Approximate solution to the CGHS field equations for two-dimensional evaporating black holes

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Abstract

Callan, Giddings, Harvey and Strominger (CGHS) previously introduced a two-dimensional semiclassical model of gravity coupled to a dilaton and to matter fields. Their model yields a system of field equations which may describe the formation of a black hole in gravitational collapse as well as its subsequent evaporation. Here we present an approximate analytical solution to the semiclassical CGHS field equations. This solution is constructed using the recently-introduced formalism of flux-conserving hyperbolic systems. We also explore the asymptotic behavior at the horizon of the evaporating black hole.

1 Introduction

The semiclassical theory of gravity treats spacetime geometry at the classical level but allows quantum treatment of the various fields which reside in spacetime. This theory asserts that a quantum field living on a black-hole (BH) background will usually be endowed with non-trivial fluxes of energy-momentum. These fluxes, represented by the renormalized stress-Energy tensor \( \hat{T}_{\alpha\beta} \), originate from the field’s quantum fluctuations, and typically they do not vanish even in the (incoming) vacuum state. The Hawking radiation
and the consequent black-hole (BH) evaporation, are perhaps the most dramatic manifestations of these quantum fluxes.

In the framework of semiclassical gravity the spacetime reacts to the quantum fluxes via the Einstein equations, which now receive the extra quantum contribution $\hat{T}_{\alpha\beta}$ at their right-hand side. The mutual interaction between geometry and quantum fields thus takes its usual General-Relativistic schematic form: The renormalized stress-Energy tensor $\hat{T}_{\alpha\beta}$ (say in the vacuum state) is dictated by the background geometry, and the latter is affected by $\hat{T}_{\alpha\beta}$ through the Einstein equations. In principle, this evolution scheme allows systematic investigation of the spacetime of an evaporating BH.

It turns out, however, that the calculation of $\hat{T}_{\alpha\beta}(x)$ for a prescribed background metric $g_{\alpha\beta}(x)$ is an extremely hard task in four dimensions. Nevertheless, in two-dimensional (2D) gravity the situation is remarkably simpler. There are two energy-momentum conservation equations, and the trace $\hat{T}^{\alpha}_{\alpha}$ is also known (the "trace-anomaly"). These three pieces of information are just enough for determining the three unknown components of $\hat{T}_{\alpha\beta}$. It is therefore possible to implement the evolution scheme outlined above in 2D-gravity, and to formulate a closed system of field equations which describe the combined evolution of both spacetime and quantum fields.

Callan, Giddings, Harvey and Strominger (CGHS) \[2\] introduced a formalism of 2D gravity in which the metric is coupled to a dilaton field $\phi$ and to a large number $N$ of identical massless scalar fields. They added to the classical action an effective term $\propto N$ which gives rise to the semiclassical trace-anomaly contribution, and thereby automatically incorporates the renormalized stress-Energy tensor $\hat{T}_{\alpha\beta}$ into spacetime dynamics. The evolution of spacetime and fields is then described by a closed system of field equations. They considered the scenario in which a BH forms in the gravitational collapse of a thin massive shell, and then evaporates by emitting Hawking radiation. Their main goal was to reveal the end-state of the evaporation process, in order to address the information puzzle.

The general analytical solution to the CGHS field equations is not known. Nevertheless, these equations can be explored analytically \[3\] as well as numerically \[4, 5, 6\]. The global structure which emerges from these studies is depicted in Fig. 1: The shell collapse leads to the formation of a BH. A spacelike singularity forms inside the BH, at a certain critical value of the dilaton \[3\]. (The BH interior may be identified as the set of all events from which all future-directed causal curves hit the singularity.) The BH interior
and exterior are separated by an outgoing null ray, which serves as an event horizon. Also an apparent horizon forms along a timelike line outside the BH (it is characterized by a local minimum of $R \equiv e^{-2\phi}$ along outgoing null rays). The event and apparent horizons are denoted by "EH" and "AH" in Fig. 1. Both horizons steadily "shrink" it time (namely $R$ decreases), exhibiting the BH evaporation process. At a certain point the apparent horizon intersects with the spacelike singularity (and with the event horizon). This intersection event (denoted "P" in Fig. 1) appears to be a naked singularity, visible to far asymptotic observers. It may be regarded as the "end of evaporation" point.

The spacelike singularity which develops inside a CGHS evaporating BH was first noticed by Russo, Susskind, and Thorlacius [3]. Its local structure was recently studied in some detail [7], by employing the homogeneous approximation. This singularity marks the boundary of predictability of the semiclassical CGHS formalism. About two years ago Ashtekar, Taveras and Varadarajan [8] proposed a quantized version of the CGHS model, in which the dilaton and metric are elevated to quantum operators. This quantum formulation of the problem seems to resolve the semiclassical singularity inside the BH [8], and thereby to shed new light on the information puzzle. In a very recent paper [9] we applied a more simplified, "Minisuperspace-like", quantum approach to the spacetime singularity inside a CGHS evaporating BH. We obtained a (bounce-like) extension of semiclassical spacetime beyond the singularity.

The goal of this paper is to construct an approximate analytical solution to the semiclassical CGHS field equations, which will satisfactorily describe the BH formation and evaporation. This approximate solution applies as long as the BH is macroscopic, and as long as the spacetime region in consideration is "macroscopic" too—namely, $R$ is sufficiently large (i.e. not too close to the singularity). In such macroscopic regions, the semiclassical effects are weak in a local sense. Our solution is "first-order accurate" in this local sense; Namely, the local errors in the second-order field-equation operators (applied to the approximate solution) are quadratic in the magnitude of local semiclassical effects (this is further discussed in Sec. 8).

The method we use here for constructing our approximate solution is based on the formalism of flux-conserving systems [10]. This formalism deals with a special class of semi-linear second-order hyperbolic systems in two dimensions. This class was first introduced in Ref. [11], and was subsequently explored more systematically in Ref. [12] (see also [13]). In particular, a
flux-conserving hyperbolic system admits a rich family of single-flux solutions, to which we refer as Vaidya-like solutions (see Sec. 3), and for which the field equations reduce to an ordinary differential equation (ODE). Very recently we demonstrated [10] that after transforming to new field variables, the semiclassical CGHS equations take a form which is approximately flux-conserving. Here we take advantage of this property and use the formalism of flux-conserving systems to construct the approximate analytic solution to the CGHS field equations.

Our approximate solution is presented in Sec. 6. It involves a single function denoted by \( H \), which is defined through a certain ODE. In order to make practical use of this approximate solution, one must have at his disposal the solution for this ODE. In Appendix A we provide an approximate analytical solution to this ODE.

In Sec. 2 we briefly summarize the CGHS model, expressing the semiclassical field equations in convenient variables \( R, S \) (already used in Refs. [11, 7, 10]). In particular we discuss the classical model of a collapsing shell—which in fact provides the initial conditions to the semiclassical problem of BH evaporation. In Sec. 3 we introduce the new field variables \( (W, Z) \), which turn the field equations into a more standard form (with no first-order derivatives). Then we observe the approximate flux-conserving system obtained in these new variables at the large-\( R \) limit. In Sec. 4 we construct the ingoing Vaidya-like solution (a "single-flux" ingoing solution) of the flux-conserving system, which constitutes the core to our approximate solution. This core solution is complemented in Sec. 5 by adding to it a weak outgoing component. The extra outgoing component must be added in order to correctly satisfy the initial conditions at the collapsing shell (and it is this component which eventually gives rise to the outgoing Hawking radiation). In Sec. 6 we summarize our approximate solution, and also present an alternative approximate expression for \( S \). In Sec. 7 we introduce two useful gauges: the "shifted-Kruskal" gauge, and the semiclassical Eddington gauge. The validity of the constructed solution is verified in Sec. 8. We first describe the magnitude of the local error in the field equations, which turns out to be quadratic in the local magnitude of semiclassical effects (assumed to be a small quantity). Subsequently we verify that the initial data at the collapsing shell are precisely matched by our approximate solution, and the same for the initial data at past-null-infinity (PNI).

In Sec. 9 we discuss the horizon of the semiclassical BH and a few related issues. The horizon (or "event-horizon") is the outgoing null line which
separates the BH interior and exterior. We present the behavior of $R$ and $S$ along the event horizon and also compute the influx $T_{vv}$ into the BH. We also discuss the location of the apparent horizon (defined by a minimum of $R$ along an outgoing null ray as mentioned above). Then in Sec. 10 we analyze the asymptotic behavior at PNI, and also remark briefly on the asymptotic behavior at future null infinity (FNI). Finally, in Sec. 11 we summarize our main results.

## 2 Background: The CGHS model

### 2.1 Action and field equations

The CGHS model [2] consists of 2D gravity coupled to a dilaton $\phi$ and to a large number $N \gg 1$ of identical (free, minimally-coupled, massless) scalar fields $f_i$. Throughout this paper we express the metric in double-null coordinates $u,v$ for convenience, namely

$$ds^2 = -e^{2\rho}dudv.$$  \hspace{1cm} \text{(1)}

The action then takes the form

$$\frac{1}{\pi} \int d^2\sigma \left[ e^{-2\phi} \left( -2\rho_{,uv} + 4\phi_{,u}\phi_{,v} - \lambda^2 e^{2\rho} \right) - \frac{1}{2} \sum_{i=1}^{N} f_{i,u}f_{i,v} + K\rho_{,u}\rho_{,v} \right],$$  \hspace{1cm} \text{(2)}

where $K \equiv N/12$. The term $\lambda^2$ denotes a cosmological constant. We shall set $\lambda = 1$ throughout. This is achieved by a change of variable $\rho \to \rho' = \rho + \ln(\lambda)$, which annihilates $\lambda$ but does not affect the field equations otherwise. This setting actually amounts to a choice of basic length unit.

The scalar fields all satisfy the trivial field equation

$$f_{i,u} = 0.$$  \hspace{1cm} \text{(3)}

The CGHS scenario consists of an imploding thin massive shell of mass $M_0$ which moves along an ingoing null line $v = \text{const} \equiv v_0$, forming a black hole. This shell is composed of ($v$-derivatives of) the scalar fields $f_i$, which are concentrated as a $\delta$-function-like distribution at $v = v_0$. The solution at $v < v_0$ is the trivial, flat, vacuum solution (see below). The main objective of this paper is the semiclassical solution which takes place at $v > v_0$. By
assumption, in this region too no incoming scalar waves are present. Therefore the solution of Eq. (3) throughout the relevant domain \( v > v_0 \) (as well as at \( v < v_0 \)) is

\[ f_i(u, v) = 0. \]  

(4)

The remaining field equations consist of two evolution equations for the fields \( \phi, \rho \), as well as two constraint equations (see [2]). To bring these equations to a simpler form we define new field variables

\[ R \equiv e^{-2\phi}, \quad S \equiv 2(\rho - \phi), \]  

(5)

following Ref. [11]. The field equations for \( R \) and \( S \) then take the form

\[ R_{,uv} = -e^S - K \rho_{,uv}, \]  

(6)

\[ S_{,uv} = K \rho_{,uv}/R, \]  

(7)

where

\[ \rho = (S - \ln R)/2 \]  

(8)

is to be substituted. The constraint equations become (after substituting \( f_i = 0 \) for the scalar-fields)

\[ R_{,uu} - R_{,u}S_{,u} + \hat{T}_{uu} = 0, \]  

(9)

\[ R_{,vv} - R_{,v}S_{,v} + \hat{T}_{vv} = 0, \]  

(10)

where \( \hat{T}_{uu}, \hat{T}_{vv} \) are the semiclassical fluxes in the two null directions, given by

\[ \hat{T}_{uu} = K \left[ \rho_{,uu} - \rho^2_{,u} + z_u(u) \right], \]  

(11)

\[ \hat{T}_{vv} = K \left[ \rho_{,vv} - \rho^2_{,v} + z_v(v) \right]. \]  

(12)

The functions \( z_u(u), z_v(v) \) are initial functions which encode the information about the system’s quantum state. Following CGHS we consider here an incoming vacuum state at PNI (apart from the imploding massive shell, which is presumably encoded in the fields \( f_i \) already at the classical level). Correspondingly the functions \( z_u(u), z_v(v) \) are determined by the requirement that \( \hat{T}_{uu} \) and \( \hat{T}_{vv} \) vanish at PNI.

\[ ^1 \text{In a 2D spacetime there are two sectors of PNI, a "right PNI" and a "left PNI". } z_u(u) \text{ is defined by the demand } \hat{T}_{uu} = 0 \text{ at left PNI, and } z_v(v) \text{ by demanding } \hat{T}_{vv} = 0 \text{ at right PNI. Similarly there are two sectors of FNI, a right one and a left one. Throughout this paper, by "PNI" and "FNI" we shall always refer to the right sectors of PNI and FNI (unless stated otherwise), as is also illustrated in Fig. 1.} \]
It will sometimes be useful to re-express the system of evolution equations (6,7) in its standard form, in which \( R_{,uv} \) and \( S_{,uv} \) are explicitly given in terms of lower-order derivatives:

\[
R_{,uv} = -e^S \frac{2R - K}{2(R - K)} - R_{,u}R_{,v} \frac{K}{2R(R - K)},
\]

\[
S_{,uv} = e^S \frac{K}{2R(R - K)} + R_{,u}R_{,v} \frac{K}{2R^2(R - K)}.
\]

This form makes it obvious that the evolution equations become singular when \( R = K \), and also at \( R = 0 \). This singularity was studied in some detail in Ref. [7].

**Gauge freedom**

In a coordinate transformation \( u \rightarrow u'(u), v \rightarrow v'(v) \) the dilaton scalar field \( \phi \) is unchanged, but \( \rho \) changes as

\[
\rho' = \rho - \frac{1}{2} \left( \ln \frac{du'}{du} + \ln \frac{dv'}{dv} \right).
\]

(as may be deduced from the coordinate transformation of the metric component \( g_{uv} = -(1/2)e^{2\rho} \)). From the definition (5) of \( R \) and \( S \) it is obvious that \( R \) is a scalar, and \( S \) changes in a coordinate transformation like 2\( \rho \):

\[
R' = R, \quad S' = S - \ln \frac{du'}{du} - \ln \frac{dv'}{dv}.
\]

Below we shall often provide expressions for \( S \) in certain specific gauges. In such cases we shall use the notation \( S_{[\ldots]} \), with the specific \( u, v \) coordinates specified in the squared brackets. The same notation will apply to \( \rho \) (and also to the gauge-dependent quantity \( Z \) introduced in the next section).

**2.2 Classical solutions**

The classical solutions are obtained by setting \( K = 0 \), leading to \( \hat{T}_{uu} = \hat{T}_{vv} = 0 \). The vacuum field equations then reduce to the evolution equations

\[
R_{,uv} = -e^S, \quad S_{,uv} = 0
\]
and the constraint equations

\[ R_{uu} - R_u S_{,u} = R_{vv} - R_v S_{,v} = 0. \]  \hspace{1cm} (18)

The general solution of these equations may be easily constructed. It takes the form

\[ R(u, v) = M - R_u(u) R_v(v), \quad S(u, v) = \ln(R_{u,u} R_{v,v}), \]  \hspace{1cm} (19)

where \( M \) is an arbitrary constant, and \( R_u(u) \) and \( R_v(v) \) are any monotonically-increasing functions of their arguments (with non-vanishing derivatives). The classical solution may thus look at first glance as a rich class depending on two arbitrary functions. However, these two functions merely reflect the gauge freedom. To fix this freedom we may use the Kruskal-like coordinates \( U \equiv R_u(u), V \equiv R_v(v) \), after which the general solution takes the simple explicit form

\[ R = M - UV, \quad S_{[U,V]} = 0. \]  \hspace{1cm} (20)

The sub-index "\([U,V]\)" recalled, indicates that this expression for \( S \) only applies in a specific gauge, the one associated with the Kruskal \( U,V \) coordinates.

The representation (20) makes it obvious that the classical vacuum solution is a one-parameter family, parametrized by the mass \( M \). We shall refer to it as the Schwarzschild-like solution.

For \( M > 0 \) the spacetime contains a BH, whose causal structure resembles that of the four-dimensional Schwarzschild spacetime (see [2]). The event horizon is located at \( U = 0 \), and the past (or "white-hole") horizon at \( V = 0 \). Inside the BH \((U,V > 0)\) there is a spacelike \( R = 0 \) singularity (\( \phi, \rho \) diverge) at \( UV = M \). For negative \( M \) there is a naked, timelike, \( R = 0 \) singularity instead of a BH. (The \( M = 0 \) case is considered below.)

Another useful gauge is the Eddington-like gauge, obtained by the transformation\(^2\)

\[ u_e \equiv -\ln(-U), \quad v_e \equiv \ln V \quad (U < 0, V > 0). \]  \hspace{1cm} (21)

The solution then takes the form

\[ R = M + e^{v_e - u_e}, \quad S_{[u_e,v_e]} = v_e - u_e. \]  \hspace{1cm} (22)

---

\(^2\)The coordinate \( u_e \) introduced here is the classical outgoing Eddington coordinate. It should not be confused with the semiclassical outgoing Eddington coordinate \( \hat{u} \) introduced in Sec. [7]. On the other hand the ingoing Eddington coordinate \( v_e \) (denoted later by \( v \)) is common to the classical and semiclassical solutions.
Note that these coordinates only cover the BH exterior. Analogous Eddington-like coordinates (with appropriate sign changes) may also be constructed for the BH interior, but we shall not need these internal coordinates here.

The special case $M = 0$ yields the *Minkowski-like* solution. In Kruskal coordinates it takes the form

$$R = -UV, \quad S_{[U,V]} = 0.$$  \hspace{1cm} (23)

In Eddington coordinates one obtains $R = e^{v_e - u_e}, S_{[u_e,v_e]} = v_e - u_e$ and hence $\rho_{[u_e,v_e]} = 0$. This demonstrates that the $M = 0$ solution is *flat*, and also indicates that the Eddington-like coordinates $(u_e, v_e)$ correspond in this case to the standard, flat, null coordinates of 2D Minkowski spacetime.

In the general case $M \neq 0$ one finds from Eqs. (22) and (8)

$$\rho_{[u_e,v_e]} = -\frac{1}{2} \ln \left(1 + \frac{M}{e^{v_e - u_e}}\right),$$  \hspace{1cm} (24)

and the associated metric $ds^2 = -e^{2\rho}du^2 dv$ yields non-vanishing curvature. However, at large $v_e - u_e$ this expression reduces to

$$\rho_{[u_e,v_e]} \approx -\frac{M}{2e^{v_e - u_e}} \approx -\frac{M}{2R},$$  \hspace{1cm} (25)

which vanishes as $v_e - u_e \to \infty$. That is, in Eddington coordinates $\rho$ vanishes at (right) PNI ($u_e \to -\infty$), FNI ($v_e \to \infty$), and spacelike infinity ($u_e \to -\infty, v_e \to \infty$). The Schwarzschild-like solution is thus *asymptotically-flat*, and $(u_e, v_e)$ serve as asymptotically-flat coordinates in this spacetime. In particular, $v_e$ and $u_e$ respectively coincide with the affine parameter along PNI and FNI.

**Classical collapsing shell**

The physical scenario which concerns us here is the formation of a BH by the collapse of a thin shell, carrying a mass $M_0 > 0$, which propagates along an ingoing null line $v = v_0$. The classical solution describing this scenario is obtained by continuously matching a Minkowski-like solution at $v \leq v_0$ with a Schwarzschild-like solution at $v \geq v_0$. To describe the matching we start with the Kruskal-gauge solution (20), along with its $M = 0$ analog (23). Obviously the $R$-functions in these two expressions do not properly match (at any $V$). To allow for continuous matching, we shift the $U$ coordinate in
the Minkowski-like domain \( v < v_0 \), defining \( \hat{U} \equiv U + \Delta U \), where \( \Delta U \) is a constant. Note that \( S \) is unchanged in this coordinate transformation. We obtain the Minkowski-like solution in its new form:

\[
R = (\Delta U - \hat{U}) V, \quad S_{[\hat{U},V]} = 0 \quad (v < v_0).
\]

This solution can now be matched to Eq. \( (20) \), by equating \( V \Delta U \) to \( M_0 \) at the null shell. Omitting the hat from \( \hat{U} \) and re-naming this new Minkowski-like Kruskal coordinate by \( U \), we obtain the matched solution

\[
S_{[U,V]} = 0, \quad R(U,V) = \begin{cases} (\Delta U - U)V & V < V_0, \\ M_0 - UV & V > V_0, \end{cases}
\]

where \( V_0 \) is the shell’s \( V \)-value (we assume \( V_0 > 0 \)), and continuity implies \( \Delta U = M_0/V_0 \).

To formulate the semiclassical initial data (next subsection) we find it most convenient to work with Eddington \( v \) and Kruskal \( U \). The collapsing-shell solution then takes the form

\[
S_{[U,v]} = v_e, \quad R(U,v_e) = \begin{cases} (\Delta U - U)e^{v_e} & v_e < v_0, \\ M_0 - Ue^{v_e} & v_e > v_0, \end{cases}
\]

where \( v_0^e \equiv \ln V_0 \).

**Fixing the \( v \)-gauge**

Throughout the semiclassical analysis below, we find the Eddington \( v_e \) to be the most convenient \( v \)-coordinate, and no other \( v \)-coordinate will be used. To simplify the notation, in the rest of this paper we shall simply denote \( v_e \) by \( v \).

For the \( u \)-gauge we shall occasionally use several \( u \)-coordinates at various stages of the construction. The value of \( S \) (like \( \rho \) and \( Z \)) in specific \( u \)-gauges will be denoted by \( S_{[..]} \)—with the specific \( u \)-coordinate only specified in the squared brackets—and it will always refer to the Eddington \( v \) coordinate. No confusion should arise, because no \( v \)-coordinate other than Eddington will be used throughout the rest of the paper.

To practice the new notational rules we re-write the above classical collapsing-shell solution \( (28) \) in the new notation:

\[
S_v = v, \quad R(U,v) = \begin{cases} (\Delta U - U)e^v & v < v_0, \\ M_0 - Ue^v & v > v_0, \end{cases}
\]
where \(v_0 = \ln V_0\) denotes the shell’s location in Eddington-\(v\), and

\[
\Delta U = M_0 e^{-v_0}.
\]

The collapsing-shell spacetime (29) is asymptotically-flat, just like the pure Schwarzschild-like and Minkowski-like solutions. Also, the Eddington coordinate \(v\) serves as an affine parameter at PNI. [This may be verified by switching to the Eddington coordinate \(u_e \equiv -\ln(-U)\) and noting that \(\rho_{(ue)}\) vanishes at the PNI limit \(u_e \to -\infty\).]

2.3 Initial data for the semiclassical solution

Initial conditions at the Shell

We return now to the semiclassical problem of collapsing shell. The semiclassical effects vanish at the portion \(v < v_0\) of the collapsing-shell spacetime, owing to its flatness. Therefore, the portion \(v < v_0\) is correctly described by Eq. (29) even in the semiclassical problem.

The portion \(v > v_0\) of the collapsing-shell spacetime will be profoundly modified by semiclassical effect (as expressed for example by the BH evaporation). However, by continuity, the initial conditions for \(R\) and \(S\) at \(v = v_0\) will still be determined by matching to the same, unmodified, Minkowski-like solution at \(v < v_0\), and will therefore be exactly the same as in the classical solution (29). Using again the Kruskal-\(U\) coordinate (and, recall, the Eddington-\(v\) coordinate as usual), these shell initial conditions take the form

\[
R_0(U) \equiv R(U, v_0) = M_0 - U e^{v_0}
\]

and

\[
S_0^{[U]}(U) \equiv S_{[U]}(U, v_0) = v_0.
\]

Initial conditions at Past null infinity

The semiclassical evolution equations (6,7) also require initial conditions at an outgoing null ray, which is most conveniently taken to be the PNI boundary. The situation here is somewhat similar to that of the initial data at the shell: Owing to asymptotic flatness of the collapsing-shell spacetime [and

\footnote{Flatness means that the term \(\rho_{,uv}\) in the semiclassical evolution equations (6,7) vanish, so these equations reduce to the classical ones.}]

11
to the lack of any strong-field region at the causal past of PNI (unlike the situation at FNI)], the asymptotic behavior at PNI is just the classical one. Specifically one finds the initial conditions

\[ R \cong M_0 - Ue^v , \quad S_{[v]} \cong v \quad (PNI). \]  

(32)
in the domain \( v > v_0 \). In particular, the influx \( \hat{T}_{uv} \) should vanish at PNI (corresponding to an incoming vacuum state).

2.4 The role of \( K \) in semiclassical dynamics

The semiclassical effects originate from the last term \( K \rho_{uv} \rho_{uv} \) in the CGHS action (2). The parameter \( K = N/12 \) thus determines the overall magnitude of semiclassical effects. It is important to note, however, that \( K \) may be factored out from CGHS dynamics by a simple shift/rescaling of variables. In the transformation

\[ N \to cN , \quad K \to cK , \quad \phi \to \phi - \frac{1}{2} \ln c \]  

(33)

[with \( \lambda, \rho \) and \( f_i \) unchanged (and with all scalar fields \( f_i \) identical)], the action is merely multiplied by \( c \), hence the field equations are unaffected. One can therefore factor out \( K \) this way by choosing \( c = 1/K \).

In the relevant domain \( v > v_0 \) the scalar fields \( f_i \) vanish anyway, so the term \( -\frac{1}{2} \sum_{i=1}^{N} f_{i,u} f_{i,v} \) is absent from the action (2). In terms of the new variables \( R, S \), the rescaling law takes the form

\[ K \to cK , \quad R \to cR , \quad S \to S + \ln c . \]  

(34)

This scaling makes it obvious that the relative magnitude of semiclassical effects in various regions of spacetime will not be determined by \( K \) or \( R \) separately, but only through (dimensionless \(^4\)) combinations like \( K/R \) or \( K/e^S \), which are invariant to the rescaling. \(^5\)

One can easily verify that in the above scaling transformation the BH original mass \( M_0 \) is multiplied by \( c \). It is a common wisdom that when a BH

\(^4\)Note that after \( \lambda \) is set to 1 [see discussion following Eq. (2)] all model’s quantities become dimensionless. This includes \( R \), and also the BH mass \( M_0 \). (The latter is naturally linked to \( R \)—for example, through the horizon’s \( R \) value.)

\(^5\)It is important not to confuse here between two different issues related to the magnitude of \( K \): First, whether the semiclassical treatment is applicable or not. Second, \textit{within
is "macroscopic", semiclassical effects will be locally weak (except in the neighborhood of the singularity). The above scaling law makes it obvious that whether a BH may be regarded as "macroscopic" or not, would only depend on the ratio of $M_0$ and $K$: A CGHS BH should be regarded as macroscopic if $M_0 \gg K$—which we indeed assume throughout this paper.

In principle, the above scaling allows us to set $K = 1$ in the field equations. We find it more convenient, however, to leave $K$ in the equations untouched. The parameter $K$ serves as a "flag" marking the terms of semiclassical origin in the various equations. Also, in the equations below $K$ always appears through combinations like $K/R$, $K/R_0$, $K/M_0$ (or sometimes with $R$ replaced by the variable $W \sim R$ introduced below). We shall assume throughout this paper that the BH is macroscopic ($M_0 \gg K$), and deal with spacetime regions satisfying $R \gg K$. This will allow us to expand various expressions to first order in small quantities such as $K/R$ or alike. All these expansions may conveniently be handled formally as expansions in $K$ (or in the parameter $q = K/4$ introduced below)—though one should bear in mind that the small parameter in the expansion is not $K$ itself, but the combinations $K/R$, $K/M_0$, etc.

3 Field redefinition and flux-conserving formulation

3.1 Field redefinition

The evolution equations (6,7) may look quite simple at first glance. However, to close the system one must substitute Eq. (8) for $\rho$, which makes the equations rather messy. Bringing these equations to their standard form, one ends up with Eqs. (13,14). In addition to their rather complicated form, the semiclassical formulation, $K$ determines the magnitude of semiclassical effects. CGHS pointed out [2] that the semiclassical treatment is only valid if $K \gg 1$. Here we address the second issue. Namely, we assume that the condition $K \gg 1$ is satisfied, and explore the scaling law which characterizes the magnitude of semiclassical effects (and its relation to the scale of other variables like $R$).

6Here, again, it is important to distinguish between two different aspects of "macroscopicity": (i) Whether the semiclassical theory is applicable to the BH evaporation; and (ii) whether, within the semiclassical formulation, the semiclassical effects are locally weak. The discussion here pertains to the second aspect. To satisfactorily deal with (i) one has to further assume $K \gg 1$ (see also previous footnote).
these second-order equations also have the inconvenient property of being explicitly dependent on first-order derivatives \( R_{,u}, R_{,v} \).

To get rid of this undesired dependence upon \( R_{,u} \) and \( R_{,v} \), we transform from \( R \) and \( S \) to new variables \( W, Z \) defined by \[ W(R) \equiv \sqrt{R(R-K)} - K \ln(\sqrt{R} + \sqrt{R-K}) + K \left( \frac{1}{2} + \ln 2 \right) \] (35)

and

\[ Z \equiv S + \Delta Z(R), \] (36)

where

\[ \Delta Z(R) \equiv \frac{2}{K}(R - W) - \ln R. \] (37)

With these new variables the evolution equations take the schematically-simpler form:

\[ W_{,uv} = e^Z V_W(W), \quad Z_{,uv} = e^Z V_Z(W), \] (38)

with certain "potentials" \( V_W(W) \) and \( V_Z(W) \). But the simplification does not come without a cost: These potentials are explicitly obtained as functions of \( R \) rather than \( W \). One finds

\[ V_W = -\frac{R - K/2}{\sqrt{R(R-K)}} e^{-\Delta Z(R)} \] (39)

and

\[ V_Z = \frac{2}{K} \left[ \frac{R - K/2}{\sqrt{R(R-K)}} - 1 \right] e^{-\Delta Z(R)}. \] (40)

Despite this disadvantage, the form (38) allows an effective treatment of the semiclassical dynamics, as will be demonstrated below.

Notice that the gauge transformation (16) of \( R \) and \( S \) carries over to \( W \) and \( Z \) respectively:

\[ W' = W, \quad Z' = Z - \ln \frac{du'}{du} - \ln \frac{dv'}{dv}. \] (41)

The value of \( Z \) in a specific \( u \)-gauge will be denoted by \( Z_{[\ldots]} \), in full analogy with our notation for \( S \).
3.2 Large-\( R \) asymptotic behavior

From now on we shall consider a macroscopic BH, \( M_0 \gg K \), and restrict the analysis to spacetime regions where \( R \gg K \). We expand Eqs. (35,37) to first order in the small quantity \( K/R \):

\[
W = R \left[ 1 - \frac{K}{2R} \ln R + O \left( \frac{K}{R} \right)^2 \right],
\]

(42)

\[
\Delta Z = -\frac{K}{4R} + O \left( \frac{K}{R} \right)^2.
\]

(43)

The inverse function \( R(W) \) is given at this order by

\[
R = W \left[ 1 + \frac{K}{2W} \ln W + O \left( \frac{K}{W} \right)^2 \right].
\]

(44)

The potentials \( V_W(W), V_Z(W) \) can now easily be expanded to first order in \( K/W \):

\[
V_W = -1 - \frac{K}{4W} + O \left( \frac{K}{W} \right)^2, \quad V_Z = \frac{1}{W} \left[ \frac{K}{4W} + O \left( \frac{K}{W} \right)^2 \right].
\]

(45)

3.3 The flux-conserving system

We now proceed with the above large-\( R \) approximation, keeping only terms up to first order in \( K/R \) (or \( K/W \)); Thus we analyze the hyperbolic system

\[
W_{,uv} = e^Z V_W(W), \quad Z_{,uv} = e^Z V_Z(W)
\]

(46)

with

\[
V_W = -1 - \frac{q}{W}, \quad V_Z = \frac{q}{W^2},
\]

(47)

where

\[
q \equiv \frac{K}{4}.
\]

Since \( V_Z = dV_W/dW \), Eqs. (46,47) constitute a flux-conserving system. This concept was first introduced in Ref. [11] and was later described in more...
detail in Refs. [12] and [10]. Our system corresponds to $F = V_W = -1 - q/4$, hence the generating function [15] is

$$h_0(W) = W + q \ln W.$$  \hfill (48)

In conjunction with the system (46,47) we shall also use the transformation between $(R, S)$ and $(W, Z)$ chopped at first order in $q$, namely

$$W = R - 2q \ln R,$$

$$R = W + 2q \ln W,$$

$$Z = S - \frac{q}{R} = S - \frac{q}{W},$$

$$S = Z + \frac{q}{R} = Z + \frac{q}{W}.$$  \hfill (50)

We also rewrite here the shell initial conditions (30,31) in the new variables $W$ and $Z$, to first order in $q$:

$$W_0(U) \equiv W(U, v_0) = R_0 - 2q \ln R_0$$

and

$$Z_0^{[U]}(U) \equiv Z_{[U]}(U, v_0) = v_0 - \frac{q}{R_0}.$$  \hfill (54)

### Ingoing Vaidya-like solutions

Like any flux-conserving system, the system (46,47) admits Vaidya-like solutions—namely, solutions with a single flux [15]. Each Vaidya-like solution is endowed with a mass-function—a function of one of the null coordinates which encodes the information about the flux. As it turns out, at the leading order an evaporating BH may be approximated by an ingoing Vaidya-like solution with a linearly-decreasing mass function. [However, a first-order outgoing component has to be superposed on it in order to precisely match the initial conditions, as we discuss below.] Assuming that the BH was created by the collapse of a shell of mass $M_0$, which propagated along the null orbit $v = v_0$, the appropriate mass function takes the form $\bar{M}(v) = M_0 - q(v - v_0)$.

This choice is motivated by the well-known fact [2] that a 2D macroscopic BH evaporates at a constant rate $q = N/48$; and the suitability of (the approximate solution derived from) this mass function is verified in Sec. 8 below. By
shifting the origin of \( v \) such that \( v_0 = -M_0/q \), we obtain the mass function in a more compact form:

\[
\bar{M}(v) = -qv \equiv m_v(v).
\] (55)

The restriction to a Vaidya-like solution leads to a great simplification, because the problem now reduces to that of solving an ODE (rather than a system of PDEs). Thus, as described in Ref. [10], \( W(u,v) \) is now determined from the ODE \( W,_{v} = h_0(W) - \bar{M}(v) \) (applied along each line \( u = \text{const} \)), or, more explicitly,

\[
W,_{v} = (W + q \ln W) +qv.
\] (56)

The other unknown \( Z(u,v) \) is then given by

\[
Z = \ln(-W,_{u}).
\] (57)

Alternatively [10] \( Z \) may be obtained from the ODE

\[
Z,_{v} = 1 + \frac{q}{W}.
\] (58)

As was mentioned above, we set the origin of \( v \) such that the collapsing shell is placed at \( v_0 = -M_0/q \). Therefore the parameter \( v_0 \) is negative. Also, throughout this paper we assume that the BH is macroscopic, namely \( M_0 \gg q \), and this implies \( v_0 \ll -1 \).

**Weakly-perturbed Vaidya-like solution**

The above mentioned ingoing Vaidya-like solution well approximates many aspects of the CGHS spacetime. However, it fails to precisely match the initial conditions at the shell. This mismatch is small, \( \propto (q/W) \), yet it must be fixed in order to satisfactorily handle some of the more subtle aspects of the solution (most importantly, the Hawking outflux at FNI). Thus, we must fix the ingoing Vaidya-like solution by adding to it a small outgoing component, seeded by the mismatch at the shell. Nevertheless, owing to the small \( \propto (q/W) \) magnitude of the mismatch, it will be possible to treat this outgoing component as a small (linear) perturbation on top of the ingoing Vaidya-like solution discussed above (to which we shall refer as the "core solution").

In the next section we shall proceed with analyzing the ingoing Vaidya-like core solution. Then in Sec. 5 we shall construct the perturbing outgoing component and thereby complete the construction of the approximate solution.
4 Constructing the ingoing Vaidya-like core solution

4.1 Processing the ODE for $W$ and introducing $H$

The term $q \ln W$ in the right-hand side of Eq. (56) makes this equation hard to analyze. In order to ease the analysis we define the auxiliary variable $H \equiv W, v + q$, namely

$$H = W + q \ln W - m_v + q.$$  \hfill (59)

The inverse function $W(H)$ cannot be expressed in a closed exact form; however, restricting the analysis to first order in $q$ we may use the relation

$$W = H + m_v - q[\ln(H + m_v) + 1].$$  \hfill (60)

Differentiating now $H$ using Eq. (59) and $W, v = H - q$ we find

$$H, v = q + \frac{\partial H}{\partial W} W, v = q + \left(1 + \frac{q}{W}\right)(H - q).$$

Further substituting Eq. (60) in the right-hand side and omitting all $\propto q^2$ terms, we obtain the ODE for $H$ in its more compact form:

$$H, v = H \left(1 + \frac{q}{H - qv}\right).$$  \hfill (61)

The initial conditions for $H$ is to be specified at the shell’s orbit, the line $v = v_0$ (this line serves as a characteristic initial surface for the non-trivial piece $v > v_0$ of the CGHS spacetime). We denote it $H_0(u) \equiv H(u, v_0)$. In Sec. 4.4 we calculate $H_0$ and show that it decreases linearly with the Kruskal coordinate $U$. Then in Sec. 7 we express $H_0(u)$ in a few other useful gauges. Note that the dependence of $H$ on $u$ only emerges through the initial condition $H_0(u)$. Correspondingly we shall often express this parametric dependence in the form $H(v; u)$, and sometimes ignore it altogether and use the abbreviated notation $H(v)$, for convenience.

We are unable to solve the ODE (61) analytically. However, an approximate solution is given in Appendix A. Furthermore, several exact key properties of the solutions of Eq. (61) are described below.
Note the advantage of the ODE of $H$ over Eq. (56) for $W, v$, particularly at large $H$. The latter corresponds to future or past null infinity, where, as it turns out, $H$ (like $W$) grows exponentially in $v$. The ODE then reduces to the trivial one $H, v \cong H + q$, which is easily solved (leading to the above-mentioned exponential growth). In the ODE for $W$, on the contrary, one faces the $\ln W$ term, which complicates the asymptotic behavior at future/past null infinity.

4.2 Some exact properties of $H(v; u)$

Given the ODE (61), the function $H(v; u)$ is uniquely determined by the initial-value function $H_0(u) \equiv H(u, v_0)$ [this function is explicitly constructed below; cf. Eqs. (67, 68)]. We therefore start our discussion here by mentioning two key properties of $H_0(u)$ which are important for the present analysis: First, $H_0(u)$ is monotonically decreasing. It is positive at early $u$ but becomes negative afterwards. It vanishes at a certain $u$ value which we denote $u^{hor}$ (in Kruskal gauge it corresponds to $U = -qe^{v_0}$). Second, $H_0(u) + M_0$ is positive at any $u$.

We first note that the ODE (61) admits a trivial solution $H(v) = 0$. It immediately follows that $H$ vanishes along the line $u = u^{hor}$ (but nowhere else).

Next, we define (off the line $u = u^{hor}$ where $H$ vanishes)

$$l(v; u) \equiv \ln |H(v; u)|. \quad (62)$$

It satisfies the ODE

$$l, v = 1 + \frac{q}{H + m_v} = 1 + \frac{q}{\pm e^\ell - qv}, \quad (63)$$

where the "±" sign reflects the sign of $H$.

Both equations (63) and (61) develop a singularity whenever $H + m_v$ vanishes, but are regular otherwise. Since $H_0 + M_0 > 0$, the quantity $H + m_v$ is always positive at $v = v_0$ and its neighborhood. In principle there could be two possibilities, which may depend on $u$ [through the initial value $H_0(u)$]: (i) $H(v; u)$ is regular throughout $v > v_0$, or (ii) $H(v; u)$ becomes singular ($H + m_v$ vanishes) at a certain finite $v = v_{sing}(u) > v_0$. 

\footnote{As will become obvious later, option (i) occurs at $u < u^{hor}$ and option (ii) at $u > u^{hor}$.}
and well-defined throughout the domain $v_0 \leq v < v_f(u)$, where $v_f(u) = \infty$ in case (i) and $v_f(u) = v_{\text{sing}}(u)$ in case (ii).

We can now deduce the following exact properties of $H(v; u)$, which hold throughout the domain of regularity $v < v_f(u)$:

(a) As was already mentioned above, $H$ vanishes along the line $u = u_{\text{hor}}$. This holds throughout the range $v_0 \leq v < 0$. [At $(u = u_{\text{hor}}, v \to 0)$ the ODE becomes singular because $H + m_v$ vanishes.] $H$ cannot vanish at any other value of $u$, because otherwise $H_0(u)$ would have to vanish too at that specific $u \neq u_{\text{hor}}$ (which is not the case).

(b) Consequently, $H(u, v)$ has the same sign as $H_0(u)$ at any $u$. As was mentioned above, this sign is positive at $u < u_{\text{hor}}$ and negative at $u > u_{\text{hor}}$. These two domains are separated by the line $u = u_{\text{hor}}$ on which $u$ vanishes.

(c) Since $H + m_v$ is positive at $v = v_0$, it remains positive throughout $v_0 < v < v_f(u)$.

(d) Correspondingly, the right-hand side in Eq. (63) is strictly positive, and in fact $> 1$. Thus, $l(v)$ is monotonically increasing, and the same for $|H(v)|$.

(e) The quantity $H_u$ satisfies the ODE

$$
\frac{d}{dv} \ln |H_u| = 1 - \frac{q^2 v}{(H + m_v)^2}
$$

[cf. Eq. (87) below]. The right-hand side is regular at any $v < v_f(u)$, and positive throughout $v \leq 0$. Since $H_0,u$ is negative for all $u$, it immediately follows from this ODE that $H_u$ is everywhere negative. Furthermore, at least at $v < 0$, $|H_u|$ is an increasing function of $v$; therefore, at fixed $u$ (and $v < 0$) $H_u$ is bounded above by the parameter $H_{0,u} < 0$.

(f) The same obviously applies to $(H + m_v)_u$ (in particular, this quantity is everywhere negative). It then follows that if a singularity $H + m_v = 0$ occurs at some $u = u_1$ (with finite $v_{\text{sing}} > v_0$), then such a singularity must occur at any $u > u_1$, and $v_{\text{sing}}(u)$ must be a non-increasing function of $u$ (throughout $u > u_1$). Furthermore, in the range where $v_{\text{sing}}(u) < 0$ [see (g)], $v_{\text{sing}}(u)$ must be a strictly-decreasing function of $u$.

(g) As was mentioned above, at $u > u_{\text{hor}}$ $H_0(u)$ is negative, and the same for $H(v; u)$. [Furthermore, since $|H(v)|$ is monotonically increasing, at each line $u = \text{const}$ in this domain $H$ is bounded above by the parameter $H_{0,u}$ decreases linearly with $U$; and we only consider here $u$-gauges satisfying $du/dU > 0$.]

20
It then follows that $H + m_v$ (which is positive at $v = v_0$) must vanish before $m_v$ vanishes. Thus, all lines $u > u_{\text{hor}}$ run into an $H + m_v = 0$ singularity at a certain $v = v_{\text{sing}}(u) < 0$. Property (f) then implies that $v_{\text{sing}}$ is a strictly-decreasing function of $u$, indicating that this is a spacelike singularity.

(h) In the domain $u < u_{\text{hor}}$, $H$ is strictly positive, and no singularity may form at $v \leq 0$ (where $m_v$ is positive too). However, at $v > 0$ $m_v$ is negative, and one might be concerned about the possibility of vanishing $H + m_v$, which would lead to a singularity. A closer examination reveals that such a singularity does not occur in this domain. This may be deduced from each of the following arguments (though a complete mathematical proof is still lacking): (i) It is easy to show that at least a locally-monotonic singularity of this type (namely, a singularity of vanishing $H + m_v$ at some finite $v = v_{\text{sing}}$, such that $H + m_v$ is monotonic in $v$ throughout some neighborhood of $v = v_{\text{sing}}$) is not possible in the domain $u < u_{\text{hor}}$: $H + m_v$ starts positive at $v = v_0$, and in order to vanish it must decrease on approaching $v = v_{\text{sing}}$. However, since $H > 0$ (and increasing), when $H + m_v$ approaches zero from above Eq. (61) yields $H \rightarrow +\infty$, and therefore $H + m_v$ must increase, so it cannot vanish. It is harder to mathematically exclude the possibility of an oscillatory approach to an $H + m_v = 0$ singularity, but such an oscillatory behavior seems very unlikely. (ii) Consider the limiting function $H(v; u_{\text{hor}})$, defined to be the limit $u \rightarrow u_{\text{hor}}$ of $H(v; u)$. At $v < 0$ this function vanishes, just like $H(v; u_{\text{hor}})$, by continuity. At $v \geq 0$ [where $H(v; u_{\text{hor}})$ is not defined; see (a)], this function becomes a non-trivial solution of the ODE (61). Numerical examination shows that $H(v; u_{\text{hor}})$ continuously increases from zero at $v = 0$ to infinity at $v \rightarrow \infty$, with $H + m_v > 0$ at any $v > 0$. From (f) it now follows that $H + m_v > 0$ throughout $u < u_{\text{hor}}$. (iii) Direct numerical simulations of $H(v; u)$ at various $u < u_{\text{hor}}$ values further confirm this conclusion.

Summarizing the above discussion on the exact properties of $H(u, v)$, and briefly re-stating it in more physical/geometrical terms: All lines $u < u_{\text{hor}}$ make it to FNI, whereas all lines $u > u_{\text{hor}}$ run into a spacelike singularity.

\footnote{The exact semiclassical CGHS spacetime admits a very similar global structure: A BH, with a spacelike singularity inside it. We point out, however, that the local properties of the spacelike singularity in our approximate solution are different from those of the precise CGHS spacetime. In particular, in our approximate solution (when literally applied to the $H + m_v = 0$ singularity) $R$ diverges logarithmically to $-\infty$, whereas in the CGHS solution $R = K$ at the singularity \cite{7}. This remarkable difference is not a surprise, because the parameter $q/R$ is no longer small when we get close to the singularity, hence our
at \( v = v_{\text{sing}}(u) < 0 \).

Based on these causal properties of \( H(u, v) \) and its singularity, we shall refer to the ranges \( u < u^\text{hor} \) and \( u > u^\text{hor} \) as the BH exterior and interior, respectively. Note that the interior is confined to the range \( v_0 \leq v < v_{\text{sing}}(u) < 0 \) (and \( u > u^\text{hor} \)), whereas the exterior extends in the entire domain \( v_0 \leq v < \infty \) (for \( u < u^\text{hor} \)).

4.3 Expressing the ingoing solution in terms of \( H \)

The function \( H(u, v) \) may be obtained by solving the ODE (61) numerically, or by analytic approximate solutions like the one given in Appendix A. Once \( H \) is known, \( W \) is given by Eq. (60), and \( Z \) in turn by Eq. (57). The former equation yields

\[
W_{,u} = (1 - \frac{q}{H + m_v}) H_{,u},
\]

hence, up to first order in \( q \),

\[
Z = \ln(-H_{,u}) - \frac{q}{H + m_v}.
\]  

One can easily verify the consistency of this expression with the \( u \)-gauge transformation of \( Z \), given in Eq. (41) (note that \( H \) is unchanged in such a transformation).

4.4 Calculating \( H_0(u) \)

To calculate \( H_0(u) \) we evaluate Eq. (64) at \( v = v_0 \) and equate it to the desired initial condition \( Z_0(u) \). It is convenient to carry out this calculation in the Kruskal gauge. Using Eqs. (54,30) we obtain the following equation for \( H_0 \):

\[
\ln(-H_{0,u}) - \frac{q}{H_0 + M_0} = v_0 - \frac{q}{M_0 - U e^{v_0}}.
\]  

A simple solution immediately suggests itself: \( H_0 = -U e^{v_0} \), but we need here the general solution for this ODE, which spans a one-parameter family. For our purpose it will be sufficient, however, to derive an \textit{approximate} general solution (up to order \( q \)), which is an easy task. Such a one-parameter family of approximate solutions is

\[
H_0(U) = -U e^{v_0} + pq.
\]  

approximate solution becomes invalid there.
where $p$ is a yet-arbitrary constant. This constant may be fixed by solving the ODE (61) for $H(v; U)$ in the PNI asymptotic limit [with the above expression for $H_0(U)$ as initial condition], constructing $W, Z$ from $H$ and then $R, S$, and comparing them to the desired initial data at PNI. In Appendix B we carry out this analysis and find that the appropriate value is $p = -1$, namely

$$H_0(U) = -U e^{v_0} - q. \tag{67}$$

We may also use Eq. (30) to re-write $H_0$ in a form which is explicitly gauge-invariant (for arbitrary $u$ coordinate): $H_0(u) = R_0(u) - M_0 - q.$ \tag{68}

5 The weak perturbing outflux

The above ingoing Vaidya-like solution was constructed (through an appropriate choice of $H_0(u)$) such that it properly matches the initial function $Z_0(u)$ at the shell. However, the other initial function $W_0(u)$ does not exactly coincide with $W$ of the above constructed ingoing solution. To fix this mismatch we shall add a weak, outgoing component as a perturbation on top of the ingoing core solution.

To this end we write the overall $W, Z$ functions as

$$W(u, v) = W^{in}(u, v) + \delta W(u, v), \tag{69}$$

$$Z(u, v) = Z^{in}(u, v) + \delta Z(u, v), \tag{70}$$

where $W^{in}, Z^{in}$ denote the Vaidya-like ingoing core solution constructed in the previous section [namely Eqs. (60) and (64)], and $\delta W, \delta Z$ denote the additional perturbing component. \footnote{Note that in a coordinate transformation $Z^{in}$ transforms like $Z$, hence $\delta Z$ is invariant.}

We first calculate the mismatch in the initial condition for $W$. Equations (60) and (67) yield for $W^{in}$ at the shell

$$W^{in}_0(U) = M_0 - U e^{v_0} - q[\ln(M_0 - U e^{v_0}) + 2] \tag{71}$$

(we have omitted the $q$ in the log argument, being a higher-order term; and we do this occasionally in the equations below). This is to be compared to the actual initial data for $W$, Eq. (53), which, together with Eq. (30), reads

$$W_0(U) = M_0 - U e^{v_0} - 2q \ln(M_0 - U e^{v_0}). \tag{72}$$
The difference is thus
\[ \delta W_0(U) = -q[\ln(M_0 - U e^{u_0}) - 2]. \] (73)

Note that no mismatch is present in the shell data for \( Z \), because we have chosen \( H_0(U) \) in the first place so as to properly match \( Z_0(U) \); therefore,
\[ \delta Z_0(U) = 0. \] (74)

Next we substitute Eqs. (69, 70) in the full flux-conserving system (46), to obtain field equations for \( \delta W, \delta Z \). Since we are only interested in the solution up to first order in \( q \), and the mismatch initial data (73) are already \( O(q) \), we may treat \( \delta W, \delta Z \) as small perturbations, satisfying the linearized equations
\[ \delta W_{,uv} = e^Z [V_W \delta W + V_{W,W} \delta W], \] (75)
\[ \delta Z_{,uv} = e^Z [V_Z \delta W + V_{Z,Z} \delta Z]. \] (76)
Furthermore, we only need to consider here the coefficients \((V_W, V_{W,W}, V_Z, V_{Z,Z})\) at zero order in \( q \)—namely, \( V_W = -1 \) and \( V_{W,W} = V_Z = V_{Z,Z} = 0 \), yielding the trivial system
\[ \delta W_{,uv} = -e^Z \delta Z, \quad \delta Z_{,uv} = 0. \] (77)

The initial conditions are Eqs. (73, 74) at the shell, and no contribution from PNI.\footnote{Since the perturbation we add here is an outgoing component, it is assumed to be seeded at the shell only (any ingoing component would be absorbed in the ingoing core solution in the first place).} For \( \delta Z \) we immediately obtain
\[ \delta Z(u, v) = 0. \] (78)

In turn \( \delta W \) satisfies the trivial equation \( \delta W_{,uv} = 0 \), yielding
\[ \delta W(U, v) = \delta W_0(U) = -q[\ln(M_0 - U e^{u_0}) - 2]. \]
Thus, the overall solution is
\[ W(U, v) = W^{in}(U, v) + \delta W_0(U) = H + m_v - q[\ln(H + m_v) + \ln(M_0 - U e^{u_0}) - 1] \] (79)
and
\[ Z(u, v) = Z^{in}(u, v) = \ln(-H_{,u}) - \frac{q}{H + m_v}. \] (80)

Finally we re-write Eq. (79) in a form which includes no specific reference to the Kruskal coordinate \( U \), by replacing \( M_0 - U e^{u_0} \) (in the log argument) with \( H_0 + M_0 \), using Eq. (67):
\[ W(u, v) = H + m_v - q[\ln(H + m_v) + \ln(H_0 + M_0) - 1]. \] (81)
6 The final approximate solution

Transforming the above results \(W, Z\) from \(W, Z\) to the original variables \(R, S\), using Eqs. \(50, 52\), we obtain our approximate solution in its final form:

\[
R(U, v) = H + m_v + q[\ln(H + m_v) - \ln(H_0 + M_0) + 1]
\]

(82)

and

\[
S(u, v) = \ln(-H_u),
\]

(83)

where \(q = K/4\). The function \(H(u, v)\), recall, is determined by the ODE

\[
H_v = H\left(1 + \frac{q}{H - qv}\right),
\]

(84)

with initial conditions \(H_0(u) \equiv H(u, v_0)\) given by

\[
H_0(u) = R_0(u) - M_0 - q,
\]

(85)

or, more explicitly (in the Kruskal gauge)

\[
H_0(U) = -Ue^{v_0} - q.
\]

(86)

Approximate analytic expressions for \(H(u, v)\) are given in Appendix A, cf. Eqs. \(107, 108\).

Note that this approximate solution precisely matches the required initial conditions at the shell, namely (using Kruskal \(U\)-gauge) \(R = M_0 - Ue^{v_0}\) and \(S[U] = \ln(-H_U) = v_0\).

The expression \(83\) for \(S\) requires \(H_u\). The approximate analytic expressions \(107, 108\) can be directly differentiated to yield \(H_u\). However, when \(H(v; u)\) is obtained by numerically solving the ODE \(84\), a direct numerical \(u\)-differentiation may be inconvenient. In this case it is easier to obtain \(H_u\) by numerically integrating the ODE it satisfies:

\[
\frac{d}{dv}(H_u) = \left[1 - \frac{q^2v}{(H - qv)^2}\right] H_u
\]

(87)

(this can be done simultaneously with the numerical integration of the ODE of \(H\) itself). The initial condition at the shell is obviously

\[
H_u(u, v_0) = \frac{d}{du} H_0(u).
\]

(88)

\(^{12}\)Note that the parameter \(q\) may easily be factored out of this ODE: Defining \(\hat{H} \equiv H/q\), we obtain the ODE in its universal form \(\hat{H}_v = \hat{H}[1 + 1/(\hat{H} - v)]\).
6.1 Alternative approximate expression for $S$

As was already mentioned in Sec. 3, the $Z$ function in the ingoing Vaidya-like solution can be obtained by either of the equations (57) or (58). The above analysis was based on the former equation, and it led to the expression (83). If one uses Eq. (58) instead, one can derive an alternative expression for $S$:

$$S_{\text{alt}}(u,v) = v_0 + \ln \frac{H}{H_0(u)} + q \left[ \frac{1}{H + m_v} - \frac{1}{H_0(u) + M_0} \right] - \ln \frac{du}{d\tilde{U}}.$$  

This expression for $S$ looks more complicated, but it has the advantage that it does not require $H_{\text{\tiny{u}}}$. It should be emphasized that the two expressions (83,89) for $S$ are not exactly identical. Yet the difference appears to be compatible with the anticipated error characterizing the entire approximation scheme used here. At the same time we also point out that so far the error in the approximation (89) for $S$ has not been explored as thoroughly as that in the original approximation (83) (cf. Sec. 8).

7 Some useful gauges

Our approximate solution was presented in Eqs. (82-85) in a fully $(u)$-gauge-covariant form. The only reference to a specific gauge was made in Eq. (86), which explicitly gave $H_0$ in terms of the Kruskal $U$-coordinate.

In this section we shall introduce a few additional useful $u$-gauges: The shifted-Kruskal gauge, and the (external as well as internal) semiclassical Eddington gauge. These new gauges slightly simplify the functional form of $H_0(u)$. More importantly, they are better adopted to the global structure of the evaporating-BH spacetime (e.g. the location of the horizon).

Note that in all these gauges, we use for the $v$-gauge the same Eddington coordinate $v$ (originally denoted $v_e$), as we do throughout this paper.

7.1 Shifted-Kruskal gauge ($\tilde{U}$)

We define the shifted Kruskal coordinate

$$\tilde{U} \equiv U + q e^{-v_0}.$$  

Since this is a constant shift, $S$ is unchanged: $S_{[\tilde{U}]} = S_{[U]}$.  

26
The expression for $H_0$ slightly simplifies in this gauge:

$$H_0(\bar{U}) = -\bar{U} e^{v_0}. \quad (91)$$

Note that $\bar{U}^{hor} = 0$.

We point out that the solution in $(\bar{U}, v)$ coordinates [just like in $(U, v)$ coordinates] covers the entire BH spacetime. As may be obvious from the discussion in subsection 4.2, the BH exterior and interior correspond to $\bar{U} < 0$ and $\bar{U} > 0$, respectively.

### 7.2 Semiclassical Eddington gauge ($\bar{u}$)

In the range $\bar{U} < 0$ (the BH exterior) we define the Semiclassical Eddington coordinate $\bar{u}$ by

$$\bar{u} \equiv -\ln(-\bar{U}) \quad (\bar{U} < 0). \quad (92)$$

$S$ is modified by this transformation according to $S_{[\bar{u}]} = S_{[\bar{U}]} - \bar{u}$. The initial function for $H$ now reads

$$H_0(\bar{u}) = e^{v_0 - \bar{u}}. \quad (93)$$

In the Semiclassical Eddington gauge the alternative expression (89) for $S$ reduces to the simpler form

$$S_{alt}^{[\bar{u}]}(\bar{u}, v) = \ln H + q \left[ \frac{1}{H + m_v} - \frac{1}{H_0 + M_0} \right]. \quad (94)$$

Note that the coordinate $\bar{u}$ only covers the BH exterior.

### 7.3 Internal semiclassical Eddington gauge ($\bar{u}$)

In the range $\bar{U} > 0$ (the BH interior) we define the internal semiclassical Eddington coordinate $\bar{u}$ by

$$\bar{u} \equiv \ln(\bar{U}) \quad (\bar{U} > 0). \quad (95)$$

Now $S$ is modified according to $S_{[\bar{u}]} = S_{[\bar{U}]} + \bar{u}$. The initial function for $H$ takes the form $H_0(\bar{u}) = e^{v_0 + \bar{u}}$. 

27
8 Verification of the approximate solution

Our approximate solution \((82, 85)\) was constructed here through a rather indirect process, which involved the transformation to new field variables \(W, Z\), the large-\(R\) approximation, the formalism of flux-conserving systems, and their Vaidya-like solutions. It is therefore important to directly examine the validity of the resultant expressions.

Naturally this examination involves two independent parts: (i) checking compliance with the field equations; and (ii) checking compatibility with initial conditions, both at the shell \((v = v_0)\) and at PNI. These two parts will be carried out in the next two subsections.

8.1 Compliance with the field equations (error estimate)

To define the local error in the evolution equations we substitute the approximate expressions \((82, 83)\) in the field equations \((13, 14)\) and evaluate the error—namely, the deviation of \(R_{uv}\) and \(S_{uv}\) from their respective values (specified at the right-hand side of these two equations). The error defined in this way is obviously gauge-dependent, and we find it convenient to employ the semiclassical Eddington coordinate \(\tilde{u}\) (along with Eddington \(v\)) for this task. Using the MATHEMATICA software we find that the local errors in the two equations indeed scale as \(q^2\), as anticipated. More specifically, the errors scale as \(R (q/R)^2\) for \(R_{uv}\) and as \((q/R)^2\) for \(S_{uv}\)—both multiplied by certain functions of \(m_v/R\). This local error estimate applies to typical off-horizon strong-field regions, namely, regions for which \(m_v/R\) is of order unity but not too close to 1. The error decays exponentially in \(|v - \tilde{u}|\) both at weak-field regions \((m_v/R \rightarrow 0, \text{corresponding to } v - \tilde{u} \rightarrow \infty)\) and near-horizon regions \((m_v/R \rightarrow 1, \text{corresponding to } v - \tilde{u} \rightarrow -\infty)\).

The global, accumulated, long-term error is harder to analyze. It may be evaluated by comparing the approximate solution \((82, 83)\) to numerical simulations, but this is beyond the scope of the present paper. In the next section, however, we shall evaluate the accumulated error in both \(R\) and \(S\) in the neighborhood of the horizon \((\tilde{u} \rightarrow \infty)\).

The error in the constraint equations \((9, 10)\) is also found to be proportional to \(q^2\), as may be expected (based on the mutual consistency of the evolution and constraint equations). However, the functional dependence of the pre-factor on \(m_v/R\) is more subtle and will not be addressed here.
8.2 Compatibility with the initial conditions

Initial data at the shell

At \( v = v_0 \) the expressions (82) and (83) reduce to \( R = H_0 + M_0 + q \) and \( S = \ln(-H_{0,w}) \) respectively. By virtue of Eq. (86), one obtains (using the Kruskal \( U \) coordinate) \( R(U, v_0) = M_0 - U e^{v_0} \) and \( S(U, v_0) = v_0 \), which exactly match the desired initial data (30,31).\(^{13}\)

Initial data at past null infinity

In Sec. [10] we analyze the asymptotic behavior of our approximate \( R \) and \( S \) at PNI, and verify that they satisfy the required asymptotic behavior (32). We also show that the influx \( T_{vv} = R,v S,v - R,vv \) vanishes at PNI, as it should.

9 Horizon

In Sec. [4] we analyzed the behavior of \( H(u,v) \) along \( u = \text{const} \) lines, and found that spacetime is divided into two domains by a certain outgoing null ray which (for a general \( u \)-gauge) we denoted \( u = u^{\text{hor}} \): All lines \( u < u^{\text{hor}} \) run to FNI in a regular manner (with steadily growing \( H \)), whereas all lines \( u > u^{\text{hor}} \) crush into a spacelike singularity. The spacetime thus contains a BH, and the domains \( u < u^{\text{hor}} \) and \( u > u^{\text{hor}} \) correspond to the BH exterior and interior, respectively. We shall therefore regard the critical null ray \( u = u^{\text{hor}} \) as the event horizon (or sometimes just horizon) of the BH .\(^{14}\)

The horizon is characterized by the vanishing of \( H \), hence its location \( u = u^{\text{hor}} \) is determined by requiring \( H_0(u) = 0 \). Referring to some specific gauges, the horizon’s location is \( \tilde{U} = 0 \) in the shifted Kruskal gauge, and \( U = -q e^{-v_0} \) in the original Kruskal gauge. In the semiclassical Eddington gauge the horizon is located at the asymptotic boundary \( \tilde{u} \to \infty \).

Recall that in the classical solution (with the same initial data) the horizon is located at \( U = 0 \). The inclusion of semiclassical effects thus shifts the horizon in \( U \) (or \( \tilde{U} \)) by an amount \( q e^{-v_0} \).

\(^{13}\)Obviously, this also implies exact matching of \( R \) and \( S \) to the Minkowski-like solution at \( v \leq v_0 \) in any other gauge.

\(^{14}\)This involves some abuse of standard terminology, because the line \( u = u^{\text{hor}} \) actually contains a naked singularity at \( v = 0 \) (the point denoted "P" in Fig. 1), where \( H + m_v \) vanishes. Note that it is only the section \( v < 0 \) of this line which separates the BH interior and exterior. The portion \( v > 0 \) of \( u = u^{\text{hor}} \) is in fact a Cauchy horizon (see Fig. 1).
9.1 Behavior of $R$ and $S$ at the horizon

The behavior of $R$ along the horizon is obtained by setting $H_0 = H = 0$ in Eq. (82), yielding $m_v + q \ln(m_v/M_0) + 1$. However, the term $q \ln(m_v/M_0)$ in this expression cannot be trusted, as may be deduced from simple error estimate. To this end we evaluate the error in $R_v$ as a function of $v$ along the horizon, using semiclassical Eddington coordinates for simplicity. We do this by integrating the local error in $R_{\tilde{u} v}$ along a line $v = \text{const} < 0$, from $\tilde{u} = -\infty$ (PNI) to $\tilde{u} = \infty$ (horizon). From the discussion in Sec. 8 it follows that along such a $v = \text{const}$ line the local error gets a maximal value of order $q^2/R \sim q^2/m_v$ at intermediate $\tilde{u} \sim -v$ values, and it decays exponentially in $\tilde{u}$ in both directions. The effective integration interval (in $\tilde{u}$) is of order unity, hence the integrated error in $R_v$ is also $\sim q^2/m_v = q/|v|$. Consequently, the integrated error in $R(v)$ along the horizon is of order $\sim q \ln(v/v_0) = q \ln(m_v/M_0)$.\(^{15}\) We therefore re-write the above result for $R(v)$ as\(^{16}\)

$$R_{\text{hor}}(v) = m_v + q + O\left(q \ln \frac{m_v}{M_0}\right) = -qv + q + O\left(q \ln \frac{v}{v_0}\right).$$ (96)

Next we analyze the behavior of $S$ in the horizon’s neighborhood, in the semiclassical Eddington gauge, using Eq. (83) which now reads $S_{[\bar{u}]} = \ln(-H_{\bar{u}})$. To this end we divide Eq. (87) by $H_{[u]}$, substitute $H = 0$ in the right-hand side, and re-write this equation as

$$\frac{d}{dv} \ln(-H_{\bar{u}}) = 1 - \frac{1}{v}. $$ (97)

The initial condition at $v = v_0$ is obtained from Eq. (93) which yields $\ln(-H_{0,\bar{u}}) = v_0 - \bar{u}$. Integrating Eq. (97) we find

$$S_{[\bar{u}]} = v - \bar{u} + \ln \frac{v_0}{v}. $$

The error estimate for $S$ parallels the one carried out above for $R$. It yields an accumulated error $\sim (q/m_v)^2 = v^{-2}$ in the value of $S_v$ at the horizon,\(^{15}\) It should be emphasized that despite this $O(q)$ integrated error, it is crucial to keep the $O(q)$ term in the approximate expression (82) for $R$. Without this term, the local error in $R_{\tilde{u} v}$ will grow from $O(q^2)$ to $O(q)$.\(^{16}\) In fact one can use the constraint equation for $T_{\bar{u} v}$ to obtain the correct coefficient of the log term in Eq. (96), but this is beyond the scope of the present paper.
and hence an integrated error of order \( \sim 1/|v| = q/m_v \) in \( S \) itself. \(^{17}\) We therefore re-write our result as

\[
S_{\text{hor}}^{\text{hor}} = v - \tilde{u} + \ln \frac{v_0}{v} + O(1/v) = v - \tilde{u} + \ln \frac{M_0}{m_v} + O(q/m_v). \quad (98)
\]

Another way to obtain this result is by integrating Eq. (63) for \( l(v) \). In the horizon’s neighborhood this equation reduces to \( l_v = 1 - 1/v \), which is easily integrated (with the appropriate initial conditions) to yield \( l = v - \tilde{u} + \ln(v_0/v) \), or

\[
H_{\text{hor}} = \frac{v_0}{v} e^{v-\tilde{u}}. \quad (99)
\]

Differentiating this expression with respect to \( \tilde{u} \) and substituting in Eq. (83), one recovers Eq. (98). \(^{18}\)

### 9.2 The apparent horizon

Motivated by the terminology used for conventional 4D spherically-symmetric BHs, we define the apparent horizon to be the locus of the points where \( R_{,v} = 0 \) (or equivalently, \( \phi_{,v} = 0 \)). By virtue of Eq. (82) this implies \( (H + m_v)_{,v} = 0 \), or \( H_v = q \). Utilizing Eq. (84), and restricting the analysis to first order in \( q \), we find that at the apparent horizon \( H \approx q \) is satisfied.

We shall consider here the properties of the apparent horizon during the macroscopic phase \( m_v \gg q \), namely \( v \ll -1 \). Throughout this phase \( H_{\text{hor}} \ll R \approx m_v \) and hence the horizon approximation \( H \approx H_{\text{hor}} \) applies. Setting \( H \approx q \) in Eq. (99) we find the apparent-horizon’s location

\[
\tilde{u} \approx v + \ln(v_0/v) - \ln q. \quad (100)
\]

Thus, \( d\tilde{u}/dv = 1 - 1/v \approx 1 \) along the apparent horizon. We find that the apparent horizon is a timelike line, which is approximately vertical in the \((\tilde{u}, v)\) coordinates. In Fig. 1 the apparent horizon is denoted "AH".

Consider now the behavior of \( H \) and \( R \) along an outgoing null geodesic located outside the BH though fairly close to the event horizon—namely, sufficiently-large fixed \( \tilde{u} \). To be more specific, let us assume that \( v_0 < \tilde{u} < 0 \), such that \( \tilde{u} \ll -1 \) but \( \tilde{u} - v_0 \gg 1 \). (As a typical example one may take \( \tilde{u} \approx \)

\(^{17}\)To be more precise, the integrated error is \( \sim (1/|v| - 1/|v_0|) = (q/m_v - q/M_0) \).

\(^{18}\)The alternative approximation \( S^{\text{alt}} \) yields the same result. To see this one substitutes Eq. (99) in (94) (and omit the unsecured \( O(q) \) term).
Recall that throughout this paper we assume $v_0 \ll -1$, corresponding to a macroscopic BH.) Then $H_0 = e^{v_0 - \bar{a}}$ is exponentially small and may be neglected in Eq. (82). Initially, near $v = v_0$, $H$ is also exponentially small and therefore (neglecting terms of order $\sim q$ compared to $m_v$), $R \approx m_v$. In this range $R$ shrinks linearly in $v$, like $m_v$ itself (and like $R^{hor}$). During this stage $H \cong H^{hor}$ grows (approximately) exponentially in $v$, but initially this does not have much effect on $R(v)$ because $H$ is still too small. However at some point the exponentially-growing $H$ starts to slow the decrease rate of $R(v)$. When $H$ approaches $q$ this exponential growth just balances the linear decrease of $m_v$. This is the point of intersection with the apparent horizon. Note that at this point $R$ is still $\approx m_v$, the difference being $O(q)$.

Soon afterwards the exponentially-growing $H$ overtakes $m_v$.

On the other hand, for earlier outgoing geodesics with sufficiently large $H_0$, the exponential growth of $H$ will dominate over $m_v$ everywhere, and $R$ will grow monotonically all the way from $v_0$ to $\infty$. Since $H_v > 0$ outside the BH, and the apparent horizon satisfies $H \cong q$, this monotonic growth of $R$ will occur at the outgoing null geodesics for which $H_0 > q$.

The evaporating-BH spacetime may thus be divided into three domains in $u$. For concreteness let us use here the coordinate $\bar{U}$ to characterize these domains: In the early domain, $\bar{U} < \bar{U}_{ah}$, $R$ grows monotonically with $v$ throughout $v_0 < v < \infty$. From Eq. (91) we find (equating $H_0$ to $q$ as explained above)

$$\bar{U}_{ah} = -qe^{-v_0}. \quad (101)$$

In the second domain, $\bar{U}_{ah} < \bar{U} < 0$, $R$ first decreases along an outgoing null ray (taking values fairly close to $m_v$) until it intersects the apparent horizon at certain $v < 0$, and then $R$ starts to increase with $v$. In the third domain $\bar{U} > 0$ (the BH interior), $H$ is everywhere decreasing (and the same for $m_v$), therefore $R$ decreases monotonically until the outgoing null ray hits the spacelike singularity.

From the discussion above it follows that at a given $v < 0$ the event and apparent horizons have roughly the same $R$ value (the difference being approximately $q$). In other words, the apparent and event horizons shrink (in $R$) in the same standard rate, $dR/dv = -q$.

It may be interesting to compare three points along the worldline of the collapsing shell: (1) The intersection of $v = v_0$ with the apparent horizon ($\bar{U} = \bar{U}_{ah}$); (2) its intersection with the event horizon ($\bar{U} = 0$); and (3) its intersection with the (would-be) event horizon of the classical CGHS
BH (namely $U = 0$, or $\tilde{U} = q e^{-v_0}$). One finds that these three points are equally-separated in $\tilde{U}$ (or in $U$), the separation being $q e^{-v_0}$. We find it more illuminating, however, to express these three points by their respective $R = R_0(u)$ values. Noting that

$$R_0(\tilde{U}) = M_0 - \tilde{U} e^{v_0} + q,$$

one finds that $R_0$ is $M_0 + 2q$ at point 1, $M_0 + q$ at point 2, and $M_0$ in point 3, so these points are equally-separated in $R$ too. Notice, however, that for a macroscopic BH ($M_0 \gg q$, which we assume throughout) the relative separation $q/R_{1,2,3} \approx q/M_0$ is $\ll 1$.

9.3 Influx at the horizon

We proceed now to calculate $T_{vv}$ at the horizon, using the constraint equation \[10\] which now reads

$$T_{\text{hor,}\;vv} = R_{\text{hor,}\;v} S_{\text{hor,}\;v} - R_{\text{hor,}\;vv}.$$

We shall restrict here the calculation to the leading order, namely first order in $q/m_v$. From Eqs. \(96, 98\) we find $S_{\text{hor,}\;v} = 1 + O(q/m_v)$, $R_{\text{hor,}\;v} = -q + O(q^2/m_v)$, and $R_{\text{hor,}\;vv} = O(q^3/m_v^3)$ will not contribute. We obtain at the leading order

$$T_{\text{hor,}\;vv} = -q = -\frac{K}{4} = -\frac{N}{48}.$$

Thus, in the macroscopic limit the influx into the horizon is constant and independent of the mass—a well-known result for a two-dimensional BH \[2\].

In principle it is possible to use Eq. \(12\) for $\hat{T}_{vv}$ to obtain the first-order correction to the fixed influx \(102\) (and thereby to fix the $O[q \ln(m_v/M_0)]$ term in $R_{\text{hor}}$), but this is beyond the scope of the present paper.

10 Null infinity

10.1 Past null infinity

The limit $U \to -\infty$ (also $\tilde{U}, \tilde{u} \to -\infty$) and finite $v$ corresponds to PNI. In this asymptotic boundary $m_v$ is finite but $H$ diverges (this immediately follows from the divergence of $H_0 \propto -\tilde{U}$, combined with the monotonic
growth of $|H|$ with $v$). At this limit the ODE (84) for $H$ reduces to $H_v \approx H + q$ and its solution, corresponding to the initial conditions (86), is

$$H^{\text{pni}} \approx -U e^v - q. \quad (103)$$

The expression (83) for $S$ then yields

$$S^{\text{pni}}_{[U]} \approx v. \quad (104)$$

(For the other gauges we find $S_{[u]}^{\text{pni}} = v$ and $S_{[\tilde{U}]}^{\text{pni}} = v - \tilde{u}$.) For $R$ Eq. (82) yields $R \approx m_v - U e^v + q \ln(H/H_0)$, and setting $\ln(H/H_0) \approx v - v_0$ we obtain

$$R^{\text{pni}} \approx M - U e^v. \quad (105)$$

These expressions for $R^{\text{pni}}$ and $S^{\text{pni}}$ properly match the desired initial conditions (32) at PNI.

By assumption the influx $T_{vv}$ should vanish at PNI. We would like to verify this by applying the constraint equation $T_{vv} = R_v S_{vv} - R_{vv}$ to our approximate solution and taking the limit $U \to -\infty$. Doing so we observe that Eqs. (104,105) yield $R^{\text{pni}}_v = R^{\text{pni}}_{vv} = -U e^v$ and $S^{\text{pni}}_v = 1$, which yields the desired result

$$T^{\text{pni}}_{vv} = 0. \quad (106)$$

Note the following subtlety, however: Because $R^{\text{pni}}_v \propto U$ diverges at PNI, the term $R^{\text{pni}}_v S^{\text{pni}}_v$ might receive a nonvanishing contribution from $O(1/U)$ corrections to $S$, if such existed. To address this issue we must carry the calculation of $S^{\text{pni}}_{[U]}$ to order $1/U$. This in turn requires a more detailed examination of $H^{\text{pni}}$. At PNI the ODE (84) takes the form $H_v = H + q + O(1/H)$, and the last term introduces $O(1/U)$ corrections to $H$, namely $H^{\text{pni}} \approx -U e^v - q + O(1/U)$. However, this only leads to $O(1/U^2)$ corrections in $S^{\text{pni}}_{[U]}$, which leave Eq. (106) intact.

### 10.2 Future null infinity

The limit $v \to \infty$ (for $\tilde{U} < 0$) corresponds to FNI. The analysis of this asymptotic region is more complicated than that of PNI, because this time we have to integrate the ODE (84) through strong-field regions (where $H$ is comparable to $m_v$). The thorough investigation of this asymptotic region is beyond the scope of the present paper, and we hope to address it elsewhere. Here we
shall merely mention two key properties: (1) Spacetime is asymptotically-flat; In other words, for appropriate choice of \( u \)-coordinate, which we denote \( \hat{u} \) (and which does not exactly coincide with \( \tilde{u} \)), \( \rho[\hat{u}] \) vanishes at \( v \to \infty \). (2) At the leading order one obtains a constant, mass-independent, Hawking outflux \( T_{\hat{u}\hat{u}} = q = N/48 \) (a well-known result [2]).

11 Summary

Our approximate solution for the semiclassical variables \( R \) and \( S \) is described in Eqs. (82,83). The determination of the original CGHS variables \( \phi, \rho \) from \( R \) and \( S \) is straightforward. The function \( H \) involved in this solution is determined by the ODE (84), along with the initial conditions (85) or (86). Approximate expressions for \( H \) are given in Eqs. (107,108).

In the CGHS formalism all semiclassical effects originate from a term in the action which is proportional to \( K \equiv N/12 \). Our approximation scheme is restricted to macroscopic BHs, namely, those with original mass \( M_0 \gg q \equiv K/4 \). Furthermore, it only applies to spacetime regions where \( R \gg q \). It nevertheless holds both outside and inside the BH, though obviously not too close to the singularity (where the condition \( R \gg q \) is violated).

Throughout this paper we use double-null coordinates \((u,v)\). Our construction explicitly preserves the gauge-freedom in \( u \), though not in \( v \). Our coordinate \( v \) coincides with the affine parameter along PNI. (We have set the origin of \( v \) such that the collapsing shell is placed at \( v = v_0 \equiv -M_0/q \).)

For the \( u \)-gauge we find two particularly useful choices: the shifted-Kruskal coordinate \( \tilde{U} \), and the semiclassical Eddington coordinate \( \tilde{u} \) (both defined in Sec. 7).

At the horizon’s neighborhood we find that \( R \) shrinks linearly with \( v \), as may be anticipated: \( R \approx -qv + q \). The other variable \( S \) behaves as \( S \approx v + \ln(v_0/v) + c \), where \( c \) is \( v \)-independent though it depends upon the \( u \)-gauge being used. (For example, \( c = 0 \) in the shifted-Kruskal gauge and \( c = -\tilde{u} \) in the semiclassical Eddington gauge.) The term \( v \) merely reflects the classical behavior of \( S \) at the horizon, whereas the log term is of semiclassical origin.

Our approximate solution is "locally first-order accurate"; Namely, the deviation of \( R_{\tilde{u}\tilde{u}} \) and \( S_{\tilde{u}\tilde{u}} \) from their respective values, as specified in the evolution equations (67) or (13,14), is proportional to \( q^2 \). (More specifically, the deviations in both \( R_{\tilde{u}\tilde{u}}/R \) and \( S_{\tilde{u}\tilde{u}} \) are \( \sim (q/R)^2 \) at typical (off-horizon)
strong-field regions, and smaller elsewhere. On the other hand, the accumulated error in $S$, and accumulated relative error in $R$, are of first order in $q/R$. This change in the power of $q$ is because the effective accumulation intervals (in $v$ and/or $\tilde{u}$) are typically of order of the evaporation time, which is $M_0/q$.

It will be interesting to check this approximate solution against numerical simulations of the CGHS field equations. Such numerical simulations are currently being conducted by several groups [5, 6]. A preliminary comparison with the numerical results [5] shows nice agreement, though a more comprehensive check still needs to be carried out.

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A Approximate solution for $H(v; u)$

A useful approximate solution of the ODE (84) for $H$ is

$$H(v; \tilde{U}) \approx -\tilde{U} e^v \frac{\ln(e^{-v_0} - \tilde{U}/q)}{\ln(e^{-v} - \tilde{U}/q)}, \quad (107)$$

where $\tilde{U} \equiv U + q e^{-v_0}$ is the shifted Kruskal coordinate defined in Sec. 7. This expression holds both inside and outside the BH (corresponding to $\tilde{U} > 0$ and $\tilde{U} < 0$ respectively).

Note that the initial condition $H_0(\tilde{U}) = -\tilde{U} e^{v_0}$ at $v = v_0$ is precisely satisfied. Also recall that the dependence of $H$ on $\tilde{U}$ only emerges through this initial condition at the shell. It is thus possible to substitute $\tilde{U} = -H_0 e^{-v}$ in Eq. (107), to obtain an expression for $H$ in terms of $H_0(u)$ but with no explicit reference to any specific $u$-gauge.

We also translate the above expression for $H$ to the semiclassical Eddington coordinate $\tilde{u}$, valid in the external world ($\tilde{U} < 0$):

$$H(v; \tilde{u}) \approx e^{v - \tilde{u}} \frac{\ln(e^{-v_0} + e^{-\tilde{u}}/q)}{\ln(e^{-v} + e^{-\tilde{u}}/q)}. \quad (108)$$
Preliminary error estimate suggests the following behavior of the relative error in the above expression for $H$: We assume $v_0 \ll -1$ throughout. For $u \sim v_0$ or $v_0 < u \ll -1$ the error scales as $1/u$, though with some logarithmic corrections. This applies to both (i) the relative error in $H$ itself, and (ii) the relative local error in $H, v$ [compared to its respective value in the ODE (84)]. For $u < v_0$ the error further scales as $e^{u-v_0}$ and quickly becomes negligible.

B Calculating the constant $p$

In this Appendix we determine the constant $p$ in Eq. (66) by analyzing the asymptotic behavior of $R$ at PNI and comparing it to the desired initial conditions.

PNI is characterized by finite $v$ but $U \to -\infty$, yielding finite $m_v$ but $H \to \infty$ (the latter follows from the divergence of $H_0$ combined with the monotonic growth of $|H|$ with $v$). Therefore, in the ODE (84) we may replace the term $H - qv$ in the denominator by $H$. We are left with the ODE $H, v \approx H + q$, whose general solution is

$$H(u, v) = -q + [H_0(u) + q]e^{v-v_0}. \tag{109}$$

Using the general expression (66) for $H_0(U)$ we get

$$H(U, v) = [-Ue^{v_0} + (p + 1)q]e^{v-v_0} - q. \tag{110}$$

We now substitute this in Eq. (66) for $W$ (or more precisely, $W^{\text{in}}$; see below). In doing so, we may approximate $q \ln(H + m_v)$ at PNI by $q \ln H$, which by virtue of Eq. (109) may be approximated by $q[v + \ln(-U)] = -m_v + q \ln(-U)$, omitting $O(q^2)$ contributions. (The same will apply to terms like $q \ln W$ and $q \ln R_0$ below.) We obtain

$$W^{\text{in}}(U, v) = [-Ue^{v_0} + (p + 1)q]e^{v-v_0} + 2m_v - 2q - q \ln(-U). \tag{110}$$

We used here the symbol $W^{\text{in}}$ to make it clear that this is the $W$-function associated with the ingoing Vaidya-like core solution (as opposed to the full-$W$ function, considered below; See discussion in Sec. 5). At $v = v_0$ this yields

$$W^{\text{in}}(U, v_0) = -Ue^{v_0} + 2M_0 - q \ln(-U) + (p - 1)q.$$
This should be compared to the full-$W$ initial data at $v = v_0$, given by Eqs. \((53,30)\):

$$W_0(U) = M_0 - U e^{v_0} - 2q[v_0 + \ln(-U)] = 3M_0 - U e^{v_0} - 2q\ln(-U).$$

[Here, again, in processing the term $2q\ln R_0 = 2q\ln(M_0 - U e^{v_0})$ in Eq. \((53)\) we have neglected $M_0$ compared to $U e^{v_0}$.] The difference is

$$\delta W(U) \equiv W_0(U) - W^{\text{in}}(U,v_0) = M_0 - q\ln(-U) - (p - 1)q. \quad (111)$$

The full function $W(U,v)$ is obtained by simply adding the outgoing perturbation $\delta W(U)$ to the ingoing solution $W^{\text{in}}(U,v)$ (this is shown in detail in Sec. \[5\] based on linear perturbation analysis). Summing Eqs. \((110,111)\) we obtain for the full-$W$ function near PNI

$$W = [-U e^{v_0} + (p + 1)q]e^{v - v_0} + M_0 + 2m_v - 2q\ln(-U) - (p + 1)q.$$ 

Finally we transform from $W$ to $R$, using Eq. \((50)\). The term $2q\ln W$ therein [in which we approximate $\ln W \simeq v + \ln(-U)$ as explained above] just cancels the terms $2m_v - 2q\ln(-U)$ in the last expression for $W$, and we obtain

$$R = M_0 - U e^v + (p + 1)q(e^{v - v_0} - 1).$$

This expression should agree with the presumed PNI initial conditions \((32)\), namely $R = M_0 - U e^v$, which dictates $p = -1$.

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Figure 1: Penrose diagram describing the formation and evaporation of a two-dimensional CGHS black hole. The black hole forms by the collapse of a thin shell of macroscopic mass $M_0$. The collapsing shell is located at $v = v_0 \equiv -M_0/q$. The lines denoted "EH", "AH", and "CH" respectively represent the event horizon, apparent horizon, and Cauchy horizon.