Extensions and variations of Andrews–Merca identities

Darlison Nyirenda¹² · Beaullah Mugwangwavari²

Received: 14 December 2022 / Accepted: 9 July 2023 / Published online: 19 July 2023
© The Author(s) 2023

Abstract
Recently, Andrews and Merca have given a new combinatorial interpretation of the total number of even parts in all partitions of \( n \) into distinct parts. We generalise this result and consider many more variations of their work. We also highlight some connections with the work of Fu and Tang.

Keywords Partition · Bijection · Generating function · Congruence

Mathematics Subject Classification 05A15 · 05A17 · 05A19 · 05A30 · 11P81

1 Introduction

A partition of a positive integer \( n \) is a representation \( \lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_r \) where \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1 \) and \( \sum_{i=1}^{r} \lambda_i = n \). A more compact notation for \( \lambda \) is \( (\mu_1^{m_1}, \mu_2^{m_2}, \ldots) \) where \( \mu_1 > \mu_2 > \cdots \) and \( m_i \) is the multiplicity of the summand \( \mu_i \). For instance, \( 14 + 14 + 10 + 10 + 7 + 7 + 7 + 1 + 1 + 1 + 1 \) can be written as \( (14^2, 10^2, 7^3, 1^4) \). The summands in a partition are called parts. The union of two partitions \( \lambda \) and \( \beta \) is simply the multiset union \( \lambda \cup \beta \) where \( \lambda \) and \( \beta \) are treated as multisets. Partition identities involving various classes of partitions have been studied, for example, see [1, 11]. We recall a popular one: For a positive integer \( t \), the number of partitions of \( n \) into parts appearing at most \( t - 1 \) times is equal to the number of partitions of \( n \) into parts not divisible by \( t \). Glaisher’s bijection [7] establishes a one-to-one correspondence between the aforesaid classes of partitions. Finite versions of this identity were studied in [2, 5, 10].

We shall let \( \phi_t \) denote Glaisher’s map, mapping the set of partitions into parts appearing at most \( t - 1 \) times to partitions into parts not divisible by \( t \). This map is described as follows:
Suppose that \( \lambda = (\lambda_1^{c_1}, \lambda_2^{c_2}, \ldots) \) is a partition of \( n \) in which each part appears at most \( t - 1 \) times. Write \( \lambda_i = t^{c_i} \cdot b_i \) where \( t \nmid b_i \) and then map \( \lambda_i \) to \( (b_i)^{t^{c_i}} \) for each \( i \), where \( b_i \) is now a part with multiplicity \( t^{c_i} \). We define \( \phi_t(\lambda) \) to be the partition

\[
\phi_t(\lambda) = \bigcup_{i \geq 1} (b_i)^{t^{c_i}}.
\]

Note that the parts of this partition are not multiples of \( t \). The inverse \( \phi_t^{-1} \) is given as follows.

Let \( \mu = (\mu_1^{m_1}, \mu_2^{m_2}, \ldots, \mu_r^{m_r}) \) be a partition of \( n \) in which no part is divisible by \( t \). The notation for \( \mu \) implies that \( \mu_1 > \mu_2 > \cdots \) are parts having multiplicities \( m_1, m_2, \ldots \), respectively. For each \( m_i \), we find its \( t \)-ary expansion, i.e.

\[
m_i = b_{i0} + b_{i1} t + b_{i2} t^2 + \cdots + b_{i\ell_i} t^{\ell_i}
\]

where \( 0 \leq b_{ij} \leq t - 1 \) and \( 0 \leq j \leq \ell_i \).

We then map \( \mu_i^{m_i} \) to \( \bigcup_{j=0}^{\ell_i} (t^{j} \mu_i)^{b_{ij}} \), where \( t^{j} \mu_i \) is now a part with multiplicity \( b_{ij} \). The image of \( \mu \), i.e. \( \phi_t^{-1}(\mu) \), is given by

\[
\bigcup_{i=1}^{r} \bigcup_{j=0}^{\ell_i} (t^{j} \mu_i)^{b_{ij}}.
\]

It is clear that this is a partition of \( n \) in which each part appears at most \( t - 1 \) times.

The case of \( t = 2 \), also known as Euler’s “Distinct v.s. Odd” partition theorem, has posed interesting questions and various studies conducted in many directions. Recently, Andrews and Merca [3] considered the following partition functions:

- \( a(n) \): the total number of even parts in all partitions of \( n \) into distinct parts;
- \( c_e(n) \) (resp. \( c_o(n) \)): the number of partitions of \( n \) in which exactly one part is repeated and this part is even (resp. odd);
- \( b_o(n) \): the number of partitions of \( n \) into an odd number of parts in which the set of even parts has only one element;
- \( b_e(n) \): the number of partitions of \( n \) into an even number of parts in which the set of even parts has only one element;
- \( b(n) = b_o(n) - b_e(n) \) and \( c(n) = c_o(n) - c_e(n) \).

They obtained the following result.

**Theorem 1** For all \( n \geq 1 \),

\[
a(n) = c(n) = (-1)^n b(n).
\]

Note that, if \( b'(n) \) denotes the number of partitions of \( n \) in which the set of even parts is singleton, then \( b'(n) = b_o(n) + b_e(n) \). Hence, one of the implications of the theorem is that

\[
2 \mid a(n) - b'(n).
\] (1)

For related work, see [4, 9].

In this paper, we generalize the identity \( a(n) = c(n) \) of Theorem 1 and also generalize (1). We then consider some variations. As a consequence, some connections with the work of Fu and Tang [6] are highlighted and bijective proofs are provided in such cases.

We recall the \( q \)-series notation:
For $a, q \in \mathbb{C}$, we have $(a; q)_n = \prod_{i=0}^{n-1} (1 - aq^i)$ for $n \geq 1$ and $(a; q)_0 = 1$, and also for $|q| < 1$,

$$(a; q)_\infty = \prod_{i=0}^{\infty} (1 - aq^i).$$

Below are some useful $q$-identities:

$$
(q; q)_\infty = \sum_{j=-\infty}^{\infty} (-1)^j q^{j(j-1)/2} = 1 + \sum_{j=1}^{\infty} (-1)^j q^{j(j+1)/2}, \tag{2}
$$

$$
(q; q)_\infty^2 = \sum_{j=0}^{\infty} (-1)^j (2j + 1) q^{(j+1)/2}. \tag{3}
$$

2 Generalizing the identity $a(n) = c(n)$ of Theorem 1

We first observe the following.

**Theorem 2** Suppose that $p$ and $r$ are non-negative integers such that $p \geq r + 2$. Denote by $a_r(n, p)$ the total number of parts congruent to $-r \pmod{p}$ in partitions of $n$ into distinct parts. Let $g_r(n, p)$ denote the number of partitions in which exactly one part is repeated and the multiplicity of this repeated part is at least $p - r$ and congruent to $-r, -r + 1 \pmod{p}$. Let $g_{r,o}(n, p)$ (resp. $g_{r,e}(n, p)$) denote the number of $g_r(n, p)$-partitions in which the repeated part is odd (resp. even) and let $c_r(n, p) = g_{r,o}(n, p) - g_{r,e}(n, p)$. Then

$$
a_r(n, p) = c_r(n, p).
$$

It is important to realize that setting $p = 2$ and $r = 0$ yields the first equality in Andrews and Merca’s theorem, Theorem 1.

**Proof** Let $F(z, q)$ be the multivariable generating function for $c_r(n, p)$ in which $z$ tracks the repeated part. Then

$$
F(z, q) = \sum_{n=1}^{\infty} z^n (q^{(p-r)n} + q^{(p-r+1)n} + q^{(p-r+2)n} + \cdots + q^{(p-r+1)n} + q^{(p-r+1+p)n} + q^{(p-r+1+2p)n} + \cdots) \prod_{j \neq n, j=1}^{\infty} (1 + q^j)
$$

$$
= \sum_{n=1}^{\infty} z^n (q^{(p-r)n}(1 + q^{pn} + q^{2pn} + q^{3pn} + \cdots) + q^{(p-r+1)n}(1 + q^{pn}
$$

$$
+ q^{2pn} + q^{3pn} + \cdots)) \prod_{j \neq n, j=1}^{\infty} (1 + q^j)
$$

$$
= \sum_{n=1}^{\infty} \frac{z^n q^{(p-r)n}(1 + q^n)(1 + q^{pn} + q^{2pn} + q^{3pn} + \cdots)}{1 - q^{pn}} \prod_{j \neq n, j=1}^{\infty} (1 + q^j)
$$

$$
= \sum_{n=1}^{\infty} \frac{z^n q^{(p-r)n}(1 + q^n)}{1 - q^{pn}} \prod_{j \neq n, j=1}^{\infty} (1 + q^j)
$$
\[
\sum_{n=1}^{\infty} \frac{z^n q^{(p-r)n}}{1 - q^{pn}} = (-q; q) \sum_{n=1}^{\infty} \frac{z^n q^{(p-r)n}}{1 - q^{pn}}.
\]

Using the fact that \(\sum_{n=0}^{\infty} (g_{r,o}(n, p) - g_{r,e}(n, p)) q^n = -F(-1, q)\), we have

\[
\sum_{n=0}^{\infty} g_r(n, p) q^n = -F(-1, q)
\]

On the other hand,

\[
\sum_{n=0}^{\infty} a_r(n, p) q^n = \left( \frac{\partial}{\partial z} \right)_{z=1} \left( \prod_{n=1}^{\infty} (1 + z q^{p n-r}) \right) \prod_{i=1, i \neq r}^{p} \prod_{n=1}^{\infty} (1 + q^{p n-i})
\]

\[
= \left( \prod_{i=1, i \neq r}^{p} \prod_{n=1}^{\infty} (1 + q^{p n-i}) \right) \left( \prod_{n=1}^{\infty} (1 + z q^{p n-r}) \right)_{z=1}
\]

\[
\times \left( \frac{\partial}{\partial z} \right)_{z=1} \sum_{n=1}^{\infty} \log(1 + z q^{p n-r})
\]

\[
= \left( \prod_{n=1}^{\infty} (1 + q^n) \right) \sum_{n=1}^{\infty} \frac{q^{p n-r}}{1 + q^{p n-r}}
\]

\[
= (-q; q) \sum_{n=1}^{\infty} \frac{q^{-r+p n}}{1 + q^{p n-r}}.
\]
Recall the identity:

\[
\sum_{n=0}^{\infty} a_r(n, p)q^n = \sum_{n=0}^{\infty} c_r(n, p)q^n
\]

and the theorem follows. \(\square\)

The above theorem provides a new combinatorial interpretation of the total number of parts congruent to \(-r\) modulo \(p\) in partitions of \(n\) into distinct parts.

In Fu and Tang [6], recall the function \(O_{r, j}(n)\) which denotes the number of partitions of \(n\) in which there are exactly \(j\) different parts congruent to 0 modulo \(p\) (possibly repeated).

For \(j = 1\), we shall denote this function by \(o_p(n)\) which counts partitions of \(n\) in which the set of parts congruent to 0 modulo \(p\) is singleton. Define \(a(n, p)\) to be the total number of parts congruent to 0 modulo \(p\) in all partitions of \(n\) into parts appearing at most \(p-1\) times.

Clearly \(a(n, 2) = a(n) = a_0(n, 2)\) and \(o_2(n) = b'(n)\). We have the following property which extends (1).

**Theorem 3** For all \(n \geq 0\),

\[ p \mid a(n, p) - o_p(n). \]

**Proof** The generating function for \(o_p(n)\) is given by

\[
\sum_{n=0}^{\infty} o_p(n)q^n = \frac{(q^p; q^p)_{\infty}}{(q; q)_{\infty}} \sum_{n=1}^{\infty} \frac{q^{pn}}{1 - q^{pn}}
\]

and the generating function for \(a(n, p)\) is given by Herden et al. [8]:

\[
\frac{(q^p; q^p)_{\infty}}{(q; q)_{\infty}} \sum_{n=1}^{\infty} \frac{q^{pn} + 2q^{2pn} + 3q^{3pn} + \ldots + (p-1)q^{(p-1)pn}}{1 + q^{pn} + q^{2pn} + \ldots + q^{(p-1)pn}}.
\]

Recall the identity:

\[
\sum_{k=1}^{n} kx^k = \frac{x(1 - x^{n+1}) - (n + 1)(1 - x)x^{n+1}}{(1 - x)^2}.
\] (4)

Manipulating the generating function, we have

\[
\sum_{n=0}^{\infty} a(n, p)q^n = \frac{(q^p; q^p)_{\infty}}{(q; q)_{\infty}} \sum_{n=1}^{p-1} \frac{jg^{pnj}}{1 - q^{pn}}
\]

\[
= \frac{(q^p; q^p)_{\infty}}{(q; q)_{\infty}} \sum_{n=1}^{\infty} \frac{q^{pn}(1 - q^{pn}) - p(1 - q^{pn})q^{pn}}{1 - q^{2pn}} \quad \text{(by (4) with } n = p - 1, x = q^{pn})
\]

\[
= \frac{(q^p; q^p)_{\infty}}{(q; q)_{\infty}} \sum_{n=1}^{\infty} \frac{1 - q^{pn}}{1 - q^{2pn}} - q^{pn}(1 - q^{pn}) - p(1 - q^{pn})q^{pn}
\]

\[
= \frac{(q^p; q^p)_{\infty}}{(q; q)_{\infty}} \sum_{n=1}^{\infty} \frac{1 - q^{pn}}{1 - q^{2pn}} \quad \text{(mod } p)
\]
\[
\sum_{n=0}^{\infty} a_p(n)q^n = \frac{(q^p; q^p)_\infty}{(q; q)_\infty} \sum_{n=1}^{\infty} \frac{q^{pn}}{1 - q^{pn}}.
\]

It is notable that
\[
\sum_{n=0}^{\infty} a(n, p)q^n \equiv \frac{1}{(q; q)_\infty}(q^p; q^p)_\infty \sum_{n=1}^{\infty} \frac{q^{pn}}{1 - q^{pn}} \quad (\text{mod } p).
\]

As the latter factor is a series in \( q^p \), it is immediately true that if \( s(pn+j) \) vanishes modulo \( p \) for all \( n \) where \( s(m) \) counts the number of unrestricted partitions of \( m \), so does \( a(pn+j, p) \).

In particular, applying the Ramanujan’s partition congruences: \( s(5n+4) \equiv 0 \) (mod 5), \( s(7n+5) \equiv 0 \) (mod 7) and \( s(11n+6) \equiv 0 \) (mod 11), we have the following result.

**Theorem 4** For all \( n \geq 0 \),
\[
a(5n+4, 5) \equiv 0 \quad \text{(mod 5)},
a(7n+5, 7) \equiv 0 \quad \text{(mod 7)}
\]

and
\[
a(11n+6, 11) \equiv 0 \quad \text{(mod 11)}.
\]

Let \( o_{p,e}(n) \) (resp. \( o_{p,o}(n) \)) be the number of \( o_p(n) \)-partitions in which the number of parts congruent to 0 modulo \( p \) is even (resp. odd) and
\[
H(z, q) = \frac{(q^p; q^p)_\infty}{(q; q)_\infty} \sum_{n=1}^{\infty} \frac{zq^{pn}}{1 - zq^{pn}}.
\]

Then
\[
\sum_{n=0}^{\infty} o_{p,o}(n)q^n = \frac{1}{2} \left( H(1, q) - H(-1, q) \right)
\]
\[
= \frac{(q^p; q^p)_\infty}{(q; q)_\infty} \left( \sum_{n=1}^{\infty} \frac{q^{pn}}{1 - q^{pn}} + \sum_{n=1}^{\infty} \frac{q^{pn}}{1 + q^{pn}} \right)
\]
\[
= \frac{(q^p; q^p)_\infty}{(q; q)_\infty} \sum_{n=1}^{\infty} \frac{q^{pn}}{1 - q^{2pn}}
\]
\[
= \frac{(q^p; q^p)_\infty}{(q; q)_\infty} \sum_{n=1}^{\infty} q^{pn} \sum_{m=0}^{\infty} q^{2pm}
\]
\[
= \frac{(q^p; q^p)_\infty}{(q; q)_\infty} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} q^{(p+2pm)n}
\]
\[
= \frac{(q^p; q^p)_\infty}{(q; q)_\infty} \sum_{m=0}^{\infty} \frac{q^{p+2pm}}{1 - q^{2pm+p}}
\]

(5)
and similarly,

\[
\sum_{n=0}^{\infty} o_{p,e}(n)q^n = \frac{1}{2} (H(1, q) + H(-1, q)) \\
= \frac{(q^p; q^p)_{\infty}}{(q; q)_{\infty}} \sum_{n=1}^{\infty} \frac{q^{2pn}}{1 - q^{2pn}}.
\]

(6)

Thus, if for \( i = 0, p, h_i(n, p) \) is the number of partitions in which parts are congruent to 1, 2, 3, \ldots, \( p - 1 \) modulo \( p \) or congruent to \( i \) modulo \( 2p \) and the set of parts \( \equiv i \pmod{2p} \) is singleton, then

\[ o_{p,o}(n) = h_p(n, p) \quad \text{and} \quad o_{p,e}(n) = h_0(n, p). \]

### 3 Further variations

Consider the following partition functions:

- \( d_e(n) \): the number of partitions of \( n \) in which exactly one even part is repeated and odd parts are unrestricted.
- \( d_o(n) \): the number of partitions of \( n \) in which exactly one odd part is repeated and even parts are unrestricted.

For \( i = 0, 2, f_i(n) \): the number of partitions of \( n \) in which the set of parts congruent to \( i \pmod{4} \) is singleton. Then we have the following theorem:

**Theorem 5** For \( n \geq 1 \),

\[ d_e(n) = f_0(n) \quad \text{and} \quad d_o(n) = f_2(n). \]

**Proof** Note that

\[
\sum_{n=0}^{\infty} d_e(n)q^n = \frac{1}{(q; q^2)_{\infty}} \sum_{n=1}^{\infty} q^{2n+2n} \prod_{j \neq n, j=1}^{\infty} (1 + q^{2j}) \\
= \frac{1}{(q; q^2)_{\infty}} \sum_{n=1}^{\infty} q^{4n} (-q^{-2}; q^2)_{\infty} (1 - q^{2n})(1 + q^{2n}) \\
= \frac{(-q^{-2}; q^2)_{\infty}}{(q; q^2)_{\infty}} \sum_{n=1}^{\infty} \frac{q^{4n}}{1 - q^{4n}} \\
= \frac{(-q^{-2}; q^2)_{\infty}(q^2; q^2)_{\infty} (q^2; q^2)_{\infty}}{(q; q^2)_{\infty}(q^2; q^2)_{\infty}} \sum_{n=1}^{\infty} \frac{q^{4n}}{1 - q^{4n}} \\
= \frac{(q^4; q^4)_{\infty}}{(q; q)_{\infty}} \sum_{n=1}^{\infty} \frac{q^{4n}}{1 - q^{4n}} \\
= \frac{1}{(q; q^4)(q^2; q^4)(q^3; q^4)_{\infty}} \sum_{n=1}^{\infty} (q^{4n} + q^{4n+4} + q^{4n+4n+4n} + \ldots) \\
= \sum_{n=0}^{\infty} f_0(n)q^n.
\]
Similarly,
\[
\sum_{n=0}^\infty d_o(n)q^n = \frac{1}{(q^2; q^2)\infty} \sum_{n=1}^\infty \frac{q^{2n-1+2n-1}}{1 - q^{2n-1}} \prod_{j \neq n, j=1}^\infty (1 + q^{2j-1})
\]
\[
= \frac{1}{(q^2; q^2)\infty} \sum_{n=1}^\infty \frac{q^{4n-2}(-q; q^2)\infty}{(1 - q^{2n-1})(1 + q^{2n-1})}
\]
\[
= \frac{(-q; q^2)\infty}{(q^2; q^2)\infty} \sum_{n=1}^\infty \frac{q^{4n-2}}{1 - q^{4n-2}}
\]
\[
= \frac{(q^2; q^2)\infty(q; q^2)\infty}{(q^2; q^2)\infty(q; q^2)\infty} \sum_{n=1}^\infty \frac{q^{4n-2}}{1 - q^{4n-2}}
\]
\[
= \frac{1}{(q; q^4)\infty(q^3; q^4)\infty(q^4; q^4)\infty} \sum_{n=1}^\infty (q^{4n-2} + q^{2(4n-2)} + q^{3(4n-2)} + \ldots)
\]
\[
= \sum_{n=0}^\infty f_2(n)q^n.
\]
\[\square\]

We also note the following bijective proof which is uniform for both identities in the theorem.

The bijection

Let \(\lambda = (\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots, \lambda_l^{m_l})\) be a partition enumerated by \(f_2(n)/f_0(n)\). Then

\[\lambda_i^{m_i} \mapsto \begin{cases} \left(\frac{\lambda_i}{2}\right)^{2m_i}, & \lambda_i \equiv 0, 2 \pmod{4}; \\ \phi_2^{-1}(\lambda_i^{m_i}), & \lambda_i \equiv 1, 3 \pmod{4}. \end{cases}\]

To invert the mapping, let \(\mu = (\mu_1^{s_1}, \mu_2^{s_2}, \ldots, \mu_l^{s_l})\) be a partition enumerated by \(d_o(n)/d_e(n)\). Then

\[\mu_i^{s_i} \mapsto (2\mu_i)^{\left\lfloor \frac{s_i}{2} \right\rfloor} \cup \phi_2(\mu_i^{s_i-2\left\lfloor \frac{s_i}{2} \right\rfloor}).\]

Recall that

\[
\sum_{n=0}^\infty d_e(n)q^n = \frac{(q^4; q^4)\infty}{(q; q)\infty} \sum_{n=1}^\infty \frac{q^{4n}}{1 - q^{4n}}
\]

so that

\[
(q; q)\infty \sum_{n=0}^\infty d_e(n)q^n = (q^4; q^4)\infty \sum_{n=1}^\infty \frac{q^{4n}}{1 - q^{4n}}.
\]

(7)

Using (2), we have
\[
(q; q) \sum_{n=0}^{\infty} d_e(n)q^n = \left(1 + \sum_{n=1}^{\infty} (-1)^n q^{\left(\frac{3n+1}{2}\right)}\right) \sum_{n=0}^{\infty} d_e(n)q^n
\]
\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} c_k d_e(n-k)q^n
\]

where

\[
c_k = \begin{cases} 
(-1)^k, & \text{if } k = \frac{j(3j+\pm1)}{2}, j \in \mathbb{Z} \geq 0; \\
0, & \text{otherwise.} 
\end{cases}
\]

Note that the right-hand side of (8) is equal to

\[
\sum_{n=0}^{\infty} \left[ d_e(n) + \sum_{j=1}^{\left\lfloor \sqrt{\frac{24n+1}{6}} \right\rfloor} (-1)^j \left( d_e\left(n - \frac{j(3j+1)}{2}\right) + d_e\left(n - \frac{j(3j-1)}{2}\right)\right) \right] q^n.
\]

Invoking (2) and the identity

\[
\sum_{n=1}^{\infty} \frac{q^n}{1-q^n} = \sum_{n=1}^{\infty} \sigma_0(n)q^n
\]

where \(\sigma_0(n)\) is the number of positive divisors of \(n\), (7) becomes

\[
\sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} \cdot \sum_{n=0}^{\infty} d_e(n)q^n = \sum_{n=-\infty}^{\infty} (-1)^n q^{2n(3n+1)} \sum_{n=1}^{\infty} \sigma_0(n)q^{4n}.
\]

Hence, we have the following.

**Corollary 6** For all integers \(r \geq 0\), let

\[
A(r) = \left\{ j \in \mathbb{Z} \geq 0 : 2j(3j+1) \equiv r \pmod{4} \text{ and } j \leq \left\lfloor (\sqrt{6r+1} - 1)/2 \right\rfloor \right\}
\]

and

\[
B(r) = \left\{ j \in \mathbb{Z} \geq 0 : 2j(3j-1) \equiv r \pmod{4} \text{ and } j \leq \left\lfloor (1 + \sqrt{6r+1})/2 \right\rfloor \right\}.
\]

Then for a positive integer \(n\), \(d_e(n)\) is equal to

\[
\begin{cases} 
\sum_{j=1}^{\left\lfloor \sqrt{\frac{24n+1}{6}} \right\rfloor} (-1)^{j+1} \left( d_e\left(n - \frac{j(3j+1)}{2}\right) + d_e\left(n - \frac{j(3j-1)}{2}\right)\right), & \text{if } n \not\equiv 0 \pmod{4}; \\
\sum_{j=1}^{\left\lfloor \sqrt{\frac{24n+1}{6}} \right\rfloor} (-1)^{j+1} \left( d_e\left(n - \frac{j(3j+1)}{2}\right) + d_e\left(n - \frac{j(3j-1)}{2}\right)\right) + \gamma(n), & \text{if } n \equiv 0 \pmod{4},
\end{cases}
\]

where

\[
\gamma(n) = \sigma_0(n/4) + \sum_{j \in A(n)} (-1)^j \sigma_0\left(\frac{n - 2j(3j+1)}{4}\right) + \sum_{j \in B(n)} (-1)^j \sigma_0\left(\frac{n - 2j(3j-1)}{4}\right).
\]

In this case, we have a relationship between \(d_e(n)\) and \(\sigma_0(n)\).
Example 1  Consider $n = 8$.

The $d_e(8)$-partitions are:

$$(4^2), (4, 2^2), (3, 2^2, 1), (2^4), (2^3, 1^2), (2^2, 1^4).$$

We now apply the recurrence. Since $8 \equiv 0 \pmod{4}$, we have

$$d_e(8) = d_e(8 - 2) + d_e(8 - 1) - d_e(8 - 7) - d_e(8 - 5) + \gamma(8)$$

$$= d_e(6) + d_e(7) - d_e(1) - d_e(3) + \gamma(8)$$

$$= d_e(4) + 2d_e(5) - d_e(1) + d_e(6) - 2d_e(2) - d_e(1) + \gamma(8)$$

$$= d_e(2) + 3d_e(3) + 3d_e(4) + d_e(5) - 3d_e(1) + \gamma(4) + \gamma(8)$$

$$= 4d_e(1) + 6d_e(2) + 4d_e(3) + d_e(4) + 4\gamma(4) + \gamma(8)$$

$$= 10d_e(1) + 5d_e(2) + d_e(3) + 5\gamma(4) + \gamma(8)$$

$$= 6d_e(1) + d_e(2) + 5\gamma(4) + \gamma(8)$$

$$= d_e(1) + 5\gamma(4) + \gamma(8)$$

$$= 5\gamma(4) + \gamma(8).$$

Now, $A(8) = \{1\}, B(8) = \{1\}, A(4) = \{\} $ and $B(4) = \{1\}$ and so

$$d_e(8) = 5\gamma(4) + \gamma(8)$$

$$= 5[\sigma_0(1) - \sigma_0(0)] + [\sigma_0(2) + \sigma_0(0) - \sigma_0(1)]$$

$$= \sigma_0(2) + 4\sigma_0(1) - 5\sigma_0(0)$$

$$= 2 + 4(1) - 5(0)$$

$$= 6.$$

We also note that

$$\sum_{n=0}^{\infty} a_n q^n = \frac{(q^2; q^4)_{\infty}}{(q; q)_{\infty}} \sum_{n=1}^{\infty} q^{4n-2}$$

$$= \frac{1}{(-q^2; q^2)_{\infty}(q; q)_{\infty}} \sum_{n=1}^{\infty} q^{4n-2}$$

$$= \frac{1}{(q^2; q^2)_{\infty}(q; q)_{\infty}} \left( \sum_{n=1}^{\infty} q^{2n} \frac{q^{4n}}{1 - q^{2n}} - \sum_{n=1}^{\infty} q^{4n} \right) \pmod{2}$$

$$= \frac{1}{(q; q)_{\infty}^3} \left( \sum_{n=1}^{\infty} q^{2n} \frac{q^{4n}}{1 - q^{2n}} - \sum_{n=1}^{\infty} q^{4n} \right) \pmod{2}$$

so that

$$\sum_{n=0}^{\infty} a_n q^n \sum_{n=0}^{\infty} q^{n(n+1)/2} \equiv \sum_{n=1}^{\infty} q^{2n} \frac{q^{4n}}{1 - q^{2n}} - \sum_{n=1}^{\infty} q^{4n} \pmod{2} \quad (9)$$

where we have used (3) for $(q; q)_{\infty}^3$.

But the right-hand side of (9) is

$$\sum_{n=1}^{\infty} \sigma_0(n) q^{2n} - \sum_{n=1}^{\infty} \sigma_0(n) q^{4n}$$
If $n \equiv 0 \pmod{4}$, we can write $n = 2^m b$ for some positive integers $m \geq 2$ and $b$ odd. Then
\[
\sigma_0(n/2) - \sigma_0(n/4) = \sigma_0(2^m b/2) - \sigma_0(2^m b/4)
\]
\[
= \sigma_0(2^{m-1} b) - \sigma_0(2^{m-2} b)
\]
\[
= \sigma_0(2^{m-1} b) - \sigma_0(2^{m-2} b) \sigma_0(b) \quad (\sigma_0 \text{ is multiplicative})
\]
\[
= \sigma_0(2^{m-1} b) - \sigma_0(2^{m-2} b) (m - (m - 1))
\]
\[
= \sigma_0(b).
\]

On the other hand, if $n \equiv 2 \pmod{4}$, we have $n = 2b$ for some odd positive integer $b$. Thus,
\[
\sigma_0(n/2) = \sigma_0(b).
\]

Denoting by $v_2(n)$, the 2-adic valuation of $n$, we obtain the following result.

**Corollary 7** For $n > 0$, we have
\[
\sum_{j=0}^{\left\lfloor \sqrt{n+1} \right\rfloor} d_0(n - j(j+1)/2) \equiv \begin{cases} 
0 \pmod{2}, & \text{if } n \equiv 1 \pmod{2}; \\
\sigma_0(n/2^{v_2(n)}) \pmod{2}, & \text{if } n \equiv 0 \pmod{2}.
\end{cases}
\]

### 4 Generalizations

The partition function $d_0(n)$ can be generalized as follows: Let $k, p \geq 2$ and $0 \leq r < p$ be integers and define $d_p(n, k, r)$ as the number of partitions of $n$ in which exactly one part $\equiv r \pmod{p}$ appears at least $k$ times. In this definition, parts not congruent to $r \pmod{p}$ appear with unrestricted multiplicity. Similary, let $f_p(n, k, r)$ denote the number of partitions of $n$ in which parts $\equiv kr \pmod{pk}$ form a singleton set.

If $r \neq 0$, we have
\[
\sum_{n=0}^{\infty} d_p(n, k, r) q^n
\]
\[
= \frac{(q^r; q^p)_\infty}{(q; q)_\infty} \sum_{n=0}^{\infty} \frac{q^{k(pn+r)}}{1 - q^{pn+r}} \prod_{j \neq n, j=0}^{\infty} (1 + q^{pj+r} + q^{2(pj+r)} + \cdots + q^{(k-1)(pj+r)})
\]
\[
= \frac{(q^r; q^p)_\infty}{(q; q)_\infty} \sum_{n=0}^{\infty} \frac{q^{k(pn+r)}}{(1 - q^{pn+r})^k} \prod_{j=0}^{\infty} \frac{1 - q^{pj+r}}{1 - q^{pj+r}}
\]
\[
\begin{align*}
&= \frac{(q^r; q^p)_{\infty}(q^{kr}; q^{kp})_{\infty}}{(q^r; q^p)_{\infty}(q; q)_{\infty}} \sum_{n=0}^{\infty} \frac{q^{pkn+kr}}{1 - q^{pkn+kr}} \\
&= \frac{(q^{kr}; q^{kp})_{\infty}}{(q; q)_{\infty}} \sum_{n=0}^{\infty} \frac{q^{pkn+kr}}{1 - q^{pkn+kr}} \\
&= \sum_{n=0}^{\infty} f_p(n, k, r)q^n.
\end{align*}
\]

If \( r = 0 \), then
\[
\sum_{n=0}^{\infty} d_p(n, k, 0)q^n = \frac{(q^p; q^p)_{\infty}}{(q; q)_{\infty}} \sum_{n=1}^{\infty} \frac{q^{kn}}{1 - q^{pn}} \prod_{j \neq n, j=1}^{\infty} (1 + q^{pj} + q^{2pj} + \cdots + q^{(k-1)pj})
\]

which simplifies to
\[
\frac{(q^{kp}; q^{kp})_{\infty}}{(q; q)_{\infty}} \sum_{n=1}^{\infty} \frac{q^{kn}}{1 - q^{pn}} = \sum_{n=0}^{\infty} f_p(n, k, 0)q^n.
\]

Thus, we have:

**Theorem 8** Let \( p \geq 2 \). For an integer \( n \geq 1 \),
\[
f_p(n, k, r) = d_p(n, k, r).
\]

A bijection (which extends the one for \( d_0(n) / f_0(n) \) and \( d_0(n) / f_2(n) \)), is given as follows: Let \( \lambda = (\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots, \lambda_l^{m_l}) \) be a partition enumerated by \( f_p(n, k, r) \). Then
\[
\lambda_i^{m_i} \mapsto \begin{cases} 
\left( \frac{\lambda_i}{k} \right)^{km_i}, & \text{if } \lambda_i \equiv 0, k, 2k, \ldots, (p - 1)k \pmod{pk}; \\
\phi_k^{-1}(\lambda_i^{m_i}), & \text{otherwise}.
\end{cases}
\]

From \( \lambda \), note that there is only one part congruent to \( kr \pmod{pk} \) (with unrestricted multiplicity). Under the bijection, this part is mapped to the exactly one part which is congruent to \( r \pmod{p} \) repeated at least \( k \) times. For other parts of \( \lambda \) that are not congruent to \( kr \pmod{kp} \), there are two cases.

Case 1: If the part is divisible by \( k \), it can be shown that such a part must be congruent to \( jk \pmod{kp} \) where \( j \in \{0, 1, 2, \ldots, p-1\} \setminus \{r\} \). This part is mapped to a part not congruent to \( r \pmod{p} \) with multiplicity at least \( k \).

Case 2: If that part of \( \lambda \) is not divisible by \( k \), we apply \( \phi_k^{-1} \).

Conversely, let \( \mu = (\mu_1^{s_1}, \mu_2^{s_2}, \ldots, \mu_l^{s_l}) \) be a partition enumerated by \( d_p(n, k, r) \). Then
\[
\mu_i^{s_i} \mapsto (k\mu_i)^{\frac{s_i}{k}} \cup \phi_k(\mu_i^{s_i-k\frac{k}{p}})
\]

represents the inverse mapping.

**Example 2** Consider \( n = 9 \), \( p = 3 \), \( k = 4 \) and \( r = 1 \).

The \( d_3(9, 4, 1) \)-partitions are:
\[
(5, 1^4), (4, 1^5), (3, 2, 1^4), (3, 1^6), (2^2, 1^5), (2, 1^7), (1^9).
\]
To find the image of \( (19) \), we perform the transformation:
\[
\left\lfloor \frac{9}{4} \right\rfloor = 2 \quad \text{and} \quad 9 - 4 \left\lfloor \frac{9}{4} \right\rfloor = 1.
\]
Applying the inverse map in (10) yields \( 19 \mapsto (4^2, 1) \).

Similarly, we have
\[
(5, 1^4) \mapsto (5, 4) \quad (4, 1^5) \mapsto (4, 1^5) \quad (3, 2, 1^4) \mapsto (4, 3, 2) \quad (3, 1^6) \mapsto (4, 3, 1^2) \quad (2^2, 1^5) \mapsto (4, 2^2, 1) \quad (2, 1^7) \mapsto (4, 2, 1^2).
\]

### 4.1 Connection with the work of Fu and Tang

Recall that \( o_k(n) \) denotes the number of partitions of \( n \) in which the set of parts congruent to 0 modulo \( k \) is singleton. Fu and Tang (2017) showed that if \( d_k(n) \) is the number of partitions of \( n \) where exactly one part is repeated at least \( k \) times, then
\[
o_k(n) = d_k(n). \tag{11}
\]
Setting \( k = 4 \) in (11), \( o_4(n) = d_4(n) \). On the other hand, we have shown that \( f_0(n) = d_e(n) \) in Theorem 5. Since \( f_0(n) = o_4(n) \), we have the identity
\[
d_4(n) = d_e(n). \tag{12}
\]
Note that (12) is not ‘trivial’. As a preparation for a more generalized result, we describe its bijective proof.

Let \( \lambda \) be enumerated by \( d_4(n) \) and \( j \) be the part that is repeated at least 4 times. Then \( \lambda = \lambda \cup j^m \) where \( m \) is the multiplicity of \( j \) in \( \lambda \) and \( \lambda \) is the subpartition of \( \lambda \) consisting of parts that are repeated at most 3 times. Write \( m \) as
\[
m = 4q + i \quad \text{where} \quad 0 \leq i \leq 3 \quad \text{and} \quad q \geq 1.
\]
One writes \( j^m = j^4q \cup j^i \) and converts \( j^4q \) into \( (2j)^{2q} \).

Apply Glaisher map \( \phi_4 \) on \( \lambda \cup j^i \) so that the image \( \phi_4(\lambda \cup j^i) \) is a partition into parts not divisible by 4.

Decompose the partition as
\[
\phi_4(\lambda \cup j^i) = \lambda' \cup \lambda''
\]
where \( \lambda' \) is the subpartition consisting of odd parts and \( \lambda'' \) is the subpartition consisting of parts \( \equiv 2 \pmod{4} \). Divide each part of \( \lambda'' \) by 2 and apply \( \phi_2^{-1} \) on the resulting partition.

What follows is a partition into distinct parts. Then multiply every part of this partition by 2 and call the resulting partition \( \beta \).

Note that
\[
(2j)^{2q} \cup \beta \cup \lambda'
\]
is a partition enumerated by $d_c(n)$.

The inverse will be demonstrated in the general map. We require the following lemma in the subsequent work.

**Lemma 1** Let $p$, $k$ and $x$ be positive integers such that $pk \geq 2$. Suppose that $x \not\equiv 0 \pmod{pk}$ and $x \equiv 0 \pmod{p}$. Then $\frac{x}{p} \not\equiv 0 \pmod{k}$.

Now, setting $r = 0$ in Theorem 8, observe that $d_p(n, k, 0)$ is the number of partitions of $n$ in which exactly one part $\equiv 0 \pmod{p}$ appears at least $k$ times. By the above result of Fu and Tang, the following generalization of (12) follows:

**Theorem 9** For $n \geq 0$,

$$d_{pk}(n) = d_p(n, k, 0).$$

Our interest in this theorem is in its bijective construction, which extends the given mapping for the special case $k = 2$ and $p = 2$.

Let $\lambda$ be enumerated by $d_{pk}(n)$ and $j$ be the part that is repeated at least $pk$ times. Then $\lambda = \lambda \cup j^m$ where is the multiplicity of $j$ in $\lambda$ and $\lambda$ is the subpartition of $\lambda$ consisting of parts that are repeated at most $pk - 1$ times. Write $m$ as

$$m = pkq + i \quad \text{where} \quad 0 \leq i \leq pk - 1 \quad \text{and} \quad q \geq 1.$$

One rewrites $j^m$ as $j^m = j^{pkq} \cup j^i$ and converts $j^{pkq}$ into $(pj)^{kq}$.

Apply Glaisher map $\phi_{pk}$ on $\lambda \cup j^i$ so that the image $\phi_{pk}(\lambda \cup j^i)$ is a partition into parts not divisible by $pk$.

Decompose this partition as

$$\phi_{pk}(\lambda \cup j^i) = \lambda' \cup \lambda''$$

where $\lambda'$ is the subpartition consisting of parts $\not\equiv 0 \pmod{p}$ and $\lambda''$ is the subpartition consisting of parts $\equiv 0 \pmod{p}$. Divide each part of $\lambda''$ by $p$ and note that, by Lemma 1, the resulting parts are not divisible by $k$. Apply $\phi^{-1}_p$ on the resulting partition. What follows is a partition into parts that appear at most $k - 1$ times. Then multiply every part of this partition by $p$ and call the resulting partition $\beta$.

Note that

$$(pj)^{kq} \cup \beta \cup \lambda'$$

is a partition enumerated by $d_p(n, k, 0)$.

**Example 3** Consider $n = 433$, $p = 3$ and $k = 4$.

Consider the $d_{12}(433)$-partition:

$$(13^{10}, 10^5, 7^{30}, 6^2, 4^5, 1^{111})$$

which is decomposed as:

$$(13^{10}, 10^5, 6^2, 4^5, 1^{111}) \cup (7^{30})$$

where $j = 7$ and $m = 30 = 12 \cdot 2 + 6$. Thus $7^{30} = (7^{24}) \cup (7^6)$ and so

$$(7^{24}) \mapsto (21^8).$$

Applying the Glaisher’s map to the remaining parts yields

$$\phi_{12}((13^{10}, 10^5, 7^6, 6^2, 4^5, 1^{111}) = (13^{10}, 10^5, 7^6, 6^2, 4^5, 1^{111}).$$
Note that $6 \equiv 0 \pmod{3}$ and thus

$$\beta = 3 \times \phi_4^{-1} \left( \frac{6}{3}, \frac{6}{3} \right) = (6^2).$$

Taking the union of all the image parts gives

$$(21^8, 13^{10}, 10^5, 6^2, 4^5, 1^{11})$$

which is a $d_3(433, 4, 0)$-partition.

The inverse is described as follows:

Let $\mu$ be enumerated by $d_p(n, k, 0)$. Decompose the partition as

$$\mu = s^m \cup \beta \cup \lambda'$$

where $s$ is that one part $\equiv 0 \pmod{p}$ whose multiplicity $m$ is at least $k$ times, $\beta$ is the subpartition consisting of parts $\equiv 0 \pmod{p}$ whose multiplicities are $\leq k - 1$ and $\lambda'$ is the subpartition consisting of parts $\not\equiv 0 \pmod{p}$. Write $m$ as

$$m = kt + f \quad \text{where } 0 \leq f \leq k - 1 \text{ and } t \geq 1.$$

One rewrites $s^m$ as $s^m = s^{kt} \cup s^f$ and converts $s^{kt}$ into $\left( \frac{s}{p} \right)^{p^kt}$.

Divide each part of $\beta \cup s^f$ by $p$ and then apply $\phi_k$ on the resulting partition. Then multiply every part of the obtained partition by $p$ and denote the resulting partition by $\mu'$. Note that $\mu'$ is a partition whose parts are not divisible by $pk$ but divisible by $p$. Thus the parts of $\lambda' \cup \mu'$ are not divisible by $pk$. Now compute

$$\mu'' = \phi^{-1}_{pk}(\lambda' \cup \mu')$$

so that $\mu''$ is a partition with parts appearing at most $pk - 1$ times. Clearly,

$$\left( \frac{s}{p} \right)^{p^kt} \cup \mu''$$

is a partition enumerated by $d_{pk}(n)$.

Reversing the previous example, recall the $d_3(433, 4, 0)$-partition:

$$(21^8, 13^{10}, 10^5, 6^2, 4^5, 1^{11}).$$

We decompose it as follows:

$$(21^8) \cup (6^2) \cup (13^{10}, 10^5, 4^5, 1^{11}).$$

Thus,

$$\left(21^8 \right) \mapsto (7^{24}),$$

$$3 \times \phi_4 \left( \frac{6}{3}, \frac{6}{3} \right) = (6^2).$$

Applying the Glaisher’s map to $(13^{10}, 10^5, 4^5, 1^{11}) \cup (6^2)$ gives

$$\phi_{12}^{-1}(13^{10}, 10^5, 6^2, 4^5, 1^{11}) = (13^{10}, 10^5, 7^6, 6^2, 4^5, 1^{11}).$$

Now, taking the union of all the image parts yields

$$(13^{10}, 10^5, 7^{30}, 6^2, 4^5, 1^{11})$$

which is a $d_{12}(433)$-partition.
4.2 A slightly different partition function

For $\alpha \geq k$, let $g(n, \alpha, k, p)$ denote the number of partitions of $n$ in which only one part appears at least $\alpha$ times and its multiplicity is congruent to $\alpha + j$ modulo $p$ where $0 \leq j \leq k - 1$ and all other parts appear at most $k - 1$ times. Denote by $g_o(n, \alpha, k, p)$ (resp. $g_e(n, \alpha, k, p)$) the number of $g(n, \alpha, k, p)$-partitions in which the part repeated at least $\alpha$ times is odd (resp. even). Then we have:

**Theorem 10**

$$\sum_{n=0}^{\infty} g_o(n, \alpha, k, p) - g_e(n, \alpha, k, p)q^n = \frac{(q^k; q^k)_\infty}{(q; q)_\infty} \sum_{n=0}^{\infty} \frac{q^{pn+\alpha}}{1 + q^{pn+\alpha}}.$$ (13)

**Proof** If $z$ tracks the part repeated at least $\alpha$ times, observe that

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} g(n, \alpha, k, p)q^n z^m$$

$$= \sum_{n=1}^{\infty} \left[ z^n \left( \sum_{m=0}^{\infty} q^{(\alpha+m)p}n + \sum_{m=0}^{\infty} q^{(\alpha+1+m)p}n + \ldots + \sum_{m=0}^{\infty} q^{(\alpha+k-1+m)p}n \right) \right]$$

$$\times \prod_{j=1, j \neq n} (1 + q^j + q^{2j} + q^{3j} + \ldots + q^{(k-1)j})$$

$$= \sum_{n=1}^{\infty} z^n \sum_{i=0}^{k-1} \prod_{m=0}^{\infty} q^{(\alpha+i+m)p}n \prod_{j=1, j \neq n} (1 + q^j + q^{2j} + q^{3j} + \ldots + q^{(k-1)j})$$

$$= \sum_{n=1}^{\infty} z^n q^{\alpha n} \sum_{i=0}^{k-1} q^{in} \prod_{m=0}^{\infty} q^{mpn} \prod_{j=1, j \neq n} (1 + q^j + q^{2j} + q^{3j} + \ldots + q^{(k-1)j}).$$

$$= \sum_{n=1}^{\infty} z^n q^{\alpha n} \prod_{j=1}^{\infty} (1 + q^j + q^{2j} + q^{3j} + \ldots + q^{(k-1)j}).$$

$$= \sum_{n=1}^{\infty} z^n q^{\alpha n} \prod_{j=1}^{\infty} \frac{1 - q^{kj}}{1 - q^j}$$

$$= \frac{(q^k; q^k)_\infty}{(q; q)_\infty} \sum_{n=1}^{\infty} z^n q^{\alpha n} \prod_{j=1}^{\infty} \frac{1 - q^{kj}}{1 - q^j}$$

so that

$$\sum_{n=0}^{\infty} (g_o(n, \alpha, k, p) - g_e(n, \alpha, k, p))q^n = -\frac{(q^k; q^k)_\infty}{(q; q)_\infty} \sum_{n=1}^{\infty} (-1)^n q^{\alpha n} \prod_{j=1}^{\infty} \frac{1 - q^{kj}}{1 - q^j}$$

$$= -\frac{(q^k; q^k)_\infty}{(q; q)_\infty} \sum_{n=1}^{\infty} (-1)^n q^{\alpha n} \sum_{m=0}^{\infty} q^{pmn}$$
\[
= - \frac{(q^k, q^k)}{(q, q)} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} (-q^{a+pm})^n
\]
\[
= - \frac{(q^k, q^k)}{(q, q)} \sum_{m=0}^{\infty} \frac{-q^{a+pm}}{1 + q^{a+pm}}
\]
\[
= \frac{(q^k, q^k)}{(q, q)} \sum_{m=0}^{\infty} \frac{q^{a+pm}}{1 + q^{a+pm}}.
\]

We have the following relationship between \( h(n, p) \) and \( g(n, p, p, p) \).

**Proposition 11** For all \( n \geq 0 \),
\[
g_o(n, p, p, p) = h_0(n, p) \quad \text{and} \quad g_e(n, p, p, p) = h_p(n, p).
\]

**Proof** From the fact that
\[
\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} g(n, p, p, p)q^nz^m = \frac{(q^p, q^p)}{(q, q)} \sum_{n=1}^{\infty} z^n q^{pn} \frac{1}{1 - q^{pn}},
\]
recall that \( z \) tracks the part which is repeated at least \( p \) times. Then
\[
\sum_{n=0}^{\infty} g_o(n, p, p, p)q^n = \frac{(q^p, q^p)}{(q, q)} \sum_{n=1, n \equiv 1 \mod 2}^{\infty} \frac{q^{pn}}{1 - q^{pn}}
\]
\[
= \frac{(q^p, q^p)}{(q, q)} \sum_{n=1}^{\infty} q^{2pn-p} \frac{1}{1 - q^{2pn-p}}
\]
\[
= \sum_{n=0}^{\infty} h_0(n, p)q^n
\]
and on the same note,
\[
\sum_{n=0}^{\infty} g_e(n, p, p, p)q^n = \frac{(q^p, q^p)}{(q, q)} \sum_{n=1, n \equiv 0 \mod 2}^{\infty} \frac{q^{pn}}{1 - q^{pn}}
\]
\[
= \frac{(q^p, q^p)}{(q, q)} \sum_{n=1}^{\infty} q^{2pn} \frac{1}{1 - q^{2pn}}
\]
\[
= \sum_{n=0}^{\infty} h_p(n, p)q^n.
\]

**Author Contributions** The authors contributed equally to this work.

**Funding** Open access funding provided by University of the Witwatersrand.

**Data Availability** The authors can confirm that this manuscript has no associated data.

**Declarations**

**Conflict of interest** On behalf of all authors, the corresponding author states that there is no conflict of interest.
Open Access  This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

References

1. Andrews, G.E.: The Theory of Partitions. Cambridge Mathematical Library. Cambridge University Press, Cambridge (1998). (Reprint of the 1976 original)
2. Andrews, G.E.: Euler’s partition identity-finite version. Math. Stud. 85, 3–4 (2016)
3. Andrews, G.E., Merca, M.: On the number of even parts in all partitions of $n$ into distinct parts. Ann. Combin. 24, 47–54 (2020)
4. Ballantine, C., Merca, M.: Dyson’s crank and unimodal compositions. Rev. R. Acad. Cienc. Exact. Fis. Nat. Ser. A Mat. RACSAM 116(4), 182 (2022)
5. Fu, S., Hua, T.: On the slack Euler pair for vector partition. Integers 18, 7 (2018)
6. Fu, S., Tang, D.: Generalizing a partition theorem of Andrews. Math. Stud. 86, 91–96 (2017)
7. Glaisher, J.W.L.: A theorem in partitions. Messenger Math. 12, 158–170 (1883)
8. Herden, D., Sepanski, M.R., et al.: Counting the parts divisible by $k$ in all the partitions of $n$ whose parts have multiplicities less than $k$. Integers 22, 49 (2022)
9. Merca, M.: A reversal of Schur’s partition theorem. Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM 116(4), 181 (2022)
10. Nyirenda, D.: A note on a finite version of Euler’s partition identity. Australas. J. Combin. 71(2), 241–245 (2018)
11. Nyirenda, D.: On parity and recurrences for certain partition functions. Contrib. Discrete Math. 15(1), 72–79 (2020)

Publisher’s Note  Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.