FOURIER QUASICRYSTALS AND DISTRIBUTIONS ON EUCLIDEAN SPACES WITH SPECTRUM OF BOUNDED DENSITY

S. YU. FAVOROV

Faculty of Mathematics and Computer Science, Jagiellonian University, Lojasiewicza 6, 30-348 Krakow, Poland
Faculty of Mathematics and Informatics, Karazin’s Kharkiv National University, Svobody sq. 4, Kharkiv, 61022, Ukraine
e-mail: sfavorov@gmail.com

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Abstract. We consider temperate distributions on Euclidean spaces with uniformly discrete support and locally finite spectrum. We find conditions on coefficients of distributions under which they are finite sums of derivatives of generalized lattice Dirac combs. These theorems are derived from properties of families of discretely supported measures and almost periodic distributions.

1. Introduction

The Fourier quasicrystal may be considered as a mathematical model for atomic arrangements having a discrete diffraction pattern. There are a lot of papers devoted to study properties of Fourier quasicrystals or, more generally, crystalline measures. For example, we can mark collections of papers [4,24], in particular, the basic paper [15].

When studying the properties of Fourier quasicrystals, it is natural and important to describe their support. In [17–19], Lev and Olevskii considered various conditions on a discrete measure on the line and its spectrum under which the support of the measure was embedded into a finite union of translates of some discrete lattice (i.e., arithmetic progressions with the same difference set). The support of the measure is always assumed to be uniformly discrete, which means that distances between any two points are bounded from below by the same strictly positive constant. In [16] these...
results were extended to temperate distributions on the line with discrete support and spectrum.

The multidimensional case is fundamentally different from the univariate one. We need additional restrictions on the discrete measure (for example, its positivity) so that its support is a subset of a finite union of translates of the single lattice. In the general case, it is only possible to prove that the support is a finite union of translates of several incommensurable lattices, and this result is accurate. This problem is usually solved by using Cohen’s Idempotent Theorem. In [7] this is done for measures, and in [9] for temperate distributions with uniformly discrete support and spectrum.

In the present paper we extend the last result for temperate distributions with locally finite spectrum of bounded density and get their explicit representation (Theorem 7). Remark that the proof is easier than in [9]. It is based on some properties of families of measures with discrete supports (Theorem 8). Also, we present a simple sufficient condition for a crystalline measure to be a Fourier quasicrystal (Theorem 6).

2. Definitions and notations

Denote by $S(\mathbb{R}^d)$ the Schwartz space of test functions $\varphi \in C^\infty(\mathbb{R}^d)$ with the finite norms

$$N_n(\varphi) = \sup_{\mathbb{R}^d} \max \{ (1 + |x|)^n |D^k \varphi(x)|, \ n = 0, 1, 2, \ldots, \}$$

where

$$|x| = (x_1^2 + \cdots + x_d^2)^{1/2}, \ k = (k_1, \ldots, k_d) \in (\mathbb{N} \cup \{0\})^d,$$

$$|k| = k_1 + \cdots + k_d, \ D^k = \partial_{x_1}^{k_1} \cdots \partial_{x_d}^{k_d}.$$  

These norms generate the topology on $S(\mathbb{R}^d)$. Elements of the space $S^*(\mathbb{R}^d)$ of continuous linear functionals on $S(\mathbb{R}^d)$ are called temperate distributions. The Fourier transform of a temperate distribution $f$ is defined by the equality

$$\hat{f}(\varphi) = f(\hat{\varphi}) \ \text{for all} \ \varphi \in S(\mathbb{R}^d),$$

where

$$\hat{\varphi}(y) = \int_{\mathbb{R}^d} \varphi(x) \exp\{-2\pi i \langle x, y \rangle\} \, dx$$

is the Fourier transform of the function $\varphi$. Also,

$$\check{\varphi}(y) = \int_{\mathbb{R}^d} \varphi(x) \exp\{2\pi i \langle x, y \rangle\} \, dx.$$
means the inverse Fourier transform. Note that the Fourier transform is the isomorphism of \( S(\mathbb{R}^d) \) on itself and, respectively, \( S^*(\mathbb{R}^d) \) on itself.

We will say that a set \( A \subset \mathbb{R}^d \) is \textit{locally finite} if the intersection of \( A \) with any ball is finite, \( A \) is \textit{relatively dense} if there is \( R < \infty \) such that \( A \) intersects with each ball of radius \( R \), and \( A \) is \textit{uniformly discrete}, if \( A \) is locally finite and has a strictly positive separating constant

\[
\eta(A) := \inf \{ |x - x'| : x, x' \in A, \ x \neq x' \}.
\]

Also, we will say that \( A \) is \textit{polynomially discrete}, or shortly \( p\text{-discrete} \), if there are positive numbers \( c, h \) such that

\[
(1) \quad |x - x'| \geq c \min\{1, |x|^{-h}\} \quad \text{for all } x, x' \in A, \ x \neq x'.
\]

A set \( A \) is of \textit{bounded density} if it is locally finite and

\[
\sup_{x \in \mathbb{R}^d} \# A \cap B(x, 1) < \infty.
\]

As usual, \( \#E \) means a number of elements of the finite set \( E \), and \( B(x, r) \) denotes the ball with center in \( x \) and radius \( r \).

An element \( f \in S^*(\mathbb{R}^d) \) is called a \textit{crystalline measure} if \( f \) and \( \hat{f} \) are complex-valued measures on \( \mathbb{R}^d \) with locally finite supports. The support of \( \hat{f} \) for a distribution \( f \in S^*(\mathbb{R}^d) \) is called \textit{spectrum} of \( f \).

Denote by \( |\mu|(A) \) the variation of a complex-valued measure \( \mu \) on \( A \). If both measures \( |\mu| \) and \( |\hat{\mu}| \) have locally finite supports and belong to \( S^*(\mathbb{R}^d) \), we say that \( \mu \) is a \textit{Fourier quasicrystal}. A measure \( \mu = \sum_{\lambda \in \Lambda} a_\lambda \delta_\lambda \) with \( a_\lambda \in \mathbb{C} \) and countable \( \Lambda \) is called \textit{purely point}. In this case we will replace \( a_\lambda \) with \( \mu(\lambda) \).

A \textit{full-rank lattice} \( L \) is a discrete (locally finite) subgroup of \( \mathbb{R}^d \), which has the form \( T\mathbb{Z}^d \), where \( T \) is a nondegenerate linear operator on \( \mathbb{R}^d \). The lattice

\[
L^* = \{ y \in \mathbb{R}^d : \langle \lambda, y \rangle \in \mathbb{Z} \text{ for all } \lambda \in L \}
\]

is called the conjugate lattice. It follows from Poisson’s formula

\[
\sum_{n \in \mathbb{Z}^d} \hat{f}(n) = \sum_{n \in \mathbb{Z}^d} f(n), \quad f \in S(\mathbb{R}^d),
\]

that for a full-rank lattice \( L = T\mathbb{Z}^d \) we have

\[
(2) \quad \sum_{\lambda \in L} \delta_\lambda = |\det T|^{-1} \sum_{\lambda \in L^*} \delta_\lambda.
\]
Following Meyer [21], we will say that a measure $\mu$ on $\mathbb{R}^d$ is a generalized lattice Dirac comb, if it has the form

$$\mu = \sum_{j=1}^{J} \sum_{\lambda \in \lambda_j + L_j} P_j(\lambda) \delta_\lambda,$$

where $L_j$ are full-rank lattices, $\lambda_j \in \mathbb{R}^d$, and $P_j(\lambda)$ are trigonometric polynomials.

3. Previous results

We begin with the following result of Lev and Olevskii:

**Theorem 1** [17,18]. Let $\mu$ be a crystalline measure on $\mathbb{R}$ with uniformly discrete support and spectrum. Then $\text{supp} \mu$ is a subset of a finite union of translations of a lattice $L \subset \mathbb{R}$ (i.e., of arithmetic progressions with the same difference), and $\mu$ has the form

$$\mu = \sum_{j=1}^{J} \sum_{\lambda \in \lambda_j + L} P_j(\lambda) \delta_\lambda,$$

where $\lambda_j$, $j = 1, \ldots, J$ are real numbers.

Theorem 1 remains valid under the weaker assumption that $\text{supp} \mu$ is a relatively dense set of bounded density (not assumed to be uniformly discrete) [19]. On the other hand, there exist examples of crystalline measures on $\mathbb{R}$, whose supports are not contained in any finite union of translations of a lattice (see, for example, [14,19,21]).

The natural analog of Theorem 1 for temperate distributions on $\mathbb{R}$ was obtained recently by Lev and Reti.

**Theorem 2** [16]. Let

$$f = \sum_{\lambda \in \Lambda} \sum_{n} p_n(\lambda) \delta_\lambda^{(n)}$$

be a temperate distribution on $\mathbb{R}$ such that $\Lambda = \text{supp} f$ and $\Gamma = \text{supp} \hat{f}$ are uniformly discrete sets. Then there is a discrete lattice $L \subset \mathbb{R}$ such that

$$f = \sum_{\tau,\omega,l,n} c(\tau,\omega,l,n) \sum_{\lambda \in L} \lambda^l e^{2\pi i \omega \lambda} \delta_\lambda^{(n)} \delta_{\lambda+\tau},$$

where $(\tau,\omega,l,n)$ goes through a finite set of quadruples such that $\tau,\omega$ are real numbers, $l,n$ are nonnegative integers, and $c(\tau,\omega,l,n)$ are complex numbers.
The result is still valid if the support $\Lambda$ is locally finite of bounded density and the coefficients $p_n(\lambda)$ have a polynomial growth, while the spectrum $\Gamma$ is uniformly discrete.

There were several analogs of the above theorems for positive measures on $\mathbb{R}^d$ ([18,19]) and for temperate distributions on $\mathbb{R}^d$, $d > 1$, under additional conditions on the sets of differences $\Lambda - \Lambda$ and $\Gamma - \Gamma$ (see [23]), or the set of differences $\Lambda - \Lambda$ and distribution coefficients (see [5]). It was proved in all these theorems that the support of the distribution embedded in a finite union of translations of a unique lattice. Note that Representation (4) for a measure $\mu$ implies that the spectrum of the measure is uniformly discrete.

Also, Representation (5) for a distribution $f$ implies that the distribution $\hat{f}$ has a similar form and a uniformly discrete support.

But there is a signed measure on $\mathbb{R}^2$ whose support and spectrum are both uniformly discrete and simultaneously are unions of pairs incommensurable full-rank lattices ([6]). Therefore, the direct analog of Theorem 1 for $d > 1$ is not true. But the support of a measure on $\mathbb{R}^d$ is often a finite number of translations of several lattices ([2,12,13,22]). In fact, the following result was proved:

**Theorem 3.** Let $\mu$ be a measure on $\mathbb{R}^d$ with uniformly discrete support $\Lambda$. If the complex masses $\mu(\lambda)$ at points $\lambda \in \Lambda$ take values in a finite set $F \subset \mathbb{C}$, the measure $\hat{\mu}$ is purely point and satisfies the condition

\[
|\hat{\mu}|(B(0,r)) = O(r^d) \quad (r \to \infty),
\]

then $\Lambda$ is a finite union of translations of several, possibly incommensurable, full-rank lattices.

Later the finiteness condition of $F$ was significantly weakened:

**Theorem 4** [7]. Let $\mu$ be a measure on $\mathbb{R}^d$ with uniformly discrete support $\Lambda$ such that $\inf_{\lambda \in \Lambda} |\mu(\{\lambda\})| > 0$ and the measure $\hat{\mu}$ is purely point and satisfy (6). Then

\[
\Lambda = \bigcup_{j=1}^J (L_j + \lambda_j),
\]

where $L_j$ are full-rank lattices and $\lambda_j \in \Lambda$.

Moreover, in the above conditions

\[
\mu = \sum_{j=1}^N \sum_s b_{j,s} e^{2\pi i \langle y, \alpha_{j,s} \rangle} \delta_{L_j + \lambda_j}, \quad \sum_s |b_{j,s}| < \infty,
\]

where $\alpha_{j,s}$ form a bounded subset of $\mathbb{R}^d$ ([8]).

We also got the corresponding result for temperate distributions:
**Theorem 5** [9]. Let a temperate distribution

\[ f = \sum_{\lambda \in \Lambda} \sum_k p_k(\lambda) D^k \delta_{\lambda}, \quad k \in (\mathbb{N} \cup \{0\})^d \]

have both uniformly discrete support \( \Lambda \) and spectrum \( \Gamma \). If there are constants \( c, C \) such that for the distribution coefficients the inequalities

\[ 0 < c \leq \sup_k |p_k(\lambda)| \leq C < \infty \quad \text{for all } \lambda \in \Lambda, \]

hold, then \( \Lambda \) is a finite union of translations of several full-rank lattices that can be incommensurable.

Here uniformly discreteness of support and corresponding representations of a measure or a distribution do not imply uniformly discreteness of their spectrum.

**Remark.** Proposition 1 (see below in Section 5) shows that for every \( f \in S^*(\mathbb{R}^d) \) there is \( K < \infty \) such that \( p_k(\lambda) = 0 \) for all \( \|k\| > K \) and \( \lambda \in \Lambda \) in (7), therefore we may replace \( \sup_k \) by \( \sum_k \) in Theorem 5.

**4. Summary of the main results**

It is not very difficult to check that for every measure \( \mu \in S^*(\mathbb{R}^d) \) with uniformly discrete support we have \( |\mu| \in S^*(\mathbb{R}^d) \). We get some strengthening of this result:

**Theorem 6.** Let a measure \( \mu \) have \( p \)-discrete support and belong to \( S^*(\mathbb{R}^d) \). Then \( |\mu| \in S^*(\mathbb{R}^d) \) too. In particular, every crystalline measure with \( p \)-discrete support and \( p \)-discrete spectrum is Fourier quasicrystal.

The following theorem is the main result of this article.

**Theorem 7.** Let a temperate distribution

\[ f = \sum_{\lambda \in \Lambda} \sum_k p_k(\lambda) D^k \delta_{\lambda}, \quad k \in (\mathbb{N} \cup \{0\})^d \]

have a uniformly discrete support \( \Lambda \) and a spectrum \( \Gamma \) of bounded density. If there are constants \( c, C \) such that for the distribution coefficients the inequalities

\[ 0 < c \leq \sup_k |p_k(\lambda)| \leq C < \infty \quad \text{for all } \lambda \in \Lambda, \]

hold, then there are full-rank lattices \( L_j \) and \( \lambda_j, \omega_n \in \mathbb{R}^d \) such that

\[ f = \sum_{j=1}^{J} \sum_k \sum_{\lambda \in \lambda_j+L_j} P_{j,k}(\lambda) D^k \delta_{\lambda}, \]
where \( P_{j,k}(\lambda) = \sum_{n=1}^{N} b_{j,k,n} e^{2\pi i \langle \lambda, \omega_n \rangle} \) are trigonometric polynomials.

In view of Proposition 1 below, here we may replace \( \sup_k \) by \( \sum_k \).

The proof of Theorem 7 uses the following assertion, which, in our opinion, is interesting in itself.

**Theorem 8.** Let \( \mu_s, s \leq S, \) be complex measures in \( S^*(\mathbb{R}^d) \) with uniformly discrete supports \( \Lambda_s \) such that \( \hat{\mu}_s \) be pure point measures, which satisfy (6). Set \( \Lambda = \bigcup_{s \leq S} \Lambda_s \). If

\[
\inf_{\lambda \in \Lambda} \sum_s |\mu_s(\lambda)| > 0,
\]

then each \( \Lambda_s \) is contained in a finite union of translations of full-rank lattices.

If, in addition, the set \( \Lambda \) is uniformly discrete, then every \( \mu_s \) with locally finite spectrum is a generalized lattice Dirac comb (3).

The condition (9) is essential even in the case of a single measure \( \mu \).

The first part of this theorem was proved earlier ([9], Proposition 4). But the case of uniformly discrete \( \Lambda \) and necessity of condition (9) are discussed here for the first time.

### 5. Almost periodic distributions

Here we recall the definition and properties of almost periodic functions and distributions that will be used in what follows. A more complete exposition of these issues is available in [1,3,20,21,25].

**Definition 1.** A continuous function \( g \) on \( \mathbb{R}^d \) is almost periodic if for any \( \varepsilon > 0 \) the set of its \( \varepsilon \)-almost periods

\[
\{ \tau \in \mathbb{R}^d : \sup_{t \in \mathbb{R}^d} |g(t + \tau) - g(t)| < \varepsilon \}
\]

is a relatively dense set in \( \mathbb{R}^d \).

An equivalent definition follows:

**Definition 2.** A continuous function \( g \) on \( \mathbb{R}^d \) is almost periodic if for any sequence \( \{t_n\} \subset \mathbb{R}^d \) there is a subsequence \( \{t'_n\} \) such that the sequence of functions \( g(t + t'_n) \) converges uniformly in \( t \in \mathbb{R}^d \).

Using the appropriate definition, one can prove various properties of almost periodic functions:

- almost periodic functions are bounded and uniformly continuous on \( \mathbb{R}^d \),
- the class of almost periodic functions is closed with respect to taking absolute values and linear combinations of a finite family of functions,
the limit of a uniformly convergent sequence of almost periodic functions is also almost periodic,

- any finite family of almost periodic functions has a relatively dense set of common $\varepsilon$-almost periods,

- for any almost periodic function $g(x)$ on $\mathbb{R}^d$ the function $h(t) = g(x^0 + tx)$ is almost periodic in $t \in \mathbb{R}$ for any fixed $x^0, x \in \mathbb{R}^d$; in particular, $g(x_1, \ldots, x_d)$ is almost periodic in each variable $x_j \in \mathbb{R}$, $j = 1, \ldots, d$, if the other variables are held fixed.

Typical examples of almost periodic functions on $\mathbb{R}^d$ are sums of the form

$$f(t) = \sum_n a_n e^{2\pi i (t, s_n)}, \quad a_n \in \mathbb{C}, \ s_n \in \mathbb{R}^d, \sum_n |a_n| < \infty.$$

It is not hard to check that $\hat{f} = \sum_n a_n \delta_{s_n}$.

**Definition 3** [1,11,25]. A distribution $g$ is almost periodic if the function $(g(y), \varphi(t-y))$ is almost periodic in $t \in \mathbb{R}^d$ for each $C^\infty$-function $\varphi$ on $\mathbb{R}^d$ with compact support.

Clearly, every finite linear combination of almost periodic distributions is almost periodic, and each almost periodic distribution has a relatively dense support.

If $g$ is a measure on $\mathbb{R}^d$, the above definition differs from the usual definition of almost periodicity for measures (instead of $\varphi \in C^\infty$, they take continuous $\varphi$ with compact support). However these definitions coincide for nonnegative measures or translation bounded measures, i.e., measures with uniformly bounded variations in balls of radius 1 (see [1,10,20]).

**Proposition 1** [5, Proposition 1]. i) If a distribution $f \in S^*(\mathbb{R}^d)$ has locally finite support, then

$$f = \sum_{\lambda \in \Lambda} \sum_{\|k\| \leq K} p_k(\lambda) D^k \delta_{\lambda}, \quad k \in (\mathbb{N} \cup \{0\})^d,$$

where $K < \infty$ does not depend on $\lambda$.

ii) If a distribution $f \in S^*(\mathbb{R}^d)$ has $p$-discrete support, then all distribution coefficients $p_k(\lambda)$ have the polynomial growth, i.e., for some $T < \infty$

$$p_k(\lambda) = O(|\lambda|^T) \quad \text{as} \ \lambda \to \infty.$$

These results were proved in [23] for the case of uniformly discrete support.

**Remark.** It is well known (see, e.g., [26]) that every distribution with locally finite support has form (10), where $K$ depends on $\lambda$. By Schwartz’
theorem (see, e.g., [28], Ch. 2), there are \( n = n(f) \) and \( C = C(f) \) such that for all \( \varphi \in S(\mathbb{R}^d) \)
\[
|\langle f, \varphi \rangle| \leq C N_n(\varphi).
\]
Part i) of Proposition 1 follows easily from the last inequality.

**Proposition 2** (see also [7, Lemma 1]). Let \( f \) be a distribution from \( S^*(\mathbb{R}^d) \) with \( \hat{f} \) be a pure point measure such that \( |\hat{f}(B(0,r))| = O(r^T) \) as \( r \to \infty \) with some \( T < \infty \). Then \( f \) is an almost periodic distribution.

**Proof.** Let \( \hat{f} = \sum_n a_n \delta_{\gamma_n} \) and \( F(r) = |\hat{f}(B(0,r))| = \sum_{n:|\gamma_n| < r} |a_n| \). For any \( \varphi \in S^*(\mathbb{R}^d) \) we have \( \varphi(x) = o(|x|^{-T-1}) \), therefore,
\[
\langle f(x), \varphi(t-x) \rangle = \langle \hat{f}(y), e^{2\pi i \langle y, t \rangle} \hat{\varphi}(y) \rangle = \sum_n a_n \hat{\varphi}(\gamma_n) e^{2\pi i \langle \gamma_n, t \rangle}
\]
and
\[
\sum_n |a_n| |\hat{\varphi}(\gamma_n)| \leq C + \sum_{n:|\gamma_n| > r_0} |a_n| |\gamma_n|^{-T-1}
\]
\[
\leq C + (T + 1) \int_{r_0}^{\infty} F(r) r^{-T-2} dr < \infty.
\]
Consequently, \( f \) is an almost periodic distribution. \( \square \)

**Proposition 3** [5, Proposition 3]. Let \( f \in S^*(\mathbb{R}^d) \) be a distribution given by (10) with both support \( \Lambda \) and spectrum \( \Gamma \) that are locally finite. We assume that
\[
|p_k(\lambda)| = O(r^{d+M}), \quad M \geq 0 \quad (r \to \infty).
\]
Then \( \hat{f} \) has the form
\[
\hat{f} = \sum_{\gamma \in \Gamma} \sum_{m \in \mathbb{N} \cup \{0\}^d} q_m(\gamma) D^m \delta_\gamma, \quad m \in (\mathbb{N} \cup \{0\})^d,
\]
and, for \( m = M \in \mathbb{N} \cup \{0\} \),
\[
q_m(\gamma) = O(|\gamma|^K) \quad \text{as} \quad \gamma \to \infty
\]
with the same \( K \) as in (11). If \( \Gamma \) is uniformly discrete, then this estimate is valid for all \( q_m(\gamma) \).
In particular, if \( f \) is a measure with uniformly bounded masses, support of bounded density, and locally finite spectrum, then \( K = M = 0 \) and \( f \) is a measure with uniformly bounded masses as well.

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Proposition 4. Let $f_k, k \in (\mathbb{N} \cup \{0\})^d$ be almost periodic temperate distributions. Set $f(y) = \sum_{\|k\| \leq K} y^k f_k(y)$, where, as usually, $y^k = y_1^{k_1} \ldots y_d^{k_d}$. If $\text{supp } f$ is of bounded density, then the same is valid for all $\text{supp } f_k, \|k\| \leq K$.

For proving this result, we need the following lemma.

Lemma 1. Let $g_0(y), g_1(y), \ldots, g_N(y)$ be temperate distributions on $\mathbb{R}^d$ such that for every $C^\infty$-function $\varphi(y)$ with compact support the functions

$$(g_n(y), \varphi(t - y)), \quad t = (t_1, t_2, \ldots, t_d), \quad n = 0, \ldots, N,$$

are almost periodic in $t_1 \in \mathbb{R}$ for every fixed $(t_2, \ldots, t_d) \in \mathbb{R}^{d-1}$. Put

$$F(y) = \sum_{n=0}^{N} y_1^n g_n(y), \quad y = (y_1, \ldots, y_d).$$

If the distribution $F$ has a support of bounded density, then all the distributions $g_n$ have supports of bounded densities.

Proof. Note that for every $C^\infty$-function $\varphi(y)$ with compact support we have

$$(F(y), \varphi(t - y)) = \sum_{n=0}^{N} (g_n(y), y_1^n \varphi(t - y)) = \sum_{m=0}^{N} \sum_{n=m}^{N} \Phi_{n,m}^\varphi(t),$$

where

$$\Phi_{n,m}^\varphi(t) = \binom{n}{m} (g_n(y), (y_1 - t_1)^{n-m} \varphi(t - y)), \quad n \geq m$$

are almost periodic functions in the variable $t_1 \in \mathbb{R}$.

Let $F$ has support of bounded density. Check that $\text{supp } g_n$ is of bounded density for each $n$.

First consider the case $n = N$. Set

$$\Gamma = \text{supp } F, \quad T = \max_{x \in \mathbb{R}^d} \# [B(x, 1) \cap \Gamma].$$

Suppose that for some $x \in \mathbb{R}^d$ there are different points $x^1, x^2, \ldots, x^{T+1} \in \text{supp } g_N \cap B(x, 1/2)$. Put

$$\alpha = \left(1/2\right) \min_{i,j,i \neq j} |x^i - x^j|,$$

and let $\varphi_1, \varphi_2, \ldots, \varphi_{T+1}$ be $C^\infty$-functions with supports in the ball $B(0, \alpha)$ such that

$$(g_j(y), \varphi_j(x^j - y)) \neq 0, \quad j = 1, \ldots, T + 1.$$
Set \( \varepsilon = (1/2) \min_j \left| (g_N(y), \varphi_j(x^j - y)) \right| \). Since the functions
\[
(g_N(y), \varphi_j(x^j + \tau e_1 - y)), \quad e_1 = (1, 0, \ldots, 0),
\]
are almost periodic in \( \tau \in \mathbb{R} \), we may consider their common \( \varepsilon \)-almost periods and take an arbitrarily large \( \tau \) with the property
\[
\left| (g_N(y), \varphi_j(x^j + \tau e_1 - y)) \right| > \varepsilon, \quad j = 1, \ldots, T + 1.
\]
On the other hand, by (13), we get for every \( j \)
\[
(F(y), \varphi_j(x^j + \tau e_1 - y)) = (g_N(y), \varphi_j(x^j + \tau e_1 - y)) + \sum_{m=0}^{N-1} [(x^j)_1 + \tau]^{m-N} \sum_{n=m}^{N} \Phi_{n,m}^{\varphi_j}(x^j + \tau e_1).
\]
All functions
\[
\Phi_{n,m}^{\varphi_j}(x^j + \tau e_1), \quad j = 1, \ldots, T + 1, \quad n, m = 0, 1, \ldots, N, \quad m \leq n,
\]
are almost periodic, therefore they are uniformly bounded in \( \tau \). Hence for \( \tau \) large enough
\[
(F(y), \varphi_j(x^j + \tau e_1 - y)) \neq 0.
\]
Therefore there are \( t^j \in B(x^j + \tau e_1, \alpha) \cap \Gamma, \quad j = 1, \ldots, T + 1, \) and these points are distinct. Since \( B(x^j + \tau e_1, \alpha) \subset B(x + \tau e_1, 1) \), we get
\[
\# [\Gamma \cap B(x + \tau e_1, 1)] > T.
\]
This is impossible, therefore \( \# \left[ \text{supp} g_N \cap B(x, 1/2) \right] \leq T \) and \( \text{supp} g_N \) is of bounded density.

Suppose that every distribution \( g_n, n > l \) has support of bounded density. Prove that the same assertion is true for \( n = l \). Set
\[
\Gamma' = \Gamma \cup \text{supp} g_N \cup \text{supp} g_{N-1} \cup \cdots \cup \text{supp} g_{l+1}, \quad T' = \max_{x \in \mathbb{R}^d} \# [B(x, 1) \cap \Gamma'].
\]
Suppose that for some \( x \in \mathbb{R}^d \) there are different points \( x^1, x^2, \ldots, x^{T'+1} \in \text{supp} g_l \cap B(x, 1/2) \). Put
\[
\alpha = (1/2) \min_{i \neq j} |x^i - x^j|,
\]
and let \( \varphi_1, \varphi_2, \ldots, \varphi_{T'+1} \) be \( C^\infty \)-functions with supports in the ball \( B(0, \alpha) \) such that
\[
(g_l(y), \varphi_j(x^j - y)) \neq 0, \quad j = 1, \ldots, T' + 1.
\]
Set $\varepsilon = (1/2) \min_j |(g_l(y), \varphi_j(x^j - y))|$. Since the functions 

$$(g_l(y), \varphi_j(x^j + \tau e_1 - y))$$

are almost periodic in $\tau \in \mathbb{R}$, we see that there are arbitrarily large $\tau$ with the property

$$|(g_l(y), \varphi_j(x^j + \tau e_1 - y))| > \varepsilon, \quad j = 1, \ldots, T' + 1.$$ 

On the other hand, by (13), we get for every $j$

$$[(x^j)_1 + \tau]^{-l}(F(y), \varphi_j(x^j + \tau e_1 - y)) = (g_l(y), \varphi_j(x^j + \tau e_1 - y))$$

$$+ [(x^j)_1 + \tau]^{-l} S_{>l}(x^j + \tau e_1) + [(x^j)_1 + \tau]^{-l} S_{<l}(x^j + \tau e_1),$$

where

$$S_{>l}(x^j + \tau e_1) = \sum_{m=\lambda+1}^{N} [(x^j)_1 + \tau]^m \sum_{n=m}^{N} \Phi_{n,m}^\varphi_j(x^j + \tau e_1)$$

and

$$S_{<l}(x^j + \tau e_1) = \sum_{m=0}^{\lambda-1} [(x^j)_1 + \tau]^m \sum_{n=m}^{N} \Phi_{n,m}^\varphi_j(x^j + \tau e_1).$$

Hence for $\tau$ large enough

$$[(x^j)_1 + \tau]^{-l}(F(y), \varphi_j(x^j + \tau e_1 - y)) - [(x^j)_1 + \tau]^{-l} S_{>l}(x^j + \tau e_1) \neq 0.$$ 

Therefore there are $t_j \in B(x^j + \tau e_1, \alpha) \cap \Gamma'$, $j = 1, \ldots, T' + 1$, and these points are distinct. Since $B(x^j + \tau e_1, \alpha) \subset B(x + \tau e_1, 1)$, we get

$$\#\left[\Gamma' \cap B(x + \tau e_1, 1]\right] > T'.$$

This is impossible, therefore $\#[\text{supp} \, g_l \cap B(x, 1/2)] \leq T'$ and $\text{supp} \, g_l$ are of bounded density. Carrying out this argument successively for $l = N - 1, N - 2, \ldots, 1$, we obtain the assertion of the Lemma. $\Box$

**Proof of Proposition 4.** We have

$$f(y) = \sum_{k_1=0}^{K} y_1^{k_1} g_{k_1}(y), \quad \text{where} \quad g_{k_1}(y) = \sum_{k_2 + \cdots + k_d \leq K-k_1} y_2^{k_2} \cdots y_d^{k_d} f_{k_1,\ldots,k_d}(y).$$
For any \( \varphi \in C^\infty \) with compact support we get
\[
(g_{k_1}(y), \varphi(t - y)) = \sum_{k_2 + \cdots + k_d \leq K - k_1} (f_{k_1, \ldots, k_d}(y), y_2^{k_2} \cdots y_d^{k_d} \varphi(t - y)).
\]

Note that each term of the last sum can be rewritten as
\[
\sum_{m_2 \leq k_2, \ldots, m_d \leq k_d} c_{m,k} t_2^{k_2 - m_2} \cdots t_d^{k_d - m_d} [(f_k(y), (t_2 - y_2)^{m_2} \cdots (t_d - y_d)^{m_d} \varphi(t - y))]
\]
with some constants \( c_{m,k} \). Since \( f_k \) are almost periodic distributions, we get that the expressions in square brackets are almost periodic functions in \( t \in \mathbb{R}^d \), and hence in \( t_1 \in \mathbb{R} \). Therefore the functions \( (g_{k_1}(y), \varphi(t - y)) \) are almost periodic in \( t_1 \in \mathbb{R} \) for any fixed \( (t_2, \ldots, t_d) \in \mathbb{R}^{d-1} \). Apply the Lemma to the distributions \( g_{k_1}, k_1 = 0, \ldots, K \). If \( f \) has support of bounded density, we get that all \( g_k \) have supports of bounded densities.

For a fixed \( k_1 \) we have
\[
g_{k_1}(y) = \sum_{k_2 = 0}^{K - k_1} y_2^{k_2} g_{k_1, k_2}(y),
\]
where
\[
g_{k_1, k_2}(y) = \sum_{k_3 + \cdots + k_d \leq K - k_1 - k_2} y_3^{k_3} \cdots y_d^{k_d} f_{k_1, \ldots, k_d}(y).
\]
The functions \( (g_{k_1, k_2}(y), \varphi(t - y)) \) are almost periodic in \( t_2 \in \mathbb{R} \) for any fixed \( (t_1, t_3, \ldots, t_d) \in \mathbb{R}^{d-1} \). Apply the Lemma to distributions \( g_{k_1, k_2}, k_2 = 0, \ldots, N - k_1 \) with respect to the variable \( y_2 \). If \( f \) has support of bounded density, that these distributions have supports of bounded densities. After a finite number of steps we obtain the statement of the Proposition. \( \square \)

6. Proofs of the Theorems

**Proof of Theorem 6.** First estimate the number \( n(r) = \# \{ \text{supp} \mu \cap B(0, r) \} \). Consider the sets
\[
A_s = \{ x \in \mathbb{R}^d : s - 1 \leq |x| < s \}, \quad s \in \mathbb{N}.
\]
By (1),
\[
B(\lambda, (c/2)s^{-h}) \cap B(\lambda', (c/2)s^{-h}) = \emptyset \quad \text{for} \quad \lambda, \lambda' \in A_s \cap \text{supp} \mu, \lambda \neq \lambda'.
\]

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Set \( s_0 > (c/2)^{1/h} \). Then for \( s \geq s_0 \) we have \((c/2)s^{-h} < 1\), and the sum of volumes of balls \( B(\lambda, (c/2)s^{-h})\), \( \lambda \in A_s \cap \text{supp } \mu \), does not exceed the volume of the annulus \( A_{s-1} \cup A_s \cup A_{s+1} \). Therefore we have

\[
\#(\text{supp } \mu \cap A_s) \leq \frac{(s + 1)^d - (s - 2)^d}{[(c/2)s^{-h}]^d} \leq Cs^{dh+d-1}, \quad C < \infty
\]

and

\[
n(r) \leq \sum_{s_0-1 \leq s \leq r+1} \#(\text{supp } \mu \cap A_s) + n(s_0) = O(r^{d(h+1)}).
\]

By Proposition 1\(\text{iii})\) we get

\[
|\mu(\lambda)| \leq \max\{1, |\lambda|^T\} \quad \text{for all } \lambda \in \text{supp } \mu,
\]

with some \( T < \infty \). Take any \( \varphi \in S(\mathbb{R}^d) \). We have \( \varphi(x) = o(|x|^{-T-\omega(h+1)-1}) \) as \( x \to \infty \). Hence,

\[
\left| \int \varphi(\lambda)|\mu|(d\lambda) \right| \leq \sum_{\lambda \in \text{supp } \mu} |\varphi(\lambda)||\mu(\lambda)|
\]

\[
\leq C_0 + \int_{r_0}^\infty \frac{n(dr)}{r^{d(h+1)+1}} = C_0 + C_1 \int_{r_0}^\infty \frac{n(r)dr}{r^{d(h+1)+2}}.
\]

Since the last integral is finite for any choice of \( \varphi \), we see that \( |\mu| \in S^*(\mathbb{R}^d) \).

**Proof of Theorem 7.** By Proposition 1, part i), \( f \) has form (10). Set \( \mu_k = \sum_{\lambda \in \Lambda} P_k(\lambda)\delta_\lambda \). We get

\[
f(x) = \sum_{\|k\| \leq K} D^k \mu_k(x), \quad \hat{f}(y) = \sum_{\|k\| \leq K} (2\pi i)^\|k\|y^k \hat{\mu}_k(y).
\]

Clearly, the measure \( \mu_k \) satisfies the condition

\[
|\mu_k|(B(0, r)) = O(r^d), \quad r \to \infty.
\]

By Proposition 2, the distributions \( \hat{\mu}_k \) are almost periodic. Then, by Proposition 4, they have supports of bounded densities and, by Proposition 3, they are measures with uniformly bounded masses. Hence, \( |\hat{\mu}_k|(B(0, r)) = O(r^d) \) as \( r \to \infty \). Applying Theorem 8 to the measures \( \mu_k \), we obtain the representation with trigonometric polynomials \( P_{j,k}(\lambda) \)

\[
f = \sum_{\|k\| \leq K} D^k \sum_{j=1}^J \sum_{\lambda \in \lambda_j + L_j} P_{j,k}(\lambda)\delta_\lambda,
\]

whence (8) follows. \(\square\)
Proof of Theorem 8. First part of the statement was proved in [9], Proposition 4, and based on Cohen’s Idempotent Theorem (see, e.g., [27]), and the local analog of the Wiener–Levi Theorem [7].

Let us prove the others parts. Let Λ be uniformly discrete, and let φ be a \( C^\infty \)-function such that \( \varphi(0) = 1 \) and \( \text{supp}\varphi \subset B(0, \eta) \) with

\[
\eta < (1/2) \inf \{|\lambda - \lambda'| : \lambda, \lambda' \in \Lambda\}.
\]

Suppose that \( \mu \) is such that \( \hat{\mu}_s \) has locally finite support. Put \( g_s = \varphi \ast \mu_s \). Clearly, \( g_s(\lambda) = \mu_s(\lambda) \) for \( \lambda \in \Lambda_s \) and \( g_s(\lambda) = 0 \) for \( \lambda \in \Lambda \setminus \Lambda_s \). Then for all \( t \in \mathbb{R}^d \)

\[
g_s(t) = \int \varphi(t - x)\mu_s(dx) = \int \hat{\varphi}(y)e^{2\pi i \langle t, y \rangle} \hat{\mu}_s(dy) = \sum_n \hat{\varphi}(\rho_n)q_ne^{2\pi i \langle t, \rho_n \rangle},
\]

whenever \( \hat{\mu}_s = \sum_n q_n\delta_{\rho_n} \). Further, \( \hat{\varphi} \in S(\mathbb{R}^d) \), therefore, \( \hat{\varphi}(y) = o(|y|^{-d-1}) \) as \( y \to \infty \) and

\[
\sum_n |\hat{\varphi}(\rho_n)| |q_n| \leq C_0 + \int_{r_0}^{\infty} r^{-d-1}M(dr)
\]

\[
\leq C_0 + (d + 1) \int_{r_0}^{\infty} r^{-d-2}M(r)dr < \infty,
\]

where \( M(r) := |\hat{\mu}_s|(B(0, r)) = O(r^d) \) as \( r \to \infty \). Therefore,

\[
g_s(x) = \sum_n a_ne^{2\pi i \langle x, \rho_n \rangle} \quad \text{with} \quad \sum_n |a_n| < \infty.
\]

By the first part of the theorem, we have

\[
\mu_s = \sum_{j=1}^{J} \sum_{\lambda \in L_j + \lambda_j} g_s(\lambda)\delta_\lambda = \sum_{j=1}^{J} \sum_{x \in L_j} \sum_n a_ne^{2\pi i \langle x + \lambda_j, \rho_n \rangle} \delta_{x + \lambda_j}.
\]

For every fixed \( j \) and each \( \rho_n \in \mathbb{R}^d \) there is \( \gamma_{n,j} \) inside the closed parallelepiped \( P_j \) generated by corresponding \( L_j^* \) such that \( \rho_n - \gamma_{n,j} \in L_j^* \), therefore, \( e^{2\pi i \langle x, \rho_n \rangle} = e^{2\pi i \langle x, \gamma_{n,j} \rangle} \) for \( x \in L_j \). Rewrite the above sum in the form (14)

\[
\mu_s = \sum_{j=1}^{J} \sum_n \sum_{x \in L_j} b_{n,j}e^{2\pi i \langle x + \lambda_j, \gamma_{n,j} \rangle} \delta_{x + \lambda_j} \quad \text{with} \quad \sum_{n,j} |b_{n,j}| < \infty \quad \text{and} \quad \gamma_{n,j} \in P_j.
\]

After collecting similar terms we may suppose that \( \gamma_{n,j} \neq \gamma_{n',j} \) for every fixed \( j \) and \( n \neq n' \).
Let \( L_j = T_j \mathbb{Z}^d \), where \( T_j \) are nondegenerate linear operators on \( \mathbb{R}^d \). By (2), for each \( j \) and corresponding \( \gamma_{n,j} \) the Fourier transform of the sum
\[
\sum_{x \in L_j} e^{2\pi i (x + \lambda_j, \gamma_{n,j})} \delta_{x+\lambda_j}
\]
equals
\[
|\det T_j|^{-1} \sum_{y \in L_j^*} e^{2\pi i (y, \lambda_j)} \delta_{y-\gamma_{n,j}}.
\]
Therefore,
\[
\hat{\mu}_s = \sum_{j=1}^J |\det T_j|^{-1} \sum_{y \in L_j^*} e^{2\pi i (y, \lambda_j)} \sum_n b_{n,j} \delta_{y-\gamma_{n,j}}.
\]
Since \( \bigcup_j P_j \) is a bounded set and \( \text{supp} \hat{\mu}_s \) is locally finite, we see that there is only a finite number of nonzero coefficients \( b_{n,j} \). Hence sum (14) is a generalized lattice Dirac comb.

Let us finally show that condition (9) is necessary. Take a set of real numbers \( x_j \in (2^{j-1}, 2^j), \ j = 1, 2, \ldots \) such that for any \( j \neq i \) the number \( x_j/x_i \) is irrational, and put
\[
T_j = \begin{pmatrix} x_j & 0 \\ 0 & 2^j \end{pmatrix}, \quad L_j = T_j \mathbb{Z}^2,
\]
\[
\mu = \sum_j j^{-2} \sum_{\lambda \in L_j + (0, 1)2^{j-1}} \delta_\lambda = \sum_j j^{-2} \sum_{n,m \in \mathbb{Z}} \delta_{(mx_j,n2^{j-1})}.
\]
It is easy to check that \( L_j + (0, 1)2^{j-1} \) are disjoint translations of full-rank mutually incommensurable lattices \( L_j \subset \mathbb{R}^2 \), and \( \text{supp} \mu \) is uniformly discrete. On the other hand,
\[
L_j^* = T_j^{-1} \mathbb{Z}^2, \quad \hat{\mu} = \sum_j j^{-2} x_j^{-1} 2^{-j} \sum_{n,m \in \mathbb{Z}} e^{\pi i n} \delta_{(mx_j^{-1},n2^{-j})}.
\]
Hence,
\[
|\hat{\mu}|[B(0,1)] \leq \sum_j j^{-2} x_j^{-1} 2^{-j} \#\{\lambda \in L_j^* : |\lambda| < 1\} \leq \sum_j j^{-2} 2^{-2j} 4(2^j)^2,
\]
and the same estimates is valid for \( |\hat{\mu}|[B((mx_j^{-1},n2^{-j}),1)] \) for all \( n, m \in \mathbb{Z} \). Therefore,
\[
\sup_{x \in \mathbb{Z}^2} |\hat{\mu}|[B(x,1)] < \infty,
\]
and condition (6) is satisfied.
We see that the measure $\mu$ satisfies all the conditions of Theorem 8 except (9), but support of $\mu$ is not a finite union of translations of full-rank lattices. □

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