PROOF OF SOME CONJECTURAL CONGRUENCES INVOLVING DOMB NUMBERS

GUO-SHUAI MAO AND YAN LIU

Abstract. In this paper, we mainly prove the following conjectures of Z.-H. Sun [19]: Let $p > 3$ be a prime. If $p \equiv 1 \pmod{3}$ and $p = x^2 + 3y^2$, then we have

$$\sum_{k=0}^{p-1} \frac{D_k}{4^k} \equiv \sum_{k=0}^{p-1} \frac{D_k}{16^k} \equiv 4x^2 - 2p - \frac{p^2}{4x^2} \pmod{p^3},$$

and if $p \equiv 2 \pmod{3}$, then

$$\sum_{k=0}^{p-1} \frac{D_k}{4^k} \equiv -2 \sum_{k=0}^{p-1} \frac{D_k}{16^k} \equiv \frac{p^2}{2} \left( \frac{p-1}{p+3} \right)^{-2} \pmod{p^3},$$

where $D_n = \sum_{k=0}^{n} \binom{n}{k} \left( \frac{2k}{p} \right) \binom{2n-2k}{n-k}$ stands for the $n$th Domb number.

1. Introduction

It is known that the Domb numbers which were introduced by Domb are defined by the following sequence:

$$D_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \binom{2n-2k}{n-k}.$$ 

The $n$th Domb number also means the number of $2n$-step polygons on diamond lattice. Such sequence appears as coefficients in various series for $1/\pi$. For example, from [1] we know that

$$\sum_{n=0}^{\infty} \frac{5n + 1}{64^n} D_n = \frac{8}{\sqrt{3\pi}}.$$ 

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In [12], Rogers showed the following identity by using very advanced and complicated method,

\[ \sum_{n=0}^{\infty} D_n u^n = \frac{1}{1 - 4u} \sum_{k=0}^{\infty} \binom{2k}{k}^2 \binom{3k}{k} \left( \frac{u^2}{(1 - 4u)^3} \right)^k. \]

Y.-P. Mu and Z.-W. Sun [9] proved a congruence involving Domb numbers by telescoping method: For any prime \( p > 3 \), we have the supercongruence

\[ \sum_{k=0}^{p-1} \frac{3k^2 + k}{16^k} D_k \equiv -4p^4 q_p(2) \pmod{p^5}, \]

where \( q_p(a) \) denotes the Fermat quotient \( (a^{p-1} - 1)/p \).

Liu [4] proved some conjectures of Z.-W. Sun and Z.-H. Sun. For instance, Let \( n \) be a positive integer. Then

\[ \frac{1}{n} \sum_{k=0}^{n-1} (2k + 1) D_k 8^{n-1-k} \quad \text{and} \quad \frac{1}{n} \sum_{k=0}^{n-1} (2k + 1) D_k (-8)^{n-1-k} \]

are all positive integers.

Z.-H. Sun gave the following congruence conjecture of the Domb numbers in [19]:

**Conjecture 1.1.** Let \( p > 3 \) be a prime. Then

\[ D_{p-1} \equiv 64^{p-1} - \frac{p^3}{6} B_{p-3} \pmod{p^4}, \]

where \( \{ B_n \} \) are Bernoulli numbers given by

\[ B_0 = 1, \quad \sum_{k=0}^{n-1} \binom{n}{k} B_k = 0 \quad (n \geq 2). \]

This conjecture was confirmed by the first author and J. Wang [6]. For more researches on Domb numbers, we refer the readers to ([4, 16] and so on).

In [22], Z.-W. Sun proposed many congruence conjectures involving Domb numbers, for example [22, Conjecture 5.2]:

\[ \sum_{n=0}^{\infty} D_n u^n = \frac{1}{1 - 4u} \sum_{k=0}^{\infty} \binom{2k}{k}^2 \binom{3k}{k} \left( \frac{u^2}{(1 - 4u)^3} \right)^k. \]
**Conjecture 1.2.** Let $p > 3$ be a prime. We have

\[
\sum_{k=0}^{p-1} \frac{D_k}{4^k} - \sum_{k=0}^{p-1} \frac{D_k}{16^k} \\
\equiv \begin{cases} 
4x^2 - 2p & \text{if } p \equiv 1 \pmod{3} \& p = x^2 + 3y^2 \ (x, y \in \mathbb{Z}), \\
0 & \text{if } p \equiv 2 \pmod{3}.
\end{cases}
\]  

(1.1)

Z.-H. Sun [20, Theorem 5.1] proved this conjecture and proposed the following conjecture.

**Conjecture 1.3.** Let $p > 3$ be a prime. If $p \equiv 1 \pmod{3}$ and $p = x^2 + 3y^2$, then we have

\[
\sum_{k=0}^{p-1} \frac{D_k}{4^k} - \sum_{k=0}^{p-1} \frac{D_k}{16^k} \equiv 4x^2 - 2p - \frac{p^2}{4x^2} \pmod{p^3},
\]

and if $p \equiv 2 \pmod{3}$, then

\[
\sum_{k=0}^{p-1} \frac{D_k}{4^k} \equiv -2 \sum_{k=0}^{p-1} \frac{D_k}{16^k} \equiv \frac{p^2}{2} \left(\frac{p-1}{p-5}\right)^{-2} \pmod{p^3}.
\]

In this paper, our main goal is to prove conjecture 1.3.

**Theorem 1.1.** Conjecture 1.3 is true.

Z.-W. Sun [22] also conjectured that If $p \equiv 1 \pmod{3}$, then

\[
\sum_{k=0}^{p-1} (3k + 2) \frac{D_k}{4^k} \equiv \sum_{k=0}^{p-1} (3k + 1) \frac{D_k}{16^k} \equiv 0 \pmod{p^2}.
\]

Our second goal is to prove the following stronger result and thus prove the above conjecture:

**Theorem 1.2.** If $p \equiv 1 \pmod{3}$, then

\[
\sum_{k=0}^{p-1} (3k + 2) \frac{D_k}{4^k} \equiv 2 \sum_{k=0}^{p-1} (3k + 1) \frac{D_k}{16^k} \equiv \frac{p^2}{2} \left(\frac{p-1}{p-5}\right)^{-2} \pmod{p^3}.
\]

We also proof the following two conjectures of Z.-H. Sun in [21, Conjecture 3.5, Conjecture 3.6]: First, Sun defined that

\[
R_3(p) = \left(1 + 2p + \frac{4}{3}(2^{p-1} - 1) - \frac{3}{2}(3^{p-1} - 1)\right) \left(\frac{p-1}{|p/6|}\right)^2.
\]
Theorem 1.3. Let \( p > 3 \) be a prime. Then
\[
\sum_{k=0}^{p-1} k^2 D_k \equiv \begin{cases} 
\frac{16}{9} x^2 - \frac{8p}{9} - \frac{7p^2}{18x^2} & \text{mod } p^3, \quad \text{if } p = x^2 + 3y^2 \equiv 1 \pmod{3}, \\
-\frac{20}{9} R_3(p) & \text{mod } p^2, \quad \text{if } p \equiv 2 \pmod{3},
\end{cases}
\]
and if \( p \equiv 2 \pmod{3} \),
\[
\sum_{k=0}^{p-1} k^2 D_k \equiv \begin{cases} 
\frac{4}{9} x^2 - \frac{2p}{9} - \frac{p^2}{18x^2} & \text{mod } p^3, \quad \text{if } p = x^2 + 3y^2 \equiv 1 \pmod{3}, \\
\frac{4}{9} R_3(p) & \text{mod } p^2, \quad \text{if } p \equiv 2 \pmod{3},
\end{cases}
\]

\[
(2.1)
\]

Remark 1.1. We also can prove the other two congruences in [21, Conjecture 3.5, Conjecture 3.6], but the process of the proof is complex, so we will not give the details in this paper. Z.-H. Sun (private communication) conjectured (1.2) which was not given public.

We are going to prove Theorems 1.1 and 1.2 in Sections 2 and 3 respectively. Section 4 is devoted to proving Theorem 1.3. Our proofs make use of some combinatorial identities which can be found and proved by the package \texttt{Sigma} [13] via the software \texttt{Mathematica}. We also rely on the \( p \)-adic Gamma function, Gamma function.

2. Proof of Theorem 1.1

For a prime \( p \), let \( \mathbb{Z}_p \) denote the ring of all \( p \)-adic integers and let \( \mathbb{Z}_p^\times := \{a \in \mathbb{Z}_p : a \) is prime to \( p\} \). For each \( \alpha \in \mathbb{Z}_p \), define the \( p \)-adic order \( \nu_p(\alpha) := \max\{n \in \mathbb{N} : p^n | \alpha\} \) and the \( p \)-adic norm \( |\alpha|_p := p^{-\nu_p(\alpha)} \). Define the \( p \)-adic gamma function \( \Gamma_p(\cdot) \) by
\[
\Gamma_p(n) = (-1)^n \prod_{1 \leq k < n \atop (k,p)=1} k, \quad n = 1, 2, 3, \ldots,
\]
and
\[
\Gamma_p(\alpha) = \lim_{\nu_p(\alpha) \to 0} \Gamma_p(n), \quad \alpha \in \mathbb{Z}_p.
\]
In particular, we set \( \Gamma_p(0) = 1 \). Following, we need to use the most basic properties of \( \Gamma_p \), and all of them can be found in [10, 11]. For example, we know that
\[
\frac{\Gamma_p(x+1)}{\Gamma_p(x)} = \begin{cases} 
-x, & \text{if } |x|_p = 1, \\
-1, & \text{if } |x|_p > 1.
\end{cases}
\]
(2.1)
\[ \Gamma_p(1-x)\Gamma_p(x) = (-1)^{a_0(x)}, \]  
\hfill (2.2)

where \(a_0(x) \in \{1, 2, \ldots, p\}\) such that \(x \equiv a_0(x) \mod p\). Among the properties we need here is the fact that for any positive integer \(n\),

\[ z_1 \equiv z_2 \mod p^n \quad \text{implies} \quad \Gamma_p(z_1) \equiv \Gamma_p(z_2) \mod p^n. \]  
\hfill (2.3)

Our proof of Theorem 1.1 heavily relies on the following two transformation formulas due to Chan and Zudilin [2] and Sun [16] respectively,

\[ \sum_{k=0}^{n} \binom{n}{k} 2k \binom{2n-2k}{n-k} = \sum_{k=0}^{n} (-1)^k \binom{n}{3k} 2k \binom{3k}{k} 16^{n-k}, \]  
\hfill (2.4)

\[ \sum_{k=0}^{n} \binom{n}{k} 2k \binom{2n-2k}{n-k} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n+k}{3k} 2k \binom{3k}{k} 4^{n-2k}. \]  
\hfill (2.5)

**Lemma 2.1.** ([14, 15]) Let \(p > 5\) be a prime. Then

\[ H_{p-1}^{(2)} \equiv 0 \mod p, \quad H_{-1}^{(2)} \equiv 0 \mod p, \quad H_{p-1} \equiv 0 \mod p^2, \]

\[ \frac{1}{5} H_{\frac{p}{3}}^{(2)} \equiv H_{\frac{p}{3}}^{(2)} \equiv \frac{1}{2} \left( \frac{p}{3} \right) B_{p-2} \left( \frac{1}{3} \right) \mod p, \]

\[ H_{\frac{p}{3}} \equiv -2q_p(2) - \frac{3}{2} q_p(3) + pq_p^2(2) + \frac{3p}{4} q_p^2(3) - \frac{5p}{12} \left( \frac{p}{3} \right) B_{p-2} \left( \frac{1}{3} \right) \mod p^2, \]

\[ H_{\frac{2p}{3}}^{(2)} \equiv -3q_p(3) + 3q_p^2(3) - \frac{p}{6} \left( \frac{p}{3} \right) B_{p-2} \left( \frac{1}{3} \right) \mod p^2, \]

\[ H_{\frac{p+1}{2}}^{(2)} \equiv -2q_p(2) + pq_p^2(2) \mod p^2, \quad H_{\frac{2p+1}{3}}^{(2)} \equiv (-1)^{\frac{p-1}{4}} 4E_{p-3} \mod p, \]

\[ H_{\frac{3p}{2}}^{(2)} \equiv -3q_p(3) + 3q_p^2(3) + \frac{p}{3} \left( \frac{p}{3} \right) B_{p-2} \left( \frac{1}{3} \right) \mod p^2. \]

**Lemma 2.2.** Let \(p > 2\) be a prime and \(p \equiv 1 \mod 3\). If \(0 \leq j \leq (p-1)/2\), then we have

\[ \binom{3j}{j} \binom{p+j}{3j+1} \equiv \frac{p}{3j+1} (1 - pH_{2j} + pH_j) \mod p^3. \]
Proof. If \(0 \leq j \leq (p-1)/2\) and \(j \neq (p-1)/3\), then we have
\[
\binom{3j}{j} \binom{p+j}{3j+1} = \frac{(p+j) \cdots (p+1)p(p-1) \cdots (p-2j)}{j!(2j)!(3j+1)}
\]
\[
= \frac{pj!(1+pH_j)(-1)^{2j}(2j)!(1-pH_{2j})}{j!(2j)!(3j+1)}
\]
\[
= \frac{p}{3j+1}(1-pH_{2j}+pH_j) \pmod{p^3}.
\]
If \(j = (p-1)/3\), then by Lemma 2.1, we have
\[
\binom{p-1}{3j+1} = \binom{p-1}{3j+1} \cdot \cdots \cdot \binom{m-1}{3j+1} + \binom{m}{3j+1} \cdot \cdots \cdot \binom{m}{3j+1} \equiv m \cdot \cdots \cdot \binom{m}{3j+1} \equiv \binom{p}{3j+1}(1-pH_{2j}+pH_j) \pmod{p^3}.
\]
Now the proof of Lemma 2.2 is complete.

\[\Box\]

**Lemma 2.3.** Let \(p > 3\) be a prime. For any \(p\)-adic integer \(t\), we have
\[
\binom{2p-2}{3j+1} + pt \equiv \binom{2p-2}{3j+1} \cdot \cdots \cdot \binom{m-1}{3j+1} + \binom{m}{3j+1} \equiv 1 - \frac{p^2}{2} \left( \frac{B}{3} \right) B_{p-2} \left( \frac{1}{3} \right) \pmod{p^3}.
\]

Proof. Set \(m = (2p-2)/3\). It is easy to check that
\[
\binom{m+pt}{(p-1)/2} = \frac{(m+pt) \cdots (m+pt-(p-1)/2+1)}{(p-1)/2)!}
\]
\[
= \frac{m \cdots (m-(p-1)/2+1)}{(p-1)/2} \cdot (1+pt(H_m-H_{m-(p-1)/2}))
\]
\[
= \binom{m}{(p-1)/2}(1+pt(H_m-H_{m-(p-1)/2}) \pmod{p^2}.
\]

So Lemma 2.3 is finished.

\[\Box\]

**Proof of Theorem 1.1.** Firstly, we prove the first congruence.
Case $p \equiv 1 \pmod{3}$. With the help of (2.5), we have

$$
\sum_{k=0}^{p-1} \frac{D_k}{4^k} = \sum_{k=0}^{p-1} \frac{1}{4^k} \sum_{j=0}^{\lfloor k/2 \rfloor} \binom{k+j}{3j} \binom{2j}{j}^2 \binom{3j}{j} 4^{k-2j}
$$

$$
= \sum_{j=0}^{(p-1)/2} \frac{(p-1/2)^2 \binom{2j}{j}^2 \binom{3j}{j}}{16^j} \sum_{k=2j}^{p-1} \binom{k+j}{3j}.
$$

(2.6)

By loading the package \texttt{Sigma} in the software \texttt{Mathematica}, we have the following identity:

$$
\sum_{k=2j}^{n-1} \binom{k+j}{3j} = \binom{n+j}{3j+1}.
$$

Thus, replacing $n$ by $p$ in the above identity and then substitute it into (2.6), we have

$$
\sum_{k=0}^{p-1} \frac{D_k}{4^k} = \sum_{j=0}^{(p-1)/2} \frac{(p-1/2)^2 \binom{2j}{j}^2 \binom{3j}{j}}{16^j} \binom{p+j}{3j+1}.
$$

Hence we immediately obtain the following result by Lemma 2.2,

$$
\sum_{k=0}^{p-1} \frac{D_k}{4^k} \equiv p \sum_{j=0}^{(p-1)/2} \frac{(2j)^2 \binom{3j}{j}}{16^j} \frac{1 - pH_{2j} + pH_j}{3j+1} \pmod{p^3}.
$$

(2.7)

Since $\binom{2k}{k}/16^k \equiv \binom{p-1/2}{k} \binom{(p-1)/2 + k}{k} (-1)^k \pmod{p^2}$ for each $0 \leq k \leq (p-1)/2$, it is easy to verify that

$$
\sum_{j=0}^{p-1/2} \frac{(2j)^2}{16^j} \frac{H_j - H_{2j}}{3j+1} \equiv \sum_{j=0}^{p-1/2} \frac{(-1)^j (H_j - H_{2j})}{3j+1} \pmod{p}.
$$

By \texttt{Sigma}, we found the following identity:

$$
\sum_{k=0}^{n} \frac{n}{k} \binom{n+k}{3k+1} \frac{(H_k - H_{2k})}{3k+1} = \frac{1}{3n+1} \prod_{k=1}^{n} \frac{3k - 1}{3k - 2} \sum_{k=1}^{n} \frac{1}{k} \prod_{j=1}^{k} \frac{3j - 2}{3j - 1}.
$$

(2.8)
In view of [7], we have
\[
\sum_{k=1}^{\frac{p-1}{3}} 4^k \equiv -2 + \frac{2}{\left(\frac{p-1}{3}\right)} \equiv -2 + \frac{1}{x} \pmod{p},
\]
\[
3 \sum_{j=1}^{\frac{p-1}{3}} \frac{4^j}{(3j-1)(^{2j}_j)} \equiv -2 + \frac{1}{x} + \frac{1}{3} \left(\frac{p-1}{2}\right) \sum_{k=1}^{\frac{p-1}{3}} 4^k \pmod{p}.
\]
So by [17, Lemma 3.1], we have
\[
\sum_{k=1}^{\frac{p-1}{3}} \frac{1}{k^2\left(\frac{2k}{3}\right)} \equiv \frac{p}{3} \sum_{k=1}^{\frac{p-1}{3}} k^2\left(\frac{-2/3}{k}\right) - \sum_{k=1}^{\frac{p-1}{3}} \frac{3}{3k-1} - 3p \sum_{k=1}^{\frac{p-1}{3}} \frac{(-1)^k}{(3k-1)^2} - p\left(\frac{p-1}{2}\right) \sum_{k=1}^{\frac{p-1}{3}} k^2\left(\frac{-1/3}{k}\right) \pmod{p^2}.
\]
It is easy to check that
\[
\sum_{k=1}^{\frac{p-1}{3}} 4^k \equiv 2 \sum_{k=0}^{\frac{p-1}{3}} (-1)^k \pmod{p}.
\]
And by [23, (6)], we have
\[
\frac{1}{n+1+k} = (n+1) \sum_{r=0}^{n} \binom{n}{r} (-1)^r \frac{1}{k + r + 1}, \quad (2.9)
\]
Hence, setting \(n = \frac{p-1}{2}\) in the above identity, we have
\[
2 \sum_{k=0}^{\frac{p-1}{3}} \frac{(-1)^k}{(k+1)\left(\frac{k-1}{3}\right)} \equiv 2 \sum_{k=0}^{\frac{p-1}{3}} \frac{1}{(k+1)\left(\frac{k-1}{3}\right)} \equiv \sum_{k=0}^{\frac{p-1}{3}} \frac{1}{k+1} \sum_{r=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{r} \frac{(-1)^r}{k + 1 + r} = \sum_{k=1}^{\frac{p-1}{3}} \sum_{r=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{r} \frac{(-1)^r}{k + r} \pmod{p}.
\]
It is easy to obtain that
\[
\sum_{k=1}^{\frac{p-1}{3}} \left(\frac{1}{k} - \frac{1}{k + r}\right) \equiv - \sum_{k=1}^{r} \frac{1}{k(3k-1)} \pmod{p}.
\]
And by Sigma, we find the following identity which can be proved by induction on $n$:

$$\sum_{r=1}^{n} \binom{n}{r} \frac{(-1)^r}{r} \sum_{k=1}^{r} \frac{1}{k(3k-1)} = H_n^{(2)} - \sum_{k=1}^{n} \frac{(-1)^k}{k^2(-2/3)}.$$ 

So we have

$$\sum_{k=1}^{p-1} \frac{4^k}{k^2(2k)} \equiv 2 \sum_{k=0}^{p-4} \frac{(-1)^k}{(k+1)\binom{2k-3}{k}} \equiv H_{\frac{p-1}{3}}^{(2)} - H_{\frac{p-1}{3}}^{(2)} + \sum_{k=1}^{\frac{p-1}{3}} \frac{(-1)^k}{k^2(-2/3)} \pmod{p}.$$ 

Then by [24, Theorem 4.12] and Lemma 2.1, we have

$$\sum_{k=1}^{\frac{p-1}{3}} \frac{1}{3k-1} \prod_{j=1}^{k} 3j-2 \equiv -3 \sum_{k=1}^{\frac{p-1}{3}} \frac{1}{3k-1} - 3p \sum_{k=1}^{\frac{p-1}{3}} \frac{1}{3k-1}^2 - \frac{p}{3} H_{\frac{p-1}{3}}^{(2)}$$

$$\equiv 3 \sum_{k=1}^{\frac{p-1}{3}} \frac{1}{3k-1} - \frac{p}{3} H_{\frac{p-1}{3}}^{(2)} \equiv 0 \pmod{p^2},$$

where we used [15, Lemma 2.3, Lemma 2.6, Lemma 2.7], which help us deduce that

$$\sum_{k=1}^{\frac{p-1}{3}} \frac{1}{3k-1} = \sum_{k=1}^{\frac{p-1}{3}} \frac{1}{k} \equiv \frac{B_{\phi(p^2)\left(\frac{1}{3}\right)}}{3\phi(p^2)} - \frac{B_{\phi(p^2)\left(\frac{2}{3}\right)}}{3\phi(p^2)} + \frac{p}{9} \left(\frac{B_{2p-3\left(\frac{1}{3}\right)}}{2p-3} - 2 \frac{B_{p-2\left(\frac{1}{3}\right)}}{p-2}\right)$$

$$= 0 + \frac{p}{9} \left(\frac{B_{p-1+p-2\left(\frac{1}{7}\right)}}{p-1+p-2} - 2 \frac{B_{p-2\left(\frac{1}{7}\right)}}{p-2}\right) \equiv -\frac{p}{9} \frac{B_{p-2\left(\frac{4}{7}\right)}}{p-2} \equiv \frac{p}{18} B_{p-2\left(\frac{1}{3}\right)} \pmod{p^2}.$$ 

So it is easy to see that

$$\frac{2}{3p-1} \prod_{k=1}^{\frac{p-1}{3}} \frac{3k-1}{3k-2} \sum_{k=1}^{\frac{p-1}{3}} \frac{1}{k} \prod_{j=1}^{k} \frac{3j-2}{3j-1} \equiv 0 \pmod{p}.$$
And hence,

\[
\sum_{j=0}^{p-1} \frac{(2j)^2}{16^j} H_j - H_{2j} \equiv \sum_{j=0}^{p-1} \frac{\binom{p-1}{j} \left( \binom{p-1}{j} + j \right) (-1)^j (H_j - H_{2j})}{3j + 1} \equiv 0 \pmod{p}.
\]

(2.10)

In view of [8], we have

\[
p \sum_{j=0}^{(p-1)/2} \frac{(2j)^2}{16^j} \frac{1}{3j + 1} \equiv 4x^2 - 2p - \frac{p^2}{4x^2} \pmod{p^3}.
\]

So the case \(p \equiv 1 \pmod{3}\) is finished.

Case \(p \equiv 2 \pmod{3}\). In the same way of above, we have

\[
\sum_{k=0}^{p-1} \frac{D_k}{4^k} \equiv \sum_{j=0}^{(p-1)/2} \frac{(2j)^2}{16^j} \frac{1}{3j + 1} = p \sum_{j=0}^{(p-1)/2} \frac{(2j)^2}{16^j} \frac{1}{3j + 1} - pH_{2j} + pH_j
\]

\[
\equiv p \sum_{j=0}^{(p-1)/2} \frac{(2j)^2}{16^j} \frac{1}{3j + 1} - pH_{2j} + pH_j
\]

\[
\equiv \frac{2p}{3p - 1} \frac{(2/3)_{(p-1)/2}}{(1/3)_{(p-1)/2}} + \frac{2p^2}{3p - 1} \frac{(2/3)_{(p-1)/2}}{(1/3)_{(p-1)/2}} \sum_{k=1}^{p-1} \frac{1}{k \left( \frac{2}{3} \right)_k} \pmod{p^3},
\]

where we used the following identity and (2.8):

\[
\sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \frac{(-1)^k}{3k+1} = \frac{1}{3n+1} \prod_{k=1}^{n} \frac{3k-1}{3k-2}.
\]
It is easy to see that

\[ p \sum_{k=1}^{p-1} \frac{1}{k + \left( \frac{2}{3} \right) k} \equiv p \sum_{k=1}^{p-1} \frac{1}{(k + \left( \frac{2}{3} \right) k) k} = p \sum_{k=1}^{p-1} \frac{1}{(k + \left( \frac{5}{3} \right) k) k} = p \sum_{k=1}^{p-1} \frac{1}{\left( k + \left( \frac{5}{3} \right) k + \frac{2}{3} \right) k} \]

\[ \equiv 3 \sum_{k=1}^{p-1} \frac{1}{\left( k + \frac{2}{3} \right) k} = \frac{(1/3)(-1)^{k+2/3}(k + \frac{5}{3})!}{3} \]

\[ \equiv 3 \sum_{k=1}^{p-1} \left( \frac{-1/3}{k + \frac{2}{3}} \right) \left( k + \frac{2}{3} \right) \left( k - 1 \right) \equiv -3 \sum_{k=1}^{p-1} \left( \frac{-1/3}{k + \frac{2}{3}} \right) \left( k - 1 \right) \pmod{p} \] (2.11)

We can find and prove the following identity by Sigma:

\[ \sum_{k=1}^{n} \left( -\frac{1}{3} \right) \left( \frac{-1}{3} \right) \left( \frac{n}{2n-k} \right) \left( \frac{1}{k - 1} \right) = \frac{3n}{6n-1} \prod_{k=1}^{n} \left( 3k-2 \right) \left( 6k-1 \right). \]

So by substituting \( n = (p + 1)/6 \) into the above identity and (2.3), we have

\[ p \sum_{k=1}^{p+1} \frac{1}{k + \left( \frac{2}{3} \right) k} = 3 \sum_{k=1}^{p+1} \frac{1}{(k + \left( \frac{5}{6} \right) k) k} = 3 \sum_{k=1}^{p+1} \frac{1}{\left( k + \left( \frac{5}{6} \right) k + \frac{1}{6} \right) k} = 3 \sum_{k=1}^{p+1} \frac{1}{\left( k + \left( \frac{5}{6} \right) k + \frac{1}{6} \right) k} \]

\[ \equiv \frac{3(-1)^{p+1/6} \Gamma_p \left( \frac{2}{3} + \frac{1}{6} \right) \Gamma_p \left( \frac{1}{3} \right)}{2} \equiv \frac{3(-1)^{p+1/6} \Gamma_p \left( \frac{1}{3} \right) \Gamma_p \left( \frac{1}{3} \right)}{2} \pmod{p} \] (mod p).

Hence, by (2.2), we have

\[ p \sum_{k=1}^{p+1} \frac{1}{k + \left( \frac{2}{3} \right) k} \equiv \frac{3(-1)^{p+1/6} (-1)^{p+1/3}}{2} = -\frac{3}{2} \pmod{p}. \] (2.12)

So

\[ \sum_{k=0}^{p-1} \frac{D_k}{4^k} \equiv -\frac{1}{2} \frac{2p}{3p - 1} \frac{(2/3)(p-1)/2}{(1/3)(p-1)/2} \pmod{p^3}. \] (2.13)
And it is easy to see that
\[
\frac{2p}{3p - 1} \frac{(2/3)_{(p-1)/2}}{(1/3)_{(p-1)/2}} = -2p^2 \frac{2 \cdot \cdots \cdot (\frac{p}{3} - 1)}{3} \cdot \frac{\Gamma_{(p+1)/2}}{\Gamma_{(p-1)/2}} \quad \text{(mod } p^3)\]
\[
\equiv \frac{2p^2}{(\frac{p}{3})_{(p-1)/2}^2} \Gamma_{(p+1)/2} \Gamma_{(p-1)/2} \quad \text{(mod } p^3)\]
\[
\equiv \frac{2p^2}{(\frac{p}{3})_{(p-1)/2}^2} \Gamma_{(p+1)/2} \Gamma_{(p-1)/2} \quad \text{(mod } p^3)\]

Then by (2.1), (2.2) and (2.3) we have
\[
\frac{\Gamma_{(p+1)/2} \Gamma_{(p-1)/2}}{\Gamma_{(p+1)/2} \Gamma_{(p-1)/2}} = (-1)^{\frac{p+1}{6}} \Gamma_{(p+1)/2} \Gamma_{(p-1)/2} = \frac{(-1)^{\frac{p+5}{6}}}{6} \Gamma_{(p+1)/2} \Gamma_{(p-1)/2} \Gamma_{(p+1)/2} \Gamma_{(p-1)/2} \equiv \frac{\Gamma_{(p+1)/2}}{\Gamma_{(p+1)/2}} \quad \text{(mod } p)\]

Thus,
\[
\frac{2p}{3p - 1} \frac{(2/3)_{(p-1)/2}}{(1/3)_{(p-1)/2}} = -p^2 \left(\frac{p-1}{p-5}\right)^2 \quad \text{(mod } p^3) \quad \text{(2.14)}
\]

This, with (2.13) yields that
\[
\sum_{k=0}^{p-1} \frac{D_k}{4^k} = p^2 \left(\frac{p-1}{p-5}\right)^2 \quad \text{(mod } p^3)\]

Therefore we obtain the desired result
\[
\sum_{k=0}^{p-1} \frac{D_k}{4^k} = \begin{cases} 
4x^2 - 2p - \frac{p^2}{x^2} & \text{if } p \equiv 1 \pmod{3} \text{ & } p = x^2 + 3y^2 \ (x, y \in \mathbb{Z}), \\
\frac{p^2}{2} \left(\frac{p-1}{p-5}\right)^2 & \text{if } p \equiv 2 \pmod{3}.
\end{cases}
\]

On the other hand, in view of [19, (5.5)], we have
\[
\sum_{k=0}^{p-1} \frac{D_k}{16^k} \equiv \sum_{k=0}^{p-1} \frac{(2^k)^2}{16k+1} \frac{p}{16k+1} (1 + pH_{2k} - pH_k) \quad \text{(mod } p^3)\]

This, with (2.10) yields that if \( p \equiv 1 \pmod{3} \) and \( p = x^2 + 3y^2 \),
\[
\sum_{k=0}^{p-1} \frac{D_k}{16^k} \equiv p \sum_{j=0}^{(p-1)/2} \frac{(2j)^2}{16j+1} = 4x^2 - 2p - \frac{p^2}{4x^2} \quad \text{(mod } p^3)\]
and if $p \equiv 2 \pmod{3}$,

$$\sum_{k=0}^{p-1} \frac{D_k}{16^k} - Y_1 \equiv p \sum_{j=0}^{(p-1)/2} \frac{(\frac{p-1}{2})^2}{16^j} \frac{1 + pH_{2j} - pH_j}{3j + 1}$$

$$\equiv p \sum_{j=0}^{\frac{p-1}{2}} \frac{(\frac{p-1}{2})\left(\frac{p+1}{2}j\right)(-1)^j}{3j + 1} - p^2 \sum_{j=0}^{\frac{p-1}{2}} \frac{(\frac{p-1}{2})\left(\frac{p+1}{2}j\right)(-1)^j(H_j - H_{2j})}{3j + 1}$$

$$= \frac{2p \cdot (2/3)(p-1)/2}{3p - 1} (1/3)(p-1)/2 - \frac{2p^2 \cdot (2/3)(p-1)/2}{3p - 1} \sum_{k=1}^{p/2} (\frac{1}{3})_k$$

$$\equiv 5 \frac{2p \cdot (2/3)(p-1)/2}{23p - 1} (1/3)(p-1)/2 \pmod{p^3}, \quad (2.15)$$

where

$$Y_1 = \frac{1}{2} \frac{\left(\frac{4p-2}{3}\right)^2}{16^p - 1} (1 + pH_{4p-2} - pH_{2p-1}) = \frac{1}{2} \left(\frac{-1/2}{3}\right)^2 \left(1 + pH_{4p-2} - pH_{2p-1}\right).$$

It is easy to see that

$$\left(\frac{-1/2}{3}\right)^2 = \left(\frac{-1/2}{3}\right)^2 \left(\frac{-p/2}{3}\right)^2 = \frac{9p^2}{(p-1)^2} \pmod{p^3}.$$

And by (2.1), (2.2) and (2.3), we have

$$\left(\frac{2p-1}{3}\right)^2 \equiv \frac{4 \left(\frac{2p+2}{3}\right)^2}{\Gamma\left(\frac{2p+2}{3}\right)^2} = \frac{4 \Gamma_p\left(\frac{2p+2}{3}\right)^2}{\Gamma_p\left(\frac{2p+2}{3}\right)^2} = \frac{4 \Gamma_p\left(\frac{2p+2}{3}\right)^2}{\Gamma_p\left(\frac{2p+2}{3}\right)^2} = \frac{36\Gamma_p\left(\frac{2p+2}{3}\right)^2}{\Gamma_p\left(\frac{2p+2}{3}\right)^2} = \frac{36\left(\frac{2p+2}{3}\right)^2}{\left(\frac{2p+2}{3}\right)^2} \equiv 4 \left(\frac{p-1}{3}\right)^2 \equiv 4 \left(\frac{p-1}{3}\right)^2 \pmod{p}.$$

It is easy to see that

$$1 + pH_{4p-2} - pH_{2p-1} \equiv 2 \pmod{p}.$$  

These yield that

$$Y_1 \equiv \frac{9p^2}{4} \left(\frac{p-1}{3}\right)^2 \pmod{p^3}.$$

This, with (2.14) and (2.15) yields the desired result

$$\sum_{k=0}^{p-1} \frac{D_k}{16^k} \equiv -p^2 \left(\frac{p-1}{3}\right)^2 \pmod{p^3}.$$
Now we finish the proof of Theorem 1.1.

3. Proof of Theorem 1.2

Lemma 3.1. Let \( p > 2 \) be a prime. If \( 0 \leq j \leq (p-1)/2 \), then we have
\[
(3j + 1) \binom{3j}{j} \binom{p + 2j}{3j + 1} \equiv p(-1)^j(1 + pH_{2j} - pH_j) \pmod{p^3}.
\]
If \( (p+1)/2 \leq j \leq p-1 \), then
\[
(3j + 1) \binom{3j}{j} \binom{p + 2j}{3j + 1} \equiv 2p^2(-1)^j(H_{2j} - H_j) \pmod{p^3}.
\]

Proof. If \( 0 \leq j \leq (p-1)/2 \), then we have
\[
(3j + 1) \binom{3j}{j} \binom{p + 2j}{3j + 1} = \frac{(p + 2j) \cdots (p+1)p(p-1) \cdots (p-j)}{j!(2j)!}
\]
\[
= \frac{p(2j)!(1+PH_{2j})(-1)^j(j)!(1-PH_j)}{j!(2j)!}
\]
\[
= p(-1)^j(1 + pH_{2j} - pH_j) \pmod{p^3}.
\]
If \( (p+1)/2 \leq j \leq p-1 \), then by Lemma 2.1, we have
\[
(3j + 1) \binom{3j}{j} \binom{p + 2j}{3j + 1} = \frac{(p + 2j) \cdots (2p+1)(2p)(2p-1) \cdots (p+1)p(p-1) \cdots (p-j)}{j!(2j)!}
\]
\[
= \frac{2p^2(2j) \cdots (p+1) \left(1 + p \sum_{k=p+1}^{2j} \frac{1}{k}\right) (p-1)!(1-PH_j)}{j!(2j)!}
\]
\[
= 2p(-1)^j \left(1 + p \sum_{k=p+1}^{2j} \frac{1}{k}\right) (1-PH_j) \equiv 2p(-1)^jPH_{2j}(1-PH_j)
\]
\[
\equiv 2p^2(-1)^j(H_{2j} - pH_{2j}H_j) \equiv 2p^2(-1)^j(H_{2j} - H_j) \pmod{p^3}.
\]
Now the proof of Lemma 3.1 is complete. \( \square \)

Proof of Theorem 1.2. Similarly, by (2.5), we have
\[
\sum_{k=0}^{p-1} (3k + 2) \frac{D_k}{4^k} = \sum_{k=0}^{p-1} \frac{3k + 2}{4^k} \sum_{j=0}^{\lfloor k/2 \rfloor} \binom{k+j}{3j} \binom{2j}{j} \binom{3j}{j} 4^{k-2j}
\]
\[
= \sum_{j=0}^{\lfloor (p-1)/2 \rfloor} \frac{\binom{2j}{j} \binom{3j}{j}}{16^j} \sum_{k=2j}^{p-1} (3k + 2) \binom{k+j}{3j}.
\]
By loading the package \texttt{Sigma} in the software \texttt{Mathematica}, we have the following identity:

\[
\sum_{k=2j}^{n-1} (3k + 2) \binom{k + j}{3j} = \frac{(3n + 1)(3j + 1)}{3j + 2} \binom{n + j}{3j + 1}.
\]

Thus, replacing \(n\) by \(p\) in the above identity and then substitute it into (3.1), we have

\[
\sum_{k=0}^{p-1} (3k + 2) \frac{D_k}{4^k} = \sum_{j=0}^{(p-1)/2} \frac{(2j)_j^2 (3j + 1)}{16^j} \frac{(3p + 1)(3j + 1)}{3j + 2} \binom{p + j}{3j + 1}.
\]

Combining Lemma 2.2 we can obtain that for any \(0 \leq j \leq (p - 1)/2,\)

\[(3j + 1) \binom{p + j}{3j + 1} \equiv p(1 - pH_j + pH_j) \pmod{p^3}.
\]

Since \(p \equiv 1 \pmod{3}\), so we have

\[
\frac{1}{3p + 1} \sum_{k=0}^{p-1} (3k + 2) \frac{D_k}{4^k} \equiv p \sum_{j=0}^{(p-1)/2} \frac{(2j)_j^2}{16^j} \frac{1 - pH_{2j} + pH_{j}}{3j + 2}
\]

\[
\equiv p \sum_{j=0}^{(p-1)/2} \frac{(2j)_j^2 (3j + 1)}{3j + 2} \frac{(p + j)(3j + 1)}{3j + 2} \equiv 2p \sum_{j=0}^{(p-1)/2} \frac{(3j + 1)}{3j + 2} \left(1 - \frac{1}{3j + 2}ight) (\pmod{p^3}).
\]

By the \texttt{Sigma} again, we find the following two identities:

\[
\sum_{k=0}^{n} \binom{n}{k} \binom{n + k}{k} \frac{(-1)^k}{3k + 2} = \frac{1}{3n + 2} \prod_{k=1}^{n} \frac{3k - 2}{3k - 1},
\]

\[
\sum_{k=0}^{n} \binom{n}{k} \binom{n + k}{k} \frac{(-1)^k (H_{2j} - H_j)}{3k + 2} = -\frac{1}{3n + 2} \prod_{k=1}^{n} \frac{3k - 2}{3k - 1} \prod_{k=1}^{n} \frac{3j - 1}{3j - 2}.
\]

Hence

\[
\sum_{k=0}^{p-1} (3k + 2) \frac{D_k}{4^k} \equiv 2p \prod_{k=1}^{p-1} \frac{3k - 2}{3k - 1} \left(1 + p \sum_{k=1}^{n} \frac{1}{k} \prod_{j=1}^{n} \frac{3j - 1}{3j - 2}\right) \pmod{p^3}.
\]

(3.2)
And it is easy to see that
\[
2p^{(1/3)(p-1)/2} \equiv \frac{2p^2}{3} \frac{1}{\frac{3}{3}} \cdots (\frac{p}{3} - 1) \frac{(p/3 + 1) \cdots (p/3 + p-7)}{\left(\frac{2}{3}\right)^{p-1}} \\
= -4p^2 \left(\frac{p-1}{2}\right) \frac{\Gamma_p(\frac{2}{3}) \Gamma_p(\frac{2}{3})}{\Gamma_p(\frac{2}{3} + \frac{1}{6})} \equiv (\mod p^3).
\]

Then by (2.1), (2.2) and (2.3) we have
\[
\Gamma_p(\frac{2}{3}) \Gamma_p(\frac{2}{3}) = \frac{\Gamma(\frac{p}{3}) \Gamma(\frac{p}{3})}{\Gamma_p(\frac{2}{3})} = (-1)^{\frac{p-1}{2}} \frac{\Gamma_p(\frac{2}{3} + \frac{1}{6})}{\Gamma_p(\frac{2}{3})} \\
\equiv (-1)^{\frac{p-1}{2}} \frac{\Gamma(\frac{p+1}{2}) \Gamma(\frac{p+1}{2})}{\Gamma(\frac{2p+1}{3})} = (-1)^{\frac{p+1}{6}} \frac{\Gamma(\frac{p+1}{2}) \Gamma(\frac{p+1}{2})}{\Gamma(\frac{2p+1}{3})} \\
\equiv (-1)^{\frac{p+1}{6}} \frac{1}{\left(\frac{2p-2}{3}\right)} \equiv -\frac{1}{\left(\frac{2p-2}{3}\right)} (\mod p).
\]

Thus,
\[
2p^{(1/3)(p-1)/2} \equiv -4p^2 \left(\frac{p-1}{2}\right) \left(\frac{p-1}{2}\right)^{-2} (\mod p^3).
\]

By similar manipulation as (2.1), we have
\[
p \sum_{k=1}^{\frac{p-1}{2}} \frac{(\frac{2}{3})_k}{k(\frac{1}{3})_k} \equiv -3 \sum_{k=1}^{\frac{p-1}{2}} \left(\frac{2/3}{p-1}_k - k\right) \left(\frac{-2/3}{k-1}\right) (\mod p).
\]

We can find and prove the following identity by Sigma:
\[
\sum_{k=1}^n \left(\frac{-2/3}{2n-k}\right) \left(\frac{-2/3}{k-1}\right) = -3n \prod_{k=1}^n \frac{(3k-1)(6k-5)}{9k(2k-1)}.
\]

So by substituting \(n = (p-1)/6\) into the above identity and (2.3), we have
\[
p \sum_{k=1}^{\frac{p-1}{2}} \frac{(\frac{2}{3})_k}{k(\frac{1}{3})_k} \equiv -3 \left(\frac{2}{3}\right) \left(\frac{p-1}{6}\right)\left(\frac{p-1}{6}\right) \equiv 3(-1)^{\frac{p+5}{6}} \frac{\Gamma_p(\frac{p}{6} + \frac{1}{3}) \Gamma_p(\frac{p}{6} + \frac{1}{3})}{2 \Gamma_p(\frac{2}{3}) \Gamma_p(\frac{2}{3})} \\
\equiv 3(-1)^{\frac{p+5}{6}} \frac{\Gamma(\frac{p}{3}) \Gamma(\frac{p}{3})}{\Gamma(\frac{2}{3}) \Gamma(\frac{2}{3})} (\mod p).
\]

Hence, by (2.2), we have
\[
p \sum_{k=1}^{\frac{p-1}{2}} \frac{(\frac{2}{3})_k}{k(\frac{1}{3})_k} \equiv -3 \left(\frac{1}{2}\right) \left(\frac{p+1}{2}\right) \left(\frac{2p+1}{3}\right) = -\frac{3}{2} (\mod p). \tag{3.4}
\]
This, with (3.2) and (3.3) yields that
\[ \sum_{k=0}^{p-1} (3k + 2) \frac{D_k}{4^k} \equiv 2p^2 \left( \frac{p-1}{2} \right)^{-2} (\text{mod } p^3). \]

In the same way, by (2.4), we have
\[ \sum_{k=0}^{p-1} (3k + 1) \frac{D_k}{16^k} = \sum_{k=0}^{p-1} \frac{3k + 1}{16^k} \sum_{j=0}^{k} (-1)^j \binom{k + 2j}{3j} \binom{2j}{j} \binom{3j}{j} 16^{k-j} \]
\[ = \sum_{j=0}^{p-1} \frac{(2j)^2}{(-16)^j} \sum_{k=j}^{p-1} \binom{k + 2j}{3j}. \]

By loading the package \texttt{Sigma} in the software \texttt{Mathematica}, we have the following identity:
\[ \sum_{k=j}^{n-1} (3k + 1) \binom{k + 2j}{3j} = \frac{(3n - 1)(3j + 1)}{3j + 2} \binom{n + 2j}{3j + 1}. \]

Thus, we have
\[ \sum_{k=0}^{p-1} (3k + 1) \frac{D_k}{16^k} = \sum_{j=0}^{p-1} \frac{(2j)^2}{(-16)^j} \frac{(3p - 1)(3j + 1)}{3j + 2} \frac{(p + 2j)}{3j + 1}. \]

It is known that \( \binom{2j}{j} \equiv 0 (\text{mod } p) \) for each \( p + 1/2 \leq j \leq p - 1 \), so combining Lemma 3.1, \( p \equiv 1 (\text{mod } 3) \), (3.3) and (3.4), we can obtain that
\[ \frac{1}{3p - 1} \sum_{k=0}^{p-1} (3k + 1) \frac{D_k}{16^k} \equiv p \sum_{j=0}^{(p-1)/2} \frac{1}{16^j} \frac{(3p - 1)(3j + 1)}{3j + 2} \frac{(p + 2j)}{3j + 1} + \]
\[ 2p^2 \sum_{j=\frac{p-1}{2}}^{p-1} \frac{(2j)^2}{16^j} \frac{H_{2j} - H_j}{3j + 2} \equiv -5 \frac{4p^2}{2} \left( \frac{p-1}{p-2} \right)^2 + p \left( \frac{\frac{p-1}{2}}{\frac{p-2}{3}} \right)^2 \left( H_{\frac{4p-4}{3}} - H_{\frac{2p-2}{3}} \right) \]
\[ = -10p^2 \left( \frac{\frac{p-1}{2}}{p-2} \right)^2 + \left( \frac{\frac{p-1}{2}}{\frac{p-2}{3}} \right)^2 (\text{mod } p^3), \]

where we used
\[ \left( \frac{\frac{p-1}{2}}{\frac{p-2}{3}} \right) \equiv 0 (\text{mod } p) \quad \text{and} \quad p \left( H_{\frac{4p-4}{3}} - H_{\frac{2p-2}{3}} \right) \equiv 1 (\text{mod } p). \]
It is easy to see that
\[
\left( -\frac{1}{2} \right)^2 = \frac{\left( -\frac{1}{2} - 1 \right) \cdots \left( -\frac{1}{2} - \frac{2p-2}{3} + 1 \right)}{(2p-2)!^2}
\]
\[
= \frac{(\frac{1}{2})^2(\frac{3}{2})^2 \cdots (\frac{p}{2} - 1)^2(\frac{p}{2} + 1)^2 \cdots (\frac{p}{2} + \frac{p-7}{6})^2}{(2p-2)!^2}
\]
\[
= \frac{p^2(\frac{p}{2} - \frac{1}{2})^2 \cdots (\frac{p}{2} - 1)^2(\frac{p}{2} + 1)^2 \cdots (\frac{p}{2} + \frac{p-7}{6})^2}{(2p-2)!^2}
\]
\[
\equiv \frac{p^2(\frac{p}{2} - \frac{1}{2})^2(\frac{p}{2} - \frac{7}{6})!^2}{(2p-2)!^2} = \frac{9p^2}{(p - 1)^2} \frac{1}{(2p-2)\left(\frac{p}{2} - 1\right)!^2} \equiv \frac{9p^2}{(\frac{p}{2} - 1)^2} \pmod{p^3}.
\]
Hence
\[
\sum_{k=0}^{p-1}(3k + 1)\frac{D_k}{16^k} \equiv p^2 \left( \frac{p-1}{2} \right)^{-2} \pmod{p^3}.
\]

Therefore, we get the desired result
\[
\sum_{k=0}^{p-1}(3k + 2)\frac{D_k}{4^k} \equiv 2 \sum_{k=0}^{p-1}(3k + 1)\frac{D_k}{16^k} \equiv 2p^2 \left( \frac{p-1}{2} \right)^{-2} \pmod{p^3}.
\]

Now the proof of Theorem 1.2 is complete. \(\square\)

4. PROOF OF THEOREM 1.3

Proof of Theorem 1.3. Similar as above, by (2.5), we have
\[
\sum_{k=0}^{p-1}k^2\frac{D_k}{4^k} = \sum_{k=0}^{p-1}k^2\sum_{j=0}^{[k/2]}\left( k+j \right)\left( 2j \right)\left( 3j \right) 4^{k-2j}
\]
\[
= \sum_{j=0}^{(p-1)/2} \frac{(2j)^2(3j)!}{16^j} \sum_{k=2j}^{p-1} k^2\left( k+j \right)\left( 3j \right).
\]
By Sigma, we have the following identity:
\[
\sum_{k=2j}^{p-1} k^2\left( k+j \right)\left( 3j \right)
\]
\[
= 1 - j^2 - n(2j + 3)(3j + 1) + n^2(3j + 1)(3j + 2) \left( \frac{n+j}{3j+2}(3j+3) \right).
\]
Thus,

\[\sum_{k=0}^{p-1} k^2 D_k \frac{4^k}{4^k} = \sum_{j=0}^{\frac{p-1}{2}} \frac{(2j)^2(3j)^2(p+j)}{16^j} \frac{1 - j^2 - p(2j + 3)(3j + 1) + p^2(3j + 1)(3j + 2)}{(3j + 2)(3j + 3)}.\]

In view of Lemma 2.2, if \( p \equiv 1 \pmod{3} \), then we have

\[\sum_{k=0}^{p-1} k^2 D_k \frac{4^k}{4^k} \equiv \sum_{k=0}^{p-1} \frac{(2k)^2}{16^k} \frac{p(1 - k^2) - p^2(1 - k^2)(H_{2k} - H_k) - p^2(2k + 3)(3k + 1)}{(3k + 1)(3k + 2)(3k + 3)} \]

\[\equiv \frac{p}{9} \sum_{k=0}^{p-1} \frac{(2k)^2}{16^k} \left( \frac{4}{3k + 1} - \frac{5}{3k + 2} \right) - \frac{p^2}{3} \sum_{k=0}^{p-1} \frac{(2k)^2}{16^k} \left( \frac{5}{3k + 2} - \frac{1}{k + 1} \right) \]

\[- \frac{p^2}{9} \sum_{k=0}^{p-1} \frac{(2k)^2}{16^k} (H_{2k} - H_k) \left( \frac{4}{3k + 1} - \frac{5}{3k + 2} \right) \pmod{p^3} \].

In view of the process of proving Theorems 1.1 and 1.2, [18, (3.5)] and [24], We have

\[\sum_{k=0}^{p-1} k^2 D_k \frac{4^k}{4^k} \equiv \frac{p}{9} \sum_{k=0}^{p-1} \frac{(2k)^2}{16^k} \left( \frac{4}{3k + 1} - \frac{5}{3k + 2} \right) + \frac{5p^2}{9} \sum_{k=0}^{p-1} \frac{(2k)^2}{16^k} (H_{2k} - H_k) \]

\[\equiv \frac{4}{9} \left( 4x^2 - 2p - \frac{p^2}{4x^2} \right) - \frac{5 - 4p^2}{9} \frac{3 - 4p^2}{4x^2} + \frac{5}{2} - \frac{9}{16p^2} - \frac{7p^2}{18x^2} \pmod{p^3}.\]
If \( p \equiv 2 \pmod{3} \), then modulo \( p^2 \), we have

\[
\sum_{k=0}^{p-1} k^2 D_k \frac{4^k}{4^k} \equiv \sum_{k=0}^{p-1} \frac{(2k)^2}{16^k} \frac{p(1 - k^2) - p^2(1 - k^2)(H_{2k} - H_k) - p^2(2k + 3)(3k + 1)}{(3k + 1)(3k + 2)(3k + 3)}
\]

\[
\equiv p \sum_{k=0}^{p-1} \frac{(2k)^2}{16^k} \left( \frac{4}{3k + 1} - \frac{5}{3k + 2} \right) - \frac{p^2}{3} \sum_{k=0}^{p-1} \frac{(2k)^2}{16^k} \left( \frac{5}{3k + 2} - \frac{1}{k + 1} \right)
\]

\[
- \frac{p^2}{9} \sum_{k=0}^{p-1} \frac{(2k)^2}{16^k} (H_{2k} - H_k) \left( \frac{4}{3k + 1} - \frac{5}{3k + 2} \right)
\]

\[
\equiv - \frac{5p(1 + 3p)}{9} \sum_{k=0}^{p-1} \frac{(2k)^2}{16^k(3k + 2)} + \frac{5p^2}{9} \sum_{k=0}^{p-1} \frac{(2k)^2}{16^k(3k + 2)} (H_{2k} - H_k).
\]

And similar as above, we have

\[
\frac{2p}{3p + 1} \left( \frac{1}{3} \right) \left( \frac{2p - 1}{3} \right) \left( \frac{p - 1}{p} \right) \left( \frac{p + 1}{p - 1} \right) (-1)^{p+1} (1 + 2p - \frac{2p}{3} q_p(2)) \pmod{p^2}.
\]

(4.1)

and

\[
\sum_{k=1}^{p-1} \frac{(\frac{2}{3})k}{k(\frac{2}{3})} k \equiv -3 \sum_{k=0}^{p-1} \frac{1}{3k + 1} \equiv -3 \sum_{k=1 \pmod{3}}^{p-1} \frac{1}{k} - 3 \equiv -3 \pmod{p}.
\]

So

\[
\sum_{k=0}^{p-1} \frac{(2k)^2}{16^k(3k + 2)} (H_{2k} - H_k) \equiv 3 \sum_{k=0}^{p-1} \frac{(2k)^2}{16^k(3k + 2)} \pmod{p}.
\]

(4.2)

Hence

\[
\sum_{k=0}^{p-1} k^2 D_k \frac{4^k}{4^k} \equiv - \frac{5p}{9} \sum_{k=0}^{p-1} \frac{(2k)^2}{16^k(3k + 2)}
\]

\[
\equiv - \frac{5}{9} \left( \frac{2p - 1}{3} \right) \left( \frac{p - 1}{2} \right) (-1)^{p+1} (1 + 2p - \frac{2p}{3} q_p(2)) \pmod{p^2}.
\]
It is easy to check that
\[
\left(\frac{2p-1}{p-1}\right)\binom{p-1}{\frac{p}{2}}\left(-1\right)^{\frac{p+1}{6}} \equiv \left(\frac{p-1}{2}\right)^2 \left(1+2pq(p) - \frac{3p}{2}q^3(p)\right)
\]
\[
\equiv 4\left(\frac{p-1}{2}\right)^2 \left(1+2pq(p) - \frac{3p}{2}q^3(p)\right) \pmod{p^2}.
\] (4.3)

Thus,
\[
\sum_{k=0}^{p-1} k^2D_k 16^k \equiv -\frac{20}{9} \left(\frac{p-1}{2}\right)^2 \left(1+2pq(p) - \frac{3p}{2}q^3(p)\right) \pmod{p^2}.
\]

So we obtain the first congruence in Theorem 1.3.

Now we consider the second congruence in Theorem 1.3. Similar as above, by (2.4), we have
\[
\sum_{k=0}^{p-1} k^2D_k 16^k = \sum_{k=0}^{p-1} k^2 \sum_{j=0}^{k} (-1)^j \binom{k+j}{3j} \binom{2j}{3j} \binom{3j}{j} 16^{k-j}
\]
\[
= \sum_{j=0}^{p-1} \binom{2j}{3j} \binom{3j}{j} \sum_{k=j}^{p-1} k^2 \binom{k+j}{3j}.
\]

By Sigma, we have the following identity:
\[
\sum_{k=j}^{n-1} k^2 \binom{k+j}{3j} = 1 + 3j + 2j^2 - n(4j + 3)(3j + 1) + n^2(3j + 1)(3j + 2) \binom{n+j}{3j+1}.
\]
Thus, if \( p \equiv 1 \pmod{3} \), then modulo \( p^3 \), we have
\[
\sum_{k=0}^{p-1} k^2D_k 16^k + \frac{1}{18p(2p+1)} \left(-\frac{1}{2}\right)^{2p-2} \binom{2p-2}{2p-1} \left(p+\frac{4p-4}{3}\right) \pmod{p^3}
\]
\[
\equiv \sum_{j=0}^{p-1} \binom{2j}{3j} \binom{3j}{j} \binom{p+2j}{3j+1} \left(1 + 3j + 2j^2 - p(4j + 3)(3j + 1) + p^2(3j + 1)(3j + 2)\right) \pmod{p^3}.
\]
Hence, similar as above, we have

\[ \sum_{k=0}^{p-1} k^2 D_k 16^k + \frac{1}{18p(2p+1)} \left( \frac{-1/2}{2p-2} \right)^2 \left( \frac{2p-2}{2p-2} \right) \left( p + \frac{4p-4}{2} \right) \]

\[ \equiv p \sum_{j=0}^{p-1} \left( \frac{2j}{16^j} \right)^2 \left( \frac{1}{3j+1} + \frac{1}{3j+2} \right) \left( p + \frac{4p-4}{2} \right) \]

\[ + \frac{p^2}{9} \sum_{j=0}^{p-1} \left( \frac{2j}{16^j} \right)^2 \left( \frac{1}{3j+1} + \frac{1}{3j+2} \right) \left( H_{2j} - H_j \right) \pmod{p^3}. \]

In view of the process of proving Theorems 1.1 and 1.2, [18, (3.5)] and [24], We have

\[ \sum_{k=0}^{p-1} k^2 D_k 16^k + \frac{1}{18p(2p+1)} \left( \frac{-1/2}{2p-2} \right)^2 \left( \frac{2p-2}{2p-2} \right) \left( p + \frac{4p-4}{2} \right) \]

\[ \equiv \frac{1}{9} \left( 4x^2 - 2p - \frac{p^2}{4x^2} \right) + \frac{1}{9} \left( 13 - 4p^2 \right) + \frac{1}{9} \left( 1 - 4p^2 \right) \]

\[ \equiv \frac{4x^2 - 2p}{9} - \frac{11p^2}{36x^2} \pmod{p^3}. \]

It is easy to see that

\[ \left( \frac{2p-2}{2p-2} \right) \left( p + \frac{4p-4}{2} \right) \equiv -2p \pmod{p^3}. \]

This, with the above \( \left( \frac{-1/2}{2p-2} \right)^2 \equiv 9p^2/(4x^2) \pmod{p^3}, \) we immediately get that

\[ \sum_{k=0}^{p-1} k^2 D_k 16^k \equiv \frac{4x^2 - 2p}{9} - \frac{11p^2}{36x^2} - \left( - \frac{p^2}{4x^2} \right) = \frac{4x^2 - 2p}{9} - \frac{p^2}{18x^2} \pmod{p^3}. \]

If \( p \equiv 2 \pmod{3}, \) then modulo \( p^2, \) we have

\[ \sum_{k=0}^{p-1} k^2 D_k 16^k \equiv \frac{2j}{16^j} \left( \frac{2j}{3j+1} \right) \left( p + 2j \right) \left( 3j+1 \right) + p(4j+3)(3j+1) + p^2(3j+1)(3j+2) \]

\[ \equiv \frac{2j}{16^j} \frac{2j}{3j+1} \left( 3j+1 \right) + \frac{2j}{3j+1} \left( 3j+1 \right) + \frac{2j}{3j+1} \left( 3j+2 \right) \left( 3j+3 \right). \]
Hence, similar as above, we have

\[
\sum_{k=0}^{p-1} k^2 \frac{D_k}{16^k} = \frac{p}{9} \sum_{j=0}^{p-1} \frac{(2j)^2}{16^j} \left( \frac{1}{3j+1} + \frac{1}{3j+2} \right) - \frac{p^2}{3} \sum_{j=0}^{p-1} \frac{(2j)^2}{16^j} \left( \frac{1}{3j+2} + \frac{1}{j+1} \right) \\
+ \frac{p^2}{9} \sum_{j=0}^{p-1} \frac{(2j)^2}{16^j} \left( \frac{1}{3j+1} + \frac{1}{3j+2} \right) (H_{2j} - H_j) \\
\equiv \frac{p}{9} \sum_{j=0}^{p-1} \frac{(2j)^2}{(3j+2)16^j} - \frac{p^2}{3} \sum_{j=0}^{p-1} \frac{(2j)^2}{(3j+2)16^j} + \frac{p^2}{9} \sum_{j=0}^{p-1} \frac{(2j)^2}{(3j+2)16^j} (H_{2j} - H_j) \\
\equiv \frac{p}{9} \sum_{j=0}^{p-1} \frac{(2j)^2}{(3j+2)16^j} = -\frac{1}{5} \sum_{k=0}^{p-1} k^2 \frac{D_k}{4^k} = \frac{4}{9} R_3(p) \pmod{p^2}.
\]

Now the proof of the second congruence in Theorem 1.3 is complete.

*Proof of (1.2).* Similarly,

\[
\sum_{k=0}^{p-1} k^2 \frac{D_k}{4^k} = \sum_{k=0}^{p-1} k^2 \frac{k}{3j} \sum_{j=0}^{k/2} \left( \frac{k+j}{3j} \right) \left( \frac{2j}{j} \right) \left( \frac{3j}{j} \right) 4^{k-2j} \\
= \sum_{j=0}^{p-1} \frac{(2j)^2 (3j)}{16^j} \sum_{k=2j}^{p-1} \left( \frac{k+j}{3j} \right).
\]

By *Sigma*, we find the following identity which can be proved by induction on \( n \):

\[
\sum_{k=2j}^{n} k \left( \frac{k+j}{3j} \right) = \frac{3nj + n - j - 1}{3j+1} \left( \frac{n+j}{3j+1} \right).
\]

Hence

\[
\sum_{k=0}^{p-1} k \frac{D_k}{4^k} = \sum_{j=0}^{p-1} \frac{(2j)^2 (3j)}{16^j} \left( 3pj + p - j - 1 \right) \left( \frac{p+j}{3j+1} \right).
\]
In view of Lemma 2.2, and $p \equiv 2 \pmod{3}$, then modulo $p^2$ we have

$$\sum_{k=0}^{p-1} \frac{D_k}{4^k} \equiv \sum_{k=0}^{p-1} \frac{(2k)^2}{16^k} \frac{p(-1-k) + p^2(1+k)(H_{2k} - H_k) + p^2(3k+1)}{(3k+1)(3k+2)}$$

$$\equiv - \frac{p}{3} \sum_{k=0}^{p-1} \frac{(2k)^2}{16^k} \left( \frac{2}{3k+1} - \frac{1}{3k+2} \right) + p^2 \sum_{k=0}^{p-1} \frac{(2k)^2}{(3k+2)16^k}$$

$$+ \frac{p^2}{3} \sum_{k=0}^{p-1} \frac{(2k)^2}{16^k} \frac{H_{2k} - H_k}{3k+1} \left( \frac{2}{3k+1} - \frac{1}{3k+2} \right)$$

$$\equiv \frac{p^2}{3} \sum_{k=0}^{p-1} \frac{(2k)^2}{(3k+2)16^k} + p^2 \sum_{k=0}^{p-1} \frac{(2k)^2}{(3k+2)16^k} - \frac{p^2}{3} \sum_{k=0}^{p-1} \frac{(2k)^2}{(3k+2)16^k} (H_{2k} - H_k).$$

By (4.1)-(4.3), we have

$$\sum_{k=0}^{p-1} k D_k \equiv \frac{p}{3} \sum_{k=0}^{p-1} \frac{(2k)^2}{(3k+2)16^k}$$

$$\equiv \frac{1}{3} \left( \frac{2p-1}{3} \right) \left( \frac{p-1}{2} \right) (-1)^{\frac{p-1}{6}} \left( 1 + 2p - \frac{2p}{3} q_p(2) \right)$$

$$\equiv \frac{4}{3} R_3(p) \pmod{p^2}.$$
It is known that \( \binom{2k}{k} \equiv 0 \pmod{p} \) for each \( (p+1)/2 \leq k \leq p-1 \), then by Lemma 3.1 and \( p \equiv 2 \pmod{3} \), we have the following modulo \( p^2 \),

\[
\sum_{k=0}^{p-1} \frac{D_k}{16^k} \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{16^k} \frac{p(-1-2k) - p^2(1+2k)(H_{2k} - H_k) + p^2(3k+1)}{(3k+1)(3k+2)}
\]

\[
\equiv -\frac{p}{3} \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{16^k} \left( \frac{1}{3k+1} + \frac{1}{3k+2} \right) + p^2 \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{(3k+2)16^k}
\]

\[
-\frac{p^2}{3} \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{(3k+2)16^k} (H_{2k} - H_k) \left( \frac{1}{3k+1} + \frac{1}{3k+2} \right)
\]

\[
\equiv -\frac{p}{3} \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{(3k+2)16^k} + p^2 \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{(3k+2)16^k} - \frac{p^2}{3} \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{(3k+2)16^k} (H_{2k} - H_k).
\]

By (4.2), we have

\[
\sum_{k=0}^{p-1} \frac{D_k}{16^k} \equiv -\frac{p}{3} \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{(3k+2)16^k} \equiv -p \sum_{k=0}^{p-1} \frac{D_k}{4^k} \pmod{p^2}.
\]

Therefore,

\[
\sum_{k=0}^{p-1} \frac{p}{k} \frac{D_k}{4^k} \equiv -\sum_{k=0}^{p-1} \frac{D_k}{16^k} \equiv \frac{4}{3} R_3(p) \pmod{p^2}.
\]

Now we finish the proof of Theorem 1.3. \( \square \)

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