OSCILLATORY BANDED HESSENBERG MATRICES, MULTIPLE ORTHOGONAL POLYNOMIALS AND RANDOM WALKS

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ABSTRACT. A spectral Favard theorem for oscillatory bounded banded lower Hessenberg matrices is found. To motivate the relevance of the oscillatory character, the spectral Favard theorem for bounded Jacobi matrices is revisited and it is shown that after an adequate shift of the Jacobi matrix one gets an oscillatory matrix. The large knowledge on the spectral and factorization properties of oscillatory and totally nonnegative matrices leads to a spectral representation, a spectral Favard theorem for these Hessenberg matrices, in terms of sequences of multiple orthogonal polynomials of types II and I with respect to a set of positive Lebesgue–Stieltjes measures. This is achieved in two different circumstances. Firstly, using bidiagonal factorization when certain nonnegative continued fraction is positive, the regular oscillatory case. It is shown that oscillatory Toeplitz banded Hessenberg matrices are regular. Moreover, it is proven that oscillatory banded Hessenberg matrices are organized in rays, being the origin of the ray nonregular oscillatory and all the interior points of the ray regular oscillatory. Secondly, even though the Hessenberg matrix happens to be nonregular, multiple orthogonal polynomials and a corresponding set of positive measures determined by a semi-infinite oscillatory bounded banded Hessenberg matrix always exists —this matrix is constructed in terms of a simple transformation of the first banded Hessenberg matrix. In the finite case, discrete measures supported in the zeros of the type II recursion polynomials, these zeros are the eigenvalues of the leading principal submatrices of the oscillatory banded Hessenberg matrix, are discussed.

It is shown how the objects in this theory connects with Aptekarev, Kalyagin and Van Iseghem genetic sums and Stieltjes moment problems and with simple Darboux–Christoel transformations. A spectral interpretation and a complete identification of the Darboux transformations of the banded Hessenberg matrices in terms of Christoel transformations of the spectral measures is found.

The spectral Favard theorem for regular oscillatory bounded banded Hessenberg matrices is applied to Markov chains with tetradial transition matrices, i.e. beyond birth and death. In the finite case, the Karlin–McGregor spectral representation is given, it is shown that the random walks are recurrent and explicit expressions in terms of the orthogonal polynomials for the stationary distributions are not necessarily recurrent, it is characterized in terms of the first measure. Ergodicity of the Markov chain is discussed in terms of the existence of a mass at 1, which is an eigenvalue and corresponding right and left eigenvectors are given. Finally, the fact that any oscillatory matrix is $LU$ factorizable in terms of bidiagonal matrices leads to a stochastic factorization of the Markov transition matrix.

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In functional analysis of linear operators is known that a self-adjoint operator defined in a Hilbert space with a cyclic vector \( u \) can be represented as a symmetric Jacobi operator in a suitable orthonormal basis, see for instance [39, Theorem 5.3.1]. The infinite matrix representative of these Jacobi operators is tridiagonal which leads to a three term recurrence relations definition of an orthonormal polynomial basis with respect to the spectral measure of the Jacobi operator. In this way the theory of orthogonal polynomials is instrumental to understand the spectrality of a self-adjoint operator in a Hilbert space [39]. The construction of the spectral measure for these Jacobi operators can be reached by means of a Markov theorem, and is a uniform limit of ratio for polynomials satisfying the three term recurrence relations coming from the Jacobi operator. The spectral measure is in fact a measure of orthogonality for the sequence of orthogonal polynomials coming from the Jacobi operator, as they share its moments. Spectral theorems hold beyond self-adjointness for normal operators (the operator commutes with its adjoint).

In the case of banded Hessenberg operators, the self-adjointness or normality no more takes place. Hence, the mentioned well established spectral theory is lost. Nevertheless, these banded Hessenberg operators case can be understood as higher order (4 or more terms) recursion relations and have associated recursion polynomials, known as type I and II. Moreover, a biorthogonality can
be derived between these two systems (the ones of polynomials and the linear forms) that stands as duality relations. In fact, this is a sort of *non-spectral* Favard theorem for banded Hessenberg operators. Several authors, in particular Sorokin and Van Iseghem have studied these non spectral Favard type theorems [40, 41, 42]. Unfortunately these non-spectral Favard type theorems for high order recurrence relations only contemplate the existence of functionals (and define the moments in terms of the recursion coefficients) that happened to be biorthogonal. These sequences of functions takes part of a theory of multiple orthogonal polynomials, type I and II [32]. For more on multiple orthogonal polynomials see [22, Chapter 23] and also [1, 4].

A natural question arises from these facts for the operator theory side: Do we have a spectral Favard theorem for a subclass of bounded lower Hessenberg matrices? Or, from the orthogonal polynomial side: Do multiple orthogonal polynomials can be of any use in the spectral description of a banded Hessenberg operators? That is, do they give an interpretations for the spectral points, as well as of the spectral measures of the corresponding operator.

A first attempt has been made by Kalyagin in [24, 25] were the author defines a class of operators related with the Hermite–Padé approximants and connects their spectral analysis with the asymptotic properties of polynomials defined by systems of orthogonality relations (that coincides with the notion of multiple orthogonality). Some years later, this author in a joint work with Aptekarev and Van Iseghem [2] made the analysis of banded Hessenberg operators with one upper diagonal based on the analysis of the genetic sums formulas for the moments of the operator. Regarding these achievements Coussement and Van Assche comment in [7, Conclusions] write: “*However, finding necessary and sufficient conditions on the recurrence coefficients to have positive orthogonality measures on the real axis is still an open problem.*” See also [46].

Since the publication of the keystone monograph [18] it is well known that the feature of being oscillatory of a Jacobi matrix is important in the study of the spectral points of a Jacobi operator. They are also instrumental in the study of the orthogonal polynomials sequences used to define the spectral measure of the operator. In fact, we take the Jacobi operators as case study in this Introduction, to give an explanation of the ideas we intend to use in this paper. To this end we begin this work by an introduction to oscillatory matrices.

In this paper we give a spectral Favard theorem: For any bounded banded Hessenberg matrix that is oscillatory and regular we have a couple of spectral Lebesgue–Stieltjes measures for which the recursion polynomials happens to be multiple orthogonal polynomials. Thus, we find sufficient conditions for the existence of spectral positive measures. However, the Jacobi–Piñeiro’s multiple orthogonal polynomials show that these are sufficient but not necessary conditions.

Our motivation for this work initiated in our previous research on random walks beyond birth and death and multiple orthogonal polynomials, see the papers [3, 5], in where the transition matrices were Hessenberg matrices with more than three diagonals. In those papers we studied two specific cases, in [3] we dealt with the Jacobi–Piñeiro multiple orthogonal polynomials, see [22, Section 23.3.2] and in [5] the hypergeometric multiple orthogonal polynomials recently presented in [29].

However, the approach of those two papers was the reverse to that initially presented by Karlin and McGregor in their seminal paper [27]. From the recursion matrix of the given examples of multiple orthogonal polynomials we gave a procedure to find a transition matrix and studied the corresponding Markov chains. While in [27], for birth and death Markov chains, a tridiagonal transition matrix was presented and the previously mentioned spectral approach was used to construct a set of *random walk orthogonal polynomials* and give spectral representation formulas in
terms of the corresponding polynomials for different important probabilistic objects in the walk. Indeed, this was a very important question, pointed to the authors by Alberto Grünbaum, that deserved an answer. What about the spectral theory for our examples of random walk multiple orthogonal polynomials? The answer to this question is the spectral Favard theorem we present in this paper for regular oscillatory Hessenberg matrices. Let us notice that all the cases we have mentioned, the tridiagonal case, and the Jacobi–Piñeiro in the semi-band and hypergeometric cases fit in this family of oscillatory Hessenberg matrices. Consequently, using this new tool we start as Karlin and McGregor from the transition matrix, assuming that is an oscillatory Hessenberg matrix and deriving from it the spectral representation of the random walk.

1.1. Oscillatory matrices. Given a matrix $A$ we have the natural concepts of submatrix and minor. For two subsets of indexes $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ and $\beta = \{\beta_1, \beta_2, \ldots, \beta_s\}$ the submatrix $A[\alpha, \beta]$ is the one obtained from $A$ by keeping only those entries at the crossing of the rows indicated by $\alpha$ and columns by $\beta$. When $\alpha = \beta$, we denote the submatrix by $A[\alpha]$ and call it a principal submatrix. A contiguous subset is of the form $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_r\} \subset \{0, 1, \ldots, N - 1\}$, i.e., $\alpha_k = \alpha_1 + k - 1$ for $k \in \{1, \ldots, r\}$. Leading principal submatrices are $A[\{1, 2, \ldots, j\}]$. The determinants of these submatrices are called minors (obviously we need $r = s$). The complementary submatrix $A(\alpha, \beta)$ is obtained from $A$ by removing the rows indicated by $\alpha$ and columns by $\beta$. When $\alpha = \beta$, we denote the complementary submatrix $A(\alpha)$.

Totally nonnegative (TN) matrices are those with all their minors nonnegative [18, 11], the set of nonsingular TN matrices are denoted by $\text{InTN}$. Oscillatory matrices [18] are totally nonnegative, irreducible [21] and nonsingular. Notice that the set of oscillatory matrices is denoted by $\text{IITN}$ (irreducible invertible totally nonnegative) in [11]. An oscillatory matrix $A$ is equivalently defined as a totally non negative matrix $A$ such that for some $n$ we have that $A^n$ is totally positive (all minors are positive).

We have the important result:

**Theorem 1** (Gantmacher–Krein Criterion). [18, Chapter 2, Theorem 10]. A totally non negative matrix $A$ is oscillatory if and only if it is nonsingular and the elements at the first subdiagonal and first superdiagonal are positive.

**Theorem 2.** [18, Chapter 2, §7] and [11, Corollary 2.6.7]. If a matrix $A$ is oscillatory then so is any principal submatrix $A[\alpha]$ for any contiguous subset $\alpha$ of indexes.

For Jacobi matrices we find:

**Theorem 3** (Oscillatory Jacobi matrices). [17, Chapter XIII, §9] and [18, Chapter 2, Theorem 11].

A tridiagonal matrix is oscillatory if and only if:

i) The matrix entries of the first subdiagonal and first superdiagonal are positive.

ii) All leading principal minors are positive.

Gauss–Borel factorizations are intimately related with these concepts:

**Theorem 4.** [11, Theorem 2.4.1] The matrix $A \in \mathbb{R}^{N \times N}$ is an InTN matrix if and only if it admits a Gauss–Borel factorization $A = LU$ with $L$ and $U$ both InTN.

For further properties, we require some more definitions:

i) **Totally nonzero vectors:** A vector $u \in \mathbb{R}^n$ is totally nonzero if no entry of the vector $u$ is zero.
ii) **Total sign variation of a totally nonzero vector:** \( v(u) = \text{cardinal}\{i \in \{1, \ldots, n-1\} : u_i u_{i+1} < 0\} \). For a general vector \( u \in \mathbb{R}^n \) we define \( v_m(u) \) (\( v_M(u) \)) as the minimum (maximum) value \( v(y) \) among all totally nonzero vectors \( y \) that coincide with \( u \) in its nonzero entries. For \( v_m(u) = v_M(u) \) we write \( v(u) \) := \( v_M(u) = v_M(u) \).

iii) Let \( a_1 \geq a_2 \geq \cdots \geq a_n \) and \( b_1 \geq b_2 \geq \cdots \geq b_{n-1} \) two sequences arranged in decreasing order. We say that the sequence \( \{b_k\}_{k=1}^{n-1} \) interlaces the sequence \( \{a_k\}_{k=1}^{n} \) if

\[
a_n \geq b_{n-1} \geq a_{n-1} \geq b_{n-2} \geq \cdots \geq b_1 \geq a_1.
\]

If all the inequalities can be taken to be strict then we say that the sequence \( \{b_i\}_{i=1}^{n-1} \) strictly interlaces the sequence \( \{a_i\}_{i=1}^{n} \). This can be extended as follows. Let \( a_1 \geq a_2 \geq \cdots \geq a_n \) and \( b_1 \geq b_2 \geq \cdots \geq b_{n-k} \) two sequences arranged in decreasing order. We say that the sequence \( \{b_i\}_{i=1}^{n-k} \) interlaces the sequence \( \{a_i\}_{i=1}^{n} \) if

\[
a_{i+k} \leq b_i \leq a_i, \quad i = 1, 2, \ldots, n-k.
\]

The following theorems are extracted from [11] many of these results are in [18].

With these at hand one finds the following spectral properties:

**Theorem 5** (Eigenvalues). Given an oscillatory matrix \( A \in \mathbb{R}^{N \times N} \), then:

i) [11, Theorem 5.2.1] The eigenvalues of \( A \) are \( N \) distinct positive numbers.

ii) [11, Theorem 5.5.2] The eigenvalues of \( A \) strictly interlace the eigenvalues of the principal submatrices \( A(1) \) (obtained from \( A \) by deleting the first row and column) and of \( A(N) \) (obtained from \( A \) by deleting the last rows and column).

iii) [11, Corollary 5.5.3] For any contiguous subset \( \alpha \), the eigenvalues of \( A[\alpha] \) and \( A \) strictly interlace.

**Theorem 6** (Eigenvectors I). [11, Theorem 5.3.7] Consider the set of column vectors \( \mathcal{B} = \{u^{(1)}, \ldots, u^{(N)}\} \subset \mathbb{R}^N \) and the set of row vectors \( \mathcal{B}^* = \{w^{(1)}, \ldots, w^{(N)}\} \subset (\mathbb{R}^N)^* \) and consider the matrices

\[
U = \begin{bmatrix} u^{(1)} & \ldots & u^{(N)} \end{bmatrix}, \quad W = \begin{bmatrix} w^{(1)} \\ \vdots \\ w^{(N)} \end{bmatrix}
\]

such that \( WU = I \). i.e. \( \mathcal{B} \) and \( \mathcal{B}^* \) are biorthogonal sets. Then, the following statements are equivalent:

i) Sign changes I: For \( 1 \leq p \leq q \leq n, p - 1 \leq v_m(u) \leq v_M(u) \leq q - 1 \), for \( u \neq 0 \) in \( \mathbb{R}\{u^{(p)}, u^{(p+1)}, \ldots, u^{(q)}\} \) and \( p - 1 \leq v_m(w) \leq v_M(w) \leq q - 1 \), for \( w \neq 0 \) in \( \mathbb{R}\{w^{(p)}, w^{(p+1)}, \ldots, w^{(q)}\} \).

ii) Sign changes II: For each \( q = 1, 2, \ldots, n \), \( v_m(u) \leq q - 1 \), for \( u \neq 0 \) in \( \mathbb{R}\{u^{(1)}, \ldots, u^{(q)}\} \) and \( v_M(w) \leq q - 1 \) for \( w \neq 0 \) in \( \mathbb{R}\{w^{(1)}, \ldots, w^{(q)}\} \).

iii) Minors: For \( k \in \{1, 2, \ldots, N+1\} \) every minor of \( U \) formed with the first \( k \) columns of \( U \) (first \( k \) rows of \( W \)) has the same (nonzero) sign.

iv) If \( D = \text{diag}(\lambda_1, \ldots, \lambda_n) \), with \( \lambda_1 > \lambda_2 > \cdots > \lambda_n > 0 \) and \( A = UDW \), then there is a positive power of \( A \) that is totally positive.

v) There is a TN matrix with right eigenvectors \( u^{(1)}, \ldots, u^{(N)} \) and left eigenvectors \( w^{(1)}, \ldots, w^{(N)} \), arranged in the standard order (with corresponding eigenvalues \( \lambda_1 > \lambda_2 > \cdots > \lambda_n > 0 \)).

vi) There is an oscillatory matrix with right eigenvectors \( u^{(1)}, \ldots, u^{(N)} \) and left eigenvectors \( w^{(1)}, \ldots, w^{(N)} \), arranged in the standard order (with corresponding eigenvalues \( \lambda_1 > \lambda_2 > \cdots > \lambda_N > 0 \)).

We also have
Theorem 7 (Eigenvectors II). Let $A \in \mathbb{R}^{N \times N}$ be an oscillatory matrix, and $u(k)$ ($w(k)$) the right (left) eigenvector corresponding to $\lambda_k$, the $k$-th largest eigenvalue of $A$. Then

i) [11, Theorem 5.3.3] We have $v_m(u(k)) = v_M(u(k)) = v(w(k)) = k - 1$ ($v_m(w(k)) = v_M(w(k)) = w(k)$ and $w(N)$ are totally nonzero); the other vectors may have a zero entry.

ii) From Perron–Frobenious theorem we know that $u(1)$ ($w(1)$) is totally positive, and that no other $u(k)$ ($w(k)$), $k = 2, \ldots, n$ is totally positive. In fact, $u(N)$ ($w(N)$) strictly alternates the sign of its entries.

1.2. Revisiting bounded Jacobi matrices. To illustrate the approach of this paper to banded Hessenberg matrices we will consider first the classical spectral theory for Jacobi matrices. We will proceed emphasizing the ideas of our approach that will be extended in the next section to banded Hessenberg matrices.

1.2.1. The Jacobi matrix and recursion polynomials. Let us consider the tridiagonal semi-infinite real matrix

$$J := \begin{bmatrix} m_0 & 1 & 0 & \cdots & 0 \\ \ell_1 & m_1 & 1 & \cdots & \ell_2 \\ 0 & \ell_2 & m_2 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \ell_N & m_N \end{bmatrix},$$

(1)

and assume that $\ell_k > 0$, $k \in \mathbb{N}$. This matrix is symmetrizable, as the positive diagonal matrix $H = \text{diag}(H_0, H_1, \ldots)$, $H_0 = 1$, $H_n := \ell_1 \cdots \ell_n$, is such that $H^{-\frac{1}{2}}JH^\frac{1}{2}$ is symmetric. The recursion polynomials $\{P_n\}_{n=0}^\infty$ are monic polynomials with $\deg P_n = 0$ satisfying the recursion relation

$$\ell_n P_{n-1} + m_n P_n + P_{n+1} = x P_n, \quad n \in \mathbb{N},$$

(2)

with $P_0 = 1$. That is,

$$P = \begin{bmatrix} P_0 \\ P_1 \\ \vdots \end{bmatrix},$$

satisfies the eigenvalue property $JP = xP$. Dual to $P(x)$ we introduce $Q = [Q_0(x) \quad Q_1(x) \cdots]$, that are left eigenvectors of the semi-infinite matrix $J$, i.e., $Q(x)J = xQ(x)$, and the initial conditions, that determine these polynomials uniquely, are taken as $Q_0 = 1$. It can be easily shown that $Q_n = \frac{P_n}{H_n}$.

Let us introduce the leading principal submatrices$^1$ $J^{[N]} = J[\{0, 1, \ldots, N\}]$, i.e.

$$J^{[N]} := \begin{bmatrix} m_0 & 1 & 0 & \cdots & 0 \\ \ell_1 & m_1 & 1 & \cdots & \ell_2 \\ 0 & \ell_2 & m_2 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \ell_N & m_N \end{bmatrix} \in \mathbb{R}^{(N+1) \times (N+1)}, \quad \Delta_N := \det J^{[N]},$$

(3)

$^1$In the context of Stieltjes moment problems, they are also known as Friedrichs approximants, see [37, Section 8.4].
As \( J^{[N]} \) is symmetrizable its eigenvalues are real. Then, from the given definition of the recursion polynomials we get that they are the characteristic polynomials of these submatrices

\[
P_{N+1}(x) = \det (xI_{N+1} - J^{[N]}).
\]

To prove it we expand the determinant along the last row to see that these characteristic polynomials satisfy (2) and have the same initial condition.

### 1.2.2. Polynomials of the second kind and truncated polynomials.

Let us consider the recursion relation (2) for \( n \in \mathbb{N}_0 \) by setting \( \ell_0 = -1 \) and \( n \in \mathbb{N}_0 \). Then, the recursion polynomials \( P_n \) correspond to the choice

\[
(4) \quad P_{-1} = 0, \quad P_0 = 1.
\]

Polynomials of the second kind \( \{P_n^{(1)}\}_{n=0}^{\infty} \) are defined from (2) by the following initial conditions

\[
(5) \quad P_{-1}^{(1)} = 1, \quad P_0^{(1)} = 0,
\]

For the recursion polynomials of the second kind one finds

\[
P_{N+1}^{(1)} = e_1^T \text{adj} (xI_{N+1} - J^{[N]}) e_1 = \begin{vmatrix}
    x - m_1 & -1 & 0 & \cdots & 0 \\
-\ell_2 & x - m_2 & -1 & \cdots & 0 \\
0 & -\ell_3 & x - m_3 & -1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & -\ell_N \\
0 & \cdots & \cdots & \cdots & \cdots & x - m_N \\
\end{vmatrix}.
\]

Here \( e_1 \) is the semi-infinite vector with entries \( \delta_{1,n} \). This can be proven by expanding the determinant along the last row to get that the recursion relation (2) and recalling the initial conditions (5).

Associated with the principal submatrices

\[
(6) \quad J^{[N,k]} := \begin{bmatrix}
    m_k & 1 & 0 & \cdots & 0 \\
\ell_{k+1} & m_{k+1} & 1 & \cdots & 0 \\
0 & \ell_{k+2} & m_{k+2} & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & \ell_N \\
0 & \cdots & \cdots & \cdots & \cdots & m_N \\
\end{bmatrix} \in \mathbb{R}^{(N+1-k) \times (N+1-k)}, \quad \Delta_{N,k} := \det J^{[N,k]},
\]

we define the truncated polynomials \( P_{N+1}^{[k]} := \det (xI_{N+1-k} - J^{[N,k]}) \). An expansion of these determinants along the first row shows that they satisfy the recursion relation

\[
\ell_{k+1} P_{N+1}^{[k+2]} + m_k P_{N+1}^{[k+1]} + P_{N+1}^{[k]} = xP_{N+1}^{[k+1]}, \quad k \in \{0, 1, \ldots, N\},
\]

where we take \( P_{N+1}^{[N+1]} = 1, P_{N+1}^{[N+2]} = 0 \).
1.2.3. Eigenvalues, left and right eigenvectors. From hereon let us assume that the zeros \( \{ \lambda_n^{[N]} \}_{n=1}^{N+1} \) of the polynomial \( P_{N+1}(x) \) are simple. We introduce

\[
P^{(N)} := \begin{bmatrix} P_0 \\ P_1 \\ \vdots \\ P_N \end{bmatrix}
\]

so that

\[
J^{[N]} P^{(N)}(x) + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ P_{N+1}(x) \end{bmatrix} = xP^{(N)}(x).
\]

From this equation we get that the set of zeros the polynomial \( P_{N+1}(x) \), are the eigenvalues of \( J^{[N]} \), with corresponding right eigenvectors \( u_n^{(N)} := P^{(N)}|_{x=\lambda_n^{[N]}}, \ n \in \{1, \ldots, N+1\} \).

Similarly, we construct a set of left eigenvectors of the matrix \( J^{[N]} \). For \( Q^{(N)} := \begin{bmatrix} Q_0 & Q_1 & \cdots & Q_N \end{bmatrix} \) and \( k \in \{1, \ldots, N+1\} \), the vectors \( w_n^{(N)} := Q^{(N)}|_{x=\lambda_n^{[N]}}, \ n \in \{1, \ldots, N+1\} \), are left eigenvectors of \( J^{[N]} \). Alternatively, let us introduce the notation \( \Omega^{(N)} := \begin{bmatrix} P_1^{[N+1]} & \cdots & P_N^{[N+1]} \end{bmatrix} \). The recursion relation for these truncated polynomials implies

\[
\begin{bmatrix} P_1^{[N+1]} & \cdots & P_N^{[N+1]} \end{bmatrix} J^{[N]} + \begin{bmatrix} P_0^{[N+1]} & 0 & \cdots & 0 \end{bmatrix} = x \begin{bmatrix} P_1^{[N+1]} & \cdots & P_N^{[N+1]} \end{bmatrix},
\]

and we conclude that the vectors \( \omega_n^{(N)} := \Omega^{(N)}|_{x=\lambda_n^{[N]}}, \ n \in \{1, \ldots, N+1\} \), are left eigenvectors of \( J^{[N]} \).

1.2.4. The Christoffel–Darboux formulas. Let us consider two Christoffel–Darboux kernels. The Christoffel–Darboux (CD) kernels are defined as

\[
K_N(x, y) := \sum_{n=0}^{N} Q_n(x) P_n(y) = \sum_{n=0}^{N} \frac{P_n(x) P_n(y)}{H_n}, \quad K^{[N]}(x, y) := \sum_{n=0}^{N} P_{n+1}^{[N+1]}(x) P_n(y).
\]

Noticing that

\[
\begin{bmatrix} P_1^{[N+1]}(x) & \cdots & P_N^{[N+1]}(x) \end{bmatrix} J^{[N]} + \begin{bmatrix} P_0^{[N+1]}(x) & 0 & \cdots & 0 \end{bmatrix} = x \begin{bmatrix} P_1^{[N+1]}(x) & \cdots & P_N^{[N+1]}(x) \end{bmatrix},
\]

\[
J^{[N]} \begin{bmatrix} P_0(y) \\ \vdots \\ P_N(y) \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ P_{N+1}(y) \end{bmatrix} = y \begin{bmatrix} P_0(y) \\ \vdots \\ P_N(y) \end{bmatrix},
\]

\[\text{The standard CD kernel } K_N, \text{ that satisfies the reproducing property, is the one normally considered in the literature see [38], while } K^{[N]} \text{ was first discussed by Van Assche in [45].}\]
we deduce that
\[
\begin{bmatrix}
P_{N+1}^{[0]}(x) & 0 & \ldots & 0 \\
\vdots & \ddots & \ldots & \vdots \\
0 & \ldots & \ldots & P_{N+1}(y)
\end{bmatrix}
- \begin{bmatrix}
P_{N+1}^{[1]}(x) & \ldots & P_{N+1}^{[N+1]}(x) \\
\vdots & \ddots & \vdots \\
0 & \ldots & 0
\end{bmatrix}
= (x - y) \begin{bmatrix}
P_{N+1}^{[1]}(x) & \ldots & P_{N+1}^{[N+1]}(x) \\
\vdots & \ddots & \vdots \\
P_{N}(y)
\end{bmatrix}
\]

and we find the following Christoffel–Darboux type relation
\[
K_{N}(x, y) = \sum_{n=0}^{N} P_{N+1}^{[n+1]}(x) P_{n}(y) = \frac{P_{N+1}(x) - P_{N+1}(y)}{x - y}
\]

and, consequently, the confluent Christoffel–Darboux type formula \(\sum_{n=0}^{N} P_{N+1}^{[n+1]} P_{n} = P_{N+1}'\) holds.\(^{3}\)

Analogously, we can deduce the standard Christoffel–Darboux formula
\[
K_{N}(x, y) = \sum_{n=0}^{N} Q_{n}(x) P_{n}(y) = \frac{1}{H_{N}} \frac{P_{N}(y) P_{N+1}(x) - P_{N+1}(y) P_{N}(x)}{x - y}
\]

and the associated confluent Christoffel–Darboux relation \(\sum_{n=0}^{N} Q_{n} P_{n} = \frac{1}{H_{N}} (P_{N+1}' P_{N} - P_{N}' P_{N+1})\).

1.2.5. *Discrete orthogonality.* For \(k = 1, \ldots, N + 1\), the vectors \(u^{(N)}_{k} := P^{(N)}(\lambda^{[N]}_{k})\) (\(\tilde{w}^{(N)}_{k} := Q^{(N)}_{k} (\lambda^{[N]}_{k})\)) or \(\Omega^{(N)}_{k} (\lambda^{[N]}_{k})\) are right (left) eigenvectors of \(J^{[N]}\). Given any set of left eigenvectors \(\{\tilde{w}^{(N)}_{k}\}_{k=1}^{N+1}\), a normalized basis \(\{w^{(N)}_{k}\}_{k=1}^{N+1}\) biorthogonal to the basis of right eigenvectors \(\{u^{(N)}_{k}\}_{k=1}^{N+1}\) is given by
\[
w^{(N)}_{k,n} = \frac{\tilde{w}^{(N)}_{k,n}}{\sum_{l=1}^{N+1} \tilde{w}^{(N)}_{k,l} P_{l-1}(\lambda^{[N]}_{k})}.
\]

In particular, the following expression holds
\[
w^{(N)}_{k,n} = \frac{Q_{n-1}(\lambda^{[N]}_{k})}{\sum_{l=0}^{N} Q_{l}(\lambda^{[N]}_{k}) P_{l}(\lambda^{[N]}_{k})} = H_{N} \frac{Q_{n-1}(\lambda^{[N]}_{k})}{P_{N}(\lambda^{[N]}_{k}) P'_{N+1}(\lambda^{[N]}_{k})}.
\]

Moreover, in terms of the masses
\[
(7) \quad \mu^{[N]}_{k} := w^{(N)}_{k,1} = \frac{1}{\sum_{l=0}^{N} Q_{l}(\lambda^{[N]}_{k}) P_{l}(\lambda^{[N]}_{k})},
\]
also known as Christoffel numbers or coefficients, we can write
\[
(8) \quad w^{(N)}_{k,n} = Q_{n-1}(\lambda^{[N]}_{k}) \mu^{[N]}_{k}, \quad \mu^{[N]}_{k} := w^{(N)}_{k,1} = \frac{1}{\sum_{l=0}^{N} Q_{l}(\lambda^{[N]}_{k}) P_{l}(\lambda^{[N]}_{k})} = \frac{1}{K_{N}(\lambda^{[N]}_{k}, \lambda^{[N]}_{k})}.
\]

\(^{3}\)These formulas were found by Van Assche in [43].
This leads to the positivity of the Christoffel coefficients.\footnote{For more on Christoffel numbers see \cite{22, §2.4}.}

The alternative expression, in terms of truncated polynomials, for the components of the normalized left eigenvectors

\[ W_{k,n}^{(N)} = \frac{P_{N+1}^{[n]}(\lambda_k^{[N]})}{\sum_{l=0}^{N} P_{N+1}^{[l+1]}(\lambda_k^{[N]}) P_l(\lambda_k^{[N]})} = \frac{P_{N+1}^{[n]}(\lambda_k^{[N]})}{P_{N+1}'(\lambda_k^{[N]})}, \]

holds. Hence, the masses (7) can be expressed as\footnote{At this point we recommend \cite[Chapter 2 and Equation 5.22 in that chapter]{32}.}

\[ \mu_k^{[N]} = \frac{P_{N+1}^{(1)}(\lambda_k^{[N]})}{P_{N+1}'(\lambda_k^{[N]})}. \]

As the Christoffel coefficients are positive we conclude that recursion polynomials \( P_{N+1} \) strictly interlace its second type polynomials \( P_{N+1}^{(1)} \).

For the corresponding matrices \( U \) (with columns the right eigenvectors \( u_k \) arranged in the standard order) and \( W \) and (with rows the left eigenvectors \( w_k \) arranged in the standard order) we find \( UW = WU = I_{N+1} \) and, in terms of \( D = \text{diag}(\lambda_1^{[N]}, \ldots, \lambda_{N+1}^{[N]}), \) we have \( UD^n W = (J^{[N]})^n \). Let us consider the singular measure, with support on the zeros of \( P_{N+1} \), given by

\[ \mu^{[N]} := \sum_{j=1}^{N+1} \mu_j^{[N]} \delta(z - \lambda_j^{[N]}). \]

Recalling (8) we see that \( WU = I \) is equivalent to the following biorthogonal relations

\[ \langle Q_k \mu^{[N]}, P_l \rangle = \delta_{k,l}, \quad k, l \in \{0, \ldots, N\}, \]

and, consequently, the following orthogonality relations

\[ \langle \mu^{[N]}, x^k P_n \rangle = 0, \quad k = 0, \ldots, n - 1, \quad \langle Q_k \mu^{[N]}, x^n \rangle = 0, \quad n \in \{0, 1, \ldots, k - 1\}, \quad k \in \{1, \ldots, N\}, \]

are fulfilled. This way of writing the orthogonality relations is prepared to be extended to multiple orthogonality later in the paper. Notice also that the polynomials \( p_n := \frac{p_n}{\sqrt{\mu_n}} = \sqrt{H_n} q_n \) are orthonormal polynomials. From \( \sum_{j=1}^{N+1} w_{j,1}^{(N)} = 1 \) we see that \( \sum_{j=1}^{N+1} \mu_{j,1}^{[N]} = 1 \). Then, we find a Lebesgue–Stieltjes representation of this singular measure. In terms of the piecewise continuous function

\[ \psi^{[N]} := \begin{cases} 0, & x < \lambda_{N+1}^{[N]}, \\ \mu_1^{[N]} + \cdots + \mu_k^{[N]}, & \lambda_{k+1}^{[N]} \leq x < \lambda_k^{[N]}, \quad k \in \{1, \ldots, N\}, \\ \mu_1^{[N]} + \cdots + \mu_{N+1}^{[N]}, & x \geq \lambda_1^{[N]}, \end{cases} \]

we have \( \mu^{[N]} = \text{d} \psi^{[N]} \).

The resolvent matrix \( R_z^{[N]} \) of the leading principal submatrix \( J^{[N]} \) is

\[ R_z^{[N]} := (zI_{N+1} - J^{[N]})^{-1} = \frac{\text{adj} (zI_{N+1} - J^{[N]})}{\det(zI_{N+1} - J^{[N]})}. \]
From the spectral decomposition of the matrix $J^{[N]}$, we obtain $R_{x}^{[N]} = U(zI_{N+1} - D)^{-1}W$. The corresponding Weyl function $S^{[N]}$ defined by $S^{[N]} := e_{1}^{T}(zI_{N+1} - T^{[N]})^{-1}e_{1}$, can be expressed as follows

$$S^{[N]}(z) = \frac{P_{N+1}^{(1)}(z)}{P_{N+1}(z)} = \sum_{n=1}^{N+1} \frac{\mu_{n}^{[N]} N}{z - \lambda_{n}^{[N]}}.$$  

1.2.6. Oscillatory matrices, interlacing properties and spectral theorem. If the matrix $J$ is bounded all the possible eigenvalues of the submatrices $J^{[N]}$ belong to the disk $D(0, \|J\|)$. As all the eigenvalues are real, let us consider those that are negative, and let $b$ be the supreme of the absolute values of all negative eigenvalues. Notice that $b \leq \|J\|$.

**Theorem 8.** For $s \geq b$ the matrix $J_s = J + sI$ is oscillatory.

**Proof.** Take $s \geq b$, then $J_s$ has the eigenvalues of its leading principal submatrices $J_{s}^{[N]} = J^{[N]} + sI_{N+1}$ all positive. The corresponding characteristic polynomials are $P_{N+1}(x - s) = \det (xI_{N+1} - J_{s}^{[N]})$, so that $\det J_{s}^{[N]} = (-1)^{N+1}P_{N+1}(-s)$, but as $-s$ is a lower bound for any possible zero of this monic polynomial, we have that $(-1)^{N+1}P_{N+1}(-s) > 0$. Hence, the leading principal minors of $J_s$ are all positive and the entries on the subdiagonal a superdiagonal are positive. Thus we conclude, attending to Theorem 3, that $J_s$ is an oscillatory matrix.

A very important consequence of this fact, i.e., that there exists a positive $s$ such that $J + sI$ is oscillatory, see Theorem 5, is that all eigenvalues are simple, and that $P_{N+1}$ interlaces $P_{N}$ and $P_{N+1}^{(1)}$. Indeed, the characteristic polynomial $P_{N+1}(x - s)$ of the oscillatory matrix $J_{s}^{[N]}$ interlaces the characteristic polynomials of the submatrices $J_{s}^{[N]}(1) = J_{s}^{[N,1]}$, i.e. $P_{N+1}^{(1)}(x - s)$, and of $J_{s}^{[N]}(N+1) = J_{s}^{[N-1]}$, i.e. $P_{N}(x - s)$. Hence, from (9) may deduce the positivity of the Christoffel coefficients, i.e. $\mu_{k}^{[N]} > 0$ for all $k \in \{1, \ldots, N + 1\}$ from the oscillatory character $J_{s}^{[N]}$, appealing to (9) and the interlacing property of the recursion polynomials and its polynomial of the second type.

Hence, recognizing the fact that for the Jacobi matrix $J$ as in (1) exists a $s$ such that $J_s = sI + J$ is oscillatory gives a very quick access to important properties of the associated recursion polynomials and to Favard’s spectral theorem, see [39, §4.1].

According to Helly’s Selection Principle [6] for any uniformly bounded sequence $\{\psi^{[N]}\}_{N=0}^{\infty}$ of non-decreasing functions defined in $\mathbb{R}$, there exists a convergent subsequence converging to a non-decreasing function $\psi$ defined in $\mathbb{R}$. This leads to Helly’s second theorem. Let us assume a uniformly bounded sequence $\{\psi^{[N]}\}_{N=0}^{\infty}$ of non-decreasing functions on a compact interval $[a, b]$ with limit function $\psi$, then for any continuous function $f$ in $[a, b]$ we have

$$\lim_{N \to \infty} \int_{a}^{b} f(x) \, d\psi^{[N]}(x) = \int_{a}^{b} f \, d\psi(x).$$

Therefore, the positivity of the masses ensure that the functions $\psi^{[N]}$ are non-decreasing and uniformly bounded by unity. Hence, Helly’s results lead to the existence of a nondecreasing functions $\psi$ and corresponding positive Lebesgue–Stieltjes measures $d\psi$ with compact support $\Delta$ such that the orthogonal relations of “type I”

$$\int_{\Delta} \chi^{k} P_{n}(x) \, d\psi(x) = 0, \quad k = 0, \ldots, n - 1,$$
and of “type I"
\[ \int_{\Delta} Q_{k-1}(x) \, d\psi(x)x^n = 0, \quad n \in \{0, 1, \ldots, k-1\}, \quad k \in \{1, \ldots, N\}, \]
hold. These polynomial sequences of types II and I are biorthogonal, i.e.,
\[ \int_{\Delta} Q_k(x) \, d\psi(x)P_l(x) = \delta_{k,l}, \]
for \( k, l \in \mathbb{N}_0 \). Recall that \( H_k Q_k = P_k \), and the biorthogonality reads
\[ \int_{\Delta} P_k(x) \, d\psi(x)P_l(x) = 0. \]

Also the type I and type II are just the same thing in this standard non multiple situation. Helly’s second theorem leads to the spectral representation in terms of the spectral function \( \psi \)
\[ s_k := e_1^T J^k e_1 = \int_{\Delta} t^k \, d\psi(t), \quad \hat{\psi} := e_1^T (zI - J)^{-1} e_1 = \int_{\Delta} \frac{d\psi(t)}{z - t}, \]
of the moments \( s_k \) and of the Stieltjes–Markov function \( \hat{\psi} \). The Markov theorem ensures that the Weyl function in (10) converges uniformly in \( \mathbb{C} \setminus \Delta \) to the Stieltjes–Markov function
\[ S^{[N]} \Rightarrow \hat{\psi}, \quad N \to \infty. \]

This is the main idea we will use for banded Hessenberg matrices \( T \) beyond Jacobi matrices \( J \) in the §3. We will assume that there exists an \( s \) such that \( T_s = T + sI \) is an oscillatory matrix, i.e. all leading principal submatrices are oscillatory.

2. Banded Hessenberg matrices and recursion polynomials

2.1. Sequences of polynomials. Let us consider the following banded monic lower Hessenberg semi-infinite matrix

\[ T = \begin{bmatrix}
  c_0 & 1 & 0 & \ldots \\
  b_1 & c_1 & 1 & \ldots \\
  a_2 & b_2 & c_2 & 1 & \ldots \\
  0 & a_3 & b_3 & c_3 & 1 & \ldots \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
\end{bmatrix}, \tag{11} \]

and assume that \( a_n > 0, b_n, c_n \geq 0 \) for \( n \in \{2, 3, \ldots\} \) and that \( c_0, c_1, b_1 \geq 0 \). The term monic refers to the normalization to unity in the first superdiagonal,

Definition 1 (Recursion polynomials of type II). The type II recursion vector of polynomials

\[ B(x) = \begin{bmatrix}
  B_0(x) \\
  B_1(x) \\
  \vdots \\
\end{bmatrix}, \quad \deg B_n = n, \]
is determined by the following eigenvalue equation

\[ TB(x) = xB(x). \tag{12} \]
Uniqueness is ensured by taking as initial condition $B_0 = 1$. We call the components $B_n$ type II recursion polynomials. One obtains that $B_1 = x - c_0$, $B_2 = (x - c_0)(x - c_1) - b_1$, and higher degree recursion polynomials are constructed by means of the 4-term recurrence relation
\begin{equation}
B_{n+1} = (x - c_n)B_n - b_nB_{n-1} - a_nB_{n-2}, \quad n \in \{2, 3, \ldots \}.
\end{equation}

**Definition 2** (Recursion polynomials of type I). Dual to the polynomial vector $B(x)$ we consider the two following polynomial dual vectors
\[
A^{(1)}(x) = \begin{bmatrix} A_0^{(1)}(x) & A_1^{(1)}(x) & \cdots \end{bmatrix}, \quad A^{(2)}(x) = \begin{bmatrix} A_0^{(2)}(x) & A_1^{(2)}(x) & \cdots \end{bmatrix},
\]
that are left eigenvectors of the semi-infinite matrix $J$, i.e.,
\[
A^{(1)}(x)T = xA^{(1)}(x), \quad A^{(2)}(x)T = xA^{(2)}(x).
\]
The initial conditions, that determine these polynomials uniquely, are taken as
\[
A_0^{(1)} = 1, \quad A_1^{(1)} = \nu, \quad A_0^{(2)} = 0, \quad A_1^{(2)} = 1,
\]
with $\nu \neq 0$ being an arbitrary constant. Then, from the first the relation
\[
c_0A_0^{(a)} + b_1A_1^{(a)} + a_2A_2^{(a)} = xA_0^{(a)}, \quad a \in \{1, 2\},
\]
we get $A_2^{(a)} = \frac{x}{a_2} - \frac{c_0 + b_1 \nu}{a_2}$ and $A_2^{(2)} = -\frac{b_1}{a_2}$. The other polynomials in these sequences are determined by the following four term recursion relation
\begin{equation}
A_n^{(a)} a_n = -A_{n-1}^{(a)} b_{n-1} + A_{n-2}^{(a)} (x - c_{n-2}) - A_{n-3}^{(a)}, \quad n \in \{3, 4, \ldots \}, \quad a \in \{1, 2\}.
\end{equation}

For example, one finds
\[
A_3^{(1)} a_3 = -A_2^{(1)} b_2 + A_1^{(1)} (x - c_1) - A_0^{(1)} = -b_2 \left( \frac{x}{a_2} - \frac{c_0 + b_1 \nu}{a_2} \right) + \nu(x - c_1) - 1,
\]
\[
A_3^{(2)} a_3 = -A_2^{(2)} b_2 + A_1^{(2)} (x - c_1) - A_0^{(2)} = b_2 \frac{b_1}{a_2} + x - c_1.
\]

**Remark 1.**

i) One can check that, $\deg A_2^{(1)} = \deg A_2^{(2)} = n$, and that $\deg A_{2n-1}^{(1)} = \deg A_{2n-1}^{(2)} = n-1$.

ii) Frequently the following notation is used for these recursion polynomials of type I
\[
A_{n,n}^{(1)} := A_{2n-1}^{(1)}, \quad A_{n,n}^{(2)} := A_{2n-1}^{(2)}, \quad A_{n,n+1}^{(1)} := A_{2n}^{(1)}, \quad A_{n,n+1}^{(2)} := A_{2n}^{(2)}.
\]
These dual polynomials are, in a sense, more fundamental to the banded Hessenberg matrix $T$ than the sequence $\{B_n\}_{n \geq 0}$, as we can reconstruct this sequence with determinantal formulas involving the sequences $\{A_n^{(1)}\}_{n \geq 0}, \{A_n^{(2)}\}_{n \geq 0}$.

**Theorem 9.** [7, Lemma 2.4] For $n \in \mathbb{N}_0$, the following formula
\begin{equation}
B_n = H_n \begin{vmatrix} A_{n+1}^{(1)} & A_{n+1}^{(2)} \\ A_{n+1}^{(1)} & A_{n+1}^{(2)} \end{vmatrix}, \quad H_n := (-1)^n \left( \prod_{k=2}^{n+1} a_k \right),
\end{equation}
holds.
Proof. For \( n = 0 \), we have
\[
B_0 = 1 = \begin{vmatrix} A_0^{(1)} & A_0^{(2)} \\ A_1^{(1)} & A_1^{(2)} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ \nu & 1 \end{vmatrix}
\]
while \( n = 1 \) we find
\[
B_1 = x - c_0 = \begin{vmatrix} (x - c_0)A_0^{(1)} & (x - c_0)A_0^{(2)} \\ A_1^{(1)} & A_1^{(2)} \end{vmatrix} = a_2A_2^{(1)} + b_1A_1^{(1)} a_2A_2^{(2)} + b_1A_1^{(2)} = a_2 \begin{vmatrix} A_2^{(1)} & A_2^{(2)} \\ A_1^{(1)} & A_1^{(2)} \end{vmatrix}.
\]
Once we have proven the cases \( n = 0, 1 \), let us apply induction. Hence, we assume that (15) holds for all nonnegative integers up to \( n \) and proceed to show that it also holds for \( n + 1 \). From the direct recursion relation (13) and our induction assumption we find
\[
B_{n+1} = H_n \begin{vmatrix} (x - c_n)A_n^{(1)} & (x - c_n)A_n^{(2)} \\ A_{n+1}^{(1)} & A_{n+1}^{(2)} \end{vmatrix} - b_nH_{n-1} \begin{vmatrix} A_{n-1}^{(1)} & A_{n-1}^{(2)} \\ A_n^{(1)} & A_n^{(2)} \end{vmatrix} - a_nH_{n-2} \begin{vmatrix} A_{n-2}^{(1)} & A_{n-2}^{(2)} \\ A_{n-1}^{(1)} & A_{n-1}^{(2)} \end{vmatrix}.
\]
Now, using the dual recursion (14) for the sequences \( \{A_n^{(1)}\}_{n=0}^{\infty}, \{A_n^{(2)}\}_{n=0}^{\infty} \), we get
\[
B_{n+1} = H_{n+1} \begin{vmatrix} A_{n+1}^{(1)} & A_{n+1}^{(2)} \\ A_{n+2}^{(1)} & A_{n+2}^{(2)} \end{vmatrix} + H_{n-1}\rho_n, \quad \rho_n = \begin{vmatrix} A_{n-1}^{(1)} & A_{n-1}^{(2)} \\ a_{n+1}A_{n+1}^{(1)} + b_nA_{n+1}^{(2)} - A_{n-2}^{(1)} & a_{n+1}A_{n+1}^{(2)} + b_nA_{n+1}^{(2)} - A_{n-2}^{(2)} \end{vmatrix}.
\]
Hence, from the dual recurrence (14) we deduce that \( \rho_n = 0 \) and we get the desired result. \( \square \)

Following [24, 25], we discuss the recursion polynomials of type II of the second kind.

**Definition 3** (Recursion polynomials of type II of the second kind). Let us consider the recursion relation (13) in the form
\[
a_nB_{n-2} + b_nB_{n-1} + c_nB_n + B_{n+1} = xB_n,
\]
set \( b_0 = a_0 = a_1 = -1 \) and \( n \in \mathbb{N}_0 \). The values, initial conditions, for \( B_{-2}, B_{-1}, B_0 \) are required to get the values \( B_n \) for \( n \in \mathbb{N} \). The polynomials of type II correspond to the choice
\[
B_{-2} = 0, \quad B_{-1} = 0, \quad B_0 = 1.
\]
Two sequences of polynomials of type II of the second kind \( \{B_{n}^{(1)}\}_{n=0}^{\infty} \) and \( \{B_{n}^{(2)}\}_{n=0}^{\infty} \) are defined by the following initial conditions
\[
B_{-2}^{(1)} = 1, \quad B_{-1}^{(1)} = 0, \quad B_0^{(1)} = 0,
\]
\[
B_{-2}^{(2)} = -1 - \nu, \quad B_{-1}^{(2)} = 1, \quad B_0^{(2)} = 0.
\]
With these prescriptions we get the first few polynomials of type II of the second kind
\[
\begin{cases}
B_1^{(1)} = 1, & B_1^{(2)} = -\nu, \\
B_2^{(1)} = x - c_1, & B_2^{(2)} = -\nu(x - c_1) + 1, \\
B_3^{(1)} = (x - c_2)(x - c_1) - b_2, & B_3^{(2)} = -\nu((x - c_2)(x - c_1) - b_2) + x - c_2.
\end{cases}
\]
Given the first two couples of polynomials \( B_{1}^{(1)}, B_{1}^{(2)}, B_{2}^{(1)}, B_{2}^{(2)} \) all the others are gotten from the recursion relation (13). For the degrees of these polynomials of type II of the second kind we have \( \deg B_n^{(1)} = \deg B_n^{(2)} = n - 1 \).
2.2. Discrete multiple orthogonality.

Definition 4. Let us denote by $T^{[N]} = T\{(0, 1, \ldots, N)\} \in \mathbb{R}^{(N+1) \times (N+1)}$ the $(N + 1)$-th leading principal submatrix of the banded Hessenberg matrix $T$:

\[
T^{[N]} := \begin{bmatrix}
    c_0 & 1 & 0 & \cdots & 0 \\
    b_1 & c_1 & 1 & \cdots & 0 \\
    a_2 & b_2 & c_2 & 1 & \cdots \\
    & 0 & a_3 & b_3 & c_3 & 1 & \cdots \\
    & & & \ddots & \ddots & \ddots & \ddots & \ddots \\
    & & & & 0 & a_N & b_N & c_N & 1
\end{bmatrix}.
\]

Recall that we are taking all $a_n > 0$, $b_n, c_n > 0$. Let us denote the corresponding leading principal minors as $\delta^{[N]} := \det T^{[N]}$.

We define the vector $B^{(N)} \in \mathbb{R}^{N+1}[x]$ by

\[
B^{(N)} := \begin{bmatrix}
    B_0 \\
    B_1 \\
    \vdots \\
    B_N
\end{bmatrix}.
\]

In what follows $e_1$ denotes the semi-infinite vector with its first entry being 1 and 0 all the others. Similarly, $e_2$ has 0 all its entries but for a 1 in its second entry.

Proposition 1 (Determinantal expressions). i) For the recursion polynomials we have the determinantal expressions

\[
B_{N+1} = \det (xI_{N+1} - T^{[N]}) = \begin{bmatrix}
    x - c_0 & -1 & 0 & \cdots & 0 \\
    -b_1 & x - c_1 & -1 & \cdots & 0 \\
    -a_2 & -b_2 & x - c_2 & -1 & \cdots \\
    & 0 & -a_3 & -b_3 & x - c_3 & -1 & \cdots \\
    & & & \ddots & \ddots & \ddots & \ddots & \ddots \\
    & & & & 0 & -a_{N-1} & -b_{N-1} & x - c_{N-1} & -1 & \cdots \\
    & & & & & 0 & -a_N & -b_N & x - c_N & 1
\end{bmatrix}.
\]

Hence, they are the characteristic polynomials of the leading principal submatrices $T^{[N]}$. 

For the recursion polynomials of type II of the second kind, $B_{N+1}^{(1)}$ and $B_{N+1}^{(2)}$, we have the following adjugate and determinantal expressions

$$B_{N+1}^{(1)} = e_1^T \text{adj} (xI_{N+1} - T^{[N]}) e_1 = \begin{vmatrix} x - c_1 & -1 & 0 \ldots & 0 \\ -b_2 & x - c_2 & -1 \\ -a_3 & -b_3 & x - c_3 & -1 \\ 0 & -a_4 & -b_4 & x - c_4 & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & -a_{N-1} & -b_{N-1} & x - c_{N-1} & -1 \\ 0 & 0 & \ldots & 0 & -a_N & -b_N & x - c_N \end{vmatrix},$$

and

$$B_{N+1}^{(2)} = e_1^T \text{adj} (xI_{N+1} - T^{[N]}) (e_2 - ve_1) = B_{N+1}^{(1)} - vB_{N+1}^{(1)},$$

and

$$b_{N+1}^{(1)} = e_1^T \text{adj} (xI_{N+1} - T^{[N]}) e_2 = \begin{vmatrix} x - c_2 & -1 & 0 \ldots & 0 \\ -b_3 & x - c_3 & -1 \\ -a_4 & -b_4 & x - c_4 & -1 \\ 0 & -a_5 & -b_5 & x - c_5 & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & -a_{N-1} & -b_{N-1} & x - c_{N-1} & -1 \\ 0 & 0 & \ldots & 0 & -a_N & -b_N & x - c_N \end{vmatrix}.$$

**Proof.** Equation (22) follows from the fact that the determinant is a monic polynomial and its zeros are the eigenvalues of matrix $T^{[N]}$. Therefore, it must be $B_{N+1}^{(1)}$. Next we observe that all these determinants satisfy the recurrence (13). For that aim we expand the determinant along the last column, and the the second determinant that appears after the expression is expanded along the last row. Notice also that $\deg B_{N+1}^{(1)} = \deg B_{N+1}^{(2)} = m$. One can check that the initial conditions, that is the first determinants coincide with the expressions found in (20). Consequently, these determinantal expressions coincide with the polynomials of type II of second kind. ❑

Then, one has the following basic result:

**Proposition 2.** Let $\lambda_{[N]} \equiv \{\lambda_n^{[N]}\}_{n=1}^{N+1}$ be the set of zeros of the polynomial $B_{N+1}(x)$. Then, $\lambda_n^{[N]}$ are the eigenvalues of $T^{[N]}$ with associated right eigenvectors

$$u_n^{(N)} := B^{(N)}|_{x=\lambda_n^{[N]}}, \quad n \in \{1, \ldots, N+1\}.$$

**Proof.** According to (22) the eigenvalues of $T^{[N]}$ are the zeros of $B_{N+1}$. From

$$T^{[N]} B^{(N)}(x) + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ B_{N+1}(x) \end{bmatrix} = x B^{(N)}(x).$$

We get that $u_n^{(N)}$ is a right eigenvector. ❑
Definition 5. Let us introduce the polynomials

\[ Q_{n,N} := \begin{bmatrix} A_n^{(1)} & A_n^{(2)} \\ A_{N+1}^{(1)} & A_{N+1}^{(2)} \end{bmatrix}, \]

and the semi-infinite row matrix \( Q_N := \begin{bmatrix} Q_{0,N} & Q_{1,N} & \cdots \end{bmatrix}, \) that satisfies \( QT = xQ. \) Additionally, we consider \( Q^{(N)} := [Q_{0,N} \; Q_{1,N} \; \cdots \; Q_{N,N}] \).

Now we are ready to give a set of left eigenvectors of the matrix \( T^{[N]} \).

Proposition 3. For \( k \in \{1, \ldots, N+1\} \), the vectors

\[ w_n^{(N)} := Q^{(N)}|_{x=\lambda_n^{[N]}}, \quad n \in \{1, \ldots, N+1\}, \]

are left eigenvectors.

Proof. As \( Q_N \) is a left eigenvector of \( T \) we find for its submatrices that

\[ Q^{(N)}T^{[N]} + [0 \; \cdots \; 0 \; a_{N+2}Q_{N+2,N}] = xQ^{(N)}. \]

From (15) we have \( H_{N+1}Q_{N+2,N} = -B_{N+1} \) and evaluating at its zeros \( \lambda_n^{[N]} \) we get the result. \( \square \)

Let us consider the principal submatrices \( T^{[N,k]} = T[\{k, \ldots, N\}] \), obtained from the leading principal submatrix \( T^{[N]} \) by removing the first \( k \) columns and rows

\[ T^{[N,k]} := \begin{bmatrix} c_k & 1 & 0 & \cdots & 0 \\ b_{k+1} & c_{k+1} & 1 & \cdots & \vdots \\ a_{k+2} & b_{k+2} & c_{k+2} & 1 & \vdots \\ 0 & a_{k+3} & b_{k+3} & c_{k+3} & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & \cdots & \cdots & \cdots & \cdots & a_{N-1} & b_{N-1} & c_{N-1} & 1 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & a_N & b_N & c_N \end{bmatrix}, \]

\( \in \mathbb{R}^{(N+1-k)\times(N+1-k)}, \) \( k \in \{0, 1, \ldots, N\}, \)

\[ T^{[N,N+1]} := 1, \]

notice that \( T^{[N]} = T^{[N,0]} \). For \( j \in \{1, \ldots, N+1-k\} \), let us denote by \( \lambda_n^{[N,k]} \) the eigenvalues of \( T^{[N,k]} \). The characteristic polynomials of the matrix \( T^{[N,k]} \) is

\[ B_{N+1}^{[k]} := \begin{bmatrix} x - c_k & -1 & 0 & \cdots & 0 \\ -b_{k+1} & x - c_{k+1} & -1 & \cdots & \vdots \\ -a_{k+2} & -b_{k+2} & x - c_{k+2} & -1 & \vdots \\ 0 & -a_{k+3} & -b_{k+3} & x - c_{k+3} & -1 \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & \cdots & \cdots & \cdots & \cdots & -a_{N-1} & -b_{N-1} & x - c_{N-1} & -1 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & -a_N & -b_N & x - c_N \end{bmatrix}, \]

\( k \in \{0, 1, \ldots, N\}. \)

For convenience, we also introduce \( B_{N+1}^{[N+1]} := 1 \) and \( B_{N+1}^{[N+2]} := 0, \) with \( \deg B_{N+1}^{[k]} = N+1-k. \) Notice that \( B_{N+1} = B_{N+1}^{[0]}, B_{N+1}^{(1)} = B_{N+1}^{[1]} \) and \( b_{N+1}^{[1]} = B_{N+1}^{[2]} \). We refer to polynomials \( B_{N+1}^{[k]} \) as truncated polynomials.
Lemma 1. For the truncated polynomials the following recurrence holds

\( a_{k+2}B_{N+1}^{[k+3]} + b_{k+1}B_{N+1}^{[k+2]} + c_k B_{N+1}^{[k+1]} + B_{N+1}^{[k]} = x B_{N+1}^{[k+1]}, \quad k \in \{0, 1, \ldots, N\}. \)

Proof. An expansion of the determinant along the first row leads to

\[
B_{N+1}^{[k]} = (x - c_k) B_{N+1}^{[k+1]} + \begin{vmatrix}
- b_{k+1} & -1 & 0 & \cdots & 0 \\
- a_{k+2} & x - c_{k+2} & -1 & \cdots & 0 \\
0 & - b_{k+3} & x - c_{k+3} & \cdots & 0 \\
0 & - a_{k+4} & - b_{k+4} & x - c_{k+4} & \cdots & 0 \\
0 & \vdots & \vdots & \ddots & \vdots \\
0 & - a_{N-1} & - b_{N-1} & x - c_{N-1} & -1 \\
0 & - a_N & - b_N & x - c_N & \end{vmatrix}.
\]

Proceeding analogously along the first row we obtain that

\[
B_{N+1}^{[k]} = (x - c_k) B_{N+1}^{[k+1]} - b_{k+1} B_{N+1}^{[k+2]} + \begin{vmatrix}
- a_{k+2} & -1 & 0 & \cdots & 0 \\
0 & x - c_{k+3} & -1 & \cdots & 0 \\
0 & - b_{k+4} & x - c_{k+3} & \cdots & 0 \\
0 & - a_{k+5} & - b_{k+5} & x - c_{k+5} & \cdots & 0 \\
0 & \vdots & \vdots & \ddots & \vdots \\
0 & - a_{N-1} & - b_{N-1} & x - c_{N-1} & -1 \\
0 & - a_N & - b_N & x - c_N & \end{vmatrix},
\]

and, consequently, we deduce that

\[
a_{k+2}B_{N+1}^{[k+3]} + b_{k+1}B_{N+1}^{[k+2]} + c_k B_{N+1}^{[k+1]} + B_{N+1}^{[k]} = x B_{N+1}^{[k+1]},
\]

that holds for \( k \in \{1, \ldots, N - 3\} \). Moreover, a direct computation shows that

\[
B_{N+1}^{[N-2]} = \begin{vmatrix}
x - c_{N-2} & -1 & 0 \\
- b_{N-1} & x - c_{N-1} & -1 \\
- a_N & - b_N & x - c_N \\
\end{vmatrix} = (x - c_{N-2}) B_{N+1}^{[N-1]} - b_{N-1} B_{N+1}^{[N]} - a_N,
\]

\[
B_{N+1}^{[N-1]} = \begin{vmatrix}
x - c_{N-1} & -1 \\
- b_N & x - c_N \\
\end{vmatrix} = (x - c_{N-1}) B_{N+1}^{[N]} - b_N.
\]

\[\square\]

Definition 6. Let us introduce the notation \( \Omega^{(N)} := [B_{N+1}^{[1]} \ldots B_{N+1}^{[N+1]}]. \)

Proposition 4. The vectors

\( \omega_n^{(N)} := \Omega^{(N)}|_{x = \lambda_n^{[N]}}, \quad n \in \{1, \ldots, N + 1\}, \)

are left eigenvectors of \( T^{(N)} \).

Proof. Equation (29) implies

\[
[ B_{N+1}^{[1]} \ldots B_{N+1}^{[N+1]} ] T^{(N)} + [ B_{N+1}^{[0]} 0 \ldots 0 ] = x [ B_{N+1}^{[1]} \ldots B_{N+1}^{[N+1]} ],
\]

and the result follows. \[\square\]
We now give two dual Christoffel–Darboux formulas. Notice that (33) and (34) appeared in [7, Theorem 2.5].

**Theorem 10** (Christoffel–Darboux type formulas). i) For the truncated polynomials the following Christoffel–Darboux type relation holds

\[
\sum_{n=0}^{N} B_{N+1}^{[n+1]}(x)B_n(y) = \frac{B_{N+1}(x) - B_{N+1}(y)}{x - y}.
\]  

ii) The following confluent Christoffel–Darboux type formula is satisfied

\[
\sum_{n=0}^{N} B_{N+1}^{[n+1]}B_n = B_{N+1}.
\]

iii) For the polynomials \(Q_{n,N}\) introduced in (24) we get the following Christoffel–Darboux formula

\[
\sum_{n=0}^{N} Q_{n,N}(x)B_n(y) = \frac{1}{H_N} \frac{B_N(y)B_{N+1}(x) - B_{N+1}(y)B_N(x)}{x - y}.
\]

iv) The following confluent Christoffel–Darboux relation is fulfilled

\[
\sum_{n=0}^{N} Q_{n,N}B_n = \frac{1}{H_N} (B_{N+1}'B_N - B_N'B_{N+1}).
\]

**Proof.** i) It follows from (23) and (30) that

\[
\begin{bmatrix}
B_{N+1}^{[11]}(x) & \cdots & B_{N+1}^{[N+1]}(x)
\end{bmatrix}
T^{[N]} +
\begin{bmatrix}
B_{N+1}^{[0]}(x) & 0 & \cdots & 0
\end{bmatrix}
= x
\begin{bmatrix}
B_{N+1}^{[1]}(x) & \cdots & B_{N+1}^{[N+1]}(x)
\end{bmatrix},
\]

\[
T^{[N]}
\begin{bmatrix}
B_0(y)

0

0

B(y)

B_{N+1}(y)
\end{bmatrix}
+
\begin{bmatrix}
0

\vdots

\vdots

0
\end{bmatrix}
= y
\begin{bmatrix}
B_0(y)

B_N(y)
\end{bmatrix},
\]

so that

\[
\begin{bmatrix}
B_{N+1}^{[0]}(x) & 0 & \cdots & 0
\end{bmatrix}
- \begin{bmatrix}
B_{N+1}^{[1]}(x) & \cdots & B_{N+1}^{[N+1]}(x)
\end{bmatrix}
\begin{bmatrix}
0

\vdots

0
\end{bmatrix}
\begin{bmatrix}
B_0(y)

B_N(y)
\end{bmatrix}
= (x - y)
\begin{bmatrix}
B_{N+1}^{[1]}(x) & \cdots & B_{N+1}^{[N+1]}(x)
\end{bmatrix}
\begin{bmatrix}
B_0(y)

\vdots

B_N(y)
\end{bmatrix},
\]

and (31) follows immediately.

ii) To get (32) take the limit \(x \to y\) in (31).
iii) We use (23) and (25) to get

\[
Q^{(N)}(x) = \begin{bmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0
\end{bmatrix} - \begin{bmatrix}
a_{N+2}Q_{N+2,N}(x) \\
\vdots \\
B_N(x)
\end{bmatrix} = (x - y)Q^{(N)}(x) \begin{bmatrix}
B_0(y) \\
\vdots \\
B_N(y)
\end{bmatrix},
\]

Now, recalling (15) we obtain (33). Finally, (34) appears as a limit in (33).

\[\square\]

**Proposition 5** (Spectral properties). Assume that \(B_{N+1}\) has simple zeros at the set \(\lambda_k^{[N]}\) \(k = 1, \ldots, N + 1\). Then,

i) For \(k = 1, \ldots, N + 1\), the vectors \(u_k^{(N)} := B^{(N)}(\lambda_k^{[N]})\) \((\tilde{w}_k^{(N)} := Q^{(N)}(\lambda_k^{[N]})\) or \(\Omega_k^{(N)}(\lambda_k^{[N]})\)) are right (left) eigenvectors of \(T^{[N]}\). Given any set of left eigenvectors \(\{w_k^{(N)}\}_{k=1}^{N+1}\), a normalized basis \(\{w_k^{(N)}\}_{k=1}^{N+1}\) biorthogonal to the basis of right eigenvectors \(\{u_k^{(N)}\}_{k=1}^{N+1}\) is given by

\[
w_{k,n}^{(N)} = \frac{\tilde{w}_{k,n}^{(N)}}{\sum_{l=1}^{N+1} \tilde{w}_{k,l}^{(N)} B_{l-1}(\lambda_k^{[N]})}.
\]

ii) The following expression holds

\[
w_{k,n}^{(N)} = \frac{Q_{n-1,N}(\lambda_k^{[N]})}{\sum_{l=0}^{N} Q_{l,N}(\lambda_k^{[N]}) B_l(\lambda_k^{[N]})} = H_N \frac{Q_{n-1,N}(\lambda_k^{[N]})}{B_N(\lambda_k^{[N]}) B_{N+1}^{(N)}(\lambda_k^{[N]})}.
\]

Moreover, we can write

\[
w_{k,n}^{(N)} = A_n^{(1)}(\lambda_k^{[N]}) \mu_{k,1}^{[N]} + A_n^{(2)}(\lambda_k^{[N]}) \mu_{k,2}^{[N]}
\]

with masses or Christoffel coefficients defined as

\[
\mu_{k,1}^{[N]} := w_{k,1}^{(N)} = \frac{A_n^{(2)}(\lambda_k^{[N]})}{\sum_{l=0}^{N} Q_{l,N}(\lambda_k^{[N]}) B_l(\lambda_k^{[N]})},
\]

\[
\mu_{k,2}^{[N]} := w_{k,2}^{(N)} - \nu w_{k,1}^{(N)}(N)
\]

iii) The following alternative expression, in terms of truncated polynomials, for the components of the normalized left eigenvectors

\[
w_{k,n}^{(N)} = \frac{B_{n+1}^{[N]}(\lambda_k^{[N]})}{\sum_{l=0}^{N} B_{l+1}^{[N]}(\lambda_k^{[N]}) B_l(\lambda_k^{[N]})} = \frac{B_{n+1}^{[N]}(\lambda_k^{[N]})}{B_{N+1}^{(N)}(\lambda_k^{[N]})},
\]

holds. Moreover, for the Christoffel coefficients we have [32, Chapter 4, Equation 3.11],

\[
\mu_{k,1}^{[N]} = \frac{B_{N+1}^{(1)}(\lambda_k^{[N]})}{B_{N+1}^{(N)}(\lambda_k^{[N]})},
\]

\[
\mu_{k,2}^{[N]} = \frac{B_{N+1}^{(2)}(\lambda_k^{[N]})}{B_{N+1}^{(N)}(\lambda_k^{[N]})},
\]

iv) The corresponding matrices \(U\) (with columns the right eigenvectors \(u_k\) arranged in the standard order) and \(W\) (with rows the left eigenvectors \(w_k\) arranged in the standard order) satisfy

\[
UW = WU = I_{N+1}.
\]
Let us show that the recursion polynomials of type II, (42)
\[ S_1, S_2 \]
\[ (\text{Multiple discrete biorthogonalities}) \]
Theorem 11
\[ \text{finite sequences of multiple discrete orthogonal polynomials with respect to these measures.} \]
\[ \text{Proof. Recalling the initial conditions } A_0^{(1)} = 1, A_0^{(2)} = 0, A_1^{(1)} = \nu \text{ and } A_0^{(2)} = 1 \text{ we find} \]
\[ w_{k,1}^{(N)} = \frac{A_{N+1}^{(2)}(\lambda_k^{[N]})}{\sum_{l=1}^{N+1} \left( A_{N+1}^{(2)}(\lambda_k^{[N]}) A_{l-1}^{(1)}(\lambda_k^{[N]}) - A_{N+1}^{(1)}(\lambda_k^{[N]}) A_{l-1}^{(2)}(\lambda_k^{[N]}) \right) B_{l-1}(\lambda_k^{[N]})}, \]
\[ w_{k,2}^{(N)} = \frac{A_{N+1}^{(2)}(\lambda_k^{[N]}) \nu - A_{N+1}^{(1)}(\lambda_k^{[N]})}{\sum_{l=1}^{N+1} \left( A_{N+1}^{(2)}(\lambda_k^{[N]}) A_{l-1}^{(1)}(\lambda_k^{[N]}) - A_{N+1}^{(1)}(\lambda_k^{[N]}) A_{l-1}^{(2)}(\lambda_k^{[N]}) \right) B_{l-1}(\lambda_k^{[N]})}, \]
and, recalling Theorem 10, the result follows. \(\square\)

Remark 2.
\[ i) \text{ If we compare the expressions for the masses in (36) and (37), in this banded Hessenberg scenario, with the simpler Jacobi case, we immediately see that positivity is not ensured in this case. Hence, to analyze its positivity it seems more promising (39), and as we will show that will be the case for oscillatory matrices.} \]
\[ ii) \text{ Multiple quadrature formulas have been studied in [12, 7]. In particular, in [12] the convergence properties of simultaneous quadrature rules of a given function with respect to different weights is studied.} \]

Let us consider two singular measures, with support on the zeros of \( B_{N+1} \),
\[ \mu_1^{[N]} := \sum_{j=1}^{N+1} \mu_{j,1}^{[N]} \delta(x - \lambda_j^{[N]}), \quad \mu_2^{[N]} := \sum_{j=1}^{N+1} \mu_{j,2}^{[N]} \delta(x - \lambda_j^{[N]}). \]

Let us show that the recursion polynomials of type II, \( \{B_n\}_{n=0}^N \), and of type I, \( \{A_n^{(1)}, A_n^{(2)}\}_{n=0}^N \), are finite sequences of multiple discrete orthogonal polynomials with respect to these measures.

Theorem 11 (Multiple discrete biorthogonalities). Assume that the recursion polynomials \( B_{N+1} \) have simple zeros \( \{\lambda_k\}_{k=1}^{N+1} \). For \( k, l \in \{0, \ldots, N\} \), the following biorthogonal relations hold
\[ \left( A_k^{(1)} \mu_1^{[N]} + A_k^{(2)} \mu_2^{[N]} \right) B_l = \delta_{k,l}. \]

Proof. The matrices
\[ U = \begin{bmatrix} B_0(\lambda_1^{[N]}) & \cdots & B_0(\lambda_{N+1}^{[N]}) \\ \vdots & \ddots & \vdots \\ B_N(\lambda_1^{[N]}) & \cdots & B_N(\lambda_{N+1}^{[N]}) \end{bmatrix}, \quad W = \begin{bmatrix} w_1^{[N]} & \cdots & w_{N+1,1}^{[N]} \\ \vdots & \ddots & \vdots \\ w_{N+1,1}^{[N]} & \cdots & w_{N+1,N+1}^{[N]} \end{bmatrix}, \]
satisfy \( UW = I \), and using (35) biorthogonality follows immediately. \(\square\)

The previous discrete biorthogonality implies a set of multiple orthogonal relations of types I and II.
Corollary 1 (Multiple discrete orthogonalities). Assume that the recursion polynomials \( B_{N+1} \) have simple zeros \( \{\lambda_k\}_{k=1}^{N+1} \). For \( n \) such that \( 2n + 1 \leq N \), the following type II multiple orthogonal conditions are satisfied

\[
\begin{align*}
\langle \mu_1^{[N]}, x^k B_{2n} \rangle &= 0, \quad k = 0, \ldots, n - 1, \\
\langle \mu_2^{[N]}, x^k B_{2n} \rangle &= 0, \quad k = 0, \ldots, n - 1,
\end{align*}
\]

For the recursion polynomials of type I we have the following discrete type I multiple orthogonality

\[
\begin{align*}
\langle A_k^{(1)} \mu_1^{[N]} + A_k^{(2)} \mu_2^{[N]}, x^n \rangle &= 0, \quad n \in \{0, 1, \ldots, k - 1\}, \quad k \in \{1, \ldots, N\}.
\end{align*}
\]

Proposition 6 (Lebesgue–Stieltjes representation of the measures). In terms of piecewise continuous functions

\[
\psi_1^{[N]} := \begin{cases} 0, & x < \lambda_1^{[N]}, \\
\mu_{1,1}^{[N]} + \cdots + \mu_{k,1}^{[N]}, & \lambda_{k+1}^{N+1} \leq x \leq \lambda_k^{[N]}, \quad k \in \{1, \ldots, N\}, \\
\mu_{1,1}^{[N]} + \cdots + \mu_{N+1,1}^{[N]}, & x > \lambda_1^{[N]},
\end{cases}
\]

\[
\psi_2^{[N]} := \begin{cases} 0, & x < \lambda_1^{[N]}, \\
\mu_{1,2}^{[N]} + \cdots + \mu_{k,2}^{[N]}, & \lambda_{k+1}^{N+1} \leq x \leq \lambda_k^{[N]}, \quad k \in \{1, \ldots, N\}, \\
\mu_{1,2}^{[N]} + \cdots + \mu_{N+1,2}^{[N]}, & x > \lambda_1^{[N]},
\end{cases}
\]

we can write \( \mu_1^{[N]} = d\psi_1^{[N]} \) and \( \mu_2^{[N]} = d\psi_2^{[N]} \).

We now discuss on the resolvent matrix \( R_z^{[N]} \) of the leading principal submatrix \( T^{[N]} \); i.e.,

\[
R_z^{[N]} := (zI_{N+1} - T^{[N]})^{-1} = \frac{\text{adj} (zI_{N+1} - T^{[N]})}{\det (zI_{N+1} - T^{[N]})}.
\]

Notice that, from the spectral decomposition of the matrix \( T^{[N]} \), we obtain

\[
R_z^{[N]} = U(zI_{N+1} - D)^{-1} W.
\]

We consider two Weyl functions \( S_1^{[N]} \) and \( S_2^{[N]} \) defined by

\[
S_1^{[N]} := e_1^T (zI_{N+1} - T^{[N]})^{-1} e_1, \quad S_2^{[N]} := e_1^T (zI_{N+1} - T^{[N]})^{-1} (e_2 - ve_1).
\]

Proposition 7. The Weyl functions can be expressed as follows

\[
S_1^{[N]}(z) = \frac{B_{N+1}^{(1)}(z)}{B_{N+1}(z)} = \sum_{n=1}^{N+1} \frac{\mu_{n,1}^{[N]}}{z - \lambda_n^{[N]}}, \quad S_2^{[N]}(z) = \frac{B_{N+1}^{(2)}(z)}{B_{N+1}(z)} = \sum_{n=1}^{N+1} \frac{\mu_{n,2}^{[N]}}{z - \lambda_n^{[N]}},
\]
Proof. The first equalities follow from adjugate expressions in Proposition 1. The second expressions can be deduced from (43). Indeed, the Weyl functions are

\[
S_1^N(z) = [1 \ldots 1](zI_{N+1} - D)^{-1}
\]

\[
S_2^N(z) = [1 \ldots 1](zI_{N+1} - D)^{-1}
\]

Remark 3. The expressions (39) for the masses agree with the previous Proposition 7: These Christoffel coefficients are residues at the simple poles of the resolvents.

3. FACTORIZATION PROPERTIES

We now discuss some aspects of the Gauss-Borel factorization and positive bidiagonal factorization of oscillatory banded Hessenberg matrices. For more on Gauss–Borel factorization and orthogonal polynomials see for example [31] and references therein.

3.1. Gauss–Borel factorization. The Gauss–Borel factorization of the matrix \(T^N\) in Definition 4 is the following factorization

\[
T^N = L^N U^N
\]

with banded triangular matrices given by

\[
L^N = \begin{bmatrix}
1 & 0 & \ldots & 0 \\
m_1 & 1 & \ldots & 0 \\
\ell_2 & m_2 & \ldots & 0 \\
0 & \ell_3 & m_3 & \ldots \\
& \ddots & \ddots & \ddots \\
0 & \ldots & 0 & \ell_N & m_N & 1
\end{bmatrix},
\]

\[
U^N = \begin{bmatrix}
\alpha_1 & 1 & 0 & \ldots & 0 \\
0 & \alpha_4 & \ldots & 0 \\
& \ddots & \ddots & \ddots \\
0 & \ldots & 0 & \alpha_7 & 0 \\
& & & \alpha_{3N+1} & 1
\end{bmatrix}.
\]

Proposition 8. The Gauss–Borel factorization exists if and only if all leading principal minors \(\delta^N\) of \(T^N\) are not zero. For \(n \in \mathbb{N}\), the following expressions for the coefficients hold

\[
\ell_{n+1} = \frac{a_{n+1}\delta[n-2]}{\delta[n-1]}, \quad m_n = c_n = \frac{\delta[n]}{\delta[n-1]}, \quad \alpha_{3n-2} = \frac{\delta[n-1]}{\delta[n-2]},
\]

where \(\delta[1] = 1\) and \(a_1 = 0\), and we have the following recurrence relation for the determinants

\[
\delta[n] = a_n\delta[n-1] - b_n\delta[n-2] + c_n\delta[n-1],
\]

is satisfied.

Proof. Notice that \(\delta[N] = \det T^N = \det U^N = \alpha_1\alpha_4 \cdots \alpha_{3N+1}\). Hence, we get \(\alpha_{3N+1} = \frac{\delta[N]}{\delta[N-1]}\). From the last row of the \(LU\) factorization we get

\[
a_N = \ell_N\alpha_{3N-5}, \quad b_N = \ell_N + m_N\alpha_{3N-2}, \quad c_N = m_N + \alpha_{3N+1},
\]
so that
\[
\ell_N = \frac{\delta^{[N-3]}}{\delta^{[N-2]}} a_N, \quad \quad m_N = c_N = \frac{\delta^{[N]}}{\delta^{[N-1]}},
\]
and
\[
b_N - a_N \frac{\delta^{[N-3]}}{\delta^{[N-2]}} = \frac{\delta^{[N-1]}}{\delta^{[N-2]}} \left( c_N - \frac{\delta^{[N]}}{\delta^{[N-1]}} \right) = 0,
\]
that is (46), which is also obtained by expanding the determinant \(\delta^{[N]}\) along the last row. \(\square\)

**Proposition 9.** Let us assume that \(T^{[N]}\) given in Definition 4 is an oscillatory matrix. Then,
\[
a_n, b_n, c_n > 0.
\]

**Proof.** According to the Gantmacher–Krein Criterion, Theorem 1, [18, II.7 Theorem 10], \(b_n > 0\) for \(n \in \{1, \ldots, N\}\). Moreover, as is an invertible totally nonnegative matrix (InTN) according to [11, page 50, Chapter 2] we also need \(c_n\) to be positive. \(\square\)

We introduce some auxiliary submatrices that will be instrumental in the following developments.

**Definition 7 (Auxiliary submatrices).** Given the lower triangular factor \(L^{[N]}\), determined by the Gauss–Borel factorization (44), we consider its complementary submatrix, by deleting first row and last column, that we call the auxiliary Jacobi matrix, \(J^{[N,k]} = L^{[N]} \setminus \{1\}, \{N+1\} \in \mathbb{R}^{N \times N}\) as in (6) with \(k = 1\). For \(k \in \{0, \ldots, N-1\}\), associated with the auxiliary Jacobi matrix \(J^{[N,1]}\) we introduce, as we did with the banded Hessenberg matrix \(T^{[N]}\), the principal submatrices \(J^{[N,k+1]}\) defined in (6). Additionally, we introduce \(T_1^{[N]} = T(\{1\}, \{N+1\}) \in \mathbb{R}^{N \times N}\) as the complementary submatrix obtained by removing the first row and last column of \(T^{[N]}\), that is
\[
(47) \quad T_1^{[N]} := \begin{bmatrix}
    b_1 & c_1 & 1 & 0 & \cdots & \cdots & 0 \\
    a_2 & b_2 & c_2 & 1 & \cdots & \cdots & \cdots \\
    0 & a_3 & b_3 & c_3 & 1 & \cdots & \cdots \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
    a_{N-1} & b_{N-1} & c_{N-1} & 1 & \cdots & \cdots & \cdots \\
    0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
    a_N & b_N & c_N & 1 & \cdots & \cdots & \cdots \\
    0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
    \end{bmatrix}, \quad \quad \delta_1^{[N]} := \det T_1^{[N]}.
\]

Further auxiliary complementary submatrices that we will consider are
\[
(48) \quad T_1^{[N,k]} := \begin{bmatrix}
    b_{k+1} & c_{k+1} & 1 & 0 & \cdots & \cdots & 0 \\
    a_{k+2} & b_{k+2} & c_{k+2} & 1 & \cdots & \cdots & \cdots \\
    0 & a_{k+3} & b_{k+3} & c_{k+3} & 1 & \cdots & \cdots \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
    a_{N-1} & b_{N-1} & c_{N-1} & 1 & \cdots & \cdots & \cdots \\
    0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
    a_N & b_N & c_N & 1 & \cdots & \cdots & \cdots \\
    0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
    \end{bmatrix}, \quad \quad \delta_1^{[N,k]} := \det T_1^{[N,k]},
\]
so that $T_1^{[N,0]} = T_1^{[N]}$ and $\delta_1^{[N,0]} = \delta_1^{[N]}$, and the upper bidiagonal matrix

$$U^{[N-1,k]} = \begin{bmatrix}
\alpha_{3k+1} & 1 & 0 & \cdots & 0 \\
0 & \alpha_{3k+4} & \alpha_{3k+7} & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \alpha_{3N-2}
\end{bmatrix}.$$ 

A set of finite continued fractions will be needed in the subsequent analysis.\(^6\)

**Definition 8** (Finite continued fractions). We introduce the finite continued fractions

$$\mathcal{K}[n,k] := m_k - \frac{\ell_{k+1}}{m_{k+1}}, \quad n \in \{k+1, k+2, \ldots\}, \quad \mathcal{K}[k+1,k] := m_k,$$

We now collect together a number of results regarding factorizations, determinants and relations of the introduced matrices. The next results are relevant in subsequent developments.

**Theorem 12** (Determinants and continued fractions). Let us assume that $T^{[N]}$ as in Definition 4 is an oscillatory matrix. Then, the following holds:

i) For the triangular factors in (44) we have $L^{[N]}, U^{[N]} \in \text{In} T^N$.

ii) The matrix entries of the triangular factors of Gauss–Borel factorization (44) of the Hessenberg matrix $T^{[N]}$ are positive:

$$\ell_2, \ldots, \ell_N, m_1, \ldots, m_N, \alpha_1, \alpha_4, \ldots, \alpha_{3N-2} > 0.$$

iii) For $k \in \mathbb{N}$, the recurrence relation

$$D(n+1) = m_{k+n}D(n) - \ell_{k+n}D(n-1), \quad n \in \mathbb{N},$$

for the initial conditions $D(0) = 1, D(1) = m_k$, has as solution $D(n) = \Delta_{k+n-1,k}$, while for the initial conditions $D(0) = 0$ and $D(1) = 1$ has as solution $\Delta_{k+n-1,k+1}$. The determinants $\Delta_{N,k}$ were defined in (6).

iv) The ratio of consecutive determinants are bounded as follows:

$$\frac{\alpha_{n+2}}{\delta_{n+2}} < \frac{\delta[n]}{\delta[n-1]} < c_n, \quad \frac{\ell_{n+1}}{m_{n+1}} < \frac{\Delta_{n,1}}{\Delta_{n-1,1}} < m_n.$$

v) For $k \in \mathbb{N}$, the continued fraction given in Definition 8 is the ratio of the consecutive determinants defined in Equation (6)

$$\mathcal{K}[n,k] = \frac{\Delta_{n,k}}{\Delta_{n,k+1}}.$$

vi) For $k = 1$ the determinants in (6) are positive, i.e. $\Delta_{n,1} > 0$.

\(^6\)For continued fractions we refer the interested reader to [47, 23, 30].
Proof. i) As $T^{[N]}$ is TN, we know it has a Gauss–Borel factorization with totally nonnegative factors, see [11, Theorem 2.4.1].

ii) For $n \in \{2, 3, \ldots, N\}$ we have $a_n = \ell_n a_{3n-5}$ and, consequently, as $a_n \neq 0$ we deduce that $\ell_2, \ldots, \ell_N, a_1, a_4, \ldots, a_{3N-2} > 0$. Moreover,

$$\begin{vmatrix} m_n & 1 \\ \ell_{n+1} & m_{n+1} \end{vmatrix} \geq 0,$$

and as $\ell_{n+1} > 0$ we deduce that $m_nm_{n+1} \neq 0$, and all $m_1, \ldots, m_N > 0$.

iii) Expand the determinant $\Delta_{n,k}$ along the last row. One can check that the initial conditions lead to the sequence of determinants.

iv) From (45) and $m_n > 0$ we get $b_n\delta^{[n-2]} > a_n\delta^{[n-3]}$, $c_n\delta^{[n-1]} > \delta^{[n]}$ and the first inequality follows. For the second one we consider the Gauss–Borel factorization of $J^{[N,1]}$, i.e.

$$\begin{bmatrix} m_1 & 1 & 0 & \cdots & \cdots & 0 \\ \ell_2 & m_2 & 1 & \cdots & \cdots & 0 \\ 0 & \ell_3 & m_3 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \ell_N & m_N \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ \beta_2 & \cdots & \cdots & \cdots & 0 \\ 0 & \beta_4 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \beta_{2N-2} & 1 \end{bmatrix},$$

so that $m_1 = \beta_1$ as well as $m_n = \beta_{2n-2} + \beta_{2n-1}$ and $\ell_n = \beta_{2n-2}\beta_{2n-3}$. From the factorization we get $\Delta_{n,1} = \beta_1 \cdots \beta_{2n-1}$ and, consequently, $\beta_{2n-1} = \frac{\Delta_{n,1}}{\Delta_{n-1,1}}$ and $\beta_{2n-2} = m_n - \frac{\Delta_{n,1}}{\Delta_{n-1,1}}$. Being $J^{[N,1]}$ oscillatory, we must have $\beta_n > 0$ so that $m_n > \frac{\Delta_{n,1}}{\Delta_{n-1,1}}$. As for the oscillatory case we require $\Delta_n > 0$, the recursion relation (49), i.e., $\Delta_{n,1} = m_n\Delta_{n-1,1} - \ell_n\Delta_{n-2,1}$, implies that $m_n\Delta_{n-1,1} > \ell_n\Delta_{n-2,1}$ and the lower bound follows immediately.

v) Use the Euler–Wallis theorem for continued fractions, see for example [10, Theorem 9.2].

vi) The first two determinants are positive, then we apply induction. Let us assume that $\Delta_{n-1,1} > 0$, and that $\Delta_{n,1} = 0$. Then, for $k = 0$ Equation (49) implies that $\Delta_{n+1,1} = -\ell_{n+1}\Delta_{n-1,1} < 0$ in contradiction with the fact $\Delta_{n+1,1} \geq 0$.

□

**Theorem 13** (Factorizations and oscillatory matrices). For the submatrices and determinants introduced in Definition 7 we find that:

i) The auxiliary Jacobi matrix $J^{[N,1]}$ is oscillatory.

ii) The following factorizations are fulfilled

\[ T^{[N]}_1 = J^{[N,1]}U^{[N-1]}, \]

\[ T^{[N,k]}_1 = m_{k+1}E_{1,1} = J^{[N,k+1]}U^{[N-1,k]}. \]
Moreover, $\hat{\delta}_1^{[N]} > 0$ and the relation between determinants

\begin{equation}
\Delta_{N,1} = \frac{\delta_1^{[N]}}{\delta_{[N-1]}}.
\end{equation}

(52)

\begin{equation}
\Delta_{N,k+1} = \alpha_1 \cdots \alpha_{3k-2} \frac{\delta_1^{[N,k]} - m_{k+1} \delta_1^{[N,k+1]}}{\delta_{[N-1]}}.
\end{equation}

(53)

hold. Recall that $\Delta_{[N,k]} := \text{det} J_{[N,k]}$, $\delta_1^{[N]} := \text{det} T_1^{[N]}$ and $\hat{\delta}_1^{[N,k]} := \text{det} T_1^{[N,k]}$.

iii) The submatrix $T_1^{[N]}$ is oscillatory.

iv) The submatrices $J_{[N,k+1]}$ and $T_1^{[N,k]}$ are oscillatory. In particular, $\Delta_{N,k+1}, \delta_1^{[N,k]} > 0$.

v) The following relations are satisfied

\begin{equation}
\Delta_{N,2} \delta_{[N-1]} = c_0 \delta_1^{[N,1]} - a_2 \delta_1^{[N,2]},
\end{equation}

(54)

\begin{equation}
\frac{\Delta_{N,1}}{\Delta_{N,2}} = \frac{\delta_1^{[N]}}{c_0 \delta_1^{[N,1]} - a_2 \delta_1^{[N,2]}}.
\end{equation}

(55)

vi) The recursion relation in $k$

is satisfied.

Proof. i) According to Theorem 3, see [17, Chapter XIII, §9] and [18, Chapter 2, Theorem 11], the Jacobi matrix $J_{[N,1]}$ is oscillatory if and only if,

(a) The matrix entries $\ell_2, \ldots, \ell_N$ are positive.

(b) All leading principal minors $\Delta_{n,1}$ are positive.

As we have seen in previous points, both requirements are satisfied.

ii) Equations (50) and (51) follows directly from the Gauss–Borel factorization of $T^{[N]}$. Taking determinants and expanding the determinant along the first row we get (53). From Equation (50) we conclude that $\alpha_1 \alpha_4 \cdots \alpha_{3n-2} \Delta_n = \text{det} T_1^{[N]}$. As previously said all $\alpha_1, \alpha_4, \ldots, \alpha_{3n-2} > 0$ and $\Delta_{n,1} > 0$. Therefore, $\delta_1^{[N]} \neq 0$.

iii) Given that the matrix $T_1^{[N]}$ belongs to InTN with $\alpha_n, b_n, c_n > 0$, see Proposition 9, the Gantmacher–Krein Criterion, Theorem 1, leads to the oscillatory character of this submatrix.

iv) If a matrix $A$ is oscillatory then so is any submatrix $A[\alpha]$ for any contiguous subset of indexes $\alpha$, see [18, Chapter 2, §7] and [11, Corollary 2.6.7]. Then, $J_{[N,k+1]} = J_{[N]}(\{k + 1, \ldots, N\})$ and $T_1^{[N,k]} = T_1^{[N]}(\{k + 1, \ldots, N\})$ are oscillatory and, consequently, $\Delta_{N,k+1} = \text{det} J_{[N]}(\{k + 1, \ldots, N\}) > 0$ and $\delta_1^{[N,k]} = \text{det} T_1^{[N]}(\{k + 1, \ldots, N\}) > 0$.

v) Put $k = 1$ in (53) and recall that $\alpha_1 = c_0$ and $\alpha_1 \ell_2 = a_2$. For Equation (54) use (52).

vi) Expand the determinants along the first row.

A set of convergent infinite continued fractions are important in what follows.
**Definition 9** (Infinite continued fraction and tails). *We introduce the following infinite continued fraction*

\[ K[1] := m_1 - \frac{\ell_2}{m_2 - \frac{\ell_3}{m_3 - \cdots}} \]

*and its tails*

\[ K[k + 1] := m_{k+1} - \frac{\ell_{k+2}}{m_{k+2} - \frac{\ell_{k+3}}{m_{k+3} - \cdots}}, \quad k \in \mathbb{N}. \]

**Corollary 2.** *The infinite continued fraction in (56) can be computed as the following large N limit ratio*

\[ K[1] = \lim_{N \to \infty} \frac{\delta_1^{[N]}}{c_0 \delta_1^{[N,1]} - a_2 \delta_1^{[N,2]}} \]

*of determinants given in (47) and (48).*

**Proof.** Direct consequence of (54). \( \square \)

Now, an important result follows regarding the behavior of these infinite continued fractions.

**Theorem 14** (Infinite continued fractions). i) *For \( k \in \mathbb{N}_0 \), the sequences \( \{K[n, k]\}_{n=k+1}^{\infty} \) of the finite continued fractions given in Definition 8 are positive and strictly decreasing.

ii) *The infinite continued fraction \( K[1] \) converges and is nonnegative.

iii) *The tails converge and are positive, i.e. \( K[k + 1] > 0 \) for \( k \in \mathbb{N} \).*

**Proof.** i) *The positivity follows at once from the positivity of \( \Delta_{N,k} \). From (55) we have*

\[ \frac{\Delta_{N+1,k+1}}{\Delta_{N+1,k+2}} = m_{k+1} - \frac{\ell_{k+2}}{\Delta_{N+1,k+2}}, \quad \frac{\Delta_{N,k+1}}{\Delta_{N,k+2}} = m_{k+1} - \frac{\ell_{k+2}}{\Delta_{N,k+2}}. \]

*As \( m_k, \Delta_{N,k+1} > 0 \) the inequality*

\[ \frac{\Delta_{N+1,k+1}}{\Delta_{N+1,k+2}} < \frac{\Delta_{N,k+1}}{\Delta_{N,k+2}}, \]

*can be written*

\[ m_{k+1} - \frac{\ell_{k+2}}{\Delta_{N+1,k+2}} < m_{k+1} - \frac{\ell_{k+2}}{\Delta_{N+1,k+3}} \]

*where we have used (59). Therefore, (60) is equivalent to the inequality*

\[ \frac{\Delta_{N+1,k+2}}{\Delta_{N+1,k+3}} < \frac{\Delta_{N,k+1}}{\Delta_{N,k+2}}. \]

*Hence, if for \( k = N - 2 \) the inequality*

\[ \frac{\Delta_{N+1,N-1}}{\Delta_{N+1,N}} < \frac{\Delta_{N,N-1}}{\Delta_{N,N}} \]
is fulfilled, the inequality (60) will hold. But,

\[
\frac{\Delta_{N+1,N-1}}{\Delta_{N+1,N}} = m_N - \frac{\ell_{N+1}}{m_{N+1}}, \quad \frac{\Delta_{N,N-1}}{\Delta_{N,N}} = m_N,
\]

and (61) satisfied.

ii) Obvious from the previous result, any positive decreasing sequence is convergent to a non-negative number.

iii) For \( k \in \mathbb{N} \) we have \( K \to k \) requires \( K \to k \), \( K \to k \) \( K \to k \), so that the convergence of \( K [k + 1] \) requires \( K [k + 1] > 0 \).

\[ \square \]

3.2. **Bidiagonal factorization.** Now we discuss how the Gauss–Borel factorization can be used to find a bidiagonal factorization of the banded Hessenberg matrix. This will lead to the appearance of continued fractions in our theory.

**Lemma 2.** The factorization of any lower triangular matrix of the form

\[
L^\text{[N]} = \begin{bmatrix}
1 & 0 & \cdots & \cdots & 0 \\
\alpha_2 & 0 & \cdots & \cdots & 0 \\
\alpha_5 & \alpha_3 \nu_1 & \cdots & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & \alpha_{3N-1} & 1
\end{bmatrix}
\]

into bidiagonal factors, i.e.,

\[
(62) \quad L^\text{[N]} = L_1^\text{[N]} L_2^\text{[N]}, \quad L_1^\text{[N]} = \begin{bmatrix}
1 & 0 & \cdots & \cdots & 0 \\
\alpha_2 & 0 & \cdots & \cdots & 0 \\
\alpha_5 & \alpha_3 \nu_1 & \cdots & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & \alpha_{3N-1} & 1
\end{bmatrix}, \quad L_2^\text{[N]} = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
\alpha_3 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \alpha_{3N} & 1
\end{bmatrix},
\]

is uniquely determined in terms of \( \alpha_2 \), with

\[
(63) \quad \alpha_{3n} = m_n - \frac{\ell_n}{m_{n-1}} - \frac{\ell_{n-1}}{m_{n-1}} - \cdots - \frac{\ell_2}{m_2}, \quad \alpha_{3n-1} = m_n - \frac{\ell_n}{m_{n-1}} - \frac{\ell_{n-1}}{m_{n-1}} - \cdots - \frac{\ell_2}{m_2},
\]

The factorization exists if and only if \( \alpha_{3n} \neq 0 \) for \( n \in \{1, \ldots, N - 1\} \).

**Proof.** The factorization (62) implies that

\[
m_n = \alpha_{3n-1} + \alpha_{3n}, \quad n \in \{1, \ldots, N\}, \quad \ell_n = \alpha_{3n-1} \alpha_{3n-3}, \quad n \in \{2, \ldots, N\}.
\]
These can be solved recursively to get

\[
\begin{align*}
\alpha_3 &= m_1 - \alpha_2, \\
\alpha_6 &= m_2 - \frac{\ell_2}{m_1 - \alpha_2}, \\
\alpha_9 &= m_3 - \frac{\ell_3}{m_2 - \frac{\ell_2}{m_1 - \alpha_2}}, \\
\alpha_5 &= \frac{\ell_2}{m_1 - \alpha_2}, \\
\alpha_8 &= \frac{\ell_3}{m_2 - \frac{\ell_2}{m_1 - \alpha_2}}, \\
\alpha_{10} &= \frac{\ell_4}{m_3 - \frac{\ell_3}{m_2 - \frac{\ell_2}{m_1 - \alpha_2}}}
\end{align*}
\]

and the result follows by induction. Hence, for a given \(\alpha_2\) the factorization exists if and only if \(\alpha_{3n} \neq 0\), for \(n \in \{1, \ldots, N - 1\}\). \(\square\)

**Proposition 10.** For each \(\alpha_2 < K[N, 1]\), with \(K[N, 1]\) the finite continued fraction in Definition 8, the factorization of \(L[N]\) into bidiagonal factors

\[
L[N] = L_1[N] L_2[N], \quad L_1[N] = \begin{bmatrix} 1 & 0 & \cdots & 0 \\
\alpha_2 & 0 & \alpha_5 & \cdots \\
& \ddots & \ddots & \ddots \\
& & \ddots & \ddots \end{bmatrix}, \quad L_2[N] = \begin{bmatrix} 1 & 0 & \cdots & 0 \\
\alpha_3 & 0 & \alpha_6 & \cdots \\
& \ddots & \ddots & \ddots \\
& & \ddots & \ddots \end{bmatrix}
\]

with \(\alpha_3, \alpha_5, \alpha_6, \alpha_8, \ldots, \alpha_{3n-1}, \alpha_{3n} > 0\), exists and is unique. If \(\alpha_2 \in [0, K[N, 1])\) then \(L_1[N], L_2[N] \in \text{In}_TN\).

**Proof.** In the solution provided by Equation (63) we require that \(\alpha_3, \alpha_5, \alpha_6, \alpha_8, \ldots, \alpha_{3N-1}, \alpha_{3N} > 0\). Let us proceed step by step, firstly if \(\alpha_2 < m_1\) we see that \(\alpha_3, \alpha_5 > 0\). In the next step, we get that if \(\alpha_2 < m_1\) and \(\alpha_2 < m_1 - \frac{\ell_2}{m_2}\) we have \(\alpha_3, \alpha_5, \alpha_6, \alpha_8 > 0\). Notice that as the sequence \(K[N, 1] > 0\) is decreasing \(m_1 - \frac{\ell_2}{m_2} < m_1\) and only one condition is needed. Then, in the next step we conclude that what is needed for \(\alpha_3, \alpha_5, \alpha_6, \alpha_8, \alpha_9, \alpha_{10} > 0\) is \(\alpha_2 < m_1 - \frac{\ell_2}{m_2 - \frac{\ell_3}{m_3}}\). Finally, induction implies the result. \(\square\)

**Theorem 15** (Positive bidiagonal factorization. Finite case). Let us assume that the matrix \(T[N]\) given in Definition 4 is oscillatory. Then, each \(\alpha_2 < K[N, 1]\) —with \(K[N, 1]\) the finite continued fraction the finite continued fraction in Definition 8— determines a positive sequence \(\{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \ldots, \alpha_{3N+1}\}\) such that the factorization

\[
T[N] = L_1[N] L_2[N] U[N], \tag{64}
\]
with bidiagonal matrices given by

\[
L_1^{[N]} = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
\alpha_2 & 1 & \cdots & 0 \\
0 & \alpha_5 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \alpha_{3N-1}
\end{bmatrix}, \quad L_2^{[N]} = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
\alpha_3 & 1 & \cdots & 0 \\
0 & \alpha_6 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \alpha_{3N}
\end{bmatrix}, \quad U^{[N]} = \begin{bmatrix}
\alpha_1 & 1 & 0 & \cdots & 0 \\
0 & \alpha_4 & 1 & \cdots & 0 \\
0 & \cdots & 0 & \alpha_7 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\end{bmatrix}
\]

is satisfied. When \( \alpha_2 \in [0, \mathcal{K}[N, 1]) \) each bidiagonal factor is InTN.

Proof. Consequence of ii) in Theorem 12 and Proposition 10. \( \square \)

Associated with the banded Hessenberg semi-infinite matrix \( T \) given in (11), let us define the following auxiliary semi-infinite matrices

\[
T_1 := \begin{bmatrix}
\cdots & b_1 & c_1 & 1 & 0 & \cdots \\
\alpha_2 & a_2 & b_2 & c_2 & 1 & \cdots \\
0 & \alpha_3 & b_3 & c_3 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\end{bmatrix}, \quad J_1 := \begin{bmatrix}
\cdots & m_1 & 1 & 0 & \cdots \\
\ell_2 & m_2 & 1 & \cdots & \cdots \\
0 & \ell_3 & m_3 & 1 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots \\
\end{bmatrix}
\]

**Theorem 16** (Positive bidiagonal factorization. Semi-infinite case). Let us assume that the banded Hessenberg matrix \( T \) in (11) is oscillatory (all leading principal submatrices are oscillatory). Then, for each \( \alpha_2 < \mathcal{K}[1], \) with \( \mathcal{K}[1] \) the infinite continued fraction in (56), there exist a unique positive sequence \( \{\alpha_1, \alpha_3, \alpha_4, \ldots\} \) such that the factorization

\[
T = L_1 L_2 U,
\]

with bidiagonal matrices given by

\[
L_1 = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
\alpha_2 & 1 & \cdots & 0 \\
0 & \alpha_5 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\end{bmatrix}, \quad L_2 = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
\alpha_3 & 1 & \cdots & 0 \\
0 & \alpha_6 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\end{bmatrix}, \quad U = \begin{bmatrix}
\alpha_1 & 1 & 0 & \cdots & 0 \\
0 & \alpha_4 & 1 & \cdots & 0 \\
0 & \cdots & 0 & \alpha_7 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\end{bmatrix}
\]

holds. If \( \alpha_2 \in [0, \mathcal{K}[1]) \) then \( L_1, L_2, U \in \text{InTN} \). The following relations for the matrix entries hold

\[
\begin{aligned}
c_n &= \alpha_{3n+1} + \alpha_{3n} + \alpha_{3n-1}, \\
b_n &= \alpha_{3n}\alpha_{3n-2} + \alpha_{3n-1}\alpha_{3n-2} + \alpha_{3n-1}\alpha_{3n-3}, \\
a_n &= \alpha_{3n-1}\alpha_{3n-3}\alpha_{3n-5}.
\end{aligned}
\]

4. **Regular oscillatory banded Hessenberg matrices**

The fact that the infinite continued fraction \( \mathcal{K}[1] \) in (56) could be zero, and that \( \alpha_2 \) can not be taken as a positive number, is an important issue in what follows. The spectral representation of the banded Hessenberg matrix \( T \), the existence of a system of two measures \( \{d\psi_1, d\psi_2\} \) and corresponding sequences of multiple orthogonal polynomials given in Theorem 20 relies on \( \alpha_2 > 0 \).
Notice that in [2] this was taken as granted, and that in the hypergeometric case [29, 3] and the Jacobi–Piñeiro in the semi-band [2, 5] is also true that \( \alpha_2 > 0 \). This motivates the following notion:

**Definition 10** (Regular oscillatory Hessenberg matrices). An oscillatory banded Hessenberg matrix \( T \) is said to be regular if \( \mathcal{K}[1] > 0 \). Otherwise, i.e., \( \mathcal{K}[1] = 0 \), is said to be nonregular.

**Proposition 11.**

i) If the sequence \( \sum_{n=2}^{N+1} \log \frac{\ell_n}{m_n} + \log m_N \) converges then \( \mathcal{K}[1] > 0 \).

ii) We have \( \mathcal{K}[1] = 0 \) if and only if

\[
\lim_{N \to \infty} \left( c_0 \frac{\delta_1^{[N,1]}}{\delta_1^{[N]}} - a_2 \frac{\delta_1^{[N,2]}}{\delta_1^{[N]}} \right) = +\infty.
\]

iii) As \( c_0 \frac{\delta_1^{[N,1]}}{\delta_1^{[N]}} > a_2 \frac{\delta_1^{[N,2]}}{\delta_1^{[N]}} \) we deduce that \( \mathcal{K}[1] > 0 \) if \( \delta_1^{[N,1]} \) converges.

iv) If we are in the Śleszyński–Pringsheim class, i.e., \( \ell_n + 1 \leq m_n \), the convergence is absolute and \( 0 < \mathcal{K}[1] < 1 \). See [30, Theorems 3.25 & 3.26].

v) If \( p_n := \frac{\ell_n}{m_n} - 1 \geq 0 \), then the infinite product \( \Pi = \lim_{N \to \infty} \Pi_N = \prod_{n=2}^{N} \frac{\ell_n}{m_n} \), converges whenever does \( \sum_{n=1}^{\infty} p_n \) and \( \Pi \geq 1 \). When \( q_n := 1 - \frac{\ell_{n+1}}{m_{n+1}} \geq q_n > 0 \), then \( 0 < \Pi < 1 \). This is the Śleszyński–Pringsheim case.

4.1. **Toeplitz oscillatory matrices.** We discuss now the uniform case

\[
a_n = a > 0, \quad b_n = b \geq 0, \quad c_n = c \geq 0.
\]

That is, the Hessenberg matrix \( T \) is a banded Toeplitz matrix

\[
T = \begin{bmatrix}
c & 1 & 0 & \cdots \\
b & c & 1 \\
a & b & c & 1 \\
0 & a & b & c & 1 \\
\vdots & & & & \\
\end{bmatrix},
\]

which is clearly a bounded semi-infinite matrix.

**Proposition 12** (Edrei–Schoenberg). The Toeplitz matrix given in Equation (69) is oscillatory if and only if there exists \( \beta_1 > \beta_2 \geq \beta_3 > 0 \) such that

\[
a = \beta_1 \beta_2 \beta_3,
\]

\[
b = \beta_1 \beta_2 + \beta_1 \beta_3 + \beta_2 \beta_3,
\]

\[
c = \beta_1 + \beta_2 + \beta_3.
\]

**Proof.** According to the Edrei–Schoenberg Theorem, see [9, 36], this Toeplitz Hessenberg matrix \( T \) is TN if and only if the generating function

\[
f(t) = 1 + ct + bt^2 + at^3
\]

can be written as

\[
f(t) = (1 + \beta_1 t)(1 + \beta_2 t)(1 + \beta_3 t), \quad \beta_1 \geq \beta_2 \geq \beta_3 \geq 0.
\]

In terms of these \( \beta \)'s we find (70). As \( a > 0 \) we must have \( \beta_1 \geq \beta_2 \geq \beta_3 > 0 \). Hence \( b > 0 \) and the Gantmacher–Krein criterion, see Theorem 1, leads to the oscillatory character of the Toeplitz matrix. \( \square \)
Now we show that all oscillatory Toeplitz matrices (69) are regular. The determinants $\delta^{[n]} = \det T^{[n]}$ are subject to the recursion relation

$$
\delta^{[n]} - c\delta^{[n-1]} + b\delta^{[n-2]} - a\delta^{[n-3]} = 0,
$$

being the initial conditions: $\delta^{[-2]} = \delta^{[-1]} = 0$ and $\delta^{[0]} = 1$.

**Proposition 13.** If $T$ is a oscillatory banded Toeplitz matrix as in (69) with $\beta_1 > \beta_2 > \beta_3 > 0$, then the determinants $\delta^{[N]} = \det T^{[N]}$ are explicitly given in terms of $\beta_1, \beta_2, \beta_3$ as follows:

$$
\delta^{[n]} = \frac{\beta_1^{n+2}}{(\beta_1 - \beta_2)(\beta_1 - \beta_3)} + \frac{\beta_2^{n+2}}{(\beta_2 - \beta_1)(\beta_2 - \beta_3)} + \frac{\beta_3^{n+2}}{(\beta_3 - \beta_1)(\beta_3 - \beta_2)}.
$$

**Proof.** Following the theory of recursion relations, see for example [10], we consider the so called characteristic polynomial

$$
p(\lambda) := \lambda^3 - c\lambda^2 + b\lambda - a,
$$

and notice that $p(\lambda) = \lambda^3 f(-\frac{1}{\lambda})$; hence, the characteristic roots are $\beta_1, \beta_2, \beta_3 > 0$. If the roots are distinct, i.e., simple, then the general solution to the recursion (71) will be

$$
C_1\beta_1^n + C_2\beta_2^n + C_3\beta_3^n
$$

for some constants $C_1, C_2, C_3$ determined by the initial conditions:

$$
\begin{bmatrix}
1 & 1 & 1 \\
\beta_1 & \beta_2 & \beta_3 \\
\frac{1}{\beta_1} & \frac{1}{\beta_2} & \frac{1}{\beta_3}
\end{bmatrix}
\begin{bmatrix}
C_1 \\
C_2 \\
C_3
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
$$

so that (72) holds.

**Corollary 3.** For $T$ as in Proposition 12, for the large $N$ ratio asymptotics of the determinants we find

$$
\lim_{n \to \infty} \frac{\delta^{[n]}}{\delta^{[n-1]}} = \beta_1.
$$

**Proof.** In the case of distinct characteristic roots $\beta_1 > \beta_2 > \beta_3 > 0$ is a direct consequence of Proposition 13.

When the two smaller characteristic roots coincide $\beta_2 = \beta_3$ the general solution will be

$$
C_1\beta_1^n + (C_2 + C_3n)\beta_2^n
$$

and the large $N$ ratio asymptotics of the determinant do not change. When the largest characteristic root is degenerate, with multiplicity two or three, then asymptotically the determinant will have a dominant term $q(n)\beta_1^n$ with a polynomial $q$ such that $\deg q = 1, 2$ in the determinants and (73) is recovered.

**Theorem 17** (Infinite continued fractions and harmonic mean). The continued fraction considered in (56) for an oscillatory banded Toeplitz $T$ as (69) is a half of the harmonic mean of the two largest characteristic roots, i.e

$$
K[1] = \frac{\beta_1\beta_2}{\beta_1 + \beta_2}.
$$
Proof. In order to compute the continued fraction $\mathcal{K}[1]$ according to (58) we only require the use of the determinants $\delta_1^{[N]}$ of

$$
\mathcal{T}_1^{[N]} = \begin{bmatrix}
 b & c & 1 & 0 & \ldots & 0 \\
 a & b & c & 1 & \ddots & 0 \\
 0 & a & b & c & \ddots & 0 \\
 \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
 0 & \ldots & a & b & \ldots & c \\
 \end{bmatrix} \in \mathbb{R}^{N \times N},
$$

as in this Toeplitz case $\delta_1^{[N,k]} = \delta_1^{[N-k]}$. These determinants are subject to the following uniform recursion relation

$$
\delta_1^{[n]} - b\delta_1^{[n-1]} + ac\delta_1^{[n-2]} - a^2\delta_1^{[n-3]} = 0,
$$

as follows from an expansion along the last column. The initial conditions are $\delta_1^{[-2]} = \delta_1^{[-1]} = 0$ and $\delta_1^{[0]} = 1$. The characteristic roots $\gamma_1, \gamma_2, \gamma_3$ are the zeros of

$$
q(t) = t^3 - bt^2 + act - a^2,
$$

and we find that $p(t) = -\frac{t^2}{a^2}q(a)$ or $q(t) = -\frac{t^3}{\beta}p(\frac{t}{\beta})$. Hence, the characteristic roots are $\gamma = \frac{a}{b}$, that arranged in decreasing order can be written as follows

$$
\gamma_1 = \beta_1\beta_2, \quad \gamma_2 = \beta_1\beta_3, \quad \gamma_3 = \beta_3\beta_2.
$$

Let us assume that the roots are distinct, i.e., simple, the other degenerate cases can be treated similarly. Then, the general solution to the recursion (74) will be

$$
C_1\gamma_1^n + C_2\gamma_2^n + C_3\gamma_3^n,
$$

for some constants $C_1, C_2, C_3$ determined by the initial conditions:

$$
\begin{bmatrix}
 1 \\
 \frac{1}{\gamma_1} \\
 \frac{1}{\gamma_2} \\
 \frac{1}{\gamma_3}
\end{bmatrix}
\begin{bmatrix}
 C_1 \\
 C_2 \\
 C_3
\end{bmatrix}
= 
\begin{bmatrix}
 1 \\
 0 \\
 0
\end{bmatrix},
$$

and, proceeding as above, we find

$$
\delta_1^{[n]} = \frac{\beta_1^n\beta_2^n}{(\beta_2 - \beta_3)(\beta_1 - \beta_3)} + \frac{\beta_1^n\beta_3^n}{(\beta_3 - \beta_2)(\beta_1 - \beta_2)} + \frac{\beta_2^n\beta_3^n}{(\beta_3 - \beta_1)(\beta_2 - \beta_1)}.
$$

Hence, according to (58)

$$
\mathcal{K}[1] = \lim_{N \to \infty} \frac{\delta_1^{[N]}}{c\delta_1^{[N-1]} - a\delta_1^{[N-2]}} = \lim_{n \to \infty} \frac{\beta_1^n\beta_2^n}{(\beta_2 - \beta_3)(\beta_1 - \beta_3)} + \frac{\beta_1^n\beta_3^n}{(\beta_3 - \beta_2)(\beta_1 - \beta_2)} - \beta_1\beta_2\beta_3 - \frac{\beta_1^n\beta_2^n}{(\beta_2 - \beta_3)(\beta_1 - \beta_3)}.
$$

$$
= \frac{\beta_1\beta_2}{(\beta_1 + \beta_2 + \beta_3)\beta_1\beta_2 - \beta_1\beta_2\beta_3} = \frac{\beta_1\beta_2}{\beta_1 + \beta_2}.
$$
Corollary 4 (Oscillatory Toeplitz matrices are regular). All oscillatory banded Toeplitz matrix as in Proposition (68) have

\[ \mathcal{K}[1] > 0, \]

i.e., they are regular oscillatory.

Remark 4. Therefore, by shifting the Toeplitz matrix \( T \to T + sI \), i.e. \( c \to c + s \) we get another Toeplitz matrices for which our method ensures that two spectral positive measures exist.

4.2. Retractions, tails and regularity. We will show that from any given oscillatory banded Hessenberg matrix we can construct regular oscillatory Hessenberg matrices in several ways. Also, we will find that oscillatory matrices are organized in rays, the origin of the ray is a nonregular oscillatory matrix and all the interior points of the ray are regular oscillatory matrices.

Given a TN matrix \( A \), then \( A + sE_{1,1} \) is also TN for \( s > -\frac{\det A}{\det A(1)} \), see [11, Section 9.5] on retractions of TN matrices and in particular the proof of [11, Lemma 9.5.2]. When \( s \) is a negative number this is known as a retraction. As we will show, for any regular oscillatory banded Hessenberg matrix there is a retraction that is oscillatory but nonregular and vice versa. We use the notation \( E_{1,2}(s) := I + sE_{1,2} \) for the bidiagonal matrix with a nonzero contribution only possibly at the entry in the second row and first column.

Theorem 18 (Retractions and oscillation regularity). i) If the banded Hessenberg matrix \( T \) as in (11) is oscillatory then

\[ T_s = E_{1,2}(s)T = \begin{pmatrix} c_0 & 1 & 0 & \cdots \\ b_1 + sc_0 & c_1 + s & 1 & \cdots \\ a_2 & b_2 & c_2 & 1 \\ 0 & a_3 & b_3 & c_3 & 1 \\ \vdots & \vdots & \vdots & \ddots & \cdots \\ \end{pmatrix}, \]

is regular oscillatory for \( s > -\mathcal{K}[1] \).

ii) If the banded Hessenberg matrix \( T \) as in (11) is regular oscillatory with \( \mathcal{K}[1] > 0 \), then

\[ \tilde{T} = E_{1,2}(-\mathcal{K}[1])T = \begin{pmatrix} c_0 & 1 & 0 & \cdots \\ b_1 - \mathcal{K}[1]c_0 & c_1 - \mathcal{K}[1] & 1 & \cdots \\ a_2 & b_2 & c_2 & 1 \\ 0 & a_3 & b_3 & c_3 & 1 \\ \vdots & \vdots & \vdots & \ddots & \cdots \\ \end{pmatrix}, \]

is a nonregular oscillatory matrix, i.e. \( \mathcal{K}[1] = 0 \).

Proof. i) The Jacobi matrix \( J_{s}^{[N,1]} = J^{[N,1]} + sE_{1,1} \) is TN for \( s \geq -\frac{\Delta_{N-1}}{\Delta_{N,2}} \) and InTN for \( s > \frac{\Delta_{N-1}}{\Delta_{N,2}} = -\mathcal{K}[N, 1] \). Thus, attending to Theorem 1, is an oscillatory matrix for \( s > -\mathcal{K}[N, 1] \). Then,
the corresponding lower unitriangular matrix $L_s^{[N]}$ that has $J_s^{[N,1]}$ as complementary submatrix, $L_s^{[N]}(\{1\}, \{N + 1\}) = J_s^{[N,1]}$ (obtained by deleting first row and last column), is InTN for $s > -\mathcal{K}[N, 1]$. This is a consequence of [11, Lemma 3.3.4]. The continued fraction $\mathcal{K}_s[N, 1]$, corresponding to the oscillatory Jacobi matrix $J_s^{[N,1]}$, is $\mathcal{K}[N, 1] + s$.

Now, let us consider the banded Hessenberg matrix by defining $T_s^{[N]} = L_s^{[N]}U^{[N]}$, which is clearly InTN for $s > -\mathcal{K}[N, 1]$ as its factors are. A direct computation shows that

\[
T_s^{[N]} = \begin{bmatrix}
c_0 & 1 & 0 & \cdots & \cdots & \cdots \\
b_1 + sc_0 & c_1 + s & 1 & \cdots & \cdots & \cdots \\
a_2 & b_2 & c_2 & 1 & \cdots & \cdots \\
a_3 & b_3 & c_3 & 1 & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
0 & a_{N-1} & b_{N-1} & c_{N-1} & 1 & \cdots \\
0 & 0 & a_N & b_N & c_N & \cdots 
\end{bmatrix}.
\]

Observe that $m_1 = \frac{b_1}{c_0} = \mathcal{K}[2, 1]$, and recall that $\{\mathcal{K}[n, 1]\} \sim_{n=2}^{\infty}$ is positive and decreasing sequence and, consequently, $b_1 + sc_0 > 0$ for $s > -\mathcal{K}[N, 1]$ and $N \in \{2, 3, \ldots\}$. Therefore, using the Gantmacher–Krein Criterion, we conclude that $T_s^{[N]}$ is oscillatory. Finally, for the large $N$ limit with $s > -\mathcal{K}[1]$, the matrix $T_s$ is oscillatory and has $\mathcal{K}_s[1] > 0$.

i) The retraction $J_s^{[N,1]} = J_s^{[N,1]} - \mathcal{K}_s[1]E_{1,1}$ of the Jacobi matrix of $T^{[N]}$ is oscillatory. This is a direct consequence of the fact that $0 < \mathcal{K}_s[1] < \mathcal{K}_s[n, 1]$, for all $n \in \mathbb{N}$. The associated finite continued fraction is $\mathcal{K}_s[N, 1] = \mathcal{K}_s[N, 1] - \mathcal{K}_s[1]$. To continue, let us consider as in the previous discussion the unitriangular matrix $L^{[N]}$ such that $J^{[N,1]} = L^{[N]}(\{1\}, \{N + 1\})$ is a complementary submatrix, deleting first row and last column. This triangular matrix is InTN, and the matrix $\tilde{T}^{[N]} = L^{[N]}U^{[N]}$ is oscillatory, as well, with associated continued fraction $\tilde{\mathcal{K}}^{[N,1]} = \mathcal{K}[N, 1] - \mathcal{K}[1]$. Thus, in the large $N$ limit the semi-infinite banded Hessenberg matrix $\tilde{T}$ is oscillatory with $\tilde{\mathcal{K}}[1] = 0$, i.e. nonregular.

\[\square\]

**Corollary 5.** If the banded Hessenberg matrix $T$ given in (11) is a non regular oscillatory matrix, i.e. $\mathcal{K}[1] = 0$, then $T_s$, as in (75) is regular oscillatory for $s > 0$.

The next result is based on the fact that the tails $\mathcal{K}[2], \mathcal{K}[3], \ldots$ of the continued fraction $\mathcal{K}[1]$ are positive.

**Theorem 19** (Tails and regular oscillatory). If $T$ is an oscillatory banded Hessenberg matrix as in (11) then the matrices

\[
T^{(2)} := \begin{bmatrix}
c_1 - \frac{b_1}{c_0} & 1 & 0 & \cdots & \cdots & \cdots \\
b_2 - \frac{a_2}{c_0} & c_2 & 1 & \cdots & \cdots & \cdots \\
a_3 & b_3 & c_3 & 1 & \cdots & \cdots \\
a_4 & b_4 & c_4 & 1 & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
0 & a_{N-1} & b_{N-1} & c_{N-1} & 1 & \cdots \\
0 & 0 & a_N & b_N & c_N & \cdots 
\end{bmatrix},
\]

...
are regular oscillatory matrices with associated continued fractions \( K[2] \) and \( K[k + 1], k \in \{2, 3, \ldots \} \), respectively.

**Proof.** For \( k \in \mathbb{N} \), the tail \( K[k + 1] \) is the continued fraction of the Jacobi matrix

\[
J^{(k+1)} := \begin{bmatrix}
\alpha_{3k+1} & 1 & 0 & \cdots \\
(c_{k+1} + \alpha_{3k+4}) \alpha_{3k+1} & c_{k+1} & 1 & \cdots \\
ak_{k-2} & b_{k+2} & c_{k+1} & 1 & \cdots \\
0 & a_{k+3} & b_{k+3} & c_{k+3} & 1 & \cdots \\
\end{bmatrix}, \quad k \in \{2, 3, \ldots \}
\]

(77)

that is oscillatory and a submatrix of

\[
L^{(k+1)} = \begin{bmatrix}
1 & 0 & \cdots \\
0 & 1 & \cdots \\
\ell_{k+2} & m_{k+2} & 1 & \cdots \\
0 & \ell_{k+3} & m_{k+3} & 1 & \cdots \\
\end{bmatrix}
\]

with all its leading principal submatrices InTN. We introduce the upper triangular matrix

\[
U^{(k+1)} = \begin{bmatrix}
\alpha_{3k+1} & 1 & 0 & \cdots \\
0 & \alpha_{3k+4} & 1 & \cdots \\
0 & 0 & \alpha_{3k+7} & \cdots \\
\end{bmatrix},
\]

with all its leading principal submatrices InTN, and the corresponding banded Hessenberg matrix

\[
T^{(k+1)} = L^{(k+1)} U^{(k+1)}
\]

is

\[
T^{(k+1)} = \begin{bmatrix}
c_k - m_k & 1 & 0 & \cdots \\
b_{k+1} - \ell_{k+1} & c_{k+1} & 1 & \cdots \\
a_{k+2} & b_{k+2} & c_{k+2} & 1 & \cdots \\
0 & a_{k+3} & b_{k+3} & c_{k+3} & 1 & \cdots \\
\end{bmatrix},
\]

with \( c_k - m_k = \alpha_{3k+1} > 0 \) and \( b_{k+1} - \ell_{k+1} = m_{k+1} \alpha_{3k+1} = (c_{k+1} + \alpha_{3k+4}) \alpha_{3k+1} > 0 \) is, according to the Gantmacher–Krein Criterion, an oscillatory matrix, having its continued fraction \( K^{(k+1)}[1] \) equal to the tail \( K[k + 1] \) that is positive. To get \( T^{(2)} \) recall that \( \alpha_1 = c_0, m_1 \alpha_1 = b_1 \) and \( \ell_2 \alpha_1 = a_2 \). □

4.3. **Hypergeometric and Jacobi–Piñeiro cases.**
4.3.1. Hypergeometric multiple orthogonal polynomials. In [29] a new set of multiple hypergeometric polynomials were introduced by Lima and Loureiro. The corresponding recursion matrix \( T_{LL} \) was used in [5] to construct stochastic matrices and associated Markov chains. This construction extends beyond birth and death. For the hypergeometric case the \( T_{LL} = L_1L_2U \) bidiagonal factorization is provided in [29, Equations 107-110], that ensures the regular oscillatory character of the matrix \( T_{LL} \) for this hypergeometric case. Notice the correspondence between Lima–Loureiro’s \( \lambda_n \) and our \( \alpha_n \) is \( \lambda_{3n+2} \to \alpha_{3n+1}, \lambda_{3n+1} \to \alpha_{3n} \) and \( \lambda_{3n} \to \alpha_{3n-1} \). These coefficients were gotten in [29] from [35, Theorem 14.5] as the coefficients of a branched-continued-fraction representation for \( 3F_2 \).

4.3.2. Jacobi–Piñeiro multiple orthogonal polynomials. Jacobi–Piñeiro multiple orthogonal polynomials, associated with weights \( w_1 = x^n(1-x)^\gamma, w_2 = x^\beta(1-x)^\gamma \) with support on \([0,1]\) \( \alpha, \beta, \gamma > -1, \alpha - \beta \not\in \mathbb{Z} \), is a well study case. This system is an AT system and the corresponding orthogonal polynomials and linear forms interlace its zeros, see [22], even though, as we will discuss now, the recursion matrix \( T_{JP} \) is not oscillatory. The corresponding monic recursion matrix \( T_{JP} \) was considered in [5, Section 4.3] and we show that this recursion matrix was a positive matrix whenever the parameters \( \alpha, \beta \) lay in the strip given by \(|\alpha - \beta| < 1\).

**Lemma 3** (Jacobi–Piñeiro’s recursion matrix bidiagonal factorization). The Jacobi–Piñeiro’s recursion matrix has bidiagonal factorizations as in Equation (65) with at least the following two set of parameters:

\[
\begin{align*}
\alpha_{6n+1} &= \frac{(n + 1 + \alpha)(2n + 1 + \alpha + \gamma)(2n + 1 + \beta + \gamma)}{(3n + 1 + \alpha + \gamma)(3n + 2 + \alpha + \gamma)(3n + 1 + \beta + \gamma)}, \\
\alpha_{6n+2} &= \frac{(3n + 2 + \alpha + \gamma)(3n + 1 + \beta + \gamma)(3n + 2 + \beta + \gamma)}{(n + 1)(2n + 1 + \gamma)(2n + 2 + \beta + \gamma)}, \\
\alpha_{6n+3} &= \frac{(3n + 2 + 2 + \alpha + \gamma)(3n + 3 + \alpha + \gamma)(3n + 2 + \beta + \gamma)}{(n + 1 + \beta)(2n + 2 + \alpha + \gamma)(2n + 2 + \beta + \gamma)}, \\
\alpha_{6n+4} &= \frac{(3n + 3 + 2 + \alpha + \gamma)(3n + 3 + \alpha + \gamma)(3n + 2 + \beta + \gamma)}{(n + 1 + \beta)(2n + 2 + \alpha + \gamma)(2n + 2 + \beta + \gamma)}, \\
\alpha_{6n+5} &= \frac{(3n + 3 + \alpha + \gamma)(3n + 4 + \alpha + \gamma)(3n + 3 + \beta + \gamma)}{(n + 1 + \alpha - \beta)(2n + 2 + \gamma)(2n + 3 + \beta + \gamma)}, \\
\alpha_{6n+6} &= \frac{(3n + 4 + \alpha + \gamma)(3n + 3 + \beta + \gamma)(3n + 4 + \beta + \gamma)}{(3n + 4 + \alpha + \gamma)(3n + 3 + \beta + \gamma)(3n + 4 + \beta + \gamma)}
\end{align*}
\]

Here \( n \in \mathbb{N}_0 \).

**Proof.** We have \( \alpha_1 = c_0 \) and \( m_1 \) and \( \alpha_2 \) are gotten from \( m_1 \alpha_1 = b_1 \) and \( m_1 + \alpha_4 = c_1 \). Then, \( \ell_n, m_n, \alpha_{3(n-1)+1}, n = 2, 3, \ldots \), are determined recursively according to \( \ell_n \alpha_{3(n-2)+1} = a_n, \ell_n + m_n \alpha_{3(n-1)+1} = b_n \) and \( m_n + \alpha_{3(n-1)+1} = c_n \) (from the first relation we get \( \ell_n \), from the second \( m_n \) and from the third \( \alpha_{3(n-1)+1} \)). Now, these expressions for \( \ell^n \)’s and \( m^n \)’s lead to the remaining \( \alpha^n \)’s. Indeed, we have \( \alpha_2 + \alpha_3 = m_1 \) and \( \alpha_5 \alpha_3 = \ell_2 \) (we get \( \alpha_3 \) and \( \alpha_5 \), respectively) and then we apply the recursion, \( n \in \mathbb{N}, \alpha_{3(n-1)+2} + \alpha_{3(n-1)+1} = m_{n+1}, \alpha_{3(n)+2} \alpha_{3(n)+1} = \ell_{n+2} \) (in each iteration we obtain \( \alpha_{3(n+1)} \) and \( \alpha_{3(n)+2} \), respectively).

**Remark 5.** The bidiagonal factorization \( \{\alpha_n\}_{n=1}^\infty \) was found in [2, Section 8.1], that is why we refer to it as the Aptekarev-Kalyagin-Van Iseghem (AKV) bidiagonal factorization.

\[ \text{For more on this see the recent paper [48].} \]
For \( n \in \mathbb{N}_0 \), given these two bidiagonal factorizations, the entries of the corresponding lower unitriangular factor \( L = L_1 L_2 = \bar{L}_1 \bar{L}_2 \) of the lower factor \( L \) in the Gauss–Borel factorization of the Jacobi–Piñeiro’s Hessenberg transition matrix, can be expressed in the following two manners
\[
\begin{align*}
\ell_{2n+2} &= \alpha_{6n+5} \alpha_{6n+3} = \tilde{a}_{6n+5} \alpha_{6n+3}, \\
\ell_{2n+3} &= \alpha_{6n+8} \alpha_{6n+6} = \alpha_{6n+8} \tilde{a}_{6n+6}.
\end{align*}
\]

To better understand the dependence on the set of Jacobi–Piñeiro’s parameters \((\alpha, \beta)\) we define some regions in the plane. Let us denote by \( \mathcal{R} := \{(\alpha, \beta) \in \mathbb{R}^2, \alpha, \beta > -1, \alpha - \beta \notin \mathbb{Z}\} \), that we call the natural region –where the orthogonality is well defined, and divide it in the following four regions:
\[
\begin{align*}
\mathcal{R}_1 &:= \{(\alpha, \beta) \in \mathcal{R} : \alpha - \beta > 1\}, & \mathcal{R}_2 &:= \{(\alpha, \beta) \in \mathcal{R} : 0 < \alpha - \beta < 1\}, \\
\mathcal{R}_3 &:= \{(\alpha, \beta) \in \mathcal{R} : -1 < \alpha - \beta < 0\}, & \mathcal{R}_4 &:= \{(\alpha, \beta) \in \mathcal{R} : \alpha - \beta < -1\}.
\end{align*}
\]

We show these regions in the following figure.

**Lemma 4.**

i) For the sequence \( \{\alpha_n\}_{n \in \mathbb{N}} \) of the first bidiagonal factorization, we have

(a) In the region \( \mathcal{R}_1 \) the sequence in TN but for \( \alpha_5 \) that is negative in \( \mathcal{R}_4 \) and \( \alpha_6 \) that is negative in \( \mathcal{R}_1 \).

(b) Is a TN sequence in the strip \( \mathcal{R}_2 \cup \mathcal{R}_3 \). Excluding \( \alpha_2 = 0 \), the sequence is TP.

ii) For the AKV sequence \( \{\tilde{a}_n\}_{n \in \mathbb{N}} \), we have

(a) In the region \( \mathcal{R} \) the sequence in TP but for \( \tilde{a}_2 \) that is negative in \( \mathcal{R}_1 \cup \mathcal{R}_2 \), \( \tilde{a}_8 \) that is negative in \( \mathcal{R}_1 \) and \( \tilde{a}_3 \) that is negative in \( \mathcal{R}_4 \).

(b) Is a TP sequence in the half strip \( \mathcal{R}_3 \).

**Proof.** For the first set of bidiagonal parameters \( \{\alpha_n\}_{n \in \mathbb{N}} \), we check that all are positive in \( \mathcal{R} \), but for \( \alpha_2 = 0 \) and \( \alpha_5, \alpha_6 \). From direct inspection we get that \( \alpha_5 < 0 \) when \( 1 + \alpha - \beta < 0 \), i.e. in region \( \mathcal{R}_4 \) and \( \alpha_6 < 0 \) when \( 1 - \alpha + \beta < 0 \), i.e. in region \( \mathcal{R}_1 \). Hence, the sequence \( \{\alpha_n\}_{n \in \mathbb{N}} \) is a TN sequence, TP but for \( \alpha_2 = 0 \), in region \( \mathcal{R}_2 \cup \mathcal{R}_3 \), is TN in \( \mathcal{R} \) but for \( \alpha_5 \) in \( \mathcal{R}_4 \) and TN in \( \mathcal{R} \) but for \( \alpha_6 \) in \( \mathcal{R}_1 \). For the AKV parameters \( \{\tilde{a}_n\}_{n \in \mathbb{N}} \), all are positive in \( \mathcal{R} \), but for \( \tilde{a}_2, \tilde{a}_8 \) and \( \tilde{a}_3 \). The entry \( \tilde{a}_2 < 0 \) when \( \alpha > \beta \), that is in \( \mathcal{R}_1 \cup \mathcal{R}_2 \), \( \tilde{a}_8 < 0 \) when \( 1 - \alpha + \beta < 0 \) i.e. in \( \mathcal{R}_1 \) and \( \tilde{a}_3 \) when \( 1 + \alpha - \beta < 0 \), i.e. in \( \mathcal{R}_4 \).

**Lemma 5.** For the two first subdiagonals of the lower triangular matrix \( L \) in the Gauss–Borel factorization of the Jacobi–Piñeiro’s Hessenberg recursion matrix \( T_{JP} \) we have
i) The sequence \( \{m_n\}_{n=1}^{\infty} \) is TP in the definition region \( \mathcal{R} \).

ii) The sequence \( \{\ell_n\}_{n=1}^{\infty} \) is TP but for \( \ell_2 (\ell_3) \), that is negative in \( \mathcal{R}_4 (\mathcal{R}_1) \).

**Proof.** From (78) and the first bidiagonal factorization we get that the lower triangular entries in the two first subdiagonals \( TP \) but for \( \ell_2 (\ell_3) \), that is negative in \( \mathcal{R}_4 (\mathcal{R}_1) \), and maybe \( m_5 = \alpha_5 + \alpha_6 \). Looking now at the AKV factorization we see that \( m_5 = \tilde{\alpha}_5 + \tilde{\alpha}_6 > 0 \). □

**Proposition 14.** The Jacobi–Piñeiro’s Hessenberg recursion matrix satisfies:

i) It follows from Lemma 4.

ii) The AKV bidiagonal factorization sequence is TP in \( \mathcal{R}_3 \).

iii) We use the Gauss–Borel factorization of these retractions described in Theorem 19, that we know have a bidiagonal factorization with TP sequences.

**Proof.**

i) From the previous discussion of the Jacobi–Piñeiro’s recursion matrix it becomes clear that demanding the matrix to be regular oscillatory is sufficient but not necessary to have spectral measures. In the natural region \( \mathcal{R} \) the Jacobi–Piñeiro’s weights exist, are positive with support on \([0, 1]\). However, we know that it is an oscillatory matrix only in the strip \( \mathcal{R}_2 \cup \mathcal{R}_3 \). The associated matrix that is regular oscillatory in the natural region \( \mathcal{R} \) is the retracted complementary matrix \( T_{JP}^{(5)} \). This observation leads to the question: Is it enough to have a spectral Favard theorem and positive measures that a retracted complementary matrix of the banded Hessenberg matrix is oscillatory?

5. Spectral Theory

5.1. Interlacing properties. As \( T^{[N]} \) is assumed to be oscillatory, we conclude that all the matrices \( T^{[N,k]} \), \( k \in \{0, 1, \ldots, N + 1\} \) are oscillatory. See Theorem 2. Then, all the eigenvalues \( \lambda_1^{[N,k]} > \lambda_2^{[N,k]} > \cdots > \lambda_{N+1-k}^{[N,k]} > 0 \) of \( T^{[N,k]} \) are simple and positive. In fact, as we have seen, these eigenvalues are the zeros of the polynomial \( B_{N+1}^{[k]} \). We also have the strict interlacing properties

\[
\lambda_1^{[N,k]} > \lambda_2^{[N,k]} > \cdots > \lambda_{N-k}^{[N,k]} > \lambda_{N+1-k}^{[N,k]} > 0,
\]

\[
\lambda_1^{[N,k]} > \lambda_2^{[N,k+1]} > \cdots > \lambda_{N-k}^{[N,k+1]} > \lambda_{N+1-k}^{[N,k+1]} > 0.
\]

That is, \( B_{N+1}^{[k]} \) interlaces \( B_{N}^{[k+1]} \) and \( B_{N}^{[k]} \).

Let us now consider for each submatrix \( T^{[N]} \) a new auxiliary matrix given by the permutation of the triangular matrices in the factorization (64), \( T^{[N]} = L_2^{[N]} U^{[N]} L_1^{[N]} \), and denote by \( B_{N+1}^{(1)} \) its corresponding second kind polynomial. Consequently, the auxiliary matrices are “Darboux transformations” of the submatrices of the banded Hessenberg matrix, see §6. In fact, these auxiliary matrices are banded Hessenberg matrices with only its first two subdiagonals different from zero.
Remark 7. These auxiliary matrices \( \hat{T}^{[N]} \) are not the \( m \)-th leading principal submatrix of the matrix \( \hat{T} := L_2 U L_1 \). The difference is in the last diagonal entry. The entries of \( \hat{T} \) are

\[
\begin{align*}
\hat{c}_n &= \alpha_{3n+2} + \alpha_{3n+1} + \alpha_{3n}, \\
\hat{b}_n &= \alpha_{3n} \alpha_{3n-1} + \alpha_{3n+1} \alpha_{3n-1} + \alpha_{3n} \alpha_{3n-2}, \\
\hat{a}_n &= \alpha_{3n} \alpha_{3n-2} \alpha_{3n-4},
\end{align*}
\]

(79)

All the entries of the \((N + 1)\)-th leading principal submatrix of \( \hat{T} \) coincide with those of \( \hat{T}^{[N]} \) but for the last diagonal entry, as \((\hat{T}^{[N]}))_{N+1,N+1} = \alpha_{3N} + \alpha_{3N+1} + \alpha_{3N+2} \) while \( \hat{c}_{N+1} = \alpha_{3N} + \alpha_{3N+1} + \alpha_{3N+2} \).

Lemma 6. Let us assume that in the factorization (64) we choose \( \alpha_2 \in (0, K[N, 1]) \). Then:

i) The auxiliary matrix \( \hat{T}^{[N]} \) is oscillatory.

ii) The polynomial \( B_{N+1} \) interlaces the polynomial \( \hat{B}_{[1]} \) of \( \hat{T}^{[N]} \).

iii) If \( w \) is a left eigenvector of \( T^{[N]} \), then \( \hat{w} = w L_1^{[N]} \) is a left eigenvector of \( \hat{T}^{[N]} \).

Proof: 

i) We have \( \hat{T}^{[N]} = L_2^{[N]} U^{[N]} L_1^{[N]} \), being each factor TN its product is TN. As all the \( \alpha \)'s are strictly positive we see that the matrix \( \hat{T}^{[N]} \) is a nonsingular TN negative matrix, with its first superdiagonal and subdiagonal also totally positive. According to Gantmacher–Krein Criterion, Theorem 1, is an oscillatory matrix.

ii) As \( \hat{T}^{[N]} = (L_1^{[N]})^{-1} T^{[N]} L_1^{[N]} \) its characteristic polynomial is \( B_{N+1} \). Consequently, from the theory of oscillatory matrices, \( B_{N+1} \) also interlaces \( \hat{B}_{[1]} \).

iii) We see that

\[
\lambda_n^{[N]} \hat{w}_n = \lambda_n^{[N]} w_n L_1^{[N]} = w_n L_1^{[N]} L_2^{[N]} U^{[N]} L_1^{[N]} = \hat{w}_n \hat{T}^{[N]},
\]

and the result follows.

Lemma 7. The polynomials \( \hat{B}_{N+1}^{[k]} \) of the auxiliary matrix \( \hat{T}^{[N]} \) satisfy

\[
\begin{align*}
\hat{B}_{N+1}^{[1]}(\lambda_n^{[N]}) &= B_{N+1}^{[1]}(\lambda_n^{[N]}) + \alpha_2 B_{N+1}^{[2]}(\lambda_n^{[N]}), \\
\hat{B}_{N+1}^{[2]}(\lambda_n^{[N]}) &= B_{N+1}^{[2]}(\lambda_n^{[N]}) + \alpha_5 B_{N+1}^{[3]}(\lambda_n^{[N]}), \\
&\vdots \\
\hat{B}_{N+1}^{[N]}(\lambda_n^{[N]}) &= B_{N+1}^{[N]}(\lambda_n^{[N]}) + \alpha_{3N-1}.
\end{align*}
\]

Proof: Let us take the left eigenvectors \( \omega_n^{(N)} \) given in (30). From previous the Lemma 6 we know that

\[
\omega_n^{(N)} L_1^{[N]} = \begin{bmatrix} B_{N+1}^{[1]}(\lambda_n^{[N]}) + \alpha_2 B_{N+1}^{[2]}(\lambda_n^{[N]}) & B_{N+1}^{[2]}(\lambda_n^{[N]}) + \alpha_5 B_{N+1}^{[3]}(\lambda_n^{[N]}) & \cdots & B_{N+1}^{[N]}(\lambda_n^{[N]}) + \alpha_{3N-1} \end{bmatrix}
\]

is a left eigenvector of \( \hat{T}^{[N]} \) with eigenvalue \( \lambda_n^{[N]} \) with the last entry normalized to 1. Consequently, we deduce that it is equal to \( \omega_n^{(N)} \), and we get the result.

Proposition 15 (Second interlacing). The polynomial \( B_{N+1} \) interlaces the polynomial \( B_{N+1}^{[1]} + \alpha_2 B_{N+1}^{[2]} \).
Proposition 18 (Masses positivity). Let us assume:

i) That $T^{[N]}$, as in Definition 4, is oscillatory.

ii) That in the bidiagonal factorization (64) of the oscillatory matrix $T^{[N]}$ we take $\alpha_2 \in (0, \mathcal{K}[N,1])$ with $\mathcal{K}[N,1] > 0$ the finite continued fraction in Definition 8.

iii) That $\nu$ in Definition 2 is such that

$$-\frac{1}{\alpha_2} = \nu.$$ 

Then, the Christoffel coefficients or masses (36) and (37) of the discrete measures given in (42) for $T^{[N]}$ are positive, i.e.,

$$\mu_{n,1}^{[N]} > 0, \quad \mu_{n,2}^{[N]} > 0, \quad n \in \{1, \ldots, N+1\}.$$ 

Proof. As $B_{N+1}^{(1)}$ and $B_{N+1}^{(2)}$ are interlaced by $B_{N+1}$, from (39) we get that $\mu_{n,1}^{[N]}, \mu_{n,2}^{[N]} > 0$ for $n \in \{1, \ldots, N+1\}$. \hfill \Box

5.2. Spectral representation. At this point we are ready to give two of the main results of the paper, Theorems 20 and 21, that establish the existence of multiple orthogonal polynomials and corresponding positive Lebesgue–Stieltjes measures for a given regular oscillatory banded Hessenberg matrix.

Theorem 20 (Favard Spectral Representation). Let $T$ be a regular oscillatory bounded banded Hessenberg matrix as in (11) — i.e. with $\mathcal{K}[1] > 0$, being $\mathcal{K}[1]$ the infinite continued fraction given in (56) — and take any $\alpha_2 \in (0, \mathcal{K}[1])$. Let us consider the corresponding sequence $\{B_n\}_{n=0}^\infty$ of recursion polynomials of type II, see Definition 1, the sequences $\{A_n^{(1)}\}_{n=0}^\infty$ and $\{A_n^{(2)}\}_{n=0}^\infty$ of recursion polynomials of type I, see Definition 2, in where we take $\nu = -\frac{1}{\alpha_2}$. Then, there exists two non decreasing functions $\psi_1$ and $\psi_2$, 

Proof. The polynomial $B_{N+1}^{[1]}(B_{N+1}^{[1]} + \alpha_2 B_{N+1}^{[2]})$ at the eigenvalue $\lambda_n^{[N]}$ is the first component of a left eigenvector of $T^{[N]}$ ($\tilde{T}^{[N]}$), that we know is not zero (the first and last entry of all eigenvectors are nonzero for oscillatory matrices). Moreover, the last component is 1. According to the properties of eigenvectors of an oscillatory matrix we know that this eigenvector has $n$ sign changes. Hence,

$$\text{sgn} \left( B_{N+1}^{[1]}(z-i_n^{[N]}) \right) = \text{sgn} \left( B_{N+1}^{[1]}(z-i_n^{[N]}) + \alpha_2 B_{N+1}^{[2]}(z-i_n^{[N]}) \right) = (-1)^{n-1}.$$ 

As $B_{N+1}$ interlaces $B_{N+1}^{(1)}$ we conclude the stated interlacing result. \hfill \Box

Proposition 16 (Third interlacing). $B_{N+1}$ interlaces the polynomial $B_{N+1}^{[N+1]} + m_1 B_{N+1}^{[N+2]} + \varepsilon_2 B_{N+1}^{[N+3]}$.

Proof. As in the previous proof but now for $UL_1L_2$. \hfill \Box

Proposition 17 (Shifting the independent variable). The shift $T \rightarrow sI + T = T_\nu$, $s \in \mathbb{R}$, of the banded Hessenberg operator leads to a corresponding shift of the polynomials of type II and associated polynomials $B_n(x) \rightarrow B_n(x-s)$, $B_n^{(1)}(x) \rightarrow B_n^{(1)}(x-s)$ and $B_n^{(2)}(x) \rightarrow B_n^{(2)}(x-s)$, of the truncated polynomials $B_n^{[k]}(x) \rightarrow B_n^{[k]}(x-s)$, and of polynomials of type I, $A_n^{(1)}(x) \rightarrow A_n^{(1)}(x-s)$ and $A_n^{(2)}(x) \rightarrow A_n^{(2)}(x-s)$. All the zeros (or eigenvalues) are shifted $\lambda^{[N,k]}_n \rightarrow \lambda^{[N,k]}(x-s)$.

If $T_\nu$ is oscillatory, despite the new eigenvalues do not need to be positive, the other properties of the eigenvalues, like simplicity and interlacing, extend to those of $T$. 

Proposition 18 (Masses positivity). Let us assume:

i) That $T^{[N]}$, as in Definition 4, is oscillatory.

ii) That in the bidiagonal factorization (64) of the oscillatory matrix $T^{[N]}$ we take $\alpha_2 \in (0, \mathcal{K}[N,1])$ with $\mathcal{K}[N,1] > 0$ the finite continued fraction in Definition 8.

iii) That $\nu$ in Definition 2 is such that

$$-\frac{1}{\alpha_2} = \nu.$$ 

Then, the Christoffel coefficients or masses (36) and (37) of the discrete measures given in (42) for $T^{[N]}$ are positive, i.e.,

$$\mu_{n,1}^{[N]} > 0, \quad \mu_{n,2}^{[N]} > 0, \quad n \in \{1, \ldots, N+1\}.$$ 

Proof. As $B_{N+1}^{(1)}$ and $B_{N+1}^{(2)}$ are interlaced by $B_{N+1}$, from (39) we get that $\mu_{n,1}^{[N]}, \mu_{n,2}^{[N]} > 0$ for $n \in \{1, \ldots, N+1\}$. \hfill \Box
and corresponding positive Lebesgue–Stieltjes measures \(d\psi_1\) and \(d\psi_2\) with compact support \(\Delta\) such that the following biorthogonality holds

\[
\int_\Delta \left( A_k^{(1)}(x) d\psi_1(x) + A_k^{(2)}(x) d\psi_2(x) \right) B_l(x) = \delta_{k,l},
\]

for \(k, l \in \mathbb{N}_0\). Consequently, the multiple orthogonal relations of type II

\[
\int_\Delta x^k B_{2n}(x) d\psi_1(x) = 0, \quad k = 0, \ldots, n - 1, \quad \int_\Delta x^k B_{2n}(x) d\psi_2(x) = 0, \quad k = 0, \ldots, n - 1,
\]

\[
\int_\Delta x^k B_{2n+1}(x) d\psi_1(x) = 0, \quad k = 0, \ldots, n, \quad \int_\Delta x^k B_{2n+1}(x) d\psi_2(x) = 0, \quad k = 0, \ldots, n - 1,
\]

and of type I

\[
\int_\Delta \left( A_{k-1}^{(1)}(x) d\psi_1(x) + A_{k-1}^{(2)}(x) d\psi_2(x) \right) x^n = 0, \quad n \in \{0, 1, \ldots, k - 2\},
\]

are satisfied.

**Proof.** The sequences \(\{\psi_1^{(N)}\}^\infty_{N=0}\) and \(\{\psi_2^{(N)}\}^\infty_{N=0}\) are uniformly bounded and nondecreasing. Consequently, following the mentioned Helly’s results, see §1.2.6, there exist subsequences that converge to nondecreasing functions \(\psi_1\) and \(\psi_2\). If the zeros lay in a bounded set \(\Delta\), that is \(T\) is bounded, these measures have a compact support. \(\square\)

**Remark 8** (Shifted measures). When \(T\) is not an oscillatory matrix but \(sI + T\) is regular oscillatory, then the corresponding positive measures \(d\psi_1(x), d\psi_2(x)\), that as we know exist, gives orthogonality measures \(d\psi_1(x + s), d\psi_2(x + s)\) for the original \(T\).

**Remark 9** (Spectral representation of moments and Stieltjes–Markov functions). For \(T\) as in Theorem 20, Helly’s second theorem leads to the spectral representation for the moments and Stieltjes–Markov functions of the full banded Hessenberg matrix in terms of the spectral functions \(\psi_1\) and \(\psi_2\):

\[
e_1^T T^k e_1 = \int_\Delta t^k d\psi_1(t), \quad e_1^T (e_2 - ve_1) = \int_\Delta t^k d\psi_2(t),
\]

\[
\hat{\psi}_1 := e_1^T (zI - T)^{-1} e_1 = \int_\Delta \frac{d\psi_1(t)}{z - t}, \quad \hat{\psi}_2 := e_1^T (zI - T)^{-1} (e_2 - ve_1) = \int_\Delta \frac{d\psi_2(t)}{z - t}.
\]

For the Weyl functions in Proposition 7 we have in \(\mathbb{C} \setminus \Delta\) uniform convergence to the Stieltjes–Markov functions

\[
S^{[N]}_1 \Rightarrow \hat{\psi}_1, \quad S^{[N]}_2 \Rightarrow \hat{\psi}_2, \quad N \to \infty.
\]

**Remark 10** (2-orthogonal Chebyshev polynomials). For the oscillatory Toeplitz case described §4.1, that happens to be regular, our results show, in a constructive manner, that a pair of positive measures always exist. The recursion polynomials can be shifted to get a new Hessenberg matrix of the form

\[
T = \begin{bmatrix}
0 & 1 & 0 & \cdots & \cdots \\
b & 0 & 1 & \cdots & \cdots \\
a & b & 0 & 1 & \cdots \\
0 & a & b & 0 & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}
\]
These shifted recursion polynomials, of type II and type I, are what sometimes in the literature is referred as 2-orthogonal Chebyshev polynomials [8].

Now we discuss some semi-infinite submatrices for which we always have an spectral representation.

**Theorem 21** (Submatrices and spectral representation). *Let us assume that:*

1. *The Hessenberg matrix* $T$ in (11) *is bounded and oscillatory.*
2. *The corresponding sequence of recursion polynomials for the Hessenberg matrices*

$$
\tilde{T} \coloneqq \begin{bmatrix}
    b_1 & 1 & 0 & \cdots & \\
    a_2 c_1 & b_2 & 1 & \cdots & \\
    a_2 a_3 & a_3 c_2 & b_3 & 1 & \cdots & \\
    0 & a_3 a_4 & a_4 c_3 & b_4 & 1 & \cdots & \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \\
    b_{k+1} - m_{k+1} & 1 & 0 & \cdots & \\
    a_{k+2} c_{k+1} & b_{k+2} & 1 & \cdots & \\
    a_{k+2} a_{k+3} & a_{k+3} c_{k+2} & b_{k+3} & 1 & \cdots & \\
    0 & a_{k+3} a_{k+4} & a_{k+4} c_{k+3} & b_{k+4} & 1 & \cdots & \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \\
\end{bmatrix},
$$

$k \in \mathbb{N},$

are $\{\tilde{B}_n, \tilde{\Lambda}_n^{(1)}, \tilde{\Lambda}_n^{(2)}\}_{n=0}^{\infty}$ for $\tilde{T}$ and $\{\tilde{B}_{k,n}, \tilde{\Lambda}_{k,n}^{(1)}, \tilde{\Lambda}_{k,n}^{(2)}\}_{n=0}^{\infty}$ for $\tilde{T}_k$, $k \in \mathbb{N}$.

3. *The initial conditions for the type I polynomials given in in Definition 2 are taken as*

$$
\tilde{\psi} = \frac{c_0}{a_2}, \quad \tilde{\psi}_k = -\frac{\alpha_{3k+1}}{a_{k+2}}.
$$

Then, there exists couples of non-decreasing functions $\{\tilde{\psi}_1, \tilde{\psi}_2\}$ for $\tilde{T}$ ($\{\tilde{\psi}_{k,1}, \tilde{\psi}_{k,1}\}$ for $\tilde{T}_k$) with associated positive Lebesgue–Stieltjes measures $\{d \tilde{\psi}_1, d \tilde{\psi}_1\}$ ($\{d \tilde{\psi}_{k,1}, d \tilde{\psi}_{k,1}\}$) with compact support, such that the corresponding recursion polynomials $\{\tilde{B}_n, \tilde{\Lambda}_n^{(1)}, \tilde{\Lambda}_n^{(2)}\}_{n=0}^{\infty}$ ($\{\tilde{B}_{k,n}, \tilde{\Lambda}_{k,n}^{(1)}, \tilde{\Lambda}_{k,n}^{(2)}\}_{n=0}^{\infty}$) are, with respect to these couple of measures, multiple orthogonal polynomials of types II and I.

**Proof.** Theorem 13 ensures that the submatrix $T_1^{[N]}$ and its submatrices $T_1^{[N,k]}$ are oscillatory. Moreover, Equation (51) and the fact that $J^{[N,k]}$ is oscillatory and $U^{[N-1,k]}$ is InTN (notice that the product is InTN and the elements in the first superdiagonal and subdiagonal are positive, then use Gantmacher–Krein Criterion, Theorem 1) imply that the retraction $T_1^{[N,k]} - \ell_{k+1} E_{1,1}$ is oscillatory as well. For more on retractions see [11, §9.5]. These matrices are upper Hessenberg, so that
transposition will transform then in lower Hessenberg

\[
(T_1^{[N]})^T := \begin{bmatrix}
  b_1 & a_2 & 0 & & & & & 0 \\
  c_1 & b_2 & a_3 & & & & & \\
  1 & c_2 & b_3 & a_4 & & & & \\
  & 0 & 1 & c_3 & b_4 & a_5 & & \\
  & & & \vdots & \ddots & \ddots & \ddots & \\
  & & & & & b_{N-1} & a_N & 0 \\
  & & & & & & 0 & 1 & a_{N-1} & b_N
\end{bmatrix},
\]

\[
(T_1^{[N,k]}-m_{k+1}E_{1,1})^T := \begin{bmatrix}
  b_{k+1} - m_{k+1} & a_{k+2} & 0 & & & & & 0 \\
  c_{k+1} & b_{k+2} & a_{k+3} & & & & & \\
  1 & c_{k+2} & b_{k+3} & a_{k+4} & & & & \\
  & 0 & 1 & c_{k+3} & b_{k+4} & a_{k+5} & & \\
  & & & \vdots & \ddots & \ddots & \ddots & \\
  & & & & & b_{N-1} & a_N & 0 \\
  & & & & & & 0 & 1 & a_{N-1} & b_N
\end{bmatrix}.
\]

They are not normalize to be monic on the first superdiagonal but on the second subdiagonal. However, a similarity \( T \mapsto ATA^{-1} \) transformation by the diagonal matrix \( A = \text{diag}(1, a_2, a_2a_3, \ldots, a_2a_3 \cdots a_N) \) leads to monic banded Hessenberg matrix,

\[
\tilde{T}^{[N-1]} := A(T_1^{[N]})^T A^{-1} = \begin{bmatrix}
  b_1 & 1 & 0 & & & & & 0 \\
  a_2c_1 & b_2 & 1 & & & & & \\
  a_2a_3 & a_3c_2 & b_3 & 1 & & & & \\
  0 & a_3a_4 & a_4c_3 & b_4 & 1 & & & \\
  & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
  & & & b_{N-1} & a_N & a_{N-1}a_N & a_{NCN-1} & b_N
\end{bmatrix},
\]

which happens to be oscillatory. Analogously, a similarity \( T \mapsto ATA^{-1} \) transformation by the diagonal matrix \( A_k = \text{diag}(1, a_{k+2}, a_{k+2}a_{k+3}, \ldots, a_{k+2}a_{k+3} \cdots a_N) \) leads to monic banded Hessenberg matrix,

\[
\tilde{T}^{[N-1,k]} := A_k(T_1^{[N,k]}-m_{k+1}E_{1,1})^T A_k^{-1} = \begin{bmatrix}
  b_{k+1} - m_{k+1} & 1 & 0 & & & & & 0 \\
  a_{k+2}c_{k+1} & b_{k+2} & 1 & & & & & \\
  a_{k+2}a_{k+3} & a_{k+3}c_{k+2} & b_{k+3} & 1 & & & & \\
  0 & a_{k+3}a_{k+4} & a_{k+4}c_{k+3} & b_{k+4} & 1 & & & \\
  & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
  & & & b_{N-1} & a_N & a_{N-1}a_N & a_{NCN-1} & b_N
\end{bmatrix}.
\]
retraction that happens to be oscillatory. Moreover, according to (50) and (51) these matrices admit the factorizations
\begin{align}
\hat{T}^{[N-1]} &= \hat{L}^{[N-1]} \hat{j}^{[N]}, \\
\hat{T}^{[N-1,k]} &= \hat{L}^{[N-1,k]} \hat{j}^{[N,1]}, \\
\hat{L}^{[N-1,k]} &:= A_k (U^{[N-1,k]})^T A_k^{-1}, \\
\hat{j}^{[N,1]} &:= A_k (J^{[N,1]})^T A_k^{-1},
\end{align} \tag{81}

with (recalling that \(\alpha_1 = c_0\))

\[
\hat{L}^{[N-1]} = \begin{bmatrix}
    c_0 & 0 & \cdots & 0 \\
    a_2 & a_4 & \cdots & 0 \\
    0 & a_3 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & \cdots & 0 & a_N \\
    \end{bmatrix}, \\
\hat{j}^{[N,1]} = \begin{bmatrix}
    m_1 & \ell_2 & \cdots & 0 \\
    a_2 & m_2 & \cdots & 0 \\
    0 & a_3 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & \cdots & 0 & a_N \\
    \end{bmatrix}.
\]

\[
\hat{L}^{[N-1,k]} = \begin{bmatrix}
    \alpha_{3k+1} & 0 & \cdots & 0 \\
    a_{k+2} & \alpha_{3k+4} & \cdots & 0 \\
    0 & a_{k+3} & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & \cdots & 0 & a_N \\
    \end{bmatrix}, \\
\hat{j}^{[N,k+1]} = \begin{bmatrix}
    m_{k+1} & \ell_{k+2} & \cdots & 0 \\
    a_{k+2} & m_{k+2} & \cdots & 0 \\
    0 & a_{k+3} & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & \cdots & 0 & a_N \\
    \end{bmatrix}.
\]

Then, from previous discussions we know that the corresponding recursion polynomials of the second kind \(\hat{B}_n^{(1)}\) are interlaced by the recursion polynomials \(\hat{B}_n\). Moreover, extending the ideas in Proposition 15 and using the factorizations (81), we get that \(c_0 \hat{B}_n^{[1]} + a_2 \hat{B}_n^{[2]}\) is interlaced by \(\hat{B}_n\). Similarly, the polynomial \(\hat{B}_{k,n}\) interlaces \(\alpha_{3k+1} \hat{B}_{k,n}^{[1]} + a_{k+2} \hat{B}_{k,n}^{[2]}\). Hence, the positivity of the masses is achieved for initial conditions as given in (80). Obviously \(\hat{\nu} \neq 0\) and given that \(\alpha_{3k+1} \neq 0\) also \(\hat{\nu}_k \neq 0\). Then, as for Theorem 20, Helly’s results leads to the conclusion. \(\Box\)

6. Applications to Darboux Transformations

6.1. Darboux transformations of oscillatory banded Hessenberg matrices. We now show how our construction connects with those of the seminal paper [2] by Aptekarev, Kalyagin and Van Iseghem on genetic sums, vector convergents, Hermite–Padé approximants and and Stieltjes problems, and with the Darboux–Christoffel transformations discussed in [4]. We identify the Darboux transformations of the oscillatory banded Hessenberg matrices with Christoffel transformations of the spectral measures. Recall that \(\nu\) is the initial condition given in Definition 2.

Definition 11 (Darboux transformed Hessenberg matrices). Given an oscillatory banded lower Hessenberg matrix \(T\) as in Definition 4 and its bidiagonal factorization as in (65), we consider the semi-infinite matrices
\[
\hat{T} := L_2 U L_1, \\
\hat{T} := U L_1 L_2.
\]
We will refer to these matrices as the first and second Darboux transformations of the banded Hessenberg matrix \(T\).
Remark 11.  
i) From definition it is immediately checked that both Darboux transformed Hessenberg matrices has the same banded structure as $T$ and fit along with the Hessenberg matrices considered in Definition 4.

ii) The first Darboux transform $\hat{T}$ has been discussed previously, see Remark 7, and its entries are given in (79). For coefficients of the second Darboux transform $\hat{T}$ we have

\begin{align*}
\hat{c}_n &= \alpha_{3n+3} + \alpha_{3n+2} + \alpha_{3n+1}, \\
\hat{b}_n &= \alpha_{3n+1} + \alpha_{3n+2} + \alpha_{3n+1} \alpha_{3n-1}, \\
\hat{a}_n &= \alpha_{3n+1} \alpha_{3n-1} \alpha_{3n-3}.
\end{align*}

iii) If $T$ is regular oscillatory, and we take $\alpha_2 < 0$, then the Darboux transforms $\hat{T}$, $\hat{T}$ are also oscillatory.

Associated with these Hessenberg matrices we introduce the vectors of polynomials

\begin{equation}
\hat{B} := UB, \quad \hat{B} = L_2 \hat{B} := L_2 UB.
\end{equation}

Notice that $\hat{B}_n$ and $\hat{B}_n$ are monic with $\text{deg} \hat{B}_n = \text{deg} \hat{B}_n = n + 1$.

Lemma 8. We have

\begin{equation}
L_1 \hat{B} = xB.
\end{equation}

Proof. Equations (82) imply $L_1 \hat{B} = L_1 L_2 UB = TB = xB$. \hfill \Box

Proposition 19. The eigenvalue properties $\hat{T}B = xB$ and $\hat{T}B = x\hat{B}$ are satisfied.

Proof. Equations (82) and (83) lead by direct computation to

\begin{align*}
\hat{T}B &= L_2 UL_1 \hat{B} = xL_2 UB = x\hat{B}, \\
\hat{T}B &= UL_1 L_2 UB = UTB = xUB = x\hat{B}.
\end{align*}

\hfill \Box

Let us denote by if $\hat{T}^{[n]}$ and $\hat{T}^{[n]}$ the $(n+1)$-th leading principal submatrices of $\hat{T}$ and $\hat{T}$.

Lemma 9. The polynomials $\hat{B}_n$ and $\hat{B}_n$ can be expressed as $\hat{B}_n = x\hat{B}_n$ and $\hat{B}_n = x\hat{B}_n$, with the monic polynomials $\hat{B}_n$, $\hat{B}_n$ having degree $n$.

Proof. One has that $\hat{B}_0 = \hat{B}_0 = \alpha_1 + \alpha_1 + x + c_0 = x$. Then, as the sequences of polynomials are found by the recurrence determined by the banded Hessenberg matrices $\hat{T}$ and $\hat{T}$, respectively, we find that the desired result. \hfill \Box

We call these polynomials $\hat{B}_n$ and $\hat{B}_n$ as Darboux transformed polynomials of type II.

Proposition 20. The entries of the Darboux transformed polynomial sequences of type II

\begin{equation}
\hat{B} = \frac{1}{x} L_2 UB, \quad \hat{B} := \frac{1}{x} UB,
\end{equation}

read

\begin{equation}
\hat{B}_n = \frac{1}{x} (B_{n+1} + (\alpha_{3n+1} + \alpha_{3n})B_n + \alpha_{3n+1} \alpha_{3n-2} B_{n-1}), \quad \hat{B}_n = \frac{1}{x} (B_{n+1} + \alpha_{3n+1} B_n)
\end{equation}
were we take $\alpha_k = 0$ for $k \in \mathbb{Z}_-$. The following determinantal expressions hold

$$
\tilde{B}_{n+1} = \det (xI_{n+1} - \tilde{T}[n]), \quad \tilde{B}_{n+1} = \det (xI_{n+1} - \tilde{T}[n]).
$$

**Proof.** Equation (85) appears as the entries of the defining equations. The determinantal expressions follow from the fact that its expansions along the last row satisfy the recursion relations with adequate initial conditions. \qed

Following definitions given in (26) and (28) we consider similar objects in this context. That is, we denote by $\tilde{T}^{[n,k]}$ ($\tilde{\hat{T}}^{[n,k]}$) the matrix obtained from $\tilde{T}^{[n]}$ ($\tilde{\hat{T}}^{[n]}$) by erasing the first $k$ rows and columns. The corresponding characteristic polynomials are

$$
\tilde{\hat{B}}_{n+1}^{[k]} = \det (xI_{n+1-k} - \tilde{T}^{[n,k]}), \quad \tilde{\hat{B}}_{n+1}^{[k]} = \det (xI_{n+1-k} - \tilde{T}^{[n,k]}).
$$

These polynomials $\tilde{\hat{B}}_n^{[k]}$ ($\tilde{\hat{B}}_n^{[k]}$) satisfy the same recursion relations, determined by $\tilde{T}$ ($\tilde{\hat{T}}$) as do $\tilde{B}_n$ ($\tilde{\hat{B}}_n$) but with different initial conditions. Following ii) in Proposition 1 we have the transformed recursion polynomials of type II

$$
\tilde{\hat{B}}^{(1)}_{n+1} = \hat{B}_{n+1}, \quad \tilde{\hat{B}}^{(1)}_{n+1} = \hat{B}_{n+1}, \quad \tilde{\hat{B}}^{(2)}_{n+1} = \hat{B}_{n+1} - v\hat{B}_{n+1}, \quad \tilde{\hat{B}}^{(2)}_{n+1} = \hat{B}_{n+1} - v\hat{B}_{n+1}.
$$

Then we consider the following vectors of polynomials

$$
\hat{B}_{n+1}^{(1)} = x\hat{B}_{n+1}^{(1)}, \quad \hat{B}_{n+1}^{(2)} = x\hat{B}_{n+1}^{(2)}, \quad \hat{B}_{n+1}^{(1)} = x\hat{B}_{n+1}^{(1)}, \quad \hat{B}_{n+1}^{(2)} = x\hat{B}_{n+1}^{(2)}.
$$

**Proposition 21** (Vector Convergents). These recursion polynomials correspond to the vector convergent $y^1_n = (A_{n,0}, A_{n,1}, A_{n,2})$ discussed in [2] as follows

$$
B_n = A_{3n,0}, \quad \hat{B}_n = A_{3n,1,0}, \quad \hat{B}_n = A_{3n+2,0},
$$

$$
B_n^{(1)} = A_{3n,1}, \quad \hat{B}_n^{(1)} = A_{3n+1,1}, \quad \hat{B}_n^{(1)} = A_{3n+2,1},
$$

$$
-\frac{1}{v}B_n^{(2)} = A_{3n,2}, \quad -\frac{1}{v}\hat{B}_n^{(2)} = A_{3n+1,2}, \quad -\frac{1}{v}\hat{B}_n^{(2)} = A_{3n+2,2}.
$$

**Proof.** It follows from the fact that they satisfy the recursion relation [2, Equation (23)] and adequate initial conditions. \qed

Then, following this dictionary the important [2, Lemma 5] states for $x \geq 0$ that

$$
\begin{array}{c|c|c|c|c|c}
| \hat{B}_n & \hat{B}_n^{(1)} | & | \hat{B}_n & \hat{B}_n^{(2)} | & | \hat{B}_n & \hat{B}_n^{(1)} | & | \hat{B}_n & \hat{B}_n^{(2)} | \\
| \hat{B}_n & \hat{B}_n^{(1)} | & | \hat{B}_n & \hat{B}_n^{(2)} | & | \hat{B}_n & \hat{B}_n^{(1)} | & | \hat{B}_n & \hat{B}_n^{(2)} | \\
| \hat{B}_n & \hat{B}_n^{(1)} | & | \hat{B}_n & \hat{B}_n^{(2)} | & | \hat{B}_n & \hat{B}_n^{(1)} | & | \hat{B}_n & \hat{B}_n^{(2)} | \\
| \hat{B}_n & \hat{B}_n^{(1)} | & | \hat{B}_n & \hat{B}_n^{(2)} | & | \hat{B}_n & \hat{B}_n^{(1)} | & | \hat{B}_n & \hat{B}_n^{(2)} |
\end{array}
$$

**Remark 12.** Using these facts, Aptekarev, Kalyagin and Van Iseghem in [2, Lemmata 6 & 7] deduce the degree of polynomials, simplicity of zeros and interlacing properties of $B_n$ with $B_{n-1}$, $B_n^{(1)}$ and $B_n^{(2)}$. Notice that we derive the same result by just using the spectral properties of regular oscillatory matrices.
For recursion polynomials of type I, we introduce the following polynomials $\hat{A}^{(2)} := A^{(1)} L_1$, $\hat{A}^{(1)} := A^{(2)} L_1$, $\hat{A}^{(1)} := A^{(1)} L_2$, and $\hat{A}^{(2)} := A^{(2)} L_2$.

**Proposition 22.** Vectors $\hat{A}^{(1)}$, $\hat{A}^{(2)}$ are left eigenvectors of $\hat{T}$ and $\hat{A}^{(1)}$, $\hat{A}^{(2)}$ are left eigenvectors of $\hat{T}$.

**Proof.** A direct computation shows that $\hat{T} A^{(2)} = A^{(1)} L_2 U L_1 = A^{(1)} T L_1 = x A^{(1)} L_1 = x \hat{A}^{(2)}$. The other cases are proven similarly. □

**Lemma 10.** Let us assume that $1 + \alpha_2 v = 0$. Then, $\hat{A}^{(2)}_0 = 0$ and $\hat{A}^{(2)}_1 = \frac{1}{a_3 a_1} x$.

**Proof.** Let us consider the vector $\hat{A}^{(2)} = A^{(1)} L_1$ with components $\hat{A}_n^{(2)} = A_n^{(1)} + \alpha_3 n + 2 A_{n+1}^{(1)}$, $n \in \mathbb{N}_0$. The first two entries are

\[
\begin{align*}
\hat{A}_0^{(2)} &= A_0^{(1)} + \alpha_2 A_1^{(1)} = 1 + \alpha_2 v, \\
\hat{A}_1^{(2)} &= A_1^{(1)} + \alpha_3 A_2^{(1)} = v - \alpha_3 \frac{c_0 + b_1 v}{a_2} + \frac{\alpha_5}{a_2} x = v - \alpha_5 \frac{\alpha_1 + (\alpha_3 + \alpha_2) \alpha_1 v}{\alpha_5 \alpha_3 a_1} + \frac{\alpha_5}{\alpha_5 \alpha_3 a_1} x \\
&= -1 + \alpha_2 v + \frac{1}{\alpha_3 a_1} x.
\end{align*}
\]

Here we have used Definition 2 and $c_0 = \alpha_1$, $b_1 = (\alpha_3 + \alpha_2) \alpha_1$ and $a_2 = \alpha_5 \alpha_3 a_1$. Then, as $1 + \alpha_2 v = 0$, we find the stated result. □

**Proposition 23.** If $1 + \alpha_2 v = 0$, we can write $\hat{A}_n^{(2)} = x \hat{A}_n^{(2)}$ and $\hat{A}_n^{(1)} = x \hat{A}_n^{(1)}$, for some polynomials $\tilde{A}_n^{(2)}$, $\tilde{A}_n^{(1)}$.

**Proof.** The recursion relation $\tilde{c}_0 \tilde{A}_0^{(2)} + \tilde{b}_1 \tilde{A}_1^{(2)} + \tilde{a}_2 \tilde{A}_2^{(2)} = x \tilde{A}_0^{(2)}$ and Lemma 10 gives $\tilde{A}_2^{(2)} = -\frac{\tilde{b}_1}{a_2 a_3 a_1} x$. Hence, induction leads to the conclusion that $\hat{A}_n^{(2)} = x \hat{A}_n^{(2)}$, for some polynomial $\tilde{A}_n^{(2)}$. □

**Lemma 11.** We have $\hat{A}_0^{(2)} = 0$ and $\hat{A}_1^{(2)} = \frac{1}{a_4} x$.

**Proof.** Let us consider the vector $\hat{A}^{(1)} = A^{(2)} L_1$ with components $\hat{A}_n^{(1)} = A_n^{(2)} + \alpha_3 n + 2 A_{n+1}^{(2)}$, $n \in \mathbb{N}_0$. The first three entries are

\[
\begin{align*}
\hat{A}_0^{(1)} &= A_0^{(2)} + \alpha_2 A_1^{(2)} = \alpha_2, \\
\hat{A}_1^{(1)} &= A_1^{(2)} + \alpha_5 A_2^{(2)} = 1 - \alpha_5 \frac{b_1}{a_2} = 1 - \alpha_5 \frac{(\alpha_3 + \alpha_2) \alpha_1}{\alpha_5 \alpha_3 a_1} = -\frac{\alpha_2}{a_3}, \\
\hat{A}_2^{(1)} &= A_2^{(2)} + \alpha_8 A_3^{(2)} = -\frac{b_1}{a_2} + \alpha_8 \left( \frac{b_1}{a_2} + \frac{b_1}{a_2} + x - c_1 \right) = \frac{b_1}{a_2} \left( \frac{\alpha_3 b_1}{a_3} - 1 \right) - \frac{\alpha_8 c_1}{a_3} + \frac{\alpha_8}{a_3} x \\
&= \frac{\alpha_3 + \alpha_2}{a_5 a_3} \left( \frac{\alpha_6 a_4 + \alpha_5 a_4 + \alpha_5 a_3}{a_5 a_3} - 1 \right) - \frac{\alpha_4 + \alpha_3 + \alpha_2}{\alpha_6 a_4} + \frac{1}{\alpha_6 a_4} x \\
&= \frac{\alpha_3 + \alpha_2}{a_5 a_3} \frac{\alpha_6 a_4}{a_5 a_3} - \frac{\alpha_4 + \alpha_3 + \alpha_2}{\alpha_6 a_4} + \frac{1}{\alpha_6 a_4} x = \frac{1}{\alpha_6 a_4} \left( \frac{\alpha_3 + \alpha_2}{\alpha_3} (\alpha_4 + \alpha_3) - \alpha_4 - \alpha_3 - \alpha_2 + x \right) \\
&= \frac{\alpha_2}{\alpha_6 a_4} + \frac{1}{\alpha_6 a_4} x.
\end{align*}
\]
Then, we consider \( \hat{A}^{(2)} = \hat{A}^{(1)} L_2 \) with components \( \hat{A}_n^{(2)} = \hat{A}_n^{(1)} + \alpha_{3n+3} \hat{A}_{n+1}^{(1)} \), \( n \in \mathbb{N}_0 \). The first two components being

\[
\hat{A}_0^{(2)} = \hat{A}_0^{(1)} + \alpha_3 A_1^{(1)} = \alpha_2 + \alpha_3 \left( - \frac{\alpha_2}{\alpha_3} \right) = 0, \quad \hat{A}_1^{(2)} = \hat{A}_1^{(1)} + \alpha_6 \hat{A}_2^{(1)} = \frac{\alpha_2}{\alpha_3} + \alpha_6 \left( \frac{\alpha_2}{\alpha_6 \alpha_3} + \frac{1}{\alpha_6 \alpha_4} \right) = \frac{1}{\alpha_4},
\]

and the result follows. \( \square \)

**Proposition 24.** There are polynomials \( \tilde{A}_n^{(2)} \) such that \( \hat{A}_n^{(2)} = x \tilde{A}_n^{(2)} \).

**Proof.** It holds for the two first entries \( \hat{A}_0^{(2)} \) and \( \hat{A}_1^{(2)} \). Hence, from the recursion relation \( \hat{A}^{(2)} T = x \hat{A}^{(2)} \) we get that it holds for any natural number \( n \). \( \square \)

We name the polynomials \( \tilde{A}_n^{(1)} \), \( \tilde{A}_n^{(1)} \), \( \tilde{A}_n^{(2)} \) and \( \tilde{A}_n^{(2)} \) as Darboux transformed polynomials of type I.

**Proposition 25.** Let us assume that \( 1 + \nu \alpha_2 = 0 \). The entries of the Darboux transformed polynomials sequences of type I

\[
\tilde{A}^{(2)} = \frac{1}{x} A^{(1)} L_1, \quad \tilde{A}^{(1)} = \frac{1}{x} A^{(1)} L_2, \quad \tilde{A}^{(2)} = A^{(2)} L_1, \quad \tilde{A}^{(2)} = A^{(2)} L_2
\]

are given by

\[
\tilde{A}_n^{(2)} = \frac{1}{x} \left( A_n^{(1)} + \alpha_{3n+2} A_{n+1}^{(1)} \right), \quad \tilde{A}_n^{(1)} = \frac{1}{x} \left( A_n^{(1)} \right) + (\alpha_{3n+2} + \alpha_{3n+3}) A_{n+1}^{(1)} + \alpha_{3n+5} \alpha_{3n+3} A_{n+2}^{(1)},
\]

\[
\tilde{A}_n^{(1)} = A_n^{(2)} + \alpha_{3n+2} A_{n+1}^{(2)} , \quad \tilde{A}_n^{(2)} = \frac{1}{x} \left( A_n^{(2)} \right) + (\alpha_{3n+2} + \alpha_{3n+3}) A_{n+1}^{(2)} + \alpha_{3n+5} \alpha_{3n+3} A_{n+2}^{(2)}.
\]

### 6.2. Spectral representation and Christoffel transformations.

We identify the entries in the bidiagonal factorization (65) with simple rational expressions in terms of the recursion polynomials evaluated at the origin.

**Theorem 22** (Parametrization of the bidiagonal factorization). *The \( \alpha \)'s in the bidiagonal factorization (65) can be expressed in terms of the recursion polynomials evaluated at \( x = 0 \) as follows:*

\[
\alpha_{3n+1} = - \frac{B_{n+1}(0)}{B_n(0)}, \quad \alpha_{3n+2} = - \frac{A_n^{(1)}(0)}{A_{n+1}^{(1)}(0)}, \quad \alpha_{3n+3} = \frac{A_n^{(1)}(0) A_{n+1}^{(2)}(0) - A_{n+1}^{(1)}(0) A_n^{(2)}(0) A_{n+2}^{(1)}(0)}{A_{n+1}^{(1)}(0) A_{n+2}^{(2)}(0) - A_{n+2}^{(1)}(0) A_{n+1}^{(2)}(0) A_{n+1}^{(1)}(0)}.
\]

The relations

\[
\begin{bmatrix}
\alpha_{3n+2} + \alpha_{3n+3} & \alpha_{3n+5} \alpha_{3n+3}
\end{bmatrix} = - \begin{bmatrix}
A_n^{(1)}(0) & A_n^{(2)}(0)
\end{bmatrix} \begin{bmatrix}
A_{n+1}^{(1)}(0) & A_{n+1}^{(2)}(0)
A_{n+2}^{(1)}(0) & A_{n+2}^{(2)}(0)
\end{bmatrix}^{-1}
\]

are satisfied as well.
**Proof.** Equation (82) and the fact that \( \hat{B}(0) = 0 \) gives that \( UB(0) = 0 \). Hence, we get \( \alpha_{3n+1}B_n(0) + B_{n+1}(0) = 0 \) and (86) follow. Now, as \( \hat{A}^{(1)} = A^{(1)}L_1 \) and \( \hat{A}^{(1)}(0) = 0 \) implies \( A^{(1)}(0)L_1 = 0 \). Hence, \( A^{(1)}(0) + A^{(1)}(0)\alpha_{3n+2} = 0 \) and we find (87). To prove (89) we observe that

\[
A^{(1)}(0) + (\alpha_{3n+2} + \alpha_{3n+3})A^{(1)}_{n+1}(0) + \alpha_{3n+5}\alpha_{3n+3}A^{(1)}_{n+2}(0) = 0,
\]

so that

\[
\begin{bmatrix}
A^{(1)}(0) & A^{(2)}(0)
\end{bmatrix} + \begin{bmatrix}
\alpha_{3n+2} + \alpha_{3n+3} & \alpha_{3n+5}\alpha_{3n+3}
\end{bmatrix} \begin{bmatrix}
A^{(1)}_{n+1}(0) & A^{(2)}_{n+1}(0)
A^{(1)}_{n+2}(0) & A^{(2)}_{n+2}(0)
\end{bmatrix} = 0,
\]

and Equation (89) follows.

This equation implies component-wise the following relations

\[
\alpha_{3n+2} + \alpha_{3n+3} = -\frac{A^{(1)}_n(0)A^{(2)}_{n+2}(0) - A^{(1)}_{n+2}(0)A^{(2)}_n(0)}{A^{(1)}_{n+1}(0)A^{(2)}_{n+2}(0) - A^{(1)}_{n+2}(0)A^{(2)}_{n+1}(0)},
\]

\[
\alpha_{3n+5}\alpha_{3n+3} = -\frac{A^{(1)}_{n+1}(0)A^{(2)}_{n+2}(0) - A^{(1)}_{n+2}(0)A^{(2)}_{n+1}(0)}{A^{(1)}_{n+1}(0)A^{(2)}_{n+2}(0) - A^{(1)}_{n+2}(0)A^{(2)}_{n+1}(0)}.
\]

Thus, we get Equation (88). \( \square \)

With the previous identification we are ready to show the complete correspondence of the described Darboux transformations of the oscillatory banded Hessenberg matrix \( T \) with Christoffel perturbations of the corresponding pair of positive Lebesgue–Stieltjes measures \((d\psi_1, d\psi_2)\).

**Theorem 23** (Darboux vs Christoffel transformations). For \( \alpha_2 = -\frac{1}{v} > 0 \), the multiple orthogonal polynomial sequences \( \{\tilde{B}_n, \tilde{A}^{(1)}_n, \tilde{A}^{(2)}_n\} \) and \( \{\tilde{B}_n, \tilde{A}^{(1)}_n, \tilde{A}^{(2)}_n\} \) correspond to the Christoffel transformations given in [4, Theorems 4 & 6] of the multiple orthogonal polynomial sequence \( \{B_n, A^{(1)}_n, A^{(2)}_n\} \). If the original couple of Lebesgue–Stieltjes measures is \((d\psi_1, d\psi_2)\), then the corresponding transformed pairs of measures are \((d\psi_2, x d\psi_1)\) and \((x d\psi_1, x d\psi_2)\), respectively.

**Proof.** Recalling (87), that \( c_n = \alpha_{3n+1} + \alpha_{3n} + \alpha_{3n-1} \) and that \( a_{n+1} = \alpha_{3n+2}\alpha_{3n}\alpha_{3n-2} \) we write

\[
\alpha_{3n+1} + \alpha_{3n} = \frac{A^{(1)}_{n-1}(0)}{A^{(1)}_n(0)} + c_n,
\]

\[
\alpha_{3n+2}\alpha_{3n+3} = -\frac{A^{(1)}_{n+1}(0)}{A^{(1)}_n(0)}.
\]

Then, using Theorem 22 and the first equation in (85) we get

\[
\tilde{B}_n = \frac{1}{x} \left( B_{n+1} + \frac{A^{(1)}_{n-1}(0)}{A^{(1)}_n(0)} + c_n \right) - \frac{A^{(1)}_{n+1}(0)}{A^{(1)}_n(0)} a_{n+1} B_{n-1},
\]

\[
\tilde{A}^{(1)}_n = A^{(2)}_n - \frac{A^{(1)}_{n+1}(0)}{A^{(1)}_n(0)} A^{(2)}_n,
\]

\[
\tilde{A}^{(2)}_n = \frac{1}{x} \left( A^{(1)}_n - \frac{A^{(1)}_{n-1}(0)}{A^{(1)}_n(0)} \right).
\]

These three equations are the Christoffel formulas in [4, Theorem 4] for the permuting Christoffel transformation \((d\psi_1, d\psi_2) \rightarrow (d\psi_2, x d\psi_1)\). Also, using again Theorem 22, the second equation
in (85) and (89) we get
\[
\tilde{B}_n = \frac{1}{x} \left( B_n - \frac{B_{n+1}(0)}{B_n(0)} B_{n+1} \right),
\]
\[
\tilde{A}_n^{(1)} = \frac{1}{x} \left( A_n^{(1)} - \begin{bmatrix} A_{n+1}^{(1)}(0) & A_{n+2}^{(2)}(0) \\ A_{n+1}^{(2)}(0) & A_{n+2}^{(2)}(0) \end{bmatrix} \right) \left[ \begin{bmatrix} A_{n+1}^{(1)}(0) \\ A_{n+2}^{(2)}(0) \end{bmatrix} \right]^{-1} \left[ \begin{bmatrix} A_{n+1}^{(1)} \\ A_{n+2}^{(2)} \end{bmatrix} \right],
\]
\[
\tilde{A}_n^{(2)} = \frac{1}{x} \left( A_n^{(2)} - \begin{bmatrix} A_{n+1}^{(1)}(0) & A_{n+2}^{(2)}(0) \\ A_{n+1}^{(2)}(0) & A_{n+2}^{(2)}(0) \end{bmatrix} \right) \left[ \begin{bmatrix} A_{n+1}^{(1)} \\ A_{n+2}^{(2)} \end{bmatrix} \right]^{-1} \left[ \begin{bmatrix} A_n^{(1)} \\ A_n^{(2)} \end{bmatrix} \right].
\]
These three equations are the Christoffel formulas in [4, Theorem 6] for the Christoffel transformation \((d \psi_1, d \psi_2) \rightarrow (x d \psi_1, x d \psi_2)\).

7. Applications to Random walks

We will discuss on the applications of the spectral representation we have given to oscillatory bounded banded Hessenberg matrices to tetradiagonal stochastic matrices of the type

\[
P_{II} = \begin{bmatrix}
0 & s_0 & 0 & \ldots \\
s_0 & r_0 & s_1 & \ldots \\
p_2 & q_2 & r_2 & s_2 \\
p_3 & q_3 & r_3 & s_3 \\
\vdots & \vdots & \vdots & \ddots \\
0 & p_3 & q_3 & r_3 & s_3 \\
\end{bmatrix}, \quad P_I = \begin{bmatrix}
0 & s_0 & 0 & \ldots \\
q_0 & r_0 & s_0 & \ldots \\
p_1 & q_1 & r_1 & s_1 \\
p_2 & q_2 & r_2 & s_2 \\
\vdots & \vdots & \vdots & \ddots \\
0 & p_3 & q_3 & r_3 & s_3 \\
\end{bmatrix}.
\]

Here all the entries are nonnegative and satisfy \(p_n + q_n + r_n + s_n = 1\), and in particular for the type II case we also have \(r_0 + s_0 = 1\) and \(q_1 + r_1 + s_1 = 1\), and for the type I, \(q_0 + r_0 + s_0 = 1\). These stochastic matrices of Hessenberg type are connected to the monic banded Hessenberg given in (11). In fact, assuming that \(s_n > 0\), we have

\[
P_{II} = H_{II} T H_{II}^{-1}, \quad H_{II} = \text{diag}(1, H_{II,1}, H_{II,2}, \ldots), \quad H_{II,n} = \frac{1}{s_0 s_1 \cdots s_{n-1}},
\]

here the entries of the monic Hessenberg matrix in terms of the stochastic matrix entries are

\[
c_n = r_n, \quad b_{n+1} = s_n q_{n+1}, \quad a_{n+2} = s_n s_{n+1} p_{n+2}.
\]

Assuming that \(p_n > 0\), then

\[
P_I = H_I^{-1} T^+ H_I, \quad H_I = \text{diag}(1, H_{I,1}, H_{I,2}, \ldots), \quad H_{I,n} = \frac{1}{p_1 p_2 \cdots p_n},
\]

now the entries of the monic Hessenberg matrix in terms of the stochastic matrix \(P_I\) entries are

\[
c_n = q_n, \quad b_{n+1} = r_n p_{n+1}, \quad a_{n+2} = s_n p_{n+1} p_{n+2}.
\]

Hence, the corresponding Markov chains can be described by the spectral methods we have constructed for the monic Hessenberg matrix \(T\).

The corresponding diagrams for these Markov chains of types II and I, respectively, are
7.1. **Finite Markov chains.** First, we discuss the finite Markov chains, in which the tetradiagonal stochastic matrices are finite, say

\[
P_{lH}^{[N]} = \begin{bmatrix}
  r_0 & s_0 & 0 & \cdots & 0 \\
  q_1 & r_1 & s_1 & \cdots & 0 \\
  p_2 & q_2 & r_2 & s_2 & 0 \\
  0 & p_3 & q_3 & r_3 & s_3 \\
  \vdots & \vdots & \vdots & \vdots & \vdots \\
  0 & \cdots & 0 & p_{N-1} & q_{N-1} \\
  0 & \cdots & 0 & p_N & q_N & r_N \\
\end{bmatrix}, \quad s_0s_1 \cdots s_{N-1} > 0.
\]

with \( p_N + q_N + r_N = 1 \); and

\[
P_{l}^{[N]} = \begin{bmatrix}
  q_0 & r_0 & s_0 & 0 & \cdots & 0 \\
  p_1 & q_1 & r_1 & s_1 & \cdots & 0 \\
  0 & p_2 & q_2 & r_2 & s_2 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  0 & \cdots & 0 & p_{N-1} & q_{N-1} & r_{N-1} \\
  0 & \cdots & 0 & p_N & q_N & r_N \\
\end{bmatrix}, \quad p_1p_2 \cdots p_N > 0,
\]

with \( p_{N-1} + q_{N-1} + r_{N-1} = 1 \) and \( p_N + q_N = 1 \). In fact, assuming that \( s_n > 0 \), we have

\[
P_{\text{osc}}^{[N]} = H_{lH}^{[N]} T^{[N]} (H_{lH}^{[N]})^{-1}, \quad H_{lH}^{[N]} = \text{diag}(1, H_{lH,1}^{[N]}, H_{lH,2}^{[N]}, \ldots, H_{lH,N}^{[N]}), \quad H_{lH,n}^{[N]} = \frac{1}{s_0s_1 \cdots s_{n-1}},
\]

while, assuming that \( p_n > 0 \), then

\[
P_{l}^{[N]} = (H_{l}^{[N]})^{-1} (T^{[N]})^{\top} H_{l}^{[N]}, \quad H_{l}^{[N]} = \text{diag}(1, H_{l,1}^{[N]}, H_{l,2}^{[N]}, \ldots, H_{l,N}^{[N]}), \quad H_{l,n}^{[N]} = \frac{1}{p_1p_2 \cdots p_n}.
\]

**Proposition 26.** Let us assume that the tetradiagonal stochastic matrices \( P_{lH}^{[N]}, P_{l}^{[N]} \) are oscillatory. Then, there is an oscillatory monic Hessenberg matrix \( T^{[N]} \) such that \( P_{lH}^{[N]} = H_{lH}^{[N]} T^{[N]} (H_{lH}^{[N]})^{-1}, P_{l}^{[N]} = (H_{l}^{[N]})^{-1} (T^{[N]})^{\top} H_{l}^{[N]} \), for some unique matrices \( H_{lH}^{[N]} = \text{diag}(1, H_{lH,1}^{[N]}, H_{lH,2}^{[N]}, \ldots, H_{lH,N}^{[N]}) \) and \( H_{l}^{[N]} = \text{diag}(1, H_{l,1}^{[N]}, H_{l,2}^{[N]}, \ldots, H_{l,N}^{[N]}) \), with diagonal entries greater than zero.
Proof. According to the Gantmacher–Krein Criterion, Theorem 1, [17, 18, 11] the stochastic matrices are InTN matrices and \( q_1 \cdots q_N > 0 \) for the type II case and that \( r_0 \cdots r_{N-1} > 0 \) for the type I situation. Then, the matrix \( T^{[N]} \) is the product of TN nonsingular matrices and satisfies the Gantmacher–Krein Criterion again. Consequently, its is oscillatory.

According to Perron–Frobenius theorem [34, 13, 14, 15] the largest eigenvalue of a stochastic matrix is 1 and the corresponding right eigenvector can be taken as

\[
1^{[N]} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^{N+1}.
\]

Hence, given that the similarity transformation preserves the eigenvalues, the largest eigenvalue \( \lambda_1^{[N]} \) of \( T^{[N]} \) is \( \lambda_1^{[N]} = 1 \).

When the monic banded Hessenberg matrix \( T^{[N]} \) is oscillatory we know that the first right and left eigenvectors

\[
u_1 = \begin{bmatrix} 1 \\ B_1(1) \\ \vdots \\ B_N(1) \end{bmatrix}, \quad \Omega_1 = \begin{bmatrix} 1 & \Omega_{1,2} & \cdots & \Omega_{1,N+1} \end{bmatrix}, \quad \Omega_{1,n} = \frac{B_{n+1}^{[n]}(1)}{B_1^{[1]}(1)} = A_n^{(1)}(1) + A_n^{(2)}(1) \frac{\mu_{1,1}^{[N]}}{\mu_{1,1}^{[N]}}
\]

are totally positive vectors, respectively.

As \( P_{II}^{[N]} 1^{[N]} = 1^{[N]} \) we find that \( T^{[N]} H_{II}^{-1} 1^{[N]} = H_{II}^{-1} 1^{[N]} \), so that \( H_{II}^{-1} 1^{[N]} \) and \( u_1 \) are proportional. Consequently, as the first entry of both vectors is normalized to 1, we deduce that \( H_{II,n}^{[N]} = \frac{1}{B_n(1)} \), so that

\[B_{n+1}(1) = s_0 s_1 \cdots s_n, \quad s_n = \frac{B_{n+1}(1)}{B_n(1)}.
\]

Similarly, \( P_{I}^{[N]} 1^{[N]} = 1^{[N]} \) we find that \( (T^{[N]})^\top H_I 1^{[N]} = H_I 1^{[N]} \), so that \( H_I 1^{[N]} \) and \( \Omega_1^\top \) are proportional. Thus, as the first entry of both vectors is normalized again to 1, we are lead to

\[H_{I,n}^{[N]} = \frac{B_{n+1}^{[n+1]}(1)}{B_{N+1}^{[N]}(1)} = A_n^{(1)}(1) + A_n^{(2)}(1) \frac{\mu_{1,2}^{[N]}}{\mu_{1,1}^{[N]}}
\]

Hence, \( \frac{B_{n+1}^{[n+1]}(1)}{B_{N+1}^{[N]}(1)} = A_n^{(1)}(1) + A_n^{(2)}(1) \frac{\mu_{1,2}^{[N]}}{\mu_{1,1}^{[N]}} = \frac{1}{p_1 \cdots p_n} \). In particular,

\[p_n = \frac{B_{n+1}^{[n]}(1)}{B_{N+1}^{[N]}(1)} = \frac{A_n^{(1)}(1) \mu_{1,1}^{[N]} + A_n^{(2)}(1) \mu_{1,2}^{[N]}}{A_n^{(1)}(1) \mu_{1,1}^{[N]} + A_n^{(2)}(1) \mu_{1,2}^{[N]}}.\]
We introduce the notation
\[
\Pi_{i,j}^{[N]} = \frac{H_{i,j}^{[N]}}{H_{i,i}^{[N]}} = \frac{B_i(1)}{B_k(1)} = \begin{cases} 
\frac{1}{s_l \cdots s_{k-1}}, & l < k, \\
1, & l = k, \\
\frac{1}{s_k \cdots s_{l-1}}, & l > k,
\end{cases}
\]

\[
\Pi_{i,j}^{[N]} = \frac{H_{i,j}^{[N]}}{H_{i,i}^{[N]}} = \frac{B_{i+1}(1)}{B_{k+1}(1)} = \frac{A_l^{(1)}(1) \mu_1^{[N]} + A_l^{(2)}(1) \mu_2^{[N]}}{A_k^{(1)}(1) \mu_1^{[N]} + A_k^{(2)}(1) \mu_2^{[N]}} = \begin{cases} 
\frac{p_{l+1} \cdots p_k}{p_{k+1} \cdots p_l}, & l < k, \\
1, & l = k, \\
\frac{1}{p_{k+1} \cdots p_l}, & l > k,
\end{cases}
\]

We now present three results regarding the spectral representation of important probability elements of the finite random walks involved.

**Proposition 27** (Karlin–McGregor representation formula). The iterated probabilities have the following spectral representation
\[
\left( (P_{i,i}^{[N]})^n \right)_{k,l} = \Pi_{i,i}^{[N]} \left( A_l^{(1)} \mu_1^{[N]} + A_l^{(2)} \mu_2^{[N]} , x^n B_k \right), \quad \left( (P_{i,j}^{[N]})^n \right)_{k,l} = \Pi_{i,j}^{[N]} \left( A_k^{(1)} \mu_1^{[N]} + A_k^{(2)} \mu_2^{[N]} , x^n B_l \right).
\]

**Proposition 28** (Spectral representation of generating functions). For \(|s| < 1\), the corresponding transition probability generating functions are
\[
(P_{i,i}^{[N]}(s))_{k,l} = \Pi_{i,i}^{[N]} \left( A_l^{(1)} \mu_1^{[N]} + A_l^{(2)} \mu_2^{[N]} , \frac{B_k(x)}{1-sx} \right), \quad (P_{i,j}^{[N]}(s))_{k,l} = \Pi_{i,j}^{[N]} \left( A_k^{(1)} \mu_1^{[N]} + A_k^{(2)} \mu_2^{[N]} , \frac{B_l(x)}{1-sx} \right).
\]

For \(k \neq l\), the first passage generating functions are
\[
(F_{i,i}^{[N]}(s))_{k,l} = \Pi_{i,i}^{[N]} \left( A_l^{(1)} \mu_1^{[N]} + A_l^{(2)} \mu_2^{[N]} , \frac{B_k(x)}{1-sx} \right), \quad (F_{i,j}^{[N]}(s))_{k,l} = \Pi_{i,j}^{[N]} \left( A_k^{(1)} \mu_1^{[N]} + A_k^{(2)} \mu_2^{[N]} , \frac{B_l(x)}{1-sx} \right).
\]

For \(k = l\) the first passage generating functions are the same for type I and II, namely
\[
F_{i,i}^{[N]}(s) = 1 - \frac{1}{A_l^{(1)} \mu_1^{[N]} + A_l^{(2)} \mu_2^{[N]} , \frac{B_l(x)}{1-sx}}.
\]

Let us show that the corresponding finite Markov chain is recurrent.

**Theorem 24** (Recurrent process). Any finite Markov chain with a transition matrix being an oscillatory Hessenberg (transpose Hessenberg) with two possibly non zero subdiagonals (superdiagonals) is recurrent.

**Proof.** Notice that
\[
\lim_{s \to 1^-} F_{i,i}(s) = 1,
\]
and all the states are recurrent. This is a consequence of the presence of a mass at 1 in the linear form \(A_l^{(1)} \mu_1^{[N]} + A_l^{(2)} \mu_2^{[N]} A_l^{(1)}\), then the contribution at 1, which has all its factors strictly positive numbers, will have a term proportional to \(\frac{1}{s} \to \infty\) as \(s \to 1^-\) in the denominator of the second term in the right hand side and, consequently, \(F_{i,i} \to 1\).

Both Markov chains have the same stationary state:
Theorem 25 (Stationary states). The stationary distribution $\pi^{[N]} = \begin{bmatrix} \pi_1^{[N]} & \ldots & \pi_{N+1}^{[N]} \end{bmatrix}$, of both stochastic matrices $P_{II}^{[N]}$ and $P_I^{[N]}$ has the following components

$$
\pi_n^{[N]} = u_1^{[N]} w_{1,n}^{[N]} = \left( A_{n-1}^{(1)}(1) \mu_{1,1}^{[N]} + A_{n-1}^{(2)}(1) \mu_{1,2}^{[N]} \right) B_{n-1}(1) = \frac{B_{n+1}^{[N]}(1) B_{n-1}(1)}{B_{N+1}^{[N]}(1)}.
$$

Proof. We peek the right and left eigenvectors, $u_1$ and $w_1$, respectively, normalized by $w_1 u_1 = 1$. If we take

$$
\pi = u_1^T \text{diag}(w_{1,1}, \ldots, w_{1,N+1}) = w_1 \text{diag}(u_{1,1}, \ldots, u_{1,N+1}).
$$

we deduce that

$$
\pi^{[N]} P_{II}^{[N]} = w_1^{[N]} \text{diag}(u_{1,1}, \ldots, u_{1,N+1}) \text{diag}(u_{1,1}, \ldots, u_{1,N+1})^{-1} T^{[N]} \text{diag}(u_{1,1}, \ldots, u_{1,N+1}) = \pi^{[N]},
$$

$$
\pi^{[N]} P_I^{[N]} = (u_1^{[N]})^T \text{diag}(w_{1,1}, \ldots, w_{1,N+1}) \text{diag}(w_{1,1}, \ldots, w_{1,N+1})^{-1} (T^{[N]})^T \text{diag}(w_{1,1}, \ldots, w_{1,N+1}) = \pi^{[N]}.
$$

Notice that $\pi_n^{[N]} > 0$, $n \in \{1, \ldots, N + 1\}$ and that $\pi^{[N]} 1^{[N]} = \sum_{n=1}^{N+1} \pi_n^{[N]} = \sum_{n=1}^{N+1} u_1 w_{1,n} = 1$.

7.2. Countable infinite Markov chains. We now discuss the case of a countable random walk with an infinite countable set of states. In previous papers [5, 3] we discuss the construction of random walks beyond birth and death Markov chains, and give as explicit examples the well known Jacobi–Piñeiro multiple orthogonal polynomials, see [46], and the recently found hypergeometric multiple orthogonal polynomials [29]. For both mentioned examples the recursion matrix $T$ is an oscillatory banded Hessenberg matrix.

The basic idea of [5] is to construct a random walk beyond birth and death for a given sequence of multiple orthogonal polynomials in the step line. The recursion matrix of these multiple orthogonal polynomials happens to be a banded Hessenberg matrix, for which we find a suitable similarity and scaling to get two stochastic matrices. In those papers the spectral question was not touched. This is an important issue as now the stochastic matrices are essentially non normal operators and are beyond the well known spectral theory for symmetric or normal operators. We address this issue now.

We have two oscillatory stochastic matrices $P_{II}$ and $P_I$ describing countable infinite Markov chains. There are similarity transformations so that (90) and (91) hold, i.e. both Markov chains are described by some oscillatory monic banded Hessenberg matrix $T$. Let us call this matrix the associated monic matrix. The leading principal submatrices $T^{[N]}$ of $T$, as described in the previous subsection, lead to stochastic matrices $P_{II}^{[N]}$ and $P_I^{[N]}$, which are not leading principal submatrices of $P_{II}$ and $P_I$, but recover them in the large $N$-limit, $P_{II}^{[N]} \xrightarrow{N \to \infty} P_{II}$ and $P_I^{[N]} \xrightarrow{N \to \infty} P_I$.

Moreover, for the diagonal matrices we have $H_{II}^{[N]} \xrightarrow{N \to \infty} H_{II}$ and $H_I^{[N]} \xrightarrow{N \to \infty} H_I$. It is clear that $H_{II,n} = B_{n-1}(1)$; however, the type I case is more subtle. If we assume that $d \psi_2 \ll d \psi_1$, i.e., $d \psi_2$ is absolutely continuous with respect $d \psi_1$ ($d \psi_2(A) = 0$ whenever $d \psi_1(A) = 0$); then, the Radon–Nikodym derivative is a non-negative function $F = \frac{d \psi_2}{d \psi_1}$ such that $H_{II,n} = A_n^{(1)}(1) + F(1) A_n^{(2)}(1) = \frac{1}{p_1 \ldots p_n} > 0$.

\footnote{See [20] for further developments regarding Jacobi–Piñeiro random walks.}
For this large $N$-limit we obtain

$$\Pi_{II,k,l} := \frac{H_{II,k}}{H_{II,l}} = \frac{B_l(1)}{B_k(1)} = \begin{cases} 
\frac{1}{s_l \cdots s_{k-1}}, & l < k, \\
1, & l = k, \\
\frac{1}{s_k \cdots s_{l-1}}, & l > k,
\end{cases}$$

$$\Pi_{I,l,k} := \frac{H_{I,l}}{H_{I,k}} = \frac{A_l^{(1)}(1) + A_l^{(2)}(1)F}{A_k^{(1)}(1) + A_k^{(2)}(1)F} = \begin{cases} 
\frac{1}{p_{l+1} \cdots p_k}, & l < k, \\
1, & l = k, \\
p_{k+1} \cdots p_l, & l > k,
\end{cases}$$

As we did with the finite case we proceed to state the main results regarding countable Markov chains with tetradiagonal transition matrices being oscillatory Hessenberg matrices (or its transpose).

**Theorem 26** (Spectral representation). *Let us consider a countable Markov chain with transition matrix an oscillatory tetradiagonal matrix such that the associated monic Hessenberg matrix is as in Theorem 20 or 21. Then, there is a sequence of multiple orthogonal polynomials of type II, $\{B_n\}_{n=0}^\infty$, and of type I, $\{A_n^{(1)}, A_2^{(n)}\}_{n=0}^\infty$, associated with positive Lebesgue–Stieltjes measures $d \psi_1$ and $d \psi_2$ such that:

i) **Karlin–McGregor representation formula:** The iterated probabilities have the following spectral representation

$$\left( (P_{II})^n \right)_{k,l} = \Pi_{II,k,l} \int_0^1 (A_l^{(1)}(1) d \psi_1(x) + A_l^{(2)}(1) d \psi_2(x)) x^n B_k(x),$$

$$\left( (P_{II})^n \right)_{k,l} = \Pi_{I,l,k} \int_0^1 (A_k^{(1)}(1) d \psi_1(x) + A_k^{(2)}(1) d \psi_2(x)) x^n B_l(x).$$

ii) **Spectral representation of generating functions:** For $|s| < 1$, the corresponding transition probability generating functions are

$$(P_{II}(s))_{k,l} = \Pi_{II,k,l} \int_0^1 (A_l^{(1)}(1) d \psi_1(x) + A_l^{(2)}(1) d \psi_2(x)) \frac{B_k(x)}{1 - sx},$$

$$(P_I(s))_{k,l} = \Pi_{I,l,k} \int_0^1 (A_k^{(1)}(1) d \psi_1(x) + A_k^{(2)}(1) d \psi_2(x)) \frac{B_l(x)}{1 - sx}.$$

For $k \neq l$, the first passage generating functions are

$$(F_{II}(s))_{k,l} = \Pi_{II,k,l} \int_0^1 (A_l^{(1)}(1) d \psi_1(x) + A_l^{(2)}(1) d \psi_2(x)) \frac{B_k(x)}{1 - sx},$$

$$(F_I(s))_{k,l} = \Pi_{I,l,k} \int_0^1 (A_k^{(1)}(1) d \psi_1(x) + A_k^{(2)}(1) d \psi_2(x)) \frac{B_l(x)}{1 - sx}.$$  

For $k = l$ the first passage generating functions are the same for type I and II, namely

$$F_{II}^N(s) = 1 - \int_0^1 (A_l^{(1)}(1) d \psi_1(x) + A_l^{(2)}(1) d \psi_2(x)) \frac{B_l(x)}{1 - sx}.$$
iii) **Recurrent random walks:** The Markov chain is recurrent if and only if the integral

$$\int_0^1 \frac{d\psi_1(x)}{1-x}$$

diverges. Otherwise is transient.

Regarding stationary states we find:

**Theorem 27** (Ergodic random walks). *The Markov chain described in previous Theorem 26 is ergodic (or positive recurrent) if and only 1 is a mass point of $d\psi_1$ and $d\psi_2$ with masses $m_1 > 0$ and $m_2 \geq 0$, respectively.* The corresponding stationary distribution is

$$\pi = [\pi_1 \pi_2 \ldots], \quad \pi_{n+1} = (A_n^{(1)}(1)m_1 + A_n^{(2)}(1)m_2)B_n(1).$$

**Proof.** Following [27], see also [16], for a recurrent chain, the expected first passage times are all finite if $0 < \lim_{n \to \infty} P_{00}^{2n} < +\infty$, that is

$$\int_0^1 d\psi_1(x)x^{2n} < +\infty.$$  

The argument goes as in [27]. Since $x^{2n} \to 0$ monotonically in $0 < x < 1$ the chain is ergodic if and only if $\psi_1$ has a jump at $x = 1$, and [27] shows that the jump occurs at $x = 1$. Therefore, from Theorem 25 we see that

$$\lim_{N \to \infty} \pi_{n+1}^{[N]} = \left(A_n^{(1)}(1) \lim_{N \to \infty} \mu_{1,1}^{[N]} + A_n^{(2)}(1) \lim_{N \to \infty} \mu_{1,2}^{[N]}\right)B_n(1).$$

and the large $N$-limits of the Christoffel coefficients $\lim_{N \to \infty} \mu_{1,1}^{[N]} = m_1$ and $\lim_{N \to \infty} \mu_{1,2}^{[N]} = m_2$ exists due to fact that 1 is a mass point.

As we have a mass at 1, we have a simple pole at 1 in the resolvent; i.e. 1 is an eigenvalue. Hence,

$$\sum_{n=1}^{\infty} \pi_n^2 < \infty.$$  

**Rem. 13.** Using Theorem 27 and results from [5, 3] we conclude that for the Jacobi–Piñeiro, in the semi-band $0 < \alpha - \beta < 1$, and for the hypergeometric case, as there are no masses, the random walks when recurrent are null recurrent, they are not ergodic, and the return time is infinity.

**Rem. 14.** Krein–Rutman [28] seminal theorem on the existence of a positive eigenvector with eigenvalue given by the spectral radius for a compact positive operator, extended previous results in finite dimensions of Perron [34] and Frobenius [13, 14, 15]. Recall that our banded operators are compact whenever $p_n, q_n, r_n, s_n \to 0$, see [44]. This is clearly incompatible with the stochastic requirement $p_n + q_n + r_n + s_n = 1$. Hence, the Krein–Rutman theorem is not applicable for the case under consideration, oscillatory banded Hessenberg matrices.

In [26] the results of Krein and Rutman where extended to bounded positive operators in the quasi compact class, i.e., those operators $T$ such that for some integer $k$ we have $\|T^k - K\| < 1$ for some compact operator $K$.

Hence, if the bounded positive matrix is quasi compact, then 1 is an eigenvalue with totally positive right and left eigenvectors in $\mathbb{C}^2$. Following [33] we know that a strongly positive matrix, a positive matrix $A$ taking totally positive vectors into totally positive vectors, with spectral radius (1 in our case) strictly bigger than the essential spectral radius (the supreme of those $\lambda$ such that $\lambda I - A$ being not a Fredholm operator),

---

9This is equivalent to the existence of a sequence $K_n$ of compact operators such that $\lim_{n \to \infty} \|T_n - K_n\| = 0$. 

then 1 is a simple eigenvalue with a totally positive eigenvector and all the other eigenvalues absolute values less than unity. The existence of these spectral gaps is equivalent to quasi compactness.

For the Jacobi–Piñeiro and hypergeometric cases mentioned earlier, we know that [5, 3] the stochastic matrices of type II (a similar discussion holds for type I) can be written as \( P = \Theta + K \) with

\[
\Theta := \frac{1}{27} \begin{pmatrix}
12 & 8 & 0 & 0 & 0 & \cdots \\
6 & 12 & 8 & 0 & 0 & 0 \\
1 & 6 & 12 & 8 & 0 \\
0 & 1 & 6 & 12 & 8 & \cdots
\end{pmatrix}, \quad K := \begin{pmatrix}
\delta r_0 & \delta s_0 & 0 & 0 & 0 & \cdots \\
\delta q_1 & \delta r_1 & \delta s_1 & 0 & 0 & 0 \\
\delta p_2 & \delta q_2 & \delta r_2 & \delta s_2 & 0 & \cdots \\
0 & \delta p_3 & \delta q_3 & \delta r_3 & \delta s_3 \\
& & & \ddots & & \ddots
\end{pmatrix},
\]

with \( K \) a compact operator, i.e., \( \delta p_n, \delta q_n, \delta r_n, \delta s_n \xrightarrow{n \to \infty} 0 \), and \( \Theta \) a Toeplitz matrix with \( \|\Theta^n\| = 1 \) for any \( n \in \mathbb{N} \) and the previous extension to quasi-compact operators can not be applied.

**Remark 15.**

i) For the positive recurrent case the operator \( T \) has 1 as an eigenvalue and a corresponding totally positive right eigenvector

\[
u_1 = \begin{bmatrix}
1 \\
B_1(1) \\
B_2(1) \\
\vdots
\end{bmatrix}, \quad Tu_1 = u_1,
\]

with finite norm \( \|u_1\|^2 = \sum_{n=0}^{\infty} (B_n(1))^2 < \infty \), and totally positive left eigenvector

\[
w_1 = \begin{bmatrix}
m_1 \\
v m_1 + m_2 \\
A_2^{(1)}(1)m_1 + A_2^{(2)}(1)m_2 \\
A_3^{(1)}(1)m_1 + A_3^{(2)}(1)m_2 \\
\vdots
\end{bmatrix}, \quad w_1 T = w_1,
\]

with \( \|w_1\|^2 = \sum_{n=0}^{\infty} (A_n^{(1)}(1)m_1 + A_n^{(2)}(1)m_2)^2 < \infty \).

ii) Notice that \( \pi_n > 0, n \in \mathbb{N} \) and that \( \lim_{N \to \infty} \sum_{n=1}^{N+1} \pi_n = 1 \) (proof: it holds for each \( N \) and the large \( N \)-limit exists). Observe also that for the masses we have

\[
m_1 = \lim_{N \to \infty} \frac{B_{N+1}^{(1)}}{B_{N+1}^{(1)'}}, \quad m_2 = \lim_{N \to \infty} \frac{B_{N+1}^{(2)}}{B_{N+1}^{(2)'}}.
\]

Being the banded Hessenberg matrix oscillatory we know that it has the \( LU \) factorization \( T = L_1L_2U \), with oscillatory bidiagonal factors given in (66). This simple \( LU \) factorization induces, through (92) and (93) a corresponding \( LU \) factorization with stochastic factors of the stochastic matrices \( P_{II} \) and \( P_I \), i.e. stochastic Gauss–Borel factorizations. The following theorem is a translation to this more general scenario of oscillatory matrices, of a similar result in [3, Theorem 11], we do not include the proof as it runs as that presented in [3, Theorem 11].

To end the paper we give the following result regarding the stochastic bidiagonal factorization in terms of simple pure death and pure birth stochastic matrices.\(^{10,11}\)

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\(^{10}\)This stochastic factorization appeared in [19] in where an urn model was proposed for the Jacobi random walks and in [20] for the Jacobi–Piñeiro situation, see [3]. In those papers a factorization \( P_{II} = P_LP_U \) with \( P_U \) being a stochastic upper triangular matrix with only the first superdiagonal nonzero, i.e. an stochastic matrix describing a pure birth Markov chain, and a matrix \( P_L \) a stochastic lower triangular matrix, with zero as an absorbing state and only the two first subdiagonals nonzero.

\(^{11}\)See [3] for the hypergeometric case.
**Theorem 28** (Stochastic bidiagonal factorization). i) The stochastic matrix $P_{II}$ has the following stochastic bidiagonal factorization

$$P_{II} = P_{II,1}^L P_{II,2}^L P_{II}^U$$

in terms of the stochastic matrices

$$P_{II,1}^L := H_{II} L_1 D_{II,2}^{-1}, \quad P_{II,2}^L := D_{II,2} L_2 D_{II,1}^{-1}, \quad P_{II}^U := D_{II,1} U H_{II}^{-1},$$

where

$$D_{II,i} = \text{diag} \left( \frac{1}{d_{II,i}^{(0)}}, \frac{1}{d_{II,i}^{(1)}}, \ldots \right), \quad i \in \{1, 2\},$$

with

$$d_{II,1}^{(n)} = \alpha_{3n+2} B_n(1) + B_{n+1}(1), \quad n \in \mathbb{N}_0$$

$$d_{II,2}^{(n)} = \alpha_{3n+1} \alpha_{3n-1} B_{n-1}(1) + (\alpha_{3n+1} + \alpha_{3n+2}) B_n(1) + B_{n+1}(1), \quad n \in \mathbb{N},$$

and $d_{II,1}^{(0)} = d_{II,2}^{(0)}$.

ii) The stochastic matrix $P_I$ has the following stochastic bidiagonal factorization

$$P_I = P_I^L P_I^U P_I^U$$

in terms of stochastic matrices

$$P_I^L := H_I^{-1} U^T D_{I,2}^{-1}, \quad P_I^U := D_{I,2} L_2^T D_{I,1}^{-1}, \quad P_I^U := D_{I,1} L_1^T H_I,$$

where

$$D_{I,i} = \text{diag} \left( \frac{1}{d_{I,i}^{(0)}}, \frac{1}{d_{I,i}^{(1)}}, \ldots \right), \quad i \in \{1, 2\},$$

with

$$d_{I,1}^{(n)} = \frac{1}{H_{I,n+1}} + \frac{\alpha_{3n+3}}{H_{I,n+2}}, \quad d_{I,2}^{(n)} = \frac{1}{H_{I,n+1}} + \frac{\alpha_{3n+3} + \alpha_{3n+4}}{H_{I,n+2}} + \frac{\alpha_{3n+4} \alpha_{3n+6}}{H_{I,n+3}},$$

for $n \in \mathbb{N}_0$.

**Remark 16.** The stochastic factorization provided here for the oscillatory situation in three simple stochastic oscillatory factors is in terms of a pure birth factor and two pure death factors for the type II, and in terms of one pure death factor and two pure birth factors for the type I case. The construction of the corresponding urn models, once the stochastic factorization is provided, will be given by an appropriate choice of three urns, one urn per factor, with three different experiments.

**Conclusions and outlook**

The main result we achieve in this paper is that banded Hessenberg matrices that are oscillatory have, generically, a set positive measures, and can be spectrally described by multiple orthogonal polynomials. This extends to the non-normal scenario the spectral Favard theorem for Jacobi matrices, see [39], that we show that for an adequate shift becomes oscillatory. We have given examples of this construction, the Toeplitz case leading to 2-orthogonal Chebyshev polynomials, and the hypergeometric recursion matrix, described in [29], also fits in our theory. We have applied these results to Markov chains beyond birth and death with tetradiagonal stochastic matrices. The Jacobi–Piñeiro multiple orthogonal polynomials illustrates the fact that this theory do not exhaust
all possibilities, as the weights exist beyond the oscillatory situation. We suggest that maybe a more refined requirement is oscillation for a retracted complementary matrix. Other open questions are:

i) What happens when the banded recursion matrix has several superdiagonals as well as subdiagonals? What about the corresponding random walks?

ii) Chebyshev (T) systems appear in [18] in relation with influence kernels and oscillatory matrices. Is there any connection between the AT property and the oscillation of the matrix or some submatrix of it?

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