Sampled-data Suboptimal State Estimation over Lossy Networks

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Abstract

This paper proposes a sampled-data state estimator which is suboptimal with respect to the error variance, over lossy networks. When signal loss is detected, the estimator switches gains and continues to update the estimate based upon recently received measurement. The gains are designed by means of a common solution of linear matrix inequalities.

1 Introduction

In networked/remote control systems we often encounter situations where measured signals are sometimes lost due to transmission error or temporal sensing failure. The authors’ group have proposed a gain switching state observer that is robustly stable against such irregular signal loss [1]. The observer gains are designed by means of switched quadratic Lyapunov functions (SQLF) assuming that the maximum length of signal loss is known. The switching gains are obtained by solving linear matrix inequalities (LMIs) [2]. The discrete-time decay rate can also be introduced in the SQLF.

In practice, however, it is desirable to guarantee some stochastic performance, further to stability [3]. In this paper we aim at designing a gain switching state estimator in consideration of such performance.

Notations. \( \mathbb{E} \) denotes expectation. \( \mathbb{R} \) and \( \mathbb{N} \) are the sets of all real numbers and all nonnegative integers, respectively. Dirac’s delta function is denoted by \( \delta(\cdot) \) and Kronecker delta is \( \delta_{ij} \). The floor function is defined by \( \lfloor t \rfloor = k \in \mathbb{N} \) if \( t \in [k, k+1) \). \( A^\top, \text{tr} A, \) and det \( A \) respectively mean the transpose, trace, and determinant of matrix \( A \). We write \( A^{-\top} = (A^\top)^{-1} \) and \( A^\top = (A^\top)^j \) for \( j = 1, 2, \ldots \), with slight abuse. \( I_n \) stands for the \( n \)-th order identity. We use the notation

\[
\begin{bmatrix}
A \\
C \\
B \\
D
\end{bmatrix} = C(sI_n - A)^{-1}B + D.
\]

2 Problem Formulation

Let a linear state space model be

\[
\dot{x}(t) = Ax(t) + Gw(t), \quad y(t) = Cx(t) + Hv(t),
\]

where \( x \in \mathbb{R}^n \) is the state to be estimated, \( y \in \mathbb{R}^p \) is the measured signal, \( w \in \mathbb{R}^r \) and \( v \in \mathbb{R}^p \) are white Gaussian signals such that

\[
\mathbb{E}[w(t)] = 0, \quad \mathbb{E}[v(t)] = 0,
\]

\[
\mathbb{E}\left[\begin{pmatrix} w(t) \\ v(t) \end{pmatrix} \begin{pmatrix} w(t') \\ v(t') \end{pmatrix}^\top \right] = I_{r+p}\delta(t - t')
\]

for any \( t, t' \in \mathbb{R} \). We assume that \( A, C, G, \) and \( H \) are known matrices of appropriate sizes, the pair \((A, C)\) is observable, \( G \) has rank \( r \), and \( H \) is nonsingular. Let \( x(0) = 0 \) for simplicity.

If we can use continuous-time state estimator

\[
\dot{\hat{x}}(t) = A\hat{x}(t) + L(y(t) - C\hat{x}(t)), \quad \hat{x}(0) = 0,
\]

then it is well known that the Kalman gain \( L \) is optimal in the sense that it minimizes

\[
\lim_{t \to \infty} \mathbb{E}[\|x(t)\|^2],
\]

where the estimation error is defined by

\[
ex(t) = x(t) - \hat{x}(t).
\]

It is also known that this optimization is equivalent to minimizing the \( H_2 \) norm of the closed-loop by \( \zeta = L\eta \) applied to the generalized plant (Fig. 1)

\[
\begin{pmatrix} \varepsilon \\ \eta \end{pmatrix} = P(s) \begin{pmatrix} w \\ v \end{pmatrix},
\]

where

\[
P(s) = \begin{bmatrix}
A & G & I_p \\
I_p & 0 & 0 \\
-C & 0 & -H \\
\end{bmatrix}.
\]

In contrast, we consider sampled-data measurement over a lossy channel in this paper, where \( y \) is not available continuously but we receive it through a (not necessarily periodic) sampler \( S \) and a zero-th order hold \( H \).
To be more specific, let a given small number $\Delta > 0$ be a basic period, and we receive measurement signal $y(t_k), k \in \mathbb{N}$, in intervals that are (unknown but bounded) integral multiples of $\Delta$ due to irregular signal loss (Fig. 2). Hence

$$t_0 = 0 < t_1 < t_2 < \cdots ,$$

$$t_{k+1} - t_k = h_k \Delta, \quad h_k \in \{1, \cdots , \hat{h} + 1\}$$

for $k \in \mathbb{N}$. Here $\hat{h} \in \mathbb{N}$ is the maximal number of successive signal loss and assumed to be fixed and known. In this setting we assume, instead of (2), that

$$\begin{align*}
\mathbb{E}[w(t)] &= 0, \quad \mathbb{E}[v(k\Delta)] = 0, \\
\mathbb{E} &\left[ \begin{pmatrix} w(t) \\ v(k\Delta) \end{pmatrix} \begin{pmatrix} w(t') \\ v(k'\Delta) \end{pmatrix}^\top \right] \\
&= \begin{pmatrix} I \delta(t-t') & 0 \\
0 & I_p \delta_{k,k'} \end{pmatrix}
\end{align*}$$

for any $t,t' \in \mathbb{R}$ and $k, k' \in \mathbb{N}$.

The signal arrival pattern (7) is unknown when designing the estimator, but we can detect signal arrival/loss in real time. In the latter case, we may want to reuse a previously received signal. The problem is that it is unpredictable how long the signal loss continues. In other words, at time $t \in (t_k, t_{k+1})$ in (7), $t_k$ and $t$ is available while $t_{k+1}$ is not.

We address this issue by switching the gains depending on the length of signal loss so far. Namely, instead of (3) we adopt

$$\begin{align*}
\dot{\hat{x}}(t) &= A \hat{x}(t) + L_{\sigma(t)}(y(t_k) - C \hat{x}(t_k)), \\
\sigma(t) &= \lfloor (t - t_k)/\Delta \rfloor, \quad t \in [t_k, t_{k+1}).
\end{align*}$$

$L_{\sigma(t)}$ depends on the number $\sigma(t)$ of how many times the signal $y$ is lost in succession before time $t$ (Figs. 2 and 3). Hence $\sigma(t)$ is a function of $t \in \mathbb{R}$ taking values in $\{0, \cdots , \hat{h}\}$. Stability of the above estimator is discussed in [1]. On the other hand, our problem in the present paper is to find suboptimal gains $L_0, \cdots , L_{\hat{h}}$ such that, for given $\gamma > 0$, the estimation error in (5) satisfies the performance

$$\limsup_{k \to \infty} \mathbb{E}[\|\epsilon(t_k)\|^2] < \gamma^2.$$  

for any arrival pattern (7).

### 3 Main Results

It is known that if $\hat{h} = 0$ in particular, then the error signal is governed by a (pure) discrete-time system and the $H_2$ norm approach carries over to this case ([3, 2]). If $\hat{h} > 0$, however, then the error system becomes time-variant and hence the $H_2$ norm as in the continuous-time case is not valid any more.
satisfies recurrence formula
\[
(A^{b_k} - T_{h_k} L_{h_k} C) X_k (A^{b_k} - T_{h_k} L_{h_k} C)^\top
\]
\[-X_{k+1} + Q_{h_k} + R_{h_k} = 0, \quad k = 0, 1, \cdots . \quad (18)
\]

**Proof.** We first show the following.

**Claim 1.** By taking \( t \uparrow t_{k+1} \) in (16) we obtain recurrence formula
\[
\varepsilon(t_{k+1}) = (A^{b_k} - T_{h_k} L_{h_k} C) \varepsilon(t_k)
\]
\[+ \int_{t_k}^{t_{k+1}} e^{A(t_{k+1} - \tau)} G w(\tau) d\tau - T_{h_k} L_{h_k} H v(t_k). \quad (19)
\]

**Proof.** Substitute \( t = t_{k+1} \) in the right-hand side of (16). Then the first term becomes \( A^{b_k} \varepsilon(t_k) \). To compute the second term, we divide the integral interval as
\[
\int_{t_k}^{t_{k+1}} e^{A(t_{k+1} - \tau)} L_\sigma(\tau) d\tau = \sum_{j=0}^{h_k - 1} \int_{t_{k+j}}^{t_{k+(j+1)}} e^{A(t_{k+1} - \tau)} d\tau L_j, \quad (20)
\]
By putting \( \tau' = t_k + (j + 1)\Delta - \tau \),
\[
\int_{t_k}^{t_{k+1}} e^{A(t_{k+1} - \tau)} L_\sigma(\tau) d\tau = \sum_{j=0}^{h_k - 1} \int_{t_k}^{t_{k+j}} e^{A(t_{k+1} - \tau')} d\tau' L_j, \quad (21)
\]
for \( j = 0, \cdots, h_k - 1 \). Then the right-hand side of (21) is
\[
\Gamma(A^{b_k - 1} L_0 + \cdots + A L_{h_k - 2} + L_{h_k - 1}) = T_{h_k} L_{h_k}
\]
This shows (19), as expected. □

Claim 1 enables us to show Lemma 1. By using (19) and (2) we compute
\[
X_{k+1} = (A^{b_k} - T_{h_k} L_{h_k} C) X_k (A^{b_k} - T_{h_k} L_{h_k} C)^\top
\]
\[+ \int_{t_k}^{t_{k+1}} e^{A(t_{k+1} - \tau)} G G^\top e^{A^\top(t_{k+1} - \tau)} d\tau
\]
\[+ T_{h_k} L_{h_k} H H^\top L_{h_k}^\top T_{h_k}. \quad (22)
\]
By changing the variable as \( \tau' = t_{k+1} - \tau \), the second and third terms of the right-hand side are equal to
\[
\int_{0}^{h_k \Delta} e^{A\tau} G G^\top e^{A^\top \tau} d\tau = Q_{h_k}
\]
and \( R_{h_k} \), respectively; see definitions (14), (15). Hence (18) is obtained. □

We should note that (18) depends upon signal arrival pattern (7). We need to consider every possible pattern when evaluating the estimation error. This is achieved by means of a common solution of Lyapunov inequalities as follows.

**Lemma 2.** Let \( X > 0 \) be such that
\[
(A^{b_k} - T_{h_k} L_{h_k} C) X (A^{b_k} - T_{h_k} L_{h_k} C)^\top - X + Q_{h_k} + R_{h_k} \prec 0 \quad \text{for } h = 1, \cdots, \hat{h} + 1.
\]
(23)
For any signal arrival pattern (7), define the error covariance matrix \( X_k \) by (17). Then
\[
X_k \prec X \quad \text{for all } k \in \mathbb{N}
\]
and hence
\[
\limsup_{k \to \infty} \mathbb{E} \left[ \|\varepsilon(t_k)\|^2 \right] < \text{tr} X. \quad (24)
\]

**Proof.** We subtract (18) from (23) with substituting \( h = h_k \) and obtain
\[
F_k (X - X_k) F_k^\top - (X - X_{k+1}) \prec 0, \quad k \in \mathbb{N},
\]
where
\[
F_k = A^{b_k} - T_{h_k} L_{h_k} C.
\]
Hence we have
\[
X - X_{k+1} \succ F_k (X - X_k) F_k^\top \succ \cdots \succ F_k \cdots F_0 X F_0^\top \cdots F_{\hat{k}}^\top \geq 0.
\]
Here we have used \( X_0 = \mathbb{E} \left[ \varepsilon(0) \varepsilon(0)^\top \right] = 0 \).

The above matrix inequality means that \( X > X_k \) for any \( k \in \mathbb{N} \). By properties of the trace, we obtain
\[
\text{tr} X > \text{tr} X_k = \mathbb{E} \left[ \|\varepsilon(t_k)\|^2 \right].
\]
Hence (24) holds by the above bound. □
We are ready to give our main result.

**Theorem.** Assume that LMI
\[
\begin{pmatrix}
-Y & Y A^h & -K_h C & Y Q_{h}^{1/2} & -K_h H \\
* & -Y & 0 & 0 \\
* & * & -I_n & 0 \\
* & * & * & -I_p
\end{pmatrix} < 0,
\]
\(h = 1, \ldots, \bar{h} + 1\)

(25)

have solutions \(W, Y, K_1, \ldots, K_{\bar{h} + 1}\). Then the switching state estimator (9) with the gains
\[L_0 = \Gamma^{-1} Y^{-1} K_1,\]
\[L_{h-1} = \Gamma^{-1} Y^{-1} K_h - A^{h-1} L_0 - \cdots - A L_{h-2},\]
\(h = 2, \ldots, \bar{h}\)

satisfies the performance (10) for any signal arrival pattern (7).

**Proof.** By (27) we have
\[Y^{-1} K_h = \Gamma (A^{h-1} L_0 + \cdots + L_{h-1}) = T_h L_h, \quad h = 1, \ldots, \bar{h}.
\]

Put \(X = Y^{-1}\). Then (25) becomes
\[
\begin{pmatrix}
-X^{-1} & X^{-1} (A^h - T_h L_h C) & X^{-1} Q_{h}^{1/2} & * & * \\
* & -X^{-1} & 0 & * & * \\
* & * & -I_n & * & * \\
* & * & * & -I_p \\
0 & 0 & 0 & 0 & -I_p
\end{pmatrix} < 0.
\]

Then by Schur complement we have
\[
X^{-1} (A^h - T_h L_h C) X (A^h - T_h L_h C) X^{-1} - X^{-1} + X^{-1} (Q_h + R_h) X^{-1} < 0.
\]

Pre- and postmultiplying this matrix inequality both by \(X\), we obtain (23). By (26) we have \(\text{tr} \ X < \gamma^2\). Hence (10) is satisfied by Lemma 2. \(\square\)

4 Numerical Simulation

Numerical simulation has been carried out to illustrate the effectiveness of the proposed method. We put
\[A = \begin{pmatrix} 0 & -0.01 \\ 1 & -0.1 \end{pmatrix}, \quad G = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\]
\[C = \begin{pmatrix} 0 & 1 \end{pmatrix}, \quad H = 0.1\]
in (1), \(\Delta = 0.1\), and \(\bar{h} = 2\). LMI (25) and (26) are solved for \(\gamma = 0.8\), which is as small as possible (in fact, the LMI are unsolvable for \(\gamma = 0.7\)). Then switching gains \(L_0, L_1, L_2\) are computed by Theorem 1.

The proposed method is effective for any signal loss pattern where the maximal length of successive loss is \(\bar{h}\) (or smaller). For simplicity, we take a specific loss pattern that repeats RLRLL (with R, L denoting received and lost, respectively) for simulation; see Table 1 for the resulting receiving times.

Fig. 4 illustrates sample paths of the state variables and their estimations (i.e., single trial). Note that the horizontal axis represents number \(k\), not real time \(t\). It appears that the estimation by the proposed method successfully follows the true values in spite of the signal loss, compared with the Kalman filter. This feature is further verified by Fig. 5 which indicates the mean square errors of the estimations under a thousand trials and theoretical bounds.

5 Concluding Remarks

In this paper we have assumed that the maximal number \(\bar{h} \in \mathbb{N}\) of successive signal loss is fixed and known. Such a hard bound is necessary for solving the problem theoretically, but it might sound inconvenient in practice. The authors will give a countermeasure against this drawback in another occasion.
Fig. 5: Mean square errors by a thousand trials for the both methods and their theoretical bounds

References

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