Generalised Functions and Distributional Curvature of Cosmic Strings

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Abstract A new method is presented for assigning distributional curvature, in an invariant manner, to a space-time of low differentiability, using the techniques of Colombeau’s “new generalised functions”. The method is applied to show that curvature of a cone is equivalent to a delta function. The same is true under small enough perturbations.

1. Introduction

For some time it has been recognised that there is an important place in relativity for metrics whose curvature has to be regarded as a distribution; that is, the components of the Riemann tensor have to be interpreted not as ordinary functions, but as functionals (Gelfand and Shilov, 1963) or as ideal limits of functions. This has been applied successfully to such examples as surface-distributions of matter (Israel, 1966; Clarke and Dray, 1987) and to gravitational radiation (Kahn and Penrose, 1971). In these cases, however, it is possible to formulate the field equations so as to avoid operations that are not well defined in ordinary distribution theory: specifically, multiplying distributions, or multiplying, say, a delta-function by a discontinuous function. The potential ambiguity of such operations was established by Schwarz (1954), who pointed out that it was impossible consistently to define an associative multiplication on distributions, together with an operation of differentiation, which coincided with the usual definition of these on continuous and $C^1$ functions, respectively, and for which there was a non-zero distribution $\delta$ satisfying $x\delta(x) = 0$.

The curvature tensor is a non-linear function of the metric, so that if one is to avoid these illegitimate operations, then there are strong constraints on the sort of metric that can be considered. Specifically, Geroch and Traschen (1987, Theorem 1) showed that in order for the components of the Riemann tensor and its contractions to be well defined as distributions, their singular parts have to have support on a submanifold of dimension of at least three. This unfortunately excludes many space-times which one expects to have a distributional curvature, for example the cone-like space-time

$$ds^2 = -dt^2 + dr^2 + A^2 r^2 d\phi^2 + dz^2, \quad |A| < 1.$$  \hspace{1cm} (1)

Various attempts (Raju, 1982; Balasin and Nachbagauer, 1993) have been made to give prescriptions for multiplying distributions in order to overcome the problem posed by the nonlinearity of Einstein’s equations, but they have tended to depend on regularization procedures whose invariance and general applicability were uncertain. There has been until recently no general theory within which it could be shown that the result was invariantly defined, independently of the particular regularization adopted.

Our aim in this paper is to describe within the setting of relativity theory the formalism, due to Colombeau (1983, 1990), in which distributional curvatures can be rigorously and unambiguously defined. We will illustrate the method in the case of the conical metric (1), where one expects there to be a distributional energy momentum tensor of the “thin string” type (Kibble, 1976). The illustration will be extended to appropriately smooth perturbations of the metric.

Roughly speaking, the method involves extending the Schwartz space of distributions to a much larger space, whose elements we will call generalised functions, within which the operations for computing the curvature can always be defined. A subspace of the generalised functions can then be defined whose elements correspond, in a many-one manner, to classical distributions. For those metrics whose curvature components lie in this subspace we can assign a well defined distributional curvature. This procedure circumvents the result of Schwarz quoted above by violating both the main conditions of Schwartz’ theorem: in the space of generalised functions $x\delta(x) \neq 0$, and multiplication does not coincide with ordinary multiplication for continuous functions (although it does for $C^\infty$ functions). These properties of generalised functions (which would usually be regarded as undesirable) are resolved when the generalised functions are mapped onto ordinary distributions, where this is possible. When this is done, $x\delta(x)$ corresponds to the zero distribution;
2. An overview of Colombeau’s generalised functions

2.1. Smoothing distributions
As we shall be concerned both with space-time, locally $\mathbb{R}^4$, and with a 2-dimensional plane transverse to the conical singularity, it will be convenient to work in $\mathbb{R}^n$ for general $n$. The basic tool will be the operation of smoothing distributions. Suppose $\Phi$ is a member of the space $\mathcal{D}(\mathbb{R}^n)$ of test functions: smooth (i.e. $C^\infty$) $\mathbb{C}$-valued functions on $\mathbb{R}^n$ with compact support; and that

$$\int \Phi(x) \, dx = 1.$$ 

Given $\epsilon > 0$, we define

$$\Phi^\epsilon(x) = \frac{1}{\epsilon^n} \Phi \left( \frac{x}{\epsilon} \right)$$

so that $\Phi^\epsilon$ has a support scaled by $\epsilon$ and an amplitude adjusted so that its integral is still unity. If $f: \mathbb{R}^n \to \mathbb{C}$ is a function, not necessarily continuous, then by a smoothing of $f$ we mean one of the convolutions

$$\tilde{f}(x) := \int f(y + x) \Phi(y) \, dy = \int f(z) \Phi(z - x) \, dz.$$ 

or

$$\tilde{f}_\epsilon(x) := \int f(y + x) \Phi^\epsilon(y) \, dy = \int f(z) \Phi^\epsilon(z - x) \, dz.$$ 

(Smoothed functions will always depend implicitly on $\Phi$ as well as on the explicit arguments $x$ and $\epsilon$.)

Smoothing is defined in the same way for distributions, but with some notational changes. A distribution $\tilde{R}$ is regarded as a $\mathbb{C}$-valued functional

$$\mathcal{D}(\mathbb{R}^n) \ni \phi \mapsto (R, \phi) \in \mathbb{C}$$

on the space $\mathcal{D}(\mathbb{R}^n)$ of test functions, and the convolution is defined by

$$\tilde{R}(x) = (R, \Phi(\cdot - x)),$$

$$\tilde{R}_\epsilon(x) = (R, \Phi^\epsilon(\cdot - x)).$$

An intuitively plausible procedure for defining the product $Rf$ of, say, a distribution $R$ and a discontinuous function $f$ would then be to define the action of the product on a test function $\psi$ by first defining the corresponding action of the product of the smoothed quantities $\tilde{R}_\epsilon$ and $\tilde{f}_\epsilon$, and then taking the limit as the smoothing is made progressively finer, with $\epsilon \to 0$:

$$(Rf, \psi) = \lim_{\epsilon \to 0} \int \tilde{R}_\epsilon(x) \tilde{f}_\epsilon(x) \psi(x) \, dx.$$ 

For example, if $R = \delta$, the Dirac $\delta$-function, and $f$ is the Heaviside function, then this prescription yields the attractive solution $\delta f = \delta/2$, provided that $\Phi(-x) = \Phi(x)$. If one considers products involving more complicated distributions, then the dependence of the answer on the nature of $\Phi$ becomes more detailed. For example, writing $x^{-1}$ for the distribution defined by taking the Cauchy principal value in integrals, $x^{-1}\delta = k\delta'$ where $k$ is a $\Phi$-dependent constant.

A further complication arises in the context of relativity, where one would like the results of multiplication to be invariant under $C^\infty$ coordinate changes. The underlying problem here is that the operation of convolution depends on the linear structure of $\mathbb{R}^4$ and so is not invariant under such changes. This suggests that smoothing by convolution should be regarded as a special case of a more general sort of smoothing, invariant under coordinate changes.

Colombeau’s definition of generalised functions therefore starts from a space $\mathcal{E}_M(\mathbb{R}^n)$ of functions depending on both the position $x$ and a smoothing kernel $\Phi$, on which are imposed conditions that reflect the special case of the smoothing of a distribution, but which are more general. Multiplication is defined on these pointwise. This produces a space with some of the required properties, but in which multiplication does not coincide with the ordinary multiplication even for $C^\infty$ functions. This is rectified by defining an equivalence relation on $\mathcal{E}_M(\mathbb{R}^n)$ and passing to a space $\mathcal{G}$ of equivalence classes. At the expense of minor complications, the whole of the construction may be applied to a domain which is a subset $\Omega$ of $\mathbb{R}^n$, in which case we denote the space of functions $\mathcal{E}_M(\Omega)$ and the space of equivalence classes $\mathcal{G}(\Omega)$. 

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2.2. Specification of the algebra

We now give a precise definition of the space of generalised functions \( \mathcal{G}(\mathbb{R}^n) \). We shall use standard multi-index notation

\[ i = (i_1, \ldots, i_n), \quad |i| = i_1 + \cdots + i_n. \]

so

\[ x^i = x_1^{i_1} \cdots x_n^{i_n} \]

and

\[ D^i = \frac{\partial^{|i|}}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}}. \]

An essential role in defining both the basic space of functions \( E^M(\mathbb{R}^n) \) and the equivalence relation used to define the final generalised functions is played by a classification of smoothing kernels into subsets \( A_q(\mathbb{R}^n) \).

**Definition 1.** For \( q \in \mathbb{N} \) we define \( A_q(\mathbb{R}^n) \) to be the set of functions \( \Phi \in D(\mathbb{R}^n) \) such that

(i) \( \int_{\mathbb{R}^n} \Phi(x) \, dx = 1 \)

(ii) \( \int_{\mathbb{R}^n} \Phi(x) x^i \, dx = 0 \quad \forall i \in \mathbb{N}^n \quad \text{such that} \quad |i| \leq q \)

We then define \( E^M(\mathbb{R}^n) \) in two steps. First, we set

\[ E(\mathbb{R}^n) = \{ R : A_1 \times \mathbb{R}^n \to \mathbb{C} \mid x \mapsto R(\Phi, x) \text{ is } C^\infty \} \]

This is an algebra (a vector space furnished with a multiplication) under the operation of pointwise multiplication

\[ (RS)(\Phi, x) := R(\Phi, x)S(\Phi, x). \]

The derivatives of these functions with respect to \( x \) will be denoted by \( D^\alpha R(\Phi, x) \).

The second step is as follows:

**Definition 2.** The subalgebra \( E^M(\mathbb{R}^n) \) (functions of moderate growth), of \( E(\mathbb{R}^n) \) is defined to be the set of functions \( R \) such that for all compact \( K \subseteq \mathbb{R}^n \) and for all \( \alpha \in \mathbb{N}^n \), there is some \( N \in \mathbb{N} \) such that: If \( \Phi \in A_N \), \( \exists c, \eta > 0 \) such that\n
\[ |D^\alpha R(\Phi', x)| \leq c \epsilon^{-N} \quad (x \in K, \ 0 < \epsilon < \eta) \]

It is easily verified that, for any distribution \( T \) with support in \( \mathbb{R}^n \), the corresponding smoothed function \( \tilde{T} \) is in \( E^M(\mathbb{R}^n) \).

Finally, we need to define the equivalence relation that will give a multiplication that coincides with the usual one on \( C^\infty \) functions. The key idea is to note that there are two different ways in which ordinary functions \( f \) can be mapped into elements of \( E^M(\mathbb{R}^n) \): we can (for any continuous \( f \)) form by smoothing the element

\[ R_f(x, \Phi) = \int f(y + x)\Phi(y) \, dy \]

or (for \( C^\infty \) functions \( f \) only) we can form the element

\[ S_f(x, \Phi) = f(x). \]

Multiplication of the \( S_f \) trivially coincides with ordinary multiplication of the functions \( f \), whereas this does not hold for the \( R_f \). Colombeau therefore defines an equivalence relation that identifies \( S_f \) with \( R_f \). This ensures that for \( C^\infty \) functions there is a single mapping of functions into generalised functions (equivalence classes), which coincides with the mapping by smoothing used for distributions, and on which multiplication coincides with ordinary multiplication for \( C^\infty \) functions. The equivalence is defined by means of an ideal \( \mathcal{N}(\mathbb{R}^n) \) (i.e. a subalgebra such that if \( R \in \mathcal{N}(\mathbb{R}^n) \) and \( S \in E^M(\mathbb{R}^n) \) then \( RS \in \mathcal{N}(\mathbb{R}^n) \)) so that elements differing by a member of \( \mathcal{N}(\mathbb{R}^n) \) are identified.
**Definition 3.** The ideal \( \mathcal{N}(\mathbb{R}^n) \), of \( \mathcal{E}_M(\mathbb{R}^n) \) is defined to be the set of functions \( R \) such that for all compact \( K \subseteq \mathbb{R}^n \) and for all \( \alpha \in \mathbb{N}^n \), there is some \( N \in \mathbb{N} \) and some increasing and unbounded sequence \( \{\gamma_n\} \) such that: If \( \Phi \in \mathcal{A}_q \), for \( q \geq N \), \( \exists c, \eta > 0 \) such that

\[
|D^{\alpha} R(\Phi, x)| \leq c e^{\gamma_q - N} \quad (x \in K, \ 0 < \epsilon < \eta)
\]

For the smoothed functions used here, \( \gamma_q \) will be simply \( q \); the more general form is used so as to ensure invariance under coordinate changes.

The key result is then that, for \( C^\infty \) functions \( f, R_f - S_f \) is in \( \mathcal{N}(\mathbb{R}^n) \).

**Definition 4 (Generalised functions).** The algebra of generalised functions is defined to be

\[
\mathcal{G}(\mathbb{R}^n) = \mathcal{E}_M(\mathbb{R}^n) / \mathcal{N}(\mathbb{R}^n).
\]

The generalised function corresponding to the product of two distributions \( T \) and \( U \), or of a distribution \( T \) and a continuous function \( f \), is then taken to be the equivalence class of the product \( \tilde{T} \tilde{U} \), or \( \tilde{T} \tilde{f} \) respectively.

**2.3. Partial correspondence with distributions**

In order to associate a generalised function with a distribution we need to give a meaning to the expression \( \int G(x) \phi(x) \, dx \) with \( G \) a generalised function and \( \phi \) a test function. Colombeau therefore introduces a concept of the integral of a generalised function.

Given a generalised function \( G = [R(\Phi, x)] \), given as the equivalence class of an element \( R \in \mathcal{E}_M(\mathbb{R}^n) \), the quantity

\[
\rho := \int_{\mathbb{R}^n} R(\Phi, x) \, dx
\]

is a function of \( \Phi \) that depends on the choice of \( R \). More specifically, it is a function \( \rho : \mathcal{A}_1 \rightarrow \mathbb{C} \) satisfying the condition:

there is some \( N \in \mathbb{N} \) such that: If \( \Phi \in \mathcal{A}_N \), \( \exists c, \eta > 0 \) such that

\[
|\rho(\Phi')| \leq c e^{-N} \quad (0 < \epsilon < \eta)
\]

**Definition 5.** The algebra \( \mathcal{E}_M \) is defined to be the set of functions \( \rho \) satisfying the above condition.

Different choices of the representative \( R \) for \( G \) result in functions \( \rho \) differing by a function in a subalgebra \( I \) playing the same role as \( \mathcal{N}(\mathbb{R}^n) \):

**Definition 6.** The ideal \( I \) of \( \mathcal{E}_M \) is defined to be the set of functions \( R \) such that there is some \( N \in \mathbb{N} \) and some increasing and unbounded sequence \( \{\gamma_n\} \) such that: If \( \Phi \in \mathcal{A}_q \) for \( q \geq N \), \( \exists c, \eta > 0 \) such that

\[
|\rho(\Phi')| \leq c e^{\gamma_q - N} \quad (0 < \epsilon < \eta)
\]

Thus if we take the quotient of \( \mathcal{E}_M \) by \( I \) we will obtain an equivalence class that is independent of the representative \( R \), and that can therefore be regarded as a value of the integral of the generalised function \( G \). Whereas the integral of an ordinary complex function is a complex number, an element of this quotient is called a generalised complex number.

**Definition 7 (Generalised Numbers).** The algebra of generalised numbers is defined to be

\[
\mathbb{C} = \mathcal{E}_M / I.
\]

For each ordinary complex number \( z \) we can form the generalised complex number \( \tilde{z} \) defined as the equivalence class \( [\rho_z] \) of the constant function \( \rho_z(\Phi) = z \). It follows then that \( z \) is associated to \( \tilde{z} \), in the following sense:
Definition 8. We say that $\bar{z} \in \bar{C}$ is associated to the classical number $z \in C$ (written as $\bar{z} \vdash z$) if there is some representative $\rho \in E_M$ of $\bar{z}$ such that for $\Phi \in A_q$ with $q$ large enough,

$$\lim_{\epsilon \to 0} \rho(\Phi^{\epsilon}) = z.$$ 

Definition 9. We say that $\bar{z}_1, \bar{z}_2 \in \bar{C}$ are associated to each other if and only if $\bar{z}_1 - \bar{z}_2 \vdash 0 \in C$.

Note that $\bar{z} \vdash z$ does not necessarily imply $\bar{z} = [\rho z]$. It is, however, this weaker notion of association that turns out to be the important one in assigning corresponding distributions to generalised functions.

To do this, we first make the following:

Definition 10 (Weak equivalence). We say that $G_1, G_2 \in G(\mathbb{R}^n)$ are associated to each other (written as $G_1 \approx G_2$) if and only for each $\Psi \in D$,

$$\int_{\mathbb{R}^n} (G_1(x) - G_2(x))\Psi(x) \, dx \vdash 0 \in C.$$

Then we say that a generalised function $G$ corresponds to a distribution $T$ if $G \approx \tilde{T}$. For physical purposes, we are interested in those generalised functions, defined by replacing the terms in the definition of the Riemann tensor components by smoothed functions and distributions, which correspond to distributions.

3. Smoothing the cone

In this section we will look at the curvature of the spacetime given by the metric (1). In this example the singular part of the curvature arises because of a conical singularity in the 2-dimensional plane transverse to the axis. We will therefore examine the curvature of the 2-dimensional metric

$$ds^2 = dr^2 + A^2 r^2 d\phi^2, \quad |A| < 1.$$ (2)

Since this is the metric of a surface the above example has the advantage that the curvature may be given by a scalar quantity, the Gaussian curvature, rather than having to use the curvature tensor. This simplifies the notation and the details of the smoothing but plays no essential role in the calculation. We explore the implications of this further in the final section.

We want to think of the cone metric as being defined on the whole of $\mathbb{R}^2$ so that we start by expressing (2) in coordinates which are regular at the origin. In Cartesian coordinates ($x = r \cos \phi, y = r \sin \phi$) the metric is given by

$$g_{ab} = \frac{1}{2}(1 + A^2)\delta_{ab} + \frac{1}{2}(1 - A^2)h_{ab}$$

where

$$h_{ab} = \begin{bmatrix}
    x^2 - y^2 & 2xy \\
    x^2 + y^2 & x^2 + y^2 \\
    2xy & 2xy \\
    x^2 + y^2 & y^2 - x^2
\end{bmatrix}$$

Note that $h_{ab}$ represents the ‘singular’ i.e. the non-smooth part of the metric.

We now wish to regard the components of the metric as elements of $E_M(\mathbb{R}^2)$ so that for $\Phi \in A_1(\mathbb{R}^2)$,

$$\tilde{g}_{ab}(x, y) = \int_{\mathbb{R}^2} g_{ab}(u, v)\Phi^c(u - x, v - y) \, du \, dv,$$

are smooth functions which we regard as the components of a smoothed metric. We can therefore calculate the Ricci curvature $\tilde{R}$ of this metric which may also be regarded as an element of $E_M(\mathbb{R}^2)$. In this section we show that the curvature is given by a delta function in the sense that $[\tilde{R}, \sqrt{\tilde{g}}] \approx 4\pi(1 - A)\delta^{(2)}$.

Remark W. We shall regard the delta function as a scalar density, and the functions in $D$ as scalars. Thus a factor of $\sqrt{\tilde{g}}$ is inserted to make the left hand side a density.
It is important to note that although the calculation is carried out using the Cartesian components of the metric for convenience, the definitions of the Colombeau algebra ensure that the result does not depend upon the choice of coordinates used provided they are related by a smooth transformation.

In order to calculate \( \tilde{g}_{ab} \) we first note that constant functions are unchanged on smoothing so that
\[
\tilde{g}_{ab}(x, y) = \frac{1}{2}(1 + A^2)\delta_{ab} + \frac{1}{2}(1 - A^2)\tilde{h}_{ab}.
\]
We next note that
\[
\frac{x^2 - y^2}{x^2 + y^2} + i\frac{2xy}{x^2 + y^2} = e^{2i\phi}
\]
So that we may calculate \( \tilde{h}_{ab} \) by smoothing the complex valued function \( h(x, y) = e^{2i\phi} \) to obtain
\[
\tilde{h}_e(x, y) = \frac{1}{c^2} \int_{\mathbb{R}^2} h(u + x, v + y)\Phi(u/e, v/e) \, du \, dv
\]
In order to calculate the integral we express it in terms of polar coordinates \((r, \phi)\) and expand \( \Phi \) in terms of circular harmonics. Writing
\[
\Phi(r, \phi) = \sum_{n \in \mathbb{Z}} \Phi_n(r)e^{in\phi}
\]
we have
\[
\tilde{h}_e(r, \phi) = \frac{1}{c^2} \sum_{n \in \mathbb{Z}} \int_0^\infty I_n(r', r, \phi)\Phi_n(r'/\epsilon) \, r' \, dr'
\]
where
\[
I_n = \int_0^{2\pi} \frac{(re^{i\phi} + r'e^{i\phi})^2}{r^2 + r'^2 + 2rr'\cos(\phi - \phi')} e^{in\phi'} \, d\phi'.
\]
Setting \( w = e^{i\phi} \) and \( z = e^{i\phi'} \), we may integrate out \( \phi' \) by considering the integral of the complex function
\[
\Gamma(z) = -i \left( \frac{r'z + rz'}{rz + r'z} \right) wz^n
\]
around the circular contour \( \kappa : |z| = 1 \).

If \( r' < r \), a single pole occurs at \( z = -r'w/r \) which has a residue of \( -i(-r'/r)^n(1 - r'^2/r)e^{i(n+2)\phi} \).

If \( n = -1 \), a single pole will occur at \( z = 0 \) which has a residue of \( -i(r'/r)e^{i\phi} \).

If \( n \leq -2 \), a \(|n|\)-fold pole will occur at \( z = 0 \) which has a residue of \( i(-r'/r)^n(1 - r'^2/r)e^{i(n+2)\phi} \).

Using the residue theorem, we find that
\[
I_n(r', r, \phi) = \begin{cases}
2\pi \left( 1 - \frac{r'^2}{r^2} \right) \left( -\frac{r'}{r} \right)^n e^{i(n+2)\phi} & \text{if } r' < r \text{ and } n \geq 0, \\
-2\pi \left( 1 - \frac{r'^2}{r^2} \right) \left( -\frac{r'}{r} \right)^n e^{i(n+2)\phi} & \text{if } r' > r \text{ and } n \leq -2, \\
2\pi \min\{r'/r, r/r'\}e^{i\phi}, & \text{if } n = -1, \\
0 & \text{otherwise}.
\end{cases}
\]
So if we expand \( \tilde{h}_e(r, \phi) \) as
\[
\tilde{h}_e(r, \phi) = \sum_{n \in \mathbb{Z}} H_{n,e}(r)e^{in\phi},
\]
we obtain
\[
H_{n,e}(r) = \begin{cases}
2\pi(-1)^n \int_0^{r/e} \left( \frac{z}{r} \right)^{n-2} \left( 1 - \frac{z^2}{r^2} \right) \Phi_{n-2}(r')r' \, dr', & \text{if } n \geq 2, \\
2\pi \int_0^{r/e} \left( \frac{z}{r} \right) \Phi_{-1}(r')r' \, dr' + 2\pi \int_{r/e}^{\infty} \left( \frac{z}{r} \right) \Phi_{-1}(r')r' \, dr', & \text{if } n = 1, \\
-2\pi(-1)^n \int_{r/e}^{\infty} \left( \frac{z}{r} \right)^{n-2} \left( 1 - \frac{z^2}{r^2} \right) \Phi_{n-2}(r')r' \, dr', & \text{if } n \leq 0.
\end{cases}
\]
We next obtain estimates for the \( r \) dependence of \( H_{n,e}(r) \). We will be interested in the behaviour both for small and large \( r/e \). Let \( R_0 = \sup \{ r \mid |\Phi_n(r)| > 0 \text{ for some } n \in \mathbb{Z} \} \) then we consider exterior case \( r > \epsilon R_0 \) first.
Case 1 \((r > \epsilon R_0)\). In polar coordinates the condition that \(\Phi \in \mathcal{A}_q\) may be expressed as
\[
\int_0^{\infty} \left(r \Phi_0(r) dr = 1 \quad (4)\right.
\]
\[
\int_0^{\infty} r^{c+1} \Phi_n(r) dr = 0, \quad |n| \leq c \leq q
\]
\(c + n\) is even

Fixing \(r > 0\) and choosing \(\epsilon < r/R_0\), we find that by (4)
\[
H_{2,\epsilon}(r) = 1 - 2\pi \int_{r/\epsilon}^{\infty} r' \Phi_0(r') dr' - 2\pi \frac{\epsilon^2}{r^2} \int_0^{r/\epsilon} r'^3 \Phi_0(r') dr'
\]
Note that the first integral gives zero because \(r/\epsilon > R_0\). So that
\[
H_{2,\epsilon}(r) = 1 + O(\epsilon^2/r^2)
\]
However if \(\Phi \in \mathcal{A}_q\) with \(q \geq 2\), we may use (5) to express \(H_{2,\epsilon}\) as
\[
H_{2,\epsilon}(r) = 1 - 2\pi \int_{r/\epsilon}^{\infty} r' \Phi_0(r') dr' + 2\pi \frac{\epsilon^2}{r^2} \int_{r/\epsilon}^{\infty} r'^3 \Phi_0(r') dr'
\]
in which both integrals vanish for \(\epsilon < r/R_0\). A similar calculation shows that for \(n = 1\) and for \(n > 2\)
\[
H_{n,\epsilon}(r) = O(\epsilon^n/r^n)
\]
with a coefficient that vanishes if \(\Phi \in \mathcal{A}_q\) with \(q \geq n\).

On the other hand, by the definition of \(R_0\)
\[
H_{n,\epsilon}(r) = 0 \quad \text{for} \quad n < 0.
\]

Putting these results together we have in this case for \(\Phi \in \mathcal{A}_q\),
\[
\tilde{h}_\epsilon(r, \phi) = e^{2i\phi} + O\left(\frac{\epsilon^{q+1}}{r^{q+1}}\right).
\]

Case 2 \((r < \epsilon R_0)\). In the following \(C_1, C_2\) etc. denote positive constants.

For \(n \geq 2\) or \(n \leq -3\) we use the fact that \(|\Phi| \leq K = \sup |\Phi|\) to obtain
\[
|H_{n,\epsilon}| \leq 2\pi K \frac{r^2}{\epsilon^2} \left|\frac{1}{n} - \frac{1}{n+2}\right| = O(r^2/\epsilon^2)
\]
The cases \(n = -2, -1, 0, 1\) are more delicate. Here we expand \(H_{n,\epsilon}\) in a Taylor series (with remainder).

For the case of \(H_0,\epsilon\) we have
\[
H_0,\epsilon(r) = 2\pi \int_{r/\epsilon}^{\infty} \left(1 - \frac{r^2}{\epsilon^2}\right) \Phi_{-2}(r') dr'
\]
\[
H_0',\epsilon(r) = \frac{2\pi}{\epsilon} \int_{r/\epsilon}^{\infty} \frac{-2r}{r'} \Phi_{-2}(r') dr'
\]
\[
H_0'',\epsilon(r) = -\frac{2\pi}{\epsilon^2} \int_{r/\epsilon}^{\infty} \frac{2}{r'} \Phi_{-2}(r') dr' + \frac{2\pi}{\epsilon} [2\Phi_{-2}(r/\epsilon)]
\]
and so
\[
H_0,\epsilon(0) = C_1 \neq 0,
\]
\[
H_0',\epsilon(0) = 0
\]
\[
H_0'',\epsilon(0) = \frac{C_2}{\epsilon^2}
\]
giving $H_{0,\epsilon}(r) = C_3 + O(r^2/\epsilon^2)$. A similar method gives

\[ H_{1,\epsilon}(r) = O(r/\epsilon), \]
\[ H_{-1,\epsilon}(r) = O(r/\epsilon), \]
\[ H_{-2,\epsilon}(r) = O(r^2/\epsilon^2). \]

Therefore in this case we have

\[ \tilde{h}_\epsilon(r, \phi) = \alpha_0 + \alpha_1 \frac{r}{\epsilon} e^{-i\phi} + \alpha_2 \frac{r}{\epsilon} e^{i\phi} + O(r^2/\epsilon^2) \]

where $\alpha_n$ are constants.

The smoothed metric may now be expressed by (3), as

\[ \tilde{h}_{\epsilon} = \begin{bmatrix} \frac{x^2 - y^2}{x^2 + y^2} & \frac{2xy}{x^2 + y^2} & \frac{2xy}{x^2 + y^2} \\ \frac{2xy}{x^2 + y^2} & \frac{2y^2}{x^2 + y^2} & \frac{2y^2}{x^2 + y^2} \\ \frac{2xy}{x^2 + y^2} & \frac{2y^2}{x^2 + y^2} & \frac{2x^2}{x^2 + y^2} \end{bmatrix} + O\left(\frac{\epsilon^{q+1}}{(x^2 + y^2)^{\frac{q}{2}+1}}\right), \]

for $r > \epsilon R_0$, and

\[ \tilde{h}_{\epsilon} = \begin{bmatrix} \beta_1 + \frac{1}{\epsilon}(\beta_3 x + \beta_4 y) & \beta_2 + \frac{1}{\epsilon}(\beta_5 x + \beta_6 y) \\ \beta_2 + \frac{1}{\epsilon}(\beta_5 x + \beta_6 y) & -\beta_1 - \frac{1}{\epsilon}(\beta_3 x + \beta_4 y) \end{bmatrix} + O\left(\frac{x^2 + y^2}{\epsilon^2}\right) \]

for $r < \epsilon R_0$.

We may now calculate the the Ricci scalar $\tilde{R}_\epsilon$ of the smoothed metric.

\[ \tilde{R}_\epsilon = \begin{cases} O(1/\epsilon^2) & \text{if } r < \epsilon R_0 \\ O(\epsilon^{q+1}) & \text{if } r > \epsilon R_0 \end{cases} \]

Note that if in Definition 3 we only consider compact sets $K$ not containing the origin, then the second case in the above equation holds for small enough $\epsilon$, and $\tilde{R}_\epsilon$ then satisfies the conditions for membership of $N$. This can be interpreted as meaning that the curvature is concentrated at the origin.

We now show that $[\tilde{R}\sqrt{g}] \approx 4\pi(1 - A)\delta^{(2)}$. It is sufficient to show that for each $\Phi \in \mathcal{D}(\mathbb{R}^2),$

\[ \lim_{\epsilon \to 0} \int_K \tilde{R}_\epsilon \sqrt{g}(x,y) \Phi(x,y) \, dx \, dy = 4\pi(1 - A)\Phi(0,0) \]

for all $\Phi \in \mathcal{A}_m$ for some $m \in \mathbb{N}^+$, where $K = \text{supp} \Psi$.

By the mean value theorem we may express the left hand side as $\Phi(0,0)I_1 + I_2$ where

\[ I_1 = \int_K \tilde{R}_\epsilon \sqrt{g} \, dx \, dy \]
\[ I_2 = \int_K \tilde{R}_\epsilon \sqrt{g} r \frac{\partial \Phi}{\partial r}(\xi x, \xi y) \, dx \, dy \]

for some $\xi \in [0,1]$. Letting

\[ C = \sup \left\{ \frac{\partial \Phi}{\partial r}(\xi x, \xi y) \right\} \]
\[ B_\epsilon = \left\{ (x,y) \in \mathbb{R}^2 \mid (x^2 + y^2)^{1/2} < \epsilon R_0 \right\} \]

we may write

\[ |I_2| \leq C \left| \int_{B_\epsilon} \tilde{R}_\epsilon \sqrt{g} r \, dx \, dy \right| + C \left| \int_{K-B_\epsilon} \tilde{R}_\epsilon \sqrt{g} r \, dx \, dy \right| \]
For the first integral
\[ \left| \int_{B_{\varepsilon}} \tilde{R}_e \sqrt{\tilde{g}_e} \, dx \, dy \right| \leq C_4 \varepsilon R_0^3 \varepsilon^2 \]
and for the second integral
\[ \left| \int_{K-B_{\varepsilon}} \tilde{R}_e \sqrt{\tilde{g}_e} \, dx \, dy \right| \leq C_5 \varepsilon^{q+1} \left( \frac{1}{\varepsilon R_0^q} - \frac{1}{R_K^q} \right) \]
where \( R_K \) is the maximum radius of \( K \).

This gives
\[ |I_2| \leq C_6 \varepsilon \]

To calculate \( I_1 \) we let \( D = \{ (x, y) \mid x^2 + y^2 \leq \mu^2 \} \) be a disc such that \( D \subseteq K \) and split up the integral into two parts.

\[ I_1 = \int_{K-D} \tilde{R}_e \sqrt{\tilde{g}_e} \, dx \, dy + \int_D \tilde{R}_e \sqrt{\tilde{g}_e} \, dx \, dy \]

For the first integral we have
\[ \int_{K-D} \tilde{R}_e \sqrt{\tilde{g}_e} \, dx \, dy = O \left( \varepsilon^{q+1} / \mu^{q+1} \right) \]

So that in the limit as \( \varepsilon \to 0 \) the only non-zero contribution comes from the integral over the disc. To calculate this we apply the Gauss-Bonnet theorem to convert it into an integral around the boundary.

\[ \frac{1}{2} \int_D \tilde{R}_e \sqrt{\tilde{g}_e} \, dx \, dy = 2\pi - \int_{\partial D} \kappa_{\tilde{g}_e} \, ds \]

where \( \kappa_{\tilde{g}_e} \) is the geodesic curvature of the smoothed metric. Since \( D \) lies in the exterior region for small enough \( \varepsilon \) we get
\[ \int_{\partial D} \kappa_{\tilde{g}_e} \, ds = 2\pi A + O \left( \frac{\varepsilon^{q+1}}{\mu^{q+1}} \right) \]

so that
\[ I_1 = 4\pi - 4\pi A + O \left( \frac{\varepsilon^{q+1}}{\mu^{q+1}} \right) \]

Therefore
\[ \lim_{\varepsilon \to 0} (\Psi(0, 0)I_1 + I_2) = 4\pi(1 - A) \]

as claimed.

4. Perturbations

If we introduce a perturbation on the metric that is small enough, we still obtain the same conclusion. Let us suppose we introduce an \( r^2 \) perturbation so that the metric takes the following form.

\[ ds^2 = dr^2 + A^2 r^2 (1 + kr^2) d\phi^2 \]

with the additional perturbation term being
\[ l_{ab} = \begin{bmatrix} x^2 - y^2 & 2xy \\ 2xy & y^2 - x^2 \end{bmatrix} \]

Since a smooth function may be regarded as a generalised function with no \( \Phi \) dependence, an element of \( \mathcal{E}(\mathbb{R}^2) \) equivalent to the smoothed metric may be defined by
\[ \tilde{g}_{ab}(x, y) = \frac{1}{2} (1 + A^2 + A^2 kr^2) \delta_{ab} + \frac{1}{2} (1 - A^2) \tilde{\eta}_{ab} - \frac{1}{2} A^2 kl_{ab} \]
We now consider the Ricci scalar of this metric. The main difference between this case and a flat cone is that the curvature does not vanish over away from the origin. In fact

\[ R = -2k \frac{3 + 2k r^2}{(1 + k r^2)^2}. \]

Using the smoothed metric,

\[ \tilde{R} = \begin{cases} 
O(1/\epsilon^2) & \text{if } r < \epsilon R_0 \\
R + O \left( \frac{\epsilon^{q+1}}{\mu^q} \right) & \text{if } r > \epsilon R_0
\end{cases} \]

We might expect to find that, in the sense of generalised functions, \([\tilde{R}\sqrt{\tilde{g}}] \approx R\sqrt{g} + 4\pi(1 - A)\delta^{(2)}. \]

We can verify this by performing a calculation similar to the one above for the flat cone, but by replacing \(\tilde{R}\sqrt{\tilde{g}}\) by \(\tilde{R}\sqrt{\tilde{g}} - R\sqrt{g}\) throughout.

The calculation of \(I_2\) is unchanged. To calculate \(I_1\) we write the integral as

\[ I_1 = \int_D \tilde{R}\sqrt{\tilde{g}} \, dx \, dy - \int_J D R\sqrt{g} \, dx \, dy + \int_{K-D} (\tilde{R}\sqrt{\tilde{g}} - R\sqrt{g}) \, dx \, dy \]

\[ = 4\pi - 4\pi A \frac{1 + 2k\mu^2}{(1 + k\mu^2)^{1/2}} + 4\pi A \left[ \frac{1 + 2k\mu^2}{(1 + k\mu^2)^{1/2}} - 1 \right] + O \left( \frac{\epsilon^{q+1}}{\mu^q} \right). \]

where we have applied the Gauss-Bonnet theorem (6) to evaluate the first integral, and explicitly computed the second. Therefore

\[ \lim_{\epsilon \to 0} (\Psi(0,0)I_1 + I_2) = 4\pi(1 - A). \]

and hence

\[ [\tilde{R}\sqrt{\tilde{g}}] \approx R\sqrt{g} + 4\pi(1 - A)\delta^{(2)} \]

It is clear from the above that replacing \(kr^2\) by a more general perturbation of the form \(k(r)r^2\) where \(k(r)\) is a smooth function would give the same result. Indeed any perturbation for which \(l_{ab}\) is a smooth function would give a similar result.

5. Conclusion

We have shown that the curvature of a 2-dimensional cone at \(r = 0\), when regarded as a generalised function is weakly equivalent to the delta distribution and the conclusion remains the same for a suitably perturbed metric. To simplify the presentation we have confined our attention to the two dimensional case however similar results hold for the case of the the 4-dimensional cone-like space-time (1). In the four dimensional case one must use a more general smoothing function with both \(t\) and \(z\) dependence to calculate the transverse components of \(g_{ab}\). However after integrating out in the \(t\) and \(z\) directions the calculation is much the same. One may then calculate the curvature tensor \(\tilde{R}^a_{\ bcd}\) of the smoothed metric and regard the components as elements of \(E_M(\mathbb{R}^4)\). The only terms that might correspond to distributions are those which are the sectional curvatures of the 2-surface transverse to the string. By applying the methods of §3 but using a holonomy argument rather than the Gauss-Bonnet theorem to calculate the final integral one may show that these terms regarded as generalised functions are weakly equivalent to delta functions, thus confirming the results of Vickers (1987).

Currently work is in progress to extend these calculations to more general four dimensional space-times in general relativity which admit quasi-regular singularities. One result would be to derive a new definition of the mass of a string per unit length, which Geroch and Traschen showed was ambiguous when defined by limiting sequences of metrics.
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