Particle Radiation From Gibbons-Maeda Black Hole

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Abstract

This paper investigates the particle radiation from Gibbons-Maeda black hole. Taking into account the self-gravitation of the particle, we calculate the tunnelling rate of the massless particle across the horizon, then we promote the work to the radiation of the charged particle. The calculations prove that the rate of tunnelling equals precisely the exponent of the difference of the black hole entropy before and after emission and the radiation spectrum deviates from exact thermal. The conclusion supports the viewpoint of information conservation.

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I. INTRODUCTION

For a long time, people believed that the black hole was such an object that things approaching it should be absorbed, while anything inside it could not escape, the black hole could only grow bigger and bigger. However, in 1970’s, S.W.Hawking declared\[1\] his astonishing discovery that the black hole can radiate particles. Hawking’s idea is that when a virtual particle pair are created near the horizon, the negative energy particle is absorbed by the black hole while the positive tunnels out of the horizon, then it materializes as a real one, appearing as Hawking radiation. The black hole radiation is purely of a quantum effect, the emission is a procedure of particle tunnelling across the potential barrier. Gibbons and Hawking also demonstrated\[2\] that the energy spectrum of the particle radiation from a collapsing star is of exactly thermal. From then on, so many studies about the Hawking radiation from various black holes gave the same conclusion that the radiation spectrum is of black body spectrum. But there are two puzzles. One is the barrier, where the barrier exists is not clear. The other is the radiation spectrum, because we can get only one parameter from a thermal spectrum, the temperature. So if the energy spectrum is exactly thermal, the radiation should not bring us any macroscopic information about the material in the black hole, even that once the black hole has evaporated out, there would be no any marks left, all the information including the unitary would be lost. In fact, the argument about the information loss has lasted many years.

M.K.Parikh and F.Wilczek\[3, 4\] claimed that there is no a barrier existing before the particle tunnels out of the black hole, instead, the barrier is created by the outgoing particle itself. The total energy of a stationary space-time is conservational during the emission, this demands that, when a particle is radiated, the left mass of the black hole decrease, so the radius of the horizon contracts from its original radius to a new, smaller radius, and the scale of the contraction depends on the energy of the radiated particle, the separation between the initial and final radius is just the barrier for tunnelling. During the course of the radiation, if the self-gravitation of the particle is taken into account , the space-time geometry is dynamic, the black hole can be regarded as an excited metastable state. M.K.Parikh and F.Wilczek presented a derivation of massless particle radiation and obtained that the emission spectrum of schwarzschild black hole deviates from the exact thermal. Enlightened by M.K.Parikh, many works\[5, 6, 7, 8, 9\] were performed about the particle
radiation from different black holes. In [7, 8], the radiation of massive and charged particle
were investigated, to get the velocity of a massive particle, the authors used the Laudau’s
condition of the clock synchronization, for the radiation of charged particle, they regarded
the electromagnetic potential as an ignorable coordinate and considered the matter-gravity
energy and electro-magnetic field energy separately.

In this paper, we promote the work of Parikh and Wilczek to the radiation of Gibbons-
Maeda Black hole, calculate the spectrum of particle radiation in a completely different way.
The four-velocity normalization equation $U^a U_a = -1$ are utilized to get the velocity of
the massive particle. The total energy of the space-time is considered. Both of the calculations
for the radiation of massless particle and charged particles give the same conclusion that
the rate of tunnelling equals precisely the exponent of the difference of the entropy of the
black hole before and after emission, this implies that the information is preserved during
the black hole radiation.

II. A CONVENIENT COORDINATE SYSTEM FOR GM BLACK HOLE

The line element of Gibbons-Maeda(GM) black hole is described[10] by

$$ds^2 = -\frac{(r - r_+)(r - r_-)}{R^2}dt^2 + \frac{R^2}{(r - r_+)(r - r_-)}dr^2 + R^2(d\theta^2 + \sin^2 \theta d\varphi^2),$$

(1)

where $R^2 = r^2 - D^2$, $r_+$ and $r_-$ are respectively the event horizon and the inner horizon,
satisfy

$$r_\pm = M \pm \sqrt{M^2 + D^2 - P^2 - Q^2},$$

(2)

and the axion dilaton charge $D$ is

$$D = \frac{P^2 - Q^2}{2M},$$

(3)

the parameters $P$ and $Q$ represent the black hole magnetic charge and the electric charge,
respectively.

It is easy to find that one of the components of the metric in (1) is singular at $r = r_+$. In
order to describe the particle tunnelling across the horizon, we need a coordinate in which
the components of the metric and the inverse metric do not diverge at both sides of the
horizon. Perform such a coordinate transformation that let $dT = dt + f(r)dr$, then the line
element of GM black hole is written as

$$ds^2 = -(1 - g)\Delta dt^2 + 2(1 - g)\Delta f(r)dt dr + \left[\frac{1}{(1 - g)\Delta} - (1 - g)\Delta f^2(r)\right]dr^2 + R^2d\Omega^2,$$

(4)
where \( g = \frac{r_+}{r} \), and \( \Delta = \frac{r^2(1 - \frac{r_-}{r})}{R^2} \). So in coordinate system \((T, r, \theta, \varphi)\),

\[
g_{rr} = \frac{1}{(1-g)\Delta} - (1-g)\Delta f^2(r) = \frac{1-(1-g)\Delta^2 f^2(r)}{(1-g)\Delta}. \tag{5}
\]

In order to eliminate the singularity of \( g_{rr} \) at \( r = r_+ \), taking into account the dimension, we suppose that

\[1-(1-g)\Delta^2 f^2(r) = k(1-g), \tag{6}\]

then we have

\[f(r) = \frac{\sqrt{(1-k)+kg}}{(1-g)\Delta}, \tag{7}\]

for the simplicity, let \( k = 1 \), get

\[f(r) = \frac{\sqrt{g}}{(1-g)\Delta}. \tag{8}\]

Substitute (8) to (4), we get that

\[ds^2 = -(1-g)\Delta dT^2 + 2\sqrt{g}dTdr + \frac{1}{\Delta}dr^2 + R^2d\Omega^2. \tag{9}\]

From (9) we find that the new coordinate system \((T, r, \theta, \varphi)\) has many attractive features. First, each component of the metric and the inverse metric is well behaved at the horizon \( r = r_+ \), it is important for the calculation of the particle tunnelling across the horizon. Second, in the global space-time there exists a Killing vector \( \partial/\partial T \), so we can construct the constant-time slices which are Euclidean flat space.

### III. THE RADIATION OF MASSLESS PARTICLES

From equation (9), a massless particle moves along the radial null geodesic obeys

\[
\dot{r} = \frac{dr}{dT} = -\Delta \sqrt{g} \pm \Delta = \frac{r^2}{R^2}(1 - \frac{r_-}{r})(\pm1 - \sqrt{\frac{r_+}{r}}), \tag{10}
\]

where the upper(lower) sign corresponds the outgoing(ingoing) geodesic under the implicit assumption that \( T \) increase towards the future, here we take the upper sign. Because the metric is spherical symmetry, so regarding the outgoing particle as a s-wave i.e. a shell of energy is acceptable. For GM space-time, the ADM mass is just the parameter mass, \( M \), it is conservational. When a shell of energy \( \omega \) tunnels out of the horizon, the leftover mass of
the black hole becomes $M - \omega$. To the radiated particle, the geometry of the space-time is changed, so the equation (10) should be modified by replacing $r_\pm(M)$ with $r_\pm(M - \omega)$.

Suppose that the outgoing wave were traced back towards the horizon, its wave-length, measured by local observers, would be blue-shifted, near the horizon, the radial wave number approaches infinity, so that the geometrical optics limit, or WKB approximation is valid. In the semi-classical limit, according to the WKB approximation, the $s$-wave of the outgoing positive energy particle can be expressed as $\Psi(r) = e^{iI(r)}$, and the probability of the tunnelling $\Gamma$ takes the form

$$\Gamma \sim |\Psi(r)|^2 = \exp(-2\text{Im } I), \quad (11)$$

here $I$ is the action, it can be obtained from

$$I = \int L dt = \int p_r dr - \int H dt, \quad (12)$$

where $L$ and $H$ are respectively the lagrangian and the Hamiltonian of the particle, $p_r$ is the radial momentum. Because $H$ is real, from (11) we know that it has no contribution to $\Gamma$, so the second term of the right hand side of equation (12) is ignored in the following discussion, that

$$I = \int_{r_i}^{r_f} p_r dr = \int_{r_i}^{r_f} \int_0^{p_r} dp_r dr, \quad (13)$$

where the initial radius $r_i$ corresponds the site of pair-creation, which should be slightly inside the initial horizon, $r_i \approx r_+(M_i)$, while the final radius $r_f$, to be slightly outside the final position of the horizon, $r_f \approx r_+(M_f)$, $M_i$ and $M_f$ are respectively the initial and final mass of the black hole, $r_f$ is actually less than $r_i$. The separation between $r_i$ and $r_f$ forms the barrier. Changing variable from momentum to energy by using Hamilton’s equation

$$\frac{dH}{dp} \bigg|_r = \frac{\partial H}{\partial p} = \dot{r},$$

we get

$$I = \int_{r_i}^{r_f} \int_0^{r_f} \frac{dH'}{\dot{r}} dr = \int_{r_i}^{r_f} \int_{M_i}^{M_f} \frac{dM}{\dot{r}} dr = \int_{M_i}^{M_f} \int_{r_i}^{r_f} \frac{R^2 dM dr}{r^2(1 - \frac{r}{r_-})(1 - \sqrt{\frac{r}{r_+}})}. \quad (14)$$

Substituting $u = \sqrt{r}$ to (14), we get

$$I = \int_{M_i}^{M_f} \int_{\sqrt{r_i}}^{\sqrt{r_f}} \frac{2(u^4 - D^2)dMd u}{(u^2 - r_-)(u - \sqrt{r_+})}. \quad (15)$$
It is easy to find that the integrand has a pole at \( u = \sqrt{r_+} \). In order to integrate across the singularity \( r = r_+ \), replacing \( \sqrt{r_+} \) by \( \sqrt{r_+} - i\epsilon \), we have

\[
I = \lim_{\epsilon \to 0} \int_{M_i}^{M_f} \int_{\sqrt{r_+}}^{\sqrt{r_+} - i\epsilon} \frac{2(u^4 - D^2) dM du}{(u^2 - r_+)(u - \sqrt{r_+} + i\epsilon)} = \lim_{\epsilon \to 0} \int_{M_i}^{M_f} \int_{\sqrt{r_+}}^{\sqrt{r_+} - i\epsilon} \frac{f(u) dM du}{u - \sqrt{r_+} + i\epsilon}, \tag{16}
\]

where \( f(u) = \frac{2(u^4 - D^2)}{(u^2 - r_-)} \). From (11), we know that the real part of \( I \) only contributes a phase while the imaginary part contributes the amplitude of the tunnelling rate, so what we are interested in is only the imaginary part of \( I \). According to the Feynman prescription, the integral can be evaluated by deforming the contour around the pole \( \sqrt{r_+} - i\epsilon \). So we get

\[
\text{Im} I = -\pi \int_{M_i}^{M_f} f(\sqrt{r_+}) dM = -\pi \int_{M_i}^{M_f} \frac{2(r_+^2 - D^2)}{(r_+ - r_-)^2} \frac{dM}{dM}. \tag{17}
\]

From (2), we have

\[
\text{Im} I = -\pi \int_{M_i}^{M_f} \frac{(M^2 - MD - Q^2 + M\sqrt{(M - D)^2 - 2Q^2})}{\sqrt{(M - D)^2 - 2Q^2}} dM = -\int_{M_i}^{M_f} K(M) dM, \tag{18}
\]

where

\[
K(M) = \pi \frac{\left(M^2 - MD - Q^2 + M\sqrt{(M - D)^2 - 2Q^2}\right)}{\sqrt{(M - D)^2 - 2Q^2}}. \tag{19}
\]

For Gibbons-Maeda black hole, the Bekenstein-Hawking entropy \( S \) is

\[
S = \pi (r_+^2 - D^2). \tag{20}
\]

So

\[
\frac{\partial S}{\partial M} = \pi \left[ 2r_+ \frac{\partial r_+}{\partial M} - 2D \frac{\partial D}{\partial M} \right] = \frac{2\pi}{\sqrt{M^2 + D^2 - P^2 - Q^2}} \left(M^2 - MD - Q^2 + M\sqrt{(M - D)^2 - 2Q^2}\right) \tag{21}
\]

Comparing (19) with (21), we are surprised to find that \( \frac{\partial S}{\partial M} = 2K(M) \). So that

\[
\text{Im} I = -\frac{1}{2} \int_{M_i}^{M_f} \frac{\partial S}{\partial M} dM = -\frac{1}{2} \int_{S_i}^{S_f} dS = -\frac{1}{2} \Delta S \tag{22}
\]

Then the rate of emission satisfies

\[
\Gamma \sim \exp(-2 \text{Im} I) = \exp(\Delta S). \tag{23}
\]

This is a familiar result, it can be obtained by many other ways.
The integrand in (18) is quite complicate, it is not easy to integrate directly, expanding $K(M)$ at the near field of $M_i$ as a Taylor series

$$K(M) = K(M_i) + K'(M_i)(M - M_i) + \cdots,$$

we have

$$\text{Im } I = - \left[ K(M_i)\Delta M + \frac{K'(M_i)}{2}\Delta M^2 + \cdots \right] = K(M_i)\omega - \frac{K'(M_i)}{2}\omega^2 + \cdots,$$

where $\Delta M = M_f - M_i$ is the variance of the mass of the black hole, which should be negative, and $\omega = -\Delta M$ is the energy of the radiated particle. Then the tunnelling rate

$$\Gamma \sim \exp[-2\text{Im } I] = \exp \left\{-2 \left[ K(M_i)\omega - \frac{K'(M_i)}{2}\omega^2 + \cdots \right] \right\}.$$ 

If we only take the first term while neglect the higher order terms in the expression (26), that is

$$\Gamma \sim \exp[-2K(M_i)\omega].$$

We find that the the rate of tunnelling, as is expected, does take the form of the Boltzmann factor $\exp(-\beta\omega)$ with the inverse of temperature $\beta \equiv 1/T = 2K(M)$. This is a familiar result and agrees with Hawking’s theory of the thermal radiation, so we confirm that Hawking radiation do can be viewed as a process of particle tunnelling. But if we take into account the higher order terms, thus the emission spectrum is not precisely thermal, while yields corrections. This is a exciting news, because whether the information is lost during the black hole radiation depends in part on the fact that whether the spectrum is exactly thermal. The spectrum is not thermal may open the way to looking for information-carrying correlations in the spectrum.

IV. RADIATION OF CHARGED PARTICLES

A. The velocity of a massive particle

Since the trajectory followed by a massive particle is not light-like, it does not obey the equation of radial null geodesic, so equation (10) is no longer available. But for a massive particle, its 4-velocity has unit length. In relativity, the tangent vector $U^a$ to a time-like curve parameterized by the proper time $\tau$ is called the 4-velocity, it is

$$U^a = \left( \frac{\partial}{\partial \tau} \right)^a.$$
The energy-momentum 4-vector, $p^a$, of the particle of “rest mass” $m$ is defined by

$$ p^a = mU^a, \quad (29) $$

and its covariant form (the dual vector)

$$ p_a = g_{ab}p^b = mg_{ab}U^b. \quad (30) $$

For a massive particle moves in the space-time described by (1), the “time component” of $p_a$ is

$$ p_0 = mg_{0\nu}U^\nu = mg_{00}U^t = -E' \quad (31) $$

that is

$$ U^t \equiv \frac{dt}{d\tau} = \frac{-E'}{mg_{00}} = \frac{-E}{g_{00}}, \quad (32) $$

where $E$ is recognized to be the energy of per unit mass of the particle measured by the stable observer at infinity.

According to the definition of the proper time $\tau$, the 4-velocity has unit length, that is

$$ U^aU_a = -1, \quad (33) $$

from (1), it can be written as

$$ g_{00} \left( \frac{dt}{d\tau} \right)^2 + g_{11} \left( \frac{dr}{d\tau} \right)^2 = -1. \quad (34) $$

From (32) and (31) we have

$$ U^r = \frac{dr}{d\tau} = \sqrt{g_{00} + E^2}, \quad (35) $$

and

$$ \frac{dr}{dt} = \frac{U^r}{U^t} = \frac{g_{00}\sqrt{g_{00} + E^2}}{-E}. \quad (36) $$

From (8) we get that

$$ \frac{dT}{dt} = 1 + f(r) \frac{dr}{dt} = 1 + \frac{\sqrt{g}}{1 - g} \frac{dr}{dt}. \quad (37) $$

Thus we get the radial velocity in coordinate $(T, r, \theta, \varphi)$

$$ \dot{r} = \frac{dr}{dT} = \frac{dr}{dt} \frac{dt}{dT} = \frac{g_{00}\sqrt{g_{00} + E^2}}{-E - \sqrt{g}\sqrt{g_{00} + E^2}} = -\frac{1 - \frac{r^2}{r}}{r} \frac{\Delta g}{g_{00} + E^2}. \quad (38) $$
B. Energy of GM space-time

Obviously, the metric in (1) is static, and the time-like vector \( \xi^a = \frac{\partial}{\partial t} \) is a killing vector, it’s dual vector \( \xi_a = g_{ab} \xi^b = g_{00} (dt)_a \). For GM space-time, the determinant of the metric is

\[
g = -\left(r^2 - D^2\right)^2 \sin^2 \theta. \tag{39}
\]

The total energy (Komar energy \([11,12]\)) of a static, asymptotically flat space-time which is vacuum in the exterior region is defined as

\[
E = -\frac{1}{8\pi} \int_S \varepsilon_{abcd} \nabla^c \xi^d, \tag{40}
\]

where \( S \) is the boundary of a space-like hypersurface. The independence of the right-hand side of equation (40) on the choice of \( S \) depends on the fact that \( \xi^a \) is a killing vector field. For GM space-time

\[
\nabla_a \xi_b = \nabla_a (g_{00} (dt)_b) = (\nabla_a g_{00}) (dt)_b + g_{00} (\nabla_a (dt)_b)
\]

\[
= (\nabla_a g_{00}) (dt)_b + g_{00} (\partial_a (dt)_b - \Gamma^c_{ab} (dt)_c)
\]

\[
= \left(\frac{d}{dr} g_{00}\right) (dr)_a (dt)_b - g_{00} \Gamma^c_{ab} (dt)_c. \tag{41}
\]

The second term

\[
g_{00} \Gamma^c_{ab} (dt)_c = g_{00} \Gamma^0_{\mu\nu} \left(\frac{\partial}{\partial x^\alpha}\right)^c (dx^\mu)_a (dx^\nu)_b (dt)_c
\]

\[
= g_{00} \Gamma^0_{\mu\nu} (dx^\mu)_a (dx^\nu)_b. \tag{42}
\]

The nonvanishing components of the Christoffel symbol needed in (42) are

\[
\Gamma^0_{01} = \Gamma^0_{10} = \frac{1}{2} g^{00} \frac{d}{dr} g_{00}, \tag{43}
\]

so that

\[
g_{00} \Gamma^c_{ab} (dt)_c = \frac{1}{2} g_{00} g^{00} \frac{d}{dr} g_{00} [(dr)_a (dt)_b + (dt)_a (dr)_b]
\]

\[
= \frac{1}{2} \frac{d}{dr} g_{00} [(dr)_a (dt)_b + (dt)_a (dr)_b]. \tag{44}
\]

That

\[
\nabla_a \xi_b = \left(\frac{d}{dr} g_{00}\right) (dr)_a (dt)_b - \frac{1}{2} \frac{d}{dr} g_{00} [(dr)_a (dt)_b + (dt)_a (dr)_b]
\]

\[
= \frac{1}{2} \left(\frac{d}{dr} g_{00}\right) [(dr)_a (dt)_b - (dt)_a (dr)_b]
\]

\[
= -\frac{[MD^2 + M^2 - (P^2 + Q^2)r]}{(r^2 - D^2)^2} [(dr)_a (dt)_b - (dt)_a (dr)_b]. \tag{45}
\]
and
\[ \nabla^c \xi^d = -\frac{[MD^2 + Mr^2 - (P^2 + Q^2)r]}{(r^2 - D^2)^2} (\frac{\partial}{\partial r})^c \wedge (\frac{\partial}{\partial t})^d. \] (46)

The volume element is
\[ \varepsilon_{abcd} = \sqrt{-g} (dt)_a \wedge (dr)_b \wedge (d\theta)_c \wedge (d\phi)_d. \] (47)

So from equation (40), we get that the total energy of Gibbons-Maeda space-time is
\[ E = -\frac{1}{8\pi} \int_S -\frac{[MD^2 + Mr^2 - (P^2 + Q^2)r]}{(r^2 - D^2)^2} (\frac{\partial}{\partial r})^c \wedge (\frac{\partial}{\partial t})^d \times \sqrt{-g} (dt)_a \wedge (dr)_b \wedge (d\theta)_c \wedge (d\phi)_d \]
\[ = -\frac{1}{8\pi} \int_S -2 \frac{[MD^2 + Mr^2 - (P^2 + Q^2)r]}{(r^2 - D^2)^2} (d\theta)_a \wedge (d\varphi)_b = \frac{MD^2 + Mr^2 - (P^2 + Q^2)r}{r^2 - D^2}. \] (48)

From (48), we can see that the matter-gravity field and the electromagnetic field have combined contribution to the total energy.

C. Charged particle tunnelling across the horizon of GM black hole

The GM black hole includes an axion dilaton charge and an electric charge, so the study of the charged particle radiation is necessary. During the tunnelling of the charged particle, the conservation of the total energy and charge in the space-time plays an important role, that is to say, when a particle with static mass \( \omega \), electro charge \( q \) and magnetic charge \( p \) tunnels across the event horizon, to the radiated particle, the background metric was modified with \( M, Q, P \) replaced by \( M - \omega, Q - q, P - p \). For the radiation of charged particle, the action
\[ I = \int_{r_i}^{r_f} \int_{M_i}^{M_f} \frac{dH}{\dot{r}} dr = \int_{r_i}^{r_f} \int_{M_i}^{M_f} \frac{dE(r)}{\dot{r}} dr 
= \int_{r_i}^{r_f} \int_{M_i}^{M_f} \left( -E - \frac{\sqrt{1 + \sqrt{g_{00} + E^2}}}{\Delta} \right) r dE(r) dr 
= \int_{M_i}^{M_f} \int_{r_i}^{r_f} \frac{f(r) dE(r)}{(r - r_+)} dr, \] (49)
where
\[ f(r) = \frac{-E - \frac{\sqrt{1 + \sqrt{g_{00} + E^2}}}{\Delta} r}{\Delta \sqrt{g_{00} + E^2}}. \] (50)
The integrand in (49) is singular at \( r = r_+ \), so just like equation (15)-(17), we get the imaginary part of the action
\[ \text{Im} I = -\pi \int_{M_i}^{M_f} f(r_+) dE(r_+). \] (51)
From (48), we get that
\[ dE(r) = \frac{\partial E}{\partial M} dM + \frac{\partial E}{\partial Q} dQ + \frac{\partial E}{\partial P} dP \]
\[ = \left( dM - \frac{2Q}{r - D} dQ - \frac{2P}{r + D} dP \right) - \frac{[MD^2 + Mr^2 - (P^2 + Q^2)r]}{M(r^2 - D^2)^2} \left( 2D^2 dM + 2DQ dQ - 2DP dP \right). \] (52)

and
\[ dE(r_+) = \left( dM - \frac{2Q}{r_+ - D} dQ - \frac{2P}{r_+ + D} dP \right) - \frac{\sqrt{M^2 + D^2 - (P^2 + Q^2)}}{M(r_+^2 - D^2)} \left( 2D^2 dM + 2DQ dQ - 2DP dP \right). \] (53)

From equation (50), we have
\[ f(r_+) = \left( -E - \sqrt{E^2} \right) \left( \frac{r_+^2 - D^2}{\sqrt{M^2 + D^2 - (P^2 + Q^2)}} \right). \] (54)

In equation (32), because \( t \) and \( \tau \) all increase towards the future, so that \( U^t > 0 \), while \( g_{00} < 0 \), so we have \( E > 0 \). The fact that the energy of the particle \( E > 0 \) is also reasonable in physics, because only in the case \( E > 0 \) the particle can have enough energy to move to infinity. Then we have
\[ f(r_+) = \frac{2(r_+^2 - D^2)}{(r_+ - r_-)} = \frac{r_+^2 - D^2}{\sqrt{M^2 + D^2 - (P^2 + Q^2)}}. \] (55)

From (51), (53), (55) we have
\[ \text{Im} I = -\pi \int \frac{r_+^2 - D^2}{\sqrt{M^2 + D^2 - (P^2 + Q^2)}} \left( dM - \frac{2Q}{r_+ - D} dQ - \frac{2P}{r_+ + D} dP \right) \]
\[ + \pi \int \frac{1}{M} \left( 2D^2 dM + 2DQ dQ - 2DP dP \right). \] (56)

In this equation, the second term of the right-hand-side
\[ \pi \int \frac{1}{M} \left( 2D^2 dM + 2DQ dQ - 2DP dP \right) = -\pi \int \left( \frac{M, P, Q}{M, P, Q} \right) d \left( D^2 \right) = 0, \] (57)

here in the last step, we suppose that during the course of emission, because of the interaction between the matter field and the electro-magnetic field, the radiation of the mass, the electric
charge and the magnetic charge is not arbitrary, while obeys the regulation that keeping
\[ D = \frac{P^2 - Q^2}{2M} \]
as a constant. Then we get
\[ \text{Im} I = -\pi \int \frac{r_+^2 - D^2}{\sqrt{M^2 + D^2 - (P^2 + Q^2)}} \left( dM - \frac{2Q}{r_+ - D} dQ - \frac{2P}{r_+ + D} dP \right). \]  
(58)

From (20), we have
\[ \frac{\partial S}{\partial M} = \frac{2\pi (r_+^2 - D^2)}{\sqrt{M^2 + D^2 - (P^2 + Q^2)}}, \]  
(59)
\[ \frac{\partial S}{\partial Q} = -\frac{2\pi (r_+ + D)Q}{\sqrt{M^2 + D^2 - (P^2 + Q^2)}}, \]  
(60)
\[ \frac{\partial S}{\partial P} = -\frac{2\pi (r_+ - D)P}{\sqrt{M^2 + D^2 - (P^2 + Q^2)}}. \]  
(61)

So we have that
\[ \text{Im} I = -\frac{1}{2} \int \left( \frac{\partial S}{\partial M} dM + \frac{\partial S}{\partial Q} dQ + \frac{\partial S}{\partial P} dP \right) = -\frac{1}{2} \int_{S_i}^{S_f} dS = -\frac{1}{2} \Delta S. \]  
(62)

Then for the radiation of charged particles, the tunnelling rate
\[ \Gamma \sim \exp(-2 \text{Im} I) = \exp(\Delta S). \]  
(63)

The result is just as same as that of the massless particle radiation.

V. CONCLUSION

The calculations of the radiation of massless and charged particle from GM black hole give
the same conclusion that the self-gravitation of the particle causes the radiation spectrum
deviates from exact thermality and the tunnelling rate satisfies \( \Gamma \sim \exp(-2 \text{Im} I) = \exp(\Delta S) \).
The later is a familiar result which was obtained by many other methods. From quantum
mechanics, the rate is expressed(Feynmn Golden-law) as
\[ \Gamma(i \rightarrow f) = |M_{fi}|^2 \cdot \text{(phase space factor)}, \]  
(64)
where the first term on the right hand side is the square of the amplitude for the process.
The phase space factor is obtained by summing over final states and averaging over initial
states. Since the number of initial(final) states is just the exponent of the initial entropy.
So
\[ \Gamma \sim \frac{e^{S_f}}{e^{S_i}} = e^{\Delta S}. \]  
(65)
This implies that the radiation spectrum carries the microscopic information of the black hole, the information is conservational.

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