NEGATIVELY REINFORCED BALANCED URN SCHEMES

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ABSTRACT. We consider weighted negatively reinforced urn schemes with finitely many colours. An urn scheme is called negatively reinforced, if the selection probability for a colour is proportional to the weight $w$ of the colour proportion, where $w$ is a non-increasing function. Under certain assumptions on the replacement matrix $R$ and weight function $w$, such as, $w$ is differentiable and $w(0) < \infty$, we obtain almost sure convergence of the random configuration of the urn model. In particular, we show that if $R$ is doubly stochastic the random configuration of the urn converges to the uniform vector, and asymptotic normality holds, if the number of colours in the urn are sufficiently large.

1. Introduction

1.1. Background and Motivation. The classical Pólya urn model was originally introduced by Pólya [20] and since then many generalization of the classical Pólya urn scheme have been studied [1, 19, 13, 11, 8, 9, 10]. One such generalization where the selection of a colour at every step is done according to an increasing weight function, with a random replacement rule was studied by Laurelle and Pages [15]. For such non linear urn models they obtained results on the almost sure convergence and central limit theorem of the random configuration. A different class of urn models, namely, linear negatively reinforced urn models was introduced in [2], where selection is done according to a weight function which is linear but non-increasing. In this paper, we investigate a generalization of these later class of models, for general non-increasing weight functions. The main tool used in this paper is stochastic approximation method, which is a powerful tool to study recursive algorithms. Recently, Zhang [21] has provided asymptotic normality for a stochastic approximation algorithm under certain assumptions, which we use in this work.

As mentioned in [2], resource constrain modelling problems is one of the main motivation to study such models. In particular, multi-server queueing systems with capacity constrains [16, 17] are good examples. For such models a desirable outcome is a balancing of the loads. In other words, it is desirable to obtain uniform load distribution at the limit. We will see later, that limiting uniform distribution can only be achieved, if we choose a doubly stochastic replacement matrix. As a result we mainly focus on the doubly stochastic matrices, and show that the almost sure convergence to uniform distribution holds under fairly general conditions.
assumptions on the weight function. We further establish the corresponding central limit theorems.

1.2. Model. In this work, we will only consider balanced urn schemes with \( k \)-colours, index by \( S := \{1, \ldots, k\} \). We essentially work under the framework introduced in [2]. For the sake of completeness, we provide here the complete description of the model, which is exactly similar to what is presented in [2], except we use more general non-increasing weight functions.

We denote by \( R := ((R_{i,j}))_{1 \leq i, j \leq k} \) the replacement matrix, that is, \( R_{i,j} \geq 0 \) is the number of balls of colour \( j \) to be placed in the urn when the colour of the selected ball is \( i \). The model will be called balanced, if all the row sums of \( R \) are constant. In that case, dividing the entries of \( R \) by the common row total, without loss we may assume \( R \) is a stochastic matrix. We will also assume that the starting configuration \( U_0 := (U_{0,j})_{1 \leq j \leq k} \) is a probability distribution on the set of colours \( S \). As we will see from the proofs of our main results, this apparent loss of generality can easily be removed. For simplicity, in this work we also assume \( U_{0,j} > 0 \) for every \( 1 \leq j \leq k \).

Denote by \( U_n := (U_{n,j})_{1 \leq j \leq k} \in [0, \infty)^k \) the random configuration of the urn at time \( n \). Also let \( \mathcal{F}_n := \sigma(U_0, U_1, \cdots, U_n) \) be the natural filtration. The \( (n+1) \)-th randomly selected colour will be denoted by \( Z_n \), which has the conditional distribution given \( \mathcal{F}_n \) as

\[
P(Z_n = j \mid \mathcal{F}_n) \propto w\left(\frac{U_{n,j}}{\sum_{i=1}^{k} U_{n,i}}\right), \quad 1 \leq j \leq k.
\]

where \( w : [0,1] \to \mathbb{R}^+ \) is a non-increasing function.

Starting with \( U_0 \) we define \( (U_n)_{n \geq 0} \) recursively as follows:

\[
U_{n+1} = U_n + \chi_{n+1}R.
\]

where \( \chi_{n+1} := (1(Z_n = j))_{1 \leq j \leq k} \).

We call the process \( (U_n)_{n \geq 0} \) a negatively reinforced urn scheme with initial configuration \( U_0 \) and replacement matrix \( R \). In this work, we will be interested in studying the asymptotic properties of the following two processes:

**Random configuration of the urn:** Observe that for all \( n \geq 0 \),

\[
\sum_{j=1}^{k} U_{n,j} = n + 1.
\]

This holds because \( R \) is a stochastic matrix and \( U_0 \) is a probability vector. Thus the random configuration of the urn, namely, \( \frac{U_n}{n+1} \) is a probability mass function.
Colour count statistics: Let $N_n := (N_{n,1}, \ldots , N_{n,k})$ be the vector of length $k$, whose $j$-th element is the number of times colour $j$ was selected in the first $n$ trials, that is
\begin{equation}
N_{n,j} = \sum_{m=0}^{n-1} 1 (Z_m = j), \quad 1 \leq j \leq k.
\end{equation}

It is easy to note that from (2) it follows
\begin{equation}
U_{n+1} = U_0 + N_{n+1} R.
\end{equation}

1.3. Outline. In Section 2 we first establish a relation between the urn model and the stochastic approximation algorithm, then in Section 3 and Section 4 we present our main results. In Section 5 we give the technical results and proofs of the main results are given in Section 6. In Section 7 some examples are given.

2. Stochastic Approximation and Urn

In this section we first define a stochastic approximation algorithm. A stochastic approximation algorithm $(X_n)_{n \geq 0}$ is defined as a stochastic process in $\mathbb{R}^k$, given by
\begin{equation}
X_{n+1} = X_n + \gamma_{n+1} h (X_n) + \gamma_{n+1} M_{n+1}, \quad n \geq 0
\end{equation}
where $h : \mathbb{R}^k \to \mathbb{R}^k$ and
\begin{enumerate}
\item[(i)] $(\gamma_n)_{n \geq 1}$ is a sequence of positive real numbers, such that,
\begin{equation}
\sum_{n=1}^{\infty} \gamma_n = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \gamma_n^2 < \infty.
\end{equation}
\item[(ii)] $(M_n)_{n \geq 1}$ is a square integrable martingale difference sequence with respect to the filtration $\mathcal{G}_n = \sigma \{X_m, M_m, m \leq n\}$ and there exists a constant $C > 0$, such that
\begin{equation}
E \left[ \|M_{n+1}\|^2 | \mathcal{G}_n \right] \leq C \left( 1 + \|X_n\|^2 \right) \text{ a.s.}
\end{equation}
for $n \geq 0$.
\end{enumerate}
In the next two subsections we will show that the vector of colour proportions and colour count proportions can be written as a stochastic approximation algorithm.

2.1. Stochastic Approximation Algorithm for the Random Urn Configuration.

We put
\begin{equation}
Y_n := \frac{U_n}{n+1},
\end{equation}
which is the vector of colour proportions at time $n$. Observe that, for the urn model defined in equation (2), we have
\begin{equation}
E [U_{n+1} - U_n | \mathcal{F}_n] = \frac{w(Y_n)}{S_w(Y_n)} R
\end{equation}
where $w(Y_n) = (w(Y_{n,1}), w(Y_{n,2}), \ldots , w(Y_{n,k}))$, and $S_w(Y_n) = \sum_{i=1}^{k} w(Y_{n,i})$. 
Therefore the recurrence relation in equation (2) can be written as
\[
U_{n+1} = U_n + E[\chi_{n+1}|\mathcal{F}_n]R + \left[\chi_{n+1} - E[\chi_{n+1}|\mathcal{F}_n]\right]R
= U_n + \frac{w(Y_n)}{S_w(Y_n)}R + M_{n+1}R
\]
(11)
where \(M_{n+1} = \chi_{n+1} - E[\chi_{n+1}|\mathcal{F}_n]\) is an \(\mathcal{F}_n\) martingale difference. Now observe that
\[
\frac{U_{n+1}}{n+2} = \frac{U_n}{n+2} + \frac{1}{n+2} \frac{w(Y_n)}{S_w(Y_n)}R + \frac{1}{n+2} M_{n+1}R
\]
\[
\Rightarrow Y_{n+1} = Y_n \frac{n+1}{n+2} + \frac{1}{n+2} \frac{w(Y_n)}{S_w(Y_n)}R + \frac{1}{n+2} M_{n+1}R
\]
\[
\Rightarrow Y_{n+1} = Y_n + \frac{1}{n+2} \left( \frac{w(Y_n)}{S_w(Y_n)}R - Y_n \right) + \frac{1}{n+2} M_{n+1}R
\]
which is exactly of the form given in the equation (6), that is the urn configuration \(Y_n\) can be written as a \(k\)-dimensional stochastic approximation algorithm given by:
\[
Y_{n+1} = Y_n + \gamma_{n+1} h(Y_n) + \gamma_{n+1}M_{n+1}R
\]
(12)
where \(\gamma_n = \frac{1}{n+1}\), and \(h : \mathbb{R}^k \rightarrow \mathbb{R}^k\) is given by
\[
h(y) = \frac{w(y)}{S_w(y)}R - y.
\]
(13)
where we extend the function \(w\) continuously to whole of \(\mathbb{R}\), by making it a constant function outside the interval \([0,1]\), that is, \(w(y) = w(0)\) for \(y \leq 0\) and \(w(y) = w(1)\) for \(y \geq 1\). Also note that \(\gamma_n \sim O(n^{-1})\) satisfies the required conditions given in (ii) and \((M_nR)_{n \geq 1}\) is a martingale difference sequence that is
\[
E [M_{n+1}R|\mathcal{F}_n] = 0, \ \forall n \geq 0.
\]
(14)
and since \(R\) is a stochastic matrix we get
\[
\|M_{n+1}R\|^2 = \sum_{i=1}^{k} \left| \sum_{j=1}^{k} M_{n+1,j} R_{j,i} \right|^2
\]
\[
\leq \sum_{i=1}^{k} \sum_{j=1}^{k} |M_{n+1,j}|^2
\]
\[
= k \sum_{j=1}^{k} |\chi_{n+1,j} - E[\chi_{n+1,j}|\mathcal{F}_n]|^2
\]
\[
\leq k \sum_{j=1}^{k} |\chi_{n+1,j}|^2 + E[\chi_{n+1,j}|\mathcal{F}_n]^2
\]
Now since $\chi_{n+1,j}^2 = \chi_{n+1,j}$ (as it only takes value 0 or 1) and $\sum_{j=1}^k \chi_{n+1,j} = 1$, therefore we get

$$E \left[ \| M_{n+1} R \|_2^2 | \mathcal{F}_n \right] \leq k \left( 1 + \sum_{j=1}^k E[\chi_{n+1,j} | \mathcal{F}_n] \right)^2 \leq k (1 + k).$$

Thus $(M_n R)_{n \geq 1}$ also satisfies the conditions given in equation (8). Therefore, the ODE associated to (12) is

$$\dot{y} = h(y) \quad (15)$$

where $h$ is given in equation (13).

2.2. Stochastic Approximation Algorithm for the Colour Count Statistics. Recall that from the definition $\sum_{i=1}^k N_{n,i} = n$, so we denote the colour count proportions by

$$\tilde{Y}_n := \frac{N_n}{n}$$

Note that we can write

$$N_{n+1} = N_n + \chi_{n+1} = N_n + E \left[ \chi_{n+1} | \mathcal{F}_n \right] + (\chi_{n+1} - E \left[ \chi_{n+1} | \mathcal{F}_n \right])$$

$$= N_n + \frac{w(Y_n)}{S_w(Y_n)} + M_{n+1}$$

$$\frac{N_{n+1}}{n+1} = \frac{N_n}{n} + \frac{1}{n+1} \left[ \frac{w(Y_n)}{S_w(Y_n)} - \frac{N_n}{n} \right] + \frac{1}{n+1} M_{n+1}$$

$$\Rightarrow \quad \tilde{Y}_{n+1} = \tilde{Y}_n + \frac{1}{n+1} \left[ \frac{w(Y_n)}{S_w(Y_n)} - \tilde{Y}_n \right] + \frac{1}{n+1} M_{n+1} \quad (16)$$

Using (5) we get

$$Y_n = \frac{1}{n+1} Y_0 + \frac{n}{n+1} \tilde{Y}_n R =: \tilde{Y}_n R + \delta_n \quad (17)$$

for

$$\delta_n = \frac{1}{n+1} Y_0 - \frac{1}{n+1} \tilde{Y}_n R$$

Therefore we can rewrite equation (16) as

$$\tilde{Y}_{n+1} = \tilde{Y}_n + \frac{1}{n+1} \left[ \frac{w(\tilde{Y}_n R + \delta_n)}{S_w(\tilde{Y}_n R + \delta_n)} - \tilde{Y}_n \right] + \frac{1}{n+1} M_{n+1}$$

$$= \tilde{Y}_n + \frac{1}{n+1} \left[ \frac{w(\tilde{Y}_n R)}{S_w(\tilde{Y}_n R)} - \tilde{Y}_n \right] + \frac{1}{n+1} \epsilon_n + \frac{1}{n+1} M_{n+1} \quad (18)$$
where
\[ \epsilon_n = \frac{w(\tilde{Y}_n R + \delta_n)}{S_w(\tilde{Y}_n R + \delta_n)} - \frac{w(\tilde{Y}_n R)}{S_w(\tilde{Y}_n R)}. \]

Therefore \( \tilde{Y}_n \) can also be written as a stochastic approximation algorithm as given in equation (6). Since \( \delta_n \to 0, \epsilon_n \to 0, \) as \( n \to \infty \), the ODE associated to (18) is
\[ \dot{\tilde{y}} = \tilde{h}(\tilde{y}) \quad (19) \]
where \( \tilde{h} : \mathbb{R}^k \to \mathbb{R}^k \) is such that
\[ \tilde{h}(\tilde{y}) = \frac{w(\tilde{y}R)}{S_w(w(\tilde{y}R))} - \tilde{y}. \quad (20) \]

For stochastic approximation technique, we will mostly refer to the work of Benaïm [5], Kushner-clark [14] and Borkar [7]. In the next two sections we state our main results for the non-linear weight functions.

3. Almost sure convergence

In this section, the almost sure convergence of the random processes \((Y_n)_{n \geq 0}\) and \((\tilde{Y}_n)_{n \geq 0}\) are obtained under different sufficient conditions. Before stating our main results we need the following two definitions:

**Definition 1.** A function \( f : \mathbb{R}^k \to \mathbb{R}^k \) is called Lipschitz, if there exists a finite real number \( C' \) such that
\[ \|f(x) - f(y)\| \leq C'|x - y|; \quad \forall x, y \in \mathbb{R}^k. \quad (21) \]

For a Lipschitz function \( f \), the Lipschitz constant is defined as
\[ M := \sup_{x \neq y} \frac{\|f(x) - f(y)\|}{\|x - y\|} \quad (22) \]
and such functions will be referred as Lip\((M)\). The function \( f \) is called a contraction if \( M < 1 \).

**Definition 2.** An equilibrium point \( y^* \) of the differential equation \( \dot{y}(t) = h(y(t)) \) is a point for which \( h(y^*) = 0 \).

Note that, for the \( h \) function given in equation (13), \( y^* \) is an equilibrium point if
\[ h(y^*) = 0 \iff w(y^*)R = S_w(y^*)y^*. \quad (23) \]

To start with, we need the ODE in equation (15) and (19) to have a unique solution. A sufficient condition for the ODEs to have a unique solution, is when \( h \) and \( \tilde{h} \) are Lipschitz functions. We will assume throughout this paper that the function \( w \) is continuously differentiable, which implies that the function \( h \) and \( \tilde{h} \) are both Lipschitz and this ensures that the associated ODEs have unique solution for any initial vector \( Y_0 \).
The equilibrium points of $h$ are important as they are possible limit points for the solution of the ODEs. For a nonlinear weight function $w$ the unique equilibrium point is guaranteed assuming that the function $F : \mathbb{R}^k \to \mathbb{R}^k$ defined as

$$F(y) := \frac{w(y)}{S_w(y)} R,$$

is a contraction map. We now present the results depending on whether $F$ is a contraction.

3.1. $F$ is a contraction.

**Theorem 1.** Suppose $w$ is a non-increasing weight function and $F$ is a contraction map then

$$Y_n \to y^* \text{ a.s., and } \tilde{Y}_n \to \tilde{y}^* \text{ a.s.}$$  \hfill (25)

where $y^*$ is the unique fixed point of $F$ and

$$\tilde{y}^* = \frac{w(y^*)}{S_w(y^*)}.$$  \hfill (26)

In particular, convergence in (25) holds, whenever non-increasing function $w$ is a Lip(M) function and $\sqrt{k} > \frac{2M}{w(1)}$.

**Corollary 1.** From equation (17) we get

$$y^* = \tilde{y}^* R,$$

or

$$\tilde{y}^* = \frac{w(y^*)}{S_w(y^*)}$$

where $y^*$ and $\tilde{y}^*$ are given in Theorem [7].

3.2. $F$ is not a contraction. In the case when $F$ is not a contraction, we will only consider doubly stochastic replacement matrices. We start with the following observation.

**Proposition 1.** The uniform vector $\frac{1}{k} \mathbf{1}$ is an equilibrium point of the ODE in equation (15), if and only if, $R$ is a doubly stochastic matrix.

**Proof.** Note that,

$$h\left(\frac{1}{k} \mathbf{1}\right) = 0 \iff \frac{w\left(\frac{1}{k}\mathbf{1}\right)}{S_w\left(\frac{1}{k} \mathbf{1}\right)} R = \frac{1}{k} \mathbf{1}$$

$$\iff \frac{1}{k} \mathbf{1} R = \frac{1}{k} \mathbf{1}$$

Thus, uniform is an equilibrium point, if and only if, $R$ is a doubly stochastic matrix. \hfill $\square$
Assuming that $R$ is doubly stochastic, $\frac{1}{k} \mathbf{1}$ is an equilibrium point for both the ODEs given in equation (15) and (19), that is

$$h\left(\frac{1}{k} \mathbf{1}\right) = 0 \text{ and } \tilde{h}\left(\frac{1}{k} \mathbf{1}\right) = 0$$

(27)

where $h$ and $\tilde{h}$ are defined in equation (13) and (20). In the next theorem, we show that for a doubly stochastic replacement matrix $R$ the random urn configuration converges almost surely.

**Theorem 2.** Let $w$ be a non-increasing weight function and $R$ be a doubly stochastic replacement matrix, such that for every eigenvalue $\lambda$ of $R$

$$\Re(\lambda) > \frac{kw\left(\frac{1}{k}\right)}{w'\left(\frac{1}{k}\right)},$$

(28)

where $\Re(\lambda)$ denotes the real part of the eigenvalue $\lambda$, then as $n \to \infty$

$$Y_n \longrightarrow \frac{1}{k} \mathbf{1} \text{ a.s. and } \tilde{Y}_n \longrightarrow \frac{1}{k} \mathbf{1} \text{ a.s.}$$

(29)

### 4. Scaling Limits

In this section, we will state the central limit theorems for $(Y_n)_{n \geq 0}$ and $(\tilde{Y}_n)_{n \geq 0}$. Throughout this section we will consider the following two assumptions

(A1) $w$ is a differentiable function.

(A2) $Y_n$ converges almost surely to the uniform vector $\frac{1}{k} \mathbf{1}$.

We will again use the stochastic approximation method to obtain central limit theorems. The rate of convergence of the discrete stochastic approximation process depends on the eigenvalues of the Jacobian matrix when evaluated at the limiting vector. For the ODE associated with $Y_n$, the Jacobian matrix of $h$ at the equilibrium point $\frac{1}{k} \mathbf{1}$ is given by

$$\frac{\partial h(y)}{\partial y} = \frac{\partial}{\partial y} \frac{w(y)}{S_w(y)} R - I$$

(30)
where,
\[
\frac{\partial \mathbf{w}(y)}{\partial y S_w(y)} = \begin{bmatrix}
\frac{w'(y_1)}{S_w(y)} - \frac{w(y_1)w'(y_1)}{S_w(y)^2} & -\frac{w(y_1)w'(y_2)}{S_w(y)^2} & \cdots & -\frac{w(y_1)w'(y_k)}{S_w(y)^2} \\
\frac{w'(y_2)}{S_w(y)^2} & \frac{w(y_2)w'(y_2)}{S_w(y)} - \frac{w(y_2)w'(y_2)}{S_w(y)^2} & \cdots & -\frac{w(y_2)w'(y_k)}{S_w(y)^2} \\
\vdots & \vdots & \ddots & \vdots \\
-\frac{w(y_k)w'(y_1)}{S_w(y)^2} & -\frac{w(y_k)w'(y_2)}{S_w(y)^2} & \cdots & \frac{w(y_k)w'(y_k)}{S_w(y)} - \frac{w(y_k)w'(y_k)}{S_w(y)^2}
\end{bmatrix}
\]

That is,
\[
\frac{\partial \mathbf{w}(y)/S_w(y)}{\partial y} = \text{diag}\left(\frac{w'(y)}{S_w(y)}\right) + \left(\frac{-w(y_i)w'(y_j)}{S_w(y)^2}\right)_{i,j=1,2,\ldots,k}.
\] (31)

Therefore
\[
\left.\frac{\partial h(y)}{\partial y}\right|_{y=1} = \left(bI - \frac{b}{k}J\right) R - I = bR - \frac{b}{k}J - I
\] (32)

where \( J \equiv J_k \equiv 1^T 1 \) and \( I \equiv I_k \) is the \( k \times k \) identity matrix and
\[
b := \frac{w'(1/k)}{kw(1/k)}.
\] (33)

Note that \( b \leq 0 \) as \( w \) is a non-increasing function. Now, since \( R \) is a stochastic matrix, it has maximal eigenvalue 1 and suppose the remaining \( s \) distinct eigenvalues are \( \lambda_1, \lambda_2, \ldots, \lambda_s \). By Perron Frobenius Theorem, the stochastic matrix \( R \) has maximal eigenvalue 1. That is the absolute real part of all eigenvalue of a stochastic matrix \( R \) is less than 1, so without loss we assume \( 1 > \Re(\lambda_1) \geq \Re(\lambda_2) \geq \cdots \geq \Re(\lambda_s) \geq -1 \). Note that the right eigenvector corresponding to the maximal eigenvalue 1 of \( R \) is \( 1^T \) and
\[
Dh\left(\frac{1}{k} 1\right) 1^T = \left(bI - \frac{b}{k}J\right) R 1^T - 1^T = -1^T
\]

Thus, \(-1\) is an eigenvalue of \( Dh\left(\frac{1}{k} 1\right) \). Now, for an eigenvalue \( \lambda_i(\neq 1) \) of \( R \), and the corresponding right eigenvector \( v_i^T \) which is orthogonal to \( 1^T \), we have
\[
Dh\left(\frac{1}{k} 1\right) v_i^T = \left(bI - \frac{b}{k}J\right) R v_i^T - v_i^T = (b\lambda_i - 1)v_i^T
\]
Therefore the Jacobian matrix \( Dh \left( \frac{1}{k} 1 \right) \) has eigenvalues \( b\lambda_i - 1 \) for every \( i = 1, \cdots, s \). Now define

\[
\rho := \max \{ 0, 1 - b\Re(\lambda_s) \}. \tag{34}
\]

For the ODE associated to the colour count proportions \( \tilde{Y}_n \),

\[
\frac{\partial \tilde{h}(\tilde{y})}{\partial \tilde{y}} = \frac{\partial w(y)}{\partial y} S_w(y) R - I = \frac{\partial h(y)}{\partial y}. \tag{35}
\]

Note that the \( Dh \left( \frac{1}{k} 1 \right) \) is a diagonal matrix, if and only if,

\[-bR_{i,j} + b/k = 0 \quad \forall \ i \neq j \iff R = \frac{1}{k} J.\]

In fact, for this choice of \( R \), we have

\[U_{n+1,i} = U_{0,i} + \frac{n+1}{k}, \quad \forall \ i = 1, 2, \cdots, k\]

so that

\[Y_{n+1,i} = \frac{U_{0,i}}{n+1} + \frac{1}{k}\]

Thus for any weight function \( w \), we have

\[Y_{n+1,i} \to \frac{1}{k} \quad \text{as} \ n \to \infty, \ \forall \ i.\]

We now use the CLT results obtained for general Jacobian matrix \( Dh \left( \frac{1}{k} 1 \right) \), using stochastic approximation by Zhang [21]. For the results stated in the next section, we recall that the exponential of a matrix \( A \) is defined as, \( e^A := \sum_{l=0}^\infty \frac{A^l}{l!} \) and for \( x \in \mathbb{R} \) and a matrix \( A \), \( x^A \) is defined as \( \exp ((\log x)A) \).

4.1. **The case** \( \rho > 1/2 \).

**Theorem 3.** Suppose \( w \) is a non-increasing function and \( R \) is a doubly stochastic matrix such that \( \rho > 1/2 \), then under assumptions (A1) and (A2),

\[
\sqrt{n} \left( Y_n - \frac{1}{k} 1 \right) \implies N(0, \Sigma_1) \tag{36}
\]

and

\[
\sqrt{n} \left( \tilde{Y}_n - \frac{1}{k} 1 \right) \implies N \left( 0, \tilde{\Sigma}_1 \right) \tag{37}
\]

with

\[
\tilde{\Sigma}_1 = \frac{1}{k} \left[ \Lambda_1 - \frac{1}{k(1-2b)} J \right] \quad \text{and} \quad \Sigma_1 = R^T \tilde{\Sigma}_1 R \tag{38}
\]

where \( \Lambda_1 \) is the unique solution of the Sylvester’s equation (see [6])

\[A\Lambda_1 - \Lambda_1 A^T = I \tag{39}\]
for $A = \frac{1}{2}I - bR^T$. In particular, if $R$ is a normal matrix then

$$\Sigma_1 = \frac{1}{k} \left[ R^T (I - b(R^T + R))^{-1} R - \frac{1}{k(1 - 2b)} J \right]$$

(40)

where $b$ is defined in equation (33).

Remark 1. Note that for a Pólya type urn, that is when $R = I$, assumption (A2) holds and $\rho = 1 - b > \frac{1}{2}$, therefore under assumption (A1), Theorem 3 holds with

$$\Sigma_1 = \frac{1}{k(1 - 2b)} \left[ I - \frac{1}{k} J \right] = \frac{1}{1 - 2b} \Gamma,$$

(41)

where $\Gamma = \frac{1}{k}I - \frac{1}{k^2}J$.

4.2. The case $\rho = 1/2$. Note that

$$\rho = \frac{1}{2} \iff \Re(\lambda_s) = \frac{kw(\frac{1}{k})}{2w'(\frac{1}{k})}$$

and since $\Re(\lambda_s) \geq -1$ thus, $\rho = \frac{1}{2}$ case is possible only when $kw(\frac{1}{k}) \leq -2w'(\frac{1}{k})$.

Let $\nu$ be the multiplicity of eigenvalue $\lambda_s$.

Theorem 4. Let $w$ be a non-increasing, twice differentiable weight function such that $\rho = 1/2$, then under assumption (A2),

$$\frac{\sqrt{n}}{(\log n)^{\nu - 1/2}} \left( Y_n - \frac{1}{k} \right) \Rightarrow N(0, \Sigma_2)$$

(42)

and

$$\frac{\sqrt{n}}{(\log n)^{\nu - 1/2}} \left( \tilde{Y}_n - \frac{1}{k} \right) \Rightarrow N(0, \tilde{\Sigma}_2)$$

(43)

where

$$\tilde{\Sigma}_2 = \frac{1}{k} \Lambda_2, \quad \text{and} \quad \Sigma_2 = R^T \tilde{\Sigma}_2 R,$$

(44)

and

$$\Lambda_2 = \lim_{n \to \infty} \frac{1}{(\log n)^{2\nu - 1}} \int_0^{\log n} e^{-u} e^{buR^T} e^{buR} du.$$

(45)

4.3. The case $\rho < 1/2$. Note that

$$\rho < \frac{1}{2} \iff \Re(\lambda_s) < \frac{kw(\frac{1}{k})}{2w'(\frac{1}{k})}$$

and thus, $\rho < \frac{1}{2}$ case is not possible whenever $kw(\frac{1}{k}) > -2w'(\frac{1}{k})$, which is true for sufficiently large $k$, assuming that $w(0)$ and $w'(0)$ are both finite. Therefore for a negatively
reinforced urn scheme, \( \rho < \frac{1}{2} \) is a rare case, and in this case we have the following convergence result.

**Theorem 5.** Let \( w \) be a non-increasing weight function which is twice differentiable and \( R \) be a doubly stochastic matrix, such that \( 0 < \rho < 1/2 \), then under assumption \((A2)\), there are complex random variables \( \xi_1, \cdots, \xi_n \) such that

\[
\frac{n^\rho}{\log n^{\nu-1}} \left( Y_n - \frac{1}{k} \right) - X_n \longrightarrow 0 \ a.s. \tag{46}
\]

where,

\[
X_n = \sum_{i: \Re(\lambda_i) = (1-\rho)/b} e^{-i(1-b\mathrm{Im}(\lambda_i))\log n} \xi_i v_i
\]

and \( v_i \) is a right eigenvector of \( Dh \left( \frac{1}{k} \right) \) with respect to the eigenvalue \( b\lambda_i - 1 \).

5. **Technical Results**

Since we study the convergence of \( (Y_n)_{n \geq 0} \) and \( (\tilde{Y}_n)_{n \geq 0} \) through the stochastic approximation method, we consider the equilibrium points of \( h \) as possible limit points. Now, as mentioned earlier, a unique equilibrium point is guaranteed assuming that \( F \) is a contraction. In the next Proposition, we obtain sufficient conditions under which \( F \) is a contraction map.

**Proposition 2.** Suppose \( w \) is a non-increasing \( \text{Lip}(M) \) function, then \( F \) is a contraction whenever

(i) \( w(1) > 0 \) and \( \sqrt{k} > \frac{2M}{w(1)} \); or

(ii) \( w \) is a convex function and \( \sqrt{k}w(1/k) > 2M \).

**Remark 2.** If \( w \) is a non-increasing convex weight function and \( w(0) < \infty \), then \( F \) is a contraction whenever \( \sqrt{k} > \frac{4M}{w(0)} \), for \( k \) sufficiently large such that \( w(1/k) > w(0)/2 \).

**Proof.**

\[
\|F(x) - F(y)\| = \left\| \frac{w(x)}{S_w(x)} - \frac{w(y)}{S_w(y)} \right\| = \left\| \frac{w(x)S_w(y) - w(y)S_w(x)}{S_w(x)S_w(y)} \right\| = \left\| \frac{(S_w(y) - S_w(x)) w(x) - S_w(x)(w(y) - w(x))}{S_w(x)S_w(y)} \right\| \leq \frac{|S_w(y) - S_w(x)| \|w(x)\| + S_w(x) \|w(y) - w(x)\|}{S_w(x)S_w(y)}
\]
Note that
\[ \|w(x)\|^2 = \sum_{i=1}^{k} |w(x_i)|^2 \leq \left( \sum_{i=1}^{k} w(x_i) \right)^2 = S_w(x)^2 \]
\[ \implies \|w(x)\| \leq S_w(x). \] (47)

Therefore,
\[ \|F(x) - F(y)\| \leq \frac{|S_w(y) - S_w(x)| + \|w(x) - w(y)\|}{S_w(y)} \] (48)

Now since \( w \) is a \( Lip(M) \) function we get
\[ \|w(x) - w(y)\|^2 = \sum_{i=1}^{k} |w(x_i) - w(y_i)|^2 \leq M^2 \sum_{i=1}^{k} |x_i - y_i|^2 = M^2 \|x - y\|^2 \] (49)

and
\[ |S_w(y) - S_w(x)| = \left| \sum_{j=1}^{k} w(y_j) - w(x_j) \right| \leq \sum_{j=1}^{k} |w(y_j) - w(x_j)| \leq M \sum_{j=1}^{k} |y_j - x_j| = M \|y - x\|_1 \]
\[ \leq M \sqrt{k} \|x - y\| \] (50)
The last inequality follows by Cauchy-Schwartz inequality. Finally from equations (49), (50) and (48), we get
\[ \|F(x) - F(y)\| \leq \frac{M \left( 1 + \sqrt{k} \right)}{S_w(y)} \|x - y\| \] (51)

Case (i): If \( w(1) > 0 \), then we can write \( S_w(y) \geq kw(1) \) and therefore from equation (51) we get
\[ \|F(x) - F(y)\| \leq \frac{M \left( 1 + \sqrt{k} \right) \|x - y\|}{kw(1)} \leq \frac{2M}{\sqrt{kw(1)}} \|x - y\| \] (52)

Thus \( F \) is a contraction if \( \sqrt{k} > \frac{2M}{w(1)} \).

Case (ii): Assuming that \( w \) is a convex function then for \( \Delta_k = \{ y \in \mathbb{R}^k : \sum_{i=1}^{k} y_i = 1 \} \)
\[ S_w(y) \geq kw(1/k), \forall y \in \Delta_k. \]
Therefore from equation (51) we get

$$\|F(x) - F(y)\| \leq \frac{M (1 + \sqrt{k})}{kw(1/k)} \|x - y\| \leq \frac{2M}{\sqrt{k}w(1/k)} \|x - y\|$$

Thus $F$ is a contraction if $\sqrt{k}w(1/k) > 2M$. \hfill \Box

The following proposition gives the sufficient condition for the stability of the equilibrium point $\frac{1}{k} \mathbf{1}$, in case of a doubly stochastic replacement matrix.

**Proposition 3.** Suppose $w$ is a non-increasing function on $[0, 1]$ and $R$ is doubly stochastic matrix with eigenvalues $1, \lambda_1, \cdots, \lambda_s$ then, $\frac{1}{k} \mathbf{1}$ is a stable equilibrium if

$$\Re(\lambda_i) > \frac{kw \left( \frac{1}{k} \right)}{w' \left( \frac{1}{k} \right)}, \ \forall i = 1, 2, \cdots, s.$$  

(54)

**Proof.** Since the eigenvalues of the Jacobian matrix are $-1$ and $b\lambda_i - 1$ for $i = 1, 2, \cdots, s$. Thus equilibrium point $\frac{1}{k} \mathbf{1}$ is stable, if only if

$$\Re(b\lambda_i - 1) < 0, \ \forall i = 1, 2, \cdots, s.$$  

(55)

$$\iff \Re(\lambda_s) > \frac{1}{b} = \frac{kw \left( \frac{1}{k} \right)}{w' \left( \frac{1}{k} \right)}.$$  

(56)

This completes the proof. \hfill \Box

**Corollary 2.** Notice that $\Re(\lambda_s) \geq -1$. Thus another sufficient condition for the stability is

$$k > -\frac{w' \left( \frac{1}{k} \right)}{w \left( \frac{1}{k} \right)}.$$  

(57)

**Remark 3.** Assuming that $w(0), w'(0+)<\infty$, equation (54) or (57) hold for $k$ sufficiently large.
6. Proofs

Proof of Theorem 1. Suppose $F$ is a contraction, then there exists a unique fixed point $y^*$ of $F$, such that $F(y^*) = y^*$. Then $h(y^*) = 0$.

that is, $y^*$ is also a unique equilibrium. Now using Theorem 2. and Corollary 3. from [7] (page 126) we get

$$Y_n \to y^*, \quad \text{as } n \to \infty.$$  

Now if $\tilde{y}^*$ is an equilibrium point of the ODE in equation (19) that is $\tilde{y}^*$ satisfies

$$\tilde{y}^* = \frac{w(\tilde{y}^* R)}{S_w(\tilde{y}^* R)}$$

then

$$\tilde{Y}_n \to \tilde{y}^*, \quad \text{as } n \to \infty.$$  

□

Proof of Theorem 2. We consider the linearized version of the non-linear ODE associated with $Y_n$, that is $\dot{y} = h(y)$, as in equation (15) around its equilibrium point $\frac{1}{k}1$, that is

$$\dot{x} = \frac{\partial h}{\partial x} \left( \frac{1}{k}1 \right) = Hx$$  

(58)

where $H = bR - \frac{b}{k}J - I$. Assuming that $\lambda \neq \frac{1}{b}$ for any eigenvalue $\lambda$ of $R$, then $(I - bR)$ is invertible and then $x^* = \frac{-b}{k} (I - bR)^{-1}$ is the unique equilibrium point of the linearized differential equation (58). By Hartman-Grobman Theorem (see [12] Chapter 9 and [11]) there exists a homeomorphism $f$ from a neighbourhood $U$ of $\frac{1}{k}1$ to a neighbourhood $V$ of $x^*$, such that $x(0) = f(y(0))$ implies $x(t) = f(y(t)) \forall t > 0$, where $x(t)$ is the solution of the linearized ODE, and $y(t)$ is the solution of the non-linear ODE. In particular, if the real part of all the eigenvalues of $H$ are positive then $\frac{1}{k}1$ is asymptotically stable, that is

$x(t) \to x^*$ almost surely and thus assuming the stability of $\frac{1}{k}1$ that is, if equation (54) holds then we get

$$y(t) \to \frac{1}{k}1 \quad \text{a.s.}$$

This completes the proof for $Y_n$. This also proves the convergence for $\tilde{Y}_n$, since the Jacobian matrices for $h$ and $\tilde{h}$ at $\frac{1}{k}1$ are same.

□

Proof of Theorem 3. Suppose $\rho > 1/2$ then (see appendix Theorem 7),

$$\sqrt{n} \left( Y_n - \frac{1}{k}1 \right) \implies N_k \left( 0, \Sigma_1 \right)$$
where
\[ \Sigma_1 = \int_0^\infty (e^{Hu})^T \Gamma_1 (e^{Hu}) \, du, \]

with
\[ H = \left( bI - \frac{b}{k} J \right) R - \frac{1}{2} I. \]

and
\[ \Gamma_1 = \lim_{n \to \infty} R^T E \left[ M_{n+1}^T M_{n+1} \big| \mathcal{F}_n \right] \]
\[ = \lim_{n \to \infty} R^T E \left[ \left( \chi_{n+1} - \frac{w(Y_n)}{S_w(Y_n)} \right)^T \left( \chi_{n+1} - \frac{w(Y_n)}{S_w(Y_n)} \right) \right] \mathcal{F}_n \]
\[ = \lim_{n \to \infty} R^T \left[ E \left[ \chi_{n+1} \chi_{n+1} \big| \mathcal{F}_n \right] - \frac{w(Y_n)^T w(Y_n)}{S_w(Y_n)^2} \right] R \]
\[ = R^T \left[ \frac{1}{k} I - \frac{1}{k^2} J \right] R. \]

Now observe that \( JR = RJ = J \), because \( R \) is doubly stochastic. Therefore
\[ e^{uH} = e^{buR - \frac{bu}{k} J} e^{-\frac{1}{2} I} = e^{buR - \frac{bu}{k} J - \frac{1}{2} I} \]

Again since \( R \) commutes with \( J \) and \( I \), we can write
\[ e^{uH} = e^{buR} e^{-\left( \frac{bu}{k} \right) J} e^{-\left( \frac{u}{2} \right) I} \]
\[ = e^{-\frac{u}{2}} \sum_{j=0}^{\infty} \left( \frac{-bu}{k} J \right)^j \frac{j!}{j!} e^{buR} \]
\[ = e^{-\frac{u}{2}} \left[ I + \frac{e^{-bu}}{k} - 1 \right] e^{buR} \] (59)

Now
\[ e^{uH T} \Gamma_1 = e^{-\frac{u}{2}} e^{buR T} \left[ I + \frac{e^{-bu}}{k} - 1 \right] R^T \left[ \frac{1}{k} J - \frac{1}{k^2} J \right] R \]
\[ = e^{-\frac{u}{2}} e^{buR T} \left[ R^T + \frac{e^{-bu}}{k} - 1 \right] \left[ \frac{1}{k} R - \frac{1}{k^2} J \right] \]
\[ = e^{-\frac{u}{2}} e^{buR T} \left[ \frac{1}{k} R^T R - \frac{1}{k^2} J \right] \] (60)
\[ e^{uH} \Gamma_1 e^{uH} = e^{-u} e^{buR^T} \left[ \frac{1}{k} R^T R - \frac{1}{k^2} J \right] \left[ I + \frac{e^{-bu} - 1}{k} J \right] e^{buR} \]
\[ = e^{-u} e^{buR^T} \left[ \frac{1}{k} R^T R - \frac{1}{k^2} J \right] e^{buR} \]
\[ = e^{-u} \left[ \frac{1}{k} e^{buR^T} R^T R e^{buR} - \frac{e^{2bu}}{k^2} J \right] \]
\[ = e^{-u} R^T \left[ \frac{1}{k} e^{buR^T} e^{buR} - \frac{e^{2bu}}{k^2} J \right] R \]  \hfill (61)

The last step follows as \( R \) and \( e^{buR} \) commute. Now we can rewrite the last expression as
\[ \frac{1}{k} R^T \left[ e^{-u} e^{buR^T} e^{buR} - \frac{e^{-u(1-2b)}}{k} J \right] R \]
\[ = \frac{1}{k} R^T \left[ e^{-u(1-2b)} e^{u(2bR-I)} - \frac{e^{-u(1-2b)}}{k} J \right] R \]  \hfill (62)

Thus,
\[ \int_0^\infty e^{uH} \Gamma_1 e^{uH} du = \frac{1}{k} R^T \left[ \Lambda_1 - \frac{1}{k(1-2b)} J \right] R \]  \hfill (63)

where
\[ \Lambda_1 = \int_0^\infty \exp(-u(1/2I - bR^T)) \exp(-u(bR - 1/2I)) du \]  \hfill (64)

which satisfies the Sylvester's equation:
\[ AA_1 - \Lambda_1 B = I. \]  \hfill (65)

for \( A = \frac{1}{2} I - bR^T \) and \( B = bR - \frac{1}{2} I = -A^T \). In particular, if \( R \) is a normal matrix then
\[ \Lambda_1 = (A - B)^{-1} = (I - b(R + R^T))^{-1} \]
satisfies the Sylvester's equation, if and only if
\[ I = A(A - B)^{-1} - (A - B)^{-1} B \]
\[ \iff A - B = (A - B)A(A - B)^{-1} - B \]
\[ \iff A = (A - B)A(A - B)^{-1} \]
\[ \iff A(A - B) = (A - B)A \]
\[ \iff AB = BA \]
\[ \iff AA^T = A^T A \]
\[ \iff R^T R = RR^T. \]

Therefore for a normal matrix \( R \)
\[ \Sigma_1 = \frac{1}{k} R^T \left[ (I - b(R + R^T))^{-1} - \frac{1}{k(1-2b)} J \right] R. \]
Similarly, if $\rho > 1/2$ we get
\[
\sqrt{n} \left( \bar{Y}_n - \frac{1}{k} \mathbf{1} \right) \implies N(0, \tilde{\Sigma}_1)
\]
where
\[
\tilde{\Sigma}_1 = \int_0^\infty (e^{suH})^T \tilde{\Gamma}_1 (e^{suH}) \, du,
\]
for
\[
H = \left. \frac{\partial h}{\partial y} \right|_{y = \frac{1}{k} \mathbf{1}} + \frac{1}{2} I = bR - \frac{b}{k} J - \frac{1}{2} I
\]
and
\[
\tilde{\Gamma}_1 = \lim_{n \to \infty} E \left[ M_{n+1}^T M_{n+1} \big| \mathcal{F}_n \right] = \left[ \frac{1}{k} I - \frac{1}{k^2} J \right].
\]
Now similar to the expression obtained in equation (61) we get
\[
e^{uH^T \tilde{\Gamma}_1 e^{uH}} = e^{-u} \left[ \frac{1}{k} e^{buR^T} e^{buR} - \frac{e^{2bu}}{k^2} J \right]
\]
and therefore,
\[
\tilde{\Sigma}_1 = \int_0^\infty e^{uH^T \tilde{\Gamma}_1 e^{uH}} \, du = \frac{1}{k} \left[ \Lambda_1 - \frac{1}{k(1-2b)} J \right].
\]
where $\Lambda_1$ satisfies the Sylvester’s equation (65).

Proof of Theorem 4. Suppose $\rho = 1/2$, then the result holds under the following two assumptions (see appendix Theorem 7)
(1)
\[
\frac{1}{n} \sum_{m=1}^n E \left[ \|M_m R\|^2 I \{ \|M_m R\| \geq \epsilon \sqrt{n} \} \big| \mathcal{F}_{m-1} \right] \to 0.
\]
a.s. or in $L^1$, for all $\epsilon > 0$.
(2) For some $\epsilon > 0$, as $y \to \frac{1}{k} \mathbf{1}$
\[
h(y) = h \left( \frac{1}{k} \mathbf{1} \right) + \left( y - \frac{1}{k} \mathbf{1} \right) Db \left( \frac{1}{k} \mathbf{1} \right) + o \left( \left\| y - \frac{1}{k} \mathbf{1} \right\|^{1+\epsilon} \right)
\]
The Linderberg condition in equation (68) holds as $\|M_m R\| \leq k(k+1)$ for all $m$, and for $\sqrt{n} > \frac{k(k+1)}{\epsilon}$, $I \{ \|M_m R\| \geq \epsilon \sqrt{n} \} = 0$ for all $m$. The second condition (69) is also satisfied for a twice differentiable function $h$. Thus for $\rho = 1/2$, (see appendix Theorem 7), we have
\[
\frac{\sqrt{n}}{\log n^{\nu-1/2}} \left( Y_n - \frac{1}{k} \mathbf{1} \right) \implies N_k(0, \Sigma_2)
\]
\[
\Sigma_2 = \lim_{n \to \infty} \frac{1}{(\log n)^{2\nu-1}} \int_0^{\log n} e^{uhT} \Gamma_1 e^{uH} du.
\]

Now from equation (62), we get
\[
\Sigma_2 = \frac{1}{k} R^T \left[ \Lambda_2 - \lim_{n \to \infty} \frac{1 - n^{-(1-2b)}}{(\log n)^{2\nu-1}} J \right] R
\]
\[
= \frac{1}{k} R^T \Lambda_2 R
\]
where
\[
\Lambda_2 = \lim_{n \to \infty} \frac{1}{(\log n)^{2\nu-1}} \int_0^{\log n} \exp \left( -u(1/2I - bR^T) \right) \exp \left( -u(bR - 1/2I) \right) \, du.
\]
\[\text{(70)}\]

Similarly for \( \tilde{Y}_n \) the required Linderberg condition holds that is
\[
\frac{1}{n} \sum_{m=1}^{n} E \left[ \|M_m\|^2 I \{ \|M_m\| \geq \epsilon \sqrt{n} \} \mid \mathcal{F}_{m-1} \right] \to 0.
\]
\[\text{(71)}\]
and therefore by Theorem 7 in the appendix we get
\[
\tilde{\Sigma}_2 = \frac{1}{k} R^T \left[ \Lambda_2 - \lim_{n \to \infty} \frac{1 - n^{-(1-2b)}}{(\log n)^{2\nu-1}} J \right] = \frac{1}{k} \Lambda_2
\]
where \( \Lambda_2 \) is as given in equation (70).
\[\square\]

**Proof of Theorem 5.** Proof of this theorem follows from Theorem 7 in the appendix.
\[\square\]

7. **Examples**

7.1. **Linear weight function.**

Let \( w : [0, 1] \to \mathbb{R}^+ \) be such that:
\[
w(y) = \theta - y; \quad \text{for } \theta \geq 1, \quad \text{and } y \in [0, 1].
\]
Then the stochastic approximation algorithm in equation (6) holds with
\[
h(y) = y (AR - I)
\]
\[\text{(72)}\]
for
\[
A_{k \times k} = \frac{1}{k\theta - 1} (\theta J - I).
\]

Notice that, an equilibrium point of the associated ODE is also a stationary distribution of \( AR \). Thus if \( AR \) is irreducible and \( y^* \) is its unique stationary distribution then by Theorem 1 we get
\[
Y_n \longrightarrow y^* \text{ a.s.}
\]
The above almost sure convergence result was also proved in Theorem 1 of [2]. In fact, in [2] a necessary and sufficient condition for \( AR \) to be irreducible is given and also convergence
for the case when $AR$ is reducible is obtained.

For the central limit theorem, consider a doubly stochastic matrix $R$. In this case, the constant $b$, as defined in equation (33) is

$$b = -\frac{1}{k\theta - 1}$$

and $\rho$ as defined in equation (34) is

$$\rho = 1 + \frac{R(\lambda_s)}{k\theta - 1}.$$ 

Now we separately consider $k = 2$ and $k \geq 3$, in order to identify the possible values of $\rho$. First let $k = 2$, and $R = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$, where $p \in [0, 1]$. Then $R$ has eigenvalues 1 and $2p - 1$ and

$$\rho = 1 + \frac{2p - 1}{2\theta - 1}.$$ 

Thus,

$$\rho \geq \frac{1}{2} \iff 2p - 1 \geq \frac{1 - 2\theta}{2}.$$ 

(73)

Therefore, by Theorem 3 and Theorem 4, we get

$$\sigma_n \left(Y_{n,1} - \frac{1}{2} \right) \Rightarrow N \left(0, \sigma^2 \right)$$ 

(74)

where $\sigma_n = \begin{cases} \sqrt{n} \log n & \text{if the eigenvalue of } R \text{ (other than 1) is equal to } \frac{1 - 2\theta}{2}, \\
\sqrt{n} & \text{if the eigenvalue of } R \text{ (other than 1) is } > \frac{1 - 2\theta}{2}. \end{cases}$

As shown in the figure below, $\rho$ is equal to $\frac{1}{2}$ on the highlighted straight line, and below this line $\rho$ is less than $\frac{1}{2}$. Note that for a large range of minimum eigenvalue of $R$ and parameter $\theta$, $\rho$ is greater than $\frac{1}{2}$ for which the asymptotic normality holds with scaling factor $\sqrt{n}$. 
Now for $k \geq 3$, using the fact that $\Re(\lambda_s) \geq -1$, we get
\[
\rho \geq 1 - \frac{1}{k\theta - 1}
\]
and therefore for $k \geq 3$, $\rho \geq 1/2$. Thus, there is no non-Gaussian limiting behavior. In fact, in this case we always have Gaussian limit with $\sqrt{n}$ scaling, except when $\rho = \frac{1}{2}$, which can only happen when $k = 3$ and then
\[
\rho = 1/2 \quad \Leftrightarrow \quad \Re(\lambda_s) = \frac{3\theta - 1}{2}.
\]
Which is possible only if $\theta = 1$ and $\Re(\lambda_s) = -1$, and for a $3 \times 3$ stochastic matrix, there can only be at most one eigenvalue with real part equal to $-1$.

The above result for $k \geq 2$ and $\rho \geq \frac{1}{2}$ has also been obtained in Theorem 2 of [2]. In fact, central limit theorem for a general class of replacement matrices is given in [2].

7.2. **Inverse power law weight function.**
Let
\[
w(x) = (\theta + x)^{-\alpha}, \quad \text{for } \theta, \alpha > 0
\]
and $R$ be a doubly stochastic matrix. Then $b = -\frac{\alpha}{k\theta + 1}$ and therefore by Proposition 3, $\frac{1}{k}$ is a stable equilibrium point if
\[
\Re(\lambda_i) > -\frac{k\theta + 1}{\alpha} \quad \text{for } i = 1, 2, \ldots, s.
\]
In particular, the above condition for stability holds if \( R = I \) or if \( \alpha < k\theta + 1 \). Also,
\[
\rho = 1 + \frac{\alpha}{k\theta + 1} \Re(\lambda_s).
\]
and then
\[
\rho \geq 1/2 \iff \Re(\lambda_s) \geq -\frac{k\theta + 1}{2\alpha}.
\]
Therefore, the scaling for central limit theorems depend on the values of \( \alpha \) and \( \theta \).

In particular, for \( \alpha = 1 \) the condition for stability in equation (76) holds for \( k \geq 2 \) and thus by Theorem 2 we get
\[
Y_n \longrightarrow \frac{1}{k^1} \text{ a.s.}
\]

For the central limit theorem, as shown in the figure below \( \rho \) takes value more than \( \frac{1}{2} \) in the shaded region and thus for given \( \theta \) and \( R \), the central limit theorems hold accordingly.

Figure 2. Range of \( \rho \) for given \( \theta \) and minimum eigenvalue of \( R \)

Note that, above a critical value for \( \theta \), we observe asymptotic normality with the scaling factor of \( \sqrt{n} \) for any choice of replacement matrix \( R \). Further, the region for \( \rho < \frac{1}{2} \) decreases as we increase the number of colours \( k \).
Now we show that, the uniform vector $\frac{1}{k} \mathbf{1}$ is not necessarily a stable equilibrium for any doubly stochastic matrix. Let

$$R = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix},$$

then the condition for stability in equation (76) is not satisfied for $\alpha > k\theta + 1$, and for this choice of $\alpha$ the equilibrium point $\frac{1}{4} \mathbf{1}$ is an unstable point. In fact, by Theorem 1 of [18] one can show that in this case

$$P \left( Y_n \to \frac{1}{4} \mathbf{1} \right) = 0.$$

### 7.3. Exponential weight function.

Let

$$w(x) = \exp \left( -\frac{x}{\theta} \right), \text{ for } \theta > 0$$

then

$$b = -\frac{1}{k\theta} \quad \text{and} \quad \rho = 1 + \frac{\Re(\lambda_s)}{k\theta}$$

and thus $\frac{1}{k} \mathbf{1}$ is a stable equilibrium for a doubly stochastic matrix if

$$\Re(\lambda_i) > -k\theta; \quad \text{for } i = 1, 2, \ldots, s.$$ 

and

$$\rho \geq 1/2 \iff \Re(\lambda_s) \geq -\frac{k\theta}{2}.$$ 

As shown in the following graph, $\rho > \frac{1}{2}$ in the shaded region and equal to $\frac{1}{2}$ on the straight line.
In this section we state some of the general results in the stochastic approximation theory, for the discrete stochastic approximation algorithm $X_n$ in $\mathbb{R}^d$ (as defined in Section 2), satisfying

$$X_{n+1} = X_n + \gamma_{n+1} h(X_n) + \gamma_{n+1} M_{n+1} R.$$ 

**Definition 3.** A set $A$ is called stable (or attractor) if for each $\epsilon > 0$ there is a $\delta > 0$ such that all trajectories starting in $N_\delta(A)$ never leave $N_\epsilon(A)$.

When $h$ is differentiable, an equivalent definition of a stable (or attractor) equilibrium point is given below.

**Definition 4.** (Stable/unstable equilibrium point). An equilibrium $x^*$ is called stable (or attractor) if all the eigenvalues of $Jh(x^*)$ have negative real part, and it is called unstable otherwise.

**Theorem 6** (Almost sure convergence). Assume that $h$ is a Lipschitz function and

$$\sup_{n \geq n_0} \mathbb{E} \left[ \|M_{n+1}\|^2 | \mathcal{F}_n \right] < \infty \text{ a.s.,}$$

and $\gamma_n$ is a positive sequence satisfying

$$\sum_{n=1}^{\infty} \gamma_n = \infty, \quad \text{and} \quad \sum_{n=1}^{\infty} \gamma_n^2 < \infty$$

On the event $A_\infty = \{ \omega \mid h(X_n(\omega))_{n \geq 0} \text{ is bounded} \}$, $\mathbb{P}(dw) - \text{a.s. then}$

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**Figure 3.** Range of $\rho$ for given $\theta$ and minimum eigenvalue of $R$
(a) (Theorem 5.7 [5]) The limit set of $X(t)$ that is,

$$L(X(t)) = \cap_{t \geq 0} X[t, \infty)$$

is almost surely an internally chain transitive set for the unique solution $\phi(x_0, t)$ of the mean limit ODE

$$\dot{x} = h(x). \quad (77)$$

(b) If the only internally chain transitive invariant set are isolated equilibrium points of $h$ then $X_n$ converges a.s. to the set of equilibrium points of $h$.

(c) If there is a unique equilibrium point that is $\{h = 0\} = \{x^*\}$ and $\phi(x_0, t) \to x^*$ as $t \to \infty$ locally uniformly in $x_0$, then

$$X_n \to x^* \text{ as } n \to \infty \text{ a.s.}$$

(c) If $k = 2$ and $\{h = 0\}$ is locally finite, then

$$L(X(t)) \subset \{h = 0\}$$

i.e.

$$X_n \to X_\infty \in \{h = 0\}.$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_p$ be the eigenvalues of $Dh(x^*)$ with $\dim(J_t) = \nu_t$ for every $t = 1, 2, \ldots, p$. Let

$$\rho = \min_{1 \leq i \leq p} \{-Re(\lambda_i)\} \quad \text{and} \quad \nu = \max\{\nu_t : Re(\lambda_t) = \rho\}.$$

We will need the following assumptions to state the central limit theorems for different values of $\rho$.

A1 $x^*$ is a stable equilibrium.

A2 The Lindeberg condition

$$\frac{1}{n} \sum_{m=1}^{n} E \left[ \|M_m\|^2 I\{\|M_m\| \geq \epsilon \sqrt{n}\} \middle| \mathcal{F}_{m-1} \right] \to 0 \text{ a.s.}$$

A3 $\frac{1}{n} \sum_{m=1}^{n} E \left[ M_{m+1}^T M_{m+1} \middle| \mathcal{F}_m \right] \to \Gamma$ \text{ a.s. or in } L_1$

**Theorem 7** (Scaling Limits [21]). Suppose $X_n \to x^*$ a.s. then

(1) if $\rho > 1/2$, and the following two conditions are satisfied where $\Gamma$ is deterministic symmetric positive semidefinite matrix, and assumptions A1, A2, and A3 hold then

$$\sqrt{n} (X_n - x^*) \overset{d}{\to} N(0, \Sigma_1)$$

where

$$\Sigma_1 = \int_{0}^{\infty} (e^{-(Dh(x^*)+1/2I)u})^T \Gamma e^{-(Dh(x^*)+1/2I)u} du \quad (78)$$
Suppose $\rho = 1/2$ and $w$ is a twice differentiable function and assumption A2 holds then

$$\sqrt{n} \frac{1}{\log n^{\nu-1/2}} (X_n - x^*) \Rightarrow N(0, \Sigma_2).$$

where

$$\Sigma_2 = \lim_{n \to \infty} \frac{1}{(\log n)^{2\nu-1}} \int_0^{\log n} \left( e^{-(Dh(x^*)-1/2)u} \right)^T \Gamma e^{-(Dh(x^*)-1/2)u} du$$

(3) Suppose $0 < \rho < 1/2$, $w$ is twice differentiable and assumption A3 holds then

$$\frac{n^\rho}{\log n^{\nu-1}} (X_n - x^*) - X_n \to 0$$

where $X_n$ is random vector defined as

$$X_n = \sum_{i: \text{Re}(\lambda_i) = \rho} e^{-i\text{Im}(\lambda_i)\log n} \xi_i v_i$$

and $v_i$ is the left eigenvector of $Dh(x^*)$ with respect to the eigenvalue $\lambda_i$.

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