Empirical least-squares fitting of parametrized dynamical systems

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Abstract Given a set of response observations for a parametrized dynamical system, we seek a parametrized dynamical model that will yield uniformly small response error over a range of parameter values yet has low order. Frequently, access to internal system dynamics or equivalently, to realizations of the original system is either not possible or not practical; only response observations over a range of parameter settings might be known. Respecting these typical operational constraints, we propose a two phase approach that first encodes the response data into a high fidelity intermediate model of modest order, followed then by a compression stage that serves to eliminate redundancy in the intermediate model. For the first phase, we extend non-parametric least-squares fitting approaches so as to accommodate parameterized systems. This results in a (discrete) least-squares problem formulated with respect to both frequency and parameter that identifies “local” system response features. The second phase uses an $H_2$-optimal model reduction strategy accommodating the specialized parametric structure of the intermediate model obtained in the first phase. The final compressed model inherits the parametric dependence of the intermediate model and maintains the high fidelity of the intermediate model, while generally having dramatically smaller system order. We provide a variety of numerical examples demonstrating our approach.

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1 Introduction

In many areas of study in science and engineering, the dynamics that govern processes of interest are expected to vary with respect to a given set of parameters describing, say, viscosity, temperature distribution, reaction and flow rates, etc. Often these parameters may be deduced from dimensional analysis or dynamic similarity considerations while detailed dynamics of the system may be inaccessible to direct modeling. Although access to internal dynamics may be lacking, there may be an abundance of accurate frequency response measurements available, which may further be classified according to what system parameter values were in effect when the system responses were observed. The problem that we address here involves building up an empirical parsimonious parameterized dynamical system model that fits the observed system response with high fidelity over a desired range of parameter values. We interpret parsimonious in this context to mean low system order. The derived dynamical system may then be used as an efficient surrogate to predict operational behavior or to produce effective control and optimization strategies.

1.1 Basic problem formulation

Since linear time invariant dynamical systems have a frequency domain response representation that involves rational functions, a natural formulation for the task at hand leads one to a data fitting problem using rational functions; this will be our principal focus. For simplicity, we assume that the original dynamics vary with respect to a single parameter \( p \), and arise as a single-input/single-output (SISO) linear time-invariant system associated with a \((p\text{-dependent})\) transfer function, \( H(s,p) \), that is unknown but is nonetheless accessible to sampling in the sense that for a variety of operating conditions distinguished by parameter values \( p = \mu_1, \ldots, \mu_{m_p} \), system response measurements can be measured or computed at predetermined frequency points, \( s = \xi_1, \ldots, \xi_{m_s} \). In the context of our working hypotheses, we will assume that the point sets, \( \{\mu_1, \ldots, \mu_{m_p}\} \) and \( \{\xi_1, \ldots, \xi_{m_s}\} \), respectively comprise \( m_p \) and \( m_s \) distinct points in \( \mathbb{C} \), and that the magnitude and phase of the complex frequency response \( \mathcal{H}(\xi_i, \mu_j) \) is available for \( i = 1, \ldots, m_s \) and \( j = 1, \ldots, m_p \). These values of \( \mathcal{H}(\xi_i, \mu_j) \) will be the only information presumed to be available for the system of interest. For projection-based parametric model reduction, which requires access to internal dynamics of the underlying system, we refer the reader to \[8,29,41\] and the references therein. Based on the samples \( \mathcal{H}(\xi_i, \mu_j) \), we will seek a bivariate function \( \hat{H}(s,p) \) such that \( \hat{H}(\xi_i, \mu_j) \approx \mathcal{H}(\xi_i, \mu_j) \) and such that \( \hat{H}(s,p) \) itself represents a parameterized transfer function. We measure closeness in the least-squares sense: Given data \( \{\xi_i, \mu_j, \mathcal{H}(\xi_i, \mu_j)\} \) with \( 1 \leq i \leq m_s \) and
We require that

\[ 1 \leq j \leq m_p, \text{ find a stable bivariate rational function } \hat{H} \text{ such that} \]

\[
\sum_{i=1}^{m_s} \sum_{j=1}^{m_p} \left| \hat{H}(\xi_i, \mu_j) - \mathcal{H}(\xi_i, \mu_j) \right|^2 \to \min. \tag{1}
\]

What particular structure should one enforce on \( \hat{H}(s, p) \)?

1.2 The solution framework

We require that \( \hat{H}(s, p) \) be a proper rational function with respect to \( s \) for each \( p \) with poles in the left-half plane, and that for each \( s \) at which \( \hat{H}(s, p) \) is finite, \( \hat{H}(s, p) \) have a functional dependence on \( p \) that may be polynomial or more general.

In order to make this more precise, for any integer \( n_s \geq 1 \), denote by \( \mathcal{R}_{n_s} \) the set of strictly proper, stable, rational functions of order no greater than \( n_s \) (i.e., for any \( g \in \mathcal{R}_{n_s} \), \( g \) is the ratio of polynomials that may be assumed to be relatively prime, with denominator that has polynomial order no greater than \( n_s \) and roots in the open left half-plane; the numerator has polynomial order strictly less than that of the denominator). \( \mathcal{R}_{n_s} \) is not itself a vector space, though it is a differentiable manifold containing all subspaces having the form

\[
\mathcal{M}_{n_s} = \text{span}\{h_1(s), h_2(s), ..., h_{r_s}(s)\}
\]

where \( \{h_k\}_{k=1}^{r_s} \) are proper, stable rational functions with orders summing to a quantity no larger than \( n_s \): \( \sum_{k=1}^{r_s} \text{order } h_k(s) \leq n_s \). We assume for convenience that \( \{h_k\}_{k=1}^{r_s} \) have mutually distinct poles, so that for some choice, \( \{\lambda_k\}_{k=1}^{n_s} \subset \mathbb{C} \), distinct points chosen in the open left half-plane,

\[
\mathcal{M}_{n_s} \subset \text{span } \{ (s - \lambda_1)^{-1}, (s - \lambda_2)^{-1}, ..., (s - \lambda_{n_s})^{-1} \} \subset \mathcal{R}_{n_s}
\]

We allow the parameter \( p \) to vary over a convex, compact subset, \( \mathcal{P} \subset \mathbb{C} \), containing our given parameter control values, \( \{\mu_1, \ldots, \mu_{m_p}\} \subset \mathcal{P} \). Denote by \( \mathcal{F}(\mathcal{P}) \) the set of continuous functions mapping \( \mathcal{P} \) to \( \mathbb{C} \). For \( r_p \geq 1 \) and some choice of functions, \( \{P_1, P_2, \ldots, P_{r_p}\} \subset \mathcal{F}(\mathcal{P}) \), let \( \mathcal{P}_{r_p} = \text{span}\{P_1, P_2, \ldots, P_{r_p}\} \subset \mathcal{F}(\mathcal{P}) \).

Consider the tensor product space, \( \mathcal{M}_{r_s} \otimes \mathcal{P}_{r_p}, \) defined formally as the span of pairwise elementary products of functions drawn from bases of \( \mathcal{M}_{r_s} \) and \( \mathcal{P}_{r_p} \):

\[
\sum_{k=1}^{r_s} \sum_{\ell=1}^{r_p} x_{k\ell} h_k(s) P_\ell(p), \text{ with } 1 \leq k \leq r_s, 1 \leq \ell \leq r_p \text{ and } \{x_{k\ell}\} \subset \mathbb{C} \].

Observe that any function

\[
\hat{H}(s, p) = \sum_{k=1}^{r_s} \sum_{\ell=1}^{r_p} x_{k\ell} h_k(s) P_\ell(p) \in \mathcal{M}_{r_s} \otimes \mathcal{P}_{r_p} \tag{2}
\]

will have the properties:

1. For any fixed \( \hat{\mu} \in \mathcal{P} \), \( \hat{H}(\cdot, \hat{\mu}) \) is a stable transfer function of order \( r_s \) (or less) with poles contained in the set \( \{\lambda_1, \lambda_2, \ldots, \lambda_{n_s}\} \).
2. For any fixed \( \hat{s} \neq \lambda_\ell, \ell = 1, \ldots, r_s, \hat{H}(\hat{s}, \cdot) \in \mathcal{P}_{r_p} \). That is, \( \hat{H}(\hat{s}, p) \) has a parametric dependence on \( p \) described by \( \{P_1, P_2, \ldots, P_{r_p}\} \).

We begin by considering the problem:
Given data \( \{\xi_i, \mu_j, \mathcal{H}(\xi_i, \mu_j)\} \) with \( 1 \leq i \leq m_s \) and \( 1 \leq j \leq m_p \), and subspaces \( \mathfrak{R}_s \) and \( \mathfrak{Q}_p \) as described above, find a stable bivariate rational function that fits the data in a least squares sense:

Find \( \hat{\mathcal{H}} \in \mathfrak{R}_s \otimes \mathfrak{Q}_p \) as in (2) that solves

\[
\sum_{i=1}^{m_s} \sum_{j=1}^{m_p} |\hat{\mathcal{H}}(\xi_i, \mu_j) - \mathcal{H}(\xi_i, \mu_j)|^2 \rightarrow \min. \tag{3}
\]

This problem is straightforwardly solved, at least in principle. Define the matrices

\[
A = [h_j(\xi_i)] \in \mathbb{C}^{m_s \times r_s}, \quad B = [P_j(\mu_i)] \in \mathbb{C}^{m_p \times r_p},
\]

and \( \mathbb{H} = [\mathcal{H}(\xi_i, \mu_j)] \in \mathbb{C}^{m_s \times m_p} \).

Then with \( X = [x_{ij}] \) and \( \hat{\mathcal{H}}(s, p) = \sum_{k,l} x_{kl} h_k(s) P_l(p) \) as in (2) we observe

\[
\sum_{i=1}^{m_s} \sum_{j=1}^{m_p} |\hat{\mathcal{H}}(\xi_i, \mu_j) - \mathcal{H}(\xi_i, \mu_j)|^2 = \| A X B^\top - \mathbb{H} \|^2_F = ||(B \otimes A) \text{vec}(X) - \text{vec}(\mathbb{H})||^2_F
\]

where we have made use of the Frobenius norm of a matrix: \( \|M\|_F^2 = \text{trace}(M^\top M) \), the stacking operator \( \text{vec}(\cdot) \), and the matrix Kronecker product, \( \otimes \) (see, e.g., Chapter 4 of [32]). The minimal norm least squares solution to (3) is given by \( \hat{\mathcal{H}}(s, p) = \sum_{k,l} x_{kl} h_k(s) P_l(p) \), where \( X = [x_{ij}] \) is obtained from

\[
\text{vec}(\hat{X}) = (B \otimes A)^\top \text{vec}(\mathbb{H}) = (B^\top \otimes A^\top) \text{vec}(\mathbb{H}) \iff \hat{X} = A^\top \cdot \mathbb{H} \cdot (B^\top)^\top. \tag{5}
\]

where \((\cdot)^\top\) denotes the matrix pseudo inverse. This discussion can be easily extended to allow weighting factors \( w_{ij} \geq 0 \) that assign varying relevance to each data input pair \( \xi_i \mapsto \mu_j \) in the optimization (3). The objective function then becomes

\[
\sum_{i=1}^{m_s} \sum_{j=1}^{m_p} w_{ij} |\hat{\mathcal{H}}(\xi_i, \mu_j) - \mathcal{H}(\xi_i, \mu_j)|^2 \rightarrow \min. \tag{6}
\]

Let \( W = (\sqrt{w_{ij}}) \). Then, using the above notation and the Hadamard (elementwise) product \( \circ \), we obtain a weighted least squares problem:

\[
\sum_{i=1}^{m_s} \sum_{j=1}^{m_p} w_{ij} |\hat{\mathcal{H}}(\xi_i, \mu_j) - \mathcal{H}(\xi_i, \mu_j)|^2 = \| (A X B^\top - \mathbb{H}) \circ W \|^2_F
\]

\[
= \| \text{diag}(\text{vec}(W))((B \otimes A) \text{vec}(X) - \text{vec}(\mathbb{H})) \|_2^2.
\]

Notice in particular that the choice of weight \( w_{ij} = 0 \) could be appropriate if no observation is available for the case \( s = \xi_i \) and \( p = \mu_j \) or the data are irrelevant.

One difficulty that has not yet been addressed here lies in representation, that is: (a) what choice to make for frequency response basis functions, \( \{h_k\}_{k=1}^{r_s} \), and more generally, the associated pole locations, \( \{\lambda_k\}_{k=1}^{r_s} \), which as yet have only been constrained to lay in the open left half-plane; and (b) what choice to make for parametric bases, \( \{P_j\}_{j=1}^{r_p} \), which as yet are only constrained to be continuous functions on \( \mathcal{P} \). These choices will make up the first of two major concerns in the discussion.
that follows; our second concern will focus on eliminating (nearly) redundant
information that may be implicit in the expression, \( \hat{H}(s, p) = \sum_{k, \ell} \hat{x}_{k\ell} \hat{h}_k(s) P_\ell(p) \), in
order to arrive at a more compact representation.

We propose a two phase approach to address these two difficulties: the first
phase encodes the response data into a high fidelity model of modest order. To
accomplish this, we extend the well-known non-parametric least-squares fitting ap-
proaches so as to accommodate parameterized systems. This results in a (discrete)
least-squares problem formulated with respect to both frequency and parameter
that identifies “local” system response features. Our approach produces effective
choices for the subspaces \( R_{rs} \) and \( \mathcal{P}_{rp} \) described above. The second phase that
follows constitutes a compression stage that serves to eliminate redundancy in
the intermediate model that was obtained in the first phase. This is accomplished
using an \( H_2 \)-optimal model reduction strategy well-suited for the specialized para-
metric structure of the intermediate model obtained in the first phase. Our final
compressed model inherits the parametric dependence of the intermediate model
while maintaining high fidelity, yet our final model will generally have dramatically
smaller system order.

The strategies that we propose have straightforward extensions to the MIMO
(multiple-input / multiple-output) setting as well as to multi parameter settings,
however for clarity we restrict the discussion here to the SISO - scalar parameter
setting, adding a brief discussion of multi parameter case in Section 5.

2 Local Models

For any fixed parameter, \( \hat{\mu} \in \mathcal{P} \), a parametrically localized model or more briefly, a
local model, will refer to a transfer function, \( \hat{h}(s) \), that is intended to model the
response of the parameterized system in a neighborhood of \( p = \hat{\mu} \). Within the
empirical framework we have adopted here, a local model, \( \hat{h}(s) \), should match the
data well for some choice of \( \hat{\mu} \in \{ \mu_1, \mu_2, \ldots, \mu_m \} \): \( \hat{h}(\xi_i) \approx H(\xi_i, \hat{\mu}) \) for \( i = 1, \ldots, m \).

One might reasonably expect that a parameterized model \( \hat{H} \in R_{rs} \otimes \mathcal{P}_{rp} \) that is
an effective solution to (3), i.e., a solution that produces a small residual, then
\( \hat{H}(s, p) \), will also determine implicitly a family of effective local models as well:
\( \hat{H}(\cdot, \mu_1), \hat{H}(\cdot, \mu_2), \ldots, \hat{H}(\cdot, \mu_m) \) each should be effective local models. Since each such
local model must also be contained in \( R_{rs} \), we may choose to generate an initial
subspace \( R_{rs} \), by finding an effective local model for each \( \{ \mu_1, \mu_2, \ldots, \mu_m \} \). Toward
that end, for each \( j = 1, 2, \ldots, m_p \), pick an integer \( \nu_j \geq 1 \) such that \( \sum \nu_j \leq n_s \) and define

\[
\hat{h}_j := \arg \min_h \sum_{i=1}^{m_s} w_i \left| h(\xi_i) - H(\xi_i, \mu_j) \right|^2, \quad (7)
\]

where the minimization is taken over strictly proper, stable rational functions
having order \( \nu_j \) or less. As in (6), the positive weights \( \{ w_1^m \} \), provide flexibility
in either balancing or focussing the importance of each summand in the objective
function, but further also allow one to approximate conditions leading to near-\( H_2 \)
optimality (see, e.g., [18]). This process defines a total of \( m_p \) local models, so we take \( r_s = m_p \) and

\[
R_{m_p} = \text{span}\{ \hat{h}_1(s), \hat{h}_2(s), \ldots, \hat{h}_{m_p}(s) \} \subset R_{ns}
\]
will be our (initial) frequency-domain factor space that can be used in solving \( (3) \). We provide an illustration of the full process in Section 2.2 after discussing how one may obtain effective local models.

2.1 Local models via Vector Fitting

The Sanathanan-Koerner iteration \([42]\) and its later refinement, Vector Fitting \([27]\), provide effective tools for solving each of the nonlinear least squares problem that are implicit in \( (7) \). There are other strategies that can be followed as well, e.g., a recent development that is compatible with our solution framework is the \( \text{rkfit} \) algorithm as described in \([10]\). We use a Vector Fitting approach, which we summarize here.

A strictly proper rational function, \( h(s) \), of order \( \nu \geq 1 \) may be represented in barycentric form as

\[
h(s) = \frac{n(s)}{d(s)} = \sum_{k=1}^{\nu} \frac{\psi_k}{s - \lambda_k} + \sum_{k=1}^{\nu} \frac{\phi_k}{s - \lambda_k},
\]

(8)

As \( (8) \) suggests, \( n(s) \) is taken to be the numerator of the rightmost expression and \( d(s) \) is the denominator. The nodes \( \{\lambda_k\}_1^{\nu} \subset \mathbb{C} \) are presumed to be distinct but otherwise are arbitrary, at least in principle. Evidently, they constitute the poles of both \( n(s) \) and \( d(s) \) (though not of \( h(s) \)). \( \psi_k \) and \( \phi_k \) are the associated residues of \( n(s) \) and \( d(s) \), respectively. The barycentric representation of rational functions has numerical advantages over representations that involve polynomial ratios, \([11,30,45]\). Moreover, the value of \( h(s) \) at any \( \lambda_k \) is directly provided in terms of the residues: \( h(\lambda_k) = \frac{\psi_k}{\phi_k} \) for \( k = 1, 2, \ldots, \nu \). For any fixed choice of distinct nodes \( \{\lambda_k\}_1^{\nu} \), the minimization in \( (7) \) is a nonlinear least squares problem with respect to the residues \( \{\psi_k\}_1^{\nu} \) and \( \{\phi_k\}_1^{\nu} \).

Observe that the objective function of \( (7) \), for a fixed \( \mu_j \), can be expressed as

\[
\sum_{i=1}^{m_s} w_i \left| \frac{n(\xi_i)}{d(\xi_i)} - \mathcal{H}(\xi_i, \mu_j) \right|^2 = \sum_{i=1}^{m_s} \frac{w_i}{|d(\xi_i)|^2} \left| n(\xi_i) - \mathcal{H}(\xi_i, \mu_j)d(\xi_i) \right|^2
\]

and that the parenthesized quantity is linear with respect to the residues \( \{\psi_k\}_1^{\nu} \) and \( \{\phi_k\}_1^{\nu} \). Following \([42]\), we are led to recast the nonlinear least-squares problem as a sequence of weighted linear least-squares problems:

\[
\sum_{i=1}^{m_s} \frac{w_i}{|d(k-1)(\xi_i)|^2} \left| n(k)(\xi_i) - d(k)(\xi_i)\mathcal{H}(\xi_i, \mu_j) \right|^2 \rightarrow \min, \quad k = 1, 2, \ldots
\]

(9)

which can be formulated in standard form as

\[
\left\| \Delta^{(k-1)} \left( Ax^{(k)} - b \right) \right\|_2^2 \rightarrow \min, \quad k = 1, 2, \ldots
\]

(10)
where \( \Delta^{(k-1)} = \text{diag}\{\sqrt{\mathcal{M}_i}/|d^{(k-1)}(\xi_i)|\}_{i=1}^{m_x} \in \mathbb{C}^{m_x \times m_x} \),

\[
\mathbf{x}^{(k)} = \begin{bmatrix} \psi_1^{(k)} & \ldots & \psi_\nu^{(k)} & \varphi_1^{(k)} & \ldots & \varphi_\nu^{(k)} \end{bmatrix}^\top \in \mathbb{C}^{2\nu}, \text{ and }
\]

\[
\mathbf{A} := \begin{bmatrix}
1 & 1 \\
\xi_1 - \lambda_1 & \xi_1 - \lambda_\nu \\
\vdots & \vdots \\
\xi_{m_x} - \lambda_1 & \xi_{m_x} - \lambda_\nu \\
\end{bmatrix} \begin{bmatrix}
\mathcal{H}(\xi_1, \lambda_j) & \mathcal{H}(\xi_1, \lambda_j) \\
\vdots & \vdots \\
\mathcal{H}(\xi_{m_x}, \lambda_j) & \mathcal{H}(\xi_{m_x}, \lambda_j) \\
\end{bmatrix} \in \mathbb{C}^{m_x \times 2\nu}
\]

(11)

This describes a step of the Sanathanan-Koerner iteration.

The key innovation of Vector Fitting (VF), as developed in [27], that distinguishes it from the Sanathanan-Koerner formulation lies in a clever change of the nodes \( \{\lambda_j^{(k)}\}_{j=1}^\nu \) at iteration step, \( k \), leading to a different representation within each step. Let the matrix \( \mathbf{A}^{(k)} \) be defined analogously to \( \mathbf{A} \) in (11), but with the previously fixed nodes, \( \lambda_j \), replaced by \( \lambda_j^{(k)} \) that will vary from step to step. At each step \( \mathbf{b}^{(k)}(s) = n^{(k)}(s)/d^{(k)}(s) \); both \( n^{(k)}(s) \) and \( d^{(k)}(s) \) are defined using the nodes \( \lambda_j^{(k)} \) that are then modified in the following way.

Consider advancing [9] from step \( k \) to step \( k + 1 \). Since the most recent \( d^{(k)}(s) \) is available, compute its zeros, \( \{\lambda_j^{(k+1)}\}_{j=1}^\nu \), and obtain an equivalent representation:

\[
d^{(k)}(s) = 1 + \sum_{j=1}^\nu \varphi_j^{(k+1)} = \frac{\prod_{j=1}^\nu (s - \lambda_j^{(k+1)})}{\prod_{j=1}^\nu (s - \lambda_j^{(k)})}.
\]

(12)

This factorization can then be used to rewrite the objective (9) as

\[
\sum_{i=1}^{m_x} \frac{w_i}{|d^{(k)}(\xi_i)|^2} \left| \sum_{j=1}^\nu \frac{\psi_j^{(k+1)} - \mathcal{H}(\xi_i)}{\xi_i - \lambda_j^{(k+1)}} \right|^2 + \sum_{i=1}^{m_x} \frac{w_i}{|d^{(k)}(\xi_i)|^2} \left| \frac{\tilde{\phi}^{(k+1)}(\xi_i) - \mathcal{H}(\xi_i)}{\xi_i - \lambda_j^{(k+1)}} \right|^2
\]

(13)

where \( \tilde{\phi}^{(k+1)} \) and \( \tilde{\phi}^{(k+1)} \) are polynomials of degrees \( \nu - 1 \) and \( \nu \), respectively.

Continuing with similar algebraic manipulations, one obtains

\[
\ldots = \sum_{i=1}^{m_x} w_i \left| \frac{\tilde{\phi}^{(k+1)}(\xi_i)}{\prod_{j=1}^\nu (\xi_i - \lambda_j^{(k+1)})} - \mathcal{H}(\xi_i) \right|^2 + \sum_{i=1}^{m_x} \frac{w_i}{|d^{(k)}(\xi_i)|^2} \left| \frac{\tilde{\phi}^{(k+1)}(\xi_i) - \mathcal{H}(\xi_i)}{\xi_i - \lambda_j^{(k+1)}} \right|^2
\]

(14)

\[
= \sum_{i=1}^{m_x} w_i \left| \frac{\tilde{\phi}^{(k+1)}(\xi_i) - \mathcal{H}(\xi_i)}{\prod_{j=1}^\nu (\xi_i - \lambda_j^{(k+1)})} \right|^2 \rightarrow \min, \quad D_w = \text{diag}(\sqrt{w})_{j=1}^{m_x}
\]
where \( \hat{x}^{(k+1)} = \left( \hat{\phi}_1^{(k+1)} \hat{\phi}_2^{(k+1)} \cdots \hat{\phi}_\nu^{(k+1)} \hat{\phi}_1^{(k+1)} \hat{\phi}_2^{(k+1)} \cdots \hat{\phi}_\nu^{(k+1)} \right)^\top \) with \( \hat{\phi}_j^{(k+1)} \) and \( \hat{\phi}_j^{(k+1)} \) as defined in \cite{14} denoting the coefficients of \( h^{(k+1)}(s) \) in a barycentric representation having nodes \( \lambda_j^{(k+1)} \), \( j = 1, \ldots, \nu \). VF may be concisely described as a representation of the Sanathanan-Koerner iteration in barycentric form with moving nodes. Upon convergence, the nodes \( \lambda_j^{(k)} \) become a fixed point for the iteration, the residues \( \phi_j^{(k)} \) approach zero and at some \( k_* \), \( h^{(k_*)}(s) \approx n^{(k_*)}(s) \) is returned in a convenient pole-residue form.

More details on implementation and variants of VF can be found in \cite{12,13,16,18,26}. It is worth noting that convergence of VF in general remains an open problem \cite{23,25,28,31,13,14}. Nonetheless, in practice we typically observe convergence in relatively few iterations, even with poorly selected initial nodes.

### 2.2 Using Local Models for the Coupled Data Fitting Problem

Once a set of local models, \{\( h_1(s), h_2(s), \ldots, h_{m_p}(s) \)}, has been obtained associated with the given parameter sampling, \{\( \mu_1, \mu_2, \ldots, \mu_{m_p} \)}, these local models can be used to define a factor space, \( \mathcal{H}_{m_p} = \text{span}\{h_1(s), h_2(s), \ldots, h_{m_p}(s)\} \), that may be expected to describe the range of variability of the data with respect to frequency. In order to solve \cite{3}, we need also to posit a parameter dependence through the specification of a parameter factor space, \( \mathcal{P}_{r_p} \), that serves to knit together the local models that we have developed in \cite{2}. The simplest model of parameter dependence involves choosing an \textit{a priori} fixed subspace spanned by polynomials or another fixed basis: \( \mathcal{P}_{r_p} = \text{span}\{P_1, P_2, \ldots, P_{r_p}\} \). This is discussed in \cite{1,2} we summarize the aggregated strategy as Algorithm \cite{1} for the unweighted case \( (w_{ij} = 1) \).

**Algorithm 1** Parametric Fitting – Fixed Parametric Basis

**INPUT:**
- Observed response data: \( \{\xi_i, \mu_j, \mathcal{H}(\xi_i, \mu_j)\}_{1 \leq i, j \leq m_p} \)
- Target model order: \( n_s \)
- Parametric basis functions \( \{P_\ell\}_{\ell=1}^{r_p} \)

**OUTPUT:**
- Bivariate function \( \hat{H}(s, p) \) that fits the data and solves \cite{3}.

1. Choose local model orders, \( \nu_j \geq 1 \) such that \( \sum_{j=1}^{m_p} \nu_j = n_s \).
2. Construct local models, \( h_j(s) \), of order \( \nu_j \) for each parameter \( \mu_j \) following \cite{7}.
3. Solve the least squares problem \cite{3} using \cite{5,6} to obtain the coefficients, \( \hat{X} = [\hat{x}_{ij}] \).
4. Return the parametrized intermediate model: given by

\[
\hat{H}(s, p) = \sum_{k=1}^{m_p} \sum_{\ell=1}^{r_p} \hat{x}_{ki} P_\ell(p) h_k(s).
\]

Combining local reduced models with various parametric bases \( \mathcal{P}_{r_p} \) has been considered in \cite{5,6}. However, these works focused on interpolatory bases, i.e., \( \hat{H}(s, p) \) interpolated the local model \( h_k(s) \) at the parameter sample \( p = \mu_k \), for \( k = 1, 2, \ldots, m_p \). Here we focus on parametric bases to minimize a discrete joint least-squares measure. And more importantly, in Section \cite{3} we will allow these
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bases functions in $p$ to adaptively vary. For the concept of local reduced models in a projection-based setting, we refer the reader to, e.g., [1, 8, 15, 36, 39] and the references therein.

Example 1 We illustrate Algorithm [1] on data generated from an idealized vibration model of a cantilevered Timoshenko beam [38] with proportional damping:

$$H(s, p) = c^T \left( M s^2 + \left( \frac{1}{2} M + p K \right) s + K \right)^{-1} b.$$ \hspace{1cm} (15)

The matrices $M$ and $K$ represent the distributed mass and stiffness of a Timoshenko beam approximated with finite elements; for the model considered here, we have 2300 degrees of freedom. The parameter $p$ represents a damping parameter and we consider $p \in P = [0, 0.8]$. We sample at $m_s = 80$ logarithmically spaced frequency points between $10^{-3}$ and $10^3$ on the imaginary axis, capturing the main dynamic range of the model and consider $m_p = 10$ parameter sampling points, equally spaced across the interval $[0.01, 0.8]$.

Local models are constructed at each of the parameter sampling points using VF as described above. Each local model has order $\nu_j = 10$ producing an aggregate rational model, $\hat{H}(s, p)$, having $s$-order $n_s = 100$.

We take for $P_{r_p}$ the subspace of polynomials having order up to 5 (so that $r_p = 6$); we use Bernstein polynomials as basis elements, $P_\ell(p)$; see [3, 19].

In Figure 1 we show the Bode amplitude and error plots at two representative parameter values, $p = 0.01$ and $p = 0.22$. The original sampled function is shown in blue, our approximation is shown with dashed red lines, and (absolute) pointwise error appears in green. To illustrate the quality of the parametric reduced model over the whole parameter domain $p \in P = [0, 1]$, in Figure 2 we show the relative error between $H(s, p)$ and the original sampled model $H(s, p)$ with respect to the $H_2$ and $H_\infty$ norm for every parameter value, i.e., for $\hat{p} \in P = [0, 1]$, we plot

$$\left\| H(\cdot, \hat{p}) - \hat{H}(\cdot, \hat{p}) \right\|_{H_2} \leq \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| H(i\omega, \hat{p}) - \hat{H}(i\omega, \hat{p}) \right|^2 d\omega \right)^{1/2},$$

and

$$\left\| H(\cdot, \hat{p}) - \hat{H}(\cdot, \hat{p}) \right\|_{H_\infty} = \sup_{\omega \in \mathbb{R}} \left| H(i\omega, \hat{p}) - \hat{H}(i\omega, \hat{p}) \right| .$$

We observe that even away from the neighborhood of sampling points, the approximation performs quite well. Indeed, over almost the entire parameter domain, we obtain relative accuracy in both measures below $10^{-3}$, with the largest relative error being approximately $5 \times 10^{-3}$.
Fig. 1: Frequency responses of $\mathcal{H}(s,p)$ (---), $\tilde{\mathcal{H}}(s,p)$ (•••), and the error function $\mathcal{H}(s,p) - \tilde{\mathcal{H}}(s,p)$ (——) at $p = 0.01$ and $p = 0.22$.

Fig. 2: Relative $\mathcal{H}_2 / \mathcal{H}_\infty$ errors over the parameter range $[0.01, 0.8]$ for parametric fitting using polynomial $P_p$. Here $r_p = 10$.

3 Coupled Parametric Fitting

We have formulated a solution strategy to the underlying least squares data fitting problem (3) that assumes the form of a solution to be given as a linear combination of products of optimized local models, as defined in (7), with a fixed set of parametric basis functions, $\{P_1, P_2, \ldots, P_{r_p}\}$, that nominally have been defined independently of the data. This solution strategy has been formalized as Algo-
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In this section, we consider an important further refinement that allows the parametric basis functions also to be adapted (iteratively) to the data.

3.1 Parametric Bases via Variable Projection

Assume that the family of parametric functions, \{P_1(p), P_2(p), \ldots, P_r(p)\} depend smoothly on a vector of ancillary (meta-)parameters, \(\pi \in \mathcal{D}\) for some bounded domain \(\mathcal{D} \subset \mathbb{C}^{r_p}\); we denote this dependence as \(\{P^{\pi}_1(p), P^{\pi}_2(p), \ldots, P^{\pi}_r(p)\}\). Our focus will be on the case that \(\{P^{\pi}_j\}\) are rational functions with pole locations given by the components of \(\pi \in \mathcal{D}\), but more general settings can be considered, so we keep our discussion general for the time being. Note that the factor matrix, \(B\), defined in (4) will now depend smoothly on \(\pi\) as well:

\[
B(\pi) = [P^{\pi}_1(\mu_1), \ldots, P^{\pi}_r(\mu_r)] \in \mathbb{C}^{m_p \times r_p}.
\]

We suppose that \(\text{rank}[B(\pi)] = \dim P^{\pi}_r\) is constant for \(\pi \in \mathcal{D}\). The pseudoinverse, \(B(\pi)^\dagger\) and the orthogonal projection onto \(\text{Ran}[B(\pi)]\), \(Q(\pi) = B(\pi)B(\pi)^\dagger\), also vary smoothly with respect to \(\pi \in \mathcal{D}\), as a consequence.

The data fitting problem (3) can now be reformulated as:

Find \(\hat{H}(s,p) = \sum_{k,\ell} \hat{x}_{k\ell} h_k(s) P^{\pi}_{\ell}(p)\),

where \(\hat{X} = [\hat{x}_{ij}]\) and \(\hat{\pi} \in \mathcal{D}\) solves

\[
(\hat{X}, \hat{\pi}) = \arg \min_{X,\pi} \|AXB(\pi)^\top - H\|^2_F. \tag{16}
\]

This is a separable least squares problem that is linear with respect to \(X\) and nonlinear with respect to \(\pi\). The linear variables, \(X\), can be eliminated, leaving an equivalent nonlinear optimization problem of greatly reduced dimension expressed solely with respect to \(\pi\). Writing \(P = AA^\dagger\) for the orthogonal projection onto \(\text{Ran}[A]\), we have the equivalent problem:

\[
\hat{X} = A^\dagger \cdot H \cdot (B(\pi)^\dagger)^\top, \quad \text{where}
\]

\[
\hat{\pi} = \arg \min_{\pi \in \mathcal{D}} \|P \cdot H \cdot (I - (Q(\pi) \otimes P)) \cdot \text{vec}(H)\|^2_F. \tag{17}
\]

Note that \(Q(\pi) \otimes P\) is an orthogonal projector that is dependent on \(\pi\), and so, the residual vector, \(r(\pi) = (I - (Q(\pi) \otimes P)) \cdot \text{vec}(H)\), is the orthogonal projection of the data, \(\text{vec}(H)\), onto the orthogonal complement of the tensor product space \(\text{Ran}[B(\pi)] \otimes \text{Ran}[A]\). The map \(\pi \mapsto r(\pi)\) is a nonlinear, albeit smooth, map and the minimization appearing in (17) is thus a nonlinear least squares problem typical of the method of variable projection developed by Golub and Pereyra [21]. The key observation of Golub and Pereyra was that many nonlinear least squares problems were in fact linear with respect to a substantial number of variables and nonlinear only with respect to a comparative few. The resulting method of variable projection stemming from this has proven to be quite useful in a wide variety of contexts and there have been substantial subsequent refinements; see e.g., [14, 20, 31, 37].

This leads us to our second algorithm which can be viewed simply as a refinement of Algorithm 1.
Algorithm 2 Parametric Fitting - Adaptive Parametric Basis

INPUT: Observed response data: \( \{ \xi_i, \mu_j, H(\xi_i, \mu_j) \}_{i=1, j=1}^{m, p} \), Target model order: \( n_s \), Parametric family of \( \pi \)-dependent basis functions \( \{ P_{\pi}^{\ell} \}_{\ell=1}^{r_p} \).

OUTPUT: Bivariate function \( \hat{H}(s, p) \) that fits the data and solves (3).

1. Choose local model orders, \( \nu_j \geq 1 \) such that \( \sum_{j=1}^{m_p} \nu_j = n_s \).
2. Construct local models, \( h_j(s) \), of order \( \nu_j \) for each parameter \( \mu_j \) following (7).
3. Using the method of variable projection (see [14,37]) find \( \hat{X} = [\hat{x}_{ij}] \) and \( \hat{\pi} \) that solve (16).
4. Return the parametrized intermediate model given by \( \hat{H}(s,p) = \sum_{k,\ell} \hat{x}_{k\ell} h_k(s) P_{\pi}^{\ell}(p) \).

3.2 Adaptive Rational Parametric Bases

One simple and often very effective choice for a parametric family of \( \pi \)-dependent basis functions is to choose \( P_{\pi}^{\ell}(p) \) to be rational functions with respect to \( p \), with pole locations given by the components of \( \pi \in \mathcal{D} \). We consider this in more detail here and assume for simplicity that \( \pi = [\pi_1 \cdots \pi_{r_p}]^{\top} \in \mathcal{D} \subset \mathbb{C}^{r_p} \) and \( P_{\pi}^{\ell}(p) = \frac{1}{p - \pi_{\ell}} \) for \( \ell = 1, \ldots, r_p \).

Assume further that \( \mathcal{D} \cap \mathcal{P} = \emptyset \), so that all functions \( P_{\pi}^{\ell}(p) \) are smoothly varying with respect to \( p \) throughout the parameter range \( \mathcal{P} \). For any \( \pi = [\pi_1 \cdots \pi_{r_p}]^{\top} \in \mathcal{D} \), using the definition \( B \) in (4), we can define the factor matrix,

\[
B(\pi) := \begin{bmatrix}
1 \\
\mu_i - \pi_j
\end{bmatrix} \in \mathbb{C}^{m_p \times r_p}.
\]

In the following three examples, we illustrate the performance of Algorithm 2 using rational parametric bases as described above. In these and subsequent examples, we use an implementation of Variable Projection for Step 3 of Algorithm 2 following [14], incorporating a Gauss-Newton strategy to solve the (reduced) nonlinear least squares problem.

Example 2 We construct a parametric transfer function with rational-dependence in \( p \):

\[
H(s,p) = \sum_{k=1}^{6} \frac{\phi_k}{p - \pi_k} G_k(s), \quad \pi_k \in \mathbb{C},
\]

where the parameter poles are chosen as \( \pi_1 = 0.4, \pi_{2,3} = 2 \pm 1.5i, \pi_{4,5} = 4 \pm 0.8i, \) and \( \pi_6 = 5.1 \). The non-parametric rational functions \( G_k(s) \) in (18) are based on a variation of the Penzl’s example [40, Ex. 3]. Let

\[
A_1 := \begin{bmatrix}
-1 & 100 \\
-100 & -1
\end{bmatrix}, \quad A_2 := \begin{bmatrix}
-1 & 200 \\
-200 & -1
\end{bmatrix}, \quad A_3 := \begin{bmatrix}
-1 & 400 \\
-400 & -1
\end{bmatrix},
\]

\[
A_4 := \text{diag} \begin{bmatrix} -1 & -2 & \cdots & -20 \end{bmatrix}, \quad A(\zeta) := \text{diag} \begin{bmatrix} (\zeta + 1)^2 A_1 A_2 A_3 \end{bmatrix},
\]

\[
b(\zeta) := \begin{bmatrix}
10 & 10 & \cdots & 10 & 0 & \cdots & \zeta + 1
\end{bmatrix}, \quad \text{and} \quad c(\zeta) := b(\zeta)^{\top},
\]
and define the transfer function

$$G(s, \zeta) := c(\zeta)^T (sE - A(\zeta))^{-1} b(\zeta).$$ \hspace{1cm} (20)

We evaluate $G(s, \zeta)$ at 6 fixed $\zeta_k$ values to construct $G_k(s)$ in (18):

$$G_k(s) := G(s, \zeta_k), \quad \text{where} \quad [\zeta_1 \, \zeta_2 \cdots \, \zeta_6] = [0.29 \, 0.57 \, 0.86 \, 1.14 \, 1.43 \, 1.71 \, 2].$$

We sample $H(s, p)$ in (18) at 100 frequency samples, logarithmically spaced between $10^{-1}$ and $10^5$ on the imaginary axis and 8 parameter samples linearly spaced on $P = [1, 5]$. We apply both the fixed polynomial basis (Algorithm 1) and adaptive rational basis (Algorithm 2) approaches to $H(s, p)$. Figure 3 and Figure 4 show the amplitude Bode plot comparisons at two of the sampled points of $p = 1.57$ and $p = 3.86$, respectively, where the green line represents the error function.

![Fig. 3: Frequency comparisons of $H(s, p)$ (---), $\hat{H}(s, p)$ (•••), and the error function $H(s, p) - \hat{H}(s, p)$ (—) at $p = 1.57$.](image1)

![Fig. 4: Frequency comparisons of $H(s, p)$ (---), $\hat{H}(s, p)$ (•••), and the error function $H(s, p) - \hat{H}(s, p)$ (—) at $p = 3.86$.](image2)

To illustrate this more clearly, we show the $H_2$ and $H_{\infty}$ approximation errors over the entire parameter domain $P = [1, 5]$ in Figure 5. We observe that, except
for very few points, adaptive rational basis functions using Algorithm 2 yields superior approximation of $\hat{H}(s,p)$.

![Approximation Quality over Parameter Range](image)

**Fig. 5:** Relative $H_2$ and $H_\infty$ errors for Algorithm 2 and Algorithm 1

**Example 3** We revisit the beam model from Example 1. We used the same samples in $s$ and $p$. However, in contrast to Example 1 here we apply Algorithm 2 using rational basis functions with order $r_p = 6$.

![Frequency responses](image)

**Fig. 6:** Frequency responses of $\mathcal{H}(s,p)$ (---), $\hat{\mathcal{H}}(s,p)$ (---), and the error function $\mathcal{H}(s,p) - \hat{\mathcal{H}}(s,p)$ (---) at $p = 0.01$ and $p = 0.02$ using rational basis functions.

The amplitude plots of $\mathcal{H}(s,p)$ and $\hat{\mathcal{H}}(s,p)$ due to Algorithm 2 are shown, in Figure 6, for two representative sampling points, illustrating the accuracy of the approximation. To give a better overall picture, once again we show the approx-
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Approximation error over the continuous parameter interval in Figure 7, illustrating an approximation with a relative error of order $10^{-3}$ for the whole parameter space.

Fig. 7: Relative $H_2$ and $H_\infty$ errors over $\mathcal{P}$ for rational basis functions with $r_p = 6$.

Example 4 In this example, we illustrate outlier-resilience of the least squares approximation for fitting parametrized dynamical systems in the framework we have developed. We used the model and setup as in Example 1 and Example 3. In order to construct outliers, we chose poor local models at two of the ten sampling points. Figure 8 shows the performance of Algorithm 1 and Algorithm 2 at the sampling points together with the accuracy of the local VF approximants. Observe that both least-squares solutions have a nearly uniform error across the parameter domain and avoids the outlier. If one interpolates among the local models, instead, the resulting parametric reduced model will suffer large excursions due to the outliers.

Fig. 8: Comparison of relative $H_2$ errors at sampled parameter values.
Our observations so far can be summarized as follows:

1. Using rational functions that are adapted to the data for the parametric basis appears to produce approximation quality that is at least as good as what may be obtained by using polynomial functions for the parametric basis, independent of the structure of the original model, see Example [3].

2. If the underlying model has, in fact, a rational parametrization, then using data-adapted rational functions for the parametric basis may be expected to outperform polynomial bases, see Example [2].

3. Combining local models with a least-squares measure helps improve resilience with respect to outliers in the data, caused by few poor local models.

Real Parametric Systems: For a rational transfer function, \( \mathcal{H}(s) \), to be real, it must have a real realization, say, \( \mathcal{H}(s) = e^{\mathbf{c}^\top (s\mathbf{I} - \mathbf{A})^{-1} \mathbf{b}} \) for \( \mathbf{A} \in \mathbb{R}^{n \times n} \) and \( \mathbf{b}, \mathbf{c} \in \mathbb{R}^n \). This can be assured if \( \mathcal{H}(\pi) = \mathcal{H}(s) \) for all \( s \in \mathbb{C} \) where \( \mathcal{H}(s) \) is defined. Analogously we find for the parametrized case that \( \mathcal{H}(s,p) \) is real if (a) \( \mathcal{P} \) is closed under conjugation \( (\mathcal{P} = \mathcal{P}) \), and (b) \( \mathcal{H}(\pi,p) = \mathcal{H}(s,p) \) for all \( s \in \mathbb{C}, p \in \mathcal{P} \). The local models, \( \mathbf{h}_k(s) \), used in constructing the intermediate parametrized model \( \hat{\mathcal{H}}(s,p) \) in (2) have been generated so that \( \mathbf{h}_k(s) = \mathbf{h}_k(\pi) \), for \( k = 1, \ldots, m_p \). In order to guarantee that \( \hat{\mathcal{H}}(s,p) \) is also real, we investigate the parametric dependence.

Let \( \rho : I \to I \) be a permutation of the index set, \( I := \{1, \ldots, r_p\} \), so that \( P_\ell(p) = \bar{P}_\rho_\ell(p) \), for \( p \in \mathcal{P} \) and each \( \ell = 1, \ldots, r_p \). If \( \hat{x}_{k,\ell} = \tilde{x}_{k,\rho_\ell} \) for \( k = 1, \ldots, m_p \) and \( \ell = 1, \ldots, r_p \) then a direct calculation shows \( \mathcal{H}(s,p) \) must be real.

On the other hand, if \( \hat{\mathcal{H}}(s,p) \) is real then

\[
\hat{\mathcal{H}}(\pi,p) - \mathcal{H}(s,p) = \sum_{k=1}^{m_p} \mathbf{h}_k(s) \sum_{\ell=1}^{r_p} \left( \hat{x}_{k,\ell} - \tilde{x}_{k,\rho_\ell} \right) \bar{P}_\rho_\ell(p) = 0.
\]

Since the sets \( \{\mathbf{h}_k\} \) and \( \{P_\ell(p)\} \) are linearly independent, we conclude \( \hat{x}_{k,\ell} = \tilde{x}_{k,\rho_\ell} \), \( k = 1, \ldots, m_p \) and \( \ell = 1, \ldots, r_p \) and so, \( \hat{\mathcal{H}}(s,p) \) in (2) is real if and only if \( \hat{x}_{k,\ell} = \tilde{x}_{k,\rho_\ell} \) for \( k = 1, \ldots, m_p \) and \( \ell = 1, \ldots, r_p \).

As a consequence, if the parametric basis functions are polynomials with real coefficients, then \( \hat{x}_{k,\ell} \) must be real for all \( \{k, \ell\} \). If the parametric basis functions are elementary rational functions, \( P_\ell(s) = \frac{1}{s - \pi_\ell} \), then the associated poles, \( \{\pi_1, \ldots, \pi_{r_p}\} \) must be closed under conjugation. That is, there is a permutation \( \rho \) such that \( \pi_\ell = \pi_{\rho_\ell} \), and as a consequence \( P_\ell(p) = \bar{P}_{\rho_\ell}(p) \). As a practical matter, the algorithms that are deployed in solving (16) or (17) will not exactly preserve the conjugate symmetry, \( \hat{x}_{k,\ell} = \tilde{x}_{k,\rho_\ell} \) for all \( k, \ell \), and so we enforce the condition explicitly, thus guaranteeing a real system as a final outcome.

4 Optimal Compression of the Intermediate Parameterized Model

In Sections [2] and [3] we described how to produce a parameterized intermediate model, \( \hat{\mathcal{H}}(s,p) \), that provides a least squares fit to the given data by combining parametrically localized system models with appropriate parametric bases. Since the number of local models grows with the number of parameter samples, the \( s \)-order of \( \hat{\mathcal{H}}(s,p) \) also grows rapidly as the number of parameter samples grows.
For example, suppose \( m_p = 20 \) parameter samples are used with uniform local model order, \( \nu = \nu_k = 50 \). The resulting \( s \)-order of \( \hat{H}(s,p) \) may be as large as \( n_s = m_p \cdot \nu = 1000 \). Could we do better?

One might expect that adjacent parameter values could produce system responses that have many common features, leading one to suspect that there could be significant redundancy among the local models. Thus, we might reasonably expect that a lower order system could be found that might also fit the data well.

In this section, we describe a second phase to our solution process that compresses \( \hat{H}(s,p) \). We produce another parameterized model, \( \hat{H}_{\text{red}}(s,p) \), that approximates \( \hat{H}(s,p) \) (and hence the data) quite well (indeed, optimally in a sense we specify); it has a parameterization that conforms with the original model parameterization in the sense that for each \( \hat{s} \in \mathbb{C} \), \( \hat{H}_{\text{red}}(\hat{s}, \cdot) \in \mathcal{P}_{\nu} \) as well; but we anticipate that \( \hat{H}_{\text{red}}(s,p) \) will have \( s \)-order substantially smaller than \( n_s \).

Our approach uses an \( H_2 \)-optimal model reduction strategy similar to one introduced by Baur et al. [4] but specifically tailored for the parametric structure of the intermediate model obtained in the first phase of our solution process described in Section 3. The compressed model coming out of this second phase retains the parametric structure of the intermediate model while producing a locally optimal approximation with respect to a continuous system-theoretic error measure which we describe below.

### 4.1 Optimal compression with respect to an \( H_2 \otimes L_2(P) \) error

For each parameter sample \( \mu_k, k = 1, \ldots, m_p \), the corresponding local model, \( h_k(s) \), can be realized as

\[
 h_k(s) = c_k^\top (sI - A_k)^{-1} b_k, \quad k = 1, \ldots, m_p,
\]

where \( A_k \in \mathbb{C}^{\nu_k \times \nu_k} \), and \( b_k, c_k \in \mathbb{C}^{\nu_k} \). Define,

\[
 A := \text{diag}(A_1, A_2, \ldots, A_{m_p}) = \begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_{m_p} \end{bmatrix} \in \mathbb{C}^{n_s \times n_s}, \quad B := \begin{bmatrix} b_1 \\ \vdots \\ b_{m_p} \end{bmatrix} \in \mathbb{C}^{n_s}.
\]

and the block diagonal matrix:

\[
 D = \text{diag}(c_1, c_2, \ldots, c_{m_p}) = \begin{bmatrix} c_1 & 0 & \ldots & 0 \\ 0 & c_2 & & \\ & \ddots & \ddots & \\ 0 & & \ldots & c_{m_p} \end{bmatrix} \in \mathbb{C}^{n_s \times m_p}
\]

(note that \( D \) is not a square matrix and moreover each \( c_k \) will be of different order generally). For the parametric basis defining the factor space, \( \mathcal{P}_{\nu} \) (or \( \mathcal{P}_{\nu}^\pi \) if it was chosen adaptively as in Section 3), define

\[
 V(p) := [P_1(p) \ldots P_{n_s}(p)]^\top \in \mathbb{C}^{n_s \times \nu_p}
\]
The parametric model resulting from either Algorithm 1 or 2 may be written

\[ \hat{H}(s, p) = \sum_{k, \ell} \hat{x}_{k\ell} b_k(s) P_{k\ell}(p) = \sum_{k, \ell} P_{k\ell}(p) \hat{x}_{k\ell} c_k^\top (sI - A_k)^{-1} b_k = V(p)^\top \mathcal{G}(s), \]  

(24)

where \( \mathcal{G}(s) = (D\hat{X})^\top (sI - \hat{A})^{-1} B \). Notice that for any \( s \in \mathbb{C}, \mathcal{G}(s) \in \mathbb{C}^n \), which can then be viewed as a parameter-free SIMO (single-input/multiple-output) system mapping scalar inputs to \( r_p \)-dimensional vector outputs.

We seek a parameterized model, \( \hat{H}_{\text{red}}(s, p) \), with a compatible parameterization to \( \hat{H}(s, p) \) and with (potentially) significantly smaller \( n_s \)-order, say, \( n_{\text{red}} \ll n_s \). This invites the ansatz that \( \hat{H}_{\text{red}}(s, p) \) must have the form

\[ \hat{H}_{\text{red}}(s, p) = V(p)^\top \mathcal{G}_{\text{red}}(s) \quad \text{with} \quad \mathcal{G}_{\text{red}}(s) = C_{\text{red}}(sI - A_{\text{red}})^{-1} B_{\text{red}}, \]

and

\[ A_{\text{red}} \in \mathbb{C}^{n_{\text{red}} \times n_{\text{red}}}, \quad B_{\text{red}} \in \mathbb{C}^{n_{\text{red}}}, \quad \text{and} \quad C_{\text{red}} \in \mathbb{C}^{p \times n_{\text{red}}}. \]  

(25)

The error with which \( \hat{H}_{\text{red}}(s, p) \) approximates \( \hat{H}(s, p) \) can be measured through a joint norm defined for bivariate functions \( \tilde{H} \in \mathcal{H}_2 \otimes L_2(\mathcal{P}) \) as

\[ \| \tilde{H} \|_{\mathcal{H}_2 \otimes L_2(\mathcal{P})} := \sqrt{\frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{\mathcal{P}} |H(i\omega, p)|^2 \, dA(p) \, d\omega}. \]  

(26)

where \( dA(p) \) is either planar Lebesgue measure defined on our parameter set, \( \mathcal{P} \subset \mathbb{C} \), or linear Lebesgue measure, if \( \mathcal{P} \) is a line segment.

In particular, note that

\[ \| \hat{H} - \hat{H}_{\text{red}} \|_{\mathcal{H}_2 \otimes L_2(\mathcal{P})}^2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{\mathcal{P}} |\hat{H}(i\omega, p) - \hat{H}_{\text{red}}(i\omega, p)|^2 \, dA(p) \, d\omega \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{\mathcal{P}} |V(p)^\top (\mathcal{G}(i\omega) - \mathcal{G}_{\text{red}}(i\omega))|^2 \, dA(p) \, d\omega \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \| \mathbf{R} (\mathcal{G}(i\omega) - \mathcal{G}_{\text{red}}(i\omega)) \|_2^2 \, d\omega \]

\[ = \| \mathbf{R} \mathcal{G} - \mathbf{R} \mathcal{G}_{\text{red}} \|_{\mathcal{H}_2}^2 \]

where \( \mathbf{R} \in \mathbb{C}^{n_{\text{red}} \times n_{\text{red}}} \) is the (upper triangular) Cholesky factor of the Gram matrix of the parametric basis with respect to \( L_2(\mathcal{P}) \); \( \mathbf{R}^\top \mathbf{R} = \int_{\mathcal{P}} V(p)^\top V(p) \, dA(p) \). The last line recognizes the usual definition of the \( \mathcal{H}_2 \) norm of a SIMO dynamical system.

This shows that finding a compressed parameterized model, \( \hat{H}_{\text{red}}(s, p) \), that is an optimal approximation to \( \hat{H}(s, p) \) with respect to the error measure [26] is equivalent to finding a \( \mathcal{H}_2 \)-optimal reduced order model, \( G_{\text{red}}(s) \), approximating a (preweighted) system, \( \mathbf{R} \mathcal{G}(s) \), and then unweighting the result by defining \( \mathcal{G}_{\text{red}}(s) = \mathbf{R}^{-1} G_{\text{red}}(s) \). Notice that once we have determined \( \mathcal{G}_{\text{red}}(s) \), our optimally compressed parameterized model is available as \( \hat{H}_{\text{red}}(s, p) = V(p)^\top \mathcal{G}_{\text{red}}(s) \).

The equivalence between an \( \mathcal{H}_2 \otimes L_2(\mathcal{P}) \)-optimal parameterized model reduction problem and a weighted parameter-free \( \mathcal{H}_2 \)-optimal model reduction problem is
an innovation introduced in [4] that allows one to solve $\mathcal{H}_2 \otimes \mathcal{L}_2 (P)$ approximation problems by using well-established, numerically efficient tools for $\mathcal{H}_2$ model reduction, e.g., the iterative rational Krylov method (IRKA) of [24]. While [4] considered only affine parameter dependence, we allow more general parametric families. For more details on the $\mathcal{H}_2$ optimal model reduction problem and in particular on IRKA, see [2,24].

We summarize this discussion and describe an overall two-phase procedure below:

Algorithm 3 Aggregate two-phase algorithm

**INPUT:** Measurements $\{\xi_i, \mu_j, \mathcal{H}(\xi_i, \mu_j)\}_{i=1,j=1}^{m_s, m_p}$

Target model order: $n_{\text{red}}$

**OUTPUT:** Parametrized model $\hat{\mathcal{H}}_{\text{red}}(s,p)$ with $s$-order $n_{\text{red}}$

**1. Phase 1:** Apply Algorithm 1 or 2 to construct an intermediate model $\tilde{\mathcal{H}}(s,p)$.

**2. Phase 2:** $\mathcal{H}_2$-compression of intermediate model

(a) Compute the Cholesky factor, $R$, of the Gram matrix associated with the parametric basis from Phase 1.

(b) Find an $\mathcal{H}_2$-optimal reduced model, $G_{\text{red}}(s)$, of order $n_{\text{red}}$ approximating the (preweighted) SIMO model, $\mathcal{R}G(s)$, where $G(s)$ is constructed as in (24) from local models derived in Phase 1.

**3. Return a final parametrized reduced model**

$$\hat{\mathcal{H}}_{\text{red}}(s,p) := V(p)^\top R^{-1} G_{\text{red}}(s)$$

4.2 Asymptotic stability over the entire parameter domain

In most applications, the transfer function $\mathcal{H}(s,p)$ is asymptotically stable for every $p \in P$, i.e., for each fixed $\hat{p}$, all the poles of $\mathcal{H}(s, \hat{p})$ have negative real parts. The structure of the derived model $\tilde{\mathcal{H}}(s,p)$ in [24] provides an assurance of asymptotic stability for every $p \in P$. Indeed, if the local models, $\{h_k\}_{k=1}^{m_p}$, are all asymptotically stable, then our parametrized reduced model $\hat{\mathcal{H}}(s,p)$ resulting from either Algorithm 1 or 2 is guaranteed to be asymptotically stable over the entire parameter domain. This is one of the main advantages of knitting together asymptotically stable local reduced models with appropriate parametric basis functions (a point that has also been pointed out in [5]). Here we use VF to construct the local models. Even though VF is not guaranteed to produce asymptotically stable systems, in almost all cases it is sufficient to add to VF a “pole-flipping” step, which involves reflecting intermediate unstable poles that may be encountered back to the left-half plane. In all of our examples we enforced stability on all local models, and so we can guarantee asymptotic stability of $\hat{\mathcal{H}}(s,p)$ in every case. This is in contrast to other parametric data-driven approaches where a two-variable barycentric-form for $\hat{\mathcal{H}}(s,p)$ is used either to enforce interpolation as in the parametric-Loewner formulation of [33,35] or to minimize a least-squares error as in [22] (the same performance criterion considered here). Allowing poles to vary with $p$ might allow one to cover a wider range of dynamics with lower order realizations — but the potential cost is that for these approaches asymptotic stability over the entire parameter domain cannot be guaranteed in general, and for any particular pa-
rameter $p = \hat{\mu}$, a stability-correcting post-processing might be required. With our approach, as we illustrate in various examples, a sufficiently rich local basis, $\mathcal{A}_{mp}$, allows us to cover wide-ranging dynamics accurately over the entire parameter domain. Another advantage of using the form (24), as revealed in this section, is that it allows an optimal parametric-reduction of $\hat{\mathcal{H}}(s, p)$ to reduce the dimension further for the cases of large number of sampling points. An $\mathcal{H}_2 \otimes L_2(\mathcal{P})$ optimal reduced model $\hat{\mathcal{H}}_{red}(s, p)$ must be, by definition, asymptotically stable. One can either enforce this by introducing a pole-flipping step in IRKA as done in VF, or can use a trust-region variant such as [7].

Example 5 We revisit the beam model from Example 1.

We chose 200 logarithmically spaced frequency samples in $[10^{-3}, 10^3]$ and $m_p = 20$ parameter samples, logarithmically spaced in $[0.0001, 1]$. In Step 1. of Algorithm 3 we use Algorithm 1 where the order of the local models is $n_s = 25$ and the polynomial order is $r_p = 20$. This results in the intermediate parametric model $\hat{\mathcal{H}}(s, p)$ of order $n_s = 500$. Then, in Step 3. of Algorithm 3 we perform $\mathcal{H}_2 \otimes L_2(\mathcal{P})$ reduction with $n_{red} = 20$. In Figure 9 we show the frequency response comparison for two representative parameter values of $p = 0.001$ and $p = 0.22$. The figure illustrates that the reduced model $\hat{\mathcal{H}}_{red}(s, p)$ of Algorithm 3 matches the approximation quality of the reduced model $\hat{\mathcal{H}}_{red}(s, p)$ of Algorithm 1 despite having a much smaller order.

Fig. 9: Frequency responses of $\mathcal{H}(s, p)$ (---), $\hat{\mathcal{H}}(s, p)$ (●●●●), and the error function $\mathcal{H}(s, p) - \hat{\mathcal{H}}(s, p)$ (→) at $p = 0.001$ and $p = 0.22$

To further illustrate the success of the $\mathcal{H}_2$-compression step in Algorithm 3 we pick three levels of reduced orders, namely $n_{red} = 10$, $n_{red} = 20$, and $n_{red} = 30$. For each of these three cases and for the intermediate model, we compute the relative $\mathcal{H}_2$ error for the entire parameter domain. The results, depicted in Figure 10 show that with $n_{red} = 20$, and $n_{red} = 30$, Algorithm 3 matches the approximation quality of Algorithm 1 over the entire domain. In Figure 11 for $p = 0.001$, we show the frequency response error plots. Observe that as $n_{red}$ increases to 30, the error due to Algorithm 3 matches very closely the error due to Algorithm 1.
5 Parametric Fitting with Several Parameters

Up to this point, we have assumed access to frequency response data associated with a single scalar parameterization. A moment reflection suggests that the approach we have proposed should have a trivial extension to the case of several parameters, \( p = [p_1, p_2, \ldots, p_d] \); we describe briefly one possible way this extension could proceed for the case of two parameters \( (d = 2) \), labeled as \( p = [p q] \).
We choose parametric basis functions in \( p \) and \( q \) independently: Take \( P_k(p), k = 1, \ldots, r_p \) to be basis functions associated with the first parameter \( p \) spanning a subspace \( \mathcal{P}_{r_p} \), and \( Q_l(q), l = 1, \ldots, r_q \) be basis functions associated with the second parameter \( q \) spanning a subspace \( \mathcal{Q}_{r_q} \). Suppose frequency response observations are taken at frequencies, \( s = \xi_1, \ldots, \xi_m \), for parameter samplings \( p = \mu_1, \ldots, \mu_m \), and \( q = \eta_1, \ldots, \eta_{m_q} \); these observations may be indexed as \( \mathcal{H}(\xi_i, \mu_j, \eta_{j_2}) = \mathcal{H}_{i,j_1,j_2} \) with \( i = 1, \ldots, m_s, j_1 = 1, \ldots, m_p, \) and \( j_2 = 1, \ldots, m_q \). We proceed much as we did in (2), assuming a separable intermediate model:

\[
\hat{\mathcal{H}}(s, p, q) = \sum_{k=1}^{r_p} \sum_{l_1=1}^{r_q} \sum_{l_2=1}^{r_q} x_{k, (l_1, l_2)} h_k(s) P_{l_1}(p) Q_{l_2}(q) \in \mathcal{R}_{r_p} \otimes \mathcal{P}_{r_p} \otimes \mathcal{Q}_{r_q}
\]  

(27)

and seek an \( \hat{X} = [\hat{x}_{k, (l_1, l_2)}] \) that solves the least squares problem:

\[
\hat{X} = \arg \min_X \sum_{i=1}^{m_s} \sum_{j_1=1}^{m_p} \sum_{j_2=1}^{m_q} \left| \hat{\mathcal{H}}(\xi_i, \mu_{j_1}, \eta_{j_2}) - \mathcal{H}(\xi_i, \mu_{j_1}, \eta_{j_2}) \right|^2
\]

While \( \mathcal{H} = [\mathcal{H}(\xi_i, \mu_{j_1}, \eta_{j_2})] \) may be viewed as a tensor with (tensor) rank 3 and dimension \( m_s \times m_p \times m_q \), it will be useful to flatten \( \mathcal{H} \) along indices associated with parameters, so without changing notation, we think of \( \mathcal{H} \) as a two-dimensional array, \( \mathcal{H} \in \mathbb{C}^{m_s \times (m_p m_q)} \), with each row associated with observations at a particular frequency, \( \xi \), and column entries stored in \( q \)-major order (that is, the observations \( \mathcal{H}_{i,j_1,j_2} \) are stored consecutively in row \( i \), with \( j_1 = 1, \ldots, m_p, j_2 = 1, \ldots, m_q, \) and with the \( q \)-index, \( j_2 \), varying most rapidly). Similarly, we flatten \( \hat{X} \) along indices associated with parameters, thinking of \( \hat{X} \) as a two-dimensional array, \( \hat{X} \in \mathbb{C}^{r_s \times (r_p r_q)} \) also without a change in notation. Defining

\[
A = [h_j(\xi_i)] \in \mathbb{C}^{m_s \times r_s}, \quad B_p = [P_j(\mu_i)] \in \mathbb{C}^{m_p \times r_p}, \quad \text{and} \quad B_q = [Q_j(\eta_i)] \in \mathbb{C}^{m_q \times r_q},
\]  

(28)

we may reformulate our least squares problem concisely as:

\[
\hat{X} = \arg \min_X \left\| AX(B_p \otimes B_q)^T - H \right\|_F^2
\]

One may follow the steps previously discussed for the single parameter case in Sections 2, 3 and 4 with minor changes now for the multiparameter case. In principle, the extension is trivial though potentially tedious. It is worth a cautionary note, however, that the usual computational issues, common to most parametric model reduction approaches, arising from the need to sample a high-dimensional parameter space will occur here as well.

The straightforward tensor product/grid-sampling strategy outlined here is an elementary extension of the framework put forward in Sections 2 and 3 and is expected to work comparably well in the multiparameter case for a small number of parameters. However, for even a modest number of parameters more subtle strategies may be necessary. Note first that the total number of local models that are used may increase dramatically as the number of parameters is increased; for \( d \) parameters, \( |P| = |p_1 p_2 \ldots p_d| \), with each parameter sampled at \( \tilde{m} \) values, say, the number of local models generated is \( \tilde{m}^d \) which grows explosively as \( \tilde{m} \) increases.
if \( d \) is large, a common problem for approaches that make use of local models. Suppose the parametric dependence with respect to each parameter is represented uniformly with \( \hat{r} \) basis vectors. Then the total number of unknowns to be determined in Phase 1 of Algorithm 3 is \((\hat{m}\hat{r})^d\) (number of elements of \( \hat{X} \)). Moreover, if \( \hat{m}^d > m_s \) then the resulting least squares problem is rank deficient, and computational strategies must take this into account, potentially at a significant additional cost. Therefore, different sampling approaches such as adaptive sparse sampling or greedy sampling will likely be necessary when \( d \) is large; see [8, Section 3.4] for a brief discussion. Such an approach leaves the framework of Algorithm 3 largely unchanged. A potential alternative that departs somewhat from the framework of Algorithm 3 is a multilevel approach that knits together the Phase 1 and Phase 2 steps, allowing smaller subsets of the \( \hat{m}^d \) local models to be hierarchically aggregated. Strategies such as these that could be suitable for modeling systems with large numbers of parameters will not be pursued further here. We note that parametric modeling problems involving a large number of parameters often produce staggering computational challenges; we anticipate that strategies such as what we offer in Algorithm 3 can play an important role in solving such problems, but are not likely to suffice themselves.

**Example 6** In this model, taken from [4], we consider the convection-diffusion model on unit square \( \Omega = [0,1] \times [0,1] \):

\[
\frac{\partial \phi(t,z)}{\partial t} = \Delta \phi(t,z) + p \cdot \nabla \phi(t,z) + b(z)u(t) \quad z \in \Omega, \ t \in (0,\infty),
\]

with homogeneous Dirichlet boundary conditions \( \phi(t,z) = 0 \), for \( z \in \partial \Omega \), where \( b(z) \) represents the characteristic function of the domain where the forcing function \( u(\cdot) \) acts. The parameter \( p = [p q]^\top \) represents convection in both directions. Discretizing (29) with a finite difference scheme yields

\[
\mathcal{H}(\mathbf{s}, \mathbf{p}) = \mathbf{c}^\top (s\mathbf{I} - (A_0 + pA_1 + qA_2) \mathbf{b}),
\]

where \( A_0, A_1, A_2 \in \mathbb{R}^{n \times n} \), and \( \mathbf{b}, \mathbf{c} \in \mathbb{R}^n \) with \( n = 10000 \). We sample \( \mathcal{H}(\mathbf{s}, \mathbf{p}) \) with a uniform \( 6 \times 6 \) grid in the parameter space \( \Omega \) together with 100 frequency points logarithmically spaced in \([10^2, 10^6]\) on the imaginary axis for each of the parameter pair. We use polynomial bases and choose \( r_p = r_q = 12 \). The approximation quality of the two-variable parametric approximation is shown in Figure 12 for a variety of parameter points, illustrating a high-quality parametric approximant.
6 Conclusions

We have presented a two phase approach to construct a parsimonious parametrized model that fits in a least-squares sense frequency response data arising from observations of a parametrized dynamical system. Using parametrized basis functions, the first phase of the proposed algorithm combines local models derived from the observed system response across parameter samplings, in order to solve a coupled least-squares data fitting problem taken with respect to both frequency and parameter samples. We consider both fixed parameter bases and varying parametric bases that have been adapted to the given data. The second-phase of our approach uses $H_2$-optimal model reduction strategies to eliminate potential redundancy that may exist among the local models obtained in the parametrized intermediate model from the first phase. Several examples illustrated the performance of our framework.

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