Matrix multinomial distribution

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Abstract

In this article, we define a matrix multinomial distribution. We prove some properties of the matrix multinomial distribution. We prove that the matrix Poisson distribution can be used as an approximation to the matrix multinomial distribution under certain conditions. We prove that the matrix normal distribution can be used as an approximation to the matrix multinomial distribution under certain conditions.

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1 Introduction

A generalization of the binomial distribution is the multinomial distribution. The multinomial distribution of a random vector \( X = (X_1, \ldots, X_k) \in \mathbb{N}^k \) can be written in the following notation:

\[
\text{Multi}_k(n, p)
\]
or

\[
\text{Multi}(n, p),
\]

where \( n \in \mathbb{N} \) and \( p \in [0, 1]^k \).

All basic information about multivariate discrete distributions, and in particular multinomial distribution, is presented in the book "Discrete multivariate distributions" [3].

Definition 1.1. \( X \sim \text{Multi}(n, p) \) if and only if the probability mass function of the random vector \( X = (X_1, \ldots, X_k)^T \in \mathbb{N}_0^k \) has the following forms:

\[
f_X(x) = P(X = x) = n! \left( 1 - \sum_{i=1}^k p_i \right)^{n-\sum_{i=1}^k x_i} \frac{\left( n - \sum_{i=1}^k x_i \right)!}{\prod_{i=1}^k x_i!} \prod_{i=1}^k p_i^{x_i}.
\]
or

\[ f_X(x) = P(X = x) = n! \frac{(1 - \text{tr } \text{diag } P)^{n - \text{tr } \text{diag } x}}{(n - \text{tr } \text{diag } x)!} \det \left( (\text{diag } x)!^{-1} (\text{diag } P)^{\text{diag } x} \right), \]

where \( 1 - \sum_{i=1}^{k} p_i \geq 0, \) \( n - \sum_{i=1}^{k} x_i \geq 0, \) \( X! := \Gamma(X + I) \) and \( \Gamma(X) \) is the matrix gamma function [1].

**Proof.** By definition [3],

\[ f_X(x) = n! \prod_{i=1}^{k+1} \frac{p_{x_i}}{x_i!}. \]

Therefore,

\[ f_X(x) = n! \frac{(1 - \sum_{i=1}^{k} p_i)^{n - \sum_{i=1}^{k} x_i}}{(n - \sum_{i=1}^{k} x_i)!} \prod_{i=1}^{k} \frac{p_{x_i}^{x_i}}{x_i!}. \]

Thus,

\[ f_X(x) = n! \frac{(1 - \text{tr } \text{diag } P)^{n - \text{tr } \text{diag } x}}{(n - \text{tr } \text{diag } x)!} \det \left( (\text{diag } x)!^{-1} (\text{diag } P)^{\text{diag } x} \right). \]

The matrix multinomial distribution is a discrete probability distribution that is a generalization of the multinomial distribution to matrix-valued random variables. The matrix multinomial distribution of a random \( k \times m \) matrix \( X = (X_{i,j}) \in \mathbb{N}_0^{k \times m} \) can be written in the following notation:

\[ \text{MMulti}_{k \times m}(n, P) \]

or

\[ \text{MMulti}(n, P), \]

where \( n \in \mathbb{N} \) and \( P \in [0,1]^{k \times m} \).

All basic information about matrix continuous distributions is presented in the book “Matrix variate distributions” [2]. All basic information about the matrix Poisson distribution is presented in the article “Matrix Poisson distribution” [4].

## 2 Definitions

### 2.1 Probability mass function

**Definition 2.1.** \( X \sim \text{MMulti}(n, P) \) if and only if the probability mass function of the random matrix \( X \in \mathbb{N}_0^{k \times m} \) has the following forms:

\[ f_X(x) = P(X = x) = n! \frac{(1 - \sum_{i=1}^{k} \sum_{j=1}^{m} p_{i,j})^{n - \sum_{i=1}^{k} \sum_{j=1}^{m} x_{i,j}}}{(n - \sum_{i=1}^{k} \sum_{j=1}^{m} x_{i,j})!} \prod_{i=1}^{k} \prod_{j=1}^{m} \frac{p_{x_{i,j}}^{x_{i,j}}}{x_{i,j}!}. \]
or

\[ f_X(x) = P(X = x) = n! \frac{(1 - \text{tr diag vec } P)^{n - \text{tr diag vec } x}}{(n - \text{tr diag vec } x)!} \]
\[ \det \left( (\text{diag vec } x)!^{-1} (\text{diag vec } P)^{\text{diag vec } x} \right), \]

where \((1 - \sum_{i=1}^{k} \sum_{j=1}^{m} P_{i,j}) \geq 0, \left( n - \sum_{i=1}^{k} \sum_{j=1}^{m} x_{i,j} \right) \geq 0, X! := \Gamma(X + I) \) and \( \Gamma(X) \) is the matrix gamma function \(^{[1]}\).

### 2.2 Relationship to multinomial distribution

**Definition 2.2.** \( X \sim \text{MMulti}_{k \times m}(n, P) \Leftrightarrow \text{vec}(X) \sim \text{Multi}_{km}(n, \text{vec } P). \)

**Proof.** By Definition 2.1

\[ X \sim \text{MMulti}(n, P) \Leftrightarrow f_X(x) = n! \frac{(1 - \text{tr diag vec } P)^{n - \text{tr diag vec } x}}{(n - \text{tr diag vec } x)!} \]
\[ \det \left( (\text{diag vec } x)!^{-1} (\text{diag vec } P)^{\text{diag vec } x} \right). \]

By Definition 1.1

\[ X \sim \text{MMulti}(n, \text{vec } P) \Leftrightarrow f_{\text{vec } X}(\text{vec } x) = n! \frac{(1 - \text{tr diag vec } P)^{n - \text{tr diag vec } x}}{(n - \text{tr diag vec } x)!} \]
\[ \det \left( (\text{diag vec } x)!^{-1} (\text{diag vec } P)^{\text{diag vec } x} \right). \]

Thus,

\[ X \sim \text{MMulti}_{k \times m}(n, P) \Leftrightarrow \text{vec}(X) \sim \text{Multi}_{km}(n, \text{vec } P). \]

### 2.3 Characteristic function

**Definition 2.3.** Let \( t \in \mathbb{R}^{k \times m} \). \( X \sim \text{MMulti}(n, P) \) if and only if the characteristic function of the random matrix \( X \in \mathbb{N}_{0}^{k \times m} \) has the following form:

\[ \varphi_X(t) = (\text{tr } (\text{diag vec } P) e^{i \text{diag vec } t})^n. \]

**Proof.** By Definition 2.2

\[ X \sim \text{MMulti}_{k \times m}(n, P) \Leftrightarrow \text{vec}(X) \sim \text{Multi}_{km}(n, \text{vec } P). \]

Therefore,

\[ \varphi_X(t) = \varphi_{\text{vec } X}(\text{vec } t). \]

By definition \([3]\),

\[ \varphi_{\text{vec } X}(\text{vec } t) = \left( \sum_{j=1}^{km} (\text{vec } P)_{i,j} e^{i(\text{vec } t)_{j}} \right)^n. \]
Thus,
\[ \varphi_X(t) = (\text{tr} \left((\text{diag vec } P) e^{i\text{vec } t}\right))^n. \]

### 2.4 Probability-generating function

**Definition 2.4.** Let \( t \in \mathbb{C}^{k \times m} \). \( X \sim \text{MMulti} (n, P) \) if and only if the probability-generating function of the random matrix \( X \in \mathbb{N}^{k \times m}_0 \) has the following form:
\[ G_X(t) = (\text{tr} \left((\text{diag vec } P)(\text{diag vec } t)\right))^n. \]

**Proof.** By Definition 2.2,
\[ X \sim \text{MMulti}_{k \times m} (n, P) \Leftrightarrow \text{vec } (X) \sim \text{Multi}_{km} (n, \text{vec } P). \]
Therefore,
\[ G_X(t) = G_{\text{vec } X}(\text{vec } t). \]
By definition 3,
\[ G_{\text{vec } X}(\text{vec } t) = \left( \sum_{j=1}^{km} (\text{vec } P)_j (\text{vec } t)_j \right)^n. \]
Thus,
\[ G_X(t) = (\text{tr} \left((\text{diag vec } P) (\text{diag vec } t)\right))^n. \]

### 3 Properties

#### 3.1 Mean or expected value

**Property 3.1.** If \( X \sim \text{MMulti} (n, P) \), then the expected value of the random matrix \( X \in \mathbb{N}^{k \times m}_0 \) has the following form:
\[ \mathbb{E}[X] = nP. \]

**Proof.** By Definition 2.2,
\[ X \sim \text{MMulti}_{k \times m} (n, P) \Leftrightarrow \text{vec } (X) \sim \text{Multi}_{km} (n, \text{vec } P). \]
Therefore,
\[ \mathbb{E}[\text{vec } X] = \text{vec } \mathbb{E}[X]. \]
By property 3,
\[ \mathbb{E}[\text{vec } X] = n \text{ vec } P. \]
Thus,
\[ \mathbb{E}[X] = nP. \]
3.2 Variance-covariance matrix

Property 3.2. If \( X \sim \text{MMulti} (n, P) \), then the variance-covariance matrix of the random matrix \( X \in \mathbb{N}_{0}^{k \times m} \) has the following form:

\[
K_{XX} = n \left( \text{diag vec} P - \text{vec} P \left( \text{vec} P \right)^{T} \right).
\]

Proof. By Definition [2.2]

\( X \sim \text{MMulti}_{k \times m} (n, P) \Leftrightarrow \text{vec} (X) \sim \text{Multi}_{km} (n, \text{vec} P) \).

Therefore,

\[
K_{XX} = K_{\text{vec} \ X, \text{vec} \ X}.
\]

By definition [3],

\[
K_{\text{vec} \ X, \text{vec} \ X} = n \left( \text{diag vec} P - \text{vec} P \left( \text{vec} P \right)^{T} \right).
\]

Thus,

\[
K_{XX} = n \left( \text{diag vec} P - \text{vec} P \left( \text{vec} P \right)^{T} \right).
\]

3.3 Moment-generating function

Property 3.3. Let \( t \in \mathbb{R}^{k \times m} \). If \( X \sim \text{MMulti} (n, P) \), then the moment-generating function of the random matrix \( X \in \mathbb{N}_{0}^{k \times m} \) has the following form:

\[
M_{X} (t) = \left( \text{tr} \left( \left( \text{diag vec} P \right) e^{\text{diag} \text{vec} t} \right) \right)^{n}.
\]

Proof. By Definition [2.2]

\( X \sim \text{MMulti}_{k \times m} (n, P) \Leftrightarrow \text{vec} (X) \sim \text{Multi}_{km} (n, \text{vec} P) \).

Therefore,

\[
M_{X} (t) = M_{\text{vec} \ X} (\text{vec} \ t).
\]

By definition [3],

\[
M_{\text{vec} \ X} (\text{vec} \ t) = \left( \sum_{j=1}^{km} (\text{vec} P)_{j} e^{(\text{vec} t)_{j}} \right)^{n}.
\]

Thus,

\[
M_{X} (t) = \left( \text{tr} \left( \left( \text{diag vec} P \right) e^{\text{diag} \text{vec} t} \right) \right)^{n}.
\]
4 Related distributions

**Theorem 4.1.** The matrix Poisson distribution $\text{MPois}_{k \times m}(nP)$ can be used as an approximation to the matrix multinomial distribution $\text{MMulti}_{k \times m}(n, P)$ if $P \to 0 (P_{i,j} \to 0 \forall i, j)$ and $n \to \infty$.

**Proof.** By Definition 4.2

$$X \sim \text{MMulti}_{k \times m}(n, P) \Leftrightarrow \text{vec}(X) \sim \text{Multi}_{km}(n, \text{vec P}).$$

By definition [4],

$$X \sim \text{MPois}_{k \times m}(nP) \Leftrightarrow \text{vec}(X) \sim \text{Pois}_{km}(n, \text{vec P}).$$

The multivariate Poisson distribution $\text{Pois}_{km}(n, \text{vec P})$ can be used as an approximation to the multinomial distribution $\text{Multi}_{km}(n, \text{vec P})$ if $\text{vec P} \to 0 ((\text{vec P})_i \to 0) \forall i$ and $n \to \infty$ [3] p. 124.

Thus, the matrix Poisson distribution $\text{MPois}_{k \times m}(nP)$ can be used as an approximation to the matrix multinomial distribution $\text{MMulti}_{k \times m}(n, P)$ if $P \to 0 (P_{i,j} \to 0 \forall i, j)$ and $n \to \infty$. \hfill \Box

**Theorem 4.2.** Let $P \in [0, 1]^{k \times m}$, $P_r \in [0, 1]^{k \times 1}$, $P_c \in [0, 1]^{m \times 1}$ and $x \in \mathbb{N}^{k \times m}$. The matrix normal distribution $\mathcal{MN}_{k \times m}(nP, \sqrt{n} \text{ diag } P_r, \sqrt{n} \text{ diag } P_c)$ can be used as an approximation to the matrix multinomial distribution $\text{MMulti}_{k \times m}(n, P)$ if $\text{vec } nP = (\sqrt{n} P_r) \otimes (\sqrt{n} P_c)$ and $nP \to \infty (nP_{i,j} \to \infty \forall i, j)$.

**Proof.** By Theorem 4.1 the matrix Poisson distribution $\text{MPois}_{k \times m}(nP)$ can be used as an approximation to the matrix multinomial distribution $\text{MMulti}_{k \times m}(n, P)$ if $P \to 0 (P_{i,j} \to 0 \forall i, j)$ and $n \to \infty$.

The matrix normal distribution $\mathcal{MN}_{k \times m}(nP, \sqrt{n} \text{ diag } P_r, \sqrt{n} \text{ diag } P_c)$ can be used as an approximation to the matrix Poisson distribution $\text{MPois}_{k \times m}(nP)$ if $\text{vec } nP = (\sqrt{n} P_r) \otimes (\sqrt{n} P_c)$ and $nP \to \infty (nP_{i,j} \to \infty \forall i, j)$ [4].

Thus, the matrix normal distribution $\mathcal{MN}_{k \times m}(nP, \sqrt{n} \text{ diag } P_r, \sqrt{n} \text{ diag } P_c)$ can be used as an approximation to the matrix multinomial distribution $\text{MMulti}_{k \times m}(n, P)$ if $\text{vec } nP = (\sqrt{n} P_r) \otimes (\sqrt{n} P_c)$ and $nP \to \infty (nP_{i,j} \to \infty \forall i, j)$. \hfill \Box

5 Conclusions

In this article, we have proved some properties, for example, the probability mass function, expected value, variance - covariance matrix, moment - generating function, characteristic function and probability - generating function. We have proved that the matrix Poisson distribution can be used as an approximation to the matrix multinomial distribution under certain conditions. We have proved that the matrix normal distribution can be used as an approximation to the matrix multinomial distribution under certain conditions. All our results are consistent with each other.

Based on all of the above, the matrix multinomial distribution is a generalization of the multinomial distribution and binomial distribution.
References

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