Quantum clocks driven by measurement.

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(Dated: September 14, 2021)
Abstract

In classical physics, clocks are open dissipative systems driven from thermal equilibrium and necessarily subject to thermal noise. We describe a quantum clock driven by entropy reduction through measurement. The mechanism consists of a superconducting transmon qubit coupled to an open co-planar resonator. The cavity and qubit are driven by coherent fields and the cavity output is monitored with homodyne detection. We show that the measurement itself induces coherent oscillations, with fluctuating period, in the conditional moments. The clock signal can be extracted from the observed measurement currents and analysed to determine the noise performance. The model demonstrates a fundamental principle of clocks at zero temperature: good clocks require high rates of energy dissipation and consequently entropy generation.

I. INTRODUCTION

All clocks are constrained by the laws of thermodynamics that limit their stability and accuracy[1]. In many cases the clock is driven from thermal equilibrium by external forces that do work upon it thus increasing its free energy. A pendulum clock, for example, is driven by a continuous supply of work from a falling mass, while a quartz clock is driven by electrical power supplied to an astable oscillator. In such cases the non linearity inherent in the mechanism enables a stable limit cycle to form and counting oscillations on this cycle constitutes the clock.

At finite temperatures, phase diffusion on limit cycles constrains the accuracy of classical oscillatory clocks. The rate of this phase diffusion is proportional to temperature and inversely proportional to the rate of energy dissipation on the limit cycle. At very low temperatures the classical model is inadequate and a fully quantum mechanical model is required. In that case, the accuracy is still limited by phase diffusion on the limit cycle but the noise arises from spontaneous emission that opens the system to measurement. In a laser, for example, the rate of phase diffusion determines the laser line-width and is inversely proportional to the average number of photons, $\bar{n}$, in the cavity.

Doing work, $\Delta W > 0$, on the clock forces it from thermal equilibrium and increases its Helmholtz free energy by $\Delta W$. Alternatively we can increase the free energy by decreasing

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the entropy $\Delta S < 0$ of the clock by measurement, and in general we can do both $\Delta F = \Delta W - T\Delta S$. This provides an alternative route to implement a thermal clock driven by measurement, following the pioneering work of Ercker et al. [2].

A classical clock will fail as the temperature goes to zero. In the absence of noise, a system at a Hopf bifurcation cannot switch to the stable limit cycle. In any case as the temperature goes to zero we need a quantum description. In that case the Helmholtz free energy is not the appropriate quantity to describe a measurement driven clock. Instead one should replace $T\Delta S$ with the energy dissipated $E_{\text{dis}}$ which remains defined even at zero temperature (for example via spontaneous emission) and of course the energy dissipated implies entropy production in the environment. We show that measurement, by decreasing the entropy of a dissipative system in a local steady state, can be used to drive a clock. The measurement itself drives the system from its zero temperature steady state, thereby creating the quantum coherence responsible for (noisy) oscillations that can serve as a clock signal. We can say that the measurement itself, by extracting information, replaces the fuel necessary to power a work driven clock.

A simple harmonic oscillator, subject to dissipation, thermal noise and accurate measurements of position, provides an example of such a measurement driven clock. The ensemble averaged steady-state distribution of the system is simply the Gibbs-Boltzmann Gaussian distribution. However, if the position or momentum of the oscillator is continuously and accurately monitored in time, one sees a noisy oscillation at the oscillator frequency. The measurement itself pushes the system away from thermal equilibrium by decreasing its entropy. As shown in [3], it remains the case that such a clock will require large dissipation for a fixed quality factor, $Q$. At low temperatures the classical noise becomes small and the fluctuations responsible for driving the clock are disabled. As the temperature is reduced to zero the likely outcome of a position measurement approaches zero on the elliptic fixed point at the origin, and no oscillations will be seen.

Turning to the quantum description at low temperatures, we find that there is no such thermal limit: quantum noise will continue to drive the clock. In this case an essential role is played by quantum back-action noise arising from continuous observation of the clock variable, say position in the case of a simple harmonic oscillator. Quantum clocks are necessarily open systems as they implicitly assume a time series of discrete or continuous measurements. In the weak measurement regime the clock is driven by measurement alone, which restores
quantum coherence and thereby enables quantum coherent dynamics; the quantum noise here simply imparts phase noise rather than driving the clock itself. In the strong measurement regime the driven qubit operates as an incoherent clock via a combination of the Zeno effect and quantum noise.

In this paper we describe a quantum clock driven from thermal equilibrium by weak driving and continuous measurement that decreases the entropy, even for systems interacting with a zero temperature environment with energy dissipated through spontaneous emission. As we will demonstrate, it remains the case that accurate clocks require large dissipation. The proposal is based on a superconducting quantum circuit and transmon qubit.

Quantum clocks represent one of the most important objectives of quantum technology and state of the art atomic clocks have reached astonishing levels of accuracy\cite{4}. The complexity of these systems means that many diverse physical phenomenon limit their performance\cite{5} but at the bottom all are limited by quantum fluctuations and measurement noise (usually in a driving laser). This is the subject of this paper.

Time is a purely relation quantity and a clock is a device used to coordinate coincidences. The relational character of time is most evident in general relativity but it is universal. Consider three local events, $A$, $B$, $C$. If an event $A$ is coincident with event $B$ and event $A$ is coincident with event $C$, then event $B$ and $C$ are simultaneous \cite{6}. If event $A$ is the tick of a clock then we see how clocks coordinate coincidences.

In this view, clock time is dimensionless; an integer corresponding to a count. The necessary fluctuations that arise due to the dissipative nature of clocks implies that two identical clocks will not necessarily agree on the counts between two events. Suppose there are two local events, $B$ and $C$ that are not simultaneous in the rest frame of the clocks. Given two independent and identical local clocks let us agree that when event $B$ occurs both clocks are initialised with a count of zero. That is to say, the events $A_1$ and $A_2$, corresponding to a count of zero, are simultaneous with event $B$. Due to fluctuations in the period of each clock, a later, local event, $C$, will not necessarily be coincident with the same count of each clock. We discuss the implications of this in the final section of the paper.
II. A MEASUREMENT-DRIVEN QUBIT CLOCK

It is known that a continuously measured qubit can exhibit persistent but noisy Rabi oscillations or random quantum jumps [7–9] that are encoded in the measurement record. Here we propose to use this as the basis for either an oscillatory or non oscillatory quantum clock, respectively. We will show that, while necessarily noisy, the performance of the clock improves the more dissipative it is, even if the dissipation occurs at close to zero temperature.

To introduce the subject consider a very simple model; a weakly driven optical dipole in a two-level system (qubit) with spontaneous emission at rate $\gamma$. We will assume that this system is operating at zero temperature and thus fluctuations are due entirely to quantum noise not thermal noise. For simplicity we will also assume that all the radiation emitted from the dipole is collected (perhaps by using wave-guide resonator with a line width much greater than the spontaneous emission rate) and subject to homodyne or heterodyne detection.

Let the ground and excited states be denoted $|g\rangle$, $|e\rangle$ respectively and the atomic transition frequency is $\omega_a$. An external laser at resonant carrier frequency $\omega_L = \omega_a$ drives the qubit coherently and provides a local oscillator for homodyne or heterodyne detection. Ignoring measurement for now, the unconditional irreversible dynamics is described by the master equation in the interaction picture as

$$
\frac{d\rho}{dt} = -i\frac{\Omega}{2}[\sigma_x, \rho] + \gamma D[\sigma_-]\rho
$$

where $D[A]\rho = A\rho A - \frac{1}{2}(A\dagger A\rho + \rho A\dagger A)$, $\Omega$ is the Rabi frequency and $\gamma$ is the rate of spontaneous emission.

The state of the qubit at any time can be written as

$$
\rho(t) = \frac{1}{2}(1 + x(t)\sigma_x + y(t)\sigma_y + z(t)\sigma_z)
$$

where $x(t) = \langle \sigma_x(t) \rangle$ etc. Substituting this representation into the master equation Eq.(1) gives the well known equations of motion for the qubit. These are easily solved and in the case $\Omega > \gamma/4$ give damped oscillations for all variables, approaching a steady state in the long time limit given by

$$
\begin{align*}
  x_{ss} &= 0 \\
  y_{ss} &= -\frac{2\Omega/\gamma}{1 - 2\Omega^2/\gamma^2} \\
  z_{ss} &= -\frac{1}{1 - 2\Omega^2/\gamma^2}
\end{align*}
$$
Note that the maximum value of $z_{ss}$ is zero achieved in the limit of strong driving $\Omega \gg \gamma/\sqrt{2}$. This is called saturation. In this limit the average rate of energy dissipation is maximum and is given by $\hbar \omega A \gamma$.

Clearly a time-independent steady state does not suggest a clock. However, any real clock is necessarily a measured system and the actual dynamics of a quantum clock is conditioned upon the measurement results (due to the projection postulate). We now show that a clock signal does arise in this model if we consider such conditional dynamics of the qubit, that is, the dynamics conditioned upon the photocurrent obtained from homodyne detection of the radiated field. This is described by two coupled equations: the stochastic master equation for the conditional state and a classical stochastic equation for the homodyne current, $J(t)$\textsuperscript{[10]}.

These are given by

$$d\rho_c = -\frac{i}{2}[\sigma_x, \rho_c]dt + \gamma D[\sigma_-]\rho_c dt + \sqrt{\gamma}dW(t)\mathcal{H}[\sigma_-e^{i\phi}]\rho_c$$

$$J(t)dt = \gamma\langle \sigma_y \rangle_c dt + \sqrt{\gamma}dW(t)$$

where $\mathcal{H}[A]\rho = A\rho + \rho A^\dagger - \text{tr}(A\rho + \rho A^\dagger)\rho$, $dW(t)$ is the Wiener process and $\phi$ is the phase of the local oscillator. We will set this to $\phi = \pi/2$ so that the measured current is $J(t)dt = \gamma\langle \sigma_y \rangle_c dt + \sqrt{\gamma}dW$. The subscript $c$ indicates that $\rho_c$ is the conditional state and $\langle A \rangle_c = \text{tr}(A \rho_c(t))$. Using the Pauli matrix expansion (Eq.(2)) for the conditional density matrix we see that the stochastic master equation is equivalent to the system of Ito stochastic differential equations,

$$dx_c = -\frac{\gamma}{2}x_c dt - \sqrt{\gamma}dW x_c y_c$$

$$dy_c = -\Omega z_c dt - \frac{\gamma}{2}y_c dt + \sqrt{\gamma}dW(1 + z_c - y_c^2)$$

$$dz_c = \Omega y_c dt - \gamma(z_c + 1)dt - \sqrt{\gamma}dW(1 + z_c)y_c$$

The radial variable, $r_c^2 = x_c^2 + y_c^2 + z_c^2$, is equal to unity for a pure state. Using the Ito calculus one can verify that $r_c$ is constant and so, if the system starts in a pure state (as we assume), it remains in a pure state under stochastic conditional dynamics. No information is lost in an ideal homodyne measurement.

Averaging over the noise we obtain the equations that govern the unconditional evolution of the system. These are known as the optical Bloch equations and can be solved analytically\textsuperscript{[11]}. In the limit $\Omega \gg \gamma/2$ the unconditional dynamics is under-damped and exhibits damped oscillations in the $y-z$ plane, if we start in the ground state. Below we
discuss how an oscillatory (non-oscillatory) clock arises with this toy model in the weak (strong) measurement regimes.

The conditional dynamics due to homodyne detection is treated in \[10\]. The effect of measuring $\sigma_y$ tends to localise the system in this variable. The free Hamiltonian due to the driving then causes this state to rotate around the $x-$ axis. The noise due to the measurement conditioning leads to phase diffusion in this oscillation. This means that the homodyne current can be used to extract a noisy clock signal.

In Fig(1) we plot the conditional stochastic mean of $y_c$ versus time for two sample trajectories, starting in the ground state of the system. There is a discernible and persistent noisy oscillation. To extract a signal that can be used to count clock periods we compute

\begin{figure}[h]
\centering
\includegraphics[width=\columnwidth]{figure1.png}
\caption{Top: Two simulations of the conditional mean of $y_c(t)$. Bottom: the corresponding record of the clock signal sign($y_c(t)$). The parameters are: $\gamma = 1.0$, $\Omega = 10.0$.}
\end{figure}

the sign of the conditional mean $s(t) = \text{sign}(y_c)$. This is also plotted in Fig(1). There is a clear clock signal with noisy phase.

We can estimate the phase noise by moving to the dimensionless time $\tau = \Omega t$ and assume $\Omega \gg \gamma$. This enables us to neglect the damping in the systematic dynamics. Under those conditions we define $z_c = \cos \theta(\tau)$, $y_c = -\sin \theta(\tau)$ so that the stochastic differential equation for the phase becomes

$$d\theta = \tau - \sqrt{\frac{\gamma}{\Omega}}(1 + \cos \theta)dW(\tau)$$

The noise is non linear and goes to zero in the ground state but the diffusion matrix remains positive. We then average the diffusion matrix over one period to approximate the non
linear diffusion by linear diffusion. The resulting Ito stochastic differential for $\theta$ is then

$$d\theta \approx \tau + \sqrt{\frac{3\gamma}{2\Omega}} dW(\tau)$$

(12)

The average period is $2\pi$ and the noise is purely diffusive.

To estimate the probability distribution for the period we note that this problem is a first passage time problem: what is the probability for the phase to go from zero to $2\pi$ for the first time. The resulting probability distribution is the inverse Gaussian distribution or Wald distribution[12],

$$P(x) = \sqrt{\frac{\lambda}{2\pi x^3}} \exp \left[ -\frac{\lambda(x - \mu)^2}{2\mu^2x} \right]$$

(13)

where the mean and variance are given by

$$E(x) = \mu$$

$$Var[x] = \frac{\mu^3}{\lambda}$$

(14)

(15)

In our case $\mu = 2\pi$ and $\lambda = 8\pi^2\Omega/\gamma$ giving the variance in the clock period in terms of dimensionless time as $3\gamma\pi/\Omega << 2\pi$. This is valid in the limit that $\Omega >> \gamma$ in which case the fluctuations in the period become small. In the same limit we saw from the steady state analysis that the maximum rate of energy dissipation is achieved (saturation). This is a general feature: Good clocks require a high rate of average energy dissipation.

In summary we see that while the ensemble average of a weakly driven and damped two-level system simply decays to a steady state, the conditional state, conditioned on a homodyne measurement current, does not. The measurement results carry a noisy clock signal. The noise is due to measurement noise at zero temperature. Homodyne detection is a phase dependent measurement and self consistently this leads to a persistent noisy oscillation. The resulting fluctuations in the clock period scale as $\gamma/\Omega$ and thus a good clock results when the driving power is large and the rate of energy dissipation is maximised.

This simple model can be used to introduce some thermodynamic properties that will be important for the experimental scheme we discuss in this paper. The energy of the undriven qubit is given by the Hamiltonian $H_0 = \hbar\omega_a\sigma_z/2$. The coherent harmonic drive does work on the system and spontaneous emission causes energy to be dissipated into the environment. In the interaction picture the unconditional average rate of change of the energy of the system is given by

$$\frac{d\langle H_0(t) \rangle}{dt} = \frac{\hbar\omega_a}{2} (\Omega y - \gamma(z + 1))$$

(16)
The first term is the power applied by the driving field while the second term is the rate of energy dissipation. At steady state these terms cancel.

In the case of conditional dynamics there is no steady state and power supplied by the drive is dissipated over each noisy cycle of oscillation defined by $z_c(t) = z_c(t + T)$ where $T$ is a random variable. The rate of energy dissipation on a stochastic trajectory is given by

$$\dot{E}_{diss} = \frac{-\hbar \omega_A \gamma (1 + z_c(t))}{2}$$

This implies that the energy dissipated over a noisy cycle is a random variable related to the period, also a random variable, by

$$E_{diss}/\hbar \omega_A = \frac{\gamma T}{2}$$

As the period follows an inverse Gaussian distribution so too does the energy dissipated on each cycle. This energy is ultimately dissipated in the photo detectors and electronic circuits that comprise the homodyne detection system. In Fig. 2 we plot the energy dissipated as a function of time superimposed on the clock signal. The linear relationship between energy dissipated and clock period are clearly evident with a slope of $\gamma/2 = 1/2$.

There is another way to use a weakly driven two-level system to define a noisy clock based purely on quantum jumps between the states of the qubit. We call this a non-oscillatory clock. While the idea of a quantum clock based on random quantum jumps seems paradoxical, keep in mind that radio-carbon dating clocks are of this kind.

To enable a quantum jump clock the two-level system must be subject to continuous measurement of energy, that is to say, of $\sigma_z$. This requires a different irreversible coupling to
the environment that commutes with the system operator $\sigma_z$, leading to a master equation of the form

$$\frac{d\rho}{dt} = -i \frac{\Omega}{2} [\sigma_x, \rho] + \Gamma D[\sigma_z] \rho$$

The conditional master equation is given by \[13\]

$$d\rho_c = -i \frac{\Omega}{2} [\sigma_x, \rho] dt + \Gamma D[\hat{\sigma}_z] \rho_c dt + \sqrt{\Gamma} H[\hat{\sigma}_z] \rho_c dW(t)$$

and the corresponding (dimensionless) measurement current satisfies the Ito stochastic differential equation

$$dy(t) = \langle \hat{\sigma}_z(t) \rangle_c dt + (8\Gamma)^{-1/2} dW(t)$$

The Ito stochastic differential equations for the Bloch sphere variables now take the form

$$dx_c = -2\Gamma x_c dt - 2\sqrt{\Gamma} dW x_c$$

$$dy_c = -\Omega z_c dt - 2\Gamma y_c dt - 2\sqrt{\Gamma} dW y_c$$

$$dz_c = \Omega y_c dt + 2\sqrt{\Gamma} dW (1 - z_c^2)$$

In Fig.(3) we plot the conditional mean $z_c(t)$ subject to a low pass filter to simulate a real detector. Quantum jumps are clearly evident in what is an effective random telegraph signal with equal transition rates. In order to make a clock one simply counts the number of times the signal changes sign. The transition rate in the limit $\Gamma \gg \Omega$ can be found by adiabatically eliminating the coherence terms ($\langle \langle \langle e|\rho|g \rangle \rangle$) from the unconditional master equation. This implies that the occupation probabilities of the energy eigenstates approximately satisfy the classical master equation

$$\frac{dp_1}{dt} = -\frac{\Omega^2}{4\Gamma} p_1 + \frac{\Omega^2}{4\Gamma} p_0$$

FIG. 3. The conditional mean $z_c(t)$ (after a low pass filter) for a weakly driven two-level system subject to a continuous measurement of $\sigma_z$ in the quantum jump regime $\Gamma \gg \Omega$. Here $\Gamma = 8.0$, $\Omega = 1.0$. 
The transition rates are thus given by \( r = \frac{\Omega^2}{4\Gamma} \). For a given epoch \( T \) the mean and variance of the count \( N_T \) are both given by \( rT \). Given a count \( N \) we estimate an elapsed time of \( t = N/r \) in SI units. The relative error in this estimate goes to zero as \( r^{-1/2} \). This is like a quantum version of a Mach thermal clock[14].

Thus we see that this quantum jump clock exhibits the same energy-accuracy relation as the oscillatory quantum clock above: the limit of maximum energy dissipation (\( \Omega \gg \gamma \)) coincides with the highest clock accuracy. This is in spite of the fact that the quantum jump clock has no corresponding limit cycle.

Of course there are many examples of classical stochastic processes described by a master equation like Eq. (25). A double well model with thermal activation is an example if one only monitors which well the particle occupies. However in this case the transition rates go to zero as the temperature goes to zero unlike Eq. (25) in which the transition rates do not depend on temperature. They only depend on the signal-to-noise ratio of the measurement itself.

III. A SUPERCONDUCTING QUANTUM CLOCK.

Above we explained how the toy model of a coherently driven qubit with only a single amplitude damping channel represents a quantum clock if the damping channel is continuously measured. The clock oscillations/ticks are encoded in the current of the detector that is used for the continuous measurement. Here we model a superconducting circuit system that, while somewhat more complex than the single qubit model in the previous section, is experimentally amenable and gives rise to either a measurement-driven oscillatory or non-oscillatory quantum clock, depending on the strength of the measurement.

Consider a superconducting coplanar waveguide resonator (“cavity”) dispersively coupled to a transmon qubit. We model the cavity as a simple harmonic oscillator with amplitude damping only and the transmon qubit as a two-level system (TLS) also with amplitude damping only; we neglect the damping of the cavity through the drive port. The cavity and qubit are independently driven on resonance. The X quadrature of the cavity output is detected via homodyne detection. See Fig. [4].

In a frame rotating at the frequencies of the cavity drive and qubit drive the equation of
The homodyne measurement scheme used to extract the clock signal from the source cavity. A transmon qubit is dispersively coupled to a superconducting microwave cavity. Both the cavity and qubit are driven coherently. The output field is subject to a homodyne measurement, resulting in conditional dynamics of the cavity field that, with further signal processing, produces the clock signal for counting.

\[
d \rho_c = -i E [a + a^\dagger, \rho_c] dt - i \Omega [\sigma_x, \rho_c] dt - i \chi [a^\dagger a \sigma_z, \rho_c] dt \tag{26}
\]

\[
+ \gamma D[\sigma_-] \rho + \kappa D[a] \rho_c dt + \sqrt{\eta \kappa} H[a] \rho_c dW(t),
\]

where \( dW(t) \) is a Wiener processes, \( E \) is the cavity drive amplitude, \( \Omega \) is the qubit drive amplitude, \( \chi \) is the dispersive shift strength, \( \gamma \) is the qubit amplitude damping rate, \( \kappa \) is the cavity damping rate through the output port, and \( \eta \) is the efficiency of the photodetector. \( D[A] \) and \( H[A] \) are the superoperators defined in the previous section.

The measurement current for the homodyne signal is

\[
J_x(t) dt = \sqrt{\eta \kappa} \langle \hat{x}(t) \rangle_c dt + dW(t),
\]

where \( \hat{x} = a + a^\dagger \). This obviously introduces extra noise into the clock signal produced from \( \langle \hat{x}(t) \rangle_c \). An actual experiment would need to use a quantum noise limited parametric amplifier so that \( J_x(t) \) would instead follow \( \langle \hat{x}(t) \rangle_c \) with high fidelity. However the parametric amplifier scheme would yield the same dynamics for \( \langle \hat{x}(t) \rangle_c \) as the homodyne scheme. For simplicity we therefore proceed by considering \( \langle \hat{x}(t) \rangle_c \) under the homodyne scheme only.

The unconditional dynamics are obtained by averaging the conditional master equation over the noise. This gives

\[
\frac{d \rho}{dt} = -i E [a + a^\dagger, \rho] - i \Omega [\sigma_x, \rho] - i \chi [a^\dagger a \sigma_z, \rho] \tag{28}
\]

\[
+ \gamma D[\sigma_-] \rho + \kappa \rho_c D[a] \rho
\]
FIG. 5. A plot of $x(t)$ versus time for different values of $\chi$ showing the emergence of the limit cycle and chaotic oscillations. Parameters are $E = \pi/4$, $\Omega = 1.0$, $\kappa = 1.0$, $\gamma = 0.2$.

Defining $\langle a \rangle = x + iy$ and $X = \langle \sigma_x \rangle$, $Y = \langle \sigma_y \rangle$, $Z = \langle \sigma_z \rangle$, the semiclassical equations of motion for the unconditional dynamics are obtained by factoring all moments. It is

\[
\begin{align*}
\dot{x} &= -\kappa x/2 + 2\chi yZ \\
\dot{y} &= -\kappa y/2 - E - 2\chi xZ \\
\dot{X} &= -\gamma X/2 - \chi(x^2 + y^2)Y \\
\dot{Y} &= -\gamma Y/2 - \Omega Z + \chi(x^2 + y^2)X \\
\dot{Z} &= -\gamma(1 + Z) + \Omega Y
\end{align*}
\]

When $\chi = 0$ these are the optical Bloch equations[11] with spontaneous emission neglected. This approximation ignores correlations between the cavity field and the qubit. Strictly speaking, this is only a valid approximation when the cavity field is strongly driven and remains close to a coherent state. However we will use it as a heuristic to explain how a limit cycle arises in this model.

The semiclassical dynamics exhibits a fixed point to limit cycle bifurcation as the Rabi frequency increases for a sufficiently large value of $\chi/\kappa$. An example of a limit-cycle is shown in Fig.(5).

The dynamics of the qubit under continuous measurement yields the oscillation/ticking of
the clock just as in the single qubit model in the previous section, while the cavity forms part of the measurement apparatus for the qubit. Due to the dispersive coupling between cavity and qubit, the phase of the cavity field is dependent on the state of the qubit; continuously measuring a quadrature of the cavity output serves as a way to continuously measure the state of the qubit. A stronger drive on the cavity leads to more information about the qubit leaving the cavity per unit time (i.e. a stronger continuous measurement), and a weaker cavity drive leads to weaker continuous measurement.

The continuous measurement counteracts decoherence so that the qubit state remains in a pure state if initialized in a pure state. For the system under consideration, we show that this results in persistent (but noisy) Rabi oscillations of the qubit when: (1) $\Omega > \gamma$, (2) $\chi > \Omega$, (3) $\chi \gtrsim \kappa$, (4) $\kappa \gtrsim \Omega$, and (5) $E$ is sufficiently small. Condition (1) is the strong driving condition that arose in the toy model above and is due to the fact that the qubit can exhibit coherent oscillations only if its oscillation frequency is larger than its decoherence rate, condition (2) arises because the information about the qubit state must be transferred to the cavity faster than the rate of change of the qubit state in order for the cavity to faithfully track the qubit state, condition (3) is due to the fact that the dispersive shift is easily resolved when it is at least as large as the cavity linewidth, condition (4) arises from the fact that the cavity state must be able to change at least as quickly as the qubit state in order to track it, and condition (5) arises because the continuous measurement strength must be sufficiently weak in order to avoid a measurement-induced suppression (i.e. quantum Zeno effect) on the qubit oscillations. By the same token, it follows that the quantum jump clock will arise instead of the Rabi oscillation clock if (1)-(3) are satisfied but $E$ is sufficiently large. Thus, a single such superconducting circuit can be tuned between the oscillatory and telegraph clock modes discussed in the introduction simply by adjusting the cavity drive strength.

Below we present simulations wherein we satisfy the aforementioned constraints by choosing the experimentally realistic parameters given in Table III. We numerically integrate Eq. (26) and apply low-pass filtering to resolve the clock signal. The maximum cavity occupation in the simulations is set at 10 photons. The simulations are performed with QuTiP. 

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TABLE I. Parameters used in the simulations of the superconducting cavity-qubit system modeled by Eq. (26). The model includes continuous homodyne detection of the cavity output.

| Parameter                                  | Value                      |
|--------------------------------------------|----------------------------|
| photodetector efficiency ($\eta$)          | 0.2                        |
| dispersive coupling rate ($\chi/2\pi$)     | $2/2\pi$ MHz               |
| qubit damping rate ($\gamma/2\pi$)         | $0.2/2\pi$ MHz             |
| cavity damping rate ($\kappa/2\pi$)        | $1/2\pi$ MHz               |
| qubit drive ($\Omega/2\pi$)                | $1/2\pi$ MHz               |
| cavity drive ($E/2\pi$)                    | 0.25 to 0.60 MHz           |

1. Rabi oscillation regime: oscillatory clock.

We obtain the Rabi oscillation clock regime by setting $E/2\pi = 0.25$. Fig. 6 shows that the resulting conditional dynamics of the qubit population $\langle \sigma_z \rangle_c(t)$ has noisy oscillations, and the cavity field $\hat{x}$-quadrature $\langle a + a^\dagger \rangle_c(t)$ tracks the evolution of $\langle \sigma_z \rangle_c(t)$. The cavity emission power spectrum $S(\omega) = \int \langle a^\dagger(\tau)a(0) \rangle e^{-i\omega\tau} d\tau$ shown in Fig. 7 verifies that these oscillations persist in the steady state.

A noisy telegraph signal can be obtained from $\langle a + a^\dagger \rangle_c(t)$ by applying first a low-pass filter and then a sign() filter, just as in the toy model example above. For the low pass filter we use a second order Butterworth filter with a cutoff frequency of $\Omega$. The statistics of the telegraph signal are shown in Fig. 9). The mean period is 5.8 $\mu$s. This is as expected from the Rabi oscillation period of $1/\Omega$.

The rate of energy dissipation by this superconducting clock is given by ($\hbar = 1$)

$$\dot{E}_{\text{diss}} = \gamma(1 + z_c(t)) + \kappa \langle a^\dagger a \rangle_c(t)$$

Integrating this over a single run of the clock gives the total energy dissipated over that run. As in the toy model example above, integrating $z_c(t)$ over any single period necessarily gives zero. The average dissipated energy per period is

$$\bar{E}_{\text{diss}} = (\gamma + \kappa \langle a^\dagger a \rangle_c) T,$$

where $T$ is the average period length and $\langle a^\dagger a \rangle_c$ is the conditional average occupation of the cavity; the latter is simply equal to the unconditional steady state occupation. So we expect
FIG. 6. When the cavity output of the superconducting device is continuously measured with the scheme of Fig. 4, as simulated with Eq. 26, the qubit exhibits noisy oscillations in its conditional steady state dynamics for a cavity drive strength of $E/2\pi = 0.25$ and other system parameters as in Table III. The expectation value of the monitored cavity quadrature, $\langle a + a^\dagger \rangle_c(t)$, closely tracks the qubit population dynamics, $\langle \sigma_z \rangle_c(t)$.

FIG. 7. The cavity emission power spectrum clearly shows the Rabi oscillations of the qubit imprinted on the cavity field in the steady state.
FIG. 8. Total energy dissipated (blue) by the superconducting circuit clock as a function of time for a particular clock run, shown together with the clock signal (red) from the same clock run. The clock here is in the oscillatory regime of operation.

\[ E_{\text{diss}}(t) \] to have an average slope of \( \gamma + \kappa \langle a^\dagger a \rangle_c \approx 1.3 \). The data shown in Fig. (8) from a single run of the clock confirms this.
FIG. 9. Distribution of clock periods from the telegraph signal derived from \(\langle a + a^\dagger \rangle_c(t)\) in the Rabi oscillation regime \((E = 0.25 \times 2\pi)\) with homodyne detection. Ten trials lasting 200 \(\mu s\) each are used to collect the data. The low pass filter cutoff frequency used is \(\Omega\). The mean period length is about 5.8 microseconds, which is close to the pure Rabi oscillation period of \(1/\Omega = 2\pi\) microseconds.

2. **Quantum jump regime: non-oscillatory clock.**

It is well known that with sufficiently strong measurement (i.e. \(\chi\) and \(E\) sufficiently large) the cavity field output amplitude will exhibit quantum jumps. This is because the measurement will pin the qubit to one of its \(\sigma_z\) eigenstates due to the quantum Zeno effect. For superconducting circuits this was first experimentally observed in Ref. \[17\]. We can enter this regime in our system by increasing the cavity drive strength to \(E/2\pi = 0.6\) while keeping the other parameters the same. An illustration is shown in Fig. \[10\]. The corresponding cavity emission power spectrum \(S(\omega) = \int \langle a^\dagger(\tau)a(0)\rangle e^{-i\omega\tau} d\tau\) shown in Fig. \[11\] reveals that the spectral weight is peaked around zero, confirming the suppression of the Rabi oscillations.

As in the Rabi oscillation regime we again apply low-pass filtering, but this time with a cutoff frequency of 2.5 \(\Omega\), and perform the sign() operation on \(\langle a + a^\dagger \rangle_c\) to get a telegraph signal. The clock period distribution from the telegraph signal is shown in Fig. \[12\].
FIG. 10. When the cavity drive is sufficiently large (here $E/2\pi = 0.6$) the measurement on the qubit becomes strong enough to give rise to the quantum Zeno effect, and the qubit exhibits quantum jumps between its ground and excited states.

FIG. 11. In contrast to the case of Rabi oscillations (see Fig. [7]), in the quantum jump regime the spectral weight of the cavity field output is peaked around zero.
FIG. 12. Distribution of clock periods from the telegraph signals derived from \( \langle a + a^\dagger \rangle \) in the quantum jump regime \( (E = 0.6 \ast (2\pi)) \) with homodyne detection. Ten trials lasting 200 \( \mu s \) each are used to collect the data. The low pass filter cutoff frequency used is 2.5 \( \Omega \). The mean clock period is about 15 \( \mu s \).

IV. DISCUSSION AND CONCLUSION.

We have shown how a clock signal can be extracted from open quantum systems operating near absolute zero though appropriate measurements and, further, proposed a feasible experiment using superconducting quantum circuits. The measurement is critical to the operation. While the unconditional master equation describing the system can be solved to give a steady state solution it does not exhibit long term dynamics. To extract a clock signal the right kind of continuous measurement needs to be preformed. The resulting measurement record does exhibit noisy oscillations from which a clock signal can be extracted. The measurement itself drives the conditional state away from the ensemble steady state, thus lowering the entropy.

In the case of the superconducting circuit clock proposal, there are two measurement regimes that lead to different clock signals, one oscillatory and one non oscillatory. Homodyne detection of the output field leads to a clock signal with phase diffusion on the semiclassical limit cycle. In this case the rate at which information is extracted from the transmon coupled to the field is low. However, using stronger driving of the field, it is
possible to enter the quantum jump regime wherein quantum transitions between the qubit states are observed leading to a non oscillatory clock based on a random telegraph signal. These jumps are enabled by strong measurements that effectively monitor the energy of the qubit at a rate faster than it can change by coherent interactions alone. This is an analogue of a two-state classical clock based on thermally induced transitions between two states. However, unlike the classical case, the quantum jump clock operates even as the temperature approaches zero.

The ability to extract a clock signal by making local measurements on open quantum systems has important implications for a number of areas of physics. In developing quantum technologies it will be necessary to implement clocks that operate at very low temperatures. Such clocks will be needed to drive circuit gates or, more generally, transitions in finite state machines\[18\]. The kinds of clocks we have discussed can be matched to the appropriate technology platform used, for example ion traps or superconducting quantum circuits.

On a more fundamental level we see that these kinds of clocks could provide a way to solve the problem of time in quantum gravity. The difficulty here is to find a local observable that can play the role of a local clock. Rovelli and Connes\[19\] have suggested that this is always possible even at zero temperature. Our model is an illustration of how this can be done in practical terms. A localised quantum system, such as a two-level atom, necessarily becomes entangled with the non local degrees of freedom of the surrounding field when it undergoes a local excitation: this is how spontaneous emission arises. We can construct a clock from a quantum field or many-body system in its ground state (zero temperature) by a local quench (or excitation) followed by continuous measurement. Recent work on the energy eigenstate thermalisation hypothesis\[20\] in many-body systems uses quenches like this. It does however require the right kind of interactions. The ability to construct measurement driven quantum clocks at zero temperature may provide insights for a theory of quantum gravity.

ACKNOWLEDGEMENTS

We acknowledge helpful discussions with Arkady Fedorov and Eric He. We acknowledge the support of FQXi program on Information as Fuel. This research was supported by the Australian Research Council Centre of Excellence for Engineered Quantum Systems (EQUS,
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