REVIEW OF RAINER WÜST’S ‘MATHEMATIK FÜR PHYSIKER UND MATHEMATIKER’

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Abstract. In this article I review Wüst’s recent handbook on mathematical physics from a philosophical standpoint. It emerges a structural approach to mathematics which evidences the utility of logic in the expression of the main concepts of this discipline.

1. Comments

1.1. Logic. Recently, Rainer Wüst published for the Wiley-VCH an handbook of Mathematics for mathematicians and physicians in two volumes: the first devoted to real analysis and linear algebra, the second to the multidimensional analysis, to the differential equations and to the partial differential equations with some complements on Hilbert’s spaces and on Fourier’s series. I will review this book purely from a didactical and philosophical standpoint.

Very interesting is the statement of the author in the introduction to the first volume:

Jeder kann Mathematik verstehen, man brauchte keine Sonderbegabung (...). Aber das Verstehen ist Arbeit (...). Und man braucht am Anfang (...) auch Geduld.

Anyone is able to understand Mathematics, one needs no particular ability or predisposition (...). But ‘understanding’ means ‘fatigue’ (...). And patience, at the beginning, is necessary (...).

The main idea is so, that mathematics in itself is not difficult; it is sufficient to formalize it in an adequate manner to grasp its main concepts, also if some apparent difficulty appears (at least at the start). For this reason, Wüst devotes a section to the logical tools needed to formalize the mathematical concepts and to prove the various sentences. This permits a matchless precision. I refer, particularly, to the spreaded use of the quantifiers (at the first order); this eliminate any possible misunderstanding in the reader. For example, let us state, the definition of uniforme [gleichmässig] continuity:

Definition. Let be $f$ a function whatever. $f$ is called uniformly continue, if for any $\varepsilon > 0$ there is a $\delta > 0$, such that for every $x_1, x_2 \in \mathcal{D}(f)$:

\begin{equation}
|f(x_1) - f(x_2)| < \varepsilon; \quad |x_1 - x_2| < \delta
\end{equation}
with quantifiers:

\[ \bigwedge_{\varepsilon>0} \bigvee_{\delta>0} \bigwedge_{x_1 \in \mathbb{D}(f)} \bigwedge_{x_2 \in \mathbb{D}(f)} (|x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon) \]

As you can easily see, the use of the quantifiers makes clear the dependence of \( \delta \) from \( \varepsilon \) and not the contrary, albeit at a first sight it could be reasonable. Moreover, in this definition it is clear the distinction \textit{language/meta-language}. The implication in (2) is not formulated in the language, but in the meta-language and states a relation between two linguistic-mathematical facts; i.e. one disequation is in a relation \( \Rightarrow \) with the other.

This is not a detail. First of all, as regards to the relationship obtaining in the equation above, it is not pedantry to make clear that \( \bigwedge \bigvee A \) is not the same as \( \bigvee \bigwedge A \). The discrepancy language/meta-language is present everywhere in the domain of mathematics. One example could be that of a derivative of a function. The derivative is \textit{not} a straight. Otherwise, we have an equation stating the identity of a straight with a number. Simply, the derivative is on a meta-level as regards to the tangent in the graph of the function. It affirms something \textit{on} an objective (‘linguistic’) geometric situation. Given the omnipresence of such difference in the mathematical area, it is right to put it in evidence, whenever possible; or, at least, to make the reader able to distinguish in the various cases if we are in a object-level or in a meta-level.

At any rate, the amazing feature of these books is the tentative to give an unitary presentation of a lot of apparent (almost) irrelated results. There is not a trivial sequence of important facts in order of complexity, but a larger structure in which they are derived. So, for example, it absent a section devoted to the trigonometrical functions, which come out as solutions of particular differential equations. It lacks also an independent treatment of Dirac’s \( \delta \) function. It is seen in a larger context of \textit{distributive} functions\(^5\) and of the exchange of limits. The name of Dirac is not made. It is a sort of \textit{gestaltisch} interpretation of mathematics where the accent is put on the totality to which the mathematical concepts belong.

The text is rich of \textit{observations} [Bemerkungen] in which Rainer Wüst aids the reader to reckon the same thing in different disguises\(^7\). In the observation of pages 425–426\(^8\), soon after the definition of \textit{Dual Space}, Wüst notes that the integral is a linear functional in the space of the Riemann-integrable functions in the interval \([a,b]\). This is a bridge between analysis and linear algebra and it evidences the priority of the concept of functional in itself as regards its possible ‘incarnations’. What Wüst doesn’t note in this paragraph is that a functional is in a meta-level as regards to the function itself. In other words, the symbol \( \int \), does not belong to the mathematical language, but to a meta-mathematical language. Another example, not mentioned, of functionals which makes clear this discourse is represented by

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\(^5\) [Rai09a, p. 82].

\(^6\) [Rai09a, p. 315].

\(^7\) The Italian mathematician Vailati wrote that \textit{mathematics is the art to call with the same name different things}. A brutal simplification, but right, properly grasped.

\(^8\) [Rai09a, pp. 425–426].
the so-called Skolem-functions, which are not properly functions, but functions of functions.

One can imagine the language of mathematics as compounded of many layers, any of which it is function of its immediate predecessor. At the base we have the reality. So, going back to the case of Riemann-integrals, we start with the geometric reality of the graph of a function, then we find the function of the graph, the integral of the function (a function of functions), and so on.

1.2. Rigour and Insight. Now, one could put a question: what is clear in itself is it also clear for our human beings? Surely, no. It is often easy to manipulate the symbols, but they in itself cannot render more clear an argument, if their introduction, meaning and scope is note yet clear. Wüst’s book is auto-sufficient; no extra mathematical notion or symbol is needed to know. He gives nothing for granted. In this sense, it is self-consistent and adapt as an handbook, which you need to consult many many times. Here you find at the right place the right concept, and once understood the matter, you don’t need another source to explain it.

Unfortunately, this is the main advantage of these books, but also their flaw. The reader is not an universal Turing machine; a rigid sequential exposition is necessary for an artificial mind, but not for a being endowed with insight. By the way, in the 18th century de Morgan observed that teaching mathematics requires a compromise between precision and intuition. Sometimes, one must simplify an involved concept, deleting also importants aspects. But, once the alum has understood its reason of being, the teacher can refnay the definition. Then, the parts of pure text are important inasmuch they prepare in an intuitive way the reader to grasp formally a sentence. In this Wüst is not always convincing; he leaves much space to the proofs, but this is not sufficient (I speak, obviously, not for a mathematician, but for a student). A proof is necessary, because, often a sentence can sound ’strange’, not so ’evident’. The proof justifies this sentence relying on simpler concepts. In this sense, after reading a proof, one can go back to the original sentence and feel it (at least) more clear. A proof must have so a rhetorical character; it must convince. The reader must feel him compelled to accept the thesis.

On the other hands, there are many sentences more than evident which need a very involved proof. In this case, is this proof a real proof? I think of no. For this motive, it is desiderable that an author at least in the delicate situations justifies and makes clear the reader how a proof is built up and what in the proof is the pivot. This is said in a little preface from Kurt Tucholsky:

Mathematik is eine Schule des Denkens. (...) Mathematics is a School of Thought. (...) Mathematics affects your thinking and your doing. (...) It trains your ability to distinguish the Important from the Un-important, the Difficult from the Ease, the Involved from the Trivial.

\[9\text{Rai09a, p. viii.} \text{ The emboldening is mine.}\]
Often in the two volumes, in effects, Rainer Wüst explains the main ideas underlying the proofs. But he lacks what sort of insight which I found, for example, in Gilardi’s handbook of analysis. It is a pity that this text was not translated in English. Gilardi in the first introduction of the concept of Riemann-integral (without any drawing illustrating the geometric meaning of the integral) speaks immediately of multiple integrals and multi-dimensional integrals. This way, the reader understands the general concept of integral, of which the integral on intervals of $\mathbb{R}$ is only a particular case. Moreover, the reader is so in the condition to understand Gauss’ and Stoke’s integrals. Wüst maintains separated the one-dimensional integral from the multi-dimensional one. This compells him to partly re-write the one-dimensional sentences in the multi-dimensional. This is not only awkward, and perhaps a little tedious, but it draws a sharp boundary between integrals in $\mathbb{R}$ and in $\mathbb{R}^m$ which doesn’t exist in the essence of the concept.

1.3. **Lebesgue.** Another gap is the absence of the Lebesgue integration. Wüst limit himself to an hint at page 340 of the first volume, just before abandoning the real analysis. No word on the meaning of the concept of Lebesgue integral and on the necessity to introduce it. Does not Riemann-integral suffice? What is it wrong with it? The German author writes:

\[
(...) \text{Jede absolut (eigentlich oder uneigentlich) Riemann-integrierbare Funktion ist auch Lebesgue-integrierbar, und die Integrale haben den gleichen Wert.}
\]

\[
(...) \text{Any absolute (proper or improper) Riemann-integrable function is Lebesgue-integrable too, and both integrals have the same value.}
\]

This is true but it cannot be sufficient to explain the absence of Lebesgue-integration. It reminds me of a curious comment of another German author:

\[
\text{Das Lebesgue-Integral ist als echte Erweiterung gedacht, um gewisse pathologische Funktionen doch integrabel zu machen (\ldots).}
\]

The Lebesgue-integral is thought of as a proper generalization [of the Riemann integration], to make integrable also pathologic functions (\ldots).

Note the expression ‘pathologische’ referred to functions. Gottlob Frege would speak of *psychologism*. The motivation of Nolting leaves without words, but it is better than Wüst. Obviously, we must take in account that Wüst and Nolting wrote for physicians and Gilardi for mathematicians, but this is a gap notwithstanding.

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10 [Gil01].

11 Gauss’ and Stoke’s theorems are at page 139 of [Gil01], while Gauss’ theorem is at page 868 of [Rai09a] and Stoke’s theorem at page 894.

12 [Rai09a, pp. 227–341].

13 [Rai09b, pp. 747–904].

14 [Rai09a, p. 339].

15 [Nol09, p. 147].

16 The translation is mine.
1.4. **Topology.** One thing that I am not able to understand is a sentence like that: the limit of $f(x)$ is $b$ when $x$ tends to $0$. What does it mean ’to tend’? I have in mind a person who grasps with one hand a book on a bookcase. Jänich in his book \[Kla05, p. 21\] gives the concept of limit as granted. In a similar way, Fischer and Kaul introduce the concept of limit without a word of explanation. \[FK07, p. 41\] What lacks is a topological background. Without the concepts of *neighbourhood* [Umgebung], *open* and *closed* set, *closure*, etc. how can we introduce the basic concepts of analysis (limit, derivative, integral, etc.)? For this reason, Wüst devotes a preliminary section in the first volume of his book (and in the first part of the second volume) to at least the more important notions of topology.

Gilardi don’t leave much space in his book to these notions, inasmuch he think that from a didactical point it is not useful to confront the alum with topology at the start of a first course in analysis. Perhaps, he is right, but I believe that an appropriate mixture of topology and formalism permits a precise definition of the concept of limit, for example.

1.5. **Jänich.** Throughout the two volumes Wüst makes many references to Jänich’s books in linear algebra, in analysis and in function theory. This is in a sense ’puzzling’ because Jänich is a sort of anti-Wüst \[Kla05, iv\] trying to eliminate any proof whenever possible, presenting in an intuitive way the concepts, with the help of a lot of drawings. He explains his attitude this way:

> Wenn Ihnen ohne die *volle* mathematische Sicherheit unwohl ist, dann wechseln Sie nur gleich das Fach und studieren lieber Mathe-matik, denn diese Sicherheit Können wir bieten.\[Kla05, p. 101\]

*If you are at discomfort without the complete precision of mathematics, then it is best that you change faculty and study mathematics [in place of physics], because there you find the rigour you need.*\[Kla05, p. 101\]

And he insists:

> Das ganze Gebiet [of physics] würde nicht existieren, wenn die Physiker nicht gewagt hätten, ohne das Halteseil der absoluten mathematischen Gewissheit weiter zu steigen.\[Kla05, iv\]

*The entire field of physics would not exists, if the physicians have not dared to improve their knowledge without the support of an absolute mathematical certainty.*

This is true, but the proofs are the soul of mathematics; otherwise it would be only a computational theory.\[Kla05, p. 101\] The proofs are needed in order to rightly understand a sentence; they are not virtuosism. How can one understand a sentence, without...
knowing the roots upon which it lays? Moreover, we cannot substitute the certainty of the proofs with the ‘plausibility’ of the drawings. A drawing can speak more than a formula, but it is only of aid. I think to the page 245 of Jänich’s book where it is stated the Gram-Schmidt process and a picture illustrating it. I find it not understandable.

Wüst doesn’t devote much space to this process; he introduces it, without demonstration, but he uses it proving the subsequent corollary:

**Theorem.** (Schmidt’s process of orto-normalization) Let be \( (V, \langle \cdot, \cdot \rangle) \) a pre-Hilbert space. Let be also \( n \in \mathbb{N} \) and \( a_1, \ldots, a_n \in V \) linear independent. Then it exists an Ortonormal system \( \{b_1, \ldots, b_n\} \) with

\[
\begin{align*}
    b_j & \in \text{span}(\{a_1, \ldots, a_j\}), \\
    a_j & \in \text{span}(\{b_1, \ldots, b_j\})
\end{align*}
\]

and \( \text{span}(\{a_1, \ldots, a_j\}) = \text{span}(\{b_1, \ldots, b_j\}) \).

**Corollary.** If, with the presuppositions and symbols of the preceeding theorem, \( \dim V = n \), and \( \{a_1, \ldots, a_n\} \) is a base for \( V \), then \( \{b_1, \ldots, b_n\} \) is an ortonormal base in \( V \).

At this point, it starts the proof for recursion on the complexity of the hypothetical base. And here it play its pivotal rôle the GS-process. Again, Wüst see this process not in its singularity but in the larger context of the search for an ortonormal basis. Furthermore, the proof by induction explains the apparent involved structure of the GS-process. This helps again the reader in distinguishing the important from the un-important, an ability which mathematics teaches to students.

### 1.6. Differential calculus

Another gap in Wüst’s work is his presentation of the concept of derivative. What is a derivative? Obviously not a tangent, but a mathematical concept describing the behaviour of a function in a point. In the usual definition of derivative we have not a mathematical representation of a straight in a *point*, but in an *interval* little at pleasure. The straight in the graph lays over a fragment of function, and so it follows its inclination. The trick is, then, to put equal to 0 this little segment. In a sense, this infinitesimal segment exists and don’t exists in two different moments of the computation. But this is another question.

I would stress the fact that the derivative is an approximation of the behaviour of a function. For this reason, Gilardi writes:

**Definition.** Let be \( x_0 \) a point in \( \mathbb{R}^n \) and \( f \) a function defined at least in a neighbourhood of \( x_0 \) with values in \( \mathbb{R}^m \). We say that \( f \) is differentiable, when it exists

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24. [Rai05a, p. 373].

25. I repeat that I am adressing to a potential student-reader.

26. The derivative cannot represent a tangent in a point, because we want know the increment of a function. So we need to know how the functions behaves in a fraction of space (time), and, accordingly, we must substitute the point with an infinitesimal segment.

27. A tangent, as straight, can at best only approximate a portion of a curved graph.
The best linear approximation of the increment of $f$ from $x_0$; i.e. when it exists a linear map $L$ from $\mathbb{R}^n$ in $\mathbb{R}^m$ such that

$$f(x_0 + h) = f(x_0) + Lh + o(|h|) \text{ for } h \text{ which tends to } 0.$$ 

In the preceding equation we are able to distinguish in its right side the linear approximation of $f$ and the rest, $o(|h|)$, that is the margin of error in the approximation. This lacks in Wüst. For the little ‘$o$’ we must wait for the multidimensional case. Landau’s symbol is introduced at pages 610–611. This concurs again to a sharp distinction between the one-dimensional and the multi-dimensional. We agree that a difference exists, but we would put the finger on the generality of the concept, and then adapt it to the singular circumstances.

1.7. **Exercises.** The exercises lack of the right answer. Often, the author makes an hint as regards to the solving of the exercises (in the cases of proofs, he lays down a possible track). On the contrary, the usual answer to simple calculations is clear [klar]. Both the books have a well done index which permits a quick finding of a theorem or definition. The index is equal in both the volumes. To facilitate further the reader, some references are italicized. For example, in the first volume the word *Gradient* is followed by the italicized number of page (629), because the concept of gradient will be introduced in the second volume at page 629. On the contrary, in the second volume the item *Rolle’s Theorem* is followed from the italicized number 168, referring to the first volume. The same applies also to the index of symbols. Only a gap: at page 604 of the second volume we find this equation:

$$x_0 := \langle \hat{x}_1, \ldots, \hat{x}_m \rangle \in \mathcal{D}(f)$$

There is no trace in the list of symbols of this ‘ring’ on the variables. It was introduced in the first volume to denote an open-set. Finally, there is no definition of *normal* and this expression is absent in the index.

2. **Conclusion**

Summing up, Wüst books on mathematical physics are amazing handbooks. Apart some exception, there is everything you must know to understand the mathematics used in physics. And all this material is in perfect order to facilitate the use of these books as handbooks. The German is clear, without the slang occuring sometimes in Jänich’s works; it is not dry, neither pedant, but pleasant. Wüst is interesting also for the philosophy of mathematics expressed. A philosophy which searches for the structural facets of mathematics, an holistic totality which realizes itself in its concepts, as the absolute in Hegel realizes itself in the multidimensionality of reality.

With the words of N. Latz from the back cover:

(…) ich bin begeistert!

(…) I am enthusiast!  

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28[Gil01, p. 55]. The translation is mine. The symbol $o$ stands for Landau’s little $o$.

29The translation is mine.
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