Acoustic radiation force on a cylindrical particle near a planar rigid boundary

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Abstract
The aim of this investigation is to provide improved mathematical series expansions of the longitudinal and transverse acoustic radiation forces for a rigid cylindrical particle in 2D of arbitrary cross-section located near a planar rigid wall. Incident plane progressive waves with variable angle of incidence are considered in a non-viscous fluid. The multiple scattering effects occurring between the particle and the rigid boundary are described using the partial-wave decomposition in cylindrical coordinates, the method of images and the translational addition theorem. Initially, an effective acoustic field incident on the particle is defined, which includes the primary incident field, the reflected waves from the flat wall and the scattered field from the image object. Subsequently, the incident effective field along with the scattered field from the object are utilized to obtain closed-form mathematical expressions for the longitudinal and transverse radiation force functions, based on a scattering approach in the far-field. The radiation force vector components are formulated in partial-wave series in cylindrical coordinates, which involve the incidence angle, the expansion coefficients of the scatterer and its image, and the distance from the center of mass of the particle to the boundary. Numerical examples for a rigid circular cylinder are considered. Computations for the longitudinal and transverse non-dimensional radiation force functions are performed. Emphasis is given on varying the size of the particle, the incidence angle of the source field and the particle-wall distance. Depending on the particle-wall distance and incidence angle, zero-longitudinal and transverse force components arise, thus, the particle becomes unaffected by the linear momentum transfer. Moreover, pushing or pulling forces between the particle and wall are predicted depending on the particle-wall distance, the incidence angle and size parameter. The results may find possible applications in the development of acousto-fluidic devices, acoustic levitation of particles nearby a boundary, cloaking/invisibility, and underwater acoustics to name a few areas, where most investigations resort initially to numerical simulations to guide the experimental design processes.

1. Introduction

The robust and optimized design of resonators used as acousto-fluidic devices in lab-on-a-chip applications relies on the accurate numerical/computational prediction of the acoustic radiation forces on a particle or multiple particles [1] in small channels [2, 3]. Several analytical [4–8] and numerical [9, 10] models exist and have been used for that purpose. Nonetheless, computations disregarding the channel wall effects on the force can lead to significant inaccuracies. Recognizing the importance of the presence of a boundary near a particle, an extended formalism for the longitudinal force component has been developed initially for an elastic spherical shell of finite thickness close to a porous impedance boundary in a non-viscous fluid [11]. Particularly, the effects of the multiple scattering between the wall and the particle have been examined. Discussions on how these effects have impacted the radiation force [11] were provided, where pulling or pushing forces have been predicted on the spherical shell centered on the axis of the incident waves at normal incidence with respect to the boundary. Later, analytical modeling and numerical simulations based on the finite volume method (limited to
the long-wavelength approximation; i.e., small particle compared to the wavelength, known also as the Rayleigh limit) to compute the longitudinal and transverse components of the radiation force vector on a rigid circular cylindrical particle nearby a rigid wall have been presented \cite{12}, where some comments seem to suggest an apparent difficulty in using the far-field scattering approach for a particle near a (chamber) wall to derive the related analytical expressions.

The aim of this work is to resolve this apparent difficulty and demonstrate the validity of the far-field scattering approach to calculate efficiently the components of the radiation force vector for a cylindrical particle of arbitrary cross-section near a flat rigid boundary, substantiated with numerical computations for the components of the radiation force vector without restriction to the small particle (Rayleigh) limit. The improved formalism presented here is valid for any scattering regime (i.e., Rayleigh, Mie or geometrical/ray acoustics). It also accounts for the multiple interactions occurring between the particle and the rigid boundary \cite{13–16}, described using the modal expansion method in cylindrical coordinates \cite{17}, the method of images \cite{18, 19} and the translational addition theorem \cite{20}.

Initially, an effective acoustic field incident on the particle is defined, which includes the principal incident field, the field corresponding to the waves reflected from the boundary, and the scattered field from the image object. Then, the expression for the incident effective field in conjunction with that of the scattered field from the object are utilized to obtain the closed-form mathematical expressions for the longitudinal and transverse components of the radiation force vector, based on the scattering in the far-field. This procedure leads to exact expressions without any approximations. The example of a rigid (fixed/immovable) circular cylindrical particle of arbitrary radius is considered, and numerical results for the longitudinal and transverse components of the acoustic radiation force vector are developed, assuming plane progressive waves with an arbitrary angle of incidence. The rigid particle case is relevant in various fluid dynamics applications (for example in a reduced gravity environment, or air) where suspended objects may act as sound impenetrable due to the large acoustic impedance mismatch between the object and its host fluid medium. In the computations, several parameters (such as the distance from the boundary, the non-dimensional size parameter and the angle of incidence of the incoming waves) are varied. This analysis may have some implications and relevance in potential applications involving particle manipulation and handling using acoustical \cite{21, 22} and elastic \cite{23} sheets (i.e. waves in 2D) and other related areas in underwater acoustics. Furthermore, the obtained radiation force expressions are valid for any particle material (fluid/liquid, elastic, viscoelastic etc.) provided the suitable boundary conditions allowing to obtain the scattering coefficients are imposed.

It is noteworthy that apart from their practical importance, in most cases 2D scattering and radiation force problems provide rigorous, convenient and efficient ways of displaying emergent physical phenomena (such as cloaking \cite{24, 25} and radiation force cancellation effects \cite{1, 26, 27}) without involving the mathematical/algebraic manipulations encountered in solving three-dimensional problems, and this analysis should assist in the development of analytical solution for the spherical, spheroidal \cite{28–33} and other arbitrary-shaped geometries in 3D.

2. Method

2.1. Multiple scattering formalism

Consider an incident plane progressive wave field upon a particle with arbitrary cross-section, with an incidence angle $\alpha$ as shown in figure 1. The host fluid is assumed non-viscous with a density $\rho$ and speed of sound $c$. The incident velocity potential field is expressed as

$$\Phi_{\text{inc}} = \phi_0 e^{-i\omega t} e^{ikr},$$  \hspace{1cm} (1)

where $\phi_0$ is the velocity potential amplitude, $\omega$ is the angular frequency, the parameter $k$ is the wave-vector, and $r$ is the vector position.

In a cylindrical coordinates system ($r$, $\theta$) located at the center of mass of the object, the incident velocity potential field given by equation (1) is expressed as a partial-wave series expansion (PWSE) such that

$$\Phi_{\text{inc}}(r, \theta, t) = \phi_0 e^{-i\omega t} e^{ikr \cos(\theta - \alpha)}$$

$$= \phi_0 e^{-i\omega t} \sum_{n=-\infty}^{+\infty} i^n e^{-in\alpha} J_n(kr) e^{in\theta},$$  \hspace{1cm} (2)

where $J_n(\cdot)$ is the cylindrical Bessel function of the first kind, and $k = \omega/c$ is the wavenumber.

The velocity potential field for the incident plane waves reflected from the rigid/fixed boundary is also expressed as a PWSE as
where \( d \) is the distance from the center of mass of the particle to the boundary along the direction \( \theta = 0 \) (Figure 1).

The scattered field resulting from the interaction of the primary incident plane progressive wave field with the object is expressed as,

\[
\Phi_{\text{object}}(r, \theta, t) = \phi_0 e^{-i\omega t} e^{ikr} \cos(\theta - \pi + \alpha) e^{i2kd \cos \alpha} = \phi_0 e^{-i\omega t} e^{i2kd \cos \alpha} \sum_{n=-\infty}^{\infty} i^n e^{-i(\pi - \alpha)} J_n(kr) e^{in\theta},
\]

(3)

where \( d \) is the distance from the center of mass of the particle to the boundary along the direction \( \theta = 0 \) (Figure 1).

The scattered field resulting from the interaction of the primary incident plane progressive wave field with the object is expressed as,

\[
\Phi_{\text{scat}}^{\text{object}}(r, \theta, t) = \phi_0 e^{-i\omega t} \sum_{n=-\infty}^{\infty} C_n H_n^{(1)}(kr) e^{in\theta},
\]

(4)

where \( H_n^{(1)}(\cdot) \) is the cylindrical Hankel function of the first kind of order \( n \), and \( C_n \) is a modal coefficient to be determined by applying suitable boundary conditions.

The method of images is now applied, which consists of replacing the multiple scatterings between the object and rigid boundary by a scattering field from the image object. The scattered field contributed by the image object is expressed as

\[
\Phi_{\text{scat}}^{\text{image}}(r', \theta', t) = \phi_0 e^{-i\omega t} \sum_{n=-\infty}^{\infty} D_n H_n^{(1)}(kr') e^{in\theta'},
\]

(5)

where \( D_n \) is the modal coefficient in \((r', \theta')\).

Notice that equations (4) and (5) are mathematical expressions valid for a non-circular object with a smooth surface and its image (in 2D). Similar expressions using the mathematical basis of cylindrical wave functions were previously utilized in elastic wave [34–36] and acoustic scattering [37–40] based on the formalism of the T-matrix [41]. (Related discussions can be also found in [39, 40, 42–45]).

Equation (5) shows that the scattered velocity potential field is written in the coordinates system \((r', \theta')\). To determine the modal coefficients \( C_n \) and \( D_n \), suitable boundary conditions should be satisfied, either at the planar rigid boundary surface [16, 46], or at the surface of the particle and its corresponding image in the two systems of coordinates [15]. Both methods are equivalent. Using the latter approach, the translational addition theorem is used.

Nearby the object (i.e. \( r < d \)) the following relationship holds [47]

\[
H_n^{(1)}(kr') e^{in\theta'}|_{r < 2d} = \sum_{m=-\infty}^{\infty} J_m(kr) H_{n-m}^{(1)}(2kd) e^{in\theta},
\]

(6)

while nearby the image object, it is expressed as,

\[
H_n^{(1)}(kr) e^{in\theta}|_{r < 2d} = \sum_{m=-\infty}^{\infty} J_m(kr') H_{n-m}^{(1)}(2kd) e^{in\theta'}. \]

(7)

Subsequently, the total velocity potential field in the host non-viscous fluid nearby the object or its image is rewritten as
\[ \Phi_{\text{tot}}(r, \theta, t)_{|r < 2d} = \Phi_{\text{inc}}(r, \theta, t) + \Phi_{R}(r, \theta, t) + \Phi_{\text{object}}(r, \theta, t) + \Phi_{\text{image}}(r, \theta, t), \]

where the orthogonality property is used to express \( \Phi_{\text{image}} \) in the system of coordinates \((r, \theta)\) as,

\[ \Phi_{\text{image}}(r, \theta, t) = \phi_0 e^{-ikr} \left( \sum_{n=-\infty}^{+\infty} \left( \sum_{m=-\infty}^{+\infty} D_m n H_{n-m}^{(1)}(2kd) \right) f_0(kr)e^{i\alpha t} \right). \]

Similarly, the expression for \( \Phi_{\text{tot}}(r', \theta', t)_{|r' < 2d} \) is given by

\[ \Phi_{\text{tot}}(r', \theta', t) = \Phi_{\text{inc}}(r', \theta', t) + \Phi_{R}(r', \theta', t) + \Phi_{\text{object}}(r', \theta', t) + \Phi_{\text{image}}(r', \theta', t), \]

where,

\[ \Phi_{\text{inc}}(r', \theta', t) = \phi_0 e^{-ikr'} \sum_{n=-\infty}^{+\infty} i^n e^{-i(n+1)\alpha} J_n(kr')e^{i\alpha t}, \]

\[ \Phi_{R}(r', \theta', t) = \phi_0 e^{-ikr'} 2kd \sum_{n=-\infty}^{+\infty} i^n e^{-i(n+1)\alpha} J_n(kr')e^{i\alpha t}, \]

and,

\[ \Phi_{\text{object}}(r', \theta', t) = \phi_0 e^{-ikr'} \sum_{n=-\infty}^{+\infty} \left( \sum_{m=-\infty}^{+\infty} C_m n H_{n-m}^{(1)}(2kd) \right) J_n(kr')e^{i\alpha t}. \]

For a rigid fixed/inmovable particle, the Neumann boundary condition (pp. 21–22 in [48]) should be imposed on its surface (and its image), such that

\[ \nabla \Phi_{\text{tot}}(r, \theta, t) \cdot \mathbf{n}_{|r=A_0} = 0, \]

\[ \nabla \Phi_{\text{tot}}(r', \theta', t) \cdot \mathbf{n}_{|r'=A_0} = 0, \]

where \( \mathbf{n} \) and \( A_0 \) are, respectively, the normal vector and the surface-shape function of the object of arbitrary geometry. Similar boundary conditions were previously used for the case of a rigid (sound-impenetrable) elliptical cylindrical particle in quasi-Gaussian cylindrically-focused beams [39] and plane waves [38, 40, 42, 43]. Moreover, at the planar rigid boundary where \( r = r' \) and \( \theta = (\pi - \theta) \), the Neumann boundary condition (used previously for a rigid sphere near a rigid flat surface [46]) still holds, such that, \( \partial \Phi_{\text{tot}}(r, \theta, t) / \partial n_{|r=r'} = \nabla \Phi_{\text{tot}}(r, \theta, t) \cdot \mathbf{n}_{|r=r'} = 0. \)

2.2. Analytical expressions for the longitudinal and transverse acoustic radiation force components

The particle located close to a boundary illuminated by the incident plane progressive wave field experiences a total force composed of a primary force due to the interaction with the principal incident field, and another force induced by the cross-scattering with the planar rigid boundary. To obtain the expressions for the longitudinal and transverse radiation force components, an incident ‘effective’ field on the particle should be determined first. This effective field accounts for the primary incident field, the reflected waves from the boundary, and the scattered waves from the image object. This procedure reduces the problem in hand to that of the single object case, but with an effective field incident upon its surface.

Considering the fact that the cylindrically-outgoing scattered waves cannot be incident on the surface of the probed object, the incident effective field in \((r, \theta)\) is determined from equation (8) such that,

\[ \Phi_{\text{eff}}(r, \theta, t) = \phi_0 e^{-ikr} \sum_{n=-\infty}^{+\infty} i^n (e^{-i(n+1)\alpha} J_n(kr) + e^{-i(n+1)\alpha} J_n(kr) e^{i\alpha t} + \sum_{m=-\infty}^{+\infty} D_m n H_{n-m}^{(1)}(2kd) J_n(kr)e^{i\alpha t}). \]

The formulation to compute the force components based on the far-field scattering approach [49, 50] is now used. In that limit, the cylindrical Bessel and Hankel functions of the first kind can be approximated such that,

\[ J_n(kr) \approx \frac{\sqrt{\pi kr}}{2} \cos \left( kr - \frac{m}{2} - \frac{\pi}{4} \right) \quad \text{and} \quad \frac{H_n^{(1)}(kr)}{kr} \approx \frac{2}{\sqrt{\pi kr}} e^{i(kr - \frac{m}{2} - \frac{\pi}{4})}. \]

Then, the radiation (particle-wall) force expressions can be derived based on the integration of the radiation stress tensor over a surface at a large radius [51–53]. As an alternative, integration on the surface of the particle using the near-field scattering can be performed, however, in a non-viscous fluid, the near-field or far-field scattering approaches are commensurate.
with the same result [50, 54] since the divergence of the radiation stress tensor is zero. Notice, however, that the use of the far-field scattering approach requires fewer algebraic manipulations.

Based on the integral expression given by equation (12) in [52], the equation for the acoustic radiation force vector is cast in a simplified form after substitution of the expressions for the incident/scattered pressures and velocities, leading to the compact form,

\[
\langle \mathbf{F} \rangle = \frac{1}{2} \rho k^2 \int_0^{2\pi} \Re \{ \Phi_h \} \, dS,
\]

where \( \Re \{ \cdot \} \) is the real part of a complex number, \( \Phi_h \mid_{kr \to \infty} = \Phi_{\text{scat}} \mid_{kr \to \infty} (i/k) \partial_{\lambda} \Phi_{\text{inc}} - \Phi_{\text{inc}} - \Phi_{\text{scat}} \}, \partial_r = \partial/\partial r, \) and \( dS = dS_{\text{eff}} \), where \( dS = r \, d\theta \) is the scalar differential element of a cylindrical surface \( S \) with unit-length. The normal unit vector (pointing outwardly to the object surface) is \( \mathbf{e}_n = \cos \theta \mathbf{e}_x + \sin \theta \mathbf{e}_y \), where \( \mathbf{e}_x \) and \( \mathbf{e}_y \) are the unit vectors in the Cartesian coordinates system. The symbol \( \langle \cdot \rangle \) denotes time-averaging, and the superscript * denotes a complex conjugate.

The particle nearby the wall experiences a force vector having a longitudinal (i.e. along the \( x \)-axis) and a transverse (i.e. along the \( y \)-axis) component. Those are defined, respectively, as

\[
\begin{align*}
\begin{bmatrix} E_x \\ E_y \end{bmatrix} &= \langle \mathbf{F} \rangle \cdot \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{bmatrix} = \begin{bmatrix} Y_x \\ Y_y \end{bmatrix} S_e E_0, \\
\end{align*}
\]

where \( S_e \) is the cross-sectional surface of the object, \( E_0 = \frac{1}{2} \rho k^2 |\phi_h|^2 \) is a characteristic energy density parameter, and \( Y_x \) and \( Y_y \) are the longitudinal and transverse non-dimensional radiation force functions.

Recognizing the property of the integrals such that,

\[
\int_0^{2\pi} e^{i(\alpha - \beta)\theta} \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} \, d\theta = \pi \begin{bmatrix} \delta_{n,n+1} + \delta_{n,n-1} \\ i(\delta_{n,n+1} - \delta_{n,n-1}) \end{bmatrix},
\]

where \( \delta_{ij} \) is the Kronecker delta function, the substitution of equations (4) and (16) (in the far-field) into equation (17), leads after algebraic manipulation to the mathematical expressions for the longitudinal and transverse force function components, respectively, as

\[
\begin{align*}
Y_x &= \frac{2}{kS_e} \Re \left\{ \sum_{n=-\infty}^{+\infty} (i^n e^{-i(n+2)k \alpha}) (e^{-(n+2)k \alpha} + e^{-i(n+2)k \alpha}) \right\} + C_n + \sum_{m=-\infty}^{+\infty} D_m H_{m-n}(2kd) \left( C_{n+1}^m - C_{n-1}^m \right), \\
Y_y &= -\frac{2}{kS_e} \Re \left\{ \sum_{n=-\infty}^{+\infty} (i^n e^{-i(n+2)k \alpha}) (e^{-(n+2)k \alpha} + e^{-i(n+2)k \alpha}) \right\} + C_n + \sum_{m=-\infty}^{+\infty} D_m H_{m-n}(2kd) \left( C_{n+1}^m + C_{n-1}^m \right),
\end{align*}
\]

where \( \Re \{ \cdot \} \) is the imaginary part of a complex number.

### 2.3. The rigid circular cylinder example

Assuming a rigid cylinder of circular cross-section of arbitrary radius \( a \), the boundary condition given by equation (14) at \( r = a \) using equation (8) and the orthogonality of the functions \( e^{i\alpha \theta} \), leads to the following coupled system of linear equations,

\[
\begin{align*}
\mathbf{i} \mathbf{r} ' (e^{-i\alpha} + e^{-i(\alpha - \alpha) + 2k \alpha} \mathbf{J} (ka) + C_e H_{[\mathbf{\alpha}]} (ka) + D_e H_{[\mathbf{\alpha}]} (ka) \sum_{n=-\infty}^{+\infty} D_n H_{\mathbf{\alpha}+n}(2kd) = 0, \\
(22)
\end{align*}
\]

where the primes in equation (22) designate the derivatives of the cylindrical wave functions.

Next, the boundary condition given by equation (15) in the image space at the surface of the image object \( r' = a \) using equation (10) and the orthogonality of \( e^{i\alpha \theta} \), also leads to another coupled system of linear equations as,

\[
\begin{align*}
\mathbf{i} \mathbf{r} ' (e^{-i\alpha} + 2k \alpha) \mathbf{J} (ka) + D_e H_{[\mathbf{\alpha}]} (ka) + D_e H_{[\mathbf{\alpha}]} (ka) \sum_{n=-\infty}^{+\infty} C_n H_{\mathbf{\alpha}+n}(2kd) = 0. \\
(23)
\end{align*}
\]

The coupled modal coefficients \( C_n \) and \( D_n \) can be determined by solving numerically equations (22) and (23) and truncating the PWSNs at a maximum limit \( N_{\text{max}} = \max(\max(ka, kb), kd) + 25 \), which ensures adequate convergence and validation of the results.
After the expansion coefficients $C_n$ and $D_n$ are obtained, the radiation force functions given by equations (20) and (21) can be calculated numerically.

3. Results and discussions

A MATLAB numerical program/code is constructed to implement and solve the coupled system of equations given by equations (22) and (23) for a rigid circular cylinder near a rigid boundary. Subsequently, computation of the acoustic radiation force functions given by equations (20) and (21) is accomplished. In the simulations, the cross-sectional surface for a unit-length circular cylinder is $S_c = 2a$.

Initially, validation and verification of the results is performed by evaluating the longitudinal radiation force function given by equation (20) for a Rayleigh circular cylinder assuming an angle of incidence $\alpha = 0^\circ$. In this test, the opposite longitudinal radiation force function $-Y_x$ is computed so as to correlate the results with those of the normalized force component shown in panels (a) and (b) of figure 6 in [12], which used a different geometrical configuration. The results are displayed in panels (a) and (b) of figure 2 for two different values of $ka = 0.063$ and 0.126, respectively. Those plots are in agreement with those shown in panels (a) and (b) of figure 6 in [12] obtained using exact and numerical finite volume methods.

Next, systematic cases are considered to illustrate the analysis. First, the effect of varying the size parameter $ka$ is examined by considering a Rayleigh circular cylinder near a boundary having $ka = 0.1$, and varying the angle of incidence of the plane progressive wave field and the dimensionless distance, respectively, in the chosen ranges $-90^\circ \leq \alpha \leq 90^\circ$ and $ka < kd \leq 15$. Panels (a) and (b) of figure 3 display the results for the longitudinal and transverse radiation force functions, respectively. Panel (a) shows that $Y_x$ exhibits variations versus $kd$ as the angle of incidence varies in the range $-60^\circ \leq \alpha \leq 60^\circ$, while outside this bandwidth, the longitudinal force function variations are less pronounced. This clearly shows that within the range $-60^\circ \leq \alpha \leq 60^\circ$, multiple reflections between the particle and boundary are significant, and can cause attraction (i.e., $Y_x < 0$), repulsion (i.e., $Y_x > 0$) and neutrality (i.e., $Y_x = 0$) forces depending on the dimensionless distance $kd$ far from the boundary. Typically, this is not the case for the transverse radiation force function, as shown in panel (b) of
Figure 3. The plots for the longitudinal (panel (a)) and transverse (panel (b)) radiation force functions versus $kd$ and $\alpha$ for a small (subwavelength) circular rigid cylinder near a rigid boundary at $ka = 0.1$.

Figure 4. The same as in figure 3 but for a larger circular cylindrical particle with $ka = 5$. 
figure 3, where \( Y_y \) vanishes for \( \alpha = 0^\circ \) by symmetry considerations (figure 1), and the variations in the plot are larger in the ranges \(-90^\circ \leq \alpha \leq -60^\circ \) and \( 60^\circ \leq \alpha \leq 90^\circ \).

The effect of increasing the size parameter to \( ka = 5 \) is also considered, and the corresponding plots are displayed in panels (a) and (b) of figure 4. As \( ka \) increased, the radiation force function amplitudes are larger. Moreover, depending on the angle of incidence \( \alpha \) and the dimensionless interparticle distance \( kd \), the function \( Y_y \) in panel (a) alternates between positive, zero, and negative values, as shown previously in panel (a) of figure 3. At the crossing point where \( Y_y \) vanishes, the circular cylinder yields complete unresponsiveness to the linear transfer of momentum caused by multiple scattering effects. Moreover, the transverse radiation force function \( Y_y \) remains asymmetric with respect to the incidence angle value \( \alpha = 0^\circ \).

Another case of interest is to investigate the effect of varying the angle of incidence while varying the size parameter and the dimensionless distance, respectively, in the chosen ranges \( 0 < ka < 5 \) and \( ka < kd \leq 15 \). Because of practical considerations, the constraint \( kd > ka \) should always be satisfied so as to lead physical results. After imposing this condition, the longitudinal and transverse radiation force functions are computed and the results are displayed, respectively, in panels (a) and (b) of figure 5, assuming an angle of incidence \( \alpha = 0^\circ \). Panel (a) shows that at low \( ka \) values, the longitudinal force function component shows sinusoidal oscillations between negative (i.e., pulling force towards the incident waves) and positive (i.e., pulling force towards the wall) values as \( kd \) increases, similarly to the plots shown in figure 2. As \( ka \) increases, the amplitude of the undulations become less pronounced. In panel (b), the transverse radiation force function \( Y_y \) vanishes as required by symmetry at \( \alpha = 0^\circ \).

Increasing the angle of incidence to \( \alpha = 45^\circ \) is considered, and panels (a) and (b) of figure 6 display the corresponding results for the plots of \( Y_x \) and \( Y_y \), respectively, while varying the size parameter and the dimensionless distance similarly to figure 5. Panel (a) shows that \( Y_x \) displays irregular (non-sinusoidal) variations, which are sensitive to the values of \( ka \) and \( kd \). Moreover, the plot for \( Y_y \) computed at \( \alpha = 45^\circ \) displays smaller amplitude variations than those observed when \( \alpha = 0^\circ \). This occurs since \( Y_y \) is maximal when \( \alpha = 0^\circ \) since the field acts as a standing and all the waves reflected from the boundary back toward the incident field interact with the particle manifold. As \( \alpha \) deviates from zero, \( Y_y \) is weakened as the reflected waves from the boundary are deviated from the direction of the incident field. Moreover, \( Y_y \) is non-zero as the symmetry is
Figure 6. The same as in figure 5 but the angle of incidence is $\alpha = 45^\circ$. The transverse radiation force function displayed in panel (b) is non-zero as the symmetry is broken for this value of the angle of incidence.

Figure 7. The plots for the longitudinal (panel (a)) and transverse (panel (b)) radiation force functions versus $ka$ and $\alpha$ for a circular rigid cylinder near a rigid boundary at $kd = 5.5$. 
broken, and varies with positive values suggesting a pushing transverse component towards the positive \( y \)-direction.

Finally, the effect of the non-dimensional particle-boundary distance \( kd \) is examined while varying the angle of incidence of the plane progressive wave field and the dimensionless size parameter, respectively, in the chosen ranges \( -90^\circ \leq \alpha \leq 90^\circ \) and \( 0 < ka < 5 \). Panels (a) and (b) of figure 7 display the plots for \( Y_x \) and \( Y_y \), respectively, at \( kd = 5.5 \). Panel (a) shows that \( Y_x \) is maximal in the region delimited by \( \sim -30^\circ \leq \alpha \leq 30^\circ \) and \( 0 < ka < 5 \), such that the circular cylinder experiences an attractive longitudinal force acting towards the rigid boundary. Moreover, \( Y_x \) exhibits positive, negative and zero values depending on the values of the selected parameters \( \alpha \) and \( ka \). The plot for \( Y_y \) in panel (b) shows the asymmetric behavior versus the axis \( \alpha = 0^\circ \). It is positive for values of \( \alpha > 0^\circ \) and negative for \( \alpha < 0^\circ \).

Increasing the dimensionless distance to \( kd = 15 \) alters the longitudinal and transverse radiation force functions, as shown in panels (a) and (b) of figure 8. As \( kd \) increases, the undulations occurrence increases as they are more pronounced in the plots.

4. Conclusion

This contribution presented a complete formalism and exact PWSEs without any approximations for the longitudinal and transverse radiation force functions on a particle of arbitrary shape in 2D located nearby a rigid boundary, subjected to incident plane progressive waves with an arbitrary angle of incidence. The coupling effects of the reflected waves from the flat rigid boundary, and those scattered by the particle and its image are taken into account in deriving the radiation force function expressions. The modal expansion method with boundary matching in cylindrical coordinates, the method of images, and the translational addition theorem are utilized. Systematic computational examples for a rigid circular cylinder are examined, where a series of undulations are manifested in the plots. Depending on the particle-boundary non-dimensional distance \( kd \), the incidence angle \( \alpha \) as well as the dimensionless size parameter \( ka \), conditions have been predicted where the cylinder yields complete unresponsiveness to the linear momentum transfer of the effective incident field. Moreover, pushing forces towards either the incident waves or the rigid planar boundary can arise by judicious selection of \( kd \), \( \alpha \), and \( ka \). The results presented here may be helpful in applications involving multiple wave interacting systems for a particle near a chamber wall, particularly, in acoustofluidics, fluid dynamics, or

![Figure 8](image.png)

*Figure 8.* The same as in figure 7 but for a larger dimensionless distance factor \( kd = 15 \).
underwater acoustics to name a few examples. The results obtained here from a particle in 2D can serve in predicting the behavior of a 3D particle nearby a boundary, and this will be the subject of a forthcoming investigation.

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