Growth and nodal current of complexified horocycle eigenfunctions

Mikhail Dubashinskiy

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Abstract. We study horocycle eigenfunctions at Lobachevsky plane. These are functions $u : \mathbb{H} = \mathbb{C}^+ = \{ z \in \mathbb{C} : \Im z > 0 \} \rightarrow \mathbb{C}$ such that
\[
-\frac{\partial^2}{\partial x^2} + 2i\tau y \frac{\partial}{\partial x} u(x + iy) = s^2 u(x + iy), \quad x + iy \in \mathbb{C}^+, \quad \tau, s \in \mathbb{R}, \quad \tau \text{ large and } s/\tau \text{ small.}
\]
In other words, we study eigenfunctions of magnetic quantum Hamiltonian on hyperbolic plane. By semiclassical correspondence principle, asymptotic behavior of such functions is related to horocycle flow on $TH$. If a sequence of horocycle functions possesses microlocal quantum unique ergodicity at the admissible energy level (with $\hbar = 1/\tau$) then we may find asymptotic distribution of divisor of $u$ analytically continued to the complexified Lobachevsky plane $\mathbb{H}^C$. This is done by establishing the asymptotic estimates on $|u|$ in $\mathbb{H}^C$. The growth of functions $u$ as $\tau \rightarrow \infty$ turns to be governed by the growth of complexified gauge factor occurring in $\tau$-automorphic kernels for functions on $\mathbb{H}$.

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1 Introduction

Let $\tau \in \mathbb{R}$. In hyperbolic Lobachevsky plane $\mathbb{H}$ implemented as upper half-plane $\mathbb{C}^+$ consider differential operator
\[
D^\tau := -\Delta_{\mathbb{H}} + 2i\tau y \frac{\partial}{\partial x};
\]
here \( \Delta_{\mathbb{H}} := y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \) is the hyperbolic Laplacian, \( x + iy \in \mathbb{C}^+ \). We study asymptotic properties of solutions of eigenfunction equation \( D^\tau u = s^2 u, \ u: \mathbb{H} \rightarrow \mathbb{C} \), for \( \tau \) large with \( s/\tau \) small.

Let \( \tau_n, s_n \in \mathbb{R} \ (n = 1, 2, \ldots) \). Suppose that functions \( u_n: \mathbb{H} \rightarrow \mathbb{C} \) are such that

\[
D^{\tau_n} u_n = s_n^2 u_n
\]

and also \( \tau_n, \tau_n/s_n \xrightarrow{n \to \infty} \infty \). We mostly drop subscript \( n \) in what follows.

If we take Planck constant \( \hbar = 1/\tau \) then principal symbol of \( \tau^{-2} D^\tau \) is \( 2H_1 - 1 \) where, for \( b \in \mathbb{R} \), we define magnetic Hamiltonian

\[
H_b(x, y, \xi_1, \xi_2) := \frac{(y\xi_1 - b)^2 + (y\xi_2)^2}{2}: T^*\mathbb{H} \rightarrow \mathbb{C}
\]

\((z = x + iy \in \mathbb{H}, (\xi_1, \xi_2)\) are cotangent coordinates conjugate to \((x, y)\)). Thus, local frequencies of function \( u \) with \( D^\tau u = s^2 u \) and \( s/\tau \) small have to concentrate near null level set \( \{H_1 = 1/2\} \subset T^*\mathbb{H} \) of the symbol. Notice that, on this set, \( H_1 \) understood as classical Hamiltonian, generates right horocycle flow. If we fold \( \mathbb{H} \) into a compact hyperbolic surface by means of an action of a group of isometries then horocycle flow is known to possess unique ergodicity property — unlike geodesic flow which is only ergodic (but is of hyperbolic Anosov type instead). Bohr semiclassical correspondence principle then leads to different conclusions on quantizations of these flows.

**Definition 1.1.** We say that \( \{u_n\}_{n=1}^{\infty} \) is Quantum Uniquely Ergodic (QUE) sequence if, for any \( a \in C_0^\infty(T^*\mathbb{H}) \) understood as a symbol of order \(-\infty\), we have

\[
\langle (\text{Op}_{1/\tau_n} a)u_n, u_n \rangle_{L^2(\mathbb{H})} \xrightarrow{n \to \infty} \int_{\{H_1 = 1/2\}} a \, d\mu_L.
\]

Here, \( \mu_L \) is horocycle Liouville measure supported by \( \{H_1 = 1/2\} \), see Section 2. Pseudodifferential operator (PDO) \( \text{Op}_{1/\tau_n} a: L^2_{\text{loc}}(\mathbb{H}) \rightarrow L^2_{\text{loc}}(\mathbb{H}) \) is any of quantizations of classical observable \( a \), see [3,4] for details.

QUE property of functions \( u_n \) means that their local frequencies scaled \( 1/\tau_n \) times become uniformly distributed at the admissible energy level \( \{H_1 = 1/2\} \).

Frequency equidistribution of functions \( u_n \) leads to consequences on their complexifications. Any real-analytic manifold admits a complexification which is not unique. Any two of such complexifications are biholomorphically equivalent near the original manifold. This is known as Bruhat–Whitney Theorem, see [BW59]. So we may just take \( \mathbb{H}^C := \mathbb{C} \times \mathbb{C} \) as complexified hyperbolic plane \( \mathbb{H} \). This set is endowed with Euclidean coordinates \((\mathbb{R}X, \mathbb{R}X, \mathbb{R}Y, \mathbb{R}Y), (X, Y) \in \mathbb{C} \times \mathbb{C} \).

In Section 2 we define complex horocycle parametrization mapping

\[
\mathbb{R} \times \mathbb{R}^+ \times (0, 1) \times (\mathbb{R} \text{ mod } 2\pi) \ni (x, y, t, \theta) \mapsto h_{-it}(x + iy, \theta) \in \mathbb{H}^C.
\]

Parameter \( \theta \) here is responsible for the slope of a horocycle starting from a point \( x + iy \in \mathbb{H} \) and, further, evaluated at imaginary time \(-it\) by analytic (with respect to time) continuation. Mapping \((x, y, t, \theta) \mapsto h_{-it}(x + iy, \theta)\) with domain as above is injective onto a set of the form \( \mathcal{G}_1 \setminus \mathbb{H}, \mathcal{G}_1 \subset \mathbb{H}^C \) being an open vicinity of \( \mathbb{H} \) in
\( \mathbb{H}^C \) (Proposition 2.2). This set \( \mathcal{G}_1 \) is called \textit{radius 1 horocycle Grauert tube}. Bijective mapping
\[
\mathbb{R} \times \mathbb{R}^+ \times (0,1) \times (\mathbb{R} \text{ mod } 2\pi) \ni (x, y, t, \theta) \mapsto h_{-t}(x + iy, \theta) \in \mathcal{G}_1 \setminus \mathbb{H}
\]
gives \textit{horocycle coordinates} \((x, y, t, \theta)\) for punctured Grauert tube \( \mathcal{G}_1 \setminus \mathbb{H} \). Sometimes we write \( t(P) \) and \( \theta(P) \) for the latter two coordinates of \( P \in \mathcal{G}_1 \setminus \mathbb{H} \).

It is easy to see that functions \( u_n \) can be analytically continued to \( \mathcal{G}_1 \) (Lemma 3.2). Our first result is on the growth of these complexifications. In horocycle coordinates, define a function \( B_0 = B_0(x, y, t, \theta) : \mathcal{G}_1 \to \mathbb{R} \) as
\[
B_0 := \log \left( \frac{2 + (t^2 - 2t) \cdot (1 + \cos \theta)}{2 + (t^2 + 2t) \cdot (1 + \cos \theta)} \right) \quad \text{on} \quad \mathcal{G}_1 \setminus \mathbb{H}, \quad B_0|_\mathbb{H} := 0. \tag{1}
\]
This function is responsible for the growth of \( u \) in the following sense:

**Theorem 1.2.** Suppose that \( D^{\tau_n} u_n = s_n^2 u_n \) with \( \tau_n, s_n \in \mathbb{R}, \ s_n/\tau_n \xrightarrow{n \to \infty} 0 \) and \( \tau_n \xrightarrow{n \to \infty} \infty \). Assume also that \( \sup_{n \in \mathbb{N}, z \in \mathbb{H}} \|u_n\|_{L^1(B_H(z, 1))} < +\infty \). Here, \( B_H(z, r) \subset \mathbb{H} \) is the open ball in hyperbolic metric centered in \( z \in \mathbb{H} \) and having radius \( r > 0 \).

Suppose also that \( \{u_n\}_{n=1}^{\infty} \) is a \textit{QUE} sequence in the sense of Definition 1.1.

Under these conditions, we have
\[
|\tau_n|^{1/2} \cdot |u_n|^2 \cdot \exp(|\tau_n|B_0) \xrightarrow{\tau \to \infty}^* b \quad \text{in} \quad \mathcal{D}'(\mathcal{G}_1 \setminus \mathbb{H}).
\]
Here, \( b \) is a smooth function separated from zero on compacts in \( \mathcal{G}_1 \setminus \mathbb{H} \) and not depending on \( \{u_n\}_{n=1}^{\infty} \).

**Remark.** If \( D^{\tau} u = s^2 u \) then, for complex conjugate we have \( D^{-\tau} \bar{u} = s^2 \bar{u} \). This allows us to reduce the case \( \tau < 0 \) to \( \tau > 0 \). In what follows we assume that \( \tau \geq 0 \).

Later on in this Introduction, we will discuss the role and meaning of the function \( B_0 \) giving the answer in Theorem 1.2.

Now, consider nodal set \( \mathcal{Z}_n := \{P \in \mathcal{G}_1 : u_n(P) = 0\} \subset \mathbb{H}^C \). Some singularities are possible at this set, but they are always negligible. In all its \textit{non-singular} points set \( \mathcal{Z}_n \) is an analytic submanifold of complex dimension 1 and thus is canonically endowed with orientation. For any non-singular point \( P \in \mathcal{Z}_n \) there is integer multiplicity of zero of \( u_n \) at \( P \), denote this multiplicity by \( m_n(P) \). Therefore, \( m_n \) and \( \mathcal{Z}_n \) naturally give rise to de Rham current \( \mathcal{Z}_n \) of dimension 2: \( \mathcal{Z}_n(\omega) := \int_{\mathcal{Z}_n} m_n \omega \) for smooth compactly supported 2-form \( \omega \) in \( \mathcal{G}_1 \). \( \mathcal{Z}_n(\cdot) \) denotes application of current \( \mathcal{Z}_n \) to a test form. This current is known to be well-defined.

In a more analytic way, nodal current given by \( u_n \) is equal to the de Rham current defined as
\[
\mathcal{Z}_n(\omega) = \frac{i}{\pi} \int_{\mathcal{G}_1} \partial \bar{\partial} \log |u_n| \wedge \omega
\]
for test form \( \omega \) in \( \mathcal{G}_1 \). This is known as \textit{Lelong–Poincaré formula}. Function \( \log |u_n| \) is understood as 4-current therein. Operators \( \partial, \bar{\partial} \) on currents are permanent to those on forms and are given by the complex structure in \( \mathbb{H}^C \). See more in [Ch], [LG].

In Section 6, we take logarithm of the asymptotic relation from Theorem 1.2 and derive our second result:
Theorem 1.3. In the assumptions of Theorem 1.2 for nodal currents given by functions $u_n$, we have

$$\frac{Z_n}{|\tau_n|} \xrightarrow{n \to \infty} \frac{1}{2\pi i} \partial \bar{\partial} B_0 \quad \text{in } D'(G_1).$$

Our main example of horocycle QUE sequence is as follows. Denote by Isom$^+(\mathbb{H})$ the group of orientation-preserving isometries of hyperbolic Lobachevsky plane $\mathbb{H}$. If $\mathbb{H}$ is implemented as upper complex half-plane $\mathbb{C}^+$ then any $\gamma \in$ Isom$^+(\mathbb{H})$ can be written in the canonical form $\mathbb{H} \ni z \mapsto \gamma z = \frac{az + b}{cz + d}$ for $a, b, c, d$ real with $ad - bc = 1$.

Let $\Gamma$ be a discrete torsion-free subgroup in Isom$^+(\mathbb{H})$. A function $u : \mathbb{H} \to \mathbb{C}$ is called $\tau$-form with respect to $\Gamma$ ($\tau \in \mathbb{R}$) if $u(\gamma z) = \left(\frac{cz + d}{cz + d}\right)^\tau u(z)$ for any $z \in \mathbb{H}$ and $\gamma \in \Gamma$ of the form $\gamma z = \frac{az + b}{cz + d}$, this relation has to be valid for some fixed choice of branches of factor $\left(\frac{cz + d}{cz + d}\right)^\tau$ consistent with the group action. Space of $\tau$-forms with respect to a group $\Gamma <$ Isom$^+(\mathbb{H})$, of course, can be understood as the space of sections of an appropriate line bundle over $\Gamma \setminus \mathbb{H}$; for $\tau \in \mathbb{Z}$, the latter bundle is $\tau$-th tensor power of the same bundle for $\tau = 1$.

In [Ze92], [D21] the following has been proven for integer $\tau_n$’s, but is also true for real ones:

Theorem 1.4. Let $\Gamma <$ Isom$^+(\mathbb{H})$ be a discrete torsion-free group with a compact fundamental domain $F$, whereas $\tau_1, \tau_2, \ldots$ be real numbers.

Suppose that functions $u_n : \mathbb{H} \to \mathbb{C}$, $n = 1, 2, \ldots$, are such that $u_n$ is a $\tau_n$-form with respect to $\Gamma$, normed as $\int_F |u_n|^2 dA_2 = 2\pi A_2(F)$, and such that $D^{\tau_n} u_n = s_n^2 u_n$ in $\mathbb{H}$ with $s_n \in \mathbb{R}$ ($A_2$ denotes hyperbolic area measure on $\mathbb{H}$).

If $\tau_n, \frac{\tau_n}{s_n} \xrightarrow{n \to \infty} \infty$ then sequence $\{u_n\}_{n=1}^\infty$ is quantum uniquely ergodic. (Observables for function $u_n$ are quantized with Planck constant $\hbar = 1/\tau_n$.)

In fact, this is a quantization of Furstenberg Theorem on unique ergodicity of horocycle flow over a compact hyperbolic surface ([Furst73], [Ma75]), up to some calculations on gauge invariance. Thus, Theorems 1.2 and 1.3 can be applied to functions from Theorem 1.2 and give control on their growth and on the behavior of nodal sets of their complexifications.

Now, let us outline the proof of Theorem 1.2. We generally follow Zelditch ([Ze07]). In his paper, he studies similar questions on free-particle quantum wavefunction. Geodesic flow is then instead of horocycle flow. Consequently, instead of horocycle Grauert tube, there is (the most usual) geodesic Grauert tube. See also recent papers [CR21], [CR22] on Schwartz kernels of Toeplitz truncated Hamiltonian derivatives tangent to boundaries of such tubes, also on spectral projectors acting at the latter boundaries.

In our paper, from physicist’s viewpoint, we quantize magnetic particle on $\mathbb{H}$. As well as [Ze07], our paper fits into the idea of Boutet de Monvel Theorem. The latter claims that growth of complexified eigenfunction $u$, when we move away from the original real manifold into its complexification, is governed by microlocal distribution of $u$ in the real part of manifold under consideration. This theorem, in a particular case
of Laplacian on a real-analytic manifold, was stated in [Bou79], and has been proved much later in [Ze11], [Leb13], [St14].

Unfortunately, our setting is not covered by the existing results in the spirit of Boutet de Monvel Theorem. In this paper, we first write, using [Fay77], analytic continuation of $u$ to $G_1$ via an integral operator. For $t \in (0,1)$ understood as horocycle coordinate, we study kernel $K_t^r(z_1, z_2) = \left( \frac{z_1 - \bar{z}_2}{\bar{z}_1 - z_2} \right)^r \cdot \exp(-\tau c_t \cosh \text{dist}(z_1, z_2))$ ($z_1, z_2 \in \mathbb{H}$) with certain $c_t \in \mathbb{R}^+$ (Section 3). Then $v(z_2) := \int_{\mathbb{H}} K_t^r(z_1, z_2) u(z_1) \, d\mathcal{A}_2(z_1)$ is a scalar multiple of $u$ whenever $u$ is an eigenfunction of $D^r$. As it is provided by [Fay77], any kernel of the form

$$K(z_1, z_2) = \left( \frac{z_1 - \bar{z}_2}{\bar{z}_1 - z_2} \right)^r \cdot (\text{function of dist}(z_1, z_2)).$$

has such a property. Term $G_t^r(z_1, z_2) := \left( \frac{z_1 - \bar{z}_2}{\bar{z}_1 - z_2} \right)^r$ is understood as gauge factor which also makes these kernels automorphic with respect to isometries of $\mathbb{H}$. The presence of gauge factor is the principal feature making our considerations different from that of [Ze11].

Our kernel is also such that $\mathbb{H} \ni z \mapsto K_t^r(z_1, z)$ can be continued analytically to $G_1$ (the same concerns mapping $z \mapsto G(z_1, z)$ for gauge factor). This leads to explicit integral formula for $u$ on $G_1$. Then, we may put this formula to left-hand side of the limit relation in Theorem 1.2. We see that weighted averaging of $|u|^2$ over $G_1$ leads us to a composition of operators.

Fix $t \in (0,1)$. Put $\Sigma_t := \{ h_{-it}(x + iy, \theta) : x + iy \in \mathbb{H}, \theta \in \mathbb{R} \, \text{mod} \, 2\pi \}$, this set is homeomorphic to (co)spherical bundle over $\mathbb{H}$ and thus is naturally endowed with invariant Liouville measure $dS_t$, see Section 2. Define diffeomorphism $M_t : \{ H_1 = 1/2 \} \to \Sigma_t$:

$$M_t \left( \text{covektor} \left( \frac{(1 + \cos \theta) \, dx + \sin \theta \, dy}{y} \right) \right. \text{ at } x + iy := h_{-it}(x + iy, \theta)$$

for $x + iy \in \mathbb{H}$, $\theta \in \mathbb{R} \, \text{mod} \, 2\pi$; any point in $\{ H_1 = 1/2 \}$ can be parametrized as as at the left. Operator given by $K_t^r(z, P)$ ($z \in \mathbb{H}$, whereas this time $P \in \Sigma_t$) should be, intuitively and very roughly speaking, understood as semiclassical ($h = 1/\tau$) Fourier Integral Operator with complex phase and canonical graph

$$\{ ((z, \xi), (M_t(z, \xi), \text{some covektor at } M_t(z, \xi)) : (z, \xi) \in \{ H_1 = 1/2 \} \} \subset \{ H_1 = 1/2 \} \times T^*\Sigma_t \subset T^*\mathbb{H} \times T^*\Sigma_t. \; (2)$$

To hit the level set $\{ H_1 = 1/2 \}$ supporting semiclassical measure of functions $u$, we have to adjust $c_t$.

Unfortunately, to author’s best knowledge, there is no theory of operators of such a kind. To calculate a "composition" we apply complex stationary phase method ([TrII], [HorII]). To this end, we need a global maximum property given by Lemma 3.1.

In this manner, in Section 4 we construct smooth functions $b_{1,t}(z, \xi) : T^*\mathbb{H} \to (0, +\infty)$, $B(P) : G_1 \setminus \mathbb{H} \to (0, +\infty)$ with the following property. For any $a \in C^\infty_0(\Sigma_t)$, there exists a smooth symbol $a : T^*\mathbb{H} \to \mathbb{R}$ such that, first,
\(a\) coincides to \(b_{1,t} \cdot (a \circ M_t)\) on \(\{H_1 = 1/2\}\), second, for pseudodifferential operator \(A := \text{Op}_{1/\tau} a\), we have

\[
\int_{\Sigma_t} dS_t(P) a(P) B(P) |u(P)|^2 \sim \tau^{-3} \cdot \langle Au, u \rangle_{L^2(H)} \quad \text{as } \tau \to \infty
\]

(see Propositions 4.2 and 4.3 for more precise statement). Function \(B\) is given by an expression depending on \(\tau\) and frequency \(s\) of \(u\) but not on \(u\) itself. This is a sliced and more precise version of Theorem 1.2.

To arrive to Theorem 1.2, it remains to calculate asymptotics for \(B\) as \(\tau \to \infty\) (Section 5). It requires more applications of stationary phase and Laplace method.

Function \(B_0\) figuring at the answers in Theorem 1.2 and Theorem 1.3 is obtained in the following manner. Any \(P \in G_1\) can be written as \(P = h_{-it}(z, \theta)\) for some \(z \in \mathbb{H}\), \(t \in (0, 1)\) and \(\theta \in \mathbb{R} \mod 2\pi\). Then

\[
B_0(P) = -\log |G(z, P)|.
\]

We thus may give a brief and qualitative reformulation of Theorem 1.2:

Growth of a complexified horocycle eigenfunction is given by the growth of kernel gauge factor restricted to the canonical graph.

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2 Coordinates and flows

In this paper, we denote by \(\mathbb{H}\) the standard upper-halfplane model of Lobachevsky hyperbolic plane. Metric tensor in \(\mathbb{H}\) is given by \((dx^2 + dy^2) \cdot y^{-2}\), \(x + iy \in \mathbb{H}\), \(y > 0\).

Point \((X, Y) \in \mathbb{H}^\mathbb{C} = \mathbb{C} \times \mathbb{C}\) will be generally denoted by \(P\), we write \(X(P)\) for \(X\) and \(Y(P)\) for \(Y\). Complex structure in \(\mathbb{H}^\mathbb{C}\) is that of \(\mathbb{C}^2\). Thus, mappings \(P \mapsto X(P)\) and \(P \mapsto Y(P)\) are analytic on \(\mathbb{H}^\mathbb{C}\). We also use \(Z(P) := X(P) + iY(P)\) and \(\bar{Z}(P) := X(P) - iY(P)\), the analytic continuations of functions \(z\) and, respectively, \(\bar{z}\) from \(\mathbb{H}\) to \(\mathbb{H}^\mathbb{C}\).

Recall that for \(z, w \in \mathbb{H}\) we have, in the hyperbolic metric,

\[
\text{dist}(z, w) = \text{arccosh} \left(1 + \frac{|z - w|^2}{2 \Re z \Im w}\right).
\]

Thus, for \(z = x + iy \in \mathbb{H}\) and \(P \in \mathbb{H}^\mathbb{C} (Y(P) \neq 0)\), we may put

\[
\cosh \text{dist}(z, P) := 1 + \frac{(x - X(P))^2 + (y - Y(P))^2}{2yY(P)},
\]

and the latter is single-valued function holomorphic with respect to \(P\).
Any orientation-preserving isometry having canonical form \( \gamma = \frac{ax + \delta}{cz + \delta}, \)
a, \( \delta, c, \delta \in \mathbb{R},\) \( a \delta - bc = 1,\) \( z \in \mathbb{H},\) can be extended analytically to \( \mathbb{C} \times \mathbb{C},\) up to possible zeroes in the denominator:

\[
\mathbb{C} \times \mathbb{C} \ni (X, Y) \mapsto \gamma(X, Y) = \left( \frac{(aX + \delta)(cX + \delta) + acY^2}{(cX + \delta)^2 + (cY)^2}, \frac{Y}{(cX + \delta)^2 + (cY)^2} \right).
\]

(As it will be seen soon, zeroes at the denominator do not really occur in our considerations since we restrict our interest only to the horocycle Grauert tube.) We have \( Z(\gamma(P)) = \gamma(Z(P)) \) and \( \bar{Z}(\gamma(P)) = \gamma(\bar{Z}(P)) \). Obviously, such isometries preserve complexified \( \cosh \text{ dist}(\cdot, \cdot). \) For complexification of gauge factor \( \frac{z_1 - \bar{z}_2}{z_1 - \bar{z}_2} \) the following relation is used for calculations:

\[
\frac{\gamma z - \gamma \bar{Z}}{\gamma \bar{z} - \gamma Z} = \frac{(c \bar{z} + \delta)(cZ + \delta)}{(cz + \delta)(c \bar{Z} + \delta)} \cdot \frac{z - \bar{Z}}{\bar{z} - Z}. \tag{3}
\]

We need one more relation. If \( \gamma z = \frac{a_\gamma z + b_\gamma}{c_\gamma z + d_\gamma}, \) \( \gamma^{-1} z = \frac{a_{\gamma^{-1}} z + b_{\gamma^{-1}}}{c_{\gamma^{-1}} z + d_{\gamma^{-1}}}, (z \in \mathbb{H}) \) are isometries written in the canonical form then, for \( P \in \mathbb{H}^c, \) we have

\[
\frac{c_\gamma Z(P) + d_\gamma}{c_\gamma \bar{Z}(P) + d_\gamma} \cdot \frac{c_{\gamma^{-1}} Z(\gamma P) + d_{\gamma^{-1}}}{c_{\gamma^{-1}} \bar{Z}(\gamma P) + d_{\gamma^{-1}}} = 1. \tag{4}
\]

This is consistent with possibility to put \( \gamma^{-1} \) instead of \( \gamma \) to the definition of \( \tau \)-form given before Theorem 1.4 and can be verified directly.

Among all the isometries of \( \mathbb{H} \) we widely use the following two types of them. The first is \( z \mapsto y_0 z + x_0 (z \in \mathbb{H}) \) with \( x_0 \in \mathbb{R}, y_0 > 0 \) fixed. Most of our constructions are obviously invariant with respect to them. The second kind is the set of rotations of \( \mathbb{H} \) around \( i \) by some angle \( \theta \in \mathbb{R} \mod 2\pi:\)

\[
R_{\theta z} := \frac{z \cos(\theta/2) + \sin(\theta/2)}{-z \sin(\theta/2) + \cos(\theta/2)}.
\]

A (right) horocycle on Lobachevsky plane \( \mathbb{H} \) is a parametrized curve of constant geodesic curvature 1 curving to the right and passed with the unit speed. An equivalent definition is: 1. the curve \( t \mapsto (-t, 1), t \in \mathbb{R}, \) in \((x, y)\)-coordinates in \( \mathbb{H} \) is a right horocycle, 2. any shift of this curve by an isometry of \( \mathbb{H} \) is also a horocycle.

We widely use horocycle coordinates in subsets in \( \mathbb{H}^c. \) Let \( z = x + iy \in \mathbb{H}, \theta \in \mathbb{R} \mod 2\pi, t \in \mathbb{R}. \) Let

\[
v = y \cdot \left( \cos \theta \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial y} \right) \in T_z \mathbb{H} \tag{5}
\]

be unit vector based in \( z. \) There exists a unique horocycle parametrized as \( t \mapsto \phi(t), \) \( t \in \mathbb{R}, \) with \( \phi'(0) = v, \phi(0) = z. \) Put \( h_t(z, \theta) := \phi(t) \in \mathbb{H}. \) Obviously, \( \Re h_t(z, \theta), \Im h_t(z, \theta) \) depend analytically on \( t. \) Therefore, mapping \( t \mapsto h_t(z, \theta) \) with \( z, \theta \) fixed admits an analytic by \( t \) continuation for complex \( t \) with \( \Im t \) small.
More precisely, let \( t \in \mathbb{R} \). If \( \theta = \pi \) then \( h_t(x + iy, \pi) = x - ty + iy, t \in \mathbb{R} \). In the latter formula, put \( x + iy = i \) and apply inversion \( z \mapsto -1/z \). We see that any other right horocycle can be parametrized as

\[
t \mapsto x_0 + y_0 \cdot \frac{1}{t - i} = x_0 + \frac{yt_0}{t^2 + 1} + i \cdot \frac{y_0}{t^2 + 1}
\]  
(6)

with some \( x_0 \in \mathbb{R}, y_0 \in \mathbb{R}^+ \), up to time shift. Form these parametrizations, we derive the following

**Proposition 2.1.** For \( |\Im t| < 1 \), mapping \( t \mapsto h_t(z, \theta) \in \mathbb{H}^C \) can be defined correctly such that \( X(h_t(z, \theta)), Y(h_t(z, \theta)) \) depend analytically on \( t \) when \( z = x + iy \in \mathbb{H} \) and \( \theta \in \mathbb{R} \mod 2\pi \) are fixed.

Further, we have

**Proposition 2.2.** Mapping \( \mathbb{H} \times (0, 1) \times (\mathbb{R} \mod 2\pi) \ni (z, t, \theta) \mapsto h_{-it}(z, \theta) \in \mathbb{H}^C \) is a diffeomorphism onto a set of the form \( U \setminus \mathbb{H} \) with \( U \subset \mathbb{H}^C \) being an open neighbourhood of \( \mathbb{H} \).

**Definition 2.3.** For \( \tilde{t} \in (0, 1) \), set

\[
\mathcal{G}_{\tilde{t}} := \{h_{-it}(z, \theta): t \in [0, \tilde{t}), z \in \mathbb{H}, \theta \in \mathbb{R} \mod 2\pi \} \subset \mathbb{H}^C
\]

is called horocycle Grauert tube of radius \( \tilde{t} \). (Notice that we may take \( t = 0 \) and thus \( \mathbb{H} \subset \mathcal{G}_{\tilde{t}} \) for any \( \tilde{t} \).

Define also \( \Sigma_{\tilde{t}} := \{h_{-it}(z, \theta): z \in \mathbb{H}, \theta \in \mathbb{R} \} \), this is the boundary of \( \mathcal{G}_{\tilde{t}} \), and, for \( \theta \in \mathbb{R} \mod 2\pi \), put \( \Sigma_{\tilde{t}, \theta} := \{h_{-it}(z, \theta): z \in \mathbb{H} \} \).

Notice that \( \mathcal{G}_{t_1} \subset \mathcal{G}_{t_2} \) for \( t_1 < t_2 \) and \( \bigcap_{t \text{ small}} \mathcal{G}_t = \mathbb{H} \). Of course, factor \( (0, 1) \times (\mathbb{R} \mod 2\pi) \) in the domain of mapping in Proposition 2.2 should be understood as a punctured disk, so that any \( \mathcal{G}_{t} \setminus \mathbb{H}, t \in (0, 1) \), is homeomorphic to \( \mathbb{H} \times (\text{punctured disk}) \). The set of the latter punctures is \( \mathbb{H} \). Thus, we may think about \( \mathcal{G}_t \) as about \( \mathbb{H} \times (\text{ball}) \), the (co)ball bundle over \( \mathbb{H} \).

**Proof of Proposition 2.2.** For \( x_0 + iy_0 \in \mathbb{H} \), put

\[
l^{(1)}_{x_0 + iy_0} := \left\{(x_0 - \frac{yt_0}{t^2 + 1}, \frac{y_0}{t^2 + 1}) : t \in \mathbb{C}, 0 < \Im t < 1\right\} \subset \mathbb{H}^C
\]

(see (6)) and

\[
l^{(2)}_{x_0 + iy_0} := \{(x_0 + iy_0t, y_0) : t \in \mathbb{R}, 0 < t < 1\} \subset \mathbb{H}^C.
\]

To prove injectivity from our statement it is enough to show that any of two sets of the form \( l^{(1)}_{x_0 + iy_0}, l^{(2)}_{x_0 + iy_0} \) are disjoint. We consider the case of two sets of the first kind, the other cases are simpler. Suppose that \( x \in \mathbb{R}, y > 0, t = t_1 + it_2 \) \((t_1 \in \mathbb{R}, t_2 \in (0, 1))\), \( X = X_1 + iX_2, Y = Y_1 + iY_2 \) \((X_1, X_2, Y_1, Y_2 \in \mathbb{R})\) and \( x - \frac{yt}{t^2 + 1} = X, \frac{y}{t^2 + 1} = Y \). Then \( X + tY \in \mathbb{R}, X_2 + t_2 Y_2 + t_1 Y_2 = 0 \), also \( Y(1 + t^2) \in \mathbb{R} \) and \( y^2(Y^2 + Y_2^2) + 2Y_1 t_1 t_2 = 0 \). Substituting \( t_1 = -(X_2 + t_2 Y_2)/Y_2 \) to the latter, we find \( t_2^2 = (X_2^2 + Y_2^2)/(Y_2^2 + Y^2) \) which allows to recover \( t_2 \) from \( X \) and \( Y \). (The case \( Y_2 = 0 \) is simpler.) Then \( t_1, y \) and \( x \) are also defined uniquely by \( X \) and \( Y \).
In the remaining cases we also have injectivity. Now, let us prove that $G_1$ contains a neighbourhood of $\mathbb{H}$ in $\mathbb{H}^C$. Recall that $h_{-i\theta}(i, \pi) = (it, 1)$. Application of complexified rotation by angle $\pi + \theta$ around $i$ and also of mapping $z \mapsto x_0 + y_0 \cdot z, z \in \mathbb{H}$, for fixed $x_0 + iy_0 \in \mathbb{H}$ lead to coordinate expressions for $(X, Y) = h_{-i\theta}(x + iy, \theta)$:

$$\Re X = x + y \cdot \frac{(t^4 - 2t^2 + (t^4 - 4t^2) \cos \theta) \sin \theta}{t^4 + (t^4 - 4t^2) \cos^2 \theta + 2(t^4 - 2t^2) \cos \theta + 4}$$

and three similarly cumbersome expressions for $\Im X, \Re Y, \Im Y$. From these expressions one can derive the following: if we take new variables $x, y, v_1 = t \cos \theta, v_2 = t \sin \theta$ such that $(x, y, v_1 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y})$ runs $T\mathbb{H}$, then mapping $(x, y, v_1, v_2) \mapsto (X, Y)$ is $C^1$-smooth and

$$\left. \frac{\partial (\Re X, \Im X, \Re Y, \Im Y)}{\partial (x, y, v_1, v_2)} \right|_{v_1=v_2=0} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -y & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -y \end{pmatrix}.$$

This matrix is non-degenerate. It follows that $G_1$ indeed contains a neighbourhood of $\mathbb{H}$.

It remains to show that Jacobian of

$$\frac{\partial (\Re X, \Im X, \Re Y, \Im Y)}{\partial (x, y, t, \theta)}$$

with $(X, Y) = h_{-i\theta}(x + iy, \theta)$ is non-zero when $t > 0$. Applying isometry of $\mathbb{H}$ we may assume that $x + iy = i, \theta = 0$. Notice that mapping

$$\mathbb{H} \times \mathbb{R} \ni (x_0 + iy_0, t_0) \mapsto \left( h_{t_0}(x_0 + iy_0, 0), \arg \frac{\partial}{\partial t_0} h_{t_0}(x_0 + iy_0, 0) \right) \in \mathbb{H} \times ((\mathbb{R} \setminus (2\pi \mathbb{Z} + \pi)) \text{ mod } 2\pi)$$

is diffeomorphic (argument of a unit tangent vector is $\theta$ as in [5]). It follows that it is enough to check that if $(X, Y) = h_{t_1+i\theta}(x_0 + iy_0, 0), t_1 + i\theta_2 \in \mathbb{C}$, then

$$\text{det} \left. \frac{\partial (\Re X, \Im X, \Re Y, \Im Y)}{\partial (x_0, y_0, t_1, t_2)} \right|_{x_0+iy_0=i, t_1=0} \neq 0.$$

But this is done in by a straightforward computation. Proof is complete. $\blacksquare$

**Remark.** From (6), we see that

$$\Re Y > |\Im X| \quad (7)$$

on $G_1$.

Hyperbolic plane $\mathbb{H}$ is endowed with Riemann area $dA_2 = \frac{dx \, dy}{y^2}$. Tangent spherical bundle $S^1\mathbb{H}$ is endowed with Liouville measure $\tilde{\mu}_L$: if vectors from this bundle are parametrized as in (5) then $d\tilde{\mu}_L = \frac{dx \, dy \, d\theta}{y^2} = dA_2 \, d\theta$.

Function $H_1$ defined at Introduction and understood as a Hamiltonian generates bijective identification $\psi_1 : T\mathbb{H} \rightarrow T^*\mathbb{H}$ given by

$$\psi_1(x, y, v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y}) = \left( x, y, \frac{v_x + y}{y^2} \, dx + \frac{v_y}{y^2} \, dy \right) \in T^*_{x+iy}\mathbb{H}$$
for $x + iy \in \mathbb{C}^+$, $v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} \in T_{x+iy} \mathbb{H}$ (see also [Takh]). Horocycle Liouville measure $\mu_L$ on the set $\{H_1 = 1/2\}$ mentioned in the Introduction is given by $\mu_L := (\psi_1)_* \tilde{\mu}_L$, this is the push-forward of $\tilde{\mu}_L$ by mapping $\psi_1$.

Push-forward of measure $\tilde{\mu}_L$ by the mapping $S_1 H \cap T_{x+iy} \mathbb{H} \ni y \cdot \left(\cos \theta \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial y}\right) \mapsto h^{-it}(x + iy, \theta) \in \Sigma_t$

will be denoted by $S_t$, this is the uniform measure on slice $\Sigma_t$.

3 Construction of an automorphic kernel

In this Section we construct analytic continuation of $u$ to the horocycle Grauert tube $G_1$ via an integral operator with kernel $K_t(\cdot, \cdot)$. For studying the growth of $u$ at $\Sigma_t$ ($t \in (0, 1)$) we need a global maximum property of this kernel.

Lemma 3.1 (on global maximum of absolute value). For $t \in (0, 1)$, put $c_t := \frac{4}{4t - t^3}$ and

$K_t(z, P) := \frac{z - \hat{Z}(P)}{z - Z(P)} e^{-c_t \cosh \text{dist}(z, P)}$, $z \in \mathbb{H}$, $P \in G_1$.

1. Function $\Phi_t(z, P) = \log |K_t(z, P)|$ is single-valued when $z \in \mathbb{H}$ and $P \in G_1$.

2. For $z \in \mathbb{H}$ and $\theta \in \mathbb{R}$ fixed,

$$\max_{P \in \Sigma_t, \theta} |K_t(z, P)|$$

is attained at $P = h^{-it}(z, \theta)$.

3. Hesse matrix

$$\begin{pmatrix}
\frac{\partial}{\partial x} \Re \Phi_t(z_0, h^{-it}(x + iy, \theta)) & \frac{\partial}{\partial y} \Re \Phi_t(z_0, h^{-it}(x + iy, \theta)) \\
\frac{\partial}{\partial y} \Re \Phi_t(z_0, h^{-it}(x + iy, \theta)) & \frac{\partial}{\partial y} \Re \Phi_t(z_0, h^{-it}(x + iy, \theta))
\end{pmatrix}$$

is non-degenerate when $x + iy = z_0$.

Proof. First claim follows from (7): it implies that $z_0 - \hat{Z}(P), \bar{z}_0 - Z(P)$ are non-zero.

Now we sketch the proof of the second assertion. We claim that it is enough to check that

$$\arg \max_{z \in \mathbb{H}} \log |K_t(z, h^{-it}(i, 0))| = i.$$  \hspace{1cm} (8)

Let us do this reduction, once in this paper. First, we claim that (8) implies that

$$\arg \max_{z \in \mathbb{H}} |K_t(z, h^{-it}(i, \theta))| = i$$  \hspace{1cm} (9)

for any $\theta \in \mathbb{R} \mod 2\pi$. Indeed, for such $\theta$, write $R_\theta$ defined at Section 2 in the canonical form as $\mathbb{H} \ni z \mapsto R_\theta z = \frac{az + b}{cz + d}$, $a, b, c, d \in \mathbb{R}$, $ad - bc = 1$. If (8) is already checked
then, by (3), we have, for any $z \in \mathbb{H}$, that

$$
|K_t(z, h_{-it}(i, \theta))| = |K_t(R_\theta R_{-it} z, R_\theta R_{-it} h_{-it}(i, \theta))| =
$$

$$
= \left| \frac{cR_\theta z + \delta}{cR_\theta z + \delta} \cdot \frac{cZ(R_\theta h_{-it}(i, \theta)) + \delta}{cZ(R_\theta h_{-it}(i, \theta)) + \delta} \cdot |K_t(R_\theta z, h_{-it}(i, 0))| \leq
$$

$$
\leq \left| \frac{cZ(h_{-it}(i, 0)) + \delta}{cZ(h_{-it}(i, 0)) + \delta} \right| |K_t(i, h_{-it}(i, 0))| =
$$

$$
= |K_t(R_\theta i, R_\theta h_{-it}(i, 0))| = |K_t(i, h_{-it}(i, \theta))|,
$$

the desired.

Now, having (9), take any $z_0 = x_0 + iy_0 \in \mathbb{H}$, put $\gamma z := x_0 + y_0 \cdot z$ for $z \in \mathbb{H}$. Notice that

$$
|K_t(i, h_{-it}(z, \theta))| = |K_t(\gamma^{-1} i, \gamma^{-1} h_{-it}(z, \theta))| = |K_t(\gamma^{-1} i, h_{-it}(\gamma^{-1} z, \theta))| =
$$

$$
= |K_t(\gamma^{-1} i, h_{-it}(i, \theta))| \leq |K_t(i, h_{-it}(i, \theta))|
$$

by (9). A similar application of an isometry like the latter one allows to replace $z = i$ in the second assertion of our Lemma by any other point in $\mathbb{H}$.

Now, denote $P_0 := h_{-it}(i, 0) = \left( -\frac{it}{1 - t^2}, \frac{1}{1 - t^2} \right)$. To check (8) it is enough to show that $|K_t(x + iy, P_0)|$ with $y$ fixed decreases by $x$ for $x \geq 0$ and increases by $x$ for $x \leq 0$ and also that $|K_t(iy, P_0)|$ attains its maximum over $y \in \mathbb{R}^+$ at $iy = i$. But both monotonicities are proved by straightforward differentiation and opening the brackets. The third assertion of Lemma is proved in the similar spirit. ■

**Remark.** Our considerations below remain valid if we replace $\exp(-ct \cdot \cosh \text{dist}(z, P))$ in $K_t(z, P)$ by any other function of the form $f_t(\text{dist}(z, P))$ with certain properties. First, such $f_t$ should depend holomorphically on $\text{dist}(z, P)$ and be single-valued whenever $z \in \mathbb{H}$, $P \in \mathcal{G}_1$. Second, kernel with such $f_t$ has to possess global maximum property as in Lemma 3.1.1 together with some second-order non-degenerateness conditions. The former maximum property provides all the further first-order identities related to $f'(\text{dist}(z, h_{-it}(z, \theta)))$. Third, we need to impose some summability restrictions on $K_t(z, P)$ as $\text{dist}(z, i) \to \infty$, e.g., as in Lemma 3.2 below, and the corresponding requirements on $f_t$ have to be satisfied.

**Lemma 3.2.** For $u: \mathbb{H} \to \mathbb{C}$ put $v(P) := \int_{\mathbb{H}} u(z) K_t^2(z, P) \, d\mathcal{A}_2(z)$ ($P \in \mathcal{G}_1$).

1. If $u \in L^\infty(\mathbb{H})$ then integral above converges absolutely together with any of its derivatives with respect to coordinates of $P$, also uniformly when $P$ ranges a compact set in horocycle Grauert tube $\mathcal{G}_1$ and $t$ ranges a compact set in $(0, 1)$.

2. If $u: \mathbb{H} \to \mathbb{C}$ is such that $D^{\tau} u = \left( -\Delta_\mathbb{H} + 2it \frac{\partial}{\partial x} \right) u = s^2 u$ then $v(z) = S(t, \tau, s) u(z)$ for some $S$ not depending neither on $z \in \mathbb{H}$ nor on $u$.

3. Function $v(P)$ is analytic for $P \in \mathcal{G}_1$.

Thus, $v$ is, up to a constant factor, an analytic continuation of $u$ to horocycle Grauert tube $\mathcal{G}_1$. 


Remark. Under conditions of Theorem 1.2, we have, in particular, 
\[ \sup_{z \in \mathbb{H}} \|u_n\|_{L^1(B_H(z,1))} < +\infty \] for any given \( n \). This implies that each \( u_n \) belongs to \( L^\infty(\mathbb{H}) \) (not necessarily uniformly by \( n \in \mathbb{N} \)). This can be seen by appropriate averaging the relation from [Fay77, Theorem 1.2]. Lemma 3.2 is therefore applicable to functions from Theorem 1.2.

Proof. The first assertion is somewhat technical. We have to prove that, for any \( k \in \mathbb{N} \), integral
\[
\int_{\mathbb{H}} (1 + |x|^k + y^k) \exp \Re \left(-\tau c_l \frac{(x - X(P))^2 + (y - Y(P))^2}{2gY(P)} \right) \, dA_2(x + iy)
\]
is bounded from the above uniformly when \( P \) ranges a compact set in \( \mathcal{G}_1 \). This is done by a straightforward estimation\(^1\). By the way, during the calculations we see that we really need to impose the condition \( |t| < 1 \) to succeed.

To prove the second assertion, we use [Fay77, Theorem 1.5]. In our case, \( g(cosh r) = \exp(-\tau c_l \cosh r) \). For the majorant \( g \) we may take \( g \) itself. If \( z, z' \) are as in [Fay77, Theorem 1.5] then \( A_2 \{ z'' \in B_H(z', \delta) : \text{dist}(z'', z) < \text{dist}(z', z) \} \) increases when \( \text{dist}(z', z) \) increases; the desired requirement on majorant follows from non-increase of \( g \).

The third assertion of our Lemma follows from analyticity of \( K_t(z, P) \) with respect to \( X(P) \) and \( Y(P) \). Proof of Lemma is complete. \( \blacksquare \)

Now we perturb \( t \) replacing it in \( c_t \) but reserving in \( \Sigma_{t,\theta} \) (which is, recall, \( \{ h_{-t}(z, \theta) : z \in \mathbb{H} \} \) with \( t, \theta \) fixed). The reason to do so is explained at Remark before Proposition 4.1. We are constructing certain operator, and we need to regularize it in order to get a smooth symbol of quantum observable for \( u \).

Lemma 3.3. Let \( t \in (0, 1) \), \( \eta \) be close enough to \( t \), \( \theta \in \mathbb{R} \mod 2\pi \) and \( z \in \mathbb{H} \). Function
\[ P \mapsto |K_\eta(z, P)| \]
has the unique maximum point at \( \Sigma_{t,\theta} \). We denote this maximizer by \( Q(z, t, \eta, \theta) \) and also put
\[ \varphi(t, \eta, \theta) := \log \max_{P \in \Sigma_{t,\theta}} |K_\eta(z, P)| = \log |K_\eta(z, Q(z, t, \eta, \theta))| \].

Proof. Follows from the third assertion of Lemma 3.1 and Implicit Function Theorem. \( \blacksquare \)

We need to hit energy level \( \{ H_1 = 1/2 \} \) in the canonical graph (2) since microlocal mass of \( u \) (that is, measure as in the relation from Definition 1.1) is concentrated near this set. This is provided by the following Lemma 3.4. Energy level \( \{ H_{-1} = 1/2 \} \) obtained in this Lemma will be finally replaced by \( \{ H_1 = 1/2 \} \) by a certain flipping in quadratic form in Proposition 4.2. In the proof of Lemma 3.4 we, in particular, apply rotations of \( \mathbb{H}^C \) around basepoint \( z_0 \) from the statement. This also allows to calculate the dependence of \( \varphi(t, \eta, \theta) \) on \( \theta \).

Lemma 3.4. Let \( t \in (0, 1) \) be fixed.

\(^1\)Isometries of \( \mathbb{H} \) allow to consider only the case when, say, \( X(P) \) is close to \( it_0 \) for some \( t_0 \in [0, 1) \) and \( Y(P) \) is close to 1.
1. For $z_0 \in \mathbb{H}$, mapping

$$\mathcal{T}_{z_0,t}(\eta, \theta) := \Im d_z \Phi_\eta(z, Q(z_0, t, \eta, \theta))|_{z=z_0}$$

is a diffeomorphism of (some neighbourhood of $t$) \times ($\mathbb{R}$ mod $2\pi$) onto a neighborhood of circle $T_{z_0}^* \mathbb{H} \cap \{H_{-1} = 1/2\}$.

2. For any $\eta$ close enough to $t$, $\theta \in \mathbb{R}$ mod $2\pi$, we have, for $B_0$ defined at Introduction,

$$\varphi(t, \eta, \theta) = \varphi(t, \eta, \pi) - \frac{1}{2} \cdot \log \left( \frac{(1 + \cos \theta) \cdot (t^2 - 2t) + 2}{(1 + \cos \theta) \cdot (t^2 + 2t) + 2} \right) = \varphi(t, \eta, \pi) - \frac{B_0}{2}.$$

Clearly, $\mathcal{T}_{z,t}$ is degree $-1$ homogeneous with respect to $\Im z$. If $\mathcal{T}_{z,t}(\eta, \theta) = (\xi_1, \xi_2) \in T_{z,t}^* \mathbb{H}$ then we write $\theta =: \Theta_{z,t}(\xi_1, \xi_2)$ and $\eta =: H_{z,t}(\xi_1, \xi_2)$. The latter mappings are defined near $\{H_{-1} = 1/2\}$. On this level set, if $z \in \mathbb{H}$, $\xi_1 dx + \xi_2 dy \in T_{z,t}^* \mathbb{H}$, $H_{-1}(z, \xi_1, \xi_2) = 1/2$ with $\xi_1 = (-1 - \cos \theta)/3z$, $\xi_2 = -\sin \theta/3z$, $\theta \in \mathbb{R}$, then

$$\Theta_{z,t}(\xi_1, \xi_2) = \theta,$$

$$H_{z,t}(\xi_1, \xi_2) = t.$$

This is seen from the calculations from the proof below.

What concerning the second assertion, the logarithm at the right-hand side will finally lead us to the asymptotics in Theorem 1.2 and answer in Theorem 1.3.

**Proof of Lemma 3.4** Let’s start with the first assertion. To begin, we consider the case $\theta = \pi$. Denote by $iy(t, \eta)$ the point $\arg\max_{z \in \mathbb{H}} |K_\eta(z, h_{-it}(i, \pi))|$, then $y(t, \eta) > 0$ automatically. By routine differentiation, we see that $d_{iy}|_{z=t} \Phi_\eta(x + iy, Q(i, t, \eta, \pi)) = i \cdot f(t, \eta) dy$ with $f$ smooth, $f(t, t) = 0$ and

$$\frac{\partial f}{\partial \eta} \bigg|_{\eta=t} \neq 0. \quad (10)$$

To make $\theta \neq \pi$, we apply rotation around $i$ by angle $\pi + \theta$. Put $R := R_{\pi - \theta}$. Similarly to the second assertion of Lemma 3.3 we conclude that

$$R^{-1} h_{-it}(i, \pi) = Q(R^{-1}(iy(t, \eta)), t, \eta, \theta) =: Q_\theta.$$

By (3),

$$\Phi_\eta(z, Q_\theta) =$$

$$= \Phi_\eta(Rz, RQ_\theta) - \log \left( \frac{z \cos(\theta/2) - \sin(\theta/2)}{z \cos(\theta/2) - \sin(\theta/2)} \right) - \log \left( \frac{Z(Q_\theta) \cos(\theta/2) - \sin(\theta/2)}{Z(Q_\theta) \cos(\theta/2) - \sin(\theta/2)} \right) =$$

$$= \Phi_\eta(Rz, h_{-it}(i, \pi)) + \log \left( \frac{z \cos(\theta + 1) - \sin \theta}{z \cos(\theta + 1) - \sin \theta} \right) - \log \left( \frac{Z(Q_\theta) \cos(\theta/2) - \sin(\theta/2)}{Z(Q_\theta) \cos(\theta/2) - \sin(\theta/2)} \right). \quad (11)$$
Since \( d_z \Phi_\eta(z, h_{-\ii}(i, \pi)) \mid_{z=i} = 0 \), we have
\[
d_z \Phi_\eta(z, Q_\theta) \mid_{z=i} = d_z \log \left( \frac{\cos \theta + 1 - \sin \theta}{\cos \theta + 1 + \sin \theta} \right) \bigg|_{z=i} = -(1 + \cos \theta) \cdot i \, dx - \sin \theta \cdot i \, dy. \tag{12}
\]

Next, suppose that \( F(z) \) is some function and that \( \gamma z = \frac{\alpha z + \beta}{cz + \delta} \). Suppose that \( d_z F \mid_{z=\gamma z_0} = i \alpha \, dx + i \beta \, dy \) for some \( \alpha, \beta \in \mathbb{C} \). Put \( F_1(z) := F(\gamma z) + \log \left( \frac{cz + \delta}{c^2 z + \delta^2} \right) \). If \( dF_1 \mid_{z_0} = i \alpha_1 \, dx + i \beta_1 \, dy \) then
\[
(\alpha \Im(\gamma z) + 1)^2 + (\beta \Im(\gamma z))^2 = (\alpha_1 \Im z + 1)^2 + (\beta_1 \Im z)^2 \tag{13}
\]
(energy conservation under automorphic change of variables). This can be checked by a direct calculation.

By (12) we see that
\[
\frac{\partial}{\partial \theta} \Im d_z \Phi_\eta(z, Q(z_0, t, \eta, \theta)) \bigg|_{z=z_0, \eta=t}
\]
is tangent to \( T_{z_0}^* \mathbb{H} \cap \{H_{-1} = 1/2\} \). Further, (13) and (10) imply that
\[
\frac{\partial}{\partial \eta} \Im d_z \Phi_\eta(z, Q(z_0, t, \eta, \theta)) \bigg|_{z=z_0, \eta=t}
\]
is transverse to \( T_{z_0}^* \mathbb{H} \cap \{H_{-1} = 1/2\} \). Thus, Jacobian of
\[
(\eta, \theta) \mapsto d_z \Im \Phi_\eta(z, Q(z_0, t, \eta, \theta)) \bigg|_{z=z_0}
\]
is non-zero near \( \eta = t \). Together with (12), this concludes the proof of the first assertion of our Lemma.

For the second assertion, notice that \( Z(h_{-\ii}(i, \pi)) = i(t+1), \bar{Z}(h_{-\ii}(i, \pi)) = i(t-1) \). Now apply (11) with \( \gamma = R, P = Q_\theta = R^{-1}h_{-\ii}(i, \pi) \) to the last logarithm in (11), then it is
\[
-\log \left( \frac{i(t+1) \cdot \cos(\theta/2) + \sin(\theta/2)}{i(t-1) \cdot \cos(\theta/2) + \sin(\theta/2)} \right).
\]
We have \( \varphi(t, \eta, \theta) = \Re \Phi_\eta(R^{-1}(iy(t, \eta)), Q_\theta); \) also, \( \varphi(t, \eta, \pi) = \Re \Phi_\eta(iy(t, \eta), h_{-\ii}(i, \pi)) \).

To conclude the calculation for the second assertion of our Lemma, it remains to put \( z = R^{-1}(iy(t, \eta)) \) to (11). \( \blacksquare \)

**Remark.** Since \( h_{-\ii}(i, \pi) = (it, 1) \), the second assertion of Lemma 3.4 implies, by a calculation, that
\[
\varphi(t, \eta, \theta) = \frac{1}{2} \log \left( \frac{(1 + \cos \theta) \cdot (t^2 + 2t) + 2}{(1 + \cos \theta) \cdot (t^2 - 2t) + 2} \right) + \log \left( \frac{2 - t}{2 + t} \right) - \frac{4 - 2t^2}{4t - t^3}. \tag{14}
\]

### 4 Kernel \( L_t(\cdot, \cdot) \) gives a semiclassical PDO

In this Section we reduce certain weighted quadratic mean of \( u \) on \( \Sigma_t \) to a quadratic form given by a pseudodifferential operator and evaluated on \( u \).
Take some $t_1, t_2 < 1$ positive and close enough one to another, $t_1 < t_2$; take $t \in (t_1, t_2)$. Pick $g : \mathbb{R} \to \mathbb{R}^+$ smooth, nonnegative and supported by $[t_1, t_2]$. Take any $a \in C^\infty_0(\Sigma_t)$ with supp $a$ small enough. Consider operator with kernel

$$L_t(z_1, z_2) := \int_{t_1}^{t_2} d\eta g(\eta) \int_{\Sigma_t} dS_t(P) \left( \frac{z_1 - \tilde{Z}(P)}{z_1 - Z(P)} \right)^\tau e^{-\tau c_\eta \cosh \text{dist}(z_1, P)a(P) \times}$$

$$\times e^{-\tau c_\eta \cosh \text{dist}(z_2, P)} \left( \frac{z_2 - \tilde{Z}(P)}{z_2 - Z(P)} \right)^\tau e^{-2\tau \varphi(t, \eta, \theta(P))} =$$

$$= \int_{t_1}^{t_2} d\eta g(\eta) \int_{\Sigma_t} dS_t(P) K^\tau_{\eta}(z_1, P)a(P)\overline{K^\tau_{\eta}(z_2, P)} \cdot e^{-2\tau \varphi(t, \eta, \theta(P))}, \quad z_1, z_2 \in \mathbb{H}.$$

Here, for $P \in \mathcal{G}_1 \setminus \mathbb{H}$, we write $\theta(P)$ for angular coordinate of $P$ in horocycle coordinates $(x, y, t, \theta)$; recall also that $dS_t(\cdot)$ is invariant Liouville measure transferred to $\Sigma_t$ by horocycle parametrization from Section 2. We assume that supp $g$ is small enough such that $T_{z, t}$ is a diffeomorphism of supp $g \times (\mathbb{R} \mod 2\pi)$ onto some closed neighborhood of $T_z \mathbb{H} \cap \{H_{-1} = 1/2\}$.

**Remark.** We want $L_t(\cdot, \cdot)$ to give a semiclassical PDO. If we drop mollification by $g$ in its definition then remaining

$$\tilde{L}(z_1, z_2) := \int_{\Sigma_t} dS_t(P) K^\tau_{\eta}(z_1, P)a(P)\overline{K^\tau_{\eta}(z_2, P)} \cdot e^{-2\tau \varphi(t, \eta, \theta(P))}$$

will not be such an operator: the symbol is too singular. That is why we perturbed $t$ by taking $\eta$ close to $t$ and took average by $\eta$ as above.

Recall that $S(\cdot, \cdot, \cdot)$ is defined in Lemma 3.2. From the second assertion of that Lemma we derive the following

**Proposition 4.1.** We have

$$\int_\mathbb{H} \int_\mathbb{H} u(z_1) \tilde{u}(z_2) L_t(z_1, z_2) \, dA_2(z_1) \, dA_2(z_2) =$$

$$= \int_{\Sigma_t} |u(P)|^2 a(P) \left( \int_{t_1}^{t_2} d\eta g(\eta)|S(\eta, \tau, s)|^2 \cdot e^{-2\tau \varphi(t, \eta, \theta(P))} \right) \, dS_t(P).$$

We thus put

$$B(P) := \int_{t_1}^{t_2} d\eta g(\eta)|S(\eta, \tau, s)|^2 \cdot e^{-2\tau \varphi(t, \eta, \theta(P))} \quad (15)$$

for $P \in \Sigma_t$ so that $B^{-1/2}$ will finally govern the asymptotics of $u$.

**Remark.** We may right now notice that $B(P) \neq 0$ if $g \neq 0$ is non-negative. Indeed, otherwise $S(\eta, \tau, s)$ vanishes at a non-degenerate interval of $\eta$'s. Using [Fay77] we have

$$S(\eta, \tau, s) = \int_1^{+\infty} e^{-\tau c_\eta \cosh \rho \mathcal{P}_{s, \tau}(r)} d \cosh r,$$

$$\mathcal{P}_{s, \tau}(r) = (1 - \tanh^2 r/2)^{\frac{s}{2}} \cdot F_1(s - \tau, \tilde{s} + \tau, 1; \tanh^2 r/2), \quad \tilde{s}(\tilde{s} - 1) = -s^2.$$
In any case, for \( s, \tau \) fixed, we have \( P_{s, \tau}(r) = O(e^{Nr}) \) as \( r \) approaches \(+\infty\) for some \( N \) large enough (see [DLMP §15.4(ii)]). We then conclude that \( S(\eta, \tau, s) \) is analytic in \( c_\eta \) and thus has at most a discrete set of zeroes. An asymptotics for \( \tilde{B} \) will be derived in Proposition 5.2 below.

The following Proposition is our main assertion relating distribution of \( |u|^2 \) at \( \mathcal{G}_1 \) to microlocal distribution of \( u \) at \( T^*\mathbb{H} \).

**Proposition 4.2.** There exists a smooth function \( b_{1,t}(z, \xi_1, \xi_2) \in C^\infty(T^*\mathbb{H}) \) depending smoothly also on \( t \) but not depending on \( u \) with the following property:

Let \( a \in C^\infty_0(\Sigma_u) \) be smooth with support small enough whereas \( g \in C^\infty_0(\mathbb{R}) \) being supported by \([t_1, t_2] \). Put

\[
\sigma(z, \xi_1, \xi_2) := b_{1,t}(z, \xi_1, \xi_2) \cdot g(H_{z,t}(-\xi_1, -\xi_2)) \cdot a(h_{t,t}(z, \Theta_{z,t}(-\xi_1, -\xi_2))).
\]

If \( A := \text{Op}_{1/\tau} \sigma \) is semiclassical PDO with symbol \( \sigma \) then

\[
\int_{\mathbb{H}} \int_{\mathbb{H}} u(z_1) \bar{u}(z_2) L_t(z_1, z_2) dA_2(z_1) dA_2(z_2) = O(1/\tau^4) + \tau^{-3} \cdot (Au, u)_{L^2(\mathbb{H})}.
\]

Notice that \( H_{z,t}(-\xi_1, -\xi_2), \Theta_{z,t}(-\xi_1, -\xi_2) \) are initially defined for \((z, \xi_1, \xi_2)\) near \( \{H_1 = 1/2\} \). By making supp \( g \) sufficiently small we can make symbol \( \sigma \) well defined for all \( \xi_1, \xi_2 \).

The proof of Proposition 4.2 is obtained by standard tools giving Composition Theorem in the theory of Fourier Integral Operators. This theory does not cover our case of semiclassical operators with complex phase. If the corresponding canonical graph calculus is established then we will be able to argue as follows. Let \( A_1 \): (functions on \( \mathbb{H} \)) \( \rightarrow \) (functions on \( \Sigma_\epsilon \)) be operator with kernel \( K_1^\tau(z, P) \), and \( A_2 \): (functions on \( \Sigma_\epsilon \)) \( \rightarrow \) (functions on \( \Sigma_\epsilon \)) be multiplication by \( a \). Then, in the left-hand side of relation claimed in Proposition 4.2 we have quadratic form given by \( A_1^*A_2A_1 \). Considering all the three factors as Fourier Integral Operators, we may compose their graphs (see Introduction) and arrive to the identical graph for the whole \( A_1^*A_2A_1 \). Together with symbol multiplication, this would allow us to obtain a semiclassical PDO at the right-hand side of the relation from Proposition 4.2. Mollification (\( \eta \)-regularization as above) may then be implemented in a bit another way.

But since there is no Composition Theorem for our case, we apply perturbed complex stationary phase directly. Let us just outline the scheme of the argument.

**Sketch of the proof of Proposition 4.2.** By a direct estimation one can see that the contribution to quadratic forms in the statement of Proposition 4.2 of \((z_1, z_2)\) with \( z_1 \) or \( z_2 \) far enough from supp \( a \) is small for \( \tau \) large. The same concerns the case when \( z_1 \) separated from \( z_2 \) (Lemma 3.1 in fact, provides strict non-degenerate maximum of \( |K_1^\tau| \) at the given point).

We thus may assume that \( z_1 \) is close enough to \( z_2 \) and both range a compact set. Next step is to slice \( \Sigma_\epsilon \) into \( \bigcup_{\theta \in \mathbb{R} \mod 2\pi} \Sigma_{\epsilon, \theta} \), see Definition 2.3. Two-dimensional set \( \Sigma_{\epsilon, \theta} \) is endowed with measure \( A_{2,\epsilon} \) which is the push-forward of hyperbolic area \( A_2 \) under parametrisation \( \mathbb{H} \ni z \mapsto h_{-it}(z, \theta) \in \Sigma_{\epsilon, \theta} \). We have estimate \( |K_\eta(z_j, P)e^{-\varphi(t, \eta, \theta)}| \leq 1 \) for \( P \in \Sigma_{\epsilon, \theta}, j = 1, 2, \) turning to the equality at \( P = Q(z_j, t, \eta, \theta) \); both points are \( h_{-it}(z, \theta) \) as \( \eta = t \) and \( z_1 = z_2 \) are equal to some \( z \in \mathbb{H} \). This allows us to apply complex stationary phase method as stated in [11] to

\[
\int_{\Sigma_{\epsilon, \theta}} dA_{2,\epsilon}(P) K_\eta^\tau(z_1, P)a(P)K_\eta^\tau(z_2, P) \cdot e^{-2\tau \varphi(t, \eta, \theta)}
\]
with \( \eta, \theta \) fixed. That is, we make use of almost-analytic continuations of amplitude and phase. This leads to the asymptotics of the form\(^2\)

\[
1/\tau \cdot a_1(z_1, z_2, t, \eta, \theta) e^{\tau \cdot \Psi(z_1, z_2, t, \eta, \theta)}
\]

for (17). Here, \( \Psi|_{z_1 = z_2} = 0 \). Also, a calculation shows that

\[
d_{z_1}|_{z_1 = z_2} \Psi(z_1, z_2, t, \eta, \theta) = i T_{z_2, t}(\eta, \theta).
\]

For estimation, it is useful to notice that \( \Re \Psi \leq 0 \) due to [111] Lemma X.2.5.

Now replace \( L_t(z_1, z_2) \) by \( I := 1/\tau \cdot \int_\mathbb{R} g(\eta) \int_0^{2\pi} d\theta a_1(z_1, z_2, t, \eta, \theta) e^{\tau \cdot \Psi(z_1, z_2, t, \eta, \theta)} \).

Localization in \( I \) and repeated integration by parts show that contribution of \((z_1, z_2)\) with \(|z_1 - z_2| \geq \tau^{-2/3}\) to

\[
\int_\mathbb{R} \int \mathbb{H} L_t(z_1, z_2) u(z_1) \bar{u}(z_2) d\mathcal{A}_2(z_1) d\mathcal{A}_2(z_2)
\]

is negligible; to find an appropriate direction of this integration by parts, we may apply non-degenerateness provided by Lemma [3.3].

Now, assume that \(|z_1 - z_2| \leq \tau^{-2/3}\). In \( I \), using Lemma [3.4] again, change variables as \((\eta, \theta) \mapsto T_{z_2, t}(\eta, \theta)\); for this, we have, of course, assume that \( \eta \) is close enough to \( t \).

Take long enough Taylor expansions over \( z_1 - z_2 \) for phase and amplitude. Main term

\[
1/\tau \cdot \int_\mathbb{R} g(\eta) \int_0^{2\pi} d\theta a_1(z_1, z_2, t, \eta, \theta) e^{\tau \cdot i T_{z_2, t}(\eta, \theta)|z_1 - z_2|}
\]

leads to PDO from the statement. (Square brackets mean application of a covector to a vector.) All the other terms are negligible in the sense of quadratic forms by Calderon–Vailliancourt Theorem. Finally, sign before \( \xi_1 \) and \( \xi_2 \) appears during examination of quadratic form given by reduced kernel. \(\blacksquare\)

Now recall that functions \( u \) are uniformly distributed at \( \{H_1 = 1/2\} \). This implies that, for \( A \) as in Proposition [4.2], we have

\[
\langle Au, u \rangle_{L^2(\mathbb{H})} \to \int_{\{H_1 = 1/2\}} b_{1, t}(z, \xi_1, \xi_2) \cdot g(H_{z, t}(-\xi_1, -\xi_2)) \cdot a(h \cdot \Theta_{z, t}(-\xi_1, -\xi_2)) \, d\mu_L(z, \xi_1, \xi_2) =
\]

\[
g(t) \cdot \int_{\{H_1 = 1/2\}} b_{1, t}(z, \xi_1, \xi_2) \cdot a(h \cdot \Theta_{z, t}(-\xi_1, -\xi_2)) \, d\mu_L(z, \xi_1, \xi_2)
\]

(18)

as \( \tau \to \infty \). Recall that \( \mu_L \) is appropriately normed Liouville measure on \( \{H_1 = 1/2\} \), see Section [2]. Since we may take arbitrary \( a \), we conclude that \( \tau^3 \cdot |u(P)|^2 \cdot B(P) \cdot dS_t(P) \) converge to a measure mutually absolutely continuous with respect to \( dS_t(P) \). In other words, \( \tau^3 \cdot |u(P)|^2 \cdot B(P) \) become equidistributed on \( \Sigma_t \) — up to a smooth non-vanishing factor.

Convergence in (18) is uniform when \( a \) ranges some compact set of symbols, namely, when all the derivatives of \( a \) up to some sufficient order are bounded. Also, this convergence is uniform when \( t \) ranges a compact subset in \((0, 1)\). The same concerns limit relation from Proposition [4.2]. Thus, integration of result of that Proposition over \( t \) leads us to the following

\(^2\)To be perfect, we need more terms of asymptotics — up to \( O(1/\tau^4) \) remainder.
Proposition 4.3. Let $0 < t_1 < t_2 < 1$ with $t_1$ close enough to $t_2$. There exists a smooth strictly positive function $b_2: \mathcal{G}_{t_2} \setminus \text{clos} \mathcal{G}_{t_1} \to \mathbb{R}^+$ with the following property:

Let $g \in C_0^\infty([t_1, t_2])$ and $B$ be as defined in (15). For such $B$ we have

$$\tau^3|u(P)|^2 \cdot B(P) \xrightarrow{\tau \to +\infty} g(t(P)) \cdot b_2(P).$$

Here, both sides of weak* convergence are understood as densities of measures with respect to some (say, Euclidean) coordinates in $\mathcal{G}_{t_2} \setminus \text{clos} \mathcal{G}_{t_1}$ whereas the limit relation is understood in the sense of $\mathcal{D}'(\mathcal{G}_{t_2} \setminus \text{clos} \mathcal{G}_{t_1})$.

Remark. In the following Section, we will ensure that $B(P)/g(t(P))$ asymptotically does not depend on $g$ as $\tau \to +\infty$, this is natural to expect.

5 Asymptotics for $B$

Now we calculate asymptotics for $B$ when $\tau$ is large. Notice, by the way, that this is not necessary to prove Theorem 1.3. Since we are going to apply Lelong–Poincaré formula to arrive to that Theorem, we may just prove that

$$2 \log g \to +\infty$$

as $\tau \to +\infty$ and $\tau \to +\infty$,

(see Lemma 6.1 below; the argument can be modified for rather inexplicit $B$). Then it remains to find asymptotics for $\frac{\partial B(P)}{\partial t(P)}$, $\frac{\partial B(P)}{\partial \theta(P)}$. If $g \geq 0$ then this can be done by differentiating (15) or rather only the exponential function therein since the integrand is non-negative in this case.

We start with asymptotic expression for $S(\eta, \tau, s)$. To this end, formulae (16) seem to be unuseful. Indeed, we may try to represent hypergeometric function therein by an integral expression in the spirit of [DLMF §15.6], then we have double integral for $S$. The maximum of exponential expression therein seems to be always on the boundary of 2-dimensional contour of integration; also, this maximum does not lead to the correct answer which is strictly less: boundary asymptotics should necessarily cancel, and this cannot be eliminated by a deformation of the contour.

Instead, we make use of the spectral nature of $S$ and of geometric intuition elaborated by now:

Proposition 5.1. As $\tau \to +\infty$ and $s = o(\tau)$, we have

$$|S(\eta, \tau, s)| \sim \tau^{-1} \cdot b_3(\eta) \cdot \exp \tau \varphi(\eta, \eta, \pi)$$

with some $b_3$ smooth and separated from zero for $\eta$ strictly inside of $(0, 1)$. The quotient of left- and right-hand sides tends uniformly to 1 for such $\eta$.

Proof. Notice that if $s = \sqrt{s^2 - 1/4}$, $v(z) = (3z)^{1+i\delta}$ ($z \in \mathbb{C}$) then $D^\tau v = s^2v$. Then, since $v(i) = 1$ and by the definition of $S$ (Lemma 3.2), we have

$$S(\eta, \tau, s) = \int_H v(z)K_\eta\tau(z, i)\, dA_2(z).$$

The integrand admits an analytic continuation to $\mathcal{G}_1$. Thus, we are able to make use of high-dimensional steepest descent method as stated in [Fe]. The above integral is

$$\int_{\Sigma_{\eta, \pi}} \left(\frac{i - \bar{Z}(P)}{i - Z(P)}\right)^\tau \cdot \exp(-\tau c_2 \cosh(d(i, P))) \, dX(P) \wedge dY(P)$$
with an appropriate orientation of $\Sigma_{\eta,\pi}$. Indeed, the homotopy between $\mathbb{H}$ and $\Sigma_{\eta,\pi}$ is given by $\Gamma(z, t) \mapsto h_{-\theta}(z, \pi)$, $z \in \mathbb{H}$, $t$ ranges from 0 to $\eta$. An estimation shows that there is no problems at infinity during homotopy $\Gamma$. (Just replace $\mathbb{H}$ in domain of integration by $[-R^{100}, R^{100}] \times [1/R, R]$, $R \to +\infty$.) Also, homotopy $\Gamma$ has image in $\mathcal{G}_i$ and therefore there no singularities caused by zeroes of $K_\tau$ or by denominators therein.

By a direct calculation we check that $P_0 = h_{-\theta}(i, \pi) = (i\eta, 1)$ is a stationary point of phase $\Phi_\eta(-, i) = \log K_\eta(-, i)$ at the whole 4-dimensional complexified Lobachevsky plane. This point is also non-degenerate: the determinant of

\[
\begin{pmatrix}
\frac{\partial^2 \Phi_\eta(P, i)}{\partial X(P)^2} & \frac{\partial^2 \Phi_\eta(P, i)}{\partial X(P)\partial Y(P)} \\
\frac{\partial^2 \Phi_\eta(P, i)}{\partial X(P)\partial Y(P)} & \frac{\partial^2 \Phi_\eta(P, i)}{\partial Y(P)^2}
\end{pmatrix}
\]

is, by a calculation, non-zero.

The last difficulty is that we have double asymptotics: besides $\tau$, there is also (possibly large) $\bar{s}$ in our integral. But since $s = o(\tau)$, we may replace $Y(P)\frac{1}{2} + i\bar{s}$ by $Y(P_0)\frac{1}{2} + i\bar{s}$. Indeed, near $P_0$, deform surface of integration $\Sigma_{\eta,\pi}$ to the canonical steepest descent contour $W$ as in proof of [P61 Chapter V, §1.3, Theorem 1.1]; then the asymptotics by $\tau$ (with $s, \bar{s}$ being fixed) is calculated by Laplace method. For any $\tau$, $s$ take $r > 0$ such that $1/\tau = o(r)$, $r = o(1/s)$ as $\tau \to \infty$, this is possible by the assumptions from Theorem 12. The contribution to the integral of $P \in W$ with $\text{dist}(P, P_0) > r$ is negligible. (The distance is understood in some, say, Euclidean coordinates in $\mathbb{H}^i$.) For the remaining part, recall that $Y(P_0) = 1$. We have $\left|Y(P)\frac{1}{2} + i\bar{s} - 1\right| = O(sr) = o(1)$ as $\text{dist}(P, P_0) \leq r$. Since there is no oscillations in Laplace-type integral along $W$, it is indeed safe to put $Y(P_0)(= 1)$ instead of $Y(P)$ in $\int_{\mathcal{G}} Y(P)\frac{1}{2} + i\bar{s} K^\tau_\eta(P, i)\,dX \wedge dY$. This concludes the proof. ■

Now we compute the asymptotics for $B$. Remark after Proposition 43 suggests that $\eta = t$ should be a Laplace point in integral 15 for $B$. This leads to the proof of the following

**Proposition 5.2.** Let $t \in (0, 1)$, $\theta \in \mathbb{R} \mod 2\pi$, and also $g: \mathbb{R} \to \mathbb{R}$ be smooth non-negative function with support close enough to $\{t\}$, $g(t) > 0$ inside of supp $g$.

As $\tau \to +\infty$ and $s = o(\tau)$, for $P \in \Sigma_{t,\theta}$ we have $B(P) \sim \tau^{-5/2} \cdot b_4(t)\,g(t)\cdot\exp(\tau B_0)$.

Here, $b_4$ is some smooth function on $\mathcal{G}_i \setminus \mathbb{H}$ separated from 0. The quotient of left- and right-hand sides of this relation tends to 1 uniformly by $t$ strictly inside of supp $g$.

**Proof.** Notice that integrand in relation 15 defining $B$ is non-negative since $g$ is such. Thus we may put asymptotics obtained in Proposition 5.1 to 15. Using also the second assertion of Lemma 3.3 we get

\[
B(P) \sim \tau^{-2} \cdot \int_{\mathbb{R}} g(\eta) b_3^2(\eta) e^{2\tau\varphi(\eta, \eta, \pi) - 2\tau\varphi(t, \eta, 0)} \,d\eta = \\
= \tau^{-2} \cdot \exp(\tau B_0) \cdot \int_{\mathbb{R}} g(\eta) b_3^2(\eta) e^{2\tau\varphi(\eta, \eta, \pi) - 2\tau\varphi(t, \eta, \pi)} \,d\eta.
\]

Surprisingly, here we may not replace $\pi$ by an arbitrary $\theta \in \mathbb{R} \mod 2\pi$. Also, for $\theta = \pi$, we may prove stationarity for imaginary horocycle time $-i\eta$ by checking the same for real time $\eta$ instead and then by analytic continuation to imaginary time.
To prove the required asymptotics, we thus need to show the following: if $t$ is fixed and $f(\eta) = \varphi(\eta, \eta, \pi) - \varphi(t, \eta, \pi)$ then $f'(t) = 0$, $f''(t) < 0$. Indeed, then, if supp $g$ is small enough then Laplace method leads to the desired.

Recall that $Q = Q(z, t, \eta, \theta)$ has been defined at Lemma 3.3. To take

$$\frac{\partial}{\partial \eta} \varphi(t, \eta, \pi) = \frac{\partial}{\partial \eta} \log |K_{\eta}(i, Q(i, t, \eta, \pi))|,$$

notice that

$$\frac{\partial}{\partial \eta} \log |K_{\eta}(i, Q(i, t, \eta_1, \pi))| = 0$$

since $Q(i, t, \eta, \pi) = \arg \max_{P \in \Sigma_{t, \pi}} |K_{\eta}(i, P)|$ and $Q(i, t, \eta_1, \pi) \in \Sigma_{t, \pi}$ for any $\eta_1$ close enough to $\eta$. We thus find

$$\frac{\partial}{\partial \eta} \varphi(t, \eta, \pi) = \frac{\partial}{\partial \eta_1} \log |K_{\eta_1}(i, Q(i, t, \eta, \pi))| = -\frac{dc_\eta}{d\eta} \cdot \Re \cosh \text{dist}(i, Q(i, t, \eta, \pi)).$$

(19)

If $\eta = t$ then we may proceed calculations using (14) and arrive to $f'(t) = 0$.

To find $f''(t)$ we still use (19). For $\eta$ close enough to $t$ we may write $Q(i, t, \eta, \pi) = h_{-it}(x(\eta) + iy(\eta), \pi)$ with some $x(\eta) + iy(\eta) \in \mathbb{H}$ depending smoothly on $\eta$, $x(t) + iy(t) = i$. Calculating Hesse matrix of $(x, y) \mapsto \log |K_{\eta}(i, h_{-it}(x + iy, \pi))|$ we find $\frac{dx(\eta)}{d\eta} = 0$, $\frac{dy(\eta)}{d\eta} \bigg|_{\eta=t} = -\frac{t(4 - 3t^2)}{2(4 - 3t^2 + t^4)}$, and, by (19),

$$f''(t) = \frac{3t^2 - 4}{t(t^4 - 3t^2 + 4)} < 0.$$

This concludes our computational proof. ■

Theorem 1.2 now is an immediate consequence of Proposition 4.3 since both sides of the limit relation therein are non-negative.

### 6 Logarithm of weak* convergence

To derive Theorem 1.3 from Theorem 1.2, we have to take the logarithm of the result of the latter one. This is done by a rather standard trick with plurisubharmonic dichotomy.

Recall that $u = u_n$, $\tau = \tau_n$, $s = s_n$ depend on $n = 1, 2, \ldots$.

**Lemma 6.1.** We have

$$\frac{2}{\tau_n} \cdot \log |u_n| + B_0 \xrightarrow{n \to \infty} 0 \text{ in } L^1_{\text{loc}}(G_1).$$

**Proof.** We mostly follow Zelditch ([Ze07]).

By the definition of $\mathcal{S}$ (Lemma 3.1),

$$\mathcal{S}(1/2, \tau_n, s_n)u_n(P) = \int_{\mathbb{H}} u_n(z)K_{1/2}^\tau (z, P) dA_2(z), \ P \in G_1.$$
We may estimate the integral using the condition \( \sup_{n \in \mathbb{N}, z \in \mathbb{H}} \| u_n \|_{L^1(B_{2r}(z,1))} < +\infty \) required in the statement of Theorem 1.2. In such a way, one is able to see that, for any compact set \( K \subset G_1 \),
\[
\sup_{n \in \mathbb{N}} \sup_{P \in K} \frac{\log | S(1/2, \tau_n, s_n) u_n(P) |}{\tau_n} < +\infty.
\]
Since we already have asymptotics for \( S(1/2, \tau_n, s_n) \) given by Proposition 5.1, we may conclude that
\[
\sup_{n \in \mathbb{N}} \sup_{P \in K} \frac{\log | u_n(P) |}{\tau_n} < +\infty, \quad (20)
\]
\( K \) being any fixed compact in \( G_1 \).

Consider plurisubharmonic functions \( \frac{\log | u_n |}{\tau_n} \), \( n = 1, 2, \ldots \). From (20) we see that these functions are bounded from the above on any compact set in \( G_1 \) uniformly by \( n \). By [HörI, Theorem 4.1.9], we have the following plurisubharmonic dichotomy: either \( \frac{\log | u_n |}{\tau_n} \xrightarrow{n \to \infty} -\infty \) uniformly on each compact subset in \( G_1 \); or, up to subsequence of indices \( n \), functions \( \frac{\log | u_n |}{\tau_n} \) converge in \( L^1_{\text{loc}}(G_1) \) as \( n \to \infty \).

The first case is impossible. Indeed, this would contradict Theorem 1.2 since \( B_0 \) is bounded from the below on compacts in \( G_1 \).

We thus may suppose, up to subsequence, that \( \frac{\log | u_n |}{\tau_n} \xrightarrow{n \to \infty} f \) in \( L^1_{\text{loc}}(G_1) \) for some function \( f \in L^1_{\text{loc}}(G_1) \). Let \( f^* \) be upper-semicontinuous regularization of \( f \) ([HörI, Theorems 4.1.11, 4.1.8]). Then \( f^* \) is plurisubharmonic and equals \( f \) almost everywhere in \( G_1 \) with respect to Euclidean coordinates therein.

First, we are going to prove that \( 2f^* + B_0 = 0 \) in \( G_1 \setminus \mathbb{H} \).

Let us show that \( 2f^* + B_0 \leq 0 \) almost everywhere in \( G_1 \setminus \mathbb{H} \). Indeed, otherwise, by F. Riesz and D. Egorov Theorems, we may assume that, up to subsequence,
\[
\lim_{n \to \infty} \left( B_0 + \frac{1}{\tau_n} \log | u_n |^2 \right) \text{ exists, is uniform and is greater or equal than some } \delta > 0 \text{ on a measurable set } E \subset G_1 \setminus \mathbb{H} \text{ of a positive measure. Then } | u_n |^2 \cdot \exp(\tau_n B_0) \geq \exp(\tau_n \delta / 2) \text{ on } E \text{ for } n \text{ large. We then arrive to contradiction to the weak* convergence from Theorem 1.2.}
\]

Now prove that \( 2f^* + B_0 \geq 0 \) in \( G_1 \setminus \mathbb{H} \). Suppose that \( 2f^*(P_0) + B_0(P_0) < -\delta \) for some \( \delta > 0 \) and for some \( P_0 \in G_1 \setminus \mathbb{H} \). Then, since \( f^* \) is upper-semicontinuous and \( B_0 \) is continuous, we have \( 2f^*(P) + B_0(P) < -\delta \) for \( P \) in some neighborhood \( U \) of \( P_0 \) precompact in \( G_1 \setminus \mathbb{H} \). Then, due continuity of \( B_0 \) again and by [HörI, Theorem 4.1.9(b)],
\[
\limsup_{n \to \infty} \sup_{\text{clos } U} \left( \frac{\log | u_n |^2}{\tau} + B_0 \right) \leq \sup_{\text{clos } U} (2f^* + B_0) < -\delta,
\]
and \( | u_n |^2 \cdot \exp(\tau_n B_0) < \exp(-\tau_n \delta) \) on \( U \) for \( n \) large enough. This again contradicts Theorem 1.2.

So, by now, from upper-semicontinuity of \( f^* \) and continuity of \( B_0 \), we have \( f^* = -\frac{B_0}{2} \) in \( G_1 \setminus \mathbb{H} \). But the above considerations do not provide any information on the behavior of functions \( u_n \) near \( \mathbb{H} \). (In all the preceding arguments we had
to assume that \( t \) is separated from zero to get uniform estimates of reminders.) In Theorem 1.3 we do not cut \( \mathbb{H} \) from \( \mathcal{G}_1 \).

Function \( -B_0/2 \) is plurisubharmonic on the whole \( \mathcal{G}_1 \). Indeed, it is such near any point in \( \mathcal{G}_1 \setminus \mathbb{H} \) since it coincides to \( f^* \) therein. Also, \( B_0 \) is continuous everywhere in \( \mathcal{G}_1 \). Finally, denote by \( B_{C_0}(0,r) \) the disc in \( C \) centered in 0 and having radius \( r > 0 \); let also \( \mathcal{H}^2 \) be area measure on \( C \). If \( P_0 \in \mathbb{H} \subset \mathbb{H} \times \mathbb{H} \), \( v \in \mathbb{C} \times \mathbb{C} \) is a vector, \( r > 0 \) is small enough then

\[
-B_0(P_0)/2 = 0 \leq -\frac{1}{\pi r^2} \int_{B_{C_0}(0,r)} B_0(P_0 + \zeta v) \, d\mathcal{H}^2(\zeta)
\]

because the integrand at the right-hand side is non-positive (see (1)). Thus, \( -B_0 \) is also plurisubharmonic in any point on \( \mathbb{H} \); by localization ([HörSV, Theorem 1.6.3]) we conclude that \( -B_0 \) is plurisubharmonic in \( \mathcal{G}_1 \).

Thus, both \( f^* \) and \( -B_0/2 \) are (pluri)subharmonic on \( \mathcal{G}_1 \) and they coincide on a set \( \mathcal{G}_1 \setminus \mathbb{H} \) having full Euclidean measure therein. Then they generate the same distribution. But a (pluri)subharmonic function is uniquely defined by its distribution ([HörI, 4.1.8]), therefore \( f^* = -B_0/2 \) everywhere at the whole \( \mathcal{G}_1 \). Proof of Lemma is complete. ■

Now, to derive Theorem 1.3 from Lemma 6.1, it remains to apply Lelong–Poincaré formula to the obtained weak* convergence.

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