Critical behavior of the correlation function of three-dimensional $O(N)$
models in the symmetric phase

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We present new strong-coupling series for $O(N)$ spin models in three dimensions, on the cubic and diamond lattices. We analyze these series to investigate the two-point Green’s function $G(x)$ in the critical region of the symmetric phase. This analysis shows that the low-momentum behavior of $G(x)$ is essentially Gaussian for all $N$ from zero to infinity. This result is also supported by a large-$N$ analysis.

1. INTRODUCTION

Three-dimensional $O(N)$-symmetric spin models describe many important critical phenomena in nature: the case $N = 3$ describes ferromagnetic materials, where the order parameter is the magnetization; the case $N = 2$ describes the helium superfluid transition, where the order parameter is the quantum amplitude; the case $N = 1$ (Ising model) describes liquid-vapor transitions, where the order parameter is the density.

In the following we will focus on the low-momentum behavior of the Fourier-transformed correlation function $\tilde{G}(k)$ in the critical region of the symmetric phase, i.e., for

$$ |k| \lesssim 1/\xi, \quad 0 < T/T_c - 1 \ll 1. $$

2. LATTICE MODELS

Let us consider an $O(N)$-symmetric lattice spin models described by the nearest-neighbor action

$$ S = -N\beta \sum_{\text{links}} \vec{s}_{x_l} \cdot \vec{s}_{x_r}, $$

where $\beta = 1/T$, $\vec{s}$ is an $N$-component real vector, and $x_l$, $x_r$ are the endpoints of the link. The two-point correlation function is defined by

$$ G(x) = \langle \vec{s}_x \cdot \vec{s}_0 \rangle. $$

In order to simplify the study the critical behavior of $G(x)$, we introduce the dimensionless RG-invariant function

$$ L(k; \beta) \equiv \frac{\tilde{G}(0; \beta)}{G(k; \beta)}. $$

In the critical region of the symmetric phase, $L(k, \beta)$ is a function only of the ratio $y \equiv k^2/M_G^2$, where $M_G \equiv 1/\xi_G$; the second-moment correlation length $\xi_G$ is defined by

$$ \xi_G^2 \equiv \frac{1}{6} \sum_x \frac{x^2 G(x)}{\sum_x G(x)}. $$

$M_G$ is the mass-scale which can be directly observed in scattering experiments. $L(y)$ can be expanded in powers of $y$ around $y = 0$:

$$ L(y) = 1 + y + l(y), \quad l(y) = \sum_{i=2}^{\infty} c_i y^i. $$

$l(y)$ parameterizes the difference from a generalized Gaussian propagator. The coefficients $c_i$ can be expressed as the critical limit of appropriate dimensionless RG-invariant ratios of the spherical moments

$$ m_2 = \sum_x x^2 G(x). $$

Another interesting quantity related to the low-momentum behavior of $G$ is the ratio $s = M^2/M_G^2$, where $M$ is the mass-gap of the theory. Its critical value is $s^* = -y_0$, where $y_0$ is the zero of $L(y)$ closest to the origin.
In the large-$N$ limit, $l(y)$ is depressed by a factor of $1/N$. The coefficients $c_i$ can be obtained from a $1/N$ expansion in the continuum:

$$
c_2 \simeq -\frac{0.0044486}{N}, \quad c_3 \simeq -\frac{0.0001344}{N},
$$

$$
c_4 \simeq -\frac{0.00000658}{N}, \quad c_5 \simeq -\frac{0.0000040}{N} \ldots
$$

We are presently computing the order $1/N^2$ of the expansion. We expect that the pattern established by the $1/N$ expansion

$$c_1 \ll c_2 \ll 1, \quad i \geq 3
$$

will be followed by all models with sufficiently large $N$. This implies $s^* - 1 \simeq c_2$; indeed, in the large-$N$ limit,

$$s^* - 1 \simeq -\frac{0.0045900}{N}.
$$

The coefficients $c_i$ can also be computed from an $\varepsilon$-expansion of the corresponding $\phi^4$ theory around $d = 4$:

$$c_i \simeq \varepsilon^2 \frac{N + 2}{(N + 8)^2} e_i,
$$

where $\varepsilon = 4 - d$ and

$$e_2 \simeq -0.007520, \quad e_3 \simeq 0.0001919.
$$

### 3. STRONG-COUPLING EXPANSION

We computed the strong-coupling expansion of $G(x)$ up to 15th order on the cubic lattice, and up to 21st order on the diamond lattice. Our technique for the strong-coupling expansion of $O(N)$ spin models was presented in Ref. [3].

We took special care in the choice of estimators for the “physical” quantities $c_i$ and $s^*$. This step is very important from a practical point of view: better estimators can greatly improve the stability of the extrapolation to the critical point. Our search for optimal estimators was guided by the requirement of a regular strong-coupling expansion (e.g., no $\ln \beta$ terms) and by the knowledge of the large-$N$ limit (we chose estimators which are “perfect” for $N = \infty$).

The strong-coupling series of the estimators were analyzed by Padé approximants, Dlog-Padé approximants and first-order integral approximants (see Ref. [4] for a review of the resummation techniques; see also Ref. [5]). For diamond lattice models with $N \neq 0$, $\beta_c$ was not known, and we estimated it from the strong coupling series of the magnetic susceptibility.

Our strong-coupling results on cubic and diamond lattices are compared with the results of the $1/N$ expansion and of the $\varepsilon$-expansion in Table 1. One may notice that universality between cubic and diamond lattice is always confirmed; furthermore, the agreement with the $\varepsilon$-expansion and with the $1/N$ expansion is satisfactory.

The predicted pattern $c_3 \ll c_2 \ll 1$ is verified for all $N$. We can conclude that the two-point Green’s function is essentially Gaussian for all momenta with $|k^2| \lesssim M^2_G$, and that the small corrections are dominated by the $(k^2)^2$ term.

### 4. APPROACH TO CRITICALITY

We investigated the approach to criticality, with special attention devoted to anisotropy (violation of rotational invariance). Let us introduce the anisotropy estimators

$$l_4 = \sum_{x,y,z} [f_4(x, y) + f_4(y, z) + f_4(z, x)] G(x, y, z),
$$

$$f_4(x, y) = (x^2 + y^2)^2 - 8x^2y^2;
$$

$$l_{6,1} = \sum_{x,y,z} [f_6(x, y) + f_6(y, z) + f_6(z, x)]
$$

$$\times G(x, y, z),
$$

$$f_6(x, y) = (x^2 + y^2)^3 - 8x^4y^2 + x^2y^4);
$$

$$l_{6,2} = \sum_{x,y,z} [x^6 + y^6 + z^6 - 45x^2y^2z^2] G(x, y, z).
$$

In the critical limit, $l_{2j}$ are depressed with respect to the spherical moments $m_{2j}$. In the large-$N$ limit one can show that

$$A_{2j,i} \equiv \frac{l_{2j,i}}{m_{2j}} \sim \xi_G^{-2}.
$$

We analyzed the strong-coupling series of

$$B_{2j,i} \equiv \frac{l_{2j,i}}{m_{2j-2}};
$$

$$c_i \simeq -\frac{0.0045900}{N}.
$$

The predicted pattern $c_3 \ll c_2 \ll 1$ is verified for all $N$. We can conclude that the two-point Green’s function is essentially Gaussian for all momenta with $|k^2| \lesssim M^2_G$, and that the small corrections are dominated by the $(k^2)^2$ term.
Table 1
Comparison of strong-coupling expansion on cubic and diamond lattices with $1/N$ and $\varepsilon$-expansion

| $N$ | lattice | $10^4 c_2$ | $10^4 c_3$ | $10^4 (s^* - 1)$ |
|-----|---------|------------|------------|------------------|
| 0   | cubic   | $|10^4 c_2| \lesssim 2$ | 1.2(1)     | 1.2(3)           |
|     | diamond | $|10^4 c_2| \lesssim 1$ | 1.0(1)     | 1.0(5)           |
|     | $\varepsilon$-expansion | $-2.35$  | 0.60       |                  |
| 1   | cubic   | $-2.9(2)$  | 1.1(1)     | $-2.3(5)$        |
|     | diamond | $-3.1(2)$  | 1.0(2)     | $-2.2(3)$        |
|     | $\varepsilon$-expansion | $-2.78$  | 0.71       |                  |
| 2   | cubic   | $-3.8(3)$  | 1.1(1)     | $-3.5(5)$        |
|     | diamond | $-4.2(3)$  | 1.1(3)     | $-3.5(2)$        |
|     | $\varepsilon$-expansion | $-3.01$  | 0.77       |                  |
| 3   | cubic   | $-4.0(2)$  | 1.1(2)     | $-4.0(4)$        |
|     | diamond | $-4.2(3)$  | 1.1(3)     | $-3.5(2)$        |
|     | $\varepsilon$-expansion | $-3.11$  | 0.79       |                  |
| 4   | cubic   | $-4.1(2)$  | 1.2(1)     | $-4.0(4)$        |
|     | diamond | $-4.7(2)$  | 1.0(2)     | $-4.0(2)$        |
|     | $\varepsilon$-expansion | $-3.13$  | 0.80       |                  |
|     | $1/N$   | $-11.12$   | 3.36       | $-11.48$         |
| 8   | cubic   | $-3.5(2)$  | 1.0(2)     | $-3.7(3)$        |
|     | diamond | $-4.0(1)$  | 0.7(5)     | $-4.0(4)$        |
|     | $\varepsilon$-expansion | $-2.94$  | 0.75       |                  |
|     | $1/N$   | $-5.56$    | 1.18       | $-5.74$          |
| 16  | cubic   | $-2.4(2)$  | 0.7(5)     | $-2.7(2)$        |
|     | diamond | $-2.65(5)$ | 0.5(5)     | $-2.9(2)$        |
|     | $\varepsilon$-expansion | $-2.35$  | 0.60       |                  |
|     | $1/N$   | $-2.78$    | 0.84       | $-2.87$          |

for all values of $N$, we found that $B_{2j,i}$ have a finite (but non-universal) $T \to T_c$ limit. This supports the validity of Eq. (15) for all $N$.

Ratios of $A_{2j,i}$ are universal quantities; we found that at criticality $A_{6,1}/A_4 \simeq 0.95$ and $A_{6,2}/A_{6,1} \simeq 0.75$ (within one per mill) for all $N$.

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