Deepening the vector coherent state analysis:
Revisiting the harmonic oscillator

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Abstract

Vector coherent states (VCS) viewed as a generalization of ordinary coherent states for higher rank tensor Hilbert spaces are investigated. We consider a systematic way of generating classes of VCS which are solvable (i.e., in the present context, normalizable states satisfying a resolution of the identity) on the Hilbert space of 2D and 3D harmonic oscillators. Thanks to the type of construction, these VCS are classified according to specific criteria. Furthermore, in many cases, the found classes of VCS are continuously deformable one onto another, still remaining solvable.

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1 Introduction

Vector coherent states (VCS) are well-known objects in mathematical physics when they are particularly defined as orbits of vectors under operators of unitary representations of groups and used in a variety of symmetry problems in quantum mechanics [1]. In some earlier works, a fairly systematic method has been introduced for constructing VCS over various types of matrix domains [2, 3] in analogy with the canonical coherent states (CS), under the additional assumption of the existence of a resolution of the identity. Besides, VCS are also formulated for quantum optical models with spin-orbit interactions among which the Jaynes-Cummings model [4, 5, 6] and its deformed versions [7, 8, 9, 10]. Furthermore, in [5], the study of the Landau levels has been achieved and different classes of VCS have been rigorously defined by taking into account the degeneracy. Among precursor works on some closely related topics, one also quotes the multidimensional generalization of CS introduced in [11] defined for Hamiltonians with non-degenerate discrete spectrum [12]. These multidimensional CS serve for the determination of the thermodynamic potential of a 2D electron gas in a perpendicular magnetic field. At the theoretical level, the latter work extends the results given in [12] to a system with several degrees of freedom. An analogous procedure was used in [13] in order to obtain the CS for a free magnetic Schrödinger operator, and in [4] by introducing a class of VCS derived with matrices viewed as simple vectors in an enlarged Hilbert space. The present work deals with an extension of these three contributions by Gazeau and Novaes [11] and Thirulogasanthar et al [4, 13].

Let us come back for the moment on basic facts on CS. For any given multidimensional system with associated quantum Hilbert space spanned by some basis \{ |n⟩ \}, [n] being some multi-valued index labeling the eigenvalues of some commuting observables, there is a straightforward way to generate a CS for the r-th degree of freedom [11]:

\[ |J_r, \gamma_r, [n]\rangle = \mathcal{N}_r(J_r)^{-\frac{1}{2}} \sum_{n_r} \frac{J_r^{n_r}}{\sqrt{\rho_r(n_r)}} e^{-i\gamma_r e_r([n])} |n_r\rangle, \]  

(1)

where \((J_r, \gamma_r)\) are Gazeau-Klauder action angle variables, \(\mathcal{N}_r(J_r)\) is a normalization factor, \(e_r([n])\) some eigenvalue of r-th observable, \(\rho_r\) a free quantity at the moment which may depend on the remaining indices in [n] and so on the label of the energy level. In fact, even for the simple harmonic oscillator, there is always a freedom in the definition of CS. Indeed, in obvious notations, we have: \(|z\rangle = \mathcal{N}(|z|, \rho)^{-1/2} \sum_{n=0}^{\infty} z^n / \sqrt{\rho(n)} |n\rangle\), where \(\rho(n)\) is still a free function of the energy level n. In the specific instance of canonical CS, \(\rho(n) = n!\). We will call these functions \(\rho_r\), generalized factorials. Dealing with a generalized version of CS [1], the function \(\rho_r([n])\) may have different forms and, moreover, its dependency on the indices of [n] may entail drastic consequences on the solvability of the CS with respect to some set of axioms. Hence an issue worthwhile to be investigated is the definition of particular classes of functions \(\rho\) making the CS solvable.

In the particular instance of [13], the authors treat some CS in a rank two Hilbert space, i.e. a Hilbert space of the tensored form \(\mathcal{H} \otimes \mathcal{H}'\). Peculiar classes of CS were defined by a procedure which takes into account some particular generalized factorials in such a way that the consequent states fulfill Gazeau-Klauder axioms [12]. By scrutinizing that procedure,
one realizes that the set of CS that the authors consider is not complete and therefore can be enlarged and, more to the point, even systematized. This is the bottom line of our investigations.

It is then valuable to investigate how the above scheme can be extended and systematized to more involved Hilbert spaces provided one could make a sense of it for some basic example at first. This is what we propose to investigate here. Any systematic approach foreseeing technicalities, we will consider simple harmonic oscillators as toy models for which the same above questions could be naturally asked and indeed find nontrivial issues as we will see.

In the present study, based on the prime scheme developed in [11, 4, 13], we perform a systematic analysis of VCS associated with the harmonic oscillator in 2D and then in 3D. The VCS are built using different generalized factorials, are normalizable and have a resolution of unity. We implement a way to classify these VCS which is given by, roughly speaking, their increasing number of complex parameters (called degrees of freedom) and increasing complexity of their generalized factorials. It turns out that the VCS classes can be also understood, from another point of view, as continuous deformed classes of one into another by different frequency limits, providing a possible second type of classification. All the VCS highlighted in this work can be extended without ambiguity to VCS of more complex systems having at least two tensor copies of the harmonic oscillator as an underlying system (Landau problem with harmonic potential, n-level system with at least a two-bosonic modes such as the n-mode Jaynes-Cummings model) or matrix VCS [3].

In addition, we emphasize that

(a) we perform the investigations not at the CS but at the VCS level. The reason for that is the following: the resolution of the identity of VCS is, in a sense, weaker than the one of ordinary CS. Hence, we expect to solve more classes.

(b) as far as we are concerned with our current analysis, we restrict the sense of solvable VCS to normalizable VCS, namely with $\mathcal{N}(J) < \infty$, and satisfying a partial resolution of the identity on the Hilbert space. The continuity in label will be obvious. However, if the procedure only ensures that the VCS satisfy these basic requirements, it is not excluded at all that their properties could be improved with respect to Gazeau-Klauder physical axioms [12]. For instance, both temporal stability could be implemented with extra parameters, taken case by case, and action angle constraints investigated afterwards.

The outline of paper is the following. Section 2 is devoted to a pedagogical review of the main aspects of solvable classes of CS and VCS associated with the harmonic oscillator. These aspects give farther motivations for this work. Next, we briefly recall the work performed in [4, 13] fitting it with our specific notations and main objectives. Then Section 3 initiates the analysis of VCS with one degree and two degrees of freedom for the 2D harmonic oscillator. Starting first with simple cases of one degree of freedom, we go into the analysis in depth, revealing some internal deformation structure and symmetries between the classes of VCS. The connection with the Landau problem is discussed also therein. Section 4 focuses on the VCS of the 3D harmonic oscillator. The discussion is directly settled on two degrees of freedom while the case of three degrees of freedom is slightly mentioned, for the sake of brevity. Section 5 provides a summary of our results and an outlook of this work. Finally, an appendix collects complementary proofs and identities used in the text.
2 Revisiting harmonic oscillators: the VCS method

One of the notorious forms of coherent states of the quantum harmonic oscillator in 1D can be written

$$|z\rangle = \mathcal{N}^{-\frac{1}{2}}(z) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\rho(n)}} |n\rangle,$$

where $z \in \mathbb{C}$, $\mathcal{N}(z)$ is a normalization factor, $\{|n\rangle, n \in \mathbb{N}\}$ forms an eigenstate basis of the number operator $N|n\rangle = n|n\rangle$, $N = a^\dagger a$ associated with the Heisenberg operators $a$ and $a^\dagger$ obeying $[a, a^\dagger] = \mathbb{I}$. The function $\rho(n) = \prod_{i=0}^{n} x_i$, $\rho(0) := 1$, goes under the name of generalized factorial, with arguments $x_i = x(i)$ which are energy-built quantities related to the spectrum of the system.

Gazeau and Klauder [12] proposed a set of axioms that can be implemented on the states (2) before calling these states CS. We will focus on two of them: (a) a normalization condition, meaning that the normalization factor

$$\mathcal{N}(z) = \mathcal{N}(|z|) = \sum_{n=0}^{\infty} \frac{|z|^{2n}}{\rho(n)}$$

should be finite and (b) CS should satisfy a resolution of the identity, i.e. there exists a measure $d\mu(z)$ such that, on a complex domain $\mathcal{D} \subset \mathbb{C}$,

$$\int_{z \in \mathcal{D}} d\mu(z) |z\rangle \langle z| = \sum_{n=0}^{\infty} |n\rangle \langle n|.$$  

These two axioms are clearly mathematical statements that one can roughly summarize as to be a proof of existence (a) and the fact that this set is an overcomplete basis (b).

It is then striking that to satisfy both axioms intimately depends on the content of $\rho(n)$. Indeed, requiring (a) is equivalent to have $\lim_{n \to \infty} n \sqrt{\rho(n)} = R \neq 0$ meanwhile, using for instance polar coordinates to parameterize the complex plane $z = re^{i\theta}$, and a measure factorized as $d\mu(z) = (1/\pi)\mathcal{N}(r) rdrd\theta \varrho(r)$, with $\varrho(r)$ a positive density function of unit weight, what boils down in (b) is simply a Stieljes like moment problem

$$\int_{r \in [0,R]} 2r dr \varrho(r) r^{2n} = \varrho(n).$$

Dealing with the well known canonical CS, we set $x_n = n$ representing directly the energy level itself, the generalized factorial $\rho(n) = n!$, the radius of convergence of the norm series is infinite and the resolution of the identity turns out to be solved by $\varrho(r) = \exp[-r^2]$. Note that $x_n = n$ is intimately rooted in group theoretical considerations since this choice appears to be the one associated with the expansion of CS using a displacement type operator $|z\rangle = e^{za^\dagger - \bar{z}a}|0\rangle$. A natural question is then: What can one put in $\rho(n)$ without breaking the normalizability and integrability of the CS?

Remarkably, in the search of solvable classes of CS even for simple harmonic oscillator or related systems invoking only\(^2\) Heisenberg algebras and their deformations as an underlying

\(^2\)We exclude here any general group theoretical consideration à la Perelomov [14].
group theoretical framework, only a few number of classes was achieved. In a broad view, models with solvable sets of CS include the model of a particle in a plane subject to a magnetic field also called the Landau problem [15], atomic $n$-level systems and quantum optics models such as the Jaynes-Cummings model [16], to mention but a few. Let us emphasize some other instructive systems. The $f$-[17], $q$-[18, 19] and $(q,p)$-[20] deformed harmonic oscillators have been solved and their CS studied. Besides, CS of the Landau problem [21], Jaynes-Cummings model [6, 22] and its deformed versions were also exactly solved ([23, 24, 25] and see more references therein). In a quantum deformed framework, $\rho(n)$ can be related to the eigenvalue \{n\} of the deformed number operator \{N\}. Hence, deforming the algebra in a well controlled way is an acknowledged efficient way to map a set of solvable non deformed CS to a set of solvable deformed CS.

The inception of VCS [1] for a higher rank tensor-like Hilbert space has enable to extend the notion of CS and thereby to achieve more in the quest of solvable classes of CS of physical model. Indeed, it has been highlighted a significant number of physical appearances of VCS [5]. Moreover, the VCS formalism has been put forward for quantum systems with many degrees of freedom including two-level systems with possible degeneracy [5, 22]. The resulting states remain integrable when fully deformed and prove to preserve regular properties of CS [7, 8, 9, 10]. More theoretically, they have opened the door to a wide range of applications by extending the notion of CS defined with a unique complex variable $z$ to the notion of CS defined over complex matrix $Z$ (shortly called matrix VCS or MVCS) [3], quaternions and complex tensor domains [4]. By extending the Barut-Girardello eigenvalue problem $a|z\rangle = z|z\rangle$, to a matrix eigenvalue problem of the type $a|Z\rangle = Z|Z\rangle$, the associated with MVCS have meaningful consequences at the group representation level. In a nutshell, the VCS formalism gives a new point of view of the CS definition: it considers each set of CS as embedded in a Hilbert subspace $\mathcal{H}_k$ of a larger Hilbert space $\mathcal{H} = \otimes_k \mathcal{H}_k$, reaping the benefit of the higher rank structure of the latter.

More closely related to our present concern, it has been unraveled in [4, 13] new classes of VCS. Let us give a digest of these results which will be at the basis of our ensuing construction. Consider the operator (in units such that $c = 1$)

$$H = \frac{1}{2m}(\vec{p} - e\vec{A})^2, \quad \vec{p} = -ih\vec{\nabla},$$

(6)

describing the motion in an infinite layer of width $d$, namely $\Sigma = \mathbb{R}^2 \times [0, d]$, of a particle of mass $m$ and charge $e$ subject to a magnetic field of vector potential taken in the symmetric gauge form $\vec{A} = (1/2)\vec{B} \times \vec{r}$, where $\vec{B} = (0, 0, B)$ is the magnetic field, $\vec{r} = (x, y, z)$ the coordinate position. Moreover, the state function $\psi(x, y, z) \in L^2(\Sigma)$ satisfy Dirichlet boundary conditions $\psi(x, y, z = d) = 0 = \psi(x, y, z = 0)$, $(x, y) \in \mathbb{R}^2$, $z \in [0, d]$ being the height. The Hamiltonian (6) proves to be diagonalizable with energies given by (considering only null or negative angular momentum modes $l \leq 0$, the spectrum becoming infinitely degenerate)

$$E_{k, (l \leq 0), n} =: E_{k, n} = \omega_1 (2k + 1) + \omega_2 \left(\frac{\pi(n + 1)}{d}\right)^2$$

(7)

where $e|B|/(2m) = \omega_1$ is the cyclotron frequency, $k$ labels the Landau levels, $\hbar^2/2m = \omega_2$ and $n$ are simply the frequency and quantum number associated with the motion of a particle
on a segment under the same boundary conditions. We will omit to report the orthonormal
eigenfunctions keeping only the formal expression \( \psi_{k,n} := \psi_{k,(i\leq 0),n} \) for sake of simplicity.

The next stage was to define classes of CS. Keeping \( n_2 \) fixed and writing

\[
E_{n_1,n_2} = (2\omega_1) \left[ n_1 + \frac{1}{2} + \frac{\omega_2}{2\omega_1} \left( \frac{\pi (n_2 + 1)}{d} \right)^2 \right],
\]

\[
\rho(n_1, n_2) = (2\omega_1)^n_1 (\gamma)_{n_1}, \quad \gamma = 1 + \frac{\omega_1 d^2 + \omega_2 \pi (n_2 + 1)^2}{2\omega_1 d^2}, \quad (a)_n = \frac{\Gamma[a + n]}{\Gamma[a]},
\]

where \( \Gamma \) is the Euler gamma function and \( (a)_n \) the Pochhammer symbol, and using just one
variable \( z \), the first kind of state can be defined with one summation in \( n_1 \)

\[
|z, n_2\rangle = N^{-\frac{1}{2}}(z, n_2) \sum_{n_1=0}^{\infty} \frac{z_1^{n_1}}{\sqrt{\rho_1(n_1, n_2)}} |\psi_{n_1,n_2}\rangle.
\]

Due to the presence of the vector index \( n_2 \), one can actually call \( |z, n_2\rangle \) as VCS provided it is
normalizable and satisfies a resolution of the identity. The normalization factor is reduced to
a hypergeometric function \( _{1}F_{1}(1, \gamma; |z|^2/(2\omega_1)) \) converging everywhere in \([0, \infty)\). The state \( |z, n_2\rangle \) also obeys a resolution of identity in the sense of VCS, i.e. a partial resolution of the
identity of the entire Hilbert space:

\[
\int d\mu(z,n_2)|z,n_2\rangle\langle z,n_2| = \sum_{n_1=0}^{\infty} |\psi_{n_1,n_2}\rangle\langle \psi_{n_1,n_2}|,
\]

\[
d\mu(z,n_2) = N(r,n_2)rdrd\theta \rho(r, n_2), \quad \rho(r, n_2) = \frac{r^{2(\gamma - 1)}}{(2\omega_1)^{\gamma}\Gamma[\gamma]} \exp \left\{ -\frac{r^2}{2\omega_1} \right\},
\]

where in the last equation we have introduced a polar parametrization of \( z = re^{i\theta} \).

A second kind of VCS was introduced by switching the role of \( n_1 \) and \( n_2 \), viz fixing \( n_1 \)
and then summing \( n_2 \), and defining

\[
E_{n_1,n_2} = \omega_2 \left( \frac{\pi}{d} \right)^2 \left[ \frac{id}{\pi} \sqrt{\omega_1 \omega_2} \sqrt{2n_1 + 1} + n_2 + 1 \right] \left[ \frac{-id}{\pi} \sqrt{\omega_1 \omega_2} \sqrt{2n_1 + 1} + n_2 + 1 \right],
\]

\[
\rho(n_1, n_2) = \left( \frac{\pi}{d} \right)^{n_2} (\beta)_{n_2} (\beta)_{n_1} \beta = 2 + \frac{i d}{\pi} \sqrt{\omega_1 (2n_1 + 1)}.
\]

The resulting state which can be naturally written as \( |z, n_1\rangle \) is normalizable and, if integrated,
gives another partial identity in the second sector. These CS were the first types of CS, VCS
in fact, defined with one degree of freedom issued from the system. Indeed, one can proceed
farther and introduces the second species of CS (these states cannot be called VCS since
they do not possess a vector dependence), by considering two degrees of freedom associated
with each sector \( n_1 \) and \( n_2 \), respectively. To make matters worse, one can couple the sectors
each to other: the sums performed on a unique label \( n_i \) of one sector become dependent on the
label \( n_j \) of another sector through the generalized factorial \( \rho(n_1, n_2) \). It becomes a non

\footnote{Note that the authors used action-angle variables \((J, \alpha)\) defining their CS. We reformulate all their
results using our notations and our considerations pertaining to the only two mentioned axioms.}
trivial issue to prove the CS axioms in this case. The following cases prove to be solvable: for *independent* sums, (i.e. the sum over the label of one sector does not depend on another sector label),

\[ \rho_1(n_1) = (2\omega_1)^{n_1} \left( \frac{3}{2} \right)_{n_1}, \text{ by factorizing the partial energy } e_{n_1} = 2\omega_1 \left( n_1 + \frac{1}{2} \right), \]

\[ \rho_2(n_2) = \left( \frac{\pi}{d} \right)^{2n_2} \omega_1^2 (2)_{n_2}^2 (2)_{n_2}, \text{ by factorizing the partial energy } e_{n_2} = \left( \frac{\pi}{d} \right)^2 (n_2 + 1), \]

(13)

whereas for *dependent* sums, the following quantities lead to well defined CS:

\[ \rho_1(n_1, n_2) = \left( \frac{\pi}{d} \right)^{2n_2} \beta_{n_2} (\beta)_{n_2}, \text{ by factorizing the total energy,} \]

(14)

\[ \rho_2(n_1) = (2\omega_1)^{n_1} \left( \frac{3}{2} \right)_{n_1}, \text{ by factorizing the partial energy } e_{n_1} = 2\omega_1 \left( n_1 + \frac{1}{2} \right). \]

Using these states, the authors then discussed the axioms of temporal stability and action-identity. The same ideas can be found in [4] dealing with another kind of physical model: the two-mode Jaynes-Cummings model.

Clearly, by simple combinatorics, the picture is far to be complete: there are many cases which remain to be studied. Furthermore, for a more simple situation and not even for the above mentioned Landau-like problem, similar ideas could be applied and might lead to results not yet investigated to the best of our knowledge. Indeed, we can simplify the analysis by considering the simple harmonic oscillator in 2D and write its dimensionless energy spectrum as

\[ E_{n_1, n_2} = e_{n_1, n_2} - \text{const.} = \omega_1 (n_1 + \frac{\omega_2}{\omega_1} n_2) = \omega_2 (n_2 + \frac{\omega_1}{\omega_2} n_1). \]

(15)

According to the formalism so far, we can built four generalized factorials associated with the two harmonic subsystems

\[ \rho_{1,2}(n_{1,2}) = (\omega_{1,2})^{n_{1,2}} n_{1,2}!, \quad \rho_{1,2}(n_{1,2}, n_{2,1}) = (\omega_{1,2})^{n_{1,2}} (\gamma_{1,2})_{n_{1,2}}, \quad \gamma_{1,2} = 1 + \frac{\omega_{2,1}}{\omega_{1,2}} n_{2,1}. \]

(16)

Unexpectedly, the number of solvable VCS classes which can be built from these quantities is really significant. Then arises a question: Is there a definite way to understand these classes of CS and to give them a substantive structure?

The specific purpose of this paper is the following: Using a rigorous combinatorics, provide the largest possible set of VCS of the harmonic oscillator in 2D and 3D fulfilling the normalizability and resolution of the identity requirements. We also investigate a way to classify the VCS, to explore possible links between them and thereby giving them a sense on their own. Our formulation completes in a more precise way the aforementioned study and furthermore improve the formulation of both [11] and [14] when restricted to the harmonic oscillator.
3 2D Harmonic oscillator

Consider the Hilbert space of the quantum harmonic oscillator in 2D:

\[ \mathcal{H}_{2D} = \text{span}\ \{|n_1, n_2\}, \ n_i \in \mathbb{N}\}, \]

(17)

where \(|n_1, n_2\) is the two-mode eigenstates of the bosonic number operators \(N_i = a_i^\dagger a_i\) of two decoupled Heisenberg algebras obeying \([a_i, a_i^\dagger] = \mathbb{I}_i\). The dimensionless Hamiltonian \(H_{2D}\) associated with this system, including different frequencies \(\omega_i\) for each sector, and its eigenvalues in this basis can be written as

\[ H_{2D} = \frac{1}{\hbar} \tilde{H}'_{2D} = \sum_{i=1,2} \omega_i (a_i^\dagger a_i + \frac{1}{2}), \quad e_{n_1,n_2} = \omega_1 n_1 + \omega_2 n_2 + \frac{1}{2}(\omega_1 + \omega_2). \]

(18)

One can shift the Hamiltonian \(H_{2D}\) by the constant \(-(1/2)(\omega_1 + \omega_2)\) giving the operator \(\tilde{H}_{2D}\) with eigenvalues \(E_{n_1,n_2} = \omega_1 n_1 + \omega_2 n_2\). At the end, we will come back on the consequences of having an unshifted spectrum. We will construct various solvable classes of VCS spanning \(\mathcal{H}_{2D}\) by scrutinizing the two-tower structure of the eigenstates regarding the energies \(E_{n_1,n_2}\).

In the sequel, Subsection 3.1.1 is quite well-known but for completeness purpose it is convenient to include it as a starting point. Subsection 3.1.2 follows ideas of \[4\] and \[11\] studying multidimensional CS (i.e. CS with many parameters) on the same type of Hilbert spaces. We also introduce therein some taxonomy. Subsection 3.1.3 starts our analysis: we improve the formulation of the above works by noting some useful facts shaded in these prime studies which will enable us to systematize the determination and classification of the VCS with two degrees of freedom in Subsection 3.2.

3.1 VCS with one degree of freedom

We recall some terminology: a VCS degree of freedom is a variable \(Z\) belonging to some continuous domain in terms of which the VCS is expanded. Dealing with the harmonic oscillator in this section, VCS will be defined with one degree of freedom \(Z = z\) which simply stands for a complex variable. Two distinct classes of VCS are introduced below. They are built on the Hilbert subspace spanned by one tower \(i = 1\) or 2, the other sector being maintained fixed. The ensuing calculations are performed by selecting the tower with label \(n_1\), and, obviously, to each highlighted class corresponds another set of VCS obtained by choosing instead the tower coined by \(n_2\) and doing the calculation. We will review, in a pedagogical spirit, the elementary constructions in order to prepare the reader to more combinatorial developments induced by an increasing number of degrees of freedom.

3.1.1 First class: Canonical CS

This class (and the similar one in the case of higher number of degrees of freedom) corresponds to a straightforward extension of ordinary canonical CS associated with annihilation operator eigenvalue problem for the 1D harmonic oscillator. The set of VCS is therefore merely built with pure factorials and, in particular, for one degree of freedom, with a unique factorial.
We define
\[ \rho(n_1) = (\omega_1)^{n_1} n_1! , \] (19)
and consider the set of states
\[ |z, n_2\rangle = \mathcal{N}(z)^{-\frac{1}{2}} \sum_{n_1=0}^{\infty} \rho(n_1)^{-\frac{1}{2}} z^{n_1} |n_1, n_2\rangle , \] (20)
z being a complex variable. The next stage is to normalize these states and to find a
resolution of the identity that they should satisfy.

The normalization to unity of the states (20) is fulfilled under the condition
\[ \langle z, n_2 | z, n_2 \rangle = 1 \Leftrightarrow \mathcal{N}(|z|) = \exp \left\{ \frac{|z|^2}{\omega_1} \right\} . \] (21)

The states (20) must form also an overcomplete basis of states and so we seek for a
measure \( d\mu(z) \) such that a partial resolution of the identity\(^4\) should be satisfied:
\[ \int_{\mathcal{D}} |z, n_2\rangle \langle z, n_2 | d\mu(z) = I_{n_2} , \] (22)
where \( I_{n_2} \) is the projector onto the subspace of \( \mathcal{H} \) obtained by keeping \( n_2 \) fixed, namely
\[ I_{n_2} = \sum_{n_1=0}^{\infty} |n_1, n_2\rangle \langle n_1, n_2| . \] (23)

Using polar coordinates for the variable \( z = re^{i\theta} \in \mathbb{C} \), the measure is of the form
\[ d\mu(z) = \frac{1}{\pi} \mathcal{N}(r) \varrho(r) r \, drd\theta . \] (24)

The integration domain is \( \mathcal{D} = \mathbb{C} \) since the norm converges everywhere. The relation (22)
translates into the Stieltjes moment problem
\[ 2 \int_{0}^{\infty} r^{2n_1+1} \varrho(r) dr = (\omega_1)^{n_1} n_1! , \] (25)
which is solved by the density
\[ \varrho(r) = \frac{1}{\omega_1} \exp \left\{ -\frac{r^2}{\omega_1} \right\} . \] (26)

The above scenario is straightforward from what one could expect for getting canonical CS. As
was claimed at the very beginning, the class of VCS (20) can be simply viewed as CS
of a harmonic oscillator in 1D attached (i.e. tensored) to some fixed vector of an abstract
Hilbert space: \( |z, n_2\rangle = |z\rangle \otimes |n_2\rangle \).

\(^4\)In order to obtain the resolution of the identity on the entire Hilbert space \( \mathcal{H} \) one should sum over the
index \( n_2 \) in (22).
3.1.2 Second class: $\gamma$-deformed CS

The second class VCS cannot be simply associated with ordinary canonical CS as it was the case for the first class VCS. To be properly defined, here the system of VCS requires at least a harmonic oscillator in 2D with unbalanced frequencies in each direction. On the computational side, the states do not involve a simple factorial but a generalized factorial: the Pochhammer symbol. Dealing with one degree of freedom, we just have one such a symbol.

One starts by observing that, setting $\omega_i \neq 0$, the eigenenergies can be factorized as

$$E_{n_1, n_2} = \omega_1 \left[ n_1 + \frac{\omega_2}{\omega_1} n_2 \right],$$

and fixing again the tower labeled by $n_2$, the following quantities can be defined

$$\rho(n_1) = \prod_{k=1}^{n_1} \omega_1 \left[ k + \frac{\omega_2}{\omega_1} n_2 \right] = (\omega_1)^{n_1} (\gamma)^{n_1},$$

$$\gamma = 1 + \frac{\omega_2}{\omega_1} n_2, \quad (\gamma)^{n_1} = \frac{\Gamma(n_1 + \gamma)}{\Gamma(\gamma)},$$

where $(\gamma)^{n}$ stands for the Pochhammer symbol. Note that, implicitly, $\gamma$ depends on $n_2$.

We introduce the set of vectors

$$|z, n_2\rangle = \mathcal{N}(z, n_2)^{-\frac{1}{2}} \sum_{n_1=0}^{\infty} \rho(n_1)^{-\frac{1}{2}} z^{n_1} |n_1, n_2\rangle$$

which can be normalized to unity according to

$$\langle z, n_2 | z, n_2 \rangle = 1, \quad \mathcal{N}(z, n_2) = \sum_{n_1=0}^{\infty} \frac{1}{(\gamma_1)^{n_1} \omega_1^n} = \,_{1}F_{1} \left( 1; \gamma; \frac{|z|^2}{\omega_1} \right),$$

where $_1F_1(\cdot)$ denotes the ordinary confluent hypergeometric function. The convergence radius of the series $_1F_1$, as for any other hypergeometric function, can be determined by a simple ratio test (or by comparison test since $1/\Gamma[\gamma_1 + n_1] \leq 1/n_1!$ for $\gamma \geq 1.$). It can be checked that (31) converges everywhere in the complex plane. The form for this series can be given in general by

$$\,_{1}F_{1} \left( 1; a; z \right) = e^z z^{-a} (\Gamma[a + 1] - a\Gamma[a, z]),$$

where $\Gamma[a, z] = \int_{z}^{\infty} t^{a-1} e^{-t} dt$ is the incomplete Euler-gamma function with ordinary conditions on the complex number $a$.

On the domain $D = \mathbb{C}$, consider the measure

$$d\mu(z, n_2) = \frac{1}{\pi} \mathcal{N}(z, n_2) \varrho(r, n_2) r dr d\theta.$$

---

5 We will always use the same notation for different VCS made with the same dependencies. For instance the VCS (20) and (30) are both denoted $|z, n_2\rangle$. This is to avoid useless proliferation of notations and, as noted, the sense they will refer to remains unambiguous.
The set of states (30) ought to satisfy the partial resolution of the identity
\[ \int_{\mathcal{D}} \langle z, n_2 \rangle \langle z, n_2 \rangle d\mu(z, n_2) = I_{n_2}, \] (34)
where \( I_{n_2} \) is again the projector (23). From (34), one infers the moment problems
\[ 2 \int_0^\infty r^{2n_1+1} \varrho(r, n_2) dr = \rho(n_1), \] (35)
solved by
\[ \varrho(r, n_2) = \frac{1}{\Gamma(\gamma)(\omega_1)^{\gamma}} r^{2(\gamma-1)} \exp \left\{ -\frac{r^2}{\omega_1} \right\}. \] (36)
Hence the states (30) define a different class of VCS.

3.1.3 Lessons from the construction with one degree of freedom

Let us start our deepening analysis of these results that will enable us to improve the above procedure of building the VCS and, from that, extending them for more degrees of freedom.

There is a dual picture to the above construction of VCS, as performed in Subsections 3.1.1 and 3.1.2, that is interesting to point out and to investigate for its properties. Indeed, the role of \( n_1 \) and \( n_2 \) being interchangeable, this implies that the other generalized factorial
\[ \rho(n_2) = (\omega_2)^n_2 n_2! \] (37)
could equally serve to construct another set of VCS with one degree of freedom and meeting all requirements, that we call dual class associated with (20) and that we denote
\[ |z, n_1\rangle = N(z) - \frac{1}{2} \sum_{n_2=0}^{\infty} \rho(n_2) - \frac{1}{2} z^{n_2} |n, n_2\rangle. \] (38)
One notes that the initial class and its dual are both of first class. The first class therefore contains two canonical sets of VCS in addition with the similar one when we will be dealing with higher number of degrees of freedom.

In analogy with the dual VCS (38), a second dual class of the set of VCS (30) can be built. Again by switching \((1 \leftrightarrow 2)\), the number \( \gamma \) (29) has a dual counterpart
\[ \gamma_1 = 1 + \frac{\omega_2}{\omega_1} n_2, \quad \gamma_2 = 1 + \frac{\omega_1}{\omega_2} n_1. \] (39)
Within this framework, since all derivations remain the same, under \((1 \leftrightarrow 2)\), a set of VCS dually associated with (30) can be generated. We have:
\[ |z, n_2\rangle_\gamma = |z, n_1\rangle_\gamma, \quad |z, n_1\rangle_\gamma = N(z, n_1)^{-\frac{1}{2}} \sum_{n_2=0}^{\infty} \rho(n_2)^{-\frac{1}{2}} z^{n_2} |n, n_2\rangle, \] (40)
where we add an index \( \gamma \) in order to distinguish the above VCS second class from the first class one.
One notices the following interesting fact which has been never discussed so far in the literature, to the best of our knowledge. At the limit $\gamma_{1,2} \rightarrow 1$ (limit when one of the two frequencies becomes much greater than the other, namely $\omega_{1,2} \gg \omega_{2,1}$), the first class set of VCS $\text{(30)}$ and its dual $\text{(40)}$ smoothly tend to the set of VCS $\text{(20)}$ and its dual $\text{(38)}$, respectively, with smooth measure deformations:

$$
\lim_{\gamma_{1,2} \rightarrow 1} \frac{1}{\omega_{1,2}^{\gamma_{1,2}} \Gamma(\gamma_{1,2})} r^{2(\gamma_{1,2}-1)} \exp \left\{ -\frac{r^2}{\omega_{1,2}} \right\} = \frac{1}{\omega_{1,2}} \exp \left\{ -\frac{r^2}{\omega_{1,2}} \right\}.
$$

(41)

Hence the name of $\gamma$-deformed VCS. For one degree of freedom, each of the two classes of VCS may define a unique set of VCS in that particular limit.

Two VCS classes are said to be of the same type if there exists a continuous limit under which one of them can be mapped onto the other. The first on which the limit is performed will be called ancestor and the second resulting state descendant. Thus, an ancestor and a descendant are of the same type. For example, above, the second classes are ancestors while the first class limits are descendants.

**Remark 1** - The construction of some VCS classes starting from building at first the second class VCS (ancestor) might be more efficient since, by a large frequency limit, one could deduce the corresponding first class (descendant) of the same type.

Another relevant remark on the above construction is the following: consider the moment problem for the second class that is given by

$$
\int_0^\infty r^{2n_1+1} \varrho(r, n_2) dr = (\omega_1)^{n_1} (\gamma_1)_{n_1},
$$

(42)

with solution

$$
\varrho(r, n_2) = \frac{1}{\Gamma(\gamma_1) (\omega_1)^{\gamma_1}} r^{2(\gamma_1-1)} \exp \left\{ -\frac{r^2}{\omega_1} \right\}.
$$

(43)

A closer look on this expression (43) shows that $\gamma_1 - 1 = \frac{\omega_2}{\omega_1} n_2$. Then, we would like to trade an extra factor in the measure, say $r^{2(\gamma_1-1)}$, for an extra variable in the VCS. In order to do so, let us introduce the new and modified generalized factorial (to be compared with (28))

$$
\rho(n_1, n_2) = \omega_1^{n_1 + \frac{\omega_2}{\omega_1} n_2} \Gamma(\gamma_1)(\gamma_1)_{n_1} = \omega_1^{n_1 + \frac{\omega_2}{\omega_1} n_2} \Gamma(\gamma_1 + n_1)
$$

(44)

for which a second class of VCS can be defined as well as

$$
|z, n_2\rangle' = N(z, n_2) \sum_{n_1=0}^{\infty} \rho(n_1, n_2)^{-\frac{1}{2}} z^{n_1 + \frac{\omega_2}{\omega_1} n_2} |n_1, n_2\rangle,
$$

(45)

$$
N(|z|, n_2) = \frac{1}{\Gamma(\gamma_1)} \left[ \frac{|z|^2}{\omega_1} \right]^{\frac{\omega_2}{\omega_1} n_2} \frac{1}{\Gamma(\gamma_1 + n_1)} _1F_1 \left( 1; \frac{|z|^2}{\omega_1} \right).
$$

(46)

The measure density integrating to unity these states at fixed $n_2$ has to be solution of

$$
2 \int_0^\infty r^{2(n_1 + \frac{\omega_2}{\omega_1} n_2)+1} \varrho(r, n_2) dr = \omega_1^{n_1 + \frac{\omega_2}{\omega_1} n_2} \Gamma(\gamma_1 + n_1)
$$

(47)
yielding a simpler formula

\[ \varrho(r) = \frac{1}{\omega_1} \exp \left\{ -\frac{r^2}{\omega_1} \right\}. \]  

(48)

Note that the two VCS, (45) and (30), are simply connected by a factor

\[ |z, n_2\rangle' = \left( \frac{z}{\sqrt{\omega_1}} \right)^{\frac{1}{2} \omega_1 n_2^2} |z, n_2\rangle. \]  

(49)

Finally, the choice (45) for defining the second class of VCS does have the advantage to display how explicitly, in the limit \( \gamma_{1,2} \to 1 \) generated by \( \kappa_{1,2} \to 0 \), the class of VCS (45) converges to the class (20). This is our

**Remark 2** - Defining the second class of VCS, use the \( \gamma \)-modified generalized factorial (44).

Finally, there can be variant forms of the previous VCS remaining still integrable to unity that, for completeness purpose, one should also address and list. So far, we emphasized the generalized factorials of the form (19) or (44) for building the VCS. However, for one degree of freedom and still summing only on \( n_1 \), there is some freedom in the choice of the prefactor \( \omega_1 \) and the complex variable \( z \). Indeed, comparing (19), (28) and (44), we see that the exponent of \( \omega_1 \) changes of form. The same observation holds for the exponent of the variable \( z \) on which the different classes of VCS are based. We can think of them as new classes, called below *sub-classes*, of VCS defined as (still summing on the tower \( n_1 \))

\[ |z, n_2\rangle = \mathcal{N}(z, n_2)^{-\frac{1}{2}} \sum_{n_1=0}^{\infty} a(n_1, n_2, z) |n_1, n_2\rangle, \]  

(50)

with general term \( a(n_1, n_2, z) = \omega_1^{-\frac{1}{2}(n_1+\bullet)} R^{-\frac{1}{2}}(n_1, n_2) z^{n_1+\bullet'} \) where \( \bullet, \bullet' \in \{0, \kappa_1 n_2\} \) and \( R(n_1, n_2) \) is a (generalized) factorial. One may wonder if having introduced these sub-classes is not in contradiction with the improvement procedure of the previous remark. Clearly, doing so will have again the effect to modify the exponents which could have led to a simple density solution of the moment problem for these states. Hence, at this point, the answer is yes. However, in general, we will see that proceeding in the same manner, when one has more degrees of freedom, will have an effect during the integration and will lead to new classes.

The measure \( d\mu(z, n_2) = (1/\pi) \mathcal{N}(|z|, n_2) r dr d\theta \varrho(r, n_2) \) integrating to unity these variant states (4 for each class) can be determined by solving the following generalized moment problem with parameters \( \alpha, \beta, \alpha', \beta' \), (to be fixed later),

\[ 2 \int r dr \varrho(r, n_2) \frac{r^{\alpha n_1 + \beta \kappa_1 n_2}}{\omega_1^{\alpha n_1 + \beta' \kappa_1 n_2}} = R(n_1, n_2). \]  

(51)

Its solutions are given by:

\[ R(n_1, n_2) = n_1!, \quad \varrho_1(n_1, n_2) = \frac{\alpha}{\omega_1^{\alpha}} \left( \frac{\omega_1^\beta r^{-\beta}}{\omega_1^{\beta'} r^{\alpha'} 2(\alpha-\beta)} \right)^{\frac{\kappa_1 n_2}{2}} e^{-\frac{2\alpha}{r^{2(1-\alpha)} \omega_1^\alpha}}, \]

\[ R(n_1, n_2) = \Gamma[n_1 + \gamma_1], \quad \varrho_1(n_1, n_2) = \frac{\alpha}{\omega_1^{\alpha}} \left( \frac{\omega_1^\beta (\omega_1^\beta - \omega_1^{\beta'}) r^{2(\alpha-\beta)}}{\omega_1^{\beta'} 2(\alpha-\beta)} \right)^{\frac{\kappa_1 n_2}{2}} e^{-\frac{2\alpha}{r^{2(1-\alpha)} \omega_1^\alpha}}. \]  

(52)
so that (50) defined with free parameters
\[ a(n_1, n_2, z) = \omega_1^{-\frac{1}{2}(\alpha' n_1 + \beta' n_2)} R(n_1, n_2) z^{\alpha n_1 + \beta n_2} \]
determines the most extended class of VCS generating all sub-classes characterized by \( \bullet, \bullet' \in \{0, \kappa_1 n_2\} \). The parameters are to be fixed as \( \alpha, \beta, \alpha', \beta' \in \{0, 1\} \). Given \( f(r, \omega) = (1/\omega)e^{-r^2/\omega} \), the following tables yield the different sub-classes generated by this restriction:

### First class (1): \( R(n_1) = n_1! \)

| A | \( a(n_1, n_2, z) = [n_1!]^{-1/2}(z/\omega_1^{1/2})^{n_1} \) | \( g(r) = f(r, \omega_1) \) |
| B | \( a(n_1, n_2, z) = [(\omega_1)^{\kappa_1 n_2}n_1!]^{-1/2}(z/\omega_1^{1/2})^{n_1} \) | \( g(r, n_2) = \omega_1^{\kappa_1 n_2} f(r, \omega_1) \) |
| C | \( a(n_1, n_2, z) = [n_1!]^{-1/2}(z/\omega_1^{1/2})^{n_1} z^{\kappa_1 n_2} \) | \( g(r, n_2) = (r^{2\kappa_1 n_2})^{-1} f(r, \omega_1) \) |
| D | \( a(n_1, n_2, z) = [n_1!]^{-1/2}(z/\omega_1^{1/2})^{n_1 + \kappa_1 n_2} \) | \( g(r, n_2) = [\omega_1/r^{2\kappa_1 n_2}] f(r, \omega_1) \) |

### Second class (\( \gamma_1 \)-deformed): \( R(n_1, n_2) = \Gamma[\gamma_1 + n_1] \)

| A | \( a(n_1, n_2, z) = [\Gamma[\gamma_1 + n_1]]^{-1/2}(z/\omega_1^{1/2})^{n_1 + \kappa_1 n_2} \) | \( g(r) = f(r, \omega_1) \) |
| B | \( a(n_1, n_2, z) = [(\omega_1)^{\kappa_1 n_2}\Gamma[\gamma_1 + n_1]]^{-1/2}(z/\omega_1^{1/2})^{n_1} \) | \( g(r, n_2) = r^{2\kappa_1 n_2} f(r, \omega_1) \) |
| C | \( a(n_1, n_2, z) = [\Gamma[\gamma_1 + n_1]]^{-1/2}(z/\omega_1^{1/2})^{n_1} z^{\kappa_1 n_2} \) | \( g(r, n_2) = \omega_1^{-\kappa_1 n_2} f(r, \omega_1) \) |
| D | \( a(n_1, n_2, z) = [\Gamma[\gamma_1 + n_1]]^{-1/2}(z/\omega_1^{1/2})^{n_1} \) | \( g(r, n_2) = [r^{2}/\omega_1^{\kappa_1 n_2} f(r, \omega_1) \propto (30) \)

As expected, the first and second sub-classes B, C and D (denoted by (1)B, (1)C, (1)D and \( \gamma_1 \)B, \( \gamma_1 \)C, \( \gamma_1 \)D, respectively) are not so enlightening: they simply appear as factors of the VCS (first and second) sub-class of the kind A (denoted by (1)A and \( \gamma_1 \)A, respectively), the latter being at the basis of the previous analysis. Indeed, to get the sub-class (1)B, (1)C and (1)D, one has just to multiply the VCS sub-class (1)A by the factor \( \omega_1^{\kappa_1 n_2/2} \), \( z^{\kappa_1 n_2} \) and \( (z/\omega_1^{1/2})^{\kappa_1 n_2} \), respectively. Similar relations hold for the second sub-classes \( \gamma_1 \)A, \( \gamma_1 \)B, \( \gamma_1 \)C and \( \gamma_1 \)D, as it can be easily checked. Note also that the second class VCS (30) coincides with the second sub-class \( \gamma_1 \)D up to a \( \Gamma(\gamma_1) \) factor. Hence defining (45) or (30) as a second class does not have any importance: they only differ by a factor.

In the present instance, we introduce the following definition: Given a number of degrees of freedom and a VCS class, a sub-class of states is called irrelevant or a factor if any of its VCS representative can be explicitly written as a product involving a previous VCS of some different sub-class. Otherwise, a sub-class is called relevant and will matter in our classification. Note that it becomes a matter of choice to identify a prime set of states from which one determines if other sub-classes are factors of this set or not.

Dual classes can be introduced again by \( (1 \leftrightarrow 2) \) and will share similar properties. Last, all sub-classes introduced so far are of the same type as sub-class (1)A (20) by observing the limit \( \kappa_1 \to 0 \).

**Remark 3** - A special combinatorics has to be taken on the exponents of the frequencies \( \omega_i \) and degree of freedom \( z_j \), as these could generate relevant VCS sub-classes.

Figure [I] gives a diagrammatical summary of the VCS classes studied so far as well as their dependence.
3.2 VCS with two degrees of freedom

In this subsection, the classes of VCS are equipped with two complex variables, so according to our definition, two degrees of freedom. There is another subtlety here due to the fact that the sectors 1 and 2 can be coupled or not in the definition of the generalized factorial.

3.2.1 Second and first classes: Generators of bi-CS

These classes are direct generalization of VCS first and second classes as defined in Subsections 3.1.1 and 3.1.2. By Remark 1, we notice that the first class can be deduced from the second one, and so we will start by building the second class. Besides, the following construction is made using both the towers \( i = 1 \) and 2 but in an asymmetric way. The dual construction can be easily recovered by switching the role of 1 and 2. In the present case, computations involve one simple and one generalized factorials.

**Second class:** \((\gamma, 1)\)-deformed VCS - Consider \( n_1 \) and \( n_2 \) fixed by the energy spectrum, \( \rho_{1}(n_1, n_2) \) given by (44) that is, in new notations,

\[
\rho_1(n_1, n_2) = (\omega_1)^{n_1 + \kappa_1 n_2} \Gamma(\gamma_1 + n_1), \quad \gamma_1 = \gamma_1(n_2) = 1 + \kappa_1 n_2, \quad \kappa_1 = \frac{\omega_2}{\omega_1},
\]

and \( \rho_2(n_2) \) given by a simple factorial (19). Then, the set of states with two degrees of freedom

\[
|z_1, z_2, n_2\rangle = N(z_1, z_2, n_2)^{-\frac{1}{2}} \sum_{n_1=0}^{\infty} \left[ \rho_1(n_1, n_2) \rho_2(n_2) \right]^{-\frac{1}{2}} z_1^{n_1 + \kappa_1 n_2} z_2^{n_2} |n_1, n_2\rangle, \tag{55}
\]

where \( z_i \in \mathbb{C}, i = 1, 2 \), will draw our attention.

A direct inspection shows us that this state is a factor of the second class VCS with one degree of freedom (45)

\[
|z_1, z_2, n_2\rangle = z_2^{n_2} [(\omega_2) n_2!]^{-\frac{1}{2}} |z_1, n_2\rangle. \tag{56}
\]

However, having more degrees of freedom, it then defines another relevant class. In fact, due to this extra degree of freedom, (55) generates sub-classes which are not factors of any of the sub-classes previously defined.

The states (55) satisfy the normalization condition

\[
\langle z_1, z_2, n_2 | z_1, z_2, n_2 \rangle = 1, \quad N(|z_1|, |z_2|, n_2) = \frac{1}{\Gamma(\gamma_1) n_2! \left( (\omega_1) n_1 \right)^{\kappa_1} \left( (\omega_2) n_2 \right)^{\kappa_2}} \frac{1}{\Gamma(\gamma_1) n_2! \left( (\omega_1) n_1 \right)^{\kappa_1} \left( (\omega_2) n_2 \right)^{\kappa_2}} \frac{1}{\Gamma(\gamma_1) n_2! \left( (\omega_1) n_1 \right)^{\kappa_1} \left( (\omega_2) n_2 \right)^{\kappa_2}} 1 F_1 \left( 1; \gamma_1; \frac{|z_1|^2}{\omega_1} \right). \tag{57}
\]
which converges everywhere in the complex plane. Defining the measure, in polar coordinate $z_k = r_k e^{i\theta_k}$, 
\[ d\mu(z_1, z_2) = \frac{1}{\pi^2} \mathcal{N}(z_1, z_2) \rho_1(r_1, n_2) r_1 dr_1 d\theta_1 \rho_2(r_2) r_2 dr_2 d\theta_2 \]  
(58)
on $D_1 \times D_2 = \mathbb{C}^2$, the VCS satisfy the partial resolution of the identity:
\[ \int_{D_1 \times D_2} |z_1, z_2, n_2\rangle\langle z_1, z_2, n_2| d\mu(z_1, z_2, n_2) = I_{n_2}. \]  
(59)

The moment problems issued from (59) are of two forms: one satisfied by $\rho_1(r_1, n_2)$ which is of the kind (47), therefore the corresponding density solution $\rho_1(r_1, n_2)$ does not actually depend on $n_2$ and coincides with (48); another moment problem for $\rho_2(r_2)$ which is of the kind (25) and so is solved by (26). Thus the relation (55) forms a VCS class that we call $(\gamma_1, 1)$-class.

First class: (1,1)-generators of bi-CS - In order to obtain the VCS first class, one performs the continuous limit $\kappa_1 \to 0$ in (55) and gets the set of VCS:
\[ |z_1, z_2, n_2\rangle = \mathcal{N}(z_1, z_2, n_2)^{-\frac{1}{2}} \sum_{n_1=0}^{\infty} [\rho_1(n_1) \rho_2(n_2)]^{-\frac{1}{2}} z_1^{n_1} z_2^{n_2} |n_1, n_2\rangle. \]  
(60)

Summing on the remaining index $n_2$, the states (60) generate the so called bi-CS as constructed in [5]. Hence, $|z_1, z_2, n_2\rangle \propto z_2^{n_2} |z_1\rangle \otimes |n_2\rangle$, with $|z_1\rangle$ the canonical CS.

The norm series
\[ \mathcal{N}(|z_1|, |z_2|, n_2) = \frac{1}{n_2!} \left[ \frac{|z_2|^2}{\omega_2} \right]^{n_2} \frac{1}{\omega_1} \exp \left\{ \frac{|z_1|^2}{\omega_1} \right\} \]  
(61)

converges everywhere in $\mathbb{C}$. The following measure
\[ d\mu(z_1, z_2) = \frac{1}{\pi^2} \mathcal{N}(z_1, z_2) \prod_{k=1}^{2} \rho_k(r_k) r_k dr_k d\theta_k, \]  
(62)
on $D_1 \times D_2 = \mathbb{C}^2$, is considered. The class of VCS (60) satisfies a partial resolution of the identity like (59); its moment problems are identical to (25) and again its densities $\rho_k(r_k)$ are given by (26). As a result, the states (60) consist in a $(1,1)$-nondeformed VCS class.

3.2.2 Solvable sub-classes

Let us now discuss on sub-classes which occur in the present study. As previously performed, we proceed in three phases: (I) to solve the most general class of VCS with deformation parameters; (II) to restrict these parameters to be valued in $\{0, 1\}$ in order to get the simplest sub-classes and to specify which of these classes are relevant in the sense that we have already defined. The following discussion will be valid for the remaining subsections.
(I) Solving the generalized moment problem. Writing the generalized state parametrized by the real numbers $\alpha_i \neq 0, \beta_i, \alpha'_i \neq 0, \beta'_i, i = 1, 2,$

$$|z_1, z_2, n_2\rangle = N(z_1, z_2, n_2)^{-\frac{1}{2}} \times \sum_{n_1=0}^{\infty} \frac{1}{[\omega_1^{\alpha'_1 n_1+\beta'_1 n_2}] R_1(n_1, n_2) [\omega_2^{\alpha'_2 n_2+\beta'_2 n_1}] R_2(n_2)^{\frac{1}{2}}} z_1^{\alpha_1 n_1+\beta_1 n_2} z_2^{\alpha_2 n_2+\beta_2 n_1} |n_1, n_2\rangle,$$

where $R_1(n_1, n_2) \in \{\Gamma[\gamma_1 + n_1], n_1\}$ and $R_2(n_2) = n_2!$, one can check that the state (63) is normalizable with a norm series of infinite radius of convergence.

The moment problem associated with the generalized class (63) is given by (we work up to unessential $\alpha_i^{(r)}, \beta_i^{(r)}$ constant dependencies in the measure $d\mu$ obtained after phase integrations)

$$2.2 \int r_1 dr_1 r_2 dr_2 \chi(r_1, r_2, n_2) \frac{r_1^{2(\alpha_1 n_1+\beta_1 n_2)}}{\omega_1^{\alpha'_1 n_1+\beta'_1 n_2}} \frac{r_2^{2(\alpha_2 n_2+\beta_2 n_1)}}{\omega_2^{\alpha'_2 n_2+\beta'_2 n_1}} = R_1(n_1, n_2) R_2(n_2)$$

where $\chi(r_1, r_2, n_2)$ is the generalized measure density that we have to determine. Passing to square variables $u_i = r_i^2, i = 1, 2$, the moment problem takes the form:

$$\int du_1 du_2 \chi(u_1, u_2, n_2) \left[ \frac{u_1^{\alpha_1} u_2^{\beta_2}}{\omega_1^{\alpha'_1} \omega_2^{\beta'_2}} \right]^{n_1} \left[ \frac{u_1^{\beta_1}}{\omega_1^{\beta'_1}} \right]^{n_2} = R_1(n_1, n_2) R_2(n_2).$$

Interesting properties now emerge from this multivariate moment problem that we ought to underline. Indeed, one realizes that there exist many solutions $\chi(u_1, u_2, n_2)$ of (65) and therefore one is led to the non unicity of the measure which integrates the VCS. The simple reason why this holds is the freedom afforded by the index $n_2$ present in the density. This non unicity could be studied for its own interest using a weaker version of the multivariate Carleman-Nussbaum criterion [26][27].

To find a density solution of (65) can be tackled in different ways (see Appendix A). The main issue here is to find an efficient and non singular change of variables pertaining to (65), so that this problem is reduced to a simpler one yielding solutions in such a way that our classification may be still achieved.

The following densities given in radial variables solve the problem (65) for $R_1(n_1) = n_1!$ and $R_2(n_2) = n_2!$ describing a first class of VCS of the kind (60) (Appendix A provides details on this result):

$$\varrho_1(r_1, r_2, n_2) = \alpha_1 \frac{r_1^{2(\alpha_1-1)} r_2^{2\beta_2}}{\omega_1^{\alpha'_1} \omega_2^{\beta'_2}} \left[ \frac{\beta_1}{\beta'_1} \right] \left[ \frac{r_1^{\beta_1}}{\omega_1^{\beta'_1}} \right]^{n_2} e^{-\frac{2\alpha_1}{\omega_1^2} \frac{2\beta_2}{\omega_2^2} r_2^2}, \quad \varrho_2(r_2) = \alpha_2 \frac{1}{\omega_2^2} r_2^{2(\alpha_2-1)} e^{-\frac{2\alpha_2}{\omega_2^2} \frac{2\beta_2}{\omega_2^2} r_2^2}.$$

$$\chi(r_1, r_2, n_2) = \alpha_1 \alpha_2 \frac{r_1^{2(\alpha_1-1)} r_2^{2(\alpha_2+\beta_2)}}{\omega_1^{\alpha'_1} \omega_2^{\beta'_2}} \left[ \frac{\beta_1}{\beta'_1} \right] \left[ \frac{r_1^{\beta_1}}{\omega_1^{\beta'_1}} \right]^{n_2} e^{-\frac{2\alpha_1}{\omega_1^2} \frac{2\beta_2}{\omega_2^2} r_2^2 - \frac{2\alpha_2}{\omega_2^2} \frac{2\beta_2}{\omega_2^2} r_2^2}.$$

Considering on the contrary $R_1(n_1, n_2) = \Gamma[\gamma_1 + n_1]$ and $R_2(n_2) = n_2!$ being data for second class VCS [55], one gets the solutions (see Appendix A):

$$\varrho_1(r_1, r_2, n_2) = \alpha_1 \frac{r_1^{2(\alpha_1-1)} \beta_2}{\omega_1^{\alpha'_1} \omega_2^{\beta'_2}} \left[ \frac{(\beta'_1-\alpha'_1) r_1^{\beta_2}}{\omega_1^{\beta'_1} \omega_2^{\beta'_2}} \right] \left[ \frac{r_1^{\beta_1}}{\omega_1^{\beta'_1}} \right] \left[ \frac{r_1^{\beta_1}}{\omega_1^{\beta'_1}} \right]^{n_2} e^{-\frac{2\alpha_1}{\omega_1^2} \frac{2\beta_2}{\omega_2^2} r_2^2},$$

$$\varrho_2(r_1, r_2, n_2) = \alpha_2 \frac{1}{\omega_2^2} r_2^{2(\alpha_2-1)} e^{-\frac{2\alpha_2}{\omega_2^2} \frac{2\beta_2}{\omega_2^2} r_2^2}.$$
\( \varrho_2(r_2) = \alpha_2 \frac{1}{\omega_2^2 r_2} e^{-\frac{2\omega_2}{r_2}}. \tag{67} \)

(II) Extracting the relevant sub-classes. We can first analyze these solutions and con-
sequently organize the sub-classes. First a quick checking shows that the VCS second class
(55) corresponds to the sub-class called A, (denoted by \( (\gamma_1, 1)A \)), defined by the 8-tuple
\[
(\alpha_1, \beta_1, \alpha'_1, \beta'_1, \alpha_2, \beta_2, \alpha'_2, \beta'_2) = (1, 1, 1, 1, 0, 1, 0) \tag{68}
\]
which yields the correct solutions of the moment problem, \( \varrho_1 \) and \( \varrho_2 \), substituting these
parameters in (67). The first class limit (60), also named sub-class A, (denoted by \( (1, 1)A \)),
is defined by
\[
(\alpha_1, \beta_1, \alpha'_1, \beta'_1, \alpha_2, \beta_2, \alpha'_2, \beta'_2) = (1, 0, 1, 0, 1, 0, 1, 0) \tag{69}
\]
and here \( \varrho_1 \) and \( \varrho_2 \) (66) resolve the moment problem for this class, given these parameters.

Henceforth, we can restrict to the situation where \( \alpha_i^{(0)}, \beta_i^{(0)} \in \{0, 1\} \) and, since \( \alpha_i^{(0)} \) should
be always fixed to 1, we have just to analyze different cases for the quadruple \( (\beta_1, \beta'_1, \beta_2, \beta'_2) \).
The sub-classes which are irrelevant are just factors of the first or second class of this section.
These include \( (1, 0, 0, 0), (0, 1, 0, 0) \). The relevant sub-classes are defined by tuples which
contain an exponent \( n_1 \) in the sector \( (\omega_2, z_2) \), since they are getting involved in the series.
The sub-classes are defined by
\[
(0, 0, 0, 1), (0, 0, 1, 0), (1, 1, 0, 1), (1, 1, 1, 1). \tag{70}
\]
Hence it remains six tuples which can be listed as:
\[
(1, 0, 0, 1), (1, 0, 1, 0), (1, 0, 1, 1), \tag{72}
(0, 1, 0, 1), (0, 1, 1, 0), (0, 1, 1, 1). \tag{73}
\]
Each VCS defined by (72) is a factor of a sub-class in (70) and each VCS defined by a tuple
in (73) becomes also a factor of some sub-class already listed in (71). To be even more
precise, we do not need to compute six sub-classes for each class: only the quadruples in
(70) are relevant with respect to the first class (55) whereas the quadruples in (71) are the
only relevant ones for the second class (60).

Given a general form of the VCS with two degrees of freedom
\[
|z_1, z_2, n_2\rangle = N(z_1, z_2, n_2)^{-\frac{1}{2}} \sum_{n_1=0}^{\infty} a(n_1, n_2, z_1, z_2) |n_1, n_2\rangle, \tag{74}
\]
the following table gives the values of \( a \equiv a(n_1, n_2, z_1, z_2) \) and measure densities \( \varrho_k, k = 1, 2 \),
corresponding to the relevant first sub-classes:
First class \((1,1)\)-generator of bi-CS: \(R_1(n_1) = n_1!, \ R_2(n_2) = n_2!\)

\[
A \quad a = \left[ \frac{z_1^2}{\omega^2} + \frac{\omega^2}{z_2} \right]^{n_1} \left[ \frac{z_2^2}{\omega^2} + \frac{\omega^2}{z_1} \right]^{n_2}; \quad \varrho_k(r_k) = f(r_k, \omega_k), \ k = 1, 2 \quad (60)
\]

\[
B \quad a = \left[ \frac{z_1^2}{\omega^2} + \frac{\omega^2}{z_2} \right]^{n_1} \left[ \frac{z_2^2}{\omega^2} + \frac{\omega^2}{z_1} \right]^{n_2}; \quad \varrho_1(r_1) = f(r_1, \omega_1 \omega_2^2); \ \varrho_2(r_2) = f(r_2, \omega_2)
\]

\[
C \quad a = \left[ \frac{z_1^2}{\omega^2} + \frac{\omega^2}{z_2} \right]^{n_1} \left[ \frac{z_2^2}{\omega^2} + \frac{\omega^2}{z_1} \right]^{n_2}; \quad \varrho_1(r_1, r_2) = r_2^{2\kappa_2} f(r_1 r_2^{\kappa_2}, \omega_1); \ \varrho_2(r_2) = f(r_2, \omega_2)
\]

\[
D \quad a = \left[ \frac{z_1^2}{\omega^2} + \frac{\omega^2}{z_2} \right]^{n_1} \left[ \frac{z_2^2}{\omega^2} + \frac{\omega^2}{z_1} \right]^{n_2}; \quad \varrho_1(r_1, r_2) = r_2^{2\kappa_2} f(r_1 r_2^{\kappa_2}, \omega_1 \omega_2^2); \ \varrho_2(r_2) = f(r_2, \omega_2)
\]

where the function \(f\) is given by \(f(r, \omega) = (1/\omega)e^{-r^2/\omega}\). As a quick inspection in order to be certain that the tuples \((71), (72)\) or \((73)\) do not define new classes, we can check that quadruples included in \((71)\), taken in that order, the elements of the list \((72)\), in that order, and those of the list \((73)\), in that order, determine equivalently the classes \((1,1)B, (1,1)C\) and \((1,1)D\), respectively, up to the factors \((z_1/\omega_1^1)^{\kappa_1 n_2}, z_1^{\kappa_1 n_2} and \omega_1^{\kappa_2 n_2/2}\), respectively. These factors can be reabsorbed in the measure density \(\varrho_1(r_1, r_2, n_2)\) without complication.

The following table gives the relevant second sub-classes:

Second class \((\gamma_1, 1)\)-deformed VCS: \(R_1(n_1, n_2) = \Gamma[\gamma_1 + n_1], \ R_2(n_2) = n_2!\)

\[
A \quad a = \left[ \frac{z_1^2}{\omega^2} + \frac{\omega^2}{z_2} \right]^{n_1 + n_1 n_2} \left[ \frac{z_2^2}{\omega^2} + \frac{\omega^2}{z_1} \right]^{n_2}; \quad \varrho_k(r_k) = f(r_k, \omega_k), \ k = 1, 2 \quad (55)
\]

\[
B \quad a = \left[ \frac{z_1^2}{\omega^2} + \frac{\omega^2}{z_2} \right]^{n_1 + n_1 n_2} \left[ \frac{z_2^2}{\omega^2} + \frac{\omega^2}{z_1} \right]^{n_2}; \quad \varrho_1(r_1, n_2) = \frac{1}{n_2} f(r_1, \omega_1 \omega_2^2); \ \varrho_2(r_2) = f(r_2, \omega_2)
\]

\[
C \quad a = \left[ \frac{z_1^2}{\omega^2} + \frac{\omega^2}{z_2} \right]^{n_1 + n_1 n_2} \left[ \frac{z_2^2}{\omega^2} + \frac{\omega^2}{z_1} \right]^{n_2}; \quad \varrho_1(r_1, n_2) = r_2^{2(\kappa_2 + n_2)} f(r_1 r_2^{\kappa_2}, \omega_1); \ \varrho_2(r_2) = f(r_2, \omega_2)
\]

\[
D \quad a = \left[ \frac{z_1^2}{\omega^2} + \frac{\omega^2}{z_2} \right]^{n_1 + n_1 n_2} \left[ \frac{z_2^2}{\omega^2} + \frac{\omega^2}{z_1} \right]^{n_2}; \quad \varrho_1(r_1, n_2) = r_2^{2(\kappa_2 + n_2)} \frac{\omega_1^{\kappa_2 n_2/2}}{\omega_2} f(r_1 r_2^{\kappa_2}, \omega_1 \omega_2^2); \ \varrho_2(r_2) = f(r_2, \omega_2)
\]

One notes that all these sets of states are new classes of VCS that our systematic analysis has allowed to generate. They consist mainly, on the non singular change of variables in the VCS \((55)\) and \((60)\) without breaking the solvability of the VCS. Each of them will also span new generalized or \((\gamma, 1)\)-deformed bi-CS by summing on the remaining index \(n_2\).

Let us discuss the type of the above VCS. As \(\kappa_2 \to 0\), all sub-classes denoted by \((1,1)B, (1,1)C\) and \((1,1)D\) tend to the sub-class \((1,1)A\). The sub-class \((\gamma_1, 1)A\) is also of this type but under the other limit \(\kappa_1 \to 0\). However, the sub-classes \((\gamma_1, 1)B, (\gamma_1, 1)C\) and \((\gamma_1, 1)D\) are not well-defined under the limits \(\kappa_1 \to 0\) or \(\kappa_2 \to 0\), and so are not of the type of the sub-class \((1,1)A\).
3.2.3 Third class: The fake dual

Third class: (1, γ)-deformed VCS - In the above construction, the indices 1 and 2 do not play a symmetric role since, at least, the index \( n_1 \) is summed and the index \( n_2 \) is not. In the following, we still assume that \( n_1 \) is summed but use instead different (generalized) factorials. We choose \( \rho_1(n_1) \) given by a simple factorial \((19)\) and \( \rho_2(n_2, n_1) \) given by \((44)\) with

\[
\gamma_2 = \gamma_2(n_1) = 1 + \kappa_2 n_1, \quad \kappa_2 = \frac{\omega_1}{\omega_2}. \tag{75}
\]

Then, we get a different set of states

\[
|z_1, z_2, n_2\rangle = \mathcal{N}(z_1, z_2, n_2)^{-\frac{1}{2}} \sum_{n_1=0}^{\infty} [\rho_1(n_1)\rho_2(n_2, n_1)]^{-\frac{1}{2}} z_1^{n_1} z_2^{n_2+\kappa_2 n_1} |n_1, n_2\rangle, \tag{76}
\]

normalized provided that the factor

\[
\mathcal{N}(|z_1|, |z_2|, n_2) = \left[ \frac{|z_2|}{\omega_2} \right]^{n_2} \sum_{n_1=0}^{\infty} \frac{1}{n_1! \Gamma(\gamma_2)(\gamma_2)_{n_2}} \left[ \frac{|z_1(z_2)^{\kappa_2}|^2}{\omega_1(\omega_2)^{\kappa_2}} \right]^{n_1} \tag{77}
\]

is converging. We can use the ratio test in order to check that \( \mathcal{N}(z_1, z_2, n_2) \) is absolutely convergent everywhere in \( \mathbb{C} \). Indeed, since \( \gamma_2 = 1 + \kappa_2 n_1 \geq 1 \) and using a comparison test, we can bound each term of this series by the term of an exponential series,

\[
\frac{1}{n_1! \Gamma(\gamma_2)(\gamma_2)_{n_2}} \left[ \frac{|z_1(z_2)^{\kappa_2}|^2}{\omega_1(\omega_2)^{\kappa_2}} \right]^{n_1} \leq \frac{1}{n_1! n_2!} \left[ \frac{|z_1(z_2)^{\kappa_2}|^2}{\omega_1(\omega_2)^{\kappa_2}} \right]^{n_1}. \tag{78}
\]

The latter is a term of a convergent series for all \( |z_1(z_2)^{\kappa_2}| > 0 \), implying \( |z_1| \) and \( |z_2| > 0 \). A resolution of the identity can be found making use of the measure

\[
d\mu(z_1, z_2, n_2) = \frac{1}{\pi^2} \mathcal{N}(z_1, z_2, n_2) \rho_1(r_1) r_1 dr_1 d\theta_1 \rho_2(r_2) r_2 dr_2 d\theta_2 \tag{79}
\]

and

\[
\int_{D_1 \times D_2} |z_1, z_2, n_2\rangle \langle z_1, z_2, n_2| d\mu(z_1, z_2, n_2) = I_{n_2} \tag{80}
\]

is satisfied for the solutions to the moment problems \((25)\) and \((47)\) for \( \rho_1(r_1) \) and \( \rho_2(r_2) \), respectively.

This VCS third class should not be confused with the dual class (hence the nickname of fake dual) obtained from the VCS second class \((55)\) by switching completely the role of 1 and 2. Note also that this third class has its own dual partner. Finally, from \((76)\), by taking the limit \( \kappa_2 \to 0 \), one generates the VCS first class \((1, 1)A \) \((60)\).

Solvable sub-classes - Let us investigate sub-classes associated with \((76)\). As customary, we introduce the general state

\[
|z_1, z_2, n_2\rangle = \mathcal{N}(z_1, z_2, n_2)^{-\frac{1}{2}} \times \sum_{n_1=0}^{\infty} \frac{1}{[\omega_1^{\alpha_1 n_1+\beta_1 \kappa_1 n_2}] R_1(n_1) \omega_2^{\alpha_2 n_2+\beta_2 \kappa_2 n_1}] R_2(n_2, n_1)} z_1^{\alpha_1 n_1+\beta_1 \kappa_1 n_2} z_2^{\alpha_2 n_2+\beta_2 \kappa_2 n_1} |n_1, n_2\rangle. \tag{81}
\]
The formalism for solving the associated generalized moment problem has been already introduced. Given \( \chi(r_1, r_2, n_2) = \varrho_1(r_1, r_2, n_2) \varrho_2(r_2) \), one solves the problem for (see Appendix A):

\[
\varrho_1(r_1, r_2, n_2) = \alpha_1 - \frac{2^{2(\alpha_1-1)} r_1^{2(\beta_2 - \alpha_2) \kappa_2}}{\omega_1^2 \omega_2^{2 \alpha_2}} \left( \frac{\beta_1}{r_1^{2 \beta_1}} \right)^{\kappa_1 n_2} e^{-\frac{1}{2} \frac{2^{2(\beta_2 - \alpha_2) \kappa_2}}{\omega_1^{2 \alpha_2}, \omega_2^{2 \alpha_2}}} ,
\]

\[
\varrho_2(r_2) = \alpha_2 - \frac{1}{\omega_2^{2 \alpha_2}} r_2^{2(\alpha_2-1)} e^{-\frac{1}{2} \frac{2^{2(\beta_2 - \alpha_2) \kappa_2}}{\omega_1^{2 \alpha_2}, \omega_2^{2 \alpha_2}}} .
\]

Replacing \( (\alpha_1, \beta_1, \alpha'_1, \beta'_1, \alpha_2, \beta_2, \alpha'_2, \beta'_2) = (1, 0, 1, 0, 1, 1, 1, 1) \), one gets the correct densities associated with (76).

Discussing the relevant classes by considering the quadruple \( (\beta_1, \beta'_1, \beta_2, \beta'_2) \) and keeping fixed \( \alpha_1 \) = 1, note first that \( (0, 0, 1, 1) \) defines (76). We need to consider as a basic sub-class the one defined by \( (0, 0, 0, 0) \). Focusing on states involving at least an exponent \( n_1 \) in the tower 2, only the following quadruples are relevant

\[
(0, 0, 1, 0), (0, 0, 0, 1).
\]

The left over tuples determine nothing but factors of the sub-classes defined by the above set of quadruples. In the following table are collected the relevant third sub-classes (using previous notations):

| Third class \((1, \gamma_2)\)-deformed VCS: \( R_1(n_1) = n_1!\), \( R_2(n_2, n_1) = \Gamma[\gamma_2 + n_2] \) |
|---|
| **A** | \( a = \left[ \frac{1}{\omega_2} \right]^{n_1} \left[ \frac{1}{\omega_2} \right]^{n_2 + n_2 n_1} \); \( \varrho_k(r_k) = f(r_k, \omega_k) \), \( k = 1, 2 \) | (76) |
| **B** | \( a = \left[ \frac{1}{\omega_2} \right]^{n_1} \left[ \frac{1}{\omega_2} \right]^{n_2 + n_2 n_1} \); \( \varrho_1(r_1) = f(r_1, \omega_1 \omega_2^{\kappa_2}); \varrho_2(r_2) = f(r_2, \omega_2) \) |
| **C** | \( a = \left[ \frac{1}{\omega_2} \right]^{n_1} \left[ \frac{1}{\omega_2} \right]^{n_2 + n_2 n_1} \); \( \varrho_1(r_1, r_2) = r_2^{-2\kappa_2} f(r_1 r_2^{\kappa_2}, \omega_1); \varrho_2(r_2) = f(r_2, \omega_2) \) |
| **D** | \( a = \left[ \frac{1}{\omega_2} \right]^{n_1} \left[ \frac{1}{\omega_2} \right]^{n_2 + n_2 n_1} \); \( \varrho_1(r_1, r_2) = r_2^{-2\kappa} f(r_1 r_2^{\kappa_2}, \omega_1 \omega_2^{\kappa_2}); \varrho_2(r_2) = f(r_2, \omega_2) \) |

As \( \kappa_2 \to 0 \), these states are all of the type of \((1,1)A\). Meanwhile, all the factors associated with that third class involving an additional factor \( \kappa_1 \) will be not of that type.

### 3.2.4 Fourth class: \((\gamma_1, \gamma_2)\)- or doubly-deformed CS

We pursue the analysis by introducing doubly dependent generalized factorials \( \rho_{1,2}(n_{1,2}, n_{2,1}) \) still given by (44) with

\[
\gamma_{1,2} = \gamma_{1,2}(n_{1,2}) = 1 + \kappa_{1,2} n_{2,1}, \quad \kappa_{1,2} = \frac{\omega_{2,1}}{\omega_{1,2}} .
\]

\(20\)
The corresponding set of states can be built as

\[ |z_1, z_2, n_2\rangle = \mathcal{N}(z_1, z_2, n_2)^{-\frac{1}{2}} \sum_{n_1=0}^{\infty} \rho_1(n_1, n_2) \rho_2(n_2, n_1)^{-\frac{1}{2}} \langle z_1^{n_1+k_{1}n_2} z_2^{n_2+k_{2}n_1} | n_1, n_2 \rangle, \tag{85} \]

with the normalization factor

\[ \mathcal{N}(\|z_1\|, \|z_2\|, n_2)^{-\frac{1}{2}} = \sum_{n_1=0}^{\infty} \frac{1}{\Gamma[\gamma_1 + n_1] \Gamma[\gamma_2 + n_2]} \left( \frac{\|z_1\|^2}{\omega_1} \right)^{n_1+k_{1}n_2} \left( \frac{\|z_2\|^2}{\omega_2} \right)^{n_2+k_{2}n_1} \tag{86} \]

converging everywhere in \( \mathbb{C} \) since the following inequality holds

\[ \frac{1}{\Gamma[\gamma_1 + n_1] \Gamma[\gamma_2 + n_2]} \left( \frac{r_1}{\omega_1} \right)^{n_1+k_{1}n_2} \left( \frac{r_2}{\omega_2} \right)^{n_2+k_{2}n_1} \leq \frac{1}{n_1!n_2!} \left( \frac{r_1}{\omega_1 \omega_2^\kappa_2} \right)^{n_1} \left( \frac{r_2}{\omega_2 \omega_1^\kappa_1} \right)^{n_2}. \tag{87} \]

The resolution of the identity of these states uses the measure

\[ d\mu(z_1, z_2, n_2) = \frac{1}{\pi^2} \mathcal{N}(z_1, z_2, n_2) \rho_1(r_1) r_1 dr_1 d\theta_1 \rho_2(r_2) r_2 dr_2 d\theta_2 \tag{88} \]

so that

\[ \int d\mu(z_1, z_2) |z_1, z_2, n_2\rangle \langle z_1, z_2, n_2| = I_{n_2} \tag{89} \]

holds for \( \rho_1(r_1) \) and \( \rho_2(r_2) \) satisfying the moment problems of the kind \( \text{(47)} \) and hence with the solutions given by \( \text{(48)} \).

**Solvable sub-classes** - Determining sub-classes of \( \text{(85)} \) can be discussed by introducing the parameters \( \alpha_i^{(\prime)} \) and \( \beta_i^{(\prime)} \) and the generalized state

\[ |z_1, z_2, n_2\rangle = \mathcal{N}(z_1, z_2, n_2)^{-\frac{1}{2}} \times \sum_{n_1=0}^{\infty} \frac{1}{[\omega_1^{\alpha_1^{(\prime)}n_1+\beta_1^{(\prime)}n_2}] R_1(n_1, n_2) \omega_2^{\alpha_2^{(\prime)}n_2+\beta_2^{(\prime)}n_1}] R_2(n_2, n_1)^{-\frac{1}{2}} \langle z_1^{\alpha_1n_1+\beta_1n_2} z_2^{\alpha_2n_2+\beta_2n_1} | n_1, n_2 \rangle. \tag{90} \]

The associated moment problem is written, with \( u_i = r_i^2 \),

\[ \int du_1du_2 \chi_1(u_1, u_2, n_2) \left[ \frac{u_2^{\alpha_2^{(\prime)}}}{\omega_2} \right]^{n_2+k_{2}n_1} \left[ \frac{u_1^{\alpha_1^{(\prime)}}}{\omega_1} \right]^{n_1+k_{1}n_2} \left[ \frac{u_1^{\alpha_1^{(\prime)}}}{\omega_1} - \frac{(\beta_2^{(\prime)}-\alpha_2^{(\prime)})u_2}{\omega_2} \right]^{\frac{\kappa_{1}n_2}{2}} \left[ \frac{u_1^{\alpha_1^{(\prime)}}}{\omega_1} - \frac{(\beta_2^{(\prime)}-\alpha_2^{(\prime)})u_2}{\omega_2} \right]^{\frac{\kappa_{1}n_2}{2}} \right] \tag{91} \]

and is solved by the following densities (see Appendix [A])

\[ \rho_1(r_1, r_2, n_2) = \alpha_1^2 \rho_1^{(2(\alpha_1-1))} \frac{2(\beta_2^{(\prime)}-\alpha_2^{(\prime)})u_2}{\omega_2} \left[ \frac{u_1^{\alpha_1^{(\prime)}}}{\omega_1} - \frac{(\beta_2^{(\prime)}-\alpha_2^{(\prime)})u_2}{\omega_2} \right]^{\frac{\kappa_{1}n_2}{2}} \left[ \frac{u_1^{\alpha_1^{(\prime)}}}{\omega_1} - \frac{(\beta_2^{(\prime)}-\alpha_2^{(\prime)})u_2}{\omega_2} \right]^{\frac{\kappa_{1}n_2}{2}} e^{-\frac{2(\alpha_1-1)}{\omega_1^2}} \]  

\[ \rho_2(r_2) = \alpha_2 \frac{1}{\omega_2^2} e^\frac{-r_2^2}{\omega_2^2}. \tag{92} \]
For the present study, the quadruple $(\beta_1, \beta_1', \beta_2, \beta_2')$ such that $(1,1,1,1)$ defines (85). We choose another sub-class defined by $(1,1,0,0)$. Proceeding as previously, the following quadruples

$$(1,1,0,1), (1,1,1,0)$$

are relevant. The ingredients defining the relevant classes, above listed, are given by the following table (in anterior notations):

| Fourth class $(\gamma_1, \gamma_2)$-deformed VCS: $R_1(n_1, n_2) = [\gamma_1 + n_1], R_2(n_2, n_1) = \Gamma[\gamma_2 + n_2]$ |
|---------------------------------------------------------------|
| $A$ $a = \left[ \frac{z_1}{\omega_1} \right]^{n_1+n_1n_2} \left[ \frac{z_2}{\omega_2} \right]^{n_2+n_2n_1}$; $\varrho_k(r_k) = f(r_k, \omega_k), k = 1, 2$ (85) |
| $B$ $a = \left[ \frac{\chi_1}{\omega_1} \right]^{n_1+n_1n_2} \left[ \frac{\chi_2}{\omega_2} \right]^{n_2+n_2n_1}$; $\varrho_1(r_1) = \frac{1}{r_2} \bar{r}_2 e^{-\gamma_2} f(r_1 \bar{r}_2, \omega_1); \varrho_2(r_2) = f(r_2, \omega_2)$ |
| $C$ $a = \left[ \frac{\rho_1}{\omega_1} \right]^{n_1+n_1n_2} \left[ \frac{\rho_2}{\omega_2} \right]^{n_2+n_2n_1}$; $\varrho_1(r_1, r_2) = \omega_2^{n_2} f(r_1, \omega_1 \omega_2^{-n_2}); \varrho_2(r_2) = f(r_2, \omega_2)$ |
| $D$ $a = \left[ \frac{\sigma_1}{\omega_1} \right]^{n_1+n_1n_2} \left[ \frac{\sigma_2}{\omega_2} \right]^{n_2+n_2n_1}$; $\varrho_1(r_1, r_2) = \frac{\omega_2^{n_2}}{r_2} f(r_1 \bar{r}_2^{-n_2}, \omega_1 \omega_2^{-n_2}); \varrho_2(r_2) = f(r_2, \omega_2)$ |

These states are not of the type of any previous classes.

We recapitulate these results on two degrees of freedom by a diagram given by Figure 2.
3.3 VCS without energy shift and Landau level VCS

There is another way to generalize the above VCS to classes built out of energy levels with or without energy shift by a fixed amount:

\[ \mathcal{E}_{n_1,n_2} = \omega_1 (n_1 + \alpha_1) + \omega_2 (n_2 + \alpha_2). \]  

(94)

Typically, of course, \( \alpha_i \) is simply the ground state energy of the harmonic oscillator, namely \( \alpha_i = 1/2 \) and the states built out of these energy levels can be called unshifted. The VCS of this system are still solvable in full generality for any value of \( \alpha_i \geq 0 \), giving another full set of \( \alpha_i \)-deformed classes of VCS such that at the limit \( \alpha_i \to 0 \), one continuously recovers earlier computed classes.

We sketch how the main quantities involved in the construction of the VCS get modified by these shifts. We introduce the modified set of generalized factorials as

\[ \tilde{\rho}_i(n_i) = (\omega_i)^n_i \prod_{k=1}^{n_i} (k + \alpha_i) = (\omega_i)^n_i (1 + \alpha_i)^{n_i}, \quad i = 1, 2, \]  

(95)

which at the limit \( \alpha_i \to 0 \) converge indeed to plain factorial \( (\omega_i)^{n_i} n_i! \). Meanwhile, the previous Pochhammer symbols find the generalization:

\[ \tilde{\rho}_{1,2}(n_{1,2}) = (\omega_{1,2})^{n_{1,2}} (\gamma_{1,2})^{n_{1,2}}, \quad \gamma_{1,2} = 1 + \alpha_{1,2} + \frac{\omega_{2,1}}{\omega_{1,2}} (n_{2,1} + \alpha_{2,1}), \]  

(96)

so that all solvable VCS classes can be directly extended using these quantities.

Some comments on another particular physical system called the Landau problem which can be exactly implemented with the above construction are in order.

The Landau problem is the quantum mechanical system describing the motion of a charged particle in a plane subjected to a magnetic field perpendicular to that plane [15, 28, 29]. The Hamiltonian of this model which can be written as (6) has been successfully applied to various condensed matter system and gave rise to interesting results among which one quotes the (simple, fractional, spin) quantum Hall effect [29]. By adding further a harmonic potential to its Hamiltonian, the Landau model is still solvable and the previous model with its infinite degeneracy for each energy level becomes lifted.

In symbol, after diagonalization, the dimensionless Landau Hamiltonian including a harmonic potential with frequency \( \omega \) is given by

\[ \frac{1}{\hbar} H_L = \Omega_{+} (N_{+} + \frac{1}{2}) + \Omega_{-} (N_{-} + \frac{1}{2}) + \text{const.}, \quad N_{\pm} = a_{\pm}^\dagger a_{\pm}, \quad [a_{\pm}, a_{\pm}^\dagger] = \mathbb{I}_{\pm}, \]  

(98)

where \( \Omega_{\pm} = \pm \omega + \sqrt{\omega^2 + \omega^2}, \omega = e|B|/m \) is the cyclotron frequency, \( a_{\pm}, a_{\pm}^\dagger \) are two bosonic Heisenberg modes (generating \( N_{\pm} \) as number operators) describing the particle helicity or the possible winding orientations of the particle. Note that turning off the harmonic potential (\( \omega \to 0 \)), the spectrum of operator becomes infinitely degenerate with (+) as a remaining helicity sector.
The VCS formalism can be obviously applied here. The above model is indeed nothing but a harmonic oscillator in 2D with only a particular feature to be degenerate at a certain limit. So all the previous sets of VCS are valid for the Landau problem with shifted or unshifted spectrum. Here, the VCS possess a vector index given by one or other helicity sector either corresponding to 1 or 2 in the anterior study. Hence, besides of the VCS as computed by Ali and Bagarello \[5\] and then improved in \[4\], we have shown that other VCS classes can be defined from the same model.

4 Harmonic oscillator in 3D

In this section, we consider the harmonic oscillator in 3D,

\[
H_{3D} = \frac{1}{\hbar} H'_{3D} = -\sum_{i=1}^{3} \omega_i (a_i^\dagger a_i + \frac{1}{2}), \quad e_{n_1,n_2,n_3} = \sum_{i=1}^{3} \omega_i (n_i + \frac{1}{2}).
\]  (99)

Our aim is to find an extension of the above VCS classes in the higher rank Hilbert space of the 3D harmonic oscillator system, namely \( \mathcal{H}_{3D} = \text{span}\{|n_1,n_2,n_3\rangle, \; n_i \in \mathbb{N}\} \). Here \(|n_1,n_2,n_3\rangle\) are again the eigenstates of the number operators of three Heisenberg algebras \([a_i,a_i^\dagger] = \mathbb{I}_i\).

Clearly, it can easily establish that all VCS classes with one degree and two degrees of freedom as worked out in Section 3 can be reported here. Concerning the case of one degree of freedom, assuming again the sum is performed on \(n_1\), the VCS become tensor coherent states labeled by \(n_2,n_3\). Dealing with two degrees of freedom and one sum on \(n_1\), the whole discussion so far is again valid in the present situation. We will not deal with these states being already well listed in the anterior study since they can be simply obtained by tensoring the ket \(|n_3\rangle\) to all these classes and consequently modifying the resolution of the identity. However, it should be emphasized that the classes of states contains more than the above and include any symmetric one obtained by interchanging the role of 1, 2 and 3.

The interesting goal here is to investigate other classes of VCS labeled by only one index, for example \(n_3\) in the following, whereas the two other indices, say \(n_1\) and \(n_2\), are summed. We are led to new classes of VCS made with two degrees of freedom. Indeed, many of the techniques worked out before do no longer be applied. Specifically, when summing on both indices, by the likely dependence of generalized factorials on many indices, this generates what is usually referred to as VCS with dependent or independent sums. It might become difficult to find measure densities solving the VCS since they should not be dependent on two indices \(n_1\) and \(n_2\). The analysis becomes more involved but still a systematic approach for listing these states remains possible.

The case of three degrees of freedom will be not treated in length here. As we will show, the number of cases which may lead to solvable VCS is really significant due to the freedom in the remaining index \(n_3\). Only two simplified situations will be exhibited.

4.1 Preliminaries: Class counting, strategy and normalization

Due to the increasing number tensor indices, the number of classes rapidly proliferate. We have to perform some combinatorics in order to identify the meaningful cases which have to
be studied.  

**Class counting** -  The definition of the VCS rests on the generalized factorials. Combinatorially, the energy \( E_{n_1,n_2,n_3} = \omega_1 n_1 + \omega_2 n_2 + \omega_3 n_3 \) can be dissected in different ways each of which generating a different generalized factorial associated with a degree of freedom. Seeking for solvable VCS with two and three degrees of freedom, the following energy-built quantities will define the possible generalized factorials

\[
\begin{align*}
\omega_i n_i, & \quad i = 1, 2, 3, \\
\omega_i (n_i + \kappa_{ii} n_i), & \quad \kappa_{ii} = \frac{\omega_i}{\omega_i}, \quad i = 1, 2, 3, \quad i \neq \bar{i}, \\
\omega_i (n_i + \kappa_{ii} n_i + \kappa_{i\bar{i}} n_{\bar{i}}), & \quad i, \bar{i}, \bar{\bar{i}} = 1, 2, 3, \quad \bar{i} \neq i \neq \bar{\bar{i}},
\end{align*}
\]

(100)

so that each generalized factorial \( \rho_i(n_i, n_{\bar{i}}, n_{\bar{\bar{i}}}) \) can be of 4 different forms:

\[
\rho_i(n_i, n_{\bar{i}}, n_{\bar{\bar{i}}}) \in \left\{ \omega_i^{n_i} n_i!, \quad \omega_i^{n_i + \kappa_{ii} n_i} \Gamma[\gamma_{ii} + n_i], \quad \omega_i^{n_i + \kappa_{ii} n_i + \kappa_{i\bar{i}} n_{\bar{i}}} \Gamma[\gamma_{ii} + n_i + n_{\bar{i}}], \quad \omega_i^{n_i + \kappa_{ii} n_i + \kappa_{i\bar{i}} n_{\bar{i}} + \kappa_{i\bar{\bar{i}}} n_{\bar{\bar{i}}}} \Gamma[\gamma_{ii} + n_i + n_{\bar{i}} + n_{\bar{\bar{i}}}] \right\},
\]

\[
\gamma_{ii} = 1 + \kappa_{ii} n_i, \quad \gamma_i = 1 + \kappa_{ii} n_i + \kappa_{i\bar{i}} n_{\bar{i}},
\]

(101)

with the same convention for the triple \((i, \bar{i}, \bar{\bar{i}})\) as above.

For definiteness, we will assume that \( n_1 \) and \( n_2 \) are summed while \( n_3 \) is kept fixed as mentioned earlier. Any over set of VCS induced by another choice taking two over the three indices in \( \{n_1, n_2, n_3\} \), will be considered as another representative of a class which will be listed in the sequel. The notion of class takes now a broader but justified sense.  

Coming back to our particular choice, \( n_1 \) and \( n_2 \) play now a completely symmetric role, and, discussing a class, we implicitly include in that class any symmetric of any representative. Using two degrees of freedom \( z_i \) and \( z_j \) associated with two different towers of states, one has two possibilities: either the VCS is expanded versus \( z_1 \) and \( z_2 \) (tower 1 and 2) or versus \( z_{1,2} \) and \( z_3 \) (tower \( i = 1 \) or 2 and 3). Focusing on the \((z_1, z_2)\) case with a total of \( 4^2 \) classes, there are 10 non symmetric (i.e. any of these classes cannot be recovered from another class in this list by just renaming \( 1 \leftrightarrow 2 \), hence are unequivalent) classes according to the distribution of 4 different generalized factorials to each variable. Besides, for the series expansion in \((z_i, z_3)\), one has again a total of \( 4^2 \) classes corresponding to the number of all pairs \((\rho_i, \rho_3)\). However, not all these pairs lead to a sensible notion of VCS class, because, as the sum is performed on towers labeled by \( n_1 \) and \( n_2 \), only a VCS series expansion dependent on both \( n_1 \) and \( n_2 \) might lead to a normalizable VCS. Fixing the pair \((\rho_1, \rho_3)\) and having a closer look on which kind of generalized factorials \((\rho_1, \rho_3)\) could lead \textit{a priori} to that condition, one agrees with that

- \( \rho_1 \equiv \omega_1^{n_1} n_1! \) could only have 2 partners involving in their definition the index \( n_2 \), which are \( \rho_3 \in \{ \omega_3^{n_3 + n_3} \Gamma[\gamma_{23} + n_3], \omega_3^{n_3 + n_3 - 1} \Gamma[\gamma_{3} + n_3] \}; \)
- \( \rho_1 \in \{ \omega_1^{n_1 + n_1 n_2} \Gamma[\gamma_{12} + n_1], \omega_1^{n_1 + n_1 - 1} \Gamma[\gamma_{1} + n_1] \} \) can be associated with any \( \rho_3 \), since it already contains a \( n_2 \); we have here \((4+4)\) cases;
- \( \rho_1 \equiv \omega_1^{n_1 + n_1 n_3} \Gamma[\gamma_{13} + n_1] \) has the same partners has \( \rho_1(n_1) = \omega_1^{n_1} n_1! \) and so 2 cases should be studied here.

Thus, the number of classes to be studied for two degrees of freedom is \( 10 + 12 = 22 \).

\footnote{This was not the case in Section \[3] a summation being performed only on \( n_1 \).}
The number of unequivalent classes with three degrees of freedom is much greater than
the latter and the total number of possible classes is $4^3$. The two integers $n_1$ and $n_2$ playing
a symmetric role, we expect a number of unequivalent classes less than this total number.
We have seen that the number of unequivalent cases expanding the VCS in terms of $z_1, z_2$ is
10. Then it remains to connect these cases to the 4 provided by the last variable $z_3$. A rapid
checking show that these cases are all unequivalent. Hence, for three degrees of freedom, we
obtain a priori 40 classes.

**Strategy and omissions** - Needless to emphasize that an optimal way to study these states
is necessary. The above combinatorics have already introduced some simplifications. The
exchange $1 \leftrightarrow 2$ becomes now a symmetry and therefore should be extensively used in order
to enlarge the notion of class. Furthermore, the idea of defining some most general class
and then continuously tuning the $\kappa$’s parameters in order to get simpler cases will be used.
Hence, a key ingredient is the notion of type. Nevertheless, at each limit, the basic axioms
should be checked (we will only reveal the intermediate steps when they have not been
treated earlier). Henceforth, we will not organize the remaining part of the text centered on
the notion of class, but more using the notion of type. Practically, we will start by a general
deformation class, the *ancestor*, prove that it is solvable and then derive by continuous limit
its descendant states (checking implicitly that they remain solvable).

The notion of sub-classes could be also introduced along ideas of Section 3 but will be
omitted for the sake of simplicity. VCS with three degrees of freedom will be not treated
in detail also because of their important number. The resulting VCS may be understood as
extensions of the VCS treated with two degrees of freedom for the 2D harmonic oscillator.
In that study, three complex variables will be introduced and only a slight discussion in the
furthest situations will be done: when the state is maximally deformed and when it is not
at all.

**On the normalization condition** - Most importantly, the axioms of VCS, namely the
normalization condition and resolution of the identity, are to be checked here. Due to our
particular choice of VCS construction, i.e. by fitting the correct exponent of the complex
variables in order to match with the generalized factorials, the resolution of the identity
will be trivialized showing thereby the robustness of our formalism using a unique kind of
moment problem. Nevertheless, and interestingly, we discover that the complications have
migrated: the convergence of the norm series, the latter becoming a double power series in
$\mathbb{C}^2$, is far from a trivial problem whenever one replaces ordinary factorials by generalized
factorials. Indeed, the issue of convergence of double series is a whole subject of investigation
on its own [30, 31]. The following theorems give some criteria for convergence of a double
series that will be extensively used in the remaining part of the text. To start with, let us
introduce some basic definitions.

Let $(a_{k,\ell}) \equiv (a_{k,\ell})_{k,\ell \in \mathbb{N}}$ be a double sequence of nonzero real numbers. The double series
$\sum_{k,\ell=0}^{\infty} a_{k,\ell}$ defines two partial series: one called row series defined by $\sum_{k=0}^{\infty} a_{k,\ell}$ and the
other column series defined by $\sum_{\ell=0}^{\infty} a_{k,\ell}$. We recall a statement that will be considered as a
definition:

**Proposition 1** (Lemma 2.1 in [30]). A double series $(a_{k,\ell})$ is absolutely convergent if and
only if the following conditions hold:
(i) There are $(k_0, \ell_0) \in \mathbb{N}$ and $\alpha_0 > 0$ such that

$$\sum_{k=k_0}^{m} \sum_{\ell=\ell_0}^{n} |a_{k,\ell}| \leq \alpha_0, \quad \forall (m, n) \geq (k_0, \ell_0).$$

(ii) Each row series as well as each column series is absolutely convergent.

We use the shorthand notation $(k, l) \leq (m, n)$ for the partial order in $\mathbb{N}^2$, $k \leq m$ and $l \leq n$. A well known consequence of this statement is the following:

**Corollary 1** (Comparison test). Let $(a_{k,\ell})$ and $(b_{n,k})$ be double sequences of nonzero numbers. Assume that $\exists K_0, L_0 \in \mathbb{N}$ such that $\forall (k, \ell) \geq (K_0, L_0)$, $|a_{k,\ell}| \leq |b_{n,k}|$. If $\sum_{k,\ell=0}^{\infty} b_{k,\ell}$ is absolutely convergent then so is $\sum_{k,\ell=0}^{\infty} a_{k,\ell}$.

**Proof.** Let us consider $k_0, \ell_0 \in \mathbb{N}$ be such that $\exists \alpha_0 > 0$ and $\sum_{k=k_0}^{m} \sum_{\ell=\ell_0}^{n} |b_{k,\ell}| \leq \alpha_0$. Then, four cases may occur:

(i) $(k_0, \ell_0) \geq (K_0, L_0)$, then for $\forall (k, \ell) \geq (k_0, \ell_0)$, we have $|a_{k,\ell}| \leq |b_{n,k}|$ and therefore

$$\forall (m, n) \geq (k_0, \ell_0) \geq (K_0, L_0) \Rightarrow \sum_{k=k_0}^{m} \sum_{\ell=\ell_0}^{n} |a_{k,\ell}| \leq \sum_{k=k_0}^{m} \sum_{\ell=\ell_0}^{n} |b_{k,\ell}| \leq \alpha_0 \quad \text{(103)}$$

(ii) $k_0 \geq K_0$ and $\ell_0 < L_0$ (resp. $\ell_0 \geq L_0$ and $k_0 < K_0$), and so

$$\forall m \geq k_0 \geq K_0, \quad \forall n \geq L_0 > \ell_0 \Rightarrow \sum_{k=k_0}^{m} \sum_{\ell=L_0}^{n} |a_{k,\ell}| \leq \sum_{k=k_0}^{m} \sum_{\ell=L_0}^{n} |b_{k,\ell}| \leq \alpha_0 \quad \text{(104)}$$

(resp. $\forall m \geq K_0 > k_0, \quad \forall n \geq L_0 \geq \ell_0 \Rightarrow \sum_{k=K_0}^{m} \sum_{\ell=\ell_0}^{n} |a_{k,\ell}| \leq \sum_{k=K_0}^{m} \sum_{\ell=\ell_0}^{n} |b_{k,\ell}| \leq \alpha_0$)

(iii) $(k_0, \ell_0) < (K_0, L_0)$, and then

$$\forall (m, n) \geq (K_0, L_0) \geq (k_0, \ell_0) \Rightarrow \sum_{k=K_0}^{m} \sum_{\ell=L_0}^{n} |a_{k,\ell}| \leq \sum_{k=K_0}^{m} \sum_{\ell=L_0}^{n} |b_{k,\ell}| \leq \alpha_0. \quad \text{(105)}$$

Finally, the ordinary theorem of comparison for simple series proves that row and column series of $\sum_{k,\ell}|a_{k,\ell}|$ are convergent if row and column series of $\sum_{k,\ell}|b_{k,\ell}|$ are, respectively. One concludes by Proposition 1.

More involved ratio tests are also useful.

**Theorem 1** (Ratio test, Theorem 2.7 in [30]). Let $(a_{k,\ell})$ be a double sequence of nonzero numbers such that either $|a_{k,\ell+1}|/|a_{k,\ell}| \to a$ or $|a_{k+1,\ell}|/|a_{k,\ell}| \to \tilde{a}$ as both $k \to \infty$ and $\ell \to \infty$, where $a, \tilde{a} \in \mathbb{R} \cup \{\infty\}$.

(i) Suppose each row-series as well as each column-series corresponding to $\sum_{k,\ell=0}^{\infty} a_{k,\ell}$ is absolutely convergent. If $a < 1$ or $\tilde{a} < 1$, then $\sum_{k,\ell=0}^{\infty} a_{k,\ell}$ is absolutely convergent.

(ii) If $a > 1$ or $\tilde{a} > 1$, then $\sum_{k,\ell=0}^{\infty} a_{k,\ell}$ is divergent.
Furthermore the following statement holds

**Theorem 2** (Ratio-Comparison test, Theorem 2.9 in [30]). Let \((a_{k,\ell})\) and \((b_{k,\ell})\) be double sequences with \(b_{k,\ell} > 0\) for all \((k,\ell) \in \mathbb{N}\).

(i) Suppose each row-series as well as each column-series corresponding to \(\sum_{k,\ell=0}^{\infty} a_{k,\ell}\) is convergent. If \(\left|a_{k,\ell+1}\right|b_{k,\ell} \leq |a_{k,\ell}|b_{k,\ell+1}\) and \(\left|a_{k+1,\ell}\right|b_{k,\ell} \leq |a_{k,\ell}|b_{k+1,\ell}\) whenever \(k\) and \(\ell\) are large, and if \(\sum_{k,\ell=0}^{\infty} b_{k,\ell}\) is convergent, then so is \(\sum_{k,\ell=0}^{\infty} a_{k,\ell}\).

(ii) If \(\left|a_{k,\ell+1}\right|b_{k,\ell} > |a_{k,\ell}|b_{k,\ell+1}\) whenever \(\ell\) is large and \(k \in \mathbb{N}\), and \(\left|a_{k+1,\ell}\right|b_{k,\ell} > |a_{k,\ell}|b_{k+1,\ell}\) whenever \(k\) is large and \(\ell \in \mathbb{N}\), and if \(\sum_{k,\ell=0}^{\infty} b_{k,\ell}\) is divergent, then so is \(\sum_{k,\ell=0}^{\infty} a_{k,\ell}\).

### 4.2 VCS with two degrees of freedom

The VCS classes studied here possess two degrees of freedom and they are generated by a double series coined by two towers, \(n_1\) and \(n_2\), (over the three) of the Hilbert space \(\mathcal{H}_{3D}\).

There are, in conformity with the discussion of Subsection 4.1, two kinds of classes either introduced by the couple \((z_1, z_2)\) or by the couple \((z_1, z_3)\) (the case \((z_2, z_3)\) will fall into the same classes of this latter).

**Case (12)** - There is two main situations: first, the generalized factorials \(\rho_1\) or \(\rho_2\) do not depend on both \(n_1\) and \(n_2\), then the sums in \(n_1\) and \(n_2\) become independent of one another; this case occurs when

\[
\begin{align*}
\rho_1(n_1) &= \omega_1^{n_1} n_1! \\
\rho_2(n_2) &= \omega_2^{n_2} n_2!
\end{align*}
\]

second, one of \(\rho_1\) or \(\rho_2\) does involve both variables \(n_1\) and \(n_2\), then the sums in these indices become dependent; this situation concerns the following factorials:

\[
\begin{align*}
\rho_1(n_1) &= \omega_1^{n_1} n_1! & \rho_1(n_1, n_2) &= \omega_1^{n_1 + \kappa_{12} n_2} \Gamma[\gamma_{12} + n_1] \\
\rho_2(n_2, n_3) &= \omega_2^{n_2 + \kappa_{23} n_3} \Gamma[\gamma_{23} + n_2] & \rho_2(n_2, n_3) &= \omega_2^{n_2 + \kappa_{23} n_3 + \kappa_{21} n_1} \Gamma[\gamma_{23} + n_2] \\
\rho_1(n_1, n_2) &= \omega_1^{n_1 + \kappa_{12} n_2} \Gamma[\gamma_{12} + n_1] & \rho_1(n_1, n_2) &= \omega_1^{n_1 + \kappa_{12} n_2 + \kappa_{13} n_3} \Gamma[\gamma_{13} + n_1] \\
\rho_1(n_1, n_2, n_3) &= \omega_1^{n_1 + \kappa_{12} n_2 + \kappa_{13} n_3} \Gamma[\gamma_{13} + n_1] & \rho_1(n_1, n_2, n_3) &= \omega_1^{n_1 + \kappa_{12} n_2 + \kappa_{13} n_3 + \kappa_{21} n_1} \Gamma[\gamma_{13} + n_1]
\end{align*}
\]

**Case (i3) \(i = 1, 2\)** - Again the same ideas hold here, either sums can be decoupled or mutually dependent given a choice \((\rho_1, \rho_3)\). The following factorials generate independent sums:

\[
\begin{align*}
\rho_1(n_1) &= \omega_1^{n_1} n_1! & \rho_3(n_3, n_2) &= \omega_3^{n_3 + \kappa_{32} n_2} \Gamma[\gamma_{32} + n_3] \\
\rho_1(n_1, n_3) &= \omega_1^{n_1 + \kappa_{13} n_3} \Gamma[\gamma_{13} + n_1] & \rho_3(n_3, n_2) &= \omega_3^{n_3 + \kappa_{32} n_2} \Gamma[\gamma_{32} + n_3]
\end{align*}
\]
Meanwhile, the following one involve dependent sums

\[ \rho_1(n_1) = \omega_1^{n_1} n_1! \]
\[ \rho_1(n_1, n_2) = \omega_1^{n_1+\kappa_1 n_2} \Gamma[\gamma_{13} + n_1] \]
\[ \rho_1(n_1, n_2) = \omega_1^{n_1+\kappa_1 n_2} \Gamma[\gamma_{12} + n_1] \]
\[ \rho_1(n_1, n_2) = \omega_1^{n_1+\kappa_1 n_2} \Gamma[\gamma_{12} + n_1] \]
\[ \rho_1(n_1, n_2) = \omega_1^{n_1+\kappa_1 n_2} \Gamma[\gamma_{12} + n_1] \]
\[ \rho_1(n_1, n_2, n_3) = \omega_1^{n_1+\kappa_1 n_2+\kappa_3 n_3} \Gamma[\gamma_{13} + n_1] \]
\[ \rho_1(n_1, n_2, n_3) = \omega_1^{n_1+\kappa_1 n_2+\kappa_3 n_3} \Gamma[\gamma_{13} + n_1] \]
\[ \rho_1(n_1, n_2, n_3) = \omega_1^{n_1+\kappa_1 n_2+\kappa_3 n_3} \Gamma[\gamma_{13} + n_1] \]

and

\[ \rho_3(n_3, n_1, n_2) = \omega_3^{\rho_3 + \kappa_3 n_1 + \kappa_3 n_2} \Gamma[\gamma_{3} + n_3] \]
\[ \rho_3(n_3, n_1, n_2) = \omega_3^{\rho_3 + \kappa_3 n_1 + \kappa_3 n_2} \Gamma[\gamma_{3} + n_3] \]
\[ \rho_3(n_3, n_1, n_2) = \omega_3^{\rho_3 + \kappa_3 n_1 + \kappa_3 n_2} \Gamma[\gamma_{3} + n_3] \]
\[ \rho_3(n_3, n_1, n_2) = \omega_3^{\rho_3 + \kappa_3 n_1 + \kappa_3 n_2} \Gamma[\gamma_{3} + n_3] \]
\[ \rho_3(n_3, n_1, n_2) = \omega_3^{\rho_3 + \kappa_3 n_1 + \kappa_3 n_2} \Gamma[\gamma_{3} + n_3] \]
\[ \rho_3(n_3, n_1, n_2) = \omega_3^{\rho_3 + \kappa_3 n_1 + \kappa_3 n_2} \Gamma[\gamma_{3} + n_3] \]

We are now in position to define VCS classes.

4.2.1 \((\gamma_{13}, \gamma_{23})\)- and \((\gamma_{13}, \gamma_{32})\)-deformed CS and descendants

We treat first the so-called VCS classes generated by independent sums. According to (109), it proves to be judicious to work with the last quantities \(\rho_1(n_1, n_3)\) and \(\rho_2(n_2, n_3)\), since the previous one can be recovered by some limits. Hence, we consider

\[ \rho_1(n_1, n_3) = \omega_1^{n_1+\kappa_1 n_3} \Gamma[\gamma_{13} + n_1], \quad \rho_2(n_2, n_3) = \omega_2^{n_2+\kappa_2 n_3} \Gamma[\gamma_{23} + n_2]. \]  

Introduce two complex variables \(z_1 = r_1 e^{i\theta_1}\) and \(z_2 = r_2 e^{i\theta_2}\), and the \((\gamma_{13}, \gamma_{23})\)-deformed class given by the state

\[ |z_1, z_2, n_3\rangle = \mathcal{N}(z_1, z_2, n_3)^{-\frac{1}{2}} \sum_{n_1, n_2=0}^{\infty} \frac{z_1^{n_1+\kappa_1 n_3} z_2^{n_2+\kappa_2 n_3}}{[\omega_1^{n_1+\kappa_1 n_3} \omega_2^{n_2+\kappa_2 n_3} \Gamma[\gamma_{13} + n_1] \Gamma[\gamma_{23} + n_2]]^{\frac{1}{2}}} |n_1, n_2, n_3\rangle, \]

where \(n_3\) is the vector index. The state is normalizable provided

\[ \mathcal{N}(z_1, z_2, n_3) = \left[ \frac{|\omega_1^{n_1+\kappa_1 n_3} \omega_2^{n_2+\kappa_2 n_3}|}{\Gamma[\gamma_{13}] \Gamma[\gamma_{23}]} \right]^{n_3} \sum_{n_1, n_2=0}^{\infty} \frac{|z_1|^{2n_1} |z_2|^{2n_2}}{[\omega_1^{n_1} \omega_2^{n_2} \Gamma[\gamma_{13} + n_1] \Gamma[\gamma_{23} + n_2]]^{\frac{1}{2}}} \]

is convergent. This is indeed the case as one easily can check by a ratio test for each sector \(n_1\) and \(n_2\) which, in this case, factorize. Alternatively, one can surely perform a comparison test, as long as \(n_i \geq 1\),

\[ \frac{1}{\Gamma[\gamma_{13} + n_1] \Gamma[\gamma_{23} + n_2]} \leq \frac{1}{n_1! n_2!}, \quad \gamma_{ij} \geq 1, \]

and then conclude by Corollary \[1\] since the r.h.s is the term of a product of exponential series.

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The resolution of the identity requires that

$$\int d\mu(z_1, z_2, n_3) |z_1, z_2, n_3\rangle\langle z_1, z_2, n_3| = I_{n_3} = \sum_{n_1, n_2=0}^{\infty} |n_1, n_2, n_3\rangle\langle n_1, n_2, n_3|, \quad (114)$$

where

$$d\mu(z_1, z_2, n_3) = \frac{1}{\pi^2} N(|z_1|, |z_2|, n_3) \chi(r_1, r_2, n_3) \prod_{i=1}^{2} r_i dr_i d\theta_i. \quad (115)$$

The resulting moment problem is of the form, using $u_i = r_i^2$,

$$\int du_1 du_2 \chi(r_1, r_2, n_3) \frac{u_1^{n_1+\kappa_1 n_3} u_2^{n_2+\kappa_2 n_3}}{\omega_1^{n_1+\kappa_1 n_3} \omega_2^{n_2+\kappa_2 n_3}} = \Gamma[\gamma_{13} + n_1] \Gamma[\gamma_{23} + n_2] \quad (116)$$

and is solved by $\chi(r_1, r_2, n_3) = \varrho_1(r_1) \varrho_2(r_2)$ where $\varrho_i(r_i)$ are given by (48).

Remark that (as already discussed in Section 4.1) the exponents of the complex variables $z_i$ and frequency $\omega_i$ have been adjusted in order to exactly match with the argument of the gamma function $\Gamma[\bullet + 1]$ in the moment problem. The latter will be solved by a simple exponential as given by (48). All the classes here and after, being designed in this particular form, possess a resolution of the identity of this sort. We will not discuss the requirement of the resolution of the identity which should be obvious provided that (a) the phase integration in $\theta_i$ yields enough constraints to project the resolution of the identity on the real domain, and (b) one ensures at least that the bounds of integration of the moment problem are the same as (48). This means that all states should be of infinite radius of convergence in $|z_i|$ under some conditions on the parameters $\kappa_{ij}$. (a) will be only discussed when it leads to nontrivial facts whereas to prove (b) will be our main goal for the remaining part.

Taking the limit $\kappa_{13} \to 0$ in (111), we get another class of VCS

$$|z_1, z_3, n_3\rangle = N(z_1, z_2, n_3)^{-\frac{1}{2}} \sum_{n_1, n_2=0}^{\infty} \frac{z_1^{n_1} z_2^{n_2+\kappa_{23} n_3}}{[\omega_1^{n_1} \omega_2^{n_2+\kappa_{23} n_3} n_1! \Gamma[\gamma_{23} + n_2]]^{\frac{1}{2}}} |n_1, n_2, n_3\rangle, \quad (117)$$

and taking both limits $\kappa_{13} \to 0$ and $\kappa_{23} \to 0$, one gets the straightforward extension of the $(1,1)$-class:

$$|z_1, z_2, n_3\rangle = \mathcal{N}^{-\frac{1}{2}} \sum_{n_1, n_2=0}^{\infty} \frac{z_1^{n_1} z_2^{n_2}}{[\omega_1^{n_1} \omega_2^{n_2} n_1! n_2!]^{\frac{1}{2}}} |n_1, n_2, n_3\rangle = |z_1\rangle \otimes |z_2\rangle \otimes |n_3\rangle, \quad (118)$$

where in the last expression $|z_i\rangle$ stands for the CS for the harmonic oscillator in 1D. Indeed (118) can be written, in sloppy symbols, $|z_1, z_2, n_3\rangle = \mathcal{N}^{-\frac{1}{2}} \sum (\mathcal{N}_{\frac{1}{2}}^2 |z_1, n_2\rangle) \otimes |n_3\rangle$, where $\mathcal{N}^{\frac{1}{2}} |z_1, n_2\rangle$ is the non-normalized state associated with the $(1,1)$-class (60). Furthermore, (117) is nothing but the state generated by $|z_1, z_2, n_3\rangle = \mathcal{N}^{-\frac{1}{2}} (\mathcal{N}_{\frac{1}{2}}^2 |z_1\rangle) \otimes \sum (\mathcal{N}_{\frac{1}{2}}^2 |z_2, n_3\rangle)$, where $\mathcal{N}_{\frac{1}{2}}^2 |z_2, n_3\rangle$ is the non-normalized state associated with (53) namely the $(\gamma_{23}, 1)$-VCS class. Both classes (117) and (118) are normalized (by a comparison procedure as was performed in (113)) and solved by the same densities as for (111). Thus, surprisingly, summing
an infinite tower of VCS like was shown before yields again a VCS. We will denote [117] and [118] by (1, γ23)- and (1, 1)-VCS classes. Note that taking another limit κ23 → 0 in [111] yields then a (γ13, 1)-VCS class (with a similar meaning but on the sector 1,3) which gives after κ31 → 0 the same (1, 1)-class [118].

Considering now independent sums provided by Case (13), we have the quantities given by [108]:

\[ \rho_1(n_1, n_3) = \omega_1^{n_1 + \kappa_{13} n_3} \Gamma[\gamma_{13} + n_1], \quad \rho_3(n_3, n_2) = \omega_3^{n_3 + \kappa_{32} n_2} \Gamma[\gamma_{32} + n_3] \]  

which define the (γ13, γ32)-deformed CS as

\[ |z_1, z_3, n_3\rangle = \mathcal{N}(z_1, z_3, n_3)^{-\frac{1}{2}} \sum_{n_1, n_2=0}^{\infty} \frac{z_1^{n_1 + \kappa_{13} n_3} z_3^{n_3 + \kappa_{32} n_2} \Gamma[\gamma_{13} + n_1] \Gamma[\gamma_{32} + n_3]}{\omega_1^{n_1} \omega_3^{n_3} \Gamma[\gamma_{13} + n_1] \Gamma[\gamma_{32} + n_3]} |n_1, n_2, n_3\rangle. \]  

The norm of this vector is unity if

\[ \mathcal{N}(z_1, z_3, n_3) = \left[ \frac{|z_1|^{2\kappa_{13}} |z_3|^2}{\omega_1^{\kappa_{13} \omega_3}} \right]^{n_3} \sum_{n_1, n_2=0}^{\infty} \frac{|z_1|^{2n_1} |z_3|^2 \Gamma[\gamma_{13} + n_1] \Gamma[\gamma_{32} + n_3]}{\omega_1^{n_1} \omega_3^{n_3} \Gamma[\gamma_{13} + n_1] \Gamma[\gamma_{32} + n_3]} \]  

\[ = \left[ \frac{|z_1|^{2\kappa_{13}} |z_3|^2}{\omega_1^{\kappa_{13} \omega_3}} \right]^{n_3} \frac{1}{\Gamma[\gamma_{13}]} \, _1F_1 \left( 1; \gamma_{13}; \frac{|z_1|^2}{\omega_1} \right) \sum_{n_2=0}^{\infty} \frac{|z_3|^{2\kappa_{32} n_2}}{\omega_3^{n_3} \Gamma[\gamma_{32} + n_3]}. \]  

This double series factorizes: the series in \(n_1\) is again bounded by an exponential and so we only have to prove that the series in \(n_2\) is convergent. Using the \(\Gamma\)-Stirling approximation

\[ \Gamma[z] \sim \sqrt{\frac{2\pi}{z}} \left( \frac{z}{e} \right)^z \]

valid at large \(\Re(z)\) and for \text{arg}(z) < π − ε for \(ε > 0\), the following relation holds as long as \(κ_{32} > 0\):

\[ \lim_{n_2 \to \infty} \frac{\Gamma[1 + \kappa_{32} n_2 + 3]}{\Gamma[1 + \kappa_{32}(1 + n_2) + 3]} = \lim_{n_2 \to \infty} \frac{\Gamma[1 + \kappa_{32}(1 + n_2) + 3]}{\Gamma[1 + \kappa_{32} n_2 + 3]} - \kappa_{32} = 0. \]

This means that the radius of convergence of the series is infinite again. Note the important fact that at \(κ_{32} = 0\), the state is non normalizable, hence the VCS is not defined at that limit.

The resolution of the identity is again of the form [114] having an integration measure like [115] including a factor of \(κ_{32}\) (canceling a contribution to a phase integration in \(\kappa_{32}\theta_3\)) and with moment problem, similar to [116] hence solvable by the same kind of densities. Henceforth, we will not mention the constant corrections to the measure up to factors of \(\kappa_{ii}\) due to unessential phase integrations which may occur.

We also have here a unique class which can be obtained at \(κ_{13} \to 0\), i.e.

\[ |z_1, z_3, n_3\rangle = \mathcal{N}(z_1, z_3, n_3)^{-\frac{1}{2}} \sum_{n_1, n_2=0}^{\infty} \frac{z_1^{n_1} z_3^{n_3 + \kappa_{32} n_2}}{\omega_1^{n_1} \omega_3^{n_3 + \kappa_{32} n_2} n_1! \Gamma[\gamma_{32} + n_3]} |n_1, n_2, n_3\rangle, \]  

\[ \mathcal{N}(z_1, z_3, n_3) = \left[ \frac{|z_3|^2}{\omega_3} \right]^{n_3} \exp \left\{ \frac{|z_1|^2}{\omega_1} \right\} \sum_{n_2=0}^{\infty} \frac{|z_3|^{2\kappa_{32} n_2}}{\omega_3^{n_3} \Gamma[\gamma_{32} + n_3]}, \]  

\[ \text{(125)} \]
which, as expected, consists in the VCS given by the other set of generalized factorial in \((108)\) generating independent sums. Note that \(\kappa_{32} \to 0\) is yet forbidden. We cannot recovered the first class VCS \((118)\) hence the type of these VCS classes \((120)\) and \((124)\) is not of any type yet encountered so far. This can be easily understood given the fact that the VCS \((118)\) is a continuous limit deformation of another VCS class which can be only recovered at the limit \(\kappa_{23} \to 0\) and so at \(\kappa_{32} \to \infty\). We mention also that \((124)\) can be simply viewed as \(|z_1, z_3, n_3\rangle = \mathcal{N}^2 \langle \mathcal{N}^2 | z_1 \rangle \otimes \sum (\mathcal{N}^2 | z_3, n_3\rangle)\) where \((\mathcal{N}^2 | z_3, n_3\rangle)\) is the unnormalized fake dual \((1, \gamma_{32})\) \((16)\) for the sector 2,3.

4.2.2 \((\gamma_i, \gamma_{kl})\)-deformed CS and descendants

This section deals with the analysis of dependent sums. We study the most general states and derive its descendants by continuous limits.

The generalized factorials

\[
\rho_1(n_1, n_3) = \omega_1^{n_1+\kappa_{13}n_3}\Gamma[\gamma_{13} + n_1] \quad \rho_2(n_2, n_1, n_3) = \omega_2^{n_2+\kappa_{23}n_3+\kappa_{21}n_1}\Gamma[\gamma_2 + n_2] \quad (126)
\]
yield the states

\[
|z_1, z_2, n_3\rangle = \mathcal{N}(z_1, z_2, n_3)^{-\frac{1}{2}} \sum_{n_1, n_2=0}^{\infty} \frac{z_1^{n_1+\kappa_{13}n_3}z_2^{n_2+\gamma_{21}-1}}{\omega_1^{n_1+\kappa_{13}n_3}\omega_2^{n_2+\gamma_{21}-1}\Gamma[\gamma_{13} + n_1]\Gamma[\gamma_2 + n_2]} |n_1, n_2, n_3\rangle \quad (127)
\]
with normalization factor

\[
\mathcal{N}(z_1, z_2, n_3) = \left[ \frac{|z_1|^{2\kappa_{13}}|z_2|^{2\kappa_{21}}}{\omega_1^{\kappa_{13}}\omega_2^{\kappa_{21}}} \right]^{n_3} \sum_{n_1, n_2=0}^{\infty} \frac{|z_1 z_2^{\kappa_{21}}|^{2n_1}|z_2|^{2n_2}}{(\omega_1\omega_2^{\kappa_{21}})^{n_1}\omega_2^{n_2}\Gamma[\gamma_{13} + n_1]\Gamma[\gamma_2 + n_2]} \quad (128)
\]

In this situation, the sums are dependent and do not factorize. Therefore we need a criterion to analyze the convergence of the double series \((128)\). Calculating the convergence of row and column series, one finds that they are indeed of infinite radius of convergence, \(|z_1 z_2^{\kappa_{21}}| > 0\) and \(|z_2| > 0\) which entail also that \(|z_1| > 0\). Corollary \(1\) is sufficient to prove the absolute convergence here since, fortunately, a similar inequality as \((113)\) can be applied for \(n_i \geq 1\) and \(\gamma \geq 1\). Thus, the normalization factor converges everywhere in \(\mathbb{C}^2\).

Treating the VCS defined through \((106)\) and \((107)\), Corollary \(1\) will be always sufficient in order to prove the convergence of the double series in all cases provided the fact that the row and column series will be convergent. Indeed, all VCS normalizations defined through the generalized factorials as defined by these cases, can be all bounded by or compared to the double exponential series with general term \(z_1^{n_1} z_2^{n_2}/(n_1!n_2!)\). In contradistinction, with cases which will be defined by \((108)\) and \((109)\), a greater effort will be required to prove the convergence of the series.

The following successive limits yield relevant VCS: taking \(\kappa_{13} \to 0\) from \((126)\) yields

\[
\rho_1(n_1) = \omega_1^{n_1}n_1!, \quad \rho_2(n_2, n_1, n_3) = \omega_2^{n_2+\kappa_{23}n_3+\kappa_{21}n_1}\Gamma[\gamma_2 + n_2] \quad (129)
\]
and, considering again \(\kappa_{23} \to 0\), one has

\[
\rho_1(n_1) = \omega_1^{n_1}n_1!, \quad \rho_2(n_2, n_1) = \omega_2^{n_2+\kappa_{21}n_1}\Gamma[\gamma_{21} + n_2]. \quad (130)
\]
Then it still remains a last limit which is $\kappa_{21} \to 0$ giving
\[
\rho_1(n_1) = \omega_1^{n_1} n_1!, \quad \rho_2(n_2) = \omega_2^{n_2} n_2!.
\]
leading to the (1,1)-VCS. Still from (126), but now considering $\kappa_{21} \to 0$, we get
\[
\rho_1(n_1, n_3) = \omega_1^{n_1 + \kappa_{13} n_3} \Gamma[\gamma_{13} + n_1], \quad \rho_2(n_2, n_3) = \omega_2^{n_2 + \kappa_{23} n_3} \Gamma[\gamma_{23} + n_2]
\]
generating the $(\gamma_{13}, \gamma_{23})$-VCS class (111) and from these, other limits can be performed. Finally, by the limit $\kappa_{23} \to 0$ in (126), we have
\[
\rho_1(n_1, n_3) = \omega_1^{n_1 + \kappa_{13} n_3} \Gamma[\gamma_{13} + n_1], \quad \rho_2(n_2, n_1) = \omega_2^{n_2 + \kappa_{21} n_1} \Gamma[\gamma_{21} + n_2],
\]
then, we perform a symmetry $1 \leftrightarrow 2$ in order to recover another element of the list (107):
\[
\rho_2(n_2, n_3) = \omega_2^{n_2 + \kappa_{23} n_3} \Gamma[\gamma_{23} + n_2], \quad \rho_1(n_1, n_2) = \omega_1^{n_1 + \kappa_{12} n_2} \Gamma[\gamma_{12} + n_1].
\]
The number of VCS classes involving dependent sums and their possible link by deformation belonging to Case (12) and having (126) as ancestor has been exhausted.

Let us now discuss Case (13). Starting by the factorials
\[
\rho_1(n_1, n_2, n_3) = \omega_1^{n_1 + \kappa_{12} n_2 + \kappa_{13} n_3} \Gamma[\gamma_1 + n_1], \quad \rho_3(n_3, n_2) = \omega_3^{n_3 + \kappa_{32} n_2} \Gamma[\gamma_{32} + n_3],
\]
one can build the following states
\[
|z_1, z_3, n_3\rangle = \mathcal{N}(z_1, z_3, n_3)^{-\frac{1}{2}} \sum_{n_1, n_2 = 0}^{\infty} \frac{z_1^{n_1 + \gamma_1 - 1} z_3^{n_3 + \kappa_{32} n_2}}{\omega_1^{n_1} \Gamma[\gamma_1 + n_1] \Gamma[\gamma_{32} + n_3]} |n_1, n_2, n_3\rangle,
\]
with normalization condition
\[
\mathcal{N}(z_1, z_3, n_3) = \left[ \frac{|z_1|^{2 \kappa_{13}|z_3|^2}}{\omega_1^{k_{13}} \omega_3} \right]^{n_3} \sum_{n_1, n_2 = 0}^{\infty} \frac{|z_1|^{2 n_1} |z_3|^{2 n_2} \Gamma[\gamma_1 + n_1] \Gamma[\gamma_{32} + n_3]}{\omega_1^{n_1} (\omega_3^{k_{32}} \omega_1^{k_{12}})^{n_2} \Gamma[\gamma_{12} + n_2]} 1F_1 \left( 1; \gamma_1; \frac{|z_1|^2}{\omega_1} \right).
\]
The hypothesis of a convergence theorem can be verified in this case also. The row series in $n_1$ is convergent whereas the sum over $n_2$ is only convergent for $\kappa_{32} > 0$ or $\kappa_{12} > 0$. Moreover, using $n_1 \geq 1$,
\[
\frac{1}{\Gamma[\gamma_1 + n_1] \Gamma[\gamma_{32} + n_3]} \leq \frac{1}{n_1! \Gamma[\gamma_{32} + n_3]},
\]
we can infer, following Corollary 1 and the same steps from (123), that the norm series is convergent for $\kappa_{32} > 0$. A double checking of this statement using Theorem 2 is given in Appendix B.

We can perform the limit $\kappa_{12} \to 0$ in (135) and get
\[
\rho_1(n_1, n_3) = \omega_1^{n_1 + \kappa_{13} n_3} \Gamma[\gamma_{13} + n_1], \quad \rho_3(n_3, n_2) = \omega_3^{n_3 + \kappa_{32} n_2} \Gamma[\gamma_{32} + n_3]
\]
which generates \((\gamma_{13}, \gamma_{32})\)-class \([120]\); and then again a unique limit is allowed which is \(\kappa_{13} \to 0\) providing the class \([124]\). Conversely taking first the limit \(\kappa_{13} \to 0\) in \([135]\), one gets

\[
\rho_1(n_1, n_2) = \omega_1^{n_1+\kappa_{12}n_2} \Gamma[\gamma_{12} + n_1], \quad \rho_3(n_3, n_2) = \omega_3^{n_3+\kappa_{32}n_2} \Gamma[\gamma_{32} + n_3];
\]

then the limit \(\kappa_{12} \to 0\) yields

\[
\rho_1(n_1, n_2) = \omega_1^{n_1} n_1!, \quad \rho_3(n_3, n_2) = \omega_3^{n_3} n_3!
\]

which finally give the same end-point limit \([124]\).

Starting from \([135]\), a third limit can be performed. As \(\kappa_{32} \to 0\), one has

\[
\rho_1(n_1, n_2, n_3) = \omega_1^{n_1+\kappa_{12}n_2+\kappa_{13}n_3} \Gamma[\gamma_1 + n_1], \quad \rho_3(n_3, n_2) = \omega_3^{n_3} n_3!
\]

and then \(\kappa_{13} \to 0\) implies

\[
\rho_1(n_1, n_2) = \omega_1^{n_1+\kappa_{12}n_2} \Gamma[\gamma_{12} + n_1], \quad \rho_3(n_3, n_2) = \omega_3^{n_3} n_3!.
\]

All the above states are properly normalized by the same normalization procedure as that for their ancestor. But the state defined by \([142]\) differ drastically from the form of its ancestor \([136]\). Its normalizability will not follow from the same recipe because it involves \(\kappa_{32} = 0\) prohibited so far. Hence, defining

\[
|z_1, z_3, n_3\rangle = \mathcal{N}(z_1, z_3, n_3)^{-\frac{1}{2}} \sum_{n_1,n_2=0}^{\infty} \frac{z_1^{n_1+\gamma_1-1} z_3^{n_3}}{\omega_1^{n_1+\gamma_1-1} \omega_3^{n_3} \Gamma[\gamma_1 + n_1] n_3!} |n_1, n_2, n_3\rangle,
\]

\[
\mathcal{N}(z_1, z_3, n_3) = \left[\frac{|z_3|^2}{\omega_3^{n_3}}\right]^{\frac{1}{2}} \sum_{n_1,n_2=0}^{\infty} \frac{|z_1|^{2(n_1+\gamma_1-1)}}{\omega_1^{n_1+\gamma_1-1} \Gamma[\gamma_1 + n_1]},
\]

and \(a_{n_1,n_2}\) standing for the general term of the series \([145]\), the ratio tests of the row and column series are such that

\[
\lim_{n_1 \to \infty} \frac{a_{n_1+1,n_2}}{a_{n_1,n_2}} = 0, \quad \lim_{n_2 \to \infty} \frac{a_{n_1,n_2+1}}{a_{n_1,n_2}} = w_2 \lim_{n_2 \to \infty} \left[\kappa_{12} n_2 + \gamma_{13} + n_1\right]^{-\kappa_{12}} \sim 0,
\]

the last equality holding only for \(\kappa_{12} > 0\). Hence, row and column series converge everywhere.

Neither the simple comparison test of Corollary \([1]\) nor the ratio comparison given by Theorem \([2]\) using some exponentials, can help here for extracting the largest solvable class (see Appendix \([3]\)). Directly evaluating the ratio tests, one has

\[
\lim_{n_1,n_2 \to \infty} \frac{a_{n_1+1,n_2}}{a_{n_1,n_2}} = w_1 \lim_{n_1,n_2 \to \infty} \frac{1}{\gamma_1 + n_1} < 1,
\]

\[
\lim_{n_1,n_2 \to \infty} \frac{a_{n_1,n_2+1}}{a_{n_1,n_2}} \sim w_2 \lim_{n_1,n_2 \to \infty} \frac{1}{\kappa_{12}(n_2+1) + \gamma_{13} + n_1} \sim 1,
\]

used in the last inequality has been made of the \(\Gamma\)-Stirling approximation and \(\kappa_{12} > 1\). The first inequality, through Theorem \([1]\) allows us to infer the convergence. Other numerical evidences for that convergence even for \(0 < \kappa_{12} \leq 1\) can be found in Appendix \([3]\).
The resolution of the identity of the state (144) is not totally straightforward as one may expect. Given four integers \((n_1, n_2, n_1', n_2')\) defined by twice the state double series, phase integrations will require a unique constraint

\[
(n_1 - n_1') + \kappa_{12}(n_2 - n_2') = 0
\]

which could possess many solutions. However, the most interesting would be the one such that

\[
\kappa_{12} \neq \frac{n_1 - n_1'}{n_2 - n_2'}
\]

which may occur for instance for any irrational value of \(\kappa_{12}\). In that situation, one is led to the unique solutions \(n_i = n_i', i = 1, 2\). Then the resolution of the identity can be directly inferred from our ordinary method. Let us comment that the derivation of the properties of the states defined by (143) is completely similar to that of the VCS (144).

Another set of factorials generates a different class compared to what occurs before. Consider, still in Case (13), the following

\[
\rho_1(n_1, n_2) = \omega_1^{n_1 + \kappa_{12} n_2} \Gamma[\gamma_{12} + n_1], \quad \rho_3(n_3, n_1, n_2) = \omega_3^{n_3 + \kappa_{31} n_1 + \kappa_{32} n_2} \Gamma[\gamma_3 + n_3].
\]

From these, we define the set of VCS

\[
|z_1, z_3, n_3\rangle = \mathcal{N}(z_1, z_3, n_3)^{-\frac{1}{2}} \sum_{n_1, n_2=0}^{\infty} \frac{z_1^{n_1 + \kappa_{12} n_2} z_3^{n_3 + \gamma_3 - 1}}{\omega_1^{n_1 + \kappa_{12} n_2} \omega_3^{n_3 + \gamma_3 - 1} \Gamma[\gamma_{12} + n_1] \Gamma[\gamma_3 + n_3]} |n_1, n_2, n_3\rangle
\]

normalized with

\[
\mathcal{N}(z_1, z_3, n_3) = \left[ \frac{|z_1|^{2 n_1} |z_3|^{2 n_3}}{\omega_3} \sum_{n_1, n_2=0}^{\infty} \frac{|z_1 z_3^{\kappa_{31}}|^{2 n_1} |z_3^{\kappa_{32} \kappa_{12}}|^{2 n_2}}{(\omega_1^{\kappa_{31} n_1} n_1 (\omega_3^{\kappa_{32} \kappa_{12}}) n_2 \Gamma[\gamma_{12} + n_1] \Gamma[\gamma_3 + n_3])} \right]^{\frac{1}{2}}
\]

We verify the normalization factor convergence using similar notations as above. The row and column series satisfy

\[
\lim_{n_1 \to 0} \frac{a_{n_1 + 1, n_2}}{a_{n_1, n_2}} = w_1 \lim_{n_1 \to 0} \frac{\Gamma[\kappa_{31} n_1 + \gamma_3 + n_3]}{(\gamma_{12} + n_1) \Gamma[\kappa_{31} (n_1 + 1) + \gamma_3 + n_3]} = 0,
\]

\[
\lim_{n_2 \to 0} \frac{a_{n_1, n_2 + 1}}{a_{n_1, n_2}} = w_2 \lim_{n_2 \to 0} \frac{\Gamma[\kappa_{12} n_2 + 1 + n_1] \Gamma[\kappa_{32} n_2 + \gamma_3 + n_3]}{(\gamma_{12} + n_2 + 1 + n_1) \Gamma[\kappa_{12} (n_2 + 1) + \gamma_3 + n_3]} = 0,
\]

where in the second limit, we require either \(\kappa_{12} > 0\) or \(\kappa_{32} > 0\). Furthermore, one writes using the monotony of the Gamma function for positive large arguments

\[
\frac{1}{\Gamma[\gamma_{12} + n_1] \Gamma[\gamma_3 + n_3]} \leq \frac{1}{n_1! \Gamma[1 + \kappa_{32} n_2 + \kappa_{31} n_1 + n_3]} \leq \frac{1}{n_1! \Gamma[\gamma_{32} + n_3]}
\]

and therefore the previous analysis for (138) holds again and ensures the convergence of the norm series. This result can be differently checked (see Appendix B).

At the limit \(\kappa_{12} \to 0\), we get from (150)

\[
\rho_1(n_1, n_2) = \omega_1^{n_1 n_1!}, \quad \rho_3(n_3, n_1, n_2) = \omega_3^{n_3 + \kappa_{31} n_1 + \kappa_{32} n_2} \Gamma[\gamma_3 + n_3]
\]

35
which, again taking $\kappa_{31} \to 0$, leads to the factorials defining (124). The convergence of the norm for this case follows from (154) given $\kappa_{32} > 0$.

A last VCS class has to be studied. This is the one generated by

$$\rho_1(n_1, n_2) = \omega_1^{n_1 + \kappa_{12} n_2} \Gamma[\gamma_{12} + n_1], \quad \rho_3(n_3, n_1) = \omega_3^{n_3 + \kappa_{31} n_1} \Gamma[\gamma_{31} + n_3]$$

(156)

which entail the class of states of the form

$$|z_1, z_3, n_3\rangle = \mathcal{N}(z_1, z_3, n_3)^{-\frac{1}{2}} \sum_{n_1, n_2=0}^{\infty} \frac{z_1^{n_1 + \kappa_{12} n_2} z_3^{n_3 + \kappa_{31} n_1}}{\omega_1^{n_1} \omega_3^{n_3} \Gamma[\gamma_{12} + n_1] \Gamma[\gamma_{31} + n_3]} |n_1, n_2, n_3\rangle.$$

A normalization condition can be formulated as

$$\mathcal{N}(z_1, z_3, n_3) = \left[ \frac{|z_3|^2}{\omega_3} \right]^{n_3} \sum_{n_1, n_2=0}^{\infty} \frac{|z_1|^{n_1} |z_3|^{n_2} |z_1 z_3|^{2n_1+2n_2}}{(\omega_1 \omega_3)^{n_1+n_2} \Gamma[\gamma_{12} + n_1] \Gamma[\gamma_{31} + n_3]}.$$

(157)

The latter series converges on $\mathbb{C}^2$ for $\gamma_{12} > 0$ since

$$\frac{1}{\Gamma[\gamma_{12} + n_1] \Gamma[\gamma_{31} + n_3]} \leq \frac{1}{\Gamma[\gamma_{12} + n_1] n_3!}$$

(159)

and the r.h.s term is nothing but a part of (143) which has been already studied. Thus, for $\kappa_{31} \to 0$, (157) tends to the VCS defined by (143).

4.2.3 (\gamma_1, \gamma_2)- and (\gamma_1, \gamma_3)- deformed CS and descendants

We pursue the analysis on dependent sums (107). Focusing on

$$\rho_1(n_1, n_2, n_3) = \omega_1^{n_1 + \kappa_{12} n_2 + \kappa_{13} n_3} \Gamma[\gamma_1 + n_1], \quad \rho_2(n_2, n_1, n_3) = \omega_2^{n_2 + \kappa_{23} n_3 + \kappa_{21} n_1} \Gamma[\gamma_2 + n_2],$$

(160)

we construct the set of states with two deformation parameters $(\gamma_1, \gamma_2)$ as follows:

$$|z_1, z_2, n_3\rangle = \mathcal{N}(z_1, z_2, n_3)^{-\frac{1}{2}} \sum_{n_1, n_2=0}^{\infty} \frac{z_1^{n_1+\gamma_1-1} z_2^{n_2+\gamma_2-1}}{\omega_1^{n_1+\gamma_1-1} \omega_2^{n_2+\gamma_2-1} \Gamma[\gamma_1 + n_1] \Gamma[\gamma_2 + n_2]} |n_1, n_2, n_3\rangle$$

(161)

with the normalization factor

$$\mathcal{N}(z_1, z_2, n_3) = \sum_{n_1, n_2=0}^{\infty} \frac{|z_1|^{2(n_1+\gamma_1-1)} |z_2|^{2(n_2+\gamma_2-1)}}{\omega_1^{n_1+\gamma_1-1} \omega_2^{n_2+\gamma_2-1} \Gamma[\gamma_1 + n_1] \Gamma[\gamma_2 + n_2]}$$

$$= \left[ \frac{|z_1|^{2\kappa_{13}} |z_2|^{2\kappa_{23}}}{\omega_1^{\kappa_{13}} \omega_2^{\kappa_{23}}} \right]^{n_3} \sum_{n_1, n_2=0}^{\infty} \frac{|z_1|^{2\kappa_{21}} |z_2|^{2\kappa_{12}}}{(\omega_1 \omega_2)^{n_1(n_2 + \gamma_2 - 1) + n_2(n_1 + \gamma_1 - 1)}} \Gamma[\gamma_1 + n_1] \Gamma[\gamma_2 + n_2].$$

(162)

Even though the series does not factorize, by simple comparison $\Gamma[\gamma_i + n_i] \geq n_i!$ for $\gamma_i \geq 1$ and for $n_i \geq 1$, it is direct to prove that (162) is convergent for all $|z_1 z_2^{\kappa_{12}}| > 0$, and all $|z_2 z_1^{\kappa_{21}}| > 0$, and therefore absolutely convergent for all $z_i \in \mathbb{C}$ $i = 1, 2$. 36
Addressing the resolution of the identity of these states in the form (114), we find that the preliminary phase integration in \( (\theta_1, \theta_2) \) gives a kind of consistency condition

\[
n_1 - n'_1 + \kappa_{12}(n_2 - n'_2) = 0 \quad \Leftrightarrow \quad n_2 - n'_2 + \kappa_{21}(n_1 - n'_1) = 0
\]

(163)

which is indeed trivially satisfied, and we are led to the moment problem solved by our ordinary technique.

Let us study the solvable classes induced by limit procedures from (161). We can perform the limit \( \kappa_{13} \to 0 \), and find the corresponding to the generalized factorials

\[
\rho_1(n_1, n_2) = \omega_1^{n_1 + \kappa_{12}n_2} \Gamma[\gamma_{12} + n_1], \quad \rho_2(n_2, n_1, n_3) = \omega_2^{n_2 + \kappa_{21}n_3} \Gamma[\gamma_2 + n_2].
\]

(164)

From these quantities, take the limit \( \kappa_{23} \to 0 \) and get

\[
\rho_1(n_1, n_2) = \omega_1^{n_1 + \kappa_{12}n_2} \Gamma[\gamma_{12} + n_1], \quad \rho_2(n_2, n_1) = \omega_2^{n_2 + \kappa_{21}n_1} \Gamma[\gamma_2 + n_2]
\]

(165)

defining another VCS class and then the procedure stops. The class (165) corresponds to an extension of the \((\gamma_{12}, \gamma_2)\)-doubly deformed VCS (85) where, in addition, a sum is carried out on the second index \( n_2 \). Thus (85) are generators of the class defined by (165).

Note that we could have done first the second limit \( \kappa_{13} \to 0 \) yielding a different intermediate step which defines nothing but a symmetric (and so not a new) class of VCS determined by the symmetric generalized factorials of (164) under \( (1 \leftrightarrow 2) \). From this state, then perform \( \kappa_{13} \to 0 \) giving the same final VCS class determined by (165). All these VCS limits have a convergent normalization factor since the above comparison criterion does not depend on the \( \kappa \) parameters.

We have another set of generalized factorials in Case (13) (109) yet defining another set of \((\gamma_{13}, \gamma_3)\)-deformation of VCS that can be reported here also. These are given by

\[
\rho_1(n_1, n_2, n_3) = \omega_1^{n_1 + \kappa_{12}n_2 + \kappa_{13}} \Gamma[\gamma_{1} + n_1], \quad \rho_2(n_3, n_1, n_2) = \omega_3^{n_3 + \kappa_{31}n_1 + \kappa_{32}n_2} \Gamma[\gamma_3 + n_3]
\]

(166)

from which we define the set of states

\[
|z_1, z_3, n_3\rangle = \mathcal{N}(z_1, z_3, n_3)^{-\frac{1}{2}} \sum_{n_1, n_2=0}^{\infty} \frac{z_1^{n_1+\gamma_1-1} z_3^{n_3+\gamma_3-1}}{\omega_1^{n_1+\gamma_1-1} \omega_3^{n_3+\gamma_3-1} \Gamma[\gamma_1+n_1] \Gamma[\gamma_3+n_3]} |n_1, n_2, n_3\rangle.
\]

(167)

The normalization factor can be computed as

\[
\mathcal{N}(z_1, z_2, n_3) = \sum_{n_1, n_2=0}^{\infty} \frac{|z_1|^{2(n_1+\gamma_1-1)} |z_3|^{2(n_3+\gamma_3-1)}}{\omega_1^{n_1+\gamma_1-1} \omega_3^{n_3+\gamma_3-1} \Gamma[\gamma_1+n_1] \Gamma[\gamma_3+n_3]} \\
= \left[ \frac{|z_1|^{2\kappa_{13}} |z_3|^{2\kappa_{13}}}{\omega_1^{\kappa_{13}} \omega_3^{\kappa_{13}}} \right]^{n_3} \sum_{n_1, n_2=0}^{\infty} \frac{|z_1 z_3^{\kappa_{31}} |z_3|^{2n_1} |z_3^{\kappa_{32}} z_1^{\kappa_{12}}|^{2n_2}}{\omega_1^{n_1} \omega_3^{n_3} \Gamma[\gamma_1+n_1] \Gamma[\gamma_3+n_3]}. 
\]

(168)

Corollary [1] can be applied here noting that at large arguments

\[
\frac{1}{\Gamma[\gamma_1+n_1] \Gamma[\gamma_3+n_3]} \leq \frac{1}{n! \Gamma[\gamma_3+n_3]}
\]

(169)
Figure 3: 3D Harmonic oscillator solvable VCS classes with two degrees of freedom organized through the relation “ancestor and descendant”. Classes highlighted (in red) have been already listed: $(1, \gamma_{23}) \equiv (\gamma_{13}, 1)$, and $(\gamma_1, \gamma_{21}) \equiv (\gamma_{12}, \gamma_2)$, by symmetry $(1 \leftrightarrow 2)$. Red arrows are undefined limits.

and the r.h.s. is part of the series defined by \[ (155) \] which has been already determined to be convergent everywhere using \[ (154) \] and $\kappa_{32} > 0$.

The resolution of the identity passes again through a well defined $(\theta_1, \kappa_{31}\theta_3)$ phase integrations yielding an obvious statement:

\[
 n_1 - n_1' + \kappa_{12}(n_2 - n_2') = 0 \iff n_1 - n_1' + \kappa_{13}\kappa_{32}(n_2 - n_2') = 0 \tag{170}
\]

and the resolution of the moment problem can be carried out through the same routine with solution given by \[ (48) \].

The limit $\kappa_{12} \to 0$ yields

\[
 \rho_1(n_1, n_3) = \omega_1^{n_1 + \kappa_{13}n_3}\Gamma[\gamma_{13} + n_1], \quad \rho_3(n_3, n_1, n_2) = \omega_3^{n_3 + \kappa_{31}n_1 + \kappa_{32}n_2}\Gamma[\gamma_3 + n_3]. \tag{171}
\]

Note that we cannot go further since any other limits would be undefined: $\kappa_{31} \to 0$ implies $\kappa_{13} \to \infty$ (or vice-versa) and $\kappa_{32} \to 0$ leads to an infinite series summing over an integrand free of the index summation $n_2$. Nevertheless, from the beginning, we can take the limit $\kappa_{32} \to 0$ such that the quantities become

\[
 \rho_1(n_1, n_2, n_3) = \omega_1^{n_1 + \kappa_{12}n_2 + \kappa_{13}}\Gamma[\gamma_1 + n_1], \quad \rho_3(n_3, n_1, n_2) = \omega_3^{n_3 + \kappa_{31}n_1}\Gamma[\gamma_{31} + n_3] \tag{172}
\]

which, for similar reasons, are an end-point of finding a well defined limit.

Figure [3] gives a summary of different classes and their descendants.
4.3 Some VCS with three degrees of freedom

We sketch here the construction of VCS with three degrees of freedom.
Consider the generalized factorials

\[ \rho_1(n_1, n_2, n_3) = \omega_1^{n_1+\kappa_1 n_2+\kappa_1 n_3} \Gamma[\gamma_1 + n_1], \quad \rho_2(n_2, n_1, n_3) = \omega_2^{n_2+\kappa_2 n_1+\kappa_2 n_3} \Gamma[\gamma_2 + n_2], \]
\[ \rho_3(n_3, n_1, n_2) = \omega_3^{n_3+\kappa_3 n_1+\kappa_3 n_2} \Gamma[\gamma_3 + n_3] \]  
(173)

from which we define the set of states maximally \((\gamma_1, \gamma_2, \gamma_3)\)-deformed given by

\[ |z_1, z_2, z_3, n_3\rangle = \mathcal{N}(z_1, z_2, z_3, n_3)^{-\frac{1}{2}} \times \]
\[ \sum_{n_1, n_2=0}^{\infty} \frac{z_1^{n_1+\gamma_1-1} z_2^{n_2+\gamma_2-1} z_3^{n_3+\gamma_3-1} \Gamma[\gamma_1 + n_1] \Gamma[\gamma_2 + n_2] \Gamma[\gamma_3 + n_3]}{\omega_1^{n_1+\kappa_1 n_2+\kappa_1 n_3} \omega_2^{n_2+\kappa_2 n_1+\kappa_2 n_3} \omega_3^{n_3+\kappa_3 n_1+\kappa_3 n_2}} |n_1, n_2, n_3\rangle \]  
(174)

with normalization factor

\[ \mathcal{N}(z_1, z_2, n_3) = \left( \frac{\left| z_1 \right|^{2\kappa_1} z_1^{2\kappa_1} |z_2|^{2\kappa_2} |z_3|^{2\kappa_3}}{\omega_1^{\kappa_1} \omega_2^{\kappa_2} \omega_3^{\kappa_3}} \right)^{\frac{1}{2}} \sum_{n_1, n_2=0}^{\infty} \frac{z_1^{n_1} z_2^{n_2} z_3^{n_3} \Gamma[\gamma_1 + n_1] \Gamma[\gamma_2 + n_2] \Gamma[\gamma_3 + n_3]}{(\omega_1^{n_1} \omega_2^{n_2} \omega_3^{n_3})} \]  
(175)

converging because bounded by the double exponential series up to some factor function of \(n_3\). One can show that the class \([174]\) is a VCS class which are integrated to unity according to our formalism.

Finally, the straightforward generalization of the \((1, 1)\) VCS \([55]\) and \([118]\) becomes a \((1, 1, 1)\) VCS made with simple factorials in the three sector as follows:

\[ |z_1, z_2, z_3, n_3\rangle = \mathcal{N}(z_1, z_2, z_3)^{-\frac{1}{2}} \sum_{n_1, n_2=0}^{\infty} \frac{z_1^{n_1} z_2^{n_2} z_3^{n_3} \Gamma[\gamma_1 + n_1] \Gamma[\gamma_2 + n_2] \Gamma[\gamma_3 + n_3]}{(\omega_1^{n_1} \omega_2^{n_2} \omega_3^{n_3})} |n_1, n_2, n_3\rangle, \]  
(176)

\[ \mathcal{N}(z_1, z_2, z_3, n_3) = \left( \frac{|z_3|^2}{\omega_3} \right)^{n_3} \exp \left\{ \frac{1}{2} (|z_1|^2 + |z_2|^2) \right\} \].  
(177)

This is just a state proportional to \(|z_1\rangle \otimes |z_2\rangle \otimes |n_3\rangle\). The classes \([174]\) and \([176]\) are the one which can be simply inferred for the harmonic oscillator and always remain solvable in any dimension. Between these states, corresponding to the maximally deformed and not deformed one, a number of intermediate states with a less number of deformation parameters occurs, but their solvability is not guaranteed.

5 Conclusion

This work has been devoted to an extension of solvable classes of VCS for the harmonic oscillator in 2D and 3D. By a combinatorics involving different possible partitions of the energy of the system and by assigning to each partition a couple of variables and thereby building the corresponding set of VCS, we improve some partial results known in the literature \([114] [13]\).
We focus on two basic requirements that the VCS ought to satisfy: a normalizability condition and a resolution of unity on the Hilbert space. The first requirement has mainly involved recent techniques providing criteria for double series convergence, and an optimization of the computations: all VCS states of this kind are normalizable. The second was performed in a row by noting an interesting fact appearing at the very basic level: the exponent of the continuous variable can be always cast in a way in order to conspire with generalized factorial for solving the Stieltjes moment problem in a particular simple way. Moreover, that resolution of the identity is proved to be valid for a wide range of the frequency parameters of the VCS, making these latter maximal in the space of these parameters. Interestingly, this work has found new connections with the (non)unicity of the measure integrating the VCS to unity \cite{26,27}. Only the aforementioned basic axioms have been proved, nevertheless the so-called Gazeau-Klauder properties could be certainly implemented from our results. In addition, a primary classification scheme has been investigated according to the number of degrees of freedom and complexity of the $\gamma$ parameters. Another way to understand these classes is through a deformed theory: VCS classes are consistent frequency dependent deformations of one another built out of ordinary CS living in a subspace of a higher rank tensor Hilbert space. The links are realized in the deformation parameter space.

An attempt of classification of these classes of VCS has been provided, though, one has to acknowledge, deserves to be definitely refined and precise. This work is under current investigations \cite{32}. Furthermore, we have mainly focused the existence proof of extended classes of VCS however, and notably, one should investigate the statistical properties of these classes of states. It is known that canonical VCS of the $(1,1)$-class form satisfy the ordinary properties of CS, i.e. that they are intelligent. This same question has to be addressed for the remaining VCS classes here. Yet more extensions of previous results could be investigated also. For instance, it will be interesting to make a sense of statistical properties of these VCS according to Gazeau-Novaes formalism \cite{11} using Berezin-Lieb inequalities for multidimensional CS.

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A Solving generalized moment problems

We consider the generalized moment problem as given in \cite{65}

$$\int du_1 du_2 \chi(u_1,u_2,n_2) \left[ \frac{u_1^{\alpha_1} u_2^{\alpha_2}}{\omega_1^{\alpha_1} \omega_2^{\alpha_2}} \right]^{n_1} \left[ \frac{u_1^{\beta_1}}{\omega_1^{\beta_1}} \right]^{n_2} \left[ \frac{u_2^{\alpha_2}}{\omega_2^{\alpha_2}} \right]^{n_2} = R_1(n_1,n_2)R_2(n_2). \quad (A.1)$$
A density solution of (A.1) is not uniquely defined. We have however a single constraint on \( \chi \); it should not depend on \( n_1 \). Hence, we are led to the following choices:

(a) All dependencies in \( n_2 \) can be simplified using the fact that \( \chi \) may depend on \( n_2 \) and all variables \( u_1, u_2 \). Then introduce, by hand, the correct dependence in \( n_2 \) which could generate the generalized factorial \( R(1,n_1, n_2)R(2,n_2) \). For instance, assume that the density \( \chi \) is of the form

\[
\chi(u_1, u_2, n_2) = \chi'(u_1, u_2, n_2) \left[ \frac{\beta_1}{\omega_1} \right]^{-n_2} \left[ \frac{\alpha_2}{\omega_2} \right]^{-n_2} r_2^{n_2} e^{-r_2^{n_2}}.
\] (A.2)

The remaining density \( \chi'(u_1, u_2, n_2) \) has the unique purpose to integrate the combined new variable \( \tilde{u}_1 = u_1^{\alpha_1} u_2^{\beta_2 n_2} \) in order to get at first \( R(1,n_1, n_2) \); the quantity \( R(2,n_2) \) can be recovered without ambiguity and then the problem (A.1) will be solved. However, considering this option boils down to cancel \( z_1^{n_2}/\omega_2^{n_2} \) at the very beginning in the VCS and thereby to redefine them as an one degree of freedom VCS with new variable \( z_1 z_2^\kappa \). Therefore, using blindly this option may lead to already known VCS. Moreover, implementing this option, one may need to introduce the same kind of terms that have been already simplified, then this method might be not very efficient.

(b) All dependencies in \( n_2 \), are not simplified and one tries to carry out a strict change of variables in order to solve a decoupled moment problem. This option does have an advantage: it is the one relevant when discussing VCS in general and when, in particular, the second index \( n_2 \) is summed. In that situation, the density \( \chi \) should depend on the variables \( u_1 \) and \( u_2 \) but not on \( n_2 \). For instance, keeping in mind (A.1), one considers the following change of variables

\[
u_1 \rightarrow \tilde{u}_1 = u_1^{\alpha_1} u_2^{\beta_2 \kappa_2} \quad \text{and} \quad u_2 \rightarrow \tilde{u}_2 = u_2^{\alpha_2} u_1^{\beta_1 \kappa_1}\]

(A.3)

with Jacobian

\[
J = (\alpha_1 \alpha_2 - \beta_1 \beta_2) u_1^{\alpha_1 + \beta_1 \kappa_1 - 1} u_2^{\alpha_2 + \beta_2 \kappa_2 - 1}
\] (A.4)

which could vanish without further assumption on the parameters. For instance the case \( \alpha_1 = \beta_1 = 1 \) which is relevant for our discussion is not allowed. Consequently, this option has a disadvantage: it is dependent on the change of variable (which could appear singular) and so restricts the kind of solvable moment problems.

(c) A third option is to use a mixed formalism: one can choose to simplify or not the dependence in \( n_2 \) but always in such a way that the change of variable (A.4) will appear non singular. The resolution of the moment problem with its generalized factorials strongly depends on the kind of integrand. Simplifications have to be chosen appropriately. In a specific instance, consider that the moment integral yields \( n_1!n_2! \). Using the dependence on \( n_2 \) of \( \chi \), remove all dependencies as \( u_1^{\beta_1 \kappa_1 n_2} \) so that we end up with a triangular change of variables:

\[
u_1 \rightarrow \tilde{u}_1 = u_1^{\alpha_1} u_2^{\beta_2 \kappa_2} \quad \text{and} \quad u_2 \rightarrow \tilde{u}_2 = u_2^{\alpha_2}
\] (A.5)

which is always invertible for \( \alpha_i \neq 0 \), \( i = 1, 2 \). However, if the product of generalized factorials is of the form \( \Gamma[\gamma_1 + n_1]n_2! \), simplifying \( u_1^{\beta_1 \kappa_1 n_2} \) will be harmful since only a power \( X_1^{\gamma_1 + \beta_1 \kappa_1 n_2} X_2^{n_2} \) with \( X_{1,2} \) some variables, could produce such a result. Hence reintroducing by hand the missing term is the only way out. In that above situation, then do not perform a simplification and still there is a change of variables which is regular (we will give more
precision afterwards). It should be emphasized also that this method cannot be reported for higher order degrees of freedom or summing over \( n_2 \), but preserves the two degrees of freedom in the VCS (the issue of option (a) is cured) and offers always a solution of the moment problem (hence the disadvantage of option (b) is avoided).

In the following, guided by the order of efficiency in finding solution of the moment problem though providing non listed VCS, we will use the third option.

We introduce the following terminology: to recombine a term \( x \) is to perform a change of variable in another variable, say \( y \) and \( y \rightarrow yx \), in order to simplify \( x \). The main steps for solving any moment problem \([A.1]\) for Subsection 3.2 (VCS with two degrees of freedom summing over \( n_1 \) built with generalized factorial \( R_1(n_1, n_2) R_2(n_2, n_1) \)) using option (c) are the following:

(0) Always consider a variable as \( r_i^{(\bullet)}/\omega_i^{(\bullet)} \), i.e. a ratio between a variable and its frequency before integrating it. Even though, the frequencies \( \omega_i \) could be regarded just as dressing factors, these provide actual continuous deformation parameters giving a sense of the VCS classification.

(1) - If \( R_i(n_i, n_i) = \Gamma[\gamma_i + n_i] \), for all \( i = 1, 2 \), then do not simplify or recombine \textit{a priori} any of the variables.

- Given a couple \( i, i \in \{1, 2\}, i \neq i \), if \( R_i(n_i, n_i) = \Gamma[\gamma_i + n_i] \) and \( R_i(n_i) = n_i! \), then recombine \((i = 2)\) or simplify \((i = 1)\) only \( u_i \).

- If \( R_i(n_i) = n_i! \) for \( i = 1, 2 \), then recombine \( u_2 \) and simplify \( u_1 \).

(2) Use the ansatz \( \chi(u_1, u_2, n_2) = \varrho_1(u_1, u_2, n_2) \varrho_2(u_2) \) for solving the moment problem where \( \varrho_1(u_1, u_2, n_2) \) will be used to integrate the single variable \( u_1 \) (and simplifying the maximum of factors) and \( \varrho_2(u_2) \) for the second variable \( u_2 \).

(3) As a convention, all extra Jacobian factors coming from the change of variables \( u_1 \rightarrow \tilde{u}_1 \) (resp. \( u_2 \rightarrow \tilde{u}_2 \)) should be reabsorbed by \( \varrho_1 \) (resp. by \( \varrho_2 \)).

This program will not give of course a direct solution of the generalized moment problem but, at least, it provides an unique way for dealing with the combinatorics of extra factors generated by the variables \( u_1 \) and \( u_2 \).

We now apply this program to the problem \([A.1]\). First, we need to specify the generalized factorials. Consider \( R_1(n_1) = n_1! \) and \( R_2(n_2) = n_2! \) describing a first class of VCS of the kind \([60]\). Then, by step (1), we simplify the extra factor in \( u_1^{\gamma_1} n_2 \), by considering the density

\[
\chi(u_1, u_2, n_2) = \varrho_1'(u_1, u_2, n_2) \left[ \frac{u_1^{\gamma_1}}{\omega_1^{\beta_1} n_1} \right]^{-n_2} \varrho_2(u_2), \tag{A.6}
\]

and substituting this in the problem and recombining \( u_2^{\beta_2 n_1} \), we get

\[
\int du_1 du_2 \varrho_1'(u_1, u_2, n_2) \varrho_2(u_2) \left[ \frac{u_1^{\alpha_1}}{\omega_1^{\alpha_1}} \right]^{n_1} \left[ \frac{u_2^{\alpha_2}}{\omega_2^{\alpha_2}} \right]^{n_2} = R_1(n_1, n_2) R_2(n_2). \tag{A.7}
\]

Perform a change of variables \( u_1 \rightarrow \tilde{u}_1 = u_1^{\alpha_1} u_2^{\beta_2 n_2} \) and \( u_2 \rightarrow \tilde{u}_2 = u_2^{\alpha_2} \) with minor Jacobians given by \([\alpha_1 u_1^{\alpha_1-1} u_2^{\beta_2 n_2}] \) and \([\alpha_2 u_2^{\alpha_2-1}] \), respectively. Use \( \varrho_1' \) and \( \varrho_2 \) to absorb these terms, respectively. The following densities given in radial variables solve the problem \([A.1]\)

\[
\varrho_1(r_1, r_2, n_2) = \alpha_1 \frac{r_1^{2(\alpha_1-1)} r_2^{2\beta_2 n_2}}{\omega_1^{\alpha_1} \omega_2^{\beta_2 n_2}} \left[ \frac{r_1^{\beta_1 n_1}}{\omega_1^{\beta_1 n_1}} \right]^{n_2} \varrho_1(r_1, r_2) = \alpha_1 \frac{1}{\omega_2^{\beta_2 n_2}} r_2^{2(\alpha_1-1)} e^{-\frac{\alpha_1}{\omega_1^{\beta_1 n_1}}}, \quad \varrho_2(r_2) = \alpha_2 \frac{1}{\omega_2^{\alpha_2}} r_2^{2(\alpha_2-1)} e^{-\frac{\alpha_2}{\omega_2^{\alpha_2}}},
\]

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\[\chi(r_1, r_2, n_2) = \alpha_1 \alpha_2 \frac{r_1^{2(\alpha_1 - 1)} r_2^{2(\beta_2 - \alpha_2 \kappa_2 - 1)}}{\omega_1^{\alpha_1'} \omega_2^{\beta_2'} \kappa_1} \left[ \frac{\beta_1' \kappa_1}{\omega_1} \right] \left( \frac{\beta_1' \kappa_1}{\omega_1} \right)^n_2 e^{-\frac{r_1^{2\alpha_1} r_2^{2\beta_2 + \kappa_2}}{\omega_1^{\alpha_1'} \omega_2^{\beta_2'} \kappa_1}}. \]  

(A.8)

The calculations are more involved using instead \(R_1(n_1, n_2) = \Gamma[\gamma_1 + n_1]\) and \(R_2(n_2) = n_2!\) which are the data for second class VCS \([55]\). By step (1), we should not simplify \(u_1^{\beta_1 \kappa_1 n_2}\) and only recombine \(u_2^{\beta_2 \kappa_1 n_1}\). Coming back to (A.7), we have

\[\int du_1 du_2 \chi_1(u_1, u_2, n_2) \left[ \frac{u_1^{\alpha_1} u_2^{\beta_2 \kappa_1 n_2}}{\omega_1^{\alpha_1'} \omega_2^{\beta_2'} \kappa_1} \right] e^{-\frac{r_1^{2\alpha_1} r_2^{2\beta_2 + \kappa_2}}{\omega_1^{\alpha_1'} \omega_2^{\beta_2'} \kappa_1}} \left[ \frac{u_1^{\alpha_1} u_2^{\beta_2 \kappa_1 n_2}}{\omega_1^{\alpha_1'} \omega_2^{\beta_2'} \kappa_1} \right] \left( \frac{u_1^{\alpha_1} u_2^{\beta_2 \kappa_1 n_2}}{\omega_1^{\alpha_1'} \omega_2^{\beta_2'} \kappa_1} \right)^{n_2} e^{-\frac{r_1^{2\alpha_1} r_2^{2\beta_2 + \kappa_2}}{\omega_1^{\alpha_1'} \omega_2^{\beta_2'} \kappa_1}} \left[ \frac{u_1^{\alpha_1} u_2^{\beta_2 \kappa_1 n_2}}{\omega_1^{\alpha_1'} \omega_2^{\beta_2'} \kappa_1} \right]. \]  

(A.9)

Clearly, in order to avoid problems with Jacobians in a double change of variables, we should simplify the ratio appearing as \(u_1^{(\beta_1 - \alpha_1) \kappa_1} / u_1^{(\beta_1' - \alpha_1') \kappa_1}\) using \(\chi\). However, we have still two choices for the variable \(u_2\): either to recombine all dependencies in \(u_2\) in one variable \(u_2^{\beta_2 - \beta_2} / \omega_2^{\alpha_2 - \beta_2}\), or to simplify again \(u_2^{\beta_2 - \beta_2} / \omega_2^{\beta_2 - \beta_2}\) using \(\chi\). The first choice will lead us to \(u_2^{\beta_2 - \beta_2} / \omega_2^{\beta_2 - \beta_2}\), which could vanish and so it is not the best option for seeking general solutions. We will therefore simplify the factor \(u_2^{\beta_2 - \beta_2} / \omega_2^{\beta_2 - \beta_2}\), set

\[\chi(u_1, u_2, n_2) = \phi_1(u_1, u_2, n_2) \left[ \frac{u_1^{\alpha_1} u_2^{\beta_2 \kappa_1 n_2}}{\omega_1^{\alpha_1'} \omega_2^{\beta_2'} \kappa_1} \right] \phi_2(u_2) \]  

(A.10)

and come to a similar problem as obtained previously for which one gets the solutions

\[\phi_1(r_1, r_2, n_2) = \alpha_1 \frac{r_1^{2(\alpha_1 - 1)} r_2^{2\beta_2 \kappa_2}}{\omega_1^{\alpha_1'} \omega_2^{\beta_2'} \kappa_2} \left[ \frac{\beta_1' \kappa_1}{\omega_1} \right] \left( \frac{\beta_1' \kappa_1}{\omega_1} \right)^n_2 e^{-\frac{r_1^{2\alpha_1} r_2^{2\beta_2 + \kappa_2}}{\omega_1^{\alpha_1'} \omega_2^{\beta_2'} \kappa_2}},\]  

\[\phi_2(r_2) = \alpha_2 \frac{1}{\alpha_2} r_2^{2(\alpha_2 - 1)} e^{-\frac{r_2^{2\alpha_2}}{\omega_2}}.\]  

(A.11)

Next, according to the same formalism, we solve the moment problem of the states \([81]\) determined by \(R_2(n_2, n_1) = \Gamma[\gamma_2 + n_2]\) and \(R_1(n_1) = n_1!\). Then \(u_2\) should not be recombined or simplified without more considerations whereas \(R_1(n_1) = n_1!\) implies that \(u_1^{\beta_1 \kappa_1 n_2}\) should be simplified. Using the same routine, we write the moment problem associated with these states:

\[\int du_1 du_2 \chi_1(u_1, u_2, n_2) \left[ \frac{u_2^{\alpha_2}}{\omega_2} \right]^{n_2 + \kappa_1 n_2} e^{-\frac{r_1^{2\alpha_1} r_2^{2\beta_2 - \alpha_2 \kappa_2}}{\omega_1^{\alpha_1'} \omega_2^{\beta_2'}} \kappa_1} \left[ \frac{u_1^{\alpha_1} u_2^{(\beta_2 - \alpha_2) \kappa_2}}{\omega_1^{\alpha_1'} \omega_2^{\beta_2'} \kappa_2} \right] \left[ \frac{u_1^{\kappa_1 n_2}}{\omega_1^{\kappa_1}} \right] \right]. \]  

(A.12)

Changing variables as \(\tilde{u}_2 = u_2^{\alpha_2}\) and \(\tilde{u}_1 = u_1^{\alpha_1} u_2^{(\beta_2 - \alpha_2) \kappa_2}\), one can solve the problem by setting \(\chi(r_1, r_2, n_2) = \phi_1(r_1, r_2, n_2) \phi_2(r_2)\) such that

\[\phi_1(r_1, r_2, n_2) = \alpha_1 \frac{r_1^{2(\alpha_1 - 1)} r_2^{2(\beta_2 - \alpha_2) \kappa_2}}{\omega_1^{\alpha_1'} \omega_2^{\beta_2'} \kappa_2} \left[ \frac{\beta_1' \kappa_1}{\omega_1} \right] \left( \frac{\beta_1' \kappa_1}{\omega_1} \right)^{\kappa_1 n_2} e^{-\frac{r_1^{2\alpha_1} r_2^{2(\beta_2 - \alpha_2) \kappa_2}}{\omega_1^{\alpha_1'} \omega_2^{\beta_2'} \kappa_2}},\]  

\[\phi_2(r_2) = \alpha_2 \frac{1}{\alpha_2} r_2^{2(\alpha_2 - 1)} e^{-\frac{r_2^{2\alpha_2}}{\omega_2}}.\]  

(A.13)
B Checking the convergence of norm series

We provide here other arguments justifying the norm series convergence by using different criteria than the one used in the main text. These additional tests remain instructive for they offer the correct way to manipulate and optimize the convergence criteria afforded by the different theorems. This will enable us to achieve less constraints on the parameters $\kappa$ and thus to get larger VCS classes.

Checking Eq. (137) As a first double checking, let us come back on the convergence of the norm series (137). We can evaluate comparison test ratios of Theorem 2, by defining $w_i > 0, i = 1, 2, b_{n_1,n_2}$ the general term of the exponential series $e^{w_1+n_2}$. We have

$$b_{n_1,n_2} = \frac{w_1^{n_1} w_2^{n_2}}{n_1! n_2!} > 0, \quad a_{n_1,n_2} = \frac{w_1^{n_1} w_2^{n_2}}{\Gamma[\gamma_1 + n_1] \Gamma[\gamma_32 + n_3]} > 0. \quad (B.14)$$

We have

$$\frac{a_{n_1+1,n_2} b_{n_1,n_2}}{a_{n_1,n_2} b_{n_1+1,n_2}} = \frac{1 + n_1}{\gamma_1 + n_1} \leq 1, \quad \frac{a_{n_1,n_2} b_{n_1,n_2+1}}{a_{n_1,n_2} b_{n_1+1,n_2}} = \frac{(n_2 + 1) \Gamma[\kappa_{12} n_2 + \gamma_{13} + n_1] \Gamma[\kappa_{32} n_2 + 1 + n_3]}{\Gamma[\kappa_{12} (n_2 + 1) + \gamma_{13} + n_1] \Gamma[\kappa_{32} (n_2 + 1) + 1 + n_3]} \quad (B.15)$$

Using the $\Gamma$-Stirling asymptote at large arguments, one gets

$$e^\kappa_{12} + \kappa_{32} (n_2 + 1) \frac{(\kappa_{12} n_2 + \gamma_{13} + n_1) \kappa_{12} n_2 + \gamma_{13} + n_1 (\kappa_{32} n_2 + 1 + n_3)}{(\kappa_{12} (n_2 + 1) + \gamma_{13} + n_1) \kappa_{12} (n_2 + 1) + \gamma_{13} + n_1 (\kappa_{32} (n_2 + 1) + 1 + n_3)} \sim \frac{\kappa_{12} (n_2 + 1) + \gamma_{13} + n_1 \kappa_{32} (n_2 + 1) + 1 + n_3}{\kappa_{32}}. \quad (B.16)$$

The latter is bounded by 1 at large $n_1$ and $n_2$ for $\kappa_{32} > 0$ (condition induced by $n_1 \to \infty$) and $\kappa_{12} + \kappa_{32} > 1$ (condition induced by $n_2 \to \infty$). Thus, by Theorem 2 providing a sufficient condition, by insisting to compare the exponential series with the present norm series, we could miss some possible solutions.

We can optimize the test, by comparing now $a_{n_1,n_2}$ with

$$b'_{n_1,n_2} = \frac{w_1^{n_1} w_2^{n_2}}{n_1! \Gamma[\gamma_{32} + n_3]} \quad (B.17)$$

which is the term of a previous norm series which proves to be convergent for $\kappa_{32} > 0$. One finds the ratios

$$\frac{a_{n_1+1,n_2} b'_{n_1,n_2}}{a_{n_1,n_2} b'_{n_1+1,n_2}} = \frac{1 + n_1}{\gamma_1 + n_1} \leq 1, \quad \frac{a_{n_1,n_2} b'_{n_1,n_2+1}}{a_{n_1,n_2} b'_{n_1+1,n_2}} = \frac{\Gamma[\kappa_{12} n_2 + \gamma_{13} + n_1] \Gamma[\kappa_{32} n_2 + 1 + n_3] \Gamma[1 + \kappa_{32} (n_2 + 1) + n_3]}{\Gamma[\kappa_{12} (n_2 + 1) + \gamma_{13} + n_1] \Gamma[\kappa_{32} (n_2 + 1) + 1 + n_3] \Gamma[\kappa_{32} n_2 + 1 + n_3]} \sim 0 \leq 1 \quad (B.18)$$
which hold at large $n_i$ using the $\Gamma$-Stirling approximation and in the last, we only require that $\kappa_{12} > 0$ turning out to be less stringent than the above condition $\kappa_{12} + \kappa_{32} > 1$ (we will come back on the second inequality in the next verification). Theorem 2 can be applied now.

Let us inspect the limit case $\kappa_{12} = 0$ by unfolding the same calculations: one ends up with two ratios (B.18), at large $n_i$ limits, bounded by 1. Then Theorem 2 is again applied and the case $\kappa_{12} = 0$ is not to exclude and will lead to relevant VCS.

Checking Eq. (145) A second verification is in order for the norm (145). Let us evaluate the ratio-comparison with double exponential term $b_{n_1,n_2}$

$$\frac{a_{n_1+1,n_2}b_{n_1,n_2}}{a_{n_1,n_2}b_{n_1+1,n_2}} \leq 1,$$

(B.19)

$$\frac{a_{n_1,n_2}b_{n_1+1,n_2}}{a_{n_1,n_2+1}b_{n_1,n_2+1}} \sim \frac{(n_2 + 1)\Gamma[k_{12}(n_2 + 1) + \gamma_{13} + n_1]}{[\kappa_{12}(n_2 + 1) + \gamma_{13} + n_1]^\kappa_{12}} \leq 1,$$

in the last inequality used has been made of the $\Gamma$-Stirling approximation and $\kappa_{12} > 1$. In fact, $\kappa_{12} > 0$ is enough for proving the convergence of the series (again comparison with exponential is a too strong requirement). We have to prove

$$\lim_{n_1,n_2 \to \infty} \frac{a_{n_1+1,n_2}}{a_{n_1,n_2}} = w_1 \lim_{n_1,n_2 \to \infty} \frac{1}{\gamma_{13} + \kappa_{12} n_2 + n_1} < 1,$$

$$\lim_{n_1,n_2 \to \infty} \frac{a_{n_1,n_2+1}}{a_{n_1,n_2}} = w_2 \lim_{n_1,n_2 \to \infty} \frac{\Gamma[\gamma_{13} + \kappa_{12}(n_2 + 1) + n_1]}{[\gamma_{13} + \kappa_{12} (n_2 + 1) + n_1]^{\kappa_{23}}} < 1,$$

(B.20)

and use Theorem 1. The bound of the first ratio is obvious and this should be sufficient to end the proof. However, let us check the second ratio bound, because also it is involved in the proof of (B.18). The second test ratio can be shown using again our favorite $\Gamma$ approximation and

$$\lim_{n_1,n_2 \to \infty} \left[ \frac{\Gamma[\gamma_{13} + \kappa_{12} n_2 + n_1]}{\Gamma[\gamma_{13} + \kappa_{23}(n_2 + 1) + n_1]} - \frac{1}{[\gamma_{13} + \kappa_{23}(n_2 + 1) + n_1]^{\kappa_{23}}} \right] \sim 0$$

(B.21)

which holds for all $\kappa_{12} > 0$. The main problem being at low $\kappa_{12} \sim 0$ but $\kappa_{12} \neq 0$, some numerics show that this is indeed the case (see Figure 1).

Checking Eq. (152) We perform the checking of the convergence of the series (152), writing

$$a_{n_1,n_2} = \frac{w_1^{n_1} w_2^{n_2}}{\Gamma[\gamma_{12} + n_1] \Gamma[\gamma_{3} + n_2]}, \quad b_{n_1,n_2} = \frac{w_1^{n_1} w_2^{n_2}}{n_1! \Gamma[1 + \kappa_{32} n_2 + \kappa_{31} n_1 + n_3]}.$$

(B.22)

First, we check that the intermediate term yields an everywhere convergent series:

$$\lim_{n_1 \to \infty} \frac{b_{n_1+1,n_2}}{b_{n_1,n_2}} = w_1 \lim_{n_1 \to \infty} \frac{\Gamma[1 + \kappa_{32} n_2 + \kappa_{31} n_1 + n_3]}{(n_1 + 1) \Gamma[1 + \kappa_{32} n_2 + \kappa_{31} (n_1 + 1) + n_3]} = 0,$$

$$\kappa_{32} > 0,$$

$$\lim_{n_2 \to \infty} \frac{b_{n_1+1,n_2}}{b_{n_1,n_2}} = w_2 \lim_{n_2 \to \infty} \frac{\Gamma[1 + \kappa_{32} (n_2 + 1) + \kappa_{31} n_1 + n_3]}{\Gamma[1 + \kappa_{32} n_2 + \kappa_{31} n_1 + n_3]} = 0,$$

$$\lim_{n_1,n_2 \to \infty} \frac{b_{n_1+1,n_2}}{b_{n_1,n_2}} \sim w_1 \lim_{n_1,n_2 \to \infty} \frac{1}{(n_1 + 1) \Gamma[1 + \kappa_{32} (n_2 + 1) + \kappa_{31} (n_1 + 1) + n_3]} \sim 0 < 1,$$

$$\lim_{n_1,n_2 \to \infty} \frac{b_{n_1+1,n_2}}{b_{n_1,n_2}} \sim w_2 \lim_{n_1,n_2 \to \infty} \frac{1}{\Gamma[1 + \kappa_{32} (n_2 + 1) + \kappa_{31} n_1 + n_3] \kappa_{32}} \sim 0 < 1.$$

(B.23)
Figure 4: The difference \[ \frac{\Gamma[\gamma_{13} + \kappa_{12}(n_2 + n_1)]}{\Gamma[\gamma_{13} + \kappa_{23}(n_2 + 1) + n_1]} - [\gamma_{13} + \kappa_{23}(n_2 + 1) + n_1]^{-\kappa_{12}} \] at large \( n_1 = m \) and \( n_2 = n \) for different parameters \( \kappa_{12} \in \{1, 1/2, 1/10, 1/10^6\} \).

The first inequality is valid for all \( \kappa \)'s whereas the second is only valid for \( \kappa_{32} > 0 \) which is consistent with the initial constraint on the column series. Theorem 1 ensures that the series defined by \( b_{n_1,n_2} \) is convergent everywhere.

Now, we can verify the hypothesis of Theorem 2 using (B.22). At large \( n_1 \) and \( n_2 \), the following holds

\[
\begin{align*}
\frac{a_{n_1+1,n_2}b_{n_1,n_2}}{a_{n_1,n_2}b_{n_1+1,n_2}} &= \frac{(n_1 + 1)}{(\gamma_{12} + n_1)} \leq 1, \\
\frac{a_{n_1,n_2+1}b_{n_1,n_2}}{a_{n_1,n_2}b_{n_1+1,n_2+1}} &= \frac{\Gamma[\kappa_{12}n_2 + 1 + n_1] \Gamma[\kappa_{23}n_2 + \gamma_{31} + n_3] \Gamma[\kappa_{32}(n_2 + 1) + \gamma_{31} + n_3]}{\Gamma[\kappa_{12}(n_2 + 1) + 1 + n_1] \Gamma[\kappa_{23}(n_2 + 1) + \gamma_{31} + n_3] \Gamma[\kappa_{32}(n_2 + 1) + \gamma_{31} + n_3]} \\
&= \frac{\Gamma[\kappa_{12}n_2 + 1 + n_1]}{\Gamma[\kappa_{12}(n_2 + 1) + 1 + n_1]} \sim \frac{1}{\kappa_{12}(n_2 + 1) + 1 + n_1} \sim 0 \leq 1, \tag{B.24}
\end{align*}
\]

valid for \( \kappa_{12} > 0 \). The proof for the case \( \kappa_{12} = 0 \) is the same as the convergence for the series defined by \( b_{n_1,n_2} \) (B.22). Hence, the norm series (152) converges everywhere.

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