Edge colouring models for the Tutte polynomial and related graph invariants

A.J. Goodall

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Abstract

For integer \( q \geq 2 \), we derive edge \( q \)-colouring models for (i) the Tutte polynomial of a graph \( G \) on the hyperbola \( H_q \), (ii) the symmetric weight enumerator for group-valued \( q \)-flows of \( G \), and (iii) a more general vertex colouring model partition function that includes these polynomials and the principal specialization order \( q \) of Stanley’s symmetric monochrome polynomial. We describe the general relationship between vertex and edge colouring models, deriving a result of Szegedy and generalizing a theorem of Loebl along the way. In the second half of the paper we exhibit a family of non-symmetric edge \( q \)-colouring models defined on \( k \)-regular graphs, whose partition functions for \( q \geq k \) each evaluate the number of proper edge \( k \)-colourings of \( G \) when \( G \) is Pfaffian.

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1 Introduction

1.1 Background

Partition functions of edge colouring and vertex colouring models are graph invariants with a definition motivated by statistical physics, where the vertices or edges of a graph $G$ are “sites” and colours are “states”. The states interact with each other along the edges in vertex colouring models and at vertices in edge colouring models. A configuration (assignment of states to all sites) is given a weight dependent on the interactions between states and possibly also on the states themselves. The partition function of the model is obtained by summing the weights over all possible configurations. For example, the partition function of the $q$-state Potts model \cite{23} is defined by a vertex colouring model, where the interaction along an edge depends on whether the endpoint states are the same or different.

Freedman, Lovász and Schrijver \cite{9} have shown that a graph parameter can be realized as the partition function of a real-valued vertex colouring model if and only if it is vertex reflection positive and has exponential rank-connectivity. Szegedy \cite{32} proved the parallel statement, that graph parameters arise as partition functions of real-valued edge colouring models if and only if they are edge reflection positive and multiplicative over disjoint graphs. (To be reflection positive means, approximately, to take non-negative values on graphs with mirror symmetry.) Vertex and edge colouring models thus play an important rôle in the algebra of quantum graphs, for which see the papers already cited and for example \cite{6}, \cite{18}, \cite{17}, \cite{16}.

Schrijver \cite{26}, in the more general context of hypergraphs, extends the scope of vertex colouring models to directed graphs (where the order in which colours appear on an edge now matters) and edge colouring models to graphs where the edges incident with a vertex come in a given order. Examples of partition functions of vertex colouring models on directed graphs include the number of oriented colourings, or the number of weak oriented colourings (monochromatic edges allowed) that feature in Stanley’s \cite{28} evaluation of the chromatic polynomial at negative integers.

An embedding of a graph on an orientable surface is defined by its vertex rotations (a clockwise order of incident edges as seen from a fixed side of the surface). For embedded graphs it would be natural to consider edge
colouring models that depend on the order up to cyclic permutation in which colours appear at a vertex. Given a 4-regular plane graph with a chequerboard colouring of its faces, the Viergruppe (elementary 2-group order 4) fixes the property of vertex transitions being black, white or crossing. Edge colouring models with this symmetry might feature in partition functions for evaluations of a transition polynomial. (For transition polynomials, of which the Penrose polynomial is an instance, see for example [1], [2], [12], [13], [24].)

Edge colouring models that depend only on the order up to even permutation in which colours appear at a vertex play a rôle in proper edge \( k \)-colouring \( k \)-regular Pfaffian graphs [7], [20]. The overall parity of the permutations determined by the distinct colours appearing at each vertex turns out to be fixed for such graphs. This property allows us to derive some non-symmetric edge \( q \)-colouring models with a partition function that gives the number of proper edge \( k \)-colourings of the graph.

1.2 Outline

The first purpose of this article is to give a general account of vertex and edge colouring models and their interrelation, and in particular to provide examples of such models for some important graph polynomials, supplementing those given for example in [6], [17], [32].

The second purpose of this article is to present some examples of edge colouring models that depend on the order in which colours appear around a vertex, while still giving a partition function which is an interesting graph parameter.

We begin in Section 2 by introducing relevant notation and concepts.

In Section 3 edge colouring models are derived for a family of partition functions of vertex colouring models, including two branches descending from the chromatic polynomial. This family includes the Tutte polynomial\(^1\) in its specialization to the \( q \)-state Potts model, treated in Section 3.2 along with its generalization to the symmetric weight enumerator of abelian-group-valued \( q \)-flows (Theorem 3.4). Szegedy’s result [32] that any symmetric real-valued vertex colouring model has an edge colouring model is included in Section 3.3. The family described in Section 3.4 includes the principle specializations of finite order of Stanley’s symmetric monochrome polynomial [30] (the latter a generalization of his symmetric chromatic function [29]). Theorem 3.9 is a generalization of a theorem of Loebl [15], itself a generalization of van der Waerden’s eulerian subgraph expansion of the Ising model [37]. We finish in Theorem 3.10 with an edge colouring model that unifies the aforementioned polynomials.

\(^{1}\)See for example [40], [41], [42], [27] for an account of the significance of this polynomial in combinatorics.
In Section 4 we derive the already advertised example of a family of non-symmetric edge colouring models. The partition function of the edge \( q \)-colouring model given in Theorem 4.10 evaluates a particular coefficient of Petersen’s graph monomial \( 22 \) of the line graph of \( G \), considered more recently by Alon and Tarsi \( 4 \), Matiyasevich \( 19 \) and Ellingham and Goddyn \( 7 \). For the restricted class of \( k \)-regular graphs that admit Pfaffian labellings \( 20, 36 \) (a class that includes planar graphs and the Petersen graph), this coefficient is up to sign equal to the number of proper edge \( k \)-colourings of \( G \). The two main theorems are to be found in Section 4.3 after the preparatory work of Sections 4.1 and 4.2.

1.3 Cubic graphs

The main theorems of this paper are most easily described at this stage by giving their statements for a cubic (3-regular) graph \( G \). These special cases give the flavour of the more substantial results that are proved in the body of the paper without the need for much preparation or notation. For the purpose of illustration we shall focus on \( F(G; 4) \), the number of nowhere-zero \( \mathbb{F}_4 \)-flows (or \( \mathbb{Z}_4 \)-flows) of a cubic graph \( G \).

Let \( Q \) be an abelian group of order \( q \). For a graph \( G \), the number of nowhere-zero \( Q \)-flows is independent of the structure of \( Q \), equal to \( F(G; q) \), the flow polynomial evaluated at \( q \). It is well known (see e.g. \( 40 \)) that, for any graph \( G \),

\[
F(G; q) = (-1)^{|E|} q^{-|V|} \sum_{\text{vertex } q\text{-colourings of } G} (1 - q)^{\# \text{monochrome edges}},
\]

where an edge is monochrome if its endpoint vertices have the same colour. This is an example of the partition function of a vertex \( q \)-colouring model.

A consequence of Corollary 3.5 (by setting \( s = 0 \)) is that \( F(G; q) \) is also the partition function of an edge colouring model, which for cubic \( G \) has a particularly concise expression:

**Proposition 1.1.** For a 3-regular graph \( G = (V, E) \) and \( q \geq 2 \),

\[
F(G; q) = q^{-|E|/2|V|} \sum_{\text{edge } q\text{-colourings of } G} (1-q)^{\# \text{monochrome vertices}} (1-q/2)^{|V|}\# \text{rainbow vertices},
\]

where a vertex is monochrome (rainbow) in an edge \( q \)-colouring if the colours on its incident edges are the same (different).

Theorem 3.4 gives an edge \( q \)-colouring model for a generalization of \( F(G; q) \), namely the symmetric weight enumerator of the set of \( Q \)-flows of \( G \). (The Tutte polynomial on the hyperbola \( H_q \) is the Hamming weight enumerator of the set of \( Q \)-flows. An edge \( q \)-colouring model for this is given in Corollary 3.5.)
Vertex colouring models are uniquely determined [32], in the sense that different vertex colouring models give partition functions that differ for some graph \( G \). Edge colouring models on the other hand may be rotated by an orthogonal transformation and yet preserve the partition function (Szegedy [32], and Theorem 3.6 below). A reverse phenomenon is seen to occur here, however, for when \( G \) is cubic \( F(G;4) \) has an infinite number of vertex 4-colouring models all of which, on applying Theorem 3.4, come from the same edge colouring model \((q = 4\) in Proposition 1.1).

**Proposition 1.2.** For a 3-regular graph \( G = (V,E) \) and indeterminates \( s, t \),

\[
(st)^{|E|/3} F(G; 4) = 4^{-|V|} \sum_{\text{vertex } \mathbb{F}_4\text{-colourings}} (1+s+t)^{#0}(1-s-t)^{#1}(-1-s+t)^{#\omega}(-1+s-t)^{#\overline{\omega}},
\]

where the exponent \(#a\) for \( a \in \mathbb{F}_4 = \{0, 1, \omega, \overline{\omega}\}\) counts the number of edges whose endpoint colours differ by \( a \).

[This proposition is not proved below, but is a simple application of MacWilliams duality identity (given as equation 5 later) on observing that all nowhere-zero \( \mathbb{F}_4 \)-flows of \( G \) have an equal number of each non-zero element.]

The vertex colouring model in Proposition 1.2 depends on the structure of \( \mathbb{F}_4 \) in order to define the weight it gives to an edge in a vertex 4-colouring. It loses this dependence on setting \( s = t = u/2 \), when it can be written as a vertex \( \mathbb{Z}_4 \)-colouring model:

\[
u^{|V|} F(G; 4) = (-1)^{|E|/2} \sum_{\text{vertex } \mathbb{Z}_4\text{-colourings}} (-1 - u)^{#0}(-1 + u)^{#2},
\]

where \(#0\) is the number of monochrome edges and \(#2\) denotes the number of edges with colour difference 2.

So far all our vertex and edge colouring models have been symmetric. They do not depend on the order in which colours appear on the endpoints of an edge (for vertex colouring models) or the order of colours appearing on the edges incident with a vertex (for edge colouring models). This is common to all the vertex colouring models for which edge colouring models are derived in Section 3.

In Section 4 we consider edge colouring models that depend on the order of colours up to even permutation. For cubic graphs this coincides with order up to cyclic permutation, and this has a natural interpretation in terms of clockwise and anticlockwise rotations.

Suppose a cubic graph \( G \) is embedded in an orientable surface. Proper edge 3-colourings of \( G \) are the same as nowhere-zero \( \mathbb{F}_4 \)-flows of \( G \), and from this comes Tait’s [31], [33] equivalent statement of the Four Colour Theorem, that every planar cubic graph has a proper edge 3-colouring. By Vizing’s
theorem [39] every \( k \)-regular graph has a proper edge \((k+1)\)-colouring. Theorem 4.11 relates proper edge \( k \)-colourings to proper edge \((k+1)\)-colourings of \( k \)-regular graphs, its special case for plane cubic graphs being the proposition that follows here. Theorem 4.10 more generally gives for any \( q \geq k \) an edge \( q \)-colouring model for the number of proper edge \( k \)-colourings of a \( k \)-regular graph that admits a Pfaffian labelling [20], [30].

An orientable embedding of \( G \) is described by its vertex rotations, giving a clockwise order of edges around a vertex. Let \( 0 < 1 < 2 < 3 \) be ordered up to cyclic permutation, i.e. as the cycle \((0 \ 1 \ 2 \ 3)\). Given a proper edge 4-colouring of \( G \) with colours \( \{0, 1, 2, 3\} \), the three colours that appear at a vertex come either in a clockwise sense, i.e. consistently with the cyclic order \((0 \ 1 \ 2 \ 3)\), or in an anticlockwise sense, i.e. in the reverse order \((3 \ 2 \ 1 \ 0)\). Colours can appear in a clockwise order (e.g. \(0, 2, 3\)), or in an anticlockwise order (e.g. \(0, 3, 2\)).

**Proposition 1.3.** Let \( G = (V, E) \) be a plane cubic graph. Then

\[
(-4)^{|E|/3} F(G; 4) = \#\{\text{even proper edge } 4\text{-colorings of } G\} - \#\{\text{odd proper edge } 4\text{-colorings of } G\},
\]

where a proper edge 4-colouring of \( G \) is even (odd) if there are an even (odd) number of vertices at which the colours appear in an anticlockwise order.

## 2 Preliminaries

For more on elementary Fourier analysis see for example [35]. The assumed graph theory is standard. For boundaries and coboundaries, and also for a related perspective on vertex colouring models, see [5].

### 2.1 Abelian groups and the Fourier transform

Let \( Q \) be a finite additive abelian group order \( q \), which we assume also has the structure of a commutative ring with unity.

The set \( \mathbb{C}^Q \) of all functions \( f : Q \to \mathbb{C} \) is an inner product space with Hermitian inner product

\[
\langle f, g \rangle = \sum_{a \in Q} f(a) \overline{g(a)},
\]

where the bar denotes complex conjugation. The subspace \( \mathbb{R}^Q \) has Euclidean inner product

\[
(f, g) = \sum_{a \in Q} f(a) g(a),
\]

and we may also use the notation \((, )\) in the larger space \( \mathbb{C}^Q \).
The pointwise product of $f$ and $g$ is defined by $f \cdot g(a) = f(a)g(a)$ and their convolution by $f \ast g(a) = \sum_{b \in Q} f(a - b)g(b)$.

Elements of $\mathbb{C}^Q$ will be regarded interchangeably as functions and as column vectors indexed by elements of $Q$. The indicator function of $P \subseteq Q$ is denoted by $1_P$.

A character of $Q$ is a homomorphism $\chi : Q \to \mathbb{C}^\times$ from $Q$ to the multiplicative of $\mathbb{C}$. The set of characters $\overline{Q}$ forms a group isomorphic to $Q$. A character $\chi$ is a generating character for $Q$ if the isomorphism $Q \to \overline{Q}$ can be realized by the map $a \mapsto \chi_a$ defined by $\chi_a(b) = \chi(ab)$. Given a generating character $\chi$, the matrix

$$F := q^{-1/2}(\chi(ab))_{a,b \in Q}$$

is the Fourier transform on the space $\mathbb{C}^Q$. The following are three key properties of the Fourier transform:

- $F$ is a unitary matrix, i.e. $F^\dagger F = I$, so that $\langle Ff, Fg \rangle = \langle f, g \rangle$.
- Pointwise products are transformed by $F$ into convolutions, $F(f \cdot g) = q^{-1/2}Ff \ast Fg$.
- If $P$ is a subgroup of $Q$ and $1_P$ is the indicator function of $P$ then $F1_P = |P|1_{PS}$, where $PS = \{a \in Q : \forall b \in P \chi(ab) = 1\}$.

The $d$-fold Cartesian product $Q^d$ is itself an abelian group and a module over $Q$. Multiplication on $Q^d$ is componentwise and the dot product of elements $a = (a_1, \ldots, a_d)$ and $b = (b_1, \ldots, b_d)$ is defined by $a \cdot b = a_1b_1 + \cdots + a_db_d$. If $\chi$ is a generating character for $Q$ then $\chi^{\otimes d}$, defined for $(a_1, \ldots, a_d) \in Q^d$ by $\chi^{\otimes d}(a_1, \ldots, a_d) = \chi(a_1) \cdots \chi(a_d)$, is a generating character for $Q^d$ and $\chi^{\otimes d}(ab) = \chi(a \cdot b)$. The matrix $F^{\otimes d}$ is the matrix for the Fourier transform on the space $\mathbb{C}^{Q^d}$, which, as above, satisfies $(F^{\otimes d}f, F^{\otimes d}g) = (f, g)$ and $F^{\otimes d}(f \cdot g) = q^{-d/2}Ff \ast Fg$ for functions $f, g \in \mathbb{C}^{Q^d}$. Unitary transformations are isometries of the Hermitian inner product space $\mathbb{C}^{Q^d}$, and orthogonal transformations isometries of the Euclidean inner product space $\mathbb{R}^{Q^d}$. If $U$ is orthogonal, i.e. $UU^T = I$, then $(U^{\otimes d}f, U^{\otimes d}g) = (f, g)$ for $f, g \in \mathbb{C}^{Q^d}$.

For $C \subseteq Q^d$ the orthogonal submodule to $C$ is defined by

$$C^\perp = \{a \in Q^d : \forall c \in C \ a \cdot c = 0\},$$

and, given that $Q$ has a generating character $\chi$, we have $C^\perp = C^\sharp = \{a \in Q^d : \forall c \in C \ \chi(a \cdot c) = 1\}$, so that

$$F^{\otimes d}1_C = q^{-d/2}|C|1_{C^\perp}.$$

Define the set of all sequences on $Q$ by

$$Q^* = \bigcup_{d \in \mathbb{N}} Q^d,$$
and the following subsets:

\[
{\text{MONOCHROME}} = \bigcup_{d \in \mathbb{N}} \{(a_1, a_2, \ldots, a_d) \in \mathbb{Q}^d : a_1 = a_2 = \cdots = a_d\},
\]

\[
{\text{ZERO-SUM}} = \bigcup_{d \in \mathbb{N}} \{(a_1, a_2, \ldots, a_d) \in \mathbb{Q}^d : a_1 + a_2 + \cdots + a_d = 0\}.
\]

Note that, for each \(d \in \mathbb{N}\), \(\text{ZERO-SUM} \cap \mathbb{Q}^d = (\text{MONOCHROME} \cap \mathbb{Q}^d)^\perp\), so that

\[
\mathbb{Q}^d_{d-1} = q^{1-d/2} \mathbb{Q}^d_{d-1} \cap \mathbb{Q}^d.
\]

Finally, it will be useful to have the following notation. Suppose that

\(U\) is a linear transformation of \(\mathbb{C}^Q\), so that the \(d\)-fold tensor product \(U \otimes \mathbb{C}^d\)

is a linear transformation of \(\mathbb{C}^Q\) for each \(d \in \mathbb{N}\). Then the map \(f \mapsto fU\)

is defined to be the unique algebra homomorphism \(\mathbb{C}^Q \to \mathbb{C}^Q\) satisfying

\(fU = Uf\) for \(f \in \mathbb{C}^Q\). In other words, \(fU\) is defined for all \(d \in \mathbb{N}\) and \(z \in \mathbb{Q}^d\)

by \(fU(z) = U \otimes d f(z)\). In this way, for example, given a function \(f \in \mathbb{C}^Q\),

the function \(f^F\) is the Fourier transform of \(f\) taken in the appropriate space according to the argument of \(f\).

### 2.2 Graphs and half-edges

Let \(G = (V, E)\) be a graph with set of vertices \(V\) and set of edges \(E\). Edges \(e \in E\) are subsets of \(V\) of size 2 (a multiset of size 2 on 1 vertex if \(e\) is a loop). Given \(e\) we shall write \(v \in e\) when \(v\) is an endpoint of \(e\), and, given \(v\), we shall write \(e \ni v\) when \(e\) is incident with \(v\). The number of connected components of \(G\) is denoted by \(k(G)\) and its rank by \(r(E) = |V| - k(G)\). For \(A \subseteq E\) the rank of the induced subgraph \((V, A)\) is denoted by \(r(A)\).

The set of half-edges of \(G\) is defined by \(H = \{(v, e) : v \in e\}\). There are two natural ways to partition the set of half-edges. The first is according to incidence with vertices \(v \in V\), with blocks \(H(v) := \{(u, e) \in H : u = v\}\) of size \(|H(v)| = d(v)\) the degree of \(v\). The second is according to incidence with edges \(e \in E\), and here a block is a set \(H(e) := \{(v, f) \in H : f = e\}\) of size 2. If \(e = \{u, v\}\) then \(H(e) = \{(u, e), (v, e)\}\) while if \(e = \{v\}\) is a loop then \(H(e)\) contains two copies of the half-edge \((v, e)\).

Since \(\{H(e) : e \in E\}\) is a partition of \(H\) we have the isomorphism

\[
\mathbb{C}^Q_H \cong \bigotimes_{e \in E} \mathbb{C}^{Q^{H(e)}},
\]

where \(\mathbb{C}^{Q^{H(e)}} \cong \mathbb{C}^{Q^2}\). In other words, if \(g \in \mathbb{C}^Q_H\) then we can write \(g = \bigotimes_{e \in E} g_e\) for functions \(g_e \in \mathbb{C}^{Q^{H(e)}}\), i.e., for \(z = (z_h : h \in H) \in Q^H\),

\[
g(z) = \prod_{e \in E} g_e(z_h : h \in H(e)).
\]
Likewise, since \( \{ H(v) : v \in V \} \) is a partition of \( H \) we also have the isomorphism

\[
\mathbb{C}^{Q^H} \cong \bigotimes_{v \in V} \mathbb{C}^{Q^{H(v)}},
\]

where \( \mathbb{C}^{Q^{H(v)}} \cong \mathbb{C}^{Q^d} \) when \( v \) is a vertex of degree \( d \).

In the other direction, a function \( f : Q^* \to \mathbb{C} \) defines a function \( f \otimes V : Q^H \to \mathbb{C} \) given by

\[
f \otimes V(z) = \prod_{v \in V} f(z_h : h \in H(v)).
\]

Similarly, a function \( g : Q^2 \to \mathbb{C} \) extends to a function \( g \otimes E : Q^H \to \mathbb{C} \) defined for \( z \in Q^H \) by

\[
g \otimes E(z) = \prod_{e \in E} g(z_h : h \in H(e)).
\]

### 2.3 The boundary and coboundary

Let \( \sigma \) be an orientation of \( G = (V, E) \), defined by

\[
\sigma_{v,e} = \begin{cases} 
+1 & e \text{ is directed into } v, \\
-1 & e \text{ is directed out of } v, \\
0 & v \notin e.
\end{cases}
\]

The boundary operator \( \partial : Q^E \to Q^V \) is defined by

\[
(\partial y)_v = \sum_{e \in E} \sigma_{v,e} y_e.
\]

The submodule \( \ker(\partial) \) is the set of \( Q \)-flows of \( G \). The coboundary operator \( \delta : Q^V \to Q^E \) is defined by

\[
(\delta x)_e = \sum_{v \in V} \sigma_{v,e} x_v,
\]

equal to \( x_v - x_u \) when \( e = \{u, v\} \) and \( u \) is directed towards \( v \).

The submodule \( \im(\delta) \) is the set of \( Q \)-tensions of \( G \). The \( Q \)-submodules \( \ker(\partial) \) and \( \im(\delta) \) are orthogonal.

Suppose that \( \{ f_v : v \in V \} \subset \mathbb{C}^Q \) and \( \{ g_e : e \in E \} \subset \mathbb{C}^Q \) are collections of functions defining \( f \in \mathbb{C}^{Q^V} \) and \( g \in \mathbb{C}^{Q^E} \) by

\[
f(x) = \prod_{v \in V} f_v(x_v), \quad g(y) = \prod_{e \in E} g_e(y_e).
\]
In [10, Theorem 12] it is shown that
\[ q^{-1/2} |V| \sum_{x \in Q^V} \prod_{v \in V} f_v(x_v) \prod_{e \in E} g_e((\delta x)_e) = q^{-1/2} |E| \sum_{y \in Q^E} \prod_{v \in V} f^F_v((\partial y)_v) \prod_{e \in E} g^F_e(y_e), \]
where the bar denotes complex conjugation and \( F \) is the unitary Fourier transform. In more compact notation,
\[ q^{-1/2} |V| \langle f, g \circ \delta \rangle = q^{-1/2} |E| \langle f^F \circ \partial, g^F \rangle. \]
This identity is a generalization of the Poisson summation formula (the case \( f = 1_{Q^V} \)) and will be useful in Section 3.4.

3 Edge and vertex colouring models

3.1 A general definition. Orthogonal symmetry of edge colouring models

We define partition functions of vertex and edge colouring models in as great a generality as required for this paper. (An example not covered by this definition is the general \( q \)-state Potts model where interactions at an edge \( e \) depend on \( e \) as well as the colours on its endpoint vertices.)

**Definition 3.1.** Let \( G = (V, E) \) be a graph and \( Q \) a set of size \( q \).

A partition function of a vertex \( Q \)-colouring model with weight functions \( f \in \mathbb{C}^Q \) and \( g \in \mathbb{C}^{Q^2} \) is a sum of the form
\[ \sum_{x \in Q^V} \prod_{v \in V} f(x_v) \prod_{e \in E} g(x_v : v \in e). \]

A partition function of an edge \( Q \)-colouring model with weight functions \( f \in \mathbb{C}^{Q^*} \) and \( g \in \mathbb{C}^Q \) is a sum of the form
\[ \sum_{y \in Q^E} \prod_{v \in V} f(y_e : e \ni v) \prod_{e \in E} g(y_e). \]

Up until Section 3.4 we shall work only with uniform vertex \( Q \)-colouring models, namely those for which \( f = 1_Q \), i.e. only with partition functions of the form
\[ \sum_{x \in Q^V} \prod_{v \in E} g(x_v : v \in e) = (1_{\text{MONOCROM,}}^{\otimes V}, g^{\otimes E}). \]

\footnote{Using the Fortuin-Kasteleyn representation of the \( q \)-state Potts model, this is the multivariate Tutte polynomial on the hyperbola \( H_q \). See e.g. [27].}
Likewise, a uniform edge $Q$-colouring model is one for which $g = 1_Q$, i.e. with partition function of the form
\[
\sum_{y \in Q^E} \prod_{v \in V} f(y_e : e \ni v) = (f^\otimes V, 1^\otimes E_{\text{Monochrome}}).
\]

The chromatic polynomial of $G$ evaluated at $q = |Q|$ (the number of proper vertex $q$-colourings) has a uniform vertex $Q$-colouring model with $g(a, b)$ equal to 1 if $a \neq b$ and 0 if $a = b$. The number of perfect matchings of $G$ has a uniform edge $\mathbb{Z}_2$-colouring model with weight function defined by $f(a_1, \ldots, a_d) = 1$ if $\#\{i : a_i = 1\} = 1$ and $f(a_1, \ldots, a_d) = 0$ otherwise.

The weight function $f$ in the partition function of a uniform edge colouring model is not uniquely determined. Indeed, we begin by deriving the result of Szegedy \[32\], given as Corollary 3.3 below, that partition functions of uniform edge colouring models are invariant under the action of the group of orthogonal transformations of $\mathbb{C}^Q$ (extended to transformations of $\mathbb{C}^Q^*$) on the weight function $f$.

**Lemma 3.2.** If $U$ is an orthogonal transformation $\mathbb{C}^Q \to \mathbb{C}^Q$, represented as an orthogonal matrix with rows and columns indexed by $Q$, then $U \otimes U$ is an orthogonal transformation of $\mathbb{C}^Q^2$ and
\[
(U \otimes U)1_{\text{Monochrome}}^{\otimes Q^2} = 1_{\text{Monochrome}}^{\otimes Q^2}.
\]

**Proof.** For an $m \times n$ matrix $A = (a_{i,j})$, vec$(A)$ is the $mn \times 1$ vector obtained by stacking columns of $A$ one on top of the other, with $k$th entry $a_{i,j}$ where $i = k - \left\lfloor \frac{k-1}{m} \right\rfloor m$ and $j = \left\lfloor \frac{k-1}{m} \right\rfloor + 1$. It is easy to verify that for matrices $A, B, C$ of compatible dimensions $(B^T \otimes A)\text{vec}(C) = \text{vec}(ACB)$.

The function $1_{\text{Monochrome}}$ on $\mathbb{C}^Q^2$ when represented by a column vector indexed by $Q^2$ is equal to vec$(I)$, where $I$ is the identity matrix with rows and columns indexed by $Q$. Since $(U \otimes U)\text{vec}(I) = \text{vec}(UIU^T) = \text{vec}(I)$, the statement of the lemma follows.

Lemma 3.2 along with the fact that an orthogonal transformation preserves euclidean inner products, yields the invariance of partition functions of uniform edge colouring models under orthogonal transformations of the vertex weight.

**Corollary 3.3.** \[32\] Proposition 2.3] If $U$ is an orthogonal transformation of $\mathbb{C}^Q$ then
\[
(f^\otimes V, 1^\otimes E_{\text{Monochrome}}) = ((f^U)^\otimes V, 1^\otimes E_{\text{Monochrome}}).
\]

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3.2 Uniform vertex and edge colouring models: the Tutte polynomial and the symmetric weight enumerator of $Q$-flows

The Tutte polynomial on the hyperbola $H_q := \{(s, t) : (s - 1)(t - 1) = q\}$ is defined by

$$(s - 1)^{|E| - r(E)} T(G; s, \frac{s - 1 + q}{s - 1}) = \sum_{A \subseteq E} q^{|A| - r(A)} (s - 1)^{|E| - |A|}.$$  

Alternatively

$$(t - 1)^{r(E)} T(G; \frac{t - 1 + q}{t - 1}, t) = \sum_{A \subseteq E} q^{r(E) - r(A)} (t - 1)^{|A|}.$$  

Given $y \in Q^E$, the Hamming weight of $y$ is defined by $|y| := \# \{e \in E : y_e \neq 0\}$.

The Hamming weight enumerator of the set of $Q$-flows is a specialization of the Tutte polynomial to the hyperbola $H_q$:

$$hwe(\ker(\partial); s) := \sum_{y \in \ker(\partial)} s^{|E| - |y|} = (s - 1)^{|E| - r(E)} T(G; s, \frac{s - 1 + q}{s - 1}). \quad (2)$$

Dually, the Tutte polynomial on $H_q$ is also given by

$$hwe(\im(\delta); t) := \sum_{y \in \im(\delta)} t^{|E| - |y|} = (t - 1)^{r(E)} T(G; \frac{t - 1 + q}{t - 1}, t). \quad (3)$$

The latter specialization of the Tutte polynomial to $H_q$ comes in the more familiar guise of the monochrome (or bad colouring) polynomial (see e.g. [40]),

$$q^k(G) hwe(\im(\delta); t) = \sum_{x \in Q^V} t^{\# \{uv \in E : x_u = x_v\}}, \quad (4)$$

so called because the exponent of $t$ is equal to the number of monochromatic edges (which are bad if proper is good) in the vertex colouring $x$.

Taking $X$ to be a random variable with uniform distribution on $Q^V$, by (3) the identity (4) gives the Tutte polynomial on $H_q$ as a uniform vertex $Q$-colouring model,

$$(t - 1)^{r(E)} T(G; \frac{t - 1 + q}{t - 1}, t) = q^{r(E)} E[t^{\# \{uv \in E : X_u = X_v\}}].$$

A complete weight enumerator of a set $S \subseteq Q^E$ evaluated at a function $h \in \mathbb{C}^Q$ is defined by

$$\text{cwe}(S; h) := \sum_{y \in S} \prod_{e \in E} h(y_e) = \sum_{y \in S} \prod_{c \in Q} h(c)^{\# \{e \in E : y_e = c\}}.$$
The complete weight enumerator may be regarded as a multivariate polynomial over \(\mathbb{Z}\) by letting \(h\) take indeterminate values, in which case we denote it by \(\text{cwe}(\mathcal{S}; t_c : c \in Q)\) when \(h(c) = t_c\) for \(c \in Q\). The Hamming weight enumerator \(\text{hwe}(\mathcal{S}; t)\) is the specialisation \(h(0) = t\) and \(h(c) = 1\) for \(c \neq 0\). If \(\mathcal{S}\) is a \(Q\)-submodule of \(Q^E\) and \(\mathcal{S}^\perp\) is its orthogonal submodule, then by the Poisson summation formula (see e.g. [11]),

\[
\text{cwe}(\mathcal{S}; h) = q^{-|E|/2} |\mathcal{S}| \text{cwe}(\mathcal{S}^\perp; h^F),
\]

where \(F\) is the unitary Fourier transform.

Since \((\ker(\partial))^\perp = \text{im}(\delta)\), by (5) the complete weight enumerator of \(\ker(\partial)\) evaluated at a function \(h\) corresponds to a uniform vertex \(Q\)-colouring model with weight function \(g(a, b) = h^F(b - a)\) depending only on the difference \(b - a\) (in other words \(g(a, b) = g(a + c, b + c)\) for each \(c \in Q\)):

\[
\text{cwe}(\ker(\partial); h) = q^{\frac{1}{2}|V|-|E|} \sum_{x \in Q^V} \prod_{(u,v) \in E} h^F(x_u - x_v).
\]

Note that \(h^F(x_u - x_v) = h^{FN}((x_u)_c),\) where \(N\) defined by \(h^N(c) = h(-c)\) commutes with \(F\) since \(F^2 = N\). The next theorem says that if \(h^N = h\) (in other words, \(g(a, b) = h^F(b - a)\) has the further property that \(g(a, b) = g(b, a)\)), then \(\text{cwe}(\ker(\partial); h)\) has an edge colouring model. In particular, we will deduce an edge colouring model for the Hamming weight enumerator (Tutte polynomial on \(H_q\)) as a corollary.

**Theorem 3.4.** Let \(g \in \mathbb{C}^Q, F\) the unitary Fourier transform on \(\mathbb{C}^Q\) and \(N\) the linear transformation defined by \(g^N(a) = g(-a)\). The complete weight enumerator of the set of \(Q\)-flows of \(G\) has the following uniform vertex colouring model and uniform edge colouring model:

\[
\text{cwe}(\ker(\partial); g \cdot g^N) = q^{-|V|} \sum_{x \in Q^V} \prod_{e \in E} \sum_{b \in Q} \prod_{v \in e} g^F(x_v - b)
\]

\[
= q^{-|V|} \sum_{y \in Q^E} \prod_{v \in V} \sum_{a \in Q} \prod_{e \in \mathcal{E}} g^F(a - y_e).
\]

In other words, if \(X\) has a uniform distribution on \(Q^V\) and \(Y\) a uniform distribution on \(Q^E\) then

\[
\text{cwe}(\ker(\partial); g \cdot g^N) = q^{|E|} \mathbb{E}\left[ \prod_{(v,e) \in H} g^F(X_v - Y_e) \right].
\]

Similarly, for \(f \in \mathbb{C}^Q\),

\[
\text{cwe}(\text{im}(\delta); f \cdot f^N) = q^{|E|+r(E)} \mathbb{E}\left[ \prod_{(v,e) \in H} f(X_v - Y_e) \right].
\]
Proof. Given a graph $G$, let $G' = (V', E')$ to be the 2-stretch of $G$, defined by $V' = V \cup E$ and $E' = \{(v, e) : v \in e\}$, i.e. $G'$ is obtained from $G$ by replacing each edge of $G$ by a path of length 2. The edges of $G'$ are in one-one correspondence with the half-edges of $G$. Also, $r(E') = |E| + r(E)$.

The set of $Q$-flows of $G'$ is in one-one correspondence with the set of $Q$-flows of $G$ as indicated by Figure 1. The orientation of an edge $(v, e)$ in $G'$ is chosen so that $v$ is directed toward $e$, as illustrated. (This makes explicit the correspondence between direct edges $(v, e)$ of $G'$ and half-edges of $G$.)

![Figure 1: Edge $e = (u, v)$ in $G$ with flow value $c$ corresponds to edges $(u, e)$ and $(v, e)$ in $G'$ with flow values $c$ and $-c$](image)

If $C'$ is the set of $Q$-flows of $G'$ and $C$ the set of $Q$-flows of $G$ (which is $\ker(\partial)$ in the statement of the theorem), then by the illustrated correspondence between $C$ and $C'$ we have

$$cwe(C; g \cdot g^N) = cwe(C'; g) = q^{\frac{1}{2} |E'| - r(E')}cwe(C'^\perp; g^F)$$

the last line by (5). With $\frac{1}{2} |E'| - r(E') = |E| - (r(E) + |E|) = -r(E)$ and $k(G') = k(G)$, the 1-to-$q^{k(G)}$ correspondence between $C'^\perp$ and vertex $Q$-colourings of $G'$ means this last line can be written as the partition function of a vertex colouring model on $G'$:

$$cwe(C; g \cdot g^N) = q^{-|V|} \sum_{z \in Q^{V'}} \prod_{(v, e) \in E'} g^F(z_e - z_v)$$

$$= q^{-|V|} \sum_{(x, y) \in Q^V \times Q^E} \prod_{(v, e) \in H} g^F(y_e - x_v)$$

using $Q' = Q^{V \cup E} \cong Q^V \times Q^E$ and $E' \cong H$ to get to the last line.

If $(X, Y)$ is uniform on $Q^V \times Q^E$, then this yields the expected value of the product of $g^F(X_v - Y_e)$ over half-edges $(v, e)$ as presented in the theorem statement. The partition functions of vertex and edge colouring models arise by conditioning on $Y$ and $X$ respectively.
First
\[
\mathbb{E}[ \prod_{(v,e) \in H} g^F(X_v - Y_e) \mid Y] = q^{-|E|} \sum_{y \in Q^E} \prod_{(v,e) \in H} g^F(X_v - y_e)
= q^{-|E|} \prod_{e \in E} \sum_{y \in Q} g^F(X_v - b).
\]

Hence
\[
\mathbb{E}[ \prod_{(v,e) \in H} g^F(X_v - Y_e)] = \mathbb{E}(\mathbb{E}[ \prod_{(v,e) \in H} g^F(X_v - Y_e) \mid Y])
= q^{-|V|-|E|} \sum_{x \in Q^V} \prod_{e \in E} \sum_{y \in Q} g^F(x_v - b).
\]

Second, conditioning on \(X\),
\[
\mathbb{E}[ \prod_{(v,e) \in H} g^F(X_v - Y_e)] = \mathbb{E}(\mathbb{E}[ \prod_{(v,e) \in H} g^F(X_v - Y_e) \mid X])
= q^{-|V|-|E|} \sum_{y \in Q^E} \prod_{v \in V} \sum_{a \in Q} g^F(a - y_e).
\]

For the last part of the theorem, by identity (5) and \((f \ast f^N)^F = q^{1/2} f^F. f^{NF}\),
\[
cwe(\text{im}(\delta); f \ast f^N) = q^{r(E)} \text{cwe}(\text{ker}(\delta); f^F \cdot f^{NF}),
\]
from which the desired result follows on writing \(g = f^F\) and noting that \(f^{NF} = g^N\) since \(NF = FN\).

The Tutte polynomial on the hyperbola \(H_q\), given by equation (2) as the Hamming weight enumerator of the set of \(Q\)-flows of \(G\) (for abelian group \(Q\) order \(q\)), inherits an edge colouring model from Theorem 3.4.

**Corollary 3.5.** Let \(G = (V, E)\) be a graph, \(Q\) a set of size \(q\), and suppose \(X\) takes values uniformly at random from \(Q^V\) and \(Y\) takes values uniformly at random from \(Q^E\). Then
\[
(s^2 - 1)^{|E|-r(E)} T(G; s, \frac{s^2 - 1 + q}{s^2 - 1}) = \mathbb{E}[(s-1+q)^{\#\{v \in e: X_v = Y_e\}} (s-1)^{\#\{v \in e: X_v \neq Y_e\}]}.
\]

In particular, the Tutte polynomial on the hyperbola \(H_q\) is the partition function of a uniform edge \(q\)-colouring model:
\[
(s^2 - 1)^{|E|-r(E)} T(G; s, \frac{s^2 - 1 + q}{s^2 - 1}) = q^{-|E|-|V|(s-1)^2|E|} \sum_{y \in Q^E} \prod_{v \in V} \sum_{a \in Q} \left(\frac{s-1+q}{s-1}\right)^{\#\{v \in e: y_e = a\}}.
\]

**Proof.** Take \(g = s_1 + 1_Q\) in Theorem 3.4 noting that \(g^N = g\) and \(g^F = q^{-\frac{1}{2}}[(s-1+q)1_Q + (s-1)1_Q\_0].\)
3.3 Real symmetric vertex colouring models as edge colouring models

A consequence of Theorem 3.4 is that if $g \in \mathbb{C}^Q^2$ is symmetric, i.e. $g(a, b) = g(b, a)$, and satisfies $g(a+c, b+c) = g(a, b)$ for each $c \in Q$, then the partition function of a uniform vertex colouring model with weight function $g$ is also given by the partition function of an edge $Q$-colouring model.

If we restrict attention to real-valued vertex $Q$-colouring models then the condition that $g \in \mathbb{R}^Q^2$ is symmetric is sufficient. The key property of such a function in this regard is that by the spectral theorem for real symmetric matrices there is a function $h \in \mathbb{C}^Q^2$ such that

$$g(a, b) = \sum_{c \in Q} h(a, c)h(b, c). \quad (7)$$

Furthermore, if the rank of the matrix $[g(a, b)]_{a,b \in Q}$ is equal to $r$ then there are $q - r$ values of $c$ for which $h(a, c) = 0$ for all $a \in Q$. If the symmetric matrix $[g(a, b)]_{a,b \in Q}$ is positive semi-definite, i.e. has non-negative eigenvalues, then the function $h$ is real-valued. Otherwise $h$ may take purely imaginary values as well as real values (and only purely imaginary values if the matrix $[g(a, b)]_{a,b \in Q}$ is negative definite).

**Theorem 3.6.** (Szegedy [32, Section 3.1].) Suppose that $g \in \mathbb{R}^Q^2$ satisfies $g(a, b) = g(b, a)$ for all $a, b \in Q$.

Then there is $h \in \mathbb{C}^Q^2$ such that

$$\sum_{x \in Q^V} \prod_{v \in V} f(x_v) \prod_{e \in E} g(x_v : v \in e) = \sum_{y \in Q^E} \prod_{v \in V} \sum_{a \in Q} f(a) \prod_{e \in v} h(a, y_e).$$

Furthermore, if the matrix $[g(a, b)]_{a,b \in Q}$ has rank $r$ then $h(a, b)$ is identically zero for $q - r$ values of $b \in Q$ (reducing the right-hand side to the partition function of an edge $r$-colouring model).

**Proof.** Let $g(a, b) = \sum_{c \in Q} h(a, c)h(b, c)$ be a spectral decomposition of $g$. Then

$$\sum_{x \in Q^V} \prod_{v \in V} f(x_v) \prod_{e \in E} g(x_v : v \in e) = \sum_{x \in Q^V} \prod_{v \in V} f(x_v) \prod_{e \in E} \sum_{c \in Q} h(x_v, c)$$

$$= \sum_{x \in Q^V} \prod_{v \in V} f(x_v) \sum_{y \in Q^E} \prod_{e \in E} \prod_{e \in v} h(x_v, y_e)$$

$$= \sum_{y \in Q^E} \sum_{x \in Q^V} \prod_{v \in V} f(x_v) \prod_{e \in E} \prod_{e \in v} h(x_v, y_e)$$

$$= \sum_{y \in Q^E} \prod_{v \in V} \sum_{a \in Q} f(a) \prod_{e \in v} h(a, y_e).$$

$\square$
Remark 3.7. An alternative proof of Theorem 3.4 is to mimic the proof of Theorem 3.3 applied to the case $\text{cwe}(\text{im}(\delta); f \ast f^{N})$, the latter the partition function of a uniform vertex $Q$-colouring model with weight function $g(a,b)$ given by
\[ g(a,b) = f \ast f^{N}(b-a) = \sum_{c \in Q} f(c-a)f(c-b), \]
which takes the form displayed in equation (7) with $h(a,b) = f(b-a)$.

3.4 A family of chromatically defined graph polynomials

So far we have seen how the Tutte polynomial on $H_{q}$ (or monochrome polynomial) and its generalization to a symmetric weight enumerator of $Q$-flows (a complete weight enumerator $\text{cwe}(\ker(\partial); h)$ satisfying $h^{N} = h$) have uniform edge $Q$-colouring models.

A generalization of the chromatic polynomial $P(G; q)$ in a direction different to that of the Tutte polynomial and complete weight enumerator is the symmetric function analogue of the monochrome polynomial (4). This is a polynomial in commuting indeterminates $s_{0}, s_{1}, \ldots$ defined [30] by
\[ X(G; s_{0}, s_{1}, \ldots, t) = \sum_{x \in \mathbb{N}^{V}} t^{\# \{uv \in E : x_{u} = x_{v}\}} \prod_{a \in \mathbb{N}} s_{a}^{\# \{v \in V : x_{v} = a\}}. \tag{8} \]

This function is invariant under permutations of $s_{0}, s_{1}, \ldots$ and specializes to Stanley’s generalized symmetric chromatic function [29] $X(G; s_{0}, s_{1}, \ldots)$ upon setting $t = 0$.

Consider the specialization $0 = s_{q} = s_{q+1} = \cdots$ which has the effect of restricting the range of summation in (8) to vertex $q$-colourings:
\[ X_{q}(G; s_{0}, \ldots, s_{q-1}; t) := \sum_{x \in \{0, \ldots, q-1\}^{V}} t^{\# \{uv \in E : x_{u} = x_{v}\}} \prod_{a \in \{0, 1, \ldots, q-1\}} s_{a}^{\# \{v \in V : x_{v} = a\}}. \]

Note that
\[ X_{q}(G; 1, 1, \ldots, 1; t) = q^{k(G)}(t-1)^{r(E)}T(G; \frac{t-1+q}{t-1}, t) \]
is the Tutte polynomial on $H_{q}$, and in particular $X_{q}(G; 1, 1, \ldots, 1; 0) = P(G; q)$. More generally, for an abelian group $Q$ of order $q$, define
\[ X_{Q}(G; \langle s_{a} : a \in Q \rangle, \langle t_{b} : b \in Q \rangle) = \sum_{x \in \mathbb{Q}^{V}} \prod_{a \in \mathbb{Q}} s_{a}^{\# \{v \in V : x_{v} = a\}} \prod_{b \in \mathbb{Q}} t_{b}^{\# \{e \in E : (\delta x)_{e} = b\}}. \tag{9} \]

This is simultaneously a generalization of the complete weight enumerator of $\text{im}(\delta)$, which has $s_{a} = 1$ for each $a \in Q$, and of the truncated symmetric monochrome polynomial $X_{q}(G; s_{0}, \ldots, s_{q-1}; t)$, which has $Q = \{0, \ldots, q-1\}$, $t_{0} = t$ and $t_{b} = 1$ for each $0 \neq b \in Q$. Indeed, the polynomial (9) may be
thought of as a multivariate generating function for pairs \((x, \delta x) \in Q^V \times Q^E\), and in Lemma 3.8 below we shall see that it may dually be regarded as a generating function for pairs \((\partial y, y) \in Q^V \times Q^E\).

Although the polynomial defined in (9) is invariant under permutations of \(\{s_a : a \in Q\}\), in general it is not invariant under permutations of \(\{t_b : b \in Q\}\), may also depend on the structure of \(Q\) as an abelian group, and may further depend on the orientation of \(G\). For example, take \(G\) to be the graph on two vertices joined by two parallel edges. For \(Q = \mathbb{F}_4 = \{0, 1, \omega, \overline{\omega}\}\), the specialization \(\text{cwe}(\text{im}(\delta); (0, t_1, t_\omega, t_\overline{\omega}))\) is equal to \(t_1^2 + t_\omega^2 + t_\overline{\omega}^2\) for all orientations of \(G\), while for \(Q = \mathbb{Z}_4 = \{0, 1, 2, 3\}\), the specialization \(\text{cwe}(\text{im}(\delta); (0, t_1, t_2, t_3))\) is equal to \(t_2^2 + 2t_1t_3\) for cyclic orientations of \(G\) and equal to \(t_1^2 + t_2^2 + t_3^2\) for the remaining, acyclic orientations of \(G\).

The relationship between the various polynomials is summarized in Figure 2 (a Hasse diagram with the ordering relation being that of specialization).

**Figure 2:** Some polynomials from vertex \(q\)-colourings of a graph \(G\).

| Vertex \(Q\)-colouring model |
|-----------------------------|
| \(X(G; s_0, s_1, \ldots)\) | \(X_Q(G; (s_a : a \in Q), (t_b : b \in Q))\) |
| \(X_q(G; s_0, \ldots, s_{q-1}; t)\) | \(\text{cwe}(\text{im}(\delta); (t_b : b \in Q))\) |
| \(X(G; s_0, s_1, \ldots)\) | \(\text{hwe}(\text{im}(\delta); t)\) |
| \(hwe(\text{im}(\delta); t)\) | \((t - 1)^{r(E)}T(G; \frac{t-1+q}{t-1}, t)\) |
| \(P(G; q)\) |

**Lemma 3.8.** The partition function \(X_Q(G; (s_a : a \in Q), (t_b : b \in Q))\) of a vertex colouring model whose edge weights depend only on the boundary \(\delta\) has the following expansion over edge colourings with vertex weights depending
only on the coboundary $\partial$:

$$X_Q(G; (s_a : a \in Q), (t_b : b \in Q)) = q^{-|E|} \prod_{y \in Q^E} \hat{s}_a \prod_{a \in Q} \hat{s}_a \{v \in V : (\partial y)_v = a\} \prod_{b \in Q} \hat{t}_b \{e \in E : y_e = b\},$$

where

$$\hat{s}_a = \sum_{c \in Q} \chi(c) s_c, \quad \hat{t}_b = \sum_{c \in Q} \chi(c) t_c$$

for a generating character $\chi$ of $Q$.

**Proof.** Apply the duality identity (1) with $f_v(x_v) = s_{x_v}$ and $g_e(y_e) = t_{y_e}$.

The polynomial $X_q(G; 1, s, \ldots, s^{q-1}; t)$ is called the principal specialization of order $q$ of the symmetric monochrome polynomial. Loebl [15] Theorem 3] proved that the principal specialization of order 2 has an edge colouring model analogous to Van der Waerden’s eulerian sub graph expansion of the Ising model [37]. Loebl’s result is the case $q = 2$ of Theorem 3.9 below, Van der Waerden’s expansion the case $q = 2$, $s = 1$.

**Theorem 3.9.**

(i) When $s^q \neq 1$,

$$X_q(G; 1, s, \ldots, s^{q-1}; t) = q^{-|E|} \prod_{y \in Q^E} \left( \frac{t-1+q}{t-1} \right)^{|E|-|y|} \prod_{v \in V} \left( \frac{s^q-1}{s e^{2\pi i (\partial y)_v} q} - 1 \right).$$

(ii) When $s = e^{-2\pi i c/q}$ for some $c \in \{0, \ldots, q-1\}$,

$$X_q(G; 1, e^{-2\pi i c/q}, \ldots, e^{-2\pi i (q-1)c/q}; t) = q^{V-|E|} \prod_{y \in Q^E} \left( \frac{t-1+q}{t-1} \right)^{|E|-|y|} \sum_{\forall v \in V : (\partial y)_v = c} \left( \frac{t-1+q}{t-1} \right)^{|E|-|y|}.$$

**Proof.** In Lemma 3.8 take $Q = \mathbb{Z}_q \cong \{0, 1, \ldots, q-1\}$, $\chi(c) = e^{2\pi i c/q}$, $s_a = s^a$ and $t_0 = t, t_b = 1$ for $b \neq 0$.

Apart from the case $q = 2$, the expansion of the principal specialization order $q$ of Stanley’s symmetric monochrome polynomial given in Theorem 3.9 is not an edge colouring model partition function, but rather a weighted sum of edge colourings with the weight at a vertex in a given colouring depending not only on the colours of incident edges but also on their orientation (in order to determine the boundary of the edge colouring at the vertex).

From Theorem 3.9 every symmetric weight enumerator of $Q$-tensions has a uniform edge colouring model. By the proof of Theorem 3.9 it is easy to
see that this property extends to the non-uniform vertex colouring model partition function $X_Q(G; (s_a : a \in Q), (t_b : b \in Q))$, provided of course that the edge weights are symmetric (independent of the orientation of $G$), i.e. $t_b = t_{-b}$.

**Theorem 3.10.** If $t_{-b} = t_b$ for each $b \in Q$ then

$$X_Q(G; (s_a : a \in Q), (t_b : b \in Q)) = \sum_{y \in Q^E} \prod_{v \in V} \sum_{a \in Q} s_a \prod_{b \in Q} u_b \# \{e \in E : ye = a+b\},$$

where $u_b$ is defined by $t_b = \sum_{a \in Q} u_a u_{a-b}$.

In particular, the principal specialization order $q$ of the symmetric monochrome function has edge colouring model given by

$$(t^2-1)^{|E|} X_q(G; 1, s, \ldots, s^{q-1}; \frac{t^2-1+q}{t^2-1})$$

$$= q^{-|V|} |E| (t-1)^{|E|} \sum_{y \in \{0,1,\ldots,q-1\}^E} \prod_{v \in V} \sum_{a \in \{0,1,\ldots,q-1\}} s_a \left(\frac{t-1+q}{t-1}\right)^{|E|} \# \{e \in E : ye = a\}.$$  

**Proof.** Follow the proof of Theorem 3.6 using the decomposition of the weight $t_b$ as a convolution $(u * u^N)_b$, i.e. as a spectral decomposition of the form needed for the proof of this earlier theorem to go through. Cf. Remark 3.7.

In order to obtain the special case, take $t_0 = t^2 - 1 + q$, $t_b = t^2 - 1$ for $b \neq 0$, for which $q^{1/2} u_0 = t - 1 + q$ and $q^{1/2} u_b = t - 1$ for $b \neq 0$.  

4 Non-symmetric edge colouring models for proper edge colourings

4.1 Oriented (near) 2-factorizations of $k$-regular graphs, parity and proper edge colourings

In this section $G = (V, E)$ will be a $k$-regular graph, with line graph $L(G) = (E, L)$ whose edge set is defined by setting $\{e, f\} \in L$ if $e$ and $f$ are incident with a common vertex in $G$. To simplify exposition we shall also take $Q = \mathbb{Z}_q$ for some $q \geq k$, although the results of this section can easily be adapted to other abelian groups of order $q$.

An arbitrary linear order is put on $\mathbb{Z}_q$, say the usual integer order $0 < 1 < 2 < \cdots < q-1$. A subset $K \subseteq \mathbb{Z}_q$ inherits a linear order from $\mathbb{Z}_q$ that enables us to assign a parity to a permutation of $K$, even (sign +1) if alternating, and odd (sign −1) otherwise. It will be helpful to make a more general definition.
Definition 4.1. Given an injection $\beta : X \to Y$ between two linearly ordered sets $(X, \langle \rangle)$ and $(Y, \langle \rangle)$, the sign of $\beta$ is defined by

$$\text{sgn}(\beta) = (-1)^{\# \{\ell, m \in X : \ell < m, \beta(\ell) > \beta(m)\}}.$$ 

If $\beta$ is not injective set $\text{sgn}(\beta) = 0$.

We shall also suppose that a linear order $<$ has been put on the half-edge set $H(v) = \{(v, e) : e \ni v\}$ for each $v \in V$. A half-edge colouring $z \in Z_q^H$ with the property that the restricted map $z_v : H(v) \to Z_q$ is injective for each $v \in V$ has sign defined by

$$\text{sgn}(z) = \prod_{v \in V} \text{sgn}(z_v) = \prod_{v \in V} (-1)^{\# \{(v, e) < (v, f) : z(v, e) > z(v, f)\}}.$$ 

A similar definition of sign holds for an edge colouring $y \in Z_q^E$, namely $\text{sgn}(y) := \text{sgn}(z)$ where $z \in Z_q^H$ is defined by $z_{(v, e)} = y_e$ for each $(v, e) \in H$.

Signs of proper edge $k$-colourings of $G$ have been considered for $k = 3$ by many authors, the earliest (implicit) example being Vigner [38]. Penrose in [21] first explicitly stated that the sign of a proper edge 3-colouring of a plane cubic graph is constant and independently Scheim [25] proved it. Other accounts of this property of plane cubic graphs can be found in [12], [14], [1]. More recently, Ellingham and Goddyn [7] proved that a similar result held for $k$-regular plane graphs for $k \geq 3$, and Norine and Thomas [20], [36] extended this to the larger class of $k$-regular graphs that admit Pfaffian labellings (a generalization of Pfaffian orientations —see below).

We use the fact that $k$-regular graphs admitting Pfaffian labellings have proper edge $k$-colourings of constant sign in order to construct a function $f \in C_{Z_q^E}$ for which $(f \otimes V, 1_{\text{ZEO-SUM}})_{\otimes E}$ is up to sign equal to $P(L(G); k)$. The following lemma then converts this inner product into the desired edge $q$-colouring model.

Lemma 4.2. For graph $G = (V, E)$ and $f \in C_{Z_q^E}$,

$$(f \otimes V, 1_{\text{ZEO-SUM}})_{\otimes E} = ((f^F) \otimes V, 1_{\text{MONOCHROME}})_{\otimes E}.$$ 

Proof. This follows since the Fourier transform $F$ is unitary, and $(\text{ZERO-SUM} \cap Q^2) \perp = \text{MONOCHROME} \cap Q^2$. 

A proper edge $k$-colouring of a $k$-regular graph is a partition of the edge set into $k$ edge-disjoint perfect matchings. The union of any two of these perfect matchings is a 2-factor of $G$ with the property that all its components have even size.

Let $P \subseteq Z_q$ have the property that $|P \cup (-P)| = k$ and that $P \cap (-P) \subseteq \{0\}$ if $q$ is odd or $P \cap (-P) \subseteq \{0, \frac{q}{2}\}$ if $q$ is even. Thus either $0 \in P$ or $\frac{q}{2} \in P$ if $k$ is odd while either $\{0, \frac{q}{2}\} \subseteq P$ or $P \subseteq Z_q \setminus \{0, \frac{q}{2}\}$ if $k$ is even.
For example, when \( q = k \) we can take \( P = \{0, 1, \ldots, \frac{k-1}{2}\} \) for \( k \) odd and \( P = \{0, 1, \ldots, \frac{k}{2}\} \) for \( k \) even. Set \( K = P \cup (-P) \), a subset of \( \mathbb{Z}_q \) of size \( k \).

An ordered (near) 2-factorization of \( G = (V,E) \) is an ordered partition \( \mathcal{F} = (F_a : a \in P) \) of \( E \), where each \( F_a \) is a 2-factor of \( G \) if \( a \neq -a \) and a 1-factor of \( G \) if \( a = -a \). (So if \( k \) is odd then there is one 1-factor labelled either \( F_0 \) or \( F_{q/2} \), and if \( k \) is even there are either no 1-factors or \( F_0 \) and \( F_{q/2} \) are both 1-factors.) An orientation of \( \mathcal{F} \) is an orientation of \( G \) with the property that each 2-factor \( F_a \) in \( \mathcal{F} \) is a union of directed circuits; the 1-factor(s) \( F_0 \) (and \( F_{q/2} \)) can be arbitrarily oriented. In an ordered bipartite (near) 2-factorization \( (F_a : a \in P) \) each 2-factor \( F_a \), \( a \neq -a \), is a union of even length circuits. In other words, the 2-factor \( F_a \) is a union of two edge-disjoint 1-factors.

Recall that for each \( v \in V \) there is a linear order \( < \) on the set \( H(v) = \{(v,e) : e \ni v \} \), the \( k \) half-edges incident with the vertex \( v \). If \( G \) is embedded in an orientable surface, this order might be taken in a clockwise sense around \( v \), for example. The order up to even permutation of half-edges incident with \( v \) will be used to distinguish the bipartite ordered (near) 2-factorizations of \( G \) from the non-bipartite. Two linear orders on \( H(v) = \{(v,e) : e \ni v \} \) differ by either an even permutation or an odd permutation. Likewise, two sets of orders on \( \{H(v) : v \in V\} \) differ altogether either by an even or odd permutation. By the phrase “the order up to even permutation of half-edges around vertices” we shall mean that two sets of linear orders which belong to the same one of these two parity classes are equivalent.

Define the edge colouring \( y \in P^E \) by setting \( y_e = a \) if \( e \in F_a \). When the ordered (near) 2-factorization \( \mathcal{F} \) is oriented by \( \sigma \), the map \( H(v) \to K \), \((v,e) \mapsto \sigma_{v,e} y_e \) is an injection, and we define

\[
\text{sgn}(\mathcal{F}) = \prod_{v \in V} \text{sgn}((v,e) \mapsto \sigma_{v,e} y_e).
\]

The colours \( (\sigma_{v,e} y_e : e \ni v) \) are a \( k \)-tuple \((b_0, b_1, \ldots, b_{k-1})\) of distinct elements of \( K \). Two such \( k \)-tuples may differ either by an even or an odd permutation, and we define two parity classes accordingly. Let

\[
\text{Even}(K) = \{(b_0, b_1, \ldots, b_{k-1}) \in K^k : \text{sgn}(\ell \mapsto b_\ell) = +1\},
\]

and define \( \text{Even} \subset \mathbb{Z}_q^k \) by

\[
\text{Even} = \bigcup_{K \subset \mathbb{Z}_q, |K|=k} \text{Even}(K).
\]

\footnote{When \( k \) is odd, each way of taking the half-edges \( \{(v,e) : e \ni v\} \) incident with a vertex \( v \) in a clockwise order is related to any other clockwise order by an even permutation (a \( k \)-cycle), and to anticlockwise orders by an odd permutation. When \( k \) is even, taking the half-edges in a clockwise direction alternates parity according to which half-edge starts the order. Thus, for even \( k \), in order to relate the linear order up to even permutation on \( \{(v,e) : e \ni v\} \) to the embedding of \( G \) an additional topological aspect of the embedding needs to be used to determine where to start the clockwise order (the “0-consistent” order taken in \cite{7} for example).}
The sets \( \text{ODD}(K) \) and \( \text{ODD} \) are defined similarly. The set \( \text{EVEN} \cup \text{ODD} \) comprises all \( k \)-tuples \((b_0, b_1, \ldots, b_{k-1})\) such that \( b_0, b_1, \ldots, b_{k-1} \) are distinct elements of \( \mathbb{Z}_q \).

**Lemma 4.3.** Let \( G = (V, E) \) be a \( k \)-regular graph and \( q \geq k \). Suppose \( K \subseteq \mathbb{Z}_q \) is of size \(|K| = k\) and satisfies \(-K = K\). Then

\[
\sum_{\mathcal{F}} \text{sgn}(\mathcal{F}) = \pm ((1_{\text{EVEN}(K)} - 1_{\text{ODD}(K)}) \otimes V, 1_{\text{Zero-sum}}),
\]

where the sum is over all oriented ordered bipartite (near) 2-factorizations \( \mathcal{F} \) of \( G \) and the sign on the right-hand side depends on the order up to even permutation of half-edges around vertices.

**Proof.** Fix an orientation \( \tau \) of \( G \). Suppose we are given the unique representation of \( \mathcal{F} \) as an orientation \( \sigma \) cyclic on its 2-factors together with an edge colouring \( y \in P^E \) such that \( \{e : y_e = a\} = F_a \). The \( k \)-tuple \((\sigma_{v,e} y_e : e \ni v)\) is then a permutation of \( K \). There is a unique \( z \in K^e \) such that \((\tau_{v,e} z_e : e \ni v)\) is a permutation of \( K \), namely \( z \) is the edge colouring defined by \( z_e = y_e \) if \( \tau \) preserves the direction of \( \sigma \) on the edge \( e \), and \( z_e = -y_e \) otherwise. (So \( z_e = \tau_{v,e} \sigma_{v,e} y_e \) for either choice of vertex \( v \in e \).) Hence the oriented ordered (near) 2-factorizations are counted by \( \pm ((1_{\text{EVEN}(K)} + 1_{\text{ODD}(K)}) \otimes V, 1_{\text{Zero-sum}}) \).

Given \( \mathcal{F} = (F_a : a \in P) \) and \( y \in P^E \) defined as above by \( y_e = a \) when \( e \in F_a \), switching the orientation \( \sigma \) on a component circuit \( C \) of a 2-factor \( F_a \) corresponds to switching the sign of \( y \) on the circuit \( C \). Under the fixed orientation \( \tau \) of \( G \), the corresponding operation is to switch the sign of \( z \in K^e \) on the circuit \( C \), where \( z \) is defined as above (negating \( y \) when \( \tau \) reverses \( \sigma \)).

For a vertex \( v \) belonging to \( C \), the effect on the \( k \)-tuple \((\tau_{v,e} z_e : e \ni v)\) is to transpose \( a \) and \(-a \), thus switching the sign of the injection \(( (v, e) : e \ni v ) \to K, ( v, e ) \mapsto \tau_{v,e} z_e \).

Hence, if a component circuit of \( F_a \) has an odd number of vertices the factorization \( \mathcal{F} \) to which it belongs will not be counted in the sum \((1_{\text{EVEN}(K)} - 1_{\text{ODD}(K)}) \otimes V, 1_{\text{Zero-sum}}) \), and the lemma is proved.

\[ \square \]

### 4.2 The Fourier transform of the parity function

Our goal now is to calculate the Fourier transform of the function \( 1_{\text{EVEN}(K)} - 1_{\text{ODD}(K)} \) for \( K \subseteq \mathbb{Z}_q \) of size \( k \) satisfying \(-K = K\). First we shall do so for \( q = k \) and \( K = \mathbb{Z}_k \), thereby establishing that the sum in Lemma 4.3 is also equal to a sum over proper edge \( k \)-colourings of \( G \) (Corollaries 4.5 and 4.6 below).

**Lemma 4.4.** The matrix \( q^{1/2} F = [e^{2\pi i \ell m/q}]_{0 \leq \ell, m \leq q-1} \) has determinant

\[
\det[e^{2\pi i \ell m/q}]_{0 \leq \ell, m \leq q-1} = i^{(q-1)(3q-2)/2} q^{l/2}.
\]

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Proof. Since \( F^2 = N \) (recall that \( N = [1 \ell(-m)]_{0 \leq \ell, m \leq q-1} \)),
\[
\det((q^{1/2}F)^2) = q^q \det(N) = q^q(-1)^{q-1}
\]
so that \( \det[e^{2\pi i\ell m/q}]_{0 \leq \ell, m \leq q-1} = \pm i^{q-1}q^{q/2} \). To determine the sign, note that
\[
e^{2\pi im/q} - e^{2\pi i\ell/q} = 2i(e^{\pi i/q})^{\ell+m} \sin \left( \frac{(m-\ell)\pi}{q} \right),
\]
and
\[
\sum_{0 \leq \ell < m \leq q-1} \ell + m = \frac{q(q-1)^2}{2}.
\]
Using the well known expansion of Vandermonde determinants, we obtain
\[
\det[e^{2\pi i\ell m/q}]_{0 \leq \ell, m \leq q-1} = \prod_{0 \leq \ell < m \leq q-1} (e^{2\pi im/q} - e^{2\pi i\ell/q})
\]
\[
= (e^{\pi i/q})^{q(q-1)^2/2} q^{(q-1)/2} \prod_{0 \leq \ell < m \leq q-1} 2 \sin \left( \frac{(m-\ell)\pi}{q} \right),
\]
and \( \sin \frac{\pi(m-\ell)}{q} > 0 \) for \( 0 \leq \ell < m \leq q-1 \). This yields the required sign and the expression for the determinant given in the statement of the lemma now follows.

Corollary 4.5. For a \( k \)-regular graph \( G = (V,E) \),
\[
\left( (1_{\text{Even}}(Z_k) - 1_{\text{Odd}}(Z_k)) \right)^{\otimes V}, \left( 1^{\otimes E}_{\text{Zero-sum}} \right) = \pm \left( (1_{\text{Even}}(Z_k) - 1_{\text{Odd}}(Z_k)) \right)^{\otimes V}, \left( 1^{\otimes E}_{\text{Monochrome}} \right),
\]
where the sign is given by
\[
\begin{cases} 
(-1)^{\frac{k-1}{2}|E|} & \text{k odd,} \\
(-1)^{\frac{k}{2}|E| + \frac{1}{4}(|V| - |E|)} & \text{k even.}
\end{cases}
\]

Proof. With a view to applying Lemma 4.3 we calculate \((1_{\text{Even}}(Z_k) - 1_{\text{Odd}}(Z_k))^F \).

For \((b_0, \ldots, b_{k-1}) \in \mathbb{Z}_k^k\), the matrix \( B := [k^{-1/2} e^{2\pi ib \ell/q}]_{0 \leq \ell, k-1} \) is zero if \( b_\ell = b_m \) for some \( 0 \leq \ell < m \leq k-1 \) and \( b_k = b_{k-1} \). Otherwise \((b_0, b_1, \ldots, b_{k-1})\) is a permutation \( \beta \) of \((0,1,\ldots,k-1)\) and the matrix \( B \) is equal to the Fourier matrix \( F = k^{-1/2} [e^{2\pi i\ell m/k}]_{0 \leq \ell, m \leq k-1} \) with its columns permuted by \( \beta \), so that by Lemma 4.3
\[
\det[k^{-1/2} e^{2\pi ib \ell/q}]_{0 \leq \ell \leq q-1} = \text{sgn}(\beta) \ell^{(q-2)(q-1)/2}.
\]
Hence
\[
(1_{\text{Even}}(Z_k) - 1_{\text{Odd}}(Z_k))^F(b_0, \ldots, b_{k-1}) = k^{-k/2} \sum_{\rho \in \text{Sym}\{0,1,\ldots,k-1\}} \text{sgn}(\rho) \prod_{0 \leq \ell \leq k-1} e^{2\pi i \rho(\ell)b_\ell/q}
\]
\[
= i^{(k-1)(3k-2)/2} (1_{\text{Even}}(Z_k) - 1_{\text{Odd}}(Z_k)).
\]
With $(k - 1)(3k - 2)/2 \equiv (k^2 - k + 1 + (-1)^k)/2 \pmod{4}$, and \(k|V| = 2|E|\) since \(G\) \(k\)-regular, when taking the product over all vertices this makes the exponent of \(i\) equal to \((k - 1)|E| + |V|\) when \(k\) is even and \((k - 1)|E|\) when \(k\) is odd. Note that although \((k - 1)|E| + |V|\) can be odd when \(|V|\) is odd, in this case the graph \(G = (V, E)\) does not have a proper edge \(k\)-colouring since it cannot have a 1-factor, i.e. the inner products in the statement of the lemma are both zero.

From Corollary 4.5 and Lemma 4.3 (with \(q = k, K = \mathbb{Z}_k\)) we deduce the existence of a sign-preserving bijection, exhibited by Alon and Tarsi in [4], [3], between ordered oriented bipartite (near) 2-factorizations of \(G\) and proper edge \(k\)-colourings of \(G\).

**Corollary 4.6.** Let \(G = (V, E)\) be a \(k\)-regular graph and \(q \geq k\). Suppose \(K \subseteq \mathbb{Z}_q\) is of size \(|K| = k\) and satisfies \(-K = K\). Then
\[
\sum_{\mathcal{F}} \text{sgn}(\mathcal{F}) = \pm((1_{\text{Even}(K)} - 1_{\text{Odd}(K)}) \otimes V, 1 \otimes E_{\text{Monochrome}}),
\]
where the sum is over all oriented ordered bipartite (near) 2-factorizations \(\mathcal{F}\) of \(G\) and the sign on the right-hand side depends on the order up to even permutation of half-edges around vertices.

Corollary 4.6 enables the derivation of an edge \(q\)-colouring model for \(P(L(G); k)\) whenever \(G\) is a \(k\)-regular graph with a so-called a Pfaffian labelling [20]. For a definition and discussion of Pfaffian orientations and labellings of a graph see [36]. All planar graphs have a Pfaffian orientation.

A graph admits a Pfaffian \(\mathbb{Z}_2\)-labelling if and only if it has a Pfaffian orientation. The complete bipartite graph \(K_{3,3}\) and the Petersen graph are not Pfaffian, but the Petersen graph admits a Pfaffian \(\mathbb{Z}_4\)-labelling.

In this context all that needs to be known about \(k\)-regular graphs with Pfaffian labellings is that their proper edge \(k\)-colourings all have the same sign [36 Theorem 8.3], i.e. the right-hand side of equation (10) is up to sign equal to \(P(L(G); k)\).

From Lemma 4.3 and Corollary 4.6 we deduce the following.

**Corollary 4.7.** Let \(G = (V, E)\) be a \(k\)-regular graph that admits a Pfaffian labelling. Suppose \(K \subseteq \mathbb{Z}_q\) has size \(|K| = k\) and that \(-K = K\). Then
\[
((1_{\text{Even}(K)} - 1_{\text{Odd}(K)}) \otimes V, 1 \otimes E_{\text{Zero-sum}}) = \pm P(L(G); k),
\]
where the sign on the right-hand side depends on the order up to even permutation of half-edges around vertices.

As an illustration of Corollary 4.7, if \(G = (V, E)\) is a plane 3-regular graph and the order of half-edges \(\{(v, e) : e \ni v\}\) around a vertex \(v\) up to
cyclic permutation is clockwise in the plane then [25], [21], [38] the sign of a proper edge $k$-colouring is always $(-1)^{|E|}$, so that the sign in equation (11) is $+1$. In [4], it is shown more generally that when $k \geq 3$ is odd, a $k$-regular plane graph with a clockwise cyclic order of half-edges around vertices always has a positive sign for its oriented ordered near 2-factorizations, which by Corollary 4.5 implies that the sign of a proper edge $k$-colouring is always $(-1)^{k/2 |E|}$. (A similar statement [4, Lemma 3.6] can be given for even $k$, only now, with $k$-cycles odd permutations, a clockwise order of half-edges needs to be consistently rooted by reference to the embedding of $G$ in order to ensure that all oriented ordered 2-factorizations of $G$ have positive sign.)

In the next lemma and its special case, Corollary 4.9, we prepare for the main theorem of this section, Theorem 4.10, which gives an edge $q$-colouring model for $P(L(G); k)$ for $q \geq k$ when $G$ has a Pfaffian labelling.

**Lemma 4.8.** Let $k$ be odd, $q > k$, and set $K = \{0, \pm 1, \ldots, \pm \frac{k-1}{2}\} \subset \mathbb{Z}_q$. Then, for $(b_0, \ldots, b_{k-1}) \in \mathbb{Z}_q^k$,

$$(1_{\text{Even}(K)} - 1_{\text{Odd}(K)})^F (b_0, \ldots, b_{k-1}) = q^{-k/2} \prod_{0 \leq \ell < m \leq k-1} 2 \sin \frac{\pi (b_m - b_\ell)}{q}.$$

Similarly, when $k$ is even and $K = \{\pm 1, \ldots, \pm \frac{k}{2}\} \subset \mathbb{Z}_q$, then for $(b_0, \ldots, b_{k-1}) \in \mathbb{Z}_q^k$,

$$q^{-k/2} \prod_{0 \leq \ell < m \leq k-1} 2 \sin \frac{\pi (b_m - b_\ell)}{q} \sum_{S \subseteq \{0, 1, \ldots, k-1\} \atop |S| = k/2} \cos \frac{\pi (\sum_{m \in S} b_m - \sum_{\ell \notin S} b_\ell)}{q}.$$

**Note.** To avoid ambiguities in sign in the equations given in this lemma assume that $b_0, b_1, \ldots, b_{k-1}$ take integer values from some fixed system of residue classes modulo $q$, say $\{0, 1, \ldots, q-1\}$.

**Proof.** Set $f = 1_{\text{Even}(K)} - 1_{\text{Odd}(K)}$.

First, take $k$ odd and $K = \{0, \pm 1, \ldots, \pm \frac{k-1}{2}\}$. For $(b_0, b_1, \ldots, b_{k-1}) \in \mathbb{Z}_q^k$,

$$f^F (b_0, \ldots, b_{k-1}) = q^{-k/2} \sum_{\rho \in \text{Sym} \left\{ \frac{k-1}{2}, \ldots, 0, \ldots, \frac{k-1}{2} \right\}} \text{sgn}(\rho) \prod_{-\frac{k-1}{2} \leq \ell \leq \frac{k-1}{2}} e^{2\pi i \rho(\ell) b_{\ell+k-1}/q}$$

$$= q^{-k/2} \det [e^{2\pi i \rho(m) b_{m+k-1}/q}]_{-\frac{k-1}{2} \leq \ell, m \leq \frac{k-1}{2}}$$

$$= q^{-k/2} e^{-\pi i (k-1)(b_0 + \ldots + b_{k-1})/q} \prod_{0 \leq \ell < m \leq k-1} (e^{2\pi i b_m/q} - e^{2\pi i b_\ell/q}).$$
where the range of $\ell, m$ has been translated to $0, 1, \ldots, k - 1$ in the last line and the resulting Vandermonde determinant evaluated using

$$\det[x^m_{\ell}]_{0 \leq \ell, m \leq k - 1} = \prod_{0 \leq \ell < m \leq k - 1} (x_m - x_\ell).$$

With

$$e^{2\pi ib_m/q} - e^{2\pi ib_\ell/q} = 2i e^{\pi (b_\ell + b_m)/q} \sin \frac{\pi (b_m - b_\ell)}{q},$$

and

$$\sum_{0 \leq \ell < m \leq k - 1} b_\ell + b_m = (k - 1) \sum_{0 \leq \ell \leq k - 1} b_\ell$$

this yields

$$f^F(b_0, b_1, \ldots, b_{k-1}) = q^{-k/2} k(k-1)/2 \prod_{0 \leq \ell < m \leq k - 1} 2 \sin \frac{\pi (b_m - b_\ell)}{q}.$$

Second, take $k$ even, $K = \{\pm 1, \pm 2, \ldots, \pm k/2\} \subset \mathbb{Z}_q.$ The matrix

$$[e^{2\pi imb_\ell/q}]_{\ell \in \{0, 1, \ldots, k - 1\}}^{m \in \{1, \ldots, \pm k/2\}}$$

takes the form

$$[x^{m-k/2}_\ell]_{\ell \in \{0, 1, \ldots, k - 1\}}^{m \in \{\pm 1, \ldots, \pm k/2\}}$$

where $x_\ell = e^{2\pi ib_\ell/q}.$ The determinant of the matrix $[x^m_{\ell}]$ (with $m \in \{0, 1, \ldots, k\} \setminus \{k/2\}$) has total degree $0 + 1 + \cdots + k - k/2 = k^2/2$ and is divisible by $x_m - x_\ell$ for each $0 \leq \ell < m \leq k - 1;$ the remaining factor is a homogeneous symmetric polynomial of total degree $k/2$ in $x_0, x_1, \ldots, x_{k-1},$ and we find that

$$\det[x^m_{\ell}]_{\ell \in \{0, 1, \ldots, k - 1\}}^{m \in \{\pm 1, \ldots, \pm k/2\}} = (x_0 x_1 \cdots x_{k-1})^{-k/2} \prod_{0 \leq \ell < m \leq q-1} (x_m - x_\ell) \prod_{S \subset \{0,1,\ldots,k-1\}} \prod_{|S| = k/2} x_\ell.$$ 

Here $(x_0 x_1 \cdots x_{k-1})^{-k/2} = e^{-\pi ik(b_0 + \cdots + b_{k-1})/q}.$ As was seen above,

$$\prod_{0 \leq \ell < m \leq k - 1} (e^{2\pi imb_m/q} - e^{2\pi imb_\ell/q}) = e^{\pi i(k-1)(b_0 + \cdots + b_{k-1})/q} \prod_{0 \leq \ell < m \leq k - 1} 2 \sin \frac{\pi (b_m - b_\ell)}{q},$$

so that the required determinant is given by

$$e^{-\pi i(b_0 + \cdots + b_{k-1})/q} \prod_{0 \leq \ell < m \leq k - 1} 2 \sin \frac{\pi (b_m - b_\ell)}{q} \sum_{S \subset \{0,1,\ldots,k-1\}} e^{2\pi i(S \cup b_m)/q}$$

and the equation given in the lemma statement follows as a result.\hfill $\square$
Corollary 4.9. (i) When \( k \) is odd and \( K = \mathbb{Z}_{k+1} \setminus \{\frac{k+1}{2}\} \),
\[
(1_{\text{Even}(K)} - 1_{\text{Odd}(K)})^F = (k + 1)^{-1/2} i^{k(k-1)/2} (1_{\text{Even}} - 1_{\text{Odd}}).
\]

(ii) When \( k \) is even and \( K = \mathbb{Z}_{k+1} \setminus \{0\} \),
\[
(1_{\text{Even}(K)} - 1_{\text{Odd}(K)})^F = (k + 1)^{-1/2} i^{k(k+1)/2} (1_{\text{Even}} - 1_{\text{Odd}}).
\]

Proof. Identify \( \mathbb{Z}_{k+1} \) with \( \{0, 1, \ldots, k\} \), with order \( 0 < 1 < \cdots < k \), and suppose that \( \beta : \ell \mapsto b_\ell \) is an bijection from \( \{0, 1, \ldots, k-1\} \) to \( \{0, 1, \ldots, k\} \setminus \{b\} \).

(i) First consider \( k \) odd. By Lemma 4.8 when \( K = \mathbb{Z}_{k+1} \setminus \{\frac{k+1}{2}\} \),
\[
(1_{\text{Even}(K)} - 1_{\text{Odd}(K)})^F (b_0, \ldots, b_{k-1}) = (k + 1)^{-k/2} \prod_{0 \leq \ell < m \leq k-1} 2i \sin \frac{\pi (b_m - b_\ell)}{k + 1}.
\]
We have
\[
\prod_{0 \leq \ell < m \leq k-1} 2i \sin \frac{\pi (b_m - b_\ell)}{k + 1} = \text{sgn}(\beta) \prod_{0 \leq \ell < m \leq k} 2i \sin \frac{\pi (m - \ell)}{k + 1} (-1)^b \prod_{0 \leq \ell \leq k} 2i \sin \frac{\pi (\ell - b)}{k + 1}.
\]
The product in the numerator is given by Lemma 4.4 as
\[
\prod_{0 \leq \ell < m \leq k} 2i \sin \frac{\pi (m - \ell)}{k + 1} = i^{k(k+1)/2} (k + 1)^{-1/2},
\]
and the product in the denominator by
\[
\prod_{0 \leq \ell \leq k} 2i \sin \frac{(\ell - b)\pi}{k + 1} = (-1)^b \prod_{1 \leq m \leq k} e^{-\pi im/(k+1)} (e^{2\pi im/(k+1)} - 1) = (-1)^b e^{-\pi i k(k+1)/2(k+1)} (-1)^k (k + 1) = (-1)^b i^k (k + 1),
\]
using the identity \( \prod_{1 \leq \ell \leq q-1} (1 - e^{2\pi i \ell/q}) = q \) (the sum of the coefficients in the polynomial \( 1 + t + \cdots + t^{q-1} \) with roots \( e^{2\pi i /q} \)). Hence
\[
(1_{\text{Even}(K)} - 1_{\text{Odd}(K)})^F (b_0, \ldots, b_{k-1}) = \text{sgn}(\beta) i^{k(k-1)/2} (k + 1)^{-1/2}.
\]

(ii) Second consider \( k \) even, \( K = \mathbb{Z}_{k+1} \setminus \{0\} \). By the calculation for odd \( k \) we see that
\[
(1_{\text{Even}(K)} - 1_{\text{Odd}(K)})^F (b_0, b_1, \ldots, b_{k-1}) = \text{sgn}(\beta) i^{k(k-1)/2} (k + 1)^{-1/2} \sum_{S \subseteq \{0, 1, \ldots, k\}} e^{\pi i (\sum_{m \in S} b_m - \sum_{\ell \notin S} b_\ell) / (k+1)}.
\]

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Since
\[ \sum_{m \in S} b_m - \sum_{\ell \not\in S} b_\ell = \sum_{m \in S} (b_m - b) - \sum_{\ell \not\in S} (b_\ell - b) \]
it suffices to consider the case where \( b = 0 \), i.e. \( \{b_0, b_1, \ldots, b_{k-1}\} = \{1, 2, \ldots, k\} \), in which case \( \sum_{m \in S} b_m - \sum_{\ell \not\in S} b_\ell \equiv 2 \sum_{m \in S} b_m \pmod{k+1} \). Thus we wish to evaluate
\[ \sum_{A \subset \{1, 2, \ldots, k\} \mid |A| = k/2} e^{2\pi i (\sum_{a \in A} a)/(k+1)}. \]
This sum is equal to the evaluation at \( s = e^{2\pi i/(k+1)} \) of the coefficient of \( t^{k/2} \) in the generating functions for subset sums:
\[ [t^{k/2}] \prod_{a \in \{1, 2, \ldots, k\}} (1 + s^a t) \pmod{s^{k+1} - 1} \]
\[ = \sum_{c \in \{0, 1, \ldots, k\}} \#\{A \subset \{1, 2, \ldots, k\} : |A| = k/2, \sum_{a \in A} a = c \pmod{k+1}\} s^c. \]
With
\[ \prod_{a \in \{1, 2, \ldots, k\}} (1 + e^{2\pi i/(k+1)} t) = \prod_{a \in \{1, 2, \ldots, k\}} (-t - e^{-2\pi i/(k+1)}) \]
\[ = \frac{t^{k+1} + 1}{t + 1} , \]
picking out the coefficient of \( t^{k/2} \) yields
\[ \sum_{A \subset \{1, 2, \ldots, k\} \mid |A| = k/2} e^{2\pi i (\sum_{a \in A} a)/(k+1)} = (-1)^{k/2}. \]
Hence
\[ (1_{\text{Evens}}(K) - 1_{\text{Odds}}(K))^F (b_0, b_1, \ldots, b_{k-1}) = \text{sgn}(\beta) t^{k(k-1)/2} (k+1)^{-1/2} (-1)^{k/2}. \]

4.3 Proper \( k \)-colourings by proper \( q \)-colourings \((q \geq k)\)

Finally, we can now write down the edge \( q \)-colouring model for proper edge \( k \)-colourings of \( k \)-regular graphs admitting Pfaffian labellings.

**Theorem 4.10.** Let \( k \) be odd and \( G = (V, E) \) a \( k \)-regular graph that admits a Pfaffian labelling. For each \( v \in V \), suppose the half-edges \( \{(v, e) : e \ni v\} \)
around $v$ are linearly $<$-ordered. Then for $q \geq k$ the number of proper $k$-colourings of $G$ is the partition function of a uniform edge $q$-colouring model, given by

$$\pm P(L(G); k) = q^{-|E|} \sum_{y \in \{0, 1, \ldots, q-1\}^E} \prod_{v \in V} \prod_{(v,e)<(v,f)} 2 \sin \frac{\pi (y_f - y_e)}{q},$$

where the sign depends on the order up to even permutation of half-edges around vertices.

**Proof.** Piece together Corollary 4.7 and Lemma 4.8.

Theorem 4.10 has an alternative expression in terms of vertex $q$-colouring models. Suppose the line graph $L(G) = (E, L)$ of the $k$-regular graph $G$ ($k$ odd) has an orientation of each edge $\{e, f\}$ in $L$ defined by directing $e$ towards $f$ whenever $(v, e) < (v, f)$ as half-edges, writing $(e, f)$ for this edge orientation. Then, for all $q \geq k$,

$$\pm P(L(G); k) = q^{-|E|} \sum_{y \in \{0, 1, \ldots, q-1\}^E} \prod_{(e,f) \in L} 2 \sin \frac{\pi (y_f - y_e)}{q},$$

where the choice of sign depends on the orientation of $L(G)$ (up to an even number reversals of direction on edges).

There is of course a parallel statement to Theorem 4.10 for even $k$, only the edge colouring model has a more complicated vertex weight (given in Lemma 4.8).

When $q = k$ Theorem 4.10 is tautologous. When $q = k + 1$ the edge $(k+1)$-colouring model for proper edge $k$-colourings is particularly simple, and in this case both for odd and even values of $k$ by using the result of Corollary 4.9.

**Theorem 4.11.** Let $G = (V, E)$ be a $k$-regular graph that admits a Pfaffian labelling. For each $v \in V$, suppose the half-edges $\{(v,e) : e \ni v\}$ around $v$ are linearly ordered. Then

$$\pm P(L(G); k) = (k+1)^{-|V|} \sum_{y \in \mathbb{Z}_{k+1}^E} \text{sgn}(y),$$

where $\text{sgn}(y) = +1$ if there are an even number of vertices $v \in V$ such that the injection $((v,e) : e \ni v) \mapsto 0 < 1 < \cdots < k$, $(v, e) \mapsto y_e$ has sign $-1$, and $\text{sgn}(y) = -1$ if there are an odd number. The sign on the left-hand side depends on the order up to even permutation of half-edges around vertices.

A statement of this theorem for plane cubic graphs was given as Proposition 1.3.
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