The nonlinear evolution equations of fourth-order for two surface gravity waves including the effect of air flowing over water

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Abstract. The nonlinear evolution equations of fourth-order have been established for two surface gravity waves in infinite depth of water including the effect of air flowing over water. In the present paper, we have applied a general approach depending upon the integral equation due to Zakharov. Based on these equations, the stability analysis has been studied in the appearance of a uniform gravity wave packet with another wave packet of the identical group velocity. Graphs have been drawn for the instability growth rate of the uniform wave packet with shorter wave number versus the perturbed wave number for several values of the amplitude of the wave packet of larger wave-number and several values of wind velocity. We have observed from the figures that the instability growth rate of the second wave packet enhances with the enhancement of nondimensional wind velocity for the settled value of the amplitude of the first wave packet.

1. Introduction

The instability problem due to wave train on water has been interesting ever since a gravity wave train was exhibited to be unstable in the appearance of sidebands. A comprehensive methodology to analyse the instability of finite-amplitude gravity waves in infinite depth of water is employing the application of the nonlinear Schrodinger equation. This investigation is appropriate in case of small wave steepness along with long-wavelength perturbations. For small values of amplitudes, the nonlinear evolution equation of the third order was established by Benny and Newell [1] and Zakharov [2]. Now for wave steepness larger than 0.15 estimations from the third-order nonlinear Schrodinger equation is misleading and fails to match precisely with the accurate results of Longuet-Higgins [3,4]. Dysthe [5] has demonstrated that a stability analysis starting from nonlinear evolution equation of fourth-order provides outcomes consistent with the accurate consequences of Longuet-Higgins [3,4] and with the experimental outcomes of Benjamin and Feir [6] for wave steepness up to 0.25. The higher-order (fourth-order) effects produce a correction compared to third-order nonlinear Schrodinger effects in numerous ways and a few of these instances have been discussed by Janssen [7]. The powerful new effect that is present in the higher-order is the influence of wave-induced mean flow and this creates a notable change in the stability property. From the aforesaid discussions, it can be inferred that a nonlinear evolution equation of the fourth-order is a better beginning point for discussing nonlinear influences in surface waves. Starting from nonlinear evolution equations of fourth order for surface gravity waves as well as capillary gravity waves of deep water and taking into account several physical aspects, discussions on stability based on those evolution equations were performed by a number of authors [8-14]. An evolution equation of fourth-order has been established by Debsarma and Das [15] in case of two counterpropagating capillary gravity wave
trains in deep water. Further Dhar and Mondal [16] have extended the paper made by Debsarma and Das [15] in the case of air blowing over water. Zakharov [2] also established the evolution equation of the lowest order i.e., the third-order beginning from an integral equation under the supposition of a narrow band of waves. In this case, it was required to incorporate the second-order terms in bandwidth. Starting from Zakharov’s integral equation Stiassnie [17] has established the evolution equation of fourth-order under the same assumption including the third-order terms.

Unlike Dhar and Das [10,11], the present paper deals with two coupled nonlinear evolution equations of fourth-order based on Zakharov’s integral equation under the supposition of a narrow band of waves in infinite depth of water for two surface-gravity wave trains having different wave-numbers in a situation of air blowing over water. Hogan [18] also employed the integral equation due to Zakharov under the supposition of narrow wavebands to derive the nonlinear evolution equations of higher-order (i.e., fourth-order) for a capillary-gravity wave train. We have made an extension of the paper of Dhar and Das [11] considering the effect of air blowing over the water.

In the present paper, the relative changes in the phase velocity for two wave packets have been established. The stability analysis has been then made from two derived evolution equations. The instability criterion and the rate of growth of instability have been derived for a gravity wave packet in the appearance of other gravity wave packet. Stable-unstable regions and the rate of growth of instability versus perturbation wave-number have been plotted for two different sets of values of wave-numbers along with different values of wind velocity.

2. Derivation of coupled nonlinear evolution equations

The Zakharov’s integral equation for two coupled evolution equations of fourth-order is given by

\[
\frac{dA(k, t)}{dt} = \int \int \int_{-\infty}^{\infty} T(k, k_1, k_2, k_3) A^*(k_1, t) A(k_2, t) A(k_3, t) \delta(k + k_1 - k_2 - k_3) \exp[i(\omega(k_1) + \omega(k_2) - \omega(k_3))] \, dk_1 \, dk_2 \, dk_3
\]

(1)

where \(A(k, t)\) is connected to \(\beta(x, t)\), the envelope of the free surface height which is as follows

\[
\beta(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |k|^{-1/2} \{ A(k, t) \exp[i(\omega(k_1) + \omega(k_2) - \omega(k_3))] + c. c. \} \, dk
\]

(2)

In the above, the wave vector and the horizontal spatial vector are respectively given by \(k = (k, l)\) and \(x = (x, y)\) and the kernel \(T(k, k_1, k_2, k_3)\) was used by Krasitskii [19].

\[
\omega(k) \text{ which is the linearized wave frequency (Dhar and Das [10]) is as follows}
\]

\[
\omega(k) = \frac{|k|^{1/2}}{1+\gamma} \left\{ \gamma |k|^{1/2} u + \left[ (1 - \gamma^2) g - \gamma |k| u^2 \right]^{1/2} \right\},
\]

(3)

in which \(\gamma\) and \(u\) denote the ratio of air to water and wind velocity respectively. A nonzero result is attained from (1) for

\[
k + k_1 - k_2 - k_3 = 0
\]

(4)

Here we take two surface gravity wave trains of little breadth centred around the wave vectors \(k_p\) and \(k_q\) for two wave packets. With \(k = k_p\), (4) is fulfilled for \(k_p\) and \(k_q\) in the following three cases:

(a) \(k_1 = k_q, k_2 = k_q, k_3 = k_p\)
(b) \(k_1 = k_3 = k_q, k_2 = k_p\)
(c) \(k_1 = k_2 = k_3 = k_p\)

We consider \(k = k_p + e\) in equation (1) and inserting new variables \(B_1(e, t)\) and \(B_2(e, t)\) given by
\[ B_1(e, t) = A(k_p + e, t) \exp[-i\{\omega(k_p + e) - \omega(k_p)\}t] \]
\[ B_2(e, t) = A(k_q + e, t) \exp[-i\{\omega(k_q + e) - \omega(k_q)\}t], \] (5)
equation (1) can be expressed as
\[ \frac{\partial B_1(e, t)}{\partial t} - B_1(e, t)[\omega(k_p + e) - \omega(k_p)] \]
\[ = \iiint \limits_{-\infty}^{\infty} T(k_p + e, k_q + e_1, k_p + e_2, k_p + e_3) B_2^*(e_1, t) B_2(e_2, t) B_1(e_3, t) \]
\[ \times \delta(e + e_1 - e_2 - e_3) de_1 de_2 de_3 \]
\[ + \iiint \limits_{-\infty}^{\infty} T(k_p + e, k_q + e_1, k_p + e_2, k_p + e_3) B_2^*(e_1, t) B_2(e_2, t) B_2(e_3, t) \]
\[ \times \delta(e + e_1 - e_2 - e_3) de_1 de_2 de_3 \]
\[ + \iiint \limits_{-\infty}^{\infty} T(k_p + e, k_p + e_1, k_p + e_2, k_p + e_3) B_2^*(e_1, t) B_2(e_2, t) B_2(e_3, t) \]
\[ \times \delta(e + e_1 - e_2 - e_3) de_1 de_2 de_3 \] (6)
in which we take \( k_1 = k_q + e_1, k_2 = k_q + e_2, k_3 = k_p + e_3 \) for the first triple integral, \( k_1 = k_q + e_1, k_2 = k_p + e_2, k_3 = k_p + e_3 \) for the second triple integral and finally \( k_1 = k_p + e_1, k_2 = k_p + e_2, k_3 = k_p + e_3 \) for the last triple integral. \( \eta_1(x, t) \) and \( \eta_2(x, t) \), which represent the surface elevations respectively for both the wave trains in terms of \( B_1(e, t) \) and \( B_2(e, t) \), become
\[ \eta_1(x, t) = \sqrt{1 + \gamma} \frac{1}{2\pi} \exp[i(k_p \cdot x - \omega(k_p) t)] \int_{-\infty}^{\infty} B_1(e, t) \exp(i.e.x) \]
\[ \times \left| k_p + e \right|^\frac{1}{2} \left| \gamma u |k_p + e|^2 + (1 - \gamma^2) g - \gamma u^2 |k_p + e| \right|^{\frac{1}{2}} \]
\[ \eta_2(x, t) = \sqrt{1 + \gamma} \frac{1}{2\pi} \exp[i(k_q \cdot x - \omega(k_q) t)] \int_{-\infty}^{\infty} B_2(e, t) \exp(i.e.x) \]
\[ \times \left| k_q + e \right|^\frac{1}{2} \left| \gamma u |k_q + e|^2 + (1 - \gamma^2) g - \gamma u^2 |k_q + e| \right|^{\frac{1}{2}} \] (7)

Fig.1 Waves of wave-numbers \( k_1 \) and \( k_2 \) are propagating on the \( xy \) plane
We now consider the case for which two wave packets having wave numbers \( k_1 \) and \( k_2 \), where \( k_1 > k_2 \), both travelling along the \( x \)-axis so that \( \theta = 0 \) and we take two wave vectors \( k_p \) and \( k_q \) as
\[
k_p = k_1 \hat{x}, \quad k_q = k_2 \hat{x},
\]
(9)
where \( \hat{x} \) is a unit vector towards the \( x \)-axis. Again, we shall take modulational perturbation only towards the \( x \)-axis, so that we have \( e = e \hat{x} \). Using \( k_p = k_1 \hat{x} \) and \( e = e \hat{x} \) in equations (7), (8) and considering Taylor-expansion in powers of \( e \) and taking only the linear terms in \( e \), we get the following expressions for \( \eta_1(x,t) \) and \( \eta_2(x,t) \)
\[
\eta_1(x,t) = \alpha_1(x,t) \exp[i(k_1 x - \omega(k_1) t)] + c.c.
\]
(10)
\[
\eta_2(x,t) = \alpha_2(x,t) \exp[i(k_2 x - \omega(k_2) t)] + c.c.,
\]
(11)
where
\[
\alpha_1(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lambda_1(e,t) \exp(i e x) \, de
\]
(12)
\[
\alpha_2(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lambda_2(e,t) \exp(i e x) \, de
\]
(13)
and
\[
\lambda_1(e,t) = \sqrt{\frac{1+\gamma}{2}} \left( \frac{k_1}{b_1^2} \right)^{\frac{1}{2}} (1 + b_1^2 e) B_1(e,t)
\]
(14)
\[
\lambda_2(e,t) = \sqrt{\frac{1+\gamma}{2}} \left( \frac{k_2}{b_2^2} \right)^{\frac{1}{2}} (1 + b_2^2 e) B_2(e,t)
\]
(15)
In equations (14) and (15) \( b_1, b_1', b_2, b_2' \) are different expressions which are related to \( \gamma, u, k_1 \) and \( k_2 \).

Integrating equation (6) with respect to \( e \) and using the relations (10), (12) and (14) the evolution equation (6) assumes the following form
\[
\frac{i}{2\pi} \frac{\partial}{\partial t} \int_{-\infty}^{\infty} \lambda_1(e,t) \exp(i e x) \, de - \frac{1}{2\pi} \int_{-\infty}^{\infty} [\omega(k_1 + e) - \omega(k_1)] \lambda_1(e,t) \exp(i e x) \, de
\]
\[
= \sqrt{\frac{1+\gamma}{2}} \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} T(k_1 + e_2 + e_3 - e_1, k_2 + e_1, k_2 + e_2) \lambda_2^*(e_1,t) \lambda_1(e_2,t) \lambda_1(e_3,t)
\times \frac{\alpha_1}{k_2} \left(1 - b_1 e_1 - e_2\right) \exp[i(e_2 + e_3 - e_1 ) \cdot x] \, de_1 \, de_2 \, de_3
\]
\[
+ \sqrt{\frac{1+\gamma}{2}} \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} T(k_1 + e_2 + e_3 - e_1, k_2 + e_1, k_1 + e_2) \lambda_2^*(e_1,t) \lambda_1(e_2,t) \lambda_2(e_3,t)
\times \frac{\alpha_2}{k_2} \left(1 - b_1 e_1 - e_3\right) \exp[i(e_2 + e_3 - e_1 ) \cdot x] \, de_1 \, de_2 \, de_3
\]
\[
+ \sqrt{\frac{1+\gamma}{2}} \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} T(k_1 + e_2 + e_3 - e_1, k_1 + e_1, k_1 + e_2) \lambda_2^*(e_1,t) \lambda_1(e_2,t) \lambda_2(e_3,t)
\times \frac{\alpha_1}{k_1} \left(1 - 2b_1 e_1 \right) \exp[i(e_2 + e_3 - e_1 ) \cdot x] \, de_1 \, de_2 \, de_3
\]
(16)
We next introduce the following dimensionless variables \( \alpha_1' = k_1 \alpha_1, \alpha_2' = k_2 \alpha_2, \chi' = k_2 x, \tau' = \omega_2 t, v' = \frac{k_2}{\sqrt{g}} \) \( u \) and then omitting the primes and finally taking the Fourier inversion integrals we
obtain the coupled nonlinear evolution equation for the first wave packet in the appearance of a second wave packet.

\[
i \left( \frac{\partial \alpha_1}{\partial t} + \delta_1^{(1)} \frac{\partial \alpha_1}{\partial x} \right) + \delta_2^{(1)} \frac{\partial^2 \alpha_1}{\partial x^2} + i \delta_3^{(1)} \frac{\partial^3 \alpha_1}{\partial x^3} = \gamma_1^{(1)} \alpha_1^* \alpha_1 + i \gamma_2^{(1)} \alpha_1 \frac{\partial \alpha_1}{\partial x} + i \gamma_3^{(1)} \alpha_1^* \frac{\partial \alpha_1^*}{\partial x} + \gamma_4^{(1)} \alpha_1 \frac{\partial}{\partial x} \left( \alpha_1 \alpha_1^* \right) + \xi_1^{(1)} \alpha_1 \alpha_2^* \alpha_1^* + i \xi_2^{(1)} \alpha_2 \alpha_2^* \frac{\partial \alpha_1}{\partial x} + i \xi_3^{(1)} \alpha_2 \alpha_2^* \frac{\partial \alpha_1^*}{\partial x} + i \xi_4^{(1)} \alpha_1 \alpha_2 \frac{\partial \alpha_2^*}{\partial x} + i \xi_5^{(1)} \alpha_1 H \left[ \frac{\partial}{\partial x} \left( \alpha_2 \alpha_2^* \right) \right]
\]

in which \( H \) is the one-dimensional Hilbert transform specified by

\[
H(\psi) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\psi(\zeta)}{\zeta-x} d\zeta
\]

The coefficients \( \delta_i^{(1)} (i = 1, 2, 3) \), \( \gamma_i^{(1)} (i = 1, 2, 3, 4) \), \( \xi_i^{(1)} (i = 1, 2, 3, 4, 5) \) arising in equation (17) are available in the appendix. For \( \nu = \rho = 0 \) and omitting the second wave train, the equation (17) is equivalent to an equation (2.20) of Hogan [18] for \( S = 0 \). Again, omitting the second wave train, we recover the evolution equation (34) of Dhar and Das [10] for a single gravity wave packet.

Following a similar procedure and considering an exchange of the suffixes \( p \) and \( q \) in equation (6), we obtain another nonlinear evolution equation of fourth-order similar to (17) for the second wave packet when the first wave packet is present.

\[
i \left( \frac{\partial \alpha_2}{\partial t} + \delta_1^{(2)} \frac{\partial \alpha_2}{\partial x} \right) + \delta_2^{(2)} \frac{\partial^2 \alpha_2}{\partial x^2} + i \delta_3^{(2)} \frac{\partial^3 \alpha_2}{\partial x^3} = \gamma_1^{(2)} \alpha_2^* \alpha_2 + i \gamma_2^{(2)} \alpha_2 \alpha_2^* \frac{\partial \alpha_2}{\partial x} + i \gamma_3^{(2)} \alpha_2^* \frac{\partial \alpha_2^*}{\partial x} + \gamma_4^{(2)} \alpha_2 \frac{\partial}{\partial x} \left( \alpha_2 \alpha_2^* \right) + \xi_1^{(2)} \alpha_1 \alpha_2 \alpha_1^* \alpha_2^* + i \xi_2^{(2)} \alpha_1 \alpha_2 \frac{\partial \alpha_2}{\partial x} + i \xi_3^{(2)} \alpha_1 \alpha_2 \alpha_2^* \frac{\partial \alpha_2^*}{\partial x} + i \xi_4^{(2)} \alpha_2 \alpha_2^* \frac{\partial \alpha_2}{\partial x} + i \xi_5^{(2)} \alpha_2 H \left[ \frac{\partial}{\partial x} \left( \alpha_1 \alpha_1^* \right) \right]
\]

Here, the coefficients \( \delta_i^{(2)} (i = 1, 2, 3) \), \( \gamma_i^{(2)} (i = 1, 2, 3, 4) \), \( \xi_i^{(2)} (i = 1, 2, 3, 4, 5) \) arising in equation (19) are available in the appendix.

3. Stability analysis of two uniform wave trains

The uniform wave train solutions of the coupled equations (17) and (19) are considered as

\[
\alpha_1 = \eta_{01} \exp (-i \Delta \omega_1 t)
\]

\[
\alpha_2 = \eta_{02} \exp (-i \Delta \omega_2 t)
\]

in which, \( \eta_{01} \) and \( \eta_{02} \) represent fixed real values. Inserting equations (20) and (21) in equations (17) and (19) successively, the nonlinear frequency shifts of the two waves \( \Delta \omega_1 \) and \( \Delta \omega_2 \) which depend on both amplitudes we obtain

\[
\Delta \omega_1 = \gamma_1^{(1)} \eta_{01}^2 + \xi_1^{(1)} \eta_{02} \eta_{01} \quad (22)
\]

\[
\Delta \omega_2 = \gamma_1^{(2)} \eta_{02}^2 + \xi_1^{(2)} \eta_{01} \eta_{02} \quad (23)
\]
Here the nondimensional wave-numbers for the two waves are respectively $k_1/k_2$ and 1 respectively. Thus, the phase speed shifts $\Delta v_1$ and $\Delta v_2$ for the first and second waves, which depend on amplitudes are as follows

$$\Delta v_1 = \frac{\Delta \omega_1}{k_1/k_2} = r\left(\gamma_1^{(1)}\eta_{01}^2 + \xi_1^{(1)}\eta_{02}^2\right), \quad \Delta v_2 = \Delta \omega_2 = \gamma_1^{(2)}\eta_{02}^2 + \xi_1^{(2)}\eta_{01}^2,$$

where $r = k_2/k_1$.

Again, the phase speeds for two waves are $k_2\omega_1/k_1\omega_2$ and 1, respectively when we omit the nonlinearity. Thus, the relative change for phase speed $\Delta v_1^\prime$ of the first wave in the appearance of the second wave is given by

$$\Delta v_1^\prime = \frac{1}{2}r^{1/2}\xi_1^{(1)}\eta_{02}^2$$

Again, the relative change for phase speed $\Delta v_2^\prime$ of the second wave in the appearance of the first wave is

$$\Delta v_2^\prime = \xi_1^{(2)}\eta_{01}^2$$

Fig 2. (a) Relative change in phase speed for first wave packet, (b) Relative change in phase speed for second wave packet.

Now the graphs corresponding to $\Delta v_1^\prime$ versus $\eta_{02}$ and $\Delta v_2^\prime$ versus $\eta_{01}$ are shown in figures 2(a) and 2(b) respectively, for $(k_1,k_2) = (1,0.71)$, $(k_1,k_2) = (0.5,0.34)$ and different values of $\nu$. 
In order to investigate the modulational stability of two wave trains, we consider the following perturbations in the uniform solutions

\[ \alpha_1 = \eta_0 \left[ 1 + \eta_1(\xi, t) \right] \exp(-i\Delta \omega_1 t), \quad \alpha_2 = \eta_0 \left[ 1 + \eta_2(\xi, t) \right] \exp(-i\Delta \omega_2 t) \]  

(26)

where \( \eta_1(\xi, t) \), \( \eta_2(\xi, t) \) are small perturbations of amplitudes.

Inserting equation (26) in two evolution equations (17) and (19) respectively, then linearizing for \( \eta_1 \) and \( \eta_2 \) and setting \( \eta_1 = \eta_r(1) + i\eta_i(1) \) and \( \eta_2 = \eta_r(2) + i\eta_i(2) \), where \( \eta_r(1), \eta_i(1), \eta_r(2), \eta_i(2) \) are real, we obtained the following equations for \( \eta_r(1), \eta_i(1), \eta_r(2), \eta_i(2) \).

\[ A_1 \eta_r^{(1)} + B_1 \eta_r^{(2)} - i \left( \Omega - C_1^{(+)} \right) \eta_i^{(1)} - D_1^{(-)} \eta_i^{(2)} = 0 \]  

(27)

\[ i \left( \Omega - C_1^{(+)} \right) \eta_i^{(1)} + iD_1^{(+)} \eta_i^{(2)} + E_1 \eta_i^{(1)} = 0 \]  

(28)

\[ A_1 \eta_r^{(2)} + B_1 \eta_r^{(1)} - i \left( \Omega - C_2^{(+)} \right) \eta_i^{(2)} - D_2^{(-)} \eta_i^{(1)} = 0 \]  

(29)

\[ i \left( \Omega - C_2^{(-)} \right) \eta_i^{(2)} + iD_2^{(+)} \eta_i^{(2)} + E_2 \eta_i^{(2)} = 0 \]  

(30)

In the above, we assumed the space-time dependence of \( \eta_r(1), \eta_r(2), \eta_i(1), \eta_i(2) \) to be of the form \( \exp(i\lambda x - \Omega t) \). The criterion for the existence after removing the effect of higher-order terms for a nontrivial solution to equations (27) to (30) gives the nonlinear dispersion relation

\[ [(\Omega - C_1)^2 - A_1 E_1] [(\Omega - C_2)^2 - A_2 E_2] = H_1(\Omega - C_1)(\Omega - C_2) - F_1(\Omega - C_1) - F_2(\Omega - C_2) + H_2. \]  

(31)

We consider our stability analysis for approximately same group velocities of the two waves so that we take \( \delta_1(1) \approx \delta_2(2) \). From equations (17) and (19) we have \( \Omega - \delta_1(1) \lambda = O(\varepsilon^2) \) and \( \Omega - \delta_2(2) \lambda = O(\varepsilon^2) \), where \( \varepsilon \) is a small ordering parameter of \( \eta_0, \eta_0 \) and \( \lambda \).

The above equation (31) for both the wave packets reduces to

\[ [(\Omega - C_2) + 0.5F_2/((\Omega - C_1)^2 - A_1 E_1)]^2 = A_2 E_2 + [H_2 - F_1(\Omega - C_2 - \delta_1(1) \lambda)]/(\Omega - C_1)^2 - A_1 E_1, \]  

(32)

The solutions of equation (31) for both the wave packets at the minimum order are \( \Omega^{(1)} \) and \( \Omega^{(2)} \) given by

\[ \Omega^{(j)} = \delta_1^{(j)} \lambda \pm \left\{ \delta_2^{(j)} \lambda^2 / 2 \right\}^{1/2}, \quad (j = 1, 2) \]  

(33)

The condition of instability from (32) for both the wave packets is the following

\[ A_2 E_2 + [H_2 - F_1(\Omega^{(2)} - \delta_1(1) \lambda)]/(\Omega^{(2)} - C_1)^2 - A_1 E_1 < 0 \]  

(34)

After omitting the first wave packet equation (34) reduces to \( A_2 E_2 < 0 \) which can be written as

\[ \delta_2^{(2)} \lambda^2 / 2 \{ \delta_2^{(2)} \lambda^2 + 2 \gamma_1^{(2)} \lambda \delta_1(1) \} \eta_0 \eta_2 < 0 \]  

(35)

The condition (35) is similar to the instability condition (57) of Dhar and Das [10] for a single gravity wave train. Also, for \( \nu = \gamma = 0 \), the condition (35) is identical to equation (3.8) of Dysthe [5]. From (34) the region of the instability of the second wave packet in the appearance of the first wave packet are
demonstrated in figures 3(a) and 3(b) for two distinct sets of values of wave-numbers \((k_1, k_2) = (1, 0.71)\) and \((k_1, k_2) = (0.5, 0.34)\). In the said figures, we have drawn curves for marginal stability of a second wave packet having a shorter wave number. Further, we have drawn in figures 3(a) and 3(b) the similar curves of second wave packet when the first wave packet is absent.

From those figures, it has been observed that the region of the instability of the second wave packet spreads out with the appearance of the first wave packet. We have also found that the region of instability decreases slightly when the fourth-order terms are included. Again, with the enhancement of velocity of wind the region of instability is further decreased. The instability growth rate \(I_G\) of the second wave packet of larger wavelength is

\[
I_G = \left[ \frac{\delta_1(1) - \Omega(2) - \Omega(2) - \Omega(1) \lambda_H}{(1 + \delta_1)^2} - A_2E_2 \right]^{1/2}
\]

We have drawn the instability growth rate \(I_G\) of the second wave train as a function of the perturbation wave-number for two values of the amplitude of the first wave packet and for two different values of wind velocity in figures 4, 5 for two sets of values of \((k_1, k_2) = (1, 0.71)\) and \((k_1, k_2) = (0.5, 0.34)\) respectively. In the said figures, we have also plotted the growth rate of instability curves for the second wave packet when the first wave packet is absent and have drawn the similar curves which can be found from the nonlinear evolution equations of the third order.

Fig.3 Instability regions of second wave packet for distinct values of non-dimensional wind velocity \(v\) shown on the graphs. Here (a): \((k_1, k_2) = (1, 0.71)\) , (b): \((k_1, k_2) = (0.5, 0.34)\) depict results obtained from the fourth-order equation and ............. depict results obtained from the third-order equation.
Fig. 4 Instability growth rate $I_G$ of second wave packet versus the perturbed wave number $\lambda$ for distinct values of non-dimensional wind velocity $v$ shown on graphs. Here $(k_1, k_2) = (1, 0.71)$, $\eta_{02} = 0.1$ depict results obtained from the fourth-order equation and ............. depict results obtained from the third-order equation.

Fig. 5 Instability growth rate $I_G$ of second wave packet against the perturbed wave number $\lambda$ for distinct values of non-dimensional wind velocity $v$ shown on graphs. Here $(k_1, k_2) = (0.5, 0.34)$, $\eta_{02} = 0.1$, depict results obtained from the fourth-order equation and ............. depict results obtained from the third-order equation.

From the said figures, it is observed that the rate of growth of instability of the second wave packet enhances in the appearance of the first wave packet and it enhances with the enhancement of the amplitude of first wave packet. Again, we observed that the influence of higher-order term i.e. the fourth-order term is to enhance the instability growth rate. Also, the instability growth rate enhances with the enhancement of wind velocity.
4. A discussion along with conclusion

From the paper made by Dysthe [5], it has been established that a fourth-order nonlinear evolution equation is a better beginning point for investigating nonlinear influences on surface waves. In view of the paper as mentioned earlier, we have established two coupled nonlinear evolution equations of fourth-order for two surface-gravity wave packets in infinite depth of water considering the effect of air blowing over water. In the present paper, we have employed the Zakharov integral equation approach to arrive at the nonlinear evolution equations of fourth-order. Based on these nonlinear evolution equations of fourth-order, regions of instability have been plotted in figures 3(a) and 3(b) for two sets of values of $k_1$, $k_2$ and for distinct values of non-dimensional wind velocity $v$. When the first gravity wave packet is present it has been observed from graphs that the regions of instability for the second gravity wave packet get expanded with the enhancement of wave steepness of the first gravity wave packet for a settled value of the wind velocity. Again, the regions of instability become shorter with the enhancement of the wind velocity for a settled value of wave steepness of the first gravity wave packet. From figures 4, 5 it has been observed that the appearance of a wave packet of shorter wavelength enhances the rate of growth of instability of a wave packet of longer wavelength for a settled value of the wind velocity. Furthermore, the rate of growth of instability enhances with the enhancement of wind velocity for a settled value of the amplitude of the first wave packet. We have also compared the results found from evolution equations of third and fourth-order in figures 3-5. We have observed that the region of instability gets shortened slightly and the instability growth rate gets enhanced slightly at fourth-order.

Appendix:

Different coefficients of equation (17)
$$\delta_1^{(1)} = \frac{\omega_1 (r + y v)}{2 \omega_2}, \quad \delta_2^{(1)} = -\frac{\omega_1 (r^2 - y v^2)}{8 \omega_2}, \quad \delta_3^{(1)} = -\frac{\omega_1 r^3 ((1 + y) - y v^2)}{16 \omega_2}$$
$$Y_1^{(1)} = \frac{\omega_1 (y v + (1 - y^2) + y v^2)}{2 \omega_2}, \quad Y_2^{(1)} = -\frac{3 \omega_1 r (y v + (1 - y^2))}{2 \omega_2}, \quad Y_3^{(1)} = -\frac{\omega_1 r (y v + 2 (1 - y^2))}{4 \omega_2},$$
$$Y_4^{(1)} = \frac{\omega_1 r (1 + 2 y v - y^2)}{2 \omega_2}$$
$$\xi_1^{(1)} = \frac{1}{r^2} \left[ -\frac{r^2 (y v + 7 (1 - y^2) + y v^2)}{2} + \frac{r (1 - r) p_2}{4} - \frac{r (1 + r) p_3}{2} \right]$$
$$\xi_2^{(1)} = \frac{1}{r^2} \left[ -\frac{r^2 (y v + 7 (1 - y^2))}{2} + r p_1 \frac{1}{8} - \frac{r (1 - r) p_2}{8} + \frac{r (5 + 3 r) p_3}{4} - \frac{\omega_2 r}{4 \omega_1} - \frac{p_3 q_1 (-r)}{2} \right]$$
$$\xi_3^{(1)} = \frac{1}{r^2} \left[ -\frac{3 r^2 (y v + 2 (1 - y^2))}{8} - \frac{3 r}{8} + \frac{(1 - r^2) p_2}{8} + \frac{(1 + 5 r + 2 r^2) p_3}{4} + \frac{p_1 q_3 (-r)}{2} \right]$$
$$\xi_4^{(1)} = \frac{1}{r^2} \left[ -\frac{3 r^2 (y v + 2 (1 + y^2))}{8} + r p_1 - \frac{\omega_2 r}{4 \omega_1} + \frac{(-1 + 2 r - r^2) p_2}{8} + \frac{(-1 + r + 2 r^2) p_3}{4} \right]$$
$$\xi_5^{(1)} = \frac{1}{2 r}$$
Different coefficients of equation (19)

\[
\delta_1^{(2)} = \frac{1 + \gamma \nu}{2}, \quad \delta_2^{(2)} = -\frac{1 + \gamma \nu^2}{8}, \quad \delta_3^{(2)} = -\frac{(1 + \gamma) - \nu^2}{16},
\]

\[
\gamma_1^{(2)} = 1 + \{\nu + 7(1 - \gamma^2) + \nu^2\}, \quad \gamma_2^{(2)} = -\frac{3[1 + \nu + 7(1 - \gamma^2) + \nu^2]}{2}, \quad \gamma_3^{(2)} = -\frac{1 + \nu + 2(1 - \gamma^2)}{4},
\]

\[
\xi_1^{(2)} = \frac{\omega_1 r}{\omega_2} \left\{ \frac{r^2 [\nu + 7(1 - \gamma^2) + \nu^2]}{2} - \frac{r(1 - r)p_2}{4} - \frac{r(1 + r)p_3}{2} \right\}
\]

\[
\xi_2^{(2)} = \frac{\omega_1 r}{\omega_2} \left\{ -r \left[ \nu [\nu + 7(1 - \gamma^2)] - rp_1 + \frac{3(1 - r)p_2}{8} + \frac{(3 + 5r)p_3}{4} + \frac{\omega_2 r}{4\omega_1} + \frac{p_1 q_2^{(-)} r}{2} \right] \right. 
\]

\[
\xi_3^{(2)} = \frac{\omega_1 r}{\omega_2} \left\{ -\frac{r^2 [\nu + 2(1 - \gamma^2)]}{8} - \frac{5r}{8} + \frac{(1 - r)p_2}{4} + \frac{(2 + 5r + r^2)p_3}{4} + \frac{p_1 q_3^{(-)} r}{2} - \frac{p_3 q_3^{(+)} r}{2} \right\}
\]

\[
\xi_4^{(2)} = \frac{\omega_1 r}{\omega_2} \left\{ \frac{r(-5 + r)(1 + 2\nu - \gamma^2)}{8} - rp_1 + \frac{\omega_2 r}{4\omega_1} + \frac{(1 - r)p_2}{4} + \frac{(2 + r - r^2)p_3}{4} + \frac{p_1 q_1^{(-)} r}{2} \right. 
\]

\[
\xi_5^{(2)} = \frac{\omega_1 r^2}{2\omega_2},
\]

where

\[
r = \frac{k_2}{k_1}
\]

\[
p_1 = \frac{\omega_2 \omega_1^2 - (\omega_1 - \omega_2)^2}{\omega_1^2 - (\omega_1 - \omega_2)^2},
\]

\[
p_2 = \frac{\omega_2 [\omega_1^2 + (\omega_1 - \omega_2)^2]}{\omega_1 [\omega_1^2 - (\omega_1 - \omega_2)^2]},
\]

\[
p_3 = \frac{\omega_1 [\omega_1^2 + (\omega_1 - \omega_2)^2]}{\omega_1 [\omega_1^2 - (\omega_1 - \omega_2)^2]}
\]

\[
q_1^{(\pm)} = \frac{2(k_1 \pm k_2)}{\omega_1^2 - (\omega_1 \pm \omega_2)^2} \left[ (cg)_{1 \pm 2} - (\omega_1 \pm \omega_2)(cg)_1 \right]
\]

\[
q_2^{(\pm)} = \frac{2(k_1 \pm k_2)}{\omega_1^2 - (\omega_1 \pm \omega_2)^2} \left[ (cg)_{1 \pm 2} - (\omega_1 \pm \omega_2)(cg)_2 \right]
\]

\[
q_3^{(\pm)} = \frac{2(k_1 \pm k_2)(\omega_1 \pm \omega_2)}{[\omega_1^2 - (\omega_1 \pm \omega_2)^2]} \left[ (cg)_1 - (cg)_2 \right]
\]

and

\[
\omega_{i\pm j} = \omega(k_{i\pm j}) = \omega(k_i \pm k_j)
\]
\( (c g)_i = \left[ \frac{d \omega(k)}{d k} \right]_{k=k_i} \)

\( \omega(k) = \frac{k^{1/2}}{1 + \gamma} [\gamma k^{1/2}u + \{(1 - \gamma^2)g - \gamma ku^2\}]^{1/2} \).

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