STABLE AND EFFICIENT PETROV-GALERKIN METHODS FOR A KINETIC FOKKER-PLANCK EQUATION

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ABSTRACT. We propose a stable Petrov-Galerkin discretization of a kinetic Fokker-Planck equation constructed in such a way that uniform inf-sup stability can be inferred directly from the variational formulation. Inspired by well-posedness results for parabolic equations, we derive a lower bound for the dual inf-sup constant of the Fokker-Planck bilinear form by means of stable pairs of trial and test functions. The trial function of such a pair is constructed by applying the kinetic transport operator and the inverse velocity Laplace-Beltrami operator to a given test function. For the Petrov-Galerkin projection we choose an arbitrary discrete test space and then define the discrete trial space using the same application of transport and inverse Laplace-Beltrami operator. As a result, the spaces replicate the stable pairs of the continuous level and we obtain a well-posed numerical method with a discrete inf-sup constant identical to the inf-sup constant of the continuous problem independently of the mesh size. We show how the specific basis functions can be efficiently computed by low-dimensional elliptic problems, and confirm the practicability and performance of the method for a numerical example.

1. INTRODUCTION

In this manuscript we develop a stable and efficient Petrov-Galerkin approximation scheme for a kinetic Fokker-Planck equation of the form

\[ \partial_t u((t,x),v) + v \cdot \nabla_x u((t,x),v) = \Delta_v \left( \frac{n((t,x),v)}{q(x,v)} \right) \quad \text{in} \quad \Omega = I_t \times \Omega_x \times \Omega_v \]

with suitable inflow boundary conditions. The equation describes a particle density \( u \) dependent on time \( t \in I_t \), position \( x \in \Omega_x \subset \mathbb{R}^d \), \( d \in \{2,3\} \), and velocity \( v \in \Omega_v = S^{d-1} \).

Formulations for particle densities governed by kinetic equations arise in various contexts. Beyond the classical applications of radiative transfer and kinetic gas theory (see e.g. [17, 20]), kinetic equations are, for instance, also used to describe densities of tumor cells in multiscale descriptions of tumor spreading [34, 25]. The latter is the application we are mainly interested in in this manuscript. More

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More precisely, we consider more general differential operators on \( \Omega_v \) associated with bilinear forms that are either coercive or satisfy a Gårding inequality.
precisely, we focus on a discretization of a glioma tumor equation described in [34], for which (1) is a prototype.

We aim for a finite element discretization with guaranteed stability. Therefore, we focus on a Petrov-Galerkin discretization based on a stable variational formulation of (1), since in such a framework the well-posedness of the discrete scheme can often be inferred from respective results on the continuous level, see e.g. [44, 41, 15, 19].

First, we establish a full-dimensional variational formulation for (1) based on Bochner-type spaces mapping the combined space-time domain \( \Omega_{t,x} = I_t \times \Omega_x \) to a Sobolev space defined on the velocity domain \( \Omega_v \) similar to spaces defined in [11, 1]. Taking the viewpoint that the Fokker-Planck equation could be interpreted as a “generalization” of a parabolic equation with a \((d + 1)\)-dimensional kinetic transport operator \( \partial_t + v \cdot \nabla_x \) instead of a one-dimensional time derivative \( \partial_t \), we analyze the well-posedness of the variational formulation for (1) by combining respective approaches developed for parabolic equations [26, 41, 44] and for transport equations [15, 19, 10]. We show existence of a weak solution by verifying the dual inf-sup condition: To that end, similarly to [26, 41] specific function pairs in the trial and test spaces are constructed: We associate a test space function \( p \) to a trial space function roughly defined as \( w_p = p - (\Delta_v)^{-1}(\partial_t p + v \cdot \nabla_x p) \). Then the bilinear form evaluated in \( w_p \) and \( p \) can be bounded from below by the respective norms of \( w_p \) and \( p \), which leads to a lower bound for the dual inf-sup constant. This approach is a generalization of proofs for parabolic equations using a variant of \( w_p \) containing only the time derivative instead of the kinetic transport operator [26, 41] and of proofs for transport equations, where a “stable function pair” consists roughly of \(-b \cdot \nabla p\) and \( p\), see [15, 19, 10]. Under an additional assumption on the global traces of certain considered functions, we also show uniqueness of the solution similar to proofs for parabolic equations [26] and transport equations [4], and have a stability estimate dependent on the inf-sup constant which is similar to the respective estimates for parabolic equations.

To design the Petrov-Galerkin discretization, we use problem-specific trial spaces ensuring stability: We first choose an arbitrary discrete test space \( Y_{\delta} \) and then define the discrete trial space roughly as \( X_{\delta} = Y_{\delta} + (\Delta_v)^{-1}(\partial_t + v \cdot \nabla_x)Y_{\delta} \). The spaces thus consist of pairs \( w_{\delta,p},p_{\delta} \) that are the discrete counterparts of the pairs \( w_p,p \) used in the proof for the lower bound of the dual inf-sup constant. This approach automatically yields a well-posed discrete problem with the same stability constant as for the continuous problem independently of the choice of the test space and thus of the mesh size. The strategy to use an application of the transport operator for defining a stable trial space was already used for linear first-order transport equations [10] and for the wave equation [31] as an alternative to computing stable test spaces by approximately inverting the transport operator [19, 15]. Our choice ensures that the spaces can be efficiently computed in the course of the numerical scheme, where we apply the high-dimensional transport operator and only solve low-dimensional elliptic problems in the velocity domain due to the inverse Laplace-Beltrami operator. As a result, we can guarantee the stability of the method with low-dimensional computations that are not dominant in the computational costs of the full solution process.

Weak solutions and variational formulations for different types of kinetic Fokker-Planck equations have been defined and analyzed in various works, see e.g. [18, 11,
However, these approaches focus on the properties of the weak solution without an orientation towards a subsequent discretization. On the other hand, discretizations of kinetic Fokker-Planck equations are often not based on the direct connection to a weak solution or do not specifically consider stability estimates. In [37], a finite element discretization of a kinetic Fokker-Planck equation is described, where the well-posedness of the discrete problem is however not analyzed. Applying the framework of [22], a mixed variational formulation with a subsequent discretization for a generalized Fokker-Planck equation is proposed in [29]. In the context of neuronal networks, a Fokker-Planck equation is discretized with finite differences in [14]. Another well-established approach to discretize kinetic equations is the method of moments, applied to Fokker-Planck equations, for instance, in [27, 40], while a related approach in the context of hierarchical model reduction is proposed in [9]. For the related Vlasov-Fokker-Planck system there are, for instance, works based on finite differences [39, 46] and streamline-diffusion discontinuous Galerkin approximations [2, 3]. For the more general class of equations with nonnegative characteristic form, discontinuous Galerkin methods [33, 32] and also sparse tensor approximations [42] have been developed.

This paper is structured as follows. After a more detailed description of the considered Fokker-Planck equation in section 2, we introduce the suitable Bochner-type function spaces and establish density and trace properties in section 3. We then derive the variational formulation and prove the existence and uniqueness results in section 4. In section 5, we introduce the discrete scheme, show well-posedness and describe an efficient computation. These properties of the proposed method are finally confirmed for a numerical example in section 6.

2. The kinetic Fokker-Planck equation

In this paper we consider a simplified version of the kinetic Fokker-Planck equation developed in [34, sect. 2.4.2] that gives a mesoscopic description of the density of glioma tumor cells. Let \( \Omega_x \subset \mathbb{R}^d \), \( d \in \{2, 3\} \) be the spatial domain with piecewise \( C^1 \) boundary that is globally Lipschitz and let \( I_t := (0, T) \) be the time interval. Moreover, let the velocity domain be the \((d-1)\)-dimensional unit sphere \( \Omega_v := S^{d-1} \), which corresponds to the assumption of particles with constant speed but varying direction. As we will often treat space and time variables simultaneously, we denote by \( \Omega_{t,x} := I_t \times \Omega_x \) the space-time domain. The full domain is defined as \( \Omega := \Omega_{t,x} \times \Omega_v \).

Boundary conditions have to be prescribed at the inflow part of \( \partial \Omega \). To that end, we first define the spatial in- and outflow domains \( \Gamma_{-}^{+}(v) := \{ x \in \partial \Omega_x : n(x) \cdot v \geq 0 \} \subset \partial \Omega_x \), where \( n(x) \) is the unit outer normal to \( \partial \Omega_x \) at \( x \). The full in- and outflow domains \( \Gamma_- \) and \( \Gamma_+ \) are then defined as

\[
\Gamma_{-} = \{(t,x),v) \in \partial \Omega_{t,x} \times \Omega_v : (v^t) \cdot n(t,x) \geq 0 \} \subset \partial \Omega,
\]

where \( n(t,x) \) is the unit outer normal to \( \partial \Omega_{t,x} \) at \( (t,x) \). \( \Gamma_{-} \) thus contain both the temporal and the spatial boundaries, i.e., \( \Gamma_- \) contains the "initial boundary" and the \((v\text{-dependent})\) spatial inflow boundary whereas \( \Gamma_+ \) contains the final time boundary and the spatial outflow boundary.
The strong form of the Fokker-Planck equation then reads
\begin{equation}
\partial_t u((t,x),v) + v \cdot \nabla_x u((t,x),v) = \Delta_v \left( \frac{u((t,x),v)}{q(x,v)} \right) \quad \text{in } \Omega,
\end{equation}
\begin{equation}
u((t,x),v) = g((t,x),v) \quad \text{on } \Gamma_-,
\end{equation}
where $\Delta_v$ is the Laplace-Beltrami operator on the unit sphere $\Omega_v = S^{d-1}$, $q : \Omega_x \times \Omega_v \rightarrow \mathbb{R}$ is the so-called “tissue fiber orientation distribution” satisfying $q(x,v) \geq \alpha_q > 0$ for all $(x,v) \in \Omega_x \times \Omega_v$ and $\int_{\Omega_v} q(x,v) \, dv = 1$ for all $x \in \Omega_x$, and $g : \Gamma_- \rightarrow \mathbb{R}$ is the inflow boundary condition that contains the initial condition $g|_{t=0}$ as well as the spatial inflow boundary condition $g|_{\Gamma_-^x(v)}, v \in \Omega_v$.

In section 4, we develop a variational formulation for this equation, where we allow for a more general differential operator on $\Omega_v$, and give specific conditions on $q$ and $g$ leading to well-posedness.

3. Function spaces

To develop a variational formulation for (2) we first introduce the necessary function spaces. Since we aim for a full-dimensional (i.e., space-time-velocity) formulation, we use Bochner spaces mapping the space-time domain $\Omega_{t,x}$ to a space of functions on $\Omega_v$.

We start with the function space for the velocity variable: Since the equation contains a Laplace-Beltrami operator on the velocity domain $\Omega_v = S^{d-1}$, we define $V := H^1(\Omega_v) \subset L^2(\Omega_v)$ as the Sobolev space of weakly differentiable functions on the surface $\Omega_v = S^{d-1}$ with norm $\|\phi\|_V^2 = \|\phi\|_{L^2(\Omega_v)}^2 + \|\nabla_v \phi\|_{L^2(\Omega_v)}^2$. For details on the definition of Sobolev spaces on manifolds, see [21, 30]. We denote the dual space of $V$ by $V' := H^{-1}(\Omega_v)$. $V$ is a dense subspace of $L^2(\Omega_v)$ and we will make use of the Gelfand triple $V \hookrightarrow L^2(\Omega_v) \hookrightarrow V'$, where we denote the dual pairing by $\langle \cdot, \cdot \rangle_{V',V}$.

As function space for the full domain, we will use the space $L^2(\Omega_{t,x}; V)$ with norm
\begin{equation}
\|w\|_{L^2(\Omega_{t,x}; V)}^2 = \int_{\Omega_{t,x}} \|w(t,x)\|_V^2 \, d(t,x)
\end{equation}
for functions without space or time derivatives. To incorporate the kinetic space-time transport operator, we define (using from now on $(\frac{1}{t}) \cdot \nabla_{t,x} p := \partial_t p + v \cdot \nabla_x p$)
\begin{equation}
H_{FP}^1(\Omega) := \{ p \in L^2(\Omega_{t,x}; V) : (\frac{1}{t}) \cdot \nabla_{t,x} p \in L^2(\Omega_{t,x}; V') \},
\end{equation}
with norm
\begin{equation}
\|p\|_{H_{FP}^1(\Omega)}^2 := \|p\|_{L^2(\Omega_{t,x}; V)}^2 + \|(\frac{1}{t}) \cdot \nabla_{t,x} p\|_{L^2(\Omega_{t,x}; V')}^2.
\end{equation}
This definition is similar to the spaces used for other variants of the kinetic Fokker-Planck equation e.g. in [1, 11, 5]. We use ideas from [1] to show the following:

Proposition 3.1. The set $C^\infty(\Omega_{t,x} \times \Omega_v) \cap H_{FP}^1(\Omega)$ is dense in $H_{FP}^1(\Omega)$.

Proof. See Appendix A. □

To discuss the boundary behavior of functions in $H_{FP}^1(\Omega)$, we introduce weighted $L^2$-spaces as usually used for transport and kinetic equations e.g. [6, 12], [16, XXI,
§2) and also for different versions of the kinetic Fokker-Planck equation [11, 1]. For any $\Gamma \subseteq \partial \Omega$ we introduce $L^2(\Gamma, [(1, v)^T \cdot n])$ with norm
\[
\|w\|_{L^2(\Gamma, [(1, v)^T \cdot n])} := \int_{\Gamma} w^2 \frac{1}{v} \cdot n \, ds.
\]

Then, we can show first that functions in $H_{FP}^1(\Omega)$ admit local traces in $\partial \Omega \setminus \Gamma_0$:

**Proposition 3.2.** For every compact set $K \subset \Gamma_+$ (resp. $K \subset \Gamma_-$), the trace operator $w \mapsto w\varepsilon_K$ from $C^\infty(\Omega)$ to $L^2(K, [(1, v)^T \cdot n])$ extends to a continuous linear operator on $H_{FP}^1(\Omega)$.

For the proof we need to estimate the product of $H_{FP}^1(\Omega)$ functions with different test functions in the following way, where the proof can be found in Appendix A.

**Lemma 3.3.** Let $\phi \in C^1(\bar{\Omega})$. Then, the mapping $f \mapsto \phi f$ is continuous in $H_{FP}^1(\Omega)$ with the estimate
\[
\|\phi f\|_{H_{FP}^1(\Omega)} \leq C\|\phi\|_{C^1(\Omega)} \|f\|_{H_{FP}^1(\Omega)}.
\]

**Proof of Proposition 3.2.** We use ideas of the proof of a similar result for transport equations e.g. in [16, Chap. XXI, Thm. 1, p. 220]. Analogous results for spaces similar to $H_{FP}^1(\Omega)$ are also given in [1, Proofs of Lemmas 4.3, 7.6].

Given a compact set $K \subset \Gamma_+$, let $\eta_K \in C^1(\Omega)$ with $\eta_K = 1$ on $K$ and supp $\eta_K \cap \Gamma_- = \emptyset$. We then obtain by integrating by parts for $w \in C^\infty(\Omega)$
\[
\int_K w^2 \frac{1}{v} \cdot n \, ds = \int_K (\eta_K w)^2 \frac{1}{v} \cdot n \, ds \leq \int_{\partial \Omega} (\eta_K w)^2 \frac{1}{v} \cdot n \, ds
\]
\[
= \int_{\partial \Omega} (\eta_K w)^2 \frac{1}{v} \cdot n \, ds = 2 \int_{\Omega} \eta_K w \frac{1}{v} \cdot \nabla_{t,x}(\eta_K w) \, d((t, x), v)
\]
\[
\leq 2 \|\eta_K w\|_{L^2(\Omega_{t,x}, V)} \|\frac{1}{v} \cdot \nabla_{t,x}(\eta_K w)\|_{L^2(\Omega_{t,x}, V^*)}
\]
\[
\leq 2 \|\eta_K w\|_{H_{FP}^1(\Omega)}^2 \leq C\|\eta_K\|_{C^1(\Omega)}^2 \|w\|_{H_{FP}^1(\Omega)}^2.
\]

We thus have continuity of the mapping $w \mapsto w\varepsilon_K$ for all $w \in C^\infty(\Omega)$, and by density (Proposition 3.1) the mapping extends to a continuous operator $H_{FP}^1(\Omega) \to L^2(K, [(1, v)^T \cdot n])$. For $K \subset \Gamma_-$ the claim can be shown analogously using $\frac{1}{v} \cdot n = -\frac{1}{v} \cdot n$ on supp $\eta_K$ in (*).

This result ensures that $H_{FP}^1(\Omega)$ functions have a trace on the non-characteristic boundary $\Gamma_+ \cup \Gamma_-$. However, from the local existence of traces we cannot directly deduce that these generally lie in global trace spaces as e.g. $L^2(\partial \Omega, [(1, v)^T \cdot n])$.

To include the boundary condition treatment in the function space, we define
\[
H_{FP, \Gamma_+}^1(\Omega) := \text{clos}_{\|w\|_{H_{FP}^1(\Omega)}} \{f \in C^\infty(\Omega) : f \equiv 0 \text{ on } \Gamma_+\},
\]
which will be used as the test space for our variational formulation. With the restriction of functions in $H_{FP, \Gamma_+}^1(\Omega)$ on the outflow boundary and the definition through the closure, we can show that these functions have a trace in $L^2(\Gamma_-, [(1, v)^T \cdot n])$:

**Proposition 3.4.** There exists a linear continuous mapping $\gamma_- : H_{FP, \Gamma_+}^1(\Omega) \to L^2(\Gamma_-, [(1, v)^T \cdot n])$ such that
\[
\|\gamma_-(w)\|_{L^2(\Gamma_-, [(1, v)^T \cdot n])} \leq C\|w\|_{H_{FP}^1(\Omega)} \quad \forall w \in H_{FP, \Gamma_+}^1(\Omega).
\]
Furthermore, the integration by parts formula
\[
\int_{\Omega} \langle \left( \frac{1}{v} \right) \cdot \nabla_{t,x} w, w \rangle_{V',V} \, dt \, dx = \frac{1}{2} \int_{\Gamma_-} w^2 \left( \frac{1}{v} \right) \cdot n \, ds
\]
holds for all \( w \in H_{FP,G_-}^1(\Omega) \).

Proof. The proof is similar to the respective result for transport equations e.g. in [10, Prop. 2.4], see also [1, sect. 4]. Let \( w \in C^\infty(\bar{\Omega}) \) with \( w \equiv 0 \) on \( \Gamma_+ \). Performing integration by parts we obtain
\[
\int_{\Omega} w \left( \frac{1}{v} \right) \cdot \nabla_{t,x} w \, dt \, dx = - \int_{\Omega} \nabla_{t,x} w \cdot \left( \frac{1}{v} \right) w \, dt \, dx + \int_{\Gamma_-} w^2 \left( \frac{1}{v} \right) \cdot n \, ds,
\]
and thus
\[
\|w\|_{L^2(\Gamma_-, (1,v)^T \cdot n))}^2 = \int_{\Gamma_-} w^2 \left( \frac{1}{v} \right) \cdot n(t,x) \, ds = 2 \int_{\Omega} \left( - \left( \frac{1}{v} \right) \cdot \nabla_{t,x} w \right) w \, dt \, dx
\]
\[
\leq 2 \|w\|_{L^2(\Omega_+, v)} \|w\|_{L^2(\Omega_+, v)} \leq 2 \|w\|_{H_{FP}^1(\Omega)}^2.
\]

By density (due to the definition of \( H_{FP,G_+}^1(\Omega) \)) the integration by parts formula and the bound for \( \|w\|_{L^2(\Gamma_-, (1,v)^T \cdot n))} \) hold for all \( w \in H_{FP,G_+}^1(\Omega) \).

Remark 3.5. Similarly, it can be shown that the space \( H_{FP,G_-}^1(\Omega) \) defined analogously to (7) admits a continuous trace operator \( \gamma_+: H_{FP,G_-}^1(\Omega) \to L^2(\Gamma_+, (1,v)^T \cdot n) \).

While the global existence of the trace and the integration by parts formula can be easily shown for functions in the closure of smooth functions vanishing on the outflow boundary in the same way as for the respective spaces for transport equations, we need a more general result, as well. To later show the uniqueness of the weak solution in section 4, we also need to verify the existence of a global trace and the integration by parts formula for certain functions in \( H_{FP}^1(\Omega) \) with vanishing trace on \( \Gamma_- \), but not necessarily in \( H_{FP,G_-}^1(\Omega) \).

This is established for spaces where the advective or kinetic terms lie in \( L^2(\Omega) \) (see e.g. [6, Thm. 2.2, Prop. 2.5], [16, Chap. XXI, Remark 3]). Similar or even stronger results for respective functions in \( H_{FP}^1(\Omega) \) are claimed to be proven in [1, 11, 5], however, we believe the arguments to be incomplete, for more details see the supplementary materials.

Since we were not able to prove the existence of a global trace for \( H_{FP}^1(\Omega) \) functions with vanishing trace on the inflow or the outflow boundary, we will formulate the exact result needed for uniqueness of the weak solution as an assumption in section 4.

4. Variational formulation

In this section, we develop a variational formulation for (2) and show its well-posedness.
Let \( a_v : \Omega_{t,x} \times V \times V \to \mathbb{R} \) be a potentially \((t,x)\)-dependent bilinear form defined on the velocity space \( V \). Moreover, let \( a_v \) satisfy the following assumptions:

(8) the map \((t,x) \mapsto a_v((t,x); \phi, \psi)\) is measurable on \( \Omega_{t,x} \) for all \( \phi, \psi \in V \);

(9) \( a_v((t,x); \cdot, \cdot) \) is bilinear for a.e. \((t,x) \in \Omega_{t,x},\)

(10) \( a_v((t,x); \phi, \psi) \leq c_v \parallel \phi \parallel_V \parallel \psi \parallel_V \) with \( c_v < \infty \) \( \forall \phi, \psi \in V \), a.e. \((x,t) \in \Omega_{t,x},\)

(11) \( a_v((t,x); \phi, \phi) + \lambda_v \parallel \phi \parallel_{L^2(\Omega_x)}^2 \geq \alpha_v \parallel \phi \parallel_V^2 \) with \( \lambda_v \in \mathbb{R}, \alpha_v > 0 \)

\( \forall \phi \in V, \) a.e. \((x,t) \in \Omega_{t,x} \).

Note that \( c_v, \lambda_v, \) and \( \alpha_v \) are assumed to be independent of \((x,t)\).

**Example 4.1.** For the strong form of the Fokker-Planck equation (2), \( a_v \) is given for all \( \phi, \psi \in V, \) a.e. \( x \in \Omega_x \) by

\[
a_v(x; \phi, \psi) = \left( \nabla_v \left( q(x,v)^{-1} \phi(v) \right), \nabla_v \psi(v) \right)_{L^2(\Omega_x)}
\]

\[
= \left( q(x,v)^{-1} \nabla_v \phi(v), \nabla_v \psi(v) \right)_{L^2(\Omega_x)} + \left( \nabla_v q(x,v)^{-1} \phi(v), \nabla_v \psi(v) \right)_{L^2(\Omega_x)},
\]

where \( \nabla_v \) is the tangential gradient on \( \Omega_v \), see e.g. [21] for a formal definition.

If \( q^{-1} \in L^\infty(\Omega_x \times \Omega_v) \) with \( \nabla_v q^{-1} \in L^\infty(\Omega_x \times \Omega_v) \) and \( q^{-1}(x,v) \geq l_q > 0 \) for a.e. \((x,v)\), then \( a_v \) fulfills the conditions (8)–(11), for instance, with \( c_v = \| q^{-1} \|_{L^\infty} + \| \nabla_v q^{-1} \|_{L^\infty}, \) \( \alpha_v = \frac{1}{2} l_q, \) and \( \lambda_v = \| \nabla_v q^{-1} \|_{L^\infty}^2 / (2 l_q) + \frac{1}{2} l_q. \) Depending on \( q, \) other estimates might be better, e.g. for \( q = q(x) \) and thus \( \nabla_v q = 0 \) we can get \( \alpha_v = \lambda_v = l_q. \)

Recalling the function spaces introduced in (3) and (7), we define the full-dimensional trial and test spaces as

(12) \( \mathcal{X} := L^2(\Omega_{t,x}, V), \) \( \mathcal{Y} := H^1_{FP, \Gamma_x}(\Omega). \)

We then define the full bilinear form \( b : \mathcal{X} \times \mathcal{Y} \to \mathbb{R} \) for \( w \in \mathcal{X}, p \in \mathcal{Y} \) by

(13) \[ b(w, p) := \int_{\Omega_{t,x}} \left< w(t,x), -\left( \frac{1}{2} \cdot \nabla_t x p(t,x) \right)_{V,V} + a_v((t,x); w(t,x), p(t,x)) \right> d(t,x). \]

The functional \( f : \mathcal{Y} \to \mathbb{R} \) containing the boundary condition \( g \in L^2(\Gamma_-, |(\frac{1}{v}) \cdot n|) \) is given as

\[ f(p) := \int_{\Gamma_-} gp |(\frac{1}{v}) \cdot n| d((t,x), v) \quad \forall p \in \mathcal{Y}, \]

which is well-defined due to Proposition 3.4, and we thus have \( f \in \mathcal{Y}' \).

We call \( u \in \mathcal{X} \) a weak solution of (2), if

(14) \[ b(u, p) = f(p) \quad \forall p \in \mathcal{Y}. \]

In the following, we examine the well-posedness of the variational formulation, using the Banach-Nečas-Babuška (or inf-sup) Theorem (see e.g. [26, Thm. 2.6]). We first prove existence of a weak solution in subsection 4.1. Then, in subsection 4.2 we also show uniqueness of the weak solution under an additional assumption on the trace of certain \( H^1_{FP}(\Omega) \) functions.
4.1. **Existence of a weak solution.** We show the existence of a weak solution \( u \) to (14) by verifying a dual inf-sup condition. To that end, we construct stable pairs of trial and test space functions such that the application of the bilinear form to the function pairs can be estimated from below by the respective norms of the functions. In these pairs, the trial space functions are derived from the test space functions by the application of the kinetic transport operator and the inverse elliptic velocity operator. We thus generalize similar proofs for parabolic equations \([26, 41]\) using a time derivative instead of the kinetic transport operator and for transport equations using only an application of the transport operator \([15, 19, 10]\).

**Theorem 4.2.** The bilinear form \( b \) satisfies the dual inf-sup condition
\[
\inf_{p \in Y} \sup_{w \in X} \frac{b(w, p)}{\|w\|_Y \|p\|_Y} \geq \beta
\]
with an inf-sup constant
\[
\beta \geq \frac{\alpha_v}{\sqrt{2} \max \{1, c_v\}}, \quad \text{if } \alpha_v \text{ is coercive, i.e., } \lambda_v \leq 0,
\]
\[
\beta \geq \frac{\alpha_v}{\sqrt{2} \max \{1, c_v + \lambda_v\} \sqrt{\max \{1 + 2\lambda_v^2, 2\} \}} e^{-\lambda_v T}, \quad \text{if } \lambda_v > 0.
\]
Consequently, the variational formulation (14) has at least one weak solution \( u \in X \).

**Remark 4.3.** The estimates for \( \beta \) are not worse than possible estimates for space-time variational formulations for parabolic equations. In fact, for the coercive case and assuming \( \alpha_v \leq 1 \) and \( c_v \geq 1 \) the estimate in \([41]\) roughly translates to \( \beta_{\text{parab}} \geq \alpha_v/(\sqrt{2}c_v^2) \), while we here have \( \beta \geq \alpha_v/\sqrt{2}c_v \). The exponential dependence on the final time \( T \) for the non-coercive case is the same for both types of equations.

**Proof of Theorem 4.2.** We start with the case of \( \alpha_v \) being coercive, i.e., \( \lambda_v \leq 0 \); the non-coercive case will be treated afterwards via a temporal transformation.

To show the inf-sup condition we combine ideas from well-posedness results for parabolic equations as e.g. in \([26, 41]\) and for transport equations as e.g. in \([10]\). To that end, we take \( \not= p \in Y \) arbitrary, but fixed. We want to construct a suitable \( w_p \in X \) and show \( b(w_p, p) \geq \beta \|w_p\|_X \|p\|_Y \) for a constant \( \beta \) independent of \( p \), which makes \( \beta \) a lower bound for the inf-sup constant.

Since \( p \in Y \), we have \( f_p := -\left(\frac{1}{\varepsilon^2}\right) \cdot \nabla_t x p \in L^2(\Omega_t,x; V') = X' \). Similar to \([38, \text{pp. 235}]\), we define the bilinear form \( m : X \times X \rightarrow \mathbb{R} \) by
\[
m(w_1, w_2) := \int_{\Omega_t,x} a_v((t, x); w_1(t, x), w_2(t, x)) d(t, x), \quad \forall w_1, w_2 \in X.
\]
Since the function \( (t, x) \mapsto a_v((t, x); \phi, \psi) \) is assumed to be measurable for all \( \phi, \psi \in V \) (see (8)) and \( a((t, x), \cdot, \cdot) \) is continuous and coercive with constants \( c_v, \alpha_v \) independent of \( (t, x) \) \((10)\) and \((11)\) with \( \lambda_v \leq 0 \), \( m \) is well-defined and continuous and coercive over \( X \times X \) with constants \( c_v \) and \( \alpha_v \). Therefore, by the Lax-Milgram theorem there exists a unique \( z_p \in X \) with
\[
m(z_p, w) = (f_p, w)_X, \forall w \in X.
\]
Due to the definitions of \( z_p, f_p, \) and \( m \), there holds
\[
\int_{\Omega_t,x} a_v(z_p, w) d(t, x) = \int_{\Omega_t,x} \langle -\left(\frac{1}{\varepsilon^2}\right) \cdot \nabla_t x p, w \rangle_{V', V} d(t, x), \quad \forall w \in X.
\]
We now define \( w_p := p + z_p \in \mathcal{X} \). To bound \( b(w_p, p) \) from below we use (16) for
\( w = z_p \) and \( w = p \), and the integration by parts formula from Proposition 3.4:

\[
(17) \quad b(w_p, p) = \int_{\Omega_{t,x}} \langle p + z_p, -(\frac{1}{v}) \cdot \nabla_{t,x} p \rangle_{V,V'} + a_v(p + z_p, p) \, d(t, x)
= \int_{\Omega_{t,x}} \langle p, -(\frac{1}{v}) \cdot \nabla_{t,x} p \rangle_{V,V'} + a_v(z_p, z_p) + a_v(p, p) + \langle -(\frac{1}{v}) \cdot \nabla_{t,x} p, p \rangle_{V,V} \, d(t, x)
\geq \alpha_v(\|p\|_{\mathcal{X}}^2 + \|z_p\|_{\mathcal{X}}^2) + 2 \int_{\Omega_{t,x}} \langle -(\frac{1}{v}) \cdot \nabla_{t,x} p, p \rangle_{V,V} \, d(t, x).
= \alpha_v(\|p\|_{\mathcal{X}}^2 + \|z_p\|_{\mathcal{X}}^2) + \int_{\Omega_{t,x}} p^2 |(\frac{1}{v}) \cdot n| \, ds \geq \alpha_v(\|p\|_{\mathcal{X}}^2 + \|z_p\|_{\mathcal{X}}^2).
\]

Since we have \( \langle f_p, w \rangle_{\mathcal{X}, \mathcal{X}} = m(z_p, w) \leq c_v\|z_p\|_{\mathcal{X}}\|w\|_{\mathcal{X}} \) for all \( w \in \mathcal{X} \), there holds

\[
(18) \quad \|f_p\|_{\mathcal{X}} \leq c_v\|z_p\|_{\mathcal{X}}.
\]

Using the definition of \( w_p, f_p \), and the norm of \( \mathcal{Y} \) as defined in (5), we can then estimate

\[
\|w_p\|_{\mathcal{X}} \|p\|_{\mathcal{Y}} = \|p + z_p\|_{\mathcal{X}}(\|p\|_{\mathcal{X}}^2 + \|f_p\|_{\mathcal{X}}^2)^{1/2}
\leq \left[ \|p + z_p\|_{\mathcal{X}}^2 + \|f_p\|_{\mathcal{X}}^2 \right]^{1/2} \leq 2 \left( \|p\|_{\mathcal{X}}^2 + \|z_p\|_{\mathcal{X}}^2 \right) \left( \|p\|_{\mathcal{X}}^2 + \|z_p\|_{\mathcal{X}}^2 \right)^{1/2}
\leq \sqrt{2} \max\{1, c_v\} \left( \|p\|_{\mathcal{X}}^2 + \|z_p\|_{\mathcal{X}}^2 \right) \leq \frac{\sqrt{2} \max\{1, c_v\}}{\alpha_v} b(w_p, p).
\]

Since \( p \in \mathcal{Y} \) was chosen arbitrarily, we thus have

\[
(20) \inf_{p \in \mathcal{Y}} \sup_{w \in \mathcal{X}} b(w, p) \geq \beta := \frac{\alpha_v}{\sqrt{2} \max\{1, c_v\}},
\]
i.e., the claim for coercive \( a_v \).

To address the case that \( a_v \) fulfills the Gårding inequality (11) with \( \lambda_v > 0 \), we use a standard temporal transformation of the full problem as proposed e.g. in [41, 44]. We set \( \tilde{w} := e^{-\lambda_v t}w \) for \( w \in \mathcal{X} \), \( \tilde{p} = e^{\lambda_v t}p \) for \( p \in \mathcal{Y} \), and define the bilinear form \( b : \mathcal{X} \times \mathcal{Y} \to \mathbb{R} \) by

\[
(21) \quad \tilde{b}(\tilde{w}, \tilde{p}) := \int_{\Omega_{t,x}} (\tilde{w}, -(\frac{1}{v}) \cdot \nabla_{t,x} \tilde{p})_{V,V'} + a_v((t, x); \tilde{w}, \tilde{p}) + \lambda_v(\tilde{w}, \tilde{p})_{L^2(\Omega_v)} \, d(t, x).
\]

Then it holds \( b(w, p) = \tilde{b}(\tilde{w}, \tilde{p}) \) for all \( w \in \mathcal{X}, p \in \mathcal{Y} \). The transformed bilinear form \( \tilde{b} \) satisfies the definition of \( b \) with the transformed velocity bilinear form \( \tilde{a}_v : V \times V \to \mathbb{R} \) defined by \( \tilde{a}_v((t, x); \phi, \psi) = a_v((t, x); \phi, \psi) + \lambda_v(\phi, \psi)_{L^2(\Omega_v)} \) for \( \phi, \psi \in V \).

Due to the Gårding inequality (11) and continuity (10) of \( a_v, \tilde{a}_v \) is coercive with constant \( \tilde{\alpha}_v = \alpha_v \) and continuous with constant \( \tilde{c}_v = c_v + \lambda_v \). As in [41], we can estimate the norms of \( \tilde{w} \in \mathcal{X} \) and \( \tilde{p} \in \mathcal{Y} \) by

\[
|\tilde{w}|_{\mathcal{X}} \geq e^{-\lambda_v T} \|w\|_{\mathcal{X}}, \quad |\tilde{p}|_{\mathcal{Y}} \geq \left( \max\{1 + 2\lambda_v^2, 2\} \right)^{-\frac{1}{2}} \|p\|_{\mathcal{Y}},
\]
where we use \( \|\psi\|_{V'} \leq \|\psi\|_{L^2(\Omega_v)} \leq \|\psi\|_{V} \) for the estimation of the \( \mathcal{Y} \)-norm.
Then, the dual inf-sup constant of $b$ can be bounded from below as follows

$$\inf_{\tilde{p}\in Y} \sup_{w\in X} \frac{b(w, p)}{\|w\|_X \|p\|_Y} = \inf_{\tilde{p}\in Y} \sup_{w\in X} \frac{\tilde{b}(\tilde{w}, \tilde{p})}{\|\tilde{w}\|_X \|\tilde{p}\|_Y} \geq \frac{\alpha_\alpha}{\sqrt{2} \max\{1, c_\nu + \lambda_\nu\} \sqrt{\max\{1 + 2\lambda_\nu^2, 2\}}}.$$ 

Since the dual inf-sup condition implies surjectivity of the operator $B : \mathcal{X} \to \mathcal{Y}$ defined by $(B, \cdot)_{\mathcal{Y}', \mathcal{Y}} = b(\cdot, \cdot)$ and thus existence of a weak solution to (14) (see for instance [26, Lemma A.40, Remark A.41]), this concludes the proof. □

4.2. **Uniqueness of the weak solution.** As already mentioned in section 3, we were not able to prove all necessary trace results in our specific function space. To show uniqueness of the weak solution, we therefore assume the following:

**Assumption 4.4.** Let $w \in H^1_{FP}(\Omega)$ with $w = 0$ a.e. on $\Gamma_-$, i.e., $w|_K = 0$ for all compact $K \subset \Gamma_-$. Moreover, assume that $b(w, p) = 0$ for all $p \in \mathcal{Y}$. Then, we have $w \in L^2(\partial\Omega, ((1, v)^T \cdot n))$ and the integration by parts formula

$$\int_{\Omega_{t,x}} \langle (\frac{1}{v}) \cdot \nabla_{t,x} w, w \rangle_{\mathcal{V}', \mathcal{V}} \, dt \, dx = \frac{1}{2} \int_{\partial\Omega} \hat{w}^2 (\frac{1}{v}) \cdot n \, ds$$

holds.

As discussed in more detail in the supplementary materials, we do not know how to prove Assumption 4.4, since, for instance, ideas from existing approaches for the related space $H^1_{NT}(\Omega) = \{w \in L^2(\Omega) : (\frac{1}{v}) \cdot \nabla_{t,x} w \in L^2(\Omega)\}$ cannot readily be transferred to the $H^1_{FP}(\Omega)$ case. We therefore leave it as an open problem. We emphasize that the respective trace and integration by parts results hold for all $H^1_{NT}(\Omega)$ functions with zero inflow or outflow trace (cf. [6, 12, 13],[16, Chap. XXI]), and also for all $H^1_{FP}(\Omega)$ functions that can be approximated by smooth functions vanishing on the inflow or outflow boundary (Proposition 3.4). Additionally, Assumption 4.4 is limited to $H^1_{FP}(\Omega)$ functions with vanishing trace on $\Gamma_-$ and satisfying a weak form of the differential equation with zero boundary condition. This additional condition on the considered functions might make it possible to show and exploit a higher regularity of the considered functions to prove existence of suitable traces and (22).

We now show uniqueness of the weak solution in the form of surjectivity of the dual operator. To that end, we follow the general structure of respective proofs for parabolic equations [26, Thm 6.6, p. 283] and transport equations [4, Thm. 16]: We take a function $w \in \mathcal{X}$ solving (14) with zero right-hand side and prove that $w = 0$ step by step by showing that $w$ possesses space- and time derivatives, that $w$ has trace zero on the outflow boundary, and finally that $w$ must therefore vanish on the whole domain.

**Theorem 4.5.** If Assumption 4.4 holds, then for all $0 \neq w \in \mathcal{X}$ we have

$$\sup_{p \in \mathcal{Y}} b(w, p) > 0.$$ 

**Proof.** Let $w \in \mathcal{X}$ such that

$$b(w, p) = 0 \quad \forall p \in \mathcal{Y}. \tag{23}$$

To prove the claim, we need to show that $w = 0$. First, we show that $w$ has a weak derivative $-(\frac{1}{v}) \cdot \nabla_{t,x} w \in \mathcal{X}' = L^2(\Omega_{t,x}; \mathcal{V}')$. To that end, let $\psi \in C_0^\infty(\Omega_{t,x})$ and
\( \phi \in V \) be arbitrary. Then \( \psi \phi = 0 \) on \( \partial \Omega \), and by approximating \( \phi \) in \( C^\infty(\Omega_w) \) we see that \( \psi \phi \in \mathcal{Y} \). Using the definition of the weak \( (t,x) \)-derivative and testing (23) with \( p = \psi \phi \) we obtain

\[
\int_{\Omega_{t,x}} \langle (\frac{1}{v}) \cdot \nabla_{t,x} w(t,x), \phi \rangle_{V, V'} \psi(t,x) d(t,x) \\
= -\int_{\Omega_{t,x}} \langle w(t,x), (\frac{1}{v}) \cdot \nabla_{t,x} \psi(t,x) \phi \rangle_{V, V'} d(t,x) \\
= -\int_{\Omega_{t,x}} a_v((x,t); w(t,x), \psi(t,x) \phi) d(t,x) \\
= -\int_{\Omega_{t,x}} (A_v(t,x)w(t,x), \phi)_{V, V} \psi(t,x) d(t,x),
\]

where the operator \( A_v(t,x) \in \mathcal{L}(V, V') \) is defined as \( (A_v(t,x) \phi, \rho)_{V, V'} = a_v((x,t); \phi, \rho) \) for all \( \phi, \rho \in V \), a.e. \( (t,x) \in \Omega_{t,x} \). Due to the density of \( C_0^\infty(\Omega_{t,x}) \) in \( L^2(\Omega_{t,x}) \) have

\[
(24) \quad - (\frac{1}{v}) \cdot \nabla_{t,x} w = A_v w \in \mathcal{X},
\]

which especially means that \( w \in H_{FP}^1(\Omega) \).

Next, let \( K \subset \subset \Gamma_- \) be an arbitrary but fixed compactly embedded subset of \( \Gamma_- \). Moreover, let \( z \in C^\infty(\bar{\Omega}) \) with \( z = 0 \) on \( \partial \Omega \setminus K \). We show \( wz \in \mathcal{Y} \): Since \( w \in H_{FP}^1(\Omega) \), due to Proposition 3.1 there is a sequence \( (w_n)_{n \in \mathbb{N}} \subset C^\infty(\bar{\Omega}) \) with \( \|w_n - w\|_{H_{FP}^1(\Omega)} \xrightarrow{n \to \infty} 0 \). Therefore, we have \( w_n z \in C^\infty(\Omega) \) with \( w_n z = 0 \) on \( \Gamma_+ \).

Due to Lemma 3.3, it holds

\[
\|wz - w_n z\|_{H_{FP}^1(\Omega)} \leq C \|z\|_{C^1(\Omega)} \|w - w_n\|_{H_{FP}^1(\Omega)}
\]

and thus \( w_n z \to wz \) in \( H_{FP}^1(\Omega) \) as \( n \to \infty \). Invoking the definition of \( \mathcal{Y} \) in (12),(7) we obtain \( wz \in \mathcal{Y} \).

Since \( K \subset \subset \Gamma_- \) is compact, we may apply Proposition 3.2 to infer that \( w \) has a trace on \( K \) and \( w|_K \in L^2(K, [(1,v)^T \cdot n]) \). Thanks to \( z|_{\partial \Omega} \in L^\infty(\partial \Omega) \) and \( \text{supp } z|_{\partial \Omega} \subset K \), we have

\[
\int_{\partial \Omega} w^2 z |(\frac{1}{v}) \cdot n| \, ds = \int_{K} w^2 z |(\frac{1}{v}) \cdot n| \, ds \leq \|z\|_{L^\infty(K)} \|w\|_{L^2(K, [(1,v)^T \cdot n])}^2 < \infty.
\]

As a consequence we can apply the linear functional in (24) to \( wz \in \mathcal{Y} \subset \mathcal{X} \), perform integration by parts, since the boundary integral exists, and use (23):

\[
0 = \int_{\Omega_{t,x}} \langle (\frac{1}{v}) \cdot \nabla_{t,x} w + A_v w, wz \rangle_{V, V'} d(t,x) \\
= \int_{\Omega_{t,x}} \langle w, (\frac{1}{v}) \cdot \nabla_{t,x} (wz) \rangle_{V, V'} + a_v(w, wz) d(t,x) + \int_{\partial \Omega} w^2 z |(\frac{1}{v}) \cdot n| \, ds \\
= b(w, wz) - \int_{K} w^2 z |(\frac{1}{v}) \cdot n| \, ds = - \int_{K} w^2 z |(\frac{1}{v}) \cdot n| \, ds.
\]

Since \( z|_K \in C^\infty_0(K) \) can be chosen arbitrarily and \( |(\frac{1}{v}) \cdot n| > 0 \) on \( K \), the fundamental lemma of calculus of variations yields \( w = 0 \) a.e. on \( K \). As also \( K \subset \subset \Gamma_- \) was chosen arbitrarily, we have \( w = 0 \) a.e. on \( \Gamma_- \).

Thanks to Assumption 4.4, it therefore holds \( w \in L^2(\partial \Omega, [(1,v)^T \cdot n]) \). We can thus use integration by parts for (24) applied to \( w \). Assuming first that \( a_v \) is
coercive, i.e., \( \lambda_v \leq 0 \), we obtain

\[
0 = \int_{\Omega_{t,x}} \left( (1) \cdot \nabla_{t,x} w + A_v(w, w) \right)_{V,V'} d(t,x) \\
= \int_{\Omega_{t,x}} \left( (1) \cdot \nabla_{t,x} w, w \right)_{V,V'} d(t,x) + \int_{\Omega_{t,x}} a_v(w, w) d(t,x) \\
\geq \frac{1}{2} \int_{\Gamma_{t,x}} w^2 \left( \frac{1}{v} \cdot n \right) ds + \alpha_v \|w\|_{X}^2,
\]

which implies \( w = 0 \).

If \( a_v \) is not coercive, we use the temporal transformation described in the proof of Theorem 4.2. Setting \( \hat{w} = e^{-\lambda_v t} w \) and using the definition of \( \hat{b} \) in (21), we see that (23) is equivalent to \( \hat{b}(\hat{w}, \hat{p}) = 0 \) for all \( \hat{p} \in Y \). Since \( \hat{a} \) is coercive, we have proven that \( \hat{w} = 0 \) and thus also \( w = 0 \). \( \square \)

We summarize our findings in the following theorem.

**Theorem 4.6 (Well-posedness).** There exists a solution \( u \in X \) to the variational problem (14). If Assumption 4.4 holds, the solution is unique and satisfies the stability estimate

\[
\|u\|_X \leq \frac{1}{\beta} \|f\|_{Y'}
\]

for \( \beta \) as defined in Theorem 4.2.

**Proof.** As the bilinear form \( b \) is continuous, the corresponding operators \( B \in L(X', Y') \) and \( B^* \in L(Y, X') \) defined as \( (Bw, p)_{Y', X} = b(w, p)_{X, X'} \) are continuous as well. The dual inf-sup condition shown in Theorem 4.2 is equivalent to the surjectivity of the operator \( B \), which means that there exists a solution to the variational problem (14).

If additionally Assumption 4.4 holds, Theorem 4.5 shows injectivity of \( B \) and thus uniqueness of the solution. The bijectivity of \( B \) is equivalent to the bijectivity of \( B^* \) with \( \|B^{-1}\|_{L(X', Y)} = \|B^*\|_{L(Y', X')} = \beta^{-1} \). With \( \|u\|_X = \|B^{-1}f\|_X \leq \|B^{-1}\|_{L(Y', X')}\|f\|_{Y'} \), the stability estimate follows. \( \square \)

### 5. Discretization

We now design a stable and efficient discretization scheme for (14). To that end, we use a Petrov-Galerkin projection onto problem-dependent discrete spaces realizing the stable function pairs with test functions \( p \in Y \) and trial functions \( w_p \in X \) developed in the proof of Theorem 4.2. As a result, the discrete inf-sup stability and thus the well-posedness of the discrete problem follow analogously to the continuous results with the same stability constant. We then illustrate for a class of data functions how the trial space functions \( w^\delta_p \) can be efficiently computed by solving low-dimensional elliptic problems in the velocity domain.

#### 5.1. Stable Petrov-Galerkin schemes.

To define an approximation of the solution \( u \in X \) of (14), we use a Petrov-Galerkin projection onto suitable discrete spaces: Given discrete trial and test spaces \( X^\delta \subset X \) and \( Y^\delta \subset Y \), the Petrov-Galerkin approximation \( u^\delta \in X^\delta \) is defined by

\[
b(u^\delta, v^\delta) = f(v^\delta) \quad \forall v^\delta \in Y^\delta.
\]
Well-posedness then depends on the discrete inf-sup stability of the discrete problem. To find a pair of spaces leading to a stable scheme, we transfer ideas from [10] to our setting. In [10], a stable discretization with a discrete inf-sup constant of one was built for a transport equation by fixing a discrete test space and defining a problem dependent trial space with optimal stability properties. In this manuscript we will use the same strategy: We start with a discrete test space and define the corresponding trial space based on the trial space functions used in the proof of Theorem 4.2.

To that end, we first define a discrete space \( V_h \subset V \) for the discretization in the velocity direction. Since the \( Y \)-norm contains a term in the \( X' = L^2(\Omega_t, V') \) norm which is not computable, we consider the norm

\[
\|w\|^2_{L^2(\Omega_t, V_h')} := \int_{\Omega_t} \|w(t, x)\|^2_{V_h'} \, d(t, x), \quad \|\psi\|_{V_h'} := \sup_{\phi_h \in V_h} \frac{\langle \psi, \phi_h \rangle_{V', V}}{\|\phi_h\|_V}
\]

instead of \( \|\cdot\|_{L^2(\Omega_t, V')} \) where necessary.

Let \( \mathcal{Y}_h \subset Y \) be a discrete space for which we assume \( \psi^\delta(t, x) \in V_h \) for all \( \psi^\delta \in \mathcal{Y}_h \) and a.e. \((t, x) \in \Omega_t, x\). \( \mathcal{Y}_h \) will be used as test space for the Petrov-Galerkin approximation. We define the discrete version of the \( Y \)-norm by

\[
\|w\|_{\mathcal{Y}_h}^2 := \|w\|^2_{L^2(\Omega_t, V)} + \|\nabla t, x w\|^2_{L^2(\Omega_t, V')}.
\]

Since we will make use of the function pairs developed in the proof of Proposition 4.2, we assume for the discretization that the velocity bilinear form \( a_v \) is coercive, i.e., \( \lambda_v \leq 0 \). For problems, where \( a_v \) only satisfies the G˚arding inequality (11) with \( \lambda_v > 0 \), a temporal transform of the problem as described in section 4 can be performed, then the transformed problem with a coercive bilinear form \( \tilde{a}_v \) can be discretized.

We now define a problem-dependent discrete trial space. For each \( p^\delta \in \mathcal{Y}_h \), we denote \( f_p := \left(\frac{1}{\hat{t}}\right) \cdot \nabla t, x p^\delta(t, x) \in X' \). We then define the function \( z_p^\delta \in X \) as the solution of

\[
a_v(z_p^\delta(t, x), \phi^h) = \langle f_p^\delta(t, x), \phi^h \rangle_{V', V}, \quad \forall \phi^h \in V_h, \text{ a.e. } (t, x) \in \Omega_t, x.
\]

\( z_p^\delta \) is the discrete counterpart of \( z_p \) defined in (15), but is here defined pointwise in \( \Omega_t, x \) due to the discrete setting. Then, the discrete trial space \( X_h \subset X \) is defined as

\[
X_h := \{ p^\delta + z_p^\delta : p^\delta \in \mathcal{Y}_h \}.
\]

**Proposition 5.1.** If \( a_v \) is coercive, i.e., \( \lambda_v \leq 0 \) in (11), and if the discrete trial and test spaces \( X_h \) and \( \mathcal{Y}_h \) are chosen according to (29), then there exists a unique solution \( u^\delta \in X_h \) to (25).

**Remark 5.2.** For non-coercive \( a_v \) the respective result holds for the discretization of the transformed problem according to (21) with \( \tilde{a}_v \) being coercive.

**Proof.** We can reuse all essential parts of the proof of the inf-sup constant for the continuous problem to also prove discrete inf-sup stability of (25).

Let \( 0 \neq u^\delta \in X_h \) be fixed. Then, by definition of \( X_h \) there is \( p^\delta \in \mathcal{Y}_h \) such that \( u^\delta = p^\delta + z_p^\delta \) with \( z_p^\delta \) defined as in (28). By using (28) and the same arguments as in (17) we obtain

\[
b(u^\delta, p^\delta) = b(p^\delta + z_p^\delta, p^\delta) \geq \alpha_v \left( \|p^\delta\|^2_X + \|z_p^\delta\|^2_X \right).
\]
As we have
\[
\langle f_p^\delta(t, x), \phi^h \rangle_{\mathcal{V}, V} = a_v(z_p^\delta(t, x), \phi^h) \leq c_v \|z_p^\delta(t, x)\|_V \|\phi^h\|_V \quad \forall \phi^h \in V_h, \text{ a.e. } (t, x) \in \Omega_{t,x}
\]
we can infer that
\[
\|f_p^\delta\|_{L^2(\Omega_{t,x}, V')} \leq c_v \|z_p^\delta\|_X.
\]
Therefore, we obtain analogously to (19), but using the discrete \(Y\)-norm,
\[
\|w_p^\delta\|_X \|p^\delta\|_{\mathcal{Y}} = \|p^\delta + z_p^\delta\|_X \left(\|p^\delta\|_X^2 + \|f_p\|_{L^2(\Omega_{t,x}, V')}^2\right)^{1/2}
\leq \left(\|p^\delta + z_p^\delta\|_X^2 + c_v^2 \|z_p^\delta\|_X^2\right)^{1/2}
\leq \left[2 \left(\|p^\delta\|_X^2 + \|z_p^\delta\|_X^2\right) \left(\|p^\delta\|_X^2 + c_v^2 \|z_p^\delta\|_X^2\right)\right]^{1/2}
= \sqrt{2} \max\{1, c_v\} \left(\|p^\delta\|_X^2 + \|z_p^\delta\|_X^2\right) \leq \frac{\sqrt{2} \max\{1, c_v\} b(w_p^\delta, p^\delta)}{\alpha_v}.
\]
This means that \(b\) is inf-sup stable on the spaces \((X, \| \cdot \|_X), (Y, \| \cdot \|_{\mathcal{Y}})\) with constant \(\beta_b \geq \alpha_v (\sqrt{2} \max\{1, c_v\})^{-1}\). Since for all \(0 \neq p^\delta\) it holds \(b(w_p^\delta, p^\delta) > 0\) and thus \(w_p^\delta \neq 0\), we have \(\dim(X) = \dim(Y)\). Therefore, inf-sup stability already guarantees well-posedness of the discrete problem (25). \(\square\)

**Remark 5.3.** Due to the finite-dimensional spaces, the Petrov-Galerkin approximation \(u^\delta \in X\) is unique even if Assumption 4.4 does not hold.

### 5.2. Efficient numerical scheme

Regarding the computational realization of the Petrov-Galerkin approximation, we have to take into account the specific choice of the discrete spaces according to (29). To assemble the linear system and to represent the discrete solution, the nonstandard parts of the \(X\)-basis functions, i.e., the functions \(z_p^\delta\) defined by (28), have to be computed for all basis functions of \(Y\). We illustrate how this can be done very efficiently for the case where \(a_v\) is coercive and has the separable form
\[
a_v((t, x), \phi, \psi) = d(t, x)\tilde{\alpha}_v(\phi, \psi),
\]
where \(d \in L^\infty(\Omega_{t,x})\) satisfies \(d(t, x) \geq \alpha^d > 0\) for a.e. \((t, x) \in \Omega_{t,x}\) and \(\tilde{\alpha}_v : V \times V \rightarrow \mathbb{R}\) is a coercive bilinear form.

To build the discrete test space, let first \(\mathcal{Y}_0^{t,x} \subset H^1(\Omega_{t,x})\) be a discrete space in the space-time domain with basis \((\psi_i^{t,x,\delta}(t, x))_{i=1}^{n_{t,x}}\) and let \(V_h \subset V\) be the already defined velocity discrete space with basis \((\psi_i^h(v))_{j=1}^{n_{v}}\). Denoting the tensor product of these spaces by \(\mathcal{Y}_0 := \mathcal{Y}_0^{t,x} \otimes V_h\), we then set
\[
\mathcal{Y} := \text{span}\{\mathcal{Y} \delta_{i,j} = \mathcal{Y}^{t,x,\delta} \psi_j^h : \mathcal{Y}^{t,x,\delta} \mathcal{Y} \psi_j^h |_{\mathcal{Y}^{t,x,\delta}} = 0\} \subset \mathcal{Y} \cap \mathcal{V}.
\]
We may then use this tensor product structure to efficiently solve (28): Fixing a basis function \(\mathcal{Y}^{i,j} \in \mathcal{Y}_{i,j} \psi_j^h \) of \(\mathcal{Y}\), the right-hand side of (28) reads
\[
\langle -\left(\frac{d}{\nu} \right) \cdot \nabla_{t,x} \mathcal{Y}^{i,j}(t, x), \phi^h \rangle_{V', V} = -\partial_{t} \psi_i^{h}(v) \phi^h(v) dv
- \sum_{k=1}^{d} \partial_{x_k} \psi_i^{h}(v) \phi^h(v) dv.
\]
for all $\phi^h \in V_h$, a.e. $(t, x) \in \Omega_{t,x}$. Using the separable form of $a_v$ (33), we can rewrite (28) as follows: Find $z_{i,j}^\delta := z_{p_{i,j}}^\delta \in X$, such that

$$
\begin{aligned}
&d(t, x)\bar{a}_v(z_{i,j}^\delta(t, x), \phi^h) = -\partial_t p^{t,x,\delta}_{i}(t, x) \int_{\Omega_v} \psi^h_j(v) \phi^h(v) \, dv \\
&\quad - \sum_{k=1}^{d} \partial_{x_k} p^{t,x,\delta}_{i}(t, x) \int_{\Omega_v} v_k \psi^h_j(v) \phi^h(v) \, dv \\
&\quad \forall \phi^h \in V_h, \text{ a.e. } (t, x) \in \Omega_{t,x}.
\end{aligned}
$$

Hence, the computation of all $z_{i,j}^\delta$ can be separated in the following way: We first compute the solutions $\rho_{j}^1, \rho_{j}^2, \ldots, \rho_{j}^d \in V_h$ to the problems

$$
\begin{aligned}
&\tilde{a}_v(\rho_{j}^1, \phi^h) = \int_{\Omega_v} \psi_j^1(v) \phi^h(v) \, dv, \quad \forall \phi^h \in V_h, \\
&\tilde{a}_v(\rho_{j}^k, \phi^h) = \int_{\Omega_v} v_k \psi_j^1(v) \phi^h(v) \, dv, \quad \forall \phi^h \in V_h, k = 1, \ldots, d,
\end{aligned}
$$

for all basis functions $\psi_j^1 \in V_h$, $j = 1, \ldots, n_v$. Then, the $z_{i,j}^\delta$ are given by

$$
(35) \quad z_{i,j}^\delta(t, x, v) = -d(t, x)^{-1} \left( \partial_t p^{t,x,\delta}_{i}(t, x) \rho_{j}^1(v) + \sum_{k=1}^{d} \partial_{x_k} p^{t,x,\delta}_{i}(t, x) \rho_{j}^k(v) \right).
$$

The full solution process thus consists of the following steps:

1. Precompute $\rho_{j}^1, \rho_{j}^2, \ldots, \rho_{j}^d$, i.e., solve $(d + 1) \times n_v$ problems of size $n_v$, which can be done in parallel.
2. Assemble the stiffness matrix $[b(p_{k,l}^\delta + z_{i,j}^\delta, p_{k,l}^\delta)](k,l,(i,j))$, using (35), and assemble the load vector $[f(p_{k,l}^\delta)](k,l)$.
3. Solve the linear system of equations to obtain the coefficient vector $[u_{i,j}^\delta]((i,j))$.
4. Compose the solution $u^\delta = \sum_{i,j} u_{i,j}^\delta (p_{i,j}^\delta + z_{i,j}^\delta) \in X$ again using (35).

Compared to a naive approach using finite element spaces without any stabilization, the additional costs thus only lie in the $n_v$-sized problems (step 1) and possibly more nonzero elements in the stiffness matrix. These effects only depend on the dimension $n_v$ of $V_h$. Therefore, the proposed discretization strategy is especially well-suited for using specific spaces $V_h$ of low dimension, which can be achieved for example by using polynomial bases or a hierarchical model reduction approach as proposed in [9].

In order to efficiently compute the problem-dependent basis functions, we heavily rely on the separable form of the bilinear form $a_v$ given in (33). To consider more general bilinear forms, the method could for example be combined with low-rank approximations to efficiently compute a discrete trial space as done in a related setting in [7]. More generally, due to the high-dimensionality of the problem, it is especially desirable to combine the discretization with further approximations as the already mentioned hierarchical model reduction [9] or tensor-based methods that have already been used in similar Petrov-Galerkin settings [7, 31] and to discretize kinetic equations like the radiative transfer equation [45, 28] or the Vlasov equation [36, 23, 24].
6. Numerical experiments

We investigate the properties of the method developed in section 5 by implementing the discretization for a basic model problem. We are especially interested in analyzing how sharp the lower bound for the inf-sup constant is and examining the efficiency in light of the nonstandard discrete spaces. The source code to reproduce all results is provided in [8].

6.1. Test Case. We consider the time-independent model problem

\[ v \cdot \nabla_x u(x, v) + c u(x, v) = d \Delta u(x, v) + f_0(x, v) \]

on the domain \( \Omega = \Omega_x \times \Omega_y \) for \( \Omega_x = (0, 1)^2 \) and \( \Omega_y = S^1 \) and with reaction and velocity diffusion constants \( c, d \in \mathbb{R}, \) \( c, d > 0. \) We assume zero inflow boundary conditions on \( \Gamma_\infty \subset \partial \Omega \) and define a source function \( f_0 \in L^2(\Omega) \) as a substitute for the initial condition of the time-dependent equation. We parametrize \( \Omega_v \) for \( \Omega = \Omega \times \Omega_v \)

\[ \Omega_2(S^1) \]

by the angle \( \phi \in [0, 2\pi], \) leading to \( v = \left( \frac{\cos \phi}{\sin \phi} \right) \) and \( \Delta_v u = \frac{\partial \phi}{\partial x} u. \)

Then, we have \( V = H^1(\Omega_v) \) and the bilinear form \( a_v : V \times V \to \mathbb{R} \) reads

\[ a_v(\psi, \rho) = \int_0^{2\pi} d\psi'(\phi)\rho'(\phi) + c \psi(\phi)\rho(\phi) \ d\phi \ \forall \psi, \rho \in V. \]

Thanks to the definition of the \( H^1(\Omega_v) \)-norm, the bilinear form \( a_v \) is coercive with constant \( a_v = \min(c, d) > 0 \) and continuous with constant \( \gamma_v = \max(c, d). \) We then have \( \mathcal{X} := L^2(\Omega_v; H^1(\Omega_v)) \), and \( \mathcal{Y} = \text{close}_{\| \cdot \|_{X'}} \{ w \in C^1(\Omega) : w = 0 \text{ on } \Gamma, \} \), where

\[ \Gamma_+ = \{ (x, v) \in \partial \Omega_x \times \Omega_y : \left( \frac{\cos \phi}{\sin \phi} \right) \cdot n_x > 0 \} \subset \partial \Omega, \]

\[ \| w \|_{X'}^2 = \| w \|_X^2 + \left\| \left( \frac{\cos \phi}{\sin \phi} \right) \cdot \nabla_x w \right\|_V^2. \]

The full bilinear form is

\[ b(w, p) := \int_{\Omega_y} (w(x), -\left( \frac{\cos \phi}{\sin \phi} \right) \cdot \nabla_x p(x))_{V', V} + a_v(w(x), p(x)) \ dx, \quad \forall w \in \mathcal{X}, p \in \mathcal{Y} \]

and the functional describing the source term is defined as

\[ f(p) := \int_{\Omega_y} \int_0^{2\pi} f_0(x, \phi)p(x, \phi) \ d\phi \ dx \quad \forall p \in \mathcal{Y}. \]

Well-posedness of the weak formulation of (36) follows completely analogously to the time-dependent case, as \( a_v \) is coercive and \( f \in \mathcal{Y}'. \)

For the discretization we choose \( V_h \subset V = H^1(\Omega_v) \) as the continuous linear FE space on \([0, 2\pi]\) with periodic boundary condition and uniform mesh with size \( h_v = 2\pi/n_v. \) The space \( \mathcal{X}_h^\infty \subset H^1(\Omega_x) \) is chosen as the continuous \( \mathbb{Q}_2 \) FE space on a uniform rectangular mesh with size \( h_x = \sqrt{2}/n_x \). The trial space \( \mathcal{X}_h \) is computed as described in subsection 5.2 by first solving \( 2n_v \) problems of dimension \( n_v. \) From the definition we see that \( \mathcal{X}_h \subset \mathcal{X}_h^\infty \otimes V_h, \) with \( \mathcal{X}_h^\infty \subset L^2(\Omega_x) \) being the discontinuous \( \mathbb{Q}_2 \) FE space.

6.2. Numerical results. To investigate whether the estimate for the discrete inf-sup constant from section 5 is sharp, we compute the constants for different mesh sizes and reaction and diffusion constants \( c \) and \( d; \) see Table 1. The estimate established in section 5 is given in our test case as \( \beta_\delta \geq \min\{c, d\}/(\sqrt{2}\max\{1, c, d\}), \)

which is \( \min\{c, d\}/\sqrt{2} \) for all considered data values in Table 1. As can be seen in the table, the computed inf-sup constants tend to \( \min\{c, d\} \) with increasing mesh.
Table 1. Computed discrete inf-sup constants for varying mesh sizes and values for the diffusion and reaction constants \( d \) and \( c \).

| \( h_x \) | \( h_\varphi \) | \( d = 0.4, \ c = 1 \) | \( d = 0.1, \ c = 1 \) | \( d = 0.1, \ c = 0.1 \) |
|---------|-------------|----------------|----------------|----------------|
| \( \sqrt{2} \) | \( \frac{2\pi}{4} \) | 0.61855 | 0.41087 | 0.30579 |
| \( \sqrt{2} \) | \( \frac{2\pi}{8} \) | 0.44891 | 0.18628 | 0.14924 |
| \( \frac{1}{16} \) | \( \frac{2\pi}{16} \) | 0.40915 | 0.11688 | 0.10585 |
| \( \frac{1}{32} \) | \( \frac{2\pi}{32} \) | 0.40202 | 0.1033 | 0.10041 |
| \( \frac{1}{64} \) | \( \frac{2\pi}{32} \) | 0.40088 | 0.10137 | 0.10008 |

Figure 1. Sparsity pattern of the stiffness matrix for \( h_{x1} = h_{x2} = \frac{1}{16} \), \( h_\varphi = \frac{\pi}{16} \), \( \dim \mathcal{Y}_b = 16256 \).

Figure 2. Plots of the solution \( u \) for \( d = c = 0.1 \), \( f_0 = \chi_{[0.4,0.6]} \), \( h_{x1} = h_{x2} = 1/48 \), \( h_\varphi = 2\pi/48 \), \( \dim \mathcal{Y}_b = 441984 \). Left: \( u(\cdot,\cdot,\varphi) \) for different angles \( \varphi \). Right: moment \( \int_0^{2\pi} u(\cdot,\cdot,\varphi) \, d\varphi \), i.e., the spatial density then shows the overall picture of the particle dynamics. Indeed the non-standard trial space leads to a realistic solution. We also see that there are no oscillations that would indicate instabilities of the method. However, we observe small artifacts in the corners of the domain in the moment plots. These arise at the “outflow corners” due to the specific choice of spaces: By definition, the test space functions vanish on the outflow boundary. Due to the specific choice of tensor product spaces and the definition of the trial space dependent on the test space functions, the trial space inherits an unphysical restriction to zero at the outflow corners. For remedies of this issue we refer to [10].

7. Conclusions

In this paper, we present a stable Petrov-Galerkin discretization of a kinetic Fokker-Planck equation. Based on an estimate for the dual inf-sup constant of the bilinear form, where “stable pairs” of trial and test functions are introduced,
we propose a discretization where these pairs are directly built into the spaces: By defining the discrete trial space dependent on the chosen discrete test space through the application of the kinetic transport and the inverse velocity Laplace-Beltrami operator, we obtain a well-posed numerical scheme with the same lower bound of the discrete inf-sup constant as for the continuous problem independently of the mesh size. We show that under suitable conditions on the data functions these spaces can be computed efficiently. Numerical experiments show that for the examined test case the estimate of the discrete inf-sup constant is sharp up to a factor of $\sqrt{2}$.

The new method is especially beneficial for spaces with few degrees of freedom in the velocity domain. Therefore, a promising application might be a combination with a hierarchical model order reduction scheme such as [9], which realizes small spaces in the velocity domain and has stability problems that might be resolved using the new method.

\section*{Appendix A. Proofs of function space results}

\textbf{Proof of Proposition 3.1.} The claim is only a slight variant of [1, Prop. 7.1], where the respective density result is shown for the space

\begin{equation}
\tilde{H}^1_{FP}(\Omega) := \{ p \in L^2(\Omega_{t,x}, \tilde{V}) : \partial_t p - v \cdot \nabla_x p \in L^2(\Omega_{t,z}, \tilde{V}') \}
\end{equation}

with $\tilde{V} = H^1_0(\mathbb{R}^d)$ being the Sobolev space on $\mathbb{R}^d$ with standard Gaussian measure. The space $\tilde{H}^1_{FP}(\Omega)$ is used to describe a Fokker-Planck equation similar to (2), but on $\tilde{\Omega}_\epsilon = \mathbb{R}^d$ and with a reverse sign for the transport term. We will therefore reuse the proofs of [1, Prop. 7.1] (and [1, Prop. 2.2], which treats the time-independent case) and modify only the parts dependent on $V$ and $\Omega_\epsilon$.

In step 1 of the proofs it is shown that we can assume without loss of generality that for every $z := (t, x) \in \Omega_{t,x} \subset \mathbb{R}^{d+1}$ and $\epsilon \in (0, 1]$ we have $B((1-\epsilon)z, \epsilon) \subset \Omega_{t,x}$, where $B(z, r)$ is the open ball with radius $r$ around $z$.

Let then $f \in H^1_{FP}(\Omega)$. As in step 2 of the proofs we take $\zeta \in C^\infty_c(\mathbb{R}^{d+1}; \mathbb{R})$ as a smooth function with compact support in $B(0, 1)$ such that $\int_{\mathbb{R}^{d+1}} \zeta = 1$. For each $\epsilon > 0$ and $z \in \mathbb{R}^{d+1}$ we write $\zeta(z) := \epsilon^{-1} \zeta(\epsilon^{-1} z)$, and define for $\epsilon \in (0, 1]$, $z \in \Omega_{t,x}$, and $v \in \Omega_\epsilon$ the mollification $f_\epsilon(z, v) := \int_{\mathbb{R}^{d+1}} f((1-\epsilon)z + z', v) \zeta(z') \, dz'$, so that we have $f_\epsilon \in C^\infty(\Omega_{t,x}; V)$. We may then show completely analogous to step 2 of the proofs of [1, Prop. 2.2 and 7.1] that $f$ belongs to the closed convex hull of the set $\{ f_\epsilon : \epsilon \in (0, 1/2] \}$ by just changing the spaces of all dual pairings and norms from $\tilde{V} = H^1_0(\mathbb{R}^d)$ to $V = H^1(S^{d-1})$ and from $L^2_\gamma(\mathbb{R}^d)$ to $L^2(S^{d-1})$.

It then remains to be shown that for fixed $\epsilon \in (0, 1/2]$ the function $f_\epsilon$ belongs to $\text{clos } \| u_{h_{FP}} \|^1_{C^\infty(\Omega_{t,x} \times \Omega_\epsilon)}$ by approximating $f_\epsilon$ also in the $v$-variable.

We choose a basis of $V = H^1_0(\Omega_{t,x})$ that is contained in $C^\infty(\Omega_{t,x})$: Since $V$ as a subspace of $L^2(\Omega_{t,x})$ is separable and $C^\infty(\Omega_{t,x}) \subset V$, there exists a dense countable set in $(C^\infty(\Omega_{t,x}), \| \cdot \|_V)$, from which we can obtain an orthonormal basis $(\psi_i)_{i \in \mathbb{N}}$ by the Gram-Schmidt algorithm. Since $\text{span}(\psi_i)_{i \in \mathbb{N}}$ is dense in $C^\infty(\Omega_{t,x})$ which is again dense in $V$ (cf. e.g. [30, Thm 2.4, p. 25]), $(\psi_i)_{i \in \mathbb{N}}$ is also an orthonormal basis of $V$.

For $k \in \mathbb{N}$, we define $f_{\epsilon,k} : \Omega_{t,x} \times \Omega_\epsilon \to \mathbb{R}$ as $f_{\epsilon,k}(z, v) := \sum_{i=1}^k (f_\epsilon(z, \cdot), \psi_i) v \psi_i(v)$.

Since we have $f_\epsilon \in C^\infty(\Omega_{t,x}; V)$, the map $z \mapsto (f_\epsilon(z, \cdot), \psi_i)_V$ is in $C^\infty(\Omega_{t,x})$. As $\psi_i \in C^\infty(\Omega_{t,x})$ for all $i \in \mathbb{N}$, we have $f_{\epsilon,k} \in C^\infty(\Omega_{t,x} \times \Omega_\epsilon)$ for all $k \in \mathbb{N}$.
Next, we compute \( \lim_{k \to \infty} \| f_x - f_{x,k} \|_{L^2(\Omega_{t,x}; V)} \). First, fix \( z \in \bar{\Omega}_{t,x} \). Since \( (\psi_i)_{i \in \mathbb{N}} \) is an orthonormal basis of \( V \) we have \( f_x(z) = \sum_{i=1}^{\infty} (f_x(z), \psi_i)_V \psi_i \) and thus
\[
\| f_x(z) - f_{x,k}(z) \|_V = \left\| \sum_{i=k+1}^{\infty} (f_x(z), \psi_i)_V \psi_i \right\|_V \xrightarrow{k \to \infty} 0.
\]
As this holds for all \( z \in \bar{\Omega}_{t,x} \) and \( \| f_x(z) - f_{x,k}(z) \|_V \leq 2\| f_x(z) \|_V \), we obtain by the dominated convergence theorem that \( \lim_{k \to \infty} \| f_x - f_{x,k} \|_{L^2(\Omega_{t,x}; V)} = 0 \). To determine \( \lim_{k \to \infty} \| (\frac{1}{v}) \cdot \nabla z (f_x - f_{x,k}) \|_{L^2(\Omega_{t,x}; V^*)} \), we first consider the partial derivatives separately: Since \( f_x \in C^\infty(\Omega_{t,x}; V) \), all first \( z \)-partial derivatives of \( f_x \) lie in \( L^2(\Omega_{t,x}; V) \) and we know that
\[
\| \partial_z f_x(z) - \partial_z f_{x,k}(z) \|_V = \left\| \sum_{i=k+1}^{\infty} (\partial_z f_x(z), \psi_i)_V \psi_i \right\|_V \xrightarrow{k \to \infty} 0
\]
for \( j = 1, \ldots, d + 1 \), and all \( z \in \bar{\Omega}_{t,x} \). Since \( |(\frac{1}{v})| \) is bounded on \( \Omega_v = S^{d-1} \), we thus have
\[
\| (\frac{1}{v}) \cdot \nabla z (f_x - f_{x,k}) \|_{L^2(\Omega_{t,x}; V^*)} \leq \sum_{j=1}^{d+1} \| (\frac{1}{v})_j \|_{L^\infty(\Omega_v)} \| \partial_z f_x(z) - \partial_z f_{x,k}(z) \|_V \xrightarrow{k \to \infty} 0,
\]
and again by the dominated convergence theorem that
\[
\lim_{k \to \infty} \| (\frac{1}{v}) \cdot \nabla z (f_x - f_{x,k}) \|_{L^2(\Omega_{t,x}; V^*)} = \lim_{k \to \infty} \| (\frac{1}{v}) \cdot \nabla z (f_x - f_{x,k}) \|_{L^2(\Omega_{t,x}; L^2(\Omega_{t,x})}) = 0.
\]
Hence, \( f_{x,k} \) converges to \( f_x \) in \( H^1_{FP}(\Omega) \), which completes the proof of Proposition 3.1.

\[\text{Proof of Lemma 3.3.} \] We estimate \( \| \phi f \|_{H^1_{FP}(\Omega)} \). Using the definition of the \( V \)-norm and the product rule we obtain for the first term\(^2\)
\[
\| \phi f \|_{L^2}^2 = \| \phi f \|_{L^2}^2 + \| (\nabla_v \phi) f + \phi \nabla_v f \|_{L^2}^2 \\
\leq \| \phi \|^2_{L^\infty} \| f \|_{L^2}^2 + 2 \| \nabla_v \phi \|^2_{L^\infty} \| f \|_{L^2} \| \nabla_v f \|_{L^2} + 2 \| \phi \|^2 \| f \|_{L^2} \| \nabla_v f \|_{L^2} \\
\leq 2 \left( \| \phi \|^2_{L^\infty} + \| \nabla_v \phi \|_{L^\infty} \right) \| f \|_{L^2}^2.
\]
By using the product rule, the characterization \( (\cdot, \cdot)_{X', X} = (\cdot, \cdot)_{L^2(\Omega)} \), and the continuity of \( C^\infty(\Omega) \) in \( H^1_{FP}(\Omega) \) we see that for arbitrary \( \psi \in X \) it holds
\[
\langle (\frac{1}{v}) \cdot \nabla_{t,x} (\phi f), \psi \rangle_{X', X} = \langle (\frac{1}{v}) \cdot \nabla_{t,x} f, \psi \rangle_{X', X} + \langle f ((\frac{1}{v}) \cdot \nabla_{t,x} \phi), \psi \rangle_{L^2(\Omega)} \\
\leq \| (\frac{1}{v}) \cdot \nabla_{t,x} f \|_{X'} \| \phi \|_{X} + \| f ((\frac{1}{v}) \cdot \nabla_{t,x} \phi) \|_{L^2(\Omega)} \| \psi \|_{L^2(\Omega)} \\
\leq \sqrt{2} \left( \| \phi \|^2_{L^\infty} + \| \nabla_v \phi \|_{L^\infty} \right)^{\frac{1}{2}} \| (\frac{1}{v}) \cdot \nabla_{t,x} f \|_{X'} \| \psi \|_{X} \\
+ \| (\frac{1}{v}) \cdot \nabla_{t,x} \phi \|_{L^\infty} \| f \|_{L^2(\Omega)} \| \psi \|_{L^2(\Omega)} \\
\leq \sqrt{2} \left( \| \phi \|^2_{L^\infty} + \| \nabla_v \phi \|_{L^\infty} \| \nabla_v \phi \|_{L^\infty} \right) \| f \|_{H^1_{FP}(\Omega)} \| \psi \|_{X}.
\]
\(^2\)As introduced in section 4, we write \( X = L^2(\Omega_{t,x}, V) \).
We thus have
\[
\left\| \left( \frac{1}{v} \right) \cdot \nabla_{t,x} (\phi f) \right\|_{X'} \leq 2 \sqrt{2} \left( \left\| \phi \right\|_{L^\infty(\Omega)} + \left\| \nabla v \phi \right\|_{L^\infty(\Omega)} + \left\| \left( \frac{1}{v} \right) \cdot \nabla_{t,x} \phi \right\|_{L^\infty(\Omega)} \right) \left\| f \right\|_{H^1_{Fp}(\Omega)}.
\]
(39)

Combining (38) and (39) and using that \( \left\| \frac{1}{v} \right\| \) is bounded in \( \Omega \), we thus have
\[
\left\| \phi f \right\|_{H^1_{Fp}(\Omega)} \leq C \left\| \phi \right\|_{C^1(\Omega)} \left\| f \right\|_{H^1_{Fp}(\Omega)}.
\]
\[\square\]

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PETROV-GALERKIN METHODS FOR A KINETIC FOKKER-PLANCK EQUATION

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