Quantum DNF Learnability Revisited

(preliminary version)

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Abstract

We describe a quantum PAC learning algorithm for DNF formulae under the uniform distribution with a query complexity of $\tilde{O}\left(\frac{s^3}{\epsilon} + \frac{s^2}{\epsilon^2}\right)$, where $s$ is the size of DNF formula and $\epsilon$ is the PAC error accuracy. If $s$ and $1/\epsilon$ are comparable, this gives a modest improvement over a previously known classical query complexity of $\tilde{O}(ns^2/\epsilon^2)$. We also show a lower bound of $\Omega\left(\frac{s \log n}{n}\right)$ on the query complexity of any quantum PAC algorithm for learning a DNF of size $s$ with $n$ inputs under the uniform distribution.

1 Introduction

In this abstract we describe a quantum learning algorithm for DNF formulae under the uniform distribution using quantum membership queries. Although Bshouty and Jackson [2] have shown that it is possible to adapt Jackson’s Harmonic Sieve algorithm [9] to the quantum setting, our goal is different. We will focus on reducing the number of quantum membership queries used by the DNF learning algorithm whereas their motivation was in showing that quantum examples are sufficient for learning DNF.

The Harmonic Sieve HS algorithm combines two crucial independent algorithms. The first algorithm is an inner algorithm for finding parity functions that weakly approximate the target DNF function. The second algorithm used in the Harmonic Sieve is an outer algorithm that is a boosting algorithm. A weak learning algorithm is an algorithm that produces hypotheses whose accuracy are slightly better than random guessing. Boosting is a method for improving the accuracy of hypotheses given by a weak learning algorithm.

For the inner algorithm, a Fourier-based algorithm given in [1] (called the KM algorithm) is used in HS for finding the weak parity approximators. The KM algorithm is based on a similar method given by Goldreich and Levin [1] in their seminal work on hardcore bits in cryptography. Subsequently, Levin [2] and Goldreich [1], independently, gave highly improved methods for solving this so-called Goldreich-Levin problem. Their ideas were adapted by Bshouty et al. [3] to obtain a weak DNF learning algorithm with query and time complexity of $\tilde{O}(n/\gamma^2)$, where $\gamma$ is the weak advantage of the parity approximator. By a result of Jackson [9], $\gamma = O(1/s)$ for DNF formula of size $s$.

For the outer algorithm, the original HS used a boosting method of Freund [4] called F1 that has various nice features. Recently, Klivans and Servedio [10] observed that a construction of

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Impagliazzo\cite{8} gave a smoother boosting algorithm called IHA. It was shown that IHA is a $O(1/\epsilon)$-smooth $O(\gamma^{-2}\epsilon^{-2})$-stage boosting algorithm, where $\gamma$ is the weak advantage of the weak learning algorithm and $\epsilon$ is the target accuracy. In contrast, F1 is a $O(1/\epsilon^3)$-smooth $O(\gamma^{-2}\log(1/\epsilon))$-stage boosting algorithm.

The fastest known algorithm for learning DNF is obtained by combining the two improved independent components that results in a total running time of $\tilde{O}(ns^3/\epsilon^2)$ and a query complexity of $\tilde{O}(ns^2/\epsilon^2)$\cite{3,10}.

We describe an efficient quantum DNF learning algorithm by combining a quantum Goldreich-Levin algorithm QGL of Adcock and Cleve\cite{1} with a well-known highly efficient boosting algorithm of Freund called $B_{\text{Comb}}$\cite{4}. The quantum algorithm of Adcock and Cleve used only $O(1/\gamma)$ queries (beating a classical lower bound of $\Omega(n/\gamma^2)$ proved also in\cite{1}). Freund’s $B_{\text{Comb}}$ algorithm is a $\tilde{O}(1/\epsilon)$-smooth $O(\gamma^{-2}\log(1/\epsilon))$-stage boosting algorithm. After adapting both algorithms for quantum PAC learning, we obtain a quantum Harmonic Sieve algorithm $QHS$ with a sample complexity of $\tilde{O}(s^3/\epsilon + s^2/\epsilon^2)$. In contrast to the best known classical upper bound of $\tilde{O}(ns^2/\epsilon^2)$, this gives a modest improvement if $s$ and $1/\epsilon$ are comparable.

As shown in \cite{1}, the quantum Goldreich-Levin algorithm has applications to quantum cryptography. In this work, we show one of its applications in computational learning theory.

For the sake of exposition, in this abstract we will describe our quantum DNF PAC learning algorithm using a conceptually simpler boosting algorithm $\text{SmoothBoost}$ given by Servedio\cite{13}. We describe a boost-by-filtering version of Servedio’s $\text{SmoothBoost}$ that is a $O(1/\epsilon)$-smooth $O(\gamma^{-2}\epsilon^{-1})$-stage boosting algorithm. So, we incur an extra $1/\epsilon$ factor in the sample complexity. We defer the details of using $B_{\text{Comb}}$ in $QHS$ to the final version of this paper.

Finally, we prove a query lower bound of $\Omega(s \log n/n)$ on any quantum PAC learning algorithm for DNF under the uniform distribution with (quantum) membership queries.

\section{Preliminaries}

We are interested in algorithms for learning approximations to an unknown function that is a member of a particular class of functions. The specific function class of interest in this paper is that of DNF expressions, that is, Boolean functions that can be expressed as a disjunction of terms, where each term is a conjunction of Boolean variables (possibly negated). Given a target DNF expression $f : \{0,1\}^n \rightarrow \{-1,+1\}$ having $s$ terms along with an accuracy parameter $0 < \epsilon < 1/2$ and a confidence parameter $\delta > 0$, the goal is to with probability at least $1-\delta$ produce a hypothesis $h$ such that $\Pr_{x \sim U_n}[f(x) \neq h(x)] < \epsilon$, where $U_n$ represents the uniform distribution over $\{0,1\}^n$. We will sometimes refer to such an $h$ as an $\epsilon$-approximator to $f$, or equivalently say that $h$ has $\frac{1}{2} - \epsilon$ advantage (this represents the advantage over the agreement between $f$ and a random function, which is $1/2$). A learning algorithm that can guarantee only $\gamma > 0$ advantage in the hypothesis produced but can do so with arbitrarily small probability of failure $\delta$ is called a weak learning algorithm, and the hypothesis produced is a weak approximator.

The information our learning algorithm is given about the target function varies. One form is a sample, that is, a set $S$ of input/output pairs for the function. We often use $x$ to denote an input and $f(x)$ the associated output, and $x \in S$ to denote that $x$ is one of the inputs of the pairs in $S$. Another type of information we sometimes use is a membership oracle for $f$, MEM$_f$. Such an oracle is given an input $x$ and returns the function’s output $f(x)$.  

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Input: \( \text{Parameters } 0 < \epsilon < 1/2, 0 \leq \gamma < 1/2 \)  
Sample \( S \) of target \( f \)  
Weak learning algorithm \( \text{WL} \)

Output: Hypothesis \( h \)

1. \( U_S \equiv \text{the uniform distribution over } S \)
2. \( M_1(x) \equiv 1, \forall x \in S \)
3. \( N_0(x) \equiv 0, \forall x \in S \)
4. \( \theta \leftarrow \gamma/(2 + \gamma) \)
5. \( t \leftarrow 1 \)
6. while \( \mathbb{E}_{x \sim U_S}[M_t(x)] > \epsilon \) do
7. \( D_t(x) \equiv M_t(x)/(m\mathbb{E}_{x \sim U_S}[M_t(x)]), \forall x \in S \)
8. \( h_t \leftarrow \text{WL}(S, D_t, \delta = \Omega(\epsilon^{-1}\gamma^{-2})) \)
9. \( N_t(x) \equiv N_{t-1}(x) + f(x)h_t(x) - \theta, \forall x \in S \)
10. \( M_{t+1}(x) \equiv [N_t(x) < 0] + (1 - \gamma)^{N_t(x)/2}[N_t(x) \geq 0], \forall x \in S \)
11. \( t \leftarrow t + 1 \)
end while
12. \( T \leftarrow t - 1 \)
13. \( H \equiv \frac{1}{T} \sum_{i=1}^{T} h_i \)
14. return \( h \equiv \text{sign}(H) \).

Figure 1: The SmoothBoost algorithm of Servedio [13].

3 A smoother Boost-by-Filtering algorithm

A modification of Servedio’s SmoothBoost boosting algorithm [13] is described in this section. A special case (discrete weak hypotheses and fixed margin) version of SmoothBoost sufficient for our purposes is shown in Figure 1. SmoothBoost is a boosting-by-sampling method that can be applied to a weak learning algorithm in order to produce a hypothesis that closely approximates the sample. Specifically, SmoothBoost receives as input a sample \( S \) of size \( m \) as well as accuracy parameter \( \epsilon \). It is also given a weak learning algorithm \( \text{WL} \). The boosting algorithm defines a series of distributions \( D_t \) over \( S \) and successively calls the weak learning algorithm, providing it with the sample \( S \) and with one of the distributions \( D_t \). In the end, the algorithm combines the weak hypotheses returned by the calls to the weak learner into a single hypothesis \( h \).

Servedio proves three key properties of SmoothBoost:

Lemma 1 (Servedio) Let \( f \) be a target function, and let \( S, \epsilon, \gamma, h, \) and \( D_t \) be as defined in Figure 1. Then

1. If every weak hypothesis \( h_t \) returned by \( \text{WL} \) has advantage at least \( \gamma \) with respect to \( D_t \), then SmoothBoost will terminate after \( T = O(\epsilon^{-1}\gamma^{-2}) \) stages.

2. If SmoothBoost terminates, then \( \Pr_{x \sim U_S}[f(x) \neq h(x)] < \epsilon \), where \( U_S \) represents the uniform distribution over \( S \) (Servedio actually proves a stronger margin result that implies this).

3. \( L_\infty(mD_t) \leq 1/\epsilon \) for all \( t \), where \( m = |S| \) (this is the smoothness property of SmoothBoost).
Here we adapt this algorithm to obtain a boosting-by-filtering algorithm that will be used by the Harmonic Sieve. First, notice that Lemma 1 holds for the special case $S = \{0,1\}^n$. However, there are potential problems with running the SmoothBoost algorithm directly on such a large $S$. First, it is not computationally feasible to exactly compute $E_{x \sim U_n}[M_t(x)]$, where $U_n$ represents the uniform distribution over $\{0,1\}^n$. So instead we must estimate this quantity by sampling. This has a small impact on both the form of the loop condition for the algorithm (line 6), but also on the “distributions” $D_t$ passed to the weak learner (line 6). In fact, the $D_t$ that will be passed to the weak learner will generally not be a true distribution at all, but instead a constant multiplied by a distribution due to the constant error in our estimate of $E_{x \sim U_n}[M_t(x)]$.

We will deal with the weak learner later, so for now let us assume that the weak learner produces the same hypothesis $h_t$ given an approximation to $D_t$ as it would given the actual distribution. Then notice that the computations for $N_t$ and $M_{t+1}$ are unchanged, so the only impact on the boosting algorithm has to do with the loop condition at line 6. This is easily addressed: let $E_t$ represent an estimate of $E_{x \sim U_n}[M_t(x)]$ to within additive error $\epsilon/3$ and change the loop condition to $E_t > 2\epsilon/3$. Then if the loop terminates it must be that $E_{x \sim U_n}[M_t(x)] \leq \epsilon$, as before. It is easily verified that given this condition, Servedio’s proof implies that $h$ is an $\epsilon$-approximator to $f$ with respect to the uniform distribution. Furthermore, since $E_{x \sim U_n}[M_t(x)] \geq \epsilon/3$ if the algorithm terminates, the other statements of Lemma 1 change only by constant factors. In particular, the smoothness condition of the lemma now becomes $L_\infty(2^n D_t) \leq 3/\epsilon$ for all $t$.

Finally, because $O \leq M_t(x) \leq 1$ for all $t$ and $x$, the Hoeffding bound gives that taking the sample mean of $M_t(x)$ over a sample of size $\Omega(\epsilon^{-2})$ will, with constant probability, produce an estimate with additive error at most $\epsilon/3$. Furthermore, if the algorithm terminates in $T$ steps, then a single uniform random sample $R$ of size $\Omega(\log(T)/\epsilon^2)$ guarantees, with constant probability, that estimating the expected value of $M_t(x)$ by the sample mean over $R$ at every step $t$ will produce an $\epsilon/3$ accurate estimate at every step.

Figure 2 presents the modified SmoothBoost algorithm. Notice that in place of a sample $S$ representing the target function $f$, we are assuming that we are given a membership oracle $MEM_f$. We will subsequently consider quantum versions of this algorithm and of the membership oracle. For this reason, we show the definitions of $M$ and $N$ as being over all of $\{0,1\}^n$, although for a classical algorithm the only values that would actually be used are those corresponding to $x \in R$.

While the SmoothBoost algorithm has been presented for illustration, Klivans and Servedio have shown that one of Freund’s boosting algorithms, which they call $B_{Comb}$, is actually slightly superior to SmoothBoost for our purposes. Specifically, they note that $B_{Comb}$ has properties similar to those of SmoothBoost given in Lemma 1, with the change that the number of stages $T$ improves from $O(\epsilon^{-1}\gamma^{-2})$ to $O(\log(1/\epsilon)/\gamma^2)$ while the smoothness of each of the distributions $D_t$ passed to the weak learner satisfies (when learning over all of $\{0,1\}^n$) $L_\infty(2^n D_t) = O(\log(1/\epsilon)/\epsilon)$. We will continue to use SmoothBoost in our analysis here, since $B_{Comb}$ and its analysis are noticeably more complicated than SmoothBoost and its analysis. However, our final sample size bounds will be stated as if $B_{Comb}$ is being used, and the final version of this paper will include details of the $B_{Comb}$ analysis.

4 A query-efficient quantum $WDNF$ algorithm

In this section we describe a quantum weak learning algorithm $WDNF$ for finding parity approximators of non-Boolean functions under smooth distributions. This algorithm is based on a quantum Goldreich-Levin algorithm given by Adcock and Cleve. For completeness we describe the quantum Goldreich-Levin algorithm in the following. This algorithm utilizes the Pauli $X$ (complement)
Input: Parameters $0 < \epsilon < 1/2$, $0 \leq \gamma < 1/2$
Membership oracle $MEM_f$
Weak learning algorithm $WL$

Output: Hypothesis $h$

1. Draw uniform random sample $R$ of $\Omega((\log(1/\epsilon - 1)) \cdot \epsilon^2)$ instances $x$ and label using $MEM_f$
2. $U_R \equiv$ the uniform distribution over $R$
3. $M_1(x) \equiv 1$, $\forall x \in \{0, 1\}^n$
4. $N_0(x) \equiv 0$, $\forall x \in \{0, 1\}^n$
5. $\theta \leftarrow \gamma/(2 + \gamma)$
6. $t \leftarrow 1$
7. while $E_{x \sim U_R}[M_t(x)] > 2\epsilon/3$ do
8. $D_t(x) \equiv M_t(x)/(2^n E_{x \sim U_R}[M_t(x)])$
9. $h_t \leftarrow WL(MEM_f, D_t, \delta = \Omega(\epsilon^{-1}\gamma^{-2}))$
10. $N_t(x) \equiv N_{t-1}(x) + f(x)h_t(x) - \theta$, $\forall x \in \{0, 1\}^n$
11. $M_{t+1}(x) \equiv \lfloor N_t(x) \leq 0 \rfloor + (1 - \gamma)^{N_t(x)/2} \lfloor N_t(x) \geq 0 \rfloor$, $\forall x \in \{0, 1\}^n$
12. $t \leftarrow t + 1$
13. end while
14. $T \leftarrow t - 1$
15. $H \equiv \frac{1}{T} \sum_{i=1}^{T} h_i$
16. return $h \equiv \text{sign}(H)$

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Figure 2: The SmoothBoost modified for boost-by-filtering.

and $Z$ (controlled phase flip) gates and the Hadamard gate $H$ defined as follows.

$$ X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. $$

Let $H_n = H^\otimes_n$ ($n$-fold tensor of $H$ with itself) be the Walsh-Hadamard transform on $n$ qubits and let $U_0(\sum_x \alpha_x |x\rangle) = \sum_{x \neq 0} \alpha_x |x\rangle - \alpha_0 |0\rangle$ be the unitary transformation that flips the phase of the all-zero state.

The $U_{MQ}$ transformation that represents a noisy membership oracle with respect to a parity function $\chi_A$ defined in $\mathbb{F}$ is given by

$$ U_{MQ}|x\rangle|0_m\rangle = \alpha_x|x, u_x, A \cdot x\rangle + \beta_x|x, v_x, A \cdot x\rangle, $$

where $\sum_x \alpha_x^2 \geq 1/2 + \gamma$ and $\sum_x \beta_x^2 \leq 1/2 - \gamma$. By a result of Jackson $\mathbb{F}$, for any DNF formula $f$ with $s$ terms, there is a parity function $A$ such that $Pr[f(x) = \chi_A(x)] \geq 1/2 + \gamma/2$, for $\gamma = 1/(2s + 1)$. Thus, a noiseless DNF oracle $QM_{Qf}$ is a noisy oracle $U_{MQ}$ for some parity function $\chi_A$. Thus, we may assume that $U_{MQ}$ is a unitary transformation that represents a quantum membership oracle $QM_{Qf}$ for a DNF formula $f$ that maps $|x\rangle|0_m\rangle$ to $|x\rangle|u_x, f(x)\rangle$, for some string $u_x \in \{0, 1\}^{m-1}$ that represents the work space of the oracle.

The quantum algorithm $QGL$ of Adcock and Cleve is represented by the following unitary transformation

$$ C = (H_n \otimes I_{m+1})(U_{MQ}^\dagger \otimes I_1)(I_{n+m-1} \otimes Z)(U_{MQ} \otimes I_1)(H_n \otimes I_m \otimes X) \quad (1) $$

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Input: Parameters \( n, \gamma \in (0, 1/2), \delta > 0 \)
Quantum membership oracle \( QMQ_f \) for Boolean function \( f \)
represented by a unitary transformation \( U_{MQ} \)
Random uniform sample \( R \) of size \( \tilde{\Omega}(\gamma^{-2}\log(1/\delta)) \).

Output: A coefficient \( A \) with the property that \( \Pr_D[f = \chi_A] \geq \frac{1}{2} + \gamma \) with probability at least \( 1 - \delta \).

1. Let \( C \) be defined as in Equation 1.
2. Label \( R \) using \( QMQ_f \).
3. Define a sampling-based \( U_{EQ} \) as in Equation 2.
4. \( |\varphi\rangle \leftarrow C|0_n\rangle|0_m\rangle_A|0\rangle_B \)
5. for \( k = 1, \ldots, O(1/\gamma) \) do
6. \( |\varphi\rangle \leftarrow -CU_0C^\dagger U_{EQ}|\varphi\rangle \)
7. end for
8. Measure and return the contents of register \( I \).

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Figure 3: The quantum weak learning algorithm \( QWDNF \) for uniform distribution.

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applied to the initial superposition of \( |0_n, 0_m, 0\rangle \). In \cite{1} it was proved that the quantum algorithm \( QGL \) prepares a superposition of all \( n \)-bit strings such that the probability of observing the coefficient \( A \) is \( 4\gamma^2 \). By repeating this for \( O(1/\gamma^2) \) stages, we can recover \( A \) with constant probability.

The number of stages can be reduced to \( O(1/\gamma) \) by using a technique called amplitude amplification. This amplification technique uses an iterate of the form

\[
G = (-CU_0C^\dagger U_{EQ})^k C|0_n, 0_m, 0\rangle,
\]

where \( k \) is approximately \( O(1/\gamma) \), and \( U_{EQ} \) is a unitary transformation that represents a quantum equivalence oracle \( QEQ_f \). The transformation \( U_{EQ} \) is defined as

\[
U_{EQ}|a\rangle = \begin{cases} 
-|a\rangle & \text{if } |E[f\chi_a]| \geq \theta \\
|a\rangle & \text{otherwise}
\end{cases}
\] (2)

For the purpose of learning DNF, we need to simulate \( U_{EQ} \) using a sampling algorithm that has access to \( QMQ_f \). A classical application of Hoeffding sampling requires \( \Omega(1/\gamma^2) \) queries to \( QMQ_f \). To simulate \( U_{EQ} \), we will simply use a sample \( R \) of size \( \Omega(1/\gamma^2 \log(1/\delta)) \) to obtain a good estimate with probability at least \( 1 - \delta \).

Finally, recall that we will be applying boosting to this weak learning algorithm, which means that \( QWDNF \) will be called a number of times. However, it is not necessary to draw a new random sample \( R \) each time \( QWDNF \) is called, as the boosting algorithm merely wants a guarantee that the algorithm succeeds with high probability and does not require independence. The resulting quantum weak learning algorithm for DNF, which we denote \( QWDNF \), is described in Figure 3.

4.1 Non-Boolean Functions over Smooth Distributions

Recall that in the Harmonic Sieve algorithm \cite{9}, we need to find weak Parity approximators for non-Boolean functions \( g \) that is based on the DNF formula \( f \) and the current boosting distribution

\footnote{Grover has proposed a quantum algorithm for estimating the mean that requires \( O(\frac{1}{\gamma} \log \log \frac{1}{\delta}) \) queries. However, in our setting, we will use fewer queries if we estimate this value classically because we can use a single sample for all estimates, as discussed below.}
\[ D \text{ in SmoothBoost, i.e., we need to consider expressions of the form (we have dropped subscripts for convenience)} \]

\[
E_D[f(x)\chi_A(x)] = \sum_x D(x)f(x)\chi_A(x)
\]

\[
= \sum_x \frac{M(x)}{2^n}f(x)\chi_A(x)
\]

\[
= \frac{E[M(x)f(x)\chi_A(x)]}{E[M(x)]}.\]

This shows a reduction from finding a coefficient \( A \) such that \( |E_D[f\chi_A]| \) is large to finding a coefficient \( A \) so that \( |E_U[g\chi_A]| \), where \( g(x) = M(x)f(x) \), is large. Assuming that \( E[M(x)] \geq \epsilon/3 \), we will use the algorithm \( \text{QWDNF} \) to find a coefficient \( A \) such that for some constant \( c_2 \)

\[
|E[M(x)f(x)\chi_A(x)]| \geq \frac{c_2\epsilon}{3(2s + 1)} = \Gamma.
\]

Note that \( 0 < M(x) \leq 1 \), for all \( x \). Thus we can use a technique of Bshouty and Jackson \cite{BshoutyJackson} that transforms the problem to the individual bits of \( M(x) \). Let \( d = \log(3/\Gamma) \), where \( \Gamma \) is as above. Let \( \alpha(x) = [2^d M(x)]/2^d \), i.e., \( M(x) \) truncated to include only \( d \) of its most significant bits. Assume that \( \alpha = \sum_{j=1}^d \alpha_j 2^{-j} + k 2^{-d} \), where \( \alpha_j \in \{-1, 1\} \) and \( k \in \{-1, 0, 1\} \). Thus

\[
|E[M(x)f(x)\chi_A(x)]| - \frac{\Gamma}{3} \leq |E[\alpha(x)f(x)\chi_A(x)]| \leq \max_j |E[\alpha_j(x)f(x)\chi_A(x)]| + \frac{\Gamma}{3}
\]

thus there exists \( j \) so that \( |E[\alpha_j(x)f(x)\chi_A(x)]| \geq \Gamma/3 \), assuming \( |E[M(x)f(x)\chi_A(x)]| \geq \Gamma \).

Note that to simulate \( U_{\text{EQ}} \) for verifying that the non-Boolean function \( g(x) = M(x)f(x) \) has a \( \Gamma \)-heavy coefficient at \( A \), i.e., \( |\hat{g}(A)| \geq \Gamma \), we need a sample of size at least \( 1/\Gamma^2 \sim (s/\epsilon)^2 \).

5 A quantum Harmonic Sieve algorithm

In this section, we describe a quantum version of the Harmonic Sieve algorithm obtained by combining the quantum Goldreich-Levin algorithm and the SmoothBoost boosting algorithm (see Figure \[ \text{[figure]} \]).

The top level part of this algorithm involves \( O(s^2/\epsilon) \) boosting rounds\footnote{This could be improved to \( O(s^2\log(1/\epsilon)) \) rounds if Freund’s \( \text{BComb} \) algorithm is used.} and each round requires invoking the algorithm \( \text{QWDNF} \) that uses \( \tilde{O}(s/\epsilon) \) queries. The “oracle” \( \text{QMQ}_f \cdot D_t \) represents the procedure that will produce Boolean functions representing the bits of \( M_t f \) and simulate quantum membership oracles to be passed to \( \text{QWDNF} \). There is an additional cost of a random sample of size \( \tilde{O}(s^2/\epsilon^2) \) for estimating the expression \( E[M_t] \) to within \( O(\epsilon) \) and for simulating the equivalence oracle \( U_{\text{EQ}} \) used by \( \text{QWDNF} \). The latter step requires estimating the expression \( E[M_t f\chi_A] \) to within \( O(\epsilon/s) \) accuracy. This random sample is shared among all boosting stages and all calls to \( \text{QWDNF} \). The key property exploited here is the oblivious nature of the sampling steps.

Thus the overall algorithm, if \( \text{BComb} \) is used as the boosting algorithm, requires \( \tilde{O}(s^3/\epsilon + s^2/\epsilon^2) \) sample complexity. The best classical algorithm (also based on \( \text{BComb} \)) has complexity \( \tilde{O}(ns^2/\epsilon^2) \). Thus, for \( s = \Theta(1/\epsilon) \), the quantum algorithm is an improvement by a factor of \( n \).
**Input:** Parameters $0 < \epsilon, \delta < 1$, $n$, a quantum membership oracle $QM_Qf$ for a DNF formula $f$, $s$ (the size of DNF $f$),

**Output:** $h$ so that $\Pr[f \neq h] < \epsilon$.

1. Draw a uniform random sample $R$ of $\Omega(s^2/\epsilon^2)$ instances $x$ and label using $QM_Qf$
2. $\gamma \leftarrow 1/(8s + 4)$ (weak advantage)
3. $k \leftarrow c_1\gamma^{-2}\epsilon^{-1}$ (number of boosting stages)
4. $M_1 \equiv 1$ (all-one function)
5. $N_0 \equiv 0$ (all-zero function)
6. for $t = 1, \ldots, k$ do
   7. $E_t \leftarrow E_x \sim U_R[M_t(x)]$
   8. if $E_t \leq 2\epsilon/3$ then
      9. break
   10. end if
   11. $D_t \equiv M_t/(2^n E_t)$
   12. $h_t \leftarrow QWDNF(n, \gamma\epsilon, \delta/2k, QM_Qf \cdot D_t, R)$ where $\Pr[D_t[h_t(x) \neq f(x)] \leq \frac{1}{2} - \gamma$.
   13. $N_t \equiv N_{t-1} + f h_t - \theta$
   14. $M_{t+1} \equiv [N_t < 0] + (1 - \gamma)N_{t/2}[N_t \geq 0]$
15. end do
16. $T = t - 1$
17. $H(x) \equiv \frac{1}{T} \sum_{t=1}^{T} h_t(x)$
18. return $h(x) = \text{sign}(H(x))$.

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**Figure 4:** The new QHS algorithm.

### 6 Lower bounds

In this section, we prove a lower bound on the query complexity of any quantum PAC learning algorithm for DNF formulæ.

**Theorem 2** Let $s \geq n / \log n$. Then any quantum PAC learning algorithm requires $\Omega(s \log n/n)$ queries to learn a DNF formula of size $s$ over $n$ variables under the uniform distribution, given $\epsilon < 1/4$ and any constant $\delta > 0$.

**Proof** We use a construction given in Bshouty et al. [3]. Let $t = \log s$ and $u = n - t$. Consider the following class $C$ of DNF formulæ over the variable set of $V = \{x_1, \ldots, x_t\} \cup \{y_1, \ldots, y_u\}$,

$$C = \left\{ \bigvee_{a \in \{0,1\}^t} x^a y_1^{a_1} \cdots y_u^{a_u} : \langle y_a \rangle_{a \in \{0,1\}^t} \right\},$$

where $x^a = \bigwedge_{i=1}^{t} x_i^{a_i}$, with the convention $x_0^0 = x_i$ and $x_1^1 = \overline{x_i}$, and for each $a \in \{0,1\}^t$, $y_a$ is a constant (0 or 1) or one of the variables $y_i$ or its negation. Each $f \in C$ is specified uniquely by a word $y \in \Sigma^*$ over the alphabet $\Sigma = \{0,1,y_1,\overline{y_1},\ldots,y_u,\overline{y_u}\}$, i.e., we may denote $f_y$ to be the DNF specified by the word $y \in \Sigma^*$. By the Gilbert-Varshamov bound, there is a code $L \subset \Sigma^*$ with minimum distance $\alpha s$ of size at least

$$\frac{|\Sigma|^s}{\sum_{k=0}^{\lfloor s \rfloor} \binom{\lfloor s \rfloor}{k} (|\Sigma| - 1)^k} \geq \left( \frac{(2u + 2)^{1-\alpha}}{2} \right)^s$$
We focus on $C_L \subset C$ where the words $y$ are taken from $L$. Note that for any distinct $y, z \in L$ we have $\Pr_U[f_y \neq f_z] = \mathbb{E}_U[f_y \oplus f_z] \geq \alpha s/2$, where the probability is taken over the uniform distribution on $V$. Letting $2\epsilon = \alpha s/2$, this implies that any two distinct DNF functions $f_y, f_z$, where $y, z \in L$, are $(2\epsilon)$-separated. So any $(\epsilon, \delta)$-PAC algorithm for $C_L$ must return exactly the unknown target function.

Now let $A$ be any quantum $(\epsilon, \delta)$-PAC algorithm with access to a quantum membership oracle $QMQ_f$ associated with a target DNF function $f$. Suppose that $A$ makes $T$ queries for any function $f \in C_L$. Following the notation in [7], let $X^f$ be the truth table of the DNF function $f$, i.e., $X^f$ is a binary vector of length $N = 2^n$. Let $P_h(X^f)$ be the probability function of $A$ of returning as answer a DNF function $h$ when the oracle is $QMQ_f$, for $h, f \in C_L$. By the PAC property of $A$, we have

- $P_f(X^f) \geq 1 - \delta$
- $\sum_{h: h \neq f} P_h(X^f) < \delta$

It is known that $P_f$ is a multivariate polynomial of degree $2T$ over $X^h$, for any $f, h$. Let $N_0 = \sum_{t=0}^{2T} \binom{2T}{t}$. For $X \in \{0,1\}^N$, let $\tilde{X} \in \{0,1\}^{N_0}$ be the vector obtained by taking all $\ell$-subsets of $[N]$, $\ell \leq 2T$. The coefficients of $P_h$ can be specified by a real vector $V_h \in \mathbb{R}^{N_0}$ and $P_h(X^f) = V_h^T X^f$. Let $M$ be a matrix of size $|C_L| \times N_0$ whose rows are given by the vectors $V_h^T$ for all $h \in C_L$. Let $N$ be a matrix of size $|C_L| \times |C_L|$ whose columns are given by the vectors $MV_g$ for all $g \in C_L$. Observe that the $(h, f)$ entry in the matrix $N$ is given by $P_h(X^f)$. As in [7], we argue that since $N$ is diagonally dominant (from the PAC conditions on $\delta$ above), it has full rank. Thus $N_0 \geq |C_L|$, which implies that

$$N^{2T} \geq |C_L| \geq \left(\frac{(2u + 2)^{1-\alpha}}{2}\right)^s.$$ 

This implies that $4nT \geq s \log(n)(1-o(1))$ which gives $T \geq \Omega(s \log n/n)$. 

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