HYDRODYNAMIC SCALING LIMIT OF CONTINUUM SOLID-ON-SOLID MODEL

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Abstract. A fourth order nonlinear evolution equation is derived from a microscopic model for surface diffusion, namely, the continuum solid-on-solid model. We use the method developed by Varadhan for the computation of the hydrodynamic scaling limit of nongradient models. What distinguishes our model from other models discussed so far is the presence of two conservation laws for the dynamics in a nonperiodic box and the complex dynamics that is not nearest-neighbor. Along the way, a few steps has to be adapted to our new context. As a byproduct of our main result we also derive the hydrodynamic scaling limit of a perturbation of continuum solid-on-solid model, a model that incorporates both surface diffusion and surface electromigration.

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1991 Mathematics Subject Classification. Primary 60F99; secondary 60H30, 60G07.

Key words and phrases. Law of large numbers, scaling limit of stochastic processes, nongradient model, central limit theorem variance.

Research supported by an Ontario Graduate Studies Fellowship and the University of Toronto Fellowship.
1. Introduction

A process of great technological importance, molecular beam epitaxy (MBE), is used to manufacture computer chips and semiconductor devices, see Barabasi and Stanley [1]. In general the chip is constructed by spraying a beam of atoms on a flat surface. There are three phenomena that take place in the construction process namely deposition, diffusion and desorption. The atoms arrive or deposit on the surface and do not stick on the first contact point, but diffuse or walk on the surface. When an atom reaches the edge of another wandering atom, the two atoms meet or glue together, forming islands. Sometimes an atom may jump out of the surface. Smaller islands may develop into larger islands affecting the roughness of the surface on the macroscopic scale. A rough surface does not have very good contact properties and engineers would like to understand the basic mechanisms affecting the morphology in general.

How deposition and desorption affect the morphology of the surface, it is quite well understood. It is of great importance to know how the profile of the surface evolves on the macroscopic scale if the atoms that make up the surface diffuse. We assume that no atoms arrive or leave the surface.

The present paper discusses a model for surface diffusion the continuum solid-on-solid model, known also as the fourth-order Ginzburg-Landau model. The system has a complex interaction that is not a nearest-neighbor interaction and has two conserved quantities in a nonperiodic box, namely the total slope of the surface and the linear mean of the surface slope. Under the assumption that the molecules of the surface follow the dynamics of the continuum solid-on-solid model, we will prove that the dynamics of the surface slope profile on the macroscale is a fourth-order nonlinear equation. Since the model is nongradient, the derivation of the limit is not trivial and we shall use the method developed in Varadhan [17] and further extended in Quastel [12], Varadhan and Yau [18]. Even though we have an interaction that is not of nearest-neighbor type, the microscopic current still decomposes as the Laplacian of the slope field and the fluctuations. Usually in nearest-neighbor models after subtracting the fluctuations from the current we end up with the gradient of some local function. We use the result that for continuum solid-on-solid model the space of exact functions has codimension one inside the space of closed functions, see Savu [14].

We also include a discussion of a perturbation of continuum solid-on-solid model, where the evolution of the surface is driven by both diffusion and electric field. The electric field will add one extra second-order term to the final nonlinear equation. The continuum solid-on-solid model is an approximation of the discrete solid-on-solid model and hence is not considered a truly microscopic model. Unfortunately at the time of writing of this paper, we don’t have the required techniques to solve the discrete solid-on-solid model. However, partial rigorous results and numerical analysis are available for the discrete case, see Krug, Doobs and Majaniemi [9].

Finally the paper is organized as follows: section 2 contains the description of continuum solid-on-solid model and of the model for surface electromigration, the statement of the main results and a note on similar models considered in the literature so far; section 3 outlines the proof of the main result, the computation of the scaling limit of continuum solid-on-solid model; section 4 shows how the final nonlinear equation is identified; in section 5 we calculate the asymptotics of central limit variance, the main ingredient used in section 4 to prove that the microscopic
current can be replaced by a multiple of the field Laplacian; in section 6 we calculate
the scaling limit of the modified model to incorporate surface electromigration.

2. The models

The continuum solid-on-solid model is a 1-dimensional lattice system with con-
tinuous order parameter, used to study the evolution of an interface. The model
describes the movement of the interface at the mesoscopic level and hence is not
considered a truly microscopic model that captures all the aspects of the particle
motion, but it has the advantage of being more suitable for computations.

The height model. Let \( T \) be the torus represented as the interval \([0, 1]\) with
0 and 1 identified. For each positive integer \( N \), there are \( N \) scaled periodic lattice
points located at sites \( i/N \) in \( T \), \( i = 1, \ldots, N \). We shall denote by \( h_i(t) \) the height of
the surface at the site \( i/N \), at time \( t \). Also because of the periodicity of the lattice
points, \( h_{N+1}(t) = h_1(t) \). The energy function \( H_N(h) \) of a height configuration \( h \)
chosen to be invariant under the global translation \( h_i(t) \mapsto h_i(t) + c \) and has the
form

\[
H_N(h) = \sum_{i=1}^{N} V(h_{i+1} - h_i).
\]

In this paper we shall assume that the potential is quadratic \( V(x) = x^2 \). We shall
require the evolution in time of the surface to preserve the sum of the heights, to
have as invariant measure the infinite mass measure \( e^{-H_N(h)} dh \), and to be reversible.
All these properties are satisfied by the solution of the stochastic differential system:

\[
dh_i(t) = \frac{N^4}{2}(w_i - w_{i-1}) dt + N^2(\sqrt{a_i} dB_i - \sqrt{a_{i-1}} dB_{i-1}), \quad 1 \leq i \leq N.
\]

Above, \( a_i(h) = a(h_{i-1} - h_i, h_{i+1} - h_i, h_{i+2} - h_{i+1}) \) where \( a \) is a function with bounded
continuous first derivatives that satisfies \( 0 < 1/a^* \leq a(x_1, x_0, x_1) \leq a^* < \infty \). The
\( n \) copies of the Brownian motion \( B_i \), \( i = 1, \ldots, n \) are independent. We define the
instantaneous current \( w_i(h) \) of particles over the bond \( \{i, i+1\} \)

\[
w_i(h) = (\partial_{i-1} a - 2\partial a + \partial_i a)(h_i - h_{i-1}, h_{i+1} - h_i, h_{i+2} - h_{i+1}) + a_i(h)(V'(h_i - h_{i-1}) - 2V'(h_{i+1} - h_i) + V'(h_{i+2} - h_{i+1})).
\]

Because the height process preserves the sum of the heights of the surface, this
dynamics models surface diffusion.

The slope process. A crucial property of the dynamics of the heights is the
gauge property, namely the dynamics invariance under the action of the group \( G \)
of translation in the \((1, \ldots, 1)\) direction,

\[
G = \{T : \mathbb{R}^N \to \mathbb{R}^N \mid T(x_1, \ldots, x_N) = (x_1 + c, \ldots, x_N + c), c \in \mathbb{R}\}.
\]

Hence there exists an induced dynamics on the quotient space \( \mathbb{R}^N/G \) of equivalence
classes. A representative of an equivalence class is the slope configuration \( x_i(t) = h_{i+1}(t) - h_i(t) \), \( 1 \leq i \leq N-1 \) and \( x_N(t) = h_1(t) - h_N(t) \). Note that \( \sum_{i=1}^{N} x_i(t) = 0 \).
As a function of the slope configuration of the surface, the energy becomes \( H_N(x) = \sum_{i=1}^{N} V(x_i) \).

In the sequel, we shall study the slope process rather than the height process.
The slope process is reversible and has as equilibrium distribution, the product
probability measure $d\nu^N = e^{-H_N(x)}/Z^N\, dx$. Below we write down the stochastic differential system, the generator, and the Dirichlet form of the slope process:

$$dx_i(t) = \frac{N^4}{2}(w_{i+1} - 2w_i + w_{i-1})dt + N^2(\sqrt{a_{i-1}} dB_{i-1} - 2\sqrt{a_i} dB_i + \sqrt{a_{i+1}} dB_{i+1}),$$

(2.2) 

$$N^4 L_N(f) = \frac{N^4}{2} \sum_{i=1}^N a_i(x)(\partial_{i+1} - 2\partial_i + \partial_{i-1})^2 f + w_i(x)(\partial_{i+1} - 2\partial_i + \partial_{i-1})f,$$

(2.3) 

$$N^4 D_N(f) = \frac{N^4}{2} \int_{\mathbb{R}^N} \sum_{i=1}^N a_i(x)(\partial_{i+1} - 2\partial_i + \partial_{i-1})^2 f \frac{e^{-H_N(x)}}{Z^N} dx.$$ 

(2.4) 

The factor $N^4$ in the generator with the lattice spacing of 1/$N$ represents the scaling of space and time. This scaling is needed to observe a nontrivial motion in the limit.

The diffusion (2.2) is driven in the direction of the linear vector fields $X_i = \partial_{i+1} - 2\partial_i + \partial_{i-1}$, $1 \leq i \leq N$, therefore is not ergodic in the whole space $\mathbb{R}^N$. It becomes ergodic when restricted to the hyperplane $x_1 + \cdots + x_N = N\bar{x}$ of average slope $\bar{x}$. The unique equilibrium probability measure of the dynamics restricted to the hyperplane is the conditional probability $e^{-H_N(x)}/Z^N\, dx$ given $x_1 + \cdots + x_N = N\bar{x}$.

Instantaneously, the slope profile of the surface decreases at some site $i$ twice as much as increases at the adjacent sites $i-1, i+1$. We see that any update of the slope configuration affects the slopes at three sites. This type of interaction, known as the three site interaction, is quite complex and has been very rarely studied, so far.

We should note an integration by parts property of the current $w_i$, for all bounded and smooth functions $f$:

$$E^\text{eq}[w_i \cdot f] = E^\text{eq}[w_i \cdot (\partial_{i+1} - 2\partial_i + \partial_{i-1})f].$$

(2.5) 

The expected value above is with respect to the equilibrium measure $d\nu^N$.

**The dynamics in a nonperiodic box.** Although we are concerned with the study of the slope process defined on a periodic space, we make use of a similar slope process evolving in a box with nonperiodic boundary. Below we describe this new dynamics and its main properties. Suppose $\Lambda$ is a box with finitely many sites of the lattice $\mathbb{Z}$. The infinitesimal generators,

$$L_\Lambda(f) = \frac{1}{2} \sum_{i \in \Lambda, i+1, i-1 \in \Lambda} a_i(x)(\partial_{i+1} - 2\partial_i + \partial_{i-1})^2 f + w_i(x)(\partial_{i+1} - 2\partial_i + \partial_{i-1})f,$$

(2.6) 

respectively,

$$L_\infty(f) = \frac{1}{2} \sum_{i \in \mathbb{Z}} a_i(x)(\partial_{i+1} - 2\partial_i + \partial_{i-1})^2 f + w_i(x)(\partial_{i+1} - 2\partial_i + \partial_{i-1})f,$$

(2.7) 

produce two diffusion processes.

In the case of the first dynamics, generated by $L_\Lambda$, there is no transport of particles over the boundary of the box, so the box does not have periodic boundary. If the box $\Lambda = [-l,l] \cap \mathbb{Z}$ or $\Lambda = [l-l, l] \cap \mathbb{Z}$ we use the shorter notation $L_l$, respectively, $L_{i,l}$ for the generator $L_\Lambda$. The second dynamics generated by $L_\infty$ is a dynamics on the infinite lattice $\mathbb{Z}$. The dynamics $L_{i,l}$ preserves the average slope,
$y_{i,l}^1 = \frac{x_{i-l} + \cdots + x_{i+l}}{2l+1}$, and the linear mean of the slopes, $y_{i,l}^2 = \frac{(-l)x_{i-l} + \cdots + lx_{i+l}}{l(l+1)}$, inside the box $[i-l, i + l] \cap \mathbb{Z}$. The second conserved quantity, $y_{i,l}^2$, should be understood as a boundary condition that is preserved in time. The linear mean slope, $y_{i,l}^2$, is conserved in time because we have a model for surface diffusion and the total height of the surface does not change as time passes by. We will use two different notation $\bar{x}_{i,l}$ and $y_{i,l}^1$ for the mean slope of the field in the box centered at $i$, of size $l$. Also $y_{i,l}$ stands for the vector of the conserved quantities, $(y_{i,l}^1, y_{i,l}^2)$. As a convention we will drop the subscript $i$, meaning the center of the box, if the box is centered at the origin.

**Equilibrium measures.** We proceed to describe next the equilibrium measures of the dynamics that we have introduced. For the dynamics $L_{i,l}$, the grand canonical measure is the product probability measure $\nu^g_{\alpha, i,l} = \bigotimes_{j=i-l}^{i+l} \frac{\mu_{x_j}^{\alpha}}{Z(\alpha)} \, dx_j$, whereas the canonical measure is the conditional probability measure $\nu^c_{\alpha, i,l} = \nu^g_{\alpha, i,l}( \cdot \mid y_{i,l})$ given the level set

$$\{x \in \mathbb{R}^{2l+1} \mid \frac{x_{i-l} + \cdots + x_{i+l}}{2l+1} = y_{i,l}, \frac{(-l)x_{i-l} + \cdots + lx_{i+l}}{l(l+1)} = y_{i,l}^2\}.$$

The canonical measure is the unique stationary probability measure for the restricted dynamics on this set. The Dirichlet forms of the operators $L_1, L_{i,l}$, with respect to the grand canonical measure with $\alpha = 0$, are denoted by $D_1(f)$, respectively, $D_{i,l}(f)$. The Dirichlet form of the operator $L_1$ with respect to the canonical measure is denoted by $D^c_{\nu^c}(f)$. For $L_{\infty}$, the product measures $\nu^g_{\alpha} = \bigotimes_{i \in \mathbb{Z}} \frac{\mu_{x_i}^{\alpha}}{Z(\alpha)} \, dx_i$ are equilibrium measures. We shall use the notation $\nu^c_{N} = \bigotimes_{i=1}^{N} \frac{\mu_{x_i}^{\alpha}}{Z} \, dx_i$

**A model for surface electromigration.** We consider also a perturbation of the continuum solid-on-solid model. The new system describes the evolution of a one-dimensional surface driven by both surface diffusion and surface electromigration. The surface electromigration refers to the motion of atoms on a solid surface that is caused by an electric current in the material. The electric field interacts with the atoms of the surface as the wind blows the sand particles, and a ripple pattern is observed in the long run. Electromigration along interfaces is believed to play a crucial role in the failure of metallic circuits, see Schimschak and Krug [10], for further details.

We assume the electric field is a continuous function $E(t, \theta)$ defined on $[0, T] \times \mathbb{T}$. As before $x_i$ represents the slope of the surface at the site $i/N$. The generator of the system on a periodic lattice, that incorporates the action of the electric field, is

$$N^4 L_{N,E}(f) = N^4 L_N(f) + \frac{N^2}{2} \sum_{i=1}^{N} E \left( t \frac{i}{N} \right) a(x_{i-1}, x_i, x_{i+1})(\partial_{x_{i-1}} - 2\partial_i + \partial_{x_{i+1}}) f.$$

**Hydrodynamic scaling limit of the models.** We shall call $P^\text{neq}_{N,T}$ and $P^\text{eq}_{N,T}$ the law up to time $T$ of the slope process started in some nonequilibrium distribution $\nu^\text{neq}_{N}$, equilibrium distribution $\nu^\text{eq}_{N}$, respectively. The law up to time $T$
of the slope process \( \mathcal{L} \), driven by the electric field, started in the nonequilibrium measure \( \nu^{\text{neq}}_N \) shall be called \( P^{\text{neq}}_{N,E,T} \).

Under \( P^{\text{neq}}_{N,T} \) and \( P^{\text{eq}}_{N,T} \) the random variable

\[
\pi_N(t) = \frac{1}{N} \left( x_1(t)\delta_{\frac{1}{N}} + \cdots + x_N(t)\delta_{\frac{N}{N}} \right)
\]

has distributions \( Q^{\text{neq}}_{N,T} \) and \( Q^{\text{eq}}_{N,T} \), respectively. We also refer to the random variable (2.9) as the empirical distribution. Every realization of this random variable is a measure-valued continuous path and \( Q^{\text{eq}}_{N,T} \) is a distribution on the space \( \mathcal{X} = \bigcup\{C([0,T],\mathcal{M}_l) \} \). The space \( \mathcal{X} \) is endowed with the inductive limit topology, the strongest topology that makes all the inclusions of \( C([0,T],\mathcal{M}_l) \) continuous. The space of signed measures \( \mathcal{M}_l \), with total variation not exceeding \( l \), is a metrizable space with the weak topology.

We say that a model has hydrodynamic scaling limit if under certain assumptions the sequence of laws of empirical distributions has a limit that is supported on the solution of an initial value problem.

**Definition 2.1.** A sequence of initial distributions \( \nu^{\text{neq}}_N \) on \( \mathbb{R}^N \) is said to correspond to the macroscopic slope profile \( m_0 \in L^1(\mathbb{T}) \) if the random variable \( \pi_N \) converges weakly in probability to \( \delta_{m_0(\theta)d\theta} \), i.e., for any continuous function \( \phi \in C(\mathbb{T}) \) and any \( \epsilon > 0 \),

\[
\limsup_{N \to \infty} \nu^{\text{neq}}_N \left\{ \left| \frac{1}{N} \sum_{i=1}^{N} \phi\left( \frac{i}{N} \right) x_i - \int_{\mathbb{T}} \phi(\theta) m_0(\theta)d\theta \right| > \epsilon \right\} = 0.
\]

Our first result of the paper says that the continuum solid-on solid model has a scaling limit and provide the form of the limiting evolution equation. More precisely,

**Theorem 2.1.** Assume that the potential \( V(x) \) is equal to \( x^2/2 \). Let \( m_0 \in L^1(\mathbb{T}) \) be a macroscopic slope profile such that \( \int_{\mathbb{T}} m_0(\theta)d\theta = 0 \). Assume that the sequence of initial distributions \( \{\nu^{\text{neq}}_N\}_N \) corresponds to the profile \( m_0 \) and the initial relative entropy \( H(\nu^{\text{neq}}_N|\nu^{\text{eq}}_N) \) is of order \( O(N) \). Then the sequence of probability measures \( \{Q^{\text{eq}}_{N,T}\}_{N \geq 0} \) is tight in \( \mathcal{X} \).

Any possible limit, \( Q_T \), of a convergent subsequence of \( \{Q^{\text{eq}}_{N,T}\}_{N \geq 0} \) is concentrated on the weak solutions \( (m(t,\theta)d\theta)_{t \in [0,T]} \) of the Cauchy problem with periodic boundary conditions

\[
\partial_t m = -\frac{1}{2} \partial^2_{\theta} (\bar{a}(m) \partial_{\theta} m), \quad m(0,\theta) = m_0(\theta), \quad \theta \in \mathbb{T}.
\]

The transport coefficient \( \bar{a} \) is a nonrandom continuous function on \( \mathbb{R} \), given by the following variational formula,

\[
\bar{a}(\alpha) = \inf_g E_{\nu^{\text{eq}}_{\alpha}} \left[ a(x_{-1},x_0,x_1) \left( 1 + (\partial_{i+1} - 2\partial_i + \partial_{i-1}) \left( \sum_{j \in \mathbb{Z}} \tau^j g \right) \right)^2 \right].
\]

The infimum on the line above is taken over all local functions \( g(x_{-N}, \ldots, x_N, y_{0,1}) \) of the slope configuration. The shift \( \tau^j \) acts on configurations \( (\tau^j x)_k = x_{k+j} \) and on local functions, \( (\tau^j g)(x) = g(\tau^j x) \). The expectation \( E_{\nu^{\text{eq}}_{\alpha}} \) is with respect to the grand canonical measure \( \nu^{\text{eq}}_{\alpha} \).
Note. It can be shown that the transport coefficient $\hat{a}$ is a continuous, bounded above and below function (see Kipnis and Landim [3]). Also we expect that the methods of Landim, Olla and Varadhan [10] can show that $\hat{a}$ is smooth but we will not pursue it here.

Note. By a weak solution of the Cauchy problem (2.10) we mean a path $m \in X$ such that for each time $T \geq 0$, the value of the path $m$ at the time $T$ is a Lebesgue absolutely continuous measures on $T$, satisfying the energy estimate

$$\int_0^T \int_T (\partial_\theta^2 m(t, \theta))^2 d\theta dt < \infty.$$  

Moreover for each $0 \leq T < \infty$ and for each test function $\phi \in C^{1,2}([0,T] \times \mathbb{T})$,

$$\int_T m(T, \theta)\phi(T, \theta) d\theta - \int_T m(0, \theta)\phi(0, \theta) d\theta - \int_0^T \int_T m(s, \theta)\partial_s \phi(s, \theta) d\theta ds + \frac{1}{2} \int_0^T \int_T \hat{a}(m)\partial_\theta^2 m(s, \theta)\partial_\theta^2 \phi(s, \theta) d\theta ds = 0.$$  

If we assume that the initial condition $m_0$ has the property that $\int_T m_0(\theta) d\theta = 0$ then for each time $t \geq 0$ the solution satisfies $\int_T m(t, \theta) d\theta = 0$.

Uniqueness of weak solutions of the Cauchy problem (2.10), that satisfy the energy estimate (2.12) has not been proved yet. If the transport coefficient $\hat{a}$ does not depend on the field $m$, the uniqueness of the Cauchy problem is known and can be found in Eidelman book [4].

As will be explained later the fluctuation dissipation equation for the continuum solid-on-solid model follows from the direct sum decomposition of a Hilbert space to be defined next.

We define the Hilbert space of closed functions, $C_X$ to be the space of those $\xi \in L^2(\mathbb{R}^N)$ that satisfy in the weak sense the equations $X_i(\tau^j \xi) = X_i(\tau^j \xi)$ for all integers $i$ and $j$. It is not hard to see that a subspace of $C_X$ is the closed linear span in $L^2(\mathbb{R}^N)$ of functions $\xi_g = X_0(\sum_{j \in \mathbb{Z}} \tau^j g)$, where $g$ is a bounded local function with bounded first derivatives. Even though the infinite sum $\sum_{j \in \mathbb{Z}} \tau^j g$ does not make sense, the function $\xi_g$ is well defined because the vector field kills all but finitely many terms of the infinite sum. We shall call this space the space of exact functions and we shall use the notation $E_X$. The space of exact functions has codimension one inside the space of closed functions. In this paper we do not include the proof of this result, since it is very technical and is not of probabilistic nature, however we include the statement, see Lemma 2.1. Lemma 2.1 is discussed in Savu [14].

**Lemma 2.1.** Let $1$ denote the constant function $1$. The direct sum decomposition holds:

$$C_X = \mathbb{R}1 \bigoplus E_X.$$  

The second result of the paper proves that the model for surface electromigration (2.8) has a scaling limit as well, and calculates the limiting evolution equation.

**Theorem 2.2.** Suppose the hypothesis of Theorem 2.1 are satisfied. Then the sequence of probability measures $\{Q_{N,E,T}^{\text{ren}}\}_{N \geq 0}$ is tight in $X$, and any possible limit $Q_{E,T}$ is supported on the weak solutions of the Cauchy problem

$$\partial_t m = -\frac{1}{2} \partial_\theta^2 (\hat{a}(m)(\partial_\theta^2 m + E)), \quad m(0, \theta) = m_0(\theta), \quad \theta \in \mathbb{T}. $$
The transport coefficient $\hat{a}$ is given by the same variational formula as in the statement of Theorem 2.1.

**Similar models.** The continuum solid-on-solid model belongs to a large class of Ginzburg-Landau models. The slope model is of nongradient type and has an unusual dynamics because two neighboring exchanges occur always simultaneously. Nishikawa [11], and Bertini, Olla and Landim [2] have investigated the hydrodynamic scaling limit in higher dimension of the gradient version (i.e., $a$ is a constant function) of the slope model. They have found that on the macroscale the interface follows a fourth-order nonlinear evolution equation.

Another similar model, the second-order Ginzburg-Landau model, where the sum of the heights is not conserved, was the subject of extensive discussions in the literature: the hydrodynamic scaling limit was derived by Fritz [6] and Guo, Papanicolaou and Varadhan [7] for the gradient version, and by Varadhan [17] for the nongradient case, whereas the nonequilibrium fluctuations have been proved by Chang and Yau [3]. The second-order Ginzburg-Landau model and the continuum solid-on-solid model correspond to Glauber, respectively, Kawasaki dynamics in the context of interacting particle systems. As expected, a different dynamics at the mesoscopic level causes different dynamics at the macroscopic level, a second-order parabolic differential equation in the case of second-order Ginzburg-Landau model versus a fourth-order parabolic differential equation for the continuum solid-on-solid model.

### 3. Hydrodynamic Scaling Limit of Continuum Solid-on-Solid Model

In this section we give a sketch of the main result, Theorem 2.1. We follow a standard scheme to derive the hydrodynamic scaling limit of the continuum solid-on-solid model. The existence of the limit follows from the tightness of the sequence of probability measures $\{Q_{N,T}^{\text{neq}}\}_N$. Let $Q_T$ be the limit of some weakly convergent subsequence of $\{Q_{N,T}^{\text{neq}}\}_N$. We proceed to characterize the limit $Q_T$, showing that it is supported on continuous paths with certain regularity property, known as the energy estimate (2.12). The most involved part of the argument is the identification of the possible weak limit $Q_T$ as some probability measure supported on the weak solution of the Cauchy problem (2.10). We shall make the assumption that the sequence of initial distributions $\{\nu_N^{\text{neq}}\}_N$ corresponds to some macroscopic profile $m_0 \in L^1(T)$.

**Note on notations.** Throughout the paper we shall make use of the shorter notation $\limsup_{z_1 \to i_1, \ldots, z_n \to i_n} f(z_1, \ldots, z_n)$ for the sequence of limits $\limsup_{z_1 \to i_1} \ldots \limsup_{z_n \to i_n} f(z_1, \ldots, z_n)$.

**Tightness.** As a consequence of Prohorov theorem and Arzela-Ascoli theorem, the tightness of the sequence $Q_{N,T}^{\text{neq}}$ follows from the next two Lemmas.

**Lemma 3.1.** For any test function $\phi \in C^2(T)$, any finite time $T$, and any $\epsilon > 0$, (3.1) \[
\limsup_{\delta \to 0, N \to \infty} P_{N,T}^{\text{neq}} \left\{ \sup_{|s-t| \leq \delta, 0 \leq s, t \leq T} \left| \frac{1}{N} \sum_{i=1}^{N} \phi \left( \frac{i}{N} \right) x_i(t) - \frac{1}{N} \sum_{i=1}^{N} \phi \left( \frac{i}{N} \right) x_i(s) \right| > \epsilon \right\} = 0.
\]
\[ (3.2) \lim_{l \to \infty, N \to \infty} P_{N,T}^{\text{neq}} \left\{ \sup_{0 \leq t \leq T} \frac{1}{N} \sum_{i=1}^{N} |x_i(t)| > l \right\} = 0. \]

**Proof.** The first convergence (3.1) follows from Garsia-Rodemich-Rumsey inequality. To prove the second convergence (3.2) we can use estimates on the moment generating function of hitting time of the diffusion process (2.2). The reader may consult Kipnis and Landim [8] or Guo, Papanicolau and Varadhan [7] or Savu [14] for a complete proof.

It is interesting to note that the stronger superexponential estimates can be established for the process in equilibrium,

\[ (3.3) \lim_{\delta \to 0, N \to \infty} -\frac{1}{N} \log P_{N,T}^{\text{neq}} \left\{ \sup_{0 \leq t, s \leq T, t-s \leq \delta} \left| \int_{t}^{s} \frac{1}{N} \sum_{i=1}^{N} N^2 w_i(u) \phi \left( \frac{i}{N} \right) du \right| \geq \epsilon \right\} = -\infty. \]

(3.4) \[ \lim_{l \to \infty, N \to \infty} -\frac{1}{N} \log P_{N,T}^{\text{neq}} \left\{ \sup_{0 \leq t \leq T} \frac{1}{N} \sum_{i=1}^{N} |x_i(t)| \geq l \right\} = -\infty. \]

**Energy estimate.** Any limiting point \( Q_T \) of the measure-valued sequence \( \{Q_{N,T}^{\text{neq}}\}_{N} \) is supported on paths \( \mu \in \mathcal{X} \) such that at each time \( t \), \( \mu(t) \) is a Lebesgue absolutely continuous measure on the torus \( T \) with density \( m(t, \theta) \). Moreover for each finite time \( T \), the density \( m(t, \theta) \) satisfies the energy estimate

\[ (3.5) \int_{0}^{T} \int_{T} (\partial^2_{\theta} m(t, \theta))^2 d\theta dt < \infty. \]

We note that the energy estimate (3.5) is equivalent to the inequality

\[ (3.6) \sup_{\phi \in C^{1,2}([0,T] \times T)} \int_{0}^{T} \int_{T} (2m(t, \theta)\partial^2_{\theta} \phi - C\phi^2) d\theta dt < \infty, \]

where \( C \) is a constant not depending on \( \phi \).

The entropy inequality (4.10), Feynman-Kac formula (4.12), and Lemma 4.2 can be used to derive the estimate

\[ (3.7) \lim_{l \to \infty, N \to \infty} \sup_{\phi \in C^{1,2}([0,T] \times T)} E_{t,T}^{\text{neq}} \left[ \frac{1}{N} \sum_{i=1}^{N} \int_{t}^{T} 2\phi \left( t, \frac{i}{N} \right) N^2 \text{Av}_{j=i-l+1}^{j+1-l} (\Delta x) \right] - \nabla_{i,T}(\Delta x, y)\phi^2 \left( t, \frac{i}{N} \right) dt < \infty. \]

Here, the cylinder function \( \Delta x \) is the discrete Laplacian of the slope field, \( x_{-1} - 2x_0 + x_1 \), and the variance \( \nabla_{i,T}(\Delta x, y) \) is defined later in section 4, see definition (??). As will be proved in Lemma 5.2, the variance, \( \nabla_{i,T}(\Delta x, y) \), has a uniform-in-\( y \) lower bound, therefore the estimate (3.7) follows from (3.5), after integrating by parts. Moreover, we can conclude that at any time \( t \) the weak second derivative of the measure \( \mu(t) \) is in \( L^2(T, d\theta) \) and hence \( \mu(t) \) is absolutely continuous with respect to Lebesgue measure on the torus \( T \).
Identification of the equation. That any limiting point of the measure-valued sequence \( \{Q^\text{neq}_{N,T}\}_{N>0} \) is supported on the weak solutions of (2.10) follows if the event corresponding to the violation of the limiting equation has probability zero in the limit.

For each test function \( \phi \in C^{1,2}([0, T] \times \mathbb{R}) \) that are differentiable in time and twice differentiable in space, and each finite time \( T \) we define the function

\[
V(t) = \frac{1}{N} \sum_{i=1}^{N} \phi \left( t, \frac{i}{N} \right) x_i(t) - \frac{1}{N} \sum_{i=1}^{N} \phi \left( 0, \frac{i}{N} \right) x_i(0) - \\
\quad - \int_{0}^{t} \frac{1}{N} \sum_{i=1}^{N} \partial_s \phi(s, \frac{i}{N}) \ddot{x}_{i,aN}(s) ds + \frac{1}{2} \int_{0}^{t} \frac{1}{N} \sum_{i=1}^{N} \phi'' \left( s, \frac{i}{N} \right) \dddot{a}(\dddot{x}_{i,aN}(s)) \times \\
\quad \times \dddot{b}^{-2}(\dddot{x}_{i-bN,cN}(s)) - 2\dddot{x}_{i,cN}(s) + \dddot{x}_{i+bN,cN}(s) ds,
\]

and the event

\[
O_{\phi}^{\phi} = \left\{ \sup_{0 \leq t \leq T} |V(t)| > \epsilon \right\}.
\]

We will prove in the next sections that for each \( \epsilon > 0 \) we have

\[
\limsup_{a,b,c \rightarrow 0, N \rightarrow \infty} P^\text{neq}_{N,T}(O_{a,b,c,\epsilon}^{\phi}) = 0.
\]

The proof of the result \( 3.10 \) is complicated and is divided into several steps. The function that defines this event can be written as a sum of functions, see the beginning of section \( 4 \). We will deal separately with each function in the sum and show that it converges to zero in probability.

4. Identification of the limiting equation

In this section we establish that the event \( 3.9 \) is negligible in the limit, and hence any weak limit \( Q_T \) is supported on the solutions of the Cauchy problem (2.10). To save space, we are suppressing the time dependence of the test function \( \phi \) that defines the event \( 3.9 \).

Note on notations. Assume \( f \) is some local function. We denote by \( \text{Av}_{j=i-l}^{i+l} f \) the average of shifts of \( f \), namely

\[
\frac{\tau^{i-l} f + \ldots + \tau^{i+l} f}{2l+1}.
\]
We write $V(t) =$

$$
(4.1) \quad \frac{1}{N} \sum_{i=1}^{N} \phi \left( \frac{i}{N} \right) x_i(t) - \frac{1}{N} \sum_{i=1}^{N} \phi \left( \frac{i}{N} \right) x_i(0) - \int_{0}^{t} \frac{1}{N} \sum_{i=1}^{N} \phi \left( \frac{i}{N} \right) N^4 L_N(x_i) \, ds +
$$

$$
(4.2) \quad + \frac{1}{2} \int_{0}^{t} \frac{1}{N} \sum_{i=1}^{N} \left\{ N^2 \left[ \phi \left( \frac{i-1}{N} \right) - 2 \phi \left( \frac{i}{N} \right) + \phi \left( \frac{i+1}{N} \right) \right] - \phi'' \left( \frac{i}{N} \right) \right\} w_i \, ds +
$$

$$
(4.3) \quad + \frac{1}{2} \int_{0}^{t} \frac{1}{N} \sum_{i=1}^{N} \phi'' \left( \frac{i}{N} \right) N^2 \left( w_i - Av_j^{i+l_i} w_j \right) \, ds +
$$

$$
(4.4) \quad + \int_{0}^{t} \frac{1}{N} \sum_{i=1}^{N} \phi'' \left( \frac{i}{N} \right) N^2 \left[ Av_j^{i+l_i} w_j - \hat{a}(\bar{x}_{i,l}) Av_j^{i+l_i} (\Delta x)_j - Av_j^{i+l_i} \tau^J L_\infty f_r \right] \, ds +
$$

$$
(4.5) \quad + \frac{1}{2} \int_{0}^{t} \frac{1}{N} \sum_{i=1}^{N} \phi'' \left( \frac{i}{N} \right) N^2 \left( Av_j^{i+l_i} \tau^J L_\infty f_r \right) \, ds
$$

$$
(4.6) \quad + \frac{1}{2} \int_{0}^{t} \frac{1}{N} \sum_{i=1}^{N} \phi'' \left( \frac{i}{N} \right) N^2 \left[ \hat{a}(\bar{x}_{i,l}) Av_j^{i+l_i} (\Delta x)_j - \hat{a}(\bar{x}_{i,aN}) b^{-2} (\bar{x}_{i-bN,cN} - 2\bar{x}_{i,cN} + \bar{x}_{i+bN,cN}) \right] \, ds.
$$

We proceed to prove that each term in the sum above converges to 0 in probability.

**Martingale estimate (the term (4.1)).** We call $M_N(t)$ the term (4.1). From Ito formula we know that the process $\{M_N(t)\}_{t \geq 0}$ is a martingale and

$$
E^{\text{neq}}[M_N^2(T)] = E^{\text{neq}} \left[ \int_{0}^{T} \frac{1}{N} \sum_{i=1}^{N} N^4 \left[ \phi \left( \frac{i-1}{N} \right) - 2 \phi \left( \frac{i}{N} \right) + \phi \left( \frac{i+1}{N} \right) \right] a_i \, ds \right].
$$

The test function $\phi$ is chosen to have continuous second derivatives, therefore $E^{\text{neq}}[M_N^2(T)]$ is of order $O(1/N)$. We can use the Doob’s inequality

$$
P_{N,T}^{\text{neq}} \left\{ \sup_{0 \leq t \leq T} |M_N(t)| \geq \epsilon \right\} \leq \frac{1}{\epsilon^2} E[M_N^2(T)], \quad \epsilon > 0,
$$

to conclude that the martingale is negligible in the limit, i.e.,

$$
\lim_{N \to \infty} P_{N,T}^{\text{neq}} \left\{ \sup_{0 \leq t \leq T} |M_N(t)| \geq \epsilon \right\} = 0, \quad \epsilon > 0.
$$

**The term (4.2).** A straightforward computation involving the Chebyshev inequality and the entropy inequality (4.10) proves that the term (4.2) converges in probability to 0. The test function needs to have continuous fourth derivative.

**A technical Lemma.** We shall prove a Lemma that reduces the problem of establishing the negligibility of an event to finding that the largest eigenvalue of a
Lemma 4.1. Let \( \{x(t)\}_{t \geq 0} \) be the slope process with generator \( \mathcal{L} \). Under the assumption

\[
\limsup_{N \to \infty} \text{supspec}_{L^2(\nu_N^\text{eq})} \left( \alpha g + \frac{N^4}{N} \mathcal{L} \right) =
\]

\[
\limsup_{N \to \infty} \sup_{\rho, \mathcal{E}^{\text{eq}}[\rho^2] = 1} \left[ \alpha \mathcal{E}^{\text{eq}}[\rho^2] - \frac{N^4}{N} D_N(\rho) \right] \leq 0, \quad \alpha \neq 0
\]

it follows that the event \( \{ \left| \int_0^T g(x(s))ds \right| \geq \epsilon \} \) has negligible probability or:

\[
\lim_{N \to \infty} P_N^\text{neq} \left( \left| \int_0^T g(x(t))dt \right| > \epsilon \right) = 0, \quad \epsilon > 0.
\]

**Proof.** We can use Chebyshev inequality to reduce the proof of (4.8) to

\[
\lim_{N \to \infty} E_N^{\text{neq}} \left[ \left| \int_0^T g(x(t))dt \right| \right] = 0.
\]

Since we do not have much information about the initial nonequilibrium distribution we use the entropy inequality to replace the nonequilibrium distribution in (4.9) by the equilibrium distribution.

Before we continue, we remind that given two probability measures \( \nu \) and \( \mu \) on the same probability space such that \( \nu \) is absolutely continuous with respect to \( \mu \), we define the relative entropy of \( \nu \) with respect to \( \mu \) by \( H(\nu|\mu) = E_\mu \left[ \log \frac{d\nu}{d\mu} \right] \) where \( \frac{d\nu}{d\mu} \) is the Radon-Nikodym derivative of \( \nu \) relative to \( \mu \). The entropy \( H(\nu|\mu) \), always a positive quantity, is the optimal constant that makes the entropy inequality

\[
E_\nu[f] \leq \frac{1}{\alpha} \left\{ H(\nu|\mu) + \log E_\mu[e^{\alpha f}] \right\},
\]

true for any bounded, measurable function \( f \) and \( \alpha > 0 \). A trivial consequence of the entropy inequality (4.10) helps us to estimate \( \nu(A) \), where \( A \) is some event,

\[
\nu(A) \leq \frac{\log(2) + H(\nu|\mu)}{\log(1 + \frac{1}{\mu(A)})}.
\]

In our context we use the entropy inequality for the distribution of the process started in nonequilibrium and the distribution of the process started in the equilibrium. We have,

\[
E_N^{\text{neq}} \left[ \left| \int_0^T g(x(t))dt \right| \right] = E_N^{\text{eq}} \left[ \frac{1}{N\alpha} \left| \int_0^T N\alpha g(x(t))dt \right| \right] \leq
\]

\[
\leq \frac{1}{N\alpha} H(\nu_N^{\text{eq}}|\nu_N^{\text{eq}}) + \frac{1}{N\alpha} \log E_N^{\text{eq}} \left[ \exp \left( \alpha \left| \int_0^T N\alpha g(x(t))dt \right| \right) \right] \leq
\]

\[
\leq \frac{C}{\alpha} + \frac{1}{N\alpha} \log E_N^{\text{eq}} \left[ \exp \left( N\alpha \int_0^T g(x(t))dt \right) \exp \left( - N\alpha \int_0^T g(x(t))dt \right) \right].
\]

As a consequence of the inequality

\[
\log(a + b) \leq \max(\log(2a), \log(2b)) \leq \log(2) + \max(\log(a), \log(b))
\]
for two positive numbers \(a\) and \(b\), follows as soon as we have
\[
\limsup_{N \to \infty} \frac{1}{N} \log E^{eq} \left[ \exp \left( N \alpha \int_0^T g(x(t)) dt \right) \right] \leq 0, \quad \alpha \neq 0.
\]
A trivial consequence of Feynman-Kac proves our Lemma,
\[
(4.12) \quad \frac{1}{N} \log E^{eq} \left[ \exp \left( N \alpha \int_0^T g(x(t)) dt \right) \right] \leq T \text{supspec}_{L^2(\nu_N^x)}(\alpha g + \frac{N^4}{N} L_N).
\]

\[\square\]

The microscopic current \(w\) can be replaced by local average of currents (the term (4.3)). We shall show that the term (4.3) converges to zero in probability. As a consequence the current \(w\) is replaced by a local average of \(w\), that is closer to a deterministic value. \(w\) by itself is a single fluctuating random variable.

We check that the hypothesis of Lemma 4.11 is valid for the function \(g_{l,N} = \frac{1}{N} \sum_{i=1}^N \phi\left(\frac{j}{N}\right) N^2 (w_i - A_{j=i-1}^t w_j)\). Recall that if the test function \(\phi\) has continuous second-order derivative the quantity
\[
A_{j=i-1}^t \phi\left(\frac{j}{N}\right) - \phi\left(\frac{i}{N}\right) = \frac{\phi\left(\frac{i+1}{N}\right) - \phi\left(\frac{i}{N}\right) + \cdots + \phi\left(\frac{i-1}{N}\right) - \phi\left(\frac{i-2}{N}\right)}{2l+1}
\]
is of order \(O\left(\frac{1}{N^2}\right)\). Now, on integrating by parts twice, it follows for a fixed function \(\rho\), with \(E^{eq}[\rho^2] = 1\), that,
\[
\left| E^{eq} \left[ \rho^2 \frac{1}{N} \sum_{i=1}^N \phi\left(\frac{i}{N}\right) N^2 (w_i - A_{j=i-1}^t w_j) \right] \right|^2 = 4 \left| E^{eq} \left[ \frac{N^2}{N} \sum_{i=1}^N \sqrt{a_i} X_i(\rho) \sqrt{a_i} \left( A_{j=i-1}^t \phi\left(\frac{j}{N}\right) - \phi\left(\frac{i}{N}\right) \rho \right) \right] \right|^2 \leq 4 E^{eq} \left[ \sum_{i=1}^N \frac{N^4}{N^2} a_i \left( A_{j=i-1}^t \phi\left(\frac{j}{N}\right) - \phi\left(\frac{i}{N}\right) \right)^2 \rho^2 \right] D_N(\rho) \leq C \frac{t^4}{N^4} \frac{N^4}{N^2} D_N(\rho).
\]
Therefore,
\[
\limsup_{l,N \to \infty} \sup_{\rho, E^{eq}[\rho^2] = 1} \frac{E^{eq} \left[ \rho^2 \frac{N^2}{N} \sum_{i=1}^N \left( A_{j=i-1}^t \phi\left(\frac{j}{N}\right) - \phi\left(\frac{i}{N}\right) \right) w_i \right] \rho^2 D_N(\rho)}{D_N(\rho)} \to 0,
\]
and hence (4.11) is satisfied. Moreover,
\[
\lim_{l,N \to \infty} P_{N,T}^{\text{eq}} \left[ \left| \int_0^T \frac{1}{N} \sum_{i=1}^N \phi\left(\frac{i}{N}\right) N^2 (w_i - A_{j=i-1}^t w_j) dt \right| \geq \epsilon \right] = 0, \quad \epsilon > 0.
\]

Inserting the fluctuations (the term (4.5)). Let \(f(x_0, \ldots, x_s, \bar{x}_{0,l}) \in C^2(\mathbb{R}^{2s+1})\) be a local function that depends on the slope configuration in a box of size \(s\) and on the mean slope in a box of a large size \(l\). We use the notation \(l_1 = l - \sqrt{T}\) for a slightly smaller \(l\). We want to show that the fluctuations approach zero, in the limit or that
\[
(4.13) \quad \lim_{l,N \to \infty} P_{N,T}^{\text{eq}} \left[ \left| \int_0^T \frac{N^2}{N} \sum_{i=1}^N \phi\left(\frac{i}{N}\right) L_N(A_{j=i-1}^t \tau_j^T f) dt \right| \geq \epsilon \right] = 0, \quad \epsilon > 0.
\]
We apply Itô formula,
\[
\int_0^T \frac{N^2}{N} \sum_{i=1}^N \phi \left( \frac{i}{N} \right) L_N(Av_{j=i-l}^{i+l_1} \tau^j f) \, dt = 
\]
\[
= \frac{1}{N^2} \sum_{i=1}^N \phi \left( \frac{i}{N} \right) \left[ (Av_{j=i-l}^{i+l_1} \tau^j f)(x(T)) - (Av_{j=i-l}^{i+l_1} \tau^j f)(x(0)) \right] + \frac{1}{N^2} M_N(t). 
\]

The summand in (4.14) converges to zero, as the function \( f \) is bounded. The second part of (4.14) approaches zero because the \( L^2 \) norm of \( \frac{M_N(T)}{N^2} \) is of order \( \mathcal{O}(\frac{L}{N}) \), as we can see below. Let \( g = \frac{1}{N} \sum_{i=1}^N \phi \left( \frac{i}{N} \right) (Av_{j=i-l}^{i+l_1} \tau^j f)(x) \), then
\[
E_{\text{eq}} \left[ \frac{M_N(T)^2}{N^4} \right] = E_{\text{eq}} \left[ \int_0^T L_N g - 2gL_N g \, dt \right] = E_{\text{eq}} \left[ \int_0^T \sum_{i=1}^N a_i (X_i g)^2 \, dt \right] = 
\]
\[
= \frac{1}{N^2} E_{\text{eq}} \left[ \int_0^T \sum_{i=1}^N a_i \left( \sum_{i=1}^N \phi \left( \frac{k}{N} \right) X_i (Av_{j=k-l}^{k+l_1} \tau^j f) \right)^2 \, dt \right].
\]
The function \( Av_{j=k-l}^{k+l_1} \tau^j f \) is a cylinder function that depends just on the sites \( n \) such that \( k - l \leq n \leq k + l \). Therefore the vector field \( X_i \) is zero when acting on most of the summands inside \( Av_{j=k-l}^{k+l_1} \tau^j f \). There are no more than \( 2l \) sites \( k \) such that \( X_i (Av_{j=k-l}^{k+l_1} \tau^j f) \neq 0 \). We put all these arguments together to conclude that \( E_{\text{eq}}[M^2(t)/N^4] \leq C l^2 / N \).

**Replacing the current by the Laplacian of the slope field (the term (4.4)).** In our model the instantaneous current, \( w \), cannot be written as the discrete Laplacian \( \tau h - 2h - \tau^{-1} h \) of some local function \( h \), thus we use the method of Varadhan \cite{17} for computing the hydrodynamic scaling limit of our model. The main idea is that the current decomposes as
\[
w = \hat{a} (\bar{x}_t) \Delta x + L_\infty f
\]
for a suitable coefficient \( \hat{a}(\bar{x}_t) \), where \( \bar{x}_t \) is the average slope in a cube centered at the origin of microscopic side \( l \). The equation (4.15) is known in the literature as the fluctuation-dissipation equations. We have explained before that terms of the form \( L_\infty f \) have no effect on the macroscopic scale. A new feature is characteristic to our model due to the complex interaction of the system: after filtering off the fluctuations from the current we are left with the Laplacian of some function and not a gradient, as happened for models previously considered in the literature. The precise meaning of (4.15) is given below.

Our aim is to prove the existence of a sequence \( \{ f_r \}_{r \geq 0} \) of local functions and of the transport coefficient \( \hat{a} \) such that
\[
\lim_{r,l,N \to \infty} \sup_{r,l,N} P_{N,A} \left\{ \left| \int_0^T \frac{N^2}{N} \sum_{i=1}^N \phi \left( \frac{i}{N} \right) \left[ Av_{j=i-l}^{i+l_1} \tau^j w_j - \hat{a}(\bar{x}_{i,t}) Av_{j=i-l}^{i+l_1} (\Delta x)_j - Av_{j=i-l}^{i+l_1} \tau^j L_\infty f_r \right] \, dt \right| \geq \epsilon \right\} = 0.
\]

The local function \( f_r(x_{s-l}, \ldots, x_s, \bar{x}_{0,l}) \) depends on two arguments, the mean slope \( \bar{x}_{0,l} \) in a box of size \( l \) and the slope field inside a box of size \( s \), the size \( s \) being
much smaller than \( l \). We can assume that the operator \( L_\infty \) does not act on the first argument \( \bar{x}_{0,j} \), since we can show that the action of the operator \( L_\infty \) at the boundary sites is negligible. To be more precise, \( A_{\lambda}^{j+1} \) stands for
\[
\frac{L_\infty f_{r}(\bar{x}_{i,l}, \tau_{i-l,i} x) + \cdots + L_\infty f_{r}(\bar{x}_{i,l}, \tau_{i+1,i} x)}{2l_1 + 1}.
\]
Note that the function \( A_{\lambda}^{j+1} \) depends just on the value of the field inside the box centered at \( i \) and of size \( l \).

As before we use Lemma \( \ref{lem:4.2} \) to conclude that the event \( \\{ \lambda_\nu < \nu \} \) has negligible probability in the limit. An additional difficulty shows up. If \( \lambda_\nu \) is the largest eigenvalue of the perturbation, \( L + \epsilon W \), of a negative operator \( L \) with principal eigenvalue 0, the eigenvalue \( \lambda_\nu \) has the formal series expansion,
\[
\lambda_\nu = 0 + \epsilon E_\nu[W] + \epsilon^2 < W, (-L)^{-1}W >_\nu + O(\epsilon^3),
\]
Hence if the potential \( W \) has mean zero, one expects \( \lim_{\nu \to 0} \lambda_\nu \epsilon^{-2} = < W, (-L)^{-1}W >_\nu \). Fortunately for suitable potential \( W \) the central limit variance \( < W, (-L)^{-1}W >_\nu \) converges to zero.

We shall need in our context, a particular result about the largest eigenvalue of a perturbation operator, whose proof is found in Quastel \( \cite{quastel} \).

**Lemma 4.2.** Let \( W \) be a real potential that satisfies
\[
< u, W u >_\nu \leq l^{-1/2} D_{l_1} (u)^{1/2} ||u||_2, \quad l < C \epsilon^{-2/5}
\]
for some \( C \) small enough, or
\[
||W||_\infty \leq C, \quad K \leq (C \epsilon)^{-1/5}.
\]
Provided that the generator \( l^4 L_l \) has spectral gap of order one the following estimate holds
\[
(4.17) \quad \epsilon^{-2} l^{-5} \sup_{vL^2(\nu)} (l^4 L_l - \epsilon l^5 W) \leq l < W, (-L_l)^{-1}W >_\nu + O(1).
\]

**Spectral gap.** Indeed the generator \( l^4 L_l \) of our model, defined by \( \ref{eq:4.6} \), has a spectral gap of order 1. The proof is standard by the method of Bakry-Emery (see Chang and Yau \( \cite{chang-yau} \) or Deuchel and Stroock \( \cite{deuchel-stroock} \)).

The operator \( L_l \) is an unbounded operator defined on the subspace \( C_0^\infty(\mathbb{R}^{2+l}) \) of the Hilbert space \( L^2(\nu_{y_0,l}^g) \) and is negative definite, with spectrum included in the negative semiaxis of the real line. 0 is an eigenvalue of the operator \( L_l \) but the eigenspace corresponding to this eigenvalue is quite large, being infinite dimensional.

It is not hard to see that we can write the Hilbert space \( L^2(\nu_{y_0,l}^g) \) as the direct sum \( \bigoplus_{g \in R^2} L^2(\nu_{y_0,l}^g) \). Moreover because of the ergodicity of the dynamics \( \ref{eq:2.6} \) on the level sets of the function \( y_{0,l} = (y_{0,l}^1, y_{0,l}^2) \), we know that the eigenspace corresponding to zero of the restriction of the operator \( L_l \) onto each Hilbert subspace \( L^2(\nu_{y_0,l}^g) \), is one dimensional. The next eigenvalue of the restriction \( L_l |_{L^2(\nu_{y_0,l}^g)} \) is a negative number. The distance between the largest eigenvalue and the next largest eigenvalue of the operator \( L_l |_{L^2(\nu_{y_0,l}^g)} \) is called the spectral gap of the operator, because it is the gap in the spectrum of the operator.

**Lemma 4.3.** (Spectral gap) There is a universal constant \( C \) that does not depend on the conserved quantities \( y_{0,l} = (y_{0,l}^1, y_{0,l}^2) \) \( \in \mathbb{R}^2 \) such that
\[
(4.18) \quad C |l|^2 E_{\nu_{y_0,l}^g} [\rho^2] \leq < (-L_l) \rho, \rho >_{\nu_{y_0,l}^g}.
\]
for any mean-zero function $\rho$, $E_{\nu_{y,s}}[\rho] = 0$.

Since the generator $L_1$ of our model has a spectral gap, see Lemma 4.3, we can introduce the central limit theorem variance in our context.

**Definition 4.1.** Suppose we have a cylinder function $f(x-s, \ldots, x_s)$ such that $E_{\nu_{y,s}}[f] = 0$ for all possible values of $y \in \mathbb{R}^2$. Recall that $\nu_{y,s}$ is the canonical measure in a box centered at 0 and size $s$ defined in section 2. The central limit theorem variance of $f$ on the box $\Lambda_{i,l}$ is defined to be:

$$V_{i,l}(f, y) = 2(2l)^2 \sum_{j=l}^{\infty} \left( \lambda_1 f + \lambda_1 \Delta f \right) > \nu_{y,i,l}.$$

If the box $\Lambda_{i,l}$ is centered at 0 then we use the shorter notation $V_{l}(f, y)$ for the CLT-variance. At this point we stress that $V_{i,l}(f, y)$ is a local function depending on the field inside of $\Lambda_{i,l}$, more precisely depending on the conserved quantities $y_{i,l} = (y_{i,l}^1, y_{i,l}^2)$, the mean slope and the linear mean of the slope field.

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The strategy is to give a bound for the largest eigenvalue of a perturbation operator in terms of the CLT-variance $V_{i,l}(f, y)$. Extra care must be taken because the CLT-variance $V_{i,l}(f, y)$ is uniformly-in-$y$ small on bounded sets and not on unbounded sets.

The canonical measure $\nu_{i,l,y}$ for our model has been obtained by conditioning the grand canonical measure on the configurations with fixed mean slope and fixed linear mean slope in a box centered at $i$ and of size $l$. The second conditioning makes the canonical measure not to have identical marginals. Actually the expected values of the marginals depends linearly on the site. However the finite-dimensional marginals of the canonical measure converges toward the finite-dimensional marginals of the grand canonical distribution as the size of the box approaches infinity, see Lemma 5.1. To benefit of this fact we will replace the CLT-variance $V_{i,l}(f, y)$ in a box of size $l$ with its expectation $E^{\nu_{l}}[V_{i,l}(f, y)|y_k]$ with respect to the canonical measure $\nu_{l,k}$ in a box of larger size $k$. We let $k$ go first to infinity. We formalize below.

Let us define the function $g$ as

$$g = \frac{N^2}{N} \sum_{i=1}^{N} \phi \left( \frac{i}{N} \right) \left[ \lambda v_{i+l}^{j-l} \lambda w_j - \hat{a}(\pi_{i,l}) \lambda v_{i+l}^{j-l} (\Delta x) \lambda - \lambda v_{i+l}^{j-l} \lambda L_{\infty} f \right].$$

Thanks to Lemma 4.1 the event (4.16) has negligible probability if

$$\limsup_{r,i,N \to \infty} \sup \text{spec}_{L^2(\nu_{N})}(g + 2\beta \frac{N^4}{N} L_N) \leq 0, \quad \beta > 0.$$

We write the operator $g + 2\beta \frac{N^4}{N} L_N$ as a sum of operators and we estimate the size of the principal eigenvalue of each operator in the sum.

$$g + 2\beta \frac{N^4}{N} L_N = \Omega_1 + \Omega_2 + \Omega_3 + \Omega_4$$
where,

\[
\begin{align*}
\Omega_1 &= g - \frac{1}{\beta N} \sum_{i=1}^{N} \phi \left( \frac{i}{N} \right)^2 \mathcal{V}_{i,l}(w - \hat{a}(\mathbf{x}_i,t)\Delta x - L_\infty f_r, y) + \beta \frac{N^4}{N} L_N \\
\Omega_2 &= \frac{1}{\beta N} \sum_{i=1}^{N} \phi \left( \frac{i}{N} \right)^2 \mathcal{V}_{i,l}(w - \hat{a}(\mathbf{x}_i,t)\Delta x - L_\infty f_r, y) - \\
&\quad - \mathcal{E} \left[ \mathcal{V}_{i,l}(w - \hat{a}(\mathbf{x}_i,t)\Delta x - L_\infty f_r, y) | y_{i,k} \right] + \beta \frac{N^4}{N} L_N \\
\Omega_3 &= \frac{1}{\beta N} \sum_{i=1}^{N} \phi \left( \frac{i}{N} \right)^2 \mathcal{E} \left[ \mathcal{V}_{i,l}(w - \hat{a}(\mathbf{x}_i,t)\Delta x - L_\infty f_r, y) | y_{i,k} \right] 1_{|y_{i,k}| \geq \delta} \\
\Omega_4 &= \frac{1}{\beta N} \sum_{i=1}^{N} \phi \left( \frac{i}{N} \right)^2 \mathcal{E} \left[ \mathcal{V}_{i,l}(w - \hat{a}(\mathbf{x}_i,t)\Delta x - L_\infty f_r, y) | y_{i,k} \right] 1_{|y_{i,k}| \leq \delta}.
\end{align*}
\]

The operators \( \Omega_2, \Omega_3 \) and \( \Omega_4 \) are understood as multiplication operators.

The operator \( \Omega_1 \). Assume that \( M \) is an upper bound for the test function \( |\phi| \). Define

\[
g_{i,l} = A_{r+1} \mathcal{V}_{j=i-l, j} - \hat{a}(\mathbf{x}_i,t) A_{r+1} \mathcal{V}_{j=i-l, j} (\Delta x)_j - A_{r+1} \mathcal{V}_{j=i-l, j} \tau L_\infty f_r.
\]

We have

\[
\text{sup} \text{sup}_{L^2(\mathbb{R}^2)} (\Omega_1) \leq \\
\leq \sup_{|\lambda| \leq M} \left( \mathcal{E} \left[ \frac{\lambda N^2 \mathcal{V}_{l}(w - \hat{a}(\mathbf{x}_t)\Delta x - L_\infty f_r, y) \rho^2}{\beta} \right] - \frac{\beta N^4}{2l + 1} D_{r, l} (\rho) \right) \leq \\
\leq \sup_{|\lambda| \leq M} \sup_{y \in \mathbb{R}^2} \left( \sup_{|\lambda| \leq M} \mathcal{E} \left[ \frac{\lambda N^2 \mathcal{V}_{l}(w - \hat{a}(\mathbf{x}_t)\Delta x - L_\infty f_r, y) \rho^2}{\beta} \right] - \frac{\beta N^4}{2l + 1} D_{r, l} (\rho) \right) \leq \\
\leq \frac{\lambda^2}{\beta} \mathcal{V}_{l}(w - \hat{a}(\mathbf{x}_t)\Delta x - L_\infty f_r, y).
\]

Integrating by parts we can show that there is a constant \( C \), not depending on \( l \) and the values of the conserved quantities, such that for each density \( \rho \):

\[
\mathcal{E} \left[ g_{0,l} \rho^2 \right] \leq C \frac{\sqrt{D_{r, l}(\rho)}}{\sqrt{2l + 1}}.
\]

Hence the hypothesis of Lemma 4.2 is satisfied and

\[
\lim_{N \to \infty} \sup_{\mathcal{E} \left[ \lambda N^2 \mathcal{V}_{l}(w - \hat{a}(\mathbf{x}_t)\Delta x - L_\infty f_r, y) \rho^2 \right]} = \\
= \frac{1}{(2l + 1)^{l^4}} \lim_{N \to \infty} \sup_{\mathcal{E} \left[ g_{0,l} \rho^2 \right]} \left[ \frac{\lambda (2l + 1)^l t^4}{N^2} D_{r, l} (\rho^2) \right] \leq \\
\leq \frac{\lambda^2}{\beta} \mathcal{V}_{l}(w - \hat{a}(\mathbf{x}_t)\Delta x - L_\infty f_r, y).
\]
The convergence on the line above is uniform over all possible values of the conserved quantity \( y_1 = (y_1^1, y_1^2) \in \mathbb{R}^2 \), therefore the principal eigenvalue of the operator \( \Omega_2 \) becomes negative as \( N \to \infty \).

**The operator \( \Omega_2 \).** Let us call

\[
v_{i,k} = V_{i,l}(w - \hat{a}(\mathbf{r}, l)) \Delta x - L_{\infty,f_r}, y) - E^{eq}[V_{i,l}(w - \hat{a}(\mathbf{r}, l)) \Delta x - L_{\infty,f_r}, y]|y_{i,k}].
\]

We observe that \( \Omega_2 \) contains \( v_{i,k} \) without being multiplied with a factor \( N^2 \). Then,

\[
\sup_{N \to \infty} \sup_{\rho \in \mathbb{R}^2} E^{eq}\left[N^4 \beta v_{0,k} \rho^2 \left( \frac{1}{\beta} \sigma_{i,k} \right) - \frac{\beta N^4}{2k+1} D_{\sigma_{i,k}}(\rho) \right] = 0
\]

Moreover, \( v_{i,k} \) is a bounded function (see Lemma 3.2) and \( E^{eq}[v_{0,k}|y_k] = 0 \). We can apply Lemma 3.2

\[
\lim_{N \to \infty} N^4 \sup_{\rho \in \mathbb{R}^2} E^{eq}\left[N^4 \beta v_{0,k} \rho^2 \left( \frac{1}{\beta} \sigma_{i,k} \right) - \frac{\beta N^4}{2k+1} D_{\sigma_{i,k}}(\rho) \right] = 0.
\]

The convergence on the line above is uniform over all possible values of the conserved quantities \( y_1 = (y_1^1, y_1^2) \); therefore \( \lim_{N \to \infty} \sup_{\rho \in \mathbb{R}^2} E^{eq}[v_{0,k}|y_k] = 0 \).

**The operator \( \Omega_3 \).** We refer to Lemma 4.2. Rather than proving that

\[
\lim_{N \to \infty} \sup_{\rho \in \mathbb{R}^2} E^{eq}\left[N^4 \beta v_{0,k} \rho^2 \left( \frac{1}{\beta} \sigma_{i,k} \right) - \frac{\beta N^4}{2k+1} D_{\sigma_{i,k}}(\rho) \right] = 0
\]

we will show

(4.22)

\[
\lim_{\delta \to 0} \frac{1}{N} \log E^{eq}\left[\exp\left(T \Omega_3 \right) ds\right] = 0,
\]

where

\[
\Omega_3 = \sum_{i=1}^{N} \frac{1}{\beta N} \phi \left( \frac{i}{N} \right) E^{eq}[V_i(w - \hat{a}(\mathbf{r}) \hat{\Delta x} - L_{\infty,f_r}, y)|y_k]|y_{i,k} \geq \delta]
\]

The equation (4.22) follows from the estimations

\[
\frac{1}{N} \log E^{eq}\left[\exp\left(NT \Omega_3 \right) ds\right] \leq \frac{1}{N} \log E^{eq}\left[\exp\left(NT \Omega_3 \right) ds\right] = \frac{1}{N} \log E^{eq}\left[\exp\left(\sum_{i=1}^{N} \phi \left( \frac{i}{N} \right) E^{eq}[V_i(w - \hat{a}(\mathbf{r}) \Delta x - L_{\infty,f_r}, y)|y_k]|y_{i,k} \geq \delta]\right) \leq \frac{1}{N} \log E^{eq}[\exp(M^2 E^{eq}[V_i(w - \hat{a}(\mathbf{r}) \Delta x - L_{\infty,f_r}, y)|y_k]|y_{i,k} \geq \delta)]
\]

The dominated convergence theorem can be applied (see the note at the end of section 5) and gives us,

\[
\lim_{\delta \to 0} \log E^{eq}[\exp(M^2 E^{eq}[V_i(w - \hat{a}(\mathbf{r}) \Delta x - L_{\infty,f_r}, y)|y_k]|y_{i,k} \geq \delta)] = 0
\]

**The operator \( \Omega_4 \).** The main purpose of sections 6 is to show that there exists a sequence of functions \( \{f_r\}_{r \geq 0} \) such that for any \( \delta > 0 \)

\[
\lim_{r \to \infty} \sup_{\rho \in \mathbb{R}^2} E^{eq}[V_i(w - \hat{a}(\mathbf{r}) \Delta x - L_{\infty,f_r}, y)|y_k] = 0,
\]

and
The term (4.6) It follows that the term (4.6) converges to zero in probability from the following two lemmas. We refer the reader to Bertini, Olla and Landim [2] or Guo, Papanicolaou and Varadhan [7] for the proof. The proof uses mainly the entropy inequality [1,10] and consequence of Feynman-Kac formula [4,12].

Lemma 4.4. (Local ergodicity) Let \( f \) be a cylinder function. Define \( \bar{f} : \mathbb{R} \to \mathbb{R} \) to be the function \( \bar{f}(\alpha) = E_{\nu^{eq}}[f] \). Then for any \( \delta > 0 \), \( \phi : \mathbb{T} \to \mathbb{R} \) a smooth function,

\[
\lim_{k,N \to \infty} P_{N,T}^{\text{eq}} \left\{ \int_0^1 \left| \frac{1}{N} \sum_{i=1}^N \phi \left( \frac{i}{N} \right) \left( \tau^i f(x(s)) - \bar{f}(\text{Av}^{i+k}_{j-1-k} x_j(s)) \right) \right| ds \geq \delta \right\} = 0.
\]

Lemma 4.5. (Two-Block estimate) For any continuous function \( g : \mathbb{R} \to \mathbb{R} \), let

\[
F_{k,a,N} = \left\{ \int_0^t \text{Av}^{i+k}_{j-1} \text{Av}^{i+aN}_{j-i-k} (g(\text{Av}^{i+k}_{j-1-k} x_j(s)) - g(\text{Av}^{i+k}_{j-1-k} x_j(s)))^2 ds \geq \delta \right\}.
\]

Then for any \( \delta > 0 \),

\[
\lim_{k \to \infty, a \to 0, N \to \infty} P_{N,T}^{\text{eq}} \{ F_{k,a,N} \} = 0.
\]

5. Computation of the central limit theorem variances

In this section we compute the value of the limit:

\[
\lim_{l \to \infty} \sup_{|\alpha| \leq \delta} E_{\nu^{eq}}[V_l(f,y)]
\]

for a particular class of cylinder functions \( f \) to be described later.

We shall need a result that relates canonical and grand canonical measures, known as the equivalence of ensemble. The equivalence of ensemble says that asymptotically the marginal in a fixed box of the canonical measure is the marginal of the grand canonical measure.

Lemma 5.1. (Equivalence of ensemble) Let \( f(x_-, \ldots, x_a) \) be a bounded, local function. Then, for any \( \epsilon > 0 \), there exist \( N_1 \in \mathbb{N} \) and \( \delta_1 > 0 \) such that

\[
|E_{\nu^{eq}_{\alpha,k}}[f] - E_{\nu^{gc}_{\alpha}}[f]| \leq \epsilon
\]

as long as \( k > N_1 \), \( l > N_1 \) and \( |y_k^1 - \alpha| < \delta_1, y_k^2 \in \mathbb{R} \).

We shall need a new notation, namely \( X^*_0 \) for the adjoint of the vector field \( X_0 = \partial_{x_1} - 2\partial_{x_0} + \partial_t \) with respect to the inner-product of the Hilbert space \( L^2(\alpha \nu^{eq}_N) \). The adjoint is given by the formula,

\[
X^*_0(h) = -X_0(ah) - (V'(x_{-1}) - 2V'(x_0) + V'(x_1))ah.
\]

Note that the current \( w \) is equal to \( X^*_0(2) \), and the slope Laplacian \( \Delta x \) is equal to \( X^*_0\left(\frac{1}{2}\right) \).

Lemma 5.2. Let \( h(x_-, \ldots, x_a) \) be a bounded, cylinder function such that \( f = X^*_0(h) \in L^2(\nu^{eq}_{\alpha,s}) \) for all \( \alpha \in \mathbb{R} \). Then there exists \( C(h) < \infty \) that depends just on the function \( h \) such that

\[
\sup_{l,y_0,1 \in \mathbb{R}^2} \frac{1}{l} < (-L_l)^{-1} \left( \sum_{|\mathbf{j}| \leq l - \sqrt{l}} \tau^j f, \sum_{|\mathbf{i}| \leq l - \sqrt{l}} \tau^i f \right) \leq C(h).
\]
In addition if the function $h$ is bounded away from zero, $h \geq C > 0$, we have the lower bound, $C_1(h) > 0$ that depends just on $h$,

$$C_1(h) \leq \sup_{l, y_0 \in \mathbb{R}^2} \frac{1}{l} \left< (-L_l)^{-1}(\sum_{|j| \leq l-\sqrt{T}} \tau^j f), \sum_{|i| \leq l-\sqrt{T}} \tau^i f > \nu_{\alpha, i}^c \right>.$$  

Proof. Call $V_l(f, y) = \frac{1}{l} < (-L_l)^{-1}(\sum_{|j| \leq l-\sqrt{T}} \tau^j f), \sum_{|i| \leq l-\sqrt{T}} \tau^i f > \nu_{\alpha, i}^c$. We use the variational formula

$$V_l(f, y) = \sup_u \frac{\left< u, \sum_{|j| \leq l-\sqrt{T}} \tau^j f \right>^2 \nu_{\alpha, i}^c}{\|D\nu_{\alpha, i}^c(u)\|}.$$  

We have

$$\frac{\left< u, \sum_{|j| \leq l-\sqrt{T}} \tau^j f > \nu_{\alpha, i}^c \right>}{\|D\nu_{\alpha, i}^c(u)\|} \leq \frac{E\nu_{\alpha, i}^c \left[ \sum_{|j| \leq l-\sqrt{T}} a_j (\tau^j f)^2 \right]}{\|D\nu_{\alpha, i}^c\|} \leq 2 l C(h) \|D\nu_{\alpha, i}^c(u)\|,$$

therefore the upper bound for the variance is established.

For the lower bound, we may chose a particular function $u$ in (5.3), namely $u = \sum_{i=-l}^{l} \frac{1}{2} x_i$. Note that $X_i(u) = 1$ for any $-l + 1 \leq i \leq l - 1$. It follows,

$$\frac{\left< u, \sum_{|j| \leq l-\sqrt{T}} \tau^j f > \nu_{\alpha, i}^c \right>}{\|D\nu_{\alpha, i}^c(u)\|} = \frac{E\nu_{\alpha, i}^c \left[ \sum_{|j| \leq l-\sqrt{T}} \alpha \tau^j f \right]}{\|D\nu_{\alpha, i}^c\| \|\sum_{|j| \leq l-\sqrt{T}} \alpha \tau^j f \|} \geq C_1(h).$$

We have just used that $a$, and $h$ are functions bounded away from zero, and $a$ is a bounded function. \hfill \Box

Note. Lemma 5.2 is true for any local function $f = \sum_{i=-s}^{s} X_i^a(h_j)$, where the functions $h_j$ are bounded, local functions.

**Definition 5.1.** For a bounded, local functions $f$ such that $E\nu_{\alpha, i}^c[f] = 0$, for all possible values of $y \in \mathbb{R}^2$, we define the semi-norm:

$$\langle \langle f \rangle \rangle_{\alpha} = \limsup_{l \rightarrow \infty, y_k \rightarrow \alpha} \frac{1}{l} E\nu_{\alpha, i}^c \left[ < (-L_l)^{-1}(\sum_{|j| \leq l-\sqrt{T}} \tau^j f), \sum_{|i| \leq l-\sqrt{T}} \tau^i f > \nu_{\alpha, i}^c \right].$$

We saw in Lemma 5.2 that $\langle \langle f \rangle \rangle_{\alpha}$ is a finite number as long as $f$ is equal to $X_0^a(h)$, where $h$ is a bounded, local function. By polarization we can extend the semi-norm $\langle \langle \cdot \rangle \rangle_{\alpha}$ to a semi-inner product $\langle \langle \cdot, \cdot \rangle \rangle_{\alpha}$. For the remaining part of this section we compute the value of the semi-norm $\langle \langle f \rangle \rangle_{\alpha}$ for certain functions $f$.

**Lemma 5.3.** Assume that $g$ is the bounded cylinder function with bounded first derivatives, $f = L_\infty g$, $w = X_0(a) - (V'(x_1) - 2V'(x_0) + V'(x_1))a$ and $\Delta x = x_1 - x_0 + x_1$, then the following identities hold,

a) $\langle \langle L_\infty g \rangle \rangle_{\alpha}^2 = E\nu_{\alpha}^c \left[ a(x_1, x_0, x_1) \left( X_0(\sum_{j \in Z} \tau^j g) \right)^2 \right],$

b) $\langle \langle w \rangle \rangle_{\alpha}^2 = 4 E\nu_{\alpha}^c [a(x_1, x_0, x_1)],$

c) $\langle \langle L_\infty g, w \rangle \rangle_{\alpha} = 2 E\nu_{\alpha}^c [a(x_1, x_0, x_1) X_0(\sum_{j \in Z} \tau^j g)],$

d) $\langle \langle L_\infty g, \Delta x \rangle \rangle_{\alpha} = 0,$
e) $\langle (u, \Delta x) \rangle_\alpha = 4$.

**Proof.** One checks directly using equivalence of ensemble lemma 5.1 and the asymptotic shift invariance of $\nu_{y, k}^\alpha$ that the relations a)-e) hold. In particular, for d) and e), it is important to notice that $\Delta x = x_{-1} - 2x_0 + x_1 = X_0^b(\frac{1}{\alpha})$. □

**Definition 5.2.** Define the Hilbert space $\mathcal{H}_\alpha$ to be the closed linear span in $L^2(\alpha d\nu_{b, \alpha}^\alpha)$ of the function 1 and functions $\xi_g = X_0(\sum_{j \in \mathbb{Z}} \tau^j g)$, where $g$ is a bounded local function with bounded first derivatives.

It is not hard to see that if $f$ is equal to $X_0^b(h)$ then,

$$\langle (f, L_\infty g) \rangle_\alpha = E_{\nu_{b, \alpha}^\alpha} \left[ a\text{Proj}_{\mathcal{H}_\alpha}(2h)X_0 \left( \sum_{j \in \mathbb{Z}} \tau^j g \right) \right], \quad \langle (f, w) \rangle_\alpha = E_{\nu_{b, \alpha}^\alpha} [a\text{Proj}_{\mathcal{H}_\alpha}(2h)2].$$

On both lines above $\text{Proj}_{\mathcal{H}_\alpha}$ stands for the projection operator in the subspace $\mathcal{H}_\alpha$. We are left to calculate $\langle (f) \rangle_\alpha$ for a function $f = X_0^b(h)$. As we will show in the following lemma, $\langle (f) \rangle_\alpha = E_{\nu_{b, \alpha}^\alpha}[a(\text{Proj}_{\mathcal{H}_\alpha}(2h))^2]$.

**Lemma 5.4.** Suppose $f$ is equal to $X_0^b(h)$, where $h$ is a bounded cylinder function. The value of the semi-norm of $f$,

$$\langle (f) \rangle_\alpha = E_{\nu_{b, \alpha}^\alpha}[a(\text{Proj}_{\mathcal{H}_\alpha}(2h))^2].$$

**Proof.** Using Cauchy-Schwartz inequality it follows that,

$$\langle (f) \rangle_\alpha = \lim_{l \to \infty} \inf \left. \frac{1}{l} E_{\nu_{b, \alpha}^\alpha} \left[ (-L_l)^{-1} \left( \sum_{|i| \leq l - \sqrt{l}} \tau^i f \right) \left( \sum_{|j| \leq l - \sqrt{l}} \tau^j f > \nu_{b, \alpha}^\alpha \right) \right] \geq E_{\nu_{b, \alpha}^\alpha}[a(\text{Proj}_{\mathcal{H}_\alpha}(2h))^2].$$

Consider $g = \sum_{|j| \leq l - \sqrt{l}} \tau^j f$, where $f = X_0^b(h)$ then

$$\frac{1}{l} < (-L_l)^{-1} \left( \sum_{|i| \leq l - \sqrt{l}} \tau^i f \right), \sum_{|j| \leq l - \sqrt{l}} \tau^j f > \nu_{b, \alpha}^\alpha \leq \frac{\sup_{\rho, D\nu_{b, \alpha}^\alpha}(\rho)}{l} \frac{\sum_{|j| \leq l - \sqrt{l}} \tau^j f > \nu_{b, \alpha}^\alpha}{2l^2} = \frac{\rho_l \sum_{|j| \leq l - \sqrt{l}} \tau^j f > \nu_{b, \alpha}^\alpha}{2l^2} = \frac{1}{2l^2} E_{\nu_{b, \alpha}^\alpha} \left[ \sum_{|j| \leq l - \sqrt{l}} a(x_{-j}, x_j, x_j)X_j(\rho_l)\tau^j h \right]^2.$$
\[
\leq 2E_{\nu_{a,b}^{\infty}}[uau] = 2E_{\nu_{a,b}^{\infty}}[au\text{Proj}_{H_{a}}h] \leq 2E_{\nu_{a,b}^{\infty}}[au^{2}E_{\nu_{a,b}^{\infty}}[a(\text{Proj}_{H_{a}}h)^{2}]] \leq E_{\nu_{a,b}^{\infty}}[a(\text{Proj}_{H_{a}}2h)^{2}].
\]

The key point that has allowed us to write the above inequalities is that the function \(u\) has the property \(X_{a}(\tau^{b}u) = X_{b}(\tau^{a}u)\) for all integers \(a\) and \(b\), and a function with this property belongs to \(H_{a}\), (see Lemma 2.1 or Savu [14, 15]). The next estimate shows that \(X_{a}(\tau^{b}u) = X_{b}(\tau^{a}u)\) is valid in a weak sense. Consider a smooth test function \(\phi\), with bounded first derivatives. Assume \(a > b\). We have

\[
< X_{a}(\tau^{b}u_{1}), \phi >_{\nu_{a,b}^{\infty}} = < \frac{1}{2l} \sum_{|j| \leq l-1/\sqrt{l}} \tau^{b-j}(\rho_{j}), X_{a}^{*}X_{b}^{*}\phi >_{\nu_{a,b}^{\infty}},
\]

and

\[
< X_{a}(\tau^{b}u_{1}) - X_{b}(\tau^{a}u_{1}), \phi >_{\nu_{a,b}^{\infty}} = < \frac{1}{2l} \sum_{j=b+1+l-1/\sqrt{l}}^{a+l-1/\sqrt{l}} \tau^{j}(\rho_{j}) - \frac{1}{2l} \sum_{j=b-l+1/\sqrt{l}}^{a-l+1/\sqrt{l}} \tau^{j}(\rho_{j}), X_{a}^{*}X_{b}^{*}\phi >_{\nu_{a,b}^{\infty}} =
\]

\[
< \frac{1}{2l} \sum_{j=b+1+l-1/\sqrt{l}}^{a+l-1/\sqrt{l}} \tau^{j}X_{b-j}(\rho_{j}) - \frac{1}{2l} \sum_{j=b-l+1/\sqrt{l}}^{a-l+1/\sqrt{l}} \tau^{j}X_{b-j}(\rho_{j}), X_{a}^{*}X_{b}^{*}\phi >_{\nu_{a,b}^{\infty}}.
\]

By Cauchy-Schwartz inequality, we obtain

\[
< X_{a}(\tau^{b}u_{1}) - X_{b}(\tau^{a}u_{1}), \phi >_{\nu_{a,b}^{\infty}} \leq E_{\nu_{a,b}^{\infty}} \left[ \left( \frac{1}{2l} \sum_{j=b+1+l-1/\sqrt{l}}^{a+l-1/\sqrt{l}} \tau^{j}X_{b-j}(\rho_{j}) + \frac{1}{2l} \sum_{j=b-l+1/\sqrt{l}}^{a-l+1/\sqrt{l}} \tau^{j}X_{b-j}(\rho_{j}) \right) \right] E_{\nu_{a,b}^{\infty}}[(X_{a}^{*}\phi)^{2}]
\]

\[
\leq C(\alpha - b) \left( \frac{1}{(2l)\sqrt{l}} \sum_{j=b-a+1+l-1/\sqrt{l}}^{-1} \sum_{j=b-a+1+l-1/\sqrt{l}}^{l-1+1/\sqrt{l}} E_{\nu_{a,b}^{\infty}}[(X_{a}^{*}\rho_{j})] + \tau^{j}X_{b-j}(\rho_{j}) \right)^{2} \leq C(\alpha - b) \frac{E_{\nu_{a,b}^{\infty}}[(X_{a}^{*}\rho_{j})]^{2}}{2l} \leq C(\alpha - b) \frac{E_{\nu_{a,b}^{\infty}}[(X_{a}^{*}\rho_{j})]^{2}}{2l}.
\]

As \(l\) converges to infinity, the sequence \(\{u_{i}\}_{1}\) approaches \(u\) in the weak sense. Combining this fact with (5.3) we can establish \(X_{a}(\tau^{b}u) = X_{b}(\tau^{a}u)\).

Now we can conclude that if we have a local function \(f\) such that \(f = X_{a}^{*}(h)\) then

\[
\langle (f, f) \rangle_{\alpha} = \lim_{l \to \infty, y_{k}^{i} \to \alpha} \frac{1}{l} E_{\nu_{a,b}^{\infty}} \left[ < (-L_{i})^{-1} \left( \sum_{|j| \leq l-1/\sqrt{l}} \tau^{j}f \right), \sum_{|j| \leq l-1/\sqrt{l}} \tau^{j}f >_{\nu_{a,b}^{\infty}, i} \right] =
\]

\[
= E_{\nu_{a,b}^{\infty}}[a(\text{Proj}_{H_{a}}(2h)^{2})].
\]

The results proved so far in this section allow us to conclude that if \(g\) is a local function equal to \(X_{b}^{*}(h)\) and \(b(y_{k}^{i})\) is a coefficient that depends on the mean-slope in a box of size \(k\) then

\[
\langle (g + bw + L_{\infty}f, g + bw + L_{\infty}f) \rangle_{\alpha} = \lim_{l \to \infty, y_{k}^{i} \to \alpha} E_{\nu_{a,b}^{\infty}}[\nu_{i}(g + bw + L_{\infty}f, y)] =
\]

\[
= E_{\nu_{a,b}^{\infty}}[(\text{Proj}_{H_{a}}(2h) + 2b + X_{b}(\sum_{i \in Z} \tau^{i}f))^{2}].
\]
Proof.

We calculate

Let us introduce the notation

Therefore we can find for each \( \alpha \)

Let

It is important to notice that the transport coefficient is also given by the formula:

\[
\partial_{\tau t} x_i \leq \mathcal{A}(\tau l, B) \quad \forall (\tau t) \in \mathbb{E}.
\]

\[
\lim_{\tau \rightarrow \infty} \langle (\Delta x + bw + L_\infty f) \rangle \alpha = 0.
\]

\[
\lim_{l,B \rightarrow \infty} \sup_{|\alpha| \leq \delta + 1} A(l,B)(\alpha) = 0.
\]

Therefore we can find for each \( \alpha \) a cylinder function \( f(\alpha) \) such that \( A(f(\alpha), \alpha) \leq \epsilon \), if \( |\alpha| \leq \delta + 1 \). We extend \( f \) to be zero on \( |\alpha| > \delta + 1 \). Then on \( |\alpha| > \delta + 1 \) we have that \( A(f(\alpha), \alpha) = E_{v^\infty}(a(1 + X(\sum_{i \in \mathbb{Z}} \tau f)^2)) - \hat{a}(\alpha) \) and \( A_l,B(\alpha) = \inf_{f \in \mathcal{F}_{l,B}} A(f, \alpha) \). The function \( A_l,B(\alpha) \) is upper semicontinuous and nonincreasing in \( l \) and \( B \), and for each \( \alpha \in \mathbb{R} \), \( \lim_{l,B \rightarrow \infty} A_l,B(\alpha) = 0 \), therefore,

\[
\lim_{l,B \rightarrow \infty} \sup_{|\alpha| \leq \delta + 1} A_l,B(\alpha) = 0.
\]

Note. From the equivalence of ensemble lemma we know that for any bounded cylinder function \( f \) and any \( \epsilon > 0 \), there exist \( N_1 \in \mathbb{N} \) and \( \delta_1 > 0 \) such that

\[
|\mathcal{A}(\tau l, B)| = |\mathcal{A}(\tau l, B)| = \mathcal{A}(\sum_{i \in \mathbb{Z}} \tau f) \leq \epsilon,
\]

as long as \( k > N_1, l > N_1 \), and \( |\sum_{i \in \mathbb{Z}} \tau f| \leq \delta_1 \). Then Lemma helps us to conclude that for any \( \epsilon > 0 \) there exists a bounded cylinder function \( f \) such that

\[
\limsup_{l,k \rightarrow \infty} \sup_{|y_k| \leq \delta} \mathcal{A}(\tau l, B)| = \mathcal{A}(\sum_{i \in \mathbb{Z}} \tau f) \leq 2\epsilon,
\]

and

\[
\mathcal{A}(\tau l, B)| = \mathcal{A}(\sum_{i \in \mathbb{Z}} \tau f) \leq 3\epsilon.
\]

\( a^* \) on the line above is the upper bound for the function \( a \) in the hypothesis of Lemma 5.5.

Properties of the transport coefficient. The transport coefficient \( \hat{a} \) has been defined as the unique real number such that

\[
\inf_{f} \langle (w - \hat{a}(\alpha) \Delta x - L_\infty f) \rangle_a = 0.
\]

It is important to notice that the transport coefficient is also given by the formula:

\[
\hat{a}(\alpha) = \frac{\langle (w, \Delta x) \rangle_a}{\langle (\Delta x, \Delta x) \rangle_a} = \frac{E_{v^\infty}[2\hat{a}]}{E_{v^\infty}[(\text{Proj}_H(\hat{a}))^2]a} = \frac{1}{E_{v^\infty}[(\text{Proj}_H(\hat{a}))^2]a} =
\]
= \inf_{\gamma, \theta} \mathcal{E}_{\alpha} \left[ (\gamma + X_0(\sum_{j \in \mathbb{Z}} \tau_j g))^2 a \right] = \inf_{\gamma, \theta} \mathcal{E}_{\alpha} \left[ \frac{1}{a} (\gamma + X_0(\sum_{j \in \mathbb{Z}} \tau_j g))^2 a \right] = \\
(5.6) = \inf_{\gamma} \mathcal{E}_{\alpha} \left[ a(x_{-1}, x_0, x_1)(1 + X_0(\sum_{j \in \mathbb{Z}} \tau_j g))^2 \right].

6. HYDRODYNAMIC LIMIT OF THE MODEL FOR SURFACE ELECTRO Migration

In this section we will show that the model for surface electromigration has a hydrodynamic scaling limit as well. We will prove Theorem 2.2. The proof is inspired by Quastel [13]. From Cameron-Martin-Girsanov formula we can find the Radon-Nikodym derivative inspired by Quastel [13]. From Cameron-Martin-Girsanov formula we can find the Radon-Nikodym derivative

\[
\frac{dP_{\text{eq}}}{dP_{\text{eq}}} = \frac{d\nu_{\text{eq}}}{d\nu_{\text{eq}}} \exp \left( \frac{1}{2} \sum_{i=1}^{N} \int_{0}^{T} E \left( t, \frac{i}{N} \right) \sqrt{a(x_{i-1}, x_i, x_{i+1})} dB_i - \frac{1}{8} \sum_{i=1}^{N} \int_{0}^{T} E^2 \left( t, \frac{i}{N} \right) a(x_{i-1}, x_i, x_{i+1}) dt \right).
\]

and the relative entropy

\[
H(P_{\text{eq}} | P_{\text{eq}}) = H(\nu_{\text{eq}} | \nu_{\text{eq}}) + E_{P_{\text{eq}}} \left[ \frac{1}{8} \sum_{i=1}^{N} \int_{0}^{T} E^2 \left( t, \frac{i}{N} \right) a(x_{i-1}, x_i, x_{i+1}) dt \right].
\]

Hence there exists a constant C such that \(H(P_{\text{eq}} | P_{\text{eq}}) \leq CN\). We can use the inequality \(\phi_{\text{eq}}\alpha\) and the superexponential estimates \(\phi_{\text{eq}}\alpha\) to conclude that the sequence of probability measures \(\{P_{\text{eq}}\}_{N \geq 0}\) is tight in the topology of the space \(\mathcal{X}\). Denote by \(P_{E,T}\) a weak limit of a subsequence of \(\{P_{\text{eq}}\}_{N \geq 0}\).

We proceed further to identify the limit \(Q_{E,T}\). To save space we ignore the time dependence of the test function \(\varphi\).

\[
\frac{1}{N} \sum_{i=1}^{N} \varphi \left( \frac{i}{N} \right) x_i(T) - \frac{1}{N} \sum_{i=1}^{N} \varphi \left( \frac{i}{N} \right) x_i(0) = \\
= \int_{0}^{T} \frac{1}{2N} \sum_{i=1}^{N} \left( N^2 w_i + E \left( t, \frac{i}{N} \right) a_i \right) \phi'' \left( \frac{i}{N} \right) dt + M_N(T) = \\
= \int_{0}^{T} \frac{N^2}{2N} \sum_{i=1}^{N} (w_i - \dot{a}(\bar{x}, \tau_i)(\Delta x)_i - L_\infty \tau_i f_i) \phi'' \left( \frac{i}{N} \right) dt + M_N(T)
\]

(6.1) \(+ \int_{0}^{T} \frac{N^2}{2N} \sum_{i=1}^{N} L_{N,E,T} \tau_i f_i \phi'' \left( \frac{i}{N} \right) dt + \)

(6.2) \(+ \int_{0}^{T} \frac{1}{2N} \sum_{i=1}^{N} \left[ E \left( t, \frac{i}{N} \right) (a_i - \dot{a}(\bar{x}, \tau_i)) + N^2 (L_\infty \tau_i f_i - L_{N,E,T} \tau_i f_i) \right] \phi'' \left( \frac{i}{N} \right) dt + \)

\(+ \int_{0}^{T} \frac{1}{2N} \sum_{i=1}^{N} \dot{a}(\bar{x}, \tau_i) \left[ N^2 (\Delta x)_i + E \left( t, \frac{i}{N} \right) \right] \phi'' \left( \frac{i}{N} \right) dt \)

Above \(\{M_N(t)\}_{t \geq 0}\) is a martingale and as in section 4 we can prove that is negligible in the limit. Because of the entropy inequality \(\alpha)\) and the superexponential
Recall that in the proof of Lemma 5.5 we introduced equation (6.7) sup
\[ \lim \sup r,l,N \to \infty \frac{1}{N} \log E^q [\exp \left( \left| \int_0^T N^2 \sum_{i=1}^N \left[ \text{Av}_{i+l}^j - \hat{a}(\bar{x}_{i,l}) \text{Av}_{i+l}^j \text{d}s \right] \right| \right) ] \leq 0, \]
the event on line (6.10) is negligible under \( P_{N,E,T}^{\text{neq}} \). For the new model that we have the negligible fluctuations are \( L_{N,E} f \) and not \( L_{\infty} f \) (i.e., the term (6.1) has no contribution towards the limit of the model). The contribution coming from nontrivial fluctuations \( L_{N,E} f - L_{\infty} f \) are gathered in the coefficient in front of the vector field \( E \) in the nonlinear equation (2.14). Because of this reason, the coefficient turns out to be the transport coefficient \( \hat{a} \), defined by the variational formula (2.11) and not \( \hat{a}(\alpha) = E_{\nu^c_{\alpha}} [a] \)

We shall show that the sequence of smooth local functions \( \{ f_r \}_{r \geq 0} \) introduced in the Note right after the Lemma (5.5) can be chosen to have the additional property that the term (6.2) is negligible in the limit. Note that the term (6.2) is negligible if \( \limsup_{r,k,N \to \infty} P_{N,E,T}^{\text{neq}} \{ |F(T)| > \epsilon \} = 0 \) where,

\[ F(T) = \int_0^T \frac{1}{2N} \sum_{i=1}^N E \left( t, \frac{i}{N} \right) \left[ a_i g_i - \hat{a}(\bar{x}_{i,k}) \right] dt, \quad g_i = 1 + X_i \left( \sum_{j \in \mathbb{Z}} \tau^j f_r \right) \]

We can write

\[ F(T) = \int_0^T \frac{1}{2N} \sum_{i=1}^N E \left( t, \frac{i}{N} \right) (a_i g_i - E_{\nu^c_{\alpha^*}} [a_i g_i]) + \]

\[ + \int_0^T \frac{1}{2N} \sum_{i=1}^N E \left( t, \frac{i}{N} \right) (E_{\nu^c_{\alpha^*}} [a_i g_i] - \hat{a}(\bar{x}_{i,k})) 1_{|\tilde{g}_{i,k}| \leq \delta} + \]

\[ + \int_0^T \frac{1}{2N} \sum_{i=1}^N E \left( t, \frac{i}{N} \right) (E_{\nu^c_{\alpha^*}} [a_i g_i] - \hat{a}(\bar{x}_{i,k})) 1_{|\tilde{g}_{i,k}| \geq \delta} \]

Recall that in the proof of Lemma (5.5) we introduced \( A(f, \alpha) \) to be the difference \( E_{\nu^c_{\alpha^*}} [a(1 + X_0 (\sum_{i \in \mathbb{Z}} r^i f)^2)] \) - \( \hat{a}(\alpha) \). We will show that we can modify the sequence \( \{ f_r \}_{r \geq 0} \) to have the properties

\[ \sup_{\alpha^*, r} A(f_r(\alpha), \alpha) \leq a^*, \quad \sup_{\alpha^*, r} |E_{\nu^c_{\alpha^*}} [ag_0] - \hat{a}(\alpha)| \leq 5a^*, \]

\[ \lim_{r \to \infty} \sup_{|\alpha| \leq \delta} A(f_r(\alpha), \alpha) = 0, \text{ and} \]

\[ \lim_{r \to \infty} \sup_{|\alpha| \leq \delta} |E_{\nu^c_{\alpha^*}} [ag_0] - \hat{a}(\alpha)| = 0. \]

We write,

\[ E_{\nu^c_{\alpha^*}} [ag_0] - \hat{a}(\alpha) = (E_{\nu^c_{\alpha^*}} [ag_0] - E_{\nu^c_{\alpha^*}} [ag_0^2]) + (E_{\nu^c_{\alpha^*}} [ag_0^2] - \hat{a}(\alpha)) = \]

\[ = E_{\nu^c_{\alpha^*}} [ag_0 (g_0 - 1)] + A(f_r(\alpha), \alpha). \]
Since $|E_{\nu^g}[\alpha g_0(g_0 - 1)]| \leq 4a^*$ and $A(f_r(\alpha), \alpha) \leq a^*$ uniform in $\alpha$ and $r$, (6.7) follows. Consider $f^*(\alpha)$ to be the minimizer of $A(f, \alpha)$ among local functions of $x_l$ through $x_r$. $f^*$ has the property that for any $f_r$, 

$$E_{\nu^g}[a \left( 1 + X_0(\sum_{j \in \mathbb{Z}} \tau^j f^*) \right) X_0(\sum_{j \in \mathbb{Z}} \tau^j f_r)] = 0.$$ 

Now, thanks to Cauchy-Schwartz inequality,

$$|E_{\nu^g}[\alpha g_0(g_0 - 1)]| = \left| E_{\nu^g}[a X_0(\sum_{j \in \mathbb{Z}} \tau^j f_r) X_0(\sum_{j \in \mathbb{Z}} \tau^j (f^* - f_r))] \right| \leq E_{\nu^g}[a (X_0(\sum_{j \in \mathbb{Z}} \tau^j f_r))^2]^{1/2} E_{\nu^g}[a (X_0(\sum_{j \in \mathbb{Z}} \tau^j (f^* - f_r)))^2]^{1/2}.$$

We modify $f_r$ such that

$$\sup_{|\alpha| \leq \delta} E_{\nu^g}[a \left( X_0(\sum_{j \in \mathbb{Z}} \tau^j (f^* - f_r)) \right)^2] \leq \frac{\epsilon^2}{6a^*}.$$

To conclude (6.7) is negligible because of local ergodicity lemma (6.8) is negligible because for any fixed $\delta$ we have (6.9) and (6.9) shrinks to zero once $\delta$ becomes very large. Also it is important to have property (6.7). Our result follows.

\[ \square \]

**Acknowledgement**

I would like to thank Professor Jeremy Quastel for proposing me this problem and for introducing me to interacting particle systems.

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