Abstract

In 1966, Cummins introduced the “tree graph”: the tree graph $T(G)$ of a graph $G$ (possibly infinite) has all its spanning trees as vertices, and distinct such trees correspond to adjacent vertices if they differ in just one edge, i.e., two spanning trees $T_1$ and $T_2$ are adjacent if $T_2 = T_1 - e + f$ for some edges $e \in T_1$ and $f \notin T_1$. The tree graph of a connected graph need not be connected. To obviate this difficulty we define the “forest graph”: let $G$ be a labeled graph of order $\alpha$, finite or infinite, and let $\mathcal{F}(G)$ be the set of all labeled maximal forests of $G$. The forest graph of $G$, denoted by $\mathcal{F}(G)$, is the graph with vertex set $\mathcal{F}(G)$ in which two maximal forests $F_1, F_2$ of $G$ form an edge if and only if they differ exactly by one edge, i.e., $F_2 = F_1 - e + f$ for some edges $e \in F_1$ and $f \notin F_1$.

Using the theory of cardinal numbers, Zorn’s lemma, transfinite induction, the axiom of choice and the well-ordering principle, we determine the $\mathcal{F}$-convergence, $\mathcal{F}$-divergence, $\mathcal{F}$-depth and $\mathcal{F}$-stability of any graph $G$. In particular it is shown that a graph $G$ (finite or infinite) is $\mathcal{F}$-convergent if and only if $G$ has at most one cycle of length 3. The $\mathcal{F}$-stable graphs are precisely $K_3$ and $K_1$. The $\mathcal{F}$-depth of any graph $G$ different from $K_3$ and $K_1$ is finite. We also determine various parameters of $\mathcal{F}(G)$ for an infinite graph $G$, including the number, order, size, and degree of its components.

Keywords: Forest graph operator; Graph dynamics.

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1 Introduction

A graph dynamical system is a set $X$ of graphs together with a mapping $\phi : X \rightarrow X$ (see Prisner [12]). We investigate the graph dynamical system on finite and infinite graphs defined by the forest graph operator $\mathcal{F}$, which transforms $G$ to its graph of maximal forests.
Let $G$ be a labeled graph of order $\alpha$, finite or infinite. (All our graphs are labeled.) A **spanning tree** of $G$ is a connected, acyclic, spanning subgraph of $G$; it exists if and only if $G$ is connected. Any acyclic subgraph of $G$, connected or not, is called a forest of $G$. A forest $F$ of $G$ is said to be maximal if there is no forest $F'$ of $G$ such that $F$ is a proper subgraph of $F'$. The tree graph $T(G)$ of $G$ has all the spanning trees of $G$ as vertices, and distinct such trees are adjacent vertices if they differ in just one edge \([12, 15]\); i.e., two spanning trees $T_1$ and $T_2$ are adjacent if $T_2 = T_1 - e + f$ for some edges $e \in T_1$ and $f \notin T_1$. The **iterated tree graphs** of $G$ are defined by $T^0(G) = G$ and $T^n(G) = T(T^{n-1}(G))$ for $n > 0$.

There are several results on tree graphs. See \([1, 18, 11]\) for connectivity of the tree graph, \([8, 13, 16, 19, 4, 7, 10, 3, 6]\) for bounds on the order of trees, and \([2, 14]\) for Hamilton circuits in a tree graph.

There is one difficulty with iterating the tree graph operator. The tree graph of an infinite connected graph need not be connected \([2, 14]\), so $T^2(G)$ may be undefined. For example, $T(K_{8_0})$ is disconnected (see Corollary \([2, 5]\) in this paper; $8_0$ denotes the cardinality of the set $\mathbb{N}$ of natural numbers); therefore $T^2(K_{8_0})$ is not defined. To obviate this difficulty with iterated tree graphs, and inspired by the tree graph operator $T$, we define a forest graph operator. Let $\mathfrak{N}(G)$ be the set of all maximal forests of $G$. The forest graph of $G$, denoted by $F(G)$, is the graph with vertex set $\mathfrak{N}(G)$ in which two maximal forests $F_1$, $F_2$ form an edge if and only if they differ by exactly one edge. The forest graph operator (or maximal forest operator) on graphs, $G \mapsto F(G)$, is denoted by $F$. Zorn’s lemma implies that every connected graph contains a spanning tree (see \([5]\)); similarly, every graph has a maximal forest. Hence, the forest graph always exists. Since when $G$ is connected, maximal forests are the same as spanning trees, then $F(G) = T(G)$; that is, the tree graph is a special case of the forest graph. We write $F^2(G)$ to denote $F(F(G))$, and in general $F^n(G) = F(F^{n-1}(G))$ for $n \geq 1$, with $F^0(G) = G$.

**Definition 1.1.** A graph $G$ is said to be $F$-convergent if $\{F^n(G) : n \in \mathbb{N}\}$ is finite; otherwise it is $F$-divergent.

A graph $H$ is said to be an $F$-root of $G$ if $F(H)$ is isomorphic to $G$, $F(H) \cong G$. The **$F$-depth** of $G$ is

$$\sup\{n \in \mathbb{N} : G \cong F^n(H)\}.$$ 

The $F$-depth of a graph $G$ that has no $F$-root is said to be zero.

The graph $G$ is said to be **$F$-periodic** if there exists a positive integer $n$ such that $F^n(G) = G$. The least such integer is called the $F$-periodicity of $G$. If $n = 1$, $G$ is called $F$-stable.

This paper is organized as follows. In Section 2 we give some basic results. In later sections, using Zorn’s lemma, transfinite induction, the well ordering principle and the theory of cardinal numbers, we study the number of $F$-roots and determine the $F$-convergence, $F$-divergence, $F$-depth and $F$-stability of any graph $G$. In particular we show that: i) A graph $G$ is $F$-convergent if and only if $G$ has at most one cycle of length 3. ii) The $F$-depth of any graph $G$ different from $K_3$ and $K_1$ is finite. iii) The $F$-stable graphs are precisely $K_3$ and $K_1$. iv) A graph that has one $F$-root has innumerably many, but only some $F$-roots are important.
2 Preliminaries

For standard notation and terminology in graph theory we follow Diestel [5] and Prisner [12]. Some elementary properties of infinite cardinal numbers that we use are (see, e.g., Kamke [9]):

1. $\alpha + \beta = \alpha \cdot \beta = \max(\alpha, \beta)$ if $\alpha, \beta$ are cardinal numbers and $\beta$ is infinite. In particular, 
   $2 \cdot \beta = \aleph_0 \cdot \beta = \beta$.
2. $\beta^n = \beta$ if $\beta$ is an infinite cardinal and $n$ is a positive integer.
3. $\beta < 2^\beta$ for every cardinal number.
4. The number of finite subsets of an infinite set of cardinality $\beta$ is equal to $\beta$.

We consider finite and infinite labeled graphs without multiple edges or loops. An isthmus of a graph $G$ is an edge $e$ such that deleting $e$ divides one component of $G$ into two of $G - e$. Equivalently, an isthmus is an edge that belongs to no cycle. Each isthmus is in every maximal forest, but no non-isthmus is.

Let $\mathcal{C}(G)$ and $\mathfrak{R}(G)$ denote the set of all possible cycles and the set of all maximal forests of a graph $G$, respectively. Note that a maximal forest of $G$ consists of a spanning tree in each component of $G$. A fundamental fact, whose proof is similar to that of the existence of a maximal forest, is the following forest extension lemma:

Lemma 2.1. In any graph $G$, every forest is contained in a maximal forest.

Lemma 2.2. If $G$ is a complete graph of infinite order $\alpha$, then $|\mathfrak{R}(G)| = 2^\alpha$.

Proof: Let $G = (V, E)$ be a complete graph of order $\alpha$ ($\alpha$ infinite), i.e., $G = K_\alpha$. Let $v_1, v_2$ be two vertices of $G$ and $V' = V \setminus \{v_1, v_2\}$. Then for every $A \subseteq V'$ there is a spanning tree $T_A$ such that every vertex of $A$ is adjacent only to $v_1$ and every vertex of $V' \setminus A$ is adjacent only to $v_2$. It is easy to see that $T_A \neq T_B$ whenever $A \neq B$. As the cardinality of the power set of $V'$ is $2^\alpha$, there are at least $2^\alpha$ spanning trees of $G$. Since $G$ is connected, the maximal forests are the spanning trees; therefore $|\mathfrak{R}(G)| \geq 2^\alpha$. Since the degree of each vertex is $\alpha$ and $G$ contains $\alpha$ vertices, the total number of edges in $G$ is $\alpha \cdot \alpha = \alpha$. The edge set of a maximal forest of $G$ is a subset of $E$ and the number of all possible subsets of $E$ is $2^\alpha$. Therefore, $G$ has at most $2^\alpha$ maximal forests, i.e., $|\mathfrak{R}(G)| \leq 2^\alpha$. Hence $|\mathfrak{R}(G)| = 2^\alpha$.

For two maximal forests of $G$, $F_1$ and $F_2$, let $d(F_1, F_2)$ denote the distance between them in $\mathbf{F}(G)$. We connect this distance to the number of edges by which $F_1, F_2$ differ; the result is elementary but we could not find it anywhere in the literature. We say $F_1, F_2$ differ by $l$ edges if $|E(F_1) \setminus E(F_2)| = |E(F_2) \setminus E(F_1)| = l$.

Lemma 2.3. Let $l$ be a natural number. For two maximal forests $F_1, F_2$ of a graph $G$, if $|E(F_1) \setminus E(F_2)| = l$, then $|E(F_2) \setminus E(F_1)| = l$. Furthermore, $F_1$ and $F_2$ differ by exactly $l$ edges if and only if $d(F_1, F_2) = l$.
We cannot apply to an infinite graph the simple proof for finite graphs, in which the number of edges in a maximal forest is given by a formula. Therefore, we prove the lemma by edge exchange.

**Proof:** We prove the first part by induction on \( l \). Let \( F_1, F_2 \) be maximal forests of \( G \) and let \( E(F_1) \setminus E(F_2) = \{ e'_1, e'_2, \ldots, e'_k \} \), \( E(F_2) \setminus E(F_1) = \{ e_1, e_2, \ldots, e_l \} \). If \( l = 0 \) then \( k = 0 = l \) because \( F_2 = F_1 \). Suppose \( l > 0 \); then \( k > 0 \) also. Deleting \( e_l \) from \( F_2 \) divides a tree of \( F_2 \) into two trees. Since these trees are in the same component of \( G \), there is an edge of \( F_1 \) that connects them; this edge is not \( e_l \) so it is not in \( F_2 \); therefore, it is an \( e'_i \), say \( e'_k \). Let \( F'_2 = F_2 - e_l + e'_k \). Then \( E(F_1) \setminus E(F'_2) = \{ e'_1, e'_2, \ldots, e'_{k-1} \} \), \( E(F_2) \setminus E(F_1) = \{ e_1, e_2, \ldots, e_{l-1} \} \). By induction, \( k - 1 = l - 1 \).

We also prove the second part by induction on \( l \). Assume \( F_1, F_2 \) differ by exactly \( l \) edges and define \( F'_2 \) as above. If \( l = 0, 1 \), clearly \( d(F_1, F_2) = l \). Suppose \( l > 1 \). In a shortest path from \( F_1 \) to \( F_2 \), whose length is \( d(F_1, F_2) \), each successive edge of the path can increase the number of edges not in \( F_1 \) by at most \( 1 \). Therefore, \( F_1 \) and \( F_2 \) differ by at most \( d(F_1, F_2) \) edges. That is, \( l \leq d(F_1, F_2) \). Conversely, \( d(F_1, F'_2) = l - 1 \) by induction and there is a path in \( F(G) \) from \( F_1 \) to \( F'_2 \) of length \( l - 1 \), then continuing to \( F_2 \) and having total length \( l \). Thus, \( d(F_1, F_2) \leq l \).

From the above lemma we have two corollaries.

**Corollary 2.4.** For any graph \( G \), \( F(G) \) is connected if and only if any two maximal forests of \( G \) differ by at most a finite number of edges.

**Corollary 2.5.** If \( G = K_\alpha \), \( \alpha \) infinite, then \( F(G) \) is disconnected.

**Lemma 2.6.** Let \( G \) be a graph with \( \alpha \) vertices and \( \beta \) edges and with no isolated vertices. If either \( \alpha \) or \( \beta \) is infinite, then \( \alpha = \beta \).

**Proof:** We know that \( |E(G)| \leq |V(G)|^2 \), i.e., \( \beta \leq \alpha^2 \) so if \( \beta \) is infinite, \( \alpha \) must also be infinite. We also know, since each edge has two endpoints, that \( |V(G)| \leq 2|E(G)| \), i.e., \( \alpha \leq 2 \beta \) so if \( \alpha \) is infinite, then \( \beta \) must be infinite. Now assuming both are infinite, \( \alpha^2 = \alpha \) and \( 2 \beta = \beta \), hence \( \alpha = \beta \).

The following lemmas are needed in connection with \( F \)-convergence and \( F \)-divergence in Section 5 and \( F \)-depth in Section 4.

**Lemma 2.7.** Let \( G \) be a graph. If \( K_n \) (for finite \( n \geq 2 \)) is a subgraph of \( G \), then \( K_{\lfloor n^2/4 \rfloor} \) is a subgraph of \( F(G) \).

**Proof:** Let \( G \) be a graph such that \( K_n \) (\( n \geq 2 \), finite) is a subgraph of \( G \) with vertex labels \( v_1, v_2, \ldots, v_n \). Then there is a path \( L = v_1, v_2, \ldots, v_n \) of order \( n \) in \( G \). Let \( F \) be a maximal forest of \( G \) such that \( F \) contains the path \( L \). In \( F \) if we replace the edge \( v_{[n/2]} v_{[n/2]+1} \) by any other edge \( v_i v_j \) where \( i = 1, \ldots, [n/2] \) and \( j = [n/2] + 1, \ldots, n \), we get a maximal forest \( F_{ij} \). Since there are \( \lfloor n^2/4 \rfloor \) such edges \( v_i v_j \), there are \( \lfloor n^2/4 \rfloor \) maximal forests \( F_{ij} \) (of which one is \( F \)). Any two forests \( F_{ij} \) differ by one edge. It follows that they form a complete subgraph in \( F(G) \). Therefore \( K_{\lfloor n^2/4 \rfloor} \) is a subgraph of \( F(G) \).

**Lemma 2.8.** If \( G \) has a cycle of (finite) length \( n \) with \( n \geq 3 \), then \( F(G) \) contains \( K_n \).
Proof: Suppose that $G$ has a cycle $C_n$ of length $n$ with edge set $\{e_1, e_2, \ldots, e_n\}$. Let $P_i = C_n - e_i$ for $i = 1, 2, \ldots, n$ and let $F_1$ be a maximal forest of $G$ containing the path $P_1$. Define $F_i = F_1 \setminus P_i \cup P_i$ for $i = 2, 3, \ldots, n$. These $F_i$’s are maximal forests of $G$ and any two of them differ by exactly one edge, so they form a complete graph $K_n$ in $\mathbf{F}(G)$.

In particular, $\mathbf{F}(C_n) = K_n$.

Lemma 2.9. Suppose that $G$ contains $K_n$, where $n \geq 3$. Then $\mathbf{F}^2(G)$ contains $K_{n^2-2}$.

Proof: Cayley’s formula states that $K_n$ has $n^{n-2}$ spanning trees. Cummins [2] proved that the tree graph of a finite connected graph is Hamiltonian. Therefore, $\mathbf{F}(K_n)$ contains $C_{n^{n-2}}$. Let $F_{T_0}$ be a spanning tree of $G$ that extends one of the spanning trees $T_0$ of the $K_n$ subgraph. Replacing the edges of $T_0$ in $F_{T_0}$ by the edges of any other spanning tree $T$ of $K_n$, we have a spanning tree $F_T$ that contains $T$. The $F_T$’s for all spanning trees $T$ of $K_n$ are $n^{n-2}$ spanning trees of $G$ that differ only within $K_n$; thus, the graph of the $F_T$’s is the same as the graph of the $T$’s, which is Hamiltonian. That is, $\mathbf{F}(G)$ contains $C_{n^{n-2}}$. By Lemma 2.8, $\mathbf{F}^2(G)$ contains $K_{n^{n-2}}$.

We do not know exactly what graphs $\mathbf{F}(K_n)$ and $\mathbf{F}^2(K_n)$ are.

Lemma 2.10. If $G$ has two edge disjoint triangles, then $\mathbf{F}^2(G)$ contains $K_9$.

Proof: Suppose that $G$ has two edge disjoint triangles whose edges are $e_1, e_2, e_3$ and $f_1, f_2, f_3$, respectively. The union of the triangles has exactly 9 maximal forests $F'_{ij}$, obtained by deleting one $e_i$ and one $f_j$ from the triangles. Extend $F'_{11}$ to a maximal forest $F_{11}$ and let $F_{ij}$ be the maximal forest $F_{11} \setminus E(F_{11}) \cup F_{ij}$, for each $i, j = 1, 2, 3$. The nine maximal forests $F_{ij}$, and consequently the maximal forests $F_{ij}$ in $\mathbf{F}(G)$, form a Cartesian product graph $C_3 \times C_3$, which contains a cycle of length 9. By Lemma 2.8, $\mathbf{F}^2(G)$ contains $K_9$.

We now show that repeated application of the forest graph operator to many graphs creates larger and larger complete subgraphs.

Lemma 2.11. If $G$ has a cycle of (finite) length $n$ with $n \geq 4$ or it has two edge disjoint triangles, then for any finite $m \geq 1$, $\mathbf{F}^m(G)$ contains $K_{m^2}$.

Proof: We prove this lemma by induction on $m$.

Case 1: Suppose that $G$ has a cycle $C_n$ of length $n$ ($n \geq 4$, $n$ finite). By Lemma 2.8, $\mathbf{F}(G)$ contains $K_n$ as a subgraph, which implies that $\mathbf{F}(G)$ contains $K_4$. By Lemma 2.9, $\mathbf{F}^3(G)$ contains $K_{16}$ and in particular it contains $K_{3^2}$.

Case 2: Suppose that $G$ has two edge disjoint triangles. By Lemma 2.10, $\mathbf{F}^2(G)$ contains $K_9$ as a subgraph. It follows by Lemma 2.7 that $\mathbf{F}^3(G)$ contains $K_{[9^2/4]} = K_{20}$ as a subgraph. This implies that $\mathbf{F}^3(G)$ contains $K_{3^2}$ as a subgraph.

By Cases 1 and 2 it follows that the result is true for $m = 1, 2, 3$. Let us assume that the result is true for $m = l \geq 3$, i.e., that $\mathbf{F}^l(G)$ contains $K_{l^2}$ as a subgraph. By Lemma 2.7, it follows that $\mathbf{F}(\mathbf{F}^l(G))$ has a subgraph $K_{[l^4/4]}$. Since $[l^4/4] > (l + 1)^2$, it follows that $\mathbf{F}^{l+1}(G)$ contains $K_{(l+1)^2}$. By the induction hypothesis $\mathbf{F}^m(G)$ contains $K_{m^2}$ for any finite $m \geq 1$.

With Lemma 2.9 it is clearly possible to prove a much stronger lower bound on complete subgraphs of iterated forest graphs, but Lemma 2.11 is good enough for our purposes.
Lemma 2.12. A forest graph that is not $K_1$ has no isolated vertices and no isthmi.

**Proof:** Let $G = F(H)$ for some graph $H$. Consider a vertex $F$ of $G$, that is, a maximal forest in $H$. Let $e$ be an edge of $F$ that belongs to a cycle $C$ in $H$. Then there is an edge $f$ in $C$ that is not in $F$ and $F' = F - e + f$ is a second maximal forest that is adjacent to $F$ in $G$. Since $C$ has length at least 3, it has a third edge $g$. If $g$ is not in $F$, let $F'' = F - e + g$. If $g$ is in $F$, let $F'' = F - g + f$. In both cases $F''$ is a maximal forest that is adjacent to $F$ and $F'$. Thus, $F$ is not isolated and the edge $FF'$ in $G$ is not an isthmus.

Suppose $F, F' \in \mathcal{N}(H)$ are adjacent in $G$. That means there are edges $e \in E(F)$ and $e' \in E(F')$ such that $F' = F - e + e'$. Thus, $e$ belongs to the unique cycle in $F + e'$. As shown above, there is an $F'' \in \mathcal{N}(H)$ that forms a cycle with $F$ and $F'$. Therefore the edge $FF'$ of $G$ is not an isthmus.

Let $F \in \mathcal{N}(H)$ be an isolated vertex in $G$. If $H$ has an edge $e$ not in $F$, then $F + e$ contains a cycle so $F$ has a neighboring vertex in $G$, as shown above. Therefore, no such $e$ can exist; in other words, $H = F$ and $G$ is $K_1$.

3 Basic Properties of an Infinite Forest Graph

We now present a crucial foundation for the proof of the main theorem in Section 5. The cyclomatic number $\beta_1(G)$ of a graph $G$ can be defined as the cardinality $|E(G) \setminus E(F)|$ where $F$ is a maximal forest of $G$.

**Proposition 3.1.** Let $G$ be a graph such that $|\mathcal{C}(G)| = \beta$, an infinite cardinal number. Then:

i) $\beta_1(G) = \beta$ and $\beta_1(F(G)) = 2^\beta$.

ii) Both the order of $F(G)$ and its number of edges equal $2^\beta$. Both the order and the number of edges of $G$ equal $\beta$, provided that $G$ has no isolated vertices and no isthmi.

iii) $F(G)$ is $\beta$-regular.

iv) The order of any connected component of $F(G)$ is $\beta$, and it has exactly $\beta$ edges.

v) $F(G)$ has exactly $2^\beta$ components.

vi) Every component of $F(G)$ has exactly $\beta$ cycles.

vii) $|\mathcal{C}(F(G))| = 2^\beta$.

**Proof:** Let $G$ be a graph with $|\mathcal{C}(G)| = \beta$ ($\beta$ infinite).

i) Let $F$ be a maximal forest of $G$. The number of cycles in $G$ is not more than the number of finite subsets of $E(G) \setminus E(F)$. This number is finite if $E(G) \setminus E(F)$ is finite, but it cannot be finite because $|\mathcal{C}(G)|$ is infinite. Therefore $E(G) \setminus E(F)$ is infinite and the number of its finite subsets equals $|E(G) \setminus E(F)| = \beta_1(G)$. Thus, $\beta_1(G) \geq |\mathcal{C}(G)|$. The number of cycles is at least as large as the number of edges not in $F$, because every such edge makes
a different cycle with $F$. Thus, $|\mathfrak{C}(G)| \geq \beta_1(G)$. It follows that $\beta_1(G) = |\mathfrak{C}(G)| = \beta$. Note that this proves $\beta_1(G)$ does not depend on the choice of $F$.

The value of $\beta_1(F(G))$ follows from this and part (vii).

ii) For the first part, let $F$ be a maximal forest of $G$ and let $F_0$ be a maximal forest of $G \setminus E(F)$. As $G \setminus E(F)$ has $\beta_1(G) = \beta$ edges by part (i), it has $\beta$ non-isolated vertices by Lemma 2.6. $F_0$ has the same number of non-isolated vertices, so it too has $\beta$ edges.

Any edge set $A \subseteq F_0$ extends to a maximal forest $F_A$ in $F \cup A$. Since $F_A \setminus F = A$, the $F_A$’s are distinct. Therefore, there are at least $2^\beta$ maximal forests in $F_0 \cup F$. The maximal forest $F$ consists of a spanning tree in each component of $G$; therefore, the vertex sets of components of $F$ are the same as those of $G$, and so are those of $F_0 \cup F$. Therefore, a maximal forest in $F_0 \cup F$, which consists of a spanning tree in each component of $F_0 \cup F$, contains a spanning tree of each component of $G$.

We conclude that a maximal forest in $F_0 \cup F$ is a maximal forest of $G$ and hence that there are at least $2^\beta$ maximal forests in $G$, i.e., $|\mathfrak{N}(G)| \geq 2^\beta$. Since $G$ is a subgraph of $K_\beta$, and since $|\mathfrak{N}(K_\beta)| = 2^\beta$ by Lemma 2.2, we have $|\mathfrak{N}(G)| \leq 2^\beta$. Therefore $|\mathfrak{N}(G)| = 2^\beta$. That is, the order of $F(G)$ is $2^\beta$. By Lemmas 2.12 and 2.7, that is also the number of edges of $F(G)$.

For the second part, note that $G$ has infinite order or else $\beta_1(G)$ would be finite. If $G$ has no isolated vertices and no isthmi, then $|V(G)| = |E(G)|$ by Lemma 2.6. By part (i) there are $\beta$ edges of $G$ outside a maximal forest; hence $\beta \leq |E(G)|$.

Since every edge of $G$ is in a cycle, by the axiom of choice we can choose a cycle $C(e)$ containing $e$ for each edge $e$ of $G$. Let $\mathfrak{C} = \{C(e) : e \in E(G)\}$. The total number of pairs $(f, C)$ such that $f \in C \in \mathfrak{C}$ is no more than $\aleph_0 |\mathfrak{C}| \leq \aleph_0 |\mathfrak{C}(G)| = \aleph_0 \beta = \beta$. This number of pairs is not less than the number of edges, so $|E(G)| \leq \beta$. It follows that $G$ has exactly $\beta$ edges.

iii) Let $F$ be a maximal forest of $G$. By part (i), $|E(G) \setminus E(F)| = \beta$. By adding any edge $e$ from $E(G) \setminus E(F)$ to $F$ we get a cycle $C$. Removing any edge other than $e$ from the cycle $C$ gives a new maximal forest which differs by exactly one edge with $F$. The number of maximal forests we get in this way is $\beta_1(G)$ because there are $\beta_1(G)$ ways to choose $e$ and a finite number of edges of $C$ to choose to remove, and $\beta_1(G)$ is infinite. Thus we get $\beta$ maximal forests of $G$, each of which differs by exactly one edge with $F$. Every such maximal forest is generated by this construction. Therefore, the degree of any vertex in $F(G)$ is $\beta$.

iv) Let $A$ be a connected component of $F(G)$. As $F(G)$ is $\beta$-regular by part (iii), it follows that $|V(A)| \geq \beta$. Fix a vertex $v$ in $A$ and define the $n$th neighborhood $D_n = \{v' : d(v, v') = n\}$ for each $n$ in $\mathbb{N}$. Since every vertex has degree $\beta$, $|D_0| = 1$, $|D_1| = \beta$, and $|D_k| \leq \beta |D_{k-1}|$. Thus, by induction on $n$, $|D_n| \leq \beta$ for $n > 0$.

Since $A$ is connected, it follows that $V(A) = \bigcup_{n \in \mathbb{N}} D_n$, i.e., $V(A)$ is the countable union of sets of order $\beta$. Therefore $|A| = \beta$, as $|\mathbb{N}| \beta'^* = \beta'$. Hence any connected component of $F(G)$ has $\beta$ vertices. By Lemma 2.6 it has $\beta$ edges.

v) By parts (ii, iv) the order of $F(G)$ is $2^\beta$ and the order of each component of $F(G)$ is $\beta$. Since $|F(G)| = 2^\beta$, $F(G)$ has at most $2^\beta$ components. Suppose that $F(G)$ has $\beta'$ components where $\beta' < 2^\beta$. As each component has $\beta$ vertices, it follows that $F(G)$ has order at most $\beta' \beta = \max\{\beta', \beta\}$. This is a contradiction to part (ii). Therefore $F(G)$ has
exactly $2^\beta$ components.

vi) Let $A$ be a component of $F(G)$. Since it is infinite, by part (iv) it has exactly $\beta$ edges. Suppose that $|\mathcal{C}(A)| = \beta'$. Then $\beta'$ is at most the number of finite subsets of $E(A)$, which is $\beta$ since $|E(A)| = \beta$ is infinite; that is, $\beta' \leq \beta$. By the argument in part (iii) every edge of $F(G)$ lies on a cycle. The length of each cycle is finite. Thus $A$ has at most $\aleph_0, \beta' = \max\{\beta', \aleph_0\} = \beta'$ edges if $\beta'$ is infinite and it has a finite number of edges if $\beta'$ is finite. Since $|E(A)| = \beta$, which is infinite, $\beta' \geq \beta$. We conclude that $\beta' = \beta$.

vii) By parts (v, vi) $F(G)$ has $2^\beta$ components and each component has $\beta$ cycles. Since every cycle is contained in a component, $|\mathcal{C}(F(G))| = \beta.2^\beta = 2^\beta$.

From the above proposition it follows that an infinite graph cannot be a forest graph unless every component has the same infinite order $\beta$ and there are $2^\beta$ components. A consequence is that the infinite graph itself must have order $2^\beta$. Hence,

**Corollary 3.2.** Any infinite graph whose order is not a power of 2, including $\aleph_0$ and all other limit cardinals, is not a forest graph.

**Corollary 3.3.** For a graph $G$ the following statements are equivalent.

i) $F(G)$ is connected.

ii) $F(G)$ is finite.

iii) The union of all cycles in $G$ is a finite graph.

**Proof:** (i) $\implies$ (iii). Suppose that $F(G)$ is connected. If $G$ has infinitely many cycles then by Proposition 3.1(v) $F(G)$ is disconnected. Therefore $G$ has finitely many cycles. Let $A = \{e \in E(G) : \text{edge } e \text{ lies on a cycle in } G\}$. Then $|A|$ is finite because the length of each cycle is finite. That proves (iii).

(iii) $\implies$ (ii). As every maximal forest of $G$ consists of a maximal forest of $A$ and all the edges of $G$ which are not in $A$, $G$ has at most $2^n$ maximal forests where $n = |A|$. Hence $F(G)$ has a finite number of vertices and consequently is finite.

(ii) $\implies$ (i). By identifying vertices in different components (Whitney vertex identification; see Section 1) we can assume $G$ is connected so $F(G) = T(G)$. Cummins [2] proved that the tree graph of a finite graph is Hamiltonian; therefore it is connected. $\blacksquare$

## 4 F-Roots

In this section we establish properties of $F$-roots of graphs. We begin with the question of what an $F$-root should be.

Since any graph $H'$ that is isomorphic to an $F$-root $H$ of $G$ is immediately also an $F$-root, the number of non-isomorphic $F$-roots is a better question than the number of labeled $F$-roots. We now show in some detail that a still better question is the number of non-isomorphic $F$-roots without isthmus.

Let $t_\beta$ be the number of non-isomorphic rooted trees of order $\beta$. We note that $t_{\aleph_0} \geq 2^{\aleph_0}$, by a construction of Reinhard Diestel (personal communication, July 10, 2015). (We do not
know a corresponding lower bound on $t_{\beta}$ for $\beta > \aleph_0$.) Let $P$ be a one-way infinite path whose vertices are labelled by natural numbers, with root 1; choose any subset $S$ of $\mathbb{N}$ and attach two edges at every vertex in $S$, forming a rooted tree $T_S$ (rooted at 1). Then $S$ is determined by $T_S$ because the vertices in $S$ are those of degree at least 3 in $T_S$. (If $2 \in S$ but $1 \notin S$, then vertex 1 is determined only up to isomorphism by $T_S$, but $S$ itself is determined uniquely.) The number of sets $S$ is $2^{\aleph_0}$, hence $t_{\aleph_0} \geq 2^{\aleph_0}$.

**Proposition 4.1.** Let $G$ be a graph with an $F$-root of order $\alpha$. If $\alpha$ is finite, then $G$ has infinitely many non-isomorphic finite $F$-roots. If $\alpha$ is finite or infinite, then $G$ has at least $t_{\beta}$ non-isomorphic $F$-roots of order $\beta$ for every infinite $\beta \geq \alpha$.

**Proof:** Let $G$ be a graph which has an $F$-root $H$, i.e., $F(H) \equiv G$, and let $\alpha$ be the order of $H$. We may assume $H$ has no isthmus and no isolated vertices unless it is $K_1$.

Suppose $\alpha$ is finite; then let $T$ be a tree, disjoint from $H$, of any finite order $n$. Identify any vertex $v$ of $H$ with any vertex $w$ of $T$. The resulting graph $H_T$ also has $G$ as its forest graph since $T$ is contained in every maximal forest of $H_T$. As the order of $H_T$ is $\alpha + n - 1$ and $n$ can be any natural number, the graphs $H_T$ are an infinite number of non-isomorphic finite graphs with the same forest graph up to isomorphism.

Suppose $\alpha$ is finite or infinite and $\beta \geq \alpha$ is infinite. Let $T$ be a rooted tree of order $\beta$ with root vertex $w$; for instance, $T$ can be a star rooted at the star center. Attach $T$ to a vertex $v$ of $H$ by identifying $v$ with the root vertex $w$. Denote the resulting graph by $H_T$; it is an $F$-root of $G$ and it has order $\beta$ because it has order $\alpha + \beta$, which equals $\beta$ because $\beta$ is infinite and $\beta \geq \alpha$. As $H$ has no isthmus, $T$ and $w$ are determined by $H_T$; therefore, if we have a non-isomorphic rooted tree $T'$ with root $w'$ (that means there is no isomorphism of $T$ with $T'$ in which $w$ corresponds to $w'$), $H_{T'}$ is not isomorphic to $H_T$. (The one exception is when $H = K_1$, which is easy to treat separately.) The number of non-isomorphic $F$-roots of $G$ of order $\beta$ is therefore at least the number of non-isomorphic rooted trees of order $\beta$, i.e., $t_{\beta}$.

Proposition 4.1 still does not capture the essence of the number of $F$-roots. Whitney’s 2-operations on a graph $G$ are the following [17]:

1. **Whitney vertex identification.** Identify a vertex in one component of $G$ with a vertex in another component of $G$, thereby reducing the number of components by 1. For an infinite graph we modify this by allowing an infinite number of vertex identifications; specifically, let $W$ be a set of vertices with at most one from each component of $G$, and let $\{W_i : i \in I\}$ be a partition of $W$ into $|I|$ sets (where $I$ is any index set); then for each $i \in I$ we identify all the vertices in $W_i$ with each other.

2. **Whitney vertex splitting.** The reverse of vertex identification.

3. **Whitney twist.** If $u, v$ are two vertices that separate $G$—that is, $G = G_1 \cup G_2$ where $G_1 \cap G_2 = \{u, v\}$ and $|V(G_1)|, |V(G_2)| > 2$, then reverse the names $u$ and $v$ in $G_2$ and then take the union $G_1 \cup G_2$ (so vertex $u$ in $G_1$ is identified with the former vertex $v$ in $G_2$ and $v$ with the former vertex $u$). Call the new graph $G'$. For an infinite graph we allow an infinite number of Whitney twists.
It is easy to see that the edge sets of maximal forests in $G$ and $G'$ are identical, hence $F(G)$ and $F(G')$ are naturally isomorphic. It follows by Whitney vertex identification that every graph with an $F$-root has a connected $F$-root, and it follows from Whitney vertex splitting that every graph with an $F$-root has an $F$-root without cut vertices.

We may conclude from Proposition 4.1 that the most interesting question about the number of $F$-roots of a graph $G$ that has an $F$-root is not the total number of non-isomorphic $F$-roots (which by Proposition 4.1 cannot be assigned any cardinality); it is not the number of a given order; it is not even the number that have no isthmi; it is the number of non-2-isomorphic, connected $F$-roots with no isthmi and (except when $G = K_1$) no isolated vertices.

We do not know which graphs have $F$-roots, but we do know two large classes that cannot have $F$-roots.

**Theorem 4.2.** No infinite connected graph has an $F$-root.

**Proof:** This follows by Corollary 3.3.

**Theorem 4.3.** No bipartite graph $G$ has an $F$-root.

**Proof:** Let $G$ be a bipartite graph of order $p$ ($p \geq 2$) and let $H$ be a root of $G$, i.e., $F(H) \cong G$. Suppose $H$ has no cycle; then $F(H)$ is $K_1$, which is a contradiction. Therefore $H$ has a cycle of length $\geq 3$. It follows by Lemma 2.8 that $F(H)$ contains $K_3$, a contradiction. Hence no bipartite graph $G$ has a root.

## 5 F-Convergence and F-Divergence

In this section we establish the necessary and sufficient conditions for $F$-convergence of a graph.

**Lemma 5.1.** Let $G$ be a finite graph that contains a $C_n$ (for $n \geq 4$) or at least two edge disjoint triangles; then $G$ is $F$-divergent.

**Proof:** Let $G$ be a finite graph. By Lemma 2.11 $F^m(G)$ contains $K_{m^2}$ as a subgraph. Therefore, as $m$ increases the clique size of $F^m(G)$ increases. Hence $G$ is $F$-divergent.

**Lemma 5.2.** If $|\mathcal{C}(G)| = \beta$ where $\beta$ is infinite, then $G$ is $F$-divergent.

**Proof:** Assume $|\mathcal{C}(G)| = \beta$ ($\beta$ infinite). By Proposition 3.1 vii), as $2^3 < 2^8 < 2^{2^8} < \cdots$, it follows that $|\mathcal{C}(F(G))| < |\mathcal{C}(F^2(G))| < |\mathcal{C}(F^3(G))| < \cdots$. Therefore, as $n$ increases $|\mathcal{C}(F^n(G))|$ increases. Hence $G$ is $F$-divergent.

**Theorem 5.3.** Let $G$ be a graph. Then,

i) $G$ is $F$-convergent if and only if either $G$ is acyclic or $G$ has only one cycle, which is of length 3.
ii) If $G$ is $F$-convergent, then it converges in at most two steps.

**Proof:** i) If $G$ has no cycle, then it is a forest and $F(G) = K_1$. If $G$ has only one cycle and that cycle has length 3, then $F(G) = K_3$. Therefore in each case $G$ is $F$-convergent.

Conversely, suppose that $G$ has a cycle of length greater than 3 or has at least two triangles. If $G$ has infinitely many cycles, then it follows by Lemma 5.2 that $G$ is $F$-divergent. Therefore we may assume that $G$ has a finite number of cycles. If $G$ has a finite number of vertices, then it is finite and by Lemma 5.1 it is $F$-divergent. Therefore $G$ has an infinite number of vertices. However, it can have only a finite number of edges that are not isthmi, because each cycle is finite. Thus $G$ consists of a finite graph $G_0$ and any number of isthmi and isolated vertices. Since $F(G)$ depends only on the edges that are not isthmi and the vertices that are not isolated, $F(G) = F(G_0)$ (under the natural identification of maximal forests in $G_0$ with their extensions in $G$ by adding all isthmi of $G$). Therefore, $G$ is $F$-divergent.

ii) If $G$ has no cycle, then $G$ is a forest and $F(G) \cong F^2(G) \cong K_1$. If $G$ has only one cycle, which is of length 3, then $F(G) \cong F^2(G) \cong K_3$. Therefore $G$ converges in at most 2 steps.

**Corollary 5.4.** A graph $G$ is $F$-stable if and only if $G = K_1$ or $K_3$.

6 **F-Depth**

In this section we establish results about the $F$-depth of a graph.

**Theorem 6.1.** Let $G$ be a finite graph. The $F$-depth of $G$ is infinite if and only if $G$ is $K_1$ or $K_3$.

**Proof:** Let $G$ be a finite graph. Suppose that $G$ is $K_1$ or $K_3$. Then by Corollary 5.4 it follows that $G$ is $F$-stable. Therefore, the $F$-depth of $G$ is infinite.

Conversely, suppose that $G$ is different from $K_1$ and $K_3$.

**Case 1:** Let $|V| < 4$. Then $G$ has no $F$-root so its $F$-depth is zero.

**Case 2:** Let $|V| = 4$. Suppose $G$ has an $F$-root $H$ (i.e., $F(H) \cong G$). Then $H$ should have exactly 4 maximal forests. That is possible only when $H$ has only one cycle, which is of length 4. By Lemma 2.8 it follows that $F(H)$ contains $K_4$, hence it is $K_4$. Therefore $G$ has an $F$-root if and only if it is $K_4$. Hence the $F$-depth of $G$ is zero, except that the depth of $K_4$ is 1.

**Case 3:** Let $|V| = n$ where $n > 4$. Suppose that $G$ has infinite $F$-depth. Then for every $m$ there is a graph $H_m$ such that $F^m(H_m) = G$. If $H_m$ does not have two triangles or a cycle of length greater than 3, then $H_m$ has only one cycle which is of length 3, or no cycle and $H_m$ converges to $K_1$ or $K_3$ in at most two steps, a contradiction. Therefore $H_m$ has two triangles or a cycle of length greater than 3. By Lemma 2.11 it follows that $F^m(H_m)$ contains $K_{m^2}$ for each $m \geq 2$, so that in particular $F^n(H_n)$ contains $K_{n^2}$. That is, $G$ contains $K_{n^2}$. This is impossible as $G$ has order $n$. Hence the $F$-depth of $G$ is finite.
Theorem 6.2. The $F$-depth of any infinite graph is finite.

Proof: Let $G$ be a graph of infinite order $\alpha$. If $G$ has an $F$-root, then $G$ is without isthmi or isolated vertices.

If $G$ is connected, Theorem 4.2 implies that $G$ has no root. Therefore its $F$-depth is zero.

If $G$ is disconnected, assume it has infinite depth. Then for each natural number $n$ there exists a graph $H_n$ such that $G \cong F^n(H_n)$. Let $\beta_n$ denote the order of $H_n$. Since $F(H_1) \cong G$, by Proposition 3.1(ii) $\alpha = 2^{\beta_1}$, from which we infer that $\beta_1 < \alpha$. This is independent of which root $H_1$ is, so in particular we can take $H_1 = F(H_2)$ and conclude that $\beta_1 = 2^{\beta_2}$, hence that $\beta_2 < \beta_1$. Continuing in like manner we get an infinite decreasing sequence of cardinal numbers starting with $\alpha$. The cardinal numbers are well ordered \cite{9}, so they cannot contain such an infinite sequence. It follows that the $F$-depth of $G$ must be finite.

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