Many-body wave scattering problems in the case of small scatterers

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Abstract

Formulas are derived for solutions of many-body wave scattering problems by small particles in the case of acoustically soft, hard, and impedance particles embedded in an inhomogeneous medium. The case of transmission (interface) boundary conditions is also studied in detail. The limiting case is considered, when the size $a$ of small particles tends to zero while their number tends to infinity at a suitable rate. Equations for the limiting effective (self-consistent) field in the medium are derived. The theory is based on a study of integral equations and asymptotics of their solutions as $a \to 0$. The case of wave scattering by many small particles embedded in an inhomogeneous medium is also studied.

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1 Introduction

There is a large literature on wave scattering by small bodies, starting from Rayleigh’s work (1871), [31], [5], [3]. For the problem of wave scattering by one body an analytical solution was found only for the bodies of special shapes, for example, for balls and ellipsoids. If the scatterer is small then the scattered field can be calculated analytically for bodies of arbitrary shapes, see [9], where this theory is presented.

The many-body wave scattering problem was discussed in the literature mostly numerically, if the number of scatterers is small, or under the assumption that the influence of the waves, scattered by other particles on a
particular particle is negligible (see [7], where one finds a large bibliography, 1386 entries). This corresponds to the case when the distance $d$ between neighboring particles is much larger than the wavelength $\lambda$, and the characteristic size $a$ of a small body (particle) is much smaller than $\lambda$. By $k = \frac{2\pi}{\lambda}$ the wave number is denoted.

This paper is a review of the author’s results. The results of papers [27] and [30] are presented. The presentation follows closely the above papers.

The basic results of this paper consist of:

i) Derivation of analytic formulas for the scattering amplitude for the wave scattering problem by one small ($ka \ll 1$) body of an arbitrary shape under the Dirichlet, impedance, and Neumann boundary condition (acoustically soft, impedance, and hard particle), and the transmission (interface) boundary condition;

ii) Solution to many-body wave scattering problem by small particles, embedded in an inhomogeneous medium, under the assumptions $a \ll d$ and $a \ll \lambda$, where $d$ is the minimal distance between neighboring particles;

iii) Derivation of the equations for the limiting effective (self-consistent) field in an inhomogeneous medium in which many small particles are embedded, when $a \to 0$ and the number $M = M(a)$ of the small particles tends to infinity at an appropriate rate;

iv) Derivation of linear algebraic systems for solving many-body wave scattering problems. These systems are not obtained by a discretization of boundary integral equations, and they give an efficient numerical method for solving many-body wave scattering problems in the case of small scatterers;

v) Theory of wave scattering by small bodies of an arbitrary shape under the transmission (interface) boundary condition, and a derivation of the equation for the effective field in the limiting medium consisting of very many small bodies.

The derivations of the results, presented in this paper, are rigorous. The order of the error estimates as $a \to 0$ is obtained. Our methods give powerful numerical methods for solving many-body wave scattering problems in the case when the scatterers are small (see [1], [2]).

In Sections 1-4 wave scattering by small bodies under the Dirichlet, Neumann, and impedance boundary conditions is developed. In Sections 5-7 wave scattering by small bodies under the transmission boundary condition is presented. In Section 8 Conclusions are briefly stated.

Let us formulate the wave scattering problems we deal with. First, let us consider a one-body scattering problem. Let $D_1$ be a bounded domain in $\mathbb{R}^3$ with a sufficiently smooth boundary $S_1$. The scattering problem consists
of finding the solution to the problem:

\[(\nabla^2 + k^2)u = 0 \text{ in } D'_1 := \mathbb{R}^3 \setminus D_1, \quad (1)\]

\[\Gamma u = 0 \text{ on } S_1, \quad (2)\]

\[u = u_0 + v, \quad (3)\]

where

\[u_0 = e^{ik\alpha \cdot x}, \quad \alpha \in S^2, \quad (4)\]

\[S^2 \text{ is the unit sphere in } \mathbb{R}^3, \quad u_0 \text{ is the incident field, } v \text{ is the scattered field satisfying the radiation condition}
\]

\[v_r - ikv = o\left(\frac{1}{r}\right), \quad r := |x| \to \infty, \quad v_r := \frac{\partial v}{\partial r}, \quad (5)\]

\[\Gamma u \text{ is the boundary condition (bc) of one of the following types}
\]

\[\Gamma u = \Gamma_1 u = u \quad \text{(Dirichlet bc)}, \quad (6)\]

\[\Gamma u = \Gamma_2 u = u_N - \zeta_1 u, \quad \text{Im}\zeta_1 \leq 0, \quad \text{(impedance bc)}, \quad (7)\]

where \(\zeta_1\) is a constant, \(N\) is the unit normal to \(S_1\), pointing out of \(D_1\), and

\[\Gamma u = \Gamma_3 u = u_N, \quad \text{(Neumann bc)}. \quad (8)\]

The transmission bc are defined in Section 5, see (100) there.

It is well known (see, e.g., [8]) that problem (1)-(3) has a unique solution. We now assume that

\[a := 0.5 \text{ diam } D_1, \quad ka \ll 1, \quad (9)\]

and look for the solution to problem (1)-(3) of the form

\[u(x) = u_0(x) + \int_{S_1} g(x,t)\sigma_1(t)dt, \quad g(x,y) := \frac{e^{ik|x-y|}}{4\pi |x-y|}, \quad (10)\]

where \(dt\) is the element of the surface area of \(S_1\). One can prove that the unique solution to the scattering problem (1)-(3) with any of the boundary conditions (6)-(8) can be found in the form (10), and the function \(\sigma_1\) in equation (10) is uniquely defined from the boundary condition (2). The scattering amplitude \(A(\beta, \alpha) = A(\beta, \alpha, k)\) is defined by the formula

\[v = \frac{e^{ikr}}{r} A(\beta, \alpha, k) + o\left(\frac{1}{r}\right), \quad r \to \infty, \quad \beta := \frac{x}{r}. \quad (11)\]
The equations for finding $\sigma_1$ are:

$$\int_{S_1} g(s,t)\sigma_1(t)dt = -u_0(s),$$ (12)

$$u_{0N} - \zeta_1 u_0 + \frac{A\sigma_1 - \sigma_1}{2} - \zeta_1 \int_{S_1} g(s,t)\sigma_1(t)dt = 0,$$ (13)

$$u_{0N} + \frac{A\sigma_1 - \sigma_1}{2} = 0,$$ (14)

respectively, for conditions (6)-(8). The operator $A$ is defined as follows:

$$A\sigma := 2 \int_{S_1} \frac{\partial}{\partial N_s} g(s,t)\sigma_1(t)dt.$$ (15)

Equations (12)-(14) are uniquely solvable, but there are no analytic formulas for their solutions for bodies of arbitrary shapes. However, if the body $D_1$ is small, $ka \ll 1$, one can rewrite (10) as

$$u(x) = u_0(x) + g(x,0)Q_1 + \int_{S_1} [g(x,t) - g(x,0)]\sigma_1(t)dt,$$ (16)

where

$$Q_1 := \int_{S_1} \sigma_1(t)dt,$$ (17)

and $0 \in D_1$ is the origin.

If $ka \ll 1$, then we prove that

$$|g(x,0)Q_1| \gg \left| \int_{S_1} [g(x,t) - g(x,0)]\sigma_1(t)dt \right|, \quad |x| > a.$$ (18)

Therefore, the scattered field is determined outside $D_1$ by a single number $Q_1$. This number can be obtained analytically without solving equations (12)-(13). The case (14) requires a special approach by the reason discussed in detail later.

Let us give the results for equations (12) and (13) first. For equation (12) one has

$$Q_1 = \int_{S_1} \sigma_1(t)dt = -Cu_0(0)[1 + o(1)], \quad a \to 0,$$ (19)

where $C$ is the electric capacitance of a perfect conductor with the shape $D_1$. For equation (13) one has

$$Q_1 = -\zeta|S_1|u_0(0)[1 + o(1)], \quad a \to 0,$$ (20)
where \(|S_1|\) is the surface area of \(S_1\). The scattering amplitude for problem (1)-(3) with \(\Gamma = \Gamma_1\) (acoustically soft particle) is

\[
A_1(\beta, \alpha) = -\frac{C}{4\pi}[1 + o(1)],
\]

since

\[
u_0(0) = e^{i\kappa \alpha \cdot x}|_{x=0} = 1.
\]

Therefore, in this case the scattering is isotropic and of the order \(O(a)\), because the capacitance \(C = O(a)\).

The scattering amplitude for problem (1)-(3) with \(\Gamma = \Gamma_2\) (small-impedance particles) is:

\[
A_2(\alpha, \beta) = -\frac{\zeta_1 |S_1|}{4\pi}[1 + o(1)],
\]

since \(u_0(0) = 1\).

In this case the scattering is also isotropic, and of the order \(O(\zeta |S_1|)\).

If \(\zeta_1 = O(1)\), then \(A_2 = O(a^2)\), because \(|S_1| = O(a^2)\). If \(\zeta_1 = O\left(\frac{1}{a^\kappa}\right)\), \(\kappa \in (0, 1)\), then \(A_2 = O(a^{2-\kappa})\). The case \(\kappa = 1\) was considered in [11].

The scattering amplitude for problem (1)-(3) with \(\Gamma = \Gamma_3\) (acoustically hard particles) is

\[
A_3(\beta, \alpha) = -\frac{k^2|D_1|}{4\pi}(1 + \beta_{pq}\beta_p\alpha_q), \quad \text{if} \ u_0 = e^{i\kappa \alpha \cdot x}.
\]

Here and below summation is understood over the repeated indices, \(\alpha_q = \alpha \cdot e_q\), \(\alpha \cdot e_q\) denotes the dot product of two vectors in \(\mathbb{R}^3\), \(p, q = 1, 2, 3\), \(\{e_p\}\) is an orthonormal Cartesian basis of \(\mathbb{R}^3\), \(|D_1|\) is the volume of \(D_1\), \(\beta_{pq}\) is the magnetic polarizability tensor defined as follows ([9], p.62):

\[
\beta_{pq} := \frac{1}{|D_1|} \int_{S_1} t_p\sigma_{1q}(t)dt,
\]

\(\sigma_{1q}\) is the solution to the equation

\[
\sigma_{1q}(s) = A_0\sigma_{1q} - 2N_q(s),
\]

\(N_q(s) = N(s) \cdot e_q\), \(N = N(s)\) is the unit outer normal to \(S_1\) at the point \(s\), i.e., the normal pointing out of \(D_1\), and \(A_0\) is the operator \(A\) at \(k = 0\). For small bodies \(\|A - A_0\| = o(ka)\).

If \(u_0(x)\) is an arbitrary field satisfying equation (1), not necessarily the plane wave \(e^{i\kappa \alpha \cdot x}\), then

\[
A_3(\beta, \alpha) = \frac{|D_1|}{4\pi} \left(ik\beta_{pq} \frac{\partial u_0}{\partial x_q} \beta_p + \triangle u_0\right).
\]
The above formulas are derived in Section 2. In Section 3 we develop a theory for many-body wave scattering problem and derive the equations for effective field in the medium, in which many small particles are embedded, as \( a \to 0 \).

The results, presented in this paper, are based on the earlier works of the author (\([10]-[25]\)). Our presentation and some of the results are novel. These results and methods of their derivation differ much from those in the homogenization theory (\([4], [6]\)). The differences are:

i) no periodic structure in the problems is assumed,

ii) the operators in our problems are non-selfadjoint and have continuous spectrum,

iii) the limiting medium is not homogeneous and its parameters are not periodic,

iv) the technique for passing to the limit is different from one used in homogenization theory.

Let us summarize the results for one-body wave scattering.

**Theorem 1.1** The scattering amplitude for the problem (1)-(4) for small body of an arbitrary shape are given by formulas (25), (26), (27), for the boundary conditions \( \Gamma_1, \Gamma_2, \Gamma_3 \), respectively.

### 2 Derivation of the formulas for one-body wave scattering problems

Let us recall the known result (see e.g., [8])

\[
\frac{\partial}{\partial N_s} \int_{S_1} g(x,t)\sigma_1(t) dt = \frac{A\sigma_1 - \sigma_1}{2} \tag{27}
\]

centering the limiting value of the normal derivative of single-layer potential from outside. Let \( x_m \in D_m, t \in S_m, S_m \) is the surface of \( D_m \), \( a = 0.5 \text{diam}D_m \).

In this Section \( m = 1 \), and \( x_m = 0 \) is the origin.

We assume that \( ka \ll 1, ad^{-1} \ll 1 \), so \( |x - x_m| = d \gg a \). Then

\[
\frac{e^{ik|x-t|}}{4\pi|x-t|} = \frac{e^{ik|x-x_m|}}{4\pi|x-x_m|} e^{-ik(x-x_m)\cdot(t-x_m)} \left(1 + O(ka + \frac{a}{d})\right), \tag{28}
\]

\[
k|x-t| = k|x - x_m| - k(x - x_m)\cdot(t - x_m) + O\left(\frac{ka^2}{d}\right), \tag{29}
\]
where
\[ d = |x - x_m|, \quad (x - x_m)^0 := \frac{x - x_m}{|x - x_m|}, \]
and
\[ \frac{|x - t|}{|x - x_m|} = 1 + O \left( \frac{a}{d} \right). \tag{30} \]

Let us derive estimate (19). Since \(|t| \leq a\) on \(S_1\), one has
\[ g(s, t) = g_0(s, t)(1 + O(ka)), \]
where \(g_0(s, t) = \frac{1}{4\pi|s - t|}\). Since \(u_0(s)\) is a smooth function, one has \(|u_0(s) - u_0(0)| = O(a)\). Consequently, equation (12) can be considered as an equation for electrostatic charge distribution \(\sigma_1(t)\) on the surface \(S_1\) of a perfect conductor \(D_1\), charged to the constant potential \(-u_0(0)\) (up to a small term of the order \(O(ka)\)). It is known that the total charge \(Q_1 = \int_{S_1} \sigma_1(t)dt\) of this conductor is equal to
\[ Q_1 = -Cu_0(0)(1 + O(ka)), \tag{31} \]
where \(C\) is the electric capacitance of the perfect conductor with the shape \(D_1\).

Analytic formulas for electric capacitance \(C\) of a perfect conductor of an arbitrary shape, which allow to calculate \(C\) with a desired accuracy, are derived in [9]. For example, the zeroth approximation formula is
\[ C^{(0)} = \frac{4\pi|S_1|^2}{\int_{S_1} \int_{S_1} \frac{dsdt}{r_{st}}}, \quad r_{st} = |t - s|, \tag{32} \]
and we assume in (32) that \(\epsilon_0 = 1\), where \(\epsilon_0\) is the dielectric constant of the homogeneous medium in which the perfect conductor is placed. Formula (31) is formula (19). If \(u_0(x) = e^{ik\alpha \cdot x}\), then \(u_0(0) = 1\), and \(Q_1 = -C(1 + O(ka))\). In this case
\[ A_1(\beta, \alpha) = \frac{Q_1}{4\pi} = -\frac{C}{4\pi}[1 + O(ka)], \]
which is formula (21).

Consider now wave scattering by an impedance particle.

Let us derive formula (20). Integrate equation (13) over \(S_1\), use the divergence formula
\[ \int_{S_1} u_0N ds = \int_{D_1} \nabla^2 u_0 dx = -k^2 \int_{D_1} u_0 dx = k^2 |D_1| u_0(0)[1 + o(1)], \tag{33} \]
where \(|D_1| = O(a^3)\), and the formula

\[-\zeta_1 \int_{S_1} u_0 ds = -\zeta_1 |S_1| u_0(0)[1 + o(1)]. \quad (34)\]

Furthermore \(|\int_{S_1} g(s, t) ds| = O(a)\), so

\[-\zeta_1 \int_{S_1} ds \int_{S_1} g(s, t) \sigma_1(t) dt = O(aQ_1). \quad (35)\]

Therefore, the term (35) is negligible compared with \(Q_1\) as \(a \to 0\). Finally, if \(ka \ll 1\), then \(g(s, t) = g_0(s, t)(1 + ik|s - t| + \ldots)\), and

\[\frac{\partial}{\partial N_s} g(s, t) = \frac{\partial}{\partial N_s} g_0(s, t)[1 + O(ka)]. \quad (36)\]

Denote by \(A_0\) the operator

\[A_0 \sigma = 2 \int_{S_1} \frac{\partial g_0(s, t)}{\partial N_s} \sigma_1(t) dt. \quad (37)\]

It is known from the potential theory that

\[\int_{S_1} A_0 \sigma_1 ds = -\int_{S_1} \sigma_1(t) dt, \quad 2 \int_{S_1} \frac{\partial g_0(s, t)}{\partial N_s} ds = -1, \quad t \in S_1. \quad (38)\]

Therefore,

\[\int_{S_1} ds \frac{A \sigma_1 - \sigma_1}{2} = -Q_1[1 + O(ka)]. \quad (39)\]

Consequently, from formulas (33)-(39) one gets formula (22).

One can see that the wave scattering by an impedance particle is isotropic, and the scattered field is of the order \(O(\zeta_1 |S_1|)\). Since \(|S_1| = O(a^2)\), one would have \(O(\zeta_1 |S_1|) = O(a^{2-\kappa})\) if \(\zeta_1 = O\left(\frac{1}{a^\kappa}\right), \kappa \in (0, 1)\).

Consider now wave scattering by an acoustically hard small particle, i.e., the problem with the Neumann boundary condition.

In this case we will prove that:

i) The scattering is anisotropic,

ii) It is defined not by a single number, as in the previous two cases, but by a tensor,

and

iii) The order of the scattered field is \(O(a^3)\) as \(a \to 0\), for a fixed \(k > 0\), i.e., the scattered field is much smaller than in the previous two cases.
When one integrates over $S_1$ equation (13), one gets
\[
Q_1 = \int_{D_1} \nabla^2 u_0 dx = \nabla^2 u_0(0)|D_1|[1 + o(1)], \quad a \to 0. \tag{40}
\]
Thus, $Q_1 = O(a^3)$. Therefore, the contribution of the term $e^{-ikx^\alpha t}$ in formula (28) with $x_\alpha = 0$ will be also of the order $O(a^3)$ and should be taken into account, in contrast to the previous two cases. Namely,
\[
u(x) = u_0(x) + g(x, 0) \int_{S_1} e^{-ik\beta^t} \sigma_1(t) dt, \quad \beta := \frac{x}{|x|} = x^\alpha. \tag{41}
\]
One has
\[
\int_{S_1} e^{-ik\beta^t} \sigma_1(t) dt = Q_1 - ik\beta_p \int_{S_1} t_p \sigma_1(t) dt, \tag{42}
\]
where the terms of higher order of smallness are neglected and summation over index $p$ is understood. The function $\sigma_1$ solves equation (14):
\[
\sigma_1 = A\sigma_1 + 2u_0N = A\sigma_1 + 2ik\alpha q Nq u_0(s), \quad s \in S_1 \tag{43}
\]
if $u_0(x) = e^{ik\alpha x}$.

Comparing (43) with (25), using (24), and taking into account that $ka \ll 1$, one gets
\[
-ik\beta_p \int_{S_1} t_p \sigma_1(t) dt = -ik\beta_p |D_1|\beta_{pq}(-i\alpha q u_0(0))[1 + O(ka)]
\]
\[
= -k^2 |D_1|\beta_{pq}\beta_p\alpha q u_0(0)[1 + O(ka)]. \tag{44}
\]
From (40), (42) and (44) one gets formula (23), because $\nabla^2 u_0 = -k^2 u_0$.

If $u_0(x)$ is an arbitrary function, satisfying equation (1), then $ik\alpha q$ in (13) is replaced by $\frac{\partial}{\partial x_q}$, and $-k^2 u_0 = \triangle u_0$, which yields formula (26).

This completes the derivation of the formulas for the solution of scalar wave scattering problem by one small body on the boundary of which the Dirichlet, or the impedance, or the Neumann boundary condition is imposed.

3 Many-body scattering problem

In this Section we assume that there are $M = M(a)$ small bodies (particles) $D_m$, $1 \leq m \leq M$, $a = 0.5 \max \text{diam} D_m$, $ka \ll 1$. The distance $d = d(a)$ between neighboring bodies is much larger than $a$, $d \gg a$, but we do not assume that $d \gg \lambda$, so there may be many small particles on the distances
of the order of the wavelength $\lambda$. This means that our medium with the embedded particles is not necessarily diluted.

We assume that the small bodies are embedded in an arbitrary large but finite domain $D$, $D \subset \mathbb{R}^3$, so $D_m \subset D$. Denote $D' := \mathbb{R}^3 \setminus D$ and $\Omega := \bigcup_{m=1}^{M} D_m$, $S_m := \partial D_m$, $\partial \Omega = \bigcup_{m=1}^{M} S_m$. By $N$ we denote a unit normal to $\partial \Omega$, pointing out of $\Omega$, by $|D_m|$ the volume of the body $D_m$ is denoted.

The scattering problem consists of finding the solution to the following problem

\begin{align}
(\nabla^2 + k^2)u &= 0 \text{ in } \mathbb{R}^3 \setminus \Omega, \quad (45) \\
\Gamma u &= 0 \text{ on } \partial \Omega, \quad (46) \\
u &= u_0 + v, \quad (47)
\end{align}

where $u_0$ is the incident field, satisfying equation (45) in $\mathbb{R}^3$, for example, $u_0 = e^{ik\alpha \cdot x}$, $\alpha \in S^2$, and $v$ is the scattered field, satisfying the radiation condition (5). The boundary condition (46) can be of the types (6)-(8).

In the case of impedance boundary condition (7) we assume that

$$u_N = \zeta_m u \text{ on } S_m, \quad 1 \leq m \leq M, \quad (48)$$

so the impedance may vary from one particle to another. We assume that

$$\zeta_m = \frac{h(x_m)}{a^\kappa}, \quad \kappa \in (0, 1), \quad (49)$$

where $x_m \in D_m$ is a point in $D_m$, and $h(x)$, $x \in D$, is a given function, which we can choose as we wish, subject to the condition $\text{Im} h(x) \leq 0$. For simplicity we assume that $h(x)$ is a continuous function.

Let us make the following assumption about the distribution of small particles: if $\Delta \subset D$ is an arbitrary open subset of $D$, then the number $N(\Delta)$ of small particles in $\Delta$, assuming the impedance boundary condition, is:

$$N_\zeta(\Delta) = \frac{1}{a^{2-\kappa}} \int_\Delta N(x)dx[1 + o(1)], \quad a \to 0, \quad (50)$$

where $N(x) \geq 0$ is a given function. If the Dirichlet boundary condition is assumed, then

$$N_D(\Delta) = \frac{1}{a} \int_\Delta N(x)dx[1 + o(1)], \quad a \to 0. \quad (51)$$

The case of the Neumann boundary condition will be considered later.
We look for the solution to problem (45)-(47) with the Dirichlet boundary condition of the form

$$u = u_0 + \sum_{m=1}^{M} \int_{S_m} g(x,t)\sigma_m(t)dt,$$  \hspace{1cm} (52)

where $\sigma_m(t)$ are some functions to be determined from the boundary condition (46). It is proved in [11] that problem (45)-(47) has a unique solution of the form (52). For any $\sigma_m(t)$ function (52) solves equation (45) and satisfies condition (47). The boundary condition (46) determines $\sigma_m$ uniquely. However, if $M \gg 1$, then numerical solution of the system of integral equations for $\sigma_m$, $1 \leq m \leq M$, which one gets from the boundary condition (46), is practically not feasible.

To avoid this principal difficulty we prove that the solution to scattering problem (45)-(47) is determined by $M$ numbers

$$Q_m := \int_{S_m} \sigma_m(t)dt,$$  \hspace{1cm} (53)

rather than $M$ functions $\sigma_m(t)$.

This is possible to prove if the particles $D_m$ are small. We derive analytical formulas for $Q_m$ as $a \to 0$.

Let us define the effective (self-consistent) field $u_e(x) = u_e^{(j)}(x)$, acting on the $j$-th particle, by the formula

$$u_e(x) := u(x) - \int_{S_j} g(x,t)\sigma_j(t)dt, \hspace{1cm} |x - x_j| \sim a.$$  \hspace{1cm} (54)

Physically this field acts on the $j$-th particle and is a sum of the incident field and the fields acting from all other particles:

$$u_e(x) = u_e^{(j)}(x) := u_0(x) + \sum_{m \neq j} \int_{S_m} g(x,t)\sigma_m(t)dt.$$  \hspace{1cm} (55)

Let us rewrite (55) as follows:

$$u_e(x) = u_0(x) + \sum_{m \neq j}^{M} g(x,x_m)Q_m + \sum_{m \neq j}^{M} \int_{S_m} [g(x,t) - g(x,x_m)]\sigma_m(t)dt.$$  \hspace{1cm} (56)

We want to prove that the last sum is negligible compared with the first one as $a \to 0$. To prove this, let us give some estimates. One has $|t - x_m| \leq a,$
\[ d = |x - x_m|, \]
\[ |g(x, t) - g(x, x_m)| = \max \left\{ O\left(\frac{a}{d^2}\right), O\left(\frac{ka}{d}\right) \right\}, \quad |g(x, x_m)| = O(1/d). \]  

(57)

Therefore, if \(|x - x_j| = O(a)|\), then
\[
\left| \int_{S_m} [g(x, t) - g(x, x_m)] \sigma_m(t) dt \right| \leq O(ad^{-1} + ka). \]  

(58)

One can also prove that
\[
J_1/J_2 = O(ka + ad^{-1}), \]  

(59)

where \(J_1\) is the first sum in (56) and \(J_2\) is the second sum in (55). Therefore, at any point \(x \in \Omega' = \mathbb{R}^3 \setminus \Omega\) one has

\[ u_e(x) = u_0(x) + \sum_{m=1}^{M} g(x, x_m)Q_m, \quad x \in \Omega'. \]  

(60)

where the terms of higher order of smallness are omitted.

### 3.1 The case of acoustically soft particles

If (46) is the Dirichlet condition, then, as we have proved in Section 2 (see formula (31)), one has

\[ Q_m = -C_m u_e(x_m). \]  

(61)

Thus,
\[ u_e(x) = u_0(x) - \sum_{m=1}^{M} g(x, x_m)C_m u_e(x_m), \quad x \in \Omega'. \]  

(62)

One has
\[ u(x) = u_e(x) + o(1), \quad a \to 0, \]  

(63)

so the full field and effective field are practically the same.

Let us write a linear algebraic system (LAS) for finding unknown quantities \(u_e(x_m)\):

\[ u_e(x_j) = u_0(x_j) - \sum_{m \neq j}^{M} g(x_j, x_m)C_m u_e(x_m). \]  

(64)
If $M$ is not very large, say $M = O(10^3)$, then LAS (64) can be solved numerically, and formula (62) can be used for calculation of $u e(x)$.

Consider the limiting case, when $a \to 0$. One can rewrite (64) as follows:

$$u e(\xi_q) = u_0(\xi_q) - \sum_{p \neq q} g(\xi_q, \xi_p) u e(\xi_p) \sum_{x_m \in \Delta_p} C_m, \quad (65)$$

where $\{\Delta_p\}_{p=1}^P$ is a union of cubes which forms a covering of $D$,

$$\max_p diam \Delta_p := b = b(a) \gg a,$$

where $|\Delta_p|$ we denote the volume (measure) of $\Delta_p$, and $\xi_p$ is the center of $\Delta_p$, or a point $x_p$ in an arbitrary small body $D_p$, located in $\Delta_p$. Let us assume that there exists the limit

$$\lim_{a \to 0} \sum_{x_m \in \Delta_p} C_m \mid \Delta_p \mid = C(\xi_p), \quad \xi_p \in \Delta_p. \quad (67)$$

For example, one may have

$$C_m = c(\xi_p)a \quad (68)$$

for all $m$ such that $x_m \in \Delta_p$, where $c(x)$ is some function in $D$. If all $D_m$ are balls of radius $\alpha$, then $c(x) = 4\pi$. We have

$$\sum_{x_m \in \Delta_p} C_m = C_p a N(\Delta_p) = C_p N(\xi_p)|\Delta_p|[1 + o(1)], \quad a \to 0, \quad (69)$$

so limit (67) exists, and

$$C(\xi_p) = c(\xi_p)N(\xi_p). \quad (70)$$

From (65), (68)-(70) one gets

$$u e(\xi_q) = u_0(\xi_q) - \sum_{p \neq q} g(\xi_q, \xi_p) c(\xi_p) N(\xi_p) u e(\xi_p)|\Delta_p|, \quad 1 \leq p \leq P. \quad (71)$$

Linear algebraic system (71) can be considered as the collocation method for solving integral equation

$$u(x) = u_0(x) - \int_D g(x, y)c(y) N(y)u(y)dy. \quad (72)$$
It is proved in [26] that system (71) is uniquely solvable for all sufficiently small $b(a)$, and the function

$$u_P(x) := \sum_{p=1}^{P} \chi_p(x)u_e(\xi_p)$$  \hspace{1cm} (73)

converges in $L^\infty(D)$ to the unique solution of equation (72). The function $\chi_p(x)$ in (73) is the characteristic function of the cube $\Delta_p$: it is equal to 1 in $\Delta_p$ and vanishes outside $\Delta_p$. Thus, if $a \to 0$, the solution to the many-body wave scattering problem in the case of the Dirichlet boundary condition is well approximated by the unique solution of the integral equation (72).

Applying the operator $L_0 := \nabla^2 + k^2$ to (72), and using the formula $L_0 g(x,y) = -\delta(x-y)$, where $\delta(x)$ is the delta-function, one gets

$$\nabla^2 u + k^2 u - q(x)u = 0 \text{ in } \mathbb{R}^3, \quad q(x) := c(x)N(x).$$  \hspace{1cm} (74)

The physical conclusion is:

If one embeds $M(a) = O(1/a)$ small acoustically soft particles, which are distributed as in (51), then one creates, as $a \to 0$, a limiting medium, which is inhomogeneous, and has a refraction coefficient $n^2(x) = 1 - k^{-2}q(x)$.

It is interesting from the physical point of view to note that the limit, as $a \to 0$, of the total volume of the embedded particles is zero.

Indeed, the volume of one particle is $O(a^3)$, the total number $M$ of the embedded particles is $O(a^3M) = O(a^2)$, and $\lim_{a \to 0} O(a^2) = 0$.

The second observation is: if (51) holds, then on a unit length straight line there are $O(\frac{1}{a^{1/3}})$ particles, so the distance between neighboring particles is $d = O(a^{1/3})$. If $d = O(a^\gamma)$ with $\gamma > \frac{1}{3}$, then the number of the embedded particles in a subdomain $\Delta_p$ is $O\left(\frac{1}{d}\right) = O(a^{-3\gamma})$. In this case, for $3\gamma > 1$, the limit in (69) is $C(\xi_p) = \lim_{a \to 0} c_p aO(a^{-3\gamma}) = \infty$. Therefore, the product of this limit by $u$ remains finite only if $u = 0$ in $D$. Physically this means that if the distances between neighboring perfectly soft particles are smaller than $O(a^{1/3})$, namely, they are $O(a^\gamma)$ with any $\gamma > \frac{1}{3}$, then $u = 0$ in $D$.

On the other hand, if $\gamma < \frac{1}{3}$, then the limit $C(\xi_p) = 0$, and $u = u_0$ in $D$, so that the embedded particles do not change, in the limit $a \to 0$, properties of the medium.

This concludes our discussion of the scattering problem for many acoustically soft particles.
3.2 Wave scattering by many impedance particles

We assume now that (49) and (50) hold, use the exact boundary condition (46) with $\Gamma = \Gamma_2$, that is,
\begin{equation}
 u_e N - \zeta m u_e + \frac{A_m \sigma_m - \sigma_m}{2} - \zeta m \int_{S_m} g(s,t) \sigma_m(t) dt = 0,
\end{equation}
and integrate (75) over $S_m$ in order to derive an analytical asymptotic formula for $Q_m = \int_{S_m} \sigma_m(t) dt$.

We have
\begin{equation}
\int_{S_m} u_e N ds = \int_{D_m} \nabla^2 u_e dx = O(a^3),
\end{equation}
\begin{equation}
\int_{S_m} \zeta m u_e(s) ds = h(x_m) a^{-\kappa} |S_m| u_e(x_m) [1 + o(1)], \quad a \to 0,
\end{equation}
\begin{equation}
\int_{S_m} \frac{A_m \sigma_m - \sigma_m}{2} ds = -Q_m [1 + o(1)], \quad a \to 0,
\end{equation}
and
\begin{equation}
\zeta m \int_{S_m} \int_{S_m} g(s,t) \sigma_m(t) dt = h(x_m) a^{1-\kappa} Q_m = o(Q_m), \quad 0 < \kappa < 1.
\end{equation}

From (75)-(79) one finds
\begin{equation}
Q_m = -h(x_m) a^{2-\kappa} |S_m| a^{-2} u_e(x_m) [1 + o(1)].
\end{equation}

This yields the formula for the approximate solution to the wave scattering problem for many impedance particles:
\begin{equation}
 u(x) = u_0(x) - a^{2-\kappa} \sum_{m=1}^{M} g(x,x_m) b_m h(x_m) u_e(x_m) [1 + o(1)],
\end{equation}
where
\begin{equation}
b_m := |S_m| a^{-2}
\end{equation}
are some positive numbers which depend on the geometry of $S_m$ and are independent of $a$. For example, if all $D_m$ are balls of radius $a$, then $b_m = 4\pi$.

A linear algebraic system for $u_e(x_m)$, analogous to (64), is
\begin{equation}
u_e(x_j) = u_0(x_j) - a^{2-\kappa} \sum_{m=1, m \neq j}^{M} g(x_j, x_m) b_m h(x_m) u_e(x_m).
\end{equation}
The integral equation for the limiting effective field in the medium with embedded small particles, as $a \to 0$, is

$$u(x) = u_0(x) - b \int_D g(x, y) N(y) h(y) u(y) dy,$$  \hspace{1cm} (83)

where

$$u(x) = \lim_{a \to 0} u_e(x),$$  \hspace{1cm} (84)

and we have assumed in (83) for simplicity that $b_m = b$ for all $m$, that is, all small particles are of the same shape and size.

Applying operator $L_0 = \nabla^2 + k^2$ to equation (83), one finds the differential equation for the limiting effective field $u(x)$:

$$(\nabla^2 + k^2 - bN(x)h(x))u = 0 \text{ in } \mathbb{R}^3,$$  \hspace{1cm} (85)

and $u$ satisfies condition (47).

The conclusion is: the limiting medium is inhomogeneous, and its properties are described by the function

$$q(x) := bN(x)h(x).$$  \hspace{1cm} (86)

Since the choice of the functions $N(x) \geq 0$ and $h(x)$, Im$h(x) \leq 0$, is at our disposal, we can create the medium with desired properties by embedding many small impedance particles, with suitable impedances, according to the distribution law (50) with a suitable $N(x)$. The function

$$1 - k^{-2}q(x) = n^2(x)$$  \hspace{1cm} (87)

is the refraction coefficient of the limiting medium. Given a desired refraction coefficient $n^2(x)$, Im$n^2(x) \geq 0$, one can find $N(x)$ and $h(x)$ so that (87) holds, that is, one can create a material with a desired refraction coefficient by embedding into a given material many small particles with suitable boundary impedances.

This concludes our discussion of the wave scattering problem with many small impedance particles.

### 3.3 Wave scattering by many acoustically hard particles

Consider now the case of acoustically hard particles, i.e., the case of Neumann boundary condition. The exact boundary integral equation for the function $\sigma_m$ in this case is:

$$u_eN + \frac{A_m \sigma_m - \sigma_m}{2} = 0.$$  \hspace{1cm} (88)
Arguing as in Section 2, see formulas (40)-(44), one obtains

\[ u_\epsilon(x) = u_0(x) + \sum_{m=1}^{M} g(x, x_m) \left[ \Delta u_e(x_m) + ik \beta_{pq}^{(m)} \frac{(x_p - (x_m)_p)}{|x - x_m|} \frac{\partial u_e(x_m)}{\partial(x)_q} \right] |D_m|, \]

(89)

where we took into account that the unit vector \( \beta \) in (44) is now the vector \( \frac{(x)_p - (x_m)_p}{|x - x_m|} \), where \( (x)_p := x \cdot e_p \) is the \( p \)-th component of vector \( x \) in the Euclidean orthonormal basis \( \{ e_p \}_{p=1}^3 \).

There are three sets of unknowns in (89): \( u_e(x_m), \frac{\partial u_e(x_m)}{\partial(x)_q}, \) and \( \Delta u_e(x_m) \), \( 1 \leq m \leq M \), \( 1 \leq q \leq 3 \). To obtain linear algebraic system for \( u_e(x_m) \) and \( \frac{\partial u_e(x_m)}{\partial(x)_q} \) one sets \( x = x_j \) in (89), takes the sum in (89) with \( m \neq j \). This yields the first set of equations for finding these unknowns. Then one takes derivative of equation (89) with respect to \( (x)_q \), sets \( x = x_j \), and takes the sum in (89) with \( m \neq j \). This yields the second set of equations for finding these unknowns. Finally, one takes Laplacian of equation (89), sets \( x = x_j \), and takes the sum in (89) with \( m \neq j \). This yields the third set of linear algebraic equations for finding \( u_e(x_m), \frac{\partial u_e(x_m)}{\partial(x)_q}, \) and \( \Delta u_e(x_m) \).

Passing to the limit \( a \to 0 \) in equation (89), yields the equation for the limiting field

\[ u(x) = u_0(x) + \int_D g(x, y) \left( \rho(y) \nabla^2 u(y) + ik \frac{\partial(u(y))}{\partial y_q} \frac{x_p - y_p}{|x - y|} B_{pq}(y) \right) dy, \]

(90)

where \( \rho(y) \) and \( B_{pq}(y) \) are defined below, see formulas (92) and (93).

Let us derive equation (90). We start by transforming the sum in (89). Let \( \{ \Delta_l \}_{l=1}^L \) be a covering of \( D \) by cubes \( \Delta_l \), \( \max_l \text{diam}\Delta_l = b = b(a) \). We assume that

\[ b(a) \gg d \gg a, \quad \lim_{a \to 0} b(a) = 0. \]

Thus, there are many small particles \( D_m \) in \( \Delta_l \). Let \( x_l \) be a point in \( \Delta_l \). One has

\[ \sum_{m=1}^{M} g(x, x_m) \left[ \Delta u_e(x_m) + \frac{\partial u_e(x_m)}{\partial(x)_q} \beta_{pq}^{(m)} \frac{(x_p - (x_m)_p)}{|x - x_m|} \right] |D_m| \]

\[ = \sum_{l=1}^{L} g(x, x_l) \sum_{x_m \in \Delta_l} |D_m| + ik \frac{\partial u_e(x_l)}{\partial(x)_q} \frac{(x_p - (x_l)_p)}{|x - x_l|} \sum_{x_m \in \Delta_l} \beta_{pq}^{(m)} |D_m|. \]

(91)

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Assume that the following limits exist:

\[
\lim_{a \to 0, y \in \Delta} \frac{\sum_{x_m \in \Delta_l} |D_m|}{|\Delta_l|} = \rho(y),
\]

(92)

and

\[
\lim_{a \to 0, y \in \Delta} \frac{\sum_{x_m \in \Delta_l} \beta_{pq}^{(m)} |D_m|}{|\Delta_l|} = B_{pq}(y),
\]

(93)

and

\[
\lim_{a \to 0} u_e(y) = u(y), \quad \lim_{a \to 0} \frac{\partial u_e(y)}{\partial(y)_q} = \frac{\partial u(y)}{\partial,y_q}, \quad \lim_{a \to 0} \nabla^2 u_e(y) = \nabla^2 u(y).
\]

(94)

Then, the sum in (91) converges to

\[
\int_D g(x, y) \left( \rho(y) \nabla^2 u(y) + ik \frac{\partial u(y)}{\partial y_q} \frac{x_p - y_p}{|x - y|} B_{pq}(y) \right) dy.
\]

(95)

Consequently, (89) yields in the limit \( a \to 0 \) equation (90). Equation (90) cannot be reduced to a differential equation for \( u(x) \), because (90) is an integrodifferential equation whose integrand depends on \( x \) and \( y \).

Let us summarize the results in the following theorem.

**Theorem 3.1.** The many-body scattering problem (45) - (47) has a unique solution for the Dirichlet, impedance, and Neumann boundary conditions. The limiting effective fields in the medium obtained by embedding many small particles of an arbitrary shape satisfy the equations (74), (85), and (90), for the Dirichlet, impedance, and Neumann boundary conditions, respectively.

4 Scattering by small particles embedded in an inhomogeneous medium

Suppose that the operator \( \nabla^2 + k^2 \) in (11) and in (45) is replaced by the operator \( L_0 = \nabla^2 + k^2 n_0^2(x) \), where \( n_0^2(x) \) is a known function,

\[
\text{Im} n_0^2(x) \geq 0.
\]

(96)

The function \( n_0^2(x) \) is the refraction coefficient of an inhomogeneous medium in which many small particles are embedded. The results, presented in Section 1-3 remain valid if one replaces function \( g(x, y) \) by the Green’s function \( G(x, y) \),

\[
|\nabla^2 + k^2 n_0^2(x)| G(x, y) = -\delta(x - y),
\]

(97)
satisfying the radiation condition. We assume that

\[ n_0^2(x) = 1 \text{ in } D' := \mathbb{R}^3 \setminus D. \] (98)

The function \( G(x, y) \) is uniquely defined (see, e.g., [11]). The derivations of the results remain essentially the same because

\[ G(x, y) = g_0(x, y)[1 + O(|x - y|)], \quad |x - y| \to 0, \] (99)

where \( g_0(x, y) = \frac{1}{4\pi|x - y|} \). Estimates of \( G(x, y) \) as \( |x - y| \to 0 \) and as \( |x - y| \to \infty \) are obtained in [11]. Smallness of particles in an inhomogeneous medium with refraction coefficient \( n_0^2(x) \) is described by the relation \( kn_0a \ll 1 \), where \( n_0 := \max_{x \in D}|n_0(x)| \), and \( a = \max_{1 \leq m \leq M} \text{diam}D_m \).

5 Wave scattering by small bodies with transmission (interface) boundary conditions

There is a large literature on "homogenization", which deals with the properties of the medium in which other materials is distributed. Quite often it is assumed that the medium is periodic, and homogenization is considered in the framework of G-convergence ([4],[6]). In most cases, one considers elliptic or parabolic problems with elliptic operators positive-definite and having discrete spectrum.

A theory of wave scattering by many small particles embedded in an inhomogeneous medium has been developed by the author ([11]-[27]). One of the practically important consequences of his theory was a derivation of the equation for the effective (self-consistent) field in the limiting medium, obtained in the limit \( a \to 0 \), \( M = M(a) \to \infty \), where \( a \) is the characteristic size of a small particle, and \( M(a) \) is the total number of the embedded particles.

The theory was developed in Sections 1-4 (see also papers [11]-[27]) for boundary conditions (bc) on the surfaces of small bodies, which include the Dirichlet bc, \( u|_{S_m} = 0 \), where \( S_m \) is the surface of the \( m \)-th particle \( D_m \), the impedance bc, \( \zeta_m u|_{S_m} = u_N|_{S_m} \), where \( N \) is the unit normal to \( S_m \), pointing out of \( D_m \); \( \zeta_m \) is the boundary impedance, and the Neumann bc, \( u_N|_{S_m} = 0 \).

In the rest of this paper the development is presented of a similar theory for the transmission (interface) bc:

\[ \rho_m u_N^+ = u_N^-, \quad u^+ = u^- \quad \text{on } S_m, \ 1 \leq m \leq M. \] (100)
Here $\rho_m$ is a constant, $+(-)$ denotes the limit of $\frac{\partial u}{\partial N}$, from inside (outside) of $D_m$.

The physical meaning of the transmission boundary conditions is the continuity of the pressure and the normal component of the velocity across the boundaries of the discontinuity of the density. One may think about problem (100)-(104) (see below) as of the problem of acoustic wave scattering by many small bodies.

The essential novelty of the theory, developed in this paper, is the asymptotically exact, as $a \to 0$, treatment of the one-body and many-body scalar wave scattering problem in the case of small scatterers on the boundaries of which the transmission boundary conditions are imposed. An analytic explicit asymptotic formula for the field scattered by one small body is derived. An integral equation for the limiting effective field in the medium, in which many small bodies are embedded, is derived in the limit $a \to 0$ and $M(a) \to \infty$, where $M(a)$ is the total number of the embedded small bodies (particles), and $M = M(a)$ tends to infinity at a suitable rate as $a \to 0$.

For the problem with the number $M$ of particles not large, say, less than 5000, our theory gives an efficient numerical method for solving many-body wave scattering problem.

For the problem with $M$ very large, say, larger than $10^5$, the solution to many-body wave scattering problem consists in numerical solution of the integral equation for the limiting field in the medium, in which small particles are embedded. The solution to this equation approximates the solution to the many-body wave scattering problem with high accuracy.

Our approach is quite different from the approach developed in homogenization theory, we do not assume periodicity in the location of the small scatterers. Our results are of interest also in the case when the number of scatterers is not large, so the homogenization theory is not applicable.

Let us formulate the scattering problem we are treating. Below condition (100) is assumed. Let

Let $\Omega := \bigcup_{m=1}^{M} D_m$, $\Omega' = \mathbb{R}^3 \setminus \Omega$,

\begin{align*}
(\nabla^2 + k^2)u &= 0 \quad \text{in } \Omega', \\
(\nabla^2 + k_m^2)u &= 0 \quad \text{in } D_m, \quad 1 \leq m \leq M, \\
u &= u_0 + v, \quad u_0 = e^{i k \alpha \cdot x}, \quad \alpha \in S^2, S^2 \text{ is a unit sphere in } \mathbb{R}^3, \\
r \left( \frac{\partial v}{\partial r} - ik v \right) &= o(1), \quad r \to \infty.
\end{align*}
We assume that $\rho_m, k$ and $k_m^2$ are fixed given positive constants, and the surfaces $S_m$ are smooth. A sufficient smoothness condition is $S_m \in C^{1, \mu}, \mu \in (0, 1)$, where $S_m$ in local coordinates is given by a continuously differentiable function whose first derivatives are Hölder-continuous with exponent $\mu$.

We assume that $x_m \in D_m$ is a point inside $D_m$, $a = \frac{1}{2}\text{diam}D_m$, $d = O(a^{\frac{1}{3}})$ is the distance between the neighboring particles, $N(\Delta) = \sum_{x_m \in \Delta} 1$, is the number of particles in an arbitrary open set $\Delta$, the domains $D_m$ are not intersecting, and

$$N(\Delta) = \frac{1}{V} \int_{\Delta} N(x)dx[1 + o(1)], \quad a \to 0,$$

(105)

where $N(x) \geq 0$ is a function which is at our disposal, $V$ is the volume of one small body, $V = O(a^3)$. If $D_m$ are balls of radius $a$, then $V = \frac{4\pi a^3}{3}$.

It is proved in [8] that problem (100)-(104) has a unique solution.

We study wave scattering by a single small body in Section 6. In other words, we study in Section 6 problem (100)-(104) with $M = 1$. The basic results of this Section are formulated in Theorem 6.1.

In section 7 wave scattering by many small bodies is considered. The basic results of this Section are formulated in Theorem 7.1. We always assume that

$$ka << 1, \quad d = O(a^{\frac{1}{3}}).$$

(106)

## 6 Wave scattering by one small body

Let us look for the solution to problem (100)-(104) with $M = 1$ of the form

$$u(x) = u_0(x) + \int_S g(x, t)\sigma(t)dt + \kappa \int_D g(x, y)u(y)dy,$$

(107)

where $S = S_1, D = D_1,$

$$\kappa := k_1^2 - k^2, \quad g(x, y) := \frac{e^{ik|x-y|}}{4\pi |x-y|},$$

(108)

and $\sigma(t)$ is to be found so that conditions (100) are satisfied. For any $\sigma \in C^{0, \mu_1}, \mu_1 \in (0, 1)$, where $C^{0, \mu_1}$ is the set of Hölder-continuous functions with Hölder’s exponent $\mu_1$, the solution to equation (107) satisfies equations (101) and (102) with $M = 1$, and equations (103) and (104). This is easily checked by a direct calculation. The second condition (100) is also satisfied.
To satisfy the first condition in equations (100) with $\rho_1 = \rho$, one has to satisfy the following equation

\[(\rho - 1)u_{0N} + \rho \frac{A\sigma + \sigma}{2} - \frac{A\sigma - \sigma}{2} + (\rho - 1)\frac{\partial}{\partial N_s}Bu = 0, \tag{109}\]

where

\[A\sigma = 2 \int_S \frac{\partial g(s,t)}{\partial N_s} \sigma(t)dt, \quad B\sigma = \kappa \int_D g(x,y)u(y)dy, \tag{110}\]

and the well-known formulas for the limiting values of the normal derivatives of the single-layer potential $T\sigma := \int_S g(x,t)\sigma(t)dt$ on $S$ from inside and outside $D$ was used.

In [8] one finds a proof of the following existence and uniqueness result. Let $H^2(D)$ denote the usual Sobolev space of functions twice differentiable in $L^2$-sense.

Proposition 1. The system of equations (107) and (109) for the unknown functions $\sigma$ on $S$ and $u(x)$ in $D$ has a solution and this solution is unique in $C^{0,\mu_1} \times H^2(D)$.

If the solution $\{\sigma, u(x) | x \in D\}$ is found, then formula (107) defines $u = u(x)$ in $\mathbb{R}^3$.

Let us rewrite (109) as

\[\sigma = \lambda A\sigma + 2\lambda B_1u + 2\lambda u_{0N}, \tag{111}\]

where

\[\lambda = \frac{1 - \rho}{1 + \rho}, \quad B_1u = \kappa \frac{\partial}{\partial N_s} \int_D g(x,y)u(y)dy. \tag{112}\]

If $\rho \in [0, \infty)$ then $\lambda \in (-1, 1)$. Let us now use the first assumption (106), that is, the smallness of $a$. One has:

\[g(s,t) = g_0(s,t)(1 + O(ka)), \quad a \to 0; \quad g_0(s,t) = \frac{1}{4\pi|s - t|}, \tag{113}\]

\[\frac{\partial}{\partial N_s} \frac{e^{ik|s-t|}}{4\pi|s - t|} = \frac{\partial g_0}{\partial N_s}(1 + O((ka)^2)), \quad a \to 0, \tag{114}\]

so $A = A_0(1 + O((ka)^2))$, $a \to 0; A_0 := A|_{k=0}$, (115)

\[B = B_0(1 + O(ka)), \quad B_0u = \kappa \int_D g_0(x,y)u(y)dy, \tag{116}\]

\[B_1u = \kappa \int_D \frac{\partial g_0(s,y)}{\partial N} u(y)dy(1 + O(k^2a^2)) := \kappa B_{10}u(1 + O(k^2a^2)). \tag{117}\]
It follows from equation (107) that
\[ u(x) = u_0(x) + \frac{e^{ik|x-x_1|}}{|x-x_1|} \left( \frac{1}{4\pi} \int_S e^{-ik\beta \cdot s} \sigma(t) dt + \frac{\kappa}{4\pi} u_1 V_1 \right), \quad |x-x_1| > a, \]
(118)
where \( V_1 \) is the volume of \( D = D_1 \), \( V_1 = vol(D_1) := |D_1|, u_1 := u(x_1) \), \( \beta := \frac{x-x_1}{|x-x_1|} \). The point \( x_1 \in D \) can be chosen as we wish. For one scatterer it is convenient to choose the origin at the point \( x_1 = 0 \).

We did not keep the factor \( e^{-ik\beta \cdot x} \) in the integral over \( D \) because \( e^{-ik\beta \cdot x} = 1 + O(ka) \), and
\[ \int_D e^{-ik\beta \cdot y} u(y) dy = u_1 V_1 (1 + O(ka)), \quad a \to 0. \]
(119)
However, it will be proved that this factor under the surface integral cannot be dropped because
\[ \int_S e^{-ik\beta \cdot t} \sigma(t) dt = \int_S \sigma(t) dt - ik\beta_p \int_S t_p \sigma(t) dt + O(a^4), \]
(120)
where over the repeated indices here and throughout this paper summation is understood, and the second integral in the right-hand side of (120) is \( O(a^3) \), as \( a \to 0 \), that is, it is of the same order of smallness as the first integral \( Q := \int_S \sigma(t) dt \). The last statement will be proved later.

With the notations
\[ Q := \int_S \sigma(t) dt, \quad Q_1 := \int_S e^{-ik\beta \cdot t} \sigma(t) dt, \]
(121)
the expression
\[ A(\beta, \alpha) := \frac{Q_1}{4\pi} + \frac{\kappa}{4\pi} u_1 V_1, \quad V_1 := V := |D|, \quad u_1 := u(x_1), \]
(122)
is the scattering amplitude, \( \alpha \) is the unit vector in the direction of the incident wave \( u_0 = e^{ik\alpha \cdot x} \), \( \beta \) is the unit vector in the direction of the scattered wave.

Let us prove that
\[ -ik\beta_p \int_S t_p \sigma(t) dt = O(a^3), \]
(123)
and therefore, the second integral in the right-hand side of equation (120) cannot be dropped.

It follows from equation (107) that
\[ u(x) \sim u_0(x) + g(x, x_1) Q_1 + \kappa g(x, x_1) u(x_1) V_1, \quad |x-x_1| \geq d \gg a, \]
(124)
where \( \sim \) means asymptotic equivalence as \( a \to 0 \).

Formula (124) can be used for calculating \( u(x) \) if two quantities \( Q_1 \) and \( u_1 := u(x_1) \) are found. Let us derive asymptotic formulas for these quantities as \( a \to 0 \). Integrate equation (111) over \( S \) and get

\[
Q = 2\lambda \int_S u_0 \, ds + \lambda \int_A \sigma \, dt + 2\lambda \int_S B_1 \, ds,
\]

Use formulas (113)-(117), the following formula (see [9], p.96):

\[
\int_S A_0 \sigma \, ds = -\int_S \sigma \, ds,
\]

and the Divergence theorem, to rewrite equation (125) as

\[
Q = 2\lambda \int_D \nabla^2 u_0 \, dx - \lambda Q + 2\lambda k \int_D \nabla x \int_D g(x, y)u(y) \, dy.
\]

Since

\[
\nabla^2 u_0 = -k^2 u_0; \quad \nabla^2 x g(x, y) = -k^2 g(x, y) - \delta(x - y),
\]

equation (127) takes the form

\[
(1 + \lambda)Q = 2\lambda \nabla^2 u_0(x_1)V_1 - 2\lambda k^2 \int_D dx \int_D g(x, y)u(y) \, dy - 2\lambda \int_D u(x) \, dx.
\]

Let us use the following estimates:

\[
\int_D u(x) \, dx = u_1 V_1 (1 + o(1)), \quad a \to 0; \quad u_1 := u(x_1),
\]

\[
\int_D dx \int_D g(x, y)u(y) \, dy = \int_D dy u(y) \int_D dx g(x, y) = O(a^5),
\]

\[
\int_D g(x, y) \, dx = O(a^2), \quad \forall y \in D.
\]

From equations (129), (130), (131), and (132) it follows that

\[
Q \sim \frac{2\lambda}{1 + \lambda} V_1 \nabla^2 u_0 - \frac{2\lambda k}{1 + \lambda} V_1 u_1, \quad a \to 0,
\]

where

\[
\nabla^2 u_0 = \nabla^2 u_0(x)|_{x = x_1}.
\]
Let us now integrate equation (107) over $D$ and use estimate (130) to obtain

$$u_1 V_1 = u_{01} V_1 + \int_S dt \sigma(t) \int_D g(x, t) dx + \zeta \int_D dy u(y) \int_D g(x, y) dx. \quad (135)$$

If $D$ is a ball of radius $a$, then one can easily check that

$$\int_D g(x, t) dx \sim \int_D g_0(x, t) dx = \frac{a^2}{3}, \quad |t| = a, \quad a \to 0. \quad (136)$$

In general, one has

$$\int_D g(x, y) dx = O(a^2), \quad y \in D, \quad a \to 0. \quad (137)$$

If $D$ is a ball of radius $a$, then equations (135)-(137) imply

$$u_1 = u_{01} + Q a^2 \frac{a^2}{3} + \zeta u_1 O(a^2), \quad a \to 0. \quad (138)$$

Consequently,

$$u_1 \sim u_{01} + O(a^2), \quad a \to 0, \quad (139)$$

because $Q = O(a^3)$.

Indeed, from equations (133) and (139) one gets

$$Q \sim V_1 (1 - \rho)[\nabla^2 u_{01} - \zeta u_{01}], \quad (140)$$

where we took into account that

$$\frac{2 \lambda}{1 + \lambda} = 1 - \rho, \quad (141)$$

the relation $u_1 \sim u_{01}$ as $a \to 0$, see equation (139), and neglected the terms of higher order of smallness. It follows from equation (140) that

$$Q = O(a^3). \quad (142)$$

From equations (139) and (140) one obtains

$$u_1 \sim u_{01}, \quad a \to 0. \quad (143)$$

Let us now estimate $Q_1$. One has

$$Q_1 = \int_S \sigma(t) dt - ik \beta_p \int_S t_p \sigma(t) dt, \quad (144)$$
up to the terms of the higher order of smallness as \( a \to 0 \), and summation is understood over the repeated indices. It turns out that the integral

\[
I := \int_S t_p \sigma(t) dt
\]

is of the same order, namely \( O(a^3) \), as \( Q = \int_S \sigma(t) dt \).

Let us check that the integral

\[
J := \int_S dt t_p \frac{\partial}{\partial N} \int_D g(t, y) u(y) dy = O(a^4)
\]

as \( a \to 0 \), and, therefore, can be neglected compared with \( I \). Indeed, \( u = O(1) \), \( \int_D \frac{\partial}{\partial N} g(t, y) dy = O(a) \), and \( \int_S t_p dt = O(a^3) \). Thus, \( J = O(a^4) \).

Define the function \( \sigma_q, q = 1, 2, 3 \), as the unique solution to the equation

\[
\sigma_q = \lambda A \sigma_q - 2\lambda N_q.
\]

Since \( \lambda = (1 - \rho)/(1 + \rho) \), and \( \rho > 0 \), one concludes that \( \lambda \in (-1, 1) \), and it is known (see, for example, \[9\]) that the operator \( A \) is compact in \( L^2(S) \) and does not have characteristic values in the interval \((-1, 1)\). This and the Fredholm alternative imply that equation (146) has a solution and this solution is unique.

Let us prove that \( \int_S \sigma_q(t) dt = O(a^3) \). To do this, integrate equation (146) over \( S \), take into account formula (126), the relation \( (A - A_0) \sigma_q = O(a^3) \), and obtain

\[
(1 + \lambda) \int_S \sigma_q(t) dt = -2\lambda \int_S N_q dt + O(a^3) = O(a^3),
\]

because \( \int_S N_q dt = 0 \) by the Divergence theorem.

Define the tensor

\[
\beta_{pq} := \beta_{pq}(\lambda) := V_1^{-1} \int_S t_p \sigma_q(t) dt, \quad p, q = 1, 2, 3.
\]

This tensor is similar to the tensor \( \beta_{pq} \) defined in \[9\], p. 62, by a similar formula with \( \lambda = 1 \). In this case \( \beta_{pq} \) is the magnetic polarizability tensor of a superconductor \( D \) placed in a homogeneous magnetic field directed along the unit Cartesian coordinate vector \( e_q \) (see \[9\], p. 62). In \[9\] analytic formulas are given for calculating \( \beta_{pq} \) with an arbitrary accuracy.

One may neglect the term \( B_1 u \) in equation (111) (because this term is \( O(a^4) \)), take into account definition (147), and get

\[
\int_S t_p \sigma(t) dt = -\beta_{pq} \frac{\partial u_0}{\partial x_q} V,
\]

(148)
where $V := V_1$, and summation is understood over $q$.

Consequently, one can rewrite (144) as

\[ Q_1 = (1 - \rho)V_1[\nabla^2(u_0(x_1) - \kappa u_0(x_1))] + i k \beta_{pq} \frac{\partial u_0(x_1)}{\partial x_q} \beta_p V_1, \quad \beta := \frac{x - x_1}{|x - x_1|}, \]

and $(x)_p := x \cdot e_p$ is the $p$-th Cartesian coordinate of the vector $x$.

Formula (118) can be written as

\[ u(x) = u_0(x) + g(x, x_1) \left( (1 - \rho)[\nabla^2 u_0(x_1) - \kappa u_0(x_1)] + i k \beta_{pq} \frac{\partial u_0(x_1)}{\partial x_1,q} \beta_p + \kappa u_0(x_1) \right) V_1. \]

Here one sums over the repeated indices, $|x - x_1| >> a$, and $\frac{\partial u_0(y)}{\partial x_q} |_{y=x_1,q} := \frac{\partial u_0(y)}{\partial x_q} |_{y=x_1,q}$.

Formulas (140), (142), (143) are valid for small $D$ of arbitrary shape. Let us formulate the results of this Section in the following theorem.

**Theorem 6.1.** Assume that $ka \ll 1$, $k_1, k$, and $\rho$ are positive constants. Then the scattering problem (100) - (104) has a unique solution. This solution has the form (107) and can be calculated by formula (150) in the region $|x - x_1| >> a$ up to the terms of order $O(a^4)$ as $a \to 0$, where $a = 0.5 \text{dim} D$, $\kappa = k_1^2 - k^2$, $V_1 = \text{vol} D$, $\beta = \frac{x - x_1}{|x - x_1|}$, $\beta_{pq}$ is defined in equation (147), and $O(a^4)$ does not depend on $x$.

### 7 Wave scattering by many small bodies

Assume that the distribution of small bodies is given by equation (105), and that there are $M = M(a)$ non-intersecting small bodies $D_m$ of size $a$. For simplicity we assume that $D_m$ is a ball of radius $a$, centered at $x_m$. There is an essential novel feature in the theory, developed in this paper compared with the one developed in [11], [18], [24], namely, the scattered field was much larger, as $a \to 0$ in the above papers. For example, for the impedance boundary condition, $u_N = \zeta u$ on $S$, the scattered field is $O(a^2)$, and for the Dirichlet boundary condition, $u = 0$ on $S$, the scattered field is $O(a)$. For the Neumann boundary condition the scattered field is $O(a^3)$. We have the same order of smallness of the scattered field, $O(a^3)$, for the problem with the transmission boundary condition because $V_1 = O(a^3)$. The basic role in Section 3 is played by formula (150). We assume that the distance $d$ between neighboring bodies (particles) is much larger than $a$, $d >> a$, but there can be many small particles on the wavelength, and
the interaction of the scattered waves (multiple scattering) is essential and cannot be neglected.

This assumption effectively means that the function \( N(x) \) in (105) has to be small, \( N(x) \ll 1 \). Indeed, if on a segment of unit length there are small particles placed at a distance \( d \) between neighboring particles, then there are \( O(1/d) \) particles on this unit segment, and \( O(1/d^3) \) in a unit cube \( C_1 \).

Since \( V = O(a^3) \), by formula (105) one gets

\[
\frac{1}{O(a^3)} \int_{C_1} N(x) dx = O\left(\frac{1}{d^3}\right).
\]

Therefore \( d >> a \) can hold only if

\[
\left(\int_{C_1} N(x) dx\right)^{1/3} = O\left(\frac{1}{a}\right) \ll 1.
\]

Let us look for the (unique) solution to problem (100)-(104) with \( 1 \leq m \leq M = M(a) \) of the form

\[
u(x) = u_0(x) + \sum_{m=1}^{M} g(x,x_m) \left[ Q_m - \frac{ik}{|x-x_m|} \int_{S_m} t_p \sigma_m(t) dt \right] +
\]

\[
+ \sum_{m=1}^{M} \kappa_m g(x,x_m) u_e(x_m)V_m, \quad Q_m := \int_{S_m} \sigma_m(t) dt, \quad a \to 0,
\]

where we have used formula (150) for the scattered field by every small particle, replaced \( u_0 \) by the effective field \( u_e \), acting on every particle, and took into account that \( \beta := \beta_m := \frac{x-x_m}{|x-x_m|} \). By \( (x-x_m)_p \) the \( p \)-th component of vector \( (x-x_m) \) is denoted.

The effective (self-consisted) field \( u_e \), acting on \( j \)-th particle, is defined as:

\[
u_e(x) = u_0(x) + \sum_{m=1,m\neq j}^{M} g(x,x_m) \left[ (1-\rho_m)[\nabla^2 u_e(x_m) - \kappa_m u_e(x_m)] +
\]

\[
i k \beta^{(m)} \frac{\partial u_e(x-x_m)_p}{\partial x_q} V_m + \sum_{m=1,m\neq j}^{M} \kappa_m g(x,x_m) u_e(x_m)V_m, \quad |x-x_j| \sim a.
\]
Setting \( x = x_j \) in equation (153), one gets a linear algebraic system for the unknowns \( u_j := u_e(x_j), 1 \leq j \leq M, \) and \( \frac{\partial u_e(x_j)}{\partial x_{jp}} \). Here \( x_{jp} \) is the \( p \)-th component of the vector \( x_j \), \( p = 1, 2, 3 \). Differentiating (153) with respect to \( x_{jp}, p = 1, 2, 3 \), and then setting \( x = x_j \), one obtains a linear algebraic system for the \( 4M \) unknowns \( u_j \) and \( \frac{\partial u_e(x_j)}{\partial x_{jp}} \), \( 1 \leq j \leq M, 1 \leq p \leq 3 \).

This linear algebraic system one gets if one solves by a collocation method the following integral equation

\[
\begin{align*}
  u(x) &= u_0(x) + \int_D g(x, y) \left[ (1 - \rho)(\nabla^2 - K^2(y) + k^2)u(y) + \right. \\
  & \left. ik\beta_{pq}(y, \lambda) \frac{\partial u(y)}{\partial y_q} \frac{(x-y)_p}{|x-y|} + (K^2(y) - k^2)u(y) \right]N(y)dy.
\end{align*}
\]

(154)

In the above equation the function \( \beta_{pq}(y, \lambda) \) is defined as

\[
\beta_{pq}(y, \lambda) = \lim_{a \to 0} \frac{\sum_{x_m \in \Delta_p} \beta_{pq}^{(m)}}{\mathcal{N}(\Delta_p)},
\]

where \( y = y_p \in \Delta_p \), and tensor \( \beta_{pq}^{(m)} = \beta_{pq}^{(m)}(\lambda) \) is defined in (147). Convergence of the collocation method was proved in [24].

Equation (154) is a non-local integrodifferential equation for the limiting effective field in the medium in which many small bodies are embedded.

This is a novel result. The original scattering problem (100)–(104) has been formulated in terms of local differential operators.

In the derivation of equation (154) from equation (153) we have assumed that \( \rho_m = \rho \) does not depend on \( m \), took into account that \( x_m^2 \) becomes in the limit \( K^2(y) - k^2 \), and denoted by \( K^2(y) \) a continuous function in \( D \) such that \( K^2(x_m) = k_m^2 \). As \( a \to 0 \) the function \( K^2(y) \) is uniquely defined because the set \( \{ x_m \}_{m=1}^{M(a)} \) becomes dense in \( D \) as \( a \to 0 \).

To derive equation (154) from equation (153) we argue as follows. Consider a partition of \( D \) into a union centered at the points \( y_p \) of \( P \) non-intersecting cubes \( \Delta_p \), of size \( b(a), b(a) >> d \), so that each cube contains many small bodies, \( \lim_{a \to 0} b(a) = 0 \). Let us demonstrate the passage to the limit \( a \to 0 \) in the sums in equation (153) using the first sum as an example.
Write the first sum in (153) as
\[
\sum_{m \neq j} g(x, x_m)(1 - \rho_m)[\nabla^2 u_e(x_m) - \kappa_m u_e(x_m)]V_m
\]
\[
= \sum_{p=1}^{P} g(x, y_p)(1 - \rho_p)[\nabla^2 u_e(y_p) - \kappa_p u_e(y_p)]V_m \sum_{x_m \in \Delta_p} 1
\]
\[
= \sum_{p=1}^{P} g(x, y_p)(1 - \rho_p)[\nabla^2 u_e(y_p) - \kappa_p u_e(y_p)]N(y_p)|\Delta_p|(1 + o(1)), \quad (155)
\]
where we have used formula (105), took into account that \( \text{diam} \Delta_p \to 0 \) as \( a \to 0 \), wrote formula (105) as
\[
V \sum_{x_m \in \Delta_p} 1 = V N(\Delta_p) = N(y_p)|\Delta_p|(1 + o(1)), \quad a \to 0, \quad (156)
\]
and used the Riemann integrability of the functions involved, which holds, for example, if these functions are continuous. By \( \rho_p \) we denote the value \( \rho(y_p) \), where \( \rho(y) \) is a continuous function.

The sum in (155) is the Riemannian sum for the integral
\[
\int_D g(x, y)(1 - \rho(y))[\nabla^2 u(y) - K^2(y)u(y) + k^2 u(y)]N(y)dy. \quad (157)
\]
Similarly one treats the other sums in (153) and obtains in the limit \( a \to 0 \) equation (154).

Let us formulate the results of this Section in the following theorem.

**Theorem 7.1.** Assume that conditions (105) and (106) hold. Then, as \( a \to 0 \), the effective field, defined by equation (153), has a limit \( u(x) \). The function \( u(x) \) solves equation (154).

### 8 Conclusions

In this paper analytic formulas for the scattering amplitudes for wave scattering by a single small particle are derived for various boundary conditions: the Dirichlet, Neumann, impedance, and transmission ones.

The equation for the effective field in the medium, in which many small particles are embedded, is derived in the limit \( a \to 0 \). The physical assumptions are such that the multiple scattering effects are not negligible, but essential. The derivations are rigorous.
On the basis of the developed theory efficient numerical methods are proposed for solving many-body wave scattering problems in the case of small scatterers.
References

[1] Andriychuk, M., Ramm, A.G., Numerical solution of many-body wave scattering problem for small particles and creating materials with desired refraction coefficient,
Chapter in the book:
"Numerical Simulations of Physical and Engineering Processes", In-Tech., Vienna, 2011, pp.1-28. (edited by Jan Awrejcewicz)
ISBN 978-953-307-620-1

[2] Andriychuk, M., Ramm, A.G., Scattering of electromagnetic waves by many thin cylinders: theory and computational modeling, Optics Communications, 285, N20, (2012), 4019-4026.

[3] Hulst, van der, Light scattering by small particles, Dover, New York, 1961.

[4] V. Jikov, S. Kozlov, O. Oleinik, Homogenization of differential operators and integral functionals, Springer-verlag, berlin, 1994

[5] L. Landau, L. Lifschitz, Electrodynamics of continuous media, Pergamon Press, Oxford, 1984.

[6] V. Marchenko, E. Khruslov, Homogenization of partial differential equations, Birkhäuser, Boston, 2006.

[7] P. Martin, Multiple scattering, Cambridge Univ. Press, Cambridge, 2006.

[8] A.G.Ramm, Scattering by obstacles, D.Reidel, Dordrecht, 1986,

[9] A.G.Ramm, Wave scattering by small bodies of arbitrary shapes, World Sci. Publishers, Singapore, 2005.

[10] A.G.Ramm, Scattering by many small bodies and applications to condensed matter physics, Europ. Phys. Lett., 80, 44001, (2007).

[11] A.G.Ramm, Many-body wave scattering by small bodies and applications, J. Math. Phys., 48, 103511, (2007).

[12] A.G.Ramm, Wave scattering by small particles in a medium, Phys. Lett. A 367, 156-161, (2007).
[13] A.G.Ramm, Wave scattering by small impedance particles in a medium, Phys. Lett. A 368, 164-172, (2007).

[14] A.G.Ramm, Distribution of particles which produces a "smart" material, Jour. Stat. Phys., 127, 915-934, (2007).

[15] A.G.Ramm, Distribution of particles which produces a desired radiation pattern, Physica B, 394, 253-255, (2007).

[16] A.G.Ramm, Creating wave-focusing materials, LAJSS (Latin-American Journ. of Solids and Structures), 5, 119-127, (2008).

[17] A.G.Ramm, Electromagnetic wave scattering by small bodies, Phys. Lett. A, 372, 4298-4306, (2008).

[18] A.G.Ramm, Wave scattering by many small particles embedded in a medium, Phys. Lett. A, 372, 3064-3070, (2008).

[19] A.G.Ramm, Preparing materials with a desired refraction coefficient and applications, In the book "Topics in Chaotic Systems: Selected Papers from Chaos 2008 International Conference", Editors C.Skiadas, I. Dimotikalis, Char. Skiadas, World Sci.Publishing, pp.265-273, (2009).

[20] A.G.Ramm, Preparing materials with a desired refraction coefficient, Nonlinear Analysis: Theory, Methods and Appl., 70, e186-e190, (2009).

[21] A.G.Ramm, Creating desired potentials by embedding small inhomogeneities, J. Math. Phys., 50, 123525, (2009).

[22] A.G.Ramm, A method for creating materials with a desired refraction coefficient, Internat. Journ. Mod. Phys B, 24, 5261-5268, (2010).

[23] A.G.Ramm, Materials with a desired refraction coefficient can be created by embedding small particles into the given material, International Journal of Structural Changes in Solids (IJSCS), 2, 17-23, (2010).

[24] A.G.Ramm, Wave scattering by many small bodies and creating materials with a desired refraction coefficient, Afrika Matematika, 22, 33-55, (2011).

[25] A.G.Ramm, Scattering by many small inhomogeneities and applications, In the book "Topics in Chaotic Systems: Selected Papers from Chaos 2010 International Conference", Editors C.Skiadas, I. Dimotikalis, Char. Skiadas, World Sci.Publishing, pp.41-52, (2011).
[26] A.G. Ramm, Collocation method for solving some integral equations of estimation theory, Internat. Journ. of Pure and Appl. Math., 62, 57-65, (2010).

[27] A.G. Ramm, Scattering of scalar waves by many small particles, AIP Advances, 1, 022135, (2011).

[28] A.G. Ramm, Scattering of electromagnetic waves by many thin cylinders, Results in Physics, 1, N1, (2011), 13-16.

[29] A.G. Ramm, Electromagnetic wave scattering by many small perfectly conducting particles of an arbitrary shape, Optics Communications, 285, N18, (2012), 3679-3683.

[30] A.G. Ramm, Wave scattering by many small bodies: transmission boundary conditions, (submitted)

[31] J. Rayleigh, Scientific papers, Cambridge, 1992.