MMD labeling of complete tripartite graphs

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Abstract. A simple, connected and undirected graph with \( n \) vertices is said to have (MMD) modular multiplicative divisor labeling if there exist a one to one and onto function \( f \) from vertices of the graph to set of all natural numbers from 1 to \( n \) and label induced on the edges by the product of labels of end vertices modulo \( n \) such that addition of all edge labels is congruent to 0 (mod \( n \)). This paper studies MMD labeling of complete tripartite graphs and some open problems.

1.  Introduction
All terms and expressions in this paper we follow Harary [1]. Every graph is finite, simple, connected and undirected. Many results about labeling of graphs are outlined in the survey paper [2]. Number theory contains more applications of graph labeling and graph coloring [6,7]. Motivated by the result of Marbun[3], MMD labeling of \( k \)- multilevel corona of path \( P_3 \) with disjoint union of \( 2m \) copies of \( K_1 \) and \( k \)- multilevel corona of \( C_n \) (adding pendant edge for each vertex of the cycle \( C_n \)) with disjoint union of \( 2m \) copies of path \( P_3 \), where \( k \) and \( m \) are positive integers have been studied [5]. Also [4] characterize certain families of MMD graphs. This paper studies MMD labeling of complete tripartite graphs and discuss open problems related to the topic.

2.  Preliminaries

**Definition 2.1.** A graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions.

**Definition 2.2.** A graph \( G \) with \( n \) vertices is said to have modular multiplicative divisor (MMD) labeling if there exist a bijection \( f \) from \( V(G) \) to \( \{1,2,...,n\} \) and the induced function \( f^* \) from \( E(G) \) to \( \{0,1,...,n-1\} \) where \( f^*(uv) = f(u)f(v) \pmod{n} \) for all \( uv \in E(G) \) such that \( n \) divides the sum of all edge labels of \( G \).

Modular multiplicative divisor labeling of a graph with \( n = 5 \) is given in figure 2.1. Sum of all edge labels = 0+0+0+0+3+2+3+2+1+4 = 15 \( \equiv 0 \pmod{5} \).
Definition 2.3. A simple graph $G$ is said to be a bipartite graph if the vertex set of $G$ can be divided into two disjoint non-empty subsets $X$ and $Y$ such that each edge joins a vertex in $X$ to a vertex in $Y$.

Definition 2.4. A bipartite graph $G$ is said to be a complete bipartite graph if every vertex of $X$ is adjacent to all vertices of $Y$.

Definition 2.5. A simple graph $G$ is said to be a complete tripartite graph if the vertex set of $G$ can be partitioned into three disjoint non-empty subsets $V_1$, $V_2$, $V_3$ such that an edge joins two vertices $u$, $v$ of $G$ if and only if $u$ and $v$ do not belong to the same $V_i$. It is denoted by $K_{l,m,n}$ if $V_1$, $V_2$, $V_3$ have $l$, $m$, $n$ elements respectively.

3. Main Results

Constructed large families of graphs using the operations corona, union and addition [5]. This section discusses MMD labeling of tripartite graphs.

Theorem 3.1 The complete tripartite graph $K_{l,m,n}$ admits MMD labeling.

Proof:

Let the vertices of $l$-vertices part, $m$-vertices part and $n$-vertices part of $K_{l,m,n}$ be $\{u_1,u_2,\ldots,u_l\}$, $\{v_1,v_2,\ldots,v_m\}$ and $\{w_1,w_2,\ldots,w_n\}$ respectively.

Let the graph $K_{l,m,n}$ be $G$ with $l+m+n=N$ (say) number of vertices and $lmn$ edges of $G$.

Case (i)

Any two vertices part of $K_{l,m,n}$ be even (This case includes $l,m,n$ all are even). Let it be $l$-vertices part and $m$-vertices part. That is $l \equiv 0 \pmod{2}$ and $m \equiv 0 \pmod{2}$. Label the vertices of $l$-vertices part and $m$-vertices part of $K_{l,m,n}$ as

$$f(u_i) = \begin{cases} 
  i & \frac{l}{2} + 1 \leq i \leq l \\
  N + \frac{l}{2} - i & 1 \leq i \leq \frac{l}{2} 
\end{cases}$$
Other vertices of $n$-vertices part can be assigned with the remaining integers. Sum of $l m n$ edge labels will be

$$f(v_i) = \left\{ \begin{array}{ll}
\frac{l}{2} + i & 1 \leq i \leq \frac{m}{2} \\
N - \frac{l}{2} - i & \frac{m}{2} + 1 \leq i \leq m
\end{array} \right.$$ 

$$f(v_i) \left[ f(u_1) + \ldots + f(u_l) \right] + f(v_j) \left[ f(u_1) + \ldots + f(u_l) \right] + \ldots + f(v_m) \left[ f(u_1) + \ldots + f(u_l) \right] + \left[ f(w_1) + \ldots + f(w_n) \right]$$

$$= \left[ \frac{lmN}{4} + \left( f(w_1) + \ldots + f(w_n) \right) \right] \left( l + m \right) N \left/ 2 \right.$$ 

which is a multiple of $N$.

**Example: 3.1** MMD labeling of complete tripartite graph $K_{4,2,3}$ is shown in figure 3.1

![Diagram](image)

**Fig. 3.1** MMD labeling of $K_{4,2,3}$

**Case (ii)**

Exactly two vertices part of $K_{l,m,n}$ be odd. Let $l \equiv 1(\mod 2)$ and $m \equiv 1(\mod 2)$ (That is $n \equiv 0(\mod 2)$). Labels of $l$-vertices part and $n$-vertices part of $K_{l,m,n}$ are
Vertices of \( m \)-vertices part which are not included above can be assigned with the remaining integers.

Sum of \( lm \) edge labels will be

\[
\begin{align*}
[f(u_1) + \ldots + f(u_l)] + [f(v_1) + \ldots + f(v_m)] + [f(w_1) + \ldots + f(w_n)] \\
= [N(l+1)/2] [f(v_1) + \ldots + f(v_m)] + [N(n/2)] [f(v_1) + \ldots + f(v_m)] + [f(u_1) + \ldots + f(u_l)]
\end{align*}
\]

which is a multiple of \( N \).

**Example: 3.2** MMD labeling of \( K_{3,3,2} \) is shown in figure 3.2

Fig. 3.2 MMD labeling of \( K_{3,3,2} \)

4. Conclusion and open problem
Studied MMD labeling of complete tripartite graphs $K_{l,m,n}$. The problem still remains open for all $l \equiv 1(\text{mod } 2), m \equiv 1(\text{mod } 2), n \equiv 1(\text{mod } 2)$.

Problem:

Does there exist MMD labeling of $K_{l,m,n}$ for all $l \equiv 1(\text{mod } 2), m \equiv 1(\text{mod } 2)$ and $n \equiv 1(\text{mod } 2)$?

References

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