A characterisation of the Calabi product of hyperbolic affine spheres

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Abstract

There exists a well known construction which allows to associate with two hyperbolic affine spheres \(f_i : M^n_i \to \mathbb{R}^{n_i+1}\) a new hyperbolic affine sphere immersion of \(I \times M_1 \times M_2\) into \(\mathbb{R}^{n_1+n_2+3}\). In this paper we deal with the inverse problem: how to determine from properties of the difference tensor whether a given hyperbolic affine sphere immersion of a manifold \(M^n \to \mathbb{R}^{n+1}\) can be decomposed in such a way.

Key words: {affine hypersphere, Calabi product, affine hypersurface}.
Subject class: 53A15.

1 Introduction

In this paper we study nondegenerate affine hypersurfaces \(M^n\) into \(\mathbb{R}^{n+1}\), equipped with its standard affine connection \(D\). It is well known that on such a hypersurface there exists a canonical transversal vector field \(\xi\), which is called the affine normal. With respect to this transversal vector field one can decompose

\[ D_X Y = \nabla_X Y + h(X, Y)\xi, \]

thus introducing the affine metric \(h\) and the induced affine connection \(\nabla\). The Pick-Berwald theorem states that \(\nabla\) coincides with the Levi Civita connection \(\hat{\nabla}\) of the affine metric \(h\) if and only if \(M\) is immersed as a nondegenerate quadric. The difference tensor \(K\) is introduced by

\[ K_X Y = \nabla_X Y - \hat{\nabla}_X Y. \]

It follows easily that \(h(K(X, Y), Z)\) is symmetric in \(X, Y\) and \(Z\). The apolarity condition states that trace \(K_X = 0\) for every vector field \(X\). The fundamental theorem of affine differential geometry, Dillen, see Ref. [7] implies that an affine hypersurface is completely determined by the metric and the difference tensor \(K\).

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Deriving the affine normal, we introduce the affine shape operator $S$ by
\[ D_X \xi = -SX. \] (3)

Here, we will restrict ourselves to the case that the affine shape operator $S$ is a multiple of the identity, i.e. $S = HI$. This means that all affine normals are parallel or pass through a fixed point. We will also assume that the metric is positive definite in which case one distinguishes the following classes of affine hyperspheres:

(i) elliptic affine hyperspheres, i.e. all affine normals pass through a fixed point and $H > 0$,

(ii) hyperbolic affine hyperspheres, i.e. all affine normals pass through a fixed point and $H < 0$,

(iii) parabolic affine hyperspheres, i.e. all the affine normals are parallel ($H = 0$).

Due to the work of amongst others Calabi [2], Pogorelov [15], Cheng and Yau [4], Sasaki [17] and Li [11], positive definite affine hyperspheres which are complete with respect to the affine metric $h$ are now well understood. In particular, the only complete elliptic or parabolic positive definite affine hyperspheres are respectively the ellipsoid and the paraboloid. However, there exist many hyperbolic affine hyperspheres.

In the local case, one is far from obtaining a classification. The reason for this is that affine hyperspheres reduce to the study of the Monge-Ampère equations. Calabi introduced a construction, called the Calabi product, which shows how to associate with one (or two) hyperbolic affine hyperspheres a new hyperbolic affine hypersphere. This construction, as well as the corresponding properties for the difference tensor are recalled in the next section.

In this paper we are interested in the reverse construction, i.e. how to determine using properties of the difference tensor whether or not a given hyperbolic affine hypersphere (with mean curvature $-1$) can be decomposed as a Calabi product of a hyperbolic affine hypersphere and a point or as a Calabi product of two hyperbolic affine hyperspheres.

In particular we show the following two theorems:

**Theorem 1** Let $\phi : M^n \to \mathbb{R}^{n+1}$ be a (positive definite) hyperbolic affine hypersphere with mean curvature $\lambda$, $\lambda < 0$. Assume that there exists two distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ such that

(i) $T_p M = \mathcal{D}_1 \oplus \mathcal{D}_2$,

(ii) $\mathcal{D}_1$ and $\mathcal{D}_2$ are orthogonal with respect to the affine metric $h$

(iii) $\mathcal{D}_1$ is a one dimensional distribution spanned by a unit length vector field $T$

(iv) there exist numbers $\lambda_1$ and $\lambda_2$ satisfying $-\lambda + \lambda_1\lambda_2 - \lambda_2^2 = 0$ such that

\[ K(T, T) = \lambda_1 T \]
\[ K(T, U) = \lambda_2 U, \]

where $U \in \mathcal{D}_2$. 

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Then $\phi : M^n \to \mathbb{R}^{n+1}$ can be decomposed as the Calabi product of a hyperbolic affine sphere $\psi : M_1^{n-1} \to \mathbb{R}^n$ and a point.

and

**Theorem 2** Let $\phi : M^n \to \mathbb{R}^{n+1}$ be a (positive definite) hyperbolic affine hypersphere with mean curvature $\lambda$, $\lambda < 0$. Assume that there exists distributions $D_1$ (of dimension 1, spanned by a unit length vector field $T$), $D_2$ (of dimension $n_2$) and $D_3$ (of dimension $n_3$) such that

\begin{align}
(i) & \quad 1 + n_2 + n_3 = n, \\
(ii) & \quad D_1, D_2$ and $D_3$ are mutually orthogonal with respect to the affine metric $h$ \\
(iii) & \quad$there exist numbers $\lambda_1$, $\lambda_2$ and $\lambda_3$ such that \\
& \quad K(T, T) = \lambda_1 T \\
& \quad K(T, V) = \lambda_2 V, \\
& \quad K(T, W) = \lambda_3 W, \\
& \quad K(V, W) = 0.
\end{align}

where $V \in D_2$, $W \in D_3$, $\lambda_1 = \lambda_2 + \lambda_3$ and $\lambda_2\lambda_3 = \lambda$.

Then $\phi : M^n \to \mathbb{R}^{n+1}$ can be decomposed as the Calabi product of two hyperbolic affine sphere immersions $\psi_1 : M_1^{n_2} \to \mathbb{R}^{n_2+1}$ and $\psi_2 : M_2^{n_3} \to \mathbb{R}^{n_3+1}$.

Note that, as explained in the next section, the converse of the above two theorems is also true.

To conclude this introduction, we remark that the basic integrability conditions for a hyperbolic affine hypersphere with mean curvature $-1$ state that:

\begin{align}
\hat{R}(X,Y)Z &= -(h(Y,Z)X - h(X,Z)Y) - [K_X, K_Y]Z, \\
(\hat{\nabla}K)(X,Y,Z) &= (\hat{\nabla}K)(Y,X,Z).
\end{align}

2 The Calabi product

Let $\psi_1 : M_1^{n_2} \to R^{n_2+1}$ and $\psi_2 : M_2^{n_3} \to R^{n_3+1}$ be hyperbolic affine hyperspheres with mean curvature $-1$. Then we define the Calabi product of $M_1$ with a point by

$$
\bar{\psi}(t, p) = (c_1 e^{\sqrt{n} t} \psi_1(p), c_2 e^{-\sqrt{n} t}),
$$

where $p \in M_1$ and $t \in \mathbb{R}$ and the Calabi product of $M_1$ with $M_2$ by

$$
\psi(t, p, q) = (c_1 e^{\sqrt{n_2+1} t} \psi_1(p), c_2 e^{\sqrt{n_3+1} t} \psi_2(q)),
$$

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where \( p \in M_1, q \in M_2 \) and \( t \in \mathbb{R} \).

We now investigate the conditions on the constants \( c_1 \) and \( c_2 \) in order that the Calabi product has constant mean curvature \(-1\). We first do so for the Calabi product of two affine spheres. We denote by \( v_1, \ldots, v_{n_3} \) local coordinates for \( M_1 \) and by \( w_1, \ldots, w_{n_3} \) local coordinates for \( M_2 \). Then, it follows that

\[
\psi_t = (c_1 \frac{\sqrt{n_2+1}}{\sqrt{n_2+1}} e^{\frac{\sqrt{n_2+1}}{\sqrt{n_3+1}}} \psi_1(p), -c_2 \frac{\sqrt{n_2+1}}{\sqrt{n_3+1}} e^{\frac{\sqrt{n_2+1}}{\sqrt{n_3+1}}} \psi_2(q)),
\]

\[
\psi_{tt} = (c_1 \frac{n_1+1}{n_2+1} e^{\frac{\sqrt{n_2+1}}{n_3+1}} \psi_1(p), -c_2 \frac{n_2+1}{n_3+1} e^{\frac{\sqrt{n_2+1}}{n_3+1}} \psi_2(q)),
\]

\[
\psi_{t v_i} = \frac{\sqrt{n_2+1}}{\sqrt{n_2+1}} (c_1 e^{\frac{\sqrt{n_2+1}}{n_3+1}} (\psi_1)_{v_i}, 0),
\]

\[
\psi_{t w_j} = -\frac{\sqrt{n_2+1}}{\sqrt{n_3+1}} (0, c_2 e^{-\frac{\sqrt{n_2+1}}{n_3+1}} (\psi_2)_{w_j}),
\]

\[
\psi_{v_i v_j} = (c_1 e^{\frac{\sqrt{n_2+1}}{n_3+1}} (\psi_1)_{v_i v_j}, 0),
\]

\[
\psi_{w_i w_j} = (0, c_2 e^{-\frac{\sqrt{n_2+1}}{n_3+1}} (\psi_2)_{w_i w_j}).
\]

If we denote by \( h_2 \) the affine metric on \( M_2 \) and by \( h_3 \) the centroaffine metric introduced on \( M_3 \) it follows from the above formulas that

\[
\psi_{tt} = \frac{n_3-n_2}{(n_2+1)(n_3+1)} \psi_t + \psi
\]

\[
\psi_{v_i v_j} = \frac{\sqrt{n_2+1)(n_3+1)}}{n_2+n_3+2} h_2(\partial v_i, \partial v_j) \psi + ...
\]

\[
\psi_{w_i w_j} = \frac{\sqrt{n_2+1)(n_3+1)}}{n_2+n_3+2} h_3(\partial w_i, \partial w_j) \psi + ...
\]

From [14] we see that \( M \) is an affine hypersphere with mean curvature \(-1\) if and only if

\[
\det[\psi, \psi_1, \psi_{v_1}, \ldots, \psi_{v_{n_2}}, \psi_{w_1}, \ldots, \psi_{w_{n_3}}] = h(\partial_1, \partial_i) \det[h(\partial v_i, \partial v_j)] \det[h(\partial w_i, \partial w_j)]
\]

Taking into account that \( \psi_1 \) and \( \psi_2 \) are already affine spheres with mean curvature \(-1\) we must have that

\[
(c_1)^{n_2+1} (c_2)^{n_3+1} = \left( \frac{\sqrt{(n_2+1)(n_3+1)}}{n_2+n_3+2} \right)^{n_2+n_3+2}.
\]

Hence we can take

\[
c_1 = \sqrt{(n_2+1)(n_3+1)} d_1,
\]

\[
c_2 = \sqrt{(n_2+1)(n_3+1)} d_2,
\]

where

\[
(d_1)^{n_2+1} (d_2)^{n_3+1} = 1.
\]
Hence by applying an equiaffine transformation we may assume that \( d_1 = d_2 = 1 \) and therefore that the Calabi product of two hyperbolic affine spheres with mean curvature \(-1\) is an hyperbolic affine sphere with mean curvature \(-1\) if and only if

\[
\psi(t, p, q) = \sqrt{\frac{(n_2+1)(n_3+1)}{n_2+n_3+2}} \left( e^{\sqrt{n_2+1} \lambda} \psi_1(p), e^{-\sqrt{n_3+1} \lambda} \psi_2(q) \right),
\]

up to an equiaffine transformation.

For the Calabi product of a hyperbolic affine sphere and a point, we proceed in the same way to deduce the following. The Calabi product of a hyperbolic affine spheres with mean curvature \(-1\) and a point is an hyperbolic affine sphere with mean curvature \(-1\) if and only if

\[
\tilde{\psi}(t, p) = \frac{\sqrt{n}}{n+1} \left( e^{\sqrt{n} \lambda} \psi_1(p), e^{-\sqrt{n} \lambda} \right),
\]

up to an equiaffine transformation.

**Remark 1** A straightforward calculation shows that the Calabi product of two hyperbolic affine spheres has parallel cubic form (with respect to the Levi Civita connection) if and only if both original hyperbolic affine spheres have parallel cubic forms. Similarly one has that the Calabi product of a hyperbolic affine sphere and a point has parallel cubic form if and only if the original affine sphere has parallel cubic form.

## 3 Characterisation of the Calabi product of two hyperbolic affine spheres and the proof of Theorem 2

Throughout this section we will assume that \( \phi : M^n \rightarrow \mathbb{R}^{n+1} \) is a hyperbolic affine hypersphere. Without loss of generality we may assume that \( \lambda = -1 \) by applying a homothety. We will now prove Theorem 2. Therefore, we shall also assume that \( M \) admits three mutually orthogonal differential distributions \( D_1, D_2 \) and \( D_3 \) of dimension 1, \( n_2 > 0 \) and \( n_3 > 0 \) respectively with \( 1 + n_2 + n_3 = n \), and, for all vectors \( V \in D_2, W \in D_3 \),

\[
K(T, T) = \lambda_1 T, \quad K(T, V) = \lambda_2 V, \\
K(T, W) = \lambda_3 W, \quad K(V, W) = 0.
\]

By the apolarity condition we must have that

\[
\lambda_1 + n_2 \lambda_2 + n_3 \lambda_3 = 0, \quad (8)
\]

Moreover, we will assume that

\[
\lambda_1 = \lambda_2 + \lambda_3 \quad (9) \\
\lambda_2 \lambda_3 = -1. \quad (10)
\]
The above conditions imply that $\lambda_1, \lambda_2$ and $\lambda_3$ are constants and can be determined explicitly in terms of the dimension $n$.

As $M$ is a hyperbolic affine sphere we have that the difference tensor is a symmetric tensor with respect to the Levi Civita connection $\hat{\nabla}$ of the affine metric. In that case, as also $h(K(X,Y),Z)$ is totally symmetric, the information of Lemma 1 and Lemma 2 of [1] remains valid and can be summarized in the following lemma:

**Lemma 1** We have

1. $\hat{\nabla}_{D_1} D_1 \subset D_1$
2. $\hat{\nabla}_{D_2} D_2 \subset D_2 \oplus D_3$
3. $\hat{\nabla}_{D_3} D_3 \subset D_2 \oplus D_3$
4. $h(\hat{\nabla}_T W, V) = h(\hat{\nabla}_W T, V) = -h(\hat{\nabla}_V T, W)$, for any $V \in D_2, W \in D_3$

Similarly using the information of the previous lemma, Lemma 3 of [1] reduces to

**Lemma 2** We have

1. $(\lambda_3 - \lambda_2) h(\hat{\nabla}_V \tilde{V}, W) = h(K(V, \tilde{V}), \hat{\nabla}_T W) = h(K(V, \tilde{V}), \hat{\nabla}_W T)$
2. $(\lambda_2 - \lambda_3) h(\hat{\nabla}_W \tilde{W}, V) = h(K(W, \tilde{W}), \hat{\nabla}_T V) = h(K(W, \tilde{W}), \hat{\nabla}_V T)$

We denote now by $\{V_1, \ldots, V_{n_2}\}$, respectively $\{W_1, \ldots, W_{n_3}\}$ an orthonormal basis of $D_2$ (resp. $D_3$) with respect to the affine metric $h$. Then, we have

**Lemma 3** Let $V, \tilde{V} \in D_2$. Then

$$h(\hat{\nabla}_V T, \hat{\nabla}_\tilde{V} T) = 0.$$ 

**Proof:** Using the Gauss equation, we have that

$$h(\hat{R}(V,T)T, \tilde{V}) = -h(V,\tilde{V}) - h(K(T,T), K(V,\tilde{V})) + h(K(T,V), K(T,\tilde{V}))$$

$$= (-1 - \lambda_1 \lambda_2 + \lambda_3^2) h(V, \tilde{V})$$

$$= (-1 - \lambda_3 \lambda_2) h(V, \tilde{V}) = 0.$$
On the other hand, by a direct computation using the previous lemmas, we have

\[ h(\hat{\mathcal{R}}(V,T)T, \tilde{V}) = h(\hat{\nabla}_V \hat{\nabla}_T T - \hat{\nabla}_T \hat{\nabla}_V T - \hat{\nabla}_{\hat{\nabla}_V T - \hat{\nabla}_T V} T, \tilde{V}) \]

\[ = h(-\hat{\nabla}_T \hat{\nabla}_V T, \tilde{V}) - \sum_{k=1}^{n_3} h(\hat{\nabla}_V T - \hat{\nabla}_T V, W_k)h(\hat{\nabla}_W_k T, \tilde{V}) \]

\[ = h(-\hat{\nabla}_T \hat{\nabla}_V T, \tilde{V}) \quad \text{by Lemma } I (iv) \]

\[ = -\sum_{k=1}^{n_3} h(\hat{\nabla}_V T, W_k)h(\hat{\nabla}_W_k T, \tilde{V}) \]

\[ = \sum_{k=1}^{n_3} h(\hat{\nabla}_V T, W_k)h(\hat{\nabla}_W T, W_k) \]

\[ = h(\hat{\nabla}_V T, \hat{\nabla}_W T). \]

Similarly, we have

**Lemma 4** Let \( W, \tilde{W} \in \mathcal{D}_3 \). Then

\[ h(\hat{\nabla}_W T, \nabla_{\tilde{W}} T) = 0. \]

Combining the two previous lemmas with Lemma 2 and Lemma I we see that the distributions determined by \( \mathcal{D}_2 \) and \( \mathcal{D}_3 \) are totally geodesic. It also implies that \( h(\hat{\nabla}_V T, W) = h(\hat{\nabla}_W T, V) = 0 \).

This is sufficient to conclude that locally \((M, h)\) is isometric with \( I \times M_1 \times M_2 \) where \( T \) is tangent to \( I \), \( \mathcal{D}_2 \) is tangent to \( M_1 \) and \( \mathcal{D}_3 \) is tangent to \( M_2 \).

The product structure of \( M \) implies the existence of local coordinates \((t,p,q)\) for \( M \) based on an open subset containing the origin of \( \mathbb{R} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3} \), such that \( \mathcal{D}_1 \) is given by \( dp = dq = 0 \), \( \mathcal{D}_2 \) is given by \( dt = dq = 0 \), and \( \mathcal{D}_3 \) is given by \( dt = dp = 0 \). We may also assume that \( T = \frac{\partial}{\partial t} \). We now put

\[ \phi_2 = -f_3 \phi + fT, \quad \phi_3 = g_3 \phi - gT, \]

where the functions \( f \) and \( g \), which depend only on the variable \( t \), are determined by

\[ f' = f(\lambda_3 - \lambda_1), \]

\[ g' = g(\lambda_2 - \lambda_1). \]

It is clear that solutions are given by

\[ f(t) = d_1 e^{(\lambda_3 - \lambda_1)t} \quad \text{and} \quad g(t) = d_2 e^{(\lambda_2 - \lambda_1)t}, \]

where \( d_1 \) and \( d_2 \) are constants. Of course, as \( \lambda_1 = \lambda_2 + \lambda_3 \) we can rewrite the above equation as

\[ f(t) = d_1 e^{-\lambda_2 t} \quad \text{and} \quad g(t) = d_2 e^{-\lambda_3 t}. \]
Computing $\lambda_1$, $\lambda_2$ and $\lambda_3$ explicitly, where if necessary by changing the sign of $E_1$ we may assume that $\lambda_2 \geq 0$ we find that

$$\lambda_2 = \sqrt[2]{\frac{n_3 + 1}{n_2 + 1}},$$

$$\lambda_3 = -\sqrt[2]{\frac{n_2 + 1}{n_3 + 1}}.$$

Solving now the above equations for the immersion $\phi$ we find that

$$\phi = \frac{1}{f(\lambda_2 - \lambda_3)}\phi_2 - \frac{1}{g(\lambda_2 - \lambda_3)}\phi_3$$

$$= \frac{1}{d_1}e^{\frac{\sqrt{n_3 + 1}}{T}}\phi_2 + \frac{1}{d_2}e^{-\frac{\sqrt{n_3 + 1}}{T}}\phi_3\left(\frac{\sqrt{(n_2 + 1)(n_3 + 1)}}{n_2 + n_3 + 2}\right).$$

A straightforward computation, using (11), now shows that

$$D_T(\phi_2) = D_T(-f\lambda_3\phi + fT)$$

$$= f(\lambda_3 - \lambda_1)(-\lambda_3\phi + T) + f(-\lambda_3 T + (K(T,T) + \phi))$$

$$= f(\lambda_3 - \lambda_1)(-\lambda_3\phi + T) + f((\lambda_1 - \lambda_3)T + \phi))$$

$$= f(\lambda_2\lambda_3 + 1)\phi = 0.$$

Similarly

$$D_W(\phi_2) = f(-\lambda_3 W + K(W,T)) = 0,$$

$$D_T(\phi_3) = 0,$$

$$D_V(\phi_3) = 0.$$

The above implies that $\phi_2$ reduces to a map of $M_1$ in $\mathbb{R}^n$ whereas $\phi_3$ reduces to a map of $M_2$ in $\mathbb{R}^n$. As we have that

$$d\phi_2(V) = D_V(\phi_2) = f(-\lambda_3 V + K(V,T)) = f(-\lambda_3 + \lambda_2)V,$$

$$d\phi_3(W) = D_W(\phi_3) = g(\lambda_2 W - K(W,T)) = g(\lambda_2 - \lambda_3)W,$$

these maps are actually immersions. Moreover, denoting by $\nabla^1$ the $D_2$ component of $\nabla$, we find that

$$D_V d\phi_2(\tilde{V}) = f(-\lambda_3 + \lambda_2)D_V \tilde{V}$$

$$= f(-\lambda_3 + \lambda_2)\nabla_V \tilde{V} + f(-\lambda_3 + \lambda_2)h(V, \tilde{V})\phi$$

$$= f(-\lambda_3 + \lambda_2)\nabla_V \tilde{V} + f(-\lambda_3 + \lambda_2)(h(K(V, \tilde{V}), T) + h(V, \tilde{V})\phi)$$

$$= d\phi_2(\nabla^1_V \tilde{V}) + f(-\lambda_3 + \lambda_2)h(V, \tilde{V})(\lambda_2 T + \phi)$$

$$= d\phi_2(\nabla^1_V \tilde{V}) + f(-\lambda_3 + \lambda_2)\lambda_2 h(V, \tilde{V})(T - \lambda_3\phi)$$

$$= d\phi_2(\nabla^1_V \tilde{V}) + (-\lambda_3 + \lambda_2)\lambda_2 h(V, \tilde{V})\phi_2.$$
metric \( h_1 = (−\lambda_3 + \lambda_2)\lambda_2 h \). Similarly, we get that \( \phi_3 \) can be interpreted as a centroaffine immersion contained in an \( n_3 + 1 \)-dimensional vector subspace of \( \mathbb{R}^{n+1} \) with induced connection \( \nabla^2 \) (the restriction of \( \nabla \) to \( D_3 \)) and affine metric \( h_2 = g(λ_3 − λ_2)λ_3 h \). Of course as both spaces are complementary, we may assume by a linear transformation that the \( n_2 + 1 \) dimensional space is spanned by the first \( n_2 + 1 \) coordinates of \( \mathbb{R}^{n+1} \) whereas the \( n_3 + 1 \) dimensional space is spanned by the last \( n_3 + 1 \) coordinates of \( \mathbb{R}^{n+1} \).

Moreover, taking \( V_1, \ldots, V_{n_2} \) as before, we find that

\[
\sum_{i=1}^{n_2} (\nabla h_1)(V_i) = \lambda_2(\lambda_2 - \lambda_3) \sum_{i=1}^{n_2} (\nabla^1 h)(V_i, V_i)
\]

\[
= -2\lambda_2(\lambda_2 - \lambda_3) \sum_{i=1}^{n_2} h(\nabla^1 V_i, V_i)
\]

\[
= -2\lambda_2(\lambda_2 - \lambda_3) \sum_{i=1}^{n_2} h(\nabla V_i, V_i)
\]

\[
= \lambda_2(\lambda_2 - \lambda_3) \sum_{i=1}^{n_2} (\nabla h)(V, V_i) = 0,
\]

as by assumption \( h(K(V, W), W) = h(K(V, T), T) \). So \( M_1 \) is an hyperbolic affine hypersphere. Choosing now the constant \( d_1 \) appropriately we may assume that \( M_1 \) has mean curvature \(-1\). A similar argument also holds for \( M_2 \).

As

\[
\phi = \frac{1}{f(λ_2 - λ_3)} \phi_2 - \frac{1}{g(λ_2 - λ_3)} \phi_3
\]

\[
= (\frac{1}{d_1} e^{\frac{1}{d_2}} ) \phi_2 + (\frac{1}{d_3} e^{\frac{1}{d_3}} ) \phi_3 (\frac{\sqrt{2n_2 + 1}}{n_2 + n_3 + 2}).
\]

We note from Section 2 that we must have that \( d_1^{n_2+1} d_2^{m_2+1} = 1 \) and that therefore \( \phi \) is given as the Calabi product of the immersions \( \phi_1 \) and \( \phi_2 \).

**Remark 2** In case that \( M \) has parallel difference tensor, i.e. if \( \nabla K = 0 \), the conditions of Theorem 2 can be weakened. Indeed we can prove:

**Theorem 3** Let \( M \) be a hyperbolic affine sphere with mean curvature \( λ \), where \( λ < 0 \). Suppose that \( \nabla K = 0 \) and there exists \( h \)-orthonormal distributions \( D_1 \) (of dimension \( 1 \)), \( D_2 \) (of dimension \( n_2 \)) and such that \( D_3 \) (of dimension \( n_3 \)) such that

\[
K(T, T) = λ_1 T,
\]

\[
K(T, V) = λ_2 V,
\]

\[
K(T, W) = λ_3 W,
\]

where \( T \) is a unit vector spanning \( D_1 \) and \( V \in D_2 \), \( W \in D_3 \). Moreover we suppose that \( λ_2 \neq λ_3 \) and \( 2λ_2 \neq λ_1 \neq 2λ_3 \). Then \( \phi : M^n \rightarrow \mathbb{R}^{n+1} \) can be decomposed as the Calabi
product of two hyperbolic affine sphere immersions \( \psi_1 : M_1^{n_1} \to \mathbb{R}^{n_1+1} \) and \( \psi_2 : M_2^{n_2} \to \mathbb{R}^{n_2+1} \) with parallel cubic form.

**Proof:** By applying an homothety we may choose \( \lambda = -1 \). As \( \hat{\nabla}K = 0 \), we also have that \( \hat{R}K = 0 \). This means that

\[
\hat{R}(X,Y)K(Z,U) = K(\hat{R}(X,Y)Z,U) + K(Z,\hat{R}(X,Y)U).
\]

So, taking \( X = Z = U = T \) and \( Y = V \), we find that

\[
\hat{R}(T,V)T = V - K_TK_VT + K_VK_TT = (1 - \lambda_2^2 + \lambda_1\lambda_2)V.
\]

Hence we deduce that

\[
(\lambda_1 - 2\lambda_2)(-1 - \lambda_1\lambda_2 + \lambda_2^2) = 0.
\]

Similarly we have

\[
(\lambda_1 - 2\lambda_3)(-1 - \lambda_1\lambda_3 + \lambda_3^2) = 0.
\]

In view of the conditions, we must have that \( \lambda_2 \) and \( \lambda_3 \) are the two different roots of the equation

\[
-1 - \lambda_1x + x^2 = 0.
\]

Consequently \( \lambda_2 + \lambda_3 = \lambda_1 \) and \( \lambda_2\lambda_3 = -1 \).

Finally we take \( Z = U = T \), \( X = V \) and \( Y = W \). Then we find that

\[
\lambda_1\hat{R}(V,W)T = 2K(\hat{R}(V,W)T,T) = -2K(K_VK_WT,T) + 2K(K_WK_VT,T)
\]

\[
= -2(\lambda_3 - \lambda_2)K_TK_VW.
\]

Hence

\[
\lambda_1(\lambda_2 - \lambda_3)K_VW = 2K(\hat{R}(V,W)T,T) = -2K(K_VK_WT,T) + 2K(K_WK_VT,T)
\]

\[
= -2(\lambda_3 - \lambda_2)K_TK_VW.
\]

This implies that \( K_VW \) is an eigenvector of \( K_T \) with eigenvalue \( \frac{1}{2}\lambda_1 \). Given the form of \( K_T \) we deduce that \( K(V,W) = 0 \). We are now in a position to apply Theorem 2 and deduce that \( M \) can be obtained as the Calabi product of the hyperbolic affine spheres. □

## 4 Characterisation of the Calabi product of a hyperbolic affine sphere and a point and the proof of Theorem 1

Throughout this section we will assume that \( \phi : M^n \to \mathbb{R}^{n+1} \) is a hyperbolic affine hypersphere with mean curvature \(-1\) and we will prove Theorem 1. Therefore, we shall also assume that \( M \) admits two mutually orthogonal differential distributions \( D_1 \) and \( D_2 \)
of dimension 1 and \( n_2 > 0 \), respectively, with \( 1 + n_2 = n \), and, for unit vector \( T \in D_1 \) and all vectors \( V \in D_2 \),

\[
K(T, T) = \lambda_1 T, \quad K(T, V) = \lambda_2 V.
\]

By the apolarity condition we must have that

\[
\lambda_1 + n_2 \lambda_2 = 0, \quad \text{(12)}
\]

Moreover, we will assume that

\[
1 + \lambda_1 \lambda_2 - \lambda_2^2 = 0. \quad \text{(13)}
\]

The above conditions imply that \( \lambda_1 \) and \( \lambda_2 \) are constant and can be determined explicitly in terms of the dimension \( n \). Indeed, if necessary by replacing \( T \) with \(-T\), we have that

\[
\lambda_2 = \frac{1}{\sqrt{n}}, \\
\lambda_1 = -\frac{n-1}{\sqrt{n}}.
\]

We now proceed as in the previous case. Using the fact that \( \hat{\nabla} K \) is totally symmetric it follows that

**Lemma 5** We have

1. \( \hat{\nabla}_T T = 0 \),
2. \( \hat{\nabla}_V T = 0 \),
3. \( h(\hat{\nabla}_V \bar{V}, T) = 0 \).

The previous lemma tells us that the distributions determined by \( D_1 \) and \( D_2 \) are totally geodesic. This is sufficient to conclude that locally \((M, h)\) is isometric with \( I \times M_1 \) where \( T \) is tangent to \( I \) and \( D_2 \) is tangent to \( M_1 \).

The product structure of \( M \) implies the existence of local coordinates \((t, p)\) for \( M \) based on an open subset containing the origin of \( \mathbb{R} \times \mathbb{R}^{n_2} \), such that \( D_1 \) is given by \( dp = 0 \) and \( D_2 \) is given by \( dt = 0 \). We may also assume that \( T = \frac{\partial}{\partial t} \). We now put

\[
\phi_2 = f \frac{1}{\lambda_2} \phi + f T, \quad \phi_3 = g \lambda_2 \phi - g T, \quad \text{(14)}
\]

where the functions \( f \) and \( g \), which depend only on the variable \( t \), are determined by

\[
f' = -f \lambda_2 = -\frac{1}{\sqrt{n}}, \\
g' = g(\lambda_2 - \lambda_1) = \sqrt{n}.
\]
It is clear that solutions are given by
\[ f(t) = d_1 e^{-\frac{1}{\sqrt{n}}t} \quad \text{and} \quad g(t) = d_2 e^{\sqrt{n}t}. \]

A straightforward computation, now shows that
\[
D_T(\phi_2) = D_T(f\sqrt{n} \phi + fT) \\
= -f(\phi + \frac{1}{\sqrt{n}}T) + f(\sqrt{n}T + (K(T, T) + \phi)) \\
= fT(-\frac{1}{\sqrt{n}} + \sqrt{n} - \frac{n-1}{\sqrt{n}}) \\
= 0.
\]

Similarly
\[
D_T(\phi_3) = 0, \\
D_V(\phi_3) = 0.
\]

The above implies that \( \phi_2 \) reduces to a map of \( M_1 \) in \( \mathbb{R}^n \) whereas \( \phi_3 \) is a constant vector in \( \mathbb{R}^n \). As we have that
\[ d\phi_2(V) = D_V(\phi_2) = f(\sqrt{n}V + K(V, T)) = f(\sqrt{n} + \frac{1}{\sqrt{n}})V, \]
the map \( \phi_2 \) is actually immersions. Moreover, denoting by \( \nabla^1 \) the \( D_2 \) component of \( \nabla \), we find that
\[
D_Vd\phi_2(\tilde{V}) = f(\sqrt{n} + \frac{1}{\sqrt{n}})D_V\tilde{V} \\
= f(\sqrt{n} + \frac{1}{\sqrt{n}})\nabla_V\tilde{V} + f(\sqrt{n} + \frac{1}{\sqrt{n}})h(V, \tilde{V})\phi \\
= f(\sqrt{n} + \frac{1}{\sqrt{n}})\nabla^1_V\tilde{V} + f(\sqrt{n} + \frac{1}{\sqrt{n}})(h(K(V, \tilde{V}), T)T + h(V, \tilde{V})\phi) \\
= d\phi_2(\nabla^1_V\tilde{V}) + f(\sqrt{n} + \frac{1}{\sqrt{n}})h(V, \tilde{V})(\frac{1}{\sqrt{n}}T + \phi) \\
= d\phi_2(\nabla^1_V\tilde{V}) + \frac{n+1}{n}h(V, \tilde{V})\phi_2.
\]

The above formulas imply that \( \phi_2 \) can be interpreted as a centroaffine immersion contained in an \( n_2 + 1 \)-dimensional vector subspace of \( \mathbb{R}^{n+1} \) with induced connection \( \nabla^1 \) and affine metric \( h_1 = \frac{n+1}{n}h \). Of course as the vector \( \phi_3 \) is transversal to the immersion \( \phi_2 \), we may assume by a linear transformation that the \( \phi_2 \) lies in the space spanned by the first \( n \) coordinates of \( \mathbb{R}^{n+1} \) whereas the constant vector lies in the direction of the last coordinate, and by choosing \( d_2 \) appropriately we may assume that \( \phi_2 = (0, \ldots, 0, 1) \).

As before we get that \( M_1 \) satisfies the apolarity condition and hence is a hyperbolic affine hypersphere. Choosing now the constant \( d_1 \) appropriately we may assume that \( M_1 \) has mean curvature \(-1\).

As
\[ \phi = (\frac{1}{d_1} e^{\sqrt{n}t} \phi_2 + \frac{1}{d_2} e^{-\sqrt{n}t} \phi_3)(\sqrt{(n)(n+1)}). \]

We note from Section 2 that we must have that \( d_1^{n+1}d_2 = 1 \) and that therefore \( \phi \) is given as the Calabi product of the immersions \( \phi_1 \) and a point.
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