FULL CROSS-DIFFUSION LIMIT IN THE STATIONARY
SHIGESADA-KAWASAKI-TERAMOTO MODEL

KOUSUKE KUTO†

ABSTRACT. This paper studies the asymptotic behavior of coexistence steady states of the
Shigesada-Kawasaki-Teramoto model as both cross-diffusion coefficients tend to infinity at the
same rate. In the case when either one of two cross-diffusion coefficients tends to infinity, Lou
and Ni [18] derived a couple of limiting systems, which characterize the asymptotic behavior
of coexistence steady states. Recently, a formal observation by Kan-on [10] implied the existence
of a limiting system including the nonstationary problem as both cross-diffusion coefficients
tend to infinity at the same rate. This paper gives a rigorous proof of his observation as far as
the stationary problem. As a key ingredient of the proof, we establish a uniform
$L^\infty$ estimate
for all steady states. Thanks to this a priori estimate, we show that the asymptotic profile of
coexistence steady states can be characterized by a solution of either of two limiting systems.

1. Introduction

This paper is concerned with the following Neumann problem of nonlinear elliptic equations:

\[
\begin{cases}
\Delta [(d_1 + \alpha v)u] + f(u,v) = 0 & \text{in } \Omega, \\
\Delta [(d_2 + \beta u)v] + g(u,v) = 0 & \text{in } \Omega, \\
u \geq 0, \ v \geq 0 & \text{in } \Omega, \\
\partial_\nu u = \partial_\nu v = 0 & \text{on } \partial \Omega,
\end{cases}
\]  

(1.1)

where

\[
f(u,v) := u(a_1 - b_1 u - c_1 v), \quad g(u,v) := v(a_2 - b_2 u - c_2 v).
\]  

(1.2)

Here $\Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial \Omega$; $\Delta := \sum_{j=1}^{N} \partial^2 / \partial x_j^2$ is the usual Laplace operator; $\nu(x)$ is the outer unit normal vector at $x \in \partial \Omega$, and $\partial_\nu u = \nu(x) \cdot \nabla u$ represents the out-flux of $u$; coefficients $a_i$, $b_i$, $c_i$ and $d_i$ ($i = 1, 2$) are positive constants; $\alpha$ and $\beta$ are nonnegative constants. System (1.1) is the stationary problem of a Lotka-Volterra competition model in which unknown functions $u(x)$ and $v(x)$ represent the stationary population densities of two competing species in the habitat $\Omega$. In the reaction terms, $a_i$ represent the birth rates of the respective species, $b_i$ and $c_i$ denote the intra-specific competition coefficients, and $d_1$ and $d_2$ denote the inter-specific competition coefficients. In the diffusion terms, $d_1 \Delta u$ and $d_2 \Delta v$ represent the linear diffusion determined by the dispersive force associated with random movement of each species, whereas $\alpha \Delta (uv)$ and $\beta \Delta (uv)$ denote the nonlinear diffusion caused by the population pressure resulting from interference between different species. The interaction term

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† Department of Applied Mathematics, Waseda University, 3-4-1 Ohkubo, Shinjuku-ku, Tokyo 169-8555, Japan.

E-mail: kuto@waseda.jp.
\(\Delta (uv)\) is often referred to as the cross-diffusion. See a book by Okubo and Levin [27] for modellings of the biological diffusion. Such a Lotka-Volterra competition system with cross-diffusion (and additional) terms was proposed by Shigesada, Kawasaki and Teramoto [28]. Beyond their bio-mathematical aim to realize segregation phenomena of two competing species observed in ecosystems, a lot of pure mathematicians have studied a class of Lotka-Volterra systems with cross-diffusion as a prototype of diffusive interactions. Today, such a class of Lotka-Volterra system with cross-diffusion is referred as the SKT model celebrating the authors of [28]. See e.g., the book chapters by Jüngel [7], Ni [25], and Yamada [32, 33] as surveys for mathematical works relating to the SKT model.

Since there are a lot of papers studying the stationary SKT model like (1.1), we just give a brief history of studies. Immediately after the proposal by [28], the group of Mimura began to study (1.1). Their main methods in 1980s are the bifurcation ([22]) and the singular perturbation ([21, 23]), and moreover, Kan-on [8] identified some criteria for ensuring the stability/instability of nonconstant solutions obtained by [23]. After the middle of 1990s, a couple of papers by Lou ([21, 23]), and moreover, Kan-on [8] identified some criteria for ensuring the stability/instability study (1.1). Their main methods in 1980s are the bifurcation ([22]) and the singular perturbation ([21, 23]).

Theorem 1.1 ([18]). Suppose that \(N \leq 3\), \(a_1/a_2 \neq b_1/b_2, a_1/a_2 \neq c_1/c_2\) and \(a_2/d_2\) is not equal to any eigenvalue of \(-\Delta\) with homogeneous Neumann boundary condition on \(\partial \Omega\). Let \(\{(u_n, v_n)\}\) be any sequence of positive nonconstant solutions of (1.1) with \(\alpha = \alpha_n \to \infty\). Let \(\{u_n, v_n\}\) be a positive number \(\delta = \delta(a_i, b_i, c_i, d_i) > 0\) such that \(\beta \leq \delta\), either of the following (i) or (ii) occurs:

(i) there are a positive function \(v \in C^2(\Omega)\) and a positive number \(\tau\) such that \((u_n, v_n)\) converges uniformly to \((\tau/v, v)\) by passing to a subsequence if necessary, and \((v, \tau)\) satisfies

\[
\begin{cases}
  d_2 \Delta v + v(a_2 - c_2 v) - b_2 \tau = 0 & \text{in } \Omega, \\
  \partial_n v = 0 & \text{on } \partial \Omega,
\end{cases}
\]

(ii) there are positive functions \(u, w \in C^2(\Omega)\) such that \((u_n, \alpha_n v_n)\) converges uniformly to \((u, w)\) by passing to a subsequence if necessary, and \((u, w)\) satisfies

\[
\begin{cases}
  \Delta [(d_1 + w)u] + u(a_1 - b_1 u) = 0 & \text{in } \Omega, \\
  \Delta [(d_2 + \beta u)w] + w(a_2 - b_2 w) = 0 & \text{in } \Omega, \\
  \partial_n u = \partial_n w = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Thanks to Theorem 1.1, one can expect that nonconstant solutions of (1.1) can be characterized by those of (1.3) or (1.4) if \(\alpha > 0\) is sufficiently large and \(\beta > 0\) is sufficiently small. Indeed, such perturbations were verified by [16, 18, 20, 26, 29, 30, 31] in various senses. In the first limiting behavior stated in (i) of Theorem 1.1, \(u_n v_n\) approaches a positive constant \(\tau\) uniformly in \(\Omega\), and thereby, it is natural to expect that the first limiting system (1.3) can realize
the segregation phenomena of two competing species when one of cross-diffusive abilities of two species is very strong. Once Theorem 1.1 was revealed by [18], there has been a great progress of study of the first limiting system (1.3) (e.g., [11, 19, 20, 24, 26, 29, 30, 31, 34]). Among other things, Lou, Ni and Yotsutani [19] obtained a global bifurcation structure of positive solutions in the one-dimensional case. In the second limiting behavior stated in (ii) of Theorem 1.1, the stationary density $v_n$ of the species with small cross-diffusive ability shrinks with the order $O(1/\alpha_n)$ as $\alpha_n \to \infty$ since $\alpha_n v_n$ tends to a positive function $w$. The author [13] obtained a global bifurcation structure of positive nonconstant solutions of (1.4) in a special case when $\beta = 0$ and $\Omega$ is a one-dimensional interval Furthermore, Li and Wu [16] investigated the instability of positive nonconstant solutions near the bifurcation point. By gathering information on solutions of (1.3) or (1.4) obtained in above mentioned papers, we have a reasonable conjecture on the bifurcation structure of (1.1) with large $\alpha$ and small $\beta \geq 0$ that the set of positive nonconstant solutions form bifurcation branches of saddle node type, and moreover, the upper branches can be approximated by solutions of the first limiting system (1.3), whereas the lower branches can be characterized by solutions of the second limiting system (1.4) by regarding $a_2$ as a bifurcation parameter (see [13, Figure 1]). In addition, we note that only the second limiting situation as (ii) occurs under homogeneous Dirichlet boundary conditions (14, 15).

The purpose of this paper is to study the asymptotic behavior of solutions of (1.1) as both $\alpha$ and $\beta$ tend to infinity with $\alpha/\beta$ approaching a positive number. Ecologically, we expect that such a study can reveal the mathematical mechanism of segregation of two competing species when the cross-diffusive abilities of both species are strong. To this end, we obtain the a priori $L^\infty$ estimate of all solutions of (1.1) as follows:

**Theorem 1.2.** For any small $\eta > 0$, there exists a positive constant $C = C(\eta, d_1, a_i, b_i, c_i)$ such that if $\alpha > 0$ and $\beta > 0$ satisfy $\eta \leq \alpha/\beta \leq 1/\eta$, then any solution $(u, v)$ of (1.1) satisfies

$$\max_{x \in \Omega} u(x) \leq C \quad \text{and} \quad \max_{x \in \Omega} v(x) \leq C.$$ 

Our approach of the proof is based on the maximum principle. In view of some papers studying (1.1), it can be said that a usual method in considering the a priori $L^\infty$ estimate is to employ the following change of variables

$$\phi(x) = \left(1 + \frac{\alpha}{d_1} v\right) u, \quad \psi(x) = \left(1 + \frac{\beta}{d_2} u\right) v, \quad (1.5)$$

which reduces the quasilinear system (1.1) to the semilinear one as follows

$$\begin{align*}
d_1 \Delta \phi + f(u, v) &= 0 \quad \text{in } \Omega, \\
d_2 \Delta \psi + g(u, v) &= 0 \quad \text{in } \Omega, \\
\phi &\geq 0, \quad \psi \geq 0 \quad \text{in } \Omega, \\
\partial_\nu \phi &= \partial_\nu \psi = 0 \quad \text{on } \partial \Omega, \quad (1.6)
\end{align*}$$

where $(u, v)$ in reaction terms is regarded as a function of $(\phi, \psi)$ determined by (1.5). A typical application of the maximum principle to the first equation of (1.6) enables us to know the nonnegativity of $f(u, v)$ at the maximum point of $\phi$. However, obviously this maximum point is different from a maximum point of $u$, and then, such a difference often makes our construction of an $L^\infty$ bound of solutions be difficult. In [13, Theorem 2.3], an exquisite combination of the above maximum principle approach and the Harnack inequality established the uniform $L^\infty$ estimate of any solution $(u, v)$ in the case when $N \leq 3$ and $\alpha > 0$ is arbitrary but $\beta \geq 0$ is restricted to be small. The restriction $N \leq 3$ comes from the Sobolev embedding theorem for the use of the Harnack inequality.
In this paper, in order to get the uniform $L^\infty$ estimate of any solution in a case when $\alpha > 0$ and $\beta > 0$ are arbitrary as long as $\eta \leq \alpha/\beta \leq 1/\eta$, we employ a different approach (without the change of variables (1.5)) to reduce (1.1) to the following form:

\[
\begin{aligned}
& (d_1 d_2 + d_1 \beta u + d_2 \alpha v) \Delta u + 2 d_2 \alpha \nabla u \cdot \nabla v + u F(u, v; \alpha, \beta) = 0 \quad \text{in } \Omega, \\
& (d_1 d_2 + d_1 \beta u + d_2 \alpha v) \Delta v + 2 d_1 \beta \nabla u \cdot \nabla v + v G(u, v; \alpha, \beta) = 0 \quad \text{in } \Omega, \\
& u \geq 0, \quad v \geq 0 \\
& \partial_\nu u = \partial_\nu v = 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

where

\[
\begin{aligned}
F(u, v; \alpha, \beta) &:= (d_2 + \beta u)(a_1 - b_1 u - c_1 v) - \alpha v(a_2 - b_2 u - c_2 v), \\
G(u, v; \alpha, \beta) &:= -\beta u(a_1 - b_1 u - c_1 v) + (d_1 + \alpha v)(a_2 - b_2 u - c_2 v).
\end{aligned}
\]

For (1.7), as explained in the next section, the maximum principle leads to the nonnegativity of $F$ (resp. $G$) at the maximum point of $u$ (resp. $v$). In the proof, we make use of a fact that if $(u, v) \in \mathbb{R}^2_+$ satisfies $F(u, v, \alpha, \beta) \geq 0$ and $(d_2 b_1 + d_1 b_2)u + (d_2 c_1 + d_1 c_2)v > d_2 a_1 + d_1 a_2$, then $G(u, v, \alpha, \beta) < 0$. By the combination of this fact and a levelset analysis for $F$ and $G$, the proof of Theorem 1.2 will be carried out. Since our proof does not use the Harnack inequality as well as the Sobolev embedding theorem, then Theorem 1.2 does not require any restriction on the dimension number $N$.

Thanks to Theorem 1.2 we can treat the asymptotic analysis for solutions of (1.1) as $\alpha \to \infty$ and $\beta \to \infty$ with $\alpha/\beta \to \gamma$ for some $\gamma > 0$. We obtain the following limiting systems in such a full cross-diffusion limit.

**Theorem 1.3.** Suppose that $a_1/a_2 \neq b_1/b_2$ and $a_1/a_2 \neq c_1/c_2$. Let $\{(u_n, v_n)\}$ be any sequence of positive nonconstant solutions of (1.1) with $\alpha = \alpha_n \to \infty$, $\beta = \beta_n \to \infty$ and $\gamma_n := \alpha_n/\beta_n \to \gamma > 0$ as $n \to \infty$. Then either of the following two situations occurs, passing to a subsequence if necessary:

(i) there exist a positive function $u \in C^2(\overline{\Omega})$ and a positive number $\tau$ such that

\[
\lim_{n \to \infty} (u_n, v_n) = \left( u, \frac{\tau}{u} \right) \quad \text{in } C^1(\overline{\Omega}) \times C^1(\overline{\Omega}),
\]

and $w_n(x) := d_1 u_n(x) - \gamma_n d_2 v_n(x)$ satisfies

\[
\lim_{n \to \infty} w_n = w \quad \text{in } C^1(\overline{\Omega})
\]

with some function $w \in C^2(\overline{\Omega})$ satisfying

\[
\begin{aligned}
\Delta w + f \left( \frac{\sqrt{w^2 + 4\gamma d_1 d_2 \tau} + w}{2d_1}, \frac{\sqrt{w^2 + 4\gamma d_1 d_2 \tau} - w}{2d_2} \right) - \gamma g \left( \frac{\sqrt{w^2 + 4\gamma d_1 d_2 \tau} + w}{2d_1}, \frac{\sqrt{w^2 + 4\gamma d_1 d_2 \tau} - w}{2d_2} \right) = 0 \quad \text{in } \Omega, \\
\partial_\nu w = 0 \quad \text{on } \partial \Omega, \\
\int_\Omega f \left( \frac{\sqrt{w^2 + 4\gamma d_1 d_2 \tau} + w}{2d_1}, \frac{\sqrt{w^2 + 4\gamma d_1 d_2 \tau} - w}{2d_2} \right) = 0
\end{aligned}
\]

and

\[
\left( u, \frac{\tau}{u} \right) = \left( \frac{\sqrt{w^2 + 4\gamma d_1 d_2 \tau} + w}{2d_1}, \frac{\sqrt{w^2 + 4\gamma d_1 d_2 \tau} - w}{2d_2} \right);
\]
(ii) there exist nonnegative functions \(u, v \in C(\Omega)\) such that \(uv = 0\) in \(\Omega\),
\[
\lim_{n \to \infty} (u_n, v_n) = (u, v) \quad \text{uniformly in } \Omega
\]
and \(w_n(x) := d_1 u_n(x) - \gamma_n d_2 v_n(x)\) satisfies
\[
\lim_{n \to \infty} w_n = w \quad \text{in } C^1(\overline{\Omega})
\]
with some sign-changing function \(w\) satisfying
\[
\begin{cases}
\Delta w + f\left(\frac{w_+}{d_1}, \frac{w_-}{\gamma d_2}\right) - \gamma g\left(\frac{w_+}{d_1}, \frac{w_-}{\gamma d_2}\right) = 0 & \text{in } \Omega, \\
\partial_\nu w = 0 & \text{on } \partial \Omega,
\end{cases}
\]
(1.10)
and
\[(u, v) = \left(\frac{w_+}{d_1}, \frac{w_-}{\gamma d_2}\right),
\]
where \(w_+ := \max\{w, 0\}\) and \(w_- := -\min\{w, 0\} \geq 0\).

It should be noted that a formal observation by Kan-on \[10\] implied the existence of a nonstationary version of the limiting system (1.9). Thus it can be said that Theorem 1.3 supports his observation by a rigorous proof as far as the stationary problem. In both situations (i) and (ii) of Theorem 1.3, \(u_n v_n\) approaches some constant \(\tau\) as \(\alpha_n, \beta_n \to \infty\) and \(\alpha_n/\beta_n \to \gamma\). Ecologically, this fact enables us to expect the segregation of competing species occurs when cross-diffusive abilities of both species are strong to the same degree. In the limiting case (i) of Theorem 1.3 since \(\tau > 0\), a typical expected ecological situation is so that the high (resp. low) density area of \(u\) is the low (resp. high) density area of \(v\) (the incomplete segregation). In the other limiting case (ii) of Theorem 1.3 since \(\tau = 0\), living areas of two competing species completely segregate each other (the complete segregation). It is known that (1.10) appears also in the fast reaction limit of the Lotka-Volterra competition model (namely, in the limiting case as \(c_1, b_2 \to \infty\) and \(c_1/b_2\) tends to some positive number in (1.1) with \(\alpha = \beta = 0\), and then, there are several papers discussing the existence of nonconstant solutions of (1.10) and related issues (e.g., [2, 3, 4, 6, 9]).

The contents of this paper is as follows: In Section 2, we derive a uniform \(L^\infty\) estimate of all solutions of (1.1) to prove Theorem 1.2. In Section 3, we study the asymptotic behavior of solutions of (1.1) as \(\alpha_n, \beta_n \to \infty\) and \(\alpha_n/\beta_n \to \gamma\) to prove Theorem 1.3. In Section 4, we discuss the existence of nonconstant solutions of the limiting system (1.9) from the bifurcation viewpoint.

Throughout this paper, the usual norms of the spaces \(L^p(\Omega)\) for \(p \in [1, \infty)\) and \(L^\infty(\Omega)\) are denoted by
\[
\|u\|_p := \left(\int_\Omega |u(x)|^p dx\right)^{1/p}, \quad \|u\|_\infty := \text{ess. sup}_{x \in \overline{\Omega}} |u(x)|.
\]
Hence \(\|u\|_\infty = \max_{x \in \Omega}\) in a case when \(u \in C(\overline{\Omega})\).

2. Uniform boundedness of steady states

This section is devoted to the proof of Theorem 1.2. Our strategy of the proof is to employ a maximum principle approach for a reduction form (1.7). We begin with the reduction.

Lemma 2.1. If \((u, v)\) is a solution of (1.1), then \((u, v)\) is a solution of (1.7).
Proof. Let \((u, v)\) be any solution of (1.1). By expanding the cross-diffusion terms, one can see that the elliptic equations of (1.1) are expressed as
\[
\begin{cases}
(d_1 + \alpha u)\Delta u + 2\alpha \nabla u \cdot \nabla v + \alpha u \Delta v + f(u, v) = 0 & \text{in } \Omega, \\
(d_2 + \beta u)\Delta v + 2\beta \nabla u \cdot \nabla v + \beta v \Delta u + g(u, v) = 0 & \text{in } \Omega.
\end{cases}
\]
Plugging the expression of \(\Delta v\) from the second equation into the first equation, we obtain the first equation of (1.7). A similar procedure also gives the second equation of (1.7). \(\square\)

Applications of the following maximum principle to (1.7) will be useful in the proof of Theorem 1.2.

Lemma 2.2 (e.g., [18]). Suppose that \(h \in C(\overline{\Omega} \times \mathbb{R})\) and \(B \in C(\overline{\Omega}; \mathbb{R}^N)\). Then the followings (i) and (ii) hold true:

(i) If \(\underline{u} \in C^2(\Omega) \cap C^1(\overline{\Omega})\) satisfies
\[
\Delta \underline{u} + B(x) \cdot \nabla \underline{u} + h(x, \underline{u}) \geq 0 \quad \text{in } \Omega, \quad \partial_{\nu} \underline{u} \leq 0 \quad \text{on } \partial \Omega,
\]
and \(\underline{u}(x_0) = \|\underline{u}\|_{\infty}\), then \(h(x_0, \underline{u}(x_0)) \geq 0\).

(ii) If \(\overline{u} \in C^2(\Omega) \cap C^1(\overline{\Omega})\) satisfies
\[
\Delta \overline{u} + B(x) \cdot \nabla \overline{u} + h(x, \overline{u}) \leq 0 \quad \text{in } \Omega, \quad \partial_{\nu} \overline{u} \geq 0 \quad \text{on } \partial \Omega,
\]
and \(\overline{u}(x_0) = \min_{x \in \overline{\Omega}} \overline{u}(x)\), then \(h(x_0, \overline{u}(x_0)) \leq 0\).

For the application of Lemma 2.2 to (1.7), we need to know the profile of \(F(u, v; \alpha, \beta)\) defined by (1.8). The next lemma yields information on the zero level set of \(F(u, v; \alpha, \beta)\).

Lemma 2.3. Suppose that \(\alpha > 0\) and \(\beta > 0\). Then the followings (i) and (ii) hold true.

(i) If \(u > a_1/b_1\), then there exists a positive number \(V(u; \alpha, \beta)\) such that
\[
F(u, v; \alpha, \beta) \begin{cases}
< 0 & \text{for } 0 < v < V(u; \alpha, \beta), \\
= 0 & \text{for } v = V(u; \alpha, \beta), \\
> 0 & \text{for } v > V(u; \alpha, \beta).
\end{cases}
\tag{2.1}
\]

(ii) Define
\[
\bar{v}_0(\alpha) := \begin{cases}
0 & \text{if } c_1/c_2 < a_1/a_2 \text{ and } \alpha < \sqrt{\frac{1}{\alpha}}, \\
\frac{aa_2 + d_2c_1 + \sqrt{(aa_2 + d_2c_1)^2 - 4a_1d_2c_1c_2}}{2ac_2} & \text{otherwise},
\end{cases}
\tag{2.2}
\]
where
\[
\alpha := \frac{d_2(2a_1c_2 - a_2c_1) + 2d_2\sqrt{a_1c_2(a_1c_2 - a_2c_1)}}{a_2^2}, \quad \overline{\alpha} := \frac{d_2(2a_1c_2 - a_2c_1) - 2d_2\sqrt{a_1c_2(a_1c_2 - a_2c_1)}}{a_2^2}.
\]
If \(v > \bar{v}_0(\alpha)\), then there exists a positive number \(U(v; \alpha, \beta)\) such that
\[
F(u, v; \alpha, \beta) \begin{cases}
> 0 & \text{for } 0 < u < U(v; \alpha, \beta), \\
= 0 & \text{for } u = U(v; \alpha, \beta), \\
< 0 & \text{for } u > U(v; \alpha, \beta).
\end{cases}
\tag{2.3}
\]
Proof. (i) We first observe the sign of $F$ on the half line $\{(u,0) : u > 0\}$ on $u$ axis as follows:

$$F(u,0;\alpha,\beta) = (d_2 + \beta u)(a_1 - b_1 u) \begin{cases} > 0 & \text{for } 0 < u < a_1/b_1, \\ < 0 & \text{for } u > a_1/b_1. \end{cases}$$

Next, for each fixed $u > a_1/b_1$, we investigate the profile of the function $v \mapsto F(u,v;\alpha,\beta)$ (regarded as a function with respect to $v > 0$). By the form of the quadratic function

$$F(u,v;\alpha,\beta) = \alpha c_2 v^2 - \{c_1(d_2 + \beta u) + \alpha(a_2 - b_2 u)\} v + (d_2 + \beta u)(a_1 - b_1 u)$$

and the fact that $F(u,0;\alpha,\beta) < 0$ for any fixed $u > a_1/b_1$, we obtain (2.1) with

$$V(u;\alpha,\beta) = \frac{c_1(d_2 + \beta u) + \alpha(a_2 - b_2 u) + \sqrt{c_1(d_2 + \beta u) + \alpha(a_2 - b_2 u)^2 - 4\alpha c_2(d_2 + \beta u)(a_1 - b_1 u)}}{2\alpha c_2}.$$

(ii) Following a similar argument, we first check the sign of $F$ on the half line $\{(0,v) : v > 0\}$ on $v$ axis. By virtue of

$$F(0,v;\alpha,\beta) = \alpha c_2 v^2 - (\alpha a_2 + d_2 c_1)v + d_2 a_1,$$

a straightforward computation enables us to check that if $c_1/c_2 < a_1/a_2$ and $\alpha < \alpha < \bar{\alpha}$, then $F(0,v;\alpha,\beta) > 0$ for any $v > 0$; otherwise,

$$F(0,v;\alpha,\beta) \begin{cases} > 0 & \text{for } v \in (0,\tilde{v}_0(\alpha)) \cup (\bar{v}_0(\alpha),\infty), \\ < 0 & \text{for } v \in (\tilde{v}_0(\alpha),\bar{v}_0(\alpha)), \end{cases}$$

where

$$\tilde{v}_0(\alpha) = \frac{\alpha a_2 + d_2 c_1 - \sqrt{(\alpha a_2 + d_2 c_1)^2 - 4\alpha d_2 a_1 c_2}}{2\alpha c_2},$$

$$\bar{v}_0(\alpha) = \frac{\alpha a_2 + d_2 c_1 + \sqrt{(\alpha a_2 + d_2 c_1)^2 - 4\alpha d_2 a_1 c_2}}{2\alpha c_2}.$$

Hence it follows that $F(0,v;\alpha,\beta) > 0$ for $v > \tilde{v}_0(\alpha)$, where

$$\tilde{v}_0(\alpha) := \begin{cases} 0 & \text{if } c_1/c_2 < a_1/a_2 \text{ and } \alpha < \alpha < \bar{\alpha}, \\ \tilde{v}_0(\alpha) & \text{otherwise.} \end{cases}$$

Next, for any fixed $v > \tilde{v}_0(\alpha)$, we check the profile of $u \mapsto F(u,v;\alpha,\beta)$. Since $F(0,v;\alpha,\beta) > 0$ for $v > \tilde{v}_0(\alpha)$, then we fix such $v$ arbitrarily and regard

$$u \mapsto F(u,v;\alpha,\beta) = -\beta b_1 u^2 + \{\beta(a_1 - c_1 v) + \alpha b_2 v - d_2 b_1\} u + \alpha c_2 v^2 - (\alpha a_2 + d_2 c_1)v + d_2 a_1$$

as a quadratic function with respect to $u > 0$ to obtain (2.3) with

$$U(v;\alpha,\beta) = \frac{1}{2\beta b_1} \left( (\alpha b_2 - \beta c_1) v + \beta a_1 - d_2 b_1 \\
+ \sqrt{(\alpha b_2 - \beta c_1)^2 v + \beta a_1 - d_2 b_1^2 + 4\beta b_1 (\alpha c_2 v^2 - (\alpha a_2 + d_2 c_1)v + d_2 a_1)} \right).$$

Then we complete the proof of Lemma 2.3. \hfill \Box

The next lemma is an elementary but a key property for the proof of Theorem 1.2. To state the property, we define an unbounded region $\Sigma$ by

$$\Sigma := \{(u,v) \in \mathbb{R}_+^2 : d_2 a_1 + d_1 a_2 - (d_2 b_1 + d_1 b_2) u - (d_2 c_1 + d_1 c_2) v < 0\}.$$

It should be noted that $\Sigma$ is independent of $\alpha$ and $\beta$. 
Lemma 2.4. If \((u, v) \in \Sigma\) satisfies \(F(u, v; \alpha, \beta) \geq 0\), then \(G(u, v, \alpha, \beta) < 0\).

Proof. Since (1.8) yields
\[
F(u, v; \alpha, \beta) + G(u, v; \alpha, \beta) = d_2a_1 + d_1a_2 - (d_2b_1 + d_1b_2)u - (d_2c_1 + d_1c_2)v,
\]
the desired property follows. \(\square\)

By Lemmas 2.2, 2.4, we shall accomplish the proof of Theorem 1.2.

Proof of Theorem 1.2. For any small \(\eta > 0\), let \(\alpha > 0\) and \(\beta > 0\) satisfy
\[
\eta \leq \frac{\alpha}{\beta} \leq \frac{1}{\eta}.
\]
(2.5)

We first discuss the case when \(0 < \alpha \leq \eta\). In this case, (2.5) implies \(\beta \leq 1\). Then we can use an estimate obtained by Lou and Ni [17, Lemma 2.3] to know
\[
\|u\|_\infty \leq C_1 \left(1 + \frac{\alpha}{d_1}\right) \leq C_1 \left(1 + \frac{\eta}{d_1}\right) \text{ and } \|v\|_\infty \leq C_1 \left(1 + \frac{\beta}{d_2}\right) \leq C_1 \left(1 + \frac{1}{d_2}\right)
\]
with some positive constant \(C_1 = C_1(a_i, b_i, c_i)\). Similarly, also in the case \(0 < \beta \leq \eta\), their result leads to
\[
\|u\|_\infty \leq C_1 \left(1 + \frac{\alpha}{d_1}\right) \leq C_1 \left(1 + \frac{1}{d_1}\right) \text{ and } \|v\|_\infty \leq C_1 \left(1 + \frac{\beta}{d_2}\right) \leq C_1 \left(1 + \frac{\eta}{d_2}\right).
\]
Then, for the sake of the proof of Theorem 1.2, we may assume
\[
\alpha > \eta \text{ and } \beta > \eta,
\]
(2.6)
in addition to (2.5). Our first aim is to prove that any solution \((u, v)\) of (1.1) satisfies
\[
\|u\|_\infty \leq \max \left\{ \frac{a_1}{b_1}, U\left(\max \left\{ \frac{d_2a_1 + d_1a_2}{d_2b_1 + d_1b_2}, \tilde{v}_0(\alpha) \right\}; \alpha, \beta \right) \right\},
\]
(2.7)
where \(U > 0\) and \(\tilde{v}_0(\alpha) \geq 0\) are numbers represented as (2.4) and (2.2), respectively. Here we recall Lemma 2.2 to note that if \(u > \max\{a_1/b_1, U(\tilde{v}_0(\alpha); \alpha, \beta)\}\), then \(u = U(v; \alpha, \beta)\) and \(v = V(u; \alpha, \beta)\) are inverses of each other, and these functions are monotone increasing with
\[
\lim_{u \to \infty} U(v; \alpha, \beta) = \infty \text{ and } \lim_{v \to \infty} U(v; \alpha, \beta) = \infty,
\]
where \(U(\tilde{v}_0(\alpha); \alpha, \beta) := \lim_{v \to \infty} U(v; \alpha, \beta)\). In order to show (2.7) for any solution \((u, v)\) of (1.1), we employ a proof by contradiction. Suppose for contradiction that (1.1) admits a solution \((u, v)\) of (1.1) satisfying
\[
\|u\|_\infty > \max \left\{ \frac{a_1}{b_1}, U\left(\max \left\{ \frac{d_2a_1 + d_1a_2}{d_2b_1 + d_1b_2}, \tilde{v}_0(\alpha) \right\}; \alpha, \beta \right) \right\}.
\]
(2.8)
Let \(x^* \in \overline{\Omega}\) be a maximum point of \(u\), that is, \(\|u\|_\infty = u(x^*)\). Since \((u, v)\) satisfies (1.7), then we know
\[
F(u(x^*), v(x^*); \alpha, \beta) \geq 0
\]
by applying (i) of Lemma 2.2 to the first equation of (1.7). Together with \(u(x^*) > a_1/b_1\) from (2.8), we know from (i) of Lemma 2.3 that
\[
v(x^*) \geq V(u(x^*); \alpha, \beta).
\]
(2.9)
If \(\tilde{v}_0(\alpha) \geq (d_2a_1 + d_1a_2)/(d_2b_1 + d_1b_2)\), then it follows from (2.8) that
\[
u(x^*) \geq \max \left\{ \frac{a_1}{b_1}, U(\tilde{v}_0(\alpha); \alpha, \beta) \right\}.
\]
(2.10)
Here we recall that the function \( v \mapsto V(v; \alpha, \beta) \) is monotone increasing for \( v > \tilde{v}_0(\alpha) \) as well as the function \( u \mapsto U(u; \alpha, \beta) \) is monotone increasing for \( u > \max\{a_1/b_1, U(\tilde{v}_0(\alpha); \alpha, \beta)\} \) since \( V(u; \alpha, \beta) \) is an inverse function of \( U(v; \alpha, \beta) \). Then (2.10) leads to \( V(u(x^*); \alpha, \beta) > \tilde{v}_0(\alpha) \), and thereby,

\[
v(x^*) > \tilde{v}_0(\alpha) \geq \frac{d_2a_1 + d_1a_2}{d_2b_1 + d_1b_2}.
\]

(2.11)

Then, in the case when \( \tilde{v}_0(\alpha) \geq (d_2a_1 + d_1a_2)/(d_2b_1 + d_1b_2) \), by virtue of (2.10), (2.11) and the monotone increasing property of \( U(v; \alpha, \beta) \) as well as \( V(u; \alpha, \beta) \), we know from Lemma 2.3 that the strip region

\[
\mathcal{R} := \{(u, v) \in \mathbb{R}^2_+ : 0 < u \leq u(x^*), \ v \geq v(x^*)\}
\]

is contained in the positive region of \( F \), that is,

\[
\mathcal{R} \subset \{F > 0\},
\]

where \( \{F > 0\} := \{(u, v) \in \mathbb{R}^2_+ : F(u, v; \alpha, \beta) > 0\} \). On the other hand, if \( \tilde{v}_0(\alpha) < (d_2a_1 + d_1a_2)/(d_2b_1 + d_1b_2) \), then (2.5) implies

\[
u(x^*) > a_1/b_1 \quad \text{and} \quad u(x^*) > U\left(\frac{d_2a_1 + d_1a_2}{d_2b_1 + d_1b_2}; \alpha, \beta\right).
\]

Since \( u \mapsto V(u; \alpha, \beta) \) is monotone increasing for \( V > \tilde{v}_0(\alpha) \) and it is an inverse function of \( U(v; \alpha, \beta) \), then \( V(u(x^*); \alpha, \beta) > (d_2a_1 + d_1a_2)/(d_2b_1 + d_1b_2) \). With (2.9), one can see

\[
v(x^*) > \frac{d_2a_1 + d_1a_2}{d_2b_1 + d_1b_2} > \tilde{v}_0(\alpha).
\]

(2.12)

Similarly, the monotone increasing property of \( U(v; \alpha, \beta) \) or \( V(u; \alpha, \beta) \) implies \( \mathcal{R} \subset \{F > 0\} \). Together with (2.11) and (2.12), we can see that our assumption (2.8) leads to

\[
\mathcal{R} \subset \{F > 0\} \cap \Sigma.
\]

(2.13)

Next, let \( y^* \in \Sigma \) be a maximum point of \( v \), namely, \( ||v||_\infty = v(y^*) \). Then applying (i) of Lemma 2.2 to the second equation of (1.7), one can see that

\[
G(u(y^*), v(y^*); \alpha, \beta) \geq 0.
\]

(2.14)

Since \( 0 \leq u(y^*) \leq u(x^*) = ||u||_\infty \) and \( v(x^*) \leq v(y^*) = ||v||_\infty \), then \( (u(y^*), v(y^*)) \in \mathcal{R} \). Then (2.13) implies that

\[
F(u(y^*), v(y^*); \alpha, \beta) > 0 \quad \text{and} \quad (u(y^*), v(y^*)) \in \Sigma.
\]

Therefore, Lemma 2.4 leads to \( G(u(y^*), v(y^*); \alpha, \beta) < 0 \). Hence this contradicts (2.14). Consequently, the above proof by contradiction enables us to conclude that all solutions of (1.1) satisfy (2.7).

Next we shall find a positive constant \( C_2 = C_2(\eta, d_i, a_i, b_i, c_i) \) such that

\[
U\left(\max\left\{\frac{d_2a_1 + d_1a_2}{d_2b_1 + d_1b_2}, \tilde{v}_0(\alpha)\right\}; \alpha, \beta\right) \leq C_2
\]

(2.15)

for any \( \alpha, \beta > \eta \) with \( \eta \leq \alpha/\beta \leq 1/\eta \). To this end, we recall \( U \) and \( \tilde{v}_0(\alpha) \) defined by (2.4) and (2.2) are expressed as

\[
U(v; \alpha, \beta) = \frac{1}{2b_1}\left((rb_2 - c_1)v + a_1 - \frac{d_2b_1}{\beta}\right)
\]

\[
\sqrt{\left((rb_2 - c_1)v + a_1 - \frac{d_2b_1}{\beta}\right)^2 + 4b_1\left(r(c_2v^2 - \left(\frac{d_2c_1}{\beta} + a_2\right)v + \frac{d_2a_1}{\beta}\right)}
\]
where \( r := \alpha/\beta \) and \( \bar{v}_0(\alpha) = 0 \) or
\[
\bar{v}_0(\alpha) = \frac{1}{2c_2} \left\{ a_2 + \frac{d_2c_1}{\alpha} + \sqrt{\left( a_2 + \frac{d_2c_1}{\alpha} \right)^2 - \frac{4d_2a_1c_2}{\alpha}} \right\}.
\]

Hence these expressions ensure a desired positive constant \( C_2 = C_2(\eta, d_1, a_1, b_1, c_1) \) satisfying (2.15) if \( \alpha, \beta > r \) and \( \eta \leq r \leq 1/\eta \) (recall (2.6)). Consequently, we obtain a positive constant \( C = C(\eta, d_1, a_1, b_1, c_1) \) such that \( \|u\|_\infty \leq C \) for any solution \((u, v)\) of (1.1). Obviously, by the same argument replacing the first equation by the second one in (1.1), we can also obtain the desired estimate of \( \|v\|_\infty \) for any solution \((u, v)\) of (1.1). Then we complete the proof of Theorem 1.3.

3. Full cross-diffusion limit

In this section, we study the asymptotic behavior of nonconstant solutions of (1.1) as \( \alpha \to \infty \) and \( \beta \to \infty \) with \( \alpha/\beta \to \gamma > 0 \) to prove Theorem 1.3. In the proof, the following lemma by Lou and Ni [IS 2000, Lemma 2.4] will be used.

**Lemma 3.1** ([IS 2000]). Suppose that \( a_1/a_2 \neq b_1/b_2, a_1/a_2 \neq c_1/c_2 \) and \( \{(u_n, v_n)\} \) are positive solutions of (1.1) with \( (d_1, d_2, \alpha, \beta) = (d_{1,n}, d_{2,n}, \alpha_n, \beta_n) \). Assume that \( (u_n, v_n) \to (u^*, v^*) \) uniformly in \( \overline{\Omega} \) as \( n \to \infty \) for some nonnegative constants \( u^* \) and \( v^* \). Then, either
\[
\frac{b_2}{b_1} > \frac{a_2}{a_1} > \frac{c_2}{c_1} \quad \text{or} \quad \frac{b_2}{b_1} < \frac{a_2}{a_1} < \frac{c_2}{c_1},
\]
moreover, \( (u^*, v^*) \) is the unique root of \( a_1 - b_1u - c_1v = a_2 - b_2u - c_2v = 0 \).

**Proof of Theorem 1.3** Suppose that \( \{\{(\alpha_n, \beta_n)\}\} \) is any positive sequence satisfying \( \alpha_n \to \infty, \beta_n \to \infty \) and \( \gamma_n := \alpha_n/\beta_n \to \gamma \) with some positive number \( \gamma \). Let \( \{(u_n, v_n)\} \) be positive solutions of (1.1) with \( (\alpha, \beta) = (\alpha_n, \beta_n) \). Multiplying the second equation of (1.1) by \( \gamma_n \) and subtracting the resulting expression from the first equation, we see that
\[
w_n(x) := d_1u_n(x) - \gamma_n d_2v_n(x)
\]
satisfies
\[
- \Delta w_n = f(u_n, v_n) - \gamma_n g(u_n, v_n) \quad \text{in} \quad \Omega, \quad \partial_\nu w_n = 0 \quad \text{on} \quad \partial \Omega.
\]
In view of the diffusion part of the first equation of (1.1), we set
\[
z_n(x) := d_1\alpha_n u_n(x) + u_n(x)v_n(x),
\]
which satisfies
\[
- \Delta z_n = \frac{1}{\alpha_n} f(u_n, v_n) \quad \text{in} \quad \Omega, \quad \partial_\nu z_n = 0 \quad \text{on} \quad \partial \Omega.
\]
It is possible to check that the correspondence of \((u_n, v_n)\) to \((w_n, z_n)\) defined by (3.1) and (3.3) is one-to-one, and more precisely, \((u_n, v_n)\) is expressed as
\[
\begin{align*}
\left\{ \begin{array}{l}
\quad u_n = \frac{1}{2d_1} \left( \sqrt{(w_n - \frac{d_1d_2}{\beta_n})^2 + 4\gamma_n d_1 d_2 z_n + w_n - \frac{d_1d_2}{\beta_n}} \right), \\
\quad v_n = \frac{1}{2\gamma_n d_2} \left( \sqrt{(w_n + \frac{d_1d_2}{\beta_n})^2 + 4\gamma_n d_1 d_2 z_n - \frac{4d_1d_2w_n}{\beta_n}} - (w_n + \frac{d_1d_2}{\beta_n}) \right).
\end{array} \right.
\end{align*}
\]
Owing to Theorem 1.2 there exists a positive constant \( C_3 = C_3(\eta, d_1, a_1, b_1, c_1) \) such that
\[
\|f(u_n, v_n)\|_\infty, \quad \|g(u_n, v_n)\|_\infty, \quad \|w_n\|_\infty, \quad \|z_n\|_\infty \leq C_3
\]
for all $n \in \mathbb{N}$. By applying the elliptic regularity theory (e.g., [5]) to (3.2) and (3.4), for any $p > 1$, we find a a positive constant $C_4 = C_4(\eta, d_1, a_i, b_i, c_i, p)$ such that
\[
\|w_n\|_{W^{2,p}}, \quad \|z_n\|_{W^{2,p}} \leq C_4
\]
for all $n \in \mathbb{N}$. Therefore, the Sobolev embedding theorem and the elliptic regularity theory ensure $w, z \in W^{2,p}(\Omega) \cap C^1(\overline{\Omega})$ for sufficiently large $p > 1$ such that
\[
\lim_{n \to \infty} (w_n, z_n) = (w, z) \quad \text{strongly in } C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) \quad \text{and weakly in } W^{2,p}(\Omega) \times W^{2,p}(\Omega),
\]
passing a subsequence if necessary. Since $\{\|f(u_n, v_n)\|_{\infty}\}$ is uniformly bounded with respect to $n \in \mathbb{N}$, then setting $n \to \infty$ in (3.4) implies that $z(x)$ is a harmonic function in $\Omega$ with $\partial_{\nu} z = 0$ on $\partial \Omega$, and therefore, $z(x) = \tau$ in $\Omega$ with some nonnegative constant $\tau$. Simultaneously, (3.3) with $\|u_n\|_{\infty} \leq C$ leads to
\[
\lim_{n \to \infty} u_n v_n = \tau \quad \text{uniformly in } \overline{\Omega}. \tag{3.7}
\]
In a case when $\tau > 0$, one can deduce from (3.5) and (3.6) that
\[
\lim_{n \to \infty} (u_n, v_n) = \left( \frac{\sqrt{w^2 + 4\gamma d_1 d_2 \tau} + w}{2d_1}, \frac{\sqrt{w^2 + 4\gamma d_1 d_2 \tau} - w}{2\gamma d_2} \right) \quad \text{in } C^1(\overline{\Omega}) \times C^1(\overline{\Omega}). \tag{3.8}
\]
Therefore, together with (3.6), we set $n \to \infty$ in (3.2) to verify that $w \in W^{2,p}(\Omega)$ satisfies
\[
\begin{cases}
-\Delta w = f\left( \frac{\sqrt{w^2 + 4\gamma d_1 d_2 \tau} + w}{2d_1}, \frac{\sqrt{w^2 + 4\gamma d_1 d_2 \tau} - w}{2\gamma d_2} \right) \\
-\gamma g\left( \frac{\sqrt{w^2 + 4\gamma d_1 d_2 \tau} + w}{2d_1}, \frac{\sqrt{w^2 + 4\gamma d_1 d_2 \tau} - w}{2\gamma d_2} \right)
\end{cases} \quad \text{in } \Omega,
\begin{cases}
\partial_{\nu} w = 0
\end{cases} \quad \text{on } \partial \Omega. \tag{3.9}
\]
In this case, the Schauder estimate for elliptic equations ensures that $w$ is a classical solution of (3.9). Since $\tau > 0$, then (3.7) and (3.8) imply $w(x) := \lim_{n \to \infty} u_n(x) > 0$ and $v(x) := \lim_{n \to \infty} v_n(x) > 0$ for all $x \in \overline{\Omega}$. Integrating (3.4) over $\Omega$, we obtain $\int_{\Omega} f(u_n, v_n) = 0$ for $n \in \mathbb{N}$.

By (3.8), the Lebesque convergence theorem ensures that
\[
\int_{\Omega} f\left( \frac{\sqrt{w^2 + 4\gamma d_1 d_2 \tau} + w}{2d_1}, \frac{\sqrt{w^2 + 4\gamma d_1 d_2 \tau} - w}{2\gamma d_2} \right) = 0.
\]
In the other case when $\tau = 0$, (3.3) and (3.6) ensure
\[
\lim_{n \to \infty} (u_n, v_n) = \left( \frac{|w| + w}{2d_1}, \frac{|w| - w}{2\gamma d_2} \right) = \left( \frac{w_+}{d_1}, \frac{w_-}{\gamma d_2} \right) \quad \text{uniformly in } \overline{\Omega},
\tag{3.10}
\]
where $w_+ := \max\{w, 0\}$ and $w_- := -\min\{w, 0\} \geq 0$. Setting $n \to \infty$ in (3.2), we know that $w \in W^{2,p}(\Omega) \cap C^1(\overline{\Omega})$ satisfies
\[
\begin{cases}
-\Delta w = f\left( \frac{w_+}{d_1}, \frac{w_-}{\gamma d_2} \right) - \gamma g\left( \frac{w_+}{d_1}, \frac{w_-}{\gamma d_2} \right) \\
\partial_{\nu} w = 0
\end{cases} \quad \text{in } \Omega,
\begin{cases}
\partial_{\nu} w = 0
\end{cases} \quad \text{on } \partial \Omega.
\]
To accomplish the proof of the second limiting case (ii) stated in Theorem 1.3, it remains to prove that $u(x) := \lim_{n \to \infty} u_n(x)$ and $v(x) := \lim_{n \to \infty} v_n(x)$ (obtained by (3.10)) are not constant. Suppose for contradiction that $u$ (resp. $v$) is a positive constant. Since $wv = 0$ in $\Omega$ by (3.7), one can see that $v = 0$ (resp. $u = 0$) in $\Omega$. Namely, (3.10) implies that $(u_n, v_n) \to (u, 0)$ (resp. $(u_n, v_n) \to (0, v)$) uniformly in $\overline{\Omega}$. This contradicts Lemma 5.1. Obviously, $(u, v) = (0, 0)$ also contradicts Lemma 3.1. Consequently, we deduce that $w_+$ and $w_-$ are not identically zero in $\Omega$, in other words, $w$ is sign-changing. Then we complete the proof of Theorem 1.3.
4. Existence of nonconstant solutions of limiting systems

In this section, as a beginning of study for the limiting system (1.9) of incomplete segregation, an existence result of nonconstant solutions will be shown. In order to state the result, we note that (1.1) admits a unique positive constant solution

\[
(u^*, v^*) := \frac{1}{b_2 c_1 - b_1 c_2}(a_2 c_1 - a_1 c_2, a_1 b_2 - a_2 b_1)
\]

in the weak competition case \(c_1/c_2 < a_1/a_2 < b_1/b_2\) or the strong competition case \(b_1/b_2 < a_1/a_2 < c_1/c_2\), and therefore, (1.9) with

\[
\tau = \tau^* := u^*v^*
\]

admits a constant solution \(w^* := d_1 u^* - \gamma d_2 v^*\). In our analysis for (1.9) based on a framework of the bifurcation theory, \(w^*\) and \(\tau^*\) will be regarded as unknowns, \(d_1\) will play a role in a bifurcation parameter, and any other coefficients will be fixed as far as the weak or the strong competition case. The next result gives a local curve of nonconstant solutions of (1.9), which bifurcate from \(w^*, \tau^*\) when the bifurcation parameter \(d_1\) passes a threshold number. In what follows, all eigenvalues of \(-\Delta\) with homogeneous Neumann boundary condition on \(\partial\Omega\) will be denoted by \(0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots\) (counting multiplicity).

**Theorem 4.1.** Suppose that \(c_1/c_2 < a_1/a_2 < b_1/b_2\) or \(b_1/b_2 < a_1/a_2 < c_1/c_2\). Furthermore, assume that \(\lambda_j\) is a positive eigenvalue whose eigenspace is one-dimension. There exists a small \(\eta_j > 0\) such that if \(0 < b_1, c_2, d_2 < \eta_j\), then there exists \(\delta_j > 0\) such that nonconstant solutions of (1.9) bifurcate from the branch of positive constant solutions

\[
\{(d_1, w, \tau) : d_1 > 0, w = d_1 u^* - \gamma d_2 v^* (=: w^*(d_1)), \tau = \tau^*\}
\]

when \(d_1\) passes \(\delta_j\). More precisely, in a neighbourhood of \((d_1, w, \tau) = (\delta_j, w^*(\delta_j), \tau^*) \in \mathbb{R} \times W^2_\nu(\Omega) \times \mathbb{R}\), the set of nonconstant solutions of (1.9) form a curve represented by

\[
(d_1, w, \tau) = (d_1(s), w^*(d_1(s)) + s(\Phi_j + \psi(\cdot, s)), \tau(s)) \quad \text{for } s \in [-\sigma, \sigma] \tag{4.3}
\]

with some small \(\sigma > 0\), where \(d_1(s), \tau(s) \in \mathbb{R}_+\), \(\Phi_j(\cdot) \in C^1(\Omega, \mathbb{R})\) and \(\psi(\cdot, s) \in W^{2, p}_\nu(\Omega)\) satisfy

\[
-\Delta \Phi_j = \lambda_j \Phi_j \quad \text{in } \Omega, \quad \partial_\nu \Phi_j = 0 \quad \text{on } \partial\Omega, \quad \|\Phi_j\|_2 = 1, \tag{4.4}
\]

and \(\psi(\cdot, s) \in W^{2, p}_\nu(\Omega)\) satisfies \(\psi(\cdot, 0) = 0\) and \(\int_\Omega \psi(x, s) = \int_\Omega \Phi_j(x) \psi(x, s) = 0\) for any \(|s| \leq \sigma\).

**Proof.** Suppose that \(c_1/c_2 < a_1/a_2 < b_1/b_2\) or \(b_1/b_2 < a_1/a_2 < c_1/c_2\). Our aim is to construct a local curve \(\{(w, \tau)\}\) of nonconstant solutions of

\[
\begin{cases}
\Delta w + f(u(w, \tau), v(w, \tau)) - \gamma g(u(w, \tau), v(w, \tau)) = 0 & \text{in } \Omega, \\
\partial_\nu w = 0 & \text{on } \partial\Omega, \\
\int_\Omega f(u(w, \tau), v(w, \tau)) = 0,
\end{cases} \tag{4.5}
\]

where

\[
w(w, \tau) = \frac{\sqrt{w^2 + 4\gamma d_1 d_2 \tau} + w}{2d_1} \quad \text{and} \quad v(w, \tau) = \frac{\tau}{u(w, \tau)} = \frac{\sqrt{w^2 + 4\gamma d_1 d_2 \tau} - w}{2\gamma d_2}.
\]

Since \((w^*(d_1), \tau^*)\) is a positive constant solution of (4.5) for any \(d_1 > 0\), we shift \((w^*(d_1), \tau^*)\) to the origin by the change of variables

\[
\phi := w - w^*(d_1) \quad \text{and} \quad \xi = \tau - \tau^*.
\]

(4.6)
Hereafter we shall construct the solution curve so that \( \phi \) lies in the Banach space \( X := \{ \phi \in W^{2,p}_{\nu}(\Omega) : \int_{\Omega} \phi = 0 \} \). To this end, we define an operator \( F(d_1, \phi, \xi) : \mathbb{R} \times X \times \mathbb{R} \rightarrow L^p(\Omega) \times \mathbb{R} \) associated with (1.9) by

\[
F(d_1, \phi, \xi) = \begin{bmatrix}
F^{(1)}(d_1, \phi, \xi) \\
F^{(2)}(d_1, \phi, \xi)
\end{bmatrix},
\]

where

\[
F^{(1)}(d_1, \phi, \xi) := \Delta \phi + f(w^*(d_1) + \phi, \tau^* + \xi), v(w^*(d_1) + \phi, \tau^* + \xi)) - \gamma g(u^*(d_1) + \phi, \tau^* + \xi), v(w^*(d_1) + \phi, \tau^* + \xi))
\]

and

\[
F^{(2)}(d_1, \phi, \xi) := \int_{\Omega} f(u^*(d_1) + \phi, \tau^* + \xi), v(w^*(d_1) + \phi, \tau^* + \xi)).
\]

In order to find bifurcation points of nonconstant solutions of \( F(d_1, \phi, \xi) = 0 \) on the trivial solution branch \( \{(d_1,0) : d_1 > 0\} \), we first seek for degenerate points of the linearized operator of \( F \) around \( (\phi, \xi) = (0,0) \), which will be denoted by

\[
L(d_1) := F_{(\phi,\xi)}(d_1,0,0) \in \mathcal{L}(X \times \mathbb{R}, L^p(\Omega) \times \mathbb{R}),
\]

that is,

\[
L(d_1) = \begin{bmatrix}
L_{11}(d_1) & L_{12}(d_1) \\
L_{21}(d_1) & L_{22}(d_1)
\end{bmatrix} = \begin{bmatrix}
F^{(1)}_{\phi}(d_1,0,0) & F^{(1)}_{\xi}(d_1,0,0) \\
F^{(2)}_{\phi}(d_1,0,0) & F^{(2)}_{\xi}(d_1,0,0)
\end{bmatrix}.
\]

Since \( f(u^*, v^*) = g(u^*, v^*) = 0 \) and

\[
(w^*)^2 + 4\gamma d_1 d_2 \tau^* = (d_1 u^* - \gamma_2 v^*)^2 + 4\gamma d_1 d_2 v^* = (d_1 u^* + \gamma_2 v^*)^2,
\]

then a straightforward computation yields

\[
\begin{bmatrix}
f_{u}^* & f_{v}^* \\
g_{u}^* & g_{v}^*
\end{bmatrix} = \begin{bmatrix}
b_1 u^* & c_1 u^* \\
b_2 v^* & c_2 v^*
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
1 & u^* \\
-1 & -v^*
\end{bmatrix} \begin{bmatrix}
1/(4d_1) \\
1/(4d_2)
\end{bmatrix},
\]

where \( f_{u}^* := f_u(u^*, v^*), \ u_{w}^* := u_w(w^*(d_1), \tau^*) \) and other notations are defined by the same manner. It follows from (4.11) that orthogonal entries of \( L(d_1) \) are computed as follows

\[
L_{11}(d_1) = \Delta + f_{u}^* u_{w}^* + f_{v}^* v_{w}^* - \gamma (g_{w}^* u_{w}^* + g_{v}^* v_{w}^*)
\]

\[
= \Delta + \frac{(c_1 + \gamma b_2) \tau^* - b_1 (u^*)^2 - \gamma c_2 (v^*)^2}{d_1 u^* + \gamma_2 v^*}.
\]

and

\[
L_{22}(d_1) = (f_{u}^* u_{\tau}^* + f_{v}^* v_{\tau}^*) |\Omega| = -\frac{u^* |\Omega|}{4(d_1 u^* + \gamma d_2 v^*)} \left( \frac{b_1}{d_1} + \frac{c_1}{d_2} \right).
\]

Here we remark that

\[
L_{21}(d_1) \phi = (f_{u}^* u_{\tau}^* + f_{v}^* v_{\tau}^*) \int_{\Omega} \phi = 0 \quad \text{for any } \phi \in X,
\]

that is to say, \( L_{21}(d_1) = 0 \) for any \( d_1 > 0 \). Since (4.12) implies that \( L_{22}(d_1) < 0 \) for any \( d_1 > 0 \), we have only to investigate the degeneracy of \( L_{11}(d_1) \in \mathcal{L}(X, L^p(\Omega)) \). In view of the potential term of \( L_{11}(d_1) \) in (4.11), we use (4.1) and (4.2) to see that

\[
\lim_{b_1 \to 0, c_2 \to 0} \frac{(c_1 + \gamma b_2) \tau^* - b_1 (u^*)^2 - \gamma c_2 (v^*)^2}{d_1 u^* + \gamma_2 v^*} = \frac{(\gamma b_2 + c_1) a_1 a_2}{d_1 a_2 c_1 + \gamma d_2 a_1 b_2} > 0.
\]
Then, for any positive eigenvalue $\lambda_j$ whose eigenspace is one dimension, there exists a small positive number $\eta_j$ such that if $0 < b_1, c_2, d_2 < \eta_j$, then

$$\frac{(c_1 + \gamma b_2)\tau^* - b_1(u^*)^2 - \gamma c_2(v^*)^2}{d_1u^* + \gamma d_2v^*} \begin{cases} > \lambda_j & \text{for } 0 < d_1 < \delta_j, \\ = \lambda_j & \text{for } d_1 = \delta_j, \\ < \lambda_j & \text{for } d_1 > \delta_j \end{cases}$$

(4.13)

with some $\delta_j > 0$. Since $L_{11}(\delta_j) = \Delta + \lambda_j$, then $\text{Ker} L_{11}(\delta_j) = \text{Span}\{\Phi_j\} \subset X$, where $\Phi_j$ is an eigenfunction satisfying (4.4). It is noted that $\int_{\Omega} \Phi_j = 0$. Together with $L_{21}(\delta_j) = 0$ and $L_{22}(\delta_j) < 0$, we know that $\text{Ker} L(\delta_j) = \{ (\phi, \xi) = t(\Phi_j, 0) : t \in \mathbb{R} \}$. In order to use the local bifurcation theorem [1, Theorem 1.7], we have to check the following transversality condition

$$\mathcal{F}_{(\phi,\xi),d_1}(\delta_j,0,0) \begin{bmatrix} \Phi_j \\ 0 \end{bmatrix} \not\in \text{Ran} L(\delta_j).$$

(4.14)

To this end, it obviously suffices to show $\mathcal{F}_{(\phi,d_1)}^{(1)}(\delta_j,0,0)\Phi_j \not\in \text{Ran} L_{11}(\delta_j)$. Suppose for contradiction that $\mathcal{F}_{(\phi,d_1)}^{(1)}(\delta_j,0,0)\Phi_j \in \text{Ran} L_{11}(\delta_j)$. It is possible to verify that

$$\mathcal{F}_{(\phi,d_1)}^{(1)}(\delta_j,0,0)\Phi_j = \frac{-u^*\{(c_1 + \gamma b_2)\tau^* - b_1(u^*)^2 - \gamma c_2(v^*)^2\}}{d_1u^* + \gamma d_2v^*} \Phi_j = -\frac{u^*\lambda_j}{d_1u^* + \gamma d_2v^*} \Phi_j,$$

where the last equality comes from (4.13). By virtue of the Fredholm alternative theorem, one can see $\text{Ran} L_{11}(\delta_j) = \{ \psi \in L^p(\Omega) : \int_{\Omega} \psi = \int_{\Omega} \psi \Phi_j \, dx = 0 \}$. Then our assumption is equivalent to

$$-\frac{u^*\lambda_j}{d_1u^* + \gamma d_2v^*} \| \Phi_j \|^2 = 0.$$

This obviously contradicts $\| \Phi_j \|^2 = 1$. Therefore, the transversality condition (4.14) holds true. Consequently, we have checked all conditions for use of the local bifurcation theorem [1, Theorem 1.7] to obtain the bifurcation curve of nonconstant solutions expressed as (4.3) by way of (4.6). We complete the proof of Theorem 4.1.

\[ \square \]

**Remark 4.2.** By the bifurcation theorem by Krasnoselski [12], we can show that $(d_1, u, \tau) = (\delta_j, w^*, \tau^*)$ is still a bifurcation point in some sense under a weaker assumption on $\lambda_j$ that its multiplicity is odd.

Concerning the other limiting system (1.10) of complete segregation, since it is also a fast reaction limiting system of the Lotka-Volterra competition model with linear diffusion terms (see e.g., [2, 3, 4, 6, 9]). In particular, for the one-dimensional case, Dancer, Hilhorst, Mimura and Peletier [3, Theorem 4.1] obtained the following detailed structure of nontrivial solutions of (1.10).

**Theorem 4.3** ([3]). Suppose that $\Omega = (0,1)$. If $\sqrt{d_1/a_1} + \sqrt{d_2/a_2} \geq 2/\pi$, there is no nonconstant solution of (1.10). For each $n \in \mathbb{N}$, if $\sqrt{d_1/a_1} + \sqrt{d_2/a_2} < 2/(n\pi)$, then (1.10) has $n$-time(s) sign-changing solutions $w^{(n)}_{fg}, w^{(n)}_g \in C^2(\Omega)$ in the sense that the number of zeros of each is $n$. More precisely, $w^{(n)}_{fg}$ is expressed as

$$w^{(n)}_{fg}(x) = \begin{cases} \sum_{j=1}^{[n/2]} \psi(x - 2j/n) + \phi(x - (n - 1)/n) & \text{if } n \text{ is odd}, \\ \sum_{j=1}^{[n/2]} \psi(x - 2j/n) & \text{if } n \text{ is even} \end{cases}$$
for \( x \in \Omega \). Here, \( \phi \) is a unit part determining the profile of \( w_{gf}^{(n)} \) defined by

\[
\phi(x) = \begin{cases} 
  d_1 u(x) > 0 & \text{if } x \in [0, \theta_n), \\
  -\gamma d_2 v(x) < 0 & \text{if } x \in (\theta_n, 1/n], \\
  0 & \text{otherwise}
\end{cases}
\]

with some \( \theta_n \in (0, 1/n) \), and \( \psi(x) = \phi(x) + \phi(2/n - x) \), where \( u(x) \ (x \in [0, \theta_n]) \) and \( v(x) \ (x \in [\theta_n, 1/n]) \) are solutions of

\[
\begin{align*}
  d_1 u'' + u(a_1 - b_1 u) &= 0, & u > 0 > u' & \text{in } (0, \theta_n), \\
  d_2 v'' + v(a_2 - c_2 v) &= 0, & v, v' > 0 & \text{in } (\theta_n, 1/n), \\
  u(\theta_n) &= v(\theta_n) = 0, & d_1 u'(\theta_n) &= -\gamma d_2 v'(\theta_n), \\
  u'(0) &= v'(1/n) = 0.
\end{align*}
\]

On the other hand, \( w_{gf}^{(n)} \) is expressed by

\[
w_{gf}^{(n)}(x) = \begin{cases} 
  w_{gf}^{(n)}(1 - x) & \text{if } n \text{ is odd,} \\
  w_{gf}^{(n)}(x + 1/n) + \phi(x - (n - 1)n) & \text{if } n \text{ is even.}
\end{cases}
\]

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