Confluent KZ equations for $\mathfrak{sl}_N$ with Poincaré rank 2 at infinity

Hajime Nagoya and Juanjuan Sun

Graduate School of Mathematical Sciences, The University of Tokyo, Tokyo 153-8914, Japan
E-mail: nagoya@math.kobe-u.ac.jp and sunjuan@ms.u-tokyo.ac.jp

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Abstract
We construct confluent KZ equations with Poincaré rank 2 at infinity for the case of $\mathfrak{sl}_N$ and the integral representation for the solutions. Hamiltonians of these confluent KZ equations are derived from suitable quantization of $d \log \tau$ constructed in the theory of monodromy preserving deformation (MPD) in Jimbo et al (1981 Physica D 2 306–52). Our confluent KZ equations can be viewed as a quantization of MPD with Poincaré rank 2 at infinity.

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1. Introduction
The KZ equation is a system of linear partial differential equations with regular singularities, and it has integral formulas of hypergeometric type for solutions [18]. Further, the KZ equation is a quantization of the Schlesinger equation, which describes monodromy preserving deformation (MPD) of linear differential equations with regular singularities [8, 16] and [20].

Irregular singular versions of the KZ equation have been considered in some cases. Generalized KZ equations with Poincaré rank 1 at infinity for $\mathfrak{sl}_2$ were presented in [2] and later for any simple Lie algebra in [6]. In [9], confluent KZ equations with an arbitrary Poincaré rank were obtained for $\mathfrak{sl}_2$ and the quantum Painlevé equations QPJ (J = I, II, III, IV, V) proposed in [9, 13] and [14] were derived formally from the Heisenberg version of confluent KZ equations. In all cases mentioned above, the solutions to the equations are expressed by integral formulas of confluent hypergeometric type. In [15], we provided a method of constructing integral formulas of confluent hypergeometric type as solutions to confluent KZ equations for any simple Lie algebra. However, we did not give explicit integral formulas and all confluent KZ equations, except for the $\mathfrak{sl}_2$ case.

1 Present address: Department of Mathematics, Kobe University, Kobe 657-8501, Japan, Research Fellow of the Japan Society for the Promotion of Science.
In this paper, we construct confluent KZ equations for all independent variables with Poincaré rank 2 at infinity for \( \mathfrak{sl}_N \) and give explicit integral formulas of confluent hypergeometric type for solutions. Further, we discuss a connection between our confluent KZ equations and a quantization of MPD with Poincaré rank 2 at infinity.

Hamiltonians for our confluent KZ equations are obtained assuming suitable quantization of the function \( d \log \tau \). The function \( \tau \) was given in the study of MPD [11]. The function \( d \log \tau \) for regular singularities gives the Hamiltonians of the Schlesinger equation. Moreover, it was shown in [21] that Hamiltonians of MPD with Poincaré rank 2 at infinity for \( \mathfrak{sl}_2 \) are derived from \( d \log \tau \). Therefore, we expect that the exact formulas of Hamiltonians of MPD are obtained from \( d \log \tau \). Indeed, the compatibility condition of the confluent KZ equation implies that Hamiltonians of MPD (classical case) with Poincaré rank 2 at infinity are derived from \( d \log \tau \).

Hamiltonians for the KZ equations are called the Gaudin Hamiltonians. The problem of diagonalization of the Gaudin Hamiltonians is called the Gaudin model [7]. It was known that the eigenvalues and eigenvectors of the Gaudin model are derived from the integral formulas for solutions to the KZ equation [1, 17]. Irregular singular versions of the Gaudin model were also studied as well as the standard Gaudin model. In [5], higher Gaudin Hamiltonians were constructed using non-highest weight representations of affine algebras, and eigenvectors of these Hamiltonians were constructed using Wakimoto modules of critical level. It was explained in [5] that there is a connection between the irregular singular version of the Gaudin model and the geometric Langlands correspondence.

The remainder of this paper is organized as follows. In section 2, we present notations of the truncated Lie algebras and define a confluent Verma module. In section 3, we define Hamiltonians directly and give confluent KZ equations. In section 4, we present an integral formula

\[ \int_{\Gamma} \Phi^z \omega \]  

as a solution to the confluent KZ equation. Here, \( \Phi \) is a scalar multi-valued master function, \( \Gamma \) is an appropriate cycle, and \( \omega \) is a differential form taking values in a tensor product of confluent Verma modules. The master function \( \Phi \) and the differential form \( \omega \) are straightforward generalizations of the respective parts of the integral formulas for the general KZ equations in [6] and confluent KZ equations for \( \mathfrak{sl}_2 \) in [9]. The differential form \( \omega \) is represented in terms of a Poincaré–Birkhoff–Witt basis, as shown in [12]. In section 5, recalling the definition of Jimbo–Miwa–Ueno’s tau function \( \tau \), we explain that our Hamiltonians are obtained assuming a suitable quantization of \( d \log \tau \). Moreover, we discuss a connection between our confluent KZ equations and a quantization of MPD with Poincaré rank 2 at infinity.

2. Preliminary

2.1. Notation

Let \( \mathfrak{g} \) be the complex simple Lie algebra \( \mathfrak{sl}_N \) and let \( \mathfrak{h} \) be the Cartan subalgebra. We denote by \( \Pi, \Delta, \Delta_+, Q \) and \( Q_+ \) the set of simple roots of \( \mathfrak{g} \), the root system, the set of positive roots, the set of the root lattice, and the set of the positive part of the root lattice, respectively. Explicitly, we have

\[ \Pi = \{ \alpha_1, \ldots, \alpha_{N-1} \}, \]

\[ \Delta = \{ \pm (\alpha_i + \cdots + \alpha_j) | 1 \leq i \leq j \leq N - 1 \}, \]

\[ \Delta_+ = \{ \alpha_i + \cdots + \alpha_j | 1 \leq i \leq j \leq N - 1 \}. \]
Here, a generalization of a Verma module and will be used in subsequent sections. For

2.2. Module

Verma module for any Poincaré rank was defined in [9].

\[ \frac{d}{dx}, e_\alpha, \quad (\alpha \in \Delta, \alpha \not\in \Delta, \alpha \not\in \Delta \cup \{0\}, \alpha, \beta \in \Delta, \alpha + \beta \not\in \Delta \cup \{0\} \]  

e_{\alpha}, e_{\beta} = \epsilon(\alpha, \beta)e_{\alpha+\beta} \]  

The Casimir operator is defined by

\[ \Omega = \sum_{\alpha \in \Delta} (e_{\alpha} \otimes e_{-\alpha} + e_{-\alpha} \otimes e_{\alpha}) + \sum_{p=1}^{N-1} h_p \otimes w_p. \]  

The homomorphism defined below gives a natural representation of \( \mathfrak{g} \) on \( \mathbb{C}^N \): for \( \alpha = \alpha_p + \alpha_{p+1} + \cdots + \alpha_q, e_\alpha = E_{p,q}, e_{-\alpha} = E_{q+1,p}; h_p = E_{p,q} - E_{p+1,q+1} \) and

\[ w_p = \sum_{a=1}^{N} \left( 1 - \frac{p}{N} \right) E_{a,a} - \frac{p}{N} \sum_{a=p+1}^{N} E_{a,a}, \]  

where \( E_{p,q} \) is the matrix with the \( (p, q) \)-entry 1 and others 0.

2.2. Module

In this subsection, we define a confluent Verma module, which may be viewed as a natural generalization of a Verma module and will be used in subsequent sections. For \( \mathfrak{sl}_2 \), a confluent Verma module for any Poincaré rank was defined in [9].

Let \( V_i \) be Verma modules of \( \mathfrak{g} \) with respect to highest weights \( \Lambda^{(i)} \) and highest weight vectors \( v_i \) for \( i = 1, \ldots, n \).

Let \( \mathfrak{g}_{(2)} \) be the truncated Lie algebra \( \mathfrak{g} \otimes [t^i / t^j g(t)] \), where \( g(t) = g \otimes \mathbb{C} [t] \). We denote \( x \otimes t^p \) by \( x \otimes t^p \). Let \( V^{(\infty)} \) be the polynomial ring \( \mathbb{C}[x_a | a \in \Delta]. \) For parameters \( \gamma_i, \mu_i \) \( (i = 1, 2, \ldots, N - 1) \), we define the action of \( \mathfrak{g}_{(2)} \) on \( V^{(\infty)} \) as

\[ e_{\alpha}[1] = \mu_\alpha \frac{1}{\partial x_\alpha}, \quad e_{-\alpha}[1] = \mu_\alpha^{-1} x_\alpha, \quad h_\alpha[1] = \gamma_\alpha, \quad h_\alpha[2] = \mu_\alpha, \quad e_{\pm\alpha}[2] = 0, \]
where $\alpha = \alpha_p + \alpha_{p+1} + \cdots + \alpha_q \in \Delta_+$, and $\gamma_a = \gamma_p + \gamma_{p+1} + \cdots + \gamma_q$ and $\mu_a = \mu_p + \mu_{p+1} + \cdots + \mu_q$.

We call $V^{(\infty)}$ a confluent Verma module with Poincaré rank 2 at infinity. The element 1 in $V^{(\infty)}$ is the highest weight, and weights $\Lambda_1^{(\infty)}$ and $\Lambda_2^{(\infty)}$ defined by $(\Lambda_1^{(\infty)}, \alpha_p) = \gamma_p$ and $(\Lambda_2^{(\infty)}, \alpha_p) = \mu_p (p = 1, \ldots, N - 1)$ are the highest weights.

We consider the $(\bigotimes_{i=1}^n \mathfrak{g} \oplus \mathfrak{g}(2))$-module

$$V = V_1 \otimes \cdots \otimes V_\infty \otimes V^{(\infty)}$$

with $\mathfrak{v} = v_1 \otimes \cdots \otimes v_n \otimes 1$. For each $i = 1, \ldots, n$, denote by $x^{(i)} : V \rightarrow V (x \in \mathfrak{g})$ the linear operator acting on the $i$th tensor factor $V_i$ and as identities on the others. For $x[\mathfrak{g}] \in \mathfrak{g}(2)$, denote by $x^{(\infty)}[\mathfrak{g}] : V \rightarrow V$ the linear operator acting as $x_\infty$ on $V^{(\infty)}$ and as identities on the others.

Let $\Lambda = \sum_{i=1}^n \Lambda^{(i)}$, $\mathbf{m} = (m_1, \ldots, m_{N-1}) \in (\mathbb{Z}_{\geq 0})^{N-1}$, and a map $\alpha : (\mathbb{Z}_{\geq 0})^{N-1} \rightarrow Q_+$ be defined as $\alpha(\mathbf{m}) = \sum_{i=1}^{N-1} m_i \alpha_p$. We denote by $V_\mathbf{m}$ the weight space of $V$ with weight $\Lambda - \alpha(\mathbf{m})$ corresponding to $\sum_{i=1}^n h^{(i)} + h^{(\infty)}[0]$, $(h \in \mathfrak{h})$, that is,

$$V_\mathbf{m} = \left\{ x \in V \left| \left( \sum_{i=1}^n h^{(i)} + h^{(\infty)}[0] \right)(x) = \left( \Lambda - \alpha(\mathbf{m}) \right)(h)x, \ h \in \mathfrak{h} \right. \right\}.$$

Here, for $h \in \mathfrak{h}$, $h^{(\infty)}[0]$ is defined as

$$h^{(\infty)}[0] = - \sum_{a \in \Delta_+} \frac{\alpha(h)}{\mu_a} e_a^{(\infty)}[1] e_a^{(\infty)}[1].$$

3. Confluent KZ equation

In this section, we give Hamiltonians and confluent KZ equations with Poincaré rank 2 at infinity directly. Hamiltonians for our confluent KZ equations are obtained assuming suitable quantization of a 1-form $\omega = d \log \tau$ given in the study of MPD [11], which is explained in section 5.

**Definition 3.1.** We define the Hamiltonians $\mathcal{H}_p^{(1)}$ and $\mathcal{H}_p^{(2)}$ ($p = 1, \ldots, N - 1$) in $\text{End}(V)$ as follows:

$$\mathcal{H}_p^{(1)} = - \sum_{i=1}^n z_i^p w_i^p - \sum_{a \in \mathfrak{h}_p} \frac{1}{\mu_a} \left( e_a^{(\infty)}[1] E_a + e_a^{(\infty)}[1] E_{-a} \right) - \sum_{a \in \mathfrak{h}_p} \frac{\gamma_a}{\mu_a} e_a^{(\infty)}[1] e_a^{(\infty)}[1]$$

$$+ \sum_{a \in \mathfrak{h}_p, \ b \neq a} \frac{\epsilon(\alpha, \beta)}{\mu_a \mu_b} \left( e_a^{(\infty)}[1] e_a^{(\infty)}[1] e_{-a}^{(\infty)}[1] e_{-b}^{(\infty)}[1] + e_a^{(\infty)}[1] e_{-a}^{(\infty)}[1] e_{-b}^{(\infty)}[1] e_a^{(\infty)}[1] \right),$$

$$2 \mathcal{H}_p^{(2)} = - \sum_{i=1}^n z_i^p w_i^p + \sum_{a \in \mathfrak{h}_p} \frac{1}{\mu_a} \left( e_a^{(\infty)}[1] E_{-a}(-1) + e_{-a}^{(\infty)}[1] E_{-a}(-1) \right) + \sum_{a \in \mathfrak{h}_p} \frac{1}{\mu_a} E_{-a} E_a$$

$$+ \sum_{a \in \mathfrak{h}_p, \ b \neq a} \frac{1}{\mu_a \mu_b} \left( e_a^{(\infty)}[1] e_a^{(\infty)}[1] H_a + \sum_{a \in \mathfrak{h}_p} \frac{\gamma_a}{\mu_a} \left( e_a^{(\infty)}[1] E_a + e_{-a}^{(\infty)}[1] E_{-a} \right) - \sum_{a \in \mathfrak{h}_p, \ b \neq a} \frac{\epsilon(\alpha, |\alpha - \beta|)}{\mu_a \mu_b} \right)$$

$$\times \left( e_{-a}^{(\infty)}[1] e_{-a}^{(\infty)}[1] e_{-b}^{(\infty)}[1] + e_a^{(\infty)}[1] e_{-a}^{(\infty)}[1] e_{-b}^{(\infty)}[1] + e_a^{(\infty)}[1] e_{-a}^{(\infty)}[1] e_{-b}^{(\infty)}[1] e_{-b}^{(\infty)}[1] \right),$$

$$+ \sum_{a \in \mathfrak{h}_p} \frac{1}{\mu_a} \left( e_a^{(\infty)}[1] e_a^{(\infty)}[1] \right) - \sum_{a \in \mathfrak{h}_p, \ b \neq a} \frac{\epsilon(\alpha, |\alpha - \beta|)}{\mu_a \mu_b} \left( \frac{\gamma_a}{\mu_a} + \frac{\gamma_b}{\mu_b} \right) e_a^{(\infty)}[1] e_{-a}^{(\infty)}[1] e_{-b}^{(\infty)}[1] e_{-b}^{(\infty)}[1].$$
Example 3.2. Let \( J. \ Phys. \ A: \ Math. \ Theor. \) Now we give the following differential equations, which we call a confluent KZ equation:

3.1. Confluent KZ equation

\[ X_\alpha(\kappa) = X_\alpha(\kappa - \gamma), \]

where \( X_\alpha = \sum_{i=1}^{\infty} x^{(i)}_\alpha z_i \) (\( \alpha \in \Delta \)), and

\[ g(\alpha, \gamma) = \begin{cases} 
\alpha - \beta & \text{if } \alpha - \beta \in Q_+ \\
\beta - \alpha & \text{if } \beta - \alpha \in Q_+, \\
-1 & \text{others}.
\end{cases} \]

3.2. Let \( g = sl_3 \) and \( n = 0 \). Then, the Hamiltonians \( \mathcal{H}^{(1)}_1 \) and \( \mathcal{H}^{(2)}_1 \) are

\[ \mathcal{H}^{(1)}_1 = -\frac{\gamma_1}{\mu_1^2} (\epsilon_\alpha^{(\infty)}(1)[1] \epsilon_\alpha^{(\infty)}(1)[1]) - \frac{\gamma_1 + \gamma_2 (\mu_1 + \mu_2)^2}{} \epsilon_\alpha^{(\infty)}(1)[1] \epsilon_\alpha^{(\infty)}(1)[1] + \frac{1}{\mu_1 (\mu_1 + \mu_2)} \epsilon_\alpha^{(\infty)}(1)[1] \epsilon_\alpha^{(\infty)}(1)[1] + \epsilon_\alpha^{(\infty)}(1)[1] \epsilon_\alpha^{(\infty)}(1)[1], \]

and

\[ 2\mathcal{H}^{(2)}_1 = -\frac{\gamma_1^2}{\mu_1^2} (\epsilon_\alpha^{(\infty)}(1)[1] \epsilon_\alpha^{(\infty)}(1)[1]) - \frac{1}{\mu_1 (\mu_1 + \mu_2)} \epsilon_\alpha^{(\infty)}(1)[1] \epsilon_\alpha^{(\infty)}(1)[1] + \frac{\gamma_1 + \gamma_2 (\mu_1 + \mu_2)^2}{} \epsilon_\alpha^{(\infty)}(1)[1] \epsilon_\alpha^{(\infty)}(1)[1] - \frac{1}{\mu_1 (\mu_1 + \mu_2)} \epsilon_\alpha^{(\infty)}(1)[1] \epsilon_\alpha^{(\infty)}(1)[1] + \epsilon_\alpha^{(\infty)}(1)[1] \epsilon_\alpha^{(\infty)}(1)[1] + \epsilon_\alpha^{(\infty)}(1)[1] \epsilon_\alpha^{(\infty)}(1)[1]. \]

3.1. Confluent KZ equation

Now we give the following differential equations, which we call a confluent KZ equation:

\[ \kappa \frac{\partial u}{\partial z_i} = G^{(i)}_{-1} u \quad (i = 1, \ldots, n), \]

\[ \kappa \frac{\partial u}{\partial y_p} = \mathcal{H}^{(p)}_{-1} u \quad (p = 1, \ldots, N - 1), \]

\[ \kappa \frac{\partial u}{\partial \mu_p} = \mathcal{H}^{(p)}_{-1} u \quad (p = 1, \ldots, N - 1), \]

where \( \kappa \in \mathbb{C} \) and the unknown function \( u(z_1, \ldots, z_n, y_1, \ldots, y_{N-1}, \mu_1, \ldots, \mu_{N-1}) \) takes value in \( V \) and the Gaudin Hamiltonians \( G^{(i)}_{-1} \) are given by

\[ G^{(i)}_{-1} = \sum_{i \neq j, j \neq i} \frac{\gamma^{(i,j)}}{z_i - z_j} - \sum_{p=1}^{N-1} \gamma_p w_p^{(i)} - \sum_{a \in \Delta} \epsilon^{(\infty)}_a[1] \epsilon^{(i)}_a - \sum_{p=1}^{N-1} \mu_p w_p^{(i)} \quad (i = 1, \ldots, n). \]
Conjecture 3.3. The confluent KZ equation (3.1)–(3.3) satisfies the compatibility condition, that is, we have

\[
\begin{align*}
  \frac{\kappa}{\partial z_i} - G_i^{(i)} - \frac{\kappa}{\partial z_j} - G_j^{(j)} &= 0 \quad (i, j = 1, \ldots, n), \\
  \frac{\kappa}{\partial z_i} - G_i^{(i)} - \frac{\kappa}{\partial \gamma_p} - H_p^{(1)} &= 0 \quad (i = 1, \ldots, n, \quad p = 1, \ldots, N - 1), \\
  \frac{\kappa}{\partial z_i} - G_i^{(i)} - \frac{\kappa}{\partial \mu_p} - H_p^{(2)} &= 0 \quad (i = 1, \ldots, n, \quad p = 1, \ldots, N - 1), \\
  \frac{\kappa}{\partial \gamma_p} - H_p^{(1)} - \frac{\kappa}{\partial \gamma_q} - H_q^{(1)} &= 0 \quad (p, q = 1, \ldots, N - 1), \\
  \frac{\kappa}{\partial \gamma_p} - H_p^{(2)} - \frac{\kappa}{\partial \mu_q} - H_q^{(2)} &= 0 \quad (p, q = 1, \ldots, N - 1).
\end{align*}
\]  

Remark 3.4. Using the software SINGULAR [19], we can confirm that the above conjecture is true when \( N \) is less than 7.

4. Integral formula

In this section, we present integral formulas taking values in \( \mathbb{V}_m \) and show that these integral formulas are solutions to our confluent KZ equations. Let \( S_p \) be an index set \( \{1, \ldots, m_p\} \) for each \( p \) (\( 1 \leq p < N - 1 \)). We prepare the following integration variables:

\[
\{t_a^{(p)}\} \quad 1 \leq p \leq N - 1, \quad a \in S_p. \tag{4.1}
\]

We define the master function of an integrable formula as follows:

\[
\Phi(z, t) = \prod_{1 \leq i < j \leq n} (z_i - z_j)^{(\Lambda^{(i)}, \Lambda^{(j)})} \prod_{1 \leq p < q \leq N - 1} \prod_{1 \leq a \in S_p} \left( t_a^{(p)} - z_j \right)^{-(\alpha_p, \Lambda^{(a)})} \\
\times \prod_{1 \leq p < q \leq N - 1} \prod_{1 \leq a \in S_p} \prod_{1 \leq b \in S_q} \left( t_a^{(p)} - t_b^{(q)} \right)^{\alpha_a \alpha_b} \prod_{p=1}^{N-1} \prod_{1 \leq a < b \leq m_p} \left( t_a^{(p)} - t_b^{(p)} \right)^{\alpha_a \alpha_b} \\
\times \exp \left( -\sum_{i=1}^{n} \left( \left( A_1^{(\infty)}, \Lambda^{(i)} \right) z_i + \left( A_2^{(\infty)}, \Lambda^{(i)} \right) \frac{z_i^2}{2} \right) \right) \\
\times \exp \left( \sum_{p=1}^{N-1} \sum_{a=1}^{m_p} \left( \left( A_1^{(\infty)}, \alpha_p \right) t_a^{(p)} + \left( A_2^{(\infty)}, \alpha_p \right) \left( t_a^{(p)} \right)^2 \right) \right). 
\]

Next, we establish the \( \omega \) part consisting of vectors \( e^{(i)}_{-\omega}, e^{(\infty)}_{-\omega} [1] \) (\( i = 1, \ldots, n, \omega \in \Delta_+ \)). For each \( \omega = \alpha_i + \cdots + \alpha_j \in \Delta_+ \) and \( a = (a_1, \ldots, a_j) \) (\( a_k \in S_i, i \leq k \leq j \)), we set

\[
\begin{equation}
\begin{align*}
  e^{-\omega}_{-\alpha} (a) &= \frac{1}{\left( t_j^{(i)} - t_{j-1}^{(i)} \right) \cdots \left( t_{a_1+1}^{(i)} - t_{a_1}^{(i)} \right)} \left( \sum_{k=1}^{n} e^{(i)}_{-\omega} (k) \left( \frac{t_k^{(i)} - t_j^{(i)}}{t_k^{(i)} - z_k} - e^{(\infty)}_{-\omega} [1] \right) \right). \tag{4.2}
\end{align*}
\end{equation}
\]
Let \( K = (k_α_1, k_α_1+α_2, \ldots, k_α_{N-1}) \in (\mathbb{Z}_{≥0})^{2N(N-1)} \) and
\[
S(m) = \left\{ K \mid \sum_{α \in J_p} k_α = m_p \ (p = 1, \ldots, N-1) \right\}.
\]
(4.3)

For \( K \in S(m) \), let
\[
A(K) = (a(α_1, 1), \ldots, a(α_1, k_α_1), a(α_1+α_2, 1), \ldots, a(α_{N-1}, k_α_{N-1})),
\]
where
\[
a(α,l) = (a_1(α, l), \ldots, a_j(α, l)) \ (α = α_1 + \cdots + α_j, a_k(α, l) \in S_k, i ≤ k ≤ j)
\]
such that \( a(α, l) ≠ a(β, l') \) for any \( (α, l) ≠ (β, l') \ (α, β \in J_p, 1 ≤ l ≤ k_α \) and \( 1 ≤ l' ≤ k_β \) and \( \{a_p(α, l) | α \in J_p, 1 ≤ l ≤ k_p\} = S_p \).

We assign \( a(α,l) \) to \( e^{-α}(t(α,l)) \) and set
\[
f_K = \sum_{A(K)} \prod e^{-α}(t(α,l)),
\]
(4.4)
where the summation is over all \( A(K) \) and the order for the factor in each product is the same as the linear order \( α_1 > α_1 + α_2 > \cdots > α_{N-1} \).

We define \( ω_m \) as
\[
ω_m = \sum_{k \in S(m)} f_K \nu.
\]
(4.5)

**Remark 4.1.** If \( N = 2 \), then \( Φ(z, t) \) and \( ω_m \) are exactly those defined in [9]. If \( e_{(t)}^{(∞)}[1] (α \in Δ) \) and \( Λ(∞) \) are zero, then \( ω_m \) and \( Φ(z, t) \) are equivalent to those defined in [6].

**Example 4.2.** We give an example for the case of \( sl_3 \). Let \( m = (1, 1) \) and \( n = 0 \); then,
\[
Φ = (t^{(1)} - t^{(2)})^{-1} \exp \left( γ_1 \frac{t^{(1)}_2}{2} + γ_2 \frac{t^{(2)}_2}{2} \right),
\]
(4.6)
and
\[
ω_m = \left( e^{-α_1}[1]e^{-α_2}[1] - \frac{e^{-α_1-α_2}[1]}{t^{(2)}_1 - t^{(1)}_1} \right) \nu.
\]
(4.7)

**Theorem 4.3.** With an appropriate choice of cycles \( Γ \), the function
\[
u = \int_{Γ} \prod_{1 ≤ p ≤ N-1} dt_a^{(p)} Φ^{1/κ}(z, t)ω_m
\]
(4.8)

taking values in \( V_m \) is a solution to the confluent KZ equation (3.1)–(3.3).
\[
\int \prod_{1 \leq p \leq N-1, 1 \leq \sigma_p \leq m_p} \text{d} t^{(p)}_u \Phi^{1/k}(z, t) \varphi
\]  
(4.10)
is invariant under the action of \( \sigma = (\sigma_1, \ldots, \sigma_{N-1}) \) (\( \sigma_p \in \mathbb{S}_{m_p}, 1 \leq p \leq N - 1 \)).

We also assume that for any variable \( t^{(q)}_b \),
\[
\int \prod_{1 \leq p \leq N-1, 1 \leq \sigma_p \leq m_p} \text{d} t^{(p)}_u \frac{\partial}{\partial t^{(q)}_b} \left( \Phi^{1/k}(z, t) \varphi \right) = 0.
\]  
(4.11)

**Outline of a proof of theorem 4.3.** We prove theorem 4.3 by direct computations. Let us explain briefly an outline of the computation. First, we compute the left-hand side of the confluent KZ equation (3.1), (3.2) and (3.3). We rewrite the result after taking the derivations repeatedly and the invariance under the action of \( \mathfrak{f}_K \). Note that the elements \( e^{(i)}_\alpha \) appeared in the expression of \( f_K \) for \( K \in S(m) \) are ordered by the linear order defined in definition 2.1. We rewrite the action of the Hamiltonians on each \( f_K \) according to the linear order. Third, we identify the parts of the left-hand side with the parts of the right-hand side.

The computation is straightforward but complicated and lengthy; hence, we compute the case of \( n = 0 \) and \( N = 3 \) only in this paper. Even in this case, the result has not been known in the literature to the authors’ knowledge. In a similar way below, we can compute the other cases.

Before proceeding the computation, we prepare some notations. Let
\[
\nabla_a^{(p)} = \frac{\partial}{\partial t^{(p)}_u} + \frac{1}{\kappa} \frac{\partial}{\partial t^{(p)}_u} \left( \log(\Phi) \right) \quad (p = 1, 2, a = 1, \ldots, m_p),
\]
because, for \( q = 1, 2 \) and \( b = 1, \ldots, m_q \),
\[
\int \prod_{1 \leq p \leq z, 1 \leq \sigma_p \leq m_p} \text{d} t^{(p)}_u \frac{\partial}{\partial t^{(q)}_b} \left( \Phi^{1/k} \varphi \right) = \int \prod_{1 \leq p \leq z, 1 \leq \sigma_p \leq m_p} \text{d} t^{(p)}_u \Phi^{1/k} \nabla^{(q)}_b (\varphi).
\]
Let rational functions \( \varphi_k \) (\( k = 1, \ldots, \min\{m_1, m_2\} \)) be defined as
\[
\varphi_k = \prod_{a=1}^k \frac{1}{t^{(a)}_u - t^{(a+1)}_u},
\]
and \( \varphi_0 = 1 \). For a rational function \( \varphi(t) \), denote by \( \langle \varphi(t) \rangle \) the integral formula
\[
\int \prod_{1 \leq p \leq z, 1 \leq \sigma_p \leq m_p} \text{d} t^{(p)}_u \Phi^{1/k} \varphi(t).
\]
Then, the function \( u \in V_m \) (\( m = (m_1, m_2) \)) (4.8) is represented as
\[
u = (-1)^{m_1+m_2} \sum_{k=0}^{\min\{m_1, m_2\}} \frac{m_1! m_2!}{(m_1-k)!(m_2-k)!} \langle \varphi_k \rangle (e^{(∞)}_{-a_1[1]})^{m_1-k} (e^{(∞)}_{-a_2[1]})^{k} (e^{(∞)}_{-a_2[1]})^{m_2-k} \nu.
\]
A proof of theorem 4.3 (3.2) in the case of \( n = 0 \) and \( N = 3 \). The left-hand side of (3.2) for \( p = 1 \) is computed as

\[
\kappa \frac{\partial u}{\partial y_1} = \sum_{b=1}^{m_1} \langle t_b^{(1)} \rangle \omega_m. \tag{4.13}
\]

We rewrite the result (4.13) in terms of \( \langle \phi_k \rangle \). By using lemma 4.4, (4.13) is computed as

\[
\sum_{b=1}^{m_1} \langle t_b^{(1)} \rangle \omega_m = (-1)^{m_1+m_2} \sum_{k=0}^{\min\{m_1, m_2\}} \frac{m_1!m_2!}{(m_1-k)!(m_2-k)!} \times \left\{ \begin{array}{c}
\frac{k}{\mu_1 + \mu_2} \left[ - (\gamma_1 + \gamma_2) \langle \phi_k \rangle + \mu_2 \langle \phi_{k-1} \rangle \right] \\
+ \frac{m_1-k}{\mu_1} \left[ (m_2-k) \langle \phi_{k+1} \rangle - \gamma_1 \langle \phi_k \rangle \right] \\
\times \left( e^{t^\infty_m} \right)^{m_1-k} \left( e^{(\infty_m)} \right)^{m_2-k} v.
\end{array} \right. \tag{4.14}
\]

On the other hand, the right-hand side of (3.2) for \( p = 1 \), \( t_1^{(1)} u \), is easily calculated and coincides with (4.14). For the case of \( p = 2 \), it can be verified in a similar manner.

**Lemma 4.4.** For \( 0 \leq k \leq \min\{m_1, m_2\} \) and \( 1 \leq b \leq k \), we have

\[
(\mu_1 + \mu_2) \langle t_b^{(1)} \phi_k \rangle = - (\gamma_1 + \gamma_2) \langle \phi_k \rangle + \mu_2 \langle \phi_{k-1} \rangle, \tag{4.15}
\]

and

\[
(\mu_1 + \mu_2) \langle t_b^{(2)} \phi_k \rangle = - (\gamma_1 + \gamma_2) \langle \phi_k \rangle - \mu_1 \langle \phi_{k-1} \rangle. \tag{4.16}
\]

For \( 0 \leq k \leq \min\{m_1, m_2\} \) and \( k+1 \leq b \leq m_1 \), we have

\[
\mu_1 \langle t_b^{(1)} \phi_k \rangle = (m_2-k) \langle \phi_{k+1} \rangle - \gamma_1 \langle \phi_k \rangle. \tag{4.17}
\]

For \( 0 \leq k \leq \min\{m_1, m_2\} \) and \( k+1 \leq b \leq m_2 \), we have

\[
\mu_2 \langle t_b^{(2)} \phi_k \rangle = -(m_1-k) \langle \phi_{k+1} \rangle - \gamma_2 \langle \phi_k \rangle. \tag{4.18}
\]

**Proof.** In order to prove (4.15), we compute \( \kappa \left( \nabla_b^{(1)} + \nabla_b^{(2)} \right) \phi_k \) as follows. From the definition, we have

\[
\kappa \left( \nabla_b^{(1)} + \nabla_b^{(2)} \right) \phi_k = \left( \sum_{c=1, c \neq b}^{m_1} \frac{2}{t_b^{(1)} - t_c^{(1)}} + \sum_{c=1}^{m_2} \frac{-1}{t_b^{(1)} - t_c^{(2)}} + \gamma_1 \langle t_b^{(1)} \rangle + \gamma_1 \langle t_b^{(1)} \rangle \\
+ \sum_{c=1, c \neq b}^{m_2} \frac{2}{t_b^{(2)} - t_c^{(2)}} + \sum_{c=1}^{m_1} \frac{-1}{t_b^{(2)} - t_c^{(1)}} + \gamma_2 \langle t_b^{(2)} \rangle \right) \phi_k = 0.
\]

Let \( X_i (i = 1, 2, 3, 4) \) be defined as

\[
X_1 = \left( \sum_{c=1, c \neq b}^{m_1} \frac{2}{t_b^{(1)} - t_c^{(1)}} \right) \phi_k, \quad X_2 = \left( \sum_{c=1}^{m_2} \frac{-1}{t_b^{(1)} - t_c^{(2)}} \right) \phi_k, \\
X_3 = \left( \sum_{c=1, c \neq b}^{m_2} \frac{2}{t_b^{(2)} - t_c^{(2)}} \right) \phi_k, \quad X_4 = \left( \sum_{c=1}^{m_1} \frac{-1}{t_b^{(2)} - t_c^{(1)}} \right) \phi_k.
\]
By the invariance under the action of $\mathcal{G}_{m_1} \times \mathcal{G}_{m_2}$, we have

$$X_1 = \frac{1}{2} \left( \sum_{c=1}^{k} \frac{-1}{t_b^{(1)} - t_c^{(1)}} \psi_k \right) + \frac{1}{2} \left( \sum_{c=1}^{k} \frac{-1}{t_b^{(2)} - t_c^{(2)}} \right).$$

Hence, we obtain

$$X_1 + X_4 = \left( \sum_{c=1}^{k} \frac{-1}{t_b^{(1)} - t_c^{(1)}} \right).$$

In a similar way, we obtain

$$X_2 + X_3 = \left( \sum_{c=1}^{k} \frac{1}{t_b^{(1)} - t_c^{(1)}} \right).$$

Consequently, by the invariance under the action of $\mathcal{G}_{m_1} \times \mathcal{G}_{m_2}$, we have

$$X_1 + X_2 + X_3 + X_4 = 0.$$

Therefore, we obtain

$$\langle \kappa (\nabla_b^{(1)} + \nabla_b^{(2)}) \psi_k \rangle = (\gamma_1 + \gamma_2) \langle \psi_k \rangle + (\mu_1 + \mu_2) \langle t_b^{(1)} \psi_k \rangle - \mu_2 \langle \psi_{k-1} \rangle = 0,$$

which finishes the proof for relation (4.15).

The other relations (4.16), (4.17), and (4.18) can be verified in a similar manner, by computing $\langle \kappa (\nabla_b^{(1)} + \nabla_b^{(2)}) \psi_k \rangle$, $\langle \kappa \nabla_b^{(1)} \psi_k \rangle$, and $\langle \kappa \nabla_b^{(2)} \psi_k \rangle$, respectively. $\square$

A proof of theorem 4.3 (3.3) in the case of $n = 0$ and $N = 3$. Because for $k = 0, 1, \ldots, \min\{m_1, m_2\}$,

$$\frac{\partial}{\partial \mu_1} \left( (\theta_{a_1}^{(1)} \theta_{a_2}^{(2)})^{m_1-k} (\theta_{a_2}^{(1)} \theta_{a_2}^{(2)})^{k} \right) \left( \theta_{a_2}^{(1)} \theta_{a_2}^{(2)} \right)^{m_2-k} v \right),$$

the left-hand side of (3.3) for $p = 1$ is computed as

$$\kappa \frac{\partial u}{\partial \mu_1} = \sum_{h=1}^{m_1} \left( \frac{1}{2} \frac{\psi_k}{o_{2h}} \right) + (-1)^{m_1+m_2} \sum_{k=0}^{\min\{m_1, m_2\}} \frac{m_1 m_2!}{2(m_1-k)! (m_2-k)! k!}$$

$$\times \left( \frac{m_1 - k}{\mu_1} + \frac{k}{\mu_1 + \mu_2} \right) \langle \psi_k \rangle \left( \theta_{a_2}^{(1)} \theta_{a_2}^{(2)} \right)^{m_1-k} \left( \theta_{a_2}^{(1)} \theta_{a_2}^{(2)} \right)^{k} \left( \theta_{a_2}^{(1)} \theta_{a_2}^{(2)} \right)^{m_2-k} v. \tag{4.19}$$

We rewrite the result (4.19) in terms of $\langle \psi_k \rangle$. By using lemma 4.5, (4.19) is computed as

$$\kappa \frac{\partial u}{\partial \mu_1} = \sum_{k=0}^{\min\{m_1, m_2\}} \frac{m_1 m_2!}{(m_1-k)! (m_2-k)! k!}$$

$$\times \left( \frac{k}{\mu_1 + \mu_2} - \left( \frac{(m_1-1)}{\mu_1} + \frac{\mu_2}{\mu_1 + \mu_2} \right) \left( \gamma_1 + \gamma_2 \right) \right) \langle \psi_k \rangle$$

$$\times \left( \frac{m_1 - k}{\mu_1} \langle \psi_{k+1} \rangle + \frac{m_1 - k}{\mu_1 + \mu_2} \langle \psi_{k-1} \rangle \right)$$

$$\times \left( (\theta_{a_2}^{(1)})^{m_1-k} (\theta_{a_2}^{(1)} \theta_{a_2}^{(2)})^{k} (\theta_{a_2}^{(1)})^{m_2-k} v. \tag{4.20}$$
On the other hand, the computation of the right-hand side of (3.3) is straightforward and we see that as a result, the right-hand side of (3.3) is equal to (4.20). The case of \( p = 2 \) can be proved in a similar way.

**Lemma 4.5.** For \( 0 \leq k \leq \min(m_1, m_2) \) and \( 1 \leq b \leq k \), we have

\[
(\mu_1 + \mu_2)\left( \langle b_t^{(1)} \rangle^2 \phi_k \right) = \left( -m_1 + 1 - \kappa + \frac{\mu_2}{\mu_1} \right) \left( m_1 - k + 1 \right) + \frac{(\gamma_1 + \gamma_2)^2}{\mu_1 + \mu_2} \langle \phi_k \rangle
\]

\[
- \mu_2 \left( \frac{\gamma_1}{\mu_1} + \frac{\gamma_1 + \gamma_2}{\mu_1 + \mu_2} \right) \langle \phi_{k-1} \rangle,
\]

(4.21)

and

\[
(\mu_1 + \mu_2)\left( \langle b_t^{(2)} \rangle^2 \phi_k \right) = \left( -m_2 + 1 - \kappa + \frac{\mu_1}{\mu_2} \right) \left( m_2 - k + 1 \right) + \frac{(\gamma_1 + \gamma_2)^2}{\mu_1 + \mu_2} \langle \phi_k \rangle
\]

\[
+ \mu_1 \left( \frac{\gamma_2}{\mu_2} + \frac{\gamma_1 + \gamma_2}{\mu_1 + \mu_2} \right) \langle \phi_{k-1} \rangle.
\]

(4.22)

For \( 0 \leq k \leq \min(m_1, m_2) \) and \( k + 1 \leq b \leq m_1 \), we have

\[
\mu_1\left( \langle b_t^{(1)} \rangle^2 \phi_k \right) = -(m_2 - k) \left( \frac{\gamma_1}{\mu_1} + \frac{\gamma_1 + \gamma_2}{\mu_1 + \mu_2} \right) \langle \phi_{k+1} \rangle
\]

\[
+ \left( m_2 - k \right) \frac{\mu_2}{\mu_1 + \mu_2} + \frac{\gamma_2^2}{\mu_1 + \mu_2} - m_1 + 1 - \kappa \langle \phi_k \rangle.
\]

(4.23)

For \( 0 \leq k \leq \min(m_1, m_2) \) and \( k + 1 \leq b \leq m_2 \), we have

\[
\mu_2\left( \langle b_t^{(2)} \rangle^2 \phi_k \right) = (m_1 - k) \left( \frac{\gamma_2}{\mu_2} + \frac{\gamma_1 + \gamma_2}{\mu_1 + \mu_2} \right) \langle \phi_{k+1} \rangle
\]

\[
+ \left( m_1 - k \right) \frac{\mu_1}{\mu_1 + \mu_2} + \frac{\gamma_1^2}{\mu_2} - m_2 + 1 - \kappa \langle \phi_k \rangle.
\]

(4.24)

**Proof.** In order to prove (4.21), we compute \( \kappa (\nabla_b^{(1)} t_b^{(1)} + \nabla_b^{(2)} t_b^{(2)}) \phi_k \) as follows. Let \( X_i \) be defined as

\[
X_1 = \sum_{c=1}^{m_1} \left( \frac{2 t_b^{(1)}}{t_b^{(1)} - t_c^{(1)}} \right) \phi_k,
\]

\[
X_2 = \sum_{c=1}^{m_2} \left( \frac{-t_b^{(1)}}{t_b^{(1)} - t_c^{(1)}} \right) \phi_k,
\]

\[
X_3 = \sum_{c=1}^{m_2} \left( \frac{2 t_b^{(2)}}{t_b^{(2)} - t_c^{(2)}} \right) \phi_k,
\]

\[
X_4 = \sum_{c=1}^{m_2} \left( \frac{-t_b^{(2)}}{t_b^{(2)} - t_c^{(2)}} \right) \phi_k.
\]

Then, we have

\[
(\nabla_b^{(1)} t_b^{(1)} + \nabla_b^{(2)} t_b^{(2)}) \phi_k = X_1 + X_2 + X_3 + X_4 + \{ \gamma_1 t_b^{(1)} \phi_k \} + \{ \gamma_2 t_b^{(2)} \phi_k \}
\]

\[
+ (\mu_1 + \mu_2)\left( \langle b_t^{(1)} \rangle^2 \phi_k \right) + \left( \langle t_b^{(1)} \rangle^2 \phi_{k-1} \right) + \kappa \langle \phi_k \rangle.
\]

(4.25)

We need to compute \( \sum_{i=1}^{k} X_i \) only, due to lemma 4.4. By the invariance under the action of \( \mathcal{S}_{m_1} \times \mathcal{S}_{m_2} \), we have

\[
X_1 \equiv (k - 1) \langle \phi_k \rangle + \sum_{c=1}^{m_1} \left( -t_b^{(1)} \right) \phi_k + \sum_{c=1}^{m_2} \left( -t_b^{(1)} \right) \frac{1}{t_c^{(1)} - t_b^{(1)}} \prod_{a=1}^{k} \frac{1}{t_a^{(1)} - t_b^{(1)}}.
\]

Hence, we obtain

\[
X_1 + X_4 \equiv (k - 1) \langle \phi_k \rangle + \sum_{c=1}^{m_1} \left( -t_b^{(2)} \right) \phi_k + \sum_{c=1}^{m_2} \left( -t_b^{(2)} \right) \frac{1}{t_c^{(2)} - t_b^{(2)}}.
\]
In a similar way, we obtain

\[ X_2 + X_3 = (k - 1)\langle \psi_k \rangle + \left( \sum_{c=1}^{k} t_{b(1)}^{(1)} t_{c(2)}^{(2)} \psi_k \right). \]

Consequently, by the invariance under the action of \( S_{m_1} \times S_{m_2} \), we have

\[ \sum_{i=1}^{4} X_i = (k - 2)\langle \psi_k \rangle. \]  \hspace{1cm} (4.26)

Substituting the result (4.26) into (4.25), we obtain relation (4.21).

The other relations (4.22), (4.23), and (4.24) can be verified in a similar manner, by computing \( \langle \kappa \nabla (1) t_b^{(1)} + \nabla (2) t_b^{(2)} \psi_k \rangle \), \( \langle \kappa \nabla (1) t_b^{(1)} \psi_k \rangle \), and \( \langle \kappa \nabla (2) t_b^{(2)} \psi_k \rangle \), respectively. \( \Box \)

5. Monodromy preserving deformation

In this section, we explain that our Hamiltonians \( H^{(i)}_p \) \((i = 1, 2, \ p = 1, 2, \ldots, N - 1)\) are obtained assuming a suitable quantization of 1-form \( \omega = d \log \tau \) given in the study of MPD [11].

Let

\[ \frac{\partial Y}{\partial x} = A(x)Y \]

be a system of linear ordinary differential equations, where \( A(x) \) is a rational matrix. In [11] and [10], Jimbo, Miwa and Ueno developed a general theory of MPD. They derived nonlinear deformation equations and proved their complete integrability. They also gave an explicit formula for a 1-form \( \omega \) expressed in terms of the coefficients of \( A(x) \), with the property \( d\omega = 0 \) for each solution of the deformation equations. In [3], Boalch showed that the deformation equations are Hamiltonian systems. However, the explicit formula for the Hamiltonian has not been given. Here, we show that after an appropriate ordering the 1-form \( \omega \) indeed provides the Hamiltonians in the quantum case at least up to Poincaré rank 2. In the following, we give the concrete construction of the Hamiltonians. For that purpose, we recall the procedure of [11].

Let us consider the following system of linear ordinary differential equations for an \( N \times N \) matrix \( Y(z) \) on \( \mathbb{P}^1 \),

\[ \frac{dY}{dz} = A(z)Y, \]  \hspace{1cm} (5.1)

where

\[ A(z) = \sum_{i=1}^{n} \sum_{p=0}^{r_i} \frac{A^{(i)}_p}{(z - z_i)^{p+1}} - \sum_{p=1}^{r} z^{p-1} B_p \]

with \( A^{(i)}_p \) \((i = 1, \ldots, n)\) and \( B_p \) \((p = 1, \ldots, r)\) are \( N \times N \) matrices, and \( r_i, r \) are non-negative integers.

We assume

\[ B_r = \begin{pmatrix} t_1 & \cdots & \cdots & t_N \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ t_N & \cdots & \cdots & t_1 \end{pmatrix} \]

with \( t_j \neq t_i \) for \( i \neq j \). For simplicity, we consider the case where all \( r_i = 0 \) \((i = 1, \ldots, n)\) and \( r \) is a positive integer.
Now, we consider the local solution of (5.1) at $z = \infty$.

**Proposition 5.1 ([11]).** There exists a unique formal series $Y(z)$ at $z = \infty$ of the following form:

$$Y(z) = \hat{Y}(z) e^{\hat{T}(z)}$$

which solves (5.1), i.e.

$$\frac{d}{dz} \hat{Y}(z) = A(z) \hat{Y}(z) - \hat{Y}(z) \frac{d}{dz} T(z).$$

Here, $T(z)$ is a diagonal matrix of the form

$$T(z) = - \sum_{p=1}^{\infty} T_p z^p / p - T_0 \log z,$$

with $T_r = B_r$ and $\hat{Y}(z)$ is a formal power series at $z = \infty$,

$$\hat{Y}(z) = \sum_{p=0}^{\infty} Y_p z^{-p},$$

with $Y_0 = 1$.

The formal power series $\hat{Y}(z) = 1 + \sum_{p=1}^{\infty} Y_p z^{-p}$ is uniquely factorized into

$$\hat{Y}(z) = F(z) D(z),$$

where

$$F(z) = 1 + \sum_{p=1}^{\infty} F_p z^{-p}, \quad D(z) = 1 + \sum_{p=1}^{\infty} D_p z^{-p}.$$

We have that $F(z) - 1$ is diagonal free, and $D(z)$ is diagonal. Substituting (5.6) into (5.3) we obtain

$$\frac{d}{dz} F(z) + F(z) \frac{d}{dz} (\log D(z) + T(z)) = A(z) F(z).$$

(5.7)

Taking the diagonal part, we obtain

$$\frac{d}{dz} (\log D(z) + T(z)) = (A(z) F(z))_D.$$

(5.8)

Then, (5.7) reads

$$\frac{d}{dz} F(z) + F(z) (A(z) F(z))_D = A(z) F(z).$$

(5.9)

Here and below, we denote by $X_D = (\delta_{i,j} X_{jj})_{i,j=1,...,N}$ (resp. $X_{OD} = X - X_D$) the diagonal part (resp. off-diagonal part) of a matrix $X = (X_{i,j})_{i,j=1,...,N}$.

The authors of [11] derived complete integrable nonlinear deformation equations whose deformation parameters are $t^{(k)}_{i,j}$, where

$$T_k = \begin{pmatrix} t^{(k)}_{1,1} & \cdots & t^{(k)}_{1,N} \\ \vdots & \ddots & \vdots \\ t^{(k)}_{N,1} & \cdots & t^{(k)}_{N,N} \end{pmatrix} \quad (k = 1, \ldots, r).$$
They also gave an explicit 1-form \( \omega \) with the property \( d\omega = 0 \) for each solution of the deformation equations. The formula for \( \omega \) reads

\[
\omega = -\text{res}_{z=\infty} \text{tr} \left( \hat{Y}^{-1}(z) \frac{\partial \hat{Y}(z)}{\partial z} d'T(z) \right),
\]

(5.10)

where \( d' \) is the exterior differentiation with respect to the parameters \( t \). The tau function is introduced by \( \omega = d \log \tau \).

For \( k = 1, \ldots, r \), we set

\[
\omega^{(k)} = \sum_{p=1}^{N} H^{(k)}_p \text{dr}^{(k)}_p
\]

and

\[
\omega = \sum_{k=1}^{r} \sum_{p=1}^{N} H^{(k)}_p \text{dr}^{(k)}_p = \sum_{k=1}^{r} \omega^{(k)}.
\]

Set \( T_p = 0 \) for \( p < 0 \), and \( B_{-p} = \sum_{i=1}^{n} A^{(i)}_{-p} \) for \( p \geq 0 \). Rewriting (5.7) in terms of \( F_p, D_p, B_p \) and \( T_p \), we obtain the form of \( \omega \) which are presented by the matrix elements of \( B_p \). We give the explicit form for the case \( r = 2 \). We have

\[
\omega^{(1)} = \sum_{p=1}^{N} \left( \sum_{k \neq p, s \neq p} \frac{(B_1)_{pk}(B_1)_{ks} (B_1)_{sp}}{(t_p^{(2)} - t_k^{(2)})(t_p^{(2)} - t_k^{(2)})^2} - (B_{-1})_{pp} \right.
\]

\[
- \sum_{k \neq p} \frac{(B_1)_{pk} (B_0)_{kp} + (B_0)_{pk} (B_1)_{kp}}{t_p^{(2)} - t_k^{(2)}} + \sum_{k \neq p} \frac{(B_1)_{pk} (B_1)_{kp} + (B_0)_{pk} (B_1)_{kp} + (B_1)_{pk} (B_1)_{kp}}{t_p^{(2)} - t_k^{(2)}}^2 \left) \text{dr}^{(1)}_p, \right.
\]

(5.11)

and

\[
\omega^{(2)} = \frac{1}{2} \sum_{p=1}^{N} \left( - \sum_{k \neq p, s \neq p} \frac{(B_1)_{pk} (B_0)_{ks} (B_1)_{sp}}{(t_p^{(2)} - t_k^{(2)})(t_p^{(2)} - t_k^{(2)})^2} + \sum_{k \neq s} \frac{(B_0)_{pk} (B_1)_{kp}}{t_p^{(2)} - t_k^{(2)}} \right.
\]

\[
- \sum_{k \neq p} \frac{(B_1)_{pk} (B_1)_{ks} (B_1)_{sp}}{(t_p^{(2)} - t_k^{(2)})^2} - \sum_{k \neq s, s' \neq p} \frac{(B_1)_{pk} (B_1)_{ks} (B_0)_{sp}}{(t_p^{(2)} - t_k^{(2)})^2} + \sum_{k \neq s} \frac{(B_0)_{pk} (B_1)_{kp}}{t_p^{(2)} - t_k^{(2)}} \right.
\]

\[
+ \sum_{k \neq p} \frac{(B_1)_{pk} (B_0)_{kp} + (B_0)_{pk} (B_1)_{kp}}{t_p^{(2)} - t_k^{(2)}} + \sum_{k \neq p} \frac{(B_1)_{pk} (B_1)_{kp} + (B_1)_{pk} (B_1)_{kp}}{t_p^{(2)} - t_k^{(2)}}^2 \left) \text{dr}^{(2)}_p. \right.
\]

(5.12)

Here, \( (B_i)_{pq} \) are the \((p, q)\)-entries of the matrix \( B_i \).

In the following, we consider the quantum case with restricting to \( s t_\nu \) for \( r = 2 \).
For \( p = 1, \ldots, N \), we define \( \tilde{H}_p^{(1)} \) (resp. \( \tilde{H}_p^{(2)} \)) in terms of \( H_p^{(1)} \) (resp. \( H_p^{(2)} \)) as follows. First, we put \((B_i)_{pq}\) with \( p \leq q \) to the right and the others to the left. Second, for \( \alpha = \alpha_1 + \cdots + \alpha_q \), we substitute \( e_{\alpha}^{(i)}, \bar{h}^{(i)}_{\alpha}, \bar{e}_{\alpha}^{(i)} \) and \( e_{\alpha}^{(\infty)} \), \( \bar{h}_{\alpha}^{(\infty)} \) and \( \bar{e}_{\alpha}^{(\infty)} \) for the matrix realizations \( \left( A_{\alpha}^{(i)} \right)_{p,q+1}, \left( A_{\alpha}^{(i)} \right)_{q+1,p}, \left( A_{\alpha}^{(i)} \right)_{pp} - \left( A_{\alpha}^{(i)} \right)_{p+1,p}, \left( B_1 \right)_{p,q+1}, \left( B_1 \right)_{q+1,p}, \left( B_1 \right)_{pq} - \left( B_2 \right)_{pq}, \right) \) respectively. Consequently, \( \tilde{H}_p^{(1)} \) and \( \tilde{H}_p^{(2)} \) consist of actions of elements in the algebra \( \mathfrak{g}^{\infty} \) on the modules \( V \otimes \cdots \otimes V \otimes V^{(\infty)} \).

**Proposition 5.2.** For \( 1 \leq p \leq N - 1 \), we have

\[
\mathcal{H}_p^{(1)} = \sum_{j=1}^{p} \left( 1 - \frac{p}{N} \right) \bar{H}_j^{(1)} - \frac{p}{N} \sum_{j=p+1}^{N} \bar{H}_j^{(1)},
\]

\[
\mathcal{H}_p^{(2)} = \sum_{j=1}^{p} \left( 1 - \frac{p}{N} \right) \bar{H}_j^{(2)} - \frac{p}{N} \sum_{j=p+1}^{N} \bar{H}_j^{(2)} + \sum_{\alpha = \alpha_1 + \cdots + \alpha_q} \frac{\mu_{\alpha} - \beta}{\mu_{\alpha}^{\infty}} e_{\alpha}^{(\infty)} \left[ e_{\alpha}^{(\infty)} \right]_{1} e_{\alpha}^{(\infty)} \left[ 1 \right],
\]

where the last term in the second line is called the complementary term.

**Proof.** Using the definitions of \( \tilde{H}_p^{(1)} \) and \( \tilde{H}_p^{(2)} \), we can prove (5.13) and (5.14) by direct computations.

As mentioned in the introduction, our confluent KZ equations may be viewed as a quantization of MPD. We give the explicit correspondence following the process in [9]. Below we use the parameter \( \hbar = 1/\kappa \) in place of \( \kappa \).

Recall that the confluent KZ equations (3.1)–(3.3) are defined from the following data: a collection of Verma modules \( V^{(i)} \) attached to each \( z_i \) \( (i = 1, \ldots, n) \), and a confluent Verma module \( V^{(\infty)} \) at \( \infty \). Let \( \bar{U} \) be an invertible matrix solution to this system. We extend these data by adjoining the natural representation \( C^N \) of \( \mathfrak{sl}_N \) at the point \( z = z_0 \) with Poincaré rank 0. Let \( \tilde{\bar{U}} \) be the matrix solution to the corresponding system (3.1)–(3.3). Let us consider the quantity \( Y(z) = U^{-1} \tilde{\bar{U}} \). The following equations immediately follow from the confluent KZ equations:

\[
\frac{\partial}{\partial z} Y = A(z) Y,
\]

\[
\frac{\partial}{\partial z_i} Y = B^{(i)}_{-1} Y,
\]

\[
\frac{\partial}{\partial \gamma_p} Y = B_{0,p}^{(\infty)} Y,
\]

\[
\frac{\partial}{\partial \mu_p} Y = B_{1,p}^{(\infty)} Y.
\]

Here, we have set

\[
A(z) = \hbar U^{-1} \tilde{\bar{U}} G^{-1}_{-1} \mathcal{O}^{(0)} U
\]

\[
= \hbar U^{-1} \left( \sum_{j=1}^{n} \frac{\Omega^{(i)}_{j}}{z - z_j} - \sum_{p=1}^{N-1} \gamma_{p} w_{p} - \sum_{\alpha \in \Delta} e_{\alpha}^{(\infty)} [1] e_{-\alpha} - \sum_{p=1}^{N-1} \mu_p w_p \right) U,
\]
\[ B_{-1}^{(i)} = \hbar U^{-1} (\bar{G}_{-1}^{(i)} - G_{-1}^{(i)}) U \]
\[ = -\hbar U^{-1} \frac{\Omega_{-1}^{(i)}}{z - z_i} U, \]
\[ B_{1,p}^{(\infty)} = \hbar U^{-1} (-\tilde{H}_p^{(1)} + \tilde{H}_p^{(1)}) U \]
\[ = \hbar U^{-1} \left( -zw_p - \sum_{a \in J_p} \frac{1}{\mu_a} (e_a^{(1)}[1]e_a^{(1)}[1]e_a) \right) U, \]
\[ B_{2,p}^{(\infty)} = \hbar U^{-1} (-\tilde{H}_p^{(2)} + \tilde{H}_p^{(2)}) U \]
\[ = \frac{\hbar}{2} U^{-1} \left( -z^2 w_p - \sum_{a \in J_p} \frac{z}{\mu_a} (e_a^{(2)}[1]e_a^{(2)}[1]e_a) + \sum_{a \in J_p} \frac{1}{\mu_a} (e_a^{(2)}[1]e_a^{(2)}[1]e_a) \right) U. \]

The integrability condition for (5.15)–(5.18) gives rise to a system of nonlinear differential equations with respect to the ‘time’ variables \( z \) and \( \gamma, \mu \). These are the quantization of the irregular Schlesinger equations.

**Lemma 5.3.** The quantized Schlesinger equations given above are Hamiltonian equations with the time-dependent Hamiltonians
\[ H_{-1}^{(i)} = \hbar U^{-1} G_{-1}^{(i)} U \]
for \( i = 1, \ldots, n \), and
\[ H_{1,p}^{(\infty)} = \hbar U^{-1} H_p^{(1)} U, \]
\[ H_{2,p}^{(\infty)} = \hbar U^{-1} H_p^{(2)} U \]
for \( p = 1, \ldots, N - 1 \).

**Proof.** The proofs follow from the integrability conditions of (5.15)–(5.18) and the integrability condition of the confluent equations (3.1)–(3.3).

We consider the case \( \lambda = i \) (\( i = 1, \ldots, n \)). The integrability condition of the confluent equations (3.1)–(3.3) gives \( [\tilde{G}_{-1}^{(i)}, \tilde{G}_{-1}^{(i)}] = 0 \); thus, we obtain \( [\tilde{G}_{-1}^{(i)}, \tilde{G}_{-1}^{(i)} - G_{-1}^{(i)}] = [\tilde{G}_{-1}^{(i)}, G_{-1}^{(i)}] \). Conjugating with \( U^{-1} \) we obtain
\[ [H_{-1}^{(i)}, A(z)] = -[B_{-1}^{(i)}, A(z)]. \]

The integrability condition of (5.15) and (5.16) implies
\[ 0 = \left[ \frac{\partial}{\partial z} - A(z), \frac{\partial}{\partial z_i} - B_{-1}^{(i)} \right] = \frac{\partial A(z)}{\partial z_i} - \frac{\partial B_{-1}^{(i)}}{\partial z} + [A(z), B_{-1}^{(i)}]. \]

We note that
\[ \frac{\partial B_{-1}^{(i)}}{\partial z} = \frac{\hbar U^{-1} \Omega_{-1}^{(i)} U}{(z - z_i)^2} = D_z A(z), \]
where \( D_z \) means \( \frac{\partial}{\partial z} \) acting only on the time variables, regarding the dynamical variables as constant. In summary, we obtain
\[
\frac{\partial}{\partial z_i} A(z) = D_{z_i} A(z) + \left[ A(z), H^{(i)} \right],
\]
showing that \(H^{(i)}\) is the Hamiltonian for the time variable \(z_i\).

The proofs for \(\gamma_p\) and \(\mu_p\) are similar, and we note that for \(\mu_p\), \(D_{\mu_p} A(z) = \hbar U^{-1} \frac{\partial G(0)}{\partial \mu_p} U\).

\(\square\)

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