We predict a \(p\)-wave Cooper pairing of the spin-polarized fermions in a binary fermion-boson mixture due to the exchange of density fluctuations of the bosonic medium. We then examine the dependence of the Cooper paring temperature on the parameters of the system. We finally estimate the effect of combining the boson-induced interaction with other pairing mechanisms, e.g., the Kohn-Luttinger one, and find that the critical temperature of \(p\)-wave Cooper pairing can be realistic for experiment.

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In general (apart from the case of resonance scattering) the harmonics of the scattering amplitude for slow particles are proportional to \( f_t \sim a (\varepsilon_F)^{2\ell} \), where \( a \) is a length of the order of the 3-scattering length. For example, the \( \ell = 1 \) bare contribution goes like \( U_{FF}^2 \nu_F \sim (\varepsilon_F)^3 \). This contribution is very small and can in general be neglected if some other triplet pairing mechanism is present.

In the case of fermions in two spin states for instance, the indirect interaction by polarisation of the fermions in the other spin state (Kohn-Luttinger mechanism) \([15]\), provides a contribution of order \((\varepsilon_F)^2\), and is therefore more important than the bare one.

A standard procedure \([16]\) yields the critical temperature for pairing with given angular momentum \( \ell \):

\[
T_{c\ell} = \xi_F \exp \left\{ -\frac{1}{|\bar{\nu}_F| \bar{\Gamma}_\ell} \right\}, \tag{2}
\]

with \( \bar{\Gamma}_\ell < 0 \) being the \( \ell \)-th spherical harmonic of the irreducible vertex, \( \xi_F \) is some energy parameter of the order of the Fermi energy, and \( \nu_F \) the density of states on the Fermi level. The real transition corresponds to the angular momentum with the maximum allowed temperature. The effective interaction between two Fermi particles \( \bar{\Gamma} \) is the sum of the bare one \( U_{FF} \), the interaction via polarisation of the bosonic medium (exchange of density fluctuations) \( U_{BFBF} \), and possibly that via polarisation of the other fermionic species \( U_{FFBF} \) if fermions in more than one spin orientation are present.

We assume temperatures much smaller than those of degeneracy. The correctness of this hypothesis will be confirmed by the results found. The effective interaction of Fermi quasi-particles on mass-surface with zero transfer energy is given by \( U_{BFBF}(q) = U_{BF}^2 \chi(q, \omega = 0) \), where \( \chi(q, \omega) \) is the density-density response function for an almost ideal Bose-gas at zero temperature \([17]\). Since we are interested in the low density limit of Bose and Fermi-gases and \( U_{BB} \gg U_{BF} \sim (a_{BF}p_B/U_{BF}) \), we can neglect the renormalization of boson density-density correlation function due to Bose-Fermi interaction, and we can write to first order in the gas parameter \([13]\):

\[
\chi(q, \omega) = \frac{1}{\omega^2 - \varepsilon_0(q)(\varepsilon_0 + 2n_B U_{BB})} \frac{n_B q^2}{m_B}, \tag{3}
\]

So the effective interaction of Fermi atoms with zero transfer energy reads:

\[
U_{BFBF}(q, 0) = -\frac{U_{BF}^2}{U_{BB}} \left( 1 + \left( \frac{q}{2m_B s} \right)^2 \right)^{-1}, \tag{4}
\]

where \( s = (n_B U_{BB}/m_B)^{1/2} \) is the sound velocity in the boson gas. We recall that stability requires \( U_{BB} > 0 \).

A direct calculation of the first three partial components of \( U_{BFBF} \) gives the following results:

\[
\nu_F U_{BFBF}^\ell - \nu_F U_{BF}^2 R_\ell (p_F/m_B s), \tag{5}
\]

with

\[
R_0(x) = \frac{\ln(1+x^2)}{x^2}, \quad R_1(x) = \frac{2}{x^2} \left[ \left( \frac{1}{x^2} + \frac{1}{2} \right) \ln(1+x^2) - 1 \right], \quad R_2(x) = \frac{6}{x^4} \left[ \frac{1}{x^2} + 1 + \frac{x^2}{6} \right] \ln(1+x^2) - \left( 1 + \frac{x^2}{2} \right). \]

FIG. 1. Functions \( R_0(x) \) – solid line; \( R_1(x) \) – dotted and \( R_2(x) \) – dashed.

The functions \( R_0(x), R_1(x) \) and \( R_2(x) \) are shown in Fig 1. The strongest interaction is in the channel with orbital momentum \( \ell = 0 \). For large \( \ell \) one can show that \( R_\ell \) drops off exponentially in \( \ell \). Therefore this contribution to the effective interaction for \( \ell > 2 \) can be neglected. The functions for \( \ell \neq 0 \) are strongly non-monotonic in contrast with zero angular momentum case. For instance, the maximum of \( R_1(x) \) is obtained for \( x_{opt} = 1.86 \) (\( R_1(1.86) = 0.1 \)). The maximum gives the optimal ratio of Bose and Fermi-components for given scattering lengths.

Let us consider a binary gas consisting of one fermionic and one bosonic species. The Cooper pairing in the s-wave channel is prohibited by the Pauli principle. It can exist owing to density fluctuations of bosonic medium with effective attractive interaction in p-wave channel given by formula \([8]\). Note that the value of the ratio \( U_{BF}^2/U_{BB} \) and the densities of the gases cannot be arbitrary. The restriction is associated with the phase separation at high densities of the binary mixture into two regions: a Fermi–Bose mixture (with densities \( n_B^{sep} \) and \( n_{F1} \)) and a pure fermionic region (with density \( n_{F2} \)). This phase separation into two large regions is a full analog of that observed in the mixtures of \(^3\)He–\(^4\)He. To check the stability of the mixture against phase separation we rewrite the expression for \( U_{BFBF} \) in dimensionless parameters \( \lambda, \alpha \) and \( \beta \) in the spirit of ref. \([20]\):

\[
\lambda = \frac{\nu_F U_{BF}^2}{U_{BB}} = \frac{2}{\pi} m_{BF} a_{BF} \frac{m_B a_{BF} a_{PF}}{m_{BF}}. \tag{6}
\]
The expression for $\lambda$ is exactly the factor in front of $R_\ell$ in (5).

$$n_B^{sep} = \frac{\varepsilon_F}{U_{BF}}(y^2 - 1) = \frac{(6\pi^2 n_F)^{2/3} m_B}{8\pi a_{BF}} m_F (y^2 - 1), \quad (7)$$

where $y \geq 1$ is solution to the equation

$$-\frac{15}{\lambda}(y + 1)^2 + 8y^3 + 16y^2 + 24y + 12 = 0. \quad (8)$$

We then find

$$\frac{1}{x^2} = \frac{\beta(y(\lambda)^2 - 1)}{\lambda}, \quad (9)$$

where

$$\beta = \frac{\alpha}{\pi} \frac{m_B}{m_F} p_F a_{BF}, \quad \alpha = \frac{n_B}{n_B^{sep}(\lambda)}.$$

The physical meaning of the introduced variables is the following. In the case of phase separation $y$ is the ratio of fermionic densities in the two regions $y = n_{F2}/n_{F1} \geq 1$ and $n_B^{sep}$ is the density of bosonic component in bosonic-fermionic mixture region. The problem of phase separation of binary mixtures of boson and fermion gases was investigated in ref. [20]. The authors have shown that there are three possibilities: a) a single uniform phase; b) a purely fermionic phase coexisting with a mixed phase; c) a purely fermionic phase coexisting with a purely bosonic. Let us examine all three possibilities.

The single uniform phase is stable provided the conditions $\lambda \leq 1$ and $\alpha \leq 1$ are fulfilled. This immediately gives $\nu_F U_{BF}/U_{BB} < 1$ and the value of the effective interaction is restricted by $U_{BF} < 0.1$, which corresponds to temperatures of Cooper pairing in the binary mixtures about 5 orders of magnitude less than the Fermi energy. In figure 2 we plot the critical temperature as a function of $\lambda$ for given $\beta$. We see, that the maximum of the critical temperature increases with increasing of $\beta$. Note that for given scattering lengths the maximum is in region of the parameters close to the phase separation ($\alpha = n_B/n_B^{sep} \rightarrow 1$).

If the total density of the boson gas is larger than $n_B^{sep}$ ($\alpha > 1$), the binary mixture undergoes phase separation into two phases: a purely fermionic phase and the mixed Fermi-Bose phase. In this case in the mixed phase the density of the bosonic gas is $n_B = n_B^{sep}$ and the density of fermi gas $n_{F1}$. These are determined by the system of the equations (5)–(9). Our result obtained for the single uniform phase is still within the mixed phase valid if the appropriate densities of fermi and boson gases are used.

The third possibility is the coexistence of a purely fermionic phase with a purely bosonic one, which exists for much higher total densities of bosons and fermions. In this case of course there is no effective interaction between fermions due to the exchange of boson density fluctuations.

We can conclude that the contribution of the exchange density fluctuations of the bosonic medium has its maximum for the set of parameters close to those of phase separation of a single uniform phase into two coexisting phases: a mixed phase and a pure fermionic one.

Let us make some estimates for real systems. Take a binary mixture of fermionic $^6$Li and bosonic $^{87}$Rb.

![FIG. 2. $T_c/\varepsilon_F$ versus $\lambda$ for binary boson-fermion mixture for different values of the coefficient $\beta$. The solid curve is for $\beta = 5$, the dashed one for $\beta = 3$, and the dotted one for $\beta = 1$.](image)

![FIG. 3. Optimal scattering length $a_{BF}$ and corresponding $T_c$ for Cooper pairing, versus fermionic density in a binary boson-fermion mixture of $^6$Li and $^{87}$Rb.](image)
arameters of the system and the corresponding critical temperatures for $p$-wave pairing in a binary mixture.

There is a possibility of increasing $T_c$ by combining the above mentioned mechanism with either $p$-wave quasi-bound resonance for the scattering of Fermi-atoms, or by considering a mixture of two spin states of fermions with one of bosons. In the former case the irreducible vertex in (3) is determined by the sum of the interactions due to polarisation of bosons and the (now large) bare $p$-wave attractive scattering of Fermi-atoms [23].

In the case of two species of fermions with one of bosons, again the effective interaction has two contributions: from boson density fluctuations and from the Kohn-Luttinger mechanism. The effective interaction in the latter is a non-monotonic function of the ratio of the densities of the different hyperfine components (see [10,11]). Its maximum is $\nu_F U_{\text{eff}} \sim 0.058(a_F)^2$ which corresponds to a ratio $n_1 \sim 2.8n_2$.

For optimal parameters [24] and a density $n_1 = 10^{14}\text{cm}^{-3}$ the critical temperature reads:

$$T_c \sim \bar{\epsilon}_F \exp \left\{ -\frac{1}{\nu_F U_{BF} + 0.058(a_F^* p_F)^2} \right\},$$

$T_c \sim 1\text{mK}$ and $T_c \sim 20\text{mK}$ respectively for bare $s$-wave Fermi-Fermi scattering lengths $|a_F^*| = 500a_0$ and $1000a_0$. Note that the value of the critical temperature obtained is valid for $a_F^* > 0$ as well as for $a_F^* \leq 0$. Pure Kohn-Luttinger mechanism would give $T_c \sim 10^{-3}\text{mK}$ and $10^{-5}\text{mK}$ respectively for the given scattering lengths, so that the main contribution comes from the boson-induced term. For $^6\text{Li}$ however the $s$-wave scattering length between two different hyperfine states is $a_F^* = -2160a_0$. The critical temperature with this scattering length is $\sim 0.5\mu\text{K}$. For pure Kohn-Luttinger mechanism it would have been $\sim 0.1\mu\text{K}$, which shows the strong effect that the bosons have also in this case. The full analogy with mixtures $^3\text{He} - ^4\text{He}$ says that here also both single uniform phase and phase separated states are possible, and explicit calculations for the case of two-Fermi species and a Bose one need to be carried out.

In conclusion we showed that the fermions in a (typical) dilute binary mixtures of Fermi and Bose gases are unstable towards $p$-wave Cooper pairing. This is due to their effective attraction arising from boson polarisation. We then calculated how the associated $T_c$ can be maximized. Although the highest $T_c$'s found don't seem presently experimentally observable, we showed that the mechanism may be used to enhance pairing when combined with others.

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