Multi-mode oscillations of a mechanical system with higher-order singularity

V S Kalnitsky
St.Petersburg State University, Universitetskaya nab., 7-9, St.Petersburg, 199034, Russian Federation
E-mail: st006987@spbu.ru

Abstract. This article is based on earlier author’s works on the behavior of mechanical systems with a singular configuration space. We obtained exact analytical expressions for the constraints providing multi-mode oscillations of a double mathematical pendulum. It was found that there are infinitely many modes of oscillations determined by the degree of degeneracy at a singular point in the configuration space of the pendulum. The problem of a double pendulum motion with an elliptical constraint, posed by the author earlier, received a complete solution, both theoretical and experimental. For such a constraint, there are exactly two different oscillation modes: symmetric and asymmetric. Each mode can be realized by selecting the system parameters and is completely determined by the smoothness of the trajectory in the configuration space with a singularity. In the study we used only standard tools of classical analytical mechanics.

1. Introduction
In work [1], the author proposed a mechanical system with a 0-singular configuration space. It is a one-dimensional mathematical double pendulum with a specially designed holonomic constraint. There are two different types of possible movement through the point of equilibrium. This means that theoretically we have geometric uncertainty from the point of view of classical mechanics, since it is possible to choose the parameters so that the generalized velocities of the system coincide for different types of motion. However, in a natural experiment, the dynamic certainty of the pendulum behavior was shown [2]. The configuration space of the double pendulum with the constraint in local angular coordinates has two one-dimensional branches that have one common point corresponding to the equilibrium position. With a special choice of parameters, these branches touch each other with different orders of tangency.

There are various conceptual approaches to expand the applicability of classical mechanics laws to the singular case. These approaches rely on a generalization of the smooth manifold concept and smooth tensor objects on it. A differential calculus over singularities one can find in the works on Diffiety spaces [3], on Diffeology [4], the theory of differential spaces and Frölicher spaces [5], gluing manifolds [6], morphisms algebras of pseudodifferential operators over a singularity [7].

In this work, our calculations do not go beyond the scope of classical mechanics, but the results obtained can offer a conceptual resolution of the arising geometric uncertainty. Using the methods of classical mechanics, we study singular case, write down the equation of motion for a smooth branch, and confirm by a series of experiments its realizability. We found that
there are potentially infinitely many modes of oscillation, completely determined by the degree of degeneracy of the configuration space. The author comes to the conclusion that, due to the differential consequences of conservation laws, the oscillation mode of the mechanical system is completely determined by the choice of a trajectory of the greatest class of smoothness.

2. Double mathematical pendulum with symmetrical constraint

Consider a plane double mathematical pendulum consisting of two rigid rods $OA = 2$ m and $AB = 1$ m of negligible weight, hinged at the point $O$ (fig. 1). Two loads considered as a material points $A$ and $B$ have equal masses $m_A = m_B = 1$ kg. Point $B$ is held on an immutable constraint. In the frame of reference with origin at the point located under the point $O$ at a distance of 1 m, the constraint is given by the graph of a convex even function. We require that this graph, tangent to the unit circle centered at the point $O$, lies strictly below it at least in the punctured neighborhood of the point of tangency. This requirement will be satisfied if in the Taylor series expansion of the constraint function the first term differing from the corresponding term of the Taylor series defining the circle has a smaller coefficient. Under the described conditions, the origin is the stable equilibrium point of the system.

Consider the series expansion of the function $f(x) = 1 - \sqrt{1-x^2}$ which graph is the arc of the circle.

$$f(x) = \sum_{k=1}^{\infty} a_k x^{2k} = \sum_{k=1}^{\infty} \frac{(k - \frac{1}{2})!}{2 \sqrt{\pi} k!} x^{2k} = \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{16} + \frac{5x^8}{128} + \frac{7x^{10}}{256} + \frac{21x^{12}}{1024} + o(x^{12})$$

If the constraint function $y(x)$ satisfies the above conditions its series expansion has only even degrees. All constraints are split into indexed classes of functions, which turned out determine the mode of the oscillation:

$$y_1(x) = \lambda x^2 + o(x^2), 0 \leq \lambda < \frac{1}{2}; \quad y_s(x) = \sum_{k=1}^{s-1} a_k x^{2k} + \lambda x^{2s} + o(x^{2s}), \lambda < a_s, s \geq 2.$$  

For a given angle $\varphi$ of displacement of the link $OA$ from the vertical, two positions of the link $AB$ are possible: “external” (further from the $Oy$ axis) with an angle $\psi_1$ and ”internal” (closer to the $Oy$ axis) with an angle $\psi_2$. The dependence of the angles $\psi_{1,2}$ on $\varphi$ is completely determined by the constraint function. Thus, the system evolution in the configuration space can be carried out along two branches that have a common point at the equilibrium position $\varphi = 0, \psi_{1,2} = 0$. The principal term of the series expansion of the function $\psi_{1,2}$ in $\varphi$ completely determines which of the motion branches, “external-external” or ”external-internal”, is smooth at the equilibrium point. For an odd degree of the principal term, the ”external-external” and ”internal-internal” branches are smooth, for an even degree — the ”external-internal” and ”internal-external” ones. In the first case, we talk about symmetric oscillations, in the second, about asymmetric oscillations.

The $x$-coordinate of point $B$ is the solution of the equation

$$(x - 2\sin \varphi)^2 + (y(x) - (1 - 2\cos \varphi))^2 = 1. \quad (1)$$
For $y = f(x)$ points $B(x, f(x))$ and $C(\sin \varphi, 1 - \cos \varphi)$ coincide and the only solution of the equation (1) for given $\varphi$ is $x = \sin \varphi$. Moreover, this equation has two solutions in the neighborhood of $x = \sin \varphi$ for sufficiently small $\varphi$.

Since $AB = AC = 1$, using the Cosine Theorem, we can write an explicit relationship between the angle $\psi$ and $x$.

$$BC^2 = 2(1 - \cos \psi), \quad BC^2 = (x - \sin \varphi)^2 + (y(x) - (1 - \cos \varphi))^2.$$  

Applying relation (1) and eliminating $\varphi$, we get

$$\cos \psi = 1 + \frac{y(x)}{2} - \frac{y^2(x)}{4} - \frac{x^2}{4}. \quad (2)$$

The rest of this section is devoted to the analysis of equation (2). We need to find the dependence of the angle $\psi$ on angle $\varphi$. For this, using equation (1), we will derive an explicit dependence of $x(\varphi)$ on $\varphi$ for several classes of constraints $y_s(x)$.

Consider the series expansion of the function $x(\varphi) = \sum_{i=1}^{\infty} b_i \varphi^i$ and the operator $D^{(n)}(f) = \frac{d^n f}{d x^n} |_{x=0}, D^{(0)}(f) = f(0)$. Applying this operator to both sides of equation (1) for different $n$ we get the system of equations on coefficients $b_i$. Note, that applying of $D^0$ does not give any relation. Using the general Leibniz rule and Faà di Bruno’s formula [8] we can write the explicit expressions of this equations for each $n, n \geq 2$,

$$\sum_{r=1}^{n-1} C_n^r \left( x^{(r)}(0) - 2Re i^{r-1} \right) \left( x^{(n-r)}(0) - 2Re i^{n-r-1} \right) + 4Re i^r + 2D^{(n)}(y(x)) +$$

$$+ \sum_{r=1}^{n-1} C_n^r \left( D^{(r)}(y(x)) + 2Re i^r \right) \left( D^{(n-r)}(y(x)) + 2Re i^{n-r} \right) = 0, \quad (3)$$

where

$$D^{(n)}(y(x)) = \sum_{r=1}^{n} y_x^{(r)}(0) \cdot B_{n,r}(x'(0), x''(0), \ldots, x^{(n-r+1)}(0)),$$

$$B_{n,r}(u_1, \ldots, u_{n-r+1}) = \sum_{j_1 + \ldots + j_{n-r+1} = n-r+1} \frac{n!}{j_1! \ldots j_{n-r+1}!} \left( \frac{u_1}{j_1} \right)^{j_1} \ldots \left( \frac{u_{n-r+1}}{(n-r+1)!} \right)^{j_{n-r+1}}.$$  

for $\lambda = 1/2$ and $\varphi = b_1 \varphi + o(\varphi)$.

Thus, we have two solutions

$$b_1^+ = \frac{1}{1/2 + \lambda} \left( 1 \pm \sqrt{1/2 - \lambda} \right), \quad x(\varphi) = b_1^+ \varphi + o(\varphi).$$

For the critical value $\lambda = 1/2$ we get $b_1 = 1$.

**Class 2.** For constraint $y_2(x) = \frac{1}{2} x^2 + \lambda x^4 + o(x^4), \lambda < 1/8, x(\varphi) = \varphi + b_2 \varphi^2 + \ldots, n = 4$, the equation (3) has the form

$$-8(6b_3 + 2) + 24b_2^2 + 2(24\lambda + 12b_2^2 + 24b_4 + 2) + 6 = 0, \quad \text{or} \quad b_2^2 + \lambda - \frac{1}{8} = 0.$$
Hence, \( x(\varphi) = \varphi \pm \sqrt{\frac{1}{8} - \lambda\varphi^2 + o(\varphi^2)} \).

**Class 3.** For constraint \( y_3(x) = \frac{1}{2}x^2 + \frac{1}{8}x^4 + \lambda x^6 + o(x^6) \), \( \lambda < 1/16 \), \( x(\varphi) = \varphi + b_3\varphi^3 + \ldots \), \( n = 6 \), the equation (3) has the form

\[
18 \left( b_3 + \frac{1}{6} \right)^2 = \frac{1}{16} - \lambda, \quad x(\varphi) = \varphi - \left( \frac{1}{6} \pm \frac{1}{3\sqrt{2}}\sqrt{\frac{1}{16} - \lambda} \right) \varphi^3 + o(\varphi^3).
\]

The following conclusion seems plausible, that for even \( y_{2s} \) and odd \( y_{2s+1} \) classes the solution has the form

\[
x(\varphi) = \varphi - \frac{1}{6}\varphi^3 + \cdots \pm \mu_{2s} \sqrt{a_{2s} - \lambda\varphi^{2s} + o(\varphi^{2s})},
\]

\[
x(\varphi) = \varphi - \frac{1}{6}\varphi^3 + \cdots + \left( (-1)^s \frac{1}{(2s + 1)!} \pm \mu_{2s+1} \sqrt{a_{2s+1} - \lambda} \right) \varphi^{2s+1} + o(\varphi^{2s+1}).
\]

This assumption for \( s \geq 2 \) will not be used in further reasoning. Knowing the exact relationships for the first three classes of constraints, we can apply a similar analysis to formula (2). Analogously let us consider the series expansion of \( \psi(\varphi) = \sum_{k=1}^{\infty} c_k \varphi^k \) and \( \cos \psi(\varphi) \) up to the sixth degree

\[
\cos \psi(\varphi) = 1 - \frac{c_1}{2} \varphi^2 - c_1 c_2 \varphi^3 + \left( \frac{c_1^2}{24} - c_1 c_3 - \frac{c_2^2}{2} \right) \varphi^4 + \\
+ \left( \frac{c_1 c_2}{6} - c_1 c_4 - c_2 c_3 \right) \varphi^5 + \left( -\frac{c_1^6}{1728} + \frac{c_1^3 c_3}{6} + \frac{c_2^4}{4} - c_1 c_5 - c_2 c_4 - \frac{c_3^2}{2} \right) \varphi^6 + o(\varphi^6).
\]

**Class 1.** \( y_1(x) = \lambda x^2 + \ldots \), \( x(\varphi) = b_1 \varphi + \ldots \)

\[
c_1^2 = b_1^2 \left( \frac{1}{2} - \lambda \right), \quad \psi(\varphi) = \pm b_1 \sqrt{\frac{1}{2} - \lambda\varphi + o(\varphi)}.
\]

**Class 2.** \( y_2(x) = \frac{1}{2}x^2 + \lambda x^4 + \ldots \), \( x(\varphi) = \varphi + b_2 \varphi^2 + \ldots \)

\[
c_1 = 0, \quad c_2^2 = \frac{1}{8} - \lambda, \quad \psi(\varphi) = \pm \sqrt{\frac{1}{8} - \lambda\varphi^2 + o(\varphi^2)}.
\]

**Class 3.** \( y_3(x) = \frac{1}{2}x^2 + \frac{1}{8}x^4 + \lambda x^6 + \ldots \), \( x(\varphi) = \varphi + b_3 \varphi^3 + \ldots \)

\[
c_1 = 0, \quad c_2 = 0, \quad c_3^2 = \frac{1}{16} - \lambda, \quad \psi(\varphi) = \pm \sqrt{\frac{1}{16} - \lambda\varphi^3 + o(\varphi^3)}.
\]

Based on the obtained expressions, it can be assumed that the general form of the dependence for the constraint class \( s \), \( s \geq 4 \), has the form

\[
\psi(\varphi) = \pm \sqrt{a_s - \lambda\varphi^s + o(\varphi^s)}.
\]

Let us now consider the constraint formed by an inextensible string, the ends of which are fixed at two different points on the \( Oy \) axis. In this case, the constraint is an ellipse with foci at the indicated points. The vertex of the ellipse must be at the origin. In the selected coordinate system, the constraint is locally defined by the function

\[
y(x) = a - a \sqrt{1 - \frac{x^2}{b^2}}, \quad y(x) = \frac{a}{2b^2}x^2 + \frac{a}{8b^4}x^4 + \frac{a}{16b^6}x^6 + o(x^6),
\]
where $a$ and $b$ are the major and the minor semiaxes of an ellipse. Let us apply the above results to this constraint.

**Class 1.** Two-parameter family of elliptic constraints with restriction $a < b^2$. In this case the symmetric motion of pendulum has the smooth trajectory in configuration space.

**Class 2.** One-parameter family of constraints with $a = b^2$, $b > 1$. Here the smooth trajectory is formed by the asymmetric oscillation.

Other classes of constraints are not implemented, since in the case $b = 1$ the constraint is a circle. The question of the nature of the true motion of the pendulum, posed by the author in [1], was resolved in the following form: the true motion of the pendulum is realized along a smooth circle. The question of the nature of the true motion of the pendulum, posed by the author in [1], was fully confirmed in full-scale experiments.

### 3. Equations of motion

Let us find the equations of motion of the pendulum for the first and second classes of constraint. For this we need to express the coordinates of points $A$ and $B$ through the angle $\varphi$ in the selected coordinate system (Fig. 1).

$$A(2\sin \varphi, 1 - 2\cos \varphi), \quad B(2\sin \varphi - \sin(\varphi - \psi(\varphi)), 1 - 2\cos \varphi + \cos(\varphi - \psi(\varphi))).$$

The Euler-Lagrange equations for the internal variable $\varphi$ have the form

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\varphi}} \right) - \frac{\partial T}{\partial \varphi} + \frac{\partial \Pi}{\partial \dot{\varphi}} = 0,$$

where $T = \frac{1}{2}m(\bar{v}_A^2(t) + \bar{v}_B^2(t))$ is the kinetic energy and $\Pi(\varphi) = mg(-4\cos \varphi + \cos(\varphi - \psi) + 3)$, $m = 1$, $\Pi(0) = \Pi'(0) = 0$, is the potential energy of the system.

Let us make standard transformations: $\bar{v}_A = \frac{dA}{d\varphi}$, $\bar{v}_B = \frac{dB}{d\varphi}$.

$$T = \frac{1}{2} \dot{\varphi}^2F(\varphi), \quad F(\varphi) = \left( \frac{dA}{d\varphi} \right)^2 + \left( \frac{dB}{d\varphi} \right)^2 + \left( \frac{dA}{d\varphi} \right)^2 = 4.$$

For further conclusions we need only three summands in the series expansion of $F(\varphi)$

$$F(\varphi) = F(0) + F'(0)\varphi + \frac{F''(0)}{2}\varphi^2 + o(\varphi^2),$$

where

$$F(0) = 5 + 2\psi'(0) + \psi'^2(0); \quad F'(0) = 2(1 + \psi'(0))\psi''(0);$$

$$F''(0) = 2\psi''(0) + 2(1 + \psi'(0))(-1 - 2\psi'(0) + \psi'^2(0) + \psi''(0)) + 2(1 + 2\psi'(0) - \psi'^2(0))^2.$$  

On the other hand $\Pi'(\varphi) = \Pi''(0)\varphi + o(\varphi), \Pi''(0) = g(3 + 2\psi'(0) - \psi'^2(0))$. Now we can write the Euler-Lagrange equation (4) in these terms for the motion law $\varphi(t)$

$$\dot{\varphi}F(\varphi) + \frac{\dot{\varphi}^2}{2}F'(\varphi) + \Pi'(\varphi) = 0.$$

**Class 1.** $\psi(\varphi) = b_1\varphi + o(\varphi), \quad F(0) = 5 + 2b_1 + b_1^2; \quad \Pi'(\varphi) = g(3 + 2b_1 - b_1^2)\varphi + \ldots$. We get the classical equation of the free harmonic motion

$$\ddot{\varphi} + \frac{g(3 + 2b_1 - b_1^2)}{5 + 2b_1 + b_1^2}\varphi = 0.$$

**Class 2.** $\psi(\varphi) = b_2\varphi^2 + o(\varphi^2), \quad F(0) = 5, \quad F'(0) = 4b_2, \quad \Pi'(\varphi) = 3g\varphi + \ldots$. The approximation of the equation has the form of the second-order nonlinear ordinary differential equation

$$5\ddot{\varphi} + 3g\varphi + 2b_2(2\dot{\varphi}^2 + \varphi^2) = 0.$$  

The plots of the numerical solution of this equation are shown in Figure 2.
Figure 2. Plots of the equation (5) numerical solution for $b_2 = 1/2$, $g = 10$, and initial data $\varphi(0) = 0$, $\dot{\varphi}(0) = 4$.

4. Conclusion

The study of equations of the type (5) has a long history [10]. Equation (5) defines an asymmetrical oscillation, which seems unexpected, since both the mechanical system itself and the potential are symmetric. Moreover, choosing the constraint parameters, we can switch vibration modes from symmetric to asymmetric and there are infinitely many such modes. However, the question of the physical realizability of the found modes remains open. In the case of elliptical constraint, the only two possible modes were realized experimentally. The fact, that the cause of asymmetric oscillations can be the singularity of the configuration space, is apparently obtained for the first time.

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