Running coupling constant of ten-flavor QCD with the Schrödinger functional method

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Abstract

Walking technicolor theory attempts to realize electroweak symmetry breaking as the spontaneous chiral symmetry breakdown caused by the gauge dynamics with slowly varying gauge coupling constant and large mass anomalous dimension. Many-flavor QCD is one of the candidates owning these features. We focus on the SU(3) gauge theory with ten flavors of massless fermions in the fundamental representation, and compute the gauge coupling constant in the Schrödinger functional scheme. Numerical simulation is performed with $O(a)$-unimproved lattice action, and the continuum limit is taken in linear in lattice spacing. We observe evidence that this theory possesses an infrared fixed point.

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I. INTRODUCTION

While the standard model has been established through a number of experiments, unnatural hierarchies are present between the electroweak scale and the Planck scale and also among the fermion masses. Large Hadron Collider (LHC) is expected to give a new insight into these hierarchies. Among various new physics models proposed so far, Technicolor (TC) model \[1\] is one of the most attractive ones in these regards, as it does not require any fundamental scalar particles, which cause the former hierarchy, and its extension, Extended TC model \[2\], has a possibility to generate the Yukawa hierarchy in a dynamical way. For recent review articles, see, for example, Refs. \[3\].

TC should be a strongly coupled vector-like gauge system, which triggers spontaneous chiral symmetry breaking (S\(\chi\)SB). It is widely known, however, that the simplest TC models obtained by rescaling ordinary QCD have already been ruled out by the \(S\)-parameter \[4\] and FCNC \[5\] constraints. Refs. \[6\] suggested a series of TC models to circumvent the FCNC problem. Those TC models appeal to the gauge dynamics in which the effective gauge coupling constant runs slowly (i.e. “walks”) at a relatively large value over a wide range of energy scale above the S\(\chi\)SB scale, and in which the chiral condensate gets large anomalous dimension. Such TC is called walking TC (WTC), and possible candidates have been enumerated through semi-quantitative analyses \[7\]. Since the dynamics that underlie WTC significantly differ from those of two or three-flavor QCD, the naive scaling argument in \(N_c\) or \(N_f\) to estimate the \(S\)-parameter would not work, and any quantitative predictions from WTC require solving nonperturbative dynamics explicitly. Lattice gauge theory provides a unique way to study this class of models from the first principles at present.

Search for candidate theories of WTC is frequently linked to the \(N_f\)-dependent phase structure of the gauge theories. Let us take SU(3) gauge theory with \(N_f\) flavors of fermions in the fundamental representation as an example. According to the analysis of the perturbative \(\beta\)-function, the system with large enough \(N_f\) (\(N_f > 16.5\)) is asymptotically non-free and trivial unless non-trivial ultraviolet fixed point exists. On the other hand, if \(N_f\) is sufficiently small (\(N_f \leq 3\)) the dynamics is QCD-like and thus in the chirally broken phase. It is believed that for the in-between \(N_f\) there exists a so-called conformal phase, where the coupling constant reaches an infrared fixed point (IRFP) without S\(\chi\)SB set in, but confinement may take place \[8\]. The range of \(N_f\) in which the conformal phase is realized is called conformal
window, and is represented by $N_{f}^{\text{crit}} < N_{f} < 16.5$. It is then natural to speculate that the
gauge dynamics slightly below $N_{f}^{\text{crit}}$ exhibit the features required for WTC; slow running
of the gauge coupling constant and $S_{\chi} \bar{S}$. The first goal in the search for WTC is thus to
identify $N_{f}^{\text{crit}}$.

In the past years, many groups have used techniques of lattice simulations to search for
$N_{f}^{\text{crit}}$ and/or WTC through hadron spectrum, eigenvalue distribution of Dirac operator, the
behavior of running coupling constant, or renormalization group analysis of candidate theo-
ries [9]. For non-lattice studies, see, for example, Refs. [10, 11]. Among various candidates,
many flavor QCD [12–19], sextet QCD [20–25], and two-color adjoint QCD [26–31] have been
intensively studied. In this work, we focus on many flavor QCD with $N_{c} = 3$ and fermions in
the fundamental representation. In a seminal work [12], the running coupling constants were
calculated for eight- and twelve-flavor QCD using the Schrödinger functional (SF) scheme on
the lattice [32]. They concluded that twelve-flavor QCD has an IRFP at $g_{SF}^{2} \sim 5$ while eight-
flavor QCD does not. In practice, the study of the running coupling alone is supposed to be
unable to fully exclude the possibility of a large IRFP because it requires lattice simulations
at arbitrarily large coupling. Even worse, the unphysical, bulk first-order phase transition
was found to occur in strong coupling regime of several gauge theories [23, 33, 34]. In such
simulations, there exists an upper limit on the bare coupling at which lattice calculation is
sensible. Nevertheless, because of the supports from the spectroscopy studies [14, 16, 18],
the conclusion in Ref. [12] that the eight-flavor QCD is QCD-like, i.e. $N_{f}^{\text{crit}} > 8$, seems to
be established nowadays.

After the work of Ref. [12], one group [15] has presented an evidence of the conformality
of twelve-flavor QCD. The opposite conclusion, however, has also been reported by the
other groups [16, 18]. Therefore $N_{f}^{\text{crit}} < 12$ is still under debate. Clearly the observed
contradiction must be clarified before going further. While in the spectroscopy study of
twelve-flavor QCD many sources of systematic uncertainties due to finite volume, taste
breaking, chiral extrapolation, lack of continuum limit, etc., remain to be quantified, the
calculation of the SF coupling constant of Ref. [12] appears, at present, to be less ambiguous.
In such a circumstance, we are tempted to explore the dynamics of ten-flavor QCD. In this
paper, we investigate, as a first step, the running coupling constant of ten-flavor QCD on
the lattice to see whether it shows conformal behavior. We find that the running slows down
and observe evidence that this theory possesses an infrared fixed point.
The paper is organized as follows. In sec. II, we give remarks on how we identify IRFP on the lattice. Sec. III summarizes the coefficients relevant to the perturbative calculation of the running coupling constant for later use. In sec. IV, the simulation setup including the definition of the running coupling constant in the Schrödinger functional scheme is presented. In sec. V, we describe analysis method and present the numerical results. Sec. VI is devoted to the summary and outlook.

II. REMARKS ON SEARCHING FOR IRFP ON THE LATTICE

Since there exists a subtlety in proving the existence of IRFP with lattice gauge theory, in this section we briefly explain what is actually calculated and then give how to identify the existence of IRFP. Here we focus on the concept only. For further details of the calculational and analysis method that we take, see the following sections.

In this work, we calculate the renormalized coupling constant in Schrödinger functional scheme at two different length scales, $L$ and $s \cdot L$. In practice, this is realized by repeating the calculation on two different volumes, $l^4$ and $(s \cdot l)^4$, at a common lattice bare coupling $g_0^2$, where $l = L/a$. We denote those couplings by $u$ (or $g^2(g_0^2, l)$) and $g^2(g_0^2, s \cdot l)$, respectively. Using those, we define the discrete beta function (DBF) by $B(u, s, l) = 1/g^2(g_0^2, s \cdot l) - 1/u$, where the rescaling factor $s$ is arbitrary but is fixed to 2. If the DBF is free from lattice discretization errors, the sign of this quantity may directly tell whether the coupling constant increases or decreases against the scale change by $s$ at the scale $L$, which is implicitly set by the value of $u$ that we can choose. Since discretization errors do exist, however, we need to take the continuum limit. The $a \to 0$ limit is taken for a fixed $L$, i.e. for a fixed $u$, by varying lattice spacing $a$. A series of the DBF thus obtained is then a function of $l$, and the $l = L/a \to \infty$ limit is expected to give the continuum limit. In summary, the DBF is constructed from a pair of lattice volumes ($l^4$, $(s \cdot l)^4$), and choice of larger $l$ results in the DBF closer to the continuum limit.

In practice, lattice spacing is varied by changing the lattice bare coupling $g_0^2$. If $g^2(g_0^2, l_2)$ turns out to be always larger than $g^2(g_0^2, l_1)$ with $l_2 > l_1$, $B(u, s, l) < 0$ should hold for any $l$ and $s > 1$. In this case, the bare coupling at which $g^2(g_0^2, l_1)$ is equal to a fixed value $u$ becomes small as lattice size $l_1$ increases or one approaches the continuum limit. Thus the $a \to 0$ limit is realized in the $g_0^2 \to 0$ limit. This is the case for asymptotically free theories.
with no IRFP such as ordinary QCD, and no subtlety is present. Even if an IRFP exists in such theories, the situation does not change as long as the input \( u \) is smaller than the IRFP, \( g^2_{\text{IRFP}} \). In other words, if the DBF extrapolated to \( l \to \infty \) (or equivalently \( 1/l \to 0 \)) is negative, the limiting value is interpreted as the continuum limit and the possibility that an IRFP exists below \( u \) is excluded.

When the DBF extrapolated to \( l \to \infty \) is positive, interpretation of numerical results becomes ambiguous. In this case, in the vicinity of \( 1/l = 0 \), \( g^2(g^2_{0}, s \cdot l) < g^2(g^2_{0}, l) \), i.e. \( B(u, s, l) > 0 \). Indeed, it happens below \( \beta = 4.4 \) in Fig. 4 of Ref. [13], for example. Then, one may expect that the \( l \to \infty \) limit is realized by \( g^2_{0} \to \infty \) on first sight. However, recalling \( \phi^4 \) theory, this expectation turns out to be too naive. In \( \phi^4 \) theory, the continuum limit exists only in the trivial case unless the theory possesses a non-trivial UV fixed point. Since the situation is similar to this case, the most plausible interpretation is that, when \( u > g^2_{\text{IRFP}} \), the continuum limit does not exist unless a nontrivial UV fixed point exists. Since no nontrivial UV fixed points has been established so far, it is not suitable to call the extrapolated value the continuum limit when it is positive. Nevertheless, we can still infer that \( u > g^2_{\text{IRFP}} \) because no other possibility remains.

We investigate the sign of the DBF, starting with the weak coupling regime \( u \sim 1 \) where the perturbative calculation is reliable and predicts a negative value. We keep monitoring the sign of the DBF with increasing \( u \). The identification of the IRFP is then made by sign-flip of the DBF extrapolated to \( l \to \infty \). Notice that, when the extrapolated value is positive, the extrapolation does not make sense and hence we do not insist that the continuum limit is determined.

**III. PERTURBATIVE ANALYSIS**

We start with defining the \( \beta \) function of an effective gauge coupling constant in a mass-independent renormalization scheme, which should have the following expansion in the perturbative regime

\[
\beta(g^2(L)) = L \frac{\partial g^2(L)}{\partial L} = b_1 g^4(L) + b_2 g^6(L) + b_3 g^8(L) + b_4 g^{10}(L) + \cdots ,
\] (1)
where $L$ denotes a length scale. The first two coefficients on the right hand side are scheme-independent, and given by

$$b_1 = \frac{2}{(4\pi)^2} \left[ 11 - \frac{2}{3} N_f \right], \quad b_2 = \frac{2}{(4\pi)^4} \left[ 102 - \frac{38}{3} N_f \right]. \quad (2)$$

The remaining coefficients are scheme-dependent and known only in the limited schemes and orders. The third coefficient takes the following form in the Schrödinger functional scheme;

$$b_3^{SF} = \bar{b}_3^{MS} + \frac{b_2 c_2^\theta - b_1 (c_3^\theta - c_2^\theta)}{2\pi}, \quad (3)$$

where $\bar{b}_3^{MS}$ is a coefficient in the $\overline{\text{MS}}$ scheme,

$$\bar{b}_3^{MS} = \frac{2}{(4\pi)^6} \left[ \frac{2857}{2} - \frac{5033}{18} N_f + \frac{325}{54} N_f^2 \right], \quad (4)$$

and the calculable quantities $c_2^\theta$ and $c_3^\theta$ depend on the spatial boundary condition imposed on the fermion fields in the SF setup, i.e., so-called $\theta$. Those for $\theta = \pi/5$ and $c_2^\theta$ for $\theta = 0$ are known to be

$$c_2^{\theta=\pi/5} = 1.25563 + 0.039863 \times N_f, \quad (5)$$

$$c_2^{\theta=0} = (c_2^{\theta=\pi/5})^2 + 1.197(10) + 0.140(6) \times N_f - 0.0330(2) \times N_f^2, \quad (6)$$

$$c_3^{\theta=0} = 1.25563 + 0.022504 \times N_f, \quad (7)$$

but $c_3^{\theta=0}$ has not been calculated yet. Although $\theta = 0$ is chosen in our simulation as described in sec. [V], the coefficients for $\theta = \pi/5$ are used only to see the situation of conformal windows inferred just from the perturbative analysis, and the potential size of difference between the two- and three-loop calculations.

The perturbative estimates of the infrared fixed point (IRFP) for SU(3) gauge theory with $N_f$ flavors of fundamental fermion are summarized in Tab. I. We note that in the three-loop perturbative analysis the existence of IRFP is determined only by the sign of $b_3$, which is always negative for the range of $N_f$ shown in Tab. I. Therefore, the existence of IRFP as well as its value may be unstable against including higher orders. Nevertheless, for $N_f \geq 14$ the difference between the two- and three-loop results is reasonably small, and one may expect that higher order corrections do not spoil the existence of IRFP or even do not change its value by much for such a large $N_f$.

According to the analysis based on Schwinger-Dyson equation, $S\chi SB$ is expected to occur when the running coupling constant reaches $g^2 \sim \pi^2$ in SU(3) gauge theories [36]. In spite
TABLE I: The perturbative IRFP obtained from the two-loop universal and the three-loop SF scheme analyses.

| $N_f$   | 4   | 6   | 8   | 10  | 12  | 14  | 16  |
|---------|-----|-----|-----|-----|-----|-----|-----|
| two-loop universal | -   | -   | -   | 27.74 | 9.47 | 3.49 | 0.52 |
| three-loop SF with $\theta = \pi/5$ | 43.36 | 23.75 | 15.52 | 9.45 | 5.18 | 2.43 | 0.47 |

of the scheme-dependence of the running coupling constant and the value of IRFP, those results motivate us to speculate that ten-flavor QCD may exhibit strongly coupled walking dynamics, and thus deserves full nonperturbative calculation.

IV. SIMULATION DETAILS

A. Schrödinger functional

We employ the Schrödinger functional (SF) method [32] to study the scale dependence of the running coupling constant. Unimproved Wilson fermion action and the standard plaquette gauge action are used without any boundary counter terms as described below.

The SF on the lattice is defined on a four dimensional hypercubic lattice with a volume $(L/a)^3 \times (T/a)$ in the cylindrical geometry. Throughout this work, the temporal extent $T/a$ is chosen to be equal to the spatial one $L/a$. Periodic boundary condition in the spatial directions with vanishing phase factor ($\theta = 0$) and Dirichlet one in the temporal direction are imposed for both gauge ($U_\mu(x)$) and fermion ($\psi(x)$ and $\bar{\psi}(x)$) fields. The boundary values for gauge and fermion fields are represented by three-by-three color matrices, $C$ and $C'$, and spinors, $\rho$, $\rho'$, $\bar{\rho}$ and $\bar{\rho}'$, respectively. The partition function of this system is given by

$$Z_{\text{SF}}(C', \bar{\rho}', \rho'; C, \bar{\rho}, \rho) = e^{-\Gamma(C', \bar{\rho}', \rho'; C, \bar{\rho}, \rho)} = \int D[U, \psi, \bar{\psi}] e^{-S[U, \psi, \bar{\psi}, C, C', \rho, \rho', \bar{\rho}, \bar{\rho}']},$$

where $\Gamma$ is the effective action, and

$$S[U, \psi, \bar{\psi}, C, C', \rho, \rho', \bar{\rho}, \bar{\rho}'] = S_g[U, C, C'] + S_q[U, \psi, \bar{\psi}, \rho, \rho', \bar{\rho}, \bar{\rho}'].$$

For the pure gauge part, we employ the plaquette action,

$$S_g[U, C, C'] = \frac{\beta}{6} \sum_x \sum_{\mu=0}^3 \sum_{\nu=0}^3 \bar{\delta}_{\mu,\nu} w_{\mu,\nu}(x_0) \text{Tr} [1 - P_{\mu,\nu}(x)],$$

where $P_{\mu,\nu}(x)$ is the plaquette operator.
where $\beta = 6/g_0^2$ denotes the inverse of the bare coupling constant, $\bar{\delta}_{\mu,\nu} = 0$ when $\mu = \nu$ otherwise 1, and $P_{\mu,\nu}(x)$ denotes a $1 \times 1$ Wilson loop on the $\mu$-$\nu$ plane starting and ending at $x$. The spatial link variables on the boundaries, the hypersurfaces at $x_0 = 0$ and $L/a$, are all set to the diagonal, constant $SU(3)$ matrices as

$$U_k(x)|_{x_0=0} = \exp[C], \quad C = \frac{ia}{L} \begin{pmatrix} \eta - \frac{\pi}{3} & 0 & 0 \\ 0 & -\frac{1}{2} \eta & 0 \\ 0 & 0 & -\frac{1}{2} \eta + \frac{\pi}{3} \end{pmatrix},$$

$$U_k(x)|_{x_0=L/a} = \exp[C'], \quad C' = \frac{ia}{L} \begin{pmatrix} -\eta - \pi & 0 & 0 \\ 0 & \frac{1}{2} \eta + \frac{\pi}{3} & 0 \\ 0 & 0 & \frac{1}{2} \eta + \frac{2\pi}{3} \end{pmatrix},$$

where $k = 1, 2, 3$, and $\eta$ is parameterizing the gauge boundary fields. The weight $w_{\mu,\nu}(x_0)$ in eq. (10) is given by

$$w_{\mu,\nu}(x_0) = \begin{cases} c_t & \text{for } (t = 0 \text{ or } t = (L/a) - 1) \text{ and } (\mu \text{ or } \nu = 0) \\ 0 & \text{for } (t = (L/a)) \text{ and } (\mu \text{ or } \nu = 0) \\ \frac{1}{2}c_s & \text{for } (t = 0 \text{ or } t = (L/a)) \text{ and } (\mu \neq 0 \text{ and } \nu \neq 0) \\ 1 & \text{for all the other cases} \end{cases} \quad (13)$$

By tuning $c_t$, $O(a)$ errors induced from the boundaries in the time direction can be removed perturbatively, but in this work we simply take its tree level values, $c_t = 1$. With this setup, the value of $c_s$ can be arbitrarily chosen because the spatial plaquettes on the boundaries do not contribute to the action. We thus set $c_s = 0$.

The fermion fields are described by the unimproved Wilson fermion action,

$$S_q[U, \psi, \bar{\psi}] = N_f \sum_{x,y} \bar{\psi}(x)D(x, y; U)\psi(y) = N_f \sum_{x,y} \bar{\psi}^{\text{lat}}(x)D^{\text{lat}}(x, y; U)\psi^{\text{lat}}(y), \quad (14)$$

$$D^{\text{lat}}(x, y; U) = \delta_{xy} - \kappa \sum_{\mu} \left\{ (1 - \gamma_\mu) U_\mu(x)\delta_{x+\hat{\mu},y} + (1 + \gamma_\mu) U_\mu^+(x - \hat{\mu})\delta_{x-\hat{\mu},y} \right\}, \quad (15)$$

where

$$\psi^{\text{lat}}(x) = \frac{1}{\sqrt{2\kappa}} \psi(x), \quad \bar{\psi}^{\text{lat}}(x) = \frac{1}{\sqrt{2\kappa}} \bar{\psi}(x), \quad D^{\text{lat}}(x, y; U) = 2\kappa D(x, y; U). \quad (16)$$

The hopping parameter $\kappa$ is related to the bare mass $m_0$ through $2\kappa = 1/(am_0 + 4)$. The dynamical degrees of freedom of the fermion field $\psi(x)$ and anti-fermion fields $\bar{\psi}(x)$ reside
on the lattice sites $x$ with $0 < x_0 < T$. On both boundaries ($x_0 = 0$ and $T$), the half of the Dirac components are set to zero and the remaining components are fixed to some prescribed values, $\rho, \bar{\rho}, \rho'$ and $\bar{\rho}'$, as

$$P_+\psi(x)|_{x_0=0} = \rho(x), \quad P_-\psi(x)|_{x_0=0} = 0,$$

$$P_-\psi(x)|_{x_0=T} = \rho'(x), \quad P_+\psi(x)|_{x_0=T} = 0,$$

$$\bar{\psi}(x)P_-|_{x_0=0} = \bar{\rho}(x), \quad \bar{\psi}(x)P_+|_{x_0=0} = 0,$$

$$\bar{\psi}(x)P_+|_{x_0=T} = \bar{\rho}'(x), \quad \bar{\psi}(x)P_-|_{x_0=T} = 0,$$

where $P_\pm = (1 \pm \gamma_0)/2$. In this work, the boundary values for the fermion fields are set to zero, i.e.

$$\rho = \rho' = \bar{\rho} = \bar{\rho}' = 0.$$

\section*{B. Definition of the running coupling}

With the gauge boundary conditions (11) and (12), the absolute minimum of the action is given by a color-electric background field denoted by $B(x)$. Then, the effective action can be defined as a function of $B$ by

$$\Gamma[B] = -\ln Z_{SF}(C', \bar{\rho}', \rho'; C, \bar{\rho}, \rho),$$

which has the following perturbative expansion in the bare coupling constant,

$$\Gamma = \frac{1}{g_0^2} \Gamma_0 + \Gamma_1 + O(g_0^4),$$

and, in particular, the lowest-order term

$$\Gamma_0 = \left[g_0^2 S_g[B]\right]_{g_0=0},$$

is exactly the classical action of the induced background field. The SF scheme coupling is then defined in the massless limit for fermions by

$$\left.\frac{\partial \Gamma}{\partial \eta}\right|_{\eta=0} = \frac{1}{g_{SF}^2(g_0^2, l=L/a)} \left.\frac{\partial \Gamma_0}{\partial \eta}\right|_{\eta=0} = \frac{k}{g_{SF}^2(g_0^2, l)},$$

where the normalization constant $k$ is determined such that $g_{SF}^2 = g_0^2$ holds in the leading order of the perturbative expansion, and is found to be

$$k = \left.\frac{\partial \Gamma_0}{\partial \eta}\right|_{\eta=0} = 12 \left(\frac{L}{a}\right)^2 \sin(2\gamma) + \sin(\gamma) = k \quad \text{with} \quad \gamma = \frac{\pi}{3} \left(\frac{a}{L}\right)^2.$$
Because of the absence of the clover term, only the $\eta$-derivative of the gauge action contributes to $1/g_{\text{SF}}^2(g_0^2, l)$.

C. Parameters

The simulation was performed on the lattice sizes of $l^4 = (L/a)^4 = 4^4, 6^4, 8^4, 12^4, \text{and } 16^4$ with the inverse of bare gauge coupling constant $\beta = 6/g_0^2$ in the range, $4.4 \leq \beta \leq 96.0$. However, the data from $l = 4$ lattices are not used in the following analysis because it was found that they have large discretization errors. We calculated the SF coupling on $18^4$ lattice with a single $\beta$ ($\beta = 4.55$), and the result is used to check the scaling violation at a specific value of $g_{\text{SF}}^2$.

The algorithm to generate the gauge configuration follows the standard HMC with five pseudo-fermion fields introduced to simulate the ten flavors of dynamical fermions. The numerical simulations were carried out on several different architectures including GPGPU, PC cluster and supercomputers. In order to achieve high performance on each architecture, the HMC code, especially the fermion solver part, were optimized depending on each architecture. In particular, mixed precision solver using multiple GPUs enables us to obtain high statistics on $g_{\text{SF}}^2$ at $l^4 = 12^4$ and $16^4$ [37]. Acceptance ratio is kept to around 80% by adjusting the molecular dynamics step size ($\delta \tau$).

Since the Wilson fermion explicitly breaks chiral symmetry, the value of $\kappa$ is tuned, for every pair of ($\beta$, $L/a$), to its critical value $\kappa_c$ realizing the massless fermion by monitoring the corresponding PCAC mass. The values of $\beta$, $\kappa$, the number of trajectories, $\delta \tau$ and the results for $l = L/a = 6, 8, 12, 16, \text{and } 18$ lattices are tabulated in Tabs. II–VI, respectively.

D. Comment on $O(a)$-unimprovement

In our pilot study, we employed the $O(a)$-improved fermion action with the perturbatively determined counter terms. With this setup, we encountered a sudden change of the plaquette and the PCAC mass at $l = 6$ and $\beta = 3.6$ when $\kappa$ was decreased from 0.1517, and we could not realize the vanishing PCAC mass. The expected SF coupling constant is about $3 \sim 4$ there. The same phenomenon also occurs on $l = 4$ lattices at almost the same value of bare coupling constant. Since the observed behavior looks similar to those reported in Refs. [23, 33, 34], we
infer that this is a bulk, first order phase transition. In order to cover the region \( g_{SF}^2 \sim O(10) \), we omitted any \( O(a) \) improvements. Thus the leading discretization error in our result is linear in lattice spacing.

Even without \( O(a) \) improvements, the bulk, first order phase transition is observed for \( \beta = 6/g_0^2 \sim 4.4 \). However, this time it happens at the renormalized coupling constant greater than the \( O(a) \)-improved case, typically \( g_{SF}^2 \sim O(10) \). Since this bulk phase transition is considered as a lattice artifact, whenever this happens we discard the gauge configurations at such \( \beta \). Thus the position of the critical \( \beta \) (\( \sim 4.4 \)) sets the lower limit on our exploration of \( \beta \).

V. ANALYSIS METHOD AND RESULTS

A. Raw data

The SF coupling constant \( (g_{SF}^2) \) and the PCAC mass \( (M) \) obtained on each \( (\beta, \kappa, l) \) are shown in Tabs. III–VII. \( g_0^2/g_{SF}^2 \) is plotted as a function of the bare coupling constant \( g_0^2 \) in Fig. 1. The figure shows that \( g_{SF}^2 \) increases with \( l = L/a \) at a fixed \( g_0^2 \), but the change between the data from \( l = 12 \) and \( l = 16 \) is tiny. For later use, we fit the data of \( g_0^2/g_{SF}^2 \) to an interpolating formula as a function of the bare coupling constant \( g_0^2 \). Among various functional forms we examined, the following form

\[
\frac{g_0^2}{g_{SF}^2(g_0^2, l)} = \frac{1 - a_{l,1} g_0^4}{1 + p_{1,l} \times g_0^2 + \sum_{n=2}^{N} a_{l,n} \times g_0^{2n}},
\]

(27)

turned out to give the minimum \( \chi^2/dof \) for a fixed number of free parameters, \( N \). We thus employ eq. (27). In eq. (27), \( p_{1,l} \) is the \( l \)-dependent coefficient and we have calculated them perturbatively in the SF scheme

\[
p_{1,l} = \begin{cases} 
0.4477107831 & \text{for } l = 6 \\
0.4624813408 & \text{for } l = 8 \\
0.4756888260 & \text{for } l = 12 \\
0.4833079203 & \text{for } l = 16 \\
0.4864767958 & \text{for } l = 18 
\end{cases} \quad (28)
\]

The other coefficients \( a_{l,n} \)'s are determined for each \( l \) independently. We optimize the degree of polynomial \( N \) in the denominator of eq. (27) by monitoring \( \chi^2/dof \), and take \( N = 5 \) for
$l = 6$ and $12$, and $N = 4$ for $l = 8$ and $16$. Tab. VII shows the fit results for the coefficients in eq. (27). The fit results are also shown as the region sandwiched by a pair of solid curves in Fig. 1.

FIG. 1: $g_0^2$ dependence of $g_{0}^2/g_{SF}^2$ for $l = L/a=6, 8, 12$ and $16$. The right panel magnifies the region of $g_0^2 \in [1.1, 1.40]$.

Hereafter we denote the SF coupling obtained at a bare coupling constant $g_0^2$ and at a lattice length of $l$ by $g_{SF}^2(g_0^2, l)$ and its continuum counterpart by $g_{SF}^2(L)$.

B. Discrete $\beta$ function

In order to see the scale dependence of the SF coupling constant, we analyze the discrete $\beta$ function (DBF) introduced in Refs. [20, 23]. The whole procedure is described below.

First, we choose an initial value of the running coupling constant, denoted by $u$. This implicitly sets the initial length scale $L_0$ through $g_{SF}^2(L_0) = u$. Using the interpolating formula (27) for the lattice size $l (= L/a)$, the bare coupling constant $g_0^2$ is numerically obtained by solving the equation $g_{SF}^2(g_0^2, l) = u$. $l$ is identified with $L_0/a$, so that the lattice spacing at $g_0^2$ is found to be $a(g_0^2, l) = L_0/l$. Now we choose a rescaling factor, $s$. The lattice step scaling function $\Sigma_0(u, s, l)$ is then defined as the SF coupling for $l' = s \cdot l$ at the same bare coupling $g_0^2$, i.e.

$$\Sigma_0(u, s, l) \equiv g_{SF}^2(g_0^2(s \cdot l)|_{g_{SF}^2(g_0^2, l)} = u) \quad (29)$$

The meaning of the subscript “0” becomes clear soon. Of course, both $l$ and $s \cdot l$ must be equal to one of $6, 8, 12$ and $16$, and hence the possible values for the rescaling factor $s$ are
| $\beta$ | $\kappa$ | Trajs. | pLq. | $\delta \tau$ | Acc. | $g_{SF}^2$ | $M$  |
|---------|---------|--------|------|-------------|------|----------|------|
| 96.0000 | 0.1267030 | 39,700 | 0.979268(0.000002) | 0.0076 | 0.827(0.002) | 0.06431(0.00006) | 0.00012(0.00003) |
| 96.0000 | 0.1267070 | 49,900 | 0.979267(0.000002) | 0.0076 | 0.826(0.002) | 0.06428(0.00005) | -0.00004(0.00003) |
| 48.0000 | 0.1276060 | 40,100 | 0.958852(0.000005) | 0.0098 | 0.857(0.002) | 0.13221(0.00010) | -0.00143(0.00002) |
| 48.0000 | 0.1276100 | 41,100 | 0.958846(0.000004) | 0.0098 | 0.857(0.002) | 0.13209(0.00010) | -0.00166(0.00002) |
| 24.0000 | 0.1295180 | 24,700 | 0.917566(0.000009) | 0.0149 | 0.848(0.002) | 0.28079(0.00015) | -0.00006(0.00002) |
| 24.0000 | 0.1295200 | 60,300 | 0.917562(0.000005) | 0.0152 | 0.838(0.002) | 0.28086(0.00010) | -0.00006(0.00001) |
| 12.0000 | 0.1339640 | 48,700 | 0.833056(0.000014) | 0.0250 | 0.812(0.002) | 0.64450(0.00076) | -0.00004(0.00005) |
| 9.6000  | 0.1365680 | 160,300| 0.789765(0.000012) | 0.0256 | 0.806(0.002) | 1.30413(0.00194) | -0.00006(0.00006) |
| 7.4000  | 0.1410690 | 120,500| 0.724148(0.000015) | 0.0270 | 0.854(0.002) | 1.51024(0.00244) | -0.00025(0.00008) |
| 6.3000  | 0.1451400 | 17,400 | 0.673231(0.000076) | 0.0333 | 0.817(0.002) | 1.93684(0.00069) | 0.00044(0.00021) |
| 6.0000  | 0.1466380 | 33,600 | 0.655993(0.000034) | 0.0333 | 0.837(0.002) | 1.93605(0.00036) | 0.00004(0.00010) |
| 5.5000  | 0.1497590 | 50,300 | 0.622923(0.000025) | 0.0370 | 0.817(0.002) | 2.38340(0.01092) | 0.00042(0.00020) |
| 5.5000  | 0.1497610 | 36,000 | 0.622942(0.000027) | 0.0357 | 0.827(0.002) | 2.36232(0.00634) | -0.00018(0.00022) |
| 5.5000  | 0.1497620 | 140,300| 0.622977(0.000023) | 0.0357 | 0.831(0.001) | 2.37542(0.00963) | 0.00023(0.00014) |
| 5.2000  | 0.1521330 | 220,300| 0.600097(0.000019) | 0.0380 | 0.812(0.001) | 2.80668(0.01246) | -0.00015(0.00014) |
| 5.0000  | 0.1539800 | 59,900 | 0.583463(0.000049) | 0.0400 | 0.806(0.002) | 3.28837(0.06618) | -0.00005(0.00046) |
| 4.6000  | 0.1585140 | 33,800 | 0.545776(0.000055) | 0.0400 | 0.813(0.002) | 5.47008(0.13064) | 0.00092(0.00043) |
| 4.6000  | 0.1585150 | 150,000| 0.545680(0.000041) | 0.0400 | 0.813(0.001) | 5.41263(0.09891) | 0.00123(0.00042) |
| 4.5000  | 0.1599020 | 100,300| 0.535280(0.000061) | 0.0400 | 0.813(0.001) | 7.02516(0.24479) | 0.00111(0.00069) |
| 4.5000  | 0.1599030 | 100,300| 0.535305(0.000066) | 0.0400 | 0.813(0.002) | 6.70575(0.19622) | 0.00033(0.00061) |
| 4.4215  | 0.1610680 | 105,900| 0.526357(0.000087) | 0.0385 | 0.825(0.001) | 8.88882(0.36944) | 0.00238(0.00097) |
| 4.4215  | 0.1610820 | 92,400 | 0.526692(0.000066) | 0.0385 | 0.826(0.001) | 8.90139(0.32355) | 0.00073(0.00075) |
| 4.4000  | 0.1614210 | 249,500| 0.524331(0.000060) | 0.0400 | 0.811(0.001) | 9.60163(0.19661) | 0.00051(0.00050) |
| 4.4000  | 0.1614220 | 182,500| 0.524342(0.000091) | 0.0400 | 0.812(0.001) | 10.17980(0.33990) | 0.00119(0.00073) |
| 4.4000  | 0.1614230 | 250,500| 0.524387(0.000062) | 0.0400 | 0.811(0.001) | 10.07713(0.25379) | 0.00049(0.00053) |

**TABLE II:** Simulation parameters and results obtained at $L/a=6$. 

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| β     | κ       | Trajs. | p|q| | δτ   | Acc.   | $g^2_{SF}$ | M        |
|-------|---------|--------|---|---|---|-------|--------|-----------|---------|
| 96.0000 | 0.1263270 | 22,500 | 0.979420(0.000002) | 0.0056 | 0.811(0.004) | 0.06434(0.00004) | 0.00001(0.00001) |
| 48.0000 | 0.1272250 | 18,300 | 0.958843(0.000003) | 0.0100 | 0.818(0.007) | 0.13247(0.00017) | 0.00002(0.00002) |
| 24.0000 | 0.1291450 | 42,300 | 0.917260(0.000004) | 0.0125 | 0.804(0.003) | 0.28282(0.00023) | -0.00004(0.00002) |
| 12.0000 | 0.1335850 | 68,500 | 0.832266(0.000007) | 0.0167 | 0.828(0.005) | 0.65380(0.00071) | 0.00004(0.00002) |
| 9.6000  | 0.1361800 | 21,820 | 0.788830(0.000013) | 0.0200 | 0.828(0.006) | 0.88751(0.00287) | 0.00008(0.00005) |
| 7.4000  | 0.1406600 | 63,330 | 0.723081(0.000010) | 0.0250 | 0.818(0.003) | 1.34182(0.00071) | 0.00004(0.00016) |
| 6.8000  | 0.1426200 | 41,500 | 0.697409(0.000013) | 0.0250 | 0.816(0.002) | 1.56232(0.00023) | -0.00004(0.00011) |
| 6.3000  | 0.1447000 | 28,000 | 0.672208(0.000021) | 0.0250 | 0.816(0.003) | 1.81987(0.00034) | 0.00034(0.00014) |
| 6.0000  | 0.1462000 | 47,000 | 0.654999(0.000012) | 0.0250 | 0.820(0.003) | 2.01248(0.00016) | 0.00042(0.00011) |
| 5.5000  | 0.1492700 | 35,900 | 0.622016(0.000021) | 0.0286 | 0.797(0.003) | 2.48139(0.00016) | -0.00021(0.00015) |
| 5.0000  | 0.1533600 | 27,900 | 0.582458(0.000038) | 0.0250 | 0.825(0.004) | 3.46930(0.00034) | 0.00094(0.00034) |
| 4.8000  | 0.1554270 | 114,500| 0.564464(0.000020) | 0.0250 | 0.860(0.001) | 4.35348(0.00026) | 0.00026(0.00024) |
| 4.7000  | 0.1565500 | 35,400 | 0.554789(0.000040) | 0.0256 | 0.854(0.002) | 4.87595(0.21035) | 0.00027(0.00051) |
| 4.6200  | 0.1575500 | 86,300 | 0.546856(0.000030) | 0.0312 | 0.783(0.001) | 6.23744(0.25321) | -0.00023(0.00033) |
| 4.5500  | 0.1588500 | 56,570 | 0.536316(0.000058) | 0.0278 | 0.827(0.002) | 8.28029(0.57687) | -0.00010(0.00059) |
| 4.5000  | 0.1591300 | 107,100| 0.534108(0.000036) | 0.0250 | 0.859(0.001) | 8.40630(0.37369) | -0.00007(0.00038) |
| 4.4800  | 0.1594000 | 41,555 | 0.531781(0.000085) | 0.0250 | 0.827(0.002) | 8.57214(0.57202) | 0.00027(0.00070) |
| 4.4215  | 0.1602640 | 160,900| 0.525143(0.000050) | 0.0263 | 0.837(0.001) | 12.21877(0.49625) | -0.00012(0.00041) |
| 4.4215  | 0.1602700 | 127,500| 0.525149(0.000058) | 0.0250 | 0.861(0.001) | 12.62365(0.68980) | -0.00059(0.00048) |
| 4.4200  | 0.1602700 | 29,700 | 0.524651(0.000132) | 0.0278 | 0.828(0.002) | 13.15085(0.99774) | 0.00214(0.00075) |
| 4.4000  | 0.1606000 | 229,500| 0.522502(0.000057) | 0.0278 | 0.819(0.002) | 15.00764(0.69115) | 0.00020(0.00042) |

TABLE III: Simulation parameters and results obtained at $L/a=8$. 

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TABLE IV: Simulation parameters and results obtained at $L/a=12$.

| $\beta$ | $\kappa$ | Trajs. | plq. | $\delta\tau$ | Acc. | $g_{SF}^2$ | $M$ |
|---------|-----------|--------|------|-------------|------|------------|-----|
| 48.0000 | 0.1269700 | 11,200 | 0.958648(0.000002) | 0.0056 | 0.815(0.0003) | 0.13304(0.00033) | −0.00014(0.00003) |
| 24.0000 | 0.1288929 | 54,620 | 0.916777(0.000002) | 0.0083 | 0.798(0.0002) | 0.28432(0.00036) | −0.00008(0.00002) |
| 12.0000 | 0.1333359 | 68,955 | 0.831306(0.000003) | 0.0125 | 0.808(0.0002) | 0.66007(0.00119) | −0.00012(0.00003) |
| 9.6000  | 0.1359350 | 86,700 | 0.787681(0.000003) | 0.0133 | 0.806(0.0002) | 0.90325(0.00233) | −0.00001(0.00003) |
| 7.4000  | 0.1404060 | 106,050 | 0.721824(0.000003) | 0.0154 | 0.795(0.0002) | 1.36896(0.00543) | −0.00001(0.00004) |
| 6.8000  | 0.1423250 | 45,150 | 0.696157(0.000006) | 0.0167 | 0.819(0.0002) | 1.59998(0.00983) | 0.00091(0.00006) |
| 6.3000  | 0.1444050 | 23,500 | 0.671015(0.000014) | 0.0182 | 0.867(0.0002) | 2.10612(0.00284) | 0.00075(0.00012) |
| 6.0000  | 0.1459000 | 43,296 | 0.653900(0.000008) | 0.0182 | 0.820(0.0002) | 2.58933(0.00496) | 0.00056(0.00009) |
| 5.8000  | 0.1470200 | 43,400 | 0.641479(0.000007) | 0.0182 | 0.799(0.0002) | 3.06212(0.00321) | 0.00017(0.00017) |
| 5.5000  | 0.1489400 | 45,200 | 0.621162(0.000012) | 0.0167 | 0.842(0.0003) | 3.39969(0.00892) | 0.00015(0.00013) |
| 5.2000  | 0.1512000 | 68,000 | 0.598557(0.000009) | 0.0200 | 0.781(0.0002) | 3.99690(0.00892) | 0.00004(0.00004) |
| 5.0800  | 0.1522350 | 25,480 | 0.588819(0.000013) | 0.0167 | 0.848(0.0002) | 4.84805(0.01450) | 0.00011(0.00011) |
| 5.0000  | 0.1529700 | 90,921 | 0.582135(0.000019) | 0.0167 | 0.826(0.0002) | 5.67356(0.00898) | 0.00044(0.00020) |
| 4.8000  | 0.1549700 | 194,300 | 0.564213(0.000006) | 0.0182 | 0.810(0.0002) | 6.48050(0.11594) | 0.00009(0.00011) |
| 4.6000  | 0.1572300 | 127,922 | 0.544423(0.000012) | 0.0182 | 0.818(0.0002) | 7.98853(0.38589) | 0.00075(0.00017) |
| 4.5500  | 0.1578500 | 57,260 | 0.539025(0.000021) | 0.0192 | 0.802(0.0002) | 10.15231(1.11827) | 0.00125(0.00027) |
| 4.5000  | 0.1585500 | 104,570 | 0.534014(0.000049) | 0.0250 | 0.701(0.0007) | 13.03915(1.33994) | −0.00237(0.00039) |

limited. The difference between $\Sigma_0(u, s, l)$ and $u$ gives the scale dependence through the scale change from $L$ to $s \cdot L$, up to lattice artifacts.

Since the raw data of $1/g_{SF}^2(g_0^2, l)$ fluctuate around zero in the strong coupling region, converting from $1/g_{SF}^2(g_0^2, l)$ to $g_{SF}^2(g_0^2, l)$ sometimes induces huge statistical uncertainty. To avoid this we treat the inverse coupling constant, $1/g_{SF}^2(g_0^2, l)$, directly. Then, to see the scale dependence of the inverse coupling constant, we introduce the lattice DBF [20, 23] by

$$B_0(u, s, l) = \frac{1}{\Sigma_0(u, s, l)} - \frac{1}{u}. 
$$

We calculate the continuum limit of this function for various initial values of the coupling constant, $u$. If the sign of the DBF in the continuum limit turns out to flip at a certain renormalized coupling constant $u$, it indicates the existence of IRFP.
C. improving discretization errors

Since $O(a)$ discretization errors are not improved at all in the lattice actions, it is important to remove the scaling violation as much as possible. To do this, we perform the following improvements on the step scaling function and the DBF before taking the continuum limit.

First let $\sigma(u, s)$ be the continuum limit of $\Sigma_0(u, s, l)$, i.e. $\sigma(u, s) = g_{SF}^2(sL)$ with $u = \ldots$.
TABLE VII: The results for the coefficients in the fit function \( g_{\text{SF}}^2(L) \). Its perturbative expression is given by

\[
\sigma(u,s) = u + s_0 u^2 + s_1 u^3 + s_2 u^4 + \cdots ,
\]

\( s_0 = b_1 \ln(s) , \)
\( s_1 = \ln(s) \left( b_1^2 \ln(s) + b_2 \right) , \)
\( s_2 = \ln(s) \left( b_1^3 \ln^2(s) + \frac{5}{2} b_1 b_2 \ln(s) + b_3 \right) , \)

where \( b_i \)'s are the coefficients of the \( \beta \)-function introduced in sec. III. Recalling the parametric form of the discretization error \[35\], the error normalized by \( \sigma(u,s) \), denoted by \( \delta_0(u,s,l) \), is written as

\[
\delta_0(u,s,l) = \frac{\Sigma_0(u,s,l) - \sigma(u,s)}{\sigma(u,s)} = \delta^{(1)}(s,l) u + \delta^{(2)}(s,l) u^2 + O(u^3) .
\]

With eq. \[31\], the discretization error at the lowest order in \( u \) is found to be

\[
\delta^{(1)}(s,l) = \left( p_{1,s,l} - b_1 \ln(s \cdot l) \right) - \left( p_{1,l} - b_1 \ln(l) \right) = p_{1,s,l} - p_{1,l} - b_1 \ln(s) .
\]

Now by replacing \( \Sigma_0(u,s,l) \) in eq. \[35\] with \( \Sigma_1(u,s,l) = \Sigma_0(u,s,l)/(1 + \delta^{(1)}(s,l) u) \), the discretization error reduces to \( O(u^2) \). Using \( \Sigma_1(u,s,l) \), the one-loop improved DBF is defined

| \( L/a \) | \( aN \) | \( \chi^2/\text{dof} \) | \( a_{L/a,1} \) | \( a_{L/a,2} \) | \( a_{L/a,3} \) | \( a_{L/a,4} \) | \( a_{L/a,5} \) |
|---------|------|-----------------|------------|------------|------------|------------|------------|
| 6 3     |      | 9.0(1.3) 0.4906(0.0025) -0.2749(0.0105) -0.1897(0.0151) |
| 6 4     |      | 1.4(0.5) 0.5048(0.0014) -0.3993(0.0119) 0.1136(0.0283) -0.2042(0.0184) |
| 6 5     |      | 1.3(0.5) 0.5015(0.0032) -0.4240(0.0256) 0.2538(0.1301) -0.4043(0.1815) 0.0899(0.0808) |
| 8 3     |      | 6.3(1.7) 0.5068(0.0018) -0.2308(0.0104) -0.2412(0.0150) |
| 8 4     |      | 6.4(0.3) 0.5153(0.0019) -0.3410(0.0260) 0.0405(0.0629) -0.1852(0.0390) |
| 8 5     |      | 6.5(0.8) 0.5153(0.0051) -0.3419(0.1697) 0.0444(0.7500) -0.1904(0.9672) 0.0021(0.3906) |
| 12 3    |      | 3.0(1.0) 0.5239(0.0047) -0.1923(0.0118) -0.3019(0.0198) |
| 12 4    |      | 3.1(0.6) 0.5400(0.0038) -0.3614(0.0376) 0.1063(0.0884) -0.2671(0.0550) |
| 12 5    |      | 1.0(0.6) 0.5438(0.0039) -0.2783(0.0726) -0.2779(0.2977) 0.2457(0.3815) -0.2165(0.1582) |
| 16 3    |      | 4.9(1.4) 0.5308(0.0055) -0.1881(0.0266) -0.3057(0.0375) |
| 16 4    |      | 4.8(0.8) 0.5520(0.0039) -0.4948(0.0663) 0.4387(0.1516) -0.4762(0.0903) |
| 16 5    |      | 4.9(0.9) 0.5538(0.0050) -0.4403(0.1324) 0.1801(0.5648) -0.1283(0.7332) -0.1457(0.3025) |
This completes the one-loop improvement.

The above procedure can be repeated to an arbitrarily higher order in \( u \), but it requires the perturbative coefficients like \( p_{1,l} \) and the perturbative expression of \( \sigma(u, s) \) to the corresponding order in \( u \). All the coefficients necessary for the two-loop improvement are not available at this moment. Instead, we follow an alternative prescription proposed in Ref. [38].

After the one-loop improvement, the scaling violation is written as

\[
\delta_1(u, s, l) = \Sigma_1(u, s, l) - \sigma(u, s) = \delta^{(2)}(s, l) u^2 + O(u^3). \tag{38}
\]

If one can somehow know \( \delta^{(2)}(s, l) \), the scaling violation can be reduced to \( O(u^3) \) by replacing \( \Sigma_0(u, s, l) \) in eq. (35) with

\[
\Sigma_2(u, s, l) = \Sigma_0(u, s, l)/(1 + \delta^{(1)}(s, l) u + \delta^{(2)}(s, l) u^2). \tag{39}
\]

\( \delta^{(2)}(s, l) \) can be determined by fitting our data for \( \delta_1(u, s, l) \) in eq. (38) to the function quadratic in \( u \). Notice that in order for this fitting to make sense, the perturbative series of \( \sigma(u, s) \) must be known through \( O(u^3) \). Since the first two coefficients, \( b_1 \) and \( b_2 \), are available, the correct value of \( \sigma(u, s) \) can be calculated to \( O(u^3) \) as seen from eq. (31).

\( \delta_1(u, s, l) \) is fitted to the form of eq. (38), neglecting \( O(u^3) \) or higher order terms, for all possible pairs of \((s, l)\) as shown in Fig. 2. The fit has to be performed in a weak coupling region where the perturbative expansion is reliable. We examine two fit ranges, \( 0 \leq u \leq 1.6 \) and \( 0 \leq u \leq 2.0 \) to see the fit range dependence. The extracted values for \( \delta^{(2)}(s, l) \) are tabulated in Tab. VIII together with \( \delta^{(1)}(s, l) \) defined in eq. (36).

The table shows that the values of \( \delta^{(1)}(s, l) \) and \( \delta^{(2)}(s, l) \) lie between \( 10^{-2} \) and \( 10^{-3} \), and \( \delta^{(2)}(s, l) \) turns out not to depend on the fit range. In the following analysis, we employ \( \delta^{(2)}(s, l) \) from the shorter fit range. It is also seen from the table that generally the coefficients for \((s, l) = (4/3, 12)\) are the smallest among others. This is anticipated because the improvement coefficient vanish as \( s \) approaches to unity or \( l \) becomes large. An exception is the one-loop coefficient \( \delta^{(1)}(4/3, 6) \). Since two-loop coefficient \( \delta^{(2)}(4/3, 6) \) is, however, much larger than \( \delta^{(1)}(4/3, 6) \), the smallness of \( \delta^{(1)}(4/3, 6) \) is probably by accident. In the data sets we have, the data with \((s, l) = (4/3, 6)\) is the coarsest one. As we will show in the
TABLE VIII: Coefficients for perturbative correction, \( \delta^{(1)}(s, l) \) and \( \delta^{(2)}(s, l) \), for each pair of \((s, l)\).

The square brackets in the first column indicate the fit range in \( u \).

following subsections, this data turns out to suffer from non-linear scaling violation larger than the linear one in the strong coupling region. Thus, we omit this data point throughout the analysis. Using \( \delta^{(2)}(s, l) \) thus obtained, we define the two-loop improved step scaling function \( \Sigma_2(u, s, l) \) in eq. (39), and in turn the two-loop improved DBF

\[
B_2(u, s, l) = \frac{1}{\Sigma_2(u, s, l)} - \frac{1}{u}.
\] (40)

D. strategy

The continuum limit is taken for a fixed rescaling factor \( s \) and a fixed input length scale \( L \) varying a lattice spacing \( a \) (\( = L/l \)). As described in the preceding subsections, an input length scale is fixed by choosing a particular value of input coupling \( u \). However, for a given \( s \) the number of data sets with different \( a \) in this work is, at most, two; \((s, l) = (2, 6)\) and \((2, 8)\) for \( s = 2 \). While it is still possible to employ these two sets of data to evaluate the continuum limit, the validity of the linear extrapolation can not be tested. Alternatively, we may supplement a data set with a desired \( s \) by interpolating or extrapolating data of \( g_{SF}^2(g_0^2, l) \) in \( l \). However, a lack of guiding principles in the interpolation or extrapolation may cause a systematic uncertainty. In this work, we use the two available data sets, \((s, l) = (2, 6)\) and \((2, 8)\) to evaluate the continuum limit by linear extrapolation, and the other data sets are used to monitor the validity of the linear extrapolation. For this purpose, we introduce a relation which approximately converts the DBF for \( s' \) into that for \( s \), as follows.
FIG. 2: Fit of $\delta_1$ to a quadratic function of $u$. The solid and dashed curves show the fit results and the fit ranges.

We start with a closer look at the discretization error. The discretization error of the lattice DBF, i.e. $B_i(u, s, l) - B(u, s)$ ($i=0, 1, 2$), can be expressed in terms of an asymptotic expansion in $1/l$ [35] as

$$B_i(u, s, l) - B(u, s) = \left( \frac{1}{l} - \frac{1}{s l} \right) e_i(u) + O(l^{-2}),$$

(41)
where $e_i(u)$ is an unknown coefficient of $O(a)$ error and is a function of $u$. We then define the rescaled lattice DBF by

$$B'_i(u, s, l, s') = \frac{\ln(s)}{\ln(s')} B_i(u, s', l).$$

(42)

In addition, using the continuum counterpart of eq. (42), we define

$$\delta B(u, s, s') = B(u, s) - \frac{\ln(s)}{\ln(s')} B(u, s'),$$

(43)

which represents the difference between the true continuum DBF and the rescaled continuum DBF. Combining eqs. (41), (42) and (43) together and introducing

$$\xi(s, l, s') = \ln(s) \ln(s') \left( \frac{1}{l} - \frac{1}{s'l} \right),$$

(44)

we arrive at

$$B'_i(u, s, l, s') = B(u, s) + \xi(s, l, s') e_i(u) - \delta B(u, s, s') + O(l^{-2}).$$

(45)

Therefore, if $\delta B(u, s, s')$ and $O(l^{-2})$ (or higher order) discretization errors are negligible compared to the statistical error of $B'_i(u, s, l, s')$, the numerical data of $B'_i(u, s, l, s')$ plotted against $\xi$ will line up on a single line, and even two unknown coefficients in eq. (45), $B(u, s)$ and $e_i(u)$, for given $u$ and $s$ can be extracted from that behavior. Instead, if both or one of them is large, the data will not align. Thus, whether $B'_i(u, s, l, s')$ plotted against $\xi$ aligns or not tests the validity of the linear extrapolation within the statistical uncertainty.

We comment on the size of $\delta B(u, s, s')$. Solving eq. (11) perturbatively, the continuum DBF is found to be

$$B(u, s) = -\ln(s) \left\{ \frac{\beta(u)}{u^2} + u^2 \ln(s) \left( \frac{1}{2} b_1 b_2 + \frac{1}{3} b_1^2 b_2 \ln(s) + b_1 b_3 + \frac{1}{2} b_2^2 \right) \right\} + O(u^4 \ln^2(s)), $$

(46)

and thus the perturbative expression of $\delta B(u, s, s')$ is

$$\delta B(u, s, s') = u^2 \ln(s) \ln\left( \frac{s}{s'} \right) \left[ -\frac{1}{2} b_1 b_2 + u \left\{ -\frac{1}{3} b_1^2 b_2 \ln(s s') - \left( b_1 b_3 + \frac{1}{2} b_2^2 \right) \right\} \right] + O(u^4 \ln(s) \ln(s/s')).$$

(47)

Since the numerical values of $b_i$’s are small, e.g. $b_1 \sim 0.055$, $b_2 \sim -0.002$, $b_3^{\text{SF}} \sim O(10^{-4})$, $\delta B(u, s, s')$ is also small in the perturbative regime as $10^{-5} \times u^2 (1.5 + 0.6 u)$, $10^{-5} \times u^2 (1.1 +
0.4 u) and $10^{-5} \times a^2 \left(-1.1 - 0.5 u\right)$ for $(s, s') = (2, 16/12), (2, 12/8)$ and $(s, s') = (2, 16/6)$, respectively. As $u$ becomes large, $\delta B(u, s, s')$ may become sizable and at some point exceed the statistical error of $B'_i(u, s, l, s')$. Then, the alignment will be deformed. Notice that, the smaller $B(u, s')$ is, the smaller $\delta B(u, s, s')$ is, and in particular, when $B(u, s') = 0$ for a certain $s'$, $\delta B(u, s, s') = 0$ holds exactly.

We extract the continuum DBF $B(u, s)$ as follows. First, we assume linear scaling violation and calculate $B(u, s)$ for $s=2$ by extrapolating the two data sets, $(s', l) = (2, 6)$ and $(2, 8)$, to $\xi = 0$. Since $s' = s$, $B'_i(u, s, l, s') = B_i(u, s', l)$ and $\delta B(u, s, s') = 0$ by construction. Thus we do not have to rely on the smallness of $\delta B(u, s, s')$. Then, to test the linearity of the scaling violation, we calculate the rescaled lattice DBF $B'_i(u, s, l, s')$ with $s=2$ from the other data sets and plot them as a function of $\xi(s, l, s')$. If the data align within the statistical error of $B'_i(u, 2, l, s')$, the assumption of the linear scaling violation is valid, $\delta B(u, s, s')$ is negligible and then the value of $B(u, s)$ thus obtained is reliable. Alternatively, once the linearity is confirmed, we can even determine the continuum limit by taking the linear extrapolation of $B'_i(u, 2, l, s')$. Since $\delta B(u, s, s')$ is negligible in perturbative regime, the linearity can be tested more rigorously in such a regime. When the data do not align, either or both of the linear violation dominance and small $\delta B(u, s, s')$ are invalid and the result for $B(u, s)$ becomes uncertain.

E. extraction of the continuum DBF

Extrapolation to the continuum limit described in the following is carried out for every jack-knife ensemble, and the statistical error in the continuum limit is estimated by the single elimination jack-knife method.

We begin with analysis at relatively weak coupling. Figure 3 shows the continuum limit of $B'_i(u, s, l, s')$ for $s = 2$ ($i=1, 2$) at the four representative values of $1/u$ corresponding to $u = 1.0, 2.0, 10/3, 5.0$, where the data with $s' = 2$ are shown in filled symbols and the other in open symbols and two data of the one-loop improved lattice DBF ($B'_i$) with $s' = 2$ (filled squares) are linearly extrapolated to $\xi = 0$. The two-loop improvement described in sec. V C is equivalent to tuning the improvement coefficients such that the resulting DBF reproduces the perturbative DBF in the region $0 < u < 1.6$. Indeed, the constant fit of the two-loop improved DBF with $s' = 2$ (filled diamonds) gives the value consistent with
FIG. 3: Linear extrapolation of $B_1$ (filled squares) and constant fit of $B'_2$ (filled diamonds) to the continuum limit. The extrapolation and fit use the data with $s' = 2$ (filled symbols). The data with $s' \neq 2$ (open symbols) are also shown to see whether they align or not. The values of $(s', l)$ of the data shown are $(4/3, 6)$, $(2, 6)$, $(8/3, 6)$, $(3/2, 8)$, $(2, 8)$, and $(4/3, 12)$ from right to left. The data points are slightly shifted in horizontal direction for clarity. The perturbative predictions including the 2-loop (plus) and 3-loop (star) effects are also shown.

The deviation from the perturbative prediction appears at $1/u = 0.3$, where the linear extrapolation gives the value closer to and consistent with zero. It is important to note that the data of the one-loop improved DBF align down to $1/u = 0.2$ with a slope increasing with
$u$. From this observation, we conclude that, in the region $1/u \geq 0.2 \ (u \leq 5)$, $\delta B(u,s,s')$ is small, the scaling violation is linear for the one-loop improved DBF and hence the extracted continuum limit is reliable.

Next let us move on to the result at a stronger coupling shown in Fig. 4. As seen from the figure, first the data of $B_1'$ except for the one with $(s',l)=(4/3, 6)$ (right-most point) remains to align within the statistical uncertainty. Thus, the linear extrapolation of $B_1$ is reliable at $1/u = 0.15$. Secondly, the linear extrapolation of $B_1$ and the constant fit of $B_2$ lead to different continuum DBF. It appears that the constant fit of $B_2$ is no longer valid and the linear fit appears to be more reasonable. Indeed, the linear fit of $B_2$ (solid line and open diamond at $\xi = 0$) turns out to give the consistent limit as shown in the figure.

From the alignment of $B_1'$, we infer that both $\delta B$ and non-linear scaling violation remain small. This is consistent with the fact that the continuum DBF obtained by linear extrapolation of $B_1$ is consistent with zero and thus $\delta B$ should be small as well.

The deviation of the coarsest data from the linear behavior indicates that the linear discretization error no longer dominates others in the data with $(s',l)=(4/3, 6)$. Since in general non-linear scaling violation can be large for small $l$, the data with $l = 6$ may suffer from this though it is not visible in the figure. To evaluate the potential uncertainty due to the $O(l^{-2})$ discretization error, we performed a linear fit without the $l = 6$ data. The fit result is shown as open square at $\xi = 0$ and the dashed line in Fig. 4. The result is consistent with that using the $s' = 2$ data only.

From Figs. 3 and 4, it turns out that for $1/u \lesssim 0.3$ the extracted continuum DBF is consistent with zero. This indicates that in this region the running coupling constant reaches an infrared fixed point or, at least, the running appreciably slows down. In order to further investigate the existence of the infrared fixed point or, at least, the running appreciably slows down. In order to further investigate the existence of the infrared fixed point, we include the data obtained from $l=18$ lattice at $\beta = 4.55$ into analysis. This data is combined with the data with $l = 12$ to construct $B_1'$ with $(s',l)=(3/2, 12)$. At $\beta = 4.55$, the inverse SF coupling for $l=12$ turns out to be $1/u = 0.107$. On $l=6$ lattice, this value of $1/u$ is realized at $\beta \sim 4.4$. In such a small $\beta$, the SF couplings are not calculated on $l=12, 16$ lattices, and hence the following analysis is carried out without the data from $l=6$ lattices.

$B_1'$ constructed from the $l=18$ data is shown in Fig. 4 (filled circle). Since the four data points shown align well, we take the linear extrapolation using all of them and obtain the positive value in $\xi = 0$. Interpretation of this result needs care as mentioned sec. 4. The most
FIG. 4: Same as Fig. 3 but the result at the stronger coupling is plotted. Solid lines denote the linear extrapolation using the data with $s'=2$ (filled symbols). Dashed line shows the linear extrapolation using the data with the three smallest $\xi$.

FIG. 5: Same as Fig. 4 but the data point obtained with $(s', l)=(3/2, 12)$ (filled circle) is included in the analysis at $1/u=0.107$. The dashed line and the open square at $\xi=0$ are the result of the linear fit.

A plausible explanation for this observation is that an IRFP exists in $u_{\text{IRFP}} < 1/0.107 = 9.35$.

Figure 6 shows the $1/u$ dependence of the continuum DBF, where the results are compared with the perturbative calculations. It is seen that the running starts to slow down at around $1/u \sim 0.5$, and eventually the coupling constant reaches a fixed point in the range of $0.107 < 1/u < 0.3$. When the DBF is positive, it is non-trivial for the continuum limit.
VI. SUMMARY AND OUTLOOK

In this work, the running coupling constant of ten-flavor QCD is numerically investigated using lattice technique. The extrapolation of the DBF to the continuum limit is taken linearly assuming that the $O(a)$ scaling violation dominates the higher order ones. The DBF extrapolated approaches zero from below as the SF coupling constant $u$ increases and when $u \gtrsim 10/3$ the DBF becomes consistent with zero. Further investigation at one particular strong coupling $u = 9.3$ ($1/u = 0.107$) is made using the data from the large lattice ($l = 18$), and suggests that the continuum DBF at this coupling is not negative. This indicates the existence of the infrared fixed point $10/3 \lesssim g_{\text{IRFP}}^2 \lesssim 9.3$. The linear extrapolation is reasonably justified within the statistical error, but further rigorous check is clearly preferable. Combining our result with that of Ref. [12], the critical number of flavors which separates the conformal phase and the broken phase is $8 < N_f^{\text{crit}} < 10$.

In order to confirm the existence of IRFP or even determine the value of the fixed point more precisely, data from larger lattices with high statistics are necessary. It is, however, difficult to do with machines currently available to us, and probably more efficient methods or different approaches are necessary to go further. As mentioned in sec. [1] the conformal
window can also be studied by looking at hadrons’ spectroscopy or renormalization group analysis on the lattice. Currently the conclusions based on various methods are not consistent among them. In order to pin down $N_f^{\text{crit}}$, these contradictions must be clarified with further studies.

What is really important in the context of the WTC is the anomalous dimension of the $\bar{\psi}\psi$ operator. The calculation of the anomalous dimension in ten-flavor QCD is on-going. The result will be published elsewhere.

Once one has fixed an attractive candidate for WTC, the next important step would be the calculation of the $S$-parameter. The calculational method has been established in Ref. [39], where the QCD $S$-parameter is calculated on the lattice for the first time and is correctly reproduced. Later, the method was applied to three-flavor QCD [40], sextet QCD [41] and six-flavor QCD [42]. In Ref. [42], the evidence of the reduction of $S$-parameter is reported. Another important quantity which should be calculated is obviously the mass spectrum of the candidate theory, including vector and scalar resonances, the decay constant of the NG boson and the chiral condensate. Although the precise determinations of these quantities are challenging, the direct comparison with the upcoming LHC results is extremely interesting and hence we believe that such calculations are worth a lot of efforts.

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