ON A FACTORIZATION OF RIEMANN’S $\zeta$ FUNCTION WITH RESPECT TO A QUADRATIC FIELD AND ITS COMPUTATION

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Abstract. Let $K$ be a quadratic field, and let $\zeta_K$ its Dedekind zeta function. In this paper we introduce a factorization of $\zeta_K$ into two functions, $L_1$ and $L_2$, defined as partial Euler products of $\zeta_K$, which lead to a factorization of Riemann’s $\zeta$ function into two functions, $p_1$ and $p_2$. We prove that these functions satisfy a functional equation which has a unique solution, and we give series of very fast convergence to them. Moreover, when $\Delta_K > 0$ the general term of these series at even positive integers is calculated explicitly in terms of generalized Bernoulli numbers.

1. Introduction

Let $K$ be a quadratic field and let $\chi$ be the Dirichlet character attached to $K/\mathbb{Q}$. Its Dedekind’s zeta function can be written as

$$\zeta_K(s) = \zeta(s)L(s, \chi),$$

where $\zeta$ is Riemann’s zeta function and $L$ is the $L$-function associated with $\chi$ (see, for example, [2]). Hence, an alternative factorization, for $\Re(s) > 1$, is the one given by the partial products

$$\zeta_K(s) = \prod_{p|d}(1 - p^{-s})^{-1}L_1(s)L_2(s),$$

where $d = |\Delta_K|$ is the absolute value of the discriminant of $K$, and

$$L_1(s) = \prod_{\chi(p) = 1} (1 - p^{-s})^{-2}, \quad L_2(s) = \prod_{\chi(p) = -1} (1 - p^{-2s})^{-1}.$$  

Note that $L_1$ and $L_2$ are obtained as partial Euler products of $\zeta(s)^2$ and $\zeta(2s)$ respectively, so they converge and are non-zero for $\Re(s) > 1$ and $\Re(s) > 1/2$ respectively.

Define now

$$(1) \quad p_1(s) = \prod_{\chi(p) = 1} (1 - p^{-s})^{-1} \quad \text{and} \quad p_2(s) = \prod_{\chi(p) = -1} (1 - p^{-s})^{-1}. $$

Then, we have that

$$L_1(s) = p_1(s)^2, \quad L_2(s) = p_2(2s),$$

Key words and phrases. Riemann’s $\zeta$ function, factorization, functional equation, quadratic field.
and thus it is equivalent to study $L_1$ and $L_2$ or $p_1$ and $p_2$. Note that
\[
\zeta(s) = \prod_{p|d}(1 - p^{-s})^{-1}p_1(s)p_2(s),
\]
and hence, $p_1$ and $p_2$ give a factorization of Riemann’s zeta function.

The plan of the paper is as follows. In section 2 we see that $p_1$ and $p_2$ satisfy a functional equation. More precisely, we prove

**Theorem 1.** The functions $p_1$ and $p_2$ satisfy the functional equations

\[
\frac{p_i(2s)}{p_i(s)^2} = q_i(s), \quad \lim_{\Re(s) \to +\infty} p_i(s) = 1, \quad \text{for} \quad i = 1, 2,
\]

where

\[
q_1(s) = \frac{\zeta(2s)}{\zeta(s)L(s, \chi)} \prod_{p|d}(1 + p^{-s}), \quad q_2(s) = \frac{L(s, \chi)}{\zeta(s)} \prod_{p|d}(1 - p^{-s})^{-1}.
\]

Furthermore, these functional equations have a unique solution, so they completely determine the functions $p_1$ and $p_2$.

Moreover, we shall see that the logarithm of the solution of this functional equation can be written as a series

\[
\log p_i(s) = -\sum_{n=0}^{+\infty} \frac{\log q_i(2^n s)}{2^n + 1}, \quad i = 1, 2,
\]

and hence, we will have an alternative expression of $p_1$ and $p_2$.

In section 3 we will see that the series given by (4) are of very fast convergence. We shall prove

**Theorem 2.** Let $s$ be a complex number such that $\Re(s) \geq 1$. Then,

\[
p_1(2s) = \exp \left\{ -\sum_{k=1}^{n} \frac{1}{2^k} \log q_1(2^k s) \right\} + o\left(2^{-2^n}\right),
\]

and

\[
p_2(2s) = \exp \left\{ -\sum_{k=1}^{n} \frac{1}{2^k} \log q_2(2^k s) \right\} + o\left(2^{-2^n}\right).
\]

As a consequence, we will have a way to evaluate $p_1$ and $p_2$ at even positive integers when $\Delta_K$ is positive. This will be done by calculating explicitly the general term of the series in this case.

2. The functional equation of $p_1$ and $p_2$

First we prove that the functional equation appearing in Theorem 1 has a unique solution and that this solution can be written as an infinite series. The statement of the result is the following.

**Proposition 3.** Let $\Omega = \{ s \in \mathbb{C} | \Re(s) > 1 \}$, and $q$ an holomorphic function defined in $\Omega$, with $q(s) \neq 0$ for all $s \in \Omega$ and $\lim_{\Re(s) \to +\infty} q(s) = 1$. Then, the functional equation

\[
\frac{p(2s)}{p(s)^2} = q(s), \quad \lim_{\Re(s) \to +\infty} p(s) = 1
\]
has a unique solution \( p(s) \). In addition, the solution can be written as

\[
p(s) = \exp \left\{ - \sum_{n \geq 0} \frac{\log q(2^n s)}{2^{n+1}} \right\},
\]

and this series is absolutely convergent for all \( s \) in \( \Omega \).

Proof. Suppose that \( p(s) \) satisfies the functional equation. Then, \( p(s) \neq 0 \) for all \( s \in \Omega \). This is because \( p(s) = 0 \) implies \( p(2s) = 0 \) and \( p(2^k s) = 0 \) for \( k = 1, 2, \ldots \), which contradicts the hypothesis \( \lim_{\Re(s) \to +\infty} p(s) = 1 \). Thus, we can define

\[
f(s) = \frac{\log p(s)}{s}, \quad g(s) = \frac{\log q(s)}{2s},
\]

where \( \log \) is the principal branch of the complex logarithm. Taking logarithms to our functional equation and dividing by \( 2s \), we have that

\[
f(2s) = f(s) + g(s), \quad \lim_{\Re(s) \to +\infty} f(s) = 0.
\]

Writing this last equation for \( s, 2s, 4s, 8s, \ldots, 2^N s \), and adding them, we obtain that

\[
f(2^{N+1}s) = f(s) + \sum_{n=0}^{N} g(2^n s).
\]

Since \( \Re(s) > 1 \), then \( \Re(2^{N+1}s) \to +\infty \) when \( N \to \infty \), so

\[
f(s) + \sum_{n=0}^{\infty} g(2^n s) = \lim_{N \to \infty} f(2^{N+1}s) = 0,
\]

and

\[
\log p(s) = - \sum_{n \geq 0} \frac{\log q(2^n s)}{2^{n+1}}.
\]

Since

\[
\lim_{\Re(s) \to +\infty} \log q(s) = 0,
\]

the sequence \( \{\log q(2^n s)\}_{n \in \mathbb{N}} \) converges (it tends to 0), and in particular it is bounded. Hence, there exists \( M > 0 \) such that \( |\log q(2^n s)| < M \), and then

\[
\sum_{n \geq 0} \left| \frac{\log q(2^n s)}{2^{n+1}} \right| \leq \sum_{n \geq 0} \frac{M}{2^{n+1}} = M,
\]

so the series is absolutely convergent for all \( s \in \Omega \).

Let us see that this function satisfies the functional equation. We have that

\[
\log p(2s) - 2 \log p(s) = - \sum_{n \geq 0} \frac{\log q(2^{n+1} s)}{2^{n+1}} + 2 \sum_{n \geq 0} \frac{\log q(2^n s)}{2^{n+1}}
\]

\[
= - \sum_{n \geq 1} \frac{\log q(2^n s)}{2^n} + \sum_{n \geq 0} \frac{\log q(2^n s)}{2^n}
\]

\[
= \log q(s),
\]

and then,

\[
\frac{p(2s)}{p(s)^2} = q(s).
\]
We now have to see that \( \lim_{\Re(s)\to+\infty} \log p(s) = 0 \).

For it, fix \( \varepsilon > 0 \). Since \( \lim_{\Re(s)\to+\infty} \log q(s) = 0 \), and exists \( \sigma > 0 \) such that

\[
|\log q(s)| < \varepsilon \quad \text{for all } s \text{ with } \Re(s) \geq \sigma.
\]

Hence, if \( \Re(s) \geq \sigma \), then

\[
|\log p(s)| \leq \sum_{n \geq 0} \left| \log q(2^n s) \right| \leq \sum_{n \geq 0} \frac{2^n \varepsilon}{2n+1} = \varepsilon,
\]

and \( \lim_{\Re(s)\to+\infty} \log p(s) = 0 \), as claimed.

Note that, in fact, the branch of the logarithm is irrelevant, since when we take exponentials, we will have

\[
p(s) = \exp \left\{ - \sum_{n \geq 0} \log q(2^n s) \right\},
\]

independently of the chosen branch. \( \square \)

We can now give the:

Proof of Theorem 1. On the one hand, it is clear that \( \lim_{\Re(s)\to+\infty} p_i(s) = 1 \), \( i = 1, 2 \).

On the other hand, we have that

\[
p_1(s)p_2(2s) = \prod_{\chi(p)=1} (1-p^{-s})^{-1} \prod_{\chi(p)=1} \left( \frac{1-p^{-2s}}{1-p^{-s}} \right)^{-1}
\]

\[
= \prod_{\chi(p)=1} (1-p^{-s})^{-1} \prod_{\chi(p)=1} \left( \frac{1-p^{-2s}}{1-p^{-s}} \right)^{-1}
\]

\[
= \prod_{\chi(p)=1} (1-p^{-s})^{-1} \prod_{\chi(p)=1} \left( 1+p^{-s} \right)^{-1}
\]

\[
= L(s, \chi),
\]

and since

\[
p_1(s) = \frac{1}{p_2(s)} \zeta(s) \prod_{p \mid d} (1-p^{-s}),
\]

then

\[
\frac{p_2(2s)}{p_2(s)^2} = \frac{L(s, \chi)}{\zeta(s)} \prod_{p \mid d} (1-p^{-s})^{-1}.
\]

Using now that

\[
p_2(s) = \frac{1}{p_1(s)} \zeta(s) \prod_{p \mid d} (1-p^{-s}),
\]

we obtain

\[
\frac{p_1(2s)}{p_1(s)^2} = \frac{p_2(s)^2}{p_2(2s)} \cdot \frac{\zeta(2s) \prod_{p \mid d} (1-p^{-2s})}{\zeta(s) \prod_{p \mid d} (1-p^{-s})^2} = \frac{\zeta(2s)}{\zeta(s) L(s, \chi)} \prod_{p \mid d} (1+p^{-s}).
\]
The fact that these functional equations have an unique solution follows from Proposition 3. □

As a consequence of Proposition 3 and Theorem 1, we obtain the following expression for \( p_1(s) \) and \( p_2(s) \).

**Corollary 4.** Let \( p_1 \) and \( p_2 \) be given by (1). Then,

\[
p_i(s) = \exp \left\{ -\frac{1}{2} \sum_{n \geq 0} \frac{\log q_i(2^n s)}{2^n} \right\} \quad \text{for} \quad i = 1, 2,
\]

where

\[
q_1(s) = \frac{\zeta(2s)}{\zeta(s) L(s, \chi)} \prod_{p|d} (1 + p^{-s}), \quad \text{and} \quad q_2(s) = \frac{L(s, \chi)}{\zeta(s)} \prod_{p|d} (1 - p^{-s})^{-1}.
\]

These expressions will be used in the next section.

### 3. Evaluating \( p_1 \) and \( p_2 \)

In this section we will calculate the order of convergence of the series given by Corollary 4. We will see that this convergence is of order \( 2^{-2^n} \), i.e.,

\[
p_i(2s) = \exp \left\{ -\sum_{k=1}^{n} \frac{1}{2^k} \log q_i(2^k s) \right\} + o \left( 2^{-2^n} \right),
\]

and therefore this will be a better way to evaluate the functions \( p_1 \) and \( p_2 \) than the one given by the infinite products

\[
p_1(s) = \prod_{\chi(p) = 1} (1 - p^{-s})^{-1} \quad \text{and} \quad p_2(s) = \prod_{\chi(p) = -1} (1 - p^{-s})^{-1}.
\]

Moreover, we will provide the general term of these series at even positive integers in the case \( \Delta_K > 0 \). For it, we will use generalized Bernoulli numbers.

**Remark 5.** Recall that

\[
f(n) = o(g(n)) \quad \text{means that} \quad \lim_{n \to +\infty} \frac{f(n)}{g(n)} = 0,
\]

and

\[
a(n) = b(n) + o(g(n)) \quad \text{means that} \quad a(n) - b(n) = o(g(n)).
\]

In order to prove Theorem 2 we will need two lemmata.

**Lemma 6.** Let \( \sigma \) be a real number, \( \sigma > 1 \). Then,

\[
\frac{2^\sigma - 1}{2^\sigma - 2} < \zeta(\sigma) < \frac{2^\sigma}{2^\sigma - 2}.
\]

**Proof.** We make a partition of \( \mathbb{N} \) in the sets \( A_k = \{ n \in \mathbb{N} : 2^k \leq n < 2^{k+1} \} \), \( k \geq 1 \).

It is clear that \( |A_k| = 2^k \), and that if \( n \in A_k \), then \( n^{-\sigma} \leq 2^{-k\sigma} \). Hence,

\[
\zeta(\sigma) = \sum_{n \in \mathbb{N}} n^{-\sigma} = \sum_{k \geq 0} \sum_{n \in A_k} n^{-\sigma} < \sum_{k \geq 0} \sum_{n \in A_k} 2^{-k\sigma} = \sum_{k \geq 0} |A_k| \cdot 2^{-k\sigma} = \sum_{k \geq 0} 2^k \cdot 2^{-k\sigma} = \sum_{k \geq 0} (2^{1-\sigma})^k = \frac{1}{1 - 2^{1-\sigma}} = \frac{2^\sigma}{2^\sigma - 2}.
\]
Using that if $n \in A_k$ then $n^{-\sigma} \leq 2^{-(k+1)\sigma}$, we obtain the other side of the inequality. □

**Lemma 7.** Let $s = \sigma + it$, with $\sigma \geq 2$, and let $q_1$ and $q_2$ be given by (3). Then,

$$|\log q_i(s)| \leq \frac{16}{2\sigma - 2}$$

for $i = 1, 2$,

where $\log$ denotes the principal branch of the complex logarithm.

**Proof.** First we claim that

$$|\log(1 + z)| \leq -\log(1 - |z|),$$

for each $|z| < 1$. To see it, it suffices to compare its power series:

$$|\log(1 + z)| = |z - \frac{z^2}{2} + \cdots| \leq |z| + |\frac{z^2}{2} + \cdots| = -\log(1 - |z|).$$

Now, using (5) and that

$$\left| \frac{1 - p^{-s}}{1 + p^{-s}} - 1 \right| = \frac{2p^{-\sigma}}{1 - p^{-\sigma}},$$

we get

$$|\log q_i(s)| = \left| \log \prod_{\chi(p) = \pm 1} \frac{1 - p^{-s}}{1 + p^{-s}} \right| \leq \sum_{\chi(p) = \pm 1} \left| \log \frac{1 - p^{-s}}{1 + p^{-s}} \right| \leq \sum_{\chi(p) = \pm 1} -\log \left( 1 - \frac{2p^{-\sigma}}{1 - p^{-\sigma}} \right) = \sum_{\chi(p) = \pm 1} \log \left( \frac{1 - p^{-\sigma}}{1 - 3p^{-\sigma}} \right).$$

Moreover, since $\log(1 + x) \leq x$ for each $x > 0$, then

$$|\log q_i(s)| \leq \sum_{\chi(p) = \pm 1} \left( \frac{1 - p^{-\sigma}}{1 - 3p^{-\sigma}} - 1 \right) = \sum_{\chi(p) = \pm 1} \frac{2}{p^\sigma - 3}.$$

But since $\sigma \geq 2$ then

$$p^\sigma - 3 \geq \frac{1}{4} p^\sigma$$

for each $p \geq 2$, and therefore

$$|\log q_i(s)| \leq 8 \sum_{\chi(p) = \pm 1} p^{-\sigma}, \quad i = 1, 2.$$

Finally, by Lemma 6 we have that

$$|\log q_i(s)| \leq 8 \sum_{n \geq 2} n^{-\sigma} \leq \frac{16}{2\sigma - 2}, \quad i = 1, 2,$$

and we are done. □
By using the last Lemma, we will be able to bound the general term of the series which give \( p_1 \) and \( p_2 \), and from this, we will deduce Theorem 2.

**Proof of Theorem 2.** Let \( x_n \) and \( y_n \) be the general term of the series which give \( \log p_1(2^s) \) and \( \log p_2(2^s) \), i.e.

\[
x_n = \frac{1}{2n+1} \log q_1(2^n s), \quad y_n = \frac{1}{2n+1} \log q_2(2^n s).
\]

By Lemma 4 we have that

\[
|x_n| = \frac{1}{2n+1} |\log q_1(2^n s)| \leq \frac{1}{2n+1} \frac{16}{2^{\sigma n} - 2} = o \left( 2^{-2n} \right).
\]

Analogously,

\[
y_n = o \left( 2^{-2n} \right).
\]

Thus,

\[
p_i(2s) = \exp \left\{ -\sum_{k=1}^{n} x_k - \sum_{k=n+1}^{\infty} o \left( 2^{-2^k} \right) \right\}
\]

\[
= \exp \left\{ -\sum_{k=1}^{n} x_k - o \left( \sum_{k=n+1}^{\infty} 2^{-2^k} \right) \right\}
\]

\[
= \exp \left\{ -\sum_{k=1}^{n} x_k + o \left( 2^{-2n} \right) \right\}
\]

\[
= \exp \left\{ -\sum_{k=1}^{n} x_k \right\} \exp \left\{ o \left( 2^{-2n} \right) \right\}
\]

\[
= \exp \left\{ -\sum_{k=1}^{n} x_k \right\} \left( 1 + o \left( 2^{-2n} \right) \right)
\]

\[
= \exp \left\{ -\sum_{k=1}^{n} x_k \right\} + o \left( 2^{-2n} \right),
\]

and we are done.

Let us see now how can we evaluate the general term \( 2^{-n-1} \log q_1(2^n s) \) of the series at even positive integers when \( \Delta_K > 0 \).

Recall that given a Dirichlet character \( \chi \mod d \), the generalized Bernoulli numbers \( \{B_n, \chi\} \) are given by

\[
\sum_{a=1}^{d} \chi(a) \frac{t e^{at}}{e^{at} - 1} = \sum_{n=0}^{\infty} B_{n, \chi} \frac{t^n}{n!},
\]

Moreover,

\[
L(1-n, \chi) = -\frac{B_{n, \chi}}{n},
\]

and using the functional equation of the \( L \)-function one can evaluate \( L \) at some positive integers, as given in the following Theorem.

**Theorem 8.** Let \( \chi \) be a nontrivial primitive character modulo \( d \), and let \( a \) be 0 if \( \chi \) is even and 1 if \( \chi \) is odd. Then, if \( n \equiv a \mod 2 \),

\[
L(n, \chi) = (-1)^{1+\frac{a-\chi}{2}} \frac{\vartheta(\chi)}{2^{\chi}} \left( \frac{2\pi}{m} \right)^{\frac{n}{2}} B_{\frac{n+1}{2}} \frac{n!}{n!}.
\]
where $g(\chi)$ is the Gauss sum of the character.

Let now be $d = \Delta_K > 0$. Then, $\chi$ is an even quadratic character mod $d$. Therefore, for each $n \in \mathbb{N}$ even, one has

\begin{equation}
L(n, \chi) = (-1)^{1 + \frac{n}{2}} \frac{\sqrt{d}}{2} \left( \frac{2\pi}{d} \right)^n B_{n,\chi} \frac{B_n}{n!},
\end{equation}

and

\begin{equation}
\zeta(n) = (-1)^{1 + \frac{n}{2}} \frac{(2\pi)^n}{2} B_n \frac{B_n}{n!}.
\end{equation}

From these equalities, we deduce the following.

**Proposition 9.** Assume that $d = \Delta_K > 0$. Then, for each even natural number $n \geq 2$, we have

\begin{equation}
q_1(n) = \frac{2d^n}{(2^n)\sqrt{d}} B_{2n} B_{n,\chi} \prod_{p|d} (1 + p^{-n}),
\end{equation}

and

\begin{equation}
q_2(n) = \frac{\sqrt{d} B_{n,\chi}}{d^{n}} B_n \prod_{p|d} (1 - p^{-n})^{-1}.
\end{equation}

**Proof.** It follows immediately from (6), (7), and the definition of $q_1$ and $q_2$. □

Hence, by using Proposition 9 and Theorem 2 we obtain series of very fast convergence to evaluate $p_1$ and $p_2$ at even positive integers.

To see an example, let $\chi$ be the primitive character modulo 5, and let us evaluate $p_1(2)$. One the one hand, Taking the first 10 terms of the infinite product one obtains 2 correct digits. On the other hand, taking also the first 10 terms in our series one obtains 619 correct digits. The following table shows the approximate error when taking $n$ terms of our series.

| $N$ | $p_1(2) - \exp \left\{ - \sum_{k=1}^{N} \frac{1}{k} \log q_1(2^k) \right\}$ |
|-----|-------------------------------------------------|
| 1   | $10^{-2}$                                       |
| 2   | $10^{-3}$                                       |
| 3   | $10^{-6}$                                       |
| 4   | $10^{-11}$                                      |
| 5   | $10^{-21}$                                      |
| 6   | $10^{-41}$                                      |
| 7   | $10^{-79}$                                      |
| 8   | $10^{-157}$                                     |
| 9   | $10^{-311}$                                     |
| 10  | $10^{-629}$                                     |
| 11  | $10^{-1237}$                                    |
| 12  | $10^{-2470}$                                    |

**Acknowledgements**

The author thanks Joan-C. Lario for all his comments and suggestions.
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