ON ANNIHILATORS OF BOUNDED \((g, \mathfrak{t})\)-MODULES

ALEXEY PETUKHOV

Abstract. Let \(g\) be a semisimple Lie algebra and \(\mathfrak{t} \subset g\) be a reductive subalgebra. We say that a \(g\)-module \(M\) is a bounded \((g, \mathfrak{t})\)-module if \(M\) is a direct sum of simple finite-dimensional \(\mathfrak{t}\)-modules and the multiplicities of all simple \(\mathfrak{t}\)-modules in that direct sum are universally bounded.

The goal of this article is to show that the “boundedness” property for a simple \((g, \mathfrak{t})\)-module \(M\) is equivalent to a property of the associated variety of the annihilator of \(M\) (this is the closure of a nilpotent coadjoint orbit inside \(g^*\)) under the assumption that the main field is algebraically closed and of characteristic 0. In particular this implies that if \(M_1, M_2\) are simple \((g, \mathfrak{t})\)-modules such that \(M_1\) is bounded and the associated varieties of the annihilators of \(M_1\) and \(M_2\) coincide then \(M_2\) is also bounded. This statement is a geometric analogue of a purely algebraic fact due to I. Penkov and V. Serganova and it was posed as a conjecture in my Ph.D. thesis.

Key words: \((g, \mathfrak{t})\)-modules, spherical varieties, symplectic geometry.

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1. Introduction

A notion of admissible \((g, \mathfrak{t})\)-module for a generic pair of Lie algebras \((g, \mathfrak{t})\) is a quite straightforward generalization of both Harish-Chandra modules and \(g\)-modules of category \(\mathcal{O}\). Bounded \((g, \mathfrak{t})\)-modules \([PS2]\) are a very specific subclass of the admissible \((g, \mathfrak{t})\)-modules and it turns out that this class is related to the notion of spherical variety \([14]\). The structure of this subcategory is still very mysterious in general and few examples are evaluated explicitly, see \([GS, PSZ2, Pe3]\).

The set works on both Harish-Chandra modules and category \(\mathcal{O}\) seems to be infinite and one can find some introduction to these subjects in \([KV, Hu]\). We would like to mention explicitly works \([Pe1, Pe2, PZ, GS, PSZ1, Zu, Fe, Ma, Mi]\).

The goal of this article is to show that this “boundedness” property for a simple \((g, \mathfrak{t})\)-module \(M\) is equivalent to a property of the associated variety of the annihilator of \(M\) (this is the closure of a nilpotent coadjoint orbit inside \(g^*\)) under the assumption that the main field is algebraically closed and of characteristic 0. Roughly speaking, a simple \((g, \mathfrak{t})\)-module \(M\) is bounded if and only if the associated variety of the annihilator of \(M\) is \(K\)-coisotropic with respect to the action of an algebraic group \(K\) attached to the pair \((g, \mathfrak{t})\).

In particular this implies that if \(M_1, M_2\) are simple \((g, \mathfrak{t})\)-modules such that \(M_1\) is bounded and the associated varieties of the annihilators of \(M_1\) and \(M_2\) coincide then \(M_2\) is also bounded. This statement is a geometric analogue of \([PS1\) Theorem 4.3] and it was posed as a conjecture in my Ph.D. thesis \([Pe3\ Conjecture 2.1]\).

2. Definitions

2.1. \((g, \mathfrak{t})\)-modules. Let \(g\) be a Lie algebra over an algebraically closed field \(\mathbb{F}\) of characteristic 0 and \(\mathfrak{t}\) be a subalgebra of \(g\). We say that a \(g\)-module \(M\) is a \((g, \mathfrak{t})\)-module if

\[
\forall m \in M (\dim(\mathfrak{u}(\mathfrak{t}) \cdot m) < \infty)
\]

where \(\mathfrak{u}(\mathfrak{t})\) is the universal enveloping algebra of \(\mathfrak{t}\).

It is clear that any \((g, \mathfrak{t})\)-module is isomorphic to the direct limit of a directed set of finite-dimensional \(\mathfrak{t}\)-modules. Thus for any given simple \(\mathfrak{t}\)-module \(V\) we can define the Jordan-Hölder multiplicity \([M : V]\). If such multiplicities \([M : V]\) are finite for all simple \(\mathfrak{t}\)-modules \(V\) then \(M\) is called admissible. If there exists \(C_M \in \mathbb{N}\) such that all multiplicities \([M : V]\) are bounded by \(C_M\) then \(M\) is called bounded.

2.2. Associated varieties. Let \(M\) be a finitely generated \(g\)-module \(M\) (for example \(M\) can be a simple \(g\)-module). We consider a finite-dimensional generating subspace \(M_0\) of \(M\) and define spaces \(M_i\) \((i \geq 0)\) inductively using formula

\[
M_{i+1} := M_i + g \cdot M_i.
\]

The associated graded object

\[
\text{gr} M := \oplus_{i \geq 0} M_{i+1}/M_i
\]

is a module of the associated graded algebra \(S(g)\) of \(U(g)\). Let \(J\) be the annihilator of \(M\) in \(S(g)\). We denote by \(\text{Var}(M)\) the set of points \(\chi \in g^*\) such that \(f(\chi) = 0\) for all \(f \in J\).
It is well known that \( \text{Var}(M) \) is the same for all choices of \( M_0 \), see for example [Pe3, p. 19], and therefore \( \text{Var}(M) \) is a proper invariant of \( M \). Next, we consider the annihilator \( \text{Ann}_{U(g)} M \) of \( M \) in \( U(g) \) as a left \( g \)-module and put

\[
\text{GVar}(M) := \text{Var}(\text{Ann}_{U(g)} M).
\]

We denote by \( G \cdot \text{Var}(M) \) the closure of the \( G \)-translation of \( \text{Var}(M) \). Note that in many interesting cases

\[
\text{GVar}(M) = G \cdot \text{Var}(M).
\]

Put \( \sqrt{J} := \{ f \in S(g) \mid \exists n \in \mathbb{Z}_{>0} : (f^n \in J) \} \). The ideal \( J \) is involutive, i.e. \( \{ \sqrt{J}, \sqrt{J} \} \subseteq \sqrt{J} \) where \( \{ \cdot, \cdot \} \) is the standard Poisson bracket on \( S(g) \), see [Ga].

Next, if \( M \) is a \( \langle g, t \rangle \)-module then one can choose filtration \( \{ M_i \} \) to be \( t \)-stable and hence \( t \subseteq J \). This condition immediately implies

\[
\text{Var}(M) \subset t^\perp
\]

where \( t^\perp := \{ \chi \in g^* \mid \chi|_t = 0 \} \).

3. **Main theorem**

We would use notions and terminology on symplectic geometry and algebraic groups actions of \[TZh\]. Fix a semisimple Lie algebra \( g \) together with the adjoint group \( G \) of \( g \), and also fix a subalgebra \( t \subset g \). Assume that \( t \) is algebraically reductive in \( g \), that is there exists a connected reductive algebraic subgroup \( K \) of \( G \) such that \( t \subset g \) is the Lie algebra of \( K \subset G \). If such a subgroup \( K \) exists then it is unique.

**Theorem 3.1.** Assume that \( g \) is a semisimple Lie algebra and \( t \) is an algebraically reductive subalgebra of \( g \). Then, for a simple \( \langle g, t \rangle \)-module \( M \), the following conditions are equivalent:

1. \( M \) is a bounded \( \langle g, t \rangle \)-module,
2. \( G \cdot \text{Var}(M) \) is \( K \)-coisotropic.

It turns out that it is possible to reduce the case of any subalgebra \( t \) of a semisimple Lie algebra \( g \) to the case of algebraically reductive subalgebra, see Propositions 4.8 4.9.

4. **Algebraic subalgebras**

Let \( t \) be a subalgebra of a semisimple Lie algebra \( g \), \( G \) be the adjoint group of \( g \), and let \( \hat{K} \) be the least algebraic subgroup of \( G \) such that the Lie algebra of \( \hat{K} \) contains \( t \). Denote by \( \hat{t} \) the Lie algebra of \( \hat{K} \). Then

- \( \hat{K} \) is connected and \( \hat{K} \) normalizes \( t \),
- \( \hat{t}/t \) is abelian and \( [\hat{t}, \hat{t}] = [t, t] \),

see [OV, Subsection 3.3.3].

**Definition 4.1.** A subalgebra \( t \) is called algebraic in \( g \) if \( t = \hat{t} \). A subalgebra \( t \) is called reductive in \( g \) if \( \hat{K} \) is reductive.

It is clear that if \( t \) is algebraically reductive in \( g \) then \( t \) is reductive in \( g \). The inverse statement is false.

**Example 4.2.** Let \( g \cong sl(3) \) be the Lie algebra of traceless \( 3 \times 3 \) matrices and let \( t \) be the one-dimensional subalgebra spanned by

\[
\begin{pmatrix}
\alpha & 0 & 0 \\
0 & \beta & 0 \\
0 & 0 & -(\alpha + \beta)
\end{pmatrix},
\]

\( \alpha, \beta \in \mathbb{F} \). If

\[
\alpha, \beta \in \mathbb{F}\backslash\mathbb{Q} \text{ and } \frac{\alpha}{\beta} \in \mathbb{F}\backslash\mathbb{Q}
\]

then \( \hat{K} \) is an algebraic torus consisting of the diagonal \( 3 \times 3 \) matrices with determinant 1, and therefore \( \hat{t} \) is the space of traceless diagonal \( 3 \times 3 \) matrices. In particular, this implies that \( t \) is reductive in \( sl(3) \), but \( t \) is not algebraically reductive in \( sl(3) \).

The following lemmas show that \( \hat{K} \)-modules, \( \hat{t} \)-modules and \( t \)-modules are close to each other.

**Lemma 4.3.** Let \( V_1, V_2 \) be \( \hat{K} \)-modules. Then the following conditions are equivalent.

1. \( V_1 \cong V_2 \),
2. \( (V_1)|_k \cong (V_2)|_k \),
3. \( (V_1)|_t \cong (V_2)|_t \).
Proof. Conditions (1) and (2) are equivalent for a connected group in characteristic 0. It is clear that (2) implies (3). We left to show that (3) implies (1).

Assume (3). Consider vector space Hom_{\mathbb{F}}(V_1, V_2). We have that Hom_{\mathbb{F}}(V_1, V_2) \neq 0. Fix (1) \[0 \neq \phi \in \text{Hom}_{\mathbb{F}}(V_1, V_2)\]
such that \(\phi\) defines the isomorphism between \(V_1\) and \(V_2\). Let \(H\) be the stabilizer of \(\phi\) in \(K\). Condition (11) implies that \(\mathfrak{g}\) is a subalgebra of the Lie algebra of \(H\). The definition of \(\hat{K}\) implies that \(H = \hat{K}\). Therefore \(\phi\) is an isomorphism of \(\hat{K}\)-modules.

**Lemma 4.4.** Let \(V\) be a \(K\)-module and let \(W \subset V\) be an \(\mathbb{F}\)-subspace of \(V\). The following conditions are equivalent.

1. \(W\) is \(\hat{K}\)-stable,
2. \(W\) is \(\mathfrak{k}\)-stable,
3. \(W\) is \(\mathfrak{g}\)-stable.

**Proof.** Conditions (1) and (2) are equivalent for a connected group in characteristic 0. It is clear that (2) implies (3). We left to show that (3) implies (1).

Assume (3). Let \(d\) be the dimension of \(W\). Consider variety \(\text{Gr}(d; V)\) of \(d\)-dimensional subspaces of \(V\) and denote by \([W]\) the point of \(\text{Gr}(d; V)\) defined by \(W\). Let \(H\) be the stabilizer of \([W]\) in \(K\). Condition (3) implies that \(\mathfrak{g}\) is a subalgebra of the Lie algebra of \(H\). The definition of \(\hat{K}\) implies that \(H = \hat{K}\). Therefore \(W\) is a \(\hat{K}\)-submodule of \(V\).

**Corollary 4.5.** Let \(V\) be a finite-dimensional \(\hat{K}\)-module. Then the lattices of \(\hat{K}\)-submodules of \(V\), \(\mathfrak{k}\)-submodules of \(V\) and \(\mathfrak{g}\)-submodules of \(V\) coincide.

**Corollary 4.6.** Let \(M\) be a direct limit of finite-dimensional \(\hat{K}\)-modules. Then

1. \(M\) is a bounded \((\mathfrak{g}, \hat{\mathfrak{k}})\)-module if and only if \(M\) is a bounded \((\mathfrak{k}, \mathfrak{g})\)-module,
2. \(M\) is an admissible \((\mathfrak{g}, \hat{\mathfrak{k}})\)-module if and only if \(M\) is an admissible \((\mathfrak{k}, \mathfrak{g})\)-module.

**Proof.** This statement is implied by Lemma 4.3 and Corollary 4.5.

We wish to establish a connection between the admissible \((\mathfrak{g}, \hat{\mathfrak{k}})\)-modules and the admissible \((\mathfrak{g}, \mathfrak{g})\)-modules. The first step is as follows.

**Lemma 4.7.** Let \(M\) be an admissible \((\mathfrak{g}, \hat{\mathfrak{k}})\)-module. Then \(M\) is a \((\mathfrak{g}, \hat{\mathfrak{k}})\)-module.

**Proof.** Let \(M_0\) be a \(\mathfrak{k}\)-stable subspace of \(M\). Consider the \(\hat{\mathfrak{k}}\)-submodule \(\hat{M}_0\) of \(M\) generated by \(M_0\). It is enough to show that \(\dim \hat{M}_0 < \infty\). Set \(M_{i+1} := \hat{\mathfrak{k}}M_i + M_i, i \geq 0\).

It is clear that each \(M_i\) is finite dimensional and \(\cup_i M_i = \hat{M}_0\). The definition of \(M_i\) implies that there exists a surjective \(\mathfrak{k}\)-morphism \(\phi : (\mathfrak{g}/\mathfrak{k}) \otimes M_i \twoheadrightarrow M_{i+1}/M_i\) of \(\mathfrak{k}\)-modules. Theorem 3 of \(\text{OV}\) Subsection 3.3.3 implies that \([\mathfrak{g}, \hat{\mathfrak{k}}] = [\mathfrak{g}, \mathfrak{g}] = [\hat{\mathfrak{k}}, \mathfrak{g}]\) and thus that \((\mathfrak{g}/\mathfrak{k})\) is a trivial \(\mathfrak{k}\)-module. Therefore the list of simple \(\mathfrak{k}\)-subquotients of \(M_{i+1}\) equals the list of simple \(\mathfrak{g}\)-subquotients of \(M_i\) for all \(i \geq 0\), and hence these lists equal to the list of simple \(\mathfrak{g}\)-subquotients of \(M_0\). If \(\lim \dim M_i = \infty\) then the multiplicity of at least one such a subquotient would tend to infinity with \(i \to \infty\).

That is incompatible with the assumption that \(M\) is an admissible \((\mathfrak{g}, \mathfrak{g})\)-module.

**Proposition 4.8.** Let \(\mathfrak{g}\) be a semisimple Lie algebra, let \(M\) be a simple \(\mathfrak{g}\)-module, and let \(\mathfrak{g} \subset \mathfrak{g}\) be a subalgebra. Then

(a) \(M\) is a bounded \((\mathfrak{g}, \hat{\mathfrak{k}})\)-module if and only if \(M\) is a bounded \((\mathfrak{g}, \mathfrak{g})\)-module,
(b) \(M\) is an admissible \((\mathfrak{g}, \hat{\mathfrak{k}})\)-module if and only if \(M\) is an admissible \((\mathfrak{g}, \mathfrak{g})\)-module.
Therefore it is enough to verify the following facts:

Proposition 4.9. These statements are implied by Corollary 4.6. □

Proof of Proposition 5.1. (to the condition Proposition 5.2. Proposition 5.3.) If Proposition 5.1. is reducible \[Jo\]. Theorem 3.1 is implied by the following propositions.

(1) \[\text{if and only if}\]

Proof. It is straightforward to argue that \(M\) is a bounded \((g, \hat{t})\)-module (respectively bounded \((g, t)\)-module) if and only if \(\mathbb{F}[\text{Var}(M)]\) is a bounded \((\hat{t}, \hat{t})\)-module (respectively bounded \((t, t)\)-module), see \[Pe1\] Proof of Proposition 2.6. In the same way we have that \(M\) is an admissible \((g, \hat{t})\)-module (respectively admissible \((g, t)\)-module) if and only if \(\mathbb{F}[\text{Var}(M)]\) is an admissible \((\hat{t}, \hat{t})\)-module (respectively admissible \((t, t)\)-module). Therefore it is enough to verify the following facts:

(a) \(\mathbb{F}[\text{Var}(M)]\) is a bounded \((t, t)\)-module if and only if \(\mathbb{F}[\text{Var}(M)]\) is a bounded \((t, t)\)-module.

(b) \(\mathbb{F}[\text{Var}(M)]\) is an admissible \((\hat{t}, \hat{t})\)-module if and only if \(\mathbb{F}[\text{Var}(M)]\) is an admissible \((t, t)\)-module.

These statements are implied by Corollary 4.6. □

Let \(L\) be a maximal connected reductive subgroup of \(\hat{K}\). Such a subgroup is unique up to conjugacy. Denote by \(l \subset g\) the Lie algebra of \(L\). It is clear that \(l\) is algebraically reductive in \(g\). We conclude this section with the following.

Proposition 4.9. Let \(M\) be a simple \((g, \hat{t})\)-module. Then

(a) \(M\) is an admissible \((g, \hat{t})\)-module if and only if \(M\) is an admissible \((g, l)\)-module,

(b) \(M\) is a bounded \((g, t)\)-module if and only if \(M\) is a bounded \((g, l)\)-module.

Proof. This is implied by two facts:

(a) the restriction of a simple finite-dimensional \(\hat{K}\)-module to \(L\) is simple,

(b) the restrictions of two nonisomorphic simple finite-dimensional \(\hat{K}\)-modules to \(L\) are nonisomorphic. □

5. Proof of Theorem 3.1

We use notation of Section 3 and of Theorem 3.1. Fix a simple \((g, t)\)-module \(M\). Then \(G\text{Var}(M)\) is irreducible \[Jo\]. Theorem 3.1 is implied by the following propositions.

Proposition 5.1. If \(M\) is a bounded \((g, t)\)-module then \(G\text{Var}(M)\) is \(K\)-coisotropic.

Proposition 5.2. If \(G\text{Var}(M)\) is \(K\)-coisotropic then \(M\) is an admissible \((g, t)\)-module.

Proposition 5.3. If \(M\) is an admissible \((g, \hat{t})\)-module and \(G\text{Var}(M)\) is \(K\)-coisotropic then \(M\) is a bounded \((g, \hat{t})\)-module.

Proof of Proposition 5.1. Proposition 2.6 of \[Pe1\] implies that if \(M\) is a bounded \((g, \hat{t})\)-module then all irreducible components of \(\text{Var}(M)\) are \(K\)-spherical varieties.

The Gelfand-Kirillov dimension of \(U(g)/\text{Ann}_{U(g)} M\) equals \(\dim G\text{Var}(M)\). Theorem 9.11 of \[KL\] implies that the Gelfand-Kirillov dimension of \(M\) is at least \(\frac{1}{2}\dim G\text{Var}(M)\). Hence there exists an irreducible component \(X\) of \(\text{Var}(M)\) such that

\[
\dim X \geq \frac{1}{2}\dim G\text{Var}(M).
\]

Let \(O\) be the nilpotent coadjoint which is dense in \(G \cdot X\). Proposition 2.6(b) of \[Pe1\] implies that \(X \cap O\) is Lagrangian in \(O\) and hence that

\[
\dim X = \frac{1}{2}\dim O \geq \frac{1}{2}\dim G\text{Var}(M).
\]

On the other hand we have that \(O \subset G\text{Var}(M)\). The variety \(G\text{Var}(M)\) is irreducible \[Jo\] and thus

\[
G\text{Var}(M) = \overline{O}.
\]

Recall that \(X \cap O\) is Lagrangian in \(O\), and therefore \(X \cap O \subset O\) is special in the sense of \[TZh\] Definition 4. Next, \[TZh\] Theorem 7 implies that

\[
2c(X) = \text{cork} M + 2\dim X - \dim O
\]

(notation of \[TZh\]), and thus \(2c(X) = \text{cork} M\). The variety \(X\) is \(K\)-spherical and therefore \(c(X) = 0\). Hence \(\text{cork} M = 0\) and \(M\) is \(K\)-coisotropic. □

Proof of Proposition 5.2. It is enough to show that \(M\) is an admissible \((g, t)\)-module, or equivalently that \(\mathbb{F}[\text{Var}(M)]\) is an admissible \((t, t)\)-module, see \[Pe1\] Proof of Proposition 2.6. The last statement is equivalent to the condition

\[
\dim(\mathbb{F}[\text{Var}(M)]^K) < \infty,
\]

see \[PV\] Theorem 3.24.

The embedding \(\hat{t} \hookrightarrow g\) defines the sequence of maps

\[S(\hat{t}) \hookrightarrow S(g) \rightarrow \mathbb{F}[G\text{Var}(M)] \rightarrow \mathbb{F}[\text{Var}(M)].\]
Further we have
\[ S(\mathfrak{t})^K \hookrightarrow S(\mathfrak{g})^K \to \mathbb{F}[\text{GVar}(M)]^K \to \mathbb{F}[\text{Var}(M)]^K. \]

Next, we need the following lemma.

**Lemma 5.4.** For any maximal ideal \( m \in S(\mathfrak{t})^K \) the algebra \( \mathbb{F}[\text{GVar}(M)]^K/m\mathbb{F}[\text{GVar}(M)]^K \) is finite dimensional.

**Proof.** The proof is based on [Lo, Theorem 1.2.1], see also [Lo, Corollary 5.9.1]. We use the notation of [Lo]. The variety \( \text{GVar}(M) \) is irreducible [Lo] and we have
\[ \text{trdeg} \mathbb{F}(\text{GVar}(M))^K = \dim \mathbb{F}(\text{GVar}(M)) - m_K(\mathbb{F}(\text{GVar}(M))) \]
where \( \text{trdeg} \mathbb{F}(\text{GVar}(M))^K \) is the transcendence degree of \( \mathbb{F}(\text{GVar}(M))^K \). This implies that the Gelfand-Kirillov dimension of \( \mathbb{F}[X]^K \) is at most \( \dim X - m_K(X) \) and therefore
\[ \dim X/K \leq \dim X - m_K(X). \]

We fix a maximal ideal \( m \) of \( S(\mathfrak{t})^K \) and denote by \( [m] \) the respective point in the maximal spectrum \( \mathfrak{t}^*/K \) of \( S(\mathfrak{t})^K \). The algebra \( \mathbb{F}[\text{GVar}(M)]/m\mathbb{F}[\text{GVar}(M)] \) is finite dimensional if and only if the fiber of the map
\[ X/K \to \mathfrak{t}^*/K \]
is at most 0-dimensional. Theorem 1.2.1 of [Lo] implies that the dimension of this fiber is at most
\[ \dim X/K - \text{def}_K X \leq \dim X - m_K(X) - \text{def}_K X. \]
The variety \( \text{GVar}(M) \) is generically symplectic in the sense of [Lo, Definition 2.2.3] and therefore
\[ \dim X = m_K(X) + \text{def}_K X, \]
see [Lo, Subsection 5.9]. This implies that the dimension of the fiber under consideration is at most 0. \( \square \)

Put \( m_0 := S(\mathfrak{t})^K \cap \{ \mathfrak{t} S(\mathfrak{g}) \} \). It is clear that \( m_0 \) is a maximal ideal of \( S(\mathfrak{t})^K \). Next,
\[ \text{Var}(M) \subset \mathfrak{t}^+ \]
implies that \( \mathbb{F}[\text{Var}(M)] \) is annihilated by \( \mathfrak{t} S(\mathfrak{g}) \). Therefore \( \mathbb{F}[\text{Var}(M)]^K \) is also annihilated by \( m_0 \). This implies that \( \mathbb{F}[\text{Var}(M)]^K \) is a quotient of
\[ \mathbb{F}[\text{GVar}(M)]^K/m_0 \mathbb{F}[\text{GVar}(M)]^K. \]

By Lemma 5.3 we have
\[ \dim \mathbb{F}[\text{GVar}(M)]^K/m_0 \mathbb{F}[\text{GVar}(M)]^K < \infty, \]
and therefore \( \dim \mathbb{F}[\text{Var}(M)]^K < \infty. \) \( \square \)

**Proof of Proposition 5.3.** Let \( X_1, ..., X_n \) be the irreducible components of \( \text{Var}(M) \subset \text{GVar}(M) \). Proposition 2.6 of [Pe1] implies that it is enough to show that all \( X_i \) are \( K \)-spherical varieties.

Fix \( i \). The variety \( G \cdot X_i \) contains unique dense orbit \( O_i \). Moreover, \( X_i \cap O_i \) is a coisotropic subvariety of \( O_i \) [Ga]. On the other hand, \( X_i \) is \( K \)-isotropic [Pe2, Theorem 2]. This implies that \( X_i \) is \( K \)-special in a sense of [TZh, Definition 4]. Applying formula (2) we have
\[ 2c(X_i) = \text{cork}(O_i). \]

Next, \( c(X_i) = 0 \) if and only if \( X_i \) is \( K \)-spherical and thus we need to check that
\[ \text{cork} O_i = 0, \]
i.e. that the action of \( K \) on \( O_i \) is coisotropic. This is a consequence of [AP, Proposition 2.7] and the facts that \( O_i \subset \text{GVar}(M) \) and \( \text{GVar}(M) \) is \( K \)-coisotropic, see also [Lo, Theorem 1.2.4]. \( \square \)

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