Galilean Classification of Curves

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Abstract

In this paper, we classify space–time curves up to Galilean group of transformations with Cartan’s method of equivalence. As an aim, we elicit invariants from action of special Galilean group on space–time curves, that are, in fact, conservation laws in physics. We also state a necessary and sufficient condition for equivalent Galilean motions.

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1 Introduction

Galilean transformation group has an important place in classic and modern physics for instance: in quantum theory, gauge transformations in electromagnetism, in mechanics [1], and conductivity tensors in fluid dynamics [2], also, in mathematical fields such as Lagrangian mechanics, dynamics and control theory [4], and so on. In physics, when we study a curve in Galilean space–time \( \mathbb{R}^3 \times \mathbb{R} \), it is very important that we know about invariants of the curve, that are conservation laws. In [1] for example, a Hamiltonian vector field with some conditions, was introduced as a Galilean invariant of special Galilean transformations on \( T^*\mathbb{R}^3 \). But in this paper, by Cartan’s method of equivalence problem, we will find all invariants. We show that there are two functionally independent invariants for a curve in a Galilean space–time up to the action of special Galilean transformation group, such that other invariants are functions of these invariants and their derivations. Then, we use of this invariants to classify space–time curves, in respect to special Galilean transformations. In the next section, we state Cartan’s theorem, that is the main key for the classification. In section 3, we propound the definition of Galilean group as a matrix group and its properties. In the latest section, we determine the invariants and classify space–time curves up to special Galilean group. Finally, we prove that this invariants are a necessary and sufficient condition for specification of space–time curves. Then, we infer a physical result for Galilean motions, from this classification.

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2 Preliminaries

Let $G \subset \GL(n, \mathbb{R})$ be a matrix Lie group with Lie algebra $\mathfrak{g}$ and $P : G \to \text{Mat}(n \times n)$ be a matrix-valued function which embeds $G$ into $\text{Mat}(n \times n)$ the vector space of $n \times n$ matrices with real entries. Its differential is $dP_B : T_B G \to T_{P(B)} \text{Mat}(n \times n) \simeq \text{Mat}(n \times n)$.

**Definition 2.1** The following form of $G$ is called Maurer-Cartan form:

$$\omega_B = \{P(B)\}^{-1} . dP_B$$

that it is often written $\omega_B = P^{-1} . dP$. The Maurer-Cartan form is the key to classifying maps into homogeneous spaces of $G$, and this process need to this theorem (for proof refer to [3]):

**Theorem 2.2 (Cartan)** Let $G$ be a matrix Lie group with Lie algebra $\mathfrak{g}$ and Maurer-Cartan form $\omega$. Let $M$ be a manifold on which there exists a $\mathfrak{g}$-valued 1-form $\phi$ satisfying $d\phi = -\phi \wedge \phi$. Then for any point $x \in M$ there exist a neighborhood $U$ of $x$ and a map $f : U \to G$ such that $f^* \omega = \phi$. Moreover, any two such maps $f_1, f_2$ must satisfy $f_1 = L_B \circ f_2$ for some fixed $B \in G$ ($L_B$ is the left action of $B$ on $G$).

**Corollary 2.3 ([3])** Given maps $f_1, f_2 : M \to G$, then $f_1^* \omega = f_2^* \omega$, that is, this pull-back is invariant, if and only if $f_1 = L_B \circ f_2$ for some fixed $B \in G$.

By corollary 2.3, one can conclude that in the view of Cartan’s theorem, the relation $f_1^* \omega = f_2^* \omega$ offers the invariant functions. In fact, these functions that we call them invariants, when $f_1 = L_B \circ f_2$ for some fixed $B \in G$, will remain permanent for maps $f_1$ and $f_2$ under the pull-back action on Maurer-Cartan form $\omega$.

3 Galilean Transformation Group

Let $\mathbb{R}^3 \times \mathbb{R}$ be a standard Galilean space–time. A Galilean transformation is a transformation of $\mathbb{R}^3 \times \mathbb{R}$ as follows:

**Definition 3.1** A map $\phi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3 \times \mathbb{R}$ with the following definition

$$\begin{pmatrix} \mathbf{X} \\ t \end{pmatrix} \mapsto \begin{pmatrix} \mathbf{R} & \mathbf{v} \\ 0^T & 1 \end{pmatrix} \begin{pmatrix} \mathbf{X} \\ t \end{pmatrix} + \begin{pmatrix} \mathbf{y} \\ s \end{pmatrix}$$

is called a Galilean transformation, where $\mathbf{R} \in \text{O}(3, \mathbb{R})$, $s \in \mathbb{R}$, and $\mathbf{y}, \mathbf{v} \in \mathbb{R}^3$.

The set of Galilean transformations is a 10-dimensional group [3]. We call this group as Galilean transformation group or in brief, Galilean group, and
denote it by $\text{Gal}(4, \mathbb{R})$. We can also identify this group with the following matrix group

$$
\text{Gal}(4, \mathbb{R}) = \left\{ \begin{pmatrix}
1 & 0 & s \\
v & R & y \\
0 & 0 & 1
\end{pmatrix} \mid R \in O(3, \mathbb{R}), \ s \in \mathbb{R}, \text{ and } y, v \in \mathbb{R}^3 \right\}, \quad (1)
$$

that with the matrix multiplication, is a 10-dimensional group. Galilean group is a subgroup of affine transformation group $A(5, \mathbb{R})$ and so a subgroup of $\text{GL}(5, \mathbb{R})$. This group also has a smooth structure and so is a smooth manifold. Hence, with the smooth action of matrix multiplication, it is a Lie group with Lie algebra $\mathfrak{gal}(4, \mathbb{R})$. By its representation, we can find its Maurer–Cartan forms that provide a base for Lie algebra $\mathfrak{gal}(4, \mathbb{R})$.

**Definition 3.2** An element of Galilean transformation group is called *special galilean transformation*, if in representation (1), $R$ be in $\text{SO}(3, \mathbb{R})$. The group of all special Galilean transformations is called *special Galilean transformation group* (or special Galilean group in brief), and denoted by $\text{SGal}(4, \mathbb{R})$. So, we have

$$
\text{SGal}(4, \mathbb{R}) = \left\{ \begin{pmatrix}
1 & 0 & s \\
v & R & y \\
0 & 0 & 1
\end{pmatrix} \mid R \in \text{SO}(3, \mathbb{R}), \ s \in \mathbb{R}, \text{ and } y, v \in \mathbb{R}^3 \right\}.
$$

$\text{SGal}(4, \mathbb{R})$ is a connected component of $\text{Gal}(4, \mathbb{R})$, and a Lie group with Lie algebra $\mathfrak{sgal}(4, \mathbb{R})$. In next section, we consider the special Galilean group for classifying space–time curves, and similar to Galilean group’s Maurer–Cartan form will be computed, it provides a base for Lie algebra $\mathfrak{sgal}(4, \mathbb{R})$.

### 4 Classification of Space–time Curves

Let $c : [a, b] \to \mathbb{R} \times \mathbb{R}^3$ be a curve with following definition:

$$
c(t) := (t, X(t)) = (t, x_1(t), x_2(t), x_3(t)),
$$

in which, the space coordinate $X$, is a smooth vector-valued function with values in $\mathbb{R}^3$, and $x_i$ s for $i = 1, 2, 3$, are a smooth scalar functions.

**Definition 4.1** By ST–curve, we mean a curve of class $C^5$ that is in four dimensional space–time $\mathbb{R} \times \mathbb{R}^3$, with this condition that it has no singular point, i.e. $\det(X', X'', X''') = X' \cdot (X'' \times X''') \neq 0$. We may assume that this value be positive.

If $c(t) = (t, X(t))$ be a ST–curve, for all point $t \in [a, b]$ we have $X'(t) \neq 0$, and the curve $X : t \mapsto X(t)$ will be regular and one can reparameterize it with arc length parameter $s$, so that for each point $s$, we have $||X'(s)|| = 1$. 

3
Definition 4.2 We call \( c(t) = (t, X(t)) \) as regular, if the curve \( X(t) \) be regular. Also, we say that the parameter of \( c \) is arc length parameter, if the parameter be an arc length parameter of \( X \).

The group of Galilean transformation can act on a ST–curve at each point of the domain, when we equate \( \mathbb{R}^3 \times \mathbb{R} \) with

\[
\mathbb{R}^5 = \left\{ \begin{pmatrix} t \\ X \\ 1 \end{pmatrix} \right| t \in \mathbb{R}, \ X \in \mathbb{R}^3 \right\},
\]

hence the action can be defined. Therefore, we say that two ST–curves are equivalent if, their representations in \( \mathbb{R}^5 \) be Galilean equivalent.

Convention 4.3 Henceforth, we consider that image of ST–curve \( c \), be in \( \mathbb{R}^5 \) as above.

We may consider a new curve \( \alpha_c : [a, b] \rightarrow \text{Gal}(4, \mathbb{R}) \) rather than \( c \), in the following form:

\[
\alpha_c(t) := \begin{pmatrix}
1 & 0 & 0 & 0 & t \\
X' & X'' & X'' & X''' & X \\
\frac{||X''||}{||X' \times X'||} & \frac{||X'' \times X'||}{||X' \times X'||} & \frac{||X'' \times X'''||}{||X' \times X'||} & \frac{||X' \times X'''||}{||X' \times X'||} & X
\end{pmatrix}
\]

where \( X \) is assumed as column matrix, and \( || \cdot || \) is the Euclidean norm. Obviously, for every time \( t \in [a, b] \), \( \alpha_c(t) \) is an element of \( \text{Gal}(4, \mathbb{R}) \) and so \( \alpha_c \) is well-defined.

We can study \( \alpha_c \) instead of \( c \), up to the action of Galilean transformation group as following:

Let \( c, \tilde{c} : [a, b] \rightarrow \mathbb{R}^5 \) be two ST–curves with definitions \( t \mapsto (t, X(t), 1) \) and \( \tilde{t} \mapsto (\tilde{t}, \tilde{X}(\tilde{t}), 1) \) respectively. If \( c \) be equivalent to \( \tilde{c} \) up to \( \text{Gal}(4, \mathbb{R}) \), that is, \( \tilde{c} = A \circ c \) for \( A \in \text{Gal}(4, \mathbb{R}) \), we have

\[
\begin{pmatrix}
\tilde{t} \\
\tilde{X}
\end{pmatrix} = A \cdot \begin{pmatrix}
t \\
X
\end{pmatrix} = \begin{pmatrix}
1 & 0 & s \\
v & R & y \\
0 & 0 & 1
\end{pmatrix} \cdot \begin{pmatrix}
t \\
X
\end{pmatrix}
\]

then, we conclude that

\[
\tilde{t} = t + s, \quad \text{and} \quad \tilde{X} = R \cdot X + tv + y \tag{2}
\]

First, second, and third differentiations of (2) are in the following form

\[
\begin{align*}
\dot{X}' & = R \cdot X' + v \\
\ddot{X}' & = R \cdot X'' \\
\dddot{X}' & = R \cdot X'''
\end{align*}
\]
From above relations we have

\[
\alpha_{\tilde{c}} = \begin{pmatrix}
1 & 0 & 0 & 0 & \hat{t} \\
\dot{X}' & \frac{\ddot{X}''}{||\ddot{X}''||} & \frac{\dddot{X}'' \times \dddot{X}'''}{||\dddot{X}'' \times \dddot{X}'''||} & \frac{\dddot{X}'' \times (\dddot{X}'' \times \dddot{X}''')}{||\dddot{X}'' \times (\dddot{X}'' \times \dddot{X}'''')||} \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

\[= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
\frac{R \cdot \dot{X}''}{||R \cdot \dot{X}''||} & \frac{R \cdot \dddot{X}'' \times R \cdot \dddot{X}'''}{||R \cdot \dddot{X}'' \times R \cdot \dddot{X}'''||} & \frac{R \cdot \dddot{X}'' \times (R \cdot \dddot{X}'' \times R \cdot \dddot{X}''')}{||R \cdot \dddot{X}'' \times (R \cdot \dddot{X}'' \times R \cdot \dddot{X}'''')||} \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

\[= A \cdot \alpha_c,
\]

since the equation \(=^*\) is concluded by knowing that for every vectors \(X\) and \(Y\) in \(\mathbb{R}^3\), and any element \(R \in SO(3, \mathbb{R})\) we have \(R \cdot (X \times Y) = R \cdot X \times R \cdot Y\) and \(||R \cdot X|| = \det(R)||X|| = ||X||\). So we have

**Theorem 4.4** Two ST-curves \(c, \tilde{c} : [a, b] \to \mathbb{R}^3\) are equivalent up to \(A \in \text{SGal}(4, \mathbb{R})\) that is \(\tilde{c} = A \circ c\); if and only if, the associated curves \(\alpha_c\) and \(\alpha_{\tilde{c}}\) are equivalent up to \(A\) that is \(\alpha_{\tilde{c}} = A \circ \alpha_c\).

So our acclaim of working with \(\alpha_c\) instead of \(c\), does not reduce the problem, with this benefit that we can use of Cartan’s theorem for \(\alpha_c\) and then find its invariants. These invariants in the view of theorem 4.4, are invariants of \(c\). Henceforth, we classify new curves \(\alpha_c\) s up to \(\text{SGal}(4, \mathbb{R})\).

From Cartan’s theorem, a necessary and sufficient condition for \(\alpha_{\tilde{c}} = B \circ \alpha_C = L_B \circ \alpha_C\) by \(B \in \text{SGal}(4, \mathbb{R})\), is that for any left invariant 1-form \(\omega^i\) on \(\text{SGal}(4, \mathbb{R})\) we have \(\alpha^*_{\tilde{c}}(\omega^i) = \alpha^*_C(\omega^i)\), that is equivalent with \(\alpha^*_{\tilde{c}}(\omega) = \alpha^*_C(\omega)\), for natural \(\text{sgal}(4, \mathbb{R})\)-valued 1-form \(\omega = P^{-1} \cdot dP\), where \(P\) is the Maurer–Cartan form.

Thereby, we must compute the \(\alpha^*_{\tilde{c}}(P^{-1} \cdot dP)\), which is invariant under special Galilean transformations, that is, its entries are invariant functions of ST-curves. This \(5 \times 5\) matrix form, consists of arrays that are coefficients of \(dt\).

Since \(\alpha^*_{\tilde{c}}(P^{-1} \cdot dP) = \alpha^{-1}_C \cdot d\alpha_C\), so for finding the invariants, it is sufficient that we calculate the matrix \(\alpha^{-1}_C \cdot d\alpha_C\). By differentiating of determinant, we
have

\[
d\alpha_c(t) = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
X'' & A_1 & A_2 & A_3 & X' \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\, dt
\]

where we suppose that

\[
A_1 = \frac{X''||X'||^2 - X''(X''\cdot X''')}{||X'||^3},
\]

\[
A_2 = \frac{(X''(X''\cdot X'''))||X''||^2 - (X''\cdot X''')(X''\cdot X''')}{||X''||^2},
\]

\[
A_3 = \frac{X''(X''\cdot X''')(X''\cdot X''')||X''||^2}{||X''||^3} - \frac{(X''\cdot X''')(X''\cdot X''')(X''\cdot X''')}{||X''||^3}.
\]

We assumed that \( X \) is in the form \((x_1 \ x_2 \ x_3)^T\). By knowing that \( \det \alpha_c = 1 \), so the inverse matrix of \( \alpha_c \), \( \alpha_c(t)^{-1} \), is in the form of following matrix

\[
\begin{pmatrix}
\frac{1}{X'' \cdot X''} & 0 & 0 & 0 & 0 \\
\frac{X'' \cdot X''}{||X''||^2} & \frac{X''}{||X''||} & 0 & 0 & 0 \\
\frac{X'' \cdot X''}{||X''||^2} & \frac{X'' \cdot X''}{||X''||^2} & \frac{X''}{||X''||} & 0 & 0 \\
\frac{X'' \cdot X''}{||X''||^2} & \frac{X'' \cdot X''}{||X''||^2} & \frac{X'' \cdot X''}{||X''||^2} & \frac{X''}{||X''||} & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

where \( A = ||X'' \times (X'' \times X''')|| \), and the notation \( T \) means the transpose of a column matrix to be as a row matrix.

After some straight computations, we find \( \alpha_c^{-1} \cdot d\alpha_c \) as the multiplication of following matrix by \( dt \):

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
||X''|| & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{B \cdot (X'' \times C)}{A ||X'' \times X''||} & 0 \\
0 & \frac{A (X'' \times B) \cdot X'''}{||X''||^3 ||X'' \times X''||^2} & \frac{A (X'' \times B) \cdot C}{||X''||^2 ||X'' \times X''||^3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

where \( B = X'' \times X''' \) and \( C = X'' \times X''' \). We may assume that the parameter of ST–curve be the natural parameter, the arc length \( s \). Henceforth, We adopt the problem with this assumption. Thus we have \( ||X'|| = 1 \), then

\[
X' \cdot X'' = 0.
\] (3)
Also if we assume that

\[ |\mathbf{X}'| = \text{constant}, \]
\[ |\mathbf{X}''| = \text{constant}, \]  

(4)

(5)

By using of (3) and (4), we have \( \mathbf{X}' \cdot \mathbf{X}'' = -|\mathbf{X}''| = \text{constant}. \) By differentiating of (4) and (5) in respect to \( s \), we achieve that \( \mathbf{X}'' \cdot \mathbf{X}''' = 0 \) and \( \mathbf{X}''' \cdot \mathbf{X}'''' = 0 \), respectively. Since

\[ |\mathbf{X}' \times \mathbf{X}''|^2 = |\mathbf{X}''|^2 |\mathbf{X}'''|^2 - (\mathbf{X}'' \cdot \mathbf{X}''')^2, \]

(6)

(7)

From (4) and (6), we conclude that

\[ A = |\mathbf{X}'' \times (\mathbf{X}'' \times \mathbf{X}''')| \]

is also constant.

Since \( \mathbf{X}'' \) is perpendicular to \( \mathbf{X}''' \), so \( \mathbf{X}'' \times (\mathbf{X}'' \times \mathbf{X}''') = -\mathbf{X}''' \) and then we find that

\[ (\mathbf{X}'' \times \mathbf{B}) \cdot \mathbf{X}''' = -|\mathbf{X}'''| \]

is constant,

\[ (\mathbf{X}'' \times \mathbf{B}) \cdot \mathbf{C} = \mathbf{B} \cdot (\mathbf{X}'' \times \mathbf{C}) \]

\[ = \mathbf{X}'' \cdot (\mathbf{X}''' \times \mathbf{X}''') \]

\[ = \sqrt{\det(\mathbf{X}(i) \cdot \mathbf{X}(j))}_{2 \leq i, j \leq 4} \]

\[ = \sqrt{|\mathbf{X}'''|^2 (|\mathbf{X}'''| + |\mathbf{X}'''|^2)} \]

\[ = \text{constant}, \]

(8)

(9)

where in the latest relation, the relation (8) comes from this fact that: for every vectors \( \mathbf{X}_1, \mathbf{X}_2, \) and \( \mathbf{X}_3 \) in \( \mathbb{R}^3 \), we have \( \{\mathbf{X}_1 \cdot (\mathbf{X}_2 \times \mathbf{X}_3)\}^2 = \det(\mathbf{X}_1 \cdot \mathbf{X}_j). \) The equation (9) acquired from assuming that \( \mathbf{X}'' \cdot (\mathbf{X}'' \times \mathbf{X}''') \) be positive (one can consider the negative case), because the lengths \( |\mathbf{X}''|, |\mathbf{X}'''|, \) and \( |\mathbf{X}''''| \) do not vanish, therefore \( \mathbf{X}'' \cdot (\mathbf{X}''' \times \mathbf{X}''') \) is not zero. Finally, we have

\[
\alpha_c^{-1}(s) \cdot d\alpha_c(s) =
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
|\mathbf{X}''| & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{\mathbf{B} \cdot (\mathbf{X}'' \times \mathbf{C})}{A \ |\mathbf{X}'' \times \mathbf{X}'''|} & 0 & 0 \\
0 & -A |\mathbf{X}'''| & |\mathbf{X}'' \times \mathbf{X}'''|^2 & \frac{A (\mathbf{X}'' \times \mathbf{B}) \cdot \mathbf{C}}{|\mathbf{X}'' \times \mathbf{X}'''|^2 |\mathbf{X}'' \times \mathbf{X}'''|^3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

ds.

Thereupon, by using of \( s \), the arc length as parameter, and assuming that the second and third derivation of \( \mathbf{X} \) (the space coordinate of curve \( c(t) = (t, \mathbf{X}) \))
have invariant lengths, then the entries of the multiplications matrix in $\alpha^{-1} \cdot d\alpha$ will be all invariant. By previous description, although these invariants calculated for the curve $\alpha_c$, but they are in fact invariants of the curve $c$. We can summarize above results in following theorem:

**Theorem 4.5** Let $c : [a, b] \to \mathbb{R} \times \mathbb{R}^3$ be a ST-curve with definition $c(t) := (t, X(t))$, then $\omega_1 = \|X''\|$ and $\omega_2 = \|X'''\|$ are differential invariants of $c$ up to special Galilean group $\operatorname{SGal}(4, \mathbb{R})$. In general, every other differential invariant of $c$, is functionally dependency to $\omega_1$, $\omega_2$, and their derivations in respect to the parameter.

**Remark 4.6** If we consider a ST-curve, the dimension of its image is 1, but the dimension of $\mathbb{R} \times \mathbb{R}^3$ is 4, hence the number of invariants must be 3. In spite of finding two fundamental invariants $\omega_1$ and $\omega_2$, one can add another invariant for instance $\omega_3 := \|X' \times X''\|$, to complete set of essential invariants.

**Theorem 4.7** Let $c, \tau : [a, b] \to \mathbb{R} \times \mathbb{R}^3$ be two ST-curves. $c$ and $\tau$ are locally equivalent up to special Galilean group $\operatorname{SGal}(4, \mathbb{R})$, if and only if, $\omega_1 = \overline{\omega}_1$ and $\omega_2 = \overline{\omega}_2$.

**Proof:** Formerly, we proved that two curves which are locally equivalent up to special Galilean transformation, have same differential invariants. We prove the converse. Let $c$ and $\tau$ be two ST-curves on $[a, b]$ with representations respectively $(t, X)$ and $(t, \tilde{X})$. We assume that $\omega_1 = \overline{\omega}_1$ and $\omega_2 = \overline{\omega}_2$, we prove that there is a special Galilean transformation $A \in \operatorname{SGal}(4, \mathbb{R})$, such that $c$ and $\tau$ in the view of convention 4.3, will be special Galilean equivalent.

We suppose that images of $c$ and $\tau$ be in $\mathbb{R}^5$ by the convention. There is an element in $\operatorname{SGal}(4, \mathbb{R})$ that transform one point of $c$ to one of $\tau$, because if we consider arbitrary points $(t_0, X_0, 1)$ of $c$ and $(\tilde{t}_0, \tilde{X}_0, 1)$ of $\tau$, then there are unique $R \in \operatorname{SO}(3, \mathbb{R})$ and $y \in \mathbb{R}^3$ so that $\tilde{X}_0 = R \cdot X_0 + y$. Thus, there is the following element of $\operatorname{SGal}(4, \mathbb{R})$ that transforms the first point to the second:

$$
\begin{pmatrix}
1 & 0 & \tilde{t}_0 - t_0 \\
0 & R & y \\
0 & 0 & 1
\end{pmatrix}.
$$

So, we can assume that $c_1 := (t, X_1, 1)$ be a special Galilean transformation of $c$, that intersects $\tau$ in the time $t_0 \in [a, b]$, that is $c_1(t_0) = \tau(t_0)$. Let this special Galilean transformation be in the following form

$$
\begin{pmatrix}
1 & 0 & s_1 \\
0 & \operatorname{Id}_3 & y_1 \\
0 & 0 & 1
\end{pmatrix}.
$$

Then, since there are unique $\tilde{R} \in \operatorname{SO}(3, \mathbb{R})$ and $\tilde{y} \in \mathbb{R}^3$ so that transfer the
We suppose that the parameters of \( \alpha \), introduced with the action of following matrix of \( \text{SGal}(4, \mathbb{R}) \) and \( \alpha_t \in 1 \), then we will have the initial condition \( \alpha_t \in \mathcal{A} + \mathcal{B} \), and so, proof is complete.

Henceforth, we show that \( \alpha_{\vec{c}} = \alpha_{\hat{c}} \). Moreover, we have

\[
\alpha_{\hat{c}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \hat{R} & \hat{y} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & s_1 \\ 0 & \hat{R} & y_1 \\ 0 & 0 & 1 \end{pmatrix} \alpha_c
\]

\[
\alpha_{\hat{c}} = \begin{pmatrix} 1 & 0 & s_1 \\ 0 & \hat{R} & y_1 + \hat{y} \\ 0 & 0 & 1 \end{pmatrix} \alpha_c,
\]

hence \( \alpha_c \) and \( \alpha_{\hat{c}} \) are equivalent with an element of \( \text{SGal}(4, \mathbb{R}) \). Thereby, \( \alpha_c \) and \( \alpha_{\hat{c}} \) will be equivalent, and by theorem 4.4, the proof will be completed. Henceforth, we show that \( \alpha_{\vec{c}} = \alpha_{\hat{c}} \).

For curves \( \vec{c} \) and \( \hat{c} \) we have following equations, respectively

\[
d\alpha_{\vec{c}} = \alpha_{\vec{c}} \cdot \vec{b}
\]

\[
d\alpha_{\hat{c}} = \alpha_{\hat{c}} \cdot \hat{b},
\]

when \( \vec{b}, \hat{b} \in \text{sgal}(4, \mathbb{R}) \). But for \( c \) and \( \hat{c} \) from assumption, in every point of domain we have \( \omega_1 = \overline{c}_1 \) and \( \omega_2 = \overline{c}_2 \). Furthermore, in each point of \( [a, b] \),

\[
\overline{\omega}_1 := ||\overline{X}'''|| = ||\hat{R} \cdot X'''|| = \text{det}(\hat{R}) ||X'''|| = \omega_1
\]

\[
\overline{\omega}_2 := ||\overline{X}'''|| = ||\hat{R} \cdot X'''|| = \text{det}(\hat{R}) ||X'''|| = \omega_2.
\]

So, we have \( \overline{\omega}_1 = \overline{c}_1 \) and \( \overline{\omega}_2 = \overline{c}_2 \). Then, with above expressions we conclude that for all points of \( [a, b] \), \( \vec{b} \) and \( \hat{b} \) are same that we call it \( b \). Now, \( \alpha_{\vec{c}} \) and \( \alpha_{\hat{c}} \) are satisfied in first order equations (resp.) \( d\alpha_{\vec{c}} = \alpha_{\vec{c}} \cdot b \) and \( d\alpha_{\hat{c}} = \alpha_{\hat{c}} \cdot b \), with the initial condition \( \alpha_{\vec{c}}(t_0) = \alpha_{\hat{c}}(t_0) \). Therefore, we have \( \alpha_{\vec{c}}(t) = \alpha_{\hat{c}}(t) \) for all \( t \in [a, b] \), and so, proof is complete. \( \Box \)
Corollary 4.8  In the physical sense, we can consider that each ST–curve be a trace of a particle with mass \( m \) and under influence of a force \( F \). By theorem 4.7, we conclude that: 1. Two particles with same masses \( m = \tilde{m} \), under influence of forces \( F \) and \( \tilde{F} \) (resp.), have same trajectory, if and only if, the norms of \( F \) and its derivative \( F' \) be equal to the corresponding norms of \( \tilde{F} \) and its derivative \( \tilde{F}' \).

1. In particular, we suppose that two observers \( O \) and \( \tilde{O} \) move with accelerations \( a \) and \( \tilde{a} \) (resp.), in an inertial coordinate system. If we consider the paths of a particle (as ST–curves) with mass \( m \) in respect to observers \( O \) and \( \tilde{O} \), and under the effect of forces \( F \) and \( \tilde{F} \) (resp.), then the paths are equal under a special Galilean transformation, if and only if, \( ||F|| = ||\tilde{F}|| \) and \( ||F'|| = ||\tilde{F}'|| \).

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