Arithmetics of homogeneous spaces over $p$-adic function fields

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Abstract
Let $K$ be the function field of a smooth projective geometrically integral curve over a finite extension of $\mathbb{Q}_p$. Following the works of Harari, Scheiderer, Szamuely, Izquierdo, and Tian, we study the local–global and weak approximation problems for homogeneous spaces of $\text{SL}_{n,K}$ with geometric stabilizers extension of a group of multiplicative type by a unipotent group. The tools used are arithmetic (local and global) duality theorems in Galois cohomology, in combination with techniques similar to those used by Harari, Szamuely, Colliot-Thélène, Sansuc, and Skorobogatov. As a consequence, we show that any finite abelian group is a Galois group over $K$, rediscovering the positive answer to the abelian case of the inverse Galois problem over $\mathbb{Q}_p(t)$. In the case where the curve is defined over a higher dimensional local field instead of a finite extension of $\mathbb{Q}_p$, coarser results are also given.

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1 | INTRODUCTION

1.1 | Context of the problem

Let $k$ be a field and let $\Omega$ be a set of places (i.e., equivalence classes of non-trivial absolute values) of $k$. For each $v \in \Omega$, we denote by $k_v$ the corresponding completion. We say that a class of smooth $k$-varieties satisfy the local–global principle (with respect to $\Omega$) if for every variety $X$ in this class, $\prod_{v \in \Omega} X(k_v) \neq \emptyset$ implies $X(k) \neq \emptyset$.

Suppose that $X$ is a $k$-variety with $X(k) \neq \emptyset$. For each $v \in \Omega$, the set $X(k_v)$ is equipped with a local topology induced by the topology of $k_v$ (see, e.g., [19, Proposition 3.1]). We say that $X$ has weak approximation in $\Omega$ if the diagonal embedding $X(k) \hookrightarrow \prod_{v \in \Omega} X(k_v)$ has dense image (where the product space is equipped with the product of local topologies; hence, it is enough to work with finite subsets $S \subseteq \Omega$). For example, affine spaces have weak approximation (Artin–Whaples).

These arithmetic properties (local–global principle and weak approximation) are $k$-stable birational invariants of smooth complete $k$-varieties, thanks to Serre’s generalized version of the implicit function theorem [63, Part II, Chapter III, §10.2] combined with the theorem of Nishimura–Lang [55]. The classical case is when $k$ is a number field (or its geometric counterpart, a global function field). In this case, Manin [50] defined an obstruction to the existence of rational points using the Brauer–Grothendieck group $Br X := H_2^{\text{ét}}(X, \mathbb{G}_m)$ and the global reciprocity law (Albert–Brauer–Hasse–Noether). These were later used by Colliot-Thélène and Sansuc to define an obstruction to weak approximation, which is now known as the Brauer–Manin obstruction.

The present article is motivated by the recent interest in studying these two arithmetic problems over fields of cohomological dimension greater than 2. Of particular interest is the case of a function field $K$ of a (smooth, complete, geometrically integral) variety $V$ over a field $k$, and where the set of places is that of points $v \in V$ of codimension 1. In this article, we focus on the case where $k$ is a finite extension of $\mathbb{Q}_p$ (except in Section 4, where $k$ is a higher dimensional local field), and where $V$ is a curve (of course, the classical case of global function fields corresponds to the case where $k$ is finite). The corresponding function field $K$ is then a field of cohomological dimension 3.

The main tool for our approach will be Galois cohomology. Over a $p$-adic function field $K$ as above, the group $H^3_{\text{ét}}(X, \mathbb{Q}/\mathbb{Z}(2))$ (and its variants) would play the role of $Br X$ in the classical case. Using (local and global) duality theorems for tori, the works of Harari, Scheiderer, and Szamuely gave us a certain understanding of the local–global principle [34] and weak approximation [31] problems for tori. Tian [68, 69] extended their ideas to study weak approximation for connected reductive groups $G$ with the property that the universal cover $G^{sc}$ of its derived subgroup $G^{ss}$ has weak approximation and contains a quasi-split maximal torus. Tian’s work relies on the powerful machinery of Borovoi called abelianization of nonabelian Galois cohomology [3–5]. Izquierdo also provided some results over function fields of curves over higher dimensional local fields [39, Théorème 0.1].

It should be noted that if we consider the completions coming from all the rank 1 discrete valuations on $K$ (not just those coming from the closed points of the curve), then many other local–global principles have been established. For these results, the readers are invited to consult the recent survey of Wittenberg [72, §4].

In this article, the varieties under consideration will be homogeneous spaces of $SL_n$ (or more generally, a special, simply connected semisimple algebraic group that has weak approximation).
We shall see that in the heart of the proof of each of the main theorems, there lies the use of a Poitou–Tate style duality theorem and the Poitou–Tate sequence. Aside from them, the techniques used here are the reinterpretation of the Brauer–Manin pairing by Harari–Szamuely [33, 34] for the existence of rational points, and by Colliot-Thélène–Sansuc–Skorobogatov [16, 66] for the weak approximation property.

1.2 Statement of the main results

Here are the main results of the article.

**Theorem A** (Theorems 3.4 and 3.14). Let $K$ be the function field of a smooth projective geometrically integral curve $\Omega$ over a $p$-adic field, $X$ a homogeneous space of $\text{SL}_{n,K}$ whose geometric stabilizers are extensions of a group of multiplicative type by a unipotent group. Then, the unramified first obstruction to the local–global principle for $X$ (i.e., the obstruction relative to the subgroup of $H^3(K(X), \mathbb{Q}/\mathbb{Z}(2))$ consisting of elements unramified over $K$ and whose restriction to $H^3(K_v(X), \mathbb{Q}/\mathbb{Z}(2))$ comes from $H^3(K_v, \mathbb{Q}/\mathbb{Z}(2))$ for all closed points $v \in \Omega$) is the only one.

**Theorem B** (Theorems 3.13 and 3.15). Let $K$ be the function field of a smooth projective geometrically integral curve $\Omega$ over a $p$-adic field, and $X$ a homogeneous space of $\text{SL}_{n,K}$ whose geometric stabilizers are extensions of a group of multiplicative type by a unipotent group. Assume that $X(K) \neq \emptyset$. Then, the reciprocity obstruction to weak approximation for $X$, relative to the subgroup of $H^3(K(X), \mathbb{Q}/\mathbb{Z}(2))$ consisting of elements unramified over $K$ and whose restriction to $H^3(K_v(X), \mathbb{Q}/\mathbb{Z}(2))$ comes from $H^3(K_v, \mathbb{Q}/\mathbb{Z}(2))$ for all but finitely many closed points $v \in \Omega$, is the only one.

Using Theorem B, we also give a positive answer to the abelian case of the inverse Galois problem over $p$-adic function fields (see Corollary 3.18).

Actually, for each of the two above theorems, we present two proofs. The first ones (Theorems 3.4 and 3.13) rely on the observation that the two arithmetic problems (the local–global principle and weak approximation) for homogeneous spaces of $\text{SL}_{n}$ is closely related to the question of universal torsors over their smooth compactifications. This is part of a “descent theory” for torsors under tori, which (over number fields) originated from the formidable work of Colliot–Thélène and Sansuc [16, §3], and to which Skorobogatov then made some complements [67, §6]. Hence, in the course of proving Theorems A and B, we shall develop this descent theory for varieties over $p$-adic function fields. In particular, the following analog of [16, Théorème 3.8.1, Proposition 3.8.7] (see also [67, Corollary 6.1.3]) is proposed.

**Theorem C** (Theorem 2.1). Let $K$ be the function field of a smooth projective geometrically integral curve $\Omega$ over a $p$-adic field. Let $X$ be a smooth proper geometrically integral variety over $K$ such that $\text{Pic } X_K$ is a finitely generated free abelian group. If the universal torsors $Y \to X$ (see their definition at the beginning of Section 2) satisfy the local–global principle (resp. the local–global principle and weak approximation), then the reciprocity obstruction (i.e., the obstruction relative to the subgroup of $H^3(K(X), \mathbb{Q}/\mathbb{Z}(2))$ consisting of elements unramified over $K$) to the local–global principle (resp. the local–global and weak approximation) on $X$ is the only one.
The second proofs of Theorems A and B (Theorems 3.14 and 3.15, respectively) are inspired by the following remark, communicated to the author by Jean-Louis Colliot-Thélène: Every homogeneous space of \( \text{SL}_{n,K} \), whose geometric stabilizers are extensions of a group of multiplicative type by a unipotent group, is \( K \)-stably birational to a \( K \)-torsor under a torus. This allows us to deduce Theorems 3.14 and 3.15 from [34, Theorem 0.2] and [31, Theorem 1.2], respectively.

For function fields of curves over higher dimensional local fields, we propose weaker versions of Theorems A and B. If \( k \) is a \( d \)-dimensional local field \( k \) (see Section 1.4 below), consider the following condition.

\[
k = \mathbb{C}(t); \text{ or } d \geq 1 \text{ and the 1-local field associated with } k \text{ has characteristic } 0. \quad (\star)
\]

In particular, \( k \) has characteristic 0. Note that the case where \( k \) is \( p \)-adic corresponds to \( d = 1 \).

**Theorem D** (Theorem 4.2). Let \( K \) be the function field of a smooth projective geometrically integral curve \( \Omega \) over a \( d \)-dimensional local field \( k \) satisfying (\( \star \)) and \( X \) a homogeneous space of \( \text{SL}_{n,K} \) with finite abelian geometric stabilizers. Then the adelic first obstruction to the local–global principle for \( X \) (i.e., the obstruction relative to the subgroup of \( H^{d+2}(X, \mathbb{Q}/\mathbb{Z}(d+1)) \) consisting of elements whose restriction to \( H^{d+2}(X_{K_v}, \mathbb{Q}/\mathbb{Z}(d+1)) \) comes from \( H^{d+2}(K_v, \mathbb{Q}/\mathbb{Z}(d+1)) \) for all closed points \( v \in \Omega \)) is the only one.

**Theorem E** (Theorem 4.3). Let \( K \) be the function field of a smooth projective geometrically integral curve \( \Omega \) over a \( d \)-dimensional local field \( k \) satisfying (\( \star \)) and \( X \) a homogeneous space of \( \text{SL}_{n,K} \) with finite abelian geometric stabilizers. Assume that \( X(K) \neq \emptyset \). Then, the generalized Brauer–Manin obstruction to weak approximation for \( X \) (i.e., the obstruction relative to the subgroup of \( H^{d+2}(X, \mathbb{Q}/\mathbb{Z}(d+1)) \) consisting of elements whose restriction to \( H^{d+2}(X_{K_v}, \mathbb{Q}/\mathbb{Z}(d+1)) \) comes from \( H^{d+2}(K_v, \mathbb{Q}/\mathbb{Z}(d+1)) \) for all but finitely many closed points \( v \in \Omega \)) is the only one.

Theorems D and E do not hold for stabilizers of multiplicative type; counterexamples in the case where \( d = 0 \) and where the stabilizers are tori are given in Example 4.6.

The condition (\( \star \)) is crucial to establish duality theorems for finite modules. It can be slightly weakened by allowing \( k \) (if \( d = 0 \)) or the one-dimensional local associated with \( k \) (if \( d \geq 1 \)) to have characteristic \( p > 0 \), provided that \( p \) does not divide the order of the stabilizers.

The article is organized as follows. In Sections 1.3, 1.4, and 1.5, we recall the notations, conventions, and known results that will be used in the proofs of our main theorems. In Section 2, we develop a version of Colliot-Thélène–Sansuc descent theory for varieties over \( p \)-adic function fields and prove Theorem C. Theorems A and B will be proved in Section 3, as consequences of the results already established in Section 2. Here we also prove that any finite abelian group is a Galois group over any \( p \)-adic function field. Finally, we prove Theorems D and E (which are results over function fields over higher dimensional local fields) in Section 4.

### 1.3 Notations and conventions

The following conventions shall be deployed throughout the article.

**Cohomology.** Unless stated otherwise, all (hyper-)cohomology groups will be \( \text{étale} \) or \( \text{Galois} \). We abusively identify each object of an abelian category to the corresponding 1-term complex

\[\text{1-term complex}
\]

\[\text{If the author is grateful to Colliot-Thélène for allowing him to include these results in the present article.}\]
concentrated in degree 0. Over a field $K$, a $K$-variety is a separated scheme of finite type $X \to \text{Spec } K$. We use $\overline{K}$ to denote a fixed separable closure of $K$, $\Gamma_K = \text{Gal}(\overline{K}/K)$ to denote its absolute Galois group, and $\overline{X} = X \times_K \overline{K}$. We denote by $D^+(X)$ (resp. $D^+(K)$, resp. $D^+(\text{Ab})$) the bounded-below derived category of étale sheaves over $X$ (resp. of discrete $\Gamma_K$-modules, resp. of abelian groups). If $\nu$ is a place of $K$, $K_{\nu}$ denotes the corresponding completion, $X_{\nu} = X \times_K K_{\nu}$, and $\text{loc}_\nu : H^i(X, -) \to H^i(X_{\nu}, -)$ denotes the localization map in cohomology. Whenever $K$ is the function field of a smooth projective geometrically integral curve $\Omega$ over a field $k$, $\Omega(1)$ denotes the set of closed points of $\Omega$. For each $\nu \in \Omega(1)$, $K_{\nu}$ (resp. $\mathcal{O}_{\Omega,\nu}$) denotes the $\nu$-adic completion of $K$ (resp. of the local ring $\mathcal{O}_{\Omega,\nu}$), whereas $K^h_{\nu}$ and $\mathcal{O}^h_{\nu}$ denote the corresponding henselizations, and $k(\nu)$ denotes the residue field of $\nu$.

**Motivic complexes.** Let $X$ be a smooth variety over a field $K$. Lichtenbaum defined arithmetic complexes $\mathbb{Z}(i)$ over $X_{\text{ét}}$ for $i = 0, 1, 2$ [46, 47] (we shall write $\mathbb{Z}_X(i)$ if we want to emphasize the variety $X$). We have quasi-isomorphisms $\mathbb{Z}(0) \cong \mathbb{Z}$ and $\mathbb{Z}(1) \cong \mathbb{G}_m^{-1}$. The complex $\mathbb{Z}(2)$ is concentrated in degrees 1 and 2. Also, there is a pairing

$$\mathbb{Z}(1) \otimes \mathbb{Z}(1) \to \mathbb{Z}(2). \quad (1.3.1)$$

If $F$ is a sheaf on $X_{\text{ét}}$, let $F(i) = F \otimes_{\mathbb{Z}} \mathbb{Z}(i)$. For $n$ invertible in $K$, there are quasi-isomorphisms $\mathbb{Z}/n(i) \cong \mu_{n^i}$, $i = 1, 2$. It follows that $\mathbb{Q}/\mathbb{Z}(i) \cong \lim_{\longrightarrow} \mu_{n^i}$ if $K$ has characteristic 0 and $i \in \{1, 2\}$. We also use this as the definition of the sheaf $\mathcal{Q}/\mathbb{Z}(i)$ for $i \notin \{1, 2\}$.

**Abelian groups.** For a topological abelian group $A$ (the topology is understood to be discrete if not specified), $A^D = \text{Hom}^c_{\text{cts}}(A, \mathbb{Q}/\mathbb{Z})$ denotes its Pontrjagin dual. For $n \geq 1$, we denote by $A_n$ the $n$-torsion subgroup of $A$, and $A_{\text{tors}} = \varprojlim_n nA$.

**Tori.** If $G$ is a smooth algebraic group over a field $K$, we denote by $\hat{G} = \mathcal{H}om_K(G, \mathbb{G}_m)$ its $\Gamma_K$-module of geometric characters, that is, $\hat{G} = \text{Hom}_K(G, \mathbb{G}_m)$ equipped with the Galois action defined by the formula

$$\forall \sigma \in \Gamma_K, \forall \chi \in \text{Hom}_K(\overline{G}, \mathbb{G}_m), \forall g \in G(\overline{K}), \quad (\sigma \chi)(g) := \sigma(\chi(\sigma^{-1} g)).$$

If $T$ is a torus over a field $K$, its dual torus is defined to be the torus $T'$ whose character module is the cocharacter module $\tilde{T} = \mathcal{H}om_K(G_m, T) = \text{Hom}(\tilde{T}, \mathbb{Z})$ of $T$. Since $T = \tilde{T} \otimes_\mathbb{Z} \mathbb{G}_m = \tilde{T} \otimes_{\mathbb{G}_m} \mathbb{G}_m$ and $T' = \tilde{T} \otimes_\mathbb{G}_m = \tilde{T} \otimes \mathbb{G}_m$, we obtain a pairing

$$T \otimes T' \to \mathbb{G}_m \otimes \mathbb{G}_m \cong \mathbb{Z}(1)[1] \otimes \mathbb{Z}(1)[1].$$

Then, (1.3.1) induces a pairing

$$T \otimes T' \to \mathbb{Z}(2)[2] \quad (1.3.2)$$

in $D^+(K)$. We say that $T$ is quasi-split if it is isomorphic to $\text{Res}_{A/K} \mathbb{G}_{m,A}$ for some étale $K$-algebra $A$, where $\text{Res}_{A/K}$ denotes the restriction of scalars à la Weil. This is equivalent to saying that $\hat{T}$ is a permutation module (i.e., it has a $\Gamma_K$-invariant $\mathbb{Z}$-basis). In this case, $H^1(K, T) = 0$ by Shapiro's lemma and Hilbert's Theorem 90. Also, $T'$ is quasi-split.

† The category of $\Gamma_K$-modules is equivalent to that of sheaves over $(\text{Spec } K)_{\text{ét}}$, and we abusively identify these two.
Tate–Shafarevich groups. Let $K$ be the function field of a smooth projective geometrically integral curve $\Omega$ over a field $k$. Let $S \subseteq \Omega^{(1)}$ be a finite set of closed points, and $C$ a complex of $\Gamma_K$-modules. For $i \in \mathbb{Z}$, we define the groups

$$\Sha_i(K, C) = \ker \left( H^i(K, C) \to \prod_{v \notin S} H^i(K_v, C) \right),$$

$$\Sha_i(K, C) = \Sha_i(\emptyset, C),$$

$$\Sha_i(\emptyset, C) = \lim_{\text{horizontal}}^{\text{vertical}} \Sha_i(S, C).$$

Unramified cohomology. Let $X$ a smooth integral variety over a field $K$, $n \geq 1$ an integer invertible in $K$, $i \geq 0$, and $j \in \mathbb{Z}$. Let $\mu_n^{(-j)} = \mathbb{Q}(n)^{(-j)}$, $\mu_n^{(0)} = \mathbb{Q}/\mathbb{Z}$. One defines the unramified part $H^i_{nr}(K(X)/K, \mu_n^{(-j)})$ to be the subgroup of $H^i(K(X), \mu_n^{(-j)})$ consisting of elements $A$ that lift to $H^i(\mathcal{O}, \mu_n^{(-j)})$ for every discrete valuation ring $\mathcal{O} \supseteq K$ with field of fractions $K(X)$ (such a lifting is necessarily unique by the injectivity property, see [12, Theorem 3.8.1]). If $X$ is proper, this amounts to requiring that $A$ comes from $H^i(\mathcal{O}_{X,v}, \mu_n^{(-j)})$ for every point $v \in X$ of codimension $1$ (see Theorem 4.1.1 in loc. cit.). The group $H^i_{nr}(K(X)/K, \mu_n^{(-j)})$ is a $K$-stable birational invariant [12, Proposition 4.1.4]). We define the “evaluation” pairing

$$H^i_{nr}(K(X)/K, \mu_n^{(-j)}) \times X(L) \to H^i(L, \mu_n^{(-j)}),$$

for any overfield $L/K$, as follows. By Bloch–Ogus theorem (Gersten's conjecture for étale cohomology) [2] (see also [12, Theorem 4.1.1]), every class $A \in H^i_{nr}(K(X)/K, \mu_n^{(-j)})$ comes from $H^i(\mathcal{O}(P), \mu_n^{(-j)})$ for any point $P \in X$. Let $Q : Spec L \to X$ be an $L$-point with image $P \in X$. Lift $A$ to a unique element of $H^i(\mathcal{O}(P), \mu_n^{(-j)})$, and define $A(Q)$ to be its image by the pullback $H^i(\mathcal{O}(P), \mu_n^{(-j)}) \to H^i(L, \mu_n^{(-j)})$.

Assume in addition that $K$ has characteristic $0$ and that $X$ is proper. For $i \geq 3$, there is a natural map (for the case where $i = 3$, see [41, Proposition 2.9] and [31, (19)])

$$H^{i+1}(X, \mathbb{Z}(2)) \to H^i_{nr}(K(X)/K, \mathbb{Q}/\mathbb{Z}(2)),$$

defined as follows. First, let $Z(2)$ be the complex concentrated in degrees $\leq 2$ defined in a similar way to $\mathbb{Z}(2)$, but over the small Zariski site $X_{zar}$. Let $\alpha : X_{et} \to X_{zar}$ denote the change-of-sites map. Then, the adjunction $Q(2)_{zar} \to \mathbb{R}\alpha_\ast Q(2)$ is an isomorphism [41, Théorème 2.6(c)]. Since $Q(2)_{zar}$ is concentrated in degrees $\leq 2$, one has $\mathbb{R}^i\alpha_\ast Q(2) = 0$, hence $\mathbb{R}^{i+1}\alpha_\ast \mathbb{Q}/\mathbb{Z}(2) \cong \mathbb{R}^i\alpha_\ast \mathbb{Q}/\mathbb{Z}(2)$. The Leray spectral sequence for $\alpha$ yields an edge map

$$H^{i+1}(X, \mathbb{Z}(2)) \to H^i_{zar}(X, \mathbb{R}^{i+1}\alpha_\ast \mathbb{Q}/\mathbb{Z}(2)) \cong H^i_{zar}(X, \mathbb{R}^i\alpha_\ast Q/\mathbb{Z}(2)).$$

Since $X$ is proper, we have $H^i_{zar}(X, \mathbb{R}^i\alpha_\ast Q/\mathbb{Z}(2)) \cong H^i_{nr}(K(X)/K, \mathbb{Q}/\mathbb{Z}(2))$ by the Gersten resolution [12, Theorem 4.1.1]. The map (1.3.4) is defined. For $X = Spec K$, this yields an isomorphism $H^{i+1}(K, \mathbb{Z}(2)) \cong H^i(K, \mathbb{Q}/\mathbb{Z}(2))$. According to its construction, the map (1.3.4) is functorial in $X$ and in $K$. Applying this functoriality to the natural morphism $Spec(K(X)) \to X$ yields a commutative diagram

$$\begin{array}{ccc}
H^{i+1}(X, \mathbb{Z}(2)) & \to & \mathbb{H}^i_{nr}(K(X)/K, \mathbb{Q}/\mathbb{Z}(2)) \\
\downarrow & & \downarrow \\
H^{i+1}(K(X), \mathbb{Z}(2)) & \cong & H^i(K(X), \mathbb{Q}/\mathbb{Z}(2)).
\end{array}$$
Algebraic groups. Let $K$ be a field, $X$ a smooth $K$-variety, and $G$ a smooth $K$-group scheme. The (nonabelian) étale cohomology (pointed) set $H^1(X, G)$ classifies $X$-torsors under $G$. Let $f : Y \to X$ be such a torsor, with class $[Y] \in H^1(X, G)$. For each Galois cocycle $c : \Gamma_K \to G(\overline{K})$, we may twist $G$ to obtain an inner $K$-form $cG$ of $G$, and a torsor $c_\cdot f : c_\cdot Y \to X$ under $c_\cdot G$. For each $P \in X(K)$, we have the following equivalence:

$$[Y](P) = [c] \in H^1(K, G) \iff [c_\cdot Y](P) = 1 \in H^1(K, cG) \iff P \in c_\cdot f(\cdot Y(K)) \quad (1.3.6)$$

(see [67, p. 22]). If $G$ is commutative, then $c_\cdot G = G$ and $[c_\cdot Y] = [Y] - \pi^*[c] \in H^1(X, G)$, where $\pi : X \to \text{Spec } K$ denotes the structure morphism. The following continuity result is perhaps well known. We present a proof here for lack of a reference.

**Lemma 1.1.** Let $K$ be a topological field (i.e., its addition, multiplication and inversion are continuous), for example, a field equipped with an absolute value. Let $X$ be a smooth integral variety over $K$ and consider the natural topology on $X(K)$ [19, Proposition 3.1].

(i) Let $G$ be a smooth, quasi-projective, commutative group scheme over $K$. Then, for any $i \geq 0$ and $A \in H^i(X, G)$, the induced evaluation $X(K) \to H^i(G, K)$ is locally constant.

(ii) For any $n \geq 1$, $i \geq 0$, $j \in \mathbb{Z}$, and $A \in H^i_{nr}(K(X)/K, \mu_n^\otimes j)$, the induced evaluation $X(K) \to H^i(K, \mu_n^\otimes j)$ is locally constant.

(iii) Suppose that $K$ is complete with respect to an absolute value. Let $G$ be a (not necessarily commutative) smooth $K$-group scheme. Then, for any torsor $Y \to X$ under $G$ with class $[Y] \in H^1(X, G)$, the induced evaluation $X(K) \to H^1(K, G)$ is locally constant.

Proof. We prove (i), the proof of (ii) being similar. Denote by $\pi : X \to \text{Spec } K$ the structure morphism. Fix a $K$-point $P_0 \in X(K)$ (if there is any). Replacing $A$ by $A - \pi^* A(P_0)$, we may assume that $A(P_0) = 0$. Since $H^i(K, G) \cong H^i(\mathcal{O}_{X,P_0}^h, G)$ [51, Chapter III, Remark 3.11], there is an étale neighborhood $f : X' \to X$ of $P_0$ and a point $P'_0 \in X'(K)$ such that $f(P'_0) = P_0$ and $f^* A = 0$. Now, $f$ induces a homeomorphism between a neighborhood (for the $K$-topology) $U' \subseteq X'(K)$ of $P'_0$ and a neighborhood $U \subseteq X(K)$ of $P_0$. Each point $P \in U'$ then has a factorization $\text{Spec } K \to X' \xrightarrow{f} X$, thus the evaluation-at-$P$ factorizes through the pullback map $f^* : H^i(X, G) \to H^i(X', G)$. Since $f^* A = 0$, it follows that $A(P) = 0$.

Let us now show (iii). Fix a $K$-point $P_0 \in X(K)$ and let $c : \Gamma_K \to G(\overline{K})$ be a Galois cocycle representing $[Y](P_0) \in H^1(K, G)$. Twisting by $c$ yields a torsor $c_\cdot Y$ under the inner form $c_\cdot G$, which contains a $K$-point lying over $P_0$ (in view of (1.3.6)). Since $K$ is complete with respect to an absolute value, we may apply Serre’s generalized version of the implicit function theorem [63, Part II, Chapter III, §10.2]. This assures the existence of a neighborhood $U \subseteq X(K)$ of $P_0$, whose $K$-points $P$ can be lifted to $K$-points of $c_\cdot Y$. In other words, $[Y](P) = [c] = [Y](P_0)$ (again, by (1.3.6)) for all $P \in U$.

Following [18, §4.2], we shall say that $G$ is special if any torsor under $G$ over a $K$-variety is locally trivial for the Zariski topology. This implies $H^1(L, G) = 1$ for any overfield $L/K$. It follows that if $H \subseteq G$ is a Zariski closed subgroup and $X = H \setminus X$ (hence the projection $G \to X$ is a torsor under $H$), then the evaluation map

$$X(L) \to H^1(L, H), \quad P \mapsto [G](P)$$
induces a bijection $X(L)/G(L) \cong H^1(L, H)$. Serre proved that any special group is linear and connected [62, §4.1, Théorème 1]. Examples of special groups are the general linear group $GL_n$ (or more generally, $GL_A$ for a central simple algebra $A$ over $K$), the special linear group $SL_n$, or the symplectic group $Sp_{2n}$.

1.4 Reciprocity obstruction

The generalized Weil reciprocity law is the key to defining the reciprocity obstructions to the local–global principle and weak approximation for varieties over fields with arithmetico-geometric nature. In this paper, we work with function fields of curves over $p$-adic fields, and more generally, higher dimensional local fields. We shall recall their definition and properties here. By definition, a \textit{zero-dimensional local field} is either a finite field or the field $\mathbb{C}((t))$ for some algebraically closed field $\mathbb{C}$ of characteristic 0. These fields have absolute Galois group $\hat{\mathbb{Z}}$ (for $\mathbb{C}((t))$, this is the celebrated Puiseux’s theorem), and hence have cohomological dimension 1. For $d \geq 1$, a \textit{d-dimensional local field} is a complete discretely valued field whose residue field is a $(d - 1)$-dimensional local field.

Proposition 1.2. Let $k$ be a $d$-local field satisfying the condition $(\ast)$ from page 3.

(i) The field $k$ has cohomological dimension $\text{cd}(k) = d + 1$.

(ii) We have a canonical trace isomorphism $H^{d+1}(k, \mathbb{Q} / \mathbb{Z}(d)) \cong \mathbb{Q} / \mathbb{Z}$. More precisely, if $k_{d-i}$ denotes the residue field of $k_i$ for $i \in \{1, \ldots, d\}$ and $k_d = k$, then this isomorphism is given by the composite of the “residue” maps

$$H^{d+1}(k, \mu_{n}^{\otimes d}) \cong H^{d}(k_{d-1}, \mu_{n}^{\otimes (d-1)}) \cong \cdots \cong H^{1}(k_0, \mathbb{Z}/n) \cong \mathbb{Z}/n.$$  \hfill (1.4.1)

(iii) For any finite extension $\ell / k$, the trace isomorphisms fit into a commutative diagram

$$\begin{array}{ccc}
H^{d+1}(k, \mathbb{Q} / \mathbb{Z}(d)) & \xrightarrow{\cong} & \mathbb{Q} / \mathbb{Z} \\
\downarrow \text{res}_{\ell/k} & & \downarrow [\ell : k], \\
H^{d+1}(\ell, \mathbb{Q} / \mathbb{Z}(d)) & \xrightarrow{\cong} & \mathbb{Q} / \mathbb{Z},
\end{array} \hfill (1.4.2)
$$

where $\text{res}_{\ell/k} : H^{d+1}(k, \mathbb{Q} / \mathbb{Z}(d)) \rightarrow H^{d+1}(\ell, \mathbb{Q} / \mathbb{Z}(d))$ denotes the restriction map. Consequently, we have a commutative diagram

$$\begin{array}{ccc}
H^{d+1}(\ell, \mathbb{Q} / \mathbb{Z}(d)) & \xrightarrow{\cong} & \mathbb{Q} / \mathbb{Z} \\
\downarrow \text{cor}_{\ell/k} & & \\
H^{d+1}(k, \mathbb{Q} / \mathbb{Z}(d)) & \xrightarrow{\cong} & \mathbb{Q} / \mathbb{Z},
\end{array} \hfill (1.4.3)
$$

where $\text{cor}_{\ell/k} : H^{d+1}(\ell, \mathbb{Q} / \mathbb{Z}(d)) \rightarrow H^{d+1}(k, \mathbb{Q} / \mathbb{Z}(d))$ denotes the corestriction map.

Proof. Note that the commutativity of (1.4.3) follows from that of (1.4.2), since the composite $\text{cor}_{\ell/k} \circ \text{res}_{\ell/k} : H^{d+1}(k, \mathbb{Q} / \mathbb{Z}(d)) \rightarrow H^{d+1}(k, \mathbb{Q} / \mathbb{Z}(d))$ is the multiplication by $[\ell : k]$, and since
\( \mathbb{Q}/\mathbb{Z} \) is divisible. We prove (i)–(iii) by induction on \( d \). For \( d = 0 \), one has \( \Gamma_k \cong \hat{\mathbb{Z}} \), so (i) and (ii) are obvious. Furthermore, for any finite extension \( \ell/k \), the inclusion \( \Gamma_\ell \subset \Gamma_k \) is the multiplication by \([ \ell : k ]\) on \( \hat{\mathbb{Z}} \), which implies (iii).

Suppose that \( d \geq 1 \), and let \( \kappa = k_{d-1} \) denote the residue field of \( k \) (which is a \((d-1)\)-local field). We recall from [64, Chapitre II, Annexe, (2.2)] that for any \( n \geq 1 \), \( i \geq 0 \) and \( j \in \mathbb{Z} \), the Hochschild–Serre spectral sequence yields an exact sequence

\[
0 \to H^{i+1}(\kappa, \mu_n^{\otimes j}) \to H^{i+1}(k, \mu_n^{\otimes j}) \overset{\delta}{\to} H^i(\kappa, \mu_n^{\otimes(j-1)}) \to 0, \tag{1.4.4}
\]

where \( \delta \) is the residue map. Since \( \text{cd}(\kappa) = d \), setting \( i = j = d \) in (1.4.4) yields

\[
H^{d+1}(k, \mu_n^{\otimes d}) \overset{\delta}{\cong} H^d(\kappa, \mu_n^{\otimes(d-1)}) \cong \mathbb{Z}/n.
\]

Furthermore, one has \( \text{cd}(k) = d + 1 \) by [64, Chapitre II, §4.3, Proposition 12]. It remains to establish diagram (1.4.2). For this, we make use of the following standard property of \( \delta \). If \( \ell \) is a finite extension of \( k \) with residue field \( \lambda \), then we have a commutative diagram

\[
\begin{array}{ccc}
H^{i+1}(\ell, \mu_n^{\otimes j}) & \overset{\delta}{\longrightarrow} & H^i(\ell, \mu_n^{\otimes(j-1)}) \\
\downarrow \text{res}_{\ell/k} & & \downarrow \text{res}_{\ell/k} \\
H^{i+1}(\ell', \mu_n^{\otimes j}) & \overset{\delta}{\longrightarrow} & H^i(\lambda, \mu_n^{\otimes(j-1)}),
\end{array}
\]

where \( e \) is the ramification index (this follows easily from the description of the residue map in [42, §1, (1.3)(i)]). Since \([ \ell : k ] = e \cdot [ \lambda : \kappa ]\), we deduce the commutativity of (1.4.2) from that of the same diagram for the extension \( \lambda/\kappa \).

From now on, let \( k \) be a \( d \)-dimensional local field satisfying the condition (*) from page 3 and let \( K \) be the function field of a smooth projective geometrically integral curve \( \Omega \) over \( k \). For each closed point \( v \in \Omega^{(1)} \), the \( v \)-adic completion \( K_v \) is a \((d+1)\)-dimensional local field, a field of cohomological dimension \( \text{cd}(K_v) = d + 2 \). The field \( K \) itself has cohomological dimension \( \text{cd}(K) \leq d + 2 \). Indeed, by [30, Proposition 5.10], it is enough to show that \( \text{cd}(k(t)) \leq d + 2 \). Let \( A \) be any torsion \( \Gamma_{k(t)} \)-module. Since \( \Gamma_k \cong \text{Gal}(\bar{k}(t)/k(t)) \), we have a spectral sequence \( H^p(k, H^q(\bar{k}(t), A)) \Rightarrow H^{p+q}(k(t), A) \). But \( \text{cd}(\bar{k}(t)) \leq 1 \) by Tsen’s theorem [27, Proposition 6.2.3, Theorem 6.2.8] and \( \text{cd}(k) = d + 1 \), hence \( H^i(k(t), A) = 0 \) for \( i > d + 2 \), or \( \text{cd}(k(t)) \leq d + 2 \).

**Proposition 1.3** (Generalized Weil reciprocity law). We have a complex

\[
H^{d+2}(K, \mathbb{Q}/\mathbb{Z}(d + 1)) \overset{(\text{loc}_v)_{v \in \Omega^{(1)}}}{\longrightarrow} \bigoplus_{v \in \Omega^{(1)}} H^{d+2}(K_v, \mathbb{Q}/\mathbb{Z}(d + 1)) \overset{\sigma}{\longrightarrow} \mathbb{Q}/\mathbb{Z}, \tag{1.4.5}
\]

where \( \sigma \) is the sum of the isomorphisms \( H^{d+2}(K_v, \mathbb{Q}/\mathbb{Z}(d + 1)) \cong \mathbb{Q}/\mathbb{Z} \) from Proposition 1.2(ii).
Proof. We prove this by applying the reciprocity law for cycle modules. The family of Galois cohomology groups $\{H^i(-, \mu_n^{\otimes j})\}_{i,j}$ forms a cycle module, that is, a cycle premodule in the sense of [60, Definition 1.1] satisfying the axioms (FD) and (C) of Definition 2.1 in loc. cit. For the proofs, we refer to [60, Remarks 1.1 and 2.5] and [42, Proposition 1.7]. For each $n \geq 1$, applying the property (RC) of [60, Proposition 2.2], we obtain a complex

$$H^{d+2}(K, \mu_n^{\otimes (d+1)} \to \bigoplus_{v \in \Omega(1)} H^{d+1}(k(v), \mu_n^{\otimes d}) \to H^{d+1}(k, \mu_n^{\otimes d})$$

where $\delta_v$ is the composite of the residue map $H^{d+2}(K_v, \mu_n^{\otimes (d+1)}) \to H^{d+1}(k(v), \mu_n^{\otimes d}) \cong \mathbb{Z}/n$ from (1.4.1) and loc. cit. Taking (1.4.3) into account, one obtains (1.4.5). □

Let $X$ be a smooth geometrically integral $K$-variety such that $\prod_{v \in \Omega(1)} X(K_v) \neq \emptyset$. The evaluation pairings defined in (1.3.3) for the extensions $K_v/K$ reassemble into a pairing

$$H^{d+2}_{nr}(K(X)/K, \mathbb{Q}/\mathbb{Z}(d+1)) \times \prod_{v \in \Omega(1)} X(K_v) \to \mathbb{Q}/\mathbb{Z}, \ (A, (P_v)_{v \in \Omega(1)}) \mapsto \sum_{v \in \Omega(1)} A(P_v)$$

(the above sum is finite by [15, Proposition 2.5(i)], see also [31, Lemma 5.1] for the case where $k$ is $p$-adic). Thanks to (1.4.5), the above pairing vanishes on the image of $H^{d+2}(K, \mathbb{Q}/\mathbb{Z}(d+1))$ in $H^{d+2}_{nr}(K(X)/K, \mathbb{Q}/\mathbb{Z}(d+1))$. Hence, it induces a pairing

$$H^{d+2}_{nr}(K(X)/K, \mathbb{Q}/\mathbb{Z}(d+1)) \to \prod_{v \in \Omega(1)} X(K_v) \to \mathbb{Q}/\mathbb{Z}, \ (A, (P_v)_{v \in \Omega(1)}) \mapsto \sum_{v \in \Omega(1)} A(P_v)$$

which vanishes on the diagonal image of $X(K)$ on $\prod_{v \in \Omega(1)} X(K_v)$ (by (1.4.5) again). The absence of a family $(P_v)_{v \in \Omega(1)} \in \prod_{v \in \Omega(1)} X(K_v)$ orthogonal to $H^{d+2}_{nr}(K(X)/K, \mathbb{Q}/\mathbb{Z}(d+1))$ is an obstruction to the existence of $K$-rational points on $X$. We refer to this as the reciprocity obstruction to the local–global principle for $X$. We are also interested in the following coarser obstruction. Restricting (1.4.6) to locally constant classes yields a map

$$\rho_X : Ker\left(\frac{H^{d+2}_{nr}(K(X)/K, \mathbb{Q}/\mathbb{Z}(d+1))}{\text{Im } H^{d+2}(K, \mathbb{Q}/\mathbb{Z}(d+1))} \to \prod_{v \in \Omega(1)} X(K_v) \to \mathbb{Q}/\mathbb{Z}\right).$$

If $X(K) \neq \emptyset$, then $\rho_X = 0$. We refer to the nonvanishing of $\rho_X$ as the (unramified) first obstruction to the local–global principle for $X$.

Suppose that $X(K) \neq \emptyset$. By Lemma 1.1(ii), the pairing (1.4.6) vanishes on the closure (for the product of $v$-adic topologies) of $X(K)$ in $\prod_{v \in \Omega(1)} X(K_v)$. The nonorthogonality to $H^{d+2}_{nr}(K(X)/K, \mathbb{Q}/\mathbb{Z}(d+1))$ of a family $(P_v)_{v \in \Omega(1)} \in \prod_{v \in \Omega(1)} X(K_v)$ is then an obstruction to approximating this family by $K$-rational points. We refer to this as the reciprocity obstruction to weak approximation for $X$. We are also interested in the following coarser obstruction to weak approximation. For any finite set $S \subseteq \Omega(1)$, (1.4.6) restricts to a pairing

$$(-, -)_S : Ker\left(\frac{H^{d+2}_{nr}(K(X)/K, \mathbb{Q}/\mathbb{Z}(d+1))}{\text{Im } H^{d+2}(K, \mathbb{Q}/\mathbb{Z}(d+1))} \to \prod_{v \in S} H^{d+2}_{nr}(K_v(X)/K_v, \mathbb{Q}/\mathbb{Z}(d+1)) \to \text{Im } H^{d+2}(K_v, \mathbb{Q}/\mathbb{Z}(d+1)) \times \prod_{v \in S} X(K_v) \to \mathbb{Q}/\mathbb{Z},$$

(1.4.8)
which vanishes on the closure of \( X(K) \). The nonorthogonality of a family \((P_v)_{v \in S}\) to the subgroup of “constant-outside-\(S\)” elements is then an obstruction to approximating this family by \( K \)-rational points.

We also note that in the case where \( d = 1 \) (e.g., when \( k \) is \( p \)-adic), by the Gersten resolution (1.3.4), the pairing (1.4.6) induces a pairing

\[
\frac{H^4(X, \mathbb{Z}(2))}{\text{Im } H^4(K, \mathbb{Z}(2))} \times \prod_{v \in \Omega^{(1)}} X(K_v) \to \mathbb{Q}/\mathbb{Z}.
\]

(1.4.9)

Similarly, the map (1.4.7) induces a map

\[
\rho_X : \text{Ker} \left( \frac{H^4(X, \mathbb{Z}(2))}{\text{Im } H^4(K, \mathbb{Z}(2))} \to \prod_{v \in \Omega^{(1)}} \frac{H^4(X_v, \mathbb{Z}(2))}{\text{Im } H^4(K_v, \mathbb{Z}(2))} \right) \to \mathbb{Q}/\mathbb{Z}.
\]

(1.4.10)

Actually, the present article only deals with the (unramified) reciprocity obstruction in the case where \( k \) is \( p \)-adic. For function fields of curves over higher dimensional fields, we do not prove that the obstruction is unramified. Instead, we shall work with the following adapted version of the pairing (1.4.6). Again, let \( K \) be the function field of a smooth projective geometrically integral curve \( \Omega \) over a \( d \)-dimensional local field \( k \) satisfying the condition (\( \ast \)) from page 3. Let \( X \) be a smooth geometrically integral \( K \)-variety. For a nonempty open subset \( U \subseteq \Omega \), we shall call a smooth, separated, finitely presented scheme \( \mathfrak{X} \to U \) such that \( \mathfrak{X} \times_U K = X \) an integral model of \( X \) over \( U \). Such an integral model exists when \( U \) is sufficiently small. For \( v \in U^{(1)} \), we have \( \mathcal{O}_v \subseteq X(K_v) \) by the valuative criterion for separatedness. We define the set \( X(A_K) \) of adelic points of \( X \) as the subset of families \((P_v)_{v \in \Omega^{(1)}} \in \prod_{v \in \Omega^{(1)}} X(K_v) \) such that \( P_v \in \mathfrak{X}(\mathcal{O}_v) \) for all but finitely many \( v \in U^{(1)} \). If \( X(A_K) \neq \varnothing \), we consider the pairing

\[
\frac{H^{d+2}(X, \mathbb{Q}/\mathbb{Z}(d + 1))}{\text{Im } H^{d+2}(K, \mathbb{Q}/\mathbb{Z}(d + 1))} \times X(A_K) \to \mathbb{Q}/\mathbb{Z}, \quad (A, (P_v)_{v \in \Omega^{(1)}}) \mapsto \sum_{v \in \Omega^{(1)}} A(P_v),
\]

which is well defined. Indeed, \( A \) comes from \( H^{d+2}(\mathfrak{X}_V, \mathbb{Q}/\mathbb{Z}(d + 1)) \) for some nonempty open subset \( V \subseteq U \) (where \( \mathfrak{X}_V = \mathfrak{X} \times_U V \)), and \( P_v \) comes from \( \mathfrak{X}(\mathcal{O}_v) \) for all but finitely many \( v \in V^{(1)} \). It follows that for these \( v \), \( A(P_v) \) comes from \( H^{d+2}(\mathcal{O}_v, \mathbb{Q}/\mathbb{Z}(d + 1)) \). By [51, Chapter III, Remark 3.11], one has \( H^{d+2}(\mathcal{O}_v, \mathbb{Q}/\mathbb{Z}(d + 1)) \cong H^{d+2}(k(v), \mathbb{Q}/\mathbb{Z}(d + 1)) = 0 \) since \( cd(k(v)) = d + 1 \). Thus, \( A(P_v) = 0 \) for all but finitely many \( v \in \Omega^{(1)} \) (note that \( \Omega \setminus V \) is finite). The above pairing vanishes on the image of \( H^{d+2}(K, \mathbb{Q}/\mathbb{Z}(d + 1)) \to H^{d+2}(X, \mathbb{Q}/\mathbb{Z}(d + 1)) \) by virtue of the generalized Weil reciprocity law (1.4.5). Thus, we obtain a pairing

\[
\frac{H^{d+2}(X, \mathbb{Q}/\mathbb{Z}(d + 1))}{\text{Im } H^{d+2}(K, \mathbb{Q}/\mathbb{Z}(d + 1))} \times X(A_K) \to \mathbb{Q}/\mathbb{Z}, \quad (A, (P_v)_{v \in \Omega^{(1)}}) \mapsto \sum_{v \in \Omega^{(1)}} A(P_v),
\]

(1.4.11)

which vanishes on \( X(K) \) (again, by (1.4.5)). Restricting (1.4.11) to locally constant classes yields a map

\[
\rho_X : \text{Ker} \left( \frac{H^{d+2}(X, \mathbb{Q}/\mathbb{Z}(d + 1))}{\text{Im } H^{d+2}(K, \mathbb{Q}/\mathbb{Z}(d + 1))} \to \prod_{v \in \Omega^{(1)}} \frac{H^{d+2}(X_v, \mathbb{Q}/\mathbb{Z}(d + 1))}{\text{Im } H^{d+2}(K_v, \mathbb{Q}/\mathbb{Z}(d + 1))} \right) \to \mathbb{Q}/\mathbb{Z},
\]

(1.4.12)
which vanishes whenever \( X(K) \neq \emptyset \). We refer to the nonvanishing of \( \rho_X \) as the (adelic) first obstruction to the local–global principle for \( X \).

Now, suppose that \( X(K) \neq \emptyset \). For any finite set \( S \subseteq \Omega^{(1)} \), (1.4.11) restricts to a pairing

\[
(-, -)_S : \text{Ker} \left( \prod_{v \in S} \text{Im} \text{H}^{d+2}(X_v, \mathbb{Q}/\mathbb{Z}(d+1)) \right) \times \prod_{v \in S} X(K_v) \to \mathbb{Q}/\mathbb{Z}, \tag{1.4.13}
\]

which vanishes on the closure (for the product of \( v \)-adic topologies) of \( X(K) \) in \( \prod_{v \in S} X(K_v) \) (this uses Lemma 1.1(i)). The nonvanishing of a family \((P_v)_{v \in S}\) to the group on the left-hand side of (1.4.13) is an obstruction to approximating this family by \( K \)-rational points. We refer to this as the generalized Brauer–Manin obstruction to weak approximation for \( X \) in \( S \).

### 1.5 Arithmetic duality theorems

This section reassembles the tools for proving our main theorems, namely, the local and global duality theorems for Galois cohomology of curves over higher dimensional local fields.

We start with the case where \( k \) is a \( d \)-dimensional local field satisfying the condition (\( \ast \)) from page 3. Recall that for \( n \geq 1 \), we have an isomorphism \( \text{H}^{d+1}(k, \mu_n^\otimes d) \cong \mathbb{Z}/n \) from (1.4.1). By [52, Chapter I, Theorem 2.17], if \( F \) is a finite \( n \)-torsion \( \Gamma_k \)-module and \( F' = \text{Hom}_k(F, \mu_n^\otimes d) \), the cup-product pairing

\[
\text{H}^i(k, F) \times \text{H}^{d+1-i}(k, F') \to \text{H}^{d+1}(k, \mu_n^\otimes d) \cong \mathbb{Z}/n
\]

is a perfect duality of finite groups for \( 0 \leq i \leq d + 1 \). Suppose that \( d \geq 1 \) and that \( F \) extends to a group scheme \( F' \) over the ring of integers \( \mathcal{O} \) of \( k \) (thus, \( F' \) also extends to \( F'' = \text{Hom}_{\mathcal{O}}(F, \mu_n^\otimes d) \)). Then, \( \text{H}^i(\mathcal{O}, F') \) is a subgroup of \( \text{H}^i(k, F) \), \( \text{H}^{d+1-i}(\mathcal{O}, F'') \) is a subgroup of \( \text{H}^{d+i-1}(k, F') \), and these subgroups are exact annihilators of each other under the above duality pairing [38, Proposition 2.5].

Let \( K \) be the function field of a smooth projective geometrically integral curve \( \Omega \) over \( k \). If \( j : U \hookrightarrow \Omega \) is a nonempty open subset and \( F \) is a complex of sheaves on \( U_{\text{ét}} \), we define the hypercohomology groups with compact support \( \text{H}^i_c(U, F) := \text{H}^i(\Omega, j_! F) \), where \( j_! \) denotes the extension by zero [51, p. 93]. From the proof of [36, Lemme 1.3], one has a canonical isomorphism \( \text{H}^{d+3}_c(U, \mathcal{O}/\mathcal{Z}(d+1)) \cong \mathbb{Q}/\mathbb{Z} \).

Let \( F \) be a finite \( \Gamma_K \)-module. For a sufficiently small nonempty open subset \( U \subseteq \Omega \), \( F \) extends to a finite étale group scheme \( F' \to U \). Let \( F'' = \text{Hom}_U(F, \mathcal{O}/\mathcal{Z}(d+1)) \), which extends the finite \( \Gamma_K \)-module \( F' = \text{Hom}_k(F, \mathcal{O}/\mathcal{Z}(d+1)) \). For \( 0 \leq i \leq d + 3 \), one has the Yoneda product pairing for cohomology with compact support (see [51, p. 168])

\[
\cdot : \text{Ext}^i_U(F', \mathcal{O}/\mathcal{Z}(d+1)) \times \text{H}^{d+3-i}_c(U, F') \to \text{H}^{d+3}_c(U, \mathcal{O}/\mathcal{Z}(d+1)) \cong \mathbb{Q}/\mathbb{Z}.
\]

On the other hand, since \( F \cong \text{Hom}_U(F', \mathcal{O}/\mathcal{Z}(d+1)) \), the spectral sequence

\[
\text{H}^p(U, \mathcal{O}/\mathcal{Z}(d+1)) \Rightarrow \text{Ext}^{p+q}_U(F', \mathcal{O}/\mathcal{Z}(d+1))
\]
yields an edge map $H^i(U, \mathcal{F}) \to \text{Ext}^i_U(F', \mathbb{Q}/\mathbb{Z}(d + 1))$. Hence, we obtain a pairing
\[
\langle -, - \rangle_{AV} : H^i(U, \mathcal{F}) \times H^{d + 3 - i}_c(U, \mathcal{F}') \to H^{d + 3}_c(U, \mathbb{Q}/\mathbb{Z}(2)) \cong \mathbb{Q}/\mathbb{Z},
\]
(1.5.1)
which is, in fact, a perfect duality of finite groups [38, Proposition 2.1]. We refer to it as the Artin–Verdier duality pairing. This pairing induces a perfect duality pairing
\[
\langle -, - \rangle_{PT} : \mathbb{H}^i(K, \mathcal{F}) \times \mathbb{H}^{d + 3 - i}(K, \mathcal{F'}) \to \mathbb{Q}/\mathbb{Z},
\]
(1.5.2)
of finite groups [38, Théorème 2.4], as follows. Let $\eta \in \mathbb{H}^i(K, \mathcal{F})$ and $\alpha \in \mathbb{H}^{d + 3 - i}(K, \mathcal{F'})$. If $U$ is sufficiently small, we may lift $\eta$ to an element $\eta_U \in H^i(U, \mathcal{F})$ and $\alpha$ to an element $\alpha_U \in H^{d + 3 - i}(U, \mathcal{F'})$. By the localization exact sequence for cohomology with compact support
\[
H^{d + 3 - i}_c(U, \mathcal{F}') \to H^{d + 3 - i}(U, \mathcal{F}') \to \bigoplus_{v \notin U} H^{d + 3 - i}(K_v, \mathcal{F}') \cong \mathbb{Q}/\mathbb{Z},
\]
(1.5.3)
(which can be proved by exactly the same argument as in [52, Chapter II, Lemma 2.4]), $\alpha_U$ comes from an element $\alpha^c_U \in H^{d + 3 - i}_c(U, \mathcal{F'})$. Then, $\langle \eta, \alpha \rangle_{PT} = \langle \eta_U, \alpha^c_U \rangle_{AV}$. The nondegeneracy of (1.5.2) will serve in the proof of Theorem D.

In order to prove Theorem E, one needs the exact sequence [39, Lemme 1.2]
\[
H^i(K, \mathcal{F}) \to \prod_{v \in S} H^i(K_v, \mathcal{F}) \xrightarrow{\psi} \mathbb{H}^{i + 2 - i}_S(K, \mathcal{F'})^D \to \mathbb{H}^{i + 2 - i}(K, \mathcal{F'})^D \to 0,
\]
(1.5.4)
for any finite subset $S \subseteq \Omega^{(1)}$ and $1 \leq i \leq d + 1$, which is established in the course of establishing the Poitou–Tate sequence for finite modules. Here, the map $\psi$ is defined by
\[
\forall (f_v)_{v \in S} \in \prod_{v \in S} H^i(K_v, \mathcal{F}), \forall \alpha \in \mathbb{H}^{i + 2 - i}_S(K, \mathcal{F')}, \quad \psi((f_v)_{v \in S})(\alpha) = \sum_{v \in S} f_v \cup \text{loc}_v(\alpha),
\]
where $f_v \cup \text{loc}_v(\alpha) \in H^{i + 2}(K_v, \mathbb{Q}/\mathbb{Z}(d + 1)) \cong \mathbb{Q}/\mathbb{Z}$ for each $v \in S$.

Let us now focus on the case where $k$ is a $p$-adic field, that is, a finite extension of $\mathbb{Q}_p$ (hence $d = 1$). Let $T$ be a $K$-torus. Recall that the dual torus $T'$ is the torus with character module $\widehat{T} = \overline{T}$. For each $v \in \Omega^{(1)}$, local duality between the tori $T$ and $T'$ over the two-dimensional local field $K_v$ asserts that the pairing (1.3.2) induces a cup-product pairing
\[
H^1(K_v, T) \times H^1(K_v, T') \to H^3(K_v, \mathbb{Z}(2)) \cong H^3(K_v, \mathbb{Q}/\mathbb{Z}(2)) \cong \mathbb{Q}/\mathbb{Z},
\]
(1.5.5)
which is a perfect duality of finite groups. For a sufficiently small nonempty open subset $U \subseteq \Omega^{(1)}$, $T$ (resp. $T'$) extends to a $U$-torus $T$ (resp. $T'$). Then, $H^1(\mathcal{O}_v, T)$ is a subgroup of $H^1(K_v, T)$, $H^1(\mathcal{O}_v, T')$ is a subgroup of $H^1(K_v, T')$, and these subgroups are exact annihilators of each other under the above duality pairing (see [34, Lemma 2.1, Proposition 2.2] and [31, Proposition 2.3(a)])

The pairing (1.3.2) extends to a pairing $\mathcal{F} \otimes L \mathcal{T} \to \mathcal{G}_m \otimes L \mathcal{G}_m \to \mathcal{Z}(2)[2]$ in $D^+(U)$. For $0 \leq i \leq 3$, we have a pairing
\[
\text{Ext}^i_U(\mathcal{T}', \mathcal{T} \otimes L \mathcal{T}') \times H^{3 - i}_c(U, \mathcal{T}') \to H^3_c(U, \mathcal{T} \otimes L \mathcal{T}') \to H^5_c(U, \mathbb{Z}(2)) \cong \mathbb{Q}/\mathbb{Z},
\]
(1.5.6)
where the first arrow is the Yoneda product pairing for cohomology with compact support \([51, p. 168]\), the second arrow is induced by the above pairing, and the isomorphism \(H^5_c(U, \mathbb{Z}(2)) \cong \mathbb{Q}/\mathbb{Z}(2)\) is \([34, Lemma 1.1]\). Furthermore, we have a composite map

\[
H^i(U, \mathcal{T}) \to H^i(U, \mathcal{H}om_U(\mathcal{T}', \mathcal{T} \otimes^L \mathcal{T}')) \to Ext^i_U(\mathcal{T}', \mathcal{T} \otimes^L \mathcal{T}'),
\]

where the first arrow is induced by the natural morphism \(\mathcal{T} \to \mathcal{H}om_U(\mathcal{T}', \mathcal{T} \otimes^L \mathcal{T}')\) in \(D^+(U)\), and the second arrow is the edge map from the spectral sequence

\[
H^p(U, Ext^q_U(\mathcal{T}', \mathcal{T} \otimes^L \mathcal{T}')) \Rightarrow Ext^{p+q}_U(\mathcal{T}', \mathcal{T} \otimes^L \mathcal{T}').
\]

From (1.5.6) and (1.5.7), we obtain a pairing

\[
\langle -, - \rangle_{AV} : H^i(U, \mathcal{T}) \times H^{3-i}(U, \mathcal{T}') \to H^5_c(U, \mathbb{Z}(2)) \cong \mathbb{Q}/\mathbb{Z}.
\]

We refer to it as the Artin–Verdier pairing (for tori). Unlike in the finite case, this is not a perfect duality (nor are the concerning cohomology groups finite). Nevertheless, for \(i = 1\) (and by exchanging \(\mathcal{T}\) and \(\mathcal{T}'\)), this induces a perfect duality pairing

\[
\langle -, - \rangle_{PT} : \mathbb{III}^2(K, T) \times \mathbb{III}^1(K, T') \to \mathbb{Q}/\mathbb{Z}
\]

of finite groups, as follows (see Theorem 1.3 and the proof of Theorem 4.1 in \([34]\)). Let \(\eta \in \mathbb{III}^2(K, T)\) and \(\alpha \in \mathbb{III}^1(K, T')\). If \(U\) is sufficiently small, we may lift \(\eta\) to an element \(\eta_U \in H^2(U, \mathcal{T})\) and \(\alpha\) to an element \(\alpha_U \in H^1(U, \mathcal{T}')\). By the localization exact sequence

\[
\cdots \to H^i_c(U, \mathcal{T}) \to H^i(U, \mathcal{T}) \to \bigoplus_{v \notin U} H^i(K_v, T) \to H^{i+1}_c(U, \mathcal{T}) \to \cdots
\]

starting from degree 1 \([34, Corollary 3.2]\), \(\eta_U\) comes from an element \(\eta^c_U \in H^2_c(U, \mathcal{T})\). Then, \(\langle \eta, \alpha \rangle_{PT} = \langle \eta^c_U, \alpha_U \rangle_{AV}\). In this article, we shall need the fact that (1.5.9) is also induced by (1.5.8) for \(i = 2\) (without exchanging \(\mathcal{T}\) and \(\mathcal{T}'\)). To see this, it suffices to show the following

**Lemma 1.4.** We have a commutative diagram of pairings

\[
\begin{array}{ccc}
H^2_c(U, \mathcal{T}) \times H^1(U, \mathcal{T}') & \xrightarrow{(\langle -, - \rangle_{AV})} & H^5_c(U, \mathbb{Z}(2)) \\
\downarrow \quad & & \downarrow \\
H^2(U, \mathcal{T}) \times H^1_c(U, \mathcal{T}') & \xrightarrow{(\langle -, - \rangle_{AV})} & H^5(U, \mathbb{Z}(2)),
\end{array}
\]

that is, for all \(\eta_U \in H^2(U, \mathcal{T})\) coming from \(\eta^c_U \in H^2_c(U, \mathcal{T})\) and \(\alpha_U \in H^1(U, \mathcal{T}')\) coming from \(\alpha^c_U \in H^1_c(U, \mathcal{T}')\), one has \(\langle \eta_U, \alpha_U \rangle_{AV} = \langle \eta^c_U, \alpha^c_U \rangle_{AV}\).

**Proof.** As in \([34, (26)]\), for any \(n \geq 1\), one has two commutative diagrams of pairings

\[
\begin{array}{ccc}
H^2_c(U, n \mathcal{T}) \times H^2(U, n \mathcal{T}') & \xrightarrow{(\langle -, - \rangle_{AV})} & H^4_c(U, \mu^n \otimes^L \mathbb{Z}(2)) \\
\downarrow \quad & & \downarrow \\
H^2_c(U, \mathcal{T}) \times H^1(U, \mathcal{T}') & \xrightarrow{(\langle -, - \rangle_{AV})} & H^5_c(U, \mathbb{Z}(2))
\end{array}
\]

for any \(n \geq 1\).
and

\[
\begin{array}{c}
\text{H}^2(U, n\mathcal{T}) \times \text{H}_c^2(U, n\mathcal{T}') \xrightarrow{(-,\cdot)_{AV}} \text{H}_c^4(U, \mu_n^{\otimes 2}) \\
\downarrow \delta_n \quad \uparrow \delta_n \\
\text{H}^2(U, \mathcal{T}) \times \text{H}_c^2(U, \mathcal{T}') \xrightarrow{(-,\cdot)_{AV}} \text{H}_c^4(U, \mathbb{Z}(2)),
\end{array}
\]

where the top rows are the Artin–Verdier pairing for finite modules (1.5.1), and the maps \(\delta_n\), \(\delta\) are induced by the respective morphisms \(\mathbb{Z}/n \to \mathbb{Z}(1)[1]\), \(\mathbb{Z}(1)[1] \to \mathbb{Z}/n(1)[1]\), and \(\mathbb{Z}/n(2) \to \mathbb{Z}(2)[1]\) in the derived category. On the other hand, by the same argument as in Lemma 4.2 and the proof of Lemma 4.7(3) in [21], we have a commutative diagram of pairings

\[
\begin{array}{c}
\text{H}^2_c(U, n\mathcal{T}) \times \text{H}^2(U, n\mathcal{T}') \xrightarrow{(-,\cdot)_{AV}} \text{H}_c^4(U, \mu_n^{\otimes 2}) \\
\downarrow \delta_n \quad \uparrow \delta_n \\
\text{H}^2(U, n\mathcal{T}) \times \text{H}_c^2(U, n\mathcal{T}') \xrightarrow{(-,\cdot)_{AV}} \text{H}_c^4(U, \mu_n^{\otimes 2}).
\end{array}
\]

Let \(\eta^c_U \in \text{H}^2_c(U, \mathcal{T})\) (with image \(\eta_U \in \text{H}^2(U, \mathcal{T})\)) and \(\alpha^c_U \in \text{H}^1_c(U, \mathcal{T})\) (with image \(\alpha_U \in \text{H}^1(U, \mathcal{T})\)). Since the group \(\text{H}^2_c(U, \mathcal{T})\) is torsion by [34, Corollary 3.3], one has \(\eta^c_U = \delta_n(\eta^c_{U,n})\) for some \(n \geq 1\) and some \(\eta^c_{U,n} \in \text{H}^2(U, n\mathcal{T})\). Denote by \(\eta_{U,n} \in \text{H}^2(U, n\mathcal{T})\) the image of \(\eta^c_{U,n}\). From the commutativity of the above diagrams, one has

\[
\langle \eta^c_U, \alpha_U \rangle_{AV} = \delta(\langle \eta^c_{U,n}, \delta_n(\alpha_U) \rangle_{AV}) = \delta(\langle \eta_{U,n}, \delta_n(\alpha^c_U) \rangle_{AV}) = \langle \eta_U, \alpha_U^c \rangle_{AV}.
\]

The lemma is now proved. \(\square\)

It follows that the pairing (1.5.9) has the following alternative description. Let \(\eta \in \text{H}^2_c(K, T)\) and \(\alpha \in \text{H}^1_c(K, T')\). Lift \(\eta\) to an element \(\eta_U \in \text{H}^2(U, \mathcal{T})\) and \(\alpha \) to an element \(\alpha_U \in \text{H}^1(U, \mathcal{T}')\) (shrinking \(U\) if necessary). By the localization sequence (1.5.10), \(\alpha_U\) comes from an element \(\alpha_U^c \in \text{H}^1_c(U, \mathcal{T}')\). Then, \(\langle \eta_U, \alpha_U \rangle_{PT} = \langle \eta_U, \alpha_U^c \rangle_{AV}\).

The nondegeneracy of (1.5.9) will serve in the proof of Theorem A. As for Theorem C, one needs the following part of the Poitou–Tate exact sequence for tori [31, Proposition 3.5]:

\[
\text{H}^1(K, T) \to \mathbb{P}^1(K, T) \xrightarrow{\partial} \text{H}^1(K, T')^D.
\]

This is an exact sequence of \emph{topological abelian groups}. The groups \(\text{H}^1(K, T)\) and \(\text{H}^1(K, T')\) are discrete. The group \(\mathbb{P}^1(K, T)\) is, by definition, the topological restricted product of the finite groups \(\text{H}^1(K_v, T)\) relative to the subgroups \(\text{H}^1(\mathcal{O}_v, \mathcal{T})\), \(v \in U^{(1)}\) (recall that \(\mathcal{T}\) is a \(U\)-torus extending \(T\)).

The map \(\partial\) is defined by

\[
\forall (t_v)_{v \in \Omega^{(1)}} \in \mathbb{P}^1(K, T), \forall \alpha \in \text{H}^1(K, T'), \quad \partial((t_v)_{v \in \Omega^{(1)}})(\alpha) = \sum_{v \in \Omega^{(1)}} t_v \cup \text{loc}_v(\alpha).
\]

Here, \(t_v \cup \text{loc}_v(\alpha) \in \text{H}^4(K_v, \mathbb{Z}(2)) \cong \mathbb{Q}/\mathbb{Z}\) via the local cup-product (1.5.5).
In order to prove Theorem B, we require the following exact sequence. Let $S \subseteq \Omega^{(1)}$ be any finite set. Since any element of $\prod_{v \in S} H^1(K_v, T)$ can completed by 0 into an element of $\mathcal{H}^1(K, T)$, (1.5.11) restricts to the three last terms of the exact sequence

$$0 \to \mathcal{H}^1(K, T) \to \mathcal{H}^1_S(K, T) \to \prod_{v \in S} H^1(K_v, T) \to H^1(K, T')^D$$

of discrete abelian groups. Dualizing this sequence and exchanging $T$ and $T'$, one obtains an exact sequence

$$H^1(K, T) \to \prod_{v \in S} H^1(K_v, T) \xrightarrow{\partial} \mathcal{H}^1_S(K, T')^D \to \mathcal{H}^1(K, T')^D \to 0,$$

where the map $\partial$ is defined by

$$\forall (t_v)_{v \in S} \in \prod_{v \in S} H^1(K_v, T), \forall \alpha \in \mathcal{H}^1_S(K, T'), \quad \partial((t_v)_{v \in S})(\alpha) = \sum_{v \in S} t_v \cup \text{loc}_v(\alpha).$$

We conclude this section with the following lemma, which is [31, Lemma 4.2(a)], whose proof relies on Tate–Lichtenbaum duality for $p$-adic curves. It is crucial for the constructions used in the proofs of Theorems A–C.

**Lemma 1.5.** If $Q$ is a quasi-split torus over a $p$-adic function field $K$, then $\mathcal{H}^2_\omega(K, Q) = 0$.

This lemma is used not only to establish the unramified nature of the obstructions but also the Poitou–Tate sequence for tori. For $K$ a function field of a curve over a $d$-dimensional local field (where $d \geq 2$), one would need the vanishing of $\mathcal{H}^d_\omega(K, Z(d)) = \mathcal{H}^d_\omega(K, Z(d))$, where $Z(d)$ is the shifted weight $d$ cycle complex $z(\bullet, -)[-2d]$ defined by Bloch [1], or equivalently, the vanishing of $\mathcal{H}^2(K, \mathbb{G}_m)$ [36, Lemme 3.15]. Unfortunately, this is not always the case (this problem was studied by Izquierdo in [37, §4 and §5]).

**Remark 1.6.** An independent but interesting consequence of Lemma 1.5 is the following local–global principle. Let $X$ be a Severi–Brauer variety over the function field $K$ of a $p$-adic curve $\Omega$. If $X(K_v) \neq \emptyset$ for all but finitely many $v \in \Omega^{(1)}$, then $X(K) \neq \emptyset$.

## 2 | DESCENT THEORY

This section is devoted to the proof of Theorem C.

### 2.1 | Preliminary remarks

We recall some facts. Let $X$ be a smooth geometrically integral variety over a field $K$ of characteristic 0. Following Colliot-Thélène and Sansuc (cf. [67, Theorem 2.3.4, Definition 2.3.5]), we define the *elementary obstruction* $e_X \in \text{Ext}^2_K(\text{Pic} X, K[X]^\times)$ to be the inverse class of the twofold extension

$$1 \to K[X]^\times \to K(X)^\times \xrightarrow{\text{div}} \text{Div} X \to \text{Pic} X \to 0$$

(2.1.1)
of $\Gamma_K$-modules. If $M$ is a $K$-group of multiplicative type, recall that $\hat{M} = \mathcal{H}om_K(M, G_m)$. The type of a torsor $Y \to X$ under $M$ is by definition the $\Gamma_K$-equivariant homomorphism

$$\hat{M} \to \text{Pic} \bar{X} = H^1(\bar{X}, G_m), \quad \chi \mapsto \chi_*[\bar{Y}].$$

Conversely, for any given $\Gamma_K$-equivariant homomorphism $\lambda : \hat{M} \to \text{Pic} \bar{X}$, the existence of $X$-torsors under $M$ of type $\lambda$ is equivalent to the vanishing of $\lambda^* e_X \in \text{Ext}^2_K(\hat{M}, K[X]^\times)$. In the case where $X(K) \neq \emptyset$, we have $e_X = 0$ and torsors of any type exist. When $K[X]^\times = K^\times$, the spectral sequence $H^p(K, \mathcal{E}xt^q_K(\hat{M}, G_m)) \Rightarrow \text{Ext}^{p+q}_K(\hat{M}, G_m)$ yields the edge maps

$$H^p(K, M) = H^p(K, \mathcal{H}om_K(\hat{M}, G_m)) \cong \text{Ext}^p_K(\hat{M}, G_m), \quad (2.1.2)$$

which are isomorphisms for $p > 0$ [67, Lemma 2.3.7]. In particular, we may regard $\lambda^* e_X$ as an element of $H^2(K, M)$. The equivalence between the vanishing of this element and the existence of $X$-torsors of type $\lambda$ is part of the “fundamental exact sequence” of Colliot-Thélène and Sansuc (see Theorem 2.3.6 and Corollary 2.3.9 in loc. cit.), which reads

$$H^1(K, M) \to H^1(X, M) \xrightarrow{\text{type}} \mathcal{H}om_K(\hat{M}, G_m) \to H^2(K, M) \to H^2(X, M). \quad (2.1.3)$$

If $\text{Pic} \bar{X}$ is finitely generated as an abelian group and $M$ is the $K$-group of multiplicative type such that $\hat{M} = \text{Pic} \bar{X}$ (i.e., an isomorphism $\hat{M} \cong \text{Pic} \bar{X}$ is fixed), we call a torsor $Y \to X$ under $M$ universal if its type is the identity morphism of $\hat{M} ^\dagger$. Indeed, the existence of such a torsor is equivalent to $e_X = 0$.

Assume furthermore that $\text{Pic} \bar{X}$ is free, then the Néron–Severi torus of $X$ is by definition the $K$-torus $T$ such that $\bar{T} = \text{Pic} \bar{X}$. For example, this is the case when $X$ is projective and rationally connected (combine [9, §8.4, Theorem 1], [20, Corollary 4.18], and [43, Théorème 5.1]), that is, for any algebraically closed overfield $K'/K$ and two general points $P_0, P_1 \in X(K')$, there exists a morphism $\gamma : \mathbb{P}^1_{K'} \to X_{K'}$, such that $\gamma(0) = P_0$ and $\gamma(1) = P_1$. Examples of such varieties are smooth compactifications of geometrically unirational varieties (such as homogeneous spaces of connected linear algebraic groups; indeed, a celebrated theorem of Chevalley asserts that connected linear algebraic groups are geometrically rational, even $K$-unirational [11]).

Before starting, let us restate the main result of this section (i.e., Theorem C).

**Theorem 2.1.** Let $K$ be the function field of a smooth proper geometrically integral curve $\Omega$ over a $p$-adic field $k$, and let $X$ be a smooth proper geometrically integral variety over $K$ such that the abelian group $\text{Pic} \bar{X}$ is finitely generated and free (e.g., $X$ is projective and rationally connected). Let $T$ be the Néron–Severi torus of $X$. There exists a homomorphism

$$u : H^1(K, T^\prime) \to \frac{H^4(X, \mathbb{Z}(2))}{\text{Im} H^4(K, \mathbb{Z}(2))}$$

with the following properties. Suppose that $\prod_{v \in \Omega(1)} X(K_v) \neq \emptyset$, then

---

1 This differs slightly from the usual convention, where a torsor is defined to be universal if its type is any isomorphism of Galois module (not just a fixed one from the beginning).
(i) universal $X$-torsors exist (i.e., $e_X = 0$) if and only if there exists a family of $\prod_{v \in \Omega(1)} X(K_v)$ that is orthogonal to $u(\text{III}^1(K, T'))$ relative to the pairing (1.4.9);
(ii) if $(\prod_{v \in \Omega(1)} X(K_v))^{\text{Im}(u)}$ denotes the subset of $\prod_{v \in \Omega(1)} X(K_v)$ consisting of the families orthogonal to $\text{Im}(u)$ relative to the pairing (1.4.9), then
$$\left( \prod_{v \in \Omega(1)} X(K_v) \right)^{\text{Im}(u)} = \bigcup_{f: Y \to X \text{ type}(Y) = \text{id}} f \left( \prod_{v \in \Omega(1)} Y(K_v) \right);$$

(iii) there are only finitely many isomorphism classes of universal torsors $Y \to X$ such that $\prod_{v \in \Omega(1)} Y(K_v) \neq \emptyset$;
(iv) if the universal torsors $Y \to X$ satisfy the local–global principle (resp. the local–global principle and weak approximation), then the reciprocity obstruction (1.4.9) to the local–global principle (resp. the local–global principle and weak approximation) on $X$ attached to $\text{Im}(u)$ is the only one.

Let $K$ be a field of cohomological dimension $\text{cd}(K) \leq 3$ and let $\pi: X \to \text{Spec} K$ be a smooth proper geometrically integral variety such that the abelian group $\text{Pic} \overline{X}$ is finitely generated and free. Let $T$ be the Néron–Severi torus of $X$ (i.e., $\hat{T} = \text{Pic} \overline{X}$). We construct a map
$$u: H^1(K, T') \to \frac{H^4(X, \mathbb{Z}(2))}{\text{Im} H^4(K, \mathbb{Z}(2))} \quad (2.1.4)$$
as in the statement of Theorem 2.1, as follows. First, since $K[X]^X = \overline{K}^X$, we have the following distinguished triangle in the category $D^+(K)$:
$$G_m \to \tau_{\leq 1} \mathbb{R} \pi_* G_{m,X} \to \hat{T}[-1] \to G_{m}[1]. \quad (2.1.5)$$

Applying the exact functor $- \otimes \mathbb{L} \mathbb{G}_m$ yields a distinguished triangle
$$G_m \otimes^\mathbb{L} G_m \to (\tau_{\leq 1} \mathbb{R} \pi_* G_{m,X}) \otimes^\mathbb{L} G_m \to T'[−1] \to G_{m} \otimes^\mathbb{L} G_{m}[1]. \quad (2.1.6)$$

Let $\theta_1: \mathbb{R} \pi_* (G_{m,X} \otimes^\mathbb{L} G_{m,X}) \to \mathbb{R} \pi_* \mathbb{Z}_X(2)[2]$ be the map induced by the pairing (1.3.1). Next, denote by $\theta_2$ the composite
$$(\tau_{\leq 1} \mathbb{R} \pi_* G_{m,X}) \otimes^\mathbb{L} G_m \to (\tau_{\leq 1} \mathbb{R} \pi_* G_{m,X}) \otimes^\mathbb{L} \mathbb{R} \pi_* G_{m,X} \to \mathbb{R} \pi_* ((\mathbb{R} \pi_{\leq 1} \mathbb{R} \pi_* G_{m,X}) \otimes^\mathbb{L} G_{m,X}) \to \mathbb{R} \pi_* (G_{m,X} \otimes^\mathbb{L} G_{m,X}),$$

where
- the first arrow is induced by the natural map $G_m \to \mathbb{R} \pi_* G_{m,X}$,
- the second arrow is the canonical “base change” morphism constructed in [26, p. 306],
- the third arrow is induced by the natural map $\tau_{\leq 1} \mathbb{R} \pi_* G_{m,X} \to \mathbb{R} \pi_* G_{m,X}$, and
- the last arrow is induced by the adjunction $\pi^* \mathbb{R} \pi_* G_{m,X} \to G_{m,X}$.

Finally, let
$$\theta = \theta_1 \circ \theta_2 : (\tau_{\leq 1} \mathbb{R} \pi_* G_{m,X}) \otimes^\mathbb{L} G_m \to \mathbb{R} \pi_* \mathbb{Z}_X(2)[2]. \quad (2.1.7)$$
By the functoriality of the pairing (1.3.1), \( \vartheta \) fits into a commutative diagram

\[
\begin{array}{cccc}
G_m \otimes \mathbb{L} G_m & \rightarrow & (\tau_{\leq 1} \mathbb{R} \pi_* G_{m,X}) \otimes \mathbb{L} G_m & \rightarrow \\
\downarrow & & \downarrow \vartheta & \\
\mathbb{Z}(2)[2] & \rightarrow & \mathbb{R} \pi_* \mathbb{Z}(2)[2]. & \\
\end{array}
\]

Let \( Z_{X/K}(2) \) denote the cone of \( \mathbb{Z}(2) \rightarrow \mathbb{R} \pi_* \mathbb{Z}(X)(2) \). It follows from the axioms of triangulated categories that there exists a dashed arrow making the diagram

\[
\begin{array}{cccc}
G_m \otimes \mathbb{L} G_m & \rightarrow & (\tau_{\leq 1} \mathbb{R} \pi_* G_{m,X}) \otimes \mathbb{L} G_m & \rightarrow \\
\downarrow & & \downarrow \vartheta & \\
\mathbb{Z}(2)[2] & \rightarrow & \mathbb{R} \pi_* Z_{X}(2)[2] & \rightarrow \\
& & \mathbb{Z}_{X/K}(2)[2] & \rightarrow \\
& & \mathbb{Z}(2)[3] & \\
\end{array}
\]

commute (the top row being (2.1.6)). Since \( H^5(K, \mathbb{Z}(2)) \cong H^4(K, \mathbb{Q}/\mathbb{Z}(2)) = 0 \) under the assumption \( \text{cd}(K) \leq 3 \), taking cohomology of the bottom row of (2.1.8) yields an identification

\[
H^4(K, \mathbb{Z}_{X/K}(2)) \cong H^4(X, \mathbb{Z}(2)) \cap \mathbb{Z}(2). \tag{2.1.9}
\]

We take the map \( u \) in (2.1.4) to be the composite

\[
H^1(K, T') \xrightarrow{\lambda} H^4(K, Z_{X/K}(2)) \cong \frac{H^4(X, \mathbb{Z}(2)) \cap \mathbb{Z}(2)}{\text{im} H^4(K, \mathbb{Z}(2))}.
\]

### 2.2 Existence of universal torsors

In this section, we prove Theorem 2.1(i). Let \( K \) be the function field of a smooth projective geometrically integral curve \( \Omega \) over a \( p \)-adic field \( k \). The point is to relate the first obstruction (1.4.10) and the global Poitou–Tate duality pairing (1.5.9), as in the following analog of [16, Lemme 3.3.3] (see also [67, (6.4)]).

**Proposition 2.2.** Let \( \pi : X \rightarrow \text{Spec} \ K \) be a smooth proper geometrically integral variety such that the abelian group \( \text{Pic} \; \overline{X} \) is finitely generated and free. Let \( T \) be the Néron–Severi torus of \( X \). Assume in addition that \( \prod_{v \in \Omega(1)} X(K_v) \neq \emptyset \). In particular, the class \( \eta \in H^2(K, T) \) corresponding to the elementary obstruction \( e_X \in \text{Ext}^2_T(K, \mathbb{G}_m) \) (under the identification (2.1.2)) belongs to \( \mathfrak{W}^2(K, T) \). Then, for all \( \alpha \in \mathfrak{W}^1(K, T') \), one has the equality

\[
\rho_X(u(\alpha)) = -\langle \eta, \alpha \rangle_{PT}.
\]

Here, the map \( \rho_X \) was defined in (1.4.10), and \( \langle -, - \rangle_{PT} \) is the pairing (1.5.9).

**Proof.** We follow the argument in [33, §3] and [34, Proposition 5.3]. First, we inspect the pairing \( \langle \eta, \cdot \rangle_{PT} \). By [7, Lemma 2.3], the object \( \tau_{\leq 1} \mathbb{R} \pi_* G_{m,X} \) in (2.1.5) is represented by the complex

\[
[\overline{K}(X)^{\times} \xrightarrow{\text{div}} \text{Div} \; \overline{X}] \text{ concentrated in degree } -1 \text{ and } 0.
\]

It follows that the class \( -e_X \in \text{Ext}^2_T(K, \mathbb{G}_m) \)
of the twofold extension (2.1.1) (note that $K[X]^\times = K^\times$) is also represented by a morphism $\hat{T} \to G_m[2]$ in $D^+(K)$ associated with triangle (2.1.5).

Let $\pi^U : \mathcal{X} \to U$ be an integral model of $X$ over some nonempty open subset $U \subseteq \Omega$. We may assume that $T$ extends to a $U$-torus $T$. Then, $\hat{T}$ (resp. $T'$) extends to the finitely presented locally constant group scheme $\hat{T} = \mathcal{H}om_U(T, G_m)$ (resp. the $U$-torus $T' = \hat{T} \otimes G_m$). For $U$ sufficiently small, $\pi^U_{*}G_{m,\mathcal{X}} = G_m$ and $\mathcal{R}^1\pi^U_{*}G_{m,\mathcal{X}} = \hat{T}$, hence one has a distinguished triangle

$$G_m \to \tau_{\leq 1}\mathcal{R}\pi^U_{*}G_{m,\mathcal{X}} \to \hat{T}[-1] \to G_m[1].$$

(2.2.1) in $D^+(U)$, which extends (2.1.5). The inverse class $e_U \in \mathcal{E}xt^2_U(\hat{T}, G_m)$ of the morphism $\hat{T} \to G_m[2]$ associated with (2.2.1) is a lifting of $e_X \in \mathcal{E}xt^2_K(\hat{T}, G_m)$.

We claim that there is a commutative diagram

$\begin{array}{ccc}
\mathcal{H}^2(U, \mathcal{H}om_U(\hat{T}, G_m)) & \xrightarrow{r_1} & \mathcal{E}xt^2_U(\hat{T}, G_m) \\
\mathcal{H}^2(U, \mathcal{H}om_U(T, \mathcal{T} \boxtimes \mathcal{T})) & \xrightarrow{r_3} & \mathcal{E}xt^2_U(T', \mathcal{T} \boxtimes \mathcal{T}') \\
\mathcal{H}^2(U, \mathcal{H}om_U(T', \mathcal{T} \boxtimes \mathcal{T}')) & \xrightarrow{r_1} & \mathcal{E}xt^2_U(T', \mathcal{T} \boxtimes \mathcal{T}') \\
\mathcal{H}^2(U, \mathcal{H}om_U(\hat{T}, T \boxtimes \hat{T})) & \xrightarrow{\gamma_1} & \mathcal{H}om_U(\hat{T}, G_m) \\
\mathcal{H}^2(U, \mathcal{H}om_U(T', T' \boxtimes T')) & \xrightarrow{\gamma_1} & \mathcal{H}om_U(T', G_m) \\
\mathcal{H}om_U(\hat{T}, G_m) & \xrightarrow{-\otimes^L G_m} & \mathcal{H}om_U(T', G_m) \\
\mathcal{H}om_U(\hat{T}, \mathcal{T}) & \xrightarrow{-\otimes^L G_m} & \mathcal{H}om_U(T', \mathcal{T} \boxtimes \mathcal{T}') \\
\mathcal{H}om_U(T', \mathcal{T} \boxtimes \mathcal{T}') & \xrightarrow{\gamma_3} & \mathcal{H}om_U(T', G_m) \\
\end{array}$

(2.2.2)

The two left triangles of (2.2.2) are obtained by applying $\mathcal{H}^2(U, -)$ to the commutative diagram

$\begin{array}{ccc}
\mathcal{H}om_U(\hat{T}, G_m) & \xleftarrow{\gamma_1} & \mathcal{H}om_U(T', G_m) \\
\mathcal{H}om_U(\hat{T}, \mathcal{T} \boxtimes \hat{T}) & \xleftarrow{-\otimes^L G_m} & \mathcal{H}om_U(T', \mathcal{T} \boxtimes \mathcal{T}') \\
\mathcal{H}om_U(T', \mathcal{T} \boxtimes \mathcal{T}') & \xleftarrow{\gamma_3} & \mathcal{H}om_U(T', G_m) \\
\end{array}$

in $D^+(U)$. The maps $r_1, r_2, r_3$ in (2.2.2) are the edge maps from the spectral sequences

$$\mathcal{H}^p(U, \mathcal{E}xt^q_U(\hat{T}, P)) \Rightarrow \mathcal{E}xt_U^{p+q}(\hat{T}, P),$$

for $P = G_m, T \boxtimes \hat{T}, T \boxtimes T'$, respectively. The two middle squares of (2.2.2) commute by the functoriality of these spectral sequences. The maps $\gamma_1, \gamma_2, \gamma_3$ in (2.2.2) are induced by the respective pairings $T \boxtimes \hat{T} \to G_m, T \boxtimes T' \to G_m \otimes^L G_m$, and (1.3.1). The other triangle and square of (2.2.2) obviously commute. Since $r_1$ is an isomorphism by [67, Lemma 2.3.7], the class $e_U \in \mathcal{E}xt^2_U(\hat{T}, G_m)$ comes from an element $\eta_U \in \mathcal{H}^2(U, \mathcal{T})$ lifting $\eta \in \mathcal{H}^2(K, T)$. Let $e_U, e'_U$ denote its respective images in $\mathcal{E}xt^2_U(\hat{T}, T \boxtimes \hat{T})$ and $\mathcal{E}xt^2_U(T', T \boxtimes T')$ by (2.2.2). Then, $-\gamma_2(e'_U)$ is represented by the morphism $T' \to G_m \otimes^L G_m[2]$ associated with the distinguished triangle obtained by applying $- \otimes^L G_m$ to (2.2.1), that is,

$$G_m \otimes^L G_m \to (\tau_{\leq 1}\mathcal{R}\pi^U_{*}G_{m,\mathcal{X}}) \otimes^L G_m \to T'[−1] \to G_m \otimes^L G_m[1].$$

(2.2.3)
and \(-\gamma_3(\varepsilon'_U)\) is represented by the composite \(\mathcal{T}' \to \mathbb{Z}(2)[4]\) of (1.3.1) with this morphism. On the other hand, we have a commutative diagram of pairings

\[
\begin{array}{c}
H^2(U, \mathcal{T}) 	imes H^1(U, \mathcal{T}') \longrightarrow H^3_c(U, \mathcal{T} \otimes \mathcal{T}') \\
\downarrow \\
\text{Ext}^1_U(\mathcal{T}', \mathcal{T} \otimes \mathcal{T}') 	imes H^1_c(U, \mathcal{T}') \longrightarrow H^3_c(U, \mathcal{T} \otimes \mathcal{T}') \\
\downarrow \gamma_2 \\
\text{Ext}^2_U(\mathcal{T}', \mathbb{G}_m \otimes \mathbb{G}_m) 	imes H^1_c(U, \mathcal{T}') \longrightarrow H^3_c(U, \mathbb{G}_m \otimes \mathbb{G}_m) \\
\downarrow \gamma_3 \\
\text{Ext}^3_U(\mathcal{T}', \mathbb{Z}(2)) 	imes H^1_c(U, \mathcal{T}') \longrightarrow H^3_c(U, \mathbb{Z}(2)) \cong \mathbb{Q}/\mathbb{Z},
\end{array}
\]

(2.2.4)

where \(\cdot\) means the Yoneda product, and where the top square commutes thanks to the construction of the cup-product (Artin–Verdier) pairing for cohomology with compact support (see (1.5.6) and (1.5.7)). Hence, we have the following equality for all \(\alpha'_U \in H^1_c(U, \mathcal{T}')\):

\[
\gamma_3(\varepsilon'_U) \cdot \alpha'_U = \langle \eta_U, \alpha_U \rangle_{AV} \in H^3_c(U, \mathbb{Z}(2)) \cong \mathbb{Q}/\mathbb{Z}.
\]  

(2.2.5)

Let \(\alpha \in \mathbb{III}^1(K, \mathcal{T}')\). By the localization sequence (1.5.10), when \(U\) is sufficiently small, \(\alpha\) lifts to an element \(\alpha'_U \in H^1_c(U, \mathcal{T})\). The right-hand side of (2.2.5) is \(\langle \eta, \alpha \rangle_{PT}\) by the discussion following the construction of (1.5.9) in Section 1.5.

The next step is to inspect the element \(\rho_X(u(\alpha))\). Consider the commutative diagram

\[
\begin{array}{cccccccc}
H^4(K, \mathbb{Z}(2)) & \longrightarrow & H^4(X, \mathbb{Z}(2)) & \longrightarrow & H^4(X, \mathbb{Z}(2)) / \text{Im } H^4(K, \mathbb{Z}(2)) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \prod_{v \in \Omega^{(0)}} H^4(K_v, \mathbb{Z}(2)) & \longrightarrow & \prod_{v \in \Omega^{(0)}} H^4(X_v, \mathbb{Z}(2)) & \longrightarrow & \prod_{v \in \Omega^{(0)}} H^4(X_v, \mathbb{Z}(2)) / \text{Im } H^4(K_v, \mathbb{Z}(2))
\end{array}
\]

(2.2.6)

with exact rows (each map \(H^4(K_v, \mathbb{Z}(2)) \to H^4(X_v, \mathbb{Z}(2))\) is injective since \(X(K_v) \neq \emptyset\)). Since \(\alpha \in \mathbb{III}^1(K, \mathcal{T}')\), \(u(\alpha)\) lies in the kernel of the right vertical arrow in (2.2.6). Let \(\beta \in \prod_{v \in \Omega^{(0)}} H^4(K_v, \mathbb{Z}(2)) / \text{Im } H^4(K, \mathbb{Z}(2))\) be its image by the snake lemma construction.

**Lemma 2.3.** We have \(\beta \in \bigoplus_{v \in \Omega^{(1)}} H^4(K_v, \mathbb{Z}(2)) / \text{Im } H^4(K, \mathbb{Z}(2))\), and its image by the sum map

\[
\sigma : \bigoplus_{v \in \Omega^{(1)}} H^4(K_v, \mathbb{Z}(2)) / \text{Im } H^4(K, \mathbb{Z}(2)) \longrightarrow \mathbb{Q}/\mathbb{Z},
\]

is precisely \(\rho_X(u(\alpha))\).

**Proof.** Let \(A \in H^4(X, \mathbb{Z}(2))\) be a lifting of \(u(\alpha)\). For each \(v \in \Omega^{(1)}\), choose any point \(P_v \in X(K_v)\). Then, the constant element \(\text{loc}_v(A) \in H^4(X_v, \mathbb{Z}(2))\) comes from \(A(P_v) \in H^4(K_v, \mathbb{Z}(2))\). By definition of the snake lemma construction, the family \((A(P_v))_{v \in \Omega^{(1)}}\) is a lifting of \(\beta\). Thanks to [31,
Lemma 5.1], we have $A(P_v) = 0$ for all but finitely many $v \in \Omega^{(1)}$, so that $\beta \in \bigoplus_{v \in \Omega^{(1)}} H^4(X_v, \mathbb{Z}(2)) / H^4(K_v, \mathbb{Z}(2))$. Finally, $\sigma(\beta) = \sum_{v \in \Omega^{(1)}} A(P_v) = \rho_X(u(\alpha))$. 

Return to the proof of Proposition 2.2. Now we study the element $\beta$ by repeating the argument in [34, Lemma 5.4]. Diagram (2.1.8) extends to a commutative diagram

\[
\begin{array}{cccccc}
G_m \otimes \mathbb{L} G_m & \xrightarrow{(\tau \otimes \pi^U_*)} & G_m \otimes \mathbb{L} G_m & \xrightarrow{\mathcal{T}'[-1]} & G_m \otimes \mathbb{L} G_m[1] \\
\downarrow & & \downarrow \lambda_U & & \downarrow \\
\mathbb{Z}(2)[2] & \xrightarrow{\mathbb{R}\pi^U_* Z_\mathcal{X}(2)[2]} & \mathbb{Z}_\mathcal{X}/U(2)[2] & \xrightarrow{\mathbb{Z}(2)[3]} & \\
\bigoplus_{v \in \mathcal{U}} j_v \ast j_v^U Z(2)[2] & \xrightarrow{\bigoplus_{v \in \mathcal{U}} j_v \ast j_v^U Z_\mathcal{X}(2)[2]} & \bigoplus_{v \in \mathcal{U}} j_v \ast j_v^U Z_\mathcal{X}/U(2)[2], & & \\
\end{array}
\]

in $D^+(\mathcal{U})$, with distinguished rows (the top row being (2.2.3)); we recall that $\mathcal{T}$ is a $\mathcal{U}$-torus extending $T$, and the arrow $\mathcal{T}' \rightarrow G_m \otimes \mathbb{L} G_m[2]$ represents $-\gamma_2(\varepsilon'_U)$. Remember that $\alpha_U^c \in H^1_c(U, \mathcal{T}')$ is a lifting of $\alpha \in \text{III}^1(K, T')$. For $v \in \Omega^{(1)}$, denote by $j_v : \text{Spec} K^h_v \rightarrow U$ be the natural map, and consider the commutative diagram

\[
\begin{array}{cccccc}
\mathbb{Z}(2)[2] & \xrightarrow{\mathbb{R}\pi^U_* Z_\mathcal{X}(2)[2]} & \mathbb{Z}_\mathcal{X}/U(2)[2] \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_{v \in \mathcal{U}} j_v \ast j_v^U Z(2)[2] & \xrightarrow{\bigoplus_{v \in \mathcal{U}} j_v \ast j_v^U Z_\mathcal{X}(2)[2]} & \bigoplus_{v \in \mathcal{U}} j_v \ast j_v^U Z_\mathcal{X}/U(2)[2], & & \\
\end{array}
\]

whose rows are parts of exact triangles. Let $C_l$ (resp. $C_r$) denote the cone of the left (resp. right) vertical arrow of (2.2.8). Using the localization sequence

\[
\cdots \rightarrow H^4_c(U, Z_\mathcal{X}/U(2)) \rightarrow H^4(U, Z_\mathcal{X}/U(2)) \rightarrow \bigoplus_{v \in \mathcal{U}} H^4(K^h_v, j_v^* Z_\mathcal{X}/U(2)) \rightarrow \cdots
\]

(see, e.g., [34, Proposition 3.1]), we identify $\lambda_U \ast \alpha_U^c \in H^4_c(U, Z_\mathcal{X}/U(2))$ to an element of $\mathcal{H}^1(C_r)$. Taking cohomology of (2.2.7) yields a commutative diagram

\[
\begin{array}{cccccc}
H^4_c(U, \mathcal{T}') & \xrightarrow{\lambda_U} & H^4_c(U, G_m \otimes \mathbb{L} G_m) \\
\downarrow \lambda_U & & \downarrow \\
H^4_c(U, Z_\mathcal{X}/U(2)) & \xrightarrow{\lambda_U} & H^4_c(U, Z(2)) \cong \mathcal{O} / \mathbb{Z}. & & \\
\end{array}
\]

In particular, the bottom arrow maps $\lambda_U \ast \alpha_U^c$ to $\beta_U^c = -\gamma_3(\varepsilon'_U) \cdot \alpha_U^c$, which is equal to $-\langle \eta_U, \alpha_U^c \rangle_{\lambda_V}$ by (2.2.5). On the other hand, $\beta_U^c$ can be identified to an element of $\mathcal{H}^2(C_l)$. Passing to the direct limit over $U$ smaller and smaller, $\lambda_U \ast \alpha_U^c$ becomes $\lambda \ast \alpha = u(\alpha)$, and $\beta_U^c$ becomes the image of $u(\alpha)$ by the snake lemma construction applied to

\[
\begin{array}{cccccc}
H^4(K, Z(2)) & \xrightarrow{u(\alpha)} & H^4(X, Z(2)) & \xrightarrow{\text{Im } H^4(K, Z(2))} & 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \prod_{v \in \Omega^{(1)}} H^4(K^h_v, Z(2)) & \rightarrow & \prod_{v \in \Omega^{(1)}} H^4(X^h_v, Z(2)) & \rightarrow & \prod_{v \in \Omega^{(1)}} \frac{H^4(K^h_v, Z(2))}{\text{Im } H^4(K^h_v, Z(2))},
\end{array}
\]
with exact rows (where $X^h := X \times_K K^h$). Let us show that we may replace henselizations by completions in the above construction. Indeed, by [51, Chapter III, Remark 3.11],
$$H^3(O^h_{\Omega, v}, \mathbb{Q}/\mathbb{Z}(2)) \cong H^3(O_v, \mathbb{Q}/\mathbb{Z}(2)) \cong H^3(k(v), \mathbb{Q}/\mathbb{Z}(2)) = 0$$
since $cd(k(v)) = 2$. It follows, by the localization sequence in étale cohomology, that we have a chain of isomorphisms
$$H^4(K^h_v, \mathbb{Z}(2)) \cong H^3(K^h_v, \mathbb{Q}/\mathbb{Z}(2)) \cong H^2(k_v, \mathbb{Q}/\mathbb{Z}(2)) \cong H^3(K_v, \mathbb{Q}/\mathbb{Z}(2)) \cong H^4(K_v, \mathbb{Z}(2)).$$
It follows that $\beta^c_U$ becomes $\sigma(\beta)$ by taking limit, where $\beta \in \prod_{v \in \Omega(T)} H^4(K_v, \mathbb{Z}(2))$ is the image of $u(\alpha)$ by the snake lemma construction. By Lemma 2.3, one has
$$\rho_X(\alpha) = \sigma(\beta) = -\langle \eta, \alpha \rangle_{PT},$$
which concludes the proof of Proposition 2.2.

Proof of Theorem 2.1(i). If there exists a family $(P_v)_{v \in \Omega(T)}$ orthogonal to $u(\Xi^1(K, T'))$ relative to the pairing (1.4.9), then the $\rho_X(\alpha) = 0$ for all $\alpha \in \Xi^1(K, T')$. By Proposition 2.2 and the nondegeneracy of $\langle \cdot, \cdot \rangle_{PT}$, one has $e_X = 0$. The converse is obvious.

2.3 | Description of the obstruction using universal torsors

In this section, we prove Theorem 2.1(ii). When universal torsors exist, they give an explicit description of the map $u$ in (2.1.4) as in the following analog of [16, Lemme 3.5.2] (see also [66, Lemma 3]).

Proposition 2.4. Let $\pi : X \to \text{Spec} K$ be a smooth proper geometrically integral variety such that the abelian group $\text{Pic} \bar{X}$ is finitely generated and free. Let $T$ be the Néron–Severi torus of $X$. Suppose that $Y \to X$ is a universal torsor† under $T$. Then the map $u$ constructed in (2.1.4) is equal to the composite
$$H^1(K, T') \xrightarrow{[Y] \cup \pi^*(-)} H^4(X, \mathbb{Z}(2)) \to H^4(K, \mathbb{Z}(2)), $$
where the cup product $H^1(X, T) \times H^1(X, T') \cup \to H^4(X, \mathbb{Z}(2))$ is induced by the pairing (1.3.2).

Proof. Let $\alpha \in H^1(K, T') = \text{Ext}^1_K(\mathbb{Z}, T')$, which can be represented by a morphism $\mathbb{Z} \to T'[1]$ in $D^+(Ab)$. This morphism gives rise to the vertical arrows in the following commutative diagram in $D^+(Ab)$:

$$\begin{array}{c}
\mathbb{R} \text{Hom}_K(T', \mathbb{R} \pi_*Z_X(2)) \\
\downarrow \quad \downarrow \\
\mathbb{R} \text{Hom}_K(T', Z_{X/K}(2)) \\
\downarrow \quad \downarrow \\
\mathbb{H}(K, \mathbb{R} \pi_*Z_X(2))[1] \\
\downarrow \\
\mathbb{H}(K, Z_{X/K}(2))[1].
\end{array}$$

The horizontal arrows in (2.3.1) are induced by the map $\mathbb{R} \pi_*\mathbb{Z}(2) \to Z_{X/K}(2)$. In what follows, we shall make use of the fact that $\mathbb{R} \text{Hom}_K(T', -) = \mathbb{R} \text{Hom}_K(T', -) \circ \mathbb{R} \pi_*$ and $\mathbb{H}(X, -) =$

† By our convention, this means that its type is the identity of $\tilde{T}$.
$H(\mathbb{K},-)$ (see [71, Corollary 10.8.3]). We claim that there is a commutative diagram

$$
\begin{array}{ccc}
H^1(X, T) & \xrightarrow{\cong} & H^1(X, T) \\
\downarrow \cong & & \downarrow \cong \\
\Ext^1_X(\hat{T}, G_m) & \xleftarrow{\cong} & \Ext^1_X(\hat{T}, \tau_{\leq 1} \mathcal{R}\pi_* G_{m,X}) \\
\downarrow \otimes^L G_m & & \downarrow \otimes^L G_m \\
\Ext^1_X(T', G_m \otimes^L G_m) & \xleftarrow{\beta_*} & \Ext^1_X(T', (\tau_{\leq 1} \mathcal{R}\pi_* G_{m,X}) \otimes^L G_m) \\
\downarrow \alpha & & \downarrow \alpha \\
H^4(X, \mathbb{Z}(2)) & \xrightarrow{\cdot \pi^* \alpha} & H^4(K, \mathcal{R}\pi_* \mathbb{Z}(2)) \\
\end{array}
$$

where $\cdot$ means the Yoneda product, and where the maps $\beta_1, \beta_2$ were defined in the course of constructing the map $\beta$ from (2.1.7). To see this, let us consider the four rectangles of (2.3.2) from the top to the bottom. As for the first rectangle, the left bottom horizontal arrow is induced by the natural map $\tau_{\leq 1} \mathcal{R}\pi_* G_{m,X} \rightarrow \mathcal{R}\pi_* G_{m,X}$ (keeping in mind that $\Ext^1_K(\hat{T}, \mathcal{R}\pi_* G_{m,X}) = \Ext^1_X(\hat{T}, G_m)$), and the right bottom horizontal arrow is induced by the map from triangle (2.1.5). The commutativity of this rectangle and the established isomorphisms are well known; see, for example, [32, Proof of Proposition 8.1, Appendix B]. The second rectangle commutes by the functoriality of $- \otimes^L -$ (bearing in mind that $\Ext^1_K(T', \mathcal{R}\pi_* (G_{m,X} \otimes^L G_{m,X})) = \Ext^1_X(T', G_m \otimes^L G_m)$). As for the third rectangle, the left square obviously commutes (noting that, of course, $\Ext^3_X(T', \mathcal{R}\pi_* \mathbb{Z}(2)) = \Ext^3_X(T', \mathbb{Z}(2))$), and the right square is induced by diagram (2.1.8). As for the fourth rectangle, the left square obviously commutes, and the right square is obtained by taking cohomology of (2.3.1).

Let $Y \rightarrow X$ be a universal torsor. By our convention, its type is the identity of $\hat{T}$. Hence, the image of $[Y] \in H^1(X, T)$ in $H^4(K, \mathbb{Z}_X/K(2))$ by (2.3.2) is precisely $\lambda_* \alpha$. Under the identification (2.1.9), this is the same as $\mu(\alpha)$. Thus, in order to prove Proposition 2.4, it remains to show that the image of $[Y]$ in $H^4(X, \mathbb{Z}(2))$ by (2.3.2) is precisely $[Y] \cup \pi^* \alpha$. To this end, we argue as in the proof of Proposition 2.2 to obtain a commutative diagram

$$
\begin{array}{ccc}
H^1(X, \mathcal{H}om_X(\hat{T}, G_m)) & \xrightarrow{\cong} & \Ext^1_X(\hat{T}, G_m) \\
\downarrow \cong & & \downarrow \gamma_1 \\
H^1(X, T) & \xrightarrow{\gamma_1} & \Ext^1_X(\hat{T}, T \otimes^L \hat{T}) \\
\downarrow \cong & & \downarrow \gamma_1 \\
H^1(X, \mathcal{H}om_X(T', T \otimes^L T')) & \xrightarrow{\gamma_1} & \Ext^1_X(T', T \otimes^L T') \\
\downarrow \cong & & \downarrow \gamma_1 \\
\Ext^1_X(T', \mathbb{Z}(2)) & & \Ext^1_X(T', G_m \otimes^L G_m) \\
\end{array}
$$

where $\cdot$ means the Yoneda product, and where the maps $\gamma_1, \gamma_2$ were defined in the course of constructing the map $\gamma$ from (2.1.7).
similar to (2.2.2). Denote by \( \varepsilon \) the image of \([Y]\) in \( \text{Ext}_1^X(T', T \otimes \mathcal{L} T') \) by (2.3.3). Then, the image of \([Y]\) in \( \text{Ext}_1^X(T', G_m \otimes \mathcal{L} G_m) \) by (2.3.2) is precisely \( \gamma_2(\varepsilon) \). Now, we have a commutative diagram of pairings

\[
\begin{array}{c}
\text{H}^1(X, T) \\
\downarrow \\
\text{Ext}_X^1(T', T \otimes \mathcal{L} T') \\
\downarrow \gamma_1 \\
\text{Ext}_X^1(T', G_m \otimes \mathcal{L} G_m) \\
\downarrow \theta_1 \\
\text{Ext}_X^1(T', Z(2)) \\
\downarrow \\
\text{H}^1(X, T') \\
\end{array} \xrightarrow{\cup} \text{H}^2(X, T \otimes \mathcal{L} T') \xrightarrow{\cup} \text{H}^2(X, T \otimes \mathcal{L} T')
\]

(2.3.4)

This yields the identity \( \theta_1 \gamma_2(\varepsilon) \cdot \pi^* \alpha = [Y] \cup \pi^* \alpha \in \text{H}^4(X, Z(2)) \), which is exactly what we need. Proposition 2.4 is hence proved.

\[\square\]

Proof of Theorem 2.1(ii). We start with the inclusion “\( \subseteq \)”. Suppose that there exists a family \((P_v)_{v \in \Omega(1)}\) orthogonal to \( \text{Im}(u) \) relative to the pairing (1.4.9). By (i), we know that there exists a universal torsor \( f : Y \to X \). In the light of Proposition 2.4, we have

\[
\sum_{v \in \Omega(1)} [Y](P_v) \cup \text{loc}_v(\alpha) = \sum_{v \in \Omega(1)} ([Y] \cup \pi^* \alpha)(P_v) = 0 \in \mathbb{Q}/\mathbb{Z}
\]

(2.3.5)

for all \( \alpha \in \text{H}^1(K, T') \). Note that if \( \mathcal{X} \to U \) is a proper integral model of \( X \) over some nonempty open subset \( U \subseteq \Omega \), then \( X(K_v) = \mathcal{X}(\mathcal{O}_v) \) for all \( v \in U(1) \) by the valuative criterion for properness. Furthermore, shrinking \( U \) if necessary, we may assume that \( Y \) extends to a torsor \( Y \to \mathcal{X} \) under a \( U \)-torus \( \mathcal{T} \) extending \( T \). Thus, \([Y](P_v)\) comes from \([Y](P_v) \in \text{H}^1(\mathcal{O}_v, \mathcal{T}) \) for all \( v \in U(1) \), or \( ([Y](P_v))_{v \in \Omega(1)} \in \text{P}^1(K, T) \). By virtue of (2.3.5) and the exact sequence (1.5.11), there exists \( t \in \text{H}^1(K, T) \) such that \( \text{loc}_v(t) = [Y](P_v) \) for all \( v \in \Omega(1) \). Twisting by a Galois cocycle representing \( t \) yields a torsor \( f : Y \to X \) (i.e., \([t, Y] = [Y] - \pi^* t \in \text{H}^1(X, T) \)) such that \( P_v \in f(\{t, Y(K_v)\}) \) (see (1.3.6)) for all \( v \in \Omega(1) \). The torsor \( Y \) is again universal by the fundamental exact sequence (2.1.3). This proves the inclusion “\( \subseteq \)”. Conversely, if \( f : Y \to X \) is a universal torsor such that \((P_v)_{v \in \Omega(1)} \in f(\bigoplus_{v \in \Omega(1)} Y(K_v))\), then \([Y](P_v) = 0 \in \text{H}^1(K_v, T) \) for all \( v \in \Omega(1) \). This obviously implies the identity (2.3.5), which means \((P_v)_{v \in \Omega(1)}\) orthogonal to \( \text{Im}(u) \) by Proposition 2.4. This proves the inclusion “\( \supseteq \)”. \[\square\]

2.4 End of the proof of Theorem C

In this section, we finish the proof of Theorem 2.1 (i.e., Theorem C).

Proof of Theorem 2.1(iii). Suppose that there is a universal torsor \( Y \to X \) (otherwise, there would be nothing to prove). In view of the fundamental exact sequence (2.1.3)\(^\dagger\), we have to show that

\[\dagger\] Recall that a torsor is universal if its type is the identity of \( \mathcal{T} \).
there are only finitely many classes \( t \in H^1(K, T) \) for which \( \prod_{v \in \Omega(1)} t(Y_{K_v}) \neq \emptyset \). Equivalently, by (1.3.6), we have to show that the property

\[
\text{“there exists } (P_v)_{v \in \Omega(1)} \in \prod_{v \in \Omega(1)} X(K_v) \text{ with } [Y](P_v) = \text{loc}_v(t) \text{ for all } v \in \Omega(1)” \tag{2.4.1}
\]

holds for only finitely many classes \( t \in H^1(K, T) \). Let \( \mathcal{X} \to U \) be a proper integral model of \( X \) over some nonempty open subset \( U \subseteq \Omega \). Shrinking \( U \) if necessary, we may assume that \( T \) extends to a \( U \)-torus \( T’ \) and \( Y \) extends to a torsor \( Y’ \to \mathcal{X} \) under \( T’ \). Suppose that \( t \in H^1(K, T) \) satisfies (2.4.1).

For all \( v \in U(1) \), since \( X(K_v) = \mathcal{X}(\mathcal{O}_v) \) by the valuative criterion for properness, the class \( \text{loc}_v(t) \) comes from \( [Y](P_v) \in H^1(\mathcal{O}_v, T) \). On the other hand, we have an exact sequence

\[
H^1(U, T) \to \prod_{v \notin U} H^1(K_v, T) \times \prod_{v \in U(1)} H^1(\mathcal{O}_v, T) \to H^1(K, T’)^D,
\]

obtained in the course of establishing the exact sequence (1.5.11) (see [31, Proof of Proposition 3.5]). Since \( H^1(K, T) \) is orthogonal to \( H^1(K, T’)^D \) by the generalized Weil reciprocity law (1.4.5), we see that \( t \) comes from \( H^1(U, T) \). Hence, it suffices to show that \( X(K_v) = \mathcal{X}(\mathcal{O}_v) \) by the valuative criterion for properness, the class \( \text{loc}_v(t) \) comes from \( [Y](P_v) \in H^1(\mathcal{O}_v, T) \).

We conclude this section with the following interesting.

**Remark 2.5.** In fact, the image of the map \( u \) from (2.1.4) is contained in \( \text{Im}(H^4(X, \mathbb{Q}/\mathbb{Z}(2)) \to H^4(X, \mathbb{Q}/\mathbb{Z}(2))) \). Indeed, since the group \( H^1(K, T’) \) has finite exponent by Hilbert’s Theorem 90 and restriction-corestriction, it would be enough to show that the torsion elements of \( H^4(K, \mathbb{Q}/\mathbb{Z}(2)) \) come from \( H^4(X, \mathbb{Q}/\mathbb{Z}(2)) \). To see this, let \( A \in H^4(X, \mathbb{Z}(2)) \) and \( n \geq 1 \) such that \( nA = \pi^*c \) for some \( c \in H^4(K, \mathbb{Z}(2)) \), where \( \pi : X \to \text{Spec} K \) denotes the structure morphism. One has \( H^4(K, \mathbb{Z}/n(2)) = H^4(K, \mu_n^\otimes) = 0 \) because \( \text{cd}(K) \leq 3 \), hence \( c = nc’ \) for some \( c’ \in H^4(K, \mathbb{Z}(2)) \). It follows that the element \( A - \pi^*c’ \in H^4(X, \mathbb{Z}(2)) \) is \( n \)-torsion, and hence it comes from
H^3(X, \mathbb{Z}/n(2)), a fortiori from H^3(X, \mathbb{Q}/\mathbb{Z}(2)). Since H^4(K, \mathbb{Z}(2)) \cong H^3(K, \mathbb{Q}/\mathbb{Z}(2)), we conclude that A itself comes from H^3(X, \mathbb{Q}/\mathbb{Z}(2)).

Consequently, the statements (i), (ii), and (iv) of Theorem 2.1 can be refined as follows.

(i) Universal X-torsors exist if and only if the map \(\rho_X\) from (1.4.12) is the zero map.

(ii) A family \((P_v)_{v \in \Omega(1)} \in \prod_{v \in \Omega(1)} X(K_v)\) can be lifted to a universal torsor \(f : Y \to X\) if it is orthogonal to \(H^3(X, \mathbb{Q}/\mathbb{Z}(2))\).

(iv) If the universal torsors \(Y \to X\) satisfy the local–global principle (resp. the local–global principle and weak approximation), then the obstruction to the local–global principle (resp. the local–global principle and weak approximation) on \(X\), defined by the pairing (1.4.11), is the only one.

3 LOCAL–GLOBAL PRINCIPLE AND WEAK APPROXIMATION

This section is devoted to the proof of Theorems A and B. For each of these results, we offer two proofs. The first proofs invoke the results from Section 2 (Theorem 2.1(i) for the local–global principle and Proposition 2.4 for the weak approximation property). They are presented in Sections 3.1 and 3.2, respectively. The second proofs, which use the fibration methods, rely on an observation communicated to the author by Jean-Louis Colliot-Thélène. They will be presented in Section 3.3.

We also discuss some questions related to weak approximation in Section 3.4. In particular, we show that any finite abelian group is a Galois group over any \(p\)-adic function field, rediscovering the positive answer to the abelian case of the inverse Galois problem over \(\mathbb{Q}_p(t)\).

3.1 Local–global principle for stabilizers of type umult

We establish Theorem A in this section. First, recall some facts.

As a rule, the problem of existence of rational points on homogeneous spaces is harder than that of weak approximation. It requires the general machinery of liens (or bands, kernels) and nonabelian Galois cohomology, which has been systematically studied in the last 30 years [3, 22]. We refer to [25, §1] for a complete exposition. Let \(X\) be a homogeneous space of a smooth algebraic group \(G\) over a field \(K\). Let \(\overline{H}\) denote the stabilizer of a \(K\)-point of \(X\), which is supposed to be smooth. If \(\overline{H}\) is commutative, it has natural \(K\)-form \(H\). Otherwise, \(\overline{H}\) need not be defined over \(K\). Nevertheless, one can always define the associated Springer \(K\)-lien \(L_X\) (grosso modo, it is the \(K\)-group \(\overline{H}\) equipped with a natural outer Galois action, that is, a Galois action modulo conjugation), the set \(H^2(K, L_X)\) of nonabelian Galois 2-cohomology, and the Springer class \(\eta_X \in H^2(K, L_X)\).

The class \(\eta_X\) is neutral if and only if \(X\) is dominated by a principal homogeneous space of \(G\) (if \(H^1(K, G) = 1\), for example, when \(G\) is special, this is equivalent to \(X(K) \neq \emptyset\)).

Since the derived subgroup \([\overline{H}, \overline{H}]\) is characteristic in \(\overline{H}\), the canonical outer Galois action induces an action on \(\overline{H}^{ab} = \overline{H}/[\overline{H}, \overline{H}]\). Thus, we obtain a \(K\)-form \(H^{ab}\) of \(\overline{H}^{ab}\). Since every character of \(\overline{H}\) factors through \(\overline{H}^{ab}\), the group \(\text{Hom}_K(\overline{H}, \mathbb{G}_m) = \text{Hom}_K(\overline{H}^{ab}, \mathbb{G}_m)\) is equipped with a structure of \(\Gamma_K\)-module via this \(K\)-form of \(\overline{H}^{ab}\).

When \(H\) is a \(K\)-group, we denote by \text{lien}(H) the canonical \(K\)-lien associated with \(H\). If \(H\) is abelian, \(H^2(K, \text{lien}(H))\) is just the usual Galois cohomology group \(H^2(K, L)\), and its only neutral class is 0. Finally, a morphism \(L \to L'\) of algebraic \(K\)-liens induces a relation \(H^2(K, L) \to H^2(K, L')\).
This turns out to be a map if either the underlying $\overline{K}$-group of $L'$ is commutative or the underlying morphism between $\overline{K}$-groups is surjective.

The following description of the Picard groups of homogeneous spaces is due to Popov [58, Corollary to Theorem 4], see also [6] and [8, Theorem 5.8].

**Lemma 3.1.** Let $X$ be a homogeneous space a smooth, simply connected semisimple linear algebraic group $G$ over a field $K$, with smooth geometric stabilizers $H$. Then, as a $\Gamma_K$-module, $\text{Pic} X$ is isomorphic to the character group of $H$ (the Galois action on this group was defined above). The isomorphism is given by pushing forward the class $[G] \in H^0(K, \text{H}^1(X, H))$ of the torsor $G \to X$ under $H$.

**Proof.** Here, we used the fact that $\overline{K}[X]^\times = \overline{K}^\times$ because $\overline{K}[G]^\times = \overline{K}^\times$ by Rosenlicht’s lemma [59, Proposition 3], and $\text{Pic} \overline{G} = 0$ since $G$ is simply connected semisimple [70, §4.3].

Let $K$ be the function field of a smooth projective geometrically integral curve $\Omega$ over a $p$-adic field $k$, and let $X$ be a homogeneous space of a simply connected semisimple linear algebraic group $G$ over $K$, with geometric stabilizers $H$ of type $\text{unmult}$, hence an extension of a group of multiplicative type $M$ by a unipotent group $U$. Since $U$ (the unipotent radical of $H$) is characteristic in $H$, we have a natural Galois action on $M = H/U$ (hence a $K$-form $M$ of $M$). Since $U$ does not have any nontrivial characters, the character module of $H$ is just $\hat{M}$, hence $\text{Pic} \overline{X} = \hat{M}$ by Lemma 3.1. Let $X^c$ be a smooth projective compactification of $X$. Since $X^c$ is smooth, projective, and geometrically unirational, the abelian group $\text{Pic} \overline{X}^c$ is finitely generated and free (see the discussion preceding Theorem 2.1). There is an exact sequence

$$0 \to \text{Div}_{\infty} \overline{X} \to \text{Pic} \overline{X} \to \text{Pic} \overline{X}^c \to 0,$$

where $\text{Div}_{\infty} \overline{X}^c$ denotes the group of Weil divisors on $\overline{X}$ supported in $\overline{X} \setminus X$ (it is a permutation $\Gamma_K$-module). Note that the injectivity of $\text{Div}_{\infty} \overline{X}^c \to \text{Pic} \overline{X}^c$ follows from the fact that $\overline{K}[X]^\times = \overline{K}[G]^\times = \overline{K}^\times$ [59, Proposition 3]. Let $T$ (resp. $Q$) be the $K$-torus with character module $\text{Pic} \overline{X}$ (resp. $\text{Div}_{\infty} \overline{X}^c$). Then, $Q$ is quasi-split. We have exact sequences

$$0 \to \hat{Q} \to \hat{T} \to \hat{M} \to 0 \quad (3.1.1)$$

and

$$1 \to M \to T \to Q \to 1. \quad (3.1.2)$$

Let $M' = \hat{M} \otimes \mathbb{Z}(1)$. Applying the functor $- \otimes \mathbb{Z}(1)$ to (3.1.1) yields a distinguished triangle

$$M' \to Q' \to T' \to M'[1] \quad (3.1.3)$$

in $D^+(K)$. In particular, $M$ (resp. $M'$) is quasi-isomorphic to the complex $[T \to Q]$ (resp. $[Q' \to T']$) concentrated in degrees 0 and 1.

**Remark 3.2.** If $M = F$ is finite abelian, the map $\hat{T} \to \hat{Q}$ on cocharacter modules is injective. Hence, there is an exact sequence

$$1 \to F' \to Q' \to T' \to 1,$$

where $F' = \mathcal{H}om_K(F, \mathbb{Q}/\mathbb{Z}(2))$. In this case, we have a quasi-isomorphism $M' \cong F'$. 

We construct a map

$$\tau : \Sha^2_\omega(K, M') \to \frac{H^3_{nr}(K(X)/K, \mathbb{Q}/\mathbb{Z}(2))}{\text{Im } H^3(K, \mathbb{Q}/\mathbb{Z}(2))}$$

(3.1.4)
as follows. First, by the Gersten resolution (1.3.4), we have a map

$$\frac{H^4(X_c, \mathbb{Z}(2))}{\text{Im } H^4(K, \mathbb{Z}(2))} \to \frac{H^3_{nr}(K(X)/K, \mathbb{Q}/\mathbb{Z}(2))}{\text{Im } H^3(K, \mathbb{Q}/\mathbb{Z}(2))}.$$  

(3.1.5)

Furthermore, since the torus $Q'$ is quasi-split, one has $H^1(L, Q') = 0$ for any overfield $L/K$ and $\Sha^2_\omega(K, Q') = 0$ by Lemma 1.5. The long exact sequence associated with (3.1.3) gives

$$\Sha^2_\omega(K, M') \cong \Sha^1_\omega(K, T') \subseteq H^1(K, T').$$

(3.1.6)

More generally, one has

$$\Sha^2_S(K, M') \cong \Sha^1_S(K, T')$$

(3.1.7)

for any finite set $S \subseteq \Omega^{(1)}$. We define the map $\tau$ in (3.1.4) as the composite of (3.1.5), the map $u : H^1(K, T') \to \frac{H^4(X_c, \mathbb{Z}(2))}{\text{Im } H^4(K, \mathbb{Z}(2))}$ constructed in (2.1.4), and (3.1.6). This map $\tau$ will serve as an obstruction to the local–global principle and weak approximation for $X$.

**Remark 3.3.** According to Remark 2.5, the image of the map $\tau$ from (3.1.4) is contained in $\text{Im}(\frac{H^4(X_c, \mathbb{Q}/\mathbb{Z}(2))}{\text{Im } H^4(K, \mathbb{Q}/\mathbb{Z}(2))})$. We can now state the following

**Theorem 3.4 (Theorem A).** Let $K$ be the function field of a smooth projective geometrically integral curve $\Omega$ over a $p$-adic field $k$, and $X$ a homogeneous space of a special, simply connected semisimple algebraic group $G$ over $K$, with geometric stabilizers $H$ of type $\text{umult}$. We keep the above notations; in particular, there is a map $\tau$ as in (3.1.4). If there exists a family $(P_v)_{v \in \Omega^{(1)}} \in \prod_{v \in \Omega^{(1)}} X(K_v)$ orthogonal to $\tau(\Sha^2_\omega(K, M'))$ relative to the pairing (1.4.6), then $X(K) \neq \emptyset$. In particular, the unramified first obstruction (1.4.7) to the local–global principle for $X$ is the only one.

First, we deal with unipotent stabilizers using the following well-known result. It shall also serve in the proof of Theorem B.

**Lemma 3.5.** Let $K$ be a field of characteristic 0 and $G$ a special algebraic group over $K$. Then, homogeneous spaces of $G$ with unipotent geometric stabilizers have $K$-rational points. They have weak approximation if $G$ does.

**Proof.** Let $X$ be such a homogeneous space. The Springer class $\eta_X$ is neutral by [23, Chapitre IV, Théorème 1.3] (see also [3, Corollary 4.2]), and hence $X$ is dominated by a principal homogeneous space of $G$. Since $G$ is special, this means that there exists a $G$-equivariant morphism $\phi : G \to X$. In particular, $X(K) \neq \emptyset$.

Let $S$ be a finite set of places of $K$, $(P_v)_{v \in S} \in \prod_{v \in S} X(K_v)$, and $U_v \subseteq X(K_v)$ a neighborhood (for the local topology) of $P_v$, $v \in S$. Each fiber $\phi^{-1}(P_v)$ is a torsor under a unipotent $K_v$-group,
and hence has a $K_v$-point $Q_v$ by [61, Lemme 1.13]. If $G$ has weak approximation, we find a point $Q \in G(K)$ that belongs to $\prod_{v \in S} \phi^{-1}(\mathcal{U}_v)$. Then, $\phi(Q) \in X(K)$ belongs to $\prod_{v \in S} \mathcal{U}_v$. □

**Proof of Theorem 3.4.** Keep the notations as above. Denote by $\eta \in H^2(K, M)$ the element corresponding to the elementary obstruction $e_X \in \text{Ext}^2_K(\text{Pic} \overline{X}, \mathbb{G}_m) = \text{Ext}^2_K(\hat{M}, \mathbb{G}_m)$ under the identification (2.1.2) (recall that $\overline{K}[X]^\times = \overline{K}[G]^\times = \overline{K}^\times$ by [59, Proposition 3]). The projection $H \to \hat{M}$ induces a surjective morphism $L_X \twoheadrightarrow \text{lien}(M)$ of algebraic $K$-liens (recall that $L_X$ denotes the Springer lien of $X$), which, in turn, induces a map $H^2(K, L_X) \to H^2(K, M)$.

**Lemma 3.6.** The map $H^2(K, L_X) \to H^2(K, M)$ sends $\eta_X$ to $\eta$.

**Proof.** Actually, this result is valid for any ambient group $G$. It suffices to follow the proof of [67, Theorem 9.5.1]. This requires the description of the relation $H^2(K, L_X) \to H^2(K, M)$ in terms of gerbes (see, e.g., [22, §2.2]). According to [28, Chapitre IV, §3.2, Chapitre V, Propositions 3.1.6 and 3.2.1], $\eta$ is represented by the gerbe $G$ of universal $X$-torsors under $M$, that is, for every finite extension $L/K$, the fiber category $Q(L)$ is the groupoid of universal $X_L$-torsors under $M$. On the other hand, $\eta_X$ is represented by the gerbe $G_X$, whose fiber category $G_X(L)$ is for every finite extension $L/K$ the groupoid of $L$-torsors under $G$ dominating $X_L$ [28, Chapitre IV, §5.1]. If $Y$ is such an $L$-torsor (equipped with a $G_L$-equivariant dominant morphism $Y \to X_L$), let $H_Y = \text{Aut}_{G_L}(Y/X_L)$. Then, $H_Y$ is an algebraic subgroup of $G_L$, and $Y \to X_L$ is a torsor under $H_Y$ (see [67, §9.2]). In particular, $H_Y$ is an $L$-form of $H$. The contracted product $Z := Y \times^H_L M_L$ is an $X_L$-torsor under $M$, and the map $H^1(X, H) \to H^1(X, M)$ sends $[\mathcal{G}] = [\overline{Y}]$ to $[\overline{Z}]$. Combining with Lemma 3.1, we see that the identification $\text{Pic} \overline{X} = \hat{M}$ is given by pushing forward the class $[\overline{Z}] \in H^0(K, H^1(X, M))$, that is, the torsor $Z \to X_L$ has type id. The construction $Y \mapsto Z$ defines a morphism $\mathcal{G}_X \to \mathcal{G}$ of algebraic $K$-gerbes, and thus, $H^2(K, L_X) \to H^2(K, M)$ maps $\eta_X$ to $\eta$. □

Return to the proof of Theorem 3.4. Since (2.1.1) is functorial in $X$, the map $\text{Ext}^2_K(\hat{F}, \mathbb{G}_m) \to \text{Ext}^2_K(\hat{F}, \mathbb{G}_m)$ sends $e_X$ to $e_{X'}$. It follows that the map $H^2(K, M) \to H^2(K, T)$ sends $\eta$ to an element $\eta'$ corresponding to $e_{X'}$ (under the identification $H^2(K, T) \cong \text{Ext}^2_K(\hat{T}, \mathbb{G}_m)$ of (2.1.2)). If $(P_v)_{v \in \Omega(1)} \in \prod_{v \in \Omega(1)} X(K_v)$ is a family orthogonal to $\tau(\text{III}_X(K, M'))$, then, as a family in $\prod_{v \in \Omega(1)} X_v^0(K_v)$, it is orthogonal to $u(\text{III}_X(K, T'))$ by the construction of $\tau$ (where $u$ is the map constructed in (2.1.4)). Theorem 2.1(i) then implies that $e_{X'} = 0$, or $\eta' = 0$. On the other hand, since $H^1(K, Q) = 0$ (the torus $Q$ being quasi-split), the long exact sequence associated with (3.1.2) assures that $H^2(K, M) \to H^2(K, T)$ is injective. It follows that $\eta = 0$. By Lemma 3.6, the map $H^2(K, L_X) \to H^2(K, M)$ sends $\eta_X$ to the neutral class $0$. Now, [22, Théorème 3.4] provides a diagram

$$
\begin{array}{ccc}
\phi & & \psi \\
\downarrow & & \downarrow \\
X_1 & \leftrightarrow & X_2 \\
\end{array}
$$

where
- $X_1$ is a homogeneous space of $G \times_K \text{SL}_n$ with Springer lien $L_{X_1} \cong L_X$ and Springer class $\eta_{X_1} = \eta_X$,
- $X_2 = M \text{SL}_n$, for some embedding $M \hookrightarrow \text{SL}_n$ and some $n$,
- $\phi$ is a torsor under $\text{SL}_n$. 

\[30\]
– the fibers of $\psi$ are homogeneous spaces of $G$ with geometric stabilizers $\text{Ker}(H \to M) = U$.

Since $X_2(K) \neq \emptyset$, we have $X_1(K) \neq \emptyset$ by Lemma 3.5, hence $X(K) \neq \emptyset$. 

\section{Weak approximation for stabilizers of type umult}

In this section, we establish Theorem B. We start by recalling the following well-known result, which already appeared in [10] (see also [29, 49] for finite subgroups of $\text{SL}_n$). We give a proof here for the sake of reference.

**Lemma 3.7.** Let $G$ be a smooth algebraic group over a field $K$, $H$ a smooth Zariski closed subgroup of $G$, and $X = H \backslash G$. The projection $G \to X$ is then a torsor under $H$. For any finite set $S$ of places of $K$, if a family $(P_v)_{v \in S} \subseteq \prod_{v \in S} X(K_v)$ lies in the closure (for the product of local topologies) of the diagonal image of $X(K)$, then $[(G)(P_v)]_{v \in S}$ belongs to the image of the localization $H^1(K, H) \to \prod_{v \in S} H^1(K_v, H)$. The converse holds if $G$ is special and has weak approximation.

**Proof.** Suppose that $(P_v)_{v \in S}$ lies in the closure of $X(K)$. By Lemma 1.1(iii), for each $v \in S$, there exists a neighborhood $\mathcal{U}_v \subseteq X(K_v)$ of $P_v$ such that $[G](P'_v) = [G](P_v)$ for all $P'_v \in \mathcal{U}_v$. Let $P \in \prod_{v \in S} \mathcal{U}_v$ be a $K$-point. Then, the element $[G](P) \in H^1(K, H)$ satisfies $[G](P_v) = \text{loc}_v([G](P))$ for all $v \in S$.

Conversely, suppose that there exists $h \in H^1(K, H)$ such that $\text{loc}_v(h) = [G](P_v)$ for all $v \in S$. Since $G$ is special, the evaluation-at-$[G]$ map $X(L) \to H^1(L, H)$ induces a bijection $X(L)/G(L) \cong H^1(L, H)$ for any overfield $L/K$. In particular, we may write $h = [G](P)$ for some $P \in X(K)$. For each $v \in S$, let $\mathcal{U}_v \subseteq X(K_v)$ be a neighborhood of $P_v$, and let $\mathcal{V}_v$ denote its preimage by the continuous map

$$G(K_v) \to X(K_v), \quad g_v \mapsto P \cdot g_v.$$ 

Since $\text{loc}_v([G](P)) = \text{loc}_v(h) = [G](P_v)$, there exists $g_v \in G(K_v)$ such that $P \cdot g_v = P_v$, hence $\mathcal{V}_v \neq \emptyset$. Under the hypothesis that $G$ has weak approximation, there exists $g \in G(K)$ that belongs to $\prod_{v \in S} \mathcal{V}_v$. Then $P \cdot g \in X(K)$ belongs to $\prod_{v \in S} \mathcal{U}_v$. □

**Remark 3.8.** Taking $G = \text{SL}_n$ in Lemma 3.7, we see that weak approximation for the quotient $H/\text{SL}_n$ is an intrinsic property of the algebraic $K$-group $H$ (independent of the embedding $H \hookrightarrow \text{SL}_n$). In fact, if $H \hookrightarrow \text{SL}_n$ and $H \hookrightarrow \text{SL}_m$ are two embeddings, the quotient varieties $H/\text{SL}_n$ and $H/\text{SL}_m$ are $K$-stably birational by the “no-name lemma” [18, §3.2].

Let $K$ be the function field of a smooth projective geometrically integral curve $\Omega$ over a $p$-adic field $k$. Let $H$ be a $K$-group of type umult, hence an extension of a $K$-group of multiplicative type $M$ by a unipotent group $U$. Let $X = H \backslash G$ for some embedding $H \hookrightarrow G$ into a simply connected semisimple linear algebraic group $G$ over $K$. By Lemma 3.1, $\text{Pic} \overline{X} = \hat{H} = \hat{M}$ as $\Gamma_K$-modules. As in Section 3.1, let $X^c$ be a smooth projective compactification of $X$, $T$ the $K$-torus with $\hat{T} = \text{Pic} \overline{X}^c$, and $Q = T/M$ (it is a quasi-split $K$-torus). Finally, let $M' = M \otimes^L \mathbb{Z}(1)$ (it is represented by a 2-term complex of tori).
Since \( X^c(K) \neq \emptyset \), by the fundamental exact sequence (2.1.3), there exists a universal torsor \( Y^c \to X^c \) under \( T \). By Proposition 2.4, the map \( \tau \) from (3.1.4) is the composite
\[
\begin{array}{c}
\text{III}_\omega^2(K, M') \cong \text{III}_\omega^1(K, T') \\
\begin{array}{c}
\xrightarrow{[Y^c] \cup (-)_{X^c}} \xrightarrow{H^4(X^c, \mathbb{Z}(2))} \xrightarrow{H^3_{nr}(K(X)/K, \mathbb{Q}/\mathbb{Z}(2))} \xrightarrow{\text{Im} H^3(K, \mathbb{Q}/\mathbb{Z}(2))} \xrightarrow{\text{Im} H^3(K, \mathbb{Q}/\mathbb{Z}(2))}
\end{array}
\end{array}
\] (3.2.1)

where the subscript \( X^c \) denotes pullback along the structure morphism \( X^c \to \text{Spec} K \), and the last arrow is induced by (1.3.4). Using the alternative description (3.2.1), we shall prove the following “functoriality” property of \( \tau \), which is required when we apply the fibration method.

**Lemma 3.9.** The map \( \tau \) enjoys the following properties.

(i) \( \tau \) does not depend on the choice of the universal torsor \( Y^c \).

(ii) \( \tau \) does not depend on the choice of the smooth projective compactification \( X^c \).

(iii) Let \( G, G_1 \) be simply connected semisimple linear algebraic groups over \( K \). Let \( H, H_1 \) be \( K \)-groups of type \text{unmult}, equipped with respective embeddings into \( G, G_1 \), and let \( X = H \setminus G \), \( X_1 = H_1 \setminus G_1 \). Let \( M, M_1 \) be the respective parts of multiplicative type of \( H, H_1 \), and let \( \tau, \tau_1 \) be the respective maps constructed in (3.1.4). Assume that there exists a morphism \( \varphi : G \to G_1 \) and a \( \varphi \)-equivariant dominant morphism \( \phi : X \to X_1 \). Then, there is a commutative diagram

\[
\begin{array}{c}
\text{III}_\omega^2(K, M') \xrightarrow{\tau_1} \xrightarrow{H^3_{nr}(K(X)/K, \mathbb{Q}/\mathbb{Z}(2))} \xrightarrow{\text{Im} H^3(K, \mathbb{Q}/\mathbb{Z}(2))} \xrightarrow{\text{Im} H^3(K, \mathbb{Q}/\mathbb{Z}(2))}
\end{array}
\]

**Proof.** We prove (i). If \( Y^c_1 \to X^c \) is a second universal torsor, then \( [Y^c_1] = [Y^c] + t_{X^c} \) in \( H^1(X^c, T) \) for some \( t \in H^1(K, T) \) by virtue of the fundamental exact sequence (2.1.3). This yields a map \( H^1(K, T') \to H^4(X^c, \mathbb{Z}(2)) \) given by \( \alpha \mapsto [Y^c] \cup \alpha_{X^c} + (t \cup \alpha)_{X^c} \). Since \( t \cup \alpha \) is an element of \( H^4(K, \mathbb{Z}(2)) \cong H^3(K, \mathbb{Q}/\mathbb{Z}(2)) \), by the description (3.2.1), we see that \( \tau \) remains unchanged when \( Y \) is replaced by \( Y_1 \).

We prove (ii). Let \( X^c_1 \) be a second smooth projective compactification of \( X \). Let \( T_1 \) be the \( K \)-torus with \( \hat{T}_1 = \text{Pic} \overline{X}^c_1 \), and \( Y_1^c \to X^c_1 \) a universal torsor. Then, there is an exact sequence
\[
0 \to Q_1 \to \hat{T}_1 \to \hat{M} \to 0
\]
similar to (3.1.1), where \( Q_1 \) is a quasi-split torus. Thus, there are \( K \)-tori \( R, R_1, T_2 \) such that \( R, R_2 \) are quasi-split and \( T \times_K R \cong T_1 \times R_1 \cong T_2 \), and the diagram

\[
\begin{array}{c}
\hat{T}_2 \\
\xrightarrow{\hat{T}_1} \\
\hat{M} = \text{Pic} \overline{X}
\end{array}
\]

(3.2.2)

\(^\dagger\) Recall our convention: a torsor under \( T \) is universal if its type is exactly the identity of \( \hat{T} \) (not just an isomorphism).
commutes. Let $Y \to X, Y_1 \to X$ denote the respective restrictions of $Y^c \to X^c$ and $Y_1^c \to X_1^c$, which are torsors whose types are, respectively, given by the bottom arrows of (3.2.2). Since $X(K) \neq \emptyset$, by the fundamental exact sequence (2.1.3), there exists a torsor $Y_2 \to X$ under $T_2$ whose type $\tilde{T}_2 \to \text{Pic } \overline{X}$ is given by either of the two composites in (3.2.2). By the very same sequence, the image of $[Y_2]$ in $H^1(X, T)$ (resp. in $H^1(X, T_1)$) has the form $[Y] + t_X$ (resp. $[Y_1] + t_1_{X'}$) for some $t \in H^1(K, T)$ (resp. $t_1 \in H^1(K, T_1)$). Twisting the $X^c$-torsor $Y^c$ (resp. $Y_1^c$) by $t$ (resp. by $t_1$) yields another universal torsor; this is something allowed by (i). Hence, we may assume that the image of $[Y_2]$ in $H^1(X, T)$ (resp. in $H^1(X, T_1)$) is precisely $[Y]$ (resp. $[Y_1]$). One then obtains a commutative diagram

$$
\begin{array}{cccccc}
H^1(K, T') & \xrightarrow{[Y^c] \cup (-)_{X^c}} & H^4(X^c, Z(2)) & \xrightarrow{H^3_{\text{nr}}(K(X)/K, Q/Z(2))} \\
\downarrow \cong \hspace{1cm} \downarrow \cong & \hspace{1cm} & \downarrow \cong & \hspace{1cm} \downarrow \cong \\
H^1(K, T'_1) & \xrightarrow{[Y_1^c] \cup (-)_{X_1^c}} & H^4(X_1^c, Z(2)) & \xrightarrow{H^3_{\text{nr}}(K(X)/K, Q/Z(2))} \\
\end{array}
$$

(the two squares on the right-hand side are (1.3.5)). By the description (3.2.1), this shows that the compactifications $X^c$ and $X_1^c$ yield the same map $\tau$.

We prove (iii). By Nagata’s Theorem [53], there exists a compactification $\phi^c : X^c \to X_1^c$ of $\phi$. Let $T$ (resp. $T_1$) be the $K$-torus with $\tilde{T} = \text{Pic } \overline{X}$ (resp. $\tilde{T}_1 = \text{Pic } \overline{X}_1$). Then, $\phi^c$ induces a morphism $\psi : T \to T_1$ of $K$-tori. Let $Y^c \to X^c$ (resp. $Y_1^c \to X_1^c$) be a universal torsor under $T$ (resp. under $T_1$). Then both the contracted product $Y^c \times^T \overline{X}_1$ and the pullback $Y_1^c \times_{X_1^c} X^c$ are $X$-torsors under $T_1$ of type $\psi^* : \tilde{T}_1 \to \tilde{T}$. Again, by the fundamental exact sequence (2.1.3), the images in $H^1(X^c, T_1)$ of $[Y^c] \in H^1(X^c, T)$ and $[Y_1^c] \in H^1(X_1^c, T_1)$ differ by a class $t_{1_{X^c}}$, where $t_1 \in H^1(K, T_1)$. Twisting the $X_1^c$-torsor $Y_1^c$ by $t_1$ (which does not modify the map $\tau$, thanks to (i)), we may assume that these images coincide. Cup-products with this common value then yield the oblique arrow in the diagram

$$
\begin{array}{cccccc}
H^1(K, T'_1) & \xrightarrow{[Y_1^c] \cup (-)_{X_1^c}} & H^4(X_1^c, Z(2)) & \xrightarrow{H^3(K(X_1)/K, Q/Z(2))} \\
\downarrow \phi^* & \hspace{1cm} & \downarrow \phi^* \\
H^1(K, T') & \xrightarrow{[Y^c] \cup (-)_{X^c}} & H^4(X^c, Z(2)) & \xrightarrow{H^3(K(X)/K, Q/Z(2))} \\
\end{array}
$$

Since the two triangles commute, the square on the left-hand side also commutes. The square on the right-hand side commutes thanks to the functoriality of the map (1.3.4). By the description (3.2.1), the commutative diagram in the statement of (iii) is established. \hfill \square

In the course of establishing duality theorems between $M$ and $M'$, Izquierdo defined a pairing

$$
M \otimes^\phi M' \to Z(2)[1] \quad (3.2.3)
$$
(see [36, Lemme 4.3]). By its very construction, the pairing (3.2.3) is functorial and generalizes (1.3.2). In particular, these pairings (for $Q, T,$ and $M$) are compatible with respect to the exact sequence (3.1.2) and triangle (3.1.3).

To establish Theorem B in the case of stabilizers of multiplicative type (i.e., when $H = M$), the first step is to reinterpret the alternative description (3.2.1) of the map $\tau$.

**Lemma 3.10.** Suppose that $H = M$ is a $K$-group of multiplicative type, so we have a class $[G] \in H^1(X, M)$ of the torsor $G \to X$ under $M$. Consider the cup-product

$$H^1(X, M) \times H^2(X, M') \to H^4(X, \mathbb{Z}(2))$$

induced by the pairing (3.2.3). Then (up to a sign) for all $\alpha \in \mathbb{W}_2(\omega, M')$, the image of the class $[G] \cup \alpha_X \in H^4(X, \mathbb{Z}(2))$ in $H^4(K, \mathbb{Z}(2)) \cong H^3(K, \mathbb{Q}/\mathbb{Z}(2))$ coincides with $\tau(\alpha)$.

**Proof.** Let $Y^c \to X^c$ be a universal torsor† and let $Y \to X$ be its restriction to $X$, which is a torsor under $T$ whose type is the map $\hat{T} \to \hat{M} = \text{Pic}\overline{X}$ from (3.1.1). Since the (universal) torsor $G \to X$ under $M$ has type id by Lemma 3.1, the fundamental exact sequence (2.1.3) assures that the map $H^1(X, M) \to H^1(X, T)$ sends $[G]$ to $[Y] + t_X$ for some $t \in H^1(K, T)$. Twisting $Y^c$ by $t$ (which does not change the map $\tau$, by Lemma 3.9(i)), we may assume that the image of $[G]$ in $H^1(X, T)$ is $[Y]$. Since the pairing (3.2.3) is functorial, the diagram

$$
\begin{array}{ccc}
H^1(K, T') & \xrightarrow{[Y^c] \cup (-)_{X^c}} & H^4(X^c, \mathbb{Z}(2)) \\
\downarrow & & \downarrow \\
H^2(K, M') & \xrightarrow{[G] \cup (-)_{X}} & H^4(X, \mathbb{Z}(2)) \\
\downarrow & & \downarrow \\
\end{array}
$$

$$
\begin{array}{ccc}
\cong H^3(K(X), \mathbb{Q}/\mathbb{Z}(2)) \\
\cong H^3(K(X), \mathbb{Q}/\mathbb{Z}(2)) \\
\end{array}
$$

commutes (except the left triangle, which commutes up to a sign). Indeed, the square on the right-hand side is (1.3.5). By the description (3.2.1) of $\tau$, the lemma is proved. 

The next step is to establish following analog of the exact sequences (1.5.4) and (1.5.12).

**Lemma 3.11.** For each finite set $S \subseteq \Omega^{(1)}$ of closed points, there is an exact sequence

$$H^1(K, M) \to \prod_{v \in S} H^1(K_v, M) \xrightarrow{\theta} \mathbb{W}_S^2(K, M')^D \to \mathbb{W}_S^2(K, M')^D \to 0,$$

where the map $\theta$ is defined by $\theta((m_v)_{v \in S})(\alpha) = \sum_{v \in S} (m_v \cup \text{loc}_v(\alpha))$. Here, the local cup-products $H^1(K_v, M) \times H^2(K_v, M') \to H^4(K_v, \mathbb{Z}(2)) \cong \mathbb{Q}/\mathbb{Z}$ are induced by the pairing (3.2.3).

† By our convention, this means that its type is the identity of $\hat{T}$.
Proof. Consider the commutative diagram

\[
\begin{array}{ccccccc}
Q(K) & \longrightarrow & H^1(K, M) & \longrightarrow & H^1(K, T) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\prod_{v \in S} Q(K_v) & \longrightarrow & \prod_{v \in S} H^1(K_v, M) & \longrightarrow & \prod_{v \in S} H^1(K_v, T) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\{\Xi^2(K, M')\}^D & \cong & \{\Xi^1(K, T')\}^D & & \Xi^2(K, M') & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & & & 
\end{array}
\]

(3.2.5)

with exact rows (the two top rows are the exact sequences associated with (3.1.2), noting that $H^1(L, Q) = 0$ for any overfield $L/K$ since $Q$ is quasi-split). The two bottom horizontal arrows are isomorphisms since $H^1(L, Q') = 0$ for any overfield $L/K$ and $\Xi^2(K, Q') = \Xi^2(K, Q') = 0$ by Lemma 1.5. That (3.2.5) commutes follows from the functoriality of (3.2.3). The right column is exact because it is (1.5.12). The left vertical arrow has dense image (the torus $Q$ has weak approximation because it is $K$-rational). For each $v \in S$, the map $Q(K_v) \to H^1(K_v, M)$ is the evaluation induced by the class $[T] \in H^1(Q, M)$ of the torsor $T \to Q$ under $M$, and hence, it is locally constant by Lemma 1.1(iii). A diagram chasing then shows that the left column (i.e., the sequence (3.2.4)) is exact.

\[\square\]

**Theorem 3.12.** Let $K$ be the function field of a smooth projective geometrically integral curve $\Omega$ over a $p$-adic field $k$, $M$ a $K$-group of multiplicative type, and let $X = M \setminus G$, where $M \hookrightarrow G$ is some embedding into a special, simply connected semisimple algebraic group $G$ over $K$ that has weak approximation. Let $M' = \hat{M} \otimes^L \mathbb{Z}(1)$, $\tau$ the map constructed in (3.1.4), and $S \subseteq \Omega^{(1)}$ a finite set of closed points. Then any family $(P_v)_{v \in S} \in \prod_{v \in S} X(K_v)$ orthogonal to $\tau(\{\Xi^2_S(K, M')\})$ relative to the pairing (1.4.8) lies in the closure of the diagonal image of $X(K)$. Moreover, $X$ has weak approximation in $S$ if and only if $\Xi^2_S(K, M') = \Xi^2(K, M')$.

**Proof.** Let $(P_v)_{v \in S} \in \prod_{v \in S} X(K_v)$ be a family orthogonal to $\tau(\{\Xi^2_S(K, M')\})$ relative to the pairing (1.4.8). This means $\sum_{v \in S} \alpha_v = 0$ for any $\alpha \in \Xi^2_S(K, M')$ and any lifting $A \in H^3_{nr}(K(X)/K, Q/\mathbb{Z}(2))$ of $\tau(\alpha)$. By virtue of Lemma 3.10, this is equivalent to

\[
\sum_{v \in S} [G](P_v) \cup \text{loc}_v(\alpha) = \sum_{v \in S} ([G] \cup \alpha_X)(P_v) = 0
\]

for all $\alpha \in \Xi^2_S(K, M')$. Using the exact sequence (3.2.4), we see that $([G](P_v))_{v \in S}$ lies in the image of the localization $H^1(K, M) \to \prod_{v \in S} H^1(K_v, M)$. But then $(P_v)_{v \in S}$ lies in the closure of the diagonal image of $X(K)$ by Lemma 3.7.

Any family $(P_v)_{v \in S} \in \prod_{v \in S} X(K_v)$ is orthogonal to $\tau(\{\Xi^2(K, M')\})$ because this group consists of everywhere locally constant classes and $X(K) \neq \emptyset$. It follows that $X$ has weak approximation.
in $S$ whenever $\Pi^2(K, M') = \Pi^2_S(K, M')$. Conversely, suppose that $\Pi^2(K, M') \subsetneq \Pi^2_S(K, M')$. Then the map $\Pi^2_S(K, M')^D \to \Pi^2(K, M')^D$ has nontrivial kernel. Again, exactness of $(3.2.4)$ implies the existence of a family $(m_v)_{v \in S} \in \prod_{v \in S} H^1(K_v, M)$ whose image in $\Pi^2_S(K, M')^D$ is nonzero, that is, $(m_v)_{v \in S}$ does not come from $H^1(K, M)$. But since $G$ is special, for each $v \in S$, there exists $P_v \in X(K_v)$ such that $m_v = [G](P_v)$. By Lemma 3.7, the family $(P_v)_{v \in S}$ does not lie in the closure of $X(K)$; thus, $X$ fails weak approximation in $S$. This concludes the proof of the theorem. 

Finally, we extend Theorem 3.12 to the main theorem of this section by allowing the stabilizers to have a unipotent part. The proof uses the fibration method.

Theorem 3.13 (Theorem B). Let $K$ be the function field of a smooth projective geometrically integral curve $\Omega$ over a $\mathfrak{p}$-adic field $k$, $H$ a linear algebraic $K$-group extension of a group multiplicative type $M$ by a unipotent group $U$, and $X = H \backslash G$ for some embedding $H \hookrightarrow G$ into a special, simply connected semisimple linear algebraic group $G$ that has weak approximation. Let $M' = \hat{M} \otimes \mathbb{Z}(1)$, the map constructed in (3.1.4), and $S \subseteq \Omega^{(1)}$ a finite set of closed points. Then, any family $(P_v)_{v \in S} \in \prod_{v \in S} X(K_v)$ orthogonal to $\tau(\Pi^2_S(K, M'))$ relative to the pairing (1.4.8) lies in the closure of the diagonal image of $X(K)$. In particular, the reciprocity obstruction to weak approximation for $X$ is the only one. Moreover, $X$ has weak approximation in $S$ if and only if $\Pi^2_S(K, M') = \Pi^2(K, M')$.

Proof. Choose an embedding $M \hookrightarrow SL_n$ for some $n$. The embedding $H \hookrightarrow G$ and the composite $H \to M \hookrightarrow SL_n$ yield a diagonal embedding $H \hookrightarrow G \times_K SL_n$. We have a diagram

$$\begin{array}{ccc}
X_1 & \xrightarrow{\phi} & X_2 \\
\downarrow \psi & & \downarrow \phi \\
X & & X
\end{array}$$

(3.2.6)

where $X_1 = H \backslash (G \times_K SL_n)$, $X_2 = M \backslash SL_n$, $\phi$ is a torsor under $SL_n$, and the fibers of $\psi$ are homogeneous spaces of $G$ with geometric stabilizers $\operatorname{Ker}(H \to M) = \overline{U}$. Let $\tau, \tau_1, \tau_2$ be the respective maps associated with $X, X_1, X_2$ via the construction (3.1.4). By Lemma 3.9(iii), diagram (3.2.6) yields a commutative diagram

$$\begin{array}{ccc}
\Pi^2_S(K, M') & \xrightarrow{\psi} & \Pi^2(K, M') \\
\downarrow \tau_1 & & \downarrow \tau \\
H^1_\omega(K(X)/K, Q/\mathbb{Z}(2)) & \xrightarrow{\phi^*} & H^1_\omega(K(X)/K, Q/\mathbb{Z}(2)) \\
\downarrow \psi^* & & \downarrow \psi^* \\
H^1_\omega(K(X)/K, Q/\mathbb{Z}(2)) / \operatorname{Im} H^1(K, Q/\mathbb{Z}(2)) & & H^1_\omega(K(X)/K, Q/\mathbb{Z}(2)) / \operatorname{Im} H^1(K, Q/\mathbb{Z}(2)).
\end{array}$$

(3.2.7)

Since $H^1(K(X), SL_n) = 1$ by a variant of Hilbert’s Theorem 90, the generic fiber of $\phi$ has a section. The extension $K(X_1)/K(X)$ is thus purely transcendental, and hence $\phi^*$ is an isomorphism.

Let $(P_v)_{v \in S} \in \prod_{v \in S} X(K_v)$ be orthogonal to $\tau(\Pi^2_S(K, M'))$, and let $\mathcal{U}_v \subseteq X(K_v)$ be a $v$-adic neighborhood of $P_v$ for each $v \in S$. Since $H^1(K_v, SL_n) = 1$, each fiber $\phi^{-1}(P_v)$ has a $K_v$-point $Q_v$. In view of (3.2.7) (note that $\phi^*$ is an isomorphism), the family $(Q_v)_{v \in S} \in \prod_{v \in S} X_1(K_v)$ is orthogonal to $\tau_1(\Pi^2_S(K, M'))$. Then, $(\psi(Q_v))_{v \in S} \in \prod_{v \in S} X_2(K_v)$ is orthogonal to $\tau_2(\Pi^2_S(K, M'))$. By Serre’s generalized version of the implicit function theorem [63, Part II, Chapter III, §10.2], we find for each $v \in S$ a $v$-adic neighborhood $\mathcal{V}_v \subseteq X_2(K_v)$ of $\psi(Q_v)$ whose $K_v$-points can be lifted...
to $K_v$-points in $\phi^{-1}(\mathcal{U}_v) \subseteq X_1(K_v)$. Applying Theorem 3.12 to $X_2 = M \setminus \text{SL}_n$ yields a $K$-point $R \in X_2(K)$ belonging to $\prod_{v \in S} \mathcal{U}_v$. Then, the fiber $\phi^{-1}(R)$ contains a family $(Q'_v)_{v \in S} \in \prod_{v \in S} \phi^{-1}(\mathcal{U}_v)$. By Lemma 3.5, $\psi^{-1}(R)$ contains a $K$-point $Q \in \prod_{v \in S} \phi^{-1}(\mathcal{U}_v)$. Then $\phi(Q) \in X(\mathcal{K})$ belongs to $\prod_{v \in S} \mathcal{U}_v$.

Since $X(K) \neq \emptyset$, any family $(P_v)$ is orthogonal to $\tau(\Omega_2^2(K,M'))$ (a subgroup consisting of everywhere locally constant classes). It follows that $X$ has weak approximation in $S$ whenever $\Omega_2^2(S(K, M')) = \Omega_2^2(K, M')$. Conversely, suppose that $X$ has weak approximation in $S$. We show that it is also the case for $X_2$. Indeed, let $(R_v)_{v \in S} \in \prod_{v \in S} X_2(K_v)$ and let $\mathcal{U}_v \subseteq X_2(K_v)$ be a $v$-adic neighborhood of $R_v$ for each $v \in S$. By Lemma 3.5, each fiber $\phi^{-1}(R_v)$ contains a $K_v$-point $Q_v$. By Serre’s generalized version of the implicit function theorem [63, Part II, Chapter III, §10.2], we find for each $v \in S$ a $v$-adic neighborhood $\mathcal{U}_v \subseteq X(\mathcal{K}_v)$ of $\phi(Q_v)$ whose $K_v$-points can be lifted to $K_v$-points in $\psi^{-1}(\mathcal{U}_v) \subseteq X_1(K_v)$. By our assumption on $X$, there is a $K$-point $P \in X(\mathcal{K})$ belonging to $\prod_{v \in S} \mathcal{U}_v$. Then, the fiber $\phi^{-1}(P)$ contains a family $(Q'_v)_{v \in S} \in \prod_{v \in S} \phi^{-1}(\mathcal{U}_v)$. Since $G$ is special and has weak approximation, the fiber $\phi^{-1}(P)$ has a $K$-point $Q \in \prod_{v \in S} \phi^{-1}(\mathcal{U}_v)$. Then, $\phi(Q) \in X_2(K)$ belongs to $\prod_{v \in S} \mathcal{U}_v$. Hence, $X_2$ has weak approximation in $S$. By Theorem 3.12, one has $\Omega_2^2(S(K, M')) = \Omega_2^2(K, M')$. □

### 3.3 Alternative proofs

The idea of the “fibration method” at the end of the proofs of Theorems 3.4 and 3.13 can be applied in an alternative way. They can be used to show that any homogeneous space of a special, $K$-rational algebraic group is $K$-stably birational to a $K$-torsor under a torus. If such a homogeneous space has a $K$-rational point, it is $K$-stably birational to a torus. This observation allows us to obtain Theorems 3.14 and 3.15 in this section, which are variants of Theorems 3.4 and 3.13, respectively (of course, they also imply Theorems A and B, respectively).

**Theorem 3.14 (Theorem A).** Let $K$ be the function field of a smooth projective geometrically integral curve $\Omega$ over a $p$-adic field $k$, and $X$ a homogeneous space of a special, $K$-rational algebraic group $G$, with geometric stabilizers $H$ of type $\text{umult}$. Let $M$ denote the natural $K$-form of the part of multiplicative type $\widehat{M}$ of $H$ and $M' = \widehat{M} \otimes \mathbb{Z}(1)$. Then, $X$ is $K$-stably birational to a $K$-torsor under a torus. Moreover, there exists a map

$$\tau_1 : \Omega_2^2(K, M') \to \frac{H^3_{\text{nr}}(K(X)/K, \mathcal{O}/\mathbb{Z}(2))}{\text{Im} \left( H^4(K, \mathcal{O}/\mathbb{Z}(2)) \right)}$$

(3.3.1)

with the following property. If there exists a family $(P_v)_{v \in \Omega(1)} \in \prod_{v \in \Omega(1)} X(K_v)$ orthogonal to $\tau_1(\Omega_2^2(K, M'))$ relative to the pairing (1.4.6), then $X(K) \neq \emptyset$. In particular, the unramified first obstruction (1.4.7) to the local–global principle for $X$ is the only one.

**Proof.** Consider the Springer $K$-lien $L_X$ and the Springer class $\eta_X \in H^2(K, L_X)$. As in the proof of Theorem 3.4, we have a map $H^3(K, L_X) \to H^2(K, M)$. Let $\eta \in H^2(K, M)$ denote the image of $\eta_X$ by this map. The first step is to find an embedding $i : M \hookrightarrow Q$ into a quasi-split torus such that $i_* \eta = 0 \in H^2(K, Q)$. This can be done as follows. Let $L/K$ be a finite extension such that $X(L) \neq \emptyset$, then the restriction of $\eta_X$ to $H^2(L, L_X)$ is neutral. Consequently, the restriction of $\eta$ to $H^2(L, M)$ is 0. Choose an embedding $i : M_L \hookrightarrow Q_1$ into a quasi-split $L$-torus, and let $\alpha : M \hookrightarrow \text{Res}_{L/K} M_L$ be
the canonical inclusion (where Res\(_{L/K}\) denotes the restriction of scalars à la Weil). The diagram

\[
\begin{array}{ccc}
H^2(K, M) & \xrightarrow{\text{can}} & H^2(K, \text{Res}_{L/K} M_L) \\
\downarrow\cong & & \downarrow(\text{Res}_{L/K} i)_* \\
H^2(L, M) & \xrightarrow{i_*} & H^2(L, Q_1)
\end{array}
\]

where the vertical arrows are the isomorphisms from Shapiro’s lemma, commutes. Indeed, its triangle commutes by [54, Proposition 1.6.5]. Let Q = Res\(_{L/K}\) Q\(_1\) (which is a quasi-split K-torus) and \(\iota = (\text{Res}_{L/K} i)_*\) can : M ↪ Q, then \(\iota_* \eta = 0\) as desired.

Let T be the cokernel of \(\iota\) (it is a K-torus). By the long exact sequence associated with

\[
1 \rightarrow M \xrightarrow{\iota} Q \rightarrow T \rightarrow 1,
\]

we know that \(\eta\) comes from \(H^1(K, T)\). If \(\{m_{\sigma, \tau}\}_{\sigma, \tau}\) is a Galois 2-cocycle representing \(\eta\), then there is a 1-cochain \(\{q_\sigma\}\) with coefficients in \(Q(K)\) such that \(\iota(m_{\sigma, \tau}) = q_\sigma q_\tau^{-1}\) for all \(\sigma, \tau \in \Gamma_K\). Its image \(\{t_\sigma\}\) in \(T\) is a 1-cocycle, whose class \([t]\) \(\in H^1(K, T)\) maps to \(\eta \in H^2(K, M)\). Let \(Z = -_T T\) be the K-torsor under \(T\) corresponding to the cocycle \(-\iota\), that is, \(Z(\overline{K}) = T(\overline{K})\) and the twisted Galois action on \(Z(\overline{K})\) is given by

\[
\cdot : \Gamma_K \times Z(\overline{K}) \rightarrow Z(\overline{K}), \quad (\sigma, z) \mapsto \sigma \cdot z := e^{\sigma z t_\sigma}.
\]

The action of \(T\) on \(Z\) (by multiplication in \(T(\overline{K}) = Z(\overline{K})\)) makes \(Z\) a homogeneous space of \(Q\) with geometric stabilizers \(\overline{M}\). We shall show that the Springer lien \(L_Z\) is isomorphic to liem\(_\sigma(M)\), and the Springer class \(\eta_Z \in H^2(K, M)\) is precisely \(\eta\). To this end, we invoke the description of \(L_Z\) and \(\eta_Z\) in terms of cocycles (see [22, §2.2.2] or [25, §5]). Indeed, for each \(\sigma \in \Gamma_K\), its action on \(Q(\overline{K})\) restricts to the usual Galois action on \(M(\overline{K})\), so that \(L_Z = \text{lien}(M)\). Next, fix the point \(1 \in T(\overline{K}) = Z(\overline{K})\). Then (3.3.3) yields

\[
\sigma \cdot 1 = 1 t_\sigma = 1 \cdot q_\sigma,
\]

where \(\cdot : Z \times Q \rightarrow Z\) denotes the action of \(Q\) on \(Z\) induced by that of \(T\). We have \(q_\sigma q_\tau q_\tau^{-1} = \iota(m_{\sigma, \tau})\) for all \(\sigma, \tau \in \Gamma_K\), and hence, \(\eta_Z\) is represented by the 2-cocycle \(\{m_{\sigma, \tau}\}_{\sigma, \tau}\), that is, \(\eta_Z = \eta\).

To conclude, the map \(H^2(K, L_X) \rightarrow H^2(K, M)\) sends \(\eta_X\) to \(\eta_Z\). Applying [22, Théorème 3.4], we obtain a diagram

\[
\begin{array}{ccc}
& X_1 & \\
\phi & \downarrow & \psi \\
X & & Z,
\end{array}
\]

where

- \(X_1\) is a homogeneous space of \(G \times_K Q\) with Springer lien \(L_{X_1} \cong L_X\) and Springer class \(\eta_{X_1} = \eta_X\),
- \(\phi\) is a torsor under \(Q\),
- the fibers of \(\psi\) are homogeneous spaces of \(G\) with geometric stabilizers \(\operatorname{Ker}(H \rightarrow M) =: \overline{U}\).
Since $G$ is special, $\phi$ has a $K$-rational section. Since $G$ is $K$-rational, the extension $K(X_1)/K(X)$ is purely transcendental. Thus, $X$ is $K$-stably birational to $X_1$. Next, by Lemma 3.5, the generic fiber of $\psi$ is isomorphic to $U\setminus G_{K(Z)}$, where $U \subseteq G_{K(Z)}$ is a unipotent Zariski closed subgroup. Since both $U$ and $G_{K(Z)}$ are $K(Z)$-rational (for $U$, this is because its exponential map is a birational isomorphism onto an affine space), the field extensions $K(G)/K(X_1)$ and $K(G)/K(Z)$ are purely transcendental. It follows that $X_1$ (hence also $X$) is $K$-stably birational to $Z$. In particular, $\phi$ and $\psi$ induce isomorphisms between $H^3_{nr}(K(X)/K, \mathbb{Q}/\mathbb{Z}(2))$, $H^3_{nr}(K(X_1)/K, \mathbb{Q}/\mathbb{Z}(2))$, and $H^3_{nr}(K(Z)/K, \mathbb{Q}/\mathbb{Z}(2))$.

In the course of establishing their theorem on the local–global principle for torsors under $K$-tori [34, Theorem 5.1], Harari and Szamuely constructed the first arrow in the composite

$$
\tau_2 : H^2(K, T') \to \frac{H^3(Z, Q/Z(2))}{\text{Im} H^3(K, Q/Z(2))} \to \frac{H^3(K(Z), Q/Z(2))}{\text{Im} H^3(K, Q/Z(2))},
$$

and Tian showed in his thesis that $\tau_2(\mathfrak{II}^2_\omega(K, T'))$ lies in the subgroup $\frac{H^3_{nr}(K(Z)/K, Q/Z(2))}{\text{Im} H^3(K, Q/Z(2))}$ [68, Corollary 1.4.5]. The map $\tau_2$ enjoys the following property. If there is a family of local points of $Z$ orthogonal to $\tau_2(\mathfrak{II}^2_\omega(K, T'))$ relative to the pairing (1.4.6), then $Z(K) \neq \emptyset$.

Applying the functor $- \otimes \mathbb{Z}(1)$ to the exact sequence

$$0 \to \hat{T} \to \hat{Q} \to \hat{M} \to 0$$

dual to (3.3.2), one obtains a distinguished triangle

$$M' \to T' \to Q' \to M'[1]. \quad (3.3.6)$$

Since $H^1(L, Q') = 0$ for any overfield $L/K$ and since $\mathfrak{II}^2_\omega(K, Q') = 0$ by Lemma 1.5, the long exact sequence associated with (3.3.6) yields an isomorphism

$$\mathfrak{II}^2_\omega(K, M') \cong \mathfrak{II}^2_\omega(K, T'). \quad (3.3.7)$$

More generally, one has $\mathfrak{II}^2_\omega(K, M') \cong \mathfrak{II}^2_\omega(K, T')$ for any finite set $S \subseteq \Omega(1)$ (in particular, $\mathfrak{II}^2(K, M') \cong \mathfrak{II}^2(K, T')$). If we define the map $\tau_1$ in (3.3.1) as the composite

$$\mathfrak{II}^2_\omega(K, M') \cong \mathfrak{II}^2_\omega(K, T') \xrightarrow{\tau_2} \frac{H^3_{nr}(K/Z, Q/Z(2))}{\text{Im} H^3(K, Q/Z(2))} \cong \frac{H^3_{nr}(K(X_1)/K, Q/Z(2))}{\text{Im} H^3(K, Q/Z(2))} \cong \frac{H^3_{nr}(K(X)/K, Q/Z(2))}{\text{Im} H^3(K, Q/Z(2))}.$$ 

Then, $\tau_1$ has the stated property. Indeed, suppose that there is a family $(P_v)_{v \in \Omega(1)} \in \prod_{v \in \Omega(1)} X(K_v)$ orthogonal to $\tau_1(\mathfrak{II}^2_\omega(K, M'))$. Since $G$ is special, we may lift $(P_v)_{v \in \Omega(1)}$ to a family $(Q_v)_{v \in \Omega(1)} \in \prod_{v \in \Omega(1)} X_1(K_v)$. Then, $(\psi(Q_v))_{v \in \Omega(1)} \in \prod_{v \in \Omega(1)} Z(K_v)$ is orthogonal to $\tau_2(\mathfrak{II}^2_\omega(K, T'))$, hence $Z(K) \neq \emptyset$. By Lemma 3.5, one has $X_1(K) \neq \emptyset$, hence $X(K) \neq \emptyset$. \qed

For the problem of weak approximation, the above proof is actually simpler, because the involved homogeneous spaces already have $K$-rational points.

**Theorem 3.15 (Theorem B).** Let $K$ be the function field of a smooth projective geometrically integral curve $\Omega$ over a $p$-adic field $k$, $H$ a $K$-group of type $\text{umult}$, and $X = H \setminus G$ for some embedding $H \hookrightarrow G$ into a special, $K$-rational algebraic group $G$. Let $M$ denote the part of multiplicative type of $H$, and $M' = \hat{M} \otimes \mathbb{Z}(1)$. Then, $X$ is $K$-stably birational to a torus. Furthermore, let $\tau_1$ be the
map \( \varPi_\omega^2(K, M') \to \frac{H^3_m(K(X)/K, \mathbb{Q}/\mathbb{Z}(2))}{\text{Im } H^3(K, \mathbb{Q}/\mathbb{Z}(2))} \) constructed as in (3.3.1), and \( S \subseteq \Omega^{(1)} \) a finite set of closed points. Then any family \( (P_v)_{v \in S} \in \prod_{v \in S} X(K_v) \) orthogonal to \( \tau_1(\varPi_\omega^2(K, M')) \) relative to the pairing (1.4.8) lies the closure of the diagonal image of \( X(K) \). In particular, the reciprocity obstruction to weak approximation for \( X \) is the only one. Moreover, \( X \) has weak approximation in \( S \) if and only if \( \varPi_\omega^2(S, M') = \varPi_\omega^2(K, M') \).

**Proof** (after J.-L. Colliot-Thélène). Following the proof of Theorem 3.14, let \( M \hookrightarrow Q \) be an embedding into a quasi-split torus. The embedding \( H \hookrightarrow G \) and the composite \( H \rightarrow M \hookrightarrow Q \) yield a diagonal embedding \( H \hookrightarrow G \times_K Q \). This gives us a diagram

\[
\begin{array}{ccc}
X_1 & \xrightarrow{\phi} & X \\
\downarrow{\psi} & & \downarrow{\psi} \\
X & & T,
\end{array}
\]

where \( X_1 = H \setminus (G \times_K Q) \), \( T = Q/M \) (it is \( K \)-torus), \( \phi \) is a torsor under \( Q \), and the fibers of \( \psi \) are homogeneous spaces of \( G \) with geometric stabilizers \( \text{Ker}(H \to M) =: \overline{U} \). Note that \( X \) is \( K \)-stably birational to \( T \).

In his thesis, Tian [68, Proposition 1.3.1] showed that the restriction

\[
\varPi_\omega^2(K, T') \to \frac{H^3_m(K(T)/K, \mathbb{Q}/\mathbb{Z}(2))}{\text{Im } H^3(K, \mathbb{Q}/\mathbb{Z}(2))}
\]

of the map (3.3.5) coincides with the construction using “flasque resolution” by Harari, Scheiderer, and Szamuely, which serves in the proof of their theorem on weak approximation for \( K \)-tori [31, Theorem 5.2]. Actually, combining this with Theorem 4.3(a) in loc. cit. gives us a more precise statement, that any family \( (P_v)_{v \in S} \in \prod_{v \in S} T(K_v) \) orthogonal to \( \tau_2(\varPi_\omega^2(K, T')) \) lies in the closure of the diagonal image of \( T(K) \). Moreover, \( T \) has weak approximation in \( S \) if and only if \( \varPi_\omega^2(S, T') = \varPi_\omega^2(K, T') \). By repeating the fibration argument as in the proof of Theorem 3.13, one sees that the map \( \tau_1 \) defined in (3.3.1) has the stated property. Moreover, \( X \) has weak approximation in \( S \) if and only if \( \varPi_\omega^2(S, M') = \varPi_\omega^2(K, M') \) (since \( \varPi_\omega^2(K, T') \simeq \varPi_\omega^2(S, K, M') \) and \( \varPi_\omega^2(K, T') \simeq \varPi_\omega^2(K, M') \)).

### 3.4 Examples

Let us discuss some corollaries to Theorem 3.13. As always, let \( K \) be the function field of a smooth projective geometrically integral curve \( \Omega \) over a \( p \)-adic field \( k \). Recall that when \( M \) is a \( K \)-group of multiplicative type, \( M' = M \otimes \mathbb{Z}(1) \) is quasi-isomorphic to a 2-term complex of tori fitting into the distinguished triangle (3.1.3). If \( M = F \) is finite abelian, then \( M' \) is quasi-isomorphic to \( F' = \text{Hom}_K(F, \mathbb{Q}/\mathbb{Z}(2)) \) (see Remark 3.2). In what follows, if \( H \) is a \( K \)-group of type \texttt{umult}, then \( M \) denotes its part of multiplicative type.

**Corollary 3.16.** Let \( H \) be a \( K \)-group of type \texttt{umult} and \( X = H \setminus \text{SL}_n \) for some embedding \( H \hookrightarrow \text{SL}_n \). Then, \( X \) has weak approximation if and only if \( \varPi_\omega^2(K, M') = \varPi_\omega^2(K, M') \).

For example, by Lemma 1.5, one has \( \varPi_\omega^2(K, \mathbb{G}_m) = 0 \), hence \( \varPi_\omega^2(K, \mu_n) = \varPi_\omega^2(K, \mu_n) = 0 \) since \( \varPi_\omega^2(K, \mu_n) \hookrightarrow \varPi_\omega^2(K, \mathbb{G}_m) \) by the Kummer sequence and Hilbert’s Theorem 90. Thus, the variety
μₙ \ SLₙ is weak approximation. By Lemma 3.7, the map
H₁(K, μₙ) → ∏ᵥ∈S H₁(Kᵥ, μₙ) is surjective for all finite sets S ⊆ Ω(1). Nonetheless, this follows easily from Kummer theory and the Artin–Whaples approximation theorem.

**Proposition 3.17.** Let H be a K-group of type umult and X = H \ SLₙ for some embedding H ↪ SLₙ. There is an infinite set S₀ ⊆ Ω(1) in which X has weak approximation.

**Proof.** Let L/K be a finite extension splitting the torus T′ in the triangle (3.1.3). It corresponds to a branched cover f : Ω′ → Ω of smooth projective geometrically integral curves over k. For any nonempty open subset U ⊆ Ω, a result of Poonen [57, Corollary 2] assures the existence of a closed point w ∈ f⁻¹(U) such that k(w) = k(f(w)). Hence, there are infinitely many points w ∈ Ω′ having this property. If, moreover, f is unramified over f(w), then L_w = K_f(w). Thus, the set S₀ of closed points v ∈ Ω for which there exists a point w ∈ f⁻¹(v) with L_w = K_v is infinite.

We claim that Ω²(K, M') = Ω²(K, M') for any finite set S ⊆ S₀ (which would conclude the proof by virtue of Theorem 3.13). Indeed, let α ∈ Ω²(K, M'), that is, loc_v(α) = 0 for any v ∉ S. For v ∈ S, we have by definition of S₀ a closed point w ∈ Ω' lying over v such that L_w = K_v. We deduce from (3.1.6) and Hilbert’s Theorem 90 that Ω²(L, M') ⊆ H²(L, T') = 0; thus, the restriction of α to H²(L, M') is 0. But then loc_v(α) = 0 since H²(K_v, M') = H²(L_w, M'). It follows that α ∈ Ω²(K, M').

A theorem of Harbater [35] says that every finite group is a Galois group over Q_p(t). His original proof involves the technique of patching. There are several other proofs by Liu [48], Colliot-Thélène [13], and Kollár [44, 45]. As remarked by Colliot-Thélène himself, the inverse Galois problem for a group G (viewed as a finite constant group) over number fields can be reduced to the question of weak weak approximation (see below) on G \ SLₙ. Using this idea, we show the following.

**Corollary 3.18.** Any finite abelian group is a Galois group over K.

**Proof.** Let F be a finite abelian group, which can be seen as a finite constant K-group scheme. By Proposition 3.17, the variety F \ SLₙ (for some embedding F ↪ SLₙ) has weak approximation in some infinite set S₀ ⊆ Ω(1). By Lemma 3.7, this means that the localization map H¹(K, F) → ∏ᵥ∈S H¹(Kᵥ, F) is surjective for every finite subset S ⊆ S₀. Let v : F → S₀ be any injective map. For each x ∈ F, choose a continuous homomorphism cₓ : Γᵥ(x) → F whose image contains x; this is possible because we have surjections Γᵥ(x) → Γ_k(v(x)) → ̂Z (the residue field of k(v(x)) being a finite field, its absolute Galois group is ̂Z), then it is enough to inflate the continuous homomorphism ̂Z → F mapping 1 to x. Each homomorphism cₓ is an element of H¹(Kᵥ(x), F). Then, there exists c ∈ H¹(K, F) such that locᵥ(x)(c) = cₓ for all x ∈ F. Thus, c is a continuous homomorphism Γ_K → F whose image contains every element of F, that is, it is surjective. Its kernel is Γ_f for some finite Galois extension L of K, and Gal(L/K) = Γ_K/Γ_f ≅ F.

Nevertheless, the abelian case of the regular inverse Galois problem over Q (hence over any field of characteristic 0 by a base change argument) is known long before Harbater’s theorem (see, e.g., [65, §4.2])¹.

¹ The author would like to thank Olivier Wittenberg for this remark.
We say that a smooth \( K \)-variety \( X \) (with \( X(K) \neq \emptyset \)) satisfies the weak weak approximation property (resp. countable weak weak approximation property) if it has weak approximation away from a finite (resp. countable) set \( S_0 \subseteq \Omega^{(1)} \). This means that \( X \) has weak approximation in every finite set \( S \subseteq \Omega^{(1)} \) with \( S \cap S_0 = \emptyset \).

**Corollary 3.19.** Let \( H \) be a \( K \)-group of type \text{umult} and \( X = H \setminus \text{SL}_n \), for some embedding \( H \hookrightarrow \text{SL}_n \). Then, \( X \) satisfies the weak weak approximation property (resp. countable weak weak approximation property) if and only if \( \mathcal{I}^2_\omega(K,M') \) is finite (resp. countable).

**Proof.** Suppose that \( \mathcal{I}^2_\omega(K,M') \) is finite (resp. countable). Let \( S_0 \) be the set of closed points \( v \in \Omega^{(1)} \) such that there exists \( \alpha \in \mathcal{I}^2_\omega(K,M') \) with \( \text{loc}_v(\alpha) \neq 0 \). Then \( S_0 \) is finite (resp. countable). For any finite set \( S \subseteq \Omega^{(1)} \) with \( S \cap S_0 = \emptyset \), one has \( \mathcal{I}^2_S(K,M') = \mathcal{I}^2(K,M') \), thus \( X \) has weak approximation in \( S \) by Theorem 3.13.

Conversely, suppose that \( X \) has weak approximation away from a finite (resp. countable) set \( S_1 \subseteq \Omega^{(1)} \). By Theorem 3.13, we have \( \mathcal{I}^2_S(K,M') = \mathcal{I}^2(K,M') \) for any finite set \( S \subseteq \Omega^{(1)} \) disjoint from \( S_1 \). We deduce from this the exactness of the bottom row of the diagram (induced by the distinguished triangle \((3.1.3))

\[
\begin{array}{ccc}
0 & \longrightarrow & \mathcal{I}^1(K,T') \\
& \downarrow & \downarrow \\
& \mathcal{I}^1_\omega(K,T') & \longrightarrow & \bigoplus_{v \in S_1} \text{H}^1(K_v,T')
\end{array}
\]

Indeed, any element \( \alpha \in \mathcal{I}^2_\omega(K,M') \) lies in \( \mathcal{I}^2_S(K,M') \) for some finite set \( S \subseteq \Omega^{(1)} \). If the image of \( \alpha \) in \( \bigoplus_{v \in S_1} \text{H}^2(K_v,M') \) is 0, then \( \alpha \in \mathcal{I}^2_{S \setminus S_1}(K,M') = \mathcal{I}^2(K,M') \). Since the left and the middle vertical arrows of the above diagram are isomorphisms by \((3.1.7))\), a diagram chasing shows that the top row is also exact. Parts of local duality \((1.5.5))\) and global duality \((1.5.9))\) say that the groups \( \mathcal{I}^1(K,T') \) and \( \text{H}^1(K_v,T') \) are finite, and hence \( \mathcal{I}^2_\omega(K,M') \cong \mathcal{I}^1_\omega(K,T') \) is finite (resp. countable).

Note that in the case where \( k \) is a number field and \( F \) is a finite \( \Gamma_k \)-module, the defect of weak approximation on the quotient \( F \setminus \text{SL}_n \) (for some embedding \( F \hookrightarrow \text{SL}_n \)) is explained by the group \( \mathcal{I}^1_\omega(k,\hat{F}) \). This group is always finite by an application of Chebotarev’s density theorem and the inflation-restriction sequence, and hence weak weak approximation always holds. However, showing the finiteness (or the countability) of \( \mathcal{I}^2_\omega(K,F') \) for a \( p \)-adic function field \( K \) seems to be a difficult task in general. In fact, we have the following.

**Proposition 3.20.** There exists a finite Galois module over \( K = \mathbb{Q}_p(t) \) such that \( \mathcal{I}^2_\omega(K,F) \) is uncountable.

**Proof.** First, we find a \( K \)-torus \( Q \) such that \( \mathcal{I}^1_\omega(K,Q) \) is uncountable. It suffices to follow the proof of [31, Proposition 4.5]. Indeed, with the notations therein, one constructs a constant torus

\[\text{The term “countable” means “finite or countably infinite.”} \]
Q with $\mathbb{III}_\omega(K, Q) = H^1(K, Q)$, and, for each $b \in \mathbb{Q}_p = \mathbb{A}^1_{\mathbb{Q}_p}(\mathbb{Q}_p)$, an element $A_b \in H^1(K, Q)$ whose “residue” at $b$ (see Remark 4.6 in loc. cit.) is nonzero. On the other hand, every element of $H^1(K, Q)$ has nonzero residues at only finitely many points $b \in \mathbb{Q}_p$ (because it comes from some open subset $U \subseteq \mathbb{P}^1_{\mathbb{Q}_p}$). Since $\mathbb{Q}_p$ is uncountable, if $H^1(K, Q)$ was countable, there would exist $b \in \mathbb{Q}_p$ at which every element of $H^1(K, Q)$ has vanishing residue, which is a contradiction. Thus, $\mathbb{III}_\omega(K, Q) = H^1(K, Q)$ must be uncountable.

By Ono’s lemma [56, Theorem 1.5.1], there are an integer $n \geq 1$, quasi-split $K$-tori $R, S$, and an isogeny $R \to Q^n \times_K S$. Let $F$ be the kernel of this isogeny. Since $H^1(L, R) = H^1(L, S) = 0$ for any overfield $L/K$ and $\mathbb{III}_2^\omega(K, R) = 0$ by Lemma 1.5, the long exact sequence associated with

$$0 \to F \to R \to Q^n \times_K S \to 1$$

gives $\mathbb{III}_2^\omega(K, F) \cong \mathbb{III}_2^\omega(K, Q^n \times_K S) \cong \mathbb{III}_\omega^1(K, Q)^n$, which is uncountable. □

Let $F$ be as in Proposition 3.20, $F' = \mathbb{H}om_K(F, \mathbb{Q}/\mathbb{Z}(2))$, and $X = F' \backslash \text{SL}_n$ for some embedding $F' \hookrightarrow \text{SL}_n$. By Corollary 3.19, $X$ fails the countable weak weak approximation property, a fortiori it fails the weak weak approximation property and the weak approximation property. It would be interesting to see if weak weak approximation is strictly weaker than weak approximation, and if countable weak weak approximation is strictly weaker than weak weak approximation. By Corollaries 3.16 and 3.19, this is equivalent to the following questions.

**Question 1.** Let $K$ be a $p$-adic function field. Does there exist a finite $\Gamma_K$-module $F$ such that $\mathbb{III}_\omega^2(K, F)$ is finite but $\mathbb{III}_\omega^2(K, F) \not\subseteq \mathbb{III}_\omega^2(K, F)$?

**Question 2.** Let $K$ be a $p$-adic function field. Does there exist a finite $\Gamma_K$-module $F$ such that $\mathbb{III}_\omega^2(K, F)$ is countably infinite?

Another closely related property is the hyperweak approximation [29, §4]. Let $F$ be a finite (not necessarily commutative) $K$-group, which extends to a finite étale group scheme $F \to U$ over a nonempty open subset $U \subseteq \Omega$. Then, we say that $F$ satisfies the hyperweak approximation property if there is a nonempty open subset $V \subseteq U$ such that for every finite set $S \subseteq V^{(1)}$, the image of the localization $H^1(K, F) \to \prod_{v \in S} H^1(K_v, F)$ contains $\prod_{v \in S} H^1(O_v, F)$. By Lemma 3.7, if the variety $F \backslash \text{SL}_n$ (for some embedding $F \hookrightarrow \text{SL}_n$) has weak approximation, then $F$ satisfies the hyperweak approximation property. It turns out that the converse holds when $F$ is commutative.

**Lemma 3.21.** If $F$ is a finite $\Gamma_K$-module satisfying the hyperweak approximation property, then $F \backslash \text{SL}_n$ (for some embedding $F \hookrightarrow \text{SL}_n$) satisfies the weak weak approximation property.

**Proof.** Indeed, if $F \backslash \text{SL}_n$ fails the weak weak approximation property, then $\mathbb{III}_\omega^2(K, F')$ is infinite by Corollary 3.19. The same argument from its proof also gives an exact sequence

$$0 \to \lim_{\overline{S} \subseteq V^{(1)}} \mathbb{III}_\omega^2(S, F') \to \mathbb{III}_\omega^2(K, F') \to \bigoplus_{v \in V} H^2(K_v, F')$$

for any nonempty open subset $V \subseteq U$. Since $H^2(K_v, F')$ is finite for each $v \not\in V$ and since $\Omega \setminus V$ is finite, $\lim_{\overline{S} \subseteq V^{(1)}} \mathbb{III}_\omega^2(S, F')$ is infinite. On the other hand, there is an exact sequence

$$H^2(V, F') \to \prod_{v \in V} H^2(K_v, F') \times \prod_{v \in V^{(1)}} H^2(O_v, F') \overset{\delta}{\to} H^1(K, F)^D,$$
where the map $\theta$ is defined by $\theta((\alpha_v)_{v \in \Omega(1)})(f) = \sum_{v \in \Omega(1)} (\text{loc}_v(f) \cup \alpha_v)$ (see the proof of [38, Proposition 2.6]). By the generalized Weil reciprocity law (1.4.5), an element $\alpha \in H^2(K, F')$ satisfies $\text{loc}_v(\alpha) \in H^2(O_v, F')$ for all $v \in V(1)$ precisely when it comes from $H^2(V, F')$. Since the group $H^2(V, F')$ is finite, there exists a finite set $S \subseteq V(1)$, a closed point $v_0 \in V(1)$, and an element $\alpha \in \text{III}_S^2(K, F')$ such that $\text{loc}_{v_0}(\alpha) \notin H^2(\hat{O}_{v_0}, F')$ (in particular, $v_0 \in S$). Since $H^1(O_v, F)$ and $H^2(O_{v_0}, F')$ are exact annihilators of each other under the cup-product pairing

$$H^1(K_{v_0}, F) \times H^2(K_{v_0}, F') \to H^3(K_{v_0}, \mathbb{Q}/\mathbb{Z}(2)) \cong \mathbb{Q}/\mathbb{Z},$$

we find an element $f_{v_0} \in H^1(O_{v_0}, F)$ such that $f_{v_0} \cup \text{loc}_{v_0}(\alpha) \neq 0$. Let $f_v = 0$ for $v \in S \setminus \{v_0\}$. Then, the family $(f_v)_{v \in S} \in \prod_{v \in S} H^1(O_v, F)$ is not orthogonal to $\text{III}_S^2(K, F')$. In view of the exact sequence (1.5.4), this family does not come from $H^1(K, F)$, and hence, $F$ fails the hyperweak approximation property. □

The following vanishing result was communicated to the author by Jean-Louis Colliot-Thélène. Recall that a $K$-group of multiplicative type $M$ is said to be split over a finite extension $L/K$ if $\Gamma_L$ acts trivially on $\hat{M}$. A finite group is said to be metacyclic if all of its Sylow subgroups are cyclic.

**Proposition 3.22.** If $F$ is a finite $\Gamma_K$-module split by a metacyclic extension, then $\text{III}_\omega^2(K, F) = 0$.

**Proof.** Let $L/K$ be a finite Galois extension splitting $F$, with metacyclic Galois group $G = \text{Gal}(L/K)$. The “coflasque resolution” [17, Lemma 0.6] provides an exact sequence

$$0 \to \hat{T} \to \hat{Q} \to \hat{F} \to 0$$

of finitely generated $G$-modules, where $\hat{T}$ and $\hat{Q}$ are free as abelian groups, where $\hat{Q}$ is permutation, and where $\hat{T}$ is coflasque (i.e., $H^1(H, \hat{T}) = 0$ for all subgroups $H \subseteq G$). Since $G$ is metacyclic, a theorem of Endo–Miyata [24, Theorem 1.5] says that $\hat{T}$ is a direct factor of a permutation module. By dualizing, we obtain an exact sequence

$$1 \to F \to Q \to T \to 1,$$

where $Q$ is a quasi-split torus and $T$ is a direct factor of a quasi-split torus. Since $H^1(K', T) = 0$ for any overfield $K'/K$ and $\text{III}_\omega^2(K, Q) = 0$ by Lemma 1.5, one has $\text{III}_\omega^2(K, F) = 0$. □

**Corollary 3.23.** Let $F$ be a finite constant abelian group of exponent not divisible by 8 and $F' = \mathcal{H}om_K(F, \mathbb{Q}/\mathbb{Z}(2))$. Then, $\text{III}_\omega^2(K, F') = 0$.

**Proof.** We may assume that $F = \mathbb{Z}/n$, where $n$ is either 2, 4, or a power of an odd prime. In all cases, the extension $K(\mu_n)/K$ is cyclic and splits $F' = \mu_n^{\otimes 2}$ (since $\hat{F}' = \mu_n^{\otimes (-1)}$); hence, Proposition 3.22 implies $\text{III}_\omega^2(K, \mu_n^{\otimes 2}) = 0$. □

As for the group $\mathbb{Z}/2^m$, where $m \geq 3$, we cannot apply Proposition 3.22 unless the extension $K(\mu_{2^m})/K$ is cyclic. Nevertheless, we always have $\text{III}_\omega^2(K(\sqrt{-1}), \mu_{2^m}^{\otimes 2}) = 0$ since the extension $K(\mu_{2^m})/K(\sqrt{-1})$ is cyclic. A restriction–corestriction argument yields

$$H^1(K_{v_0}, F) \times H^2(K_{v_0}, F') \to H^3(K_{v_0}, \mathbb{Q}/\mathbb{Z}(2)) \cong \mathbb{Q}/\mathbb{Z},$$
Corollary 3.24. The group $\text{III}^2_{\omega}(K, \mu^{\otimes 2}_{2m})$ is 2-torsion. It is trivial if the extension $K(\mu_{2m})/K$ is cyclic (e.g., if $\sqrt{-1} \in K$).

Thus, we are interested in the following question, to which a negative answer is expected.

**Question 3.** Let $K$ be a $p$-adic function field. Is the group $\text{III}^2_{\omega}(K, \mu^{\otimes 2}_{2m})$ trivial for $m \geq 3$?

## 4 CURVES OVER HIGHER DIMENSIONAL LOCAL FIELDS

In this section, we work with homogeneous spaces over function fields over higher dimensional local fields. The main results here are Theorems D and E. Our approach is similar to that in Sections 3.1 and 3.2. The price we have to pay is that the obtained results are much coarser than those in the case of $p$-adic function fields. First, the geometric stabilizers are supposed to be finite abelian; the case of toric stabilizers is not treated because we do not have the corresponding duality theorems for tori at our disposal. Second, the constructed obstruction is not shown to be unramified.

### 4.1 Construction of the obstruction

A single construction suffices for both local–global and weak approximation problems. Let $d \geq 0$ and let $\pi : X \to \text{Spec } K$ be a smooth geometrically integral variety over a field $K$ of cohomological dimension $\text{cd}(K) \leq d + 2$ and characteristic 0. We construct a map

$$r : H^{d+1}(K, H^1(\overline{X}, \mathbb{Q}/\mathbb{Z}(d + 1))) \rightarrow \frac{H^{d+2}(X, \mathbb{Q}/\mathbb{Z}(d + 1))}{\text{Im } H^{d+2}(K, \mathbb{Q}/\mathbb{Z}(d + 1))} \quad (4.1.1)$$

as follows. Consider the Hochschild–Serre spectral sequence

$$E_2^{p,q} = H^p(K, H^q(\overline{X}, \mathbb{Q}/\mathbb{Z}(d + 1))) \Rightarrow H^{p+q}(X, \mathbb{Q}/\mathbb{Z}(d + 1)).$$

Under the assumption $\text{cd}(K) \leq d + 2$, the outgoing differentials from $E_i^{d+1,1}$ vanish for all $i \geq 2$; hence, there is a map $E_2^{d+1,1} \rightarrow E_{\infty}^{d+1,1} = \frac{H^{d+1}H^{d+2}}{F^{d+2}H^{d+2}} \subseteq \frac{H^{d+2}}{F^{d+2}H^{d+2}}$, where

$$H^{d+2} = F^0H^{d+2} \supseteq F^1H^{d+2} \supseteq \ldots \supseteq F^{d+2}H^{d+2}$$

is a filtration of $H^{d+2} := H^{d+2}(X, \mathbb{Q}/\mathbb{Z}(d + 1))$. On the other hand, there is a surjection $E_2^{d+2,0} \rightarrow E_{\infty}^{d+2,0}$, hence $F^{d+2}H^{d+2} = E_{\infty}^{d+2,0} = \text{Im}(E_2^{d+2,0} \rightarrow H^{d+2})$. Since $\overline{X}$ is integral, one has $H^0(\overline{X}, \mathbb{Q}/\mathbb{Z}(d + 1)) = \mathbb{Q}/\mathbb{Z}(d + 1)$, hence $E_2^{d+2,0} = H^{d+2}(K, \mathbb{Q}/\mathbb{Z}(d + 1))$, and the edge map $E^{d+2,0} \rightarrow H^{d+2}$ is the natural pullback induced by $\pi$. Thus, we obtain a map

$$r : E_2^{d+1,1} \rightarrow \frac{H^{d+2}}{\text{Im } E_2^{d+2,0}}.$$
as in (4.1.1). It fits into the commutative diagram

\[
\begin{array}{ccc}
H^{d+2}(K, \mathbb{Q}/\mathbb{Z}(d + 1)) & \rightarrow & H^{d+2}(K, \tau \ll_1 \mathbb{R} \pi_* \mathbb{Q}/\mathbb{Z}(d + 1)) \\
\downarrow & & \downarrow \\
H^{d+2}(K, \mathbb{Q}/\mathbb{Z}(d + 1)) & \rightarrow & H^{d+2}(X, \mathbb{Q}/\mathbb{Z}(d + 1)) \\
\downarrow & & \downarrow \\
& & H^{d+2}(X, \mathbb{Q}/\mathbb{Z}(d + 1))
\end{array}
\]

(4.1.2)

with exact rows, where the top row is associated with the distinguished triangle

\[
\mathbb{Q}/\mathbb{Z}(d + 1) \rightarrow \tau \ll_1 \mathbb{R} \pi_* \mathbb{Q}/\mathbb{Z}(d + 1) \rightarrow H^1(X, \mathbb{Q}/\mathbb{Z}(d + 1))[-1] \rightarrow \mathbb{Q}/\mathbb{Z}(d + 1)[1]
\]

(4.1.3)
in \(D^+(K)\). The middle vertical arrow in (4.1.2) is the composite

\[
H^{d+2}(K, \tau \ll_1 \mathbb{R} \pi_* \mathbb{Q}/\mathbb{Z}(d + 1)) \rightarrow H^{d+2}(K, \mathbb{R} \pi_* \mathbb{Q}/\mathbb{Z}(d + 1)) = H^{d+2}(X, \mathbb{Q}/\mathbb{Z}(d + 1)),
\]

where the last identification is due to the fact that \(\mathbb{H}(K, -) \circ \mathbb{R} \pi_* = \mathbb{H}(X, -)\) [71, Corollary 10.8.3].

To prove Theorem E, we shall need the following higher dimensional generalization of Proposition 2.4 (which is actually an analog of [16, Lemme 3.5.2] and [66, Lemma 3]).

**Proposition 4.1.** Keep the above notations and assume in addition that \(\overline{K}[X]^\times = \overline{K}\times\). Let \(F\) be a finite \(\Gamma_K\)-module and \(\lambda : \hat{F} \rightarrow \text{Pic} X\) a \(\Gamma_K\)-equivariant homomorphism. Then, the following claims hold.

(i) \(\lambda\) factors through a unique \(\Gamma_K\)-equivariant homomorphism \(\lambda^{(0)} : \hat{F} \rightarrow H^1(X, \mathbb{Q}/\mathbb{Z}(1))\).

(ii) Let \(F' = \hat{F} \otimes \mathbb{Q}/\mathbb{Z}(d) = \mathbb{H}om_K(F, \mathbb{Q}/\mathbb{Z}(d + 1))\) and let \(\lambda^{(d)} : F' \rightarrow H^1(X, \mathbb{Q}/\mathbb{Z}(d + 1))\) denote the twist by \(\mathbb{Q}/\mathbb{Z}(d)\) of the map \(\lambda^{(0)}\) from (i). Then, for any torsor \(Y \rightarrow X\) of type \(\lambda\) (see the discussion at the beginning of Section 2.1), the diagram

\[
\begin{array}{ccc}
H^{d+1}(K, F') & \rightarrow & H^{d+2}(X, \mathbb{Q}/\mathbb{Z}(d + 1)) \\
\downarrow \lambda^{(d)} & & \downarrow \\
H^{d+1}(K, H^1(X, \mathbb{Q}/\mathbb{Z}(d + 1))) & \rightarrow & H^{d+2}(X, \mathbb{Q}/\mathbb{Z}(d + 1))
\end{array}
\]

commutes. Here, the cup-product \(H^1(X, F) \times H^{d+1}(X, F') \rightarrow H^{d+2}(X, \mathbb{Q}/\mathbb{Z}(d + 1))\) is induced by the pairing \(F \otimes F' \rightarrow \mathbb{Q}/\mathbb{Z}(d + 1)\), and the map \(r\) was constructed in (4.1.1).

**Proof.** We prove (i). Since \(\overline{K}[X]^\times = \overline{K}\times\) is divisible, for each \(n \geq 1\), the Kummer sequence yields an identification \(H^1(K, \mu_n) = n \text{Pic} X\). Furthermore, one checks that for \(n \mid m\), the map \(H^1(X, \mu_n) \rightarrow H^1(X, \mu_m)\) (induced by the inclusion \(\mu_n \hookrightarrow \mu_m\)) is precisely the inclusion \(n \text{Pic} X \hookrightarrow m \text{Pic} X\). Taking limit yields an identification \(H^1(X, \mathbb{Q}/\mathbb{Z}(1)) = (\text{Pic} X)_{\text{tors}}\). Since \(\hat{F}\) is finite, \(\lambda\) factors through a unique morphism \(\lambda^{(0)} : \hat{F} \rightarrow H^1(X, \mathbb{Q}/\mathbb{Z}(1))\).

Let us now show (ii). Let \(\alpha \in H^{d+1}(K, F') = \text{Ext}_K^{d+1}(\mathbb{Z}, F')\) and let \(\zeta \rightarrow F'[d + 1]\) be a morphism in \(D^+(K)\) representing it. Such a morphism gives rise to the vertical arrows in the following
commutative diagram in $D^+(\mathbb{A}b)$:

\[
\begin{align*}
\mathbb{R} \text{Hom}_{K}(F', \tau_{\leq 1}\mathbb{R}\pi_* O/Z(d + 1))[1] & \longrightarrow \mathbb{R} \text{Hom}_{K}(F', \mathbb{H}^1(\mathcal{X}, O/Z(d + 1))) \\
\mathbb{H}(K, \tau_{\leq 1}\mathbb{R}\pi_* O/Z(d + 1))[d + 2] & \longrightarrow \mathbb{H}(K, \mathbb{H}^1(\mathcal{X}, O/Z(d + 1)))[d + 1].
\end{align*}
\]

(4.1.4)

The horizontal arrows in (4.1.4) are induced by triangle (4.1.3). We claim that there is a commutative diagram

\[
\begin{align*}
\mathbb{H}^1(\mathcal{X}, F) & \longrightarrow \mathbb{H}^1(\mathcal{X}, F) \text{ type} \longrightarrow \text{Hom}_{K}(\bar{F}, \text{Pic} \mathcal{X}) \\
\text{Ext}_{\mathcal{X}}^1(\bar{F}, G_m) & \leftarrow \text{Ext}_{K}^1(\bar{F}, \tau_{\leq 1}\mathbb{R}\pi_* G_{m,\mathcal{X}}) \longrightarrow \text{Hom}_{K}(\bar{F}, \text{Pic} \mathcal{X}) \\
\text{Ext}_{\mathcal{X}}^1(\bar{F}, O/Z(1)) & \leftarrow \text{Ext}_{K}^1(\bar{F}, \tau_{\leq 1}\mathbb{R}\pi_* O/Z(1)) \longrightarrow \text{Hom}_{K}(\bar{F}, \mathbb{H}^1(\mathcal{X}, O/Z(1))) \\
\mathbb{H}^{d+2}(\mathcal{X}, O/Z(d + 1)) & \leftarrow \mathbb{H}^{d+2}(K, \tau_{\leq 1}\mathbb{R}\pi_* O/Z(d + 1)) \longrightarrow \mathbb{H}^{d+1}(K, \mathbb{H}^1(\mathcal{X}, O/Z(d + 1))) \\
\mathbb{H}^{d+2}(\mathcal{X}, O/Z(d + 1)) & \longrightarrow \text{Im} \mathbb{H}^{d+1}(K, O/Z(d + 1)),
\end{align*}
\]

(4.1.5)

where $\cdot$ means the Yoneda product. For further use, we recall that for any $\Gamma_K$-module $M$ and any sheaf $F$ on $X_{\text{et}}$, the distinguished triangle

\[
\tau_{\leq 1}\mathbb{R}\pi_* F \rightarrow \mathbb{R}\pi_* F \rightarrow \tau_{\geq 2}\mathbb{R}\pi_* F \rightarrow (\tau_{\leq 1}\mathbb{R}\pi_* F)[1]
\]

in $D^+(K)$ yields an isomorphism $\text{Ext}_{K}^1(M, \tau_{\leq 1}\mathbb{R}\pi_* F) \cong \text{Ext}_{K}^1(M, \mathbb{R}\pi_* F)$, since $\tau_{\geq 2}\mathbb{R}\pi_* F$ is acyclic in degrees 0 and 1. But $\text{Ext}_{K}^1(M, \mathbb{R}\pi_* F) = \text{Ext}_{\mathcal{X}}^1(\pi^* M, F)$ since $\mathbb{R}\text{Hom}_{K}(M, -) \circ \mathbb{R}\pi_* = \mathbb{R}\text{Hom}_{\mathcal{X}}(\pi^* M, -)$ [71, Corollary 10.8.3]. Thus, we obtain an isomorphism

\[
\text{Ext}_{K}^1(M, \tau_{\leq 1}\mathbb{R}\pi_* F) \cong \text{Ext}_{\mathcal{X}}^1(\pi^* M, F).
\]

(4.1.6)

To see why (4.1.5) commutes, let us consider the four rectangles of (4.1.5) from the top to the bottom. As for the first rectangle, the right bottom horizontal arrow is induced by the map from
the distinguished triangle
\[ \mathbb{G}_m \to \tau_{\leq 1} \mathbb{R} \pi_* \mathbb{G}_m, X \to \text{Pic} \bar{X}[-1] \to \mathbb{G}_m[1], \]

in $D^+(K)$ (here we use the fact that $\bar{K}[X]^x = \bar{K}^x$). The left vertical arrow is the isomorphism (2.1.2). The left bottom horizontal arrow is the isomorphism (4.1.6) for $M = \hat{F}$ and $P' = \mathbb{G}_m$. The middle vertical arrow is the obvious isomorphism that makes the left square commute. For the commutativity of the right square, see [32, Appendix B].

As for the second rectangle, the left bottom horizontal arrow is the isomorphism (4.1.6) for $M = \hat{F}$ and $P = \mathbb{Q}/\mathbb{Z}(1)$, the right bottom horizontal arrow is induced by the map from the distinguished triangle
\[ \mathbb{Q}/\mathbb{Z}(1) \to \tau_{\leq 1} \mathbb{R} \pi_* \mathbb{Q}/\mathbb{Z}(1) \to H^1(\bar{X}, \mathbb{Q}/\mathbb{Z}(1))[-1] \to \mathbb{Q}/\mathbb{Z}[1] \]
in $D^+(K)$ (here we use the fact that $H^0(\bar{X}, \mathbb{Q}/\mathbb{Z}(1)) = \mathbb{Q}/\mathbb{Z}(1)$ since $\bar{X}$ is integral). This rectangle obviously commutes. The right vertical arrow is an isomorphism since $\hat{F}$ is finite and since $H^1(\bar{X}, \mathbb{Q}/\mathbb{Z}(1)) = (\text{Pic} \bar{X})_{\text{tors}}$.

The third rectangle obviously commutes. Its left bottom horizontal arrow is the isomorphism (4.1.6) for $M = F'$ and $P = \mathbb{Q}/\mathbb{Z}(d + 1)$ and its right bottom horizontal arrow is induced by the map from triangle (4.1.3). As for the fourth rectangle, the left square obviously commutes (bearing in mind that $H^{d+2}(X, \mathbb{Q}/\mathbb{Z}(d + 1)) = H^{d+2}(K, \mathbb{R} \pi_* \mathbb{Q}/\mathbb{Z}(d + 1))$ since $\mathbb{H}(K, -) \circ \mathbb{R} \pi_* = \mathbb{H}(X, -)$ [71, Corollary 10.8.3]), and the right square is obtained by taking cohomology of (4.1.4). Finally, the bottom square of (4.1.5) is part of (4.1.2); the triangle obviously commutes.

By a similar argument as in the proof of Proposition 2.4, we have a commutative diagram

\[ \begin{array}{ccc}
H^1(X, \mathbb{H}om_X(\hat{F}, \mathbb{G}_m)) & \xrightarrow{\cong} & \text{Ext}^1_X(\hat{F}, \mathbb{G}_m) \\
\cong
\downarrow
\gamma'_1
\downarrow
\gamma_1
\cong

H^1(X, \mathbb{H}om_X(\hat{F}, \mathbb{Q}/\mathbb{Z}(1))) & \xrightarrow{\cong} & \text{Ext}^1_X(\hat{F}, \mathbb{Q}/\mathbb{Z}(1)) \\
\cong
\downarrow
\gamma'_2
\downarrow
\gamma_2

H^1(X, F) & \xrightarrow{\cong} & \text{Ext}^1_X(\hat{F}, F \otimes \hat{F}) & \xrightarrow{\gamma'_3} & \text{Ext}^1_X(\hat{F}, \mathbb{Q}/\mathbb{Z}(1)) \\
\downarrow
\cong Q/Z(d) & & \downarrow
\cong Q/Z(d) & & \\
H^1(X, \mathbb{H}om_X(F', F \otimes F')) & \xrightarrow{\gamma'_3} & \text{Ext}^1_X(F', F \otimes F') & \xrightarrow{\gamma'_3} & \text{Ext}^1_X(F', \mathbb{Q}/\mathbb{Z}(d + 1)).
\end{array} \]

Let $Y \to X$ be a torsor under $F$ of type $\lambda \in \text{Hom}_K(\hat{F}, \text{Pic} \bar{X})$, and let $\varepsilon, \varepsilon'$ denote the respective images in $\text{Ext}^1_X(\hat{F}, F \otimes \hat{F})$ and $\text{Ext}^1_X(F', F \otimes F')$ of $[Y] \in H^1(X, F)$ by (4.1.7). The image of $[Y]$ in $\text{Ext}^1_X(\hat{F}, \mathbb{G}_m)$ by (4.1.5) is precisely $\gamma_1(\gamma_2(\varepsilon))$. The class $\gamma_2(\varepsilon) \in \text{Ext}^1_X(\hat{F}, \mathbb{Q}/\mathbb{Z}(1))$ corresponds to an element of $\text{Ext}^1_K(\hat{F}, \tau_{\leq 1} \mathbb{R} \pi_* \mathbb{Q}/\mathbb{Z}(1))$, whose image in $\text{Hom}_K(\hat{F}, H^1(\bar{X}, \mathbb{Q}/\mathbb{Z}(1)))$ is the homomorphism $\lambda^{(0)}$ from (i). Moreover, the image of $\gamma_3(\varepsilon)$ in $\text{Ext}^1(F', \mathbb{Q}/\mathbb{Z}(d + 1))$ by (4.1.5) is precisely $\gamma_3(\varepsilon')$. By exploiting the commutativity of (4.1.5), we see that the image of $\gamma_3(\varepsilon') \cdot \pi^* \alpha \in H^{d+2}(X, \mathbb{Q}/\mathbb{Z}(d + 1))$ in $H^{d+2}(X, \mathbb{Q}/\mathbb{Z}(d + 1)) \lim H^{d+2}(K, \mathbb{Q}/\mathbb{Z}(d + 1))$ is precisely $r(\lambda^{(d)} \alpha)$. Finally, we have
a commutative diagram of pairings

\[
\begin{array}{ccc}
\mathbb{H}^1(X, F) & \times & \mathbb{H}^{d+1}(X, F') \\
\downarrow & & \downarrow \\
\text{Ext}^1_X(F', F \otimes F') & \times & \mathbb{H}^{d+1}(X, F') \\
\downarrow & & \downarrow \\
\text{Ext}^1_X(F', \mathbb{Q}/\mathbb{Z}(d+1)) & \times & \mathbb{H}^{d+1}(X, F') \\
\end{array}
\]

where \( \cdot \) means the Yoneda product, and where the top square commutes by \([51, \text{Chapter V, Proposition 1.20}].\) This yields the identity \( \gamma_3(e') \cdot \pi^* \alpha = [Y] \cup \pi^* \alpha \in \mathbb{H}^{d+2}(X, \mathbb{Q}/\mathbb{Z}(d+1)) \), which proves (ii).

4.2 The main theorems

We prove Theorems D and E in this section. Let \( k \) be a \( d \)-dimensional local field satisfying the condition \((\star)\) from page 3, \( \Omega \) a smooth projective geometrically integral curve over \( k \), and \( K = K(\Omega) \) its function field. Let \( X \) be a homogeneous space of a simply connected semisimple linear algebraic group \( G \) over \( K \), with finite abelian geometric stabilizers \( \tilde{F} \). By Lemma 3.1 (and the discussion preceding it), \( \tilde{F} \) has a natural \( K \)-form \( F \), and \( \hat{F} = \text{Pic} \tilde{X} \) as \( \Gamma_K \)-modules. Recall that \( \overline{K}[X]^\times = \overline{K}[G]^\times = \overline{K}^\times \) by Rosenlicht’s lemma \([59, \text{Proposition 3}]\). Hence, we may apply Proposition 4.1(ii), which says that \( \text{id} : \hat{F} \rightarrow \text{Pic} \tilde{X} \) yields an isomorphism \( \text{id}(d) : F' \overset{\cong}{\rightarrow} H^1(\tilde{X}, \mathbb{Q}/\mathbb{Z}(d+1)) \), where \( F' = \mathcal{H}\hom_K(F, \mathbb{Q}/\mathbb{Z}(d+1)) \). Let

\[
r^{(d)} = r \circ \text{id}^{(d)} : H^{d+1}(K, F') \rightarrow \frac{H^{d+2}(X, \mathbb{Q}/\mathbb{Z}(d+1))}{\text{Im} H^{d+2}(K, \mathbb{Q}/\mathbb{Z}(d+1))}, \quad (4.2.1)
\]

where \( r \) is the map from \((4.1.1)\). We can now state the main results of this section.

**Theorem 4.2** (Theorem D). Let \( K \) be the function field of a smooth projective geometrically integral curve \( \Omega \) over a \( d \)-dimensional local field \( k \) satisfying the condition \((\star)\) from page 3, \( G \) a special, simply connected semisimple algebraic group over \( K \), and \( X \) a homogeneous space of \( G \) with finite abelian geometric stabilizers \( \tilde{F} \). Let \( F \) be the natural \( K \)-form of \( \overline{F} \), \( F' = \mathcal{H}\hom_K(F, \mathbb{Q}/\mathbb{Z}(d+1)) \), and \( r^{(d)} \) the map defined in \((4.2.1)\). If there is a point of \( X(\mathbb{A}_K) \) orthogonal to \( r^{(d)}(\mathcal{H}^{d+1}(K, F')) \) relative to the pairing \((1.4.11)\), then \( X(K) \neq \emptyset \). In particular, the first adelic obstruction \((1.4.12)\) to the local–global principle for \( X \) is the only one.

**Theorem 4.3** (Theorem E). Let \( K \) be the function field of a smooth projective geometrically integral curve \( \Omega \) over a \( d \)-dimensional local field \( k \) satisfying the condition \((\star)\) from page 3, \( F \) a finite \( \Gamma_K \)-module, equipped with an embedding \( F \hookrightarrow G \) into a special, simply connected semisimple algebraic group over \( K \) that has weak approximation, and \( X = F\backslash G \). Let \( F' = \mathcal{H}\hom_K(F, \mathbb{Q}/\mathbb{Z}(d+1)) \), \( r^{(d)} \) the map defined in \((4.2.1)\), and \( S \subseteq \Omega^{(1)} \) a finite set. Then, any family \( (P_v)_{v \in S} \in \prod_{v \in S} X(K_v) \) orthogonal to \( r^{(d)}(\mathcal{H}^{d+1}(K, F')) \) relative to the pairing \((1.4.13)\) lies in the closure of the diagonal
image of $X(K)$. In particular, the generalized Brauer–Manin obstruction to weak approximation for $X$ is the only one. Moreover, $X$ has weak approximation in $S$ if and only if $\text{III}^{d+1}_S(K, F') = \text{III}^{d+1}(K, F')$.

Theorem 4.2 would follow from the following higher dimensional generalization of Proposition 2.2 (which is actually an analog of [16, Lemme 3.3.3] and [67, (6.4)]).

**Proposition 4.4.** Keep the notations from Theorem 4.2. Let $\eta_X \in H^2(K, F)$ be the Springer class of $X$ (see the discussion at the beginning of Section 3.1). Assume in addition that $X(\mathbb{A}_K) \neq \emptyset$ (hence $\eta_X \in H^2(K, F)$). For any $\alpha \in \text{III}^2(K, F')$, there holds

$$\rho_X(r^{(d)}(\alpha)) = -\langle \eta_X, \alpha \rangle_{PT}.$$ 

Here, the map $\rho_X$ was defined in (1.4.12), and $\langle -, - \rangle_{PT}$ is the pairing (1.5.2).

**Proof.** Let $e_X \in \text{Ext}^2_K(\hat{F}, G_m)$ denote the elementary obstruction of $X$ (see the discussion at the beginning of Section 2.1). By Lemma 3.6, the isomorphism (2.1.2) sends $\eta_X$ to $e_X$. By the same argument at the beginning of the proof of Proposition 2.2, $-e_X$ is represented by a morphism $\hat{F} \to G_m[2]$ associated with the distinguished triangle

$$G_m \to \tau_{\leq 1} R \pi_\ast G_{m,X} \to \hat{F}[-1] \to G_m[1]$$

(4.2.2) in $D^+(K)$. Let $\pi^U : \mathcal{X} \to U$ be an integral model of $X$ over some nonempty open subset $U \subseteq \Omega$. We may assume that $F$ extends to a locally constant finite étale $U$-group scheme $F$. Then, $\hat{F}$ (resp. $F'$) extends to the locally constant finite étale $U$-group scheme $\hat{F} = \text{Hom}_U(F, G_m)$ (resp. $F' = \hat{F} \otimes \mathbb{Q}/\mathbb{Z}(d) = \text{Hom}(F, \mathbb{Q}/\mathbb{Z}(d+1))$). We have $\pi^U_\ast G_{m,X} = G_m$ (since the canonical map $G_m \to \pi^U_\ast G_{m,X}$ induces isomorphisms on the stalks of geometric points, by Rosenlicht's lemma [59, Proposition 3]). Similarly, $\hat{F} = \mathbb{R}^1 \pi^U_\ast G_{m,X}$ (by applying Lemma 3.1 to the stalks of geometric points). Thus, (4.2.2) extends to a distinguished triangle

$$G_m \to \tau_{\leq 1} R \pi_\ast G_{m,X} \to \hat{F}[-1] \to G_m[1]$$

(4.2.3) in $D^+(U)$. The inverse class $\lambda^* e_U \in \text{Ext}^2_U(\hat{F}, G_m)$ of the morphism $\hat{F} \to G_m[2]$ associated with (4.2.3) is a lifting of $\lambda^* e_X$. The spectral sequence $H^p(U, \mathbb{E}xU_\hat{F}(\hat{F}, G_m)) \Rightarrow \text{Ext}^{p+q}_U(\hat{F}, G_m)$ provides an edge map

$$H^2(U, F) \cong H^2(U, \text{Hom}_U(\hat{F}, G_m)) \to \text{Ext}^2_U(\hat{F}, G_m),$$

which is an isomorphism by [67, Lemma 2.3.7]. It maps a lifting $\eta_U$ of $\eta_X \in H^2(K, F)$ to $\lambda^* e_U$.

Note that $\pi^U_\ast \mathbb{Q}/\mathbb{Z}(1) = \mathbb{Q}/\mathbb{Z}(1)$ since the fibers of $\pi^U$ are geometrically integral. Furthermore, by the smooth base change theorem [51, Chapter VI, §4], $\text{id}^{(0)} : \hat{F} \cong H^1(\overline{X}, \mathbb{Q}/\mathbb{Z}(1))$ extends to
an isomorphism \( \text{id}^{(0)} : \hat{\mathcal{F}} \cong R^1\pi_*^U \mathbb{Q}/\mathbb{Z}(1) \). Thus, we have a distinguished triangle

\[
\frac{\mathbb{Q}}{\mathbb{Z}(1)} \to \tau_{\leq 1} R^1\pi_*^U \mathbb{Q}/\mathbb{Z}(1) \to \hat{\mathcal{F}}[-1] \to \frac{\mathbb{Q}}{\mathbb{Z}(1)}[1]
\]

(4.2.4)
in \( D^+(U) \), extending the distinguished triangle

\[
\frac{\mathbb{Q}}{\mathbb{Z}(1)} \to \tau_{\leq 1} \pi_* \mathbb{Q}/\mathbb{Z}(1) \to \hat{\mathcal{F}}[-1] \to \frac{\mathbb{Q}}{\mathbb{Z}(1)}[1]
\]
in \( D^+(K) \). To replace \( \mathbb{G}_m \) by \( \mathbb{Q}/\mathbb{Z}(1) \), we need the following technical lemma.

**Lemma 4.5.** The edge map \( H^2(U, \mathcal{F}) \cong H^2(U, \mathcal{H}om_U(\hat{\mathcal{F}}, \mathbb{Q}/\mathbb{Z}(1))) \to \text{Ext}^2_U(\hat{\mathcal{F}}, \mathbb{Q}/\mathbb{Z}(1)) \), induced by the spectral sequence \( H^p(U, \mathcal{E}xt^q_U(\hat{\mathcal{F}}, \mathbb{Q}/\mathbb{Z}(1))) \Rightarrow \text{Ext}^{p+q}_U(\hat{\mathcal{F}}, \mathbb{Q}/\mathbb{Z}(1)) \), sends \( \eta_U \) to the inverse class of the morphism \( \hat{\mathcal{F}} \to \mathbb{Q}/\mathbb{Z}(1)[2] \) associated with triangle (4.2.4).

**Proof.** Let \( n \geq 1 \) such that \( n\mathcal{F} = 0 \), then \( \hat{\mathcal{F}} = \mathcal{H}om_{U,Z/n}(\mathcal{F}, \mathbb{Z}/n) \), where the subscript \( U,Z/n \) means that we are taking \( \mathcal{H}om \) in the category of \( n \)-torsion sheaves over \( U_{\text{et}} \). In the spectral sequence \( H^p(U, \mathcal{E}xt^q_{U,Z/n}(\hat{\mathcal{F}}, \mathbb{Z}/n)) \Rightarrow \text{Ext}^{p+q}_{U,Z/n}(\hat{\mathcal{F}}, \mathbb{Z}/n) \), one has \( \mathcal{E}xt^p_{U,Z/n}(\hat{\mathcal{F}}, \mathbb{Z}/n) = 0 \) because \( \mathbb{Z}/n \cong \mathbb{Z}/n \) (which is an injective \( \mathbb{Z}/n \)-module) locally for the étale topology on \( U \). It follows that the induced edge map \( H^2(U, \mathcal{F}) \cong H^2(U, \mathcal{H}om_{U,Z/n}(\hat{\mathcal{F}}, \mathbb{Z}/n)) \to \text{Ext}^2_{U,Z/n}(\hat{\mathcal{F}}, \mathbb{Z}/n) \) is an isomorphism.

It follows that, in the commutative diagram

\[
\begin{array}{ccc}
\text{Ext}^2_{U,Z/n}(\hat{\mathcal{F}}, \mathbb{Z}/n) & \cong & \text{Ext}^2_U(\hat{\mathcal{F}}, \mathbb{Z}/n) \\
| & | & |
\mu_n & \cong & \mu_n[1] \\
\tau_\leq R^1\pi_*^U \mathbb{Z}/n & \cong & \tau_\leq R^1\pi_*^U \mathbb{Z}/n[1] \\
\frac{\mathbb{Q}}{\mathbb{Z}(1)} & \cong & \frac{\mathbb{Q}}{\mathbb{Z}(1)}[1] \\
| & | & |
\mathbb{G}_m & \cong & \mathbb{G}_m[1] \\
\end{array}
\]

(4.2.5)

the composite of the bottom row is an isomorphism. On the other hand, the isomorphism \( \hat{\mathcal{F}} \cong H^1(\mathcal{X}, \mathbb{Q}/\mathbb{Z}(1)) \) factors through an isomorphism \( \hat{\mathcal{F}} \cong H^1(\mathcal{X}, \mathbb{Z}/n) \), which extends to an isomorphism \( \hat{\mathcal{F}} \cong R^1\pi_*^U \mathbb{Z}/n \) by the smooth base change theorem [51, Chapter VI, §4]. Thus, we have a commutative diagram

\[
\begin{array}{ccc}
\mu_n & \cong & \mu_n[1] \\
\tau_\leq R^1\pi_*^U \mathbb{Z}/n & \cong & \tau_\leq R^1\pi_*^U \mathbb{Z}/n[1] \\
\frac{\mathbb{Q}}{\mathbb{Z}(1)} & \cong & \frac{\mathbb{Q}}{\mathbb{Z}(1)}[1] \\
\mathbb{G}_m & \cong & \mathbb{G}_m[1] \\
\end{array}
\]

in \( D^+(U) \), with distinguished rows (the middle row being (4.2.4)). The existence of this diagram implies that the map \( H^2(U, \mathcal{F}) \to \text{Ext}^2_U(\hat{\mathcal{F}}, \mathbb{Q}/\mathbb{Z}(1)) \) from (4.2.5) effectively sends \( \eta_U \) to the inverse class of the morphism \( \hat{\mathcal{F}} \to \mathbb{Q}/\mathbb{Z}(1)[2] \) associated with triangle (4.2.4). \( \square \)
Return to the proof of Proposition 4.4. Just like in the proof of Proposition 4.1, we have a commutative diagram

\[
\begin{array}{ccc}
\mathcal{H}^2(U, \mathcal{H}om_U(\hat{F}, G_m)) & \cong & \mathcal{E}xt^2_U(\hat{F}, G) \\
\cong & & \\
\mathcal{H}^2(U, \mathcal{H}om_U(\hat{F}, \mathbb{Q}/\mathbb{Z}(1))) & \rightarrow & \mathcal{E}xt^2_U(\hat{F}, \mathbb{Q}/\mathbb{Z}(1)) \\
\mathcal{H}^2(U, F) & \rightarrow & \mathcal{H}^2(U, \mathcal{H}om_U(\hat{F}, F \otimes \hat{F})) \\
\mathcal{H}^2(U, \mathcal{H}om_U(F', F' \otimes F')) & \rightarrow & \mathcal{E}xt^2_U(F', F' \otimes F') \\
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{H}^2(U, \mathcal{H}om_U(\hat{F}, \mathbb{Q}/\mathbb{Z}(1))) & \rightarrow & \mathcal{E}xt^2_U(\hat{F}, \mathbb{Q}/\mathbb{Z}(1)) \\
\mathcal{H}^2(U, \mathcal{H}om_U(F', F' \otimes F')) & \rightarrow & \mathcal{E}xt^2_U(F', F' \otimes F') \\
\mathcal{H}^2(U, \mathcal{H}om_U(F', F' \otimes F')) & \rightarrow & \mathcal{E}xt^2_U(F', F' \otimes F') \\
\end{array}
\]

Let \(\varepsilon_U, \varepsilon'_U\) denote the respective images of \(\eta_U\) in \(\mathcal{E}xt^2_U(\hat{F}, F \otimes \hat{F})\) and \(\mathcal{E}xt^2_U(F', F' \otimes F')\) by (4.2.6). By Lemma 4.5, \(-\gamma_2(\varepsilon_U)\) is the class of the morphism \(F \to \mathbb{Q}/\mathbb{Z}(d + 1)\) associated with triangle (4.2.4). The image in \(\mathcal{E}xt^2(F', \mathbb{Q}/\mathbb{Z}(d + 1))\) by (4.2.6) of \(\gamma_2(\varepsilon_U)\) is \(\gamma_3(\varepsilon'_U)\), that is, \(-\gamma_3(\varepsilon'_U)\) is the class of the morphism \(F' \to \mathbb{Q}/\mathbb{Z}(d + 1)\) from the distinguished triangle

\[
\tau_{\leq 1}/\mathbb{R}_X := \mathbb{Q}/\mathbb{Z}(d + 1) \rightarrow F'[−1] \rightarrow \mathbb{Q}/\mathbb{Z}(d + 1)[1] \quad (4.2.7)
\]

extending (4.1.3) (recall that \(F' = H^1(X, \mathbb{Q}/\mathbb{Z}(d + 1))\)).

Now, we have a commutative diagram of pairings

\[
\begin{array}{ccc}
\mathcal{H}^2(U, F) & \times & \mathcal{H}^{d+1}(U, F') \\
\mathcal{E}xt^2(U', F' \otimes F') & \times & \mathcal{H}^{d+1}(U, F') \\
\mathcal{E}xt^2(U', \mathbb{Q}/\mathbb{Z}(d + 1)) & \times & \mathcal{H}^{d+1}(U, \mathbb{Q}/\mathbb{Z}(d + 1)) \\
\end{array}
\]

where \(\cdot\) means the Yoneda product, and where the top square commutes thanks to the construction of the cup-product (Artin–Verdier) pairing for cohomology with compact support (see (1.5.1) and (1.5.7)). Let \(\alpha \in \mathcal{H}^2(K, F')\), which lifts to an element \(\alpha_U \in \mathcal{H}^{d+1}(U, F')\) (after shrinking \(U\) if necessary). By the localization sequence (1.5.3), \(\alpha_U\) comes from an element \(\alpha^c_U \in \mathcal{H}^{d+1}_c(U, F')\). The commutativity of (4.2.8) implies \(\gamma_3(\varepsilon'_U) \cdot \alpha^c_U = \langle \eta_U, \alpha^c_U \rangle_{AV}\). In other words, the map \(H^{d+1}_c(U, F') \to H^{d+1}_c(U, \mathbb{Q}/\mathbb{Z}(d + 1)) \cong \mathbb{Q}/\mathbb{Z}\), induced by triangle (4.2.7), sends \(\alpha^c_U\) to \(-\langle \eta_U, \alpha^c_U \rangle_{AV}\), which is \(-\langle \varepsilon' \rangle_{PT}\) by the construction of \(\langle - , - \rangle_{PT}\). Finally, by repeating the argument with the snake lemma construction at the end of the proof of Proposition 2.4, we see that the same map sends \(\alpha^c_U\) to an element \(\beta^c_U \in \mathbb{Q}/\mathbb{Z}\), which would become \(\rho_X(r^{(d)}(\alpha))\) after taking limit over \(U\).

\(\square\)
Proof of Theorem 4.2. If there is a point of \(X(\mathbb{A}_K)\) orthogonal to \(r^{(d)}(\mathbb{W}^{d+1}(K, F'))\), then Proposition 4.4 implies that \(\langle \eta_X, \alpha \rangle_{PT} = 0\) for all \(\alpha \in \mathbb{W}^{d+1}(K, F')\). Since the Poitou–Tate pairing is nondegenerate, we have \(\eta_X = 0\). Since \(G\) is special, this is equivalent to \(X(K) \neq \emptyset\).

\[\square\]

Proof of Theorem 4.3. We recall that the map \(G \to X\) is a torsor under \(F\) of type \(\text{id} : \hat{F} \to \text{Pic} \hat{X}\). Let \((P_v)_{v \in S} \in \prod_{v \in S} X(K_v)\) be a family orthogonal to \(r^{(d)}(\mathbb{W}^{d+1}_S(K, F'))\) relative to the pairing (1.4.13). By Proposition 4.1, this is equivalent to

\[\sum_{v \in S} [G](P_v) \cup \text{loc}_v(\alpha) = \sum_{v \in S} ([G] \cup \pi^*\alpha)(P_v) = 0\]

for all \(\alpha \in \mathbb{W}^{d+1}_S(K, M')\), where \(\pi : X \to \text{Spec} K\) is the structure morphism. Using the exact sequence (1.5.4), we see that the family \(\left([G](P_v)\right)_{v \in S} \in \prod_{v \in S} H^1(K_v, F)\) comes from \(H^1(K, F)\). By Lemma 3.7, \((P_v)_{v \in S}\) lies in the closure of the diagonal image of \(X(K)\). The claim that \(X\) has weak approximation in \(S\) if and only if \(\mathbb{W}^{d+1}_S(K, \hat{M}) = \mathbb{W}^{d+1}(K, F')\) can be proved using the same argument as in the proof of Theorem 3.12 (and by using (1.5.4) instead of (3.2.4)).

\[\square\]

We conclude by noting that both Theorems 4.2 and 4.3 fail for toric stabilizers.

Example 4.6. Let \(k = \mathbb{C}(t)\) (so that \(d = 0\)) and \(K = k(\Omega)\) for some smooth projective geometrically integral \(k\)-curve. For a torus \(T\) over \(K\), (local and global) duality theorems between \(T\) and \(\hat{T}\) are established in [14]. It is possible to construct such a torus \(T\) and a homogeneous \(X\) of \(\text{SL}_n, K\) with Springer lien \(\text{lien}(T)\) for which the Brauer–Manin obstruction attached to the group \(\mathbb{W}^1(K, \hat{T})\) (resp. \(\mathbb{W}^1_{\omega}(K, \hat{T})\)) is insufficient to explain the failure of the local–global principle (resp. weak approximation). Indeed, an example in the case of the local–global principle is given at the beginning of [40, §4]. As for weak approximation, see Zhang’s recent work [73, Proposition 4.1].

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