Approximation of functions by linear summation methods in the Orlicz-type spaces

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Abstract. Approximative properties of linear summation methods of Fourier series are considered in the Orlicz-type spaces $S_M$. In particular, in terms of approximations by such methods, constructive characteristics are obtained for the classes of functions whose moduli of smoothness do not exceed a certain majorant.

Keywords. Linear summation method, modulus of smoothness, direct approximation theorem, inverse approximation theorem, Orlicz-type spaces.

1. Introduction

Linear methods (or processes) of summation of Fourier series are an important object of research in approximation theory. In particular, this is due to the fact that most of these methods naturally generate the corresponding aggregate of approximation. These topics are well studied in classical functional spaces such as the Lebesgue and Hilbert ones, the spaces of continuous functions, etc. However, there are relatively a few papers devoted to similar topics in Banach spaces of the Orlicz type. Particularly, this concerns the direct and inverse theorems of approximation by linear summation methods.

In the present paper, the approximative properties of linear summation methods of Fourier series are studied in the Orlicz-type spaces $S_M$. The spaces $S_M$ are defined in the following way. The Orlicz function $M(t)$, $t \geq 0$ is a non-decreasing convex function and is such that $M(0) = 0$ and $M(t) \to \infty$ as $t \to \infty$. Let $S_M$ be the space of all $2\pi$-periodic Lebesgue summable functions $f$ ($f \in L_1$) such that the following quantity (which is also called the Luxemburg norm of $f$) is finite:

$$\|f\|_M := \|\{\hat{f}(k)\}_{k \in \mathbb{Z}}\|_{L_M(\mathbb{Z})} = \inf \left\{a > 0 : \sum_{k \in \mathbb{Z}} M(|\hat{f}(k)|/a) \leq 1 \right\},$$

(1.1)

where $\hat{f}(k) := [f] \sim(k) = (2\pi)^{-1} \int_0^{2\pi} f(t)e^{-ikt}dt$, $k \in \mathbb{Z}$, are the Fourier coefficients of $f$. Functions $f \in L_1$ and $g \in L_1$ are equivalent in the space $S_M$ when $\|f - g\|_M = 0$.

The spaces $S_M$ defined in this way are Banach spaces. They were considered in [6]. In particular, the direct and inverse approximation theorems in terms of the best approximations of functions and moduli of fractional smoothness were proved for the spaces $S_M$ in [6].

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In the case $M(t) = t^p$, $p \geq 1$, the spaces $S_M$ coincide with the well-known spaces $S^p$ [18] of functions $f \in L_1$ with finite norm

$$\|f\|_{S^p} = \|\{\hat{f}(k)\}_{k \in \mathbb{Z}}\|_{l_p(\mathbb{Z})} = \left(\sum_{k \in \mathbb{Z}} |\hat{f}(k)|^p\right)^{1/p}. $$

In $S^p$, the approximative properties of linear summation methods of Fourier series were studied in [16, 17]. The purpose of this paper is to continue this study of the approximative properties of linear summation methods in the spaces $S_M$. In this case, our attention is drawn to the connection of the approximative properties of these methods with the differential properties of the functions, namely, the direct and inverse theorems of approximation by the methods of Zygmund, Abel–Poisson, Taylor–Abel–Poisson are proved, and, in terms of approximations by such methods, the constructive characteristics are given for the classes of functions of $S_M$ such that the moduli of smoothness of their generalized derivatives do not exceed a certain majorant.

2. Preliminaries

For any function $f \in L_1$ with the Fourier series

$$S[f](x) := \sum_{k \in \mathbb{Z}} \hat{f}(k)e^{ikx},$$

let us consider the linear transformations $S_n$, $Z_n^{(s)}$, $P_{\varrho,s}$, and $A_{\varrho,r}$:

$$S_n(f)(x) := \sum_{k=-n}^{n} \hat{f}(k)e^{ikx}, \quad n = 0, 1, \ldots,$$

$$Z_n^{(s)}(f)(x) := \sum_{k=-n}^{n} \left(1 - \left(\frac{|k|}{n+1}\right)^s\right) \hat{f}(k)e^{ikx}, \quad s > 0,$$

$$P_{\varrho,s}(f)(x) := \sum_{k \in \mathbb{Z}} \varrho^{|k|s} \hat{f}(k)e^{ikx}, \quad s > 0, \quad \varrho \in [0,1),$$

and

$$A_{\varrho,r}(f)(x) := \sum_{k \in \mathbb{Z}} \lambda_{|k|,r}(\varrho) \hat{f}(k)e^{ikx}, \quad (2.1)$$

where, for $k = 0, 1, \ldots, r - 1$, the numbers $\lambda_{k,r}(\varrho) \equiv 1,$ and

$$\lambda_{k,r}(\varrho) := \sum_{j=0}^{r-1} \binom{k}{j} (1 - \varrho)^{j} \varrho^{k-j}, \quad k = r, r+1, \ldots, \quad \varrho \in [0,1]. \quad (2.2)$$

The expressions $S_n(f)$, $Z_n^{(s)}(f)$, and $P_{\varrho,s}(f)$ are called the partial sum of a Fourier series, Zygmund sum, and generalized Abel–Poisson sum of a function $f$, respectively. The expression $A_{\varrho,r}(f)$ is called the Taylor–Abel–Poisson sum of the function $f$. If $s = 1$, then the sum $Z_n^{(1)}(f)$ coincides with the Fejér sum of a function $f$, i.e.,

$$Z_n^{(1)}(f)(x) = \sigma_n(f)(x) := \frac{1}{n+1} \sum_{k=0}^{n} S_k(f)(x) = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) \hat{f}(k)e^{ikx}. $$
Note that the transformation $A_{\varrho,r}$ can be considered as a linear operator on $L_1$ into itself. Indeed, for $k = 0, 1, \ldots, r - 1$, the numbers $\lambda_{k,r}(\varrho) \equiv 1$, and
\[
\sum_{j=0}^{r-1} \binom{k}{j} (1 - \varrho)^j \varrho^{k-j} \leq r^q k^{r-1}, \text{ where } q = \max\{1 - \varrho, q\}.
\]

Hence, for any $f \in L_1$ and $0 < \varrho < 1$, the series on the right-hand side of (2.1) is majorized by the convergent series $2r\|f\|_{L_1} \sum_{k=r}^{\infty} q^k r^{r-1}$.

Denote by $P(f)(\varrho, x)$, $0 \leq \varrho < 1$, the Poisson integral (the Poisson operator) of $f$, i.e.,
\[
P(f)(\varrho, x) := \frac{1}{2\pi} \int_0^{2\pi} f(\varrho, x - t) dt,
\]
where $P(\varrho, t) = \frac{1 - \varrho^2}{1 - \varrho^2}$ is the Poisson kernel.

According to the decomposition of the Poisson kernel in powers of $\varrho$, for any function $f \in L_1$, its Poisson integral $P(f)(\varrho, x)$, with $\varrho \in [0, 1)$ and $x \in [0; 2\pi]$ can be written in the form
\[
P(f)(\varrho, x) = \sum_{k \in \mathbb{Z}} \varrho^{|k|} f_k e^{ikx}.
\]

The sum of the right-hand side of this equality coincides with the sum of the Abel–Poisson of the series $\sum_{k \in \mathbb{Z}} \hat{f}(k) e^{ikx}$, or, what is the same, with the sum of $P_{\varrho,1}(f)(x)$. For $x = 0$, we denote, by $F(\varrho)$, the sum of this series and consider it as a function of the variable $\varrho$. It is clear that the function $F$ is analytic on $[0, 1)$. Therefore, in the neighborhood of $\varrho \in [0, 1)$, the following Taylor formula is satisfied for the functions $F$:
\[
F(t) = \sum_{k=0}^{\infty} \frac{F(k)(\varrho)}{k!} (t - \varrho)^k.
\]

By a direct computation, we see that the partial sum of this series of order $r - 1$ for $t = 1$ coincides with the sum $A_{\varrho,r}(f)(0)$. In particular, for $r = 1$, we obtain $F(\varrho) = A_{\varrho,1}(f)(0) = P_{\varrho,1}(f)(0)$.

Consequently, on the one hand, the sum of $A_{\varrho,r}(f)(0)$ can be interpreted as the Taylor sum of order $r - 1$ of the function $F$. On the other hand, for $r = 1$, it can be interpreted as the Abel–Poisson sum.

The operators $A_{\varrho,r}$ were first studied in [15], where the author gave a structural characteristic of the Hardy–Lipschitz classes $H_p^{\alpha}$ for one-variable functions, holomorphic in a unit disk on the complex plane in the terms of these operators. Approximative properties of these operators were also considered in [13, 16]. In the general case, the operators $P_{\varrho,s}$ were, perhaps, first considered as the aggregates of approximation of functions of one variable in [3, 4]. In special cases where $r = s = 1$, the operators $A_{\varrho,1}$ and $P_{\varrho,1}$ coincide with each other and generate the Abel–Poisson summation method of Fourier series. The problem of approximation of $2\pi$-periodic functions by Abel–Poisson sums has a long history, full of many results. Here, we mention only books [1, 5, 20] which contain fundamental results on this subject.

3. Derivatives and moduli of smoothness

Let $\psi = \{\psi(k)\}_{k \in \mathbb{Z}}$ be a numerical sequence whose members are not all zero and
\[
\mathcal{Z}(\psi) := \{k \in \mathbb{Z} : \psi(k) = 0\}.
\]
In what follows, assume that the number of elements of the set $\mathcal{Z}(\psi)$ is finite.
If, for the function \( f \in L_1 \), there exists a function \( g \in L_1 \) with the Fourier series of the form

\[
S[g](x) = \sum_{k \in \mathbb{Z}} \tilde{f}(k)e^{ikx}/\psi(k),
\]

then we say that, for the function \( f \), there exists a \( \psi \)-derivative \( g \), for which we use the notation \( g = f^\psi \).

This definition of \( \psi \)-derivative is adapted to the needs of the research described in this paper, and it is not fundamentally different from the established concept of \( \psi \)-derivative by A. I. Stepanets [19, Ch. XI].

Here, we consider \( \psi \)-derivatives defined by the sequences of two following forms: 1) \( \psi(k) = |k|^{-s} \), \( k \in \mathbb{Z}, \ s > 0 \), and 2) \( \psi(k) = 0 \) for \( |k| < r - 1 \) and \( \psi(k) = (|k| - r)/(|k|!) \) for \( |k| \geq r \), where \( r \in \mathbb{N} \). In the first case, for the \( \psi \)-derivative of \( f \), we use the notation \( f^{(1)} \). In the second case, we use the notation \( f^{[r]} \). If \( r = 0 \), then we set \( f^{(0)} = f^{[0]} = f \). Note that \( f^{(1)} = f^{[1]} \).

In the terms of Poisson integrals, we give the following interpretation of the derivative \( f^{[r]} \). Assume that \( g \in [0, 1) \), then

\[
P(f^{[r]})(g, x) = g^r \frac{\partial^r}{\partial g^r} P(f)(g, x).
\]

By virtue of the well-known theorem on radial limit values of the Poisson integral (see, e.g., [14]), we have

\[
f^{[r]}(x) = \lim_{g \to 1-} \frac{\partial^r}{\partial g^r} P(f)(g, x)
\]

for almost all \( x \in [0; 2\pi] \). The modulus of smoothness of \( f \in S_M \) of the index \( \alpha > 0 \) is defined by

\[
\omega_\alpha(f, \delta)_M := \sup_{|h| \leq \delta} \| \Delta^\alpha f \|_M = \sup_{|h| \leq \delta} \left\| \sum_{j=0}^{\infty} (-1)^j \left( \frac{\alpha}{j} \right) f(x - jh) \right\|_M,
\]

where \( \delta > 0, \ (\alpha)_0 := 1, \ \left( \frac{\alpha}{j} \right) = \alpha(\alpha - 1) \ldots (\alpha - j + 1)/j!, \ j \in \mathbb{N} \).

Let \( \omega \) be a function defined on the interval \([0, 1]\). For \( \alpha > 0 \), we set

\[
S_M H^\alpha := \left\{ f \in S_M : \ \omega_\alpha(f, \delta)_M = O(\omega(\delta)), \ \delta \to 0+ \right\}.
\]

Further, we consider the functions \( \omega(t), \ 0 \leq t \leq 1, \) satisfying the following conditions 1)-4): 1) \( \omega(t) \) is continuous on \([0, 1]\); 2) \( \omega(t) \) is monotonically increasing; 3) \( \omega(t) \neq 0 \) for \( t \in (0, 1] \); 4) \( \omega(t) \to 0 \) as \( t \to 0 \); and the well-known Zygmund–Bari–Stečkin conditions \( (\mathcal{B}) \) and \( (\mathcal{B}_s) \), \( s \in \mathbb{N} \) (see, e.g., [2]):

\[
(\mathcal{B}) : \ \sum_{v=n+1}^{\infty} v^{-1} \omega(v^{-1}) = O[\omega(n^{-1})], \ \ n \to \infty;
\]

\[
(\mathcal{B}_s) : \ \sum_{v=1}^{n} v^{s-1} \omega(v^{-1}) = O[n^{s}\omega(n^{-1})], \ \ n \to \infty.
\]

**Remark 3.1.** It follows from condition \( (\mathcal{B}_s) \) that \( \lim_{\delta \to 0+} \inf(\delta^{-s}\omega(\delta)) > 0 \), or, for any \( r \geq s \), the quantity \( (1 - \rho)^{r-s}\omega(1 - \rho) \gg (1 - \rho)^{r} \) as \( \rho \to 1- \).
4. The main results

Proposition 4.1. Assume that \( f \in L_1, s > 0, \) and \( \omega \) is a function satisfying conditions 1)–4) and \((B)\). The following statements are equivalent:

1) \( \|S_n(f^{(s)})\|_M = O(n^s\omega(n^{-1})),\ n \to \infty; \)
2) \( \|f - Z_n^{(s)}(f)\|_M = O(\omega(n^{-1})),\ n \to \infty; \)
3) \( f \in S_MH_s^\omega. \)

We note that, in the case where \( s \in \mathbb{N} \) and the function \( \omega \) satisfies conditions 1)–4), \((B)\), and \((B)_s\), relation 1) of Proposition 4.1 is equivalent to the corresponding relation for the derivative \( f^{[s]} \):

\[
\|S_n(f^{[s]})\|_M = O(n^s\omega(n^{-1})),\ n \to \infty. \tag{4.1}
\]

Indeed, by definition, we have \( 0 = |\tilde{f}^{[s]}(k)| \leq |\tilde{f}^{(s)}(k)| \) for \( |k| < s \) and, for \( |k| \geq s \),

\[
|f^{[s]}(k)| = |k|(|k| - 1)\ldots(|k| - s + 1)f(k) \leq |k|^s|\tilde{f}(k)| = |\tilde{f}^{(s)}(k)|.
\]

Therefore, if statement 1) of Proposition 4.1 holds, then

\[
\|S_n(f^{[s]})\|_M \leq \|S_n(f^{(s)})\|_M = O(n^s\omega(n^{-1})),\ n \to \infty.
\]

On the other hand, for \( |k| \geq s \), we have

\[
|\tilde{f}^{[s]}(k)| = |k|^s \cdot (1 - \frac{1}{|k|}) \ldots (1 - \frac{s - 1}{|k|})|\tilde{f}(k)| \geq \frac{|k|^s}{|s|^s}|\tilde{f}(k)| = s^{-s}|\tilde{f}^{(s)}(k)|.
\]

Therefore, taking Remark 3.1 into account, we see that relation (4.1) yields statement 1):

\[
\|S_n(f^{[s]})\|_M \leq \|S_{s-1}(f^{[s]})\|_M + \left\| \sum_{s \leq |k| \leq n} |k|^s \tilde{f}(k)e^{ikx} \right\|_M
\]
\[
\leq \|S_{s-1}(f^{[s]})\|_M + s^s\|S_n(f^{[s]})\|_M = O(n^s\omega(n^{-1})),\ n \to \infty.
\]

Hence, the following assertion is valid:

Proposition 4.2. Assume that \( f \in L_1, s \in \mathbb{N}, \) and \( \omega \) is the function, satisfying conditions 1)–4), \((B)\), and \((B)_s\). The following statements are equivalent:

1) \( \|S_n(f^{[s]})\|_M = O(n^s\omega(n^{-1})),\ n \to \infty, \) where \( f^{[s]} \) is one of the derivatives \( f^{[s]} \) or \( f^{(s)}; \)
2) \( \|f - Z_n^{(s)}(f)\|_M = O(\omega(n^{-1})),\ n \to \infty; \)
3) \( f \in S_MH_s^\omega. \)

In the case when \( s = 1 \), we have \( f^{(1)} = f^{[1]} \) and \( Z_n^{(1)}(f) = \sigma_n(f). \)

Corollary 4.1. Assume that \( f \in L_1 \) and \( \omega \) is a function satisfying conditions 1)–4) and \((B)\). The following statements are equivalent:

1) \( \|S_n(f^{[1]})\|_M = O(n\omega(n^{-1})),\ n \to \infty; \)
2) \( \|f - \sigma_n(f)\|_M = O(\omega(n^{-1})),\ n \to \infty; \)
3) \( f \in S_MH_1^\omega. \)
The proof of these and other assertions will be given in Section 6. Let us give some comments. First, let us note that in the proposed assertions, the equivalence 2) ⇔ 3) is the statement of the type direct and inverse theorem for Zygmund and Fejér method [5].

In the papers [9–12], Móricz investigated properties of 2π-periodic functions represented by Fourier series, which convergent absolutely. In particular, in [9] and [12], the author found the conditions under which such functions satisfy the Lipshitz and Zygmund condition respectively.

Theorem 4.1. Assume that \( f \in L_1, s, r \in \mathbb{N}, s \leq r \) and \( \omega \) is a function satisfying conditions 1)–4), (\( B \)) and (\( B_s \)). The following statements are equivalent:

1) \( \| f - A_{\varrho,r}(f) \|_M = \mathcal{O}((1 - \varrho)^{-s} \omega(1 - \varrho)), \varrho \to 1- \);
2) \( \| P(f^{(r)})(\varrho, \cdot) \|_M = \mathcal{O}((1 - \varrho)^{-s} \omega(1 - \varrho)), \varrho \to 1- \);
3) \( \| S_n(f^{(r)}) \|_M = \mathcal{O}(n^s \omega(n^{-1})), \quad n \to \infty; \)
4) \( f^{(r-s)} \in S_M H^s_\omega \).

We note that the implication 2) ⇒ 3) is the statement of the Hardy–Littlewood-type theorems [8].

Remark 4.1. In Remark 3.1, it is noted that condition (\( B_s \)) implies that \( (1 - \varrho)^{-s} \omega(1 - \varrho) \gg (1 - \varrho)^r \) as \( \varrho \to 1- \). Therefore, if condition (\( B_s \)) is satisfied, then the quantity on the right-hand side of the relation in statement 1) decreases to zero as \( \varrho \to 1- \) not faster, than the function \((1 - \varrho)^r\). Note that the relation \( \| f - A_{\varrho,r}(f) \|_M = \mathcal{O}((1 - \varrho)^r)) \), \( \varrho \to 1- \), holds only in the trivial case where \( f(x) = \sum_{|k| \leq r-1} f_k e^{ikx} \). In such case, the theorems are easily proved. This fact is related to the so-called saturation property of the approximation method, generated by the operator \( A_{\varrho,r} \). In particular, in [15], it was shown that the operator \( A_{\varrho,r} \) generates the linear approximation method for holomorphic functions, which is saturated in the Hardy space \( H_p \) with the saturation order \((1 - \varrho)^r\) and the saturation class \( H_p^{r-1} \) Lip 1.

Consider approximative properties of the sums \( P_{\varrho,s}(f) \) in the space \( S_M \).

Let us prove that, for any function \( f \in S_M \) such that the derivative \( f^{(s)} \in S_M \), the following relation holds as \( \varrho \to 1- \):

\[
\| f - P_{\varrho,s}(f) \|_M \sim f^{(s-1)}(f^{(s-1)}) \|_M \sim (1 - \varrho) \| f^{(s)} \|_M. \tag{4.2}
\]

Show that

\[
\| f - P_{\varrho,s}(f) \|_M \sim (1 - \varrho) \| f^{(s)} \|_M, \quad \varrho \to 1- \tag{4.3}
\]

The second relation in (4.2) is proved similarly.

For any \( n \in \mathbb{N} \), we have \( 1 - \varrho^n = (1 - \varrho)(1 + \varrho + \ldots + \varrho^{n-1}) \). Then, setting \( b_1 := (1 - \varrho)\| f^{(s)} \|_M \), we get, for all \( \varrho \in (0,1) \),

\[
\sum_{k \in \mathbb{Z}} M \left( (1 - \varrho^{|k|}) \| \hat{f}(k) \| b_1 \right) \leq \sum_{k \in \mathbb{Z}} M \left( (1 - \varrho^{|k|}) \| \hat{f}(k) \| b_1 \right) \leq 1.
\]

Therefore, \( \| f - P_{\varrho,s}(f) \|_M \leq (1 - \varrho) \| f^{(s)} \|_M \).
On the other hand side, since \( f^{(s)} \in S_M \), then there exists a number \( N \in \mathbb{N} \) for any \( \varepsilon > 0 \) such that, for all \( n \geq N \)
\[
\| S_n(f^{(s)}) \|_M \geq \| f^{(s)} \|_M - \varepsilon/4.
\]
By the definition of the norm,
\[
\sum_{|k| \leq N} M\left( \frac{|k|^s |\hat{f}(k)|}{\| f^{(s)} \|_M - \varepsilon/2} \right) \geq \sum_{|k| \leq N} M\left( \frac{|k|^s |\hat{f}(k)|}{\| S_n(f^{(s)}) \|_M - \varepsilon/4} \right) > 1.
\]
Choosing \( \theta_0 \) such that, for all \( \theta \in (\theta_0, 1) \) and \( |k| \leq N \), the inequality
\[
\left( \| f^{(s)} \|_M - \varepsilon/2 \right) \left( 1 + \theta + \ldots + \theta^{|k|-1} \right) > |k|^s \left( \| f^{(s)} \|_M - \varepsilon \right)
\]
holds, we see that, for such \( \theta \) and \( b_2 := (1 - \theta)(\| f^{(s)} \|_M - \varepsilon) \),
\[
\sum_{k \in \mathbb{Z}} M\left( (1 - \theta^{|k|^s}) |\hat{f}(k)|/b_2 \right) \geq \sum_{|k| \leq N} M\left( (1 - \theta)(1 + \theta + \ldots + \theta^{|k|-1}) |\hat{f}(k)|/b_2 \right) > 1.
\]
Thus, for all \( \theta \in (\theta_0, 1) \), we have \( \| f - P_{\theta,1}(f) \|_M \geq (1 - \theta)(\| f^{(s)} \|_M - \varepsilon) \). Hence, relation (4.3) holds.

It is clear that
\[
P_{\theta,1}(f)(x) = A_{\theta,1}(f)(x).
\]
Therefore, applying Theorem 4.1 to the function \( f = g^{(s-1)} \) with \( r = 1 \) and taking relation (4.2) into account, we obtain the following result.

**Theorem 4.2.** Let \( f \in L_1, s \in \mathbb{N}, \) and let \( \omega \) be a function satisfying conditions 1)–4), (B), and (B_\delta). The following statements are equivalent:

1) \( \| f - P_{\theta,s}(f) \|_M = \mathcal{O}(\omega(1 - \theta)), \quad \theta \to 1^-; \)
2) \( \| P(f^{(s)})(\theta, \cdot) \|_M = \mathcal{O}(\omega(1 - \theta)), \quad \theta \to 1^-; \)
3) \( f^{(s-1)} \in S_M H_\omega^{1,1}. \)

We note that in the case where \( M(t) = t^p, p \geq 1 \), i.e., in the spaces \( S^p \), Proposition 4.1, Theorem 4.1 for \( s = 1 \), and Theorem 4.2 were proved in [16].

**5. The equivalence between moduli of smoothness and K-functional**

It is known that approximative properties of functions are well expressed by their K-functional. In [16], the authors showed the dependence of the order of approximation of a given function by the Taylor–Abel–Poisson means and the behavior of its modulus of smoothness in the spaces \( S^p \). In [13], the order of approximation of a given function by the Taylor–Abel–Poisson means and the behavior of K-functional of the function generated by its radial derivatives in the spaces \( L_p \) were analyzed. It is natural to study the relations between the modulus of smoothness and such K-functional of functions in the spaces \( S_M \).

In the space \( S_M \), the Petree K-functional of a function \( f \) (see, e.g., [7, Ch. 6]), which is generated by its radial derivative of order \( n \in \mathbb{N} \), is the following quantity:
\[
K_n(\delta, f)_M = \inf \left\{ \| f - g \|_M + \delta^n \| g^{[n]} \|_M : g^{[n]} \in S_M \right\}, \quad \delta > 0.
\]
Theorem 5.1. For any \( n \in \mathbb{N} \), there exist constants \( C_1(n) \) and \( C_2(n) \) > 0 such that for each \( f \in \mathcal{S}_M \) and all \( \delta > 0 \)

\[
C_1(n) \omega_n(f, \delta)_M \leq K_n(\delta, f)_M + \delta^n \left\| \sum_{0 \leq |k| \leq n-1} \hat{f}(k)e^{ikx} \right\|_M \leq C_2(n) \omega_n(f, \delta)_M. \tag{5.2}
\]

Remark 5.1. Let \( f \in \mathcal{S}_M \). For any \( \alpha > 0, h \in \mathbb{R} \), and \( k \in \mathbb{Z} \), we have

\[
[\Delta^n_h f](k) = \left[ \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} f(\cdot - jh) \right](k) = \hat{f}(k) \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} e^{-ijh} = (1 - e^{-ikh})^\alpha \hat{f}(k). \tag{5.3}
\]

For a fixed \( r = 0, 1, \ldots \), we denote, by \( f_r \), the function from \( \mathcal{S}_M \) such that \( \hat{f}_r(k) = 0 \) when \( |k| \leq r \), and \( \hat{f}_r(k) = \hat{f}(k) \) when \( |k| > r \). According to (5.3), we have \( \| \Delta^n_h f \|_M = \| \Delta^n_h f_0 \|_M \). Therefore,

\[
\omega_n(f, \delta)_M = \omega_n(f_0, \delta)_M. \tag{5.4}
\]

On the other hand, by virtue of (5.1) and the definition of a radial derivative, it is clear that the infimum on the right-hand side of (5.1) is attained at the set \( G_n \) of all functions \( g \in \mathcal{S}_M \) such that \( g^{[n]} \in \mathcal{S}_M \) and \( \tilde{g}(k) = \hat{f}(k) \) for \( |k| \leq n - 1 \). Hence,

\[
K_n(\delta, f)_M = K_n(\delta, f_{n-1})_M. \tag{5.5}
\]

Thus, in (5.2), we use the term \( \delta^n \left\| \sum_{0 \leq |k| \leq n-1} \hat{f}(k)e^{ikx} \right\|_M \) which takes the peculiarities of relations (5.4) and (5.5) into account.

6. Proof of the results

Proof of Proposition 4.1. Implication 1) \( \Rightarrow \) 2). For any \( n \in \mathbb{N} \), we have

\[
\left\| f - Z_n^{(s)}(f) \right\|_M \leq (n+1)^{-s} \left\| \sum_{|k| \leq n} |k|^s \hat{f}(k)e^{ikx} \right\|_M + \left\| \sum_{|k| > n} \hat{f}(k)e^{ikx} \right\|_M. \tag{6.1}
\]

Therefore, if relation 1) holds, then

\[
(n+1)^{-s} \left\| \sum_{|k| \leq n} |k|^s \hat{f}(k)e^{ikx} \right\|_M = (n+1)^{-s} \left\| \sum_{|k| \leq n} \hat{f}^{(s)}(k)e^{ikx} \right\|_M
\]

\[
= (n+1)^{-s} \| S_n(f^{(s)}) \|_M = O(\omega(n^{-1})), \quad n \to \infty. \tag{6.2}
\]

To estimate the second term in (6.1), we fix an integer \( N > n \) and apply the Abel transformation:

\[
\left\| \sum_{n < |k| \leq N} \hat{f}(k)e^{ikx} \right\|_M = \left\| \sum_{n < |k| \leq N} |k|^{-s} \hat{f}^{(s)}(k)e^{ikx} \right\|_M
\]

\[
= \left\| \sum_{j=n+1}^{N-1} \frac{1}{j^s} - \frac{1}{(j+1)^s} \right\| \sum_{|k| \leq j} \hat{f}^{(s)}(k)e^{ikx}
\]
\[ + N^{-s} \sum_{|k| \leq N} \hat{f}(s)(k)e^{ikx} - (n + 1)^{-s} \sum_{|k| \leq n} \hat{f}(s)(k)e^{ikx} \| M. \]

Then
\[ \left\| \sum_{n<|k|\leq N} \hat{f}(k)e^{ikx} \right\| M \leq s \sum_{j=n+1}^{N-1} j^{-s-1}\|S_j(f(s))\| M \]
\[ + N^{-s}\|S_N(f(s))\| M + (n + 1)^{-s}\|S_n(f(s))\| M. \]

If relation 1) holds, then there exists a number \( C_1 > 0 \) such that, for all integers \( N > n \),
\[ \left\| \sum_{n<|k|\leq N} \hat{f}(k)e^{ikx} \right\| M \leq C_1 \left( \sum_{j=n+1}^{N-1} \omega(j^{-1})/j + \omega(N^{-1}) + \omega(n^{-1}) \right) \]
\[ \leq C_1 \left( \sum_{j=n+1}^{\infty} \omega(j^{-1})/j + 2\omega(n^{-1}) \right). \]

In view of condition (\( \mathcal{B} \)), we get
\[ \left\| \sum_{|k|>n}\hat{f}(k)e^{ikx} \right\| M = \mathcal{O}(\omega(n^{-1})), \quad n \to \infty. \quad (6.3) \]

Combining relations (6.1)–(6.3), we obtain relation 2). Furthermore, since \( \omega(\delta) \to 0 \) as \( \delta \to 0+ \), relation 2) yields \( f \in \mathcal{S}_M \).

2) \( \Rightarrow \) 3). Let us set \( n := [1/\delta] - 1 \). By virtue of (5.3), for any \( |h| \leq \delta \) and \( |k| \leq n \), we have
\[ |[\Delta_h^s f]^- (k)| = |1 - e^{-ik|h|}| \hat{f}(k)| = \left| 2\sin \frac{hk}{2} \right| |\hat{f}(k)| \]
\[ \leq \delta^s |k|^s |\hat{f}(k)| \leq (n + 1)^{-s}|k|^s |\hat{f}(k)| \]
and \( |[\Delta_h^s f]^- (k)| \leq |\hat{f}(k)| \) for \( |k| > n \). Let \( a_1 := \|f - Z_n^{(s)}(f)\| M \). Then
\[ \sum_{k\in\mathbb{Z}} M([|\Delta_h^s f]^- (k)|/a_1) \leq \sum_{|k|\leq n} M((n + 1)^{-s}|k|^s |\hat{f}(k)|/a_1) \]
\[ + \sum_{|k|>n} M(|\hat{f}(k)|/a_1) \leq 1. \]

Therefore, for any \( |h| \leq \delta \),
\[ \|\Delta_h^s f\| M \leq \|f - Z_n^{(s)}(f)\| M = \mathcal{O}(\omega(n^{-1})) = \mathcal{O}(\omega(\delta)), \quad \delta \to 0+, \]
and, hence, \( f \in \mathcal{S}_M H_\omega^s \).

3) \( \Rightarrow \) 1). Setting \( h_n := \pi/n, n \in \mathbb{N} \), and \( a_2 := (n/2)^s|\Delta_h^s f| M \), by virtue of the inequality \( \theta_n \leq \pi \sin(\theta_n/2) \), which is valid for all \( t \in [0, n] \), we get
\[ \sum_{|k|\leq n} M(|\hat{f}(s)(k)|/a_2) = \sum_{|k|\leq n} M\left(h_n^s|k|^s |\hat{f}(k)|/(a_2 h_n^s)\right) \]

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Combining these relations and equality (3.2), we see that, for \( \rho \leq 714 \),

\[
\text{Then, for } \rho \geq 714, \text{it remains to prove the truth of Theorem 4.1.}
\]

On the other hand, by virtue of (3.2),

\[
\text{Thus,}
\]

\[
\|S_n(f^{(s)})\|_M \leq (n/2)^s \|A_{\rho} f\|_M
\]

\[
\leq (n/2)^s \omega_n(f, \pi/n) = \mathcal{O}(n^s \omega(n^{-1})), \quad n \to \infty.
\]

It should be noted that, in the case where \( M(t) = t, \omega(t) = t^\beta, \beta > 0, \) the equivalence of relations 1) and (6.3) was also proved in [9, Lemma 1].

**Proof of Theorem 4.1.** It is shown above that Theorem 4.2 follows from Theorem 4.1. Therefore, it remains to prove the truth of Theorem 4.1.

1) \( \Rightarrow \) 2). We note that

\[
\sum_{j=0}^{\nu} \binom{\nu}{j} (1 - \rho)^j \rho^{\nu-j} = ((1 - \rho) + \rho)^\nu = 1, \quad \nu = 0, 1, \ldots.
\]

Then, for \( a_3 := \|f - A_{\rho} f\|_M \), we have

\[
1 \geq \sum_{|k| \geq r} M \left( |1 - \lambda_{|k|, r}(\rho)| \frac{\|f(k)\|}{a_3} \right)
\]

\[
= \sum_{|k| \geq r} M \left( |1 - \sum_{j=0}^{r-1} \binom{|k|}{j} (1 - \rho)^j \rho^{k - j}| \frac{\|f(k)\|}{a_3} \right)
\]

\[
= \sum_{|k| \geq r} M \left( \sum_{j=r}^{|k|} \binom{|k|}{j} (1 - \rho)^j \rho^{k - j} |f(k)| / a_3 \right)
\]

\[
\geq \sum_{|k| \geq r} M \left( \binom{|k|}{r} (1 - \rho)^r \rho^{k - r} |f(k)| / a_3 \right).
\]

On the other hand, by virtue of (3.2),

\[
\|P(f^{[r]})(\rho, \cdot)\|_M = \left\| \left. \frac{\partial^r}{\partial \rho^r} P(f)(\rho, \cdot) \right|_M \right. 
\]

\[
= \inf \left\{ a > 0 : \sum_{|k| \geq r} M \left( r! \binom{|k|}{r} \rho^{k} |f(k)| / a \right) \leq 1 \right\}.
\]

Combining these relations and equality (3.2), we see that, for \( \rho \to 1^- \),

\[
\|P(f^{[r]})(\rho, \cdot)\|_M \leq r! \rho^r (1 - \rho)^{-r} \|f - A_{\rho} f\|_M = \mathcal{O}((1 - \rho)^{-s} \omega(1 - \rho)).
\]

2) \( \Rightarrow \) 3). For \( a_4 := \|P(f^{[r]})(\rho, \cdot)\|_M \) and for any numbers \( n > r \) and \( \rho \in [0, 1) \), we have

\[
1 \geq \sum_{|k| \geq r} M \left( \binom{|k|}{r} r! \rho^{k} |f(k)| / a_4 \right) \geq \sum_{r \leq |k| \leq n} M \left( \rho^{|k|} \binom{|k|}{r} r! |f(k)| / a_4 \right) = \sum_{r \leq |k| \leq n} M \left( \rho^{|k|} |f^{[r]}(k)| / a_4 \right).
\]
This yields \( \|S_n(f^{[r]\prime})\|_M \leq g^{-n}\|P(f^{[r]})(\varrho, \cdot)\|_M \). Putting \( g = 1 - 1/n \) and taking statement 2) into account, we see that

\[
\|S_n(f^{[r]\prime})\|_M \leq (1 - 1/n)^{-n}O(n^s\omega(n^{-1})) = O(n^s\omega(n^{-1})), \quad n \to \infty.
\]

3) \( \Rightarrow \) 4). Let us set \( g := f^{[r-s]} \). By definition, for \( |k| \geq r \), we have

\[
|\hat{f}^{[r]}(k)| = \frac{|k|!|\hat{f}(k)|}{(|k| - r)!} = |g^{[s]}(k)|\frac{(|k| - r + 1)(|k| - r + 2) \cdots (|k| - r + s)}{|k|(|k| - 1) \cdots (|k| - s + 1)} \geq |g^{[s]}(k)|\left(1 - \frac{r - 1}{|k|}\right)^s \geq r^{-s}|g^{[s]}(k)|.
\]

Therefore, in view of Remark 3.1, we get

\[
\|S_n(g^{[s]})\|_M \leq \|S_{r-1}(g^{[s]})\|_M + \left\| \sum_{r \leq |k| \leq n} g^{[s]}(k)e^{ikx} \right\|_M \leq \|S_{r-1}(g^{[s]})\|_M + r^s\|S_n(f^{[r]\prime})\|_M = O(n^s\omega(n^{-1})), \quad n \to \infty.
\]

Then, by virtue of Proposition 4.2, we see that \( \|g-Z_n^{(s)}(g)\|_M = O(\omega(n^{-1})), \quad n \to \infty \). Hence, \( g = f^{[r-s]} \in S_M, \ f \in S_M, \) and \( f^{[r-s]} \in S_M H^s_n \).

4) \( \Rightarrow \) 3). If \( g := f^{[r-s]} \), then, according to Proposition 4.2, we get

\[
\|S_n(g^{[s]})\|_M = O(n^s\omega(n^{-1})), \quad n \to \infty.
\]

We have \( \hat{f}^{[r]}(k) = 0 \) for \( |k| < r \) and

\[
\hat{f}^{[r]}(k) = \frac{|k|!}{(|k| - r)!} |\hat{f}(k)| \leq \frac{|k|!}{(|k| - s)!} \frac{|k|!}{(|k| - r + s)!} |\hat{f}(k)| = |g^{[s]}(k)|
\]

for \( |k| \geq r \). Thus,

\[
\|S_n(f^{[r]\prime})\|_M \leq \|S_n(g^{[s]})\|_M = O(n^s\omega(n^{-1})), \quad n \to \infty.
\]

3) \( \Rightarrow \) 1). From identity (6.4), it follows that for any \( \varrho \in [0, 1] \),

\[
\sum_{j=r}^{\nu} \binom{\nu}{j} (1 - \varrho)^j \varrho^{r-j} \leq 1, \quad \nu \geq r.
\]

This yields the relation

\[
\sum_{|k| \geq r} M \left( |1 - \lambda_{|k|, r}(\varrho)| \frac{|\hat{f}(k)|}{a_5} \right) \leq \sum_{|k| \geq r} M \left( \frac{|\hat{f}(k)|}{a_5} \right) \leq 1,
\]

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where \(a_5 := \|f\|_M\). Therefore, we have \(\|f - A_{g,r}(f)\|_M \leq \|f\|_M < \infty\). From this relation, we conclude that, for any \(\varepsilon > 0\), there exists a number \(n_0\) such that, for all \(n > n_0\) and all \(\varrho \in [0, 1)\),

\[
\|f - A_{g,r}(f)\|_M \leq \left\| \sum_{r \leq |k| \leq n} \sum_{j=r}^{|k|} \left( \begin{array}{c} |k| \\ j \end{array} \right) (1 - \varrho)^j \varrho^{|k| - j} \tilde{f}(k)e^{ikx} \right\|_M + \varepsilon. \tag{6.6}
\]

Let us use the inequality

\[
\sum_{j=r}^\nu \left( \begin{array}{c} \nu \\ j \end{array} \right) (1 - \varrho)^j \varrho^{\nu - j} \leq \left( \begin{array}{c} \nu \\ r \end{array} \right) (1 - \varrho)^r \tag{6.7}
\]

which is valid for all \(\nu \geq r\) and \(\varrho \in [0, 1]\) (see, e.g., [16]). Putting \(a_6 := (1 - \varrho)^r \|S_n(f^{[r]}i)\|_M / r!\), we get

\[
\sum_{r \leq |k| \leq n} M \left( \sum_{j=r}^{|k|} \left( \begin{array}{c} |k| \\ j \end{array} \right) (1 - \varrho)^j \varrho^{|k| - j} \frac{\tilde{f}(k)}{a_6} \right) \leq \sum_{r \leq |k| \leq n} M \left( (1 - \varrho)^r \left( \begin{array}{c} |k| \\ r \end{array} \right) \frac{\tilde{f}(k)}{a_6} \right) \leq 1.
\]

Thus,

\[
\left\| \sum_{r \leq |k| \leq n} \sum_{j=r}^{|k|} \left( \begin{array}{c} |k| \\ j \end{array} \right) (1 - \varrho)^j \varrho^{|k| - j} \tilde{f}(k)e^{ikx} \right\|_M \leq \frac{(1 - \varrho)^r}{r!} \|S_n(f^{[r]}i)\|_M. \tag{6.8}
\]

Combining relations (6.6) and (6.8) and putting \(n := n_0 = \lfloor (1 - \varrho)^{-1} \rfloor\), where \([\cdot]\) means the integer part of the number, we get

\[
\|f - A_{g,r}(f)\|_M \leq \frac{(1 - \varrho)^r}{r!} \|S_n(f^{[r]}i)\|_M + \varepsilon = (1 - \varrho)^r \mathcal{O}(n_0^\alpha \omega(n_0^{-1})) + \varepsilon = \mathcal{O}((1 - \varrho)^r \omega(1 - \varrho)) + \varepsilon,
\]

as \(\varrho \to 1-\). By virtue of the arbitrariness of \(\varepsilon\), it follows from this relation that implication 3) \(\Rightarrow 1)\) is true.

**Proof of Theorem 5.1.** Before proving Theorem 5.1, let us formulate some known auxiliary statements.

**Lemma 6.1.** [6] Assume that \(f, g \in S_M\), \(\alpha, \delta > 0\), \(h \in \mathbb{R}\). Then

(i) \(\|\Delta_h^n f\|_M \leq K(\alpha) \|f\|_M\), where \(K(\alpha) := \sum_{j=0}^{\infty} \|\alpha\| \leq 2^{\langle \alpha \rangle}\),

\(\{\alpha\} = \text{inf}\{k \in \mathbb{N} : k \geq \alpha\}\).

(ii) \(\omega_\alpha(f + g, \delta)_M \leq \omega_\alpha(f, \delta)_M + \omega_\alpha(g, \delta)_M\).

(iii) \(\omega_\alpha(f, \delta)_M \leq 2^{\langle \alpha \rangle} \|f\|_M\).

**Lemma 6.2.** [6] Assume that \(\alpha > 0\), \(n \in \mathbb{N}\) and \(0 \leq h \leq 2\pi/n\). Then, for any polynomial \(\tau_n(x) = \sum_{|k| \leq n} a_k e^{ikh}\)

\[
\left( \frac{\sin(nh/2)}{n/2} \right)^\alpha \|\tau_n(\alpha)\|_M \leq \|\Delta_h^n \tau_n\|_M \leq h^\alpha \|\tau_n(\alpha)\|_M. \tag{6.9}
\]
Lemma 6.3. [6] If \( f \in S_M \), then, for any numbers \( \alpha > 0 \) and \( m \in \mathbb{N} \), the following inequality holds:
\[
\|f - S_m(f)\|_M = E_{m+1}(f)_M \leq C(\alpha) \omega_\alpha(f, m^{-1})_M,
\]
where \( C = C(\alpha) \) is a constant that does not depend on \( f \) and \( n \).

Consider an arbitrary function \( g \) from the set \( G_{n,f} \) defined in Remark 5.1. By virtue of (5.3), if \(|h| < \delta\), then \( |\Delta^n_h g|^{\infty}(0) = 0\), for all \( 0 < |k| \leq n - 1 \),
\[
\left| \Delta^n_h g(k) \right| = \left| \frac{2\sin \frac{k|n|}{2} |\hat{g}(k)|}{\delta^{n(n - 1)\delta}} \right| \leq \delta^n(n - 1)\delta^n|\hat{g}(k)| \leq \delta^n(n - 1)^n|\hat{f}(k)|,
\]
and, for \( |k| \geq n \),
\[
\left| \Delta^n_h g(k) \right| \leq |k|^n\delta^n|\hat{g}(k)| \leq \delta^n(n - 1)^n|\hat{f}(k)| = \delta^n(n - 1)^n|\hat{g}^{(n)}(k)|.
\]
Therefore, for any \(|h| < \delta\), we have
\[
\|\Delta^n_h g\|_M \leq \delta^n(n - 1)^n \sum_{0 < |k| \leq n - 1} |\hat{f}(k)e^{i\omega x}|_M + \delta^n(n - 1)^n\|g^{(n)}\|_M
\]
and, hence,
\[
\omega_n(g, \delta) \leq \delta^n(n - 1)^n \sum_{0 < |k| \leq n - 1} |\hat{f}(k)e^{i\omega x}|_M + \delta^n(n - 1)^n\|g^{(n)}\|_M.
\]

By virtue of Lemma 6.1 (ii) and (iii) and relation (6.11), for any \( g \in G_{n,f} \), we have
\[
\omega_n(f, \delta)_M \leq \omega_n(f - g, \delta)_M + \omega_n(g, \delta)_M \\
\leq 2^n\|f - g\|_M + \delta^n(n - 1)^n \left\| \sum_{0 < |k| \leq n - 1} |\hat{f}(k)e^{i\omega x}|_M \right\|.
\]

Taking the infimum of the right-hand side of the last relation over all \( h \in G_{n,f} \), we get the left-hand side of (5.2) with the constant \( C_1 = \min\{2^{-n}, n^{-n}\} \).

Now, we shall prove the right-hand side of (5.2). Let \( S_m := S_m(f_0) \), \( m \geq n \), be the Fourier sum of \( f_0 \) defined in Remark 5.1. Then, for \( n \leq |k| \leq m \), the Fourier coefficients of the derivative \( S^{(n)}_m \)
\[
\left| \left[ S^{(n)}_m \right]^\infty k \right| = |k|(|k| - 1)\ldots(|k| - n + 1)|\hat{f}(k)| \leq |k|^n|\hat{f}(k)| = |\left[ S^{(n)}_m \right]^\infty k |,
\]
and \( \left[ S^{(n)}_m \right]^\infty k = 0 \) for \( |k| \in \mathbb{N} \setminus [n, m] \). Therefore, \( \|S^{(n)}_m\|_M \leq \|S^{(n)}_m\|_M \).

Now, let \( \delta \in (0, 2\pi) \) and \( m \in \mathbb{N} \) such that \( \pi/m < \delta < 2\pi/m \). Using Lemma 6.2 with \( h = \pi/m \) and property (i) of Lemma 6.1, we obtain
\[
\|S^{(n)}_m\|_M \leq \|S^{(n)}_m\|_M \leq (m/2)^n\|\Delta^n_{m/2} S_m\|_M \\
\leq (m/2)^n\|\Delta^n_{m/2} f\|_M \leq (\pi/\delta)^n \omega_n(f, \delta)_M
\]
and
\[
\left\| \sum_{0 < |k| \leq n - 1} |\hat{f}(k)e^{i\omega x}|_M \right\| \leq \left\| \sum_{0 < |k| \leq m} |k|^n|\hat{f}(k)e^{i\omega x}|_M \right\|.
\]
(m/2)^n \| \Delta_n^{\omega_n(f, \delta)} \|_M \leq (\pi/\delta)^n \omega_n(f, \delta)_M. \quad (6.13)

By virtue of Lemma 6.3, we have
\[
\| f_0 - S_m \|_M = E_{m+1}(f_0)_M \leq C(n) \omega_n(f_0, \delta)_M = C(n) \omega_n(f, \delta)_M. \quad (6.14)
\]

Setting \( C_2(n) := C(n) + 2\pi^n \) and combining (6.12)–(6.14), we obtain the right-hand side of (5.2):
\[
K_n(\delta, f)_M + \delta^n \sum_{0 < |k| \leq n-1} \widehat{f}(k) e^{ikx} \leq \| f_0 - S_m \|_M + \delta^n \| S_m^{(n)} \|_M + \delta^n \sum_{0 < |k| \leq n-1} \widehat{f}(k) e^{ikx} \leq C_2(\alpha) \omega_n(f, \delta)_M.
\]

\[\square\]

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