ON ASYMPTOTIC STABILITY IN ENERGY SPACE OF GROUND STATES OF NLS IN 2D

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ABSTRACT. We transpose work by K. Yajima and by T. Mizumachi to prove dispersive and smoothing estimates for dispersive solutions of the linearization at a ground state of a Nonlinear Schrödinger equation (NLS) in 2D. As an application we extend to dimension 2D a result on asymptotic stability of ground states of NLS proved in the literature for all dimensions different from 2.

§1 Introduction

We consider even solutions of a NLS

\begin{equation}
    iu_t + \Delta u + \beta(|u|^2)u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^2, \quad u(0, x) = u_0(x).
\end{equation}

We assume:

(H1) \( \beta(0) = 0, \beta \in C^\infty(\mathbb{R}, \mathbb{R}) \);

(H2) there exists a \( p_0 \in (1, \infty) \) such that for every \( k = 0, 1, \)

\[ \left| \frac{d^k}{dv^k} \beta(v^2) \right| \lesssim |v|^{p_0 - k - 1} \quad \text{if} \quad |v| \geq 1; \]

(H3) there exists an open interval \( O \) such that \( \Delta u - \omega u + \beta(u^2)u = 0 \) admits a \( C^1 \)-family of ground states \( \phi_\omega(x) \) for \( \omega \in O \);

(H4) \( \frac{d}{d\omega} \| \phi_\omega \|^2_{L^2(\mathbb{R})} > 0 \) for \( \omega \in O \);

(H5) Let \( L_+ = -\Delta + \omega - \beta(\phi_\omega^2) - 2\beta'(\phi_\omega^2)\phi_\omega^2 \) be the operator whose domain is \( H^2_{rad}(\mathbb{R}^2) \).

We assume that \( L_+ \) has exactly one negative eigenvalue.

By [ShS] the \( \omega \to \phi_\omega \in H^1(\mathbb{R}) \) is \( C^2 \) and by [W1, GSS1-2] (H4-5) yields orbital stability of the ground state \( e^{i\omega t} \phi_\omega(x) \). Here we investigate asymptotic stability.

We need some additional hypotheses.

(H6) For any \( x \in \mathbb{R}, \ u_0(x) = u_0(-x) \). That is, the initial data \( u_0 \) of (1.1) are even.
Consider the Pauli matrices $\sigma_j$ and the linearization $H_\omega$ given by:

\begin{equation}
\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix};
\end{equation}

\begin{equation}
H_\omega = \sigma_3 \left[ -\Delta + \omega - \beta(\phi^2_\omega) - \beta'(\phi^2_\omega)\phi^2_\omega \right] + i\beta'(\phi^2_\omega)\phi^2_\omega \sigma_2.
\end{equation}

Then we assume:

(H7) Let $H_\omega$ be the linearized operator around $e^{it\omega}\phi_\omega$, see (1.2). $H_\omega$ has a positive simple eigenvalue $\lambda(\omega)$ for $\omega \in \mathcal{O}$. There exists an $N \in \mathbb{N}$ such that $N\lambda(\omega) < \omega < (N + 1)\lambda(\omega)$.

(H8) The Fermi Golden Rule (FGR) holds (see Hypothesis 4.2 in Section 4).

(H9) The point spectrum of $H_\omega$ consists of 0 and $\pm \lambda(\omega)$. The points $\pm \omega$ are not resonances.

Then we prove:

**Theorem 1.1.** Let $\omega_0 \in \mathcal{O}$ and $\phi_{\omega_0}(x)$ be a ground state in a family of ground states $\phi_\omega$. Let $u(t,x)$ be a solution to (1.1). Assume (H1)–(H9). In particular assume the (FGR) in Hypothesis 4.2. Then, if (1.1) is generic, there exist an $\epsilon_0 > 0$ and a $C > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ and for any $u_0$ with $\|u_0 - e^{it\omega_0}\phi_{\omega_0}\|_{H^1} < \epsilon$, there exist $\omega_+ \in \mathcal{O}$, $\theta \in C^1(\mathbb{R}; \mathbb{R})$ and $h_\infty \in H^1$ with $\|h_\infty\|_{H^1} + |\omega_+ - \omega_0| \leq C\epsilon$ such that

\[ \lim_{t \to +\infty} \|u(t,\cdot) - e^{it\theta(t)}\phi_{\omega_+} - e^{it\Delta} h_\infty\|_{H^1} = 0. \]

Theorem 1.1 is the two dimensional version of Theorem 1.1 [CM]. The one dimensional version is in [Cu3]. We recall that results of the sort discussed here were pioneered by Soffer & Weinstein [SW1], see also [PW], followed by Buslaev & Perelman [BP1-2], about 15 years ago. In this decade these early works were followed by a number of results [BS,Cu1-2,GNT,M1,CZ,M2,P,RSS,SW1-3,TY1-3,Wd1]. It was heuristically understood that the rate of the leaking of energy from the so called "internal modes" into radiation, is small and decreasing when $N$ increases, producing technical difficulties in the closure of the nonlinear estimates. For this reason prior to Gang Zhou & Sigal [GS], the literature treated only the case when $N = 1$ in (H6). [GS] sheds light for $N > 1$. The results in [GS] deal with all spatial dimensions different from 2 under the so called Fermi Golden Rule (FGR) hypothesis. [CM,Cu3] strengthen [GS] by considering initial data in $H^1$, by showing that the (FGR) hypothesis is a consequence of what looks generic condition, Hypothesis 4.2 below, if (H8) is assumed. [CM] treats also the case when there are many eigenvalues and Hypothesis 4.2 is replaced by a more stringent hypothesis which is a natural generalization of the (FGR) hypothesis in [GS]. The same result with many eigenvalues case can be proved also here and in [Cu3], but we skip for simplicity the proof. We recall that Mizumachi [M1], resp. [M2], extends to dimension 1, resp 2, the results in [GNT] valid for small solitons obtained by by bifurcation from ground
states of a linear equation, while [CZ] extends in 2D the result in [SW2]. [Cu3] 
transposes [M1] to the case of large solitons, with the generalizations contained in 
[CM]. Here we consider the case of dimension 2. Thanks to the work by [M2], it is 
quite clear how to transpose to dimension 2 the higher dimensional arguments in 
[CM]. The nonlinear arguments in [CM] are not sensitive to the dimension except 
for the lack in 2D of the endpoint Stricharz estimate. Mizumachi [M2] shows how 
to replace it with an appropriate smoothing estimate of Kato type. The estimate 
and its proof are suggested by [M2]. In order to complete the proof of Theorem 1.1 
we need some dispersive estimates on the linearization \( H_\omega \) which in spatial dimen-

\section{Linearization, Modulation and Set up}

We will use the following classical result, [We1,GSS1-2], see also [Cu3]:
Theorem 2.1. Suppose that $e^{i\omega t} \phi_\omega(x)$ satisfies (H). Then $\exists \epsilon > 0$ and a $A_0(\omega) > 0$ such that for any $\|u(0,x) - \phi_\omega\|_{H^1} < \epsilon$ we have for the corresponding solution $\inf\{\|u(t,x) - e^{i\gamma t} \phi_\omega(x-x_0)\|_{H^1(x\in \mathbb{R}^2)} : \gamma \in \mathbb{R} \& x_0 \in \mathbb{R}\} < A_0(\omega)\epsilon$.

We can write the ansatz $u(t,x) = e^{i\Theta(t)}(\phi_\omega(t) + r(t,x))$, $\Theta(t) = \int_0^t \omega(s)ds + \gamma(t)$. Inserting the ansatz into the equation we get

$$i r_t = -r_{xx} + \omega(t)r - \beta(\phi^2_\omega(t))r - \beta'(\phi_\omega(t))\phi^2_\omega(t)r$$
$$- \beta'(\phi^2_\omega(t))\phi^2_\omega(t)\vec{r} + \gamma(t)\phi_\omega(t) - i\dot{\omega}(t)\partial_\omega \phi_\omega(t) + \gamma(t)r + O(r^2).$$

We set $iR = (r, \vec{r})$, $i\Phi = (\phi_\omega, \phi)\phi_\omega$ and we rewrite the above equation as

$$(2.1) \quad iR_t = H_\omega R + \sigma_3 \gamma R + \sigma_3 \gamma \Phi - i\dot{\omega}\partial_\omega \Phi + O(R^2).$$

Set $H_0(\omega) = \sigma_3(-d^2/dx^2 + \omega)$ and $V(\omega) = H_\omega - H_0(\omega)$. The essential spectrum is

$$\sigma_e = \sigma_e(H_\omega) = \sigma_e(H_0(\omega)) = (-\infty, -\omega] \cup [\omega, +\infty).$$

0 is an isolated eigenvalue. Given an operator $L$ we set $N_g(L) = \cup_{j \geq 1} \ker(L^j)$. [We2] implies that, if $\{\cdot\}$ means span, $N_g(H^*_\omega) = \{\Phi, \sigma_3 \partial_\omega \Phi\}$. $\lambda(\omega)$ has corresponding real eigenvector $\xi(\omega)$, which can be normalized so that $\langle \xi, \sigma_3 \xi \rangle = 1$. $\sigma_1 \xi(\omega)$ generates $\ker(H_\omega + \lambda(\omega))$. The function $(\omega, x) \in \mathcal{O} \times \mathbb{R} \rightarrow \xi(\omega, x)$ is $C^2$; $|\xi(\omega, x)| < ce^{-a|x|}$ for fixed $c > 0$ and $a > 0$ if $\omega \in K \subset \mathcal{O}$, $K$ compact. $\xi(\omega, x)$ is even in $x$ since by assumption we are restricting ourselves in the category of such functions. We have the $H_\omega$ invariant Jordan block decomposition

$$L^2 = N_g(H_\omega) \oplus (\oplus_{j, \pm} \ker(H_\omega + \lambda(\omega))) \oplus L^2_c(H_\omega) = N_g(H_\omega) \oplus N^\perp_g(H^*_\omega)$$

where we set $L^2_c(H_\omega) = \{N_g(H^*_\omega) \oplus \oplus_{\pm} \ker(H^*_\omega + \lambda(\omega))\}^\perp$. We can impose

$$(2.2) \quad R(t) = (z\xi + \bar{z}\sigma_1 \xi) + f(t) \in \left[\sum_{\pm} \ker(H^*_\omega + \lambda(\omega))\right] \oplus L^2_c(H_\omega(t)).$$

The following claim admits an elementary proof which we skip:

Lemma 2.2. There is a Taylor expansion at $R = 0$ of the nonlinearity $O(R^2)$ in (2.1) with $R_{m,n}(\omega, x)$ and $A_{m,n}(\omega, x)$ real vectors and matrices rapidly decreasing in $x$: $O(R^2) = \sum_{2 \leq m+n \leq 2N+1} R_{m,n}(\omega)z^m\bar{z}^n + \sum_{1 \leq m+n \leq N} z^m\bar{z}^n A_{m,n}(\omega)f + O(f^2 + |z|^{2N+2}).$
In terms of the frame in (2.2) and the expansion in Lemma 2.2, (2.1) becomes

\begin{equation}
if_t = (H_\omega(t) + \sigma_3 \dot{\gamma}) f + \sigma_3 \dot{\gamma} \Phi(\omega) - i \dot{\omega} \partial_\omega \Phi(t) + (z \lambda(\omega) - i \dot{\gamma}) \xi(\omega) - (\ddot{\omega} \lambda(\omega) + i \dot{\gamma}) \sigma_1 \xi(\omega) + \sigma_3 \dot{\gamma} (z \xi + \ddot{z} \sigma_1 \xi) - i \dot{\omega} (z \partial_\omega \xi - \ddot{z} \sigma_1 \partial_\omega \xi) \\
+ \sum_{2 \leq m+n \leq 2N+1} z^m z^n R_{m,n}(\omega) + \sum_{1 \leq m+n \leq N} z^m z^n A_{m,n}(\omega) f + O(f^2) + O_{loc}(|z|^{2N+2})
\end{equation}

(2.3)

where by $O_{loc}$ we mean that the there is a factor $\chi(x)$ rapidly decaying to 0 as $|x| \to \infty$. By taking inner product of the equation with generators of $N_\ast(H_\omega)$ and $\ker(H_\omega - \lambda)$ we obtain modulation and discrete modes equations:

\begin{equation}
\dot{\omega} \frac{d||\phi_\omega||^2_2}{d\omega} = (\sigma_3 \dot{\gamma} (z \xi + \ddot{z} \sigma_1 \xi) - i \dot{\omega} (z \partial_\omega \xi - \ddot{z} \sigma_1 \partial_\omega \xi) + \sum_{m+n=2}^{2N+1} z^m z^n R_{m,n}(\omega) \\
+ (\sigma_3 \dot{\gamma} + i \dot{\omega} \partial_\omega P_c + \sum_{m+n=1}^N z^m z^n A_{m,n}(\omega)) f + O(f^2) + O_{loc}(|z|^{2N+2}), \Phi
\end{equation}

(2.4)

\begin{equation}
\dot{\gamma} \frac{d||\phi_\omega||^2_2}{d\omega} = \langle \text{ same as above } , \sigma_3 \partial_\omega \Phi \rangle \\
i \dot{z} - \lambda(\omega) z = \langle \text{ same as above } , \sigma_3 \xi \rangle.
\end{equation}

§3 Spacetime estimates for $H_\omega$

We need analogues of Lemmas 2.1-3 and Corollary 2.1 in [M2]. We call admissible all pairs $(p, q)$ with $1/p = 1/2 - 1/q$ and $2 \leq q < \infty$. We set $(p', q') = (p/(p - 1), q/(q - 1))$. In the lemmas below we assume that the $H_\omega$ of the form (1.2) for which hypotheses (H3-5), (H7) and (H9) hold.

**Lemma 3.1 (Strichartz estimate).** There exists a positive number $C = C(\omega)$ upper semicontinuous in $\omega$ such that for any $k \in [0, 2]$:

(a) for any $f \in L^2_c(\omega)$ and any admissible all pairs $(p, q)$,

\[ ||e^{-itH_\omega} f||_{L^p_t W^{k,q}_x} \leq C ||f||_{H^k}. \]

(b) for any $g(t, x) \in S(\mathbb{R}^2)$ and any couple of admissible pairs $(p_1, q_1)$ $(p_2, q_2)$ we have

\[ ||\int_0^t e^{-i(t-s)H_\omega} P_c(\omega) g(s, \cdot) ds||_{L^p_t W^{k,q}_x} \leq C ||g||_{L^{p_1}_t W^{k,q_1}_x} ||g||_{L^{p_2}_t W^{k,q_2}_x}. \]

Lemma 3.1 follows immediately from Proposition 1.2 since $W$ and $Z$ intertwine $e^{-itH_\omega} P_c(H_\omega)$ and $e^{-itH_0}$. 

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Lemma 3.2. Let \( s > 1 \). \( \exists C = C(\omega) \) upper semicontinuous in \( \omega \) such that:

(a) for any \( f \in S(\mathbb{R}^2) \),
\[
\|e^{-it H_\omega} P_c(\omega)f\|_{L^2_t L^{2,-s}_x} \leq C\|f\|_{L^2_x};
\]

(b) for any \( g(t, x) \in S(\mathbb{R}^3) \)
\[
\left\| \int_{\mathbb{R}} e^{it H_\omega} P_c(\omega)g(t, \cdot) dt \right\|_{L^2_x} \leq C\|g\|_{L^2_t L^{2,s}_x}.
\]

Notice that (b) follows from (a) by duality.

Lemma 3.3. Let \( s > 1 \). \( \exists C = C(\omega) \) as above such that \( \forall g(t, x) \in S(\mathbb{R}^3) \) and \( t \in \mathbb{R} \):
\[
\left\| \int_0^t e^{-i(t-s) H_\omega} P_c(\omega)g(s, \cdot) ds \right\|_{L^2_t L^{2,-s}_x} \leq C\|g\|_{L^2_t L^{2,s}_x}.
\]

As a corollary from Christ and Kiselev [CK], Lemmas 3.2 and 3.3 imply:

Lemma 3.4. Let \( (p, q) \) be an admissible pair and let \( s > 1 \). \( \exists C = C(\omega) \) as above such that \( \forall g(t, x) \in S(\mathbb{R}^3) \) and \( t \in \mathbb{R} \):
\[
\left\| \int_0^t e^{-i(t-s) H_\omega} P(\omega)g(s, \cdot) ds \right\|_{L^p_t L^q_x} \leq C\|g\|_{L^p_t L^{q,s}_x}.
\]

Lemma 3.5. Consider the diagonal matrices \( E_+ = \text{diag}(1, 0) \) \( E_- = \text{diag}(0, 1) \). Set \( P_+ (\omega) = Z(\omega) E_+ W(\omega) \) with \( Z(\omega) \) and \( W(\omega) \) the wave operators associated to \( H_\omega \). Then we have for \( u \in L^2_c(H_\omega) \)
\[
P_+ (\omega) u = \lim_{\epsilon \to +0} \frac{1}{2\pi i} \lim_{M \to +\infty} \int_0^M \left[ R_{H_\omega}(\lambda + i\epsilon) - R_{H_\omega}(\lambda - i\epsilon) \right] ud\lambda
\]
\[
P_- (\omega) u = \lim_{\epsilon \to +0} \frac{1}{2\pi i} \lim_{M \to +\infty} \int_{-M}^{-\infty} \left[ R_{H_\omega}(\lambda + i\epsilon) - R_{H_\omega}(\lambda - i\epsilon) \right] ud\lambda
\]
and for any \( s_1 \) and \( s_2 \) and for \( C = C(s_1, s_2, \omega) \) upper semicontinuous in \( \omega \), we have
\[
\| (P_+ (\omega) - P_- (\omega) - P_c(\omega)\sigma_3) f \|_{L^{2,s_1}_x} \leq C\|f\|_{L^{2,s_2}_x}.
\]

Proof. Formulas (1) hold with \( P_\pm (\omega) \) replaced by \( E_\pm \) and \( H_\omega \) replaced by \( H_0 \) and for any \( u \in L^2(\mathbb{R}^2) \). Applying \( W(\omega) \) we get (1) for \( H_\omega \). Estimate (2) follows by the proof of inequality (3) in Lemma 5.12 [Cu3] which is valid for all dimensions.
§4 Proof of Theorem 1.1

We restate Theorem 1.1 in a more precise form:

**Theorem 4.1.** Under the assumptions of Theorem 1.1 we can express

\[ u(t, x) = e^{i\Theta(t)}(\phi_\omega(t)(x) + \sum_{j=1}^{2N} p_j(z, \bar{z})A_j(x, \omega(t)) + h(t, x)) \]

with \( p_j(z, \bar{z}) = O(z) \) near 0, with \( \lim_{t \to +\infty} \omega(t) \) convergent, with \( |A_j(x, \omega(t))| \leq Ce^{-a|x|} \) for fixed \( C > 0 \) and \( a > 0 \), \( \lim_{t \to +\infty} z(t) = 0 \), and for fixed \( C > 0 \)

\[ (1) \quad \|z(t)\|_{L^{2N+2}} + \|h(t, x)\|_{L^\infty H^1 L^3 W^{1,6}} < C\epsilon. \]

Furthermore, there exists \( h_\infty \in H^1(\mathbb{R}, \mathbb{C}) \) such that

\[ (2) \quad \lim_{t \to \infty} \|e^{t\int_0^\infty \omega(s)ds} \gamma(t) h(t) - e^{it\Delta} h_\infty\|_{H^1} = 0. \]

The proof of Theorem 4.1 consists in a normal forms expansion and in the closure of some nonlinear estimates. The normal forms expansion is exactly the same of \([CM, Cu3]\), in turn adaptations of \([GS]\).

§4.1 Normal form expansion

We repeat \([CM]\). We pick \( k = 1, 2, \ldots, N \) and set \( f = f_k \) for \( k = 1 \). The other \( f_k \) are defined below. In the ODE’s there will be error terms of the form

\[ E_{ODE}(k) = O(|z|^{2N+2}) + O(z^{N+1} f_k) + O(f_k^2) + O(\beta(|f_k|^2) f_k). \]

In the PDE’s there will be error terms of the form

\[ E_{PDE}(k) = O_{loc}(|z|^{N+2}) + O_{loc}(z f_k) + O_{loc}(f_k^2) + O(\beta(|f_k|^2) f_k). \]

In the right hand sides of the equations (2.3-4) we substitute \( \dot{\gamma} \) and \( \dot{\omega} \) using the modulation equations. We repeat the procedure a sufficient number of times until we can write for \( k = 1 \) and \( f_1 = f \)

\[ i\omega \frac{d\|\phi_\omega\|^2}{d\omega} = \langle \sum_{m+n=1}^{2N+1} z^m \bar{z}^n A^{(k)}_{m,n}(\omega) + \sum_{m+n=1}^{N} z^m \bar{z}^n A^{(k)}_{m,n}(\omega) f_k + E_{ODE}(k), \Phi(\omega) \rangle \]

\[ i\dot{z} - \lambda z = \langle \text{same as above }, \sigma_3 \xi(\omega) \rangle \]

\[ i\partial_t f_k = (H_\omega + \sigma_3 \dot{\gamma}) f_k + E_{PDE}(k) + \sum_{k+1 \leq m+n \leq N+1} z^m \bar{z}^n R^{(k)}_{m,n}(\omega), \]
with $A_{m,n}^{(k)}$, $R_{m,n}^{(k)}$ and $A_{m,n}^{(k)}(\omega, x)$ real exponentially decreasing to 0 for $|x| \to \infty$ and continuous in $(\omega, x)$. Exploiting $|(m - n)\lambda(\omega)| < \omega$ for $m + n \leq N, m \geq 0, n \geq 0$, we define inductively $f_k$ with $k \leq N$ by

$$f_{k-1} = - \sum_{m+n=k} z^m z^n R_{H,\omega}((m-n)\lambda(\omega)) R_{m,n}^{(k-1)}(\omega) + f_k.$$  

Notice that if $R_{m,n}^{(k-1)}(\omega, x)$ is real exponentially decreasing to 0 for $|x| \to \infty$, the same is true for $R_{H,\omega}((m-n)\lambda(\omega)) R_{m,n}^{(k-1)}(\omega)$ by $|(m-n)\lambda(\omega)| < \omega$. By induction $f_k$ solves the above equation with the above notifications. Now we manipulate the equation for $f_N$. We fix $\omega_1 = \omega(0)$. We write

$$i\partial_t P_c(\omega_1) f_N - \{H_{\omega_1} + (\dot{\gamma} + \omega - \omega_1)(P_+(\omega_1) - P_-(\omega_1))\} P_c(\omega_1) f_N =$$

$$+ P_c(\omega_1) \tilde{E}_{\text{PDE}}(N) + \sum_{m+n=N+1} z^m z^n P_c(\omega_1) R_{m,n}^{(N)}(\omega_1)$$

where we split $P_c(\omega_1) = P_+(\omega_1) + P_-(\omega_1)$ with $P_\pm(\omega_1)$, see Lemma 3.5, where $P_+(\omega_1)$ are the projections in $\sigma_c(H_{\omega_1}) \cap \{\lambda : \pm \lambda \geq \omega_1\}$ and with

$$\tilde{E}_{\text{PDE}}(N) = E_{\text{PDE}}(N) + \sum_{m+n=N+1} z^m z^n \left(R_{m,n}^{(N)}(\omega) - R_{m,n}^{(N)}(\omega_1)\right) + \varphi(t, x) f_N$$

$$\varphi(t, x) := (\dot{\gamma} + \omega - \omega_1) (P_c(\omega_1) \sigma_3 - (P_+(\omega_1) - P_-(\omega_1))) f_N + (V(\omega) - V(\omega_1)) f_N + (\dot{\gamma} + \omega - \omega_1) (P_c(\omega) - P_c(\omega_1)) \sigma_3 f_N.$$ 

By Lemma 3.5 for $C_N(\omega_1)$ upper semicontinuous in $\omega_0$, $\forall N$ we have

$$\|\langle x \rangle^N (P_+(\omega_1) - P_-(\omega_1) - P_c(\omega_1) \sigma_3) f \|_{L_2^2} \leq C_N(\omega_1) \|\langle x \rangle^{-N} f \|_{L_2^2}.$$ 

The term $\varphi(t, x)$ in (4.2) can be treated as a small cutoff function. We write

$$f_N = - \sum_{m+n=N+1} z^m z^n R_{H,\omega_1}((m-n)\lambda(\omega_1) + i0) P_c(\omega_1) R_{m,n}^{(N)}(\omega_1) + f_{N+1}.$$ 

Then

$$i\partial_t P_c(\omega_1) f_{N+1} = \{H_{\omega_1} + (\dot{\gamma} + \omega - \omega_1)(P_+(\omega_1) - P_-(\omega_1))\} P_c(\omega_1) f_N +$$

$$+ \sum_{\pm} O(\varepsilon|z|^{N+1}) R_{H,\omega_1}(\pm(N+1)\lambda(\omega_1) + i0) R_{\pm}(\omega_1) + P_c(\omega_1) \tilde{E}_{\text{PDE}}(N)$$

with $R_+ = R_{N+1,0}^{(N)}$ and $R_- = R_{0,N+1}^{(N)}$ and $\tilde{E}_{\text{PDE}}(N) = \tilde{E}_{\text{PDE}}(N) + O_{\text{loc}}(\varepsilon z^{N+1})$, where we have used that $(\omega - \omega_0) = O(\varepsilon)$ by Theorem 2.1. Notice that $R_{H,\omega_0}(\pm(N+1)\lambda(\omega_1) + i0) R_{\pm}(\omega_1)$
\(1) \lambda(\omega_0) + i0 R_\pm(\omega_0) \in L^\infty \) do not decay spatially. In the ODE’s with \(k = N\), by the standard theory of normal forms and following the idea in Proposition 4.1 [BS], see [CM] for details, it is possible to introduce new unknowns

\[
\begin{align*}
\tilde{\omega} &= \omega + q(\omega, z, \bar{z}) + \sum_{1 \leq m+n \leq N} z^m \bar{z}^n \langle f_N, \alpha_{mn}(\omega) \rangle, \\
\tilde{z} &= z + p(\omega, z, \bar{z}) + \sum_{1 \leq m+n \leq N} z^m \bar{z}^n \langle f_N, \beta_{mn}(\omega) \rangle,
\end{align*}
\]

(4.6)

with \(p(\omega, z, \bar{z}) = \sum p_{m,n}(\omega) z^m \bar{z}^n\) and \(q(z, \bar{z}) = \sum q_{m,n}(\omega) z^m \bar{z}^n\) polynomials in \((z, \bar{z})\) with real coefficients and \(O(|z|^2)\) near 0, such that we get

\[
\begin{align*}
i \tilde{\omega} &= \langle E_{PDE}(N), \Phi \rangle, \\
i \tilde{z} - \lambda(\omega) \tilde{z} &= \sum_{1 \leq m \leq N} a_m(\omega) |\tilde{z}|^2 \tilde{z} + \langle E_{ODE}(N), \sigma_3 \xi \rangle + \\
&+ \bar{z}^N \langle A_{0,N}(\omega) f_N, \sigma_3 \xi \rangle.
\end{align*}
\]

(4.7)

with \(a_m(\omega)\) real. Next step is to substitute \(f_N\) using (4.4). After eliminating by a new change of variables \(\tilde{z} = \hat{z} + p(\omega, \hat{z}, \bar{\hat{z}})\) the resonant terms, with \(p(\omega, \hat{z}, \bar{\hat{z}}) = \sum \hat{p}_{m,n}(\omega) z^m \bar{z}^n\) a polynomial in \((z, \bar{z})\) with real coefficients \(O(|z|^2)\) near 0, we get

\[
\begin{align*}
i \hat{\omega} &= \langle E_{PDE}(N), \Phi \rangle, \\
i \hat{z} - \lambda(\omega) \hat{z} &= \sum_{1 \leq m \leq N} \hat{a}_m(\omega) |\tilde{z}|^2 \hat{z} + \langle E_{ODE}(N), \sigma_3 \xi \rangle - \\
&- |\tilde{z}|^2 \hat{z} \langle \hat{A}_{0,N}(\omega) R_{H\omega}(N + 1) \lambda(\omega_1) + i0 \rangle P_c(\omega_0) R_{N+1,0}^{(N)}(\omega_1) \sigma_3 \xi) + \\
&+ \bar{z}^N \langle \hat{A}_{0,N}(\omega) f_{N+1}, \sigma_3 \xi \rangle
\end{align*}
\]

(4.8)

with \(\hat{a}_m, \hat{A}_{0,N}\) and \(R_{N+1,0}\) real. By \(\frac{1}{z-i0} = PV \frac{1}{z} + i \pi \delta_0(x)\) and by an elementary use of the wave operators, we can denote by \(\Gamma(\omega, \omega_0)\) the quantity

\[
\begin{align*}
\Gamma(\omega, \omega_1) &= \Im \left( \langle \hat{A}_{0,N}(\omega) R_{H\omega_1}(N + 1) \lambda(\omega_1) + i0 \rangle P_c(\omega_1) R_{N+1,0}^{(N)}(\omega_1) \sigma_3 \xi(\omega) \rangle \right) \\
&= \pi \langle \hat{A}_{0,N}(\omega) \delta(H\omega_1 - (N + 1) \lambda(\omega_1)) P_c(\omega_1) R_{N+1,0}^{(N)}(\omega_1) \sigma_3 \xi(\omega) \rangle.
\end{align*}
\]

Now we assume the following:

**Hypothesis 4.2.** There is a fixed constant \(\Gamma > 0\) such that \(|\Gamma(\omega, \omega)| > \Gamma|\). By continuity and by Hypothesis 4.2 we can assume \(|\Gamma(\omega, \omega_1)| > \Gamma/2\). Then we write
\[
\frac{d}{dt} |\hat{z}|^2 = -\Gamma(\omega, \omega_1)|z|^{2N+2} + \Im \left( \langle \tilde{A}_{0,N}(\omega)f_{N+1}, \sigma_3 \xi(\omega) \rangle \right)^{N+1} + \Im \left( \langle E_{ODE}(N), \sigma_3 \xi(\omega) \rangle \right).
\]

\[\tag{4.9}\]

\section{4.2 Nonlinear estimates}

By an elementary continuation argument, the following a priori estimates imply inequality (1) in Theorem 4.1, so to prove (1) we focus on:

\textbf{Lemma 4.3.} There are fixed constants \(C_0\) and \(C_1\) and \(\epsilon_0 > 0\) such that for any \(0 < \epsilon \leq \epsilon_0\) if we have

\[\tag{4.10} \|\hat{z}\|_{L^2_{t}} \leq 2C_0 \epsilon \quad \text{and} \quad \|f_N\|_{L^\infty_t H^1_x \cap L^2_t W^{1,6}_x \cap L^{2p_0-1}_t W^{1,2p_0}_x \cap L^2_t H^{1-s}_x} \leq 2C_1 \epsilon\]

then we obtain the improved inequalities

\[\|f_N\|_{L^\infty_t H^1_x \cap L^2_t W^{1,6}_x \cap L^{2p_0-1}_t W^{1,2p_0}_x \cap L^2_t H^{1-s}_x} \leq C_1 \epsilon, \tag{4.11}\]

\[\|\hat{z}\|_{L^2_{t}} \leq C_0 \epsilon. \tag{4.12}\]

\textit{Proof.} Set \(\ell(t) := \gamma + \omega - \omega_1\). First of all, we have:

\textbf{Lemma 4.4.} Let \(g(0, x) \in H^1_x \cap L^2_x(\omega_1)\) and let \(\omega(t)\) be a continuous function. Consider \(i g_t = \{H_{\omega_1} + \ell(t)(P_+(\omega_0) - P_-(\omega_0))\} g + P_c(\omega_1)F\). Then for a fixed \(C = C(\omega_1, s)\) upper semicontinuous in \(\omega_1\) and \(s > 1\) we have

\[\|g\|_{L^\infty_t H^1_x \cap L^2_t W^{1,6}_x \cap L^{2p_0-1}_t W^{1,2p_0}_x \cap L^2_t H^{1-s}_x} \leq C(\|g(0, x)\|_{H^1_x} + \|F\|_{L^1_t H^1_x + L^2_t H^{1-s}_x}). \]

Lemma 4.4 follows easily from Lemmas 3.1-4 and \(P_\pm(\omega_1)g(t) = \)

\[e^{-it\omega_1} e^{-i \int_0^t \ell(\tau)d\tau} P_\pm(\omega_1)g(0) - i \int_0^t e^{-i(t-s)\omega_1} e^{i \int_s^t \ell(\tau)d\tau} P_\pm(\omega_1)F(s) \, ds\]

\textbf{Lemma 4.5.} Consider equation (4.1) for \(f_N\) and assume (4.10). Then we can split \(\tilde{E}_{PDE}(N) = X + O(f^3_N) + O(f^{p_0}_N)\) such that \(\|X\|_{L^2_t H^1_x} \leq \epsilon^2\) for any fixed \(M\) and \(\|O(f^3_N) + O(f^{p_0}_N)\|_{L^1_t H^1_x} \leq \epsilon^3\).

\textit{Proof of Lemma 4.5.} In the error terms for \(k = N\) at the beginning of §4.1 we can write \(\tilde{E}_{PDE}(N) = \)

\[O(\epsilon)\psi(x)f_N + O_{loc}(\|z\|^{N+2} + O_{loc}(z f_N) + O_{loc}(f^3_N) + O(f^{p_0}_N).\]

\[10\]
with $\psi(x)$ a rapidly decreasing function, $p_0$ the exponent in (H2) and with $O(f_{N}^{p_0})$ relevant only for $p_0 > 3$. Denoting $X$ the sum of all terms except the last one, setting $f = f_N$, by (4.10) we have:

1. $\|O(\epsilon)\psi(x)f\|_{L_t^2H_x^{1, \alpha}} \lesssim \epsilon \|f\|_{L_t^2H_x^{1, -\alpha}} \lesssim \epsilon^2$;

2. $\|O_{loc}(zf)\|_{L_t^2H_x^{1, \alpha}} \lesssim \|z\|\|f\|_{L_t^2H_x^{1, -\alpha}} \lesssim \epsilon^2$;

3. $\|O_{loc}(f^2)\|_{L_t^2H_x^{1, \alpha}} \lesssim \|f\|^2_{L_t^2H_x^{1, -\alpha}} \lesssim \epsilon^2$.

This yields $\|\langle x \rangle^M f\|_{H_x^1L_x^2} \lesssim \epsilon^2$. To bound the remaining term observe:

4. $\|f\|^2_{L_t^1H_x^1} \lesssim \|f\|_{W_x^{1,6}} \|f\|_{L_t^1} \leq \|f\|^3_{L_t^2W_x^{1,6}} \lesssim \epsilon^3$;

5. $\|O(f^{p_0})\|_{L_t^1H_x^{1}} \lesssim \|f\|_{W_x^{1,2p_0}} \|f\|_{L_t^1} \leq \|f\|_{L_t^2W_x^{1,2p_0}} \|f\|_{L_t^2} \lesssim \epsilon^{p_0}$, where in the last step we use $\|f\|_{L_t^{2p_0}} \|f\|_{L_t^{2p_0}} \lesssim \|f\|^\alpha_{L_t^{2p_0}} \|f\|^{1-\alpha}_{L_t^{2p_0}}$ for some $0 < \alpha < 1$ by $p_0 > 3$, interpolation and Sobolev embedding.

**Proof of (4.11).** Recall that $f_N$ satisfies equation (4.1) whose right hand side is $P_{\epsilon}(\omega_1) \tilde{E}_{PDE}(N) + O_{loc}(z^{N+1})$. In addition to Lemma 4.5 we have the estimate $\|O_{loc}(z^{N+1})\|_{L_t^2H_x^{1, \alpha}} \lesssim \|z\|_{L_t^{N+1}} \lesssim 2C_0\epsilon$. So by Lemmas 3.1-4, for some fixed $c_2$ we get schematically

$$\|f_N\|_{L_t^\infty H_x^{1} \cap L_t^{2p_0} W_x^{1,6} \cap L_t^{2p_0+1} W_x^{1,2p_0}} \leq 2c_2C_0\epsilon + \epsilon + O(\epsilon^2)$$

where $\epsilon$ comes from initial data, $O(\epsilon^2)$ from all the nonlinear terms save for the $R_{m,n}(\omega_0)z^m \bar{z}^n$ terms which contribute the $2c_2C_0\epsilon$. Let now $f_N = g + h$ with

$$ig_t = \{H_{\omega_1} + \ell(t)(P_+(\omega_1) - P_-(\omega_1))\} g + X + O_{loc}(z^{N+1}), \quad g(0) = f_N(0)$$

$$ih_t = \{H_{\omega_1} + \ell(t)(P_+(\omega_1) - P_-(\omega_1))\} h + O(f_N^3) + O(f_N^{p_0}), \quad h(0) = 0$$

in the notation of Lemma 4.5. Then, by Lemmas 3.2 and 3.3 and by the estimates in Lemma 4.5 we get $\|g\|_{L_t^2H_x^{1, -\alpha}} \lesssim 2C_0\epsilon + O(\epsilon^2) + c_0\epsilon$ for a fixed $c_0$. Finally,

$$\int_0^\infty \|e^{-it}e^{i(t-s)}H_{\omega_1} e^{i\ell(t)} e^{i\ell(t)} dt \|_{L_t^2H_x^{1, -\alpha}} \lesssim \int_0^\infty \|O(f_N^3) + O(f_N^{p_0})\|_{H_x^{1}} \lesssim \epsilon^3.$$

So if we set $C_1 \approx 2C_0 + c_0 + 1$ we obtain (4.11). We need to bound $C_0$.

**Proof of (4.12).** We first need:
Lemma 4.6. We can decompose $f_{N+1} = h_1 + h_2 + h_3 + h_4$ with for a fixed large $M > 0$: 

1. $\|h_1\|_{L^2_t L^{2, M}_x} \leq O(\epsilon^2)$;
2. $\|h_2\|_{L^2_t L^{2, M}_x} \leq O(\epsilon^2)$;
3. $\|h_3\|_{L^2_t L^{2, M}_x} \leq O(\epsilon^2)$;
4. $\|h_4\|_{L^2_t L^{2, M}_x} \leq c(\omega_1)\epsilon$ for a fixed $c(\omega_1)$ upper semicontinuous in $\omega_1$.

Proof of Lemma 4.6. We set

$$i\partial_t h_1 = (H_{\omega_1} + \ell(t)(P_+ - P_-)) h_1$$

$$h_1(0) = \sum_{m+n=N+1} R_{H_{\omega_1}} ((m-n)\lambda(\omega_1) + i0) R^{(N)}_{m,n}(\omega_1) z^m(0)\bar{z}^n(0).$$

We get $\|h_1\|_{L^2_t L^{2, -M}_x} \leq c(\omega_1)|z(0)|^2 \sum \|R^{(N)}_{m,n}(\omega_1)\|_{L^2_x} = O(\epsilon^2)$ by the following lemma:

Lemma 4.7. There is a fixed $s_0$ such that for $s > s_0$,

$$\|e^{-iH_\omega t} R_{H_\omega}(\Lambda + i0) P_c(\omega) \varphi\|_{L^2_t L^{2, -s}_x} \leq C_s(\Lambda, \omega) \|\varphi(x)\|_{L^2_x}$$

$$\left\| \int_0^t e^{-iH_\omega (t-\tau)} R_{H_\omega}(\Lambda + i0) P_c(\omega) g(\tau) d\tau \right\|_{L^2_t L^{2, -s}_x} \leq C_s(\Lambda, \omega) \|g(t, x)\|_{L^2_t L^2_x}$$

with $C_s(\Lambda, \omega)$ upper semicontinuous in $\omega$ and in $\Lambda > \omega$.

Let us assume Lemma 4.7 for the moment, for the proof see §9. We set $h_2(0) = 0$ and

$$i\partial_t h_2 = (H_{\omega_1} + \ell(t)(P_+ - P_-)) h_2 +$$

$$+ O(\epsilon^2) R_{H_{\omega_1}} ((N+1)\lambda(\omega_1) + i0) R^{(N)}_{N+1,0}(\omega_0)$$

$$+ O(\epsilon^2) R_{H_{\omega_1}} ((N+1)\lambda(\omega_1) + i0) R^{(N)}_{0,N+1}(\omega_1).$$

Then we have $h_2 = h_{21} + h_{22}$ with $h_{2j} = \sum_{\pm} h_{2j\pm}$ with $h_{21\pm}(t) = \int_0^t e^{-iH_{\omega_1}(t-s)} e^{\pm i\int_s^t \ell(\tau) d\tau} P_\pm O(\epsilon^2) R^{(N)}_{H_{\omega_1}} ((N+1)\lambda(\omega_1) + i0) R^{(N)}_{N+1,0}(\omega_1) ds$

and $h_{22\pm}$ defined similarly but with $R_{H_{\omega_0}} ((N+1)\lambda(\omega_1) + i0) R^{(N)}_{0,N+1}$. Now by (4.13) we get

$$\|h_{2j\pm}(t)\|_{L^2_t L^{2, -M}_x} \leq C\epsilon \|\varphi\|_{L^{N+1}_x}$$

and so $\|h_2(t)\|_{L^2_t L^{2, -M}_x} = O(\epsilon^2)$. Let $h_3(0) = 0$ and
\[ i\partial_t P_c(\omega_1)h_3 = (H_{\omega_1} + \ell(t)(P_+(\omega_1) - P_-(\omega_1))) P_c(\omega_1)h_3 + P_c(\omega_1)\tilde{E}_{PDE}(N). \]

Then by the argument in the proof of (4.11) we get claim (3). Finally let \( h_4(0) = f_N(0) \) and

\[ i\partial_t P_c(\omega_1)h_4 = (H_{\omega_1} + \ell(t)(P_+(\omega_1) - P_-(\omega_1))) P_c(\omega_1)h_4. \]

Then by Lemma 3.2 \( \| \langle x \rangle^{-M} h_4 \|_{L^2_x} \lesssim \| f_N(0) \|_{L^2_x} \leq c(\omega_1)\epsilon \) we get (4).

**Continuation of proof of Lemma 4.3.** We integrate (4.9) in time. Then by Theorem 2.1 and by Lemma 4.4 we get, for \( A_0 \) an upper bound of the constants \( A_0(\omega) \) of Theorem 2.1,

\[ \| \tilde{ z} \|_{L^{2N+2}_t L^{2N+2}_x} \leq A_0 \epsilon^2 + \epsilon \| \tilde{ z} \|_{L^{N+1}_t L^{2N+2}_x} + o(\epsilon^2). \]

Then we can pick \( C_0 = (A_0 + 1) \) and this proves that (4.10) implies (4.12). Furthermore \( \tilde{ z}(t) \rightarrow 0 \) by \( \frac{d}{dt} \tilde{ z}(t) = O(\epsilon) \).

As in [CM,Cu3] in the above argument we did not use the sign of \( \Gamma(\omega,\omega_0) \). With the same argument in [CM,Cu3] one can prove

**Corollary 4.8.** If Hypothesis 4.2 holds, then \( \Gamma(\omega,\omega) > \Gamma \).

The proof that, for \( t f_N(t) = (h(t)\tilde{ h}(t)) \), \( h(t) \) is asymptotically free for \( t \rightarrow \infty \), is similar to the analogous one in [CM] and we skip it.

**§5 Limiting absorption principle and \( L^2 \) theory for \( H_\omega \)**

In sections §5- §7 we prove Proposition 1.2. We start emphasizing two consequences of hypothesis (H9), in particular (b) clarifies the absence of resonance at \( \pm \omega \):

(a) \( H_\omega \) has no eigenvalues in \( [\omega, +\infty) \cup (-\infty, -\omega) \);

(b) if \( g \in W^{2,\infty}(\mathbb{R}^2, \mathbb{C}^2) \) satisfies \( H_\omega g = \omega g \) or \( H_\omega g = -\omega g \) then \( g = 0 \).

Because of the fact that \( H_\omega \) is not a symmetric operator, we need some preparatory work to show that in fact \( H_\omega \) is diagonalizable in the continuous spectrum. This work is done in §5 which ends with a formula for the wave operator \( W \) which is the basis to develop in §6-7 a transposition of the work of Yajima [Y2].

We first need a preliminary on Schrödinger operators. We will denote by \( q(x) \) a real valued function with: \( q(x) \geq 0 \) with \( q(x) > 0 \) at some points; \( q(x) \in C^\infty_0(\mathbb{R}^2) \). We set \( h_q = -\Delta + q(x) \). Then we have:
Lemma 5.1. Let $\mathbb{C}_+ = \{ z \in \mathbb{C} : \Im z > 0 \}$. Suppose $q(x) = 0$ for $r \geq r_0 > 0$. Then we have the following facts.

1. There exists $s_0 > 0$ and $C_0 > 0$ such that for $s \geq s_0$, $R_{h_q}(z)$ extends into a function $z \rightarrow R^+_q(z)$ which is in $(L^\infty \cap C^0)(\mathbb{C}_+, B(L^{2,s}, L^{2,-s}))$.

2. For any $n_0 \in \mathbb{N}$ there exists $s_0 > 0$ such that for any $a_0 > 0$ there is a choice of $C > 0$ such that for $n \leq n_0$

$$\left\| \frac{d^n}{dz^n} R^+_q(z) : L^{2,s}(\mathbb{R}^2) \rightarrow L^{2,-s}(\mathbb{R}^2) \right\| \leq C_0(\gamma)^{\frac{1}{2}(1+n)} \forall z \in \mathbb{C}_+ \cap \{ z : |z| \geq a_0 \}.$$ 

3. The same argument can be repeated for $\mathbb{C}_- = \{ z \in \mathbb{C} : \Im z < 0 \}$ and $R^-_{h_q}(z)$.

Claim (2) follows from [Ag] and [JK] and claim (3) follows along the lines of the previous two claims. In view of (2), it is enough to prove (1) for $z \approx 0$. For $\zeta = re^{i\theta}$ with $\theta \in (-\pi, \pi)$ let $\sqrt{r} = \sqrt{re^{i\theta/2}}$. With this convention for $z \notin [0, \infty)$ for $R_0(z) = (\Delta - z)^{-1}$ we have

$$R_0(z) = \frac{1}{2\pi} K_0(\sqrt{-z}|x|) = \frac{i}{4} H_0^+(i\sqrt{-z}|x|) = -\frac{i}{4} H_0^-(i\sqrt{-z}|x|)$$

for the Macdonald function $K_0$ and the Hankel functions $H_0^\pm$. We set $G_0 = -\frac{1}{2\pi} \log |x|$, $P_0f = \int_{\mathbb{R}^2} f dx$. We have for $M(z) = (1 + \sqrt{q}R_0(z)\sqrt{q})$ the identity

$$R_{h_q}(z) = R_0(z) - R_0(z)\sqrt{q}M^{-1}(z)\sqrt{q}R_0(z).$$

From the expansion at 0 in $\mathbb{C}_+$ of $H_0^+$ and by the argument in Lemma 5 [Sc] we have in $B(L^{2,s}, L^{2,-s})$, for $s$ sufficiently large,

$$R_0(z) = c(z)P_0 - G_0 + O(-z \log \sqrt{-z}) \quad c(z) = \frac{i}{4} - \frac{\gamma}{2\pi} - \frac{1}{2\pi} \log(\sqrt{-z}/2).$$

Consider the projections in $L^2(\mathbb{R}^2)$, $P = \sqrt{\langle \cdot, \sqrt{q} \rangle}/\|q\|_{L^1}$ and $Q = 1 - P$. Let $T = 1 + \sqrt{q}G_0\sqrt{q}$. Then $QTQ$ is invertible in $QL^2(\mathbb{R}^2)$. Denote its inverse in $QL^2(\mathbb{R}^2)$ by $D_0 = (QTQ)^{-1}$. Consider the operator in $L^2 = PL^2 \oplus QL^2$ defined by

$$S = \begin{bmatrix} P & -PTQD_0Q \\ -QD_0QTPTQD_0Q & QD_0QTPTQD_0Q \end{bmatrix}$$

and $h(z) = \|q\|_{L^1} c(z) + \text{trace}(PTP - PTQD_0QTP)$. Then by [Sc]

$$R_{h_q}(z) = R_0(z) - h^{-1}(z)R_0(z)\sqrt{q}S\sqrt{q}R_0(z) - R_0(z)\sqrt{q}QD_0Q\sqrt{q}R_0(z) - R_0(z)\sqrt{q}O(-z \log \sqrt{-z})\sqrt{q}R_0(z).$$
By direct computation

\[
\begin{align*}
   h^{-1}(z)R_0(z)\sqrt{q}S\sqrt{q}R_0(z) &= \frac{c^2(z)}{h(z)}\langle \cdot,1 \rangle \sqrt{q}S\sqrt{q}\langle \cdot,1 \rangle + \frac{c(z)}{h(z)}\langle \cdot,1 \rangle \sqrt{q}S\sqrt{q}G_0 + \\
   &+ \frac{c(z)}{h(z)}G_0\sqrt{q}S\sqrt{q}\langle \cdot,1 \rangle + \frac{c(z)}{h(z)}G_0\sqrt{q}S\sqrt{q}G_0 + O(-z \log \sqrt{-z}),
\end{align*}
\]

where all terms, except the first on the right hand side, admit continuous extension in \( \mathbb{C}_+ \) at 0. We have \( \langle \cdot,1 \rangle \sqrt{q}S\sqrt{q}\langle \cdot,1 \rangle = \|q\|_{L^1}P_0 \) and so by (5)

\[
   R_0(z) = \frac{c^2(z)}{h(z)}\|q\|_{L^1}P_0
\]

admits continuous extension in \( \mathbb{C}_+ \) at 0. By direct computation

\[
   R_0(z)\sqrt{q}QD_0Q\sqrt{q}R_0(z) = G_0\sqrt{q}QD_0Q\sqrt{q}G_0 + O(-z \log \sqrt{-z})
\]

admits continuous extension in \( \mathbb{C}_+ \) at 0. So \( R_{h,q}(z) \) admits continuous extension in \( \mathbb{C}_+ \) at 0, and so on all \( \mathbb{C}_+ \).

A consequence of Lemma 5.1 is the \( h_q \) smoothness in the sense of Kato [Ka] of multiplication operators involving rapidly decreasing functions \( \psi \):

**Lemma 5.2.** Let \( \psi(x) \in L^\infty(\mathbb{R}^2) \cap L^{2,s}(\mathbb{R}^2) \) for \( s \gg 1 \) and \( q \) as in Lemma 5.1. Then the multiplication operator \( \psi \) is \( h_q \) smooth, that is, for a fixed \( C > 0 \)

\[
   \int_{\mathbb{R}} \|\psi R_{h,q}(\lambda+i\varepsilon)u\|_2^2 d\lambda < C\|u\|_2^2 \text{ for all } u \in L^2(\mathbb{R}^2) \text{ and } \varepsilon \neq 0.
\]

This follows from one of the characterizations of \( H \) smoothness in the case \( H \) is selfadjoint, see Theorem 5.1 [Ka], specifically from the fact that by Lemma 5.1 we have that for \( \psi_1, \psi_1 \in L^\infty \cap L^{2,s} \) for \( s \gg 1 \) there is a number \( C > 0 \) such that for all \( z \not\in \mathbb{R} \) we have \( \|\psi_1R_{h,q}(z)\psi_2\|_{L^2,L^2} < C \).

We consider now \( H_q = \sigma_3(-\Delta + q + \omega) \) and consider our linearization \( H_\omega \). Write

\[
   H_\omega = H_q + (V_\omega - \sigma_3q),
\]

and factorize \( V_\omega - \sigma_3q = B^*A \) with \( A, B \) smooth \( |\partial_x^\beta A(x)| + |\partial_x^\beta B(x)| < Ce^{-|x|} \forall x \), for some \( \alpha, C > 0 \) and for \( |\beta| \leq N_0, N_0 \) sufficiently large. We have \( \sigma_1H_q = -H_q\sigma_1, \sigma_1H_\omega = -H_\omega\sigma_1 \). We choose the factorization \( B^*A \) so that \( \sigma_1B^* = -B^*\sigma_1, \sigma_1A = A\sigma_1 \). By these equalities \( \sigma_1R_{H,q}(z) = -R_{H,q}(-z)\sigma_1 \) and \( \sigma_1R_{H,\omega}(z) = -R_{H,\omega}(-z)\sigma_1 \), so in some of the estimates below it is enough to consider \( z \in \mathbb{C}_{++} \) with \( \mathbb{C}_{++} = \{z : \Im z > 0, \Re z > 0\} \).
Lemma 5.3. For $z \in \overline{C_+}$ the function $R^+_{H_q}(z)$ is well defined and satisfies the following properties:

1. There exists $s_0 > 0$ and $C_0 > 0$ such that for $s \geq s_0$ the function $z \rightarrow R^+_{H_q}(z)$ is in $(L^\infty \cap C^0)(\overline{C_+}, B(L^{2,s}, L^{2,-s}))$.
2. For any $n_0 \in \mathbb{N}$ there exists $s_0 > 0$ such that for any $a_0 > 0$ there is a choice of $C > 0$ such that for $n \leq n_0$ and $x \in C_+ \cap \{z : \text{dist}(z, \pm \omega) \geq a_0\}$,
   \[
   \left\| \frac{d^n}{dz^n} R^+_{H_q}(z) : L^{2,s}(\mathbb{R}^2) \rightarrow L^{2,-s}(\mathbb{R}^2) \right\| \leq C_0(z)^{-\frac{1}{2}(1+n)}.
   \]
3. For any $\psi(x) \in L^\infty(\mathbb{R}^2) \cap L^{2,s}(\mathbb{R}^2)$ for $s \gg 1$ the multiplication operator $\psi$ is $H_q$ smooth, that is, for a fixed $C > 0$ we have
   \[
   \int_{\mathbb{R}} \| \psi R_{H_q}(\lambda + i\varepsilon)u \|_2^2 d\lambda < C \| u \|_2^2 \text{ for all } u \in L^2(\mathbb{R}^2) \text{ and } \varepsilon \neq 0.
   \]
4. Analogous statements hold for $z \in \overline{C_-}$ and the function $R^-_{H_q}(z)$.

Lemma 5.3 is a trivial consequence of Lemmas 5.1-2. The properties in Lemma 5.4 are partially inherited by $H_\omega$. Let $Q^+_{H_q}(z) = AR^+_{H_q}(z)B^*$. Then for $z \in C_+$

Lemma 5.4. Fix an exponentially decreasing bounded function $\psi$. For $z \in C_+$ the function $AR_{H_\omega}(z)\psi$ extends into a function $AR^+_{H_\omega}(z)\psi$ for $z \in C_+ \setminus \sigma_d(H_\omega)$ with the following properties:

1. $\forall a_0 > 0 \exists C_0 > 0$ such that for $X_{a_0} = \overline{C_+} \cap \{z : \text{dist}(z, \sigma_d(H_\omega)) \geq a_0\}$
   \[
   AR^+_{H_\omega}(z)\psi \in (L^\infty \cap C^0)(X_{a_0}, B(L^2, L^2))
   \]
2. For any $n_0 \in \mathbb{N}$ there exists $s_0 > 0$ such that for any $a_0 > 0$ there is a choice of $C > 0$ such that for $n \leq n_0$ and $x \in X_{a_0} \cap \{z : \text{dist}(z, \pm \omega) \geq a_0\}$,
   \[
   \left\| \frac{d^n}{dz^n} AR^+_{H_\omega}(z)\psi : L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2) \right\| \leq C_0(z)^{-\frac{1}{2}(1+n)}.
   \]
3. There is a constant $C > 0$ such that
   \[
   \int_{\mathbb{R}} \| AR_{H_\omega}(\lambda + i\varepsilon)u \|_2^2 d\lambda \leq C \| u \|_2^2 \text{ for all } u \in L^2(H_\omega) \text{ and } \varepsilon \neq 0.
   \]
4. Analogous statements hold for $z \in \overline{C_-}$ and the function $R^-_{H_\omega}(z)$.

Proof. Let us write $Q^+_{H_q}(z) = AR^+_{H_q}(z)B^*$ and for $z \in C_+$

\[
AR_{H_\omega}(z) = (1 + Q^+_{H_q}(z))^{-1} AR_{H_q}(z).
\]

By Lemma 5.3 we have $\lim_{z \rightarrow \infty} \| Q^+_{H_q}(z) \|_{L^2,L^2} = 0$. By analytic Fredholm theory $1+Q^+_{H_q}(z)$ is not invertible only at the $z \in \overline{C_+}$ where $\ker(1+Q^+_{H_q}(z)) \neq 0$. This set has 0 measure in $\mathbb{R}$. By Lemma 2.4 [CPV] if at some $z \neq \pm \omega$ we have $\ker(1+Q^+_{H_q}(z)) \neq 0$, then $z$ is an eigenvalue. By hypothesis there are no eigenvalues in $\sigma_e(H_\omega)$. Hence we get claim (2).
Lemma 5.5. If \( \ker(1 + Q^+_q(\omega)) \neq 0 \) then there exists \( g \in W^{2,\infty}(\mathbb{R}^2) \) with \( g \neq 0 \) such that \( H_\omega g = \omega g \).

Let us assume Lemma 5.5. By hypothesis such \( g \) does not exist. This yields (1). By (5), claim (4) Lemma 5.4 and Neumann expansion we get (4). Next, apply (5) to \( u \in L_c(H_\omega) \). \( AR_{H_\omega}(z)u \) is an analytic function in \( z \) with values in \( L^2(\mathbb{R}^2) \) for \( z \) near any isolated eigenvalue \( z_0 \) of \( H_\omega \) because the natural projection of \( u \) in \( N_g(H_\omega - z_0) \) is 0. Away from isolated eigenvalues of \( H_\omega \), \( (1 + Q^+_q(z))^{-1} \) is uniformly bounded. Hence (3) in Lemma 5.3 implies (3) in Lemma 5.4.

Proof of Lemma 5.5. Let \( 0 \neq \tilde{g} \in \ker(1 + Q^+_q(\omega)) \). Then

\[
B^*\tilde{g} + (V_\omega - q)R_{H_q}(\omega)B^*\tilde{g} = 0.
\]

Set \( g = R_{H_q}(\omega)B^*\tilde{g}. \) Then \( Ag = -\tilde{g} \) and so \( g \neq 0 \). By \( g + R_{H_q}(\omega)(V_\omega - q)g = 0 \) we have \( g \in H^2_{loc}(\mathbb{R}^2) \) and \( H_\omega g = \omega g \). We want now to show that \( g \in L^\infty(\mathbb{R}^2) \), contrary to the hypotheses. We have \( t^s \rightarrow (g_1, g_2) \) with \( g_2 = (\Delta - q - 2\omega)^{-1}(B^*\tilde{g})_2 \), where \( B^*\tilde{g} \in L^2(\mathbb{R}^2) \) for any \( s \), so \( g_2 \in H^2(\mathbb{R}^2) \). We have \( g_1 = R^+_{(h_q)}(0)(B^*\tilde{g})_1 \) with \( g_1 \in L^2(\mathbb{R}^2) \) for sufficiently large \( s \). We split \( L^2 = L^2_{r^s} \oplus (L^2_{r^s})^\perp \) where \( L^2_{r^s} \) are the radial functions and we are considering the standard pairing \( (r, R) = \int_{\mathbb{R}^2} q \cdot s \cdot d\mu \) given by \( \int_{\mathbb{R}^2} f(x)g(x)dx \). We decompose \( g_1 = g_{1r} + g_{1nr} \) with \( g_{1r} \in L^2_{r^s} \) and \( g_{1nr} \in (L^2_{r^s})^\perp \). In \( (L^2_{r^s})^\perp \rightarrow (L^2_{r^s})^\perp \) we have \( R^+_{(h_q)}(0) = G_0 - G_0q(1 + Q\omega q)Q^{-1}G_0 \) with \( Q = 1 - P \), for \( P = P_0q_0, q_0 = c_0^{-1}q, c_0 = \int_{\mathbb{R}^2} q dx, P_0u = \int_{\mathbb{R}^2} u dx \).

Then

\[
g_{1nr} = G_0(B^*\tilde{g})_{1nr} - G_0q(1 + Q\omega q)Q^{-1}G_0(B^*\tilde{g})_{1nr}
\]

and by asymptotic expansion for \( |x| \rightarrow \infty \) we conclude that for some constants

\[
\partial_x^\alpha \left( g_{1nr} - a - \frac{b_1x_1 + b_2x_2}{|x|^2} \right) = O(|x|^{-1-a-\epsilon})
\]

for some \( \epsilon > 0 \). Finally we look at \( g_{1r} \). We can consider solutions \( \phi(r) \) and \( \psi(r) \) of \( h_q u = 0 \) with: \( \phi(0) = 1 \) and \( \phi_\epsilon(0) = 0; \psi(r_0) = 1 \) and \( |\psi(r)| \) bounded for \( r \geq r_0 \), \( \psi(r_0) \approx c \log r \) with \( c \neq 0 \) for \( r \rightarrow 0 \). In terms of these two functions the kernel of \( R^+_{(h_q)}(0) \) in \( L^2(0, \infty), dr \) is

\[
R^+_{(h_q)}(0)(r_1, r_2) = \frac{\phi(r_1)\psi(r_2)}{W(r_2)} \text{ if } r_1 < r_2 \text{ or } \frac{\phi(r_2)\psi(r_1)}{W(r_2)} \text{ if } r_1 > r_2,
\]

with \( W(r) = [\phi(\cdot), \psi(\cdot)](r) = c/r \) for some \( c \neq 0 \). We have \( g_{1r}(r) \)

\[
= c^{-1}\psi(r) \int_{0}^{r} \phi(s)(B^*\tilde{g})_{1r}(s) s ds + c^{-1}\phi(r) \int_{r}^{+\infty} \psi(s)(B^*\tilde{g})_{1r}(s) s ds.
\]
Then for \( r \geq r_0 \), \(|g_{1r}(r)| \leq \)

\[
|e^{-1}\psi(r)| \int_0^r |\phi(t)(B^*\tilde{g})_{1r}(t)| \, dt + |e^{-1}\phi(r)| \int_r^{+\infty} |\psi(t)(B^*\tilde{g})_{1r}(t)| \, dt
\]

\[
\lesssim \| \log(x) \|_{L^2, -s}(\mathbb{R}^2) \| B^*\tilde{g} \|_{L^2, s}(\mathbb{R}^2) + \log(2 + r) \| B^*\tilde{g} \|_{L^2, s}(\{x \in \mathbb{R}^2 : |x| \geq r\}) = O(1).
\]

Then we conclude that we have a nonzero \( g \in H^2_{loc}(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \) such that \( H_\omega g = \omega g \). But this is contrary to the nonresonance hypothesis.

Analogous to Lemma 5.4 is:

**Lemma 5.6.** Fix an exponentially decreasing bounded function \( \psi \). For \( z \in \mathbb{C}_+ \) the function \( BR_{H^*_\omega}(z)\psi \) extends into a function \( BR_{H^*_\omega}(z)\psi \) for \( z \in \mathbb{C}_+ \setminus \sigma_d(H_\omega) \) with the following properties:

1. For any \( a_0 > 0 \) there exists \( C_0 > 0 \) such that \( BR_{H^*_\omega}(z)\psi \in L^\infty(X_{a_0}, B(L^2, L^2)) \) where \( X_{a_0} = \mathbb{C}_+ \cap \{ z : \text{dist}(z, \sigma_d(H_\omega)) \geq a_0 \} \).
2. For any \( n_0 \in \mathbb{N} \) there exists \( s_0 > 0 \) such that for any \( a_0 > 0 \) there is a choice of \( C > 0 \) such that for \( n \leq n_0 \) and \( \forall z \in X_{a_0} \cap \{ z : \text{dist}(z, \pm \omega) \geq a_0 \} \),

\[
\left\| \frac{d^n}{dz^n} BR_{H^*_\omega}(z)\psi : L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2) \right\| \leq C_0(z)^{-\frac{1}{2}(1+n)}.
\]

3. There is a constant \( C > 0 \) such that

\[
\int \| BR_{H^*_\omega}(\lambda + i\epsilon)u \|^2 \, d\lambda \leq C\| u \|^2_2 \text{ for all } u \in L^2_c(H^*_\omega) \text{ and } \epsilon \neq 0.
\]

4. Analogous statements hold for \( z \in \mathbb{C}_- \) and the function \( R_{H^*_\omega}(z) \).

From §2 [Ka] we conclude:

**Lemma 5.7.** There are isomorphisms \( \tilde{W} : L^2 \rightarrow L^2_c(H_\omega) \) and \( \tilde{Z} : L^2_c(H_\omega) \rightarrow L^2 \), inverses of each other, defined as follows: for \( u \in L^2, v \in L^2_c(H^*_\omega) \),

\[
\langle \tilde{W}u, v \rangle = \langle u, v \rangle + \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \langle AR_{H_\omega}(\lambda + i\epsilon)u, BR_{H^*_\omega}(\lambda + i\epsilon)v \rangle d\lambda;
\]

for \( u \in L^2_c(H_\omega), v \in L^2 \),

\[
\langle \tilde{Z}u, v \rangle = \langle u, v \rangle + \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \langle AR_{H_\omega}(\lambda + i\epsilon)u, BR_{H_\omega}(\lambda + i\epsilon)v \rangle d\lambda.
\]
We have $H_\omega \tilde{W} = \tilde{W} H_q$ and $H_q \tilde{Z} = \tilde{Z} H_\omega$, $e^{itH_\omega} \tilde{W} = \tilde{W} e^{itH_q}$ and $e^{itH_q} \tilde{Z} = \tilde{Z} e^{itH_\omega} P_c(H_\omega)$. The operators $\tilde{W}$ and $\tilde{Z}$ depend continuously on $\tilde{A}$ and $\tilde{B}^*$ and can be expressed as

$$\tilde{W} u = \lim_{t \to +\infty} e^{itH_\omega} e^{-itH_q} u \text{ for any } u \in L^2$$

$$\tilde{Z} u = \lim_{t \to +\infty} e^{itH_q} e^{-itH_\omega} u \text{ for any } u \in L^2(H_\omega).$$

In particular we remark:

**Lemma 5.8.** We have for $C(\omega)$ upper semicontinuous in $\omega$ and

$$\|e^{-itH_\omega} g\|_2 \leq C(\omega) \|g\|_2 \text{ for any } g \in L^2_c(H_\omega).$$

Having proved that $e^{-itH_\omega} P_c(H_\omega)$ are bounded in $L^2$, we want to relate $H_\omega$ to $H_0 = \sigma_3(-\Delta + \omega)^*$. Write $H = H_0 + V_\omega$, $V_\omega = B^* A$. We have $\sigma_1 H_0 = -H_0 \sigma_1$, $\sigma_1 H_\omega = -H_\omega \sigma_1$. We choose the factorization of $V_\omega$ so that $\sigma_1 B^* = B^* \sigma_1$, $\sigma_1 A = -A \sigma_1$. By these equalities $\sigma_1 R_{H_0}(z) = -R_{H_0}(-z) \sigma_1$ and $\sigma_1 R_{H_\omega}(z) = -R_{H_\omega}(-z) \sigma_1$. We have the following result about existence and completeness of wave operators:

**Lemma 5.9.** The following limits are well defined:

1. $W u = \lim_{t \to +\infty} e^{itH_\omega} e^{-itH_0} u \text{ for any } u \in L^2$
2. $Z u = \lim_{t \to +\infty} e^{itH_0} e^{-itH_\omega} u \text{ for any } u \in L^2_c(H_\omega)$.

$W(L^2) = L^2_c(H_\omega)$ is an isomorphism with inverse $Z$.

**Proof.** The existence of $P_c(H_\omega) \circ W$ follows from Cook’s method and Lemma 5.8. By an elementary argument $W u \in L^2_c(H_\omega)$ for any $u \in L^2$, so $W = P_c(H_\omega) \circ W$. We have $W = \tilde{W} \circ W_1$ with

$$W_1 u = \lim_{t \to +\infty} e^{itH_q} e^{-itH_0} u \text{ for any } u \in L^2(\mathbb{R}^2)$$

$$\tilde{W} u = \lim_{t \to +\infty} e^{itH_\omega} e^{-itH_q} u \text{ for any } u \in L^2.$$
Lemma 5.10. For \( u \in L^{2,s}(\mathbb{R}^2) \) with \( s > 1/2 \) we have

\[
Wu = u - \frac{1}{2\pi i} \int_{|\lambda| \geq \omega} R_{H_0^-}(\lambda)V_\omega \left[R_{H_0^+}(\lambda) - R_{H_0^-}(\lambda)\right] u d\lambda.
\]

Proof. \( Wu \in L^2(\mathbb{R}^2) \) by Lemma 5.9, but the above formula is meaningful in the larger space \( L^{2,-s}(\mathbb{R}^2) \). For \( v \in L^{2,s}(\mathbb{R}^2) \cap L^2_c(H_\omega^*) \) and for \( \langle u, v \rangle_2 = \int_{\mathbb{R}^2} u \cdot \bar{v} dx \) the standard \( L^2 \) pairing, we have by Plancherel

\[
\langle Wu, v \rangle_2 = \langle u, v \rangle_2 + \lim_{\epsilon \to 0^+} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \langle AR_{H_0}(\lambda + i\epsilon)u, BR_{H_0^*}(\lambda + i\epsilon)v \rangle_2 dt.
\]

By the orthogonality in \( L^2(\mathbb{R}) \) of boundary values of Hardy functions in \( H^2(\mathbb{C}_+) \) and in \( H^2(\mathbb{C}_-) \) we have for \( \epsilon > 0 \)

\[
\int_{-\infty}^{+\infty} \langle AR_{H_0}(\lambda + i\epsilon)u, BR_{H_0^*}(\lambda + i\epsilon)v \rangle_2 d\lambda =
\int_{-\infty}^{+\infty} \langle A[R_{H_0}(\lambda + i\epsilon) - R_{H_0}(\lambda - i\epsilon)]u, BR_{H_0^*}(\lambda + i\epsilon)v \rangle_2 d\lambda.
\]

By \( u \in L^{2,s}(\mathbb{R}^2) \) and \( v \in L^{2,s}(\mathbb{R}^2) \cap L^2_c(H_\omega^*) \) the limit in the right hand side for \( \epsilon \searrow 0 \) exists and we have

\[
\langle Wu, v \rangle_2 = \langle u, v \rangle_2 + 
\frac{1}{2\pi} \int_{-\infty}^{+\infty} \langle A[R_{H_0}(\lambda + i0) - R_{H_0}(\lambda - i0)]u, BR_{H_0^*}(\lambda + i0)v \rangle_2 d\lambda =
\langle u, v \rangle_2 + 
\frac{1}{2\pi} \int_{|\lambda| \geq \omega} \langle A[R_{H_0}(\lambda + i0) - R_{H_0}(\lambda - i0)]u, BR_{H_0^*}(\lambda + i0)v \rangle_2 d\lambda.
\]

This yields Lemma 5.10. The crucial part of our linear theory is the proof of the following analogue of [Y]:

Lemma 5.11. For any \( p \in (1, \infty) \) the restrictions of \( W \) and \( Z \) to \( L^2 \cap L^p \) extend into operators such that for \( C(\omega) < \infty \) semicontinuous in \( \omega \)

\[
\|W\|_{L^p(\mathbb{R}^2), L^p_c(H_\omega)} + \|Z\|_{L^p_c(H_\omega), L^p(\mathbb{R}^2)} < C(\omega).
\]

In the next two sections we will consider \( W \) only, since the proof for \( Z \) is similar. The argument in the following two sections is a transposition of [Y]. We consider diagonal matrices

\[
E_+ = \text{diag}(1, 0) \text{ and } E_- = \text{diag}(0, 1).
\]
Keeping in mind Lemma 5.10, $\sigma_1 R(z) = -R(-z)\sigma_1$ for $R(z)$ equal to $R_{H_0}(z)$ or to $R_{H_0}(z)$ and $\sigma_1 L_0^2(H_\omega) = L_0^2(H_\omega)$, it is easy to conclude that the $L^p$ boundness of $W$ is equivalent to $L^p$ boundness of

$$Uu := \int_{\lambda \geq \omega} R_{H_\omega}(\lambda)V_\omega \left[ R_{H_0}^+(\lambda) - R_{H_0}^-(\lambda) \right] ud\lambda$$

$$= \int_{\lambda \geq \omega} R_{H_\omega}(\lambda)V_\omega \left[ R_0^+(\lambda) - R_0^-(\lambda) \right] E_+ ud\lambda.$$

As in [Y] we deal separately with high, treated in §6, and low energies, treated in §7. We introduce cut-off functions $\psi_1(x) \in C_0^\infty(\mathbb{R})$, and $\psi_2(x) \in C^\infty(\mathbb{R})$, with $\psi_1(x) + \psi_2(x) = 1$, $\psi_1(-x) = \psi_1(x)$, $\psi_1(x) = 1$ for $|x| \leq C$ and $\psi_1(x) = 0$ or $|x| > 2C$ for some $C > 0$.

§6 $L^p$ BOUNDNESS OF $U$: HIGH ENERGIES

This part is almost the same of the corresponding part in [Y2]. For $\psi_1(x)$ the cutoff function introduced after Lemma 5.11, $\psi_1(H_0)$ is a convolution operator with symbol $\psi_1(|\xi|^2 + \omega)$. Both $\psi_1(H_0)$ and $\psi_2(H_0)$ are bounded operators in $L^p(\mathbb{R}^2)$ for any $p \in [1, \infty]$. In order to estimate the high frequency part (the so called high energy) $U\psi_2(H_0)$, we expand $R_{H_\omega}(\lambda)$ into the sum of few terms of Born series

$$R_{H_\omega}(\lambda) = R_{H_0}^-(\lambda) - R_{H_0}^+(\lambda)V_\omega R_{H_0}^-(\lambda) + R_{H_0}^-(\lambda)V_\omega R_{H_0}^+(\lambda)V_\omega R_{H_\omega}^-(\lambda),$$

getting by Lemma 5.10 the decomposition $U = U_1 + U_2 + U_3$ with

$$U_1 u = -\frac{1}{2\pi i} \int_{\lambda \geq \omega} R_{H_0}^-(\lambda)V_\omega R_0^+(\lambda - \omega)E_+ ud\lambda,$$

$$U_2 u = \frac{1}{2\pi i} \int_{\lambda \geq \omega} R_{H_0}^-(\lambda)V_\omega R_{H_0}^+(\lambda)V_\omega R_0^+(\lambda - \omega)E_+ ud\lambda,$$

$$U_3 u = -\frac{1}{2\pi i} \int_{\lambda \geq \omega} R_{H_0}^-(\lambda)V_\omega R_{H_0}^+(\lambda)V_\omega R_{H_\omega}^+(\lambda)V_\omega R_0^+(\lambda - \omega)E_+ ud\lambda.$$

Lemma 6.1. The operator $U_1\psi_2(H_0)$ is bounded in $L^p(\mathbb{R}^2)$ for all $1 < p < \infty$. Specifically for any $s > 1$ there exists a constant $C_s > 0$ so that for $T = U_1\psi_2(H_0)$

(1) $||Tu||_{L^p} \leq C_s \|\langle x \rangle^s V_\omega\|_{L^2} \|u\|_{L^p}$ for all $u \in L^p(\mathbb{R}^2)$.

Proof. Recall $R_0(z) = (-\Delta - z)^{-1}$ and $R_{H_0}^\pm(z) = \text{diag}(R_0^\pm(z - \omega), -R_0^\pm(z + \omega))$. For $u = (u_1, u_2)$, and for $F$ the Fourier transform, we are reduced to operators of schematic form $F(E_\pm U_1 u)(\xi) =$

$$= \int_{\lambda \geq \omega} d\lambda \int_{\mathbb{R}^2} \frac{1}{|\xi|^2 + \omega \mp \lambda + \imath 0} \widehat{a_1}(\xi - \eta)\delta(\lambda - (|\xi - \eta|^2 + \omega))\widehat{V}(\eta)d\eta,$$

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with \(\hat{V}\) the Fourier transform of the generic component of \(V_\omega\). Then

\[
E_\pm U_1 u = \int_{\mathbb{R}^2} d\eta \, \hat{V}(\eta) \, T_\eta^\pm u_{\eta} \]

where \(u_{\eta}(x) = e^{ix \cdot \eta} u_1(x)\), \(T_\eta^- u_{\eta} = \frac{1}{4\pi} K_0(\sqrt{\frac{\eta^2}{4} + \omega |x|}) \ast u_{\eta}\) and by \([\text{Y1}]\)

\[
T_\eta^+ u_{\eta}(x) = \frac{i}{2\eta} \int_0^\infty e^{it|\eta|(x + t\eta/|\eta|)} dt.
\]

By \([\text{Y2}]\) we have that \(T = E_+ U_1\) satisfies inequality (1) while for \(T = E_- U_1\) we use

\[
\|T_\eta^\pm u\|_{L^p} \leq \frac{1}{4\pi} \left\| K_0(\sqrt{\frac{\eta^2}{4} + \omega |x|}) \right\|_{L^2_x} \|u_{\eta}\|_{L^p} \leq C(\eta)^{-1} \|u_{\eta}\|_{L^p}
\]

and so \(\|E_- U_1 u\|_{L^p} \lesssim \|\hat{V}(\eta)/\langle \eta \rangle\|_{L^1} \|u_{\eta}\|_{L^p}\).

**Lemma 6.2.** The operator \(U_2 \psi_2(H_0)\) is bounded in \(L^p(\mathbb{R}^2)\) for all \(1 < p < \infty\), moreover, there exists a constant \(C_s > 0\) so that for \(T = U_2 \psi_2(H_0)\)

\[
\|T u\|_{L^p} \leq C_s \|\langle x \rangle^s V_\omega\|_{L^2_x} \|u\|_{L^p} \quad \text{for all } u \in L^p(\mathbb{R}^2).
\]

is valid, provided \(s > 1\).

**Proof.** By \([\text{Y1}]\) and with the notation of Lemma 6.1 we are reduced to a combination of operators

\[
I_{\pm, \pm} u = \int_{\mathbb{R}^2} d\eta_1 T_{\eta_1}^\pm \int_{\mathbb{R}^2} d\eta_2 \hat{V}(\eta_1) \hat{V}(\eta_1 - \eta_2) T_{\eta_2}^\pm u_{\eta_1 \eta_2}.
\]

\(T f = I_{\pm, \pm} u\) satisfies inequality (1) by Proposition 2.2 \([\text{Y2}]\). The other cases follow from Lemma 6.1. For example, for \(K(\eta_1, \eta_2) = \hat{V}(\eta_1) \hat{V}(\eta_2 - \eta_1)\) and \(\bar{K}(x, \eta_2) = \int d\eta e^{ix \cdot \eta} K(\eta, \eta_2)\),

\[
\left\| I_{\pm, \pm} u \right\|_{L^p} = \left\| \int_{\mathbb{R}^2} d\eta_2 \int_{\mathbb{R}^2} d\eta_1 K(\eta_1, \eta_2) T_{\eta_1}^- T_{\eta_2}^+ u_{\eta_1 \eta_2} \right\|_{L^p}
\]

\[
\leq \tilde{C}_s \int_{\mathbb{R}^2} d\eta_2 \|\langle x \rangle^s \bar{K}(x, \eta_2)\|_{L^2_x} \|T_{\eta_2}^+ u_{\eta_1 \eta_2}\|_{L^p}
\]

\[
\leq \tilde{C}_s \int_{\mathbb{R}^2} d\eta_2 \|\langle x \rangle^s \bar{K}(x, \eta_2)\|_{L^2_x} (\langle \eta_2 \rangle)^{-1} \|u_{\eta_1}\|_{L^p} C_s \|\langle x \rangle^s V_\omega\|_{L^2_x} \|u_{\eta_1}\|_{L^p}.
\]
Lemma 6.3. Set $T = U_3\psi_2(H_0)$. Then $T$ is bounded in $L^p(\mathbb{R}^2)$ for all $1 \leq p \leq \infty$.

Proof. Schematically

$$E_+ U_3 \psi_2(H_0) u = \int_{k \geq 0} R_0^-(k^2) V F(k^2 + \omega) V \left[ R_0^+(k^2) - R_0^-(k^2) \right] \psi_2(\lambda + \omega) u_1 kdk,$$

with $F(k^2 + \omega) = R^{H_0}_+(k) V R^-(k)$ and $V$ the generic component of $V_\omega$. By (3) Lemma 5.4 for $G_{k,y}^\pm(x) = e^{\pm ik \cdot y} G^\pm(x - y, k)$ with $G^\pm(x, k) = \pm \frac{1}{4} H^\pm_0(k|x|)$ we have the following analogue of inequality (3.5) [Y2]

$$\left| \partial_\lambda^i \langle F(k^2 + \omega) VG^\pm_{k,y}, VG^\pm_{k,x} \rangle \right| \leq \frac{C_j \|\langle x \rangle^\alpha V_\omega \|_\infty^3}{k^3 \sqrt{\langle x \rangle \langle y \rangle}}$$

and by Proposition 3.1 [Y2] this yields the desired result for $T = E_+ U_3 \psi_2(H_0)$. Since (1) continues to hold if we replace $G^\pm_{k,x}$ with $e^{-ik|x|} G_{k,x}$ with $G_{k,x}(y) = G(x - y, k)$, where $G(x, k) = K_0(\sqrt{k^2 + |\omega|^2})$, we get also the desired result for $T = E_- U_3 \psi_2(H_0)$.

§7 $L^p$ BOUNDEDNESS OF $U$: LOW ENERGIES

Set

$$T u := \int_{\lambda \geq \omega} R^{H_0}_{-\omega}(\lambda) V_\omega \left[ R_0^+(\lambda - \omega) - R_0^-(\lambda - \omega) \right] \psi_1(\lambda) E_+ u d\lambda.$$

We want to prove:

Lemma 7.1. For any $p \in (1, \infty)$ the restriction of $T$ on $L^2 \cap L^p$ extends into an operator such that $\|T\|_{L^p(\mathbb{R}^2), L^p(\mathbb{R}^2)} < C(\omega)$ for $C(\omega) < \infty$ semicontinuous in $\omega$.

Let $V_\omega = V = \{V_{\ell j} : \ell, j = 1, 2\}$, $W = \{W_{\ell j} : \ell, j = 1, 2\}$ with $W_{12} = W_{21} = 0$, $W_{22} = 1 \in \mathbb{R}$ and $W_{11}(x) = 1$ for $V_{11}(x) \geq 0$ and $W_{11}(x) = -1$ for $V_{11}(x) < 0$. Set $B^* = \langle x \rangle^{-N}$ for some large $N > 0$, and $A = \{A_{\ell j} : \ell, j = 1, 2\}$ with $A_{11}(x) = |V_{11}(x)|$, $A_{12}(x) = W_{11}(x) V_{12}(x)$ and $A_{2j}(x) = V_{2j}(x)$. Then $W^2 = 1$, $B^* W A = V$.

Let $k > 0$ be such that $k^2 = \lambda - \omega$ and set $M(k) = W + A R^{H_0}_-(\lambda) B^*$. Then

$$R^{H_0}_{-\omega}(\lambda) = R^{H_0}_{-\omega}(\lambda) - R^{H_0}_{-\omega}(\lambda) B^* M^{-1}(k) A R^{H_0}_{-\omega}(\lambda).$$

We have $M(k) = W + c^-(k) P + A G_0 B^* + O(k^2 \log k)$ where: $c^-(k) = a^- + b^- \log k$; $P$ is a projection in $L^2$ defined by

$$P = \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} \frac{\langle , B^*_{11} \rangle}{\|V_{11}\|_{L^1}};$$
Finally, an application of the Theorem VI.22, Chapter VI, in [RS], shows that
yields
\[ T = \text{product of an Hilbert-Schmidt operator with one in } L \]
only if \( \omega \) is not a resonance or an eigenvalue for \( H_\omega \) and in that case \( M^{-1}(k) = g^{-1}(k)(P - PM_0QD_0Q - QD_0QMP_0QM_0P + QD_0Q + O(k^2 \log k)) \)
with \( g(k) = c^- \log k + d^- \) for \( c^- \neq 0 \) and \( D_0 = (QM_0Q)^{-1} \) by [JN]. We claim now that \( QD_0Q - QWQ \) is a Hilbert-Schmidt operator. In fact, following the the argument in Lemma 3 [JY], we get that the operator \( L = P + QM_0Q \) is invertible in \( QL^2 \), and \( D_0 = QL^{-1}Q \). We have
\[ L = W + [A\tilde{G}_0B^* + P + PM_0P - PM_0Q - QM_0P]. \]
Set \( L := W(1 + \tilde{S}) \), the operators \( P, PM_0P, PM_0Q, QM_0P \) are of rank one while \( A\tilde{G}_0B^* \) is a Hilbert-Schmidt operator. From the fact that \( W \) is invertible, we get that also \( (1 + \tilde{S}) \) is invertible. Moreover the identity \( (1 + \tilde{S})^{-1} = 1 - \tilde{S}(1 + \tilde{S})^{-1} \) yields
\[ L^{-1} - W = -\tilde{S}(1 + \tilde{S})^{-1}W, \]
that is the product of an Hilbert-Schmidt operator with one in \( B(L^2(\mathbb{R}^2), L^2(\mathbb{R}^2)) \).
Finally, an application of the Theorem VI.22, Chapter VI, in [RS], shows that \( L^{-1} - W \) is of Hilbert-Schmidt Type.

So we are reduced to the following list of operators:
\[ T_0^+ u := \int_0^\infty R_0^-(k^2)E_+V_\omega E_+ \left[ R_0^+(k^2) - R_0^-(k^2) \right] \psi_1(\lambda)ukdk, \]
and \( T_0^- \) defined as above but with \( R_0^-(k^2)E_+ \) replaced by \( R_0(-k^2 - 2\omega)E_- \) which are bounded in \( L^p \) for \( 1 < p < \infty \) by Lemma 6.1;
\[ T_1^+ u := \int_0^\infty R_0^-(k^2)E_+N(k) \left[ R_0^+(k^2) - R_0^-(k^2) \right] \psi_1(\lambda)E_+ukdk \]
with
\[ \|d^j/dk^j N(k^2 \log k)\|_{L^2,-s,L^2,s} \leq Ck^{2-j}(\log k) \quad j = 0, 1, 2, \quad 0 < k < c \]
which is bounded in \( L^p \) for \( 1 \leq p \leq \infty \) by Proposition 4.1 [Y];
\[ T_2^+ u := \int_0^\infty R_0^-(k^2)E_+B^*(d(k)F + L + W)A \left[ R_0^+(k^2) - R_0^-(k^2) \right] \psi_1(\lambda)E_+ukdk \]
with $F$ a rank 3 operator, $L$ a Hilbert Schmidt operator in $L^2$, and $d(k) = g^{-1}(k)$. There are also operators $T_j$, for $j = 0, 1, 2$, defined as above but with $R_0^-(k^2)E_+$ replaced by $R_0^-(k^2)E_-$ and bounded in $L^p$. So $T_{2,j}^\pm = T_{2,j}^\pm d(\sqrt{-\Delta}) + T_{2,j}^\pm + T_{2,j}^\pm$ with $T_{2,j}^\pm$ for $j = 1, 2, 3$ operators bounded in $L^p$ for $1 < p < \infty$ because of the following statement proved in [Y2] (the + case is exactly that in [Y2], and the - case can be proved following the same argument):

if $K$ is an operator with integral kernel $K(x, y)$ such that for any fixed $s > 1$

$$
\|K\|_s := \int_{\mathbb{R}^2} dy \left( \int_{\mathbb{R}^2} dx (x)^{2s}|K(x, x - y)|^2 \right)^{\frac{1}{2}} < \infty
$$

then the operators

$$
Z^+ u := \int_0^\infty R_0^-(k^2)K \left[ R_0^+(k^2) - R_0^-(k^2) \right] \, u \, dk
$$

$$
Z^- u := \int_0^\infty R_0^-(k^2 + 2\omega)K \left[ R_0^+(k^2) - R_0^-(k^2) \right] \, u \, dk
$$

are bounded in $L^p$ for $1 < p < \infty$ with $\|Z^\pm\|_{L^p, L^p} < C_{s,p}\|K\|_s$.

§8 Proofs of Lemmas 3.2, 3.3 and 3.4

We mimic Mizumachi [M2]. By the limiting absorption principle we have

$$
P_c(\omega)e^{-itH_\omega} f = \frac{1}{2\pi i} \int_{-\infty}^\infty e^{-it\lambda} \chi(\omega)^2 P_c(\omega) [R_{H_\omega}^+(\omega) - R_{H_\omega}^-(\omega)] f d\omega.
$$

We consider a smooth function $\chi(x)$ satisfying $0 \leq \chi(x) \leq 1$ for $x \in \mathbb{R}$, $\chi(x) = 1$ if $x \geq 2$ and $\chi(x) = 0$ if $x \leq 1$. $\chi_M(x)$ is an even function satisfying $\chi_M(x) = \chi(x-M)$ for $x \geq 0$. Let $\tilde{\chi}_M(x) = 1 - \chi_M(x)$. We have:

Lemma 8.1. For any fixed $s > 1$ there exists a positive $C(\omega)$ upper semicontinuous in $\omega$, such that for any $u \in S(\mathbb{R}^2)$ we have

$$
\|R_{H_\omega}^\pm(\lambda)f\|_{L^2(\sigma_c(H_\omega), L^2_{s-\frac{s}{2}})} \leq C\|f\|_{L^2}.
$$

First, we prove Lemma 3.2 assuming Lemma 8.1.

Proof of Lemma 3.2. We split

$$
P_c(\omega)e^{-itH_\omega} f = P_c(\omega)e^{-itH_\omega}\chi_M(H_\omega)f + P_c(\omega)e^{-itH_\omega}\tilde{\chi}_M(H_\omega)f
$$

with

$$
P_c(\omega)\chi_M(H_\omega)e^{-itH_\omega} f = \frac{1}{2\pi i} \int_{-\infty}^\infty e^{-it\lambda} \chi(\omega)^2 P_c(\omega)[R_{H_\omega}^+(\omega) - R_{H_\omega}^-(\omega)] f d\lambda,
$$

$$
P_c(\omega)e^{-itH_\omega}\tilde{\chi}_M(H_\omega)f = \frac{1}{2\pi i} \int_{-\infty}^\infty e^{-it\lambda} \tilde{\chi}(\omega)^2 P_c(\omega)[R_{H_\omega}^+(\omega) - R_{H_\omega}^-(\omega)] f d\lambda.
$$
Integrating by parts, in $S'_x(\mathbb{R}^2)$ for any $t \neq 0$ and $f \in S_x(\mathbb{R}^2)$

$$P_c(\omega)e^{-itH_\omega}f = \frac{(it)^{-j}}{2\pi i} \int_{-\infty}^{\infty} d\lambda e^{-it\lambda} \partial^j_x P_c(\omega)\{(R_{H_\omega}^+(\lambda) - R_{H_\omega}^-(\lambda))\chi_M(\lambda)\}f.$$

Since by (3) Lemma 5.4 for high energies we have

$$\|\partial^j_x P_c(\omega)R_{H_\omega}^\pm(\lambda) : \langle x \rangle^{(j+1)/2+0}L^2 \rightarrow \langle x \rangle^{-(j+1)/2-0}L^2\| \lesssim \langle \lambda \rangle^{-(j+1)/2},$$

the above integral absolutely converges in $\langle x \rangle^{-(j+1)/2-0}L^2$ for $j \geq 2$. Let $g(t, x) \in S(\mathbb{R} \times \mathbb{R}^2)$. By Fubini and integration by parts, $j \geq 2$,

$$\langle \chi_M(\omega)e^{-itH_\omega}P_c(\omega)f, g \rangle_{t,x}$$

$$= \frac{1}{2\pi i} \int_{\mathbb{R}} dt \langle (it)^{-j} \int_{\mathbb{R}} d\lambda e^{-it\lambda} \partial^j_x \langle \chi_M(\lambda)(R_{H_\omega}^+(\lambda) - R_{H_\omega}^-(\lambda))\rangle f, \tilde{g} \rangle_x$$

$$= \frac{1}{2\pi i} \int_{\mathbb{R}} d\lambda \left\{ \partial^j_x \langle \chi_M(\lambda)(R_{H_\omega}^+(\lambda) - R_{H_\omega}^-(\lambda))\rangle P_c(\omega)f, \int_{\mathbb{R}} dt \langle (it)^{-j} \tilde{g}(t)e^{it\lambda} \rangle \right\}_x$$

$$= \frac{1}{\sqrt{2\pi i}} \int_{\mathbb{R}} d\lambda \langle \chi_M(\lambda)(R_{H_\omega}^+(\lambda) - R_{H_\omega}^-(\lambda))P_c(\omega)f, \tilde{g}(\lambda) \rangle_x.$$

Hence, by Fubini and Plancherel, we have

$$\left|\langle \chi_M(\omega)e^{-itH_\omega}P_c(\omega)f, g \rangle_{t,x}\right| \leq (2\pi)^{-1/2} \left\| \chi_M(\lambda)(R_{H_\omega}^+(\lambda) - R_{H_\omega}^-(\lambda)) \right\|_{L^2_x(\sigma_c(\omega);L^{2-s}_x)} \left\| \tilde{g}(\lambda) \right\|_{L^2_{\lambda}L^{2,s}_x}$$

$$= (2\pi)^{-1/2} \left\| \chi_M(\lambda) \right\|_{L^2_x(\sigma_c(\omega);L^{2-s}_x)} \left\| g \right\|_{L^2_{\lambda}L^{2,s}_x},$$

In a similar way we have

$$\left|\langle e^{-itH_\omega}\tilde{\chi}_M(\omega) f, g \rangle_{t,x}\right| \leq (2\pi)^{-1/2} \left\| \tilde{\chi}_M(\omega)(R_{H_\omega}^+(\lambda) - R_{H_\omega}^-(\lambda)) \right\|_{L^2_x(\sigma_c(\omega);L^{2-s}_x)} \left\| g \right\|_{L^2_{\lambda}L^{2,s}_x},$$

therefore we achieve

$$\left|\langle e^{-itH_\omega}P_c(\omega)f, g \rangle_{t,x}\right| \leq (2\pi)^{-1/2} \left\| \chi_M(\lambda) \right\|_{L^2_x(\sigma_c(\omega);L^{2-s}_x)} \left\| g \right\|_{L^2_{\lambda}L^{2,s}_x}$$

$$+ \left\| \tilde{\chi}_M(\lambda)(R_{H_\omega}^+(\lambda) - R_{H_\omega}^-(\lambda)) \right\|_{L^2_x(\sigma_c(\omega);L^{2-s}_x)} \left\| g \right\|_{L^2_{\lambda}L^{2,s}_x},$$

and by Lemma 8.1 this estimate yields Lemma 3.2.
Proof of Lemma 3.3 By Plancherel’s identity and Hölder inequalities we have

\[
\| \int_0^t e^{-i(t-s)H_\omega} P_c(\omega)g(s,\cdot) ds \|_{L^2_x L^p_t} \leq \| R_{H_\omega}^+(\lambda) P_c(\omega) \hat{\chi}_{[0,+,\infty)} \ast \hat{\gamma}(\lambda, x) \|_{L^2_x L^2_\lambda} \leq \left\| R_{H_\omega}^+(\lambda) P_c(\omega) \right\|_{L^2_x L^2_\lambda} \hat{\gamma}(\lambda, x) \|_{L^2_x L^2_\lambda} \|_{L^3_\lambda}.
\]

By Lemma 5.4 \( \sup_{\lambda \geq \omega} \| R_{H_\omega}^+(\lambda) P_c(\omega) \|_{B(L^{2,2},L^{2,-2})} \lesssim \langle \lambda \rangle^{-1/2} \), and so

\[
\sup_{\lambda \in \mathbb{R}} \| R_{H_\omega}^+(\lambda) P_c(\omega) \|_{B(L^{2,2},L^{2,-2})} \| g \|_{L^2_x L^2_\lambda} \leq C \| g \|_{L^2_x L^2_\lambda}.
\]

The above inequalities yields Lemma 3.3.

Proof of Lemma 3.4 Let \((q, r)\) be admissible and let \( T \) be an operator defined by

\[
T g(t) = \int_{\mathbb{R}} ds e^{-i(t-s)H_\omega} P_c(\omega)g(s).
\]

Using Lemmas 3.2 and 3.3 we get \( f := \int_{\mathbb{R}} ds e^{i\lambda H_\omega} P_c(\omega)g(s) \in L^2(\mathbb{R}) \) and that there exists a \( C > 0 \) such that

(1) \[
\| Tg(t) \|_{L^q_t L^p_x} \leq C \| g \|_{L^{2,2}_x L^2_\lambda}.
\]

for every \( g \in S(\mathbb{R} \times \mathbb{R}^2) \). Since \( q > 2 \), it follows from Lemma 3.1 in [SmS] (see also [Bq]) and (1) that

\[
\left\| \int_{s < t} ds e^{-i(t-s)H_\omega} P_c(\omega)g(s) \right\|_{L^q_t L^p_x} \lesssim \| g \|_{L^2_x L^{2,2}_\lambda}.
\]

This yields Lemma 3.4.

To prove Lemma 8.1 observe that it is not restrictive to prove

(8.1) \[
\| R_{H_\omega}^\pm(\lambda) f \|_{L^2_\lambda((\omega,\infty);L^{2,-2}_x)} \leq C \| f \|_{L^2}.
\]

Following the argument in §4 [M2] we need the following:

Lemma 8.2. There exists a positive constant \( C \) such that for \( s > 1 \)

\[
\| R_{H_\omega}^\pm(\lambda) f \|_{L^{2,-2}_x L^2_\lambda(\omega,\infty)} \leq C \| f \|_{L^2}.
\]
Proof. \( E_+ R_{H_0}^\pm (\lambda) \) \( f = R_{H_0}^\pm (\lambda - \omega) E_+ f \) and by Lemma 4.2 \([2]\) we get

\[
(1) \quad \| R_{H_0}^\pm (\lambda) E_+ f \|_{L_x^{2-\sigma} L_\omega^{\infty}(0, \infty)} \leq C \sup_x \| R_{H_0}^\pm (\lambda) E_+ f \|_{L_x^2(0, \infty)} \leq C \| E_+ f \|_{L_x^2}.
\]

We have \( E_- R_{H_0}^\pm (\lambda) f = -R_0(-\omega - \lambda) E_- f = -\frac{-\Delta + \omega - \lambda}{-\Delta + 2\omega + \lambda} R_{H_0}^\pm (\lambda - \omega) E_- f \). So by (1)

\[
\| E_- R_{H_0}^\pm (\lambda) f \|_{L_x^{2-\sigma} L_\omega^{\infty}(\omega, \infty)} \leq \left\| -\frac{-\Delta + \omega - \lambda}{-\Delta + 2\omega + \lambda} \right\|_{L_x^\infty((\omega, \infty), B(L_x^{2-\sigma}, L_x^{2-\sigma}))} \times \| R_{H_0}^\pm (\lambda) E_- f \|_{L_x^{2-\sigma} L_\omega^{\infty}(0, \infty)} \leq C_1 \| R_{H_0}^\pm (\lambda) E_- f \|_{L_x^{2-\sigma} L_\omega^{\infty}(0, \infty)} \leq C_1 C \| E_- f \|_{L_x^2}.
\]

Proof of inequality (8.1). We consider the operator \( h_q = -\Delta + q(x) \) introduced in §5 and \( H_q = \sigma_3 (h_q + \omega) \). We claim that

\[
(1) \quad \| R_{H_q}^\pm (\lambda) f \|_{L_x^2((\omega, \infty), L_x^{2-\sigma})} \leq C \| f \|_{L_x^2}.
\]

Indeed \( E_+ R_{H_q}^\pm (\lambda) f = R_{H_q}^\pm (\lambda - \omega) E_+ f \) and \( \| R_{H_q}^\pm (\lambda) E_+ f \|_{L_x^{2-\sigma} L_\omega^{\infty}(0, \infty)} \leq C \| f \|_{L_x^2} \) by Lemma 4.1 \([2]\). On the other hand \( E_- R_{H_q}^\pm (\lambda) f = \)

\[
= -R_{h_q} (-\lambda - \omega) E_- f = -R_0(-\lambda - \omega) E_- f + R_0(-\lambda - \omega) q R_{h_q} (-\lambda - \omega) E_- f.
\]

The bound for the first term comes from Lemma 8.2 and

\[
\| R_0(-\lambda - \omega) q R_{h_q} (-\lambda - \omega) E_- f \|_{L_x^{2-\sigma} L_\omega^{\infty}} \lesssim \| R_0(-\lambda - \omega) q R_{h_q} (-\lambda - \omega) E_- f \|_{L_x^{\infty} L_x^2}
\]

\[
\lesssim \| q R_{h_q} (-\lambda - \omega) E_- f \|_{L_x^\infty L_x^2} \leq C \| E_- f \|_{L_x^2}.
\]

Armed with inequality (1) we consider the identity

\[
R_{H_\omega}^\pm (\lambda) = (1 + R_{H_q}^\pm (\lambda)(V_\omega - \sigma_3 q))^{-1} R_{H_q}^\pm (\lambda) =
\]

\[
= R_{H_q}^\pm (\lambda) - R_{H_q}^\pm (\lambda)(V_\omega - \sigma_3 q)(1 + R_{H_q}^\pm (\lambda)(V_\omega - \sigma_3 q))^{-1} R_{H_q}^\pm (\lambda).
\]

By (1) it is enough to bound the last term in the last sum. This is bounded by

\[
\| R_{H_q}^\pm (\lambda)(V_\omega - \sigma_3 q)(1 + R_{H_q}^\pm (\lambda)(V_\omega - \sigma_3 q))^{-1} R_{H_q}^\pm (\lambda) f \|_{L_x^2 L_x^{2-\sigma}} \leq
\]

\[
\| R_{H_q}^\pm (\lambda)(V_\omega - \sigma_3 q)(1 + R_{H_q}^\pm (\lambda)(V_\omega - \sigma_3 q))^{-1} \|_{L_x^\infty B(L_x^{2-\sigma}, L_x^{2-\sigma})} \| R_{H_q}^\pm (\lambda) f \|_{L_x^{2-\sigma}} \leq
\]

\[
\| R_{H_q}^\pm (\lambda) \|_{L_x^\infty(B(L_x^{2-\sigma}, L_x^{2-\sigma}))} \| (1 + R_{H_q}^\pm (\lambda)(V_\omega - \sigma_3 q))^{-1} \|_{L_x^\infty B(L_x^{2-\sigma}, L_x^{2-\sigma})} \| f \|_{L_x^2}
\]

\[
\lesssim \| f \|_{L_x^2} \text{ by (1) and by the fact that the above } L_x^\infty(\omega, \infty) \text{ norms are bounded by Lemmas 5.1 and 5.4.}
\]

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Lemma 4.7. We have for \( \varphi(t, x) \) Schwarz functions, for \( t \in [0, \infty) \) and for fixed \( s > 1 \) sufficiently large

\[
\| e^{-iH_\omega t} R_{H_\omega}^+ (\Lambda) P_c(\omega) \varphi \|_{L^2 L^2_{x,-s}} \leq C(\Lambda, \omega) \| \varphi(x) \|_{L^2_{s,0}}
\]

\[
\left\| \int_0^t e^{-iH_\omega (t-\tau)} R_{H_\omega}^+ (\Lambda) P_c(\omega) \varphi(\tau) d\tau \right\|_{L^2_{1} L^2_{x,-s}} \leq C(\Lambda, \omega) \| \varphi(t, x) \|_{L^2_{1} L^2_{x,s}}
\]

with \( C(\Lambda, \omega) \) upper semicontinuous in \( \omega \) and in \( \Lambda > \omega \).

Proof. We consider \( \omega < a/ < a << \Lambda < b < \infty \) and the partition of unity

\[
1 = g + \tilde{g}
\]

with \( g \in C_0^\infty(\mathbb{R}) \) with \( g = 1 \) in \([a, b] \) and \( g = 0 \) in \([a/2, 2b] \). By Lemma 3.2 we get

\[
\| e^{-iH_\omega t} R_{H_\omega}^+ (\Lambda) P_c(\omega) \tilde{g}(H_\omega) \varphi \|_{L^2_{1} L^2_{x,-s}} \leq C(\omega) \| R_{H_\omega}^+ (\Lambda) P_c(\omega) \tilde{g}(H_\omega) \varphi \|_{L^2_{2}}
\]

\[
\leq C(\omega) C_0(a, b, \omega) \| \varphi \|_{L^2_{2}}.
\]

Similarly by the proof of Lemma 3.3, for any \( s > 1 \)

\[
\left\| \int_0^t e^{-i(t-s)H_\omega} R_{H_\omega}^+ (\Lambda) P_c(\omega) \tilde{g}(H_\omega) \varphi(s, \cdot) ds \right\|_{L^2_{2,-s} L^2_{t}} \leq
\]

\[
\leq \| R_{H_\omega}^+ (\Lambda) P_c(\omega) \tilde{g}(H_\omega) \chi_{[0, +\infty)} * \chi(\lambda, x) \|_{L^2_{2,-s} L^2_{\lambda}} \leq
\]

\[
\leq \left\| \| R_{H_\omega}^+ (\Lambda) P_c(\omega) \tilde{g}(H_\omega) \|_{L^2_{2,s}, L^2_{x,-s}} \| \chi_{[0, +\infty)} * \chi(\lambda, x) \|_{L^2_{2,s}} \right\|_{L^2_{\lambda}}
\]

\[
\leq C(s, a, b, \omega) \| \varphi \|_{L^2_{2,s} L^2_{t}}
\]

by \( (\lambda - \Lambda) R_{H_\omega}^+ (\Lambda) = R_{H_\omega}^+ (\Lambda) - R_{H_\omega}^+ (\Lambda), \) Lemma 5.4 and \( |\lambda - \Lambda| \geq a \wedge b \).

We consider now

\[
\langle x \rangle^{-\gamma} g(H_\omega) e^{-iH_\omega t} R_{H_\omega} (\Lambda + i\epsilon) P_c(H_\omega) \langle y \rangle^{-\gamma} =
\]

\[
(9.1_c) e^{-i\lambda t} \langle x \rangle^{-\gamma} \int_t^{+\infty} e^{-i(H_\omega - \lambda - i\epsilon)s} g(H_\omega) P_c(H_\omega) ds \langle y \rangle^{-\gamma}.
\]

We claim the following:

Lemma 9.1. There are functions \( u(x, \xi) \) defined for \( x \in \mathbb{R}^2 \) and for \( |\xi| \in [a/2, 2b] \) with values in \( C^2 \) such that for any \( \chi \in C_0^\infty(a/2, 2b) \) we have (for \( t u \sigma_3 f \) the product row column and \( t u \) the transpose of a column vector)
\[ (9.2) \quad \chi(H_\omega)f(x) = (2\pi)^{-2} \int_{\mathbb{R}^4} u(x, \xi) \overline{\alpha(y, \xi)} \sigma_3 f(y) \chi(|\xi|^2 + \omega) d\xi dy. \]

There are constants \( c_{\alpha\beta} \) such that
\[ (9.3) \quad |\partial_x^\alpha \partial_\xi^\beta u(x, \xi)| \leq c_{\alpha\beta}(x)|\beta| \quad \text{for all } x \in \mathbb{R}^2 \text{ and } |\xi| \in [a/2, 2b]. \]

Let us assume Lemma 9.1. Then we can write the kernel of operator (9.1) as
\[ (9.4) \quad \langle x \rangle^{-\gamma} g(H_\omega)e^{-iH_\omega t} R_{H_\omega}(\Lambda + i\epsilon)\langle y \rangle^{-\gamma} = (\text{constant}) \times \]
\[ \langle x \rangle^{-\gamma} \int_{\mathbb{R}^3} u(x, \xi) e^{-i(\sigma_3(\xi^2 + \omega) - \Lambda - i\epsilon)s} g(\xi^2 + \omega)^i \overline{\omega}(y, \xi) d\xi \langle y \rangle^{-\gamma}. \]

Estimates (9.3) and elementary integration by parts yields
\[ |(9.4)| \leq c \langle x \rangle^{-\gamma+r} \langle y \rangle^{-\gamma+r} s^{-r} e^{-\epsilon t} \quad \text{and so } |(9.1)_{0+}| \leq c \langle x \rangle^{-\gamma+r} \langle y \rangle^{-\gamma+r} (t)^{-r+1}. \]

For \( \gamma > r + 1 \) and \( r \geq 3 \), we obtain
\[ \|e^{-iH_\omega t} R_{H_\omega}^+(\Lambda)g(H_\omega)P_\epsilon(H_\omega)\varphi\|_{L_2(t_1, t_2; \gamma)} \leq C\|\varphi(x)\|_{L^2, \gamma}. \]

Similarly
\[ \left\| \int_0^t e^{-i(t-s)H_\omega} R_{H_\omega}^+(\Lambda)P_\epsilon(\omega)g(H_\omega)\varphi(s, \cdot) ds \right\|_{L_2^2(\gamma)} \leq \]
\[ \leq \left\| \int_0^t (t-s)^{-2} \|\varphi(s, \cdot)\|_{L^2, \gamma} ds \right\|_{L_2^2} \leq C\|\varphi\|_{L_2^2(L^2, \gamma)} \]

We need now to prove Lemma 9.1.

§10 PROOF OF LEMMA 9.1

First of all we explain how to define the \( u(x, \xi) \). We set \( V_\omega = B^* A \) with \( A(x) \) and \( B^*(x) \) rapidly decreasing and continuous. Then we have

**Lemma 10.1.** For any \( \lambda > \omega \) and any \( \xi \in \mathbb{R}^2 \) with \( \lambda = \omega + |\xi|^2 \), in \( L^2(\mathbb{R}^2) \) the system
\[ (1) \quad (1 + AR_{H_0}^+(\lambda)B^*) \bar{u} = A e^{-i\xi \cdot x} \bar{e}_1 \]

admits exactly one solution \( \bar{u}(x, \xi) \in H^2 \) such that for any \( [a, b] \subset (1, \infty) \setminus \sigma_p(H) \) there is a fixed \( C < \infty \) such that for any \( \lambda \in [a, b] \) and any \( \xi \) as above we have
\[ (2) \quad \|\bar{u}(\cdot, \xi)\|_{H^2} \leq C. \]
Proof. \( AR^+_{H_0}(\lambda)B^* \) is compact and \( \ker(1 + AR^+_{H_0}(\lambda)B^*) = \{0\} \) for \( \lambda > \omega \) by [CPV], since in that case \( \lambda \notin \sigma_p(H_\omega) \). By Fredholm alternative we get existence and uniqueness of \( \tilde{u}(x, \xi) \). Regularity theory and continuity of the coefficients of system (1) with respect to \( \xi \) yield (2)

Let now \( t^e_1 = (1, 0) \) and \( G_0(|x|, k) = \text{diag}(\frac{i}{\omega}H_0^+(k|x|), -\frac{1}{\omega}K_0(\sqrt{k^2 + 2\omega}|x|)) \) for \( k > 0 \). We have \( G_0(r, k) = \frac{i\sqrt{2}}{4\sqrt{\pi}kr}e^{ikr}e_1 + O(r^{-\frac{3}{2}}) \) and \( \partial_r G_0(r, k) = -k\frac{\sqrt{2}}{4\sqrt{\pi}r}e^{ikr}e_1 + O(r^{-\frac{3}{2}}) \). We set

\[
 u(x, \xi) = e^{-i\xi \cdot x}e_1 + v(x, \xi) = e^{-i\xi \cdot x}e_1 - R^+_{H_0}(\lambda)B^*\tilde{u}(\cdot, \xi).
\]

Then \( (H_\omega - \lambda)u(x, \xi) = B^*(Ae^{-i\xi \cdot x}e_1 - \tilde{u} - AR^+_{H_0}(\lambda)B^*\tilde{u}) = 0 \). Notice \( B^*\tilde{u} = V_\omega u \)

so \( v(x, \xi) = e^{-ix \cdot \xi}w(x, \xi) \) where \( w(x, \xi) \) is the unique solution in \( L^2_{r^{-1}} \), \( s > 1 \), of the integral equation

\[
 w(x, \xi) = -F(x, \xi) - \int_{\mathbb{R}^2} G_0(|x - z|, |\xi|)e^{i(x - z) \cdot \xi}V_\omega(z)w(z, \xi)dz,
\]

with

\[
 F(x, \xi) = \int_{\mathbb{R}^2} G_0(|x - z|, |\xi|)V_\omega(z)e^{i(x - z) \cdot \xi}e_1dz.
\]

It is elementary to show that, for \( |\xi| \in [a, b] \), then \( |\partial^\alpha_x \partial^\beta_\xi F(x, \xi)| \leq \tilde{c}_{\alpha\beta} |x|^{-1/2} \).

By standard arguments and Lemmas 5.3 and 5.4 we have \( |\partial^\alpha_x \partial^\beta_\xi w(x, \xi)| \leq \tilde{c}_{\alpha\beta} |x|^{-1/2} \).

This yields (9.3). To get (9.2) we follow the presentation in Chapter 9 [Ta]. We denote by \( R^+_{H_\omega}(x, y, k) \) the kernel of \( R^+_{H_\omega}(k^2 + \omega) \). We set

\[
 R^+_{H_\omega}(x, y, k) = G_0(|x - y|, k) + h(x, y, k)
\]

with \( h(\cdot, y, k) = -R^+_{H_0}(k^2 + \omega)V_\omega G_0(| \cdot - y |, k) \). Let \( (r, \Sigma) \) be polar coordinates on the sphere \( S^1 \), then we claim:

**Lemma 10.2.** Let \( k > 0 \). For \( r \to \infty \) we have uniform convergence on compact sets of, with \( u \cdot (1, 0) \) the raw column product between column \( u \) and raw \( (1, 0) \),

\[
 R^+_{H_\omega}(x, r\Sigma, k) = \frac{i\sqrt{2}}{4\sqrt{\pi}kr}e^{ikr}u(x, k\Sigma) \cdot (1, 0) + O(r^{-2})
\]

\[
 \frac{\partial}{\partial r} R^+_{H_\omega}(x, r\Sigma, k) = -\frac{\sqrt{2}}{4\sqrt{\pi}kr}ke^{ikr}u(x, k\Sigma) \cdot (1, 0) + O(r^{-2}),
\]

\[
 R^+_{H_\omega}(r\Sigma, y, k) = \frac{i\sqrt{2}}{4\sqrt{\pi}kr}e^{ikr} \begin{bmatrix} 1 \\ 0 \end{bmatrix} t u(y, k\Sigma)\sigma_3 + O(r^{-2}),
\]

\[
 \frac{\partial}{\partial r} R^+_{H_\omega}(r\Sigma, y, k) = -\frac{\sqrt{2}}{4\sqrt{\pi}kr}ke^{ikr} \begin{bmatrix} 1 \\ 0 \end{bmatrix} t u(y, k\Sigma)\sigma_3 + O(r^{-2}).
\]
For $R_{H,\omega}^-(x, y, k)$ the asymptotic expansion follows from $R_{H,\omega}^+(x, y, k) = R_{H,\omega}^+(x, y, k)$.

We write $R_{H,\omega}^+(x, r\Sigma, k) = G_0(|x - r\Sigma|, k) + h(x, r\Sigma, k)$ with

$$h(x, r\Sigma, k) = -R_{H,\omega}^+(k^2 + \omega)V_\omega G_0(|\cdot - r\Sigma|, k)$$

$$= -R_{H,\omega}^+(k^2 + \omega) \left[ V_\omega(x) \left( \frac{i\sqrt{2}}{4\sqrt{i \pi kr}} e^{ikr} e^{-ik\Sigma \cdot x} \text{diag}(1, 0) + O(r^{-\frac{3}{2}}) \right) \right].$$

We have

$$\|V_\omega(x)G_0(|x - r\Sigma|, k) - V_\omega(x)\frac{i\sqrt{2}}{4\sqrt{i \pi kr}} e^{ikr} e^{-ik\Sigma \cdot x} \text{diag}(1, 0)\|_{L^2, s} = O(r^{-3/2}).$$

From $v(x, \xi) = -R_{H,\omega}^+(k^2 + \omega)V_\omega(x)e^{-ik\Sigma \cdot x}e_1$, with $t^t e_1 = (1, 0)$ we get $v(x, \xi)^t e_1 = -R_{H,\omega}^+(k^2 + \omega)V_\omega(x)e^{-ik\Sigma \cdot x}\text{diag}(1, 0)$. Then we conclude for any $s > 1$

$$\|h(x, r\Sigma, k) - \frac{i\sqrt{2}}{4\sqrt{i \pi kr}} v(x, k\Sigma)^t e_1\|_{L^2, -s} = O(r^{-3/2})$$

and

$$\|R_{H,\omega}^+(x, r\Sigma, k) - \frac{i\sqrt{2}}{4\sqrt{i \pi kr}} u(x, k\Sigma)^t e_1\|_{L^2, -s} = O(r^{-3/2}).$$

Then point wise $h(x, r\Sigma, k + i0) - \frac{i\sqrt{2}}{4\sqrt{i \pi kr}} v(x, k\Sigma)^t e_1 = O(r^{-3/2})$ and

$$R_{H,\omega}^+(x, r\Sigma, k) - \frac{i\sqrt{2}}{4\sqrt{i \pi kr}} u(x, k\Sigma)^t e_1 = O(r^{-3/2}).$$

This yields (1) in Lemma 10.2. (2) can be obtained with a similar argument. (3) and (4) follow from (1) and (2) by

$$\sigma_3 R_{H,\omega}^\pm(x, y, k)\sigma_3 = R_{H,\omega}^\pm(x, y, k) = t R_{H,\omega}^+(y, x, k).$$

By Lemma 3.5 for $v \in L^2(H_\omega) \cap C_0^\infty$ and for $\varphi \in C_0^\infty(\mathbb{R})$ supported in $(\omega, \infty)$ we have

$$\varphi(H_\omega)v(x) = \frac{2}{\pi} \int_0^\infty dk \int_{\mathbb{R}^2} \varphi(k^2 + \omega) \Im R_{H,\omega}^+(x, y, k)v(y)dy.$$

We prove (here $u^t \pi$ is a raw column product between column $u$ and raw $^t\pi$)

$$\Im R_{H,\omega}^+(x, y, k) = \frac{1}{8\pi} \int_{S^1} u(x, k\Sigma)^t \pi(y, k\Sigma)\sigma_3 d\Sigma,$$

(3)
where $d\Sigma$ is the standard measure on $S^1$. By the Green theorem for $S_R = \{ z \in \mathbb{R}^2 : |z| = R \}$, $|x| < R$, $|y| < R$ and $r = |z|$

By the Green theorem for $S_R = \{ z \in \mathbb{R}^2 : |z| = R \}$, $|x| < R$ and $|y| < R$,

$$\Im R^+_{H,\omega}(x, y, k) = \frac{1}{2i} \int_{S_R} I(x, y, z, k) d\ell(z)$$

$$I(x, y, z, k) := R^+_{H,\omega}(x, z, k) \sigma_3 \partial_{|z|} R^-_{H,\omega}(z, y, k) - (\partial_{|z|} R^+_{H,\omega}(x, z, k)) \sigma_3 R^-_{H,\omega}(z, y, k)$$

By Lemma 10.2

$$\left| \Im R^+_{H,\omega}(x, y, k) - \frac{1}{8\pi} \int_{S^1} u(x, k\Sigma) \tilde{\tau}(y, k\Sigma) \sigma_3 d\Sigma \right| =$$

$$= \frac{R}{2i} \int_{S^1} I(x, y, r\Sigma, k)|_{r=R} d\Sigma - \frac{1}{8\pi} \int_{S^1} u(x, k\Sigma) \tilde{\tau}(y, k\Sigma) \sigma_3 d\Sigma \right| \leq O(R^{-\frac{3}{2}}).$$

Therefore, taking $R \to +\infty$, we arrive at (3). Moreover, we obtain

$$\varphi(H_\omega)v(x) = \frac{2}{\pi} \int_0^\infty k \, dk \int_{\mathbb{R}^2} \varphi(k^2 + \omega) \Im G(x, y, k)v(y)dy =$$

$$= \frac{1}{4\pi^2} \int_0^\infty k \, dk \int_{\mathbb{R}^2} \int_{S^1} u(x, k\Sigma) \tilde{\tau}(y, k\Sigma) \sigma_3 v(y) \varphi(k^2 + \omega) d\Sigma dy =$$

$$= (2\pi)^{-2} \int_{\mathbb{R}^4} u(x, \xi) \tilde{\tau}(y, \xi) \sigma_3 v(y) \varphi(|\xi|^2 + \omega) d\xi dy,$$

that is the integral representation (9.2). This completes the proof of Lemma 9.1.

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