Constrained Pure Exploration Multi-Armed Bandits with a Fixed Budget

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Abstract
We consider a constrained, pure exploration, stochastic multi-armed bandit formulation under a fixed budget. Each arm is associated with an unknown, possibly multi-dimensional distribution and is described by multiple attributes that are a function of this distribution. The aim is to optimize a particular attribute subject to user-defined constraints on the other attributes. This framework models applications such as financial portfolio optimization, where it is natural to perform risk-constrained maximization of mean return. We assume that the attributes can be estimated using samples from the arms' distributions and that these estimators satisfy suitable concentration inequalities. We propose an algorithm called CONstrained-SR based on the Successive Rejects framework, which recommends an optimal arm and flags the instance as being feasible or infeasible. A key feature of this algorithm is that it is designed on the basis of an information theoretic lower bound for two-armed instances. We characterize an instance-dependent upper bound on the probability of error under CONstrained-SR, that decays exponentially with respect to the budget. We further show that the associated decay rate is nearly optimal relative to an information theoretic lower bound in certain special cases.

1 Introduction
The aim of the pure exploration, stochastic, multi-armed bandit (MAB) problem is to identify, via exploration, the optimal arm among a given basket of arms. Here, each arm is associated with an a priori unknown probability distribution, and the optimal arm is classically defined as one that optimizes a certain attribute associated with its distribution (for example, the mean). However, in practical applications, there is rarely just a single arm attribute that is of interest. For example, in clinical trials, one might be interested in not just the efficacy of a treatment protocol, but also its cost and the severity of its side effects. In portfolio optimization, one is interested in not just the expected return of a candidate portfolio, but also the associated variability/risk.

The classical approach in the MAB literature for handling multiple constraints is to combine them into a single objective, often via a linear combination (Vakili and Zhao 2016; Kagrecha, Nair, and Jagannathan 2019). For example, in portfolio optimization, the optimization of a linear combination of expected return and its variance is often recommended (Sani, Lazaric, and Munos 2012). However, the main drawback of this approach is that there is typically no sound way of determining the weights for this linear combination. After all, can one equate the ‘value’ of a unit decrease in expected return of a portfolio to the ‘value’ of a unit decrease in the return variance in a scale-free manner? Given that the mean-variance landscape across the arms is a priori unknown, a certain choice of arm objective might result in the ‘optimal’ arm having either an unacceptably low expected return, or an unacceptably high variability.

An alternative approach for handling multiple arm attributes is to pose the choice of optimal arm as a constrained optimization problem. Specifically, the optimal arm is defined as the one that optimizes a certain attribute, subject to constraints on other attributes of interest. This avoids the ‘apples to oranges’ translation required in order to combine multiple attributes into a single objective. Returning to our portfolio optimization example, this approach would (potentially) define the optimal arm/portfolio as the one that optimizes expected return subject to a prescribed risk appetite.

In this paper, we analyse such a constrained stochastic MAB formulation, in the fixed budget pure exploration setting. Specifically, each arm is associated with a (potentially multi-dimensional) probability distribution. We consider two attributes, both of which are functions of the arm distribution. The optimal arm is then defined as one that minimizes one attribute (henceforth referred to as the objective attribute), subject to a prescribed constraint on the other attribute (henceforth referred to as the constraint attribute). Crucially, we make no limiting assumptions on the class of arm distributions, or on the specific attributes considered. Instead, we simply assume that the arm attributes can be estimated from samples obtained from arm pulls, with reasonable concentration guarantees (details in Section 2).

While the unconstrained (single attribute) pure exploration MAB formulation is well studied in the fixed budget setting, the algorithms and lower bounds for this case do not generalize easily to the constrained formulation described above. For example, the best known algorithms for the unconstrained case divide the exploration budget into phases, and eliminate/reject one or more arms at the end of

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1We consider only a single constraint attribute in this paper. The generalization to multiple constraint attributes is straightforward, but cumbersome.
each phase (for example, the Successive Rejects algorithm by Audibert and Bubeck 2010, and the Sequential Halving algorithm by Karnin, Koren, and Somekh 2013). The last surviving arm is then flagged as optimal. While the exact ‘rejection schedule’ differs across state of the art algorithms, the decision on which arm(s) to reject is itself straightforward, given a single, scalar arm attribute.

However, in the presence of multiple constraints, the decision on which arm(s) to reject is non-trivial, given that estimates of both attributes of each surviving arm must be taken into consideration. A naive strategy is to focus on first rejecting the arms that appear ‘infeasible’ (i.e., arms whose constraint attribute estimates violate the prescribed threshold), and then reject those arms that appear ‘feasible but suboptimal.’ However, this strategy can be far from optimal (see Section 5). Instead, the approach we propose exploits an information theoretic lower bound for two-armed instances. Specifically, this lower bound motivates the definition of certain suboptimality gaps between pairs of arms. We then reject arms sequentially based on empirical estimates of these pairwise gaps, along with a specific tie-breaking rule. This novel approach, which we formalize as the CONstrained-SR algorithm, is the main contribution of this paper.

This paper is organized as follows. After a brief survey of the related literature, we formally describe the constrained pure exploration MAB formulation in Section 2. In Section 3, we derive an information theoretic lower bound for two-armed instances, which leads to a conjecture on the lower bound for the general K-armed case. The CONstrained-SR algorithm is described and analyzed in Section 4. Finally, we provide a numerical case study in Section 5 and conclude in Section 6.

Throughout the paper, references to the appendix (mainly for proofs of certain technical results) point to the appendix in the supplementary materials document.

Related literature: There is a substantial literature on the multi-armed bandit problem. We refer the reader to excellent textbook treatments by Bubeck and Cesa-Bianchi 2012, Lattimore and Szepesvári 2020 for an overview. In this review, we restrict attention to (the few) papers that consider MABs with multiple attributes.

Drugan and Nowe 2013, Yahyaa and Manderick 2015 consider the Pareto frontier in the attribute space; the goal in these papers is to play all Pareto-optimal arms equally often. Another useful notion is lexicographic optimality, where the attributes are ‘ranked’ with ‘less important’ attributes used to break ties in values of ‘more important’ attributes (see Ehrgott 2005). Tekin and Turgay 2018, Tekin 2019 apply the notion of lexicographic optimality to contextual MABs.

The paper closest to the present paper is Kaggrecha, Nair, and Jagannathan 2020, which analyzes a similar constrained MAB formulation, but in the regret minimization setting. This paper proposes a UCB-style algorithm for this problem, and establishes information theoretic lower bounds. The follow-up paper Chang, Zhu, and Tan 2020 proposes a Thompson Sampling based variant. Special cases of the constrained MAB problem (with a risk constraint) are considered in the pure exploration fixed confidence setting in David et al.

2018, Hou, Tan, and Zhong 2022. Chang 2020 considers an average cost constraint (each arm has a cost distribution that is independent of its reward distribution), pursuing the weaker goal of asymptotic optimality. A linear bandit setting is considered in Pacchiano et al. 2021 under the assumption that there is at least one arm which satisfies the constraints. Finally, Amani, Alizadeh, and Thrampoulidis 2019, Moradinari et al. 2019 consider the problem of maximizing the reward subject to satisfying a linear ‘safety’ constraint with high probability. None of the above mentioned papers considers the fixed budget pure exploration setting considered here. Additionally, all the papers above (with the exception of Kaggrecha, Nair, and Jagannathan 2020) implicitly assume that the instance is feasible; the present paper explicitly addresses the practically relevant possibility that the learning agent may encounter an instance where no arm meets the prescribed constraint(s).

2 Problem formulation

In this section, we describe the formulation of the constrained stochastic MAB problem studied here. We consider the fixed budget, pure exploration framework: the MAB instance is parameterized by a budget of T rounds (a.k.a., arm pulls) and K arms labelled 1, . . . , K, each of which is associated with an a priori unknown probability distribution. We consider a constrained setting, wherein the optimal arm is defined to be the one that optimizes a certain attribute, subject to a constraint on another attribute. In a nutshell, the goal of the learner (a.k.a., algorithm) is to identify the optimal arm in the instance, and also to flag the instance as being feasible or infeasible (i.e., indicating whether any or none of the arms meets the constraint, respectively), using the budget of T arm pulls for exploration. The rest of this section is devoted to formalizing this problem.

Each arm i is associated with a possibly multidimensional distribution ν(i). These distributions are unknown to the learner. Let C denote the space of arm distributions, i.e., ν(i) ∈ C for all i. We define the objective and constraint attributes µ1 and µ2, respectively, to be functions from C to R. We henceforth refer to µ1(i) = µ1(ν(i)) as the value of attribute j (j ∈ {1, 2}) associated with arm i, with µ(i) denoting the vector (µ1(i), µ2(i)). The user specifies a threshold τ ∈ R, which defines an upper bound for the attribute µ2. An instance of this constrained MAB problem is specified by (ν, τ) where ν = (ν(1), . . . , ν(K)). The arms for which the constraint is satisfied, i.e., µ2(i) ≤ τ, are called feasible arms; and the set of feasible arms is denoted by K(ν). The instance (ν, τ) is said to be feasible if K(ν) ≠ ∅, and infeasible if K(ν) = ∅.

Consider a feasible instance. We define an arm to be optimal if it has the least value of µ1(·), subject to the constraint µ2(·) ≤ τ. For simplicity of exposition, we assume that there is a unique optimal arm. We formally denote the

As is well understood in the pure exploration, fixed budget setting, it is straightforward to handle the generalization where there are multiple optimal arms.
samples of each arm, we assume the following concentration properties for these estimators. Specifically, we assume that for $i \in \{1, 2\}$ and distribution $G \in C$, there exists an estimator $\bar{\mu}_{i,n}(G)$ of $\mu_i(G)$ which uses $n$ i.i.d. samples from $G$, satisfying the following concentration inequality: There exists $a_i > 0$ such that for all $\Delta > 0$, 

$$\mathbb{P}(\left| \bar{\mu}_{i,n}(G) - \mu_i(G) \right| \geq \Delta) \leq 2 \exp(-a_i n \Delta^2). \hspace{1cm} (1)$$

Such concentration inequalities are commonly used for analyzing MAB algorithms\(^5\). For instance, if the attributes can be expressed as expectations of sub-Gaussian or bounded random variables (which are themselves functions of the arm samples), concentration inequalities of the form $[1]$ would hold for the empirical average using the Cramér-Chernoff bound or the Hoeffding inequality respectively (refer Chapter 5 of [Lattimore and Szepesvári 2020]). Several risk measures like Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR) also admit estimators with concentration properties of the form $[1]$; see [Wang and Gao 2010; Cassel, Mannor, and Zeevi 2018; Kolla et al. 2019; Bhat and Prashanth 2019].

Finally, we define the notion of consistency of an algorithm. An algorithm is said to be consistent over $(\mathcal{C}, \tau)$ if, for all instances of the form $(\nu, \tau)$, where $\nu \in C^K$, $\lim_{T \to \infty} \epsilon_T = 0$.

3 Lower bound

In this section, we provide an information theoretic lower bound on the probability of error under any algorithm, for a class of two-armed Gaussian bandit instances. We then extrapolate this bound to conjecture a lower bound for the general $K$-armed case. Crucially, the lower bound for the two-armed case forms the basis for our algorithm design (see Section 4).

First, we define some sub-optimality gaps that will be used to state our lower bounds, and also later when we discuss algorithms. Given two arms $i$ and $j$, we say $i \succ j$ if $i$ is an optimal arm in a two-armed instance consisting only of arms $i$ and $j$. For $i \succ j$, define $\delta(i, j) = \delta(\mu(i), \mu(j))$ as

\(^5\)The standard practice when dealing with classical (unconstrained) MAB problems is to specify both the set $\mathcal{C}$ of arm distributions (for example, as the set of l-subGaussians) and the attribute being optimized $\mu_1$ (for example, the mean of the arm distribution). These choices then imply natural estimators $\bar{\mu}_1$ and their corresponding concentration properties. In this work, to avoid working with a specific distribution class and a specific set of arm attributes, and to emphasize the generality of the proposed approach, we simply assume that attribute estimates satisfy concentration inequalities of the form $[1]$. Moreover, this particular form for the concentration inequality is assumed only for ease of exposition; changes to this form (as might be needed, for example, if the arm distributions are sub-exponential or heavy-tailed) lead to minor modifications to our algorithms and bounds.
follows. \[ \delta(i, j) := \begin{cases} \sqrt{a_1} (\mu_1(j) - \mu_1(i)), & \text{if } i, j \in K(\nu), \\ \sqrt{a_2} (\mu_2(j) - \tau), & \text{if } i \in K(\nu) \text{ and } j \notin K(\nu) \text{ is a deceiver}, \\ \max\{\sqrt{a_2} ((\mu_2(j) - \tau), & \text{if } i \in K(\nu) \text{ and } j \notin K(\nu) \text{ is suboptimal}, \\ \sqrt{a_2} (\mu_2(j) - \mu_2(i)), & \text{if } j \notin K(\nu) \}. \end{cases} \]

Next, for \( i > j \), define \( \Delta(i, j) = (\mu(i), \mu(j)) \) as follows.

\[ \Delta(i, j) := \min\{\sqrt{a_2} |\tau - \mu_2(i)|, \delta(i, j)\}. \]

As we will see, the smaller the value of \( \Delta(i, j) \), the harder it is for a learner to identify the optimal arm \( i \) in a two-armed instance consisting of arms \( i \) and \( j \). Thus, one may interpret \( \Delta(i, j) \) as the 'suboptimality gap' of arm \( j \) (relative to arm \( i \)); note that this gap depends on the values of the objective attributes, the constraint attributes, the threshold \( \tau \), and also the concentration parameters \( a_1, a_2 \). For example, if \( i \) is feasible and \( j \) is feasible and suboptimal, \( \Delta(i, j) = \min\{\sqrt{a_2} (\mu_2(j) - \mu_2(i)), \sqrt{a_1} (\mu_1(j) - \mu_1(i))\} \).

Thus, the closer \( i \) is to the constraint boundary, and the smaller the gap between \( i \) and \( j \) in the objective attribute, the harder it is to identify \( i \) as the optimal arm in this pair. The gaps for the other cases can be interpreted in a similar manner.

We are now ready to state our information theoretic lower bound. Consider the class of arm distributions \( \mathcal{D} \), which consists of 2-dimensional Gaussian distributions with covariance matrix \( \Sigma = \text{diag}(\frac{1}{a_2}, \frac{1}{a_2}) \). Attribute \( \mu_1 \) is the mean of the first dimension, while attribute \( \mu_2 \) is the mean of the second dimension. Note that the empirical average estimator satisfies \( \mathcal{H} \) for both attributes.

**Theorem 1.** Let \( \nu \) be a two-armed bandit instance where \( \nu(i) \in \mathcal{D} \) for \( i \in \{1, 2\} \), with attribute \( \mu_1 \) being the mean of the first dimension of the arm distribution, and attribute \( \mu_2 \) being the mean of the second dimension of the arm distribution. Under any consistent algorithm,

\[ \lim sup_{T \to \infty} \frac{1}{T} \log e_T(\nu) \leq (\Delta(1, 2))^2, \]

where arm 1 is taken to be the optimal arm (without loss of generality).

The proof of Theorem 1 can be found in Appendix A. Note that Theorem 1 provides an upper bound on the (asymptotic) exponential rate of decay of the probability of error as \( T \to \infty \). Specifically, the decay rate can be at most \( \Delta^2(1, 2) \). This formalizes the interpretation of \( \Delta(1, 2) \) as a 'suboptimality gap' between arm 2 and arm 1. It is instructive to see which aspects of the arm attributes influence this suboptimality gap. For example, if both arms 1 and 2 are feasible, then \( \Delta(1, 2) \) depends on the optimality gap (i.e., \( \mu_1(2) - \mu_1(1) \)) and the feasibility gap of arm 1 (i.e., \( \tau - \mu_2(1) \)) but not on the feasibility gap of arm 2 (i.e., \( \tau - \mu_2(2) \)). On the other hand, if arm 1 is feasible and arm 2 is a deceiver, then \( \Delta(1, 2) \) depends on the feasibility gap of arm 1 (i.e., \( \tau - \mu_2(1) \)) and the infeasibility gap of arm 2 (i.e., \( \tau - \mu_2(2) \)), but not on the gap between the objective attributes. In Section 4, we design an algorithm that eliminates arms from consideration sequentially based on estimates of these (pairwise) suboptimality gaps.

Based on Theorem 1 and results from (Audibert and Bubeck 2010) on the classical (unconstrained) MAB problem, we conjecture the following extension of Theorem 1 to the case of \( K \) arms as follows. Taking arm 1 to be the optimal arm without loss of generality, define \( H_1 := \sum_{i=2}^{K} \Delta(1, i) \).

**Conjecture 1.** Let \( \nu \) be a \( K \)-armed bandit instance where \( \nu(i) \in \mathcal{D} \) for \( i \in [K] \), with attribute \( \mu_1 \) being the mean of the first dimension of the arm distribution, and attribute \( \mu_2 \) being the mean of the second dimension of the arm distribution. Under any consistent algorithm,

\[ \lim sup_{T \to \infty} \frac{1}{T} \log e_T(\nu) \leq \frac{d}{H_1}, \]

where \( d \) is a universal positive constant.

The main challenge in proving this conjecture for a general \( K \)-armed instance is that existing lower bound approaches for the unconstrained setting (Audibert and Bubeck 2010; Kaufmann, Cappé, and Garivier 2016; Carpentier and Locatelli 2016) do not generalize to the constrained setting. As per Conjecture 1, \( H_1 \) can be interpreted as a measure of the hardness of the instance under consideration. Indeed, this definition of \( H_1 \) agrees with the hardness measure that appears in lower bounds for the classical (unconstrained) MAB problem (also denoted \( H_1 \); see (Audibert and Bubeck 2010; Kaufmann, Cappé, and Garivier 2016)) when \( \tau \to \infty \).

**4 The CONSTRANED-SR algorithm**

In this section, we propose the CONSTRANED-SR algorithm for the constrained MAB problem posed in Section 2 and provide a performance guarantee via an upper bound on the probability of error under this algorithm. This upper bound compares favourably with the information theoretic lower bound conjectured in Section 3 (see Conjecture 1), suggesting that the CONSTRANED-SR algorithm is nearly optimal. Importantly, the design of the CONSTRANED-SR algorithm is motivated by our information theoretic lower bound for the two-armed case (see Theorem 1). CONSTRANED-SR rejects arms sequentially based on estimates of the same pairwise suboptimality gaps that appear in the lower bound.

**Algorithm description:** The CONSTRANED-SR algorithm is based on the well-known Successive Rejects (SR) framework proposed by (Audibert and Bubeck 2010). Informally, SR runs over \( K - 1 \) phases; at the end of each phase, one arm (the one that looks empirically 'worst') is rejected from consideration. Specifically, SR defines positive integers \( n_1, n_2, \ldots, n_{K-1} \) such that \( n_1 < n_2 < \cdots < n_{K-1} \).
and \( n_1 + n_2 + \cdots + n_{K-2} + 2n_{K-1} \leq T \) (see Algorithm 1 for the details). In phase \( k \), each of the surviving \( K - k + 1 \) arms is pulled \( n_k - n_{k-1} \) times. (This means that by the end of phase \( k \), each surviving arm has been pulled \( n_k \) times.) The sole arm that survives at the end of phase \( K - 1 \) is declared to be optimal, and the instance is flagged as feasible (respectively, infeasible) if this surviving arm ‘appears’ feasible (respectively, infeasible).

**Constrained-SR** (formal description as Algorithm 1) differs from SR in the criterion used to reject an arm at the end of each phase. Note that the classical SR algorithm is designed for a single attribute; this makes the choice of the empirically ‘worst’ arm obvious. In contrast, the elimination criterion for our constrained MAB problem should depend on estimates of both attributes for each surviving arm. The **Constrained-SR** algorithm does this as follows: Let \( \hat{J}(A_k) \) denote the arm that ‘appears’ optimal at the end of phase \( k \), where \( A_k \) denotes the set of surviving arms at the beginning of phase \( k \). Formally, letting \( \hat{\mu}_1^k(i) \) denote the estimate of attribute 1 for arm \( i \) at the end of phase \( k \),

\[
\hat{J}(A_k) = \begin{cases} 
\arg\min_{i \in A_k} \mu_1^k(i), & \{i \in A_k : \hat{\mu}_2^k(i) \leq \tau\} \neq \emptyset \\
\arg\min_{i \in A_k} \mu_2^k(i), & \{i \in A_k : \hat{\mu}_2^k(i) \leq \tau\} = \emptyset.
\end{cases}
\]

Then, the gaps \( \hat{\delta}(\hat{J}(A_k), i) \) are estimated for all arms \( i \in A_k \) as follows,

\[
\hat{\delta}(\hat{J}(A_k), i) = \min \left\{ \hat{\mu}_2^k(i), \hat{\mu}_2^k(j) \right\},
\]

where \( \hat{\mu}_2^k(i) := (\hat{\mu}_1^k(i), \hat{\mu}_2^k(i)) \). In other words, the gaps relative to the ‘seemingly optimal’ arm are estimated by replacing the (unknown) arm attributes by their available estimates.

Finally, the arm \( \arg\max_{i \in A_k} \hat{\delta}(\hat{J}(A_k), i) \), i.e., the arm with the largest estimated gap relative to \( \hat{J}(A_k) \), is rejected, with the following rule used to break ties:

\[
\hat{D}(A_k) = \{ i \in A_k : i \neq \hat{J}(A_k) \}.
\]

This tie-breaking rule plays a key role in the performance of **Constrained-SR** in contrast, the tie-breaking rule is *inconsequential* in the original SR algorithm for single attribute MABs.

The graph showing a feasible instance that motivates our tie-breaking rule. Panel (b) shows a feasible instance that motivates the use of estimates of our information theoretic suboptimality gaps to guide arm elimination.

**Remark:** While the above example might suggest that it is sound to blindly eliminate seemingly infeasible arms first, the scenario shown in Figure 2(b) (again, at the end of a generic phase) demonstrates that this is not always the case. Here, arm 2 appears optimal, but arm 1, placed slightly above the \( \tau \) boundary, might be optimal if \( \hat{\mu}_2(1) \) is a (small) overestimation of \( \mu_2(1) \). It is therefore ‘safer’ in this scenario to eliminate arm 3; this is exactly what **Constrained-SR** would do, since \( \hat{\Delta}(2, 1) < \hat{\Delta}(2, 3) \). This highlights the importance of the sophisticated elimination criterion employed by **Constrained-SR**, that captures the relative likelihoods of different arms being optimal (via estimates of information theoretic suboptimality gaps).

**Performance evaluation:** We now characterize the performance of **Constrained-SR**. For the purpose of expressing our performance guarantee, we order the arm labels as follows (without loss of generality). Arm 1 is the optimal arm, and arms 2, \ldots, \( K \) are labelled in increasing order of \( \Delta(1, \cdot) \), with ties broken in a manner that is consistent with the **Constrained-SR** algorithm. Formally, for any \( 1 < i < j < K \), if \( \Delta(1, i) = \Delta(1, j) \), then either

- \( i, j \in K(\nu) \) and \( \mu_1(i) \leq \mu_1(j) \), or
- \( i \in K(\nu) \) and \( j \notin K(\nu) \), or
- \( i, j \notin K(\nu) \) and \( \mu_2(i) \leq \mu_2(j) \).

**Theorem 2.** Under the **Constrained-SR** algorithm, the probability of error is upper bounded as:

\[
e^{-c(K) \exp \left( -\frac{\beta T}{H_2 \log(K)} \right)},
\]

where \( H_2 = \max_{i \in [K], i \neq 1} \frac{1}{1 - \frac{i}{(1, i)}} \), \( c(K) \) is a function of \( K \), and \( \beta \) is a positive universal constant.
The main takeaways from Theorem 2 are as follows.

- Theorem 2 provides an upper bound on the probability of error under CONSTRANDED-SR, that decays exponentially with the budget $T$. The associated decay rate is given by $\frac{\beta}{\log(K)}$, suggesting that the instance-dependent parameter $H_2$ captures the hardness of the instance (under the CONSTRANDED-SR algorithm); a larger value of $H_2$ implies a ‘harder’ instance, since the probability of error decays more slowly with the budget.

- The ‘hardness index’ $H_2$ agrees with the hardness index obtained for the classical SR algorithm in (Audibert and Bubeck 2010) (also denoted $H_2$) for the unconstrained MAB problem when $T \to \infty$.

- The decay rate from the upper bound for CONSTRANDED-SR can be compared with that in the information theoretic lower bound conjectured in Section 3 (see Conjecture 1). Indeed, it can be proved that $\frac{\beta}{\log(K)} \leq H_1 \leq \log(K)H_2$ (see (Audibert and Bubeck 2010)). This suggests that the decay rate under CONSTRANDED-SR is optimal up to a factor that is logarithmic in the number of arms. In other words, this suggests CONSTRANDED-SR is nearly optimal.

**Sketch of the proof of Theorem 2** In the remainder of this section, we sketch the proof of Theorem 2. The complete proof can be found in Appendix B. Note that

$$e_T = \mathbb{P} \left( \left\{ \tilde{J}([K]) \neq \hat{J}([K]) \right\} \cup \left\{ \tilde{F}([K]) \neq F([K]) \right\} \right)$$

$$= \sum_{k=1}^{K-1} \mathbb{P} (\text{Arm 1 is dismissed in round } k)$$

$$+ \mathbb{P} \left( \tilde{J}([K]) = J([K]), \tilde{F}([K]) \neq F([K]) \right).$$

Let $A_k$ denote the event that arm 1 is rejected at the end of round $k$. Noting that the event in the last term above implies that the feasibility status of arm 1 is estimated incorrectly at the end of phase $K - 1$, (1) implies

$$e_T \leq \sum_{k=1}^{K-1} \mathbb{P} (A_k) + 2 \exp \left( -a_2 n_{K-1} (|\tau - \mu_2(1)|)^2 \right)$$

$$\leq \sum_{k=1}^{K-1} \mathbb{P} (A_k) + 2 \exp \left( -n_{K-1} \Delta^2(1, 2) \right).$$

We now bound $\mathbb{P} (A_k)$. In round $k$, at least one of the $k$ ‘worst’ arms (according to the ordering defined on the arms) survives (i.e., belongs to $A_k$). Thus, for arm 1 to be dismissed at the end of round $k$, it must appear empirically ‘worse’ than this arm. Formally, we have

$$\mathbb{P} (A_k) \leq \sum_{j=K-k+1}^{K} \mathbb{P} \left( \hat{J}(A_k) = j \right)$$

$$+ \sum_{i=2}^{K-k} \sum_{j=K-k+1}^{K} \mathbb{P} \left( \hat{J}(A_k) = i, \hat{\delta}(i, j) \leq \hat{\delta}(i, 1) \right)$$

$$=: S_1 + S_2.$$

The summation $S_1$ above corresponds to the event that one of the worst $k$ arms looks empirically optimal at the end of phase $k$. On the other hand, the summation $S_2$ corresponds to the event that some other arm $i$ (not among the worst $k$ arms) looks empirically optimal at the end of phase $k$, and further that arm 1 has a greater (estimated) gap (relative to $i$) than an arm $j$, which is among the worst $k$ arms (this is necessary for the elimination of arm 1). Crucially, the terms in $S_1$ can be bounded by analysing a two-armed instance consisting only of arms 1 and $j$. Similarly, the terms in $S_2$ can be bounded by analysing a three-armed instance consisting only of arms 1, $i$ and $j$. The relevant bounds are summarized below.

**Lemma 3.** Consider a two-armed instance where the arms are labelled (without loss of generality) as per the convention described before. Under CONSTRANDED-SR, the
where \( \hat{c} \) substituting the above bound into (7), we get

\[
\tilde{c} = \max(c_2, c_3) \quad \beta = \min(\beta_2, \beta_3).
\]

Finally, substituting the above bound into (7), we get

\[
ev_T \leq \sum_{k=1}^{K-1} K \hat{c} \exp \left( -\beta n_k \Delta^2(1, K-k+1) \right)
+ 2 \exp \left( -n_{K-1} \Delta^2(1, 2) \right)
\leq (K^3 \hat{c} + 2) \exp \left( -\beta \min_{1 \leq k \leq K-1} (n_k \Delta^2(K-k+1)) \right),
\]
where \( \hat{c} = \max(c_2, c_3) \) and \( \beta = \min(\beta_2, \beta_3) \).

Using Lemmas 3 and 4 (proofs in Appendix B), \( \mathbb{P}(A_k) \) can be upper bounded as follows:

\[
\mathbb{P}(A_k) \leq kc_2 \exp \left( -\beta n_k \Delta^2(1, K-k+1) \right)
+ k(K-k-1)c_3 \exp \left( -\beta n_k \Delta^2(1, K-k+1) \right)
\leq K^2 \hat{c} \exp \left( -\beta n_k \Delta^2(1, K-k+1) \right),
\]

where \( \hat{c} = \max(c_2, c_3) \) and \( \beta = \min(\beta_2, \beta_3) \).

Consider is feasible and has three arms with the mean vectors
\([1 0.995]^T, [2 1.005]^T\) and \([12 0.001]^T\) with \( \tau = 1 \). Thus, arm 1 is optimal, arm 2 is a deceiver and arm 3 is feasible suboptimal. The third instance that we consider is also feasible and has four arms with the mean vectors \([0.3 0.45]^T, [0.35 0.45]^T, [0.2 0.8]^T\) and \([0.5 0.8]^T\) and \( \tau = 0.5 \). Thus, arm 1 is optimal, arm 2 is feasible suboptimal, arm 3 is a deceiver and arm 4 is infeasible suboptimal. The fourth instance that we consider is infeasible and has four arms with the mean vectors \([0.3 1.6]^T, [0.4 1.7]^T, [0.2 1.1]^T, [0.5 1.2]^T\) and \( \tau = 1 \). Thus, arm 3 is the optimal arm for this instance. The results of the simulations for each of these instances can be found in Figures 5a, 5b, 5c and 5d respectively.

The algorithms were run for horizons up to 10000 and averaged over 10000 runs for the feasible instances and over 10000 runs for the infeasible instance. Empirical averages were used as the attribute estimators. Figure 5 shows the variation of \( \log_{10}(ev_T) \) with the horizon \( T \) for these four instances. Note that the slope of this curve captures the (exponential) decay rate of the probability of error. In the case of the infeasible instance (Figure 3d), the performance is nearly the same. In Figures 5b and 5c, we once again observe that the decay rates of CONstrained-SR and IF are identical; the probability of error under CONstrained-SR appears to be smaller than that under IF by a constant factor in Figure 5b. However, in Figure 5d, CONstrained-SR
demonstrates a superior decay rate, since it employs a more sophisticated elimination criterion using gaps inspired by the two-armed lower bound (as noted in Section 4).

6 Concluding Remarks

This work motivates follow-ups in several directions. On the theoretical front, the main gap in this work pertains to the information theoretic lower bound. Proving Conjecture 1 would not only establish the ‘near’ optimality of the CONSTRAINED-SR algorithm, but also, quite likely, introduce a novel approach for deriving lower bounds in the fixed budget pure exploration setting. On the application front, the present work motivates an extensive case study applying the proposed algorithm in various application scenarios.

This work also motivates generalizations to constrained reinforcement learning, where the goal is to identify the optimal policy that fulfills additional constraints.

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A Proof of Theorem 1

Proof. With some abuse of notation, we denote by \((J(\nu), F(\nu))\) the correct output for instance \(\nu\). Consider
any alternative bandit model \( \nu' = (\nu'(1), \nu'(2)) \) such that its correct output, \( (J(\nu'), F(\nu')) \neq (J(\nu), F(\nu)) \). Let \( \mathcal{L} \) be a consistent algorithm. We apply Lemma 1 of (Kaufmann, Cappé, and Garivier 2016) with the stopping time \( \sigma = T \) a.s. on the event \( \mathcal{H} = \{ J([2]) = J(\nu) \} \cap \{ F([2]) = F(\nu) \} \) to get:

\[
E_{\nu'}[N_1(T)] \text{KL} (\nu'(1), \nu(1)) + E_{\nu'}[N_2(T)] \text{KL} (\nu'(2), \nu(2)) \\
\geq d(\mathbb{P}_{\nu'}(\mathcal{H}), \mathbb{P}_{\nu}(\mathcal{H})) ,
\]

(8)

where \( E_{\nu'}(\cdot) \) and \( \mathbb{P}_{\nu}(\cdot) \) denote the expectation and the probability, respectively, with respect to the randomness introduced by the interaction of the algorithm with the bandit instance \( \nu \), and \( d(\cdot, \cdot) \) denotes the binary relative entropy. Denote by \( e_T(\nu) \) the probability of error of the algorithm on the instance \( \nu \).

We have that \( e_T(\nu) = 1 - \mathbb{P}_{\nu}(\mathcal{H}) \) and \( e_T(\nu') \geq \mathbb{P}_{\nu}(\mathcal{H}) \). As algorithm \( \mathcal{L} \) is consistent, we have that for every \( \epsilon > 0, \exists T_0(\epsilon) \) such that for all \( T \geq T_0(\epsilon), \mathbb{P}_{\nu}(\mathcal{H}) \leq \epsilon \leq \mathbb{P}_{\nu}(\mathcal{H}). \) For \( T \geq T_0(\epsilon) \), we have:

\[
E_{\nu'}[N_1(T)] \text{KL} (\nu'(1), \nu(1)) + E_{\nu'}[N_2(T)] \text{KL} (\nu'(2), \nu(2)) \\
\geq d(\epsilon, 1 - e_T(\nu)) \geq (1 - \epsilon) \log \frac{1 - \epsilon}{e_T(\nu)} + \epsilon \log \epsilon.
\]

In the limit where \( \epsilon \) goes to zero, we have,

\[
\limsup_{T \to \infty} \frac{1}{T} \log e_T(\nu) \\
\leq \limsup_{T \to \infty} \frac{2}{T} E_{\nu'}[N_j(T)] \text{KL} (\nu'(j), \nu(j)) \\
\leq \max_{j = 1, 2} \text{KL} (\nu'(j), \nu(j)).
\]

(9)

Denote by \( \mathcal{M} \) the set of two-armed bandit instances whose arms belong to \( D \). Minimizing the RHS over all \( \nu' \in \mathcal{M} \) whose correct output differs from that of \( \nu \) gives us:

\[
\limsup_{T \to \infty} \frac{1}{T} \log e_T(\nu) \\
\leq \inf_{\nu' \in \mathcal{M} \setminus \{ \nu \}} \max_{j = 1, 2} \text{KL} (\nu'(j), \nu(j)).
\]

(10)

Using the formula for the KL divergence between two multivariate distributions in (9) gives:

\[
\limsup_{T \to \infty} \frac{1}{T} \log e_T(\nu) \\
\leq \frac{1}{2} \inf_{\nu' \in \mathcal{M} \setminus \{ \nu \}} \max \left\{ a_1 (\mu_1(1) - \mu'_1(1))^2 \\
+ a_2 (\mu_2(1) - \mu'_2(1))^2, a_1 (\mu_1(2) - \mu'_1(2))^2 \\
+ a_2 (\mu_2(2) - \mu'_2(2))^2 \right\}.
\]

(10)

Evaluating the RHS of (10) for each type of two-armed bandit instance gives the required result. There are broadly two cases involved here: \( \nu \) being a feasible instance and \( \nu \) being an infeasible instance. The former has three subcases for the non-optimal arm, i.e., arm 2: arm 2 is feasible suboptimal, deceiving, and infeasible suboptimal. The general methodology used here is to minimize both terms inside the maximum subject to the constraints on the arms.

**Case 1: \( \nu \) is feasible and \( \nu(2) \) is feasible suboptimal**

We first evaluate the infimum over the two cases: \( J(\nu') \neq J(\nu), F(\nu') = F(\nu) \) and \( F(\nu') \neq F(\nu) \), and then find the minimum of these two cases. In the former case, we have that \( J(\nu') \neq J(\nu), F(\nu') = F(\nu) \), i.e., both \( \nu \) and \( \nu' \) are feasible but their optimal arms are different, while in the latter case, we have that \( \nu' \) is infeasible. We first consider the former case. WLOG, we assume that \( J(\nu) = 1 \) and \( J(\nu') = 2 \).

- **Arm 1 of \( \nu' \) is feasible.**

In this case, we have that \( \mu'_2(1) \leq \tau \), and hence there are no restrictions on \( \mu'_2(1) \) and \( \mu'_2(2) \) (as long as they are below \( \tau \)). We thus set \( \mu'_2(1) = \mu_2(1) \) and \( \mu'_2(2) = \mu_2(2) \). It follows that:

\[
\inf_{\nu' \in \mathcal{M} \setminus \{ \nu \}} \max \left\{ a_1 (\mu_1(1) - \mu'_1(1))^2 \\
+ a_2 (\mu_2(1) - \mu'_2(1))^2, a_1 (\mu_1(2) - \mu'_1(2))^2 \\
+ a_2 (\mu_2(2) - \mu'_2(2))^2 \right\} \\
= \inf_{\nu' \in \mathcal{M} \setminus \{ \nu \}} \max \left\{ a_1 (\mu_1(1) - \mu'_1(1))^2, \\
\mu'_2(2) < \mu_2(1) \right\} \\
\leq a_1 (\mu_1(2) - \mu_1(1))^2,
\]

where the infimum is attained midway between \( \mu_1(1) \) and \( \mu_1(2) \).

- **Arm 1 of \( \nu' \) is infeasible.**

In this case, we have that \( \mu'_2(1) > \tau \), and hence there are no restrictions on \( \mu'_1(1), \mu'_1(2), \) and \( \mu'_2(2) \) (as long as it is below \( \tau \)). We thus set \( \mu'_1(1) = \mu_1(1) \), \( \mu'_1(2) = \mu_1(2) \) and \( \mu'_2(2) = \mu_2(2) \). Thus,

\[
\inf_{\nu' \in \mathcal{M} \setminus \{ \nu \}} \max \left\{ a_1 (\mu_1(1) - \mu'_1(1))^2 \\
+ a_2 (\mu_2(1) - \mu'_2(1))^2, a_1 (\mu_1(2) - \mu'_1(2))^2 \\
+ a_2 (\mu_2(2) - \mu'_2(2))^2 \right\} \\
= \inf_{\nu' \in \mathcal{M} \setminus \{ \nu \}} \max \left\{ a_2 (\mu_2(1) - \mu'_2(1))^2 \right\} \\
= a_2 (\tau - \mu_2(1))^2.
\]

It is enough to evaluate the infimum only for the case where \( J(\nu') \neq J(\nu), F(\nu') = F(\nu) \) because in the case where \( F(\nu') \neq F(\nu) \), the infimum is at
least $a_2 \max \{ (\tau - \mu_2(1))^2, (\tau - \mu_2(2))^2 \}$. Thus, combining the results of the cases discussed above, in the case of a feasible instance with optimal arm being arm 1 and arm 2 being a suboptimal feasible arm, we have that

$$\limsup_{T \to \infty} \frac{1}{T} \log e_T(\nu) \leq \frac{1}{2} \min \left\{ a_2 (\tau - \mu_2(1))^2, \frac{a_1 (\mu_1(2) - \mu_1(1))^2}{4} \right\}.$$ 

**Case 2: $\nu$ is feasible and $\nu(2)$ is a deceiver**

We first evaluate the infimum over the two cases: $J(\nu') \neq J(\nu), F(\nu') = F(\nu)$, and $F(\nu') \neq F(\nu)$; and then find the minimum of these two cases. In the former case, we have that $J(\nu') \neq J(\nu)$, $F(\nu') = F(\nu)$, i.e., both $\nu$ and $\nu'$ are feasible but their optimal arms are different, while in the latter case, we have that $\nu'$ is infeasible. We first consider the former case. WLOG, we assume that $J(\nu) = 1$ and $J(\nu') = 2$.

- **Arm 1 of $\nu'$ is feasible.**

  In this case, we have that $\mu'_2(1) \leq \tau$. As we also have that $\mu_1(2) \leq \mu_1(1)$ and the only constraint on the first dimensions of the arms of instance $\nu'$ is $\mu'_1(2) \leq \mu'_1(1)$, we set $\mu_2(1) = \mu'_1(2)$ and $\mu_1(1) = \mu'_1(1)$ to minimize each term inside the maximum. Thus,

  $$\inf_{\nu' \in M: \frac{J(\nu')}{2} = 2} \max \left\{ a_1 (\mu_1(1) - \mu'_1(1))^2, a_2 (\mu_2(1) - \mu'_2(1))^2, a_2 (\mu_2(2) - \mu'_2(1))^2 \right\} = \inf_{\nu' \in M: \frac{J(\nu')}{2} = 2} \max \left\{ a_2 (\mu_2(2) - \mu'_2(2))^2 \right\}$$

- **Arm 1 of $\nu'$ is infeasible.**

  In this case, we have that $\mu'_2(1) > \tau$, and hence there are no restrictions on $\mu'_1(1)$ and $\mu'_1(2)$. In this case, we set $\mu'_1(1) = \mu_1(1), \mu'_1(2) = \mu_1(2)$. Thus,

  $$\inf_{\nu' \in M: \frac{J(\nu')}{2} = 2} \max \left\{ a_1 (\mu_1(1) - \mu'_1(1))^2, a_2 (\mu_2(1) - \mu'_2(1))^2, a_2 (\mu_2(2) - \mu'_2(2))^2 \right\}$$

We now consider the case where $F(\nu') \neq F(\nu)$, i.e., $\nu'$ is an infeasible instance. Here, as there are no constraints on arm 2 of the instance $\nu'$ apart from $\mu'_2(2) \geq \mu'_2(1) \geq \tau$, we set $\mu'_2(2) = \mu(2)$. We also set $\mu'_1(1) = \mu(1)$ as the only constraints on arm 1 of the instance $\nu'$ is that $\mu'_2(2) \geq \mu'_2(1) \geq \tau$.

$$\inf_{\nu' \in M: \frac{J(\nu')}{2} = 2} \max \left\{ a_1 (\mu_1(1) - \mu'_1(1))^2, a_2 (\mu_2(1) - \mu'_2(1))^2, a_2 (\mu_2(2) - \mu'_2(2))^2 \right\}$$

Thus, combining the results of the three cases above, we have that for a two-armed feasible instance $\nu$ with arm 1 being the optimal arm and arm 2 being a deceiver arm,

$$\limsup_{T \to \infty} \frac{1}{T} \log e_T(\nu) \leq \frac{1}{2} \min \left\{ a_2 (\tau - \mu_2(1))^2, a_2 (\tau - \mu_2(2))^2 \right\}.$$ 

**Case 3: $\nu$ is feasible and $\nu(2)$ is infeasible suboptimal**

We first evaluate the infimum over the two cases: $J(\nu') \neq J(\nu), F(\nu') = F(\nu)$, and $F(\nu') \neq F(\nu)$; and then find the minimum over the two cases. In the former case, we have that $J(\nu') \neq J(\nu)$, $F(\nu') = F(\nu)$, i.e., both $\nu$ and $\nu'$ are feasible but their optimal arms are different, while in the latter case, we have that $\nu'$ is infeasible. We first consider the former case. WLOG, we assume that $J(\nu) = 1$ and $J(\nu') = 2$.

1. **Arm 1 of $\nu'$ is feasible.**

   In this case, as there are no restrictions on the second dimensions of the arms of $\nu'$ apart from them being smaller than $\tau$, we set $\mu'_2(1) = \mu_2(1), \mu'_2(2) = \tau$. Also, as $\mu'_2(1) \leq \mu'_1(1)$, to attain the infimum in (10), it is clear that $\mu'_2(2) \geq \mu_2(1)$ and $\mu'_2(1) \leq \mu_1(2)$. Hence, we also set $\mu'_2(2) = \mu_1(2)$. Thus,

   $$\inf_{\nu' \in M: \frac{J(\nu')}{2} = 2} \max \left\{ a_1 (\mu_1(1) - \mu'_1(1))^2, a_2 (\mu_2(1) - \mu'_2(1))^2, a_2 (\mu_2(2) - \mu'_2(2))^2 \right\}$$

   $$= \inf_{\nu' \in M: \frac{J(\nu')}{2} = 2} \max \left\{ a_1 (\mu_1(1) - \mu'_1(1))^2, a_2 (\mu_2(1) - \mu'_2(1))^2, a_2 (\tau - \mu_2(2))^2 \right\}.$$
where $y = \max\{y, M\}$. We also have that
\[
\frac{z^2}{4} \leq \left(\frac{y^2 + z^2}{2z}\right)^2 \leq z^2,
\]
i.e., the infimum is within a constant factor of $z^2$. As our algorithm is motivated by these gaps and to avoid comparisons between the first and the second dimensions, we have that
\[
\inf_x \max\{x^2, y^2 + (M - x)^2\} \leq z^2.
\]

2. Arm 1 of $\nu'$ is infeasible.

In this case, as there are no restrictions on the first dimensions of the arms of $\nu'$, we set $\mu'_1(1) = \mu_1(1)$, $\mu'_1(2) = \mu_1(2)$. Thus,
\[
\inf_{\nu' \in M \colon \nu'(1) \neq \nu'(2)} \max\left\{a_1 (\mu_1(1) - \mu'_1(1))^2 + a_2 (\mu_2(1) - \mu'_2(1))^2, a_1 (\mu_1(2) - \mu'_1(2))^2 + a_2 (\mu_2(2) - \mu'_2(2))^2\right\}
\]
\[= \inf_{\nu' \in M \colon J(\nu') = 2} \max\left\{a_2 (\mu_2(1) - \mu'_2(1))^2, a_1 (\mu_1(2) - \mu'_1(2))^2\right\}
\]
\[= \inf_{\nu' \in M} \min\{a_2 (\mu_2(1) - \mu'_2(1))^2, a_1 (\mu_1(2) - \mu'_1(2))^2\}
\]
\[= a_2 \max\{(\tau - \mu_2(1))^2, (\mu_2(2) - \tau)^2\}.
\]

Next, we consider the case where $F(\nu') \neq F(\nu)$, i.e., $\nu'$ is an infeasible instance. In this case, as there are no restrictions on the first dimensions of the arms of $\nu'$, we set $\mu'_1(1) = \mu_1(1)$, $\mu'_1(2) = \mu_1(2)$. Moreover, as $\mu_2(2) > \tau$ and $\mu'_2(2) > \tau$, we set $\mu_2(2) = \mu'_2(2)$. Thus,
\[
\inf_{\nu' \in M \colon J(\nu') = 2} \max\left\{a_1 (\mu_1(1) - \mu'_1(1))^2 + a_2 (\mu_2(1) - \mu'_2(1))^2, a_1 (\mu_1(2) - \mu'_1(2))^2 + a_2 (\mu_2(2) - \mu'_2(2))^2\right\}
\]
\[= \inf_{\nu' \in M} \min\{a_2 (\mu_2(1) - \mu'_2(1))^2, a_1 (\mu_1(2) - \mu'_1(2))^2\}
\]
\[= a_2 \max\{(\tau - \mu_2(1))^2, (\mu_2(2) - \tau)^2\}.
\]

Thus, combining the results of all the cases discussed above, we have that for a two-armed feasible instance $\nu$ with arm 1 being the optimal arm and arm 2 being infeasible sub-optimal,
\[
\lim_{T \to \infty} \sup_{\nu} - \frac{1}{T} \log e_{\nu}(T) \leq \min\left\{a_2 (\tau - \mu_2(1))^2, a_1 (\mu_1(2) - \mu_1(1))^2\right\}
\]
\[+ \max\{a_2 (\mu_2(2) - \tau)^2, a_1 (\mu_1(1) - \mu_1(2))^2\}.
\]
Case 4: \( \nu \) is infeasible

We first evaluate the infimum over the two cases: \( J(\nu') \neq J(\nu) \), \( F(\nu') = F(\nu) \), and \( F(\nu') \neq F(\nu) \); and then find the minimum of these two cases. In the former case, we have that \( J(\nu') \neq J(\nu) \), \( F(\nu') = F(\nu) \), i.e., both \( \nu \) and \( \nu' \) are infeasible but their optimal arms are different, while in the latter case, we have that \( \nu' \) is feasible. We first consider the former case. WLOG, we assume that \( J(\nu) = 1 \) and \( J(\nu') = 2 \).

As there are no restrictions on \( \mu_1'(1) \) and \( \mu_2'(2) \), we set \( \mu_1'(1) = \mu_1(1) \) and \( \mu_2'(2) = \mu_2(2) \). It follows that:

\[
\inf_{\nu' \in M; J(\nu') = 2} \max \left\{ a_1 (\mu_1(1) - \mu_1'(1))^2 + a_2 (\mu_2(1) - \mu_2'(1))^2, a_1 (\mu_1(2) - \mu_1'(2))^2 + a_2 (\mu_2(2) - \mu_2'(2))^2 \right\}
\]

\[
= \inf_{\nu' \in M; \tau < \mu_2(1)} a_2 (\mu_2(2) - \mu_2'(2))^2
\]

\[
= \frac{a_2 (\mu_2(2) - \mu_2'(2))^2}{4}
\]

where the infimum is attained midway between \( \mu_2(1) \) and \( \mu_2(2) \).

Next, we consider the case where \( F(\nu') \neq F(\nu) \), i.e., \( \nu' \) is a feasible instance. As the only restriction is that at least one arm of \( \nu' \) is feasible, we set \( \mu_2' = \mu_2 \) and make arm 1 feasible. Thus, we have that

\[
\inf_{\nu' \in M; \tau < \mu_2(1)} a_2 (\mu_2(1) - \mu_2'(1))^2 + a_2 (\mu_2(2) - \mu_2'(2))^2
\]

\[
= a_2 (\mu_2(1) - \tau)^2.
\]

Thus, combining the results of the two cases discussed above, we have that for a two-armed infeasible instance \( \nu \) with optimal arm being arm 1,

\[
\limsup_{T \to \infty} - \frac{1}{T} \log e_T(\nu) \leq \min \left\{ \frac{a_1 (\mu_2(2) - \mu_2'(1))^2}{4}, a_2 (\mu_2(2) - \tau)^2 \right\}.
\]

\[\square\]

B Complete proof of Theorem 2

In this section, we complete the proof for Theorem 2 by proving Lemmas 3 and 4 for feasible and infeasible instances separately.

Underlying instance is feasible

Proof of Lemma 2 Here, the feasible instance consists of two arms and each arm has been drawn \( n_1 \) times. Let \( B_k \) denote the event that arm 1 is empirically feasible at the end of round \( k \). Thus,

\[
P(A_1) = P(A_1 \cap B_1) + P(A_1 \cap B_2^c)
\]

\[
\leq P(A_1 \cap B_1) + P(\mu_2'(2) > \tau).
\]

Using (1) to bound the last term from above, we get:

\[
P(A_1) \leq P(A_1 \cap B_1) + 2 \exp \left( -a_2 n_1 (\tau - \mu_2(1))^2 \right)
\]

\[
\leq P(A_1 \cap B_1) + 2 \exp \left( -n_1 \Delta(1, 2)^2 \right) \quad (11)
\]

The event \( \{A_1 \cap B_1\} \) corresponds to the set of outcomes where arm 1 is empirically feasible at the end of round 1 and is still rejected. This would require that arm 2 be empirically feasible and also be the empirically optimal arm. Thus, we get

\[
P(A_1 \cap B_1) \leq P(\mu_2(2) \leq \tau, \mu_1'(1) > \mu_1'(2)) \quad (12).
\]

This can be bounded using (1) depending upon the nature of arm 2.

Case 1: Arm 2 is a feasible suboptimal arm. Using (12),

\[
P(A_1 \cap B_1) \leq \mu_1'(1) > \mu_2'(2)
\]

\[
= P\left( \mu_1'(1) - \mu_1(1) > \mu_1'(2) - \mu_1(2) \right)
\]

\[
> (\mu_1(2) - \mu_1(1))
\]

\[
\leq P\left( \mu_1'(1) - \mu_1(1) > \frac{\mu_1(2) - \mu_1(1)}{2} \right)
\]

\[
+ P\left( \mu_1'(1) - \mu_1(1) < - \frac{\mu_1(2) - \mu_1(1)}{2} \right),
\]

where the last step follows from the fact that both \( (\mu_1'(1) - \mu_1(1)) \) and \( (\mu_1(2) - \mu_1'(2)) \) cannot be greater than \( (\mu_1(2) - \mu_1(1)) / 2 \) and a subsequent union bounding argument. Thus, using (1) and (13),

\[
P(A_1 \cap B_1) \leq 4 \exp\left( - \frac{n_1}{4} A_1(2) - \mu_1(1))^2 \right)
\]

\[
\leq 4 \exp\left( - \frac{n_1}{4} \Delta(1, 2)^2 \right). \quad (13)
\]

Case 2: Arm 2 is a deceiver arm. Similarly, using (12),

\[
P(A_1 \cap B_1) \leq P(\mu_2'(2) \leq \tau)
\]

\[
\leq 2 \exp \left( -n_1 o_2 (\mu_2(2) - \tau)^2 \right)
\]

\[
\leq 2 \exp \left( -n_1 \Delta(1, 2)^2 \right). \quad (11)
\]

Case 3: Arm 2 is an infeasible suboptimal arm. This case follows from Case 1 if \( \delta(1, 2) \) is dictated by the suboptimality gap of arm 2, and from Case 2 if \( \delta(1, 2) \) is dictated by the infeasibility gap of arm 2.

The statement of the lemma now follows, combining these three cases with (11). □

Proof of Lemma 2 The feasible instance consists of three arms, each of which has been drawn \( n_1 \) times. Let \( B_k \) denote the event that arm 1 is empirically feasible at the end of round \( k \). Proceeding similarly as in (11),

\[
P\left( \{\hat{J}(A_k) = 2\} \cap A_1 \right)
\]
The term $P(G)$ can be bounded depending upon the nature of arm 3.

**Case 1: Arm 3 is a feasible suboptimal arm.**

Note that $P(G)$ is the probability of arm 1 being rejected at the end of round 1, arm 2 being empirically optimal, and arm 1 looking empirically feasible. Event $G$ thus implies that arm 2 is also empirically feasible, and moreover, has a lower value of the objective attribute $\mu_1(\cdot)$ than arm 1. We now further decompose $P(G)$ as follows:

$$P(G) = P\left(G \cap \{\hat{\Delta}(2, 1) = \hat{\Delta}(2, 3)\}\right) + P\left(G \cap \{\hat{\Delta}(2, 1) > \hat{\Delta}(2, 3)\}\right) =: P(G_1) + P(G_2)$$

**Bounding $P(G_1)$:** The event $G_1$ implies that arm 1 is rejected based on our tie breaking rule. This can only occur if arm 3 appears empirically feasible, and moreover, appears superior to arm 1 on the objective attribute $\mu_1(\cdot)$. Thus,

$$P(G_1) \leq P\left(\hat{\mu}_1(1) \geq \hat{\mu}_1(3)\right) \leq 4 \exp\left(-\frac{n_1}{4}\delta(1, 3)^2\right),$$

where the last step follows from (13).

**Bounding $P(G_2)$:** The event $G_2$ implies that arm 1 is rejected based on its estimated suboptimality gap alone. In this case, we must have

$$\hat{\delta}(2, 3) = \hat{\Delta}(2, 3) < \hat{\Delta}(2, 1) \leq \hat{\delta}(2, 1) = \sqrt{a_1}\left(\mu_1(1) - \mu_1(2)\right).$$

Thus, $P(G_2)$ is upper bounded by

$$P\left(\sqrt{a_1}\left(\mu_1(1) - \mu_1(2)\right) > \hat{\delta}(2, 3)\right) \cap B_1 \cap \{\hat{J}(A_k) = 2\}.\tag{14}$$

Note that for $\sqrt{a_1}(\mu_1(1) - \mu_1(2)) \geq \hat{\delta}(2, 3)$ to happen when arm 2 is empirically optimal and arm 1 is empirically feasible, it cannot be that arm 3 has a higher $\mu_1(\cdot)$ than arm 1 (i.e., arm 3 appears inferior on the objective attribute), regardless of whether arm 3 is empirically feasible or infeasible. Thus, we have that:

$$P(G_2) \leq P\left(\hat{\mu}_1(3) \leq \hat{\mu}_1(1)\right) \leq 4 \exp\left(-\frac{n_1}{4}\delta(1, 3)^2\right).$$

Thus, for a feasible instance where arm 3 is suboptimal, the probability of arm 1 being rejected at the end of the first round when arm 2 is empirically optimal can be bounded as follows, combining (14) with our bounds on $P(G_1)$ and $P(G_2)$:

$$P\left(\{\hat{J}(A_k) = 2\} \cap A_1\right) \leq 10 \exp\left(-\frac{n_1}{4}\Delta(1, 3)^2\right).\tag{15}$$

**Case 2: Arm 3 is a deceiver arm.**

Next, we consider the case where arm 3 is an infeasible arm. We bound $P(G)$ in the following way:

$$P(G) = P\left(G \cap \{\hat{\mu}_1(3) \leq \tau\}\right) + P\left(G \cap \{\hat{\mu}_1(3) > \tau\}\right) \leq P\left(\hat{\mu}_1(3) \leq \tau\right) + P\left(G \cap \{\hat{\mu}_1(3) > \tau\}\right) \leq 2 \exp(-n_1\Delta(1, 3)^2) + P\left(G \cap \{\hat{\mu}_1(3) > \tau\}\right)$$

$$= 2 \exp(-n_1\Delta(1, 3)^2) + P\left(G \cap \{\hat{\mu}_1(3) > \tau\}\right) =: 2 \exp(-n_1\Delta(1, 3)^2) + P\left(G_1\right) + P\left(G_2\right)$$

We now take two cases for the nature of arm 2.

**Case 2a: Arm 2 is suboptimal.** In this case, we use

$$\sqrt{a_1}\left(\mu_1(1) - \mu_1(2)\right) \geq \hat{\Delta}(2, 1).$$

Combining this bound with (17),

$$P(G_2) \leq P\left(\sqrt{a_1}\left(\mu_1(1) - \mu_1(2)\right) \geq \sqrt{a_2}\left(\mu_1(3) - \tau\right)\right) = P\left(\left\{\sqrt{a_1}\left(\mu_1(1) - \mu_1(2) + \mu_1(2)\right) \geq \sqrt{a_2}\left(\mu_1(3) - \mu_2(3)\right) + \delta(1, 3) + \delta(1, 2)\right\} \right) \leq 6 \exp\left(\frac{n_1}{9}\left(\delta(1, 3) + \delta(1, 2)\right)^2\right) \leq 6 \exp\left(\frac{n_1}{9}\delta(1, 3)^2\right).$$

**Case 2b: Arm 2 is a deceiver.** When arm 2 is a deceiver arm, we use

$$\sqrt{a_2}(\tau - \mu_2(2)) \geq \hat{\Delta}(2, 1).$$

Combining this bound with (17),

$$P(G_2) \leq P\left(\sqrt{a_2}(\tau - \mu_2(2)) \geq \sqrt{a_2}\left(\mu_1(3) - \tau\right)\right) = P\left(\left\{(\mu_2(3) - \mu_1(3)) - (\mu_2(2) - \mu_2(2)) \geq (\mu_2(2) - \tau) + (\mu_2(3) - \tau)\right\}\right) \leq 4 \exp\left(-\frac{n_1}{4}\delta(1, 3)^2\right) \leq 4 \exp\left(-\frac{n_1}{4}\delta(1, 3)^2\right).$$

Thus, combining the results of the two cases with (14) and (16) gives us the following bound on the probability of arm 1 being rejected at the end of the first round when arm 2 is empirically optimal:

$$P\left(\{\hat{J}(A_k) = 2\} \cap A_1\right) \leq 10 \exp\left(-\frac{n_1}{9}\Delta(1, 3)^2\right).$$

**Case 3: Arm 3 is an infeasible suboptimal arm.**

This case follows from Case 1 if $\delta(1, 3)$ is dictated by the the suboptimality gap of arm 3, and from Case 2 if $\delta(1, 3)$ is dictated by the the infeasibility gap of arm 3.
Underlying instance is infeasible

Proof of Lemma 3[5] The infeasible instance consists of two arms, each of which has been drawn $n_1$ times. Note that arm 1 is rejected either if arm 1 appears empirically feasible, or if arm 2 appears feasible relative to arm 1 on the constraint attribute. Thus,

$$
P(A_1) \leq P(\hat{\mu}_2^1(1) \leq \tau) + P(\hat{\mu}_2^1(2) \leq \hat{\mu}_2^1(1)).$$

Each of the two terms above can be bounded from above using (11), as demonstrated before, to yield the statement of the lemma.

Proof of Lemma 4[4] The infeasible instance consists of three arms, each of which has been drawn $n_1$ times. Let $B_k$ denote the event that arm 1 is empirically feasible at the end of round $k$. Proceeding similarly as in (11),

$$
P\left(\{\hat{J}(A_k) = 2\} \cap A_1\right)$$

$$\leq P\left(A_1 \cap B_k^c \cap \{\hat{J}(A_k) = 2\} \cap \{\hat{\mu}_2(2), \hat{\mu}_2(3) > \tau\}\right)$$

$$+ 6 \exp\left(-n_1 \Delta(1, 3)^2\right), \quad (18)$$

the only difference being that here, we bound the probability of arm 1, 2 or 3 being empirically feasible using (11). We also use the fact that $\mu_2(1) < \mu_2(2) \leq \mu_2(3)$. The first term in (18) is the probability of arm 1 being rejected at the end of round 1 when all three arms are empirically infeasible and arm 2 is empirically optimal. It thus implies $\hat{\mu}_2^1(1) > \hat{\mu}_2^3(3)$, yielding

$$
P\left(A_1 \cap B_k^c \cap \{\hat{J}(A_k) = 2\} \cap \{\hat{\mu}_2(2), \hat{\mu}_2(3) > \tau\}\right)$$

$$\leq P(\hat{\mu}_2^1(1) > \hat{\mu}_2^3(3))$$

$$\leq 4 \exp\left(-\frac{n_1}{4}(\mu_2(3) - \mu_2(1))^2\right)$$

$$= 4 \exp\left(-\frac{n_1}{4} \Delta(1, 3)^2\right).$$

Combining the above result and (18), we get that

$$
P\left(\{\hat{J}(A_k) = 2\} \cap A_1\right) \leq 10 \exp\left(-\frac{n_1}{4} \Delta(1, 3)^2\right).$$

C The Infeasible First algorithm

Informally, the algorithm removes the most (empirically) infeasible arm that has survived so far. If there are no infeasible arms, it removes the most (empirically) suboptimal arm. The formal version of the algorithm is given in Algorithm 2.

Algorithm 2: Infeasible First algorithm

1: procedure IF($T, K, \tau$)
2: Let $A_1 = \{1, \ldots, K\}$
3: $\log(K) := \frac{1}{2} + \sum_{k=2}^{K-1} \frac{1}{k}$
4: $n_0 = 0, n_k = \left\lceil \frac{2}{\log(K) \cdot K - 1} \right\rceil$ for $1 \leq k \leq K - 1$
5: for $k = 1, \ldots, K - 1$ do
6: For each $i \in A_k$, pull arm $i$ ($n_k - n_{k-1}$) times
7: Compute $\hat{K}(A_k) = \{i \in A_k : \hat{\mu}_2^k(i) > \tau\}$
8: Compute $K^c(A_k) = A_k \setminus \hat{K}(A_k)$
9: if $\hat{K}(A_k) \neq \emptyset$ then
10: $A_{k+1} = A_k \setminus \{\arg\max_{i \in K^c(A_k)} \hat{\mu}_1^k(i)\}$ (if there is a tie, choose randomly)
11: else
12: $A_{k+1} = A_k \setminus \{\arg\max_{i \in \hat{K}(A_k)} \hat{\mu}_1^k(i)\}$ (if there is a tie, choose randomly)
13: end if
14: end for
15: Let $J_T$ be the unique element of $A_K$
16: if $\hat{\mu}_2^{K-1}(J_T) > \tau$ then
17: $\tilde{O}(\log K) = 0$
18: else
19: $\tilde{O}(\log K) = J_T$
20: end if
21: return $\tilde{O}(\log K)$
22: end procedure