Spline wavelets use for output processes analysis of multi-dimensional non-stationary linear control systems

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Abstract. Methods for solving systems of differential equations based on the use of wavelets have become very popular recently. Debesly wavelets, coiflets were used among others. The disadvantage of such wavelets is that they do not have an analytical expression. Therefore, there are great difficulties in integrating and differentiating expressions containing these wavelets. The article presents an algorithm for the numerical solution of a system of differential equations with variable coefficients based on spline-wavelets on a interval. The algorithm presented generalizes the well-known method based on Haar wavelets which are a special case of spline-wavelets. The results of the article are used to analyze the output processes of multi-dimensional non-stationary linear control systems.

1. Introduction

Many mathematical applications require numerical approximation of solutions of differential equations. One of the methods for constructing such an approximation is based on Haar wavelets. A vast number of works have been devoted to the various applications of these wavelets. An efficient numerical method for solution of nonlinear evolution equations based on the Haar wavelets approach is proposed in the article [1]. Haar wavelets techniques for the solution of ordinary differential equations is discussed in articles [2], [4] and in the book [5]. Integral equations are considered in the article [3]. Since Haar wavelets are orthogonal the system of linear equations for finding wavelet coefficients of approximant becomes sparse. Unfortunately the approximants are piecewise constant in solving integral equations, and the smoothness class of the approximant will be one less than the order of the differential equation in solving differential equations. Applying a spline wavelet, the special case of which are Haar wavelets, can remove such shortcomings. One can construct approximants of any smoothness class using a spline wavelet. It should be noted that spline wavelets in the general case are a semiorthogonal system, but as it is shown in the article [6] the system of linear equations for determining the wavelet coefficients of approximation is pseudo-sparse i.e. contains a large number of elements close to zero. This article is devoted to the use of a spline wavelet to the construction of approximants of any smoothness class of solutions to linear systems of differential equations.

2. Problem formulation

Let there be given a linear system of differential equations
\[
\begin{align*}
\frac{dx_1}{dt} &= a_{11}(t)x_1 + a_{12}(t)x_2 + \ldots + a_{1m}(t)x_m + f_1(t); \\
\frac{dx_2}{dt} &= a_{21}(t)x_1 + a_{22}(t)x_2 + \ldots + a_{2m}(t)x_m + f_2(t); \\
\vdots \\
\frac{dx_m}{dt} &= a_{m1}(t)x_1 + a_{m2}(t)x_2 + \ldots + a_{mm}(t)x_m + f_m(t),
\end{align*}
\]

and the initial conditions are given

\[
x_i(t_0) = x_{i0}, \quad i = 1, 2, \ldots, m.
\]

We assume that \( a_{ij}, f_i \in C[t_0; t_1] \) for all \( i, j = 1, 2, \ldots, m \). The approximants of any class of smoothness are required to be constructed \( C^m[t_0; t_1] \) system solution (1), satisfying the initial condition (2).

3. Theoretical description

The approach to the construction of wavelet systems on an interval proposed in the present paper will be briefly considered in this section [7]. Let the net function \( \varphi \) belongs to the real space \( L_2(\mathbb{R}) \) satisfying the equation

\[
\varphi(x) = \sqrt{2} \sum_{k \in \mathbb{Z}} u_k \varphi(2x - k), \quad u_k \in \mathbb{R}
\]

and has a compact support \([a; b]\). Let \( \varphi_{jk}(x) = 2^{j/2} \varphi(2^j x - k), x \in [a; b] \) be defined. Function \( \varphi \) is called scaling in the theory for wavelets, and equation (3) is called scaling relation [8], [9]. It is clear that only a finite number of such functions will be different from zero on an interval \([a; b]\). Let these be functions be \( \varphi_{j,0}, \varphi_{j,1}, \ldots, \varphi_{j,n_j-1} \) for definiteness. If linear spaces

\[
V_j = \text{lin} \{ \varphi_{j,0}, \varphi_{j,1}, \ldots, \varphi_{j,n_j-1} \}, \quad \text{dim} V_j = n_j
\]

are considered then by virtue of equation (3) \( V_0 \subset V_1 \subset \ldots \subset L_2([a; b]) \) will be satisfied. So \( \varphi_{j-1,k} = \sum_{i=0}^{n_{j-1}} p_{i,k} \varphi_{j,i} \). Let us introduce the notation as in [7]

\[
\Phi_j(x) = \left( \varphi_{j,0}(x), \varphi_{j,1}(x), \ldots, \varphi_{j,n_j-1}(x) \right),
\]

\[
P_j = \left( p_{i,k} \right)_{i=0, \ldots, n_{j-1}}^{k=0, \ldots, n_{j-1}}.
\]

In this case \( \Phi_{j-1} = \Phi_j P_j \). Let us denote the orthogonal complement to the space \( V_{j-1} \) in the space \( V_j \) by \( W_{j-1} \). By virtue of the fact that \( V_j = V_{j-1} \oplus W_{j-1} \) and \( W_{j-1} \subset V_j \), then \( W_{j-1} \) is finite-dimensional space

\[
W_j = \text{lin} \{ \psi_{j,0}, \psi_{j,1}, \ldots, \psi_{j,m_j-1} \}, \quad \text{dim} W_j = m_j
\]

and \( \psi_{j-1,k} = \sum_{i=0}^{n_{j-1}} q_{i,k} \psi_{j,i} \). Functions \( \psi_{j,i} \) are called wavelets, and spaces \( W_j \) are called wavelet spaces [8], [9]. We again introduce matrices [7]

\[
\Psi_j(x) = \left( \psi_{j,0}(x), \psi_{j,1}(x), \ldots, \psi_{j,m_j-1}(x) \right),
\]

\[
Q_j = \left( q_{i,k} \right)_{i=0, \ldots, n_{j-1}}^{k=0, \ldots, n_{j-1}}.
\]
In this case \( \Psi_{j,1} = \Phi_j Q_j \). It is necessary to mention that \( n_j + m_j = n_{j+1} \). Let us assume that \( f \in L_2(X) \) and \( \Pi_j : L_2(X) \rightarrow V_j \). In this case
\[
\Pi_j f = \sum_{k=0}^{n_j} c_k \phi_k = \Pi_{j-1} f + \Pi_{j-1}^W f = \sum_{k=0}^{n_{j-1}} c_{j-1,k} \phi_{j-1,k} + \sum_{k=0}^{m_{j-1}} d_{j-1,k} y_{j-1,k}.
\]
This equation can be rewritten in a matrix form if we introduce vectors
\[
C_j = (c_{j,0}, \ldots, c_{j,n_j})^T, \quad D_j = (d_{j,0}, \ldots, d_{j,m_{j-1}})^T.
\]
The first vector governs the approximant of the function \( f \), and the second vector represents wavelet coefficients characterizing excursion \( \Pi_{j-1} f \) from \( \Pi_j f \). As it is shown in [7] there is an equation found \( C_j = P_j C_{j-1} + Q_j D_{j-1} \). This equation allows to restore the approximant \( \Pi_j f \) by more rude approximation \( \Pi_{j-1} f \) and wavelet coefficients.

Since linear operators (projectors) \( V_j \rightarrow V_{j-1}, V_j \rightarrow W_{j-1} \) are defined by some matrices \( A_j, B_j \), then \( C_{j-1} = A_j C_j, D_{j-1} = B_j C_j \).

By the wavelet decomposition of \( f \) function we mean the finding of vectors \( C_0, D_0, D_1, \ldots, D_{j-1} \). Matrices \( Q_j \) and \( P_j \) are known as synthesis filters. Matrices \( A_j \) and \( B_j \) are known as analysis filters. The multitude \( \{A_j, B_j, P_j, Q_j\} \) is known as filterbank.

As it is shown in paper [7] there is a following link between matrices \( A_j, B_j \) and \( P_j, Q_j \)
\[
\begin{bmatrix} A_j \\ B_j \end{bmatrix} = (P_j, Q_j)^{-1}.
\]

We shall consider now how to determine the matrix \( Q_j \). We introduce the following notation. If \( f = (f_1, \ldots, f_r), \quad g = (g_1, \ldots, g_r) \) are some vectors then \( [f, g](t) = (f_t, g_t) \) is a scalar product matrix. As it is shown in paper [7] matrix \( Q_j \) satisfies the following equation \( P_j^T [\Phi_j, \Phi_j] Q_j = 0 \).

We turn to spline wavelets on a segment now. We define B-splines of order \( n \) as convolution [10]
\[
N_n(x) = N_{n-1} * N_0, N_0(x) = \begin{cases} 1, & x \in [0;1), \\ 0, & x \notin [0;1). \end{cases}
\]

As it is shown in [10] if we define the function \( \phi(x) = N_n(x) \) it satisfies the equation
\[
\phi(x) = \sum_{k=0}^{n} \frac{C_{n+1}}{2^n} \phi(2x - k), \quad \text{where} \quad C_{n+1} = \frac{(n+1)!}{k!(n+1-k)!}.
\]
The article [11] represents the filterbank that corresponds to the function \( \phi(x) = N_n(x) \), that is the following results are valid.

**Lemma 1.** The function \( \phi(x) = N_n(x) \) defines a sequence of subspaces
\[
V_{a,0} \subset V_{a,1} \subset \ldots, V_{a,j} = \text{lin}\{\phi_{j,-n}, \phi_{j,-n+1}, \ldots, \phi_{j,2a(n+1)-1}\}
\]
of the space \( L_2[0; \alpha(n+1)], \alpha = 1, 2, \ldots \) such that \( \bigcup_{j=0}^{a} V_{a,j} = L_2[0; \alpha(n+1)] \).
Lemma 2. We have the par
\[
\sum_{k=-n}^{2^l n(n+1)-1} \varphi_{j,k}(x) \equiv 2^l, \ x \in [0; \alpha(n+1)].
\]

If \( V_{a,j} = V_{a,j-1} \oplus W_{a,j-1} \), then \( \dim W_{a,j-1} = 2^{j-1} \alpha(n+1) \).

Let us assume that
\[
\lambda_{m,k} = \int_{k}^{k+1} N_n(z)N_n(z-m) \, dz, \ m = -n, \ldots, n, \ k = 0, \ldots, n
\]
and \( \omega_{a,k} = \sum_{s=-n}^{n} \lambda_{s,k} \), \( \theta_{a,k} = \sum_{s=0}^{n} \lambda_{s,k} \), \( 0 \leq k \leq n \). We introduce the vector \( p \in \mathbb{R}^{2^l \alpha(n+1)+n} \)
which we define by the par
\[
p = \begin{cases} 
(C_{n+1}^{0} C_{n+1}^{1} \ldots C_{n+1}^{k} C_{n+1}^{k+1} \ldots C_{n+1}^{0} 0 \ldots 0)^T, & n = 2k; \\
(C_{n+1}^{0} C_{n+1}^{1} \ldots C_{n+1}^{k} C_{n+1}^{k+1} \ldots C_{n+1}^{0} 0 \ldots 0)^T, & n = 2k+1.
\end{cases}
\]

We define shift operator \( R_a : \mathbb{R}^m \to \mathbb{R}^m \) by the following rule
\[
R_a a = \begin{cases} 
(0, \ldots, 0, a_1, \ldots, a_{m-s})^T, & m > s \geq 0; \\
(a_{d+1}, \ldots, a_m, 0, \ldots, 0)^T, & -m < s < 0,
\end{cases}
\]
where \( a = (a_1, \ldots, a_m)^T \).

If \( |s| \geq m \) then \( R_a a = 0 \).

Lemma 3. Matrices \( P_j \) and \( [(\Phi_j, \Phi_j)] \) for the sequence of subspaces \( V_{a,j} \subset \ldots \) are as follows
\[
P_j = \frac{1}{2^l \alpha(n+1)+n} \left( R_{a} p \ R_{a+1} p \ldots R_{a+2^{l-1}\alpha(n+1)+n} p \right);
\]

\[
[(\Phi_j, \Phi_j)] = \left( d_1, \ldots, d_n, q, R_q, \ldots, R_{2^{l-1}\alpha(n+1)+n} q, u, \ldots, u_k \right)^T.
\]

where
\[
d_j = (a_1, a_2, \ldots, a_{n+1}, q_{n+1}, \ldots, q_n, 0, \ldots, 0)^T, \ u_j = (0, \ldots, 0, q_n, \ldots, q_1, \theta_1, \ldots, \theta_n)^T,
\]
\[
q = (q_n, q_{n-1}, \ldots, q_1, q_0, q_{n+1}, \ldots, q_n, 0, \ldots, 0)^T \in \mathbb{R}^{2^l \alpha(n+1)+n}, \ q_k = (N_n(c), N_n(c-k)).
\]

The matrix transposed to \( T_j = P_j^T [(\Phi_j, \Phi_j)] \) is as follows
\[
2^{n+\frac{1}{2}} \cdot T_j = \left( L_{1}, \ldots, L_{n} w, R_{2} w, \ldots, R_{2^{l-1}\alpha(n+1)+n-2} w, L_{2^{l-1}\alpha(n+1)+1}, \ldots, L_{2^{l-1}\alpha(n+1)+n} w \right),
\]
where \( w = (p^T R_{2} q, p^T R_{2} q, \ldots, p^T R_{2} q, 0, \ldots, 0)^T \in \mathbb{R}^{2^l \alpha(n+1)+n} \),
\[
L_j = \left( (R_{a+2^i-2} p)^T d_1, \ldots, (R_{a+2^i-2} p)^T d_n, 0, \ldots, 0 \right)^T + (R_{n} \circ R_{a+2^i-2} w), \ i = 1, \ldots, n,
\]
\[
L_{a+1} = \left( 0, \ldots, 0, (R_{a+2^i} p)^T u_i, \ldots, (R_{a+2^i} p)^T u_s \right)^T + (R_{n} \circ R_{a+2^i} w), \ i = 2^{i-1} \alpha(n+1), \ldots, n-1 + 2^{i-1} \alpha(n+1).
\]
Using the lemma 3 in the article [11] are found $2^{j+1} \alpha(n+1)$ of linearly independent solutions $h_j = (h_{1,j}, h_{2,j}, \ldots, h_{2^{j+1},j})^T$ of system of linear equation $T_j h = 0$. These solution represent the columns of the matrix $Q_j = \begin{pmatrix} h_1, \ldots, h_{2^{j+1},j} \end{pmatrix}$. Columns $h_j$ are chosen in such a way that the functions

$$
\psi_{j,s}(x) = \Phi_j(x) h_j = \sum_{i=1}^{2^{j+1}} h_{i,j} \cdot \varphi_{j-n(i-1)}(x),
$$

as possible would represent shifted versions of one function i.e. would have one form (except for, of course, the boundary wavelets). We introduce the abbreviated notation for matrices composed of matrix elements $T_j$:

$$
T_j \left( \begin{array}{cccc} t_{1,j} & \cdots & t_{i,j} & \cdots & t_{m,j} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ t_{1,j} & \cdots & t_{i,j} & \cdots & t_{m,j} \end{array} \right).
$$

For internal wavelets (the carrier is contained in a segment $[0, \alpha(n+1)]$)

$$
h_j = (0, \ldots, 0, h_{2^{-n-j-1},j}, \ldots, h_{2^{n+2j},0}, 0, \ldots, 0)^T, \quad s = n + 1, \ldots, 2^{j+1} \alpha(n+1) - n,
$$

where $T_j \left( \begin{array}{cccc} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \\ 1, \ldots, 2^{j+2} & \cdots & 1, \ldots, 2^{j+2} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 1, \ldots, 2^{j+2} & \cdots & 1, \ldots, 2^{j+2} \end{array} \right) (h_{2^{-n-j-1},j}, \ldots, h_{2^{n+2j},0})^T = 0.
$$

The solutions corresponding to the boundary wavelets were chosen as follows. For $s = 1, 2, \ldots, n$ we assume $h_j = (0, \ldots, 0, h_{2^{-n+1-j},j}, \ldots, h_{2^{n+2j},j}, 0, \ldots, 0)^T$, where

$$
T_j \left( \begin{array}{cccc} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \\ 1, \ldots, 2^{j+2} & \cdots & 1, \ldots, 2^{j+2} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 1, \ldots, 2^{j+2} & \cdots & 1, \ldots, 2^{j+2} \end{array} \right) (h_{2^{-n-j},j}, \ldots, h_{2^{n+2j},j})^T = 0.
$$

For $s = 2^{j+1} \alpha(n+1) - n + 1, \ldots, 2^{j+1} \alpha(n+1)$ we assume

$$
h_j = (0, \ldots, 0, h_{2^{-n-j-1},j}, \ldots, h_{2^{n+2j},j}, 0, \ldots, 0)^T,
$$

where

$$
T_j \left( \begin{array}{cccc} s - n & \cdots & s - n & \cdots & s - n \\ 2s - n & \cdots & 2s - n & \cdots & 2s - n \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \\ 1, \ldots, 2^{j+2} & \cdots & 1, \ldots, 2^{j+2} \end{array} \right) (h_{2^{-n-j},j}, \ldots, h_{2^{n+2j},j})^T = 0.
$$

The following lemma is obvious.

**Lemma 4.** Let us assume that $f \in L_2[0; n+1]$, then $\Pi_j f = \Phi_j \mathcal{C}_j$, where $\mathcal{C}_j = [(\Phi_j, \Phi_j)]^{-1}[(f, \Phi_j)]$.

**3.1. Integrals from spline wavelets**

Let us assume that $Q_j = \begin{pmatrix} h^j_1, \ldots, h^j_{2^{j+1}} \end{pmatrix}$, where $h^j_j = (h^j_{1,j}, h^j_{2,j}, \ldots, h^j_{2^{j+1},j})^T$. According to the results of the previous section

$$
\psi_{j,s}(x) = \sum_{i=1}^{2^{j+1}} h^j_{i,s} \varphi_{j-n(i-1)}(x), \quad s = 1, \ldots, n,
$$

(4)
\[ \psi_{j,l,t}(x) = \sum_{i=2s-n-1}^{2s+2n} h_{i,j} \varphi_{j,-n+i-1}(x), \quad s = n+1, \ldots, 2^{l-1}(n+1) - n, \quad (5) \]

\[ \psi_{j,l,t}(x) = \sum_{i=2s-n-1}^{2s+2n} h_{i,j} \varphi_{j,-n+i-1}(x), \quad s = 2^{l-1}(n+1) - n + 1, \ldots, 2^{l-1}(n+1). \quad (6) \]

As well as in the papers [1]-[5] for the ease of convenience we introduce the following notation

\[ w_j(x) = \varphi_{0,l,-n-1}, \quad l = 1, 2, \ldots, 2n+1, \]

\[ w_j(x) = \psi_{j,l}(x), \quad l = 2^{l}(n+1) + n + s, \quad j = 0, 1, \ldots, 2^{l}(n+1). \]

Figure 1 represents graphs of some functions \( w_l \) for the case \( n = 5 \).

![Graphs of functions \( w_l \) for \( n = 5 \)](image)

Let us assume that \( J \geq 0 \), \( \Pi_j : L_2[0; n+1] \to V_j \) is a projection and \( M = 2^l(n+1) + n \). We denote \( H_j = (w_1, \ldots, w_M) \) and introduce the matrix of scalar products \( [(H_j, H_j)] \). Lemma 3 represents matrices of scalar products \( [(\Phi_j, \Phi_j)] \) for all \( k = 0, 1, \ldots \). We notice that \( \Psi_k = \Phi_{k+1} Q_{k+1} \) and \( [(\Psi_k, \Psi_k)] = Q_{k+1}^T [(\Phi_k, \Phi_k)] Q_{k+1} \), and we obtain the matrix

\[ [(H_j, H_j)] = \begin{bmatrix} (\Phi_0, \Phi_0) & 0 & 0 & \cdots & 0 \\ 0 & Q_1^T [(\Phi_1, \Phi_1)] Q_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & Q_{k+1}^T [(\Phi_{k+1}, \Phi_{k+1})] Q_{k+1} \end{bmatrix}. \]

As \( V_j = V_0 \oplus V_1 \oplus \ldots \oplus V_{J-1} \), then for \( f \in L_2[0; n+1] \) we have \( \Pi_j f = \sum_{j=1}^{M} c_j w_j = H_j C_j \) where \( C_j = (c_1, \ldots, c_M)^T \). As it is done in papers [1]-[5] we define functions

\[ \xi_{j,l}(x) = \int_0^x w_j(t)dt, \quad \xi_{n+1,l}(x) = \int_0^x \xi_{j,l}(t)dt = \frac{1}{\nu^l} \int_0^x (x-t)^\nu w_j(t)dt, \quad \nu = 1, 2, \ldots \quad (7) \]
According to defining functions \( w_l \) and pars (4)-(6) function \( \xi_{v+t,l}(x) \) is a linear combination of functions \( \eta_{n,v}^{l,s}(x) = \int_0^1 (x-t)^v N_n(2^j t - s) \, dt \).

**Lemma 5.** There is the following recurrence equation found

\[
\eta_{n,v}^{l,s}(x) = \frac{x^{v+1}}{v+1} N_n(-s) + \frac{2^j}{v+1} \left( \eta_{n-1,v+1}^{l,s}(x) - \eta_{n-1,v+1}^{l,s+1}(x) \right),
\]

where

\[
\eta_{0,v}^{l,s}(x) = \begin{cases} 
\frac{(x-a)^{v+1} - (x-b)^{v+1}}{v+1}, & [a;b] = [0; x] \cap \left[ \frac{s}{2^j}; \frac{s+1}{2^j} \right] \neq \emptyset; \\
0, & [a;b] = [0; x] \cap \left[ \frac{s}{2^j}; \frac{s+1}{2^j} \right] = \emptyset.
\end{cases}
\]

**Proof.** By the characteristic B-splines [10] there is a par found \( N_n'(x) = N_{n-1}(x) - N_{n-1}(x-1) \). It follows that by the integrating by parts formula we obtain

\[
\eta_{0,v}^{l,s}(x) = \frac{(x-t)^v}{v+1} N_n(2^j t - s) \int_0^1 2^j (x-t)^v N_n(2^j t - s) \, dt = \\
= \frac{x^{v+1}}{v+1} N_n(-s) + \frac{2^j}{v+1} \left( \eta_{n-1,v+1}^{l,s}(x) - \eta_{n-1,v+1}^{l,s+1}(x) \right).
\]

Par (9) is obvious. The lemma is convinced.

Formulas (8), (9) allow us to find the value of a function \( \eta_{n,v}^{l,s}(x) \) at any point without integration. So for \( l=1,2,...,2n+1 \) we obtain

\[
\xi_{v+t,l}(x) = \frac{1}{v!} \eta_{v,v}^{0,l-n-1}(x), \quad l=1,2,...,2n+1.
\]

For \( l = 2^j(n+1) + n + s, \quad j = 0,1,..., \quad s = 1,...,2^j(n+1) \) we obtain

\[
\xi_{v+t,l}(x) = \frac{2^j}{v!} \sum_{i=0}^{\frac{s}{2^j}} \sum_{n=1}^{2^j} h_{v+1}^{l,i} \eta_{v,v}^{s+1,n+1}(x), \quad s = 1,...,n;
\]

\[
\xi_{v+t,l}(x) = \frac{2^j}{v!} \sum_{i=0}^{\frac{s}{2^j}} \sum_{n=1}^{2^j} h_{v+1}^{l,i} \eta_{v,v}^{s+1,n+1}(x), \quad s = n+1,...,2^j(n+1) - n;
\]

\[
\xi_{v+t,l}(x) = \frac{2^j}{v!} \sum_{i=0}^{\frac{s}{2^j}} \sum_{n=1}^{2^j} h_{v+1}^{l,i} \eta_{v,v}^{s+1,n+1}(x), \quad s = 2^j(n+1) - n + 1,...,2^j(n+1) + 1.
\]

The pars obtained are valid for all \( v=0,1,\ldots \)

### 3.2. Application of a spline wavelets to the solution of linear systems of differential equations

Two equations are considered in the projection methods for solving linear equations; the first in the complete normed space \( X \)

\[
Kx = x - \lambda Hx = f,
\]

And the second one in its complete subspace \( X_j \)
\[ K_j x_j = x_j - \lambda H_j x_j = \Pi_j f, \]

where \( H \) is a continuous linear operator in \( X \), \( H_j \) is a continuous linear operator in \( X_j \) and \( \Pi_j \) is a continuous linear operator projecting \( X \) to \( X_j \), i.e. \( \Pi_j(X) = X_j \), \( \Pi_j \circ \Pi_j = \Pi_j \). Equation (10) is called an exact equation, and equation (11) is called an approximate equation. It is assumed that the following conditions are satisfied:

1. (Condition of aboutness of operators \( H \) and \( H_j \)) \( \| \Pi_j H x_j - H_j x_j \| \leq \rho_j \| x_j \| \) is satisfied for each \( x_j \in X_j \).
2. (The condition of a good approximation of elements of the form \( H x \) by elements from \( X_j \)) There is an element \( x_j \in X_j \) for each \( x \in X \) such that \( \| H x - x_j \| \leq \rho_j \| x \| \).
3. (The condition of a good approximation of the free term of the exact equation) There is an element \( f_j \in X_j \) such that \( \| f - f_j \| \leq \rho_{2,j} \| f \| \). Unlike the previous conditions \( \rho_{2,j} \) here generally speaking, depends on \( f \).

If operator \( K \) has a continuous inverse, the equation (10) has a solution and if \( \lim_{j \rightarrow +\infty} \rho_j = 0 \), \( \lim_{j \rightarrow +\infty} \rho_{2,j} = 0 \) then \( \lim_{j \rightarrow +\infty} \| x - x_j \| = 0 \), where \( x_j \) is the solution for equation (11), and \( x \) is the solution for equation (10).

We consider a linear system of differential equations (1)-(2). Without loss of generality we can assume that \( t_0 = 0 \), \( t_i = n + 1 \), where \( n \in \mathbb{N} \). We introduce the notation \( y_i(t) = dx_i(t)/dt \). Then subject to conditions (2) we obtain \( x_i(t) = x_{i,0} + \int_0^t y_i(\tau)d\tau \). System (1) can be rewritten in the form

\[
\begin{align*}
y_1 &= \int_0^t (a_{11}(t)y_1(\tau) + a_{12}(t)y_2(\tau) + \ldots + a_{1m}(t)y_m(\tau))d\tau + g_1(t); \\
y_2 &= \int_0^t (a_{21}(t)y_1(\tau) + a_{22}(t)y_2(\tau) + \ldots + a_{2m}(t)y_m(\tau))d\tau + g_2(t); \\
&\quad \vdots \\
y_n &= \int_0^t (a_{n1}(t)y_1(\tau) + a_{n2}(t)y_2(\tau) + \ldots + a_{nm}(t)y_m(\tau))d\tau + g_m(t)
\end{align*}
\]

where \( g_j(t) = f_j(t) + \sum_{j=1}^m a_{j1}(t)x_{j,0} \). We rewritten the system obtained in the matrix form

\[
Y(t) = \int_0^t A(t)Y(\tau)d\tau + G(t), \quad \text{where} \quad Y = (y_1, \ldots, y_m)^T, \quad G = (g_1, \ldots, g_m)^T, \quad A = (a_{ij})_{i,j=1}^m. \quad \text{As a complete normed space we consider the space} \quad X = (L_2([0; n+1]))^m = L_2([0; n+1]) \times \ldots \times L_2([0; n+1]), \quad \text{with the norm} \quad \| Y \| = \sqrt{\sum_{j=1}^m \| y_j \|^2}. \quad \text{And as its full subspace we consider} \quad X_j = (V_j)^m. \quad \text{Let us assume that} \quad \Pi_j^m : X \rightarrow X_j \quad \text{is defined by a par} \quad \Pi_j^m(Y) = (\Pi_j y_1, \ldots, \Pi_j y_m)^T.
\]
We denote $A_i = (a_{i,1}, \ldots, a_{i,m})^T \in X$ as $i$ row of matrix $A$.

We define operators $K : X \rightarrow X$, $H : X \rightarrow X$ and $H_j : X_j \rightarrow X_j$ by pars

$$ (K_Y)(t) = Y(t) - \int_0^t A(t)Y(\tau)d\tau, \quad (HY)(t) = \int_0^t A(t)Y(\tau)d\tau, \quad H_j = \Pi_j^n \circ H. $$

Condition of aboutness of operators $H$ and $H_j$ is satisfied with $\rho_j = 0$. Let us assume that $Y \in X$ and

$$ Y_j(t) = \left( \int_0^t (\Pi_j^n A_i(t))^T Y(\tau)d\tau, \ldots, \int_0^t (\Pi_j^n A_m(t))^T Y(\tau)d\tau \right)^T \in (V_j)^m. $$

Then $\rho_{1,j} = (n+1) \sum_{l=1}^m \| A_l - (\Pi_j^n A_l) \|^2$. It is obvious that $\lim_{j \rightarrow +\infty} \rho_{1,j} = 0$. It follows that the condition of a good approximation of elements of the form $HY$ by elements from $(V_j)^m$ is also satisfied. Finally for the certain $G \in X, G \neq 0$ we obtain $G_j = \Pi_j^n G$, and

$$ \rho_{2,j} = \frac{\| G - G_j \|}{\| G \|}. $$

Then $\lim_{j \rightarrow +\infty} \rho_{2,j} = 0$.

We shall seek the solution of the approximate equation

$$ Y_j - \Pi_j^n \circ HY_j = \Pi_j^n G, \quad (13) $$

in the for

$$ Y_j = \left( \sum_{i=1}^M C_{i,1}w_i(t), \ldots, \sum_{i=1}^M C_{i,m}w_i(t) \right)^T, $$

where $M = 2(n+1) + n$. Then we can rewrite the equation (13) in the form of the systems of linear equations for the determination of unknown wavelet coefficients

$$ \sum_{i=1}^M C_{i,s-1,1}w_i = \sum_{i=1}^M \sum_{k=1}^m C_{i,s-1,1}(a_{i,k} \xi_{j,k}w_i) + (g_j, w_i), \quad s = 1, \ldots, m, l = 1, \ldots, M. $$

The approximate solution of system (1) has the form

$$ x_{s,j}(t) = \sum_{i=1}^M C_{i,s,1} \xi_{j,1}(t) + x_{s,0}, \quad s = 1, 2, \ldots, m. $$

4. Results of experiments

Multi-dimensional non-stationary linear systems are described by the equations of state and output

$$ dx(t)/dt = A(t)x(t) + B(t)g(t), \quad x(0) = x_0 \quad (14) $$

$$ y(t) = L(t)x(t). \quad (15) $$

where $x = (x_1, \ldots, x_m)^T$ is a m-dimensional state vector; $g = (g_1, \ldots, g_r)^T$ is a r-dimensional input stimulus vector; $y = (y_1, \ldots, y_k)^T$ is a k-dimensional output vector; $t$ is time; $A(t) = (a_{i,j}(t))_{i,j=1}^{m,m}$, $B(t) = (b_{i,j}(t))_{i,j=1}^{m,r}$, $L(t) = (l_{i,j}(t))_{i,j=1}^{k,m}$ are m×m, m×r, k×m dimention matrices respectively. The
search for the law of change of the state vector and the output vector according to the given input signal shall be understood to mean the analysis of the output process.

We shall seek an approximate solution of the system (14) in the form

$$x_{i,j}(t) = \sum_{i=1}^{M} c_{i,j} x_i(t) + x_{i,0}, s = 1, 2, \ldots, m,$$

where $x_0 = (x_{1,0}, \ldots, x_{m,0})^T$ and coefficients $c_{i,j}$ are found from the linear equation systems

$$\sum_{i=1}^{n} \sum_{k=1}^{M} (a_{i,j} - w_k) c_{i,j} = - \sum_{i=1}^{n} (a_{i,j} - w_k) x_{i,0} - \sum_{k=1}^{j} (b_{i,j} - w_k),$$

$$k = 1, \ldots, n, \alpha = 1, \ldots, M.$$

![Figure 2. Approximation graphs $x_2(t)$ (dashed line), $x_4(t)$ (continuous line) and a grid function graph \{$(t_i, x_j)$\} (points), obtained using the Runge-Kutta method.](image)

**Example 1.** Let us consider a nonstationary automatic control system whose behavior is described by a differential equation

$$\sum_{i=1}^{n} a_i(t) x^{(i)}(t) = g(t),$$

Where coefficients $a_i(t)$ are determined from the following expression:

$$\begin{bmatrix}
a_0(t) & 0.5596 & 1.8918 & 2.5825 & 1.7855 & 0.6277 & 0.0909 \\
a_1(t) & 0.7113 & 2.3843 & 3.222 & 2.1975 & 0.7588 & 0.1065 \\
a_2(t) & 0.3717 & 1.2333 & 1.6449 & 1.1038 & 0.3728 & 0.0507 \\
a_3(t) & 0.1002 & 0.3278 & 0.43 & 0.2827 & 0.093 & 0.0122 \\
a_4(t) & 0.014 & 0.0449 & 0.0576 & 0.0369 & 0.0118 & 0.0015 \\
a_5(t) & 0.0008 & 0.0025 & 0.0031 & 0.0019 & 0.006 & 0.00007 \\
\end{bmatrix} \begin{bmatrix} t \\ t^2 \\ t^3 \\ t^4 \\ t^5 \end{bmatrix}.$$
\[ g(t) = (85,7661 + 338,5984 t + 497,0437 t^2 + 406,9496 t^3 + 
+ 186,9354 t^4 + 46,7809 t^5 + 4,8258 t^6)e^{-\beta t}. \]

Initial conditions are zero. Research interval is \([0;5]\) c.

Solution. Since the initial conditions are zero, an approximate solution of this problem will be sought in the form
\[ x_j(t) = \sum_{s=1}^{M} c_s \xi_{5,s}(t), \]
where coefficients \(c_1, \ldots, c_M\) are defined from the linear equation system
\[ \sum_{s=1}^{2t(n+1)+n} c_s \left( a_s w_s + \sum_{k=0}^{s} a_{s-k} \xi_{5,s-k}(t), w_s \right) = \left( g_s, w_s \right), \quad l = 1, \ldots, 2t(n+1)+n. \]

Figure 2 shows graphs of the third and fifth approximations \(x_2(t)\) (dashed line), \(x_4(t)\) (continuous line) and a grid function graph \((\xi, \tilde{\xi})\) (points) obtained using the Runge-Kutta method.

**Example 2.** The behavior of a linear nonstationary system is described by the following system of differential equations
\[
\begin{align*}
\left( \frac{dx(t)}{dt} \right) &= \left( t^2 - 1 - t \right) x(t) + \left( t^2 - 1 \right) y(t), \\
\left( \frac{dy(t)}{dt} \right) &= \left( t^2 - 1 \right) y(t) + \left( t^2 - 1 \right) x(t)
\end{align*}
\]
Find the response of the system to the input effect
\[ g_1(t) = 0.23315158 t^6 - 3.89665 t^6 + 26.4309725 t^7 - 93.4794 t^7 + 183.95 t^8 - 200.83 t^8 + 
+ 256.386587 t^9 - 50.135386 t^9 + 13.095959 t^9 - 2.8258 t^9 + 
+ 50.135386 t^9 - 378.61242 t^7 + 336.683591 t^8 - 213.9681871 t^9 + 106.4889 t^9 - 47.3676 t^9 + 
+ 19.56997 t^9 - 3.863587 t^9 - 0.0004283.
\]
for initial conditions \(x(0) = -1, y(0) = 2\) over a time slot of \([0;2]\) c.

Solution. An approximate solution will be sought in the form
\[ y_j(t) = 2 + \sum_{s=1}^{M} c_{M,s} \xi_{5,s}(t), \]
where \(M = 2t(n+1)+n\), and coefficients \(c_s, s = 1, 2, \ldots, 2M\) are defined from the linear equation system
\[
\sum_{s=1}^{M} c_s \left( w_s, w_l \right) - \sum_{s=1}^{n+1} \left( t^2 \xi_{5,s}(t) w_l(t) dt \right) - \sum_{s=1}^{M} c_{M+s} \int_{0}^{n+1} (1-t) \xi_{5,s}(t) w_l(t) dt = \\
= \int_{0}^{n+1} (t^2 g_1(t) - t^2 + 2(1-t)) w_l(t) dt, l = 1, 2, \ldots, M;
\]
\[
\sum_{s=1}^{M} c_s \int_{0}^{n+1} (1+t) \xi_{5,s}(t) w_l(t) dt + \sum_{s=1}^{M} c_{M+s} \int_{0}^{n+1} (t^2 - t)^2 \xi_{5,s}(t) w_l(t) dt - (w_s, w_l) = \\
= \int_{0}^{n+1} (2t^2 - 1 - g_1(t) - tg_2(t)) w_l(t) dt, l = 1, 2, \ldots, M.
\]
Figure 3 shows graphs of the first and third approximations $x_0(t), y_0(t)$ (dashed line), $x_i(t), y_i(t)$ (continuous line) and a grid function graph $\{(t_i, \tilde{x}_i)\}$, $\{(t_i, \tilde{y}_i)\}$ (points) obtained using the Runge-Kutta method.

![Figure 3. Approximation graphs: (a) $x_0(t)$ (dashed line), $x_i(t)$ (continuous line) and a grid function graph $\{(t_i, \tilde{x}_i)\}$ (points) obtained using the Runge-Kutta method; (b) $y_0(t)$ (dashed line), $y_i(t)$ (continuous line) and a grid function graph $\{(t_i, \tilde{y}_i)\}$ (points) obtained using the Runge-Kutta method.](image)

5. Summary and conclusion

In this paper we have considered the use of spline wavelets for the approximate solution of linear systems of differential equations and as an application for analyzing the output processes of multidimensional linear nonstationary control systems. As a novelty of the present paper one can note the obtainment of finite formulas for approximations of any $C^n$ smoothness class of solutions of linear differential equations systems by means of application of spline-wavelets constructed on the basis of a B-spline of arbitrary order $n$. Known methods based on Haar wavelets are obtained from the methods presented in this article when $n = 0$.

References

[1] Lepik Ü 2007 Numerical solution of evolution equations by the Haar wavelet method Appl. Math. Comput 185 pp 695–704
[2] Lepik Ü 2008 Haar wavelet method for solving higher order differential equations Int. J. Math. Comput 1 pp 84–94
[3] Lepik Ü 2007 Application of the Haar wavelet transform to solving integral and differential equations Proc. Estonian Acad. Sci. Phys. Math 56 pp 28–46
[4] Lepik Ü 2005 Numerical solution of differential equations using Haar wavelets Math. Comput. Simul 68 pp 127–43
[5] Lepik Ü and Hein H 2014 Haar wavelets with applications (Berlin: Springer) p 207
[6] Blatov I A and Rogova N V 2013 Application of semiorthogonal spline wavelets and the Galerkin method to the numerical simulation of thin wire antennas Computational Mathematics and Mathematical Physics 53 pp 727–36
[7] Finkelstein A and Salesin D H 1994 Multiresolution curves ACM SIGGRAPH pp 261–268
[8] Frazier M. W 1999 An introduction to wavelets through linear algebra (New York: Springer) p 503
[9] Daubechies I 1992 Ten lectures on wavelets (Philadelphia: SIAM) p 358
[10] Chui Ch K 1991 An introduction to wavelets (Boston: Academic press) p 412
[11] Bityukov Yu I and Akmaeva V N 2016 The use of wavelets in the mathematical and computer modelling of manufacture of the complex-shaped shells made of composite materials Bulletin of the South Ural State University. Ser. Mathematical Modelling, Programming and Computer Software 9 pp 5–16