Possible consistent extra time dimensions in the early universe

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Abstract

Inflation driven by a real massless scalar inflaton field $\varphi$ whose potential is identically equal to zero is described. Assuming that inflation takes place after the Plank scale (after quantum gravity effects are important), zero inflaton potential is concomitant with an initial condition for $\varphi$ that is exponentially more probable than an initial condition that assumes an initial inflaton potential of order of the Planck mass. The Einstein gravitational field equations are formulated on an eight-dimensional spacetime manifold of four space dimensions and four time dimensions. The field equations are sourced by a cosmological constant $\Lambda$ and the real massless scalar inflaton field $\varphi$. For the case of a diagonal metric, two periodic solution classes for the coupled Einstein field equations are obtained that exhibit temporal exponential deflation of three of the four time dimensions and temporal exponential inflation of three of the four space dimensions. For brevity this phenomenon is sometimes simply called “inflation.” We show that the extra time dimensions do not generally induce the exponentially rapid growth of fluctuations of quantum fields.

Comoving coordinates for the two non-inflating/deflating dimensions are chosen to be $(x^4, x^8)$. The $x^4$ coordinate corresponds to our universe’s observed physical time dimension, while the $x^8$ coordinate corresponds to a new spatial dimension that is assumed to be compact. Quadratic $\partial_{x^8} \varphi$ terms are seen to play the role of an effective inflaton potential in the dynamical field equations. In this model, after “inflation” the observable physical macroscopic world appears to a classical observer to be a homogeneous, isotropic universe with three space dimensions and one time dimension.

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I. INTRODUCTION

Recent Planck 2013 data analysis [1] is in remarkable accord with a flat ΛCDM model with inflation, based upon a spatially flat, expanding universe whose dynamics are governed by General Relativity and sourced by cold dark matter, a cosmological constant Λ, and a slow-roll scalar inflaton field [2][4]. The main predictions of inflationary cosmology are also consistent [5][8] with other recent observational data from important experiments such as WMAP [9] and the Sloan Digital Sky Survey [10][12], to name only two.

However, standard inflationary cosmology may not represent fundamental physics. It has been shown that if one employs a canonical measure for inflation [13][14][15], then the probability for the existence of the initial conditions that are required for slow-roll chaotic inflation are extremely unlikely [13], assuming that inflation takes place after the Plank scale (after quantum gravity governs the physics).

Here we discuss a new model of inflation/deflation that overcomes this problem. This model is based on the idea that our universe has as many time dimensions as space dimensions. Everyone is familiar with arguments against the existence of extra time dimensions. While there are many aspects to this issue, we address two arguments against the existence of extra time dimensions which stand out most clearly [16]:

1. A spacetime with extra time dimensions may not carry a spin structure ⇔ spinors cannot be defined on the spacetime;

2. Momenta corresponding to the extra time dimensions induce exponentially rapid growth of quantum fluctuations of the field; the universe is unstable. This instability is associated with the very largest momenta (shortest wavelengths).

Issue [1] is not applicable here; it is well known that our model spacetime carries a spin structure [17]. The issues raised in [2] may be investigated by studying the field equation for the propagation of a massive complex scalar field ψ in the background gravitational field:

\[ 0 = \frac{1}{\sqrt{\text{det}(g_{\alpha\beta})}} \partial_\mu \left[ \sqrt{\text{det}(g_{\alpha\beta})} g^{\mu\nu} \partial_\nu \psi \right] - (\Lambda m^2 + \zeta R) \psi, \quad (1) \]

which is derivable from the Lagrangian

\[ L_\psi = \sqrt{\text{det}(g_{\alpha\beta})} \left[ - g^{\mu\nu} \partial_\mu \psi^* \partial_\nu \psi - \psi^* \psi (\Lambda m^2 + \zeta R) \right]. \quad (2) \]
The quantum fluctuations of $\psi$ satisfy a field equation similar to Eq.\[1\]. Here $R$ is the Ricci scalar, and $m$ and $\zeta$ are real input parameters; the factors of $\Lambda$ are included for convenience. We shall discuss two types of solution to Eq.\[1\], the first with nonzero coupling to the Ricci scalar and the second with $\zeta = 0$. It should be emphasized that the expected coupling of the massive complex scalar field $\psi$ with the massless real scalar inflaton field $\varphi$ is not included here. We do not wish to confuse the production/annihilation of $\psi$ “particles” due to its interaction with the inflaton field with instabilities in the $\psi$ field that are sourced by the momenta associated to the extra time dimensions. Also, it is beyond the scope of the present paper to consider any issue with the forms for the Lagrangians of Eq\[2\] and implicit in Eq\[11\] that may arise from a calculation of the one-loop effective potential and concomitant renormalization. This is an important question, but outside the scope of this work.

In Sections III and IV we shall seek “plane wave” solutions to Eq.\[1\] in terms of comoving wave vectors and coordinates of the form

$$\psi = \Psi(x^4, x^8) e^{i(\vec{k} \cdot \vec{r} - \vec{w} \cdot \vec{R})},$$

where

$$\vec{r} = (x^1, x^2, x^3)^T \quad \text{and} \quad \vec{k} = (k_1, k_2, k_3)^T;$$

$$\vec{R} = (x^5, x^6, x^7)^T \quad \text{and} \quad \vec{w} = (k_5, k_6, k_7)^T.$$ (3) 

Because some of the coefficients in Eq.\[1\] are physical-time $x^4$ dependent there is no dispersion relation, per se, relating a physical-frequency ($\omega \leftrightarrow -i \frac{\partial}{\partial x^4} \ln \psi$) to momentum wave vectors $(\vec{k}, \vec{w})$. When discussing these solutions the following shorthand is employed:

$$\Lambda k^2 = \vec{k} \cdot \vec{k} = k_1^2 + k_2^2 + k_3^2;$$

$$\Lambda w^2 = \vec{w} \cdot \vec{w} = k_5^2 + k_6^2 + k_7^2.$$ (4)

We shall give solutions of the Einstein field equations and Eq.\[1\] for which the effects of multiple time dimensions are stable.

A. Notation and conventions

This model is cast on $\mathbb{X}_{4,4}$, which is an eight-dimensional pseudo-Riemannian manifold that is a spacetime of four space dimensions, with local comoving spatial coordinates $(x^1, x^2, x^3, x^8)$, and four time dimensions, with local comoving temporal coordinates...
(x^4, x^5, x^6, x^7) (employing the usual component notation in local charts). Greek indices run from 1 to 8. It is assumed that \(-\infty < x^\alpha < \infty\) for \(\alpha = 1, 2, \ldots, 7\). The compact \(x^8\) domain is refined as the discussion progresses.

We seek solutions to the coupled Einstein field equations that exhibit temporal exponential deflation of three of the four time dimensions. We arbitrarily label the temporal direction/dimension that does not deflate (or inflate, for that matter) the \(x^4\)-axis. This is completely arbitrary. We assume that unknown quantum gravity effects sort things out shortly after the Universe is created so that the non-inflating/deflating temporal direction/dimension is the same for all values of \((x^1, x^2, x^3, x^8)\); it is then labeled the \(x^4\) dimension, and the remaining deflating temporal directions/dimensions are labeled with \((x^5, x^6, x^7)\).

This is an unsatisfying assumption, but a necessary hypothesis given our current knowledge of quantum gravity.

For brevity the compact spatial dimension is referred to as the \(x^8\) dimension. \((x^1, x^2, x^3)\) label local comoving coordinates for the three noncompact spatial dimensions. Again, for brevity, the first noncompact spatial dimension is referred to as the \(x^1\) dimension, and so on.

Let \(g\) denote the pseudo-Riemannian metric tensor on \(X_{4,4}\). The signature of the metric \(g\) is \((4,4) \leftrightarrow (+ + - - - - - -)\). The covariant derivative with respect to the symmetric connection associated to the metric \(g\) is denoted by a vertical double-bar. We employ the Landau-Lifshitz spacelike sign conventions [18]. \(g \leftrightarrow g_{\alpha\beta} = g_{\alpha\beta}(x^\mu)\) is assumed to carry the Newton-Einstein gravitational degrees of freedom. It is moreover assumed that the ordinary Einstein field equations (on \(X_{4,4}\))

\[ G_{\alpha\beta} + g_{\alpha\beta} \Lambda = 8 \pi \mathbb{G} T_{\alpha\beta} \]  \hspace{1cm} (5)

are satisfied. Here \(G_{\alpha\beta}\) denotes the Einstein tensor, \(\Lambda\) is the cosmological constant, \(\mathbb{G}\) denotes the Newtonian gravitational constant, and the reduced Planck mass is \(M_{Pl} = [8\pi \mathbb{G}]^{-1/2}\).

Natural units \(c = 1 = \hbar = 8\pi \mathbb{G}\) are used throughout. Lastly, if \(f = f(x^4, x^8)\) then

\[ f^{(1,0)} = \frac{\partial}{\partial x^4} f(x^4, x^8), \quad f^{(0,1)} = \frac{\partial}{\partial x^8} f(x^4, x^8), \quad f^{(1,1)} = \frac{\partial^2}{\partial x^4 \partial x^8} f(x^4, x^8), \quad f^{(2,0)} = \frac{\partial^2}{\partial x^4 \partial x^8} f(x^4, x^8), \]  \hspace{1cm} (6)
II. EINSTEIN FIELD EQUATIONS

A. Preface

We seek solutions to the Einstein field equations Eq.[5] for which the observable physical macroscopic world appears to a classical observer to be a homogeneous, isotropic universe with three space dimensions and one time dimension. Accordingly the three space dimensions ($x^1, x^2, x^3$) are assumed to possess equal scale factors denoted $a = a(x^4, x^8)$, and the three extra time dimensions ($x^5, x^6, x^7$) are assumed to possess equal scale factors denoted $b = b(x^4, x^8)$.

It will be seen that at least two periodic (in $x^8$) solution classes ($E_0, E_1$) to the Einstein field equations Eq.[5] on $X_{4,4}$ exist that exhibit inflation/deflation and describe a universe that is spatially flat throughout the inflation era. For ($E_0, E_1$), during “inflation”, the scale factor $a = a(x^4, x^8)$ for the three space dimensions ($x^1, x^2, x^3$) exponentially inflates as a function of $x^4$, and the scale factor $b = b(x^4, x^8)$ for the three extra time dimensions ($x^5, x^6, x^7$) exponentially deflates as a function of $x^4$; moreover, the scale factors for the $x^4$ and $x^8$ dimensions are constants equal to one.

We define the cosmological relative expansion rate $\ell$ as

$$\ell = \frac{1}{2} \frac{\partial}{\partial x^4} \ln \left( \frac{b}{a} \right).$$  \hspace{1cm} (7)

In Section [III] it is shown that $\ell$ is independent of time and equal to $\ell_0 = -\sqrt{\frac{1}{18}} \Lambda$ for solutions in the first class $E_0$; this solution corresponds to pure exponential time inflation of the scale factor $a(x^4, x^8)$ and pure exponential time deflation of the scale factor $b(x^4, x^8)$. For this case $|\ell_0|$ coincides with the Hubble parameter $H$. All solutions in the class $E_0$ have a spatial $x^8$ period of

$$0C_8 = \pi \sqrt{\frac{2}{\Lambda}}.$$  \hspace{1cm} (8)

Therefore $\ell_0$ verifies

$$0C_8 |\ell_0| = \frac{\pi}{3},$$  \hspace{1cm} (9)

which is reminiscent of a semiclassical quantization relationship.

For solutions in the second class $E_1$, which describes a transition from pure exponential “inflation,” $\ell = \ell_1$ is generally time dependent. All solutions in this class have a spatial $x^8$ period of

$$1C_8 = 2\pi \sqrt{\frac{3}{5\Lambda}} = 0C_8 \sqrt{\frac{6}{5}}.$$
Universes that possess positive relative expansion rate $\ell > 0$ for $x^4 > 0$ exist and are reported below. However they are analogous to collapsing universe solutions in the usual Friedmann cosmological model, or to universes with multiple macroscopic time dimensions, and hence are not directly relevant to the physical processes discussed in this paper.

B. Line element

The line element for inflation/deflation is assumed to be given by

$$\{ds\}^2 = \left\{a(x^4, x^8)^2 \left[ (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \right] - (dx^4)^2 \right. - \left. \{b(x^4, x^8)^2 \left[ (dx^5)^2 + (dx^6)^2 + (dx^7)^2 \right] + (dx^8)^2 \right. \right. $$

$$= a^2 \left[ (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \right] - b^2 \left[ (dx^5)^2 + (dx^6)^2 + (dx^7)^2 \right] - (dx^4)^2 + (dx^8)^2 ; \quad (10)$$

where $a = a(x^4, x^8)$ and $b = b(x^4, x^8)$ carry the metric degrees of freedom in this model. The real massless scalar inflaton field is $\varphi = \varphi(x^4, x^8)$. The action for the metric and minimally coupled inflaton degrees of freedom is assumed to be given by

$$S = \int \left( \frac{1}{8 \pi G} \right)^2 d^8x \sqrt{\det(g_{\alpha\beta})} \left[ \frac{1}{16 \pi G} (R - 2\Lambda) - \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi \right]. \quad (11)$$

Here $\Lambda$ is the cosmological constant. The inflaton potential is zero; its action is purely kinematic, although $\varphi^{(0,1)}(x^4, x^8)^2$ may be regarded as contributing to an effective inflaton potential.

C. Canonical stress-energy tensor

The canonical stress-energy tensor for the real massless scalar inflaton field is $T_{\mu\nu} = -g^{\mu\alpha} \frac{\partial}{\partial g_{\alpha\nu}} L_{\varphi}, L_{\varphi} = \sqrt{\det(g_{\alpha\beta})} \left[ -\frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi \right]$. The distinct components are

$$T_{33} = \frac{1}{2} a(x^4, x^8)^2 \left[ \varphi^{(1,0)}(x^4, x^8)^2 - \varphi^{(0,1)}(x^4, x^8)^2 \right] , \quad (12)$$

$$T_{44} = \frac{1}{2} \left[ \varphi^{(1,0)}(x^4, x^8)^2 + \varphi^{(0,1)}(x^4, x^8)^2 \right] , \quad (13)$$
(which clearly satisfies the weak energy condition),

\[ T_{55} = \frac{1}{2} b(x^4, x^8)^2 \left( \varphi^{(0,1)}(x^4, x^8)^2 - \varphi^{(1,0)}(x^4, x^8)^2 \right), \]  

(14)

\[ T_{88} = \frac{1}{2} \varphi^{(0,1)}(x^4, x^8)^2 + \frac{1}{2} \varphi^{(1,0)}(x^4, x^8)^2, \]  

(15)

and

\[ T_{18} = T_{84} = \varphi^{(0,1)}(x^4, x^8)\varphi^{(1,0)}(x^4, x^8), \]  

(16)

which is non-zero, in general. Due to the \((x^4, x^8)\) dependence of the metric the \((4, 8)\) and 
\((8, 4)\) components of the Einstein tensor \(G_{48} = G_{84}\) are also, in general, non-zero.

**D. Field Equations**

\[ G_{\mu\nu} = 8 \pi \mathbb{G} T_{\mu\nu} \]  

(17)

The distinct field equation components may be written as

\[ G_{48} = G_{84} = -\frac{3a^{(1,1)}}{a} - \frac{3b^{(1,1)}}{b} = 8\pi\mathbb{G} \varphi^{(0,1)}\varphi^{(1,0)} \]  

(18)

\[ G_{33} = \frac{1}{b^2} \left[ 3ab \left( 2a^{(0,1)}b^{(0,1)} - 2a^{(1,0)}b^{(1,0)} + a \left( b^{(0,2)} - b^{(2,0)} \right) \right) + (a^{(0,1)}^2 - a^{(1,0)}^2) + 2a \left( a^{(0,2)} - a^{(2,0)} \right) \right] b^2 + 3a^2 \left( b^{(0,1)}^2 - b^{(1,0)}^2 \right) \]  

\[ = 4\pi\mathbb{G} a^2 \left( \varphi^{(0,1)^2} + \varphi^{(1,0)^2} \right) \]  

(19)

\[ G_{44} = -\frac{3}{a^2b^2} \left[ b^2 \left( a^{(0,1)^2} - a^{(1,0)^2} + aa^{(0,2)} \right) + ab \left( 3a^{(0,1)}b^{(0,1)} - 3a^{(1,0)}b^{(1,0)} + ab^{(0,2)} \right) \right] + a^2 \left( b^{(0,1)}^2 - b^{(1,0)}^2 \right) \]  

\[ = 4\pi\mathbb{G} \left( \varphi^{(0,1)^2} + \varphi^{(1,0)^2} \right) \]  

(20)

7
\[
G_{55} = \frac{1}{a^2} \left[ 2ab \left( -3a^{(0,1)}b^{(0,1)} + 3a^{(1,0)}b^{(1,0)} + a \left( b^{(2,0)} - b^{(0,2)} \right) \right) \\
+ 3 \left( -a^{(0,1)2} + a^{(1,0)2} + a \left( a^{(2,0)} - a^{(0,2)} \right) \right) b^2 + a^2 \left( b^{(1,0)2} - b^{(0,1)2} \right) \right] \\
= 4\pi G b^2 \left( \varphi^{(0,1)2} - \varphi^{(1,0)2} \right)
\]

(21)

\[
G_{88} = -\frac{3}{a^2b^2} \left[ a^2 \left( b^{(1,0)2} - b^{(0,1)2} \right) + \left( -a^{(0,1)2} + a^{(1,0)2} + a a^{(2,0)} \right) b^2 \\
+ ab \left( -3a^{(0,1)}b^{(0,1)} + 3a^{(1,0)}b^{(1,0)} + ab^{(2,0)} \right) \right] \\
= 4\pi G \left( \varphi^{(0,1)2} + \varphi^{(1,0)2} \right)
\]

(22)

The components of \( T_{\mu\nu} \) that are not identically zero must satisfy

\[
a b T_{\mu|4}^{\mu} = 0 = 3a \left( -b^{(1,0)} \left( \varphi^{(1,0)2} \right) + b^{(0,1)} \varphi^{(0,1)} \varphi^{(1,0)} + b^{(0,1)} \right) \\
+ b \left( -3a^{(1,0)} \left( \varphi^{(1,0)2} \right) + 3a^{(0,1)} \varphi^{(0,1)} \varphi^{(1,0)} \\
+ a \left( \varphi^{(1,0)} \left( \varphi^{(0,2)} \varphi^{(2,0)} \right) \right) \right) \\
a b T_{\mu|8}^{\mu} = 0 = 3a \left( b^{(0,1)} \left( \varphi^{(0,1)2} \right) - b^{(1,0)} \varphi^{(0,1)} \varphi^{(1,0)} \right) \\
- b \left( -3a^{(0,1)} \varphi^{(0,1)2} + 3a^{(1,0)} \varphi^{(1,0)} \varphi^{(0,1)} \right) \\
- a \left( \varphi^{(0,1)} \left( \varphi^{(0,2)} \varphi^{(2,0)} \right) \right).
\]

(23)

Let \( L_1 = \frac{a^{(1,0)}(x^4,x^8)}{a(x^4,x^8)} \), \( L_3 = \frac{a^{(0,1)}(x^4,x^8)}{a(x^4,x^8)} \), \( L_2 = \frac{b^{(1,0)}(x^4,x^8)}{b(x^4,x^8)} \) and \( L_4 = \frac{b^{(0,1)}(x^4,x^8)}{b(x^4,x^8)} \). The Euler-Lagrange equation for the inflaton field yields

\[
\varphi^{(2,0)}(x^4,x^8) - \varphi^{(0,2)}(x^4,x^8) + 3\varphi^{(1,0)}(x^4,x^8) \left( L_1 + L_2 \right) - 3\varphi^{(0,1)}(x^4,x^8) \left( L_3 + L_4 \right) \\
= 0,
\]

(24)

or, equivalently,

\[
\varphi^{(2,0)}(x^4,x^8) + 3\varphi^{(1,0)}(x^4,x^8) \left( L_1 + L_2 \right) + \mu^2 \varphi(x^4,x^8) \\
= \varphi^{(0,2)}(x^4,x^8) + 3\varphi^{(0,1)}(x^4,x^8) \left( L_3 + L_4 \right) + \mu^2 \varphi(x^4,x^8),
\]

(25)

where \( \mu^2 \) is arbitrary.
Lastly, the field equations demand that the constraint equation

\[
L_1^2 + 3L_1L_2 + L_2^2 - \frac{4}{3}\pi\sigma \varphi^{(1,0)}(x^4, x^8)^2 \\
- \left[ L_3^2 + 3L_3L_4 + L_4^2 - \frac{4}{3}\pi\sigma \varphi^{(0,1)}(x^4, x^8)^2 \right] = \frac{2}{9}\Lambda,
\]

be satisfied.

To solve the field equations we use the fact that \( P = \ln (a \times b) \), \( R = \ln \left( \frac{b}{a} \right) \) and \( \varphi \) satisfy, respectively, uncoupled, linear and linear field equations of the form

\[
f^{(0,2)}(x^4, x^8) - f^{(2,0)}(x^4, x^8) = S + \\
+ 3P^{(1,0)}(x^4, x^8)f^{(1,0)}(x^4, x^8) - 3P^{(0,1)}(x^4, x^8)f^{(0,1)}(x^4, x^8),
\]

where \( f = P, R \) or \( \varphi \) and \( S = 0 \) unless \( f = P \) in which case \( S = -\frac{2}{3}\Lambda \). General solutions to these equations are substituted into the constraint Eq [26] and the remaining field equations, which are then solved; this procedure yields the solutions in the classes \( E_0 \) and \( E_1 \) that are described below.

III. CLASS \( E_0 \) SOLUTIONS: EXACT TEMPORAL EXPONENTIAL INFLATION/DEFLATION

Using the technique described above we find that this model admits the following solutions, which are exponential in \( x^4 \), periodic in \( x^8 \) and that are parameterized by \( \Lambda \): The scale factors are

\[
a = a(x^4, x^8) = a_0 e^{\pm \frac{1}{3}\sqrt{2} x^4} \sqrt{\sin^2 \left( \frac{x^8 \sqrt{2}\Lambda}{2} \right)} \\
b = b(x^4, x^8) = b_0 e^{\mp \frac{1}{3}\sqrt{2} x^4} \sqrt{\sin^2 \left( \frac{x^8 \sqrt{2}\Lambda}{2} \right)},
\]

where \( a_0 \) and \( b_0 \) are constants. The relative expansion rate and the inflaton field are given by

\[
\ell_0 = \mp \frac{1}{3}\sqrt{\frac{\Lambda}{2}} \\
\varphi = \pm \frac{5}{6} \ln \left[ \tan^2 \left( \frac{1}{2}\sqrt{2\Lambda} x^8 \right) \right].
\]

For this case the scale factors \( (a, b) \) have a spatial period equal to \( \frac{1}{2} e^{-\Lambda} \), while the three functions \( (a, b, \varphi) \) possess a common spatial periodicity in the \( x^8 \) coordinate whose nonzero
minimum value is equal to the $x^8$-dimension spatial period

$$^0C_8 = \pi \sqrt{\frac{2}{\Lambda}} = \frac{\pi}{3|\ell_0|},$$

(30)

For this case $|\ell_0|$ coincides with the Hubble parameter $H$.

Since a universe with many macroscopic times is not observed, one may identify the physical solution for inflation by choosing the solution with the $-$ sign in the equation for $b$. This solution then predicts the exponential deflation, with respect to time, of the scale factor $b$ associated with the three extra time dimensions. This coincides with the exponential inflation, with respect to time, of the scale factor $a$ associated with the three observed spatial dimensions.

The reduced volume element $d\Omega$ on the hypersurface $x^4 = x^4_0 = \text{constant}$ is

$$d\Omega = \left[ \sqrt{\det(g)} \right]_{x^4=x^4_0} \ d\tau$$

(31)

where $d\tau = |dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^5 \wedge dx^6 \wedge dx^7 \wedge dx^8|$. For this case $d\Omega = d\tau |\sin \left( x^8 \sqrt{2\Lambda} \right)|$, which clearly does not inflate. (The four dimensional spacetime submainfold of $\mathbb{K}_{4,4}$ comprised of the $(x^1, x^2, x^3, x^4)$ dimensions experiences conventional FLRW inflation.) The fact that $d\Omega$ vanishes on a set of measure zero is discussed in the Conclusion.

A. Stability of a massive complex scalar field

The question of the possible existence of instabilities in the massive complex scalar $\psi$ field that are sourced by the momenta $\vec{w}$ associated to the extra time dimensions, the issue raised in Section [2], may be investigated by studying the propagation of the complex scalar field $\psi$ in the background gravitational field, but which is not coupled to the inflaton field $\varphi$. The field equation for this scalar field is given in Eq.[1]. Substituting the background gravitational field of Eq.[28] into Eq.[1] yields

$$0 = -\Psi^{(2,0)}(x^4, x^8) + \Psi^{(0,2)}(x^4, x^8) + \sqrt{2\Lambda} \cot \left( \sqrt{2\Lambda} x^8 \right) \Psi^{(0,1)}(x^4, x^8)$$

$$+ \Lambda \left( w^2 e^{\frac{1}{4} \sqrt{2\Lambda} x^4} - k^2 e^{-\frac{1}{4} \sqrt{2\Lambda} x^4} \right) \sqrt{\csc^2 \left( \sqrt{2\Lambda} x^8 \right)} \Psi(x^4, x^8)$$

$$- \Lambda \left[ m^2 + \frac{1}{3} \zeta \left( 8 + 5 \csc^2 \left( \sqrt{2\Lambda} x^8 \right) \right) \right] \Psi(x^4, x^8).$$

(32)
Here we have used the fact that the Ricci scalar is 
\[ \frac{1}{3} \Lambda \left( 5 \csc^2 (\sqrt{2 \Lambda} x^8) + 8 \right) \] for the class 
\( E_0 \) background gravitational field. In Eq. [32] we put \( \Psi(x^4, x^8) = \psi \left( \sqrt{2 \Lambda} x^4, \sqrt{2 \Lambda} x^8 \right) \), then set \( x = \sqrt{2 \Lambda} x^8 \) (where \(-\pi \leq x \leq \pi\)), and \( t = \sqrt{2 \Lambda} x^4 \). This yields a wave equation for the massive complex scalar field given by

\[
0 = -\psi^{(2,0)}(t, x) + \psi^{(0,2)}(t, x) + \cot(x) \psi^{(0,1)}(t, x) \\
+ \frac{1}{2} \left( k^2 e^{-t/3} - w^2 e^{t/3} \right) \sqrt{\csc^2(x)} \psi(t, x) \\
- \frac{1}{2} \left[ m^2 + \frac{1}{3} \zeta \left( 8 + 5 \csc^2(x) \right) \right] \psi(t, x).
\] (33)

Because some of the coefficients in Eq. [33] are physical-time \( x^4 \leftrightarrow t \) dependent there is no simple dispersion relation that directly relates a physical-frequency \( \omega \) to the momentum wave vectors \((k, \bar{w})\). Note that

\[
\frac{1}{2} \left( w^2 e^{t/3} - k^2 e^{-t/3} \right) = kw \frac{1}{2} \left( \frac{w}{k} e^{t/3} - \frac{k}{w} e^{-t/3} \right) = kw \sinh \left[ \frac{t}{3} + \ln \left( \frac{w}{k} \right) \right].
\] (34)

We also state without proof that

\[
\sqrt{\csc^2(x)} = \frac{2 \sqrt{\pi}}{\Gamma \left( \frac{3}{2} \right) \Gamma \left( \frac{5}{6} \right)} \left( 2 F_1 \left( \frac{1}{6}, 1; \frac{5}{6}; e^{-2ix} \right) + 2 F_1 \left( \frac{1}{6}, 1; \frac{5}{6}; e^{2ix} \right) - 1 \right). 
\] (35)

almost everywhere; this identity is occasionally useful.

We study two special cases of Eq. [33], the first with \( \zeta = 3/10 \) and the second with \( \zeta = 0 \). The solution for \( \zeta = 3/10 \) is expanded into a series of Fourier modes \( \sum_{n=\infty}^\infty f(t, n) \exp(i n s x) \). The second solution with \( \zeta = 0 \) is expanded into a series of (two) Legendre polynomial modes, each of the form \( \sum_{n=0}^\infty h_{ns}(t) \left[ \sqrt{(n + \frac{1}{2})} P_{ns}(\cos x) \right] \).

There is an expansion for each of the intervals \(-\pi \leq x \leq 0\) and \(0 \leq x \leq \pi\). The two Legendre expansions must be matched at \( x = 0 \) and \( x = \pm \pi \).

1. Wave equation with \( \zeta = 3/10 \)

We take advantage of the coupling of the massive scalar field to the Ricci scalar to simplify the wave equation. In Eq. [33] we first put \( \psi(t, x) = \sin \frac{1}{2} (x) F(t, x) \). This yields

\[
0 = F^{(2,0)}(t, x) - F^{(0,2)}(t, x) \\
+ \frac{1}{2} \left( k^2 e^{-t/3} - w^2 e^{t/3} \right) \sqrt{\csc^2(x)} F(t, x) \\
+ \frac{1}{12} \left( 16 \zeta + 6 m^2 - 3 \right) F(t, x) + \frac{1}{12} (10 \zeta - 3) \csc^2(x) F(t, x).
\] (36)
In order to eliminate the $\frac{1}{12}(10\zeta - 3)\csc^2(x)F(t, x)$ term from this equation we henceforth assume that $\zeta = 3/10$.

Substituting a Fourier decomposition

$F(t, x) = \sum_{n=-\infty}^{\infty} f(t, n_{s}) \exp(i n_{s} x)$ into Eq.36 and then multiplying by $\frac{1}{2\pi} \exp(-i m_{s} x) \, dx$ and integrating from $(-\pi, \pi)$ yields

$$0 = f^{(2,0)}(t, m_{s}) + (M^2 + m_{s}^2) f(t, m_{s}) - \frac{1}{2} \left( w^2 e^{t/3} - k^2 e^{-t/3} \right) \sum_{n_{s}=-\infty}^{\infty} (c_{m_{s}n_{s}} f(t, n_{s})),$$

(37)

Here we have put $M^2 = \frac{m^2}{2} + \frac{3}{20}$. The matrix elements of $\sqrt[6]{\csc^2(x)}$ in the Fourier basis are

$$c_{m_{s},n_{s}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sqrt[6]{\csc^2(x)} \exp[i (n_{s} - m_{s}) x] \, dx$$

$$= \sqrt[6]{2} \Gamma \left( \frac{2}{3} \right) \begin{cases} \frac{(-1)^N}{\Gamma \left( \frac{5}{6} - N \right) \Gamma \left( \frac{5}{6} + N \right)} & \text{if } n_{s} - m_{s} = 2N \text{ is even} \\ 0 & \text{if } n_{s} - m_{s} \text{ is odd} \end{cases}$$

(38)

Since $c_{m_{s},n_{s}} = c_{m_{s}-n_{s},0} = c_{0,n_{s}-m_{s}}$, Eq.37 may be written as

$$0 = \frac{d^2}{dt^2} f(t; m_{s}) + \left[ (M^2 + m_{s}^2) - \frac{1}{2} \left( w^2 e^{t/3} - k^2 e^{-t/3} \right) c_{0,0} \right] f(t; m_{s}) - \frac{1}{2} \left( w^2 e^{t/3} - k^2 e^{-t/3} \right) \sum_{n_{s}=-\infty}^{\infty} c_{2n_{s},0} \left[ f(t, 2n_{s} + m_{s}) + f(t, -2n_{s} + m_{s}) \right].$$

(39)

As expected, the effective mass $m_{\text{eff}}$

$$m_{\text{eff}}^2 = (M^2 + m_{s}^2) = \frac{m^2}{2} + \frac{3}{20} + m_{s}^2$$

(40)

of the scalar field receives contributions from the spatial $x^8$ Fourier modes. Because

$$\frac{1}{\Gamma \left( \frac{5}{6} - N \right) \Gamma \left( \frac{5}{6} + N \right)} \propto \left( \frac{1}{|N|} \right)^{2/3}$$

as $N \to \pm \infty$, the series in this equation exhibits a naïve very long range coupling between modes.

We call the first line of Eq.39

$$\mathcal{D}[f](t, m_{s}; k, w) = \frac{d^2}{dt^2} f(t; m_{s}) + \left[ m_{\text{eff}}^2 - \frac{1}{2} \left( w^2 e^{t/3} - k^2 e^{-t/3} \right) c_{0,0} \right] f(t; m_{s})$$

(41)

the diagonal contribution to Eq.39.
2. Wave equation with $\zeta = 0$

We put $\zeta = 0$ in Eq.\[33\] to obtain the wave equation for a minimally coupled massive complex scalar field, which is given by

$$0 = -\psi^{(2,0)}(t, x) + \psi^{(0,2)}(t, x) + \cot(x) \psi^{(0,1)}(t, x)$$
$$+ \frac{1}{2} \left( w^2 e^{\frac{1}{2}t} - k^2 e^{-\frac{1}{2}t} \right) \sqrt{\csc^2(x)} \psi(t, x) - \frac{1}{2} m^2 \psi(t, x)$$

$$= -\psi^{(2,0)}(t, x) + \frac{1}{\sin(x)} \frac{\partial}{\partial x} \left[ \sin(x) \frac{\partial}{\partial x} \psi(t, x) \right]$$
$$+ \frac{1}{2} \left( w^2 e^{\frac{1}{2}t} - k^2 e^{-\frac{1}{2}t} \right) \sqrt{\csc^2(x)} \psi(t, x) - \frac{1}{2} m^2 \psi(t, x). \tag{42}$$

We recall that $-\pi \leq x \leq \pi$ and not $0 \leq x \leq \pi$. In order to obtain the general solution to Eq.\[42\] we expand $\psi(t, x)$ on each of the two intervals $I_{-1} = -\pi \leq x \leq 0$ and $I_1 = 0 \leq x \leq \pi$ in terms of a series of Legendre polynomials of the form

$$\psi(t, x)|_{I_{\pm 1}} = \sum_{n_{8}=0}^{\infty} h_{n_{8}}^{(\pm 1)}(t) \left[ \sqrt{\left( n_{8} + \frac{1}{2} \right)} P_{n_{8}}(\cos x) \right] \tag{43}$$

and match these expansions at $x = 0$ and $x = \pm \pi$. Substituting Eq.\[43\] into Eq.\[42\], multiplying by $\left[ \sqrt{\left( m_{8} + \frac{1}{2} \right)} P_{m_{8}}(\cos x) \right] \sin x \, dx$ and integrating over the interval yields (suppressing the superscript $(\pm 1)$ on $h_{n_{8}}^{(\pm 1)}(t)$)

$$\ddot{h}_{m_{8}}(t) + \left[ \frac{1}{2} m^2 + m_{8}(m_{8} + 1) \right] h_{m_{8}}(t) \frac{1}{2} \left( k^2 e^{-\frac{1}{2}t} - w^2 e^{\frac{1}{2}t} \right) \sum_{n_{8}=0}^{\infty} d_{m_{8}n_{8}} h_{n_{8}}(t) = 0 \tag{44}$$

where

$$d_{m_{8}n_{8}} = \pi \Gamma \left( \frac{5}{6} \right)^2 \sqrt{\left( m_{8} + \frac{1}{2} \right) \left( n_{8} + \frac{1}{2} \right)} \sum_{j=|m_{8}-m_{8}|}^{m_{8}+m_{8}, \text{ step 2}} \left[ \frac{(2j+1)(-1)^{j+m_{8}+n_{8}} \Gamma(j+m_{8}-n_{8}+1) \Gamma(j-m_{8}+n_{8}+1)}{\Gamma \left( \frac{1}{2}-\frac{1}{2} \right) \Gamma \left( \frac{5}{6}-\frac{1}{2} \right) \Gamma \left( \frac{1}{2}+1 \right) \Gamma \left( \frac{1}{2}+\frac{4}{3} \right)} \right]$$

$$\frac{\Gamma(-j+m_{8}+n_{8}+1) \Gamma \left( \frac{1}{2}(j+m_{8}+n_{8}+2) \right)^2}{\Gamma \left( \frac{1}{2}(j+m_{8}-n_{8}+2) \right)^2 \Gamma \left( \frac{1}{2}(j-m_{8}+n_{8}+2) \right)^2} \frac{1}{\Gamma \left( \frac{1}{2}(-j+m_{8}+n_{8}+2) \right)^2 \Gamma(j+m_{8}+n_{8}+2)} \right] \tag{45}$$
For this case the effective mass \( m_{\text{eff}} \) is given by

\[
m_{\text{eff}}^2 = \left( \frac{1}{2} m^2 + m_8 (m_8 + 1) \right)
\]

The diagonal contribution to Eq. 44 is

\[
D[h](t, m_8, k, w) = \ddot{h}_{m_8}(t) + \left[ m_{\text{eff}}^2 - \frac{1}{2} \left( w^2 e^{t/3} - k^2 e^{-t/3} \right) d_{m_8 m_8} \right] h_{m_8}(t),
\]

which has the same form as the diagonal contribution to Eq. 39.

Setting to zero the diagonal contribution to either Eq. 39 or Eq. 44 yields an equation of the form

\[
0 = \ddot{Y}_{m_8}(t) + \left[ m_{\text{eff}}^2 - B \frac{1}{2} \left( w^2 e^{t/3} - k^2 e^{-t/3} \right) \right] Y_{m_8}(t),
\]

where

\[
\begin{aligned}
    m_{\text{eff}}^2 &= \frac{3}{20} + \frac{1}{2} m^2 + m_8^2, \quad B = c_{0,0}, \quad \text{for } \zeta = \frac{3}{10}, \\
    m_{\text{eff}}^2 &= \frac{1}{2} m^2 + m_8 (m_8 + 1), \quad B = B(m_8) = d_{m_8 m_8}, \quad \text{for } \zeta = 0 \quad (49)
\end{aligned}
\]

3. Solution of the mode equations, neglecting mode coupling, for \( k = 0 \)

Since we are primarily interested in determining the stability of a \( \psi \) state configuration with respect to fluctuations of the extra time momenta \( \vec{w} \), we assume in this Section that our comoving frame is also a spatial rest frame for the \( \psi \) field so that \( \vec{k} = \vec{0} \). If the ordinary 3-momentum magnitude \( k = 0 \) then the solution to Eq. 48 can be expressed in terms of a linear combination of Bessel functions.

Let \( J_n(z) \) denote an ordinary Bessel function of the first type, \( I_\lambda(z) \) denote the modified Bessel function of the first type and order \( \lambda \), and \( K_\lambda(z) \) denote a modified Bessel function of the second type (also known as the modified Bessel function of the third kind). If the ordinary 3-momentum magnitude \( k = 0 \) then it is easy to verify that the solution to Eq. 48 is given by

\[
Y_{m_8}(t) = Q_1^{(m_8)} K_{6i m_{\text{eff}}} \left( e^{t/6} 3 w \sqrt{2 B} \right) + Q_2^{(m_8)} I_{6i m_{\text{eff}}} \left( e^{t/6} 3 w \sqrt{2 B} \right),
\]

(50)
The values of \( Q_1^{(m_s)} \), \( Q_2^{(m_s)} \) for the \( k = 0 \) solution to Eq. [48] subject to the initial conditions \( Y_{m_s}(0) = f(0) \) and \( \frac{d}{dt} Y_{m_s}(t) \) \( t = 0 \) are given by
\[
Q_1^{(m_s)} = \frac{1}{2} f(0) W [I_{i\nu-1}(W) + I_{i\nu+1}(W)] - 6 f'(0) K_{i\nu}(W)
\]
\[
= f(0) W \frac{\partial I_{i\nu}(W)}{\partial W} - 6 f'(0) K_{i\nu}(W),
\]
\[
Q_2^{(m_s)} = \frac{1}{2} f(0) W [K_{i\nu-1}(W) + K_{i\nu+1}(W)] + 6 f'(0) K_{i\nu}(W)
\]
\[
= -f(0) W \frac{\partial K_{i\nu}(W)}{\partial W} + 6 f'(0) K_{i\nu}(W),
\]
where \( \nu = 6 m_{\text{eff}} \) and \( W = 3 w \sqrt{2B} \). It is well known that \( K_{i\nu}(z) \) has the integral representation
\[
K_{i\nu}(z) = \int_0^\infty \cos(\nu \xi) e^{-z \cosh(\xi)} d\xi,
\]
valid for \( \arg z < \frac{\pi}{2} \). Also, asymptotically, for large argument \( |z| \), the modified Bessel functions behave as \( K_\lambda(z) \sim e^{-z} \sqrt{\frac{\pi}{2z}} + \cdots \) and \( I_\lambda(z) \sim e^z \sqrt{\frac{1}{2\pi z}} + \cdots \) [19]. We are seeking, for large \( w \), a stable, Dirac delta normalizable (on hypersurfaces \( x^4 = \text{constant} \)) solution to the equation for \( Y_{m_s}(t) \). Therefore, for each \( m_s \), the coefficient \( Q_2^{(m_s)} \) of \( I_{i\nu} \left( \frac{e^{t/6} 3 w \sqrt{2B}}{\sqrt{2B}} \right) \) must vanish. Hence, given \( w \), we require that the initial conditions for the uncoupled modes satisfy
\[
6 \frac{d}{dt} \ln(Y_{m_s}(t)) \big|_{t=0} = W \frac{d}{dW} \ln[K_{i\nu}(W)]
\]
Substitution of this result into the expressions for \( (Q_1^{(m_s)}, Q_2^{(m_s)}) \) in Eq. [51] yields
\[
(Q_1^{(m_s)}, Q_2^{(m_s)}) = \left( \frac{Y_{m_s}(0)}{K_{i\nu}(3 w \sqrt{2B})}, 0 \right) = \left( \frac{Y_{m_s}(0)}{K_{6i m_{\text{eff}}}(3 w \sqrt{2B})}, 0 \right)
\]
Accordingly the uncoupled mode function \( Y_{m_s}(t) \) that solves the \( k = 0 \) limit of Eq. [48] is
\[
Y_{m_s}(t) = Y_{m_s}(0) \frac{1}{K_{6i m_{\text{eff}}}(3 w \sqrt{2B})} K_{6i m_{\text{eff}}} \left( \frac{e^{t/6} 3 w \sqrt{2B}}{\sqrt{2B}} \right).
\]
The uncoupled \( k = 0 \) mode functions \( Y_{m_s}(t) \) may trivially be normalized at \( t = 0 \). We now make the natural assumption that the normalized \( Y_{m_s}(0) \) are either independent of \( w \) or have a power of \( w \) times a Gaussian \( w^\eta(m_s)^2 e^{-\lambda(m_s)^2 w^2} \) type dependence, for large \( w \). Using the well-known asymptotic expansions of the Bessel functions [19] it is clear that for \( t > 0 \) the \( \lim_{w \to \infty} \frac{K_{i\nu} \left( \frac{e^{t/6} 3 w \sqrt{2B}}{\sqrt{2B}} \right)}{K_{i\nu} \left( \frac{3 w \sqrt{2B}}{\sqrt{2B}} \right)} \) is well-behaved as \( w \to \infty \) for \( t > 0 \). Hence we have demonstrated the existence, for large
$w$, of a stable solution to the equation for $Y_{m_8}(t)$ in Eq.[50]. This suggests that instabilities that are sourced by the momenta $\vec{w}$ associated to the extra time dimensions, the issue raised in Section[2], may not always arise in this model. This suggests that extra time dimensions may be consistent with the physics of the early universe.

We note in passing that the behavior of the modified Bessel function of the second kind with pure imaginary order, $K_{\nu}(z)$, has been investigated by Balogh [20] and others. Balogh has proved that $K_{\nu}(\nu \frac{p}{p})$ is a positive monotone decreasing convex function of $p$ for $0 < p < 1$ and it oscillates boundedly for $p > 1$, having a countably infinite number of zeros. Balogh gives asymptotic expansions of the zeros of $K_{\nu}(z)$ and its derivative that are uniform with respect to the enumeration of the zeros.

To repeat the point, it must be emphasized that in this case one concludes that instabilities that are sourced by the momenta $\vec{w}$ associated to the extra time dimensions are not inevitable in this early universe model.

4. Solution of the mode equations, neglecting mode coupling, for $k \neq 0$

For both the $\zeta = 3/10$ case and the second case with $\zeta = 0$, one may solve the diagonal mode equations obtained from Eq.[39] and Eq.[44] using Mathieu functions. However complex manipulations involving such mode functions is limited by the state of Mathieu function science. Instead, for both cases, to solve the uncoupled mode function equations we make use of the following identity, which is easily verified using the recursion relations for the Bessel functions. First, let us define several quantities.

Let $m_{\text{eff}}^2 > 0, B > 0, \nu, \alpha$ and $\beta$ be given constants, and $n \in \mathbb{Z}$; let $J_n = J_n \left( 3 k \sqrt{2 B} \exp \left( -\frac{t}{6} \right) \right)$, $K_{\mu} = K_{\mu} \left( 3 w \sqrt{2 B} \exp \left( +\frac{t}{6} \right) \right)$ and $I_{\mu} = I_{\mu} \left( 3 w \sqrt{2 B} \exp \left( +\frac{t}{6} \right) \right)$. Then using the recursion relations for the Bessel functions [19] it is straightforward to show that
\[ 0 = \frac{\partial^2}{\partial t^2} \left[ J_n \left( \alpha K_{n+i\nu} + \beta I_{n+i\nu} \right) \right] + \left( m_{\text{eff}}^2 - \frac{1}{2} B \left( \frac{w^2}{3} e^{t/3} - k^2 e^{-t/3} \right) \right) J_n \left( \alpha K_{n+i\nu} + \beta I_{n+i\nu} \right) \\
- \left[ m_{\text{eff}}^2 + \frac{1}{9} \left( n + \frac{i\nu}{2} \right)^2 \right] J_n \left( \alpha K_{n+i\nu} + \beta I_{n+i\nu} \right) \\
+ \frac{1}{2} k w B J_{n-1} \left( -\alpha K_{n-1+i\nu} + \beta I_{n-1+i\nu} \right) \\
+ \frac{1}{2} k w B J_{n+1} \left( \alpha K_{n+1+i\nu} - \beta I_{n+1+i\nu} \right). \quad (56) \]

Employing this identity to solve the uncoupled mode function equations has its roots in a closely related technique due to Dougall \[21\], Section[15], pages 191-193, with appropriate modifications that account for the asymmetric “potential” \[\frac{1}{2} \left( w^2 e^{t/3} - k^2 e^{-t/3} \right)\] in this problem. Dougall gives the solution of Mathieu’s modified differential equation as a series of products of Bessel functions.

The general solution to Eq.\[48\],

\[ 0 = \ddot{Y}_{m8}(t) + \left[ m_{\text{eff}}^2 - B \frac{1}{2} \left( w^2 e^{t/3} - k^2 e^{-t/3} \right) \right] Y_{m8}(t), \]

may be obtained by introducing coefficient sets \(\{c_n, d_n\}, \ n = -\infty, \ldots, \infty\) and then setting \(\alpha = (-i)^n c_n\) and \(\beta = (i)^n d_n\) in Eq.\[56\]. Next we sum over \(n\) from \(n = -\infty, \ldots, \infty\) and then re-label indices and identify coefficients of \(J_n \left( K_{n+i\nu} \right)\) and \(J_n \left( I_{n+i\nu} \right)\). We find that a general solution to Eq.\[48\] is given by

\[ Y_{m8}(t) = \sum_{n = -\infty}^{\infty} (-i)^n c_n J_n \left( e^{-t/6} 3k \sqrt{2B} \right) K_{n+i\nu} \left( e^{t/6} 3w \sqrt{2B} \right) \\
+ \sum_{n = -\infty}^{\infty} i^n d_n J_n \left( e^{-t/6} 3k \sqrt{2B} \right) I_{n+i\nu} \left( e^{t/6} 3w \sqrt{2B} \right). \quad (57) \]

The coefficient sets \(\{c_n, d_n\}, \ n = -\infty, \ldots, \infty\) are solutions of the same recurrence relation

\[ \frac{1}{2} k w B \left( C_{n-1} + C_{n+1} \right) - \left[ m_{\text{eff}}^2 + \frac{1}{9} \left( n + \frac{i\nu}{2} \right)^2 \right] C_n = 0, \quad (58) \]

but with possibly distinct initial values, since a general solution to the three-term recurrence relation Eq.\[58\] possesses two arbitrary constants. Here \(\nu\) is arbitrary, but may be given as
\[ \nu = 6 m_{\text{eff}} \] in order to obtain agreement with the \( k \to 0 \) solution obtained in the previous Subsection [III A 3]. Otherwise \( \nu \) may be chosen to speed up the convergence of the recurrence relation.

Eq. [57] may find application in calculating the cross section for the creation of particle/anti-particle pairs of \( \psi \) particles through the annihilation of \( \varphi \) quanta, if one generalizes this model to include an interaction of \( \psi \) with the inflaton \( \varphi \) of the form \( \lambda \psi^* \psi \varphi \).

However, since the \( k \to 0 \) solution obtained in the previous Subsection [III A 3] has already been shown to be stable in the limit \( w \to \infty \), we shall not discuss this case further here.

5. Solution of the mode equations for \( \zeta = 3/10 \) and \( k = 0 \) with mode coupling

Let us first recall the definition of Eq. [41]

\[
D[f](t, m_8; k, w) = \frac{d^2}{dt^2} f(t; m_8) + \left[ m_{\text{eff}}^2 - \frac{1}{2} \left( w^2 e^{t/3} - k^2 e^{-t/3} \right) c_{0,0} \right] f(t; m_8)
\]

For this case \( m_{\text{eff}}^2 = (M^2 + m_8^2) = \left( \frac{3}{20} + m^2 + m_8^2 \right) \).

To investigate the stability of a massive scalar field as \( w \to \infty \) we seek a stable solution to Eq. [39] of the form \( f(t, m_8) = \lim_{j \to \infty} f^{(j)}(t, m_8) \) where

\[
D[f^{(0)}](t, m_8; k, w) = 0
\]
\[
D[f^{(j+1)}](t, m_8; k, w) = \frac{1}{2} \left( w^2 e^{t/3} - k^2 e^{-t/3} \right) \sum_{n_8 = 1}^{\infty} c_{2n_8,0} \left[ f^{(j)}(t, 2n_8 + m_8) + f^{(j)}(t, -2n_8 + m_8) \right],
\]

for \( j = 0, 1, 2, \ldots \).

(59)

We assume in this Section that our comoving frame is also a spatial rest frame for the \( \psi \) field so that \( \vec{k} = 0 \). Since \( k = 0 \) we may employ the uncoupled mode functions of Eq. [55],

\[
Y_{m_8}(t) = Y_{m_8}(0) \frac{K_{6i m_{\text{eff}}}(e^{t/6} 3 w \sqrt{2B})}{K_{6i m_{\text{eff}}}(3 w \sqrt{2B})} = Y_{m_8}(0) \frac{K_{6i \sqrt{M^2+m_{\text{eff}}^2}}(e^{t/6} 3 w \sqrt{2B})}{K_{6i \sqrt{M^2+m_{\text{eff}}^2}}(3 w \sqrt{2B})}
\]

as \( [f^{(0)}](t, m_8; 0, w) \). We again assume that the \( Y_{m_8}(0) \) are either independent of \( w \) or have a power of \( w \) times a Gaussian \( w^\eta(m_8) e^{-\lambda(m_8)^2 w^2} \) type dependence, for large \( w \).
Consider next, for large $w$, the equation for $f^{(1)}(t, m_8; 0, w)$ in Eq. [59]

\[
\mathcal{D}[f^{(1)}](t, m_8; k = 0, w) =
\frac{w^2}{2} e^{t/3/2} \sum_{n_8 = 1}^{\infty} c_0 2n_8 [Y_{2n_8+m_8}(t) + Y_{-2n_8+m_8}(t)]
\]

\[
= \frac{w^2}{2} e^{t/3/2} \Gamma \left( \frac{2}{3} \right) \sum_{n_8 = 1}^{\infty} \frac{(-1)^{n_8}}{\Gamma \left( \frac{5}{6} - n_8 \right) \Gamma \left( \frac{5}{6} + n_8 \right)} \times
\]

\[
\begin{bmatrix}
Y_{2n_8+m_8}(0) \frac{K_{6i2\sqrt{M^2+(m_8-2n_8)^2}(e^{t/6}3w \sqrt{2c_0,0})}}{K_{6i2\sqrt{M^2+(m_8-2n_8)^2}(3w \sqrt{2c_0,0})}} +
Y_{2n_8+m_8}(0) \frac{K_{6i2\sqrt{M^2+(m_8+2n_8)^2}(e^{t/6}3w \sqrt{2c_0,0})}}{K_{6i2\sqrt{M^2+(m_8+2n_8)^2}(3w \sqrt{2c_0,0})}}
\end{bmatrix}
\]

(60)

Using the well-known asymptotic expansions of the Bessel functions [19] it may be shown that, for $t > 0$, \( \lim_{w \to \infty} w^2 \frac{K_{\mu}(e^{t/6}3w \sqrt{2c_0,0})}{K_{\mu}(3w \sqrt{2c_0,0})} = 0 \). If we simply assume that the sum in Eq. [60] converges uniformly (which depends on the behavior of \( \frac{1}{\Gamma \left( \frac{5}{6} - n_8 \right) \Gamma \left( \frac{5}{6} + n_8 \right)} \times Y_{2|n_8|+m_8}(0) \sim \frac{Y_{2|n_8|+m_8}(0)}{|n_8|^{5/3}} \)) then we may interchange the limit and the sum, and therefore conclude that the right hand side of equation Eq. [60] vanishes as $w \to \infty$. In fact, all Fourier mode coupling terms, for all orders of approximation $f^{(j)}(t, m_8)$, vanish as $w \to \infty$. We emphasize that we have taken the $w \to \infty$ limit of only the coupling terms, and not the limit of the left hand side of this equation, which is then solved using the $k = 0$ uncoupled mode functions of Eq. [55]. This is obviously only a simple approximation, one which needs to be improved through further study.

We thereby have demonstrated the existence, for large $w$, of a stable approximate solution to the equation for $f(t, m_8; 0, w)$ in Eq. [59]. This demonstrates that instabilities that are sourced by the momenta $\vec{w}$ associated to the extra time dimensions, the issue raised in Section [2], are not inevitable in this model. We conclude that extra time dimensions may be consistent with the physics of the early universe. We may rationally assert that three extra time dimensions exist in our physical universe, as well as the extra $x^8$ spatial dimension, and then hope for experimental confirmation/rejection of this claim.
IV. CLASS E₁ SOLUTIONS

The model field equations admit a second class of solutions that is periodic in $x^8$ and parameterized by $\Lambda$ and $\xi$, $-\frac{1}{\sqrt{5}} < \xi < \frac{1}{\sqrt{5}}$, and which has a time dependent $\ell$ and also a larger $x^8$-dimension spatial period than the Class $E₀$ solutions:

$^{1}C₈ = 2\pi \sqrt{\frac{3}{5}} = ^{0}C₈ \sqrt{\frac{6}{5}}$

(61)

\[ \ell₁ = \mp \frac{5\sqrt{5}\xi \tanh\left(\frac{\sqrt{\Lambda}x^4}{\sqrt{5}}\right)}{3\sqrt{3}} = \pm 5\xi \sqrt{\frac{2}{3}} \ell₀ \tanh \left(x^4 \sqrt{\frac{\Lambda}{3}}\right) \]

\[ \varphi = \sqrt{1 - 5\xi^2} \ln \left[ \frac{\cosh^5 \left(\frac{\sqrt{5}x^8}{\sqrt{5}}\right) \csc^2 \left(\frac{\sqrt{5}}{3} \sqrt{\Lambda}x^8\right)}{2\sqrt{30}} \right] \pm \xi \ln \left[ \tan^2 \left(\frac{1}{2} \sqrt{\frac{5}{3} \sqrt{\Lambda}x^8}\right) \right] ; \]

(62)

when $\xi = 0$ then both $\pm \varphi$ satisfy the field equations. The scale factors are

\[ a = a₁ \left[ \cosh \left(x^4 \sqrt{\frac{\Lambda}{3}}\right) \right]^{\frac{1}{6}(1+5\xi)} \]

\[ \times \left[ \tan^2 \left(\frac{1}{2} \sqrt{\frac{5}{3} \sqrt{\Lambda}x^8}\right) \right]^{\pm \sqrt{\frac{1}{30}} \sqrt{1-5\xi^2}} \left[ \sin^2 \left(\sqrt{\frac{5}{3} \sqrt{\Lambda}x^8}\right) \right]^{\frac{1}{12}(1+\xi)} \]

(63)

\[ b = b₁ \left[ \cosh \left(x^4 \sqrt{\frac{\Lambda}{3}}\right) \right]^{\frac{1}{6}(1-5\xi)} \]

\[ \times \left[ \tan^2 \left(\frac{1}{2} \sqrt{\frac{5}{3} \sqrt{\Lambda}x^8}\right) \right]^{\pm \sqrt{\frac{1}{30}} \sqrt{1-5\xi^2}} \left[ \sin^2 \left(\sqrt{\frac{5}{3} \sqrt{\Lambda}x^8}\right) \right]^{\frac{1}{12}(1+\xi)} , \]

where $a₁$ and $b₁$ are constants. For the problem studied in this paper a physical solution to the field equations does not correspond either to a collapsing universe solution $\frac{\partial}{\partial x^4} \ln (a) < 0$ or to a physical universe with multiple macroscopic times, $\frac{\partial}{\partial x^4} \ln (b) \geq 0$. These conditions restrict $\xi$ to the interval $\frac{1}{\sqrt{5}} < \xi < \frac{1}{\sqrt{5}}$.

For this case $d\Omega = d\tau \cosh \left(x^4 \sqrt{\frac{\Lambda}{3}}\right) \left| \sin \left(x^8 \sqrt{\frac{5}{3}} \Lambda\right) \right|$, which clearly does inflate with time as expected on physical grounds; it is also independent of both $\xi$ and the choice of $\pm$ signs.
As in the previous Section, one may investigate the issue of instability raised in Section [2] by studying the propagation of a massive complex scalar field $\psi$ whose field equation is given in Eq. [1]. For this case we substitute into Eq. [3] $\Psi(x^4, x^5) = \sin^{-\frac{1}{2}} \left( \sqrt{\frac{2}{3}} \Lambda x^8 \right) F \left( \sqrt{\frac{2}{3}} \Lambda x^4, \sqrt{\frac{5}{3}} \Lambda x^8 \right)$ and set $x = \sqrt{\frac{2}{3}} \Lambda x^4$, $t = \sqrt{\frac{5}{3}} \Lambda x^4$, then use the fact that when $\xi = 0$ the Ricci scalar is $\frac{1}{5} \Lambda \left( 5 \sech^2 \left( \frac{\sqrt{2}}{\sqrt{3}} x^4 \right) + \csc^2 \left( \sqrt{\frac{5}{3}} \sqrt{\Lambda} x^8 \right) + 42 \right)$, otherwise it is $\frac{1}{5} \Lambda \left( 10 (5 \xi^2 - 1) \tan^2 \left( \frac{\sqrt{2}}{\sqrt{3}} x^4 \right) + \alpha \csc^2 \left( \sqrt{\frac{5}{3}} \sqrt{\Lambda} x^8 \right) \right)$, where $\alpha = 49 + 235 \xi^2 \pm 8 \xi \sqrt{30 - 150 \xi^2} \cos \left( \sqrt{\frac{2}{3}} \sqrt{\Lambda} x^8 \right) - (5 \xi^2 + 47) \cos \left( 2 \sqrt{\frac{5}{3}} \sqrt{\Lambda} x^8 \right)$.

For $\xi = 0$ the “plane wave” equation is

$$- F^{(2,0)}(t, x) = - F^{(0,2)}(t, x) + \frac{1}{20} \left( 28 \zeta + 12 m^2 - 5 \right) F(t, x) + \frac{1}{60} (2 \zeta - 15) \csc^2(x) F(t, x)$$

$$+ \frac{1}{6} \xi \sech^2 \left( \frac{t}{\sqrt{5}} \right) F(t, x) + \frac{\tanh \left( \frac{t}{\sqrt{5}} \right) F^{(1,0)}(t, x)}{\sqrt{5}}$$

$$+ \frac{3}{5} F(t, x) \left( k^2 \tan \frac{\pm 2 \sqrt{5}}{\sqrt{3}} \left( \frac{x}{2} \right) - w^2 \tan \frac{\pm 2 \sqrt{5}}{\sqrt{3}} \left( \frac{x}{2} \right) \right) \sqrt{\sech \left( \frac{t}{\sqrt{5}} \right) | \csc(x) |}. \quad (64)$$

Both the $\vec{k}$ and the $\vec{w}$ terms in the wave equation have coefficients that vanish exponentially as $t \to \pm \infty$. In this case, for large enough $|t|$, the momenta $\vec{w}$ associated to the extra time dimensions may not introduce an instability. For finite $t$ this matter must be studied more carefully.

For $\xi \neq 0$ a typical “plane wave” equation is of the form

$$- F^{(2,0)}(t, x) = - F^{(0,2)}(t, x)$$

$$+ \frac{1}{60} F(t, x) \left( 2 \zeta (5 \xi^2 + 47) + 36 m^2 - (15 - 2 \zeta (115 \xi^2 + 1)) \csc^2(x) - 15 \right)$$

$$- \frac{2}{15} \xi \sqrt{30 (1 - 5 \xi^2)} \cot(x) \csc(x) F(t, x)$$

$$+ \frac{1}{6} \xi (5 \xi^2 - 1) \tan^2 \left( \frac{t}{\sqrt{5}} \right) F(t, x) + \frac{\tanh \left( \frac{t}{\sqrt{5}} \right) F^{(1,0)}(t, x)}{\sqrt{5}}$$

$$+ \frac{3}{5} F(t, x) \left[ k^2 \cosh \frac{\pm 2 \sqrt{5}}{\sqrt{3}} \left( \frac{t}{\sqrt{5}} \right) \sin^{-\frac{\pm 2}{3}} \left( \frac{\xi}{3} x^4 \right) \tan^{-2} \sqrt{\frac{2}{3}} \sqrt{1 - 5 \xi^2} \left( \frac{x}{2} \right) \right]$$

$$- w^2 \cosh^{-\frac{\pm 2}{3}} \left( \frac{t}{\sqrt{5}} \right) \sin^{-\frac{\pm 2}{3}} \left( \frac{\xi}{3} x^4 \right) \tan^2 \sqrt{\frac{2}{3}} \sqrt{1 - 5 \xi^2} \left( \frac{x}{2} \right). \quad (65)$$

In this case, for $\sqrt{\frac{2}{3}} < \xi < - \frac{1}{5}$, the coefficient of $\vec{k} \cdot \vec{k}$ decreases exponentially, while the coefficient of $\vec{w} \cdot \vec{w}$ grows exponentially as $t \to \pm \infty$, which is similar to behavior seen in the previous Section. For $\frac{1}{5} < \xi < \sqrt{\frac{2}{5}}$, the coefficient of $\vec{v} \cdot \vec{v}$ grows exponentially, while
the coefficient of $\vec{w} \cdot \vec{w}$ exponentially decreases as $t \to \pm \infty$. There may exist a range of $\xi$ for which the momenta $\vec{w}$ associated with the extra time dimensions do not introduce an instability. This is under study.

V. CONCLUSION

It has been shown that the extra time dimensions of this model may be consistent with some of the important physics of the early universe. One may rationally propose that three extra time dimensions exist in our physical universe, as well as the extra $x^8$ spatial dimension, and then seek an extension of this given model that naturally explains both the end of inflation and the creation of the observed/inferred early universe lepton, quark and gauge boson densities.

A. End of Inflation

To be physically meaningful, one must extend this model to incorporate the creation of $\psi$ particle/anti-particle pairs through the annihilation of $\varphi$ quanta by generalizing the Lagrangian to include an interaction of $\psi$ with the inflaton $\varphi$, possibly as simple as $\lambda \psi^* \psi \varphi$. Then one must add Standard Model degrees of freedom so that $\psi$ has something (other than $\varphi$) to decay into.

Consider the case of the complex scalar $\psi$ field non-minimally coupling to gravity ($\zeta = 3/10$). One of the many reasons that such an extension of this model is required is because the mass-squared of the complex scalar $\psi$ field can become negative. In Eq. [48], $m_{\text{eff}}$ is one of the $\psi$ eigenmode labels, and physical states may exist in which $m_{\text{eff}}$ is an observable. Now $m_{\text{eff}}^2 = m_{\text{eff}}^2(m; m_8) = \frac{3}{20} + \frac{1}{2} m^2 + m_8^2$ and nature may presumably choose to produce $\psi$ field quanta with $0 \leq m_{\text{eff}}^2 < \infty$. Therefore $m^2 = -\frac{3}{10} - 2 m_8^2 + 2 m_{\text{eff}}^2$, which will always become negative for some ranges of $m_8$ integer values. Negative mass-squared typically characterizes a so-called “uncondensed phase.” In such a configuration the $\psi$ field excitations are tachyonic and will spontaneously decay due to the instability caused by the imaginary mass [22][23]. The $\psi$ field excitations will spontaneously lower their energy by producing Standard Model particles. The final state is expected to be a “condensate” of Standard Model particles that fills the volume of the universe. In this scenario $\varphi$ will decay to $\psi$ which will decay to a
final stable configuration with no $\psi$ tachyons and a universe filled with Standard Model particles. Inflation ends through a natural sequence of decay/creation processes. No “slow roll” inflaton potential is required. Instead, an identically zero inflaton potential is required. Recall that this model describes an initially inflating/deflating universe in which the massless inflaton potential is identically equal to zero. Assuming that inflation takes place after the Plank scale (after quantum gravity governs the physics), the concomitant initial condition for this inflaton potential model is exponentially more probable [13], [14], [15] than the corresponding initial condition for a model in which the initial inflaton potential is non-zero and on the order of the Planck mass, give or take a few factors of 10 (in order for the inflationary period to persist for approximately 60-e-folds). Note that in Eq. [24], for a general separable solution $\varphi = \varphi(x^4, x^8) = \phi_4(x^4)\phi_8(x^8)$, the term $-\frac{1}{\phi_8(x^8)} \frac{\partial^2}{\partial x^8} \phi_8(x^8)$ acts as an effective mass-squared term in the $\frac{\partial^2}{\partial x^4} \phi_4(x^4)$ equation. In this case the effective mass of the inflaton comes from geometry: the magnitude of the $x^8$ component of the inflaton momentum contributes to an effective inflaton mass.

To summarize, one must generalize the Lagrangian in this model to incorporate Standard Model leptons, quarks and gauge bosons, and also to support the creation (and annihilation, to preserve unitarity) of $\psi$ particle/anti-particle pairs through the annihilation (and creation) of $\varphi$ quanta with the addition of a term like $\lambda \psi^\dagger \psi \varphi$. $\psi$ may be promoted to a Higgs-type doublet, with an added Brout-Englert-Higgs-Guralnik-Hagen-Kibble type $(\psi^\dagger \psi)^2$ term in the Lagrangian. It may also be possible to replace the $(\varphi, \psi)$ pair with a $\psi$ doublet so that $\psi$ plays the role of inflaton and Higgs scalar. This has been attempted before, but always with a non-zero inflaton potential. In this model the inflaton potential is identically zero and the inflaton is massless.

B. The scale factors vanish on a set of measure zero

Typically and approximately, inflation scenarios inflate a scale of the size of one billionth the present radius of a proton to the size of the present radius of a marble or a grapefruit in about $10^{-32}$ seconds. In virtue of the Heisenberg Uncertainty Principle, and because the comoving $(x^1, x^2, x^3)$ dimensions have undergone inflation while the $x^8$ dimension has not, present epoch quantum fields that are functions of $(x^1, x^2, x^3, x^4, x^8)$ are expected to almost uniformly sample the region of the $x^8$ dimension that they occupy, if the $x^8$ spatial dimension
is compact, or if their functional dependence on $x^8$ is periodic. The spatial $x^8$ average of functions of $x^8$ are expected to appear in effective four dimensional spacetime theories. The fact that the scale factors vanish on a set of $x^8$ values of measure zero may be handled in a straightforward manner by working with spatial $x^8$ averages.

C. Inflationary fluctuations

If inflation persists long enough for the effects of the three extra time dimensions (with exponentially deflating scale factors) to become negligible, then calculations for the “stretching” of quantum fluctuations (until they exit the horizon) in this model are quite similar to conventional calculations. The main difference is that fluctuations in correlation functions due to the periodic dependence of quantum fields on the spatial $x^8$ dimension degree of freedom may also contribute to the primordial spectra of scalar and tensor fluctuations that source curvature and density perturbations, and give rise to cosmological structure. Calculations of these effects are underway, following the ideas of [24] [25] [26], modified by the following considerations, if the $x^8$ spatial dimension is compact: It makes little physical sense to mix a compact dimension with non-compact dimensions under a general coordinate transformation (or even a special Lorentz transformation), because the domain of the image of the transformation (i.e., the new coordinate functions) is ill-defined. Either the idea that the $x^8$ dimension is compact should be abandoned, or it must be recognized that the $x^8$ dimension plays a distinguished role in the physics. One is quickly led to a classical model of the universe with a distinguished time “degree of freedom,” coordinatized by $x^4$, which carries classical observers along its axis in the direction of the “arrow of time,” plus a second distinguished dimension that is spatial and compact. At this point in our understanding it seems that allowed coordinate transformations should preserve a distinguished $x^8$ compact spatial dimension, if it physically exists. Once the allowed coordinate transformations have been restricted this metric theory of gravity superficially resembles an “induced-matter interpretation” of a Kaluza-Klein theory [27] [28], except that the present model possesses three extra time dimensions (albeit with exponentially deflated scale factors for these dimensions) and the fact that in the present model all physical fields critically depend on the $x^8$ coordinate of the compact spatial dimension, while in most Kaluza-Klein models the fields do not vary along the extra spatial dimension.
D. Observing the extra dimensions

Both the fourth space dimension, whose associated comoving coordinate is $x^8$, as well as the extra time dimensions, evidently pose a challenge to observe, if they exist. Based on present science, it is not inconsistent to assert that the extra $x^8$ spatial dimension is compact, with the properties of a one-dimensional topological space that is a (possibly disjoint) union of two distinct sets of closed one-dimensional circles ($c_0$, $c_1$) that have circumferences $0C_8$ and $1C_8$, respectively. However if the $x^8$ spatial dimension is not compact then the experimental detection of this dimension is more likely, notwithstanding the fact that the other three spatial dimensions have inflated (thereby defining laboratory length scales) and the $x^8$ spatial dimension has not. Relative to the first three space dimensions, whose comoving coordinates are $(x^1, x^2, x^3) \in \mathbb{R}^3$, the physical distance between two comoving points $(x_0, x_0 + \Delta X)$ on the $x^8$-axis is expected to be exponentially smaller by about 60 e-folds than the distance between two comoving points in $\mathbb{R}^3$ separated by the same coordinate difference $\Delta X$, but lying on, say, the $x^3$-axis; the distance between two comoving points $(x_0, x_0 + \Delta X)$ lying on, say, the $x^7$-axis is even smaller, since the extra time dimensions experience deflation.

E. Quantum gravity

The three extra time dimensions do not become unimportant physically until after inflation/deflation. Quantum gravity is important prior to inflation/deflation. The ideas underlying the foundation of quantum gravity will require some revision if the extra dimensions described mathematically in this paper are physically real. Successful calculations of the cross section for the creation of $\psi$ particle/anti-particle pairs through the annihilation of $\varphi$ quanta and the concomitant end of $\varphi$-driven inflation, as well as the subsequent decay of $\psi$ particles into Standard Model degrees of freedom would strengthen confidence in this model.

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