SYSTEMS OF QUASILINEAR ELLIPTIC EQUATIONS WITH DEPENDENCE ON THE GRADIENT VIA SUBSOLUTION-SUPERSOLUTION METHOD

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Abstract. For the homogeneous Dirichlet problem involving a system of equations driven by \((p, q)\)-Laplacian operators and general gradient dependence we prove the existence of solutions in the ordered rectangle determined by a subsolution-supersolution. This extends the preceding results based on the method of subsolution-supersolution for systems of elliptic equations. Positive and negative solutions are obtained.

1. Introduction. The aim of this paper is to study the Dirichlet boundary value problem for the following system of quasilinear elliptic equations

\[
\begin{aligned}
(P_{\mu_1, \mu_2}) &
\begin{cases}
-\Delta_{p_1} u_1 - \mu_1 \Delta_{q_1} u_1 = f_1(x, u_1, u_2, \nabla u_1, \nabla u_2) & \text{in } \Omega \\
-\Delta_{p_2} u_2 - \mu_2 \Delta_{q_2} u_2 = f_2(x, u_1, u_2, \nabla u_1, \nabla u_2) & \text{in } \Omega \\
u_1 = u_2 = 0 & \text{on } \partial \Omega
\end{cases}
\end{aligned}
\]

through the method of subsolution-supersolution on a bounded domain \(\Omega \subset \mathbb{R}^N\) with a \(C^2\) boundary \(\partial \Omega\). Here \(\Delta_{p_i}\) and \(\Delta_{q_i}\), for \(i = 1, 2\) denote the \(p_i\)-Laplacian and \(q_i\)-Laplacian, respectively, with \(1 < q_i < p_i < +\infty\). We recall that the \(p\)-Laplacian for \(1 < p < +\infty\) is the operator \(\Delta_p : W_{0}^{1,p}(\Omega) \rightarrow W^{-1,p'}(\Omega)\), with \(p' = \frac{p}{p-1}\), defined by \(\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)\). In order to unify important special cases, in the statement of problem \((P_{\mu_1, \mu_2})\) there are given the constants \(\mu_1, \mu_2 \geq 0\). If \(\mu_1 = \mu_2 = 0\), the first equation is driven by the \(p_1\)-Laplacian and the second one by the \(p_2\)-Laplacian with possibly \(p_1 \neq p_2\). When \(\mu_1 = \mu_2 = 1\), a situation essentially different occurs having the first equation governed by the \((p_1, q_1)\)-Laplacian and the second one by the \((p_2, q_2)\)-Laplacian. The mixed situations \((\mu_1 = 0, \mu_2 = 1)\) or \((\mu_1 = 1, \mu_2 = 0)\) read as systems with an equation involving a \(p_i\)-Laplacian and the other a \((p_j, q_j)\)-Laplacian. Usually, all these cases need to be treated separately with

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specific techniques, including for applying the method of subsolution-supersolution. In the present work we are able to provide a unifying approach for the general problem $(P_{\mu_1,\mu_2})$.

A relevant feature of our paper is the fact that the nonlinearity in the right-hand side of the equations in $(P_{\mu_1,\mu_2})$ depend on the solution and its gradient, which is a serious difficulty to be overcome and is rarely handled in the literature, especially in the system setting due to the interaction between different equations. More precisely, the nonlinearity in the right-hand side of the elliptic equations (called often convection terms) are introduced by means of functions $f_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, which are Carathéodory meaning that $x \mapsto f_i(x, s_1, s_2, \xi_1, \xi_2)$ is measurable for all $(s_1, s_2, \xi_1, \xi_2) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N$ and $(s_1, s_2, \xi_1, \xi_2) \mapsto f_i(x, s_1, s_2, \xi_1, \xi_2)$ is continuous for a.a. $x \in \Omega$. The variables $(s_1, s_2)$ correspond to the solution $(u_1, u_2)$, whereas the vector variables $(\xi_1, \xi_2)$ figure out for the gradients $(\nabla u_1, \nabla u_2)$. The variational methods are not applicable to this setting.

In order to compose with the state of the art for our problem, we mention that some results on existence, uniqueness and asymptotic properties with respect to $(\mu_1, \mu_2)$ for the general system $(P_{\mu_1,\mu_2})$ have been recently obtained in [11] and for the equation version in [1] without location and enclosure properties as provided by subsolution-supersolution approach. The case where in $(P_{\mu_1,\mu_2})$ one takes $\mu_1 = \mu_2 = 0$ was investigated in [3] through the method of subsolution-supersolution. For the study of equations driven by $p$-Lapacian and exhibiting gradient dependence in lower order terms we refer to [5], [6], [7], [8], [13], [14].

By a solution to system $(P_{\mu_1,\mu_2})$ we mean a weak solution, that is a pair $(u_1, u_2) \in W^{1,p_1}_0(\Omega) \times W^{1,p_2}_0(\Omega)$ such that

$$\begin{align*}
\int_{\Omega} |\nabla u_1|^{p_1-2} \nabla u_1 \nabla v_1 \, dx + \mu_1 \int_{\Omega} |\nabla u_1|^{q_1-2} \nabla u_1 \nabla v_1 \, dx \\
= \int_{\Omega} f_1(x, u_1, u_2, \nabla u_1, \nabla u_2) v_1 \, dx,
\end{align*}$$

$$\begin{align*}
\int_{\Omega} |\nabla u_2|^{p_2-2} \nabla u_2 \nabla v_2 \, dx + \mu_2 \int_{\Omega} |\nabla u_2|^{q_2-2} \nabla u_2 \nabla v_2 \, dx \\
= \int_{\Omega} f_2(x, u_1, u_2, \nabla u_1, \nabla u_2) v_2 \, dx
\end{align*}$$

for all $(v_1, v_2) \in W^{1,p_1}_0(\Omega) \times W^{1,p_2}_0(\Omega)$. In the present paper, the approach for studying problem $(P_{\mu_1,\mu_2})$ relies on the method of subsolution-supersolution for systems as presented in [2]. We recall that $(\bar{u}_1, \bar{u}_2), (\bar{\overline{u}}_1, \bar{\overline{u}}_2) \in W^{1,p_1}(\Omega) \times W^{1,p_2}(\Omega)$ is a subsolution-supersolution of problem $(P_{\mu_1,\mu_2})$ if $\underline{u}_1 \leq \bar{u}_1$ a.e. in $\Omega$, $\underline{u}_2 \leq 0 \leq \bar{u}_2$ on $\partial \Omega$ for $i = 1, 2$, and there hold

$$\begin{align*}
\int_{\Omega} \left( |\nabla u_1|^{p_1-2} \nabla u_1 \nabla v_1 + \mu_1 |\nabla u_1|^{q_1-2} \nabla u_1 \nabla v_1 - f_1(x, u_1, u_2, \nabla u_1, \nabla u_2) v_1 \right) \\
+ \int_{\Omega} \left( |\nabla u_2|^{p_2-2} \nabla u_2 \nabla v_2 + \mu_2 |\nabla u_2|^{q_2-2} \nabla u_2 \nabla v_2 - f_2(x, u_1, u_2, \nabla u_1, \nabla u_2) v_2 \right)
\leq 0
\end{align*}$$

and

$$\begin{align*}
\int_{\Omega} \left( |\nabla \overline{u}_1|^{p_1-2} \nabla \overline{u}_1 \nabla v_1 + \mu_1 |\nabla \overline{u}_1|^{q_1-2} \nabla \overline{u}_1 \nabla v_1 - f_1(x, \overline{u}_1, \overline{u}_2, \nabla \overline{u}_1, \nabla \overline{u}_2) v_1 \right)
\end{align*}$$
that the Sobolev critical exponent corresponding to
following growth: there exist constants
under more restrictive hypotheses. Indeed, in [3] the driving differential operator is
(subsolution-supersolution exist. It is worth pointing out that assuming condition
\( P \) subsolution-supersolution of problem (1) we cannot apply the known results as available in [2], [3], [11], which hold
basically along the same lines. Subsequently, we make the convention that for every
\( r \in [1, +\infty] \), its H"older conjugate is denoted by, \( r' \), i.e., \( \frac{1}{r} + \frac{1}{r'} = 1 \).

Our main abstract result is stated as Theorem 3.1 below. It requires the existence of a subsolution-supersolution \((u_1, u_2), (\overline{u}_1, \overline{u}_2)\) in \(W^{1,p_1}(\Omega) \times W^{1,p_2}(\Omega)\) of problem \((P_{p_1, p_2})\) such that the following condition is satisfied:

\( (H) \) there exist constants \( b_i \geq 0 \), \( \beta_i \in [0, \frac{p_i}{p_i'}] \) and functions \( \sigma_i \in L^{\gamma_i}(\Omega) \), with
\( \gamma_i \in [1, p_i'] \), such that

\[
|f_i(x, s_1, s_2, \xi_1, \xi_2)| \leq \sigma_i(x) + b_1(|\xi_1|^{p_1} + |\xi_2|^{\frac{p_1}{p_1'}}),
\]

\[
|f_2(x, s_1, s_2, \xi_1, \xi_2)| \leq \sigma_2(x) + b_2(|\xi_1|^{\frac{p_2}{p_2'}} + |\xi_2|^{\beta_2})
\]

for a.e. \( x \in \Omega \), all \( s = (s_1, s_2) \in [u_1(x), \overline{u}_1(x)] \times [u_2(x), \overline{u}_2(x)] \), \( \xi_1, \xi_2 \in \mathbb{R}^N \),
\( i = 1, 2 \).

Under hypothesis \((H)\), the integrals in the preceding definitions of solution and subsolution-supersolution exist. It is worth pointing out that assuming condition \((H)\) we cannot apply the known results as available in [2], [3], [11], which hold under more restrictive hypotheses. Indeed, in [3] the driving differential operator is more particular and for the nonlinearities in the reaction terms it is supposed the following growth: there exist constants \( c_i \geq 0 \) and functions \( \rho_i \in L^{\gamma_i}(\Omega) \) such that

\[
|f_i(x, s_1, s_2, \xi_1, \xi_2)| \leq \rho_1(x) + c_1(|\xi_1|^{p_1-1} + |\xi_2|^{\frac{p_1}{p_1'}}),
\]

\[
|f_2(x, s_1, s_2, \xi_1, \xi_2)| \leq \rho_2(x) + c_2(|\xi_1|^{\frac{p_2}{p_2'}} + |\xi_2|^{\beta_2-1})
\]

for a.e. \( x \in \Omega \) and all \( s = (s_1, s_2) \in [u_1(x), \overline{u}_1(x)] \times [u_2(x), \overline{u}_2(x)] \), \( \xi_1, \xi_2 \in \mathbb{R}^N \),
which is more restrictive than hypothesis \((H)\) because \( p_i - 1 = \frac{p_i}{p_i'} < \frac{p_i}{(p_i')'} \) for \( i = 1, 2 \). In [2] the setting is even more restrictive than in [3]. In [11], due to a completely different approach, there are imposed assumptions of other type, for instance a generalized sign condition.

Weakening the preceding assumptions, Theorem 3.1 below establishes the existence and location properties for the solutions of problem \((P_{p_1, p_2})\) under hypothesis \((H)\). Like in [2] and [3], the proof relies on the study of an associated auxiliary problem (see Theorem 2.1 below). In comparison with what was done before, the main contribution consists in the use of a cut-off function adapted to the general growth condition in assumption \((H)\). Finally, we show in Theorem 4.1 below the existence of a positive solution \((u_1, u_2)\) of system \((P_{p_1, p_2})\) meaning that both components are positive. Proceeding along the same lines, one can obtain a negative solution \((u_1, u_2)\), i.e., both components are negative, as well as hybrid solutions \((u_1, u_2)\), in the sense that the components \( u_1 \) and \( u_2 \) are of opposite constant sign. The essential point for obtaining these solutions is the construction of an appropriate subsolution-supersolution of problem \((P_{p_1, p_2})\) permitting to apply Theorem 3.1. Contrary to the previous works where the driving operator was the \( p \)-Laplacean,
the subsolution-supersolution in the case of the \((p,q)\)-Laplacian as driving operator cannot be achieved through the first eigenvalue, which simply does not exist for the negative \((p,q)\)-Laplacian (see [10]). However, we are able to find a verifiable criterion for getting the desired subsolution-supersolution. A major part in these arguments is played by nonlinear regularity theory and strong maximum principle.

2. Auxiliary problem. This section is devoted to solve an auxiliary problem that will permit to establish the existence and location of solutions for the original problem \((P_{\mu_1,\mu_2})\). Our goal is to show that the method of subsolution-supersolution can be worked out for the general hypothesis \((H)\) by changing appropriately the cut-off functions.

Fix \(\mu_1, \mu_2 \geq 0\) and let \((\bar{u}_1, \bar{u}_2), (\bar{\pi}_1, \bar{\pi}_2) \in W^{1,p_1}(\Omega) \times W^{1,p_2}(\Omega)\) be a subsolution-supersolution of problem \((P_{\mu_1,\mu_2})\) as required in condition \((H)\). For \(i = 1, 2\), we consider the truncation operators \(T_i : W^{1,p_i}(\Omega) \to W^{1,p_i}(\Omega)\) defined by

\[
(T_i u)(x) = \begin{cases} 
\bar{\pi}_i(x) & \text{if } u(x) > \bar{\pi}_i(x) \\
u_i(x) & \text{if } \bar{\pi}_i(x) \leq u(x) \leq \bar{\pi}_i(x) \\
_i(x) & \text{if } u(x) < n_i(x).
\end{cases}
\]

The operators \(T_i\) are continuous and bounded.

We introduce the cut-off functions related to the given subsolution-supersolution to match the growth condition in hypothesis \((H)\). Namely, with the constants \(\beta_i\) in \((H)\), we set for a.a. \(x \in \Omega\), all \(s \in \mathbb{R}\), \(i = 1, 2\):

\[
\pi_i(x, s) = \begin{cases} 
(s - \bar{\pi}_i(x))^\frac{\beta_i}{p_i} & \text{if } s > \bar{\pi}_i(x) \\
(s - \bar{\pi}_i(x))^\frac{\beta_i}{p_i} & \text{if } \bar{\pi}_i(x) \leq s \leq \bar{\pi}_i(x) \\
(s - \bar{n}_i(x))^{\frac{\beta_i}{p_i}} & \text{if } s < \bar{n}_i(x).
\end{cases}
\]

It is clear that \(\pi_i\) is a Carathéodory function satisfying the estimate

\[
|\pi_i(x, s)| \leq \rho_i(x) + c_i |s|^{\frac{\beta_i}{p_i - \beta_i}}
\]

for a.a. \(x \in \Omega\), all \(s \in \mathbb{R}\), with a constant \(c_i \geq 0\) and a function \(\rho_i \in L^{\frac{p_i}{\beta_i}}(\Omega)\).

We have that \(\rho_i \in L^{\frac{p_i}{\beta_i}}(\Omega)\) because \(\bar{u}_i, \bar{\pi}_i \in W^{1,p_i}(\Omega)\), so \(\bar{u}_i, \bar{\pi}_i \in L^{p_i}(\Omega)\) by the Sobolev embedding theorem, and \(\beta_i < \frac{\rho_i}{(p_i)}\) by hypothesis \((H)\). Moreover, the same reasoning shows the existence of positive constants \(r_1^{(i)}\) and \(r_2^{(i)}\) such that

\[
\int_{\Omega} \pi_i(x, u)udx \geq r_1^{(i)} \|u\|_{L^{\frac{p_i}{\beta_i}}(\Omega)}^{\frac{\beta_i}{p_i - \beta_i}} - r_2^{(i)} \text{ for all } u \in W^{1,p_i}(\Omega), \ i = 1, 2.
\]

From (2) we infer that the corresponding Nemitskij operators \(\Pi_i : W^{1,p_i}(\Omega) \to L^{\frac{p_i}{\beta_i}}(\Omega)\) given by \(\Pi_i u(x) = \pi_i(x, u(x))\) are completely continuous due to the compact embedding of \(W^{1,p_i}(\Omega)\) into \(L^{p_i}(\Omega)\).

For any \(\lambda > 0\), consider the auxiliary problem

\[
\begin{cases} 
-\Delta_{p_1} u_1 - \mu_1 \Delta_{q_1} u_1 + \lambda \Pi_1(u_1) = f_1(x, T_1u_1, T_2u_2, \nabla(T_1u_1), \nabla(T_2u_2)) & \text{in } \Omega \\
-\Delta_{p_2} u_2 - \mu_2 \Delta_{q_2} u_2 + \lambda \Pi_2(u_2) = f_2(x, T_1u_1, T_2u_2, \nabla(T_1u_1), \nabla(T_2u_2)) & \text{in } \Omega \\
u_1 = u_2 = 0 & \text{on } \partial \Omega.
\end{cases}
\]

The solutions of problem (4) are understood in the weak sense as for the original problem \((P_{\mu_1,\mu_2})\) (see Section 1). Their existence is given in the next statement.
Theorem 2.1. Let \((u_1, u_2), (\overline{u}_1, \overline{u}_2) \in W^{1,p_1}(\Omega) \times W^{1,p_2}(\Omega)\) be a subsolution-supersolution of problem \((P_{\mu_1, \mu_2})\) such that condition \((H)\) is satisfied. Then, for every \(\lambda > 0\) sufficiently large, problem (4) has a solution \((u_1, u_2) \in W^{1,p_1}_0(\Omega) \times W^{1,p_2}_0(\Omega)\).

Proof. Using the subsolution-supersolution \((u_1, u_2), (\overline{u}_1, \overline{u}_2)\), we introduce the ordered intervals

\[ [u_i, \overline{u}_i] = \{ u \in W^{1,p_i}(\Omega) : u_i \leq u \leq \overline{u}_i \text{ a.e. in } \Omega \}, \quad i = 1, 2. \tag{5} \]

In view of (5) and hypothesis \((H)\), we have that the Nemytskij operator

\[ N : [u_1, \overline{u}_1] \times [u_2, \overline{u}_2] \subset W^{1,p_1}(\Omega) \times W^{1,p_2}(\Omega) \]

\[ \rightarrow L^{p_i}_1(\Omega) \times L^{p_i}_2(\Omega) \hookrightarrow W^{-1,p_i}_1(\Omega) \times W^{-1,p_i}_2(\Omega) \]

defined through the functions \(f_i\) by

\[ N(u_1, u_2) = (f_1(x, u_1, u_2, \nabla u_1, \nabla u_2), f_2(x, u_1, u_2, \nabla u_1, \nabla u_2)), \tag{6} \]

is bounded and completely continuous thanks to Rellich-Kondrachov compactness embedding theorem.

At this point, with a \(\lambda > 0\) that will be selected later on, we define the nonlinear operator \(A : W^{1,p_1}_0(\Omega) \times W^{1,p_2}_0(\Omega) \rightarrow W^{-1,p_1}_1(\Omega) \times W^{-1,p_2}_2(\Omega)\) given by

\[ A(u_1, u_2) = (A_1(u_1, u_2), A_2(u_1, u_2)) \]

\[ := \left( -\Delta_{p_1} u_1 - \mu_1 \Delta_{q_1} u_1 + \lambda \Pi_1 u_1, -\Delta_{p_2} u_2 - \mu_2 \Delta_{q_2} u_2 + \lambda \Pi_2 u_2 \right) \]

\[ - N(T_1 u_1, T_2 u_2). \]

According to (2), (5), (6) and hypothesis \((H)\), the operator \(A\) is well defined, bounded and continuous.

The next step in the proof is to show that the operator \(A\) is pseudomonotone. Toward this, let \((u_{1,n}, u_{2,n}) \rightharpoonup (u_1, u_2) \in W^{1,p_1}_0(\Omega) \times W^{1,p_2}_0(\Omega)\) and

\[ \limsup_{n \to +\infty} \langle A(u_{1,n}, u_{2,n}), (u_{1,n} - u_1, u_{2,n} - u_2) \rangle \leq 0. \tag{7} \]

Since \(\frac{p_i}{p_i - \beta_i} < p_i^*\) as known from the assumption on \(\beta_i\) in \((H)\), it follows the strong convergence \(u_{i,n} \rightharpoonup u_i\) in \(L^{p_i} (\Omega)\), so taking into account (2) we get

\[ \lim_{n \to +\infty} \int_{\Omega} \pi_i (x, u_{i,n}(x)) (u_{i,n} - u_i) dx = 0, \quad i = 1, 2. \tag{8} \]

Similarly, through Hölder’s inequality and Rellich-Kondrachov compactness embedding theorem, the weak convergence \((u_{1,n}, u_{2,n}) \rightharpoonup (u_1, u_2)\) in \(W^{1,p_1}_0(\Omega) \times W^{1,p_2}_0(\Omega)\) enables us to derive that

\[ \int_{\Omega} |\sigma_i| |u_{i,n} - u_i| dx \leq \|\sigma_i\|_{L^{p_i}_1(\Omega)} \|u_{i,n} - u_i\|_{L^{q_i}(\Omega)} \rightarrow 0 \text{ as } n \to +\infty. \tag{9} \]

We claim that

\[ \int_{\Omega} |\nabla (T_i u_{i,n})|^{\beta_i} |u_{i,n} - u_i| dx \rightarrow 0 \text{ as } n \to +\infty. \tag{10} \]
Indeed, writing explicitly the expression of the truncation operator, it turns out
\[ \int_\Omega |\nabla (T_j u_{i,n})|^{\beta_i} |u_{i,n} - u_i| dx = \int_{\{u_{i,n} < \underline{u}_i\}} |\nabla u_i|^{\beta_i} |u_{i,n} - u_i| dx + \int_{\{\underline{u}_i \leq u_{i,n} \leq \overline{u}_i\}} |\nabla u_{i,n}|^{\beta_i} |u_{i,n} - u_i| dx + \int_{\{u_{i,n} > \overline{u}_i\}} |\nabla u_{i,n}|^{\beta_i} |u_{i,n} - u_i| dx. \]

Because of \( \frac{p_j}{p_i - \beta_i} < p_1^* \), we have
\[ \int_{\{u_{i,n} < \underline{u}_i\}} |\nabla u_i|^{\beta_i} |u_{i,n} - u_i| dx \leq \|\nabla u_i\|_{L^{p_i}(\Omega)}^{\beta_i} \|u_{i,n} - u_i\|_{L^{\frac{p_j}{p_i - \beta_i}}(\Omega)} \to 0, \]
\[ \int_{\{\underline{u}_i \leq u_{i,n} \leq \overline{u}_i\}} |\nabla u_{i,n}|^{\beta_i} |u_{i,n} - u_i| dx \leq \|\nabla u_{i,n}\|_{L^{p_i}(\Omega)}^{\beta_i} \|u_{i,n} - u_i\|_{L^{\frac{p_j}{p_i - \beta_i}}(\Omega)} \to 0, \]
\[ \int_{\{u_{i,n} > \overline{u}_i\}} |\nabla u_{i,n}|^{\beta_i} |u_{i,n} - u_i| dx \leq \|\nabla u_{i,n}\|_{L^{p_i}(\Omega)}^{\beta_i} \|u_{i,n} - u_i\|_{L^{\frac{p_j}{p_i - \beta_i}}(\Omega)} \to 0, \]
whence (10) follows.

In the same way, for \( i \neq j \), we can prove
\[ \int_\Omega |\nabla (T_j u_{j,n})|^{\beta_j} |u_{j,n} - u_i| dx \leq \|\nabla (T_j u_{j,n})\|_{L^{p_j}(\Omega)}^{\beta_j} \|u_{j,n} - u_i\|_{L^{\frac{p_i}{p_j - \beta_j}}(\Omega)} \to 0. \] (11)

Combining (9), (10), (11) with hypothesis (H) results in
\[ \lim_{n \to +\infty} \int_\Omega f_i(x, T_1 u_{1,n}, T_2 u_{2,n}, \nabla (T_1 u_{1,n}), \nabla (T_2 u_{2,n}))(u_{i,n} - u_i) dx = 0, \quad i = 1, 2. \] (12)

If one substitutes (8) and (12) in (7), we arrive at
\[ \limsup_{n \to +\infty} (\langle -\Delta_{p_1} u_{1,n} - \mu_1 \Delta_{q_1} u_{1,n}, u_{1,n} - u_1 \rangle + \langle -\Delta_{p_2} u_{2,n} - \mu_2 \Delta_{q_2} u_{2,n}, u_{2,n} - u_2 \rangle) \leq 0. \] (13)

Let us show that (13) implies
\[ \limsup_{n \to +\infty} (\langle -\Delta_{p_i} u_{i,n} - \mu_i \Delta_{q_i} u_{i,n}, u_{i,n} - u_i \rangle) \leq 0, \quad i = 1, 2. \] (14)

Arguing by contradiction, we may assume that
\[ \lim_{n \to +\infty} (\langle -\Delta_{p_1} u_{1,n} - \mu_1 \Delta_{q_1} u_{1,n}, u_{1,n} - u_1 \rangle) > 0, \]
\[ \lim_{n \to +\infty} (\langle -\Delta_{p_2} u_{2,n} - \mu_2 \Delta_{q_2} u_{2,n}, u_{2,n} - u_2 \rangle) < 0. \]

The second inequality from above and the \((S)_+\)-property of \(-\Delta_{p_2} - \mu_2 \Delta_{q_2}\) on \(W^{1,p_2}_0(\Omega)\) (see [9, p. 39]) yield \(u_{2,n} \to u_2\) in \(W^{1,p_2}_0(\Omega)\), which leads to a contradiction with (13), thus proving (14).

By (14) and the \((S)_+\)-property of \(-\Delta_{p_i} - \mu_i \Delta_{q_i}\) on \(W^{1,p_i}_0(\Omega)\) we derive that \(u_{i,n} \to u_i\) in \(W^{1,p_i}_0(\Omega)\) as \(n \to +\infty, i = 1, 2\). Consequently, for every \((v_1, v_2) \in W^{1,p_1}_0(\Omega) \times W^{1,p_2}_0(\Omega)\) it holds
\[ \lim_{n \to +\infty} \langle A(u_{1,n}, u_{2,n}), (u_{1,n} - v_1, u_{2,n} - v_2) \rangle = \langle A(u_1, u_2), (u_1 - v_1, u_2 - v_2) \rangle \]
establishing that the operator \(A\) is pseudomonotone.
We show that the operator $A$ is coercive, which means that for every sequence $(u_{1,n}, u_{2,n}) \subset W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$ such that $\|(u_{1,n}, u_{2,n})\| \to +\infty$, we have
\[
\lim_{n \to +\infty} \frac{\langle A(u_{1,n}, u_{2,n}), (u_{1,n}, u_{2,n}) \rangle}{\|(u_{1,n}, u_{2,n})\|} = +\infty. \tag{15}
\]
To this end, by using (3) and hypothesis (H), we note that
\[
\langle A_1(u_{1,n}, u_{2,n}), u_{1,n} \rangle = \|\nabla u_{1,n}\|_{L^{p_1}(\Omega)}^{p_1} + \mu_1 \|\nabla u_{1,n}\|_{L^{p_1}(\Omega)}^{p_1} + \lambda \int_\Omega f_1(x, T_1 u_{1,n}, T_2 u_{2,n}, \nabla(T_1 u_{1,n}), \nabla(T_2 u_{2,n})) u_{1,n} dx
\]
\[
\geq \|\nabla u_{1,n}\|_{L^{p_1}(\Omega)}^{p_1} + \lambda \left( r_1^{(1)} \|u_{1,n}\|_{L^{p_1}(\Omega)}^{p_1 - \beta_1} - r_2^{(1)} \right) - \int_\Omega \sigma_1(x) |u_{1,n}(x)| dx
\]
\[
- \lambda_1 \int_\Omega |\nabla(T_1 u_{1,n})|^{\beta_1} |u_{1,n}| dx - \lambda_2 \int_\Omega |\nabla(T_2 u_{2,n})|^{\beta_2} |u_{1,n}| dx.
\]
We estimate the three integral terms listed above. First, observe that
\[
\int_\Omega \sigma_1(x) |u_{1,n}(x)| dx \leq \|\sigma_1\|_{L^{\gamma_1}(\Omega)} \|u_{1,n}\|_{L^{\gamma_1}(\Omega)} \leq C_1 \|\nabla u_{1,n}\|_{L^{p_1}(\Omega)},
\]
with a constant $C_1 > 0$. Second, using Young’s inequality with any $\varepsilon > 0$, we get
\[
\int_\Omega |\nabla(T_1 u_{1,n})|^{\beta_1} |u_{1,n}(x)| dx \leq \varepsilon \|\nabla(T_1 u_{1,n})\|_{L^{p_1}(\Omega)}^{p_1} + C_1(\varepsilon) \|u_{1,n}\|_{L^{p_1}(\Omega)}^{p_1 - \beta_1} \|u_{1,n}\|_{L^{p_1}(\Omega)} \]
\[
\leq \varepsilon \|\nabla u_{1,n}\|_{L^{p_1}(\Omega)}^{p_1} + \varepsilon \|\nabla u_{2,n}\|_{L^{p_1}(\Omega)}^{p_1} + C_1(\varepsilon) \|u_{1,n}\|_{L^{p_1}(\Omega)}^{p_1 - \beta_1}
\]
and similarly
\[
\int_\Omega |\nabla(T_2 u_{2,n})|^{\beta_2} |u_{1,n}(x)| dx \leq \varepsilon \|\nabla(T_2 u_{2,n})\|_{L^{p_2}(\Omega)}^{p_2} + C_2(\varepsilon) \|u_{1,n}\|_{L^{p_2}(\Omega)}^{p_2 - \beta_2} \|u_{1,n}\|_{L^{p_2}(\Omega)} \]
\[
\leq \varepsilon \|\nabla u_{2,n}\|_{L^{p_2}(\Omega)}^{p_2} + \varepsilon \|\nabla u_{2,n}\|_{L^{p_2}(\Omega)}^{p_2} + C_2(\varepsilon) \|u_{1,n}\|_{L^{p_2}(\Omega)}^{p_2 - \beta_2}
\]
for some constants $C_1(\varepsilon), C_2(\varepsilon) > 0$. Gathering all these estimates renders
\[
\langle A_1(u_{1,1}, u_{2,1}), u_{1,n} \rangle \geq (1 - \varepsilon) \|\nabla u_{1,n}\|_{L^{p_1}(\Omega)}^{p_1} - C_1 \|u_{1,n}\|_{L^{p_1}(\Omega)}^{p_1}
\]
\[
+ (\lambda_1^{(1)} - C(\varepsilon)) \|u_{1,n}\|_{L^{p_1}(\Omega)}^{p_1 - \beta_1} \|u_{1,n}\|_{L^{p_1}(\Omega)}^{p_1 - \beta_1} - \varepsilon \|\nabla u_{2,n}\|_{L^{p_2}(\Omega)}^{p_2} - D(\varepsilon),
\]
with positive constants $C(\varepsilon), D(\varepsilon)$. An analogous computation provides
\[
\langle A_2(u_{1,1}, u_{2,1}), u_{2,n} \rangle \geq (1 - \varepsilon) \|\nabla u_{2,n}\|_{L^{p_2}(\Omega)}^{p_2} - C_2 \|u_{2,n}\|_{L^{p_2}(\Omega)}^{p_2}
\]
\[
+ (\lambda_1^{(2)} - C'(\varepsilon)) \|u_{2,n}\|_{L^{p_2}(\Omega)}^{p_2 - \beta_2} \|u_{2,n}\|_{L^{p_2}(\Omega)}^{p_2 - \beta_2} - \varepsilon \|\nabla u_{1,n}\|_{L^{p_1}(\Omega)}^{p_1} - D'(\varepsilon),
\]
with positive constants $C'_1, C''(\varepsilon), D'(\varepsilon)$.

Summing up the last two inequalities, we notice that (15) holds true because $p_1, p_2 > 1$ and by choosing $\varepsilon > 0$ sufficiently small and $\lambda > 0$ sufficiently large. Therefore the operator $A : W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega) \to W_0^{-1, p'_1}(\Omega) \times W_0^{-1, p'_2}(\Omega)$ is pseudomonotone, bounded, and coercive. Consequently, we may apply the main theorem on pseudomonotone operators (see [2, Theorem 2.99]), which ensures that a solution $(u_1, u_2) \in W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$ of the equation
\[
A(u_1, u_2) = 0
\]
exists. According to the definition of the operator $A$, equation (16) reads as system (4). We can thus conclude that the solution $(u_1, u_2)$ of (16) is a solution to the auxiliary problem (4), which completes the proof.

**3. Existence and enclosure result.** Here we present our abstract result on the existence of a solution to problem $(P_{\mu_1, \mu_2})$ within the ordered rectangle determined by a subsolution-supersolution provided hypothesis $(H)$ is verified. The proof relies on Theorem 2.1 dealing with the auxiliary problem (4) and on the cut-off functions in (1), which are suitable for comparison with the given subsolution-supersolution for problem $(P_{\mu_1, \mu_2})$.

**Theorem 3.1.** Let $(u_1, u_2), (\overline{u}_1, \overline{u}_2) \in W^{1,p_1}(\Omega) \times W^{1,p_2}(\Omega)$ be a subsolution-supersolution of problem $(P_{\mu_1, \mu_2})$ such that condition $(H)$ is fulfilled. Then there exists a solution $(u_1, u_2) \in W^{1,p_1}_0(\Omega) \times W^{1,p_2}_0(\Omega)$ of problem $(P_{\mu_1, \mu_2})$ satisfying the enclosure property $\underline{u}_i \leq u_i \leq \overline{u}_i$ a.e. in $\Omega$, $i = 1, 2$.

**Proof.** Assumption $(H)$ enables us to apply Theorem 2.1, which provides a solution $(u_1, u_2) = (u_1(\lambda), u_2(\lambda))$ to the auxiliary problem (4) for every $\lambda > 0$ sufficiently large.

Let $(u_1, u_2) \in W^{1,p_1}_0(\Omega) \times W^{1,p_2}_0(\Omega)$ be a solution of problem (4) as given by Theorem 2.1. We claim that $(u_1, u_2) \in [\underline{u}_1, \overline{u}_1] \times [\underline{u}_2, \overline{u}_2]$ (see (5)).

We start by checking that $u_1 \leq \overline{u}_1$ a.e. in $\Omega$. For this, let us utilize as test function $v = (u_1 - \overline{u}_1)^+ = \max\{u_1 - \overline{u}_1, 0\} \in W^{1,p_1}_0(\Omega)$ in the definitions of solution to problem (4) and of supersolution to problem $(P_{\mu_1, \mu_2})$, which gives

$$
\langle -\Delta_p u_1 - \mu_1 \Delta q, u_1, (u_1 - \overline{u}_1)^+ \rangle + \lambda \int_\Omega \overline{u}_1(x, u_1)(u_1 - \overline{u}_1)^+ dx
$$

and

$$
\langle -\Delta_p \overline{u}_1 - \mu_1 \Delta q, \overline{u}_1, (u_1 - \overline{u}_1)^+ \rangle \geq \int_\Omega f_1(x, \overline{u}_1, \overline{u}_2, \nabla \overline{u}_1, \nabla \overline{u}_2)(u_1 - \overline{u}_1)^+ dx
$$

whenever $u_2 \in W^{1,p_2}(\Omega)$ with $\underline{u}_2 \leq u_2 \leq \overline{u}_2$ a.e. in $\Omega$. Thanks to the definition of the truncation operator $T_2$ there holds $u_2 \leq T_2 u_2 \leq \overline{u}_2$, so we are able to insert $u_2 = T_2 u_2$ in (18) resulting in

$$
\langle -\Delta_p \overline{u}_1 - \mu_1 \Delta q, \overline{u}_1, (u_1 - \overline{u}_1)^+ \rangle \geq \int_\Omega f_1(x, \overline{u}_1, T_2 u_2, \nabla \overline{u}_1, \nabla (T_2 u_2))(u_1 - \overline{u}_1)^+ dx.
$$

Subtracting (19) from (17) yields

$$
\langle -\Delta_p u_1 - \mu_1 \Delta q, u_1 - \langle -\Delta_p \overline{u}_1 - \mu_1 \Delta q, \overline{u}_1 \rangle, (u_1 - \overline{u}_1)^+ \rangle
$$

$$
+ \lambda \int_\Omega \overline{u}_1(x, u_1)(u_1 - \overline{u}_1)^+ dx
$$

$$
\leq \int_\Omega [f_1(x, T_1 u_1, T_2 u_2, \nabla (T_1 u_1), \nabla (T_2 u_2))
$$

$$
- f_1(x, \overline{u}_1, T_2 u_2, \nabla \overline{u}_1, \nabla (T_2 u_2))](u_1 - \overline{u}_1)^+ dx
$$

$$
= 0,
$$

where the last equality follows from the fact that $T_1 u_1 = \overline{u}_1$ on the set $\{u_1 > \overline{u}_1\}$, in view of the definition of the truncation operator $T_1$. This amounts to saying that
Then, the inequalities
\[(|\zeta|^{p_i-2}\zeta - |\eta|^{p_i-2}\eta)(\zeta - \eta) > 0, \quad (|\zeta|^{q_i-2}\zeta - |\eta|^{q_i-2}\eta)(\zeta - \eta) > 0\] (20)
for all \(\zeta, \eta \in \mathbb{R}^N\) with \(\zeta \neq \eta\), ensure that \(u_1 \leq \overline{u}_1\) a.e. in \(\Omega\).

On the same pattern, making use of adequate test functions for needed comparison, we can show that \(u_1 \leq u_1\) and \(u_2 \leq u_2\) a.e. in \(\Omega\). Hence the claim \((u_1, u_2) \in [\overline{u}_1, \overline{u}_1] \times [\overline{u}_2, \overline{u}_2]\) is verified.

On the basis of the above claim, we see that \(T_i u_i = u_i\) and \(\Pi_i u_i = 0\) for \(i = 1, 2\).

Therefore problem (4) reduces to \((P_{\mu_1, \mu_2})\), which renders that \((u_1, u_2)\) solves system \((P_{\mu_1, \mu_2})\). The proof is thus complete. \(\square\)

4. Constant-sign solutions. We illustrate the application of Theorem 3.1 by showing the existence of at least one positive solution of system \((P_{\mu_1, \mu_2})\). Suppose that:

(H1) For \(i = 1, 2\), there exist \(a_i \in L^{\alpha_i}(\Omega)\), with \(\alpha_i > N\) and \(a_i(x) > 0\) on a set of positive measure, and a Carathéodory function \(g_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}\) satisfying
\[0 \leq a_i(x) \leq f_i(x, s_1, s_2, \xi_1, \xi_2) \leq g_i(x, s_1, \xi_1) \leq \sigma_i(x) + b_i|\xi_1|^{\beta_i}\]
for a.a. \(x \in \Omega\), all \(s_1, s_2 \geq 0, \xi_1, \xi_2 \in \mathbb{R}^N\), with \(b_i, \beta_i, \sigma_i\) as in \((H)\), and such that the Dirichlet problem
\[
\begin{cases}
  -\Delta_{p_i} u - \mu_i \Delta_{q_i} u = g_i(x, u, \nabla u) & \text{in } \Omega \\
  u = 0 & \text{on } \partial\Omega
\end{cases}
\]
has a supersolution \(\overline{u}_i \in W^{1, p_i}(\Omega) \cap C(\Omega)\).

Our result in this direction is as follows.

**Theorem 4.1.** Assume that condition \((H1)\) is satisfied. Then there exists a solution \((u_1, u_2) \in W^{1, p_1}(\Omega) \times W^{1, p_2}(\Omega)\) of problem \((P_{\mu_1, \mu_2})\), which is positive, meaning that \(u_i > 0\) a.e. in \(\Omega\), \(i = 1, 2\).

**Proof.** In view of \((H1)\), for \(i = 1, 2\) it holds that \(a_i \in W^{-1, p_i}(\Omega)\), hence the Dirichlet problem
\[
\begin{cases}
  -\Delta_{p_i} u - \mu_i \Delta_{q_i} u = a_i & \text{in } \Omega \\
  u = 0 & \text{on } \partial\Omega
\end{cases}
\]
possesses a unique solution \(\overline{u}_i \in W^{1, p_i}_0(\Omega)\). The assumption \(a_i \in L^{\alpha_i}(\Omega)\) allows us to invoke [4, Theorem 3.1] for deducing that \(\overline{u}_i\) is bounded. Then, through the nonlinear regularity theory, it follows that \(\overline{u}_i \in C^1(\Omega)\).

Let us prove that
\[\overline{u}_i > 0 \quad \text{in } \Omega, \quad i = 1, 2.\] (21)

To this end, we act with the test function \(-\overline{u}_i^- = -\max\{-\overline{u}_i, 0\}\), that is
\[
\langle -\Delta_{p_i} \overline{u}_i - \mu_i \Delta_{q_i} \overline{u}_i, -\overline{u}_i^- \rangle = -\int_{\Omega} a_i(x) \overline{u}_i^-(x) \leq 0.
\]
We arrive at
\[ \int_{\Omega} (|\nabla u|^p_i + \mu_i |\nabla u|^q_i) \, dx \leq 0, \]
thereby \( u_i \geq 0 \) a.e. in \( \Omega \). Taking into account that \( u_i \) is nontrivial, the strong maximum principle in [12, Theorem 5.4.1] implies (21).

We further claim that
\[ u_i \leq \overline{u}_i \ a.e. \ in \ \Omega, \ i = 1, 2. \tag{22} \]
We only show that \( u_1 \leq \overline{u}_1 \) because the other inequality can be verified analogously. Arguing by contradiction, assume that the open set \( \{ u_1 > \overline{u}_1 \} \) is nonempty. By means of hypothesis (H1), acting with \( (\overline{u}_1 - u_1)^+ \) yields
\[
\langle -\Delta_{p_1} u_1 - \mu_1 \Delta_{q_1} u_1, (u_1 - \overline{u}_1)^+ \rangle = \int_{\Omega} a_1(x)(u_1 - \overline{u}_1)^+ \, dx \\
\leq \int_{\Omega} g_1(x, \overline{u}_1(x), \nabla \overline{u}_1(x))(u_1 - \overline{u}_1)^+ \, dx \leq \langle -\Delta_{p_1} \overline{u}_1 - \mu_1 \Delta_{q_1} \overline{u}_1, (u_1 - \overline{u}_1)^+ \rangle,
\]

hence
\[
\int_{\{ u_1 > \overline{u}_1 \}} \langle \nabla u_1 \rangle^{p_1-2} \nabla u_1 - \langle \nabla \overline{u}_1 \rangle^{p_1-2} \nabla \overline{u}_1 \rangle \langle \nabla u_1 - \nabla \overline{u}_1 \rangle \, dx \\
+ \mu_1 \int_{\{ u_1 > \overline{u}_1 \}} \langle |\nabla u_1|^{q_1-2} \nabla u_1 - |\nabla \overline{u}_1|^{q_1-2} \nabla \overline{u}_1 \rangle \langle \nabla u_1 - \nabla \overline{u}_1 \rangle \, dx \leq 0.
\]

From the strict inequality in (20), it follows that the function \( \overline{u}_1 - u_1 \) is locally constant on \( \{ u_1 > \overline{u}_1 \} \). Letting a path \( \gamma : [0, 1] \to \overline{\Omega} \) such that \( \gamma(0) \in \{ u_1 > \overline{u}_1 \} \) and \( \gamma(1) \in \partial \Omega \), and noting that \( u_1 \leq \overline{u}_1 \) on \( \partial \Omega \), we find \( t_0 \in (0, 1) \) such that \( \gamma(t_0) \notin \{ u_1 > \overline{u}_1 \} \) and \( \gamma(t) \in \{ u_1 > \overline{u}_1 \} \) for all \( t \in [0, t_0) \). The continuity of \( u_1, \overline{u}_1 \) and the fact that \( u_1 - \overline{u}_1 \) is locally constant on \( \{ u_1 > \overline{u}_1 \} \) yield a contradiction. So the desired conclusion in (22) ensues.

Next we prove that \( (u_1, u_2), (\overline{u}_1, \overline{u}_2) \) is a subsolution-supersolution for system \((P_{\mu_1, \mu_2})\). By hypothesis (H1) we derive that
\[
\langle -\Delta_{p_1} u_1 - \mu_1 \Delta_{q_1} u_1, v \rangle = \int_{\Omega} a_1(x)v(x) \, dx \leq \int_{\Omega} f_1(x, u_1, w_1, \nabla u_1, \nabla w_1) \, v \, dx
\]
and
\[
\int_{\Omega} f_1(x, \overline{u}_1, w_1, \nabla \overline{u}_1, \nabla w_1) \, v \, dx \leq \int_{\Omega} g_1(x, \overline{u}_1, \nabla \overline{u}_1) \, v \, dx \\
\leq \langle -\Delta_{p_1} \overline{u}_1 - \mu_1 \Delta_{q_1} \overline{u}_1, v \rangle
\]
for all \( v \in W^{1,p}_0(\Omega) \) with \( v \geq 0 \) a.e. in \( \Omega \) and all \( w_2 \in W^{1,p_2}_0(\Omega) \) with \( u_2 \leq w_2 \leq \overline{u}_2 \) a.e. in \( \Omega \). Notice that due to (22), it makes sense to consider \( w_2 \in W^{1,p_2}_0(\Omega) \) such that \( \overline{u}_2 \leq w_2 \leq \overline{u}_2 \) a.e. in \( \Omega \). Similarly, from hypothesis (H1) it is seen that
\[
\langle -\Delta_{p_2} u_2 - \mu_2 \Delta_{q_2} u_2, v \rangle = \int_{\Omega} a_2(x)v(x) \, dx \leq \int_{\Omega} f_2(x, u_1, w_1, \nabla u_1, \nabla w_2) \, v \, dx
\]
and
\[
\int_{\Omega} f_2(x, \overline{u}_1, w_1, \nabla w_1, \nabla w_2) \, v \, dx \leq \int_{\Omega} g_2(x, \overline{u}_1, \nabla \overline{u}_2) \, v \, dx \\
\leq \langle -\Delta_{p_2} \overline{u}_2 - \mu_1 \Delta_{q_2} \overline{u}_2, v \rangle
\]
for all \( v \in W^{1,p_2}_0(\Omega) \) with \( v \geq 0 \) a.e. in \( \Omega \) and all \( w_1 \in W^{1,p_1}_0(\Omega) \) with \( u_1 \leq w_1 \leq \bar{u}_1 \) a.e. in \( \Omega \). Altogether we have that \((u_1, u_2), (\bar{u}_1, \bar{u}_2) \) is a subsolution-supersolution for system \((P_{\mu_1, \mu_2})\).

Now we are in a position to apply Theorem 3.1, which ensures the existence of a solution \((u_1, u_2) \in W^{1,p_1}_0(\Omega) \times W^{1,p_2}_0(\Omega) \) of system \((P_{\mu_1, \mu_2})\) satisfying \( u_i \leq u_1 \leq \bar{u}_1 \) a.e. in \( \Omega, i = 1, 2 \) (see (22)). Then, in view of (21), we infer that \( u_i > 0 \) a.e. in \( \Omega, i = 1, 2 \). The proof is thus complete.

\[ \square \]

**Example 1.** The simplest situation to fulfill hypothesis \((H1)\) is that there exist \( a_i, a'_i \in L^{\alpha_i}(\Omega) \), with \( \alpha_i > N \) and \( a_i(x) > 0 \) on a set of positive measure, such that

\[ 0 \leq a_i(x) \leq f_i(x, s_1, s_2, \xi_1, \xi_2) \leq a'_i(x) \]

for a.a. \( x \in \Omega \) and all \( s_1, s_2 > 0, \xi_1, \xi_2 \in \mathbb{R}^N \).

**Remark 1.** Contrary to the case of \( p \)-Laplacian, in our framework of \((p, q)\)-Laplacian driving operator, the subsolution-supersolution for system \((P_{\mu_1, \mu_2})\) cannot be constructed through the classical procedure based on the first eigenvalue. We recall that the first eigenvalue of \(-\Delta_p\) on \( W^{1,p}_0(\Omega) \) is given by

\[ \lambda_{1,p} = \inf_{u \in W^{1,p}_0(\Omega), u \neq 0} \frac{\|\nabla u\|^p_{L^p(\Omega)}}{\|u\|^p_{L^p(\Omega)}}, \]

which is not possible for the negative \((p, q)\)-Laplacian \(-\Delta_p - \Delta_q\) (see [10]).

Symmetrically to \((H1)\), we can formulate the condition:

\((H2)\) For \( i = 1, 2 \), there exist \( a_i \in L^{\alpha_i}(\Omega) \), with \( \alpha_i > N \) and \( a_i(x) < 0 \) on a set of positive measure, and a Carathéodory function \( g_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R} \) satisfying

\[ \sigma_i(x) - b_i|\xi|^{\beta_i} \leq g_i(x, s_1, \xi) \leq f_i(x, s_1, s_2, \xi_1, \xi_2) \leq a_i(x) \leq 0 \]

for a.a. \( x \in \Omega \) and all \( s_1, s_2 > 0, \xi_1, \xi_2 \in \mathbb{R}^N \), with \( b_i, \beta_i, \sigma_i \) as in \((H)\), and such that the Dirichlet problem

\[ \begin{cases} -\Delta_p u - \mu_i \Delta_q u &= g_i(x, u, \nabla u) & \text{in } \Omega \\ u &= 0 & \text{on } \partial \Omega \end{cases} \]

has a subsolution \( u_i \in W^{1,p_i}(\Omega) \cap C(\overline{\Omega}) \).

On the pattern of Theorem 4.1 we can prove the following sufficient condition to have negative solutions.

**Theorem 4.2.** Assume that condition \((H2)\) is satisfied. Then there exists a solution \((u_1, u_2) \in W^{1,p_1}_0(\Omega) \times W^{1,p_2}_0(\Omega) \) of system \((P_{\mu_1, \mu_2})\), which is negative, meaning that \( u_i < 0 \) a.e. in \( \Omega, i = 1, 2 \).

In order to obtain solutions to system \((P_{\mu_1, \mu_2})\) whose components are of opposite constant sign, we state the condition:

\((H3)\) For \( i, j = 1, 2, i \neq j \), there exist \( a_i \in L^{\alpha_i}(\Omega), a_j \in L^{\alpha_j}(\Omega) \) with \( \alpha_i, \alpha_j > N \) and \( a_i(x) < 0, a_j(x) > 0 \) on sets of positive measure, and Carathéodory functions \( g_i, g_j : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R} \) satisfying

\[ \sigma_i(x) - b_i|\xi|^{\beta_i} \leq g_i(x, s_1, \xi) \leq f_i(x, s_1, s_2, \xi_1, \xi_2) \leq a_i(x) \leq 0 \]

for a.a. \( x \in \Omega \) and all \( s_1, s_2 < 0, \xi_1, \xi_2 \in \mathbb{R}^N \), with \( b_i, \beta_i, \sigma_i \) as in \((H)\), and

\[ 0 \leq a_j(x) \leq f_j(x, s_1, s_2, \xi_1, \xi_2) \leq g_j(x, s_j, \xi) \leq \sigma_j(x) + b_j|\xi|^{\beta_j} \]
for a.a. \( x \in \Omega \) and all \( s_1, s_2 > 0, \xi_1, \xi_2 \in \mathbb{R}^N \), with \( b_j, \beta_j, \sigma_j \) as in (H), and such that the Dirichlet problem
\[
\begin{cases}
-\Delta_{p_i} u - \mu_i \Delta_{q_i} u = g_i(x, u, \nabla u) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]
has a subsolution \( u_0 \in W^{1,p_i}(\Omega) \cap C(\overline{\Omega}) \), and the Dirichlet problem
\[
\begin{cases}
-\Delta_{p_j} u - \mu_j \Delta_{q_j} u = g_j(x, u, \nabla u) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]
has a supersolution \( \overline{u}_j \in W^{1,p_j}(\Omega) \cap C(\overline{\Omega}) \).

Our final result provides solutions to problem \((P_{\mu_1, \mu_2})\) with a negative component and a positive component.

**Theorem 4.3.** Assume that condition (H3) is satisfied. Then there exists a solution \((u_1, u_2) \in W^{1,p_1}(\Omega) \times W^{1,p_2}(\Omega)\) of system \((P_{\mu_1, \mu_2})\) with \( u_i < 0 \) a.e. in \( \Omega \) and \( u_j > 0 \) a.e. in \( \Omega \), where \( i, j = 1, 2 \), \( i \neq j \), correspond to the statement in (H3).

Overall, the proof can be done using the ideas in the proof of Theorem 4.1.

**Example 2.** Hypothesis (H3) is fulfilled if there exist \( a_i, a_i', a_j, a_j' \in L^\alpha(\Omega) \setminus \{0\} \), with \( \alpha > N \), such that
\[
a_i'(x) \leq f_i(x, s_1, s_2, \xi_1, \xi_2) \leq a_i(x) \leq 0
\]
for a.a. \( x \in \Omega \) and all \( s_1, s_2 < 0, \xi_1, \xi_2 \in \mathbb{R}^N \), and
\[
0 \leq a_j(x) \leq f_j(x, s_1, s_2, \xi_1, \xi_2) \leq a_j'(x)
\]
for a.a. \( x \in \Omega \) and all \( s_1, s_2 > 0, \xi_1, \xi_2 \in \mathbb{R}^N \).

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