HAUSDORFF DIMENSION AND QUASI-SYMMETRIC UNIFORMIZATION OF CANTOR CIRCLE JULIA SETS

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Abstract. We calculate the Hausdorff dimension of the Julia sets which are Cantor circles, and prove that for any Cantor circle hyperbolic component \( H \) in the space of rational maps, the infimum of the Hausdorff dimensions of the Julia sets of the maps in \( H \) is equal to the conformal dimension of the Julia set of any representative \( f_0 \in H \), and that the supremum of the Hausdorff dimensions is equal to 2.

For each Cantor circle Julia set of a hyperbolic rational map, we prove that this Julia set is quasi-symmetrically equivalent to a standard Cantor circle (each connected component is a round circle). This gives a quasi-symmetric uniformization of all hyperbolic Cantor circle Julia sets. We also give an explicit formula of the numbers of the Cantor circle hyperbolic components in the moduli space of rational maps with any fixed degree.

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1. Introduction

The study of topological and geometric properties of the Julia sets of holomorphic functions is one of the important topics in complex dynamics. In this paper we study a class of Julia sets of rational maps with special topology: they are all homeomorphic to the Cartesian product of the middle third Cantor set and the unit circle, i.e. the Cantor circles. McMullen is the first one who constructed such kind of Julia sets [McMM8], and his family of rational maps was referred as McMullen maps later (see [DLU08], [Ste06], and [QWY12]).

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Besides the McMullen maps, one can find the Cantor circle Julia sets in some other families. For example, see [HP12], [XQY14], [FY15], [QYY15], [QYY16] and [WYZL18]. In particular, in the sense of topological conjugacy on the Julia sets, all the Cantor circle Julia sets have been found in [QYY15].

Except [HP12], only few work has been done on the geometric properties of the Cantor circle Julia sets. In this paper we focus our attention on the two aspects of the Cantor circle Julia sets: quasisymmetric classification and the Hausdorff dimensions (including conformal dimensions). We will give all hyperbolic Cantor circle Julia sets a quasi-symmetric uniformization and calculate the infimum and the supremum of the Hausdorff dimensions of the Julia sets in each Cantor circle hyperbolic component.

1.1. Statement of the results. Let \((X, d_X)\) and \((Y, d_Y)\) be two metric spaces. Suppose that there exist two homeomorphisms \(f : X \to Y\) and \(\psi : [0, +\infty) \to [0, +\infty)\) such that

\[
d_Y(f(x), f(y)) \leq \psi(d_X(x, y)) \leq d_Y(f(x), f(z))
\]

for every distinct points \(x, y, z \in X\). Then \((X, d_X)\) and \((Y, d_Y)\) is called quasisymmetrically equivalent to each other.

From the topological point of view, all Cantor circle Julia sets are the same since they are all topologically equivalent (homeomorphic) to the ‘standard’ Cantor circle. Hence a natural problem is to give a uniformization of the Cantor circle Julia sets in the sense of quasi-symmetric equivalence. Actually, some similar kind of this work has been done in [QYY16]. In that paper each Cantor circle Julia set has been given a quasi-symmetrically equivalent model, which is also a Cantor circle Julia set. In this paper, we prove the following result.

**Theorem 1.1.** Let \(f\) be a hyperbolic rational map whose Julia set is a Cantor circle. Then \(f\) is quasi-symmetrically equivalent to a standard Cantor circle.

The explicit definition of the ‘standard’ Cantor circles will be given in [2]. Roughly speaking, a standard Cantor circle is the Cartesian product of a Cantor set and the unit circle, where this Cantor set is generated by an iterated function system whose elements are affine transformations. For the study of quasi-symmetric uniformization of Cantor circles of McMullen maps, one may refer to [QYY18].

Recently, the quasi-symmetric geometries of some other types of the Julia sets of rational maps have been studied. For example, the critically finite rational maps with Sierpiński carpet Julia sets was studied in [BLM16], and the critically infinite case was studied in [QYZ19]. Moreover, the group of all quasi-symmetric self-maps of the Julia sets of \(z \mapsto z^2 - 1\) (i.e. the basilica) has been studied in [LM18].

Let \(\text{Rat}_d = \mathbb{C}P^{2d+1} \setminus \{\text{Resultant} = 0\}\) be the space of rational maps of degree \(d \geq 2\). The moduli space of \(\text{Rat}_d\) is \(\mathcal{M}_d = \text{Rat}_d/\text{PSL}_2(\mathbb{C})\), where \(\text{PSL}_2(\mathbb{C})\) is the complex projective special linear group. The Möbius conjugate class of \(f \in \text{Rat}_d\) in \(\mathcal{M}_d\) is denoted by \(\langle f \rangle\). By abuse of notations, we also use \(f\) to denote the equivalent class \(\langle f \rangle\) for simplicity. A rational map is called hyperbolic if all its critical orbits are attracted by the attracting periodic cycles. Each connected component of all hyperbolic maps in \(\mathcal{M}_d\) is called a hyperbolic component.

Let \(f_1, f_2\) be two rational maps. We say that \((f_1, J(f_1))\) and \((f_2, J(f_2))\) are topologically conjugate on their corresponding Julia sets \(J(f_1)\) and \(J(f_2)\) if there is an orientation preserving homeomorphism \(\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}\) such that \(\phi(J(f_1)) = J(f_2)\) and \(\phi \circ f_1 = f_2 \circ \phi\) on \(J(f_1)\). It is known from Mañé-Sad-Sullivan [MSSS83] that if \(f_1, f_2\) are in the same hyperbolic component of \(\mathcal{M}_d\), then \(f_1\) and \(f_2\) are topologically
Conjugate on their corresponding Julia sets. In this paper we prove that the converse of this statement is also true when the Julia sets are Cantor circles.

**Theorem 1.2.** Let $f_1, f_2$ be two hyperbolic rational maps whose Julia sets are Cantor circles. Then $f_1$ and $f_2$ lie in the same hyperbolic component of $\mathcal{M}_d$ if and only if they are topologically conjugate on their corresponding Julia sets.

Theorem 1.2 leads to the calculation of the number $N(d)$ of the Cantor circle hyperbolic components in $\mathcal{M}_d$. We can obtain an explicit formula of $N(d)$ as stated in the following theorem.

**Theorem 1.3.** The number of Cantor circle hyperbolic components in $\mathcal{M}_d$ is a finite number $N(d)$ depending only on the degree $d \geq 5$, which can be calculated by

$$N(d) = \sum_{n \geq 2} \sharp \left\{ (d_1, \ldots, d_n) \in \mathbb{N}^n \left| \sum_{i=1}^n d_i = d, \sum_{i=1}^n \frac{1}{d_i} < 1 \right. \right\}$$

$$+ \sum_{\text{odd } n \geq 3} \sharp \left\{ (d_1, \ldots, d_n) \in \mathbb{N}^n \left| \sum_{i=1}^n d_i = d, \sum_{i=1}^n \frac{1}{d_i} < 1, (d_1, \ldots, d_n) = (d_n, \ldots, d_1) \right. \right\}.$$  \hspace{1cm} (1.1)

We will show that the Julia set of a rational map cannot be a Cantor circle if its degree is less than 5 (see Proposition 4.1). See Table 1 in §4 for the list of $N(d)$ with $5 \leq d \leq 36$. For example, $N(5) = \sharp\{(2, 3), (3, 2)\} = 2$, $N(6) = \sharp\{(2, 4), (3, 3), (4, 2)\} = 3$ and $N(10) = \sharp\{(2, 8), (3, 7), (4, 6), (5, 5), (6, 4), (7, 3), (8, 2)\} + \sharp\{(3, 3, 4), (3, 4, 3), (4, 3, 3)\} + \sharp\{(3, 4, 3)\} = 11$. For the characterization of the global topological structure of Cantor circle hyperbolic components, see [WY17].

The conformal dimension $\dim_C(X)$ of a compact set $X$ is the infimum of the Hausdorff dimensions of all metric spaces which are quasi-symmetrically equivalent to $X$. For a given hyperbolic component $\mathcal{H}$ in $\mathcal{M}_d$, it follows from [MSS83] that all the Julia sets of the maps in $\mathcal{H}$ are quasi-symmetrically equivalent to each other and hence they have the same conformal dimension. There is a natural question:

**Question.** Let $\mathcal{H}$ be a hyperbolic component in $\mathcal{M}_d$ with $d \geq 2$ containing a map $f_0$. Does one have: $\inf_{f \in \mathcal{H}} \dim_H(J(f)) = \dim_C(J(f_0))$?

In this paper we give an affirmative answer to this question for Cantor circle hyperbolic components. We prove the following result.

![Figure 1: The Julia set of the McMullen map $z \mapsto z^2 + 10^{-5}/z^3$ (which is a Cantor set of circles) and its corresponding standard Cantor circle, which is generated by the iterated function system $\{e^{-3}/z^3, z^2\}$.](image)
Theorem 1.4. Let $\mathcal{H}$ be a Cantor circle hyperbolic component containing a rational map $f_0$. Then
\[ \inf_{f \in \mathcal{H}} \dim_H(J(f)) = \dim_C(J(f_0)) \quad \text{and} \quad \sup_{f \in \mathcal{H}} \dim_H(J(f)) = 2. \]

In fact, we can show that the conformal dimension of $J(f_0)$ is $1 + \alpha$, where $\alpha$ is the unique positive root of $\sum_{i=1}^{n} d_i^{\alpha} = 1$, where $(d_1, \cdots, d_n)$ is determined by the combinatorial information of the maps in the Cantor circle hyperbolic component $\mathcal{H}$ (see Proposition 5.2). Moreover, we believe that $\sup_{f \in \mathcal{H}} \dim_H(J(f)) = 2$ holds for any hyperbolic component in the space of rational maps with any fixed degree.

For the study of the Hausdorff dimension of Cantor circle Julia sets (or their subsets) of McMullen maps, one may refer to [WY14] and [BW15, Theorem C(b)]. Haisinsky and Pilgrim has constructed two quasi-symmetrically inequivalent hyperbolic Cantor circle Julia sets from McMullen maps by studying their conformal dimensions [HP12].

Corollary 1.5. The Hausdorff dimension of any Cantor circle Julia set lies in the interval $(1, 2)$. Moreover, for each given $1 < s < 2$, there exists a Cantor circle Julia set $J$ such that the Hausdorff dimension of $J$ is exactly $s$.

1.2. Organization of the paper and the sketch of the proofs. In §2 we will give the Cantor circle Julia sets three types of models. Each of these models is based on the combinations of the dynamics of Cantor circle rational maps. The combinatorial information allows us to define an associated iterated function system (IFS) whose attractor is the so called standard Cantor circle. We prove the quasisymmetric uniformation by constructing a quasiconformal homeomorphism which maps the hyperbolic Cantor circle Julia set to the attractor of the associated IFS.

Let $f$ and $g$ be two rational maps with Cantor circle Julia sets on which the dynamics are conjugate to each other. The idea of proving Theorem 1.2 is to make the deformations on the critical annuli and obtain a continuous path $(f_t)_{t \in [0, 1]}$, $(g_t)_{t \in [0, 1]}$ of hyperbolic rational maps such that $f_0 = f$, $g_0 = g$ and $f_1 = g_1$ (see Theorem 3.2). In order to state the procedure more clearly, the deformations are made on the standard annuli, which lie in the dynamical plane of a quasi-regular map $\tilde{F}$ whose restriction on some annuli is exactly the IFS associated to $f$ (and $g$). This section is the most important part of this paper.

Based on Theorem 1.2 we can obtain the computational formula of the Cantor circle hyperbolic component by considering the different topological conjugate class of Cantor circle Julia sets and hence prove Theorem 1.3. This will be done in §4.

Still by Theorem 1.2 we can find a specific rational map $f_{d_1, \cdots, d_n}$ with a Cantor circle Julia set in each Cantor circle hyperbolic component (see Theorem 3.1 and Corollary 3.3). For the infimum of the Hausdorff dimensions of the Cantor circle Julia sets, we will study the specific $f_{d_1, \cdots, d_n}$, decompose the dynamics of $f_{d_1, \cdots, d_n}$ to obtain an iterated function system. By estimating the contracting factors of the inverse of $f_{d_1, \cdots, d_n}$ in the log-plane, we prove the first part of Theorem 1.4 by using Falconer’s criterion on the calculation of Hausdorff dimensions.

For the supremum of the Hausdorff dimension of the Cantor circle Julia sets, we will use a theorem on Hausdorff dimensions established by Shishikura. Then the second part of Corollary 1.5 can be obtained immediately by the continuous dependence of the Hausdorff dimension of hyperbolic rational maps.

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Notations. We will use the following notations throughout the paper. Let \( \mathbb{C} \) be the complex plane and \( \hat{\mathbb{C}} = \mathbb{C} \cup \{ \infty \} \) the Riemann sphere. Let \( \mathbb{D}_r := \mathbb{D}(0, r) \) be the disk centered at the origin with radius \( r \) and \( \mathbb{T}_r := \partial \mathbb{D}_r \) the boundary of \( \mathbb{D}_r \). In particular, \( \mathbb{D} := \mathbb{D}_1 \) is the unit disk. For \( 0 < r < R < +\infty \), let \( A(r, R) := \{ z \in \mathbb{C} : r < |z| < R \} \) be the annulus centered at the origin. Moreover, we denote by \( A_r := A(r, 1) \) for all \( 0 < r < 1 \).

2. Quasi-symmetric uniformalization

From the topological point of view, all Cantor circles are the same since they are all homeomorphic to the Cartesian product of the middle third Cantor set and the unit circle. In this section we study the Cantor circle Julia sets in the sense of quasi-symmetric equivalence. This will give all hyperbolic Cantor circle Julia sets a more rich geometric classification.

2.1. Combinatorics of Cantor circle Julia sets. In this subsection we give a sketch of all the possible combinatorics of the Cantor circle Julia sets. For more details, one may refer to \[\text{[QYY15, QYY16 and WY17].}\]

Let \( f \) be a hyperbolic rational map of degree \( d \geq 2 \) whose Julia set is a Cantor set of circles. Note that the complement of any Cantor circle Julia set (i.e. the Fatou set) consists of two simply connected components and countably many doubly connected components. In the following, we always make the following assumption:

**Assumption**: \( f \) is chosen in the moduli space of the rational maps such that the two simply connected Fatou components of \( f \), denoted by \( D_0 \) and \( D_\infty \), contain 0 and \( \infty \) respectively.

Note that all the doubly connected Fatou components of \( f \) are iterated to \( D_0 \) or \( D_\infty \) eventually. For \( n \geq 2 \), let \( D_1, \ldots, D_{n-1} \) be the annular components such that \( f^{-1}(D_0 \cup D_\infty) = D_0 \cup D_\infty \cup \bigcup_{i=1}^{n-1} D_i \), where \( \{ D_i \}_{1 \leq i \leq n-1} \) are labeled such that \( D_i \) separates \( D_0 \) from \( D_{i'} \) for all \( 0 \leq i' < i < i'' = \infty \). The annuli \( \{ D_i : 1 \leq i \leq n-1 \} \) are called critical annuli and \( \{ D_i : i = 0, 1, \ldots, n-1, \infty \} \) are called critical Fatou components. Let \( A_i \) be the annulus (which is a closed set) between \( D_{i-1} \) and \( D_i \), where \( 1 \leq i \leq n-1 \) and \( A_\infty \) the annulus between \( D_{n-1} \) and \( D_\infty \). Then \( f^{-1}(A) = \bigcup_{i=1}^{n} A_i \), where \( A = \hat{\mathbb{C}} \setminus (D_0 \cup D_\infty) \). See Figure 2.

![Figure 2: The structure of the Cantor circle Julia sets on the Riemann sphere. All the critical Fatou components \( D_i \) with \( i = 0, 1, \ldots, n-1, \infty \) have been marked and all the non-critical annuli \( \{ A_i \}_{1 \leq i \leq n} \) have been colored by yellow.](image)

Note that \( f|_{A_i} : A_i \to A \) is a covering map and we suppose that \( \deg(f|_{A_i} : A_i \to A) = d_i \), where \( 1 \leq i \leq n \). Then \( \deg(f|_{D_i} : D_i \to D_0 \text{ or } D_\infty) = d_i + d_{i+1} \), where \( 1 \leq i \leq n-1 \). Moreover, \( \deg(f|_{D_\infty}) = d_1 \) and \( \deg(f|_{D_0}) = d_n \). Up to a conjugacy of Möbius transformations, all the Cantor circle Julia sets can be divided into the following three types.
Type I: $f(D_0) = D_\infty$, $f(D_\infty) = D_\infty$ and $n \geq 2$ is even. Moreover, 
\[
    f^{-1}(D_0) = \bigcup_{i=1}^{n/2} D_{2i-1} \quad \text{and} \quad f^{-1}(D_\infty) = D_0 \cup D_\infty \cup \bigcup_{i=1}^{(n-2)/2} D_{2i}.
\]

Type II: $f(D_0) = D_0$, $f(D_\infty) = D_\infty$ and $n \geq 3$ is odd. Moreover, 
\[
    f^{-1}(D_0) = D_0 \cup \bigcup_{i=1}^{(n-1)/2} D_{2i} \quad \text{and} \quad f^{-1}(D_\infty) = D_\infty \cup \bigcup_{i=1}^{(n-1)/2} D_{2i}.
\]

Type III: $f(D_0) = D_\infty$, $f(D_\infty) = D_0$ and $n \geq 3$ is odd. Moreover, 
\[
    f^{-1}(D_0) = D_\infty \cup \bigcup_{i=1}^{(n-1)/2} D_{2i-1} \quad \text{and} \quad f^{-1}(D_\infty) = D_0 \cup \bigcup_{i=1}^{(n-1)/2} D_{2i}.
\]

Note that $f^{-1}(A) = \bigcup_{i=1}^{n} A_i$ and each $A_i$ is essentially contained in $A$. It follows from Grötzsch’s module inequality that 
\[
    \sum_{i=1}^{n} d_i = d \quad \text{and} \quad \sum_{i=1}^{n} \frac{1}{d_i} < 1. \tag{2.1}
\]

**Definition** (Combinations of Cantor circles). Let $\mathcal{C}$ be the collection of all the combinations with the form $\mathcal{C} = (\kappa; d_1, \ldots, d_n)$, where $\kappa \in \{I, II, III\}$ is the type, the positive integers $(d_1, \ldots, d_n)$ satisfies (2.1), and 
\[
    n \geq 2 \quad \text{is} \quad \begin{cases} 
        \text{even} & \text{if } \kappa = I, \\
        \text{odd} & \text{if } \kappa = II \text{ or } III.
    \end{cases}
\]

For a hyperbolic rational map $f$ with Cantor circle Julia set, there exists at least one combinatorial data $\mathcal{C}(f) = (\kappa; d_1, \ldots, d_n) \in \mathcal{C}$ corresponding to $f$.

**Lemma 2.1.** Let $f$ be a hyperbolic rational map whose Julia set is a Cantor set of circles. Then $\mathcal{C}(f)$ has exactly one element if and only if $f$ is of
(a) type I; or 
(b) type II or III with $(d_1, \ldots, d_n) = (d_n, \ldots, d_1)$.

**Proof.** Note that if $f$ has combination $(\kappa; d_1, \ldots, d_n)$ with $\kappa \in \{II, III\}$, then $1/f(1/z)$ has combination $(\kappa; d_n, \ldots, d_1)$. If further $(d_1, \ldots, d_n) \neq (d_n, \ldots, d_1)$, then $\mathcal{C}(f)$ consists of exactly two elements $(\kappa; d_1, \ldots, d_n)$ and $(\kappa; d_n, \ldots, d_1)$. □

**Remark.** Actually, all the classifications and definitions in this subsection are valid for parabolic Cantor circle Julia sets. That means at least one of $D_0$ and $D_\infty$ is a parabolic periodic Fatou component. However, any parabolic Cantor circle Julia sets cannot be quasi-symmetrically equivalent to the standard Cantor circles (see the definitions in the next subsection) since such kind of Cantor circle Julia sets always contain some Julia components with cusps. See [QYY16].

2.2. Standard Cantor circles and quasi-symmetric uniformization. For each given $\mathcal{C} = (\kappa; d_1, \ldots, d_n) \in \mathcal{C}$, we define an iterated function system (IFS in short) associated to $\mathcal{C}$. Let 
\[
    -1 = b_-^1 < b_+^1 < b_-^2 < b_+^2 < \cdots < b_-^n < b_+^n = 0 \tag{2.2}
\]
be a partition of the unit interval $I = [-1, 0]$, where $b_+^i - b_-^i = 1/d_i$ with $1 \leq i \leq n$ (This is always possible since $\sum_{i=1}^{n} 1/d_i < 1$). For $1 \leq i \leq n$, we define 
\[
    L_i^+(x) := \pm d_i(x - b_+^i), \quad \text{where } x \in [b_-^i, b_+^i].
\]
We denote a symbol function \( \chi(\pm 1) := \pm \) and define the following iterated function system\footnote{In general an IFS is the collection of finitely many contraction maps. For convenience we say \( \tilde{L}(C) \) is an IFS since it is the inverse of a genuine IFS. See the definition in [Fal03, §7.1 and Theorem 9.3], we have the following immediate result.} according to \( C \):

\[
\tilde{L}(C) = \begin{cases} 
L_i^{\chi((-1)^i)} : 1 \leq i \leq n & \text{if } \kappa = I \text{ or III}, \\
L_i^{\chi((-1)^{-i})} : 1 \leq i \leq n & \text{if } \kappa = II.
\end{cases}
\]

It is easy to see that the attractor of \( \tilde{L}(C) \) is a Cantor set \( A(C) \subset [-1,0] \) which has strict self-similarity.

**Definition** (Standard Cantor circles). Let \( J(C) := \{ z \in \mathbb{C} : \log z \in A(C) \times \mathbb{R} \} \) be the standard Cantor circle associated to the combination \( C \). Then \( J(C) \) is contained in a closed annulus \( \mathbb{A}(1/e,1) \). For \( 1 \leq i \leq n \), we define

\[
\varphi_i^{\pm}(z) := z^{\pm d_i/e} \pm b_i^{\pm} d_i.
\]

By a coordinate transformation, it is straightforward to verify that an equivalent definition of \( J(C) \) is that it is the attractor of the following iterated function system:

\[
L(C) = \begin{cases} 
\varphi_i^{\chi((-1)^i)} : 1 \leq i \leq n & \text{if } \kappa = I \text{ or III}, \\
\varphi_i^{\chi((-1)^{-i})} : 1 \leq i \leq n & \text{if } \kappa = II.
\end{cases}
\]

See Figure 3.

![Figure 3: Two standard Cantor circles](image)

Figure 3: Two standard Cantor circles \( J(I; 3, 3) \) and \( J(II; 4, 4, 4) \), which are generated by two IFS \( \{z^{-3}/e^3, z^3\} \) and \( \{e^3 z^4, z^{-4}/e^{5/2}, z^4\} \) respectively.

Let \( d_1, \ldots, d_n \geq 2 \) be positive integers satisfying \((2.1)\). We use \( \alpha = \alpha_{d_1, \ldots, d_n} \in (0,1) \) to denote the unique positive root of

\[
\sum_{i=1}^{n} \left( \frac{1}{d_i} \right)^\alpha = 1.
\]

(2.3)

According to \([Fal03, §7.1 and Theorem 9.3]\), we have the following immediate result.

\[
\text{Let } d_1, \ldots, d_n \geq 2 \text{ be positive integers satisfying } (2.1). \text{ We use } \alpha = \alpha_{d_1, \ldots, d_n} \in (0,1) \text{ to denote the unique positive root of }
\]

\[
\sum_{i=1}^{n} \left( \frac{1}{d_i} \right)^\alpha = 1.
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\]

\[
\sum_{i=1}^{n} \left( \frac{1}{d_i} \right)^\alpha = 1.
\]

(2.3)

According to \([Fal03, §7.1 and Theorem 9.3]\), we have the following immediate result.
Lemma 2.2. A standard Cantor circle $J(C)$ with $C = (\kappa; d_1, \cdots, d_n) \in \mathcal{C}$ has Hausdorff dimension $1 + \alpha d_1, \cdots, d_n$.

Now we give the quasi-symmetric uniformization of the hyperbolic Cantor circle Julia sets.

**Theorem 2.3.** Every hyperbolic Cantor circle Julia set is quasi-symmetrically equivalent to a standard Cantor circle.

**Proof.** Let $f$ be a hyperbolic rational map whose Julia set $J(f)$ is a Cantor set of circles with the combinatorial data $C = (\kappa; d_1, \cdots, d_n) \in \mathcal{C}$. In the following we prove that $J(f)$ is quasi-symmetrically equivalent to the attractor $J(C)$ of the IFS $\mathcal{L}(C)$. The idea is to extend the IFS $\mathcal{L}(C)$ to a quasi-regular map $F$ and then prove that $f : J(f) \to J(f)$ is conjugated to $F : J(C) \to J(C)$ by the restriction of a quasiconformal mapping. For convenience we only prove the case $\kappa = I$. The cases for $\kappa = II, III$ are completely similar. Parts of the idea in the proof are similar to [QYY15, Theorem 3.2].

**Step 1:** Extend the IFS $\mathcal{L}(C)$ to a quasi-regular map $F$. Since $\kappa = I$, it means that $n \geq 2$ is even and we have

$$\mathcal{L}(C) = \{ \varphi_n^-, \varphi_n^+ \},$$

$$\varphi_n^-(z) = z^{-d_1/e^{d_1}}, z^{d_2/e^{d_2}}, \ldots, z^{-d_n/e^{d_n}}, z_{d_n},$$

$$\varphi_n^+(z) = z_{d_n}.$$

The IFS $\mathcal{L}(C)$ is defined on the annuli $\bigcup_{i=1}^{n} A_i(e^{b_i^-}, e^{b_i^+})$ by

$$\varphi_n^{(i)}(z) : A_i(e^{b_i^-}, e^{b_i^+}) \to A_i(1/e, 1),$$

where $1 \leq i \leq n$. We extend $F$ by setting

$$F(z) := \begin{cases} \varphi_n^{-}(z) = z^{-d_1/e^{d_1}} & \text{if } z \in D(0, 1/e), \\ \varphi_n^{+}(z) = z_{d_n} & \text{if } z \in C \setminus \mathbb{D}, \\ \text{quasi-regular interpolation} & \text{if } z \in \mathbb{A}(e^{b_i^-}, e^{b_i^+}), \end{cases}$$

where $1 \leq i \leq n - 1$. In particular, the interpolations are chosen such that $F(\mathbb{A}(e^{b_i^-}, e^{b_i^+})) = D(0, 1/e)$ if $i$ is odd and $F(\mathbb{A}(e^{b_i^-}, e^{b_i^+})) = \mathbb{C} \setminus \mathbb{D}$ if $i$ is even.

Then it is straightforward to see that $F : C \to C$ is a quasi-regular mapping of degree $d = \sum_{i=1}^{n} d_i$.

Similar to the notations used in [2.1] (see also Figure 2), we denote $D_0 := D(0, 1/e), D_{\infty} := \mathbb{C} \setminus \mathbb{D}, A_i := \mathbb{A}(e^{b_i^-}, e^{b_i^+})$ with $1 \leq i \leq n$, and $D_i := \mathbb{A}(e^{b_i^-}, e^{b_i^+})$ with $1 \leq i \leq n - 1$. Then we have $F(D_0) = D_{\infty}, F(D_{\infty}) = D_0$ and

$$F^{-1}(D_0) = \bigcup_{i=1}^{n/2} D_{2i-1} \text{ and } F^{-1}(D_{\infty}) = D_0 \cup D_{\infty} \cup \bigcup_{i=1}^{(n-2)/2} D_{2i}.$$
Since both \( f : A_i \to A \) and \( F : A_i' \to A' \) are covering mappings of degree \( d_i \), where \( 1 \leq i \leq n \), there exists a lift \( \phi_1 : A_i \to A_i' \), which is a quasiconformal mapping\(^5\) such that the following diagram is commutative:

\[
\begin{array}{ccc}
A_i & \xrightarrow{\phi_1} & A_i' \\
\downarrow f & & \downarrow F \\
A & \xrightarrow{\phi_0} & A'.
\end{array}
\]

Note that \( \phi_0 \circ F = F \circ \phi_0 \) on \( \partial D_0 \cup \partial D_\infty \). One can choose \( \phi_1 \) such that \( \phi_1|_{\partial D_0} = \phi_0|_{\partial D_0} \) and \( \phi_1|_{\partial D_\infty} = \phi_0|_{\partial D_\infty} \). The choices of the lifts \( \phi_1 : A_i \to A_i' \) for \( 2 \leq i \leq n-1 \) are not unique. We fix any one of them.

Define \( \phi_1 := \phi_0 \) on \( D_0 \cup D_\infty \). Then \( \phi_1 \) is defined on \( \hat{C} \) except \( \bigcup_{i=1}^{n-1} D_i \). Since all components of \( f^{-1}(\partial D_0 \cup \partial D_\infty) \) are quasi-circles, one can extend \( \phi_1 \) continuously to the annuli \( \{D_i\}_{1 \leq i \leq n-1} \) by \( \phi_1 : D_i \to D_i' \), to obtain a quasiconformal mapping \( \hat{\phi}_1 : \hat{C} \to \hat{C} \) such that (1) \( \hat{\phi}_1|_A \) is homotopic to \( \phi_0|_A \) rel \( \partial A = \partial D_0 \cup \partial D_\infty \); (2) \( \hat{\phi}_0 \circ f = F \circ \phi_0 \) on \( \bigcup_{i=1}^{n-1} A_i \); and (3) \( \hat{\phi}_1 \circ f = F \circ \phi_1 \) on \( f^{-1}(\partial D_0 \cup \partial D_\infty) \).

Now we define \( \phi_2 \). First, let \( \phi_2|_{D_i} = \phi_i \) for \( i \in \{0, 1, \ldots, n-1, \infty\} \). Since \( \phi_1|_A \) and \( \phi_0|_A \) are homotopic to each other rel \( \partial A \), it follows that there exists a lift \( \phi_2 : A_i \to A_i' \) of \( \phi_1 : A \to A' \) for each \( 1 \leq i \leq n \), such that \( \phi_2 : \hat{C} \to \hat{C} \) is continuous. In particular, \( \phi_2 : \hat{C} \to \hat{C} \) is a quasiconformal mapping which satisfies (1) \( \phi_2|_A \) is homotopic to \( \phi_1|_A \) rel \( \partial A \); (2) The dilatation \( K(\phi_2) = K(\phi_1) \); (3) \( \phi_2|_{f^{-1}(\partial D_0 \cup \partial D_\infty)} = \phi_1 \); (4) \( \phi_1 \circ f = F \circ \phi_2 \) on \( \bigcup_{i=1}^{n-1} A_i \); and (5) \( \phi_0 \circ f = F \circ \phi_2 \) on \( f^{-2}(\partial D_0 \cup \partial D_\infty) \).

Suppose that we have obtained \( \phi_{k-1} \) for some \( k \geq 2 \), then \( \phi_k \) can be obtained completely similarly to the procedure above. Inductively, one can obtain a sequence of quasiconformal mappings \( \{\phi_k : \hat{C} \to \hat{C}\}_{k \geq 0} \) such that (1) \( K(\phi_k) = K(\phi_1) \) for \( k \geq 2 \); (2) \( \phi_k(z) = \phi_{k-1}(z) \) for \( z \in f^{-(k-1)}(D_0 \cup D_\infty) \); (3) \( \phi_{k-1} \circ f = F \circ \phi_k \) on \( \bigcup_{i=1}^{n-1} A_i \); and (4) \( \phi_k \circ f = F \circ \phi_k \) on \( f^{-2}(\partial D_0 \cup \partial D_\infty) \).

**Step 3:** The limit conjugates the dynamics on the Julia set to the attractor. One can see that the sequence \( \{\phi_k : \hat{C} \to \hat{C}\}_{k \geq 0} \) forms a normal family. Taking any convergent subsequence of \( \{\phi_k : \hat{C} \to \hat{C}\}_{k \geq 0} \), we denote the limit by \( \phi_\infty \). Then \( \phi_\infty : \hat{C} \to \hat{C} \) is a quasiconformal mapping satisfying \( \phi_\infty \circ f = F \circ \phi_\infty \) on \( \bigcup_{k \geq 0} f^{-k}(\partial D_0 \cup \partial D_\infty) \). Since \( \phi_\infty \) is continuous, it follows that \( \phi_\infty \circ f = F \circ \phi_\infty \) holds on the closure of \( \bigcup_{k \geq 0} f^{-k}(\partial D_0 \cup \partial D_\infty) \), which is the Julia set of \( f \). Since \( \phi_\infty(J(f)) = J(\mathcal{C}) \), it means that \( J(f) \) is quasi-symmetrically equivalent to \( J(\mathcal{C}) \). This completes the proof of Theorems 2.3 and 1.1.

**Remark.** Note that in Step 2 of the proof of Theorem 2.3 \( \phi_0 : \hat{C} \to \hat{C} \) can be chosen such that it is \( C^1 \)-continuous in \( D_0 \cup D_\infty \). Similarly, \( \phi_1 : D_i \to D_i' \) can be chosen such that it is \( C^1 \)-continuous for all \( 1 \leq i \leq n-1 \). Then by induction, \( \phi_\infty \) is \( C^1 \)-continuous in the Fatou set \( F(f) = \bigcup_{k \geq 0} f^{-k}(\bigcup_{i=1}^{n-1} D_i) \) of \( f \) since \( \phi_k(z) = \phi_{k-1}(z) \) for \( z \in f^{-(k-1)}(D_0 \cup D_\infty) \) and \( \phi_{k-1} \circ f = F \circ \phi_k \) on \( \bigcup_{i=1}^{n-1} A_i \), where \( k \geq 2 \). This observation will be used in the next section.

As an immediate corollary of Theorem 2.3 we have

**Corollary 2.4 ([QYY18, Theorem 1.1(b)])**. If the Julia set \( J_\lambda \) of \( f_\lambda(z) = z^q + \lambda/z^p \) is a Cantor circle, then \( J_\lambda \) is quasi-symmetrically equivalent to the standard Cantor circle \( J(1; p, q) \), which is the attractor generated by the IFS \( \{z^{-p}/e^p, z^q\} \).

\(^5\)Usually a quasiconformal map is defined in an open domain. Here we mean that \( \phi_1 : A_i \to A_i' \) is the restriction of a quasiconformal map defined in an open annulus containing \( A_i \).
For each given combination $C \in \mathcal{C}$, the definition of the standard Cantor circle $J(C)$ depends on the partition of the unit interval $[-1,0]$ (if $n \geq 3$). See \((2.2)\). However, from the proof of Theorem \ref{thm:2.5} we have the following immediate result.

**Corollary 2.5.** All standard Cantor circles with the same combination $C \in \mathcal{C}$ (the partitions of $[-1,0]$ in \((2.2)\) are allowed to be different) are in the same quasi-symmetrically equivalent class.

From Corollary \ref{cor:2.5} we know that the classes of quasi-symmetrically equivalent Cantor circles are determined by the combinatorial data but not the geometric information. However, on the other hand a quasi-symmetrically equivalent class may contain different types of Cantor circle Julia sets.

**Proposition 2.6.** The standard Cantor circles with the combinations $C_1 = (\Pi; d_1, \ldots, d_n)$ and $C_2 = (\Pi; d_1, \ldots, d_n)$ are quasi-symmetrically equivalent provided $d_i = d_n+1-i$ for all $1 \leq i \leq n$.

**Proof.** A direct calculation shows that the two IFS $\mathcal{L}(C_1)$ and $\mathcal{L}(C_2)$ have the same attractor $J(C_1) = J(C_2)$, and hence these two standard Cantor circles are quasi-symmetrically equivalent (but not conjugate with respect to the dynamics).

We end this section with the following

**Questions.** (1) Let $C_1 = (I; d_1, d_2)$ and $C_2 = (I; d_2, d_1) \in \mathcal{C}$ with $d_1 \neq d_2$. Is $J(C_1)$ quasi-symmetrically equivalent to $J(C_2)$?

(2) Let $C_1 = (\kappa; d_1, d_3, d_2)$ and $C_2 = (\kappa; d_1, d_3, d_2) \in \mathcal{C}$, where $\kappa \in \{\Pi, \Pi\}$ and $d_2 \neq d_3$. Is $J(C_1)$ quasi-symmetrically equivalent to $J(C_2)$?

Note that in the above two questions $J(C_1)$ and $J(C_2)$ have the same conformal dimension (see the definition in \((5.1)\), which is a necessary condition of quasi-symmetric equivalence.

### 3. Topological Conjugacy and Hyperbolic Components

In order to find all (in the sense of topological conjugacy on the Julia sets) rational maps whose Julia sets are Cantor circles, the following Theorem \ref{thm:3.1} was proved in \cite{QYY15}.

**Theorem 3.1 (QYY15).** For every $\varrho \in \{0, 1\}$ and positive integers $d_1, \ldots, d_n$ with $n \geq 2$ satisfying $\sum_{i=1}^{n}(1/d_i) < 1$, there exist suitable parameters $a_1, \ldots, a_{n-1}$ such that the Julia set of

$$f_{\varrho, d_1, \ldots, d_n}(z) = z^{(-1)^{n-\varrho}}a_1 \prod_{i=1}^{n-1}(z^{d_i+d_{i+1}} - a_i^{d_i+d_{i+1}})^{(-1)^{n-\varrho}} \quad (3.1)$$

is a Cantor set of circles. Moreover, any rational map whose Julia set is a Cantor set of circles must be topologically conjugate to $f_{\varrho, d_1, \ldots, d_n}$ for some $\varrho$ and $d_1, \ldots, d_n$ on their corresponding Julia sets with suitable parameters $a_1, \ldots, a_{n-1}$.

Theorem \ref{thm:3.1} gives a complete topological classification of the Cantor circle Julia sets of rational maps under the dynamical behaviors. To study the hyperbolic components of Cantor circle type, we hope to find a representative map with the form \((3.1)\) in each Cantor circle hyperbolic component. This is one of the motivations for us to prove the following result.

**Theorem 3.2.** Let $f$, $g$ be two hyperbolic rational maps with degree $d$ whose Julia sets are Cantor circles on which the dynamics are topologically conjugate. Then $f$ and $g$ lie in the same hyperbolic component of the moduli space $\mathcal{M}_d$. 


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Proof. The proof will be divided into several steps. Since \( f \) and \( g \) are conjugate on their Julia sets, they have the same combinatorial data. Without loss of generality, we assume that they have the same combination \( C = (1; d_1, \ldots, d_n) \subset \mathcal{C} \). The rest two types can be treated completely similarly. The idea of the proof can be summed up as following: For \( f \) we assume that the attracting cycle is super-attracting. Then we prove that \( f \) is quasiconformally conjugated to a quasi-regular map \( \hat{F} \) whose restriction on some annuli is exactly the IFS \( \mathscr{L}(C) \) (see the definition in (2.2)). Next we deform the map \( \hat{F} \) and construct a continuous path \( (\hat{F}_s)_{s \in [0,1]} \) of quasi-regular maps such that \( \hat{F}_0 = \hat{F} \), and \( \hat{F}_1 = F \). From this we can obtain a continuous path \( (f_t)_{t \in [0,1]} \) of hyperbolic rational maps such that \( f_0 = f \) and \( f_1 = \xi_1 \circ F \circ \xi_1^{-1} \) for some quasiconformal mapping \( \xi_1 : \hat{C} \to \hat{C} \).

Similarly, the same construction guarantees the existence of continuous path \( (gt)_{t \in [0,1]} \) of hyperbolic rational maps such that \( g_0 = g \) and \( g_1 = \xi_1 \circ F \circ \xi_1^{-1} = f_1 \). Note that the map \( F \) here is seen to be the same map in the previous paragraph. Then the theorem follows since one can connect \( f \) with \( g \) by a continuous path in the hyperbolic component. Now we make the proof precisely.

Step 1: Transferring attracting to super-attracting (multi-critical to unicritical). Let \( \mathcal{H} \) be the Cantor circle hyperbolic component containing \( f \). According to [BF11] Chap. 4], by performing a standard quasiconformal surgery, there exists a continuous path in \( \mathcal{H} \) connecting \( f \) with \( \hat{f} \), such that \( \hat{f} \) has a super-attracting basin \( D_\infty \) with super-attracting fixed point \( \infty \) on which the dynamics is conjugate to \( z \mapsto z^{d_1} \), and moreover, \( \hat{f} : D_\infty \setminus \{0\} \to D_\infty \setminus \{\infty\} \) is a covering map of degree \( d_1 \), such that \( \hat{f} \) (note that \( \hat{f} \) has the same combination as \( f \)).

Step 2: Deformations from rational maps to quasi-regular maps. For saving the notations, we assume that the given \( f \) is exactly \( \hat{f} \). We continue using the notations, such as \( D_1, A_1, \) and \( A \) etc., for a rational map with Cantor circle Julia set (and hence \( f \)) as in (2.1). Since \( \kappa = 1 \), it means that \( n \geq 2 \) is even. Recall that

\[
\mathscr{L}(C) = \{\varphi_1, \varphi_2, \ldots, \varphi_n^\star\} = \{z^{-d_1}/e^{d_1}, z^{d_2}/e^{b_2}, \ldots, z^{d_n}\}
\]

is the IFS defined in (2.4). In this step we construct a quasiconformal conjugacy between \( f \) and a quasi-regular map \( \hat{F} : \hat{C} \to \hat{C} \), such that the restriction of \( \hat{F} \) on the union of the annuli \( \bigcup_{i=1}^n e^{(b_i^\pm)} \) is exactly the IFS \( \mathscr{L}(C) \), where \((b_i^\pm)_{i=1}^n\) are numbers given in (2.2). As before, we denote \( D_0' := D(0,1/e) \), \( D_\infty' := \hat{C} \setminus D_0' \), \( A'_i := e^{(b_i^+)} \) with \( 1 \leq i \leq n \), and \( D_i' := A(e^{b_i^+}, e^{-b_i^+}) \) with \( 1 \leq i \leq n-1 \).

Let \( F : \hat{C} \to \hat{C} \) be the quasi-regular map defined in (2.5). Moreover, we assume that \( F \) is \( C^1 \)-continuous on \( \hat{C} \) except on \( \bigcup_{i=1}^n \partial D_i' \). There exists a quasiconformal mapping \( \phi_0 : \hat{C} \to \hat{C} \) such that \( \phi_0(D_0) = D_0' \), \( \phi_0(D_\infty) = D_\infty' \) and \( \phi_0 \circ f = F \circ \phi_0 \) holds on \( D_0 \cup D_\infty \). Moreover, \( \phi_0 \) is chosen such that it is conformal in \( D_0 \) and \( D_\infty \). By using a completely similar argument as in the proof of Theorem 2.3, one obtains a sequence of quasiconformal mappings \((\phi_k : \hat{C} \to \hat{C})_{k \geq 0}\) which satisfies (1) \( K(\phi_k) = K(\phi_1) \) for \( k \geq 2 \); (2) \( \phi_k(z) = \phi_{k-1}(z) \) for \( z \in f^{-1} \cdot (D_0 \cup D_\infty) \); (3) \( \phi_{k-1} \circ f = F \circ \phi_0 \circ \phi_{k-1} \) on \( \bigcup_{i=1}^n A_i \); and (4) \( \phi_k \circ f = F \circ \phi_k \circ f^{-k} \). Note that \( (\phi_k : \hat{C} \to \hat{C})_{k \geq 0} \) is a normal family. Taking a convergence subsequence of \((\phi_k)_{k \geq 0}\) whose limit is denoted by \( \phi_\infty : \hat{C} \to \hat{C} \), we obtain that \( \phi_\infty \circ f = F \circ \phi_\infty \) on \( D_0 \cup D_\infty \). The map

\[
\hat{F}(z) := \phi_\infty \circ f \circ \phi_\infty^{-1}(z) : \hat{C} \to \hat{C}
\]

\[6\text{Note that in the following F is seen to be fixed.} \]
is quasi-regular and the restriction of $\tilde{F}$ on $\bigcup_{i=1}^{n} A'_i$ is exactly the IFS $\mathcal{L}(\mathcal{C})$. In particular, we have $\tilde{F} = F$ on $D'_0 \cup D'_\infty \cup \bigcup_{i=1}^{n} A'_i$.

By the construction of $\phi_n$ (see the remark following the proof of Theorem 2.3), we can choose the sequence $\{\phi_k\}_{k \in \mathbb{N}}$ such that the limit $\phi_\infty : \hat{\mathcal{C}} \rightarrow \hat{\mathcal{C}}$ is $C^1$-continuous in the Fatou set of $f$. This means that $\tilde{F}$ is holomorphic in the interior of $\hat{\mathcal{C}} \setminus \bigcup_{i=1}^{n-1} D'_i$ and $C^1$-continuous in $\bigcup_{i=1}^{n-1} D'_i$.

**Step 3:** Twist deformations on the annuli. Although both $\tilde{F}$ and $F$ are quasi-regular extensions of the IFS $\mathcal{L}(\mathcal{C})$ on $\bigcup_{i=1}^{n} A'_i$, $\tilde{F}$ needs not to be homotopic to $F$ rel $\partial D'_i$ for some $1 \leq i \leq n - 1$. Indeed, for $1 \leq i \leq n - 1$, it turns out that up to a homotopy with fixed end points, $\tilde{F}|_{D'_i}$ and $F|_{D'_i}$ differ several compositions of the twist map along $D'_i$:

$$T_i(z) := \begin{cases} z & \text{if } z \in \hat{\mathcal{C}} \setminus D'_i, \\ z e^{2\pi i (|z| - r_i)} & \text{if } z \in D'_i, \\ z e^{2\pi i r_i} & \text{if } z \in A'_1. \end{cases}$$

where $s_i = e^{r_i^+}$ and $r_i = e^{r_i^{-1}}$.

Suppose that $\tilde{F} \circ T_i^{k_i}|_{D'_i}$ is homotopic to $F|_{D'_i}$ rel $\partial D'_i$ for some $k_i \in \mathbb{Z}$, where $1 \leq i \leq n - 1$. For every $t \in [0, 1]$, we define a family of mappings by setting

$$F^i_t(z) := \begin{cases} \tilde{F}(z) & \text{if } z \in D'_0 \cup A'_1, \\ \tilde{F}(ze^{-2\pi i (|z| - r_i)t}) & \text{if } z \in D'_1, \\ \tilde{F}(ze^{-2\pi i r_i t}) & \text{other.} \end{cases}$$

It is straightforward to check that $F^i_t$ depends continuously on $t \in [0, 1]$ and each $F^i_t$ is quasi-regular. In particular, for all $t \in [0, 1]$, $F^i_t$ is holomorphic in the interior of $\hat{\mathcal{C}} \setminus \bigcup_{i=1}^{n-1} D'_i$ and $C^1$-continuous in $\bigcup_{i=1}^{n-1} D'_i$. Moreover,

- $F^i_0 = \tilde{F}$;
- $F^i_1 = \tilde{F}$ on $\hat{\mathcal{C}} \setminus D'_1$; and
- $F^i_t|_{D'_i}$ is homotopic to $F|_{D'_i}$ rel $\partial D'_i$.

Inductively, for $2 \leq i \leq n - 1$ and $t \in [0, 1]$, we define

$$F^i_t(z) := \begin{cases} F^{i-1}_t(z) & \text{if } z \in \bigcup_{j=0}^{i-1} D'_j \cup \bigcup_{j=1}^{i} A'_j, \\ F^{i-1}_t(ze^{-2\pi i (|z| - r_i)t}) & \text{if } z \in D'_i, \\ F^{i-1}_t(ze^{-2\pi i r_i t}) & \text{other.} \end{cases}$$

Then $F^i_t$ depends continuously on $t \in [0, 1]$ and each $F^i_t$ is quasi-regular. In particular, for all $t \in [0, 1]$, $F^i_t$ is holomorphic in the interior of $\hat{\mathcal{C}} \setminus \bigcup_{i=1}^{n-1} D'_i$ and $C^1$-continuous in $\bigcup_{i=1}^{n-1} D'_i$. Moreover,

- $F^i_0 = F^{i-1}_1$;
- $F^i_1 = \tilde{F}$ on $\hat{\mathcal{C}} \setminus \bigcup_{j=1}^{i-1} D'_j$; and
- $F^i_t$ is homotopic to $F$ on $\bigcup_{j=0}^{i} D'_j \cup \bigcup_{j=1}^{i+1} A'_j$ rel $\partial D'_j$ for all $1 \leq j \leq i$.

For $t \in [0, 1]$, we define

$$F_t(z) := \begin{cases} \tilde{F}(z) & \text{if } t = 0, \\ F^i_{(n-1)t-(i-1)} & \text{if } t \in (\frac{i}{n-1}, \frac{i+1}{n-1}] \text{ for } 1 \leq i \leq n - 1. \end{cases}$$

Then $F_t$ depends continuously on $t \in [0, 1]$ and each $F_t$ is quasi-regular. In particular, for all $t \in [0, 1]$, $F_t$ is holomorphic in the interior of $\hat{\mathcal{C}} \setminus \bigcup_{i=1}^{n-1} D'_i$ and $C^1$-continuous in $\bigcup_{i=1}^{n-1} D'_i$. Moreover,

- $F_1 = F^{n-1}_1 = \tilde{F}$ on $\hat{\mathcal{C}} \setminus \bigcup_{k=1}^{n-1} D'_k$; and
- $F_t$ is homotopic to $F$ on $\hat{\mathcal{C}}$ rel $\partial D'_i$ for all $1 \leq i \leq n - 1$. 

Finally, let \((\hat{F}_t : \mathbb{C} \to \mathbb{C})_{t \in [0, 1]}\) be a continuous path of quasi-regular maps such that \(\hat{F}_0 = F_1\), \(\hat{F}_1 = F\) and \(\hat{F}_t = F_1 v C \setminus \bigcup_{i=1}^{n-1} D'_i\). In particular, the path can be chosen such that for all \(t \in [0, 1]\), \(\hat{F}_t\) is holomorphic in the interior of \(C \setminus \bigcup_{i=1}^{n-1} D'_i\) and \(C^1\)-continuous in \(\bigcup_{i=1}^{n-1} D'_i\). Denote by
\[
\hat{F}_s(z) := \begin{cases} 
F_{2s}(z) & \text{if } s \in [0, 1/2], \\
\hat{F}_{2s-1}(z) & \text{if } s \in (1/2, 1]. 
\end{cases}
\]
Then each \(\hat{F}_s\) is quasi-regular and depends continuously on \(s \in [0, 1]\). In particular, for all \(s \in [0, 1]\), \(\hat{F}_s\) is holomorphic in the interior of \(C \setminus \bigcup_{i=1}^{n-1} D'_i\) and \(C^1\)-continuous in \(\bigcup_{i=1}^{n-1} D'_i\). Moreover, we have
\begin{itemize}
  \item \(\hat{F}_0 = \hat{F}, \hat{F}_1 \equiv F\); and
  \item \(\hat{F}_1 = \hat{F}\) on \(C \setminus \bigcup_{i=1}^{n-1} D'_i\).
\end{itemize}

**Step 4:** The continuous paths in the hyperbolic component. Let \(\sigma_0\) be the standard conformal structure on \(\hat{C}\) represented by the zero Beltrami differential. For each \(s \in [0, 1]\) we define a measure conformal structure function
\[
\sigma_s(z) := \begin{cases} 
\sigma_0 & \text{if } z \in D'_0 \cup D'_{\infty}, \\
(\hat{F}_s)^{\ast}(\sigma_0) & \text{if } z \in \hat{F}_{-s}^{(t-1)}(\bigcup_{i=1}^{n-1} D'_i) \text{ for some } t \geq 1, \\
\sigma_0 & \text{other}.
\end{cases}
\]
Since each \(\hat{F}_s\) is holomorphic in \(C \setminus \bigcup_{i=1}^{n-1} D'_i\), it is easy to see that \(\sigma_s\) has bounded dilatation and invariant under the action of \(F\). According to Measurable Riemann Mapping Theorem, there exists a quasiconformal map \(\xi : C \to \hat{C}\) which solves the Beltrami equation \(\xi_{\ast}(\sigma_0) = \sigma_s\). Moreover, \(\xi_s\) is unique if it normalized by fixing 0, 1 and \(\infty\). Note that \(\sigma_s\) depends continuously on \(s \in [0, 1]\) (since each \(\hat{F}_s\) is holomorphic in the interior of \(C \setminus \bigcup_{i=1}^{n-1} D'_i\) and \(C^1\)-continuous in \(\bigcup_{i=1}^{n-1} D'_i\)). By Ahlfors-Bers theorem \([AB69]\), the map
\[
\tilde{f}_s := \xi_s \circ \hat{F}_s \circ \xi_{s}^{-1}
\]
is a rational map which depends continuously on \(s \in [0, 1]\). Moreover, \(\tilde{f}_0 = \xi_0 \circ \hat{F} \circ \xi_{0}^{-1}\), \(\tilde{f}_1 = \xi_1 \circ F \circ \xi_{1}^{-1}\) and each \(\tilde{f}_s\) with \(s \in [0, 1]\) is a hyperbolic rational map with a Cantor circle Julia set.

Since \(\hat{F} = \phi_{\infty} \circ f \circ \phi_{\infty}^{-1}\) and \(\tilde{f}_0 = \xi_0 \circ \hat{F} \circ \xi_{0}^{-1}\), we have
\[
\tilde{f}_0 = \phi \circ f \circ \tilde{\phi}^{-1},
\]
where \(\tilde{\phi} := \xi_0 \circ \phi_{\infty} : \hat{C} \to \hat{C}\) is quasiconformal.

For \(t \in [0, 1]\), define a conformal structure \(\sigma_t = t \tilde{\phi}^{\ast}(\sigma_0)\). Since \(f\) is a rational map, \(\sigma_t\) is preserved by \(f\). By the Measurable Riemann Mapping Theorem, there exists a unique quasiconformal mapping \(\zeta_t : \hat{C} \to \hat{C}\) which solves the Beltrami equation \(\zeta_t^{\ast}(\sigma_0) = \sigma_t\) and fixes 0, 1 and \(\infty\). Define \(\tilde{f}_t := \zeta_t \circ f \circ \zeta_{t}^{-1}\). Then each \(\tilde{f}_t\) is a hyperbolic rational map with a Cantor circle Julia set. According to Ahlfors-Bers \([AB69]\), \((\tilde{f}_t)_{t \in [0, 1]}\) is a continuous path connecting \(\tilde{f}_0\) with \(\tilde{f}_1\).

For \(t \in [0, 1]\), we define
\[
f_t(z) := \begin{cases} 
\tilde{f}_{2t}(z) & \text{if } t \in [0, 1/2], \\
\tilde{f}_{2t-1}(z) & \text{if } t \in (1/2, 1]. 
\end{cases}
\]
Then \(f_t\) depends continuously on \(t \in [0, 1]\) and each \(f_t\) is a hyperbolic rational map with a Cantor circle Julia set. In particular, \((f_t)_{t \in [0, 1]}\) is a continuous path in the hyperbolic component \(H\) connecting \(f_0 = f\) with \(f_1 = \xi_1 \circ F \circ \xi_{1}^{-1}\). The reason is that both \(F_1\) and \(F\) are holomorphic in the interior of \(C \setminus \bigcup_{i=1}^{n-1} D'_i\) and \(C^1\)-continuous in \(\bigcup_{i=1}^{n-1} D'_i\).
Step 5: The conclusion. If we begin with the rational map \( g \) whose combination is also \( C = (I; d_1, \cdots, d_n) \), then as above one can find a continuous path \((g_t)_{t \in [0,1]} \) in a hyperbolic component connecting \( g_0 = g \) with \( g_1 = \xi_1 \circ F \circ \xi_1^{-1} \) (Careful: not \( \xi_1 \circ G \circ \xi_1^{-1} \) for some \( G \) since \( F \) is the given quasi-regular mapping depending only on the combination \( C \)). Therefore, \( f \) and \( g \) can be connected by a continuous path in the hyperbolic component \( \mathcal{H} \). This completes the proof of Theorem 3.2 and hence Theorem 1.2.

□

Remark. (1) Let \( f_1 \) and \( f_2 \) be two hyperbolic rational maps with degree \( d \) whose Julia sets are Cantor circles. If \( f_1 \) and \( f_2 \) has the same combinatorial data, then they lie in the same hyperbolic component of the moduli space \( \mathcal{M}_d \).

(2) Another idea of proving Theorem 3.2 is to modify the argument in Step 2. Note that all \( \phi_k \) with \( k \geq 0 \) (including \( \phi_\infty \)) are homotopic rel \( \partial A' \) (see the proof of Theorem 2.3). Hence one can choose a suitable quasiconformal mapping \( \phi_\infty : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) such that \( F = \phi_\infty \circ f \circ \phi_\infty^{-1} \) is homotopic to \( F \) rel \( D'_i \) for every \( 1 \leq i \leq n - 1 \). Then the rest argument is similar.

Recall that \( f_{g,d_1,\cdots,d_n} \) is the family introduced in (3.1). One can see that at least one of 0 and \( \infty \) lies in a super-attracting basin of \( f_{g,d_1,\cdots,d_n} \). Let \( D_0 \) and \( D_\infty \) be the Fatou components containing 0 and \( \infty \) respectively. There are following four cases (Here we denote by \( f : f_{g,d_1,\cdots,d_n} \) for simplicity):

- if \( g = 1 \) and \( n \) is even, then \( f(D_0) = D_\infty \) and \( f(D_\infty) = D_\infty \);
- if \( g = 1 \) and \( n \) is odd, then \( f(D_0) = D_0 \) and \( f(D_\infty) = D_\infty \);
- if \( g = 0 \) and \( n \) is odd, then \( f(D_0) = D_\infty \) and \( f(D_\infty) = D_0 \);
- if \( g = 0 \) and \( n \) is even, then \( f(D_0) = D_0 \) and \( f(D_\infty) = D_0 \).

Note that up to topological conjugacies, we only need to consider the first three cases since case (d) is conjugate to case (a) (compare (2.1)). In particular, cases (a), (b) and (c) have the combinations \( (I; d_1, \cdots, d_n), (II; d_1, \cdots, d_n) \) and \( (III; d_1, \cdots, d_n) \) respectively (see [QYY15] §2). As an immediate corollary of Theorems 3.2 we have

**Corollary 3.3.** Any Cantor circle hyperbolic component \( \mathcal{H} \) in \( \mathcal{M}_d \) contains at least one map \( f_{g,d_1,\cdots,d_n} \) with suitable parameters \( a_1, \cdots, a_{n-1} \).

Remark. The parameters \( a_1, \cdots, a_{n-1} \) in Theorem 3.1 are chosen such that

\[ 0 < |a_1| < |a_2| < \cdots < |a_{n-1}| \leq 1. \]

See [QYY15] Theorem 2.5 or Theorem 5.1.

If a hyperbolic component of rational maps of degree \( d \geq 2 \) has compact closure in \( \mathcal{M}_d \), then this hyperbolic component is called bounded. A theorem of Makienko asserts that if the Julia set of a hyperbolic rational map is disconnected, then the hyperbolic component containing this rational map is unbounded (see [Mak00]). Note that each Cantor circle Julia set is disconnected. Therefore, we have

**Corollary 3.4.** All Cantor circle hyperbolic components in \( \mathcal{M}_d \) are unbounded.
4. NUMBER OF CANTOR CIRCLE HYPERBOLIC COMPONENTS

The aim of this section is to calculate the number of the Cantor circle hyperbolic components in the moduli space $\mathcal{M}_d$, where $d \geq 2$.

**Proposition 4.1.** Let $f$ be a rational map whose Julia set is a Cantor set of circles. Then $\deg(f) \geq 5$.

**Proof.** If $d \leq 4$, then (2.1) has no solution.

Note that Proposition 4.1 is also valid for parabolic rational maps. In the following we use $\sharp A$ to denote the cardinal number of a finite set $A$.

**Theorem 4.2.** For each $d \geq 5$, the number $N(d)$ of different Cantor circle hyperbolic components in $\mathcal{M}_d$ is calculated by (1.1).

**Proof.** According to Theorem 1.2 it is sufficient to calculate the different topologically conjugate classes on the Cantor circle Julia sets. On the other hand, it is convenient for us to consider the combinations of the Cantor circle Julia sets. There are three types in all (see (2.1)). Obviously, the dynamics on these three types of Cantor circle Julia sets are not topologically conjugate to each other.

For each given $d \geq 5$, we define

$$N^1 := \left\{(1; d_1, \ldots, d_n) \mid \sum_{i=1}^n d_i = d, \sum_{i=1}^n \frac{1}{d_i} < 1 \text{ and } n \geq 2 \text{ is even} \right\}.$$ 

For each given $d \geq 5$ and $\kappa \in \{I, II, III\}$, we define

$$N^\kappa_1 := \left\{(\kappa; d_1, \ldots, d_n) \mid \sum_{i=1}^n d_i = d, \sum_{i=1}^n \frac{1}{d_i} < 1 \right\},$$

$$N^\kappa_2 := \left\{(\kappa; d_1, \ldots, d_n) \mid \sum_{i=1}^n d_i = d, \sum_{i=1}^n \frac{1}{d_i} < 1 \right\},$$

Note that $\sharp N^I_1 = \sharp N^I_{II}$ and $\sharp N^I_2 = \sharp N^I_{III}$. By noticing Lemma 2.1, the number of different topologically conjugate classes on the Cantor circle Julia sets of maps in $\mathcal{M}_d$ is calculated by

$$N(d) = \sharp N^I + \sharp N^I_{II} + \frac{\sharp N^I_{III}}{2} + \frac{\sharp N^I}{2} + \frac{\sharp N_{III}}{2}$$

This ends the proof of Theorem 1.2 and hence Theorem 1.3.

An enumerative method can be easily used to calculate $N(d)$ for each given $d \geq 5$ by Theorem 1.3. See Table 1.

5. HAUSDORFF DIMENSION OF CANTOR CIRCLES: THE INFIMUM

Recall that $f_{\varphi, d_1, \ldots, d_n}$ is the family defined in Theorem 3.1. The aim of this section is to find the infimum of the Hausdorff dimension of the Julia sets of the rational maps in the Cantor circle hyperbolic components. Since the lower bound of the Hausdorff dimensions of the Cantor circle Julia sets can be obtained easily (see Proposition 5.2), according to Corollary 3.3 it is sufficient to work with the family $f_{\varphi, d_1, \ldots, d_n}$ and prove that it can produce a sequence of Hausdorff dimensions which approach the lower bound. Then the lower bound becomes the infimum.

Let

$$K = \max\{d_1, \ldots, d_n\} \text{ and } \eta := \sum_{i=1}^n \frac{1}{d_i} < 1.$$ 

The parameters $a_1, \ldots, a_{n-1}$ in Theorem 3.1 can be chosen more specifically as in the following theorem.
Table 1: The list of the number $N(d)$ of Cantor circle hyperbolic components in the moduli space of rational maps of degree $d$, where $5 \leq d \leq 36$. An interesting problem is to give an estimation of the order of $N(d)$ in $d$ by using $\frac{\ln 1}{\ln d}$.

| $d$  | 5   | 6   | 7   | 8   | 9   | 10  | 11  | 12  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $N(d)$ | 2   | 3   | 4   | 5   | 6   | 11  | 22  | 37  |
| $d$  | 13  | 14  | 15  | 16  | 17  | 18  | 19  | 20  |
| $N(d)$ | 46  | 57  | 68  | 81  | 110 | 159 | 228 | 290 |
| $d$  | 21  | 22  | 23  | 24  | 25  | 26  | 27  | 28  |
| $N(d)$ | 410 | 519 | 716 | 872 | 1070| 1323| 1722| 2258|
| $d$  | 29  | 30  | 31  | 32  | 33  | 34  | 35  | 36  |
| $N(d)$ | 3066| 4227| 5566| 6950| 8604| 10483| 12916| 15838|

Theorem 5.1 ([QYY15] Theorem 2.5]). Let $u_1 = sK^{-5}$, $v_1 = sK^{-2}$; and $u_0 = s^{1+1/d_0+2(1-\eta)/3}$, $v_0 = s^{1/d_0+(1-\eta)/3}$.

(a) For $\varrho = 1$, set $|a_{n-1}| = v_1^{1/d_0}$ and $|a_i| = u_1^{1/d_{i+1}} |a_{i+1}|$ for $1 \leq i \leq n - 2$;
(b) For $\varrho = 0$, set $|a_{n-1}| = v_0^{1/d_n}$ and $|a_i| = u_0^{1/d_{i+1}} |a_{i+1}|$ for $1 \leq i \leq n - 2$.

Then $J(f_{g,d_1,\ldots,d_n})$ is a Cantor circle if $s > 0$ is small enough.

In fact, if $s > 0$ is small enough, then we have

$$0 < |a_1| \ll |a_2| \ll \cdots \ll |a_{n-1}| \ll 1.$$ (5.1)

Since at least one of 0 and $\infty$ (or both) lies in the super-attracting basins of $f_{g,d_1,\ldots,d_n}$, we can define the corresponding $A_i$ with $1 \leq i \leq n$ and $D_i$ with $1 \leq i \leq n - 1$ for $f_{g,d_1,\ldots,d_n}$ (see [2.1]). From [QYY15] Lemma 2.4] we know that $D_i$ contains the circle $T_{|a_i|}$ and $d_i + d_{i+1}$ critical points for all $1 \leq i \leq n - 1$. In the following, we always assume that $\alpha$'s are chosen like in Theorem 5.1 such that the Julia set of $f_{g,d_1,\ldots,d_n}$ is a Cantor set of circles.

5.1 Conformal dimension of the Cantor circle Julia sets. Let $X$ be a metric space. The conformal dimension $\dim_C(X)$ of $X$ is the infimum of the Hausdorff dimensions of all metric spaces which are quasi-symmetrically equivalent to $X$. Note that the conformal dimension is an invariant of the quasi-symmetric class of a metric space. Recall that $\alpha_{d_1,\ldots,d_n} \in (0, 1)$ is a number determined by [2.3].

Proposition 5.2. Let $H$ be a Cantor circle hyperbolic component whose combination is $C = (\kappa; d_1, \ldots, d_n) \in \mathcal{C}$. Then $\dim_C(J(f)) = 1 + \alpha_{d_1,\ldots,d_n}$ for all $f \in H$.

Proof. According to Theorem 2.3, the Julia set of each $f \in H$ is quasi-symmetrically equivalent to a standard Cantor circle $J(C)$. To prove this proposition we will use the following fact (see [Pan89] Proposition 2.9 or [Har99] Proposition 3.7): if $X$ is a $\lambda$-Ahlfors regular metric space, then $X \times [0, 1]$ equipped with the product metric has conformal dimension $1 + \lambda$. Note that the standard Cantor set $A(C)$ is an $\alpha$-Ahlfors regular metric space with $\alpha = \alpha_{d_1,\ldots,d_n}$ (see Lemma 2.2). Hence the conformal dimension of the Julia set of $f$ is $1 + \alpha$. \qed

Let $J_{g,d_1,\ldots,d_n}$ be the Julia set of $f_{g,d_1,\ldots,d_n}$ for $n \geq 2$. The following result is an immediate corollary of Proposition 5.2 and Corollary 3.3.

Corollary 5.3. The conformal dimension of $J_{g,d_1,\ldots,d_n}$ is $1 + \alpha_{d_1,\ldots,d_n}$.
Remark. If $d_i = d_0 > n$ for all $1 \leq i \leq n$, then $\sum_{i=1}^{n} 1/d_i = n/d_0 < 1$ and
\[
\dim_H(J_{g, d_1, \ldots, d_n}) = 1 + \frac{\log n}{\log d_0}.
\]

5.2. Decomposition of the dynamical planes. For calculating the Hausdorff dimension of $J_{g, d_1, \ldots, d_n}$, we need to decompose the dynamical planes and estimate the expanding factor near the Julia sets. In the rest of this section, we always assume that the parameters $a_1, \ldots, a_{n-1}$ are chosen as positive numbers.

For small $\alpha > 0$, $s > 0$ and every $1 \leq i \leq n-1$, we define the following numbers:
\[
R_0 = R_0^+, \quad R_1^- = s^a a_i, \quad \text{and} \quad R_i^+ = s^{-n} a_i, \quad R_\infty = R_0^-(2/s)^{1/d_0}.
\]

Recall that the disks $D_0$, $D_\infty$ and the annuli $D_i$ with $1 \leq i \leq n-1$, $A_i$ with $1 \leq i \leq n$ are defined for $f_s := f_{g, d_1, \ldots, d_n}$ above. For $0 < r_1 < r_2 < \infty$, recall that $\mathcal{A}(r_1, r_2) := \{ z \in \mathbb{C} : r_1 < |z| < r_2 \}$. The following result has been included in the proof of [QYY15] Lemma 2.4.

Lemma 5.4. There exists a small $\alpha > 0$ such that if $s > 0$ is small enough, then
\[
\mathbb{D}_{R_0} \subset D_0, \quad \mathcal{A}(R_1^-, R_i^+) \subset D_i \quad \text{with} \quad 1 \leq i \leq n-1, \quad \text{and} \quad \hat{\mathbb{C}} \setminus \mathbb{D}_{R_\infty} \subset D_\infty, \quad A_i \subset \mathcal{A}(R_{i-1}^+, R_i^-) \quad \text{with} \quad 1 \leq i \leq n.
\]

Moreover, every $f_s(\mathcal{A}(R_{i-1}^-, R_i^-))$ contains $\mathcal{A}(R_0, R_\infty)$, where $1 \leq i \leq n$. All the critical values of $f_s$ are contained in $\mathbb{D}_{R_0} \cup (\hat{\mathbb{C}} \setminus \mathbb{D}_{R_\infty})$.

5.3. IFS and logarithm coordinates. Let $\Omega$ be a closed subset of $\mathbb{R}^k$, where $k \geq 1$. A map $\psi : \Omega \to \Omega$ is called a contraction on $\Omega$ if there exists a real number $c \in (0, 1)$ such that $|\psi(x) - \psi(y)| \leq c|x - y|$ for all $x, y \in \Omega$. A finite family of contractions $\{\psi_1, \psi_2, \ldots, \psi_m\}$ defined on $\Omega \subset \mathbb{R}^k$, with $m \geq 2$, is called an iterated function system (or IFS in short). The IFS $\{\psi_1, \psi_2, \ldots, \psi_m\}$ is said to satisfy the open set condition if there exists a non-empty bounded open set $V$ such that $V \supset \bigcup_{i=1}^{m} \psi_i(V)$ with the union disjoint.

Note that $f_s$ is a real rational map and $f_s^{-1}(\mathcal{A}(R_0, R_\infty) \setminus \mathbb{R}^+)$ consists of $d_i$ components in $\mathcal{A}(R_{i-1}^+, R_i^-) \setminus \mathbb{R}^+$ for every $1 \leq i \leq n$. We label the closure of the $d_i$ components of $f_s^{-1}(\mathcal{A}(R_0, R_\infty) \setminus \mathbb{R}^+)$ in $\mathcal{A}(R_{i-1}^+, R_i^-) \setminus \mathbb{R}^+$ counter-clockwise as $S_{1,i}, S_{2,i}, \ldots, S_{d_i,i}$, such that $S_{1,i}$ lies above of $\mathbb{R}^+$ and $S_{d_i,i}$ lies below of $\mathbb{R}^+$.

Let $\Sigma_i = \{(i, t) : t = 1, 2, \ldots, d_i\}$ for $1 \leq i \leq n$ and
\[
\Sigma := \Sigma_1 \cup \Sigma_2 \cup \ldots \cup \Sigma_n
\]
be the index sets. We denote $S := \mathcal{A}(R_0, R_\infty) \setminus \mathbb{R}^+$ and treat every $z \in [R_0, R_\infty]$ as two different points in $S$, i.e. $S$ will be seen as a simply connected closed domain.

Based on the convenience introduced above, one can see that $f_s|_{S_\sigma} : S_\sigma \to S$ is a homeomorphism for every $\sigma \in \Sigma$. Let $\varphi_\sigma : S \to S_\sigma$ be the inverse of $f_s|_{S_\sigma}$. Then every $\varphi_\sigma$ is a contracting mapping and $\{\varphi_\sigma : \sigma \in \Sigma\}$ forms an iterated function system. The attractor of $\{\varphi_\sigma : \sigma \in \Sigma\}$ is exactly the Julia set $J_s$ of $f_s$.

To compute the Hausdorff dimension of $J_s$, we need the following result (see [Fal03] Chapter 9).

---

8When $g, d_1, \ldots, d_n$ are given, the parameters $a_1, \ldots, a_{n-1}$ are functions with variable $s > 0$.

9In [QYY15] Lemma 2.4, $A_i$ is an annulus containing the critical circle. But in this paper we use $A_i$ to denote the annulus between every two adjacent critical annuli.
Theorem 5.5 (Falconer). Let \( \{\psi_1, \psi_2, \ldots, \psi_m\} \) be an IFS on a closed set \( \Omega \subset \mathbb{R}^k \) satisfying the open set condition and the bi-Lipschitz condition

\[
|b_i(x - y)| \leq |\psi_i(x) - \psi_i(y)| \leq c_i|x - y|,
\]

where \( 0 < b_i \leq c_i < 1 \) and \( 1 \leq i \leq m \). Then the attractor \( J \) of \( \{\psi_1, \psi_2, \ldots, \psi_m\} \) has Hausdorff dimension satisfying

\[
\beta_- \leq \dim_H(J) \leq \beta_+,
\]

where \( \beta_- \) and \( \beta_+ \) satisfy

\[
\sum_{i=1}^{m} b_i^{\beta_-} = 1 \quad \text{and} \quad \sum_{i=1}^{m} c_i^{\beta_+} = 1.
\]

From Lemma 5.4 one can see that the IFS \( \{\varphi_\sigma : \sigma \in \Sigma\} \) satisfies the open condition. In order to estimate the constants of the bi-Lipschitz property, we lift the map \( f_s \) (and the IFS) to the logarithm coordinate.

Note that \( S := \mathring{K}(R_0, R_\infty) \setminus \mathbb{R}^+ \) is seen as a simply connected closed domain. We lift \( S \) (and \( S_\sigma \)) under

\[
\xi : Z \mapsto z = e^Z
\]

to obtain \( \tilde{S} \) (and \( \tilde{S}_\sigma \)) such that \( \tilde{S} = \log S \subset \{Z : 0 \leq \Im Z \leq 2\pi\} \). Hence this lift is unique determined. For every \( \sigma \in \Sigma \), we define \( F_\sigma(Z) \) on \( \tilde{S}_\sigma \) by

\[
F_\sigma(Z) := \xi^{-1} \circ f \circ \xi(Z) = \log f(e^Z) = (-1)^{n-\varrho} \left( d_1 Z + \sum_{i=1}^{n-1} (-1)^i \log(e^{d_i + d_{i+1}} Z - a_i^{d_i + d_{i+1}}) \right) \tag{5.3}
\]

Then \( F_\sigma \) is a homeomorphism from \( \tilde{S}_\sigma \) to \( \tilde{S} \).

Let \( \Phi_\sigma : \tilde{S} \rightarrow \tilde{S}_\sigma \) be the inverse of \( F_\sigma \). Then \( \{\Phi_\sigma : \sigma \in \Sigma\} \) forms an IFS defined on \( \tilde{S} \) which is conjugated by log to the IFS \( \{\varphi_\sigma : \sigma \in \Sigma\} \). Hence the attractor of \( \{\Phi_\sigma : \sigma \in \Sigma\} \) is \( \tilde{J}_\sigma := \{Z = \log z : z \in J_s\} \). Hence we have

\[
\dim_H(J_s) = \dim_H(\tilde{J}_\sigma). \tag{5.4}
\]

Obviously, each \( \Phi_\sigma \) can be extended to be a univalent function on a neighborhood of \( \tilde{S} \) and \( \Phi_\sigma(\tilde{S}) = \tilde{S}_\sigma \).

Proof of the first part of Theorem 1.4. We first estimate the asymptotic behavior of \( F'_\sigma(Z) \) as \( s \to 0 \). By (5.3) we have

\[
F'_\sigma(Z) = (-1)^{n-\varrho} \left( d_1 + \sum_{i=1}^{n-1} (-1)^i \frac{(d_i + d_{i+1}) z^{d_i + d_{i+1}}}{z^{d_i + d_{i+1}} - a_i^{d_i + d_{i+1}}} \right), \tag{5.5}
\]

where \( Z \in \tilde{S}_\sigma \) and \( z = e^Z \in S_\sigma \).

By Lemma 5.4 if \( s > 0 \) is sufficiently small, we have

\[
0 < R_0 = R_0^+ \ll R_1^- \ll a_1 \ll R_1^+ \ll \cdots \ll R_{n-1}^- \ll a_{n-1} \ll R_{n-1}^+ \ll 1 \ll R_n^- = R_\infty
\]

and

\[
\lim_{s \to 0} \frac{R_i^-}{a_i} = \lim_{s \to 0} \frac{a_i}{R_i^+} = 0, \quad \text{where} \quad 1 \leq i \leq n - 1. \tag{5.6}
\]

If \( \sigma = (i, \ell) \) with \( 1 \leq i \leq n \) and \( 1 \leq \ell \leq d_i \), then \( z \in S_\sigma \) implies

\[
R_{i-1}^+ < |z| < R_i^-.
\]
Therefore, by (5.5) and (5.6), if \( z \in S_{(i, \ell)} \) we have

\[
\tilde{c}_i(s) \leq \left| F'_\sigma(Z) \right| - \left| d_i + \sum_{j=1}^{i-1} (-1)^j(d_j + d_{j+1}) \right| = \left| F'_\sigma(z) \right| - d_i \leq \tilde{b}_i(s),
\]

where \( \tilde{b}_i(s) \) and \( \tilde{c}_i(s) \) are positive numbers depending on \( s \) (also on \( z \)) which satisfies

\[
\lim_{s \to 0} \tilde{b}_i(s) = \lim_{s \to 0} \tilde{c}_i(s) = 0 \tag{5.7}
\]

uniformly on \( S_{(i, \ell)} \), where \( 1 \leq i \leq n \) and \( 1 \leq \ell \leq d_i \). Hence if \( \sigma = (i, \ell) \) with \( 1 \leq \ell \leq d_i \) we have

\[
\tilde{c}_i(s) \leq \left| F'_\sigma(Z) \right| \leq \tilde{b}_i(s),
\]

where

\[
\tilde{b}_i(s) := d_i + \tilde{b}_i(s) \quad \text{and} \quad \tilde{c}_i(s) := d_i - \tilde{c}_i(s).
\]

Set \( b_i = 1/\tilde{b}_i(s) \) and \( c_i = 1/\tilde{c}_i(s) \), where \( 1 \leq i \leq n \). It is easy to see that \( \Phi_\ast \) is bi-Lipschitz on \( S \) with the corresponding constants for all \( \sigma \in \Sigma \). By Theorem 5.5 and [5.4], we have

\[
\beta_- \leq \dim_H(J_s) = \dim_H(J_\ast) \leq \beta_+,
\]

where \( \beta_- \) and \( \beta_+ \) satisfy

\[
\sum_{i=1}^{n} d_i b_i^{\beta_-} = 1 \quad \text{and} \quad \sum_{i=1}^{n} d_i c_i^{\beta_+} = 1.
\]

By the definition of \( b_i \) and \( c_i \), we have

\[
\sum_{i=1}^{n} \frac{d_i}{(d_i + \tilde{b}_i(s))^{\beta_-}} = 1 \quad \text{and} \quad \sum_{i=1}^{n} \frac{d_i}{(d_i - \tilde{c}_i(s))^{\beta_+}} = 1.
\]

Let \( \alpha_d, \ldots, d_n \in (0, 1) \) be the number determined by (2.3). From the above equations, we see that

\[
\lim_{s \to 0} \beta_- = 1 + \alpha_d, \ldots, d_n = \lim_{s \to 0} \beta_+.
\]

This means that \( \lim_{s \to 0} \dim_H(J_s) = 1 + \alpha_d, \ldots, d_n \). By noting Proposition 5.2 this completes the proof of the first assertion of Theorem 1.4. \( \square \)

6. Hausdorff dimension of Cantor circles: The supremum

In this section we study the supremum the Hausdorff dimensions of the Cantor circle Julia sets. The idea is to perturb some parabolic rational maps with Cantor circle Julia sets to the hyperbolic ones and then use Shishikura’s result on the Hausdorff dimension of parabolic bifurcations. In fact, we will prove the second part of Theorem 1.4 for some more general hyperbolic components. For the study of parabolic Cantor circle Julia sets, one may refer to [QYY15] and [QYY16].

The following theorem is a weak version of [Shi98, Theorem 2].

**Theorem 6.1** (Shishikura). Suppose that a rational map \( f_0 \) of degree \( d \geq 2 \) has a parabolic fixed point \( z_0 \) with multiplier 1 and that the immediate parabolic basin of \( z_0 \) contains only one critical point of \( f_0 \). Then for any \( \varepsilon > 0 \) and \( b > 0 \), there exist a neighborhood \( N \) of \( f_0 \) in the space of rational maps of degree \( d \), a neighborhood \( V \) of \( z_0 \) in \( \hat{C} \), positive integers \( N_1 \) and \( N_2 \) such that if \( f \in N \), and if \( f \) has a fixed point in \( V \) with multiplier \( e^{2\pi i\alpha} \), where

\[
\alpha = \pm \frac{1}{a_1 \pm \frac{1}{a_2 + \beta}} \tag{6.1}
\]

with integers \( a_1 \geq N_1, a_2 \geq N_2 \) and \( \beta \in \mathbb{C}, 0 \leq \text{Re}\beta < 1, |\text{Im}\beta| \leq b \), then

\[
\dim_H(J(f)) > 2 - \varepsilon.
\]
For the characterization of the region for $\alpha$ satisfying (6.1), see \cite{Shi98} Figure 3] or \cite{Yan18} Lemma 3.3 and Figure 2).

**Theorem 6.2.** Let $H$ be a hyperbolic component in $M_d$ with $d \geq 2$. Suppose that every $f \in H$ has a simply connected periodic Fatou component whose closure is disjoint with any other Fatou components. Then

$$\sup_{f \in H} \dim_H(J(f)) = 2.$$  

**Proof.** By the assumption, every $f \in H$ has a cycle of attracting periodic Fatou components $U_0 \to U_1 \to \cdots \to U_{p-1} \to U_0$ which are all simply connected, where $p \geq 1$. Moreover, $\overline{U_i} \cap \overline{U_j} = \emptyset$ for any $i \neq j$. By performing a quasiconformal surgery, it is easy to see that $H$ contains at least one map $f_0$ such that $U_0 \to U_1 \to \cdots \to U_{p-1} \to U_0$ is a cycle of super-attracting basins of $f_0$ and $f_{0}^{\text{op}} : U_0 \to U_0$ contains exactly one critical point 0 (counted without multiplicity). By a standard quasiconformal surgery \cite[Chap. 4]{BF14}, one can construct a continuous path $(f_t : \hat{C} \to \hat{C})_{t \in [0,1)}$ of hyperbolic rational maps in $H$ such that $f_t^{\text{op}}$ has a geometrically attracting fixed point 0 with multiplier $t$ whose immediately attracting basin $U_0^t$ contains exactly one critical point (counted without multiplicity).

According to \cite{}, $(f_t : \hat{C} \to \hat{C})_{t \in [0,1)}$ can be chosen as a pinching path and the limit $f_1 := \lim_{t \to 0} f_t$ exists, where $f_1$ is a parabolic rational map having the following properties (see also \cite{McM00} and \cite{Kaw03}):

(a) $f_1^{\text{op}}$ has a parabolic fixed point 0 at 0 with multiplier 1 whose immediately attracting parabolic basin $U_0^1$ contains exactly one critical point $1$.

(b) $f_1^{\text{op}}|_{J(f_1)}$ is topologically conjugate to $f_t^{\text{op}}|_{J(f_t)}$ for all $t \in [0,1)$ (actually topologically conjugate to $f_t^{\text{op}}|_{J(f_t)}$ for all $f \in H$);

(c) The Julia set $J(f_1)$ is homeomorphic to the Julia set of $J(f_0)$.

By Theorem 6.1, for any $\varepsilon > 0$, there exist a small neighborhood $N_\varepsilon$ of $f_1^{\text{op}}$ in the moduli space $M_d$ with $d' = d^p$ and a subset $N'_\varepsilon \subset N_\varepsilon$, such that every $f \in N'_\varepsilon \cap H$ has a cycle of geometrically attracting periodic point with multiplier satisfying $f_1^{\text{op}}$, and the Hausdorff dimension of $J(f)$ is at least $2 - \varepsilon$. Therefore we have $\sup_{f \in H} \dim_H(J(f)) = 2$.

If $H$ is a Cantor circle hyperbolic component, then the closures of any two different Fatou components of $f \in H$ are disjoint. Moreover, every $f \in H$ has a cycle of simply connected periodic Fatou components. This ends the proof of Theorem 6.2 and the second part of Theorem 1.4.

**Remark.** (i) The Julia sets in Theorem 6.2 could be disconnected. Indeed, the maps in this theorem are only required to contain one simply connected attracting basin but the other attracting basins may be infinitely connected.

(ii) Theorem 6.2 can be used to study the Hausdorff dimension of some other kind of Julia sets. For example, for any Sierpiński carpet hyperbolic component $H$ (i.e. every map in $H$ has a Sierpiński carpet Julia set), one has $\sup_{f \in H} \dim_H(J(f)) = 2$ (see \cite{}).

**Proof of Corollary 1.5.** Let $H$ be a Cantor circle hyperbolic component in $M_{2d}$ such that each $f \in H$ has the combination $(1; d, d) \in \mathcal{C}$. By Theorem 1.4 we have

$$\inf_{f \in H} \dim_H(J(f)) = 1 + \frac{\log 2}{\log d} \quad \text{and} \quad \sup_{f \in H} \dim_H(J(f)) = 2.$$  

\footnote{Note that $f_1^{\text{op}}$ has exactly one petal (contained in $U_0^1$) at the parabolic fixed point 0. This is the reason we assumed that each $f \in H$ has a simply connected periodic Fatou component whose closure is disjoint with any other Fatou components.}
\footnote{One can perturb $f_1$ along horocycles to obtain the required multipliers, see \cite{McM00} §12}.
Note that $f \mapsto \dim_H(J(f))$ is a continuous function as $f$ moves in $\mathcal{H}$ (see [Rue82]). For each $s \in (\log 2/\log d, 2)$, there exists a map $f \in \mathcal{H}$ such that $\dim_H(J(f)) = s$. Since $d$ can be chosen such that it is arbitrarily large, the second statement of Corollary 1.5 follows.

Let $f$ be a rational map with a Cantor circle Julia set $J(f)$. According to [QYY15], $f$ is hyperbolic or parabolic. By [Urb91] or [Yin00], we have $\dim_H(J(f)) < 2$. If $f$ is hyperbolic, then $f$ is contained in some Cantor circle hyperbolic component and we have $\dim_H(J(f)) = \dim_H(J(f)) > 1$ by Proposition 5.2. Suppose that $f$ is parabolic. By the continuity of the Hausdorff dimension of the Julia sets (see [McM00 Theorem 11.2]), there exists a sequence of hyperbolic rational maps $f_n$ such that $\dim_H(J(f_n)) = \lim_{n \to \infty} \dim_H(J(f_n))$, where $\{f_n\}_{n \in \mathbb{N}}$ are contained in the same Cantor circle hyperbolic component. This means that $\dim_{\mathcal{C}}(J(f)) = \inf_{n \in \mathbb{N}} \dim_{\mathcal{C}}(J(f_n)) = \dim_{\mathcal{C}}(J(f_n)) > 1$. Therefore, we have $1 < \dim_H(J(f)) < 2$ if $f$ is a Cantor set of circles.

Note that Shishikura’s criterion in Theorem 6.1 only works for the maps whose immediate basins contain exactly one critical point. If all attracting basins are multi-connected for the maps in a hyperbolic component, then we cannot use the criterion in Theorem 6.1 to obtain the same result as in Theorem 6.2. However, we would like to address the following

Question. Let $\mathcal{H}$ be any hyperbolic component in $\mathcal{M}_d$ with $d \geq 2$. Is it always true: $\sup_{f \in \mathcal{H}} \dim_H(J(f)) = 2$?

For quadratic rational maps, there exists exactly one unbounded hyperbolic component in $\mathcal{M}_2$ which is Cantor locus $\mathcal{C}_2$ (i.e. every map in $\mathcal{C}_2$ has a Cantor Julia set). According to Shishikura [Shi98] Remark 1.1(i), $\sup_{f \in \mathcal{C}_2} \dim_H(J(f)) = 2$. For any other hyperbolic component $\mathcal{H}$ in $\mathcal{M}_2$, the corresponding Julia sets are all connected and hence $\sup_{f \in \mathcal{H}} \dim_H(J(f)) = 2$. Therefore, the answer to the above question is yes if $d = 2$ (pointed out by Kevin Pilgrim).

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