WEAK INTERMITTENCY AND SECOND MOMENT BOUND OF A FULLY DISCRETE SCHEME FOR STOCHASTIC HEAT EQUATION

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Abstract. In this paper, we first prove the weak intermittency, and in particular the sharp exponential order $C\lambda^4 t$ of the second moment of the exact solution of the stochastic heat equation with multiplicative noise and periodic boundary condition, where $\lambda > 0$ denotes the level of the noise. In order to inherit numerically these intrinsic properties of the original equation, we introduce a fully discrete scheme, whose spatial direction is based on the finite difference method and temporal direction is based on the $\theta$-scheme. We prove that the second moment of numerical solutions of both spatially semi-discrete and fully discrete schemes grows at least as $\exp\{C\lambda^2 t\}$ and at most as $\exp\{C\lambda^4 t\}$ for large $t$ under natural conditions, which implies the weak intermittency of these numerical solutions. Moreover, a renewal approach is applied to show that both of the numerical schemes could preserve the sharp exponential order $C\lambda^4 t$ of the second moment of the exact solution for large spatial partition number.

1. Introduction

In this paper, we study numerically the preservation of the weak intermittency and the sharp exponential order of the second moment of the exact solution of the following stochastic heat equation (SHE) with periodic boundary condition:

$$\begin{align*}
\partial_t u(t, x) &= \partial_x^2 u(t, x) + \lambda \sigma(u(t, x)) \dot{W}(t, x), \\
u(0, x) &= u_0(x),
\end{align*}$$

(1.1a)

$$u(t, 0) = u(t, 1), \quad t \geq 0,$$  

(1.1b)

$$u(0, x) = u_0(x), \quad 0 \leq x \leq 1.$$  

(1.1c)

Here, $\dot{W}(t, x), t \geq 0, x \in [0, 1]$ denotes the space-time white noise with respect to some given filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, $\lambda > 0$ denotes the level of the noise, $\sigma : \mathbb{R} \to \mathbb{R}$ is a globally Lipschitz function, and $u_0$ is a bounded, non-negative, non-random and measurable function. Eq. (1.1a) characterizing the evolution of a field in a random media, arises in several settings, for example generalized Edwards-Wilkinson models for the roughening of surfaces, continuum limits of particle processes and continuous space parabolic Anderson models (PAMs) (see [26] and references therein).

For a random field of multiplicative type, intermittency is a universal phenomenon (see [30]). It originates from the physics literature on turbulence and refers to the chaotic behavior of a random field that develops unusually high peaks over small areas (see [31, 32]). Recall that the upper $p$th moment Lyapunov exponent of $u$ at $x$ is defined as $\tilde{\gamma}_p(x) := \limsup_{t \to \infty} \frac{\log[u(t, x)]^p}{t}, \forall p > 0$. The random field $u$ is called intermittent (also called fully intermittent) if for all $x$, the mapping $p \to \tilde{\gamma}_p(x)/p$ is strictly increasing on $p \in [1, \infty)$. This

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mathematical definition implies that the appearance of high peaks giving the main contribution to the statistical moments of the solution, leads to the non-trivial exponential behaviors of the moments of the solution. The existing research on the intermittency usually begins with a Feynmann-Kac type formula to calculate the explicit expression of the $p$th moment Lyapunov exponent of the solution. For example, in the case of $\sigma(u) = u$, which refers to the famous PAM, it is shown in [6, 10] that the solution of PAM is intermittent both in the continuous case and in the spatially discrete case. In the nonlinear case, it is difficult to obtain the explicit expression of the $p$th moment Lyapunov exponent, so there comes a notion called weak intermittency, which means for all $x$, $\bar{\gamma}_2(x) > 0$ and $\bar{\gamma}_p(x) < \infty \ (\forall p \geq 2)$. It is shown in [17] that the weak intermittency implies intermittency whenever the comparison principle holds.

For Eq. (1.1a) in the whole space, the weak intermittency of the solution both on the real line ([17]) and on the lattice ([19, 20]) has been studied. For the continuous Eq. (1.1a) with various boundary conditions in a bounded domain, the weak intermittency and in particular the effects of the noise level $\lambda$ on the second moment of the solution have been extensively studied (see [19, 24, 25, 29]). More precisely, for the case with Dirichlet boundary condition, it is proved in [20] that the second moment of the solution grows exponentially fast if the noise level $\lambda$ is large enough, and decays exponentially if $\lambda$ is small. While for the case with Neumann boundary condition, it is shown in [19] that the second moment of the solution grows exponentially fast no matter what $\lambda$ is. A fine result is proved in [24], which suggests that the second moment of the solution has sharp exponential order $C\lambda^4t$. As for the case with periodic boundary condition, based on the analysis of the Green function of [11], it is proved in Section 2 that (1) is weakly intermittent and the second moment of the solution has the sharp exponential order $C\lambda^4t$.

In general, the solutions of stochastic partial differential equations can not be solved exactly, thus numerical methods provide a qualitative and quantitative approach to investigate the properties of the exact solution, which have been developed in the past three decades. Many spatially and temporally discrete schemes, for instance the finite difference method, the finite element method, explicit and implicit Euler method and exponential integrator method, have been well studied (see e.g. [1, 5, 7, 8, 11, 12, 13, 14, 15, 21, 22, 28]). It is natural to ask:

(Q$_1$) Is there a numerical scheme inheriting the weak intermittency of (1)?
(Q$_2$) Furthermore, can the numerical scheme preserve the sharp exponential order of the second moment of the exact solution?

Considering the above two questions, we apply the finite difference method to (1) to obtain a spatially semi-discrete scheme, which is convergent to the exact solution in the mean square sense with order $1/2$. By finding the explicit expression of the semi-discrete Green function, the continuous version solution of the semi-discrete scheme can be written into a compact form, which plays a key role in the analysis of the weak intermittency. With the detailed analysis on the integral properties of the semi-discrete Green function, the a priori estimation of the numerical solution gives an intermittent upper bound $(Cp^3\lambda^4)$ for the upper $p$th moment Lyapunov exponent. The semi-discrete Green function, whose point-wise property is slightly different from the continuous one, is proved to be positive when time is large. This positivity, combining with a modified reverse Grönnwall’s inequality reveals an intermittent lower bound $(C\lambda^2)$ of the upper $2$th moment Lyapunov exponent under natural conditions. These imply that the numerical solution of this semi-discretization is weakly intermittent. To enhance the exponential order of the second moment of the semi-discrete numerical solution, a renewal approach depending on the finer integral lower estimate of the semi-discrete Green function
on the spatial grid points is applied. We prove that the second moment of the semi-discrete numerical solution on the spatial grid points has sharp exponential order $C\lambda^4 t$ provided additionally that the initial data is a positive constant and the partition number is large.

For the full discretization, we apply further the $\theta$-scheme to get a fully discrete scheme, which is convergent to the exact solution in the mean square sense with order $1/2$ in the spatial direction and $1/4$ in the temporal direction. The compact integral form is formulated by presenting the explicit expressions of the fully discrete Green functions. The prerequisite for the proof of the weak intermittency is the technical estimates of the fully discrete Green functions. We prove that the numerical solution of this fully discrete scheme is weakly intermittent with an intermittent upper bound ($Cp^3\lambda^4t$) for the upper $p$th moment Lyapunov exponent and an intermittent lower bound ($\log(1 + C\lambda^2\tau)\tau$ with $\tau$ being the time step size) for the upper $2$th moment Lyapunov exponent. This implies that the second moment of the numerical solution of this fully discrete scheme grows at most as $\exp\{C\lambda^4 t\}$ and at least as $\exp\{C\lambda^2 t\}$ for sufficiently large $t := m\tau$. To fill the gap of the index of $\lambda$, a discrete renewal method is implemented, which essentially depends on the finer estimate of the fully discrete Green function. Under some coupling condition between the space and time step sizes, we prove that the second moment of this fully discrete scheme has sharp exponential order $C\lambda^4 t_m$.

This paper is organized as follows. In Section 2, the weak intermittency of the mild solution of (1) is established. In Section 3, we apply the finite difference method to (1) for spatial discretization, and prove the weak intermittency and the sharp exponential order of the second moment of the numerical solution of this spatially semi-discrete scheme. The convergence order in the mean square sense of the spatially semi-discrete scheme is given. Section 4 is devoted to the analysis of the fully discrete scheme on the preservation of the weak intermittency and the sharp exponential order of the second moment of the exact solution. Moreover, we give the mean square convergence order of the fully discrete scheme. In Section 5, we give our conclusions and propose several open problems for future study. At last, some proofs are given in the appendix.

2. Weak intermittency of exact solution

The goal of this section is to investigate the weak intermittency of the mild solution of (1). The goal of this section is to investigate the weak intermittency of the mild solution of (1). Before that, we first present the definitions of Lyapunov exponent and intermittency, which can be found in [23]. Throughout this paper, we let $i^2 = -1$, and constant $C$ may be different from line to line.

Definition 2.1. Fix some $x \in [0, 1]$, define the upper $p$th moment Lyapunov exponent of $u$ at $x$ as

$$\bar{\gamma}_p(x) := \limsup_{t \to \infty} \frac{1}{t} \log \mathbb{E}( |u(t, x)|^p ),$$

for all $p \in (0, \infty)$.

Definition 2.2. (i) We say that $u$ is fully intermittent if for all $x \in [0, 1]$, the map $p \to \frac{\bar{\gamma}_p(x)}{p}$ is strictly increasing for $p \in [2, \infty)$.

(ii) We say that $u$ is weakly intermittent if for all $x \in [0, 1]$, $\bar{\gamma}_2(x) > 0$ and $\bar{\gamma}_p(x) < \infty$ for each $p > 2$.

Remark 2.3. (i) The full intermittency can be implied by the weak intermittency on some certain circumstances, for example, $\sigma(0) = 0$ and $u_0(x) \geq 0$. For its proof, we refer to [6].
Theorem 3.1.2.
(ii) All the results in this paper are still valid if we choose the lower $p$th moment Lyapunov exponent, whose definition is
\[
\gamma_p(x) := \liminf_{t \to \infty} \frac{1}{t} \log \mathbb{E}(|u(t, x)|^p).
\]

The mild solution of (1) can be written as
\[
u(t, x) = \int_0^1 \mathcal{G}(t, x, y) u_0(y) \, dy + \lambda \int_0^t \int_0^1 \mathcal{G}(t-s, x, y) \sigma(u(s, y)) \, dW(s, y),
\]
where the Green function $\mathcal{G}(t, x, y)$ is defined as (see [16])
\[
\mathcal{G}(t, x, y) = \frac{1}{\sqrt{4\pi t}} \sum_{m=\pm \infty} e^{-\frac{(x-y-m)^2}{4t}}, \quad t > 0, \quad x, y \in [0,1],
\]
and its spectral decomposition is
\[
\mathcal{G}(t, x, y) = \sum_{j=\pm \infty} e^{-4\pi^2 j^2 t} e^{2\pi i j (x-y)}, \quad t > 0, \quad x, y \in [0,1].
\]

In order to investigate the weak intermittency of the exact solution of (1), we make the following assumption on the initial data and diffusion coefficient.

**Assumption 1.** Let $I_0 := \inf_{x \in [0,1]} u_0(x)$, and $J_0 := \inf_{x \in \mathbb{R} \setminus \{0\}} |\sigma(x)|$. We assume that $I_0 > 0, J_0 > 0$.

**Theorem 2.4.** Under Assumption 1, the solution of (1) is weakly intermittent.

Before giving the proof of Theorem 2.4, we show some properties of the Green function to (1).

**Lemma 2.5.** $\mathcal{G}(t, x, y)$ has the following properties:
(i) $\mathcal{G}(t, x, y) \geq 0$ for $t > 0, x, y \in [0,1]$, and $\int_0^1 \mathcal{G}(t, x, y) \, dy = 1$ for $t > 0, x \in [0,1]$.
(ii) $\int_0^1 \mathcal{G}^2(t, x, y) \, dy = \mathcal{G}(2t, x, x) \geq \frac{1}{\sqrt{8\pi t}}$ for $t > 0, x \in [0,1]$.
(iii) $\int_0^1 \mathcal{G}^2(t, x, y) \, dy \leq C \left( \frac{1}{\sqrt{t}} + 1 \right)$ with a positive constant $C$ for all $t > 0, x \in [0,1]$.

**Proof.** It is obvious that (i) holds. We prove (ii) by the use of the spectral decomposition (5) of the Green function.
\[
\int_0^1 \mathcal{G}^2(t, x, y) \, dy = \int_0^1 \sum_{r, j=-\infty}^{+\infty} e^{-4\pi^2 (r^2+j^2) t} e^{2\pi i (r+j) (x-y)} \, dy = \sum_{\{r, j \in \mathbb{Z}; \ r+j=0\}} e^{-4\pi^2 (r^2+j^2) t}
\]
\[
= \sum_{r=-\infty}^{+\infty} e^{-8\pi^2 r^2 t} = \mathcal{G}(2t, x, x).
\]

By (4), we have
\[
\mathcal{G}(2t, x, x) = \frac{1}{\sqrt{8\pi t}} \sum_{m=-\infty}^{+\infty} e^{-\frac{m^2}{8\pi t}} \geq \frac{1}{\sqrt{8\pi t}}.
\]
As for (iii), combining (ii) and [25, Lemma B.1], we can get the desired result. The proof is finished. \qed
Proof of Theorem 2.4

Proof. The following intermittent upper bound is a direct consequence of [25] Proposition 4.1:
\[ \sup_{x \in [0,1]} \gamma_p(x) \leq C L_\sigma^4 \lambda^4 p^3, \]
with some constant \( C > 0 \) for all \( p \in [2, \infty) \), where \( L_\sigma \) is the Lipschitz constant of \( \sigma \).

Following the approach presented by Khoshnevisan et al. in [24] Section 2.2, and combining Lemma 2.5 (i) (ii), the intermittent lower bound is given below. Under Assumption 1
\[ \inf_{x \in [0,1]} \gamma_2(x) \geq \frac{\lambda^4 f_2^4}{8} > 0. \]
Hence, the proof is finished.

At the end of this subsection, let’s intuitively see the information that the weak intermittency can bring to us. Suppose \( \gamma_2(x) = \gamma_2(x) := \gamma_2(x) \). Take constants \( \alpha_1, \alpha_2 \) satisfying
\[ 0 < \alpha_1 \lambda^4 < C_1 \lambda^4 \leq \frac{\gamma_2(x)}{2} \leq C_2 \lambda^4 < \alpha_2 \lambda^4. \]
Set \( B_1(t) := \{ \omega \in \Omega : |u(t,x)(\omega)| > e^{\alpha_2 \lambda^4 t} \} \) and \( B_2(t) := \{ \omega \in \Omega : |u(t,x)(\omega)| < e^{\alpha_1 \lambda^4 t} \} \).

By Chebyshev’s inequality,
\[ P(B_1(t)) \leq e^{-2\alpha_2 \lambda^4 t} \mathbb{E}[u(t,x)]^2 \approx e^{-2\alpha_2 \lambda^4 \gamma_2(x) t} \leq e^{-Ct} \]
with some \( C > 0 \), where \( f(t) \approx g(t) \) means \( \lim_{t \to \infty} (\log f(t) - \log g(t))/t = 0 \). This implies that the random field \( u \) may take very large values with exponentially small probabilities, and therefore it develops high peaks when \( t \) is large.

Moreover,
\[ \mathbb{E}(|u(t,x)|^2; B_2(t)) \leq e^{2\alpha_1 \lambda^4 t} \ll e^{\gamma_2(x) t} \approx \mathbb{E}[u(t,x)]^2, \]
where \( f(t) \ll g(t) \) denotes \( \lim_{t \to \infty} f(t)/g(t) = 0 \). This means the contribution to the second moment of \( u \) at \( x \) comes from \( (B_2(t))^c \) where may appear the high peak for large \( t \).

When the random field \( u \) is fully intermittent, the main contribution to each moment of \( u \) is carried by higher and higher, more and more widely spaced peaks. The above analysis is also valid for numerical solution. For more details, we refer to [1, 25].

3. Intrinsic property-preserving spatial semi-discretization

In this section, we apply the finite difference method to [1] to get a spatially semi-discrete scheme, whose continuous version solution can be written into a compact integral form by the use of explicit expression of the semi-discrete Green function. The spatially semi-discrete scheme is convergent to the exact solution in the mean square sense with order \( \frac{1}{2} \). Based on the detailed analysis on the semi-discrete Green function and reverse Grönwall’s inequality, we prove that the numerical solution of this semi-discretization is weakly intermittent. Moreover, this semi-discrete scheme preserves the sharp exponential order of the second moment of the exact solution.

3.1. Spatially semi-discrete scheme. We introduce the uniform partition on the spatial domain \([0,1]\) with step size \( \frac{1}{n} \) for a fixed integer \( n \geq 3 \). Let \( u^n(t, \frac{k}{n}) \) be the approximation of \( u(t, \frac{k}{n}) \), \( k = 0, 1, \ldots, n - 1 \). The spatially semi-discrete scheme based on the finite difference
The method is given by:

\[
\begin{aligned}
du^n(t, \frac{k}{n}) &= n^2 \left( u^n(t, \frac{k+1}{n}) - 2u^n(t, \frac{k}{n}) + u^n(t, \frac{k-1}{n}) \right) dt + \lambda \sqrt{n} \sigma (u^n(t, \frac{k}{n})) dW^n_k(t), \\
u^n(t, 0) &= u^n(t, 1), \quad u^n(t, -\frac{1}{n}) = u^n(t, \frac{n-1}{n}), \quad t \geq 0, \\
u^n(0, \frac{k}{n}) &= u_0 \left( \frac{k}{n} \right), \quad k = 0, 1, \ldots, n - 1,
\end{aligned}
\]

where \( W^n_k(t) := \sqrt{n} \left( W(t, \frac{k+1}{n}) - W(t, \frac{k}{n}) \right) \). By the linear interpolation with respect to the space variable, it follows from Appendix \( 6.1 \) that the mild form of \( u^n \) is given by:

\[
u^n(t, x) = \int_0^1 G^n(t, x, y)u^n(0, (\kappa_n(y))) dy + \lambda \int_0^t \int_0^1 G^n(t - s, x, y) \sigma (u^n(s, \kappa_n(y))) dW(s, y),
\]

almost surely for all \( t \geq 0 \) and \( x \in [0, 1] \), where \( G^n(t, x, y) := \sum_{j=0}^{n-1} e^{\lambda_j t} e_j^n(x) \bar{e}_j(\kappa_n(y)) \) with \( \lambda_j^n := -4n^2 \sin^2 \left( \frac{j\pi}{n} \right) \), \( \kappa_n(y) := \left\lfloor \frac{\lfloor ny \rfloor}{n} \right\rfloor \), \( \lceil \cdot \rceil \) being the greatest integer function, \( e_j(x) = e^{2\pi i j x} \), \( \bar{e}_j(\cdot) \) representing the complex conjugate of \( e_j(\cdot) \) and

\[
e_j^n(x) := e_j(\kappa_n(x)) + (nx - n\kappa_n(x)) \left[ e_j \left( \kappa_n(x) + \frac{1}{n} \right) - e_j(\kappa_n(x)) \right], \quad \forall x \in [0, 1].
\]

Nevertheless, based on the periodicity of \( \lambda_j^n \) and \( e_j \) with respect to \( j \), \( G^n(t, x, y) \) can be rewritten into two cases:

\[
G^n(t, x, y) = \begin{cases} 
\sum_{j=-\lfloor \frac{n}{2} \rfloor}^{\lceil \frac{n}{2} \rceil} e^{\lambda_j^n t} e_j^n(x) \bar{e}_j(\kappa_n(y)), & n \text{ is odd}, \\
\sum_{j=-\frac{n}{2}+1}^{\frac{n}{2}} e^{\lambda_j^n t} e_j^n(x) \bar{e}_j(\kappa_n(y)), & n \text{ is even}.
\end{cases}
\]

By expanding the real and imaginary parts of \( G^n \), it is not difficult to observe that \( G^n \) is a real function (see Appendix \( 6.2 \)). Now we give the main result of this subsection.

**Theorem 3.1.** Under Assumption \( 1 \), the solution of the spatially semi-discrete scheme is weakly intermittent.

The proof of Theorem 3.1 follows from Sections \( 3.2 \) and \( 3.3 \). Before that, we prove the following properties of the semi-discrete Green function \( G^n \), which is essential in establishing the weak intermittency of \( G \).

**Lemma 3.2.** \( G^n(t, x, y) \) has the following properties:

(i) \( \int_0^1 G^n(t, x, y) dy = 1 \) for \( t > 0, \ x \in [0, 1] \).

(ii) For \( t > 0, \ x \in [0, 1] \), the following equalities hold:

\[
\int_0^1 (G^n(t, x, y))^2 dy = \sum_{j=0}^{n-1} e^{2\lambda_j^n t} |e_j^n(x)|^2 = \begin{cases} 
\sum_{j=-\lfloor \frac{n}{2} \rfloor}^{\lceil \frac{n}{2} \rceil} e^{2\lambda_j^n t} \left| e_j^n(x) \right|^2, & n \text{ is odd}, \\
\sum_{j=-\frac{n}{2}+1}^{\frac{n}{2}} e^{2\lambda_j^n t} \left| e_j^n(x) \right|^2, & n \text{ is even}.
\end{cases}
\]

Moreover, \( \int_0^1 (G^n(t, x, y))^2 dy \geq 1 \) for \( t > 0, \ x \in [0, 1] \).

(iii) \( \int_0^1 (G^n(t, x, y))^2 dy \leq 1 + \sqrt{\frac{n}{8t}} \) for all \( t > 0, \ x \in [0, 1] \).

(iv) For each fixed \( n \geq 3 \), there exists a number \( t(n) > 0 \) depending on \( n \), such that \( G^n(t, x, y) \geq \frac{1}{2} > 0 \) for all \( t > t(n), \ x, y \in [0, 1] \).
Proof. (i) For all $t \geq 0, x \in [0, 1]$, we get
\[
\int_0^1 G^n(t, x, y) \, dy = \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} e^{\lambda^n_j t} e^n_j(x) e^{-2\pi i j \pi/k} = 1 + \sum_{j=1}^{n-1} e^{\lambda^n_j t} e^n_j(x) \sum_{k=0}^{n-1} \cos \left(2\pi j \frac{k}{n}\right) = 1,
\]
where we have used the fact that $\sum_{k=0}^{n-1} \cos \left(2\pi j \frac{k}{n}\right) = 0$ for $j \notin n\mathbb{Z}$.

(ii) For all $t \geq 0, x \in [0, 1]$, taking advantage of the orthogonality of $\{e_j\}_{j=0,1,\ldots,n-1}$, we get
\[
\int_0^1 (G^n(t, x, y))^2 \, dy = \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} e^{2\lambda^n_j t} |e^n_j(x)|^2 \left| \tilde{e}_j \left(\frac{k}{n}\right) \right|^2 + \frac{1}{n} \sum_{j \neq l} e^{(\lambda^n_j + \lambda^n_l) t} e^n_j(x) \sum_{k=0}^{n-1} \tilde{e}_j \left(\frac{k}{n}\right) e_l \left(\frac{k}{n}\right)
\]
\[
= \sum_{j=0}^{n-1} e^{2\lambda^n_j t} |e^n_j(x)|^2.
\]
Similarly, we can get the result in the case of $n$ being odd and even.

(iii) By (ii), we have
\[
\int_0^1 (G^n(t, x, y))^2 \, dy \leq 1 + 4 \sum_{j=1}^{\left\lfloor \frac{n}{2} \right\rfloor} e^{2\lambda^n_j t} + 1 + 4 \sum_{j=1}^{\left\lfloor \frac{n}{2} \right\rfloor} e^{-8j^2 \pi^2 c^n_j t} \leq 1 + 4 \sum_{j=1}^{\left\lfloor \frac{n}{2} \right\rfloor} e^{-32j^2 t}
\]
\[
\leq 1 + 4 \int_0^{\left\lfloor \frac{n}{2} \right\rfloor} e^{-32z^2 t} \, dz \leq 1 + 4 \int_0^{\infty} e^{-32z^2} \, dz \leq 1 + \sqrt{\frac{\pi}{8t}}.
\]
where we have used the fact that $c^n_j := \sin^2 \left(\frac{j\pi}{n}\right)/\left(\frac{j\pi}{n}\right)^2 \in \left[\frac{1}{n^2}, 1\right]$ for $j = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor$.

(iv) Since
\[
4n^2 \sin^2 \left(\frac{j\pi}{n}\right) t \geq 4n^2 \sin^2 \left(\frac{\pi}{n}\right) t, \quad j = 1, 2, \ldots, n-1,
\]
and the right hand side of (8) converges to infinity as $t \to \infty$, we get
\[
\left| \sum_{j=1}^{n-1} e^{\lambda^n_j t} e^{2\pi i j \frac{k}{n}} \right| \leq \sum_{j=1}^{n-1} e^{\lambda^n_j t} \to 0 \quad \text{as} \quad t \to \infty
\]
uniformly for all $q, l = 0, 1, \ldots, n-1$. Hence, for each fixed $n$, there is a positive constant $t(n)$ such that when $t > t(n), -\frac{1}{2} \leq \sum_{j=1}^{n-1} e^{\lambda^n_j t} e^{2\pi i j \frac{q}{n}} \leq \frac{1}{2}$ holds for all $q, l = 0, 1, \ldots, n-1$. Therefore, when $t > t(n)$ and for all $q, l = 0, 1, \ldots, n-1$, we have
\[
G^n \left(t, \frac{q}{n}, \frac{l}{n}\right) = 1 + \sum_{j=1}^{n-1} e^{\lambda^n_j t} e^{2\pi i j \frac{q}{n}} \geq \frac{1}{2}.
\]
This will lead to our desired result after linear interpolation with respect to the space variable. Hence the proof is completed.
3.2. Intermittent upper bound. To give the a priori estimation of the mild solution to (7), we introduce norms on the space of random fields,

\[ N_{\beta,p}(u) := \sup_{t \geq 0} \sup_{x \in [0,1]} \left\{ e^{-\beta t} \| u(t,x) \|_p \right\}, \quad \forall \beta > 0, \ p \geq 2, \]

where \( \| \cdot \|_p \) denotes the \( L^p(\Omega) \)-norm. Let \( L^{\beta,p} \) be the completion of simple random fields in \( N_{\beta,p} \)-norm. For more details, we refer to [28, Chapter 4].

**Proposition 3.3.** There exists a random field \( u^n \in \bigcup_{\beta > 0} L^{\beta,p} \) solving (7) for each \( n \geq 3, \ p \geq 2 \). Moreover, \( u^n \) is a.s.-unique among all random fields satisfying

\[ \sup_{x \in [0,1]} \mathbb{E} \left( |u^n(t,x)|^p \right) \leq C_1^p \exp \left\{ C_2 L_\sigma^4 \lambda^3 p^3 t \right\}, \quad \text{for} \ p \geq 2, \ t \geq 0, \]

with some constants \( C_1, C_2 > 0 \).

**Proof.** We apply Picard’s iteration by defining

\[ u^n(0,t,x) := u(0,x), \]
\[ u^n(q+1)(t,x) := \int_0^t G^n(t,s) u^n(s,x) ds + \int_0^t (e^n(t,s,x) - e^n(s,x)) \, dW(s) \]

Using Lemma 3.2 (ii) (iii), combining the linear growth of \( \sigma \), Minkowski inequality and Burkholder-Davis-Gundy inequality, we obtain

\[ \left\| u^n(q+1)(t,x) \right\|_p^2 \leq 2 \sup_{x \in [0,1]} |u^n(0,x)|^2 \times \int_0^1 (G^n(t,x,y))^2 dy \]
\[ + 8p^2 \lambda^2 \int_0^t \int_0^1 (G^n(t-s,x,y))^2 \left\| \sigma(u^n(s,x)) \right\|_p^2 ds \]
\[ \leq 2 \sup_{x \in [0,1]} |u^n(0,x)|^2 \times \left( 2 \sum_{j=0}^{n-1} e^{2\lambda_j^2 t} \right) \]
\[ + C L_\sigma^2 p \lambda^2 \left( \sqrt{t} + 1 \right) + C L_\sigma^2 p \lambda^2 \int_0^t \left( \frac{1}{\sqrt{t-s}} + 1 \right) \sup_{y \in [0,1]} \left\| u^n(s,y) \right\|_p^2 ds. \]

(9)

Multiplying \( e^{-2\beta t} \) with \( 2\beta \geq 1 \) on both sides of (9), taking supremum over \( x \in [0,1], t \geq 0 \), and noticing \( 2\sum_{j=0}^{n-1} e^{2\lambda_j^2 t} \leq 2n \), we get

\[ \mathcal{N}_{\beta,p}^2 \left( u^n_{q+1} \right) \leq 4 \sup_{x \in [0,1]} |u^n(0,x)|^2 \times + \frac{CL_\sigma^2 p \lambda^2}{\sqrt{2\beta}} + \frac{3CL_\sigma^2 p \lambda^2}{\sqrt{2\beta}} \left( \sqrt{\frac{\pi}{2\beta}} + \frac{1}{2\beta} \right) \mathcal{N}_{\beta,p}^2 \left( u^n_q \right) \]

\[ \leq 4 \sup_{x \in [0,1]} |u^n(0,x)|^2 \times + \frac{3CL_\sigma^2 p \lambda^2}{\sqrt{2\beta}} \mathcal{N}_{\beta,p}^2 \left( u^n_q \right), \]

where in the last step we have used \( \sqrt{\frac{\pi}{2\beta}} + \frac{1}{2\beta} \leq \frac{3}{\sqrt{2\beta}} \) for \( \beta \geq \frac{1}{2} \).

There exists a \( \beta \) such that

\[ \frac{3CL_\sigma^2 p \lambda^2}{2\beta} \leq \frac{1}{2} \quad \text{and} \quad \beta \geq \frac{1}{2}. \]

(10)
For example, one can choose $\beta = 18C^2L^2\sigma^2p^2\lambda^4 + \frac{1}{2}$. For such $\beta$, we have

$$
\mathcal{N}_{\beta,p}^2\left(u_{(q+1)}^n\right) \leq 4 \sup_{x \in [0,1]} |u_0(x)|^2 \times n + \frac{1}{2} + \frac{1}{2} \mathcal{N}_{\beta,p}^2\left(u_{(q)}^n\right)
$$

which yields $u_{(q+1)}^n \in \mathcal{L}^{\beta,p}$. Eq. (11) implies that for all $t \geq 0$, $x \in [0,1]$ and $\beta$ satisfying (10),

$$
\mathbb{E}\left(|u_{(q+1)}^n(t, x)|^p\right) \leq C_1^p \exp\{\beta t\}
$$

for each $p \geq 2$, $q \geq 0$.

Similarly, using the technique as before, we can prove

$$
\mathcal{N}_{\beta,p}^2\left(u_{(q+1)}^n - u_{(q)}^n\right) \leq \frac{3CL^2p\lambda^2}{\sqrt{2\beta}} \mathcal{N}_{\beta,p}^2\left(u_{(q)}^n - u_{(q-1)}^n\right).
$$

By choosing $\beta$ satisfying (10), we obtain that $\{u_{(q)}^n(t, x)\}_{q \geq 0}$ is a Cauchy sequence in $\mathcal{N}_{\beta,p}$-norm, i.e. $\{u_{(q)}^n(t, x)\}_{q \geq 0}$ converges to some random field $u^n$ in $\mathcal{N}_{\beta,p}$-norm for each fixed $p \geq 2$. Since $\mathcal{L}^{\beta,p}$ is complete, we deduce that $u^n \in \mathcal{L}^{\beta,p}$. Moreover, $u^n$ satisfies the integral equation (7) in $\mathcal{N}_{\beta,p}$-norm.

The uniqueness of the numerical solution in $\mathcal{N}_{\beta,p}$-norm can be shown in a similar way as above. Thus the proof is completed.

Based on Proposition 3.3, we can give the upper bound of the upper $p$th moment Lyapunov exponent of numerical solution of the spatially semi-discrete scheme.

**Proposition 3.4.** There exists a positive constant $C$, such that for each $n \geq 3$, $p \in [2, \infty)$, we have

$$
\sup_{x \in [0,1]} \tilde{\gamma}_p^n(x) := \sup_{x \in [0,1]} \limsup_{t \to \infty} \frac{1}{t} \log \mathbb{E}\left(|u^n(t, x)|^p\right) \leq CL_\sigma^4\lambda^4 p^3.
$$

### 3.3. Intermittent lower bound.

It remains to investigate the lower bound for the upper 2th moment Lyapunov exponent. Before that, we give the following reverse Grönwall’s inequality.

**Lemma 3.5.** (Reverse Grönwall’s inequality) Let $\phi$ be nonnegative and satisfy

$$
\phi(t) \geq \alpha + \beta \int_0^t \phi(s) \, ds
$$

for $t > a > 0$, where $\alpha, \beta > 0$ are constants, then for $t > a$,

$$
\phi(t) \geq e^{\beta(t-a)} \left(\alpha + \beta \int_0^a \phi(s) \, ds\right).
$$

**Proof.** Note that $\phi$ satisfies $\phi(t) \geq (\alpha + \beta \int_0^a \phi(s) \, ds) + \beta \int_a^t \phi(s) \, ds$ for $t > a$, we can easily get the desired result.

**Proposition 3.6.** Under Assumption 4, we have

$$
\inf_{x \in [0,1]} \tilde{\gamma}_2^n(x) \geq \lambda^2J_0^2 > 0.
$$
Applying Lemma 3.5 with $\alpha$, taking infimum over the second moment of the solution of (1) has the sharp exponential order which leads to 3.4.

Sharp exponential order of the second moment. □

Hence we finish the proof.

By applying a renewal approach, we can get the same kind of result for the numerical solution of the semi-discrete scheme for large $n$. Before giving the proof of Theorem 3.7, we present the refined property of the semi-discrete Green function and a probability density function for the renewal approach.

Lemma 3.8. For $t > 0, x \in [0, 1]$, we have

$$\int_0^1 (G^n(t, \kappa_n(x), y))^2 dy \geq \frac{1 - e^{-2n^2\pi^2 t}}{\sqrt{32\pi t}}.$$

Proof. For each fixed $n \geq 3$, taking the second moment on both sides of (7), combining Walsh isometry and Lemma 3.2 (i) (ii) (iv), we get when $t > t(n)$,

$$\mathbb{E} \left[ |u^n(t, x)|^2 \right] = \left| \int_0^1 G^n(t, x, y) u^n(0, \kappa_n(y)) dy \right|^2 + \lambda^2 \int_0^t \int_0^1 (G^n(t - s, x, y))^2 \mathbb{E} \left[ |\sigma(u^n(s, \kappa_n(y)))|^2 \right] ds dy \geq I_0^2 \left| \int_0^1 G^n(t, x, y) dy \right|^2 + \lambda^2 J_0^2 \int_0^t \int_0^1 (G^n(t - s, x, y))^2 \mathbb{E} \left[ |u^n(s, \kappa_n(y))|^2 \right] ds dy \geq I_0^2 + \lambda^2 J_0^2 \int_0^t \int_{[0, 1]} (G^n(t - s, x, y))^2 dy \inf_{y \in [0, 1]} \mathbb{E} \left[ |u^n(s, y)|^2 \right] ds.

Taking infimum over $x \in [0, 1]$, we have

$$\inf_{x \in [0, 1]} \mathbb{E} \left[ |u^n(t, x)|^2 \right] \geq I_0^2 + \lambda^2 J_0^2 \int_0^t \inf_{y \in [0, 1]} \mathbb{E} \left[ |u^n(s, y)|^2 \right] ds.

Applying Lemma 3.5 with $\alpha = I_0^2, \beta = \lambda^2 J_0^2, a = t(n)$, we obtain

$$\inf_{x \in [0, 1]} \mathbb{E} \left[ |u^n(t, x)|^2 \right] \geq I_0^2 e^{\lambda^2 J_0^2 (t - t(n))}, \text{ for } t > t(n),

which leads to

$$\inf_{x \in [0, 1]} \tilde{\gamma}_2^n(x) = \lim_{t \to \infty} \frac{1}{t} \log \mathbb{E} \left[ |u^n(t, x)|^2 \right] \geq \lambda^2 J_0^2 > 0.

Hence we finish the proof. □

3.4. Sharp exponential order of the second moment. It is shown in Section 2 that the second moment of the solution of (11) has the sharp exponential order $C \lambda^4 t$ under Assumption 1. By applying a renewal approach, we can get the same kind of result for the numerical solution of the semi-discrete scheme for large $n$, provided additionally that the initial data is a positive constant.

Assumption 2. We assume that $u_0 := I_0 > 0$, and the spatial partition number $n$ satisfies $n \geq \zeta^2$ with some constant $\zeta > 0$.

Theorem 3.7. Under Assumptions 7 and 8 for each $n$, we have

$$\inf_{x \in [0, 1]} \mathbb{E} \left[ |u_n(t, \kappa_n(x))|^2 \right] \geq C e^{C_2 J_0^4 \lambda^4 t}, \quad t > T,

where $T := T(n) > 0$, $C_1 = \frac{8\pi \zeta l_0^2}{J_0^2 + 8\pi \zeta}$, $C_2 = \frac{2\zeta^2 \pi^2}{(J_0^2 + 8\pi \zeta)^2}$.

Before giving the proof of Theorem 3.7, we present the refined property of the semi-discrete Green function and a probability density function for the renewal approach.
Proof. Since \( e_j^n(\kappa_n(x)) \) = 1, we obtain
\[
\int_0^1 (G^n(t, \kappa_n(x), y))^2 \, dy = \sum_{j=0}^{n-1} e^{2\lambda_j t} \geq \sum_{j=0}^{[\frac{n}{4}]} e^{-8j^2 \pi^2 t} \geq \int_0^{\frac{n}{2}} e^{-8z^2 \pi^2 t} \, dz
\]
\[
= \sqrt{\frac{1}{4} \int_0^{\frac{\pi}{2}} \int_0^{\frac{n}{2}} e^{-8(z^2+u^2)\pi^2 t} \, dw \, dz} = \sqrt{\frac{1}{4} \int_0^{\frac{\pi}{2}} \int_0^{\frac{n}{2}} e^{-8(z^2+u^2)\pi^2 t} \, dw \, dz} \geq \sqrt{\frac{1}{32\pi} \frac{1}{t} - \frac{1}{\sqrt{32\pi} t}} \geq \frac{1}{e^{2n^2 \pi^2 t}},
\]
where we have used the polar coordinate transformation in the last line. The proof is finished.

\[\Box\]

**Lemma 3.9.** Let \( b := \frac{\lambda^2 J^2}{\sqrt{32\pi}} \) and \( n \geq \zeta \lambda^2 \). Then \( g(t) = be^{-\mu^2 b^2 t} \times \frac{1-e^{-2n^2 \pi^2 t}}{\sqrt{t}} \) is a probability density function on \([0, \infty)\) with some suitable \( \mu \geq \frac{8\pi \zeta}{J^2 + 8\pi \zeta} > 0 \).

**Proof.** It suffices to find some \( \mu > 0 \) such that
\[
\int_0^\infty be^{-\mu^2 b^2 t} \times \frac{1-e^{-2n^2 \pi^2 t}}{\sqrt{t}} \, dt = 1,
\]
or equivalently, to prove that the continuous function
\[
h(\mu) := \frac{b}{\mu^2 b^2 + 2n^2 \pi} \left( \frac{1}{\mu} - 1 \right)
\]
has a zero point \( \mu > 0 \). Since \( n \geq \zeta \lambda^2 \), so \( h(\mu) \leq \sqrt{\frac{b^2}{\mu^2 b^2 + 2n^2 \pi} - \left( \frac{1}{\mu} - 1 \right)} \leq \sqrt{\frac{J^2}{64\pi^2}} - \left( \frac{1}{\mu} - 1 \right) \), which implies \( h(0^+) < 0 \). It is obvious that \( h(1^+) > 0 \) for each fixed \( n \). Hence, there exists a \( \mu \in (0,1) \) such that \( h(\mu) = 0 \), and \( g(t) \) is a probability density function with this \( \mu \). Moreover, \( \mu = 1/(\sqrt{\frac{b^2}{2n^2 \pi + \mu b^2} + 1}) \geq \frac{8\pi \zeta}{J^2 + 8\pi \zeta} \). The proof is finished.

\[\Box\]

**Proof of Theorem 3.7.**

**Proof.** Taking the second moment on both sides of (7) with the space variable being \( \kappa_n(x) \), combining Walsh isometry, Lemma 3.2 (i) (ii) and Lemma 3.8 we get
\[
\mathbb{E} \left( |u^n(t, \kappa_n(x))|^2 \right) \geq I_0^2 + \lambda^2 J_0^2 \int_0^t \int_0^1 (G^n(t-s, \kappa_n(x), y))^2 \, dy \inf_{y \in [0,1]} \mathbb{E} \left( |u^n(s, \kappa_n(y))|^2 \right) \, ds
\]
\[
\geq I_0^2 + \frac{\lambda^2 J_0^2}{\sqrt{32\pi}} \int_0^t 1 - \frac{e^{-2n^2 \pi^2 (t-s)}}{\sqrt{t-s}} \inf_{y \in [0,1]} \mathbb{E} \left( |u^n(s, \kappa_n(y))|^2 \right) \, ds. \tag{13}
\]
Taking infimum over \( x \in [0, 1] \), then multiplying \( e^{-\pi \mu^2 b^2 t} \) on both sides of (13) with \( b := \frac{\lambda^2 J^2}{\sqrt{32\pi}} \) and \( \mu \) being a parameter that will be determined later, and denoting
\[
M^n(t) := e^{-\pi \mu^2 b^2 t} \inf_{x \in [0, 1]} \mathbb{E} \left( |u^n(t, \kappa_n(x))|^2 \right),
\]
we obtain
\[
M^n(t) \geq e^{-\pi \mu^2 b^2 t} I_0^2 + \int_0^t be^{-\pi \mu^2 b^2 (t-s)} \times \frac{1-e^{-2n^2 \pi^2 (t-s)}}{\sqrt{t-s}} M^n(s) \, ds.
\]
Consider
\[ e^{-\pi \mu^2 b^2 t} f(t) = e^{-\pi \mu^2 b^2 t} I_0^2 + \int_0^t g(t-s) e^{-\pi \mu^2 b^2 s} f(s) ds, \]  \tag{14}
where \( g(t) \) is defined as in Lemma 3.9 and is a probability density function. Hence, Renewal Theorem (see [2, Theorem 8.5.14]) ensures
\[ \lim_{t \to \infty} e^{-\pi \mu^2 b^2 t} f(t) = \frac{\int_0^\infty e^{-\pi \mu^2 b^2 t} I_0^2 dt}{\int_0^\infty b \sqrt{t} e^{-\pi \mu^2 b^2 t} dt} = 2 \mu I_0^2. \]
Therefore, there exists \( T := T(n) > 0 \), such that
\[ f(t) \geq \mu I_0^2 e^{\pi \mu^2 b^2 t}, \quad \forall t > T. \]  \tag{15}
Observing that \( M^n(t) \) is a super-solution to (14) and applying [23, Theorem 7.11], we have
\[ M^n(t) \geq e^{-\pi \mu^2 b^2 t} f(t), \quad \forall t > 0, \]
which together with (15) implies
\[ \inf_{x \in [0,1]} \mathbb{E} \left( |u^n(t, \kappa_n(x))|^2 \right) \geq \mu I_0^2 e^{\pi \mu^2 b^2 t}, \quad \forall t > T. \]
Moreover, by Lemma 3.9 we have \( \mu^2 b^2 \geq \frac{2 \zeta_2^2 \pi \lambda_4^4}{(I_0^2 + 8 \kappa)}. \) This leads to (12). Hence we complete the proof of the theorem.

Theorem 3.7 and Proposition 3.3 indicate that the second moment of numerical solution to the spatially semi-discrete scheme grows at most and at least as \( \exp \{C \lambda^4 t\} \) as \( t \to \infty \).

3.5. Error estimations of semi-discrete scheme. In this subsection, we present the convergence result of the spatially semi-discrete scheme. It is based on the error estimates of \( G^n(t, x, y) \) and \( G(t, x, y) \), whose proofs are postponed to Appendix 6.2.

Lemma 3.10. (i) There exists a constant \( C > 0 \) such that
\[ \int_0^\infty \int_0^1 |G(t, x, y) - G^n(t, x, y)|^2 \, dy \, dt \leq \frac{C}{n} \]
for all \( x \in [0,1] \) and \( n \geq 3 \).
(ii) For any \( \frac{1}{2} < \alpha < 1 \), there exists a constant \( C := C(\alpha) > 0 \) such that
\[ \int_0^1 |G(t, x, y) - G^n(t, x, y)|^2 \, dy \leq C n^{1-2\alpha} t^{-\alpha} \]
for all \( x \in [0,1] \), \( t > 0 \) and \( n \geq 3 \).

Based on Lemma 3.10, we can establish the convergence theorem of the spatially semi-discrete scheme. Here, we omit its proof since it can be proved in a similar way as in [21].

Theorem 3.11. For every \( 0 < \alpha < \frac{1}{2} \), \( p \geq 1 \) and for every \( t > 0 \), there is a constant \( C := C(\alpha, p, t) > 0 \) such that
\[ \sup_{x \in [0,1]} \|u^n(t, x) - u(t, x)\|_{2p} \leq C n^{-\alpha}. \]
4. Intrinsic property-preserving full discretization

In this section, we discretize (6) in temporal direction by the \( \theta \)-scheme to get a fully discrete scheme, whose solution can be written into a compact integral form by finding explicit expressions of the fully discrete Green functions. The fully discrete scheme is convergent to the exact solution in the mean square sense with order \( \frac{1}{2} \) in the spatial direction and order \( \frac{1}{4} \) in the temporal direction. Based on the technical estimates of the fully discrete Green functions, the numerical solution of this full discretization is proven to be weakly intermittent and to preserve the sharp exponential order of the second moment of the exact solution.

4.1. Fully discrete scheme. We fix the uniform time step size \( 0 < \tau < 1 \). In the sequel, we always assume \( n \geq 3 \). By using the \( \theta \)-scheme to discretize (6), we obtain the following fully discrete scheme:

\[
\begin{aligned}
 & \left\{ 
 u^{n,\tau}(t_{i+1}, x_j) = u^{n,\tau}(t_i, x_j) + (1 - \theta) \tau \Delta_n u^{n,\tau}(t_i, \cdot)(x_j) + \theta \tau \Delta_n u^{n,\tau}(t_{i+1}, \cdot)(x_j) \\
 & \quad + \lambda \tau \sigma(u^{n,\tau}(t_i, x_j)) \square_{n,\tau} W(t_i, x_j), \\
 & \right.
\end{aligned}
\]

\[
\begin{aligned}
 & u^{n,\tau}(t_i, 0) = u^{n,\tau}(t_i, 1), \\
 & u^{n,\tau}(t_i, \frac{1}{n}) = u^{n,\tau}(t_i, \frac{n-1}{n}), \quad i = 0, 1, \ldots, \\
 & (16)
\end{aligned}
\]

where \( u^{n,\tau} \) is an approximation of \( u^n \), \( t_i := i \tau, x_j := \frac{j}{n} \), and

\[
\begin{aligned}
 & \Delta_n u^{n,\tau}(t_i, \cdot)(x_j) := n^2 (u^{n,\tau}(t_i, x_{j+1}) - 2u^{n,\tau}(t_i, x_j) + u^{n,\tau}(t_i, x_{j-1})), \\
 & \square_{n,\tau} W(t_i, x_j) := n\tau^{-1} (W(t_{i+1}, x_{j+1}) - W(t_{i+1}, x_j) - W(t_i, x_{j+1}) + W(t_i, x_j)).
\end{aligned}
\]

By the linear interpolation with respect to the space variable, i.e., for \( i = 0, 1, \ldots, \)

\[
\begin{aligned}
 & u^{n,\tau}(t_i, x) := u^{n,\tau}(t_i, \kappa_n(x)) + n(x - \kappa_n(x)) \left[ u^{n,\tau}(t_i, \kappa_n(x) + \frac{1}{n}) - u^{n,\tau}(t_i, \kappa_n(x)) \right],
\end{aligned}
\]

the mild form of \( u^{n,\tau} \) is given by:

\[
\begin{aligned}
 & u^{n,\tau}(t, x) = \int_0^1 G_1^{n,\tau}(t, x, y) u_0(\kappa_n(y)) dy \\
 & \quad + \lambda \int_0^t \int_0^1 G_2^{n,\tau}(t - \kappa_{\tau}(s) - \tau, x, y) \sigma(u^{n,\tau}(\kappa_{\tau}(s), \kappa_n(y))) dW(s, y), \quad (17)
\end{aligned}
\]

almost surely for every \( t = i \tau, x \in [0, 1] \), where the fully discrete Green functions

\[
\begin{aligned}
 & G_1^{n,\tau}(t, x, y) := \sum_{l=0}^{n-1} (R_{1,t}R_{2,t})^{\left\lfloor \frac{n}{2} \right\rfloor} \bar{e}_l^n(x) \bar{e}_l(\kappa_n(y)), \\
 & G_2^{n,\tau}(t, x, y) := \sum_{l=0}^{n-1} (R_{1,t}R_{2,t})^{\left\lfloor \frac{n}{2} \right\rfloor} R_{1,t} \bar{e}_l^n(x) \bar{e}_l(\kappa_n(y))
\end{aligned}
\]

with \( R_{1,t} := (1 - \theta \tau \lambda)^{-1}, R_{2,t} := 1 + (1 - \theta) \tau \lambda \), \( \kappa_{\tau}(s) := \left\lceil \frac{s}{\tau} \right\rceil \tau \). For the derivation of (17), we refer to Appendix 6.3

Moreover, \( G_i^{n,\tau}, i = 1, 2 \) can be rewritten as

\[
G_1^{n,\tau}(t, x, y) = \left\{ \begin{array}{ll}
\sum_{l=-\left\lceil \frac{n}{2} \right\rceil}^{\left\lfloor \frac{n}{2} \right\rfloor} (R_{1,t}R_{2,t})^{\left\lfloor \frac{n}{2} \right\rfloor} \bar{e}_l^n(x) \bar{e}_l(\kappa_n(y)), & \text{if } n \text{ is odd}, \\
\sum_{l=-\left\lceil \frac{n}{2} \right\rceil + 1}^{\left\lfloor \frac{n}{2} \right\rfloor} (R_{1,t}R_{2,t})^{\left\lfloor \frac{n}{2} \right\rfloor} \bar{e}_l^n(x) \bar{e}_l(\kappa_n(y)), & \text{if } n \text{ is even},
\end{array} \right.
\]

In this section, we discretize (6) in temporal direction by the \( \theta \)-scheme to get a fully discrete scheme, whose solution can be written into a compact integral form by finding explicit expressions of the fully discrete Green functions. The fully discrete scheme is convergent to the exact solution in the mean square sense with order \( \frac{1}{2} \) in the spatial direction and order \( \frac{1}{4} \) in the temporal direction. Based on the technical estimates of the fully discrete Green functions, the numerical solution of this full discretization is proven to be weakly intermittent and to preserve the sharp exponential order of the second moment of the exact solution.
Hence, we divide \( \theta \)

Moreover, 2

(ii) The numerical solution \( u^{n,\tau} \) of a fully discrete scheme, its upper \( p \)-th moment Lyapunov exponent at \( x \in [0,1] \) is defined by

\[
\bar{\gamma}^{n,\tau}_p(x) := \limsup_{m \to \infty} \frac{1}{m \tau} \log \mathbb{E} (|u^{n,\tau}(m \tau, x)|^p),
\]

for \( p \in (0, \infty) \).

(ii) The numerical solution \( u^{n,\tau} \) is called weakly intermittent if for all \( x \in [0,1] \), \( \bar{\gamma}^{n,\tau}_2(x) > 0 \) and \( \bar{\gamma}^{n,\tau}_p(x) < \infty \) for \( p > 2 \).

Before we investigate the weak intermittency of the fully discrete scheme, we first present some conditions on step sizes to ensure the well-posedness of the fully discrete Green functions. That is to say, the step sizes are chosen to such that \( |R_{1j} R_{2j}| < 1 \), \( j = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \). Note that \( R_{1j} R_{2j} < 1 \), so what we need is to find conditions such that

\[
R_{1j} R_{2j} = \frac{1 + (1 - \theta) \tau \lambda_j^n}{1 - \theta \tau \lambda_j^n} \geq -1 + \epsilon, \quad j = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor
\]

for some fixed \( \epsilon > 0 \). It is equivalent to

\[
-4(1 - 2 \theta + \epsilon) n^2 \tau \sin^2 \frac{j \pi}{n} \geq -2 + \epsilon.
\]

Hence, we divide \( \theta \) into the following three cases.

Case 1: \( \theta \in [0, \frac{1}{2}) \). For such \( \theta \), we have \( 1 - 2 \theta + \epsilon \theta > 0 \), hence, (19) is equivalent to

\[
n^2 \tau \sin^2 \frac{j \pi}{n} \leq \frac{2 - \epsilon}{4(1 - 2 \theta + \epsilon \theta)}.
\]

Suppose \( n^2 \tau \leq r \leq \frac{2 - \epsilon}{4(1 - 2 \theta + \epsilon \theta)} \), then \( \epsilon \leq 2 - \frac{4r}{1 + 4 \theta} \) holds for \( j = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \). Moreover, \( 2 - \frac{4r}{1 + 4 \theta} > 0 \) implies \( r < \frac{1 - \theta}{2 - \theta} \).

Case 2: \( \theta = \frac{1}{2} \). We suppose \( n^2 \tau \leq \frac{2 - \epsilon}{4(1 - 2 \theta + \epsilon \theta)} = \frac{1}{\epsilon} - \frac{1}{2} \), then (19) holds with \( \theta = \frac{1}{2} \).

Case 3: \( \theta \in (\frac{1}{2}, 1] \). For such \( \theta \), we can choose \( \epsilon > 0 \) small enough, e.g., \( \epsilon := \min \left\{ -\frac{1 - 2 \theta}{2 - \theta}, \frac{1}{2} \right\} \), such that (19) holds for all \( n \geq 3 \), \( 0 < \tau < 1 \), \( j = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \).

To sum up, we make the following assumptions on the spatial step size \( \frac{1}{n} \) and the temporal step size \( \tau \) when \( \theta \) takes different values.

Assumption 3. (i) For \( 0 \leq \theta < \frac{1}{2} \), suppose \( n^2 \tau \leq r < \frac{1 - \theta}{2 - 4 \theta} \) with some constant \( r > 0 \).
(ii) For \( \theta = \frac{1}{2} \), suppose \( n^2 \tau \leq \frac{1}{\epsilon} - \frac{1}{2} \) with some \( \epsilon \in (0, \frac{1}{2}) \).
(iii) For \( \frac{1}{2} < \theta \leq 1 \), there is no coupled requirement for \( n, \tau \).

Below, we give the main result of this subsection.

Theorem 4.2. Under Assumptions (i) and (ii), the solution of the fully discrete scheme is weakly intermittent.
The proof of Theorem 4.2 follows from the intermittent upper bound (Sections 4.2) and
the intermittent lower bound (Section 4.3). Before that, we prove some properties of the fully
discrete Green functions, which play a key role in the estimates of the intermittent upper and
lower bounds. In the following, we define \( R_{3,j} := (R_{1,j}R_{2,j})^{-1} - 1 = -\frac{\lambda_j \tau}{1 + (1 - \theta)\tau}, \) where 
\( \lambda_j \) is odd
\( \lambda_j \) is even.

**Lemma 4.3.** For \( n \geq 3, 0 < \tau < 1, G_1^{n,\tau}(t,x,y), i = 1, 2 \) have the following properties:

1. For \( t > 0, x \in [0,1], \) the following equalities hold:
   \[
   \int_0^1 (G_2^{n,\tau}(t,x,y))^2 \, dy = \sum_{j=0}^{n-1} (R_{1,j}R_{2,j})^{2[\frac{n}{2}]} R_{1,j}^2 |e_j^n(x)|^2
   \]
   \[
   = \begin{cases}
   \sum_{j=[\frac{n}{2}]}^{[\frac{n}{2}]} (R_{1,j}R_{2,j})^{2[\frac{n}{2}]} R_{1,j}^2 |e_j^n(x)|^2, & n \text{ is odd,} \\
   \sum_{j=-[\frac{n}{2}]+1}^{[\frac{n}{2}]} (R_{1,j}R_{2,j})^{2[\frac{n}{2}]} R_{1,j}^2 |e_j^n(x)|^2, & n \text{ is even.}
   \end{cases}
   \]
   Moreover, we have \( \int_0^1 (G_2^{n,\tau}(t,x,y))^2 \, dy \geq 1. \)

2. Under Assumption 3, \( \int_0^1 (G_2^{n,\tau}(t,x,y))^2 \, dy \leq 1 + \frac{C}{\sqrt{\frac{n}{2}} + \tau} \) with some constant \( C := C(\theta) > 0 \) for all \( t > 0, x \in [0,1]. \)

3. Under Assumption 3 for each fixed \( n \geq 3 \) and \( 0 < \tau < 1, \) there exists a number \( t(n,\tau) > 0 \) depending on \( n, \tau, \) such that \( G_1^{n,\tau}(t,x,y) \geq \frac{1}{2} > 0 \) for all \( t > t(n,\tau), x, y \in [0,1]. \)

**Proof.** The proofs of (i) (ii) are similar to those in Lemma 3.2 so we only prove (iii) (iv).

(iii) We split the set \( \{ j : 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor \} \) into two parts, i.e.,
   \[
   \{ j : 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor \} = \left\{ j : R_{1,j}R_{2,j} \geq \frac{1}{2} \right\} \cup \left\{ j : -1 + \epsilon \leq R_{1,j}R_{2,j} < \frac{1}{2} \right\}
   \]
   \[
   = A_1 \cup A_2.
   \]

In the sequel, we always use the fact that for \( j \in A_1, \frac{1}{2} < R_{2,j} < 1 \) and \(-\lambda_j \tau \leq R_{3,j} \leq -2A_j \tau, \) and for \( j \in A_2, |R_{1,j}R_{2,j}| \leq 1 - \epsilon. \) Moreover, we observe that \( A_1 \subset \{ j : 1 \leq j \leq \frac{1}{2}\sqrt{\frac{1}{(2-\theta)\tau}} \} \) and \( A_2 \subset \{ j : \frac{1}{2\pi} \sqrt{\frac{1}{(2-\theta)\tau}} < j \leq \lfloor \frac{n}{2} \rfloor \}. \)

Hence,
   \[
   \int_0^1 (G_2^{n,\tau}(t,x,y))^2 \, dy \leq 1 + 4 \sum_{j=1}^{[\frac{n}{2}]} (R_{1,j}R_{2,j})^{2[\frac{j}{2}]} R_{1,j}^2
   \]

\[
= 1 + 4 \sum_{j \in A_1} (R_{1,j}R_{2,j})^{2[\frac{j}{2}]} R_{1,j}^2 + 4 \sum_{j \in A_2} (R_{1,j}R_{2,j})^{2[\frac{j}{2}]} R_{1,j}^2
   \]

\[
\leq 1 + 16 \sum_{j \in A_1} (1 + R_{3,j})^{-2[\frac{j}{2}]} - 2 + 4 \sum_{j \in A_2} (1 - \epsilon)^{2[\frac{j}{2}]} R_{1,j}^2 =: 1 + J_1 + J_2.
   \]

We split \( J_1 \) further as follows,
   \[
   J_1 = 16 \sum_{j \in A_1} (1 + R_{3,j})^{-2[\frac{j}{2}]} - 2
   \]
For the term $J_{1.2}$,

\[
J_{1.2} \leq 16 \sum_{j \in A_1} e^{-32j^2 \tau ([\frac{t}{\tau}] + 1)} \leq 16 \sum_{1 \leq j \leq \frac{\pi}{\sqrt{2(2-\theta)\tau}}} e^{-32j^2 \tau ([\frac{t}{\tau}] + 1)}
\]

\[
\leq 16 \int_0^\infty e^{-32z^2 \tau ([\frac{t}{\tau}] + 1)} \, dz \leq C \left( \frac{t}{\tau} \right) \tau + \tau \right)^{-\frac{1}{2}}.
\]

As for $J_{1.1}$,

\[
J_{1.1} \leq 16 \sum_{j \in A_1} \exp \left\{ -2 \left( \left[ \frac{t}{\tau} \right] + 1 \right) \ln (1 + R_{3,j}) \right\}
\times \left( 1 - \exp \left\{ 2 \left( \left[ \frac{t}{\tau} \right] + 1 \right) (-R_{3,j} + \ln (1 + R_{3,j})) \right\} \right)
\]

\[
\leq 16 \sum_{j \in A_1} \exp \left\{ -2 \left( \left[ \frac{t}{\tau} \right] + 1 \right) \ln (1 - \lambda_j^\theta \tau) \right\}
\times \left( 1 - \exp \left\{ 2 \left( \left[ \frac{t}{\tau} \right] + 1 \right) (2\lambda_j^\theta \tau + \ln (1 - 2\lambda_j^\theta \tau)) \right\} \right)
\]

\[
\leq 16 \sum_{1 \leq j \leq \frac{\pi}{\sqrt{2(2-\theta)\tau}}} \exp \left\{ -2 \left( \left[ \frac{t}{\tau} \right] + 1 \right) \ln (1 - \lambda_j^\theta \tau) \right\}
\times \left( 1 - \exp \left\{ 2 \left( \left[ \frac{t}{\tau} \right] + 1 \right) (2\lambda_j^\theta \tau + \ln (1 - 2\lambda_j^\theta \tau)) \right\} \right)
\]

\[
\leq 16 \sum_{1 \leq j \leq \frac{\pi}{\sqrt{2(2-\theta)\tau}}} \exp \left\{ 2 \left( \left[ \frac{t}{\tau} \right] + 1 \right) C_2 \lambda_j^\theta \tau \right\} \times \left( 2 \left( \left[ \frac{t}{\tau} \right] + 1 \right) C_1 (2\lambda_j^\theta \tau)^2 \right)
\]

\[
\leq \sum_{1 \leq j \leq \frac{\pi}{\sqrt{2(2-\theta)\tau}}} C \left( \left[ \frac{t}{\tau} \right] + 1 \right)^2 \tau^{-\frac{3}{2}} \left( \left[ \frac{t}{\tau} \right] + 1 \right)^4 \tau^2
\]

\[
\leq \sum_{1 \leq j \leq \frac{\pi}{\sqrt{2(2-\theta)\tau}}} C \frac{t}{\tau} \tau^{-\frac{1}{2}} \tau^{-\frac{1}{2}} \leq C(\theta) \left( \left[ \frac{t}{\tau} \right] \tau + \tau \right)^{-\frac{1}{2}},
\]

where we have used the fact that $\lambda_j^\theta \tau \in [-4\pi^2 j^2 \tau, -16j^2 \tau]$ for $j = 1, 2, \ldots, \left[ \frac{n}{\tau} \right]$ and $j^2 \tau < \frac{1}{16(2-\theta)}$, so $z := -\lambda_j^\theta \tau \in \left( 0, \frac{\pi^2}{4(2-\theta)} \right]$, for such $z$, we have $-C_1 z^2 \leq -z + \ln(1 + z) \leq 0$ and $\ln(1 + z) \geq C_2 z$ for some $C_1, C_2 > 0$. The inequalities $-z + \ln(1 + z) \leq 0$ with $z \geq 0$, $1 - e^{-z} \leq z$ with $z \geq 0$ and $e^{-z^2} \leq C(\alpha)z^{-\alpha}$ with $\alpha > 0, z > 0$ are also used, here we choose $\alpha = 3$. 
For the term $J_2$,
\[
J_2 \leq \sum_{\frac{1}{2\pi}}^{\frac{1}{2\pi}} 4(1 - \epsilon)^2\left(1 - \theta \lambda \tau^2\right)^{-2} \leq \sum_{\frac{1}{2\pi}}^{\frac{1}{2\pi}} 4(1 - \epsilon)^2\left(1 + 16\theta^2 \tau^2\right)^{-2}
\]
\[
\leq 4 \int_{\frac{1}{2\pi}}^{\frac{1}{2\pi}} (1 - \epsilon)^2\left(1 + 16\theta x^2 \tau^2\right)^{-2} dx \quad \text{(let } y = x\sqrt{\tau})
\]
\[
\leq \frac{4}{\sqrt{\tau}} \int_{\frac{1}{2\pi}}^{\frac{1}{2\pi}} (1 - \epsilon)^2\left(1 + 16\theta y^2\right)^{-2} dy
\]
\[
\leq \frac{4}{\sqrt{\tau}} \times e^{-2\left(\frac{\tau}{\pi} + 1\right)\ln(1 - \epsilon)^{-1}} \times (1 - \epsilon)^{-2} \int_{\frac{1}{2\pi}}^{\frac{1}{2\pi}} (1 + 16\theta y^2)^{-2} dy
\]
\[
\leq C(\theta) \left(\left[\frac{t}{\tau}\right] \tau + \tau\right)^{-\frac{1}{2}} \int_{\frac{1}{2\pi}}^{\frac{1}{2\pi}} (1 + 16\theta y^2)^{-2} dy,
\]
where in the last line we use the inequality $e^{-z^2} \leq C(\alpha) z^{-\alpha}$, for $\alpha > 0$, $z > 0$, and $\alpha$ is chosen to be 1. Therefore, it remains to prove $\int_{\frac{1}{2\pi}}^{\frac{1}{2\pi}} (1 + 16\theta y^2)^{-2} dy \leq C$ for some $C > 0$.

For the Case 1 and Case 2, because $n^2 \tau$ is bounded, so $\int_{\frac{1}{2\pi}}^{\frac{1}{2\pi}} (1 + 16\theta y^2)^{-2} dy \leq C$.

For the Case 3,
\[
\int_{\frac{1}{2\pi}}^{\frac{1}{2\pi}} (1 + 16\theta y^2)^{-2} dy \leq \int_{0}^{\infty} (1 + 16\theta y^2)^{-2} dy \leq C.
\]

Combining these three cases, we finish the proof of (iii).

(iv) We only prove the case of $n$ being odd since the proof is similar when $n$ is even. For each fixed $n \geq 3, 0 < \tau < 1$, $R_{1,j}R_{2,j}$ is a decreasing sequence of $j$. Hence, under Assumption 3, for all $j = 1, 2, \ldots, \left[\frac{n}{2}\right]$, we have $-1 + \epsilon \leq R_{1,j}R_{2,j} \leq R_{1,1}R_{2,1} < 1$. Therefore, for each $n, \tau$, we can choose $\epsilon' := \min\{\epsilon, 1 - R_{1,1}R_{2,1}\} > 0$, such that $|R_{1,j}R_{2,j}| \leq 1 - \epsilon'$ for $j = 1, 2, \ldots, \left[\frac{n}{2}\right]$. Then
\[
2 \sum_{j=1}^{\left[\frac{n}{2}\right]} (R_{1,j}R_{2,j})^{\left[\frac{a}{2}\right]} e_j(\kappa_n(x))e_j(\kappa_n(y)) \leq 2 \sum_{j=1}^{\left[\frac{n}{2}\right]} (R_{1,j}R_{2,j})^{\left[\frac{a}{2}\right]} \leq 2 \sum_{j=1}^{\left[\frac{n}{2}\right]} (1 - \epsilon')^{\left[\frac{a}{2}\right]} \to 0
\]
as $t \to \infty$ for all $x, y \in [0, 1]$. So there exists a $t := t(n, \tau) > 0$ large enough, such that when $t > t(n, \tau)$,
\[
-\frac{1}{2} \leq 2 \sum_{j=1}^{\left[\frac{n}{2}\right]} (R_{1,j}R_{2,j})^{\left[\frac{a}{2}\right]} e_j(\kappa_n(x))e_j(\kappa_n(y)) \leq \frac{1}{2},
\]
which implies
\[
G_{1^n,\tau}(t, \kappa_n(x), y) = 1 + 2 \sum_{j=1}^{\left[\frac{n}{2}\right]} (R_{1,j}R_{2,j})^{\left[\frac{a}{2}\right]} e_j(\kappa_n(x))e_j(\kappa_n(y)) \geq \frac{1}{2}
\]
for all \( x, y \in [0, 1] \) and \( t > t(n, \tau) \). This will lead to our desired result after linear interpolation with respect to the space variable.

4.2. Intermittent upper bound.

**Proposition 4.4.** Under Assumption 3, there exists a random field \( u^{n, \tau} \in \bigcup_{\beta > 0} \mathcal{L}^{\beta,p} \) solving (17) for each \( n \geq 3, 0 < \tau < 1, p \geq 2 \). Moreover, \( u^{n, \tau} \) is a.s.-unique among all random fields satisfying

\[
\sup_{x \in [0,1]} \mathbb{E} \left( |u^{n, \tau}(t,x)|^p \right) \leq C_1^p \exp \left\{ C_2 L_\alpha^4 \lambda^3 t \right\}, \quad \text{for } p \geq 2, \ t = m\tau, \ m \geq 0
\]

with \( C_1 := C_1 \left( \sup_{x \in [0,1]} u_0(x), n \right) > 0 \) and \( C_2 > 0 \).

**Proof.** We apply Picard’s iteration again by defining

\[
u^{n, \tau}_{(0)}(t,x) := u_0(x),
\]

\[
u^{n, \tau}_{(q+1)}(t,x) := \int_0^t G^{n, \tau}_1(t, x, y) u_0(\kappa_n(y)) \, dy + \lambda \int_0^t \int_0^1 G^{n, \tau}_2(t - \kappa_\tau(s) - \tau, x, y) \sigma \left( u^{n, \tau}_{(q)}(\kappa_\tau(s), y) \right) \, dW(s, y).
\]

Using Lemma 4.3 (ii) (iii), combining the linear growth of \( \sigma \), Minkowski inequality and Burkholder-Davis-Gundy inequality, we obtain

\[
\left\| \nu^{n, \tau}_{(q+1)}(m\tau, x) \right\|_p^2 \leq 2 \sup_{x \in [0,1]} |u_0(x)|^2 \times \left( 1 + 4 \sum_{j=1}^{[\frac{m}{2}]} (R_{1,j} R_{2,j})^{2m} \right)
\]

\[+ C L_\alpha^2 p \lambda^2 \int_0^{m\tau} \left( \frac{1}{\sqrt{m\tau - \kappa_\tau(s)}} + 1 \right) \left( 1 + \sup_{y \in [0,1]} \left\| u^{n, \tau}_{(q)}(\kappa_\tau(s), y) \right\|_p^2 \right) \, ds
\]

\[
\leq 2 \sup_{x \in [0,1]} |u_0(x)|^2 \times \left( 1 + 4 \sum_{j=1}^{[\frac{m}{2}]} (R_{1,j} R_{2,j})^{2m} \right) + C L_\alpha^2 p \lambda^2 \int_0^{m\tau} \left( \frac{1}{\sqrt{m\tau - \kappa_\tau(s)}} + 1 \right) \sup_{y \in [0,1]} \left\| u^{n, \tau}_{(q)}(\kappa_\tau(s), y) \right\|_p^2 \, ds.
\]

Under Assumption 3 we have \( |R_{1,j} R_{2,j}| < 1 \), so \( 1 + 4 \sum_{j=1}^{[\frac{m}{2}]} (R_{1,j} R_{2,j})^{2m} \leq 1 + 2n \leq 3n \). Therefore,

\[
\left\| u^{n, \tau}_{(q+1)}(m\tau, x) \right\|_p^2 \leq 6n \sup_{x \in [0,1]} |u_0(x)|^2 + C L_\alpha^2 p \lambda^2 (\sqrt{m\tau} + m\tau)
\]

\[+ C L_\alpha^2 p \lambda^2 \int_0^{m\tau} \left( \frac{1}{\sqrt{m\tau - \kappa_\tau(s)}} + 1 \right) \sup_{y \in [0,1]} \left\| u^{n, \tau}_{(q)}(\kappa_\tau(s), y) \right\|_p^2 \, ds. \quad (21)
\]

Multiplying \( e^{-2\beta m\tau} \) with \( 2\beta \geq 1 \) on both sides of (21) and taking supremum over \( m \geq 0 \), we obtain

\[
\sup_{m \geq 0} \sup_{x \in [0,1]} \left\{ e^{-2\beta m\tau} \left\| u^{n, \tau}_{(q+1)}(m\tau, x) \right\|_p^2 \right\}
\]
\[ \leq 6n \sup_{x \in [0,1]} |u_0(x)|^2 + CL_\alpha^2 p \lambda^2 \left( \frac{1}{\sqrt{4\beta e}} + \frac{1}{2\beta e} \right) + CL_\alpha^2 \beta \lambda^2 \]

\[
\times \sup_{m \geq 0} \sum_{j=0}^{m-1} e^{-2\beta(m\tau - j\tau)} \int_{j\tau}^{(j+1)\tau} \left( \frac{1}{\sqrt{m\tau - j\tau}} + 1 \right) ds \sup_{j \geq 0} \sup_{y \in [0,1]} \left\{ e^{-2\beta j\tau} \left\| u_{(q)}^{n,\tau}(j\tau, y) \right\|_p^2 \right\}
\]

\[
\leq 6n \sup_{x \in [0,1]} |u_0(x)|^2 + CL_\alpha^2 p \lambda^2 \left( \frac{1}{\sqrt{4\beta e}} + \frac{1}{2\beta e} \right)

+ CL_\alpha^2 p \lambda^2 \int_0^\infty e^{-2\beta r} \left( \frac{1}{\sqrt{r}} + 1 \right) dr \sup_{j \geq 0} \sup_{y \in [0,1]} \left\{ e^{-2\beta j\tau} \left\| u_{(q)}^{n,\tau}(j\tau, y) \right\|_p^2 \right\}
\]

\[
\leq 6n \sup_{x \in [0,1]} |u_0(x)|^2 + CL_\alpha^2 p \lambda^2 \left( \frac{1}{\sqrt{4\beta e}} + \frac{1}{2\beta e} \right)

+ CL_\alpha^2 p \lambda^2 \frac{\sqrt{\pi}}{2\beta} \sup_{j \geq 0} \sup_{y \in [0,1]} \left\{ e^{-2\beta j\tau} \left\| u_{(q)}^{n,\tau}(j\tau, y) \right\|_p^2 \right\}
\]

where in the last step we have used \( \frac{\sqrt{\pi}}{2\beta} + \frac{1}{2\beta} \leq \frac{3}{2\beta} \) for \( 2\beta \geq 1 \).

The remaining part of the proof is similar to that of Proposition 3.3 by choosing \( \beta = 18C^2L_\alpha^4 p^2 \lambda^4 + \frac{1}{2} \) that satisfies \( \frac{3CL_\alpha^2 p \lambda^2}{\sqrt{2\beta}} \leq \frac{1}{2} \) and \( \beta \geq \frac{1}{2} \). Hence, we can get

\[ E \left( \left\| u_{(q+1)}^n(m\tau, x) \right\|^p \right) \leq C_1^p \exp \{ \beta m\tau \}, \quad p \geq 2, \]

where \( C_1 = (12n + 1) \sup_{x \in [0,1]} |u_0(x)|^2 + 1. \) Moreover, by the similar technique as in Proposition 3.3, one can prove the convergence of \( \left\{ u_{(q)}^n \right\}_{q \geq 0} \) and the uniqueness of the solution of (17). We omit the details. The proof is completed. \( \square \)

Based on Proposition 4.4, we give the following result, which shows the upper bound for the upper \( p \)th moment Lyapunov exponent.

**Proposition 4.5.** Under Assumption 3, there exists a positive constant \( C \) such that for each fixed \( 0 < \tau < 1, \ n \geq 3 \) and \( p \in [2, \infty) \), we have

\[ \sup_{x \in [0,1]} \gamma_{n,\tau}^p(x) \leq CL_\alpha^4 \lambda^4 p^3. \]

### 4.3. Intermittent lower bound.

It remains to investigate the lower bound of \( \gamma_{2,\tau}^n \). Before that, we give the following reverse discrete Grönwall type inequality.

**Lemma 4.6.** (Reverse Discrete Grönwall type inequality) Let \( \left\{ y_n \right\}_{n \geq 0} \) be nonnegative sequence and satisfy

\[ y_n \geq \alpha + \sum_{0 \leq k \leq n-1} \beta y_k \]

for \( n \geq N, \) where \( \alpha, \beta > 0. \) Then for \( l = 0, 1, 2, \ldots, \)

\[ y_{N+l} \geq \left( \alpha + \beta \sum_{0 \leq k \leq N-1} y_k \right) (1 + \beta)^l. \]
Proof. We prove (23) by induction. When \( l = 0 \), (22) with \( n = N \) implies
\[
y_N \geq \alpha + \sum_{0 \leq k \leq N-1} \beta y_k,
\]
which is (23) with \( l = 0 \).

Suppose that (23) holds for all \( l \leq q \), now we prove it in the case of \( l = q + 1 \). By (22) with \( n = N + q + 1 \) and the case of \( l \leq q \), we get
\[
y_{N+q+1} \geq \alpha + \sum_{0 \leq k \leq N-1} \beta y_k + \sum_{N \leq j \leq N+q} \beta y_j = \alpha + \sum_{0 \leq k \leq N-1} \beta y_k + \sum_{0 \leq j \leq q} \beta y_{N+j}
\]
\[
= \alpha + \sum_{0 \leq k \leq N-1} \beta y_k + \sum_{0 \leq j \leq q} \beta \left( \alpha (1 + \beta)^j + \beta (1 + \beta)^j \right) \sum_{0 \leq k \leq N-1} y_k
\]
\[
= \alpha \left( 1 + \sum_{0 \leq j \leq q} \beta (1 + \beta)^j \right) + \beta \left( 1 + \sum_{0 \leq j \leq q} \beta (1 + \beta)^j \right) \sum_{0 \leq k \leq N-1} y_k.
\]
It suffices to prove
\[
1 + \sum_{0 \leq j \leq q} \beta (1 + \beta)^j = (1 + \beta)^{q+1}, \quad q \geq 1.
\]
(24)

To this end, we show it by induction again.

Obviously, (24) holds for \( q = 1 \). Suppose that it holds for \( q = r - 1 \), we check it for \( q = r \),
\[
(1 + \beta)^{r+1} = \left( 1 + \sum_{0 \leq j \leq r-1} \beta (1 + \beta)^j \right) (1 + \beta)
\]
\[
= 1 + \sum_{0 \leq j \leq r-1} \beta (1 + \beta)^j + \beta \left( 1 + \sum_{0 \leq j \leq r-1} \beta (1 + \beta)^j \right)
\]
\[
= 1 + \sum_{0 \leq j \leq r-1} \beta (1 + \beta)^j + \beta (1 + \beta)^r = 1 + \sum_{0 \leq j \leq r} \beta (1 + \beta)^j.
\]
Hence we finish the proof.

Proposition 4.7. Under Assumptions \([A] \) and \([B] \) for each fixed \( n \geq 3 \) and \( 0 < \tau < 1 \),
\[
\inf_{x \in [0,1]} \tilde{\gamma}^{n,\tau}_{1/2} (x) \geq \frac{\log(1 + \lambda^2 J_0^2 \tau)}{\tau} > 0.
\]

Proof. For each fixed \( n \geq 3 \), \( 0 < \tau < 1 \), Lemma 4.3 (iv) implies that there is a \( t(n, \tau) > 0 \), such that \( G_1^{n,\tau}(t, x, y) > 0 \) for \( t > t(n, \tau) \). Hence, taking the second moment on both sides of (17), combining Walsh isometry and Lemma 4.3 (i) (ii), we get when \( m\tau > t(n, \tau) \),
\[
\mathbb{E} \left( |u^{n,\tau}(m\tau, x)|^2 \right)
\]
\[
\geq I^2_0 + \lambda^2 J_0^2 \int_0^{m\tau} \int_0^1 G_2^{n,\tau}(m\tau - \kappa(s) - \tau, x, y)^2 \mathbb{E} \left( |u^{n,\tau}(\kappa(s), \kappa(y))|^2 \right) ds dy
\]
\[
\geq I^2_0 + \lambda^2 J_0^2 \int_0^{m\tau} \int_0^1 G_2^{n,\tau}(m\tau - \kappa(s) - \tau, x, y)^2 dy \inf_{y \in [0,1]} \mathbb{E} \left( |u^{n,\tau}(\kappa(s), y)|^2 \right) ds
\]
\[ \geq I_0^2 + \lambda^2 J_0^2 \int_0^{m \tau} \inf_{y \in [0,1]} \mathbb{E} \left( |u_{n,\tau}(\kappa_\tau(s), y)|^2 \right) \, ds. \]

Taking infimum over \( x \in [0,1] \) yields

\[ \inf_{x \in [0,1]} \mathbb{E} \left( |u_{n,\tau}(m \tau, x)|^2 \right) \geq I_0^2 + \lambda^2 J_0^2 \int_0^{m \tau} \inf_{y \in [0,1]} \mathbb{E} \left( |u_{n,\tau}(\kappa_\tau(s), y)|^2 \right) \, ds, \]

which is equivalent to

\[ \inf_{x \in [0,1]} \mathbb{E} \left( |u_{n,\tau}(m \tau, x)|^2 \right) \geq I_0^2 + \lambda^2 J_0^2 \sum_{j=0}^{m-1} \inf_{y \in [0,1]} \mathbb{E} \left( |u_{n,\tau}(j \tau, y)|^2 \right) \tau. \]

Applying Lemma 4.6 with \( \alpha = I_0^2, \beta = \lambda^2 J_0^2 \tau, N = \left\lceil \frac{t(n, \tau)}{\tau} \right\rceil + 1 \) and omitting the last term on the right hand side of (23), we obtain

\[ \inf_{x \in [0,1]} \mathbb{E} \left( |u_{n,\tau}(N \tau + l \tau, x)|^2 \right) \geq I_0^2 (1 + \lambda^2 J_0^2 \tau)^l. \]

This leads to

\[ \inf_{x \in [0,1]} \gamma_2^{n,\tau}(x) = \inf_{x \in [0,1]} \limsup_{l \to \infty} \frac{\log \mathbb{E} \left( |u_{n,\tau}(N \tau + l \tau, x)|^2 \right)}{N \tau + l \tau} \geq \frac{\log(1 + \lambda^2 J_0^2 \tau)}{\tau} > 0. \]

The proof is finished. \( \square \)

**Remark 4.8.** (i) By Theorem 4.4, we have

\[ \inf_{x \in [0,1]} \liminf_{\tau \to 0} \gamma_2^{n,\tau}(x) \geq \lim_{\tau \to 0} \frac{\log(1 + \lambda^2 J_0^2 \tau)}{\lambda^2 J_0^2 \tau} \lambda^2 J_0^2 = \lambda^2 J_0^2, \]

where this lower bound is equal to that of the spatial semi-discretization.

(ii) As for the exponential integrator method (see [1]), whose continuous version can be written into the following mild form:

\[ u_{n,\tau}^E(t, x) := \int_0^1 G^n(t, x, y) u_0(\kappa_n(y)) \, dy \]

\[ + \lambda \int_0^t \int_0^1 G^n(t - \kappa_\tau(s), x, y) \sigma \left( u_{n,\tau}^E(\kappa_\tau(s), \kappa_n(y)) \right) W(d s d y), \]

for \( t = i \tau, x \in [0,1] \), where

\[ G^n(t, x, y) = \sum_{j=0}^{n-1} e^{\lambda_j t} \tilde{e}_j^n(x) \tilde{e}_j(\kappa_n(y)), \]

we can get the weak intermittency of this fully discrete scheme similarly.

### 4.4. Sharp exponential order of the second moment.

In this subsection, by applying a discrete renewal method, the second moment of the numerical solution of the fully discrete scheme is proved to have sharp exponential order \( C \lambda^4 \tau \).

**Theorem 4.9.** Let Assumptions [7] and [2] hold. Then for \( n, \tau \) satisfying \( \frac{J_0 n^2 \tau}{16 \pi^2} + 16 \pi^2 n^2 \tau < 1 \), we have

\[ \inf_{x \in [0,1]} \mathbb{E} \left( |u_{n,\tau}(m \tau, \kappa_n(x))|^2 \right) \geq C_1 e^{C_2 J_0^3 \lambda^4 m \tau}, \quad m \tau > T, \quad (25) \]
Proof. It suffices to find some constant $\mu > 0$ to be the zero point of the function $\tilde{h}(\mu) := \frac{1}{b} \left( \sum_{r=1}^{\infty} \tilde{g}(r) - 1 \right) = \sum_{r=1}^{\infty} \frac{e^{-\pi \mu^2 \tilde{b}_r^2 \tau}}{\sqrt{r \tau}} - \sum_{r=1}^{\infty} \frac{e^{-(\pi \mu^2 \tilde{b}_r^2 + 4\pi^2 \tau^2) r \tau}}{\sqrt{r \tau}} \tau - \frac{1}{\mu b}. \quad (26)

On the one hand,
\[
\sum_{r=1}^{\infty} \frac{e^{-\pi \mu^2 \tilde{b}_r^2 \tau}}{\sqrt{r \tau}} \leq \int_{0}^{\infty} \frac{e^{-\pi \mu^2 \tilde{b}_r^2 \tau}}{\sqrt{z}} \sqrt{\tau} \, dz = \frac{1}{\mu b}. \quad (27)
\]

On the other hand,
\[
\sum_{r=1}^{\infty} \frac{e^{-\pi \mu^2 \tilde{b}_r^2 \tau}}{\sqrt{r \tau}} \geq \int_{1}^{\infty} \frac{e^{-\pi \mu^2 \tilde{b}_r^2 \tau}}{\sqrt{z}} \sqrt{\tau} \, dz \geq \int_{0}^{1} \frac{e^{-\pi \mu^2 \tilde{b}_r^2 \tau}}{\sqrt{z}} \sqrt{\tau} \, dz \geq \frac{1}{\mu b} - 2 \sqrt{\tau}. \quad (28)
\]

where in the last line we use the same technique as in Lemma 3.8. The proof is finished. \(\square\)

Lemma 4.11. Let $\tilde{g}(r) := \tilde{b} e^{-\pi \mu^2 \tilde{b}_r^2 \tau} \frac{1 - e^{-4\pi^2 \tau^2 r^2}}{\sqrt{r \tau}} \tau$, where $\tilde{b} := \frac{\lambda^2 J_0^2}{8 \sqrt{\pi}}$. Suppose that $n, \tau$ satisfy $\frac{J_0^2}{16 \pi^3 \zeta} + 16 \pi n^2 \tau < 1$ and $n \geq \zeta \lambda^2$ for some $\zeta > 0$, then \{\tilde{g}(r)\}_{r \geq 1} is a discrete probability density function with some suitable $\mu \geq \frac{16 \pi \zeta}{J_0^2 + 32 \pi \zeta} > 0$.

Proof. It suffices to find some constant $\mu > 0$ to be the zero point of the function

Proof. Under conditions in Lemma 4.10 we have $\frac{1}{2} < R_{2,j} \leq 1, R_{3,j} < -2 \lambda_j^n \tau$, hence,
\[
\int_{0}^{1} (G_{2r}^{n, \tau}(t, \kappa_n(x), y))^2 \, dy \geq \sum_{j=0}^{\left[ \frac{n}{\tau} \right]} \left( 1 + R_{3,j} \right)^{2[n_j] - 2} (R_{2,j})^{-2}
\]
\[
\geq \sum_{j=0}^{\left[ \frac{n}{\tau} \right]} \exp \left\{ -2 \left( \left[ \frac{n}{\tau} \right] + 1 \right) \ln \left( 1 + R_{3,j} \right) \right\} \geq \sum_{j=0}^{\left[ \frac{n}{\tau} \right]} \exp \left\{ -2 \left( \left[ \frac{n}{\tau} \right] + 1 \right) \right\}
\]
\[
geq \sum_{j=0}^{\left[ \frac{n}{\tau} \right]} \exp \left\{ 4\tau \lambda_j^n \left( \left[ \frac{n}{\tau} \right] + 1 \right) \right\} \geq \sum_{j=0}^{\left[ \frac{n}{\tau} \right]} \exp \left\{ -16\lambda_j^n \left( \left[ \frac{n}{\tau} \right] + 1 \right) \right\}
\]
\[
\geq \frac{1 - \exp \left\{ -4\pi^2 \tau^2 \left( \left[ \frac{n}{\tau} \right] + 1 \right) \right\}}{8 \sqrt{\pi} \left( \left[ \frac{n}{\tau} \right] + 1 \right) \tau},
\]

where in the last line we use the same technique as in Lemma 3.8. The proof is finished. \(\square\)
Similarly, we have
\[
\sqrt{\frac{1}{\mu^2 b^2 + 4n^2\pi}} - 2\sqrt{\tau} \leq \sum_{r=1}^{\infty} \frac{e^{-(\pi\mu^2 b^2 + 4n^2\pi^2)\tau}}{\sqrt{r\tau}} \leq \sqrt{\frac{1}{\mu^2 b^2 + 4n^2\pi}}. 
\] (30)
Combining (27) (28) and (30), we obtain
\[
\frac{1}{\mu b} - 2\sqrt{\tau} - \sqrt{\frac{1}{\mu^2 b^2 + 4n^2\pi}} - \frac{1}{b} \leq h(\mu) \leq \frac{1}{\mu b} - \left(\sqrt{\frac{1}{\mu^2 b^2 + 4n^2\pi}} - 2\sqrt{\tau}\right) - \frac{1}{b}.
\]
Hence, for any \( \varepsilon > 0 \), the right hand side of (26) converges uniformly to a continuous function on \( \mu \in [\varepsilon, 1] \), we still denote it by \( \tilde{h}(\mu) \).

Because of \( n \geq \zeta \lambda^2 \), by choosing \( \varepsilon = \frac{1}{\sqrt{4\pi^2 \lambda^4 + 2}} = \frac{16\pi \zeta}{J_0^2 + 32\pi \zeta} \leq \frac{1}{\sqrt{4\pi^2 \lambda^4 + 2}} \) that is independent of \( \tilde{b} \), we have
\[
\left(\frac{1}{\varepsilon} - 1\right) \geq \frac{\tilde{b}^2}{4n^2\pi} + 1 > \frac{\tilde{b}^2}{4n^2\pi} + 2\tilde{b}\sqrt{\tau} > \sqrt{\frac{\tilde{b}^2}{4n^2\pi}} + 2\tilde{b}\sqrt{\tau},
\]
which yields \( \tilde{h}(\varepsilon) > 0 \).

Since \( \frac{h_n^2}{10\pi^2} + 16\pi n^2\tau < 1 \) implies \( 1 > 4\tilde{b}^2\tau + 16\pi n^2\tau \), so \( \tilde{h}(1) \leq - \left(\sqrt{\frac{1}{\tilde{b}^2 + 4n^2\pi}} - 2\sqrt{\tau}\right) < 0 \).
Therefore, there is a \( \mu := \mu(n, \tau, \tilde{b}) \in (\varepsilon, 1) \) satisfying \( \tilde{h}(\mu) = 0 \), and \( \mu \geq \varepsilon = \frac{16\pi \zeta}{J_0^2 + 32\pi \zeta} \). The proof is finished.

Proof of Theorem 4.9.

Proof. Taking the second moment on both sides of (17) with space variable being \( \kappa_n(x) \) and time variable being \( m\tau \), combining Walsh isometry, Lemma 4.3 (i) (ii) and Lemma 3.10, we get
\[
E \left( \left| u^{n,\tau}(m\tau, \kappa_n(x)) \right|^2 \right) \geq I_0^2 + \frac{\lambda^2 J_r^2}{8\sqrt{\pi}} \sum_{j=0}^{m-1} \frac{1 - e^{-4\pi^2\tau^2(m-j)^2}}{\sqrt{(m-j)^2}} \inf_{y \in [0,1]} E \left( \left| u^{n,\tau}(m\tau, \kappa_n(y)) \right|^2 \right) \tau.
\]
Taking infimum over \( x \in [0,1] \), then multiplying both sides by \( e^{-\pi \mu^2 \tilde{b}^2 m\tau} \) with \( \tilde{b} = \frac{\lambda^2 J_r^2}{8\sqrt{\pi}} \), we see that
\[
M^{n,\tau}(m\tau) := e^{-\pi \mu^2 \tilde{b}^2 m\tau} \inf_{x \in [0,1]} E \left( \left| u^{n,\tau}(m\tau, \kappa_n(x)) \right|^2 \right)
\]
satisfies
\[
M^{n,\tau}(m\tau) \geq e^{-\pi \mu^2 \tilde{b}^2 m\tau} I_0^2 + \sum_{j=0}^{m-1} b e^{-\pi \mu^2 \tilde{b}^2 (m-j)^2} \frac{1 - e^{-4\pi^2\tau^2(m-j)^2}}{\sqrt{(m-j)^2}} M^{n,\tau}(m\tau) \tau
\]
\[
= e^{-\pi \mu^2 \tilde{b}^2 m\tau} I_0^2 + \sum_{j=1}^{m} b e^{-\pi \mu^2 \tilde{b}^2 j^2} \frac{1 - e^{-4\pi^2\tau^2 j^2}}{\sqrt{j^2}} M^{n,\tau}((m-j)\tau) \tau.
\]
By Lemma 4.11 \( \tilde{g}(\tau) = \frac{b e^{-\pi \mu^2 \tilde{b}^2 \tau} - e^{-4\pi^2\tau^2\tau}}{\sqrt{\tau}} \) is a discrete probability density function. Hence, applying the discrete Renewal Theorem (see [2 Theorem 8.5.13]) and the discrete
Lemma 4.13. (i) Under conditions in Assumption 3, there is a constant $C_{\lambda}$ of the fully discrete scheme has sharp exponential order $C_{\lambda} t_m$ for large $n$ and $t_m = m\tau$. 

Remark 4.12. As for the exponential integrator method, the second moment of the numerical solution also has sharp exponential order $C_{\lambda} t$ for large $n$ and $t$. 

4.5. Error estimations of fully discrete scheme. In this subsection, we show the convergence of the fully discrete scheme. It is based on the error estimates on the fully discrete Green functions and the exact one, whose proofs are postponed to Appendix 6.4. In the sequel, we assume $n \geq 3$, $0 < \tau < 1$.

Lemma 4.13. (i) Under conditions in Assumption 3, there is a constant $C := C(\theta) > 0$ such that for all $x \in [0, 1]$,

\[
\int_0^\infty \int_0^1 |G(t, x, y) - G_{\alpha}^\tau(t, x, y)|^2 dy dt \leq C \left( \frac{1}{n} + \sqrt{\tau} \right). \tag{31}
\]

(ii) Under conditions in Assumption 3 (i) or Assumption 3 (ii) or $\theta = 1$ or $\theta \in (\frac{1}{2}, 1)$ with $u_0 \in C^2([0, 1])$, for any $\frac{1}{2} < \alpha < 2$, there is a positive constant $C := C(\alpha, \theta)$ such that

\[
\left| \int_0^1 (G^\alpha(t, x, y) - G_{\alpha}^\tau(t, x, y)) u_0(\kappa_n(y)) dy \right|^2 \leq C r^{\alpha - \frac{1}{2}} \left( \frac{t}{\tau} \right)^\alpha \tag{32}
\]

for all $x \in [0, 1]$, $t \geq \tau > 0$.

Based on Lemma 4.13, we can establish the convergence theorem of the fully discrete scheme. The proof is omitted since it can be proved in a similar way as [22].

Theorem 4.14. Under conditions in Lemma 4.13 (ii), for every $0 < \alpha < \frac{1}{2}$, $0 < \beta < \frac{1}{2}$, $p \geq 1$ and for every $t \in (0, T]$ with the fixed $T > 0$, there is a constant $C := C(\alpha, p, T) > 0$ such that

\[
\sup_{x \in [0, 1]} \| u_n^\tau(t, x) - u(t, x) \|_{2p} \leq C(\tau^{\alpha} + n^{-\beta}). \tag{33}
\]
5. Conclusions and future aspects

In this paper, in order to investigate the numerical schemes that could inherit the weak intermittency of the SHE and preserve the sharp exponential order of the second moment of the exact solution, we implement an approach based on the compact integral form of the numerical scheme and the detailed analysis of the discrete Green function. It is shown that the semi-discrete scheme and the fully discrete scheme are both weakly intermittent. Furthermore, both of them could preserve the sharp exponential order of the second moment of the exact solution. In fact, there are still many problems that remains to be solved. We list several possible aspects for future work:

(1) Is there a criterion that is easy to check, to judge whether a numerical scheme can inherit the weak intermittency of the original equation?

(2) If a numerical scheme can inherit the weak intermittency of the original equation, how to estimate the error of the Lyaponuv exponents?

The above two problems are very challenging. Generally, the expression of the discrete Green function for a numerical scheme can not be written explicitly, so it is difficult to analyze the detailed point-wise and integral estimates of the discrete Green function. We leave these problems as open problems, and attempt to study them in our future work.

6. Appendix

6.1. Proof of (7). Using notations

\[ [U^n(t)]_k = U^n_k(t) := u^n(t, \frac{k}{n}), \quad [W^n(t)]_k = W^n_k(t) := \sqrt{n} \left( W\left(t, \frac{k+1}{n}\right) - W\left(t, \frac{k}{n}\right) \right), \]

it follows from (6) that

\[ [dU^n(t) - n^2 DU^n(t)dt]_k = \lambda \sqrt{n} [\text{diag}(\sigma(U^n_0(t)), \ldots, \sigma(U^n_{n-1}(t))) dW^n(t)]_k, \]

where \( D = (D_{ki}) \) is an \( n \times n \) matrix

\[
\begin{pmatrix}
-2 & 1 & 1 \\
1 & -2 & \ddots \\
& \ddots & \ddots & 1 \\
1 & 1 & -2
\end{pmatrix}.
\]

The eigenvalues of \( n^2 D \) are \( \lambda^n_j := -4n^2 \sin^2\left(\frac{2\pi j}{n}\right), j = 0, 1, \ldots, n - 1 \), and the corresponding complex eigenvectors are denoted by \( f_j \), whose \( k \)th component is \( [f_j]_k := \frac{1}{\sqrt{n}} e^{2\pi i j \frac{k}{n}}, j, k = 0, 1, \ldots, n - 1 \). Denote \( e_j(x) := e^{2\pi i j x} \). Moreover, \( f_j, j = 0, 1, \ldots, n - 1 \) form an orthogonal normal basis in \( \mathbb{C}^n \) (see [16]).

Simple computation yields

\[ [U^n(t)]_k = \left[ e^{n^2 D t} U^n(0) \right]_k + \lambda \sqrt{n} \left[ \int_0^t e^{n^2 D (t-s)} \text{diag}(\sigma(U^n_0(s)), \ldots, \sigma(U^n_{n-1}(s))) dW^n(s) \right]_k. \]

Note that

\[
[e^{n^2 D t} U^n(0)]_k = \sum_{j=0}^{n-1} a_j e^{\lambda^n_j t} f_j \left( \frac{k}{n} \right) = \sum_{j=0}^{n-1} a_j \frac{1}{\sqrt{n}} e^{\lambda^n_j t} e_j \left( \frac{k}{n} \right)
\]
where
\[ G^n(t, \frac{k}{n}, y) = \sum_{j=0}^{n-1} e^{\lambda_j t} \tilde{e}_j(\kappa_n(y)), \quad u^n(0) = \sum_{j=0}^{n-1} a_j f_j, \quad a_j = \sum_{l=0}^{n-1} u_0 \left( \frac{l}{n} \right) \frac{1}{\sqrt{n}} \tilde{e}_j \left( \frac{l}{n} \right). \]

Similarly,
\[
\lambda \sqrt{n} \left[ \int_0^t e^{nD(t-s)} \text{diag}(\sigma(U^n_0(s)), \ldots, \sigma(U^n_{n-1}(s))) dW^n(s) \right]_k
\]
\[
= \lambda \int_0^t \int_0^1 \sum_{l=0}^{n-1} \frac{1}{\sqrt{n}} e^{\lambda_j t} \sigma(U^n_l(s)) \tilde{e}_j \left( \frac{l}{n} \right) dW^n(s)
\]
\[
= \lambda \int_0^t \int_0^1 G^n(t-s, \frac{k}{n}, y) \sigma(u^n(s, \kappa_n(y))) dW(s, y). \tag{36}
\]

Combining equations \[34\] and \[35\] and \[36\], we get
\[
u^n(t, \frac{k}{n}) = \int_0^1 G^n(t, \frac{k}{n}, y) u_0(\kappa_n(y)) dy + \lambda \int_0^t \int_0^1 G^n(t-s, \frac{k}{n}, y) \sigma(u^n(s, \kappa_n(y))) dW(s, y). \tag{37}
\]

We construct the continuous version of \[37\] by the linear interpolation:
\[
u^n(t, x) := u^n(t, \kappa_n(x)) + (nx - n\kappa_n(x)) \left[ u^n(t, \kappa_n(x) + \frac{1}{n}) - u^n(t, \kappa_n(x)) \right], \quad x \in [0, 1].
\]

Denote
\[ e^n_j(x) := e_j(\kappa_n(x)) + (nx - n\kappa_n(x)) \left[ e_j(\kappa_n(x) + \frac{1}{n}) - e_j(\kappa_n(x)) \right], \quad x \in [0, 1], \]
then
\[ G^n(t, y) = \sum_{j=0}^{n-1} e^{\lambda_j t} e^n_j(x) \tilde{e}_j(\kappa_n(y)), \quad t \geq 0, \, x, y \in [0, 1]. \]

Obviously \( u^n \) satisfies the equation
\[
u^n(t, x) = \int_0^1 G^n(t, x, y) u^n(0, \kappa_n(y)) dy + \lambda \int_0^t \int_0^1 G^n(t-s, x, y) \sigma(u^n(s, \kappa_n(y))) dW(s, y),
\]
almost surely for all \( t \geq 0 \) and \( x \in [0, 1] \). Hence we get \[7\].

6.2. Proof of Lemma 3.10

Proof. (i) \( G^n(t, x, y) \) can be rewritten as follows by expanding its real and imaginary parts:
If \( n \) is odd,
\[
G^n(t, x, y) = 1 + 2 \sum_{j=1}^{[\frac{n}{2}]} e^{\lambda_j t} \left( \phi^n_{c,j}(x) \varphi_{c,j}(\kappa_n(y)) + \phi^n_{s,j}(x) \varphi_{s,j}(\kappa_n(y)) \right).
\]
If $n$ is even,

$$G^n(t, x, y) = 1 + 2 \sum_{j=1}^{\frac{n}{2}-1} e^{\lambda_j^nt} \left( \varphi^{n}_{c,j}(x) \varphi_{c,j}(\kappa_n(y)) + \varphi^{n}_{s,j}(x) \varphi_{s,j}(\kappa_n(y)) \right)$$

$$+ e^{\frac{\lambda_j^nt}{2}} \left( \varphi^{n}_{c,\frac{n}{2}}(x) \varphi_{c,\frac{n}{2}}(\kappa_n(y)) + \varphi^{n}_{s,\frac{n}{2}}(x) \varphi_{s,\frac{n}{2}}(\kappa_n(y)) \right)$$

$$+ i \varphi^{n}_{s,\frac{n}{2}}(x) \varphi_{c,\frac{n}{2}}(\kappa_n(y)) - i \varphi^{n}_{c,\frac{n}{2}}(x) \varphi_{s,\frac{n}{2}}(\kappa_n(y))$$

$$= 1 + 2 \sum_{j=1}^{\frac{n}{2}-1} e^{\lambda_j^nt} \left( \varphi^{n}_{c,j}(x) \varphi_{c,j}(\kappa_n(y)) + \varphi^{n}_{s,j}(x) \varphi_{s,j}(\kappa_n(y)) \right)$$

$$+ e^{-4n^2t} \varphi^{n}_{c,\frac{n}{2}}(x) \varphi_{c,\frac{n}{2}}(\kappa_n(y)),$$

where $\varphi_{c,j}(x) := \cos(2\pi j x)$, $\varphi_{s,j}(x) := \sin(2\pi j x)$, and

$$\varphi^{n}_{c,j}(x) := \varphi_{c,j}(\kappa_n(x)) + (nx - n\kappa_n(x)) \left[ \varphi_{c,j}(\kappa_n(x) + \frac{1}{n}) - \varphi_{c,j}(\kappa_n(x)) \right],$$

$$\varphi^{n}_{s,j}(x) := \varphi_{s,j}(\kappa_n(x)) + (nx - n\kappa_n(x)) \left[ \varphi_{s,j}(\kappa_n(x) + \frac{1}{n}) - \varphi_{s,j}(\kappa_n(x)) \right].$$

Let’s rewrite the spectral decomposition of $G(t, x, y)$ by using notations above as follows:

$$G(t, x, y) = 1 + 2 \sum_{j=1}^{\infty} e^{-4\pi^2 j^2 t} \cos(2\pi j(x - y))$$

$$= 1 + 2 \sum_{j=1}^{\infty} e^{-4\pi^2 j^2 t} \left( \cos(2\pi j x) \cos(2\pi j y) + \sin(2\pi j x) \sin(2\pi j y) \right)$$

$$= 1 + 2 \sum_{j=1}^{\infty} e^{-4\pi^2 j^2 t} \left( \varphi_{c,j}(x) \varphi_{c,j}(y) + \varphi_{s,j}(x) \varphi_{s,j}(y) \right).$$

We first show the result by supposing $n$ is odd. It is clear that

$$I := \int_0^\infty \int_0^1 |G(t, x, y) - G^n(t, x, y)|^2 \, dy \, dt \leq 8 \sum_{k=1}^{4} (I_k^c + I_k^s),$$

where

$$I_1^c := \int_0^\infty \int_0^1 \left| 2 \sum_{r=\left[\frac{n}{2}\right]+1} e^{-4\pi^2 r^2 t} \varphi_{c,r}(x) \varphi_{c,r}(y) \right|^2 \, dy \, dt,$$

$$I_2^c := \int_0^\infty \int_0^1 \left| 2 \sum_{r=1}^{\left[\frac{n}{2}\right]} e^{-4\pi^2 r^2 t} \varphi^{n}_{c,r}(x) \left( \varphi_{c,r}(y) - \varphi_{c,r}(\kappa_n(y)) \right) \right|^2 \, dy \, dt,$$

$$I_3^c := \int_0^\infty \int_0^1 \left| 2 \sum_{r=1}^{\left[\frac{n}{2}\right]} \left( e^{-4\pi^2 r^2 t} - e^{\lambda r^nt} \right) \varphi^{n}_{c,r}(x) \varphi_{c,r}(\kappa_n(y)) \right|^2 \, dy \, dt,$$

$$I_4^c := \int_0^\infty \int_0^1 \left| 2 \sum_{r=1}^{\left[\frac{n}{2}\right]} e^{-4\pi^2 r^2 t} \left( \varphi_{c,r}(x) - \varphi^{n}_{c,r}(x) \right) \varphi_{c,r}(y) \right|^2 \, dy \, dt.$$
Moreover, \( I_k, k = 1, 2, 3, 4 \) are defined in a similar way via replacing \( \cos(\cdot) \) by \( \sin(\cdot) \).

Simple computation yields that when \( n \geq 3 \),

\[
I_1 = 4 \int_0^\infty \int_0^1 \sum_{r=[\frac{n}{2}]+1}^{\infty} e^{-8\pi^2r^2t} \cos^2(2\pi r x) \cos^2(2\pi r y) \, dy \, dt
\]

\[
= 2 \int_0^\infty \sum_{r=[\frac{n}{2}]+1}^{\infty} e^{-8\pi^2r^2t} \cos^2(2\pi r x) \, dt \leq 2 \int_0^\infty \sum_{r=[\frac{n}{2}]+1}^{\infty} e^{-8\pi^2r^2t} \, dt
\]

\[
\leq 2 \sum_{r=[\frac{n}{2}]+1}^{\infty} \frac{1}{8\pi^2r^2} \leq \frac{C}{n},
\]

\[
I_4 = 2 \int_0^\infty \sum_{r=1}^{[\frac{n}{2}]} e^{-8\pi^2r^2t} \left( \varphi_{c,r}(x) - \varphi_{c,r}^n(x) \right)^2 \, dt
\]

\[
\leq 4 \int_0^\infty \sum_{r=1}^{[\frac{n}{2}]} e^{-8\pi^2r^2t} \left( \left( \varphi_{c,r}(x) - \varphi_{c,r}(\kappa_n(x)) \right)^2 + \left( \varphi_{c,r}^n(\kappa_n(x)) - \varphi_{c,r}^n(x) \right)^2 \right) \, dt
\]

\[
\leq 8 \int_0^\infty \sum_{r=1}^{[\frac{n}{2}]} e^{-8\pi^2r^2t} \times \left( \frac{2\pi r}{n} \right)^2 \, dt \leq \frac{32\pi^2}{n^2} \times \frac{1}{8\pi^2} \times \left[ \frac{n}{2} \right] \leq \frac{C}{n}
\]

Moreover,

\[
I_3 \leq 2 \int_0^\infty \sum_{r=1}^{[\frac{n}{2}]} e^{-4\pi^2r^2t} - e^{-c_r^2t} \right)^2 \, dt \leq 2 \int_0^\infty \sum_{r=1}^{[\frac{n}{2}]} e^{-8\pi^2r^2t} c_r^n t \times \left( 1 - e^{-4\pi^2r^2(1-c_r^2)t} \right)^2 \, dt
\]

\[
\leq 2 \int_0^\infty \sum_{r=1}^{[\frac{n}{2}]} e^{-r^2t} \times \left( 4\pi^2r^2(1-c_r^2)t \right)^2 \, dt \leq \frac{C}{n^4} \int_0^\infty \sum_{r=1}^{[\frac{n}{2}]} e^{-r^2t} \times r^8t^2 \, dt \leq \frac{C}{n^4} \sum_{r=1}^{[\frac{n}{2}]} r^2 \leq \frac{C}{n},
\]

where \( c_r^n := \sin^2 \frac{r\pi}{n}/(\frac{r\pi}{n})^2 \) in \([\frac{4}{8\pi^2}, 1]\) for \( r = 1, 2, \ldots, [\frac{n}{2}] \). Here we have used the inequalities \( 1 - e^{-z} \leq e^{-z} \leq 1 - \frac{e^{-z}}{z} \leq \frac{z}{3} \) for \( z > 0 \).

In the sequel we use the notation \( \tilde{G}_n(t, x, y) := 1 + 2 \sum_{r=1}^{[\frac{n}{2}]} e^{-4\pi^2r^2t} \varphi_{c,r}(x) \varphi_{c,r}(y) \).

Note that for every function \( u \in C^1([0, 1]) \),

\[
\int_0^1 \left| u(y) - u(\kappa_n(y)) \right|^2 \, dy = \int_0^1 \left| \int_{\kappa_n(y)}^y \frac{d}{dt} u(t) \, dt \right|^2 \, dy \leq \frac{1}{n} \int_0^1 \int_{\kappa_n(y)}^y \left| \frac{d}{dt} u(t) \right|^2 \, dt \, dy
\]

\[
\leq \frac{1}{n} \int_t^{t+\frac{1}{n}} \left| \frac{d}{dt} u(t) \right|^2 \, dy \, dt \leq \frac{1}{n^2} \int_0^1 \left| \frac{d}{dt} u(t) \right|^2 \, dt.
\]

Thus,

\[
I_2 = \int_0^\infty \int_0^1 \left| \tilde{G}_n(t, x, y) - \tilde{G}_n(t, x, \kappa_n(y)) \right|^2 \, dy \, dt \leq \frac{1}{n^2} \int_0^\infty \int_0^1 \left| \frac{d}{dy} \tilde{G}_n(t, x, y) \right|^2 \, dy \, dt
\]

\[
= \frac{1}{n^2} \int_0^\infty \int_0^1 \left| 2 \sum_{r=1}^{[\frac{n}{2}]} e^{-4\pi^2r^2t} \varphi_{c,r}^n(x) \times 2\pi r \sin(2\pi ry) \right|^2 \, dy \, dt
\]
Similarly, we can get $I_k^i \leq \frac{C}{n^\alpha}, k = 1, 2, 3, 4$.

When $n$ is even, there is one more term denoted by $I_5$ to be estimated,

$$I_5 \leq C \int_0^\infty \int_0^1 \left| e^{-4n^2 \theta^2} \cdot \cos(2\pi \times \frac{n}{2} \times \kappa_n(y)) \right|^2 dy dt \leq \frac{C}{n^2}.$$  

Combining the estimations of $I_k^c, I_k^s, k = 1, 2, 3, 4$ and $I_5$, we finish the proof of (i).

(ii) The proof follows the same line as that of (i). Making use of the inequality $e^{-z^2} \leq C(\alpha)z^{-\alpha}$ for $z > 0, \alpha > 0$, we get

$$\sum_{r=\lfloor \frac{n}{2} \rfloor + 1}^\infty e^{-8\pi^2 r^2 t} \leq C(\alpha) \sum_{r=\lfloor \frac{n}{2} \rfloor + 1}^\infty r^{-2\alpha}t^{-\alpha} \leq C(\alpha) \int_{\lfloor \frac{n}{2} \rfloor + 1}^\infty x^{-2\alpha} dx \leq C(\alpha) n^{1-2\alpha}t^{-\alpha}$$

for $\alpha > \frac{1}{2}$, and

$$\sum_{r=1}^{\lfloor \frac{n}{2} \rfloor} e^{-8\pi^2 r^2 t} \leq C(\alpha) \sum_{r=1}^{\lfloor \frac{n}{2} \rfloor} r^{-2\alpha}t^{-\alpha} n^{-2} \leq C(\alpha) \int_0^{\lfloor \frac{n}{2} \rfloor + 1} x^{-2\alpha} dx \times t^{-\alpha} n^{-2} \leq C(\alpha) n^{1-2\alpha}t^{-\alpha}$$

for $\alpha < 1$.

Similarly, we have

$$\sum_{r=1}^{\lfloor \frac{n}{2} \rfloor} e^{-2 \pi \times \frac{n}{2} \times \kappa_n(y)} \leq C \sum_{r=1}^{\lfloor \frac{n}{2} \rfloor} r^{-2\gamma}t^{-\gamma} n^{-4} \leq C n^{\beta-2\gamma}t^{-\gamma} =: C(\alpha) n^{1-2\alpha}t^{-\alpha}$$

for $\alpha < 2$, and $e^{-8n^2 t} \leq C n^{-2\alpha}t^{-\alpha}$ for $\alpha > 0$.

Combining the terms above, we finish the proof.

\[\square\]

6.3. **Proof of (17).** It is clear that (16) is equivalent to

$$u^{n,\tau}(t_{i+1}, x_j) = (1 - \theta \tau \Delta_n)^{-1} (1 + (1 - \theta)\tau \Delta_n) u^{n,\tau}(t_i, \cdot)(x_j) + (1 - \theta \tau \Delta_n)^{-1} \lambda \tau \sigma(u^{n,\tau}(t_i, \cdot)) \square_{n,\tau} W(t_i, \cdot)(x_j).$$

It is easy to check that

$$\Delta_n e_j(x_k) = \lambda_j^n e_j(x_k), \quad k, j = 0, 1, \ldots, n - 1.$$
Let $R_1 := (1 - \theta \tau \Delta_n)^{-1}, R_2 := (1 + (1 - \theta) \tau \Delta_n), R_{1,l} := (1 - \theta \tau \lambda^p_l)^{-1}, R_{2,l} := (1 + (1 - \theta) \tau \lambda^p_l)$.

By iteration, we get

$$u^{n,\tau}(t_i, \cdot)(x_j) = (R_1 R_2)^i u^{n,\tau}(0, \cdot)(x_j) + \lambda \sum_{k=0}^{i-1} (R_1 R_2)^k R_1 \tau \sigma (u^{n,\tau}(t_{i-1-k}, \cdot) \bigtriangleup_{n,\tau} W(t_{i-1-k}, \cdot)(x_j)$$

$$= \sum_{l=0}^{n-1} \sum_{k=0}^{n-1} \frac{1}{n} (R_{1,l} R_{2,l})^i u_{(n,t)} \left( \frac{k}{n} \right) \bar{e}_l \left( \frac{k}{n} \right) e_l \left( \frac{j}{n} \right)$$

$$+ \lambda \sum_{k=0}^{i-1} \sum_{l=0}^{n-1} \sum_{q=0}^{n-1} \frac{1}{\sqrt{n}} (R_{1,l} R_{2,l})^{i-1-k} R_{1,l} \sigma \left( u^{n,\tau}\left( t_{k, \frac{q}{n}} \right) \right) (W^n_q(t_{k+1}) - W^n_q(t_k)) \bar{e}_l \left( \frac{q}{n} \right) e_l \left( \frac{j}{n} \right)$$

$$= : \int_0^1 G^{n,\tau}_1(t_i, x_j, y) u_0(\kappa_n(y)) dy$$

$$+ \lambda \int_0^t \int_0^1 G^{n,\tau}_2(t_i - \kappa_\tau(s) - \tau, x_j, y) \sigma (u^{n,\tau}(\kappa_\tau(s), \kappa_n(y))) dW(s, y),$$

where $\kappa_\tau(s) := \left[ \frac{s}{\tau} \right] \tau$ and

$$G^{n,\tau}_1(t, x_j, y) := \sum_{l=0}^{n-1} (R_{1,l} R_{2,l})^{\left[ \frac{q}{n} \right]} e_l(x_j) \bar{e}_l(\kappa_n(y)),$$

$$G^{n,\tau}_2(t, x_j, y) := \sum_{l=0}^{n-1} (R_{1,l} R_{2,l})^{\left[ \frac{q}{n} \right]} R_{1,l} e_l(x_j) \bar{e}_l(\kappa_n(y)).$$

By the linear interpolation with respect to the space variable, and denoting

$$G^{n,\tau}_1(t, x, y) := \sum_{l=0}^{n-1} (R_{1,l} R_{2,l})^{\left[ \frac{q}{n} \right]} e_l^n(x) \bar{e}_l(\kappa_n(y)),$$

$$G^{n,\tau}_2(t, x, y) := \sum_{l=0}^{n-1} (R_{1,l} R_{2,l})^{\left[ \frac{q}{n} \right]} R_{1,l} e_l^n(x) \bar{e}_l(\kappa_n(y)),$$

it is clear that $u^{n,\tau}$ satisfies the integral equation

$$u^{n,\tau}(t, x) = \int_0^1 G^{n,\tau}_1(t, x, y) u_0(\kappa_n(y)) dy$$

$$+ \lambda \int_0^t \int_0^1 G^{n,\tau}_2(t - \kappa_\tau(s) - \tau, x, y) \sigma (u^{n,\tau}(\kappa_\tau(s), \kappa_n(y))) dW(s, y),$$

almost surely for every $t = i\tau, x \in [0, 1]$. Hence we get (17).

6.4. Proof of Lemma 4.13

Proof. (i) By expanding the real and imaginary parts, the fully discrete Green functions can be written as follows,

$$G^{n,\tau}_1(t, x, y) = 1 + 2 \sum_{l} (R_{1,l} R_{2,l})^{\left[ \frac{q}{n} \right]} (\varphi_{c,l}(x) \varphi_{c,l}(\kappa_n(y)) + \varphi_{s,l}(x) \varphi_{s,l}(\kappa_n(y)))$$

$$+ \left( R_{1,\frac{q}{n}} R_{2,\frac{q}{n}} \right)^{\left[ \frac{q}{n} \right]} g_n(x, y),$$
\[ G_{2}^{n,\tau}(t, x, y) = 1 + 2 \sum_{l} (R_{1,l}R_{2,l}) \left[ \varphi_{c,l}(x)\varphi_{c,l}(\kappa_{n}(y)) + \varphi_{s,l}(x)\varphi_{s,l}(\kappa_{n}(y)) \right] \\
+ (R_{1,\frac{1}{2}}R_{2,\frac{1}{2}}) \left[ \varphi_{c,\frac{1}{2}}(x)\varphi_{c,\frac{1}{2}}(\kappa_{n}(y)) \right] R_{1,\frac{1}{2}} g_{n}(x, y), \]

where

\[ g_{n}(x, y) := \begin{cases} 0, & n = 2k + 1, \\
\varphi_{c,\frac{1}{2}}(x)\varphi_{c,\frac{1}{2}}(\kappa_{n}(y)), & n = 2k + 2, \quad k = 1, 2, \ldots. \]

It suffices to prove

\[ \int_{0}^{\infty} \int_{0}^{1} \left| G_{n}(t, x, y) - G_{2}^{n,\tau}(t, x, y) \right|^{2} dy \, dt \leq C \sqrt{\tau}. \]

Let’s first assume that \( n \) is odd. It is obvious that

\[ \int_{0}^{\infty} \int_{0}^{1} \left| G_{n}(t, x, y) - G_{2}^{n,\tau}(t, x, y) \right|^{2} dy \, dt \leq 4 \sum_{j=1}^{\left[ \frac{n}{2} \right]} \int_{0}^{\infty} \left| e^{\lambda_{j}^{n} t} - (R_{1,j}R_{2,j}) \left[ \frac{t}{\tau} \right] R_{1,j} \right|^{2} dt. \]

We use the idea and notations in Lemma 4.3 (iii). We split the set \( \{ j : 1, 2, \ldots, \left[ \frac{n}{2} \right] \} \) into two parts, i.e.

\[ \left\{ j : 1, 2, \ldots, \left[ \frac{n}{2} \right] \right\} = \left\{ j : R_{1,j}R_{2,j} \geq \frac{1}{2} \right\} \cup \left\{ j : -1 + \epsilon \leq R_{1,j}R_{2,j} < \frac{1}{2} \right\} \\
= A_{1} \cup A_{2}. \]

Recall that for \( j \in A_{1}, \frac{1}{2} < R_{2,j} < 1 \) and \(-\lambda_{j}^{n}\tau \leq R_{3,j} \leq -2\lambda_{j}^{n}\tau, \) and for \( j \in A_{2}, |R_{1,j}R_{2,j}| \leq 1 - \epsilon. \) Moreover, we have

\[ A_{1} \subset \left\{ j : 1 \leq j \leq \frac{1}{4} \sqrt{\frac{1}{(2 - \theta)\tau}} \right\} \text{ and } A_{2} \subset \left\{ j : \frac{1}{2\pi} \sqrt{\frac{1}{(2 - \theta)\tau}} < j \leq \left[ \frac{n}{2} \right] \right\}. \]

Hence, for \( \tau < 1, \) we have

\[ \sum_{j \in A_{1}} \int_{0}^{\infty} \left| e^{\lambda_{j}^{n} t} - (R_{1,j}R_{2,j}) \left[ \frac{t}{\tau} \right] R_{1,j} \right|^{2} dt \leq \sum_{j \in A_{1}} 4 \int_{0}^{\infty} \left| e^{\lambda_{j}^{n} t} - \exp \left\{-R_{3,j} \frac{t}{\tau} \right\} \right|^{2} dt \]

\[ + \sum_{j \in A_{1}} 4 \int_{0}^{\infty} \left| \exp \left\{-R_{3,j} \frac{t}{\tau} \right\} - \exp \left\{-R_{3,j} \left[ \frac{t}{\tau} \right] \right\} \right|^{2} dt \]

\[ + \sum_{j \in A_{1}} 4 \int_{0}^{\infty} \left| \exp \left\{-R_{3,j} \left[ \frac{t}{\tau} \right] \right\} - \exp \left\{-R_{3,j} \left[ \frac{t}{\tau} \right] \right\} \right|^{2} dt \]

\[ + \sum_{j \in A_{1}} 4 \int_{0}^{\infty} \left| \exp \left\{-R_{3,j} \left[ \frac{t}{\tau} \right] \right\} R_{1,j} - (R_{1,j}R_{2,j}) \left[ \frac{t}{\tau} \right] R_{1,j} \right|^{2} dt \]

\[ = 4(I_{1} + I_{2} + I_{3} + I_{4}). \]
For the term $I_1$,

$$I_1 \leq \sum_{j \in A_1} \int_0^\infty e^{2\lambda_j^* t} \times \left| 1 - \exp \left\{ \lambda_j^* \left( \frac{1}{1 + (1 - \theta) \lambda_j} - 1 \right) \frac{t}{\tau} \right\} \right|^2 dt$$

$$= \sum_{j \in A_1} \int_0^\infty e^{2\lambda_j^* t} \times \left| 1 - \exp \left\{ - \left(1 - \theta \right) \left( \frac{\tau \lambda_j^*}{1 + (1 - \theta) \lambda_j} \right) \frac{t}{\tau} \right\} \right|^2 dt$$

$$\leq \sum_{1 \leq j \leq \frac{1}{\tau} \sqrt{\frac{1}{2 - \eta} \tau}} C \int_0^\infty e^{2\lambda_j^* t} \times \left( \frac{t}{\tau} \right)^2 \left( \lambda_j^* \tau \right)^4 dt \leq \sum_{1 \leq j \leq \frac{1}{\tau} \sqrt{\frac{1}{2 - \eta} \tau}} C \int_0^\infty e^{-32j^2 t} \times j^8 \tau^2 t^2 dt$$

$$\leq \sum_{1 \leq j \leq \frac{1}{\tau} \sqrt{\frac{1}{2 - \eta} \tau}} C \tau^2 j^2 \leq C \tau^2 \int_0^\frac{1}{\tau} \sqrt{\frac{1}{2 - \eta} \tau} + 1 x^2 dx \leq C(\theta) \sqrt{\tau},$$

where we have used the inequality $1 - e^{-z} \leq z, z > 0$.

For the term $I_2$, by applying the mean value theorem, we get

$$I_2 \leq \sum_{j \in A_1} \int_0^\infty \exp \left\{ -2R_{3,j} \left[ \frac{t}{\tau} \right] \right\} |R_{3,j}|^2 dt$$

$$\leq \sum_{j \in A_1} C \int_0^\infty e^{-32j^2 t} \times j^4 \tau^2 dt \leq \sum_{1 \leq j \leq \frac{1}{\tau} \sqrt{\frac{1}{2 - \eta} \tau}} C \int_0^\infty e^{-32j^2 t} \times j^4 \tau^2 dt$$

$$\leq \sum_{1 \leq j \leq \frac{1}{\tau} \sqrt{\frac{1}{2 - \eta} \tau}} C(\theta) \int_0^\infty e^{-32j^2 t} \times j^4 \tau^2 dt \leq C(\theta) \tau^2 \times \sum_{1 \leq j \leq \frac{1}{\tau} \sqrt{\frac{1}{2 - \eta} \tau}} j^2 \leq C(\theta) \sqrt{\tau},$$

For terms $I_3$ and $I_4$,

$$I_3 \leq \sum_{j \in A_1} \int_0^\infty \exp \left\{ -2R_{3,j} \left[ \frac{t}{\tau} \right] \right\} \left| 1 - \frac{1}{1 - \theta \tau \lambda_j^*} \right|^2 dt$$

$$\leq \sum_{1 \leq j \leq \frac{1}{\tau} \sqrt{\frac{1}{2 - \eta} \tau}} C(\theta) \int_0^\infty e^{-32j^2 t} \times j^4 \tau^2 dt \leq C(\theta) \tau^2 \times \sum_{1 \leq j \leq \frac{1}{\tau} \sqrt{\frac{1}{2 - \eta} \tau}} j^2 \leq C(\theta) \sqrt{\tau},$$

$$I_4 \leq \sum_{j \in A_1} \int_0^\infty \exp \left\{ -2 \left[ \frac{t}{\tau} \right] \ln \left( 1 + R_{3,j} \right) \right\} \times \left| 1 - \exp \left\{ \left[ \frac{t}{\tau} \right] \left( -R_{3,j} + \ln \left( 1 + R_{3,j} \right) \right) \right\} \right|^2$$

$$\leq \sum_{j \in A_1} \int_0^\infty \exp \left\{ -2 \left[ \frac{t}{\tau} \right] \ln \left( 1 - \lambda_j^* \right) \right\} \times \left| 1 - \exp \left\{ \left[ \frac{t}{\tau} \right] \left( 2\lambda_j^* + \ln \left( 1 - 2\lambda_j^* \right) \right) \right\} \right|^2 dt$$
\[ \leq \sum_{1 \leq j \leq \frac{1}{\sqrt{1 - 2 \theta^2}}} \int_0^\infty \exp \left\{ 2C_2 \lambda_j^* \left[ \frac{1}{\tau} \right] \right\} \times \left| 1 - \exp \left\{ - \left[ \frac{t}{\tau} \right] C_1 (-2\lambda_j^* \tau)^2 \right\} \right|^2 dt \]

\[ \leq \sum_{1 \leq j \leq \frac{1}{\sqrt{1 - 2 \theta^2}}} C(\theta) \int_0^\infty e^{-32C_2 j^2 t} \times t^2 j^2 \tau^2 dt \leq C(\theta) \tau^2 \times \sum_{1 \leq j \leq \frac{1}{\sqrt{1 - 2 \theta^2}}} j^2 \leq C(\theta) \sqrt{\tau}, \]

where we have used the fact that \( z := -\lambda_j^* \tau \in \left( 0, \frac{\pi^2}{4(2 - \theta)} \right], j = 1, 2, \ldots \left[ \frac{n}{2} \right] \) because of \( j^2 \tau \leq \frac{1}{16(2 - \theta)}. \) For such \( z \), we have \(-C_1 z^2 \leq -z + \ln(1 + z) \leq 0\) and \( \ln(1 + z) \geq C_2 z \) for some \( C_1, C_2 > 0. \)

It remains to prove

\[ I_5 := \sum_{j \in A_2} \int_0^\infty e^{2\lambda_j^* t} dt \leq \sum_{j \leq \left[ \frac{n}{2} \right]} \int_0^\infty e^{-32j^2 t} dt \leq \sum_{j \leq \left[ \frac{n}{2} \right]} \frac{C}{j^2} \leq C(\theta) \sqrt{\tau} \]

and

\[ I_6 := \sum_{j \in A_2} \int_0^\infty (R_{1,j}R_{2,j})^\left[ \frac{4}{7} \right] R_{1,j}^2 dt \leq \sum_{j \in A_2} \int_0^\infty (1 - \epsilon)^2 \left[ \frac{4}{7} \right] \times (1 + 16\theta j^2 \tau)^{-2} dt \]

\[ \leq \int \frac{\left[ \frac{4}{7} \right]}{2\pi} \sqrt{\frac{1}{1 - 2 \theta}} \int_0^\infty (1 - \epsilon)^2 \left[ \frac{4}{7} \right] \times (1 + 16\theta x^2 \tau)^{-2} dt dx \quad \text{(let } r = \frac{t}{\tau}, y = x\sqrt{\tau}) \]

\[ = \sqrt{\tau} \int \frac{\left[ \frac{4}{7} \right]}{2\pi} \sqrt{\frac{1}{1 - 2 \theta}} \int_0^\infty (1 - \epsilon)^2 \left[ \frac{4}{7} \right] \times (1 + 16\theta y^2)^{-2} dr dy \]

\[ = \sqrt{\tau} \sum_{k=0}^\infty (1 - \epsilon)^{2k} \int \frac{\left[ \frac{4}{7} \right]}{2\pi} \sqrt{\frac{1}{1 - 2 \theta}} (1 + 16\theta y^2)^{-2} dy \leq C(\theta) \sqrt{\tau}, \]

where in the last line we use the same analysis as in Lemma 13 (iii).

When \( n \) is even, what we need to prove is the difference of the term of \( j = \frac{n}{2} \) in the expansion of \( G^n_2 \) and \( G^n_2, \) i.e.,

\[ \int_0^\infty \int_0^1 \left| \left( e^{-4n^2} - \left( R_{1,\frac{n}{2}}R_{2,\frac{n}{2}} \right)^\left[ \frac{4}{7} \right] R_{1,\frac{n}{2}} \right) g_n(x, y) \right|^2 dy dt \]

\[ \leq \int_0^\infty \left| e^{-4n^2} - \left( R_{1,\frac{n}{2}}R_{2,\frac{n}{2}} \right)^\left[ \frac{4}{7} \right] \right|^2 dt \]

\[ \leq 2 \int_0^\infty e^{-8n^2} \, dt + 2 \int_0^\infty (1 - \epsilon)^2 \left[ \frac{4}{7} \right] (1 + 4\theta n^2 \tau)^{-2} dt \]

\[ \leq \frac{C}{n^2} + C \int_0^\infty (1 - \epsilon)^2 \left[ \frac{4}{7} \right] \, dt \leq \frac{C}{n^2} + C \tau \sum_{k=0}^\infty (1 - \epsilon)^{2k} \leq \frac{C}{n^2} + C \tau. \]

Hence the proof of (i) is completed.

(ii) Let's suppose that \( n \) is odd since the even case can be proved similarly.

For the cases of \( \theta \in \left[ 0, \frac{1}{2} \right] \) and \( \theta = 1 \), it suffices to prove for any \( \frac{1}{2} < \alpha < 2, \)

\[ \int_0^1 \left| G^n(t, x, y) - G^n_{1,\tau}(t, x, y) \right|^2 dy \leq C\tau^{\alpha - \frac{1}{2}} \left( \left[ \frac{1}{\tau} \right] \tau \right)^{-\alpha} \tag{38} \]
with some $C := C(\alpha, \theta) > 0$ for all $x \in [0,1], t \geq \tau$. The proof of (38) follows the same order of the terms $I_i, i = 1, 2, \ldots, 6$ in (i) after replacing $G_{i, \tau}^{n_i, \tau}$ by $G_{1, \tau}^{n_i, \tau}$ and removing the integral with $t$, we still denote them by $I_i, i = 1, 2, \ldots, 6$.

When $t \geq \tau$, we have for $\alpha < 2$,

$$I_1 = \sum_{j \in A_1} e^{2\lambda_j t} \times \left| 1 - \exp \left\{ \lambda_j^{\tau} \left( \frac{1}{1 + (1 - \theta)\tau \lambda_j^{\tau}} - 1 \right) \frac{t}{\tau} \right\} \right|^2 \leq \sum_{1 \leq j \leq \frac{1}{2} \sqrt{(2 - \theta)\tau}} C e^{2\lambda_j^{\left[\frac{4}{3}\right]} \tau} \times j^8 \tau^4 \left( \frac{t}{\tau} \right)^2 \left( \frac{1}{\tau} + 1 \right)^2 \leq \sum_{1 \leq j \leq \frac{1}{2} \sqrt{(2 - \theta)\tau}} C(\gamma) \left( j^2 \tau \left[ \frac{t}{\tau} \right] \right)^{-\gamma} j^8 \tau^4 \left[ \frac{t}{\tau} \right]^2 \quad \text{(let } \alpha = \gamma - 2) \leq C(\alpha, \theta) \tau^{\alpha - \frac{1}{2}} \left( \frac{t}{\tau} \right)^{-\alpha},$$

and for $\alpha < 2$,

$$I_2 = \sum_{j \in A_1} \exp \left\{ -2R_{3, j} \left[ \frac{t}{\tau} \right] \right\} \left| R_{3, j} \right|^2 \leq \sum_{1 \leq j \leq \frac{1}{2} \sqrt{(2 - \theta)\tau}} C(\alpha) \left( j^2 \tau \left[ \frac{t}{\tau} \right] \right)^{-\alpha} j^4 \tau^2 \leq C(\alpha, \theta) \tau^{\alpha - \frac{1}{2}} \left( \frac{t}{\tau} \right)^{-\alpha}.$$

Terms $I_3$ and $I_4$ can be estimated in a similar way. By using the inequality $e^{-z^2} \leq C(\beta)z^{-\beta}, z > 0, \beta > 0$, we get for $\alpha > \frac{1}{2}$,

$$I_5 = \sum_{\frac{1}{2\tau} \sqrt{(2 - \theta)\tau} < j \leq \left[ \frac{n}{2} \right]} e^{2\lambda_j^{\tau}} \leq \sum_{\frac{1}{2\tau} \sqrt{(2 - \theta)\tau} < j \leq \left[ \frac{n}{2} \right]} e^{-32j^{\left[\frac{4}{3}\right]} \tau} \leq \sum_{\frac{1}{2\tau} \sqrt{(2 - \theta)\tau} < j \leq \left[ \frac{n}{2} \right]} C(\alpha) \left( j^2 \tau \left[ \frac{t}{\tau} \right] \right)^{-\alpha} \leq C(\alpha) \int_{\frac{1}{2\tau} \sqrt{(2 - \theta)\tau} < j \leq \left[ \frac{n}{2} \right]} x^{-2\alpha} \, dx \times \left( \frac{t}{\tau} \right)^{-\alpha} \leq C(\alpha, \theta) \tau^{\alpha - \frac{1}{2}} \left( \frac{t}{\tau} \right)^{-\alpha},$$

and the remaining term $I_6$ can be estimated as follows. For $\theta \in [0, \frac{1}{2}]$, because $n^2 \tau$ is bounded, we have for $\alpha > 0$,

$$\sum_{j \in A_2} (R_{1, j} R_{2, j})^{-2[\frac{4}{3}]} \leq \frac{1}{\sqrt{\tau}} \int_{\sqrt{\tau}}^{\left[\frac{\tau}{\sqrt{\tau}}\right]} (1 - e^{2[\frac{4}{3}]} \, dy \leq \frac{C(\theta, \alpha)}{\sqrt{\tau}} e^{-2[\frac{4}{3}] \ln(1 - e)^{-1}} \leq \frac{C(\theta, \alpha)}{\sqrt{\tau}} \left[ \frac{t}{\tau} \right]^{-\alpha}.$$

For $\theta = 1$, we have for $\alpha > 0, t \geq \tau$,

$$\sum_{\frac{1}{2\tau} \sqrt{\tau} < j \leq \left[ \frac{n}{2} \right]} (1 - \tau \lambda_j^{\tau})^{-2[\frac{4}{3}]} \leq \sum_{\frac{1}{2\tau} \sqrt{\tau} < j \leq \left[ \frac{n}{2} \right]} (1 + 16j^2 \tau)^{-2[\frac{4}{3}]} \leq \frac{1}{\sqrt{\tau}} \int_{\frac{1}{\sqrt{\tau}}}^{\infty} (1 + 16y^2)^{-2[\frac{4}{3}]} \, dy \leq \frac{1}{\sqrt{\tau}} \left( 1 + \frac{4}{\pi^2} \right)^{\left[\frac{4}{3}\right]} \times \int_{\frac{1}{\sqrt{\tau}}}^{\infty} (1 + 16y^2)^{-\alpha} \, dy \leq C(\alpha) \frac{1}{\sqrt{\tau}} \left[ \frac{t}{\tau} \right]^{-\alpha}. $$
For the case of $\theta \in (\frac{1}{2}, 1)$, we split the integral
\[
\int_0^1 (G^n(t, x, y) - G^{n,\tau}_1(t, x, y)) \, u_0(\kappa_n(y)) \, dy
\]
into two parts, i.e.,
\[
\int_0^1 (G^n(t, x, y) - G^{n,\tau}_1(t, x, y)) \, u_0(\kappa_n(y)) \, dy
= \int_0^1 (G^n(t, x, y) - G^{n,\tau}_1(t, x, y)) \, u_0(y) \, dy
+ \int_0^1 (G^n(t, x, y) - G^{n,\tau}_1(t, x, y)) \, (u_0(\kappa_n(y)) - u_0(y)) \, dy
= : Q_1 + Q_2.
\]

We only consider the space grid points $\kappa_n(x)$ because the result of other points can be obtained by the inequality $(a + b)^2 \leq 2a^2 + 2b^2$. Taking $x = \frac{i}{n}, i = 0, 1, \ldots, n - 1$, then we have $\varphi_{c,j}(x) = \varphi_{c,j}(x) = \cos(2\pi j x)$, $\varphi_{s,j}(x) = \varphi_{s,j}(x) = \sin(2\pi j x)$. So
\[
Q_1 = \int_0^1 \sum_{j=1}^{\left[\frac{n}{2}\right]} \left( e^{\lambda_j^2 t} - (R_{1,j}R_{2,j})^{[\frac{1}{2}]} \right) \left( \varphi_{c,j}(x) \varphi_{c,j}(y) + \varphi_{s,j}(x) \varphi_{s,j}(y) \right) \, u_0(y) \, dy
= \sum_{j=1}^{\left[\frac{n}{2}\right]} \left( e^{\lambda_j^2 t} - (R_{1,j}R_{2,j})^{[\frac{1}{2}]} \right) \int_0^1 \cos(2\pi j (x - y)) \, u_0(y) \, dy.
\]
Suppose that the initial data $u_0 \in C^2([0, 1])$, then by the integration by parts formula and $u_0(0) = u_0(1)$, we get for $j = 1, \ldots, \left[\frac{n}{2}\right]$,
\[
\int_0^1 \cos(2\pi j (x - y)) \, u_0(y) \, dy = -\frac{1}{2\pi j} \int_0^1 u_0(y) \, d\sin(2\pi j (x - y))
= -\frac{1}{2\pi j} \left( u_0(1) \sin(2\pi j (x - 1)) - u_0(0) \sin(2\pi j x) \right) + \frac{1}{2\pi j} \int_0^1 \sin(2\pi j (x - y)) \, u_0(y) \, dy
= \frac{1}{(2\pi j)^2} \int_0^1 u_0(y) \, d\cos(2\pi j (x - y))
= \frac{1}{(2\pi j)^2} \left( u_0'(1) \cos(2\pi j (x - 1)) - u_0'(0) \cos(2\pi j x) - \int_0^1 \cos(2\pi j (x - y)) \, u_0''(y) \, dy \right) \leq \frac{C}{j^2}.
\]
Hence,
\[
Q_1 \leq C \sum_{j=1}^{\left[\frac{n}{2}\right]} \left| e^{\lambda_j^2 t} - (R_{1,j}R_{2,j})^{[\frac{1}{2}]} \right| \frac{1}{j^2}
\leq \sum_{j \in A_1} C \left| e^{\lambda_j^2 t} - (R_{1,j}R_{2,j})^{[\frac{1}{2}]} \right| \frac{1}{j^2} + \sum_{j \in A_2} C \left| e^{\lambda_j^2 t} - (R_{1,j}R_{2,j})^{[\frac{1}{2}]} \right| \frac{1}{j^2} =: Q_{11} + Q_{12}.
\]
For the term $Q_{12}$,
\[
Q_{12} \leq \sum_{j \in A_2} C e^{\lambda_j^2 t} \times \frac{1}{j^2} + \sum_{j \in A_2} C |R_{1,j}R_{2,j}|^{[\frac{1}{2}]} \frac{1}{j^2}
\]
For the term $Q_{11}$, by Cauchy-Schwarz inequality, we have for any $\frac{1}{2} < \alpha < 2$,

$$Q_{11}^2 \leq \sum_{j \in A_1} C |e^{j\alpha^{1/2}} - (R_{1,j} R_{2,j})|^{1/2} | \int_{0}^{1} \frac{1}{n^2} \left[ e^{j\alpha^{1/2}} - (R_{1,j} R_{2,j})|^{1/2} \right] dx \leq C(\alpha, \theta) \left( \frac{t}{\tau} \right)^{-\alpha}$$

where the last step can be obtained as before.

For the term $Q_2$, since $u_0 \in C^2([0,1])$, we have

$$Q_2^2 \leq \sum_{j \in A_1} C |e^{j\alpha^{1/2}} - (R_{1,j} R_{2,j})|^{1/2} | \int_{0}^{1} \frac{1}{n^2} \left[ e^{j\alpha^{1/2}} - (R_{1,j} R_{2,j})|^{1/2} \right] dx \leq C(\alpha, \theta) \left( \frac{t}{\tau} \right)^{-\alpha}$$

for $\alpha > 0$, and $Q_{21} \leq C(\alpha, \theta) \left( \frac{t}{\tau} \right)^{-\alpha}$ for $\frac{1}{2} < \alpha < 2$.

Combining the above terms, we have for any $\frac{1}{2} < \alpha < 2$,

$$\left| \int_{0}^{1} \left( G^{n}(t, x, y) - G_{1}^{n, \tau}(t, x, y) \right) u_0(\kappa_n(y)) dy \right|^2 \leq 2Q_1^2 + 2Q_2^2 \leq C(\alpha, \theta) \left( \frac{t}{\tau} \right)^{-\alpha}$$

The proof is finished. \(\square\)

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