SEPARATION AND APPROXIMATE SEPARATION OF
MULTIPARTITE QUANTUM GATES

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Abstract. The number of qubits of current quantum computers is one of the most
dominating restrictions for applications. So it is naturally conceived to use two or
more small capacity quantum computers to form a larger capacity quantum com-
puting system by quantum parallel programming. To design the parallel program
for quantum computers, the primary obstacle is to decompose quantum gates in the
whole circuit to the tensor product of local gates. In the paper, we first devote to
analyzing theoretically separability conditions of multipartite quantum gates on finite
or infinite dimensional systems. Furthermore, we perform the separation experiments
for \( n \)-qubit quantum gates on the IBM’s quantum computers by the software Q\(|SI\rangle\).
Not surprisingly, it is showed that there exist few separable ones among multipartite
quantum gates. Therefore, we pay our attention to the approximate separation prob-
lems of multipartite gates, i.e., how a multipartite gate can be closed to separable
ones.

1. Introduction

Motivated by development of quantum hardware, programming for quantum com-
puters had been an urgent task ([1]-[4]). Extensive research on quantum programming
has become conducted in the last decade, as surveyed in [1], [5], [6] and [7]. Several
quantum programming platforms have been developed in the last two decades. The first
quantum programming environment can be backed to the project ‘QCL’ proposed by
Ömer [8,9] in 1998. In 2003, Bettelli et al. [2] defined a quantum language called Q lan-
guage as a C++ library. Furthermore, in recent years, some more scalable and robust
quantum programming platforms have emerged. In 2013, Green et al. [10] proposed
a scalable functional quantum programming language, called Quipper, using Haskell
as the host language. JavadiAbhari et al. [11] defined Scafford in 2014, presenting its
accompanying compilation system ScaffCC in [12]. In the same year, Wecker and Svore
from QuArc (the Microsoft Research Quantum Architecture and Computation team)
developed LIQU|i\rangle\) as a modern tool-set embedded within F# [14]. At the end of 2017,
QuARC announced a new programming language and simulator designed specifically
for full stack quantum computing, called Q\#, which represents a new milestone in
quantum programming. Also in the same year, one of the authors released the quantum programming [13], namely QSI, supporting a more complicated loop structure. Up to now, current programming language or tools are mainly focus on the sequential ones.

However, beyond the constraints of quantum hardware, there are still several barriers to developing practical applications for quantum computers. One of the most serious issues is the number of physical qubits that physical machines provide. For example, IBMQ makes two 5 qubits quantum computing [18] and one 16 qubits quantum computer [19] available to programmers through the cloud, but with far fewer, qubits than a practical quantum algorithm requires. Today, quantum hardware is in its infancy. But as the number of available qubits gradually increases, many scholars are beginning to wonder whether the various quantum hardware could be united to work as a single entity and, as a result, bring about a bloom of growth in the number of qubits. Along with the motivation to increase accessible qubits of quantum hardware, one approach is the concurrent or parallel quantum programming. Although recently quantum specific environments only focus on the sequential structure, some researchers exploit the possibility of parallel or concurrent quantum programming on the general programming platform form different aspects. Vizzotto and Costa [21] applied mutually exclusive accesses to global variables for concurrent programming in Haskell to the case of concurrent quantum programming. Yu and Ying [20] carefully studied the termination of concurrent programs. And the papers [22, 23, 24, 25] provide mathematics tools of process algebras for the description of interaction, communications and synchronization.

When implementing parallel programs, the very first obstacle is to separate multipartite quantum gates into the tensor products of local gates. If separation is possible, a potential parallel execution will result naturally. Here, we provide the sufficient and necessary conditions for the separability of multipartite gates. Unsurprisingly, multipartite quantum gates seldom exist that can be separated simply. However, we can confirm there is always a separable gate close to a non-separable gate in certain approximate conditions.

Moreover, we show an approximate separable example in a two-qubit system.

2. Criteria for separation of quantum gates and IBMQ experiments

In this analysis, let $\mathcal{H}_k$ be a separable complex Hilbert space of finite or infinite dimension, and let $\otimes_{k=1}^n \mathcal{H}_k$ be the tensor product of $\mathcal{H}_k$s. Denote by $\mathcal{B}(\otimes_{k=1}^n \mathcal{H}_k), \mathcal{U}(\otimes_{k=1}^n \mathcal{H}_k)$ and $\mathcal{B}_s(\otimes_{k=1}^n \mathcal{H}_k)$ respectively the set of all bounded linear operators, the set of all unitary operators, and the set of all self-adjoint operators on the underline space $\otimes_{k=1}^n \mathcal{H}_k$. 
Let $U$ be a multipartite gate on the composite system $\otimes_{k=1}^{n} \mathcal{H}_k$. We call that $U$ is separable (local or decomposable) if there exist quantum gates $U_k$ on $\mathcal{H}_k$ such that

$$
U = \otimes_{k=1}^{n} U_k.
$$

Next, we establish the separation problem for multipartite gates as follows.

The Separation Problem: Consider the multipartite system $\otimes_{k=1}^{n} \mathcal{H}_k$. If $U = \exp[i\mathbf{H}]$ with $\mathbf{H} = \sum_{i=1}^{N_H} A_i^{(1)} \otimes A_i^{(2)} \otimes ... \otimes A_i^{(n)}$ for a multipartite unitary gate $U$, do any unitary operators $U_k$ on $\mathcal{H}_k$ exist such that $U = \otimes_{k=1}^{n} U_k$? Further, how does the structure of each $U_k$ depend on the exponents of $A_k^{(j)}$, $i = 1, 2, ..., n$?

Remark 2.1. Note that when the dimension of $\otimes_{k=1}^{n} \mathcal{H}_k$ is finite, every unitary gate $U$ has the form $U = \exp[i\mathbf{H}]$ with $\mathbf{H} = \sum_{i=1}^{N_H} A_i^{(1)} \otimes A_i^{(2)} \otimes ... \otimes A_i^{(n)}$ and $N_H < \infty$. Generally speaking, in the decomposition of $\mathbf{H} = \sum_{i=1}^{N_H} A_i^{(1)} \otimes A_i^{(2)} \otimes ... \otimes A_i^{(n)}$ with $N_H < \infty$, many selections of the operator set $\{A_i^{(j)}\}_{i,j}$ (even $A_i^{(j)}$ exist that may not be self-adjoint). However, for an arbitrary (self-adjoint or non-self-adjoint) decomposition $\mathbf{H} = \sum_{i=1}^{N_H} B_i^{(1)} \otimes B_i^{(2)} \otimes ... \otimes B_i^{(n)}$, there exists a self-adjoint decomposition $\mathbf{H} = \sum_{i=1}^{N_H} A_i^{(1)} \otimes A_i^{(2)} \otimes ... \otimes A_i^{(n)}$ such that (29)

$$
\text{span}\{B_1^{(j)}, B_2^{(j)}, ..., B_n^{(j)}\} = \text{span}\{A_1^{(j)}, A_2^{(j)}, ..., A_n^{(j)}\}.
$$

So in the following, we always assume that $\mathbf{H}$ takes its self-adjoint decomposition.

To answer the separation question, we begin the discussion with a simple case: the length $N_H$ of $\mathbf{H}$ is 1, i.e., $\mathbf{H} = A_1 \otimes A_2 \otimes ... \otimes A_n$. Let us first deal with a case where $n = 2$.

**Theorem 2.2.** Let $\mathcal{H}_1 \otimes \mathcal{H}_2$ be a bipartite system of any dimension. For a quantum gate $U = \exp[i\mathbf{H}] \in U(\mathcal{H}_1 \otimes \mathcal{H}_2)$ with $\mathbf{H} = A \otimes B$, the following statements are equivalent:

(I) There exist unitary operators $C, D$ such that $U = C \otimes D$;

(II) One of $A, B$ belongs to $\mathbb{R}I$, and there exist real scalars $\alpha, \beta$ such that either $C = \exp[i(tA + \alpha I)], D = I$ if $B = tI$, or $D = \exp[i(sB + \beta I)], C = I$ if $A = sI$.

Before giving the proof of Theorem 2.2, recall the following lemma concerning the separate vectors of operator algebras. Let $\mathcal{A}$ be a $C^*$-algebra on a Hilbert space $\mathcal{H}$. A vector $|x_0\rangle \in \mathcal{H}$ is called a separate vector of $\mathcal{A}$ if, for any $T \in \mathcal{A}$, $T(|x_0\rangle) = 0 \Rightarrow T = 0$. The following lemma is needed to complete the proof of Theorem 2.2 for the infinite dimensional case.

**Lemma 2.3.** Every Abel $C^*$-algebra has separate vectors.

**Proof of Theorem 2.2.** (II) $\Rightarrow$ (I) is obvious. We only need to check (I) $\Rightarrow$ (II).
Assume (I). Then, for any unit vectors \( |x \rangle, |x' \rangle \) in the first system and \( |y \rangle, |y' \rangle \) in the second system, one has

\[
U |xy \rangle \langle x'y'| = \exp[iA \otimes B] |xy \rangle \langle x'y'| \\
(2.2) = |xy \rangle \langle x'y'| + iA \otimes B |xy \rangle \langle x'y'| - \frac{A^2 \otimes B^2 |xy \rangle \langle x'y'|}{2!} - ... - i^k \frac{A^k \otimes B^k |xy \rangle \langle x'y'|}{k!} + ... \\
\]

and,

\[
(2.3) U |xy \rangle \langle x'y'| = C \otimes D |xy \rangle \langle x'y'|.
\]

Connecting Eq. 2.2 and 2.3 and taking a partial trace of the second (first) system respectively, we obtain that

\[
\langle y | D | y' \rangle C |x \rangle \langle x' | = \langle y | y' \rangle |x \rangle \langle x' | + \langle y | B | y' \rangle A |x \rangle \langle x' | \\
- \langle y | B^2 | y' \rangle \frac{A^2}{2!} |x \rangle \langle x' | - ... + i^k \langle y | B^k | y' \rangle \frac{A^k}{k!} |x \rangle \langle x' | + ... \\
\]

and

\[
\langle x | C | x' \rangle D |y \rangle \langle y' | = \langle x | x' \rangle |y \rangle \langle y' | + \langle x | A | x' \rangle B |y \rangle \langle y' | \\
- \langle x | A^2 | x' \rangle \frac{B^2}{2!} |y \rangle \langle y' | - ... + i^k \langle x | A^k | x' \rangle \frac{B^k}{k!} |y \rangle \langle y' | + ... .
\]

Then it follows from the arbitrariness of \( |x' \rangle \) and \( |y' \rangle \) that

\[
(2.4) \langle y | D | y' \rangle C |x \rangle = \langle y | y' \rangle |x \rangle \langle x' | + \langle y | B | y' \rangle A |x \rangle \langle x' | - \langle y | B^2 | y' \rangle \frac{A^2}{2!} |x \rangle \langle x' | - ... + i^k \langle y | B^k | y' \rangle \frac{A^k}{k!} |x \rangle + ... \\
\]

and

\[
(2.5) \langle x | C | x' \rangle D |y \rangle = \langle x | x' \rangle |y \rangle \langle y' | + \langle x | A | x' \rangle B |y \rangle \langle y' | - \langle x | A^2 | x' \rangle \frac{B^2}{2!} |y \rangle \langle y' | - ... + i^k \langle x | A^k | x' \rangle \frac{B^k}{k!} |y \rangle + ... .
\]

There are the three cases that we should deal with.

Case 1. \( B = tI \). In this case, by taking \( y' = y \) in Eq. 2.4 we see that

\[
\langle y | D | y' \rangle C |x \rangle = I |x \rangle + A |x \rangle - t^2 \frac{A^2}{2!} |x \rangle - ... + i^k t^k \frac{A^k}{k!} |x \rangle + ... = \exp[itA] |x \rangle \\
\]

holds for all \( |x \rangle \). Note that \( C \) and \( \exp[itA] \) are unitary, so there exists some \( \alpha \in \mathbb{R} \) such that \( C = \exp[i\alpha] \exp[itA] = \exp[i(tA + \alpha I)] \). It follows that \( U = \exp[i(tA + \alpha I)] \otimes I \).

Case 2. \( A = sI \). Similar to Case 1, in this case we have \( D = \exp[i\beta] \exp[isB] = \exp[i(sB + \beta I)] \) for some \( \beta \in \mathbb{R} \). It follows that \( U = I \otimes \exp[i(sB + \beta I)] \).

Case 3. \( A, B \notin \mathbb{R}I \). In this case, a contradiction is induced, so that Case 3 may not happen. Dividing the two subcases, have

Subcase 3.1. Both \( A \) and \( B \) have two distinct eigenvalues. It follows that there exist two real numbers \( t_1, t_2 \) with \( t_1 \neq t_2 \) such that \( A |x_1 \rangle = t_1 |x_1 \rangle \) and \( A |x_2 \rangle = t_2 |x_2 \rangle \),
and \( s_1, s_2 \) with \( s_1 \neq s_2 \) such that \( B|y_1\rangle = s_1|y_1\rangle \) and \( B|y_2\rangle = s_2|y_2\rangle \). Taking \( |x\rangle = |x'\rangle = |x_1\rangle \) and \( |x\rangle = |x\rangle = |x_2\rangle \) in Eq. 2.5 respectively, and \( |y\rangle = |y'\rangle = |y_1\rangle \) and \( |y\rangle = |y'\rangle = |y_2\rangle \) in Eq. 2.4 respectively, we have that

\[
\langle x_1|C|x_1\rangle D = \exp[t_1B], \quad \langle x_2|C|x_2\rangle D = \exp[t_2B],
\]

and

\[
\langle y_1|D|y_1\rangle C = \exp[s_1A], \quad \langle y_2|D|y_2\rangle C = \exp[s_2A].
\]

It follows that

\[
\langle x_1|C|x_1\rangle = \frac{\exp[s_1t_1]}{\langle y_1|D|y_1\rangle} \quad \text{and} \quad \langle x_2|C|x_2\rangle = \frac{\exp[s_1t_2]}{\langle y_1|D|y_1\rangle}.
\]

So one gets

\[
\frac{\langle y_1|D|y_1\rangle \exp[t_1B]}{\exp[s_1t_1]} = D = \frac{\langle y_1|D|y_1\rangle \exp[t_2B]}{\exp[s_1t_2]}.
\]

Taking the inner product for \( |y_2\rangle \) on both sides of the above equation, we have

\[
\frac{\exp[t_1s_2]}{\exp[t_1s_1]} = \frac{\exp[t_2s_2]}{\exp[t_2s_1]}.
\]

It follows that \( \exp[t_1s_2 - t_1s_1] = \exp[t_2s_2 - t_2s_1] \), which leads to \( t_1 = t_2 \) as \( s_1 - s_2 \neq 0 \). This is a contradiction.

**Subcase 3.2.** At least one of \( A \) and \( B \) has no distinct eigenvalues.

In this case, we must have \( \dim \mathcal{H}_1 \otimes \mathcal{H}_2 = \infty \) and at least one of \( \sigma(A) \) and \( \sigma(B) \), respectively the spectrum of \( A \) and \( B \), is an infinite closed subset of \( \mathbb{R} \). With no loss of generality, say \( \sigma(A) \) has infinite many points. Let \( A = \text{cl span}\{I, A, A^2, \ldots, A^n, \ldots\} \), then \( A \) is a Abelian C*-algebra. By Lemma 2.2, \( A \) has a separate vector \( |x_0\rangle \). Replacing \( |x\rangle \) with \( |x_0\rangle \) and taking vectors the \( |y\rangle, |y'\rangle \) satisfying \( \langle y|D|y'\rangle = 0 \) in Eq. 2.4, we see that

\[
0 = \langle y|D|y'\rangle C|x_0\rangle
= \langle y|y'\rangle I|x_0\rangle + \langle y|B|y'\rangle A|x_0\rangle - \langle y|B^2|y'\rangle \frac{A^2}{2!}|x_0\rangle - \ldots

+ i^k \langle y|B^k|y'\rangle \frac{A^k}{k!}|x_0\rangle + \ldots

= (\sum_k \lambda_k A^k)|x_0\rangle,
\]

where \( \lambda_k = \frac{i^k \langle y|B^k|y'\rangle}{k!} \). As \( |x_0\rangle \) is a separate vector, we must have \( \sum_k \lambda_k A^k = 0 \).

We claim that each \( \lambda_k = 0 \). For any fixed \( |y\rangle, |y'\rangle \), note that the function \( f(z) = \sum_k \lambda_k z^k \) is analytic. Since \( f(A) = 0 \), the spectrum \( \sigma(f(A)) \) of \( f(A) \) contains the unique element 0. So, by the spectrum mapping theorem, we have

\[
\{0\} = \sigma(f(A)) = \{f(\lambda)|\lambda \in \sigma(A)\}.
\]
Note that, by the assumption of this subcase, \( \sigma(A) \) is an infinite set and has at most one isolated point. So the analytic function \( f(z) \) must by zero. Then each \( \lambda_k = 0 \). It follows that, for each \( k = 0, 1, 2, ..., n, ... \),

\[
\langle y | B^k | y' \rangle = 0
\]

holds for any vectors \( |y\rangle, |y'\rangle \) satisfying \( \langle y | D | y' \rangle = 0 \). Particularly, for the case \( k = 0 \), we have that, for any vectors \( |y\rangle, |y'\rangle \), \( \langle y | D | y' \rangle = 0 \Rightarrow \langle y | y' \rangle = 0 \). This ensures that \( D \in \mathbb{R}I \). Now consider the case \( k = 1 \), one obtains that, for any vectors \( |y\rangle, |y'\rangle \), \( \langle y | D | y' \rangle = 0 \Rightarrow \langle y | B | y' \rangle = 0 \). This implies that \( B \) is linearly dependent to \( D \). So we get \( B \in \mathbb{R}I \), which is a contradiction.

This completes the proof. \( \square \)

Next, we extend Theorem 2.2 to the multipartite systems. Before stating the result, let us give some notations. Let \( A_i \) be self-adjoint operators on \( \mathcal{H}_i \), \( i = 1, 2, ..., n \) such that \( H = A_1 \otimes A_2 \otimes ... \otimes A_n \). If there exists at most one element in the set \( \{ A_1, A_2, ..., A_n \} \) that does not belong to the set \( \mathbb{R}I \), we can define a scalar

\[
\delta(A_j) = \begin{cases} 
\prod_{k \neq j} \lambda_k, & \text{if } A_j \notin \mathbb{R}I; \\
0, & \text{if } A_j \in \mathbb{R}I
\end{cases}
\]

where \( A_k = \lambda_k I \) if \( A_k \in \mathbb{R}I \).

Based on Theorem 2.2 we reach the following conclusion in the multipartite case.

**Theorem 2.4.** Let \( \otimes_{i=1}^n \mathcal{H}_i \) be a multipartite system of any dimension. For a multipartite quantum gate \( U = \exp(iH) \in U(\otimes_{i=1}^n \mathcal{H}_i) \) with \( H = A_1 \otimes A_2 \otimes ... \otimes A_n \), the following statements are equivalent:

(I) There exist unitary operators \( C_i \in U(\mathcal{H}_i) \) \( (i = 1, 2, ..., n) \) such that \( U = \otimes_{i=1}^n C_i \);

(II) At most one element in \( \{ A_i \}_{i=1}^n \) does not belong to \( \mathbb{R}I \), and there is a unit-model number \( \lambda \) such that

\[
U = \lambda \otimes_{j=1}^n \exp[i\delta(A_j)A_j],
\]

where \( \delta(A_j)s \) are as that defined in Eq. 2.7.

**Proof.** (II) \( \Rightarrow \) (I) is obvious. To prove (I) \( \Rightarrow \) (II), we use induction on \( n \).

According to Theorem 2.2, (I) \( \Rightarrow \) (II) is true for \( n = 2 \). Assume that the implication is true for \( n = k \). Now let \( n = k + 1 \). We have that

\[
\exp[iA_1 \otimes A_2 \otimes ... \otimes A_{k+1}] = \exp[iH] = C_1 \otimes C_2 \otimes ... \otimes C_k \otimes C_{k+1} = T \otimes C_{k+1}.
\]

It follows from Theorem 2.2 that either \( A_{k+1} \in \mathbb{R}I \) or \( A_1 \otimes A_2 \otimes ... \otimes A_k \in \mathbb{R}I \). If \( A_1 \otimes A_2 \otimes ... \otimes A_k \in \mathbb{R}I \), then each \( A_i \) belongs to \( \mathbb{R}I \). According to the induction
Theorem 2.5. For a multipartite quantum gate \( U \in \mathcal{U}(\otimes_{k=1}^n \mathcal{H}_k) \), if \( U = \exp[-iH] \) with \( H = \sum_{i=1}^{N_H} T_i \) and \( T_i = A_i^{(1)} \otimes A_i^{(2)} \otimes \ldots \otimes A_i^{(n)} \), the product of homogeneous Lie polynomials in Eq. (2.10) \( \prod_{k=1}^N \mathcal{P}_z(\sum_{j=k+1}^N A_j) \) is a homogeneous Lie polynomial in variables \( A, B, \ldots \), i.e., \( \mathcal{P}_z(A,B) \) is a linear combination (with rational coefficients) of commutators of the form \([V_1,[V_2, \ldots, [V_{m-1},V_m]]] \) with \( V_i \in \{A, B\} \) (32 [34]). Especially, \( \mathcal{P}_z(A,B) = -\frac{1}{2}[A,B] \) and \( \mathcal{P}_3(A,B) = \frac{1}{3}[B,[A,B]] + \frac{1}{6}[A,[A,B]] \). As it is seen, if \( \prod_{i=1}^N \mathcal{P}_z(A_i) = \Pi \) \( \mathcal{P}_z(A,B) \) is a multiple of the identity, then \( \exp[A] \exp[B] = \lambda \exp[A + B] \) for some scalar \( \lambda \). Particularly, if \( AB = BA \), then \( \prod_{i=1}^N \mathcal{P}_z(A_i) \in \mathbb{C}I \). Furthermore, for the multi-variable case, we have

\[
\exp[\sum_{i=1}^N A_i] = \prod_{i=1}^N \exp[A_i] \mathcal{P}_z(A_{N-1}, A_N) \cdot \mathcal{P}_z(A_{N-2}, A_{N-1} + A_N) \ldots \mathcal{P}_z(A_1, \sum_{j=2}^N A_j) = \prod_{i=1}^N \exp[A_i] \prod_{k=1}^N \mathcal{P}_z(A_k, \sum_{j=k+1}^N A_j).
\]

Assume that a multipartite quantum gate \( U = \exp[-iH] \) with \( H = \sum_{i=1}^{N_H} T_i \) and \( T_i = A_i^{(1)} \otimes A_i^{(2)} \otimes \ldots \otimes A_i^{(n)} \). If at most one element in each set \( \{A_i^{(1)}, A_i^{(2)}, \ldots, A_i^{(n)}\} \) does not belong to the set \( \mathbb{R}I \), we define a function:

\[
\delta(A_k^{(i)}) = \begin{cases} 
\prod_{k \neq i} \lambda_j^{(k)}, & \text{if } A_j^{(i)} \notin \mathbb{R}I; \\
0, & \text{if } A_j^{(i)} \in \mathbb{R}I,
\end{cases}
\]

where we denote \( A_j^{(k)} = \lambda_j^{(k)}I \) if \( A_j^{(k)} \in \mathbb{R}I \).

Theorem 2.5. For a multipartite quantum gate \( U \in \mathcal{U}(\otimes_{k=1}^n \mathcal{H}_k) \), if \( U = \exp[-iH] \) with \( H = \sum_{i=1}^{N_H} T_i \) and \( T_i = A_i^{(1)} \otimes A_i^{(2)} \otimes \ldots \otimes A_i^{(n)} \), the product of homogeneous Lie polynomials in Eq. (2.10) \( \prod_{k=1}^N \mathcal{P}_z(T_k, \sum_{j=k+1}^N T_j) \in \mathbb{C}I \) and at most one element in each set \( \{A_i^{(1)}, A_i^{(2)}, \ldots, A_i^{(n)}\} \) does not belong to the set \( \mathbb{R}I \), then up to a unit modular scalar,

\[
U = U^{(1)} \otimes U^{(2)} \otimes \ldots \otimes U^{(n)},
\]
where \( U^{(i)} \) is the local quantum gate on \( H_i \),

\[
U^{(i)} = \prod_{k=1}^{N_H} \exp[\iota\delta^{(i)}(A^{(i)}_k)A^{(i)}_k],
\]

where \( \delta^{(i)}_k \) is defined by Eq. 2.14.

Remark 2.6. In Theorem 2.5 we provided a sufficient condition for the separability of a multipartite gate. However, this condition is not easy to check since the product of homogeneous Lie polynomials \( \prod_{k=1}^{N} \mathcal{P}_z(T_k, \sum_{j=k+1}^{N} T_j) \) in Eq. 2.10 is complicated and difficult to be presented. We observe that if \( [T_k, T_l] \in \mathbb{C}I \) for each pair \( k, l \), then \( \prod_{k=1}^{N} \mathcal{P}_z(T_k, \sum_{j=k+1}^{N} T_j) \in \mathbb{C}I \). To make this easier to check, if \( [T_k, T_l] \in \mathbb{C}I \) and there exists at most one element in \( \{A^{(1)}_i, A^{(2)}_i, \ldots, A^{(n)}_i\} \) that does not belong to the set \( \mathbb{R}I \), then \( U \) has the tensor product decomposition in Eq 2.12. An impressive fact is mentioned here that, as \( T_k \)'s are bounded, \( [T_k, T_l] \in \mathbb{C}I \) implies that \( [T_k, T_l] = 0 \).

Proof of Theorem 2.5 Let us first observe that for any real number \( r \), \( \exp[rT] = (\exp[T])^r \). Furthermore, \( \exp[rT \otimes S] = \exp[rT] \otimes \exp[rS] \) if \( \exp[T \otimes S] = \exp[T] \otimes \exp[S] \). Indeed, for arbitrary positive integer \( N \), it follows from Baker formula that \( \exp[NT] = (\exp[T])^N \). In addition, \( \exp[T] = \exp[\frac{T}{M} \cdot M] \) gives \( \exp[\frac{T}{M}] = (\exp[T])^{\frac{1}{M}} \). So, for any rational number \( a \), we have \( \exp[aT] = (\exp[T])^a \). As \( \phi(a) = \exp[aT] \) is continuous in \( a \in [0, \infty) \) and \( \exp[-T] = (\exp[T])^{-1} \), one sees that \( \exp[aT] = (\exp[T])^a \) holds for any real number \( a \).

According to the assumption and the definition of \( \delta^{(i)}_j \), by writing \( \prod_{k=1}^{N} \mathcal{P}_z(T_k, \sum_{j=k+1}^{N} T_j) = \lambda I \), it follows from Theorem 2.4 that

\[
U = \exp[\iota t H] = \exp[\iota t (\sum_{i=1}^{N_H} T_i)] = \prod_{i=1}^{N_H} \exp[\iota t T_i] \prod_{k=1}^{N} \mathcal{P}_z(T_k, \sum_{j=k+1}^{N} T_j)
\]

\[
= \lambda \prod_{i=1}^{N_H} \exp[\iota t T_i] = \lambda \prod_{i=1}^{N_H} \exp[\iota t A^{(1)}_i \otimes A^{(2)}_i \otimes \ldots \otimes A^{(n)}_i]
\]

\[
= \lambda \prod_{k=1}^{N_H} \exp[\iota \delta^{(1)}_k A^{(1)}_k] \otimes \prod_{k=1}^{N_H} \exp[\iota \delta^{(2)}_k A^{(2)}_k] \otimes \ldots \otimes \prod_{k=1}^{N_H} \exp[\iota \delta^{(n)}_k A^{(n)}_k].
\]

Now absorbing the unit modular scalar \( \lambda \) and letting \( U^{(i)} = \prod_{k=1}^{N_H} \exp[\iota \delta^{(i)}_k A^{(i)}_k] \), we complete the proof. \( \square \)

In the following we devote to designing an algorithm to check whether or not a multipartite gate is separable in \( n \)-qubit case (see Algorithm 2.1). We perform the experiments on the IBM quantum processor *ibmqx4*, while generate the circuits by \( Q|SI \) (the key code segments can be obtained in https://github.com/klinus9542).
Algorithm 2.1 Check whether a unitary is separable or not

Require: \( U \)

Ensure: \( \text{Status}, \text{NonIndentiIndex} \)

1: \( \text{function } \left[ \text{Status, NonIndentiIndex}\right] = \text{CheckSeper}(U) \triangleright \text{If separable, it can tell the status; otherwise it will answer nothing about the status} \)

2: \( H \leftarrow \text{Hermitian value of } U \)

3: \( \text{for index}=1: \text{Number of System do} \)

4: \( \text{if PosChecker}(H, \text{index}) \text{ then} \)

5: \( \quad \text{return } \text{Status} \leftarrow \text{Separable} \)

6: \( \quad \text{return } \text{NonIndentiIndex} \leftarrow \text{index} \)

7: \( \text{function } \text{Status} = \text{PosChecker}(H, \text{index}) \quad \triangleright \text{Recurse solve this problem} \)

8: \( \text{if } \text{index} == 1 \text{ then} \)

9: \( \quad \text{Status} \leftarrow \text{CheckPosLastDimN}(H) \)

10: \( \text{else} \)

11: \[
\begin{pmatrix}
C_{11} & C_{12} \\
C_{21} & C_{22}
\end{pmatrix} = H
\]

where \( \text{dim}(C_{11}) = \text{dim}(C_{12}) = \text{dim}(C_{21}) = \text{dim}(C_{22}) = \frac{1}{2} \times \text{dim}(H) \)

12: \( \quad \text{if } C_{12} \text{ and } C_{21} \text{ is NOT all 0 matrix then} \quad \triangleright \text{Counter-diagonal matrix is all 0} \)

13: \( \quad \quad \text{Status} \leftarrow 0 \)

14: \( \quad \text{else if } C_{11} \text{ is NOT equal to } C_{22} \text{ then} \quad \triangleright \text{Ensure } C_{11} \text{ is a repeat of } C_{22} \)

15: \( \quad \quad \text{Status} \leftarrow 0 \)

16: \( \quad \text{else} \)

17: \( \quad \quad \text{if PosChecker}(C_{11}, \text{index} - 1) \text{ then} \quad \triangleright \text{Recursion process sub-matrix} \)

18: \( \quad \quad \quad \text{return } \text{Status} \leftarrow 0 \)

19: \( \quad \quad \text{else} \)

20: \( \quad \quad \quad \text{return } \text{Status} \leftarrow 1 \)

21: \( \text{else} \)

22: \( \text{function } \text{Status} = \text{CheckPosLastDimN}(H) \quad \triangleright \text{If the dimension of input matrix great or equal to 4, conduct this process; otherwise return true} \)

23: \[
\begin{pmatrix}
C_{11} & C_{12} \\
C_{21} & C_{22}
\end{pmatrix} = H
\]

where \( \text{dim}(C_{11}) = \text{dim}(C_{12}) = \text{dim}(C_{21}) = \text{dim}(C_{22}) = \frac{1}{2} \times \text{dim}(H) \)

24: \( \quad \text{if } C_{11}, C_{12}, C_{21} \text{ and } C_{22} \text{ are all diagonal matrix with only 1 element then} \)

25: \( \quad \quad \text{Status} \leftarrow 0; \)

26: \( \quad \text{else} \)

27: \( \quad \quad \text{Status} \leftarrow 1; \)
3. Approximate separation of multipartite gates

In this section, we turn to the approximate separation problem of multipartite gates.

\textbf{\textit{\(\varepsilon\)-approximate separation question}} Given a positive scalar \(\varepsilon\) and a multipartite quantum gate \(U \in \mathcal{U}(\bigotimes_{k=1}^{n} \mathcal{H}_k)\), whether or not there are local gates \(U_i \in \mathcal{U}(\mathcal{H}_i)\) such that

\begin{equation}
    d(U, \bigotimes_{i=1}^{n} U_i) < \varepsilon,
\end{equation}

where \(d(\cdot, \cdot)\) is a distance of two operators. We call \(U\) is \(\varepsilon\)-approximate separable if Eq. 3.1 holds true. Further, how to find these local gates \(U_i\)?

Remark 3.1. Note that the set of tensor products of local unitary gates \(\mathcal{U}_l = \{ \bigotimes_{i=1}^{n} U_i | U_i \in \mathcal{U}(\mathcal{H}_i) \}\) is closed. It follows that there exists some positive number \(\varepsilon_0\) \(d(U, \mathcal{U}_l) = \varepsilon_0 > 0\) if \(U\) is not separable. So Eq. 3.1 holds true only if \(\varepsilon\) is greater than \(\varepsilon_0\). This implies that \(\varepsilon\) can not be chosen freely.

To answer the \(\varepsilon\)-approximate separation question, we need to estimate the upper bound of the distance \(d(U, \bigotimes_{i=1}^{n} U_i)\). In the following theorem, we pay our attention to this task.

**Theorem 3.2.** For any real number \(t\), let \(U = \exp[it\hat{H}] \in \mathcal{U}(\bigotimes_{k=1}^{n} \mathcal{H}_k)\) be a multipartite quantum gate with \(\hat{H} \in \mathcal{B}_s(\bigotimes_{k=1}^{n} \mathcal{H}_k)\) and \(U_k = \exp[it\hat{H}_k] \in \mathcal{U}(\mathcal{H}_k)\) with \(\hat{H}_k \in \mathcal{B}_s(\mathcal{H}_k)\). Then,

(I)

\begin{equation}
    \| U - \bigotimes_{k=1}^{n} U_k \| \leq M \| \hat{H} - \sum_k \hat{H}_k \|,
\end{equation}

where \(\hat{H}_k = I_1 \otimes I_2 \otimes \ldots I_{k-1} \otimes \hat{H}_k \otimes I_{k+1} \otimes \ldots I_n\), \(I_j\) is the identity on \(\mathcal{H}_j\), \(M = |t| \| \exp[-it\sum_{k=1}^{n} \hat{H}_k] \| \| \exp[-it\hat{H}] \|\) and \(\| \cdot \|\) is arbitrary a given norm of the operator.

(II) If the norm is chosen as the uniform operator norm \(\| \cdot \|_o\), then

\begin{equation}
    \| U - \bigotimes_{k=1}^{n} U_k \|_o \leq |t| \| \hat{H} - \sum_k \hat{H}_k \|_o.
\end{equation}

Remark 3.3. The norm \(\| \cdot \|\) in Eq. 3.2 can be selected freely. For example in Eq. 3.2 when we choose the uniform operator norm \(\| \cdot \|_o\) defined by \(\| A \|_o = \sup_x \| Ax \|\), then \(M = |t|\), since \(\| \exp[-it\sum_{i=1}^{n} \hat{H}_i] \|_o = 1 = \| \exp[-it\hat{H}] \|_o\). So Eq. 3.2 can be simplified as Eq. 3.3. In the finite dimensional case, the norm \(\| \cdot \|\) can be selected as arbitrary a matrix norm, including the trace norm and the Hilbert-Schmidt norm.
To prove Theorem 3.2, we need two lemmas. The first lemma is obvious by Theorem 2.4.

**Lemma 3.4.** For self-adjoint operators $A_i$s and real number $t$, $\otimes_{i=1}^{n} \exp[-itA_i] = \exp[\sum_{i=1}^{n}(-it\hat{A}_i)]$, where $\hat{A}_i = I_1 \otimes I_2 \otimes \ldots I_{i-1} \otimes A_i \otimes I_{i+1} \otimes \ldots I_n$.

**Lemma 3.5.** $\exp[A + B] - \exp[A] = \int_{0}^{1} \exp[(1 - x)A]B \exp[x(A + B)]dx$.

**Proof of Theorem 3.2** According to the assumptions, it follows from Lemma 3.4 and 3.5 that

$$\|U - \otimes_{i=1}^{n} U_i\| = \|\exp[-it\hat{H}] - \otimes_{i=1}^{n} \exp[-it\hat{H}_i]\|$$

$$= \|\exp[-it\hat{H}] - \exp[\sum_{i=1}^{n}(-it\hat{H}_i)]\|$$

$$= \|\int_{0}^{1} \exp[(1 - x)(-it\hat{H}_i)](-it\hat{H} - \sum_{i=1}^{n}(-it\hat{H}_i))\exp[x(-it\hat{H})] \, dx\|$$

$$\leq \|\exp[-it\hat{H}_i]\|\| - it\hat{H} - \sum_{i=1}^{n}(-it\hat{H}_i)\|\|\exp[-it\hat{H}]\|$$

$$= |t|\|\exp[-it\sum_{i=1}^{n}\hat{H}_i]\|\|\hat{H} - \sum_{i=1}^{n}\hat{H}_i\|\|\exp[-it\hat{H}]\|. $$

Let $M = |t|\|\exp[-it\sum_{i=1}^{n}\hat{H}_i]\|\|\exp[-it\hat{H}]\|$, we complete the proof. □

Theorem 3.2 will be helpful to answer the $\epsilon$-approximate separation question. To arrive at the approximate separation for a given approximate bound $\epsilon$ and a multipartite gate $U = \exp[it\hat{H}] \in \mathcal{U}(\otimes_{k=1}^{n} H_k)$ with $\dim(H_k) < +\infty$, we need to find self-adjoint operators $\hat{H}_i$ such that in Eq. 3.2

$$\|\hat{H} - \sum_{i=1}^{n}\hat{H}_i\| < \frac{\epsilon}{M}. $$

(3.4)

Next we propose another kind of answers to the $\epsilon$-approximate separation question of multipartite unitary gates in the finite dimensional case. This result refines that in Theorem 3.2.

**Theorem 3.6.** For given positive scalar $\epsilon$ and multipartite quantum gate $U = \exp[it\hat{H}] \in \mathcal{U}(\otimes_{k=1}^{n} H_k)$ with $\dim(H_k) = m_k < \infty$, there exist unitary operators $U_k = \exp[it\hat{H}_k] \in \mathcal{U}(H_k)$ such that $\|U - \otimes_{i=1}^{n} U_i\|_o < \epsilon$ if

$$\text{tr}(M_j|x_j\rangle\langle x_j|M_j^\dagger) < \left(\frac{\epsilon}{|t| \prod_{k=1}^{n} m_k}\right)^2, \ j = 1, 2, \ldots,$$

(3.5)
where \( \{|x_j\rangle\}_{j=1}^{\prod_{k=1}^{n} m_k} \) is the orthonormal basis of \( \bigotimes_{k=1}^{n} H_k \) consists of all eigenvectors of \( U \) and \( U|x_j\rangle = e^{it\lambda_j}|x_j\rangle \), \( M_j = \lambda_j I - \sum_{i=1}^{n} \hat{H}_i, \hat{H}_i = I_1 \otimes I_2 \otimes \ldots \otimes I_{i-1} \otimes H_i \otimes I_{i+1} \otimes \ldots I_n \), \( I_j \) is the identity on \( H_j \), \( \| \cdot \|_o \) denotes the uniform operator norm and \( \dagger \) means the composition of the conjugation and transpose.

Remark 3.7. As \( \text{tr}(M|x_j\rangle\langle x_j|M^\dagger) = \| (\lambda_j I - \sum_{i=1}^{n} \hat{H}_i)|x_j\rangle \|^2 \), so Eq. (3.5) is equivalent to
\[
(3.6) \quad \| (\lambda_j I - \sum_{i=1}^{n} \hat{H}_i)|x_j\rangle \| < \frac{\epsilon}{\| \prod_{k=1}^{n} m_k \|}, \ j = 1, 2, \ldots, \prod_{k=1}^{n} m_k.
\]
Moreover, different from Theorem 3.2, to answer the \( \epsilon \)-approximate separation question based on Theorem 3.6, it does not need to find the \( H \). This may help to reduce the computational complexity.

To prove Theorem 3.6, we need some more lemma. Let us recall some notations on the matrix norms. A matrix norm \( \| \cdot \| \) is unitary invariant if \( \| UAV \| = \| A \| \) holds for any unitary matrices \( U, V \) and any matrix \( A \); and is called unitary similarity invariant if \( \| UAU^\dagger \| = \| A \| \) holds for any unitary matrix \( U \) and any matrix \( A \). The matrix norm \( \| \cdot \|_c \) is called a cross norm if \( \| \cdot \|_c \) is unitary invariant and \( \| A \otimes B \|_c = \| A \|_c \| B \|_c \) holds for all matrices \( A, B \). Recall that the Schatten-\( p \) norm of \( A \) is defined by
\[
\| A \|_p = \text{tr}((A^\dagger A)^{\frac{p}{2}})^{\frac{1}{p}}.
\]
The Schatten-\( p \) norm and uniform operator norm are examples of cross norms.

Lemma 3.8. For any bounded linear operator \( X \) and self-adjoint operators \( A, B \), we have \( \| \exp[iA]X - X \exp[iB]\|_c \leq \| AX - XB \|_c \).

Proof. It is not difficult to show that for any bounded linear operator \( X \) and self-adjoint operators \( A, B \) on the Hilbert space \( H \), we have
\[
\exp[iA]X \exp[-iB] - X = \int_0^1 i \exp[itA](AX - XB) \exp[-itB]dt
\]
(see [37]). Since the cross norm is unitarily invariant,
\[
\| \exp[iA]X \exp[iB] \|_c = \| \exp[iA]X \exp[-iB] - X \|_c
\]
\[
= \| \int_0^1 i \exp[itA](AX - XB) \exp[-itB]dt \|_c
\]
\[
\leq \| \exp[itA](AX - XB) \exp[-itB] \|_c
\]
\[
= \| AX - XB \|_c,
\]
completing the proof.

Proof of Theorem 3.6. To complete the proof, it is enough to check the following implication: Eq. (3.6) \( \Rightarrow (1) \Rightarrow (2) \Rightarrow (3) \). Where
We complete the proof. □

Furthermore, note that for arbitrary unit vector \(|x_j\rangle\in\mathbb{C}^d\), that (1) holds true. To check (1) \(\Rightarrow\) (3). To prove Eq. 3.6 \(\Rightarrow\) (1), assume that

\[||\lambda_j|x_j\rangle - \left(\sum_{i=1}^{n} \hat{H}_i\right)|x_j\rangle|| < \frac{\epsilon}{|t|\prod_{k=1}^{m_k}};\]

then

\[\|H|x_j\rangle\langle x_j| - \left(\sum_{i=1}^{n} \hat{H}_i\right)|x_j\rangle\langle x_j||o = \|H|x_j\rangle\langle x_j| - \left(\sum_{i=1}^{n} \hat{H}_i\right)|x_j\rangle|| \]

\[< \frac{\epsilon}{|t|\prod_{k=1}^{m_k}}.\]

Furthermore, note that \(|x_j\rangle\) is the eigenvector of \(H\). So

\[\|(x_j\rangle\langle x_j|H - \left(\sum_{i=1}^{n} \hat{H}_i\right)|x_j\rangle\langle x_j||o = \|H|x_j\rangle\langle x_j| - \left(\sum_{i=1}^{n} \hat{H}_i\right)|x_j\rangle\langle x_j||o \]

It follows from Lemma 3.8 that

\[\|\exp[\imath H]|x_j\rangle - \exp[\sum_{i=1}^{n} \imath \hat{H}_i]|x_j\rangle|| \]

\[= \|x_j\rangle\langle x_j|\exp[\imath H] - \exp[\sum_{i=1}^{n} \imath \hat{H}_i]|x_j\rangle\langle x_j||o \]

\[< \frac{\epsilon}{|t|\prod_{k=1}^{m_k}};\]

that is, (1) holds true.

To check (1) \(\Rightarrow\) (2), note that \(\{|x_j\rangle\}_{j=1}^{n} \subset H_k\) is a orthonormal basis of \(\otimes_{k=1}^{n} H_k\). So for arbitrary unit vector \(|x\rangle\in\otimes_{k=1}^{n} H_k\), it can be represented as \(|x\rangle = \sum_{j=1}^{n} \alpha_j|x_j\rangle\).

Obviously, \(|\alpha_j| \leq 1\) as \(||x|| = 1\). Then, it follows from (1) that

\[\|\exp[\imath H]|x\rangle - \exp[\sum_{i=1}^{n} \imath \hat{H}_i]|x\rangle|| \]

\[= \sum_{j=1}^{n} \alpha_j \exp[\imath \hat{H}_i]|x_j\rangle - \sum_{j=1}^{n} \alpha_j \exp[\sum_{i=1}^{n} \imath \hat{H}_i]|x_j\rangle || \]

\[\leq \sum_{j=1}^{n} \alpha_j \||x_j\rangle - \exp[\sum_{i=1}^{n} \imath \hat{H}_i]|x_j\rangle || \]

\[\leq (\prod_{k=1}^{n} m_k)||x\rangle - \exp[\sum_{i=1}^{n} \imath \hat{H}_i]|x\rangle || < \epsilon.\]

We complete the proof.

4. Conclusion and Discussion

We established a number of evaluation criteria for the separability of multipartite gates. These criteria demonstrate that almost all \(A \in \{A_i\}_{i=1}^{n}\) should belong to \(\mathbb{R}I\) for a separable multipartite gate \(U = \exp[\imath H]\), where \(H = A_1 \otimes A_2 \otimes \ldots \otimes A_n\). Most
of random multipartite gates cannot fundamentally satisfy the separability condition in Theorem 2.4. We devoted to the existing of the infimum of the gap between $U$ and local gate $U_i$ and illustrated the search algorithm approaching to arbitrary unitary gate using local gates. Moreover, as examples, the very practical two-qubits composite spin-$\frac{1}{2}$ system is introduced and used for checking the criteria.

This work reveals that there are very few quantum computational tasks (quantum circuits) that can be automatically parallelized. Concurrent quantum programming and parallel quantum programming still needs to be researched for a greater understanding of quantum specific features concerning the separability of quantum states, local operations and classical communication and even quantum networks.

The further interesting task is to generalize Algorithm 2.1 to the higher dimensional case and design the algorithms for approximate separation of multipartite gates.

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