Generalized T-splines and VMCR T-meshes
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Abstract
The paper deals with the extension of the T-spline approach to the generalized
B-splines (GB-splines), a relevant class of non-polynomial splines. This requires
both to deepen the study of some basic properties of the GB-splines, in particular
of the knot insertion formulae, and the formalization of the concept of T-splines
constructed using GB-splines. The study of the linear independence of the so-
obtained functions leads to the definition of a class of T-meshes which guarantees the
linear independence of the associated T-spline functions (both for the polynomial
and non-polynomial case) and properly includes the class of analysis-suitable T-
meshes for any order. We refer to this class as VMCR T-meshes.

Keywords: T-spline, T-mesh, GB-spline, analysis-suitable, dual-compatible, linear
independence.

1 Introduction

In the last years, the introduction of the so-called T-splines and of the spline spaces
deﬁned over T-meshes introduced signiﬁcant advancements for the use of polynomial
spline functions in the CAD and CAGD techniques. The main idea of this approach,
in the basic case of surface modelling in $\mathbb{R}^3$, is to free the control points of the surface
from the constraint to lie, topologically, on a rectangular grid whose edges intersect only
at “cross junctions”, and allow instead partial lines of control points, which leads to the
possibility to have “T-junctions” between the edges of the grid. Such a framework gave
some important improvements in CAD and CAGD methods: the possibility to locally
refine the surfaces, a considerable reduction of the quantity of control points needed, the
ability to easily avoid gaps between surfaces to be joined (see, e.g., \cite{11} and \cite{12}), just to
name a few. All these advantages became even more important in the applications, such
as the isogeometric approach for the analysis problems represented by partial differential
equations (see, e.g., \cite{6}, \cite{7} and \cite{1}).

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The T-spline idea has been applied mainly to polynomial splines, while we know that several type of non-polynomial splines are used for certain applications because of their particular properties. For this reason, recently we proposed a generalization of the T-spline approach to the trigonometric GB-splines (see [3]), a particularly relevant class of non-polynomial splines because of their adaptability and their application to the already mentioned isogeometric analysis (see, e.g., [8] and [10]). Roughly speaking, the GB-splines are a basis of spaces of piecewise functions, locally spanned both by polynomials and by non-polynomial functions, which in the trigonometric case are \( \sin(\omega s) \) and \( \cos(\omega s) \), with a given frequency \( \omega \). Such functions have been successfully used to construct tensor-product surfaces (see, e.g., [10] and references therein) with control points on rectangular grids.

The first goal in this paper is to provide a natural extension of the results obtained in [3] to any type of GB-spline, in order to be able to take full advantage of the features of the GB-splines while, at the same time, using the T-spline approach. In order to achieve this goal, the first step will be to clarify some key issues in the univariate case, in particular about the knot insertion formula and the conditions needed to be able to get it: we will explicitly show that the choice of the non-polynomial functions corresponding to the new intervals obtained after a knot insertion is not free. This allows us to introduce the new concept of Generalized T-splines (GT-splines) and to study their main properties, including the fundamental issue of their linear independence. Similarly to the trigonometric case (see [3]), we will prove that there exists a strong relation between the GT-splines of bi-order \((p, q)\) and the polynomial T-splines of the same bi-order. During this study we will achieve our second main goal, that is, introducing the class of VMCR T-meshes (Void Matrix after Column Reduction T-meshes): such class, whose basic concept was implicitly contained in [3], guarantees the linear independence of the associated GT-spline and T-spline blending functions. Such a class and its relationship with the well-known analysis-suitable (equivalently, dual-compatible) T-meshes (see, e.g., [5] and [9]) will be analyzed, eventually showing that it strictly includes the one of analysis-suitable T-meshes for any bi-order \((p, q)\).

The paper is organized as follows. In Section 2 we recall the definition and the basic properties of the univariate GB-splines, and we deal with the conditions needed to get a knot insertion formula. In Section 3, after having recalled the definition of T-mesh, we introduce the GT-splines and we give some properties following directly from their definition. In Section 4 we study the linear independence of the GT-spline blending functions and, more importantly, the class of VMCR T-meshes. Finally, Section 5 contains some concluding remarks.

## 2 Univariate generalized B-splines

### 2.1 Definition and main properties

Let \( n, p \in \mathbb{N}, p \geq 2 \), and let \( \Sigma = \{s_1 \leq \ldots \leq s_{n+p}\} \) be a non-decreasing knot sequence (knot vector); we associate to \( \Sigma \) two vectors of functions \( \Omega_u = \{u_1(s), \ldots, u_{n+p-1}(s)\} \) and \( \Omega_v = \{v_1(s), \ldots, v_{n+p-1}(s)\} \), where, for \( i = 1, \ldots, n+p-1 \), \( u_i, v_i \) belong to \( C^{p-2}[s_i, s_{i+1}] \).
and are such that the space $W$ spanned by the derivatives

$$U_i(s) = \frac{d^{p-2} u_i(s)}{ds^{p-2}}, \quad V_i(s) = \frac{d^{p-2} v_i(s)}{ds^{p-2}}$$

is a Chebyshev space, that is, any function belonging to it has at most one solution in $[s_i, s_{i+1}]$. Let, for $i = 1, ..., n + p$, $m_i$ be the multiplicity of $s_i$ in $\Sigma$, that is, the cardinality of the set

$$\{k : 1 \leq k \leq n + p, s_k = s_i\}.$$ 

Note that $m_i = m_j$ if $s_i = s_j$. We assume that $1 \leq m_i \leq p$, for $i = 1, ..., n + p$. We consider the generalized spline space spanned, in each interval $[s_i, s_{i+1}]$, by $\{u_i(s), v_i(s), 1, s, ..., s^{p-3}\}$ for $p \geq 3$ and by $\{u_i(s), v_i(s)\}$ for $p = 2$. For this space we can define a basis of compactly-supported splines, which are called Generalized B-splines (GB-splines).

The definition of such basis is usually given in a recursive form, which we briefly recall (see also [8] and [10]). Since we required that the space spanned by $U_i$ and $V_i$, denoted by $W = \langle U_i, V_i \rangle$, is a Chebyshev space, then it is not restrictive to choose, as generating functions of $W$, two functions $U_i(s)$ and $V_i(s)$ such that

$$U_i(s_i) > 0, \quad U_i(s_{i+1}) = 0, \quad V_i(s_i) = 0, \quad V_i(s_{i+1}) > 0. \quad (1)$$

Note that, since $W$ is a Chebyshev space, $U_i(s) > 0$ for any $s \in [s_i, s_{i+1}]$ and $V_i(s) > 0$ for any $s \in (s_i, s_{i+1})$. We will call the selected functions $U_i(s)$ and $V_i(s)$ generating functions associated to $[s_i, s_{i+1}]$. In what follows, we will consider only generating functions $U_i(s)$, $V_i(s)$. Then, it can be shown (see [8], [10] and references therein) that we can define a basis of compactly-supported spline functions for the generalized spline space in the following way: for $p = 2$

$$N_{i}^{(2)}(s) = \begin{cases} 
V_i(s), & \text{if } s_i \leq s < s_{i+1}, \\
\frac{V_i(s_{i+1})}{U_i(s_{i+1})}, & \text{if } s_{i+1} \leq s < s_{i+2}, \\
0, & \text{otherwise,}
\end{cases} \quad (2)$$

while, for $p \geq 3$,

$$N_{i}^{(p)}(s) = \int_{-\infty}^{s} \left( \delta_{i}^{(p-1)} N_{i}^{(p-1)}(r) - \delta_{i+1}^{(p-1)} N_{i+1}^{(p-1)}(r) \right) dr, \quad i = 1, ..., n, \quad (3)$$

where

$$\delta_{i}^{(p)} = \left[ \int_{-\infty}^{\infty} N_{i}^{(p)}(r) dr \right]^{-1}, \quad i = 1, ..., n. \quad (4)$$

Moreover, if $N_{i}^{(p)}(s) = 0$, we set

$$\int_{-\infty}^{s} \delta_{i}^{(p)} N_{i}^{(p)}(r) dr = \begin{cases} 
1, & s \geq s_{i+p}, \\
0, & s < s_{i+p},
\end{cases} \quad (5)$$

**Remark.** The assumption that $\langle U_i, V_i \rangle$ is a Chebyshev space is needed for several reasons: it is necessary to prove that the functions defined in [2] and [3] are a basis of
the generalized spline space (see [10]) and it guarantees the positivity of such functions. Moreover, it is essential to have a well-posed definition of the basis: however the generating function $U_i(s)$ ($V_i(s)$) belonging to $W$ is chosen, the normalized function $U_i(s)/U_i(s_i)$ ($V_i(s)/V_i(s_{i+1})$) does not change, since, being $W$ a Chebyshev space, there is a unique element of $W$ taking values 1 (0, resp.) at $s_i$ and 0 (1, resp.) at $s_{i+1}$.

The GB-splines have essentially the same properties of the classical polynomial splines.

**Property 2.1.** The GB-splines satisfy the following properties.

1. Continuity: each $N_{i}^{(p)}$ is $(p-m_j-1)$ times continuously differentiable at the knot $s_j$, where $m_j$ is the multiplicity of $s_j$ in the knot vector $\{s_i,...,s_{i+p}\}$, with $1 \leq m_i \leq p$.
2. Positivity: $N_{i}^{(p)}(s) \geq 0$ for $s \in \mathbb{R}$, $i = 1,...,n$ and $p \in \mathbb{N}$, $p \geq 2$.
3. Local support: if $s \notin [s_i,s_{i+p}]$ $N_{i}^{(p)}(s) = 0$, $i = 1,...,n$ and $p \in \mathbb{N}$, $p \geq 2$.
4. Partition of unity: for $p \geq 3$ and $s \in [s_p,s_{n+1}]$, $\sum_{i=1}^{n} N_{i}^{(p)}(s) = 1$.
5. Linear independence: for any $p \geq 2$ $N_{1}^{(p)},...,N_{n}^{(p)}$ are linearly independent.

### 2.2 Knot insertion formulae

One of the main reasons to introduce the T-spline approach to the construction of spline surfaces is, as already mentioned, the possibility to apply local refinement techniques. Therefore, it is crucial to have reliable knot insertion formulae, which of course must be constructed starting from the univariate case. Differently from the classical polynomial case (see Boehm’s seminal work [2]), in order to give a knot insertion formula for GB-splines, we have to deal with the issue that adding a knot requires the insertion of new functions locally spanning the spline space too.

Therefore, in this section our first goal is to determine the minimum requirements to be satisfied by the newly inserted functions in order to be able to obtain a knot insertion formula. Let us consider the interval $[s_i,s_{i+1}]$, in which the generalized spline space is spanned by the monomials $1, s, ..., s^{p-3}, u_i(s), v_i(s)$, with associated generating functions $U_i(s) = \frac{d^{p-2}u_i(s)}{ds^{p-2}}$ and $V_i(s) = \frac{d^{p-2}v_i(s)}{ds^{p-2}}$. Let us now insert a new knot $\bar{s}$ between $s_i$ and $s_{i+1}$, $i \geq 1$: the interval is split into $[s_i, \bar{s}]$ and $[\bar{s}, s_{i+1}]$, and then in each of these subintervals the generalized spline space is spanned by the monomials $1, s, ..., s^{p-3}$ and by two new functions $\bar{u}_i(s), \bar{v}_i(s)$ and $\bar{u}_{i+1}(s), \bar{v}_{i+1}(s)$, respectively. As a consequence, we must consider the new generating functions $\bar{U}_i(s), \bar{V}_i(s)$ and $\bar{U}_{i+1}(s), \bar{V}_{i+1}(s)$, respectively. If we use the notations $N_j^{(p)}(s)$ and $\bar{N}_j^{(p)}(s)$ for the GB-splines of order $p$, respectively before and after the knot insertion, obtaining a knot insertion formula would mean proving a relation of type

$$N_j^{(p)}(s) = a\bar{N}_j^{(p)}(s) + b\bar{N}_{j+1}^{(p)}(s), \quad j = 1,...,n, \quad (5)$$

with $a$ and $b$ suitable real constants depending on $i$, $j$ and $p$. The formula (5), if true for any $j$, implies that the generalized spline space, coinciding with $\{1, s, ..., s^{p-3}, u_i(s), v_i(s)\}$
on $[s_i, s_{i+1}]$, also coincides with the space $(1, s, ..., s^{p-3}, \bar{u}_i(s), \bar{v}_i(s))$ on $[s_i, \bar{s}]$ and with 
$(1, s, ..., s^{p-3}, \bar{u}_{i+1}(s), \bar{v}_{i+1}(s))$ on $[\bar{s}, s_{i+1}]$. In particular, this means that

$$u_i(s)|_{[s_i, \bar{s}]} = \alpha_0 + \alpha_1 s + ... + \alpha_{p-3,i} s^{p-3} + \alpha_{p-2,i} \bar{u}_i(s) + \alpha_{p-1,i} \bar{v}_i(s),$$
$$v_i(s)|_{[s_i, \bar{s}]} = \beta_0 + \beta_1 s + ... + \beta_{p-3,i} s^{p-3} + \beta_{p-2,i} \bar{u}_i(s) + \beta_{p-1,i} \bar{v}_i(s),$$

for suitable coefficients $\alpha_{k,i}, \beta_{k,i}, \alpha_{k,i+1}, \beta_{k,i+1}$. Since $u_i$ and $v_i$ are differentiated $p - 2$ times for constructing the GB-spline functions, it is not restrictive to consider $\alpha_{k,i} = \beta_{k,i} = \alpha_{k,i+1} = \beta_{k,i+1} = 0$ for $h = 0, ..., p - 3$:

$$u_i(s)|_{[s_i, \bar{s}]} = \alpha_{p-2,i} \bar{u}_i(s) + \alpha_{p-1,i} \bar{v}_i(s),$$
$$v_i(s)|_{[s_i, \bar{s}]} = \beta_{p-2,i} \bar{u}_i(s) + \beta_{p-1,i} \bar{v}_i(s),$$

or, equivalently,

$$\begin{bmatrix} u_i(s) \\ v_i(s) \end{bmatrix} = A_i \begin{bmatrix} \bar{u}_i(s) \\ \bar{v}_i(s) \end{bmatrix}, \quad \begin{bmatrix} u_i(s) \\ v_i(s) \end{bmatrix} = A_{i+1} \begin{bmatrix} \bar{u}_{i+1}(s) \\ \bar{v}_{i+1}(s) \end{bmatrix},$$

where

$$A_i = \begin{bmatrix} \alpha_{p-2,i} & \alpha_{p-1,i} \\ \beta_{p-2,i} & \beta_{p-1,i} \end{bmatrix}, \quad A_{i+1} = \begin{bmatrix} \alpha_{p-2,i+1} & \alpha_{p-1,i+1} \\ \beta_{p-2,i+1} & \beta_{p-1,i+1} \end{bmatrix}.$$

Since, of course, we suppose that $u_i(s)$ and $v_i(s)$, $\bar{u}_i(s)$ and $\bar{v}_i(s)$, $\bar{u}_{i+1}(s)$ and $\bar{v}_{i+1}(s)$ are linearly independent, we have that $det(A_i), det(A_{i+1}) \neq 0$ and therefore, we can obtain

$$\begin{bmatrix} \bar{u}_i(s) \\ \bar{v}_i(s) \end{bmatrix} = A_i^{-1} \begin{bmatrix} u_i(s) \\ v_i(s) \end{bmatrix} \quad \begin{bmatrix} \bar{u}_{i+1}(s) \\ \bar{v}_{i+1}(s) \end{bmatrix} = A_{i+1}^{-1} \begin{bmatrix} u_i(s) \\ v_i(s) \end{bmatrix},$$

that is, the new non-polynomial functions locally spanning the generalized spline space must be linear combinations of the ones used before the knot insertion. In other words, if we want to obtain a knot insertion formula the generalized spline space cannot be changed when inserting a new knot. We can therefore assume $\bar{u}_i(s) = \bar{u}_i(s) = u_i(s)$ and $\bar{v}_i(s) = \bar{v}_i(s) = v_i(s)$. Note that this doesn’t imply that $\bar{U}_i(s) = \bar{U}_{i+1}(s) = U_i(s)$ and $\bar{V}_i(s) = \bar{V}_{i+1}(s) = V_i(s)$. In fact, from the conditions $\square$ it is evident that $\bar{U}_i(s) \neq U_i(s)$, since $\bar{U}_i(\bar{s}) = 0$ while $U_i(s) > 0$; for an analogous reason, $\bar{V}_{i+1}(s) \neq V_i(s)$. On the contrary, it is not restrictive to assume $\bar{V}_i(s) = V_i(s)$, since the only requirements for $\bar{V}_i(s)$ are that it belongs to the space $W = \langle \frac{d^{p-2}u_i}{ds^{p-2}}, \frac{d^{p-2}v_i}{ds^{p-2}} \rangle$ and that it is 0 at $s_i$ and strictly positive at $\bar{s}$, which are satisfied by $\bar{V}_i(s)$. Similarly, we can set $\bar{U}_{i+1}(s) = U_i(s)$.

With this assumptions, we will now prove the knot insertion formula, similarly to what has been done in $\square$.

**Theorem 2.2.** Let $\Sigma = \{s_1, ..., s_{n+1}\}$ be a knot vector, $\bar{\Sigma} = \{\bar{s}_1, ..., \bar{s}_{n+p+1}\}$ the knot vector obtained by inserting a new knot $\bar{s}$, $s_i \leq \bar{s} < s_{i+1}$. Let $\Omega_u = \{u_1(s), ..., u_{n+p-1}(s)\}$,
\( \Omega_v = \{v_1(s), \ldots, v_{n+p-1}(s)\} \) and \( \tilde{\Omega}_u = \{\tilde{u}_1(s), \ldots, \tilde{u}_{n+p}(s)\} \), \( \bar{\Omega}_v = \{\bar{v}_1(s), \ldots, \bar{v}_{n+p}(s)\} \) be the corresponding vectors of functions, where

\[
\begin{align*}
\bar{u}_j(s) &= u_j(s) \quad \text{and} \quad \bar{v}_j(s) = v_j(s) \quad \text{if} \quad j \leq i \\
\bar{u}_j(s) &= u_{j-1}(s) \quad \text{and} \quad \bar{v}_j(s) = v_{j-1}(s) \quad \text{if} \quad j > i.
\end{align*}
\]

(6)

If we denote by \( N_i^{(p)}(s) \) and \( \tilde{N}_i^{(p)}(s) \) the GB-splines of order \( p \), respectively before and after the knot insertion, and by \( r+1 \) the multiplicity of \( \bar{s} \) in \( \Sigma \), then we obtain

\[
N_j^{(p)}(s) = \alpha_{j,p}N_j^{(p)}(s) + \beta_{j+1,p}\tilde{N}_{j+1}^{(p)}(s),
\]

(7)

with, for \( p > 2 \),

\[
\alpha_{j,p} = \begin{cases} 
1 & j \leq i - p, \\
\frac{\delta_j^{(p-1)}}{\delta_j^{(p-1)}\alpha_{j,p-1}} & i - p < j < i - r + 1, \\
0 & j \geq i - r + 1
\end{cases}
\]

\[
\beta_{j,p} = \begin{cases} 
0 & j \leq i - p + 1, \\
\frac{\delta_j^{(p-1)}}{\delta_{j+1}^{(p-1)}\beta_{j+1,p-1}} & i - p + 1 < j < i - r + 2, \\
1 & j \geq i - r + 2
\end{cases}
\]

and, for \( p = 2 \),

\[
\alpha_{j,2} = \begin{cases} 
1 & j < i, \\
\frac{V_i(\bar{s})}{V_i(s_{i+1})} & j = i, \\
0 & j \geq i + 1
\end{cases}
\]

\[
\beta_{j,2} = \begin{cases} 
0 & j < i, \\
\frac{U_i(\bar{s})}{U_i(s_i)} & j = i, \\
1 & j \geq i + 1
\end{cases}
\]

where \( \delta_j^{(p-1)} \) and \( \tilde{\delta}_j^{(p-1)} \) are the constants defined by (1) for \( \Sigma \) and \( \bar{\Sigma} \) respectively, and \( U_i(s) \) and \( V_i(s) \), \( \bar{U}_i(s) \) and \( \bar{V}_i(s) \), \( U_{i+1}(s) \) and \( \bar{V}_{i+1}(s) \) are the generating functions associated to \([s_i, s_{i+1}], [s_i, \bar{s}], [\bar{s}, s_{i+1}]\) respectively and such that

\[
\bar{V}_i(s) = V_i(s), \\
\bar{U}_{i+1}(s) = U_i(s).
\]

**Proof.** We will prove this result by induction. Let us first consider the case of the functions \( N_i^{(2)}(s) \) defined in (2). Since the \( \bar{N}_j^{(2)} \) and \( \tilde{N}_{j+1}^{(2)} \) are given by (2), in the cases where \( \bar{s} \notin [s_j, s_{j+2}] \), that is \( j < i - 1 \) or \( j \geq i + 1 \), clearly we obtain \( N_j^{(2)}(s) = \bar{N}_j^{(2)}(s) \) or \( N_j^{(2)}(s) = \tilde{N}_{j+1}^{(2)}(s) \), and then \( \alpha_{j,2} = 1 \) and \( \beta_{j+1,2} = 0 \) or \( \alpha_{j,2} = 0 \) and \( \beta_{j+1,2} = 1 \) respectively.
In the case \( j = i - 1 \), since \( \bar{s} \not\in [s_j, s_{j+1}] \) we get \( N^{(2)}_{i-1}(s) |_{[s_{i-1}, s_i]} = \bar{N}^{(2)}_{i-1}(s) |_{[s_{i-1}, s_i]} \), which implies \( \alpha_{i-1} = 1 \), while \( \beta_{i,2} \) is given by the equality

\[
N^{(2)}_{i-1}|_{[s_{i-1}, s_i]} = \beta_{i,2} \bar{N}^{(2)}_{i-1}(s)|_{[s_{i-1}, s_i]} = U_i(s) \beta_{i,2} \frac{U_i(s)}{U_{i+1}(s)} \Rightarrow \beta_{i,2} = \frac{U_i(s)}{U_{i+1}(s)},
\]

where we used the fact that \( \bar{U}_{i+1}(s) = U_i(s) \). We have to verify that just obtained coefficients provide us a true formula on \([s_i, \bar{s}]\) as well:

\[
N^{(2)}_{i-1}|_{[s_i, \bar{s}]} = \bar{N}^{(2)}_{i-1}(s)|_{[s_i, \bar{s}]} + \frac{U_i(\bar{s})}{U_i(s_i)} \bar{N}^{(2)}_{i}(s)|_{[s_i, \bar{s}]}.
\]

More explicitly, using again \( \bar{U}_{i+1}(s) = U_i(s) \) and \( \bar{V}_i(s) = V_i(s) \), we have

\[
\frac{U_i(s)}{U_i(s_i)} = \frac{U_i(s)}{U_i(s_i)} + \frac{U_i(\bar{s})V_i(s)}{U_i(s_i)V_i(\bar{s})},
\]

that is,

\[
\frac{U_i(s)}{U_i(s_i)} = \frac{U_i(s)}{U_i(s_i)} - \frac{U_i(\bar{s})V_i(s)}{U_i(s_i)V_i(\bar{s})}.
\]

In order to show that \( \text{(3)} \) is true, it’s sufficient to note that the left-hand side is the function involved in the definition \( \text{(2)} \) and then is the only function in the space spanned by \( \frac{d^{p-2}u}{ds^{p-2}} \) and \( \frac{d^{p-2}u}{ds^{p-2}} \) which takes values 1 at \( s_i \) and 0 at \( \bar{s} \), and the right-hand side is a function belonging to the same space and taking values 1 at \( s_i \) and 0 at \( \bar{s} \). Therefore, the equality must be true. The coefficients for the case \( j = i \) can be analogously obtained, and the case \( p = 2 \) is then proved.

The second part of the proof is essentially the same given for the knot insertion formula in the trigonometric case (see \([23]\)). We suppose that the theorem holds for \( p = m - 1 \), and then show that it is true for \( p = m \) as well. If \( j \leq i - m \), \( \bar{s} \not\in [s_j, s_{j+m}] \), and therefore \( N^{(m)}_j(s) = \bar{N}^{(m)}_j(s) \), which means that \( \alpha_{j,m} = 1 \) and \( \beta_{j+1,m} = 0 \). Similarly, if \( j \geq i - r + 1 \), we get \( N^{(m)}_j(s) = \bar{N}^{(m)}_{j+1}(s) \) (\( \alpha_{j,m} = 0 \) and \( \beta_{j+1,m} = 1 \)) because the sequences of knots \( \{s_j, ..., s_{j+m}\} \) and \( \{\bar{s}_{j+1}, ..., \bar{s}_{j+m+1}\} \) coincide. Therefore, we only need to consider the case \( i - m + 1 \leq j \leq i - r \). By expanding \( \text{(3)} \) and using the induction hypothesis we obtain

\[
N^{(m)}_j(s) = \int_{-\infty}^{s} (\delta^{(m-1)}_j N^{(m-1)}_j(v) - \delta^{(m-1)}_{j+1} \bar{N}^{(m-1)}_{j+1}(v))dv
\]

\[
= \int_{-\infty}^{s} \left( \delta^{(m-1)}_j (\alpha_{j,m-1} \bar{N}^{(m-1)}_j(v) + \beta_{j+1,m-1} \bar{N}^{(m-1)}_{j+1}(v)) - \delta^{(m-1)}_{j+1} (\alpha_{j+1,m-1} \bar{N}^{(m-1)}_{j+1}(v) + \beta_{j+2,m-1} \bar{N}^{(m-1)}_{j+2}(v)) \right)dv
\]

\[
=A_1(s) + A_2(s) + A_3(s),
\]
Finally, we get where

\[
A_1(s) = \delta_j^{(m-1)} \left( \delta_j^{(m-1)} \right)^{-1} \alpha_{j,m-1} \int_{-\infty}^{s} \left( \delta_j^{(m-1)} \tilde{N}_j^{(m-1)}(v) - \delta_{j+1}^{(m-1)} \tilde{N}_{j+1}^{(m-1)}(v) \right) dv
\]

\[
A_2(s) = \delta_{j+1}^{(m-1)} \left( \delta_{j+1}^{(m-1)} \right)^{-1} \beta_{j+2,m-1} \int_{-\infty}^{s} \left( \delta_{j+1}^{(m-1)} \tilde{N}_{j+1}^{(m-1)}(v) - \delta_{j+2}^{(m-1)} \tilde{N}_{j+2}^{(m-1)}(v) \right) dv
\]

\[
A_3(s) = \lambda \int_{-\infty}^{s} \tilde{N}_{j+1}^{(m-1)}(v) dv,
\]

for some real number \( \lambda \in \mathbb{R} \); such constant \( \lambda \) turns out to be null: in fact, since \( N_j^{(m)}(w) = A_1(w) = A_2(w) = 0 \) and \( \int_{-\infty}^{w} \tilde{N}_j^{(m-1)}(v) dv = 1 \) for \( w \geq s_{i-r+m} \), we have

\[
0 = N_j^{(m)}(w) = A_1(w) + A_2(w) + A_3(w) = \lambda \int_{-\infty}^{w} \tilde{N}_{j+1}^{(m-1)}(v) dv = \lambda.
\]

Finally, we get

\[
N_j^{(m)}(s) = \delta_j^{(m-1)} \delta_j^{(m-1)} \alpha_{j,m-1} \tilde{N}_j^{(m)}(s) + \delta_{j+1}^{(m-1)} \beta_{j+2,m-1} \tilde{N}_{j+1}^{(m)}(s),
\]

which concludes the proof of the theorem. \( \square \)

## 3 Generalized T-splines

### 3.1 T-mesh

In order to define the GT-splines, we need to briefly recall some definitions and notations about the T-meshes, which is the same used for the classical polynomial case (see, e.g., [4] and [5]).

Let \( \Sigma^s = \{ s_{-\lfloor p/2 \rfloor + 1}, ..., s_{\mu+\lfloor p/2 \rfloor} \} \) and \( \Sigma^t = \{ t_{-\lfloor q/2 \rfloor + 1}, ..., t_{\nu+\lfloor q/2 \rfloor} \} \) be two index vectors, where \( \mu, \nu \in \mathbb{Z}, p, q \in \mathbb{Z} \) are equal to or greater than 2 and, for any real number \( k, \lfloor k \rfloor \) is the largest integer smaller than or equal to \( k \). Analogously, \( \Omega_u^s = \{ u_1^s(s), ..., u_{n+p-1}^s(s) \} \) and \( \Omega_v^s = \{ v_1^s(s), ..., v_{n+p-1}^s(s) \} \), \( \Omega_u^t = \{ u_1^t(t), ..., u_{n+q-1}^t(t) \} \) and \( \Omega_v^t = \{ v_1^t(t), ..., v_{n+q-1}^t(t) \} \) are the associated vectors of functions.

An index T-mesh \( M \) is a rectangular partition of the index domain \( [\lfloor \lfloor p/2 \rfloor + 1, \mu + \lfloor p/2 \rfloor] \times \lfloor \lfloor q/2 \rfloor + 1, \nu + \lfloor q/2 \rfloor \] such that the vertices have integer coordinates (see Figure 1(a)). In other words, \( M \) is the collection of all the elements of such partition, which are called cells. Note that, since the elements are rectangular, T-junctions are allowed but L-junctions or I-junctions are not. We call edge any segment, either horizontal or vertical, linking two vertices of the mesh. We denote the set of vertices by \( V \) and by \( hE, vE \) and \( E \) the sets containing only horizontal, only vertical and all the edges respectively. We
assume that the edges are open, and we denote by $\partial e$ the set of the endpoints of an edge $e \in E$. The valence of a vertex $P$ is the number of edges $e \in E$ such that $P \in \partial e$. We call horizontal (vertical) skeleton the union of all horizontal (vertical) edges and all vertices, and we denote it by $hS$ ($vS$). The skeleton is instead defined as $S = hS \cup vS$.

We define the active region $AR_{p,q}$ and frame region $FR_{p,q}$ (see Figure 1(b)) as

$$AR_{p,q} = [1, \mu] \times [1, \nu],$$

and

$$FR_{p,q} = \left(\left[-\lfloor p/2 \rfloor + 1, 1\right] \cup \left[\mu, \mu + \lfloor p/2 \rfloor\right]\right) \times \left[\left[-\lfloor q/2 \rfloor + 1, 1\right] \cup \left[\nu, \nu + \lfloor q/2 \rfloor\right]\right].$$

Figure 1: A T-mesh (a) in the case $p = q = 4$, $\mu = 4$ and $\nu = 3$, and (b) the corresponding active region highlighted in gray, with the remaining part representing the frame region.

**Definition 3.1.** A T-mesh $M$ is admissible for the bi-order $(p, q)$ if $S \cap FR_{p,q}$ includes the segments

$$\{l\} \times \left[\left[-\lfloor q/2 \rfloor + 1, 1\right] \cup \left[\mu, \mu + \lfloor p/2 \rfloor\right]\right]$$

for $l = -\lfloor p/2 \rfloor + 1, \ldots, 1$, and $l = \mu, \ldots, \mu + \lfloor p/2 \rfloor$,

$$\left[-\lfloor p/2 \rfloor + 1, \mu + \lfloor p/2 \rfloor\right] \times \{l\}$$

for $l = -\lfloor q/2 \rfloor + 1, \ldots, 1$, and $l = \nu, \ldots, \nu + \lfloor q/2 \rfloor$,

and all vertices belonging to $(-\lfloor p/2 \rfloor + 1, \mu + \lfloor p/2 \rfloor) \times (-\lfloor q/2 \rfloor + 1, \nu + \lfloor q/2 \rfloor) \cap FR_{p,q}$ have valence 4 (see Fig. 2). $AD_{p,q}$ will denote the set of admissible T-meshes for the bi-order $(p, q)$.

**Definition 3.2.** A T-mesh $M \in AD_{p,q}$ belongs to $AD_{p,q}^+$ if, for any couple of vertices $P_1 = (i_1, j_1), P_2 = (i_2, j_2) \in \mathcal{V}$ both belonging to the boundary of a cell and such that $i_1 = i_2$ ($j_1 = j_2$, resp.), the segment $i_1 \times \{j_1\} \times (i_1, i_2), \text{resp.}$) belongs to $S$. 

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In other words, a T-mesh satisfying the definition 3.2 does not have any “facing” T-junctions. While considering this additional requirement is not necessary now, we will need it later to guarantee the equivalence between analysis-suitable and dual-compatible T-meshes (see [5]).

The so-called anchors, which are basic to the construction of T-splines, are defined as follows.

**Definition 3.3.** Given T-mesh $M \in AD_{p,q}$, the set of anchors $A_{p,q}(M)$ is defined in the following way (see Figure 3 for some examples):

- if both $p$ and $q$ are even, $A_{p,q}(M) = \{ A \in \mathcal{Y} : A \subset AR_{p,q} \}$;
- if $p$ is odd and $q$ is even, $A_{p,q}(M) = \{ A \in hE : A \subset AR_{p,q} \}$;
- if $p$ is even and $q$ is odd, $A_{p,q}(M) = \{ A \in vE : A \subset AR_{p,q} \}$;
- if both $p$ and $q$ are odd, $A_{p,q}(M) = \{ A \in M : A \subset AR_{p,q} \}$.

Note that an anchor $A$ can be expressed in the form $a \times b$, where $a$ and $b$ can be: i) two integers with $1 \leq a \leq \mu$, $1 \leq b \leq \nu$, if both $p$ and $q$ are even; ii) two open segments with integer endpoints, if both $p$ and $q$ are odd; iii) an open segment and an integer, if $p$ is odd and $q$ is even; iv) an integer and an open segment, if $p$ is even and $q$ is odd. The case (i) describes a vertex, (ii) describes a cell of the T-mesh $M$, and (iii) (iv), respectively) describes a vertical (horizontal, respectively) edge.

![Figure 2](image1.png)  
![Figure 2](image2.png)

**Figure 2:** Examples of (a) admissible and (b) non-admissible T-mesh with $p = q = 4$, $\mu = \nu = 5$. 

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Figure 3: Two examples of T-mesh with their anchors: in (a), where $p = q = 4$ and $\mu = \nu = 5$, the anchors are the vertices of the active region, while in (b), where $p = q = 3$ and $\mu = \nu = 7$, the anchors are the cells of the active region.

We define the ordered sets

\[
\begin{align*}
  hJ(a) &= \{i \in \mathbb{Z} : \{i\} \times a \subset vS\} \\
  vJ(a) &= \{j \in \mathbb{Z} : a \times \{j\} \subset hS\},
\end{align*}
\]

where $a$ is either a single integer or an open interval with integer endpoints.

**Definition 3.4.** For each anchor $A = a \times b \in \mathcal{A}_{p,q}(M)$, the global horizontal (vertical) index vector is

\[
\begin{align*}
  \mathbf{I}^h(A) &= hJ(b) = \{i^h_1(A), \ldots, i^h_p(A)\} \\
  \mathbf{I}^v(A) &= vJ(a) = \{i^v_1(A), \ldots, i^v_q(A)\},
\end{align*}
\]

where $p + 1 \leq \bar{p} \leq \mu + 2\lfloor p/2 \rfloor$ and $q + 1 \leq \bar{q} \leq \nu + 2\lfloor q/2 \rfloor$. Moreover, for each anchor $A = a \times b \in \mathcal{A}_{p,q}(M)$, the local horizontal index vector is the subset $\mathbf{I}^l(A)$ defined by

- if $p$ is even, $\mathbf{I}^l(A) = (i^l_{i,1}(A), \ldots, i^l_{i,p+1}(A))$ contains the $p + 1$ consecutive indices in $hJ(b)$ such that $i^l_{i,p/2+1}(A) = a$;
- if $p$ is odd, $\mathbf{I}^l(A) = (i^l_{i,1}(A), \ldots, i^l_{i,p+1}(A))$ contains the $p + 1$ consecutive indices in $hJ(b)$ such that $(i^l_{i,(p+1)/2}, i^l_{i,(p+1)/2+1}) = a$.

The local vertical index vector $\mathbf{I}^v_l(A)$ is analogously defined (see Figure 4 for an example).

The T-mesh in parameter space (see Figure 5(b)) is defined as the partition of the domain $[s_{-\lfloor p/2 \rfloor+1}, s_{\mu+\lfloor p/2 \rfloor}] \times [t_{-\lfloor q/2 \rfloor+1}, t_{\nu+\lfloor q/2 \rfloor}]$ obtained by considering the elements of the form

\[(s_{i_1}, s_{i_2}) \times (t_{j_1}, t_{j_2}) \neq \emptyset,
\]

where $(i_1, i_2) \times (j_1, j_2) \in M$. 

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Figure 4: (a) Local index vector and (b) global index vector for the anchors $A$ and $B$ belonging to a T-mesh with $p = q = 4$ and $\mu = \nu = 5$: the gray circles represent the indices of $A$’s vectors, while the black squares denote the indices of $B$’s.

Finally, we associate to each anchor $A \in \mathcal{A}_{p,q}(M)$ the following global (local, respectively) knot and functions vectors:

$$
\Sigma^a(A) = \{s_i \in \Sigma^a : i \in I^a(A)\},
$$

$$
\Sigma^t(A) = \{t_j \in \Sigma^t : j \in I^t(A)\},
$$

$$
\Omega^a_{u}(A) = \{u^a_i(s) \in \Omega^a_{u} : i \in I^a(A) \setminus \{i^a_p(A)\}\},
$$

$$
\Omega^a_{v}(A) = \{v^a_i(s) \in \Omega^a_{v} : i \in I^a(A) \setminus \{i^a_p(A)\}\},
$$

$$
\Omega^t_{u}(A) = \{u^t_j(t) \in \Omega^t_{u} : j \in I^t(A) \setminus \{j^t_p(A)\}\},
$$

$$
\Omega^t_{v}(A) = \{v^t_j(t) \in \Omega^t_{v} : j \in I^t(A) \setminus \{j^t_p(A)\}\},
$$

$$
\Sigma^a_t(A) = \{s_i \in \Sigma^a : i \in I^a_t(A)\},
$$

$$
\Sigma^t_t(A) = \{t_j \in \Sigma^t : j \in I^t_t(A)\},
$$

$$
\Omega^a_{u,1}(A) = \{u^a_i(s) \in \Omega^a_{u,1} : i \in I^a_t(A) \setminus \{i^a_{p+1}(A)\}\},
$$

$$
\Omega^a_{v,1}(A) = \{v^a_i(s) \in \Omega^a_{v,1} : i \in I^a_t(A) \setminus \{i^a_{p+1}(A)\}\},
$$

$$
\Omega^t_{u,1}(A) = \{u^t_j(t) \in \Omega^t_{u,1} : j \in I^t_t(A) \setminus \{j^t_{q+1}(A)\}\},
$$

$$
\Omega^t_{v,1}(A) = \{v^t_j(t) \in \Omega^t_{v,1} : j \in I^t_t(A) \setminus \{j^t_{q+1}(A)\}\}
$$

The T-mesh is very often represented in the index-parameter space, where lines corresponding to a repeated knot do not coincide (see Figure 5).

Before formally defining the generalized T-splines, we introduce some additional notations we will use the in the next sections. In order to make more explicit the dependence of the GB-splines on the knot vector $\Sigma$ and on the vectors of functions $\Omega_u$ and $\Omega_v$, we set

$$
N_j[\Sigma, \Omega_u, \Omega_v](s) \equiv N^{(p)}_j(s), \quad j = 1, \ldots, n,
$$
where \( N_j(s) \) is the \( j \)-th GB-spline defined in (2) and (3) with vectors \( \Sigma, \Omega_u \) and \( \Omega_v \). Given a local knot vector \( \Sigma_l = \{ s_1, \ldots, s_{p+1} \} \) of length \( p + 1 \) and two local vectors of functions \( \Omega_{u,l} = \{ u_1(s), \ldots, u_p(s) \} \) and \( \Omega_{v,l} = \{ v_1(s), \ldots, v_p(s) \} \) of length \( p \), we denote by \( N[\Sigma_l, \Omega_{u,l}, \Omega_{v,l}](s) \) the only GB-spline of order \( p \) we can define by setting \( \Sigma = \Sigma_l, \Omega_u = \Omega_{u,l} \) and \( \Omega_v = \Omega_{v,l} \) in the equations (2) and (3). We will use analogous notations for the classical univariate polynomial B-splines: \( P_j[\Sigma](s) \) is the \( j \)-th polynomial B-spline function of degree \( p - 1 \) defined with the knot vector \( \Sigma \) (see, e.g., [2] and [1]), and \( P[\Sigma_l](s) \) is the only polynomial B-spline function of degree \( p - 1 \) defined on the knot vector \( \Sigma_l \) of length \( p + 1 \).

### 3.2 Generalized T-splines: definition and basic properties

Then, we define, for each anchor, a bivariate Generalized T-spline (GT-splines):

\[
N_A(s,t) = N[\Sigma^s(A), \Omega^s_{u,l}(A), \Omega^s_{v,l}(A)](s)N[\Sigma^t(A), \Omega^t_{u,l}(A), \Omega^t_{v,l}(A)](t),
\]

where \( N[\Sigma^s(A), \Omega^s_{u,l}(A), \Omega^s_{v,l}(A)](s) \) and \( N[\Sigma^t(A), \Omega^t_{u,l}(A), \Omega^t_{v,l}(A)](t) \) are univariate GB-splines in the variables \( s \) and \( t \). Of course, the Trigonometric Generalized T-splines (TGT-splines) introduced in [3] are a particular case of the just defined GT-splines, obtained by setting \( u^*_i(s) = \cos(\omega^*_i(s)) \) and \( v^*_i(s) = \sin(\omega^*_i(s)) \) and \( u^*_j(t) = \cos(\omega^*_j(t)) \) and \( v^*_j(t) = \sin(\omega^*_j(t)) \), where \( \omega^*_i \) and \( \omega^*_j \) are frequencies such that \( 0 < \omega^*_i < \pi/(s_{i+1} - s_i) \) and \( 0 < \omega^*_j < \pi/(t_{j+1} - t_j) \) for any \( i, j \).

Several properties holding for the polynomial T-splines are also satisfied by the GT-splines.

**Property 3.5.** The GT-splines enjoy the following properties, as direct consequence of their definition.

1. **Continuity:** each blending function \( N_A(s,t) \), for any \( A \in \mathcal{A}_{p,q}(M) \), is \((p - m^s_i - 1)\) times continuously differentiable with respect to \( s \) and \((q - m^t_j - 1)\) times continuously differentiable with respect to \( t \) at the point \((s_i, t_j)\), where \( m^s_i \) and \( m^t_j \) are the multiplicities of \( s_i \) and \( t_j \) in the knot vectors \( \Sigma^s(A) \) and \( \Sigma^t(A) \), respectively.
2. **Positivity:** $N_A(s, t) \geq 0$ for $(s, t) \in \mathbb{R}^2$, $A \in \mathcal{A}_{p,q}(M)$ and $p, q \in \mathbb{N}$, $p, q \geq 2$.

3. **Local support:** if $(s, t) \notin [s_i^{(A)}, s_{i+1}^{(A)}] \times [t_j^{(A)}, t_{j+1}^{(A)}]$, then $N_A(s, t) = 0$, $A \in \mathcal{A}_{p,q}(M)$ and $p, q \in \mathbb{N}$, $p, q \geq 2$.

4. **Linear independence for tensor-product case:** If $M$ is a tensor-product mesh, that is, all the vertices have valence 4, then the corresponding blending functions are linearly independent.

5. **Partition of unity for tensor-product case:** If $M$ is a tensor-product mesh, then the corresponding blending functions form a partition of unity.

We can use the blending functions (9) to construct a spline surface in the same way as in the polynomial case:

$$T(s, t) = \sum_{A \in \mathcal{A}_{p,q}(M)} T_A w_A N_A(s, t),$$

where $T_A \in \mathbb{R}^3$ are given control points and $w_A \in \mathbb{R}^+$ are the weights.

Note that the constant function 1 may not belong to $\text{span}\{N_A(s, t) : A \in \mathcal{A}_{p,q}(M)\}$, and then considering the rational form (10) allows to get the partition of unity property. If $1 \in \text{span}\{N_A(s, t) : A \in \mathcal{A}_{p,q}(M)\}$, the rational form is not needed and we can use

$$T(s, t) = \sum_{A \in \mathcal{A}_{p,q}(M)} T_A N_A(s, t),$$

which allows to combine the features of the classical T-splines and the reproduction properties of the GB-splines. If instead $1 \notin \text{span}\{N_A(s, t) : A \in \mathcal{A}_{p,q}(M)\}$ and we want to avoid the rational form, we can consider T-meshes constructed starting from a tensor-product one and apply knot insertion formula repeatedly. In fact, as already mentioned between their properties, the GT-splines associated to a tensor-product mesh are a partition of unity, and it can be easily shown, as in the polynomial case (see, e.g., [11] and [12]), that applying the knot insertion formula allows us to preserve the partition of unity property, that is, to get a set of GT-splines $\{N_A(s, t)\}_{A \in \mathcal{A}_{p,q}(M)}$ such that

$$\sum_{A \in \mathcal{A}_{p,q}(M)} k_A N_A(s, t) = 1,$$

where the $k_A$ are non-negative coefficients. In short, the concepts of standard and semi-standard T-splines (see [11] and [12]) can be extended to our non-polynomial setting.

## 4 Linear independence of the GT-splines and VMCR T-meshes

### 4.1 GT-splines and tensor-product splines

The linear independence of the T-splines is a key point for at least one of their main applications, that is, isogeometric analysis (see, e.g., [1]). Therefore, the study of linear
independence is basic for the theory of the just introduced GT-splines as well, which is the reason why we devote Section 4 to this topic.

In general, the situation about linear independence of GT-splines may not coincide with the one of the classical polynomial T-splines. For instance, it has been shown that there are examples where the arguments used to prove the linear dependence of the T-splines do not hold in the case of the TGT-splines (see [3]). We will then study the linear independence of the GT-splines by examining the relation between them and the tensor-product spline functions associated to the so-called underlying tensor product mesh (which are linearly independent, as already mentioned).

**Definition 4.1.** Given a T-mesh $M$, its underlying tensor-product mesh $\hat{M}$ is the T-mesh with the same index domain of $M$ and obtained by adding to $M$ vertices and edges such that all the vertices have valence 4 and the knot and functions vectors $\Sigma^s$, $\Sigma^t$, $\Omega_u$, $\Omega_v$, $\Omega_\hat{u}$ and $\Omega_\hat{v}$ are unvaried.

Then, we can get the underlying tensor-product mesh $\hat{M}$ of a T-mesh $M$ by adding edges. Therefore, there is a linear relation between the two sets of GT-splines associated to $\Sigma$ belonging to $M$ mesh with the same index domain of $A \in A_\Sigma$.

**Theorem 4.2.** A necessary and sufficient condition for the GT-spline blending functions $\{N_i(s,t)\}_{i=1}^n$ to be linearly independent is that $C$ is full rank.

**Proof.** See the analogous Theorem in [3].

\[ N_A(s,t) = \sum_{B \in A_{p,q}(M)} c_{A,B} \hat{N}_B(s,t), \quad A \in A_{p,q}(M), \]  

where $\{N_A(s,t)\}_{A \in A_{p,q}(M)}$ and $\{\hat{N}_B(s,t)\}_{B \in A_{p,q}(\hat{M})}$ are the sets of GT-splines associated to $M$ and $\hat{M}$, respectively. If we denote the sets of the anchors of $M$ and $\hat{M}$ by $A_{p,q}(M) = \{A_1, ..., A_n\}$ and $A_{p,q}(\hat{M}) = \{\hat{A}_1, ..., \hat{A}_n\}$, (11) can be also written in the form

\[ N = C \hat{N}, \]  

where $N = [N_{A_1}(s,t), ..., N_{A_n}(s,t)]^T$, $\hat{N} = [\hat{N}_{A_1}(s,t), ..., \hat{N}_{A_n}(s,t)]^T$, and $C$ is an $n \times \hat{n}$ matrix $C = (c_{ij})_{i=1, ..., n, j=1, ..., \hat{n}}$, whose elements are obtained by re-labeling the coefficients $c_{A,B}$ in (11). The linear independence of the GT-spline blending functions is equivalent to $C$ being a full-rank matrix.
Given the same T-mesh $M$, the same knot vectors $\Sigma^s$ and $\Sigma^t$, and as a consequence the same anchors and the same global and local knot vectors. We denote by $\{P_{A_1}(s, t)\}_{i=1,...,n}$ the polynomial T-spline blending functions of bi-degree $(p-1, q-1)$, and by $\{\hat{P}_{A_1}(s, t)\}_{i=1}^n$ the tensor-product B-spline functions associated to the underlying tensor-product mesh $\bar{M}$. We can obtain also in this case, by repeatedly applying Boehm’s knot insertion formula for the polynomial splines (see, e.g., [4]), the relation

$$\mathbf{P} = \mathbf{D}\hat{\mathbf{P}},$$

where $\mathbf{P} = [P_{A_1}(s, t), ..., P_{A_n}(s, t)]^T$, $\hat{\mathbf{P}} = [\hat{P}_{A_1}(s, t), ..., \hat{P}_{A_n}(s, t)]^T$ and $\mathbf{D}$ is an $n \times n$ matrix.

**Theorem 4.3.** A necessary and sufficient condition for the T-spline blending functions $\{P_{A_i}(s, t)\}_{i=1}^n$ to be linearly independent is that $\mathbf{D}$ is full rank.

**Proof.** See, e.g., [9].

In both cases, the nonpolynomial and the polynomial one, the linear independence of the blending functions is equivalent to the matrices $\mathbf{C}$ and $\mathbf{D}$ being full rank, respectively. We will prove that there is a strong connection between the two cases, and in particular between the two matrices. Since, as we’ve already mentioned, the elements of the two matrices $\mathbf{C}$ and $\mathbf{D}$ are obtained by a repeated application of the respective knot insertion formulae, in order to find the link between the two cases we need to understand the relation between their knot insertion formulae, stated in the following Lemma.

**Lemma 4.4.** Let be given a knot vector $\Sigma = \{s_1, ..., s_{n+p}\}$ and another knot vector $\Sigma = \{\bar{s}_1, ..., \bar{s}_{n+p+1}\}$ obtained by inserting a new knot $\bar{s}$ between $s_i$ and $s_{i+1}$. Moreover, let $\Omega_u = \{u_1(s), ..., u_{n+p}\}$, $\Omega_v = \{v_1(s), ..., v_{n+p}\}$ and $\bar{\Omega}_u = \{\bar{u}_1(s), ..., \bar{u}_{n+p}\}$, $\bar{\Omega}_v = \{\bar{v}_1(s), ..., \bar{v}_{n+p}\}$ be their respective vectors of functions, where $\bar{u}_j(s) = u_j(s)$ and $\bar{v}_j(s) = v_j(s)$ if $j \leq i$ or $\bar{u}_j(s) = u_{i+1}(s)$ and $\bar{v}_j(s) = v_{i+2}(s)$ if $j > i$. Let us consider, for any order $p \geq 2$, the knot insertion formulas for the univariate generalized B-splines and for the univariate polynomial B-splines:

$$N_j[\Sigma, \Omega_u, \Omega_v](s) = \alpha_{j,p}\bar{N}_j[\Sigma, \bar{\Omega}_u, \bar{\Omega}_v](s) + \beta_{j+1,p}\bar{N}_{j+1}[\Sigma, \bar{\Omega}_u, \bar{\Omega}_v](s), \quad j = 1, ..., n,$$

$$P_j(s)[\Sigma] = \gamma_{j,p}\bar{P}_j[\Sigma](s) + \eta_{j+1,p}\bar{P}_{j+1}[\Sigma](s), \quad j = 1, ..., n,$$

If we assume that the multiplicity of $\bar{s}$ is $r$ in $\Sigma$ and $r + 1$ in $\bar{\Sigma}$, the coefficients for the univariate generalized B-splines are defined as follows: for $p > 2$,

$$\alpha_{j,p} = \begin{cases} 1, & j \leq i - p, \\ \frac{\delta^{(p-1)}_j}{\delta^{(p-1)}_{j+1}} \alpha_{j,p-1}, & i - p < j < i - r + 1, \\ 0, & j \geq i - r + 1. \end{cases}$$

$$\beta_{j,p} = \begin{cases} 0, & j \leq i - p + 1, \\ \frac{\delta^{(p-1)}_j}{\delta^{(p-1)}_{j+1}} \beta_{j+1,p-1}, & i - p + 1 < j < i - r + 2, \\ 1, & j \geq i - r + 2. \end{cases}$$
and, for $p = 2$,

$$
\alpha_{j,2} = \begin{cases} 
1, & j < i, \\
\frac{V_i(s_i)}{V_i(s_{i+1})}, & j = i, \\
0, & j \geq i + 1
\end{cases},
\beta_{j,2} = \begin{cases} 
0, & j < i, \\
\frac{U_i(s_i)}{U_i(s_{i+1})}, & j = i, \\
1, & j \geq i + 1
\end{cases}
$$

where $\delta^{(p-1)}_j$ and $\bar{\delta}^{(p-1)}_j$ are the constants defined by (4) for $\Sigma$ and $\bar{\Sigma}$ respectively, and $U_i(s)$ and $V_i(s)$, $\bar{U}_i(s)$ and $\bar{V}_i(s)$ are the generating functions associated to $[s_i, s_{i+1}]$, $[s_i, \bar{s}]$, $[\bar{s}, s_{i+1}]$, respectively and such that

$$
\bar{V}_i(s) = V_i(s), \\
\bar{U}_{i+1}(s) = U_i(s).
$$

The coefficients for the polynomial univariate B-splines are instead

$$
\gamma_{j,p} = \begin{cases} 
1, & j \leq i - p, \\
\frac{u - \bar{s}_j}{s_{j+p} - s_j}, & i - p < j < i - r + 1, \\
0, & j \geq i - r + 1
\end{cases},
\eta_{j,p} = \begin{cases} 
0, & j \leq i - p + 1, \\
\frac{\bar{s}_{j+p} - u}{s_{j+p} - s_j}, & i - p + 1 < j < i - r + 2, \\
1, & j \geq i - r + 2
\end{cases}
$$

Then, for $j = 1, \ldots, n$, we have

$$
\alpha_{j,p}, \beta_{j,p}, \gamma_{j,p}, \eta_{j,p} \geq 0, \\
\alpha_{j,p} = 0 \iff \gamma_{j,p} = 0, \beta_{j,p} = 0 \iff \eta_{j,p} = 0.
$$

**Proof.** The result follows from the expressions of $\alpha_{j,p}, \beta_{j,p}, \gamma_{j,p}, \eta_{j,p}$, $j = 1, \ldots, n$. □

The connection between the matrices $C$ and $D$ will be stated in Theorem 4.5 and Corollary 4.6 which can be proved, starting from Lemma 4.4, by using essentially the same arguments used in [3] to study the relation between the polynomial T-splines and the Trigonometric Generalized T-splines (TGT-splines). In fact, the key point to prove the following results is that the two types of splines involved have knot insertion formulae with coefficients satisfying (14). For this reason, here we will just state the results in the more general case of GT-splines without the proofs, which can be found in [3].

**Theorem 4.5.** Let $M \in AD_{p,q}$, let $\Sigma^u$ and $\Sigma^v$ be the associated knot vectors, and $\Omega^u$, $\Omega^v$ and $\Omega^u_i$, $\Omega^v_i$ their corresponding vectors of functions. Let us denote by $\{P_{Ai}\}_{j=1, \ldots, n}$ and $\{N_{Ai}\}_{j=1, \ldots, n}$ the sets of the $T$-spline blending functions and of the GT-spline blending
functions associated to \( M \), respectively. Moreover, let us suppose that we want to complete the vertical lines belonging to \( E^v \subset \{ e \} \times [q/2] \) and the horizontal lines belonging to \( E^h \subset \{ f \} \times \{ p/2 \} \), which corresponds to inserting a set of horizontal knots \( K^h = \{ s_e : e \} \times [q/2]+1, \nu+q/2] \in E^h \) and a set of vertical knots \( K^v = \{ t_f : [-p/2]+1, \mu+p/2] \times \{ f \} \in E^v \) in the horizontal and vertical global knot vectors of the anchors of \( M \), respectively (see Figure 7 for a basic example). If we accordingly apply to the functions \( N_{A_j} \) and \( P_{A_j}, j = 1, ..., n \), the respective knot insertion formulae we get the relations

\[
N = C\tilde{N}, \quad P = D\tilde{P},
\]

where \( \tilde{N} = \tilde{N}_1(s,t), ..., \tilde{N}_n(s,t) \)^T and \( \tilde{P} = \tilde{P}_1(s,t), ..., \tilde{P}_n(s,t) \)^T are vectors containing tensor-product GB-splines and polynomial B-splines of type

\[
\tilde{N}_h(s,t) = N_h[\tilde{\Sigma}_h^s, \tilde{\Sigma}_h^v, \tilde{\Omega}_h^s, \tilde{\Omega}_h^v](s)N_h[\tilde{\Sigma}_h^s, \tilde{\Sigma}_h^v, \tilde{\Omega}_h^s, \tilde{\Omega}_h^v](t), \quad \tilde{P}_h(s,t) = P_h[\tilde{\Sigma}_h^s]P_h[\tilde{\Sigma}_h^v](t),
\]

where, for each \( h = 1, ..., \tilde{n}, h^s \) and \( h^v \) are suitable indices, \( \tilde{\Sigma}_h^s \) and \( \tilde{\Sigma}_h^v \) are vectors obtained by inserting the knots belonging to \( K^h \) and \( K^v \) in the global knot vectors \( \Sigma^s(A) \) and \( \Sigma^v(A) \), for a certain \( A \in A_{p,q}(M) \), and \( \tilde{\Omega}_h^s \) and \( \tilde{\Omega}_h^v \) are the corresponding vectors of functions. If we adopt the notation \( C = (c_{ij})_{i=1,...,n,j=1,...,\tilde{n}} \) and \( D = (d_{ij})_{i=1,...,n,j=1,...,\tilde{n}} \), we have that

\[
c_{ij}, d_{ij} \geq 0, \quad c_{ij} = 0 \iff d_{ij} = 0, \quad i = 1, ..., n, j = 1, ..., \tilde{n}.
\]

Figure 7: T-mesh \( M \) (a) in the index space, with \( p = q = 4, \mu = 5 \) and \( \nu = 4 \), and (b) the same T-mesh in the index-parameter space. The dashed line represents the edge to be added in order to complete the line \( \{ 2 \} \times [1, 6] \) (\( \{ s_2 \} \times [t_{-1}, t_6] \) in the index-parameter space); the insertion of the edge corresponds to inserting \( s_2 \) in the horizontal global knot vectors of the anchors \( A_1, A_2, A_3, A_4, A_5, A_6 \).
Corollary 4.6. Let $M \in AD_{p,q}$, and let $\{N_{A_j}\}_{j=1,\ldots,n}$ and $\{P_{A_j}\}_{j=1,\ldots,n}$ be the sets of the GT-spline blending functions and of the T-spline blending functions associated to $M$, respectively. Moreover, let $\{\hat{N}_{A_j}\}_{j=1,\ldots,\hat{n}}$ and $\{\hat{P}_{A_j}\}_{j=1,\ldots,\hat{n}}$ be the sets of the GT-spline blending functions and of the T-spline blending functions associated to the underlying tensor-product mesh $\hat{M}$. If we denote by $C$ and $D$ the matrices expressing the relation between the functions $\{N_{A_j}\}_{j=1,\ldots,n}$ and $\{\hat{N}_{A_j}\}_{j=1,\ldots,\hat{n}}$, and between $\{P_{A_j}\}_{j=1,\ldots,n}$ and $\{\hat{P}_{A_j}\}_{j=1,\ldots,\hat{n}}$, defined in (12) and (13), we have that

$$c_{ij} = 0 \iff d_{ij} = 0, \quad i = 1,\ldots,n, \quad j = 1,\ldots,\hat{n}.$$ 

(15)

4.2 VMCR T-meshes

Using Corollary\ref{corollary:4.6} we will now show that it’s possible to define a class of T-meshes which guarantees the linear independence both for the classical polynomial T-splines and for the GT-splines.

First of all, let us recall what the procedure of column reduction is (see, e.g., \cite{ appendix}). Given a matrix $C$, if all the elements of the $i$-th row are zeros except the $j$-th one, then we call the $j$-th column innocuous. The column reduction procedure consists of removing from $C$ all the innocuous columns and all the zero rows left after the column removal. The following lemma provides a sufficient condition on the result of column reduction and the rank of the considered matrix.

Lemma 4.7. Given an $m \times n$ matrix $Q$ ($m \leq n$), if the column reduction procedure applied to $Q^T$ gives as result the void matrix, then $Q$ is a full rank matrix.

Proof. See, e.g., \cite{ appendix}. $\square$

If we apply the column reduction to the matrices $C^T$ and $D^T$ defined in (12) and (13), we can state the following result.

Corollary 4.8. Let $C$ and $D$ be the matrices defined in (12) and (13), and let $C_{CR}$ and $D_{CR}$ be the matrices obtained by applying the column reduction procedures to $C^T$ and $D^T$, respectively. Then we have

$$C_{CR} = \emptyset \iff D_{CR} = \emptyset,$$

where $\emptyset$ stands for the void $0 \times 0$ matrix.

Proof. The equivalence (16) is a direct consequence of (15) and of the definition of column reduction, which depends only on whether or not the elements of matrix are zero. $\square$

As a consequence, the following Corollary holds.

Corollary 4.9. There exists a class of T-meshes for which both the associated GT-spline blending functions of bi-order $(p,q)$ and the T-spline blending functions of bi-degree $(p-1,q-1)$ are linearly independent. This class is defined as the class of T-meshes such that the matrix $C_{CR}$ obtained by applying the column reduction procedure to $C^T$ (equivalently, the matrix $D_{CR}$ obtained by applying the column reduction procedure to $D^T$) is the void matrix. We will call it the class of VMCR T-meshes (Void Matrix after Column Reduction). All the tensor-product meshes belong to this class.
Proof. Let $C_{CR}$ and $D_{CR}$ be the matrices obtained by applying the column reduction procedure to $C^T$ and $D^T$, defined in (12) and (13), respectively. Let us consider the class of T-meshes such that the matrix $C_{CR}$ is the void matrix or, equivalently, such that the matrix $D_{CR}$ is the void matrix. In fact, if one of these conditions is satisfied for a T-mesh $M$, by Corollary 4.8 also the other is satisfied, and by Lemma 4.7 $C$ and $D$ are full rank. Finally, this implies, by Theorems 4.2 and 4.3 that the GT-spline and the T-spline blending functions associated to $M$ are linearly independent.

Note that, since for any two matrices $C$ and $D$ the conditions $C_{CR} = \emptyset$ and $D_{CR} = \emptyset$ are sufficient, but not necessary, to ensure that $C$ and $D$ are full rank, Corollary 4.8 doesn’t imply the equivalence of the linear independence of the GT-spline and T-spline blending functions associated to a T-mesh.

We will now study the class of VMCR T-meshes. In particular, we will investigate its relationship with the class of analysis-suitable T-meshes, the most known class of T-meshes guaranteeing the linear independence of the associated T-spline blending functions. Then, let us recall the definition and some basic facts about it.

First, we need the definition of T-junction extension, introduced in [5] and generalized to any bi-order in [5]. Given a T-mesh $M$ (in the index space), let us consider a T-junction $T = (i, j)$ belonging to the active region $AR_{p,q}$ and with valence 3, and assume it is of type “+”, that is, two opposite vertical edges and one horizontal edge from left intersects at $T$. Moreover, let us consider the set of indices $hJ(j)$ and let $i_1, \ldots, i_p$ be the $p$ consecutive indices extracted from $hJ(j)$ such that $i_k = i$, with $k = \lceil p/2 \rceil$. Then the horizontal extension $hext_{p,q}(T)$, with respect to the bi-order $(p, q)$, is defined as the union of the face extension $hext_{f_{p,q}}(T)$ and of the edge extension $hext_{e_{p,q}}(T)$, which are determined as follows:

$$hext_{f_{p,q}}(T) = [i, i_p] \times \bar{j}, \quad hext_{e_{p,q}}(T) = [i_1, \bar{i}] \times \bar{j}.$$  

We can define analogously the extensions for the other types of T-junctions (see Figure 8 for an example).

![Figure 8](image_url)

(a)  (b)

Figure 8: A T-mesh (a), with $p = q = 4$, and (b) the T-node extensions of its T-nodes.
Definition 4.10. If no horizontal extension with respect to the bi-order \((p, q)\) intersects a vertical extension with respect to the bi-order \((p, q)\), the T-mesh is called analysis suitable with respect to the bi-order \((p, q)\) (see Figure 9 for an example).

Figure 9: An example of (a) analysis-suitable and (b) non-analysis-suitable T-mesh (with \(p = q = 4\)).

The class of analysis-suitable T-meshes coincides with another one: dual-compatible T-meshes. This class was introduced in [4] and its equivalence to analysis-suitable T-meshes was proved, for a general bi-degree, in [5]. Since this equivalence will be the key to study the relationship between the analysis-suitable and the VMCR T-meshes, let us recall the definition of dual-compatible T-mesh.

Let \(M \in AD_{p,q}^+\) and let \(A_1\) and \(A_2\) be two anchors with local horizontal index vectors 
\[ I_h^*(A_1) = \{i_{h+1}^*(A_1), \ldots, i_{p+1}^*_h(A_1)\} \] and 
\[ I_h^*(A_2) = \{i_{h+1}^*_1(A_2), \ldots, i_{p+1}^*_h(A_2)\} \]. We say that \(A_1\) and \(A_2\) overlap horizontally if
\[
\forall k \in I_h^*(A_1), \quad i_{h+1}^*(A_2) \leq k \leq i_{p+1}^*(A_2) \Rightarrow k \in I_h^*(A_2),
\]
\[
\forall k \in I_h^*(A_2), \quad i_{h+1}^*(A_1) \leq k \leq i_{p+1}^*(A_1) \Rightarrow k \in I_h^*(A_1).
\]

Analogously, if \(I_t^*(A_1) = \{i_t^*(A_1), \ldots, i_{q+1}^*(A_1)\} \) and \(I_t^*(A_2) = \{i_t^*(A_2), \ldots, i_{q+1}^*(A_2)\} \) are the vertical index vectors of \(A_1\) and \(A_2\), we we say that \(A_1\) and \(A_2\) overlap vertically if
\[
\forall h \in I_t^*(A_1), \quad i_t^*(A_2) \leq h \leq i_{q+1}^*(A_2) \Rightarrow h \in I_t^*(A_2),
\]
\[
\forall h \in I_t^*(A_2), \quad i_t^*(A_1) \leq h \leq i_{q+1}^*(A_1) \Rightarrow h \in I_t^*(A_1).
\]

Moreover, the anchors \(A_1\) and \(A_2\) are said to partially overlap if they overlap either horizontally or vertically. We can now give the definition of dual compatible T-meshes.

Definition 4.11. A T-mesh \(M \in AD_{p,q}^+\) is dual compatible with respect to the bi-order \((p, q)\) if any two anchors \(A_1, A_2 \in \mathcal{A}_{p,q}(M)\) partially overlap.
In [9] Li and his co-authors proved that the following fundamental result holds.

**Lemma 4.12.** Let $M \in AD_{4,4}$, and let $D_{CR}$ be the matrix obtained by applying the column reduction procedure to the transpose of the matrix $D$ defined in (13) for the case of T-spline blending functions of bi-degree $(3,3)$. If $M$ is analysis suitable, then we have that $D_{CR}$ is the void matrix.

As a consequence, the following result about analysis-suitable T-meshes holds.

**Theorem 4.13.** For the bi-order $(4,4)$, the class of analysis-suitable T-meshes is included in the class of VMCR T-meshes.

**Proof.** By Lemma 4.12 the matrix $D_{CR}$ obtained by applying the column reduction to the transpose of the matrix $D$ defined in (13) is void and therefore, by Corollary 4.8, so is also the matrix $C_{CR}$ obtained by applying the column reduction to the transpose of the matrix $C$ defined in (15).

This implies that, at least in the case of bi-order $(4,4)$, the class of VMCR T-meshes contains other T-meshes besides the tensor-product ones, since the class of the analysis-suitable ones is included in it. Our next goal in this paper will be to show that such inclusion is true for a general bi-order $(p,q)$, which will be proved by using a generalization of some of the concepts used by Li and his co-authors in [9] and the equivalence between analysis-suitable and dual-compatible T-meshes.

First, we introduce the notion of influence sub-matrix of a set of anchors $A$. We will give the definitions and the following results referring to the GT-spline blending functions, but it can be easily verified, by using Corollary 4.6, that they hold for the polynomial T-splines as well.

**Definition 4.14.** Given a set of anchors $A \subset A_{p,q}(M)$, the influence submatrix of $A$ (for the GT-spline blending functions), denoted by $C(A)$, is obtained from the matrix $C$ defined in (12) by removing the rows corresponding to the anchors not belonging to $A$ and all the zero columns left after the rows removal.

We observe that, since each $C(A)$ is essentially a submatrix of $C$ defined in (12), each of its rows corresponds to an anchor in $A_{p,q}(M)$ and each of its columns corresponds to an anchor in $A_{p,q}(M)$, where $M$ is the underlying tensor product mesh of $M$.

**Definition 4.15.** The influence submatrix $C(A)$ of a given set of anchors $A$ is called a 2-influence submatrix if each column has at least 2 non-zero elements.

**Lemma 4.16.** Given a T-mesh $M \in AD_{p,q}$, if for any set of anchors $A \subset A_{p,q}(M)$ the influence submatrix $C(A)$ for the GT-spline functions is not a 2-influence submatrix, then $M$ belongs to the class of VMCR T-meshes.

**Proof.** The matrices we obtain at each step of the procedure of column reduction applied to $C^T$ can be considered as transpose matrices of influence submatrices, since removing columns and the zero rows left after the columns removal from $C^T$ is equivalent to removing rows and the zero columns left after the rows removal from $C$. Therefore, since by hypothesis the transpose of each of these matrices is not a 2-influence submatrix, we
have that, at each step of the procedure, the obtained matrix has at least a row with no more than one non-zero element. As a consequence, a further column reduction can be always performed, until we reach the void matrix, which proves the Lemma. □

Now, by using the previous Lemma and the equivalence between analysis-suitable and dual-compatible T-meshes we will prove the following result.

**Theorem 4.17.** Any analysis-suitable T-mesh \( M \in AD_{p,q}^+ \) is a VMCR T-mesh.

**Proof.** Let us consider an analysis-suitable T-mesh \( M \), which is of course dual-compatible as well. Let us suppose that there exists a set of anchors \( \mathcal{A} \subset \mathcal{A}_{p,q}(M) \) such that \( C(\mathcal{A}) \) is a 2-influence submatrix: all the columns have at least two non-zero elements. The proof will show that this assumption leads to a contradiction: more precisely, it allows to determine in \( \mathcal{A} \) two anchors \( \bar{A}_1 \) and \( \bar{A}_2 \) not partially overlapping, which is impossible since the given T-mesh \( M \) is dual-compatible.

Let us denote by \( \hat{\mathcal{A}} \subset \mathcal{A}_{p,q}(...M) \) the set of anchors of \( \hat{M} \) corresponding to the columns of \( C(\mathcal{A}) \). Then, we can always select the anchor \( \bar{A} \in \hat{\mathcal{A}} \) such that for any \( \hat{A} \in \hat{\mathcal{A}} \) one of the following conditions is satisfied:

\[
\begin{align*}
\bar{A} &> \bar{A} = \bar{A} &\text{ and } \bar{A} < \bar{A}.
\end{align*}
\]

Roughly speaking, these two conditions means that we first choose the lowest anchor in \( \hat{\mathcal{A}} \) and then, if it is not unique, the rightmost one. The ability to choose \( \bar{A} \) is guaranteed by the fact that \( \hat{\mathcal{A}} \) is finite. Moreover, we consider in the column of \( C(\mathcal{A}) \) corresponding to \( \bar{A} \) two of the non-zero elements: they always exist since we assumed that \( C(\mathcal{A}) \) is a 2-influence submatrix, and correspond to two anchors \( \bar{A}_1, \bar{A}_2 \in \mathcal{A} \). In the following, we will show that \( \bar{A}_1 \) and \( \bar{A}_2 \) don’t partially overlap, since assuming that they do implies the existence of an anchor \( \bar{A} \in \hat{\mathcal{A}} \) satisfying none of the conditions (17)-(18).

We now need a couple of Lemmas to proceed with the proof. The first one is essentially a remark about the knot insertion formula in the one-dimensional case, while the second one gives us some information about the elements of \( \hat{\mathcal{A}} \) related to \( A_1 \) and \( A_2 \).

**Lemma 4.18.** Let \( \Sigma = \{s_1, ..., s_{n+p}\} \) be a knot vector, with the corresponding functions vectors \( \Omega_u = \{u_1, ..., u_{n+p-1}\} \) and \( \Omega_v = \{v_1, ..., v_{n+p-1}\} \). If we denote by \( N_{h}^{(p)}(s) \) and \( N_{k}^{(p)}(s) \) the GB-splines of order \( p \) constructed, respectively, on \( \Sigma, \Omega_u, \Omega_v \) and on the corresponding vectors obtained by inserting in \( \Sigma, \Omega_u, \Omega_v \) a certain number of knots and functions, then for any couple of different indices \( 1 \leq i, j \leq n \), we get

\[
\begin{align*}
N_{i}^{(p)}(s) &= \sum_{h=1}^{i_1} \sum_{k=1}^{i_2} a_{h,p}N_{h}^{(p)}(s), \\
N_{j}^{(p)}(s) &= \sum_{h=1}^{j_1} \sum_{k=1}^{j_2} b_{h,p}N_{k}^{(p)}(s),
\end{align*}
\]

where the \( a_{h,p} \) and \( b_{h,p} \) are coefficients obtained by a repeated application of the knot insertion formula. Moreover, we have either \( i_2 < j_2 \) or \( j_2 < i_2 \).

**Proof.** The Lemma is a direct consequence of the knot insertion formula. □
Lemma 4.19. Given an anchor \( A \in \mathcal{A}_{p,q}(M) \), let us denote by \( \mathbf{I}^s(A) = \{i_1(A), \ldots, i_{p+1}(A)\} \) and \( \mathbf{I}^t(A) = \{t_1(A), \ldots, t_{q+1}(A)\} \) the local index vectors of \( A \) in the T-mesh \( M \). By Definition (9), the associated GT-spline blending function can be represented in the form

\[
N_A(s, t) = N\left[\Sigma^s(\mathbf{I}^s(A)), \Omega_u^t(\mathbf{I}^t(A)), \Omega_v^t(\mathbf{I}^t(A))\right](s) \\
\times N\left[\Sigma^t(\mathbf{I}^t(A)), \Omega_u^s(\mathbf{I}^s(A)), \Omega_v^s(\mathbf{I}^s(A))\right](t),
\]

where we define

\[
\Sigma^s(\mathbf{I}^s) = \{s_i \in \Sigma^s : i \in \mathbf{I}^s\}, \\
\Sigma^t(\mathbf{I}^t) = \{t_j \in \Sigma^t : j \in \mathbf{I}^t\}, \\
\Omega_u^s(\mathbf{I}^s) = \{u_i^s \in \Omega_u^s : i \in \mathbf{I}^t\}, \\
\Omega_u^t(\mathbf{I}^t) = \{u_j^t \in \Omega_u^t : j \in \mathbf{I}^t\}, \\
\Omega_v^s(\mathbf{I}^s) = \{v_i^s \in \Omega_v^s : i \in \mathbf{I}^t\}, \\
\Omega_v^t(\mathbf{I}^t) = \{v_j^t \in \Omega_v^t : j \in \mathbf{I}^t\}.
\]

for any index vectors \( \mathbf{I}^s = \{i_1, \ldots, i_{p+1}\}, \mathbf{I}^t = \{j_1, \ldots, j_{q+1}\} \). Let us assume that

\[
N\left[\Sigma^s(\mathbf{I}^s(A)), \Omega_u^t(\mathbf{I}^t(A)), \Omega_v^t(\mathbf{I}^t(A))\right](s) = \sum_{\mathbf{P} \in H(A)} a_{\mathbf{P}}^s N\left[\Sigma^s(\mathbf{P}^s), \Omega_u^t(\mathbf{P}^t), \Omega_v^t(\mathbf{P}^t)\right](s), \\
N\left[\Sigma^t(\mathbf{I}^t(A)), \Omega_u^s(\mathbf{I}^s(A)), \Omega_v^s(\mathbf{I}^s(A))\right](t) = \sum_{\mathbf{P} \in K(A)} a_{\mathbf{P}}^t N\left[\Sigma^t(\mathbf{P}^t), \Omega_u^s(\mathbf{P}^s), \Omega_v^s(\mathbf{P}^s)\right](t),
\]

where \( H(A) \) and \( K(A) \) are suitable sets of index vectors of length \((p+1)\) and \((q+1)\), respectively, and the real coefficients \( a_{\mathbf{P}}^s \) and \( a_{\mathbf{P}}^t \) are obtained by repeatedly applying the knot insertion formula. Then, we have

\[
N_A(s, t) = \sum_{\mathbf{P} \in H(A)} \sum_{\mathbf{P} \in K(A)} a_{\mathbf{P}}^s a_{\mathbf{P}}^t N\left[\Sigma^s(\mathbf{P}^s), \Omega_u^t(\mathbf{P}^t), \Omega_v^t(\mathbf{P}^t)\right](s) N\left[\Sigma^t(\mathbf{P}^t), \Omega_u^s(\mathbf{P}^s), \Omega_v^s(\mathbf{P}^s)\right](t).
\]

In other words,

\[
N_A(s, t) = \sum_{\hat{A} \in \hat{\mathcal{A}}(A)} c_{\hat{A}} N_{\hat{A}}(s, t),
\]

where \( \hat{\mathcal{A}}(A) = \{B \in \mathcal{A}_{p,q}(\hat{M}) : \mathbf{I}^t(B) \in H(A), \mathbf{I}^s(B) \in K(A)\} \), and \( c_{\hat{A}} > 0 \) for \( \hat{A} \in \hat{\mathcal{A}}(A) \).

Proof. The lemma is a direct consequence of the knot insertion formula applied to the GT-splines associated to the T-mesh \( M \). □

Since the T-mesh \( M \) is dual-compatible, \( \hat{A}_1 \) and \( \hat{A}_2 \) must overlap either horizontally or vertically. However, this cannot happen since one of the following three possibilities occurs and for each of them it’s not difficult to show, by using Lemma 4.18 and 4.19 that we can construct \( \hat{A} \in \hat{\mathcal{A}} \) satisfying neither (17) nor (18).

- \( \hat{A}_1 \) and \( \hat{A}_2 \) overlap horizontally with the respective horizontal local index vectors \( \mathbf{I}^s(\hat{A}_1) \) and \( \mathbf{I}^s(\hat{A}_2) \) not coinciding. It’s possible to construct \( \hat{A} \in \hat{\mathcal{A}} \) such that

\[
i_{p+1}^s(\hat{A}) > i_{p+1}^s(\hat{A}) \quad \mathbf{I}^t(\hat{A}) = \mathbf{I}^t(\hat{A}),
\]
which contradicts the definition of $\tilde{A}$ given in (17)-(18). Then, we must conclude that $\tilde{A}_1$ and $\tilde{A}_2$ cannot overlap horizontally with the local index vectors $I^t(\tilde{A}_1)$ and $I^t(\tilde{A}_2)$ not coinciding.

- $A_1$ and $A_2$ overlap vertically with the respective vertical local index vectors $I^t(\tilde{A}_1)$ and $I^t(\tilde{A}_2)$ not coinciding. Then we can construct $\tilde{A} \in \tilde{A}$ such that

$$i^t_{q+1}(\tilde{A}) < i^t_{q+1}(\tilde{A})$$

which contradicts the definition of $\tilde{A}$ given in (17)-(18). Then, $\tilde{A}_1$ and $\tilde{A}_2$ cannot overlap vertically with the local index vectors $I^t(\tilde{A}_1)$ and $I^t(\tilde{A}_2)$ not coinciding.

- $\tilde{A}_1$ and $\tilde{A}_2$ overlap horizontally (vertically) and their local horizontal (vertically) vectors coincide. By the definitions of anchors and local index vectors, this implies that they must overlap also vertically (horizontally) with not coinciding local vertical (horizontal) index vectors (otherwise we would have $\tilde{A}_1 = \tilde{A}_2$), which leads to the two preceding cases. Then, also this possibility cannot occur.

The conclusion, then, is that we found two anchors, $A_1$ and $A_2$, of $M$ not partially overlapping, which contradicts the hypothesis that $M$ is an analysis-suitable/dual-compatible mesh; the Theorem is then proved.

Theorem 4.17 tells us that the class of analysis-suitable (dual-compatible) T-meshes is included in the one of VMCR T-meshes. We will now show, by presenting a detailed example, that the class of VMCR T-meshes is a proper superset of the class of analysis-suitable T-meshes.

Let us consider $p = q = \mu = \nu = 4$, the T-mesh $M$ in Figure 10 (a) and its underlying tensor-product mesh $\hat{M}$ in Figure 10 (b). We assume that all the knots have multiplicity 1.

---

**Figure 10:** An example of (a) non-analysis-suitable VMCR T-mesh and (b) its underlying tensor-product mesh (with $p = q = 4$). The active region is highlighted in light gray, while the T-junction extensions in (a) are highlighted in dark gray.
There are 14 anchors in $M$ and 16 in $\hat{M}$. The following relations hold between the GT-spline blending functions associated to $M$ and the ones associated to $\hat{M}$:

\[
\begin{align*}
N_{A_1} &= \alpha_{A_1} \hat{N}_{\hat{A}_1} + \beta_{A_1} \hat{N}_{\hat{A}_2} \\
N_{A_2} &= \alpha_{A_2} \hat{N}_{\hat{A}_2} + \beta_{A_2} \hat{N}_{\hat{A}_3} \\
N_{A_3} &= \alpha_{A_3} \hat{N}_{\hat{A}_3} + \beta_{A_3} \hat{N}_{\hat{A}_4} + \gamma_{A_3} \hat{N}_{\hat{A}_7} + \eta_{A_3} \hat{N}_{\hat{A}_8} \\
N_{A_4} &= \hat{N}_{\hat{A}_5} \\
N_{A_5} &= \hat{N}_{\hat{A}_6} \\
N_{A_6} &= \hat{N}_{\hat{A}_7} \\
N_{A_7} &= \alpha_{A_7} \hat{N}_{\hat{A}_8} + \beta_{A_7} \hat{N}_{\hat{A}_{12}} \\
N_{A_8} &= \hat{N}_{\hat{A}_9} \\
N_{A_9} &= \hat{N}_{\hat{A}_{10}} \\
N_{A_{10}} &= \hat{N}_{\hat{A}_{11}} \\
N_{A_{11}} &= \hat{N}_{\hat{A}_{13}} \\
N_{A_{12}} &= \hat{N}_{\hat{A}_{14}} \\
N_{A_{13}} &= \hat{N}_{\hat{A}_{15}} \\
N_{A_{14}} &= \alpha_{A_{14}} \hat{N}_{\hat{A}_{16}} + \beta_{A_{14}} \hat{N}_{\hat{A}_{12}} ,
\end{align*}
\]

where the coefficients are all positive and obtained by using the knot insertion formula (7). In other words, we have that $N = C \hat{N}$, where $N = [N_{A_1}, \ldots, N_{A_{14}}]^T$, $\hat{N} = [\hat{N}_{\hat{A}_1}, \ldots, \hat{N}_{\hat{A}_{16}}]^T$ and

\[
C^T = \begin{bmatrix}
\alpha_{A_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\beta_{A_1} & \alpha_{A_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \beta_{A_2} & \alpha_{A_3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \beta_{A_3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_{A_{14}} \\
\end{bmatrix}
\]

It’s easy to verify that applying the column reduction to $C^T$ gives the void matrix and therefore the T-mesh $M$ is VMCR. From Figure 10 and Definition \[\text{[2.10]}\] it’s also evident that $M$ is not analysis-suitable.
5 Conclusions

In this paper we completed the work of generalizing the T-spline approach to the GB-splines (started in [3] for the trigonometric case). The key points to reach this result were the complete study of the knot insertion formulae for GB-splines in the one-dimensional case and the study of the linear independence of the obtained GT-splines. This second issue led to another relevant achievement, that is, the introduction of VMCR T-meshes, a new class of T-meshes guaranteeing the linear independence of both the T-spline and GT-spline associated blending functions with the same bi-order. We showed that this class of T-meshes properly contains the analysis-suitable (dual-compatible) T-meshes.

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