LAGRANGE SPECTRA IN TEICHMÜLLER DYNAMICS VIA RENORMALIZATION

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ABSTRACT. We introduce Lagrange Spectra of closed-invariant loci for the action of \( SL(2, \mathbb{R}) \) on the moduli space of translation surfaces, generalizing the classical Lagrange Spectrum, and we analyze them with renormalization techniques. A formula for the values in such spectra is established in terms of the Rauzy-Veech induction and it is used to show that any invariant locus has closed Lagrange spectrum and values corresponding to pseudo-Anosov elements are dense. Moreover we show that Lagrange spectra of arithmetic Teichmüller discs contain an Hall’s ray, giving an explicit bound for it via a second formula which uses the classical continued fraction algorithm. In addition, we show the equivalence of several definitions of bounded Teichmüller geodesics and bounded type interval exchange transformations and we prove quantitative estimates on excursions to the boundary of moduli space in terms of norms of positive matrices in the Rauzy-Veech induction.

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1991 Mathematics Subject Classification. 37D40, 11J06.
Key words and phrases. Bounded-type interval exchange transformations and translation surfaces, Lagrange spectra.
1. Introduction

The classical Lagrange Spectrum is the set $L$ of values

$$L(\alpha) := \limsup_{q,p \to \infty} \frac{1}{q|q\alpha - p|} \text{ for } \alpha \in \mathbb{R}.$$ 

The quantity $L(\alpha)$ has the following interpretation in Diophantine Approximation. Recall that by Dirichlet theorem, given any irrational $\alpha \in \mathbb{R}$, there exists infinitely many integers $p,q$, with $q \neq 0$, such that $|\alpha - \frac{p}{q}| < \frac{1}{cq^2}$; we have that

$$L(\alpha) = \{ \sup c > 0 \text{ such that } |\alpha - \frac{p}{q}| < \frac{1}{cq^2} \text{ for infinitely many } p,q \in \mathbb{Z}, q \neq 0 \}.$$ 

The set $L$ has been studied in depth for many decades, see for example the survey by Cusick and Flahive [C,F] and the references therein. In particular, some of the basic properties are summarized below.

1. $L$ is a closed subset of the real line (Cusick 1975).
2. The values $L(\beta)$ for $\beta$ quadratic irrational are dense in $L$ (Cusick 1975).
3. $L$ contains an Hall’s ray, that is a positive half-line (Hall 1947).

It is also known that $L$ begins with a discrete sequence (Markoff 1879), which coincides with the Markoff Spectrum and whose smaller term is $\sqrt{5}$, which is called Hurwitz constant (Hurwitz 1891). More recently, the structure of $L$ in between the discrete part and the Hall ray has been investigated by G. Moreira in [Mor]. The continued fraction algorithm plays a crucial role in the proof of all these results on $L$, thanks to the following beautiful formula for $L(\alpha)$ (Perron 1921). Let $\alpha = a_0 + [a_1, a_2, \ldots]$ be the continued fraction expansion of $\alpha$. Then we have

$$(1.1) \quad L(\alpha) = \limsup_{n \to \infty} \left( [a_{n-1}, \ldots, a_1] + a_n + [a_{n+1}, a_{n+2}, \ldots] \right).$$

It is clear from this formula that $L(\alpha)$ is finite if and only if $\alpha$ is of bounded type, that is the sequence $(a_n)_{n \in \mathbb{N}}$ is bounded. It is well known that the continued fraction expansion is related to the coding of the geodesic flow on the modular surface (see [Sc1, Sc0]). In this geometric context there is a way to establish a correspondence between bounded type real numbers $\alpha$ and bounded geodesic rays $(g_\alpha^t)_{t>0}$. Any value $L(\alpha)$ corresponds to the following geometric quantity

$$\limsup_{t \to \infty} \text{heigth}(g_\alpha^t),$$

which gives the asymptotic depth of penetration of the ray $g_\alpha^t$ into the cusp of the modular surface, where $\text{heigth}(\cdot)$ denotes the hyperbolic height.

Many generalizations of Lagrange and Markhoff Spectra have been studied by several authors. We do not attempt here to summarize the huge literature, but we just mention a partial list of examples. A first natural generalization of the modular surface are quotients of the upper half plane by Hecke groups and more in general by Fuchsian groups with cusps. Investigations of the Hurwitz constant and the Markoff spectrum for Hecke and triangle groups were carried out in [Sc2, Ha, S, Vu1] and for Fuchsian groups with cusps in [Vu2]. Moreover, the existence of a Hall ray for penetration Markoff Spectra for any Riemann surface with cusps is shown in [S¹,S]. Discrete groups acting on higher dimensional hyperbolic spaces were also studied, in particular the Hurwitz constant and the Lagrange Spectrum are studied in [Vu3, Vu4], and the closure of the spectrum is special case of [Mau] and [P,P¹]. Lagrange and Markhoff spectra for quotients by Bianchi groups have also been studied since they yield number theoretical applications for the approximation of a complex number by elements of an imaginary quadratic number field, see [Vu5, Mau]. In [Mau], Maucourant proves more in general the closure of Lagrange Spectra for manifolds with negative sectional curvature. Other geometric generalizations of the Lagrange Spectra in negative
the equivalence being evident, since datum \((S, w, \pi)\) and conical singularities whose angles are multiples of \(2\pi\).

### 1.1. Translation surfaces

A translation surface \(X\) is a compact surface of genus \(g\), with a flat metric and conical singularities whose angles are multiples of \(2\pi\). Alternatively, a translation surface \(X\) is a datum \((S, w)\), where \(S\) is a compact Riemann surface of genus \(g\) and \(w\) is an holomorphic 1-form on \(S\) with zeros of orders \(k_1, \ldots, k_r\) at points \(p_1, \ldots, p_r\). The two definitions are equivalent, one direction of the equivalence being evident, since \(w \otimes \varpi\) is a flat metric on \(S\) with cone angle \(2(k_i + 1) \cdot \pi\) at each \(p_i\). Conversely, the data \((S, w)\) can be recovered from \(X\) as follows. Outside the conical singularities \(p_1, \ldots, p_r\), an atlas \(S\) of flat charts can be chosen identifying flat neighborhoods of \(X\) with open sets in \(\mathbb{C}\), that is introducing a local coordinate \(z\) on each flat neighborhood. Since the angles at \(p_1, \ldots, p_r\) are multiples of \(2\pi\), then the atlas \(S\) can be chosen so that \(z' = z + \text{const}\) whenever any two such charts \(z\) and \(z'\) overlap, thus it is well-defined also an holomorphic 1-form \(w\) whose expression in these charts is \(dz\).

On a small neighborhood of a conical singularity \(p_i\) the 1-form \(w\) has a local primitive \(\zeta_i := \int w\) and the latter satisfies \(d\zeta_i = z'^k dz\) for any flat chart \(z'\) overlapping with \(\zeta_i\), where \(2(k_i + 1) \cdot \pi\) is the conical angle at \(p_i\). Therefore the atlas \(S\) extends holomorphically to each \(p_i\) and the same does \(w\), admitting a zero of order \(k_i\).

For any translation surface \(X\) the associated 1-form \(w\) induces an area form \((\sqrt{-1}/2)w \wedge \varpi\), different from zero outside \(p_1, \ldots, p_r\), so that the euclidian area of \(X\) is given by \(\text{Area}(X) = (\sqrt{-1}/2) \int_X w \wedge \varpi\). The 1-form \(w\) induces also a pair of parallel vector fields \(\partial_x\) and \(\partial_y\) by \(w(\partial_x) = 1\) and \(w(\partial_y) = \sqrt{-1}\). These vector fields are not complete, since they have \((k_i + 1)\) entering trajectories and \((k_i + 1)\) outgoing trajectories at the conical singularity \(p_i\). The vertical flow \(\phi^t\) of \(X\) is the integral flow of \(\partial_y\).

#### 1.1.1. Strata and action of \(\text{SL}(2, \mathbb{R})\)

For a translation surface \(X\) the genus and the orders of zeroes satisfy the relation \(k_1 + \cdots + k_r = 2g - 2\). Fox fixed integers \(k_1, \ldots, k_r\) satisfying the last relation, denote \(\mathcal{H}(k_1, \ldots, k_r)\) the corresponding stratum of the moduli space of translation surfaces, that is the set of translation surfaces whose associated 1-form \(w\) has \(r\) zeroes with orders \(k_1, \ldots, k_r\). It is a complex orbifold with complex dimension \(2g + r - 1\). Consider a translation surface \(X = (S, w)\) in the stratum \(\mathcal{H}(k_1, \ldots, k_r)\) and \(A \in \text{SL}(2, \mathbb{R})\). A new translation surface \(A \cdot X = (A_s S, A_s w)\) is defined, where the 1-form \(A_s w\) is the composition of \(w\) with \(A\) and \(A_s\) is the complex atlas for which \(A_s w\) is holomorphic. Therefore the group \(\text{SL}(2, \mathbb{R})\) acts on \(\mathcal{H}(k_1, \ldots, k_r)\). The Teichmüller flow \(\mathcal{F}_t\) is the action of the diagonal subgroup, that is

\[
\mathcal{F}_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.
\]

The Teichmüller flow preserves the hypersurface \(\mathcal{H}^{(1)}(k_1, \ldots, k_r) \subset \mathcal{H}(k_1, \ldots, k_r)\) consisting of all \(X \in \mathcal{H}(k_1, \ldots, k_r)\) with \(\text{Area}(X) = 1\).

Other relevant subgroups are \(\mathcal{R}_\theta := \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}\) and the horocyclic flow \(\mathcal{U}_s := \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}\).

#### 1.1.2. Bounded geodesics in moduli space

A saddle connection is a geodesic segment \(\gamma\) for the flat metric of \(X\) starting and ending in two conical singularities and not containing any other conical singularity in its interior. The set \(\text{Hol}(X)\) of periods of \(X\) is the set of complex numbers \(v := \int_\gamma w\), where \(\gamma\) is a saddle
connection for $X$ and $w$ is the holomorphic one-form. Compact subsets in the stratum are characterized as follows by the systole function $X \mapsto \text{Sys}(X)$, defined as the length of the shortest saddle connection of the translation surface $X$, that is

$$\text{Sys}(X) := \min \{|v|; v \in \text{Hol}(X)\}.$$ 

According to the so-called Mahler criterion, a sequence $X_n$ is bounded in $\mathcal{H}(k_1, \ldots, k_r)$ if and only if there exists some $\delta > 0$ such that $\text{Sys}(X_n) \geq \delta$ for any $n$. We study bounded orbits for the action of the Teichmüller flow on strata. A dynamical estimate of the asymptotic maximal excursion of the positive orbit $F_t(X)$ is given by

$$(1.2) \quad s(X) := \liminf_{t \to \infty} \frac{1}{\text{Area}(X)} \text{Sys}(F_t \cdot X).$$

For any period $v$ of a translation surface $X$ we set $\text{Area}(v) = |\Re(v)| \cdot |\Im(v)|$, where $\Re(v)$ and $\Im(v)$ denote respectively the real and imaginary part of $v \in \mathbb{C}$ (geometrically, $\text{Area}(v)$ is the Euclidean area of a rectangle which has $v$ as diagonal). In terms of the flat geometry of $X$, we obtain an estimate of the asymptotic maximal excursion of $F_t(X)$ setting

$$(1.3) \quad a(X) := \liminf_{|\Im(v)| \to \infty} \frac{\text{Area}(v)}{\text{Area}(X)}.$$

For any translation surface $X$ the two quantities introduced above are related by $s(X) = \sqrt{2a(X)}$ (see Proposition (1.1) below). In particular, the forward Teichmüller geodesics $F_t(X)$ is bounded in moduli space if, equivalently, $s(X) > 0$ or $a(X) > 0$.

1.2. Interval Exchange Transformations. An alphabet is a finite set $\mathcal{A}$ with $d \geq 2$ elements. An IET (which is a shortening for interval exchange transformation), is a map $T$ from an interval $I$ to itself such that $T$ admits two partitions in subintervals

$$P_T := \{I_\alpha^I; \alpha \in \mathcal{A}\} \text{ and } P_B := \{I_\beta^B; \beta \in \mathcal{A}\}$$

and for any $\alpha$ in $\mathcal{A}$ the restriction of $T$ to $I_\alpha^I$ is the translation onto $I_\alpha^B$. For any $\alpha$ in $\mathcal{A}$ the intervals $I_\alpha^I$ and $I_\alpha^B$ have the same length, that we denote $\lambda_\alpha$. We call length datum of $T$ the vector $\lambda$ in $\mathbb{R}^d_+$ whose $\alpha$-entry is $\lambda_\alpha$ for any $\alpha$ in $\mathcal{A}$. We call combinatorial datum of $T$ the pair of bijections $\pi = (\pi^I, \pi^B)$ from $\mathcal{A}$ to $\{1, \ldots, d\}$ such that for any $\alpha$ in $\mathcal{A}$, if we count starting from the left, $I_\pi^I$ is in the $\pi(I)\alpha$-th position in $P_T$ and $I_\pi^B$ is in the $\pi(B)\alpha$-th position in $P_B$. On the interval $I := (0, \sum_{\alpha \in \mathcal{A}} \lambda_\alpha)$ the data $(\pi, \lambda)$ determine uniquely $T$. We say that the combinatorial datum $\pi$ is admissible if there is no proper subset $\mathcal{A}' \subset \mathcal{A}$ with $k < d$ elements such that $\pi^I(\mathcal{A}') = \pi^B(\mathcal{A}') = \{1, \ldots, k\}$. The parameter space of all IETs with combinatorial datum $\pi$ is $\{\pi\} \times \mathbb{R}^d_+$. IETs have been largely studied, we refer for example to reader to the lecture notes [71, 71, 72].

IETs are strictly linked to translation surfaces and to the Teichmüller flow on their moduli space. More precisely, any translation surface $X$ has an unitary constant vector field whose first return map to a transversal segment $I$ in $X$ is an IET. In [Ve1], W. Veech gave a combinatorial presentation of a translation surface in relation to IETs, known as zippered rectangles construction. If $T$ is an IET defined by the data $(\pi, \lambda)$, where the combinatorial datum $\pi$ is admissible, then there exists a suspension datum $\tau$, that is a vector $\tau \in \mathbb{R}^d$ such that the triple of data $(\pi, \lambda, \tau)$ defines a translation surface $X$, whose first return map is $T$. The details of Veech’s zippered rectangles construction are summarized in [72]. It is possible to see that if Hol($X$) does not contain pure imaginary elements, then there exist data $\pi, \lambda, \tau$ such that $X = X(\pi, \lambda, \tau)$. For this last condition to hold it is sufficient to have

$$\liminf_{t \to \infty} \text{Sys}(F_t \cdot X) < +\infty,$$

which is obviously satisfied by bounded geodesic rays.

1.2.1. Bounded Type IETs. Let $X$ be a translation surface and consider an horizontal segment $I$ embedded in $X$. Let $T$ be the IET induced as first return to $I$ of the vertical flow of $X$. Following Boshernitzan, for any positive integer $n$ we define $\mathcal{E}_n(T)$ as the minimum of the distance between two singularities of the $n$-th iterated $T^n$ of $T$. Boshernitzan’s criterion says that if $T$ is a minimal IET which is not uniquely
ergodic, then \( \lim_{n \to \infty} n \cdot \mathcal{E}_n(T) = 0 \) (see [Bos] and [Ve2]). The last criterion suggests a third way to estimate the asymptotic maximal excursion of the orbit of \( X \), which is to define

\[
\mathcal{E}(T) := \liminf_{n \to \infty} \frac{n \cdot \mathcal{E}_n(T)}{|I|}.
\]

It turns out that for any \( X \) and \( T \) related as explained above we always have \( \mathcal{E}(T) = a(X) \) (see Proposition 1.1 below). In particular, the positive orbit \( \mathcal{F}_r(X) \) is bounded in moduli space if and only if \( \mathcal{E}(T) > 0 \).

### 1.3. Lagrange spectra of invariant loci.

The notion of bounded type translation surfaces and IETs corresponds to the positivity of the quantities \( s(X) \), \( a(X) \) and \( \mathcal{E}(T) \) introduced in Equations 1.5, 1.6 and 1.7. We have actually just one notion, since according to Proposition 1.1 below these three asymptotic quantities are in fact the same. Furthermore bounded type IETs can be defined in terms of positive matrices in the so-called Kontsevich-Zorich cocycle (see §1.2) which generalize the definition of \( \alpha \) of bounded type in terms of bounded entries in the continued fraction expansion (see Corollary 4.8).

**Proposition 1.1** (Vorobets’ identity). Consider data \((\pi, \lambda, \tau)\), let \( X \) be the underlying translation surface and \( T \) be the IET corresponding to \((\pi, \lambda)\). We have

\[
\mathcal{E}(T) = a(X) = s^2(X) / 2.
\]

The group \( \text{SL}(2, \mathbb{R}) \) acts on strata of translation surfaces. Recently, A. Eskin and M. Mirzakhani [E.Mi] proved that any closed and invariant subset for such action is an affine sub-orbifold of the corresponding stratum, that is a nice moduli space itself (we refer to [E.Mi] for definitions). Therefore it is natural to study the asymptotic quantities above for any such locus.

**Definition 1.2.** Let \( \mathcal{I} \) be an invariant locus, that is a subset of some stratum, which is closed and invariant under \( \text{SL}(2, \mathbb{R}) \). We define the Lagrange Spectrum of the invariant locus \( \mathcal{I} \) as

\[
\mathcal{L}(\mathcal{I}) := \{ a^{-1}(X) ; X \in \mathcal{I} \},
\]

where \( a^{-1}(X) \) denotes the inverse \( 1/a(X) \).

An interesting question is to compute the Hurwitz constant of the spectra, that is the smallest element of any spectrum \( \mathcal{L}(\mathcal{I}) \). We can establish the following immediate lower bound (the proof is given in [3]).

**Lemma 1.3.** Let \( \mathcal{I} \) be an invariant locus contained in the stratum \( \mathcal{H}(k_1, \ldots, k_r) \) of translation surfaces of genus \( g \), where \( 2g - 2 = k_1 + \cdots + k_r \). We have

\[
\mathcal{L}(\mathcal{I}) \subset \left[ \pi \cdot \frac{2g + r - 2}{2}, +\infty \right] .
\]

### 1.3.1. Lagrange spectra of Veech surfaces.

Focusing on closed orbits of \( \text{SL}(2, \mathbb{R}) \) it is possible to see that Definition 1.2 generalizes the classical definition of Lagrange spectrum.

The Veech group of a translation surface \( X \) is the stabilizer of \( X \) under the action of \( \text{SL}(2, \mathbb{R}) \), and it is denoted \( \text{SL}(X) \). We say that a translation surface \( X \) is a Veech surface if \( \text{SL}(X) \) has finite co-volume in \( \text{SL}(2, \mathbb{R}) \). It is well-known (see Sim, We) that \( X \) is a Veech surface if and only if its orbit \( \text{SL}(2, \mathbb{R}) \cdot X \) is closed in the stratum, that is it is a closed-invariant locus \( \mathcal{I}_X \) for the action of \( \text{SL}(2, \mathbb{R}) \), and in particular it can be identified with the unitary tangent bundle of the upper half plane quotiented by \( \text{SL}(X) \).

With the notation of [1.1.1] recall that for any \( A \) in \( \text{SL}(2, \mathbb{R}) \) there are \( \theta \) in \( [0, 2\pi) \) and \( t \) and \( s \) in \( \mathbb{R} \) such that \( A = R_\theta \cdot F_t \cdot U_s \). The map \( X \mapsto s(X) \) is obviously invariant under \( F_t \). Moreover for any \( t \) and \( s \) we have \( F_t \circ U_s = U_{s^{-1} \cdot t} \circ F_t \), hence \( X \mapsto s(X) \) is also invariant under \( U_s \), since the function \( \text{Sys}(\cdot) \) is continuous. Recalling Proposition 1.1 we can state the following Lemma.
Lemma 1.4 (Symmetries). Let $X$ be any translation surface. For any $s$ and $t$ in $\mathbb{R}$ we have
$$a(F_t \cdot X) = a(U_s \cdot X) = a(X).$$

Observe that for any $\theta \in [0, 2\pi)$ we have the extra symmetry $a(R_{\theta+\pi} \cdot X) = a(R_{\theta} \cdot X)$, indeed for any $\theta$ we obviously have $\text{Hol}(R_{\theta+\pi} \cdot X) = -\text{Hol}(R_{\theta} \cdot X)$. It follows that if $X$ is a Veech surface and $\mathcal{I}_X := \text{SL}(2, \mathbb{R}) \cdot X$ is its closed orbit under $\text{SL}(2, \mathbb{R})$, then Lemma 1.4 implies
$$\mathcal{L}(\mathcal{I}_X) = \{a^{-1}(R_{\theta} \cdot X) ; \ -\frac{\pi}{2} < \theta \leq \frac{\pi}{2}\}.$$ 

For convenience we pass to the variable $\alpha = \tan(\theta) \in (-\infty, +\infty]$ and we fix the convention that the horizontal direction on the Veech surface $X$ corresponds to $\alpha = +\infty = \tan(\pi/2)$ and that it is a periodic direction. Moreover we will write simply $\mathcal{L}(X)$ instead of $\mathcal{L}(\mathcal{I}_X)$. With these convention, the Lagrange Spectrum of a Veech surface $X$ is parametrized by a function $L_X : (\mathbb{R} \setminus \mathbb{Q}) \to \mathcal{L}(X)$, that we call standard parametrization, defined by

$$L_X(\alpha) := \frac{1}{a(R_{\arctan(\alpha)} \cdot X)}.$$

We can now explain how we can recover the classical Lagrange spectrum from these definitions. Let $\mathbb{T}^2$ be the standard torus, that is $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Its Veech group $\text{SL}(\mathbb{T}^2)$ is $\text{SL}(2, \mathbb{Z})$ and the stratum $\mathcal{H}(0)$, that is the moduli space of flat tori, coincides with $\text{SL}(2, \mathbb{R}) \cdot \mathbb{T}^2 = \text{SL}(2, \mathbb{R})/\text{SL}(2, \mathbb{Z})$. Therefore in genus one the whole parameter space is the only invariant locus and the only Lagrange spectrum is $\mathcal{L}(\mathbb{T}^2)$. In term of Equation (1.5) it is easy to verify that (a more general formula is established by Lemma 5.10)

$$L_{\mathbb{T}^2}(\alpha) = \limsup_{q,p \to \infty} \frac{1}{q|\alpha p - q|}.$$ 

Thus $\mathcal{L}(\mathbb{T}^2)$ coincides with $\mathcal{L}$ and Definition 1.2 is a generalization of the classical definition of Lagrange spectrum.

1.4. Lagrange spectra via the Rauzy-Veech induction. The Rauzy-Veech induction is a generalization for translation surfaces of the continued fraction expansion. It is an invertible map acting on data $(\pi, \lambda, \tau)$ coming from the zippered rectangles construction, whose iteration gives a sequence of data $(\pi^{(r)}, \lambda^{(r)}, \tau^{(r)})_{r \in \mathbb{Z}}$. The induction was defined by W. Veech in [Ve1], as natural extension of a two-to-one induction map acting on IETs, introduced by G. Rauzy in [R]. The induction at the level of IETs has an acceleration with a finite and ergodic smooth invariant measure, which was discovered in [Z1] by A. Zorich. Details on the Rauzy and Rauzy-Veech induction can be found in [22] and [23].

Recall that a real number $\beta$ is quadratic irrational if and only if its continued fraction expansion is eventually periodic. In higher genus this notion is naturally generalized by translation surfaces whose data $(\pi, \lambda, \tau)$ have periodic Rauzy-Veech induction. Therefore, for any invariant locus $\mathcal{I}$ we are lead to consider the set $PA(\mathcal{I})$ of those translation surfaces $X$ in $\mathcal{I}$ whose data $(\pi, \lambda, \tau)$ have periodic Rauzy-Veech induction. Equivalently, such $X$ correspond to periodic points for the restriction of the Teichmüller flow $\mathcal{F}_t$ on $\mathcal{I}$, or to translation surfaces admitting a pseudo-Anosov diffeomorphism.

1.4.1. Closure of Lagrange spectra and density of periodic values. One of the main tools of this paper is the following formula for the values in Lagrange spectra $\mathcal{L}(\mathcal{I})$ in terms of the Rauzy-Veech induction (the formula is established by Theorem 1.6). If $X$ is a translation surface corresponding to combinatorial, length and suspension data $(\pi, \lambda, \tau)$ and $(\pi^{(r)}, \lambda^{(r)}, \tau^{(r)})_{r \in \mathbb{Z}}$ is its sequence of iterates under Rauzy-Veech induction, we have

$$\frac{1}{a(X)} = \limsup_{r \to \infty} \frac{1}{w(\pi^{(r)}, \lambda^{(r)}, \tau^{(r)})},$$

where $w$ is a continuous function of the data $(\pi, \lambda, \tau)$ which is explicitly defined in [41, 22] (see Definition 1.5). The formula can be seen as a generalization of the classical formula (1.1) to the Rauzy-Veech induction. We prove the following theorem.

Theorem 1.5. Let $\mathcal{I}$ be any invariant locus contained in some stratum $\mathcal{H}$ of the moduli space of translation surfaces. Then

1. The Lagrange Spectrum $\mathcal{L}(\mathcal{I})$ is a closed subset of the real line.
2. The values $a^{-1}(X)$ for $X$ in $PA(\mathcal{I})$ are dense in $\mathcal{L}(\mathcal{I})$. 

1.4.2. Relations with other results in the literature. In [Mau], F. Maucourant treats the closure of Lagrange Spectra and density of values coming from periodic orbits for the geodesic flow on the unitary tangent bundle of a non-compact Riemannian manifold $M$ with finite volume and sectional curvature not grater than $-1$. Maucourant’s result obviously provides a proof of the special case of Theorem 1.5 for the Lagrange spectrum $\mathcal{L}(X)$ of a Veech surface $X$. Nevertheless it cannot be applied in the general case to any spectrum $\mathcal{L}(\mathcal{I})$, indeed strata and their sub-loci are not the tangent bundle of the moduli space $\mathcal{M}_g$ of Riemann surfaces of genus $g$ (but just components of a stratification) and moreover the Teichmüller metric on $\mathcal{M}_g$ has vanishing curvature.

Maucourant’s proof is essentially based on the classical closing Lemma for negatively curved manifolds together with geometric approximation arguments, which also appear in Paulin-Parkonnen [PP]. Very recently, Eskin, Mirzakhani and Rafi proved in [E,Ra,Mi] a generalization of the classical closing Lemma for strata of quadratic differentials (see also [Ham] for another proof of the closing Lemma on strata and [W] where the closing lemma is formulated in each invariant locus). Combined with the geometric approach in the spirit of Maucourant/Paulin-Parkonnen [Mau,PP], the closing Lemma for strata could be used to provide a different proof of properties (1) and (2) of the classical Lagrange spectrum for the Lagrange spectrum of any invariant locus.

Our proof of Theorem 1.5 is of combinatorial flavour and is a generalization of the proof given by Cusick in [C,F] in terms of the Rauzy-Veech induction. Nevertheless, at the combinatorial level, we also essentially prove a closing and shadowing Lemma for the Teichmüller flow on closed-invariant sub-loci $\mathcal{I}$ of strata (see Appendix B). It is remarkable that, even though Rauzy-Veech induction is in principle used to study the whole connected component of a stratum, it follows a posteriori that our construction provides closed orbits which live in an invariant locus $\mathcal{I}$. This follows in virtue of the local product structure of invariant loci, recently proved in the already mentioned work by Eskin and Mirzakhani [E,Mi]. Thus, our techniques can for example be used to produce pseudo-Anosov diffeomorphism in a given locus by using Rauzy-Veech algorithm.

1.5. Hall’s ray for square-tiled surfaces. A square-tiled surface, also said origami, is a translation surface $X$ tiled by copies of the square $[0,1]^2$. Equivalently, $X$ is square-tiled if there exists a ramified covering $\rho : X \to \mathbb{R}^2/\mathbb{Z}^2$, unramified outside 0 $\in \mathbb{R}^2/\mathbb{Z}^2$ and such that $\rho^*(dz)$ is the holomorphic 1-form of $X$. Square-tiled surfaces are Veech’s surfaces. Some more detail can be found in [5.1]

In Section 5 we prove the following extension of the classical formula (1.1) to square-tiles surfaces (see Theorem 5.12). We need to consider reduced origamis, where an origami $X$ is reduced if the lattice spanned by $\text{Hol}(X)$ is $\mathbb{Z}^2$ (see [5.1]). Any arithmetic closed Teichmüller disc contains reduced origamis. If $X$ is a reduced origami with $N$ squares and $\alpha = a_0 + [a_1, a_2, \ldots]$ denotes the continued fraction expansion of $\alpha$, then (under a technical condition on $\alpha$ for which we refer to Theorem 5.12) we have

$$L_X(\alpha) = N \cdot \limsup_n \left[ \frac{a_n, \ldots, a_1 + a_{n+1} + [a_{n+2}, a_{n+3}, \ldots]}{m_X^{\ast}(p_n/q_n)} \right],$$

where $p_n/q_n$ denotes the $n$-th convergent $[a_1, \ldots, a_n]$ of $\alpha$ and the factor $m_X^{\ast}(p_n/q_n)$ is the multiplicity of $p_n/q_n$, which is defined in [5.2.1] As an application of this formula, we prove the following result.

**Theorem 1.6.** Let $X$ be any reduced square-tiled surface. Then there exists $r(X) > 0$, whose explicit value appears in the statement of Theorem 5.16, such that $\mathcal{L}(X)$ contains the half-line $[r(X), +\infty)$.

Obviously, an inclusion between invariant loci $\mathcal{I} \subset \mathcal{I}'$ implies the inclusion $\mathcal{L}(\mathcal{I}) \subset \mathcal{L}(\mathcal{I}')$ for the corresponding Lagrange spectra, thus we have the following criterion for the existence of Hall’s ray.

**Corollary 1.7** (Criterion for Hall’s ray). If an invariant locus $\mathcal{I}$ contains a square-tiled surface $X$, then its Lagrange Spectrum $\mathcal{L}(\mathcal{I})$ contains an Hall’s ray.

1.5.1. Relation with other results in the literature. Let us remark that Teichmüller disks of square tiled surfaces are hyperbolic surfaces with constant negative curvature and cusps (see Veech [Ve3]). For any such surface Sheingorn and Schmidt proved the existence of a Hall ray for the Markoff spectrum of penetration of geodesics into cusps (see [SS]). One can show that this implies the existence of Hall ray for the set of values

$$\left\{ \sup_{t>0} \frac{2}{\text{sys}^2(\mathcal{F}_t Y) \cdot X} \right\}$$
where $X$ is a Veech surface. We stress that such result does not imply that there is a Hall ray for the Lagrange spectrum $\mathcal{L}(X)$. Finally let us mention that in [P知情] Paulin et Parkkonen showed the existence of Hall rays for Lagrange spectra of hyperbolic manifolds of negative curvature, but only in dimension $n \geq 3$. Moreover S. A. Romi̇na announced the proof, obtained in his Ph-D thesis, of the existence of open intervals (but not Hall’s ray) in the Lagrange Spectrum for a generic small $C^k$-perturbation of an hyperbolic metric on a punctured surface, where $k \geq 2$ (see [R10]).

1.6. Bounded type IETs and penetration estimates. Another contribution of this paper is to present the proof of a series of equivalent characterizations of bounded-type IETs and translation surfaces whose forward Teichmüller orbit is bounded.

First of all, in addition to the equivalences stated by Vorobets’ identity (Proposition 1.1), in §4.2 we also provide two other equivalent characterizations. The first is a characterization of bounded type IETs as IETs such that the matrices of a suitable acceleration of the Rauzy-Veech induction have bounded norms (see Corollary 1.3 in Section 4.2). This generalizes the classical definition of bounded type real numbers as those $\alpha \in \mathbb{R}$ whose continued fraction entries $a_n$ are bounded. The second equivalence characterizes bounded type IETs as those whose orbit under the Rauzy-Veech map stays in a compact set. This is analogous to the fact that orbits of bounded type real numbers under the Gauss map stay away from zero.

Finally we establish estimates on excursions to the boundary of moduli space in terms of norms of positive matrices in the Rauzy-Veech induction (see Proposition 4.10 and Proposition 4.11). This generalizes the classical relation between the bound on $a_n$ and the depth of penetration to the cusp of the corresponding geodesic. We believe that these quantitative estimates (whose proofs is rather technical and postponed to the Appendix C) will be useful in future applications, since they show that Rauzy-Veech induction, which is a very well-studied combinatorial tool (see the surveys [Y, Vi, Z2]) can be actually used to deduce properties about the geometry (specifically, the depth of penetration to infinity) of a Teichmueller geodesic.

Structure of the paper. The rest of the paper is arranged as follows. In §2 we include some technical background material. In particular, we define the Rauzy-Veech induction for IETs and for translation surfaces (in §2.1 and §2.3 respectively) and recall the Veech zippered rectangles construction in §2.2.

In §3 we prove Vorobet’s identity (Proposition 1.1). We also introduce the notions of reduced periods for a translation surface (§3.1) and reduced triples for an IET (§3.2) and explain their connection via the zippered rectangles construction (§3.3).

In §4 we prove the formula which enables us to compute the Lagrange spectrum $\mathcal{L}(I)$ of any invariant locus via the Rauzy-Veech algorithm (Theorem 4.1) and then we prove Theorem 4.3 (closure of $\mathcal{L}(I)$ and density of values of periodic orbits). First, in §4.1 we prove Theorem 4.1. In §4.2 we state quantitative relation between elements $a^{-1}(X)$ in the Lagrange spectra $\mathcal{L}(I)$ and the maximal asymptotic size of positive matrices occurring in the Rauzy-Veech expansion which encodes $X$ (see Theorem 4.7). The proof of Theorem 4.7 is based on estimations in finite (uniformly bounded) time, namely Proposition 4.10 and Proposition 4.11 whose proof is postponed to the Appendix C. As consequence of the estimations in §4.2 we prove the qualitative equivalence between several notion of bounded type IET’s and translation surfaces. In §4.3 we define a sub-shift associated to a given Rauzy class, whose alphabet is the set of positive paths in the Rauzy diagram associated to the connected component $C$ which contains $I$. The language of sub-shifts provides a convenient formalism to prove Theorem 1.5. The proof of Theorem 1.5 is given in §4.4.

In §5 we prove Theorem 1.6. The main tool is the formula appearing in Theorem 5.11 which enables us to compute the Lagrange spectrum $\mathcal{L}(X)$ of a reduced square-tiled surface $X$ in terms of a skew-product over the classical continued fraction. Applying the formula, and following the classical argument given by Hall, the existence of the Hall ray for $\mathcal{L}(X)$ is established in Theorem 5.10.

In the Appendix C we give the proof of Proposition 4.10 and Proposition 4.11 which provide quantitative finite time relations between the norms of the Rauzy-Veech matrices and two geometrical quantities, namely $w(\pi, \lambda, \tau)$ appearing in the formula of Proposition 4.6 and the distortion $\Delta(T)$ of the IET associated to $(\pi, \lambda, \tau)$ (see [P知情]).

Finally, in Appendix B we describe the structure of invariant loci in the parameter space of Rauzy-Veech induction and give the proof of the combinatorial closing and shadowing lemmas for invariant loci (Propositions B.11 and B.12 which are used in the proof of Theorem 1.5).
Acknowledgements. We would like to thank U. Hamenstädt, E. Lanneau, S. Lelièvre, C. Matheus, F. Paulin and J.-C. Yoccoz for useful discussions. We would also like to thank the Hausdorff Institute for Mathematics for the hospitality during the program *Geometry and Dynamics of Teichmüller Spaces*, where this project started. Some of the research visits which made this collaboration possible were supported by the EPSRC Grant EP/I019030/1 and the ANR Project GeoDyM. P. Hubert is partially supported by the ANR Project GeoDyM. C. Ulcigrai is partially supported by EPSRC Grant EP/I019030/1. C. Ulcigrai would also like to thank the FIM Institute for Mathematical Research at ETH Zurich where part of this work was completed.

2. Background: zippered rectangles and Rauzy-Veech induction

Fix an alphabet $\mathcal{A}$ with $d \geq 2$ letters and call $\mathcal{G}(\mathcal{A})$ the set of all admissible combinatorial data $\pi$ over $\mathcal{A}$. If $T$ is an IET determined by the combinatorial-length data $(\pi, \lambda)$, we assume that it acts on the interval $I = (0, \sum \lambda_i \chi_i)$. If $\alpha$ and $\beta$ are letters in $\mathcal{A}$ with $\pi^t(\alpha) > 1$ and $\pi^b(\beta) > 1$ we set respectively

$$u^t_\alpha := \sum_{\pi(\chi) < \pi^t(\alpha)} \lambda_\chi$$

and

$$u^b_\beta := \sum_{\pi(\chi) < \pi^b(\beta)} \lambda_\chi.$$

For any such $\alpha$ and $\beta$ the points $u^t_\alpha$ and $u^b_\beta$ are the left endpoints of the subintervals $I^t_\alpha$ and $I^b_\beta$. In general $T$ is not continuous at $u^t_\alpha$ and $T^{-1}$ is not continuous at $u^b_\beta$. We will often consider the sup norm of length data, that is $||\lambda|| := \sum \lambda_\chi$.

2.1. Rauzy induction for IETs. In this paragraph we give a brief survey of the basic properties of the Rauzy induction for IETs, following [A,G,Y] [M,M,Y] [Y]. For a comprehensive introduction, we refer the reader to the lecture notes [Y].

2.1.1. Rauzy map and Rauzy elementary operations. Let $T$ be an IET defined by combinatorial-length data $(\pi, \lambda)$, acting on the interval $I = (0, \sum \lambda_i \chi_i)$. Consider the variable $\epsilon \in \{t, b\}$, where the letter $t$ stands for top and the letter $b$ for bottom. If $\epsilon = t$ we set $1 - \epsilon := b$ and if $\epsilon = b$ we set $1 - \epsilon := t$. Let $\alpha_t$ and $\alpha_b$ be the letters in $\mathcal{A}$ such that respectively $\pi^t(\alpha_t) = d$ and $\pi^b(\alpha_b) = d$. The rightmost singularity of $T$ is therefore $u^t_{\alpha_t}$ and the rightmost singularity of $T^{-1}$ is $u^b_{\alpha_b}$. When the condition

$$u^t_{\alpha_t} \neq u^b_{\alpha_b}$$

is satisfied, we say that $T$ is of type $\epsilon \in \{t, b\}$ if

$$u^\epsilon_{\alpha_\epsilon} < u^{1-\epsilon}_{\alpha_{1-\epsilon}}.$$

Consider the subinterval $\bar{I} := I \cap (0, u^{1-\epsilon}_{\alpha_{1-\epsilon}})$ of $I$ and define $\bar{T} : \bar{I} \to \bar{I}$ as the first return map of $T$ to $\bar{I}$, which is obviously an IET. The combinatorial datum $\bar{\pi} = (\bar{\pi}^t, \bar{\pi}^b)$ of $\bar{T}$ is given by:

$$\pi^t(\alpha) = \pi^t(\alpha) \text{ for any } \alpha \in \mathcal{A}$$

$$\pi^{1-\epsilon}(\alpha) = \pi^{1-\epsilon}(\alpha) \text{ if } \pi^{1-\epsilon}(\alpha) \leq \pi^{1-\epsilon}(\alpha)$$

$$\pi^{1-\epsilon}(\alpha_{1-\epsilon}) = \pi^{1-\epsilon}(\alpha_{1-\epsilon}) + 1$$

$$\pi^{1-\epsilon}(\alpha) = \pi^{1-\epsilon}(\alpha) + 1 \text{ if } \pi^{1-\epsilon}(\alpha) < \pi^{1-\epsilon}(\alpha) < d.$$

The length datum $\bar{\lambda}$ of $\bar{T}$ is given by:

$$\bar{\lambda}_\alpha = \lambda_\alpha \text{ if } \alpha \neq \alpha_\epsilon$$

$$\bar{\lambda}_{\alpha_\epsilon} = \lambda_{\alpha_\epsilon} - \lambda_{\alpha_{1-\epsilon}}.$$

When $T = (\pi, \lambda)$ satisfies condition (2.1), Equations (2.2) and (2.3) define a map $T \mapsto Q(T) := \bar{T}$, called Rauzy map. Introduce the operations $R^t$ and $R^b$ from $\mathcal{G}(\mathcal{A})$ to itself setting $R^t(\pi) := \bar{\pi}$, where $\epsilon$ and $\pi$ are respectively the type and the combinatorial datum of $T$ and $\bar{\pi}$ is the combinatorial datum of $\bar{T}$. It is easy to check that if $\pi$ is admissible then both $R^t(\pi)$ and $R^b(\pi)$ are admissible. The maps $R^t$ and $R^b$ are called the Rauzy elementary operations.

A Rauzy class is a minimal non-empty subset $\mathcal{R}$ of $\mathcal{G}(\mathcal{A})$ which is invariant under $R^t$ and $R^b$. A Rauzy diagram is a connected oriented graph $\mathcal{D}$ whose vertexes are the elements of $\mathcal{R}$ and whose oriented arcs, or arrows, correspond to Rauzy elementary operations $\pi \mapsto R^t(\pi)$. An arrow corresponding to $R^t$ is called a top arrow and we say that $\alpha_t$ is its winner and $\alpha_\epsilon$ is its loser. Conversely an arrow corresponding to $R^b$ is called a bottom arrow and we say that $\alpha_t$ is its loser and $\alpha_b$ is its winner.
A concatenation of $r$ compatible arrows $\gamma_1, \ldots, \gamma_r$ in a Rauzy diagram is called a _Rauzy path_ and is denoted $\gamma = \gamma_1 \ast \cdots \ast \gamma_r$. The set of all Rauzy paths connecting elements of $\mathcal{R}$ is denoted $\Pi(\mathcal{R})$. If a path $\gamma$ is concatenation of $r$ simple arrows, we say that $\gamma$ has length $r$. Length one paths are arrows, elements of $\mathcal{R}$ are identified with trivial (that is length-zero) paths.

### 2.1.2. Linear action and iterations of the Rauzy map

Let $\{e_\xi\}_{\xi \in \mathcal{A}}$ be the canonical basis of $\mathbb{R}^d$. For any Rauzy class $\mathcal{R}$ over $\mathcal{A}$ and any path $\gamma \in \Pi(\mathcal{R})$ define a linear map $B_\gamma \in \text{SL}(d, \mathbb{Z})$ as follows. If $\gamma$ is trivial then $B_\gamma := \text{id}$. If $\gamma$ is an arrow with winner $\alpha$ and loser $\beta$ set

$$B_\gamma e_\alpha = e_\alpha + e_\beta$$

Then extend the definition to paths in $\Pi(\mathcal{R})$ so that $B_{\gamma_1 \gamma_2} = B_{\gamma_2} B_{\gamma_1}$. For a combinatorial datum $\pi \in \mathcal{R}$ and a Rauzy path $\gamma \in \Pi(\mathcal{R})$ starting at $\pi$ define the simplicial sub-cone $\Delta_\gamma \subset \mathbb{R}_+^d$ by

$$\Delta_\gamma = t B_\gamma(\mathbb{R}_+^d),$$

where $t B_\gamma$ is the transpose matrix of $B_\gamma$. Finally, denoting by $\bar{I}$ the vector of $\mathbb{N}^d$ whose entries are all equal to 1, for any $\gamma \in \Pi(\mathcal{R})$ define a vector $q^\gamma \in \mathbb{N}^d$ by

$$q^\gamma := t B_\gamma \bar{I}.$$ 

If $T = (\pi, \lambda)$ is an IET in $\mathcal{R} \times \mathbb{R}_+^d$ admitting $r$ elementary steps $\gamma_1, \ldots, \gamma_r$ of the Rauzy map $Q$ and $\gamma = \gamma_1 \ast \cdots \ast \gamma_r$ is the corresponding Rauzy path, we also write $q^{(r)}$ instead of $q^\gamma$. For such $T$, denote $T^{(r)} := Q^r(T)$ the $r$-th step of the induction, whose combinatorial-length data are $(\pi^{(r)}, \lambda^{(r)})$. The interval where $T^{(r)}$ acts is $I^{(r)} = (0, \sum \chi^{(r)}_\chi)$, which is a sub-interval of $I$ with the same left endpoint. For $\alpha \in \mathcal{A}$ call $u^{(r)}_{\alpha,t}$ and $u^{(r)}_{\alpha,t}$ the singularity respectively of $T^{(r)}$ and of $(T^{(r)})^{-1}$ corresponding to the letter $\alpha$. The length datum of $T^{(r)}$ is given by the following Lemma (see [Y] for a proof).

**Lemma 2.1.** Fix combinatorial data $\pi$ and $\pi'$ in $\mathcal{R}$ and let $\gamma \in \Pi(\mathcal{R})$ be a path starting at $\pi$ and ending in $\pi'$ with length $r$. Then the $r$-th iterate of $Q$ is a linear isomorphism defined on $\pi \times \Delta_\gamma$ with values in $\pi' \times \mathbb{R}_+^d$ and the length datum $\lambda^{(r)}$ of $T^{(r)}$ is given by the formula

$$\lambda^{(r)} = t B_\gamma^{-1} \lambda.$$  

Condition (2.1) corresponds to an union of hyperplanes in $\mathcal{R} \times \mathbb{R}_+^d$. Similarly, for any $r$ the iterate $Q^r$ is defined outside a finite union of linear spaces, therefore IETs admitting infinitely many iterations of the map $Q$ form a set with full Lebesgue measure. The complement of such set corresponds to those $T$ such that $T^{(r)} = Q^r(T)$ eventually does not satisfy condition (2.1), that is to those $T$ such that the algorithm stops. The following combinatorial characterization holds (see [Y] for a proof).

**Lemma 2.2.** $T$ admits infinitely many steps of the Rauzy induction if any only if it does not have connections.

### 2.1.3. Return times

Let $\pi \in \mathcal{R}$ and $\gamma \in \Pi(\mathcal{R})$ be a path of length $r$. Let $T$ be an IET in $\{\pi\} \times \Delta_\gamma$ and $T^{(r)}$ be the $r$-th step of the Rauzy induction applied to $T$, whose combinatorial-length data are $(\pi^{(r)}, \lambda^{(r)})$. The sub-interval of $I^{(r)}$ where $T^{(r)}$ (respectively $(T^{(r)})^{-1}$) acts as a translation are $I^{(r),t}_\pi := (u^{(r),t}_\pi, u^{(r),t}_\pi + \lambda^{(r)}_\pi)$ (respectively $I^{(r),b}_\pi := (u^{(r),b}_\pi, u^{(r),b}_\pi + \lambda^{(r)}_\pi)$) for $\chi \in \mathcal{A}$. For such $\chi$ and $r$ let $R_r(\chi)$ be the minimal positive integer $k$ such that $T^k(I^{(r),t}_\pi)$ belongs to $I^{(r)}$. Observe that $R_r(\chi)$ is also equal to the minimal positive integer $k$ such that $T^{-k}(I^{(r),b}_\pi)$ belongs to $I^{(r)}$. For a matrix $A$ let $[A]_{\alpha\beta}$ be the entry of $A$ in row $\alpha$ and column $\beta$. A proof of the following Lemma can be found in [Y].

**Lemma 2.3.** Let $T$ be an IET admitting $r$ elementary steps $T \mapsto T^{(r)}$ of the Rauzy map, represented by the path $\gamma \in \Pi(\mathcal{R})$. For any $\alpha \in \mathcal{A}$ we have

$$R_r(\alpha) = q^{(r)}_{\alpha} = \sum_{\beta \in \mathcal{A}} [B_\gamma]_{\alpha\beta},$$

that is $q^{(r)}_{\alpha}$ is the return time to $I^{(r)}$ of the sub-interval $I^{(r),t}_\alpha$. More precisely, for any $\alpha, \beta \in \mathcal{A}$ the entry $[B_\gamma]_{\alpha\beta}$ of $B_\gamma$ is equal to the cardinality of the following two sets:

1. The integers $k$ with $0 \leq k < R_r(\alpha)$ such that $T^k(I^{(r),t}_\alpha) \subset I^{(t)}_\beta$.
2. The integers $k$ with $0 \leq k < R_r(\alpha)$ such that $T^{-k}(I^{(r),b}_\alpha) \subset I^{b}_\beta$. 


2.2. Zippered rectangles construction. Here we describe Veech’s zippered rectangles construction, following [M.M.Y]. Fix \( \pi \in \mathcal{G}(\mathcal{A}) \) and introduce the labelling \( \alpha(1), \ldots, \alpha(d) \) of the letters in \( \mathcal{A} \), according to their order in \( \pi \). Similarly, introduce the labelling \( \beta(1), \ldots, \beta(d) \) of the letters in \( \mathcal{A} \) according to their order in \( \pi \).

A suspension datum for \( \pi \) is a vector \( \tau \) in \( \mathbb{R}^d \) such that for any \( 1 \leq k \leq d - 1 \) we have

\[
\sum_{j \leq k} \tau_{\alpha(j)} > 0 \quad \text{and} \quad \sum_{j \leq k} \tau_{\beta(j)} < 0.
\]

Let \( \Theta_\pi \) be the open sub-cone of \( \mathbb{R}^d \) of suspension data \( \tau \) for \( \pi \). Observe that the vector \( \tau \) with coordinates \( \tau_\xi := \pi^b(\xi) - \pi^l(\xi) \) satisfies the inequalities above, hence \( \Theta_\pi \) is not empty.

Consider \( \lambda \in \mathbb{R}_+^d \) and \( \tau \in \Theta_\pi \) and define the complex vector \( \xi = \lambda + i\tau \in \mathbb{C}^d \), then for any \( 1 \leq k \leq d - 1 \) we introduce the complex numbers

\[
\xi_{\alpha(k)}^\lambda := \sum_{j \leq k} \zeta_{\alpha(j)} \quad \text{and} \quad \xi_{\beta(k)}^\lambda := \sum_{j \leq k} \zeta_{\beta(j)}.
\]

2.2.1. The height function \( h \). Let \( T \) be the IET defined by the data \( (\pi, \lambda) \), acting on the interval \( I = (0, \sum_\alpha \lambda_\alpha) \). Observe that the singularities \( u^b_\beta \) and \( u^l_\alpha \) of \( T \) and \( T^{-1} \) satisfy \( u^l_\alpha = \Re(\xi_{\alpha(k)}^\lambda) \) and \( u^b_\beta = \Re(\xi_{\beta(k)}^\lambda) \).

The condition \( \tau \in \Theta_\pi \) is equivalent to \( \Im(\xi_{\alpha(k)}^\lambda) > 0 \) and \( \Im(\xi_{\beta(k)}^\lambda) < 0 \) for any \( 1 \leq k \leq d - 1 \). Moreover admissibility of \( \pi \) implies \( \alpha(1) \neq \beta(1) \) and \( \alpha(d) \neq \beta(d) \), hence the vector \( h = h(\pi, \tau) \) defined by setting for \( 1 \leq k \leq d \)

\[
h_{\alpha(k)} := \sum_{j \leq k} \tau_{\alpha(j)} - \sum_{j \leq k} \tau_{\beta(j)}
\]

is a positive vector, that is \( h_{\alpha} > 0 \) for any \( \alpha \). It is useful to consider \( h \) as a piecewise constant function \( h : I \to \mathbb{R}_+ \), with constant value \( h_\alpha \) on each sub-interval \( I_\alpha \) of \( I \).

Observe that for any \( k \) with \( 1 < k \leq d \) the suspension conditions imply \( h_{\alpha}(k) \geq \tau_{\alpha(1)} + \cdots + \tau_{\alpha(k-1)} > 0 \) and \( -h_{\beta}(k) \leq \tau_{\beta(1)} + \cdots + \tau_{\beta(k-1)} < 0 \). Combining this remark with the admissibility of \( \pi \) it is not difficult to get the following estimate for the height function \( h \).

Lemma 2.4. We have

\[
\|\tau\|_\infty \leq \|h\|_\infty.
\]

2.2.2. Construction of a translation surface. We construct a translation surface \( X = X(\pi, \lambda, \tau) \) whose vertical flow is the suspension flow over \( T \) with roof function \( h \), that is the unit vertical flow \( \eta = 1 \) with respect to the equivalence relation \( (x, y) \sim (T(x), y - h(x)) \). Embed \( I \) in the complex plane, identifying it with \( (0, \sum_\alpha \lambda_\alpha) \times \{0\} \) and accordingly denote elements in \( \mathbb{C} \cap I \) just with their real coordinate. Define 2d open rectangles in \( \mathbb{C} \) setting

\[
R^v_\alpha := (u^l_\alpha, u^l_\alpha + \lambda_\alpha) \times (0, h_\alpha) \quad \text{and} \quad R^b_\beta := (u^b_\beta, u^b_\beta + \lambda_\beta) \times (-h_\beta, 0).
\]

Consider the translation datum \( \theta \), that is the vector in \( \mathbb{C}^d \) defined by \( \theta_\alpha := \xi_{\alpha}^b - \xi_{\alpha}^l \) for any \( \alpha \). In order to get a surface we "zip" together these rectangles by glueing their boundaries according to the identifications described below.

1. For each \( \alpha \) the rectangle \( R^v_\alpha \) is equivalent to the rectangle \( R^b_\beta \) via the translation by \( \theta_\alpha \).
2. For each \( k > 1 \) we paste together \( R^v_{\alpha(k-1)} \) and \( R^v_{\alpha(k)} \) along the common vertical open segment connecting \( u_{v,\alpha(k)}^l \) to \( \xi_{\alpha(k)}^l \).
3. For each \( k \) with \( k > 1 \) we paste together \( R^b_{\beta(k-1)} \) and \( R^b_{\beta(k)} \) along the vertical open segment connecting \( c_{\beta(k)}^b \) to \( u_{b,\beta(k)}^l \).
4. For any \( \alpha \) we paste \( R^v_{\alpha} \) to \( I \) along its lower horizontal boundary segment \( (u^l_\alpha, u^l_\alpha + \lambda_\alpha) \times \{0\} \).
5. For any \( \beta \) we paste \( R^b_{\beta} \) to \( I \) along its upper horizontal boundary segment \( (u^b_\beta, u^b_\beta + \lambda_\beta) \times \{0\} \).
6. If \( \sum_\alpha \tau_\alpha \geq 0 \) then we can find the vertical segment connecting \( u^l_{\beta(d)+1} + ih_{\beta(d)} \) and \( \xi_{\beta(d)+1}^l \) and the vertical segment connecting \( \sum_\alpha \lambda_\alpha \) to \( \sum_\alpha \zeta_\alpha \). This is possible because \( h_{\beta(d)} = \Im(\xi_{\beta(d)}^l) - \Im(\xi_{\beta(d)}^b) = \Im(\xi_{\beta(d)+1}^l) - \Im(\xi_{\beta(d)+1}^b) \). If \( \sum_\alpha \tau_\alpha < 0 \) then we take the vertical segment connecting \( \xi_{\alpha(d)+1}^l \) to \( u_{\alpha(d)+1}^b - ih_{\alpha(d)} \) and the vertical segment connecting \( \sum_\alpha \zeta_\alpha \) to \( \sum_\alpha \lambda_\alpha \). This is possible because \( -h_{\alpha(d)} = -\sum_\alpha \tau_\alpha + \Im(\xi_{\alpha(d)+1}) > \Im(\xi_{\alpha(d)+1}^b) \).
(7) Finally we add the points $c^k_{\alpha_1} \pm 1$ and $c^k_{\beta_1} \pm 1$ for $1 \leq k \leq d - 1$.

**Proposition 2.5.** For any connected component $C$ of any stratum of translation surfaces there exists a Rauzy class $R$ such that the following holds.

1. For any $\pi \in R$, any $\lambda \in \mathbb{R}^d$, and any $\tau \in \Theta_\pi$ the surface $X(\pi, \lambda, \tau)$ given by the zippered rectangles construction belongs to $C$.
2. If $X$ is a translation surface in $C$ such that do not exist data $\pi \in R$, $\lambda \in \mathbb{R}^d$, and $\tau \in \Theta_\pi$ with $X = X(\pi, \lambda, \tau)$, then $X$ has a vertical and an horizontal saddle connection. Therefore the zippered rectangles construction fills a subset of $C$ of co-dimension 2.
3. In particular, if $s(X) > 0$ (recall the definition 1.2), then there exist data $\pi, \lambda$ and $\tau$ such that $X = X(\pi, \lambda, \tau)$.

### 2.3. Rauzy-Veech induction.

Consider combinatorial-length suspension data $(\pi, \lambda, \tau)$. When the data $(\pi, \lambda)$ satisfy condition (2.2), we define a new suspension datum $\tilde{\tau}$ extending Equation (2.3) to $\tau$ by

\[
\begin{align*}
\tilde{\tau}_\alpha &= \tau_\alpha \quad \text{if } \alpha \neq \alpha\varepsilon, \\
\tilde{\tau}_\alpha &= \tau_\alpha - \tau_{\alpha_1 - \varepsilon},
\end{align*}
\]

where the letters $\alpha_\varepsilon$ and $\alpha_{1 - \varepsilon}$ are defined at the beginning of 2.1.1. Equation (2.4), together with Equations (2.2) and (2.3), define a map

\[
(\pi, \lambda, \tau) \mapsto \tilde{Q}(\pi, \lambda, \tau) := (\tilde{\tau}, \tilde{\lambda}, \tilde{\tau}),
\]

called Rauzy-Veech map. Let $\gamma$ be a Rauzy path starting at $\pi$ and ending in $\pi'$ and let $B_\gamma$ be the associated matrix. Define the cone $\Theta_\gamma := B_\gamma^{-1}\Theta_\pi$.

**Lemma 2.6.** If $\gamma$ starts at $\pi$ and ends in $\pi'$, then $\Theta_\gamma$ is a subcone of $\Theta_{\pi'}$.

**Proof:** It is enough to prove the Lemma for a simple arrow. In this case, if $\tilde{\tau}$ is obtained from a suspension datum $\tau$ in $\Theta_\pi$ via Equation (2.4), then is easy to check that if $\tilde{\tau}$ is a suspension datum for $\pi'$.

The iteration of the map $\tilde{Q}$ satisfies the same formalism as the Rauzy map $Q$. More precisely, for any Rauzy path $\gamma$ as above, we have a linear homeomorphism $\tilde{Q}_\gamma$ from $\{\pi\} \times \Delta_\gamma \times \Theta_\pi$ with values in $\{\pi'\} \times \mathbb{R}^d \times \Theta_\gamma$ defined by

\[
\tilde{Q}_\gamma(\pi, \lambda, \tau) := (\pi', \lambda', \tau').
\]

**Lemma 2.7.** Let $\gamma$ be a Rauzy path starting at $\pi$ and ending in $\pi'$. Consider $\lambda$ in $\Delta_\gamma$ and $\tau$ in $\Theta_\pi$, then set $\lambda' := B_\gamma^{-1}\lambda$ and $\tau' := B_\gamma^{-1}\tau$. Then the zippered rectangles construction applies to the data $(\pi', \lambda', \tau')$, moreover the translation surfaces $X(\pi, \lambda, \tau)$ and $X(\pi', \lambda', \tau')$ define the same element in the moduli space.

### 3. On Vorobet’s identity

#### 3.1. Reduced periods for translation surfaces.

Fix a translation surface $X$ and consider a saddle connection $\gamma$ for $X$. The standard orientation of $\gamma$ is the orientation induced by any smooth parametrization $t \mapsto \gamma(t)$ of $\gamma$, where $t \in [0, 1]$, such that for any $t$ we have

\[
\text{angle} \left( \frac{d}{dt} \gamma(t), \partial_y \right) < \frac{\pi}{2}.
\]

In this case we say that $\gamma$ starts at $p_{\text{out}}$ and ends in $p_{\text{in}}$ if we have respectively $\gamma(0) = p_{\text{out}}$ and $\gamma(1) = p_{\text{in}}$. Observe that if $\gamma$ has the standard orientation, then its period $v = \int_\gamma w$ belongs to the upper half plane $\mathbb{H}$. For such a period, let $R_v$ be the rectangle in $\mathbb{H}$ whose diagonal is $v$.

**Definition 3.1.** We say that a period $v \in \text{Hol}(X)$ is a reduced period if there exists an immersion $\rho : R_v \rightarrow X$ isometric with respect to the flat metric of $X$ such that $t \mapsto \rho(tv)$ for $0 \leq y \leq 1$ is a standard parametrization of $\gamma$ and $\rho(R_v)$ does not contain other conical singularities in its interior (the endpoints of $\gamma$ of course belong to the boundary of $\rho(R_v)$).

Note that in general $\rho$ cannot be an embedding, since it may not be injective.

**Lemma 3.2.** For any $X$, the value $a(X)$ can be computed taking the lim inf just on reduced periods.
Proof: Just observe that if a period \( v \in \text{Hol}(X) \) is not reduced then there exists an other reduced period \( v' \) in \( \text{Hol}(X) \) with \( |\Re(v')| < |\Re(v)| \) and \( |\Im(v')| < |\Im(v)| \), and hence \( \text{Area}(v') < \text{Area}(v) \). Then the Lemma follows because \( \text{Hol}(X) \) is a discrete subset of the plane. \( \square \)

3.2. Reduced triples for IETs. We now define the notion of reduced triples for an IET, which is the combinatorial counterpart at the level of IETs of the notion of reduced periods given in \( \text{S}\text{3.1} \) above. Let \( T \) be an IET with combinatorial and length data \((\pi, \lambda)\), acting on the interval \( I = (0, \sum \lambda_i) \). Recall that if \( \alpha \) and \( \beta \) are two letters in \( \mathcal{A} \) with \( \pi^t(\alpha) > 1 \) and \( \pi^t(\beta) > 1 \) the points \( u_\alpha^1 \) and \( u_\beta^1 \) denote the left endpoint of the subintervals \( I^t_\alpha \) and \( I^t_\beta \) respectively and in general \( T \) is not continuous in \( u_\alpha^1 \) and \( T^{-1} \) is not continuous at \( u_\beta^1 \). A connection for \( T \) is a triple \((\beta, \alpha, n)\) with \( \pi^b(\beta) > 1 \), \( \pi^t(\alpha) > 1 \) and \( n \in \mathbb{N} \) such that \( T^n(u_\beta^1) = u_\alpha^1 \). In particular one can check that if \( T \) has no connections then \( \pi \) is admissible. Consider a triple \((\beta, \alpha, n)\) with \( n \in \mathbb{N} \), \( \pi^b(\beta) > 1 \) and \( \pi^t(\alpha) > 1 \) and suppose that it is not a connection for \( T \). Then we denote \( I(\beta, \alpha, n) \) the open subinterval of \( I \) whose endpoints are \( T^n(u_\beta^1) \) and \( u_\alpha^1 \).

Definition 3.3. We say that \((\beta, \alpha, n)\) is a reduced triple for \( T \) if it is not a connection for \( T \) and moreover for any \( k \in \{0, \ldots, n\} \) the pre-image \( T^{-k}(I(\beta, \alpha, n)) \) does not contain any singularity of \( T \) or of \( T^{-1} \).

3.2.1. An alternative way to compute Lagrange Spectra. We introduce the following asymptotic quantity, which gives an alternative way to compute Lagrange Spectra.

\[
l(T) := \frac{1}{\|\Lambda\|} \liminf_{n \to \infty} n \cdot \min_{\beta,\alpha} |T^n u_\beta^b - u_\alpha^t|.
\]

Proposition 3.4. Let \( T \) be an IET. We have \( l(T) = \mathcal{E}(T) \).

The following Lemmas are useful in the proof of Proposition 3.4.

Lemma 3.5. Let \((\beta, \alpha, n)\) be a triple which is not reduced for \( T \). Then there exists a triple \((\beta', \alpha', m)\) with \( m < n \) which is reduced for \( T \) and such that

\[
|T^m(u_{\beta'}^b) - u_{\alpha'}^t| < |T^n(u_\beta^b) - u_\alpha^t|.
\]

Proof: Let \((\beta, \alpha, n)\) be as in the statement and assume without loss of generality that \( T^n(u_\beta^b) < u_\alpha^t \) and that \( I(\beta, \alpha, n) \) does not contain other singularities. Since \((\beta, \alpha, n)\) is not reduced for \( T \), then there exists \( k \) with \( 1 \leq k \leq n \) such that \( T^{-k} I(\beta, \alpha, n) \) contains either a singularity for \( T \) or a singularity for \( T^{-1} \). Let \( k \) be minimal satisfying the property above. Minimality implies that \( T^{-k} I(\beta, \alpha, n) \) contains a singularity of \( T \) (observe that singularities of \( T^{-1} \) are the image of singularities of \( T \)). Calling \( u_1^t \) such singularity, we have \( T^{n-k}(u_1^t) < u_1^t < T^{-k}(u_\alpha^t) \), hence, since \( T^k \) is an isometry on \( I(\beta, \alpha, n) \), the triple \((\beta', \chi, n-k)\) satisfies \( |T^{n-k}(u_{\beta'}^b) - u_1^t| < |T^{n-k}(u_{\beta'}^b) - T^{-k}(u_\alpha^t)| = |T^n(u_{\beta'}^b) - u_\alpha^t| \). Moreover we can assume that the interval \( T^{n-k}(u_{\beta'}^b), u_1^t \) does not contains other singularities. If the triple \((\beta', \chi, n-k)\) is reduced then the Lemma is proved, otherwise we replace \((\beta, \alpha, n)\) by \((\beta', \chi, n-k)\) and we repeat the procedure above. Such procedure admits at most \( m \) steps, hence repeating it finitely many times we end up with a triple \((\beta', \alpha', m)\) as in the statement. \( \square \)

The following Lemma is the analogous for reduced triples of Lemma 3.2 for reduced periods.

Lemma 3.6. Let \( T \) be any IET. Then \( l(T) \) can be computed taking the \( \lim \inf \) on reduced triples only.

Proof: Observe that for triples \((\beta, \alpha, n)\) with \( n \) bounded the values \( |T^n(u_{\beta'}^b) - u_1^t| \) are bounded from below by a positive constant. Thus the Lemma is an immediate consequence of Lemma 3.5. \( \square \)

Recall that a singularity of \( T^n \) has the form \( T^{-i}(u_\alpha^t) \), where \( \pi^t(\alpha) > 1 \) and \( 0 \leq i < n \). Recall also that \( u_\alpha^t = T(u_\alpha^t) \). Therefore the following Lemma is obvious.

Lemma 3.7. For any \( n \) there exists \( m = m(n) \) with \( m < n \) and a triple \((\beta, \alpha, m)\) reduced for \( T \) such that

\[
\mathcal{E}_n(T) = |T^m(u_{\beta'}^b) - u_\alpha^t|.
\]
3.2.2. Proof of Proposition 3.4. Fix $N > 0$ and $\epsilon > 0$. Consider $n$ in $\mathbb{N}$ such that $n\mathcal{E}_n(T) < \mathcal{E}(T) + \epsilon$ and $m(n) > N$, where $m = m(n)$ is given by Lemma 5.4. Let $(\beta, \alpha, m)$ be the triple reduced for $T$ associated to $n$ by the same Lemma. We have
\[ m \cdot |T^m(u_3^\beta) - u_1^\alpha| = \frac{m}{n} \cdot \mathcal{E}_n(T) \leq \mathcal{E}(T) + \epsilon, \]
hence it follows that $l(T) \leq \mathcal{E}(T)$, since $m$ is arbitrarily large. To prove the reverse inequality fix again $N > 0$ and $\epsilon > 0$. Consider a triple $(\beta, \alpha, n)$ reduced for $T$ such that $n > N$ and $n \cdot |T^n(u_3^\beta) - u_1^\alpha| < l(T) + \epsilon$. Observe that such triple exists according to Lemma 3.6. Since $(\beta, \alpha, n)$ is reduced then $T^{-(n+1)}$ is continuous on $I(\beta, \alpha, n)$. Thus, $u_3^\beta = T^{-1}(u_3^\beta)$ we have
\[ n \cdot \mathcal{E}_{n+1}(T) \leq n \cdot |u_3^\beta - T^{-n-1}(u_1^\alpha)| = n \cdot |T^n(u_3^\beta) - u_1^\alpha| \leq l(T) + \epsilon. \]
It follows that $\mathcal{E}(T) \leq l(T)$ and the Proposition is proved.

3.3. Correspondence between triples and periods. In this paragraph we use Veech’s zippered rectangles construction, following the notation introduced in [22] to explain the connection between periods (see 3.1) and triples (see 3.2). Consider combinatorial, length and suspension data $(\pi, \lambda, \tau)$, let $X$ be the underlying translation surface and $\phi^X$ be the vertical flow of $X$. Let $T$ be the IET corresponding to $(\pi, \lambda)$ and acting on the interval $I = (0, \sum \lambda_i)$, which is naturally embedded in $X$ along the horizontal direction. Recall that in this setting $\phi^X$ corresponds to the suspension flow over $T$ under the roof function $h = h(\pi, \tau)$ defined in [22].

The singularities of $T$ lie on the vertical negative time orbits of the conical singularities of $X$ and can be obtained considering the first intersections of these orbits with $I$. In particular, if $\pi^\prime(\alpha) > 1$, the vertical segment connecting $u_1^\alpha$ to the conical singularity where its orbit ends has length $\mathfrak{S}(\xi_1^b)$, where the complex number $\xi_1^b$ is defined at the beginning of §2.2. Similarly, the singularities of $T^{-1}$ can be obtained as first intersection with $I$ of the vertical orbits in positive time of the conical singularities of $X$, thus, if $\pi^\prime(\beta) > 1$, the vertical segment connecting $u_3^\beta$ to the conical singularity where its orbit starts has length $-\mathfrak{S}(\xi_1^b)$.

Lemma 3.8. Consider data $(\pi, \lambda, \tau)$ as above and let $X$ be the corresponding translation surface and IET respectively. The following holds.

1. For any triple $(\beta, \alpha, n)$ reduced for $T$ and such that $|T^n(u_3^\beta) - u_1^\alpha| < \min_{\chi \in A} \lambda_\chi$ there exists a saddle connection $\gamma$ whose period $\nu$ is reduced and satisfies $|\Re(\nu)| = |T^n(u_3^\beta) - u_1^\alpha|$ with $n = \sharp(I \cap \gamma)$.

2. For any saddle connection $\gamma$ whose period $\nu$ is reduced and satisfies $|\Re(\nu)| > \max_{\alpha, \beta} \mathfrak{S}(\xi_1^b - \xi_3^b)$ there exists a triple $(\beta, \alpha, n)$ such that $|\Re(\nu)| = |T^n(u_3^\beta) - u_1^\alpha|$ and $n = \sharp(I \cap \gamma)$.

Proof: We first prove part (1). Consider a triple $(\beta, \alpha, n)$ as in the statement. Let $V_{out}$ be the outgoing vertical half-line whose first intersection with $I$ is $u_1^\alpha$ and call $p_{out}$ its starting point. Similarly, let $V_{in}$ be the ingoing vertical half-line whose first intersection with $I$ is $u_1^\alpha$ and call $p_{in}$ its ending point. Recall that the points $\xi_3^b$ and $\xi_3^b$ defined in [22] are identified to $p_{out}$ and $p_{in}$ respectively. Our aim is to define a saddle connection $\gamma$ starting in $p_{out}$ and ending in $p_{in}$ and satisfying the properties in the statement. Assume without loss of generality that $T^n(u_3^\beta) < u_1^\alpha$, the other case being the same, then set $\delta := u_1^\alpha - T^n(u_3^\beta)$ and observe that by assumption we have $\delta < \lambda_\chi$ for any $\chi$ in $A$. In particular, $T^n(u_3^\beta)$ belongs to the subinterval $I^*_A$, where $A$ is the letter with $\pi^\prime(A) = \pi^\prime(\alpha) - 1$. Equation (C.3) implies that $\xi_3^b$ belongs to the boundary of the rectangle $R^\beta_3$. Consider the horizontal segment in $R^\beta_3$ with length $\delta$ and left endpoint $\xi_3^b$ and let $H$ be the image of the same segment under the isometric embedding of $R^\beta_3$ in $X$. In particular $H$ has length $\delta$ and its left endpoint is $p_{out}$. Since $(\beta, \alpha, n)$ is reduced for $T$, then $T^{-k}$ is defined on the subinterval $I(\beta, \alpha, n)$ of $I$ for $0 \leq k \leq n$. Recalling that $T$ is the first return of $\phi^X$ to $I$, we have instants $0 < t(0) < \cdots < t(n)$ such that $\phi_{t(k)}(H) = T^{k-n}(I(\beta, \alpha, n))$ for $0 \leq k \leq n$. Let $s$ in $(0, \delta)$ be the length coordinate on $H$ and call $p_s$ the point of $H$ at distance $s$ from $p_{out}$. Since $(\beta, \alpha, n)$ is reduced, then $\phi_s(p_s)$ is not a conical singularity for $s$ in $(0, \delta)$ and $t$ in $(0, t(n) + \tau_A)$. For $s$ and $t$ in the same intervals, set
\[ \rho(s, t) := \phi_s(p_s), \]
which is an isometric immersion from $(0, \delta) \times (0, t(n) + \tau_A)$ to $X$ which avoids singularities. Observe that $\rho(0, 0) = p_{out}$. If $\mathfrak{S}(\xi_3^b) \leq h_A$ then we set $a := t(n) + \mathfrak{S}(\xi_3^b)$ and $\gamma(t) := \rho(t, ta)$, which defines a saddle connection, since $\rho(\delta, a) = p_{in}$. Otherwise, if $\mathfrak{S}(\xi_3^b) > h_A$ then we must have $\pi^\prime(\alpha) = d$ and...
0 < \sum \tau_x < h_B \text{ where } B \text{ is the letter such that } \pi'(B) = d. \text{ Moreover the upper side of the rectangle } R^i_A \text{ is identified to the lower side of } R^i_B \text{, where the left endpoints coincide, therefore } \rho(s, t) \text{ extends to } (0, \delta) \times (0, t(n) + h_A + h_B). \text{ We have } \Im(\xi'_n) = \sum_\xi \tau_\xi + h_A, \text{ thus setting again } a := t(n) + \Im(\xi'_n) \text{ and } \gamma(t) := \rho(\delta, ta) \text{ we get a saddle connection, because } \rho(\delta, a) = p_{in}. \text{ In both cases above, the period } v = \int_\gamma w \text{ obviously satisfies the required conditions. }

Now we prove part (2). Let \gamma be a saddle connection whose period \( v \) is as in the statement. Assume that \( \Re(v) > 0 \), the other case being the same. Since \( v \) is reduced, we have an isometric immersion \( \rho : (0, \Re(v)) \times (0, \Im(v)) \to X \) which avoids the conical singularities and such that \( \rho(v) = \gamma(t) \). Since \( \Im(v) > \max_{a, b} \Im(\xi'_a - \xi'_b) \) then \( \gamma \) crosses \( I \), thus there are instants \( 0 < t(1) < \cdots < t(n) < \Im(v) \) such that \( \rho((0, \Re(v)) \times \{t(k)\}) \subseteq I \) for \( 0 \leq k \leq n \). The left and right vertical sides of the rectangle \( (0, \Re(v)) \times (0, \Im(v)) \) correspond to segment of vertical half-trajectories in \( X \) respectively starting at \( p_{out} \) and ending in \( p_{in} \).

Therefore there exist letters \( \beta \) and \( \alpha \) such that for \( 0 \leq k \leq n \) we have

\[
\rho((0, \Re(v)) \times \{t(k)\}) = (T^k(u^b_\alpha), T^{-k-n}(u^t_\beta)).
\]

The required identity follows considering \( k = n \) in the equation above. \( \square \)

**Remark 3.9.** In general, the triple \((\beta, \alpha, n)\) in part (2) of Lemma 3.8 fails to be reduced just because \((T^n(u^b_\alpha), u^t_\beta)\) may contain a singularity for \( T \) whose positive vertical orbit reaches a conical singularity in time greater than \( \Im(v) - t(n) \) and similarly \((u^b_{T^n(\alpha)}, T^{-n}(u^t_\beta))\) may contain a singularity for \( T^{-1} \) whose negative vertical orbit reaches a conical singularity in time greater than \( t(0) \).

### 3.4. Proof of Vorobets identity.

Consider data \((\pi, \lambda, \tau)\), let \( X \) be the underlying translation surface and \( T \) be the IET corresponding to \((\pi, \lambda)\). Here we prove Proposition 3.10 that is

\[
\mathcal{E}(T) = a(X) = \frac{s^2(X)}{2}.
\]

We first show separately the second equality.

**Lemma 3.10.** For any translation surface \( X \) we have

\[
a(X) = \frac{s(X)^2}{2}.
\]

**Proof:** Let \( v_n \) be a sequence of periods in \( \text{Hol}(X) \) such that \( \text{Area}(v_n) \to a(X) \cdot \text{Area}(X) \). Since any \( v_n \) belongs to a fixed vertical cone, then \( \text{angle}(v_n, \sqrt{-1}) \to 0 \), that is the angle with the vertical direction goes to zero. If \( v \) is a period of \( X \), corresponding to the saddle connection \( \gamma \), then for any \( t \) we denote \(|v|_t\) the length of \( \gamma \) with respect to the flat metric of \( F_t \cdot X \). Consider the sequence of positive instants \( \tau_n \) such that \( \text{Area}(v_n) = \frac{|v_n|_{\tau_n}^2}{2} \). Since \( \text{angle}(v_n, \sqrt{-1}) \to 0 \) then \( \tau_n \to \infty \), therefore we have

\[
\frac{s^2(X)}{2} = \lim_{t \to \infty} \frac{\text{sys}(F_t \cdot X)}{2 \cdot \text{Area}(X)} \leq \lim_{n \to \infty} \frac{\text{sys}(F_{\tau_n} \cdot X)}{2 \cdot \text{Area}(X)} \leq \frac{\text{Area}(v_n)}{2 \cdot \text{Area}(X)} = \lim_{n \to \infty} \frac{|v_n|_{\tau_n}^2}{2} = a(X).
\]

On the other hand, consider positive instants \( t_n \) with \( t_n \to +\infty \) and such that \( \text{sys}(F_{t_n} \cdot X) \to s(X) \). Consider a sequence of periods \( v_n \) in \( \text{Hol}(X) \) such that \(|v_n|_{t_n} = \text{sys}(F_{t_n} \cdot X) \). Finally, for any such \( v_n \), let \( \tau_n \) be the unique instant such that \( \text{Area}(v_n) = \frac{|v_n|_{\tau_n}^2}{2} \). Since \( t_n \to +\infty \) then \( \text{angle}(v_n, \sqrt{-1}) \to 0 \), otherwise \( \text{Area}(v_n) \to \infty \). This implies that the sequence of periods \( v_n \) eventually belongs to any fixed vertical cone. Hence we have

\[
\frac{s^2(X)}{2} = \lim_{n \to \infty} \frac{\text{sys}(F_{\tau_n} \cdot X)}{2} = \lim_{n \to \infty} \frac{|v_n|_{\tau_n}^2}{2} \geq \lim_{n \to \infty} \frac{|v_n|_{t_n}^2}{2} = \lim_{n \to \infty} \text{Area}(v_n) \geq a(X).
\]

\( \square \)

Now we prove the second equality, that is \( \mathcal{E}(T) = a(X) \). The proof is much simpler for those \( X \) whose vertical flow is not uniquely ergodic. For such \( X \), Masur’s criterion (1.11) implies that the projection of \( F_t \cdot X \) to \( \mathcal{M}_g \) diverges, hence \( F_t \cdot X \) diverges in its stratum too, therefore we have \( s(X) = 0 \), which is equivalent to \( a(X) = 0 \) according to Lemma 3.10. On the other hand, since \( T \) is not uniquely ergodic, Boshernitzan’s criterion (see 4.2.1) implies \( \mathcal{E}(T) = 0 \). Thus, we can prove the identity assuming that \( X \)
has uniquely ergodic vertical flow. Recall that if the vertical flow $\phi^t$ of $X$ is uniquely ergodic, then for any $p \in X$ we have

$$\lim_{t \to +\infty} \frac{\sharp\{s \in (0,t); \phi^s(p) \in I\}}{t} = \frac{|I|}{\text{Area}(X)}.$$  

Equation (3.1) applies in particular to any trajectory $\phi^t(p)$ emanating from a conical singularity $p$. In this case, let $\gamma$ be a saddle connection starting at $p$ whose period $v = \int_{\gamma} w$ is reduced. Let $R_w$ be the rectangle in $\mathbb{H}$ whose diagonal is $v$ and $\rho : R_w \to X$ be the corresponding isometric immersion. Then the paths $t \mapsto \rho(0,t;\Sigma(v))$ and $t \mapsto \rho(t;\Sigma(v),t;\Sigma(v))$ with $t \in [0,1]$ have the same number of intersections with the transversal segment $I$. Therefore, for any $\epsilon > 0$, if $\gamma$ is a saddle connection whose period $v$ is reduced and $|\Sigma(v)|$ is big enough we have

$$(1 - \epsilon) \cdot \frac{|I|}{\text{Area}(X)} < \frac{\sharp(\gamma \cap I)}{|\Sigma(v)|} < (1 + \epsilon) \cdot \frac{|I|}{\text{Area}(X)}.$$  

Let $(v_k)_{k \in \mathbb{N}}$ be a sequence of reduced periods such that $\text{Area}(v_k) \to a(X) \cdot \text{Area}(X)$. For any $k$ let $\gamma_k$ be the saddle connection corresponding to $v_k$ and $(\beta(k), \alpha(k), n(k))$ be the sequence of triples given by part (2) of Lemma 3.8. For $k$ big enough $|\Sigma(v_k)|$ is arbitrarily large, hence we have

$$\frac{\text{Area}(v_k)}{\text{Area}(X)} = \frac{|\Sigma(v_k)|}{|\Sigma(v)|} \geq \frac{\sharp(\gamma_k \cap I) \cdot |\Sigma(v_k)|}{|I| \cdot (1 + \epsilon)} = \frac{n(k) \cdot |T^{n(k)}u^0_{\beta(k)} - u^0_{\alpha(k)}|}{|I| \cdot (1 + \epsilon)} \geq \frac{l(T)}{(1 + \epsilon)^2}.$$  

On the other hand, let $(\beta, \alpha, n)$ be a reduced triple such that $n \cdot |T^n u^0_{\beta} - u^0_{\alpha}| \leq (1 + \epsilon)l(T)$ and $n$ is big enough. Let $\gamma$ be the saddle connection given by part (1) of Lemma 3.8. For the associated period $v$ we have

$$\frac{n \cdot |T^n u^0_{\beta} - u^0_{\alpha}|}{|I|} = \frac{\sharp(I \cap \gamma) \cdot |\Sigma(v)|}{|I|} \geq \frac{(1 - \epsilon)|\Sigma(v)| \cdot |\Sigma(v)|}{\text{Area}(X)} \geq (1 - \epsilon)^2 a(X).$$  

Proposition 1.1 is proved.

4. Closure of spectra and density of values of periodic elements

Fix a connected component $C$ of some stratum of translation surfaces let $I$ be a closed invariant locus for the action of $\text{SL}(2, \mathbb{R})$ on $C$. Let $\mathcal{R}$ be a Rauzy class representing the surfaces in $C$, as it is explained in Proposition 2.5. In this section we establish a formula (Theorem 4.6) which allows to compute values of Lagrange spectra $\mathcal{L}(I)$ via the Rauzy-Veech induction, that is in terms of the map $\hat{Q}$ defined in (2.3). Then we use the formula to give a combinatorial proof of Theorem 4.5.

We stress that the map $\hat{Q}$ is defined on data $(\pi, \lambda, \tau)$ such that the pair $(\pi, \lambda)$ satisfies condition (2.4). Therefore it can be iterated infinitely many times on data $(\pi, \lambda, \tau)$ such that the corresponding translation surface $X(\pi, \lambda, \tau)$ does not have vertical saddle connections. The latter is a full-measure subset of parameter space, and moreover it contains all data $(\pi, \lambda, \tau)$ of bounded type. The iteration of $\hat{Q}$ produces a sequence $(\pi^{(r)}, \lambda^{(r)}, \tau^{(r)}), r \in \mathbb{N}$ of combinatorial-length-suspension data. For any fixed Rauzy time $r$, the $r$-th iterate $\hat{Q}^r$ is a piecewise linear map. Linear pieces are in bijection with finite Rauzy paths. For any such path $\gamma$, starting and ending at combinatorial data $\pi$ and $\pi'$ respectively, there are non-empty open sub-cones $\Delta_\gamma \subset \mathbb{R}^d_+$ and $\Theta_\gamma \subset \Theta_\pi$ such that $\hat{Q}$ is a linear homeomorphism from $\{\pi\} \times \Delta_\gamma \times \Theta_\pi$ to $\{\pi'\} \times \mathbb{R}^d_+ \times \Theta_\gamma$.

4.0.1. Notation for Rauzy paths. Let $T$ be an IET corresponding to data $(\pi, \lambda)$. If $T$ admits $r$ steps of the map $\hat{Q}$, we denote $\gamma(0, r)$ the unique finite Rauzy path of combinatorial length $r$ such that $T \in \{\pi\} \times \Delta_\gamma(0, r)$. Similarly, fix data $(\pi, \lambda, \tau)$ admitting infinitely many steps of the map $\hat{Q}$, both in the future and in the past. For any pair of Rauzy times $r_-, r_+ < 0 < r_+$ we call $\gamma(r_-, r_+)$ the concatenation of the Rauzy path $\gamma(r_-, 0)$ ending in $\pi$, with the path $\gamma(0, r_+)$ starting at $\pi$, such that

$$\{\pi, \lambda, \tau\} \in \{\pi\} \times \Delta_\gamma(0, r_+) \times \Theta_\gamma(0, r_-).$$  

Taking the limits of the corresponding cones, the paths $\gamma(r_-, r_+)$ can be defined for $r_- = -\infty$ and/or $r_+ = +\infty$. The paths $\gamma(0, r)$ and $\gamma(r_-, r_+)$ obviously depend on $(\pi, \lambda)$ and $(\pi, \lambda, \tau)$ respectively. Therefore, when referring to these paths, we will always specify the corresponding data inducing them. In (4.2) 1.3 and 4.3 we will often denote by the letter $\gamma$ both finite, half-infinite and bi-infinite Rauzy paths, taking care to avoid ambiguities when necessary.
4.1 Renormalized formula on strata. In order to state and prove the renormalized formula (Theorem 4.6 below), we first introduce the notion of diagonals in Rauzy-Veech induction.

4.1.1. Diagonals. Fix data \((\pi, \lambda, \tau)\), let \(X\) be the underlying translation surface and \(T\) be the IET corresponding to \((\pi, \lambda)\). The interval \(I = (0, \sum_{\chi \in A} \lambda_\chi)\) where \(T\) acts naturally embedded in \(X\). For \(\chi \in A\) denote \(e_\chi\) the correspondent element of the standard basis of \(\mathbb{R}^A\). For a pair of letters \(\beta, \alpha\) such that \(\pi^b(\beta) > 1\) and \(\pi^t(\alpha) > 1\) consider the elements of \(\mathbb{Z}^A\) defined by

\[
w_{\beta, \pi}^b := \sum_{\pi^b(\beta) < \pi^t(\alpha)} e_\chi \quad \text{and} \quad w_{\alpha, \pi}^t := \sum_{\pi^t(\alpha) < \pi^t(\alpha)} e_\chi,
\]

then set \(w_{\beta, \alpha, \pi}^b := w_{\beta, \pi}^b - w_{\alpha, \pi}^t\). For any letter \(\alpha\) in \(A\), the values \(\xi^t_\alpha := \langle \lambda, w_{\alpha, \pi}^t \rangle + i \langle \tau, w_{\alpha, \pi}^t \rangle\) and \(\xi^b_\alpha := \langle \lambda, w_{\alpha, \pi}^b \rangle + i \langle \tau, w_{\alpha, \pi}^b \rangle\) are the coordinates of the conical singularities of \(X\) in the zippered rectangle construction induced by \((\pi, \lambda, \tau)\) (see [2.2]).

**Definition 4.1.** We say that a period \(v\) in \(\text{Hol}(X)\) is a diagonal with respect to \((\pi, \lambda, \tau)\) if there exist letters \(\beta, \alpha\) with \(\pi^b(\beta) > 1\) and \(\pi^t(\alpha) > 1\) such that

\[
v = \langle \lambda, \pi, \beta, \alpha \rangle + i \langle \tau, \pi, \beta, \alpha \rangle.
\]

Notice that not any pair of letters \(\beta, \alpha\) corresponds to a diagonal, indeed the vector \(v\) defined by Equation (4.1) may not be a period of \(X\). Nevertheless we have the following Lemma, whose simple proof is left to the reader.

**Lemma 4.2.** If the letters \(\beta\) and \(\alpha\) are such that

\[
|\langle \lambda, w_{\pi, \beta, \alpha} \rangle| \cdot |\langle \tau, w_{\pi, \beta, \alpha} \rangle| = \min_{\beta', \alpha' \in A} \left|\frac{\langle \lambda, w_{\pi, \beta', \alpha'} \rangle}{\langle \tau, w_{\pi, \beta', \alpha'} \rangle}\right|
\]

then Equation (4.1) defines a reduced period \(v\) of \(X\), thus a diagonal with respect to \((\pi, \lambda, \tau)\).

According to Lemma 4.2 from the sequence of data \((\pi^{(r)}, \lambda^{(r)}, \tau^{(r)})\) given by iteration of the Rauzy-Veech algorithm we can produce a sequence of reduced periods of \(X\) as diagonals. Conversely, we have the following criterion to establish whether or not a period is a diagonal with respect some renormalized data \((\pi^{(r)}, \lambda^{(r)}, \tau^{(r)})\).

**Proposition 4.3.** Consider a triple \((\beta, \alpha, n)\) reduced for \(T\) and such that \(|T^n(u_{\beta}^b) - u_{\alpha}^t| < \min_{\chi \in A} \lambda_\chi\). Let \(v\) be the period associated to \((\beta, \alpha, n)\) by Lemma 3.8. Then there exists \(r \in \mathbb{N}\) such that \(v\) is a diagonal with respect to \((\pi^{(r)}, \lambda^{(r)}, \tau^{(r)})\).

The Proposition follows immediately as a Corollary of Lemma 3.8 and the following Lemma, which was first shown by Rauzy and is proved in this form in [Mat] (see Lemma 3.3 in [Mat]).

**Lemma 4.4.** If the triple \((\beta, \alpha, n)\) is reduced for \(T = (\pi, \lambda)\) then there exists \(r \in \mathbb{N}\) such that

\[
|T^n u_{\beta}^b - u_{\alpha}^t| = |u_{\alpha}^{(r), t} - u_{\beta}^{(r), b}|,
\]

where \(u_{\beta}^{(r), b}\) and \(u_{\beta}^{(r), t}\) denote the singularities of \(T^{(r)} = (\pi^{(r)}, \lambda^{(r)})\).

4.1.2. The renormalized formula. Consider data \((\pi, \lambda, \tau)\) and let \(X\) be the underlying translation surface. Recall that \(\text{Area}(X) = \langle \lambda, \Omega_\pi \tau \rangle\) for a proper antisymmetric matrix \(\Omega_\pi\), depending on \(\pi\) (see for example [Y], page 20).

**Definition 4.5.** We define a continuous function on \((\pi, \lambda, \tau)\) setting

\[
w(\pi, \lambda, \tau) := \frac{1}{\langle \lambda, \Omega_\pi \tau \rangle} \min_{\beta, \alpha \in A} |\langle \lambda, w_{\pi, \beta, \alpha} \rangle| \cdot |\langle \tau, w_{\pi, \beta, \alpha} \rangle|.
\]

According to Lemma 4.2 for a translation surface with unitary area, \(w(\pi, \lambda, \tau)\) equals to \(\text{Area}(v)\) of some period \(v\) of \(X\) which is diagonal with respect to \((\pi, \lambda, \tau)\) in the sense of Definition 4.1.

**Theorem 4.6.** Let \(X\) be the translation surface without vertical saddle connection and corresponding to the data \((\pi, \lambda, \tau)\). We have

\[
a(X) = \lim_{r \to \infty} \inf w(\pi^{(r)}, \lambda^{(r)}, \tau^{(r)}).
\]
Proof: Let $T$ be the IET corresponding to $(\pi, \lambda)$. For any $r$ let $\beta(r)$ and $\alpha(r)$ be the letters satisfying the assumptions of Lemma 4.2 with respect to the data $(\pi(r), \lambda(r), \tau(r))$ and let

$$v_r = (\lambda(r), w_{\beta(r), \alpha(r)}, \pi(r)) + i(\tau(r), w_{\beta(r), \alpha(r)}, \pi(r))$$

be the corresponding period. We have obviously

$$\liminf_{r \to \infty} w(\pi(r), \lambda(r), \tau(r)) = \liminf_{r \to \infty} \frac{\text{Area}(v_r)}{\text{Area}(X)} \geq a(X).$$

In order to prove the Proposition it just remains to establish the reverse inequality. According to Proposition 1.1 and Proposition 3.3, it is enough to prove the required inequality replacing $a(X)$ by $l(T)$. Fix $\epsilon > 0$ and two positive integers $N > 0$ and $R > 0$. Consider a triple $(\beta, \alpha, n)$ reduced for $T$ and such that $n \geq N$ and $n \cdot |T^n u_{\beta}^h - u_{\alpha}^v | / |I| < l(T) + \epsilon$. Let $\gamma$ and $v$ be respectively the saddle connection and the period given by part (1) of Lemma 3.8. According to Proposition 4.2, $v$ is a diagonal with respect to $(\pi(r), \lambda(r), \tau(r))$ for some proper $r$. Moreover, modulo taking a bigger $N$, we can suppose $r > R$. Let $\pi = \pi(r)$ and let $\beta$ and $\alpha$ be the letters such that $v = (\lambda(r), w_{\beta, \alpha}, \pi(r))$. We have

$$l(T) + \epsilon > \frac{n \cdot |T^n u_{\beta}^h - u_{\alpha}^v |}{|I|} = \frac{\sharp(\gamma \cap I) \cdot |(\lambda(r), w_{\beta, \alpha}, \pi(r))|}{|I|}.$$ 

Let us first assume that $l(T) > 0$, which implies that the vertical flow of $X$ is uniquely ergodic. Observe that we have $|\Im(v)| = |(\pi(r), w_{\beta, \alpha}, \pi(r))|$. Therefore, according to Equation (3.1), unique ergodicity implies

$$\frac{\sharp(\gamma \cap I) \cdot |(\lambda(r), w_{\beta, \alpha}, \pi(r))|}{|I|} \geq (1 - \epsilon) \frac{|(\lambda(r), w_{\beta, \alpha}, \pi(r))|}{\text{Area}(X)} \geq (1 - \epsilon) w(\pi, \lambda(r), \tau(r)).$$

We get $a(X) + \epsilon = l(T) + \epsilon \geq (1 - \epsilon) \cdot w(\pi(r), \lambda(r), \tau(r))$, hence the required inequality follows because $\epsilon$ is arbitrarily small and $r$ is arbitrarily big.

Now assume that $l(T) = 0$. In this case, for $L := \max_{\alpha, \beta} \Im(\xi_{\alpha} - \xi_{\beta})$ we have $|\Im(v)| \leq L \cdot \sharp(\gamma \cap I)$, hence

$$\frac{\sharp(\gamma \cap I) \cdot |(\lambda(r), w_{\beta, \alpha}, \pi(r))|}{|I|} \geq \frac{|(\lambda(r), w_{\beta, \alpha}, \pi(r))|}{\text{Area}(X) \cdot L} \geq L^{-1} \cdot w(\pi(r), \lambda(r), \tau(r)).$$

It follows that $\epsilon = l(T) + \epsilon \geq L^{-1} w(\pi(r), \lambda(r), \tau(r))$, hence we get $\liminf_{r \to +\infty} w(\pi(r), \lambda(r), \tau(r)) = 0$, because $\epsilon$ is arbitrarily small and $r$ is arbitrarily big. \qed

4.2. Control of the excursions via the cocycle. Let $\gamma$ be any Rauzy path, either finite, bi-infinite, or half-infinite. We say that a finite sub-path $\nu$ of $\gamma$ is minimal positive if it is positive and it does not have proper positive sub-paths. Denote $\mathcal{M}(\gamma)$ the set of minimal positive sub-paths $\nu$ of $\gamma$, then define

$$(4.2) \quad N(\gamma) := \sup\{\|B_{\nu}\|; \nu \in \mathcal{M}(\gamma)\},$$

where the norm of the real $d \times d$ matrix $A$ is $\|A\| := \max_{\alpha, \beta \in A} |A_{\alpha, \beta}|$.

Fix data combinatorial length and suspension data $(\pi, \lambda, \tau)$ inducing a bi-infinite path $\gamma$. Let $X = X(\pi, \lambda, \tau)$ and $T = (\pi, \lambda)$ be respectively the corresponding translation surface and IET. For $r \geq 0$ the positive half-infinite path $\gamma(r, +\infty)$ just depends on $T$. Define

$$N_\infty(T) := \limsup_{r \to +\infty} N(\gamma(r, +\infty)).$$

The following Theorem 4.7 establishes a relation between $N_\infty(T)$ and the function $X \mapsto a(X)$, which gives rise to Lagrange Spectra. Its proof is given in [1,2,3] and is based on Proposition 1.10 and Proposition 4.11 in the following section.

Theorem 4.7. Fix combinatorial-length-suspension data $(\pi, \lambda, \tau)$, let $X$ be the underlying translation surface and $T$ be the IET corresponding to $(\pi, \lambda)$. We have

$$\frac{1}{\sqrt{d}} \cdot N_\infty(T)^{2(\pi - 1)/(2\pi - 1)} \leq \frac{1}{a(X)} \leq \frac{d}{2} \cdot N_\infty(T)^4,$$
From Theorem \[\text{4.10}\] we can also deduce a characterization of IETs of bounded type (defined in \[\text{§4.2.1}\]) in terms of a suitable acceleration of the Rauzy-Veech induction. Let \( T \) be the IET given by \((\pi, \lambda)\) which induces an half-infinite Rauzy path \( \gamma(0, +\infty) \). Denote \( \nu(T) \) the minimal positive Rauzy path of the form \( \gamma(0, r) \) with \( r \in \mathbb{N}_+ \). We can define an acceleration \( Q_+ \) of the Rauzy-Veech induction, that we call positive acceleration, by setting \( Q_+(T) = Q^{\nu(T)}(T) \). Finally set \( P(T) \) and let \( P_n(T) = P(Q^n_+(T)) \) be the sequence of positive matrices produced by the accelerated induction.

**Corollary 4.8.** \( T \) is an IET of bounded type if and only if it admits infinitely many iterations of the positive acceleration \( Q_+ \) and the corresponding matrices \( P_n(T) \) have norms uniformly bounded for \( n \in \mathbb{N} \).

After this paper, Kim and Marmi announced a proof of a similar characterization of IETs of bounded type in \[\text{K.M.}\].

### 4.2.1. Positive matrices and flat geometry.

The following two Propositions contain two geometrical estimates of the distortion of data \((\pi, \lambda, \tau)\) in terms of a finite and bounded number of steps of the Rauzy-Veech induction. Their proof is quite technical and is given in \[\text{C}\]. Fix combinatorial-length-suspension data \((\pi, \lambda, \tau)\) and let \( T \) be the IET corresponding to \((\pi, \lambda)\).

**Definition 4.9.** We define the distortion of \( T \) by

\[
\Delta(T) := \max_{\beta, \alpha \in \mathcal{A}} \frac{\lambda_\beta}{\lambda_\alpha}.
\]

Recall from \[\text{M.M.M.Y.}\] that a finite Rauzy path \( \gamma \) is said complete if any letter \( \alpha \) in \( \mathcal{A} \) wins in \( \gamma \) (we refer to \[\text{§2.1}\] for terminology). Then we say that a path \( \gamma \) is strongly complete if it is a concatenation of \( d \) complete paths. Let \( \gamma(-\infty, 0) \) be the half-infinite path induced by the data \((\pi, \lambda, \tau)\). Consider the minimal positive integer \( m \) such that the ending part \( \gamma(-m, 0) \) of \( \gamma(-\infty, 0) \) is strongly complete. Then set

\[
m(\pi, \lambda, \tau) := \min_{-m \leq r \leq 0} w(\pi(r), \lambda(r), \tau(r)).
\]

**Proposition 4.10.** Consider combinatorial-length-suspension data \((\pi, \lambda, \tau)\) and let \( T \) be the IET corresponding to \((\pi, \lambda)\). We have

\[
\Delta(T) \leq \frac{d!}{m(\pi, \lambda, \tau)^{d-1}}.
\]

**Proposition 4.11.** Let \( \gamma_1, \gamma_2, \gamma_3, \gamma_4 \) be positive paths that can be concatenated and let \( r, s \) be instants such that that \( \gamma(-s, 0) = \gamma_1 \circ \gamma_2 \) and \( \gamma(0, r) = \gamma_3 \circ \gamma_4 \). Then we have

\[
w(\pi, \lambda, \tau) \geq \frac{2}{d} \cdot \frac{1}{\|B_{\gamma_1}\|_\infty \cdot \|B_{\gamma_2}\|_\infty \cdot \|B_{\gamma_3}\|_\infty \cdot \|B_{\gamma_4}\|_\infty}.
\]

The proof of Proposition \[\text{4.10}\] and Proposition \[\text{4.11}\] is given in \[\text{C}\]. In the rest of the section we apply them to prove Theorem \[\text{4.10}\].

### 4.2.2. Auxiliary results.

If \( \gamma \) is a finite Rauzy path of length \( r \) and \( T \) is an IET defined by data \((\pi, \lambda)\) in \( \Delta_{\gamma} \), denote \( Q_\gamma \) the branch of the Rauzy map induced by \( \gamma \), that is \((\pi(r), \lambda(r)) = Q_\gamma(\pi, \lambda) \). If \( T(r) \) is the IET corresponding to \((\pi(r), \lambda(r))\), we also write \( T(r) = Q_\gamma(T) \).

Consider combinatorial-length data \((\pi, \lambda)\) and let \( T \) be the corresponding IET. Let \( \gamma(0, +\infty) \) be the half-infinite path induced by \((\pi, \lambda)\) and recall that \( P(T) := B_{\nu(T)} \) where \( \nu(T) \) is the smallest \( r \in \mathbb{N}_+ \) such that the Rauzy path \( \gamma(0, r) \) is positive.

**Lemma 4.12.** Let \( T \) be any IET. We have

\[
\Delta(T) \leq \|P(T)\|.
\]

**Proof:** Set \((\pi', \lambda') = Q_{\nu(T)}(\pi, \lambda) \) and \( P := P(T) \). We have \( P^TP = \lambda \). Since \( P \) is positive with integer entries then for any \( \alpha \) in \( \mathcal{A} \) we have \( \sum_{\chi \in \mathcal{A}} \lambda'_\chi \leq \lambda_\alpha \leq \|P\|\|: \sum_{\chi \in \mathcal{A}} \lambda'_\chi \). It follows that for any \( \beta \) and \( \alpha \) we have \( \lambda_\alpha \leq \|P\|\| \lambda_\beta \), thus the Lemma is proved. \( \square \)

**Lemma 4.13.** Let \( \gamma \) be a non-complete path and \((\pi, \lambda)\) be combinatorial-length data in \( \Delta_{\gamma} \). If \( T \) is the IET corresponding to the data \((\pi, \lambda)\) then we have

\[
\|B_\gamma\|_\infty \leq \Delta(T) \cdot \Delta(Q_\gamma(T)).
\]
Proof: Observe that we have $B_\gamma \lambda' = \lambda$. Let $\alpha$ be a letter which never wins in $\gamma$. The $\alpha$-column of $B_\gamma$ equals to the vector $e_\alpha$ of the standard basis of $\mathbb{R}^A$, that is $[B_\gamma]_{\alpha,\alpha} = 1$ and $[B_\gamma]_{\chi,\alpha} = 0$ for $\chi \neq \alpha$, thus $\lambda_\alpha = \lambda'_\alpha$. On the other hand, if $(\beta', \beta)$ is the pair of letters such that $\|B_\gamma\|_\infty = [B_\gamma]_{\beta, \beta'}$, we have

$$\lambda_\beta = \sum_{\chi \in A} [B_\gamma]_{\chi, \beta} \lambda'_\chi > [B_\gamma]_{\beta', \beta} \lambda'_\beta = \|B_\gamma\|_\infty \lambda'_\beta.$$

Therefore we get

$$\lambda_\alpha = \lambda'_\alpha < \Delta(Q_\gamma(T)) \cdot \lambda'_\beta < \Delta(Q_\gamma(T)) \frac{\lambda_\beta}{\|B_\gamma\|_\infty} < \Delta(Q_\gamma(T)) \frac{\Delta(T) \cdot \lambda_\alpha}{\|B_\gamma\|_\infty}.$$

\[\square\]

The following Corollary is motivated by the Lemma at page 833 in [M,M,Y], where it is proved that the concatenation of $2d-3$ maximal non-complete paths is positive.

**Corollary 4.14.** Consider a finite Rauzy path $\gamma$ that is the concatenation $\gamma = \gamma_1 * \cdots * \gamma_{2d-3}$ of $2d-3$ maximal non-complete paths $\gamma_1, \ldots, \gamma_{2d-3}$. For any $k$ with $1 \leq k \leq 2d-3$ denote $Q_k := Q_{\gamma_1 * \cdots * \gamma_k}$. If $T$ is an IET in $\Delta_\gamma$, then we have

$$\|P(T)\|_\infty \leq \Delta(T) \cdot \Delta^2(Q_1(T)) \cdot \cdots \cdot \Delta^2(Q_{2d-3}(T)).$$

**Proof:** Observe that $\|B_\gamma\|_\infty \leq \|B_{\gamma_1}\|_\infty \cdot \cdots \cdot \|B_{\gamma_{2d-3}}\|_\infty$. Observe also that $T$ belongs to $\Delta_{\gamma_k}$ and moreover, for any $1 < k \leq 2d-3$ the IET $Q_{k-1}(T)$ belongs to $\Delta_{\gamma_k}$. Then the Corollary follows applying Lemma 4.13 to each factor in the product. \[\square\]

4.2.3. **Proof of Theorem 4.7.** Fix combinatorial-length-suspension data $(\pi, \lambda, \tau)$, let $\gamma = (\infty, \infty)\gamma$ be the corresponding bi-infinite Rauzy path and $X$ be the underlying translation surface. Let $T$ be the IET corresponding to $(\pi, \lambda)$.

We first prove the second inequality. Fix $M > N_\infty(T)$. For any $r$ big enough, we can find integers $r_- < r < r_+$ and positive paths $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ with $\|B_{\gamma_i}\| < M$ for $i = 1, \ldots, 4$ such that $\gamma_1 * \gamma_2 = \gamma(r_-, r_1)$ and $\gamma_3 * \gamma_4 = \gamma(r, r_+)$.

For these big values of $r$, Proposition 4.11 implies $M^{-4} \cdot 2/d \leq w(\pi(r), \lambda(r), \tau(r))$, therefore we have

$$\frac{2}{d \cdot M^4} \leq \liminf_{r \to \infty} w(\pi(r), \lambda(r), \tau(r)).$$

According to the formula in Proposition 4.6 we have $M^{-4} \cdot 2/d \leq a(X)$, and since this last estimate holds for any $M > N_\infty(T)$, then the second inequality in Theorem 4.7 follows.

Now we prove the first inequality. Recall the formula in Proposition 4.6 and observe that we have

$$a(X) = \liminf_{r \to +\infty} w(\pi(r), \lambda(r), \tau(r)) = \liminf_{r \to +\infty} m(\pi(r), \lambda(r), \tau(r)),$$

therefore Proposition 4.10 implies $\limsup_{r \to \infty} \Delta(T(r)) \leq dt \cdot a(X)^{-d-1}$. Observe that with our notation we have $N_\infty(T) = \limsup_{r \to +\infty} \|P(T(r))\|_\infty$, thus $N_\infty(T) \leq \limsup_{r \to +\infty} \Delta(T(r))^{2(2d-3)}$, according to Corollary 4.14. Combining the last two results we get

$$N_\infty(T) \leq \left(d^2(2d-3) \cdot a(X)^{-2(d-1)(2d-3)}\right).$$

The first inequality in Theorem 4.7 follows. \[\square\]

4.3. **Subshifts in the Rauzy class.** In this subsection we consider bi-infinite Rauzy paths $\gamma$ such that the quantity $N(\gamma)$ introduced in Equation 4.2 is finite and we show that these paths can be encoded by a symbolic sequence in a subshift of finite type. This coding will be convenient to present the proof of Theorem 1.5.
4.3.1. Construction of the sub-shift. For any finite Rauzy path $\gamma$ consider the corresponding matrix $B_\gamma$ in the Kontsevich-Zorich cocycle, that is the matrix defined in \(4.12\). Recall that $\gamma$ is said to be \textit{positive} if all the entries of the matrix $B_\gamma$ are positive. Consider the countable alphabet $\Gamma$ of finite and positive Rauzy paths in $\mathcal{R}$. Let $\Gamma^\mathbb{Z}$ be the shift space of sequences in the elements of $\Gamma$. Let $\Gamma^\mathbb{Z}_\mathcal{R} \subset \Gamma^\mathbb{Z}$ be the sub-shift space consisting of all sequences $(\gamma_k)_{k \in \mathbb{Z}}$ in $\Gamma^\mathbb{Z}$ such that for all $k \in \mathbb{Z}$ the final permutation of $\gamma_k$ is the same than the initial permutation of $\gamma_{k+1}$, so that the infinite concatenation $\cdots \gamma_{k-1} \ast \gamma_k \ast \gamma_{k+1} \cdots$ is a bi-infinite path on $\mathcal{R}$. For any $M > 0$, let $\Gamma(M) \subset \Gamma$ be the subset of the alphabet of those $\gamma \in \Gamma$ such that $\|B_\gamma\| \leq M$. Clearly $\Gamma(M)$ is a finite alphabet. Let $\Gamma^\mathbb{Z}_\mathcal{R}(M)$ be the sub-shift space consisting of all bi-infinite sequences in the letters of $\Gamma(M)$ and let

$$
\Gamma^\mathbb{Z}_\mathcal{R}(M) := \Gamma^\mathbb{Z}_\mathcal{R} \cap \Gamma^\mathbb{Z}(M).
$$

Then the shift $\sigma : \Gamma^\mathbb{Z}_\mathcal{R}(M) \to \Gamma^\mathbb{Z}_\mathcal{R}(M)$ is a topological Markov chain, the transition matrix $A$ being defined setting $A_{\gamma,\gamma'} := 1$ if and only if the concatenation $\gamma \ast \gamma'$ is a path on $\mathcal{R}$, and $A_{\gamma,\gamma'} := 0$ otherwise, where $\gamma, \gamma'$ are any pair of letters in $\Gamma(M)$. Remark that while the whole shift $\sigma$ is not of finite type, for any $M > 0$ the sub-shift $\sigma : \Gamma^\mathbb{Z}_\mathcal{R}(M) \to \Gamma^\mathbb{Z}_\mathcal{R}(M)$ is of finite type. Recall the function $\gamma \to N(\gamma)$ defined by Equation (4.2). According to the following Lemma, whose proof is obvious, any bi-infinite path $\gamma$ with $N(\gamma) < M$ is decomposed into packets, each packet $\gamma_k$ being the entry of a sequence in $\Gamma^\mathbb{Z}_\mathcal{R}(M)$.

\textbf{Lemma 4.15.} Let $\gamma$ be a bi-infinite path such that $N(\gamma) < M$ for some $M > 0$. Then there exists a sequence $(\gamma_k)_{k \in \mathbb{Z}} \in \Gamma^\mathbb{Z}_\mathcal{R}(M)$ such that

$$
\gamma = \cdots \gamma_{k-1} \ast \gamma_k \ast \gamma_{k+1} \cdots
$$

4.3.2. Hilbert metric. The Hilbert pseudo-metric on $\mathbb{R}^2_+$ is

$$
dist_{\mathbb{R}^2_+}(x, y) := \log \max_{1 \leq i,j \leq 2} \frac{x_i y_j}{x_j y_i},
$$

which is invariant under linear isomorphisms of $\mathbb{R}^2_+$, that is the group of invertible diagonal matrices. More generally, if $C \subset \mathbb{R}^A \setminus \{0\}$ is an open convex cone whose closure does not contain any one-dimensional subspace of $\mathbb{R}^A$, one defines a Hilbert pseudo-metric on $C$ as follows. If $x$ and $y$ are collinear, then $\text{dist}_C(x, y) = 0$. Otherwise, $C$ intersects the subspace generated by $x$ and $y$ in a cone isomorphic to $\mathbb{R}^2_+$. We let $\text{dist}_C(x, y) := \text{dist}_{\mathbb{R}^2_+}(\psi x, \psi y)$, where $\psi$ is any such isomorphism. When $C = \mathbb{R}^A$ the formula above gives

$$
dist_{\mathbb{R}^A}(x, y) = \log \max_{\beta,\alpha} \frac{x_\alpha y_\beta}{x_\beta y_\alpha}.
$$

It is easy to check that for any two points $x$ and $y$ in $C$ and for any pair of positive real numbers $a$ and $b$ we have $\text{dist}_C(x, y) = \text{dist}_C(a \times x, b \times y)$, therefore the Hilbert pseudo-metric induces a well defined metric on the space of rays $\{tx : t \in \mathbb{R}^+_x\}$ contained in $C$, called \textit{Hilbert’s metric}.

Let $A$ be linear map such that $A(C) \subset C$. According to a classical result due to G. Birkhoff (see \[B\]), if $A(C)$ is bounded in $C$ then $A$ is a contraction of the Hilbert pseudo-metric. More precisely, denoting by $D$ the diameter of $A(C)$, for any $x, y$ in $C$ we have

$$
\text{dist}_C(A \times x, A \times y) \leq \delta \cdot \text{dist}_C(x, y), \text{ where } \delta = \frac{\sqrt{e^D} - 1}{\sqrt{e^D} + 1}.
$$

The last estimate implies the following Lemma, whose proof is left to the reader.

\textbf{Lemma 4.16.} For any $M > 0$ there exists a positive constant $\delta = \delta(M) < 1$ such that, for any Rauzy path $\gamma : \pi \to \pi'$ with $\|B_\gamma\| \leq M$ the following holds

1. If $\mathcal{R}_0^A$ has compact closure in $\mathbb{R}_+^A$ then the map $\lambda \mapsto \mathcal{R}_0^A(\lambda)$ is a $\delta$-contraction.

2. If $\mathcal{R}_\sigma^A$ has compact closure in $\Theta_\sigma$, then the map $\tau \mapsto \mathcal{R}_\sigma^A(\tau)$ is a $\delta$-contraction.

4.3.3. A continuous function for the sub-shift. Fix $M > 0$ and let $\gamma$ be a bi-infinite Rauzy path with $N(\gamma) \leq M$, then according to Lemma 4.15 decompose it as

$$
\gamma = \cdots \gamma_{k-1} \ast \gamma_k \ast \gamma_{k+1} \ast \cdots \text{ with } (\gamma_k)_{k \in \mathbb{Z}} \in \Gamma^\mathbb{Z}_\mathcal{R}(M).
$$

For any $r$ in $\mathbb{Z}$ consider the half-infinite path $\gamma(r, +\infty)$ defined in \[4.0.1\] and let $\pi^{(r)}$ be the element in $\mathcal{R}$ where $\gamma(r, +\infty)$ starts. Since any $\gamma_k$ is positive then $\mathcal{R}_0^{A_k}$ has compact closure in $\mathbb{R}^A_+$ with respect to the Hilbert’s metric. Lemma 4.3 in \[A,G,Y\] implies that the same property holds for the cone of
suspension data, modulo considering the concatenation $\gamma_k \cdots \gamma_{k+3d-5}$ of at most $3d-4$ positive paths, where $d$ denotes the cardinality of $A$. Therefore, according to Lemma 4.10, the cone $\bigcap_{r}^s (B_{\gamma(r,s)} \mathbb{R}^d_+)$ shrinks exponentially with respect to the Hilbert’s metric as $s \to +\infty$ and similarly $\bigcap_{s}^r (B_{\gamma(s,r)} \Theta_{\pi(r)})$ shrinks exponentially as $s \to -\infty$. Therefore we have two rays $\widehat{\lambda}(r)$ in $\mathbb{R}^d_+$ and $\widehat{\tau}(r)$ in $\Theta_{\pi(r)}$ defined by

$$\widehat{\lambda}(r) := \bigcap_{s=r}^\infty (B_{\gamma(r,s)} \mathbb{R}^d_+)$$

$$\widehat{\tau}(r) := \bigcap_{s=-\infty}^{r-1} (B_{\gamma(s,r-1)} \Theta_{\pi(r)}).$$

**Definition 4.17.** Fix a bi-infinite path with $N(\gamma) < +\infty$ and any $r \in \mathbb{Z}$. Define length and suspension data $\lambda(r)$ and $\tau(r)$ on the rays $\widehat{\lambda}(r)$ and $\widehat{\tau}(r)$ respectively via the normalizations $\|\lambda(r)\| = 1$ and $\text{Area}(\pi(r), \lambda(r), \tau(r)) = 1$. Then define

$$a_r(\gamma) := w(\pi(r), \lambda(r), \tau(r)).$$

**Lemma 4.18 (Approximation Lemma).** For any $\epsilon > 0$ and $M > 0$ there exists $m = m(\epsilon, M) \in \mathbb{N}$ such that the following holds. Let $\gamma', \gamma''$ be two bi-infinite paths on $\mathbb{R}$ with $N(\gamma') < M$ and $N(\gamma'') < M$ and let $r^- < r < r^+$ be any Rauzy times such that we can write

$$\gamma'(r^-, r) = \gamma''(r^-, r),$$

$$\gamma'(r, r^+) = \gamma''(r, r^+),$$

where $\gamma_1, \ldots, \gamma_{2m}$ are $2m$ positive paths in $\Gamma(M)$. Then

$$\left| \frac{a_r(\gamma')}{{\lambda}', \lambda(r), \tau(r)} - 1 \right| \leq \epsilon.$$

**Proof:** For any $r$ let $(\pi, \lambda'(r), \tau'(r))$ and $(\pi, \lambda''(r), \tau''(r))$ be the unique data associated respectively to $\gamma'$ and to $\gamma''$ as in Definition 4.17. Proposition 4.11 implies

$$a_r(\gamma') \geq \frac{2}{d \cdot M^4}$$

and $a_r(\gamma'') \geq \frac{2}{d \cdot M^4}$.

The condition $\gamma'(r, r^+) = \gamma''(r, r^+)$ implies that both $\lambda'(r)$ and $\lambda''(r)$ belong to the cone $\Delta_{\gamma_{m+1} \cdots \gamma_{2m}}$, whose projective diameter is less than $\delta^m$, according to Lemma 4.10, where $\delta$ is the constant in the Lemma. Similarly, the condition $\gamma'(r^-, r) = \gamma''(r^-, r)$ implies that both $\tau'(r)$ and $\tau''(r)$ belong to a subset of $\Theta_{\pi(r)}$ with diameter bounded by $\delta^m$. The required $m$ exists because the function $(\pi, \lambda, \tau) \mapsto w(\pi, \lambda, \tau)$ is continuous.

Consider two sequences $(\gamma'_k)$ and $(\gamma''_k)$ in $\Pi^2_{\mathbb{R}}$ and the corresponding bi-infinite Rauzy paths $\gamma' = \cdots \gamma'_1 \cdots \gamma'_{k+1} \cdots$ and $\gamma'' = \cdots \gamma''_1 \cdots \gamma''_{k+1} \cdots$ obtained by bi-infinite concatenation. Recall that any bi-infinite Rauzy path admits such decomposition.

**Corollary 4.19 (Interpolation).** For any $\epsilon > 0$ and $M > 0$, let $m = m(\epsilon, M)$ be the positive integer given by the Approximation Lemma 4.18. If there are integers $k_1 < k_2$ and finite paths $\sigma_i$ in $\Gamma(M)$ with $k_1 - m \leq i \leq k_2 + m$ such that

$$\gamma_i' = \gamma_i'', \quad \text{for any } k_1 - m \leq i \leq k_2 + m$$

then for the instants $r_1 < r_2$ such that $\gamma'(r_1, r_2) = \sigma_k \cdots \sigma_{k_2}$ and for any $r_1 \leq r < r_2$ we have

$$\left| \frac{a_r(\gamma')}{{\lambda}', \lambda(r), \tau(r)} - 1 \right| \leq \epsilon.$$

**Proof:** Let $r_1 \leq r < r_2$. Consider $k$ with $k_1 \leq k \leq k_2$ and the two instants $r_k^-$ and $r_k^+$ with $r_1 \leq r_k^- \leq r < r_k^+ \leq r_2$ such that $\gamma_k = \gamma(r_k^-, r) \gamma(r + 1, r_k^+)$. Thus we have

$$\gamma(r_1, r) = \sigma_1 \cdots \sigma_{k-1} \gamma(r_k^-, r) \gamma(r + 1, r_2) = \gamma(r + 1, r_k^+) \sigma_{k+1} \cdots \sigma_{k_2}.$$
Observe that $\tilde{\gamma}_{k-1} := \sigma_{k-1} \ast \gamma(r_k, r)$ and $\tilde{\gamma}_{k+1} := \gamma(r + 1, r_k^+) \ast \sigma_{k+1}$ are both positive paths such that $\|B_{\tilde{\gamma}_{k-1}}\| \leq M^2$ and $\|B_{\tilde{\gamma}_{k+1}}\| \leq M^2$. According to the assumptions, we have two Rauzy times $r^- < r < r^+$ such that

$$\gamma'(r^-, r) = \gamma^n(r^-, r) = \sigma_{k_1-m} \ast \cdots \ast \tilde{\gamma}_{k-1},$$

$$\gamma'(r, r^+) = \gamma^n(r, r^+) = \tilde{\gamma}_{k+1} \ast \cdots \ast \sigma_{2m},$$

where $m = m(e, M^2)$ and $\tilde{\gamma}_{k-1}, \tilde{\gamma}_{k+1}$ and all $\sigma_i$ appearing above belong to $\Gamma(M^2)$. The Corollary follows applying the Approximation Lemma 4.18.

4.3.4. Recurrence for the sub-shift. In this paragraph we show that for sub-shifts of finite type there always exists instants such that the hypothesis required by Lemma 4.18 and Corollary 4.19 are satisfied.

**Lemma 4.20** (Cantor diagonal Lemma). Consider a sequence of sequences $(\gamma_k^{(n)})_{k \in \mathbb{Z}}$ in $\Gamma^2(M)$, indexed by $n \in \mathbb{N}$. There exists a sequence $\sigma = (\sigma_k)_{k \in \mathbb{Z}}$ in $\Gamma^2(M)$ and a subsequence $(n_r)_{r \in \mathbb{N}}$ such that for any $m \in \mathbb{N}$, we have

$$\gamma_k^{(n_r)} = \sigma_k \text{ for all } -m \leq k \leq m \text{ and all } r \geq m.$$  

In other words, the sequence $(\gamma_k^{(n_r)})_{k \in \mathbb{Z}}$ provided by the Lemma consists of sequences which eventually agree on arbitrarily large central blocks with the limiting sequence $(\sigma_k)_{k \in \mathbb{Z}}$.

**Proof:** Recall that $\Gamma(M)$ is a finite set. Since the sequence of central letters $(\gamma_0^{(n)})_{n \in \mathbb{N}}$ has values in $\Gamma(M)$, then there exists a letter $\sigma_0 \in \Gamma(M)$ which occurs infinitely often, that is there exists an increasing subsequence $(n(0,j))_{j \in \mathbb{N}}$ of natural numbers such that $\gamma_0^{(n(0,r))} = \sigma_0$ for any $r \in \mathbb{N}$. Fix $m \geq 0$ and suppose that we have $2m + 1$ letters $\sigma_{-m}, \ldots, \sigma_0, \sigma_1, \ldots, \sigma_m$ in $\Gamma(M)$ and a nested family of subsequences

$$n(m, \cdot) \subset n(m-1, \cdot) \subset \cdots \subset n(0, \cdot)$$

such that for any $r$ with $0 \leq r \leq m$ and for any $j$ we have

$$\gamma_i(n(r,j)) = \sigma_i \text{ for all } -r \leq i \leq r,$$

where we stress that each $n(r, \cdot)$ is a sequence in the index $j$, that is $n(r, \cdot) = (n(r,j))_{j \in \mathbb{N}}$. The sequence of pair of letters $(\gamma_{m+1}^{(n(m,j))}, \gamma_m^{(n(m,j))})_{j \in \mathbb{N}}$ has values in $\Gamma(M) \times \Gamma(M)$, which is a finite set. Therefore there exits a pair of letters $(\sigma_{-m-1}, \sigma_{m+1})$ and a subsequence $n(m + 1, \cdot) \subset n(m, \cdot)$, where $n(m + 1, \cdot) = (n(m + 1,j))_{j \in \mathbb{N}}$, such that

$$\gamma_{m+1}^{(n(m+1,j))}, \gamma_m^{(n(m+1,j))} = (\sigma_{-m-1}, \sigma_{m+1}) \text{ for all } j \in \mathbb{N}.$$ 

Therefore the sequence $\sigma = (\sigma_j)_{j \in \mathbb{Z}}$ is defined inductively, moreover we have also an inductively defined nested sequence of subsequences $(n(m, \cdot))_{m \in \mathbb{N}}$ with $\ldots n(0, \cdot) \subset n(-1, \cdot) \ldots$. The required sequence is diagonal subsequence, that is $m_m := n(m, m)$.

**Corollary 4.21** (Recurrence). Consider a sequence $(\gamma_k)_{k \in \mathbb{Z}}$ in $\Gamma^2(M)$ and any subsequence $(k_n)_{n \in \mathbb{N}}$ with $k_n \to \infty$. For any $m \in \mathbb{N}$ there exist $2m + 1$ letters $\sigma_{-m}, \ldots, \sigma_0, \sigma_1, \ldots, \sigma_m$ in $\Gamma(M)$ such for infinitely many $n \in \mathbb{N}$ we have

$$\gamma_{k_n+i} = \sigma_i \text{ for all } -m \leq i \leq m.$$ 

In other words, the finite word $\sigma_{-m} \cdots \sigma_m$ appears infinitely often in $(\gamma_k)_{k \in \mathbb{Z}}$ and moreover it appears centered at infinitely many of the prescribed indexes $(k_n)_{n \in \mathbb{N}}$.

**Proof:** For any $n$ define a sequence $(\gamma_k^{(n)})_{k \in \mathbb{Z}}$ in the index $k$ setting $\gamma_k^{(n)} := \gamma_{k+n+k}$. Let $\sigma = (\sigma_k)_{k \in \mathbb{Z}}$ be the sequence in $\Gamma^2(M)$ given by Lemma 4.20 and $(n_r)_{r \in \mathbb{N}}$ be the subsequence defined in the same Lemma. Fix any non-negative integer $m$. Then, for any $r \geq m$ set $n = n_r$. The Corollary follows observing that for any $i$ with $-m \leq i \leq m$ we have

$$\gamma_{k_n+i} = \gamma_i^{(n)} = \sigma_i.$$  

□
4.4. Proof of closure and density of periodic elements. Recall that we consider a fixed connected component \( C \) of some stratum of translation surfaces and a closed invariant locus \( I \) for the action of \( SL(2, \mathbb{R}) \) on \( C \). Then \( R \) is a Rauzy class representing the surfaces in \( C \), as it is explained in Proposition 2.5. For any combinatorial datum \( \pi \in R \) let \( I_\pi \) be the set of data \((\lambda, \tau) \in \mathbb{R}_+^3 \times \Theta_\pi \) such that the surface \( X(\pi, \lambda, \tau) \) belongs to \( I \). The structure of \( I_\pi \) is described in Appendix B.

If \( \gamma \) is a positive loop at \( \pi \in R \), denote respectively \( \gamma^\lambda \) and \( \gamma^\tau \) the maximal and the minimal eigenvector of the matrix \( B^{-1}_\gamma \), normalized so that \( \|\gamma^\lambda\| = 1 \) and \( \text{Area}(\pi, \lambda, \tau) = 1 \). Let \( X(\pi, \lambda, \tau) \) be the translation surface corresponding to such data, which is a periodic point for \( \mathcal{F}_1 \). More precisely, for \( T := -\log \left( \| B^{-1}_\gamma \lambda^\tau \| \right) \) we have

\[
\mathcal{F}_T \cdot X(\pi, \lambda^\tau, \tau^\lambda) = X(\pi, \lambda^\tau, \tau^\lambda).
\]

If \( I \) is a closed and \( SL(2, \mathbb{R}) \)-invariant subset of the connected component corresponding to the Rauzy class \( R \), then define

\[
\text{Per}(R, I) := \{ \gamma \text{ positive loop at } \pi \in R \text{ with } X(\pi, \lambda^\tau, \tau^\lambda) \in I \}.
\]

In order to simplify the notation, in the following we will refer to elements of \( \text{Per}(R, I) \) either as positive loops \( \gamma \), or as bi-infinite paths corresponding to the infinite concatenation \( \ldots \gamma \ast \gamma \ast \gamma \ast \ldots \). Any \( \gamma \) in \( \text{Per}(R, I) \) corresponds to some closed Teichmüller geodesic in \( I \), that is an element of \( PA(I) \). Not o any element of \( PA(I) \) is represented by a positive loop in \( \text{Per}(R, I) \). Nevertheless, for any \( X \in PA(I) \) whose orbit has period \( T \), there exists a positive integer \( m \) and some \( \gamma \in \text{Per}(R, I) \) representing the orbit \( t \mapsto \mathcal{F}_{mt} \cdot X \) for \( 0 \leq t \leq T \). In particular, the set of values \( \{ a(X); X \in PA(I) \} \) can be computed just considering the elements in \( \text{Per}(R, I) \).

4.4.1. Reduction to the sub-shift. Recall that \( \mathcal{L}(I) \subset \mathcal{L}(C) \). We introduce a monotone increasing function \( \mathcal{N} : \mathcal{L}(C) \to \mathbb{R}_+ \), which is obviously defined by restriction also on \( \mathcal{L}(I) \), given explicitly by

\[
\mathcal{N}(L) := (4|L|)(2d-3) \cdot 2^d L^{1/2} L^{2d-3}.
\]

Consider data \((\pi, \lambda, \tau)\) inducing a bi-infinite path \( \gamma \) and let \( X = X(\pi, \lambda, \tau) \) be the corresponding translation surface. For \( r \in \mathbb{Z} \) consider the corresponding paths \( \gamma(r, \infty) \) defined in §4.0.1. The first inequality in Theorem 4.7 implies

\[
\limsup_{r \to \infty} N(\gamma(r, \infty)) \leq N(a(X)^{-1}).
\]

If \( N(\gamma) < \infty \) then we can consider the function \( r \mapsto a_r(\gamma) \) introduced in Definition 4.17 and in view of Theorem 4.16 we have \( a(X) = a(\gamma) \), where

\[
a(\gamma) := \lim \inf_{r \to +\infty} a_r(\gamma).
\]

In general, Condition 4.3 just holds for some positive half-infinite segment \( \gamma(r, \infty) \) of \( \gamma \) for \( r \) big enough. According to Lemma 4.22 below, for any value in \( \mathcal{L}(I) \) it is possible to find a bi-infinite path \( \gamma \) with \( N(\gamma) < +\infty \), so that Equation 4.4 applies.

**Lemma 4.22.** Consider any \( a > 0 \) such that \( a^{-1} \in \mathcal{L}(I) \). Then there exists \( \pi \in R \) and data \((\lambda, \tau) \in I_\pi \) such that the bi-infinite path \( \gamma \) induced by \((\pi, \lambda, \tau)\) satisfies

\[
N(\gamma) \leq N(a^{-1}) \text{ and } a = a(\gamma).
\]

**Proof:** Set \( M := N(a^{-1}) \). Consider \( \pi \in R \) and data \((\hat{\lambda}, \hat{\tau}) \in I_\pi \) such that the surface \( \hat{X} := X(\pi, \hat{\lambda}, \hat{\tau}) \) gives \( a(\hat{X}) = a \). Let \( \tilde{\gamma} \) be the bi-infinite path induced by \((\pi, \hat{\lambda}, \hat{\tau})\) and observe that according to Equation 4.3 we have \( N(\tilde{\gamma}(r_0, +\infty)) < M \) for some \( r_0 \in \mathbb{Z} \). Modulo applying the Rauzy-Veech induction to the data \((\pi, \hat{\lambda}, \hat{\tau})\) we can assume that \( r_0 < -m \) for some \( m > r_1(M) \), where \( r_1(M) \) is the integer appearing in the statement of Proposition 13.11. We have a decomposition

\[
\tilde{\gamma}(r_0, +\infty) = \gamma_0 \ast \gamma_1 \ast \ldots \ast \gamma_k \ast \ldots
\]

where any \( \gamma_k \) is a finite positive sub-path of \( \tilde{\gamma} \) with \( N(\gamma_k) < M \). Fix \( l \in \mathbb{N} \) and according to Corollary 4.21 let \((k_n)_{n \in \mathbb{N}} \) be a subsequence with \( k_n \to +\infty \) and \( \sigma_{-l}, \ldots, \sigma_l \) be positive paths in \( \Gamma(M) \) such that for any \( n \in \mathbb{N} \) we have

\[
\gamma_{k_n+i} = \sigma_i \text{ for } |i| \leq l.
\]
Observe that \( \gamma_{k_n+1-1} = \sigma_{-1} \) ends in the element of \( R \) where \( \gamma_{k_n} = \sigma_0 \) starts, therefore we obtain a positive loop consider the concatenation

\[
\gamma^{(n)} := \gamma_{k_n} * \cdots * \gamma_{k_{n+1}-1}.
\]

Modulo taking a larger \( l \) we can assume that the concatenation \( \sigma_{-l} * \cdots * \sigma_1 \) has length larger than \( 2m \), in terms of Rauzy elementary operations. Modulo discarding some elementary arrows we can assume that \( \sigma_{-l} * \cdots * \sigma_1 \) has length exactly \( 2m \), thus let \( \nu \) and \( \delta \) be the two paths with length \( m \) such that we have

\[
\sigma_{-l} * \cdots * \sigma_1 = \nu * \delta.
\]

Proposition B.11 implies that \( \gamma^{(n)} \in \text{Per}(R, I) \) and abusing the notation we denote with the same name the corresponding bi-infinite path. We have \( N(\gamma^{(n)}) < M \), since \( \gamma^{(n)} \) is concatenation of sub-paths of \( \tilde{\gamma}(r_0, +\infty) \). Moreover, modulo shifting the base point of the loop we have

\[
\gamma^{(n)}(-m + 1, 0) = \nu \quad \text{and} \quad \gamma^{(n)}(1, m) = \delta,
\]

and we can assume without loss of generality that the base point of the loop is \( \pi \). Modulo increasing \( l \) and repeating the construction above, we can assume that \( m \) is big enough to apply Proposition B.12 (in fact the constant \( r_1 \) appearing in Propositions B.11 is actually the same as the one appearing in Proposition B.12). Consider the infinite concatenation

\[
\gamma := \cdots \gamma^{(1)} \gamma^{(1)} \gamma^{(2)} \cdots \gamma^{(n)} \gamma^{(n+1)} \cdots,
\]

where the past half-infinite part of \( \gamma \) coincides with the periodic path with period \( \gamma^{(1)} \). We obviously have \( N(\gamma) < M \). Moreover Proposition B.12 implies that \( \gamma \) is induced by data \((\pi, \lambda, \tau)\) with \((\lambda, \tau) \in I_\pi\).

Finally observe that there is some \( r_2 > 0 \) such that \( \tilde{\gamma}(r_2, +\infty) = \gamma(r_2, +\infty) \). Therefore there exists an IET \( T \) which is a section both for the vertical flow of \( X(\pi, \hat{\lambda}, \hat{\tau}) \) and the vertical flow of \( X(\pi, \lambda, \tau) \).

According to Proposition 4.13 we have

\[
a(\gamma) = a\left(X(\pi, \lambda, \tau)\right) = \mathcal{E}(T) = a\left(X(\pi, \hat{\lambda}, \hat{\tau})\right) = a.
\]

According to Lemma 4.15 any bi-infinite path \( \gamma \) with \( N(\gamma) < M \) is decomposed into packets, each packet \( \gamma_k \) being the entry of a sequence in \( \Gamma_R^z(M) \). Lemma 4.23 provides a more accurate packing, satisfying a compatibility condition with a prescribed sequence of Rauzy times, which is a technical detail needed in the proof of Propositions 4.24 and 4.26 below.

**Lemma 4.23** (Packing Lemma). Fix \( M > 0 \) and let \( \gamma \) be a bi-infinite Rauzy path with \( N(\gamma) < M \). Consider an increasing sequence of Rauzy times \( r_n \) such that \( \gamma(r_n, r_n+1-1) \) is positive for any \( n \). Then there exists a sequence \( \gamma_k \in \Gamma_R^z(M^2) \) and an increasing subsequence of indexes \( k_n \) such that for any \( n \) we have

\[
\gamma(r_n, r_n+1-1) = \gamma_{k_n} * \gamma_{k_{n+1}} * \cdots * \gamma_{k_{n+1-1}}.
\]

**Proof:** Since \( N(\gamma) < M \) then for any \( n \), any sub-path \( \nu \) of \( \gamma(r_n, r_{n+1}) \) with \( \|B_\nu\| \geq M \) is positive. If \( \|B_\gamma(r_0, r_1-1)\| < M^2 \), then we set \( \gamma_{k_0} := \gamma(r_0, r_1 - 1) \) and \( k_1 := k_0 + 1 \). Otherwise, we choose Rauzy times \( r_0 := s_0 < s_1 < \cdots < s_m := r_1 \) such that for any \( j \) with \( 0 \leq j < m \) we have \( M \leq \|B_{\gamma(s_j, s_{j+1}-1)}\| < M^2 \) and set \( k_1 := k_0 + m \) and \( \gamma_{k_0+j} := \gamma(s_j, s_{j+1}-1) \). The required sequences \( \gamma_k \) \( k \in \mathbb{Z} \) are defined replying to the same construction iteratively on \( n \).

4.4.2. *Density of Pseudo-Anosov elements*. In this section we prove the first part of Theorem 1.5.

**Proposition 4.24.** We have

\[
\mathcal{L}(I) \subset \{a^{-1}(X) ; X \in PA(I)\}.
\]

**Proof:** Consider \( a > 0 \) such that \( a^{-1} \) belongs to \( \mathcal{L}(I) \). Then set \( M := N(a^{-1}) \) and fix \( \epsilon > 0 \). The statement follows defining a periodic path \( \tilde{\gamma} \in \text{Per}(R, I) \) such that

\[
(1 - \epsilon)^2 a < a(\tilde{\gamma}) < (1 + \epsilon)^2 a.
\]
According to Lemma 4.23, there exists $\pi \in \mathcal{R}$ and data $(\lambda, \tau) \in \mathcal{I}_z$ such that the bi-infinite path $\gamma$ induced by $(\pi, \lambda, \tau)$ satisfies $N(\gamma) < M$ and $a(\gamma) = a$. Consider a sequence of instants $(r_n)_{n \in \mathbb{N}}$ such that
\[ r_n \to +\infty \text{ and } a_{r_n}(\gamma) \to a \text{ for } n \to +\infty. \]

Up to extracting a sub-sequence, we can assume that $\gamma(r_n, r_{n+1})$ is positive for any $n$. Thus, by the Packing Lemma 4.23, there exists a sequence $(\gamma_k)_{k \in \mathbb{Z}}$ in $\Gamma^z_R(M^2)$ and a subsequence $(k_n)_{n \in \mathbb{N}}$ with $k_n \to +\infty$ for $n \to +\infty$ such that for $n$ big enough we have
\[ \gamma(r_n, r_{n+1}) = \gamma_{k_n} \cdot \gamma_{k_{n+1}} \cdots \gamma_{k_{n+1}}. \]

Fix $l \in \mathbb{N}$. Corollary 4.21 implies that there exist $2l + 1$ elements $\sigma_{-l}, \ldots, \sigma_0, \sigma_1, \ldots, \sigma_l$ in $\Gamma(M^2)$ such that, up to extracting a sub-sequence of $(k_n)_{n \in \mathbb{N}}$, and hence a subsequence of the instants $(r_n)_{n \in \mathbb{N}}$, we can assume that for any $n \in \mathbb{N}$ we have
\[ \gamma_{k_n+i} = \sigma_i \text{ for all } -l \leq i \leq l. \]

Without any loss of generality, we can also assume that $k_{n+1} - k_n > 2l$.

**Step 1:** Definition of the periodic path $\hat{\gamma} \in \text{Per}(\mathcal{R}, \mathcal{I})$. Let $N$ be such that for any $n \geq N$ we have
\[ (1 - \epsilon)a < a_{r_n}(\gamma) < (1 + \epsilon)a. \]

Let $(\hat{\gamma}_k)_{k \in \mathbb{Z}}$ be the periodic sequence in $\Gamma^z$ with period $k_{N+1} - k_N$ defined setting
\[ \hat{\gamma}_{k+N} := \gamma_{k+N} \text{ for } 0 \leq k < k_{N+1} - k_N. \]

Observe that $(\hat{\gamma}_k)_{k \in \mathbb{Z}}$ belongs to $\Gamma^z_R$, and not just to $\Gamma^z$, that is the concatenation is possible. For $k \neq 0 \mod k_{N+1} - k_N$ the concatenation $\hat{\gamma}_{k-1} \cdot \hat{\gamma}_k$ is possible because $(\gamma_k)$ is obtained via Lemma 4.23. For $k = 0 \mod k_{N+1} - k_N$ just observe that by the recurrence property $\gamma_{k_{N+1}} = \sigma_1$ and $\gamma_0 = \sigma_0$. Therefore let $\hat{\gamma}$ be the bi-infinite periodic path on $\mathcal{R}$ obtained concatenating the paths in the sequence $(\hat{\gamma}_k)_{k \in \mathbb{Z}}$, that is
\[ \hat{\gamma} := \cdots \hat{\gamma}_{k-1} \cdot \hat{\gamma}_k \cdot \hat{\gamma}_{k+1} \cdots. \]

In particular, the period $\hat{\gamma}(r_{N}, r_{N+1})$ of $\hat{\gamma}$ is a positive loop at some $\pi \in \mathcal{R}$. Moreover, modulo taking a larger $l$, we can assume that the concatenation $\sigma_{-l} \cdots \sigma_l$ has length at least $2m$ in terms of elementary Rauzy operations, where $m$ is big enough to apply Proposition 4.11. Arguing as in the proof of Lemma 4.22, we get that $\hat{\gamma}(r_{N}, r_{N+1}) \in \text{Per}(\mathcal{R}, \mathcal{I})$.

**Step 2:** Extra equalities between $(\hat{\gamma}_k)_{k \in \mathbb{Z}}$ and $(\gamma_k)_{k \in \mathbb{Z}}$ from recurrence. For any $i$ with $0 \leq i \leq l$ we have $\gamma_{k_{N}-i} = \sigma_i = \gamma_{k_{N+1}-i} = \hat{\gamma}_{k_{N+1}-i} = \hat{\gamma}_{k_{N}-i}$, where the first two equalities follow from the recurrence property of the instants $k_n$, and the last two from the definition of the sequence $(\hat{\gamma}_k)_{k \in \mathbb{Z}}$ and its periodicity. The symmetric argument shows that for any $i$ with $0 \leq i \leq l$ we also have $\gamma_{k_{N+1}+i} = \hat{\gamma}_{k_{N+1}+i}$. Resuming, the choice of instants $k_n$ implies that the sequences $(\gamma_k)_{k \in \mathbb{Z}}$ and $(\hat{\gamma}_k)_{k \in \mathbb{Z}}$ coincide not only for $k_N \leq k \leq k_{N+1}$ but also for a larger interval of instants, that is
\[ \hat{\gamma}_{k+N} = \gamma_{k+N} \text{ for } -l \leq k < k_{N+1} - k_N + l. \]

**Step 3:** Areas are well approximated. Since the path $\hat{\gamma}$ is periodic with period $k_{N+1} - k_N$, there exists an instant $r_{\min}$ with $r_N \leq r_{\min} < r_{N+1}$ such that
\[ a_{r_{\min}}(\hat{\gamma}) = \min_{r_N \leq r < r_{N+1}} a_r(\hat{\gamma}) = \liminf_{r \in \mathbb{N}} a_r(\hat{\gamma}) = a(\hat{\gamma}). \]

Modulo taking a bigger $l$, we can assume that $l > m(\epsilon, M^2)$, where $m(\epsilon, M^2)$ is the positive integer appearing in Corollary 4.19. The extra equalities between $(\hat{\gamma}_k)_{k \in \mathbb{Z}}$ and $(\gamma_k)_{k \in \mathbb{Z}}$ proved in Step 2 imply that Corollary 4.19 applies to the finite segments $\gamma(r_{N}, r_{N+1})$ and $\hat{\gamma}(r_{N}, r_{N+1})$ and since $r_N \leq r_{\min} < r_{N+1}$ then we have
\[ a_{r_{\min}}(\hat{\gamma}) \leq a_{r_{N}}(\gamma) \leq (1 + \epsilon)a_{r_{N}}(\gamma) \leq (1 + \epsilon)^2a, \]
where the first inequality follows by definition of the instant $r_{\min}$, the second by the Interpolation Corollary 4.19 and the last by the choice of the instant $r_N$. Similarly we have
\[ a_{r_{\min}}(\hat{\gamma}) \geq (1 - \epsilon)a_{r_{\min}}(\gamma) \geq (1 - \epsilon)^2a, \]
where the first equality follows again from Corollary 4.19 and the second holds because $a = \liminf_{r} a_r(\gamma)$. The Proposition follows observing that $a_{r_{\min}}(\hat{\gamma}) = a(\hat{\gamma})$ and therefore
\[ (1 - \epsilon)^2 < \frac{a}{a(\hat{\gamma})} < (1 + \epsilon)^2. \]
4.4.3. **Closure of the spectrum.** In this section we prove the Proposition below, which gives the opposite inclusion to Proposition 4.24 and allows to conclude the proof of Theorem 1.5.

**Proposition 4.25.** We have

\[
\{a^{-1}(X) : X \in PA(\mathcal{I})\} \subset \mathcal{L}(\mathcal{I}).
\]

**Proof:** Let \(a^{-1}\) be an accumulation point of the values \(a^{-1}(X)\) for \(X \in PA(\mathcal{I})\). Thus there exists a sequence \(\tilde{\gamma}^{(n)} \in \text{Per}(\mathcal{R}, \mathcal{I})\) such that

\[
a = \lim_{n \to \infty} a(\tilde{\gamma}^{(n)}).
\]

Let \(r_n\) be the period of \(\tilde{\gamma}^{(n)}\) and assume without loss of generality that any finite loop \(\tilde{\gamma}^{(n)}(0, r_n)\) is based at the same \(\pi \in \mathcal{R}\). We can assume also that \(\tilde{\gamma}^{(n)}(0, r_n)\) is a positive loop at \(\pi\). Indeed it is a complete loop, otherwise \(\tilde{\gamma}^{(n)}\) is not admissible. Then positivity follows modulo concatenating at most \(2d-3\) copies of it, according to [MMY] (the Lemma at page 833).

We show that \(a^{-1} \in \mathcal{L}(\mathcal{I})\) by explicitly constructing a bi-infinite path \(\gamma\) on \(\mathcal{R}\), induced by data \((\pi, \lambda, \tau) \in \mathcal{I}_x\) and such that \(a(\gamma) = a\). Observe that since \(a > 0\) then there exists some \(\delta > 0\) such that \(a(\tilde{\gamma}^{(n)}) > \delta\) for any \(n\). Therefore according to Equation 4.3 there exists some \(M > 0\) such that

\[
N(\tilde{\gamma}^{(n)}) < M \text{ for any } n.
\]

**Step 1: Symbolic coding.** For any \(n\), according to the Packing Lemma 4.23 applied to \(\tilde{\gamma}^{(n)}\), there exists a positive integer \(p_n\) and finite paths \(\tilde{\gamma}_0^{(n)}, \ldots, \tilde{\gamma}_{p_n-1}^{(n)}\) in \(\Gamma(M^2)\) such that

\[
\tilde{\gamma}^{(n)}(0, r_n) = \tilde{\gamma}_0^{(n)} \ast \tilde{\gamma}_1^{(n)} \ast \cdots \tilde{\gamma}_{p_n-1}^{(n)}.
\]

In other words, the periodic bi-infinite path \(\tilde{\gamma}^{(n)}\) is coded by the periodic sequence \((\tilde{\gamma}_k^{(n)})_{k \in \mathbb{Z}}\) of period \(p_n\) in the sub-shift space \(\Gamma^Z_{\mathcal{R}}(M^2)\) defined by

\[
\tilde{\gamma}_k^{(n)} := \tilde{\gamma}^{(n)} \mod p_n \text{ for any } k \in \mathbb{Z}.
\]

We stress that \(p_n\) is the period of \(\tilde{\gamma}^{(n)}\) in terms of Rauzy elementary operations, whereas \(p_n\) is the period of its coding \((\tilde{\gamma}_k^{(n)})_{k \in \mathbb{Z}}\) in terms if the sub-shift time. We assume without loss of generality that \(p_n \geq n\) for any \(n \in \mathbb{N}\).

**Step 2: Definition of the path \(\gamma\).** Apply Lemma 4.20 to the sequence of sequences \((\tilde{\gamma}_k^{(n)})_{k \in \mathbb{Z}}\) in \(\Gamma^Z_{\mathcal{R}}(M^2)\), indexed by \(n \in \mathbb{N}\). The Lemma provides a sequence \((\sigma_k)_{k \in \mathbb{Z}}\) in \(\Gamma^Z_{\mathcal{R}}(M^2)\) and a sequence of integers \(n(l)\), indexed by \(l \in \mathbb{N}\) and with \(n(l) \to +\infty\), such that for any \(l \in \mathbb{N}\) we have

\[
\tilde{\gamma}_k^{(n(l))} = \sigma_k \text{ for all } -l \leq k \leq l \text{ and all } j \geq l.
\]

In order to simplify the notation, we assume that \(n(l) = l\), that is we assume that \((\tilde{\gamma}_k^{(n)})_{k \in \mathbb{Z}}\) satisfies itself the conclusion of Lemma 4.20. Thus, resuming, for any \(l \in \mathbb{N}\) we have

\[
\tilde{\gamma}_k^{(n)} = \sigma_k \text{ for all } -l \leq k \leq l \text{ and all } n \geq l.
\]

Fix \(l \in \mathbb{N}\). We construct a sequence \((\gamma_k)_{k \in \mathbb{Z}}\) in \(\Gamma^Z_{\mathcal{R}}(M^2)\) by concatenating the periods of the sequences \((\tilde{\gamma}_k^{(n)})_{k \in \mathbb{Z}}\) with \(l \geq n\) as follows.

1. Set \(P_l := p_l\) and for \(n > l\) set \(P_n := \sum_{i=l}^n p_i\), so that \(P_n - P_{n-1} = p_n\).
2. For \(k < P_l\) define the entries \(\gamma_k\) repeating infinitely the period \(\tilde{\gamma}_0^{(l)}, \ldots, \tilde{\gamma}_{p_l-1}^{(l)}\) of \((\tilde{\gamma}_k^{(l)})_{k \in \mathbb{Z}},\) that is

\[
\gamma_k := \tilde{\gamma}_k^{(l)} \mod P_l.
\]
3. For \(n > l\) and any \(k\) with \(P_{n-1} \leq k < P_n\) define the entries \(\gamma_k\) to be equal to the period \(\tilde{\gamma}_0^{(n)}, \ldots, \tilde{\gamma}_{p_n-1}^{(n)}\) of \((\tilde{\gamma}_k^{(n)})_{k \in \mathbb{Z}},\) that is

\[
\gamma_k := \tilde{\gamma}_{k-P_{n-1}}^{(n+1)}.
\]
For any $n \geq l$ we have $\gamma_{p_n-1} = \tilde{\gamma}_{(n)} = \sigma_1 - 1$ and $\gamma_{p_n} = \tilde{\gamma}_{(n)} = \sigma_0$, since the sequence $(\tilde{\gamma}_k^{(n)})_{k \in \mathbb{Z}}$ is periodic with period $p_n$. Hence the concatenation $\gamma_{p_n-1} * \gamma_{p_n} = \sigma_1 * \sigma_0$ is admissible, because $(\sigma_k)_{k \in \mathbb{Z}}$ belongs to $\Gamma_\mathcal{R}$. Therefore we can define an bi-infinite path $\gamma$ on $\mathcal{R}$ setting

$$\gamma := \ldots * \gamma_{k+1} * \gamma_k * \gamma_{k-1} * \ldots.$$ 

Finally recall that any $\tilde{\gamma}_{(n)}$ belongs to $\text{Per} (\mathcal{R}, \mathcal{I})$ and that $N(\tilde{\gamma}_{(n)})$ is uniformly bounded. Assume without loss of generality that $l$ is such that the concatenation $\sigma_{-l} * \cdots * \sigma_1$ has even length $2m$ in terms of Rauzy elementary operations. Then let $\nu$ and $\delta$ be the two paths with length $m$ such that we have

$$\sigma_{-l} * \cdots * \sigma_1 = \nu \delta.$$

Modulo taking a bigger $l$ we can assume that $m$ is big enough to apply Proposition 4.24. Therefore the path $\gamma$ defined above is induced by data $(\pi, \lambda, \tau)$ with $(\lambda, \tau) \in \mathcal{I}_n$.

**Step 3:** Extra equalities between $\gamma$ and $(\tilde{\gamma}_{(n)})$. Here we consider any $n \in \mathbb{N}$ and we show that for all $n > l$ we have

$$\gamma_j = \tilde{\gamma}_{j-P_{n-1}}^{(n)}$$

for $P_{n-1} \leq j < P_n$. When $P_{n-1} \leq j < P_n$, the required equality follows trivially from the definition of the sequence $(\gamma_j)$ in Step 2. For $P_n \leq j < P_n + l$, setting $j = P_n + k$ we have

$$\gamma_j = \gamma_{P_n+k} = \tilde{\gamma}_{k+1}^{(n)} = \sigma_k = \tilde{\gamma}_k^{(n)} = \tilde{\gamma}_{(n)} = \tilde{\gamma}_{j-P_{n-1}}^{(n-1)},$$

where the first and last equality follow from the change of index, the second form the definition of $\gamma_{P_n+k}$, the third and the fourth from Lemma 4.20 (and since $|k| < l$) and finally the fifth holds because $(\tilde{\gamma}_{k}^{(n)})$ is periodic with period $p_n$. The proof for $P_n - 1 \leq j < P_n$ is similar.

**Step 4:** Approximation properties of $\gamma$. For any $n$ let $R_n$ be the length of the path $\gamma(0, P_n)$ in terms of elementary Rauzy operations, so that $R_n - R_{n-1} = r_n$ is the period of $\tilde{\gamma}_{(n)}$. Recall that we have $\gamma(R_{n-1}, R_n) = \tilde{\gamma}_{(n)}(0, R_n - R_{n-1})$, by definition of $\gamma$, and that $N(\tilde{\gamma}_{(n)}) < M$ and $N(\gamma) < M^2$. Fix $\epsilon > 0$. According to Step 3 the assumptions in Corollary 4.19 are satisfied, thus there exists $l = l(\epsilon, M^2)$ such that for any $n > l$ and any $R_n \leq r < R_{n+1}$ we have

$$\left| \frac{a_r(\gamma)}{a_r - R_n(\tilde{\gamma}_{(n)})} - 1 \right| \leq \epsilon.$$ 

**Step 5:** Computation of $a(\gamma)$. Each $\tilde{\gamma}_{(n)}$ has period $R_n - R_{n-1} = r_n$ in terms of elementary Rauzy operations, thus there exists an integer $r_{\min}(n)$ with $R_{n-1} \leq r_{\min}(n) < R_n$ such that

$$a_{r_{\min}(n) - R_{n-1}}(\tilde{\gamma}_{(n)}) = \min_{R_{n-1} \leq r < R_n} a_{r - R_{n-1}}(\tilde{\gamma}_{(n)}) = a(\gamma).$$

Since $a(\tilde{\gamma}_{(n)}) \to a$ for $n \to \infty$, then the approximation established at step 4 implies

$$1 - \epsilon)a \leq \liminf_n a_{r_{\min}(n)}(\gamma) \leq (1 + \epsilon)a.$$ 

and hence $a(\gamma) = \liminf_{n \to \infty} a_r(\gamma) \leq (1 + \epsilon)a$. On the other hand, for any $n$ and any $r$ with $R_{n-1} \leq r < R_n$ the approximation at Step 4 implies

$$a_r(\gamma) \geq (1 - \epsilon)a_{r - R_{n-1}}(\tilde{\gamma}_{(n)}) \geq (1 - \epsilon)a_{r_{\min}(n) - R_{n-1}}(\tilde{\gamma}_{(n)}) = (1 - \epsilon)a(\tilde{\gamma}_{(n)}).$$

Since $a(\tilde{\gamma}_{(n)}) \to a$ for $n \to \infty$, then we get $\liminf_{n \to \infty} a_r(\gamma) \geq (1 - \epsilon)a$. Moreover the two estimations for $\liminf_r a_r(\gamma)$ hold for any $\epsilon$, then the Proposition is proved.

### 4.4.4. End of the proof.

**Proof of Theorem 1.5** The inclusion in Proposition 4.24 proves the density of values of periodic elements, that is part (2) of Theorem 1.5. Proposition 4.20 gives the opposite inclusion, therefore we get

$$\mathcal{L}(\mathcal{I}) = \{a^{-1}(X) : X \in PA(\mathcal{I})\},$$

which implies in particular that $\mathcal{L}(\mathcal{I})$ is a closed subset of $\mathbb{R}$. 

\[\square\]
5. Hall’s ray for Square-tiled surfaces

5.1. Background on square-tiled surfaces. We recall some basic facts on square-tiled surfaces. We closely follow \[\text{Hu,L}\]. A translation covering is a map \(f : X_1 \to X_2\) of translation surfaces that

1. is topologically a ramified covering;
2. maps conical singularities of \(X_1\) to conical singularities of \(X_2\);
3. is locally a translation in the translation charts of \(X_1\) and \(X_2\).

A square-tiled surface, also called origami, is a translation cover of the standard torus \(\mathbb{R}^2/\mathbb{Z}^2\) marked at the origin. Such surfaces are tiled by squares and in particular are Veech surfaces. Square-tiled surfaces can be characterized by their Veech group, according to the following Theorem (for a proof see Appendix C in \[\text{Hu,L}\]).

**Theorem (Gutkin-Judge).** A translation surface \(X\) is square-tiled if and only if its Veech group \(\text{SL}(X)\) shares a finite-index subgroup with \(\text{SL}(2, \mathbb{Z})\).

The following Lemma gives a characterization of square-tiled surfaces in terms of the subgroup \((\text{Hol}(X))\) of \(\mathbb{R}^2\) spanned by the elements of \(\text{Hol}(X)\).

**Lemma 5.1** (Lemma 2.1 in \[\text{Hu,L}\]). A translation surface \(X\) is square-tiled if and only if \((\text{Hol}(X))\) is a rank two sublattice of \(\mathbb{Z}^2\).

5.1.1. Reduced origamis. A square-tiled surface \(X\) is called reduced if \((\text{Hol}(X)) = \mathbb{Z}^2\). Note that this is not always the case, for example \(X = \mathbb{Z}^2/(\mathbb{Z} + 2 \cdot \mathbb{Z})\) satisfies \((\text{Hol}(X)) = \mathbb{Z} + 2 \cdot \mathbb{Z}\), moreover many other examples can be constructed as translation covers of such \(X\). For reduced square-tiled surfaces we have a very concrete way to represent Veech groups, according to the following Lemma.

**Lemma 5.2** (Lemma 2.3 in \[\text{Hu,L}\]). For any square-tiled surface \(X\) we have

\[
\text{SL}(X) < \text{SL}(\mathbb{R}^2/(\text{Hol}(X))).
\]

In particular, if \(X\) is reduced, then \(\text{SL}(X) < \text{SL}(2, \mathbb{Z})\).

The following Lemma shows that being reduced is preserved by the action of \(\text{SL}(2, \mathbb{Z})\).

**Lemma 5.3** (Lemma 2.4 in \[\text{Hu,L}\]). The \(\text{SL}(2, \mathbb{Z})\)-orbit of a reduced square-tiled surface with \(N\) squares is the set of reduced square-tiled surfaces with \(N\) squares in its \(\text{SL}(2, \mathbb{R})\)-orbit.

5.1.2. Action of \(\text{SL}(2, \mathbb{Z})\) for reduced origamis. Consider the set of generators \(\{T, V\}\) of \(\text{SL}(2, \mathbb{Z})\), where

\[
T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad V := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.
\]

According to Lemma 5.3 the elements of \(\text{SL}(2, \mathbb{Z}) \cdot X\) are those reduced origamis in \(\text{SL}(2, \mathbb{R}) \cdot X\) with the same number of squares as \(X\), thus they are a finite set, which is identified with

\[
E(X) := \text{SL}(2, \mathbb{Z})/\text{SL}(X).
\]

The action of \(T\) and \(V\) passes to the quotient \(E(X)\) and is represented by a oriented graph \(G(X)\) whose vertices are the elements of \(E(X)\) and whose oriented edges correspond to the operations \(Y \mapsto T \cdot Y\) and \(Y \mapsto V \cdot Y\) for \(Y \in E(X)\).

5.1.3. Cusps. Let \(X\) be a reduced origami, thus in particular a Veech surface. Then \(\text{SL}(2, \mathbb{R}) \cdot X\) is isometric to the unitary tangent bundle of \(\mathbb{H}/\text{SL}(X)\). Any cusp of \(\mathbb{H}/\text{SL}(X)\) is isometric to the quotient of a strip \(\{z \in \mathbb{H}; |\Im(z)| < \delta, \Im(z) > M\}\) by the translation \(z \mapsto z + \delta\), where \(\delta > 0\) is called the width of the cusp and \(M > 0\) is any real number large enough. The cusps of \(\mathbb{H}/\text{SL}(X)\) therefore correspond to conjugacy classes under \(\text{SL}(X)\) of its primitive parabolic elements, that is the elements in \(\text{SL}(X)\) with trace equal to \(\pm 2\), where primitive means not powers of other parabolic elements of \(\text{SL}(X)\).

If \(X\) is a reduced origami the eigendirections of parabolic elements of \(\text{SL}(X)\) are exactly the elements of \(\mathbb{Q}\). Therefore the cusps of \(\mathbb{H}/\text{SL}(X)\) correspond to equivalence classes for the homographic action of \(\text{SL}(X)\) on \(\mathbb{Q}\) (see 5.1.3). According to the following Lemma (see Lemma 2.5 in \[\text{Hu,L}\]) the cusps of \(\mathbb{H}/\text{SL}(X)\) correspond to the orbits of \(T\) on the set \(E(X)\) defined in 5.1.2.

**Lemma 5.4** (Zorich). Let \(X\) be a reduced origami. Then the cusps of \(\mathbb{H}/\text{SL}(X)\) are in bijection with the \(T\)-orbits of \(\text{SL}(2, \mathbb{Z}) \cdot X\).
5.1.4. Continued fraction. Let \( \alpha = a_0 + [a_1, a_2, \ldots] \) be the continued fraction expansion of \( \alpha \), where \( a_0 \in \mathbb{Z} \) and \( a_n \in \mathbb{N}^* \). The \( n \)-th Gauss approximation of \( \alpha \) is the rational number
\[
\frac{p_n}{q_n} := a_0 + [a_1, a_2, \ldots, a_n].
\]

At the projective level, the affine action of \( \text{SL}(2, \mathbb{Z}) \) on translation surfaces corresponds to the homographic action on co-slopes, that is
\[
A \cdot \alpha := \frac{a\alpha + b}{c\alpha + d} \quad \text{for } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \alpha = \tan(\theta),
\]
where \( -\infty \leq \alpha \leq +\infty \) is the co-slope of a line which forms an angle \( -\pi/2 \leq \theta \leq \pi/2 \) with respect to the vertical direction measured clockwise. In particular, \( \theta = 0 \) correspond to the vertical direction and to the co-slope 0, while the horizontal direction on \( X \) corresponds to the co-slope \( \infty \). The following Lemma describes the relation between the continued fraction and the projective action of \( \text{SL}(2, \mathbb{Z}) \).

**Lemma 5.5.** If \( \alpha = a_0 + [a_1, a_2, \ldots] \) then the sequence of co-slopes \( p_n/q_n \) of the Gauss approximations of \( \alpha \) is given by
\[
\begin{align*}
\frac{p_n}{q_n} &= T^{a_0} \circ V^{a_1} \circ \cdots \circ V^{a_{n-1}} \circ T^{a_n} \cdot 0, \quad \text{for even } n \\
\frac{p_n}{q_n} &= T^{a_0} \circ V^{a_1} \circ \cdots \circ V^{a_{n-1}} \circ V^{a_n} \cdot \infty \quad \text{for odd } n.
\end{align*}
\]

**Proof:** Just recall that the recursive relations satisfied by the convergents (see for example \([Kl]\)) show that the sequence \( (p_n, q_n) \) is obtained by setting \( (p_{-2}, q_{-2}) = (0, 1) \) and \( (p_{-1}, q_{-1}) = (1, 0) \) and then applying for any \( k \in \mathbb{N} \) the recursive relations
\[
\begin{pmatrix} p_{2k-1} & p_{2k} \\ q_{2k-1} & q_{2k} \end{pmatrix} = \begin{pmatrix} p_{2k} & p_{2k-2} \\ q_{2k} & q_{2k-2} \end{pmatrix} \circ T^{a_{2k}} \quad \text{and} \quad \begin{pmatrix} p_{2k+1} & p_{2k} \\ q_{2k+1} & q_{2k} \end{pmatrix} = \begin{pmatrix} p_{2k-1} & p_{2k} \\ q_{2k-1} & q_{2k} \end{pmatrix} \circ V^{a_{2k+1}}.
\]

The following Lemma is also well-known (see for example Theorem 19, Chapter 6 in \([Kl]\)).

**Lemma 5.6.** Fix \( \alpha \in \mathbb{R} \) and consider a pair of integers \((q, p)\).

If \( |q\alpha - p| < \frac{1}{2q} \) then \((p, q) = (p_n, q_n)\) for some \( n \in \mathbb{N} \).

5.2. Renormalized formula for reduced square-tiled surfaces. Let \( X \) be a reduced square-tiled surface. According to Lemma 5.2 the lattice \( (\text{Hol}(X)) \) spanned by the relative periods of \( X \) equals \( \mathbb{Z}^2 \). Therefore, if \( p \) is any conical singularity of \( X \), the map \( \zeta \mapsto \int_{p}^{\zeta} w \) is defined modulo elements of \( (\text{Hol}(X)) = \mathbb{Z}^2 \) and we have a well-defined map \( \rho : X \to \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2 \) setting
\[
(5.1) \quad \rho(\zeta) := \int_{p}^{\zeta} w \mod \mathbb{Z}^2.
\]

**Lemma 5.7.** For any origami \( X \) there exists a reduced origami \( X_{\text{red}} \) in \( \text{GL}(2, \mathbb{R}) \cdot X \) and moreover we have
\[
\mathcal{L}(X) = \mathcal{L}(X_{\text{red}}).
\]

**Proof:** Let \( \{f_1, f_2\} \) be a basis of \( (\text{Hol}(X)) \) and \( G \) be the matrix sending it to the standard basis \( \{e_1, e_2\} \) of \( \mathbb{Z}^2 \). Then \( X_{\text{red}} = G \cdot X \) is reduced. Moreover we have \( G = \det(G) \cdot G' \) with \( G' \in \text{SL}(2, \mathbb{R}) \), therefore
\[
\mathcal{L}(X_{\text{red}}) = \mathcal{L}(G' \cdot X) = \mathcal{L}(X),
\]
where the first inequality follows because, on the stratum, the function \( X \mapsto a(X) \) is invariant under homothetic transformations (since in the definition \([L3]\) of \( a(X) \) we renormalize by \( \text{Area}(X) \)) and the second follows because \( G' \cdot X \) belongs to \( \text{SL}(2, \mathbb{R}) \cdot X \) (see Lemma 1.4). \( \square \)
5.2.1. **Multiplicities.** Let $X$ be a reduced origami and $\rho : X \to \mathbb{T}^2$ be the covering in Equation (5.1).

**Definition 5.8 (Multiplicities of a rational direction).** If $\gamma : [0,1] \to X$ is a saddle connection for $X$, we define its multiplicity $m(\gamma)$ as the degree of the map $t \mapsto \rho \circ \gamma(t)$. The multiplicity of a rational direction with co-slope $p/q$ over the surface $X$ is the minimal multiplicity among all saddle connections on $X$ with the same co-slope $p/q$, that is the number $m_X(p/q)$ defined by

$$m_X(p/q) := \min \{m(\gamma); \quad \gamma \text{ saddle connection with } \text{Hol}(\gamma) \cap (p+iq) = 0\}.$$ 

Recall from [5.14] that the action of $\text{SL}(2,\mathbb{Z})$ on translation surfaces induces the homographic action on co-slopes. The Remark below is a direct consequence of covariance of the multiplicity under $\text{SL}(2,\mathbb{Z})$, that is

$$m_X(p/q) = m_{A \cdot X}(A \cdot (p/q)).$$

**Remark 5.9 (Multiplicities are cusp invariant).** If the co-slopes $p/q$ and $p'/q'$ are in the same cusp then there exists some $A \in \text{SL}(X)$ such that $p'/q' = A \cdot (p/q)$ and therefore

$$m_X(p/q) = m_X(p'/q').$$

Fix a reduced origami $X$ and consider the parametrization $L_X : \mathbb{R} \to \mathcal{L}(X)$ of its Lagrange Spectrum via the function $\alpha \mapsto L_X(\alpha)$ defined by Equation (1.5). For convenience of notation we also introduce the function $a_X(\alpha) := L_X^{-1}(\alpha)$. Let $N = N(X)$ be the number of squares of $X$, so that $\text{Area}(X) = N$.

**Lemma 5.10.** Let $X$ be a reduced origami. We have

$$L_X(\alpha) = N \cdot \limsup_{q,p \to \infty} \frac{1}{m_X(p/q) \cdot q \cdot |q\alpha - p|}.$$

**Proof:** Fix a co-slope $\alpha$ in $\mathbb{R}$ and set $\theta = \arctan(\alpha)$. Remark that $R_\theta$ sends lines in direction $\theta$ on $X$ (that is lines which form an angle $\theta$ with the vertical line measured clockwise) to vertical lines on $R_\theta X$. For any period $v$ in $\text{Hol}(X)$ call $a_{X,\alpha}(v)$ its area with respect to $R_\theta X$, that is $a_{X,\alpha}(v) := |\Re(e^{i\theta} v)| \cdot |\Im(e^{i\theta} v)|$. If $\gamma$ is the saddle connection in $X$ corresponding to the period $v$ in $\text{Hol}(X)$, let $\rho(v)$ in $\text{Hol}(\mathbb{T}^2)$ be the period corresponding to the saddle connection $\rho(\gamma)$ in $\mathbb{T}^2$. By definition of multiplicity of a saddle connection, we have

$$a_{X,\alpha}(v) = m_X^2(v) \cdot a_{\mathbb{T}^2,\alpha}(\rho(v)).$$

Let us show that if $w$ varies in $\text{Hol}(\mathbb{T}^2)$ we have

$$\lim_{\Im(w) \to \infty} \frac{a_{\mathbb{T}^2,\alpha}(w)}{q \cdot |q\alpha - p|} = 1.$$ 

Given $w \in \text{Hol}(\mathbb{T}^2)$, since $w$ is a period of $\mathbb{T}^2$ we have $w = p + iq$, where $p$ and $q$ are co-prime integers. Since the line in direction $\theta$ has co-slope $\alpha$, it is parametrized by $(at, t)$ as $t \in \mathbb{R}$. Thus, in particular it contains the point $(q\alpha, q)$. This shows that $q|q\alpha - p|$ is the area of a parallelogram $P$ through the origin which has $(q\alpha, q)$ and $(p, q)$ as vertices. Since $a_{\mathbb{T}^2,\alpha}(w)$ is the area of a rectangle which can be written as the union of $P$ and two triangles of area $\frac{1}{2}|q\alpha - p| \cos \theta \sin \theta$, the asymptotics in (5.4) follows. Thus, letting $v$ vary in $\text{Hol}(X)$ and using (5.3) and (5.4), we have

$$a(R_\theta \cdot X) = \liminf_{\Im(e^{i\theta} v) \to \infty} \frac{a_{X,\alpha}(v)}{\text{Area}(X)} = \frac{1}{N} \liminf_{q,p \to \infty} m_X^2(p/q) \cdot q \cdot |q\alpha - p|.$$ 

\hfill \Box

5.2.2. **Renormalized formula for origamis.** The following Lemma, already proved by Perron in 1921, is well-known (see for example [Mor] for the proof).

**Lemma 5.11.** For any $\alpha$ in $\mathbb{R}$ we have

$$L_{\mathbb{T}^2}(\alpha) = \limsup_n \left( \left\lfloor a_{n-1}, \ldots, a_q \right\rfloor + a_n + \left\lfloor a_{n+1}, a_{n+2}, \ldots \right\rfloor \right).$$
Lemma 5.11 enables to compute the classical Lagrange spectrum via the continued fraction. In this paragraph we show that the same is possible for reduced origamis under the assumption on $\alpha$ stated in Theorem 5.12 below. To simplify the notation, set

$$L(A, n) := [a_n, \ldots, a_1] + a_{n+1} + [a_{n+2}, a_{n+3}, \ldots],$$

where $A = (a_k)_{k \in \mathbb{N}^*}$ denotes the sequence of entries of the continued fraction expansion of the irrational slope $\alpha$, that is $\alpha = a_0 + [a_1, a_2, \ldots]$. It is an easy computation (see for example Proposition 1.4, page 58 in [Moi]) to check that for the $n$-th approximation $p_n/q_n = a_0 + [a_1, \ldots, a_n]$ of $\alpha$ we have

$$L(A, n) = \frac{1}{q_n \cdot |q_n \alpha - p_n|}.$$

Set also

$$M_X(\alpha) := \max_{n \in \mathbb{N}} m_X\left(\frac{p_n}{q_n}\right).$$

**Theorem 5.12.** Let $X$ be a reduced origami and $N$ be the number of squares of $X$. If $\alpha \in \mathbb{R}$ satisfies $L_{T^2}(\alpha) > 2M_X(\alpha)^2$ then we have

$$L_X(\alpha) = N \cdot \limsup_{n \to \infty} \frac{[a_n, \ldots, a_1] + a_{n+1} + [a_{n+2}, a_{n+3}, \ldots]}{m_X^2(p_n/q_n)}.$$ 

**Proof:** Set $M := M_X(\alpha)$ and fix $\epsilon > 0$ such that $L_{T^2}(\alpha) > 2M^2 + 2\epsilon$. Since Lemma 5.11 gives $L_{T^2}(\alpha) = \limsup_n L(A, n)$ then there are arbitrary large integers $n$ such that $L(A, n) > L_{T^2}(\alpha) - \epsilon$. For any such $n$, since $1 \leq m_X(p_n/q_n) \leq M$, we have

$$\frac{N}{m_X^2(p_n/q_n) \cdot q_n \cdot |q_n \alpha - p_n|} = \frac{L(A, n)}{m_X^2(p_n/q_n)} > \frac{2N}{2M^2} > 2N.$$ 

On the other hand, consider a pair $(q, p)$ such that $p/q$ does not belong to the sequence of Gauss approximations $p_n/q_n$ of $\alpha$. Lemma 5.6 implies $|q\alpha - p| > 1/(2q)$ and therefore, since $m_X^2(p/q) \geq 1$ for any $q$ and $p$, we get

$$\frac{N}{m_X^2(p/q) \cdot q \cdot |q\alpha - p|} < 2N.$$ 

Finally we apply the factorized formula in Lemma 5.10. The last two inequalities imply

$$\limsup_{q, p} \frac{N}{m_X^2(p/q) \cdot q \cdot |q\alpha - p|} = \limsup_{p/q \in \mathbb{Q}} \frac{N}{m_X^2(p_n/q_n) \cdot q_n \cdot |q_n \alpha - p_n|}.$$ 

5.3. Hall’s Ray for square-tiled surfaces. We present here below a constructive proof for the existence of an Hall ray for origamis (see Theorem 5.10 below) which is an adaptation of the original Hall’s argument for the classical Lagrange spectrum ([H], see also Theorem 3, Chapter 4 in [CP]).

Let $X$ be a reduced origami. The maximal and the minimal multiplicity over all rational direction of $X$ are respectively

$$M^+(X) := \max_{p/q \in \mathbb{Q}} m_X(p/q) \text{ and } M^-(X) := \min_{p/q \in \mathbb{Q}} m_X(p/q).$$

**Remark 5.13.** It is easy to find a reduced origami $X$ with $M^+(X) > 1$. For example if $X$ is reduced genus 2 origami with $N(X) = 6$ or $N(X) = 7$, that is with 6 or 7 squares, then we have $M^+(X) = 2$. It is possible to check that the Lagrange Spectra of these examples are non-trivial generalizations of the classical Lagrange Spectrum $\mathcal{L}$, that is

$$\mathcal{L}(X) \neq N(X) \cdot \mathcal{L}.$$ 

On the other hand, we do not know any example of a reduced square-tiled surface $X$ such that $M^-(X) > 1$.

Consider the set of generators $\{T, V\}$ for $\text{SL}(2, \mathbb{Z})$. For any finite word $A = (a_1, \ldots, a_n)$ with $n$ elements, where $a_j \in \mathbb{N}^*$ for any $1 \leq j \leq n$, define $g(A) \in \text{SL}(2, \mathbb{Z})$ setting

$$g(A) := V^{a_1} \circ T^{a_2} \circ \cdots \circ V^{a_{n-1}} \circ T^{a_n} \text{ if } n \text{ is even}$$

$$g(A) := V^{a_1} \circ T^{a_2} \circ \cdots \circ T^{a_{n-1}} \circ V^{a_n} \text{ if } n \text{ is odd}.$$
The sequence \((a_1, \ldots, a_n)\) correspond to the rational slope \([a_1, \ldots, a_n] \in (0, 1)\). Applying Equation \eqref{eq:property} and Lemma \ref{lem:property}, one gets immediately the following Lemma.

**Lemma 5.14.** For any reduced origami \(X\) and any rational slope \([a_1, \ldots, a_n]\) we have

\[
m_X([a_1, \ldots, a_n]) = m_{g^{-1}(a_1, \ldots, a_n)}(0) \text{ if } n \text{ is even}
\]

\[
m_X([a_1, \ldots, a_n]) = m_{g^{-1}(a_1, \ldots, a_n)}(\infty) \text{ if } n \text{ is odd}.
\]

5.3.1. **The graph \(G_{\text{even}}(X)\).** Consider the finite set \(E(X)\) and let \(\text{Card}(E(X))\) be its cardinality. Recall that the cusps of \(G\) are in bijection with the orbits of \(T\) on \(E(X)\), according to Lemma \ref{lem:property}. For any \(Y \in \text{SL}(2, \mathbb{Z}) \cdot X\) denote \(w(Y)\) the minimal \(w \in \mathbb{N}^*\) such that \(T^w \in \text{SL}(Y)\). Observe that we have \(V = R \circ T^{-1} \circ R^{-1}\), where \(R\) is the counterclockwise rotation by an angle \(\pi/2\), that is

\[
R := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

It follows that \(w = w(R \cdot Y)\) is the minimal \(w \in \mathbb{N}^*\) such that \(V^w \in \text{SL}(Y)\). Let \(p(X)\) be the maximal cyclic order or elements of \(E(X)\) under the action of \(T\). Equivalently, \(p(X)\) is the maximal width of all cusps of \(\mathbb{H}/\text{SL}(X)\).

Recall from §5.1.2 that the action of \(T\) and \(V\) on \(E(X)\) is represented by the edges of the oriented graph \(G(X)\). In this section it is more useful to consider a different oriented graph, denoted \(G_{\text{even}}(X)\), whose vertices are the elements of \(E(X)\) and whose oriented edges are represented by the action of \(g(A)\) for elementary even words, that is the words of the form

\[
A = (a_1, a_2) \text{ with } 1 \leq a_1 < p(X) \text{ and } 1 \leq a_2 < p(X).
\]

The graph \(G_{\text{even}}(X)\) is a finite graph, non necessarily connected. It is easy to verify that the action of elementary even words is recurrent on \(G_{\text{even}}(X)\), that is for any \(Y_1\) and \(Y_2\) in \(E(X)\) and any finite sequence of elementary even words \(A_1, \ldots, A_k\) such that \(Y_2 = g(A_1 \ast \cdots \ast A_k) \cdot Y_1\) there exists a finite sequence of elementary even words \(B_1, \ldots, B_l\) such that \(Y_1 = g(B_1 \ast \cdots \ast B_l) \cdot 2_1\) (since if there is path from \(Y_2\) to \(Y_1\) of odd length, one can concatenate it with a loop of odd length from \(Y_1\) to itself).

Define \(E_{\text{even}}(X)\) as the subset of \(E(X)\) of the vertices of \(G_{\text{even}}(X)\) which belong to the same connected component of \(X\). Fix an element \(X_{\text{min}} \in E(X)\) such that the vertical direction has minimal multiplicity, that is \(m_{X_{\text{min}}}(0) = M^{-}(X)\). Observing that \(L(X) = L(X')\) for any \(X' \in \text{SL}(2, \mathbb{Z}) \cdot X\), modulo changing the initial origami \(X\) inside \(\text{SL}(2, \mathbb{Z}) \cdot X\) we can assume that

\[
X_{\text{min}} \in E_{\text{even}}(X).
\]

**Lemma 5.15.** For any two even length words \(A = (a_1, \ldots, a_{2i})\) and \(B = (b_1, \ldots, b_{2j})\) there exists an integer \(i \leq \text{Card}(E(X))\) and a finite even length word \(D = (d_1, \ldots, d_{2i})\) whose entries satisfy

\[
d_j < p(X) \text{ for } j = 1, \ldots, 2i
\]

such that the co-slope corresponding to the concatenated word \(A \ast D \ast B\) satisfies

\[
m_{X}([A \ast D \ast B]) = M^{-}(X).
\]

**Proof:** According to Lemma \ref{lem:property}, it is enough to find a finite word \(D\) as in the statement such that the concatenation \(A \ast D \ast B\), which has even length \(2(r + s + i)\), satisfies \(g(A \ast D \ast B) \cdot X_{\text{min}} = X\). Since \(g(A \ast D \ast B) = g(A) \cdot g(D) \cdot g(B)\) then the last condition is equivalent to

\[
g(D) \cdot (g(B) \cdot X_{\text{min}}) = g(A)^{-1} \cdot X,
\]

thus the required word \(D\) exits because both \(g(B) \cdot X_{\text{min}}\) and \(g(A)^{-1} \cdot X\) belong to the same connected component of \(G_{\text{even}}(X)\). Observe that \(G_{\text{even}}(X)\) has at most as many vertices as \(G(X)\), thus the bound on \(i\) follows. Finally the bound on the entries \(d_j\) of \(D\) is trivial in view of the considerations at the beginning of this section \ref{sec:5.3.1}. \(\square\)
5.3.2. Constructive proof of the existence of Hall’s ray.

**Theorem 5.16.** Let $X$ be a reduced origami. Then $L(X)$ contains the half-line $[r(X), +\infty)$, where

$$r(X) = \frac{N}{M^{-}(X)^2} \cdot \max \{2M^{+}(X)^2 + 1, 7, p(X) + 2 \}.$$

**Proof:** Let $x$ be any real number with $x > r(X) \cdot M^{-}(X)^2 \cdot N^{-1}$, where $r(X)$ is as in the statement. According to a classical result due to Hall ([3], see also Theorem 1, Chapter 4 in [4]), there exist $x_0 \in \mathbb{N}$ and two sequences $A = (a_k)_{k \in \mathbb{N}}$ and $B = (b_k)_{k \in \mathbb{N}}$ with $a_k, b_k \in \{1, 2, 3, 4 \}$ for any $k \in \mathbb{N}$ such that

$$x = [a_1, a_2, \ldots] + x_0 + [b_1, b_2, \ldots].$$

Observe that the assumption on $x$ and the definition of $r(X)$ imply that $x_0 \geq 2M^{+}(X)^2$. We define inductively an infinite word $C = (c_n)_{n \in \mathbb{N}}$ and a subsequence $\{n(m)\}_{m \in \mathbb{N}}$ such that for any $m$ we have

$$m \cdot L(X) = M^{-}(X) \quad \text{and} \quad c_{n(m)+1} = x_0.$$

**Step one.** Set $A := (x_0, a_1)$ and $B := (b_2, b_1)$. Let $i(1) \leq \text{Card}(E(X))$ be the positive integer and $D = (d_{1}, \ldots, d_{2i(1)})$ be the finite word with $d_j \leq p(X)$ for any $1 \leq j \leq 2i(1)$ provided by Lemma 5.15. Condition (5.5) is satisfied if we define the initial block $(c_1, \ldots, c_{n(1)})$ of $C$ setting

$$n(1) := 4 + 2i(1) \quad \text{and} \quad (c_1, \ldots, c_{n(1)}, c_{n(1)+1}) := (x_0, a_1, d_1, \ldots, d_{2i(1)}, b_1, b_2, x_0).$$

**Inductive step.** Suppose that the integer $n(m)$ is defined and $c_k$ are defined for any $1 \leq k \leq n(m) + 1$ in order to satisfy Condition (5.5). Consider the finite words

$$A := (c_1, \ldots, c_{n(m)}, x_0, a_1, \ldots, a_{2m+1})$$

$$B := (b_{2m+2}, \ldots, b_1, x_0).$$

Let $i(m+1) \leq \text{Card}(E(X))$ be the positive integer and $D = (d_{m+1}^1, \ldots, d_{m+1}^{2i(m+1)})$ be the corresponding finite word with $d_{j}^{m+1} \leq p(X)$ for any $1 \leq j \leq 2i(m+1)$ which are provided by Lemma 5.15. We set

$$n(m+1) := n(m) + 4(m+1) + 2i(m+1) \quad \text{and} \quad (c_{n(m)+1}, \ldots, c_{n(m)+1}) := (x_0, a_1, \ldots, a_{2m+1}, d_{1}^{m+1}, \ldots, d_{2i(m+1)}^{m+1}, b_{2m+2}, \ldots, b_1, x_0).$$

The finite word $(c_1, \ldots, c_{n(m)+1})$ is equal to the concatenation $(c_1, \ldots, c_{n(m)}) * (c_{n(m)+1}, \ldots, c_{n(m)+1})$ and satisfies Condition (5.5). The inductive construction of the infinite word $C$ is therefore complete and we set $\alpha := [c_1, c_2, \ldots]$, which is of course irrational and of bounded type.

**End of the proof.** Observe that $L_{T^2}(\alpha) > x_0 \geq 2M^{+}(X)^2$. Thus, Theorem 5.12 can be applied and it implies

$$L_X(\alpha) = N \cdot \limsup_{n \to \infty} \frac{L(C, n)}{m^2_X([c_1, \ldots, c_n])}.$$

Observe that for any integer $m$ Condition (5.5) implies

$$N \cdot \frac{L(C, n(m))}{m^2_X([c_1, \ldots, c_{n(m)}])} > N \cdot \frac{x_0}{M^{-}(X)^2}.$$

On the other hand, since for $n \neq n(m)+1$ we have $1 \leq c_n \leq \max\{4, p(X) - 1\}$, for all $n \neq n(m)$ we have

$$N \cdot \frac{L(C, n)}{m^2_X([c_1, \ldots, c_n])} < N \cdot \frac{2 + \max\{4, p(X) - 1\}}{M^{-}(X)^2}.$$

The assumption on $x$ and the definition of $r(X)$ imply that $x_0 \geq \max\{6, p(X) + 1\}$. Thus, it follows that

$$L_X(\alpha) = \frac{N \cdot \limsup_{m \to \infty} \frac{L(C, n(m))}{m^2_X([c_1, \ldots, c_{n(m)}])}}{M^{-}(X)^2} = \frac{N}{M^{-}(X)^2} \cdot \limsup_{m \to \infty} L(C, n(m))$$

$$= \frac{N}{M^{-}(X)^2} \cdot \limsup_{m \to \infty} \left( [c_{n(m)}] + c_{n(m)+1} + [c_{n(m)+2}, c_{n(m)+3}, \ldots] \right)$$

$$= \frac{N}{M^{-}(X)^2} \cdot \limsup_{m \to \infty} \left( [b_1, b_2, \ldots] + x_0 + [a_1, a_2, \ldots] \right)$$

$$= \frac{N}{M^{-}(X)^2} \cdot \limsup_{m \to \infty} \left( [b_1, b_2, \ldots] + x_0 + [a_1, a_2, \ldots] \right) = \frac{N \cdot x}{M^{-}(X)^2}.$$
APPENDIX A. LOWER BOUND FOR SPECTRA

Here we prove Lemma 1.3 that is for an invariant locus \( I \) contained in the stratum \( \mathcal{H}(k_1, \ldots, k_r) \) of translation surfaces of genus \( g \), where \( 2g - 2 = k_1 + \cdots + k_r \), we have
\[
\mathcal{L}(I) \subset \left[ \pi \cdot \frac{2g + r - 2}{2}, +\infty \right].
\]

Proof: Fix any \( X \in I \) and, up to rescaling, assume that \( \text{Area}(X) = 1 \). Recall that by Vorobets’ identity we have that \( a^{-1}(X) = 2s^{-2}(X) \) (see Proposition 1.1). Thus, the Lemma follows immediately from the definition of \( s(X) \) if we show that for any \( Y \) in \( \mathcal{H}(1)(k_1, \ldots, k_r) \) we have
\[
\frac{\text{Sys}^2(Y)}{2} \leq \frac{2}{\pi(2g - 2 + r)}.
\]

The proof of this upper bound for the systole function follows from the following argument given by Smillie and Weiss in [Sm,We]. For each conical singularity \( p_i, 1 \leq i \leq r \), consider the set \( B_i \subset Y \) of points which have flat distance less than \( \text{Sys}(Y)/2 \) from \( p_i \). If \( p_i \) has conical angle \( 2\pi(k_i + 1) \) then \( \text{Area}(B_i) = (k_i + 1)\pi\text{Sys}(Y)^2/4 \) and by definition of systole, \( B_i \) are pairwise disjoint. Thus we have
\[
\sum_{i=1}^r (k_i + 1)\text{Sys}(Y)^2/4 \leq 1,
\]
which gives the desired upper bound since \( \sum_{i=1}^r k_i = 2g - 2 \).

APPENDIX B. CLOSING AND SHADOWING LEMMA FOR AFFINE INVARIANT LOCI

The main results of this Appendix are Proposition B.11 and Proposition B.12 which provide a combinatorial version respectively of the closing Lemma and of the Shadowing Lemma for closed \( \text{SL}(2,\mathbb{R}) \)-invariant loci, adapted to zippered rectangles coordinates. These two results are used in §3 to prove the closure of the Lagrange spectrum and density of periodic orbits for any such locus (Theorem 1.5). They show that, given a path which describe a zippered rectangle in an invariant locus, the combinatorial version of the closing and shadowing lemmas used in §3 produce new paths in the same \( \text{SL}(2,\mathbb{R}) \)-invariant locus. In §3.B.1 we first describe the structure of \( \text{SL}(2,\mathbb{R}) \)-invariant loci in terms of zippered rectangles coordinates, which follows from Eskin-Mirzakhani’s work [E,M]. In §3.B.2 we prove some technical Lemmas, providing nice flow-boxes, or hyperbolic boxes, in the zippered rectangles coordinates. Finally in §3.B.3 we prove Proposition B.11 and Proposition B.12.

B.1. Rauzy-Veech induction and affine loci. Let \( \mathcal{H}(k_1, \ldots, k_r) \) be a stratum of translation surfaces. Up to considering a finite covering, we can assume that it is an affine manifold without orbifold points. Let \( \mathcal{C} \) be a connected component of the stratum and \( \mathcal{R} \) be a Rauzy class representing the surfaces in \( \mathcal{C} \). Rauzy paths and combinatorial data are considered in \( \mathcal{R} \). If \( \gamma \) is a Rauzy path denote \( \hat{Q}_\gamma \) the corresponding linear branch of the Rauzy-Veech map. If \( \gamma \) is a positive loop \( \gamma \) at \( \pi \in \mathcal{R} \), denote respectively \( \lambda, \lambda^* \) the maximal and the minimal eigenvector of \( t B_{\gamma}^{-1} \), normalized so that \( \|\lambda\| = 1 \) and \( \text{Area}(\pi, \lambda, \lambda^*) = 1 \). The \( \mathcal{F}_\gamma \)-orbit of such a data is a closed geodesic in the stratum. In terms of the Rauzy-Veech induction we have
\[
\hat{Q}_\gamma(\pi, \lambda^*, \lambda^*) = \|t B_{\gamma}^{-1} \lambda^* \| \cdot (\pi, \lambda^*, \lambda^*).
\]

B.1.1. Affine invariant loci. Let \( I \) be a closed subset of \( \mathcal{C} \) which is invariant under \( \text{SL}(2,\mathbb{R}) \). For any combinatorial datum \( \pi \in \mathcal{R} \) let \( I_\pi \) be the set of data \( (\lambda, \tau) \in \mathbb{R}_+^4 \times \Theta_\pi \) such that the surface \( X(\pi, \lambda, \tau) \) belongs to \( I \). Fix data \( (\lambda, \tau) \in I_\pi \) representing the surface \( X(\pi, \lambda, \tau) \). Let \( t \mapsto A_t := \begin{pmatrix} a_t & b_t \\ c_t & d_t \end{pmatrix} \) be a continuous path in \( \text{SL}(2,\mathbb{R}) \) with \( A_0 = Id \) and \( I \subset \mathbb{R} \) be an interval containing 0 such that
\[
(A_t \lambda + b_t \tau, c_t \lambda + d_t \tau) \in \mathbb{R}_+^4 \times \Theta_\pi \text{ for any } t \in I.
\]
Then the data \( (a_t \lambda + b_t \tau, c_t \lambda + d_t \tau) \) belong to \( I_\pi \) for any \( t \in I \) and represent the surface \( A_t \cdot X(\pi, \lambda, \tau) \). The celebrated result of Eskin-Mirzakhani [E, M] can be stated in terms of Rauzy-Veech induction as follows.

Theorem B.1 (Eskin-Mirzakhani). For any \( \pi \in \mathcal{R} \) there exists a countable family \( \mathcal{V}_\pi = (V_j) \) of vector subspaces of \( \mathbb{R}_+^4 \times \mathbb{R}_+^4 \) of the same dimension as \( I \) satisfying the properties above.

---

1The bound given in [Sm,We] is unfortunately incorrect because of a typo in the arguments of the proof, but the proof that we reproduce here is due to them.
(1) We have \( I_\pi = \left( \bigcup_{V_j \subset V_\pi} V_j \right) \cap \left( \mathbb{R}_+^d \times \Theta_\pi \right) \), that is any \( C_j := V_j \cap \left( \mathbb{R}_+^d \times \Theta_\pi \right) \) is a convex open subcone of \( V_j \) of the same dimension as \( I, \) providing local affine coordinates for the locus.

(2) For any \( i \) and \( j \) we have \( V_i \cap V_j \cap \left( \mathbb{R}_+^d \times \Theta_\pi \right) = \emptyset \) if \( i \neq j. \) Indeed points in a non-trivial intersection \( V_i \cap V_j \) would represent elements of \( I \) not admitting a local affine chart.

(3) For any loop \( \gamma \) at \( \pi \) (not necessarily positive) and any \( V_j \in V_\pi \) intersecting \( \Delta_\gamma \times \Theta_\pi \) there exists \( V_i \in V_\pi \) such that \( \hat{Q}_\gamma(V_j) = V_i, \) that is the Rauzy-Veech induction represents affine coordinate changes for \( I. \)

The following Lemma is immediate.

**Lemma B.2.** Consider \( V \in V_\pi \) and data \((\lambda, \tau) \in C = V \cap I_\pi. \) Let \( t \mapsto A_t \) be a continuous path in \( \text{SL}(2, \mathbb{R}) \) with \( A_0 = \text{Id} \) and \( I \) be an interval with \( 0 \in I \) satisfying condition \( [B.3]. \) Then

\[
(a_t \lambda + b_t \tau, c_t \lambda + d_t \tau) \in C \quad \text{for any} \quad t \in I.
\]

**Remark B.3.** Consider \( V \in V_\pi \) and the corresponding convex open subcone \( C = V \cap I_\pi. \) The following facts hold.

(1) If \( \gamma \) is a (not necessarily positive) loop at \( \pi \) and we have data \((\lambda, \tau) \in C \cap (\Delta_\gamma \times \Theta_\pi) \) such that

\[
(t^{-1} B^{-1}_\gamma \lambda, t^{-1} B^{-1}_\gamma \tau) \in C, \quad \text{then} \quad \hat{Q}_\gamma V = V.
\]

(2) In particular, if \( C \) contains pseudo-Anosov data \((\pi, \lambda^\gamma, \tau^\gamma) \) corresponding to a positive loop \( \gamma \) at \( \pi \) then \( \hat{Q}_\gamma V = V. \)

(3) For any \((\lambda, \tau) \in C \) and any \( t \in \mathbb{R} \) we have \((e^t \lambda, e^{-t} \tau) \in \mathbb{R}_+^d \times \Theta_\pi, \) thus Lemma \( [B.2] \) implies \((e^t \lambda, e^{-t} \tau) \in C \) for all \( t \in \mathbb{R}. \)

**B.2. Hyperbolic boxes in the space of zippered rectangles.** Let \( \gamma \) be a finite path. Let \( \Delta^{(1)}_\gamma \) be the simplex of those \( \lambda \in \Delta_\gamma \) normalized by \( \sum\chi | r_\chi | = 1. \) Let \( \Theta^{(1)}_\gamma \) be the codimension one hypersurface of those \( \tau \in \Theta_\gamma \) normalized by \( \sum\chi | r_\chi | = 1. \)

**Definition B.4.** Consider a positive path \( \nu \) ending at \( \pi \) and a positive path \( \delta \) starting at \( \pi, \) so that the concatenation \( \nu \circ \delta \) is defined. Let \( I \subset \mathbb{R} \) be any interval. An hyperbolic box \( \text{Box}_\pi(I, \delta, \nu) \) is the image of \( I \times \Delta^{(1)}_\delta \times \Theta^{(1)}_\nu \) into the area-one hyperboloid \( \{ (\pi, 1) \times \Delta_\delta \times \Theta_\nu \) via the map

\[
(t, \lambda, \tau) \mapsto \frac{1}{\text{Area}(\pi, 1, \tau)} \cdot (\pi, e^t \lambda, e^{-t} \tau),
\]

satisfying the extra property that any orbit of the Rauzy-Veech map meets \( \text{Box}_\pi(I, \delta, \nu) \) at most once.

To simplify notation we always assume that the path \( \delta \) and \( \nu \) defining an hyperbolic box have same length \( m. \) Recall that for any Rauzy path \( \gamma, \) either finite, half-infinite or bi-infinite, in Equation \((4.2) \) we define \( \text{N}(\gamma) \) as the supremum of \( \|B_\nu\| \) over all positive sub-paths \( \nu \) of \( \gamma \) not admitting a proper positive sub-path.

**Lemma B.5.** For any \( M > 0 \) there exists a positive integer \( m_1 = m_1(M) \) such that for any integers \( r \) and \( m \) such that \( r > m_1 \) and \( m_1 \leq m \leq r \) the following holds. If \( \gamma(-\infty, +\infty) \) is a bi-infinite path with \( N(\gamma(-r, +\infty)) < M \) and if we set

\[
\delta := \gamma(1, m) \quad \text{and} \quad \nu := \gamma(-m + 1, 0)
\]

then there exists some interval \( I \subset \mathbb{R} \) such that \( \text{Box}_\pi(I, \delta, \nu) \) is an hyperbolic box.

**Proof:** Given \( m \in \mathbb{N} \) and a bi-infinite path \( \gamma(-\infty, +\infty), \) let us denote by \( \delta_m := \gamma(1, m) \) and \( \nu_m := \gamma(-m + 1, 0). \) Let \( m_0 \) be a positive integer such that the path \( \delta_{m_0} \) is positive. Since \( \Delta^{(1)}_{m_0} \) is compact, there exists \( 0 < c < 1 \) such that for any \( \lambda \in \Delta^{(1)}_{m_0} \) we have \( \lambda_{\chi} > c \| \lambda \| = c \) for any \( \chi \in \mathcal{A}. \) According to Lemma \( \text{[4.18]} \) when we increase \( m \in \mathbb{N} \) the supremum of the diameters of \( \Delta^{(1)}_{m_0} \) and \( \Theta^{(1)}_{\nu_m} \) over all paths such \( N(\gamma(-m, +\infty)) < M \) shrinks exponentially as a function of \( m. \) Moreover \( \text{Area}(\pi, \lambda, \tau) \) is a continuous function of \((\pi, \lambda, \tau), \) bounded away from zero, therefore there exists \( m_1 \geq m_0 \) such that the condition

\[
N(\gamma(-r, +\infty)) < M \quad \text{for some} \quad r > m_1
\]
implies that for any $m_1 \leq m \leq r$ and any pair of data $(\lambda, \tau)$ and $(\lambda', \tau')$ in $\Delta^{(1)}_{\delta_m} \times \Theta^{(1)}_{\nu_m}$ we have

\begin{equation}
\sqrt{1-c} < \frac{\text{Area}(\pi, \lambda', \tau')}{\text{Area}(\pi, \lambda, \tau)} < \frac{1}{\sqrt{1-c}}.
\end{equation}

Thus, if we choose the interval $I$ small enough so that $e^{-|s-t|} > \sqrt{1-c}$ for any $s, t \in I$, let us show that Box$_\nu(I, \delta, \nu)$ where $\nu = \nu_m$ and $\delta = \delta_m$ is an hyperbolic box. Since by definition $m \geq m_0$, we have that $\delta$ is positive, so we just need to show that there cannot be an orbit of $Q$ meeting Box$_\nu(I, \delta, \nu)$ twice. If by contradiction this is not the case, there exists data $(\pi, \lambda, \tau)$ and $(\pi', \lambda', \tau')$ in $\{\pi\} \times \Delta^{(1)}_{\delta_m} \times \Theta^{(1)}_{\nu_m}$ and $s, t \in I$ and an integer $n \geq 1$ such that

\begin{equation}
\frac{1}{\text{Area}(\pi, \lambda', \tau')} \left(\pi, e^s \lambda', e^{-s} \tau'\right) = \frac{1}{\text{Area}(\pi, \lambda, \tau)} \left(\pi(n), e^s \lambda(n), e^{-s} \tau(n)\right).
\end{equation}

Since $\Delta^{(1)}_{\delta_m} \subset \Delta_{\delta_0}$ then $\lambda_\chi > c||\lambda|| = c$ for any $\chi \in \mathcal{A}$ and therefore $||\lambda^{(n)}|| \leq ||\lambda^{(1)}|| = 1 - \min \chi \lambda_\chi < 1 - c$. It follows that

\begin{equation}
e^{-|s-t|} \frac{\text{Area}(\pi, \lambda, \tau)}{\text{Area}(\pi, \lambda', \tau')} < 1 - c.
\end{equation}

Since $e^{-|s-t|} > \sqrt{1-c}$ by choice of $I$, this contradicts (B.2). Thus Box$_\nu(I, \delta, \nu)$ is a hyperbolic box. \(\square\)

B.2.1. Hyperbolic boxes and affine loci. Let $\mathcal{I}$ be a closed $SL(R, \mathbb{R})$ invariant subset of $\mathcal{C}$.

**Definition B.6.** An hyperbolic box Box$_\pi(I, \delta, \nu)$ is adapted to $\mathcal{I}$ if there exists an unique $V \in \mathcal{V}_\pi$ such that $V \cap \text{Box}_\pi(I, \delta, \nu) \neq \emptyset$.

If Box$_\nu(I, \delta, \nu)$ is an adapted hyperbolic box, $V \cap \text{Box}_\nu(I, \delta, \nu)$ is a convex disc of the same dimension as $\mathcal{I}^{(1)}$, that is the image of Box$_\nu(I, \delta, \nu)$ is a small open set in $\mathcal{H}^{(1)}(k_1, \ldots, k_r)$ which intersects $\mathcal{I}^{(1)}$ in an open disc.

Consider data $(\pi, \lambda, \tau) \in \mathcal{I}_\pi$ with $\text{Area}(\pi, \lambda, \tau) = 1$ and let $\gamma(-\infty, +\infty)$ be the corresponding bi-infinite path. Observe that if $\delta$ and $\nu$ have length $m$ and Box$_\nu(I, \delta, \nu)$ contains $(\pi, \lambda, \tau)$ then $\lambda \in \Delta_\delta$ and $\tau \in \Theta_\nu$ and therefore

$$\gamma(1, m) = \delta \text{ and } \gamma(-m + 1, 0) = \nu.$$  

**Lemma B.7.** For any $M > 0$ there exists $r_1 = r_1(M) \in \mathbb{N}$ such that the following holds. If $(\pi, \lambda, \tau)$ are data with $(\lambda, \tau) \in \mathcal{I}_\pi$ and $\text{Area}(\pi, \lambda, \tau) = 1$, that induce a bi-infinite path $\gamma(-\infty, +\infty)$ with

$$N(\gamma(-r, +\infty)) < M \text{ for some } r \geq r_1$$

then there exists an hyperbolic box Box$_\nu(I, \delta, \nu)$ adapted to $\mathcal{I}$ which contains $(\pi, \lambda, \tau)$, where $\delta$ and $\nu$ have length bounded by $r_1$.

**Proof:** Fix $M > 0$ and let $m_1 = m_1(M)$ be the positive integer given by Lemma B.5. Consider $r > m_1$ such that $N(\gamma(-r, +\infty)) < M$. Then, if we denote as before by $\delta_m := \gamma(1, m)$ and by $\nu_m := \gamma(-m + 1, 0)$, for any $m_1 \leq m < r$ there exists a proper interval $I$ such that $(\pi, \lambda, \tau) \in \text{Box}_\nu(I, \delta_m, \nu_m)$. So we just need to show that there exists $r_1 > m_1$ which depends only on $M$ such that if $r > r_1$ than the hyperbolic box $\text{Box}_\nu(I, \delta_r, \nu_r)$ is also adapted to $\mathcal{I}$. Since $\gamma(1, r_1)$ and $\gamma(-r_1 + 1, 0)$ are positive since $r_1 > m_1$ (see the proof of Lemma B.5, $\Delta^{(1)}_{\delta_{r_1}} \times \Theta^{(1)}_{\nu_{r_1}} \subset \mathcal{K}$ for some open sub-cone $\mathcal{K} \subset \mathbb{R}_+^d \times \Theta_{\pi}$ with compact closure with respect to the Hilbert metric of $\mathbb{R}_+^d \times \Theta_{\pi}$ (see §1.3.2), which depends only on $r_1(M)$ and hence only on $M$.

Recall that $\mathcal{I}^{(1)}$ is an affine sub-manifold of $\mathcal{H}^{(1)}(k_1, \ldots, k_r)$, thus locally it is the zero locus of some analytic function. It follows that there exists just finitely many hyperplanes $V \in \mathcal{V}_\pi$ such that $V \cap \mathcal{K} \neq \emptyset$. Hence there exists some $\epsilon > 0$ such that the $\epsilon$-neighborhoods of $V \cap \mathcal{K}$ with respect to the Hilbert metric are disjoint each other. Thus, reasoning as in the proof of Lemma B.7, there exists $r_1 = r_1(M) > r_0$ such that for any bi-infinite path $\gamma(-\infty, +\infty)$ with $N(\gamma(-r_1, +\infty)) < M$ the diameter of $\Delta^{(1)}_{\delta_{r_1}} \times \Theta^{(1)}_{\nu_{r_1}}$, a smaller than $\epsilon/2$ with respect to the Hilbert metric. In this case, the hyperbolic box Box$_\nu(I, \delta_{r_1}, \nu_{r_1})$ can intersect at most one $V \in \mathcal{V}_\pi$ and the Lemma is thus proved. \(\square\)
Lemma B.8. Fix $\pi \in \mathcal{R}$ and $V \in \mathcal{V}_\pi$ and consider the open subcone $C := V \cap \mathcal{I}_\pi$. Let $\text{Box}_\pi(\delta, \nu, I)$ be an hyperbolic box adapted to $\mathcal{I}$ with non-empty intersection with $C$, where $\delta$ and $\nu$ are positive path of length $m$. If $\gamma(0, r)$ is a loop at $\pi$ and moreover we have the extra conditions

$$\gamma(r - m + 1, r) = \nu \text{ and } \gamma(r + 1, r + m) = \delta$$

then $(\pi, \lambda^{(r)}, \tau^{(r)}) = \hat{Q}_{\gamma(0, r)}(\pi, \lambda, \tau) \in C$.

Proof: The assumption in the statement is equivalent to $\lambda^{(r)} \in \Delta_\delta$ and $\tau^{(r)} \in \Theta_\nu$, therefore there exists $t \in \mathbb{R}$ such that $(\pi, e^t \lambda^{(r)}, e^{-t} \tau^{(r)})$ belongs to $\text{Box}_\pi(I, \delta, \nu)$. On the other hand $(\lambda^{(r)}, \tau^{(r)})$ belongs to $\mathcal{I}_\pi$, since $\mathcal{I}_\pi$ is invariant under $\hat{Q}_{\gamma(0, r)}$ and contains $(\lambda, \tau)$. Moreover also $(e^t \lambda^{(r)}, e^{-t} \tau^{(r)})$ belongs to $\mathcal{I}_\pi$, since condition (B.1) is satisfied, thus we get

$$(\pi, e^t \lambda^{(r)}, e^{-t} \tau^{(r)}) \in \mathcal{I}_\pi \cap \text{Box}_\pi(I, \delta, \nu) = C.$$ 

Finally part (3) of Remark B.3 implies $(\lambda^{(r)}, \tau^{(r)}) \in C$. \hfill $\Box$

B.3. Closing and Shadowing Lemma.

B.3.1. The main geometrical Lemma.

Lemma B.9. Fix $\pi \in \mathcal{R}$ and $V \in \mathcal{V}_\pi$ and consider the convex open subcone $C = V \cap \mathcal{I}_\pi$. Consider a finite path $\eta$ ending at $\pi$ and a finite path $\gamma$ starting at $\pi$, so that the concatenation $\eta * \gamma$ is possible. If $C$ has non-empty intersection both with $\Delta_{\gamma} \times \Theta_{\pi}$ and with $\mathbb{R}_+^A \times \Theta_\eta$ then we have

$$C \cap (\Delta_{\gamma} \times \Theta_{\pi}) \neq \emptyset.$$ 

Proof: Let $(\lambda, \tau) \in C \cap (\Delta_{\gamma} \times \Theta_{\pi})$ and $(\lambda', \tau') \in C \cap (\mathbb{R}_+^A \times \Theta_\eta)$. Part (2) of Remark B.3 and convexity of $C$ imply that for any $t \in \mathbb{R}$ and $s \in (0, 1)$ we have

$$(1 - s) \cdot (\lambda, \tau) + s \cdot (e^{-t} \lambda', e^t \tau') \in C.$$ 

Since $\lambda$ and $\tau$ belong respectively to $\Delta_{\gamma}$ and to $\Theta_{\gamma}$, and the latter are open subcones of $\mathbb{R}^A$, then taking $s$ small enough and $t$ such that $se^t$ is big enough, we have

$$(1 - s) \cdot (\lambda, \tau) + s \cdot (e^{-t} \lambda', e^t \tau') \in \mathcal{I}_\pi,$$

thus the data $(1 - s) \cdot (\lambda, \tau) + s \cdot (e^{-t} \lambda', e^t \tau')$ belong to $C \cap (\Delta_{\gamma} \times \Theta_{\pi})$. \hfill $\Box$

Corollary B.10. Fix $\pi \in \mathcal{R}$ and $V \in \mathcal{V}_\pi$ and consider the convex open subcone $C = V \cap \mathcal{I}_\pi$. Let $\gamma_1, \ldots, \gamma_k$ be positive loops at $\pi$ such that for any $i = 1, \ldots, k$ the following holds

1. $C$ has non-empty loops at $\pi$ such that for any $i = 1, \ldots, k$ the following holds
2. We have $\hat{Q}_{\gamma_i}(V) = V$.

Then for any $i = 1, \ldots, k$ we have

$$C \cap (\Delta_{\gamma_1} \times \ldots \times \Delta_{\gamma_k} \times \Theta_{\gamma_1} \times \ldots \times \Theta_{\gamma_k}) \neq \emptyset.$$ 

Proof: Observe that $\hat{Q}_{\gamma_1 \ldots \gamma_k}$ is a linear bijection from $\Delta_{\gamma_1 \ldots \gamma_k} \times \Theta_{\pi}$ onto $\Delta_{\gamma_1 \ldots \gamma_k} \times \Theta_{\gamma_1 \ldots \gamma_k}$, and since assumption (2) in the statement is equivalent to $\hat{Q}_{\gamma_1 \ldots \gamma_k}(V) = V$, then it is enough to prove that for any $k > 0$ we have

(B.3) 

$$C \cap (\Delta_{\gamma_1} \times \ldots \times \Delta_{\gamma_k} \times \Theta_{\pi}) \neq \emptyset.$$ 

For $k = 1$ the required condition corresponds to assumption (1) in the statement. The general case is proved by induction on $k$. Suppose that condition (B.3) holds for $k$ positive loops $\gamma_1, \ldots, \gamma_k$ and consider an extra loop $\gamma_{k+1}$ at $\pi$ satisfying the assumptions in the statement. According to Lemma B.9 we have

$$C \cap (\Delta_{\gamma_{k+1}} \times \Theta_{\gamma_1} \times \ldots \times \Theta_{\gamma_k}) \neq \emptyset.$$ 

Moreover $\hat{Q}_{\gamma_1 \ldots \gamma_k}$ maps $\Delta_{\gamma_1 \ldots \gamma_k} \times \gamma_{k+1} \times \Theta_{\pi}$ onto $\Delta_{\gamma_{k+1}} \times \Theta_{\gamma_1} \times \ldots \times \Theta_{\gamma_k}$ and since $\hat{Q}_{\gamma_i}(V) = V$ for any $i = \ldots, k$ then condition (B.3) follows. \hfill $\Box$
B.3.2. Closing Lemma.

**Proposition B.11.** Fix $\pi \in \mathcal{R}$. For any $M > 0$ there exists a positive integer $r_1 = r_1(M)$ such that for any $m > r_1$ the following holds. If $(\pi, \lambda, \tau)$ are data with $(\lambda, \tau) \in \mathcal{I}_\pi$ that induce a bi-infinite path $\gamma(-\infty, +\infty)$ with $N(\gamma(-m, +\infty)) < M$ and for $r > 0$ the segment $\gamma := \gamma(0, r)$ is a positive loop at $\pi$ satisfying the extra condition

$$\gamma(r - m + 1, r + m) = \gamma(-m + 1, m)$$

then $(\pi, \lambda^*, \tau^*) \in \mathcal{I}_\pi$, that is the closed geodesic corresponding to $\gamma = \gamma(0, r)$ is contained in $\mathcal{I}$.

**Proof:** We can assume that $\text{Area}(\pi, \lambda, \tau) = 1$, indeed the induced bi-infinite path $\gamma(-\infty, +\infty)$ is invariant under scalar multiplication $(\pi, \lambda, \tau) \mapsto (\pi, c \cdot \lambda, c \cdot \tau)$ for $c \in \mathbb{R}_+$. Let $r_1 = r_1(M)$ be the integer in the statement of Lemma B.7. Then $(\pi, \lambda, \tau)$ is contained in some hyperbolic box $\text{Box}_\pi(I, \delta, \nu)$ adapted to $\mathcal{I}$, where the path $\delta$ and $\nu$ have length bounded by $r_1$. Moreover $\text{Box}_\pi(I, \delta, \nu)$ remains a hyperbolic box adapted to $\mathcal{I}$ replacing $\delta$ and $\nu$ by longer segments, thus we can assume that $\delta = \gamma(1, m)$ and $\nu := \gamma(-m + 1, 0)$. Consider the unique $V \in \mathcal{V}_\pi$ intersecting $\text{Box}_\pi(I, \delta, \nu)$ and let $C := V \cap \mathcal{I}$ be the corresponding open subcone of $V$. Since $(\pi, \lambda, \tau)$ belongs to $\mathcal{I}$ and to $\text{Box}_\pi(I, \delta, \nu)$, then $(\pi, \lambda, \tau) \in C$. On the other hand Lemma B.8 implies $(\pi, \lambda^*, \tau^*) \in C$. Finally $Q_\gamma$ maps $\{\pi\} \times \Delta_\gamma \times \Theta_\gamma$ onto $\{\pi\} \times \mathbb{R}_+^2 \times \Theta_\gamma$, and of course $(\pi^*(\gamma), \lambda^*(\gamma), \tau^*(\gamma)) = Q_\gamma(\pi, \lambda, \tau)$, therefore both $(\Delta_\gamma \times \Theta_\gamma) \cap C$ and $(\mathbb{R}_+^2 \times \Theta_\gamma) \cap C$ are not empty, hence Lemma B.9 implies

$$(\Delta_\gamma \times \Theta_\gamma) \cap C \neq \emptyset.$$ 

Moreover both $(\pi, \lambda, \tau)$ and $Q_\gamma(\pi, \lambda, \tau)$ belong to $C$, therefore part (1) of Remark B.3 implies $Q_\gamma(C) = C$.

The last two conditions imply that Corollary B.10 can be applied with $\gamma_1 = \gamma, \ldots, \gamma_k = \gamma$ for arbitrary $k$, thus we get

$$(\pi, \lambda^*, \tau^*) \in \bigcap_{\text{length}(\gamma_{i+1} \cdots \gamma) \to \infty} \{\pi\} \times \Delta_{\gamma_{i+1} \cdots \gamma} \times \Theta_{\gamma_{i+1} \cdots \gamma} \subset \mathcal{I}.$$ 

$\square$

B.3.3. Shadowing Lemma.

**Proposition B.12.** Fix $\pi \in \mathcal{R}$ and $M > 0$. There exists a positive integer $r_1 = r_1(M)$ such that for any $m > r_1$ the following holds. Consider a sequence $(\gamma_i)_{i \in \mathbb{Z}}$ of positive loops at $\pi$ such that $N(\gamma_i) < M$ and satisfying the extra conditions:

1. we have $(\lambda^*, \tau^*) \in \mathcal{I}_\pi$ for all $i \in \mathbb{Z}$
2. there exists positive paths $\delta$ and $\nu$, both of length $m$, such that for any $i \in \mathbb{Z}$ we have

$$\gamma_i(-m + 1, 0) = \nu \quad \text{and} \quad \gamma_i(1, m) = \delta.$$ 

Then the bi-infinite path $\gamma(-\infty, +\infty) = \cdots \gamma_1 \ast \gamma_i \ast \gamma_{i+1} \ast \cdots$ is induced by data $(\pi, \lambda, \tau)$ with $(\lambda, \tau) \in \mathcal{I}_\pi$.

**Proof:** Let $r_1 = r_1(M)$ be the constant in the statement of Lemma B.7. Since for any $i$ we have $(\lambda^*, \tau^*) \in \mathcal{I}_\pi$ and moreover $\gamma_i$ is periodic with $N(\gamma_i) < M$, then Lemma B.7 implies that any $(\pi, \lambda^*, \tau^*)$ is contained in some hyperbolic box adapted to $\mathcal{I}$, whose size is bounded by $r_1$. Assumption (2) implies that all there hyperbolic boxes contain $\text{Box}_\pi(\delta, \nu, I)$ and moreover the latter contains all data $(\pi, \lambda^*, \tau^*)$ for $i \in \mathbb{Z}$. It is also obvious that $\text{Box}_\pi(\delta, \nu, I)$ is adapted to $\mathcal{I}$. Let $V$ be the unique linear space in $\mathcal{V}_\pi$ such that the intersection $V \cap \text{Box}_\pi(\delta, \nu, I)$ is not empty and consider its convex open subcone $C := V \cap \mathcal{I}_\pi$. Lemma B.7 implies $(\lambda^*, \tau^*) \in C$ for any $i \in \mathbb{Z}$. On one hand, it follows that $C$ intersects $\Delta_{\gamma_i} \times \Theta_{\gamma_i}$ for any $i \in \mathbb{Z}$. On the other hand part (2) of Remark B.3 implies $Q_\gamma(V) = V$ for any $i \in \mathbb{Z}$. Therefore, for any $k \in \mathbb{N}$, we can apply Corollary B.10 to the family of positive loops $\gamma_{-k+1}, \ldots, \gamma_0, \ldots, \gamma_k$ and we get

$$C \cap (\Delta_{\gamma_{-k+1} \cdots \gamma_k} \times \Theta_{\gamma_{-k+1} \cdots \gamma_k}) \neq \emptyset.$$ 

Observe that $\bigcap_{k \in \mathbb{N}} \Delta_{\gamma_{-k+1} \cdots \gamma_k} \times \Theta_{\gamma_{-k+1} \cdots \gamma_k}$ is the half line spanned by the data $(\pi, \lambda, \tau)$, because $N(\gamma_i)$ is uniformly bounded. Therefore we get $(\pi, \lambda, \tau) \in C$. $\square$
Appendix C. Proof of finite-time estimates

In this section we prove Proposition 4.10 and Proposition 4.11. Some technicalities appear, mostly in the proof of Proposition C.3 due to the fact that we need to treat the combinatorics both of length data and suspension data, and the latter are less clearly understood than the former.

Fix an admissible combinatorial datum $\pi$ and recall that it consists in two bijections $\pi^t$ and $\pi^b$ from $A$ to $\{1, \ldots, d\}$. In order to simplify the notation, we introduce the labelling $\alpha(1), \ldots, \alpha(d)$ and the labelling $\beta(1), \ldots, \beta(d)$ of the letters of $A$, according to their order in $\pi^t$ and $\pi^b$ respectively. With this notation, which is the same as in \textbf{[2.2]}, we have

\begin{equation}
\pi = \begin{pmatrix} \alpha(1) & \ldots & \alpha(d) \\ \beta(1) & \ldots & \beta(d) \end{pmatrix}.
\end{equation}

Consider length-suspension data $\lambda$ and $\tau$ for $\pi$ and let $X$ be translation surface corresponding to $\langle \pi, \lambda, \tau \rangle$. We fix the normalization

\[ \text{Area}(\pi, \lambda, \tau) := \text{Area}(X) = 1. \]

C.1. Combinatorial control for $(\pi, \lambda, \tau)$.

C.1.1. Distortion of length data and singularities of IETs. Let $\gamma$ be a finite Rauzy path of length $r$ and let $T$ be an IET in $\Delta_\gamma$. Set $T^{(r)} := Q^r(T)$ and let $I^{(r)}$ be the sub-interval of $I$ where $T^{(r)}$ acts.

**Lemma C.1.** If $\gamma$ is a positive path then $I^{(r)}$ does not contain any singularity of $T$ or of $T^{-1}$ in its interior.

**Proof:** We prove the statement for the singularities of $T$, the argument for those of $T^{-1}$ being the same. For any singularity $u^{(r), t}_\alpha$ of $T^{(r)}$ we have $u^{(r), t}_\alpha = T^{-k}(u^{(r)}_\alpha)$, where $k$ is the minimal positive integer such that $T^{-k}(u^{(r)}_\alpha)$ belongs to the interior of $I^{(r)}$. Suppose that $u^{(r)}_\alpha$ belongs to the interior of $I^{(r)}$ for some $\alpha$ with $\pi^t(\alpha) > 1$. Then $u^{(r), t}_\alpha = u^{(r), t}_\alpha$, that is $u^{(r)}_\alpha$ is a singularity of $T^{(r)}$ too. Therefore $I^{(r)}_J \cap T^k(I^{(r)}, t)_I = \emptyset$ for any letter $\beta$ with $\pi^t(\beta) < \pi^t(\alpha)$ and for any $k$ with $0 \leq k < q^r_\alpha$. Since $[B_\gamma]_{\alpha \beta}$ is positive by assumption then we get an absurd, according to Lemma 2.3.

**Corollary C.2.** If $\gamma$ is a positive path then

\[ \min_{\chi \in A} \lambda^{(r)}(\chi) \leq \min_{\pi^t(\alpha) > 1, \pi^t(\beta) > 1} |u^{(r), t}_\alpha - u^{(r), t}_\beta|. \]

**Proof:** Consider $\beta$ and $\alpha$ with $\pi^t(\alpha) > 1$ and $\pi^t(\beta) > 1$. According to Lemma C.1 consider the minimum $k$ with $1 \leq k \leq r$ such that both $u^{(r), t}_\alpha$ and $u^{(r), t}_\beta$ do not belong to the interior of $I^{(r)}$. In this case, either $\alpha$ or $\beta$ is the looser of the step $T^{(k-1)} \mapsto T^{(k)}$. According to the case, we get either $\lambda^{(k)}(\alpha) \leq |u^{(r), t}_\alpha - u^{(r), t}_\beta|$ or $\lambda^{(k)}(\beta) \leq |u^{(r), t}_\alpha - u^{(r), t}_\beta|$. 

C.1.2. Distortion of suspension data and cords. Fix combinatorial-length-suspension data $(\pi, \lambda, \tau)$ and let $X$ be the corresponding translation surface. As in \textbf{[2.2]}, for any $\chi \in A$ we set $\zeta_\chi := \lambda_\chi + \sqrt{-1} \cdot \tau_\chi$. A period $v \in \text{Hol}(X)$ is a cord with respect to $(\pi, \lambda, \tau)$ if there exist integers $l, m$ with $1 \leq l \leq m \leq d$ and such that

\[ \text{either } v = \sum_{i=l}^m \zeta_{\alpha(i)} \text{ or } v = \sum_{i=l}^m \zeta_{\beta(i)}. \]

When $v$ is a cord, we say that it is a top cord if the first condition above is satisfied, and a bottom cord if the second condition is satisfied. In both cases, we denote $A(v)$ the subset of $A$ of those letters appearing in the sum, for which we have $v = \sum_{\alpha \in A(v)} \zeta_\alpha$. The same translation surface admits many different combinatorial-length-suspension data, thus there are many ways to represent the same cord. When ambiguities may arise, we write $A_{\langle \pi, \lambda, \tau \rangle}(v)$ instead of just $A(v)$.

**Notation.** Let $\gamma$ be a finite Rauzy path. A path $\gamma'$ is an end of $\gamma$ if there exists another path $\gamma''$ such that $\gamma = \gamma'' \ast \gamma'$. Consider two pairs $(\pi, \lambda, \tau)$ and $(\pi', \lambda', \tau')$ of combinatorial-length-suspension data and a finite Rauzy path $\gamma$ such that

\begin{equation}
(\pi, \lambda, \tau) = \tilde{Q}_\gamma(\pi', \lambda', \tau').
\end{equation}

For $\chi \in A$ write also $\zeta'_\chi := \lambda'_\chi + \sqrt{-1} \cdot \tau'_\chi$. Recall from \textbf{[2.3]} the following facts.
(1) If the path $\gamma$ starts at $\pi'$ and ends in $\pi$, then $\hat{Q}$ is a linear isomorphism from $\{\pi'\} \times \Delta_\gamma \times \Theta_{\pi'}$ onto $\{\pi\} \times \mathbb{R}^d_+ \times \Theta_\gamma$.

(2) Thus for data $(\pi, \lambda, \tau)$ in $\{\pi\} \times \mathbb{R}^d_+ \times \Theta_\gamma$ Equation (C.2) determines an unique triple of data $(\pi', \lambda', \tau')$ in $\{\pi'\} \times \Delta_\gamma \times \Theta_{\pi'}$.

(3) Let $\gamma'$ be a path starting at $\pi'$ and ending in $\pi$ and $\gamma$ be a path ending at $\pi$. If $\gamma'$ is an end of $\gamma$ then $\Delta_\gamma \subseteq \Delta_{\gamma'}$ and $\Theta_\gamma \subseteq \Theta_{\gamma'}$, therefore for data $(\pi, \lambda, \tau)$ in $\{\pi\} \times \mathbb{R}^d_+ \times \Theta_\gamma$ the path $\gamma'$ determines an unique triple of data $(\pi', \lambda', \tau')$ in $\{\pi'\} \times \Delta_{\gamma'} \times \Theta_{\pi'}$ via Equation (C.2).

(4) Data $(\pi, \lambda, \tau)$ and $(\pi', \lambda', \tau')$ satisfying Equation (C.2) represent the same translation surface $X$.

Recall that $\gamma$ is a strongly complete if it is concatenation of $d$ complete paths. Consider the function $(\pi, \lambda, \tau) \mapsto w(\pi, \lambda, \tau)$ defined in (4.1) (see Definition 4.5). Proposition C.3 below corresponds to Corollary C.2 in the case of negative time.

**Proposition C.3.** Consider data $(\pi, \lambda, \tau)$ and a strongly complete path $\gamma$ such that $(\pi, \lambda, \tau)$ belong to $\{\pi\} \times \mathbb{R}^d_+ \times \Theta_\gamma$. For any cord $v = \sum_{\alpha \in A(v)} \zeta_\alpha$ with respect to $(\pi, \lambda, \tau)$ there exists some end $\gamma'$ of $\gamma$ such that the data $(\pi', \lambda', \tau')$ determined by $\gamma'$ via the relation (C.2) satisfy

$$\text{Area}(v) \geq w(\pi', \lambda', \tau').$$

The rest of the paragraph is devoted to the proof of Proposition C.3.

**Assumption.** We suppose that $v$ is a top cord with respect to $(\pi, \lambda, \tau)$, the proof for bottom cords being the same. Thus in terms of the labelling in Equation (C.1) we assume $A(v) = \{\alpha(l), \ldots, \alpha(m)\}$ and $v = \zeta_{\alpha(l)} + \cdots + \zeta_{\alpha(m)}$. The letters $\alpha(l), \ldots, \alpha(m)$ in $A(v)$ have of course a different position in the top line of combinatorial data different from $\pi$. Anyway, for simplicity, in C.1.3 and C.1.4 we keep their labelling $\alpha(i)$ with $l \leq i \leq m$ induced by $\pi$.

**C.1.3. Some special cases.** In this subsection we fix two pairs of combinatorial-length-suspension data $(\pi, \lambda, \tau)$ and $(\pi', \lambda', \tau')$ and a Rauzy path $\gamma$ as in Equation (C.2). In the following Lemmas we treat special cases of paths $\gamma$ satisfying different assumptions. In C.1.4 we show that the general case can be reduced to these special cases.

**Lemma C.4.** Let $v$ be a top cord with respect to $(\pi, \lambda, \tau)$ of the simplest form $v = \zeta_\alpha$, that is $A(v) = \{\alpha\}$ for some $\alpha \in A$. The following holds.

1. If $\gamma = \nu_\alpha$ is a simple arrow with winner $\alpha$, then Area$(v) \geq w(\pi', \lambda', \tau')$, where $(\pi', \lambda', \tau')$ are the data determined by $\nu_\alpha$ via the relation (C.3).
2. Moreover $\nu_\alpha$ is a top arrow if and only if $\tau_\alpha < 0$. Similarly $\nu_\alpha$ is a bottom arrow if and only if $\tau_\alpha > 0$.

**Proof:** Part (2) follows trivially from (2.3) thus we just prove part (1). Suppose that $\nu_\alpha$ is a top arrow, the other case being the same. Let $\beta$ be the loser of $\nu_\alpha$. The pre-image of $\pi$ under $\nu_\alpha$ has the form $\pi' = \left(\ldots, \alpha, \beta \right)$, thus we have $v = \zeta_\alpha = \zeta_\alpha - \zeta_\beta = \langle \zeta', w_{\pi', \beta, \alpha} \rangle$. \hfill $\square$

A neutral path is a finite path $\gamma$ that does not contain arrows with winner $\alpha \in A(v)$, moreover the only arrows in $\gamma$ where some $\alpha \in A(v)$ loses (if they exist) are top arrows. The terminology is motivated by Lemma C.5 which says that the Rauzy operations contained in a neutral path does not affect $v$. Recall that with our notation any letter $\alpha(i) \in A(v)$ is in $i$-th position in the top line of $\pi$, that is $\pi^i(\alpha(i)) = i$ for any $l \leq i \leq m$.

**Lemma C.5.** Let $\gamma$ be a neutral path. Then the following holds.

1. There exist some integer $k$ with $0 \leq k < m - l$ such that $\alpha(i)$ is in $(i-k)$-th position in the top line of $\pi'$ for any $l \leq i \leq m$.
2. In particular, if $\alpha(m)$ is in $d$-th position in the top line of $\pi'$, then $\gamma$ is trivial.
3. Finally, $v$ is a top cord with respect to $(\pi', \lambda', \tau')$ with $A(\pi', \lambda', \tau')(v) = A(\pi, \lambda, \tau)(v)$.

**Proof:** We first prove part (1) of the Lemma. We suppose that $\gamma$ is a simple arrow, the general case follows repeating the argument finitely many times. Let $\beta$ be the winner of $\gamma$, which does not belong to $A(v)$. If $\gamma$ is a top arrow, then the statement obviously follows with $k = 0$. If $\gamma$ is a bottom arrow, then we have two possibilities. If $\pi'(\beta) > m$ then the statement follows with $k = 0$. Otherwise we must have $\pi'(\beta) < l$, because $\beta$ does not belong to $A(v)$. Moreover $\alpha(l)$ cannot be the loser of $\gamma$ by assumption,
thus $\pi^i(\beta) \leq l - 2$. In this case the statement follows with $k = 1$. In order to prove part (2), observe that if $\alpha(m)$ is in $d$-th position in the top line of $\pi'$, then part (1) implies that $\alpha(m)$ is in $d$-th position also in the top line of $\pi$, and in such position $\alpha(m)$ can't neither win nor lose, since $\gamma$ is neutral. Part (3) follows from part (1), indeed we have $\zeta'_\alpha = \zeta_\alpha$ for any $\alpha$ in $\mathcal{A}(v)$, since $\gamma$ does not contain arrows with winner in $\mathcal{A}(v)$.

Lemma C.6. Consider a path $\gamma$ of the form $\gamma = \eta_\alpha(m) \cdots \eta_\alpha(l) \gamma(1)$, where $\gamma(1)$ is a (possibly trivial) neutral path and $\eta_\alpha(m), \ldots, \eta_\alpha(l)$ are bottom arrows with losers respectively $\alpha(m), \ldots, \alpha(l)$ and with the same winner $\beta \in \mathcal{A} \setminus \mathcal{A}(v)$. Then any letter $\alpha(i)$ is in $(d - m + i)$-th position in the top line of $\pi'$ for any $l \leq i \leq m$, that is

$$\pi' = \left(\begin{array}{ccccccc} \cdots & \beta & \cdots & \alpha(l) & \cdots & \alpha(m) & \beta \\ \end{array}\right).$$

Moreover $v$ is a top cord with respect to $(\pi', \lambda', \tau')$ with $\mathcal{A}(\pi', \lambda', \tau')(v) = \mathcal{A}(v)$.

Proof: Let $(\pi(1), \lambda(1), \tau(1))$ be the data determined by $\gamma(1)$ via the relation (C.2). Lemma C.5 implies that the letters $\alpha(l), \ldots, \alpha(m)$ keep their reciprocal order in the top line of $\pi(1)$ and they are possibly translated to the left. Moreover $\pi(1)$ is the endpoint of the bottom arrow $\eta_\alpha(l)$ with winner $\beta$ and looser $\alpha(l)$, therefore we have $\pi(1) = \left(\begin{array}{cccccc} \cdots & \beta & \alpha(l) & \cdots & \alpha(m) & \cdots \\ \end{array}\right)$. The pre-image of $\pi(1)$ under the arrows $\eta_\alpha(m), \ldots, \eta_\alpha(l)$ is easily computable, hence the statement on $\pi'$ follows. Since $\pi'$ is as in the statement, then the sum $\zeta'_\alpha + \cdots + \zeta'_\alpha$ is a top cord with respect to $(\pi', \lambda', \tau')$. Moreover $\gamma$ does not contain arrows with winner in $\mathcal{A}(v)$ therefore $\zeta'_\alpha = \zeta_\alpha$ for any $\alpha$ in $\mathcal{A}(v)$. It follows that $v = \zeta_\alpha(l) + \cdots + \zeta_\alpha(m) = \zeta'_\alpha(l) + \cdots + \zeta'_\alpha(m)$, that is $v$ is a top cord with respect to $(\pi', \lambda', \tau')$. $\Box$

The Corollary below follows immediately repeating $k$ times the argument in Lemma C.6.

Corollary C.7. Consider a path $\gamma$ of the form $\gamma = \gamma_k \cdots \gamma_1$, where $\gamma_1, \ldots, \gamma_k$ are as in Lemma C.6. Then $\pi'$ and $v$ are as in the conclusion of Lemma C.6.

Lemma C.8. Fix an integer $i$ such that $l \leq i \leq m$ and consider a path $\gamma$ given by the concatenation $\gamma = \nu_\alpha(i) \eta_\alpha(i) \cdots \eta_\alpha(l) \gamma(2) \gamma(1)$, where

1. $\gamma(1)$ is a (possibly trivial) path as in Corollary C.7,
2. $\gamma(2)$ is a (possibly trivial) neutral path,
3. $\eta_\alpha(i), \ldots, \eta_\alpha(l)$ are bottom arrows with looser respectively $\alpha(i), \ldots, \alpha(l)$ and winner $\beta$ in $\mathcal{A} \setminus \mathcal{A}(v)$,
4. $\nu_\alpha(i)$ is a top arrow with winner $\alpha(i)$.

Then, denoting by $(\pi', \lambda', \tau')$ the data determined by $\gamma$ via (C.2) we have $\text{Area}(v) \geq w(\pi', \lambda', \tau')$.

Proof: Denote $(\pi(1), \lambda(1), \tau(1))$ and $(\pi(2), \lambda(2), \tau(2))$ the triples of data determined via (C.2) respectively by the paths $\gamma(1)$ and $\gamma(2) \gamma(1)$. According to Corollary C.7 applied to the path $\gamma(1)$, for $\pi(1)$ we have the same conclusion as for $\pi'$ in Lemma C.6. Then Lemma C.5 applies to the path $\gamma(2)$ and moreover $\pi(2)$ is the ending point of the bottom arrow $\eta_\alpha(l)$ with winner $\beta \in \mathcal{A} \setminus \mathcal{A}(v)$, thus we have $\pi(2) = \left(\begin{array}{cccccc} \cdots & \beta & \alpha(l) & \cdots & \alpha(m) & \beta \\ \end{array}\right)$. Observe that $\pi(2, \nu_\alpha(i)) < d$, since $\beta \notin \mathcal{A}(v)$. Denote $\chi$ the letter in $\mathcal{A}$ with $\pi(2, \nu_\alpha(i)) = \pi(2, \nu_\alpha(i)) + 1$, where the case $\chi = \beta$ is possible. It is easy to check that the pre-image of $\pi(2)$ under the concatenation of arrows $\nu_\alpha(i) \eta_\alpha(i) \cdots \eta_\alpha(l)$ has the form

$$\pi' = \left(\begin{array}{cccccc} \cdots & \beta & \alpha(l) & \cdots & \alpha(i) & \chi \\ \cdots & \alpha(i) & \cdots & \beta & \chi \\ \end{array}\right),$$

where for any $l \leq j \leq i$ the letter $\alpha(j)$ is in $(d + l - j)$-th position in the top line of $\pi'$. Since $\gamma(1)$ and $\gamma(2)$ do not contain arrows with winner in $\mathcal{A}(v)$ then $\zeta'_\alpha = \zeta_\alpha$ for any $\alpha$ in $\mathcal{A}(v)$. Moreover the winner of the arrows $\eta_\alpha(i), \ldots, \eta_\alpha(l)$ is $\beta \in \mathcal{A} \setminus \mathcal{A}(v)$ and $\nu_\alpha(i)$ has winner $\alpha(i)$ and looser $\chi$, thus we have $\zeta'_\alpha = \zeta'_\alpha - \zeta_\chi$ and $\zeta_\alpha = \zeta'_\alpha$ for any $\alpha$ in $\mathcal{A}(v) \setminus \{\alpha(i)\}$. Setting $v' := \zeta_\alpha(l) + \cdots + \zeta_\alpha(i)$ we have

$$v' = \zeta_\alpha(l) + \cdots + \zeta_\alpha(i) = \zeta'_\alpha(l) + \cdots + \zeta'_\alpha(i) = \zeta'_\alpha(l) + \cdots + \zeta'_\alpha(i) - \zeta'_\chi = (\zeta'_\chi, w_{\pi', \alpha(i), \chi}).$$

Remark that in general $v'$ is not a period of the translation surface $X$, and thus nor a top cord with respect to $(\pi, \lambda, \tau)$. In order to prove that $\text{Area}(v') \leq \text{Area}(v)$, observe first that $3(v') < 0$. Consider the data
(π, λ, τ) and the corresponding broken line in the complex plane whose vertices are ξα(j) with l ≤ j ≤ m and ξα(l) = ξα(m). The vertices ξα(l) and ξα(l) + ξα(i) are connected by v′ = ξα(l) + \ldots + ξα(i), whereas the period v = ξα(l) + \ldots + ξα(m) connects the extremal vertices ξα(l) and ξα(m). Since v is a top cord then the intermediate vertex ξα(l) + ξα(i) lies above the line through ξα(l) and ξα(m) + ξα(l); moreover such line has negative slope, because |]|v’| < 0. It follows that Ξ(ξα(l)) ≥ Ξ(ξα(l) + ξα(i)) ≥ Ξ(ξα(l) + ξα(m)) and therefore ||Ξ(v’)| ≤ ||Ξ(v)|. We have obviously R(v’) = α(i) + \ldots + α(m) = R(v), thus Area(v’) ≤ Area(v).

Lemma C.9. Consider a path γ of the form γ = να where να is a bottom arrow with winner α ∈ A(v) and let (π′, λ′, τ′) be the data determined by να via (C.3). We have the following two possibilities

1. If α = α(i) with l ≤ i < m then v is a top cord with respect to (π′, λ′, τ′) and moreover we have A(π′, λ′, τ′)(v) = A(π, λ, τ)(v) \{ α(i + 1) \}.
2. If α = α(m) then Area(v) ≥ w(π′, λ′, τ′).

Proof: The data (π, λ, τ) are the image of a bottom arrow with winner α, thus α is in d-th position in the bottom line of π, moreover part (2) of Lemma C.3 implies τα > 0 both in case (1) and in case (2).

If α = α(i) with l ≤ i < m then α(i + 1) belongs to A(v) and the pre-image of π under α(i) is

π′ = \left( \ldots \alpha(l) \ldots \alpha(i) \ldots \alpha(i+1) \ldots \right).

In this case we have ξα′(i) = ξα(i) + ξα(i+1) and therefore v = Σα∈A(v) ξα = Σα∈A(v)\{α(i+1)\} ξ′. It follows that v is a top cord with respect to the data (π′, λ′, τ′) with A(π′, λ′, τ′)(v) = A(π, λ, τ)(v) \{ α(i + 1) \}.

If α = α(m) then we must have π′(α(m)) < d, because π′(α(m)) = d. If β ∈ A \ A(v) is the letter such that π′(β) = π′(α(m)) + 1, that is the letter which loses against α(m) in να, then the pre-image of π under να is

π′ = \left( \ldots \alpha(l) \ldots \alpha(m) \ldots \beta \ldots \right),

where the position of any letter α(i) ∈ A(v) in the top line of π′ is the same as its position in the top line of π. We have ξα(m) = ξ′(m) − ξ′(i) = ξ′, w′, β, α(m) and it is enough to prove that Area(ξα(m)) ≤ Area(π′).

Consider the data (π, λ, τ) and the corresponding broken line whose vertices are ξα(l) with l ≤ j ≤ m and ξα(l) + ξα(l). The vertices ξα(l) and ξα(l) + ξα(l) are connected by the period ξα(l), whereas the period v = ξα(l) + \ldots + ξα(m) connects the extremal vertices ξα(l) and ξα(m) + ξα(l). Since v is a top cord then the intermediate vertex ξα(l) lies above the line through ξα(l) and ξα(m) + ξα(l); moreover such line has positive slope, because τα(m) > 0. It follows that Ξ(ξ′(m)) ≤ Ξ(ξ′(m)) ≤ Ξ(ξ′(m) + ξα(m)) and therefore ||Ξ(ξα(m))|) ≤ ||Ξ(v)|. We have obviously R(ξα(m)) = α(m) = R(π′), thus Area(ξα(m)) ≤ Area(v).

C.1.4. The general case: proof of Proposition C.3 Let γ be as in Proposition C.3 and decompose it as

γ = γ^activ * γ^neutral,

where γ^neutral is the maximal neutral end of γ (recall that a finite Rauzy path is said neutral if it does not contain neither arrows with winner α ∈ A(π) nor bottom arrows with loser α ∈ A(π), whereas top arrows with loser α ∈ A(π) are admitted). Let (π(1), λ(1), τ(1)) be the combinatorial-length-suspension data determined by γ^neutral via the relation (C.2), thus in particular γ^activ ends in π(1). Since γ^neutral does not contain arrows with winner in A(π), then γ^activ is the concatenation of d paths γ_1 * \ldots * γ_d, where each letter in A(π) wins at least once in each γ_i. Moreover, the last arrow γ_last of γ^activ is as in cases A or B or C as below.

A: We have γ_last = να, where να is a top arrow with winner α ∈ A(π). In this case part (2) of Lemma C.3 implies that α = α(m) and γ^neutral is the trivial path. Therefore we have

π(1) = π = \left( \ldots \alpha(l) \ldots \alpha(m) \right).
B: We have $\gamma_{last} = \eta_\alpha$, where $\eta_\alpha$ is a bottom arrow with looser $\alpha \in A(v)$ and whose winner $\beta$ does not belong to $A(v)$. In this case, part (1) of Lemma $C.3$ implies $\pi(1).t(\beta) = \pi(1).t(\alpha(l)) - 1$, thus the loser of $\eta_\alpha$ is $\alpha = \alpha(l)$ and $\pi(1)$ has the form

$$\pi(1) = \left( \ldots \beta_{\alpha(l)}, \alpha(l + 1), \ldots \right).$$

C: We have $\gamma_{last} = \nu_\alpha$, where $\nu_\alpha$ is a bottom arrow with winner in $A(v)$. In this case we consider the following two subcases.

C1: We have $\alpha = \alpha(m)$.

C2: We have $\alpha = \alpha(i)$ with $l < i < m$.

Remark C.10. According to Lemma $C.9$, $v$ is a top cord with respect to $(\pi(1), \lambda(1), \tau(1))$ and we have $A_{(\pi(1), \lambda(1), \tau(1))}(v) = A_{(\pi, \lambda, \tau)}(v)$. Therefore, in order to prove Proposition $C.3$ it is enough to replace $\gamma$ by $\gamma^{\text{activ}}$ and $(\pi, \lambda, \tau)$ by $(\pi(1), \lambda(1), \tau(1))$. This amounts to consider data $(\pi, \lambda, \tau)$ in $\{\pi\} \times \mathbb{R}^d \times \Theta_\gamma$, where $\gamma$ is a path satisfying the conditions below.

1. The path $\gamma$ is the concatenation of $d$ paths $\gamma_1 \ast \cdots \ast \gamma_d$, where each letter in $A(v)$ wins in each $\gamma_i$.

2. The last arrow $\gamma_{last}$ of $\gamma$ is as in cases A or B or C above.

Case A. Let $(\pi', \lambda', \tau')$ be the data determined by $\nu_\alpha(m)$ via Equation $C.2$. If $\beta$ is the letter such that $\pi'(\beta) = \pi'(\alpha(m)) + 1$, we have $\zeta_\alpha(m) = \zeta_\alpha'(m) - \zeta_\beta$. Since the pre-image of $\pi$ under $\nu_\alpha(m)$ is $\pi' = (\alpha(l), \ldots, \alpha(m), \ldots, \beta)$, then Proposition $C.5$ follows with $\gamma' = \nu_\alpha(m)$, indeed we have

$$v = \zeta_\alpha(l) + \cdots + \zeta_\alpha(m) = \zeta_\alpha'(l) + \cdots + \zeta_\alpha'(m) - \zeta_\beta = (\zeta_\pi', \zeta_{\beta, \alpha(m)}).$$

Case B. Decompose the path $\gamma$ as $\gamma = \gamma^{\text{start}} \ast \gamma^{\text{end}}$, where $\gamma^{\text{end}}$ is minimal containing an arrow $\nu_\alpha$ with winner in $A(v)$. Observe that such decomposition exists, because each letter in $A(v)$ wins at least once in $\gamma$. We consider separately two cases, according to the type (top or bottom) of $\nu_\alpha$.

If $\nu_\alpha$ is a top arrow, let $i$ be the integer with $l < i < m$ such that the winner of $\nu_\alpha$ is $\alpha = \alpha(i)$. It is not difficult to check that $\gamma^{\text{end}}$ decomposes as in Lemma $C.8$. Thus, according to the same Lemma, Proposition $C.3$ follows with $\gamma' := \gamma^{\text{end}}$.

If $\nu_\alpha$ is a bottom arrow, it is easy to check that we have a decomposition $\gamma^{\text{end}} = \nu_\alpha \ast \gamma^{(1)} \ast \gamma^{(2)}$, where $\gamma^{(1)}$ is neutral and $\gamma^{(2)}$ is as in Corollary $C.7$. In this case, if $(\pi'', \lambda'', \tau'')$ are the data determined by $\gamma^{(1)} \ast \gamma^{(2)}$ via $C.2$, then Lemma $C.5$ and Corollary $C.7$ imply that $v$ is a top cord with respect to $(\pi'', \lambda'', \tau'')$ with $A_{(\pi'', \lambda'', \tau'')}(v) = A_{(\pi, \lambda, \tau)}(v)$. Moreover the path $\gamma^{\text{start}} \ast \nu_\alpha$ is a in Remark $C.10$ where its last arrow is as in case C, therefore in this case we can replace respectively $\gamma$ by $\gamma^{\text{start}} \ast \nu_\alpha$ and $(\pi, \lambda, \tau)$ by $(\pi'', \lambda'', \tau'')$ and then apply the discussion of Case C.

Case C1. Let $(\pi', \lambda', \tau')$ be the data determined by $\nu_\alpha(m)$ via the relation $C.2$. Proposition $C.8$ follows with $\gamma := \nu_\alpha(m), A_{(\pi, \lambda, \tau')}(v) := \{\alpha(m)\}$ and $v' := \zeta_\alpha(m)$, according to part (2) of Lemma $C.9$.

Case C2. Decompose $\gamma$ as $\gamma = \gamma^{\text{start}} \ast \nu_\alpha(i)$ and observe that $\gamma^{\text{start}}$ is the concatenation of at least $d - 1$ complete paths, because the initial path $\gamma$ in Proposition $C.3$ is assumed strongly complete. Part (1) of Lemma $C.9$ applies, therefore $v$ is a top cord with respect to the data $(\pi', \lambda', \tau')$ determined by $\nu_\alpha(i)$ via the relation $C.2$ and moreover we have $A_{(\pi', \lambda', \tau')}(v) = A_{(\pi', \lambda, \tau)}(v) \setminus \{\alpha(i + 1)\}$. We start an iterative procedure on the cardinality of $A(v)$, replacing the data $(\pi, \lambda, \tau)$ by $(\pi', \lambda', \tau')$, the path $\gamma$ by $\gamma^{\text{start}}$ and the set $A(v)$ by $A(v) \setminus \{\alpha(i + 1)\}$ and then repeating the discussion of $C.1$ for these new data. At each step, either Proposition $C.3$ is proved, or we end up in Case C2 and we apply part (1) of Lemma $C.9$ so that the new set $A(v)$ loses an element. After at most $m - l - 1$ steps, if Proposition $C.3$ is not yet proved, then we end up with a top cord with the simple form $v = \zeta_\alpha$ for some $\alpha$ in $A$. In this case, Proposition $C.3$ follows from part (1) of Lemma $C.4$. The proof of Proposition $C.3$ is completed.

C.2. Geometric bounds for $(\pi, \lambda, \tau)$. This section is devoted to the proof of Proposition $4.10$ and Proposition $4.11$. Fix combinatorial-length-suspension data $(\pi, \lambda, \tau)$ such that $\text{Area}(\pi, \lambda, \tau) = 1$. In this section we give an estimate of the distortion $\Delta(T)$ of $T = (\pi, \lambda)$ in terms of the minimum of the values $\text{Area}(v)$ over all cords $v$ with respect to $(\pi, \lambda, \tau)$.
C.2.1. **Geometric bounds via the control on cords.** Let \( k \) be an integer with \( 1 \leq k < d \). Recall from \( \text{(2.2)} \) that since \( \tau \) is a suspension datum for \( \pi \) then we have

\[(C.3) \quad h_{\alpha(k+1)} \geq \tau_{\alpha(1)} + \cdots + \tau_{\alpha(k)} > 0 \quad \text{and} \quad -h_{\alpha(k+1)} \leq \tau_{\beta(1)} + \cdots + \tau_{\beta(k)} < 0.\]

If we have \( h_{\alpha(k)} < \tau_{\alpha(1)} + \cdots + \tau_{\alpha(k)} \) we say that the letter \( \alpha(k) \) is *special in top line*. Observe that the condition is equivalent to have \( \pi'(\alpha(k)) = d \), that is \( \alpha(k) = \beta(d) \), and \( \sum_{\chi} \tau_{\chi} > 0 \). Similarly, if we have \( -h_{\alpha(k)} > \tau_{\beta(1)} + \cdots + \tau_{\beta(k)} \) we say that the letter \( \beta(k) \) is *special in bottom line* and we observe that this second condition is equivalent to have \( \pi'(\beta(k)) = d \), that is \( \beta(k) = \alpha(d) \), and \( \sum_{\chi} \tau_{\chi} < 0 \). We remark that for any triple of data \( (\pi, \lambda, \tau) \) with \( \sum_{\chi} \tau_{\chi} \neq 0 \) there always exists either an \( \alpha(k) \) which is special in top line, or an \( \beta(k) \) which is special in bottom line. Moreover, a special letter in top line is necessarily unique, and it excludes the existence of special letters in bottom line. Similarly, a special letter in bottom line is unique, and it excludes the existence of special letters in top line.

**Lemma C.11.** Consider an integer \( k \) with \( 1 \leq k < d \). The following holds.

1. We always have \( 0 < \tau_{\alpha(1)} + \cdots + \tau_{\alpha(k)} < \left(\lambda_{\alpha(k+1)}\right)^{-1} \). Moreover if \( \alpha(k) \) is not special in top line then we have \( 0 < \tau_{\alpha(1)} + \cdots + \tau_{\alpha(k)} < \left(\lambda_{\alpha(k)}\right)^{-1} \).
2. We always have \( -\left(\lambda_{\beta(k+1)}\right)^{-1} < \tau_{\beta(1)} + \cdots + \tau_{\beta(k)} < 0 \). Moreover if \( \beta(k) \) is not special in top line then we have \( -\left(\lambda_{\beta(k)}\right)^{-1} < \tau_{\beta(1)} + \cdots + \tau_{\beta(k)} < 0 \).

**Proof:** We just prove part (1), part (2) being the same. We have \( \chi \cdot h_{\chi} < 1 \) for any \( \chi \in \mathcal{A} \), thus the first estimate follows from Equation \( (C.3) \). Since \( \alpha(k) \) is not special in top line, then \( \sum_{j \leq k} \tau_{\alpha(j)} \leq h_{\alpha(k)} \) and the second inequality follows trivially.

**Lemma C.12.** Consider \( \delta > 0 \) such that any cord \( v \) with respect to \( (\pi, \lambda, \tau) \) satisfies \( \text{Area}(v) > \delta \). For any \( 1 \leq k < d \) we have the following two conditions.

\[
(\lambda_{\alpha(1)} + \cdots + \lambda_{\alpha(k)}) \cdot |\tau_{\alpha(1)} + \cdots + \tau_{\alpha(k)}| \geq \delta,
\]

\[
(\lambda_{\beta(1)} + \cdots + \lambda_{\beta(k)}) \cdot |\tau_{\beta(1)} + \cdots + \tau_{\beta(k)}| \geq \delta.
\]

**Proof:** We prove the first equation, the second being the same. Set \( v_k := \zeta_{\alpha(1)} + \cdots + \zeta_{\alpha(k)} \) for \( 1 \leq k < d \) and observe that the \( v_k \) for which \( \text{Area}(v_k) \) is minimal is a top cord with respect to \( (\pi, \lambda, \tau) \).

**Lemma C.13.** Consider \( \delta > 0 \) such that any cord \( v \) with respect to \( (\pi, \lambda, \tau) \) satisfies \( \text{Area}(v) > \delta \). For any integer \( k \) with \( 1 \leq k < d \) there exists positive integers \( j \) and \( i \) with \( j \leq k \) and \( i \leq k \) such that

\[
\lambda_{\alpha(j)} > \frac{\delta}{k} \lambda_{\alpha(k+1)} \quad \text{and} \quad \lambda_{\beta(i)} > \frac{\delta}{k} \lambda_{\beta(k+1)}
\]

**Proof:** Part (1) of \( \text{(C.11)} \) and part (1) of Lemma \( \text{(C.12)} \) imply

\[
\frac{1}{\lambda_{\alpha(k+1)}} > \tau_{\alpha(1)} + \cdots + \tau_{\alpha(k)} \geq \frac{\delta}{\lambda_{\alpha(1)} + \cdots + \lambda_{\alpha(k)}},
\]

thus there exists \( j \leq k \) with \( \lambda_{\alpha(j)} > \delta \lambda_{\alpha(k+1)}/k \). The existence of \( \beta(i) \) follows by a similar argument.

**Lemma C.14.** Consider \( \delta > 0 \) such that any cord \( v \) with respect to \( (\pi, \lambda, \tau) \) satisfies \( \text{Area}(v) > \delta \). Then for any integers \( m, l \) with \( 2 \leq l + 1 < m \leq d \) both the following two properties hold.

1. If \( \alpha(l) \) is not special in top line, then there exists an integer \( k \) with \( l < k < m \) such that

\[
\lambda_{\alpha(k)} > \frac{\delta}{m - l - 1} \cdot \min\{\lambda_{\alpha(l)}, \lambda_{\alpha(m)}\}.
\]

2. If \( \beta(l) \) is not special in bottom line, then there exists an integer \( k \) with \( l < k < m \) such that

\[
\lambda_{\beta(k)} > \frac{\delta}{m - l - 1} \cdot \min\{\lambda_{\beta(l)}, \lambda_{\beta(m)}\}.
\]

**Proof:** We just prove part (1), the second being the same. Fix \( \epsilon > 0 \) such that \( \lambda_{\beta(l)} > \epsilon \) and \( \lambda_{\beta(m)} > \epsilon \). It is enough to show that

\[
\lambda_{\alpha(l+1)} + \cdots + \lambda_{\alpha(m-1)} > \delta \cdot \epsilon.
\]

Setting \( v := \zeta_{\alpha(l+1)} + \cdots + \zeta_{\alpha(m-1)} \), the statement is equivalent to \( \Re(v) > \delta \cdot \epsilon \). Assume by absurd that \( 0 < \Re(v) \leq \delta \cdot \epsilon \). We have \( \lambda_{\alpha(l)} \cdot h_{\alpha(l)} < 1 \) and \( \lambda_{\alpha(m)} \cdot h_{\alpha(m)} < 1 \), thus part (1) of Lemma \( \text{(C.11)} \) implies
\[ \tau_{\alpha(1)} + \cdots + \tau_{\alpha(m-1)} < 1/\epsilon \] and \[ \tau_{\alpha(1)} + \cdots + \tau_{\alpha(l)} < 1/\epsilon, \] where the second inequality holds because we assume that \( \alpha(l) \) is not special in top line. Consider the broken line whose vertices are the points \( \zeta_{\alpha(k)}^l = \zeta_{\alpha(1)} + \cdots + \zeta_{\alpha(k-1)} \) for \( l < k \leq m \). If we assume \( 0 < \Re(v) \leq \delta \cdot \epsilon \), then the last two inequalities imply that any vertex \( \zeta_{\alpha(k)}^l \) with \( l + 1 < k \leq m - 1 \) must lie above the line connecting the extremal vertices \( \zeta_{\alpha(l+1)}^l \) and \( \zeta_{\alpha(m)}^l \) indeed otherwise we would get a top cord \( v' \) with \( \text{Area}(v') < \delta \cdot \epsilon / 1/\epsilon = \delta \).

It follows that \( v \) is a top cord, but then the assumption \( 0 < \Re(v) \leq \delta \cdot \epsilon \) implies \( \text{Area}(v) < \delta \), which is absurd. \( \square \)

Recall the notation \( \| \lambda \| := \sum_{\chi \in A} \lambda_\chi = 1. \)

**Proposition C.15.** Consider \( \delta > 0 \) such that any cord \( v \) with respect to \((\pi, \lambda, \tau)\) satisfies \( \text{Area}(v) > \delta \).

Then for any letter \( \alpha \) in \( A \) we have

\[ \lambda_\alpha > \frac{\delta^{d-1}}{d!} \cdot \| \lambda \|. \]

**Proof:** There obviously exists a letter \( \chi(1) \in A \) such that \( \lambda_{\chi(1)} \geq \| \lambda \| / d \). We set \( A_1 := \{ \chi(1) \} \) and we define an increasing sequence of sub-alphabets \( A_1 \subset \cdots \subset A_i \subset \cdots \subset A_d := A \) such that for any \( 1 \leq i \leq d \) and any \( \chi \in A_i \) we have

\[ \lambda_\chi > \delta^{d-1} \cdot \frac{(d-i)!}{d!} \cdot \| \lambda \|. \]

Consider \( 1 \leq i < d \) and suppose that \( A_i \) is defined. In order to define \( A_{i+1} \) it is enough to find a letter \( \chi(i+1) \in A \setminus A_i \) such that

\[ \lambda_{\chi(i+1)} > \frac{\delta}{d-i} \cdot \min_{\chi \in A_i} \lambda_\chi. \]

The sequence of alphabets \( A_i \) is defined according to two combinatorial procedures, the discriminant to pass from the first procedure to the second being the condition

\[ \{ \alpha(1), \beta(1) \} \subset A_i. \]

Observe that \( A_i \) obviously does not satisfy (C.3). Fix \( 1 \leq i < d \), suppose that \( A_i \) is defined and moreover that the condition (C.3) is not satisfied. Assume \( \alpha(1) \not\in A_i \), the argument for the case \( \beta(1) \not\in A_i \) being the same. Let \( \alpha(k) \in A_i \) be the letter such that \( \pi^l(\chi) \geq \pi^l(\alpha(k)) \) for any other \( \chi \in A_i \). According to Lemma C.13 there exists \( j \) with \( 1 \leq j < k \) and such that \( \lambda_{\alpha(j)} > \delta \lambda_{\alpha(k)}(k-1)^{-1} \).

Observe that \( k-1 \leq d-i \), thus the letter \( \chi(i+1) := \alpha(j) \in A \setminus A_i \) satisfies the condition (C.4).

Let \( r \) be minimal such that \( A_r \) is defined and moreover the condition (C.3) is satisfied. Assume that \( A_r \) is defined for \( r \leq i < d \) and observe that such \( A_r \) still satisfies (C.3). Consider the conditions below.

1. There exist integers \( l, m \) with \( 2 \leq l+1 < m \leq d \) such that \( \alpha(l) \in A_i \) and \( \alpha(m) \in A_i \) and \( \alpha(j) \in A \setminus A_i \) for any \( l < j < m \).
2. There exist integers \( l, m \) with \( 2 \leq l+1 < m \leq d \) such that \( \beta(l) \in A_i \) and \( \beta(m) \in A_i \) and \( \beta(j) \in A \setminus A_i \) for any \( l < j < m \).
3. Condition (1) holds, moreover \( \alpha(l) \) is not special in top line.
4. Condition (2) holds, moreover \( \beta(l) \) is not special in bottom line.

We first prove that we have either condition (3) or condition (4), then we will define the required letter \( \chi(i+1) \) using one of these two conditions. Since \( r \) is admissible, then condition (C.3) implies that either condition (1) or condition (2) hold. Suppose that condition (1) holds, but (3) is no true, that is \( \alpha(l) \) is special in top line, thus in particular \( \alpha(l) = \beta(d) \). In this case, since \( A_i \) has just \( i \) elements, then there exists \( \beta(l') \) and \( \beta(m') \) satisfying condition (2), moreover \( \beta(l') \) cannot be special in bottom line, because \( \alpha(l) \) is special in top line, therefore condition (4) holds. Conversely, suppose that condition (2) holds but (4) is not true. Then arguing as above, we get that condition (3) holds. Now we define the letter \( \chi(i+1) \).

If condition (3) holds, observing that \( m-l \leq d-i \) and applying part (1) of Lemma C.14 we get \( j \) with \( l < j < m \) such that

\[ \lambda_{\alpha(j)} > \frac{\delta}{d-i} \cdot \min \{ \lambda_{\alpha(l)}, \lambda_{\alpha(m)} \}. \]

The letter \( \chi(i+1) := \alpha(j) \) satisfies (C.3). If condition (4) holds, then we apply part (2) of Lemma C.14 and the same argument. \( \square \)
C.2.2. Proof of Proposition 4.10. Consider data \((\pi, \lambda, \tau)\) with \(\text{Area}(\pi, \lambda, \tau) = 1\). In order to prove Proposition 4.10 it is enough to prove that for any \(\alpha\) in \(A\) we have
\[
\frac{\lambda_\alpha}{\|\lambda\|} \geq \frac{1}{d!} m(\pi, \lambda, \tau)^{d-1}.
\]
Let \(\delta\) be the minimum of \(\text{Area}(v)\) over all cords \(v\) with respect to \((\pi, \lambda, \tau)\). According to Proposition C.3 we have \(\delta \geq m(\pi, \lambda, \tau)\). Then the required estimate follows from Proposition C.15. The proof of Proposition 4.10 is complete.

C.2.3. Geometric bounds via the control on positive matrices. Fix data \((\pi, \lambda, \tau)\) and let \(X\) be the underlying translation surface. Let \(T\) be the IET determined by the pair \((\pi, \lambda)\), acting on the interval \(I = (0, \sum_{\chi \in A} \lambda_\chi)\), which is embedded in \(X\) along the horizontal direction and with the left endpoint in a conical singularity. The vertical flow of \(X\) corresponds to the suspension flow \(\phi^t\) over \(T\) under the roof function \(h = h(\pi, \tau)\) defined in (2.2). For letters \(\beta\) and \(\alpha\) with \(\pi^t(\alpha) > 1\) and \(\pi^t(\beta) > 1\) consider the integer vectors \(w^t_{\beta, \pi}, w^t_{\alpha, \pi}\) and \(w^t_{\beta, \alpha, \pi} = w^t_{\beta, \pi} - w^t_{\alpha, \pi}\), defined in (4.11). Recall that the singularities of \(T\) and \(T^{-1}\) are given by \(u^t\alpha = \langle w^t_{\alpha, \pi}, \lambda\rangle\) and \(u^t\beta = \langle w^t_{\beta, \pi}, \lambda\rangle\) respectively, and thus the values \(|\langle \lambda, \alpha, \beta, \tau \rangle|\) represent the distance between singularities of \(T\) and of \(T^{-1}\). When \(v := \langle \zeta, \pi, \lambda, \tau \rangle\) is a period of \(X\), then \(|\langle \lambda, \alpha, \beta, \tau \rangle| = |\mathcal{R}(v)|\). Denote \(\|\lambda\|_{\infty} := \max_\chi \lambda_\chi\).

Lemma C.16. Let \(\gamma\) and \(\gamma'\) be two positive paths that can be concatenated and such that \(\gamma \ast \gamma'\) ends in \(\pi\), then consider \(\lambda\) in \(\Delta_{\gamma \ast \gamma'}\). For any \(\beta\) and \(\alpha\) as above we have
\[
|\langle \lambda, \alpha, \beta, \tau \rangle| \geq \frac{\|\lambda\|_{\infty}}{B^\tau_{\gamma} \|B^\tau_{\gamma'}\|_{\infty}}.
\]
Proof: Let \((\pi', \lambda')\) be the image of \((\pi, \lambda)\) of the branch of the Rauzy map \(Q_\gamma\) determined by \(\gamma\). Since \(\gamma\) is positive, then Corollary C.2 implies
\[
|\langle \lambda, \alpha, \beta, \tau \rangle| \geq \min_{\chi \in A} \lambda'_\chi.
\]
On the other hand, \(\lambda'\) belongs to \(\Delta_{\gamma'}\), that is the image under \(t^tB_{\gamma'}\) of the cone \(\mathbb{R}_+^A\), thus we have
\[
\|\lambda'\|_{\infty} \leq \|B^\tau_{\gamma'}\|_{\infty} \cdot \min_{\chi \in A} \lambda'_\chi.
\]
Finally, since \(\lambda = t^t B_{\gamma} \lambda'\) then considering the norm of the linear operator \(t^t B_{\gamma}\) we get
\[
\|\lambda\|_{\infty} \leq \|B_{\gamma}\|_{\infty} \cdot \|\lambda'\|_{\infty}.
\]
The Lemma follows summing up the three results. \(\Box\)

Let \(\beta\) and \(\alpha\) be letters such that \(\pi^t(\alpha) > 1\) and \(\pi^t(\beta) > 1\). Recall that \(\langle u^t_{\alpha, \pi}, \tau \rangle\) equals the length of the vertical segment connecting \(u^t_{\alpha}\) to the conical singularity where its orbit ends. Similarly, \(\langle u^t_{\beta, \pi}, \tau \rangle\) equals the length of the vertical segment connecting (in negative time) \(u^t_{\beta}\) to the conical singularity where its orbit starts. When \(v := \langle \zeta, \pi, \lambda, \tau \rangle\) is a period of \(X\), then \(|\langle \tau, \alpha, \beta, \tau \rangle| = |\mathcal{R}(v)|\).

Lemma C.17. Let \(\gamma\) and \(\gamma'\) be two positive paths that can be concatenated and such that \(\gamma \ast \gamma'\) starts at \(\pi\), then consider \(\tau\) in \(\Theta_{\gamma \ast \gamma'}\). For any \(\beta\) and \(\alpha\) as above we have
\[
|\langle \tau, \alpha, \beta, \tau \rangle| \geq 2 \cdot \frac{\|h\|_{\infty}}{\|B^\tau_{\gamma}\|_{\infty} \cdot \|B^\tau_{\gamma'}\|_{\infty}}.
\]
Proof: Let \((\pi', \lambda', \tau')\) be the data determined by \(\gamma'\) via the relation (C.2). Let \(T'\) be the IET corresponding to the pair \((\pi', \lambda')\) and \(I'\) be the interval where \(T'\) acts. The interval \(I'\) contains \(I\) as subinterval (with the same left endpoint) and is horizontally embedded in \(X\), so that \(T'\) is the first return map to \(I'\) of the vertical flow \(\phi\) on \(X\). Let \(\phi' = h'(\pi', \tau')\) be the vector whose entries are the (piecewise constant) values of the height function for \(T'\). Lemma C.3 applies to \(T\), since \(\gamma'\) is positive, thus \(I\) does not contain any singularity of \(T'\) or of \((T')^{-1}\). Since \(u^t_{\alpha}\) is a singularity of \(T\), then its positive vertical orbit comes back to \(I'\) before reaching a conical singularity, therefore we have \(\langle u^t_{\alpha, \pi}, \tau \rangle\rangle \geq \min_{\chi \in A} h'_\chi\). The symmetric argument in negative time applies to \(u^t_{\beta}\), which is a singularity for \((T')^{-1}\). Combining the two estimates, we get
\[
|\langle \tau, \alpha, \beta, \tau \rangle| \geq 2 \cdot \min_{\chi \in A} h'_\chi.
\]
The two height functions $h$ and $h'$ are related via the co-cycle by the relation $h = B_\gamma h'$, therefore we have

$$\|h\|_\infty \leq \|B_\gamma\|_\infty \cdot \|h'\|_\infty.$$  

Moreover the same relation holds for the matrix $B_\gamma$, that is we have $h' = B_\gamma h''$ for some vector $h''$ in $\mathbb{R}_A^d$, hence positivity of $\gamma$ implies that for any $\chi \in A$ we have

$$\|h'\|_\infty \leq \|B_\gamma\|_\infty \cdot h'_\chi$$

The Lemma follow summing up the three results above. \qed

Lemma \[C.4\] and Lemma \[C.17\] immediately imply the following Corollary.

**Corollary C.18.** Let $\gamma$ and $\gamma'$ be two positive paths that can be concatenated and such that $\gamma \ast \gamma'$ start at $\pi$, then consider $\tau$ in $\Theta_{\gamma \ast \gamma'}$. For any $\beta$ and $\alpha$ as above we have

$$\langle \tau, w, \alpha, \pi \rangle \geq 2 \cdot \|\tau\|_\infty \cdot \|B_\gamma\|_\infty \cdot \|B_{\gamma'}\|_\infty.$$  

C.2.4. *Proof of Proposition \[C.14\]* Let $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ be positive paths that can be concatenated so that $\pi$ is the ending point of $\gamma_1 \ast \gamma_2$ and the starting point of $\gamma_3 \ast \gamma_4$. Consider length-suspension data $(\lambda, \tau)$ for $\pi$ with $\text{Area}(\pi, \lambda, \tau) = \langle \lambda, h \rangle = 1$ and such that $\tau$ belongs to the sub-cone $\Theta_{\gamma_1, \gamma_2}$ of $\Theta_\pi$ and $\lambda$ belongs to $\Delta_{\gamma_3, \gamma_4}$. Fix any pair of letters $\beta$ and $\alpha$ with $\pi(\alpha) > 1$ and $\pi(\beta) > 1$. Lemmas \[C.16\] and \[C.17\] imply

$$\langle \lambda, w, \beta, \alpha, \pi \rangle \cdot \|\tau\|_\infty \cdot \|B_{\gamma_3}\|_\infty \cdot \|B_{\gamma_4}\|_\infty \geq \frac{2}{d} \cdot \|\tau\|_\infty \cdot \|B_{\gamma_1}\|_\infty \cdot \|B_{\gamma_2}\|_\infty,$$

where the second inequality follows combining the inequality $\|\cdot\|_2 \leq \sqrt{d} \cdot \|\cdot\|$ between the $L_\infty$ and $L_2$ norms and the Cauchy-Schwartz inequality applied to the scalar product $\langle \lambda, h \rangle = 1$. Since the above inequality holds for any $\beta$ and $\alpha$, then it holds for $w(\pi, \lambda, \tau)$ too. Proposition \[C.14\] is proved.

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