Is the chiral U(1) theory trivial? *

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The chiral U(1) theory differs from the corresponding vector theory by an imaginary contribution to the effective action which amounts to a phase factor in the partition function. The vector theory, \textit{i.e.} QED, is known to be trivial in the continuum limit. It is argued that the presence of the phase factor will not alter this result and the chiral theory is non-interacting as well.

1. Introduction

In the continuum limit QED appears to be a trivial theory of massless fermions. Vice versa, massless fermions turn out to be non-interacting irrespective of the value of the cut-off \[ a \]. The chiral theory, \textit{i.e.} the theory of charged massless left- and right-handed fermions interacting via photon exchange, differs from the vector theory by an imaginary contribution to the effective action, while the real part of the effective action is vector-like \[ \tilde{\mathcal{A}} \]. This raises the question as to whether the imaginary part can turn a non-interacting theory into an interacting one. If not, that has interesting consequences.

The presence of the imaginary part is the main obstacle in simulating chiral fermions on the lattice. In this talk we shall follow the 'continuum fermion approach' (CFA) \[ \tilde{\mathcal{A}} \] to the problem. The idea here is to compute the fermion action in the continuum. Starting from a lattice of extent \( L \) with spacing \( a \), the original lattice on which one does the simulations, one constructs a finer lattice with spacing \( a_f \) on which one puts the fermions. The action for a single fermion of charge \( e_\alpha \) (in units of \( e = 1/\sqrt{\beta} \)) and chirality \( \epsilon_\alpha = \pm 1 \) is taken to be

\[
S_{e_\alpha, \epsilon_\alpha} = \frac{1}{2a_f} \sum_{n,\mu} \bar{\psi}(n) \gamma_\mu \left\{ \begin{array}{l}
[ P_{-\epsilon_\alpha} + P_{\epsilon_\alpha} (U_\mu(n))^\epsilon_\alpha ] \\
\times \psi(n + \hat{\mu}) - [ P_{-\epsilon_\alpha} + P_{\epsilon_\alpha} (U_\mu(n) - \hat{\mu})^\epsilon_\alpha ] \\
\times \psi(n - \hat{\mu}) \end{array} \right\} + S_W,
\]

where \( U_\mu = \exp(iA_\mu) \in U(1) \) and \( P_{\epsilon_\alpha} = (1 + \epsilon_\alpha \gamma_5)/2 \), and \( S_W \) is the ungauged Wilson term:

\[
S_W = \frac{1}{2a_f} \sum_{n,\mu} \bar{\psi}(n) \{ 2\psi(n) - \psi(n + \hat{\mu}) - \psi(n - \hat{\mu}) \}.
\]

The action \[ \tilde{\mathcal{A}} \] obeys the Golterman-Petcher shift symmetry so that in the limit \( a_f \to 0 \) it describes one interacting massless fermion of chirality \( \epsilon_\alpha \), and one free fermion of chirality \( -\epsilon_\alpha \) which decouples from the system. The lattice effective action \( \tilde{\mathcal{W}}_{e_\alpha, \epsilon_\alpha} \) (for finite \( a_f \)) follows from

\[
\exp(-\tilde{\mathcal{W}}_{e_\alpha, \epsilon_\alpha}) = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp(-S_{e_\alpha, \epsilon_\alpha}),
\]

and the continuum action is given by

\[
\tilde{\mathcal{W}}_{e_\alpha, \epsilon_\alpha} = \lim_{a_f \to 0} (\tilde{\mathcal{W}}_{e_\alpha, \epsilon_\alpha} + C),
\]

where \( C \) is a local bosonic counterterm whose purpose is to render \( \text{Re}\tilde{\mathcal{W}}_{e_\alpha, \epsilon_\alpha} \) invariant under chiral gauge transformations. The counterterm is...
known analytically. For the real part of the effective action we obtain, in agreement with the continuum result \[2\],

\[
\text{Re} \hat{W}_{e\alpha} = \frac{1}{2} (\hat{W}_{e\alpha}, V + \hat{W}_0),
\]

where \(\hat{W}_{e\alpha}, V\) and \(\hat{W}_0\) are the actions of the corresponding vector theory and the free theory, respectively, in the \(a_f \to 0\) limit. For the imaginary part we obtain

\[
\text{Im} \hat{W}_{e\alpha} = A + \pi \epsilon_\alpha \eta_{e\alpha} \ [\text{mod } 2\pi],
\]

where \(A\) is the anomalous part of the effective action, and \(\eta_{e\alpha}\), the so-called \(\eta\) invariant, is a gauge invariant quantity. The anomaly cancelling condition is

\[
\sum_\alpha \epsilon_\alpha \eta_{e\alpha}^3 = 0.
\]

The vector theory has been extensively studied for compact \[4\] and non-compact \[1\] \(U(1)\) fields. The phase diagram is shown in Fig. 1. Common to both formulations is that the theory has a line of second order phase transitions at zero bare mass, extending from (some) \(\beta_0 > 0\) to \(\beta = \infty\), on which chiral symmetry is restored and the theory

is massless. It is this second order line on which one can take the (quantum) continuum limit. In the non-compact case this has been shown \[1\] to correspond to a non-interacting theory, and it is believed that the same is true for the compact theory. The chiral theory would naturally live on the second order line. But we know of cases where a phase factor changes the properties of the theory.

We consider compact gauge fields. In the absence of monopoles the gauge fields decompose into sectors of integer magnetic flux \(m_{\mu\nu}\) and topological charge \[5\]

\[
Q = \frac{1}{8} \epsilon_{\mu\nu\rho\sigma} m_{\mu\nu} m_{\rho\sigma}.
\]

The configurations with non-vanishing flux are strongly suppressed, however, so that we do not expect to see any differences between the compact and the non-compact formulation of the theory in the chirally symmetric phase.

### 2. \(\text{Im} W\): does it make a difference?

As a first test of the method we have computed the anomaly \(A\) for a set of plane wave potentials. This allowed us to do calculations on (fine) lattices as large as \(L_f = 16\) and to compare our results with the continuum expression given by the triangle diagram in Fig. 2. We found excellent agreement between the numerical results extrapolated to \(L_f \to \infty\) and the analytical values.

We now turn to the \(\eta\) invariant. \[†\] We consider three different charges, \(e_\alpha = 1/2, 1, 2\). The calculation proceeds from \(L = 4\) (orig-

\[†\] In the chiral Schwinger model \[6\] \(\eta\) receives contributions from toron fields \(t_t\) only and assumes values between \(-1\) and \(+1\), no matter how small \(L\) is. Furthermore, it is non-analytic at \(t = 0\) and has no perturbative expansion.
We have chosen to work at $\beta = 2$. This coupling lies well in the symmetric phase of the vector model where we may hope to find massless fermions. The Monte Carlo sampling proceeds in two steps. In the first step we generate quenched gauge field configurations. On these configurations we then compute the fermion determinant which we include in the observable. This is justified if the action does not fluctuate too much. We have checked that (9) is fulfilled. In Table 1 we present the real and imaginary part of the effective action of the anomaly-free model $\epsilon_\alpha e_\alpha = +1, \alpha = 1, \ldots, 8$, $\epsilon_\alpha e_\alpha = -2, \alpha = 9$ for 6 consecutive gauge field configurations and for various levels of interpolation. We also show the extrapolation to $L_f = \infty$. The imaginary part of the effective action turns out to be surprisingly small, and it appears that the larger the value is the larger is the corresponding real part which suppresses its effect even further.

With the imaginary part being so small, the first question which comes to mind is whether $\text{Im} \hat{W}$ can be described by perturbation theory. The first few diagrams are given in Fig. 2. If so, we would expect to find for a given $\epsilon_\alpha$

$$
\epsilon_\alpha = \frac{1}{2}: \text{Im} \hat{W} = \frac{1}{2} A + \frac{1}{12} B + \frac{1}{720} C + \cdots,
$$
$$
\epsilon_\alpha = 1: \text{Im} \hat{W} = A + 2B + C + \cdots, \quad (7)
$$
$$
\epsilon_\alpha = 2: \text{Im} \hat{W} = 8A + 32B + 128C + \cdots,
$$

where $A$, $B$ and $C$ are the contributions from the 3-, 5- and 7-leg diagrams, respectively. We find indeed good agreement with the perturbative behavior with $|A| : |B| : |C| \sim 1 : 0.1 : 0.01$. We expect to obtain similar ratios on larger lattices. Let us consider an observable $O$ of even parity now. Its expectation value can be written

$$
\langle O \rangle = \frac{(\langle O e^{i\pi \eta_a} \rangle)_{\text{Re}}}{(e^{i\pi \eta_a})_{\text{Re}}}
$$
$$
= \langle O \rangle_{\text{Re}} + \frac{(\langle O \cos \pi \eta_a \rangle)_{\text{Re}} - \langle O \rangle_{\text{Re}} \cos \langle \pi \eta_a \rangle_{\text{Re}} \cos \pi \eta_a}{(\langle \cos \pi \eta_a \rangle)_{\text{Re}}}
$$
$$
= \langle O \rangle_{\text{Re}} + \langle O \cos \pi \eta_a \rangle^\text{con}_{\text{Re}}, \quad (8)
$$

where ‘Re’ means that the path integral is done over the real part of the effective action only, which corresponds to the vector theory given by the action (3) with 4 + $x$ active flavors, and ‘con’ refers to the totally connected correlation function. In Fig. 3 we show a corresponding diagram. Using Schwarz’s inequality and the results in Table 1 we estimate

$$
|\langle O \rangle - \langle O \rangle_{\text{Re}}| \leq \sigma_{\text{Re}}(O) \frac{\sigma_{\text{Re}}(\cos \pi \eta_a)}{(\cos \pi \eta_a)_{\text{Re}}} \quad (9)
$$
$$
\approx \sigma_{\text{Re}}(O) \times 3 \cdot 10^{-5}, \quad (10)
$$

where $(\sigma_{\text{Re}}(O))^2$ is the variance of the operator $O$. A similar estimate is found for the parity-odd operator. This indicates that the chiral U(1) theory is basically vector-like. We do not expect that this estimate will change significantly on larger volumes. The connected correlation function in\footnote{This behavior is quite different from what we found in the chiral Schwinger model.}
The real and imaginary part of the effective action of the anomaly-free model for 6 consecutive gauge field configurations. The errors represent the ambiguity in the interpolation.

\[ L_f \]

| #  | 4          | 5          | 6          | 7          | 8          | ∞          |
|----|------------|------------|------------|------------|------------|------------|
| Re \( \hat{W}_a \) | 17.3(23)   | 17.7(02)   | 17.7(18)   | 17.9(01)   | 19.8(36)   | 18.1(03)   |
|     | 17.1(17)   | 17.2(02)   | 17.1(03)   | 17.2(01)   | 19.1(29)   | 17.2(03)   |
|     | 16.5(17)   | 17.3(01)   | 17.7(04)   | 18.2(01)   | 19.9(36)   | 17.2(03)   |
|     | 23.3(19)   | 24.1(02)   | 24.9(04)   | 25.3(02)   | 27.2(34)   | 26.5(04)   |
|     | 25.2(18)   | 27.1(158)  | 31.8(363)  | 76.0(1100) | 25.3(26)   | 25.4(35)   |
| Im \( \pi \eta_a \) | 0.0001(6)  | 0.0005(6)  | 0.0010(7)  | 0.0013(5)  | 0.0014(2)  | 0.0019(3)  |
|     | -0.0008(3) | -0.0015(5) | -0.0020(4) | -0.0021(5) | -0.0019(1) | -0.0023(1) |
|     | -0.0029(8) | -0.0045(6) | -0.0039(3) | -0.0022(4) | -0.0002(3) | -0.0001(4) |
|     | -0.0012(3) | -0.0035(4) | -0.0060(7) | -0.0088(3) | -0.0104(8) | -0.0120(4) |
|     | -0.0016(10)| -0.0069(23)| -0.0295(34)| -0.0676(102)| -0.0805(51)| -0.0604(37)|
|     | 0.0063(10) | 0.0146(26) | 0.0207(23) | 0.0224(15) | 0.0225(10) | 0.0287(12) |

Table 1

The real and imaginary part of the effective action of the anomaly-free model for 6 consecutive gauge field configurations. The errors represent the ambiguity in the interpolation.

\[ \langle O \rangle \rightarrow \langle O \rangle_{\text{Re}} \rightarrow \langle O \rangle_{0} \]

where the subscript ‘0’ denotes the result of the free theory. This would mean that the chiral theory is trivial, like massless QED. A direct estimate of \( \langle \cos \pi \eta_a \rangle_{\text{Re}} \) leads to the same conclusion. As far as we can see, our argument could go wrong only if \( \eta_a \) is not described by perturbation theory and \( \langle \cos \pi \eta_a \rangle_{\text{Re}} \approx 0 \).

3. Conclusions

We have presented first results of the effective action of the chiral U(1) theory on small lattices, which led us to argue that the theory is trivial. To substantiate our results we obviously need to do calculations on larger lattices and at further couplings. If true, our results would perhaps explain why neutrinos are massiv, in the same way as QED tells us that electrons must be massiv.

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References

1. M. Göckeler et al., Nucl. Phys. B371 (1992) 713, Phys. Rev. Lett. 80 (1998) 4119.
2. L. Alvarez-Gaumé et al., Phys. Lett. B166 (1986) 177.
3. M. Göckeler, G. Schierholz, Nucl. Phys. B (Proc. Suppl.) 29B,C (1992) 114, ibid. 30 (1993) 609; G. ‘t Hooft, Phys. Lett. B349 (1995) 491; G. T. Bodwin, Phys. Rev. D54 (1996) 6497; P. Hernández, R. Sundrum, Nucl. Phys. B455 (1995) 287; V. Bornyakov et al., Prog. Theor. Phys. Suppl. 131 (1998) 337.
4. A. Hoferichter et al., Nucl. Phys. B (Proc. Suppl.) 63 (1998) 454.
5. A. Phillips, Annals Phys. 161 (1985) 399.
6. V. Bornyakov et al., 8.
7. M. Göckeler et al., Nucl. Phys. B404 (1993) 287.
8. M. Göckeler et al., Nucl. Phys. B487 (1997) 313.