NON-EXISTENCE OF GLOBAL SOLUTIONS TO NONLINEAR
WAVE EQUATIONS WITH POSITIVE INITIAL ENERGY

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Abstract. We consider the Cauchy problem for nonlinear abstract wave equations in a Hilbert space. Our main goal is to show that this problem has solutions with arbitrary positive initial energy that blow up in a finite time. The main theorem is proved by employing a result on growth of solutions of abstract nonlinear wave equation and the concavity method. A number of examples of nonlinear wave equations are given. A result on blow up of solutions with arbitrary positive initial energy to the initial boundary value problem for the wave equation under nonlinear boundary conditions is also obtained.

1. Introduction. The paper is devoted to the question of nonexistence of global solutions of second order abstract wave equations of the form

\[ Pu_{tt} + Au + Qu_t = F(u), \]

under the initial conditions

\[ u(0) = u_0, \quad u_t(0) = u_1 \]

in a real Hilbert space \( H \) with the inner product \( \langle \cdot, \cdot \rangle \) and the corresponding norm \( \| \cdot \| \). Here \( A \) is densely defined selfadjoint positive definite operator in a Hilbert space \( H \), \( P \) is selfadjoint densely defined positive operator and \( Q \) is selfadjoint densely defined non-negative operator in \( H \) so that

\[ D(A) \subseteq D(Q), \quad D(A) \subseteq D(P). \]

\( F(\cdot) : D(A^{\frac{1}{2}}) \to H \) is a nonlinear gradient operator with the potential \( G(\cdot) \) which satisfies the condition

\[ (F(v), v) \geq 2(1 + 2\alpha)G(v) - 2R_0, \quad \forall v \in D \]

for some \( \alpha > 0, R_0 \geq 0 \).

As far as we know first result about nonexistence of a global solution of an evolution equation of the form (1) in a Hilbert space \( H \) is a global non-existence theorem obtained by using the energy method in [34] for the equation (1) with \( P = I, Q = 0 \) and a nonlinear gradient operator \( F(\cdot) \) with potential \( G(u) \) that satisfy the conditions

\[ (F(u), u) \geq \nu G(u), \quad (F(u), u) \geq G(\|u\|^2), \quad \forall u \in D(A^{\frac{1}{2}}), \]

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where $\nu > 2$ is a given number and a continuous function $G : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the condition
\[
\int_{v_0}^{\infty} \frac{dv}{2(1 - 2\nu)G(v)} < \infty \text{ for some } v_0 > 0.
\]

One of the effective techniques which has been employed to demonstrate when a solution to a nonlinear partial differential equation blows up in a finite time is the concavity method. The idea of the concavity method introduced in [20] is based on a construction of some positive functional $\Psi(t) = \psi(u(t))$, which is defined in terms of the local solution of the problem (the local solvability of the problem is therefore required) and proving that the function $\Psi(t)$ satisfies the inequality (5) given in the following statement:

**Lemma 1.1 ([20]).** Let $\Psi(t)$ be a positive, twice differentiable function, which satisfies the inequality
\[
\Psi''(t)\Psi(t) - (1 + \alpha) \left| \Psi'(t) \right|^2 \geq 0, \quad t \geq t_0
\]
for some $\alpha > 0$. If $\Psi(t_0) > 0$ and $\Psi'(t_0) > 0$, then there exists a time $t_1 \leq t_0 + \frac{\Psi(t_0)}{\alpha \Psi'(t_0)}$ such that $\Psi(t) \to +\infty$ as $t \to t_1$.

The concavity method and its modifications were used in the study of various nonlinear partial differential equations (see e.g. [2, 11, 12, 14], [21]-[24], [15, 28, 30, 32, 33]). One of the main conditions on initial data guaranteeing nonexistence of a global solution of problems considered in these papers is negativity of the initial energy of the corresponding problem. In a number of papers employing the potential well theory introduced in [29] it is shown that some solutions of nonlinear wave equations with positive energy may blow up in a finite time.

In recent years considerable attention has been given to the question of global non-existence of solutions to initial boundary value problems for nonlinear wave equations with arbitrary positive initial energy.

The concavity method and its modifications are employed to find sufficient conditions of blow up of solutions with arbitrary large positive initial energy to the Cauchy problem and initial boundary value problems for nonlinear Klein-Gordon equation, damped Kirchhoff-type equation, generalized Boussinesq equation, quasilinear strongly damped wave equations and some other equations (see, e.g. [1, 4], [16]-[18], [25, 26, 35, 36] and references therein).

Our main goal is to show that non-existence of global solutions with arbitrary positive initial energy of the problem (1) actually can be established for wider class of nonlinear wave equations than equations considered in the preceding papers by using the Lemma 1.1 and a modification of the following theorem on growth of solutions of the problem (1), (2) with $Q = 0$.

**Theorem 1.2 ([20]).** Suppose that $P, A : D \to H$ are positive symmetric operators, $F(\cdot) : D \to H$ satisfies the condition
\[
(F(v), v) \geq 2(1 + 2\alpha)G(v), \quad \forall v \in D
\]
for some $\alpha > 0$, $u$ is a solution of the problem (1), (2) and the initial data satisfy the conditions
\[
(u_0, Pu_1)/(u_0, Pu_0) > 0, \quad \frac{1}{2}(u_0, Au_0) + \frac{1}{2}(Pu_1, u_1) - G(u_0) > 0,
\]
and
\[
\frac{1}{2}(u_0, Au_0) + \frac{1}{2}(Pu_1, u_1) - G(u_0) < \frac{1}{2}(u_0, Pu_1)^2/(u_0, Pu_0).
\]
Then,
\[
\lim_{t \to +\infty} (u(t), Pu(t)) = +\infty,
\]
if $u(\cdot)$ exists on $(0, +\infty)$.

2. Non-existence of global solutions to abstract wave equation. In this section we show that a version of the Theorem 1.2 due to [20] is valid also for the solutions of the damped equation (1). We show that if $u(t)$ is a global solution of the problem (1), (2) for initial data satisfying some conditions (including initial data with arbitrary positive initial energy), then it becomes unbounded as $t \to \infty$. Then by using this result we prove our main Theorem 2.2 about nonexistence of global solutions of the problem (1), (2).

**Theorem 2.1.** Suppose that the operators $P, A, Q, F : D \to H$ satisfy the conditions (3) and (4) and $u$ is a global solution (i.e. a solution defined on $(0, +\infty)$) of the problem (1), (2). Suppose also that the initial data satisfy one of the following conditions:

\[
B_0 := \frac{1}{2}(Au_0, u_0) + \frac{1}{2}(Pu_1, u_1) - G(u_0) + \frac{R_0}{1 + 2\alpha} < 0
\]

or

\[
B_0 \geq 0, \quad (u_0, Pu_1) > [2B_0(Pu_0, u_0)]^{\frac{1}{2}} + \frac{1}{2} \left[ \left( \frac{2 + 2\alpha}{1 + 2\alpha} \right)^{\frac{1}{2}} - 1 \right] (Pu_0, u_0).
\]

Then,
\[
\lim_{t \to +\infty} \left[ (Pu(t), u(t)) + \int_0^t (Qu(s), u(s)) ds \right] = +\infty.
\]

**Proof.** First, if we multiply the equation (1) by $u_t(t)$ in $H$, and integrate over time we get the energy identity
\[
E(t) = E(0) - \int_0^t (Qu_s(s), u_s(s)) ds, \quad t \geq 0,
\]
where as before the energy is given by
\[
E(t) = \frac{1}{2}(Pu_t(t), u_t(t)) + \frac{1}{2}(Au(t), u(t)) - G(u(t)).
\]
Set
\[
\Psi(t) := (Pu(t), u(t)) + \int_0^t (Qu(s), u(s)) ds + C_0,
\]
where $C_0 > 0$ is left to be chosen later. Then, we have
\[
\Psi'(t) = 2(Pu(t), u_t(t)) + (Qu(t), u(t)) - 2(u(t), Pu_t(t)) + 2 \int_0^t (Qu(s), u_s(s)) ds + (Qu_0, u_0)
\]

\[
= 2(u(t), Pu_t(t)) + 2 \int_0^t (Qu(s), u_s(s)) ds + (Qu_0, u_0)
\]

\[
= 2(u(t), Pu_t(t)) + 2 \int_0^t (Qu(s), u_s(s)) ds + (Qu_0, u_0)
\]

\[
= \frac{1}{2}(Pu_t(t), u_t(t)) - \frac{1}{2}(Pu_{tt}(t), u(t)) + \frac{1}{2}(Au(t), u(t)) - G(u(t)).
\]
and
\[ \Psi''(t) = 2(Pu_t(t), u_t(t)) + 2(u(t), P_{tt}(t) + Qu(t)). \] (15)
Employing the equation (1) in (15), utilizing the condition (4) and the definition of the energy (12) we obtain
\[ \Psi''(t) = 2(Pu_t(t), u_t(t)) - 2(Au(t), u(t)) + 2(u(t), F(u(t))) \geq 2(Pu_t(t), u_t(t)) - 2(Au(t), u(t)) + 4(1 + 2\alpha)E(u(t)) - 4R_0 \]
\[ = 4(1 + \alpha)(Pu_t(t), u_t(t)) + 4\alpha(Au(t), u(t)) - 4(1 + 2\alpha)E(t) - 4R_0. \]

Thanks to the energy identity (11) we deduce from this inequality the following one:
\[ \Psi''(t) \geq 4(1 + \alpha)(Pu_t(t), u_t(t)) + 4\alpha(Au(t), u(t)) + 4(1 + 2\alpha) \int_0^t (Qu_s(s), u_s(s))ds - 4(1 + 2\alpha)B_0. \] (16)
First suppose that the condition (8) holds true. Then, the last inequality implies that
\[ \Psi''(t) \geq -4(1 + 2\alpha)B_0 > 0, \quad \forall t > 0. \]
Therefore \( \Psi(t) \to +\infty \) and \( \Psi'(t) \to +\infty \) as \( t \to +\infty \).
Suppose now that (9) holds true. Then, upon multiplying the inequality (16) by \( \Psi(t) \geq 0, \) in view of the equality (14), we may write
\[
\Psi''(t)\Psi(t) - (1 + \alpha) [\Psi'(t)]^2 
\geq 4(1 + \alpha) \left[ (Pu_t(t), u_t(t)) + \int_0^t (Qu_s(s), u_s(s))ds + \frac{(Qu_0, u_0)^2}{4C_0} \right] \Psi(t)
- 4(1 + \alpha) \left[ (u(t), Pu_t(t)) + \int_0^t (Qu(s), u_s(s))ds + \frac{(Qu_0, u_0)^2}{2} \right]^2
+ 4\alpha(Au(t), u(t))\Psi(t) - \left[ \frac{1 + \alpha}{C_0} (Qu_0, u_0)^2 + 4(1 + 2\alpha)B_0 \right] \Psi(t). \] (17)
In view of the definition (13) of \( \Psi(t) \), by an application of the Cauchy-Schwarz inequality for the inner product
\[ \langle [z_1, v_1, r_1], [z_2, v_2, r_2] \rangle = \langle Pu_1, u_1 \rangle + \int_0^t (Pu_1, u_1)d\tau + r_1 r_2 \]
with \( [z_1, v_1, r_1] = [u(t), u(t), c_0] \) and \( [z_2, v_1, r_1] = [u(t), u(t), c_0] \) we obtain that, the difference composed of the first two terms on the right hand side of (17) is nonnegative. Consequently, we deduce that
\[ \Psi''(t)\Psi(t) - (1 + \alpha) [\Psi'(t)]^2 \geq 4\alpha(Au(t), u(t)) - D_0 \Psi(t) \geq -D_0 \Psi(t), \] (18)
where
\[ D_0 = (1 + \alpha)C_0^{-1}(Qu_0, u_0)^2 + 4(1 + 2\alpha)B_0. \] (19)
Following [20] we define the positive functional \( H(t) := \Psi^{-\alpha}(t) \) and multiplying the inequality (18) by \( -\alpha \Psi^{-2-\alpha}(t) \) we see that \( H(t) \) satisfies
\[ H''(t) \leq \alpha D_0 H^{-\frac{\alpha + 1}{\alpha}}(t), \quad \forall t \geq 0. \] (20)
Now, suppose that
\[ \Psi'(0) > (2D_0/(1 + 2\alpha))^{\frac{1}{2}} \Psi^\frac{1}{2}(0) \] (21)
which translates in terms of $H$ as

$$\frac{H'(0)}{\alpha} < -\alpha \left(2D_0/(1 + 2\alpha)\right)^{\frac{1}{2}} H^{\frac{1+2\alpha}{2\alpha}}(0) < 0.$$  \hspace{1cm} (22)

We claim that $H'(t) < 0$ for all $t \geq 0$. If not, then there is a smallest time $\delta > 0$ such that

$$H'(\delta) = 0, \quad \text{and} \quad H'(t) < 0, \quad \forall t \in [0, \delta).$$

Hence, upon multiplying the inequality (20) by $H'(t)$ we deduce the inequality

$$\frac{d}{dt} \left[ H'(t) \right]^2 \geq D_1^2 \frac{d}{dt} H^{\frac{1+2\alpha}{2\alpha}}(t), \quad \forall t \in [0, \delta),$$

where $D_1^2 := 2\alpha^2 D_0/(1 + 2\alpha)$. Integrating this inequality and rearranging we get

$$(H'(t) - D_1 H^{\frac{1+2\alpha}{2\alpha}}(t)) (H'(t) + D_1 H^{\frac{1+2\alpha}{2\alpha}}(t)) \geq \left( H'(0) - D_1 H^{\frac{1+2\alpha}{2\alpha}}(0) \right) \left( H'(0) + D_1 H^{\frac{1+2\alpha}{2\alpha}}(0) \right), \quad \forall t \in [0, \delta).$$

Thanks to the condition (22) the right hand side of the inequality (23) is strictly positive. Hence, the left hand side must be strictly positive for each $t \in [0, \delta]$. This contradicts our assumption $H'(\delta) = 0$. Thus, we conclude that

$$H'(t) < 0, \quad \forall t \geq 0.$$  \hspace{1cm} (23)

This, in turn, implies that the inequality (23) is valid for all $t \geq 0$, and the left hand side of the inequality (23) is strictly positive for all $t \geq 0$. Also note that, as $H'(t) < 0$, the first term in brackets on the left hand side of the inequality (23) is strictly negative. Thus, we conclude that the second term in brackets is also strictly negative for all times, that is we have

$$H'(t) < -\alpha \sqrt{\frac{2D_0}{1 + 2\alpha}} H^{\frac{1+2\alpha}{2\alpha}}(t), \quad \forall t \geq 0.$$  \hspace{1cm} (24)

Translating back this inequality equivalently reads

$$\frac{1}{2} \Psi'(t) > \alpha \sqrt{\frac{2D_0}{1 + 2\alpha}} \Psi^{\frac{1}{2}}(t), \quad \forall t \geq 0.$$  \hspace{1cm} (25)

This inequality implies that, $\Psi(t)$ grows as $t^2$, and $\Psi'(t)$ grows as $t$, as time increases. Let us note that, only the right hand side of the last inequality involves the positive parameter $C_0$. Therefore, we choose

$$C_0 = \sqrt{1 + \alpha (Qu_0, u_0)(Pu_0, u_0)^2 / [4(1 + 2\alpha)B_0]^{\frac{1}{2}}},$$

so that the right hand side of (21) is minimal. It is easy to check that, with this choice of $C_0$, the condition (21) is equivalent to the second condition in (9). The first statement of the theorem, that is

$$\Psi(t) \rightarrow +\infty \quad \text{and} \quad \Psi'(t) \rightarrow +\infty \quad \text{as} \quad t \rightarrow \infty,$$

easily follows from the inequality (24). This finishes the proof. \hfill \square

Next, suppose that $u(t)$ is a global solution to the problem (1), (2) with initial data $[u_0, u_1]$, that satisfy the condition (8). From the inequality (16) we immediately deduce that

$$\frac{1}{2} \frac{d}{dt} \Psi(t) \geq -4(1 + 2\alpha)B_0 > 0, \quad \forall t \geq 0.$$  \hspace{1cm} (26)

This implies the statement of the theorem.
**Theorem 2.2.** Suppose that the operators $P, Q, A$ and $F$ satisfy all the conditions of Theorem 2.1. Let $u(t)$ be the solution to the problem (1)-(2) with initial data satisfying one of conditions (8) or (9). Further suppose that, there exists $a_0, a_1 > 0$ such that

$$ (Av, v) \geq a_0 (Pv, v), \quad (Av, v) \geq a_1 (Qv, v), \quad \forall v \in D. \quad (26) $$

Then, there exists $t_1 > 0$ such that

$$ \lim_{t \to t_1^+} \left[ (Pu(t), u(t)) + \int_0^t (Qu(s), u(s)) ds \right] = +\infty. \quad (27) $$

Moreover, there are infinitely many initial data $[u_0, u_1]$ with arbitrary large initial energy $E(u_0, u_1)$ for which the corresponding solutions blow up in a finite time, i.e. (27) holds true for some finite $t_1 > 0$.

**Proof.** Let $u(t)$ be a local strong solution of the problem (1),(2) i.e. a solution $u$ for which all terms in (1) are elements of $L^2(0, T; H)$.

For local existence of weak and strong solutions of a Cauchy problem for second order evolution equations of the form (1), we refer, e.g. to [3][5] [13] and references therein. Let us note that under some natural restrictions on the nonlinearity and its potential, local solvability of the problem (1),(2) can be established by Galerkin method (see e.g. [5]).

Suppose that the solution $u(t)$ to the problem (1)-(2) with the chosen initial data is global. Again, set

$$ \Psi(t) = (Pu(t), u(t)) + \int_0^t (Qu(s), u(s)) ds + C_0, $$

where $C_0 > 0$ is chosen as in the proof of Theorem 2.1. Consequently, the inequality (18) derived in the proof of Theorem 2.1, which reads as

$$ \Psi''(t) \Psi(t) - (1 + \alpha) [\Psi'(t)]^2 \geq 4\alpha (Au(t), u(t)) − D_0 \Psi(t), $$

is valid for each $t \geq 0$. Moreover, by the statement of Theorem 2.1, and the conditions (26) we see that

$$(Au(t), u(t)) \to +\infty, \quad as \ t \to +\infty. $$

Hence, we deduce from the previous inequality that, there exists $t_0^* \geq 0$ such that, for all $t \geq t_0^*$ we have

$$ \Psi''(t) \Psi(t) - (1 + \alpha) [\Psi'(t)]^2 \geq 0, $$

which is the main inequality in the assumptions of the Lemma 1.1. Moreover, as stated by Theorem 2.1, we also have that $\Psi'(t) \to +\infty$ as $t \to +\infty$, which implies that we can find $t_0 \geq t_0^*$ such that $\Psi'(t_0) > 0$. Finally, both of the assumptions (8) and (9) on the initial data necessarily imply that $u_0 \neq 0$, which ensures by our choice (25) of $C_0$ that, $\Psi(t) \geq C_0 > 0$ for any $t \geq 0$, in particular $\Psi(t_0) > 0$. Hence, we see that all assumptions of the Lemma 1.1 are satisfied after the time $t = t_0$, and consequently $\Psi(t) \to +\infty$ in a finite time. This contradicts to our initial assumption that the solution was global. It remains to show that the last statement of the theorem holds true. Suppose that $u_0$ is an arbitrary element of $D$ such that

$$ U(u_0) = \frac{1}{2} (Au_0, u_0) − G(u_0) + \frac{R_0}{1 + 2\alpha} < 0. $$

Note that, the last condition is necessary to have the condition (9).
Let us take $u_1 = \eta u_0 + \mu v$, where $\eta > 0, \mu > 0$ are arbitrary numbers and $v$ is an element of $D$ such that $(Pu_0, u_0) = 0$.

The condition (7) for this choice of $u_1$ takes the form

$$\eta (Pu_0, u_0) > \left\{ \eta^2 (Pu_0, u_0) + \mu^2 (Pv, v) + 2U(u_0) \right\}^{\frac{1}{2}},$$

which can be seen to be equivalent to

$$0 > \mu^2 (Pv, v) + 2U(u_0).$$

In view of (29) this inequality is satisfied for $\mu^2$ small enough. Moreover, the initial energy has the form

$$E(u_0, u_1) = \frac{\eta^2}{2} (Pu_0, u_0) + \frac{\mu^2}{2} (Pv, v) + \frac{1}{2} (Au_0, u_0) - G(u_0),$$

and it is clear that $E(u_0, u_1)$ can take arbitrary large values for $\eta$ large enough.

Finally, let us note that the condition (29) is satisfied, for instance, if the potential $G(\cdot)$ satisfies the conditions

$$G(0) = 0, \ G(\lambda u) \geq \lambda^{4\alpha+2} G(u), \ \forall \lambda \in \mathbb{R}^+, \ \forall u \in D.$$

Clearly (29) is satisfied for $u_0 = \lambda w$ with $\lambda \gg 1$ and $w \in D$ such that $G(w) > 0$. \qed

\textbf{Remark 1.} It is also worth mentioning that, the virtue of the conditions (9) of the Theorem 2.2 lies in the fact that, it provides blowing up solutions with initial data having arbitrary large positive initial energies. For small positive initial energies however, this condition can not recapture what is already known, for instance, for the problem

$$\begin{cases}
  u_{tt} - \Delta u + mu = w^3, & x \in \Omega, \ t > 0, \\
  u(x, t) = 0, & x \in \partial \Omega, \ t > 0, \\
  u(x, 0) = u_0(x), & u_t(x, 0) = u_1(x), \ x \in \Omega,
\end{cases} \quad (30)$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial \Omega$. The potential well theory gives a full characterization of the behaviour of the solutions in terms of global existence/nonexistence for initial energies less than the depth of the potential well. For example, take $R_0 = 0$ and suppose that the initial data verify $E(0) = 0$. In this case we know from the potential well theory that, all nonzero solutions must blow up in finite time (see [29]), whereas the conditions (9) only contains initial data that satisfy the condition $(u_0, u_1) > 0$.

3. \textbf{Examples of nonlinear wave equations.}

\textbf{Example 1. Nonlinear Klein-Gordon equation.} Let $u$ be a local strong solution to the Cauchy problem

$$\begin{cases}
  u_{tt} - \Delta_x u + m^2 u = u^3, & x \in \mathbb{R}^3, \ t > 0, \\
  u(x, 0) = u_0(x), & u_t(x, 0) = u_1(x),
\end{cases} \quad (31)$$

where $m > 0$ is a given number, $u_0 \in H^1(\mathbb{R}^3), u_1 \in L^2(\mathbb{R}^3)$ are given compactly supported functions.

The equation (31) can be written in the form (1) with $P = I, \ A = -\Delta_x + m^2 I$ and $F(u) = |u|^3 u$. It is easy to see that, this nonlinearity satisfies the condition (4) with $\alpha = \frac{4}{3} > 0$ and $R_0 = 0$. Moreover, the condition (26) holds with $a_0 = m^2$. Thus, it follows from Theorem 2.2 that, if the initial energy
\[ E(u_0, u_1) = \frac{1}{2} \|u_1\|^2 + \frac{1}{2} \|\nabla u_0\|^2 + \frac{m^2}{2} \|u_0\|^2 - \frac{1}{4} \|u_0\|_{L^4}^4 \]

is nonnegative, and
\[ (u_0, u_1) > \left[ 2E(u_0, u_1) \right]^{\frac{1}{2}} \|u_0\|, \quad (32) \]
then the solution of the problem (31) can exist only on a finite interval \([0, T)\) and\(^1\)
\[ \lim_{t \to T^-} \|u(t)\| = +\infty. \]

**Remark 2.** The Theorem (2.2) holds true also for solutions of the initial boundary value problem for the nonlinear wave equation under the homogeneous Dirichlet boundary condition:
\[
\begin{aligned}
&\left\{ \begin{array}{ll}
  u_{tt} - \Delta_x u + m^2 u = \|u\|_p u, & x \in \Omega, t > 0, \\
  u(x, 0) = u_0(x), & u_t(x, 0) = u_1(x), & x \in \Omega, \\
  u(x, t) = 0, & x \in \partial \Omega, t > 0,
\end{array} \right.
\end{aligned} \tag{33}
\]
where \(\Omega \subset \mathbb{R}^n\) is a bounded domain with smooth boundary \(\partial \Omega\), \(p\) is an arbitrary positive number if \(n = 1, 2\) and \(p \in (0, \frac{n}{n-2})\) if \(n \geq 3\).

Let us note that this result easily follows from the results of T. Cazenave obtained in [8] for solutions of the problem (33) and the Theorem 1.2 of H. A. Levine.

Indeed, T. Cazenave proved that each solution of the problem (33) either blows up in a finite time or is uniformly bounded.

Thus, if the functions \(u_0, u_1\) satisfy the conditions of Theorem 1.2, that is
\[ (u_0, u_1) > \left[ \|u_1\|^2 + \|\nabla u_0\|^2 + m^2 \|u_0\|^2 - \frac{2}{p+2} \|u_0\|_{L^{p+2}}^{p+2} \right]^{\frac{1}{2}} \|u_0\|, \]

then the corresponding local solution of the problem (33) can not be continued on the whole interval \([0, \infty)\), i.e. it must blow up in a finite time. For results on local solvability of the Cauchy problem and initial boundary value problems for semilinear Klein - Gordon equations see, e.g., [7].

**Example 2. Generalized Boussinesq equation.** Similarly we can find sufficient conditions for blow up of solutions with arbitrary positive initial energy for the generalized Boussinesq equation
\[ u_{tt} - \Delta u_{tt} - \Delta u + \nu \Delta^2 u + \Delta f(u) = 0, \quad x \in \Omega, t > 0 \tag{34} \]
under the homogeneous Dirichlet boundary conditions
\[ u = \Delta u = 0, \quad x \in \partial \Omega, t > 0, \]
where \(f(u) = |u|^m u + P_{m-1}(u), \ m \geq 1\) is a given integer, \(\nu > 0\) is a given number, \(\Omega \subset \mathbb{R}^n\) is a bounded domain and \(P_{m-1}(u)\) is a polynomial of order \(\leq m - 1\). Applying \((-\Delta)^{-1}\) to (34), where \(-\Delta\) is the Laplace operator under the homogeneous Dirichlet boundary condition, we obtain an equation of the form (1) with \(P = (-\Delta)^{-1} + A\), \(A = I - \nu \Delta\) and \(F\) replaced by \(f\). It is easy to see that, there exists a nonnegative number \(R_0\) such that \(f\) satisfies (4) with \(G(u) = \int_0^u f(s)dsdx\) and \(\alpha = \frac{m}{4}\). Since \(\Omega\) is bounded, the Poincaré inequality assures that the assumption

\(\^1\)Here and in what follows we are using the notations \(\| \cdot \|\) and \((\cdot, \cdot)\) also for the norm and inner product of \(L^2\) respectively, and \(\| \cdot \|_p\) is used for the norm of \(L^p\).
(26) is verified. Hence, the conclusion of Theorem 2.2 holds provided that the assumptions of Theorem 1.2 are fulfilled, that is $u_0, u_1$ satisfy

$$(Pu_0, u_1) > 0, \quad \frac{(Pu_0, u_1)^2}{(Pu_0, u_0)} > 2E(u_0, u_1) + \frac{2R_0}{1+m},$$

where

$$E(u_0, u_1) = \frac{1}{2}(Pu_1, u_1) + \frac{1}{2} \|u_0\|^2 + \frac{\nu}{2} \|\nabla u_0\|^2 - \int_{\Omega} \int_{\Omega} f(s) ds dx.$$  

**Example 3. Nonlinear plate equations.** It is clear that we can apply Theorem 2.2 to find sufficient conditions of blow up of solutions to initial boundary value problems for the nonlinear plate equations of the form

$$u_{tt} + \Delta^2 u + \left(a + b \int_{\Omega} u_{x_1}^2 dx\right) u_{x_1} + \left(c + d \int_{\Omega} u_{x_2}^2 dx\right) u_{x_2} = 0, \quad x \in \Omega, t > 0,$$

and

$$u_{tt} + \Delta^2 u = f(u), \quad x \in \Omega, t > 0,$$

under the boundary conditions

$$u = \Delta u = 0, \quad x \in \partial \Omega,$$

where $f(\cdot) : \mathbb{R} \to \mathbb{R}$ is a continuous function which satisfies the condition

$$f(s) s - 2(1 + 2\alpha) F(s) \geq -r_0, \quad \forall s \in \mathbb{R}, \quad (35)$$

where $r_0 \geq 0, a, c \in \mathbb{R}, b > 0, d > 0$ are given numbers, $F(s) := \int_0^s f(\tau) d\tau$ and $\Omega \subset \mathbb{R}^2$ is a bounded domain with sufficiently smooth boundary $\partial \Omega$.

**Example 4. Sixth order Boussinesq equations.** Applying Theorem 2.2 we can obtain similar results on blow up of solutions to initial boundary value problem for

- sixth order generalized Boussinesq equation

  $$u_{tt} - \Delta u + \Delta^2 u - \Delta^3 u + \Delta f(u) = 0,$$

  and

- improved Boussinesq equation

  $$u_{tt} - \Delta u_{tt} + \Delta^2 u + \Delta f(u) = 0,$$

under homogeneous Dirichlet boundary conditions. Here $f(\cdot)$ is a smooth function which satisfies the condition (35).

Theorem 2.2 allows us to find sufficient conditions for blow up of solutions with arbitrary positive initial energy of the following problems

- Cauchy problem and initial boundary value for system of nonlinear Klein-Gordon equation

  $$\begin{cases} 
  u_{tt} - \Delta u + mu + bu_t = uu^2 + h_1(x), \\
  v_{tt} - \Delta v + \mu v + \beta v_t = vv^2 + h_2(x), 
  \end{cases}$$

  where $h_1, h_2 \in L^2(\mathbb{R}^3)$ are given functions, $m, \mu, b, \beta$ are positive numbers.

- Initial boundary value problem under homogeneous Dirichlet boundary condition for the strongly damped wave equation

  $$u_{tt} - \Delta u - \Delta u_t = F(u) + |u|^p u + h(x), \quad p > 0,$$

where $P$ is some polynomial of order less than $p + 1$.  

(36)
• Initial boundary value problem under homogeneous Dirichlet boundary condition for nonlinear Dirichlet wave equation with structural damping term
\[
 u_{tt} - \Delta u + (-\Delta)^s u_t = f(u), \quad s \in \left(\frac{1}{2}, 1\right],
\]
where \( f : \mathbb{R} \to \mathbb{R} \) is a continuous function that satisfies the condition (35).

For local solvability of initial boundary value problem for equations (36) and (37) we refer to [6], where the authors employed the fact that the semigroups generated by corresponding linear problems are analytic (see [9]).

• Initial boundary value problem for quasilinear strongly damped wave equation of the form
\[
 \begin{cases}
 u_{tt} - \Delta u - \nabla (|\nabla u|^{m-2} \nabla u) - \Delta u_t = |u|^{p-2} u, & x \in \Omega, t > 0, \\
 u(x, t) = 0, & x \in \partial\Omega, t > 0, \\
 u(x, 0) = u_0(x), & u_t(x, 0) = u_1(x), \quad x \in \Omega,
\end{cases}
\]
where \( p > m \geq 2 \) are given numbers. This problem can be written as an abstract Cauchy problem (1), (2) in the Hilbert space \( H = L^2(\Omega) \) with
\[
 P = I, \quad A = -\Delta, \quad F(u) = \nabla (|\nabla u|^{m-2} \nabla u) + |u|^p u,
\]
\[
 G(u) = -\frac{1}{m} \|u\|_{L^m}^m + \frac{1}{p} \|u\|_{L^p}^p.
\]
Since \( p > m \geq 2 \), the condition (4) is satisfied with \( \alpha = \frac{p}{2} - \frac{1}{2} > 0 \) and \( R_0 = 0 \).

• Initial boundary value problem for nonlinear Love equation:
\[
 \begin{cases}
 u_{tt} - u_{xx} - u_{txx} - bu_{xxx} + cu_t = a|u_x|^{m-2}u_x + f(u), & x \in (0, 1), t > 0, \\
 u(0, t) = u(1, t) = 0, & t > 0, \\
 u(x, 0) = u_0(x), & u_t(x, 0) = u_1(x), \quad x \in (0, 1),
\end{cases}
\]
Here \( f(u) = |u|^{p-2} u + P(u) + h(x) \), \( P(u) \) is a polynomial of order \( < p - 1 \), \( h \in L^2(0, 1) \) is a given function, \( m > 2, p > 2, b > 0, c > 0, a > 0 \in \mathbb{R} \), are given numbers. We assume also that \( m < p \) when \( a > 0 \). This problem can be written as a Cauchy problem (1), (2) in the Hilbert space \( H = L^2(0, 1) \) with
\[
 P = I - \frac{d^2}{dx^2}, \quad A = -\frac{d^2}{dx^2}, \quad Q = c I - b \frac{d^2}{dx^2},
\]
\[
 F(u) = a|u_x|^{m-2}u_x + f(u), \quad G(u) = \frac{a}{m} \int_0^1 |u_x|^{m-2} dx + \frac{1}{p} \int_0^1 |u|^p dx.
\]
Employing Young’s inequality we can show that the condition (4) is satisfied with some \( \alpha > 0 \) and \( R_0 > 0 \).

Local solvability of the problem (38) as well as blow up of solutions of this problem with nonpositive initial energy when \( a < 0, f(u) = |u|^{m-2} u \) is discussed in [27].

Remark 3. Theorem 2.2 allows us to find sufficient conditions on initial data guaranteeing nonexistence of a global solution of many nonlinear PDE’s which can be written in the form
\[
 A^s u_{tt} + Au + b A^s u_t = F(u)
\]
in a Hilbert space \( H = L^2(\Omega) \), \( b \geq 0, \sigma \in [-1, 1], s \in [\sigma, 1] \), \( A \) is a positive definite operator generated by an elliptic operator under homogeneous Dirichlet’s boundary condition,
\[
 F(\cdot) : D(A^\frac{1}{2}) \to H \quad \text{is a gradient operator which satisfies the condition (4)}.
\]
**Example 5. Wave equation under nonlinear boundary conditions**

Consider the following problem

\[
\begin{align*}
      u_{tt} - \Delta u + bu &= 0, \quad x \in \Omega, \ t > 0, \\
      u(x, 0) &= u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega,
\end{align*}
\]

(39)

\[
\partial u \partial n = f(u), \quad x \in \partial \Omega, \ t > 0,
\]

(41)

where \( \Omega \subset \mathbb{R}^N \) is a bounded domain with sufficiently smooth boundary, \( b > 0 \) is a given number, \( \vec{n} \) denotes the outward directed normal to \( \partial \Omega \) and \( f(\cdot) : \mathbb{R} \rightarrow \mathbb{R} \) is nonlinear term that satisfies the condition (35). For the local solvability of the problem (39)-(41) we refer to the paper [19].

The energy equality in this case has the form

\[
E(t) := \frac{1}{2} \| u_t(t) \|^2 + \frac{1}{2} \| \nabla u(t) \|^2 + b \| u(t) \|^2 - \int_{\partial \Omega} F(u(x, t)) d\sigma = E(u_0, u_1),
\]

(42)

where

\[
E(u_0, u_1) = \frac{1}{2} \| u_1 \|^2 + \frac{1}{2} \| \nabla u_0 \|^2 + b \| u_0 \|^2 - \int_{\partial \Omega} F(u_0(x)) d\sigma.
\]

Set

\[
\Psi(t) = \| u(t) \|^2, \quad t \geq 0,
\]

where \( u(t) \) is a solution of the problem (39). Then employing the equation (39), the boundary condition (41) and the condition (35) we obtain

\[
\Psi''(t) \geq 2 \| u_t(t) \|^2 - 2 \| \nabla u(t) \|^2 - 2b \| u(t) \|^2 + 4(1 + 2\alpha) \int_{\partial \Omega} F(u(x, t)) d\sigma - 2R_0,
\]

where \( R_0 = \text{mes}(\partial \Omega) r_0 \).

Utilizing the energy equality (42) from the last inequality we obtain that

\[
\Psi''(t) \geq -4(1 + 2\alpha) E(u_0, u_1) - 2R_0 + 4(1 + \alpha) || u_t(t) ||^2 + 2(1 + 2\alpha) \| \nabla u(t) \|^2 + 4\alpha b \| u(t) \|^2.
\]

(43)

Employing (43), similar to the proof of the Theorem 1.2 we can show that if

\[
\frac{(u_0, u_1)}{\| u_0 \|^2} > 2E(u_0, u_1) + \frac{R_0}{1 + 2\alpha} > 0,
\]

(44)

then

\[
\Psi(t) = \| u(t) \|^2 \rightarrow +\infty \quad \text{as} \quad t \rightarrow +\infty.
\]

Finally arguing as in the proof of the Theorem 2.2 we get the inequality

\[
\Psi''(t) \Psi(t) - (1 + \alpha) |\Psi'(t)|^2 \geq (4\alpha b \Psi(t) - 4(1 + 2\alpha) E(u_0, u_1) - 2R_0) \Psi(t).
\]

Thanks to the last inequality and the Lemma 1.1 we proved the following

**Proposition 1.** If the conditions (44) are satisfied, then the interval of existence \([0, T)\) of solution to the problem (39)-(41) is finite. Moreover

\[
\| u(t) \| \rightarrow +\infty \quad \text{as} \quad t \rightarrow T^-.
\]

**Remark 4.** Proposition 1 holds true also for the equation

\[
\begin{align*}
      u_{tt} - \Delta u &= 0, \quad x \in \Omega, \ t > 0
\end{align*}
\]

with a nonlinear boundary conditions of the form

\[
\begin{align*}
      u(x, t) &= 0, \quad x \in \Gamma_1, \quad \frac{\partial u}{\partial n} = f(u), \quad x \in \Gamma_2, \ t > 0,
\end{align*}
\]
where $\Gamma_1 \cup \Gamma_2 = \partial \Omega$, $\text{mes}(\Gamma_1) \neq 0$ and the nonlinear term $f(\cdot)$ satisfies the condition (35). Notice that the first result on blow up of solutions of the equation (39) under nonlinear boundary conditions (41) with negative initial energy was obtained in [22], in [24] it is shown that there are solutions with positive initial energy that blow up in a finite time.

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REFERENCES

[1] A. B. Aliyev and A. A. Kazimov, Global non-existence of solutions with fixed positive initial energy of the Cauchy problem for a system of Klein-Gordon equations, *Differ. Equ.*, **51** (2015), 1563–1568.

[2] A. B. Alshin, M. O. Korpusov and A. G. Sveshnikov, *Blow up in Nonlinear Sobolev Type Equations*, De Gruyter Series in Nonlinear Analysis and Applications, **15**, Walter de Gruyter and Co., Berlin, 2011. xii+648 pp.

[3] P. Aviles and J. Sandefur, Nonlinear second order equations with applications to partial differential equations, *J. Differential Equations*, **58** (1985), 404–427.

[4] B. A. Bilgin and V. K. Kalantarov, Blow up of solutions to the initial boundary value problem for quasilinear strongly damped wave equations, *J. Math. Anal. Appl.*, **403** (2013), 89–94.

[5] E. H. de Brito, Nonlinear initial-boundary value problems, *Nonlinear Anal.*, **11** (1987), 125–137.

[6] A. N. Carvalho, J. W. Cholewa and T. Dlotko, Strongly damped wave problems: bootstrapping and regularity of solutions, *J. Differential Equations*, **244** (2008), 2310–2333.

[7] T. Cazenave and A. Haraux, *An Introduction to Semilinear Evolution Equations*, Oxford Lecture Series in Mathematics and its Applications, **13**, The Clarendon Press, Oxford University Press, New York, 1998, xiv+186 pp.

[8] T. Cazenave, Uniform estimates for solutions of nonlinear Klein-Gordon equations, *J. Funct. Anal.*, **60** (1985), 36–55.

[9] Sh. P. Chen and R. Triggiani, Proof of extensions of two conjectures on structural damping for elastic systems, *Pacific J. Math.*, **136** (1988), 15–55.

[10] F. Gazzola and M. Squassina, Global solutions and finite time blow up for damped semilinear wave equations, *Ann. Inst. H. Poincare Anal. Non Lineaire*, **23** (2006), 185–207.

[11] H. A. Erbay, S. Erbay and A. Erkip, Thresholds for global existence and blow-up in a general class of doubly dispersive nonlocal wave equations, *Nonlinear Anal.*, **95** (2014), 313–322.

[12] V. A. Galaktionov and S. I. Pohozaev, Blow-up and critical exponents for nonlinear hyperbolic equations, *Nonlinear Anal.*, **53** (2003), 453–466.

[13] S. J. Jakubov, Solvability of the Cauchy problem for abstract quasilinear hyperbolic equations of second order and their applications, *Trans. Moscow Math. Soc.*, **23** (1970), 36–59.

[14] V. K Kalantarov and O. A. Ladyzhenskaya, The occurrence of collapse for quasilinear equations of parabolic and hyperbolic type, *J. Soviet Math.*, **10** (1978), 53–70.

[15] R. J. Knops, H. A. Levine and L. E. Payne, Non-existence, instability, and growth theorems for solutions of a class of abstract nonlinear equations with applications to nonlinear elastodynamics, *Arch. Rational Mech. Anal.*, **55** (1974), 52–72.

[16] M. O. Korpusov, Blow-up of the solution of a nonlinear system of equations with positive energy, *Theoret. and Math. Phys.*, **171** (2012), 725–738.

[17] M. O. Korpusov, On the blow-up of solutions of a dissipative wave equation of Kirchhoff type with a source and positive energy, *Sb. Math.*, **52** (2011), 471–483.

[18] N. Kutev, N. Kolkovska and M. Dimova, Nonexistence of global solutions to new ordinary differential inequality and applications to nonlinear dispersive equations, *Math. Methods Appl. Sci.*, **39** (2016), 2287–2297.

[19] I. Lasiecka and A. Stahel, The wave equation with semilinear Neumann boundary conditions, *Nonlinear Anal.*, **15** (1990), 39–58.

[20] H. A. Levine, Instability and nonexistence of global solutions to nonlinear wave equations of the form $Pu_{tt} = −Au + F(u)$, *Trans. Am. Math. Soc.*, **192** (1974), 1–21.
[21] H. A. Levine, Some additional remarks on the nonexistence of global solutions to nonlinear wave equations, *SIAM J. Math. Anal.*, 5 (1974), 138–146.

[22] H. A. Levine and L. E. Paine, Nonexistence theorems for the heat equations with nonlinear boundary conditions and for porous medium equation backward in time, *J. Differential Equations*, 16 (1974), 319–334.

[23] H. A. Levine, S. R. Park and J. Serrin, Global existence and global nonexistence of solutions of the Cauchy problem for a nonlinearly damped wave equation, *J. Math. Anal. Appl.*, 228 (1998), 181–205.

[24] H. A. Levine and R. A. Smith, A potential well theory for the wave equation with a nonlinear boundary condition, *J. Reine Angew. Math.*, 374 (1987), 1–23.

[25] H. A. Levine and G. Todorova, Blow up of solutions of the Cauchy problem for a wave equation with nonlinear damping and source terms and positive initial energy, *Proc. Amer. Math. Soc.*, 129 (2001), 793–805.

[26] S. A. Messaoudi and B. Said-Houari, Global nonexistence of positive initial-energy solutions of a system of nonlinear viscoelastic wave equations with damping and source terms, *J. Math. Anal. Appl.*, 365 (2010), 277–287.

[27] L. T. Ngoc and N. T. Long, Existence, blow-up and exponential decay for a nonlinear Love equation associated with Dirichlet conditions, *Appl. Math.*, 61 (2016), 165–196.

[28] S. R. Park, Nonexistence of global solutions of some quasilinear initial-boundary value problems, *J. Korean Math. Soc.*, 34 (1997), 623–632.

[29] L. E. Payne and D. H. Sattinger, Saddle points and instability of nonlinear hyperbolic equations, *Israel J. Math.*, 22 (1975), 273–303.

[30] E. Pişkin and N. Polat, Existence, global nonexistence, and asymptotic behavior of solutions for the Cauchy problem of a multidimensional generalized damped Boussinesq-type equation, *Turkish J. Math.*, 38 (2014), 706–727.

[31] P. Pucci and J. Serrin, Global nonexistence for abstract evolution equations with positive initial energy, *J. Differential Equations*, 150 (1998), 203–214.

[32] B. Straughan, Further global nonexistence theorems for abstract nonlinear wave equations, *Proc. Amer. Math. Soc.*, 48 (1975), 381–390.

[33] B. Straughan, *Explosive Instabilities in Mechanics*, Springer, 1998.

[34] M. Tsutsumi, On solutions of semilinear differential equations in a Hilbert space, *Math. Japon.*, 17 (1972), 173–193.

[35] Y. Wang, A Sufficient condition for finite time blow up of the nonlinear Klein-Gordon equations with arbitrary positive initial energy, *Proc. Amer. Math. Soc.*, 136 (2008), 3477–3482.

[36] R. Zeng, Ch. Mu and Sh. Zhou, A blow-up result for Kirchhoff-type equations with high energy, *Math. Methods Appl. Sci.*, 34 (2011), 479–486.

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