Nonfreeness of some algebras of hermitian modular forms.

Stuken Ekaterina
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Abstract

We study the algebras of hermitian automorphic forms for the lattice $L_n = \text{diag}(1, 1, \ldots, 1, -1)$ and for the field $K = \mathbb{Q}(\sqrt{-d})$ such that $p = 2$ is unramified and the ring of integers $\mathcal{O}_K$ is a p.i.d. We prove that for $d > 7$ these algebras can’t be free. When $d = 7$ and $d = 3$ we give an estimate for the dimension of the symmetric spaces for which these algebras might be free. We also compare our results with the known results for $d = 3$.

Introduction

Let $K = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field, $d$ is odd and square free. We restrict ourselves to the case when the ring of integers $\mathcal{O}_K$ is a principal ideal domain and $p = 2$ is unramified. Let $L$ be an integral hermitian lattice of signature $(n, 1)$, $n > 1$. Let $V$ be the vector space $V = L \otimes_{\mathcal{O}_K} \mathbb{C}$. Denote by $\Gamma$ the group $U(L)$. We consider the Hermitian symmetric domain $D = \{(z) \in P(V) | (z, z) < 0\}$ and a principal $\mathbb{C}^*$-bundle $A = \{z \in V | [z] \in D\}$.

Definition 1. Automorphic form of weight $k \in \mathbb{Z}$, $k > 0$ for $\Gamma$ with character $\chi : \Gamma \to \mathbb{C}^*$ is a holomorphic function $f$ in $A$ such that:
1) $f(tz) = t^{-k}f(z)$, $t \in \mathbb{C}^*$
2) $f(g(z)) = \chi(g)f(z)$, $g \in \Gamma$.

Here we prove the following theorem:

Theorem 1. Let $K = \mathbb{Q}(\sqrt{-d})$ be a quadratic field such that it’s ring of integers is a p.i.d. and $p = 2$ is unramified. Consider the lattice $L_n = \text{diag}(1, 1, \ldots, 1, -1)$ of signature $(n, 1)$. Let $A_n$ be the algebra of automorphic forms for the group $U(L_n)$. Then:

- if $d > 7$ then the algebra $A_n$ can’t be free;
- if $d = 7$ than the algebra $A_n$ can be free only for $n \leq 4$;
- if $d = 3$ than the the algebra $A_n$ can be free only for $n \leq 7$.

∗Chebyshev Laboratory, St. Petersburg State University, 14th Line V.O., 29B, Saint Petersburg 199178 Russia.
To prove this theorem we consider the invariant $K(\Gamma)$ for the group $\Gamma$:

$$K(\Gamma) := \frac{\sum_{[\pi]} \frac{n_\pi}{\Vol_{HM}(B^{n-1}/\Gamma_x)}}{\Vol_{HM}(B^n/\Gamma)}$$

where $\Gamma_x$ is the stabiliser of the mirror $\pi$ in the group $\Gamma$, the sum is taken over all $\Gamma$-conjugacy classes of the mirrors of reflections $\pi$ in the group $\Gamma$, $n_\pi$ is the order of the reflection in $\pi$.

We will need one theorem of J. H. Bruinier ([2]). In the case of interest to us and in a form convenient for us it states the following:

**Theorem A.** Let $F$ be a meromorphic automorphic form for the group $\Gamma$ of weight $K$ such that it has zeros on all mirrors of reflections in $\Gamma$ and only on them. Denote by $n_\pi$ the order of the reflection in $\pi$. If each zero has the order $n_\pi - 1$, then $K(\Gamma) = K$.

We also use another theorem of H. Aoki, T. Ibukiyama ([1]):

**Theorem B.** If the algebra of $\Gamma$-automorphic forms is free with the generators of weights $k_1, k_2, \ldots, k_{n+1}$, then there exists unique up to the proportionality $\Gamma$-automorphic form $F$ of weight $n + 1 + k_1 + k_2 + \ldots + k_{n+1}$ whose divisor is $\div(F) = \sum(n_\pi - 1)\pi$ (the sum is taken over all mirrors of reflections in the group $\Gamma$).

We apply this theorem to the lattice $L_n$ and the group $\Gamma = U(L_n)$. If the algebra $A_n$ is free then it has $n + 1$ generators of weight at least 1. Hence the weight $K$ of the form $F$ from theorem $B$ for $\Gamma$ is not less then $2n + 2$. Applying to the form $F$ theorem $A$ we get that $K(\Gamma) \geq 2n + 2$.

The idea of the proof of theorem [1] is to estimate in formula (1) the value of $K(\Gamma)$. We show that the denominator of $K(\Gamma)$ grows much faster than the numerator, thus $K(\Gamma)$ is less than $2n + 2$ when $n$ is great enough.

H. Wang and B. Williams in [5] proved that for $d = 3$ the algebra $A_n$ is free when $n \leq 4$ and computed the weights of generators. One can easily check that these weights correspond to our calculations of the value $K(\Gamma)$, i.e. $K(\Gamma) = K = n + 1 + k_1 + k_2 + \ldots + k_{n+1}$.

Let $L_n = \text{diag}(1, 1, \ldots, 1, -1)$ and $M_n = \text{diag}(1, 1, \ldots, 1, -2)$ be the lattices of signature $(n, 1)$. Consider the groups $SU(L_n, O_K)$ and $SU(M_n, O_K)$ for the ring of integers $O_K$ of the field $K$. These are discrete arithmetic groups naturally acting on a complex ball $B^n$. Their quotient spaces have finite volumes. We will write $\Vol(L_n)$ and $\Vol(M_n)$ respectively for the Hirzebruch-Mumford covolume of the spaces $B^n/SU(L_n, O_K)$ and $B^n/SU(M_n, O_K)$.

To prove theorem [1] we need to know the values of covolumes $\Vol(L_n)$ and $\Vol(M_n)$. The first one was calculated in [6], the second in [4]. Denote by $L(p)$ the $L$-function of the quadratic field $K$ with character $\chi_D(p) = \left(\frac{D}{p}\right)$ (Kronecker symbol).
Theorem 2. The Hirzebruch-Mumford covolume of $\mathbb{B}^n/SU(L_n,\mathcal{O}_K)$ is

| $n$ | $D$ | $\text{Vol}(L_n)$ |
|-----|-----|----------------|
| even | -4d | $D \frac{n^2 + 3n}{4} \cdot \prod_{j=1}^{n} \left( \frac{j!}{(2\pi)^{j+1}} \right) \cdot \zeta(2) \cdot L(3) \cdot \zeta(4) \cdot L(5) \cdot \ldots \cdot L(n+1)$ |
| even | -d  | $D \frac{n^2 + 3n}{4} \cdot \prod_{j=1}^{n} \left( \frac{j!}{(2\pi)^{j+1}} \right) \cdot \zeta(2) \cdot L(3) \cdot \zeta(4) \cdot L(5) \cdot \ldots \cdot L(n+1)$ |
| odd  | -4d | $D \frac{n^2 + 3n}{4} \cdot (1 - 2^{-(n+1)}) \cdot \prod_{p|d} \left( 1 + \left( \frac{(-1)^{(n+3)/2}}{p} \right) \cdot p^{-\frac{n+1}{2}} \right) \prod_{j=1}^{n} \left( \frac{j!}{(2\pi)^{j+1}} \right) \cdot \zeta(2) \cdot \ldots \cdot \zeta(n+1)$ |
| odd  | -d  | $D \frac{n^2 + 3n}{4} \cdot \prod_{p|d} \left( 1 + \left( \frac{(-1)^{(n+3)/2}}{p} \right) \cdot p^{-\frac{n+1}{2}} \right) \prod_{j=1}^{n} \left( \frac{j!}{(2\pi)^{j+1}} \right) \cdot \zeta(2) \cdot L(3) \cdot \ldots \cdot \zeta(n+1)$ |

Theorem 3. The Hirzebruch-Mumford covolume of $\mathbb{B}^n/SU(M_n,\mathcal{O}_K)$ is

| $n$ | $D$ | $\text{Vol}(M_n)$ |
|-----|-----|----------------|
| even | -4d | $D \frac{n^2 + 3n}{4} \cdot \prod_{j=1}^{n} \left( \frac{j!}{(2\pi)^{j+1}} \right) \cdot \zeta(2) \cdot L(3) \cdot \zeta(4) \cdot L(5) \cdot \ldots \cdot L(n+1) \cdot (2^n - 1)$ |
| even | -d  | $D \frac{n^2 + 3n}{4} \cdot \prod_{j=1}^{n} \left( \frac{j!}{(2\pi)^{j+1}} \right) \cdot \zeta(2) \cdot L(3) \cdot \zeta(4) \cdot L(5) \cdot \ldots \cdot L(n+1) \cdot 2^n \cdot \frac{1 - \left( \frac{\zeta(2)}{2} \right)^{n+1}}{1 - \left( \frac{\zeta(2)}{2} \right)^{2^{n+1}}}$ |
| odd  | -4d | $D \frac{n^2 + 3n}{4} \cdot (1 - 2^{-(n+1)}) \cdot \prod_{p|d} \left( 1 + \left( \frac{(-1)^{(n+3)/2}}{p} \right) \cdot p^{-\frac{n+1}{2}} \right) \prod_{j=1}^{n} \left( \frac{j!}{(2\pi)^{j+1}} \right) \cdot 2^n \zeta(2) \cdot \ldots \cdot \zeta(n+1)$ |
| odd  | -d  | $D \frac{n^2 + 3n}{4} \cdot \prod_{p|d} \left( 1 + \left( \frac{(-1)^{(n+3)/2}}{p} \right) \cdot p^{-\frac{n+1}{2}} \right) \prod_{j=1}^{n} \left( \frac{j!}{(2\pi)^{j+1}} \right) \cdot 2^n \frac{1 - \left( \frac{\zeta(2)}{2} \right)^{n+1}}{1 - \left( \frac{\zeta(2)}{2} \right)^{2^{n+1}}} \zeta(2) \cdot \ldots \cdot \zeta(n+1)$ |

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Bruinier formula for principal ideal rings

The ring of integers of the field $K$ is a principal ideal ring for

$$d \in \{ 1, 2, 3, 7, 11, 19, 43, 67, 163 \}.$$

We consider the case where $p = 2$ is unramified, so

$$d \in \{ 3, 7, 11, 19, 43, 67, 163 \}.$$

Definition 2. Vector $e \in L$ is called a root if $(e,e) > 0$ and the reflection in the subspace $e^\perp$ belongs to $\Gamma$. The length of the root $e$ is the number $(e,e)$.

The length of the roots of the lattice divides $2s$, where $s$ is the greatest invariant factor of the lattice. Since the lattice $L_n$ is unimodular, there only exist reflections in roots with length 1 and 2.

Definition 3. A vector $l$ in a lattice $L$ is called primitive if $K \cdot l \cap L = \mathcal{O}_K \cdot l$. 









Consider the number of orbits for vectors of a fixed length $a$:

$$N(L_n, a) = \{ l \in L_n | (l, l) = a \text{ and } l \text{ is primitive} \}/\text{Aut}(L_n).$$

We use the following theorem by D. James ([3]) and formulate it in a form, convenient to us:

**Theorem 4.** Let $K = \mathbb{Q}(\sqrt{-d})$ be a quadratic field such that $p = 2$ is unramified. Let $L$ be a unimodular lattice on an indefinite hermitian space with dimension $n \geq 3$. Let $a$ be a nonzero element in $\mathcal{O}_K$, represented by $L$. Then $|N(L, a)| = 1$.

Since in our case $\mathcal{O}_K$ is a p.i.d., all the roots of the lattice $L_n$ are primitive. Applying James’s theorem to the lattice $L_n$ we get that there is only one orbit for the roots with length 1 or 2. It means, that

- the stabiliser of the root with length 1 in the lattice $L_n$ is $U(L_n - 1, \mathcal{O}_K)$;
- the stabiliser of the root with length 2 in the lattice $L_n$ is $U(M_n - 1, \mathcal{O}_K)$.

First we consider the case $d \neq 3$. Each reflection has order 2 (the group of units is $\pm 1$). Then the invariant $K(\Gamma)$ is

$$K(\Gamma) = \frac{1}{2}\text{Vol}(L_{n-1}) + \frac{1}{2}\text{Vol}(M_{n-1}) \div \text{Vol}(L_n).$$

On the other hand, $K(\Gamma) \geq 2(n + 1)$ (because there is $n + 1$ generating function of weight at least 1 and $n + 1$ comes from the dimension). So, if

$$K(\Gamma) = \frac{1}{2}\text{Vol}(L_{n-1}) + \frac{1}{2}\text{Vol}(M_{n-1}) \div \text{Vol}(L_n) \leq 2n + 2,$$

then the corresponding algebra of automorphic forms can’t be free.

Now we consider the case $d = 3$. The difference from the previous case is that now the lattice $L_n$ is stable under the multiplication by $\epsilon E$, $\epsilon = \sqrt{3}$, $E$ is the identity matrix. It follows from the definition of an automorphic form that in this case the weight of all forms is divisible by 6.

The reflections in roots with square 1 have order 6 and the reflections in roots with square 2 have order 2. Then the invariant $K(\Gamma)$ is

$$K(\Gamma) = \frac{5}{6}\text{Vol}(L_{n-1}) + \frac{1}{6}\text{Vol}(M_{n-1}) \div \text{Vol}(L_n).$$

On the other hand $K \geq 7(n + 1)$ (because there is $n + 1$ generating function of weight at least 6 and $n + 1$ comes from the dimension). So, if

$$K(\Gamma) = \frac{5}{6}\text{Vol}(L_{n-1}) + \frac{1}{6}\text{Vol}(M_{n-1}) \div \text{Vol}(L_n) \leq 7n + 7,$$

then the corresponding algebra of automorphic forms can’t be free. To shorten the notation from now on we will use $K$ instead of $K(\Gamma)$.
The case of odd \( n \)

Consider odd \( n \). Then

\[
K = \frac{\frac{1}{2}D^{(n-1)/2} \cdot \prod_{j=1}^{n-1} \frac{d}{(2\pi)^{j+1}} \cdot \zeta(2) \cdot L(3) \cdot \zeta(4) \cdot L(5) \cdots \cdot L(n) \left( 1 + 2^{n-1} \cdot \frac{1-(-\frac{d}{d})^n}{1-(\frac{-d}{d})^{n-n}} \right)}{D^{\frac{n-1}{2}} \cdot \prod_{j=1}^{n} \frac{d}{(2\pi)^{j+1}} \cdot \zeta(2) \cdot L(3) \cdot \zeta(4) \cdot L(5) \cdots \cdot \zeta(n+1)}.
\]

We note that \( \frac{1-(-\frac{d}{d})^n}{1-(\frac{-d}{d})^{n-n}} = 1 \) for all \( d \neq 7 \) and \( \frac{-7}{2} = 1 \).

Moreover, \( \left( \frac{1-(-\frac{d}{d})^{n+1}}{d} \right) = 1 \), if \( n \equiv 1 \) (mod 4), and \( \left( \frac{1-(-\frac{d}{d})^{n+1}}{d} \right) = (-1)^{(n-1)/2} = -1 \), if \( n \equiv -1 \) (mod 4).

We get:

| \( n \) (mod 4) | \( d \) | \( K \) |
|-----------------|--------|----------------|
| \( n \equiv 1 \) | \( d \neq 7 \) | \( \frac{(2\pi)^{n+1} \left( 1 + \frac{d+1}{3} \right)}{2 \left( d^{(n+1)/2} + 1 \right) \cdot n! \cdot \zeta(n+1)} \) |
| \( n \equiv -1 \) | \( d \neq 7 \) | \( \frac{(2\pi)^{n+1} \left( 1 + \frac{d+1}{3} \right)}{2 \left( d^{(n+1)/2} - 1 \right) \cdot n! \cdot \zeta(n+1)} \) |
| \( n \equiv 1 \) | \( d = 7 \) | \( \frac{(2\pi)^{n+1} \cdot 2^n}{2 \left( d^{(n+1)/2} + 1 \right) \cdot n! \cdot \zeta(n+1)} \) |
| \( n \equiv -1 \) | \( d = 7 \) | \( \frac{(2\pi)^{n+1} \cdot 2^n}{2 \left( d^{(n+1)/2} - 1 \right) \cdot n! \cdot \zeta(n+1)} \) |

We note that \( \zeta(n+1) > 1 \), so, removing this multiplier from the denominator, we get an estimate from above for \( K \). Denote this by \( K' \):
Since when \( d \) is fixed, the numerator grows faster than the denominator, starting from some moment \( K' < 2n + 2 \).
First consider \( d = 7 \):
When \( n = 3 \) and \( n = 5 \) we get \( K' > 2n + 2 \). But for \( n > 5 \) we get \( K' < 2n + 2 \), so when \( n \) is odd and \( n > 5 \) the corresponding algebra of automorphic forms can’t be free.
Let’s compute the exact value of \( K \) for \( n = 3 \) and \( n = 5 \). We use that \( \zeta(4) = \frac{\pi^4}{90} \) and \( \zeta(6) = \frac{\pi^6}{945} \). Then:

| \( n \)   | \( d \)   | \( K' \)                                           |
|----------|----------|---------------------------------------------------|
| \( n \equiv 1 \pmod{4} \) | \( d \neq 7 \) | \( \frac{(2\pi)^{n+1} \left(1 + \frac{d}{(d(n+1)/2 + 1)}\right)}{2 \cdot d(n+1)/2 - 1} \cdot n! \) |
| \( n \equiv -1 \pmod{4} \) | \( d \neq 7 \) | \( \frac{(2\pi)^{n+1} \left(1 + \frac{d}{(d(n+1)/2 - 1)}\right)}{2 \cdot d(n+1)/2 - 1} \cdot n! \) |
| \( n \equiv 1 \pmod{4} \) | \( d = 7 \) | \( \frac{(2\pi)^{n+1} \cdot 2^n}{2 \cdot d(n+1)/2 - 1} \cdot n! \) |
| \( n \equiv -1 \pmod{4} \) | \( d = 7 \) | \( \frac{(2\pi)^{n+1} \cdot 2^n}{2 \cdot d(n+1)/2 - 1} \cdot n! \) |

The value of \( K \) isn’t integer for \( n = 5 \), so the corresponding algebra of automorphic forms isn’t free as well.
Now we consider the case \( d \geq 11 \).
We note that for \( d = 11 \) and \( n = 3 \) or \( n = 5 \) the inequality \( K' < 2n + 2 \) holds.
When \( d \) grows, the numerator of \( K' \) doesn’t change, and the denominator gets bigger. So, for the rest values of \( d \) the inequality \( K' < 2n + 2 \) holds as well. So, the corresponding algebra of automorphic forms can’t be free.
So we get that when \( n \) is odd and \( d \geq 7 \) the only algebra of automorphic forms that can be free corresponds to the values \( d = 7 \) and \( n = 3 \).

**The case of even \( n \)**

Writing down the Bruinier formula, we get:

\[
K = \frac{\frac{1}{2} \zeta(2) \cdot \ldots \cdot L(n)}{d^{n+1} \cdot \frac{n!}{(2\pi)^n} \cdot \zeta(2) \cdot L(3) \cdot \ldots \cdot L(n+1)} \left(1 + \left(\frac{(-1)^{(n+2)/2}}{d}\right) d^{-\frac{n}{2}} + (1 + \left(\frac{(-1)^{(n+2)/2}}{d}\right) d^{-\frac{n}{2}} \right) 2^{n-1} \frac{1 - \left(\frac{d}{2}\right)^{2}}{1 - \left(\frac{d}{2}\right)^{2}}
\]

\[
K = \frac{(2\pi)^{n+1} \left(1 + \left(\frac{(-1)^{(n+2)/2}}{d}\right) d^{-\frac{n}{2}} + (1 + \left(\frac{(-1)^{(n+2)/2}}{d}\right) d^{-\frac{n}{2}} \right) 2^{n-1} \frac{1 - \left(\frac{d}{2}\right)^{2}}{1 - \left(\frac{d}{2}\right)^{2}}}{2d^{n+1} \cdot n! \cdot L(n+1)}
\]
\[ K = \frac{(2\pi)^{n+1} \left( 1 + \left( \frac{(-1)^{(n+2)/2}}{d} \right) d^{-\phi} + (1 + \left( \frac{(-1)^{(n+2)/2}}{d} \right) d^{-\phi} \right) 2^n - \left( \frac{d}{2} \right)^n}{2d^{n+1} \cdot n! \cdot L(n+1)}. \]

We note that \((\frac{d}{2}) = -1\) for all \(d \neq 7\) and \((\frac{d}{2}) = 1\). So:

\[ K = \frac{(2\pi)^{n+1} \left( 1 + \left( \frac{(-1)^{(n+2)/2}}{d} \right) d^{-\phi} + (1 + \left( \frac{(-1)^{(n+2)/2}}{d} \right) d^{-\phi} \right) 2^n - 1}{2d^{n+1} \cdot n! \cdot L(n+1)}, \quad d \neq 7, \]

\[ K = \frac{(2\pi)^{n+1} \left( 1 + \left( \frac{(-1)^{(n+2)/2}}{d} \right) d^{-\phi} + (1 + \left( \frac{(-1)^{(n+2)/2}}{d} \right) d^{-\phi} \right)(2^n - 1)}{2d^{n+1} \cdot n! \cdot L(n+1)}, \quad d = 7. \]

Since \((\frac{d}{2}) = (-1)^{d-1} = -1\), \((\frac{d}{2}) = (-1)^{d-1} = -1\) for \(d \neq 7\) and \((\frac{d}{2}) = 1\) and \((\frac{d}{2}) = \left( \frac{d}{2} \right) \cdot \left( \frac{d}{2} \right)\), we get the following table:

| \(n \equiv 0 \pmod{4}\) | \(d \neq 7\) | \(K\) |
|-------------------------|--------------|-------|
| \(n \equiv 2 \pmod{4}\) | \(d \neq 7\) | \(\frac{(2\pi)^{n+1} \left( 1 - d^{-n/2} + \frac{2^{n-1}}{3} (1 + d^{-n/2}) \right)}{2 \cdot d^{(n+1)/2} \cdot n! \cdot L(n+1)}\) |
| \(n \equiv 0 \pmod{4}\) | \(d = 7\) | \(\frac{(2\pi)^{n+1} \left( 1 - d^{-n/2} + (2^n - 1)(1 - d^{-n/2}) \right)}{2 \cdot d^{(n+1)/2} \cdot n! \cdot L(n+1)}\) |
| \(n \equiv 2 \pmod{4}\) | \(d = 7\) | \(\frac{(2\pi)^{n+1} \left( 1 + d^{-n/2} + (2^n - 1)(1 + d^{-n/2}) \right)}{2 \cdot d^{(n+1)/2} \cdot n! \cdot L(n+1)}\) |

Simplifying the value of \(K\) for \(d = 7\), we get:

| \(n \equiv 0 \pmod{4}\) | \(d \neq 7\) | \(K\) |
|-------------------------|--------------|-------|
| \(n \equiv 2 \pmod{4}\) | \(d \neq 7\) | \(\frac{(2\pi)^{n+1} \left( 1 - d^{-n/2} + \frac{2^{n-1}}{3} (1 + d^{-n/2}) \right)}{2 \cdot d^{(n+1)/2} \cdot n! \cdot L(n+1)}\) |
| \(n \equiv 0 \pmod{4}\) | \(d = 7\) | \(\frac{x^{n+1} \cdot 2^{2n}(1 - d^{-n/2})}{d^{(n+1)/2} \cdot n! \cdot L(n+1)}\) |
| \(n \equiv 2 \pmod{4}\) | \(d = 7\) | \(\frac{x^{n+1} \cdot 2^{2n}(1 + d^{-n/2})}{d^{(n+1)/2} \cdot n! \cdot L(n+1)}\) |

We note that \(\frac{1}{x^{(n+1)}} \leq 2\). This gives the estimate from the above for \(K\). Denote this by \(K'\):
We estimate the numerator for \( d \neq 7 \) from the above, using that \( d^{-n/2} < \frac{1}{7} \), \( -d^{n/2} < 0 \). Denote this number by \( K'' \).

| \( n \) (mod 4) | \( d \) | \( K'' \) |
|-----------------|---------|-------------------|
| \( n \equiv 0 \) | \( d \neq 7 \) | \( \frac{(2\pi)^{n+1}(1 + \frac{2d^{n/2}}{d^{n/2} + 1})}{d^{(n+1)/2} \cdot n^1} \) |
| \( n \equiv 2 \) | \( d \neq 7 \) | \( \frac{(2\pi)^{n+1}(1 + \frac{2d^{n/2}}{d^{n/2} + 1})}{d^{(n+1)/2} \cdot n^1} \) |
| \( n \equiv 0 \) | \( d = 7 \) | \( \frac{\pi^{n+1}, 2^{2n+1}(1 - d^{-n/2})}{d^{(n+1)/2} \cdot n^1} \) |
| \( n \equiv 2 \) | \( d = 7 \) | \( \frac{\pi^{n+1}, 2^{2n+1}(1 + d^{-n/2})}{d^{(n+1)/2} \cdot n^1} \) |

Since when \( d \) is fixed, the numerator grows faster than the denominator, starting from some moment \( K'' < 2n + 2 \).

First consider the case \( d = 7 \):

When \( n = 2, n = 4 \) and \( n = 6 \) we get the inequality \( K'' > 2n + 2 \). But for even \( n > 6 \) the inequality \( K'' < 2n + 2 \) holds, so the corresponding algebra of automorphic forms can’t be free.

We compute the exact values of \( K \) for \( n = 2, n = 4 \) and \( n = 6 \), using that for the field \( \mathbb{Q}(\sqrt{-7}) \) the values of \( L \)-function are: \( L(3) = \frac{32}{21\pi^3} \sqrt{7} \pi^3 \), \( L(5) = \frac{64}{1944} \sqrt{7} \pi^5 \), \( L(7) = \frac{4672}{570944} \sqrt{7} \pi^7 \). Then:

| \( n \) | \( K \) | \( 2n + 2 \) |
|-------|--------|----------------|
| \( n = 2 \) | \( \frac{\pi^2 \cdot 2^2(1 + 7^{-1})}{7^{3/2} \cdot 2! \cdot L(3)} \) | 6 |
| \( n = 4 \) | \( \frac{\pi^4 \cdot 2^2(1 - 7^{-2})}{7^{5/2} \cdot 4! \cdot L(5)} \) | 10 |
| \( n = 6 \) | \( \frac{\pi^6 \cdot 2^2(1 + 7^{-3})}{7^{7/2} \cdot 6! \cdot L(7)} \) | 18.8 |

The value of \( K \) is not integer for \( n = 6 \), so the corresponding algebra can’t be free.

Now we consider the case \( d \geq 11 \).

We note that for \( d = 11 \) and \( n = 4 \) or \( n = 6 \) the inequality \( K'' < 2n + 2 \) holds. When \( d \) grows, the numerator of \( K'' \) doesn’t change, and the denominator gets bigger. So, for the rest values of \( d \) the inequality \( K'' < 2n + 2 \) holds as well. So, the corresponding algebra of automorphic forms can’t be free.

We compute the exact value of \( K \) for \( n = 2 \) and \( d = 11, d = 19, d = 43, d = 67, d = 163 \). We use the exact values of \( L \)-functions: \( L(3)_{11} = \frac{12}{1331} \sqrt{17} \pi^3 \), \( L(5)_{11} = \frac{64}{1944} \sqrt{7} \pi^5 \), \( L(7)_{11} = \frac{4672}{570944} \sqrt{7} \pi^7 \).
\[ L(3)_{19} = \frac{44}{329000} \sqrt{19} \pi^3, L(3)_{43} = \frac{882}{990000} \sqrt{143} \pi^3, L(3)_{67} = \frac{1004}{99000000} \sqrt{67} \pi^3, L(3)_{163} = \frac{9260}{3330000} \sqrt{163} \pi^3 \]

(L(3) \text{ for the field } \mathbb{Q}(\sqrt{-d})).

Since \( n = 2 \) then \( 2n + 2 = 6 \). We get:

| \( d \) | \( K \) |
|--------|---------|
| 11     | \( \frac{(2\pi)^3}{11^{3/2} \cdot 2 \cdot L(3)} = \frac{11}{3} < 6 \) |
| 19     | \( \frac{(2\pi)^3}{19^{3/2} \cdot 2 \cdot L(3)} = \frac{19}{11} < 6 \) |
| 43     | \( \frac{(2\pi)^3}{43^{3/2} \cdot 2 \cdot L(3)} = \frac{43}{332} < 6 \) |
| 67     | \( \frac{(2\pi)^3}{67^{3/2} \cdot 2 \cdot L(3)} = \frac{67}{251} < 6 \) |
| 163    | \( \frac{(2\pi)^3}{163^{3/2} \cdot 2 \cdot L(3)} = \frac{163}{2315} < 6 \) |

It follows that in all this cases the corresponding algebra of automorphic forms can’t be free.

We get that for even \( n \) and \( d \geq 7 \) the algebra of automorphic forms which can be free corresponds to the values \( d = 7 \) and \( n = 2 \) or \( d = 7 \) and \( n = 4 \).

\( d = 3 \)

Here we provide the same calculations as in the previous case, just the Bruinier formula looks a bit different:

\[
\frac{5}{6} \text{Vol}(L_{n-1}) + \frac{1}{2} \text{Vol}(M_{n-1}) = K \cdot \text{Vol}(L_n),
\]

where \( K \geq 7(n+1) \) (because there is \( n+1 \) generating function of weight at least 6 and \( n+1 \) comes form the dimension). So, if

\[
K = \frac{\frac{5}{6} \text{Vol}(L_{n-1}) + \frac{1}{2} \text{Vol}(M_{n-1})}{\text{Vol}(L_n)} \leq 7n + 7,
\]

then the corresponding algebra of automorphic forms can’t be free.

**The case of odd \( n \)**

Consider odd \( n \). Then

\[
K = \frac{1}{2} D^{(n-1)^2} \cdot \prod_{j=1}^{(n-1)/2} \left( \zeta(2) \cdot L(3) \cdot \zeta(4) \cdot L(5) \cdot \ldots \cdot L(n) \right) \left( \frac{5}{3} + 2^{n-1} \cdot \frac{1 - \left(\frac{-1}{2}\right)^2 2^{-n}}{1 - \left(\frac{-1}{2}\right)^2 2^{-n}} \right)
\]

\[
K = D^{\frac{n+1}{2}} \cdot \prod_{j=1}^{(n+1)/2} \left( \zeta(2) \cdot L(3) \cdot \zeta(4) \cdot L(5) \cdot \ldots \cdot \zeta(n+1) \right)
\]

\[
K = \frac{(2\pi)^{n+1}}{2 \cdot 3^{(n+1)/2} (1 + \left(\frac{-1}{2}\right)^{(n+3)/2}) \cdot n!} \cdot \zeta(n+1),
\]
We note that \((-\frac{3}{2})\) = -1. Moreover, \((-\frac{1}{3})\) = 1, if \(n \equiv 1 \pmod{4}\), and \((-\frac{1}{3})\) = -1, if \(n \equiv -1 \pmod{4}\).

We get:

| \(n\) \(\pmod{4}\) | \(K\) |
|---|---|
| \(n \equiv 1 \pmod{4}\) | \((2\pi)^{n+1} \cdot (6 + 2^n) \cdot 6 \cdot (3^{(n+1)/2} + 1) \cdot n! \cdot \zeta(n+1)\) |
| \(n \equiv -1 \pmod{4}\) | \((2\pi)^{n+1} \cdot (6 + 2^n) \cdot 6 \cdot (3^{(n+1)/2} - 1) \cdot n! \cdot \zeta(n+1)\) |

We note that \(\zeta(n+1) > 1\), so, removing this multiplier from the denominator, we get an estimate from above for \(K\). Denote this by \(K'\):

| \(n\) \(\pmod{4}\) | \(K\) |
|---|---|
| \(n \equiv 1 \pmod{4}\) | \((2\pi)^{n+1} \cdot (6 + 2^n) \cdot 6 \cdot (3^{(n+1)/2} + 1) \cdot n!\) |
| \(n \equiv -1 \pmod{4}\) | \((2\pi)^{n+1} \cdot (6 + 2^n) \cdot 6 \cdot (3^{(n+1)/2} - 1) \cdot n!\) |

Since the numerator grows faster than the denominator, starting from some moment \(K' < 7n + 7\).

For odd \(n > 9\) we get \(K' < 7n + 7\), so when \(n\) is odd and \(n > 9\) the corresponding algebra of automorphic forms can’t be free.

Let’s compute the exact value of \(K\) for \(n = 3, n = 5, n = 7\) and \(n = 9\). We use that \(\zeta(4) = \frac{\pi^4}{90}, \zeta(6) = \frac{\pi^6}{945}, \zeta(8) = \frac{\pi^8}{9450}, \text{ and } \zeta(10) = \frac{\pi^{10}}{93555}\). Then:

| \(n\) | \(\zeta(n+1)\) |
|---|---|
| \(n = 9\) | \(\frac{\pi^{10}}{93555}\) |
| \(n = 7\) | \(\frac{\pi^8}{9450}\) |
| \(n = 5\) | \(\frac{\pi^6}{945}\) |
| \(n = 3\) | \(\frac{\pi^4}{90}\) |

The value of \(K\) isn’t integer for \(n = 9\), so the corresponding algebra of automorphic forms isn’t free as well.

So we get that when \(n\) is odd and \(d = 3\) the algebras of automorphic forms that can be free correspond to the values \(n = 7, n = 5\) and \(n = 3\).
The case of even $n$

Writing down the Bruinier formula, we get:

$$K = \frac{\frac{1}{2}\zeta(2) \cdot \ldots \cdot \zeta(n)}{\zeta(2) \cdot L(3) \cdot \ldots \cdot L(n+1)} \cdot \left(\frac{\frac{1}{4}(1 + \left(\frac{(-1)^{n+2}/2}{3}\right) \cdot 3^{-\frac{3}{2}}) + (1 + \left(\frac{(-1)^{(n+2)/2}}{3}\right) \cdot 3^{-\frac{3}{2}}) \cdot 2^{n-1} \cdot \frac{n}{1 - \left(\frac{3}{2}\right)^{2-1}}}{d^{n+1} \cdot n! \cdot L(n+1)}\right),$$

$$K = \frac{(2\pi)^{n+1} \left(\frac{\frac{1}{4}(1 + \left(\frac{(-1)^{(n+2)/2}}{3}\right) \cdot 3^{-\frac{3}{2}}) + (1 + \left(\frac{(-1)^{(n+2)/2}}{3}\right) \cdot 3^{-\frac{3}{2}}) \cdot 2^{n-1} \cdot \frac{n}{1 - \left(\frac{3}{2}\right)^{2-1}}}{d^{n+1} \cdot n! \cdot L(n+1)}\right)}{2 \cdot 3^{n+1} \cdot n! \cdot L(n+1)}.$$  

We note that $\left(\frac{-3}{2}\right) = -1$. So:

$$K = \frac{(2\pi)^{n+1} \left(\frac{\frac{1}{4}(1 + \left(\frac{(-1)^{(n+2)/2}}{3}\right) \cdot 3^{-\frac{3}{2}}) + (1 + \left(\frac{(-1)^{(n+2)/2}}{3}\right) \cdot 3^{-\frac{3}{2}}) \cdot 2^{n-1} \cdot \frac{n}{1 - \left(\frac{3}{2}\right)^{2-1}}}{d^{n+1} \cdot n! \cdot L(n+1)}\right)}{2 \cdot 3^{n+1} \cdot n! \cdot L(n+1)}.$$  

Since $\left(\frac{-1}{4}\right) = \left(\frac{2}{3}\right) = -1$ and $\left(\frac{-2}{3}\right) = 1$, we get the following table:

| $n$        | $K$                                                        |
|-----------|------------------------------------------------------------|
| $n \equiv 0 \pmod{4}$ | $\frac{(2\pi)^{n+1} \left(\frac{\frac{1}{4}(1 - 3^{-\frac{3}{2}}) + (1 + 3^{-\frac{3}{2}}) \cdot \frac{3}{2}}{d^{n+1}}\right)}{2 \cdot 3^{n+1} \cdot n! \cdot L(n+1)}$ |
| $n \equiv 2 \pmod{4}$ | $\frac{(2\pi)^{n+1} \left(\frac{\frac{1}{4}(1 + 3^{-\frac{3}{2}}) + (1 - 3^{-\frac{3}{2}}) \cdot \frac{3}{2}}{d^{n+1}}\right)}{2 \cdot 3^{n+1} \cdot n! \cdot L(n+1)}$ |

Simplifying the value of $K$, we get:

| $n$        | $K$                                                        |
|-----------|------------------------------------------------------------|
| $n \equiv 0 \pmod{4}$ | $\frac{(2\pi)^{n+1} \left(5(1 - 3^{-\frac{3}{2}}) + (1 + 3^{-\frac{3}{2}})(2^n - 1)\right)}{6 \cdot 3^{n+1} \cdot n! \cdot L(n+1)}$ |
| $n \equiv 2 \pmod{4}$ | $\frac{(2\pi)^{n+1} \left(5(1 + 3^{-\frac{3}{2}}) + (1 - 3^{-\frac{3}{2}})(2^n - 1)\right)}{6 \cdot 3^{n+1} \cdot n! \cdot L(n+1)}$ |

We note that $\frac{1}{\zeta(\sigma+1)} \leq 2$. This gives the estimate from the above for $K$. Denote this by $K'$.
\[ K' = (2\pi)^{n+1} (5(1 - 3^{-\frac{n}{2}}) + (1 + 3^{-\frac{n}{2}})(2^n - 1)). \]

\[ K' = (2\pi)^{n+1} (5(1 + 3^{-\frac{n}{2}}) + (1 - 3^{-\frac{n}{2}})(2^n - 1)). \]

Since the numerator grows faster than the denominator, starting from some moment \( K' < 7n + 7 \). For odd \( n > 10 \) we get \( K' < 7n + 7 \), so when \( n \) is even and \( n > 10 \) the corresponding algebra of automorphic forms can’t be free.

Let’s compute the exact value of \( K' \) for \( n = 2, n = 4, n = 6, n = 8 \) and \( n = 10 \). We use that for the field \( \mathbb{Q}(\sqrt{-3}) \) the values of \( L \)-function are:

\[ L(3) = \frac{4}{27} \sqrt{3} \pi^3, \quad L(5) = \frac{4}{2187} \sqrt{3} \pi^5, \quad L(7) = \frac{56}{295245} \sqrt{3} \pi^7, \quad L(9) = \frac{3236}{167403915} \sqrt{3} \pi^9, \quad L(11) = \frac{14776}{7533176175} \sqrt{3} \pi^{11}. \]

Then:

| \( n \) | \( K' \) | \( 7n + 7 \) |
|------|-------|-------|
| 2    | 39    | 21    |
| 4    | 95    | 35    |
| 6    | 127   | 49    |
| 8    | \approx 118.5 | 63    |
| 10   | \approx 67.3 | 77    |

The value of \( K' \) is not integer for \( n = 8 \) and \( n = 10 \), so the corresponding algebra can’t be free. So we get that when \( n \) is even and \( d = 3 \) the algebras of automorphic forms that can be free correspond to the values \( n \leq 7 \).

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