Integrable hydrodynamic equations for initial chiral currents and infinite hydrodynamic chains from WZNW model and string model of WZNW type with

\[ SU(2), SO(3), SP(2), SU(\infty), SO(\infty), SP(\infty) \text{ constant torsions} \]

D.J. Cirilo-Lombardo

Bogoliubov Laboratory of Theoretical Physics Joint Institute for Nuclear Research,
141980, Dubna, Russian Federation

V. D. Gershun

ITP, NSC Kharkov Institute of Physics and Technology, Kharkov, UA

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Abstract

The WZNW and string models are considered in the terms of the initial and invariant chiral currents assuming that the internal and external torsions coincide (anticoincide) and they are the structure constants of the \( SU(n), SO(n), SP(n) \) Lie algebras. These models are the auxiliary problems in order to construct integrable equations of hydrodynamic type. It was shown that the WZNW and string models in terms of invariant chiral currents are integrable for the constant torsion associated with the structure constants of the \( SU(2), SO(3), SP(2) \) and \( SU(3) \) algebras only. The equation of motion for the density of the first Casimir operator was obtained in the form of the inviscid Burgers equation. The solution of this equation is presented through the Lambert function. Also, a new equation of motion for the initial chiral current was found.

The integrable infinite hydrodynamic chains obtained from the WZNW and string models are given in terms of invariant chiral currents with the \( SU(2), SO(3), SP(2) \) and with \( SU(\infty), SO(\infty), SP(\infty) \) constant torsions. Also, the equations of motion for the density of any Casimir operator and new infinite dimensional equations of hydrodynamic type for the initial chiral currents through the symmetric structure constant of \( SU(\infty), SO(\infty), SP(\infty) \) algebras are obtained.
I. **INTRODUCTION**

The integrability of the two dimensional WZNW and string models is based on the existence of the infinite number of the local and nonlocal currents and on their charges.
The $n$-dimensional WZNW model is described by the chiral left $J^L_A = g^{-1} \partial_A g$ or the chiral right $J^R_A = \partial_A g \, g^{-1}$ currents for arbitrary space-time dimension $A = 1, \ldots, n$, where $g$ is an element of the symmetry group of the model, $J_A = J^A_{\mu} t_{\mu}$ and $t_{\mu}$ are the generators of the corresponding Lie algebra. These chiral currents are related to left and right multiplication on the group space. The two dimensional models $(A = 0, 1)$ have the additional chiral currents

$$J^L_{\mu}(t, x) = J^0_{\mu} + \frac{\delta_{\mu \nu} J^\nu_1}{\sqrt{2}} = U_{\mu}(x + t)$$

$$J^R_{\mu}(t, x) = J^0_{\mu} - \frac{\delta_{\mu \nu} J^\nu_1}{\sqrt{2}} = V_{\mu}(x - t)$$

related to the dynamic on the $(t, x)$ plane. The chiral currents $U_{\mu}, V_{\mu}$ play an important role for a construction and study of the integrable systems. In the $\sigma$-model is not possible a priori, to restrict the dynamics to only one mode (left or right). The only possibility is to introduce a Witten term to the Wess-Zumino model. This term introduces a potential for the torsion tensor over the curved space of the group parameters in the addition to the metric tensor. Then, under some conditions between the constant torsion and the structure constant of Lie algebra the possibility to restrict the motion to one mode (left or right) arises.

String models with the antisymmetric background field was considered in the conformal and light-cone gauges. This antisymmetric field plays the role of potential of the external torsion in addition to the internal torsion, that is directly related to the metric tensor. Under some conditions between the both constant torsions, it is possible to introduce the chiral currents $U_{\mu}$ and $V_{\mu}$. Such currents form independent Kac-Moody algebras making possible the restriction of the string dynamics to one mode. If we assume that both torsions are precisely the structure constants of a Lie algebra, the transverse coordinates of string belong to the compact space of the corresponding Lie group. Consequently, under these assumptions, the WZNW and string models with the antisymmetric field coincide. Supersymmetric models and the integrability condition in supergroup spaces will be not treated here we refer the reader to references [67], [68], [69] and [70] for a general description of such cases. The paper is organized as follows. In Section 2 is devoted to the WZNW model: in subsection 2.1 the Lagrangian and equation of motion are considered in the repere formalism obtaining the antisymmetric field $B_{ab}$ in terms of the repere. The Hamiltonian formalism and the commutations relations for new variables are focused in Subsections 2.2 and 2.3: these variables are the chiral currents under the condition that the external torsion coincides
(anti-coincides) with the structure constants of the $SU(n)$, $SO(n)$, $SP(n)$ algebras. In Section 3 we consider integrable WZNW model with the constant torsion: in subsections 3.1, 3.2 the models with the $SU(2)$, $SO(3)$, $SP(2)$ and $SU(3)$ constant torsion are considered, the equation of motion for density of first Casimir operator is obtained as the inviscid Burgers equation and its solution expressed as the Lambert function. As a bonus a new nonhomogeneous equation for the initial chiral current is obtained. The integrable infinite dimensional hydrodynamic chains for WZNW model with the constant $SU(2)$, $SO(3)$, $SP(2)$ and $SU(\infty)$, $SO(\infty)$, $SP(\infty)$ torsions are the subject of subsection 3.3: new equations of motion of hydrodynamic type for the initial chiral currents in terms of the symmetric structure constant of the $SU(\infty)$, $SO(\infty)$, $SP(\infty)$ algebras are presented.

An integrable string model as auxiliary problem to integrable hydrodynamic equations is treated in Section 4. In subsection 4.1 the string model in the conformal and light-cone gauges is formulated obtaining the Hamiltonian and the conditions on the metric and antisymmetric tensors. In subsection 4.2 we show that the same Hamiltonian can be obtained from the flat string model in the field theory approach. The subsection 4.3 we obtain and analyze the commutation relations of the new variables, which are the chiral currents under the condition that an internal and external torsions coincide (anti-coincide) obeying the Kac-Moody algebras. In subsection 4.4 we presented a short review of the string models of the hydrodynamic type for the case when both torsions are null. In the short Section 5 a string model with constant torsions is considered. The mathematical description of the equations of motion for the invariant chiral and initial chiral currents of the string model with the constant torsion in this section coincides with the mathematical description of the equations of motion for the invariant chiral and the initial chiral currents of the WZNW model with the constant torsion in section 3. Finally some concluding remarks are given.

**II. INTEGRABLE WZNW MODEL**

In this section we will show how the Poisson brackets (PBs) and the equations of motion for the infinite hydrodynamic chains from the WZNW dynamic model are obtained. Also, new infinite dimensional PBs and new nonlinear equations of motion of the hydrodynamic type by using the symmetric structure constants of $SU(\infty)$, $SO(\infty)$, $SP(\infty)$ Lie algebras are presented and analyzed.
A. Lagrangian and equations of motion

The conformal invariant two-dimensional non-linear sigma model is described by WZNW model which is nothing more that the sigma model [1]-[8] with the Wess-Zumino term [9]-[14], [8] on the group manifold. To each point of a dimensional world-sheet one associated an element $g$ of a group $G$. We want to construct an action (Lagrangian density) with the element of volume of the two-dimensional space invariant under of a group transformations

$$S = \frac{1}{4} \int Tr(\omega \wedge dx^\alpha)(\omega \wedge dx^\beta)\eta_{\alpha\beta} + \frac{1}{2} \int Tr(\omega(d) \wedge \omega(d) \wedge \omega(d)).$$

(1)

Here $x^\alpha = (t, x)$ are coordinates of the flat two-dimensional space: $\alpha = (0, 1)$ with signature $(-1,1)$ and $\eta_{\alpha\beta}$ is diagonal metric of this space. The form $\omega(d) = \omega(d)^\mu t_\mu$ is the differential Cartan one-form which belongs to a simple Lie algebra

$$[t_\mu, t_\nu] = 2iC^\lambda_{\mu\nu}t_\lambda, \quad Tr(t_\mu t_\nu) = 2g_{\mu\nu}, \quad (\mu, \nu = 1, 2, \ldots, n).$$

(2)

In any parametrization Cartan forms $\omega(d) = (g^{-1}dg)^\mu t_\mu$ depend on the group parameters $\phi^a$: $\omega(d) = \omega(\phi, d\phi)$. The first term of the Lagrangian (1) has form

$$\frac{Tr(\omega \wedge dx^\alpha)(\omega \wedge dx^\beta)\eta_{\alpha\beta}}{\epsilon_{\lambda\rho}dx^\lambda \wedge dx^\rho} = \frac{2}{d^2x} g_{\mu\nu}\omega^\mu_a \frac{\partial \phi^a}{\partial x^\gamma} (dx^\gamma \wedge dx^\alpha) \omega^\nu_b \frac{\partial \phi^b}{\partial x^\rho} (dx^\rho \wedge dx^\beta) \eta_{\alpha\beta}$$

$$= 2g_{ab}(\phi) \frac{\partial \phi^a}{\partial x^\gamma} \frac{\partial \phi^b}{\partial x^\rho} \eta^{\alpha\beta} d^2x.$$

(3)

Here we introduce notation

$$g_{ab}(\phi) = g_{\mu\nu}\omega^\mu_a \omega^\nu_b, \quad dx^\gamma \wedge dx^\alpha = \epsilon^{\gamma\alpha}d^2x.$$

(4)

One can see, that $g_{ab}(\phi)$ is metric tensor on the curve space of local fields $\phi^a, \quad (a = 1, 2, \ldots, n).$

The $\omega^\mu_a(\phi)$ forms a repere basis on the tangent space with the metric $g_{\mu\nu}$ in arbitrary point of the curve space $\phi^a$. In the paper of [13] the antisymmetric field $B_{ab}(\phi)$ was obtained from the second term of action (1) for the metric tensor $g_{ab} = \delta_{ab} + (1 - |\phi|^2)^{-1}\phi_a\phi_b, \quad (a, b = 1, 2, 3)$.

Here we want to rewrite the WZNW model as $\sigma$ - model of string type with antisymmetric field $B_{ab}(\phi)$ in terms of the repere for the arbitrary metric $g_{ab}(\phi)$ and for any dimension $n$:

$$Tr(\omega(d) \wedge \omega(d) \wedge \omega(d)) = 2iC_{\mu\lambda\alpha}\omega^\mu(d) \wedge \omega^\nu(d) \wedge \omega^\lambda(d) = 2iC_{\mu\lambda\alpha}\Omega^\mu_A\Omega^\nu_B\Omega^\lambda_C dx^A \wedge dx^B \wedge dx^C =$$

$$= 2g_{\mu\nu}\Omega^\mu_A \partial C \Omega^\nu_B \epsilon^{ABC} d^3x = 2g_{\mu\nu}\Omega^\mu_a \partial x^A \partial x^C \partial x^B \epsilon^{ABC} d^3x =$$
\[ = g_{\mu\nu}(\Omega^\mu_{\alpha} \frac{\partial \Omega^\nu_{\beta}}{\partial x^C} - \Omega^\mu_{\beta} \frac{\partial \Omega^\nu_{\alpha}}{\partial x^C}) \frac{\partial \Phi^a}{\partial x^A} \frac{\partial \Phi^b}{\partial x^B} \epsilon^{ABC} d^3x. \]  

In first line of (5) the integrability condition of equation \( \partial_A g = g \Omega^\mu_{\alpha} t^\mu_a \) was used

\[ \partial_A \Omega^\mu_B - \partial_B \Omega^\mu_A + 2iC^{\mu\nu\lambda} \Omega^\nu_{\alpha} \Omega^\lambda_{\beta} = 0. \]

Here \( x^A (A = 0, 1, 2) \) are coordinates of three dimension space-time, \( \Omega^\mu(d) \) is the associated one form on this space. Let us to separate the second component of \( A (A = \alpha, 2; \alpha = 0, 1) \) in the last line of (5). Then, the second term of the action takes the following form:

\[ \int g_{\mu\nu} \epsilon^{\alpha\beta\gamma}(\Omega^\mu_{\alpha} \partial_2 \Omega^\nu_{\beta} - \partial_2 \Omega^\mu_{\alpha} \Omega^\nu_{\beta}) \frac{\partial \Phi^a}{\partial x^\alpha} \frac{\partial \Phi^b}{\partial x^\beta} d^3x = \int d^2x \int_0^M \epsilon^{\alpha\beta\gamma} B_{ab2} \frac{\partial \Phi^a}{\partial x^\alpha} \frac{\partial \Phi^b}{\partial x^\beta} dx^2 \]  

Here \( B_{ab2} = g_{\mu\nu}(\Omega^\mu_{\alpha} \partial_2 \Omega^\nu_{\beta} - \partial_2 \Omega^\mu_{\alpha} \Omega^\nu_{\beta}) = -B_{ba2}. \) The integration over the coordinate \( x^2 \) was performed in the limits \((0, M)\) with the following boundary conditions:

\[ \Phi^a(x^\alpha, x^2) |^{x^2=M} = \phi^a(x^\alpha), B_{ab2}(x^\alpha, x^2) |^{x^2=M} = B_{ab}(x^\alpha). \]

The integral in \( x^2 \) on the lower limit of integration equals zero, what is easy to see by using the expansion of the integrand into the Taylor series. Therefore, the second term of action becomes to:

\[ \frac{1}{2} \int \epsilon^{\alpha\beta} B_{ab} \frac{\partial \phi^a}{\partial x^\alpha} \frac{\partial \phi^b}{\partial x^\beta} dx^2. \]

Consequently, the total action is:

\[ S = \frac{1}{2} \int d^2x [g_{ab}(\phi) \eta^{\alpha\beta} + B_{ab}(\phi) \epsilon^{\alpha\beta}\frac{\partial \phi^a}{\partial x^\alpha} \frac{\partial \phi^b}{\partial x^\beta}]. \]

Here \( g_{ab}(\phi) = g_{ba}(\phi) \) is the metric tensor of a group space \( G \) and \( \phi^a(x) \) are the corresponding group parameters, \( a, b = 1, 2, \ldots n \). For compact groups of dimension \( n \), the space-time signature is \((n, 0)\). The background field \( B_{ab}(\phi) \) on the group space \( G \) is the antisymmetric tensor field \( B_{ab}(\phi(x)) = -B_{ba}(\phi(x)) \). The coordinates \( x^\alpha = (t, x), \alpha = 0, 1 \) belong to a 2-dimensional word-sheet with the constant metric tensor \( \eta_{\alpha\beta} \) and signature \((-1, 1)\). The second order equation of motion has following form:

\[ g_{ab}(\phi) \eta^{\alpha\beta} \frac{\partial^2 \phi^b}{\partial x^\alpha \partial x^\beta} + \Gamma_{abc} \eta^{\alpha\beta} \frac{\partial \phi^b}{\partial x^\alpha} \frac{\partial \phi^c}{\partial x^\beta} + H_{abc} \epsilon^{\alpha\beta} \frac{\partial \phi^b}{\partial x^\alpha} \frac{\partial \phi^c}{\partial x^\beta} = 0, \]

\[ \Gamma_{abc} = \frac{1}{2} \left( \frac{\partial g_{ab}}{\partial x^c} + \frac{\partial g_{ac}}{\partial x^b} - \frac{\partial g_{bc}}{\partial x^a} \right), H_{abc} = \frac{\partial B_{ab}}{\partial x^c} + \frac{\partial B_{ca}}{\partial x^b} + \frac{\partial B_{bc}}{\partial x^a}. \]
Here $\Gamma_{abc}(\phi)$ are the Christoffel symbols. It is a symmetric function in $b, c$. The function $H_{abc}(\phi)$ is a total antisymmetric function in $a, b, c$. If $H_{abc}(\phi)$ equals zero the antisymmetric term in the Lagrangian is a pure topological one. The case $H_{abc} \neq 0$ describes the WZNW model of string type. The antisymmetric field $B_{ab}$ in equation (18) in the light-cone variables $x^\pm$ can be considered as the antisymmetric part of a metric. It is the torsion potential [14], [15]. This form of the equation of the motion gave the possibility to introduce symplectic and Poisson structures on the loop spaces of smooth manifolds [16], [17]. Let us introduce a repere $e^a_\mu(\phi) = \omega^a_\mu(\phi)$ on the compact group space $G$ and its inverse $e^{(1)}_\mu a(\phi)$ such that the metric tensor can be written as

$$g_{ab}(\phi) = e^a_\mu(\phi) e^{(1)}_\nu b(\phi) \delta_{\mu \nu}, \quad \delta_{\mu \nu} = e^a_\mu(\phi) e^{(1)}_\nu b(\phi) g_{ab}(\phi). \quad (11)$$

Here $\delta_{\mu \nu}(\mu, \nu = 1, 2, \ldots n)$ is a constant tensor on the tangent space of the compact group space $G$ at some point $\phi_a(x)$ with the same signature as $g_{ab}(\phi)$. To introduce the Hamiltonian we rewrite the Lagrangian density and the equation of motion in terms of the world-sheet coordinates $(t, x)$

$$L = \frac{1}{2} g_{ab}(\phi) \left[ \frac{\partial \phi^a}{\partial t} \frac{\partial \phi^b}{\partial t} - \frac{\partial \phi^a}{\partial x} \frac{\partial \phi^b}{\partial x} \right] + B_{ab}(\phi) \left[ \frac{\partial \phi^a}{\partial t} \frac{\partial \phi^b}{\partial t} \right] + 2 H_{abc}(\phi) \left[ \frac{\partial \phi^a}{\partial t} \frac{\partial \phi^b}{\partial x} \right] = 0. \quad (12)$$

The equation of motion finally has the form:

$$g_{ab}(\phi) \left[ \frac{\partial^2 \phi^a}{\partial t^2} - \frac{\partial^2 \phi^a}{\partial x^2} \right] + \Gamma_{abc}(\phi) \left[ \frac{\partial \phi^b}{\partial t} \frac{\partial \phi^c}{\partial t} - \frac{\partial \phi^b}{\partial x} \frac{\partial \phi^c}{\partial x} \right] + 2 H_{abc}(\phi) \frac{\partial \phi^b}{\partial t} \frac{\partial \phi^c}{\partial x} = 0. \quad (13)$$

**B. Canonical momentum and Hamiltonian**

The canonical momentum is defined as

$$p_a(\phi(t, x)) = \frac{\delta L}{\delta (\frac{\partial \phi^a}{\partial t})} = g_{ab}(\phi) \frac{\partial \phi^b}{\partial t} + B_{ab}(\phi) \frac{\partial \phi^b}{\partial x}. \quad (14)$$

Consequently, the Hamiltonian has the following form:

$$H(\phi, p) = p_a \frac{\partial \phi^a}{\partial t} - L = \frac{1}{2} g_{ab}(\phi) \left[ p_a - B_{ac}(\phi) \frac{\partial \phi^c}{\partial x} \right] \left[ p_b - B_{bd}(\phi) \frac{\partial \phi^d}{\partial x} \right] + \frac{1}{2} g_{ab}(\phi) \frac{\partial \phi^a}{\partial x} \frac{\partial \phi^b}{\partial x}. \quad (15)$$

Let us introduce new dynamical variables

$$J_{0\mu}(\phi) = e^a_\mu(\phi) \left[ p_a - B_{ab}(\phi) \frac{\partial \phi^b}{\partial x} \right], \quad J_{1\mu}(\phi) = e^a_\mu(\phi) \frac{\partial \phi^a}{\partial x}. \quad (16)$$
We see that Hamiltonian \[15\] is factorized in these variables

\[
H = \frac{1}{2}[\delta_{\mu\nu} J_{0\mu}(\phi) J_{0\nu}(\phi) + \delta_{\mu\nu} J_1^\mu(\phi) J_1^\nu(\phi)].
\]

(17)

The equations of motion in terms of this variables are first order ones

\[
\partial_0 J_1^\mu(\phi) - \partial_1 J_0^\mu(\phi) = C_{\nu\lambda}^\mu J_0^\nu(\phi) J_1^\lambda(\phi), \quad \partial_0 J_0^\mu(\phi) - \partial_1 J_1^\mu(\phi) = -H_{\nu\lambda}^\mu (\phi) J_0^\nu(\phi) J_1^\lambda(\phi).
\]

(18)

Here \(C_{\nu\lambda}^\mu\) is the structure constant tensor which can be obtain from the Maurer-Cartan equation:

\[
C_{\nu\lambda}^\mu = \frac{\partial e^\mu_a(\phi)}{\partial x^b} [e^b_\nu(\phi) e^a_\lambda(\phi) - e^a_\nu(\phi) e^b_\lambda(\phi)] = \left[ \frac{\partial e^\mu_a(\phi)}{\partial x^b} - \frac{\partial e^\mu_b(\phi)}{\partial x^a} \right] e^b_\nu(\phi) e^a_\lambda(\phi)
\]

(19)

and similarly

\[
H_{\nu\lambda}^\mu (\phi) = g^{\mu\rho} H_{abc}(\phi) e^a_\rho(\phi) e^b_\nu(\phi) e^c_\lambda(\phi).
\]

C. Commutation relations for new variables

The starting point is the canonical Poisson bracket (PB):

\[
\{\phi^a(x), p_b(y)\} = \delta_0^a \delta(x - y).
\]

(20)

Let us consider the commutation relations for the functions \(J_{0\mu}(\phi(x)), \quad J_{1\nu}(\phi(x)) = \delta_{\mu\nu} J_1^\nu(\phi(x))\) on the phase space under the PB (20)

\[
\{J_{0\mu}(\phi(x)), J_{0\nu}(\phi(y))\} = C_{\mu\lambda}^\nu J_{0\lambda}(\phi(x)) \delta(x - y) + H_{\mu\lambda}^\nu (\phi(x)) J_{1\lambda}(\phi(x)) \delta(x - y),
\]

\[
\{J_{0\mu}(\phi(x)), J_{1\nu}(\phi(y))\} = C_{\mu\lambda}^\nu J_{1\lambda}(\phi(x)) \delta(x - y) + g_{\mu\nu} \partial_x \delta(x - y),
\]

\[
\{J_{1\mu}(\phi(x)), J_{1\nu}(\phi(y))\} = 0.
\]

(21)

Let us introduce the chiral variables

\[
U_\mu = \frac{J_{0\mu} + \delta_{\mu\nu} J_1^\nu}{\sqrt{2}}, \quad V_\mu = \frac{J_{0\mu} - \delta_{\mu\nu} J_1^\nu}{\sqrt{2}}.
\]

(22)

The chiral variables \(U_\mu(\phi(x)), V_\mu(\phi(x))\) satisfy to following commutation relations:

\[
\{U_\mu(\phi(x)), U_\nu(\phi(y))\} = \frac{1}{2\sqrt{2}} [(3C_{\mu\nu}^\lambda + H_{\mu\nu}^\lambda(\phi(x))) U_\lambda(\phi(x)) - (C_{\mu\nu}^\lambda + H_{\mu\nu}^\lambda(\phi(x))) V_\lambda(\phi(x))] \delta(x - y) + \delta_{\mu\nu} \partial_x \delta(x - y),
\]

(23)

\[
\{U_\mu(\phi(x)), V_\nu(\phi(y))\} = \frac{1}{2\sqrt{2}} [(3C_{\mu\nu}^\lambda + H_{\mu\nu}^\lambda(\phi(x))) V_\lambda(\phi(x)) - (C_{\mu\nu}^\lambda + H_{\mu\nu}^\lambda(\phi(x))) U_\lambda(\phi(x))] \delta(x - y).
\]

(24)
\{V_\mu(\phi(x)), V_\nu(\phi(y))\} = \frac{1}{2\sqrt{2}} [(3C^\lambda_{\mu\nu} - H^{\lambda}_{\mu\nu}(\phi(x)))V_\lambda(\phi(x)) - (C^\lambda_{\mu\nu} - H^{\lambda}_{\mu\nu}(\phi(x)))U_\lambda(\phi(x))] \delta(x-y) - \delta_{\mu\nu}\partial_x\delta(x-y),

\{U_\mu(\phi(x)), V_\nu(\phi(y))\} = \frac{1}{2\sqrt{2}} [(C^\lambda_{\mu\nu} + H^{\lambda}_{\mu\nu}(\phi(x)))U_\lambda(\phi(x)) + (C^\lambda_{\mu\nu} - H^{\lambda}_{\mu\nu}(\phi(x)))V_\lambda(\phi(x))] \delta(x-y).

The commutation relations (21), (23) are not Poisson brackets because the torsion $H^{\lambda}_{\mu\nu}(\phi)$ is not a smooth function. These commutation relations form an algebra, if $H^{\lambda}_{\mu\nu}(\phi)$ is a constant tensor. The interesting cases arise if $H^{\lambda}_{\mu\nu}(\phi)$ is a constant tensor. In the case where $H^{\lambda}_{\mu\nu}(\phi) = \pm C^\lambda_{\mu\nu}$ the variables $U_\mu(\phi)$ form the closed Kac-Moody algebra [18], [19] for the right chiral currents

\{V_\mu(\phi(x)), U_\nu(\phi(y))\} = C_{\mu\nu}^\lambda U_\lambda(\phi(x)) \delta(x-y) + \delta_{\mu\nu}\partial_x\delta(x-y).

Here we note the PB (24) as $PB_2$. The remaining relations are not essential.

\{V_\mu(\phi(x)), V_\nu(\phi(y))\} = C_{\mu\nu}^\lambda (2V_\lambda(\phi(x)) - U_\lambda(\phi(x))) \delta(x-y) - \delta_{\mu\nu}\partial_x\delta(x-y),

\{U_\mu(\phi(x)), V_\nu(\phi(y))\} = C_{\mu\nu}^\lambda V_\lambda(\phi(x)) \delta(x-y).

In the case of $H^{\lambda}_{\mu\nu} = C^\lambda_{\mu\nu}$ variables $V_\mu(\phi)$ form the closed Kac-Moody algebra for the left chiral currents

\{V_\mu(\phi(x)), V_\nu(\phi(y))\} = C_{\mu\nu}^\lambda V_\lambda(\phi(x)) - \delta_{\mu\nu}\partial_x\delta(x-y).

so, the remaining relations now are:

\{U_\mu(\phi(x)), U_\nu(\phi(y))\} = C_{\mu\nu}^\lambda (2U_\lambda(\phi(x)) - V_\lambda(\phi(x))) + \delta_{\mu\nu}\partial_x\delta(x-y),

\{U_\mu(\phi(x)), V_\nu(\phi(y))\} = C_{\mu\nu}^\lambda U_\lambda(\phi(x)) \delta(x-y).

Let me note that Kac-Moody algebra has considered as a hidden symmetry of a two-dimensional chiral models [20], [21]. In the 1983 one of the authors (VDG) and Volkov, Tkach [22] considered the algebra of the nonlocal charges of the $\sigma$-model in the context of its integrability. We shown that the nonlocal charges form the enveloping algebra over the Kac-Moody algebra. Let us rewrite equations of motion (13) in terms of variables $U_\mu(\phi(x)), V_\mu(\phi(x))$

\[\partial_- U_\mu(\phi(t,x)) = \frac{1}{2}(C^\nu_{\mu\lambda} + H^{\nu\lambda}_{\mu\nu})(\phi(t,x))V_\nu(\phi(t,x))U_\lambda(\phi(t,x)),\]

\[\partial_+ V_\mu(\phi(t,x)) = \frac{1}{2}(C^\nu_{\mu\lambda} - H^{\nu\lambda}_{\mu\nu})(\phi(t,x))U_\nu(\phi(t,x))V_\lambda(\phi(t,x)).\]
In the case $C_{\mu \nu}^\lambda = -H_{\mu \nu}^\lambda$ the equation of motion is
\begin{equation}
\partial_- U_\mu(\phi(t, x)) = 0, \quad \partial_+ V_\mu(\phi(t, x)) = C_{\mu \nu}^{\nu \lambda} U_\nu(\phi)V_\lambda(\phi).
\end{equation}
(27)

If $C_{\mu \nu}^\lambda = H_{\mu \nu}^\lambda$ the equation of motion is
\begin{equation}
\partial_+ V_\mu(\phi(t, x)) = 0, \quad \partial_- U_\mu(\phi(t, x)) = C_{\mu \nu}^{\nu \lambda} V_\nu(\phi) U_\lambda(\phi).
\end{equation}
(28)

We see from the equations (24) and (27) that the chiral currents $U_\mu$ form the closed system in the first case and from the equations (25), (28) we see that the chiral currents $V_\mu$ also form the closed system. Here we introduced the notation:
\begin{equation}
\partial_+ = \frac{1}{\sqrt{2}}(\partial_0 + \partial_1), \quad \partial_- = \frac{1}{\sqrt{2}}(\partial_0 - \partial_1).
\end{equation}

The chiral currents $U^\mu(x)$ are the generators of translations in the curved space of the fields $\phi^a(x)$
\begin{equation}
\delta_c \phi^a(x) = \{\phi^a(x), c^\mu U_\mu(x)(\phi(x))\} = c^\mu e^{a}_{\mu}(\phi(x)) = c^a(\phi(x)).
\end{equation}

Simultaneously, they are the generators of the group transformations (with the structure constants $C_{\lambda \mu \nu}$) in the tangent space.

III. INTEGRABLE WZNW MODEL WITH CONSTANT TORSION

One of the ways to construct an integrable dynamical system is as follows. We must to have a hierarchy of a Hamiltonians and to find a hierarchy of Poisson brackets. This way is more simple, if the dynamical system have some group structure. Let the torsion $C_{abc}$ to be the structure constants of a Lie algebra. In the bi-Hamiltonian approach to the integrable string models with the constant torsion we have considered the conserved primitive chiral invariant currents (densities of the dynamical Casimir operators) $C_n(U(x))$, as the local fields of the Riemannian manifold. The primitive and non-primitive local charges of the invariant chiral currents form the hierarchy of the new Hamiltonians. The primitive invariant currents are the densities of the Casimir operators. The non primitive currents are functions of the primitive ones. The commutation relations (24) show that the currents $U^\mu$ form the closed algebra. Therefore, we will consider PBs of the right chiral currents $U^\mu$ and the Hamiltonians constructed only from the right currents. The constant torsion will does not contributes to the equation of motion, but it gives the possibility to introduce the group structure and to
introduce the symmetric structure constants. This paper was stimulated by the papers [23], [24], [25], [26] about the local conserved charges in two dimensional models. Evans, Hassan, MacKay, Mountain [25] constructed the local invariant chiral currents as the polynomials of the initial chiral currents of the $SU(n)$, $SO(n)$, $SP(n)$ principal chiral models. Their paper was based on the paper of de Azcarraga, Macfarlane, MacKay, Perez Bueno [27] about the tensor invariants for the simple Lie algebras.

Let $t_\mu$ to be the generators of the $SU(n)$, $SO(n)$, $SP(n)$ Lie algebras:

$$[t_\mu, t_\nu] = 2i C_{\mu\nu\lambda} t_\lambda. \quad (29)$$

There are additional relations for generators of Lie algebra in the defining matrix representation. There is following relation for the symmetric double product of the generators of $SU(n)$ algebra:

$$\{t_\mu, t_\nu\} = \frac{4}{n} \delta_{\mu\nu} + 2d_{\mu\nu\lambda} t_\lambda, \mu = 1, ..., n^2 - 1. \quad (30)$$

Here $d_{\mu\nu\lambda}$ is the totally symmetric structure constant tensor. The Killing tensor $g_{\mu\nu}$ equals $\delta_{\mu\nu}$ for the compact Lie algebras. The similar relation for the totally symmetric triple product of the $SO(n)$ and $SP(n)$ algebras has the form:

$$t_{(\mu} t_\nu t_\lambda) = v^\rho_{\mu\nu\lambda} t_\rho. \quad (31)$$

Here $v_{\mu\nu\lambda\rho}$ is the totally symmetric structure constant tensor. The invariant chiral currents are the Liouville coordinates and they can be constructed as the product of the invariant symmetric tensors

$$d_{(\mu_1...\mu_n)} = d_{(\mu_1\mu_2} d^{k_2}_{k_3} ... d^{k_{n-3}}_{k_{n-1}...1\mu_n)}; \quad d_{\mu_1\mu_2} = \delta_{\mu_1\mu_2}$$

for $SU(n)$ group and the initial chiral currents $U^\mu(\phi(x))$. It is so called ”d-family” of the invariant chiral currents [28], [29], [30]:

$$C_n(U(\phi(x))) = d_{(\mu_1...\mu_n)} U_{\mu_1} U_{\mu_2} ... U_{\mu_n}, \quad C_2(U(\phi(x))) = \delta_{\mu\nu} U^\mu U^\nu. \quad (32)$$

Any of these currents satisfy the equation of motion $\partial C_n(U(\phi(t, x))) = 0$. The similar construction can be used for $SO(n)$, $SP(n)$ groups. The invariant chiral currents can be constructed as product of the invariant symmetric constant tensor

$$v_{(\mu_1...\mu_{2n})} = v^\nu_{(\mu_1\mu_2\nu_3} v^{\nu_4\nu_5} ... v^{\nu_{2n-3}}_{\nu_{2n-2}\nu_{2n-1}\mu_{2n})}; \quad v_{\mu_1\mu_2} = \delta_{\mu_1\mu_2}.\quad (32)$$
and the initial chiral currents $U^\mu$ as

$$C_{2n}(U(\phi(x))) = v_{\mu_1...\mu_{2n}} U^{\mu_1...U^{\mu_{2n}}}, \quad C_2(U(\phi(x))) = \delta_{\mu_1\mu_2} U^{\mu_1} U^{\mu_2}. \quad (33)$$

The invariant chiral currents for $SU(2)$, $SO(3)$, $SP(2)$ have form:

$$C_{2n} = (C_2)^n \quad (34)$$

Another family of the invariant symmetric currents $J_n$ based on the invariant symmetric chiral currents of simple Lie groups, is realized as the symmetric trace of the $n$ product chiral currents $U(x) = t_\mu U^\mu; \mu = 1, ..., n^2 - 1$

$$J_n(U(\phi(x))) = \text{SymTr}(U...U). \quad (35)$$

These invariant currents are polynomials of the product of the basic chiral currents $C_k, k = 2, 3, ..., k$.

$$J_2 = 2C_2, \quad J_3 = 2C_3, \quad J_4 = 2C_4 + \frac{4}{n}C_2^2, \quad J_5 = 2C_5 + \frac{8}{n}C_2C_3, \quad J_6 = 2C_6 + \frac{4}{n}C_3^2 + \frac{8}{n}C_2C_4 + \frac{8}{n^2}C_2^3, \quad J_7 = 2C_7 + \frac{8}{n}C_3C_4 + \frac{8}{n}C_2C_5 + \frac{24}{n^2}C_2^2C_3, \quad J_8 = 2C_8 + \frac{4}{n}C_4^2 + \frac{8}{n}C_3C_5 + \frac{8}{n}C_2C_6 + \frac{24}{n^2}C_2C_3^2 + \frac{24}{n^2}C_2^2C_4 + \frac{16}{n^3}C_4^2.$$

Let us introduce the PB of hydrodynamic type for the chiral currents in the Liouville form

$$\{C_m(\phi)(x), C_n(\phi(y))\} = -W_{mn}(\phi(y)) \frac{\partial}{\partial y} \delta(y - x) + W_{mn}(\phi(x)) \frac{\partial}{\partial x} \delta(x - y). \quad (36)$$

The asymmetric Hamiltonian function $W_{mn}(U(\phi(x)))$ for the finite dimensional $SU(n)$, $SO(n)$, $SP(n)$ groups has the following form

$$W_{mn}(C(U(x))) = \frac{n-1}{m+n-2} \sum_{k} a_k C_{m+n-2,k}(U(x)), \quad \sum_{k=0} a_k = mn. \quad (37)$$

This PB can be rewritten as the PB of the hydrodynamic type

$$\{C_m(U(x)), C_n(U(y))\} = -\frac{n-1}{m+n-2} \sum_{k} a_k \frac{dC_{m+n-2,k}(U((x)))}{dx} \delta(x - y) - \sum_{k} a_k C_{m+n-2,k}(U(x)) \frac{\partial}{\partial x} \delta(x - y), \quad \sum_{k=0} a_k = mn. \quad (38)$$
Here we used the following equalities \[22\], \[31\] to perform the PB of the hydrodynamic type:

\[
B(y)A(x) \frac{\partial}{\partial x} \delta(x - y) = B(y)A(y) \frac{\partial}{\partial x} \delta(x - y) - B(y) \frac{\partial A(y)}{\partial y} \delta(x - y),
\]

\[
\frac{\partial A(y)}{\partial y} \delta(x - y) + A(x) \frac{\partial}{\partial x} \delta(x - y) = A(y) \frac{\partial}{\partial x} \delta(x - y),
\]

\[
\frac{\partial}{\partial x} \delta(x - y) = -\frac{\partial}{\partial y} \delta(y - x).
\]

Here the invariant total symmetric currents \(C_{n,k}\), \(k = 1,2\ldots\) are new currents which are polynomials of the product of the basic invariant currents \(C_n, C_{n_1}\ldots C_{n_n}, n_1 + \ldots + n_n = n\). They can be obtained by the calculation of the total symmetric invariant currents \(J_n\) using the different replacements of the double product \(\{30\}\) for the \(SU(n)\) group and of the triple product \(\{31\}\) for the \(SO(n)\), \(SP(n)\) groups in the expressions for the invariant currents \(J_n\).

\[
J_6 = Tr[t(tt)(tt)t] = 2C_6 + \frac{4}{n}C_3^2 + \frac{8}{n}C_2C_4 + \frac{8}{n^2}C_2^3,
\]

\[
J_6 = Tr[(tt)(tt)(tt)] = 2C_{6,1} + \frac{12}{n}C_2C_4 + \frac{8}{n^2}C_2^3,
\]

\[
J_7 = Tr[t(tt)t(t)t] = 2C_7 + \frac{8}{n}C_3C_4 + \frac{8}{n^2}C^2C_5 + \frac{24}{n^2}C^2C_3,
\]

\[
J_7 = Tr[(tt)(tt)(tt)] = 2C_{7,1} + \frac{4}{n}C_3C_4 + \frac{12}{n^2}C^2C_5 + \frac{24}{n^2}C^2C_3,
\]

\[
J_8 = Tr[t(tt)t(t)t] = 2C_8 + \frac{4}{n}C_4^2 + \frac{8}{n}C_3C_5 + \frac{8}{n}C_2C_6 + \frac{24}{n^2}C_2C_3 + \frac{24}{n^2}C^2C_4 + \frac{16}{n^3}C_2^4,
\]

\[
J_8 = Tr[(tt)(tt)(tt)] = 2C_{8,1} + \frac{4}{n}C_4^2 + \frac{4}{n}C_3C_5 + \frac{24}{n^2}C_2C_3 + \frac{12}{n}C_2C_6 + \frac{24}{n^2}C^2C_4 + \frac{16}{n^3}C_2^4,
\]

\[
J_8 = Tr[(tt)(tt)(tt)] = 2C_{8,2} + \frac{4}{n}C_4^2 + \frac{16}{n}C_2C_{6,1} + \frac{32}{n^2}C^2C_4 + \frac{16}{n^3}C_2^4,
\]

\[
J_8 = Tr[t(tt)(tt)t] = 2C_8,3 + \frac{12}{n}C_2C_6 + \frac{8}{n}C_3C_5 + \frac{24}{n^2}C^2C_4 + \frac{24}{n^2}C^2C_3 + \frac{16}{n^2}C_2C_4 + \frac{16}{n^2}C_2C_3 + \frac{16}{n^2}C_2C_4.
\]

The new chiral currents \(C_{n,k}(U)\) have the form:

\[
C_{6,1} = d^k_{\mu \nu}d^l_{\lambda \rho}d^m_{\sigma \varphi}d^{klm}U^{\mu \nu}U^{\lambda \rho}U^{\sigma \varphi}U^{\tau},
\]

\[
C_{7,1} = d^k_{\mu \nu}d^l_{\lambda \rho}d^m_{\sigma \varphi}d^{klm}U^{\mu \nu}U^{\lambda \rho}U^{\sigma \varphi}U^{\tau},
\]

\[
C_{8,1} = \left[d^k_{\mu \nu}d^l_{\lambda \rho}d^{mn}U^{\mu \nu}U^{\lambda \rho}U^{\sigma \varphi}U^{\tau}\right],
\]

\[
C_{8,2} = \left[d^k_{\mu \nu}d^l_{\lambda \rho}d^{mn}d^{klp}U^{\mu \nu}U^{\lambda \rho}U^{\sigma \varphi}U^{\tau}\right],
\]

\[
C_{8,3} = \left[d^k_{\mu \nu}d^l_{\lambda \rho}d^{mn}d^{klp}U^{\mu \nu}U^{\lambda \rho}U^{\sigma \varphi}U^{\tau}\right].
\]

The following figures Fig1, Fig2 show the graphic images of the basic and the ”monster” invariant currents. Here are only \(l = n - 1\) primitive invariant tensors for \(SU(n)\) algebra,
$l = \frac{n-1}{2}$ for $SO(n)$ algebra and $l = \frac{n}{2}$ for $SP(n)$ algebra. Higher invariant currents $C_n$ for $n \geq l + 1$ are non-primitive currents and they are polynomials of the primitive ones. By using the formula (35) we can obtain the expression for these polynomials from the condition $J_k = 0$ for $k > l$ for the generating function

$$
det(1 - \lambda t^\mu U^\mu) = \exp Tr(ln(1 - \lambda U)) = \exp \left(-\sum_{k=2}^{\infty} \frac{\lambda^k}{k} J_k \right).
$$

The charges corresponding to the non-primitive chiral currents $C_n$ are not Casimir operators. Consequently the WZNW model is not the integrable system for the group symmetry of the finite rank $l \geq 1$.

A. Integrable WZNW models with $SU(2)$, $SO(3)$, $SP(2)$ constant torsions

There is only one primitive invariant tensor of $SU(2)$, $SO(3)$, $SP(2)$ algebras. The invariant non primitive tensors for $n \geq 2$ are functions of the primitive tensor. Let us introduce the local chiral currents based on the invariant symmetric polynomials on the $SU(2)$, $SO(3)$, $SP(2)$ Lie groups:

$$
C_2(U) = \delta_{\mu\nu} U^\mu U^\nu, C_{2n}(U) = (\delta_{\mu\nu} U^\mu U^\nu)^n,
$$
where \( n = 1, 2, \ldots \) and \( \mu, \nu = 1, 2, 3 \). The PB of Liouville coordinate \( C_2(U(x)) \) has the following forms:

\[
\{ C_2(U(x)), C_2(U(y)) \} = -2C_2(U(y))\partial_y\delta(y - x) + 2C_2(U(x))\partial_x\delta(x - y),
\]

\[
\{ C_2(U(x)), C_2(U(y)) \} = 4C_2(U(x))\partial_x\delta(x - y) + 2\frac{\partial}{\partial x}C_2(U(x))\delta(x - y).
\]

We will consider the chiral invariant \( C_2(U(x)) \) as a local field on the Riemmann space of the chiral currents. As the Hamiltonians we choose the following functions

\[
H_{2(n+1)} = \frac{1}{2(n + 1)} \int_0^{2\pi} C_{2}^{n+1}(U(y))dy, \ n = 0, 1, \ldots \infty. \quad (40)
\]

The equation of motion for the density of the first Casimir operator is as follows

\[
\frac{\partial C_2}{\partial t_{2(n+1)}} - (2n + 1)(C_2)^n\frac{dC_2}{dx} = 0. \quad (41)
\]

The equation for the currents \( C_2^n = C_{2n} \) is following:

\[
\frac{\partial C_2^n}{\partial \tau_n} + (C_2)^n\frac{dC_2^n}{dx} = 0, \ \tau_n = -(2n + 1)t_{2(n+1)}. \quad (42)
\]

This equation is the inviscid Burgers equation \([64]\). We will find the solution in the form:

\[
C_2^n(\tau_n, x) = \exp(a + i(x - \tau_n C_2^n(\tau_n, x))). \quad (43)
\]
To obtain the solution of equation (42) we rewrite this equation of motion as follows:

\[ Y_n = Z_n e^{Z_n}, \quad Y_n = i \tau_n e^{(a+ix)}, \quad Z_n = i \tau_n C_n^m. \]  \hspace{1cm} (44)

The inverse transformation \( Z_n = Z_n(Y_n) \) is defined by the periodical Lambert function \( 65 \):

\[ C_n^m(\tau_n, x) = \frac{1}{i \tau_n} W(i \tau_n e^{a+ix}). \]  \hspace{1cm} (45)

The solution for the first Casimir operator is consequently

\[ C_2(t_{2(n+1)}, x) = \left[ \frac{i}{(2n+1)t_{2(n+1)}} W(-i(2n+1)t_{2(n+1)}e^{a+ix}) \right]^\frac{1}{n}. \]  \hspace{1cm} (46)

The equation of motion for the initial chiral current \( U^\mu \) defined by the PB (24) and the Hamiltonian (40) is

\[ \frac{\partial U_\mu}{\partial t_{2(n+1)}} = \frac{\partial}{\partial x} [U_\mu(UU)^n] = C_2^n \frac{\partial}{\partial x} U_\mu + U_\mu \frac{\partial}{\partial x} C_2^n, \quad = n U_\mu C_2^{n-1} \frac{\partial}{\partial x} C_2 + C_2^n \frac{\partial}{\partial x} U_\mu, \quad \mu = 1, 2, 3. \]  \hspace{1cm} (47)

It is easy to test, that equation of motion (47) is in accordance with equation (41) by multiplication with the chiral current \( U_\mu \) on the both sides of equation (47). It is possible to rewrite this equation as the linear equation by using the solution (45) which diagonalize the equation (46)

\[ \frac{\partial U_\mu}{\partial t_{2(n+1)}} = \frac{\partial U_\mu}{\partial x} f_n + U_\mu \frac{\partial}{\partial x} f_n \]

or as the linear nonhomogeneous equation

\[ \frac{\partial z_\mu}{\partial t_{2(n+1)}} = f(t_n, x) \frac{\partial z_\mu}{\partial x} + \frac{\partial}{\partial x} f(t_n, x), \quad z_\mu = \ln U_\mu, \quad f = C_2^n, \quad \frac{\partial z_\mu}{\partial x} = \frac{1}{U_\mu} \frac{\partial U_\mu}{\partial x}, \quad \text{(not sum)}. \]  \hspace{1cm} (48)

B. Equation of motion for WZNW model with \( SU(3) \) torsion

The invariant chiral currents \( C_2(U), C_3(U) \) form a closed system. The non-primitive currents have the following form

\[ C_{2n} = C_2^n, \quad C_{2n+1} = C_2^{n-1} C_3, \]

\[ C_2 = \delta_{\mu\nu} U^\mu U^\nu, \quad C_3 = d_{\mu\nu\lambda} U^\mu U^\nu U^\lambda, \quad \mu\nu\lambda = 1, 2...8. \]

The algebra of corresponding charges is a not abelian algebra, but the charges \( C_{2n} \) form the corresponding invariant subalgebra. The currents \( C_2 \) and \( C_3 \) are the local coordinates
on the Riemann space and the invariant currents $C_{2n}$ are densities of the Hamiltonians. Equation of motion for $C_3$ is following:

$$\frac{\partial C_3(x)}{\partial t_{2(n+1)}} = -2C_2^n \frac{\partial}{\partial x} C_3 - 6C_3 \frac{\partial}{\partial x} C_2^n.$$ (49)

In terms of variables $g = \ln C_3$, $f = C_2^n$ it is linear equation

$$\frac{\partial g}{\partial t_{2(n+1)}} + 2f \frac{\partial}{\partial x} g + 6 \frac{\partial}{\partial x} f = 0.$$ (50)

C. Infinite dimensional hydrodynamic chains

The first example of the infinite dimensional hydrodynamic chains is based on the invariant chiral currents $C_{2n} = (C_2)^n$, $n = 1, 2, ..., \infty$ of the WZNW model with the $SU(2)$, $SO(3)$, $SP(2)$ constant torsions. The PB of the different degrees of the invariant chiral currents $C_2^n(x), C_2^{m}(x)$ has form:

$$\{C_2^n(x), C_2^m(y)\} = 2nm(2m-1)\frac{C_2^{m+n-1}(x)}{n + m - 1} \frac{\partial}{\partial x} \delta(x - y) - 2nm(2n-1)\frac{C_2^{m+n-1}(y)}{n + m - 1} \frac{\partial}{\partial y} \delta(y - x).$$ (51)

The equation of motion for invariant current $C_2^m$ with Hamiltonian

$$H_{2n} = \frac{1}{2n} \int_0^{2\pi} C_{2n}(y)dy$$

has form:

$$\frac{\partial C_2^m}{\partial t_{2n}} = \frac{m(2n-1)}{m + n - 1} \frac{\partial}{\partial x} C_2^{m+n-1}.$$ 

After the redefinition $C_2^n = C_{2n} = C_p$ we can obtain the standard form of the hydrodynamic chain

$$\{C_p(x), C_q(y)\} = \frac{pq(p-1)}{p + q - 2} C_{p+q-2}(x) \frac{\partial}{\partial x} \delta(x - y) - \frac{pq(q-1)}{p + q - 2} C_{p+q-2}(y) \frac{\partial}{\partial y} \delta(y - x).$$ (52)

The second example of the infinite dimensional chain is based on the invariant chiral currents of the WZNW model with the $SU(\infty)$, $SO(\infty)$, $SP(\infty)$ constant torsions. If the dimension of the matrix representation $n$ is not ended ($n \to \infty$), all the chiral currents are the primitive currents. This is easy to see from the expression for the new chiral currents $C_{m,k}$. For example:

$$C_{6,1} = C_6 + \frac{2}{n} C_3^2 - \frac{2}{n} C_2 C_4, \quad C_{7,1} = C_7 + \frac{4}{n} C_3 C_4 - \frac{4}{n} C_2 C_5,$$
\[C_{8,1} = C_8 + \frac{2}{n}C_3C_5 - \frac{2}{n}C_2C_6, \quad C_{8,3} = C_8 + \frac{2}{n}C_4^2 - \frac{2}{n}C_2C_6,\]
\[C_{8,2} = C_8 + \frac{4}{n}C_3C_5 - \frac{4}{n}C_2C_6 - \frac{4}{n^2}C_2C_3 + \frac{4}{n^2}C_2^2C_4.\]

The PB in Liouville coordinates \(C_m(x), m = 2, 3, \ldots, \infty\) has the form:

\[\{C_m(x), C_n(y)\} = - W_{mn}(C(y)) \frac{\partial}{\partial y} \delta(y - x) + W_{nm}(C(x)) \frac{\partial}{\partial x} \delta(x - y), \quad (53)\]

\[W_{mn}(C(x)) = \frac{mn(n - 1)}{m + n - 2} C_{m+n-2}(x). \quad (54)\]

This PB satisfies the skew-symmetric condition \(\{C_m(x), C_n(y)\} = - \{C_n(y), C_m(x)\}\). The Jacobi identity imposes conditions on the Hamiltonian function \(W_{mn}(C(x))\) \[31], \[32], \[56]\]

\[(W_{kp} + W_{pk}) \frac{\partial W_{mn}}{\partial C_k} = (W_{km} + W_{mk}) \frac{\partial W_{pm}}{\partial C_k}, \quad \frac{dW_{kp}}{dx} \frac{\partial W_{mn}}{\partial C_k} = \frac{dW_{km}}{dx} \frac{\partial W_{np}}{\partial C_k}. \quad (55)\]

The Jacobi identity is satisfied by the metric tensor \(W_{mn}(C(x))\) \[54\]. The PB \(53\) forms the algebra and can be rewritten as the PB of the hydrodynamic type.

\[\{C_m(x), C_n(y)\} = \frac{mn(n - 1)}{m + n - 2} \frac{dC_{m+n-2}}{dx} \delta(x - y) + mnC_{m+n-2} \frac{\partial}{\partial x} \delta(x - y). \quad (56)\]

The algebra of charges \(\int_0^{2\pi} C_n(x)dx\) is the abelian algebra. Let us choose the Casimir operators \(C_n\) as the Hamiltonians

\[H_n = \frac{1}{n} \int_0^{2\pi} C_n(x)dx, \quad n = 2, 3, \ldots. \quad (57)\]

The equations of motion for the Casimir operator densities are the following

\[\frac{\partial C_m(x)}{\partial t_n} = \frac{1}{n} \int_0^{2\pi} [- W_{mn}(C(y)) \frac{\partial}{\partial y} \delta(y - x) + W_{nm}(C(x)) \frac{\partial}{\partial x} \delta(x - y)] dy = \frac{m(n - 1)}{m + n - 2} \frac{\partial}{\partial x} C_{m+n-2}. \quad (58)\]

Thus, the invariant chiral currents with the \(SU(2), SO(3), SP(2)\) constant torsion and the invariant chiral currents with the \(SU(\infty), SO(\infty), SP(\infty)\) constant torsion form the same infinite hydrodynamic chain \[52], \[53], \[58\].

These PBs \[53], \[57\] are particular case of the \(M\)-brackets of Dorfman \[38\] and Kupershmidt \[39], \[40\] for \(M = 2\) and describe the hydrodynamic chains (see \[36], \[37\] and references therein). We can construct the new nonlinear equations of motion for the initial
chiral currents $U^\mu$ using the flat $PB_2$ \cite{21} and Hamiltonians $H_n$ \cite{58}, where $C_n(x)$ defined by the equation (32) for $SU(\infty)$ group:

$$\frac{\partial U_\mu(x)}{\partial t_n} = \frac{1}{n} \int_0^{2\pi} dy \{U_\mu(x), C_n(U(y))\}_2,$$

$$\frac{\partial U_\mu(x)}{\partial t_n} = \frac{\partial}{\partial x} \left[ d_{\nu_1\nu_2}^{k_1} d_{\nu_3\nu_4}^{k_2} \cdots d_{\nu_{n-3}\nu_{n-1}}^{k_{n-3}} U^{\nu_1}(x) \cdots U^{\nu_{n-1}}(x) \right]. \tag{59}$$

As an example we consider $n = 3$:

$$\frac{\partial U_\mu}{\partial t_3} = \frac{\partial}{\partial x} (d_{\mu\nu\lambda} U^{\nu} U^{\lambda}), \quad \mu = 1, 2, \ldots \infty. \tag{60}$$

It is easy to see that this dynamical system is bi-Hamiltonian:

$$\frac{\partial U_\mu(x)}{\partial t_3} = \frac{1}{3} \int_0^{2\pi} dy \{U_\mu(x), C_3(U(y))\}_3 = \frac{1}{2} \int_0^{2\pi} dy \{U_\mu(x), C_2(U(y))\}_3. \tag{61}$$

Here $PB_3$ has form:

$$\{U_\mu(x), U_\nu(y)\}_3 = 2d_{\mu\nu\lambda} U^{\lambda}(x) \frac{\partial}{\partial x} \delta(x - y). \tag{62}$$

Let us remind that $d_{\mu\nu\lambda}$ are the symmetric structure constant of the $SU(\infty)$ algebra in a matrix representation. This PB satisfies the Jacobi identity for ($n \to \infty$)

$$d_{\sigma\mu\nu} d_{\nu\lambda\rho} + d_{\sigma\mu\lambda} d_{\sigma\nu\rho} + d_{\sigma\rho\lambda} d_{\sigma\nu\mu} = \frac{1}{n} (\delta_{\nu\rho} \delta_{\lambda\sigma} + \delta_{\mu\lambda} \delta_{\nu\sigma} + \delta_{\nu\rho} \delta_{\nu\lambda}).$$

Similarly, we can obtain the equation of motion for the chiral currents of $SO(\infty)$, $SP(\infty)$:

$$\frac{\partial U_\mu(x)}{\partial t_n} = \frac{\partial}{\partial x} \left[ v_{\nu_1\nu_2\nu_3}^{k_1} \cdots v_{\nu_{n-3}\nu_{n-2}\nu_{n-1}}^{k_{n-3}} U^{\nu_1}(x) \cdots U^{\nu_{n-1}}(x) \right]. \tag{63}$$

As an example we consider $n = 4$:

$$\frac{\partial U_\mu}{\partial t_4} = \frac{\partial}{\partial x} (v_{\mu\nu\lambda\rho} U^{\nu} U^{\lambda} U^{\rho}), \quad \mu = 1, 2, \ldots \infty. \tag{64}$$

Also, we can obtain a solution for the metric function $W_{mn}(C(x))$ which is analog to the Dubrovin-Novikov metric tensor $W_{\mu\nu} = \frac{\partial^2 F}{\partial U^\mu \partial U^\nu}$:

$$C_m(U(x)) = m F((U(x)), \quad F(x, t_n) = g(t_n + \frac{x}{n-1})$$

and $g(t_n + \frac{x}{n-1})$ is an arbitrary function of its argument.
IV. INTEGRABLE STRING MODEL

In this section we will show that the description of the dynamics of the transverse coordinates of the string in the conformal and light-cone gauges coincides with the description of the dynamics of compact coordinates of WZNW model (see [66]). Consequently, we will show here that the string model can be considered as an auxiliary problem in order to obtain integrable equations of hydrodynamic type.

A. Lagrangian in the conformal and light-cone gauges

The conformal invariant closed string model in the background gravity and antisymmetric fields is described by the following action [41] (see also [42] and references therein)

$$S = \frac{1}{2} \int_0^{2\pi} d^2x \sqrt{g} [g^{\alpha\beta} g_{AB}(X) \partial X^A \partial X^B + \epsilon^{\alpha\beta} \sqrt{g} B_{AB}(X) \partial X^A \partial X^B]. \quad (65)$$

The action $S[g_{\alpha\beta}, X^A]$ is a functional of the worldsheet metric $g_{\alpha\beta}(x)$ and of the $n+2$ spacetime coordinate fields $X^A(x) = X^A(x + 2\pi)$. Here $x^\alpha$ ($\alpha = 0, 1$) are world-sheet coordinates, $X^A(x)$ ($A = 0, a, n+1$) are target coordinates and $a = 1, 2, ..., n$. The signature of worldsheet space is $(-, +)$ and signature of target space is $(-, +, ..., +)$. Here $g$ is the determinant world-sheet metric tensor $g_{\alpha\beta}$. The target space-time fields consist of the symmetric metric tensor $g_{AB}(X) = g_{BA}(X)$ and antisymmetric tensor field $B_{AB}(X) = -B_{BA}(X)$. The antisymmetric tensor $\epsilon^{\alpha\beta}$ is the Levi-Civita tensor such that $\epsilon^{01} = 1$. Notice that target space-time indices are Roman, worldsheet indices are Greek and Einstein convention is assumed. The world-sheet metric $g^{\alpha\beta}$ is an auxiliary field in the Lagrangian (65). The variation of Lagrangian (65) with respect to the field $g^{\alpha\beta}$ yields the classical field equation:

$$g_{AB}(X) \partial X^A \partial X^B - g_{\alpha\beta} g^{\gamma\delta} \partial X^A \partial X^B = 0. \quad (66)$$

The Lagrangian (65) is invariant under re-parametrization of the world-sheet coordinates

$$x^\alpha \rightarrow x^\alpha + \epsilon^\alpha(t, x).$$

These transformations permit to consider the conformal gauge for the world-sheet metric

$$g^{\alpha\beta} = g^{(t, x)} \eta^{\alpha\beta}. \quad (67)$$
and the light-cone gauge for the light-cone variable $X^+ = p^+ t$. There $\eta^{\alpha\beta}$ is the diagonal metric of the flat world-sheet $\eta^{\alpha\beta} = \{\eta^{00}, \eta^{11}\} = (-1, 1)$ and

$$X^\pm = \frac{X^{n+1} \pm X^0}{\sqrt{2}}.$$  \hspace{1cm} (68)

The Lagrangian (65) does not depend of the conformal field $\theta(t, x)$. The equation (66) leads to the following relations:

$$g_{AB}(X)(\partial_t X^A \partial_t X^B + \partial_x X^A \partial_x X^B) = 0, \quad g_{AB}(X)\partial_t X^A \partial_x X^B = 0. \hspace{1cm} (69)$$

These relations describe the geodesic motion of the probe body on the two dimensional surface, which is embedded in the $n + 2$ curve target space. To introduce the Hamiltonian, let us rewrite the Lagrangian density (65) in terms of the world-she et coordinates $(t, x)$ in the conformal gauge (67). It has form:

$$L = \frac{1}{2} g_{AB}(X) [\partial X^A \partial_t X^B - \partial X^A \partial_x X^B] + B_{AB}(X) \partial_t X^A \partial_x X^B. \hspace{1cm} (70)$$

The canonical momentum is the following

$$p_A(t, x) = \frac{\delta L}{\delta (\partial_t X^A)} = g_{AB}(X) \partial X^B + B_{AB}(X) \partial_x X^B. \hspace{1cm} (71)$$

By the definition, the Hamiltonian has the following form:

$$H(X, p) = p_A \frac{\partial X^A}{\partial t} - L = \frac{1}{2} g^{AB}[p_A - B_{AC}(X) \frac{\partial X^C}{\partial x}] [p_B - B_{BD}(X) \frac{\partial X^D}{\partial x}] + \frac{1}{2} g_{AB} \frac{\partial X^A}{\partial x} \frac{\partial X^B}{\partial x}. \hspace{1cm} (72)$$

Let us consider the constraints (69), the equation for canonical momentum (71) and the definition of the inverse metric tensor $g_{AB}$ $g^{BC} = \delta^C_A$ in the light-cone gauge $X^+ = p^+ t$ to delete nonphysical components of the metric tensor $g_{AB}$. From the definition of the components of the momentum $p_\perp = -p^+$, we obtain the following constraints:

$$g_\perp = g_\perp = B_\perp = B_\perp = g^+ = g^+ = B^+ = B^+ = 0, g_+ = -1. \hspace{1cm} (73)$$

The momentum component $p_\perp$ leads to constraints:

$$p_\perp = -p_\perp = -X_\perp, B_\perp = B_\perp = 0. \hspace{1cm} (74)$$

The component $p_a$ has form

$$p_a = g_{ab}(X) \partial_t X_a + B_{ab}(X) \partial_x X^b. \hspace{1cm} (75)$$
From the inverse metric we obtain the following constraints:

\[ g^{\pm b} = -g_{-a} g^{ab} = 0, \quad g^{++} = g_{--} = 0, \quad g_{+-} = g^{-} = -1. \]  \quad (76)

From the relations \( g_{+A} g^{A b} = 0 \) and \( g_{+A} g^{A -} = 0 \) we obtained constraints \( g_{+a} = g_{++} = 0 \).

These constraints were not obtained in the paper [42]. From the constraints \( g_{AB} \partial X^A \partial_x X^B = 0 \) we obtain the relation

\[ \partial_x X^- = \frac{1}{p^+} g_{ab}(X) \partial_t X^a \partial_x X^b. \]  \quad (77)

From the constraints

\[ g_{AB}(\partial_t X^A \partial_t X^B + \partial_x X^A \partial_x X^B) = 0 \]

the equation

\[ \partial_t X^- = \frac{1}{2p^+} g_{ab}(X)(\partial_t X^a \partial_t X^b + \partial_x X^a \partial_x X^b) \]  \quad (78)

is obtained. By using the equations (73), (74) and (76) we can write the Hamiltonian in the light-cone gauge:

\[ H = p^- p^+ = \frac{1}{2}[g^{ab}(p_a - B_{ac} \partial_x X^c)(p_b - B_{bd} \partial_x X^d) + g_{ab} \partial_x X^a \partial_x X^b]. \]  \quad (79)

This Hamiltonian describes the transverse coordinates of string and it differs from the Hamiltonian in paper [42] due to the constraints \( g_{--} = g_{++} = 0, g_{-a} = -g_{a+} = 0, B_{-a} = -B_{+a} = 0 \).

The motion of the longitudinal coordinate \( X^-(x, t) \) is described consequently by the equations (77), (78).

### B. String Lagrangian of field theory type

Historically, a classical field theory considers models of free fields without interaction, study the group symmetry of them and representations of this group of symmetry. Further, a theory of interactions is constructed for a chosen representation with conservation of the corresponding group symmetry. We apply this method for constructing a string model with transverse coordinates. The action of the free string has form:

\[ S = \frac{1}{2} \int_0^{2\pi} d^2 x \sqrt{g} g^{\alpha \beta} g_{AB} \partial_\alpha X^A \partial_\beta X^B. \]  \quad (80)
The difference this action from action (65) consist of the lack of the antisymmetric field $B_{AB}$ and by the constant metric tensor $g_{AB}$. In the conformal gauge (67), the Lagrangian has form:

$$L = \frac{1}{2} g_{AB} \left[ \frac{\partial X^A}{\partial t} \frac{\partial X^B}{\partial t} - \frac{\partial X^A}{\partial x} \frac{\partial X^B}{\partial x} \right].$$  \hspace{1cm} (81)

The canonical momentum is the following

$$p_A(X(t, x)) = \delta L / \delta (\partial X^A / \partial t) = g_{AB} \frac{\partial X^B}{\partial t}.$$  \hspace{1cm} (82)

By the definition, the Hamiltonian has following form:

$$H(X, p) = p_a \frac{\partial X^A}{\partial t} - L = \frac{1}{2} g^{AB} p_A p_B + \frac{1}{2} g_{AB} \frac{\partial X^A}{\partial x} \frac{\partial X^B}{\partial x}.$$  \hspace{1cm} (83)

The variation of the Lagrangian (80) with respect to the field $g^{\alpha\beta}$ yields two constraints:

$$g_{AB} (\partial X^A \partial X^B + \partial_x X^A \partial_x X^B) = 0, \quad g_{AB} \partial_t X^A \partial_x X^B = 0.$$  \hspace{1cm} (84)

Let us consider the constraints (84) in the light-cone gauge $X^+ = p^+ t$. Let us remember, that the target space fields $X^A (A = 0, a, n + 1)$ in the light-cone coordinates $(A = +, -, a)$ have form (68)

$$X^A = (X^+, X^-, X^a) = (\frac{X^{n+1} + X^0}{\sqrt{2}}, \frac{X^{n+1} - X^0}{\sqrt{2}}, X^a).$$

From the definition of the canonical momentum (82) and of the inverse metric tensor we obtain constraints for the constant metric

$$g_{--} = g_{++} = g_{a+} = g_{a-} = g^{++} = g^{--} = g^{a+} = g^{a-} = 0, \quad g_{+-} = g^{--} = -1.$$  \hspace{1cm} (87)

The constraints (84) leads to the following equations:

$$\partial_x X^- = \frac{1}{p^-} g_{ab} \partial_t X^a \partial_x X^b.$$ \hspace{1cm} (85)

$$\partial_t X^- = \frac{1}{2p^+} g_{ab} (\partial_t X^a \partial_t X^b + \partial_x X^a \partial_x X^b).$$ \hspace{1cm} (86)

By using the definition of the momentum (82), we can write the Hamiltonian in the light-cone gauge

$$H = p^- p^+ = \frac{1}{2} [g^{ab} p_a p_b + g_{ab} \partial_x X^a \partial_x X^b].$$  \hspace{1cm} (87)

Thus, we obtained the Hamiltonian which describe the transverse coordinates of the flat target space. We want to obtain the Hamiltonian which describe the interaction of the
transverse coordinates. To do this, we will consider a curved target space by the replacement $g_{ab} \to g_{ab}(X)$. An interaction with the gauge field $B_{ab}(X)\partial_x X^b$ can be introduced to the Hamiltonian by minimal replacement $p_a \to p_a - B_{ab}(X)\partial_x X^b$. Thus, both methods lead to the same Hamiltonian for the transverse target space coordinates. The moving of the longitudinal coordinate is describe by the equations (77), (78).

C. Repere formalism and commutation relations

Let us return to the subsection (4.1). In the repere formalism

$$g_{ab}(X) = e^\mu_a(X)e^\nu_b(X)g_{\mu\nu},$$

where $g_{\mu\nu}$ is constant tensor of the tangent space to the curved space of the string coordinates $X^a$. The repere $e^\mu_a$ satisfies to the condition

$$g_{\mu\nu} = e^\mu_a(X)e^\nu_b(X)g_{ab}(X).$$

To factorize the Hamiltonian (72) we introduce new variables:

$$J_{0\mu}(X) = e^\mu_a(X)[p_a - B_{ab}(X)X^b], \quad J_{1\mu}(X) = g_{\mu\nu}e^\nu_a\partial_x X^a.$$  \hspace{1cm} (88)

We see, that the Hamiltonian (72) is factorized in this variables:

$$H = \frac{1}{2}[g^{\mu\nu}J_{0\mu}(X)J_{0\nu}(X) + g_{\mu\nu}J^\mu_1(X)J^\nu_1(X)]$$ \hspace{1cm} (89)

The canonical PB is the following:

$$\{X^a(x), \ p_b(y)\} = \delta^a_b\delta(x - y).$$

The commutation relations of new variables $J_{0\mu}(X(x))$, $J_{1\mu}(X(x))$ under PB have the following form:

$$\{J_{0\mu}(X(x)), \ J_{0\nu}(X(y))\} = C^\lambda_{\mu\nu}(X(x))J_{0\lambda}(x)\delta(x - y) + H^\lambda_{\mu\nu}(X(x))J_{1\lambda}(x)\delta(x - y),$$

$$\{J_{0\mu}(X(x)), \ J_{1\nu}(X(y))\} = C^\lambda_{\mu\nu}(X(x))J_{1\lambda}(X(x))\delta(x - y) + g_{\mu\nu}\frac{\partial}{\partial x}\delta(x - y),$$

$$\{J_{1\mu}(X(x)), \ J_{1\nu}(X(y))\} = 0.$$  \hspace{1cm} (90)

Here the tensor $C^\lambda_{\mu\nu}(X)$ is the torsion related to the metric tensor:

$$C^\mu_{\nu\lambda}(X) = \frac{\partial e^\mu_a}{\partial x^b}(e^b_c e^\nu_c - e^\nu_c e^b_c) = \left(\frac{\partial e^\mu_b}{\partial x^a} - \frac{\partial e^\mu_a}{\partial x^b}\right)e^b_c e^\nu_c$$ \hspace{1cm} (91)
The tensor $H^\lambda_{\mu\nu}(X)$ is the torsion related to the background field $B_{ab}$:

$$H^\mu_{\nu\lambda}(X) = G^{\mu\rho}H_{abc}(X)\varepsilon^a_\rho(X)\varepsilon^b_\nu(X)\varepsilon^c_\lambda(X),$$

$$H_{abc}(X) = \frac{\partial B_{ab}}{\partial X^c} + \frac{\partial B_{ca}}{\partial X^b} + \frac{\partial B_{bc}}{\partial X^a}. \quad (92)$$

The difference between the commutation relations (88) of the string model with the commutation relations of the WZNW model (21) is due the different tensors $C^\lambda_{\mu\nu}$. The constant torsion $C^\lambda_{\mu\nu}$ of the WZNW model is the structure constant of the Lie algebra by simple definition. The torsion $C^\lambda_{\mu\nu}(X(x))$ is the function of the Riemann space coordinates. In some applications this torsion may be zero tensor or the constant tensor. Let us introduce the chiral variables:

$$U_\mu(X) = \frac{J_{0\mu}(X) + g_{\mu\nu}J^\nu_1(X)}{\sqrt{2}}, \quad V_\mu(X) = \frac{J_{0\mu}(X) - g_{\mu\nu}J^\nu_1(X)}{\sqrt{2}}. \quad (93)$$

The chiral variables $U_\mu(X(x)), V_\mu(X(x))$ satisfy the following commutation relations:

$$\{U_\mu(X(x)), U_\nu(X(y))\} = \frac{1}{2\sqrt{2}}[(3C^\lambda_{\mu\nu}(X(x)) + H^\lambda_{\mu\nu}(X(x)))U_\lambda(X(x)) -$$

$$-(C^\lambda_{\mu\nu}(X(x)) + H^\lambda_{\mu\nu}(X(x)))V_\lambda(X(x))]\delta(x - y) + g_{\mu\nu}\partial_\lambda\delta(x - y),$$

$$\{V_\mu(X(x)), V_\nu(X(y))\} = \frac{1}{2\sqrt{2}}[(3C^\lambda_{\mu\nu}(X(x)) - H^\lambda_{\mu\nu}(X(x)))V_\lambda(X(x)) -$$

$$-(C^\lambda_{\mu\nu}(X(x)) - H^\lambda_{\mu\nu}(X(x)))U_\lambda(X(x))]\delta(x - y) - g_{\mu\nu}\partial_\lambda\delta(x - y),$$

$$\{U_\mu(X(x)), V_\nu(X(y))\} = \frac{1}{2\sqrt{2}}[(C^\lambda_{\mu\nu}(X(x)) + H^\lambda_{\mu\nu}(X(x)))U_\lambda(X(x)) +$$

$$+(C^\lambda_{\mu\nu}(X(x)) - H^\lambda_{\mu\nu}(X(x)))V_\lambda(X(x))]\delta(x - y).$$

This commutation relations form an algebra, if tensors $C^\lambda_{\mu\nu}(X), H^\lambda_{\mu\nu}(X)$ are the constant tensors. The interesting cases, again arise if $H^\lambda_{\mu\nu} = \pm C^\lambda_{\mu\nu}$ and tensor $C^\mu\nu_\lambda$ is the structure tensor of the compact Lie algebra. In the case $H^\lambda_{\mu\nu} = -C^\lambda_{\mu\nu}$ variables $U_\mu$ form the closed Kac-Moody algebra for the right chiral currents

$$\{U_\mu(X(x)), U_\nu(X(y))\} = C^\lambda_{\mu\nu}U_\lambda(X(x))\delta(x - y) + \delta_{\mu\nu}\partial_\delta(x - y). \quad (95)$$

In the case $H^\lambda_{\mu\nu} = C^\lambda_{\mu\nu}$ variables $V_\mu$ form the closed Kac-Moody algebra for the left chiral currents

$$\{V_\mu(X(x)), V_\nu(X(y))\} = C^\lambda_{\mu\nu}V_\lambda(X(x)) - \delta_{\mu\nu}\partial_\delta(x - y). \quad (96)$$

The more complicated case arise if both torsion are null tensors.
D. Integrable string model with null torsion $C^\mu_\lambda = 0$

To construct an integrable dynamical system we must have a hierarchy of PBs and we must find a hierarchy of Hamiltonians through the bi-Hamiltonity condition. The basic idea of the hydrodynamic approach given by Dubrovin, Novikov [31, 32] to integrable systems is a construction of compatible local PB of an abelian currents from a pencil local PB on the flat space of the currents and from the local PB on the curved space of the currents. This approach was generalized by Ferapontov [46] and Mokhov, Ferapontov [47] on the non-local PBs of hydrodynamic type. The hydrodynamic type systems was considered by Tsarev [49], Maltsev [50], Ferapontov [51], Mokhov [48, 52, 53], Pavlov [54, 55], Maltsev, Novikov [50]. The Jacobi identity for compatible PB leads to the WDVV [44, 45] associativity equation for the metric tensor of a curved space. Dubrovin [57] shown that the WDVV equation is related to the Frobenius algebra of a chiral currents. This equation is precisely the associativity condition of the Frobenius algebra. Dubrovin and Zhang [58] gave many examples of the solutions of the WDVV equation. We have applied the hydrodynamic approach of the Dubrovin, Novikov and the Dubrovin solutions of the WDVV equation to the description of the integrable string model in the background gravity field with zero torsion [33, 34, 35]. Now let us to consider string model with constant torsion.

V. INTEGRABLE STRING MODEL WITH THE CONSTANT TORSION

We have considered the string model with constant torsion without antisymmetric field $B_{ab}(X)$ in the light-cone gauge in the target space [35, 59, 60, 61]. This model coincides to the principal chiral model on the compact simple Lie group. We can not separate the dynamics on the right-mode only and the left mode for the $\sigma$- model because of the initial chiral currents are not conserved

$$\partial_- U_\mu = C^\mu_\nu U_\nu V_\lambda \neq 0, \quad \partial_+ V_\mu = C^\mu_\nu V_\nu U_\lambda \neq 0.$$ 

The correspondent charges are not Casimirs. These papers were motivated by the papers [23, 24, 25]. Evans, Hassan, MacKay, Mountain [25] where they constructed the local invariant chiral currents $C_n(U)$ [31, 32], as the scalar symmetric polynomials of the initial chiral currents $U_\mu(X(x))$ of the $SU(n)$, $SO(n)$, $SP(n)$ principal chiral models. The constant
torsion $C^\mu_{\nu\lambda}$ in the commutation relations for the initial chiral currents $U^\mu(x)$ does not contribute to the equation of motion for the invariant currents $C_n(U(X))$ through the total symmetrical functions $C_n(U)$. The index $n$ of the invariant current $C_n$ is the power of the product of the initial chiral currents $U^\mu$. However, the introduction of the antisymmetric field $B_{ab}(X)$ to Lagrangian permits to obtain the equation of motion for the initial chiral current $U^\mu(X(t, x))$.

Now, we can make use of the results section 3 to describe the string model of the WZNW type with the constant torsion (see also papers [62], [63]. The mathematical description of the equations of motion of the invariant chiral and initial currents of the string model with the constant torsions in this section coincides with the case of the WZNW model with the constant torsion which was considered in Section 3.

A. Integrable string models with $SU(2)$, $SO(3)$, $SP(2)$ constant torsions and hydrodynamic chains

The PB of Liouville coordinate $C_2(U(x))$ has the following forms:

$$\{C_2(U(x)), C_2(U(y))\} = -2C_2(U(y))\partial_y\delta(y - x) + 2C_2(U(x))\partial_x\delta(x - y).$$

We will consider invariant chiral $C_2(U(x))$ as a local field on the Riemmann space of the chiral currents. As the Hamiltonians we choose the following functions

$$H_{2(n+1)} = \frac{1}{2(n+1)} \int_0^{2\pi} C_2^{n+1}(U(y))dy, \quad n = 0, 1, ... \infty.$$ 

The equation of motion for the density of the first Casimir operator is as follows

$$\frac{\partial C_2}{\partial t_{2(n+1)}} - (2n+1)(C_2)^n \frac{dC_2}{dx} = 0.$$ 

The equation for the currents $C_2^m = C_2$ is following:

$$\frac{\partial C_2^m}{\partial \tau_n} + (C_2)^n \frac{dC_2^m}{dx} = 0, \quad \tau_n = -(2n+1)t_{2(n+1)}.$$ 

This equation is inviscid Burgers equation [64]. We will find the solution in the form:

$$C_2^m(\tau_n, x) = \exp(a + i(x - \tau_n C_2^m(\tau_n, x))).$$
following:
\[ C_2(t_{2(n+1)}, x) = \left[ \frac{i}{(2n + 1)t_{2(n+1)}} W(-i(2n + 1)t_{2(n+1)} e^{a+ix}) \right]^\frac{1}{n}. \]

Here function \( W(t_{2(n+1)}, x) \) is periodical Lambert function \([65]\). The equation of motion for the initial chiral current \( U^\mu \) defined by the PB \([24]\) and the Hamiltonian \([40]\)

\[ \frac{\partial U_\mu}{\partial t_{2(n+1)}} = \frac{\partial}{\partial x} [U_\mu(U U)^n] = C_2^n \frac{\partial}{\partial x} U_\mu + U_\mu \frac{\partial}{\partial x} C_2^n, \quad \mu = 1, 2, 3. \]

It is possible to rewrite this equation as the linear equation by using the solution \([45]\) which diagonalize the equation \([46]\)

\[ \frac{\partial U_\mu}{\partial t_{2(n+1)}} = \frac{\partial U_\mu}{\partial x} f_n + U_\mu \frac{\partial}{\partial x} f_n \]

or as the linear nonhomogeneous equation

\[ \frac{\partial z_\mu}{\partial t_{2(n+1)}} = f(t_n, x) \frac{\partial z_\mu}{\partial x} + \frac{\partial}{\partial x} f(t_n, x), \quad z_\mu = \ln U_\mu, \quad f = C_2^n, \quad \frac{\partial z_\mu}{\partial x} = \frac{1}{U_\mu} \frac{\partial U_\mu}{\partial x}, \quad (not \ sum). \]

The first example of the infinite dimensional hydrodynamic chains is based on the invariant chiral currents \( C_{2n} = (C_2)^n, \quad n = 1, 2, ..., \infty \) of the string model with the \( SU(2), SO(3), SP(2) \) constant torsions. The PB of the different degrees of the invariant chiral currents \( C_2^n(x), C_2^m(x) \) has form:

\[ \{C_2^m(x), C_2^n(y)\} = \frac{2nm(n - 1)}{n + m - 1} C_2^{m-1}(x) \frac{\partial \delta(x - y)}{\partial x} - \frac{2nm(n - 1)}{n + m - 1} C_2^{m-1}(y) \frac{\partial \delta(y - x)}{\partial y}. \]

The equation of motion for invariant current \( C_2^m \) with Hamiltonian

\[ H_{2n} = \frac{1}{2n} \int_0^{2\pi} C_2^n(y) dy \]

is

\[ \frac{\partial C_2^m}{\partial t_{2n}} = \frac{m(2n - 1)}{m + n - 1} \frac{\partial C_2^{m+n-1}}{\partial x}. \]

After the redefinition \( C_2^n = C_{2n} = C_p \) we can obtain the standard form of the hydrodynamic chain \([52]\).

\[ \{C_p(x), C_q(y)\} = \frac{pq(p - 1)}{p + q - 2} C_{p+q-2}(x) \frac{\partial \delta(x - y)}{\partial x} - \frac{pq(q - 1)}{p + q - 2} C_{p+q-2}(y) \frac{\partial \delta(y - x)}{\partial y}. \]

The second example of the infinite dimensional chain is based on the invariant chiral currents of the \( \sigma - model \) with the \( SU(\infty), SO(\infty), SP(\infty) \) constant torsions. If the
dimension of the matrix representation \( n \) is not ended \((n \to \infty)\) all the chiral currents are the primitive ones. The PB in Liouville coordinates \( C_m(x), \ m = 2, 3, \ldots, \infty \) has the form (53): (54):

\[
\{C_m(x)C_n(y)\} = -W_{mn}(C(y)) \frac{\partial}{\partial y} \delta(y - x) + W_{nm}(C(x)) \frac{\partial}{\partial x} \delta(x - y),
\]

\[
W_{mn}(C(x)) = \frac{mn(n - 1)}{m + n - 2} C_{m+n-2}(x).
\]

This PB satisfies to the skew-symmetric condition \( \{C_m(x), C_n(y)\} = -\{C_n(y), C_m(x)\} \). The Jacobi identity imposes conditions on the Hamiltonian function \( W_{mn}(C(x)) \) [31], [32], [56] (look (53):

\[
(W_{kp} + W_{pk}) \frac{\partial W_{mn}}{\partial C_k} = (W_{km} + W_{mk}) \frac{dW_{pn}}{dx} \frac{\partial W_{nm}}{\partial C_k} - \frac{dW_{km}}{dx} \frac{\partial W_{np}}{\partial C_k}.
\]

The Jacobi identity satisfies for metric tensor \( W_{mn}(C(x)) \) (54). The PB (53) forms the algebra and can be rewritten as the PB of the hydrodynamic type.

\[
\{C_m(x), C_n(y)\} = \frac{mn(n - 1)}{m + n - 2} \frac{dC_{m+n-2}(x)}{dx} \delta(x - y) + mnC_{m+n-2}(x) \frac{\partial}{\partial x} \delta(x - y).
\]

The algebra of charges \( \int_0^{2\pi} C_n(x) dx \) is the abelian algebra. Let us choose the Casimir operators \( C_n \) as the Hamiltonians

\[
H_n = \frac{1}{n} \int_0^{2\pi} C_n(x) dx, \ n = 2, 3, \ldots.
\]

The equations of motion for the densities of Casimir operators are following

\[
\frac{\partial C_m(x)}{\partial t_n} = \frac{1}{n} \int_0^{2\pi} [-W_{mn}(C(y)) \frac{\partial}{\partial y} \delta(y - x) + W_{nm}(C(x)) \frac{\partial}{\partial x} \delta(x - y)] dy = \frac{m(n - 1)}{m + n - 2} \frac{\partial}{\partial x} C_{m+n-2}.
\]

Thus the invariant chiral currents with the \( SU(2), SO(3), SP(2) \) constant torsion and the invariant chiral currents with the \( SU(\infty), SO(\infty), SP(\infty) \) constant torsion form the same infinite hydrodynamic chain (52), (53), (54). These PBs (53), (54) are a particular case of the \( M \)-brackets (Dorfman [38] and Kupershmidt [39], [40] )for \( M = 2 \) and describe the hydrodynamic chains (see [36], [37] and references therein). We can construct the new nonlinear equations of motion for the initial chiral currents \( U^\mu \) using the flat \( PB_2 \) (24) and Hamiltonians \( H_n \) (58), where \( C_n(x) \) defined by the equation (32) for \( SU(\infty) \) group:

\[
\frac{\partial U^\mu(x)}{\partial t_n} = \frac{1}{n} \int_0^{2\pi} dy \{U^\mu(x), C_n(U(y))\}_2,
\]

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As an example we consider $n = 3$:

$$\frac{\partial U_\mu}{\partial t_3} = \frac{\partial}{\partial x} (d_{\mu \lambda \nu} U_\nu U_\lambda), \quad \mu = 1, 2, \ldots \infty.$$

It is easy to see that this dynamical system is bi-Hamiltonian:

$$\frac{\partial U_\mu}{\partial t_3} = \frac{1}{3} \int_0^{2\pi} dy \{ U_\mu(x), C_3(U(y)) \}_2 = \frac{1}{2} \int_0^{2\pi} dy \{ U_\mu(x), C_2(U(y)) \}_3.$$

Here $PB_3$ has form:

$$\{ U_\mu(x), U_\nu(y) \}_3 = 2 d_{\mu \lambda \nu} U^\lambda(x) \frac{\partial}{\partial x} \delta(x - y).$$

Let us remind that $d_{\mu \nu \lambda}$ are the symmetric structure constant of the $SU(\infty)$ algebra in a matrix representation. This PB satisfies to Jacobi identity for $(n \to \infty)$

$$d_{\sigma \mu \lambda} d_{\sigma \lambda \rho} + d_{\sigma \mu \lambda} d_{\sigma \nu \rho} + d_{\sigma \mu \rho} d_{\sigma \nu \lambda} = \frac{1}{n} (\delta_{\mu \nu} \delta_{\lambda \rho} + \delta_{\mu \lambda} \delta_{\nu \rho} + \delta_{\nu \rho} \delta_{\nu \lambda}).$$

By the similar manner we can obtain the equation of motion for the chiral currents of $SO(\infty), SP(\infty)$:

$$\frac{\partial U_\mu(x)}{\partial t_n} = \frac{\partial}{\partial x} [d_{\nu_1 \nu_2}^{k_1} d_{\nu_2 \nu_3}^{k_2} \ldots d_{\nu_{2n-1} \mu}^{k_{n-3}} U^{\nu_1} \ldots U^{\nu_{2n-1}}].$$

As an example we consider $n = 4$:

$$\frac{\partial U_\mu}{\partial t_4} = \frac{\partial}{\partial x} (v_{\mu \rho \lambda \nu} U^\nu U^\rho), \quad \mu = 1, 2, \ldots \infty.$$

As final remark of this Section, we stress that the Casimir $C_2$ depend on physical coordinates, having all the constraints conveniently solved.

VI. CONCLUDING REMARKS

We considered WZNW and string models as auxiliary problems to obtain integrable equations of hydrodynamic type. We show that WZNW model coincided to string mode for the transverse coordinates of string and that these models are integrable if the torsions are constant and they are the structure constants of $SU(2), SO(3), SP(2)$ algebras. Also new integrable hydrodynamic chains for $SU(\infty), SO(\infty), SP(\infty)$ are obtained.
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