SOME UNITED EXISTENCE RESULTS OF PERIODIC SOLUTIONS FOR NON-QUADRATIC SECOND ORDER HAMILTONIAN SYSTEMS

XINGYONG ZHANG

Department of Mathematics, Faculty of Science, Kunming University of Science and Technology, Kunming, Yunnan 650500, P.R. China

XIANHUA TANG

School of Mathematical Sciences and Computing Technology, Central South University, Changsha, Hunan 410083, P.R. China

Abstract. In this paper, some existence theorems are obtained for periodic solutions of second order Hamiltonian systems under non-quadratic conditions by using the minimax principle. Our results unite, extend and improve those relative works in some known literature.

1. Introduction and main results. Consider the second order Hamiltonian system

\[ \ddot{u}(t) + A(t)u(t) + \nabla F(t, u(t)) = 0, \quad \text{a.e. } t \in \mathbb{R}, \]

where \( A(t) \) is an \( N \times N \) real symmetric matrix, and is continuous and \( T \)-periodic in \( t \), and \( F : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R} \) is \( T \)-periodic in \( t \) and satisfies the following assumption:

(A) \( F(t, x) \) is measurable in \( t \) for every \( x \in \mathbb{R}^N \) and continuously differentiable in \( x \) for a.e. \( t \in [0, T] \), and there exist \( a \in C(\mathbb{R}^+, \mathbb{R}^+) \) and \( b \in L^1(0, T; \mathbb{R}^+) \) such that

\[ |F(t, x)| \leq a(|x|)b(t), \quad |\nabla F(t, x)| \leq a(|x|)b(t) \]

for all \( x \in \mathbb{R}^N \) and a.e. \( t \in [0, T] \).

When \( A(t) \equiv 0 \), system (1.1) reduces to the following second order Hamiltonian system

\[ \ddot{u}(t) + \nabla F(t, u(t)) = 0, \quad \text{a.e. } t \in \mathbb{R}. \]  

Under the subquadratic condition, there have been many existence results for system (2) (for example, see [1], [14], [20] and references therein). In 1978, Rabinowitz [17] obtained the following result under the superquadratic condition:

Theorem 1.1 (see [17]). Suppose that \( F \) satisfies the following conditions:

(F1) \( F(t, x) \geq 0, \quad \forall (t, x) \in [0, T] \times \mathbb{R}^N \);

(F2) \( F(t, x) = o(|x|^2) \) as \( |x| \to 0 \) uniformly with respect to \( t \in [0, T] \);

(F3) there exist \( \mu > 2 \) and \( L > 0 \) such that

\[ 0 < \mu F(t, x) \leq |\nabla F(t, x), x|, \quad \forall |x| \geq L, \quad \text{a.e. } t \in [0, T]. \]

Then system (2) has at least one nonconstant \( T \)-periodic solution.

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The condition (F3) has been used extensively in the literature (see [2], [4], [5], [6], [7], [13], [16] and references therein). In 2002, Fei [8] obtained the existence of periodic solutions for system (2) under a kind of new superquadratic condition which is different from (F3). Subsequently, Tao and Tang [21] gave the following result which is more general than Fei’s:

**Theorem 1.2** (see [21], Theorem 1). Suppose that $F$ satisfies (F1) and the following conditions:

(F4) \[
\lim_{|x|\to 0} \frac{F(t,x)}{|x|^2} < \frac{2\pi^2}{T^2} < \lim_{|x|\to \infty} \frac{F(t,x)}{|x|^2} \quad \text{uniformly for a.e. } t \in [0,T];
\]

(F5) there exists $\theta > 2$ such that
\[
\limsup_{|x|\to \infty} \frac{F(t,x)}{|x|^\theta} < \infty \quad \text{uniformly for a.e. } t \in [0,T];
\]

(F6) there exists $\mu > \theta - 2$ such that
\[
\liminf_{|x|\to \infty} \frac{(\nabla F(t,x),x) - 2F(t,x)}{|x|^\mu} > 0 \quad \text{uniformly for a.e. } t \in [0,T].
\]

Then system (2) has at least one nonconstant $T$-periodic solution.

In 2006, Schechter [18] also obtained some new existence results of nonconstant periodic solutions for system (2). We will not present those theorems here.

There are also lots of results on the existence of periodic solutions of system (1) under the superquadratic condition (see [12], [13], [24] and references therein). In 1995, Li and Willem [13] established so-called abstract local linking theorem and applied it to study the existence of nontrivial periodic solutions for system (1) under condition (F3). In detail, they obtained the following result:

**Theorem 1.3** (see [13], Theorem 7). Suppose that $F$ satisfies (F2) and the following conditions:

(F7) \[
\lim_{|x|\to \infty} \frac{F(t,x)}{|x|^2} = +\infty \quad \text{uniformly for a.e. } t \in [0,T];
\]

then system (1) has at least one nontrivial $T$-periodic solution.

In 2005, Luan and Mao [11] improved the abstract local linking theorem obtained by Li and Willem [13] and in [12], they also applied it to study the existence of nontrivial periodic solutions for system (1). They obtained the following different result:

**Theorem 1.4** (see [12], Theorem A). Suppose that $F$ satisfies (F2) and the following conditions:

(F8) \[
\lim_{|x|\to \infty} \frac{F(t,x)}{|x|^2} = +\infty \quad \text{uniformly for a.e. } t \in [0,T];
\]

then system (1) has at least one nontrivial $T$-periodic solution.
(F8) there exist positive constants $a_1$, $a_2$ and $R$ such that
\[
\frac{1}{2}(\nabla F(t, x), x) - F(t, x) \geq a_1|x|^2, \quad \text{if } |x| \geq R,
\]
\[
\frac{1}{2}(\nabla F(t, x), x) - F(t, x) \geq a_2\frac{\nabla F(t, x)^\sigma}{|x|^\sigma}, \quad \text{if } |x| \geq R,
\]
where $\sigma > 1$. Then the conclusions (1) and (2) in Theorem 1.3 hold.

In 2009, Ye and Tang [24] obtained the following result which is different from Theorem 1.3 and Theorem 1.4:

**Theorem 1.5** (see [24], Theorem 2). Suppose that $F$ satisfies (F2), (F5), (F6) and (F7). Then the conclusions (1) and (2) in Theorem 1.3 hold.

In 2010, in [10], Kyritsia and Papageorgiou also obtained a different result by using Morse theory and a result of Perera [15].

**Theorem 1.6** (see [10], Theorem 3.7). Suppose that $F$ satisfies (F5), (F7) and the following conditions:

(F9) there exists $\mu > \theta - 1$ such that (3) holds;

(F10) there exists $g \in L^1(0, T; \mathbb{R}^+)\setminus\{0\}$ such that
\[
\liminf_{|x| \to 0} \frac{F(t, x)}{|x|^{\mu}} > \frac{g(t)}{2} \quad \text{uniformly for a.e. } t \in [0, T];
\]

(F11) there exists $\delta_0 > 0$ such that
\[
F(t, x) \leq \lambda_2|x|^2, \quad \text{for } |x| \leq \delta_0, \text{ a.e. } t \in [0, T],
\]
where $\lambda$ is the smallest positive eigenvalue of $-d^2/dt^2 - \hat{A}$ (with periodic boundary condition). Then system (1) has one nontrivial $T$-periodic solution.

**Remark 1.** (F11) is weaker than (F2). However, (F9) and (F10) are stronger than (F6) and (4), respectively. These facts show that Theorem 1.6 is different from Theorem 1.3–Theorem 1.5.

Moreover, in 2002, a noteworthy result was given in [22], which partially inspired us.

**Theorem 1.7** (see [22], Theorem 3). Suppose that $F$ satisfies (F5), (F6) and the following condition:

(F12) there exist consecutive eigenvalues $\lambda_k$ and $\lambda_{k+1}$ of $-d^2/dt^2 - \hat{A}$ (with periodic boundary condition) such that
\[
\lim_{|x| \to 0} \frac{F(t, x)}{|x|^2} < \frac{\lambda_{k+1}}{2} < \lim_{|x| \to +\infty} \frac{F(t, x)}{|x|^2},
\]
and
\[
\frac{\lambda_k}{2} |x|^2 \leq F(t, x) \quad \text{for a.e. } t \in [0, T] \text{ and all } x \in \mathbb{R}^N.
\]

Then system (1) has one nontrivial $T$-periodic solution.

Based on these works, in this paper, we will present several different results. For the sake of convenience, denote $\hat{L}$ by the operator $-d^2/dt^2 - \hat{A} : H^1_T \to H^1_T$ (with periodic boundary condition), where
\[
H^1_T = \{ u : \mathbb{R} \to \mathbb{R}^N | u \text{ is absolutely continuous}, \quad u(t) = u(t + T) \text{ and } \dot{u} \in L^2(0, T; \mathbb{R}^N) \}.
\]
Theorem 1.8. Suppose that $F$ satisfies (A), (F7) and the following conditions:

(H1) there exist positive constants $m, \zeta, \eta$ and $\nu \in [0, 2)$ such that
$$
\left(2 + \frac{1}{\zeta + \eta|x|^{\nu}}\right) F(t, x) \leq (\nabla F(t, x), x), \quad \forall \, x \in \mathbb{R}^N, \; |x| > m, \; \text{a.e. } t \in [0, T];
$$

(H2) there exist consecutive eigenvalues $\lambda_k$ and $\lambda_{k+1}$ of $\hat{L}$ with $\lambda_k < \lambda_{k+1}$ and $l_0 > 0$ such that
$$
\frac{\lambda_k}{2}|x|^2 \leq F(t, x) \leq \frac{\lambda_{k+1}}{2}|x|^2 \quad \text{for a.e. } t \in [0, T] \quad \text{and all } |x| \leq l_0.
$$

Then system (1) has at least one nontrivial $T$-periodic solution in $H^1_T$.

Moreover, when the eigenvalues locate near 0, we will present one better result. Denote $\lambda_{-1}$ and $\lambda_{+1}$ by the largest negative eigenvalue and the smallest positive eigenvalue of $\hat{L}$, respectively.

Theorem 1.9. Suppose that $F$ satisfies (A), (F7), (H1) and the following condition:

(H3) there exists $l_1 > 0$ such that
$$
F(t, x) \leq \frac{\lambda_{+1}}{2}|x|^2 \quad \text{for a.e. } t \in [0, T] \quad \text{and all } |x| \leq l_1.
$$

Then one of the following cases occurs:

(C1) if 0 is not an eigenvalue of $\hat{L}$, $\hat{L}$ has at least one negative eigenvalue and there exists $l_2 > 0$ such that
$$
F(t, x) \geq \frac{\lambda_{-1}}{2}|x|^2 \quad \text{for a.e. } t \in [0, T] \quad \text{and all } |x| \leq l_2, \quad (7)
$$

then system (1) has at least one nontrivial $T$-periodic solution;

(C2) if 0 is not an eigenvalue of $\hat{L}$, $\hat{L}$ has no negative eigenvalues and
$$
\int_0^T F(t, 0) = 0, \quad (8)
$$

then system (1) has at least one nontrivial $T$-periodic solution;

(C3) if 0 is an eigenvalue of $\hat{L}$ and there exists $l_2 > 0$ such that
$$
F(t, x) \geq 0 \quad \text{for a.e. } t \in [0, T] \quad \text{and all } |x| \leq l_2, \quad (9)
$$

then system (1) has at least one nontrivial $T$-periodic solution;

(C4) if 0 is an eigenvalue of $\hat{L}$, $\hat{L}$ has no negative eigenvalues and there exists $l_2 > 0$ such that
$$
F(t, x) \leq 0 \quad \text{for a.e. } t \in [0, T] \quad \text{and all } |x| \leq l_2, \quad (10)
$$

and (8) holds, then system (1) has at least one nontrivial $T$-periodic solution;

(C5) if 0 is an eigenvalue of $\hat{L}$, $\hat{L}$ has at least one negative eigenvalue, there exists $l_2 > 0$ such that (7) and (10) hold, then system (1) has at least one nontrivial $T$-periodic solution.

Remark 2. It is easy to verify that condition (F3) implies (H1) and (F7). Moreover, (F2) and (4) imply (H2), (H3), (7)-(9) (or (10)). Hence, Theorem 1.8 and Theorem 1.9 improve Theorem 1.3.
Remark 3. We claim that (F5), (F6) (or (F9) in Theorem 1.6) and (F7) (or (F4)) imply (H1). In fact, by (F5), we know that there exist constants $M_1 > 0$ and $m_1 > 0$ such that

$$F(t, x) \leq M_1|x|^\theta, \quad \forall x \in \mathbb{R}^N, \ |x| > m_1, \ \text{a.e.} \ t \in [0, T].$$  \hspace{1cm} (11)

It follows from (F6) (or (F9) in Theorem 1.6) that there exist constants $M_2 > 0$ and $m_2 > 0$ such that

$$\langle \nabla F(t, x), x \rangle \geq M_2|x|^\mu + 2F(t, x), \ \forall x \in \mathbb{R}^N, \ |x| > m_2, \ \text{a.e.} \ t \in [0, T].$$  \hspace{1cm} (12)

By (F7) (or (F4)), we know that there exist constants $M_3 > 0$ and $m_3 > 0$ such that

$$F(t, x) \geq M_3|x|^2, \ \forall x \in \mathbb{R}^N, \ |x| > m_3, \ \text{a.e.} \ t \in [0, T].$$  \hspace{1cm} (13)

Let $m = \max\{m_1, m_2, m_3\}$. Then by (11), (12) and (13), for any $M_4 > 0$, we have

$$(\nabla F(t, x), x) \geq F(t, x) \left( 2 + \frac{M_2|x|^\mu}{F(t, x)} \right) \geq F(t, x) \left( 2 + \frac{M_2|x|^\mu}{M_1|x|^\theta} \right) \geq F(t, x) \left( 2 + \frac{M_2}{M_4 + M_1|x|^\theta - \mu} \right), \ \forall x \in \mathbb{R}^N, \ |x| > m, \ \text{a.e.} \ t \in [0, T].$$

By (12) and assumption (A), it is easy to see that there exist $M_5 > 0$ and $M_6 > 0$ such that

$$\langle \nabla F(t, x), x \rangle \geq M_2|x|^\mu - M_5b(t) - M_6, \ \forall x \in \mathbb{R}^N, \ \text{a.e.} \ t \in [0, T].$$  \hspace{1cm} (14)

It follows from differential mean value theorem that there exists $s \in (0, 1)$ such that

$$F(t, x) - F(t, 0) = \langle \nabla F(t, sx), x \rangle.$$  \hspace{1cm} (15)

By assumption (A), (11), (14) and (15), we have

$$TM_1|x|^\theta + M_7 \geq \int_0^T F(t, x)dt = \int_0^T \frac{1}{s} \langle \nabla F(t, sx), sx \rangle dt + \int_0^T F(t, 0)dt \geq \int_0^T \frac{1}{s} [M_2|sx|^\mu - M_5b(t) - M_6]dt - \int_0^T a(0)b(t)dt \geq s^{\mu-1}M_2T|x|^\mu - M_8, \ \forall x \in \mathbb{R}^N,$$

for some $M_7 > 0$ and $M_8 > 0$. The above inequality implies $\theta \geq \mu$. Let $\nu = \theta - \mu$. Then $\nu < 2$. Thus (H1) holds. Moreover, (F2) and (4) imply (H2), (H3) and (7)-(9) (or (10)). Hence, Theorem 1.8 and Theorem 1.9 also improve Theorem 1.5 and Theorem 1.6.

Remark 4. Note that conditions (F5) and (F6) (or (F9) in Theorem 1.6) are different from condition (F3). One can find examples satisfying Theorem 1.3 but not satisfying Theorem 1.5 and Theorem 1.6. For example, let

$$F(t, x) \equiv F(x) = (e^{||x||^2} - |x|^2 - 1)^2 \quad \text{for a.e.} \ t \in [0, T].$$  \hspace{1cm} (16)

One also can find examples satisfying Theorem 1.5 (an example corresponding to Theorem 1.6 can be seen in [10]) but not satisfying Theorem 1.3. For example, let

$$F(t, x) \equiv F(x) = |x|^2 \ln(|x|^2 + 1) \quad \text{for a.e.} \ t \in [0, T].$$  \hspace{1cm} (17)
From Remark 1, Remark 2 and Remark 3, we know that Theorem 1.8 and Theorem 1.9 unite and improve Theorem 1.3, Theorem 1.5 and Theorem 1.6. One can verify that (16), (17) and the example in [10] satisfy our Theorem 1.1 and Theorem 1.2 and there exist examples satisfying our Theorem 1.8 and Theorem 1.9 but not satisfying Theorem 1.3, Theorem 1.5 and Theorem 1.6. For example, let

\[ F(t, x) \equiv F(x) = \begin{cases} \frac{\lambda+1}{2}|x|^2, & \text{if } |x| \leq 1 \\ \left( e^{\delta|x|^2} - |x|^2 - e + 1 \right) ^2 + \frac{\lambda+1}{2}|x|^2, & \text{if } |x| > 1. \end{cases} \]  

Moreover, it is remarkable that in [3], the following condition which is similar to (H1) has been presented:

(\tilde{S}_2) there exist \( p > 2, c_1, c_2, c_3 > 0 \) and \( \nu \in (0, 2) \) such that, for all \( |z| \geq r_1, \)

\[ |\nabla H(t, z)||z| \leq c_1(\nabla H(t, z), z), \quad |\nabla H(t, z)| \leq c_2|z|^{p-1}, \]

\[ H(t, z) \leq \left( \frac{1}{2} - \frac{1}{c_3|z|^{\nu}} \right) |\nabla H(t, z), z|, \]

which is used to consider the existence of homoclinic solutions for the first order Hamiltonian system \( \dot{z} = J\nabla H(t, z). \) Ding [3] claimed that (\tilde{S}_2) implies a condition which is similar to (F8). However, our Theorem 1.8 and Theorem 1.9 present that

\[ |\nabla H(t, z)||z| \leq c_1(\nabla H(t, z), z) \quad \text{and} \quad |\nabla H(t, z)| \leq c_2|z|^{p-1} \]

are not necessary when we consider the existence of periodic solutions for system (1). The fact shows that Theorem 1.8 and Theorem 1.9 are different from Theorem 1.4. One can verify that (18) does not satisfy Theorem 1.4.

**Remark 5.** In Theorem 1.7, (5) is weaker than (F7). However, (6) is a global condition while in Theorem 1.8 and Theorem 1.9, (H2), (H3) and (7)-(10) are local conditions. Moreover, Remark 2 and Remark 3 show that (F5) and (F6) imply (H1). So Theorem 1.8 and Theorem 1.9 are different from Theorem 1.7. The example (18) dose not satisfy Theorem 1.7.

By Theorem 1.9, we can obtain two useful corollaries. We denote \( \xi_{ii}(t) (i = 1, \cdots, N) \) by the main diagonal entries of \( A(t). \)

**Corollary 1.** Suppose that \( F \) satisfies (A), (F7), (H1) and (H3). Then (C3) or one of the following cases occurs:

\( (C1)' \) if 0 is not an eigenvalue of \( \hat{L}, \) there exists a \( \xi_{ii}(t) \) such that \( \int_0^T \xi_{ii}(t) dt > 0 \) and (7) holds, then system (1) has at least one nontrivial \( T \)-periodic solution.

\( (C2)' \) if 0 is not an eigenvalue of \( \hat{L}, \) \( A(t) \) is semi-negative definite for all \( t \in [0, T] \) and (7) holds, then system (1) has at least one nontrivial \( T \)-periodic solution.

\( (C4)' \) if 0 is an eigenvalue of \( \hat{L}, \) \( A(t) \) is semi-negative definite for all \( t \in [0, T] \) and (8) and (10) hold, then system (1) has at least one nontrivial \( T \)-periodic solution.

\( (C5)' \) if 0 is an eigenvalue of \( \hat{L}, \) there exists a \( \xi_{ii}(t) \) such that \( \int_0^T \xi_{ii}(t) dt > 0 \) and (7) and (10) hold, then system (1) has at least one nontrivial \( T \)-periodic solution.

**Corollary 2.** Suppose that \( A(t) \equiv A \) which is a real symmetric constant matrix, and \( F \) satisfies (A), (F7), (H1) and (H3). Then (C3) or one of the following cases occurs:

\( (C1)'' \) if 0 is not an eigenvalue of \( \hat{L}, \) \( A \) is not semi-negative definite and (7) holds, then system (1) has at least one nontrivial \( T \)-periodic solution.
(C2)$'$ if 0 is not an eigenvalue of $\hat{L}$, $A$ is semi-negative definite and (8) holds, then system (1) has at least one nontrivial $T$-periodic solution.

(C4)$'$ if 0 is an eigenvalue of $\hat{L}$, $A$ is semi-negative definite and (8) and (10) hold, then system (1) has at least one nontrivial $T$-periodic solution.

(C5)$'$ if 0 is an eigenvalue of $\hat{L}$, $A$ is not semi-negative definite and (7) and (10) hold, then system (1) has at least one nontrivial $T$-periodic solution.

Theorem 1.10. Suppose that $F$ satisfies assumption (A), (F1), (F2), (H1) and (H4) there exist $r_1 > 0$ and $\beta > \lambda_1 + 1/2$ such that

$$F(t,x) \geq \beta |x|^2$$

for a.e. $t \in [0, T]$ and all $|x| \geq r_1$.

Then system (1) has at least one nontrivial $T$-periodic solution. Furthermore, if $A(t)$ is semipositive definite matrix for all $t \in [0, T]$, then system (1) has at least one nonconstant $T$-periodic solution.

Remark 6. Remark 3 shows that Theorem 1.10 is different from Theorem 1.7. Moreover, since (F1) is a global condition and (H4) is weaker than (F7), Theorem 1.10 is also different from Theorem 1.8 and Theorem 1.9. Another aim of Theorem 1.10 is to obtain the nonconstant solution. Remark 2 shows that Theorem 1.10 extends and improves Theorem 1.1.

Theorem 1.11. Suppose that $F$ satisfies assumption (A), (F1), (H1) and the following conditions hold:

$$\lim_{|x| \to 0} \frac{F(t,x)}{|x|^2} < \frac{2\pi^2}{T^2} - \frac{\|A\|}{2} \text{ uniformly for a.e. } t \in [0, T],$$

and

$$\lim_{|x| \to \infty} \frac{F(t,x)}{|x|^2} > \frac{2\pi^2}{T^2} \text{ uniformly for a.e. } t \in [0, T].$$

If $A(t)$ is semi-positive definite matrix for all $t \in [0, T]$ and $\|A\| < \frac{4\pi^2}{T^2}$, then system (1) has at least one nonconstant $T$-periodic solution, where

$$\|A\| = \sup_{t \in [0, T]} \max_{|x|=1, x \in \mathbb{R}^N} |A(t)x|$$

and $A^T(t)$ denotes the transpose of $A(t)$.

Remark 7. Remark 2 and Remark 3 show that Theorem 1.11 still unites and maybe improves Theorem 1.1 and Theorem 1.2 even if $A(t) \equiv 0$. Here, we use the word “maybe” before “improves” because we do not find an example satisfying Theorem 1.11 but not satisfying both Theorem 1.1 and Theorem 1.2 when $A(t) \equiv 0$. We would like to leave this problem to readers. Moreover, it is remarkable that Theorem 1.10 and Theorem 1.11 are different from those results in [18] even if $A(t) \equiv 0$ because the example (17) does not satisfy them.

2. Preliminaries. $H^1_T$ is a Hilbert space with the inner product and the norm defined by

$$\langle u, v \rangle = \left[ \int_0^T (u(t), v(t))dt + \int_0^T (\dot{u}(t), \dot{v}(t))dt \right]^{1/2}$$
and
\[ \|u\| = \left[ \int_0^T |u(t)|^2 dt + \int_0^T |\dot{u}(t)|^2 dt \right]^{1/2} \]
for each \( u, v \in H^1_T \). Let
\[ \ddot{u} = \frac{1}{T} \int_0^T u(t) dt, \quad \dot{u}(t) = u(t) - \ddot{u}. \]
Then one has
\[ \|\dot{u}\|_\infty^2 \leq \frac{T}{12} \int_0^T |\dot{u}(t)|^2 dt, \quad \text{(Sobolev’s inequality)} \]
\[ \|\ddot{u}\|_2^2 \leq \frac{T^2}{4\pi^2} \int_0^T |\ddot{u}(t)|^2 dt, \quad \text{(Wirtinger’s inequality)} \]
(see Proposition 1.3 in [14]) and
\[ \|u\|_\infty \leq C \|u\| \quad (21) \]
for some \( C > 0 \) and all \( u \in H^1_T \), where \( \|u\|_\infty = \max_{t \in [0,T]} |u(t)| \) (see Proposition 1.1 in [14]). It follows from assumption (A) that the functional \( \varphi \) on \( H^1_T \) given by
\[ \varphi(u) = \frac{1}{2} \int_0^T |\dot{u}(t)|^2 dt - \frac{1}{2} \int_0^T (A(t)u(t), u(t)) dt - \int_0^T F(t, u(t)) dt \]
is continuously differentiable. Moreover, one has
\[ \langle \varphi'(u), v \rangle = \int_0^T [(\dot{u}(t), \dot{v}(t)) - (A(t)u(t), v(t))] dt \]
for \( u, v \in H^1_T \). It is well known that the solutions of system (1.1) correspond to the critical points of \( \varphi \) (see [14]).

Similar to the proofs of Theorem 1.4 and Theorem 1.5, we shall use the local linking theorem (Theorem 2.2 in [11]) to prove Theorem 1.8 and Theorem 1.9.

Let \( X \) be a real Banach space with \( X = X^1 \oplus X^2 \) and \( X^0_j \subset X^1 \subset \cdots \subset X^j \) such that \( X^j = \bigcup_{n \in \mathbb{N}} X^n_j, j = 1, 2 \). For every multi-index \( \alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2 \), let \( X_{\alpha} = X^1_{\alpha_1} \oplus X^2_{\alpha_2} \). We say \( \alpha \leq \beta \iff \alpha_1 \leq \beta_1, \alpha_2 \leq \beta_2 \). A sequence \( (\alpha_n) \subset \mathbb{N}^2 \) is admissible if for every \( n \in \mathbb{N}^2 \), there is \( m \in \mathbb{N} \) such that \( n \geq m \Rightarrow \alpha_n \geq \alpha \). We say that \( \varphi \in C^1(X, \mathbb{R}) \) satisfies the \((C)^*\) condition if every sequence \( (u_{\alpha_n}) \) such that \( (\alpha_n) \) is admissible and satisfying
\[ u_{\alpha_n} \in X_{\alpha_n}, \quad \sup \varphi(u_{\alpha_n}) < \infty, \quad (1 + \|u_{\alpha_n}\|)\|\varphi'(u_{\alpha_n})\| \to 0 \]
contains a subsequence which converges to a critical point of \( \varphi \).

**Lemma 2.1** (see [11]). Suppose that \( \varphi \in C^1(X, \mathbb{R}) \) satisfies the following conditions:
\begin{enumerate}[(\varphi 1)]
\item \( \varphi \) satisfies \((C)^*\) condition.
\item \( \varphi \) maps bounded sets into bounded sets.
\item For every \( m \in \mathbb{N} \), \( \varphi(u) \to -\infty \) as \( \|u\| \to \infty \) on \( X^1_m \oplus X^2 \).
\item \( X^1 \neq \{0\} \) and \( \varphi \) has a local linking at 0, that is, for some \( r_0 > 0 \),
\begin{enumerate}[(i)]
\item \( \varphi(u) \geq 0, \forall u \in X^1 \text{ with } \|u\| \leq r_0 \),
\item \( \varphi(u) \leq 0, \forall u \in X^2 \text{ with } \|u\| \leq r_0 \).
\end{enumerate}
\end{enumerate}

Then \( \varphi \) has at least one nonzero critical point.
In order to prove Theorem 1.10 and Theorem 1.11, we need the following linking method in [19]. Let \((E, \| \cdot \|)\) be a Banach space and \(\Phi\) be the set of all continuous maps \(\Gamma = \Gamma(t)\) from \(E \times [0,1] \) to \(E\) such that

1. \(\Gamma(0) = I\), the identity map.
2. For each \(t \in [0,1]\), \(\Gamma(t)\) is a homeomorphism of \(E\) onto \(E\) and \(\Gamma^{-1}(t) \in C(E \times [0,1], E)\).
3. \(\Gamma(1)E\) is a single point in \(E\) and \(\Gamma(t) A\) converges uniformly to \(\Gamma(1) E\) as \(t \to 1\) for each bounded set \(A \subset E\).
4. For each \(t_0 \in [0,1]\) and each bounded set \(A \subset E\),
   \[
   \sup_{0 \leq t \leq t_0} \{ \| \Gamma(t)u \| + \| \Gamma^{-1}(t)u \| \} < \infty.
   \]

**Definition 2.2** (see [19], Definition 3.2). We say that \(A\) links \(B\) if \(A\) and \(B\) are subsets of \(E\) such that \(A \cap B = \emptyset\), and for each \(\Gamma(t) \in \Phi\), there is a \(t' \in (0,1]\) such that \(\Gamma(t')A \cap B \neq \emptyset\).

**Lemma 2.3** (see [19], Theorem 3.4). Let \(E\) be a Banach space, \(\varphi \in C^1(E, \mathbb{R})\) and \(A \subset E\) two subsets of \(E\) such that \(\varphi\) links \(B\). Assume that
   \[
   \sup_{A} \varphi \leq \inf_{B} \varphi
   \]
   \[
   c := \inf_{\Gamma \in \Phi} \sup_{t \in [0,1]} \varphi(\Gamma(t)u) < \infty.
   \]
   Let \(\psi(t)\) be a positive, nonincreasing and locally Lipschitz continuous function on \([0,\infty)\) satisfying \(\int_{0}^{\infty} \psi(r)dr = \infty\). Then there exists a sequence \(\{u_n\} \subset E\) such that \(\varphi(u_n) \to c\) and \(\varphi'(u_n) / \psi(\|u_n\|) \to 0\), as \(n \to \infty\).

**Remark 8** (see [19]). Since \(A\) links \(B\), by Definition 2.2, it is easy to know that \(c \geq \inf_{B} \varphi\). If we let \(\psi(r) = 1\), the sequence \(\{u_n\}\) coincides with (PS) sequence, that is \(\{u_n\}\) satisfying
   \[
   \varphi(u_n) \to c, \quad \varphi'(u_n) \to 0 \quad \text{as} \quad n \to \infty.
   \]
If we let \(\psi(r) = \frac{1}{1+r^2}\), the sequence \(\{u_n\}\) is the Cerami sequence, that is \(\{u_n\}\) satisfying
   \[
   \varphi(u_n) \to c, \quad (1 + \|u_n\|)\|\varphi'(u_n)\| \to 0 \quad \text{as} \quad n \to \infty.
   \]

3. **Proofs of Theorems.**

   We will verify those conditions in Lemma 2.1.

   Let \(\{\lambda_j\}_{j=1}^{+\infty}\) be eigenvalues of \(d^2/dt^2 - \hat{A}\) with \(\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{k-1} < \lambda_k < \lambda_{k+1} \leq \cdots\) and
   \[
   H^- = \left\{ u \in H_T^1 : -\ddot{u} - \hat{A}u = \lambda u \text{ with } \lambda < \lambda_k \right\},
   \]
   \[
   H^0 = \left\{ u \in H_T^1 : -\ddot{u} - \hat{A}u = \lambda_k u \right\},
   \]
   \[
   H^+ = \left\{ u \in H_T^1 : -\ddot{u} - \hat{A}u = \lambda u \text{ with } \lambda > \lambda_k \right\}.
   \]
Then \(H_T^1 = H^- \oplus H^0 \oplus H^+\), and \(H^-\) and \(H^0\) are finite dimensional.

   Let
   \[
   q(u) = \frac{1}{2} \int_{0}^{T} \left[ |\dot{u}(t)|^2 - (A(t)u(t), u(t)) \right] dt.
   \]
Then it is easy to obtain that

\[ q(u) \leq \frac{\lambda_{k-1}}{2} \int_0^T |u(t)|^2 dt \quad \text{if } u \in H^-, \quad (22) \]
\[ q(u) = \frac{\lambda_k}{2} \int_0^T |u(t)|^2 dt \quad \text{if } u \in H^0, \quad (23) \]
\[ q(u) \geq \frac{\lambda_{k+1}}{2} \int_0^T |u(t)|^2 dt \quad \text{if } u \in H^+. \quad (24) \]

Let

\[ p(u) = \frac{1}{2} \int_0^T \left[ |\dot{u}(t)|^2 - (A(t)u(t), u(t)) - \lambda_k|u(t)|^2 \right] dt. \]

Then there exists \( \delta > 0 \) such that

\[ p(u) \leq -\frac{\delta}{2} ||u||^2 \quad \text{if } u \in H^-, \quad (25) \]
\[ p(u) = 0 \quad \text{if } u \in H^0, \quad (26) \]
\[ p(u) \geq \frac{\delta}{2} ||u||^2 \quad \text{if } u \in H^+ \quad (27) \]

(see [9], [10], [14], [22]).

(1) We verify that \( \varphi \) satisfies the \((C)^*\) condition. Set \( X = H^1_T \), \( X^1 = H^+ \) with \((e_n)_{n \geq 1}\) being its Hilbertian basis, \( X^2 = H^- \oplus H^0 \) and define

\[ X_n^1 = \text{span}\{e_1, e_2, \cdots, e_n\}, \quad n \in \mathbb{N}; \]
\[ X_n^2 = X^2, \quad n \in \mathbb{N}; \]
\[ X^j = \bigcup_{n \in \mathbb{N}} X_n^j, \quad j = 1, 2. \]

Let \( (u_{\alpha_n}) \) be a sequence in \( H^1_T \) such that \( (\alpha_n) \) is admissible and satisfying

\[ u_{\alpha_n} \in X_{\alpha_n}, \quad \sup \varphi(u_{\alpha_n}) < \infty, \quad (1 + ||u_{\alpha_n}||) ||\varphi'(u_{\alpha_n})|| \to 0. \]

Then there exists a constant \( C_1 > 0 \) such that

\[ ||\varphi(u_{\alpha_n})|| \leq C_1, \quad (1 + ||u_{\alpha_n}||) ||\varphi'(u_{\alpha_n})|| \leq C_1 \quad \text{for all } n \in \mathbb{N}. \quad (28) \]

By (F7), we know that for some \( C_2 > 0 \), there exists \( L > 0 \) such that

\[ F(t, x) \geq C_2|x|^2, \quad \forall \ x \in \mathbb{R}^N, \ |x| > L, \ a.e. \ t \in [0, T]. \quad (29) \]

It follows from (29) and assumption (A) that there exists \( C_3 > 0 \) such that

\[ F(t, x) \geq C_2|x|^2 - C_3 b(t) - C_2 L^2, \quad \forall \ x \in \mathbb{R}^N, \ a.e. \ t \in [0, T]. \quad (30) \]

By (H1), we have

\[ [(\nabla F(t, x), x) - 2F(t, x)](\zeta + \eta|x|) \geq F(t, x), \forall \ x \in \mathbb{R}^N, \ |x| > m, \ a.e. \ t \in [0, T]. \quad (31) \]

Then by assumption (A) and (31), there exists a constant \( C_4 > 0 \) such that

\[ [(\nabla F(t, x), x) - 2F(t, x)](\zeta + \eta|x|) \geq F(t, x) - C_4 b(t), \forall \ x \in \mathbb{R}^N, \ a.e. \ t \in [0, T]. \quad (32) \]
It follows from assumption (A), (30) and (32) that there exist $C_5 > 0, C_6 > 0$ and $C_7 > 0$ such that

$$
(\nabla F(t, x), x) - 2F(t, x) \geq \frac{F(t, x) - C_4 b(t)}{\zeta + \eta|x|^\nu}
\geq \frac{C_2|x|^2 - C_3 b(t) - C_4 b(t) - C_2 L^2}{\zeta + \eta|x|^\nu}
\geq C_5|x|^{2-\nu} - C_6 b(t) - C_7, \forall \, x \in \mathbb{R}^N, \text{a.e. } t \in [0, T].
$$

Hence, it follows from (28) and (33) that

$$
2C_1 + C_1 \geq 2\varphi(u_{\alpha_n}) - \langle \varphi'(u_{\alpha_n}), u_{\alpha_n} \rangle
= \int_0^T [(\nabla F(t, u_{\alpha_n}(t)), u_{\alpha_n}(t)) - 2F(t, u_{\alpha_n}(t))]dt
\geq C_5 \int_0^T |u_{\alpha_n}(t)|^{2-\nu}dt - C_6 \int_0^T b(t)dt - C_7 T.
$$

This shows that $\int_0^T |u_{\alpha_n}(t)|^{2-\nu}dt$ is bounded. By (29) and (H1), we have

$$
[(\nabla F(t, x), x) - 2F(t, x)](\zeta + \eta|x|^\nu) \geq F(t, x) \geq C_2|x|^2 > 0,
\forall \, x \in \mathbb{R}^N, \, |x| > m + L, \text{a.e. } t \in [0, T].
$$

Since $A(t)$ is continuous and $T$–periodic in $t$, there exists $G > 0$ such that

$$
|(A(t)x, x)| \leq G|x|^2, \forall \, x \in \mathbb{R}^N.
$$
By assumption (A), (28), (32), (34), (35), (36) and (37), we have

\[
\frac{1}{2} \|u_{\alpha_n}\|^2 \leq \varphi(u_{\alpha_n}) + \int_0^T F(t, u_{\alpha_n}(t)) \, dt + \frac{1}{2} \int_0^T |u_{\alpha_n}(t)|^2 \, dt + \frac{1}{2} \int_0^T (A(t)u_{\alpha_n}(t), u_{\alpha_n}(t)) \, dt \\
\leq C_1 + C_4 \int_0^T b(t) \, dt + \frac{1}{2} \int_0^T |u_{\alpha_n}(t)|^2 \, dt + \frac{1}{2} \int_0^T (A(t)u_{\alpha_n}(t), u_{\alpha_n}(t)) \, dt \\
+ \int_0^T (\zeta + \eta |u_{\alpha_n}(t)|^\nu) [(\nabla F(t, u_{\alpha_n}(t)), u_{\alpha_n}(t)) - 2F(t, u_{\alpha_n}(t))] \, dt \\
\leq C_1 + C_4 \int_0^T b(t) \, dt + \frac{1}{2} \int_0^T |u_{\alpha_n}(t)|^2 \, dt \\
+ \int_0^T (\zeta + \eta |u_{\alpha_n}(t)|^\nu) [(\nabla F(t, u_{\alpha_n}(t)), u_{\alpha_n}(t)) - 2F(t, u_{\alpha_n}(t))] \, dt \\
\leq C_1 + C_4 \int_0^T b(t) \, dt + \frac{1}{2} \int_0^T |u_{\alpha_n}(t)|^2 \, dt + C_8 \int_0^T b(t) \, dt \\
+ (\zeta + \eta \|u_{\alpha_n}\|_{\infty}^\nu) \int_0^T [(\nabla F(t, u_{\alpha_n}(t)), u_{\alpha_n}(t)) - 2F(t, u_{\alpha_n}(t))] \, dt \\
- (\zeta + \eta \|u_{\alpha_n}\|_{\infty}^\nu) \int_0^T [(\nabla F(t, u_{\alpha_n}(t)), u_{\alpha_n}(t)) - 2F(t, u_{\alpha_n}(t))] \, dt \\
\leq C_1 + (C_4 + C_8) \int_0^T b(t) \, dt + \frac{1}{2} \int_0^T |u_{\alpha_n}(t)|^2 \, dt + C_9 (\zeta + \eta \|u_{\alpha_n}\|_{\infty}^\nu) \int_0^T b(t) \, dt \\
+ (\zeta + \eta \|u_{\alpha_n}\|_{\infty}^\nu) \int_0^T [(\nabla F(t, u_{\alpha_n}(t)), u_{\alpha_n}(t)) - 2F(t, u_{\alpha_n}(t))] \, dt \\
\leq C_1 + (C_4 + C_8) \int_0^T b(t) \, dt + \frac{1}{2} \|u_{\alpha_n}\|_{\infty}^{2\nu} \int_0^T |u_{\alpha_n}(t)|^{2-\nu} \, dt \\
+ C_9 (\zeta + \eta \|u_{\alpha_n}\|_{\infty}^\nu) \int_0^T b(t) \, dt + 3C_1 (\zeta + \eta \|u_{\alpha_n}\|_{\infty}^\nu) \\
\leq C_1 + (C_4 + C_8) \int_0^T b(t) \, dt + \frac{(1 + G)C^\nu}{2C^5} \|u_{\alpha_n}\|^\nu \left[ 3C_1 + C_6 \int_0^T b(t) \, dt + C_7 T \right] \\
+ C_9 (\zeta + \eta C^\nu \|u_{\alpha_n}\|^\nu) \int_0^T b(t) \, dt + 3C_1 (\zeta + \eta C^\nu \|u_{\alpha_n}\|^\nu),
\]

(38)
where $C_8$ and $C_9$ are positive constants. Since $\nu < 2$, (38) implies that $\|u_{\alpha_n}\|$ is bounded. Hence, $\varphi'(u_{\alpha_n}) \rightarrow 0$ by $(1 + \|u_{\alpha_n}\|)\|\varphi'(u_{\alpha_n})\| \rightarrow 0$. Thus we have

\[
\langle \varphi'(u_{\alpha_n}), v \rangle = \int_0^T (\dot{u}_{\alpha_n}(t), \dot{v}(t)) \, dt - \int_0^T (A(t)u_{\alpha_n}(t), v(t)) \, dt
\]

\[
- \int_0^T (\nabla F(t, u_{\alpha_n}(t)), v(t)) \, dt \rightarrow 0, \quad \forall \, v \in H^1_T.
\]

Since $H^1_T$ is a reflexive Banach space, then there is a renamed subsequence such that

\[
u_{\alpha_n} \rightharpoonup u \text{ weakly in } H^1_T.
\]

(40)

Furthermore, by Proposition 1.2 in [14], we have

\[
u_{\alpha_n} \rightarrow u \text{ strongly in } C([0, T], \mathbb{R}^N).
\]

(41)

By (39) and (40), we have

\[
\int_0^T (\dot{u}_{\alpha_n}(t) - \dot{u}(t), \dot{u}_{\alpha_n}(t) - \dot{u}(t)) \, dt
\]

\[
- \int_0^T (A(t)u_{\alpha_n}(t) - A(t)u(t), u_{\alpha_n}(t) - u(t)) \, dt
\]

\[
- \int_0^T (\nabla F(t, u_{\alpha_n}(t)) - \nabla F(t, u(t)), u_{\alpha_n}(t) - u(t)) \, dt
\]

\[
= \langle \varphi'(u_{\alpha_n}) - \varphi'(u), u_{\alpha_n} - u \rangle \rightarrow 0, \quad \text{as } n \rightarrow \infty.
\]

By the boundedness of $\{u_{\alpha_n}\}$, assumption (A) and (41), we have

\[
\int_0^T (\nabla F(t, u_{\alpha_n}(t)) - \nabla F(t, u(t)), u_{\alpha_n}(t) - u(t)) \, dt \rightarrow 0 \quad \text{as } n \rightarrow \infty.
\]

(43)

By (37) and (41), we have

\[
\int_0^T (A(t)u_{\alpha_n}(t) - A(t)u(t), u_{\alpha_n} - u) \, dt \rightarrow 0, \quad \text{as } n \rightarrow \infty.
\]

(44)

Hence, it follows from (42), (43) and (44) that

\[
\int_0^T |\dot{u}_{\alpha_n}(t) - \dot{u}(t)|^2 \, dt \rightarrow 0 \quad \text{as } n \rightarrow \infty.
\]

(45)

On the other hand, it is easy to derive from (41) that

\[
\int_0^T |u_{\alpha_n}(t) - u(t)|^2 \, dt \rightarrow 0 \quad \text{as } n \rightarrow \infty.
\]

(46)

So (45) and (46) imply that $\|u_{\alpha_n} - u\| \rightarrow 0$. Thus we conclude that $(C)^*$ condition is satisfied.

(2) We claim that $\varphi$ maps bounded sets into bounded sets.
Assume that there exists $G_1 > 0$ such that $\|u\| \leq G_1$. Then by (21), we know that $\|u\|_{\infty} \leq CG_1$. It follows from (F7) and assumption (A) that there exist $u$ and $G_4$ such that

$$\varphi(u) = \frac{1}{2} \int_0^T \|\dot{u}(t)\|^2 dt - \frac{1}{2} \int_0^T (A(t)u(t), u(t)) dt - \int_0^T F(t, u(t)) dt$$

$$\leq \frac{1}{2} \int_0^T \|\dot{u}(t)\|^2 dt + \frac{G}{2} \int_0^T |u(t)|^2 dt + \int_0^T a(|u(t)|) b(t) dt$$

$$\leq \frac{(1 + G)G_4^2}{2} + \max_{0 \leq s \leq C G_1} \int_0^T b(t) dt, \quad \forall \|u\| \leq G_1.$$

(3) We claim that for every $m \in \mathbb{N},$

$$\varphi(u) \to -\infty \quad \text{as} \quad \|u\| \to \infty \quad \text{on} \quad X_m^1 \oplus X^2 \quad (47)$$

Since $X_m^1 \oplus H^- \oplus H^0$ is finite dimensional, there exists $G_2 > 1$ such that

$$\|u\| \leq G_2\|u\|_{L^2}, \quad \forall u \in X_m^1 \oplus H^- \oplus H^0. \quad (48)$$

It follows from (F7) and assumption (A) that there exist $G_3 > [G_2(1 + G + |\lambda_k|)]/2, G_4 > 0$ and $G_5 > 0$ such that

$$F(t, x) \geq G_3|x|^2 - G_4 b(t) - G_5, \quad \forall x \in \mathbb{R}^N, \quad \text{a.e.} \quad t \in [0, T]. \quad (49)$$

By (25), (26), (48) and (49), we have, for $u = u^+ + u^- + u^0 \in X_m^1 \oplus H^- \oplus H^0,$

$$\varphi(u) = p(u^-) + p(u^+) + \frac{\lambda_k}{2} \int_0^T \|u^- - u^0(t) + u^+(t)\|^2 dt - \int_0^T F(t, u(t)) dt$$

$$\leq -\frac{\delta}{2} \|u^-\|^2 + \frac{1}{2} \int_0^T \|\dot{u}^+(t)\|^2 dt - \frac{1}{2} \int_0^T (A(t)u^+(t), u^+(t)) dt$$

$$+ \frac{\lambda_k}{2} \int_0^T |u^-|^2 dt + \frac{\lambda_k}{2} \int_0^T |u^-| dt + \frac{\lambda_k}{2} \int_0^T |u^0(t)|^2 dt$$

$$- G_3 \int_0^T |u^- - u^0(t) + u^+(t)|^2 dt + G_4 \int_0^T |b(t)| + G_5 T$$

$$\leq -\frac{\delta}{2} \|u^-\|^2 + \frac{1}{2} \int_0^T \|\dot{u}^+(t)\|^2 dt + \frac{G + |\lambda_k|}{2} \int_0^T |u^+(t)|^2 dt$$

$$- G_3 \int_0^T |u^+(t)|^2 dt - \left(G_3 - \frac{|\lambda_k|}{2}\right) \int_0^T |u^0(t)|^2 dt$$

$$- \left(G_3 - \frac{|\lambda_k|}{2}\right) \int_0^T |u^-(t)|^2 dt + G_4 \int_0^T |b(t)| + G_5 T$$

$$\leq -\frac{\delta}{2} \|u^-\|^2 + \frac{1 + G + |\lambda_k|}{2} |u^+|^2 - \frac{G_4}{G_2} \|u^+\|^2$$

$$- \frac{1}{G_2} \left(G_3 - \frac{|\lambda_k|}{2}\right) \|u^0\|^2 + G_4 \int_0^T |b(t)| + G_5 T.$$

Since $G_3 > [G_2(1 + G + |\lambda_k|)]/2$ and $G_2 > 1,$ (50) implies that (47) holds.

(4) We verify that $\varphi$ satisfies (\phi4).
For $u \in X^1 = H^+$ with $\|u\| \leq \rho_0 = l_0/C$, it follows from (21), (24) and (H2) that
\[
\varphi(u) = \frac{\lambda_{k+1}}{2} \int_0^T |u(t)|^2 dt - \int_0^T F(t, u(t)) dt \\
\geq \frac{\lambda_{k+1}}{2} \int_0^T |u(t)|^2 dt - \frac{\lambda_{k+1}}{2} \int_0^T |u(t)|^2 dt \\
= 0.
\]
For $u = u^- + u^0 \in X^2 = H^- \oplus H^0$ satisfying $\|u\| \leq \rho_0$, it follows from (21), (22), (23) and (H2) that
\[
\varphi(u) = q(u^-) + q(u^0) - \int_0^T F(t, u(t)) dt \\
\leq \frac{\lambda_{k-1}}{2} \int_0^T |u^-(t)|^2 dt + \frac{\lambda_k}{2} \int_0^T |u^0(t)|^2 dt - \frac{\lambda_k}{2} \int_0^T |u^-(t) + u^0(t)|^2 dt \\
\leq 0.
\]
Let $r_0 = \rho_0$. Then (51) and (52) imply that $(\varphi 4)$ of Lemma 2.1 holds. Thus we complete our proof.

**Proof of Theorem 1.9.** Different from Theorem 1.8, we define
\[
H^- = \{ u \in H^1_T : -\ddot{u} - \hat{A}u = \lambda u \text{ with } \lambda < 0 \}, \\
H^0 = \{ u \in H^1_T : -\ddot{u} - \hat{A}u = 0 \}, \\
H^+ = \{ u \in H^1_T : -\ddot{u} - \hat{A}u = \lambda u \text{ with } \lambda > 0 \}.
\]
Then $H_T^1 = H^- \oplus H^0 \oplus H^+$ and it is easy to obtain that
\[
q(u) = \frac{\lambda_{-1}}{2} \int_0^T |u(t)|^2 dt \text{ if } u \in H^-, \\
q(u) = 0 \text{ if } u \in H^0, \\
q(u) = \frac{\lambda_{+1}}{2} \int_0^T |u(t)|^2 dt \text{ if } u \in H^+,
\]
where $\lambda_{-1}$ and $\lambda_{+1}$ are the largest negative eigenvalue and the smallest positive eigenvalue of $\hat{L}$, respectively. Moreover, there exists $\delta_1 > 0$ such that
\[
q(u) \leq -\frac{\delta_1}{2} \|u\|^2 \text{ if } u \in H^-, \\
q(u) \geq \frac{\delta_1}{2} \|u\|^2 \text{ if } u \in H^+
\]
(see [9], [10], [14], [22]). Next, we only verify that $\varphi$ satisfies $(\varphi 4)$. The others are similar to Theorem 1.8. We omit the details.

For $u \in X^1 = H^+$ with $\|u\| \leq \rho_1 = l_1/C$, it follows from (21), (55) and (H3) that
\[
\varphi(u) = \frac{\lambda_{+1}}{2} \int_0^T |u(t)|^2 dt - \int_0^T F(t, u(t)) dt \\
\geq \frac{\lambda_{+1}}{2} \int_0^T |u(t)|^2 dt - \frac{\lambda_{+1}}{2} \int_0^T |u(t)|^2 dt \\
= 0.
\]
Case (i): $0$ is not an eigenvalue of $\hat{L}$ and $\hat{L}$ has at least one negative eigenvalue. Then $H^− \neq \{0\}$ and $H^0 = \{0\}$. For $u = u^- \in X^2 = H^−$ satisfying $\|u\| \leq \rho_2 = l_2/C$, it follows from (21), (53) and (7) that
\[
\varphi(u) = q(u^−) - \int_0^T F(t, u^−(t)) dt \\
\leq \frac{\lambda_−1}{2} \int_0^T |u^−(t)|^2 dt - \frac{\lambda_−1}{2} \int_0^T |u^−(t)|^2 dt \\
= 0.
\] (59)

Let $r_0 = \{\rho_1, \rho_2\}$. Then (58) and (59) imply that (ϕ4) of Lemma 2.1 holds.

Case (ii): $0$ is not an eigenvalue of $\hat{L}$ and $\hat{L}$ has no negative eigenvalues. Then $H^− = \{0\}$ and $H^0 = \{0\}$. Let $X^2 = H^− \oplus H^0 = \{0\}$. Then it follows from (8) that
\[
\varphi(0) = -\int_0^T F(t, 0) dt = 0.
\] (60)

Let $r_0 = \rho_1$. Then (58) and (60) imply that (ϕ4) of Lemma 2.1 holds.

Case (iii): $0$ is an eigenvalue of $\hat{L}$. Then $H^0 \neq \{0\}$. For $u = u^− + u^0 \in X^2 = H^− \oplus H^0$ satisfying $\|u\| \leq \rho_2 = l_2/C$, it follows from (21), (53), (54) and (9) that
\[
\varphi(u) = q(u^−) - \int_0^T F(t, u(t)) dt \\
\leq \frac{\lambda_1}{2} \int_0^T |u^+(t)|^2 dt - \int_0^T F(t, u(t)) dt \\
\leq 0.
\] (61)

Let $r_0 = \min\{\rho_1, \rho_2\}$. Then (58) and (61) imply that (ϕ4) of Lemma 2.1 holds.

Next, we consider case (iv) and case (v). Different from case (i)-case (iii), we let $X_1 = H^+ \oplus H^0$ and $X_2 = H^−$.

For $u = u^+ + u^0 \in X^1 = H^+ \oplus H^0$ with $\|u\| \leq \rho_2 = l_2/C$, it follows from (21), (54), (55) and (10) that
\[
\varphi(u) \geq \frac{\lambda_+1}{2} \int_0^T |u^+(t)|^2 dt - \int_0^T F(t, u(t)) dt \\
\geq \frac{\lambda_1}{2} \int_0^T |u^+(t)|^2 dt \\
\geq 0.
\] (62)

Case (iv): $0$ is an eigenvalue of $\hat{L}$ and $\hat{L}$ has no negative eigenvalues. Then $X_2 = H^− = \{0\}$. It follows from (8) that
\[
\varphi(0) = -\int_0^T F(t, 0) = 0.
\] (63)

Let $r_0 = \rho_2$. Then (62) and (63) imply that (ϕ4) of Lemma 2.1 holds.

Case (v): $0$ is an eigenvalue of $\hat{L}$ and $\hat{L}$ has at least one negative eigenvalue. Then $X_2 = H^− \neq \{0\}$. For $u = u^− \in X^2 = H^−$ satisfying $\|u\| \leq \rho_2 = l_2/C$, it
follows from (7) that
\[
\varphi(u) \leq \frac{\lambda_1}{2} \int_0^T |u(t)|^2 dt - \int_0^T F(t, u(t)) dt
\]
\[
\leq \frac{\lambda_1}{2} \int_0^T |u(t)|^2 dt - \frac{\lambda_1}{2} \int_0^T |u(t)|^2 dt
\]
\[
= 0.
\]
Let \( r_0 = \rho_2 \). Then (62) and (64) imply that \( (\varphi 4) \) of Lemma 2.1 holds. Thus we complete our proof.

In order to prove Corollary 1 and Corollary 2, we need the following Propositions.

**Proposition 1.** (a) If \( H^- = \{0\} \), then \( \int_0^T \xi_{ii}(t) dt \leq 0, \forall i = 1, \cdots, N \), where \( \xi_{ii}(t) \) (i = 1, \cdots, N) are the main diagonal entries of \( A(t) \);
(b) If \( A(t) \) is semi-negative definite for all \( t \in [0, T] \), then \( H^- = \{0\} \).

**Proof.** (a) If \( H^- = \{0\} \), then (54) and (57) imply that \( q(u) \geq 0, \forall u \in H^+_1 \).
Especially, for \( u = \alpha = (\alpha_1, 0, \cdots, 0) \in \mathbb{R}^N \) with \( \alpha_i \neq 0 \), we have
\[
q(u) = -\int_0^T (A(t)u(t), u(t)) dt = -\int_0^T (A(t)\alpha, \alpha) dt
\]
\[
= -|\alpha|^2 \int_0^T \xi_{11}(t) dt \geq 0.
\]
So \( \int_0^T \xi_{11}(t) dt \leq 0 \). Similarly, if we let \( u = \alpha = (0, \cdots, \alpha_i, \cdots, 0) \in \mathbb{R}^N \) with \( \alpha_i \neq 0, i \in \{2, \cdots, N\} \), it is easy to obtain that \( \int_0^T \xi_{ii}(t) dt \leq 0, \forall i \in \{2, \cdots, N\} \).

(b) If \( A(t) \) is semi-negative definite for all \( t \in [0, T] \), then it is easy to see that \( q(u) \geq 0, \forall u \in H^+_1 \). Then (54), (56) and (57) imply that \( H^- = \{0\} \). Thus we complete the proof.

**Proposition 2** (see [14], Proposition 4.2). Assume that \( A(t) \equiv A \) which is a constant matrix. Then \( H^- = \{0\} \) if and only if \( A \) is semi-negative definite.

**Proofs of Corollary 1 and Corollary 2.** Note that \( H^- = \{0\} \) if and only if \( \hat{L} \) has no negative eigenvalues. Then the proofs are easy to be completed by Proposition 1, Proposition 2 and Theorem 1.9.

**Proof of Theorem 1.10.** Let \( E = H^+_1 \). We use the same decomposition as in the proof of Theorem 1.9. We first construct \( A \) and \( B \) which satisfy assumptions in Lemma 2.3. It follows from (F2) that for any given \( 0 < \varepsilon_0 < \delta_1/2 \), there exists \( r > 0 \) such that
\[
F(t, x) \leq \varepsilon_0 |x|^2, \forall x \in \mathbb{R}^N, |x| \leq r, \text{ a.e. } t \in [0, T].
\]
If \( u \in H^+ \) and
\[
\|u\| = \rho = \frac{r}{C},
\]
then by (21), we have \( \|u\|_\infty \leq r \). Let \( B_\rho = \{ u \in H^+_1 : \|u\| < \rho \} \). Hence, for all \( u \in \partial B_\rho \cap H^+ \), it follows from (57) and (65) that
\[
\varphi(u) = q(u) - \int_0^T F(t, u(t)) dt \geq \frac{\delta_1}{2} \|u\|^2 - \varepsilon_0 \int_0^T |u(t)|^2 dt \geq \frac{\delta_1}{2} - 2\varepsilon_0 \|u\|^2 > 0.
\]
By (H4) and (F1), we know that there exists \( K_1 > 0 \) such that
\[
F(t, x) \geq \beta |x|^2 - K_1, \quad x \in \mathbb{R}^N, \text{ a.e. } t \in [0, T],
\]

Let \( u(t) = u^- + u^0 + sw_0(t) \), where \( s \geq 0 \) and \( w_0 \in H^+ \) is an eigenvector corresponding to \( \lambda_{+1} \). Since \( H^0 = \ker(I - K) \) is a finite dimensional space, there exist positive constants \( K_2 \) and \( K_3 \) such that

\[
K_2\|u^0\| \leq \|u^0\|_{L^2} \leq K_3\|u^0\|.
\]

Consequently, by (54), (56), (68) and the above inequality, we obtain

\[
\varphi(u) = q(u^-) + q(u^0) + \frac{1}{2} \int_0^T |sw_0(t)|^2 dt - \frac{s^2}{2} \int_0^T (A(t)w_0(t), w_0(t)) dt
\]

\[
-\int_0^T F(t, u(t)) dt
\]

\[
=q(u^-) + \frac{\lambda_{+1}s^2}{2} \int_0^T |w_0(t)|^2 dt - \int_0^T F(t, u(t)) dt
\]

\[
\leq -\frac{\delta_1}{2}\|u^-\|^2 + \frac{\lambda_{+1}s^2}{2} \int_0^T |w_0(t)|^2 dt - \beta s^2 \int_0^T |w_0(t)|^2 dt
\]

\[
- \beta \int_0^T |w_0(t)|^2 dt + K_1T
\]

\[
\leq -\frac{\delta_1}{2}\|u^-\|^2 + \left(\frac{\lambda_{+1}}{2} - \beta\right) \int_0^T |w_0(t)|^2 dt s^2 - \beta K_2^2\|u^0\|^2 + K_1T
\]

\[
\to -\infty \quad \text{as} \quad s^2 + \|u^-\|^2 + \|u^0\|^2 \to \infty.
\]

By (F1) and (53), it is easy to obtain that

\[
\varphi(u) \leq 0, \quad \forall u \in H^- \oplus H^0.
\]

For \( \varrho > \rho \), let

\[
A = \{ u \in H^- \oplus H^0 : \|u\| \leq \varrho \} \cup \{ sw_0 + u : u \in H^- \oplus H^0, s \geq 0, \|sw_0 + u\| = \varrho \},
\]

\[
B = \partial B_{\varrho} \cap H^+.
\]

By Example 3 of section 3.5 in [19], we know that \( A \) links \( B \). Moreover, we can choose sufficiently large \( \varrho \) such that

\[
\sup_A \varphi \leq 0 < \inf_B \varphi.
\]

By definitions of \( \Gamma \) and set \( A \), it is easy to know that

\[
\infty > c = \inf_{\Gamma \in \Phi} \sup_{s \in [0,1]} \varphi(\Gamma(s)u) \geq \inf_B \varphi > 0 \geq \sup_A \varphi. \tag{69}
\]

By Lemma 2.3 and Remark 8, we know there exists a sequence \( \{u_n\} \subset H_T^1 \) such that

\[
\varphi(u_n) = \frac{1}{2} \|\dot{u}_n\|_{L^2}^2 - \frac{1}{2} \int_0^T (A(t)u_n(t), u_n(t)) dt - \int_0^T F(t, u_n(t)) dt \to c, \tag{70}
\]

\[
(1 + \|u_n\|) \|\varphi'(u_n)\|_{(H_T^1)'} \to 0 \quad \text{as} \quad n \to \infty.
\]

Similar to the proof of Theorem 1.8, we can obtain a renamed subsequence such that \( u_n \to u \) strongly in \( H_T^1 \). Since \( \|\varphi'(u_n)\|_{(H_T^1)'} \to 0 \), it follows from the continuity of \( \varphi' \) in \( H_T^1 \) that \( \varphi'(u) = 0 \). So \( u \) is a \( T \)-periodic solution of system (1). Since \( \varphi \) is continuous, by (69) and (70), we know that

\[
\varphi(u) = c > 0.
\]
If $u = 0$, then by (F1) and (65), we know that $F(t, 0) = 0$. Hence,
$$\varphi(u) = 0,$$
which contradicts $\varphi(u) = c > 0$.

If $u \in \mathbb{R}^N$ and $A(t)$ is semipositive, then by (F1),
$$\varphi(u) = -\int_0^T F(t, u)dt \leq 0,$$
which contradicts $\varphi(u) = c > 0$. Thus the proof is complete. \qed

**Proof of Theorem 1.11.** Let
$$\tilde{H}^1_2 = \left\{ u \in H^1_2 \mid \int_0^T u(t)dt = 0 \right\}.$$ 

Then $H^1_2 = \mathbb{R}^N \oplus \tilde{H}^1_2$.

We first construct $A$ and $B$ which satisfy assumptions in Lemma 2.3. It follows from (19) that for any given $0 < \varepsilon_1 < \frac{2\pi^2}{T^2} - \frac{\|A\|}{2}$, there exists $r_1 > 0$ such that
$$F(t, x) \leq \varepsilon_1|t|^2, \quad \forall x \in \mathbb{R}^N, |x| \leq r_1, \ a.e. \ t \in [0, T].$$ \hspace{1cm} (71)

If $u \in \tilde{H}^1_2$ and
$$\|u\| = \rho_1 = \frac{r_1}{C},$$ \hspace{1cm} (72)
then by (21), we have $\|u\|_\infty \leq r_1$. Let $B_{\rho_1} = \{u \in H^1_2 : \|u\| < \rho_1\}$. Hence, for all $u \in \partial B_{\rho_1} \cap \tilde{H}^1_2$, it follows from (71) and Wirtinger’s inequality that
$$\varphi(u) = \frac{1}{2} \int_0^T |\dot{u}(t)|^2dt - \frac{1}{2} \int_0^T (A(t)u(t), u(t))dt - \int_0^T F(t, u(t))dt \geq \frac{1}{2} \int_0^T |\dot{u}(t)|^2dt - \frac{\|A\|}{2} \int_0^T |u(t)|^2dt - \varepsilon_1 \int_0^T |u(t)|^2dt$$
$$\geq \frac{1}{2} \int_0^T |\dot{u}(t)|^2dt - \frac{T^2}{4\pi^2} \left( \frac{\|A\|}{2} + \varepsilon_1 \right) \int_0^T |u(t)|^2dt$$
$$= \left( \frac{1}{2} - \frac{T^2\|A\|}{8\pi^2} + \frac{T^2\varepsilon_1}{4\pi^2} \right) \int_0^T |\dot{u}(t)|^2dt.$$ \hspace{1cm} (73)

Note that
$$\|A\| < \frac{4\pi^2}{T^2}, \quad \varepsilon_1 < \frac{2\pi^2}{T^2} - \frac{\|A\|}{2}$$
and $||\dot{u}||_{L^2}$ is equivalent to $\|u\|$ in $\tilde{H}^1_2$. Hence, (73) implies that
$$\varphi(u) > 0, \quad \forall u \in \partial B_{\rho_1} \cap \tilde{H}^1_2.$$ 

By (20) and (F1), there exist constants $\beta_1 > 2\pi^2/T^2$ and $K_4 > 0$ such that
$$F(t, x) \geq \beta_1|x|^2 - K_4, \quad \forall x \in \mathbb{R}^N, \ a.e. \ t \in [0, T].$$ \hspace{1cm} (74)

Let $u(t) = x + sw_1(t)$, where $x \in \mathbb{R}^N$, $s \geq 0$ and $w_1(t) = (\sin 2\pi t/T, 0, \ldots, 0) \in \tilde{H}^1_2$. Then
$$\int_0^T |w_1(t)|^2dt = \frac{T}{2}, \quad \int_0^T |\dot{w}_1(t)|^2dt = \frac{2\pi^2}{T}.$$
Since $A(t)$ is semi-positive definite, by (74), we obtain
\[
\varphi(x + sw_1) \leq \frac{1}{2} \int_0^T |sw_1'(t)|^2 dt - \int_0^T F(t, x + sw_1(t)) dt
\]
\[
\leq \frac{\pi^2 s^2}{T} - \frac{\beta_1}{T} \int_0^T |x + sw_1(t)|^2 dt + K_4 T
\]
\[
= \left( \frac{\pi^2}{T} - \frac{\beta_1 T}{2} \right) s^2 - \beta_1 T|x|^2 + K_4 T
\]
Hence, by $\beta_1 > 2 \pi^2 / T^2$, we have
\[
\varphi(x + sw_1) \to -\infty \quad \text{as} \quad s^2 + |x|^2 \to \infty.
\]
Moreover, Since $A(t)$ is semi-positive definite, by (F1), it is easy to obtain that
\[
\varphi(x) \leq 0, \quad \forall \ x \in \mathbb{R}^N.
\]
For $q_1 > \rho_1$, let
\[
A = \{ x \in \mathbb{R}^N : \| x \| \leq q_1 \} \cup \{ sw_1 + x : x \in \mathbb{R}^N, \ s \geq 0, \ \| sw_1 + x \| = q_1 \},
\]
\[
B = \partial B_{\rho_1} \cap \tilde{H}_T.
\]
By Example 3 of section 3.5 in [19], we know that $A$ links $B$. Moreover, we can choose sufficiently large $q_1$ such that
\[
\sup_A \varphi \leq 0 < \inf_B \varphi.
\]
The other arguments are similar to Theorem 1.10. We omit the details. □

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E-mail address: zhangxingyong1@163.com
E-mail address: tangxh@mail.csu.edu.cn