lambda$.

Results: fluctuation of the vortex axis. – We measure the deviation of the vortex axis from the $z$-axis to see the effect of gravity on the mobility of the entire cluster. The angle of the vortex axis and the vertical axis is given as shown in fig. 6(a), $\Theta(t) = \arccos(M(t) \cdot e_z) \in [0, \pi]$, which we transform to the normalized angle

$$\Theta_z(t) = \frac{\theta_z(t)}{\pi} - \frac{1}{2} \in [-0.5, 0.5]. \quad (9)$$

As shown in fig. 6(a), $\Theta_z(t)$ shows random motion on a long timescale for $\zeta = 0.0001$, while it almost converges to $\pm 0.5$ for $\zeta \geq 0.01$. For the intermediate value $\zeta = 0.0007$, we find flip-flops of the vortex axis between $\Theta_z \simeq 0.5$ and $-0.5$.

In order to distinguish between the random motion and the flip-flop, we measure long time-averaged $\langle|\Theta_z(t)|\rangle$, instead of $\langle\Theta_z(t)\rangle$, to quantify the mobility of the vortex axis, where $\langle \cdot \rangle$ means time-average over the interval $t \in [1500, 201500]$. Figure 6(b) shows that $\langle|\Theta_z(t)|\rangle$ increases with $\zeta$, from $\Theta_0 = \frac{1}{2} - \frac{1}{2} \simeq 1.82$ (isotropic limit) to 0.5 (vertical limit), for both $N_u = 1$ and $N_u = 3$. Note that $\Theta_0 \equiv \int_{-1}^{1} d(\cos \theta_z) P(\cos \theta_z) |\Theta_z|$, where $P(\cos \theta_z) = 1/2$ is the probability distribution of $\cos \theta_z$, means complete random motion of the vortex axis.

Discussions. – Now let us discuss some characteristic features of the collective patterns induced by the gravity. The giant rotating cylinder that emerges for $N_u = 1$ is well approximated by a cylindrical rotating cluster in terms of the rotational order parameter. On the other hand, when $N_u = 2$ and 3, an increase in $\zeta$ tends to prevent formation of a rotating cluster and instead promotes formation of a polarized school. A large $N_u$ means that each fish...
interacts with many neighbors and that a local group of fish moves cohesively. To rotate as a whole, many fish must change their moving directions simultaneously. In contrast, for $N_u = 1$, each fish sensitively follows the motion of the nearest neighbor, and change of the moving direction propagates from one fish to another. Therefore, a rotating cluster emerges more easily for smaller $N_u$. Increase of $\zeta$ causes strong damping of the vertical velocity component and relative increase of the horizontal velocity. It strengthens horizontal alignment via orientational interaction and tends to induce a polarized school. However, for $N_u = 1$, further increase of $\zeta$ causes recovery of rotating clusters (fig. 2). It is explained by increase of the lateral area of the cluster (fig. 3(c), inset). Because a rotating cluster is induced by a rapid change of moving direction by attractive interaction, and because only the fish near the cluster surface feel attraction, the increased lateral area tends to induce rotating clusters.

The polarized school obtained for strong attraction ($\lambda = 9.0$) has a high frontal density. This is in agreement with the previous result [15], but the mechanism could be different. In our model, attractive forces act on the agents near the surface of a cluster [11], but not on those inside the bulk. The attraction slows down the agents in the front, while the agents behind them keep moving forward and get jammed at the frontal part. In fact, the density becomes uniform for weak attraction ($\lambda = 0.5$, fig. 4(c)). Thus the high frontal density in our model is explained by autonomous control of attraction, which is not considered in previous models. In ref. [15], the high frontal density is explained by combination of attractive forces and a blind angle. Another interesting result is that the polarized school is elongated vertically for strong attraction and horizontally for weak attraction. This is explained as follows. When the attraction is weak, repulsive forces cause spreading of agents in the moving direction, as in the case without gravity [11]. On the other hand, when the attraction dominates, the cluster is compressed horizontally and the agents are pushed away toward the vertical direction. The agents that are displaced vertically are reoriented toward the horizontal direction by the gravitational factor, which prevents them to return to the original position. Thus the cluster is more elongated vertically for larger $\zeta$. The previous work [15] simulated a school of population less than 2000, and obtained horizontally elongated clusters, which was explained by sideward orientation. In experiments, most of the schools in tanks or with small populations are elongated horizontally [17,22], while vertically elongated schools are formed by more than several thousand anchovies (Engraulis mordax) in deep water [23].

We have checked that reducing the number of agents to $N = 300$ in our model does not significantly change the shape anisotropy (data not shown). The effective strength of attraction might depend on species and environment, and could possibly explain the different shape anisotropy in the experiments. However, it is beyond the scope of the present work and is left for future study.

The MSD in a rotating cylinder shows the typical time evolution of Brownian motion in a finite domain (see fig. 5(a)). The gravity suppresses vertical motion and therefore the diffusion constant $D_z$ becomes a decreasing function of $\zeta$ (see fig. 5(c)). The weak superdiffusion found for $\zeta = 0.1$ is probably due to fluctuation of the vortex axis from the $z$-axis, which has a long correlation time and contributes to the vertical MSD.

The vortex axis of the rotating cluster matches the $z$-axis for relatively small $\zeta$ (of the order of 0.01). This indicates that, if the timescale of the reaction to gravitational field $\tau_0/\zeta$ is sufficiently longer than the characteristic timescale $\tau_0$, the entire cluster shows macroscopic effects of the gravitational field, resulting in the vertical alignment of the vortex axis as experimentally observed [13]. A smaller $N_u$ allows an agent to move more flexibly, resulting in larger deviation of the vortex axis from the $z$-axis (see fig. 6(b)).

Finally, there are some aspects to be addressed in the future. Introducing a blind angle into our model might bring about rotating clusters in a wider range of $\zeta$, as found in [5,8]. On the other hand, a noise term will bring about a swarming (random) state as shown in [10]. Moreover,
three-body interaction acts as a restitution force from two neighbors [16]. It might facilitate periodic alignment of fish and formation of clusters undergoing rigid-body rotation. Heterogeneity of fish will be another factor that affects cluster formation [24]. Hydrodynamic interactions tend to induce a wider variety of patterns [9,10] while it is left as a theoretical challenge to incorporate flow at high Reynolds number.

Data availability statement: The data that support the findings of this study are available upon reasonable request from the authors.

Appendix A: the rotational parameter of a cylindrical rotating cluster. – The rotational parameter of a cylindrical rotating cluster of height $h$, radius $a$ and uniform density is calculated as follows. We assume that the agents move horizontally with the same radial and tangential velocities $v_r$ and $v_\phi$. Regarding the position $c = re_x + ze_z$ and the velocity $v = v_xe_x + v_\phi e_\phi$ of the agents as field variables, where $e_x$ and $e_\phi$ are the radial and tangential unit vectors, respectively, we obtain

$$ M = \frac{1}{\pi a^2 h} \int_{cylinder} dV \frac{c \times v}{|c||v|} = w(v_r, v_\phi) \Psi \left( \frac{h}{2a} \right), \quad \text{(A.1)} $$

where

$$ \Psi(\beta) = \frac{1}{3\beta} \left\{ \beta \sqrt{1 + \beta^2} + 2 \ln \left( \beta + \sqrt{1 + \beta^2} \right) + \beta^3 \left[ \ln \beta - \ln \left( 1 + \sqrt{1 + \beta^2} \right) \right] \right\}, \quad \text{(A.2)} $$

and $w(v_r, v_\phi) = |v_\phi|/\sqrt{v_r^2 + v_\phi^2}$. In the numerical analysis, we replaced $w(v_r, v_\phi)$ by its average obtained from the simulation and $h$ and $a$ by $25h$ and $2a$, respectively. We also plotted the case $w(v_r, v_\phi) = 1$ (i.e., $v_r = 0$) in fig. 3.

Appendix B: definition of the spatial distribution function. – First we define the moving orthogonal frame $(x_P, y_P, z)$ so that the polar order parameter $P(t)$ is contained in the $x_Pz$-plane. The unit vectors $e_x(P)(t)$ and $e_y(P)(t)$ along the $x_P$- and $y_P$-axis are obtained by normalization of $(I - e_z \otimes e_z) \cdot P(t)$, and $e_y(P)(t) = e_z \times e_x(P)(t)$, respectively. The spatial distribution on the $(x_P, z)$ plane is defined by

$$ f(\rho, t) = \sum_{i=1}^{N} H \left( \frac{r_c}{2} - |e_y(P)(t) \cdot c_i(t)| \right) \delta(\rho - \rho_i(t)), \quad \text{(B.1)} $$

where $H(\cdot)$ is the Heaviside step function, $\rho = (x_P, z)$, and $\rho_i(t) = (I - e_z \otimes e_z)(t) \cdot e_y(P)(t) \cdot c_i(t)$. Here we took into account the agents in a slab of thickness $r_c$ around the vertical center plane.

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