CONFORMAL HIGGS MODEL: GAUGE FIELDS CAN PRODUCE A 125 GEV RESONANCE

ROBERT K. NESBET
IBM Almaden Research Center, 650 Harry Road, San Jose, CA 95120, USA
rkn@earthlink.net

Received received date
Revised revised date

Recent cosmological observations and compatible theory offer an understanding of long-mysterious dark matter and dark energy. The postulate of universal conformal local Weyl scaling symmetry, without dark matter, modifies action integrals for both Einstein-Hilbert gravitation and the Higgs scalar field by gravitational terms. Conformal theory accounts both for observed excessive external galactic orbital velocities and for accelerating cosmic expansion. SU(2) symmetry-breaking is retained by the conformal scalar field, which does not produce a massive Higgs boson, requiring an alternative explanation of the observed LHC 125GeV resonance. Conformal theory is shown here to be compatible with a massive neutral particle or resonance $W^2$ at 125GeV, described as binary scalars $g_{\mu\nu}W^\mu W^\nu$ and $g_{\mu\nu}Z^\mu Z^\nu$ interacting strongly via quark exchange. Decay modes would be consistent with those observed at LHC. Massless scalar field $\Phi$ is dressed by the $W^2$ field to produce Higgs Lagrangian term $\lambda(\Phi^*\Phi)^2$ with the empirical value of $\lambda$ known from astrophysics.

Keywords: LHC 125GeV resonance; Higgs scalar field; Conformal theory

PACS Nos.: 04.20.Cv, 14.80.Gt, 98.80.-k

1. Introduction

In the currently accepted ΛCDM paradigm for cosmology, gravitational phenomena not explained by general relativity as formulated by Einstein are attributed to cold dark matter. Dark energy $\Lambda$ remains without an explanation. The search for tangible dark matter has continued for many years\[1] without result.

Consideration of an alternative paradigm is motivated by this situation. An alternative postulate is that of universal conformal local Weyl scaling symmetry\[2], requiring local Weyl scaling covariance\[3] for all massless elementary physical fields, without dark matter. Conformal symmetry, valid for fermion and gauge boson fields\[4], is extended to both the metric tensor field of general relativity and the Higgs scalar field of elementary-particle theory\[5,6], with no novel elementary fields. Lagrangian densities of both conformal fields include gravitational terms. Neither model requires dark matter. Conformal gravity(CG)\[7,8,9,10,11,12,13,14] fits rotation data for 138 galaxies. The
conformal Higgs model (CHM) \cite{14,15,16,17} fits observed accelerating Hubble expansion for redshifts $z \leq 1$ (7.33 Gyr) accurately with one free constant parameter\cite{15}.

The CHM evaluates parameters of the Higgs model using observed empirical data from cosmology and electroweak particle physics. The implied parameter values are inconsistent with a massive Higgs particle\cite{15,17}, requiring an alternative explanation of the observed LHC 125GeV resonance\cite{18,19,20}. The CHM, extended here to include charged gauge fields $W_{\mu}^{\pm}$, is shown to be consistent with a novel neutral 125GeV resonance.

The Higgs model of electroweak physics postulates a scalar field $\Phi$\cite{6,7,8} whose classical field equation has an exact stable solution of finite amplitude. Gauge symmetry defines covariant derivatives of $\Phi$ that couple it to neutral $Z_{\mu}$ and charged $W_{\mu}^{\pm}$ gauge boson fields. Gauge boson masses result from scalar and gauge boson field equations\cite{6,8} evaluated at a semiclassical level. Electroweak masses depend only on the existence of a spacetime solution of the Higgs scalar field equation with nonvanishing spacetime amplitude. The induced masses do not require a stable fluctuation of the scalar field, the usual definition of a massive Higgs boson\cite{8}.

The Higgs model postulates incremental scalar field Lagrangian density\cite{8} $\Delta L_{\Phi} = w^{2}\Phi^{\dagger}\Phi - \lambda(\Phi^{\dagger}\Phi)^{2}$. The conformal Higgs model\cite{14} acquires an additional term $-\frac{1}{6}R\Phi^{\dagger}\Phi$ with gravitational Ricci scalar $R = g_{\mu\nu}R^{\mu\nu}$. The modified Lagrangian, defined for neutral gauge field $Z_{\mu}$\cite{15}, determine $w^{2}$, which becomes a cosmological constant in the conformal Friedmann cosmic evolution equation\cite{14}. Lagrangian term $w^{2}\Phi^{\dagger}\Phi$ is due to induced neutral $Z_{\mu}$ field\cite{15} which dresses the bare scalar field. Finite $w^{2}$ breaks conformal symmetry, but does not determine a Higgs mass.

Empirical $\lambda < 0$ is shown here to result from dressing bare $\Phi$ by an induced neutral scalar field $W_{2}$, defined as a strongly interacting pair of gauge bosons. Empirical data implies estimated $\lambda$ consistent with a spinless particle $W_{2}$ of mass 125GeV. This might account for the observed 125GeV resonance\cite{15,19,20}. The model of neutral $W_{2}$ proposed here does not conflict with other implications of electroweak theory. The Higgs mechanism is retained by the massless Higgs scalar field of finite amplitude. Parameters $w^{2}$ and $\lambda$, required for nonzero $\Phi^{\dagger}\Phi$, both result from the $U(1) \times SU(2)$ covariant derivative of the Higgs field, which produces finite classical source density for two neutral fields. Neutral $Z_{\mu}$ dresses field $\Phi$ to produce Lagrangian term $w^{2}\Phi^{\dagger}\Phi$. Neutral $W_{2}$ dresses field $\Phi$ to produce $-\lambda(\Phi^{\dagger}\Phi)^{2}$.

Conformal gravity\cite{4,12} has recently been applied to fit anomalous galactic rotation data for 138 galaxies\cite{21,22,23}, without dark matter. Formal objections\cite{24,25,26} have been discussed and resolved in detail.\cite{27,17,28} McGaugh et al\cite{29} show for 153 galaxies that observed radial acceleration $a$ is effectively a universal function of the classical Newtonian acceleration $a_N$, computed for the observed baryonic distribution. Such a universal correlation function, $a(a_N) = \nu(a_N/a_0)a_N$, is a basic postulate of MOND\cite{30,31}. This removes uncertainty in earlier studies due to adjustment of mass-to-light ratios for individual galaxies. This conclusion is disputed\cite{13} by a purely CG fit to the same data. Resolution requires more accurate data at large
radii. Together with the CHM and the depleted halo model of galactic halos, without dark matter, CG implies a similar correlation function if Mannheim non-classical acceleration $\gamma$ is mass-independent. This strongly suggests, for an isolated galaxy, that dark matter can be eliminated and dark energy explained by consistent conformal theory.

2. Review of theory

Variational theory for fields in general relativity is a straightforward generalization of classical field theory. Given scalar Lagrangian density $L$, action integral $I = \int d^4x \sqrt{-g}L$ is required to be stationary for all differentiable field variations, subject to appropriate boundary conditions. The determinant of metric tensor $g_{\mu\nu}$ is denoted here by $g$. Riemannian metric covariant derivatives $D_\lambda$ are defined such that

$$4D_\lambda g^{\mu\nu} = 0.$$

Conformal symmetry is defined by invariance of action integral $I = \int d^4x \sqrt{-g}L$ under local Weyl scaling, such that $g_{\mu\nu}(x) \rightarrow g_{\mu\nu}(x) \Omega^2(x)$ for arbitrary real differentiable $\Omega(x)$, with fixed coordinates $x^\mu$. For any Riemannian tensor, $T(x) \rightarrow \Omega d(x) T(x) + R(x)$ defines weight $d[T]$ and residue $R[T]$. $d[\Phi] = -1$ for a scalar field. Conformal Lagrangian density $L$ must have weight $d[L] = -4$ and residue $R[L] = 0$, up to a 4-divergence.

Gravitational field equations are determined by metric functional derivative $X^{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\delta I}{\delta g_{\mu\nu}}$. Any scalar $L_a$ determines energy-momentum tensor $\Theta_{\mu\nu} = -2X_{a \mu\nu}$, evaluated for a solution of the coupled field equations. Generalized Einstein equation $\sum_a X_{a \mu\nu} = 0$ is expressed as $X^{\mu\nu} = \frac{1}{2} \sum_{a \neq g} \Theta_{a \mu\nu}$. Hence summed trace $\sum_a g_{\mu\nu}X_{a \mu\nu}$ vanishes for exact field solutions. Concurrent solution of exact equations for both Higgs field $\phi_0(t)$ and Friedmann scale $a(t)$ is needed for accurate extension to large redshifts $z$. The derivation of $a(t)$ for $z \leq 1$ and an earlier derivation for $z \leq 1090\text{(CMB)}$ treat $\dot{\phi}/\phi$ as a constant parameter.

3. Conformal scalar field

The fundamental postulate that all primitive fields have conformal Weyl scaling symmetry is satisfied by spinor and gauge fields, but not by the scalar field of the conventional Higgs model. A conformally invariant action integral is defined for complex SU(2) doublet scalar field $\Phi$ by Lagrangian density

$$L_\Phi = (\partial_\mu \Phi)^\dagger \partial^\mu \Phi - \frac{1}{6} R \Phi^4 - \lambda (\Phi^4)^2,$$

where $R$ is the gravitational Ricci scalar. The Higgs model postulates incremental Lagrangian density $\Delta L_\Phi$, which adds term $w^2 \Phi^4$ to $L_\Phi$. Because this $w^2$ term breaks conformal symmetry, universal conformal symmetry requires it to be produced dynamically.

The conformal scalar field equation including parametrized $\Delta L_\Phi$ is

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} \partial^\mu \Phi) = (-\frac{1}{6} R + w^2 - 2\lambda \Phi^4) \Phi.$$

(2)
Neglecting second-order time derivative terms for $\Phi$, $\Phi^\dagger\Phi = \phi_0^2 = (w^2 - \frac{1}{6}R)/2\lambda$ generalizes the Higgs construction if this ratio is positive. $w^2 < \frac{1}{6}R$, which implies $\lambda < 0$.

Because Ricci scalar $R$ varies on a cosmological time scale, it induces an extremely weak time dependence of $\phi_0^2$, which in turn produces source current densities for the gauge fields. The resulting coupled semiclassical field equations determine nonvanishing but extremely small parameter $w^2$, in agreement with the cosmological constant deduced from Hubble expansion data. This argument depends only on squared magnitudes of quantum field amplitudes. Conformal theory is shown here to determine biquadratic term $\lambda(\Phi^\dagger\Phi)^2$ by a similar semiclassical argument.

4. Gauge fields and Higgs parameters

In standard Higgs theory, it is assumed that charged component $\Phi_+\Phi$ of the postulated SU(2) doublet scalar field vanishes identically, while $\Phi^\dagger\Phi = \phi_0^2$ is a space-time constant. In conformal theory, Ricci scalar $R$ causes $\phi_0$ to vary on a cosmological time scale. Extending a derivation for neutral $Z^\mu$,[14] mass formulae $m^2_Z = \frac{1}{2}g^2_w\phi_0^2$, $m^2_W = \frac{1}{2}g^2_w\phi_0^2$, $m^2_\lambda = 0$. $\Delta\mathcal{L}$ implies source current density $J^0_Z = -ig_z\phi_0\Phi^\dagger\Phi$ and $Z^0 = J^0_Z/m^2_Z$, which determines Higgs parameter $w^2 = \frac{1}{4}g^2_wZ^\mu Z^\mu = (\frac{\phi_0}{\phi})^2$.

From the covariant derivative of $\Phi$, with $\Phi_+ = 0$,

$$\Delta\mathcal{L} = (-\frac{i\mu}{\sqrt{2}}W_{-\mu}\Phi^\dagger)(\frac{i\mu}{\sqrt{2}}W^\mu\Phi)$$

$$+((\partial_\mu + \frac{ig_z}{2}Z^\mu)^\dagger\Phi^\dagger)(\partial_\mu - \frac{ig_z}{2}Z^\mu)\Phi - \partial_\mu\Phi^\dagger\partial^\mu\Phi$$

$$= (\frac{1}{4}g^2_w(W^\mu_+ W^\mu_+ + W^\mu_+ W^\mu_-) + \frac{1}{4}g^2_wZ^\mu Z^\mu)\Phi^\dagger\Phi$$

$$-\partial_\mu\Phi^\dagger(\frac{1}{2}ig_zZ^\mu\Phi) + (\frac{1}{2}ig_zZ^\mu\Phi^\dagger)\partial^\mu\Phi.$$  \hspace{1cm} (3)

$W^\mu_+ - W^\mu_- = \frac{1}{2}(W^\mu_+ W^\mu_+ + W^\mu_+ W^\mu_-)$ is shown here to contribute to $-\lambda(\Phi^\dagger\Phi)^2$. Although an independent field $W^\mu_\pm$ would violate charge neutrality, there is no contradiction in treating neutral composite scalar $WW = g_{\mu\nu}W^\mu W^\nu$ as an independent field or particle, in analogy to atoms, molecules, and nuclei. Bare $WW$ must interact strongly with corresponding neutral scalar field $ZZ = g_{\mu\nu}Z^\mu Z^\nu$, through exchange of quarks and leptons. Assuming that the interacting bare fields produce relatively stable $W_2 = WW\cos\theta_x + ZZ\sin\theta_x$ and complementary resonance $Z_2 = -WW\sin\theta_x + ZZ\cos\theta_x$, $W_2$ can dress the bare $\Phi$ field while maintaining charge neutrality. This will be shown here to determine Higgs parameter $\lambda$.

$\Delta\mathcal{L}$ does not determine an explicit source density for field $WW$. However, simultaneous creation of fields $Z^\mu_+, Z^\nu_+$ can produce the dressed field $W_2$. The rate of simultaneous creation is the product of independent rates, expressed by $J_{ZZ}/\Phi^\dagger\Phi = (J^\mu_{Z\mu}/\Phi^\dagger\Phi)(J^\nu_{Z\nu}/\Phi^\dagger\Phi)$. This implies $J_{ZZ}\Phi\Phi = J^\mu_{Z\mu}J^\nu_{Z\nu} = g^2_w(\frac{\phi_0}{\phi})^2(\Phi^\dagger\Phi)^2$. Assum-
ing a $W_2$ field containing linear combination $WW \cos \theta_x + ZZ \sin \theta_x$, the effective source current is $J_{W_2} = J_{ZZ} \sin \theta_x$. Neglecting derivatives in an effective Klein-Gordon equation, the induced field amplitude is $W_2 = J_{W_2}/m_{W_2}^2$. Given $WW = W_2 \cos \theta_x - Z_2 \sin \theta_x$, $W_2$ projects onto $WW$ with factor $\cos \theta_x$. Term $\frac{1}{2} g_\mu^2 WW \Phi^\dagger \Phi$ in $\Delta L$ becomes $\frac{1}{2} g_\nu^2 \sin 2 \theta_x J_{ZZ} \Phi^\dagger \Phi/m_{W_2}^2 = -\lambda (\Phi^\dagger \Phi)^2$, where dimensionless $\lambda = -\frac{1}{4} g_\nu^2 g_\mu^2 \sin 2 \theta_x (\frac{\phi_0}{m_0})^2 h^2/m_{W_2}^2 e^4$.

If $\lambda$ is constant, parameter $\frac{\phi_0}{m_0} = -2.731 H_0$, where Hubble constant $H_0 = 67.66 \text{km/s/Mpc} = 2.197 \times 10^{-18}/s$. The $W_2$ mass must be consistent with empirical $\lambda \approx -10^{-8}$.

5. $W_2$ particle and $Z_2$ resonance

Conformal theory evaluates otherwise undetermined Higgs field parameters from accurately fitted observed cosmic expansion and galactic rotation. The resulting empirical parameter values remain consistent with the standard electroweak model, requiring only spontaneous generation of finite neutral Higgs field amplitude $\phi_0$, but are incompatible with existence of a massive Higgs scalar particle. Hence the observed LHC 125GeV resonance requires an alternative explanation.

A model Hamiltonian matrix can be defined in which indices 0,1 refer respectively to bare neutral scalar states $WW = g_{\mu\nu} W_\mu^\dagger W_\nu^\dagger$, $ZZ = g_{\mu\nu} Z_\mu^\dagger Z_\nu^\dagger$. Assumed diagonal elements are $H_{00} = 2 m_W = 160\text{GeV}$, $H_{11} = 2 m_Z = 182\text{GeV}$, for empirical masses $m_W$ and $m_Z$. Intermediate quark and lepton states define a large complementary matrix $\tilde{H}$ indexed by $i,j \neq 0,1$, with eigenvalues $\epsilon_i$, and off-diagonal elements $\tilde{A}_{i0}, \tilde{A}_{i1}$. $\tilde{H}$ determines energy-dependent increments in a $2 \times 2$ reduced matrix

$$H_{ab} - \mu_{ab} = \sum_{i \neq 0,1} \tilde{A}_{ai} (\epsilon_i - \epsilon)^{-1} \tilde{A}_{ib}. \quad (4)$$

$H_{01} - \mu_{01} = (WW|H_{red}|ZZ)$ corresponds to Feynman diagrams for quark and lepton exchange. The most massive and presumably most strongly coupled intermediate field that interacts directly would be tetraquark $T = bb\bar{b}\bar{b}$, whose mass is estimated as $c_T = 350\text{GeV}$. A simplified estimate of $W_2$ energy is obtained by restricting intermediate states to the three color-indexed tetraquark states $T = bb\bar{b}\bar{b}$, and assuming elements $\tilde{A}_{T0}, \tilde{A}_{T1}$ of equal magnitude $\alpha/\sqrt{3}$. For the reduced $2 \times 2$ matrix, matrix increments $\mu_{ab} \simeq \mu(\epsilon) = \frac{\alpha^2}{\epsilon} \pi^{-\frac{1}{2}}$ are all defined by a single parameter $\alpha^2$. Secular equation

$$(2m_W - \mu(\epsilon) - \epsilon)(2m_Z - \mu(\epsilon) - \epsilon) = \mu^2(\epsilon) \quad (5)$$

is to be solved for two eigenvalues $\epsilon = E_0, E_1$.

It is found that identifying the model diboson $W_2$ with the recently observed LHC 125GeV resonance predicts the empirical value of Higgs parameter $\lambda$. Setting $E_0 = 125\text{GeV} = 0.8644 \times 10^{14} h H_0$ for the $W_2$ state, dominated by the bare $WW$ field, determines parameters $\alpha^2 = 4878 \text{GeV}^2$, $\mu(E_0) = 21.68\text{GeV}$.
and \( \tan \theta_x = 0.6138 \). Using \( \alpha^2 \) determined by \( E_0 \), the present model predicts \( E_1 = 173 \text{GeV} \), with \( \mu(E_1) = 27.62 \text{GeV} \). This higher eigenvalue is the energy of a resonance \( Z_2 \) dominated by the bare \( ZZ \) field. \( Z_2 \) decay into bare \( WW \), two free charged gauge bosons, is allowed by energy conservation, but not into bare \( ZZ \). Composite field \( W_2 \) cannot decay spontaneously into either \( WW \) or \( ZZ \).

Identifying \( E_0 \) with the observed 125GeV resonance, and using \( g_w = 0.6312 \) and \( g_z = 0.7165 \) with computed \( \tan \theta_x = 0.6138 \), the implied value of Higgs parameter \( \lambda = \frac{-\frac{1}{4} g_w^2 g_z^2 \sin 2\theta_x (\frac{c_0}{m_0})^2 h^2}{m_{W_2}^2 c^4} \approx -0.455 \times 10^{-88} \) is consistent with its empirical value \( \lambda \approx -10^{-88} \).

6. Conclusions and implications

Postulated universal conformal symmetry modifies both general relativity and the Higgs scalar field model. Higgs parameter \( w^2 \) and all mass terms break conformal symmetry and must be generated dynamically. The conformal Higgs model remains valid for gauge boson masses, but the negative sign of parameter \( \lambda \) implied by cosmological data precludes a massive particle as a Higgs scalar field fluctuation.

The coupled semiclassical field equations of conformal theory imply a very small but nonvanishing source density for the neutral gauge field \( Z_\mu Z^\mu \). This results from the cosmological time dependence of gravitational Ricci scalar \( R \) in the conformal scalar field Lagrangian density. Bare Higgs scalar \( \Phi \) is dressed by a nonvanishing neutral gauge field, producing parameter \( w^2 \) of the correct empirical magnitude for dark energy density. Agreement with empirical data is extended here to Higgs parameter \( \lambda \). Preserving charge neutrality, double excitation of the \( Z^\mu \) field induces a previously unknown field \( W_2 \), based on strongly interacting bare fields \( WW = W^\mu W_\mu \) and \( ZZ = Z^\mu Z^\mu \), which dresses bare scalar field \( \Phi \). Implied parameter \( \lambda \) has empirically correct sign and magnitude if scalar field \( W_2 \) is identified with the recently observed 125GeV resonance.

Explanation of Higgs parameter \( \lambda \) makes it possible to carry the modified Friedmann cosmic evolution equation back to the big-bang epoch, with time-dependent parameters. The reversed sign of the conformal gravitational constant in uniform, isotropic geometry implies rapid expansion due to primordial mass and radiation density. Time variation of conformal Higgs amplitude \( \phi_0 \) is relevant to nucleosynthesis, because it directly affects the Fermi \( \beta \)-decay constant.

Dynamical models of galactic clusters should be revised to take into account the non-Newtonian gravitational effects of conformal theory. It cannot yet be concluded that dark matter is needed to explain galactic evolution.

The conformal Higgs model, backed by well-determined cosmological data, does not imply a massive Higgs particle, but supports the alternative interpretation proposed here of the observed 125GeV resonance. If Higgs parameter \( w^2 \) were large enough to produce the observed 125GeV resonance, the conformal model would imply dark energy density large enough to blow the universe apart long before the
present epoch. This implication is removed by the present theory. Moreover, without a stable fluctuation, the symmetry-breaking Higgs Φ field does not produce large zero-point energy that must somehow be suppressed. An extremely small scale parameter $\dot{\phi}/\phi_0$, unique to conformal theory, relates cosmology to electroweak physics. The Higgs scalar field breaks gauge symmetry and produces gauge boson mass through its finite spacetime amplitude. The standard electroweak model\cite{7} attributes fermion mass to direct coupling with the Higgs scalar field. This mechanism is preserved by the conformal Higgs field amplitude.

References
1. R. H. Sanders, *The Dark Matter Problem*, (Cambridge Univ. Press, New York, 2010).
2. R. K. Nesbet, *Entropy* **15**, 162 (2013).
3. H. Weyl, *Sitzungber. Preuss. Akad. Wiss.* , 465 (1918); *Math. Zeit.* **2**, 384 (1918).
4. P. D. Mannheim, *Prog. Part. Nucl. Phys.* **56**, 340 (2006).
5. B. S. DeWitt, in *Relativity, Groups, and Topology*, C. DeWitt and B. S. DeWitt, eds. (Gordon and Breach, New York, 1964).
6. P. W. Higgs, *Phys. Rev. Lett.* **13**, 508 (1964).
7. M. E. Peskin and D. V. Schroeder, *Introduction to Quantum Field Theory* (Westview Press, Boulder, CO, 1995).
8. W. N. Cottingham and D. A. Greenwood, *An Introduction to the Standard Model of Particle Physics* (Cambridge Univ. Press, New York, 1998).
9. P. D. Mannheim and D. Kazanas, *ApJ* **342**, 635 (1989).
10. P. D. Mannheim, *Gen. Rel. Grav.* **22**, 289 (1990).
11. P. D. Mannheim, *Annals N.Y. Acad. Sci.* **631**, 194 (1991).
12. P. D. Mannheim, *Found. Phys.* **42**, 388 (2012).
13. J. G. O’Brien, T. L. Chiarelli and P. D. Mannheim, *Phys. Lett. B* **782**, 433 (2018).
14. R. K. Nesbet, *Mod. Phys. Lett. A* **26**, 893 (2011).
15. R. K. Nesbet, *Eur. Phys. Lett.* **125**, 19001 (2019).
16. R. K. Nesbet, *Eur. Phys. Lett.* **109**, 59001 (2015).
17. R. K. Nesbet, *Eur. Phys. Lett.* **131**, 10002 (2020).
18. M. Della Negra, P. Jenni, and T.S. Virdee, *Science* **338**, 1560 (2012).
19. ATLAS Collaboration, *Phys. Lett. B* **716**, 1 (2012).
20. CMS Collaboration, *Phys. Lett. B* **716**, 30 (2012).
21. P. D. Mannheim and J. G. O’Brien, *Phys. Rev. Lett.* **106**, 121101 (2011).
22. P. D. Mannheim and J. G. O’Brien, *Phys. Rev. D* **85**, 124020 (2012).
23. J. G. O’Brien and P. D. Mannheim, *MNRAS* **421**, 1273 (2012).
24. E. E. Flanagan, *Phys. Rev. D* **74**, 023002 (2006).
25. Y. Brihaye and Y. Verbin, *Phys. Rev. D* **80**, 124048 (2009).
26. Y. Yoon, *Phys. Rev. D* **88**, 027504 (2013).
27. P. D. Mannheim, *Phys. Rev. D* **75**, 124006 (2007).
28. R. K. Nesbet, *MNRAS* **476**, L69 (2018).
29. S. S. McGaugh, F. Lelli and J. M. Schombert, *Phys. Rev. Lett.* **117**, 201101 (2016).
30. M. Milgrom, *ApJ* **270**, 365 (1983).
31. B. Famaey and S. McGaugh, *Living Reviews in Relativity* **15**, 10 (2012).
32. R. K. Nesbet, *Variational Principles and Methods in Theoretical Physics and Chemistry* (Cambridge Univ. Press, New York, 2003).
33. Planck Collaboration. [arXiv:1502.01589v2[astro-ph.CO]] (2015).