Isotopic Equivalence from Bézier Curve Subdivision

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Abstract

We prove that the control polygon of a Bézier curve $\mathcal{B}$ becomes homeomorphic and ambient isotopic to $\mathcal{B}$ via subdivision, and we provide closed-form formulas to compute the number of subdivision iterations to ensure these topological characteristics. We first show that the exterior angles of control polygons converge exponentially to zero under subdivision.

Keywords: Bézier curve, subdivision, piecewise linear approximation, non-self-intersection, homeomorphism, ambient isotopy.

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1. Introduction

Preserving certain topological characteristics such as homeomorphism and ambient isotopy, between an initial geometric model and its approximation, is of contemporary interest in geometric modeling \cite{1, 2, 18, 22}, with the focus here being on Bézier curves.

A Bézier curve is characterized by an indexed set of points, which form a PL approximation of the curve, called a control polygon (Definition \ref{def:control_polygon}). The de Casteljau algorithm \cite{9} is a subdivision algorithm associated to Bézier curves which recursively generates control polygons more closely approximating the curve under Hausdorff distance \cite{24}. We focus on homeomorphism and ambient isotopy between a Bézier curve and the control polygon. The control polygon homeomorphic to a simple Bézier curve is also simple,

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so homeomorphism precludes undesired self-intersections, while the control polygon ambient isotopic to a Bézier curve has the same knot type as the Bézier curve.

However, there may be substantial topological differences between Bézier curves and their control polygons. First of all, Bézier curves and their control polygons are not necessarily homeomorphic. There are examples in the literature showing simple Bézier curves with self-intersecting control polygons or self-intersecting Bézier curves with simple control polygons [15, 26, 28]. Secondly, Bézier curves and their control polygons are not necessarily ambient isotopic. There is an example showing an unknotted Bézier curve with a knotted control polygon [3, 21]. Examples of a knotted Bézier curve with an unknotted control polygon have recently appeared [15, 29].

Computationally, it is known that the convergence in Hausdorff distance is exponential [12, 25]. We show that the angular convergence rate is also exponential, and this becomes a useful tool in determining classical topological equivalence (by homeomorphism) as well as for knot equivalence. Consequently the convergence for homeomorphic and isotopic equivalence is also exponential. Furthermore, we derive closed-form formulas to compute sufficient numbers of subdivision iterations to achieve homeomorphism and ambient isotopy respectively. These formulas rely upon the constructive geometric proofs presented here.

2. Related Work

Exponential convergence in Hausdorff distance under Bézier curve subdivision has been studied in the literature [12, 25]. Morin and Goldman proved that the discrete derivatives of the control polygons converge exponentially to the derivatives of the Bézier curve, by showing that discrete differentiation commutes with subdivision [23]. Our angular convergence is based on these previous results.

The established topological equivalence by homeomorphism was given [26] by invoking the hodograph[3], but did not provide the number of subdivision iterations. We provide a constructive geometric proof for specified numbers of subdivision iterations to first produce a control polygon homeomorphic and, later, ambient isotopic to a given Bézier curve. Topologically reliable

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[1] The derivative of a Bézier curve is also expressed as a Bézier curve, known as the hodograph [3].
approximation in terms of homeomorphism of composite Bézier curves was established [6], which used algorithmic techniques that do not completely rely upon the de Casteljau algorithm, but techniques related to “significant points”. As we mentioned in the introduction, topological preservation can be used to prevent undesired self-intersections. The intersection of curves and surfaces is one of the fundamental problems in areas of geometric modeling [27]. For intersections between two Bézier curves, C. K. Yap gives a complete subdivision algorithm [31].

We construct a tubular neighborhood for a Bézier curve, with the boundary of the tubular neighborhood being a pipe surface. Pipe surfaces have been studied since the 19th century [20], but the presentation here follows a contemporary source [17]. These authors perform a thorough analysis and description of the end conditions of open spline curves. The junction points of a Bézier curve are merely a special case of that analysis.

Ambient isotopy is a stronger notion of equivalence than homeomorphism. An earlier algorithm [11] establishes an isotopic approximation over a broad class of parametric geometry, but does so at the expense of the a priori bounds provided here by restricting to subdivision on splines. Other recent papers [4, 16] present algorithms to compute isotopic PL approximation for 2D algebraic curves. Computational techniques for establishing isotopy and homotopy have been established regarding algorithms for point-cloud by “distance-like functions” [5].

Ambient isotopy under subdivision was previously established [22] for 3D Bézier curves of low degree (less than 4), where a crucial unknotting condition was trivially established for these low degrees. The results presented here extend to Bézier curves of arbitrary degree, by a more refined analysis of avoiding knots locally within the PL approximation generated. The focus on higher degree versions was motivated by applications in molecular simulation where Bézier curve models are created on input of hundreds of thousands of points, with interest in having curves that are at least $C^1$. Preserving that continuity over low degree models on this magnitude of points would be extremely tedious.

Denne and Sullivan proved that for homeomorphic curves, if their distance and angles between first derivatives are within some given bounds, then these curves are ambient isotopic [7]. We use this result to derive ambient isotopy for Bézier curves and provide formulas to compute the number of subdivision iterations, which is computationally crucial, as Bézier curves are used in many practical areas.
3. Definitions and Notation

Mathematical definitions, notation and a fundamental supportive theorem are presented in this section. More specialized definitions will follow in appropriate sections. The standard Euclidean norm will be denoted by $\| \cdot \|$. 

**Definition 3.1.** The parameterized Bézier curve, denoted as $B(t)$, of degree $n$ with control points $p_j \in \mathbb{R}^3$ is defined by

$$B(t) = \sum_{j=0}^{n} B_{j,n}(t)p_j, \quad t \in [0, 1],$$

where $B_{j,n}(t) = \binom{n}{j} t^j (1-t)^{n-j}$ and the PL curve given by the points $\{p_0, p_1, \ldots, p_n\}$ is called its control polygon. When $p_0 = p_n$, the control polygon is closed. Otherwise when $p_0 \neq p_n$, it is open.

In order to avoid technical considerations and to simplify the exposition, the class of Bézier curves considered will be restricted to those where the first derivative never vanishes.

**Definition 3.2.** A differentiable curve is said to be **regular** if its first derivative never vanishes.

**Definition 3.3.** A curve is said to be **simple** if it is non-self-intersecting.

The Bézier curve of Definition 3.1 is typically called a **single segment Bézier curve**, while a **composite Bézier curve** is created by joining two or more single segment Bézier curves at their common end points.

We use $\mathcal{B}$ to denote a simple, regular, $C^1$, compact, composite Bézier curve in $\mathbb{R}^3$, throughout the paper.

**Definition 3.4.** [24] Let $X$ and $Y$ be two non-empty subsets of a metric space $(M, d)$. The **Hausdorff distance** $\mu(X, Y)$ is defined by

$$\mu(X, Y) := \max\{\sup_{x \in X} \inf_{y \in Y} d(x, y), \sup_{y \in Y} \inf_{x \in X} d(x, y)\}.$$
Subdivision algorithms are fundamental for Bézier curves [9] and a brief overview is given here. Figure 1 shows the first step of the de Casteljau algorithm with an input value of $\frac{1}{2}$ on a single Bézier curve. For ease of exposition, the de Casteljau algorithm with this value of $\frac{1}{2}$ is assumed, but other fractional values can be used with appropriate minor modifications to the analyses presented. The initial control polygon $P$ is used as input to generate local PL approximations, $P^1$ and $P^2$, as Figure 1(b) shows. Their union, $P^1 \cup P^2$, is then a new PL curve whose Hausdorff distance is closer to the curve than that of $P$.

![Subdivision process](image1)

![Initial and resultant curves](image2)

Figure 1: A subdivision with parameter $\frac{1}{2}$

A summary is that subdivision proceeds by selecting the midpoint of each segment of $P$ and these midpoints are connected to create new segments, as Figure 1(a) shows. Recursive creation and connection of midpoints continues until a final midpoint is selected. The union of the segments from the last step then forms a new PL curve. Termination is guaranteed since $P$ has only finitely many segments.

After $i$ iterations, the subdivision process generates $2^i$ PL sub-curves, each being a control polygon for part of the original curve [9], which will be referred to as a sub-control polygon$^3$ denoted by $P^k$ for $k = 1, 2, 3, \ldots, 2^i$. Each $P^k$ has $n$ points and their union $\bigcup_k P^k$ forms a new PL curve that converges in Hausdorff distance to approximate the original Bézier curve. The Bézier curve defined by $\bigcup_k P^k$ is exactly the same Bézier curve defined by the original control points $\{p_0, p_1, \ldots, p_n\}$ [12]. So $\bigcup_k P^k$ is a new control polygon of the Bézier curve.

$^3$Note that by the subdivision process, each sub-control polygon of a simple Bézier curve is open.
Exterior angles were defined in the context of closed PL curves, but are adapted here for both closed and open PL curves. Exterior angles unify the concept of total curvature for curves that are PL or differentiable.

**Definition 3.5.** [19] The exterior angle between two oriented line segments, denoted as $\overrightarrow{p_{m-1}p_{m}}$ and $\overrightarrow{p_{m}p_{m+1}}$, is the angle formed by $\overrightarrow{p_{m}p_{m+1}}$ and the extension of $\overrightarrow{p_{m-1}p_{m}}$. Let the measure of the exterior angle to be $\alpha_{m}$ satisfying:

$$0 \leq \alpha_{m} \leq \pi.$$

**Definition 3.6.** Parametrize a curve $\gamma(s)$ with arc length $s$ on $[0, \ell]$. Then its total curvature is $\int_{0}^{\ell} ||\gamma''(s)|| \, ds$.

Total curvature can be defined for both $C^2$ and PL curves. In both cases, the total curvature is denoted by $T_{\kappa}(\cdot)$. The unified terminology is invoked in Fenchel’s theorem, which is fundamental to the work presented here.

**Definition 3.7.** [19] The total curvature of a PL curve in $\mathbb{R}^3$ is the sum of the measures of the exterior angles.

Fenchel’s Theorem [8] presented below is applicable both to PL curves and to differentiable curves.

**Theorem 3.1.** [8, Fenchel’s Theorem] The total curvature of any closed curve is at least $2\pi$, with equality holding if and only if the curve is convex.

Denote a PL curve with vertices $\{p_{0}, p_{1}, \ldots, p_{n}\}$ by $P$, and the uniform parametrization [23] of $P$ over $[0, 1]$ by $l(P)[0,1]$. That is:

$$l(P)[0,1](\frac{j}{n}) = p_{j} \text{ for } j = 0, 1, \ldots, n$$

and $l(P)[0,1]$ interpolates linearly between vertices.

**Definition 3.8.** Discrete derivatives [23] are first defined at the parameters $t_{j} = \frac{j}{n}$, where

$$l(P)[0,1](t_{j}) = p_{j}$$
for \( j = 0, 1, \ldots, n - 1 \). Let
\[
p'_j = l'(P)[0,1](t_j) = \frac{p_{j+1} - p_j}{t_{j+1} - t_j}.
\]
Denote \( P' = (p'_0, p'_1, \ldots, p'_{n-1}) \). Then define the discrete derivative for \( l(P)[0,1] \) as:
\[
l'(P)[0,1] = l(P')[0,1].
\]
For simplicity of notation, we let \( P(t) = l(P)[0,1] \) and \( P'(t) = l'(P)[0,1] \).

4. Angular Convergence under Subdivision

We use the notation established in Section 3. Also, let \( P_0(t) \) denote the original control polygon before subdivision. Let \( M \) be the maximum of the distance between two consecutive vertices of \( P_0'(t) \). Let \( P(t_{j-1}) \) and \( P(t_j) \) be any consecutive vertices of a control polygon \( P \) obtained by subdivision.

**Lemma 4.1.** For a \( C^1 \), composite Bézier curve \( B \), we have
\[
||P'(t_j) - P'(t_{j-1})|| \leq \frac{M}{2^i}.
\]

**Proof:** Morin and Goldman \[23\], Lemma 4] proved that the discrete differentiation commutes with subdivision, so \( P' \) can be viewed as being obtained by subdividing \( P_0' \). But \( P_0' \) is a control polygon of \( B' \) \[23\], Lemma 6]. Another previous result \[12\], Lemma 2.5] showed that the distance between any two consecutive vertices of a control polygon is bounded by \( \frac{M}{2^i} \). \( \square \)

**Theorem 4.1 (Angular Convergence).** For a \( C^1 \), composite Bézier curve \( B \), the exterior angles of the PL curves generated by subdivision converge uniformly to 0 at a rate of \( O(\sqrt{\frac{1}{2^i}}) \).

**Proof:** Since \( B(t) \) is assumed to be regular and \( C^1 \), the non-zero minimum of \( ||B'(t)|| \) over the compact set \( [0,1] \) is obtained. For brevity, the notations of \( u_i = P'(t_j) \), \( v_i = P'(t_{j-1}) \) and \( \alpha = \alpha_m \) are introduced. The convergence of \( u_i \) to \( B'(t_j) \) \[23\] implies that \( ||u_i|| \) has a positive lower bound for \( i \) sufficiently large, denoted by \( \lambda \).

Lemma 4.1 gives that \( ||u_i - v_i|| \to 0 \) as \( i \to \infty \) at a rate of \( O(\sqrt{\frac{1}{2^j}}) \). This implies: \( ||u_i|| - ||v_i|| \to 0 \) as \( i \to \infty \) at a rate of \( O(\frac{1}{2^i}) \).
Consider
\[
1 - \cos(\alpha) = 1 - \frac{u_i v_i}{||u_i|| ||v_i||}
\]
\[
= \frac{||u_i|| ||v_i|| - v_i v_i + v_i v_i - u_i v_i}{||u_i|| ||v_i||}
\]
\[
\leq \frac{||u_i|| - ||v_i||}{||u_i||} + \frac{||v_i - u_i||}{||u_i||} \leq \frac{||u_i|| - ||v_i||}{\lambda} + \frac{||v_i - u_i||}{\lambda} \leq 2 \frac{||v_i - u_i||}{\lambda}
\]
It follows from Lemma 4.1 that
\[
1 - \cos(\alpha) \leq \frac{M}{\lambda^{2i-1}}.
\]
It follows from the continuity of \(\arccos\) that \(\alpha\) converges to 0 as \(i \to \infty\). To obtain the convergence rate, taking the power series expansion of \(\cos\) we get
\[
1 - \cos(\alpha) \geq \alpha^2 \left(1 - \frac{\alpha^2}{4!} - \frac{\alpha^4}{6!} + \cdots\right)
\]
\[
= \alpha^2 \left(\frac{1}{2} - \alpha^2 \left|\frac{1}{4!} - \frac{\alpha^2}{6!} + \cdots\right|\right)
\]
Note that for \(1 > \alpha\),
\[
e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots > \left|\frac{1}{4!} - \frac{\alpha^2}{6!} + \cdots\right|.
\]
Combining Inequality 3 and 4 we have,
\[
1 - \cos(\alpha) > \alpha^2 \left(\frac{1}{2} - \alpha^2 e\right).
\]
For any \(0 < \tau < \frac{1}{2}\), sufficiently many subdivisions will guarantee that \(\alpha\) is small enough such that \(1 > \alpha\) and \(\tau > \alpha^2 e\). Thus
\[
1 - \cos(\alpha) > \alpha^2 \left(\frac{1}{2} - \alpha^2 e\right) > \alpha^2 \left(\frac{1}{2} - \tau\right) > 0.
\]
By Inequality 2 we have
\[
\alpha < \sqrt{\frac{2M}{\lambda(\frac{1}{2} - \tau)}} \sqrt{\frac{1}{2i}}.
\]
So \(\alpha\) converges to 0 at a rate of \(O\left(\sqrt{\frac{1}{2i}}\right)\). \(\square\)
5. Topologically reliable control polygons

We present sufficient conditions for a homeomorphism between a subdivided control polygon and its associated Bézier curve, and derive an ambient isotopy by relying on related results [7].

5.1. Homeomorphism

To obtain a homeomorphism, we first establish a local homeomorphism between a sub-control polygon and the corresponding sub-curve of $B$, and then establish a global homeomorphism between the control polygon and $B$.

Lemma 5.1. [14, Lemma 7.4] Let $P$ be an open PL curve in $\mathbb{R}^3$. If $T_k(P) = \sum_{j=1}^{n-1} \alpha_j < \pi$, then $P$ is simple.

Theorem 5.1. For a $C^1$, composite Bézier curve $B$, there exists a sufficiently large value of $i$, such that after $i$-many subdivisions, each of the sub-control polygons generated as $P^k$ for $k = 1, 2, 3, \ldots, 2^i$ will be simple.

Proof: For each $P^k$, the measures of the exterior angles of $P^k$ converge uniformly to zero as $i$ increases (Theorem 4.1). Each open $P^k$ has $n$ edges. Denote the $n - 1$ exterior angles of each $P^k$ by $\alpha_j^k$, for $j = 1, \ldots, n - 1$ and for $k = 1, 2, 3, \ldots, 2^i$. Then there exists $i$ sufficiently large such that

$$\sum_{j=1}^{n-1} \alpha_j^k < \pi,$$

for each $k = 1, 2, 3, \ldots, 2^i$. Use of Lemma 5.1 completes the proof. □

The proof techniques for homeomorphism rely upon the sub-control polygons to be pairwise disjoint, except at their common end points. Denote two generated sub-control polygons of $B$ as

$$P = (p_0, p_1, \ldots, p_n) \text{ and } Q = (q_0, q_1, \ldots, q_n).$$

Definition 5.1. The sub-control polygons $P$ and $Q$ are said to be consecutive if the last vertex $p_n$ of $P$ is the first vertex $q_0$ of $Q$, that is, $p_n = q_0$.

Remark 5.1. For $B$, the $C^1$ assumption ensures that the segments $\overrightarrow{p_{n-1}p_n}$ and $\overrightarrow{q_0q_1}$ are collinear. The regularity assumption ensures that the exterior angle can not be $\pi$. So the exterior angle at the common point is 0.
Lemma 5.2 extends to arbitrary degree Bézier curves from a previously established result that was restricted to cubic Bézier curves [30], as used in the proof of isotopy under subdivision for low-degree Bézier curves [22].

**Lemma 5.2.** Let $\Pi$ be the plane normal to a sub-control polygon at its initial vertex. If the total curvature of the sub-control polygon is less than $\frac{\pi}{2}$, then the initial vertex is the only single point where the plane intersects the sub-control polygon.

**Proof:** Denote the sub-control polygon as $Q = (q_0, q_1, \ldots, q_n)$, where Figure 2 shows an orthogonal projection of this 3D geometry. Assume to the contrary that $\Pi \cap Q$ contains a point $u$ where $u \neq q_0$. Consider the closed polygon formed by vertices $\{q_0, \ldots, u, q_0\}$. Then by Theorem 3.1 we know that the total curvature of the closed polygon is at least $2\pi$. However, excluding the exterior angles at $q_0$ (which is $\frac{\pi}{2}$), and the exterior angles at $u$ (which is at most $\pi$ by Definition 3.5), we still at least have $\frac{\pi}{2}$ left, which contradicts to $T_\kappa(Q) < \frac{\pi}{2}$.

**Lemma 5.3.** Recall that $B$ denotes a simple, regular, $C^1$, composite Bézier curve in $\mathbb{R}^3$. Let $w$ be a point of $B$ where $B$ is subdivided and let $\Pi$ be the plane normal to $B$ at $w$. Then there exists a subdivision of $B$ such that the
sub-control polygon ending at \( w \) and the sub-control polygon beginning at \( w \) intersect \( \Pi \) only at the single point \( w \).

**Proof:** The plane \( \Pi \) separates \( \mathbb{R}^3 \) into two disjoint open half-spaces, denoted as \( H_1 \) and \( H_2 \), such that \( \mathbb{R}^3 = H_1 \cup \Pi \cup H_2 \) and \( H_1 \cap H_2 = \emptyset \). By Remark 5.1, the exterior angle at \( \{w\} \) is 0.

Perform sufficient many subdivisions so that the control polygon ending at \( w \), denoted by \( P \), and the control polygon beginning at \( w \), denoted by \( Q \), each have total curvature less than \( \pi/2 \) by Theorem 4.1. Therefore, by Lemma 5.2 the only point where \( P \) or \( Q \) intersect \( \Pi \) is at \( w \). \( \square \)

This global homeomorphism will be proven by reliance upon pipe surfaces, which are defined below.

**Definition 5.2.** The **pipe surface** of radius \( r \) of a parameterized curve \( c(t) \), where \( t \in [0, 1] \) is given by

\[
p(t, \theta) = c(t) + r[\cos(\theta) \ n(t) + \sin(\theta) \ b(t)],
\]

where \( \theta \in [0, 2\pi] \) and \( n(t) \) and \( b(t) \) are, respectively, the normal and bi-normal vectors at the point \( c(t) \), as given by the Frenet-Serret trihedron. The curve \( c \) is called a spine curve.

For \( B \) and \( i \) subdivisions, with resulting sub-control polygons \( P^k \) for \( k = 1, \ldots, 2^i \), let \( S_B(r) \) be a pipe surface of radius \( r \) for \( B \) so that \( S_r(B) \) is non-self-intersecting. For each \( k = 1, \ldots, 2^i \), denote

- the parameter of the initial point of \( P^k \) by \( t^k_0 \), and that of the terminal point by \( t^k_n \)
- the normal disc of radius \( r \) centered at \( B(t) \) as \( D_r(t) \),
- the union \( \bigcup_{t \in [t^k_0, t^k_n]} D_r(t) \) by \( \Gamma_k \), and designate it as a **pipe section**.

**Theorem 5.2.** Sufficient subdivisions will yield a simple control polygon that is homeomorphic to \( B \).

**Proof:** By Theorem 4.1 we can take \( \iota_1 \) subdivisions so that \( T_\kappa(P^k) < \pi/2 \), for each sub-control polygon \( P^k \). By Lemma 5.1, this choice of \( \iota_1 \) guarantees that each \( P^k \) is simple. By the convergence in Hausdorff distance under
subdivision \([23]\), we can take \(\iota_2\) subdivisions such that the control polygon generated by \(\iota_2\) subdivision fits inside the pipe surface \(S_r(B)\). Choose \(\iota = \max\{\iota_1, \iota_2\}\). By Lemma \([5.3]\) this choice of \(\iota\) ensures that each \(P^k\) fits inside the corresponding \(\Gamma_k\). This plus the fact that \(P^k\) is simple shows that the control polygon, \(\bigcup_{k=1}^{2t} P^k\), is simple, which implies the homeomorphism. \(\square\)

5.2. Ambient isotopy

We derive the ambient isotopy following \([7, Proposition 3.1]\).

Corollary 5.2.1. Sufficient subdivisions will yield a simple control polygon that is ambient isotopic to \(B\).

Proof: By Theorem \([5.2]\) sufficiently many subdivisions will produce a homeomorphic \(P\). We can define a homeomorphism \(h\) mapping \(P(t)\) to \(B(t)\) by

\[h(p) = B(P^{-1}(p))\text{ for } p \in P.\]

Denne and Sullivan \([7, Proposition 3.1]\) proved that provided the homeomorphism, \(B\) and \(P\) are ambient isotopic \(\|B(t) - P(t)\| < \frac{r}{2}\) (where \(r\) is the radius of a pipe surface) and \(\max_{t \in [0, 1]} \theta(t) < \frac{\pi}{6}\) (where \(\theta(t)\) is the angle between \(B'(t)\) and \(P'(t)\)).

Because \(P(t)\) converges to \(B(t)\) \([25]\) and \(P'(t)\) converges to \(B'(t)\) \([23]\), the conclusion follows. \(\square\)

Remark 5.2. The result \([7, Proposition 3.1]\) contains an assumption that the limit curve is \(C^{1,1}\), to ensure the existence of a positive thickness, which is equivalent to the existence of a non-self-intersecting pipe surface here. Note that our limit curve \(B\) is assumed to be a simple, compact composite Bézier curve. So the curve is \(C^2\) except at finitely many points. It follows easily that \(B\) is actually \(C^{1,1}\).

6. Sufficient Subdivision Iterations

In this section, we shall establish closed-form formulas to compute sufficient numbers of subdivisions for small exterior angles, homeomorphism and ambient isotopy respectively.

From the previous sections we know that the homeomorphism is obtained by subdivision based on two criteria: (1) angular convergence; and (2) convergence in distance. So the speed of achieving these topological characteristics is determined by the angular convergence rate and the convergence rate
in distance which are both exponential. Here, we further find closed-form formulas to compute sufficient numbers of subdivision iterations to achieve these properties.

**Definition 6.1.** Let \( P \) denote a control polygon of a Bézier curve, and let \( P_x \) denote an ordered list of all of \( x \)-coordinates of \( P \) (with similar meaning given to \( P_y \) for the \( y \)-coordinates and to \( P_z \) for the \( z \)-coordinates). Let

\[
\| \Delta_2 P_x \|_\infty = \max_{0 < m < n} |P_{m-1,x} - 2P_{m,x} + P_{m+1,x}|
\]

be the maximum absolute second difference of the \( x \)-coordinates of control points, (with similar meanings for the \( y \) and \( z \) coordinates). Let

\[
\Delta_2 P = (\| \Delta_2 P_x \|_\infty, \| \Delta_2 P_y \|_\infty, \| \Delta_2 P_z \|_\infty),
\]

(i.e.) a vector with 3 values.

**Definition 6.2.** The distance\(^4\) between a Bézier curve \( B \) and the control polygon \( P \) generated by \( i \) subdivisions is given by

\[
\max_{t \in [0,1]} ||P(t) - B(t)||.
\]

**Lemma 6.1.** The distance between the Bézier curve and its control polygon after \( i \)th-round subdivision is bounded by

\[
\frac{1}{2^i} N_\infty(n) \| \Delta_2 P \|,
\]

where

\[
N_\infty(n) = \frac{[n/2] \cdot [n/2]}{2n}.
\]

**Proof:** A published lemma [25, Lemma 6.2] proves a similar result restricted to scalar valued polynomials. We consider coordinate-wise and apply this result to the \( x, y, \) and \( z \) coordinates respectively, so that the distance of

\(^4\)The distance here is as previously used [25]. Note that the distance is not smaller than Fréchet distance. Our following results remain true if this distance is changed to Fréchet distance.
the $x$-coordinates of the Bézier curve and its control polygon after $i$th-round subdivision is bounded by

$$\frac{1}{2^i} N_\infty(n) \| \Delta_2 P_x \|_\infty,$$

with similar expressions for the $y$ and $z$ coordinates. Taking the Euclidean norm of the indicated three $x$, $y$ and $z$ bounds yields the upper bound given by (5), an upper bound of the distance between the Bézier curve and its control polygon after the $i$th subdivision.

For convenience, denote the above bound in distance as:

$$B_{dist}(i) := \frac{1}{2^i} N_\infty(n) \| \Delta_2 P \|.$$

(6)

Lemma 6.2. After $i$ subdivision iterations, the distance between $P'$ and $B'$ is bounded by $B'_{dist}(i)$, where

$$B'_{dist}(i) := \frac{1}{2^i} N_\infty(n - 1) \| \Delta_2 P' \|,$$

(7)

and $P'$ that consists of $n - 1$ control points is the control polygon of $B'$.

Proof: A control polygon’s derivative is again a control polygon of the Bézier curve’s derivative [23, Lemma 6]. So by Lemma 6.1, we have

$$\max_{t \in [0, 1]} \| P'(t) - B'(t) \| \leq B'_{dist}(i).$$

(8)

6.1. Subdivision iterations for small exterior angles

Assume $\nu$ is a small measure of angle between 0 and $\pi$. We shall find how many subdivisions will generate a control polygon such that the measure $\alpha$ of each exterior angle satisfies

$$\alpha < \nu.$$  

(9)

Recall the proof of angular convergence (Theorem 4.1). Consider two arbitrary consecutive derivatives $u_i = P'(t_j)$ and $v_i = P'(t_{j-1})$ and the corresponding exterior angle $\alpha$. Recall that in Section 4 we had Inequalities 1 and 2

$$1 - \cos(\alpha) \leq \frac{2 \| v_i - u_i \|}{\| u_i \|} \leq \frac{M}{\| u_i \| 2^{i-1}}.$$  

(10)

14
Let $\sigma = \min\{||B'(t)|| : t \in [0, 1]\}$. The regularity of $B$ ensures that $\sigma > 0$ and the continuity of $B'$ on the compact interval $[0, 1]$ ensures that the minimum exists. Recall $u_i = P'(t_j)$ for some $t_j \in [0, 1]$. So it follows from Inequality (8) that

$$||B'(t_j)|| - ||u_i|| \leq B'_{\text{dist}}(i).$$

Solving the inequality we get

$$||u_i|| \geq ||B'(t_j)|| - B'_{\text{dist}}(i) \geq \sigma - B'_{\text{dist}}(i).$$

In order to have $u_i \neq 0$, it is sufficient to perform enough subdivisions such that

$$||u_i|| \geq \sigma - B'_{\text{dist}}(i) > 0,$$

that is $B'_{\text{dist}}(i) < \sigma$. By the definition (Equation (7)) of $B'_{\text{dist}}(i)$ we set,

$$\frac{1}{2^i} N_{\infty}(n-1) ||\triangle_2 P'|| < \sigma.$$

Therefore for $B'_{\text{dist}}(i) < \sigma$, it suffices to have\(^5\)

$$i > \frac{1}{2} \log\left(\frac{N_{\infty}(n-1) ||\triangle_2 P'||}{\sigma}\right) = N_1.$$

(11)

After the $i$ subdivision iterations, whenever $i > N_1$, then $B'_{\text{dist}}(i) < B'_{\text{dist}}(N_1)$, because $B'_{\text{dist}}(i)$ is a strictly decreasing function (Equation (7)). So it follows from Inequality (10) that whenever $i > N_1$,

$$1 - \cos(\alpha) \leq \frac{M}{2^{i-1}(\sigma - B'_{\text{dist}}(i))}.$$

To obtain $\alpha < \nu$ (Inequality (9)), it suffices to have that $1 - \cos(\alpha) < 1 - \cos(\nu)$. Now choose $i$ large enough so that

$$1 - \cos(\alpha) \leq \frac{M}{2^{i-1}(\sigma - B'_{\text{dist}}(N_1))} < 1 - \cos(\nu).$$

(12)

The second inequality of Inequality (12) implies that

$$i > \log\left(\frac{2M}{(1 - \cos(\nu))(\sigma - B'_{\text{dist}}(N_1))}\right).$$

\(^5\)Throughout this paper, we use log for $\log_2$. 

15
To simplify this expression, let

$$f(\nu) = \frac{2M}{(1 - \cos(\nu))(\sigma - B'_{\text{dist}}(N_1))}. \quad (13)$$

Then, we have

$$i > \log(f(\nu)).$$

**Theorem 6.1.** Given any \(\nu > 0\), there exists an integer \(N(\nu)\) defined by

$$N(\nu) = \lceil \max\{N_1, \log(f(\nu))\} \right\rceil \quad (14)$$

where \(N_1\), and \(f(\nu)\) are given by Equations 11 and 13 respectively, such that each exterior angle is less than \(\nu\), whenever \(i > N(\nu)\).

**Proof:** It follows from the definitions of \(N_1\) and \(f(\nu)\) and the analysis in this section.

It is worth to note that \(N\) is a logarithm depending on several parameters such as \(\sigma\), \(N_\infty(n)\) and \(\triangle_2P'\) as well as an upper bound variable \(\nu\).

### 6.2. Subdivision iterations for homeomorphism

For a regular Bézier curve \(B\) of degree 1 or 2, the control polygon is trivially ambient isotopic to \(B\). We consider \(n \geq 3\).

Given any \(\nu > 0\), Theorem 6.1 shows that there exists an integer \(N(\nu)\), such that each exterior angle is less than \(\nu\) after \(N(\nu)\) subdivisions. Furthermore, there is an explicit closed formula to compute \(N(\nu)\).

**Theorem 6.2.** There exists a positive integer, \(N\left(\frac{\pi}{n-1}\right)\) for \(n > 2\), where \(N\left(\frac{\pi}{n-1}\right)\) is defined by Equation 14, such that after \(\lceil N\left(\frac{\pi}{n-1}\right) \rceil\) subdivisions, each sub-control polygon will be simple.

**Proof:** By Theorem 6.1 after \(N\left(\frac{\pi}{n-1}\right)\) subdivisions, each exterior angle is less than \(\frac{\pi}{n-1}\). Since each sub-control polygon has a \(n - 1\) exterior angles, the total curvature of each sub-control polygon is less than \(\pi\). Lemma 5.1 implies that this is a sufficient condition for each sub-control polygon being simple. \(\square\)

\(^6\)For degree 1, both the curve and the polygon are either a point or a line segment. For degree 2, there are three points. The curve and the polygon are planar and open (otherwise the curve is not regular).
While existence of sufficiently many iterations for the control polygon to fit inside the pipe $S_r(B)$ has been established, it remains of interest to bound the number of subdivisions that are sufficient for this containment. Define $N'(r)$ by

$$N'(r) = \frac{1}{2} \log\left(\frac{N_\infty(n)||\Delta_2P||}{r}\right),$$

(15)

where $r$ is the radius of a non-self-intersecting pipe surface for $B$. By the definition of $B_{dist}(i)$ (Equation 6) and Equation 15, we have $B_{dist}(i) < r$ whenever $i > N'(r)$.

**Lemma 6.3.** The control polygon generated by $i$ subdivisions, where $i > N'(r)$ and $N'(r)$ is given by Equation 15, satisfies

$$\max_{t \in [0,1]} ||B(t) - P(t)|| < r,$$

and hence fits inside the pipe surface of radius $r$ for $B$.

**Proof:** By Lemma 6.1, $\max_{t \in [0,1]} ||B(t) - P(t)|| \leq B_{dist}(i)$. Then this lemma follows from the definition of $N'(r)$ given by Equation 15.

While Theorem 6.2 addresses each sub-control polygon, it is of interest to ensure that the union of all these sub-control polygons is also simple. In Theorem 6.3, that union is the ‘control polygon’, as the result of multiple subdivisions.

**Theorem 6.3.** Set

$$\hat{N} = \max\{N\left(\frac{\pi}{2(n-1)}\right), N'(r)\},$$

where $N(\nu)$ is defined by Equations 14 and $N'(r)$ is given by Equation 15. After $[\hat{N}]$ or more subdivisions, the control polygon will be homeomorphic.

**Proof:** The inequality $N \geq N'(r)$ implies that the control polygon generated after the $N$th subdivision lies inside the pipe. The inequality $N \geq N\left(\frac{\pi}{2(n-1)}\right)$ ensures that the total curvature of its each sub-control polygon is less than $\frac{\pi}{2}$. These two conditions are sufficient conditions for the control polygon being simple (The proof of Theorem 5.2).
6.3. Subdivision iterations for ambient isotopy

Recall, by Corollary 5.2.1, that a homeomorphic $P$ will further be ambient isotopic if $||B(t) - P(t)|| < \frac{\pi}{2}$ and $\max_{t \in [0,1]} \theta(t) < \frac{\pi}{6}$ (where $\theta(t)$ is the angle between $B(t)$ and $P'(t)$). We may produce $N'(\frac{\pi}{2})$ subdivisions to satisfy the first condition (Lemma 6.3). To guarantee the second condition, we consider:

\[
1 - \cos(\theta(t)) = 1 - \frac{B'(t) \cdot P'(t)}{||B'(t)|| \cdot ||P'(t)||}
\]

\[
= \frac{||B'(t)|| \cdot ||P'(t)|| - P'(t) \cdot P'(t) + P'(t) \cdot P'(t) - B'(t) \cdot P'(t)}{||B'(t)|| \cdot ||P'(t)||}
\]

\[
\leq \frac{||B'(t)|| - ||P'(t)||}{||B'(t)||} + \frac{||B'(t) - P'(t)||}{||B'(t)||} \leq \frac{2||B'(t) - P'(t)||}{\sigma},
\]

where $\sigma = \min\{||B'(t)|| : t \in [0,1]\}$ (Recall $\sigma > 0$.) From Inequality 8

\[
\max_{t \in [0,1]} ||B'(t) - P'(t)|| \leq B_{dist}'(i),
\]

we have

\[
1 - \cos(\theta(t)) \leq \frac{2B_{dist}'(i)}{\sigma}.
\]

To have $\theta(t) < \frac{\pi}{6}$, it suffices to set

\[
\frac{2B_{dist}'(i)}{\sigma} < 1 - \cos\left(\frac{\pi}{6}\right) = 1 - \frac{\sqrt{3}}{2}.
\]

By Equality 7

\[
B_{dist}'(i) := \frac{1}{2^{\alpha_i}} N_\infty(n - 1) ||\Delta_2 P'||,
\]

we get

\[
i \geq \frac{1}{2} \log\left(\frac{2N_\infty(n - 1) ||\Delta_2 P'||}{(1 - \frac{\sqrt{3}}{2})\sigma}\right) = N_2.
\]

(16)

So $N_2$ subdivision iterations will guarantee the second condition.

**Theorem 6.4.** Set

\[
N^* = \max\{N\left(\frac{\pi}{2(n - 1)}\right), N'(\frac{\tau}{2}), N_2\},
\]

where $N, N', N_2$ are given by Equations 14, 15 and 16 respectively. After $[N^*]$ or more subdivisions, the control polygon $P$ will be ambient isotopic to the Bézier curve $B$. 

18
Proof: The values $N\left(\frac{\pi}{2(n-1)}\right)$ and $N'(\frac{\pi}{2})$ are used to obtain a homeomorphism, by Theorem 6.3. And then $N_2$ is used to further obtain an ambient isotopy. \qed

Remark 6.1. Note that $N\left(\frac{\pi}{2(n-1)}\right) = \max\{N_1, \log f\left(\frac{\pi}{2(n-1)}\right)\}$. Comparing $N_1$ (Equation 11) and $N_2$, we find that $N_2 < N_1 + 2$. By Equation 15 we also have $N'(\frac{\pi}{2}) < N'(r) + 1$. So $N^* < \hat{N} + 2$, where $\hat{N} = \max\{N\left(\frac{\pi}{2(n-1)}\right), N'(r)\}$ is a sufficient number of subdivisions to guarantee homeomorphism. So after a homeomorphism based on Theorem 6.3 is attained, no more than 2 additional subdivision iterations will be used to produce the ambient isotopy.

7. Conclusion

We first proved the exterior angles of control polygons under subdivision converge to 0 exponentially. We then showed that sufficiently many subdivisions produce a control polygon homeomorphic to the Bézier curve and further derived the ambient isotopy by relying upon a previous isotopy result by Denne and Sullivan. We established closed-form formulas to compute a priori sufficient number of subdivisions to achieve these topological characteristics. These results are being applied in computer graphics, computer animation and scientific visualization, especially in visualizing molecular simulations.

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7 The dissertation work [13] of the first author adopted an alternative, more explicit way to construct the ambient isotopy, with iteration bound of $\max\{N\left(\frac{\pi}{2n}\right), N'(r)\}$. It was shown [13] Remark 4.2.7 that $N\left(\frac{\pi}{2n}\right) < N\left(\frac{\pi}{2(n-1)}\right) + 1$, so that no more than 1 additional subdivision iteration would be used to produce the ambient isotopy after a homeomorphism is attained from Theorem 6.3. However, the method here has advantages because of its direct use of subdivision versus specialized techniques.
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