HOMOLOGICAL ALGEBRA RELATED TO SURFACES WITH BOUNDARY

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Abstract. In this article we describe an algebraic framework which can be used in three related but different contexts: string topology, symplectic field theory, and Lagrangian Floer theory of higher genus.

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1. Introduction

The purpose of this article is to describe an algebraic framework which can be used for three related but different purposes: (equivariant) string topology
symplectic field theory, and Lagrangian Floer theory of higher genus. It turns out that the relevant algebraic structure for all three contexts is a homotopy version of involutive bi-Lie algebras, which we call IBL\(_{\infty}\)-algebras. This concept has its roots in such diverse fields as string field theory, noncommutative geometry, homotopy theory, and others. To avoid confusion, let us emphasize right away that the algebraic structure is not a topological field theory in the sense of Atiyah and Segal, see also [23, 24].

The structure we discuss encodes the combinatorial structure of certain compactifications of moduli spaces of Riemann surfaces with punctures and/or with boundary. Informally, this relation can be described as follows. To each topological type of a compact connected oriented surface, characterized by its genus \(g \geq 0\), the number \(k \geq 1\) of “incoming” boundary components and the number \(\ell \geq 1\) of “outgoing” boundary components, one associates a linear map \(C^{\otimes k} \to C^{\otimes \ell}\) between tensor powers of a given vector space (satisfying certain symmetry properties). Compositions of such maps correspond to (partial) gluing of incoming boundaries of the second to outgoing boundary components of the first map. However, in contrast to a topological field theory, we do not require compositions corresponding to the same end result of gluing to agree. Instead, the case \((k, \ell, g) = (1, 1, 0)\) gives rise to a boundary map \(p_{1,1,0}: C \to C\) squaring to zero, and in general the operation associated to the surface of signature \((k, \ell, g)\) is a chain homotopy (with respect to this boundary operator) from the zero map to the sum of all possible compositions resulting in genus \(g\) with \(k\) incoming and \(\ell\) outgoing boundaries.

**IBL\(_{\infty}\)-algebras.** To be more precise, we first introduce some notations (full details appear in §2). Let \(R\) be a commutative ring which contains \(\mathbb{Q}\) (think of \(R = \mathbb{Q}\) or \(\mathbb{R}\)). Let \(C\) be a free graded module over \(R\). Its degree shifted version \(C[1]\) has graded pieces \(C[1]^k = C^{k+1}\). We consider the symmetric product
\[
E_k C := (C[1] \otimes_R \cdots \otimes_R C[1]) / \sim,
\]
that is the quotient of the tensor product by the action of the symmetric group permuting the factors with signs. Let \(EC\) be the direct sum \(\bigoplus_{k \geq 1} E_k C\).

Next consider a series of \(R\)-module homomorphisms \(p_{k,\ell,g}: E_k C \to E_\ell C\) indexed by triples of integers \(k, \ell \geq 1, g \geq 0\). (They will also have specific degrees, which we ignore in this introduction). They canonically extend to \(R\)-linear maps \(\tilde{p}_{k,\ell,g}: EC \to EC\) (see §2). We introduce the formal sum
\[
\hat{p} := \sum_{k, \ell \geq 1, g \geq 0} \tilde{p}_{k,\ell,g} h^{k+g-1} \tau^{k+\ell+2g-2}: EC\{h, \tau\} \to EC\{h, \tau\},
\]
where \(EC\{h, \tau\}\) denotes the space of power series in the formal variables \(h, \tau\) with coefficients in \(EC\).
**Definition 1.1.** We say that \((C, \{p_{k,\ell,g}\}_{k,\ell\geq 1, g\geq 0})\) is an IBL\(_{\infty}\)-algebra if
\[
\hat{p} \circ \hat{p} = 0.
\]

Note that \(EC\) has both an algebra and a coalgebra structure. However, due to the fact that both \(k\) and \(\ell\) are allowed to be greater than 1, \(\hat{p}\) is neither a derivation nor a coderivation. In fact, one can show that \(\hat{p}_{k,\ell,g}\) is a differential operator of degree \(k\) in the sense of graded commutative rings (see [20] and [17]). The symmetric bar complex \(EC\) then inherits a structure which is a special case of what was called a BV\(_{\infty}\) structure in [20] (see also [11, 68]). This structure can also be described in terms of a Weyl algebra formalism (see [7]). Such a formulation has its origin in the physics literature (see for example [51]) and in [11, 45]. The description of symplectic field theory in [28] is of this form. The formulation in Definition 1.1 is closer to one whose origin is in algebraic topology ([49, 64] etc.). It is also close to the algebraic formulation of Lagrangian Floer theory in [34].

In the applications in the context of holomorphic curves that we have in mind, one is often faced with the following situation. The principal object of interest is geometric, for example a Lagrangian submanifold \(L \subset (W, \omega)\) in a symplectic manifold. To analyze it, one chooses as auxiliary data a suitable almost complex structure \(J\), and studies \(J\)-holomorphic curves. Due to transversality issues, there are often many additional choices that need to be made, but eventually one writes down a chain complex with operations such as the \(p_{k,\ell,g}\) described above, where \(k\), \(\ell\) and \(g\) have the expected meaning in terms of the holomorphic curves.

Then one is faced with the task of proving independence of the algebraic structure of all choices made. One of the standard methods is to again use holomorphic curves to define morphisms between the algebraic structures for two different choices, as well as chain homotopies between them. To organize such proofs, it is therefore useful to have explicit algebraic descriptions of the structures, the morphisms and the homotopies that arise.

The first goal of this paper is to develop the homotopy theory of IBL\(_{\infty}\) algebras from this point of view. We follow the standard approach via obstruction theory, which leads to fairly explicit formulas. In §2 we define IBL\(_{\infty}\)-algebras and their morphisms, and discuss the defining relations from various points of view. In §3, we prepare for the homotopy theory by identifying the obstructions to extending certain partial structures or partial morphisms inductively. In §4 we introduce the notion of homotopy and show that it defines an equivalence relation. We also prove that compositions of homotopic morphisms are homotopic.

Note that it follows from the defining equation for an IBL\(_{\infty}\)-algebra that the map \(p_{1,1,0} : C[1] \rightarrow C[1]\) is a boundary operator, i.e. we have \(p_{1,1,0} \circ p_{1,1,0} = \)
Moreover, it turns out that part of any morphism $f : (C, \{p_{k,\ell,g}\}) \to (C', \{p'_{k,\ell,g}\})$ of $\text{IBL}_\infty$-algebras is a chain map $f_{1,1,0} : (C, p_{1,1,0}) \to (C', p'_{1,1,0})$. In §5 we prove the following

**Theorem 1.2.** Let $f$ be a morphism of $\text{IBL}_\infty$-algebras such that $f_{1,1,0}$ induces an isomorphism on homology. If $R$ is a field of characteristic $0$, then $f$ is a homotopy equivalence of $\text{IBL}_\infty$-algebras.

In §6 we prove that every $\text{IBL}_\infty$-structure can be pushed onto its homology:

**Theorem 1.3.** If $R$ is a field of characteristic $0$ and $(C, \{p_{k,\ell,g}\})$ is an $\text{IBL}_\infty$-algebra over $R$, then there exists an $\text{IBL}_\infty$-structure on its homology $H = H(C, p_{1,1,0})$ which is homotopy equivalent to $(C, \{p_{k,\ell,g}\})$.

**Weyl algebras and symplectic field theory.** In §7 we discuss the relationship of $\text{IBL}_\infty$-structures with the Weyl algebra formalism which is used in the formulation of symplectic field theory [28]. In a nutshell, the relation is as follows.

In symplectic field theory, the information on moduli spaces of holomorphic curves in the symplectization of a contact manifold is packaged in a certain Hamiltonian function $H$, which is a formal power series in $p$-variables (corresponding to positive asymptotics, or “inputs”) and $h$ (corresponding to genus) with polynomial coefficients in the $q$-variables (corresponding to negative asymptotics, or “outputs”). The exactness of the symplectic form on the symplectization forces $H|_{p=0} = 0$. There is a notion of augmentation in this context, which allows one to change the structure to also achieve $H|_{q=0} = 0$. Geometrically, augmentations arise from symplectic fillings of the given contact manifold. It is this augmented part of symplectic field theory which is shown to be equivalent to the $\text{IBL}_\infty$ formalism described here.

Let us also mention that in [27] the Weyl algebra formalism was shown to be equivalent to the structure of an algebra over a certain properad, at least on the level of objects. The emergence of $\text{IBL}_\infty$-operations on $S^1$-equivariant symplectic cohomology (which in view of [12] is essentially equivalent to symplectic field theory) is outlined in [62].

**Filtrations and Maurer-Cartan elements.** For various applications one needs the more general notion of a filtered $\text{IBL}_\infty$-algebra which we introduce in §8. As is common in homotopical algebra, there is a version of the Maurer-Cartan equation for our structure. We discuss this in §9 where we show that Maurer-Cartan elements have many of the expected properties.

This concludes the basic part of the theory. The remaining part of the paper gives some ideas how $\text{IBL}_\infty$-structures arise in algebraic and symplectic topology.
The dual cyclic bar complex of a cyclic DGA. The relation of string topology to Hochschild cohomology (of, say, the de Rham complex) and to Chen’s iterated integrals [18] has already been described by various authors, see e.g. [22, 44, 53, 25, 61, 32, 67]. In particular, our operator $p_{2,1,0}$ corresponds to the Gerstenhaber bracket [37] in that case. Our description below can be regarded as a ‘higher genus analogue’ of it.

With de Rham cohomology in mind, let us restrict to $R = \mathbb{R}$. Recall that an $A_\infty$-algebra structure on a graded $\mathbb{R}$-vector space $A$ consists of a series of $\mathbb{R}$-linear maps $m_k : A[1]^\otimes k \to A[1]$ for $k \geq 1$, which are assumed to satisfy the so called $A_\infty$-relations (see §12). A cyclic $A_\infty$-algebra in addition comes with a nondegenerate pairing $\langle \cdot, \cdot \rangle : A \otimes A \to \mathbb{R}$, of degree $-n$ such that

$$\langle m_k(x_1, \cdots, x_k), x_0 \rangle = (-1)^* \langle m_k(x_0, \cdots, x_{k-1}), x_k \rangle$$

with suitable signs $(-1)^*$. The notion of an $A_\infty$-algebra was introduced by Stasheff [65] and cyclic $A_\infty$-algebras were used in a related context by Kontsevich [46]. Their relation to symplectic topology was discussed in [46, 30, 17].

We can construct a dIBL-algebra (i.e., an $IBL_\infty$-algebra whose only non-vanishing terms are $p_{1,1,0}$, $p_{1,2,0}$ and $p_{2,1,0}$) from a cyclic differential graded algebra (DGA) as follows. We let $B^\text{cyc}_k A$ be the quotient of $A[1]^\otimes k$ by the $\mathbb{Z}_k$-action which cyclically permutes the factors with signs. In §10 and §11 we prove the following two results.

**Proposition 1.4.** Let $(A, \langle \cdot, \cdot \rangle, d)$ be a finite dimensional cyclic cochain complex whose pairing has degree $-n$. Then the shifted dual cyclic bar complex $(B^\text{cyc} A)[2 - n] = \bigoplus_{k \geq 1} \text{Hom}(B^\text{cyc}_k A, \mathbb{R})[2 - n]$ carries natural operations $p_{1,1,0} = d$, $p_{2,1,0}$ and $p_{1,2,0}$ which make it a dIBL-algebra.

**Theorem 1.5.** The dIBL-structure in Proposition 1.4 is $IBL_\infty$-homotopy equivalent to the analogous structure on the dual cyclic bar complex of the cohomology $H(A, d)$.

So far, the algebra structure on $A$ was not used. In §12 we take it into account and prove

**Proposition 1.6.** Let $A$ be a finite dimensional cyclic DGA whose pairing has degree $-n$. Then its product $m_2$ gives rise to a Maurer-Cartan element $m_2^+$ for the dIBL-structure on the dual cyclic bar complex of $A$ (completed with respect to its canonical filtration).

We can twist the dIBL-structure of Proposition 1.4 by the Maurer-Cartan element from Proposition 1.6 to obtain a twisted filtered dIBL-structure on the dual cyclic bar complex $(B^\text{cyc} A)[2 - n]$. Using (the filtered version of) Theorem 1.3, this structure can be pushed to its homology with respect to
the twisted differential. Moreover, using Theorem 1.5 and general homotopy theory of IBL∞-algebras we prove

**Theorem 1.7.** Let $A$ be a finite dimensional cyclic DGA whose pairing has degree $-n$, and let $H = H(A,d)$ be its cohomology. Then there exists a filtered IBL∞-structure on $(B_cyc^\ast H)[2-n]$ which is IBL∞-homotopy equivalent to $(B_cyc^\ast A)[2-n]$ with its twisted filtered d(IBL)-structure. In particular, its homology equals Connes’ version of cyclic cohomology of $A$ as defined e.g. in [50].

**Remark 1.8.** Theorem 1.7 together with its idea of proof using summation over ribbon graphs was explained by the authors on several occasions, for example by the second named author at the conference ‘Higher Structures in Geometry and Physics’ in Paris 2007. At the final stage of completing this paper, the authors found that results in [6] and [19] seem to be closely related to Theorem 1.7.

**The de Rham complex and string topology.** Now consider the de Rham complex $(\Omega(M),d)$ of a closed oriented manifold $M$. The wedge product and the intersection pairing $\int_M u \wedge v$ give $\Omega(M)$ the structure of a cyclic DGA. However, it is not finite dimensional, so we cannot directly apply the theory in Sections §10 and §11. To remedy this, we introduce in §13 the subspaces $B_cyc^\ast_k \Omega(M)_\infty \subset B_cyc^\ast_k \Omega(M)$ of operators with smooth kernel and prove the following analogues of Proposition 1.4 and Theorem 1.5 (see §13 for the relevant definitions)

**Proposition 1.9.** Let $M$ be a closed oriented manifold of dimension $n$. Then $B_cyc^\ast \Omega(M)_\infty[2-n]$ carries the structure of a Fréchet dIBL-algebra.

**Theorem 1.10.** The Fréchet dIBL-structure in Proposition 1.9 is IBL∞-homotopy equivalent to the analogous structure on the dual cyclic bar complex of the de Rham cohomology $H_{dR}(M)$.

The triple intersection product $m_2^\ast(u,w) = (-1)^s \int_M u \wedge v \wedge w$ defines an element $m_2^\ast \in B_cyc^\ast_3 \Omega(M)$ satisfying the equations of a Maurer-Cartan element. However, $m_2^\ast$ does not have a smooth kernel, so we cannot use it directly to twist the Fréchet dIBL-structure. Nevertheless, we expect that by pushing the structure onto the (finite dimensional!) de Rham cohomology $H_{dR}(M) = H(\Omega(M),d)$ one can prove the following analogue of Theorem 1.7.
Conjecture 1.11. Let $M$ be a closed oriented manifold of dimension $n$ and $H = H_{dR}(M)$ its de Rham cohomology. Then there exists a filtered IBL$_\infty$-structure on $B^{cyc} H[2-n]$ whose homology equals Connes’ version of cyclic cohomology of the de Rham complex of $M$.

Recall that if $M$ is simply connected, then by a theorem of Jones [42] (see also Chen [18] and Goodwillie [38]) various versions of cyclic cohomology of $\Omega(M)$ are isomorphic to suitable versions of $S^1$-equivariant homology of the free loop space of $M$. It is shown in [21] that Connes’ version of cyclic cohomology is isomorphic to a “reduced” version of $S^1$-equivariant homology of the free loop space, which differs from standard equivariant homology (defined via the Borel construction) only by the equivariant homology of a point. We conjecture that under this identification the involutive Lie bialgebra structure induced by the IBL$_\infty$-structure in Conjecture 1.11 agrees with the string bracket and cobracket described by Chas and Sullivan in [15, 16]. This is supposed to be a special case of such a structure on the equivariant loop space homology of any closed oriented manifold $M$ [67, 20].

The strategy to prove Conjecture 1.11 is to mimic the proof of Theorem 1.7 in the de Rham case. Then finite sums over basis elements get replaced by multiple integrals involving the Green kernel associated to a Riemannian metric on $M$, in a way similar to [43, 47, 31] and to perturbative Chern-Simons gauge theory [72, 8]. The difficulty in making this rigorous arises from possible divergences at the diagonal where some integration variables become equal. We hope to show in future work that this problem can be resolved in a similar way as in perturbative Chern-Simons gauge theory [3].

Lagrangian Floer theory. Finally, consider an $n$-dimensional closed oriented Lagrangian submanifold $L$ of a symplectic manifold $(W, \omega)$ (closed or convex at infinity). Then holomorphic curves in $W$ with boundary on $L$ give rise to a further deformation of the IBL$_\infty$-structure in Conjecture 1.11 associated to $L$. The structure arising from holomorphic disks has been described in [32], and the general structure (in slightly different language) in [21]. In the terminology of this paper, it can be described as follows.

It is proved in [32, 33] that moduli spaces of holomorphic disks with boundary on $L$ give rise to a (filtered) cyclic $A_\infty$-structure on its de Rham cohomology $H_{dR}(L)$. Moduli spaces of holomorphic curves of genus zero with several boundary components should give rise to a solution of an appropriate version of Batalin-Vilkovisky Master equation, see §12. Moreover, this data can be further enhanced using holomorphic curves of higher genus. We prove in [12] that Proposition 1.6 carries over to such $A_\infty$-algebras, so we arrive at the following

Conjecture 1.12. Let $L$ be an $n$-dimensional closed oriented Lagrangian submanifold $L$ of a symplectic manifold $(W, \omega)$ (closed or convex at infinity)
and let $H = H_{dR}(M)$ be its de Rham cohomology. Then there exists a filtered IBL$_\infty$-structure on $B^{cyc} H[2 - n]$ whose homology equals the cyclic cohomology of the cyclic $A_\infty$-structure on $H$ constructed in [34, 33].

**Remark 1.13.** Suppose that $\pi : W \to D$ is an exact symplectic Lefschetz fibration over the disk (e.g. any Stein domain $W$ admits such a fibration) and let $L_1, \ldots, L_m$ be the vanishing cycles. A conjecture of Seidel ([63], see [10] for further evidence for this conjecture) asserts that the symplectic homology of $W$ equals the Hochschild homology of a certain $A_\infty$-category with objects $L_1, \ldots, L_m$. An equivariant version of this conjecture would equate the $S^1$-equivariant symplectic cohomology $SH^*_S1(W)$ to the cyclic cohomology of this $A_\infty$-category. Hence the IBL$_\infty$-structure on $SH^*_S1(W)$ mentioned above would follow from this conjecture and an extension of Proposition 1.6 to suitable $A_\infty$-categories.

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2. **Involutive Lie bialgebras up to infinite homotopy**

In this section we define involutive Lie bialgebras up to infinite homotopy, or briefly IBL$_\infty$-algebras, and morphisms among them.

Let $R$ be a commutative ring with unit that contains $\mathbb{Q}$. Let $C = \bigoplus_{k \in \mathbb{Z}} C^k$ be a free graded $R$-module.

It is convenient to introduce the degree shifted version $C[1]$ of $C$ by setting $C[1]^d := C^{d+1}$. Thus the degrees $\deg c$ in $C$ and $|c|$ in $C[1]$ are related by $|c| = \deg c - 1$.

We introduce the $k$-fold symmetric product

$$E_k C := (C[1] \otimes_R \cdots \otimes_R C[1]) / \sim$$

as the quotient of the $k$-fold tensor product under the standard action of the symmetric group $S_k$ permuting the factors with signs, and the reduced symmetric algebra

$$EC := \bigoplus_{k \geq 1} E_k C.$$ 

Note that we do not include a constant term in $EC$. As usual, we write the equivalence class of $c_1 \otimes \cdots \otimes c_k$ in $E_k C$ as $c_1 \cdots c_k$. 

Remark 2.1. More precisely, we set $EC := \bigoplus_{k \geq 1} C[1] \otimes^k \mathcal{I}$, where $\mathcal{I}$ is the two-sided ideal generated by all elements $c \otimes c' - (-1)^{|c||c'|} c' \otimes c$. Since $R$ contains $\mathbb{Q}$, $EC$ is canonically isomorphic as a graded $R$-module to the subspace

\[
\left( \bigoplus_{k \geq 1} C[1] \otimes^k \right)^{\text{symm}} \subset \bigoplus_{k \geq 1} C[1] \otimes^k
\]

of invariant tensors under the action of the symmetric group: An inverse of the quotient map $\left( \bigoplus_{k \geq 1} C[1] \otimes^k \right)^{\text{symm}} \to EC$ is given by the symmetrization map

\[
I(c_1 \cdots c_k) := \frac{1}{k!} \sum_{\rho \in S_k} \varepsilon(\rho) c_{\rho(1)} \otimes \cdots \otimes c_{\rho(k)}.
\]

Here the sign $\varepsilon(\rho)$ (which depends on the $c_i$) is defined by the equation

\[
c_{\rho(1)} \cdots c_{\rho(k)} = \varepsilon(\rho) c_1 \cdots c_k.
\]

We extend any linear map $\phi : E_k C \to E_{\ell} C$ to a linear map $\hat{\phi} : EC \to EC$ by $\hat{\phi} = 0$ on $E_m C$ for $m < k$ and

\[
\hat{\phi}(c_1 \cdots c_m) := \sum_{\rho \in S_m} \varepsilon(\rho) \phi(c_{\rho(1)} \cdots c_{\rho(k)}) c_{\rho(k+1)} \cdots c_{\rho(m)}
\]

(2.1)

for $m \geq k$. Note that $\hat{\phi}$ maps $E_{k+s} C$ to $E_{\ell+s} C$ for every $s \geq 0$.

Remark 2.2. The map $\hat{\phi}$ is a differential operator of order $\leq k$ and a “codifferential operator” of order $\leq \ell$. In particular, $\hat{\phi}$ is a derivation if $k = 1$ and a coderivation if $\ell = 1$.

Now we consider a series of graded $R$-module homomorphisms

\[
p_{k,\ell,g} : E_k C \to E_{\ell} C, \quad k, \ell \geq 1, \ g \geq 0
\]

degree

\[
|p_{k,\ell,g}| = -2d(k + g - 1) - 1
\]

for some fixed integer $d$. Define the operator

\[
\hat{p} := \sum_{k,\ell = 1}^{\infty} \sum_{g=0}^{\infty} \hat{p}_{k,\ell,g} t^{k+g-1} r^{k+\ell+2g-2} : EC\{h,\tau\} \to EC\{h,\tau\},
\]

where $h$ and $\tau$ are formal variables of degree

\[
|h| := 2d, \ |\tau| = 0,
\]

and $EC\{h,\tau\}$ denotes formal power series in these variables with coefficients in $EC$. 
Definition 2.3. We say that \((C, \{p_{k,\ell,g}\}_{k,\ell, g \geq 1, g \geq 0})\) is an IBL\(_\infty\)-algebra of degree \(d\) if

\[
\hat{p} \circ \hat{p} = 0.
\]

Remark 2.4. The algebra over the Frobenius properad appearing in \([13]\) is closely related to the IBL\(_\infty\)-algebra. The Koszul-ness of the former in the sense of \([70]\) is proved in \([13]\). It provides a purely algebraic reasoning for this structure being a ‘correct’ infinity version of an involutive Lie bialgebra. We however do not use this fact in this paper.

Let us explain this definition from various angles.

(1) One can write equation (2.2) more explicitly as the sequence of equations

\[
\sum_{t=2-\min(k,\ell)}^{g+1} \left( \hat{p}_{k_2,\ell_2,g_2} \circ \hat{p}_{k_1,\ell_1,g_1} \right) |_{E_k C} = 0
\]

for each triple \((k,\ell,g)\) with \(k,\ell \geq 1\) and \(g \geq 1 - \min(k,\ell)\). Indeed, equation (2.3) is the part of the coefficient of \(h^{k+\ell+2g-2}\tau^{k+\ell+2g-2}\) in (2.2) mapping \(E_k C\) to \(E_{k'} C\).

More appropriately, one should view equation (2.3) as the definition of an IBL\(_\infty\)-structure, and the formal variables \(h, \tau\) are mere bookkeeping devices that allow us to write this equation in the more concise form (2.2). Alternatively, we could also consider equation (2.2) on the space \(\prod_{k \geq 1} E_k C \{h\}\).

(2) It is instructive to think of \(p_{k,\ell,g}\) as an operation associated to a compact connected oriented surface \(S_{k,\ell,g}\) of signature \((k,\ell,g)\), i.e. with \(k\) incoming and \(\ell\) outgoing boundary components and of genus \(g\). Then the coefficient \(k + \ell + 2g - 2\) of the formal variable \(\tau\) is the negative Euler characteristic of \(S_{k,\ell,g}\). It is also useful to think of the identity \(C \to C\) as the operation

![Figure 1](image-url)

Figure 1. On the left is a pictorial representation of \(p_{5,4,3}\) by a surface with signature \((5, 4, 3)\), with incoming boundaries drawn at the top and outgoing boundaries at the bottom. The whole picture would be our graphical representation of the extension \(\hat{p}_{5,4,3} : E_{5+2} C \to E_{4+2} C\).
associated to a trivial cylinder. Then the extension \( \hat{p}_{k,\ell,g} : E_{k+\ell,C} \to E_{\ell+g,C} \)
corresponds to the disjoint union of \( S_{k,\ell,g} \) with \( r \) trivial cylinders. Define
the genus \( g \in \mathbb{Z} \) of a possibly disconnected surface with \( k \) incoming and \( \ell \)
outgoing boundary components such that its Euler characteristic equals
\( 2 - 2g - k - \ell \), so e.g., adding a cylinder lowers the genus by one. Then the
terms of the form \( \hat{p}_{k_2,\ell_2,g_2} \circ \hat{p}_{k_1,\ell_1,g_1} \) on the left hand side of (2.3)
correspond to all possible gluings of two connected surfaces of signatures \((k_1,\ell_1,g_1)\) and
\((k_2,\ell_2,g_2)\), plus the appropriate number of trivial cylinders, to a possibly
disconnected surface of signature \((k,\ell,g)\).

The following definition will be repeatedly used in inductive arguments.

**Definition 2.5.** We define a linear order on signatures by saying \((k',\ell',g') \prec (k,\ell,g)\) if one of the following conditions holds:

(i) \( k' + \ell' + 2g' < k + \ell + 2g \),
(ii) \( k' + \ell' + 2g' = k + \ell + 2g \) and \( g' > g \), or
(iii) \( k' + \ell' + 2g' = k + \ell + 2g \) and \( g' = g \) and \( k' < k \).

This choice of ordering is explained in Remark 2.7 below. The sequence of
ordered signatures starts with
\( (1,1,0) \prec (1,2,0) \prec (2,1,0) \prec (1,1,1) \prec (1,3,0) \prec (2,2,0) \prec (3,1,0) \prec \ldots \)

(3) The preceding discussion suggests that (2.3) can be reformulated in terms
of gluing to connected surfaces. For this, let us denote by \( \hat{p}_{k_2,\ell_2,g_2} \circ_s \hat{p}_{k_1,\ell_1,g_1} \)
the part of the composition where exactly \( s \) of the inputs of \( \hat{p}_{k_2,\ell_2,g_2} \) are
outputs of \( \hat{p}_{k_1,\ell_1,g_1} \).

**Lemma 2.6.** Equation (2.2) is equivalent to the sequence of equations

\[
\sum_{s=1}^{g+1} \sum_{k_1+k_2=k+1 \atop \ell_1+\ell_2=\ell+1 \atop g_1+g_2=g+1-s} (\hat{p}_{k_2,\ell_2,g_2} \circ_s \hat{p}_{k_1,\ell_1,g_1})|_{E_{k,C}} = 0, \quad k, \ell \geq 1, g \geq 0.
\]

Moreover, for \((k,\ell,g) \succ (1,1,0)\) equation (2.4) has the form

\[
\hat{p}_{1,1,0} \circ p_{k,\ell,g} + p_{k,\ell,g} \circ \hat{p}_{1,1,0} + P_{k,\ell,g} = 0,
\]

where \( P_{k,\ell,g} : E_{k,C} \to E_{\ell,C} \) involves only compositions of terms \( p_{k',\ell',g'} \) whose
signatures satisfy \((1,1,0) \prec (k',\ell',g') \prec (k,\ell,g)\).

**Remark 2.7.** The order \( \prec \) on signatures was chosen so that (2.3) for all
signatures \((k,\ell,g) \preceq (K,L,G)\) is equivalent to (2.4) for the same range of
signatures. Other choices are possible.
Figure 2. A typical term in $\hat{p}_{5,4,3} \circ \hat{p}_{5,2,2} : E_7 C \to E_3 C$, appearing on the left hand side of equation (2.4) for $(k, \ell, g) = (7, 3, 7)$.

Proof. Abbreviate the left-hand side of (2.4) by

$$q_{k,\ell,g} := \sum_{s=1}^{g+1} \sum_{\substack{k_1+k_2=k+s \\ell_1+\ell_2=\ell+s \\g_1+g_2=g+1-s}} \hat{p}_{k_2,\ell_2,g_2} \circ \hat{p}_{k_1,\ell_1,g_1} \big|_{E_k C} : E_k C \to E_\ell C.$$  

Note that the terms of the form $\hat{p}_{k_2,\ell_2,g_2} \circ \hat{p}_{k_1,\ell_1,g_1}$ on the right hand side of this definition correspond to all possible gluings of two surfaces of signatures $(k_1, \ell_1, g_1)$ and $(k_2, \ell_2, g_2)$ along $s$ boundary loops (outgoing for the first one and incoming for the second one) to a connected surface of signature $(k, \ell, g)$. Denote by $\hat{q}_{k,\ell,g}$ the usual extension. Then

$$\sum_{r=0}^{\min(k,\ell)-1} \hat{q}_{k-r,\ell-r,g+r} = \sum_{r=0}^{\min(k,\ell)-1} \sum_{s=1}^{g+1+r} \sum_{\substack{k_1+k_2=k+s-r \\ell_1+\ell_2=\ell+s-r \\g_1+g_2=g+1-s+r}} \hat{p}_{k_2,\ell_2,g_2} \circ \hat{p}_{k_1,\ell_1,g_1} \big|_{E_k C}$$

$$= \sum_{t=2-\min(k,\ell)}^{g+1} \sum_{\substack{k_1+k_2=k+t \\ell_1+\ell_2=\ell+t \\g_1+g_2=g+1-t}} \hat{p}_{k_2,\ell_2,g_2} \circ \hat{p}_{k_1,\ell_1,g_1} \big|_{E_k C}.$$  

Here the last equality follows by setting $t = s - r$ and observing that

$$\hat{p}_{k_2,\ell_2,g_2} \circ \hat{p}_{k_1,\ell_1,g_1} = -\hat{p}_{k_1,\ell_1,g_1} \circ \hat{p}_{k_2,\ell_2,g_2}.$$  

So the additional terms corresponding to $s = 0$ (which were not present in the line above) appear in cancelling pairs. Since the last expression agrees with the one in (2.3), this shows that (2.2) is equivalent to the sequence of
equations

\[ \sum_{r=0}^{\min(k,\ell)-1} q_{k-r,\ell-r,g+r} = 0, \quad k, \ell \geq 1, g \geq 1 - \min(k, \ell), \]

where summands with \( g + r < 0 \) are interpreted as 0. Clearly (2.4) implies (2.6). The converse implication follows by induction over the order \( \prec \) because (2.6) is of the form

\[ q_{k,\ell,g} + Q_{k,\ell,g} = 0, \]

where \( Q_{k,\ell,g} \) is a sum of terms \( q_{k',\ell',g'} \) with \((k', \ell', g') \prec (k, \ell, g)\).

For the last statement, consider a term \( \hat{q}_{k,\ell,g} \) appearing in equation (2.4). Denote by \( \chi_{k,\ell,g} := 2 - 2g - k - \ell \) the Euler characteristic of a surface of signature \((k, \ell, g)\). Since the Euler characteristic is additive and only nonpositive Euler characteristics occur for the admissible triples, we get

\[ \chi_{k,\ell,g} = \chi_{k,\ell,g}, \]

with all terms being nonpositive. If \( \chi_{k_1,\ell_1,g_1} < 0 \) and \( \chi_{k_2,\ell_2,g_2} < 0 \), then \((k_1, \ell_1, g_1) \prec (k_2, \ell_2, g_2)\) by definition of the ordering. If \( \chi_{k_1,\ell_1,g_1} = 0 \), then \((k_1, \ell_1, g_1) = (1, 1, 0)\) and it follows that \((k_2, \ell_2, g_2) = (k, \ell, g)\), so we find the first term in equation (2.5). Similarly, in the case \( \chi_{k_2,\ell_2,g_2} = 0 \) we find the second term in equation (2.5). \( \square \)

Remark 2.8. Note that the proof of Lemma 2.6 only uses property (i) in Definition 2.5. Moreover, the proof still works if in Definition 2.3 we allow all triples \((k, \ell, g)\) except the following ones:

\[(2.7) \quad (0,0,0), (1,0,0), (0,1,0), (0,0,1), (2,0,0), (0,2,0).\]

We call such a structure a generalized IBL\(_{\infty}\)-structure. The notion below of an IBL\(_{\infty}\)-morphism can be generalized in the same way, and all the theory in Sections 2-6 works for these generalized structures with the exception of the following discussion in (4).

(4) IBL\(_{\infty}\)-algebra stands for involutive bi-Lie algebra up to infinite homotopy. To justify this terminology, let us spell out equation (2.4) for the first few triples \((k, \ell, g)\). For \((k, \ell, g) = (1,1,0)\) we find that

\[ \mu(a, b) := (-1)^{|a|} p_{2,1,0}(a, b) \]
defines a graded Lie bracket up to homotopy on \( C \). Similarly, the relations for \((k, \ell, g) = (1, 2, 0)\) and \((1, 3, 0)\) show that

\[
\delta(a) := (\iota \otimes 1)p_{1,2,0}(a)
\]

is a chain map which defines a graded Lie cobracket up to homotopy on \( C \). Here \( \iota \) multiplies a homogeneous element \( c \in C \) by \((-1)^{|c|}\).

Now the relation with \((k, \ell, g) = (1, 1, 1)\) reads

\[
p_{1,1,0} \circ p_{1,1,1} + p_{1,1,1} \circ p_{1,1,0} + p_{2,1,0} \circ p_{1,2,0} = 0
\]

and yields involutivity of \((\mu, \delta)\) up to homotopy. Finally, the relation with \((k, \ell, g) = (2, 2, 0)\) has the form

\[
p_{1,1,0} \circ p_{2,2,0} + p_{2,2,0} \circ p_{1,1,0} + p_{1,2,0} \circ p_{2,1,0} + p_{2,1,0} \circ p_{1,2,0} = 0,
\]

yielding compatibility of \( \mu \) and \( \delta \) up to homotopy. In summary, \((\mu, \delta)\) induces the structure of an involutive bi-Lie algebra on the homology \( H(C, \partial) \).

(5) Let us consider some special cases of an IBL\(_{\infty}\)-structure. If \( p_{k,\ell,g} = 0 \) whenever \( \ell \geq 2 \) or \( g > 0 \), then (2.4) is equivalent to the sequence of equations

\[
\sum_{k_1 + k_2 = k + 1} (p_{k_2,1,0} \circ p_{k_1,1,0})|_{E_kC} = 0, \quad k = 1, 2, \ldots
\]

So we recover one of the standard definitions of an \( L_{\infty}\)-algebra (cf. [51]). Similarly, if \( p_{k,\ell,g} = 0 \) whenever \( k \geq 2 \) or \( g > 0 \) we recover the definition of a co-\( L_{\infty}\) structure.

(6) Suppose the following finiteness condition is satisfied: Given \( k \geq 1 \), \( g \geq 0 \) and \( a \in E_kC \), the term \( p_{k,\ell,g}(a) \) is nonzero for only finitely many \( \ell \geq 1 \). Then we can set \( \tau = 1 \) above and consider \( p \) as a map of \( EC[h] \). This is the case in our main examples (exact symplectic field theory and string topology); see e.g. Section [7] below.

**Morphisms.** Next we turn to the definition of morphisms. So let \((C^+, \{p_{k,\ell,g}^+\})\) and \((C^-, \{p_{k,\ell,g}^-\})\) be two IBL\(_{\infty}\)-algebras of the same degree \( d \). To a collection of linear maps \( f_i : E_k C^+ \rightarrow E_k C^- \), \( 1 \leq i \leq r \), we associate a linear map \( f_1 \odot \cdots \odot f_r : E_{k_1+\cdots+k_r} C^+ \rightarrow E_{\ell_1+\cdots+\ell_r} C^- \) by

\[
f_1 \odot \cdots \odot f_r(c_1 \cdots c_k) := \sum_{\rho \in S_k} \varepsilon(\rho) f_1(c_{\rho(1)} \cdots c_{\rho(k_1)}) \cdots f_r(c_{\rho(k_1+\cdots+k_{r-1}+1)} \cdots c_{\rho(k)})
\]

(2.8)

\[
= \sum_{\rho \in S_k} \frac{\varepsilon(\rho)}{k_1! \cdots k_r!} f_1(c_{\rho(1)} \cdots c_{\rho(k_1)}) \cdots f_r(c_{\rho(k_1+\cdots+k_{r-1}+1)} \cdots c_{\rho(k)}).
\]

(2.9)
Note that \( f_1 \circ f_2 = (-1)^{|f_1||f_2|} f_2 \circ f_1 \) if the \( f_i \) are homogeneous of degree \(|f_i|\). Now we consider a series of graded module homomorphisms

\[
\hat{f}_{k,\ell,g} : E_k C^+ \to E_\ell C^-, \quad k, \ell \geq 1, \ g \geq 0
\]

of degree

\[
|\hat{f}_{k,\ell,g}| = -2d(k + g - 1).
\]

Define the operator

\[
f := \sum_{k,\ell=1}^{\infty} \sum_{g=0}^{\infty} \hat{f}_{k,\ell,g} h^{k+g-1} r^{k+\ell+2g-2} : EC^+\{h, \tau\} \to EC^-\{h, \tau\},
\]

where each \( \hat{f}_{k,\ell,g} \) is viewed as a map \( EC^+ \to E_\ell C^- \subset EC^- \) by setting it zero on \( E_m C^+ \) for \( m \neq k \). Furthermore, we introduce the exponential series \( e^f \) with respect to the symmetric product, i.e.

\[
e^f := \sum_{r=0}^{\infty} \sum_{k_i,\ell_i,g_i} \frac{1}{r!} \hat{f}_{k_1,\ell_1,g_1} \circ \cdots \circ \hat{f}_{k_r,\ell_r,g_r} h^{\sum k_i + \sum g_i - r - 1 + \sum \ell_i + 2 \sum g_i - 2r}.
\]

**Definition 2.9.** We say that \( \{\hat{f}_{k,\ell,g}\}_{k,\ell \geq 1, g \geq 0} \) is an IBL\(_\infty\)-morphism if

\[
e^f \hat{p}^+ - \hat{p}^- e^f = 0.
\]

Again, let us explain this definition from various angles.

1. Equation \((2.11)\) is equivalent to requiring that for each triple \((k, \ell, g)\) with \( k, \ell \geq 1 \) and \( g \geq 1 - \min(k, \ell) \) the equation

\[
\sum_{r=1}^{\ell} \sum_{k_1+\cdots+k_r=k+\ell+1} \sum_{\ell_1+\cdots+\ell_r=\ell} \frac{1}{r!} \hat{f}_{k_1,\ell_1,g_1} \circ \cdots \circ \hat{f}_{k_r,\ell_r,g_r} \circ \hat{p}^+_{k^+,\ell^+,g^+} = 0.
\]

2. As before, one thinks of \( \hat{f}_{k,\ell,g} \) as an operation associated to a compact connected oriented surface of signature \((k, \ell, g)\). Then the terms \((\hat{f}_{k_1,\ell_1,g_1} \circ \cdots \circ \hat{f}_{k_r,\ell_r,g_r})\hat{p}^+_{k^+,\ell^+,g^+}\) on the left hand side of \((2.12)\) correspond to complete gluings of \( r \) connected surfaces of signatures \((k_i, \ell_i, g_i)\) at their incoming loops to the outgoing loops of a surface of signature \((k^+, \ell^+, g^+)\), plus an
appropriate number of trivial cylinders, to a *possibly disconnected* surface of signature \((k, \ell, g)\).

(3) Again, it is useful to reformulate (2.11) in terms of gluing to *connected* surfaces. For this, let us denote by

\[
(\hat{f}_{k_1, \ell_1, g_1} \circ \cdots \circ \hat{f}_{k_r, \ell_r, g_r}) \circ_{s_1, \ldots, s_r} \hat{p}_{k+\ell+.g}^+
\]

the part of the composition where precisely \(s_i\) of the inputs of \(\hat{f}_{k_i, \ell_i, g_i}\) are outputs of \(p_{k+\ell+.g}^+\), and similarly for composition with \(\hat{p}_{k-\ell-.g}^-\).

**Lemma 2.10.** Equation (2.11) is equivalent to the sequence of equations

\[
\sum_{r=1}^{\ell} \frac{1}{r!} (\hat{f}_{k_1, \ell_1, g_1} \circ \cdots \circ \hat{f}_{k_r, \ell_r, g_r}) \circ_{s_1, \ldots, s_r} \hat{p}_{k+\ell+.g}^+ = 0,
\]

\[
(2.13)
\]

\[-\sum_{r=1}^{k} \frac{1}{r!} \hat{p}_{k-\ell-.g}^- \circ_{s_1, \ldots, s_r} (\hat{f}_{k_1, \ell_1, g_1} \circ \cdots \circ \hat{f}_{k_r, \ell_r, g_r}) = 0,
\]

for \(k, \ell \geq 1\) and \(g \geq 0\), where the left hand side is viewed as a map from \(E_kC^+\) to \(E_\ell C^-.\) Equation (2.13) for a fixed triple \((k, \ell, g)\) has the form

\[
\hat{f}_{k, \ell, g} \circ \hat{p}_{k+\ell+.g}^+ - \hat{p}_{1,1,0}^+ \circ \hat{f}_{k, \ell, g} + R_{k, \ell, g}(\hat{f}, p^+, p^-) = 0,
\]

where the expression \(R_{k, \ell, g}(\hat{f}, p^+, p^-)\) contains only components \(\hat{f}_{k', \ell', g'}\) of \(\hat{f}\) with \((k', \ell', g') \prec (k, \ell, g)\) and only components \(\hat{p}_{k', \ell', g'}^+\) of \(\hat{p}^+\) with \((1, 1, 0) \prec (k', \ell', g')\). Moreover, we have

\[
R_{k, \ell, g}(\hat{f}, p^+, p^-) = \frac{1}{e!} \hat{f}_{1,1,0}^\circ \hat{p}_{k, \ell, g}^+ - \frac{1}{k!} \hat{p}_{k, \ell, g}^- \circ \hat{f}_{1,1,0}^\circ + \hat{R}_{k, \ell, g}(\hat{f}, p^+, p^-),
\]

where the expression \(\hat{R}_{k, \ell, g}(\hat{f}, p^+, p^-)\) contains only components \(\hat{p}_{k', \ell', g'}^+\) with \((k', \ell', g') \prec (k, \ell, g)\).

Before giving the proof, let us introduce the following notation. For a map \(F : EC^+\{\tau, h\} \to EC^-\{\tau, h\}\), we will denote by \(\langle F \rangle_{k, \ell, g}\) the part of the coefficient of \(h^{k+g-1}\tau^h\ell^{+2g-2}\) which corresponds to a map from \(E_kC^+\) to \(E_\ell C^-\). Then we have for example

\[
\langle e^1 \rangle_{k, \ell, g} = \sum_{r=1}^{\text{min}(k, \ell)} \frac{1}{r!} \hat{f}_{k_1, \ell_1, g_1} \circ \cdots \circ \hat{f}_{k_r, \ell_r, g_r}.
\]

(2.14)
Note that this can be non-zero for \( g \geq 1 - \min(k, \ell) \), so in general negative \( g \) are allowed here.

**Proof.** We rewrite the term of the first sum in (2.12) for fixed \( r \geq 1 \) as

\[
\frac{1}{r!} \sum_{\ell_1 + \cdots + \ell_r + k = \ell \atop g_1 + \cdots + g_r + g^+ = g + r - \ell \atop s_1 + \cdots + s_r = \ell} \left( \hat{f}_{k, \ell, g_1} \circ \cdots \circ \hat{f}_{k, \ell, g_r} \right) \circ s_1, \ldots, s_r \circ \hat{p}^+_{k, \ell^+, g^+}
\]

\[
= \sum_{r'=1}^r \frac{1}{r'!(r - r')!} \sum_{\ell' \atop k_1 + \cdots + k_{r'} + k = k' \atop g_1 + \cdots + g_{r'} + g^+ = g' + r' - \ell'} \left( \hat{f}_{k, \ell, g_1} \circ \cdots \circ \hat{f}_{k, \ell, g_{r'}} \right) \circ s_1, \ldots, s_{r'} \circ \hat{p}^+_{k, \ell^+, g^+}
\]

Here we have used the commutation properties of the product \( \circ \) as well as the identity

\[
\left( \hat{f}_{k, \ell, g_1} \circ \cdots \circ \hat{f}_{k, \ell, g_r} \right) \circ s_1, \ldots, s_{r-1}, 0 \circ \hat{p}^+_{k, \ell^+, g^+}
\]

\[
= \left[ \left( \hat{f}_{k, \ell, g_1} \circ \cdots \circ \hat{f}_{k, \ell, g_{r-1}} \right) \circ s_1, \ldots, s_{r-1} \circ \hat{p}^+_{k, \ell^+, g^+} \right] \circ \hat{f}_{k, \ell, g_r}.
\]

A similar discussion applies to the second sum. Note that the left-hand side of this equation corresponds to all gluings to possibly disconnected surfaces of signature \((k, \ell, g)\), while the terms \( (\hat{f}_{k, \ell, g_1} \circ \cdots \circ \hat{f}_{k, \ell, g_r}) \circ s_1, \ldots, s_r, \hat{p}^+_{k, \ell^+, g^+} \) on the right-hand side correspond to connected surfaces.

Abbreviate the left-hand side of (2.12) by \( G_{k, \ell, g} \) and the left-hand side of (2.13) by \( H_{k, \ell, g} \). Then we have the relation

\[
G_{k, \ell, g} = H_{k, \ell, g} + \sum_{k', \ell', g', k'' = k \atop g' + g'' = g + 1} H_{k', \ell', g'} \circ (\ell')_{k'', \ell', g''}.
\]

So clearly the sequence of equations (2.13) implies \( G_{k, \ell, g} = 0 \). The converse implication follows by induction over the order \( \prec \) because all terms in the sum on the right-hand side of the above equation involve \((k', \ell', g') \prec (k, \ell, g)\).

The last statement of the lemma follows as in the proof of Lemma 2.6. \( \square \)

(4) Let us spell out equation (2.12) for the first few triples \((k, \ell, g)\). For \((k, \ell, g) = (1, 1, 0)\) we find that

\[
\hat{f}_{1,1,0} : (C^+, \hat{p}^+_{1,1,0}) \to (C^-, \hat{p}^-_{1,1,0})
\]
is a chain map. The relations for \((k, \ell, g) = (2, 1, 0)\) resp. \((1, 2, 0)\) show that \(f_{1,1,0}\) intertwines the products \(\mu^\pm\) resp. the coproducts \(\delta^\pm\) up to homotopy. In particular, \(f_{1,1,0}\) induces a morphism of involutive Lie bialgebras on homology.

(5) In the special cases that \(p_{k,\ell,g} = 0\) and \(f_{k,\ell,g} = 0\) whenever \(\ell \geq 2\) (resp. \(k \geq 2\)) or \(g > 0\) we recover the definitions of \(L_\infty\) (resp. co-\(L_\infty\)) morphisms.

(6) As before with \(\hat{\mathbb{F}}\), by setting \(\tau = 1\) the operators \(\hat{f}\) and \(\hat{e}^f\) define maps \(EC^+\{h\} \to EC^-\{h\}\) if the following finiteness condition is satisfied: Given \(k \geq 1, g \geq 0\) and \(a \in E_kC^+\), the term \(f_{k,\ell,g}(a)\) is nonzero for only finitely many \(\ell \geq 1\). Again, this condition holds in our main examples (exact symplectic field theory and string topology).

(7) We say that an \(IBL_\infty\)-morphism \(\{f_{k,\ell,g}\}\) is linear if \(f_{k,\ell,g} = 0\) unless \((k, \ell, g) = (1, 1, 0)\). In this case its exponential is given by

\[
e^f = \sum_{r=0}^{\infty} \frac{1}{r!} e^{\hat{f}}_{1,1,0} : c_1 \cdots c_r \mapsto f_{1,1,0}(c_1) \cdots f_{1,1,0}(c_r)
\]

and equation (2.13) simplifies to

\[
\frac{1}{\ell!} e^{\hat{f}}_{1,1,0} \circ p_{k,\ell,g}^+ = p_{k,\ell,g}^- \circ \frac{1}{k!} e^{\hat{f}}_{1,1,0}.
\]

**Composition of morphisms.** Consider now two \(IBL_\infty\)-morphisms

\[
\hat{f}^+ = \{f^+_{k,\ell,g}\} : (C^+, \{p_{k,\ell,g}^+\}) \to (C^+, \{p_{k,\ell,g}\}),
\]

\[
\hat{f}^- = \{f^-_{k,\ell,g}\} : (C^+, \{p_{k,\ell,g}\}) \to (C^-, \{p_{k,\ell,g}^-\}).
\]

**Definition 2.11.** The composition \(\hat{f} = \hat{f}^- \circ \hat{f}^+\) of \(\hat{f}^+\) and \(\hat{f}^-\) is the unique \(IBL_\infty\)-morphism \(\hat{f} = \{f_{k,\ell,g}\} : (C^+, \{p_{k,\ell,g}^+\}) \to (C^-, \{p_{k,\ell,g}^-\})\) satisfying

\[
e^f = e^{\hat{f}^-} e^{\hat{f}^+}.
\]

To see existence and uniqueness of \(\hat{f}\), consider the signature \((k, \ell, g)\) part of \(e^{\hat{f}^-} e^{\hat{f}^+}\),

\[
\langle e^{\hat{f}^-} e^{\hat{f}^+} \rangle_{k,\ell,g} = \sum_{r^+_1=1}^{k} \sum_{r^-_1=1}^{r^+_1} \sum_{r^+_2=1}^{\ell} \sum_{r^-_2=1}^{r^+_2} \sum_{s=\max(r^+, r^-)}^{r^+_1+r^-_1+g-1} \sum_{r^+_3=1}^{\ell} \sum_{r^-_3=1}^{r^+_3} \sum_{s=\max(r^+, r^-)}^{r^+_2+r^-_2+g-1}
\]

\[
\frac{1}{r^+_1! r^+_2! r^-_1! r^-_2!} \cdot (f^-_{k_1,\ell_1, g_1}^+ \circ \cdots \circ f^-_{k_{r^-_1}, \ell_{r^-_1}, g_{r^-_1}}^+) \circ (f^+_{k_1,\ell_1, g_1} \circ \cdots \circ f^+_{k_{r^+_1}, \ell_{r^+_1}, g_{r^+_1}}),
\]

where \(r^+_1 + \cdots + r^+_2 = k\), \(r^-_1 + \cdots + r^-_2 = \ell\), \(r^+_2 + \cdots + r^+_3 = k\), \(r^-_2 + \cdots + r^-_3 = \ell\), \(g_1^+ + \cdots + g_{r^+_2}^+ + g_1^- + \cdots + g_{r^-_2}^- = s\), and \(r^+_3 + \cdots + r^+_4 = \ell\), \(r^-_3 + \cdots + r^-_4 = \ell\).
Note also that equation (2.14) has the form
\[
\langle e^f \rangle_{k,\ell,g} = \hat{f}_{k,\ell,g} + \sum_{r=2}^{\min\{k,\ell\}} \sum_{\substack{k_1 + \ldots + k_r = k \\ \ell_1 + \ldots + \ell_r = \ell \\ g_1 + \ldots + g_r = g + r - 1}} \frac{1}{r!} (\hat{f}_{k_1,\ell_1,g_1} \circ \cdots \circ \hat{f}_{k_r,\ell_r,g_r}),
\]
where the conditions in the last summand add up to
\[
\sum_{i=1}^{r} (k_i + \ell_i + 2g_i - 2) = k + \ell + 2g - 2.
\]
As \( r \geq 2 \), this easily implies \((k_i, \ell_i, g_i) \prec (k, \ell, g)\) for all \(i\), so we can inductively solve the equation \( \langle e^f \rangle_{k,\ell,g} = \langle e^f e^f \rangle_{k,\ell,g} \) for \( \hat{f}_{k,\ell,g} \) to find
\[
(2.16) \quad \hat{f}_{k,\ell,g} = \langle e^f e^f \rangle_{k,\ell,g} - \sum_{r=2}^{\min\{k,\ell\}} \sum_{\substack{k_1 + \ldots + k_r = k \\ \ell_1 + \ldots + \ell_r = \ell \\ g_1 + \ldots + g_r = g + r - 1}} \frac{1}{r!} (\hat{f}_{k_1,\ell_1,g_1} \circ \cdots \circ \hat{f}_{k_r,\ell_r,g_r}).
\]

Here are some explanations to this definition.

1. Recall that we think of \( \hat{f}_{k,\ell,g} \) as associated to connected Riemann surfaces with signature \((k, \ell, g)\). The first term on the right hand side of equation (2.16) describes all possible ways to obtain a (possibly disconnected) Riemann surface of Euler characteristic \(2 - 2g - k - \ell\) as a complete gluing of pieces corresponding to the \((k_i^+, \ell_i^+, g_i^+)\). The second term then subtracts all disconnected configurations.

2. For \((k, \ell, g) = (1, 1, 0)\) equation (2.16) shows that \( f_{1,1,0} \) and the \( f_{1,1,0}^\pm \) are related by
\[
\hat{f}_{1,1,0} = f_{1,1,0}^+ \circ f_{1,1,0}^-.
\]

3. In the special cases that \( p_{k,\ell,g} = 0 \) and \( \hat{f}_{k,\ell,g} = 0 \) whenever \( \ell \geq 2 \) (resp. \( k \geq 2 \)) or \( g > 0 \) we recover the definitions of composition of \( L_\infty \) (resp. \( \sigma L_\infty \)) morphisms.

4. If \( f^+ \) is linear, then (2.16) simplifies to
\[
\hat{f}_{k,\ell,g} = \frac{1}{k!} f_{k,\ell,g} \circ (f_{1,1,0}^+)^\otimes k.
\]
Indeed, if we define \( \hat{f}_{k,\ell,g} \) by this equation we find
\[
\frac{1}{k!} (f_{k_1,\ell_1,g_1} \circ \cdots \circ f_{k_r,\ell_r,g_r}) \circ (f_{1,1,0}^+)^\otimes k = \hat{f}_{k_1,\ell_1,g_1} \circ \cdots \circ \hat{f}_{k_r,\ell_r,g_r}
\]
and hence
\[
\langle e^f e^f \rangle_{k,\ell,g} = \sum_{r=1}^{\ell} \sum_{\substack{k_1 + \ldots + k_r = k \\ \ell_1 + \ldots + \ell_r = \ell \\ g_1 + \ldots + g_r = r + g - 1}} \frac{1}{r!k!} (f_{k_1,\ell_1,g_1} \circ \cdots \circ f_{k_r,\ell_r,g_r}) \circ (f_{1,1,0}^+)^\otimes k
\]
\[
= \langle e^f \rangle_{k,\ell,g}.
\]
A similar discussion applies if $f^-$ is linear. In particular, if both $f^+$ are linear then so is their composition.

Finally, we again record a useful observation for later use.

**Lemma 2.12.** For $(1, 1, 0) \prec (k, \ell, g)$, the component $f_{k, \ell, g}$ of the composition $f$ of two $\IBL_\infty$-morphisms $f^+$ and $f^-$ has the form

$$f_{k, \ell, g} = f_{k, \ell, g}^+ \circ \frac{1}{k!}(f_{1,1,0}^-)^{\otimes k} + \frac{1}{\ell!}(f_{1,1,0}^+)^{\otimes \ell} \circ f_{k, \ell, g}^- + C_{k, \ell, g}(f^-, f^+),$$

where $C_{k, \ell, g}(f^-, f^+)$ contains only components $f_{k', \ell', g'}^\pm$ with $(k', \ell', g') \prec (k, \ell, g)$. Moreover, if either $f^-$ or $f^+$ is linear, then $C_{k, \ell, g}(f^-, f^+) = 0$. □

**Proof.** The conditions in the first sum in (2.16) add up to

$$\sum_{i=1}^{r^+} (k_i^+ + \ell_i^+ + 2g_i^+ - 2) + \sum_{i=1}^{r^-} (k_i^- + \ell_i^- + 2g_i^- - 2) = k + \ell + 2g - 2,$$

from which the first statement easily follows. The last statement follows from (4) above. □

**Remark 2.13.** We leave it to the reader to check that composition of morphisms is associative.

### 3. Obstructions

In this section we prove the main technical proposition underlying the homotopy theory of $\IBL_\infty$-algebras (Proposition 3.11). All the results in the following three sections will be formal consequences of this proposition.

Given free chain complexes $(C, d^C)$ and $(D, d^D)$ over $R$, define a boundary operator $\delta$ on $\Hom(E_k C, E_\ell D)$ by

$$\delta \varphi = d^D \varphi + (-1)^{|\varphi|+1} \varphi d^C,$$

where $\hat{d}^C$ and $\hat{d}^D$ are the usual extensions of $d^C$ and $d^D$. This operation is a derivation of composition, in the sense that for $\varphi \in \Hom(E_n B, E_k C)$ and $\psi \in \Hom(E_k C, E_\ell D)$ we have

$$\delta (\psi \circ \varphi) = (\delta \psi) \circ \varphi + (-1)^{|\psi|} \psi \circ (\delta \varphi).$$

Below, we always consider the case that $B$, $C$ and $D$ are (partial) $\IBL_\infty$-algebras and the boundary operators used in the definition of $\delta$ are the corresponding structure maps $p_{1,1,0}$. The following proposition identifies the obstructions to inductive extensions of partially defined $\IBL_\infty$-structures and their morphisms.
Proposition 3.1. (1) Let \( \{p_{k,l,g} : E_k C \to E_l C\}_{(k,l,g) \prec (K,L,G)} \) be a collection of maps that satisfy the defining relation (2.4) for IBL\_\infty\_structures for all \((k,l,g) \prec (K,L,G)\). Then the expression \( P_{K,L,G} \in \text{Hom}(E_K C, E_L C) \) implicitly defined in Lemma 2.10 satisfies
\[
\delta P_{K,L,G} = 0.
\]

(2) Let \((C, \{p_{k,l,g}^C\})\) and \((D, \{p_{k,l,g}^D\})\) be IBL\_\infty\_algebras, and suppose the collection of maps \( \{p_{k,l,g} : E_k C \to E_l D\}_{(k,l,g) \prec (K,L,G)} \) satisfies the defining relation (2.12) for IBL\_\infty\_morphisms for all \((k,l,g) \prec (K,L,G)\). Then the expression \( R_{K,L,G}(f,p^C,p^D) \in \text{Hom}(E_K C, E_L D) \) implicitly defined in Lemma 2.10 satisfies
\[
\delta R_{K,L,G}(f,p^C,p^D) = 0.
\]

(3) Assume further that \((B, \{p_{k,l,g}^B\})\) is another IBL\_\infty\_algebra and the collection \( \mathfrak{g} = \{g_{k,l,g} : E_k B \to E_l C\}_{(k,l,g) \prec (K,L,G)} \) also satisfies the defining relation (2.12) for morphisms for all \((k,l,g) \prec (K,L,G)\). Then
\[
R_{K,L,G}(f \circ g, p^B, p^D) = \frac{1}{L!} \hat{\delta}_{1,1,0} \circ R_{K,L,G}(g, p^B, p^C) + R_{K,L,G}(f, p^C, p^D) \circ \frac{1}{L!} \hat{\delta}_{1,1,0} + \delta C_{K,L,G}(f, g),
\]

where \( C_{K,L,G}(f, g) \in \text{Hom}(E_K B, E_L D) \) is the expression implicitly defined in Lemma 2.12.

For the proof we will need some more notation. For three linear maps \( p_i : E_k C \to E_l C \) and integers \( s_{12}, s_{13}, s_{23} \geq 0 \) we denote by
\[
\hat{p}_3 \circ_{s_{23}, s_{13}} (\hat{p}_2 \circ_{s_{12}} \hat{p}_1) = (\hat{p}_3 \circ_{s_{23}} \hat{p}_2) \circ_{s_{13}, s_{12}} \hat{p}_1
\]
the part of the composition \( \hat{p}_3 \circ \hat{p}_2 \circ \hat{p}_1 \) where exactly \( s_{ij} \) of the inputs of \( p_j \) are outputs of \( p_i \). The following properties follow immediately from the definition:
\[
\hat{p}_3 \circ_{s} (\hat{p}_2 \circ_{s_{12}} \hat{p}_1) = \sum_{s_{13} + s_{21} = s} \hat{p}_3 \circ_{s_{23}, s_{13}} (\hat{p}_2 \circ_{s_{12}} \hat{p}_1),
\]
\[
(\hat{p}_3 \circ_{s_{23}} \hat{p}_2) \circ_{s} \hat{p}_1 = \sum_{s_{12} + s_{13} = s} (\hat{p}_3 \circ_{s_{23}} \hat{p}_2) \circ_{s_{13}, s_{12}} \hat{p}_1.
\]

Note in particular that, since \( p_i \circ_0 p_j = (-1)^{|p_i||p_j|} p_j \circ_0 p_i \), we have
\[
\hat{p}_3 \circ_{0, s_{13}} (\hat{p}_2 \circ_{s_{12}} \hat{p}_1) = (\hat{p}_3 \circ_0 \hat{p}_2) \circ_{s_{13}, s_{12}} \hat{p}_1
\]
\[
= (-1)^{|p_2||p_3|} (\hat{p}_2 \circ_0 \hat{p}_3) \circ_{s_{13}, s_{12}} \hat{p}_1
\]
\[
= (-1)^{|p_2||p_3|} \hat{p}_2 \circ_{0, s_{12}} (\hat{p}_3 \circ_{s_{13}} \hat{p}_1),
\]
and similarly
\[
(\hat{p}_3 \circ_{s_{23}} \hat{p}_2) \circ_{s_{13}, 0} \hat{p}_1 = (-1)^{|p_1||p_2|} (\hat{p}_3 \circ_{s_{13}} \hat{p}_1) \circ_{s_{23}, 0} \hat{p}_2.
\]
Applying these properties in the case where one of the $p_i$ is the boundary operator $p_{1,1,0}$, we obtain

$$\delta(\hat{p}_2 \circ s \hat{p}_1) = (\delta\hat{p}_2) \circ s \hat{p}_1 + (-1)^{|p_2|} \hat{p}_2 \circ s (\delta\hat{p}_1).$$

**Proof of Proposition 3.1.** For the proof of (1), recall from Lemma 2.6 that $P_{k,\ell,g}$ is defined by

$$P_{k,\ell,g} := \sum_{s=1}^{g+1} \sum_{\substack{k_1 + k_2 = K + s \\
\ell_1 + \ell_2 = \ell + s \\
g_1 + g_2 = G + 1 - s \\
(k_1, \ell_1, g_1) \neq (1,1,0)}} \left(\hat{p}_{k_2, \ell_2, g_2} \circ s \hat{p}_{k_1, \ell_1, g_1}\right) |_{E_kC}$$

Moreover, in our current notation equation (2.4) can be written as

$$\delta p_{k,\ell,g} = -P_{k,\ell,g}.$$ 

Since all terms in the definition of $P_{K,L,G}$ satisfy $(k_i, \ell_i, g_i) \prec (K, L, G)$, using the hypothesis we find

$$\delta P_{K,L,G} = \sum_{s=1}^{G+1} \sum_{\substack{k_1 + k_2 = K + s \\
\ell_1 + \ell_2 = \ell + s \\
g_1 + g_2 = G + 1 - s \\
(k_1, \ell_1, g_1) \neq (1,1,0)}} \left(\hat{p}_{k_2, \ell_2, g_2} \circ s \hat{p}_{k_1, \ell_1, g_1}\right) |_{E_kC}$$

Geometrically, the terms $A$ and $B$ both correspond to breaking up a connected Riemann surface of signature $(k, \ell, g)$ in all possible ways into three non-trivial pieces. To show algebraically that indeed $A = B$, in the sum $B$ we rename $(k_2, \ell_2, g_2)$ to $(k_3, \ell_3, g_3)$, $(k_1, \ell_1, g_1)$ to $(k, \ell, g)$, and we insert the
definition of $P_{k,\ell,g}$ to rewrite it as

$$B = \sum_{s=1}^{G+1} \sum_{k+k_1+L+k_1=k±1} \sum_{g+g_1=1} \sum_{t=1}^{g+1} \sum_{k_1+k_2+k_3=k±2} \sum_{g_1+g_2=g±1} \sum_{t=1}^{g+1} \hat{p}_{k_3,\ell_3,g_3} \circ_p \hat{p}_{k_2,\ell_2,g_2} \circ_t \hat{p}_{k_1,\ell_1,g_1} |E_k C$$

$$= \sum_{s,t \geq 1} \sum_{s+t \leq G} \hat{p}_{k_3,\ell_3,g_3} \circ_s \hat{p}_{k_2,\ell_2,g_2} \circ_t \hat{p}_{k_1,\ell_1,g_1} |E_k C$$

We rewrite the term in $B$ for fixed $s, t$ as

$$\sum_{s_{13}+s_{23}=s} \sum_{k_1+k_2+k_3=k±1} \sum_{\ell_1+\ell_2+\ell_3=\ell±1} \sum_{s_{13}+s_{23}=s} \hat{p}_{k_3,\ell_3,g_3} \circ_s \hat{p}_{k_2,\ell_2,g_2} \circ_t \hat{p}_{k_1,\ell_1,g_1}.$$  

By the above properties of triple compositions the terms with $s_{23} = 0$ cancel in pairs, and renaming $t = s_{12}$ we obtain

$$B = \sum_{s_{12}+s_{23} \geq 1, s_{13} \geq 0} \sum_{u=s_{12}+s_{13}+s_{23} \leq G+2} \hat{p}_{k_3,\ell_3,g_3} \circ_{s_{23}, s_{13}} \hat{p}_{k_2,\ell_2,g_2} \circ_{s_{12}} \hat{p}_{k_1,\ell_1,g_1}.$$  

By similar discussion and the above associativity of triple decompositions we see that this expression agrees with $A$, which proves $\delta P_{K,L,G} = 0$.

To prove (2), recall that $R_{K,L,G} = R_{K,L,G}(f, p^C, p^D)$ has the general form

$$R_{K,L,G} = \sum \frac{1}{n!} (\hat{f}_{k_1,\ell_1,g_1} \circ \cdots \circ \hat{f}_{k_r,\ell_r,g_r}) \circ_{s_1, \ldots, s_r} \hat{p}_{k^+, \ell^+, g^+}$$

$$- \sum \frac{1}{n!} \hat{p}_{k^-, \ell^-, g^-} \circ_{s_1, \ldots, s_r} (\hat{f}_{k_1,\ell_1,g_1} \circ \cdots \circ \hat{f}_{k_r,\ell_r,g_r}),$$

where all the terms satisfy $s_i \geq 1$, $(k^±, \ell^±, g^±) \neq (1, 1, 0)$ and $(k_1, \ell_i, g_i) \prec (K, L, G)$. Hence, using the properties of triple compositions, the hypothesis $\delta p_{k,\ell,g} + P_{k,\ell,g} = 0$ for all $(k, \ell, g)$, and the induction hypothesis $R_{k,\ell,g} = \delta f_{k,\ell,g}$.
for \((k, \ell, g) < (K, L, G)\), we find that
\[
\delta R_{K,L,G} = \sum \frac{1}{(r-1)!} (\delta f_{k_1,\ell_1,g_1} \circ \cdots \circ f_{k_r,\ell_r,g_r}) \circ_{s_1,\ldots,s_r} \hat{P}_{K+\ell+g}^+
\]
\[
+ \sum \frac{1}{r!} (f_{k_1,\ell_1,g_1} \circ \cdots \circ f_{k_r,\ell_r,g_r}) \circ_{s_1,\ldots,s_r} \delta \hat{P}_{K+\ell+g}^+-
\]
\[
- \sum \frac{1}{r!} \hat{P}_{K-\ell-g}^- \circ_{s_1,\ldots,s_r} (f_{k_1,\ell_1,g_1} \circ \cdots \circ f_{k_r,\ell_r,g_r})
\]
\[
+ \sum \frac{1}{(r-1)!} (R_{k_1,\ell_1,g_1} \circ f_{k_2,\ell_2,g_2} \circ \cdots \circ f_{k_r,\ell_r,g_r}) \circ_{s_1,\ldots,s_r} \hat{P}_{K+\ell+g}^+
\]
\[
- \sum \frac{1}{r!} (f_{k_1,\ell_1,g_1} \circ \cdots \circ f_{k_r,\ell_r,g_r}) \circ_{s_1,\ldots,s_r} P_{K+\ell+g}^+
\]
\[
+ \sum \frac{1}{r!} P_{K-\ell-g}^- \circ_{s_1,\ldots,s_r} (f_{k_1,\ell_1,g_1} \circ \cdots \circ f_{k_r,\ell_r,g_r})
\]
\[
= \sum \frac{1}{(r-1)!} (R_{k_1,\ell_1,g_1} \circ f_{k_2,\ell_2,g_2} \circ \cdots \circ f_{k_r,\ell_r,g_r}) \circ_{s_1,\ldots,s_r} \hat{P}_{K+\ell+g}^+
\]
Now observe that the first summand contains three kinds of terms:

(i) terms of the type
\[
(f_{k_1,\ell_1,g_1} \circ \cdots \circ f_{k_r,\ell_r,g_r}) \circ_{s_1,\ldots,s_r} (\hat{P}_{K+\ell+g}^+ \circ_{s'} \hat{P}_{K'+\ell'+g'}^+)
\]
with \(s > 0\) cancelling with the second sum;

(ii) terms of the type
\[
(f_{k_1,\ell_1,g_1} \circ \cdots \circ f_{k_r,\ell_r,g_r}) \circ_{s_1,\ldots,s_r} (\hat{P}_{K+\ell+g}^+ \circ_{s'} \hat{P}_{K'+\ell'+g'}^+)
\]
which appear in cancelling pairs;

(iii) terms of the type
\[
-\hat{P}_{K-\ell-g}^- \circ (f_{k_1,\ell_1,g_1} \circ \cdots \circ f_{k_r,\ell_r,g_r}) \circ_{s_1,\ldots,s_r} \hat{P}_{K+\ell+g}^+
\]
which appear with opposite sign in the forth sum.

A similar discussion applies to the forth sum, and so in total we find that
\(\delta R_{K,L,G} = 0\) as claimed.

To prove (3), recall that
\[
C_{K,L,G}(f, g) = (f \circ g)_{K,L,G} - (\frac{1}{L!} \delta_{1,1,0}^L) \circ g_{K,L,G} - f_{K,L,G} \circ (\frac{1}{K!} \delta_{1,1,0}^K)
\]
\[
= \left\langle \sum \frac{1}{r!r'} (f_{k_1,\ell_1,g_1} \circ \cdots \circ f_{k_r,\ell_r,g_r}) \circ_{\text{conn}}
\]
\[
\left( g_{k_1',\ell_1',g_1'} \circ \cdots \circ g_{k_r',\ell_r',g_r'} \right) \right\rangle_{K,L,G},
\]
where \( \circ_{\text{conn}} \) signifies that we only keep those compositions which in the geometric picture correspond to a connected end result of gluing surfaces.\(^1\) Note also that in each factor \( \mathbf{f}_{k_1, \ell_1, g_1} \circ \cdots \circ \mathbf{f}_{k_r, \ell_r, g_r} \) (respectively \( \mathbf{g}_{k'_1, \ell'_1, g'_1} \circ \cdots \circ \mathbf{g}_{k'_r, \ell'_r, g'_r} \)) at least one of the signatures is different from \((1, 1, 0)\).

Now since \( \delta \) is a derivation of composition, and using the hypothesis that \( \delta \mathbf{f}_{k, \ell, g} = R_{k, \ell, g}(\mathbf{f}) \) and \( \delta \mathbf{g}_{k, \ell, g} = R_{k, \ell, g}(\mathbf{g}) \) for all \((k, \ell, g) \prec (K, L, G)\), we find that

\[
\delta C_{K, L, G} = \left\langle \frac{1}{(r-1)!} \sum_{(k_1, \ell_1, g_1) \prec (K, L, G)} \right. \\
\left. \frac{1}{r!(r'-1)!} \sum_{(k', \ell', g') \prec (K, L, G)} \right. \\
\left( R_{k_1, \ell_1, g_1}(\mathbf{f}) \circ \cdots \circ \mathbf{f}_{k_r, \ell_r, g_r} \right) \circ_{\text{conn}} (\mathbf{g}_{k'_1, \ell'_1, g'_1} \circ \cdots \circ \mathbf{g}_{k'_r, \ell'_r, g'_r}) \\
+ \left( \mathbf{f}_{k_1, \ell_1, g_1} \circ \cdots \circ \mathbf{f}_{k_r, \ell_r, g_r} \right) \circ_{\text{conn}} \left( R_{k'_1, \ell'_1, g'_1}(\mathbf{g}) \circ \cdots \circ \mathbf{g}_{k'_r, \ell'_r, g'_r} \right) \right\rangle_{K, L, G}
\]

\[= A + B.\]

Next substitute the corresponding expressions for the \( R_{k, \ell, g} \) and observe that most of the components in \( A \) involving \( \hat{p}^{C} \) have a corresponding term with opposite sign in \( B \). The only ones remaining are of the form

\[
\left( \frac{1}{L!} \mathbf{f}_{1,1,0}^{\circ L} \circ \hat{p}^{C}_{1, \ell', g'} \circ_{s_1, \ldots, s_r} (\mathbf{g}_{k'_1, \ell'_1, g'_1} \circ \cdots \circ \mathbf{g}_{k'_r, \ell'_r, g'_r}) \right)
\]

with all \( s_i > 0 \). They appear in the summands of the form

\[
\frac{1}{(r-1)!r'} R_{k_1, \ell_1, g_1}(\mathbf{f}) \circ (\mathbf{f}_{1,1,0})^{\circ r-1} \circ_{\text{conn}} (\mathbf{g}_{k'_1, \ell'_1, g'_1} \circ \cdots \circ \mathbf{g}_{k'_r, \ell'_r, g'_r})
\]

where \((k_1, \ell_1, g_1)\) is the only signature different from \((1, 1, 0)\) in the first factor. In the claimed expression for \( R_{K, L, G}(\mathbf{f} \circ \mathbf{g}) \), these terms precisely cancel the terms in \( \frac{1}{L!} \mathbf{f}_{1,1,0}^{\circ L} \circ R_{K, L, G}(\mathbf{f}) \) involving \( \hat{p}^{C} \).

Similarly, the only terms from \( B \) involving \( \hat{p}^{C} \) that remain after cancellation with corresponding terms in \( A \) are those of the form

\[
(\mathbf{f}_{k_1, \ell_1, g_1} \circ \cdots \circ \mathbf{g}_{k_r, \ell_r, g_r}) \circ_{s_1, \ldots, s_r} \hat{p}^{C}_{K, \ell', g'} \circ (\frac{1}{K!} \mathbf{g}_{1,1,0}^{\circ K})
\]

with all \( s_i > 0 \), and they precisely cancel the contributions from \( R_{K, L, G}(\mathbf{f}) \circ \frac{1}{K!} \mathbf{g}_{1,1,0}^{\circ K} \) involving \( \hat{p}^{C} \).

Finally note that all the terms in \( A \) involving \( \hat{p}^{D} \) and all the terms in \( B \) involving \( \hat{p}^{B} \) also appear in \( R_{K, L, G}(\mathbf{f} \circ \mathbf{g}) \). Moreover, the missing pieces

\(^1\)This process is parallel to the relation between a prop and a properad as in [70, Proposition 2.1].
are again precisely those which are supplied by the corresponding terms in $R_{K,L,G}(f)\circ \frac{1}{K} g^{\otimes K}$ involving $p^D$ and the terms in $f^{\otimes L}_{1,1,0} \circ R_{K,L,G}(f)$ involving $p^B$. □

Besides these assertions, we will repeatedly make use of the following well-known observations.

**Lemma 3.2.** Let $e : (B, \partial_B) \to (C, \partial_C)$ be a homotopy equivalence of chain complexes which has a chain homotopy inverse $i : (C, \partial_C) \to (B, \partial_B)$ such that $ei = id_c$.

1. If $f : (A, \partial_A) \to (B, \partial_B)$ is a map of degree $D$ satisfying $\partial_B f - (-1)^D f \partial_A = 0$ and $ef = 0$, then there exists a map $H : (A, \partial_A) \to (B, \partial_B)$ of degree $D + 1$ such that $f + \partial_B H + (-1)^D H \partial_A = 0$ and $eH = 0$.

2. If $g : (B, \partial_B) \to (A, \partial_A)$ is a map of degree $D$ satisfying $\partial_A g - (-1)^D g \partial_B = 0$ and $gi = 0$, then there exists a map $H : (B, \partial_B) \to (A, \partial_A)$ of degree $D + 1$ such that $g + \partial_A H + (-1)^D H \partial_B = 0$ and $Hi = 0$.

**Proof.** Choose any homotopy $h$ satisfying

$$\partial_B h + h \partial_B = ie - id_B$$

and set $h' = (id_B - ie)h(id_B - ie)$. Then, since $ei = id_C$, one straightforwardly checks that $h'$ is also a homotopy between $ie$ and $id_B$, satisfying in addition that $eh' = h'i = 0$. Now for assertion (1), define $H := h'f$ and compute that

$$\partial_B H + (-1)^D H \partial_A + f = (\partial_B h' + h' \partial_B) f + f = (ief - f) + f = 0$$

and $eH = eh'f = 0$.

Similarly, for assertion (2), define $H := gh'$ and compute that

$$\partial_A H + (-1)^D H \partial_B + g = g(\partial_B h' + h' \partial_B) + g = gie - g + g = 0$$

and $Hi = gh'i = 0$ as required. □

4. Homotopy of morphisms

In this section we define homotopies between IBL$_\infty$-morphisms and prove some elementary properties of this relation. We follow the approaches of [66] and [34].

**Definition 4.1.** Let $(C, \{p_{k,\ell,g}\})$ and $(\mathfrak{C}, \{q_{k,\ell,g}\})$ be IBL$_\infty$-algebras. We say that $(\mathfrak{C}, \{q_{k,\ell,g}\})$ together with IBL$_\infty$-morphisms $\iota : C \to \mathfrak{C}$ and $\varepsilon_0, \varepsilon_1 : \mathfrak{C} \to C$ is a path object for $C$ if the following hold:

1. $\iota, \varepsilon_0$ and $\varepsilon_1$ are linear morphisms (and we denote their $(1,1,0)$ parts by the same letters).
Using the inductive assumption (3) and part (1) of Proposition 3.1, one finds

\[ q(2) \]

Proposition 4.2. For any IBL\(_\infty\)-algebra \( (C, \{ p_{k, \ell, g} \}) \) there exists a path object \( \mathcal{C} \).

Proof. Define

\[ \mathcal{C} := C \oplus C \oplus C[1], \]

with boundary operator \( q_{1,1,0} : \mathcal{C} \to \mathcal{C} \) given by

\[ q_{1,1,0}(x_0, x_1, y) = (p_{1,1,0}(x_0), p_{1,1,0}(x_1), x_1 - x_0 - p_{1,1,0}(y)). \]

We define \( \iota : C \to \mathcal{C} \) by \( \iota(x) = (x, x, 0) \), and we define \( \varepsilon_i : \mathcal{C} \to C \) and \( \varepsilon_2 : \mathcal{C} \to C \) by \( \varepsilon_i(x_0, x_1, y) = x_i \).

With these definitions it is obvious that \( \iota, \varepsilon_0 \) and \( \varepsilon_1 \) are chain maps, that \( \varepsilon_0 \circ \iota = \varepsilon_1 \circ \iota = \text{id}_C \), and that \( \varepsilon_0 \oplus \varepsilon_1 \) admits a right inverse.

To prove that \( \iota \circ \varepsilon_0 \) is homotopic to the identity of \( \mathcal{C} \), we define \( H : \mathcal{C} \to \mathcal{C} \) by \( H(x_0, x_1, y) = (0, -y, 0) \). Then one checks that

\[ (q_{1,1,0}H + Hq_{1,1,0})(x_0, x_1, y) = (0, -p_{1,1,0}(y), -y) + (0, x_0 - x_1 + p_{1,1,0}(y), 0) = (\iota \circ \varepsilon_0 - \text{id}_C)(x_0, x_1, y). \]

A similar argument works for \( \iota \circ \varepsilon_1 \).

To complete the proof, it remains to show that we can construct the higher operations \( q_{k, \ell, g} : E_k \mathcal{C} \to E_\mathcal{C} \) in such a way that \( \iota, \varepsilon_0 \) and \( \varepsilon_1 \) are IBL\(_\infty\)-morphisms.

We proceed by induction on our linear order of the signatures \( (k, \ell, g) \). So assume that \( q_{k, \ell, g} \) has been constructed for all \( (k, \ell, g) \prec (K, L, G) \), in such a way that

\[ \begin{align*}
(1) & \quad \frac{1}{1} \prod_i \varepsilon_0^i q_{k, \ell, g} = q_{k, \ell, g} \frac{1}{1} \prod_i \varepsilon_0^i : E_k C \to E_\mathcal{C} \\
(2) & \quad \frac{1}{1} \prod_i \varepsilon_0^i q_{k, \ell, g} = p_{k, \ell, g} \frac{1}{1} \prod_i \varepsilon_0^i : E_k \mathcal{C} \to E_{\mathcal{C}} \\
(3) & \quad \text{the operations } q \text{ satisfy equation (2.3)} \quad \text{for all } (k, \ell, g) \prec (K, L, G). \end{align*} \]

Define \( q_{K,L,G}'' := \frac{1}{1} \prod_i \varepsilon_0^i p_{K,L,G} \frac{1}{1} \prod_i \varepsilon_0^i K \), and consider

\[ \Gamma := Q_{K,L,G} + \hat{q}_{1,1,0} q_{K,L,G}'' + q_{K,L,G}'' \hat{q}_{1,1,0} : E_k \mathcal{C} \to E_\mathcal{C}, \]

where \( Q_{K,L,G} \) is the quadratic expression in the \( q \)'s implicitly defined in Lemma 2.6. Note that \( \Gamma \) is homogeneous of even degree \(-2d(K + G - 1) - 2\). Using the inductive assumption (3) and part (1) of Proposition 3.1, one finds that

\[ \hat{q}_{1,1,0} \Gamma - \Gamma \hat{q}_{1,1,0} = \hat{q}_{1,1,0} Q_{K,L,G} - Q_{K,L,G} \hat{q}_{1,1,0} = 0, \]
i.e. $\Gamma$ is a chain map. Moreover, it follows from assumption (1) above that

$$Q_{K,L,G} \frac{1}{K!} \varepsilon^\otimes K = \frac{1}{L!} \varepsilon^\otimes L P_{K,L,G},$$

where $P_{K,L,G}$ is the expression of Lemma 2.6 in the $p$'s. We conclude that

$$\Gamma \frac{1}{K!} \varepsilon^\otimes K = (Q_{K,L,G} + \hat{q}_{1,1,0} q'_{K,L,G} + q_{K,L,G} \hat{q}_{1,1,0}) \frac{1}{K!} \varepsilon^\otimes K$$

$$= \frac{1}{L!} \varepsilon^\otimes L P_{K,L,G} + \hat{q}_{1,1,0} q''_{K,L,G} \frac{1}{K!} \varepsilon^\otimes K + q_{K,L,G} \frac{1}{K!} \varepsilon^\otimes K \hat{p}_{1,1,0}$$

$$= \frac{1}{L!} \varepsilon^\otimes L (P_{K,L,G} + \hat{p}_{1,1,0} p_{K,L,G} + p_{K,L,G} \hat{p}_{1,1,0})$$

$$= 0.$$

Applying part (2) of Lemma 3.2 with $e = \frac{1}{K!} \varepsilon^\otimes K$, $i = \frac{1}{K!} \varepsilon^\otimes K$ and $g = \Gamma$, we get the existence of a map $H : E_K C \rightarrow E_L C$ such that

$$\Gamma + \hat{q}_{1,1,0} H + H \hat{q}_{1,1,0} = 0 \quad \text{and} \quad H \frac{1}{K!} \varepsilon^\otimes K = 0.$$

We set

$$q'_{K,L,G} := q''_{K,L,G} + H.$$

Then the collection $\{q'_{K,L,G}, q_{k,\ell,g} | (k, \ell, g) \prec (K, L, G)\}$ satisfies the inductive assertions (3) and (1) above for the triple $(K, L, G)$.

We now want to modify $q'_{K,L,G}$ in such a way that it will also satisfy the inductive assertion (2). To proceed, we define

$$\Gamma_i := \frac{1}{L!} \varepsilon^\otimes L q_{K,L,G} - p_{K,L,G} \frac{1}{K!} \varepsilon^\otimes K : E_K C \rightarrow E_L C, \quad i = 0, 1,$$

which are homogeneous of degree $-2d(K + g - 1) - 1$. These are the error terms in (2). Now

$$\hat{p}_{1,1,0} \Gamma_i + \Gamma_i \hat{q}_{1,1,0} = \hat{p}_{1,1,0} \Gamma_i + (\frac{1}{L!} \varepsilon^\otimes L q'_{K,L,G} - p_{K,L,G} \frac{1}{K!} \varepsilon^\otimes K) \hat{q}_{1,1,0}$$

$$= \hat{p}_{1,1,0} \Gamma_i + (p_{K,L,G} + \hat{p}_{1,1,0} p_{K,L,G}) \frac{1}{K!} \varepsilon^\otimes K$$

$$- \frac{1}{L!} \varepsilon^\otimes L Q_{K,L,G} + P_{K,L,G} \frac{1}{K!} \varepsilon^\otimes K$$

$$= 0$$

by inductive assumption (2). Moreover, by (1) we have

$$\Gamma_i \frac{1}{K!} \varepsilon^\otimes K = \frac{1}{L!} \varepsilon^\otimes L q_{K,L,G} \frac{1}{K!} \varepsilon^\otimes K - p_{K,L,G} = 0.$$
So again by part (2) of Lemma 3.2 above, there exist even maps \( \chi_i : E_K \mathcal{C} \to E_L C, \ i = 0, 1 \) of degree \(-2d(K+g-1)\) such that

\[
\Gamma_i + \hat{p}_{1,1,0} \chi_i - \chi_i \hat{q}_{1,1,0} = 0 \quad \text{and} \quad \chi_i \frac{1}{K!} \otimes K = 0.
\]

Choosing a right inverse \( \rho : E_L C \oplus E_L C \to E_L \mathcal{C} \) to \( \frac{1}{\prod_1} \otimes L \), we find a linear lift \( \chi = \rho \circ (\chi_0 \oplus \chi_1) : E_K \mathcal{C} \to E_L \mathcal{C} \) such that \( \chi_i = \frac{1}{L} \otimes L \circ \chi \) and \( \chi \frac{1}{K!} \otimes K = 0 \). Now we define

\[
\hat{q}_{K,L,G} := \hat{q}_{K,L,G}^i + \hat{q}_{1,1,0} \chi - \chi \hat{q}_{1,1,0},
\]

which is easily seen to satisfies all three inductive assumptions. \( \square \)

**Remark 4.3.** Note that in the inductive construction of the IBL\(_\infty\)-structure in the above proof we did not make use of the specific form of the chain complex \((\mathcal{C}, q_{1,1,0})\) satisfying (a)-(d), so one could have started equally well with any other chain model for \( \mathcal{C} \).

**Proposition 4.4.** Let \( C \) and \( D \) be IBL\(_\infty\)-algebras, and let \( \mathcal{C} \) and \( \mathcal{D} \) be path objects for \( C \) and \( D \), respectively. Let \( f : C \to D \) be a morphism. Then there exists a morphism \( \mathfrak{F} : \mathcal{C} \to \mathcal{D} \) such that the diagram

\[
\begin{array}{ccc}
C & \xrightarrow{C} & \mathcal{C} & \xrightarrow{\epsilon_1^C} & C \\
\downarrow f & & \downarrow \mathfrak{F} & & \downarrow f \\
D & \xrightarrow{D} & \mathcal{D} & \xrightarrow{\epsilon_1^D} & D
\end{array}
\]

commutes for both \( i = 0 \) and \( i = 1 \).

**Proof.** The proof is inductive and similar in structure to the proof of Propostion 4.2.

**Step 1:** We construct a chain map \( \mathfrak{F}_{1,1,0} : (\mathcal{C}, q_{1,1,0}^C) \to (\mathcal{D}, q_{1,1,0}^D) \) satisfying the required relations.

Set \( F' := \epsilon_D f_{1,1,0} \epsilon_0^C : \mathcal{C} \to \mathcal{D} \), and note that \( F' \epsilon_C - \epsilon_D f_{1,1,0} = 0 \) as required. Similarly, \( \Gamma_0 = \epsilon_D F' - f_{1,1,0} \epsilon_0^C = 0 \), but \( \Gamma_1 := \epsilon_D F' - f_{1,1,0} \epsilon_1^C \neq 0 \). One checks that

\[
p_{1,1,0} \Gamma_1 - \Gamma_1 q_{1,1,0}^C = 0, \quad \text{and} \quad \Gamma_1 \epsilon_C = 0.
\]

Hence by part (2) of Lemma 3.2 there exists a chain homotopy \( E_1 : \mathcal{C} \to \mathcal{D} \) such that

\[
\Gamma_1 + p_{1,1,0}^D E_1 + E_1 q_{1,1,0}^C = 0 \quad \text{and} \quad E_1 \epsilon_C = 0.
\]

Choosing a right inverse to \( \epsilon_0 \oplus \epsilon_1 : \mathcal{D} \to D \oplus D \), we construct a lift \( E : \mathcal{C} \to \mathcal{D} \) of \( 0 \oplus E_1 \) such that \( \epsilon_0 \circ E = 0, \epsilon_1^D \circ E = E_1 \) and \( E \circ \epsilon_C = 0 \). Then

\[
\mathfrak{F}_{1,1,0} := F' + q_{1,1,0}^D E + E q_{1,1,0}^C : \mathcal{C} \to \mathcal{D}
\]
is the required chain map.

**Step 2:** We now proceed by induction on our linear order of signatures \((k, \ell, g)\). So suppose we have already constructed maps \(\mathfrak{F}_{k, \ell, g} : E_k \mathfrak{C} \to E_L \mathfrak{D}\) for all \((k, \ell, g) \prec (K, L, G)\) such that:

1. \[ \frac{1}{L!} (\varepsilon_i^D) \circ \mathfrak{f}_{k, \ell, g} = \mathfrak{F}_{k, \ell, g} \frac{1}{K!} (\varepsilon_i^C) \circ \mathfrak{r} \]
   for all \((k, \ell, g) \prec (K, L, G)\),

2. \[ \frac{1}{L!} (\varepsilon_i^D) \circ \mathfrak{r}_{k, \ell, g} = \mathfrak{F}_{k, \ell, g} \frac{1}{K!} (\varepsilon_i^C) \circ \mathfrak{r} \]
   for \(i = 0, 1\) and for all \((k, \ell, g) \prec (K, L, G)\), and

3. the defining equation (2.12) for morphisms holds for \(\mathfrak{F}\), \(q^C\) and \(q^D\) for all \((k, \ell, g) \prec (K, L, G)\).

Consider the expression \(R_{K,L,G}(\mathfrak{F}, q^C, q^D) : E_K \mathfrak{C} \to E_L \mathfrak{D}\) as defined in the statement of Lemma 2.10. Using the inductive assumptions and part (2) of Proposition 3.1, one proves that

\[ \hat{q}_{1,1,0}^D R_{K,L,G}(\mathfrak{F}, q^C, q^D) + R_{K,L,G}(\mathfrak{F}, q^C, q^D) q_{1,1,0}^C = 0 \]

\[ \frac{1}{L!} (\varepsilon_i^D) \circ L R_{K,L,G}(\mathfrak{f}, p^C, p^D) = R_{K,L,G}(\mathfrak{F}, q^C, q^D) \frac{1}{K!} (\varepsilon_i^C) \circ K \]

\[ \frac{1}{L!} (\varepsilon_i^D) \circ L R_{K,L,G}(\mathfrak{F}, q^C, q^D) = R_{K,L,G}(\mathfrak{f}, p^C, p^D) \frac{1}{K!} (\varepsilon_i^C) \circ K \]

for \(i = 0, 1\).

Now, as in the proof of Proposition 4.2, define \(\mathfrak{F}_{K,L,G} := \frac{1}{L!} (\varepsilon_i^D) \circ L \mathfrak{f}_{K,L,G} \frac{1}{K!} (\varepsilon_i^C) \circ K : E_K \mathfrak{C} \to E_L \mathfrak{D}\), and observe that

\[ F := R_{K,L,G}(\mathfrak{F}, q^C, q^D) + \hat{q}_{1,1,0}^D \mathfrak{F}_{K,L,G} - \mathfrak{F}_{K,L,G} q_{1,1,0}^C : E_K \mathfrak{C} \to E_L \mathfrak{D} \]

is of odd degree and satisfies \(\hat{q}_{1,1,0}^D F + F q_{1,1,0}^C = 0\) and \(F \frac{1}{K!} (\varepsilon_i^C) \circ K = 0\). Hence, by part (1) of Lemma 3.2 we find \(H : E_K \mathfrak{C} \to E_L \mathfrak{D}\) of even degree such that

\[ F + q_{1,1,0}^C H - H q_{1,1,0}^C = 0, \quad \text{and} \quad H \frac{1}{K!} (\varepsilon_i^C) \circ K = 0. \]

Then the map \(\mathfrak{F}_{K,L,G} := \hat{q}_{1,1,0}^D + H : E_K \mathfrak{C} \to E_L \mathfrak{D}\) satisfies the conditions (1) and (3) in the inductive assumption.

To achieve (2), consider the maps of even degree

\[ \Gamma_i := \frac{1}{L!} (\varepsilon_i^D) \circ L \mathfrak{f}_{K,L,G} - \mathfrak{f}_{K,L,G} \frac{1}{K!} (\varepsilon_i^C) \circ K : E_K \mathfrak{C} \to E_L D, \quad i = 0, 1 \]

and compute that

\[ p_{1,1,0}^D \Gamma_i - \Gamma_i q_{1,1,0}^C = 0, \quad \text{and} \quad \Gamma_i \frac{1}{K!} (\varepsilon_i^C) \circ K = 0. \]

Hence, by part (2) of Lemma 3.2, there exist maps \(E_i : E_K \mathfrak{C} \to E_L D\) of odd degree such that

\[ \Gamma_i + p_{1,1,0}^D E_i + E_i q_{1,1,0}^C = 0 \quad \text{and} \quad E_i \frac{1}{K!} (\varepsilon_i^C) \circ K = 0. \]
With a right inverse \( \rho_L : E_LD \oplus E_LD \to E_L\mathcal{D} \) to \((\varepsilon_i^D)^{\odot L} \oplus (\varepsilon_1^D)^{\odot L}\), we define a linear extension \( E = \rho_L \circ (E_0 \oplus E_1) : E_K\mathcal{C} \to E_L\mathcal{D} \) such that
\[
\frac{\partial}{\partial t}(\varepsilon_i^D)^{\odot L} \circ E = E_i \text{ and } E \circ \frac{\partial}{\partial t}(\ell_i^C)^{\odot K} = 0.
\]

Then
\[
\mathfrak{F}_{K,L,G} := \mathfrak{F}_{K,L,G}' + q_{1,1,0}^D E + E q_{1,1,0}^C E_K\mathcal{C} \to E_L\mathcal{D}
\]
has the required properties. This completes the inductive step and hence the proof of the proposition. \(\square\)

We now come to the main definition of this section.

**Definition 4.5.** We say that two IBL\(_\infty\)-morphisms \( f_0 : C \to D \) and \( f_1 : C \to D \) are homotopic if for some path object \( \mathcal{D} \) for \( D \) there exists a morphism \( \mathfrak{F} : C \to \mathcal{D} \) such that \( \varepsilon_0 \circ \mathfrak{F} = f_0 \) and \( \varepsilon_1 \circ \mathfrak{F} = f_1 \). We call such an \( \mathfrak{F} \) a homotopy between \( f_0 \) and \( f_1 \).

**Proposition 4.6.** The notion of homotopy has the following properties:

(a) A homotopy between \( f_0 \) and \( f_1 \) exists for some path object for \( D \) if and only if it exists for all path objects for \( D \).

(b) Homotopy of morphisms is an equivalence relation.

(c) If \( f_0 : B \to C \) and \( f_1 : B \to C \) are homotopic and \( g_0 : C \to D \) and \( g_1 : C \to D \) are homotopic, then \( g_0 \circ f_0 \) and \( g_1 \circ f_1 \) are homotopic.

**Proof.** To prove (a), suppose \( \mathfrak{F} : C \to \mathcal{D} \) is a homotopy between \( f_0 : C \to D \) and \( f_1 : C \to D \), and let \( \mathcal{D}' \) be any other path object for \( D \). Applying Proposition 4.3 to the identity of \( D \) and the two path objects \( \mathcal{D} \) and \( \mathcal{D}' \), we obtain an IBL\(_\infty\)-morphism \( \mathcal{I} : \mathcal{D} \to \mathcal{D}' \). Setting \( \mathfrak{F}' := \mathcal{I} \circ \mathfrak{F} \), one verifies that
\[
\varepsilon_0' \circ \mathfrak{F}' = \varepsilon_1' \circ \mathcal{I} \circ \mathfrak{F} = \varepsilon_0 \circ \mathfrak{F} = f_i
\]
as required.

We next prove (b). To see that \( f : C \to D \) is homotopic to itself, consider any path object \( \mathcal{D} \) for \( D \) and set \( \mathfrak{F} := \iota \circ f \).

To see that the relation is symmetric, note that if \( (\mathcal{D}, \iota, \varepsilon_0, \varepsilon_1) \) is a path object for \( D \), then \( (\mathcal{D}, \iota, \varepsilon_0', \varepsilon_1') \) is also a path object, where \( \varepsilon_0' = \varepsilon_1 \) and \( \varepsilon_1' = \varepsilon_0 \).

To prove transitivity of the relation, suppose \( f_0 \) and \( f_1 \) are homotopic via a homotopy \( \mathfrak{F}^1 : C \to \mathcal{D}^1 \) and \( f_1 \) and \( f_2 \) are homotopic via a homotopy \( \mathfrak{F}^2 : C \to \mathcal{D}^2 \).

We define a new path object \( \mathcal{D} \) for \( D \) as follows. As a vector space, set
\[
\mathcal{D} := \{(d_1, d_2) \in \mathcal{D}^1 \oplus \mathcal{D}^2 | \varepsilon_1^1(d_1) = \varepsilon_2^2(d_2) \}.
\]

We define the map \( \iota : D \to \mathcal{D} \) by \( \iota(x) = (\iota^1(x), \iota^2(x)) \) and the maps \( \varepsilon_i : \mathcal{D} \to D \) by \( \varepsilon_0(d_1, d_2) := \varepsilon_0(d_1) \) and \( \varepsilon_1(d_1, d_2) = \varepsilon_1(d_2) \). To construct the structure maps \( q_{K,L,G} \), first note that the two projections \( \pi_i : \mathcal{D} \to \mathcal{D}^i \) determine a
projection $\pi_k : E_k \mathcal{D} \to E_k \mathcal{D}^1 \oplus E_k \mathcal{D}^2$ which surjects onto $\mathcal{P}_k := \{(g^1, g^2) \in E_k \mathcal{D}^1 \oplus E_k \mathcal{D}^2 | \frac{1}{k} (\varepsilon^1_0) \circ_k (g^1) = \frac{1}{k} (\varepsilon^2_0) \circ_k (g^2)\}$. In particular, $\pi_k$ admits a right inverse $\rho_k : \mathcal{P}_k \to E_k \mathcal{D}$.

Now we define $q_{k,\ell,g} : E_k \mathcal{D} \to E_\ell \mathcal{D}$ by

$$q_{k,\ell,g} := \rho_\ell \circ (q_{k,\ell,g}^1 \oplus q_{k,\ell,g}^2) \circ \pi_k.$$  

Observe that, by construction, the defining relations (2.3) for $q_{k,\ell,g}$ follow from the defining relations of $q_{i,\ell,g}$, and similarly the properties (a)-(d) of a path object can be easily checked using the corresponding properties of the $\mathcal{D}^i$. So we have proven that $\mathcal{D}$ is a path object for $D$.

We now define a homotopy between $f_0$ and $f_2$ to be the morphism $\tilde{\mathfrak{H}} : C \to \mathcal{D}$ whose component $\tilde{\mathfrak{H}}_{k,\ell,g} : E_k C \to E_\ell \mathcal{D}$ is defined as $\rho_\ell \circ (\tilde{\mathfrak{H}}_{k,\ell,g}^1 \oplus \tilde{\mathfrak{H}}_{k,\ell,g}^2)$. One straightforwardly checks that this is indeed a morphism with the required properties, and this completes the proof of part (b).

We now prove part (c). First let $\mathfrak{G} : C \to \mathcal{D}$ be a homotopy between $g_0$ and $g_1$. Then $\mathfrak{G} \circ f_0 : B \to \mathcal{D}$ is a homotopy between $g_0 \circ f_0$ and $g_1 \circ f_0$. So by part (b), it suffices to prove the claim for $g_0 = g_1 =: g$.

Applying Proposition 4.4 to $g : C \to D$, we obtain a map $\mathfrak{G} : C \to \mathcal{D}$ with $\varepsilon^D_i \circ \mathfrak{G} = g \circ \varepsilon^C_i$. Now let $\tilde{\mathfrak{H}} : B \to \mathcal{D}$ be a homotopy between $f_0$ and $f_1$ and set $\mathfrak{H} := \mathfrak{G} \circ \tilde{\mathfrak{H}} : B \to \mathcal{D}$. Then from the definitions one checks that

$$\varepsilon^D_i \circ \mathfrak{H} = \varepsilon^D_i \circ \mathfrak{G} \circ \tilde{\mathfrak{H}} = g \circ \varepsilon^C_i \circ \tilde{\mathfrak{H}} = g \circ f_i,$$

so that $\mathfrak{H}$ is the required homotopy between $g \circ f_0$ and $g \circ f_1$. □

We say that an IBL$_\infty$-morphism $\mathfrak{f} : C \to D$ is a homotopy equivalence if there exists an IBL$_\infty$-morphism $\mathfrak{g} : D \to C$ such that $\mathfrak{f} \circ \mathfrak{g}$ and $\mathfrak{g} \circ \mathfrak{f}$ are each homotopic to the respective identity map. From the above discussion, we get the following immediate consequence.

**Corollary 4.7.** A composition of homotopy equivalences is a homotopy equivalence. In particular, homotopy equivalence is an equivalence relation. □

### 5. Homotopy Inverse

In this section we prove Theorem 1.2 following the scheme of the corresponding argument in [34, §4.5] in the $A_\infty$-case.

**Lemma 5.1.** Consider IBL$_\infty$-algebras $(C, \{p_{k,\ell,g}^C\})$ and $(D, \{p_{k,\ell,g}^D\})$, and let $(\mathcal{D}, \{q_{k,\ell,g}^D\})$ be a path object for $D$. Suppose $\mathfrak{h} = \{h_{k,\ell,g} : E_k C \to E_\ell \mathcal{D}\}_{(k,\ell,g) \prec (K,L,G)}$
satisfies \((2.12)\) for all \((k, \ell, g) \prec (K, L, G)\). Then
\[
[R_{K,L,G}(\varepsilon_0 \circ h, p^C, p^D)] = [R_{K,L,G}(\varepsilon_1 \circ h, p^C, p^D)] \in H_*(\text{Hom}(E_KC, E_LD), \delta).
\]

**Proof.** Observe that, since the \(\varepsilon_i\) are linear IBL\(_{\infty}\)-morphisms, by part \((3)\) of Proposition \(3.1\) we have
\[
R_{K,L,G}(\varepsilon_i \circ h, p^C, p^D) = \frac{1}{L!} \varepsilon_i^{\odot L} \circ R_{K,L,G}(h, p^C, q^D), \quad i = 0, 1.
\]
Now the two maps \(E_i : (\text{Hom}(E_KC, E_LD), \delta) \rightarrow (\text{Hom}(E_KC, E_LD), \delta)\) given by \(E_i(\varphi) = \frac{1}{L!} \varepsilon_i^{\odot L} \circ \varphi\) are both homotopy inverses to the same map \(I : (\text{Hom}(E_KC, E_LD), \delta) \rightarrow (\text{Hom}(E_KC, E_LD), \delta)\), \(I(\psi) = \frac{1}{L!} \varepsilon_i^{\odot L} \circ \psi\), so they induce the same map in homology. This proves the claim. \(\square\)

The proof of Theorem \(1.2\) will be an easy consequence of the following observation.

**PROPOSITION 5.2.** Let \(f : D \rightarrow C\) be an IBL\(_{\infty}\)-morphism such that \(f_{1,1,0} : (D, p^D_{1,1,0}) \rightarrow (C, p^C_{1,1,0})\) is a quasi-isomorphism of chain complexes. Then there exists an IBL\(_{\infty}\)-morphism \(g : C \rightarrow D\) such that \(g \circ f\) is homotopic to the identity of \(D\).

**Proof.** Step 1: One first constructs a chain map \(g_{1,1,0} : (C, p^C_{1,1,0}) \rightarrow (D, p^D_{1,1,0})\) which is a chain homotopy inverse to \(f_{1,1,0}\), together with a homotopy \(h_{1,1,0} : D \rightarrow \mathcal{D}\) between \(g_{1,1,0} \circ f_{1,1,0}\) and the identity of \(D\). This is completely standard.

Step 2: Now we proceed by induction on our linear order of signatures \((k, \ell, g)\). Suppose we have constructed maps \(g_{k,\ell,g} : E_kC \rightarrow E_\ell D\) and \(h_{k,\ell,g} : E_kD \rightarrow E_\ell \mathcal{D}\) for all \((k, \ell, g) \prec (K, L, G)\) such that
\begin{enumerate}
  \item \(\frac{1}{L!} \varepsilon_i^{\odot L} \circ h_{k,\ell,g} = 0\) for all \((1, 1, 0) \prec (k, \ell, g) \prec (K, L, G)\) and \(\varepsilon_0 \circ h_{1,1,0} = \text{id}_D\),
  \item \(\frac{1}{L!} \varepsilon_i^{\odot L} \circ h_{k,\ell,g} = (g \circ f)_{k,\ell,g}\) for all \((k, \ell, g) \prec (K, L, G)\),
  \item \(h\) satisfies \((2.12)\) for all \((k, \ell, g) \prec (K, L, G)\)
  \item \(g\) satisfies \((2.12)\) for all \((k, \ell, g) \prec (K, L, G)\).
\end{enumerate}

By inductive assumption \((i)\), \(\varepsilon_0 \circ h\) is the identity of \(D\), which is clearly an IBL\(_{\infty}\)-morphism. So by part \((3)\) of Proposition \(3.1\)
\[
\frac{1}{L!} (\varepsilon_0^{\odot L})_* [R_{K,L,G}(h, p^D, q^D)] = [R_{K,L,G}(\varepsilon_0 \circ h, p^D, q^D)] = 0
\]
in \(H_*(\text{Hom}(E_kD, E_\ell \mathcal{D}), \delta)\). Applying part \((1)\) of Lemma \(3.2\) with \(i = \frac{1}{L!} \varepsilon_i^{\odot L} : E_kD \rightarrow E_\ell \mathcal{D}, \quad e = \frac{1}{L!} \varepsilon_0^{\odot L} : E_kD \rightarrow E_\ell D\) and \(f = R_{K,L,G}(h, p^D, q^D)\), we obtain \(S : E_kD \rightarrow E_\ell \mathcal{D}\) such that
\[
R_{K,L,G}(h, p^D, q^D) = \delta S
\]
and \( \frac{1}{L!} \varepsilon_0^\oplus \circ S = 0 \). Note that
\[
\delta(\frac{1}{L!} \varepsilon_1^\oplus \circ S) = \frac{1}{L!} \varepsilon_1^\oplus R_{K,L,G}(\varepsilon, p^D, q^D)
= R_{K,L,G}(\varepsilon_1 \circ\varepsilon, p^D, p^D)
= R_{K,L,G}(\varepsilon \circ\varepsilon, p^D, p^D)
= \frac{1}{L!} \varepsilon_1^{\oplus L} R_{K,L,G}(\varepsilon, p^D, p^D) + R_{K,L,G}(\varepsilon, p^C, p^D) \circ \frac{1}{K!} \varepsilon_1^{\bigodot K} + \delta C_{K,L,G}(\varepsilon, f),
\]
where we used part (3) of Proposition 3.1 in the last step. Since \( \varepsilon \) is an IBL\(_\infty\)-morphism, we conclude that
\[
[R_{K,L,G}(\varepsilon, p^C, p^D) \circ \frac{1}{K!} \varepsilon_1^{\bigodot K}] = 0 \in H_*(\text{Hom}(E_K D, E_L D), \delta).
\]
Since \( \frac{1}{K!} \varepsilon_1^{\bigodot K} \) is a homotopy equivalence, this implies that
\[
R_{K,L,G}(\varepsilon, p^C, p^D) = \delta T
\]
for some \( T : E_K C \to E_L D \). Now consider
\[
F = T \circ \frac{1}{K!} \varepsilon_1^{\bigodot K} + \frac{1}{L!} \varepsilon_1^{\bigodot L} \circ f_{K,L,G} + C_{K,L,G}(\varepsilon, f) - \frac{1}{L!} \varepsilon_1^{\bigodot L} \circ S : E_K D \to E_L D,
\]
and note that by construction we have \( \delta F = 0 \).

Observe also that precomposition with \( \frac{1}{K!} \varepsilon_1^{\bigodot K} \) induces a homotopy equivalence from \( (\text{Hom}(E_K C, E_L D), \delta) \) to \( (\text{Hom}(E_K D, E_L D), \delta) \). In particular, there exists \( G : E_K C \to E_L D \) with \( \delta G = 0 \) and
\[
[F + G \circ \frac{1}{K!} \varepsilon_1^{\bigodot K}] = 0 \in H_*(\text{Hom}(E_K D, E_L D), \delta)).
\]
This in turn means that we can find \( H_1 : E_K D \to E_L D \) such that
\[
\delta H_1 = F + G \circ \frac{1}{K!} \varepsilon_1^{\bigodot K}.
\]
Since \( \varepsilon_0 \oplus \varepsilon_1 : \mathcal{D} \to D \oplus D \) admits a right inverse, we find a lift \( H : E_K D \to E_L \mathcal{D} \) such that \( \frac{1}{L!} \varepsilon_0^{\bigodot L} \circ H = H_1 \) and \( \frac{1}{L!} \varepsilon_0^{\bigodot L} \circ H = 0 \). Now set
\[
\mathfrak{g}_{K,L,G} := T + G \quad \text{and} \quad \mathfrak{h}_{K,L,G} := S + \delta H.
\]
Let us check that they satisfy properties (i)-(iv) for \( (K, L, G) \). For (i), observe that by construction
\[
\frac{1}{L!} \varepsilon_0^{\bigodot L} \circ (S + \delta H) = \frac{1}{L!} \varepsilon_0^{\bigodot L} \circ S + \delta(\frac{1}{L!} \varepsilon_0^{\bigodot L} \circ H) = 0
\]
as required. For (ii), we check that
\[
\frac{1}{L!} \varepsilon_1^{\bigodot L} \circ (S + \delta H) = \frac{1}{L!} \varepsilon_1^{\bigodot L} \circ S + \delta H_1
\]
\[
=(T + G) \circ \frac{1}{K!} \varepsilon_1^{\bigodot K} + \frac{1}{L!} \varepsilon_1^{\bigodot L} \circ f_{K,L,G} + C_{K,L,G}(\mathfrak{g}, f)
\]
\[
=(\mathfrak{g} \circ f)_{K,L,G},
\]
where in the last step we used Lemma 2.12. For (iii), we check that
\[ \delta h_{K,L,G} = \delta S = R_{K,L,G}(h, p^D, q^D) \]
as required. Similarly, for (iv) we observe that
\[ \delta g_{K,L,G} = \delta T = R_{K,L,G}(g, p^C, p^D). \]
This completes the induction step and hence the proof of Proposition 5.2. □

We now conclude this section by proving Theorem 1.2. Indeed, let \( f : D \to C \)
be an IBL\(_\infty\)-morphism such that \( f_{1,1,0} \) induces an isomorphism in homology.
Then by Proposition 5.2 there exists an IBL\(_\infty\)-morphism \( g : C \to D \)
such that \( g \circ f \) is homotopic to the identity of \( D \).
As \( g_{1,1,0} \) induces the inverse isomorphism in homology, we can apply Proposition 5.2
again to construct another IBL\(_\infty\)-morphism \( f' : D \to C \)
such that \( f' \circ g \) is homotopic to the identity of \( C \). Now it follows that
\[ f \sim f' \circ g \circ f \sim f', \]
so that by Proposition 4.6(c) we conclude that \( f \circ g \) is also homotopic to the
identity of \( C \). In other words, \( f \) and \( g \) are homotopy inverses of each other.
This completes the proof of Theorem 1.2.

6. Canonical model

In this section we prove the following statement, which is Theorem 1.3 from
the Introduction. We assume that the ground ring \( R \) is a field containing \( \mathbb{Q} \).

**Theorem 6.1.** Suppose \((C, \{p_{k,\ell,g}\})\) is an IBL\(_\infty\)-algebra. Then there exist
operations \( \{q_{k,\ell,g}\} \) on its homology \( H := H_* (C, p_{1,1,0}) \) giving it the structure
of an IBL\(_\infty\)-algebra such that there exists a homotopy equivalence \( f : (H, \{q_{k,\ell,g}\}) \to (C, \{p_{k,\ell,g}\}) \).

**Proof.** Fix a cycle-choosing embedding \( f_{1,1,0} : H \to C \) and a splitting \( C = H \oplus B \oplus A \),
where we identify \( H \) with its image under \( f_{1,1,0} \) and where \( B = \text{im } p_{1,1,0}. \)
Denote by \( \pi : C \to H \) the projection along \( B \oplus A \), and by \( h : C \to C \) the map which
vanishes on \( H \oplus A \) and is equal to the inverse of \( p_{1,1,0} : A \cong B \) on \( B \). Then we have \( \pi f_{1,1,0} = \text{id}_H \)
and
\[ p_{1,1,0} h + h p_{1,1,0} = \text{id}_C - f_{1,1,0} \pi, \]
so that \( f_{1,1,0} : (H, q_{1,1,0} = 0) \to (C, p_{1,1,0}) \) is a chain homotopy equivalence.
Now as usual we argue by induction on our linear order of signatures \((k, \ell, g)\).
So assume that we have defined \( q_{k,\ell,g} \) and \( f_{k,\ell,g} \) for all \((k, \ell, g) < (K, L, G)\)
such that

(i) the \( q \)'s satisfy (2.4) for all \((k, \ell, g) < (K, L, G)\),
(ii) the \( q \)'s and the \( f \)'s satisfy (2.13) for all \((k, \ell, g) < (K, L, G)\).
Consider the expression $\tilde{R}_{K,L,G}(\hat{f}, q, p) : E_KH \to E_LC$ appearing in the second statement of Lemma 2.10 and define

$$q_{K,L,G} := \frac{1}{L!} \pi^{\otimes L} \left( p_{K,L,G} \frac{1}{K!}^{\otimes K} \tilde{R}_{K,L,G}(\hat{f}, q, p) \right).$$

We claim that with this (or any other) definition the $\{q_{k,\ell,g}\}_{(k,\ell,g) \prec (K,L,G)}$ satisfy equation (2.4) for $(K, L, G)$. In fact, since $q_{1,1,0} = 0$, we need only show that the quadratic expression $Q_{K,L,G}$ in the $\hat{q}_{k,\ell,g}$ defined in Lemma 2.6 vanishes, which does not involve $q_{K,L,G}$.

As in the proof of Proposition 3.1 we use the notation $\langle A \rangle_{k,\ell,g}$ to denote the part of the coefficient of $\hat{h}^{k+g-1} \tau^{k+\ell+2g-2}$ in some map $A : EH \otimes R\{\tau, h\} \to EH \otimes R\{\tau, h\}$ which corresponds to the part mapping $E_kH$ to $E_\ell H$. Define

$$\hat{q}' := \sum_{(k,\ell,g) \prec (K, L, G)} \hat{q}_{k,\ell,g} h^{k+g-1} \tau^{k+\ell+2g-2} : EH \to EH,$$

and note that hypothesis (i) implies that $\langle \hat{q}' \hat{q}' \rangle_{k,\ell,g} = 0$ for all $(k,\ell,g) \prec (K, L, G)$, so that the claim $Q_{K,L,G} = 0$ is equivalent to $\langle \hat{q}' \hat{q}' \rangle_{K,L,G} = 0$.

We also define

$$\hat{p}' := \sum_{(k,\ell,g) \prec (K, L, G)} \hat{p}_{k,\ell,g} h^{k+g-1} \tau^{k+\ell+2g-2} : EC \to EC$$

and

$$\hat{f}' := \sum_{(k,\ell,g) \prec (K, L, G)} \hat{f}_{k,\ell,g} h^{k+g-1} \tau^{k+\ell+2g-2} : EH \to EC.$$

Induction hypothesis (ii) implies that

$$\langle e^f \hat{q}' - \hat{p}' e^f \rangle_{k,\ell,g} = 0$$

for all $(k,\ell,g) \prec (K, L, G)$. But this, together with $\langle \hat{q}' \hat{q}' \rangle_{k,\ell,g} = 0$ for all $(k,\ell,g) \prec (K, L, G)$, implies

$$\langle \hat{q}' \hat{q}' \rangle_{K,L,G} = \frac{1}{L!} \pi^{\otimes L} \frac{1}{L!} \tilde{R}_{K,L,G}(\hat{q}' \hat{q}')_{K,L,G}$$

$$= \frac{1}{L!} \pi^{\otimes L} \langle e^f \hat{q}' \hat{q}' \rangle_{K,L,G}$$

$$= \frac{1}{L!} \pi^{\otimes L} \langle \hat{p}' \hat{f}' \hat{q}' \rangle_{K,L,G}$$

$$= \frac{1}{L!} \pi^{\otimes L} \langle \hat{p}' \hat{p}' \hat{f}' \rangle_{K,L,G}$$

$$= -\frac{1}{L!} \pi^{\otimes L} \langle \hat{p}_{1,1,0} p_{K,L,G} \hat{p}_{1,1,0} \rangle \frac{1}{L!} \tilde{R}_{1,1,0}$$

$$= 0$$

since $\pi p_{1,1,0} = p_{1,1,0} \hat{f}_{1,1,0} = 0$. This proves that the $\hat{q}'$’s satisfy relation (2.4) for $(K, L, G)$. 
Now we apply part (2) of Proposition 3.1 to find that
\[ \hat{p}_{1,0,0} \left( \frac{1}{K!} p_{K,L,G} \hat{f}_{1,1,0} - \hat{R}_{K,L,G}(f, q, p) \right) = \delta R_{K,L,G}(f, q, p) = 0. \]

Since \( \frac{1}{L!} f_{1,1,0}^\otimes \) is a chain homotopy inverse to \( \pi_L \), we can choose a chain homotopy between \( \frac{1}{L!} f_{1,1,0}^\otimes q_{K,L,G} \) and \( \frac{1}{K!} p_{K,L,G} f_{1,1,0}^\otimes - \hat{R}_{K,L,G}(f, q, p) \) and denote it by \( \tilde{f}_{K,L,G} \). Then by construction we have
\[ -\hat{p}_{1,1,0} f_{K,L,G} + \frac{1}{L!} f_{1,1,0}^\otimes q_{K,L,G} - \frac{1}{K!} p_{K,L,G} f_{1,1,0}^\otimes + \hat{R}_{K,L,G}(f, q, p) = 0, \]
which according to Lemma 2.10 proves property (ii) for the induction. \( \square \)

Suppose \((H, \{q_{k,\ell,g}\})\) is an IBL\(_\infty\)-algebra, where \( H = H_\ast(C, \partial) \) is the homology of some chain complex \((C, \partial)\), and so in particular \( q_{1,1,0} = 0 \). Let \( f = \tilde{f}_{1,1,0} : H \to C \) and \( \pi : C \to H \) be maps as in the above proof. Then we get the structure of an IBL\(_\infty\)-algebra on \( C \) by setting \( p_{1,1,0} := \partial \) and
\[ p_{k,\ell,g} := \frac{1}{\ell!} f_{\ell}^\otimes q_{k,\ell,g} \frac{1}{K!} f_{K}^\otimes \]
for \((1, 1, 0) \prec (k, \ell, g)\). Since \( \pi f = \text{id}_H \) and \( \pi p_{1,1,0} = p_{1,1,0} \hat{f} = 0 \), the identities for \((C, p_{k,\ell,g})\) easily follow from those for \( H \), and \( f \) and \( \pi \) are linear homotopy equivalences inverse to each other.

Now suppose \( f : (C, \partial C) \to (D, \partial D) \) is a chain homotopy equivalence and suppose \( C \) has the structure of an IBL\(_\infty\)-algebra \((C, \{ p_{\otimes}^C \})\) with \( p_{1,1,0}^C = \partial C \). By Theorem 6.1, this structure can be projected to an IBL\(_\infty\)-structure \((H, \{q_{\otimes}^C\})\) on the homology \( H = H(C, \partial C) \cong H(D, \partial D) \) and then lifted according to the above discussion. So we have

**Corollary 6.2.** Let \((C, \{ p_{k,\ell,g}\})\) be an IBL\(_\infty\)-algebra, \((D, \partial D)\) a chain complex, and \( f : (C, \partial C) \to (D, \partial D) \) a chain homotopy equivalence. Then there exists an IBL\(_\infty\)-structure \( \{q_{k,\ell,g}\} \) on \( D \) with \( q_{1,1,0} = \partial D \) and a chain homotopy equivalence of IBL\(_\infty\)-algebras \( \tilde{f}_{k,\ell,g} : (C, \{ p_{k,\ell,g}\}) \to (D, \{ q_{k,\ell,g}\}) \) with \( \tilde{f}_{1,1,0} = f \). \( \square \)

7. **Relation to differential Weyl algebras**

In this section we will explain the relation between the IBL\(_\infty\)-formalism and the formalism of differential Weyl algebras used to describe symplectic field theory (SFT) for contact manifolds in \([25]\).

**Objects.** Fix a ground ring \( R \) containing \( \mathbb{Q} \), and fix some index set \( \mathcal{P} \) (which corresponds to the set of periodic orbits in SFT). Consider the Weyl algebra \( \mathcal{W} \) of power series in variables \( \{ p_\gamma \}_{\gamma \in \mathcal{P}} \) and \( h \) with coefficients polynomial
over \( R \) in variables \( \{ q_\gamma \} \in \mathcal{P} \). Each variable comes with an integer grading, and we assume that
\[
|p_\gamma| + |q_\gamma| = |h| = 2d
\]
for some integer \( d \) and all \( \gamma \in \mathcal{P} \). (In SFT, \( d = n - 3 \) for a contact manifold of dimension \( 2n - 1 \).) \( W \) comes equipped with an associative product \( \star \) in which all variables commute according to their grading except for \( p_\gamma \) and \( q_\gamma \) corresponding to the same index \( \gamma \), for which we have
\[
p_\gamma \star q_\gamma - (-1)^{|p_\gamma||q_\gamma|} q_\gamma \star p_\gamma = \kappa_\gamma h
\]
for some integers \( \kappa_\gamma \geq 1 \) (which correspond to multiplicities or periodic orbits in SFT).

A homogeneous element \( \mathbb{H} \in \frac{1}{\hbar} W \) of degree \(-1\) satisfying the master equation
(7.1)
\[
\mathbb{H} \star \mathbb{H} = 0
\]
is called a Hamiltonian, and the pair \((W, \mathbb{H})\) is called a differential Weyl algebra of degree \( d \). Indeed, the commutator with \( \mathbb{H} \) is then a derivation of \((W, \star)\) of square 0.

We will impose two further restrictions on our Hamiltonians \( \mathbb{H} \), namely
(7.2)
\[
\mathbb{H}|_{p=0} = 0 \quad \text{and} \quad \mathbb{H}|_{q=0} = 0.
\]

**Remark 7.1.** In SFT, the first condition in (7.2) is always satisfied, and the second one can be arranged if the theory admits an augmentation, which for example is always the case if the corresponding contact manifold admits a symplectic filling.

Note that, under our restrictions, \( \mathbb{H} \) can be expanded as
(7.3)
\[
\mathbb{H} = \sum_{k, \ell \geq 1, g \geq 0} H_{k, \ell, g} \hbar^{g-1},
\]
where \( H_{k, \ell, g} \) is the part of the coefficient of \( \hbar^{g-1} \) which has degree \( k \) in the \( p \)'s and degree \( \ell \) in the \( q \)'s.

Consider now the free \( R \)-module \( C \) generated by the elements \( q_\gamma \) for \( \gamma \in \mathcal{P} \), and graded by the degrees \( \deg(q_\gamma) := |q_\gamma| + 1 \). Then \( EC = \bigoplus_{k \geq 1} E_k C \), defined as in Section 2, is the non-unital commutative algebra of polynomials in the variables \( \{ q_\gamma \} \in \mathcal{P} \) without constant terms. We can represent \( W \) as differential operators acting on the left on \( EC \{ \hbar \} \) by the replacements
\[
p_\gamma \longrightarrow h \kappa_\gamma \frac{\partial}{\partial q_\gamma}.
\]

Then the Hamiltonian \( \mathbb{H} \) determines operations
(7.4)
\[
p_{k, \ell, g} := \frac{1}{h^k} \overrightarrow{H_{k, \ell, g}} : E_k C \to E_\ell C.
\]
The fact that the coefficients of $H$ are polynomial in the $q_\gamma$’s translates into
\begin{equation}
(7.5)
\end{equation}
Given $k \geq 1$, $g \geq 0$ and $a \in E_k C$, the term $p_{k,\ell,g}(a)$
is nonzero for only finitely many $\ell \geq 1$.

Conversely, $H$ can be recovered from the operations $p_{k,\ell,g}$ by
\begin{equation}
(7.6)
H_{k,\ell,g} = \sum_{\gamma_1,\ldots,\gamma_k \in \mathcal{P}} \frac{1}{\kappa_{\gamma_1} \cdots \kappa_{\gamma_k}} p_{k,\ell,g}(q_{\gamma_1} \cdots q_{\gamma_k}) p_{\gamma_1} \cdots p_{\gamma_k}.
\end{equation}

**Proposition 7.2.** Equations (7.4) and (7.6) define a one-to-one correspondence between differential Weyl algebras satisfying (7.2) and $\text{IBL}_\infty$-algebras satisfying (7.5) (both of degree $d$).

**Proof.** In the present context, the operator $\hat{p}$ appearing in Definition 2.3 can be written as
\begin{equation}
\hat{p} = \sum_{k,\ell,g} H_{k,\ell,g} h^{g-1} : EC\{h\} \rightarrow EC\{h\}.
\end{equation}
(The condition (7.5) allows us to set $\tau = 1$ in $\hat{p}$.) It is easily checked that $H \ast \hat{H} = 0$ is equivalent to $\hat{p} \circ \hat{p} = 0$. \hfill \Box

**Morphisms.** Next suppose $(\mathcal{W}^+,\mathbb{H}^+)$ and $(\mathcal{W}^-,\mathbb{H}^-)$ are differential Weyl algebras of the same degree $d$ with indexing sets $\mathcal{P}^+$ and $\mathcal{P}^-$. Let $\mathcal{D}$ denote the graded commutative associative algebra of power series in the $p^+$ and $h$ with coefficients polynomial in the $q^-$. By definition, a **morphism between the differential Weyl algebras** is an element $F \in \frac{1}{h}\mathcal{D}$ satisfying
\begin{equation}
(7.7)
e^{-F}(H^- e^F - e^F H^+) = 0.
\end{equation}
Here $\mathbb{H}^+$ acts on $e^F$ from the right by replacing each $q^+_\gamma$ by $h \kappa_{\gamma} \frac{\partial}{\partial p^+_\gamma}$, and the expression is to be viewed as an equality of elements of $\frac{1}{h}\mathcal{D}$. Again, we impose the additional condition that
\begin{equation}
(7.8)F|_{p^+=0} = 0 \quad \text{and} \quad F|_{q^-=0} = 0.
\end{equation}

**Remark 7.3.** In SFT, the first condition in (7.8) is satisfied for potentials coming from exact cobordisms, and the second one can be arranged in the augmented case. Moreover, the potential of a general (augmented) symplectic cobordism can also be viewed as a morphism in the above sense by splitting off the part $A = F|_{p^+=0}$, which is treated as a Maurer-Cartan element in the $\text{IBL}_\infty$-algebra associated to the negative end; cf. [20]. The remaining part $F - A$ then gives a morphism from $(\mathcal{W}^+,\mathbb{H}^+)$ to the twisted version $(\mathcal{W}^-,\mathbb{H}^-)$. 

As with $H$ above, we expand $F$ as

$$F = \sum_{k,\ell,g} F_{k,\ell,g} \hbar^{g-1}, \tag{7.9}$$

and define operators $\overrightarrow{F_{k,\ell,g}} : E_k C^+ \to E_\ell C^-$ by substituting $p_\gamma^+ \text{ by } \hbar \kappa_\gamma \overrightarrow{\frac{\partial}{\partial q_\gamma^+}}$. In this way, we get maps

$$f_{k,\ell,g} := \frac{1}{\hbar k} F_{k,\ell,g} : E_k C^+ \to E_\ell C^- \tag{7.10}$$

satisfying the condition

$$\text{Given } k \geq 1, g \geq 0 \text{ and } a \in E_k C^+, \text{ the term } f_{k,\ell,g}(a) \text{ is nonzero for only finitely many } \ell \geq 1. \tag{7.11}$$

Again, $F$ can be recovered from the operations $f_{k,\ell,g}$ by

$$F_{k,\ell,g} = \sum_{\gamma_1,\ldots,\gamma_k \in P^+} \frac{1}{\kappa_{\gamma_1} \cdots \kappa_{\gamma_k}} f_{k,\ell,g}(q_{\gamma_1}^+ \cdots q_{\gamma_k}^+) p_{\gamma_1}^+ \cdots p_{\gamma_k}^+. \tag{7.12}$$

**Proposition 7.4.** Equations (7.10) and (7.12) define a one-to-one correspondence between morphisms of differential Weyl algebras satisfying (7.8) and morphisms of IBL$_\infty$-algebras satisfying (7.11).

**Proof.** Again one checks easily that equation (7.7) translates into equation (2.11) relating the exponential of

$$\frac{1}{\hbar k} F_{k,\ell,g} \left|_{q^+=0} \right. \tag{7.7}$$

and the operators $\overrightarrow{p^\pm}$ (where again we have set $\tau = 1$). For the computation, it is useful to observe that for any monomial $Q$ in the $q^+$ we have

$$e^f(Q) = (e^F Q) \bigg|_{q^+=0}. \tag{7.11}$$

Moreover, $e^F H^+ = e^F \circ H^+$, and similarly $H^- e^F = H^- \circ e^F$, which follows easily from the definitions. \hfill \square

The composition $F^- \circ F^+$ of morphisms $F^+$ from $(W^+, H^+)$ to $(W, H)$ and $F^-$ from $(W, H)$ to $(W^-, H^-)$ is the morphism $F$ from $(W^+, H^+)$ to $(W^-, H^-)$ defined as the unique solution of

$$e^F = (e^{F^-}) \ast (e^{F^+})|_{q=p=0}. \tag{7.7}$$

Here the star product is with respect to the middle variables $p$ and $q$, and one checks that indeed $F \in \frac{1}{\hbar} D$ as required. We leave it to the reader to check that this agrees with composition of IBL$_\infty$-morphisms.
**Homotopies.** For the discussion of homotopies it is convenient to extend the definitions of Weyl algebras and morphisms between them from ordinary ground rings \( R \) to differential graded ground rings \( (\hat{R}, d) \).

In general, a Weyl algebra \( \hat{W} \) over a differential graded ring \( (\hat{R}, d) \) consists of power series in variables \( \{ p_\gamma \} \) with coefficients polynomial over \( \hat{R} \) in variables \( \{ q_\gamma \} \), with the same grading and commutation relations as before. A Hamiltonian in this context is now a homogeneous element \( \hat{H} \in \hat{W} \) of degree \(-1\) satisfying the generalized master equation

\[
d\hat{H} + \hat{H} \cdot \hat{H} = 0,
\]

where \( d \) is the differential in the ring, as well as our standing assumption

\[
\hat{H}|_{p=0} = 0, \quad \hat{H}|_{q=0} = 0.
\]

We let \( \hat{C} \) be the free graded \( \hat{R} \)-module generated by the elements \( q_\gamma \) and \( E\hat{C} = \bigoplus_{k \geq 1} E_k\hat{C} \), where the tensor products are taken over the differential graded ring \( \hat{R} \). Representing elements of \( \hat{W} \) as differential operators as before, the generalized master equation (7.13) ensures that the operations

\[
p_{k,\ell,g} := \frac{1}{\hbar^k} \overrightarrow{\hat{H}_{k,\ell,g}} : E_k\hat{C} \to E_{\ell}\hat{C},
\]

together with the differential \( d \) on the coefficients, determine a differential \( d + \hat{p} \) on \( E\hat{C}\{\hbar\} \) which squares to zero.

Given differential graded Weyl algebras \( (\hat{W}^+, \hat{H}^+) \) over the differential graded ring \( \hat{R}^+ \) and \( (\hat{W}^-, \hat{H}^-) \) over the differential graded ring \( \hat{R}^- \), we let \( \hat{D} \) denote the power series in the \( p^+ \) with coefficients polynomial over \( \hat{R}^- \) in the \( q^- \). A morphism between the differential graded Weyl algebras now consists of a morphism of differential graded rings \( \rho : \hat{R}^+ \to \hat{R}^- \) and an element \( G \in \frac{1}{\hbar}\hat{D} \) satisfying

\[
e^{-G} \left( d e^G + \overrightarrow{\hat{H}} e^G - e^G e^{\rho(\hat{H}^+)} \right) = 0,
\]

as well as

\[
G|_{p^+=0} = 0, \quad G|_{q^-} = 0.
\]

As above, this induces a morphism \( g \) from \( (E\hat{C}^+, \hat{p}^+) \) to \( (E\hat{C}^-, \hat{p}^-) \) satisfying

\[
(d + \hat{p}^-) e^G - e^G (d + \hat{p}^+) = 0.
\]

We will apply these generalizations as follows. Associated to a given ground ring (without differential) \( R \), we now introduce the differential graded ring \( (R[s, ds], d) \) where

\[
|s| = 0, \quad |ds| = -1, \quad \text{and} \quad d(s) = ds, \quad d(ds) = 0.
\]
Thinking of elements in $R[s,ds]$ as polynomial functions $f(s,ds)$ with values in $R$, we have morphisms

$$R \xrightarrow{j} (R[s,ds], d) \xrightarrow{e_i} R, \quad i = 0, 1$$

defined by

$$j(r) = r, \quad e_i(f(s,ds)) = f(i,0).$$

They satisfy $e_i \circ j = \text{id}_R$ and $j \circ e_i \sim \text{id}_{R[s,ds]}$, where a chain homotopy $H : R[s,ds] \to R[s,ds]$ with $dH + Hd = \text{id} - je_i$ is given by the integration map $g(s) + h(s)ds \mapsto \int_s^h g(s)ds$.

Now given any differential Weyl algebra $(\mathcal{W}, \mathbb{H})$ over the ring (without differential) $R$ generated by $\{p_\gamma\}_{\gamma \in \mathcal{P}}$ and $\{q_\gamma\}_{\gamma \in \mathcal{P}}$, we consider the new differential Weyl algebra $(\hat{\mathcal{W}}, \hat{\mathbb{H}})$ over $R[s,ds]$, where $\hat{\mathcal{W}} = R[s,ds] \otimes \mathcal{W} \equiv \mathcal{W}[s,ds]$, and we view $\hat{\mathbb{H}} = \mathbb{H}$ as an element independent of $s$ and $ds$. The generalized master equation (7.13) for $\hat{\mathbb{H}}$ follows directly from the corresponding equation (7.1) for $\mathbb{H}$.

Note that $(\mathcal{W}[s,ds], \hat{\mathbb{H}})$ corresponds to an IBL$_\infty$-algebra whose underlying $R[s,ds]$-module is $C[s,ds] = R[s,ds] \otimes C$. Since we take tensor products over $R[s,ds]$, we have

$$E(C[s,ds]) = (EC)[s,ds],$$

so notations are not too ambiguous. Graded commutativity inserts the usual signs when $ds$ is moved past some $q_\gamma$ or $p_\gamma$. As already mentioned before, the differential takes the form $d + \hat{p} : EC[s,ds]\{h\} \to EC[s,ds]\{h\}$, where $\hat{p}$ is induced from $\hat{\mathbb{H}} = \mathbb{H}$ as above.

Associated to the above ring morphisms $j : R \to (R[s,ds], d)$ and $e_i : (R[s,ds], d) \to R$ there are morphisms $\hat{J} : (\mathcal{W}, \mathbb{H}) \to (\mathcal{W}[s,ds], \hat{\mathbb{H}})$ and $\hat{E}_i : (\mathcal{W}[s,ds], \hat{\mathbb{H}}) \to (\mathcal{W}, \mathbb{H})$ which act as the identity on $C$. Explicitly, the associated power series in all three cases is

$$\frac{1}{h} \sum_{\gamma \in \mathcal{P}} q_\gamma p_\gamma,$$

and the nontrivial part comes from the action on the coefficients. Note that they are linear morphisms of Weyl algebras, in the sense that they equal their $(1,1,0)$-terms and satisfy

$$e^{-J} \left( \hat{p} e^J - e^J \hat{p} \right) = 0, \quad e^{-E_i} \left( \hat{p} e^{E_i} - e^{E_i} \hat{p} \right) = 0.$$

Moreover, $E_i \circ J = \text{id}\mathcal{W}$ for $i = 0, 1$.

**Definition 7.5.** We define a homotopy between two morphisms $F_0$ and $F_1$ from $(\mathcal{W}^+, \mathbb{H}^+)$ to $(\mathcal{W}^-, \mathbb{H}^-)$ as a morphism $G$ from $(\mathcal{W}^+, \mathbb{H}^+)$ to $(\mathcal{W}^-, \mathbb{H}^-)$
which on coefficients corresponds to the inclusion \( j : R \to (\mathbb{R}[s, ds], d) \), and such that
\[
\mathcal{E}_i \circ G = F_i, \quad i = 0, 1.
\]

According to our definitions, such a \( G \) is a power series in the \( p^+ \) with coefficients polynomial in the \( q^- \), \( s \) and \( ds \). Therefore it can be written in the form
\[
(7.15) \quad G = F(q^-, s, p^+) + ds \mathbb{K}(q^-, s, p^+).
\]

It satisfies the equation
\[
0 = e^{-G} \left( de^G + \overrightarrow{\mathbb{H}^-} e^G - e^G \overrightarrow{\mathbb{H}^+} \right)
\]
\[
= e^{-F(s)}(1 - ds \mathbb{K}(s)) \left( ds \frac{\partial}{\partial s} e^{F(s)} + \overrightarrow{\mathbb{H}^-} (e^{F(s)}(1 + ds \mathbb{K}(s)))
\]
\[
- (e^{F(s)}(1 + ds \mathbb{K}(s))) \overrightarrow{\mathbb{H}^+} \right),
\]
which splits into the two equations
\[
0 = e^{-F(s)} \left( \overrightarrow{\mathbb{H}^-} e^{F(s)} - e^{F(s)} \overrightarrow{\mathbb{H}^+} \right) \quad \text{and}
\]
\[
0 = e^{-F(s)} \left( \frac{\partial}{\partial s} e^{F(s)} - \overrightarrow{\mathbb{H}^-} (\mathbb{K}(s) e^{F(s)}) - (e^{F(s)} \mathbb{K}(s)) \overrightarrow{\mathbb{H}^+} \right)
\]
\[
- e^{-F(s)} \mathbb{K}(s) \left( \overrightarrow{\mathbb{H}^-} e^{F(s)} - e^{F(s)} \overrightarrow{\mathbb{H}^+} \right)
\]
\[
= e^{-F(s)} \left( \frac{\partial}{\partial s} e^{F(s)} - [\overrightarrow{\mathbb{H}^-}, \mathbb{K}(s)] e^{F(s)} - e^{F(s)} [\mathbb{K}(s), \mathbb{H}^+] \right).
\]

Note that the second equation and the fact that the \( \mathbb{H}^\pm \) are Hamiltonians imply that
\[
\frac{\partial}{\partial s} \left( e^{-F(s)} \left( \overrightarrow{\mathbb{H}^-} e^{F(s)} - e^{F(s)} \overrightarrow{\mathbb{H}^+} \right) \right) = 0.
\]

So together with the initial condition that \( F(0) \) is a morphism it implies the first equation.

Summarizing the above discussion (and recalling \( \mathbb{H}^- = \mathbb{H}^- \)), we see:

**Lemma 7.6.** Two Weyl algebra morphisms \( F_0, F_1 : \mathcal{W}^+, \mathbb{H}^+ \to (\mathcal{W}^-, \mathbb{H}^-) \) are homotopic in the sense of Definition 7.5 if and only if there exists
\[
G = F(q^-, s, p^+) + ds \mathbb{K}(q^-, s, p^+)
\]

such that
\[
F(q^-, 0, p^+) = F_0(q^-, p^+), \quad F(q^-, 1, p^+) = F_1(q^-, p^+)
\]
and

\[ 0 = \frac{\partial}{\partial s} e^F(s) - \left[ H^{-}, K(s) \right] e^F(s) - e^F(s) \left[ K(s), H^{+} \right]. \]

**Remark 7.7.** In SFT, one works in the slightly more general context of not necessarily augmented morphisms, and equation (7.16) is taken as the definition of homotopy between morphisms of Weyl algebras, cf. \[28\], p. 629.

Now we have the following:

**Proposition 7.8.** Consider two differential Weyl algebras \((W^+, H^+)\) and \((W^-, H^-)\), and denote by \((C^+, \{p^+_{k,\ell,g}\})\) and \((C^-, \{p^-_{k,\ell,g}\})\) the corresponding \(IBL_{\infty}\)-algebras, respectively. Let \(F_0, F_1 : (W^+, H^+) \to (W^-, H^-)\) be Weyl algebra morphisms and denote by \(f^{(0)} = \{f^{(0)}_{k,\ell,g}\}\) and \(f^{(1)} = \{f^{(1)}_{k,\ell,g}\}\) the corresponding morphisms \((C^+, \{p^+_{k,\ell,g}\}) \to (C^-, \{p^-_{k,\ell,g}\})\) of \(IBL_{\infty}\)-algebras, respectively.

Then \(F_0\) is homotopic to \(F_1\) in the sense of Definition 7.5 if and only if \(f^{(0)}\) is homotopic to \(f^{(1)}\) in the sense of Definition 4.5.

The proof uses the following lemma.

**Lemma 7.9.** Let \((C, \{p_{k,\ell,g}\})\) be an \(IBL_{\infty}\)-algebra over \(R\), and let \((C[s, ds], \{p_{k,\ell,g}\})\) be the corresponding \(IBL_{\infty}\)-algebra over \((R[s, ds], d)\). Let \((\mathcal{C}, \{q_{k,\ell,g}\}), \iota, \varepsilon_i\) be a path object for \((C, \{p_{k,\ell,g}\})\). Then there exists a morphism

\[ a = \{a_{k,\ell,g}\} : (\mathcal{C}, \{q_{k,\ell,g}\}) \to (C[s, ds], \{p_{k,\ell,g}\}), \]

corresponding to \(j : R \to R[s, ds]\) on coefficients, which makes the following diagram commute:

\[
\begin{array}{c}
C \xrightarrow{\iota} \mathcal{C} \xrightarrow{\varepsilon_1} C \\
\downarrow \id \quad \downarrow a \quad \downarrow \id \\
C \xrightarrow{j} C[s, ds] \xrightarrow{\varepsilon_1} C
\end{array}
\]

Similarly, there is a morphism

\[ b = \{b_{k,\ell,g}\} : (C[s, ds], \{p_{k,\ell,g}\}) \to (\mathcal{C}, \{q_{k,\ell,g}\}), \]

corresponding to \(e_0 : R[s, ds] \to R\) on coefficients, which makes the following diagram commute:

\[
\begin{array}{c}
C \xrightarrow{j} C[s, ds] \xrightarrow{\varepsilon_1} C \\
\downarrow \id \quad \downarrow b \quad \downarrow \id \\
C \xrightarrow{\iota} \mathcal{C} \xrightarrow{\varepsilon_1} C
\end{array}
\]
Proof. $C[s, ds]$ is not a path object for $C$ in the sense of Definition 4.4 (they are even defined over different rings), but still it is true that

(a) $j$, $e_0$ and $e_1$ are linear morphisms (and we denote their $(1, 1, 0)$ parts by the same letters);
(b) $e_i \circ j = \text{id}_C$ and $j \circ e_i \sim \text{id}_{C[s, ds]}$;
(c) $j : C \to C[s, ds]$ and $e_i : C[s, ds] \to C$ are homotopy equivalences (of chain complexes over $R$ with differentials $p_{1,1,0}$ and $d + p_{1,1,0}$, respectively);
(d) the map $e_0 \oplus e_1 : C[s, ds] \to C \oplus C$ admits the linear right inverse $(c_0, c_1) \mapsto c_0 + (c_1 - c_0)s$.

So while the lemma is not a particular case of Proposition 4.4, the proof there can be adapted to the present situation. \hfill \square

Proof of Proposition 7.8. If $G = F(q^-, s, p^+) + ds \mathbb{K}(q^-, s, p^+)$ is a homotopy between $F_0$ and $F_1$ in the sense of Definition 7.5, and $G$ is the corresponding IBL$\infty$-morphism from $C^+$ to $C^-[s, ds]$, then the composition $H := b \circ G$ is the required morphism from $C^+$ to $C^-$ with $\varepsilon_i \circ H = f(i)$.

Similarly, if $H$ is a homotopy in the sense of Definition 4.5, then $G := a \circ H$ is an IBL$\infty$-morphism from $C^+$ to $C^-[s, ds]$ whose Weyl algebra translation satisfies Definition 7.5. \hfill \square

8. Filtered IBL$\infty$-structures

For many applications the notion of an IBL$\infty$-structure needs to be generalized to that of a filtered IBL$\infty$-structure. In this section we define this generalization and extend our previous results to this case. This refinement is also necessary for the discussion of Maurer-Cartan elements in Section 9.

Filtrations. A filtration on a commutative ring $R$ is a family of subrings

$$R \supset R_\lambda \supset R_\mu, \quad \lambda \leq \mu, \quad \bigcup_{\lambda \in \mathbb{R}} R_\lambda = R, \quad R_\lambda \cdot R_\mu \subset R_{\lambda + \mu}.$$ 

Such a filtration is equivalent to a valuation\footnote{Commonly, a valuation is required to satisfy $\|r\| \geq 0$ and $\|rr'\| = \|r\| + \|r'\|$, but we will not need these stronger conditions.} $\| : R \to \mathbb{R} \cup \{\infty\}$ satisfying $\|0\| = \infty$ and

$$\|r + r'\| \geq \min\{\|r\|, \|r'\|\}, \quad \|rr'\| \geq \|r\| + \|r'\|$$

for $r, r' \in R$. The two notions are related by

$$\|r\| = \sup\{\lambda \mid r \in R_\lambda\}, \quad R_\lambda = \{r \mid \|r\| \geq \lambda\}.$$
The trivial filtration on a ring $R$ is defined by the trivial valuation $\|r\| = 0$ for all $r \neq 0$. Another example of a filtered ring is the universal Novikov ring considered later in this section.

A filtration on an $R$-module $C$ is a family of $R$-linear subspaces

$$C \supseteq \mathcal{F}^\lambda C \supseteq \mathcal{F}^\mu C, \quad \lambda \leq \mu, \quad \bigcup_{\lambda \in \mathbb{R}} \mathcal{F}^\lambda = C, \quad R \cdot \mathcal{F}^\mu C \subseteq \mathcal{F}^{\lambda + \mu} C.$$

Again, this is equivalent to a valuation $\| \| : C \rightarrow \mathbb{R} \cup \{ \infty \}$ satisfying

$$\|c + c'\| \geq \min\{\|c\|, \|c'\|\}, \quad \|rc\| \geq \|r\| + \|c\|$$

for $c, c' \in C$ and $r \in R$, where the two notions are related by

$$\|c\| = \sup\{\lambda \mid c \in \mathcal{F}^\lambda C\}, \quad \mathcal{F}^\lambda C = \{c \mid \|c\| \geq \lambda\}.$$

If $C$ is an $R$-algebra, we require in addition that $\mathcal{F}^\lambda C \cdot \mathcal{F}^\mu C \subseteq \mathcal{F}^{\lambda + \mu} C$, or equivalently, $\|c \cdot c'\| \geq \|c\| + \|c'\|$.

The completion of $C$ with respect to $\mathcal{F}$ is the completion with respect to the metric $d(c, c') := e^{-\|c - c'\|}$, i.e., the $R$-module

$$\hat{C} := \left\{\sum_{i=1}^{\infty} c_i \mid c_i \in C, \lim_{i \rightarrow \infty} \|c_i\| = \infty\right\}$$

$$= \left\{\sum_{i=1}^{\infty} c_i \mid c_i \in C, \#\{i \mid c_i \notin \mathcal{F}^\lambda C\} < \infty \text{ for all } \lambda \in \mathbb{R}\right\}.$$

Note that $\hat{C}$ inherits a filtration from $C$.

For example, the completion of a direct sum $C = \bigoplus_{k \geq 0} C^k$ with respect to the filtration $\mathcal{F}^\lambda C := \bigoplus_{k \geq \lambda} C^k$ is the direct product $\hat{C} = \prod_{k \geq 0} C^k$ with the induced filtration

$$\mathcal{F}^\lambda \hat{C} = \{(c_k) \in \hat{C} \mid c_k = 0 \text{ for all } k < \lambda\}.$$

A linear map $f : C \rightarrow C'$ between filtered $R$-modules is called filtered if it satisfies

$$f(\mathcal{F}^\lambda C) \subseteq \mathcal{F}^{\lambda + K} C'$$

for some constant $K \in \mathbb{R}$. In this case we call the largest such constant the (filtration) degree of $f$ and denote it by $\|f\|$.

Filtrations on $C$ and $C'$ induce filtrations on the direct sum and tensor product by

$$\mathcal{F}^\lambda (C \oplus C') := \mathcal{F}^\lambda C \oplus \mathcal{F}^\lambda C',$$

$$\mathcal{F}^\lambda (C \otimes C') := \bigoplus_{\lambda_1 + \lambda_2 = \lambda} \mathcal{F}^{\lambda_1} C \otimes \mathcal{F}^{\lambda_2} C'.$$

3 The filtration degree $\| \|$ should not be confused with the grading $\mid \mid$ on $C$, which plays no role in this section.
Given several filtrations \( F_j^\lambda C \) on \( C \), we denote by \( \hat{C} \) the completion with respect to the filtration \( F^\lambda C := \bigcup_j F_j^\lambda C \).

A filtration on \( C \) induces filtrations on the symmetric products
\[
F^\lambda (C[1] \otimes_R \cdots \otimes_R C[1]) / \sim := \bigoplus_{\lambda_1 + \cdots + \lambda_k = \lambda} (F^{\lambda_1} C[1] \otimes_R \cdots \otimes_R F^{\lambda_k} C[1]) / \sim
\]
We denote by \( \hat{E}_k C \) the completion of the \( k \)-fold symmetric product \((C[1] \otimes_R \cdots \otimes_R C[1]) / \sim\) with respect to this filtration. We now also include the case \( \hat{E}_0 C := R \) (where \( R \) is assumed complete).

Note that the symmetric algebra \( \bigoplus_{k \geq 0} \hat{E}_k C \) has two filtrations: the one induced by \( F \), and the filtration by the sets \( \bigoplus_{k \geq \lambda} \hat{E}_k C \). We denote by \( \hat{E}C \) the completion of \( \bigoplus_{k \geq \lambda} \hat{E}_k C \) with respect to these two filtrations. Thus elements in \( \hat{E}C \) are infinite sums \( \sum_{i=1}^{\infty} c_i \) such that \( c_i \in F^{\lambda_i} \hat{E}_{k_i} C \) with
\[
\lim_{i \to \infty} \max\{k_i, \lambda_i\} = \infty.
\]

Given a map \( p : \hat{E}_k C \to \hat{E}_\ell C \) of finite filtration degree \( \|p\| \), formula (2.1) extends to the completion to define a map \( \hat{p} : \hat{E}C \to \hat{E}C \).

For the remainder of this section, \( R \) will denote a complete filtered commutative ring, and \( C \) a filtered \( R \)-module.

Filtered IBL_\infty-algebras. Consider now a collection of maps
\[
p_{k,\ell,g} : \hat{E}_k C \to \hat{E}_\ell C, \quad k, \ell, g \geq 0
\]
of finite filtration degrees satisfying
\[
\|p_{k,\ell,g}\| \geq \gamma \chi_{k,\ell,g} \text{ for all } k, \ell, g.
\]
Here \( \gamma \geq 0 \) is a fixed constant and
\[
\chi_{k,\ell,g} := 2 - 2g - k - \ell
\]
is the Euler characteristic of a Riemann surface of genus \( g \) with \( k \) positive and \( \ell \) negative boundary components. Note that, in contrast to the unfiltered case, we allow \( k = 0 \) and \( \ell = 0 \). In Section 12 we will use \( \gamma = 2 \).

Define
\[
\hat{p} := \sum_{k,\ell,g=0}^{\infty} \hat{p}_{k,\ell,g} h^{k+g-1} : \hat{E}C\{h\} \to \frac{1}{h}\hat{E}C\{h\},
\]
where \( \hat{E}C\{h\} \) denotes the space of power series in \( h \) with coefficients in \( \hat{E}C \). To see that \( \hat{p} \) is well-defined, note that elements of \( \hat{E}C\{h\} \) are given by \( c = \sum_{\ell',g' \geq 0} c_{\ell',g'} h^{g'} \) with \( c_{\ell',g'} \in \hat{E}_{\ell'} C \). Then
\[
\hat{p}(c) = \sum_{\ell'' \geq 0, g'' \geq -1} c_{\ell'',g''} h^{g''}
\]
\[c_{\ell'',g''} = \sum_{k+g+g'-1 = \ell''} \hat{p}_{k,\ell,g}(c_{\ell',g'}) \in \hat{E}_{\ell'}C.\]

We need to show that for \(\ell'', g''\) and \(\|c_{\ell',g'}\|\) bounded from above only finitely many terms can appear on the right hand side. Since \(k,g,g' \geq 0\), the relation \(k + g + g' - 1 = g''\) bounds \(k,g,g'\) in terms of \(g''\). Then the relation \(\ell + \ell' - k = \ell''\) bounds \(\ell,\ell' \geq 0\) in terms of \(g'', \ell''\). In particular, \(\chi_{k,\ell,g}\) is bounded. So

\[\|\hat{p}_{k,\ell,g}(c_{\ell',g'})\| \geq \|p_{k,\ell,g}\| + \|ce_{\ell',g'}\| \geq \gamma \chi_{k,\ell,g} + \|ce_{\ell',g'}\|\]

bounds \(\|c_{\ell',g'}\|\) from above in terms of \(\ell'', g''\) and \(\|c_{\ell',g'}\|\), so by convergence of \(c\) only finitely many such terms appear.

**Definition 8.1.** A filtered IBL\(_\infty\)-structure of bidegree \((d, \gamma)\) on a filtered graded \(R\)-module \(C\) is a collection of maps

\[p_{k,\ell,g}: \hat{E}_kC \to \hat{E}_{\ell}C, \quad k,\ell,g \geq 0\]

of grading degrees \(-2d(k + g - 1) - 1\) and filtration degrees satisfying (8.1), where the inequality is strict for the following triples \((k, \ell, g)\):

\[(8.2) (0,0,0), (1,0,0), (0,1,0), (0,0,1), (2,0,0), (0,2,0),\]

such that

\[\hat{p} \circ \hat{p} = 0.\]

A filtered IBL\(_\infty\)-structure is called strict if \(p_{k,\ell,g} = 0\) unless \(k,\ell \geq 1\).

We remark that the coefficient of \(\hbar^{-2}\) of the \(\hat{E}_0C \to \hat{E}_0C\) component of \(\hat{p} \circ \hat{p}\) is \(p_{0,0,0}^2\). So \(p_{0,0,0} = 0\) automatically. Inductively, it follows that \(p_{0,0,g} = 0\) for all \(g \geq 0\).

**Remark 8.2.** We could absorb the constant \(\gamma\) by giving the variable \(\hbar\) filtration degree

\[\|\hbar\|_{\text{new}} := 2\gamma\]

and shifting the filtration on \(C\) by

\[\|c\|_{\text{new}} := \|c\| + \gamma.\]

Then we obtain

\[\|p_{k,\ell,g}(x_1 \cdots x_k)\hbar^{k+g-1}\|_{\text{new}} = \|p_{k,\ell,g}(x_1 \cdots x_k)\| + \gamma (2k + 2g - 2 + \ell)\]

\[\geq \|x_1 \cdots x_k\| + \gamma (2 - 2g - k - \ell) + \gamma (2k + 2g - 2 + \ell)\]

\[= \|x_1 \cdots x_k\|_{\text{new}},\]

hence \(\hat{p}: \hat{EC}\{\hbar\} \to \frac{1}{\hbar} \hat{EC}\{\hbar\}\) has filtration degree \(\|\hat{p}\|_{\text{new}} \geq 0\). In this paper we will not use this filtration, but rather carry out inductive proofs explicitly by considering connected surfaces as in the discussion preceding Definition 8.1.
**Filtered $\text{IBL}_\infty$-morphisms.** Consider two filtered $R$-modules $(C^\pm, p^\pm)$ and a collection of maps

$$f_{k, \ell, g} : \widehat{E}kC^+ \to \widehat{E}\ell C^-, \quad k, \ell, g \geq 0$$

of finite filtration degrees satisfying

$$\|f_{k, \ell, g}\| \geq \gamma \chi_{k, \ell, g} \quad \text{for all} \quad k, \ell, g,$$

where the inequality is strict for the triples in (8.2). Here $\gamma \geq 0$ is the fixed constant from above, and again we allow $k = 0$ and $\ell = 0$.

**Lemma 8.3.** Given $(C^\pm, p^\pm)$ and $f_{k, \ell, g}$ as above, there exist unique collections of maps $q^\pm_{k, \ell, g} : \widehat{E}kC^+ \to \widehat{E}\ell C^-$ satisfying (8.1) such that

$$e\hat{p}^+ = \hat{q}^+, \quad \hat{p}^- e\hat{f} = \hat{q}^- : \widehat{EC}^+ \{h\} \to \widehat{EC}^- \{h, h^{-1}\}.$$ 

where $\hat{q}^\pm$ is defined by

$$\hat{q}^\pm := \sum_{r=1}^{\infty} \sum_{k_i \ell_i g_i \geq 1}^r \frac{1}{(r-1)!} f_{k_1 \ell_1 g_1} \circ \cdots \circ f_{k_r \ell_r g_r} \circ q^\pm_{k_r \ell_r g_r} h^{\sum k_i + \sum g_i - r}.$$

**Proof.** Let us consider the composition $\hat{p}^- e\hat{f} = \hat{q}^-$, the other one being analogous. According to Lemma 2.10, the map $q^-_{k, \ell, g}$ is given by the sum

$$\sum_{r=0}^{\infty} \sum_{k_1 + \cdots + k_r = k} \sum_{\ell_1 + \cdots + \ell_r = \ell} \sum_{g_1 + \cdots + g_r = g} \sum_{s_1 + \cdots + s_r = k^-} \frac{1}{r!} p^-_{k^-, \ell^-, g^-} \circ s_1, \ldots, s_r (f_{k_1 \ell_1 g_1} \circ \cdots \circ f_{k_r \ell_r g_r})$$

corresponding to complete gluings of $r$ connected surfaces of signatures $(k_i, \ell_i, g_i)$ at their outgoing ends to the ingoing ends of a connected surface of signature $(k^-, \ell^-, g^-)$, plus an appropriate number of trivial cylinders, to obtain a connected surface of signature $(k, \ell, g)$. In particular, for each term in this sum the Euler characteristics satisfy

$$\chi_{k, \ell, g} = \chi_{k^-, \ell^-, g^-} + \sum_{i=1}^r \chi_{k_i, \ell_i, g_i}.$$ 

Let us write $\{1, \ldots, r\}$ as the disjoint union $I \cup J \cup K$, where $i \in I$ iff $(k_i, \ell_i, g_i)$ is one of the triples in (8.2), $i \in J$ iff $\chi_{k_i, \ell_i, g_i} < 0$, and $i \in K$ iff $(k_i, \ell_i, g_i) = (1, 1, 0)$. Let $\delta > 0$ be such that $\|f_{k_i, \ell_i, g_i}\| - \gamma \chi_{k_i, \ell_i, g_i} \geq \delta$ for all
i ∈ I. Then the filtration conditions on p− and f imply
\[ \|q_{k,\ell,g} - \gamma \chi_{k,\ell,g} \| \geq (\|p_{k-\ell,g} - \gamma \chi_{k-\ell,g} \| + \sum_{i=1}^{r} (\|f_{k_i,\ell_i,g_i} - \gamma \chi_{k_i,\ell_i,g_i} \|) \geq \delta |I| \geq 0. \]
(8.5)

This shows that |I| is uniformly bounded, where we say that a quantity is uniformly bounded if it is bounded from above in terms of k, ℓ, g and \( \|q_{k,\ell,g} \| \).

It follows that
\[ \sum_{j \in J} -\chi_{k_j,\ell_j,g_j} = -\chi_{k,\ell,g} + \chi_{k-\ell,g} + \sum_{i \in I} \chi_{k_i,\ell_i,g_i} \]
is uniformly bounded. Since each term \( 2g_j + k_j + \ell_j - 2 \) on the left-hand side is \( \geq 1 \), this provides uniform bounds on |J| as well as all the \( g_j, k_j, \ell_j \) for \( j \in J \). Finally, the fact that each \( i \in K \) contributes 1 to the sum \( k_1 + \cdots + k_r = k \) yields a uniform bound on |K|. Hence the number of terms in the sum in (8.4) is uniformly bounded, which proves convergence of \( q_{k,\ell,g} \) with respect to the filtration. Inequality (8.5) shows that \( q_{k,\ell,g} \) satisfies (8.1).

In view of the preceding lemma, the following definition makes sense.

**Definition 8.4.** A filtered IBL∞-morphism between filtered IBL∞-algebras \((C^+, \{p^+_{k,\ell,g}\}) \) is a collection of maps
\[ f_{k,\ell,g} : \tilde{E}_k C^+ \to \tilde{E}_\ell C^-, \quad k, \ell, g \geq 0 \]
of grading degrees \(-2d(k + g - 1)\) and filtration degrees satisfying (8.3) and (8.2) such that
\[ e^\ell \tilde{p}^+ - \tilde{p} - e^\ell = 0. \]
(8.6)

A filtered IBL∞-morphism \( f \) is called strict if \( f_{k,\ell,g} = 0 \) unless \( k, \ell \geq 1 \).

Note that for a strict filtered IBL∞-morphism or structure, condition (8.2) is vacuous.

**Composition of filtered IBL∞-morphisms.** Consider now two filtered IBL∞-morphisms
\[ f^+ = \{f^+_{k,\ell,g}\} : (C^+, \{p^+_{k,\ell,g}\}) \to (C, \{p_{k,\ell,g}\}), \]
\[ f^- = \{f^-_{k,\ell,g}\} : (C, \{p_{k,\ell,g}\}) \to (C^-, \{p^-_{k,\ell,g}\}). \]
Lemma 8.5. There exists a unique filtered IBL_{\infty}-morphism \( f = \{ f_{k,\ell,g} \} : (C^+, \{ k^+_{k,\ell,g} \}) \to (C^-, \{ k^-_{k,\ell,g} \}) \) satisfying

\[ e^f = e^{-} e^{f^+} . \]

We call \( f \) the composition of \( f^+ \) and \( f^- \).

Proof. According to the discussion following Definition 2.11 the map \( f_{k,\ell,g} \) is given by the sum

\[ \sum_{k^+_1 + \cdots + k^+_r = k} \frac{1}{r!} (f^-_{k^+_1,\ell^+_1,g^+_1} \circ \cdots \circ f^-_{k^+_r,\ell^+_r,g^+_r}) \circ (f_{k^+1,\ell^+_1,g^+_1} \circ \cdots \circ f_{k^+r,\ell^+_r,g^+_r}), \]

(8.7)

corresponding to complete gluings of \( r^+ \) connected surfaces of signatures \((k^+_1, \ell^+_1, g^+_1)\) at their outgoing ends to the ingoing ends of \( r^- \) connected surfaces of signatures \((k^-_1, \ell^-_1, g^-_1)\) to obtain a connected surface of signature \((k, \ell, g)\). In particular, for each term in this sum the Euler characteristics satisfy

\[ \chi_{k,\ell,g} = \sum_{i=1}^{r^+} \chi_{k^+_i,\ell^+_i,g^+_i} + \sum_{i=1}^{r^-} \chi_{k^-_i,\ell^-_i,g^-_i}. \]

Let us write \( \{1, \ldots, r^\pm\} \) as the disjoint union \( I^\pm \cup J^\pm \cup K^\pm \), where \( i \in I^\pm \) iff \((k^+_i, \ell^+_i, g^+_i)\) is one of the triples in \([3.2]\), \( i \in J^\pm \) iff \( \chi_{k^+_i,\ell^+_i,g^+_i} < 0 \), and \( i \in K^\pm \) iff \((k^-_i, \ell^-_i, g^-_i) = (1, 1, 0) \). Let \( \delta > 0 \) be such that \( \| f_{k^+_i,\ell^+_i,g^+_i} \| - \gamma \chi_{k^+_i,\ell^+_i,g^+_i} \geq \delta \) for all \( i \in I^\pm \). Then the filtration conditions on \( f^{\pm} \) imply

\[ \| f_{k,\ell,g} \| - \gamma \chi_{k,\ell,g} \geq \sum_{i=1}^{r^+} (\| f_{k^+_i,\ell^+_i,g^+_i} \| - \gamma \chi_{k^+_i,\ell^+_i,g^+_i}) + \sum_{i=1}^{r^-} (\| f_{k^-_i,\ell^-_i,g^-_i} \| - \gamma \chi_{k^-_i,\ell^-_i,g^-_i}) \]

(8.8)

\[ \geq \sum_{i \in I^+} (\| f_{k^+_i,\ell^+_i,g^+_i} \| - \gamma \chi_{k^+_i,\ell^+_i,g^+_i}) + \sum_{i \in I^-} (\| f_{k^-_i,\ell^-_i,g^-_i} \| - \gamma \chi_{k^-_i,\ell^-_i,g^-_i}) \]

This shows that \( |I^+| \) and \( |I^-| \) are uniformly bounded, i.e. bounded from above in terms of \( k, \ell, g \) and \( \| f_{k,\ell,g} \| \). It follows that

\[ \sum_{j \in J^+} -\chi_{k^+_j,\ell^+_j,g^+_j} + \sum_{j \in J^-} -\chi_{k^-_j,\ell^-_j,g^-_j} = -\chi_{k,\ell,g} + \sum_{i \in I^+} \chi_{k^+_i,\ell^+_i,g^+_i} + \sum_{i \in I^-} \chi_{k^-_i,\ell^-_i,g^-_i} \]

is uniformly bounded. Since each term \( 2g^+_j + k^+_j + \ell^+_j - 2 \) on the left-hand side is \( \geq 1 \), this provides uniform bounds on \( |J^+| \) and \( |J^-| \) as well as all the \( g^+_j, k^+_j, \ell^+_j \) for \( j \in J^\pm \). Finally, the fact that each \( i \in K^+ \) contributes 1 to the
sum $k^+_1 + \cdots + k^+_r = k$ and each $i \in K^-$ contributes 1 to the sum $\ell^-_1 + \cdots + \ell^-_r = \ell$ yields uniform bounds on $|K^+|$ and $|K^-|$. Hence the number of terms in the sum in (8.7) is uniformly bounded, which proves convergence of $q_{k,\ell,g}$ with respect to the filtration. Inequality (8.8) shows that $f_{k,\ell,g}$ satisfies (8.3), where the inequality is strict if $I^+$ or $I^-$ is nonempty. If $I^+$ and $I^-$ are both empty, then either $\chi_{k,\ell,g} < 0$ (if $J^+$ or $J^-$ are nonempty), or (if $J^+$ and $J^-$ are both empty) $(k^+_i, \ell^+_i, g^+_i) = (1, 1, 0)$ for all $i$ and hence $(k, \ell, g) = (1, 1, 0)$ (the corresponding connected surface is a gluing of cylinders and hence a cylinder). This shows that $f_{k,\ell,g}$ also satisfies (8.2).

**Homotopies of filtered IBL$_\infty$-algebras.** Consider a subset $G$ of $\mathbb{R}_{\geq 0}$ such that

1. $g_1, g_2 \in G$ implies $g_1 + g_2 \in G$;
2. $0 \in G$;
3. $G$ is a discrete subset of $\mathbb{R}$.

We call such $G$ a **discrete submonoid**, and we will write it as

\[
G = \{\lambda_0, \lambda_1, \ldots\},
\]

where $\lambda_j < \lambda_{j+1}$ and $\lambda_0 = 0$.

**Definition 8.6.** We say that filtered IBL$_\infty$-algebra of bidegree $(d, \gamma)$ is **G-gapped** if the operations $p_{k,\ell,g}$ can be written as

\[
p_{k,\ell,g} = \sum_{j=0}^{\infty} p^j_{k,\ell,g} : \hat{E}_k C \to \hat{E}_\ell C,
\]

where the filtration degrees of $p^j_{k,\ell,g}$ satisfy

\[
\|p^j_{k,\ell,g}\| - \gamma \chi_{k,\ell,g} \geq \lambda_j \in G,
\]

and $p^0_{k,\ell,g} = 0$ for the triples $(k, \ell, g)$ in (8.2). We call an IBL$_\infty$ algebra gapped if it is $G$-gapped for some discrete submonoid $G \subset \mathbb{R}_{\geq 0}$.

We define a linear ordering on **extended signatures** $(j, k, \ell, g) \in \mathbb{N}_0^4$ by saying $(j', k', \ell', g') \prec (j, k, \ell, g)$ if either $j' < j$, or $j' = j$ and $(k', \ell', g') \prec (k, \ell, g)$ in the sense of Definition 2.5.

**Remark 8.7.** As with the original ordering in Definition 2.5, this is only one of several possible choices.

Now we have the following analogue of Lemma 2.6.
LEMMA 8.8. For a gapped filtered IBL$_\infty$-algebra $(C, \{p_{k,\ell,g}\})$ the condition $\hat{p} \circ \hat{p} = 0$ is equivalent to $p^{0}_{1,1,0} \circ p^{0}_{1,1,0} = 0$, together with the sequence of relations

$$p^{0}_{1,1,0} \circ p^{0}_{k,\ell,g} + p^{0}_{k,\ell,g} \circ p^{0}_{1,1,0} + P^{j}_{k,\ell,g} + R^{i}_{k,\ell,g} = 0$$

as maps from $\hat{E}_k C$ to $\hat{E}_\ell C$ for all extended signatures $(j, k, \ell, g) > (0, 1, 1, 0)$, where $P^{j}_{k,\ell,g} : \hat{E}_k C \to \hat{E}_\ell C$ involves only compositions of terms $p^{j'}_{k',\ell',g'}$ whose extended signatures satisfy $(0, 1, 1, 0) < (j', k', \ell', g') < (j, k, \ell, g)$, and $\|R^{i}_{k,\ell,g}\| > \lambda_j + \gamma \chi_{k,\ell,g}$.

Proof. Recall that the left hand side of relation (2.4) is a sum of terms $\hat{p}_{s_2} \circ \hat{p}_{s_1} \circ \hat{p}_{p_{2},\ell_2,g_2}$ which correspond to gluings of two connected surfaces of signatures $\sigma_i = (k_i, \ell_i, g_i)$ along $s \geq 1$ boundary components to a connected surface of signature $\sigma = (k, \ell, g)$. We fix $j \geq 0$ and combine all terms in this sum of filtration degree $> \lambda_j + \gamma \chi_{\sigma}$ into one summand, which we denote by $R^{i}_{k,\ell,g}$. Next consider a term with

$$\|\hat{p}^{j}_{s_2} \circ \hat{p}^{j}_{s_1} \circ \hat{p}^{j}_{p_{2},\ell_2,g_2}\| \leq \lambda_j + \gamma \chi_{\sigma}$$

Then

$$\lambda_{j_2} + \gamma \chi_{\sigma_2} + \lambda_{j_1} + \gamma \chi_{\sigma_1} \leq \|\hat{p}^{j}_{s_2}\| + \|\hat{p}^{j}_{s_1}\| \leq \|\hat{p}^{j}_{s_2} \circ \hat{p}^{j}_{s_1}\| \leq \lambda_j + \gamma \chi_{\sigma}.$$ 

Since $\chi_{\sigma_2} + \chi_{\sigma_1} = \chi_{\sigma}$, this implies

$$\lambda_{j_2} + \lambda_{j_1} \leq \lambda_j.$$ 

If $j = 0$, then $j_1 = j_2 = 0$ and, by the last condition in Definition (8.6), $\sigma_1$ and $\sigma_2$ are none of the triples in (8.2). Thus Lemma 2.6 implies that either $\sigma_1 = (1, 1, 0)$ and $\sigma_2 = \sigma$, or $\sigma_2 = (1, 1, 0)$ and $\sigma_1 = \sigma$, or $\sigma_1, \sigma_2 < \sigma$.

If $j > 0$, then either $j_1, j_2 < j$, or $j_1 = j$ and $j_2 = 0$, or $j_2 = j$ and $j_1 = 0$. In the first case we are done, so consider the second case (the third case is analogous). Then $(j_2, \sigma_2) < (j, \sigma)$ and $\sigma_2$ is none of the triples in (8.2). In particular, $\chi_{\sigma_2} \leq 0$, and thus

$$-\chi_{\sigma_1} \leq -\chi_{\sigma_1} - \chi_{\sigma_2} = -\chi_{\sigma}.$$ 

If $\chi_{\sigma_2} < 0$, this yields $(j_1, \sigma_1) < (j, \sigma)$ and we are done. If $\chi_{\sigma_2} = 0$, then $\sigma_2 = (1, 1, 0)$ and it follows that $(j_1, \sigma_1) = (j, \sigma)$.

For a filtered IBL$_\infty$-algebra $(C, \{p_{k,\ell,g}\})$, the composition $p_{1,1,0} \circ p_{1,1,0}$ may be nonzero due to the presence of $p_{0,1,0}$ or $p_{1,0,0}$. If the IBL$_\infty$-algebra is strict, then these terms are not present and we get a chain complex $(C, p_{1,1,0})$. We define path objects in the category of (gapped) filtered IBL$_\infty$-structures in the same way as in Definition 4.1 except that we require the morphisms $\iota$, $\varepsilon_0$ and $\varepsilon_1$ to have filtration degree 0, and in condition (c) the maps $p^{0}_{1,1,0}$ and $q^{0}_{1,1,0}$ replace $p_{1,1,0}$ and $q_{1,1,0}$. The proofs of the following two propositions are now completely analogous to those of Proposition 4.2 and Proposition 4.4, using induction over the linear order on extended signatures and Lemma 8.8.
Proposition 8.9. For any gapped filtered $\text{IBL}_{\infty}$-algebra $(C, \{p_{k,\ell,g}\})$ there exists a path object $\mathfrak{C}$ that is gapped.

Proposition 8.10. Let $C$ and $D$ be gapped filtered $\text{IBL}_{\infty}$-algebras, and let $\mathfrak{C}$ and $\mathfrak{D}$ be gapped path objects for $C$ and $D$, respectively. Let $f : C \to D$ be a gapped filtered $\text{IBL}_{\infty}$-morphism. Then there exists a gapped filtered $\text{IBL}_{\infty}$-morphism $F : C \to D$ such that the diagram

$$
\begin{array}{ccc}
C & \xrightarrow{\epsilon^C_i} & C \\
\downarrow f & & \downarrow f \\
D & \xrightarrow{\epsilon^D_i} & D
\end{array}
$$

commutes for both $i = 0$ and $i = 1$.

We define the notion of a homotopy between gapped filtered $\text{IBL}_{\infty}$-algebras in the same way as in Definition 4.5. Then Proposition 4.6 and Corollary 4.7 can be generalized to the gapped filtered case in the same way.

Now Proposition 5.2 and Theorem 6.1 have the following analogues in the strict filtered case. (More generally, they hold for gapped filtered $\text{IBL}_{\infty}$-algebras with $p_{k,\ell,g} = 0$ for the triples in (8.2).)

Proposition 8.11. Let $f : (C, p) \to (D, q)$ be a strict gapped filtered $\text{IBL}_{\infty}$-morphism such that $f_{1,1,0} : (C, p_{1,1,0}) \to (D, q_{1,1,0})$ is a chain homotopy equivalence. Then $f$ is a filtered $\text{IBL}_{\infty}$-homotopy equivalence.

Theorem 8.12. Suppose $(C, \{p_{k,\ell,g}\})$ is a strict gapped filtered $\text{IBL}_{\infty}$-algebra. Then there exist operations $\{q_{k,\ell,g}\}$ on its homology $H := H_*(C, p_{1,1,0})$ giving it the structure of a strict gapped filtered $\text{IBL}_{\infty}$-algebra such that there exists a gapped homotopy equivalence $f : (H, \{q_{k,\ell,g}\}) \to (C, \{p_{k,\ell,g}\})$.

Remark 8.13. Proposition 8.11 continues to hold in the nonstrict gapped case provided that $f_{1,1,0}^{0}$ is a chain homotopy equivalence with respect to $p_{1,1,0}^{0}$ and $q_{1,1,0}^{0}$, and similarly for Theorem 8.12.

Filtered $\text{IBL}_{\infty}$-algebras over the universal Novikov ring.

In applications to symplectic geometry (both to SFT and Lagrangian Floer theory), the $\text{IBL}_{\infty}$-algebra that is expected to appear has coefficients in a Novikov ring and has a filtration by energy (that is, the symplectic area of pseudo-holomorphic curves). Here we explain the algebraic part of this story and show that various results in the previous sections have analogues in this setting. Let $\mathbb{K}$ be a field of characteristic 0 (for example $\mathbb{Q}$).
Definition 8.14. The universal Novikov ring $\Lambda_0$ consists of formal sums

$$a = \sum_{i=0}^{\infty} a_i T^{\lambda_i},$$

where $a_i \in \mathbb{K}$, $\lambda_i \in \mathbb{R}_{\geq 0}$ such that $\lambda_{i+1} > \lambda_i$ and $\lim_{i \to \infty} \lambda_i = +\infty$. It is a commutative ring with the obvious sum and product. The $T$-adic valuation

$$\|a\|_T := \inf \{\lambda_i \mid a_i \neq 0\} \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$$

turns $\Lambda_0$ into a complete filtered ring. $\Lambda_0$ is a local ring whose unique maximal ideal $\Lambda_0^+$ is the subset of elements in (8.10) such that $\lambda_i > 0$ for all $i$ with $a_i \neq 0$. Note that $\Lambda_0/\Lambda_0^+ = \mathbb{K}$.

Let now $\overline{C}$ be a $\mathbb{K}$-vector space. Recall that the tensor product $\overline{C} \otimes_{\mathbb{K}} \Lambda_0$ consists of finite sums $\sum_{i=1}^{N} x_i \otimes a_i$, where $x_i \in \overline{C}$ and $a_i \in \Lambda_0$. In particular, it contains finite sums

$$\sum_{i=1}^{N} x_i T^{\lambda_i},$$

where $x_i \in \overline{C}$ and $\lambda_i \in \mathbb{R}_{\geq 0}$. We denote by

$$C := \overline{C} \widehat{\otimes}_{\mathbb{K}} \Lambda_0$$

the space of possibly infinite sums

$$x = \sum_{i=1}^{\infty} x_i T^{\lambda_i}$$

such that $x_i \in \overline{C}$, $\lambda_i \in \mathbb{R}_{\geq 0}$, and $\lim_{i \to \infty} \lambda_i = +\infty$.

Hereafter in this section we shall only consider $\Lambda_0$-modules that are obtained as $C = \overline{C} \widehat{\otimes}_{\mathbb{K}} \Lambda_0$ for some $\overline{C}$. Then $C$ has a valuation defined by

$$\|x\|_T := \inf \{\lambda_i \mid a_i \neq 0\},$$

which turns $C$ into a complete filtered $\Lambda_0$-module. On such $C$, the notion of a filtered $\text{IBL}_{\infty}$-structure over $\Lambda_0$ is now defined as above, with the ring $R$ replaced by $\Lambda_0$ and the constant $\gamma = 0$. Note that the operations $p_{k,\ell,g}$ descend to the quotient $\overline{C} \cong C/(\Lambda_+ \cdot C)$ to give $(\overline{C}, \overline{p}_{k,\ell,g})$ the structure of a generalized $\text{IBL}_{\infty}$-algebra over $\Lambda_0/\Lambda_+ = \mathbb{K}$, which we call the reduction of $(C, p_{k,\ell,g})$.

As above, consider a discrete sub-monoid $G = \{\lambda_0, \lambda_1, \ldots\}$, where $\lambda_j < \lambda_{j+1}$ and $\lambda_0 = 0$. Let $\overline{C}_i$ ($i = 1, 2$) be two $\mathbb{K}$-vector spaces and $C_i = \overline{C}_i \widehat{\otimes} \Lambda_0$. A $\Lambda_0$-linear map

$$F : C_1 \to C_2$$

is said to be $G$-gapped if there exist $\mathbb{K}$-linear maps $F_j : \overline{C}_1 \to \overline{C}_2$ for each $\lambda_j \in G$ such that

$$F = \sum T^{\lambda_j} F_j,$$
where we extend $F_j$ to $C_1 \to C_2$ by $\Lambda_0$-linearity. Note that the $F_j$ are uniquely determined by $F$, and a filtered IBL$_\infty$-algebra over $\Lambda_0$ is $G$-gapped in the sense of Definition 8.6 if all the operations $\hat{p}_{k,\ell,g}$ are $G$-gapped.

We define path objects in the category of filtered IBL$_\infty$-structures over $\Lambda_0$ in the same way as in Definition 4.1. Then Propositions 4.2 and 4.4 have analogues in this category. Thus we obtain a notion of homotopy between gapped filtered IBL$_\infty$-algebras over $\Lambda_0$ with the same properties as in Section 4. Now Proposition 5.2 and Theorem 6.1 have the following analogues, which improve Proposition 8.11 and Theorem 8.12 in this setting and are proved analogously.

**Proposition 8.15.** Let $f : (C,p) \to (D,q)$ be a gapped filtered IBL$_\infty$-morphism between gapped filtered IBL$_\infty$-algebras over the universal Novikov ring $\Lambda_0$. Suppose that its reduction $\bar{f}_{1,1,0} : (\bar{C},\bar{p}_{1,1,0}) \to (\bar{D},\bar{q}_{1,1,0})$ is a chain homotopy equivalence. Then $f$ is a filtered IBL$_\infty$-homotopy equivalence. □

**Theorem 8.16.** Suppose $(C,\{p_{k,\ell,g}\})$ is a gapped filtered IBL$_\infty$-algebra over the universal Novikov ring $\Lambda_0$. Set $\bar{H} := H(C,\bar{p}_{1,1,0})$ and $H := \bar{H} \otimes \Lambda_0$. Then there exist operations $\{q_{k,\ell,g}\}$ on $H$ giving it the structure of a gapped filtered IBL$_\infty$-algebra over $\Lambda_0$ such that there exists a gapped homotopy equivalence $f : (H,\{q_{k,\ell,g}\}) \to (C,\{p_{k,\ell,g}\})$. □

The main example in this paper is the dual cyclic bar complex of a cyclic DGA, which is discussed in Sections 10 and 12.

### 9. Maurer-Cartan elements

In this section we discuss Maurer-Cartan elements and the resulting twisted IBL$_\infty$-structures. With the applications in the following sections in mind, we formulate the discussion for strict filtered IBL$_\infty$-algebras. However, most statements in this section remains true if we drop the word “strict” throughout.

Let $(C,p = \{p_{k,\ell,g}\})$ be a strict filtered IBL$_\infty$-algebra of bidegree $(d,\gamma)$. Consider a collection of elements

$$m_{\ell,g} \in \tilde{E}_{\ell}C, \quad \ell \geq 1, g \geq 0$$

of grading degrees

$$|m_{\ell,g}|_{\text{grading}} = -2d(g - 1)$$

and filtration degrees $\|m_{\ell,g}\|$ satisfying

$$\|m_{\ell,g}\| \geq \gamma \chi_{0,\ell,g} \text{ for all } \ell, g,$$

where

$$\chi_{0,\ell,g} = \begin{cases} 2\ell & \text{if } \ell \leq g - 1 \\ 2g - 2 & \text{otherwise} \end{cases}$$

and

$$\tilde{E}_{\ell}C = \bigoplus_{g \geq 0} H^g(C,\tilde{p}_\ell)$$

is the filtered bar complex of $(C,p)$.
where the inequality is strict for the pairs \((\ell, g) = (1, 0)\) and \((2, 0)\). Define the grading degree zero element
\[
m := \sum_{\ell \geq 1, g \geq 0} m_{\ell, g} h^{g - 1} \in \frac{1}{\hbar} \hat{E} C \{ h \}.
\]

**Definition 9.1.** \(\{ m_{\ell, g} \}_{\ell \geq 1, g \geq 0} \) is called a Maurer-Cartan element in \((C, \{ p_{k, \ell, g} \})\) if
\[
(9.2) \quad \hat{p}(e^m) = 0.
\]

Here we view \(m\) as a filtered IBL\(_\infty\)-morphism from the trivial IBL\(_\infty\)-algebra \(0\) to \((C, p)\) whose \((0, \ell, g)\) term sends \(1 \in R = \hat{E}_0 0\) to \(m_{\ell, g} \in \hat{E}_\ell C\). Then the Maurer-Cartan equation (9.2) is just equation (8.6) for the corresponding filtered IBL\(_\infty\)-morphism. In view of this observation and Lemma 8.3, the left hand side of (9.2) converges with respect to the metric induced by the filtration.

For later reference, we record the following observation.

**Lemma 9.2.** Suppose that \((C, \{ p_{k, \ell, g} \})\) is a filtered dIBL\(_\infty\)-algebra, i.e. its only nonvanishing terms are \(d = p_{1,1,0}, p_{2,1,0}\) and \(p_{1,2,0}\), and the only nonvanishing term in \(m\) is \(m_{1,0}\). Then the Maurer-Cartan equation (9.2) is equivalent to
\[
(9.3) \quad dm_{1,0} + \frac{1}{2} p_{2,1,0}(m_{1,0}, m_{1,0}) = 0, \quad p_{1,2,0}(m_{1,0}) = 0.
\]

**Proof.** We compute
\[
\hat{p}(e^m) = (\hat{p}_{1,1,0} + \hat{p}_{2,1,0} h + \hat{p}_{1,2,0})(e^{h^{-1}m_{1,0}})
\]
\[
= \left( p_{1,1,0}(m_{1,0}) + \frac{1}{2} p_{2,1,0}(m_{1,0}, m_{1,0}) + p_{1,2,0}(m_{1,0}) \right) (h^{-1} e^m),
\]
implying the equivalence. \(\square\)

**Twisted IBL\(_\infty\)-structures.** The next proposition shows that a Maurer-Cartan element gives rise to a “twisted” IBL\(_\infty\)-structure.

**Proposition 9.3.** Let \(\{ m_{\ell, g} \}_{\ell \geq 1, g \geq 0}\) be a Maurer-Cartan element in the strict filtered IBL\(_\infty\)-algebra \((C, \{ p_{k, \ell, g} \})\). Then there exists a unique strict filtered IBL\(_\infty\)-structure \(\{ p^m_{\ell, g} \}_{k, \ell \geq 1, g \geq 0}\) on \(C\) satisfying
\[
(9.4) \quad \hat{p}^m = e^{-m} \hat{p}(e^m) : \hat{E} C \{ h \} \to \hat{E} C \{ h \}.
\]
Proof. The map \( \widehat{p}^m_{k,\ell,g} : \widehat{E}_k C \to \widehat{E}_\ell C \) is given by the sum

\[
\sum_{r \geq 0} \sum_{m_{k-\ell,g}} \frac{1}{r!} \sigma_{k-\ell,g} \left( m_{\ell_1,g_1} \cdots m_{\ell_r,g_r} \right) \text{conn}
\]

corresponding to complete gluings of \( r \) connected surfaces of signatures \((0,\ell_i,g_i)\), plus \( k \) trivial cylinders, at their outgoing ends to the ingoing ends of a connected surface of signature \((k^-,-,g^-)\), plus an appropriate number of trivial cylinders, to obtain a connected surface of signature \((k,\ell,g)\). In particular, for each term in this sum the Euler characteristics satisfy

\[
\chi_{k,\ell,g} = \chi_{k^-,-,g^-} + \sum_{i=1}^r \chi_{k_i,\ell_i,g_i}.
\]

Let us write \( \{1,\ldots, r\} \) as the disjoint union \( I \cup J \), where \( i \in I \) if \((\ell_i,g_i)\) equals \((1,0)\) or \((2,0)\), and \( i \in J \) if \( \chi_{0,\ell_i,g_i} < 0 \). Let \( \delta > 0 \) be such that \( \|m_{\ell_i,g_i}\| > \gamma \chi_{0,\ell_i,g_i} \geq \delta \) for all \( i \in I \). As in the proof of Lemma 8.3, the filtration conditions on \( p \) and \( m \) imply

\[
\|\widehat{p}^m_{k,\ell,g}\| - \gamma \chi_{k,\ell,g} \geq \delta |I| \geq 0.
\]

This shows that \( |I| \) is uniformly bounded, i.e. bounded from above in terms of \( k,\ell,g \) and \( \|\widehat{p}^m_{k,\ell,g}\| \). Then the equation for the Euler characteristics provides uniform bounds on \( |J| \) as well as all the \( \ell_j,g_j \) for \( j \in J \). Finally, the equation \( \ell_1 + \cdots + \ell_r + \ell^- - k^- = \ell^- - k \) provides a uniform bound on \( k^- \), which proves convergence of \( \widehat{p}^m_{k,\ell,g} \) with respect to the filtration. Inequality (8.6) shows that \( \widehat{p}^m_{k,\ell,g} \) satisfies (8.1). The equation \( \widehat{p}^m \circ \widehat{p}^m = 0 \) follows immediately from \( \widehat{p}^m = e^{-m} \widehat{p}(e^m) \) and \( \widehat{p} \circ \widehat{p} = 0 \). \( \square \)

Remark 9.4. (1) Note that, although \( m \) contains negative powers of \( h \), the map \( \widehat{p}^m \) does not contain negative powers of \( h \).

(2) In the case of filtered IBL\(_\infty\)-algebras over the universal Novikov ring \( \Lambda_0 \), condition (7.1) just says that \( m_{1,0}, m_{2,0} \equiv 0 \mod \Lambda_+ \).

Push-forward of Maurer-Cartan elements. Next consider a strict filtered IBL\(_\infty\)-morphism \( \hat{f} = \{f_{k,\ell,g} \} : (C, \{p_{k,\ell,g}\}) \to (D, \{q_{k,\ell,g}\}) \) between strict filtered IBL\(_\infty\)-algebras and a Maurer-Cartan element \( \{m_{\ell,g}\} \) in \((C, \{p_{k,\ell,g}\})\). The interpretation of Maurer-Cartan elements as filtered IBL\(_\infty\)-morphisms from the trivial IBL\(_\infty\)-algebra and Lemma 8.3 immediately imply

Lemma 9.5. There exists a unique Maurer-Cartan element \( \{e^i m_{\ell,g}\} \) in \((D, \{q_{k,\ell,g}\})\) satisfying

\[
e^{f_{\ell,g}}(e^m) = e^{i_{\ell,m}}.
\]

\( \square \)
We call \( \{ f_{k, \ell, g} \} \) the push-forward of the Maurer-Cartan element \( \{ m_{\ell, g} \} \) under the morphism \( \{ f_{k, \ell, g} \} \).

**Proposition 9.6.** In the situation of Lemma 9.5, there exists a unique strict filtered \( IBL_\infty \)-morphism \( \{ f_{k, \ell, g}^m \} \) from \( (C, \{ p_{k, \ell, g}^m \}) \) to \( (D, \{ q_{k, \ell, g}^{1-m} \}) \) satisfying

\[
e^{f_{k, \ell, g}^m} = e^{-f_{k, \ell, g}^m} e^{(e_{k, \ell, g}^m)} : EC\{ h \} \to ED\{ h \},
\]

Moreover, if \( \{ f_{k, \ell, g} \} \) is a strict gapped filtered \( IBL_\infty \)-homotopy equivalence, then so is \( \{ f_{k, \ell, g}^m \} \).

**Proof.** As usual, we translate the equation \( e^{f_{k, \ell, g}^m} = e^{-f_{k, \ell, g}^m} e^{(e_{k, \ell, g}^m)} \) for disconnected surfaces into one for connected surfaces. This shows that the map \( f_{k, \ell, g}^m : E_k C \to E_\ell D \) is given by the sum

\[
\sum_{r^+} \frac{1}{r^+!} \prod_{i=1}^{r^+} (f_{k_i, \ell_i, g_i}^+ \circ \cdots \circ f_{k_{r^+}, \ell_{r^+}, g_{r^+}}^+) \cdot (m_{\ell_1, g_1}^+ \cdots m_{\ell_r, g_r}^+ \cdots m_{\ell_{r^+}, g_{r^+}}^+).conn
\]

(9.7) \[\sum_{r^+} \frac{1}{r^+!} \prod_{i=1}^{r^+} (f_{k_i, \ell_i, g_i}^+ \circ \cdots \circ f_{k_{r^+}, \ell_{r^+}, g_{r^+}^+}) \cdot (m_{\ell_1, g_1}^+ \cdots m_{\ell_r, g_r}^+ \cdots m_{\ell_{r^+}, g_{r^+}}^+).conn\]

This corresponds to complete gluings of \( r^+ \) connected surfaces of signatures \((0, \ell_i^+, g_i^+)\), plus \( k \) trivial cylinders, at their outgoing ends to the ingoing ends of \( r^- \) connected surfaces of signatures \((k_i^-, \ell_i^-, g_i^-)\) to obtain a connected surface of signature \((k, \ell, g)\). In particular, for each term in this sum the Euler characteristics satisfy

\[
\chi_{k, \ell, g} = \sum_{i=1}^{r^+} \chi_{0, \ell_i^+, g_i^+} + \sum_{i=1}^{r^-} \chi_{k_i^-, \ell_i^-, g_i^-}.
\]

Let us set \( k_i^+ := 0 \) and write \( \{ 1, \ldots, r^+ \} \) as the disjoint union \( I^+ \cup J^+ \cup K^+ \), where \( i \in I^+ \) iff \((k_i^+, \ell_i^+, g_i^+)\) is one of the triples in (8.2), \( i \in J^+ \) iff \( \chi_{k_i^+, \ell_i^+, g_i^+} < 0 \), and \( i \in K^+ \) iff \( (k_i^+, \ell_i^+, g_i^+) = (1, 1, 0) \). Note that \( K^+ = \emptyset \). Let \( \delta > 0 \) be such that \( \| f_{k_i^+, \ell_i^+, g_i^+} \| - \gamma \chi_{k_i^+, \ell_i^+, g_i^+} \geq \delta \) for all \( i \in I^- \) and

\[
\| m_{\ell_i^+, g_i^+} \| - \gamma \chi_{0, \ell_i^+, g_i^+} \geq \delta \text{ for all } i \in I^+.
\]

As in the proof of Lemma 8.5 the filtration conditions on \( f \) and \( m \) imply

\[
\| f_{k, \ell, g}^m \| - \gamma \chi_{k, \ell, g} \geq \delta (|I^+| + |I^-|) \geq 0.
\]

(9.8) \[\| f_{k, \ell, g}^m \| - \gamma \chi_{k, \ell, g} \geq \delta (|I^+| + |I^-|) \geq 0.\]

This shows that \( |I^+| \) and \( |I^-| \) are uniformly bounded, i.e. bounded from above in terms of \( k, \ell, g \) and \( \| f_{k, \ell, g}^m \| \). Then the equation for the Euler characteristics provides uniform bounds on \( |J^+| \) and \( |J^-| \) as well as all the \( k_i^-, \ell_j^+, g_j^+ \) for \( j \in J^\pm \). Finally, the fact that each \( i \in K^+ \) contributes \( 1 \) to the sum \( \ell_1^- + \cdots + \ell_{r^-}^- = \ell \) yields a uniform bound on \(|K^-|\). Hence the number of terms in the sum in (9.7) is uniformly bounded, which proves convergence of \( f_{k, \ell, g}^m \) with respect to the filtration. Inequality (9.8) shows that \( f_{k, \ell, g}^m \) satisfies (8.3), where the inequality is strict if \( I^+ \) or \( I^- \) is nonempty. If
\( I^+ \) and \( I^- \) are both empty then either \( \chi_{k,\ell,g} < 0 \) (if \( J^+ \) or \( J^- \) are nonempty), or (if \( J^+ \) and \( J^- \) are both empty) \((k_i^- , \ell_i^- , g_i^-) = (1, 1, 0)\) for all \( i \) and hence \((k, \ell, g) = (1, 1, 0)\). This shows that \( f^m_{k,\ell,g} \) also satisfies (8.2). That \( f^m \) defines an \( \text{IBL}_\infty \)-morphism now follows from

\[
\hat{q}^m \cdot e^m = e^{-1} \hat{q} e^m = e^{-1} \hat{q} (e^m_{\cdot}) = e^{-1} \hat{p} (e^m_{\cdot}) = e^{-1} \hat{p} (e^m_{\cdot}).
\]

Note that

\[
f^m_{1,1,0} = f_{1,1,0} + \sum_{k=1}^{\infty} \frac{1}{k!} f_{k+1,1,0} (m_{1,0} \cdots m_{1,0}),
\]

where the \( k \)-th term in the sum has filtration degree at least \( k \delta > 0 \).

Finally, suppose that \( \{f_{k,\ell,g}\} \) is a strict gapped filtered \( \text{IBL}_\infty \)-homotopy equivalence. Then \( f_{1,1,0} \) is a chain homotopy equivalence. A standard spectral sequence argument using the filtration (cf. [52, chapter 3]) now shows that \( f^m_{1,1,0} \) is also a chain homotopy equivalence. By Proposition 8.11, this implies that \( \{f^m_{k,\ell,g}\} \) is a filtered \( \text{IBL}_\infty \)-homotopy equivalence.

**Gauge equivalence of Maurer-Cartan elements.** We conclude this section with a brief discussion of gauge equivalence, which will be important for geometric applications in SFT and Lagrangian Floer theory.

**Definition 9.7.** Let \( m_0, m_1 \) be Maurer-Cartan elements of a strict \( G \)-gapped filtered \( \text{IBL}_\infty \)-algebra \( C \), and let \( \mathcal{C} \) be a path object for \( C \). We say \( m_0 \) is gauge equivalent to \( m_1 \) if there exists a Maurer-Cartan element \( \mathcal{M} \) of \( \mathcal{C} \) such that

\[
(\varepsilon_0)_* \mathcal{M} = m_0, \quad (\varepsilon_1)_* \mathcal{M} = m_1.
\]

**Proposition 9.8.** (1) The notion of gauge equivalence is independent of the choice of the path object \( \mathcal{C} \).

(2) Gauge equivalence is an equivalence relation.

(3) Let \( f, g \) be strict \( G \)-gapped filtered \( \text{IBL}_\infty \)-morphisms from \( C \) to \( D \), and \( m_0, m_1 \) be Maurer-Cartan elements of \( C \). If \( f \) is homotopic to \( g \) and \( m_0 \) is gauge equivalent to \( m_1 \), then \( f_* m_0 \) is gauge equivalent to \( g_* m_1 \).

Using the fact that a Maurer-Cartan element is identified with a morphism from \( 0 \), Proposition 9.8 immediately follows from the \( G \)-gapped filtered version of Proposition 4.6.
Proposition 9.9. Let $m_0, m_1$ be gauge equivalent Maurer-Cartan elements of a strict $G$-gapped filtered $\mathbb{I}BL_{\infty}$-algebra $(C, \{p_{k,\ell,g}\})$. Then $(C, \{p_{k,\ell,g}^{m_1}\})$ is homotopy equivalent to $(C, \{p_{k,\ell,g}^{m_0}\})$.

Proof. Let $(C, \{q_{k,\ell,g}\})$ be a path object and $M$ as in Definition 9.7. By Proposition 9.6, the $\varepsilon_i$ induce morphisms $\varepsilon_i^{m_1} : (C, \{q_{k,\ell,g}^{m_1}\}) \to (C, \{p_{k,\ell,g}^{m_1}\})$, which are homotopy equivalences since the $\varepsilon_i$ are. The proposition now follows from the $G$-gapped filtered version of Corollary 4.7. □

10. The dual cyclic bar complex of a cyclic cochain complex

In this section we show that the dual cyclic bar complex of a cyclic cochain complex carries a natural dIBL-structure, i.e., a $\mathbb{I}BL_{\infty}$-structure such that $p_{k,\ell,g} = 0$ unless $(k,\ell,g) \in \{(1,1,0), (2,1,0), (1,2,0)\}$.

The dIBL structure on the dual cyclic bar complex. Let $(A = \bigoplus_k A^k, d)$ be a $\mathbb{Z}$-graded cochain complex over $\mathbb{R}$. We assume that dim $A$ is finite. Let $n$ be a positive integer and

$$
\langle \cdot, \cdot \rangle : \bigoplus_k A^k \otimes A^{n-k} \to \mathbb{R}
$$

a nondegenerate bilinear form, which we extend by zero to the rest of $A \otimes A$.

Definition 10.1. $(A, \langle \cdot, \cdot \rangle, d)$ is called a cyclic cochain complex if

$$
\langle dx, y \rangle + (-1)^{\deg x-1} \langle x, dy \rangle = 0,
$$

$$
\langle x, y \rangle + (-1)^{(\deg x-1)(\deg y-1)} \langle y, x \rangle = 0.
$$

We define the cyclic bar complex

$$
B^cyc_k A := A[1] \otimes \cdots \otimes A[1] / \sim
$$

as the quotient of the $k$-fold tensor product under the action of $\mathbb{Z}_k$ by cyclic permutations with signs. As explained in Remark 2.1, $B^cyc_k A$ is isomorphic to the subspace of invariant tensors under the cyclic group action. We introduce the dual cyclic bar complex

$$
B^{cyc*}_k A := Hom(B^cyc_k A, \mathbb{R}),
$$

$$
B^{cyc*} A := \bigoplus_{k=1}^{\infty} B^{cyc*}_k A.
$$

To avoid confusion, we will denote the degree in $A$ by $\deg x$ and the degree in $A[1]$ by

$$
|x| = \deg x - 1.
$$
An element $\varphi \in B^*_{cyc}A$ is homogeneous of degree $D$ if $\varphi(x_1 \otimes \cdots \otimes x_k) = 0$ whenever $\sum |x_i| \neq D$. The coboundary operator $d$ induces a boundary operator on $B^*_{cyc}A$ in the obvious way, which we denote by $d$:

$$(d\varphi)(x_1, \ldots, x_k) := \sum_{j=1}^{k} (-1)^{|x_1|+\cdots+|x_{j-1}|} \varphi(x_1, \ldots, x_{j-1}, dx_j, x_{j+1}, \ldots x_k).$$

Note that the coboundary operator on $A$ has degree $+1$, so the induced boundary operator $d$ on $B^*_{cyc}A$ has degree $-1$.

We will now construct two operations

$$\mu : B^*_{cyc}A \otimes B^*_{cyc}A \to B^*_{cyc}A,$$

$$\delta : B^*_{cyc}A \to B^*_{cyc}A \otimes B^*_{cyc}A$$

of degree $|\mu| = |\delta| = 2-n$, which together with the differential $d$ will give rise to a dIBL-structure. It suffices to define these operations on homogeneous elements in $B^*_{cyc}A = \bigoplus_{k \geq 1} B^*_{cyc}kA$, respectively

$$B^*_{cyc}A \otimes B^*_{cyc}A = \bigoplus_{k_1,k_2 \geq 1} B^*_{cyc}k_1A \otimes B^*_{cyc}k_2A \cong \bigoplus_{k_1,k_2 \geq 1} \text{Hom}(B^*_{cyc}k_1A \otimes B^*_{cyc}k_2A, \mathbb{R}).$$

Let $e_i$ be a homogeneous basis of $A$ and set

$$\eta_i := |e_i| = \deg e_i - 1.$$

Let $e^i$ be the dual basis of $A$ with respect to the pairing $\langle \ , \ \rangle$, i.e.

$$\langle e_i, e^j \rangle = \delta^j_i.$$

We set

$$g_{ij} := \langle e_i, e_j \rangle, \quad g^{ij} := \langle e^i, e^j \rangle.$$

Note that

$$g_{ij} = (-1)^{\eta_i \eta_j + 1} g^{ji}.$$

In the following we use Einstein’s sum convention. Then $g_{ik}g^{jk} = \delta^j_i$, i.e. $(g^{ij})$ is the transpose of the inverse matrix of $(g_{ij})$. Note that $\deg e_i + \deg e^i = n$, so $g^{ij} \neq 0$ implies that $\eta_i + \eta_j = n - 2$. If $\tilde{e}_j = \tau^j_i e_i$ is another basis, then its dual basis is $\tilde{e}^j = \sigma_j^i e^i$ with $(\sigma_j^i)$ the inverse matrix of $(\tau^j_i)$ and the pairing in the new basis is given by

$$\tilde{\varphi}_{\alpha\beta} = \langle \tilde{e}_\alpha, \tilde{e}_\beta \rangle = \tau^{ai}_\alpha g_{ab} \tau^b_\beta, \quad g_{ab} = \sigma^a_\alpha \tilde{\varphi}_{\alpha\beta} \sigma_b^\beta,$$

(10.1)

$$\tilde{g}_{\alpha\beta} = \langle \tilde{e}^\alpha, \tilde{e}^\beta \rangle = \sigma^{ai}_\alpha g_{ab} \sigma^b_\beta, \quad g_{ab} = \tau^{ai}_\alpha \tilde{g}_{\alpha\beta} \tau^b_\beta.$$

Finally, we introduce the notation

$$d e_i = \sum_j d^j_i e_j,$$

for the coboundary operator.
**Lemma 10.2.** The following identities hold:

(10.2) \[ de^a = (-1)^{\eta_a} d^a_c e^c \]
(10.3) \[ (-1)^{\eta_a} d^a_b g^{ab} + g^{a'b'} d^{b'}_b = 0. \]

**Proof.** To prove the first equation, we compute the coefficient of \( e^c \) in \( de^a \) as

\[ \langle e_c, de^a \rangle = (-1)^{\eta_a + 1} \langle de_c, e^a \rangle \]
\[ = (-1)^{\eta_a + 1} d^a_c \langle e_c, e^a \rangle \]
\[ = (-1)^{\eta_a + 1} d^a_c. \]

Since the degrees of \( e^c \) and \( e^a \), and hence also the degrees of \( e_c \) and \( e_a \) differ by one, the first claim follows.

To prove the second claim, we again use the cyclic relation

\[ \langle de^{a'}, e^b \rangle = (-1)^{|e^{a'}| - 1} \langle e^{a'}, de^b \rangle = (-1)^{\eta_{a'} + n - 3} \langle e^{a'}, de^b \rangle. \]

Using the first equation, we find

\[ \langle de^{a'}, e^b \rangle = (-1)^{\eta_{a'}} d^{a'}_a g^{ab} \]

and

\[ \langle e^{a'}, de^b \rangle = (-1)^{\eta_b} d^b_g g^{a'b'}. \]

Putting things together and noting that \( \eta_a + \eta_b \equiv n - 2 \), we obtain the result. \( \square \)

An element \( \varphi \in B_{k}^{\text{cyc}} A \) is determined by its coefficients

\[ \varphi_{i_1 \cdots i_k} = (-1)^{\eta_k} \sum_{j=1}^{k-1} \eta_j \varphi_{i_k i_1 \cdots i_{k-1}}. \]

The boundary operator on \( B_{1}^{\text{cyc}} A \) acts on these coefficients by

\[ (d \varphi)_{i_1 \cdots i_k} = \sum_{j=1}^{k} \sum_{a} (-1)^{\eta_1 + \cdots + \eta_{j-1} + 1} d^a_j \varphi_{i_1 \cdots i_{j-1} a i_{j+1} \cdots i_k}. \]

Now we are ready to define a bracket and a cobracket on \( B_{1}^{\text{cyc}} A \). For the bracket, let \( \varphi^1 \in B_{k_1+1}^{\text{cyc}} A, \ \varphi^2 \in B_{k_2+1}^{\text{cyc}} A \), \( k_1, k_2 \geq 0, k_1 + k_2 \geq 1 \). We define

\[ \mu(\varphi^1, \varphi^2) \in B_{k_1+k_2}^{\text{cyc}} A \]

by

(10.4) \[ \mu(\varphi^1, \varphi^2)_{i_1 \cdots i_{k_1+k_2}} := \sum_{a,b} \sum_{c=1}^{k_1+k_2} (-1)^{\eta_a+\eta_b} g^{ab} \varphi^1_{a i_c \cdots i_{k_1+c-1}} \varphi^2_{b \cdots i_{k_1+k_2}} g_{i_{k_1+c-1} \cdots i_{k_1+k_2}}. \]

where

\[ \eta = \eta(i_1, \ldots, i_{k_1+k_2}; a; b; c) = \sum_{r=1}^{c-1} \sum_{s=c}^{k_1+k_2} \eta_{i_r} \eta_{i_s} + \eta_{b} \cdot \sum_{t=c}^{c+k_1-1} \eta_{i_t}. \]
is the sign needed to move \((e_a, e_{i_1}, \ldots, e_{i_{c+k_1-1}}, e_b, e_{i_{c+k_1}}, \ldots, e_{i_{c-1}})\) to the order \((e_a, e_b, e_{i_1}, \ldots, e_{i_{k_1+k_2}})\). Here and hereafter we put \(i_{k_1+k_2+m} = i_m\) etc.

Using \(\sum_{c+k_1-1} \eta_i = |\varphi_i| - \eta_a\) one verifies

\[
\eta(i_2, \ldots, i_{k_1+k_2}, i_1; a; b; c) - \eta(i_1, \ldots, i_{k_1+k_2}; a; b; c+1) = \eta_i \sum_{s=2}^{k_1+k_2} \eta_s,
\]

so \(\mu(\varphi^1, \varphi^2)\) picks up the correct signs under cyclic permutation to define an element in \(B_{k_1+k_2}^{\text{cyc}} A\). Note that the operation \(\mu : B_{k_1}^{\text{cyc}} A \otimes B_{k_2}^{\text{cyc}} A \rightarrow B_{k_1+k_2}^{\text{cyc}} A\) has degree \(2 - n\). The independence of the basis \(e_i\) follows from the transformation law (10.1), using the equivalent definition for \(x_i \in A\).

\[\mu(\varphi^1, \varphi^2)(x_1, \ldots, x_{k_1+k_2})
\]

\[= \sum_{a,b} \sum_{c=1}^{k_1+k_2} (-1)^{\eta_a + \eta_b} \varphi^1(e_a, x_c, \ldots, x_{c+k_1-1}) \varphi^2(e_b, x_{c+k_1}, \ldots, x_{c-1}).\]

For the cobracket, note that an element \(\psi \in B_{k_1}^{\text{cyc}} A \otimes B_{k_2}^{\text{cyc}} A \cong \text{Hom}(B_{k_1}^{\text{cyc}} A \otimes B_{k_2}^{\text{cyc}} A, \mathbb{R})\) is determined by the coefficients

\[\psi_{i_1, \ldots, i_{k_1}, j_1, \ldots, j_{k_2}} = \psi((e_{i_1} \cdots e_{i_{k_1}}) \otimes (e_{j_1} \cdots e_{j_{k_2}})).\]

Now let \(\varphi \in B_{k_1}^{\text{cyc}} A, k \geq 4\). We define

\[\delta(\varphi) \in \bigoplus_{k_1+k_2=k-2} B_{k_1}^{\text{cyc}} A \otimes B_{k_2}^{\text{cyc}} A\]

by

\[\delta(\varphi)_{i_1, \ldots, i_{k_1}, j_1, \ldots, j_{k_2}} := \sum_{a,b} \sum_{c=1}^{k_1} \sum_{c'=1}^{k_2} (-1)^{\eta_a + \eta_b} g^{ab} \varphi_{a i_{c-1} e_{c'} - 1 b j_{c'} - 1}.\]
where

\[ \eta = \eta(i_1, \ldots, i_{k_1}; j_1, \ldots, j_{k_2}; a; b; c, c') \]

\begin{equation}
\sum_{r=1}^{c-1} \sum_{s=c}^{k_1} \eta_{ir} \eta_{is} + \sum_{r'=1}^{c'-1} \sum_{s'=c'}^{k_2} \eta_{j'r'} \eta_{j's'} + \eta_{b} \cdot \sum_{t=1}^{k_1} \eta_{it} \tag{10.7}
\end{equation}

is the sign needed to move \((e_a, e_{i+1}, \ldots, e_{i+k_1}, e_b, e_{j'+1}, \ldots, e_{j'+k_2})\) to the order \((e_a, e_b, e_{i_1}, \ldots, e_{i_{k_1}}, e_{j_1}, \ldots, e_{j_{k_2}})\).

Figure 4. A graphical illustration of formula (10.8) for \(\delta\). The inner boundary component is the first and is oriented clockwise, whereas the outer one is second and oriented counterclockwise.

The operation \(\delta : B^{\text{cyc}*} A \to B^{\text{cyc}*} A \otimes B^{\text{cyc}*} A\) also has degree \(2 - n\), and independence of the basis follows from the equivalent definition in terms of \(x_i \in A\)

\[ (\delta \varphi)(x_1 \cdots x_{k_1} \otimes y_1 \cdots y_{k_2}) \]

\[ = \sum_{a,b} \sum_{c=1}^{k_1} \sum_{c'=1}^{k_2} (-1)^{\eta_a + \eta_b} g^{ab} \varphi(e_a, x_c, \ldots, x_{c-1}, e_b, y_{c'}, \ldots, y_{c'-1}). \tag{10.8} \]

Denote by \(\tau : B^{\text{cyc}*} A \otimes B^{\text{cyc}*} A \to B^{\text{cyc}*} A \otimes B^{\text{cyc}*} A\) the map permuting the factors with sign, i.e.

\[ \tau(\varphi \otimes \psi) = (-1)^{|\varphi||\psi|} \psi \otimes \varphi \]

whenever \(\varphi\) and \(\psi\) are homogeneous elements of \(B^{\text{cyc}*} A\).

Lemma 10.3. The operations \(\delta\) and \(\mu\) introduced above satisfy

1. \(\mu \tau = (-1)^{n-3} \mu\),
2. \(\tau \delta = (-1)^{n-3} \delta\).
Proof. The assertions can be proven by tedious but straightforward computations. We will later see equivalent assertions within a larger “graphical calculus” (cf. Remark 10.3), so we do not give details here.

We will use these operations to define a $dIBL$-structure on $B^{cyc}A$. To fit with the conventions used in §2, we consider the degree shifted version

$$C := (B^{cyc}A)[2 - n].$$

For the shifted degrees, $\mu$ has degree $2(2 - n)$ and $\delta$ has degree 0. We define the boundary operator

$$p_{1,1,0} := d : E_1C \to E_1C$$

so that

$$(10.9) \quad (p_{1,1,0}\varphi)_{i_1...i_k} = \sum_{j=1}^{k} \sum_{a} (-1)^{h_{i_1} + \cdots + h_{i_{j-1}}} d_{i_j} a_{i_{j+1}...i_k} \varphi_{i_1...i_{j-1}a_{i_{j+1}...i_k}}.$$

Next, we define maps

$$P_k : (B^{cyc}A)^{\otimes k} \to (B^{cyc}A)[3 - n]^{\otimes k} = C[1]^{\otimes k}$$

by

$$(10.10) \quad P_k(c_1 \otimes \cdots \otimes c_k) := (-1)^{(n - 3)\sum_{i=1}^{k} (k - i)} c_1 \otimes \cdots \otimes c_k.$$

The sign can be interpreted in terms of formal variables $s_1, \ldots, s_k$ of degree $n - 3$ as the sign for the change of order

$$s_1c_1 \cdots s_kc_k \longrightarrow s_1 \cdots s_k c_1 \cdots c_k.$$

The maps $P_k$ conjugate the $(n - 3)$-twisted action of the permutation group $S_k$ on $(B^{cyc}A)^{\otimes k}$, given for $\sigma \in S_k$ by

$$\sigma \cdot (\varphi^1 \otimes \cdots \otimes \varphi^k) := (-1)^{\eta + (n - 3)\text{sgn}(\sigma)} \varphi^{\sigma(1)} \otimes \cdots \otimes \varphi^{\sigma(k)},$$

with the usual action by signed permutations on $C[1]$, i.e. the following diagram commutes:

$$\begin{align*}
(B^{cyc}A)^{\otimes k} & \xrightarrow{\alpha} (B^{cyc}A)^{\otimes k} \\
C[1]^{\otimes k} & \xrightarrow{P_k} C[1]^{\otimes k},
\end{align*}$$

where the action of $\alpha$ is $(n - 3)$-twisted on the first line (with respect to degrees in $B^{cyc}A$) and standard on the second line (with respect to degrees in $C[1] = B^{cyc}A[3 - n]$).

According to Lemma 10.3 $\mu$ and $\delta$ are symmetric operations on $BC^{cyc}A$ with respect to the $(n - 3)$-twisted action of the permutation group, so
defining
\[ p_{2,1,0} := P_1 \circ \mu \circ P_2^{-1} : C[1] \otimes C[1] \to C[1], \]
\[ p_{1,2,0} := \frac{1}{2} P_2 \circ \delta \circ P_1^{-1} : C[1] \to C[1] \otimes C[1], \]

makes these operations symmetric with respect to the usual action of permutations on \( C[1] \otimes k \). Hence \( p_{2,1,0} \) descends to a map \( p_{2,1,0} : E_2 C \to E_1 C \) of degree \(-2(n - 3) - 1\) and \( p_{1,2,0} \) descends to a map \( p_{1,2,0} : E_1 C \to E_2 C \) of degree \(-1\). Explicitly, the operations are given by
\[ p_{2,1,0}(\phi, \psi) = (-1)^{(n-3)|\phi|} \mu(\phi, \psi), \]
\[ p_{1,2,0}(\phi)(x, y) = \frac{1}{2} (-1)^{(n-3)|x|} \delta(\phi)(x, y). \] (10.11)

The following result corresponds to Proposition 1.4 from the Introduction.

**Proposition 10.4.** \( (C = (B^{cyc} A)[2 - n], p_{1,1,0}, p_{1,2,0}, p_{2,1,0}) \) is a dIBL-algebra of degree \( d = n - 3 \).

**Remark 10.5.** Proposition 10.4 was known among researchers in string topology, see e.g. [14, 40].

We will prove this proposition below, after introducing an alternative point of view on the construction of the operations. This will allow us to largely avoid messy computations and instead concentrate on describing the underlying organizing principles. We hope that will make the proofs in this section more transparent.

**Defining maps using ribbon graphs.** As promised, we now develop the graphical calculus underlying the constructions in this section. It generalizes similar constructions of \( A_\infty \) or \( L_\infty \) structures and homotopy equivalences among them, which are used to construct canonical models and are based on summation over ribbon trees (see [47], [34, Subsection 5.4.2]).

By a **ribbon graph** we mean a finite connected graph \( \Gamma \) with a cyclic ordering of the (half-)edges incident to each vertex. We denote by \( d(v) \) the **degree** of a vertex \( v \), i.e. the number of (half-)edges incident to \( v \). Let the set of vertices be decomposed as
\[ C_0(\Gamma) = C_0^{\text{int}}(\Gamma) \cup C_0^{\text{ext}}(\Gamma) \]

into **interior** and **exterior** vertices, where all exterior vertices have degree 1 (interior vertices can have degree 1 or higher). We assume that all our ribbon graphs have at least one interior vertex.

---

4The combinatorial factor \( \frac{1}{2} \) in the definition of \( p_{1,2,0} \), which seems rather unmotivated here, will be explained in a more general context below.
Such a graph $\Gamma$ can be thickened to a compact oriented surface $\Sigma_\Gamma$ with boundary in a unique way (up to orientation preserving diffeomorphism) such that $\Gamma \cap \partial \Sigma_\Gamma = C_0^{\text{ext}}(\Gamma)$. See Figure 5, where interior vertices are drawn as $\bullet$ and exterior vertices as $\circ$. We denote by $s(b)$ the number of exterior vertices on the boundary component $b$.

We assume that each graph has at least one interior vertex. Moreover, we impose the following condition:

**Each boundary component of $\Sigma_\Gamma$ contains at least one exterior vertex of $\Gamma$.**

The set of edges is decomposed as

$$C_1(\Gamma) = C_1^{\text{int}}(\Gamma) \cup C_1^{\text{ext}}(\Gamma)$$

into interior and exterior edges, where an edge is called exterior if and only if it contains an exterior vertex.

The **signature** of $\Gamma$ is $(k, \ell, g)$, where $k = |C_0^{\text{int}}(\Gamma)|$ is the number of interior vertices, $\ell$ is the number of boundary components of $\Sigma_\Gamma$, and $g$ is its genus. For example, the graph in Figure 5 has signature $(4, 3, 0)$.

An **automorphism** of a ribbon graph $\Gamma$ is an automorphism of the underlying graph which is required to preserve the cyclic ordering around vertices, but can permute vertices and edges (and hence also boundary components). For example, the ribbon graph in Figure 5 has a unique nontrivial automorphism (in the picture it is given by rotation around the center by the angle $\pi$). We denote the group of automorphisms of a ribbon graph $\Gamma$ by $\text{Aut}(\Gamma)$.

We denote by $\text{RG}_{k,\ell,g}$ the set of isomorphism classes of ribbon graphs of signature $(k, \ell, g)$. Figure 6 gives some examples of such graphs, where we use the same convention of labelling interior vertices by $\bullet$ and exterior vertices by $\circ$.

Let us make two comments on this definition.
Figure 6. Examples of ribbon graphs from $RG_{1,1,0}$, $RG_{1,2,0}$, $RG_{2,1,0}$, $RG_{1,1,1}$ and $RG_{2,2,0}$.

(1) Since all surfaces of the same signature are diffeomorphic, the set $RG_{k,\ell,g}$ can alternatively be described as follows. Fix a connected compact oriented surface $\Sigma$ of genus $g$ with $\ell$ boundary components. Then $RG_{k,\ell,g}$ is the set of isomorphism classes of (connected) graphs $\Gamma$ embedded in $\Sigma$ that satisfy the following conditions:

- $\Gamma \cap \partial \Sigma$ consists of degree 1 vertices. We call it the set of exterior vertices and write $C^0_{\text{ext}}(\Gamma)$. All the other vertices are called interior and make up a set $C^0_{\text{int}}(\Gamma)$.
- Each connected component $D_i$ of $\Sigma \setminus \Gamma$ is an open disk such that $D_i \cap \partial \Sigma$ is a (nonempty) arc.
- $\Gamma \cap \partial_b \Sigma$ is nonempty for each boundary component $\partial_b \Sigma$.

In this description, two graphs are isomorphic if there is an orientation preserving diffeomorphism of $\Sigma$ mapping one to the other.

(2) Given a ribbon graph $\Gamma$ of signature $(k,\ell,g)$, let $\Sigma^-_\Gamma$ be the surface obtained from $\Sigma_\Gamma$ by removing a small disk around each interior vertex. (In Figure 3 these small disks are indicated by dotted circles). Viewing the $\ell$ original boundary components as outgoing and the $k$ new boundary components as incoming, $\Sigma^-_\Gamma$ is a surface of signature $(k,\ell,g)$ as considered in Section 2 $\Gamma \cap \Sigma^-_\Gamma$ is a collection of disjoint arcs on $\Sigma^-_\Gamma$ starting and ending on the boundary such that every boundary component meets some arc. Identifying $\Sigma^-_\Gamma$ with a model surface $\Sigma^-$, each ribbon graph of signature $(k,\ell,g)$ thus induces such a collection of arcs on $\Sigma^-$.

To each interior edge of our ribbon graphs we will associate a decomposable 2-tensor $T = T^{ab} e_a \otimes e_b$ (using Einstein’s sum convention) which has the symmetry property

$$T^{ab} = (-1)^{\eta_a \eta_b + (n-3)} T^{ba}.$$
Remark 10.6. In this section we will only use the tensor $T^{ab} = (-1)^{n_a} g^{ab}$.

In the following section we will introduce another such tensor.

Given a ribbon graph $\Gamma \in R_{k,\ell,g}$ with such tensors associated to its interior edges, we want to define a map

$$F_\Gamma : (C[1])^{\otimes k} \to (C[1])^{\otimes \ell}.$$ 

In order to do that, we will make additional choices.

Definition 10.7. A labelling of a ribbon graph $\Gamma$ consists of

- a numbering of the interior vertices by $1, \ldots, k$;
- a numbering of the boundary components of $\Sigma_\Gamma$ by $1, \ldots, \ell$;
- a numbering of the (half)-edges incident to a given vertex $v$ by $1, \ldots, d(v)$, which is compatible with the previously given cyclic ordering;
- a numbering of the exterior vertices on the $b$-th boundary component by $1, \ldots, s(b)$, which is compatible with the cyclic order induced from the orientation of the surface $\Sigma_\Gamma$.

The first two of these choices induce a (suitable equivalence class of a) choice of ordering and orientation of the interior edges. One procedure to make a consistent such choice will be described in Definition 11.4 below. Here we only state that for every labelled tree with two vertices and one interior edge the resulting orientation of that edge points from the first to the second vertex, and for every labelled graph with one interior vertex and one interior edge (which is necessarily a loop at that vertex) the resulting orientation of the edge is such that the first boundary component is to the left of the edge. These conventions were already illustrated in Figures 3 and 4, where examples of contributing graphs are drawn on their ribbon surfaces.

Now imagine basis elements $e_{\beta(b,1)}, \ldots, e_{\beta(b,s(b))}$ feed into the exterior edges incident to each of the boundary components $b$. For an oriented interior edge labelled by $T^{ab} e_a \otimes e_b$, label the half-edge of the starting point with $e_a$ and the half-edge of the endpoint with $e_b$. Then around each interior vertex $v$, the $j$-th incident half-edge comes labelled with a basis vector $e_{\alpha(v,j)}$, and we can define

$$(F_\Gamma(\varphi^{1} \otimes \cdots \otimes \varphi^{k}))_{\beta(1,1) \cdots \beta(1,s(1)) \cdots \beta(\ell,1) \cdots \beta(\ell,s(\ell))} := \frac{1}{\ell! |\text{Aut}(\Gamma)| \prod_v d(v)} \sum (-1)^{\eta} \left( \prod_{l \in C^{\text{int}}_1(\Gamma)} T^{a_l b_l} \prod_{v \in C^{\text{int}}_0(\Gamma)} \varphi_{\alpha(v,1) \cdots \alpha(v,d(v))}^{v} \right),$$

where the sum is over all possible ways of making the choices mentioned above, and for each interior edge $l \in C^{\text{int}}_1(\Gamma)$ we also sum over all $a_l, b_l$. 


ranging in the index set of the chosen basis of $A$. The sign exponent

$$\eta = \eta_1 + \eta_2$$

is determined as follows. With all the choices that we have made, we can write all the involved basis elements $e_i$ in two different orders:

**Edge order.**

$$\prod_{t \in C^{|i|}(\Gamma)} e_{a_t} e_{b_t} \prod_{b=1}^{\ell} e_{\beta(b,1)} \cdots e_{\beta(b,s(b))}.$$  

Note that this depends on the ordering of the interior edges, the orientation of the interior edges, the ordering of the boundary components, and the ordering of the vertices on each boundary component.

**Vertex order.**

$$\prod_{v \in C^{|v|}(\Gamma)} e_{\alpha(v,1)} \cdots e_{\alpha(v,d(v))}.$$  

Note that this depends on the ordering of the interior vertices, and the ordering of the half-edges incident to each interior vertex. Now $(-1)^{\eta_1}$ is defined as the sign needed to move the edge order to the vertex order, according to the $A[1]$-degrees $\eta_i = \deg e_i - 1$. The other part of the sign exponent is determined as above by viewing the map as a composition

$$\mathbf{C}[1] \otimes_k \mathcal{P} \mathcal{P} \longrightarrow (B^{\text{cyc}} A) \otimes_k \mathcal{P} \mathcal{P} \longrightarrow (B^{\text{cyc}} A) \otimes_\ell P_\ell \longrightarrow \mathbf{C}[1] \otimes_\ell.$$

Now $\eta_2$ is the part of the sign exponent coming from the conjugation with $P_k$ and $P_\ell$ as in (10.10), i.e.

$$\eta_2 = (n - 3) \left( \sum_{v=1}^{k} (k - v)|\varphi^v| + \sum_{b=1}^{\ell} (\ell - b)|x^b| \right),$$

where $x^b = e_{\beta(b,1)} \cdots e_{\beta(b,s(b))}$ is the word associated to the $b$-th boundary component.

We now discuss some consequences of this definition, where the second one depends on additional properties of a specific choice for $T$:

1. Reversing the orientation of an interior edge yields a change in $\eta_1$ of $\eta_a \eta_b + n - 3$ from replacing $T_{ab}$ by $T_{ba}$ (cf. (10.12)), and another change in $\eta_1$ of $\eta_a \eta_b$ from interchanging $e_a$ and $e_b$. Since $\eta_2$ is unchanged, reversing the orientation of an interior edge yields the total sign $(-1)^{n-3}$.

2. With the specific choice of $T_{ab} = (-1)^{\eta_a} g_{ab}$, interchanging the order of two adjacent interior edges leads to a sign $(-1)^{n-2}$ due to the change of $\eta_1$ from interchanging the corresponding pairs of basis vectors in the edge order, because $\eta_a + \eta_b = n - 2$ whenever $g_{ab} \neq 0$.

3. Changing the ordering of the half-edges at an interior vertex by a cyclic permutation yields the same sign twice, once from the change of the coefficient $\varphi^v_{\alpha(v,1) \cdots \alpha(v,d(v))}$ in (10.13) (because $\varphi^v \in B^{\text{cyc}}_{d(v)} A$), and once from
the cyclic permutation of the corresponding basis vectors \( e_{\alpha(v,1)} \cdots e_{\alpha(v,d(v))} \). So definition \( \{10.13\} \) does not depend on the ordering (compatible with the cyclic order) of the half-edges at an interior vertex.

(4) Changing the ordering of the vertices on the \( b \)-th boundary component by a cyclic permutation yields the sign obtained from the cyclic permutation of the corresponding basis vectors \( e_{\beta(b,1)} \cdots e_{\beta(b,s(b))} \). As the sum in \( \{10.13\} \) extends over all orderings (compatible with the boundary orientations) of the exterior vertices on each boundary component, it defines a map

\[
B_{d(1)}^{\text{cyc}} A \otimes \cdots \otimes B_{d(k)}^{\text{cyc}} A \to B_{s(1)}^{\text{cyc}} A \otimes \cdots \otimes B_{s(l)}^{\text{cyc}} A.
\]

(5) Interchanging the order of two adjacent boundary components of total \( B_{d(1)}^{\text{cyc}} A \)-degrees \( t_1, t_2 \) yields a change of \( t_1 t_2 \) in \( \eta_1 \) from permuting the corresponding basis vectors in the edge order. It also yields a change of \((n - 3)(t_1 + t_2)\) in \( \eta_2 \).

(6) Interchanging the order of two adjacent interior vertices \( v, w \) yields a change of \(|\varphi^v| |\varphi^w|\) in \( \eta_1 \) from swapping the corresponding basis vectors in the vertex order. It also yields a change of \((n - 3)(|\varphi^v| + |\varphi^w|)\) in \( \eta_2 \).

Now we define \( p_{2,1,0} : C[1] \otimes C[1] \to C[1] \) by summation over all graphs \( \Gamma \in RG_{2,1,0} \), that is graphs with two interior vertices and exactly one interior edge connecting them. According to (1) and (6) above, reversing the order of the interior vertices and the orientation of the interior edge simultaneously has the effect of changing the sign exponent by \((|\varphi^v| + n - 3)(|\varphi^w| + n - 3)\), which gives exactly the correct sign for the standard action of \( S_2 \) on \( C[1] \otimes C[1] \). So \( p_{2,1,0} \) is symmetric in its inputs, and descends to a map

\[
p_{2,1,0} : E_2 C \to E_1 C.
\]

Similarly, we define \( p_{1,2,0} : C[1] \to C[1] \otimes C[1] \) by summation over all graphs \( \Gamma \in RG_{1,2,0} \), i.e. graphs with one interior vertex and one interior edge (which then necessarily is a loop at that vertex) Then according to (1) and (5) above, the map takes values in the invariant part of \( C[1] \otimes C[1] \), which we identify with \( E_2 C \) to get a map

\[
p_{1,2,0} : E_1 C \to E_2 C.
\]

**Remark 10.8.** Explicitly, the maps just defined are given by

\[
p_{2,1,0}(\varphi^1, \varphi^2)(x) = \sum_{ab} (-1)^{\eta_b|x| + (n-3)|\varphi| + \eta_a} g^{ab} \varphi^1(e_a, x(1)) \varphi^2(e_b, x(2)) + \text{cyclic},
\]

\[
p_{1,2,0}(\varphi)(x, y) = \frac{1}{2} \sum_{ab} (-1)^{\eta_b|x| + (n-3)|\varphi(1)| + \eta_a} g^{ab} \varphi(e_a, x, e_b, y) + \text{cyclic}.
\]

One sees that these definitions agree with the ones previously given. In particular, our sign considerations have validated the symmetry properties of \( p_{2,1,0} \) and \( p_{1,2,0} \), which are equivalent to the assertions of Lemma \{10.3\}. 

We are now ready to prove Proposition 10.4.

**Proof of Proposition 10.4.** The maps \( p_{1,1,0}, p_{1,2,0} \) and \( p_{2,1,0} \) extend uniquely to maps \( \hat{p}_{1,1,0} \) (both a derivation and a coderivation), \( \hat{p}_{1,2,0} \) (a derivation) and \( \hat{p}_{2,1,0} \) (a coderivation), all defined on \( EC \), respectively. It remains to prove that the following maps vanish (with \( o_s \) defined as in Section 2):

\[
\begin{align*}
\hat{p}_{1,1,0} & \circ 1 \cdot p_{2,1,0} + p_{2,1,0} \circ 1 \cdot \hat{p}_{1,1,0} : E_2 C \to E_1 C \\
\hat{p}_{1,1,0} & \circ 1 \cdot p_{1,2,0} + p_{1,2,0} \circ 1 \cdot p_{1,1,0} : E_1 C \to E_2 C \\
& \quad \quad p_{2,1,0} \circ 1 \cdot \hat{p}_{2,1,0} : E_3 C \to E_1 C \quad \text{(Jacobi)} \\
& \quad \quad \hat{p}_{1,2,0} \circ 1 \cdot p_{1,2,0} : E_1 C \to E_3 C \quad \text{(co-Jacobi)} \\
\hat{p}_{1,2,0} & \circ 1 \cdot \hat{p}_{2,1,0} + p_{2,1,0} \circ 1 \cdot \hat{p}_{1,2,0} : E_2 C \to E_2 C \quad \text{(Drinfeld)} \\
& \quad \quad p_{2,1,0} \circ 2 \cdot p_{1,2,0} : E_1 C \to E_1 C \quad \text{(involutivity)}.
\end{align*}
\]

To discuss the first equation, i.e. compatibility of the bracket with the boundary map, we write out the equation more explicitly:

\[
p_{1,1,0}(p_{2,1,0}(\phi, \psi)) + p_{2,1,0}(p_{1,1,0}\phi, \psi) + (-1)^{|\phi|+(n-3)}p_{2,1,0}(\phi, p_{1,1,0}\psi) = 0.
\]

Let us first look at corresponding terms in the first and second summands: The \( \eta_1(p_{2,1,0}) \) for the second term differs from \( \eta_1(p_{2,1,0}) \) for the first term by \( \eta_b \) because of the difference of degrees of the first argument of \( p_{2,1,0} \). For the same reason, \( \eta_2 \) of these two terms differs by \( n-3 \). Finally, the sign exponent in the application of \( p_{1,1,0} \) in these two terms differs by \( \eta_a \) because in the second term the differential has to be moved past the additional argument \( e_a \) in the first slot. This gives a total difference in sign exponents of

\[
\eta_a + \eta_b + (n - 3) = 1,
\]

so that corresponding terms cancel.

One can similarly compare corresponding terms in the first and third summands: Here \( \eta_2 \) is the same for both summands, but for the application of \( d \) there is an additional contribution of \( |\phi| - \eta_a \) in the first term (from moving \( d \) past the part of the inputs which feeds into \( \phi \)) and an additional contribution of \( \eta_b \) in the third term from moving \( d \) past the \( e_b \) in the first slot. Together with the external exponent, we again get exactly the total difference of 1 as above, meaning that corresponding terms again cancel.

Finally, there are two more terms, one each in the second and third summands, coming from applying the differential \( d \) to \( e_a \) and \( e_b \) respectively. Lumping the arguments that feed into \( \phi \) and \( \psi \) together into words \( x \) and \( y \) respectively, and ignoring the identical part of the sign coming from cyclic
permutation of the inputs, we see that the two terms are of the form
\[
(-1)^{\eta_b} |x| + (n-3)(|\phi|+1) \bar{g}^{ab} \phi(d_e a, x) \psi(e_b, y) = \\
(-1)^{\eta_b} |x| + (n-3)(|\phi|+1) \bar{g}^{ab} \phi'(e_{a'}, x) \psi(e_b, y),
\]
and
\[
(-1)^{\eta_b'} |x| + (n-3)(|\phi|+1) \bar{g}^{a'b'} \phi(e_{a'}, x) \psi(d_e b', y) = \\
(-1)^{\eta_b'} |x| + (n-3)(|\phi|+1) \bar{g}^{a'b'} \phi'(e_{a'}, x) \psi(e_b, y).
\]
Using \(\eta_b = \eta_{b'} + 1, \eta_{a'} = \eta_a + 1,\) and \(|\phi| = |x| + \eta_a + 1,\) we see that these terms cancel using equation (10.3) of Lemma 10.2.

A similar discussion proves the compatibility of \(p_{1,1,0}\) and \(p_{1,2,0}\).

The remaining four equations are the Jacobi, co-Jacobi, Drinfeld and involutivity relations, respectively. To prove them, we will argue by cut-and-paste techniques on suitable ribbon graphs and their associated surfaces.

**Jacobi and co-Jacobi.** Writing out the Jacobi identity \(p_{2,1,0} \circ_1 \hat{p}_{2,1,0} = 0\) yields the equation
\[
\begin{align*}
& p_{2,1,0}(p_{2,1,0}(\phi, \psi), \theta) + (-1)^{|\phi|+n-3}|\psi|+|\theta| p_{2,1,0}(p_{2,1,0}(\psi, \theta), \phi) + \\
& (-1)^{|\theta|+n-3}|\phi|+|\psi| p_{2,1,0}(p_{2,1,0}(\theta, \phi), \psi) = 0.
\end{align*}
\]

The possible configurations of interior edges for these compositions are depicted in Figure 7, where we have left off the exterior edges for clarity. The first term in the Jacobi identity contains contributions from the first and third graphs, the second term from the first and second graphs, and the last term from the second and third graphs. To compare the signs of the contributions to the first and second summand in the Jacobi identity corresponding to the first graph in Figure 7, we (use the symmetry properties of \(p_{2,1,0}\) to) rewrite the second summand as
\[
(-1)^{|\phi|+n-3} p_{2,1,0}(\phi, p_{2,1,0}(\psi, \theta)).
\]

The two possible compositions are depicted schematically in Figure 8. Consider first the composition corresponding to the cut separating \(\phi\) and \(\psi\) from \(\theta,\) which contributes to the first summand in (10.14). Reading the figure from top to bottom, one sees that successively applying the two operations the sum of the \(\eta_1\)-parts of the sign yield the correct sign to switch from the
Figure 8. Two of the composition in the Jacobi identity that yield the same overall operation up to sign (which corresponds to the first graph in Figure 7).

vertex order
\[ \prod_{v=1}^{3} e_{\alpha(v,1)} \cdots e_{\alpha(v,d(v))} \]
to the edge order
\[ e_{a_1} e_{b_1} e_{a_2} e_{b_1} e_{i_1} \cdots e_{i_{k_1+k_2+k_3}}, \]
where \( e_{a_1} \) and \( e_{b_1} \) are the labels of the left edge (connecting \( \phi \) and \( \psi \)) and \( e_{a_2} \) and \( e_{b_2} \) are attached to the ends of the right edge (connecting \( \psi \) and \( \theta \)).

On the other hand, consider the other possible cut. The \( \eta_1 \)-part of the sign for the first operation now corresponds to moving from the above vertex order to
\[ e_{\alpha(1,1)} \cdots e_{\alpha(1,k_1+1)} e_{a_2} e_{b_2} e_{\gamma(1)} \cdots e_{\gamma(k_1+k_2+1)}, \]
where \( e_{\gamma(i)} \) are the labels of the intermediate exterior vertices created in the cutting process. Moving \( e_{a_2} e_{b_2} \) to the front yields an extra sign of \( |\phi|(n-2) \), and then the \( \eta_1 \)-part of the sign for applying the second operation yields the reordering into
\[ e_{a_2} e_{b_2} e_{a_1} e_{b_1} e_{i_1} \cdots e_{i_{k_1+k_2+k_3}}. \]
Comparing this with the previous outcome, we need an additional exponent of \( n-2 \) to exchange \( e_{a_1} e_{b_1} \) with \( e_{a_2} e_{b_2} \). So the difference in the \( \eta_1 \)-component of the sign is \( |\phi|(n-2) + (n-2) \). The sum of the \( \eta_2 \)-terms for the two operations in the first case is
\[ (n-3)|\phi| + (n-3)(|\phi| + |\psi| + 2 - n) = (n-3)|\psi|, \]
and in the second case it is
\[ (n-3)(|\phi| + |\psi|). \]
In total, the difference in sign exponent of the two ways of producing this output is (also taking into account the “external sign” of the second term)
\[ \frac{|\phi|(n-2) + n-2 + (n-3)|\phi| + |\phi| + n-3 = 1}{\text{difference in } \eta_1} \]
so these terms cancel. Similar discussions apply to the other pairs of terms, and also to the co-Jacobi identity.

\[ \text{Drinfeld.} \quad \text{We next prove the Drinfeld compatibility between bracket and cobracket. Using the common short hand notation } p_{1,2,0} \phi = \phi(1) \otimes \phi(2) \text{ etc, it takes the explicit form} \]

\[
0 = p_{1,2,0} \circ p_{2,1,0}(\phi, \psi) \\
+ (-1)^{|\phi(1)|+n-3} \phi(1) \otimes p_{2,1,0}(\phi(2), \psi) \\
+ (-1)^{|\phi(2)|+n-3(|\psi|+n-3)} p_{2,1,0}(\phi(1), \psi) \otimes \phi(2) \\
+ (-1)^{|\phi|+n-3(|\psi|+n-3)+|\psi(1)|+n-3} \psi(1) \otimes p_{2,1,0}(\psi(2), \phi) \\
+ (-1)^{|\phi|+n-3} \psi(1) \otimes p_{2,1,0}(\phi, \psi(1)) \otimes \psi(2). 
\]

This time, the possible interior parts of the underlying graphs are shown in Figure 9. Let us consider the configuration in Figure 10 which contributes to both \( p_{1,2,0} \circ p_{2,1,0}(\phi, \psi) \) and \( \phi(1) \otimes p_{2,1,0}(\phi(2), \psi) \). The \( \eta_1 \)-part of the sign

\[ \text{Figure 9. Possible interior parts of the graph in the compositions for the Drinfeld identity. The right hand graph comes in four flavours, depending on the orientation of the edges, yielding the four terms in } p_{2,1,0} \circ p_{1,2,0}, \text{ which also appear in } p_{1,2,0} \circ p_{2,1,0}. \text{ The left hand term represents the self-cancelling part of } p_{1,2,0} \circ p_{2,1,0}. \]

\[ \text{Figure 10. A possible configuration of interior edges appearing in the Drinfeld compatibility equation and the two gluing that give rise to it.} \]
of the composition $p_{1,2,0} \circ p_{2,1,0}(\phi, \psi)$ (the horizontal cut) corresponds to moving from the vertex order

$$\prod_{v=1}^{2} e_{\alpha(v,1)} \cdots e_{\alpha(v,d(v))}$$

to the edge order

$$e_{a_1} e_{b_1} e_{a_2} e_{b_2} \prod_{b=1}^{2} e_{\beta(b,1)} \cdots e_{\beta(b,s(b))},$$

and the $\eta_2$-part of the sign exponent is simply $(n-3)(|\phi| + t_1)$, where $t_1$ is the total $B^\text{gc}*A$-degree of the first output (which turns out to be $|\phi(1)|$ from the second point of view).

Let us now consider the vertical cut, which corresponds to $(-1)^{|\phi(1)|+n-3}\phi(1) \otimes p_{2,1,0}(\phi(2), \psi)$. The $\eta_1$ part of the sign for $p_{1,2,0}(\phi)$ corresponds to moving from the above vertex order to

$$e_{a_2} e_{b_2} e_{\beta(1,1)} \cdots e_{\beta(1,s(1))} e_{\gamma(1)} \cdots e_{\gamma(r)} e_{\alpha(2,1)} \cdots e_{\alpha(2,d(2))},$$

and the $\eta_1$ part of the sign of the $p_{2,1,0}$-part of the operation allows one to move this to

$$e_{a_2} e_{b_2} e_{\beta(1,1)} \cdots e_{\beta(1,s(1))} e_{\alpha(2,1)} e_{b_1} e_{\beta(2,1)} \cdots e_{\beta(2,s(2))}.$$  

To get to the same order as in the first case, we need to move $e_{a_1} e_{b_1}$ to the front, yielding an extra contribution to the sign exponent of $(n-2)(|\phi(1)| + n - 2)$. This time, the $\eta_2$-part of the sign exponent equals $(n-3)(|\phi(1)| + |\phi(2)|)$. Together with the “external sign”, the total difference in sign exponents is

$$n - 2 + |\phi(1)| (n - 2) + (n - 3)(|\phi| + |\phi(2)|) + |\phi(1)| + n - 3 = 1,$$

because $|\phi| = |\phi(1)| + |\phi(2)| + 2 - n$. Hence the two contributions cancel. Similar discussions apply to the other three terms involving the second graph in Figure 9.

To complete the proof of Drinfeld compatibility, it remains to discuss the contributions of the first graph in Figure 9. In Figure 11, we show the two possible gluings which yield this configuration, this time with the numbering of vertices and the orientations of the edges included. The latter are chosen so that the outer boundary component is the first output in both cases. Notice that here the difference in sign comes solely from changing the order of the edges and changing the orientation of one of them, which according to points (2g) and (1) in the discussion of signs above yield sign exponents of $n-2$ and $n-3$, respectively, for a total difference of 1 as needed for cancellation.
Involutivity. The involutivity relation follows from an analogous argument, but this time applied to the underlying graph depicted in Figure 12, which gives rise to a genus one surface. Here the orientation of the first edge determines the order of the outputs of the first operation, which by our conventions forces the orientation of the second edge. Again, switching the order of the edges forces the reversal of one of the edge orientations for consistency.

This concludes our proof of Proposition 10.4.

In the above proof we used only rather elementary graphs from our “graphical calculus”. General ribbon graphs will make their appearance in the following section.

11. The dIBL structure associated to a subcomplex.

In this section we relate the dIBL structure associated to \((A, \langle \ , \rangle, d)\) to that of suitable subcomplexes \(B \subset A\) having the same homology. Our main result of this section states that the dIBL-structure on such a subcomplex
is $\text{IBL}_\infty$-homotopy equivalent to the original one. We closely follow [47, Section 6.4].

Let $(A, \langle \cdot, \cdot \rangle, d)$ be as above, i.e.

\[(11.1) \quad \langle dx, y \rangle + (-1)^{|x|} \langle x, dy \rangle = 0, \quad \langle x, y \rangle = -(-1)^{|x||y|} \langle y, x \rangle.\]

Let $B \subset A$ be a subcomplex such that the restriction of $\langle \cdot, \cdot \rangle$ to $B$ is non-degenerate. This means that $B$ is the image of a projection $\Pi: A \to A$ satisfying $\Pi^2 = \Pi$ and

\[(11.2) \quad \Pi d = d \Pi, \quad \langle \Pi x, y \rangle = \langle x, \Pi y \rangle.\]

We assume in addition the existence of a chain homotopy $G: A^* \to A^{*-1}$ such that

\[(11.3) \quad dG + Gd = \Pi - \text{id}, \quad \langle Gx, y \rangle = (-1)^{|x|} \langle x, Gy \rangle.\]

Note that conditions (11.3) imply conditions (11.2). It follows that the inclusion $i: B \to A$ and the projection $p: A \to B$ are chain homotopy inverses of each other, in particular they induce isomorphisms on cohomology. We will be mostly interested in the case that $B$ is isomorphic to the cohomology of $(A,d)$, which is possible due to the following

**Lemma 11.1.** There exists a subcomplex $B \subset \ker d \subset A$ satisfying conditions (11.2) and (11.3) such that

\[\ker d = \text{im} d \oplus B.\]

**Proof.** The proof is a straightforward exercise in linear algebra. Note first that the orthogonal complements with respect to $\langle \cdot, \cdot \rangle$ satisfy

\[(\text{im} d)^\perp = \ker d, \quad (\ker d)^\perp = \text{im} d.\]

We will construct subspaces $B \subset \ker d$ and $C \subset A$ with the following properties:

\[(11.4) \quad A = \ker d \oplus C, \quad \ker d = \text{im} d \oplus B, \quad C \perp B \oplus C.\]

Given such subspaces, it follows from $\ker d \perp \text{im} d$ that $B \perp \text{im} d \oplus C$. Let $\Pi: A \to A$ be the orthogonal projection onto $B$ and define $G: A \to A$ with respect to the decomposition $A = \text{im} d \oplus B \oplus C$ by

\[G(dz, b, c) := (0,0,-z), \quad c, z \in C, \quad b \in B.\]

Then it is easy to verify that $\Pi$ and $G$ satisfy conditions (11.2) and (11.3).

Subspaces $B, C$ satisfying (11.4) can be constructed directly. A more conceptual argument is based on the following

**Claim.** There exist linear operators $\ast: A^k \to A^{n-k}, \quad k \in \mathbb{Z},$
such that \( \langle \cdot, \cdot \rangle := \langle \cdot, \ast \cdot \rangle \) is a positive definite inner product on \( A \) and
\[
\ast^2 = (-1)^{k(n-k)+n} \text{id} : A^k \to A^k.
\]
To construct \( \ast \), suppose first \( k < n/2 \). Pick any metric \( \langle \cdot, \cdot \rangle \) on \( A^k \). It induces an isomorphism
\[
I : A^k \to (A^k)^*, \quad y \mapsto \langle \cdot, y \rangle.
\]
Similarly, \( \langle \cdot, \cdot \rangle \) induces an isomorphism
\[
J : A^{n-k} \to (A^k)^*, \quad y \mapsto \langle \cdot, y \rangle.
\]
Denote by \( \langle \cdot, \cdot \rangle \) the induced metric on \( A^{n-k} \) via the isomorphism \( I^{-1}J : A^{n-k} \to A^k \), i.e.
\[
(x, y) := (I^{-1}J x, I^{-1}J y), \quad x, y \in A^{n-k}.
\]
Define \( \ast \) on \( A^k \oplus A^{n-k} \) by
\[
\ast := J^{-1}I : A^k \to A^{n-k}, \quad \ast := (-1)^{k(n-k)+n-3}I^{-1}J : A^{n-k} \to A^k.
\]
Then \( \ast^2 = (-1)^{k(n-k)+n} \text{id} \) on \( A^k \). For \( x, y \in A^k \) we compute
\[
\langle x, \ast y \rangle = \langle x, J^{-1}I y \rangle = (JJ^{-1}Iy)(x) = (Iy)(x) = (x, y),
\]
\[
\langle \ast x, \ast y \rangle = (-1)^{k(n-k)+n} \langle x, \ast y \rangle = (y, \ast x) = (y, x) = (x, y) = \langle x, y \rangle.
\]
Thus \( \langle \cdot, \cdot \rangle = \langle \cdot, \ast \cdot \rangle \), so \( \ast \) has the desired properties.

For \( n \) even and \( k = n/2 \) we distinguish two cases. If \( k \) is even, then \( \langle \cdot, \cdot \rangle \) is symmetric on \( A^k \) and there exists a basis \( (e_i) \) of \( A^k \) with \( \langle e_i, e_j \rangle = \pm \delta_{ij} \); then \( \ast e_i := \pm e_i \) has the desired properties. If \( k \) is odd, then \( \langle \cdot, \cdot \rangle \) is symplectic on \( A^k \) and there exists a symplectic basis \( (e_i, f_i) \) of \( A^k \) with \( \langle e_i, e_j \rangle = 0 = \langle f_i, f_j \rangle \) and \( \langle e_i, f_j \rangle = \pm \delta_{ij} \); then \( \ast e_i := f_i, \ast f_i := -e_i \) has the desired properties. This proves the claim. Note that so far we have not used the operator \( d \).

From the claim the lemma follows by standard Hodge theory arguments: Define the adjoint operator \( d^* : A^{k+1} \to A^k \) of \( d \) by
\[
(d^* x, y) = (x, dy), \quad x \in A^{k+1}, y \in A^k.
\]
It follows that
\[
d^* = \pm \ast d^* : A^{k+1} \to A^k
\]
and
\[
\langle d^* x, y \rangle = \pm \langle x, d^* y \rangle, \quad x \in A^{k+1}, y \in A^{n-k}.
\]
Define the Laplace operator \( \Delta := dd^* + d^*d \) and
\[
B := \ker \Delta = \ker d \cap \ker d^*, \quad C := \text{im} d^*.
\]
Then the decomposition \( A = \text{im} d \oplus B \oplus C \) is orthogonal with respect to \( \langle \cdot, \cdot \rangle \) and satisfies \([1.1]\). Note that the operator \( G \) is explicitly given by
\[
G = -d^* \Delta^{-1} = -\Delta^{-1}d^*,
\]
where \( \Delta^{-1} \) is zero on \( B \) and the inverse of \( \Delta \) on \( \text{im} d \oplus C \). This concludes the proof of Lemma \([1.1]\) \( \square \).
Given a choice of basis $e_i$ of $A$ and the dual basis $e^i$, we set

$$G^{ab} := \langle Ge^a, e^b \rangle.$$ 

Then we have $G^{ab} \neq 0$ only if $\eta_a + \eta_b = n - 3$. Moreover, from (11.3) and Definition 10.1 one deduces the symmetry properties

(11.5) $$G^{ba} = (-1)^{\eta_a \eta_b + n - 3} G^{ab}.$$ 

**Lemma 11.2.** The equation $dG + Gd = \Pi - \text{id}$ translates into the identity

(11.6) $$d^a\eta_b^{a'b} + (-1)^{\eta_a} d_b^a G^{ab'} = (-1)^{\eta_a} \eta_b^{\bar{a} \bar{b}} - (-1)^{\eta_a} \eta_b^{ab},$$ 

where in the first term on the right hand side the bar signifies that we take the inner product of the images in the subcomplex, i.e.

$$\eta_b^{\bar{a} \bar{b}} := \langle \Pi e^a, \Pi e^b \rangle.$$ 

**Proof.** We do a straightforward computation:

$$\langle dGe^a, e^b \rangle = (-1)^{|e^a|} \langle Ge^a, de^b \rangle$$
$$= (-1)^{|e^a| + n} d_b^a \langle Ge^a, e^{b'} \rangle = d_b^a G^{ab'},$$

(since we need $|e^a| = |e_b|$ for the term to be nonzero) and

$$\langle Gde^a, e^b \rangle = (-1)^{\eta_a} \eta_a^{a'} \langle Ge^{a'}, e^b \rangle$$
$$= (-1)^{\eta_a} \eta_a^{a'} G^{ab'}. $$

Now multiplying the equation

$$\langle (dG + Gd)e^a, e^b \rangle = \langle \Pi e^a, e^b \rangle - \langle e^a, e^b \rangle$$

by $(-1)^{\eta_a}$ gives the claim. \qed

Now we return to $(A, \langle \ , \ \rangle, d)$ and a subcomplex $B \subset A$ satisfying conditions (11.2) and (11.3). We denote the induced structures on $B$ by $d^B$ and $\langle \ , \ \rangle$. They again satisfy condition (11.1). Therefore, Proposition 10.2 equips $(B^\text{cyc}* B)[2 - n]$ with a dIBL algebra structure.

The next theorem, which corresponds to Theorem 1.5 from the Introduction, is the main result of this section.

**Theorem 11.3.** There exists an IBL$_\infty$-homotopy equivalence

$$f : (B^\text{cyc}* A)[2 - n] \to (B^\text{cyc}* B)[2 - n]$$

such that $f_{1,1,0} : (B^\text{cyc}* A)[2 - n] \to (B^\text{cyc}* B)[2 - n]$ is the map induced by the dual of the inclusion $i : B \to A$.

**Proof.** We provide two proofs of Theorem 11.3. We first give a short proof. Take a chain map $j : A \to B$ which is orthogonal with respect to the inner product and is a left inverse to $i : B \to A$. Set $g_{1,1,0} := j^* : (B^\text{cyc}* B)[2 - n] \to (B^\text{cyc}* A)[2 - n]$ and $g_{k,\ell,g} := 0$ for $(k,\ell,g) \neq (1,1,0)$. 


It is easy to see from the definition that this defines an IBL\(_\infty\)-morphism 
\((B^{cyc}\cdot B)[2-n] \rightarrow (B^{cyc}\cdot A)[2-n]\) such that \(g_{1,1,0}\) induces an isomorphism on homology. Therefore, by Theorem 1.2, it has a homotopy inverse \(f\). We can arrange \(f_{1,1,0} = i^*\) by choosing \(f_{1,1,0}\) this way in the first step of the proof of Proposition 5.2. Let us emphasize that the same proof does not work in the opposite direction, with the inclusion \(i\) in place of \(j\). This explains the appearance of the nontrivial terms \(f_{k,\ell,g}\) in the second proof.

We next discuss another proof of Theorem 11.3 which will occupy most of the remainder of this section. This proof provides an explicit description of the map \(f\). We think this explicit description is interesting because of its relation to perturbative Chern-Simons theory, as we explain in Section 13 during the discussion of Conjecture 1.11. Also, it is likely to be useful for the generalization of Theorem 11.3 to the case when \(A\) has infinite dimension.

We will construct \(f\) by summation over general ribbon graphs. Similar constructions using ribbon trees are well-known, see e.g. [47] and [34, Subsection 5.4.2].

Since the \(G^{ab}\) satisfy the symmetry relation (10.12), we can apply the procedure described in the previous section to associate a map 
\[
f_{\Gamma} : (B^{cyc}\cdot A[2-n])^\otimes k \rightarrow (B^{cyc}\cdot B[2-n])^\otimes \ell
\]
to any ribbon graph \(\Gamma \in RG_{k,\ell,g}\) via the formula (10.13), i.e.
\[
(f_{\Gamma}(\varphi^1 \otimes \cdots \otimes \varphi^k))_{\beta(1)\cdots \beta(\ell)} := \frac{1}{\ell! |\text{Aut}(\Gamma)| \prod_v d(v)} \sum (-1)^g \left( \prod_{t \in C^{int}_1(\Gamma)} G^{ab|t} \prod_{v \in C^{int}_0(\Gamma)} \varphi^v_{\alpha(v,1) \cdots \alpha(v,d(v))} \right),
\]
with the conventions as before. Recall that in this definition we sum over all labelings of \(\Gamma\) in the sense of Definition 10.7.

The signs also depend on choices of an ordering and orientations for the interior edges. We will now specify these in such a way that expression (11.7) becomes independent of these additional choices.

Given a ribbon graph \(\Gamma \in RG_{k,\ell,g}\), we have its associated ribbon surface \(\Sigma_\Gamma\), which is already determined by the subgraph \(\Gamma_{\text{int}} \subset \Gamma\) of interior edges. Collapsing the boundary components of \(\Sigma_\Gamma\) to points results in a closed oriented surface \(\tilde{\Sigma}_\Gamma\) of genus \(g\), which comes with a cell decomposition whose vertices and edges correspond to the vertices and edges of \(\Gamma_{\text{int}}\), and whose 2-cells correspond bijectively to the boundary components of \(\Sigma_\Gamma\). Denote by \(\Gamma^{*}_{\text{int}}\) the dual graph, whose vertices correspond to the boundary components of \(\Sigma_\Gamma\) and whose edges are transverse to those of \(\Gamma_{\text{int}}\).
Choose a maximal tree $T \subset \Gamma_{\text{int}}$, which will have $k - 1$ edges, and a maximal tree $T^* \subset \Gamma^*_{\text{int}}$ disjoint from $T$, which will have $\ell - 1$ edges. Denote by $\Gamma' \subset \Gamma_{\text{int}}$ the subgraph of edges from $T$ and of edges dual to those of $T^*$. The graph $\Gamma_{\text{int}}$ has exactly $2g$ further edges. When added to $\Gamma'$, each one of them determines a unique cycle in $\Gamma_{\text{int}} \subset \Sigma_{\Gamma}$, and these cycles form a basis for $H_1(\hat{\Sigma}_{\Gamma})$.

**Definition 11.4 (ordering and orientation of edges).** In formula (11.7), we allow any ordering and orientation of interior edges which arises from choices of $T$ and $T^*$ according to the following rules:

- The oriented edges $e_1, \ldots, e_{k-1}$ are the edges of $T$, oriented away from vertex 1 and numbered such that $e_i$ ends at vertex $k + 1 - i$. In other words, they are numbered in decreasing order of the vertex they point to.

- We orient the edges of $T^*$ away from the first boundary component and label them in increasing order of the boundary component they point to, so that $e_{k+s-2}^*$ points to the boundary component $s$. The oriented edges $e_{k+\ell-2}, \ldots, e_{k+s-2}$ are obtained as the dual edges to the $e_i^*$, oriented so that the pair $\{e_i^*, e_i\}$ defines the orientation of the surface $\Sigma_{\Gamma}$.

- Finally we choose the order and orientation of the remaining edges $e_{k+\ell-1}, \ldots, e_{k+\ell+2g-2}$ compatible with the symplectic structure on $H_1(\hat{\Sigma}_{\Gamma})$ corresponding to the intersection pairing.

![Diagram](image)

**Figure 13.** For this particular graph $\Gamma \in RG_{2,3,0}$, the choices of trees $T$ (in red) and $T^*$ (in green) are unique (exterior edges were omitted for clarity). We also show the numbering and orientation of the interior edges resulting from the given numbering of interior vertices and boundary components.

Of course, the orientations and order of the edges obtained in this way depend on the choices of the trees $T$ and $T^*$. Note that for $g = 0$ the tree
is uniquely determined by the choice of \( T \), and that the conventions here agree with those used for the graphs in \( RG_{2,1,0} \) and \( RG_{1,2,0} \) in the definition of \( p_{2,1,0} \) and \( p_{1,2,0} \).

**Lemma 11.5.** Let \( \Gamma, \tilde{\Gamma} \in RG_{k,\ell,g} \) correspond to the same graph, with the numberings of the interior vertices and boundary components differing by permutations \( \sigma \in S_k \) and \( \tau \in S_\ell \), respectively. Consider pairs of maximal trees \((T, T^*)\) and \((\tilde{T}, \tilde{T}^*)\) as above corresponding to \( \Gamma \) and \( \tilde{\Gamma} \), respectively, and their induced orderings and orientations of edges. Let \( r \) be the number of edges whose orientations differ in the two conventions, and let \( \rho \in S_{k+\ell+2g-2} \) denote the permutation realizing the relabelling of the edges. Then

\[
\text{sgn}(\sigma)\text{sgn}(\tau)\text{sgn}(\rho) = (-1)^r.
\]

This lemma is proved in Appendix A, where our convention for the orientation and ordering of the interior edges is reinterpreted in terms of orientations on the singular chain complex of a surface.

The symmetry properties (1)-(6) described on page 71 above also apply to \( f_{\Gamma} \), with the exception of \((2_g)\), which is replaced by

\[
(2_G) \quad \text{With the specific choice of } T^{ab} = G^{ab}, \text{ interchanging the order of two adjacent edges leads to a sign } (-1)^{n-3} \text{ from interchanging the corresponding pairs of basis vectors in the edge order, because } \eta_a + \eta_b = n - 3 \text{ whenever } G^{ab} \neq 0.
\]

It follows from Lemma 11.5 (with \( \tilde{\Gamma} = \Gamma \) but different pairs of trees) and the symmetry properties (1) and \((2_G)\) that the expression for \( f_{\tilde{\Gamma}} \) is independent of the choice of maximal trees \((T, T^*)\) used to write down \eqref{eq:11.7}. We define \( f_{k,\ell,g} : (B^cyc A[3-n])^\otimes k \to (B^cyc B[3-n])^\otimes \ell \) as

\[
f_{k,\ell,g} := (-1)^{n-3} \sum_{\Gamma \in RG_{k,\ell,g}} f_{\Gamma}.
\]

The symmetry property (4) ensures that \( f_{k,\ell,g} \) indeed lands in \((B^cyc B[3-n])^\otimes \ell\), i.e., each tensor factor in the output is cyclically symmetric. It follows from Lemma 11.5 (with \( \sigma = \text{id} \) and \( \tau \) a transposition) and the symmetry properties (5) and \((2_G)\) that \( f_{k,\ell,g} \) actually lands in the invariant subspace (under the action of the symmetric group \( S_\ell \)) of \((B^cyc B[3-n])^\otimes \ell\). Similarly, it follows from Lemma 11.5 (with \( \sigma \) a transposition and \( \tau = \text{id} \)) and the symmetry properties (6) and \((2_G)\) that \( f_{k,\ell,g} \) descends to the quotient \( E_k B^cyc A \) of \((B^cyc A[3-n])^\otimes k\) under the action of \( S_k \). We now define

\[
f_{k,\ell,g} := \pi \circ f_{k,\ell,g} \circ I : E_k B^cyc A \to E_\ell B^cyc B,
\]
where as in Remark 2.1 the map $I : E_k B^{cyce} A \rightarrow (B^{cyce} A[3 - n])^{\otimes k}$ is the inverse of the projection $\pi : (B^{cyce} A[3 - n])^{\otimes k} \rightarrow E_k B^{cyce} A$, given by

$$I(c_1 \cdots c_k) = \frac{1}{k!} \sum_{\rho \in S_k} \epsilon(\rho) c_\rho(1) \otimes \cdots \otimes c_\rho(k),$$

and similarly $\pi : (B^{cyce} A[3 - n])^{\otimes k} \rightarrow E_k B^{cyce} A$ is the projection to the quotient. Note that, since $f_{k,\ell,g}$ is symmetric in the inputs, the symmetrization and the factor $1/k!$ in $I$ are actually unnecessary and will not appear in formulae below. Note also that we try to distinguish in the notation between $f$ and $\hat{f}$.

**Remark 11.6.** The global sign $(-1)^{n-3}$ in definition (11.8) will be needed for the signs to work out at the end of the proof of Claim 4 below (and similarly for Claim 5). We do not have a conceptual explanation for this sign.

We claim that $\hat{f} = \{f_{k,\ell,g}\}$ is the required IBL_{\infty}-homotopy equivalence. To understand this, we first note that $RG_{1,1,0}$ consists of trees $T_r$ with only one interior vertex and $r \geq 1$ exterior vertices, and that each such tree induces the map

$$f_{T_r} : B^{cyce} A[3 - n] \rightarrow B^{cyce} B[3 - n]$$

which is dual to the inclusion. It follows that the map

$$\hat{f}_{1,1,0} = \sum_r f_{T_r} : B^{cyce} A[3 - n] \rightarrow B^{cyce} B[3 - n]$$

is induced by the dual of the inclusion $B \rightarrow A$ and hence a chain homotopy equivalence. It remains to prove that $\hat{f} = \{f_{k,\ell,g}\}$ is an IBL_{\infty} morphism, since then it follows from Theorem 1.2 that $\hat{f}$ is a homotopy equivalence.

To prove that assertion, we start by considering the difference

$$\hat{f}_{1,1,0} \circ p_{1,1,0} - q_{1,1,0} \circ \hat{f}_{1,1,0}$$

for a fixed graph $\Gamma \in RG_{k,\ell,g}$, where for clarity we denote the restriction of the boundary operator $p_{1,1,0}$ to the subcomplex by $q_{1,1,0}$.

**Claim 1.** All the terms of $q_{1,1,0} \circ \hat{f}_{1,1,0}$ appear in $\hat{f}_{1,1,0} \circ p_{1,1,0}$ with the same sign.

Here and below, we use the notation $\alpha(v) = (\alpha(v, 1), \ldots, \alpha(v, d(v)))$ and $\beta(b) = (\beta(b, 1), \ldots, \beta(b, d(b)))$ for the indices associated to an interior vertex $v$ or a boundary component $b$, respectively.

**Proof of Claim 1.** To prove the claim, consider an exterior edge in $\Gamma \in RG_{k,\ell,g}$ going from vertex $i$ to boundary component $j$. In $(\hat{f}_{1,1,0} \circ p_{1,1,0})(\varphi^1, \ldots, \varphi^k)\beta(1), \ldots, \beta(\ell)$, this contributes

$$(-1)^{e} \sum (1)^{e} \prod_{v < i} G^{v}_{\alpha(v)} \prod_{v > i} \varphi^{v}_{\alpha(v)} \prod_{v \in i} \varphi^{i}_{\alpha(v)};$$
where $\alpha(i) = (\alpha'(i), \alpha(i, r)\alpha''(i))$, and the sign exponents are as follows:

- **the external sign exponent**, coming from the application of $\hat{\phi}_{1,1,0}$, is $\varepsilon = \sum_{b < i}(|\varphi^v| + (n - 3)) + \eta_{\alpha'}(i)$.
- $\eta = \eta_1 + \eta_2$, where $\eta_1$ is the sign exponent corresponding to the permutation

$$\prod_t e_{a_t} e_{b_t} \prod_b e_{\beta(b)} \rightarrow \prod_v e_{\alpha(v)}$$

and $\eta_2$ is given by

$$(n - 3) \left( \sum_v (k - v)|\varphi^v| + (k - i) + \sum_b (\ell - b)|x^b| \right).$$

In the corresponding term in $(\hat{\phi}_{1,1,0} \circ \hat{f}_\Gamma)(\phi^1, \ldots, \phi^k)_{\beta(1)} \ldots \beta(\ell)$, we assume that the considered edge corresponds to the point numbered $s$ on the $j$th boundary component and write $\beta(j) = (\beta'(j)\beta(j, s)\beta''(j))$. The sign exponents are as follows:

- **the external sign exponent**, coming from the application of $\hat{\phi}_{1,1,0}$, is $\varepsilon = \sum_{b < j}(|\varphi^v| + (n - 3)) + \eta_{\alpha'}(j)$.
- $\eta = \eta_1 + \eta_2$, where $\eta_1$ is the sign exponent corresponding to the permutation

$$\prod_t e_{a_t} e_{b_t} \prod_{b < j} e_{\beta'(j)} e_{a} e_{\beta''(j)} \prod_{b > j} e_{\beta(b)} \rightarrow \prod_{v < i} e_{\alpha(v)} e_{\alpha'(i)} e_{a} e_{\alpha''(i)} \prod_{v > i} e_{\alpha(v)}$$

and $\eta_2$ is given by

$$(n - 3) \left( \sum_v (k - v)|\varphi^v| + \sum_b (\ell - b)|x^b| + (\ell - j) \right).$$

To compare the $\eta_1$-part to the previous one, imagine bringing $e_a$ to the front, replacing it by $e_{a(i, r)} = e_{\beta(j, s)}$, and moving it back to its place. Doing this on both sides relates the second permutation to the first permutation, so we have

$$\eta_1(\hat{\phi}_{1,1,0} \circ \hat{f}_\Gamma) - \eta_1(\hat{f}_\Gamma \circ \hat{\phi}_{1,1,0}) \equiv (n - 3)(k + \ell - 2) + \sum_{b < j} |x^b| + \eta_{\beta'(j)} + \sum_{v < i} |\varphi^v| + \eta_{\alpha'}(i),$$

where the first summand reflects the fact that the number of edges of $\Gamma$ is $(k + \ell - 2) \mod 2$. Combining this with

$$\eta_2(\hat{f}_\Gamma \circ \hat{\phi}_{1,1,0}) - \eta_2(\hat{\phi}_{1,1,0} \circ \hat{f}_\Gamma) = (n - 3)((k - i) - (\ell - j))$$

and

$$\varepsilon(\hat{f}_\Gamma \circ \hat{\phi}_{1,1,0}) - \varepsilon(\hat{\phi}_{1,1,0} \circ \hat{f}_\Gamma)$$

$$= (i - 1)(n - 3) + \sum_{v < i} |\varphi^v| + \eta_{\alpha'}(i) - \left( (j - 1)(n - 3) + \sum_{b < j} |x^b| + \eta_{\beta'(j)} \right),$$
we conclude that the total sign exponents are congruent modulo 2, proving Claim 1.

In the remaining terms in $f_G \circ p_{1,1,0}$ the differential $d$ is applied to one of the labels at an interior vertex coming from an interior edge.

**Claim 2.** Given a graph $\Gamma$ and an interior edge $e_{t_0}$ of $\Gamma$, the contributions coming from the differential acting on the two ends of the edge $e_{t_0}$ have the correct relative signs to combine to yield the left hand side of (11.6).

**Proof.** In the proof of this claim, one needs to consider two cases: either the edge $e_{t_0}$ connects two different vertices, or it is a loop. We will treat the second case in detail, the first case is handled the same way.

So assume $e_{t_0} \in C^\text{int}_1(\Gamma)$ is a loop at the $i$th vertex leaving the vertex as the half-edge numbered $r_1$ and coming back as the half-edge $r_2 > r_1$ (the other case $r_2 < r_1$ could be handled similarly). Then the relevant terms come from

$$((-1)^{\epsilon}f_G(\varphi^1, \ldots, p_{1,1,0}\varphi^1, \ldots, \varphi^k)_{\beta(1), \ldots; \beta(\ell)} = (-1)^{\epsilon} \sum_{t} G_{p_{1,1,0}\varphi^1, \ldots, \varphi^k}^{a_{t_0}} \prod_{v < i} \varphi^v_{\alpha(i)} \prod_{v > i} \varphi^v_{\alpha(v)},$$

where the sign exponents are as follows:

- the external exponent is $\epsilon = \sum_{v < i} (|\varphi^v| + (n - 3))$.
- $\eta = \eta_1 + \eta_2$, where $\eta_1$ is the sign of the permutation moving

$$\frac{(11.9)}{\prod} e_{\alpha_1} e_{b_2} \prod_{b} e_{\beta(b)} \rightarrow \prod_{\nu} e_{\alpha(\nu)},$$

and $\eta_2$ is given by

$$(n - 3) \left( \sum_{v} (k - v) |\varphi^v| + (k - i) + \sum_{b} (\ell - b) x^b \right).$$

Freezing all other coefficients, the two relevant terms here are

$$\sum_{a_{t_0}} (-1)^{\eta} \prod_{t} G_{p_{1,1,0}\varphi^1, \ldots, \varphi^k}^{a_{t_0}} (-1) \sum_{r < r_1} \eta_{a(i,r)} d_{a_{t_0}}^{i} \varphi^i_{\alpha(i)a\alpha''(i)b\alpha''(i)b(i)} \prod_{v > i} \varphi^v_{\alpha(v)}$$

and

$$\sum_{b_{t_0}} (-1)^{\eta} \prod_{t} G_{p_{1,1,0}\varphi^1, \ldots, \varphi^k}^{b_{t_0}} (-1) \sum_{r < r_2} \eta_{a(i,r)} d_{b_{t_0}}^{i} \varphi^i_{\alpha(i)a\alpha''(i)b\alpha''(i)b(i)} \prod_{v > i} \varphi^v_{\alpha(v)}$$
where we renamed the variables for better comparison. For the purposes of computing $\eta_1$, the degrees $\eta_{\alpha(i,r_2)}$ differ by one in the two expressions, because in the first setting $\alpha(i,r_2) = b$ but in the second setting $\alpha(i,r_2) = b'$. Imagine doing the permutation (11.9) in stages,

$$\prod_{t} e_{\alpha_t} e_{b_t} \prod_{b} e_{\beta(b)} \rightarrow \prod_{v<i} e_{\alpha(v)} e_{\alpha'(i)} e_{a_{\alpha}} e_{b_{t0}} e_{\alpha''(i)} e_{\alpha'''(i)} \prod_{v>i} e_{\alpha(v)} \rightarrow \prod_{v} e_{\alpha(v)}.$$  

The first stage will give the same sign in both cases, because $\eta_a t_0 + \eta b t_0 = n - 3$ for both. The difference in the second stage will be

$$\sum_{r_1 < r < r_2} \eta_{\alpha(i,r)}.$$  

In total, we get a difference in sign exponent of $\eta_a$ (because $\alpha(i,r_1) = a$ in the second case), which is exactly what is needed to produce the right hand side of (11.6). This finishes the proof of the Claim 2 when $e_{t0}$ is a loop, the other case being similar. □

Claim 2 motivates the following definition.

**Definition 11.7.** Given an edge $e \in C_1^{\text{int}}(\Gamma)$, we define maps $f^\text{id}_{\Gamma,e}$ and $f^\text{II}_{\Gamma,e}$ by a formula analogous to (11.7), with the following modifications:

- to the edge $e$ we associate $(-1)^{\eta_a} g^{ab}$ (for $f^\text{id}_{\Gamma,e}$) resp. $(-1)^{\eta_a} g^{\bar{a}\bar{b}}$ (for $f^\text{II}_{\Gamma,e}$) in place of $G^{ab}$, with $g^{\bar{a}\bar{b}}$ defined in Lemma 11.2;
- the sign $\eta$ gets replaced by $\eta + (n-3)(t_0 - k)$, where $e = e_{t0}$ in the chosen ordering of the edges.

This choice of sign makes the definition independent of the ordering of the edges: Interchanging $e$ with an adjacent edge does not change $\eta_t$ (because the basis vectors assigned to the edge $e$ have total degree $n-2$, while those associated to the other edges have total degree $n-3$), but it yields sign $(-1)^{n-3}$ from replacing $t_0$ by $t_0 \pm 1$. So symmetry property (2.1) still holds and Lemma 11.5 yields independence of the pair of trees $(T, T^*)$ defining the edge ordering.

Summing over all interior edges $e \in C_1^{\text{int}}(\Gamma)$, we get maps

$$f^\text{id}_{\Gamma} := \sum_{e \in C_1^{\text{int}}(\Gamma)} f^\text{id}_{\Gamma,e} \quad \text{and} \quad f^\text{II}_{\Gamma} := \sum_{e \in C_1^{\text{int}}(\Gamma)} f^\text{II}_{\Gamma,e},$$

respectively. In analogy to (11.8), we sum over graphs $\Gamma \in RG_{k,\ell,g}$ to define maps

$$f^\text{id}_{k,\ell,g} := (-1)^{n-3} \sum_{\Gamma \in RG_{k,\ell,g}} f^\text{id}_{\Gamma} \quad \text{and} \quad f^\text{II}_{k,\ell,g} := (-1)^{n-3} \sum_{\Gamma \in RG_{k,\ell,g}} f^\text{II}_{\Gamma},$$

respectively.
To prove Theorem 11.3 it remains to prove equations (2.13) for each triple $(k,\ell,g)$. This is the content of the following sequence of claims.

**Claim 3.** With the definitions above, for each $\Gamma \in RG_{k,\ell,g}$ we have

$$f^\Pi_{\Gamma} - f^{id}_{\Gamma} = \hat{f}_\Gamma \circ \hat{\rho}_{1,1,0} - \hat{q}_{1,1,0} \circ f_\Gamma,$$

and so in particular

$$f^\Pi_{k,\ell,g} - f^{id}_{k,\ell,g} = f_{k,\ell,g} \circ \hat{\rho}_{1,1,0} - \hat{q}_{1,1,0} \circ f_{k,\ell,g}$$

for all $(k,\ell,g) \geq (1,1,0)$.

**Claim 4.** We have

$$f^\Pi_{k,\ell,g} = \hat{q}_{1,2,0} \circ f_{k,\ell-1,g} + \hat{q}_{2,1,0} \circ f_{k,\ell+1,g-1}$$

$$+ \frac{1}{2} \sum_{k_1+k_2=k, \ell_1+\ell_2=\ell+1, g_1+g_2=g} \hat{q}_{2,1,0} \circ q_{1,1} \left( f_{k_1,\ell_1,g_1} \circ f_{k_2,\ell_2,g_2} \right).$$

**Claim 5.** We have

$$f^{id}_{k,\ell,g} = f_{k-1,\ell,g} \circ \hat{\rho}_{2,1,0} + f_{k+1,\ell,g-1} \circ \hat{\rho}_{1,2,0}$$

$$+ \frac{1}{2} \sum_{k_1+k_2=k+1, \ell_1+\ell_2=\ell, g_1+g_2=g} \left( f_{k_1,\ell_1,g_1} \circ f_{k_2,\ell_2,g_2} \right) \circ q_{1,1} \circ \hat{\rho}_{1,2,0}.$$

**Proof of Claim 3.** Claim 3 essentially follows from Claims 1 and 2: By Claim 1, the right hand side is the sum of terms where the differential $d$ is applied to both ends of each interior edge. In view of Claim 2 we can apply Lemma 11.2 to convert it instead into the sum of terms where a particular interior edge is labelled with either $(-1)^y g^{\tilde{a}\tilde{b}}$ of $(-1)^y g^{ab}$, which correspond to the terms on the left hand side. It remains to check that the signs match.

So let $\Gamma \in RG_{k,\ell,g}$ be given and suppose the edge $e$ in $\Gamma$ runs from vertex $i$ to vertex $j$ (the case of a loop is treated similarly). For definiteness we assume $i < j$, and for simplicity we also assume that the corresponding half-edges come first in the respective orders at these vertices. Since by Claim 2 the relative signs of the two terms corresponding to these two half-edges are correct, we just consider the term coming from the half-edge at vertex $i$. The relevant term in $f_{\Gamma} \circ \hat{\rho}_{1,1,0}$ is then of the form

$$(-1)^\varepsilon f_{\Gamma}(\varphi^1, \ldots, \varphi^{1,0}_v \varphi^i, \ldots, \varphi^{k}_v)_{\beta(1),\ldots,\beta(\ell)}$$

$$= (-1)^\varepsilon \sum (\varphi^{1+\eta_2} \prod_{v} G_{\alpha b} (\prod_{v < i} \varphi^v_{\alpha(v)}) (\varphi^{1,0}_v \varphi^i_{\alpha(i)} \prod_{v > i} \varphi^v_{\alpha(v)})$$

with $\varepsilon = \sum_{v<i} (|\varphi^v| + (n-3))$, $\eta_1$ the sign for permuting

$$\prod_t e_{a_t} e_{b_t} \prod_b e_{\beta(b)} \quad \rightarrow \quad \prod_v e_{\alpha(v)},$$

where $e_{a_t} e_{b_t} e_{\beta(b)}$ is the sign for permuting $\varphi^i$.
and
\[ \eta_2 = (n - 3) \left( \sum_v (k - v)|\varphi^v| + (k - i) + \sum_b (\ell - b)|x^b| \right). \]

In \( f^\text{id}_\Gamma \) (and \( f^\Pi_\Gamma \)) we have to replace \( e_{\alpha(i,1)} \) by a basis element whose degree is smaller by 1. One easily checks that this changes \( \eta_1 \) by \( (t_0 - 1)(n - 3) + \sum_{v<i} |\varphi^v| \). At the same time, \( \eta_2 \) changes by \( (k - i)(n - 3) \), since for \( f^\text{id}_\Gamma \) we compute \( \eta_2 \) with the arguments \( \varphi^1, \ldots, \varphi^k \). Together with the external sign \( \varepsilon \), the total sign difference is \( (t_0 + k)(n - 3) \), which exactly fits the extra sign added in the definition of \( f^\text{id}_\Gamma \). This proves of Claim 3.

\[ \square \]

**Proof of Claim 4.** We start with an explanation of the combinatorial factors. Throughout this discussion, we use the notation \( \hat{\Gamma} \) for the graphs with a marked edge \( e \) appearing in \( f^\text{id}_{k,\ell,g} \) and \( f^\Pi_{k,\ell,g} \), and \( \Gamma \) or \( \Gamma_1 \) and \( \Gamma_2 \) for the graphs appearing in the expressions on the right hand side of (11.12).

**Remark 11.8.** (Automorphisms) A term associated to an edge \( e \) of a ribbon graph \( \hat{\Gamma} \in RG_{k,\ell,g} \) appears in \( f^\Pi_{k,\ell,g} \) with the combinatorial coefficient
\[ \frac{1}{\ell! |\text{Aut}(\hat{\Gamma})| \prod_v d(v)}. \]

In order to avoid considerations of how automorphisms of graphs behave under gluings, it is convenient to consider labelled ribbon graphs, i.e., ribbon graphs together with a labelling in the sense of Definition 10.7. Since a labelled graph has no automorphisms preserving the labelling, the automorphism group of an unlabelled graph \( \hat{\Gamma} \) acts freely on its labellings. So in the sum over all labellings of a graph \( \hat{\Gamma} \) each isomorphism class of labelled graphs appears \( |\text{Aut}(\hat{\Gamma})| \) times, and we can replace the sum
\[ \sum_{\hat{\Gamma} \in RG_{k,\ell,g}} \frac{1}{\text{Aut}(\hat{\Gamma})} \sum_{\text{labellings of } \hat{\Gamma}} \]
in the definition of \( f_{k,\ell,g} \) by the sum over all isomorphism classes of labelled ribbon graphs of signature \( (k,\ell,g) \) without the factor \( 1/|\text{Aut}(\hat{\Gamma})| \). This is what we will do in the following discussion.

For a graph \( \hat{\Gamma} \in RG_{k,\ell,g} \) with distinguished edge \( e \) as above we now consider the new ribbon graph \( \Gamma \) obtained by cutting \( e \) into two halves and viewing their endpoints as new exterior vertices \( v', v'' \in \Gamma \). We have three distinct cases:

(i) The dual edge \( e^* \) connects distinct boundary components (\( e \) itself could be a loop, or it could connect distinct vertices). In this case, \( \Gamma \) is necessarily still connected and \( \Gamma \in RG_{k,\ell-1,g} \), and the contribution to \( f^\Pi_\Gamma \) will correspond to a term in \( \hat{q}_{1,2,0} \circ f_{k,\ell-1,g} \). See Figure 14.
Figure 14. On the left we have an example of a graph $\hat{\Gamma} \in RG_{2,2,0}$ with a marked edge $e$ and its dual edge $e^*$ (dotted) as in case (i), both drawn on the ribbon surface $\Sigma_{\hat{\Gamma}}$. On the right one sees the graph $\Gamma \in RG_{2,1,0}$ obtained from cutting open $e$, with its ribbon surface. The dotted line connects the new exterior vertices which will be reconnected by an edge in the corresponding term in $q_{1,2,0} \circ f_{2,1,0}$.

(ii) The dual edge $e^*$ is a loop connecting some boundary component to itself, and $\Gamma$ is still connected. Then $\Gamma \in RG_{k,\ell+1,g+1}$ and the contribution to $f^H_{\Gamma}$ will correspond to a term in $q_{2,1,0} \circ f_{k,\ell+1,g+1}$. See Figure 15.

Figure 15. On the left we have an example of a graph $\hat{\Gamma} \in RG_{3,1,1}$ with a marked edge $e$ and its dual edge $e^*$ (dotted) as in case (ii), both drawn on the ribbon surface $\Sigma_{\hat{\Gamma}}$. On the right one sees the graph $\Gamma \in RG_{3,2,0}$ obtained from cutting open $e$, with its ribbon surface. The dotted line connects the new exterior vertices which will be reconnected by an edge in the corresponding term in $q_{2,1,0} \circ f_{3,2,0}$.

(iii) The dual edge $e^*$ is a loop connecting some boundary component to itself, and $\Gamma = \Gamma_1 \coprod \Gamma_2$ is disconnected. This time the contribution to $f^H_{\Gamma}$ will correspond to a term in $q_{2,1,0} \circ (f_{k_1,\ell_1,g_1} \circ f_{k_2,\ell_2,g_2})$ for suitable $(k_i, \ell_i, g_i)$ corresponding to our two graphs $\Gamma_i$. See Figure 16.
Figure 16. On the top we have an example of a graph $\hat{\Gamma} \in RG_{4,3,0}$ with a marked edge $e$ and its dual edge $e^*$ (dotted) as in case (iii), both drawn on the ribbon surface $\Sigma_{\Gamma}$. On the bottom one sees the graphs $\Gamma_1 \in RG_{1,2,0}$ and $\Gamma_2 \in RG_{3,2,0}$ obtained from cutting open $e$, with their ribbon surfaces. The dotted line connects the new exterior vertices which will be reconnected by an edge in the corresponding term in $q_{2,1,0} \circ (f_{1,2,0} \otimes f_{3,2,0})$.

We now discuss the combinatorial factors $\frac{1}{\ell! \prod_{v=1}^{k} d(v)}$. Consider first a composition $f_{\Gamma'} \circ f_{\Gamma}$ corresponding to a complete gluing of the interior vertices of $\Gamma'$ to the boundary components of $\Gamma$. Recall that in the definition of $f_{\Gamma}$ we sum over all labellings of $\Gamma$, and similarly for $f_{\Gamma'}$. Thus each term in the map $f_{\hat{\Gamma}}$ corresponding to a graph $\hat{\Gamma}$ obtained by gluing $\Gamma$ and $\Gamma'$ appears $\ell! \prod_{w=1}^{k'} d'(w)$ times in $f_{\Gamma'} \circ f_{\Gamma}$. Combining this with the combinatorial factors of $f_{\Gamma}$ and $f_{\Gamma'}$, we see that each term in $f_{\hat{\Gamma}}$ appears with the correct combinatorial factor

$$\left( \frac{1}{\ell! \prod_{v=1}^{k} d(v)} \right) \left( \frac{1}{\ell'! \prod_{w=1}^{k'} d'(w)} \right) \left( \ell! \prod_{w=1}^{k'} d'(w) \right) = \frac{1}{\ell'! \prod_{v=1}^{k} d(v)}.$$

The same discussion also applies to an incomplete gluing where one of the maps, say $f_{\Gamma}$, is replaced by an extended map $\hat{f}_{\Gamma}$. To see this, let us again abbreviate $C := B^{cyc} A[2 - n]$ and consider a map $f : C[1] \otimes_{\ell_1} C[1] \otimes_{\ell_1}$. 


Its extension to a map $\hat{f} : C[1]^\otimes k \to C[1]^\otimes \ell$, with $k = k_1 + r$ and $\ell = \ell_1 + r$ for some $r \geq 1$, is defined by

$$\hat{f}(c_1, \ldots, c_k) := \frac{\ell_1!}{\ell!} \sum_{\sigma \in S_{k_1+r}} \sum_{\tau \in S_{\ell_1+r}} \sum_{\rho \in S_r} \tau\big(f(c_{\sigma(1)} \otimes \cdots \otimes c_{\sigma(k_1)})\big)$$

\[(11.14)\]

$$\otimes c_{\sigma(k_1+\rho(1))} \otimes \cdots \otimes c_{\sigma(k_1+\rho(r))}.$$

Here $S_{p,r}$ denotes the set of $(p,r)$-shuffles, i.e., permutations $\sigma \in S_{p+r}$ with $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(p+r)$. Since the number of $(p,r)$-shuffles is $|S_{p,r}| = \binom{p+r}{r}$, the combinatorial factor can be written as

$$\frac{\ell_1!}{\ell!} = \frac{1}{r!\binom{\ell}{r}} = \frac{1}{|S_r| |S_{\ell_1,r}|}.$$

When considered as a map into the quotient under permutations $E_2C = C[1]^\otimes \ell / \sim$, the averaging over $\tau \in S_{\ell_1+r}$ and $\rho \in S_r$ in the definition of $\hat{f}$ can be dropped and we recover our earlier definition $(2.1)$. Now we apply the extension $(11.14)$ to the map $f_\Gamma$ associated to a ribbon graph $\Gamma \in RG_{k_1,\ell_1,g}$. Then the resulting map $\hat{f}_\Gamma : C[1]^\otimes k \to C[1]^\otimes \ell$ descends to the quotient $E_2C$ and lands in the invariant part of $C[1]^\otimes \ell$, a property it shares with the map $f_\Gamma$ itself. Moreover, the combinatorial factors in the definition of $f_\Gamma$ and in $(11.14)$ combine to the combinatorial factor for $\hat{f}_\Gamma$:

$$\left(\frac{1}{\ell_1! \prod_{v=1}^{k_1} d(v)}\right)\left(\frac{1}{\ell! \prod_{v=1}^{\ell_1} d(v)}\right) = \frac{1}{\ell! \prod_{v=1}^{k_1} d(v)}.$$

Using this, we see that in the composition $f_\Gamma \circ \hat{f}_\Gamma$ corresponding to an incomplete gluing each term in $f_\Gamma$ appears with the correct combinatorial factor

$$\left(\frac{1}{\ell_1! \prod_{v=1}^{k_1} d(v)}\right)\left(\frac{1}{\ell! \prod_{v=1}^{\ell_1} d(v)}\right)\left(\frac{\ell_1!}{\ell! \prod_{v=1}^{\ell_1} d(v)}\right) = \frac{1}{\ell! \prod_{v=1}^{k_1} d(v)}.$$

Here $s(b)$ is the number of vertices on the $b$-th boundary component of $\Sigma_\Gamma$ and we have used the fact that $\ell - \ell_1 = k - k_1$ factors in $\prod_{w=1}^{\ell'} d'(w)$ combine with $\prod_{v=1}^{k_1} d(v)$ to give $\prod_{v=1}^{k} d(v)$ in the denominator, while the remaining $\ell_1$ terms cancel $\prod_{b=1}^{\ell_1} s(b)$. A similar (in fact, easier) argument applies in the case of an incomplete gluing of the type $\hat{f}_\Gamma \circ f_\Gamma$.

Consider now a gluing of two labelled graphs $\Gamma, \Gamma'$. Suppose that each interior vertex of $\Gamma'$ is glued to a boundary component of $\Gamma$, but some boundary components of $\Gamma$ may remain free. Such a gluing is described uniquely in terms of the following gluing data:
• an injective map \( \lambda : \{1, \ldots, k'\} \to \{1, \ldots, \ell\} \) such that \( d'(v) = s(\lambda(v)) \) for all \( v = 1, \ldots, k' \), where \( d'(v) \) is the degree of the vertex \( v \) of \( \Gamma' \) and \( s(b) \) is the number of vertices on the \( b \)-th boundary component of \( \Gamma \);
• for each \( v = 1, \ldots, k' \) a bijection \( c_v : \{1, \ldots, d'(v)\} \to \{1, \ldots, s(\lambda(v))\} \) preserving the cyclic orders.

Note that the resulting ribbon graph \( \Gamma \# \Gamma' \) naturally inherits a labelling from those of \( \Gamma \) and \( \Gamma' \). Moreover, different gluing data give rise to different isomorphism classes of labelled graphs \( \Gamma \# \Gamma' \). This shows that the sums over isomorphism classes of labelled graphs on both sides in Claim 4 agree without further combinatorial factors due to automorphisms.

The preceding considerations show that in all three cases the combinatorial factors of the corresponding terms on both sides of (11.12) match. Here the additional factor \( 1/2 \) in case (iii) is due to the fact that for each split graph \( \Gamma = \Gamma_1 \amalg \Gamma_2 \) the two terms \( f_{\Gamma_1} \odot f_{\Gamma_2} \) and \( f_{\Gamma_2} \odot f_{\Gamma_1} \) of the right-hand side of (11.12) correspond to the same term on the left-hand side. This finishes the discussion of combinatorial factors.

**Signs.** We now discuss the signs involved in formula (11.12), starting with case (i) above. So let \( \hat{\Gamma} \in RG_{k,\ell,g} \) be given and assume for simplicity that the special edge \( e = e_{t_0} \) separates the boundary components labelled 1 and 2 and is dual to the first edge of \( T^* \). According to our edge ordering conventions in Definition 11.4 we then have \( t_0 = k \). As a consequence, the sign of the corresponding term in \( f_{\Pi}^{\hat{\Gamma}} \) according to Definition 11.7 is just the “usual” sign \((-1)^{\eta_1(\hat{\Pi}) + \eta_2(\hat{\Pi})}\), where \( \eta_1(\hat{\Pi}) \) is the sign of the permutation

\[
\prod_t e_{a_t}e_{b_t} \prod_b e_{\beta(b)} \quad \longrightarrow \quad \prod_v e_{\alpha(v)}
\]

and

\[
\eta_2(\hat{\Pi}) = (n - 3) \left( \sum_{v=1}^{k} (k - v)|\varphi^v| + \sum_{b=1}^{\ell} (\ell - b)|x^b| \right).
\]

To understand the signs in the corresponding term in \( \hat{q}_{1,2,0} \circ f_{k,\ell-1,g} \), let \( \Gamma \) as before be the graph obtained from \( \hat{\Gamma} \) by cutting \( e_{t_0} \) in half and adding exterior vertices at the new end points. We can assume that the resulting new boundary component is labelled 1, and all other boundary components have their labelling decreased by 1, and that the trees \( T(\Gamma) \) and \( T^*(\Gamma) \) agree with \( T(\hat{\Gamma}) \) and \( T^*(\hat{\Gamma}) \setminus e_{t_0}^* \). Then \( \eta_1(f_{\Gamma}) \) is the sign of the permutation

\[
\left( \prod_{t \neq t_0} e_{a_t}e_{b_t} \right) e_{\beta'(1)} \prod_{b \geq 3} e_{\beta(b)} \quad \longrightarrow \quad \prod_v e_{\alpha(v)},
\]

where \( \beta'(1) \) is the vector of labels at the new first boundary component, and \( \eta_1(q) \) is the sign of the permutation

\[
e_{a_{t_0}}e_{b_{t_0}}e_{\beta(1)}e_{\beta(2)} \quad \longrightarrow \quad e_{\beta'(1)}.
\]
Figure 17. The relevant situation for the term of $q_{1,2,0} \circ f_\Gamma$ in case (i) for which the signs discussed. We have drawn the edge $e = e_{t_0}$ in blue and the dual edge $e^*$ dotted in red. We only show the endpoints of $e$ (which could also coincide), since the remaining part of the graph $\Gamma$ is irrelevant to the discussion.

Since permuting $e_{a_{t_0}}e_{b_{t_0}}$ past the other pairs of edge elements does not introduce signs (because $\eta_{a_{t_0}} + \eta_{b_{t_0}} = n - 2$ and the sum equals $n - 3$ for all other edges), we see that the sum of these terms matches $\eta_1(f^{\Pi})$.

Denoting by $\bar{x} \in B_{\text{cyc}}^e B$ the element associated to the new first boundary component of $\Gamma$, we have

$$\eta_2(f_\Gamma) = (n - 3) \left( \sum_{v=1}^{k} (k - v)|\varphi^v| + (\ell - 2)|\bar{x}| + \sum_{b \geq 3} (\ell - b)|x^b| \right)$$

and

$$\eta_2(q) = (n - 3)|x^1|.$$

Since $|\bar{x}| = |x^1| + |x^2| + n - 2$, we see that their sum also matches $\eta_2(f^{\Pi})$, so the two terms appear on both sides with the same sign.

Finally we discuss case (iii), leaving the slightly easier case (ii) to the reader. Here we start again with $\tilde{\Gamma} \in RG_{k,\ell,g}$, and assume that cutting $e = e_{t_0}$ results in graphs $\Gamma_1 \in RG_{k_1,\ell_1,g_1}$ and $\Gamma_2 \in RG_{k_2,\ell_2,g_2}$. For simplicity, we also assume that the vertices of $\Gamma_1$ are labelled $1, \ldots, k_1$ and the boundary components it inherits from $\tilde{\Gamma}$ have labels $1, \ldots, \ell_1 - 1$, with the new one being the last component (labelled $\ell_1$), whereas the new boundary component of $\Gamma_2$ is
Figure 18. The assumptions for the term of \( \hat{q}_{2,1,0} \circ (f_{\Gamma_1} \otimes f_{\Gamma_2}) \) in case (iii). Again we have drawn the edge \( e = e_t \), but not the remaining parts of the graphs \( \Gamma_1 \) and \( \Gamma_2 \).

its first (again labelled \( \ell_1 \)), followed by the inherited boundary components labelled \( \ell_1 + 1, \ldots, \ell = \ell_1 + \ell_2 - 1 \). Note that \( e_t \) will necessarily belong to \( T \). For convenience we assume that \( e_t \) ends at vertex \( k_1 + 1 \in \Gamma_2 \subset \hat{\Gamma} \), so that according to our conventions in Definition 11.4 we have \( t_0 = k_2 \).

In particular, the sign of this term in \( f_{\Pi_{k,\ell,g}} \) according to Definition 11.7 is \( \eta_1 + \eta_2 + (n - 3)(k + k_2) \), where \( \eta_1 \) and \( \eta_2 \) are standard as before.

The corresponding term in \( \hat{q}_{2,1,0} \circ (f_{\Gamma_1} \otimes f_{\Gamma_2}) \) involves the following signs. Denoting by \( \beta'(\ell_1) \) and \( \beta''(\ell_1) \) the vectors of labels occurring at the new boundary components of \( \Gamma_1 \) and \( \Gamma_2 \), respectively, the \( \eta_1 \)-parts of the signs of \( \hat{q}_{2,1,0} \circ f_{\Gamma_1} \) and \( f_{\Gamma_2} \) are respectively the signs of the permutations

\[
e_{a_0} e_{b_0} e_{\beta(\ell_1)} \rightarrow e_{\beta'(\ell_1)} e_{\beta''(\ell_1)};
\]

\[
\prod_{t \in C_{1}^{\text{in}}(\Gamma_1)} e_{a_t} e_{b_t} \left( \prod_{b < \ell_1} e_{\beta(b)} \right) e_{\beta'(\ell_1)} \rightarrow \prod_{v \leq k_1} e_{a(v)}
\]

and

\[
\left( \prod_{t \in C_{1}^{\text{in}}(\Gamma_2)} e_{a_t} e_{b_t} \right) e_{\beta''(\ell_1)} \prod_{b > \ell_1} e_{\beta(b)} \rightarrow \prod_{v \geq k_1 + 1} e_{a(v)}.
\]
To understand the difference in the $\eta_1$-part of the sign on both sides, we write the permutation corresponding to $\eta_1(\tilde{f}^H_t)$ in stages, as

$$\prod_t e_a_t e_b_t \prod_b e_\beta(b)$$

(1) $\prod_{t \in C_1^{int}(\Gamma_2)} e_{a_t} e_{b_t} \prod_{t \in C_1^{int}(\Gamma_1)} e_{a_t} e_{b_t} e_{a_0} e_{b_0} \prod_b e_\beta(b)$

(2) $\prod_{t \in C_1^{int}(\Gamma_2)} e_{a_t} e_{b_t} \prod_{t \in C_1^{int}(\Gamma_1)} e_{a_t} e_{b_t} e_{\beta(b)} \prod_{b \leq \ell_1} e_{a_0} e_{b_0} \prod_{b > \ell_1} e_\beta(b)$

(3) $\prod_{t \in C_1^{int}(\Gamma_2)} e_{a_t} e_{b_t} \prod_{t \in C_1^{int}(\Gamma_1)} e_{a_t} e_{b_t} e_{\beta(b)} \prod_{b \leq \ell_1} e_{\beta'(\ell_1)} e_{\beta''(\ell_1)} \prod_{b > \ell_1} e_\beta(b)$

(4) $\prod_{t \in C_1^{int}(\Gamma_2)} e_{a_t} e_{b_t} \prod_{v \leq k_1} e_{\alpha(v)} e_{\beta''(\ell_1)} \prod_{b > \ell_1} e_\beta(b)$

(5) $\prod_{v \leq k_1} e_{\alpha(v)} \prod_{t \in C_1^{int}(\Gamma_2)} e_{a_t} e_{b_t} e_{\beta''(\ell_1)} \prod_{b > \ell_1} e_\beta(b)$

(6) $\prod_{v} e_{\alpha(v)}$

The sign exponents are as follows:

in (1) it is $(n - 3)(\ell_2 - 1)(k_1 + \ell_1)$ for moving the edges of $T^*(\Gamma_2)$ and $H_1(\Gamma_2)$ past the edges of $\Gamma_1$,

in (2) it is $(n - 2) \sum_{b < \ell_1} |x^b|$ for moving the edge $e_{t_0}$ past these boundary components,

in (3) it is simply $\eta_1(q)$,

in (4) it is simply $\eta_1(f_{\Gamma_1})$,

in (5) it is $(n - 3)(k_2 + \ell_2) \sum_{v \leq k_1} |\varphi''|)$ for moving the edges of $\Gamma_2$ past the inputs of $f_{\Gamma_1}$, and

in (6) it is simply $\eta_1(f_{\Gamma_2})$.

Here the sign exponent in (1) may require some explanation. Recall that according to Definition 11.4 the interior edges are ordered as follows: edges in $T_2$ (in reverse order), $e_{t_0} = e_{k_2}$, edges in $T_1$ (in reverse order), edges dual to $T_1^*$, edges dual to $T_2^*$, and finally the remaining edges generating $H_1(\Sigma_{\Gamma})$. Since the sign exponent of the variables on $e_{t_0}$ is $n - 2$ and the sign exponents on all other edges are $n - 3$, and the edges generating $H_1(\Sigma_{\Gamma})$ appear in pairs and thus do not contribute to the sign, the sign exponent for moving the edges in $\Gamma_2$ to the first position (in the correct order) comes from moving $T_2^*$ past $T_1$ and $T_2^*$, hence equals $(n - 3)(\ell_2 - 1)(k_1 + \ell_1)$. Note that for $i = 1, 2$ the orientations and orderings of the edges in $\Gamma_i$ induced by the trees $T_i, T_i^*$ according to Definition 11.4 (oriented away from the first vertex resp. boundary component) agree with those induced by the trees $T, T^*$ in
Γ because $e_{t_0}$ was assumed to end at the first vertex $k_1 + 1$ of $\Gamma_2$. Had $e_{t_0}$ ended at a different vertex $k_1 + s$, then the total change in sign comparing $T(\Gamma)$ with $T(\Gamma_2)$ would contribute a sign exponent $(n - 3)(s - 1)$, which would cancel with the same change in the extra sign in Definition 11.7.

In total, the difference in sign exponents for $\eta$ is

$$(n - 3) \left( (\ell_2 - 1)(k_1 + \ell_1) + \sum_{b < \ell_1} |x^b| + (k_2 + \ell_2) \sum_{v \leq k_1} |\varphi^v| \right) + \sum_{b < \ell_1} |x^b|.$$ 

Let us denote by $x^{\ell_1}_{\eta}$ and $x^{\ell_1}_{\overline{\eta}}$ the terms at the new boundary components of $f_{\Gamma_1}$ and $f_{\Gamma_2}$ (or equivalently, the inputs of $q_{2,1,0}$). Replacing $\sum_{v \leq k_1} |\varphi^v|$ using the relation

$$\sum_{b < \ell_1} |x^b| + |x^{\ell_1}_{\eta}| + (k_1 + \ell_1)(n - 3) \equiv \sum_{v \leq k_1} |\varphi^v| \mod 2,$$

the preceding sign exponent can be rewritten as

$$(11.15) \quad (n - 3) \left( k_2 \sum_{v \leq k_1} |\varphi^v| + (\ell_2 + 1) \sum_{b < \ell_1} |x^b| + x^{\ell_1}_{\overline{\eta}} - (k_1 + \ell_1) \right) + \sum_{b < \ell_1} |x^b|.$$ 

Next we note that

$$\eta_2(q) = (n - 3)|x^{\ell_1}_{\eta}|,$$

$$\eta_2(f_{\Gamma_1}) = (n - 3) \left( \sum_{v \leq k_1} (k_1 - v)|\varphi^v| + \sum_{b < \ell_1} (\ell_1 - b)|x^b| \right) \quad \text{and}$$

$$\eta_2(f_{\Gamma_2}) = (n - 3) \left( \sum_{v > k_1} (k - v)|\varphi^v| + (\ell - \ell_1)|x^{\ell_1}_{\eta}| + \sum_{b > \ell_1} (\ell - b)|x^b| \right).$$

It follows (using $|x^{\ell_1}_{\eta}| = |x^{\ell_1}_{\eta}| + |x^{\ell_1}_{\overline{\eta}}| + n - 2$) that the total sign difference in $\eta_2$ can be written as

$$(11.16) \quad (n - 3) \left( k_2 \sum_{v \leq k_1} |\varphi^v| + (\ell_2 - 1) \sum_{b < \ell_1} |x^b| + \ell_2 |x^{\ell_1}_{\eta}| \right).$$

Comparing with (11.15), we see that this cancels the first three terms there, so we get a total sign difference in $\eta_1 + \eta_2$ of

$$(n - 3)(k_1 + \ell_1) + \sum_{b < \ell_1} |x^b|.$$ 

Finally, note that the external sign of $\tilde{q}_{2,1,0}$ (from moving $q_{2,1,0}$ past the outputs $x^1, \ldots, x^{\ell_1-1}$) contributes $(n - 3)(\ell_1 - 1) + \sum_{b < \ell_1} |x^b|$ and the extra
sign of $f^\Pi$ contributes $(n - 3)(k + k_2)$ to exponents. We conclude that the
total difference in sign exponents in the two terms is

$$(n - 3)(\ell_1 - 1 + k + k_2 + k_1 + \ell_1) = (n - 3).$$

Now recall that the definition (11.8) of $f_{k,\ell,g}$ in terms of the $f_{11}$
involves a global sign $(-1)^{n-3}$, and similarly for the definition (11.10) of $f_{k,\ell,g}^\Pi$. Since
in (11.12) the last term is quadratic in $f$ and all other terms are linear, this
cancels the sign difference that we just computed. This concludes the proof
of Claim 4.

**Proof of Claim 5.** The proof of Claim 5 is analogous to that of Claim 4. This
time, given a graph $\hat{\Gamma} \in RG_{k,\ell,g}$ with a marked edge $e$, the graph $\Gamma$ (or more
precisely its ribbon surface $\Sigma_{\Gamma}$) is obtained by cutting out a neighborhood of
the edge $e$ in $\Sigma_{\hat{\Gamma}}$ and collapsing each resulting new “boundary” component
to a new vertex. Again there are three cases to consider:

(i) If the edge $e$ connects different vertices, then the resulting graph $\Gamma \in
RG_{k-1,\ell,g}$ is obtained simply by collapsing the edge $e$ in $\hat{\Gamma}$. Here the
contribution to $f^\Pi_{\Gamma}$ corresponds to a term in $f_{k-1,\ell,g} \circ \hat{p}_{2,1,0}$. See Figure 19.

(ii) If the edge $e$ is a loop at a vertex $v$ in $\hat{\Gamma}$ such that $\hat{\Gamma} \setminus \{v,e\}$ is connected,
then the graph $\Gamma \in RG_{k+1,\ell,g-1}$ is obtained by deleting $e$ and splitting
the vertex $v$ into two new vertices $v', v'' \in \Gamma$ whose incident half-edges
belong to the two (ordered) collections of half-edges of $\hat{\Gamma}$ incident
to $v$ which form the complement of $e$. The contribution to $f^\Pi_{\Gamma}$ will
correspond to a term in $f_{k+1,\ell,g-1} \circ \hat{p}_{1,2,0}$, $\hat{p}_{1,2,0}$. See Figure 20.

(iii) If the edge $e$ is a loop at a vertex $v$ in $\hat{\Gamma}$ such that removing $v$ from $\hat{\Gamma} \setminus e$
disconnects the remaining graph, then $\Gamma = \Gamma_1 \sqcup \Gamma_2$ is disconnected, and
the contribution to $f^\Pi_{\Gamma}$ will correspond to a term in $(f_{k_1,\ell_1,g_1} \circ \hat{f}_{k_2,\ell_2,g_2}) \circ \hat{p}_{1,2,0}$. See Figure 21.

As the discussion of combinatorial factors and signs follows the lines of
argument used in Claim 4, we leave the remaining details to the reader.

This concludes the proof of Claims 1–5, and thus of Theorem 11.3.

**The filtration on the dual cyclic bar complex.** Consider a cyclic
cochain complex $(A, \langle , \rangle, d)$ as in Section 11. The space

$$B^{\text{cyc}*}A = \bigoplus_{k \geq 1} B_k^{\text{cyc}*}A$$

carries a natural filtration by the subspaces

$$F^\lambda B^{\text{cyc}*}A := \bigoplus_{k \geq \lambda} B_k^{\text{cyc}*}A.$$

Figure 19. On the top we have the graph \( \hat{\Gamma} \in \text{RG}_{4,3,0} \) from Figure 16 with the same marked edge \( e \), which now corresponds to case (i) in Claim 5, both drawn on the ribbon surface \( \Sigma_{\hat{\Gamma}} \). On the bottom one sees the graph \( \Gamma \in \text{RG}_{3,3,0} \) obtained by contracting \( e \), drawn on its ribbon surface. The dotted circle marks the new vertex which receives the output from \( p_{2,1,0} \) in the corresponding term in \( f_{3,3,0} \circ p_{2,1,0} \).

Its completion with respect to this filtration,

\[ \overline{B^{cyc}} A = \prod_{k \geq 1} B^{cyc}_{k} A = \text{Hom}(B^{cyc} A, \mathbb{R}), \]

inherits a filtration by the subspaces

\[ \mathcal{F}^\lambda \overline{B^{cyc}} A := \{ \varphi \in \overline{B^{cyc}} A \mid \varphi|_{B^{cyc}_{k} A} = 0 \text{ for all } k < \lambda \}. \]

Recall that the operations \( p_{1,1,0}, p_{2,1,0}, p_{1,2,0} \) defining the dIBL structure and the operations \( f_{k,\ell,g} \) in Theorem 11.3 are defined by summation over certain ribbon graphs. Consider a ribbon graph \( \Gamma \) with \( k \) interior vertices of degrees \( d(1), \ldots, d(k) \) and \( s = s(1) + \cdots + s(\ell) \) exterior vertices distributed on the \( \ell \) boundary components of the corresponding ribbon surface \( \Sigma \). Then the number of exterior edges equals \( s \) and the number of interior edges equals \( (d-s)/2 \), where \( d = d(1) + \cdots + d(k) \) (since each interior edge meets precisely
Figure 20. On the left we have the graph $\hat{\Gamma} \in RG_{3,1,1}$ with a marked edge $e$ as in Figure 15 which now corresponds to case (ii) in Claim 5, drawn on its ribbon surface $\Sigma_{\hat{\Gamma}}$. On the right one sees the resulting graph $\Gamma \in RG_{4,1,0}$ obtained by removing $e$ and splitting its vertex, drawn on its ribbon surface. The dotted circles mark the new vertices which receive the output from $p_{1,2,0}$ in the corresponding term in $f_{1,1,0} \circ p_{1,2,0}$.

two interior vertices. It follows that

$$2 - 2g - \ell = \chi(\Sigma) = \chi(\Gamma) = |C^0|_\text{int} + |C^0|_\text{ext} - |C^1|_\text{int} - |C^1|_\text{ext} = k + s - \frac{d - s}{2} - s,$$

so $f_{k,\ell,g}$ has filtration degree

$$\|f_{k,\ell,g}\| = s - d = 2(2 - 2g - k - \ell) = 2\chi_{k,\ell,g}.$$

Similarly, $p_{k,\ell,g}$ has filtration degree $\|p_{k,\ell,g}\| = 2\chi_{k,\ell,g}$ for $(k, \ell, g)$ equal to $(1, 1, 0)$, $(2, 1, 0)$ or $(1, 2, 0)$. Thus the operations $p_{k,\ell,g}$ and $f_{k,\ell,g}$ are filtered in the sense of Definitions 8.1 and 8.4 with $\gamma = 2$. Note that they are $\mathbb{N}_0$-gapped in the sense of Definition 8.6 for the discrete submonoid $\mathbb{N}_0 = \{0, 1, 2, \ldots\} \subset \mathbb{R}_{\geq 0}$, where all the higher terms $p^j_{k,\ell,g}$ and $f^j_{k,\ell,g}$ vanish for $j \geq 1$. Hence Proposition 10.4, Theorem 11.3 and Proposition 8.11 imply

Corollary 11.9. The operations $p_{1,1,0}, p_{1,2,0}, p_{2,1,0}$ in Proposition 10.4 induce on $C = B^{\text{cyc}*}A[2 - n]$ a strict $\mathbb{N}_0$-gapped filtered dIBL-structure of bidegree $(d, \gamma) = (n - 3, 2)$. Moreover, the operations $f_{k,\ell,g}$ in Theorem 11.3 induce a strict $\mathbb{N}_0$-gapped filtered IBL$_\infty$-homotopy equivalence between $B^{\text{cyc}*}A[2 - n]$ and $B^{\text{cyc}*}B[2 - n]$. \hfill \Box

This corollary will be the basis for our discussion of Maurer-Cartan elements in the following section.
Figure 21. On the top we have a graph $\hat{\Gamma} \in RG_{3,3,0}$ with a marked edge $e$ corresponding to case (iii) in Case 5, both drawn on the ribbon surface $\Sigma_{\hat{\Gamma}}$. On the bottom one sees the resulting graphs $\Gamma_1 \in RG_{2,1,0}$ and $\Gamma_2 \in RG_{2,2,0}$ obtained by removing $e$ and splitting its vertex, drawn on their ribbon surfaces. The dotted circles mark the new vertices which receive the output from $p_{1,2,0}$ in the corresponding term in $(f_{2,1,0} \odot f_{2,2,0}) \circ p_{1,2,0}$.

12. The dual cyclic bar complex of a cyclic DGA

In this section we show that for a cyclic DGA the dIBL-structure on its dual cyclic bar complex comes with a natural Maurer-Cartan element. This gives rise to a twisted dIBL-structure on the dual cyclic bar complex, and thus to a twisted IBL$_\infty$-structure on its homology. In this way, we prove Proposition 1.6 and Theorem 1.7 from the Introduction.

We begin by considering cyclic $A_\infty$-structures.

Definition 12.1. Let $(A, \langle \cdot, \cdot \rangle, d)$ be a cyclic cochain complex with pairing of degree $-n$ and set $m_1 := d$. A series of operations

$$m_k : A[1]^{\otimes k} \to A[1], \quad k \geq 2$$
Remark 12.2. ∗∗ where satisfies (12.1)

\[ m \cdot \sum_{k+\ell=r+1}^{r+1-\ell} \sum_{c=1}^{\infty} (-1)^c m_k(x_1, \ldots, m_\ell(x_c, \ldots, x_{c+\ell-1}), \cdots, x_r) = 0, \]

where ∗ = deg \( x_1 + \cdots + deg x_{c-1} + c - 1 \), and

\[ (m_k(x_1, \cdots, x_k), x_0) = (-1)^{**}(m_k(x_0, \cdots, x_{k-1}), x_k), \]

where ** = (deg \( x_0 + 1 \))(deg \( x_1 + \cdots + deg x_k + k \)).

We refer to a cyclic \( A_\infty \)-algebra \( (A, \langle , \rangle, \{m_k\}) \) with \( m_k = 0 \) for \( k \geq 3 \) as a cyclic DGA. Note that, as usual, to go from \( m_2 \) to a multiplication on \( A \) which makes it a differential graded algebra in the usual sense involves adding signs; see (13.1) below for a possible convention.

For operations \( m_k : A[1]^{\otimes k} \rightarrow A[1] \) we define \( m_k^+ : A[1]^{\otimes (k+1)} \rightarrow \mathbb{R} \) by

\[ m_k^+(x_0, x_1, \cdots, x_k) := (-1)^{n-2}(m_k(x_0, \cdots, x_{k-1}), x_k). \]

Then \( m_k \) satisfies (12.2) if and only if \( m_k^+ \in B^\text{cyc}A_{k+1} \). In this case we obtain an element

\[ m^+ := \sum_{k \geq 2} m_k^+ \in \hat{\mathcal{C}} := \hat{B}^\text{cyc}A[2-n]. \]

Note that \( m^+ \) is homogeneous of degree \( n-3 \) in \( \hat{B}^\text{cyc}A \) and so it has degree \( 2(n-3) \) when viewed as an element of \( E_1 \mathcal{C} = \hat{\mathcal{C}}[1] = \hat{B}^\text{cyc}A[3-n] \). Moreover, \( m^+ = \sum_{k \geq 2} m_k^+ \) has filtration degree at least 3 with respect to the degree \( k \) in \( B^\text{cyc}A \), so it satisfies the grading and filtration conditions for a Maurer-Cartan element in the filtered dIBL-algebra (of bidegree \( (n-3, 2) \))

\[ (\mathcal{C} = B^\text{cyc}A[2-n], \mathfrak{p}_{1,1,0}, \mathfrak{p}_{1,2,0}, \mathfrak{p}_{2,1,0}) \]

from Corollary 11.9.

Proposition 12.3. Let \( \{m_k\}_{k \geq 2} \) satisfy (12.2) of Definition 12.1. Then it satisfies (12.1) if and only if

\[ \mathfrak{p}_{1,1,0}m^+ + \frac{1}{2}\mathfrak{p}_{2,1,0}(m^+, m^+) = 0 \]

in \( \hat{E}_1 \mathcal{C} \).

Proof. Consider \( k \geq 2 \) and any element \( \phi \in B^\text{cyc}_{k+1}A \), and set \( r := k + \ell \). Since \( |m^+| = n-3 \), the relation (10.11) between \( \mathfrak{p}_{2,1,0} \) and \( \mu \) yields

\[ \mathfrak{p}_{2,1,0}(m_k^+, \phi) = (-1)^{n-3} \mu(m_k^+, \phi). \]

The global sign \((-1)^{n-2}\) is inserted to make Proposition 12.3 below true. We do not have a conceptual explanation for this.
The formula (10.5) for $\mu$ then yields

$$p_{2,1,0}(m_k^+, \phi)(x_1, \ldots, x_r) = \sum_{c=1}^{r} (-1)^{\nu_c} b_k(x_c, \ldots, x_{c-1}),$$

with

$$b_k(x_1, \ldots, x_r) = \sum_{a,b} (-1)^{\eta_{a} + \eta_{b}(|x_1| + \cdots + |x_k|) + n - 3} g_{ab} m_k^+(e_a, x_1, \ldots, x_k) \phi(e_b, x_{k+1}, \ldots, x_r)$$

and the sign

$$(12.5) \quad \nu_c = (|x_1| + \cdots + |x_{c-1}|) (|x_c| + \cdots + |x_r|).$$

Abbreviating $x := (x_1, \ldots, x_k)$ and using the symmetries of $m_k^+$ and $\langle \ , \ \rangle$, we compute

$$m_k^+(e_a, x) = (-1)^{\eta_a |x|} m_k^+(x, e_a)$$

$$= (-1)^{\eta_a |x| + n - 2} \langle m_k(x), e_a \rangle$$

$$= (-1)^{\eta_a |x| + n - 2 + \eta_a (|x| + 1 + 1) + 1} \langle e_a, m_k(x) \rangle$$

$$= (-1)^{\eta_a + n - 3} \langle e_a, m_k(x) \rangle.$$ 

Using the relation

$$\sum_{a,b} g_{ab} (e_a, z) e_b = z$$

we obtain

$$\sum_{a,b} (-1)^{\eta_a + n - 3} g_{ab} m_k^+(e_a, x) e_b = m_k(x).$$

Next note that in the formula for $b_k$ we have the relations $\eta_b = \eta_a + (n - 2) = (|x| + n - 3) + (n - 2) = |x| + 1 \pmod{2}$, so the term $\eta_b |x|$ is even and can be dropped from the sign exponent. Inserting the previous formula, we obtain

$$b_k(x_1, \ldots, x_r) = \phi \left( m_k(x_1, \ldots, x_k), x_{k+1}, \ldots, x_r \right),$$

and therefore

$$p_{2,1,0}(m_k^+, \phi)(x_1, \ldots, x_r)$$

$$(12.6) \quad = \sum_{c=1}^{r} (-1)^{\nu_c} \phi \left( m_k(x_c, \ldots, x_{c+k-1}), x_{c+k}, \ldots, x_{c-1} \right).$$

Let us now insert $\phi = m_k^+$ with $\ell \geq 2$ in this formula and consider a summand with $1 \leq c \leq \ell$. Then $x_r$ appears in the argument of $m_k^+$ and we can rewrite
the summand as

\[( -1)^{n} m^{+}_{k} \left( m_{k}(x_{c}, \ldots, x_{c+k-1}), x_{c+k}, \ldots, x_{c-1} \right) \]

\[= ( -1)^{n} m^{+}_{k} \left( x_{1}, \ldots, m_{k}(x_{c}, \ldots, x_{c+k-1}), x_{c+k}, \ldots, x_{r} \right) \]

\[= ( -1)^{n-2} m^{+}_{k} \left( x_{1}, \ldots, m_{k}(x_{c}, \ldots, x_{c+k-1}), x_{c+k}, \ldots, x_{r-1} \right) , x_{r} \right) \]

with \( \ast = \nu_{c} + (|x_{1}| + \cdots + |x_{c-1}|)(|x_{c}| + \cdots + |x_{r}| + 1) = |x_{1}| + \cdots + |x_{c-1}| \)

as in Definition 12.1. For \( r + 1 \leq c \leq k + \ell \) we obtain a similar expression with the roles of \( m_{k} \) and \( m_{\ell} \) interchanged. It follows that

\[\frac{1}{2} p_{2,1,0}(m^{+}, m^{+})(x_{1}, \ldots, x_{r}) = \sum_{k+\ell=r} \frac{1}{2} p_{2,1,0}(m^{+}_{k}, m^{+}_{\ell})(x_{1}, \ldots, x_{r}) \]

\[= ( -1)^{n-2} \left( \sum_{k+\ell=r, k \geq 2} \sum_{c=1}^{r-\ell} ( -1)^{c} m^{+}_{k} \left( x_{1}, \ldots, m_{\ell}(x_{c}, \ldots, x_{c+\ell-1}), \ldots, x_{r-1} \right), x_{r} \right) \]

Note that the last sum contains all terms appearing in (12.1) with \( k, \ell \geq 2 \).

The missing terms appear in

\[ ( -1)^{n-2} p_{1,1,0} m^{+}_{r-1}(x_{1}, \ldots, x_{r}) \]

\[= ( -1)^{n-2} \sum_{c=1}^{r} ( -1)^{|x_{1}| + \cdots + |x_{c-1}|} m^{+}_{r-1}(x_{1}, \ldots, dx_{c}, \ldots, x_{r}) \]

\[= \sum_{c=1}^{r-1} ( -1)^{|x_{1}| + \cdots + |x_{c-1}|} \langle m_{r-1}(x_{1}, \ldots, m_{1}(x_{c}), \ldots, x_{r-1}), x_{r} \rangle \]

\[+ ( -1)^{|x_{1}| + \cdots + |x_{r-1}|} \langle m_{r-1}(x_{1}, \ldots, x_{r-1}, dx_{r}) \rangle \]

\[= \left( m_{1}(m_{r-1}(x_{1}, \ldots, x_{r})) + \sum_{c=1}^{r-1} ( -1)^{c} m_{r-1}(x_{1}, \ldots, m_{1}(x_{c}), \ldots, x_{r-1}), x_{r} \right) \].

Here in the last equation we have used the relation \( \langle dx, y \rangle = ( -1)^{|x|}\langle x, dy \rangle \).

Combining the preceding equations, we obtain

\[( -1)^{n-2} \left( p_{1,1,0} m^{+} + \frac{1}{2} p_{2,1,0}(m^{+}, m^{+}) \right)(x_{1}, \ldots, x_{r}) \]

\[= \left( \sum_{k+\ell=r, k \geq 1} \sum_{c=1}^{r-\ell} ( -1)^{c} m^{+}_{k}(x_{1}, \ldots, m_{\ell}(x_{c}, \ldots, x_{c+\ell-1}), \ldots, x_{r-1}), x_{r} \right) \]

and Proposition 12.3 follows. \( \Box \)

Let \( (A, \langle , \rangle, m) \) be a cyclic \( A_{\infty} \)-algebra. Proposition 12.3 implies that the twisted differential \( p^{m^{+}_{1,1,0}}: \overline{C} \to \overline{C} \),

\[p^{m^{+}_{1,1,0}}(\varphi) := p_{1,1,0}(\varphi) + p_{2,1,0}(m^{+}, \varphi) \]
squares to zero. For \( \phi \in \mathcal{C}^\text{cy} \mathfrak{A} \) and \( r = k + \ell, k \geq 1 \), the component of \( \mathfrak{p}_{1,1,0,1}^m(\varphi) \) in \( \mathcal{C}_r \mathfrak{A} \) is given explicitly by

\[
\mathfrak{p}_{1,1,0}^m(\varphi)(x_1, \ldots, x_r)
\]

(12.7) 

\[
= \sum_{c=1}^r (-1)^{\nu_c} \phi\left( m_c(x_c, \ldots, x_{c+k-1}), x_{c+k}, \ldots, x_{c-1} \right),
\]

with the sign exponent \( \nu_c \) defined in (12.5). (According to (12.6) this formula holds for \( k \geq 2 \), and one easily verifies that it also holds for \( k = 1 \).) Note that the differential \( \mathfrak{p}_{1,1,0}^m \) depends only on the \( A_\infty \)-operations \( m_k \) and not on the pairing \( \langle \ , \ \rangle \).

**Remark 12.4.** In the case of a cyclic DGA (i.e. \( m_k = 0 \) for \( k \geq 3 \)), formula (12.7) shows that the twisted differential \( \mathfrak{p}_{1,1,0}^m \) is dual to the Hochschild differential on Connes’ cyclic complex (see [50]), so its homology equals Connes’ version of cyclic cohomology. The precise relation to the definitions of cyclic cohomology appearing in the literature in the \( A_\infty \)-case (such as [36, 48, 69]) will be discussed elsewhere.

Let \((\mathfrak{a}, \langle \ , \rangle, \mathfrak{m})\) be a cyclic \( A_\infty \)-algebra. As above, let

\[
(\mathcal{C} = B^{\text{cy} \mathfrak{a}} \mathfrak{A}[2-n], \mathfrak{p}_{1,1,0,1}, \mathfrak{p}_{1,2,0,1}, \mathfrak{p}_{2,1,0})
\]

be the filtered IBL-algebra of bidegree \((n-3,2)\) from Corollary 11.9. According to Proposition 12.3, the degree \( 2(n-3) \) element \( m^+ \in E_1 \mathcal{C} = \mathcal{C}[1] = B^{\text{cy} \mathfrak{a}} \mathfrak{A}[3-n] \) satisfies the first part of the Maurer-Cartan equation (9.3) in Lemma 9.2. Let us consider the second part of the Maurer-Cartan equation, \( \mathfrak{p}_{1,2,0}(m^+) = 0 \). Using the relation \( \sum_a g^{ab} e_b = e^a \), we compute for \( k_1, k_2 \geq 1 \), \( k = k_1 + k_2 + 1 \geq 3 \):

\[
\delta(m^+_k)(x_1 \cdots x_{k_1} \otimes y_1 \cdots y_{k_2})
\]

\[
= \sum_{a,b} \sum_{c=1}^{k_1} \sum_{c'=1}^{k_2} (-1)^{n_a + \eta} g^{ab} m^+_k(e_a, x_c, \ldots, x_{c-1}, e_b, y_{c'}, \ldots, y_{c'-1})
\]

\[
= \sum_{c=1}^{k_1} \sum_{c'=1}^{k_2} \sum_a (-1)^{n_a + \eta + n - 2} (e_a, m_k(x_c, \ldots, x_{c-1}, e^a, y_{c'}, \ldots, y_{c'-1}))
\]

where \( \eta \) is given by (10.7). This expression does not vanish for a general cyclic \( A_\infty \)-algebra, so \( \mathfrak{p}_{1,2,0}(m^+) = 0 \) does not hold in general. However, \( \mathfrak{p}_{1,2,0}(m^+_k) \) vanishes for degree reasons if \( k \leq 2 \), so in view of Lemma 9.2, Proposition 9.3 and Remark 12.4 we conclude:

**Proposition 12.5.** If \((\mathfrak{a}, \langle \ , \rangle, \mathfrak{m}_1 = d, \mathfrak{m}_2)\) is a cyclic DGA, then \( m^+_2 \) defines a Maurer-Cartan element in the filtered dIBL-algebra \((\mathcal{C} = B^{\text{cy} \mathfrak{a}} \mathfrak{A}[2-n], \mathfrak{p}_{1,1,0,1}, \mathfrak{p}_{1,2,0,1}, \mathfrak{p}_{2,1,0})\).
n], p_{1,1,0} = d, p_{1,2,0}, p_{2,1,0}). It induces a twisted filtered dIBL-structure
\[ p_{\ell+2} = \{ p_{1,1,0} + p_{2,1,0}(m_{\ell}^+, \cdot), p_{1,2,0}, p_{2,1,0} \} \]
on whose homology equals Connes’ version of cyclic cohomology of \( (A, d, m_2) \).

The special case of a DGA in Proposition 12.5 will suffice for the purposes of this paper. More generally, the preceding computation shows that
\[ p_{1,2,0}(m^+) = 0 \]
holds if the \( m_k \) are “traceless” in the sense that
\[ \sum a \langle e_a, m_k(x_1, \ldots, x_k, e^a, x_{k+1}, \ldots, x_{k+k_2}) \rangle = 0 \]
for all \( x_i \) and all \( k_1, k_2 \geq 1, k = k_1 + k_2 + 1 \geq 3 \). The actual condition we need is weaker:

**Definition 12.6.** A solution of the genus zero master equation for a cyclic \( A_\infty \) algebra \( (A, \langle \cdot, \cdot \rangle, \{ m_k \}_{k=1}^\infty) \) is a sequence of elements \( m_{(\ell)}^+ \in \hat{E}_\ell(B^{cy\ast} A)[2-n] \) for \( \ell = 1, 2, \ldots \) such that

- \( m_{(1)}^+ \) coincides with \( m^+ \) defined by (12.4), and
- the following Batalin-Vilkovisky master equation is satisfied:

\[ (12.8) \quad p_{1,0} m_{(\ell)}^+ + p_{1,2,0}(m_{(\ell-1)}^+) + \frac{1}{2} \sum_{i=1}^\ell \hat{p}_{2,0}(m_{(\ell)}^+, m_{(\ell-i+1)}^+) = 0. \]

**Remark 12.7.** (1) See Barannikov-Kontsevich [7], Barannikov [4, 5], Costello [26], Kontsevich-Soibelman [48] for related results. The authors thank B. Vallette for a remark which is closely related to this point.
(2) In the context of symplectic field theory, equation (12.8) corresponds to equation (16) in [20] and to equation (44) in [28]. It also coincides with the genus zero case of equation (5.5) in [5].
(3) The relation between the “traceless” property and equation (12.8) has appeared in [6, Theorem 5].
(4) As will be discussed elsewhere, the second term \( m_{(2)}^+ \) of the solution of the genus zero master equation is related to the Hodge to de Rham degeneration in Lagrangian Floer theory; see [48]. The role of the solution of the genus zero master equation in Lagrangian Floer theory is discussed at the end of the Introduction; see also Remark 12.12.

It is straightforward to check that equation (12.8) is equivalent to the condition that \( m^+ = \{ m_{(\ell)}^+ \}_{\ell=1}^\infty \) is a Maurer-Cartan element of the dIBL-algebra \( (C = B^{cy\ast} A[2-n], p_{1,1,0}, p_{1,2,0}, p_{2,1,0}) \), so we have the following generalization of Proposition 12.5.
Proposition 12.8. Let \((A, \langle , \rangle, \{m_k\}_{k=1}^{\infty})\) be a cyclic \(A_\infty\)-algebra with a solution of the genus zero master equation \(\{m_k^{+}\}_{k=1}^{\infty}\). Then \(m^+ = \{m_k^{+}\}_{k=1}^{\infty}\) defines a Maurer-Cartan element of \((C = B^{cyc}\Lambda[2-n], p_{1,1,0}, p_{1,2,0}, p_{2,1,0})\).

Finally, we consider the induced structure on cohomology. Let \((A, \langle , \rangle, d, m_2)\) be a cyclic DGA and \(H = H(A, d)\) its cohomology. The inner product descends to cohomology, so by Corollary 11.9, it induces a \(G\)-gapped filtered IBL-structure \(\{q_{1,2,0}, q_{2,1,0}\}\) on \(B^{cyc}\Lambda[2-n]\). Moreover, due to Lemma 11.1 and Corollary 11.9 (with \(B \cong H\)), there exists a filtered IBL_{\infty}-homotopy equivalence
\[
\hat{f} : (B^{cyc}\Lambda)[2-n] \to (B^{cyc}\Lambda)[2-n]
\]
such that \(\hat{f}_{1,1,0} : \hat{B}^{cyc}\Lambda[2-n] \to \hat{B}^{cyc}\Lambda[2-n]\) is the map induced by the dual of the inclusion \(i : H \cong B \to A\) from Lemma 11.1. According to Lemma 9.5, the Maurer-Cartan element \(m_2^+\) on \(B^{cyc}\Lambda[2-n]\) from Proposition 12.5 can be pushed forward via \(\hat{f}\) to a Maurer-Cartan element \(\hat{f}_*m_2^+\) on \(B^{cyc}\Lambda[2-n]\). By Proposition 9.3, this induces a twisted filtered IBL_{\infty}-structure
\[
q^{\hat{f}_*m_2^+} = \{q_{1,1,0}^{\hat{f}_*m_2^+}, q_{2,1,0}\}_{\ell \geq 1, \rho \geq 0}
\]
on \(B^{cyc}\Lambda[2-n]\). By Proposition 9.6, this structure is homotopy equivalent to the twisted filtered dIBL-structure \(p^{m_2^+} = \{d + p_{2,1,0}(m_2^+, \cdot), p_{1,2,0}, p_{2,1,0}\}\) on \(B^{cyc}\Lambda[2-n]\) in Proposition 12.5. The situation of Proposition 12.8 can be handled in the same way.

Thus we have proved the following theorem, whose first part corresponds to Theorem 1.7 in the Introduction.

Theorem 12.9. (a) Let \((A, \langle , \rangle, d, m_2)\) be a cyclic DGA with cohomology \(H = H(A, d)\). Then \(B^{cyc}\Lambda[2-n]\) carries a \(G\)-gapped filtered IBL_{\infty}-structure which is homotopy equivalent to the twisted filtered dIBL-structure \(p^{m_2^+}\) on \(B^{cyc}\Lambda[2-n]\) in Proposition 12.5. In particular, its homology equals Connes’ version of cyclic cohomology of \((A, d, m_2)\).

(b) More generally, let \((A, \langle , \rangle, \{m_k\}_{k=1}^{\infty})\) be a cyclic \(A_\infty\)-algebra with a solution of the genus zero master equation \(\{m_k^{+}\}_{k=1}^{\infty}\) and cohomology \(H = H(A, m_1)\). Then \(B^{cyc}\Lambda[2-n]\) carries a filtered IBL_{\infty}-structure which is homotopy equivalent to the twisted filtered IBL_{\infty}-structure \(p^{m_2^+}\) on \(B^{cyc}\Lambda[2-n]\).

Remark 12.10. The construction in Section 10 gives the following explicit description of the Maurer-Cartan element \(\hat{f}_*m_2^+\) on \(B^{cyc}\Lambda[2-n]\), where \(H\) is the cohomology of a cyclic DGA. Denote by \(RG_{k,\ell,\rho}\) the set of isomorphism classes of ribbon graphs of signature \((k, \ell, g)\) all of whose interior vertices
are trivalent. Then
\[(f_\ast m^+_2)_{e,\varrho} = \sum_{k=1}^{\infty} \sum_{\Gamma \in R\mathcal{C}_k} n_\Gamma \in \widehat{E}_\ell (B^{\text{cyc}*} H)[2-n],\]
where the numbers
\[n_\Gamma (x_1^1 \cdots x_{s_1}^1 \otimes \cdots \otimes x_\ell^1 \cdots x_{s_\ell}^\ell)\]
for \(x_j^b \in H\) are defined as in Section 10 using the following assignments (compare Figure 22):

\begin{itemize}
  \item To the \(j\)-th exterior vertex on \(\partial_b \Sigma\) we assign \(x_j^b\).
  \item To each interior vertex of \(\Gamma\) we assign \(m^+_2\).
  \item To each interior edge we assign the element dual to the map \(x \otimes y \mapsto \langle Gx, y \rangle\).
\end{itemize}

There is a similar description in the case of a cyclic \(A_\infty\) algebra with a solution of the genus zero master equation.

**Remark 12.11.** The existence of a filtered \(I\mathcal{B}_{\infty}\) \(\ast\)-structure on \(B^{\text{cyc}*} H[2-n]\) in Theorem 12.9 (a) also follows from [4, Theorem 2] by using Remark 12.7 and Proposition 9.3.

**Remark 12.12.** Let \(m_k\) be a cyclic filtered \(A_\infty\) algebra \(A\) defined over the universal Novikov ring \(\Lambda_0\) with ground field \(\mathbb{K}\) (see [33, Introduction] for its definition). We decompose the differential as
\[m_1 = \overline{m}_1 + m_{+,1}\]
where \( m_{+1} \equiv 0 \mod \Lambda_+ \) and \( \overline{m}_1 \) is induced from a \( K \)-linear map \( \overline{C} \to \overline{C} \). Here \( \overline{C} \) is a \( K \)-vector space and \( C = \overline{C} \hat{\otimes}_K \Lambda_0 \). We do not assume \( m_0 = 0 \), but we do assume \( m_{1+} \equiv 0 \mod \Lambda_+ \) (see [34, 33]).

We define \( m^+ \) by using \( m_k \), \( k > 1 \), and \( m_{1+} \) in the same way as in (12.4). Let \( m^+_{(\ell)} \) be a solution of the genus zero master equation. We assume \( m^+_{(\ell)} \equiv 0 \mod \Lambda_+ \) for \( \ell \geq 2 \). It is easy to see that \( m^+_{(1)} = m^+ \) together with \( m^+_{(\ell)} \), \( \ell \geq 2 \), defines a Maurer-Cartan element of the filtered dIBL-algebra obtained from \( (\overline{C}, \langle, \rangle, \overline{m}_1) \hat{\otimes}_K \Lambda_0 \).

We consider \( H := H(C, m_1) \). (Note that \( H \) is in general different from the homology of \( (C, m_1) \), even in the case \( m_0 = 0 \).) Now we apply Theorem 11.3 to obtain a twisted filtered IBL\(_{\infty}\)-structure on \( B^{\infty}H \).

13. The dual cyclic bar complex of the de Rham complex

In this section we prove Proposition 1.10 and Theorem 1.11 from the Introduction and discuss Conjecture 1.11. Throughout this section, we work over the ring \( R = \mathbb{R} \).

Fréchet IBL\(_{\infty}\)-algebras. We start by recalling some basic facts about Fréchet spaces, see e.g. [58]. A Fréchet space \( X \) is a topological vector space whose topology is defined by a countable family of semi-norms \( \| \cdot \|_i \), \( i \in \mathbb{N} \), such that \( X \) is complete with respect to the metric

\[
\text{dist}(x, y) = \sum_{i=1}^{\infty} 2^{-i} \frac{\|x - y\|_i}{1 + \|x - y\|_i}.
\]

The basic example is the space \( C^\infty(M, \mathbb{R}) \) of smooth functions on a closed manifold \( M \) with the semi-norms \( \|f\|_k = \max_M |D^k f| \), where \( D^k f \) is the total \( k \)-th covariant derivative with respect to some connection. In the same way the space \( \Omega(M) \) of smooth differential forms also becomes a Fréchet space.

According to [59], \( \Omega(M) \) belongs to a special class of Fréchet spaces called nuclear spaces and two Fréchet spaces \( X, Y \) have a natural tensor product \( X \hat{\otimes} Y \) as a Fréchet space in case they are nuclear spaces. It is defined as a suitable completion of the algebraic tensor product \( X \otimes Y \) and characterized by the usual universal property. For two closed manifolds \( M, N \) the canonical inclusion \( \Omega(M) \otimes \Omega(N) \subset \Omega(M \times N) \) (via the wedge product of differential forms) induces an isomorphism

\[
\Omega(M) \hat{\otimes} \Omega(N) \cong \Omega(M \times N).
\]

Consider now a graded Fréchet space \( C = \bigoplus_{k \in \mathbb{Z}} C^k \) and define the degree shifted space \( C[1] \) as usual. The action of the symmetric group permuting
the factors of the algebraic tensor product $C[1] \otimes \cdots \otimes C[1]$ extends to the completion, so we can define the completed symmetric product

$$\hat{E}_k C \subset C[1] \hat{\otimes} \cdots \hat{\otimes} C[1]$$

as the (closed) subspace invariant under the action of the symmetric group with the usual signs (cf. Remark 2.1). We set

$$\hat{EC} := \bigoplus_{k \geq 1} \hat{E}_k C.$$

Note that the meaning of $\hat{E}_k C$ and $\hat{EC}$ in this section differs from that in Section 8. Consider a series of continuous linear maps

$$p_{k,\ell,g} : \hat{E}_k C \to \hat{E}_\ell C, \quad k, \ell \geq 1, \ g \geq 0$$

of degree

$$|p_{k,\ell,g}| = -2d(k + g - 1) - 1$$

for some fixed integer $d$ and define the operator

$$\hat{p} := \sum_{k,\ell=1}^{\infty} \sum_{g=0}^{\infty} \hat{p}_{k,\ell,g} h^{k+g-1} \tau^{k+\ell+2g-2} : \hat{EC}\{h,\tau\} \to \hat{EC}\{h,\tau\}$$

as before.

**Definition 13.1.** We say that $(C, \{p_{k,\ell,g}\}_{k,\ell \geq 1, g \geq 0})$ is a Fréchet IBL$_\infty$-algebra of degree $d$ if

$$\hat{p} \circ \hat{p} = 0.$$

The notions of Fréchet IBL$_\infty$-morphisms and homotopies are defined in the obvious way, requiring all maps to be continuous linear maps between completed symmetric products. All the results from Sections 2 to 9 carry over to the Fréchet case.

**Remark 13.2.** If $C$ is finite dimensional (or more generally, filtered with finite dimensional quotient spaces), then a Fréchet IBL$_\infty$-structure on $C$ is just an IBL$_\infty$-structure in the sense of Definition 2.3.

**The de Rham complex.** Let $M$ be a closed oriented manifold of dimension $n$ and $(\Omega(M), d, \wedge)$ its de Rham complex. Here $d$ is the exterior differential and $\wedge$ the wedge product on differential forms. Together with integration over $M$ this is almost a cyclic DGA in the sense of Remark 12.2, since we can define a “cyclic $A_\infty$-structure” with inner product of degree $-n$ and with $m_k = 0$ for $k \geq 3$ by setting

$$m_1(u) := du,$$

$$m_2(u, v) := (-1)^{\deg u} u \wedge v,$$

$$(u, v) := (-1)^{\deg u} \int_M u \wedge v.$$
We want to apply the arguments of Sections 10 and 11 to this situation. However, the de Rham complex is infinite dimensional and the pairing in (13.1) is not perfect. In fact, the dual to the space of smooth forms is the space of currents. In the following we will explain a method to overcome this difficulty. In the rest of this section, we will omit some sign computations.

**Definition 13.3.** A homomorphism

$$\varphi : \Omega(M)[1] \otimes \cdots \otimes \Omega(M)[1] \to \mathbb{R}$$

$k$ times

is said to have smooth kernel if there exists a smooth differential form $K_\varphi$ on $M^k = M \times \cdots \times M$ such that for $u_i \in \Omega(M)$ we have

$$\varphi(u_1 \otimes \cdots \otimes u_k) = \int_{M^k} (u_1 \times \cdots \times u_k) \wedge K_\varphi$$

where $\times : \Omega(M) \otimes \Omega(M) \to \Omega(M^2)$ is the exterior wedge product. We call $K_\varphi$ the kernel of $\varphi$. We write

$$B^\ast_k \Omega(M)_{\infty}$$

for the subspace of such $\varphi$. We set

$$B^\ast_k \Omega(M)_{\infty} := B^\ast_k \Omega(M)_{\infty} \cap B^\ast_k \Omega(M)$$

and

$$B^\ast \Omega(M)_{\infty} := \bigoplus_{k \geq 1} B^\ast_k \Omega(M)_{\infty}.$$ 

Note that the condition for an element of $B^\ast_k \Omega(M)_{\infty}$ to belong to $B^\ast_k \Omega(M)_{\infty}$ is equivalent to an appropriate symmetry of the kernel with respect to cyclic permutation of variables.

**Lemma 13.4.** The differential $p_{1,0}: B^\ast_k \Omega(M) \to B^\ast_k \Omega(M)$ induced by $m_1 = d$ preserves the subspace $B^\ast \Omega(M)_{\infty} \subset B^\ast_k \Omega(M)$.

**Proof.** For $\varphi \in B^\ast_k \Omega(M)$ with kernel $K_\varphi$ we compute

$$p_{1,0}(\varphi)(u_1 \otimes \cdots \otimes u_k) = \sum_i \pm \varphi(u_1 \otimes \cdots \otimes du_i \otimes \cdots \otimes u_k)$$

$$= \int_{M^k} \sum_i \pm (u_1 \times \cdots \times du_i \times \cdots \times u_k) \wedge K_\varphi$$

$$= \pm \int_{M^k} (u_1 \times \cdots \times u_k) \wedge dK_\varphi,$$

so $\pm dK_\varphi$ is a kernel for $p_{1,0}(\varphi)$.

We next define a completion of the algebraic tensor products $B^\ast_k \Omega(M)_{\infty} \otimes B^\ast_k \Omega(M)_{\infty}$. Note that the assignment $\varphi \mapsto K_\varphi$ defines a canonical inclusion

$$B^\ast_k \Omega(M)_{\infty} \subset \Omega(M^k).$$
This induces an inclusion of the algebraic tensor product
\[(13.4)\quad B_k^{\text{cyc}}\Omega(M)_\infty \otimes B_\ell^{\text{cyc}}\Omega(M)_\infty \subset \Omega(M^{k+\ell}).\]

By the discussion above, the \textit{completed tensor product}
\[
B_k^{\text{cyc}}\Omega(M)_\infty \hat{\otimes} B_\ell^{\text{cyc}}\Omega(M)_\infty
\]
as Fréchet spaces equals the closure of the image of the inclusion \((13.4)\) with respect to the \(C^\infty\) topology. We define the completed tensor product among three or more \(B_k^{\text{cyc}}\Omega(M)_\infty\)'s in the same way.

As above, we introduce the \textit{completed symmetric product}
\[
\widehat{E}_m\left(B^{\text{cyc}}\Omega(M)_\infty[2-n]\right)
\]
\[
:= \left( \bigoplus_{k_1,k_2,\ldots,k_m \geq 1} B_{k_1}^{\text{cyc}}\Omega(M)_\infty[3-n] \hat{\otimes} \ldots \hat{\otimes} B_{k_m}^{\text{cyc}}\Omega(M)_\infty[3-n] \right) / \sim.
\]

By Remark 2.1, we can identify this quotient with the subspace of elements in \(B_{k_1}^{\text{cyc}}\Omega(M)_\infty \otimes \ldots \otimes B_{k_m}^{\text{cyc}}\Omega(M)_\infty\) that are invariant under the action of the symmetric group on \(m\) elements (with appropriate signs). In this way it is canonically embedded into \(\Omega(M^{k_1+\cdots+k_m})\). The following result corresponds to Proposition 1.9 from the introduction.

**Proposition 13.5.** \(B_k^{\text{cyc}}\Omega(M)_\infty[2-n]\) carries the structure of a Fréchet dIBL-algebra of degree \(n-3\).

**Proof.** We will give here the proof modulo signs, and postpone the discussion of signs to Remark 13.9 below.

Consider \(\varphi \in B_{k_1+1}^{\text{cyc}}\Omega(M)_\infty\), \(\psi \in B_{k_2+1}^{\text{cyc}}\Omega(M)_\infty\) with kernels \(\mathcal{K}_\varphi\), \(\mathcal{K}_\psi\). Let \(k := k_1 + k_2\) and consider \(M^{k+1}\) with coordinates \((t,x_1,\ldots,x_k)\). We define \(p_{2,1,0}(\varphi,\psi) \in B_k^{\text{cyc}}\Omega(M)_\infty\) by the analogue of equation \((10.5)\):

\[
p_{2,1,0}(\varphi,\psi)(u_1 \cdots u_k)
\]
\[
= \sum_{c=1}^k \pm \int_{M^{k+1}} \mathcal{K}_\varphi(t,x_1,\ldots,x_{k_1}) \wedge \mathcal{K}_\psi(t,x_{k_1+1},\ldots,x_{k_2}) \wedge u_{c+1}(x_1) \wedge \cdots \wedge u_c(x_k)
\]
\[
\quad + \sum_{c=1}^k \pm \int_{M^{k+1}} \mathcal{K}_\varphi(t,x_{c+1},\ldots,x_{c+k_1}) \wedge \mathcal{K}_\psi(t,x_{c+k_1+1},\ldots,x_{c+k_2}) \wedge u_1(x_1) \wedge \cdots \wedge u_k(x_k)
\]

For \(1 \leq c \leq k\) we define \(I_c : M^{k+1} \to M^{k+2}\) by

\[
I_c(t,x_1,\ldots,x_k) := (t,x_{c+1},\ldots,x_{c+k_1},t,x_{c+k_1+1},\ldots,x_{c+k_2}).
\]

Let

\[
Pr_{11} : \Omega(M^{k+}) \to \Omega(M^k)
\]
be integration along the fiber associated to the projection \( \text{Pr}_1 : M^{k+1} \to M^k \) forgetting the first component. Then the preceding computation shows that \( \mathfrak{p}_{2,1,0}(\varphi, \psi) \) has kernel

\[
\sum_{c=1}^{k} \pm \text{Pr}_1 I_c^*(\mathfrak{R}_\varphi \times \mathfrak{R}_\psi).
\]

We have thus defined \( \mathfrak{p}_{2,1,0} : \bigwedge^{\text{cyc}} \Omega(M) \otimes \bigwedge^{\text{cyc}} \Omega(M) \to \bigwedge^{\text{cyc}} \Omega(M) \).

By (13.6) this operator extends to the completion \( \bigwedge^{\text{cyc}} \Omega(M) \hat{\otimes} \bigwedge^{\text{cyc}} \Omega(M) \).

Next consider \( \varphi \in \bigwedge^{\text{cyc}} \Omega(M) \) with kernel \( \mathfrak{R}_\varphi \). For \( k_1 + k_2 = k - 2 \) we consider \( M^{k-1} \) with coordinates \((t, x_1, \ldots, x_{k_1}, y_1, \ldots, y_{k_2})\). We define

\[
\mathfrak{p}_{1,2,0}(\varphi) \in \bigoplus_{k_1 + k_2 = k - 2} \bigwedge^{\text{cyc}}_{k_1} \Omega(M) \otimes \bigwedge^{\text{cyc}}_{k_2} \Omega(M).
\]

by the analogue of equation (10.8):

\[
\mathfrak{p}_{1,2,0}(\varphi)(u_1 \cdots u_{k_1} \otimes v_1 \cdots v_{k_2})
\]

\[
:= \sum_{c=1}^{k_1} \sum_{c'=1}^{k_2} \pm \int_{M^{k-1}} \mathfrak{R}_\varphi(t, x_{c+1}, \ldots, x_c, t, y_{c'+1}, \ldots, y_{c'}) \wedge
\]

\[
\wedge u_1(x_1) \wedge \cdots \wedge u_{k_1}(x_{k_1}) \wedge v_1(y_1) \wedge \cdots \wedge v_{k_2}(y_{k_2}).
\]

For \( 1 \leq c \leq k_1 \) and \( 1 \leq c' \leq k_2 \) we define \( I_{c,c',k_1,k_2} : M^{k-1} \to M^k \) by

\[
I_{c,c',k_1,k_2}(t, x_1, \ldots, x_{k_1}, y_1, \ldots, y_{k_2}) := (t, x_{c+1}, \ldots, x_c, t, y_{c'+1}, \ldots, y_{c'}).
\]

Then the preceding computation shows that \( \mathfrak{p}_{1,2,0}(\varphi) \) has kernel

\[
\sum_{c_1 + c_2 = k - 2} \sum_{c'=1}^{c_1} \sum_{c''=1}^{c_2} \pm \text{Pr}_1 I_{c,c',k_1,k_2}^*(\mathfrak{R}_\varphi).
\]

It is straightforward to check that this is indeed the kernel of an element of \( \bigwedge^{\text{cyc}} \Omega(M) \hat{\otimes} \bigwedge^{\text{cyc}} \Omega(M) \). Note that we need to take the completion here because the kernel defined by (13.8) is not a finite sum of exterior products.

The proof of the dIBL-relations is now analogous to the proof of Proposition 10.4 and is omitted.

The inner product in (13.11) induces one on de Rham cohomology \( H_{dR}(M) \). The de Rham cohomology thus becomes a cyclic cochain complex with trivial differential. Since \( H_{dR}(M) \) is finite dimensional, we can apply Proposition 10.4 to obtain an IBL-structure on \( \bigwedge^{\text{cyc}} H_{dR}(M)[2-n] \). Note that this structure does not use the cup product or higher products on \( H_{dR}(M) \).

For the remainder of this section, let us now fix a Riemannian metric on \( M \). We identify \( H_{dR}(M) \) with the space \( \mathcal{H}(M) \) of harmonic forms, and denote
by \( i : \mathcal{H}(M) \hookrightarrow \Omega(M) \) the inclusion. Then we have the following analogue of Theorem 11.3 for the de Rham complex, which corresponds to Theorem 1.10 from the introduction.

**Theorem 13.6.** There exists a Fréchet IBL\(_\infty\)-morphism

\[
\bar{f} : B^\text{cyc}\Omega(M)\infty[2 - n] \to B^\text{cyc}H^\text{dR}(M)[2 - n]
\]
such that \( \bar{f}_{1,1,0} : B^\text{cyc}\Omega(M)\infty[2 - n] \to B^\text{cyc}H^\text{dR}(M)[2 - n] \) is the map induced by the dual of the inclusion \( i : H^\text{dR}(M) \cong \mathcal{H}(M) \hookrightarrow \Omega(M) \).

**Proof.** We use the same notation as in the proof of Theorem 11.3. Consider a ribbon graph \( \Gamma \in RG_{k,\ell,g} \). We want to define

\[
(13.9) \quad \bar{f}_{\Gamma} : B^\text{cyc}_d(\Omega(M)\infty \otimes \cdots \otimes B^\text{cyc}_{d(k)}(\Omega(M)) \to B^\text{cyc}_s H^\text{dR}(M) \otimes \cdots \otimes B^\text{cyc}_s H^\text{dR}(M).
\]

The idea is to replace \( G \) in Section 11 by the Green kernel and summation over the basis by integration. We then define \( (13.9) \) by a formula similar to (11.7) as follows.

Let \( G(x,y) \in \Omega^{n-1}(M \times M) \) be the kernel of the Green operator \( G : \Omega(M) \to \Omega(M) \),

\[
G(u)(x) = \int_{y \in M} G(x,y) \wedge u(y)
\]

(with respect to the fixed metric) satisfying

\[
d \circ G + G \circ d = \Pi - \text{id}
\]

where \( \Pi : \Omega(M) \to \mathcal{H}(M) \) is the orthogonal projection onto the harmonic forms. (\( G \) is called the propagator in [3].)

Now let \( \varphi^v \in B^\text{cyc}_{d(v)}(\Omega(M)) \), \( v = 1, \ldots, k \) be given with kernels \( \bar{G}^v(x_1, \ldots, x_{d(v)}) \in \Omega(M^{d(v)}) \). These kernels will play a similar role as \( \varphi^{v_1, \ldots, v_d(v)} \) in Sections 10 and 11. Let \( \alpha^b_1 \cdots \alpha^b_{s(b)} \in B^\text{cyc}_{s(b)} H^\text{dR}(M), b = 1, \ldots, \ell \) be given in terms of harmonic forms \( \alpha^b_i \) associated to the exterior edges of \( \Gamma \).

We denote by \( \text{Flag}(\Gamma) \) the set of flags (or half-edges) of \( \Gamma \), where a flag is a pair \( f = (v, l) \) consisting of an interior vertex and an (interior or exterior) edge with \( v \in l \). Suppose that we are given a labelling of \( \Gamma \) and an ordering and orientations of the interior edges in the sense of Definitions 10.7 and 11.4.

Given these data, we can unambiguously associate flags

- \( f(t,1) \) and \( f(t,2) \) to the initial and end point of each interior edge \( t \);
- \( f(v,1), \ldots, f(v,d(v)) \) to the ordered half-edges around each interior vertex \( v \);
- \( f(b,1), \ldots, f(b,s(b)) \) to the ordered exterior edges ending on each boundary component \( b \).
To each flag $f$ we associate a variable $x_f$ which runs over $M$. Then we set
\begin{equation}
\Gamma(\varphi^1 \otimes \cdots \otimes \varphi^k)(\alpha_1^1 \cdots \alpha_{s(1)}^1 \otimes \cdots \otimes \alpha_1^\ell \cdots \alpha_{s(\ell)}^\ell) := \int_{(x_f) \in M \text{Flag}(\Gamma)} (-1)^n \prod_{t \in C^1_{\text{int}}(\Gamma)} \int_{M \text{Flag}(\Gamma)} \frac{\partial^n}{\partial x_{f(t,1)}^{\alpha_1^1} \cdots \partial x_{f(t,2)}^{\alpha_1^\ell}} \prod_{v \in C^0_{\text{int}}(\Gamma)} K_v(x_{f(v,1)}, \ldots, x_{f(v,d(v))}) \prod_{b=1}^{s(b)} \alpha_b^b(x_{f(b,i)}),
\end{equation}
with appropriate signs $(-1)^n$.

At this point, we could proceed to prove that (13.10) defines an IBL-infinity-morphism in a way similar to the proof of Theorem 11.3, using Stokes' theorem. Here we take a shortcut using finite dimensional approximations and reduce the proof to Theorem 11.3 itself, as we will now explain.

The fixed Riemannian metric on $M$ induces the Laplace operator $\Delta$ and the Hodge star operator $\ast$ on $\Omega(M)$. For a positive number $E$, we denote by $\Omega_E(M)$ the finite dimensional subspace of $\Omega(M)$ that is generated by eigenforms of $\Delta$ with eigenvalue $< E$. The differential $d$ and the Hodge star operator preserve $\Omega_E(M)$, so the pairing $\langle , \rangle$ is nondegenerate on $\Omega_E(M)$. Therefore, by Proposition 10.4 we obtain the structure of a dIBL-algebra on $C_E := B_{\text{cyc}} \ast \Omega_E(M)[2 - n]$.

We denote it by $p_{1,1,0; E}$, $p_{2,1,0; E}$, $p_{1,2,0; E}$. To compare this structure to the Fréchet dIBL-structure on $C_\infty = B_{\text{cyc}} \ast \Omega(M)_{\infty}[2 - n]$, we need the following

**Lemma 13.7.** There exist canonical restriction and extension maps
\[ B_{\text{cyc}} \ast \Omega_E(M) \xrightarrow{\text{ext}} B_{\text{cyc}} \ast \Omega_M(\infty) \xrightarrow{\text{rest}} B_{\text{cyc}} \ast \Omega_E(M) \]
satisfying $\text{rest} \circ \text{ext} = \text{id}$.

**Proof.** The restriction map is just dual to the inclusion $\Omega_E(M) \hookrightarrow \Omega(M)$. To define the extension map, we fix a basis $\{e_i\}$ of $\Omega_E(M)$ of pure degree. As in Section 10, we denote by $\{e^i\}$ be the dual basis with respect to the pairing $\langle , \rangle$, i.e. $\langle e_i, e^j \rangle = \delta_i^j$, and we set $g^{ij} := \langle e^i, e^j \rangle$. Then to $\varphi \in B_{k}^{\text{cyc}} \ast \Omega_E(M)$ we associate the collection of real numbers
\[ \varphi_{i_1 \cdots i_k} := \varphi(e_{i_1}, \ldots, e_{i_k}) \]
and the smooth kernel
\[ \hat{K}_\varphi(x_1, \ldots, x_k) := \sum_{i_1, \ldots, i_k} \varphi_{i_1 \cdots i_k} e^{i_1}(x_1) \cdots e^{i_k}(x_k). \]
By formula \([13.2]\), this defines an extension of \(\varphi\) to \(B^\text{cyc}_k \Omega(M)_\infty\). To check the identity \(\text{rest} \circ \text{ext} = \text{id}\), we compute for \(u_1, \ldots, u_k \in \Omega_E\):

\[
\int_{M^k} (u_1 \times \cdots \times u_k) \wedge \hat{\mathcal{R}} \varphi
= \sum_{i_1, \ldots, i_k} \varphi_{i_1 \cdots i_k} \int_{M^k} u_1(x_1) \cdots u_k(x_k) e^{i_1}(x_1) \cdots e^{i_k}(x_k)
= \sum_{i_1, \ldots, i_k} \pm \varphi_{i_1 \cdots i_k} \langle u_1, e^{i_1} \rangle \cdots \langle u_k, e^{i_k} \rangle
= \pm \varphi(u_1, \ldots, u_k),
\]

where in the last equation we have used \(\sum_j \langle u_i, e^j \rangle e_j = u_i\). \(\square\)

Using this lemma, we associate to a map \(f : E_k C_\infty \rightarrow E_\ell C_\infty\) its restriction \(f_E\) as the composition

\[
f_E : E_k C_E \xrightarrow{\text{ext}} E_k C_\infty \xrightarrow{f} E_\ell C_\infty \xrightarrow{\text{rest}} E_\ell C_E.
\]

**Lemma 13.8.** The operators \(p_{1,1,0;E}, p_{2,1,0;E}, p_{1,2,0;E}\) on \(C_E\) are the restrictions of the operators \(p_{1,1,0}, p_{2,1,0}, p_{1,2,0}\) on \(C_\infty\).

**Proof.** The operator \(p_{1,1,0;E}\) is clearly the restriction of \(p_{1,1,0}\) because both are induced by \(m_1\). Let us check that \(p_{2,1,0;E}\) is the restriction of \(p_{2,1,0}\) (the case of \(p_{2,2,0;E}\) and \(p_{1,2,0}\) is analogous). Consider \(\varphi \in B^\text{cyc}_{k_1+1} \Omega_E(M)\) and \(\psi \in B^\text{cyc}_{k_2+1} \Omega_E(M)\). We define their kernels as above and denote their extensions by the same letters. Then for \(k = k_1 + k_1\) and \(u_1, \ldots, u_k \in \Omega_E\)
we compute, using the definitions (13.5) and (10.5):

\[
p_{2,1,0}(\varphi, \psi)(u_1 \cdots u_k)
\]

\[
= \sum_{c=1}^{k} \pm \int_{M^{k+1}} \rho_{\varphi}(t, x_1, \ldots, x_{k_1}) \wedge \rho_{\psi}(t, x_{k_1+1}, \ldots, x_k) \wedge u_{c+1}(x_1) \wedge \cdots \wedge u_c(x_k)
\]

\[
= \sum_{c=1}^{k} \pm \sum_{a,b,i_1,\ldots,i_k} \pm \varphi_{a_i \cdots i_k} \phi_{b_{i_{k+1} \cdots i_k}} \int_{M^{k+1}} u_{c+1}(x_1) \cdots u_c(x_k) \wedge e^a(t)e^{i_1}(x_k) \cdots e^{i_k}(x_k) \wedge e^b(t)e^{i_{k+1}}(x_{k+1}) \cdots e^{i_k}(x_k)
\]

\[
= \sum_{c=1}^{k} \pm \sum_{a,b,i_1,\ldots,i_k} \pm g_{ab} \phi(e_a, u_{c+1}, \ldots, u_{c+k_1}) \psi(e_b, u_{c+k_1+1}, \ldots, u_c)
\]

\[
= \pm p_{2,1,0,E}(\varphi, \psi)(u_1 \cdots u_k).
\]

This proves Lemma 13.8 up to signs. Now we remark that the signs in (13.5) and (13.7) are not yet defined. So we define the signs there so that Lemma 13.8 holds with signs.

\[\square\]

**Remark 13.9.** Before completing the proof of Theorem 13.6, let us complete the sign part of the proof of Proposition 13.5. It is easy to see from the definition that \(C_E \subset C_{E'}\) for \(E < E'\) and the restrictions of \(p_{2,1,0,E'}, p_{1,2,0,E'}\) to \(C_E\) coincide with \(p_{2,1,0,E}, p_{1,2,0,E}\). Therefore, they define operators on the union \(\bigcup_E C_E\). Then formulas (13.3), (13.7) imply that they further extend continuously to the closure \(C_{\infty}\) of \(\bigcup_E C_E\) with respect to the \(C^\infty\) topology. These are the operators in Proposition 13.5. Now the signs work out because they do on \(C_E\) for each finite \(E\). The same remark applies to the rest of the proof of Theorem 13.6.

Now we return to the proof of Theorem 13.6. Denote by \(\Omega_+(M)\) the direct sum of the eigenspaces of \(\Delta\) to positive eigenvalues. By the Hodge theorem, \(\Omega_+(M) = \text{im } d \oplus \text{im } d^*\) and \(d : \text{im } d^* \to \text{im } d, d^* : \text{im } d \to \text{im } d^*\) are isomorphisms. So we can choose a basis \(\{e_i, f_i\}\) of \(\Omega_+(M)\) satisfying

\[
\Delta e_i = \lambda_i e_i, \quad \Delta f_i = \lambda_i f_i, \quad df_i = e_i, \quad de_i = 0
\]

by picking a basis \(\{e_i\}\) for \(\text{im } d\) and setting

\[
f_i := \frac{1}{\lambda_i} d^* e_i.
\]

Here \(d^* = - * d^*\). Then we can choose the Green operator \(G\) so that

\[
G|_{\mathcal{H}(M)} = 0, \quad G(e_i) = f_i, \quad G(f_i) = e_i.
\]
On the other hand, we have

\[ \langle e_i, e_j \rangle = \int_M df_i \wedge e_j = \pm \int_M f_i \wedge de_j = 0. \]

Similarly, \( \langle f_i, f_j \rangle = 0 \) and

\[ \langle e_i, f_j \rangle = \frac{1}{\lambda_j} \int_M f_i \wedge de_j = \pm \langle f_i, e_j \rangle \]

if \( \lambda_i = \lambda_j \), and \( \langle e_i, f_j \rangle = 0 \) otherwise. We set

\[ h_{ij} := \langle e_i, f_j \rangle = \pm \langle f_i, e_j \rangle. \]

Let \((h^{ij})\) be the inverse matrix of \((h_{ij})\). Then the propagator is given by

\[ G(x, y) = \sum_{i,j} \pm h^{ij} f_i(x) \wedge f_j(y), \]

where the sign depends only on the degrees of \( f_i, f_j \).

We restrict \( G \) in \((13.12)\) to \( \Omega_E(M) \) and obtain an operator \( G_E \). We use it in the proof of Theorem \( 11.3 \) to obtain an \( \text{IBL}_\infty \)-morphism \( \{f^\Gamma, E\} \) from \( C_E \) to \( B_{cyc}^\ast \Omega(M) \).

**Lemma 13.10.** The restriction of the operator \( f^\Gamma \) defined by \((13.10)\) to \( C_E \) coincides with \( f^\Gamma, E \).

**Proof.** Using \((13.12)\), \((13.13)\), \((13.14)\) and \((13.15)\) it is easy to see that the operators coincide up to sign. Now we define the sign in \((13.10)\) so that they coincide with sign. □

It is easy to see that the maps \( f^\Gamma, E \) are compatible with the inclusions \( C_E \subset C_{E'} \). So they define an \( \text{IBL}_\infty \)-morphism on the union \( \bigcup_E C_E \). Now \((13.10)\) and Lemma \( 13.10 \) imply that this morphism extends to the closure \( C^\infty \) with respect to the \( C^\infty \) topology, and the proof of Theorem \( 13.6 \) is complete. □

**String topology and Conjecture 1.11**. Now we would like to proceed as in Section \( 12 \) twist the Fréchet dIBL-structure on \( B_{cyc}^\ast \Omega(M) \) by the Maurer-Cartan element \( m^+_2 \) arising from the product \( m^+_2 \) in \((13.1)\), and use Theorem \( 13.6 \) to push the twisted \( \text{IBL}_\infty \)-structure onto \( B_{cyc}^\ast H_{dR}(M) \). However, there is one difficulty in doing so. As in Section \( 12 \) the product \( m_2(u, v) = (-1)^{\deg w} u \wedge v \) gives rise to an element

\[ m^+_2 \in B_3^\ast \Omega(M). \]

defined by

\[ m^+_2(u, v, w) := (-1)^n (m_2(u, v), w) = (-1)^{n-2+\deg v} \int_M u \wedge v \wedge w. \]
However, \( m_2^+ \) does not have a smooth kernel. In fact, its Schwartz kernel is the current represented by the triple diagonal

\[
\Delta_M = \{(x,x,x) \mid x \in M\} \subset M^3.
\]

So we cannot use \( m_2^+ \) directly to twist the Fréchet dIBL-structure on the space \( B^{\text{cyc}}\Omega(M)_\infty[2-n] \). On the other hand, we may try to formally apply the map in Theorem 13.6 to \( e^{m_2^+} \) and then show that this defines a twisted IBL\(_\infty\)-structure on \( B^{\text{cyc}}H_{dR}(M)_\infty[2-n] \). For this, we need to consider the integrals (13.10) for trees \( \Gamma \) all of whose interior vertices are trivalent and with inputs \( \phi_v = m_2^+ \) for all \( v = 1, \ldots, k \) (see Remark 12.10). Then the \( R^v \) appearing in the formula must be the current \( \Delta_M \). Hence in place of the second product of the right hand side of (13.10) (involving the kernels \( K_v \)) we restrict to the submanifold

\[
\{x_{f(v,1)} = x_{f(v,2)} = x_{f(v,3)} \text{ for all } v \in C_0^0(\Gamma)\}
\]

and perform the integration of the other differential forms (the Green kernels and the harmonic forms) over this submanifold. This integral is very similar to those appearing in perturbative Chern-Simons gauge theory [72, 8]. The difficulty with this integral comes from the singularity of the propagator \( G(x,y) \) at the diagonal \( x = y \). We believe that one can resolve this problem by using a real version of the Fulton-MacPherson compactification [35] in a similar way as in [3]. As a result, one should obtain Conjecture 1.11 from the Introduction: There exists an IBL\(_\infty\)-structure on \( B^{\text{cyc}}H_{dR}(M)[2-n] \) whose homology equals Connes’ version of cyclic cohomology of the de Rham complex of \( M \).

Lagrangian Floer theory and Conjecture 1.12. We expect that these ideas of this paper can be applied to study Lagrangian Floer theory of arbitrary genus and with an arbitrary number of boundary components in the following way. Let \( L \) be an \( n \)-dimensional closed Lagrangian submanifold of a symplectic manifold \((X,\omega)\), and \( J \) be an almost complex structure on \( X \) compatible with \( \omega \). For fixed integers \( g \geq 0, \ell \geq 1 \) and \( s_1, \ldots, s_\ell \geq 1 \) and a relative homology class \( \beta \in H_2(X,L;\mathbb{Z}) \) we consider \((\Sigma,\vec{z},u)\) such that

- \( \Sigma \) is a compact oriented Riemann surface of genus \( g \) with \( \ell \) boundary components;
- \( \vec{z} = (z_1, \ldots, z_\ell) \), where \( z_b = (z_{b,1}, \ldots, z_{b,s_b}) \) is a vector of \( s_b \) distinct points in counterclockwise order on the \( b \)-th boundary component \( \partial_b \Sigma \cong S^1 \) of \( \Sigma \);
- \( u : (\Sigma, \partial \Sigma) \to (X,L) \) is a \( J \)-holomorphic map representing the relative homology class \( \beta \).

We denote the compactified moduli space of such \((\Sigma,\vec{z},u)\) by \( \mathcal{M}_{g;\beta}(\Sigma,\vec{z}) \). There is a natural evaluation map

\[
ev = (ev_1, \ldots, ev_\ell) : \mathcal{M}_{g;\beta}(\Sigma,\vec{z}) \to L^{s_1} \times \cdots \times L^{s_\ell}.
\]
Integrating the pullback of differential forms under the evaluation map over the moduli space (using an appropriate version of the virtual fundamental chain technique) should give rise to an element

$$m_{g;(s_1,\ldots,s_\ell)}(\beta) \in B^{\text{cyc}} S(L)[2 - n] \otimes \cdots \otimes B^{\text{cyc}} S(L)[2 - n],$$

where $S(L)$ is a suitable cochain complex realizing the cohomology of $L$. Let $\Lambda_0$ be the universal Novikov ring of formal power series in the variable $T$ introduced in Section 3. The elements $m_{g;(s_1,\ldots,s_\ell)}(\beta)$ should then combine to elements

$$m_{g,\ell} := \sum_{s_1,\ldots,s_\ell} \sum_{\beta} m_{g;(s_1,\ldots,s_\ell)}(\beta) T^{\omega(\beta)} \in \widehat{E}_\ell\left( B^{\text{cyc}} S(L)[2 - n] \otimes \Lambda_0 \right),$$

where $\widehat{E}_\ell$ denotes the completed symmetric product with respect to the two natural filtrations on $B^{\text{cyc}} S(L)[2 - n] \otimes \Lambda_0$. The boundary degenerations of the moduli spaces $\mathcal{M}_{g;(s_1,\ldots,s_\ell)}(\beta)$ suggest that

$$m := \sum_{g,\ell} m_{g,\ell} h^{g-1} - \frac{1}{\hbar} \widehat{E}_\ell\left( B^{\text{cyc}} S(L)[2 - n] \otimes \Lambda_0 \right) \{ h \}$$

taxtisfies the Maurer-Cartan equation

$$\widehat{p} (e^m) = 0$$

for an appropriate IBL$_\infty$-structure $\widehat{p}$ on $B^{\text{cyc}} S(L)$.

We hope to work this out by taking for $S(L)$ de Rham complex $\Omega(L)$ and using the results of this section as follows. Recall from Proposition 13.5 that the subcomplex $B^{\text{cyc}} \Omega(L)_\infty[2 - n] \subset B^{\text{cyc}} \Omega(L)[2 - n]$ of homomorphisms with smooth kernel carries a natural Fréchet dIBL-structure. However, the elements $m_{g,\ell}$ do not have smooth kernel and thus cannot be viewed as a Maurer-Cartan element on this Fréchet dIBL-algebra. The idea is now to use the construction in Conjecture 1.11 to push forward the $m_{g,\ell}$ to a Maurer-Cartan element on $B^{\text{cyc}} H_{\text{dR}}(L)[2 - n] \otimes \Lambda_0$. It should give rise to a twisted filtered IBL$_\infty$-structure on $B^{\text{cyc}} H_{\text{dR}}(L)[2 - n] \hat{\otimes} \Lambda_0$ whose homology equals the cyclic cohomology of the cyclic $A_\infty$-structure on $H_{\text{dR}}(L)$ constructed in [34 33]. Moreover, its reduction at $T = 0$ should equal the filtered IBL$_\infty$-structure on $B^{\text{cyc}} H_{\text{dR}}(L)[2 - n]$ in Conjecture 1.11 and Conjecture 1.12 from the Introduction should follow.

Appendix A. Orientations on the homology of surfaces

In this appendix, we present a procedure to order and orient the interior edges of a labelled ribbon graph in terms of orientations on the singular chain complex of the associated surface. Relating this to the procedure using spanning trees in Definition 11.4, we prove Lemma 11.5 from Section 10.

Consider a ribbon graph $\Gamma$ with $k$ interior vertices $v_1,\ldots,v_k$ and $m$ interior edges $e_1,\ldots,e_m$. We denote by $\Sigma$ the surface with boundary associated to
Γ, and by Σ the closed connected oriented surface (of genus g) obtained by gluing disks to the ℓ boundary components of Σ. Note that

\[ 2 - 2g = \chi(\hat{\Sigma}) = \chi(\Sigma) + \ell = \chi(\Gamma) + \ell = k - m + \ell. \]

We view Γ as a graph on Σ. The ribbon condition implies that Σ \ Γ is the union of ℓ disks whose closures we denote by f₁, ..., f_ℓ. So we have a cell complex

\[
\begin{align*}
C_2 &= \langle f_1, \ldots, f_\ell \rangle \xrightarrow{\partial_2} C_1 = \langle e_1, \ldots, e_m \rangle \xrightarrow{\partial_1} C_0 = \langle v_k, \ldots, v_1 \rangle \\
\end{align*}
\]

(say, with \( \mathbb{Q} \)-coefficients) computing the homology of Σ. We pick complements \( V_i \) of ker \( \partial_i \) in \( C_i \) and \( H_i \) of im \( \partial_{i+1} \) in ker \( \partial_i \), so that the complex becomes

\[
\begin{align*}
C_2 &= V_2 \oplus H_2 \xrightarrow{\partial_2} C_1 = V_1 \oplus H_1 \oplus \text{im} \partial_2 \xrightarrow{\partial_1} C_0 = \text{im} \partial_1 \oplus H_0.
\end{align*}
\]

Note that \( \partial_i : V_i \to \text{im} \partial_i \) are isomorphisms and the \( H_i \) are isomorphic to the homology groups of \( \hat{\Sigma} \). Now equation (A.1) determines an orientation of \( C = C_0 \oplus C_1 \oplus C_2 \) once we choose a labelling of Γ in the sense of Definition 10.7, i.e.,

(i) an ordering \( v_1, \ldots, v_k \) of the interior vertices,
(ii) and orderings and orientations of the interior edges,
(iii) an ordering of the boundary components.

More precisely, given these choices we orient the chain groups as follows:

(i)' an oriented basis \( v_k, \ldots, v_1 \) of \( C_0 \) is given by the interior vertices in reverse order;
(ii)' an oriented basis \( e_1, \ldots, e_m \) of \( C_1 \) is given by the oriented interior edges in their given order;
(iii)' an oriented basis \( f_1, \ldots, f_\ell \) of \( C_2 \) is given by the 2-cells, oriented according to the orientation of \( \hat{\Sigma} \) and ordered according to the ordering of the boundary components.

We arbitrarily orient \( V_1, V_2 \) and equip \( \text{im} \partial_1, \text{im} \partial_2 \) with the induced orientations via the isomorphisms \( \partial_i : V_i \to \text{im} \partial_i \). These orientations together with the orientation of \( C \) induce via (A.2) an orientation on \( H = H_0 \oplus H_1 \oplus H_2 \). This orientation does not depend on the chosen orientations on \( V_1, V_2 \), but it does depend on the order of the direct summands in (A.2).

On the other hand, the homology \( H \) is canonically oriented: We set

\[ H_0 := \langle v_1 + \cdots + v_k \rangle, \quad H_1 := \langle f_1 + \cdots + f_\ell \rangle \]

and give \( H_1 \cong \mathbb{Q}^{2g} \) the symplectic orientation induced by the intersection form. We define the sign exponent \( \eta_3(\Gamma) \) (mod 2) as 0 if the two orientations of \( H \) coincide, and 1 if not. Here we always consider the labelling data (i)-(iii) as part of Γ.
Remark A.1. To a ribbon graph $\Gamma$ we can associate its dual graph $\bar{\Gamma}$ lying on the same closed surface $\hat{\Sigma}$ by exchanging vertices and faces. The edges of $\bar{\Gamma}$ are canonically oriented by requiring that the intersection number of each edge of $\Gamma$ with the corresponding edge of $\bar{\Gamma}$ is $+1$. The dual graph to $\bar{\Gamma}$ is $\Gamma$ with the orientation of all edges reversed, which has the same sign as $\Gamma$ iff the number of edges is even. Thus the sign of $\bar{\Gamma}$ cannot always be equal to the sign of $\Gamma$. Is there a simple criterion to decide when the signs agree. Is this duality good for anything?

Orientation via spanning trees. An orientation on the chain group $C = C_0 \oplus C_1 \oplus C_2$ associated to a labelled ribbon graph $\Gamma$ can be specified via the construction in Definition 11.4, which we first recall.

Choose a maximal tree $T \subset \Gamma_{\text{int}}$. It will have $k - 1$ edges, which we orient away from vertex 1 and label in decreasing order such that the $i$th edge ends at vertex $k + 1 - i$. Next choose a maximal tree $T^* \subset \Gamma^*_{\text{int}}$ disjoint from $T$. It will have $\ell - 1$ edges, which we orient away from the first boundary component and label in increasing order such that the edge $e^*_k, \ldots, e^*_{k+\ell-2}$ points to the boundary component $s$. The oriented edges $e_k, \ldots, e_{k+\ell-2}$ are obtained as the dual edges to the $e^*_i$, oriented so that the pair $\{e^*_i, e_i\}$ defines the orientation of the surface $\Sigma^\Gamma$. The remaining $2g$ edges of $\Gamma_{\text{int}}$ edges determine a basis for $H_1(\hat{\Sigma}^\Gamma)$ and we choose their order and orientation compatible with the symplectic structure on $H_1(\hat{\Sigma}^\Gamma)$ corresponding to the intersection pairing.

It follows from these conventions that for $s = 2, \ldots, \ell$ the edge $e_{k+s-2}$ occurs with a minus sign in $\partial f_s$, and for $i = 1, \ldots, k - 1$ the vertex $v_{i+1}$ occurs with a plus sign in $\partial e_i$. So they specify compatible bases of $V_i$ and $\text{im} \partial_i$ given by

\[
C_2 = V_2 \oplus H_2 = \langle -f_2, \ldots, -f_\ell \rangle \oplus \langle f_1 \rangle,
\]

\[
C_1 = V_1 \oplus H_1 \oplus \text{im} \partial_2 = \langle e_{k-1}, \ldots, e_1 \rangle \oplus H_1 \oplus \langle e_k, \ldots, e_{k+\ell-2} \rangle,
\]

\[
C_0 = \text{im} \partial_1 \oplus H_0 = \langle v_k, \ldots, v_2 \rangle \oplus \langle v_1 \rangle.
\]

Since the resulting orientations $\langle -f_2, \ldots, -f_\ell, f_1 \rangle = \langle f_1, \ldots, f_\ell \rangle$ of $C_2$ and $\langle v_k, \ldots, v_2, v_1 \rangle$ of $C_0$ agree with the orientation conventions above, the corresponding sign is $\eta_3 = 0$.

Lemma 11.5 from Section 10 immediately follows from this because the operations

- changing the orientation of an interior edge,
- interchanging the order of two adjacent interior edges,
- interchanging the order of two adjacent interior vertices,
- interchanging the order of two adjacent boundary components

all change $\eta_3$ by 1.
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