Relative free splitting and free factor complexes I:
Hyperbolicity

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1 Introduction

Masur and Minsky in their papers [MM99, MM00] introduced a hierarchy of connected simplicial complexes associated to a finite type surface $S$: at the top of the hierarchy is the curve complex of $S$; and at lower levels are the curve complexes of essential, connected subsurfaces of $S$. They prove hyperbolicity of the curve complexes of all finite type surfaces, which applies immediately to all levels of the hierarchy of $S$. This hierarchy of hyperbolic complexes has proved immensely useful in many applications to the large scale geometry of the mapping class group $\text{MCG}(S)$ [BF02, BM08, Man10, BKMM12, BBF10].

In [HM13] we proved hyperbolicity of the free splitting complex $\mathcal{FS}(F_n)$ of a rank $n$ free group $F_n$, originally introduced as Hatcher’s sphere complex [Hat95]. In [BF14], Bestvina and Feighn proved hyperbolicity of the complex of free factors $\mathcal{F}(F_n)$ of a rank $n$ free group $F_n$. Each of these complexes is regarded as an analogue, in different ways, of the curve complex of a finite type surface.

In this paper we study the large scale geometry of relative free factor and free splitting complexes of $F_n$, proving their hyperbolicity. These complexes are intended as analogues of curve complexes of essential connected subsurfaces of a finite type surface. Unlike the situation with surfaces, hyperbolicity of these relative complexes is not a consequence of the absolute cases covered in [HM13], [BF14], rather it is a generalization. With a bit of extra effort, we also consider relative free factor and free splitting complexes of groups in general, and prove their hyperbolicity.

While the need for relativizing [HM13] and [BF14] has been clear, what “relativization” might mean has been less clear to us. In our work on the classification of subgroups of $\text{Out}(F_n)$ [HM13a–e] we studied both subgroups and individual elements that are fully irreducible relative to a free factor system. This work helped us to formulate an appropriate concept of relativization: we propose here to study free factor and free splitting complexes of $F_n$ relative to a fixed free factor system of $F_n$. 
Relative complexes of free factor systems and of free splittings for $F_n$.

Free factor systems for $F_n$ were first used by Bestvina, Feighn, and Handel [BFH00] to analyze the dynamics of general elements of Out($F_n$). Formally a free factor system of $F_n$ is a finite set of the form $A = \{[A_1], \ldots, [A_K]\}$ such that there exists a free factorization $F_n = A_1 * \cdots * A_K * B$, ($K \geq 0$), where each $A_k$ is nontrivial, and $[\cdot]$ denotes conjugacy class of a subgroup. We refer to the elements of the set $A$ as its components. We refer to the free factor $B$ as a cofactor of $A$, with a careful emphasis that $B$ is far from unique, not even up to conjugacy, although its rank and therefore its isomorphism type is well-defined; note that $B$ may be trivial. Inclusion of free factors up to conjugacy induces a partial ordering on free factor systems which is denoted $A \sqsubseteq A'$.

Fixing one free factor system $A$ of $F_n$, the complex of free factor systems of $F_n$ relative to $A$, denoted $\mathcal{FF}(F_n; A)$, is defined to be the geometric realization of the partial ordering $\sqsubseteq$ restricted to the set of free factor systems $B$ of $\Gamma$ that are properly nested between $A$ and the “improper” free factor system $\{[F_n]\}$; that is, $A \sqsubseteq B$ and $A \neq B \neq \{[F_n]\}$. The familiar case of $\mathcal{FF}(F_n; \emptyset)$ contains the complex of free factors $\mathcal{F}(F_n)$ as the subcomplex consisting of those $B$ having but a single component; the natural inclusion is a quasi-isometry (see Proposition 6.3). The exceptional free factor systems are certain ones close to the maximum $\{[F_n]\}$ for which $\mathcal{FF}(F_n; A)$ exhibits the exceptional behavior of being either empty or 0-dimensional (see Section 2.5).

A free splitting of $F_n$ is a minimal action of $F_n$ on a nontrivial simplicial tree $T$ with trivial edge stabilizers and with finitely many edge orbits. The set of conjugacy classes of nontrivial vertex stabilizers forms a free factor system of $F_n$ denoted $\mathcal{F}(T)$ (see Section 3.2). Two free splittings which differ by an equivariant homeomorphism are equivalent. Collapsing equivariant subgraphs of free splittings defines a partial ordering on equivalence classes which is denoted $S \succ T$.

The free splitting complex of $F_n$ relative to a free factor system $A$, denoted $\mathcal{FS}(F_n; A)$, is the simplicial realization of the partial ordering $\succ$ restricted to equivalence classes of free splittings $T$ such that $A \sqsubseteq \mathcal{F}(T)$; here we allow equality $A = \mathcal{F}(T)$. The familiar case $\mathcal{FS}(F_n; \emptyset)$ is the free splitting complex of $F_n$ as studied in [HM13].

**Theorem 1.1.** For any proper free factor system $A$ of $F_n$, the complex $\mathcal{FS}(F_n; A)$ is nonempty, connected, and hyperbolic.

**Theorem 1.2.** For any nonexceptional free factor system $A$ of $\Gamma$, the complex $\mathcal{FF}(\Gamma; A)$ is positive dimensional, connected, and hyperbolic.

Theorem 1.1 can also have certain special behavior when $A$ is exceptional; see Section 4.2.

While this paper does not contain applications of Theorems 1.1 and 1.2, being focused on their proofs and the proofs of generalized versions as discussed below, we expect that the relative complexes $\mathcal{FS}(F_n; A)$ and $\mathcal{FF}(F_n; A)$ will become part of a toolkit for investigating the large scale geometry of Out($F_n$) and its subgroups.
In a companion paper [HM] we shall study the dynamics of the natural subgroup of $\text{Out}(F_n)$ that acts on $\mathcal{FF}(F_n; \mathcal{A})$ and on $\mathcal{FS}(F_n; \mathcal{A})$, namely the subgroup $\text{Out}(F_n; \mathcal{A})$ that stabilizes the free factor system $\mathcal{A}$. The results of [HM] will generalize results of [HM14] which carried out this study in the absolute case where $\mathcal{A} = \emptyset$. We shall characterize which elements of $\text{Out}(F_n; \mathcal{A})$ act loxodromically on $\mathcal{FS}(F_n; \mathcal{A})$ and which act loxodromically on $\mathcal{FF}(F_n; \mathcal{A})$, and we shall characterize the attracting and repelling endpoints at infinity of loxodromic elements, expressing these results in terms of attracting laminations. For example we will prove that $\phi \in \text{Out}(F_n; \mathcal{A})$ acts loxodromically on $\mathcal{FF}(F_n; \mathcal{A})$ if and only if $\phi$ has an attracting lamination that fills relative to $\mathcal{A}$, and $\phi$ acts loxodromically on $\mathcal{FS}(F_n; \mathcal{A})$ if and only if $\phi$ is fully irreducible relative to $\mathcal{A}$. In addition we shall prove the following result, which is new even in the absolute case:

**Theorem 1.3.** There are constants $M, L > 0$, depending only on $n$, such that for every free factor system $\mathcal{A}$ of $F_n$ and every $\phi \in \text{Out}(F_n; \mathcal{A})$, the action of $\phi$ on $\mathcal{FS}(F_n; \mathcal{A})$ satisfies one of two possibilities: either $\phi$ has an orbit of diameter $\leq M$; or $\phi$ acts loxodromically with stable translation length $\geq L$.

The analogous theorem for pseudo-Anosov homeomorphisms of a surface acting on the curve complex is due to Bowditch [Bow08, Corollary 1.5].

**Relative complexes of free factor systems and free splitting for general groups.**

As alluded to earlier, we shall generalize Theorems 1.2 and 1.1 to any group relative to any choice of free factor system of that group. As it turns out, the proofs of hyperbolicity work identically in this general context, after some preliminary work establishing basic facts which are well known in the special case of $F_n$ (see Sections 2 and 3).

The general context in which we shall work is a very natural one, identical to the context of what might be called the “relative outer spaces” as formulated by Guirardel and Levitt in Section 4 of their paper [GL07]. Although they did not use this language, their work may be regarded as a study of the outer space of a group $\Gamma$ relative to a free factor system of $\Gamma$. An explicit general definition of free factor system for any group may be found in the work of Francavigila and Martino [FM14] where they study train track maps in the context of the Guirardel–Levitt relative outer spaces.

Our general Theorems 1.4 and 1.5 are intended as a contribution to a growing mathematical study of outer automorphism groups of freely decomposable groups—both absolute, and relative to a choice of free factor system—with a goal of developing analogies between theorems about these groups and theorems about $\text{Out}(F_n)$. For other works in this genre see [Mar99], [McM96], [CT94].

The roots of free factor systems and the partial order $\sqsubseteq$ in the context of a general finitely generated group may be seen in the following fundamental theorem. Given a
group $\Gamma$ define a *Grushko decomposition* to be a free product decomposition of the form

$$(*) \quad \Gamma = A_1 \ast \cdots \ast A_K \ast B \quad (K \geq 0)$$

in which each $A_k$ is nontrivial, freely indecomposable, and not infinite cyclic, and $B$ is free of finite rank (possibly trivial).

**Grushko Decomposition Theorem** ([Chi76, Coh89]). *Every finitely generated group $\Gamma$ has a Grushko decomposition.*

Any Grushko decomposition $(*)$ of any group $\Gamma$ has certain uniqueness properties:

1. If $A' < \Gamma$ is a free factor which is not a finite rank free group then there exists $k \in \{1, \ldots, K\}$ such that $A_k$ is conjugate to a subgroup of $A'$.

2. For any other Grushko decomposition $\Gamma = A'_1 \ast \cdots \ast A'_{K'} \ast B'$, $(K' \geq 0)$, we have $K = K'$, $\text{rank}(B) = \text{rank}(B')$, and for each $k = 1, \ldots, K'$ the subgroups $A_k, A'_{\sigma(k)}$ are conjugate, where $\sigma$ is a uniquely determined index permutation.

These properties follow easily from the Kurosh subgroup theorem (see Section 2.1).

Formally, free factor systems are defined for a general group $\Gamma$ just as in the special case $\Gamma = F_n$, namely sets of the form $A = \{[A_1], \ldots, [A_K]\}$ such that there is a free factorization $\Gamma = A_1 \ast \cdots \ast A_K \ast B$ where each $A_k$ is nontrivial, $B$ is free and of finite rank (possibly trivial), and $[A_k]$ denotes the conjugacy class of the subgroup $A_k$. The “extension” relation $A \sqsubseteq A'$ is defined exactly as in the special case. Note that our definition is stricter than in [FM14], by requiring the “cofactor” $B$ to be free of finite rank; this restriction is motivated in part by the Grushko Decomposition Theorem, and in part by the requirements of our theory.

The above uniqueness properties of a Grushko decomposition $(*)$ may be expressed by saying that the associated *Grushko free factor system* $A = \{[A_1], \ldots, [A_K]\}$ is the unique minimum of the partial ordering $\sqsubseteq$ on the set of free factor systems of $\Gamma$. The converse is also true: if the free factor system $A$ is the unique minimum of $\sqsubseteq$, then $A$ is the Grushko free factor system associated to a Grushko decomposition (see Proposition 2.13).

Fix now an arbitrary group $\Gamma$ and a free factor system $A$ of $\Gamma$. We do not require that $\Gamma$ be finitely generated, nor that $A$ be a Grushko free factor system. We treat the elements of $A$ as indivisible atoms: for purposes of defining and studying the relative complexes $\mathcal{FF}(\Gamma; A)$ and $\mathcal{FS}(\Gamma; A)$ we never peer at the internal structure of a subgroup representing a component of $A$, we never ask whether those subgroups are freely indecomposable, etc. Of course for applications the internal structure of $A$ will be important, as it is in the results of [GL07] regarding virtual cohomological dimension.

The *relative outer automorphism group* $\text{Out}(\Gamma; A)$ is defined to be the subgroup of $\text{Out}(\Gamma)$ whose action on the set of free factor systems of $\Gamma$ fixes $A$. This is the
group whose virtual cohomological dimension is studied by Guirardel and Levitt [GL07, Theorem 5.2] as an application of their construction of the outer space of $\Gamma$ relative to $\mathcal{A}$. In this paper the group $\text{Out}(\Gamma; \mathcal{A})$ is mostly lurking behind the scenes: we do not do much here with $\text{Out}(\Gamma; \mathcal{A})$, but in the expectation of future applications we record basic facts about how $\text{Out}(\Gamma; \mathcal{A})$ acts on relative complexes of free factor systems and of free splittings (see Sections 3.6 and 6).

The complex of free factor systems of $\Gamma$ relative to $\mathcal{A}$, denoted $\mathcal{FF}(\Gamma; \mathcal{A})$, is defined to be the geometric realization of the partial ordering $\sqsubseteq$ restricted to the set of proper free factor systems $\mathcal{B}$ of $\Gamma$ such that $\mathcal{A} \sqsubseteq \mathcal{B}$ and $\mathcal{A} \neq \mathcal{B}$. The special case $\mathcal{FF}(\Gamma; \mathcal{A})$ when $\mathcal{A}$ is a Grushko free factor system might be thought of as the absolute complex of free factor systems of $\Gamma$. Just as for $\Gamma = F_n$ (see above), there are exceptional free factor systems closest to the maximum $\{[\Gamma]\}$ for which $\mathcal{FF}(\Gamma; \mathcal{A})$ exhibits exceptional behavior (see Section 2.5 and Proposition 6.2).

A free splitting of $\Gamma$ is a minimal action of $\Gamma$ on a nontrivial simplicial tree $T$ with trivial edge stabilizers and with finitely many edge orbits. The set of conjugacy classes of nontrivial vertex stabilizers forms a free factor system of $\Gamma$ denoted $\mathcal{F}(T)$. Two free splittings which differ by an equivariant homeomorphism are equivalent. Collapsing equivariant subgraphs of free splittings defines a partial ordering on equivalence classes which is denoted $S \succ T$.

The free splitting complex relative to $\mathcal{A}$, denoted $\mathcal{FS}(\Gamma; \mathcal{A})$, is the simplicial realization of the equivalence classes of free splittings $T$ such that $\mathcal{A} \sqsubseteq \mathcal{F}(T)$; here we allow equality $\mathcal{A} = \mathcal{F}(T)$. When $\mathcal{A}$ is a Grushko free factor system then one may think of $\mathcal{FS}(\Gamma; \mathcal{A})$ as the absolute free splitting complex of $\Gamma$. Just as happens for $\mathcal{FS}(F_n)$ (c.f. [Hat95]), in general the complex $\mathcal{FS}(\Gamma; \mathcal{A})$ may be regarded as a kind of “simplicial completion” of the Guirardel-Levitt outer space of $\Gamma$ relative to $\mathcal{A}$; more precisely, that relative outer space is naturally the complement of the subcomplex of $\mathcal{FS}(\Gamma; \mathcal{A})$ consisting of all $T$ for which the inclusion $\mathcal{A} \sqsubseteq \mathcal{F}(T)$ is proper.

**Theorem 1.4.** For any group $\Gamma$ and any proper free factor system $\mathcal{A}$ of $\Gamma$, the complex $\mathcal{FS}(\Gamma; \mathcal{A})$ is nonempty, connected, and hyperbolic.

**Theorem 1.5.** For any group $\Gamma$ and any nonexceptional free factor system $\mathcal{A}$ of $\Gamma$, the complex $\mathcal{FF}(\Gamma; \mathcal{A})$ is nonempty, connected, and hyperbolic.

These theorems can both be enhanced by descriptions of geodesics; see Theorem 5.4 for $\mathcal{FS}(\Gamma; \mathcal{A})$ and Theorem 6.7 for $\mathcal{FF}(\Gamma; \mathcal{A})$.

Just as was done in [MM99] for the action of a surface mapping class group $\text{McG}(S)$ on the curve complex of $S$, one may view these theorems as describing “weak relative hyperbolicity” of $\text{Out}(\Gamma; \mathcal{A})$ with respect to the conjugacy classes of subgroups of $\text{Out}(\Gamma; \mathcal{A})$ that stabilize simplices of $\mathcal{FS}(\Gamma; \mathcal{A})$ and of $\mathcal{FF}(\Gamma; \mathcal{A})$. 
Outline of the paper. Sections 2 and 3 develop basic concepts of free factor systems and free splittings in the context of a general group $\Gamma$. Much of the content of those two sections is well known for $\Gamma = F_n$, although even in that case certain new numerical measurements are needed in later sections. The reader interested solely in $\Gamma = F_n$ may wish to read the opening paragraphs of Sections 2 and 3 for information on how to get through those sections more quickly, before proceeding to the proofs of Theorems 1.4 and 1.5 which occupy the remainder of the paper.

Sections 4 and 5 contain the proof of Theorem 1.4. The basic method of the proof is quite similar to the proof of the absolute case of Theorem 1.1 given in [HM13] (but see below for remarks on the methods with an emphasis on the differences with [HM13]). Section 4 sets up the machinery of fold paths in $\mathcal{FS}(\Gamma; A)$, combing properties of fold paths, and combinatorial measurements along fold paths known as “free splitting units”. Section 5 uses these tools to prove Theorem 1.4 in combination with axioms for hyperbolicity developed by Masur and Minsky in [MM99], together with a “Big Diagram” argument as in [HM13]. Also as in [HM13], one side benefit of the proof is that the collection of fold paths in $\mathcal{FS}(\Gamma; A)$ is shown to be uniformly quasigeodesically parameterized by free splitting units, and any two vertices in $\mathcal{FS}(\Gamma; A)$ are (coarsely) connected by such a fold path; see Theorem 5.4.

Section 6 contains the proof of Theorem 1.5. We use a method developed by Kapovich and Rafi in [KR14] to derive hyperbolicity of $\mathcal{FF}(\Gamma; A)$ from hyperbolicity of $\mathcal{FS}(\Gamma; A)$, generalizing their derivation of hyperbolicity of $\mathcal{FF}(F_n)$ from hyperbolicity of $\mathcal{FS}(F_n)$.

Remarks on methods of proof. In modifying the arguments of [HM13] to work in this paper, there are three major areas of change. Two of these areas are accounted for by the content of Sections 2 and 3: generalizing from $F_n$ to $\Gamma$; and relativizing the absolute concepts of [HM13].

The third area of change is motivated by work of Bestvina and Feighn who in the appendix of their paper [BF12] introduced some changes in the methods of [HM13]. The fold paths used in [HM13] came with a restriction on the behavior of vertices of valence $\geq 3$ along a fold path of free splittings, what we shall call the “gate 3 condition”. From their need to get a wider class of fold paths to serve as quasigeodesics, in the paper [BF12] they dropped the gate 3 condition and introduced some changes in the methods of proof to make it work without that condition. We adopt these changes in this paper, and we will emphasize these changes in the narrative of Sections 4 and 5.

Otherwise, many absolute concepts originally encountered when studying $\mathcal{FS}(F_n)$ in [HM13] can be directly generalized and relativized by replacing the symbol $F_n$ with $\Gamma$ and by specializing the definitions to free splittings rel $A$; when such direct generalization is possible we shall do so with little comment. Also, large pieces of proof from [HM13] work without change and will not be repeated in our current more general context, but will merely be cited with at most a sketch provided in the more important cases.
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7
2 Free factor systems

Throughout this paper our convention is that Γ represents an arbitrary freely decomposable group, meaning that Γ can be expressed as a nontrivial free product of nontrivial groups (if Γ were freely indecomposable then the main objects of study of this paper—relative free factor and free splitting complexes—would be empty). In particular this convention rules out the possibility that Γ is infinite cyclic; see remarks after the definition of free factor systems in Section 2.1 and after the definition of free splittings in Section 3.2.

For readers whose primary interest is the case Γ = F_n, the contents of this section are either well known or rather evident. Any such reader, as well as anyone who wishes to proceed directly to the definitions of relative complexes of free factor systems in Section 6, may profitably do so after skimming this section.

This section contains basic material regarding free factor systems of Γ, their partial order \( \sqsubseteq \) known as “containment” or “extension”, their binary operation \( \wedge \) known as “meet”, and properties thereof. This material will be used in Sections 3, 4 and 5 regarding relative free splitting complexes of Γ, and in Section 6 regarding relative complexes of free factor systems of Γ.

The contents of this section consist for the most part of applications of the Kurosh Subgroup Theorem and, in the finitely generated case, of Grushko’s Theorem. One such application is the Extension Lemma 2.11, regarding the structure of a nested pair of free factor systems \( \mathcal{A} \sqsubseteq \mathcal{B} \). The Extension Lemma and its consequences will be used throughout the rest of the paper, even for the case of finite rank free groups.

An important application of the Extension Lemma is Lemma 2.14 which describes a formula for the depth of a free factor system with respect to the partial ordering \( \sqsubseteq \), together with various properties of depth (the depth of an element of a partially ordered set is the length of the longest ascending chain starting with the given element). Depth of free factor systems will be applied in several ways later in the paper, including in a dimension formula for relative free splitting complexes (see Proposition 6.1). Of more central importance, in Section 4.4 bounds on depth are used to derive topological and metric properties of free splittings and their fold paths, and in Section 4.5 these bounds are translated into properties of free splitting units along fold paths.

2.1 Free factorizations and free factor systems

Free factorizations. In any group Γ a free factorization is a set of nontrivial subgroups \( \mathcal{H} = \{ H_l \}_{l \in \mathcal{L}} \) satisfying the universality property that for any group K, any set of homomorphisms \( \{ f_l : H_l \to K \mid l \in \mathcal{L} \} \) is the restriction of a unique homomorphism \( \Gamma \to K \). Equivalently, every nonidentity element \( \gamma \in \Gamma \) is represented by a unique reduced word \( \gamma = \gamma_1 \cdots \gamma_I \) (\( I \geq 1 \)), meaning that there is a sequence \( l_1, \ldots, l_I \in \mathcal{L} \)
such that $\gamma_i \in H_{l_i} - \{\text{Id}\}$ for $1 \leq i \leq I$, and $l_i \neq l_{i+1}$ for $1 \leq i \leq I - 1$. Note that $H_l = H_m \iff l = m$. When a free factorization is finite—e.g. when $\Gamma$ is finitely generated (Grushko’s theorem)—we will generally pick a bijection $L \leftrightarrow \{1, \ldots, L\}$ and write $\Gamma = H_1 \ast \cdots \ast H_L$. Meanwhile, as we ponder infinite free factorizations in these early sections of the paper, we shall write $\Gamma = \ast_{l \in L} H_l$ or just $\Gamma = \ast H$. A free factor $H < \Gamma$ is any element of a free factorization, in which case by conglomerating the other free factors one obtains a free factorization of the form $\Gamma = H \ast H'$.

Note that each conjugacy class in $\Gamma$ is represented by a cyclically reduced word (meaning a reduced word that also satisfies $l_I \neq l_1$) and this representative is unique up to cyclic permutation. From this we immediately obtain the following, which incorporates the well known result that every free factor is malnormal:

**Lemma 2.1.** Every free factorization $\Gamma = \ast H$ is mutually malnormal, meaning that for each $H, H' \in H$ and $\gamma \in \Gamma$, if $\gamma H \gamma^{-1} \cap H'$ is nontrivial then $\gamma \in H = H'$.

From malnormality of a free factor $H < \Gamma$ it follows that two subgroups of $H$ are conjugate in $\Gamma$ if and only if they are conjugate in $H$. We shall make tacit use of this equivalence in what follows.

A partial free factorization of $\Gamma$ is a subset of a free factorization. Every partial free factorization $H$ has a cofactor which is a subgroup $H < \Gamma$ such that $\Gamma = \ast_{l \in L} H_l$ is a free factorization. We make no assumptions on the cofactor, but we do have the following result. Let $N(H)$ be the subgroup normally generated by the union of all the subgroups of $H$.

**Lemma 2.2.** For any partial free factorization $H$ of $\Gamma$ and any realization $\Gamma = \ast_{l \in L} H_l$ with cofactors $H, H'$, there is a retraction $\Gamma \to H$ with kernel $N(H)$. It follows that $H$ is isomorphic to the quotient $\Gamma/N(H)$.

**Proof.** The retraction on a reduced word $w = \gamma_1 \cdots \gamma_l$ erases each letter $\gamma_i$ in each element of $H$, and multiplies out the surviving letters of $H$ in order. Evidently $N(H)$ is in the kernel $K$. Conversely $w$ can be rewritten by moving the letters of $w$ lying in $H$ to the front of the word, preserving their order, at the expense of replacing the other letters by conjugates; so if $w \in K$ then after rewriting one sees that $w \in N(H)$.  

**Lemma 2.3.** Consider a group $\Gamma$ and two partial free factorizations $H = \{H_l\}_{l \in L}$ and $H' = \{H'_l\}_{l \in L}$ of $\Gamma$ with the same index set $L$. Consider also realizations $\Gamma = \ast H = \ast H'$ with cofactors $H, H'$. If $H_l$ is conjugate to $H'_l$ for all $l \in L$ then the cofactors $H, H'$ are isomorphic. Furthermore there is an isomorphism $\Gamma \to \Gamma$ which restricts to a conjugation from $H_l$ to $H'_l$ and restricts to an isomorphism from $H$ to $H'$.

**Proof.** Noting that $N(H) = N(H')$, apply Lemma 2.2 to conclude that each of $H, H'$ is isomorphic to $\Gamma/N(H)$. After choosing conjugations $H_l \to H'_l$ and an isomorphism $H \to H'$, the lemma follows by applying the universality property.
Remark. For an example which determines the extent to which cofactors can fail to be well-defined up to conjugacy, see the discussion of $\Gamma = A \ast Z$ following Proposition 6.2.

**Definition 2.4** (Free factor systems.) A *weak free factor system* of $\Gamma$ is a set of the form $A = \{[A_l]\}_{l \in L}$ such that $\{A_l\}_{l \in L}$ is a partial free factor system of $\Gamma$ with free cofactor; we make no assumption on the cardinality of the set $A$ nor on the rank of the cofactor, although by Lemma 2.3 the rank of the cofactor is well-defined. A *free factor system* $A$ is a weak free factor system which is finite and has a finite rank cofactor. Note that the definition allows $A = \emptyset$ as a possible free factor system, but only if $\Gamma$ is free of finite rank. We usually write $A = \{[A_1], \ldots, [A_L]\}$ so that $A$ is realized as $\Gamma = A_1 \ast \cdots \ast A_L \ast A'$; as usual, the cofactor $A'$ may be trivial. A free factor system $A$ is proper if $A \neq \{[\Gamma]\}$. The individual elements $[A_1], \ldots, [A_L]$ of $A$ are called its components.

Remark. Recalling our blanket assumption that $\Gamma$ is not infinite cyclic, nevertheless an infinite cyclic group does have a unique proper free factor system, namely $\emptyset$.

Remark. In the case $\Gamma = F_n$ every weak free factor system is a free factor system, and the same is true for finitely generated groups, by Grushko’s Theorem. The reader interested solely in $F_n$ or other finitely generated $\Gamma$ may therefore safely ignore the adjective “weak”, which should cut down on the technical overload of this section. Also, the Extension Lemma 2.11 will provide a relative setting in which we can also ignore “weak”, which we shall do forever afterwards, once the Extension Lemma is proved.

### 2.2 The Kurosh Subgroup Theorem

**Extension** $\sqcap$ and **meet** $\sqcap$.

The results obtained in this section by applying the Kurosh Subgroup Theorem are standard in the case $\Gamma = F_n$; see [BFH00].

The following foundational theorem can be proved using Bass-Serre theory; see for example [Coh89]. The usual expression of this theorem is in the language of double cosets. We provide a translation into the language of conjugacy of subgroups, as well as a slightly more detailed conclusion, particularly in the case of a free factor $A < \Gamma$.

**Kurosh Subgroup Theorem.** For any group $\Gamma$, any free factorization $\Gamma = \ast (H_l)_{l \in L}$, and any subgroup $A < \Gamma$, there exists for each $l \in L$ a subset $U_l \subset \Gamma$ consisting of representatives $u$ of distinct double cosets $AuH_l$, and there exists a free subgroup $C < A$, such that the following hold:

1. $A = \ast \{A \cap uH_l u^{-1} \mid l \in L, u \in U_l\} \ast C$

2. For each $(l, v) \in L \times \Gamma$:
   a. The subgroup $C \cap vH_lv^{-1}$ is trivial.
(b) The subgroup \( A \cap vH_1v^{-1} \) is nontrivial \( \iff \) there exists \( u \in U_1 \) such that the subgroups \( A \cap vH_1v^{-1} \) and \( A \cap uH_1u^{-1} \) are conjugate in \( A \iff \) there exists \( u \in U_1 \) such that \( AvH_1 = AuH_1 \).

If furthermore \( A \) is itself a free factor then:

(3) For each \((l, u), (m, v) \in \mathcal{L} \times \Delta \) such that \( u \in U_l \) and \( v \in U_m \), if the subgroups \( A \cap uH_1u^{-1} \) and \( A \cap vH_mv^{-1} \) are conjugate in \( \Delta \) then \( l = m \) and \( u = v \) (and so in particular those subgroups are equal).

**Remarks.** The statement of the Kurosh Subgroup Theorem found for example in [Coh89] incorporates only item (1), but the others are easily proved. Item (2a) is easily derived from the Bass-Serre theory proof found in [Coh89], as is the first equivalence of item (2b). The second equivalence of (2b) is a calculation: if \( AvH_1 = AuH_1 \) then \( u = avh \) for some \( a \in A \), \( h \in H_1 \) and so \( a(\cap vH_1v^{-1})a^{-1} = A \cap uH_1u^{-1} \); conversely if \( a(\cap vH_1v^{-1})a^{-1} = A \cap uH_1u^{-1} \) for \( a \in A \) then \( A \cap (av)H_1(av)^{-1} = A \cap uH_1u^{-1} \) and so, by malnormality of \( H_1 \), we have \( u^{-1}av \in H_1 \) implying that \( AvH_1 = AuH_1 \). For proving item (3), the conjugating element must be in \( A \) by malnormality of \( A \), and \( l = m \) by mutual malnormality of \( \ast \{H_1\}_{i \in \mathcal{L}} \); the rest follows from (2b).

One standard consequence of the Kurosh Subgroup Theorem is that for any partial free factorization \( \{A_i\} \) of \( \Delta \) and any subgroup \( B < \Delta \), if each \( A_i \) is a subgroup of \( B \) then \( \{A_i\} \) is a partial free factorization of \( B \). The following slight generalization, also an immediate consequence of the Kurosh Subgroup Theorem, is needed for the proof of the Extension Lemma 2.11.

**Lemma 2.5.** For any group \( \Delta \), any subgroup \( B < \Delta \), any partial free factorization \( \{A_i\}_{i \in I} \) of \( \Delta \), and any identically indexed set of subgroups \( \{A'_i\}_{i \in I} \), if \( A'_i \) is conjugate to \( A_i \) and if \( A'_i < B \) for all \( i \in I \), then \( \{A'_i\}_{i \in I} \) is a partial free factorization of \( B \). \( \diamond \)

**Extension \( \sqsubset \) of free factor systems.** Given two subgroups \( A, A' \subset \Delta \) with conjugacy classes \([A], [A']\), let \( [A] \sqsubset [A'] \) denote the well-defined relation that \( A \) is conjugate to a subgroup of \( A' \). Define a partial ordering \( \mathcal{A} \sqsubset \mathcal{A}' \) on weak free factor systems by requiring that for each \( [A] \in \mathcal{A} \) there exists \( [A'] \in \mathcal{A}' \) such that \([A] \sqsubset [A'] \). We express the relation \((this) \sqsubset (that)\) in various ways: \((this) \) is contained in \((that)\); or \((that) \) is an extension of \((this)\); or \((this) \sqsubset (that) \) is an extension; etc. An extension \( \mathcal{A} \sqsubset \mathcal{A}' \) such that \( \mathcal{A} \neq \mathcal{A}' \) is called a proper extension.

If \( \mathcal{A}, \mathcal{B} \) are free factor systems then we also express the relation \( \mathcal{A} \sqsubset \mathcal{B} \) by saying that \( \mathcal{B} \) is a free factor system relative to \( \mathcal{A} \).
Meet of free factor systems. The meet $\wedge$ is a binary operation on weak free factor systems defined by

$$A \wedge B = \{ [A \cap uBu^{-1}] \mid [A] \in A, [B] \in B, u \in \Gamma, A \cap uBu^{-1} \neq \{1\}\}$$

We shall prove the following using the Kurosh Subgroup Theorem:

**Lemma 2.6** (Weak Meet Lemma). In any group $\Gamma$, the meet of any two weak free factor systems is a weak free factor system.

We will need to strengthen the conclusion of this lemma by removing the word “weak” in various situations. One such situation, for finitely generated groups, is described in Corollary 2.8. Another “relativized” version is given in Proposition 2.12.

Before giving the proof of Lemma 2.6, here are two immediate corollaries.

**Corollary 2.7.** For any weak free factor systems $A, B$ in any group $\Gamma$, their meet $A \wedge B$ can be characterized as the unique weak free factor system having the following properties:

1. $A \wedge B \sqsubseteq A$;
2. $A \wedge B \sqsubseteq B$;
3. For every weak free factor system $C$, if $C \subseteq A$ and $C \subseteq B$ then $C \subseteq A \wedge B$.  

The next result, well known in the case of free groups from [BFH00], follows immediately by combining Lemma 2.6, Grushko’s Theorem, and Corollary 2.7.

**Corollary 2.8.** In any finitely generated group $\Gamma$, for any two free factor systems $A, B$ of $\Gamma$, their meet $A \wedge B$ is a free factor system. Furthermore if $A, B$ are free factor systems relative to a third free factor system $C$ then $A \wedge B$ is also a free factor system relative to $C$.

The second sentence of Corollary 2.8 is true in a general group; see Proposition 2.12.

**Proof of the Weak Meet Lemma 2.6.** Consider $A = \{[A_i]_{i \in I}\}$ and $B = \{[B_j]_{j \in J}\}$ with respective realizations

$$\Gamma = *(A_i)_{i \in I} * A' \quad \text{and} \quad \Gamma = *(B_j)_{j \in J} * B'$$

Applying the Kurosh Subgroup theorem to $A_i$ using the given realization of $B$, we obtain a free factorization

$$A_i = *(A_{ik})_{k \in K_i} * A'_i$$
where $A'_i$ is a free group and the subgroups $A_{ik}$ are representatives of the $\Gamma$-conjugacy classes of all nontrivial intersections of $A_i$ with conjugates of the $B_j$’s. It follows that

$$A \wedge B = \{[A_{ik}] \mid i \in I, k \in K_i\}$$

Substituting (##) into (#) we obtain a free factorization

$$\Gamma = \ast\left(\ast\{A_{ik}\}_{k \in K_i} \ast A'_i\right) \ast A' = \left(\ast\{A_{ik}\}_{i \in I, k \in K_i}\right) \ast \left(\ast\left(\ast\{A'_i\}_{i \in I}\right)\right)$$

which, the factor in brackets [·] clearly being free, shows that $A \wedge B$ is a weak free factor system. ♦

2.3 Corank and the structure of extensions of free factor systems

In this section we prove the Extension Lemma 2.11 detailing the structure of any extension $A \subset B$ of free factor systems. This will be applied in studying the depth of $\subset$ in Section 2.5, and when studying free splitting units in Sections 4.4 and 4.5.

Also, the Extension Lemma 2.11 guarantees that any weak free factor system that is an extension of a free factor system is itself a free factor system, and Proposition 2.12 guarantees that for any free factor system $A$ the meet of two free factor systems rel $A$ is also a free factor system rel $A$, which is how we generalize Corollary 2.8 to non finitely generated groups. These results allow us henceforth to ignore the adjective “weak”.

**Corank.** Define the *corank* of a free factor system $A$ of a group $\Gamma$ to be the integer

$$\text{corank}(A) = \text{rank}(\Gamma/N(A)) = \text{rank}(A') \geq 0$$

where $A'$ is the cofactor of any realization of $A$. When we wish to emphasize the ambient group we also write corank$(A; \Gamma)$. From Bass-Serre theory it follows that corank$(A)$ is equal to the topological rank of the underlying graph for any finite graph of groups representation of $\Gamma$ with trivial edge groups and with nontrivial vertex groups $A_1, \ldots, A_K$ so that $A = \{[A_1], \ldots, [A_K]\}$.

When $\Gamma = F_n$ and $A = \{[A_1], \ldots, [A_K]\}$, the free factors $A_1, \ldots, A_K$ are all free of finite rank, and we have the following *rank sum formula* for the corank of $A$:

$$\text{corank}(A) = n - \sum_{k=1}^{K} \text{rank}(A_k)$$
This formula may be useful to the reader for deriving quick proofs of results to follow in the special case $\Gamma = F_n$.

The following lemma defines what we shall call the \textit{containment function} from one free factor system to any of its extensions.

\textbf{Lemma 2.9.} Given an extension $\mathcal{A} \sqsubseteq \mathcal{B}$ of weak free factor systems of a group $\Gamma$, the relation $\sqsubseteq$ between components of $\mathcal{A}$ and components of $\mathcal{B}$ defines a function $\mathcal{A} \mapsto \mathcal{B}$, called the containment function.

\textit{Proof.} By definition, for any component $[A] \in \mathcal{A}$ there exists a component $[B] \in \mathcal{B}$ such that $[A] \sqsubseteq [B]$. By mutual malnormality of any realization of $\mathcal{B}$, this $[B]$ depends uniquely on $[A]$. $\diamond$

The following result, in the special case $\Gamma = F_n$, is an evident consequence of the rank sum formula for corank.

\textbf{Proposition 2.10.} For any nested pair of free factor systems $\mathcal{A} \sqsubseteq \mathcal{A}'$ of $\Gamma$ we have $\text{corank}(\mathcal{A}) \geq \text{corank}(\mathcal{A}')$. Equality holds if and only if the containment function $\mathcal{A} \mapsto \mathcal{A}'$ is surjective and for each $[A'] \in \mathcal{A}'$ there exists a free factorization with trivial cofactor $A'_j = A_{j1} \cdots A_{jk_j}$ so that the preimage of $[A'_j]$ under the containment function is $\{[A_{j1}], \ldots, [A_{jk_j}]\} \subseteq \mathcal{A}$.

The proof of Proposition 2.10 in the general case—where rank sum does not make sense—will be given after the statement and proof of the following Extension Lemma.

For understanding the conclusions of the Extension Lemma we refer the reader to Figure 1 which depicts those conclusions in tabular format. The proof of the Extension Lemma is similar to the proof of the Weak Meet Lemma 2.6 but with more care taken regarding cardinalities.

\textbf{Lemma 2.11} (Extension Lemma). In any group $\Gamma$, if $\mathcal{A}$ is a free factor system, if $\mathcal{A}'$ is a weak free factor system, and if $\mathcal{A} \sqsubseteq \mathcal{A}'$, then $\mathcal{A}'$ is a free factor system. Moreover, consider any realization $\Gamma = A'_1 \cdots A'_K \ast B'$ of $\mathcal{A}' = \{[A'_1], \ldots, [A'_K]\}$, with indexing chosen so that the image of the containment function $\mathcal{A} \mapsto \mathcal{A}'$ equals $\{[A'_1], \ldots, [A'_J]\}$, where $0 \leq J \leq K$. For $1 \leq j \leq J$ let $A_j \subseteq \mathcal{A}$ be the pre-image of $[A'_j]$ under the containment function, and let $k_j = |A_j|$. Then there exists a realization of $\mathcal{A}$ of the form

$$\Gamma = A_{11} \cdots A_{1k_1} \cdots \cdots A_{j1} \cdots A_{jk_j} \ast (B_1 \cdots B_j \ast A'_{j+1} \cdots A'_K \ast B')$$

such that

$$A_j = \{[A_{j1}], \ldots, [A_{jk_j}]\} \quad \text{and} \quad A'_j = A_{j1} \ast \cdots \ast A_{jk_j} \ast B_j \quad (1 \leq j \leq J)$$
The subgroups $B_1, \ldots, B_J, A'_{J+1}, \ldots, A'_K, B'$ are all free of finite rank. By abuse of notation (identifying conjugacy classes in $\Gamma$ with conjugacy classes in $A'_j$) we may regard $A_j$ as a free factor system of the group $A'_j$ realized with cofactor $B_j$.

**Proof.** Since $A$ is finite and $A \subseteq A'$, any realization of the weak free factor system $A'$ can be listed as

$$(*): \quad \Gamma = A'_1 \ast \cdots \ast A'_J \ast (\ast\{A'_k\}_{k \in K}) \ast B', \quad J \geq 0$$

so that $B'$ is the cofactor, and so that the subset $\{[A'_1], \ldots, [A'_J]\} \subseteq A'$ is the image of the containment map $A \to A'$ (we assume all free factors of $(*)$ are nontrivial, except perhaps $B'$). For $1 \leq j \leq J$, let $A_j \subseteq A$ be the preimage of $[A'_j]$ under the containment map $A \to A'$, and let $k_j = |A_j| \geq 1$. We may choose pairwise nonconjugate subgroups $A_{j1}, \ldots, A_{jk_j} < A'_j$ so that $A_j = \{[A_{j1}], \ldots, [A_{jk_j}]\}$. By Lemma 2.5 we have a free factorization

$$(**)_j: \quad A'_j = A_{j1} \ast \cdots \ast A_{jk_j} \ast B_j$$

Substituting each $(**)_j$ into $(*)$ and rearranging terms we obtain the following free factorization of $\Gamma$, which is clearly a realization of $A$:

$$\Gamma = A_{11} \ast \cdots \ast A_{1k_1} \ast \cdots \ast A_{J1} \ast \cdots \ast A_{Jk_J} \ast (B_1 \ast \cdots \ast B_J \ast (\ast\{A'_k\}_{k \in K}) \ast B'))$$

Since $B$ is a finite rank free group, it follows that $K$ is finite, and that the subgroups $A'_k$ for $k \in K$ and $B_1, \ldots, B_J, B'$ are all finite rank and free. It follows that $A'$ is a free factor system of $\Gamma$ with cofactor $B'$, and that $A_j$ may be regarded as a free factor system of $A'_j$ with cofactor $B_j$. 

**Proof of Proposition 2.10.** This is a quick application of Lemma 2.11. Following the notation of that lemma we have $\text{corank}(A) = \text{rank}(B) \geq \text{rank}(B') = \text{corank}(A')$, with equality if and only if and only if none of $B_1, \ldots, B_J, A'_{J+1}, \ldots, A'_K$ exist: nonexistence of $A'_{J+1}, \ldots, A'_K$ is equivalent to $J = K$ which is equivalent to surjectivity of $A \to A'$; and nonexistence of the cofactor $B_j$ is equivalent to existence of the desired free factorization $A'_j = A_{j1} \ast \cdots \ast A_{jk_j}$ without cofactor.

Here is the promised relativization of Corollary 2.8.

**Proposition 2.12.** For any group $\Gamma$, any free factor system $A$, and any two free factor systems $B, C$ of $\Gamma$ relative to $A$, their meet $B \wedge C$ is a free factor system relative to $A$.

**Proof.** Applying Corollary 2.7, $B \wedge C$ is a weak free factor system, and by item (iii) of that corollary we have $A \subseteq B \wedge C$. By Lemma 2.11 it follows that $B \wedge C$ is a free factor system.
Figure 1: The Extension Lemma 2.11 shows that for each extension $A \sqsubset A'$ of free factor systems, and for any realization of $A'$, there exists a free factorization with terms as depicted which simultaneously incorporates the following: the given realization of $A' = \{[A'_1], \ldots, [A'_K]\}$ with cofactor $B'$; for each $j \leq J$ a realization of a free factor system of $A'_j$, namely $A'_j = \{[A_{j1}], \ldots, [A_{jk_j}]\}$, with cofactor $B_j$; and a realization of $A = \{[A_1], \ldots, [A_{k1}], \ldots, [A_{1J}], \ldots, [A_{kJ}]\}$ with cofactor $B = B_1 \ast \cdots \ast B_J \ast A'_{J+1} \ast \cdots A'_K \ast B'$.

| $j$ | free factorization of $A'_j$ | cofactor of $A'$ |
|-----|-----------------------------|------------------|
| 1   | $A_{11} \cdots A_{1k_1}$   | $B_1$            |
| $\vdots$ | $\vdots$                  | $\vdots$        |
| $J$ | $A_{1J} \cdots A_{1k_J}$   | $B_J$            |
| $J+1$ | $A'_{J+1}$             |                  |
| $\vdots$ | $\vdots$                |                  |
| $K$ | $A'_K$                    | $B'$             |

2.4 Grushko free factor systems

Recall Grushko’s theorem, which ways that every finitely generated group has a Grushko decomposition. Grushko decompositions can also exist naturally outside of the realm of finitely generated groups: any free product of a finite rank free group and finitely many freely decomposable groups yields a Grushko decomposition. The following proposition describes the behavior of general Grushko decompositions, expressed in terms of the $\sqsubset$ relation.

**Proposition 2.13.** For any group $\Gamma$ and any free factor system $A$ of $\Gamma$, the following are equivalent:

1. Some realization $\Gamma = A_1 \ast \cdots \ast A_K \ast A'$ of $A$ is a Grushko decomposition.

2. Any realization $\Gamma = A_1 \ast \cdots \ast A_K \ast A'$ of $A$ is a Grushko decomposition.

3. $A$ is a minimum weak free factor system with respect to $\sqsubset$.

4. For any weak free factor system $B$ of $\Gamma$ we have $A \sqsubset B$. In particular $A$ is the unique minimum weak free factor system with respect to $\sqsubset$.

If these properties hold then we say that $A$ is the **Grushko free factor system** of $\Gamma$. 

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Proof. Clearly (4) \implies (3) \implies (2) \implies (1). Assuming (1), in order to prove (4) it suffices by Corollary 2.7 to prove that \( \mathcal{A} = \mathcal{A} \cap \mathcal{B} \). Let \( \mathcal{C} = \mathcal{A} \cap \mathcal{B} \subseteq \mathcal{A} \). For each \( [C] \in \mathcal{C} \), consider the unique component \( [A_k] \in \mathcal{A} \) such that \( [C] \subseteq [A_k] \). Applying the Kurosh Subgroup Theorem, after conjugation it follows that \( C \) is a nontrivial free factor of \( A_k \), but \( A_k \) is freely indecomposable, and so \( C = A_k \). This proves that \( C \) is a subset of \( \mathcal{A} \). If \( C \neq \mathcal{A} \) then there exists \( [A_k] \in \mathcal{A} \) such that \( [A_k] \not\in C \), and by the Extension Lemma 2.11 it follows that \( [A_k] \) is a free factor of some cofactor of \( C \). But cofactors are free and \( A_k \) is not free, a contradiction. \( \diamond \)

2.5 Free factor system depth of a free factor system.

In general the depth of an element \( x \) of a partially ordered set is the cardinality \( \ell \) of the longest ascending chain \( x = x_0 \sqsubseteq \cdots \sqsubseteq x_\ell \) of order relations starting with the given element. Given a group \( \Gamma \) we compute depth for the set of free factor systems of \( \Gamma \) with respect to the partial ordering \( \sqsubseteq \), and we derive some properties of this depth. These could be immediately applied to define and compute depths of complexes of free factor systems relative to a free factor system, but we shall delay that until Section 6.

Given a free factor system \( \mathcal{A} = \{[A_1], \ldots, [A_K]\} \) of \( \Gamma \) define the free factor system depth of \( \mathcal{A} \) to be

\[
D_{\text{FF}}(\mathcal{A}) = 2 \text{corank}(\mathcal{A}) + |\mathcal{A}| - 1 = 2 \text{rank}(B) + K - 1
\]

where \(|\cdot|\) denotes the cardinality, and \( B \) is any cofactor of any realization of \( \mathcal{A} \).

Assuming \( \Gamma = F_n \), for any free factor system \( \mathcal{A} = \{[A_1], \ldots, [A_K]\} \) we have

\[
D_{\text{FF}}(\mathcal{A}) = 2(\sum_{k=1}^{K} \text{rank}(A_k)) + K - 1 = (2n - 1) - \sum_{k=1}^{K} (2 \text{rank}(A_k) - 1)
\]

Part of the content of Lemma 2.14 to follow is that \( D_{\text{FF}}(\mathcal{A}) \) is indeed the depth of \( \mathcal{A} \) with respect to the partial ordering \( \sqsubseteq \). This is easily checked when \( \Gamma = F_n \).

Here are some examples. The exceptional free factor systems \( \mathcal{A} \), defined to be those for which \( D_{\text{FF}}(\mathcal{A}) \leq 2 \), can be enumerated as follows:

- \( D_{\text{FF}}(\mathcal{A}) = 0 \) if and only if \( \mathcal{A} \) is the improper free factor system \( \mathcal{A} = \{[\Gamma]\} \).
- \( D_{\text{FF}}(\mathcal{A}) = 1 \) if and only if \( \text{corank}(\mathcal{A}) = 0 \) and \( |\mathcal{A}| = 2 \), in which case \( \mathcal{A} = \{[A_1], [A_2]\} \) with \( \Gamma = A_1 \ast A_2 \). The possibility that \( \text{corank}(\mathcal{A}) = 1 \) and \( |\mathcal{A}| = 0 \) is equivalent to \( \Gamma \) being infinite cyclic, which was ruled out.
- \( D_{\text{FF}}(\mathcal{A}) = 2 \) if and only if one of the following happens: either \( |\mathcal{A}| = 1 \) and \( \text{corank}(\mathcal{A}) = 1 \), in which case \( \mathcal{A} = \{[A]\} \) with realization \( \Gamma = A \ast Z \) where the cofactor \( Z \) is infinite cyclic; or \( |\mathcal{A}| = 3 \) and \( \text{corank}(\mathcal{A}) = 0 \) in which case \( \mathcal{A} = \{[A_1], [A_2], [A_3]\} \) with realization \( \Gamma = A_1 \ast A_2 \ast A_3 \).
As we shall see in Proposition 6.2, the exceptional free factor systems $\mathcal{A}$ are those for which the complex of free factor systems relative to $\mathcal{A}$ is exceptionally simple, either empty or 0-dimensional.

We say that a proper extension $\mathcal{A} \subseteq \mathcal{A}'$ is elementary if one of the following holds:

1. $\mathcal{A}' = \mathcal{A} \cup \{[Z]\}$ where $Z < \Gamma$ is infinite cyclic; or

2. there is a realization $\Gamma = A_1 \ast \cdots \ast A_K \ast B$ of $\mathcal{A}$ and two components $[A_i], [A_j]$ ($i \neq j \in \{1, \ldots, K\}$) such that

$$\mathcal{A}' = (\mathcal{A} - \{[A_i], [A_j]\}) \cup \{[A_i \ast A_j]\}$$

Another part of Lemma 2.14 is that the statement “$\mathcal{A} \subseteq \mathcal{A}'$ is elementary” is equivalent to $D_{FF}(\mathcal{A}) = D_{FF}(\mathcal{A}') + 1$ which is equivalent to saying that no other free factor system is properly contained between $\mathcal{A}$ and $\mathcal{A}'$. Again this is easily checked when $\Gamma = F_n$.

**Lemma 2.14.** The function $D_{FF}$ on free factor systems of $\Gamma$ has the following properties:

1. If $\mathcal{A} \subseteq \mathcal{A}'$ then $D_{FF}(\mathcal{A}) \geq D_{FF}(\mathcal{A}')$ with equality if and only if $\mathcal{A} = \mathcal{A}'$. As a special case, $D_{FF}(\mathcal{A}) \geq 0$ with equality if and only if $\mathcal{A} = \{[\Gamma]\}$.

2. If $\mathcal{A} \subseteq \mathcal{A}'$ is a proper extension then $D_{FF}(\mathcal{A}) \geq D_{FF}(\mathcal{A}') + 1$ with equality if and only if $\mathcal{A} \subseteq \mathcal{A}'$ is an elementary extension.

3. For any proper extension $\mathcal{A} \subseteq \mathcal{A}'$ there exists a free factor system $\mathcal{C}$ such that $\mathcal{A} \subseteq \mathcal{C} \subseteq \mathcal{A}'$ and such that $\mathcal{A} \subseteq \mathcal{C}$ is elementary.

4. For every chain of proper extensions of the form $\mathcal{A} = \mathcal{A}_0 \subseteq \cdots \subseteq \mathcal{A}_K = \{[\Gamma]\}$, its length $K$ satisfies $K \leq D_{FF}(\mathcal{A})$. Equality holds if only if the chain is maximal, if and only if every extension $\mathcal{A}_{k-1} \subseteq \mathcal{A}_k$ is an elementary extension.

**Proof.** Noting that item (4) is a consequence of the earlier items, it remains to prove (1), (2) and (3). Assuming $\mathcal{A} \subseteq \mathcal{A}'$, in items (1) and (2) we are interested in the difference

$$D_{FF}(\mathcal{A}) - D_{FF}(\mathcal{A}') = 2(\text{corank}(\mathcal{A}) - \text{corank}(\mathcal{A}')) + |\mathcal{A}| - |\mathcal{A}'|$$

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Applying Lemma 2.11 and adopting its notation, we have

\[
\text{corank}(A) = \sum_{j=1}^{J} \text{rank}(B_j) + \sum_{j=J+1}^{K} \text{rank}(A'_j) + \text{rank}(B')
\]

\[
\text{corank}(A) - \text{corank}(A') = \sum_{j=1}^{J} \text{rank}(B_j) + \sum_{j=J+1}^{K} \text{rank}(A'_j)
\]

\[
|A| - |A'| = \sum_{j=1}^{J} |A_j| - K
\]

\[
\sum_{j=1}^{J} (|A_j| - 1) - (K - J)
\]

\[
D_{\text{FF}}(A) - D_{\text{FF}}(A') = \sum_{j=1}^{J} 2 \text{rank}(B_j) \underbrace{\sum_{j=1}^{J} (|A_j| - 1)}_{(a)} + \sum_{j=J+1}^{K} \text{rank}(A'_j) - 1 \underbrace{\sum_{j=J+1}^{K} (2 \text{rank}(A'_j) - 1)}_{(c)}
\]

From this it follows that \(D_{\text{FF}}(A) \geq D_{\text{FF}}(A')\) because each of the quantities \((a)_j, (b)_j, (c)_j\) is non-negative: for \(1 \leq j \leq J\) the quantity \((a)_j\) is a non-negative even integer, and the quantity \((b)_j\) is a non-negative integer because \(A_j \neq \emptyset\); and for \(J + 1 \leq j \leq K\) the quantity \((c)_j\) is an odd positive integer because \(A'_j\) is free of rank \(\geq 1\). Furthermore:

- \((a)_j = 0\) if and only if the free factorization \(A'_j = A_{j1} \cdots A_{jk_j} * B_j\) has trivial cofactor \(B_j\) (for \(1 \leq j \leq J\)).

- \((b)_j = 0\) if and only if \(|A_j| = k_j = 1\) if and only if \(A_j\) has exactly one component (for \(1 \leq j \leq J\)).

- \((c)_j > 0\) (for \(J + 1 \leq j \leq K\)).

Thus \(D_{\text{FF}}(A) = D_{\text{FF}}(A')\) if and only if no \((c)_j\)'s exist, i.e. \(J = K\), and \(A_j = \{|A_{j1}\}\) for each \(1 \leq j \leq J\), which happens if and only if \(A = A'\). This completes the proof of (1).

We next prove the “if” direction of item (2). Suppose that \(A \sqsubset A'\) is an elementary extension. In one case we have \(A' = A \cup \{Z\}\) where \(Z\) is infinite cyclic, and it follows that \(K = J + 1\), that \((a)_j = (b)_j = 0\) for \(1 \leq j \leq J\), and that \((c)_{J+1} = 1\). In the other case, there exists \(j_0 \in \{1, \ldots, J\}\) and two components \([A], [A'] \in A\) such that up to conjugacy we have \(A'_{j_0} = A * A'\), and \(A' = (A - \{|[A], [A']|\}) \cup \{|A'_{j_0}|\}\). It follows that each \((a)_j = 0\), that \((b)_j = 1\) if \(j = j_0\) and \((b)_j = 0\) otherwise, and that there are no \((c)_j\)'s. In either case we have \(D_{\text{FF}}(A) = D_{\text{FF}}(A') + 1\).

Suppose now that \(A \sqsubset A'\) is a proper expansion, equivalently \(D_{\text{FF}}(A) - D_{\text{FF}}(A') > 0\), equivalently at least one of the quantities \((a)_j, (b)_j, (c)_j\) is positive. In each case we
exhibit a free splitting \( C \) such that \( A \sqsupset C \sqsupset A' \), and \( A \sqsupset C \) is elementary. Item (3) and the remaining contentions of item (2) follow immediately.

**Case 1:** Some \( (a)_j > 0 \) (1 ≤ \( j \) ≤ \( J \)) which means the free factorization \( A'_j = A_{j1} \ast \cdots \ast A_{jk_j} \ast B_j \) has nontrivial cofactor \( B_j \). Let \( Z \) be rank 1 free factor of \( B \) and let \( C = A \cup \{ [Z] \} \).

**Case 2:** Some \( (b)_j > 0 \) (1 ≤ \( j \) ≤ \( J \)) which means \( A_j = \{ [A_{j1}], [A_{j2}], \ldots, [A_{jk_j}] \} \) has \( k_j \geq 2 \) components. Let \( C = (A - \{ [A_{j1}], [A_{j2}] \}) \cup \{ [A_{j1} \ast A_{j2}] \} \).

**Case 3:** Some \( (c)_j > 0 \) exists, which means \( J < K \). For \( J + 1 \leq j \leq K \) each of the groups \( A'_j \) is free of positive rank. Let \( Z < A'_{J+1} \) be a rank 1 free factor and let \( C = A \cup \{ [Z] \} \). ♦

### 3 The Free Splitting Complex and Its Relativizations

In Sections 3.1–3.3, given an arbitrary freely decomposable group \( \Gamma \) we define free splittings of \( \Gamma \) and their partial ordering \( \succ \) called the “collapse relation”. Also, using these concepts we define free splitting complexes of \( \Gamma \), both the “absolute” free splitting complex \( \mathcal{FS}(\Gamma) \) and the free splitting complex \( \mathcal{FS}(\Gamma; A) \) “relative to” a choice of free factor system \( A \). We also study a function which associates to each free splitting a free factor system called its “vertex stabilizer system”, and in Section 3.4 we study how this function relates the partial orderings \( \sqsubseteq \) and \( \succ \). In Section 3.5 we study the depth of the inverted partial ordering \( \prec \). We apply that study to obtain a formula for the dimension of \( \mathcal{FS}(\Gamma; A) \), and to obtain a finer understanding of the partial ordering as it relates to inclusion of simplices. Of particular importance is Proposition 3.6 that explains exactly which free splittings are maximal and minimal with respect to the collapse relation \( \succ \), and which chains of the relation \( \succ \) correspond to maximal simplices of \( \mathcal{FS}(\Gamma; A) \).

The proofs in this section are primarily applications of Bass-Serre theory along with basic topological manipulations of graphs and trees, and a few further applications of the Kurosh Subgroup Theorem.

For the case of \( \Gamma = F_n \), many of the results of this section, regarding basic concepts of free splittings and the collapse partial ordering may be familiar to a reader of [HM13]. Nonetheless we examine these concepts from new points of view, in order to study relative free splitting complexes. Throughout this section we try to view these points first from the vantage of the special case \( \Gamma = F_n \), before moving on the general formulation. This is done so as to enable the reader interested mostly in \( \Gamma = F_n \) to get through this section more quickly.
3.1 Basic terminology and notation regarding graphs.

A graph $G$ is a 1-dimensional simplicial complex, a tree is a contractible graph, and a subgraph of a graph $G$ is a subcomplex of some simplicial decomposition of $G$. Given $p \in G$, let $D_p G$ denote the set of directions at $p$, meaning initial germs of locally injective paths with initial point $p$. If $p$ is a vertex then each element of $D_p G$ is uniquely represented by an oriented edge with initial vertex $p$.

**Relatively natural cell structures. Subdivisions and edgelets.** Suppose that $G$ is a connected graph, and $P$ is a subset of the vertex set that includes all vertices of valence 1 and which accumulates on all isolated ends of $G$ (the only case of isolated ends that matters at all to us is when $G$ is homeomorphic to the line). In any setting where $P$ is fixed, there is a unique relatively natural cell structure on $G$ which is the CW structure on $G$ whose 0-skeleton, the set of relatively natural vertices, is the union of $P$ with all points of valence $\geq 3$; the 1-cells of this structure are called the relatively natural edges. When $P$ is understood we will often drop the adverb “relatively”.

Note that any CW structure on $G$ whose vertex set contains $P$ is a subdivision of the relatively natural cell structure. In any context where one such CW structure is specified we sometimes refer its edges as edgelets and we refer to that structure as an edgelet subdivision of the relatively natural cell structure.

3.2 Free splittings and the partial order $\succ$.

A free splitting of $\Gamma$ is a minimal simplicial action $\Gamma \curvearrowright T$ of the group $\Gamma$ on a simplicial tree $T$ such that $T$ is not a point, the stabilizer of each edge is trivial, and there are finitely many edge orbits. It follows that there are finitely many vertex orbits, and so $T/\Gamma$ is a finite graph of groups. It also follows, using minimality, that $T$ has no valence 1 vertices.

Two free splittings $S,T$ of $\Gamma$ are equivalent, denoted $S \approx T$, if there exists a $\Gamma$-equivariant homeomorphism $f : S \to T$. While this homeomorphism need not be simplicial, one can always make $f$ be simplicial by first subdividing $T$ along the image of the vertex set of $S$, and then pulling the subdivided vertices of $T$ back to obtain a subdivision of $S$. More generally a map $f : S \to T$ between free splittings is an equivariant function which, with respect to some subdivision of the domain and range, is simplicial.

For any free splitting $\Gamma \curvearrowright T$ we define the relatively natural cell structures on $T$ and on the quotient graph of groups $T/\Gamma$, so that the quotient map $T \to T/\Gamma$ is a cellular map taking relatively natural vertices to relatively natural vertices, and taking relatively natural edges to relatively natural edges. On $T$ the relatively natural vertices are the points which either have valence $\geq 3$ or have nontrivial stabilizer; and on $T/\Gamma$ the relatively natural vertices are the points which either have valence $\geq 3$ or have a nontrivial vertex group.
Remark. Note that a point \( p \in T \) has stabilizer isomorphic to \( \mathbb{Z}/2 \) if and only if \( p \) has valence 2 and the stabilizer of \( p \) is nontrivial. Using this one can show that if \( \Gamma \rhd T \) is a free splitting and if \( T \) has an isolated end then the following conclusion holds: \( \Gamma \) is the infinite dihedral group, \( T \) is a line, and \( \Gamma \rhd T \) is the Bass-Serre tree of the free decomposition \( \Gamma = \mathbb{Z}/2 * \mathbb{Z}/2 \). Also, the same conclusions hold for any free splitting of any group \( \Gamma \) that contains an infinite cyclic subgroup of finite index. Here we use our convention, from the opening of Section 2, which rules out the possibility that \( \Gamma \) is infinite cyclic (allowing \( \Gamma \) to be infinite cyclic, its unique free splitting up to equivalence is its translation action on the line).

Vertex stabilizer systems. Associated to each free splitting \( \Gamma \rhd T \) is a proper free factor system of \( \Gamma \) denoted \( \mathcal{F}(T) \) and called the vertex stabilizer system of \( T \), namely the conjugacy classes of nontrivial vertex stabilizers. The fact that \( \mathcal{F}(T) \) is indeed a free factor system follows from Bass-Serre theory, by using any isomorphism between \( \Gamma \) and the fundamental group of the quotient graph of groups \( T/\Gamma \). In the converse direction we have the following fact, which will often be invoked silently:

Lemma 3.1. For every proper free factor system \( \mathcal{A} \) of \( \Gamma \) there exists a free splitting \( \Gamma \rhd T \) such that \( \mathcal{A} = \mathcal{F}(T) \).

Proof. Choose a realization \( \Gamma = A_1 * \cdots * A_K * B \) of \( \mathcal{A} = \{[A_1], \ldots, [A_K]\} \). Construct a graph of groups with base point \( p \), attaching to \( B \) a rose with rank equal to \( \text{corank}(A) = \text{rank}(B) \), and attaching \( K \) additional edges to \( p \) with opposite vertices of valence 1 having respective vertex groups \( A_1, \ldots, A_K \). The fundamental group of this graph of groups has an isomorphism to \( \Gamma = A_1 * \cdots * A_K * B \). Letting \( T \) be the Bass-Serre tree of this graph of groups with associated \( \Gamma \) action, we obtain a free splitting of \( \Gamma \) satisfying \( \mathcal{F}(T) = \mathcal{A} \).

Collapse maps and the partial ordering \( \succ \). A partial ordering on the set of equivalence classes of free splittings of \( \Gamma \) is defined as follows. A collapse map \( f : T \to S \) is a map such that for each \( x \in S \) its inverse image \( f^{-1}(x) \) is connected. The union of those inverse images \( f^{-1}(x) \) which are not single points is called the collapse forest \( \sigma \subset T \), and so \( \sigma \) has no degenerate components, meaning no components that are single points. Letting \( T \mapsto T/\sigma \) denote the equivariant quotient map under which each component of \( \sigma \) is collapsed to a single point, it follows that \( T/\sigma \) and \( S \) are equivalent free splittings.

We sometimes incorporate \( \sigma \) into the notation by writing \( T \overset{[\sigma]}{\to} S \).

A collapse \( T \overset{[\sigma]}{\to} S \) is relatively natural if \( \sigma \) is a subcomplex of the relatively natural cell structure, equivalently \( \sigma \) is a union of relatively natural edges. Note that if a collapse map \( T \overset{[\sigma]}{\to} S \) exists then a relatively natural collapse map exists, by replacing \( \sigma \) with its unique maximal relatively natural cell subcomplex (which might be empty).
We define a relation denoted \( T \succ S \) or \( S \prec T \) to mean that there exists a collapse map \( T \mapsto S \), equivalently there exists a relatively natural collapse map. This relation is well-defined on equivalence classes. The relation \( T \succ S \) is a partial order because a composition of collapse maps is a collapse map. We express the relation \( T \succ S \) in various ways, such as \( T \) collapses to \( S \), or \( S \) expands to \( T \), or \( S \prec T \) is an expansion, or \( T \succ S \) is a collapse. If furthermore \( T \not\approx S \) then the collapse or expansion is proper, and this holds if and only if for some (all) collapse maps \( T \mapsto S \) some point pre-image contains more than one relatively natural vertex of \( T \).

Note that for any map of free splittings \( f : S \to T \), each element of \( \Gamma \) that is elliptic in \( S \) is also elliptic in \( T \), and therefore \( F(S) \subseteq F(T) \) (see Lemma 3.2 (3)). In particular, if \( T \prec S \), equivalently if \( S \succ T \), then \( F(S) \subseteq F(T) \).

**Remark on abuses of notation.** While a free splitting is formally denoted \( \Gamma \curvearrowright T \), we often use this notation to emphasize the action, also we often suppress the action from the notation and simply write \( T \). The action is always suppressed from the notation for the equivalence class \([T]\), and sometimes we write just \( T \) for the equivalence class.

### 3.3 Free splitting complexes and their relativizations.

We define the (absolute) free splitting complex of \( \Gamma \), denoted \( \mathcal{FS}(\Gamma) \), to be the simplicial complex which is the geometric realization of the set of equivalence classes of free splittings of \( \Gamma \) partially ordered by \( \prec \). Thus \( \mathcal{FS}(\Gamma) \) has a 0-simplex for each equivalence class of free splittings \( \Gamma \curvearrowright T \), denoted \([T]\). In general \( \mathcal{FS}(\Gamma) \) has a K-simplex for each \( K + 1 \)-tuple of distinct 0-simplices \([T_0],[T_1],\ldots,[T_K]\) such that \( T_0 \prec T_1 \prec \cdots \prec T_K \); this simplex is denoted \([T_0] \prec [T_1] \prec \cdots \prec [T_K]\). By our convention that \( \Gamma \) be freely indecomposable, \( \mathcal{FS}(\Gamma) \) is always nonempty.

Consider now a proper free factor system \( \mathcal{A} \) of \( \Gamma \). A free splitting of \( \Gamma \) rel \( \mathcal{A} \) is a free splitting \( \Gamma \curvearrowright T \) with the property that \( \mathcal{A} \subseteq F(T) \), equivalently \( \mathcal{A} \) is elliptic with respect to \( T \) meaning that each subgroup of \( \Gamma \) representing an element of \( \mathcal{A} \) fixes some point of \( T \). The free splitting complex of \( \Gamma \) relative to \( \mathcal{A} \), denoted \( \mathcal{FS}(\Gamma;\mathcal{A}) \), is the flag subcomplex of \( \mathcal{FS}(\Gamma) \) consisting of all simplices \([T_0]\prec\cdots\prec[T_K]\) such that \( \mathcal{A} \) is elliptic in each of the free splittings \( \Gamma \curvearrowright T_0,\ldots,T_K \); this is equivalent to requiring simply that \( \mathcal{A} \) is elliptic in \( T_K \), because \( F(T_K) \subseteq \cdots \subseteq F(T_0) \). The requirement that \( \mathcal{A} \) be proper implies that free splittings rel \( \mathcal{A} \) exist (by Lemma 3.1) and so \( \mathcal{FS}(\Gamma;\mathcal{A}) \) is nonempty. In Corollary 4.5 below we will see that \( \mathcal{FS}(F_n;\mathcal{A}) \) is connected.

Note that if \( \Gamma \) has a proper Grushko decomposition, equivalently if there exists a free factor system \( \mathcal{A} \) which is minimal with respect to \( \subseteq \) (see Proposition 2.13), then \( \mathcal{FS}(\Gamma) = \mathcal{FS}(\Gamma;\mathcal{A}) \); this holds for example whenever \( \Gamma \) is finitely generated.

**Remarks on terminology and notation.** The notation \([T]\) is used both for the equivalence class of a free splitting \( \Gamma \curvearrowright T \) and for the corresponding 0-simplex of \( \mathcal{FS}(\Gamma) \).
Sometimes we abuse notation by writing things like “\(T \in \mathcal{FS}(\Gamma; A)\)” which can be read formally either as “\(T\) is a free splitting of \(\Gamma\) rel \(A\)” or as “\([T]\) is a 0-simplex of \(\mathcal{FS}(\Gamma; A)\)”.

In [HM13] we used the notation \(\mathcal{FS}(F_n)\) a little differently, namely the complex with one \(k\)-simplex for each equivalence class of free splittings \(T\) having \(k + 1\)-orbits of natural edges, where the face inclusion is defined by the relation \(S \prec T\). Also, we used the notation \(\mathcal{FS}'(F_n)\) for the first barycentric subdivision of \(\mathcal{FS}(F_n)\) which is equivalent to the free splitting complex as defined in this section. But even in [HM13] we worked primarily with this first barycentric subdivision, and since relative free splitting complexes live naturally as subcomplexes of this first barycentric subdivision, in this current work we switch the notation and we hope that this does not cause confusion.

### 3.4 Relations between the partial orders \(\sqsubseteq\) and \(\succ\).

In the following lemma we collect properties relating the partial order \(\sqsubseteq\) on free factor systems to the partial order \(\succ\) on (equivalence classes of) free splittings. These properties are all true as well when they are specialized by choosing a free factor system \(A\) and putting in the qualifier “relative to \(A\)”.

**Lemma 3.2.** For any \(\Gamma\) the following hold:

1. For any map of free splittings \(f: T \to S\) we have an extension \(\mathcal{F}(T) \sqsubseteq \mathcal{F}(S)\) of free factor systems. In particular if \(S \prec T\) then \(\mathcal{F}(T) \sqsubseteq \mathcal{F}(S)\).

2. For any free factor system \(A\) of \(\Gamma\) and any two free splittings \(\Gamma \succ S, T\) rel \(A\) there exists a free splitting \(\Gamma \succ U\) rel \(A\) and a relatively natural collapse map \(f: U \to T\) such that \(\mathcal{F}(U) = \mathcal{F}(S) \land \mathcal{F}(T)\) and such that for each \(x \in T\), if the subgroup \(\text{Stab}_T(x)\) is nontrivial then its action on \(f^{-1}(x) \subset U\) is equivalent to its action on its minimal subtree in \(S\).

3. For any free splitting \(\Gamma \succ T\) and any free factor system \(B \sqsubseteq \mathcal{F}(T)\) there exists a free splitting \(U\) and a collapse map \(U \to T\) such that \(\mathcal{F}(U) = B\).

4. More generally, for each sequence of extensions \(A_0 \sqsubseteq A_1 \sqsubseteq \cdots \sqsubseteq A_K\) of free factor systems rel \(A\), and each free splitting \(S_K\) such that \(\mathcal{F}(S_K) = A_K\) there exists a sequence of free splittings and collapses \(S_0 \succ S_1 \succ \cdots \succ S_K\) such that \(\mathcal{F}(S_k) = A_k\) for each \(k = 0, \ldots, K\).

**Proof.** Item (1) is evident since \(\text{Stab}(x) < \text{Stab}(f(x))\). Clearly (2) \(\implies\) (3) by taking \(S\) to be any free splitting such that \(\mathcal{F}(S) = B\) and using that \(B \sqsubseteq C\) implies \(B \land C = B\). Also clearly (3) \(\implies\) (4).

The proof of item (2) is an elaboration of the Bass-Serre theory proof of the Kurosh Subgroup Theorem (see e.g. [Coh89]); here are a few details. First apply Proposition 2.12...
to conclude that $B = F(S) \cap F(T)$ is a free factor system rel $A$. Consider $x \in T$ such that $\text{Stab}_T(x)$ is nontrivial, let $S^x \subset S$ be the minimal subtree for the action $\text{Stab}_T(x) \act S$, and suppose that $S^x$ is not a point. Blow up the vertex $x$ using $S^x$: detach each of the directions of $D_x T$ from $x$, then remove $x$, then reattach the directions of $D_x T$ to a copy of the tree $S^x$ in a $\text{Stab}_T(x)$-equivariant manner. Now extend this “detachment–attachment” operation over the whole orbit of $x$, reattaching the directions in a $\Gamma$-equivariant manner. Doing this for each orbit of such points $x$ results in the desired free splitting $\Gamma \act U$.

\[\Box\]

3.5 Free splitting depth of free factor systems and dimensions of relative free splitting complexes.

The absolute free splitting complex of a rank $n$ free group $FS(F_n)$ has the following easily proved properties. Define a free splitting $F_n \act T$ to be generic if every vertex has valence 3. First, $T$ has at most $3n - 3$ natural edge orbits, the maximum being attained if and only if $T$ is generic. Also, the maximal number of natural vertex orbits is the number attained for generic $T$ which is $2n - 2$. These are proved by simple Euler characteristic calculations taking place in the quotient graph of groups $T/F_n$. Next, given a $D$-simplex $[T_0] \prec [T_1] \prec \cdots \prec [T_D]$ with corresponding sequence of relatively natural collapse maps $T_D \mapsto \cdots \mapsto T_1 \mapsto T_0$, the following are easily proved to be equivalent:

1. $D = 3n - 4$.

2. $T_D$ is generic, each map $T_d \mapsto T_{d-1}$ collapses exactly one orbit of natural edges, and $T_0$ has exactly one orbit of natural edges.

3. The simplex $[T_0] \prec [T_1] \prec \cdots \prec [T_D]$ is maximal, meaning it is not a proper face of any other simplex.

As a consequence, the dimension of $FS(F_n)$ equals $3n - 4$ and every simplex is a face of some simplex of maximal dimension $3n - 4$.

We now generalize, stating and proving analogous results for relative free splitting complexes.

**Definition 3.3.** Let $\Gamma$ be a group and $A$ any free splitting of $\Gamma$.

1. The **free splitting depth** of $A$ is defined to be the number

\[D_{FS}(A) = 3 \text{corank}(A) + 2|A| - 4\]

2. A free splitting $\Gamma \act T$ rel $A$ is generic if $F(T) = A$ and for each vertex $v$ the following holds: if $\text{Stab}(v)$ is trivial then $v$ has valence $\leq 3$; whereas if $\text{Stab}(v)$ is nontrivial then $\text{Stab}(v)$ acts transitively on $D_v T$. 

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Note that for Γ ↷ T to be generic, it is equivalent that in the quotient graph of groups
\[ G = T/F_n \] the following hold: the nontrivial vertex groups are of the form \( A_1, \ldots, A_K \) where \( A = \{[A_1], \ldots, [A_K]\} \); and for every vertex \( V \) of \( G \), if \( V \) has trivial vertex group then \( V \) has valence 2 or 3, whereas if \( V \) has nontrivial vertex group then \( V \) has valence 1. One can always choose the vertex groups to fit into a realization of \( A \) of the form
\[ \Gamma = A_1 \ast \cdots \ast A_K \ast B \]
in such a way that \( B \) is identified with a lift to \( \Gamma \) of the fundamental group of the underlying graph of \( G \).

**Proposition 3.4.** For any free splitting \( \Gamma \当事 \rel A \) the following hold:

1. The number of relatively natural edge orbits of \( T \) satisfies \( E(T) \leq D_{FS}(A) + 1 \).

2. The following are equivalent:
   (a) \( E(T) = D_{FS}(A) + 1 \).
   (b) \( T \) is generic.
   (c) \([T]\) is maximal with respect to the partial ordering \(<\), that is, for every free splitting \( \Gamma \当事 U \), if there exists a collapse map \( U \leftrightarrow T \) then \([U]\) = \([T]\).

3. The number of relatively natural vertex orbits of \( T \) satisfies
\[ V(T) \leq D_{FS}(A) + 2 - \text{corank}(A) = 2\text{corank}(A) + 2A - 2 \]
with equality if and only if \( T \) is generic.

**Proof.** In this proof we assume that all vertices and all edges of free splittings are relatively natural, equivalently no valence 2 vertex has nontrivial stabilizer; if any such vertices exist, just remove them from the 0-skeleton. Thus every vertex and every edge of the quotient graph of groups is relatively natural, meaning that no valence 2 vertex has trivial vertex group. Also, all collapse maps are relatively natural and are nontrivial if and only if they are not homeomorphisms. Having done this, for any such free splitting \( T \) with quotient \( G = T/\Gamma \) the numbers \( E = E(T) \) and \( V = V(T) \) are just the counts of edge and vertex orbits of \( T \), equivalent of edges and vertices of \( G \). Let \( V_k = V_k(T) \) be the number of valence \( k \) vertices of \( G \), equivalently the number of \( \Gamma \)-orbits of vertices \( v \in T \) at which the set \( D_1 \Gamma \) has exactly \( k \) orbits under the action of \( \text{Stab}(v) \).

We first prove \((2b) \Rightarrow (2a)\). Assuming \( T \) is generic we have \( V = V_1 + V_3 \) and \( V_1 = |A| \). We also have \( E = \frac{1}{2}(V_1 + 3V_3) \) and \( \text{corank}(A) = E - V + 1 \) (= rank\( (G) \)). Eliminating \( V \), \( V_1 \), and \( V_3 \) gives \( E = D_{FS}(A) + 1 \).

We next claim that for every free splitting \( \Gamma \当事 T \rel A \) there exists a generic free splitting \( \Gamma \当事 S \rel A \) and a relatively natural collapse map \( S \xrightarrow{[\sigma]} T \). From this claim we obtain the following consequences. First, item (1) holds because \( E(T) \leq E(S) = D_{FS}(A) + 1 \). Next, the implication \((2a) \Rightarrow (2c)\) holds, because if \( (2c) \) does not hold
then there exists \( U \) and a collapse \( U \xrightarrow{[\sigma]} T \) such that \([U] \neq [T]\), and so \( \sigma \) is nontrivial, implying by (1) that \( E(T) < E(U) \leq D_{FS}(A) + 1 \). Next, (2c) \( \implies \) (2b), because if \( T \) is not generic then the collapse map \( S \mapsto T \) is nontrivial and so \([T]\) is not maximal.

Finally, item (3) follows because the collapse map \( S \xrightarrow{[\sigma]} T \) takes the relatively natural vertices of \( S \) onto the relatively natural vertices of \( T \) and so \( V(S) \geq V(T) \), with equality if and only if \( \sigma = \emptyset \) if and only if \([S] = [T] \) if and only if \( T \) is generic, and

\[
V(S) = 1 - \text{rank}(S/\Gamma) + E(S) = 1 - \text{corank}(A) + D_{FS}(A) + 1
\]

To prove the claim we do a sequence of expansions of \( T \) one at a time to build up the properties of a generic free splitting rel \( A \). First, by applying the expansion from Lemma 3.2 (3) we may assume that \( \Gamma \bowtie T \) satisfies \( F(T) = A \).

Next, by expanding \( T \) we may assume that if \( v \in T \) is a vertex with nontrivial stabilizer, and so \([\text{Stab}(v)] \in A\), then the number \( k_v \) of \( \text{Stab}(v) \)-orbits in the set \( D_v \Gamma \) satisfies \( k_v = 1 \). Otherwise, if \( k_v \geq 2 \), choose orbit representatives \( d_1, \ldots, d_k \in D_v \Gamma \), do a simultaneous partial fold of these directions by identifying proper initial segments into a single segment \( e \), having one vertex with the same stabilizer as \( v \) and opposite vertex of valence \( k + 1 \) and with trivial stabilizer. Extending these identifications equivariantly, the resulting free splitting is an expansion of \( T \) because by collapsing the orbit of \( e \) we recover \( T \).

Finally, we may assume that if \( v \) is a vertex with trivial stabilizer and valence \( \geq 3 \) then \( v \) has valence 3, for otherwise we may group \( D_v \Gamma \) into two sets of cardinality \( \geq 2 \) and expand \( T \) by pulling these two sets apart, inserting a new edge, and extending this expansion equivariantly over the orbit of \( v \). This expansion decreases the lexicographically ordered sequence \((V_3(T), V_4(T), \ldots)\).

\[
\Box
\]

**Definition 3.5.** Let \( \Gamma \) be a group.

1. A relatively natural collapse map \( S \xrightarrow{[\sigma]} T \) of free splittings of \( \Gamma \) is **elementary** if \( \sigma \) consists of a single orbit of relatively natural edges.

2. A **one edge free splitting** is a free splitting \( \Gamma \bowtie T \) with exactly one relatively natural edge orbit.

**Proposition 3.6.** For each \( D \)-simplex \([T_0] < [T_1] < \cdots < [T_D] \) in \( FS(\Gamma; A) \) with corresponding sequence of relatively natural collapse maps \[ T_D \xrightarrow{[\sigma_D]} T_{D-1} \xrightarrow{[\sigma_{D-1}]} \cdots \xrightarrow{[\sigma_2]} T_1 \xrightarrow{[\sigma_1]} T_0 \]

the following are equivalent:
(1) \( D = D_{FS}(A) \).

(2) Each of the following holds: (a) \( T_D \) is generic relative to \( A \); (b) the collapse map \( T_d \mapsto T_{d-1} \) is elementary for each \( d = 1, \ldots, D \); (c) \( T_0 \) is a one-edge free splitting.

(3) The simplex \([T_0] \prec [T_1] \prec \cdots \prec [T_D]\) is maximal, meaning it is not a face of any other simplex.

As a consequence, the dimension of \( FS(\Gamma; A) \) equals \( D_{FS}(A) \), and every simplex is a face of a simplex of maximal dimension \( D_{FS}(A) \).

Proof. As in the proof of Proposition (3.4), we assume that all edge and vertices are relatively natural, and we continue to use the notation \( E(T) \), \( V(T) \) as in that proof.

The scheme of the proof is (1) \( \iff \) (2) \( \iff \) (3).

Assuming item (2) we shall prove (1). By applying Proposition 3.4 one concludes \( E(T_D) = D_{FS}(A) + 1 \), and then one notices that from (2) it follows that the edge orbits of \( T_D \) are collapsed one-at-a-time until only one remains, implying that the number \( D \) of collapse maps equals \( D_{FS}(A) \).

Assuming (1) we shall prove (2). For any relatively natural collapse map \( S \xrightarrow{[\sigma]} T \), letting \( E(\sigma) \) be the number of natural edge orbits of \( S \) contained in the \( \Gamma \)-equivariant natural subforest \( \sigma \), we have \( E(T) + E(\sigma) = E(S) \); recall also that \( E(\sigma) = 0 \iff \sigma = \emptyset \iff [S] = [T] \). Using that each of \( E(\sigma_D), \ldots, E(\sigma_1), E(T_0) \) is \( \geq 1 \) we have

\[
D + 1 \leq E(\sigma_D) + \cdots + E(\sigma_1) + E(T_0) = E(T_D) \leq D_{FS}(A) + 1 \quad \text{(by Proposition 3.4 (1))}
\]

and so all inequalities are equations. Applying Proposition 3.4 (2), it follows \( T_D \) is generic. It also follows that \( E(\sigma_D) = \cdots = E(\sigma_1) = E(T_0) = 1 \), which proves (2).

Assuming (2) holds, we prove (3) as follows. Since \( T(D) \) is generic, there does not exist any proper collapse map of the form \( S \mapsto T_D \) for that would imply \( E(S) > D_{FS}(A) + 1 \), contradicting Proposition 3.4 (1). Since \( T_d \mapsto T_{d-1} \) is elementary, there exist any factorization of \( T_d \mapsto T_{d-1} \) into proper collapse maps of the form \( T_d \mapsto S \mapsto T_{d-1} \) because that would imply \( E(T_d) \geq E(T_{d-1}) + 2 \), contradicting that \( E(T_d) = E(T_{d-1}) + 1 \). Nor does there exist any proper collapse map of the form \( T_0 \mapsto S \), for that would imply \( E(S) \leq E(T_0) - 1 = 1 - 1 = 0 \). It follows that the simplex \([T_0] \prec \cdots \prec [T_D]\) is maximal.

Assuming (2) fails, we prove that (3) fails as follows. One of (a), (b), or (c) must fail. If \( T_D \) is not generic then by Proposition 3.4 (2c) there exists a free splitting \( S \) and a proper relatively natural collapse map \( S \mapsto T_D \). If \( T_d \xrightarrow{[\sigma]} T_{d-1} \) is not elementary then,
first collapsing a single edge orbit of $\sigma$, there a sequence of proper relatively natural collapse maps $T_d \mapsto S \mapsto T_{d-1}$. If $T_0$ has more than one edge orbit then, collapsing just one edge orbit, there exists a proper relatively natural collapse map $T_0 \mapsto S$. In each case we obtain a simplex of one dimension higher containing the simplex $[T_0] \prec \cdots \prec [T_D]$. ◊

### 3.6 The relative outer automorphism group $\Out(\Gamma; A)$.

Now that the sets of free factor systems and free splittings rel $A$ have been defined together with various relations and operations on them, we pause here to carefully define the relative outer automorphism group $\Out(\Gamma; A)$ and its actions on those sets. We also define the action of the group $\Out(\Gamma; A)$ on the relative free splitting complex $FS(\Gamma; A)$, although the definition of its action on the complex of free factor systems rel $A$ will await the definition of that complex to be given in Section 6.1.

The group $\Out(\Gamma)$ has a canonical left action on the set of free factor systems $A$, namely: given $\phi \in \Out(\Gamma)$, choosing a representative $\Phi \in \Aut(\Gamma)$, and choosing a realization $\Gamma = A_1 \ast \cdots \ast A_K \ast B$ of $A$, one defines

$$\phi(A) = \{[\Phi(A_1)], \ldots, [\Phi(A_K)]\}$$

This action is well-defined independent of choices, the left action equations $\phi(\psi(A)) = (\phi \psi)(A)$ and $\Id(A) = A$ hold, and the action preserves the extension partial order $\sqsubset$ and the meet operation $\wedge$.

**Relative outer automorphism groups.** Given a free factor system $A$ of $\Gamma$, the subgroup of $\Out(\Gamma)$ that fixes $A$ is denoted $\Out(\Gamma; A)$ and is called the outer automorphism group of $\Gamma$ rel $A$. This is the group studied by Guirardel and Levitt in [GL07] who derive information about the virtual cohomological dimension of $\Out(\Gamma; A)$ using information about the virtual cohomological dimensions of the groups $A_k, \Aut(A_k),\text{ and }\Out(A_k), k = 1, \ldots, K$.

**Action on relative free splitting complexes.** The group $\Out(\Gamma)$ has a canonical right action on the set of equivalence classes of free splittings of $\Gamma$ as follows. Consider the equivalence class $[T]$ of a free splitting $\Gamma \cong T$ with associated homomorphism $\alpha: \Gamma \to \Aut(T)$; incorporating $\alpha$ into the notation we write $\Gamma \cong_\alpha T$. Consider also $\phi \in \Out(\Gamma)$ represented by $\Phi \in \Aut(\Gamma)$. Precomposing $\alpha$ by $\Phi$ we obtain a homomorphism $\alpha \circ \Phi: \Gamma \to \Aut(T)$ which defines a free splitting $\Gamma \cong_{\alpha \circ \Phi} T$, the equivalence class of which is defined to be $[T] \cdot \phi$. This free splitting is well-defined, the right action equations $[T] \cdot (\Phi \psi) = ([T] \cdot \phi) \cdot \psi$ and $[T] \cdot \Id = [T]$ hold, and the action preserves the collapse partial order $\succ$. We obtain thereby an induced right action of $\Out(\Gamma)$ on linear chains of free splittings as follows:

$$\left([T_0] \succ [T_1] \succ \cdots \succ [T_K]\right) \cdot \phi = [T_0] \cdot \phi \succ [T_1] \cdot \phi \succ \cdots \succ [T_K] \cdot \phi$$
Finally, for any free factor system $A$ of $\Gamma$, we obtain by restriction a right action of $\text{Out}(\Gamma; A)$ on linear chains of free splittings rel $A$. These chains define simplices of the relative free splitting complex $\mathcal{FS}(\Gamma; A)$, and so we immediately obtain the right action of $\text{Out}(\Gamma; A)$ on $\mathcal{FS}(\Gamma; A)$ by simplicial isomorphisms.

**Action on chains of relative free factor systems.** The action of $\text{Out}(\Gamma)$ on free factor systems preserving $\sqsubseteq$ induces an action on linear chains of free factor systems:

$$\phi(A_0 \sqsubseteq A_1 \sqsubseteq \cdots \sqsubseteq A_K) = \phi(A_0) \sqsubseteq \phi(A_1) \sqsubseteq \cdots \sqsubseteq \phi(A_K)$$

For any given free factor system $A$ we obtain by restriction a left action of $\text{Out}(\Gamma; A)$ on linear chains of free splittings rel $A$. Once the formal definitions are given in Section 6.1, we will immediately obtain the left action of $\text{Out}(\Gamma; A)$ on the relative complex of free factor systems $\mathcal{F}\mathcal{F}(\Gamma; A)$ by simplicial isomorphisms.

We record here one fact for later use, which is a simple consequence of the definitions:

**Lemma 3.7.** The function $[T] \mapsto \mathcal{F}[T]$ satisfies the inverted equivariance condition with respect to the actions of $\text{Out}(\Gamma)$: given an equivalence class of free splittings $[T]$ and $\phi \in \text{Out}(\Gamma)$ we have the following equation of free factor systems:

$$\mathcal{F}([T] \cdot \phi) = \phi^{-1}(\mathcal{F}[T])$$



## 4 Fold paths and free splitting units

In this section we fix a group $\Gamma$ and study fold paths in relative free splitting complexes $\mathcal{FS}(\Gamma; A)$. Section 4.1 contains the basic definitions, generalizing fold paths following [HM13] but also following [BF12] to the extent of dropping the “gate 3 condition” of [HM13]. In Section 4.2 we use fold paths to give an explicit description of $\mathcal{FS}(\Gamma; A)$ in the simplest cases where the free factor system $A$ is very close to the improper free factor system $\{[\Gamma]\}$. In Section 4.3 we generalize the concepts of combing of fold paths following [HM13]. In Section 4.4 we consider a measurement of the complexity of a $\Gamma$-invariant subforest of a free splitting $\Gamma \curvearrowright T$, and we study how this complexity can change along a fold path. In Section 4.5 we use change of complexity to define free splitting units along fold paths; in later sections these units are shown to give efficient upper and lower bounds to distance along fold paths.

### 4.1 Fold sequences

Given two free splittings $\Gamma \curvearrowright S, T$ and a map $f : S \to T$ which is injective on each edgelet, for each $p \in S$ there is an induced “derivative” $df_p : D_p S \to D_{f(p)} T$, which maps the direction of each oriented edgelet $E \subset S$ with initial vertex $p$ to the initial
direction of the path \( f \mid E \). The point pre-images of the map \( df_p \) are called the gates of \( f \) at \( p \). We say that \( f: S \to T \) is foldable if it is injective on each edgelet and has at least 2 gates at each vertex. A foldable sequence is a sequence of maps

\[
T_0 \xrightarrow{f_1} T_1 \xrightarrow{f_2} \cdots \xrightarrow{f_K} T_K
\]

such that each map \( f_i = f_j \circ \cdots f_{i+1} : T_i \to T_j \) is foldable. In discussing foldable sequences we often restrict our attention to subsequences \( T_{i_1} \mapsto \cdots \mapsto T_{i_j} \) parameterized by an integer subinterval \( [i, j] = \{k \in \mathbb{Z} \mid i \leq k \leq j\} \).

A foldable map \( f: S \to T \) is a fold if there exist initial segments \( e, e' \subset S \) of oriented natural edges such that \( e \cap e' = v \) is their common initial point, and there exists an orientation preserving simplicial homeomorphism \( h: e \leftrightarrow e' \) such that for all \( x \neq x' \in S \), \( f(x) = f(x') \) if and only if there exists \( \gamma \in F_n \) such that \( \gamma(x) \in e \), \( \gamma(x') \in e' \), and \( h(\gamma(x)) = \gamma(x') \).

We review the Bestvina-Feighn classification of folds given in [BF91] Section 2, with simplifications as applied to free splittings. Consider free splittings \( \Gamma \simeq S, T \) and a fold map \( f: S \to T \) which folds two edges \( e, e' \) as above. Let \( w, w' \) be the endpoints of \( e, e' \) opposite their common initial endpoint \( v \). Let \( \pi: S \to S/\Gamma \) be the map to the quotient graph of groups. The type I folds are as follows: \( f \) has type IA if \( \pi \) is one-to-one on \( e \cup e' \); and \( f \) has type IB if (up to interchanging \( e, e' \)) the map \( \pi \) identifies \( v \) and \( w \) and is otherwise one-to-one on \( e \cup e' \). Type II folds do not occur in our setting, as they involve nontrivial edge stabilizers. The type III folds are as follows: \( f \) has type IIIA if \( \pi \) identifies \( w, w' \) and is otherwise one-to-one on \( e \cup e' \); and \( f \) has type IIIB if \( \pi \) identifies \( v, w, w' \) and is otherwise one-to-one on \( e \cup e' \). In all cases the extension \( \mathcal{F}(S) \subset \mathcal{F}(T) \) can be described explicitly. For types IA or IB: if at least one of \( \text{Stab}(w) \), \( \text{Stab}(w') \) is trivial then \( \mathcal{F}(S) = \mathcal{F}(T) \) (and this is the only case of equality); otherwise \( [\text{Stab}(w)], [\text{Stab}(w')] \) are two components of \( \mathcal{F}(S) \), and \( \mathcal{F}(T) \) is obtained from \( \mathcal{F}(S) \) by replacing those two with the strictly larger component \( ([\text{Stab}(w) \cup \text{Stab}(w')] \cup \{g\}) \).

A fold sequence is a foldable sequence denoted as above in which each of the maps \( f_i: T_{i-1} \to T_i \) is a fold map. Lemma 4.3 to follow implies that every foldable sequence as denoted above can be interpolated by a fold sequence, meaning that each foldable map \( T_{k-1} \mapsto T_k \) in the sequence may be replaced by a fold sequence from \( T_{k-1} \) to \( T_k \), yielding a fold sequence from \( T_0 \) to \( T_K \).

**Remark on the “gate 3 condition”.** In [HM13], in the setting of \( \Gamma = F_n \), the definition of a foldable map \( f: S \to T \) had an additional requirement, the following “gate 3 condition”: for any vertex \( p \in S \) of valence \( \geq 3 \) the map \( f \) has at least three gates at \( p \). Here we follow Bestvina and Feighn [BF12] to the extent of weakening the definition of
[HM13] by dropping the “gate 3 condition”. In what follows we will occasionally explain how this change effects the proofs. For the most part these are desirable changes, but see after the statement of Lemma 5.2 for a significant exception. For the present moment, we point out two desirable effects of dropping the gate 3 condition. First, it allows for a broader collection of fold sequences in the free splitting complex; this was an important motivation for dropping that condition in [BF12]. Second, the interpolation of the previous paragraph does not generally work when foldable maps are required to satisfy the gate 3 condition.

The following commonly used relativization tool is an immediate consequence of Lemma 3.2 (3):

**Lemma 4.1.** If \( S \in \mathcal{FS}(\Gamma; A) \) and \( T \in \mathcal{FS}(\Gamma) \), and if there exists a map \( f: S \to T \), then \( T \in \mathcal{FS}(\Gamma; A) \).

---

**Lemma 4.2** (cf. Lemma 2.4 of [HM13]). For any \( S, T \in \mathcal{FS}(\Gamma; A) \) there exists \( S', S'' \in \mathcal{FS}(\Gamma; A) \) such that \( S \prec S' \succ S'' \) and such that a foldable map \( S'' \mapsto T \) exists. If \( \mathcal{F}(S) \sqsubseteq \mathcal{F}(T) \) then one can take \( S = S' \).

**Remark.** The proof of the above lemma is considerably simpler than its [HM13] analogue Lemma 2.4, due to the removal of the gate 3 condition.

**Proof.** Choose a free splitting \( \Gamma \acts S' \) such that \( S \prec S' \) and \( \mathcal{F}(S') \sqsubseteq \mathcal{F}(T) \); if \( \mathcal{F}(S) \sqsubseteq \mathcal{F}(T) \) choose \( S' = S \); otherwise, applying Lemma 3.2 (3), choose \( S' \) so that \( S \prec S' \) and \( \mathcal{F}(S') = A \sqsubseteq \mathcal{F}(T) \). In either case we have \( S' \in \mathcal{FS}(\Gamma; A) \).

There exists a map \( S' \mapsto T \) which on each edge of \( S \) is either constant or injective: for each \( v \in S' \) choose \( f(v) \in T \) in a \( \Gamma \)-equivariant manner so that \( \text{Stab}(v) < \text{Stab}(f(v)) \), and extend linearly over each edge; this is possible because \( \mathcal{F}(S') = A \sqsubseteq \mathcal{F}(T) \). For each such map, let \( S' \) be subdivided so that each edgelet maps either to a vertex or an edge of \( T \). Amongst all such maps \( S' \mapsto T \), choose \( f: S' \to T \) to minimize the number of orbits of edgelets of \( S' \) on which \( f \) is nonconstant. Factor \( f \) as \( S' \xrightarrow{f'} S'' \xrightarrow{f''} T \) where \( f' \) collapses to a point each component of the union of edgelets on which \( f \) is constant. The map \( f'' \) is injective on each edgelet.

To prove that \( f'' \) is foldable it remains to show that at each vertex \( v \in S'' \) the map \( f'' \) has at least two gates. Suppose to the contrary that \( f'' \) has only one gate at \( v \), let \( e_1, \ldots, e_I \) be the edgelets incident vertex \( v \), and let \( w_1, \ldots, w_I \) be their opposite endpoints. Let \( S'' \mapsto S''' \) be the quotient map obtained by collapsing to a point each of \( e_1, \ldots, e_I \) and all edgelets in their orbits, so we get an induced action \( \Gamma \acts S''' \).

Noting that \( w_1, \ldots, w_I \) all map to the same point in \( T \), there is an alternate description of \( S''' \) as follows: remove from \( S'' \) the point \( v \) and the interiors of \( e_1, \ldots, e_I \), identify
w_1, \ldots, w_I to a single point, and extend equivariantly. From this description it follows
that the map f''_i: S''_i \rightarrow T induces a map S'' \rightarrow T, and by construction the composition
S' \xrightarrow{f''_i} S''_i \rightarrow S''' \rightarrow T is nonconstant on a smaller number of edgelet orbits than f is
nonconstant on. This contradicts minimality of the choice of f.

Applying Lemma 4.1 we have S'' \in \mathcal{FS}(\Gamma; A). \diamond

Lemma 2.7 of [HM13] shows in the case \Gamma = F_n that any foldable map of free
splittings \rightarrow T factors into a fold sequence of free splittings, and the exact same proof
works for general \Gamma. Assuming in addition that S \in \mathcal{FS}(\Gamma; A), by applying Lemma 4.1
inductively starting with S it follows that each term in the fold sequence is in \mathcal{FS}(\Gamma; A).
This proves:

Lemma 4.3 (cf. Lemma 2.7 of [HM13]). For any S, T \in \mathcal{FS}(\Gamma; A), any foldable map
S \rightarrow T factors as a fold sequence in \mathcal{FS}(\Gamma; A). \diamond

Lemma 4.4 (cf. Lemma 2.5 of [HM13]). Given S, T \in \mathcal{FS}(F_n; A), if there is a fold
S \rightarrow T then in \mathcal{FS}(F_n; A) we have d(S, T) \leq 2.

Proof. Let e, e' \subset S be oriented segments with the same initial vertex p that are folded
by S \rightarrow T. In [HM13] Lemma 2.5 the following is proved in the case \Gamma = F_n, but the
proof applies in general: either there is an expansion S \prec T; or there is an expansion–
collapse S \prec U \succ T where \U \succ S collapses some edge e \subset U down to the point p, the
edge e has one vertex of valence 3, the opposite vertex of e has the same stabilizer as p,
and other vertex stabilizers outside the orbit of p are unaffected by this collapse. It
follows that \mathcal{F}(U) = \mathcal{F}(S), so U \in \mathcal{FS}(\Gamma; A) and d(S, T) \leq 2. \diamond

A sequence of vertices (T_i)_{i \in I} in \mathcal{FS}(\Gamma; A), parameterized by some subinterval I \subset \mathbb{Z},
is called a fold path if for each i - 1, i \in I there exists a map f_i: T_{i-1} \rightarrow T_i such that
the sequence of maps \cdots \xrightarrow{f_{i-1}} T_{i-1} \xrightarrow{f_i} T_i \xrightarrow{f_{i+1}} \cdots is a fold sequence.

By combining Lemmas 4.2, 4.3 and 4.4 (cf. remark following Theorem 3.1 of [HM13])
we have proved:

Corollary 4.5. \mathcal{FS}(F_n; A) is connected. Fold paths form an almost transitive sequence
of paths in \mathcal{FS}(F_n; A), meaning: for any S, T \in \mathcal{FS}(F_n; A) there is a fold path starting
at distance \leq 2 from S, making jumps of distance \leq 2, and ending at T. \diamond

4.2 \mathcal{FS}(\Gamma; A) in low complexity cases

Using the results of Section 4.1 we now give a complete description of free splitting
complexes \mathcal{FS}(\Gamma; A) in two low complexity cases where \mathcal{FS}(\Gamma; A) is a very specific
finite diameter tree. In all remaining cases we conjecture that \mathcal{FS}(\Gamma; A) is of infinite
diameter, indeed that the action of $\text{Out}(\Gamma; A)$ on $\mathcal{FS}(\Gamma; A)$ has loxodromic elements. In the companion work [HM] we shall prove this conjecture when $\Gamma = F_n$.

The first low complexity case is when $D_{FF}(A) = 0$, which occurs if and only if $A = \{[A_1], [A_2]\}$ and $\Gamma = A_1 \ast A_2$. The second is when $D_{FF}(A) = 1$ and $|A| \leq 1$, which occurs if and only if $A = \{[A]\}$ and $\Gamma = A \ast Z$ where $Z$ is infinite cyclic. We consider these cases separately in Propositions 4.6 and 4.7 to follow.

**Proposition 4.6.** Suppose that $D_{FF}(A) = 0$, equivalently $A = \{[A_1], [A_2]\}$ has a realization of the form $\Gamma = A_1 \ast A_2$. In this case $\mathcal{FS}(\Gamma; A)$ is a single point, corresponding to the Bass-Serre tree of the free factorization $\Gamma = A_1 \ast A_2$.

Remark. In the case that $A_1, A_2$ are free of finite rank, this proposition is contained in [BFH00] Corollary 3.2.2. The proof here is an extension of that proof.

**Proof.** We first note the fact that for any proper free factor system $A'$ of $\Gamma$, if $A \subset A'$ then $A = A'$. It follows that for any free splitting $\Gamma \succeq T$ rel $A$, we have $A = \mathcal{F}(T)$. We next note the fact that since $S$ has one edge orbit, if $S \succ S''$ then $S$ and $S''$ are equivalent.

Given a vertex $T \in \mathcal{FS}(\Gamma; A)$ we must prove that $S,T$ are equivariantly homeomorphic. Applying Lemma 4.2 and the facts noted above, it follows that there exists a foldable map $f: S \rightarrow T$. Applying Lemma 4.3, there exists a fold sequence from $S$ to $T$. However, at each vertex $v \in S$ all of the directions at $v$ are in the same orbit of the subgroup $\text{Stab}(v)$, because the quotient graph of groups $S/\Gamma$ has two vertices each of valence 1. A fold map cannot fold two directions in the same orbit. Thus the fold sequence from $S$ to $T$ has length zero and $S, T$ are equivalent. ♦

For describing the next case, we need a few definitions.

Consider a free product $\Gamma = A \ast Z$ where $Z = \langle z \rangle$ is infinite cyclic, and consider the free factor system $A = \{[A]\}$. Define a monomorphism $A \hookrightarrow \text{Out}(\Gamma; A)$ denoted $a \mapsto \phi_a$, where $\phi_a$ is represented by $\Phi_a \in \text{Aut}(\Gamma)$ which is characterized by $\Phi_a | A = \text{Id}$, $\Phi_a(z) = za$. Noting that the subgroups $\langle z \rangle$ and $\langle za \rangle$ are conjugate in $\Gamma$ if and only if $a$ is trivial, it follows that the homomorphism $a \mapsto \phi_a$ is injective.

In any 1-complex $X$, a *star point* is a 0-cell $v$ such that the closure of each component of $X - v$ is an arc, a subcomplex of $X$ called a *beam* (we do not require a beam to consist of a single edge). If a star point exists then $X$ is a *star graph*.

**Proposition 4.7.** Suppose that $D_{FF}(A) = 1$ and $|A| = 1$, equivalently $A = \{[A]\}$ has a realization of the form $\Gamma = A \ast Z$ where $Z = \langle z \rangle$ is infinite cyclic. In this case $\mathcal{FS}(\Gamma; A)$ is a star graph with star point $T$ such that each beam has the form $T \prec S \succ R$ with quotient graphs of groups as follows (assuming natural cell structures):

Loop type: $T/\Gamma$ has one vertex labelled $A$ and one edge forming a loop with both endpoints at the vertex.
Sewing needle type: $S/\Gamma$ has two vertices, one labelled $A$ and the other of valence 3 labelled with the trivial group, with one edge connecting the $A$ vertex to the valence 3 vertex, and one edge forming a loop with both ends at the valence 3 vertex.

Edge type: There exists a realization $\Gamma = A * Z$ of $A$ such that $R/\Gamma$ has two vertices, one labelled $A$ and the other labelled by the infinite cyclic group $Z$, and one edge connecting the two vertices.

Furthermore, under the monomorphism $A \hookrightarrow \text{Out}(\Gamma; A)$ given by $a \mapsto \phi_a$ described above, the induced action $A \curvearrowright \mathcal{FS}(\Gamma; A)$, is free and transitive on the set of beams, allowing beams to be enumerated as follows:

- Every free factorization of the form $\Gamma = A * Z'$ satisfies $Z' = \langle za \rangle$ for a unique $a \in A$.
- There are bijections: \{beams of $\mathcal{FS}(\Gamma; A)$\} $\leftrightarrow$ \{edge-type free splittings rel $A$\} $\leftrightarrow$ \{free factorizations $\Gamma = A * \langle za \rangle$, $a \in A$\} $\leftrightarrow$ $A$.

Since $A$ is nontrivial, there are at least two beams and the diameter of $\mathcal{FS}(\Gamma; A)$ equals 4.

Remark. As was the case for Proposition 4.6, the proof of Proposition 4.7 is an elaboration upon the proof of Corollary 3.2.2 of [BFH00] which is concerned with the case that $\Gamma$ is free of some finite rank $n$ and $A$ is free of rank $n - 1$.

Proof. The proof uses Bass-Serre theory [SW79] and the Bestvina–Feighn classification of folds [BF91] that was reviewed earlier.

For any free splitting $\Gamma \curvearrowright U$ representing a 0-simplex of $\mathcal{FS}(\Gamma; A)$, the free factor system $\mathcal{F}(U)$ satisfies either $\mathcal{F}(U) = A$, or $\mathcal{F}(U) = A \cup \{[Z]\}$ for some free factorization $\Gamma = A * Z$ with $Z$ infinite cyclic. It follows that $U$ has a unique vertex $v(U)$ such that $\text{Stab}(v(U)) = A$.

First we prove existence of a free splitting rel $A$ of loop type. From the hypotheses on $A$ it follows that there exists a free factorization $\Gamma = A * Z$ with $Z$ infinite cyclic, the Bass-Serre tree of which is an edge type free splitting $\Gamma \curvearrowright R$. Expanding $R$ by blowing up the $Z$ vertex of $R/\Gamma$ into a loop one gets a free splitting $\Gamma \curvearrowright S$ of sewing needle type. Collapsing the non-loop edge of $S/\Gamma$ one gets a free splitting of loop type.

Fix now a loop type free splitting $\Gamma \curvearrowright T$ rel $A$. Consider any free splitting $\Gamma \curvearrowright U$ rel $A$. Since $\mathcal{F}(T) \subseteq \mathcal{F}(U)$ and $T$ has one edge orbit it follows, as in the proof of Proposition 4.6, that there is a foldable map $f: T \to U$. Note that $f(v(T)) = v(U)$.

The derivative $d_{v(T)}: D_{v(T)}T \to D_{v(U)}U$ is either one-to-one or two-to-one.

In the first case where $d_{v(T)}$ is one-to-one, the map $f$ is a homeomorphism and $T \equiv U$, just as in the proof of Proposition 4.6.
In the second case where $d_v(T)$ is two-to-one, consider a fold sequence that factors the map $f$, given by $T = T_0 \xrightarrow{f_1} T_1 \rightarrow \cdots \xrightarrow{f_K} T_K = U$ with $K \geq 1$. Using the Bestvina-Feighn classification of fold types described earlier, the folds in this sequence are as follows. If $K = 1$ then $f : T \rightarrow U$ is either of type IA and $U$ is of sewing needle type, or $f$ is of type IIIA and $U$ is of edge type. If $K \geq 2$ then each of $f_1, \ldots, f_{K-1}$ is of type IA, and each of $T_1, \ldots, T_{K-1}$ is of sewing needle type; the final fold $f_K$ is either of type IA and $T_K = U$ is also of sewing needle type, or $f_K$ is of type IIIA and $T_K = U$ is of edge type.

Note in particular that if $U$ is of loop type then $d_v(T)$ is not two-to-one, and so any foldable map $T \rightarrow U$ is a homeomorphism and $T \equiv U$, so there is a unique loop-type 0-cell in $FS(F_n; A)$.

We have proved that each 0-cell in $FS(F_n; A)$ is represented by a free splitting of one of the three types described. Each 0-cell of sewing needle type collapses to exactly two other 0-cells, namely the unique one of loop type and one other of edge type. Each 0-cell of edge type expands to exactly one other 0-cell, that being of sewing needle type. This proves that the unique loop type 0-cell is a star point and each beam is as described.

To prove the “Furthermore” clause, by applying Proposition 4.6 to any free factor system of the form $\{[A], [Z']\}$ where $Z'$ is a cofactor of $A$, it follows that there is an $Out(\Gamma; A)$-equivariant bijection between the set of edge-type free splittings relative to $A$ and the set of conjugacy classes of cofactors of realizations of $A$. Each realization of $A$ is conjugate in $\Gamma$ to one of the form $\Gamma = A * Z'$. It therefore suffices to show that each realization of the latter form is conjugate to a unique one of the form $A * \langle za \rangle$, $a \in A$. Uniqueness follows from the observation that $\langle za \rangle$ is conjugate to $\langle zb \rangle$ if and only if $a = b$. To prove existence, pick a generator $Z' = \langle z' \rangle$. The two free factorizations $\Gamma = A * \langle z \rangle = A * \langle z' \rangle$ determine two loop type free splittings rel $A$, namely the Bass-Serre trees of the two HNN extensions of $A$ over the trivial group, one with stable letter $z$ and the other with stable letter $z'$. But we proved above that any two loop type free splittings rel $A$ are equivalent, and it follows that $z' = b z^{\pm 1} c$ for some $b, c \in A$. After possibly replacing $z'$ with its inverse we have $z' = b c$, which is conjugate to $z c b$, and taking $a = c b$ we are done. 

\[\diamond\]

### 4.3 Combing

Consider a foldable map $f : S \rightarrow T$ of free splittings of $\Gamma$ rel $A$. Given a $\Gamma$-invariant subgraph $\sigma_T \subset T$, a component of $\sigma_T$ is is degenerate if it consists of a single point. Assuming that $\sigma_T$ has no degenerate component, the pullback of $\sigma_T$ is the subgraph $\sigma_S$ obtained from $f^{-1}(\sigma_T)$ by removing degenerate components.

Following [HM13] Section 4.1 (but using the current definition of foldable sequences), a combing rectangle in $FS(\Gamma)$ is defined to be a commutative diagram of free splittings
of $\Gamma$ of the form

\[
\begin{array}{cccccccc}
S_0 & f_1 & \cdots & f_{i-1} & S_{i-1} & f_i & \cdots & f_{K-1} & S_K \\
[\sigma_0] & \pi_0 & [\sigma_{i-1}] & [\pi_{i-1}] & [\sigma_i] & [\pi_i] & [\sigma_K] & [\pi_K] \\
T_0 & g_1 & \cdots & g_{i-1} & T_{i-1} & g_i & T_i & g_{i+1} & \cdots & g_K & T_K
\end{array}
\]

where the top and bottom rows are foldable sequences, each vertical arrow $\pi_k: S_k \to T_k$ is a collapse map with indicated collapse forest $\sigma_k$, and each $\sigma_k$ is obtained from $(f^k_i)^{-1}(\sigma_K)$ by removing any components that degenerate to a point; we say that $\sigma_k$ is the pullback of $\sigma_K$ under the map $f^k_i$. If $A$ is a free factor system of $\Gamma$ and each $S_k, T_k$ is in $FS(\Gamma; A)$ then we also say this is a combing rectangle in $FS(\Gamma; A)$.

Denoting a combing rectangle in shorthand as $(S_i; T_i)_{0 \leq i \leq K}$, two combing rectangles $(S'_i; T'_i)_{0 \leq i \leq K'}$ and $(S''_i; T''_i)_{0 \leq i \leq K''}$ are equivalent if $K = K'$ and if there are equivariant homeomorphisms $S_i \leftrightarrow S'_i$ and $T_i \leftrightarrow T'_i$ making all resulting squares commute.

**Lemma 4.8** (Relative combing by collapse, cf. [HM13] Proposition 4.3). For any combing rectangle, if its top row is in $FS(\Gamma; A)$ then so is its bottom row. For any foldable sequence $S_0 \mapsto \cdots \mapsto S_K$ and any collapse $\pi_K: S_K \to T_K$ in $FS(\Gamma; A)$, there exists a combing rectangle with that top row and right edge, and that combing rectangle is unique up to equivalence.

*Proof.* The first sentence follows from Lemma 4.1. The existence statement in second sentence is proved in the case $\Gamma = F_n, A = \emptyset$ in [HM13] Proposition 4.3, that proof works without change to prove existence in our present setting, and the proof also gives uniqueness. In outline: define $\sigma_i \subset S_i$ uniquely as required by the definition; use $\sigma_i$ to uniquely define the collapse map $S_i \xrightarrow{[\sigma_i]} T_i$; check that there is a well-defined induced map $g_i: T_{i-1} \to T_i$ which uniquely defines the bottom row; and then check that the bottom row is a foldable sequence. \( \Diamond \)

**Lemma 4.9** (Relative combing by expansion, cf. [HM13] Proposition 4.4). For any foldable sequence $T_0 \mapsto \cdots \mapsto T_K$ and any collapse map $\pi_K: S_K \to T_K$ in $FS(\Gamma; A)$ there exists a combing rectangle in $FS(\Gamma; A)$ with that bottom row and right edge, and that combing rectangle is unique up to equivalence.

*Proof.* The existence proof in the case $\Gamma = F_n, A = \emptyset$ is found in [HM13], Proposition 4.4, “Step 1” and “Preparation for Step 2” (the work in Step 2 of that proof is entirely concerned with establishing the gate 3 condition for the $S$ row, and so is not relevant to us here). Following that proof, consider the fiber product of the two free splittings $\Gamma \times T_s, S_K$ with respect to the two maps $T_i \mapsto T_K, S_K \mapsto T_K$, namely the subset of the Cartesian product $T_i \times S_K$ consisting of ordered pairs $(x, y)$ such that the image
of $x$ in $T_K$ equals the image of $y$ in $T_K$. This fiber product is a simplicial tree on which $\Gamma$ acts with trivial edge stabilizers; take $S_k$ to be the minimal subtree for that action. The two projection maps of the Cartesian product induce maps $\pi_i: S_i \to T_i$ and $h_i^K: S_i \to S_K$. Exactly as in “Step 1”, the map $\pi_i$ is a collapse map which collapses a subforest $\sigma_i \subset S_i$, and $\sigma_i$ is the set of nondegenerate components of $(h_i^K)^{-1}(\sigma_K)$. And exactly as in “Preparation for Step 2”, the map $h_i^K$ is injective on edgelets and has $\geq 2$ gates at each vertex, and so $h_i^K$ is foldable according to our current definition. We thus have a combing diagram in $FS(\Gamma)$, and we need to check that $S_i \in FS(F_n; A)$. For each subgroup $A < \Gamma$ such that $[A] \in A$, since $A$ fixes unique points of $T_k$ and of $S_K$ it follows that $A$ fixes a unique point of the fiber product tree; since $A$ is nontrivial, that fixed point is in the minimal subtree $S_i$, and so $S_i \in FS(F_n; A)$.

Uniqueness follows by noticing that for any combing rectangle, the maps $\pi_i: S_i \to T_i$ and $f_i^K: S_i \to S_K$ embed $S_i$ in the fiber product tree of the two maps $T_i \leftrightarrow T_K$ and $S_K \leftrightarrow T_K$. Since the action of $\Gamma$ on $S_i$ is minimal it follows that $S_i$ is identified with the minimal subtree of the fiber product tree, and under this identification the maps $\pi_i, f_i^K$ are identified with the restrictions of the projection maps of the Cartesian product. The desired uniqueness property is an immediate consequence. ♦

4.4 Invariant subgraphs of free splittings, and their complexity

This section is concerned with an important technical underpinning of the proof of hyperbolicity. The key idea is that as one moves along a fold path, one studies “pullback subgraph sequences” along that path, meaning a sequence of $\Gamma$-equivariant subforests, one in each free splitting along that fold path, such that the sequence is invariant under pullback of the fold maps along that path. We focus on how the topology of the subforest varies along the sequence, and we use numerical measurements of “complexity” to measure this change of topology. These subgraphs are just forests, of course: the only aspects of their topology that concerns us are their component sets; and the maps on component sets induced by foldable maps will be the only aspects of change of topology that we consider.

The way the results of this section will be applied in what follows is to use upper and lower bounds on the change of complexity along fold paths to obtain information about upper and lower bounds on distance in $FS(\Gamma; A)$ along folds paths; see the discussion just below regarding the definition of complexity.

4.4.1 Definition of complexity.

Consider a free factor system $A$ of $\Gamma$, a free splitting $\Gamma \curvearrowright T$ relative to $A$, and a $\Gamma$-invariant proper subgraph $\beta \subset T$ with no degenerate components. We shall define a positive integer valued complexity denoted $C(\beta)$ which is a sum of several terms. This
complexity $C(\beta)$ will be dominated by a single term $C_1(\beta)$, equal to the number of $\Gamma$-orbits of components of $\beta$: indeed the difference $C(\beta) - C_1(\beta)$ is bounded above and below by constants depending only on $\Gamma$ and $A$, as we shall see in Lemma 4.10.

The definition of complexity is designed so that various upper and lower bounds on $C(\beta)$ can be used to obtain topological and metric conclusions. The most important of these conclusions are as follows:

- From upper bounds on complexity we obtain upper bounds on diameters along fold paths: see Lemma 4.13 (3a) and Lemma 4.14, and applications of those lemmas in later sections. Underlying these diameter bounds is the key technical result Lemma 4.12. The terms forming the difference $C(\beta) - C_1(\beta)$ are designed specifically to make Lemma 4.12 work.

- From lower bounds on $C(\beta)$ we obtain lower bounds on $C_1(\beta)$, from which we deduce that some component of $\beta$ is an arc in the interior of a natural edge of $T$: see Lemma 4.11 and Fact 4.16 (5b). Ultimately this leads to lower bounds on diameter along fold paths, as expressed in Theorem 5.4.

One may formally view the proof of hyperbolicity of $FS(\Gamma; A)$ as a game in which upper and lower bounds on complexity are played against each other, to obtain various upper and lower bounds on distance as needed for proving hyperbolicity.

The complexity $C(\beta)$ is defined by adding four non-negative integer summands:

$$C(\beta) = C_1(\beta) + C_2(\beta) + C_3(\beta) + C_4(\beta)$$

These summands are each tailored to cases in the proof of Lemma 4.12. For defining them, recall the free splitting $T/\beta$ obtained from $T$ by collapsing to a point each component of $\beta$. The free factor system $F(T/\beta)$ decomposes into two subsets $F(T/\beta) = F(\beta) \sqcup F(T - \beta)$ as follows: given $[B] \in F(T/\beta)$, put $[B]$ in $F(\beta)$ if $B$ stabilizes some component of $\beta$, and put $[B]$ in $F(T - \beta)$ if $B$ stabilizes some point of $T - \beta$.

- Define $C_1(\beta)$ to be the number of $\Gamma$-orbits of components of $\beta$.
- Define $C_2(\beta) = D_{FF}(F(T/\beta)) - 1$.
- Define $C_3(\beta)$ to be the number of components $[A] \in A$ satisfying the following: the component of $F(T/\beta)$ containing $[A]$ is in the set $F(T - \beta)$; equivalently $[A]$ stabilizes some vertex of $T - \beta$.

For defining $C_4(\beta)$, first apply Lemma 2.11 using $A$ and $A' = F(T/\beta)$, with the following conclusions: letting $[B_1], \ldots, [B_N]$ be those components of $F(T/\beta)$ that do not contain any component of $A$, each of $B_1, \ldots, B_N$ is free of finite rank and their free product $B_1 \ast \cdots \ast B_N$ is a free factor of a cofactor of a realization of $A$ (in the notation...
of Lemma 2.11 these components are $[A'_{J+1}], \ldots, [A'_{K}]$. Up to re-indexing there exists $M \in \{0, \ldots, N\}$ so that $[B_1], \ldots, [B_M] \in \mathcal{F}(T - \beta)$ and $[B_{M+1}], \ldots, [B_N] \in \mathcal{F}(\beta)$. Thus $[B_1], \ldots, [B_M]$ are precisely the components of $\mathcal{F}(T/\beta)$ that do not contain a component of $\mathcal{A}$ and whose representative subgroups $B_1, \ldots, B_M$ each fix some point of $T - \beta$. Since each $B_m$ is free of finite rank, it follows that the set $\mathcal{F}(T/\beta) - \{[B_1], \ldots, [B_M]\}$ is still a free factor system, and it is still true that $\mathcal{A} \sqsubseteq \mathcal{F}(T/\beta) - \{[B_1], \ldots, [B_M]\}$. Define

$$C_4(\beta) = \text{corank}\left(\mathcal{F}(T/\beta) - \{[B_1], \ldots, [B_M]\}\right) = \text{corank}(\mathcal{F}(T/\beta)) + \sum_{m=1}^{M} \text{rank}(B_m)$$

We record for later use some estimates that are evident from the definitions:

**Lemma 4.10.** The three summands $C_2(\beta), C_3(\beta), C_4(\beta)$ have the following upper bounds:

$$C_2(\beta) \leq 2 \text{corank}(\mathcal{A}) + |\mathcal{A}| - 1 \quad \text{(by Lemma 2.14 (1))}$$

$$C_3(\beta) \leq |\mathcal{A}|$$

$$C_4(\beta) \leq \text{corank}(\mathcal{A}) \quad \text{(by Corollary 2.10)} \quad \Diamond$$

### 4.4.2 Consequence of a lower bound on complexity.

Lemma 4.11 to follow gives a very simple topological consequence for a certain constant lower bound on the subgraph complexity. Further consequences of that lower bound are derived later in Proposition 4.16 (5b), and those consequences will play an important role in the central arguments of Section 5, particularly in the statement and proof of Proposition 5.3 where that constant is denoted $b_1 = 5 \text{corank}(\mathcal{A}) + 4 |\mathcal{A}| - 3$.

**Lemma 4.11.** For any free splitting $\Gamma \searrow T$ rel $\mathcal{A}$, and for any $\Gamma$-invariant subgraph $\beta \subset T$, if $C(\beta) > 5 \text{corank}(\mathcal{A}) + 4 |\mathcal{A}| - 3$ then some component of $\beta$ is an arc contained in the interior of a relatively natural edge of $T$.

**Proof.** Combining the hypothesis with Lemma 4.10 it follows that

$$C_1(\beta) > 2 \text{corank}(\mathcal{A}) + 2 |\mathcal{A}| - 2$$

The right hand side is the maximal number of relatively natural vertex orbits amongst all free splittings of $\Gamma$ rel $\mathcal{A}$, according to Proposition 3.4 (3). So $\beta$ has more than that number of component orbits, and hence one of those orbits must be disjoint from the relatively natural vertices of $T$. Each component in that orbit is therefore contained in the interior of some relatively natural edge. \quad \Diamond
4.4.3 Complexity change under a foldable map.

We now turn to a study of subgraph complexity along a foldable sequence, starting with its behavior under a single foldable map.

**Lemma 4.12.** (c.f. [HM13] Sublemma 5.3) Let \( S, T \) be free splittings of \( \Gamma \) rel \( A \), let \( f : S \to T \) be a foldable map, let \( \beta_T \subset T \) be a proper \( \Gamma \)-invariant subgraph, and let \( \beta_S \subset T \) be the pullback of \( \beta_T \) under \( f \). Then we have \( C_i(\beta_S) \geq C_i(\beta_T) \) for \( i = 1, 2, 3, 4 \), and hence \( C(\beta_S) \geq C(\beta_T) \). Furthermore, if \( C(\beta_S) = C(\beta_T) \) then \( f \) induces a bijection between the set of components of \( \beta_S \) and the set of components of \( \beta_T \).

**Proof.** The \( \Gamma \)-equivariant surjection \( f : \beta_S \to \beta_T \) induces a \( \Gamma \)-equivariant surjection of component sets \( f_* : \pi_0(\beta_S) \to \pi_0(\beta_T) \) which induces in turn a surjection of component orbit sets \( f_{**} : \pi_0(\beta_S)/\Gamma \to \pi_0(\beta_T)/\Gamma \). It follows that \( C_i(\beta_S) \geq C_i(\beta_T) \). By Lemma 4.8 the foldable map \( f : S \to T \) induces a foldable map \( f/\beta : S/\beta_S \to T/\beta_S \) and hence \( F(S/\beta_S) \subset F(T/\beta_T) \). Applying Lemma 2.14 (1) it follows that \( C_2(\beta_S) \geq C_2(\beta_T) \).

To prove the inequality \( C_3(\beta_S) \geq C_3(\beta_T) \), we use the fact that the composition of containment functions \( A \to F(S/\beta_S) \to F(T/\beta_T) \) is the containment function \( A \to F(T/\beta_T) \). It follows that for each component \([A]\) of \( A \), if the component of \( F(S/\beta_S) \) containing \([A]\) is in \( F(\beta_S) \) then the component of \( F(T/\beta_T) \) containing \([A]\) is in \( F(\beta_T) \). The inequality \( C_3(\beta_S) \geq C_3(\beta_T) \) follows. Furthermore, for the equation \( C_3(\beta_S) = C_3(\beta_T) \) to hold is equivalent to saying that for each \([A]\) \( \in A \), \([A]\) is contained in \( F(S/\beta_S) \) if and only if \([A]\) is contained in \( F(T/\beta_T) \).

We next prove the inequality \( C_4(\beta_S) \geq C_4(\beta_T) \). Consider nested components \([B] \subset [B']\) of \( F(\beta_S) \subset F(\beta_T) \), respectively. Note that if \( B' \) stabilizes a point of \( T - \beta_T \) then \( B \) stabilizes a point of \( S - \beta_S \), and so if \([B'] \in F(T - \beta_T)\) then \([B] \in F(S - \beta_S)\). Let \([B_1], \ldots, [B_M]\) be the components of \( F(S - \beta_S) \) not containing a component of \( A \), and let \([B'_1], \ldots, [B'_{M'}]\) be the components of \( F(T - \beta_T) \) not containing a component of \( A \). It follows that we have an extension of free factor systems to which we apply Lemma 2.14 (1):

\[
F(S/\beta_S) - \{[B_1], \ldots, [B_M]\} \subset F(T/\beta_T) - \{[B'_1], \ldots, [B'_{M'}]\}
\]

with equality holding in (3.2) if and only if it holds in (3.3).

Assuming that \( C(\beta_S) = C(\beta_T) \), and so \( C_i(\beta_S) = C_i(\beta_T) \) for \( i = 1, 2, 3, 4 \), it remains to prove that \( f_* \) is a bijection. From surjectivity of \( f : \beta_S \to \beta_T \) it follows that \( f_* \) is also surjective, and what is left is to show that \( f_* \) is injective. Consider a component...
$b'$ of $\beta_T$; we must prove that there is exactly one nondegenerate component of $f^{-1}(b')$. Since $C_1(\beta_S) = C_1(\beta_T)$ it follows that $f_{ss}$ is a bijection, and so all of the nondegenerate components of $f^{-1}(b')$ are in the same $\Gamma$-orbit. If $f^{-1}(b')$ has more than one nondegenerate component then any element of $\gamma$ taking one to the other is a nontrivial element of $\text{Stab}(b')$; therefore if $\text{Stab}(b')$ is trivial then $f^{-1}(b')$ has only one nondegenerate component and we are done.

We have reduced to the case that $\text{Stab}(b')$ is nontrivial; by definition we have that $[\text{Stab}(b')] \in \mathcal{F}(\beta_T)$. Since $D_{FF}(\mathcal{F}(S/\beta_S)) + 1 = C_2(\beta_S) = C_2(\beta_T) = D_{FF}(\mathcal{F}(T/\beta_T)) = 1$, and since $\mathcal{F}(S/\beta_S) \subseteq \mathcal{F}(T/\beta_T)$, by applying Lemma 2.14 (1) we have:

$$(*) \quad \mathcal{F}(S/\beta_S) = \mathcal{F}(T/\beta_T)$$

Consider the subcase that some nondegenerate component $b$ of $f^{-1}(b')$ has nontrivial stabilizer. Since $\text{Stab}(b) < \text{Stab}(b')$, it follows from $(*)$ that $\text{Stab}(b) = \text{Stab}(b')$. But since all nondegenerate components of $f^{-1}(b')$ are in the same orbit, $b$ must be the only such component, because otherwise any $\gamma \in \Gamma$ taking $b$ to a different nondegenerate component is an element of $\text{Stab}(b')$ but not of $\text{Stab}(b)$. The proof is therefore complete in this subcase.

We have further reduced to the subcase that all nondegenerate components of $f^{-1}(b')$ have trivial stabilizer, and in this subcase we shall derive a contradiction. It follows from $(*)$ that there exists $x \in S - \beta_S$ such that $\text{Stab}(x) = \text{Stab}(b') = H < \Gamma$. By definition we have $[H] = [\text{Stab}(x)] \in \mathcal{F}(S - \beta_S)$ whereas $[H] = [\text{Stab}(b')] \in \mathcal{F}(\beta_T)$. We now break into two cases, depending on whether $[H]$ contains some element of $A$.

Suppose first that $[H]$ contains some $[A] \in A$, and so up to conjugacy we have $A < H$. Since $C_3(\beta_S) = C_3(\beta_T)$, it follows that $A$ stabilizes a point of $S - \beta_S$ if and only if $A$ stabilizes a point of $T - \beta_T$ if and only if $A$ does not stabilize any component of $\beta_T$. But $A$ stabilizes the point $x$ of $S - \beta_S$ and the component $b'$ of $\beta_T$, a contradiction.

Suppose next that $[H]$ contains no element of $A$. In the notation of (3.1), up to conjugacy we have $H = \text{Stab}(x) = B_m$ for some $m = 1, \ldots, M$ and so $[H]$ is not an element of the left hand side of (3.1), although $[H] = [\text{Stab}(b')]$ is an element of the right hand side. By Lemma 2.11 applied to the extension in (3.1), it follows that the free factor system on the left hand side of (3.1) has a realization with a cofactor $B$ that freely factors into two or more nontrivial terms, one term up to conjugacy being $H = B_m$ (one of the terms denoted $A'_{f+1}, \ldots, A'_K$ in Lemma 2.11), and another term being a cofactor $B'$ for a realization of the free factor system on the right hand side. From this we obtain $C_4(\beta_S) = \text{rank}(B) \geq \text{rank}(B') + \text{rank}(B_m) > \text{rank}(B') = C_4(\beta_T)$, contradicting that $C_4(\beta_S) = C_4(\beta_T)$.

\[\Diamond\]
4.4.4 Complexity change along a foldable sequence.

Given a foldable sequence \( T_0 \xrightarrow{f_1} T_1 \xrightarrow{f_2} \cdots \xrightarrow{f_K} T_K \), a pullback subgraph sequence is a sequence of \( \Gamma \)-invariant subgraphs \( \beta_k \subset T_k \), each with no degenerate components, such that for each \( k \in \{0, \ldots, K\} \) the graph \( \beta_k \) is the pullback of \( \beta_K \) via \( f_K \). Equivalently for \( 0 \leq i \leq j \leq K \) the graph \( \beta_i \) is the pullback of \( \beta_j \) via \( f_j^i \). For example, the sequence of subgraphs occurring in a combing rectangle (see Section 4.3) is a pullback subgraph sequence. Note that the complementary sequence \( \rho_k = \text{cl}(T_k - \beta_k) \subset T_k \) is also a pullback subgraph sequence, and the subgraphs \( \beta_k, \rho_k \) decompose the tree \( T_k \) in the sense that every edgelet of \( T_k \) is in either \( \beta_k \) or \( \rho_k \); such a sequence of decompositions \( T_k = \beta_k \cup \rho_k \) is called a pullback blue–red decomposition.

The next lemma uses upper bounds on complexity to derive diameter bounds along fold sequences.

**Lemma 4.13** (cf. [HM13] Lemma 5.2). Given a pullback subgraph sequence of a foldable sequence of free splittings of \( \Gamma \) rel \( A \), as denoted above, the following holds:

1. The quantities \( C_1(\beta), C_2(\beta), C_3(\beta), C_4(\beta) \), and \( C(\beta) \) are all nonincreasing as functions of \( k \).
2. Equality \( C(\beta_{k-1}) = C(\beta_k) \) holds if and only if \( f_k : T_{k-1} \to T_k \) restricts to a bijection from components of \( \beta_{k-1} \) to components of \( \beta_k \).
3. On any subinterval \( a \leq k \leq b \) along which \( C(\beta_k) \) is constant we have:
   
   (a) The diameter of \( \{T_a, \ldots, T_b\} \) in \( FS'(\Gamma; A) \) is at most 4.
   
   (b) \( C_1(\beta_a) \leq C_1(\beta_b) + (3 \text{corank}(A) + 2 |A| - 1) \)

**Proof.** Items (1) and (2) follow from Lemma 4.12. Item (3b) is an immediate consequence of Lemma 4.10 combined with the equation \( C(\beta_a) = C(\beta_b) \) and the monotonicity of \( C_2(\beta_i), C_3(\beta_i) \) and \( C_4(\beta_i) \).

To prove (3a), first apply item (2) to conclude that each \( f_k \) induces a bijection from the component set of \( \beta_{k-1} \) to the component set of \( \beta_k \). Now apply the proof of [HM13] Lemma 5.2 (3) with little change; here is an outline. Given \( a \leq i \leq j \leq b \), there exists \( 0 \leq P \leq Q \), a fold sequence \( T_i = U_0 \leftrightarrow \cdots \leftrightarrow U_P \leftrightarrow \cdots \leftrightarrow U_Q = T_j \), free splittings \( X, Y \), and a collapse expand sequence \( T_i = U_0 \succ X \prec U_P \succ Y \prec U_Q = T_j \). To see why, the first part of the fold sequence \( U_0 \leftrightarrow \cdots \leftrightarrow U_P \) is done by folding only blue edgelet pairs until the induced foldable map \( U_P \leftrightarrow T_j \) is injective on blue edgelets. This is possible because \( f_j : T_i \to T_j \) induces a bijection from components of \( \beta_i \) to components of \( \beta_j \), and so as one applies the construction of a fold factorization of \( f_j^i \), as long as the induced foldable map to \( T_j \) is not yet injective there must exist a blue edgelet pair in some component of \( \beta_j \) which is ready to be folded, by virtue of having a common endpoint...
and having the same image in $T_j$. The second part of the fold sequence from $U_P$ to $U_Q$ may then be done by folding only red edgelet pairs. There is a single free splitting $\Gamma \searrow X$ obtained by collapsing all blue edgelets of $U_0$ or of $U_P$, and a single free splitting $\Gamma \searrow Y$ obtained by collapsing all red edgelets of $U_P$ or of $U_Q$. Applying Lemma 4.1 and using that $T_i \in \mathcal{FS}(\Gamma; A)$, it follows, in order, that $X, U_P, Y \in \mathcal{FS}(\Gamma; A)$, and hence $d(T_i, T_j) \leq 4$.

The construction in the following lemma is adapted from an argument of Bestvina and Feighn, namely Lemma 4.1 of [BF12], and translated into the language of complexity. In the context of $\mathcal{FS}(F_n)$ this construction has the simplifying effect of enfolding two upper bounds on distance from [HM13]—the “almost invariant edge bound” and the “blue–red decomposition bound”—into a single distance bound. In the current context, Lemma 4.14 will be used in the proof of Lemma 5.2 to get certain upper bounds to distance along fold paths.

Lemma 4.14. For any foldable map $f : S \to T$ of free splittings of $\Gamma$ rel $A$, and for any point $x \in T$ in the interior of some edgelet, there is a proper, $\Gamma$-invariant subgraph $\beta_T \subset T$ with pullback subgraph $\beta_S \subset S$ such that

$$C(\beta_S) \leq |f^{-1}(x)| + (3 \operatorname{corank}(A) + 2 |A| - 1)$$

Proof. Let $e$ be the edgelet whose interior contains $x$, so $f^{-1}(e)$ is a union of $|f^{-1}(x)|$ edgelets, and by subdividing further we may assume that this is a disjoint union. Letting $\beta_T = \Gamma \cdot e$, it follows that $C_1(\beta_S) = |f^{-1}(x)|$, and the conclusion follows immediately from Lemma 4.10.

4.5 Free splitting units

In [HM13] we defined free splitting units along fold paths of $\mathcal{FS}(F_n)$, and showed that they give a uniformly quasigeodesic parameterization of fold paths in $\mathcal{FS}(F_n)$ (see Theorem 5.4). We shall do the same here in the relative setting of $\mathcal{FS}(\Gamma; A)$.

The manner in which free splitting units are applied in this paper is the same as in [HM13]: any argument that bounds distance gives more information, by bounding free splitting units. One can think of free splitting units as way of taking distance estimates that are buried in the details of various proofs, bringing those estimates to the surface, and using them to give explicit quasigeodesic parameterizations of fold paths, as will be done in Theorem 5.4.

The manner in which free splitting units are defined in this paper is a little different than in [HM13], with influences from [BF12]. The definition of free splitting units along fold paths in $\mathcal{FS}(F_n)$, as given originally in [HM13] Section 5.2, involves two bounds: a “blue–red” distance bound (c.f. Lemma 4.13(3a) in our present context); and an “almost
invariant edge” distance bound. As it turns out, the two bounds can be enfolded into a single, simpler bound, a fact which we overlooked in [HM13]. A hint of this can be seen in the simplifications of certain steps of the proof of hyperbolicity of $FS(F_n)$ that can be found in [BF12] and which are taken up in various places around the paper; see for example Lemma 4.14 and the preceding discussion. Taking this hint, we present here a simplified version of free splitting units, generalized to the setting of $FS(\Gamma;A)$.

This new definition of free splitting units seems to be more powerful and to have more applications. We will apply free splitting units in the companion paper [HM] to give a proof of Theorem 1.3 from the introduction, saying that elements of $Out(\Gamma;A)$ which act loxodromically on $FS(\Gamma;A)$ have stable translation length uniformly bounded away from zero.

**Definition 4.15.** Consider a fold sequence $S_0 \xrightarrow{f_1} S_1 \xrightarrow{f_2} \cdots \xrightarrow{f_K} S_K$ in $FS(\Gamma;A)$, and consider $0 \leq i \leq j \leq K$.

1. Define a collapse–expand diagram rel $A$ over $S_i \xrightarrow{f_i+1} S_j$ to be a commutative diagram of the form

$$
\begin{array}{cccccccc}
T_i & \rightarrow & T_{i+1} & \rightarrow & \cdots & \rightarrow & T_{j-1} & \rightarrow & T_j \\
\downarrow & & \downarrow & & \uparrow & & \downarrow & & \uparrow \\
S_i' & \rightarrow & S_{i+1}' & \rightarrow & \cdots & \rightarrow & S_{j-1}' & \rightarrow & S_j' \\
\downarrow & & \downarrow & & \uparrow & & \downarrow & & \uparrow \\
S_i & \xrightarrow{f_{i+1}} & S_{i+1} & \xrightarrow{f_{i+2}} & \cdots & \xrightarrow{f_{j-1}} & S_{j-1} & \xrightarrow{f_j} & S_j \\
\end{array}
$$

where each horizontal row is a foldable sequence rel $A$ and each of the two rectangles shown is a combing rectangle rel $A$. The diagram is trivial if all vertical arrows are simplicial isomorphisms.

2. We say that $S_i, S_j$ differ by $< 1$ free splitting unit rel $A$ if there exists a collapse expand diagram over $S_i \xrightarrow{f_i+1} S_j$, denoted as above, such that on the top row $T_i \rightarrow \cdots \rightarrow T_j$ there exists a pullback subgraph sequence $\beta_k \subset T_k$ of constant complexity $C(\beta_k)$.

3. More generally, the number of free splitting units rel $A$ between $S_i$ and $S_j$ is the maximum length $\Upsilon = \Upsilon_{ij}$ of a subsequence $i \leq i(0) < \cdots < i(\Upsilon) \leq j$ such that for $u = 1, \ldots, \Upsilon$ the number of free splitting units between $S_{i(u-1)}$ and $S_{i(u)}$ is not $< 1$. Any such subsequence $i(0) < \cdots < i(\Upsilon)$ of $[i,j]$ is called a greedy sequence between $S_i$ and $S_j$ (while we do not require $i = i(0)$ and $i(\Upsilon) = j$, Proposition 4.16 (2) guarantees that such a greedy sequence exists).
(4) For $0 \leq i \leq j \leq K$, the **back greedy subsequence** between $i, j$ is the decreasing sequence $j = L_0 > L_1 > \cdots > L_U \geq i$ defined inductively as follows: if $L_u$ is defined, and if there exists $k$ with $i \leq k < u$ such that $L_k, L_u$ differ by $\geq 1$ free splitting unit rel $A$, then $L_{u+1}$ is the largest such value of $k$. The **front greedy subsequence** is the increasing sequence in $[0, K]$ defined similarly.

(5) We extend free splitting units to a symmetric function by requiring $\Upsilon_{ij} = \Upsilon_{ji}$.

The following summarizes basic properties of free splitting units. Of particular importance is item (5b), which derives from Lemma 4.11, and which plays a central role in the “big diagram argument”, the proof of Proposition 5.3.

**Proposition 4.16.** Consider a fold sequence $S_0 \overset{f_1}{\rightarrow} S_1 \overset{f_2}{\rightarrow} \cdots \overset{f_K}{\rightarrow} S_K$ in $FS(\Gamma; A)$, and let $\Upsilon_{ij}$ be the number of free splitting units rel $A$ between $S_i, S_j$. We have:

1. For $0 \leq i \leq j \leq K$ and $i', j' \in [i, j]$, if $S_i, S_j$ differ by $< 1$ free splitting unit then $S_{i'}, S_{j'}$ differ by $< 1$ free splitting unit.

2. (c.f. [HM13] after Definition 5.10) For any $0 \leq i \leq j \leq K$ there exists a greedy sequence between $S_i$ and $S_j$ with first term $i$ and with final term $j$. Also, the front and back greedy sequences between $S_i$ and $S_j$ are, indeed, greedy sequences, in particular they have length equal to $\Upsilon_{ij}$.

3. The “short triangle inequality” (c.f. [HM13] Lemma 5.12): For any $i, j, k \in \{0, \ldots, K\}$ we have

   $$\Upsilon_{ik} \leq \Upsilon_{ij} + \Upsilon_{jk} + 1$$

4. The “long triangle inequality”: For any sequence $0 \leq k_0 < k_1 < \cdots < k_L \leq K$ we have:

   $$\Upsilon_{k_0, k_1} + \cdots + \Upsilon_{k_{L-1}, k_L} \leq \Upsilon_{k_0, k_L} \leq \Upsilon_{k_0, k_1} + \cdots + \Upsilon_{k_{L-1}, k_L} + L - 1$$

5. For any collapse expand diagram as in Definition 4.15, and for any pullback sequence $\beta_k \subset T_k$ defined for $i \leq k \leq j$, we have:

   - (a) $\Upsilon_{ij} \leq C(\beta_i) - C(\beta_j)$
   - (b) If $\Upsilon_{ij} \geq 5\text{corank}(A) + 4|A| - 3$ then some component of $\beta_i$ is arc in the interior of a natural edge of $T_i$.

6. (c.f. [HM13] Lemma 5.11) The diameter of $\{S_i, \ldots, S_j\}$ is $\leq 10\Upsilon_{ij} + 8$. 

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Proof. Item (1) is an immediate consequence of Definition 4.15. Items (2), (3) follow exactly as in the references given above; their proofs are elementary.

To prove the first inequality of (4), for \( l = 1, \ldots, L \) apply (2) to obtain a subsequence of \([k_l-1, k_l]\) which is a greedy sequence between \(S_{k_l-1}\) and \(S_{k_l}\), which starts with \(k_l-1\), which ends with \(k_l\), and which has length \( \Upsilon_{k(l-1),k(l)} \). The union of these subsequences is a sequence of length equal to the sum \( \Upsilon_{k_0, k_1} + \cdots + \Upsilon_{k_{L-1}, k_L} \), because the subsequence of \([k_l-1, k_l]\) ends with \(k_l\) and the subsequence of \([k_l, k_{l+1}]\) begins with \(k_l\). Between any two terms of this sequence the number of free splitting units is \( \geq 1 \), so a greedy sequence between \(S_{k_0}\) and \(S_{k_L}\) has length no less than the sum.

The second inequality of (4) is a generalization of (3), and is proved as follows. If the second inequality is false then there exists a greedy sequence between \(S_{k_0}\) and \(S_{k_L}\) whose length is at least \( \Upsilon_{k_0, k_1} + \cdots + \Upsilon_{k_{L-1}, k_L} + L \). From the pigeonhole principle, for some \( l = 1, \ldots, L \) we obtain a greedy sequence between \(S_{k_l-1}\) and \(S_{k_l}\) of length \( \geq \Upsilon_{k_l-1, k_l} + 1 \), a contradiction.

Item (5b) follows from (5a) and Lemma 4.11 together with \(C(\beta_j) \geq 1\). To prove (5a), apply (2) to obtain a greedy sequence

\[ i = k(0) < k(1) < \cdots < k(\Upsilon_{ij}) = j \]

By Lemma 4.13(1) we can uniquely decompose the interval \([i, j] = [k(0), k(\Upsilon_{ij})]\) as a concatenation of \(M\) maximal subintervals on each of which \(C(\beta_k)\) is constant:

\[ [l(0), l(1)], [l(1)+1, l(2)], \ldots, [l(M-2)+1, l(M-1)], [l(M-1)+1, l(M)] \]

If \( \Upsilon_{ij} \geq C(\beta_i) - C(\beta_j) \) then, since \( C(\beta_i) - C(\beta_j) \geq M - 1 \), it follows \( \Upsilon_{ij} + 1 \geq M \), and so there exists \( u \in [1, \Upsilon_{ij}] \) and \( m \in [1, M] \) such that \( k(u-1), k(u) \in [l(m-1)+1, l(m)] \).

It follows further that \(C(\beta_k)\) is constant for \( k \in [k(u-1), k(u)] \), contradicting that there are \( \geq 1 \) free splitting units between \(S_{k(u-1)}\) and \(S_{k(u)}\).

Item (6) is proven just as in the reference given, except that one applies Lemma 4.4 and Lemma 4.13 (3a) in place of the analogous results of [HM13]: subdivide the interval \([i, j]\) into a concatenation of maximal subintervals on which \(C(\beta_k)\) is constant; apply Definition 4.15(1,2) and Lemma 4.13(3a) to obtain diameter \( \leq 8 \) over each subinterval; and apply Lemma 4.4 to obtain distance \( \leq 2 \) between incident endpoints of adjacent subintervals.

\[ \diamond \]

5 Hyperbolicity of relative free splitting complexes

In this section we prove hyperbolicity of the relative free splitting complex \(F\mathcal{S}(\Gamma; \mathcal{A})\) for any group \(\Gamma\) and any free factor system \(\mathcal{A}\). The proof uses the three Masur–Minsky
axioms for hyperbolicity of a connected simplicial complex, which are reviewed in Section 5.1, where one will also find specific details about how those axioms will be verified for $FS(\Gamma; A)$. Section 5.2 contains the proof of the first of those axioms, the Coarse Retract Axiom; this is the one place where we pay the piper for dropping the “gate 3 condition” on fold paths. Section 5.3 contains the statement of Proposition 5.3, which states that certain properties of fold maps and free splitting units together imply the two remaining Masur–Minsky axioms—the Coarse Lipschitz and the Strong Contraction Axioms. The proof of Theorem 1.1 is thereby reduced to Proposition 5.3. Section 5.4 also applies Proposition 5.3 to the proof of Theorem 5.4 which says that free splitting units give a quasigeodesic parameterization along a fold path. Section 5.5 contains the proof of Proposition 5.3, what we call the “Big Diagram” argument, an argument concerning the large scale behavior of certain diagrams of combing rectangles in $FS(F_n; A)$.

The structure of this proof of Theorem 1.1, in particular the Big Diagram argument, follows very closely the structure of the proof of hyperbolicity of $FS(F_n)$ given in [HM13]. But changing the definition of foldable maps by dropping the gate 3 condition has some major effects on this structure: the proof of the Coarse Retract Axiom is quite a bit more complex and so has needed to be rewritten from the beginning; and subtle changes in the Big Diagram Argument make it necessary to re-present it from the beginning. In both cases, we take up these changes from the version of the proof given by Bestvina and Feighn in [BF12].

5.1 The Masur–Minsky axioms.

Suppose one is given a connected, finite dimensional simplicial complex $X$ with the simplicial metric, a collection $P$ of finite “paths” $p: \{0, \ldots, L_p\} \rightarrow X^{(0)}$, and for each $p \in P$ a “projection map” $\pi_p: X^{(0)} \rightarrow \{0, \ldots, L_p\}$. Suppose that $P$ is almost transitive meaning that there is a constant $A$ with two properties: for each $p \in P$ and $\ell \in \{1, \ldots, L_p\}$ we have $d(p(\ell - 1), p(\ell)) \leq A$; and for each $x, y \in X^{(0)}$ there exists $p \in P$ such that $d(x, p(0)) \leq A$ and $d(p(L_p), y) \leq A$ (as noted in [HM13], “almost transitivity” yields equivalent axioms compared to “coarse transitivity” as used originally in [MM99]). Suppose furthermore that there are constants $a, b, c > 0$ such that the following three axioms hold for each $p \in P$:

Coarse Retract Axiom: For each $\ell \in \{0, \ldots, L_p\}$ the diameter of $p[\ell, \pi_p(p(\ell))]$ is at most $c$.

Coarse Lipschitz Axiom: For all $x, y \in X^{(0)}$, if $d(x, y) \leq 1$ then the diameter of $p[\pi_p(x), \pi_p(y)]$ is at most $c$.

Strong Contraction Axiom: For all $x, y \in X^{(0)}$, if $d(x, p[0, L_p]) \geq a$, and if $d(y, x) \leq b \cdot d(x, p[0, L_p])$, then the diameter of $p[\pi_p(x), \pi_p(y)]$ is at most $c$. 

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The theorem proved by Masur and Minsky [MM99] is that if these axioms hold then $X$ is hyperbolic.

For verifying these axioms and hence proving hyperbolicity of $\mathcal{FS}(\Gamma; A)$, we let $P$ be the collection of all fold sequences rel $A$, for which we have already established Coarse Transitivity in Corollary 4.5. In Section 5.2 we define the system of projection maps $\pi_p, p \in P$, and we prove the Coarse Retract Axiom (see Lemma 5.2). In Section 5.3 we shall reduce the Coarse Lipschitz and Strong Contraction Axioms to a single statement regarding fold sequences and free splitting units rel $A$ (see Proposition 5.3). In Section 5.4 we prove that fold paths are uniformly quasigeodesic when parameterized by free splitting units (see Theorem 5.4, which also uses Proposition 5.3). Finally in Section 5.5 we prove Proposition 5.3 using the “big diagram argument”.

5.2 Projection maps and the proof of the coarse retract axiom.

Given a fold path in $\mathcal{FS}(\Gamma; A)$ represented by a particular fold sequence, we now define the projection map to that fold path, as required for the formulation of the Masur–Minsky axioms. We then immediately turn to verification of the Coarse Retract Axiom, which takes up the bulk of this section.

**Definition 5.1.** Given a fold sequence $S_0 \mapsto \cdots \mapsto S_K$ and a free splitting $T$ each in $\mathcal{FS}(\Gamma; A)$, a projection diagram rel $A$ from $T$ to $S_0 \mapsto \cdots \mapsto S_K$ is defined to be a projection diagram as in [HM13] Section 4.1 in which all free splittings that occur are restricted to lie in $\mathcal{FS}(\Gamma; A)$. This means a commutative diagram of free splittings and maps of the form

$$
\begin{array}{ccc}
T_0 & \cdots & T_J & \cdots & T \\
\downarrow & & & & \downarrow \\
S'_0 & \cdots & S'_J & \cdots & S' \\
\downarrow & & & & \downarrow \\
S_0 & \cdots & S_J & \cdots & S_K
\end{array}
$$

such that each free splitting is in $\mathcal{FS}(\Gamma; A)$, each row is a foldable sequence, and each of the two rectangles shown is a combing rectangle. The integer $J \in \{0, \ldots, K\}$ is called the depth of the projection diagram. The projection of $T$ to $S_0 \mapsto \cdots \mapsto S_K$ is an integer $\pi(T) \in \{0, \ldots, K\}$ defined as follows: if there exists a projection diagram rel $A$ from $T$ to $S_0 \mapsto \cdots \mapsto S_K$ then $\pi(T)$ is the maximal depth of such diagrams; otherwise $\pi(T) = 0$.

**Lemma 5.2** (The Coarse Retract Axiom). For any fold sequence $S_0 \mapsto \cdots \mapsto S_K$ in $\mathcal{FS}(\Gamma; A)$ and any $0 \leq I \leq K$ the number of free splitting units between $S_I$ and $S_{\pi(S_I)}$, and the diameter of the fold sequence between $S_I$ and $S_{\pi(S_I)}$, are both bounded above by constants depending only on $\text{corank}(A)$ and $|A|$.
By assuming the gate 3 condition, the proof of the Coarse Retract Axiom in [HM13] was significantly simpler than the argument to be presented here. In lieu of that assumption, we instead adapt some concepts and arguments of Bestvina and Feighn from [BF12], namely the “hanging trees” of Proposition A.9; see “Claim (♯)” below.

Proof. The proof starts as in [HM13]. Note that $\pi(S_I) \geq I$, because there exists a projection diagram rel $\mathcal{A}$ from $S_I$ to $S_0, \ldots, S_K$ of depth $I$, namely the trivial diagram defined by taking $T_i = S'_i = S_i$ for $i = 0, \ldots, I$.

Choose a projection diagram rel $\mathcal{A}$ of maximal depth $J = \pi(S_I) \geq I$ from $S_I$ to $S_0 \mapsto \cdots \mapsto S_K$, as follows:

\[
\begin{array}{cccccccc}
T_0 & \rightarrow & \cdots & \rightarrow & T_I & \rightarrow & \cdots & \rightarrow & T_J & \rightarrow & S_I \\
& \downarrow & & & \downarrow & & & \downarrow & & \\
S'_0 & \rightarrow & \cdots & \rightarrow & S'_I & \rightarrow & \cdots & \rightarrow & S'_J & & \\
& \downarrow & & & \downarrow & & & \downarrow & & \\
S_0 & \rightarrow & \cdots & \rightarrow & S_I & \rightarrow & \cdots & \rightarrow & S_J & \rightarrow & \cdots & \rightarrow & S_K
\end{array}
\]

Once we have bounded the number of free splitting units between $S_I$ and $S_J = S_{\pi(S_I)}$, the diameter bound on the set $\{S_I, \ldots, S_{\pi(S_I)}\}$ follows from Proposition 4.16 (6).

The key observation is that in the foldable sequence $T_I \mapsto \cdots \mapsto T_J \mapsto S_I$, its first and last terms $T_I, S_I$ each collapse to the same free splitting, namely $S'_I$. This observation will be combined with the following:

Claim (♯): Consider a fold sequence $U_0 \xrightarrow{f_1} \cdots \xrightarrow{f_L} U_L$ in $\mathcal{F}(\Gamma; \mathcal{A})$. If there exists a free splitting $R \in \mathcal{F}(\Gamma; \mathcal{A})$ and collapse maps $U_0 \mapsto R$, $U_L \mapsto R$, then there exist integers $0 = \ell_0 \leq \ell_1 \leq \ell_2 = L$, and for $i = 1, 2$ there exist $x_i \in U_{\ell_i}$, such that the inverse image $(f_{\ell_i}^{-1}(x_i)) \subset U_{\ell_{i-1}}$ has cardinality bounded by a constant $b_\#$ depending only on corank($\mathcal{A}$) and $|\mathcal{A}|$.

Before proving Claim (♯), we apply it to finish the proof of the lemma, as follows. By replacing each individual arrow in the foldable sequence $T_I \mapsto \cdots \mapsto T_J \mapsto S_I$ by a fold sequence that factors it, we obtain a fold sequence which contains $T_I, \ldots, T_J, S_I$ as a subsequence. To that fold sequence we may then apply Claim (♯), combined with Lemma 4.14 followed by Lemma 4.13 (1), with the effect of subdividing the fold sequence between $T_I$ and $T_J$ into at most two subintervals along each of which there is pullback subgraph sequence of bounded complexity difference. Then applying Proposition 4.16 (5) we obtain a subdivision of the fold sequence between $S_I$ and $S_J$ into at most two subintervals along each of which the number of free splitting units rel $\mathcal{A}$ is bounded. Applying Proposition 4.16 (3) we obtain an upper bound to the number of free splitting units rel $\mathcal{A}$ between $S_I$ and $S_J$. 

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We turn to the proof of Claim (\#). Denote $V = U_0 \overset{f=I}{\longrightarrow} U_L = W$. Choose oriented relatively natural edges $e_V = [v_-, v_+ \in V$, $e_W = [w_-, w_+ \in W$ which map onto the same oriented relatively natural edge $e_R = [r_-, r_+ \in R$ under collapse maps $V, W \leftrightarrow R$. Decompose $V \setminus e_V = V_\llcorner \cup V_\lrcorner$ and $W \setminus e_W = W_\llcorner \cup W_\lrcorner$ so that $v_\pm$ is the frontier of $V_\pm$, and $w_\pm$ is the frontier of $W_\pm$, respectively. We have equations

$$(*) \quad f(V_+) = W_+ \cup [w_+, f(v_+)] \quad f(V_-) = W_- \cup [w_-, f(v_-)]$$

which are obtained by referring to [HM13] Lemma 5.5 and following the proof of the implication $(4) \implies (1)$, except that one may ignore the very last sentence which is the only place in that proof where the gate 3 condition was used. We briefly outline the proof of $(*)$ for $f(V_+)$. Decompose $R \setminus e_R = R_\llcorner \cup R_\lrcorner$ so that $r_\pm$ is in the frontier of $R_\pm$. Let $\Gamma_+$ be the set of elements of $\Gamma$ acting loxodromically on $R$ with axis contained in $R_+$. First one shows the inclusion $W_+ \subset f(V_+)$ by proving for each $\gamma \in \Gamma_+$ that $\gamma$ acts loxodromically on $V$ and $W$ with axes contained in $V_+$ and $W_+$ respectively, and that the union of such axes over $\gamma \in \Gamma_+$ equals $V_+, W_+$ respectively, and finally using that for each $\gamma \in \Gamma_+$ the $f$ image of the axis of $\gamma$ in $V_+$ contains the axis of $\gamma$ in $W_+$. Next one shows, by bounded cancellation, that $f(V_+)$ is contained in a finite radius neighborhood $N_r(W_\pm)$ of $W_\pm$. Finally, using that neighborhood, one shows that if $(*)$ fails then $f(V_+)$ contains a valence 1 point distinct from $f(v_+)$, that point has the form $f(x)$ for some $x \in V_+ - v_+$, and $f$ has only one gate at $x$, contradicting foldability of $f$.

Consider the oriented segment $f(e_V) = [f(v_-), f(v_+)] \subset W$. If $f(e_V)$ intersects $\text{int}(e_W)$ and preserves orientation, then $f$ is one-to-one over some point $x \in \text{int}(e_W)$, so Claim (\#) is proved with $l_1 = L$, $x_1 = x_2 = x$, and $b_\# = 1$. If $f(e_V)$ intersects $\text{int}(e_W)$ and reverses orientation, then $f$ has only one gate at $v_-$ and at $v_+$, a contradiction.

**Remark.** Under the gate 3 hypothesis on the given fold sequence the proof of Claim (\#) ends here, because the segment $f(e_V)$ must intersect $\text{int}(e_W)$; see the last lines of the proof of [HM13] Lemma 5.5. Without the gate 3 hypothesis our work continues for rather a long while.

We may assume that $f(e_V)$ is a subset of $W_-$ or of $W_+$; up to reverse of orientation we have

$$f(e_V) \subset W_+ \quad f(V_+) = W_+ \quad \text{and} \quad e_W = [w_-, w_+] \subset [w_-, f(v_-)]$$

We orient $\alpha$ with initial endpoint $w_-$ and terminal endpoint $f(v_-)$, and we parameterize $\alpha$ by simplicial distance from $w_-$, inducing a linear order on $\alpha$ which lets us speak of maxima and minima in $\alpha$. Let $\Sigma = V_- \cap f^{-1}(\alpha)$ and $\xi = \Sigma \cap f^{-1}(w_-)$. Note that $\Sigma - \xi$ is connected; otherwise the closure of some component of $\Sigma - \xi$ would not contain $v_-$, its image in $\alpha$ would have a maximum value achieved at some $x \in \text{int}(\Sigma)$, and $x$ would
have one gate, a contradiction. It follows that \( \Sigma \) is connected, that each point of \( \xi \) has valence 1 in \( \Sigma \), and that \( \xi \in \text{Fr}(\Sigma) \). Furthermore \( \text{Fr}(\Sigma) = \xi \cup \{v_-\} \), and the map \( f: \Sigma \to \alpha \) takes \( \text{Fr}(\Sigma) \) to \( \partial\alpha \), mapping \( \xi \) to \( w_- \) and \( v_- \) to \( f(v_-) \).

Consider the initial edgelets of \( \Sigma \) meaning the edgelets incident to points of \( \xi \), each of which maps to the initial edgelet of \( e_W \). It follows that the initial edgelets of \( \Sigma \) are all in different \( \Gamma \)-orbits, and so there are only finitely many of them, implying that \( \xi \) is finite and so \( \Sigma \) is a finite tree. Since each vertex of \( \Sigma - \text{Fr}(\Sigma) \) has at least two gates with respect to the map \( f \mid \Sigma \), and since \( f(\Sigma - \text{Fr}(\Sigma)) \subseteq \alpha \) it follows that each vertex of \( \Sigma - \text{Fr}(\Sigma) \) has exactly two gates and that \( f(\Sigma - \text{Fr}(\Sigma)) \cap \text{int}(\alpha) = (w_-, f(v_-)) \). Assign an orientation to each edgelet of \( \Sigma \) so as to point towards \( v_- \). By induction on distance to \( \xi \) it follows that \( f \) maps each edgelet of \( \Sigma \) to an edgelet of \( \alpha \) in an orientation preserving manner.

It follows that for each \( x \in \xi \) the map \( f \) takes \([x, v_-] \) one-to-one onto \( \alpha \). Furthermore at each \( y \in \text{int}(\Sigma) \) there is therefore a unique positive direction with respect to \( f \), namely the direction pointing towards \( v_- \), which is the unique direction at \( y \) whose image under \( f \) is the direction at \( f(y) \in \alpha \) pointing towards \( f(v_-) \). All other directions at \( y \) form the negative gate, each mapping to the direction at \( f(y) \in \alpha \) pointing back towards \( w_- \). This gives \( \Sigma \) the structure of a “hanging tree” in the terminology of \([BF12]\). It follows that every edgelet of \( \Sigma \) that maps to the initial edgelet of \( e_W \) is an initial edgelet of \( \Sigma \).

We break into two cases depending on the behavior of the following subset of \( \Gamma \):

\[
\hat{Z} = \{ \gamma \in \Gamma \mid \text{int}(\Sigma) \cap \text{int}(\gamma \cdot \Sigma) \neq \emptyset \} = \{ \gamma \in \Gamma \mid \Sigma \cap \gamma \cdot \Sigma \text{ contains at least one edgelet} \}
\]

**Case 1:** \( \hat{Z} = \{\text{Id}\} \). Consider the graph of groups \( V/\Gamma \). It follows in Case 1 that the orbit map \( V \to V/\Gamma \) restricts to an injection on \( \text{int}(\Sigma) \). The images of the initial edgelets of \( \Sigma \) are therefore all contained in distinct oriented relatively natural edges of \( V/\Gamma \). Applying Proposition 3.4 (1) it follows that the number of initial edgelets is bounded by the number \( 2D_{\text{FS}}(A) + 2 = 6 \text{corank}(A) + 4|A|-6 \). Taking this number to be \( b# \), Claim (\#) is proved with \( \ell_1 = L \) and with \( x_1 = x_2 = \) an interior point of the initial edgelet of \( e_W \).

**Case 2:** \( \hat{Z} \neq \{\text{Id}\} \). The action of each \( \gamma \in \hat{Z} \) on the tree \( W \) restricts as

\[
\tau_\gamma: (\gamma^{-1} \cdot \alpha) \cap \alpha \to \alpha \cap (\gamma \cdot \alpha)
\]

Furthermore, this map \( \tau_\gamma \) is an isometry with respect to the parameterization of \( \alpha \) described earlier, so we may speak about whether \( \gamma \) preserves or reverses orientation, and if \( \gamma \) preserves orientation we may also speak about the translation length of \( \gamma \), all by reference to what \( \tau_\gamma \) does to the parameterization of \( \alpha \).

We next show:

1. For each \( \gamma \in \hat{Z} \) the arc \( \alpha \cap \gamma \cdot \alpha \) has endpoints in the set \( \partial\alpha \cup \gamma \cdot \partial \alpha \).
This follows from the earlier description of how \( f \) maps \( \Sigma \) to \( \alpha \) and \( \text{Fr}(\Sigma) \) to \( \partial \alpha \), together with the fact that \( \text{Fr}(\Sigma \cap (\gamma \cdot \Sigma)) \subset \text{Fr}(\Sigma) \cup (\gamma \cdot \text{Fr}(\Sigma)) \).

If \( \gamma \) reverses orientation it follows from (1) that \( \gamma^2 \) fixes the arc \( \alpha \cap \gamma \cdot \alpha \) and so, since \( W \) is a free splitting, \( \gamma^2 \) is trivial, but that is a contradiction. Every element of \( \hat{Z} \) therefore preserves orientation.

We define \( \gamma \in \hat{Z} \) to be positive if \( \gamma \) has positive translation length with respect to the parameterization of \( \alpha \); for \( \gamma \) to be negative is similarly defined by requiring negative translation length. Thinking of the map \( \Sigma \overset{f}{\rightarrow} \alpha \) as a “height function”, an element of \( \hat{Z} \) is positive if and only if it increases height in \( \Sigma \), and negative if and only if it decreases height.

Letting \( Z < \Gamma \) be the group generated by \( \hat{Z} \), we shall show:

(2) There exists a positive \( \gamma \in \hat{Z} \) such that \( Z = \langle \gamma \rangle \) is infinite cyclic.

For any free splitting \( \Gamma \acts X \) in which the action of the cyclic group \( Z \) is not elliptic, let \( \text{Ax}(X) \) denote the axis of that cyclic group. We show furthermore that:

(3) The set \( H = Z \cdot \Sigma \subset V \) is a two-ended tree on which \( Z \) acts cocompactly, there is a \( Z \)-equivariant deformation retraction \( H \rightarrow \text{Ax}(V) \), and

\[
    f(H) = f(Z \cdot \Sigma) = Z \cdot f(\Sigma) = Z \cdot \alpha = \text{Ax}(W)
\]

(4) The map \( f: H \rightarrow \text{Ax}(W) \) gives \( H \) the structure of a “bi-infinite hanging tree” as follows: at each \( x \in H \) the map \( f \mid H \) has a positive gate consisting of the unique direction at \( x \) whose \( f \)-image points towards the positive end of \( \text{Ax}(W) \), and if \( x \) is not of valence 1 then all other directions at \( x \) are in a single negative gate whose \( f \)-image points towards the negative end of \( \text{Ax}(W) \).

(5) All translates of the tree \( H \) by elements of \( \Gamma - Z \) have disjoint interiors.

For the proofs of (2)–(5), pick a positive \( \gamma \in \hat{Z} \) whose translation distance on \( \alpha \) is a minimum. Given a positive \( \delta \in \hat{Z} \), note that \( \gamma^{-1}\delta \) is in \( \hat{Z} \) and is non-negative. By induction there exists \( i \geq 0 \) such that \( \gamma^{-i}\delta \in \hat{Z} \) and has translation number zero, implying that it fixes an arc of \( \alpha \), and so \( \delta = \gamma^i \). This proves (2), and (3) follow easily. Item (4) follows from the analogous properties of the map \( f: \Sigma \rightarrow [w_-, f(v_-)] \). For (5), suppose \( \delta \in \Gamma \) has the property that the interiors of \( H \) and \( \delta \cdot H \) are not disjoint. Choose integers \( i, j \) such that the interiors of \( \gamma^i \cdot \Sigma \) and \( \delta \gamma^j \cdot \Sigma \) are not disjoint, so the interiors of \( \Sigma \) and \( \gamma^{-i}\delta \gamma^j \Sigma \) are not disjoint. By (2) we have \( \gamma^{-i}\delta \gamma^j \in \hat{Z} \) and so \( \delta \in Z \), proving (5).

From properties (2)–(5) it follows that the 1-complex \( H/Z \) deformation retracts to the circle \( \text{Ax}(V)/Z \). Furthermore, the induced map \( H/Z \rightarrow V/\Gamma \) is an embedding on the complement of the valence 1 vertices. Define an initial edgelet of \( H \) to be an oriented edgelet whose initial vertex has valence 1 in \( H \), and so we have a bijection between initial

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edgelets and valence 1 vertices. Define an initial edgelet of \( H/Z \) in a similar fashion. We have a bijection between \( Z \)-orbits of initial edgelets of \( H \) and initial edgelets of \( H/Z \). Under the map \( H/Z \mapsto V/\Gamma \), the initial edgelets of \( H/Z \) all map into distinct oriented natural edges of \( V/\Gamma \). The number of \( Z \)-orbits of initial edgelets of \( H \) is therefore bounded above by \( 2D_{FS}(A) + 2 = 6 \text{corank}(A) + 4|A| - 6 \) (see Proposition 3.4 (1)), and so the number of \( Z \)-orbits of valence 1 vertices of \( H \) has the same bound. A branch of \( H \) is an oriented arc with initial endpoint at a valence 1 vertex, terminal endpoint on \( \text{Ax}(V) \), and interior disjoint from \( \text{Ax}(V) \). Letting \( \beta_v \subset H \) denote the branch with initial vertex \( v \), we have a \( Z \)-equivariant bijection \( v \leftrightarrow \beta_v \) between valence 1 vertices and branches, and so:

(6) The number of \( Z \)-orbits of branches of \( H \) is bounded by

\[
2D_{FS}(A) + 2 = 6 \text{corank}(A) + 4|A| - 6
\]

We also have, as a consequence of property (4), the following:

(7) The map \( f: V \to W \) is injective on each branch \( \beta_v \), mapping it homeomorphically to an arc of \( \text{Ax}(W) \). In particular, \( \beta_v \) is legal with respect to \( f \).

Consider now the whole fold sequence \( V = U_0 \mapsto \cdots \mapsto U_L = W \). Choose \( x_2 \in e_W \) to be in the interior of some edgelet. If \( f_0^L \) is one-to-one over \( x_2 \) then we are done with \( x_1 = x_2, \ell_1 = L, \) and \( b_\# = 1 \). Otherwise, let \( \ell_1 \in \{0, \ldots, L\} \) be the largest integer such that the map \( f_0^{\ell_1} : U_{\ell_1} \to U_L \) is not 1-to-1 over \( x_2 \). Since \( f_0^{\ell_1+1} \) is 1-to-1 over \( x_2 \), and since the fold map \( f_0^{\ell_1+1} \) is at worst 2-to-1 over the interior of each edgelet, it follows that \( f_0^{\ell_1} \) is exactly 2-to-1 over \( x_2 \). Let \( y \in U_{\ell_1+1} \) be the unique point of \( (f_0^{\ell_1+1})^{-1}(x_2) \). Note that \( y \in \text{Ax}(U_{\ell_1+1}) \), because

\[
x_2 \in \alpha \subset \text{Ax}(W) \quad \text{(see item (3))}
= \text{Ax}(U_L) \subset f_0^{\ell_1+1}(\text{Ax}(U_{\ell_1+1}))
\]

Under the fold map \( f_{\ell_1+1} : U_{\ell_1} \to U_{\ell_1+1} \) the point \( y \) has exactly 2 pre-images, exactly one of which denoted \( x_1 \) is disjoint from \( \text{Ax}(U_{\ell_1}) \). Let \( P = (f_0^{\ell_1})^{-1}(x_1) \subset U_0 \), which is disjoint from \( \text{Ax}(U_0) \). It remains to show

**Claim** (*) The cardinality of \( P \) is \( \leq \) the number of \( Z \)-orbits of branches of \( H \).

Applying this claim, it follows by (6) that the cardinality of \( P \) is bounded above by \( 2D_{FS}(A) + 2 = 6 \text{corank}(A) + 4|A| - 4 \). Claim (#) is then proved by taking \( b_\# = \max\{2, 2D_{FS}(A) + 2\} = 2D_{FS}(A) + 2 \).

For proving Claim (*), by applying item (5) we conclude that \( P \subset \text{int}(H) \), so \( P \) is the inverse image of \( x_1 \) under the restriction of \( f_0^{\ell_1} \) to \( H \). Consider \( H_{\ell_1} = f_0^{\ell_1}(H) \subset U_{\ell_1} \).
From items (3), (4), which describe the infinite hanging tree structure on $H$ with respect to the map $f = f^0_L: H \to \text{Ax}(W)$, it follows that $H_{\ell_1}$ also has an infinite hanging tree structure with respect to the map $f^0_{\ell_1}: H_{\ell_1} \to \text{Ax}(W)$. For each branch $\beta_v \subset H$, by (7) the map $f$ takes $\beta_v$ homeomorphically onto a subsegment of $\text{Ax}(W)$, from which it follows that $f^0_{\ell_1}$ maps $\beta_v$ homeomorphically onto its image $f^0_{\ell_1}(\beta_v)$, a path which therefore takes no illegal turns in $H_{\ell_1}$ with respect to the map $f^0_{\ell_1}: H_{\ell_1} \to \text{Ax}(W)$. Combining this with the infinite hanging tree structure on $H_{\ell_1}$, it follows that if $\mu \subset \beta_v$ is a subpath, with homeomorphic image subpath $f^0_{\ell_1}(\mu) \subset f^0_{\ell_1}(\beta_v)$, and if the endpoints of $f^0_{\ell_1}(\mu)$ are disjoint from $\text{Ax}(U_{\ell_1})$, then all of $f^0_{\ell_1}(\mu)$ is disjoint from $\text{Ax}(U_{\ell_1})$, because any path between points in distinct components of an infinite hanging tree minus its axis must contain an illegal turn.

If Claim (â†’) fails then there exist $b \neq b' \in P$, a branch $\beta_v \subset H$, and $\gamma^i \in Z$, such that $b \in \beta_v$ and $b' \in \gamma^i \cdot \beta_v$. The path $\mu = [\gamma^{-i}(b'), b]$ is contained in $\beta_v$ and so it is mapped homeomorphically to the path $f^0_{\ell_1}(\mu) = f^0_{\ell_1}\gamma^{-i}(b'), b]$. Also, the endpoints $\gamma^{-i}(b'), b$ of $\mu$ are mapped by $f^0_{\ell_1}$ to the endpoints $\gamma^{-i}(x_1), x_1$ of $f^0_{\ell_1}(\mu)$, neither of which are in $\text{Ax}(U_{\ell_1})$. It follows that $f^0_{\ell_1}(\mu)$ is disjoint from $\text{Ax}(U_{\ell_1})$. In the tree $U_0$ consider the bi-infinite, $\gamma^i$-invariant sequence of paths

$$
\cdots \gamma^{-2i}(b'), \gamma^{-i}(b), \gamma^{-i}(b'), \gamma^0(b'), \gamma^i(b), \cdots
$$

Since $f^0_{\ell_1}(\gamma^{-mi}(b)) = f^0_{\ell_1}(\gamma^{-mi}(b'))$ for all $m$, it follows that the image of the above sequence of paths under the map $f^0_{\ell_1}$ concatenates together to form a bi-infinite $\gamma^i$-invariant path in $U_{\ell_1}$ which is disjoint from $\text{Ax}(U_{\ell_1})$, a contradiction.

\[\diamond\]

5.3 Proof of Theorem 1.1: Reducing the coarse lipschitz and strong contraction axioms to Proposition 5.3.

In [HM13] we proved hyperbolicity of $FS(F_n)$ by using fold sequences and free splitting units to verify hyperbolicity axioms established by Masur and Minsky in [MM99]. We follow the same method here to prove hyperbolicity of $FS(\Gamma; A)$.

As alluded to earlier, Proposition 5.3 may be regarded as a translation of the Coarse Lipschitz and Strong Contraction Axioms into a single statement regarding fold sequences and free splitting units. To state it we need one more definition.

Given a fold sequence $S_0 \mapsto \cdots \mapsto S_K$ and a free splitting $T$ in $FS(\Gamma; A)$, an augmented projection diagram over $A$ of depth $J$ from $T$ to $S_0 \mapsto \cdots \mapsto S_K$ is a commutative diagram of free splittings and maps rel $A$ of the form shown in Figure 2 such that the sequence $T_J \mapsto \cdots \mapsto T_L$ is a fold sequence, and such that the two rectangles shown are combing rectangles; the diagram obtained by replacing that sequence with the composed foldable map $T_J \mapsto T_L$ is therefore an ordinary projection diagram as given in
Definition 5.1. Also, any projection diagram as given in Definition 5.1 can be converted into an augmented projection diagram by simply factoring the map $T_J \mapsto T$ as a fold sequence.

**Proposition 5.3.** [c.f. [HM13] Proposition 6.1] Let $b_1 = 5 \text{corank}(A) + 4 |A| - 3$. Let $S_0 \mapsto \cdots \mapsto S_K$ be a fold sequence rel $A$, and let $\pi: \mathcal{FS}(\Gamma; A) \to \{0, \ldots, K\}$ be its associated projection map. Let $T$ be a free splitting rel $A$, and consider any augmented projection diagram rel $A$ of depth $J = \pi(T)$ (as denoted in Figure 2). Let $\Upsilon$ be the number of free splitting units rel $A$ between $S_0$ and $S_J$. For any free splitting $R$ rel $A$, if $d(T, R) \leq \max\{2\lfloor \Upsilon/b_1 \rfloor, 1\}$ and if the number of free splitting units rel $A$ between $S_0$ and $S_J$ is at least $b_1$, then there exists $\ell \in [0, \pi(R)]$ such that the number of free splitting units between $S_\ell$ and $S_J$ is at most $b_1$.

This proposition will be proved in the next section, using the Big Diagram argument. For now we use it to prove our main results on hyperbolicity of $\mathcal{FS}(\Gamma; A)$ and on the uniform quasigeodesic parameterization of fold paths using free splitting units.

**Proof of Theorem 1.4.** The argument follows closely the proof of hyperbolicity of $\mathcal{FS}(F_n)$ given in [HM13] Section 6.1; here are a few details. We have already verified the Coarse Retract Axiom in Lemma 5.2. Fixing free splittings $T, R \in \mathcal{FS}(\Gamma; A)$ we must verify the Coarse Lipschitz Axiom and the Strong Contraction Axiom, which we do with constant $c = 10b_1 + 8$. After interchanging $T, R$ we may assume $\pi(R) \leq \pi(T) = J$. We may also assume that the number of free splitting units between $S_0$ and $S_J$ is at least $b_1$, for otherwise by applying Proposition 4.16 (6) the set $\{S_0, \ldots, S_J\}$ and its subset $\{S_{\pi(R)}, \ldots, S_J\}$ each have diameter $\leq 10b_1 + 8 = c$ and the axioms follow.

For the Coarse Lipschitz Axiom, if $d(T, R) \leq 1$ then Proposition 5.3 applies to produce $\ell \leq J$ such $\ell \leq \pi(R) \leq J$ and such that between $S_\ell$ and $S_J$ there are at most $b_1$ free splitting units, so just as above the set $\{S_\ell, \ldots, S_J\}$ and its subset $\{S_{\pi(R)}, \ldots, S_J\}$ each have diameter $\leq 10b_1 + 8 = c$ and the axiom follows.
For the **Strong Contraction Axiom**, one considers two cases. For the first case where \( \Upsilon < 2b_1 \), by Proposition 4.16 (6) we have \( d(T, T_J) \leq 20b_1 + 8 \) and so \( d(T, S_J) \leq 20b_1 + 10 \), and taking \( a = 20b_1 + 10 \) we may dispense with this case. For the second case where \( \Upsilon \geq 2b_1 \geq 1 \), let \( b = \frac{1}{20b_1} \). Then follow exactly the final part of the proof of hyperbolicity of \( \mathcal{FS}(F_n) \) given in [HM13] Section 6.1, with the conclusion that if \( d(T, R) \leq \Upsilon/b_1 \leq 2[\Upsilon/b_1] \), and then Proposition 5.3 applies just as above with the conclusion that \( \{ S_{\pi(R)}, \ldots, S_J \} \) has diameter \( \leq 10b_1 + 8 = c \), so the axiom follows.

Having verified all of the Masur–Minsky axioms, hyperbolicity of \( \mathcal{FS}(\Gamma; A) \) therefore follows from [MM99].

5.4 Theorem 5.4: Parameterizing fold paths using free splitting units

In the absolute case, [HM13] Proposition 6.2 shows that fold paths, when parameterized using free splitting units, are uniformly quasigeodesic in \( \mathcal{FS}(F_n) \). That argument relativizes with very little change to the current setting using free splitting units rel \( A \), producing Theorem 5.4 below.

The free splitting unit parameterization of a fold path can be described with either a discrete parameter or a continuous parameter. Consider a fold sequence \( S_0 \mapsto \cdots \mapsto S_M \) rel \( A \). Letting \( \Upsilon \) be the number of free splitting units rel \( A \) from \( S_0 \) to \( S_M \), one chooses an integer sequence \( 0 = m_0 < m_1 < \cdots < m_\Upsilon = M \) such that if \( 1 \leq u \leq \Upsilon \) then there is at least one free splitting unit between \( S_{m_u-1} \) and \( S_{m_u} \). The discrete parameterization of this fold path by free splitting units is the map defined on the integer interval \( [0, \Upsilon] = \{ u \in \mathbb{Z} \mid 0 \leq u \leq \Upsilon \} \mapsto X^{(1)} \) define by \( u \mapsto S_{m_u} \). The continuous parameterization is a function defined on the real interval \( \{ t \in \mathbb{R} \mid 0 \leq u \leq \Upsilon \} \), whose restriction to the subinterval \( u-1 \leq t \leq u \) parameterizes an interpolation of the fold path \( S_{m_{u-1}} \mapsto S_{m_{u-1}+1} \mapsto \cdots \mapsto S_{m_u} \mapsto S_{m_{u-1}} \), where each fold is replaced by an edge path in \( X^{(1)} \) of length at most 2 (c.f. Lemma 4.4). Since the set of vertices \( \{ S_{m_u} \mid m_u \leq \Upsilon \} \) has uniformly bounded diameter in \( \mathcal{FS}(\Gamma; A) \) (by Proposition 4.16 (6)), uniform quasi-isometry of the integer parameterization and of the real parameterization are equivalent properties.

**Theorem 5.4** (c.f. [HM13] Proposition 6.2). Parameterizations of fold paths by free splitting units are uniform quasigeodesics in \( \mathcal{FS}(\Gamma; A) \). That is, there exist constants \( k \geq 1, c \geq 0 \) depending only on corank \( A \) and \( |A| \) such that for any fold path as denoted above, the free splitting parameterization \( u \mapsto S_{m_u} \), defined for integers \( 0 \leq u \leq \Upsilon \), is a \((k, c)\) quasigeodesic in \( \mathcal{FS}(\Gamma; A) \).

**Proof.** For this proof we switch from “subscript notation” to “function notation” along the fold path, writing \( S(m) \) rather than \( S_m \). By Proposition 4.16 (6), the map \( u \mapsto \)
$S(m_u)$ is Lipschitz with constant 18. We must find constants $k, c$ such that for each $u < v$ in $[0, \Upsilon]$, letting $D = d(S(m_u), S(m_v))$, we have

$$|u - v| \leq kD + c$$

Let $\pi : FS(F_n; A) \to \{0, \ldots, M\}$ denote the projection to the fold path $S(0) \mapsto \cdots \mapsto S(M)$, and fix a projection diagram from $S(m_u)$ to that fold path of maximal depth $\pi(S(m_u))$. Fix a geodesic edge path $\rho$ in $FS(\Gamma; A)$ between $S(m_u)$ and $S(m_v)$, having length $D$. For any edge along this path, the number of free splitting units between their $\pi$-images is uniformly bounded and so, by the “long triangle inequality” for free splitting units, Proposition 4.16 (4), the number of free splitting units between $S(\pi(S(m_u)))$ and $S(\pi(S(m_v)))$ is bounded above by $kD$ for some uniform constant $k$. By the Coarse Retract Axiom, Lemma 5.2, the number of free splitting units between $S(m_u)$ and $S(m_v)$, is bounded above by some uniform constant $c'$. Again by the “long triangle inequality”, it follows that $|u - v|$, which is the number of free splitting units between $S(m_u)$ and $S(m_v)$, is bounded above by $kD + c' + 1$.

5.5 The proof of Proposition 5.3: Big Diagrams.

Through the proof we fix the constant $b_1 = 5 \text{corank}(A) + 4 |A| - 3$, the significance of which is established in Lemma 4.11.

The proof of Proposition 5.3 is, in essence, a study of the large scale geometry of certain diagrams of fold sequences and combing rectangles, diagrams that may be regarded as living in the relative free splitting complex $FS(\Gamma; A)$. We call these “big diagrams”. We begin the proof by using the hypotheses of Proposition 5.3 to set up the appropriate big diagram, and then we proceed to a study of its large scale geometry.

Constructing the Big Diagram, Step 0. The reader may refer to Figure 3 to follow this construction.

Consider a fold sequence $S_0 \mapsto \cdots \mapsto S_K$ in $FS(\Gamma; A)$ with associated projection map $\pi : FS(\Gamma; A) \to \{0, \ldots, K\}$. Consider also a free splitting $T \in FS(\Gamma; A)$ with augmented projection diagram over $A$ of depth $J = \pi(T)$ as denoted in Figure 2 with $T = T_L$. Along the foldable sequence in the top horizontal line of that augmented projection diagram, add a superscript 0, and so that sequence becomes

$$T_0^0 \mapsto \cdots \mapsto T_J^0 \mapsto \cdots \mapsto T_L^0 = T$$

Consider another free splitting $R \in FS(\Gamma; A)$ and consider also any geodesic path from $T_L^0$ to $R$ in the 1-skeleton of $FS(\Gamma; A)$. Since the concatenation of two collapse maps is a single collapse map, a geodesic necessarily has the form of a zig-zag path
alternating between collapses and expansions. It is convenient for us to slightly alter the geodesic path from $T^0_L$ to $R$ so that it begins with a collapse and ends with an expansion: in order to achieve this, prepend a trivial collapse and/or append a trivial expansion as needed. The result is a path of even length $D$ of the form

$$T = T^0_L \to T^1_L \leftarrow T^2_L \to \cdots \leftarrow T^D_L = R$$

where $d(T, R) \leq D \leq d(T, R) + 2$.

If $D = 0$ then $T = R$ and we are done. Henceforth we assume $D \geq 2$.

Construct a stack of $D$ combing rectangles atop the foldable sequence $T^0_L \to \cdots \to T^0_L$, by alternately applying relative combing by collapse, Lemma 4.8, and relative combing by expansion, Lemma 4.9, using the arrows in the path from $T^0_L$ to $T^D_L$, for a total of $D$-applications. The result is the Big Diagram Step 0 depicted in Figure 3, in which $T^d_L$ denotes the entry in the “row $D$” and “column $\ell$” of the stack of combing rectangles, and in which we have highlighted certain columns and rows.

Here is the general idea of the proof of Proposition 5.3. Notice that each column of the Big Diagram Step 0 is a zig-zag path, alternating between collapses and expansions. The diagram has the shape of a piece of corrugated aluminum. The far right edge is, by construction, a geodesic (except possibly for the first and last of its edges). The idea of the proof is that as one sweeps leftward through the Big Diagram, one discovers shorter vertical paths than the ones given in the diagram, allowing one to construct new Big Diagrams with fewer corrugations between the top and bottom rows. Eventually enough corrugations are removed to produce a projection diagram from $R$ to $S_0 \mapsto \cdots \mapsto S_K$ from which one can estimate $\pi(R)$.

For each even integer $d$ with $2 \leq d \leq D - 2$ we get a pair of collapse maps of the form $T^{d-1} \leftarrow \left[ \rho \beta \right] T^d \left[ \beta' \right] T^{d+1}$. If the subgraph $\rho \cup \beta \subset T^d$ were proper in $T^d$, then there would be a path $T^{d-2} \to T^{d-1} \left[ \beta' \right] T^h \left[ \rho' \right] T^{d+1} \leftarrow T^{d+2}$ where $T^h$ is obtained by collapsing $T^d \left[ \rho \cup \beta \right] T^h$, where $\beta'$ is the image of $\beta$ under $T^d \to T^{d-1}$ and $\rho'$ is the image of $\rho$ under $T^d \to T^{d+1}$; these images are proper, which is what allows this subpath to exist. But by concatenating the two collapse maps from $T^{d-2}$ to $T^h$ into a single collapse map, and similarly for the two collapse maps from $T^{d+2}$ to $T^h$ one obtains a shorter path between $T^0_L$ and $R$, contradicting that the chosen path was geodesic. It follows that $T^d = \rho \cup \beta$.

Letting $\Upsilon$ be the number of free splitting units rel $A$ between $T_J$ and $T_L$, and letting $\Omega = \lfloor \Upsilon / b_1 \rfloor$, consider the sequence $L = L_0 > L_1 > \cdots > L_\Omega \geq J$ which is obtained from the right greedy sequence by taking only every $b_1^{1h}$ term. By induction it follows for each $1 \leq \omega \leq \Omega$ that $L_\omega$ is the greatest integer $\leq L_{\omega-1}$ such that between $T_L \omega$ and $T_L \omega-1$ there are $\geq b_1$ free splitting units. Columns in big diagrams indexed by $L_0, L_1, \ldots, L_\Omega$ will be emphasized as those diagrams evolve.

We have seen that we have a union $T^2_{L_0} = \rho_{L_0} \cup \beta_{L_0}$. Knowing this, we may reduce to the case that this union is a blue-red decomposition meaning that $\rho_{L_0} \cap \beta_{L_0}$ contains
no edgelet: if this is not already so then we may alter the diagram to make it so, using exactly the same normalization process described in [HM13] Section 6.2. In brief, one replaces row $T^2$ by collapsing the intersection of red and blue along this row.

As in [HM13], the heart of the argument is an induction, starting with the Big Diagram step 0 and producing Big Diagrams steps 1, 2, \ldots, $(D - 2)/2$, each of which consists of a stack of combing diagrams grouped into successive pairs forming collapse–expand diagrams. At each step the number of combing rectangles decreases by 2, the final diagram at step $(D - 2)/2$ being just a stack of 2 combing diagrams forming a single collapse–expand diagram. At all stages of the induction we highlight column $T_J$, the number $J$ being the projection of $T$ onto $S_0 \mapsto \cdots \mapsto S_K$. Throughout the induction we suppress the projection diagram atop which all big diagrams are formed. In particular the foldable sequence $T_0^0 \mapsto \cdots \mapsto T_J^0$ is unaltered up until the case of the Big Diagram step $(D - 2)/2$, at which point we again highlight the projection diagram, obtaining in that case a stack of 4 combing rectangles. At that step we carry out one final alteration, producing a stack of 2 combing rectangles forming a projection diagram from $R$ to $S_0 \mapsto \cdots \mapsto S_K$ the depth of which is no more than $b_1$ free splitting units to the left of $S_J$.

For the induction step, assuming that $D \geq 4$, we adopt variations introduced by Bestvina and Feighn in [BF12] for the method of successively producing the next Big Diagram. We describe in detail the first step of the induction, going from step 0 in Figure 3 to step 1 in Figure 8; further steps of the induction are then described very briefly. The induction is complete at step $(D - 2)/2$, after which there will be one final special alteration step, to be described in detail later.

The first induction step when $D \geq 4$. Consider the collapse–expand diagram defined by the subrectangle $T^d_\ell$ for $(\ell, d) \in \{L_1, \ldots, L_0\} \times [0, 1, 2]$, along the top row of which we have an invariant blue–red decomposition $T^2_\ell = \beta_\ell \cup \rho_\ell$. The key observation that gets the construction started is that $\beta_{L_1}$, the collapse forest for the map $T^2_{L_1} \to T^3_{L_1}$, has a component $[x, y]$ which is a subarc of the interior of some natural edge of the free splitting $T^2_{L_1}$. This follows from the fact that there are $\geq b_1$ free splitting units between $T^0_{L_1}$ and $T^0_{L_0}$, by applying Proposition 4.16 (5b). Let $b \subset \beta_{L_1}$ be the blue edgelet in $[x, y]$ with endpoint $x$. Let $e$ be the red edgelet not in $[x, y]$ with endpoint $x$. Factor the collapse map $T^2_{L_1} \xrightarrow{[\beta_{L_1}]} T^3_{L_1}$ as a product of two collapse maps as follows. The first factor collapses everything in $\beta_{L_1}$ except the orbit of $b$, collapsing the subgraph $\beta'_{L_1} = \beta_{L_1} \setminus F_n \cdot b$, and taking $b$ to an edgelet $b' \subset T^b_{L_1}$. The second factor collapses the orbit of $b'$:

$$
\begin{align*}
T^2_{L_1} & \xrightarrow{[\beta_{L_1}]} T^3_{L_1} \\
[\beta'_{L_1} \setminus F_n \cdot b] & \xrightarrow{T^b_{L_1}} T^3_{L_1}
\end{align*}
$$
Figure 3: The Big Diagram, Step 0. Certain columns $L = L_0, L_1, \ldots$ are emphasized, using free splitting units along the fold path $T^D_j \to \cdots \to T^D_{L_0}$. As the Big Diagram evolves, and up until nearly the end of the evolution, the original projection diagram atop which the diagram is built, which involves the $S'$ and $S$ rows, will not change. Those rows will be suppressed in the meantime, returning only in the Penultimate Diagram of Figure 9.
Note that \( b' \) is contained in the interior of some natural edge \( \eta' \) of \( T^3_{bL_1} \). Also, letting \( e' \subset T^3_{bL_1} \) be the image of \( e \), note that arc \( e' \cup b' \) is also contained in \( \eta' \).

**Remark.** The particular way in which the edgelets \( b \) and \( e \) are used in the above paragraph is an innovation of Bestvina and Feighn in [BF12], arising from dropping the gate 3 condition on fold paths, and having the effect of simplifying the Big Diagram argument.

The collapse map \( T^3_{bL_1} \xrightarrow{[F_n,b]} T^3_{L_1} \) is equivariantly homotopic to a homeomorphism \( h: T^3_{L_1} \to T^3_{L_1} \) as follows. The homotopy is stationary off of the orbit of \( e' \cup b' \). Restricted to the arc \( e' \cup b' \), the collapse is a quotient map taking \( b' \) to a point, and that quotient map is homotopic, relative to the endpoints of the arc \( e' \cup b' \), to a homeomorphism; extend that restricted homotopy over the orbit of \( e' \cup b' \).

Using the above concatenation of two collapse maps, the combing rectangle \((\ell,d) \in [0,L_1] \times [2,3]\) factors it into a concatenation of two combing rectangles of the form shown in Figure 4, whose right side is the above factorization of the collapse map \( T^3_{L_1} \to T^3_{L_1} \) (here and later we silently apply the obvious generalizations to \( \mathcal{FS}(\Gamma;A) \) of the results of Section 4.3 of [HM13] which construct compositions and decompositions of combing rectangles).

Now we proceed from step 0 to step 0.1, depicted in Figure 5. Starting from the step 0 diagram depicted in Figure 3, discard the portion of the diagram that lies strictly below row 3 and right of column \( L_1 \), and the portion strictly above row 3 and left of column \( L_1 \). Next, replace the combing rectangle \((\ell,d) \in [0,L_1] \times [2,3]\) by inserting a certain portion of the two concatenated combing rectangles from Figure 4, namely, the lower of the two combing rectangles between row 2 and row 3, plus the collapse map \( T^3_{L_1} \to T^3_{L_1} \); do not insert any part of row 3 to the left of \( T^3_{L_1} \), nor any of the vertical arrows to the left of the collapse map \( T^3_{L_1} \to T^3_{L_1} \). And now replace the collapse map \( T^3_{L_1} \to T^3_{L_1} \) by the equivariant homeomorphism \( h: T^3_{L_1} \to T^3_{L_1} \), and using that homeomorphism identify the
free splittings $T_{L_1}^{3b} \approx T_{L_1}^3$. This ostensibly completes the construction of the Big Diagram step 0.1 shown in Figure 5.

Unfortunately, the map $h$ is not simplicial, because $h(x)$ is not a vertex of $T_{L_1}^3$. But we need the identification $h: T_{L_1}^{3b} \approx T_{L_1}^3$ to be a simplicial homeomorphism in order for the induction to continue. To do this we need only subdivide $T_{L_1}^3$ at the orbit of $h(x)$, after which $h$ is simplicial. Unfortunately, after this subdivision the maps $T_{L_1}^4 \mapsto T_{L_1}^3 \mapsto T_{L_1}^{3+1}$ are no longer simplicial. To resolve this issue once and for all, we push the subdivision of $T_{L_1}^3$ up and to the right, throughout the upper right rectangle of Figure 5 defined by $(\ell, d) \in [L_1, L_0] \times [3, D]$, restoring that all maps in this rectangle are simplicial. Do this restoration by the following procedure: first push the subdivision forward along the row $T_{L_1}^3 \mapsto \cdots \mapsto T_{L_1}^2$ using the fold maps of that row; then pull the subdivision back to the row $T_{L_1}^4 \mapsto \cdots \mapsto T_{L_1}^1$ under the collapse maps from row 4 to row 3; then push the subdivision forward to the row $T_{L_1}^5 \mapsto \cdots \mapsto T_{L_1}^0$ under the collapse maps from row 4 to row 5; etc. Using the simplicial homeomorphism $h$ we may now identify $T_{L_1}^{3b}$ and $T_{L_1}^3$, truly completing the construction of the Big Diagram step 0.1.

We must show that the following row in Figure 5 is a fold sequence:

$$T_{0}^{3b} \mapsto \cdots \mapsto T_{J}^{3b} \mapsto \cdots \mapsto T_{L_1}^{3b} \mapsto \cdots \mapsto T_{L_0}^{3b} \approx T_{L_1}^3 \mapsto \cdots \mapsto T_{L_0}^3$$
where the homeomorphism \( h \) is used to identify \( T_{L_1}^{3b} \approx T_{L_1}^3 \). By construction it is a fold sequence from \( T_0^{3b} \) to \( T_{L_1}^{3b} \) and from \( T_{L_1}^0 \) to \( T_{L_0}^3 \), and so it suffices to show that if \( 0 \leq \ell \leq L_1 \) then the map \( T_{L_1}^{3b} \to T_{L_0}^3 \) has at least two gates at each vertex \( y_\ell \in T_{L_1}^{3b} \). Let various images of \( y_\ell \) under the maps in Figure 4 be denoted \( y_{L_1} \in T_{L_1}^{3b} \), \( z_\ell \in T_{L_1}^3 \), and \( z_{L_1} \in T_{L_1}^3 \), and so we have \( z_{L_1} = h(y_{L_1}) \). There are two cases depending on whether \( y_{L_1} \in F_n \cdot b \). If \( y_{L_1} \notin F_n \cdot b \) then \( y_\ell \) is not in the collapse graph of the map \( T_{L_1}^{3b} \to T_{L_1}^3 \), and so under this collapse map the directions at \( y_\ell \) and at \( z_\ell \) correspond bijectively as do the directions at \( y_{L_1} \) and at \( z_{L_1} \). The gates at \( y_\ell \) and at \( z_\ell \) for the maps to \( T_{L_1}^3 \) therefore also correspond bijectively, but at \( z_{L_1} \) there are at least two such gates, and so at \( y_\ell \) there are also at least two such gates. If \( y_{L_1} \in F_n \cdot b \) then \( y_{L_1} \) has valence 2 as does \( z_{L_1} \), and at \( y_\ell \) the map to \( T_{L_1}^3 \) has exactly two gates, one for each direction at \( z_{L_1} \); those two directions map to two different directions in \( T_{L_0}^3 \), and so there are two gates at \( y_\ell \) for the map \( T_{L_1}^{3b} \to T_{L_0}^3 \).

Next we proceed from step 0.1 to step 0.2, depicted in Figure 6. Starting from the step 0.1 diagram depicted in Figure 5, apply relative combing by collapse, Lemma 4.8, and relative combing by expansion, Lemma 4.9. These are applied alternately to insert \( D - 3 \) combing rectangles into the upper left corner of step 0.1, between row \( 3b \) and row \( D \) and between column 0 and column \( L_1 \); we also delete everything strictly below row \( T^4 \) and right of \( T_{L_1} \); the result is shown in Figure 6, with names \( T_i \) re-used in the restored upper left corner. We note that for each \( 5 \leq d \leq D \), the rectangle between rows \( T^{d-1} \) and \( T^d \) is a combing rectangle from column 0 to \( L_1 \), and from column \( L_1 \) to \( L_0 \), and these piece together to form a single combing rectangle from column 0 to \( L_0 \), as follows by applying the uniqueness clauses in the statements of relative combing by collapse, Lemma 4.8, and relative combing by expansion, Lemma 4.9.

Next we proceed to the Big Diagram step 0.3, depicted in Figure 7. Notice that in \( T_{L_1}^2 \) we have an edgelet disjoint union

\[
T_{L_1}^2 = \rho_{L_1} \cup \beta_{L_1}' \cup (F_n \cdot b)
\]

Define a commutative “baseball diagram” of collapse maps:
Using Combing by Collapse on each of the five arrows in this diagram we obtain similar baseball diagrams replacing $L_1$ by any $i \in [0, \ldots, L_1]$. The combing diagrams that correspond to the two arrows from 2nd base $T^2_{L_1}$ to 1st and 3rd bases $T^1_{L_1}$ and $T^{3b}_{L_1}$ are the same as the two combing rectangles depicted in Figure 6 between rows $T^2$ and rows $T^1$ and $T^{3b}$. The Big Diagram step 0.3 is now constructed by replacing those two combing rectangles by the ones that correspond to the two arrows from 1st and 3rd bases to home base $T^h_{L_1}$.

Finally, the Big Diagram step 1, depicted in Figure 8, is obtained from step 0.3 by concatenating the two combing rectangles from row $T^0$ to $T^1$ and from row $T^1$ to $T^h$ into a single coming rectangle from row $T^0$ to row $T^h$, and by concatenating the two combing rectangles from row $T^2$ to row $T^{3b}$ and from row $T^{3b}$ to row $T^h$ into a single coming rectangle from row $T^4$ to row $T^h$. This completes the first step of the induction, constructing the Big Diagram step 1 from the Big Diagram step 0.

**Further induction steps.** Continuing to assume that $D \geq 4$, each further induction step for $2 \leq d \leq (D - 2)/2$ starts with the Big Diagram step $d - 1$, depicted as in Figure 8 but with column subscript $L_1$ replaced by $L_{d-1}$ and row superscript 4 replaced.
Figure 7: The Big Diagram, step 0.3

Figure 8: The Big Diagram, step 1
Figure 9: The Penultimate Diagram, aka the Big Diagram step \((D-2)/2\). The projection
diagram atop which all the Big Diagrams are constructed has been restored (except for
the portion of the \(T\) row to the right of column \(J\)). Column \(I\) is determined by requiring
that \(I\) is the largest integer \(\leq J\) such that between \(S_I\) and \(S_J\) there are exactly \(b_1\) free
splitting units.

The final step. When the induction is complete (which happens immediately if
\(D = 2\)), the Big Diagram step \((D-2)/2\) consists of a single collapse–expand diagram.
From this diagram discard everything strictly right of column \(J\) and below the top row.
Also, from the projection diagram for \(T\) depicted in Figure 2 discard everything in the
\(T\) row strictly to the right of column \(J\). Then glue these two diagrams together along
the two copies of the sequence \(T_0 \mapsto T_J\), resulting in the penultimate diagram shown in
Figure 9. In this diagram we emphasize also column \(I \in [0, \ldots, J]\) which is defined so
that \(I\) is the largest integer for which there are \(\geq b_1\) free splitting units between \(S_I\) and
\(S_J\), and hence there are exactly \(b_1\) free splitting units between \(S_I\) and \(S_J\); the existence
of \(I\) follows from the hypothesis of Proposition 5.3 that there are \(\geq b_1\) free splitting units

by \(2d\). From there one constructs the Big Diagram step \(d\), using a straightforward no-
tational variation of the construction from step 0 to step 1. The key observation which
gets the construction started is that the collapse forest for the map \(T_{L_d}^{2d} \mapsto T_{L_d}^{2d+1}\)
has a component which is contained in the interior of a natural edge of \(T_{L_d}^{2d}\). This follows by
applying Proposition 4.16 (5b) together with the fact that the number of free splitting
units between \(T_{L_d}^0\) and \(T_{L_{d-1}}^0\) is greater than or equal to \(b_1 = 5 \text{corank}(\mathcal{A}) + 4 |\mathcal{A}| - 3\).
Figure 10: The Ultimate Diagram. Notations $T_i^D$ and $T_i^h$ from Figure 9 have been reused to denote new objects.

The final construction is triggered by the observation that the collapse forest for the map from $T_I$ to $T_I^h$ has a component that is contained in the interior of a natural edge of $T_I$, which follows by applying Proposition 4.16 (5b) together with the assumption that between $S_I$ and $S_J$ there are $\geq b_1$ free splitting units. Based on this observation, we may now follow the same construction steps as above, the conclusion of which is a diagram of the form shown in Figure 10 (where the names $T_i^D, T_i^h$ for $0 \leq i \leq I$ have been reused). This is a projection diagram from $R$ to $S_0 \mapsto \cdots \mapsto S_K$ of depth $I$, and so the maximal depth of such a projection diagram, which by definition is the projection $\pi(R)$, satisfies $\pi(R) \geq I$, finishing the proof of Proposition 5.3.

6 Hyperbolicity of relative free factor complexes

In this section, given a group $\Gamma$ and a free factor system $A$ of $\Gamma$, we define the complex $\mathcal{FF}(\Gamma; A)$ of free factor systems of $\Gamma$ relative to $A$ (Section 6.1), we prove that $\mathcal{FF}(\Gamma; A)$ is connected (Section 6.2), and we prove that it is hyperbolic (Section 6.3).

Our proof of hyperbolicity follows the method of Kapovich and Rafi developed in [KR14] and used by them to derive hyperbolicity of $\mathcal{FS}(F_n)$ from hyperbolicity of $\mathcal{FS}(F_n)$. The way this method works is to derive hyperbolicity of a connected simplicial complex $Y$ from hyperbolicity of a given connected simplicial complex $X$, by exhibiting a surjective Lipschitz map $f: X \mapsto Y$ satisfying a simple geometric condition. Intuitively this condition says that if a geodesic in $X$ has its endpoints mapped near each other in $Y$ by the map $f$, then the entire $f$-image of that geodesic is bounded. We construct the required surjective Lipschitz map $\mathcal{FS}(\Gamma; A) \mapsto \mathcal{FF}(\Gamma; A)$ in Section 6.2, and we prove that it satisfies the needed condition on geodesics in Section 6.3.
6.1 The complex of free factor systems relative to a free factor system.

Fix a group \( \Gamma \). Define the \textit{unreduced complex of free factor systems of} \( \Gamma \) to be the simplicial realization of the set of free factor systems with respect to the partial ordering \( \sqsubseteq \). This complex has a 0-simplex for each free factor system \( A \), and a \( K \)-simplex for each chain of proper extensions of the form \( A_0 \sqsubseteq A_1 \sqsubseteq \cdots \sqsubseteq A_K \). The unreduced complex of free factor system is a single point if and only if \( \Gamma \) is freely indecomposable. For present purposes our interest in this unreduced complex is as a home for relative complexes of free factor systems.

Given a free factor system \( A \) of \( \Gamma \), the \textit{complex of free factor systems of} \( \Gamma \) \textit{relative to} \( A \), denoted \( \mathcal{FF}(\Gamma; A) \), is the flag subcomplex of the unreduced complex of free factor systems that contains a simplex \( A_0 \sqsubseteq \cdots \sqsubseteq A_K \) if and only if there is a proper inclusion \( A \sqsubseteq A_0 \) and a proper inclusion \( A_K \sqsubseteq \{[\Gamma]\} \).

In the case that \( \Gamma \) has a Grushko free factor system \( A \), for example when \( \Gamma \) is finitely generated, it makes sense to define the (absolute) complex of free factor system \( \mathcal{FF}(\Gamma) \) to be \( \mathcal{FF}(\Gamma; A) \). Note that this complex is obtained from the unreduced complex by “reducing” it, by which we mean removing the minimal and maximal 0-simplices \( A \) and \( \{[\Gamma]\} \) and the interiors of any incident simplices of positive dimension.

Recall the formula for the free factor system depth of a free factor system \( A \):

\[
D_{\mathcal{FF}}(A) = 2 \text{corank}(A) + |A| - 1
\]

The following is an immediate corollary of Lemma 2.14 and the definition of relative free factor systems:

**Proposition 6.1.** For any group \( \Gamma \) and any free factor system \( A \) such that \( D_{\mathcal{FF}}(A) \geq 2 \), the complex \( \mathcal{FF}(\Gamma; A) \) has dimension \( D_{\mathcal{FF}}(A) - 2 \), and any simplex is contained in a simplex of maximal dimension \( D_{\mathcal{FF}}(A) - 2 \). \( \diamond \)

For the following result, which is a corollary of Proposition 6.1 together with a straightforward case analysis, recall from Section 2.5 that a free factor system \( A \) of \( \Gamma \) is \textit{exceptional} if and only if \( D_{\mathcal{FF}}(A) \leq 2 \). This result enumerates the exceptional behavior of the topology of \( \mathcal{FF}(\Gamma; A) \) when \( A \) is exceptional. We also enumerate the 1-dimensional complexes.

**Proposition 6.2.** For any group \( \Gamma \), a free factor system \( A \) of \( \Gamma \) is exceptional if and only if \( \mathcal{FF}(\Gamma; A) \) is empty or 0-dimensional, and \( D_{\mathcal{FF}}(A) = 3 \) if and only if \( \mathcal{FF}(\Gamma; A) \) is 1-dimensional. More explicitly:

1. \( \mathcal{FF}(\Gamma; A) = \emptyset \iff D_{\mathcal{FF}}(A) \leq 1 \iff \) either \( A = \{[\Gamma]\} \), or \( A = \{[A_1], [A_2]\} \) with a realization \( \Gamma = A_1 \ast A_2 \).

2. \( \mathcal{FF}(\Gamma; A) \) is 0-dimensional \iff \( D_{\mathcal{FF}}(A) = 2 \iff \) one of the following occurs:
(a) \( A = \{[A]\} \) with realization \( \Gamma = A \ast Z \) where \( Z \) is infinite cyclic; or
(b) or \( A = \{[A_1],[A_2],[A_3]\} \) with realization \( \Gamma = A_1 \ast A_2 \ast A_3. \)

(3) \( \mathcal{FF}(\Gamma;A) \) is 1-dimensional \( \iff D_{\mathcal{FF}(A)} = 3 \iff \) one of the following occurs:

(a) \( A = \emptyset \) and with realization \( \Gamma = Z_1 \ast Z_2 \), each of \( Z_1, Z_2 \) being infinite cyclic (i.e. \( \Gamma \) is free of rank 2 and \( \mathcal{FF}(\Gamma;A) \) is its absolute complex of free factor systems); or

(b) \( A = \{[A_1],[A_2]\} \) with realization \( \Gamma = A_1 \ast A_2 \ast Z \) where \( Z \) has rank one; or

(c) \( A = \{[A_1],[A_2],[A_3],[A_4]\} \) with realization \( \Gamma = A_1 \ast A_2 \ast A_3 \ast A_4. \)

If \( \Gamma \) has a Grushko free factor system \( \mathcal{A} \), for example when \( \Gamma \) is finitely generated, then the unreduced complex of free factor systems is connected and has diameter \( \leq 2 \), because every other free factor system is an extension of \( \mathcal{A} \). If \( \Gamma \) has no Grushko free factor system then the relation \( \sqsubseteq \) has no minimum, in which case the depth of free factor systems of \( \Gamma \) is unbounded, and the dimension of the unreduced complex of free factor systems is infinite.

The complex \( \mathcal{FF}(F_n) (= \mathcal{FF}(F_n;\emptyset)) \) is related to the complex of free factors which we denote \( \mathcal{F}(F_n) \), introduced by Hatcher and Vogtmann in [HV98] (and see [BF14]). Several other closely related complexes, known to be equivariantly quasi-isometric to each other, are described for example in [KR14]. We recast the definition of \( \mathcal{F}(F_n) \) in our present setting as follows. In ranks \( n \geq 3 \), \( \mathcal{F}(F_n) \) is the subcomplex of \( \mathcal{FF}(F_n) \) consisting of all simplices \( A_0 \sqsubset \cdots \sqsubset A_K \) each of whose 0-simplices \( A_0, \ldots, A_K \) has but a single component. When \( n = 2 \) this definition would lead to a 0-dimensional complex whose simplices have the form \( \{[A]\} \) where the free factor \( A < F_2 \) has rank 1, but then \( \mathcal{F}(F_2) \) itself is obtained by attaching a 1-simplex to each pair of 0-simplices \([A],[B]\) whenever there is a free factorization \( F_2 = A \ast B \); clearly \( \mathcal{FF}(F_2) \) is the first barycentric subdivision of \( \mathcal{F}(F_2) \).

**Proposition 6.3.** The inclusion \( \mathcal{F}(F_n) \hookrightarrow \mathcal{FF}(F_n) \) is a quasi-isometry.

**Proof.** In this proof we shall abuse notation by identifying each 0-simplex \( A = \{[A]\} \) of \( \mathcal{F}(F_n) \) with its single component \([A]\).

We may assume \( n \geq 3 \). It suffices to construct a Lipschitz retract \( r \) from the 0-skeleton of \( \mathcal{FF}(F_n) \) to the 0-skeleton of \( \mathcal{F}(F_n) \). Given a 0-simplex \( A \) in \( \mathcal{FF}(F_n) \), choose any component \([A]\) \( \in \mathcal{A} \) and define \( r(A) = [A] \). If \( \mathcal{A} \) has but a single component then clearly \( r(A) = A \) (abusing notation).

Given a 1-simplex \( A \sqsubset A' \) in \( \mathcal{FF}(F_n) \), consider \( r(A) = [A]\) \( \in \mathcal{A} \) and \( r(A') = [A'] \in \mathcal{A}' \). We must bound the distance between \([A]\) and \([A']\) in \( \mathcal{F}(F_n) \). Letting \([A'']\) \( \in A' \) be the unique element such that \([A]\sqsubset[A'']\), since \([A],[A'']\) have distance at most 1 in \( \mathcal{F}(F_n) \) it suffices to bound the distance between \([A']\) and \([A'']\). If \([A'] = [A'']\) we are
done, so assume $[A'] \neq [A'']$. If necessary, rechoose $A', A''$ in their conjugacy classes so that each is a term in a realization of $A'$, and hence we have a free factorization of the form $F_n = A' * A'' * C$. Picking rank 1 free factors $B' < A', B'' < A''$, we have a free factorization $F_n = B'*B''*D$, and since $n \geq 3$ and $B', B''$ each have rank 1 it follows that the rank 2 subgroup $B'*B''$ is a proper free factor. We therefore obtain a path in $\mathcal{F}(F_n)$ of length at most 4 between $[A']$ and $[A'']$, namely $[A'] - [B'] - [B'*B''] - [B''] - [A'']$. ♦

6.2 Connectivity of $\mathcal{FF}(\Gamma; A)$; a Lipschitz map $\mathcal{FS}(\Gamma; A) \mapsto \mathcal{FF}(\Gamma; A)$.

Consider a group $\Gamma$ and a free factor system $A$ such that $D_{\mathcal{FF}}(A) \geq 3$, and so $\mathcal{FF}(\Gamma; A)$ has dimension $\geq 1$. We shall kill two birds with one stone: prove connectivity of $\mathcal{FF}(\Gamma; A)$; and describe a map $\mathcal{FS}(\Gamma; A) \mapsto \mathcal{FF}(\Gamma; A)$ which is Lipschitz with respect to simplicial metrics. The stone is Lemma 6.4, the birds are Proposition 6.5 (1) and (2).

The free factor system $A$ may be realized as $\Gamma = A_1 * \cdots * A_K * B$ with $A = \{[A_1], \ldots, [A_K]\}$, $K = |A| \geq 0$, and while $K$ is fixed we will vary such realizations as needed.

Define the projection set map $\Pi$ from the 0-skeleton of $\mathcal{FS}(\Gamma; A)$ to finite subsets of the 0-skeleton of $\mathcal{FF}(\Gamma; A)$, as follows. Consider a 0-simplex $[T] \in \mathcal{FS}(\Gamma; A)$ represented by a free splitting $\Gamma \acts T$ rel $A$. Define $\Pi[T] \subset \mathcal{FF}(\Gamma; A)$ to be the set of all 0-simplices of the form $\mathcal{F}(U)$ such that $\Gamma \acts U$ is a free splitting rel $A$, $\mathcal{F}(U) \neq A$, and there exists a collapse map $T \mapsto U$ (which one may always choose to be relatively natural). Here are a few properties of the set map $\Pi$ that we will use without comment in what follows:

- $\Pi[T]$ is well-defined within the equivalence class of $T$.
- The sets $\Pi[T]$ cover the entire 0-skeleton of $\mathcal{FF}(\Gamma; A)$, as $[T]$ varies over the 0-skeleton of $\mathcal{FS}(\Gamma; A)$.
- The inverted equivariance property: $\Pi([T] \cdot \phi) = \phi^{-1}(\Pi[T])$ for each $\phi \in \text{Out}(\Gamma; A)$.
- $\Pi[T] \neq \emptyset$.

The first item is evident, the second follows from Lemma 3.1, and the third from Lemma 3.7. The fourth follows from Proposition 3.6 (2)(c) which guarantees the existence of a collapse map $T \mapsto U$ such that $U$ is a one-edge free splitting, together with the fact that $D_{\mathcal{FF}}(A) \geq 2$ which guarantees that $\mathcal{F}(U) \neq A$.

Using Bass-Serre theory, we next translate the definition of $\Pi[T]$ into the language of graphs of groups. In the quotient graph of groups $T/\Gamma$, given a subgraph $G \subset T/\Gamma$, consider the following two properties of $G$:

1. $G$ contains every vertex of $T/\Gamma$ with nontrivial vertex group.
2. Each vertex of valence 0 or 1 in $G$ has nontrivial vertex group.
We say that $G$ is a relative core graph if both of (1) and (2) hold. If only (1) holds then $G$ contains a unique maximal relative core graph denoted core($G$); inductively remove any vertex that violates (2) together with any incident edge. Every subgraph $G \subset T/\Gamma$ satisfying (1) represents a free factor system rel $\mathcal{A}$ that we denote $[G]$; to define $[G]$, let $\tilde{G} \subset T$ be the total lift of $G$ via the Bass-Serre universal covering map $T \to T/\Gamma$, and define $[G]$ to be the set of conjugacy classes of stabilizers of components of $\tilde{G}$. Note that $[G] = \mathcal{F}(U)$ where the map $T \to U$ collapses to a point each component of $\tilde{G}$. Note also that $[G] = \{[\Gamma]\}$ if and only if $G = T/\Gamma$. A relative core graph $G \subset T/\Gamma$ is trivial if $\mathcal{A}$ has a realization $\Gamma = A_1 \ast \cdots \ast A_K \ast B$ such that $G$ consists solely of $K$ vertices $v_1, \ldots, v_K$ with vertex groups $A_1, \ldots, A_K$; a trivial relative core graph of $T/\Gamma$ exists if and only if $\mathcal{F}(T) = \mathcal{A}$. The triviality property is extended to arbitrary subgraphs $G \subset T/\Gamma$ that satisfy (1) by requiring that Core($G$) be trivial. Note that $[G] = \mathcal{A}$ if and only if $G$ and core($G$) are trivial.

To complete the Bass-Serre translation, we note that $\Pi[T]$ is equal to the following set of 0-simplices in $\mathcal{FF}(\Gamma; \mathcal{A})$:

$$\Pi[T] = \{[G] \in \mathcal{FF}(\Gamma; \mathcal{A}) \mid G < T/\Gamma \text{ is a proper, nontrivial, relative core graph}\}$$

To prove this, in the discussion above we have already proved the inclusion $\supset$. For the opposite inclusion $\subset$, given $\mathcal{F}(U) \in \Pi[T]$ and a relatively natural collapse map $T \xrightarrow{[\sigma]} U$, let $\sigma'$ be the union of $\sigma$ with all vertices of $T$ having nontrivial stabilizer, let $G$ be the image of $\sigma'$ under the quotient map $T \to T/\Gamma$, and it follows that $G$ is a proper, nontrivial relative core graph and that $[G] = \mathcal{F}(U)$.

**Lemma 6.4.** The set map $\Pi$ has the following properties:

1. For any collapse of free splittings $S \succ T$ we have $\Pi[T] \subset \Pi[S]$.

2. For each $[T] \in \mathcal{FS}(\Gamma; \mathcal{A})$ the set $\Pi[T]$ is contained in a connected subcomplex of $\mathcal{FF}(\Gamma; \mathcal{A})$ of simplicial diameter $\leq 6$.

Before proving Lemma 6.4 we apply it as follows. A function $\pi$ from the 0-skeleton of $\mathcal{FS}(\Gamma; \mathcal{A})$ to the 0-skeleton of $\mathcal{FF}(\Gamma; \mathcal{A})$ is called a projection map if $\pi[T] \in \Pi[T]$ for each $[T] \in \mathcal{FS}(\Gamma; \mathcal{A})$. Projection maps always exist by simply choosing $\pi[T] \in \Pi[T] \neq \emptyset$. If it is so desired, perhaps because of an aversion to wearing out the Axiom of Choice [Wei], for a concretely given group such as $\Gamma = F_n$ there are explicit constructions of projection maps, based on explicit enumeration of the 0-skeleta of $\mathcal{FS}(\Gamma; \mathcal{A})$ and of $\mathcal{FF}(\Gamma; \mathcal{A})$ and explicit computation of the set map $\Pi$. 

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Proposition 6.5. Assuming $D_{FF}(A) \geq 3$ the following hold:

1. $FF(\Gamma; A)$ is connected.

2. For any projection map $\pi$ from the 0-skeleton on $FS(\Gamma; A)$ to the 0-skeleton of $FF(\Gamma; A)$ we have:
   
   (a) $\pi$ is Lipschitz, with constant depending only on $\text{corank}(A)$ and $|A|$.
   
   (b) $\pi$ satisfies the “inverted coarse equivariance property”: $d(\pi(T \cdot \phi), \phi^{-1}(\pi[T]))$ has an upper bound depending only on $\text{corank}(A)$ and $|A|$, for $[T] \in FS(\Gamma; A)$ and $\phi \in \text{Out}(\Gamma; A)$.

Proof. To prove connectivity, for each 0-simplex $[T] \in FS(\Gamma; A)$ let $\Pi[1][T]$ be the union of all edge paths having endpoints in $\Pi[T]$ and having length $\leq 6$. Connectivity of $\Pi[1][T]$ follows from Lemma 6.4 (2). This is the basis step of an inductive proof of the following statement: for each edge path $S_0 \rightarrow S_1 \rightarrow \ldots \rightarrow S_L$ in $FS(\Gamma; A)$ the set $\Pi[1][S_0] \cup \ldots \cup \Pi[1][S_L]$ is connected. For the induction step one uses that either $S_{L-1} \succ S_L$ or $S_{L-1} \prec S_L$ and therefore by Lemma 6.4 (1) the set $\Pi[S_{L-1}] \cup \Pi[S_L]$ equals either $\Pi[S_{L-1}]$ or $\Pi[S_L]$ and so is connected. It follows that $\cup\{\Pi[1][T] \mid [T] \in FS(\Gamma; A)\}$ is connected, and this includes the entire 0-skeleton of $FF(\Gamma; A)$.

To prove $\pi$ is Lipschitz it suffices to prove for any 1-simplex $[S] \prec [T]$ in $FS(\Gamma; A)$ that $d(\pi(S), \pi(T))$ is bounded, but this follows from Lemma 6.4 which implies that $\Pi[S] \cup \Pi[T] = \Pi[T]$ has diameter $\leq 6$.

Inverted coarse equivariance for $\pi$ follows from inverted equivariance for $\Pi$ combined with Lemma 6.4 (2). 

Proof of Lemma 6.4. To prove item (1) choose a collapse map $S \rightarrow T$. Each element of $\Pi[T]$ has the form $F(U)$ for some collapse map $T \rightarrow U$, and since the composition $S \rightarrow T \rightarrow U$ is a collapse map it follows that $F(U) \in \Pi[S]$.

Having already proved (1), in order to prove (2) we may reduce to the case that the free splitting $T$ is generic (see Definition 3.3): by Proposition 3.6 there exists a generic free splitting $S$ such that $S \succ T$, and applying (1) we see that property (2) for $S$ implies property (2) for $T$.

Henceforth we assume that $T$ is generic. Again we may choose the realization $\Gamma = A_1 \ast \ldots \ast A_K \ast B$ of $A$ and the vertex groups of $T/\Gamma$ so that $V_{nt} = \{v_1, \ldots, v_K\}$ with vertex groups $A_1, \ldots, A_K$. The set of valence 1 vertices in $T/\Gamma$ is precisely $V_{nt}$, and every other vertex of $T/\Gamma$ has valence 2 or 3. Also, there is at least one valence 3 vertex in $T/\Gamma$ because otherwise either $K = 0$ and $\Gamma$ is a circle, or $K = 2$ and $\Gamma = A_1 \ast A_2$, and each of these is ruled out by the hypothesis $D_{FF}(A) \geq 3$. Every relatively natural vertex has valence 1 or 3 and hence is a natural vertex, so we shall drop the adverb “relatively” from the phrase “relatively natural” for the rest of the proof.
For the proof, given two proper, nontrivial relative core graphs $G_1 \neq G_2 \subset T/\Gamma$ we shall construct an edge path in $\mathcal{F}(\Gamma; A)$ with endpoints $[G_1], [G_2]$ and length $\leq 6$; the vertices along this edge path need not stay in $\Pi[T]$. We start with some easy cases:

**The Nested Case:** Suppose $G_1 \subset G_2$. In this case we have a path $[G_1] \supset [G_2]$ of length 1 and we are done.

**The Nontrivial Intersection Case:** Suppose $G_1 \cap G_2$ is nontrivial. In this case we have a path $[G_1] \supset [G_1 \cap G_2] \subset [G_2]$ of length 2 and we are also done.

To complete the proof, after applying the Nested Case it suffices to connect $[G_1], [G_2]$ by a path in $\mathcal{F}(\Gamma; A)$ of length $\leq 4$ under the following assumption:

(a) Each of $G_1, G_2 \subset T/\Gamma$ is maximal with respect to inclusion.

Furthermore, after applying the Nontrivial Intersection Case we may also assume that:

(b) The subgraph $G_1 \cap G_2$ is trivial.

From here the proof proceeds in two steps. Step 1 uses assumptions (a), (b) to show that $T/\Gamma$ with its natural cell structure is isomorphic one of three special graphs of low complexity:

**The clam:** the rank 2 graph having two valence 3 vertices and three edges each with its endpoints at distinct vertices.

**The spindle:** the rank 1 graph having two valence 3 vertices and two valence 1 vertices, consisting of a circle with two edges attached each by identifying a single endpoint of the edge to a distinct point on the circle.

**The clam with an antenna:** the rank 2 graph obtained from the clam by attaching one endpoint of an edge to an interior point of one of the clam edges.

Step 2 constructs the needed path of length $\leq 4$ in each of these three special cases.

**Step 1.** Note first that a proper relative core graph $G \subset T/\Gamma$ is maximal if and only if it is obtained from $T/\Gamma$ by removing the interior of a single natural edge having distinct endpoints. The “if” direction is clear. For the “only if” direction: if $G$ is missing the interiors of two natural edges $e_1, e_2$ and if $e_1$ has distinct endpoints then $T - \text{int}(e_1)$ is a proper relative core graph larger than $G$; and if $G$ is missing the interior of a natural edge $e$ with both ends at some natural vertex $p$ then, letting $e'$ be the edge having a single endpoint at $p$, the graph $G$ cannot contain $e'$ for otherwise $p$ would have valence 1 in $G$ and trivial vertex group, and so $T/\Gamma - \text{int}(e')$ is a proper relative core graph larger than $G$.

Let $G_i = T/\Gamma - \text{int}(e_i)$ for natural edges $e_1 \neq e_2$ each with distinct endpoints. Each set $\partial e_1, \partial e_2$ has two points, so their union $\partial e_1 \cup \partial e_2$ has four, three, or two points.
We handle those cases separately, and we also break into various subcases, in each of which we find that $T/\Gamma$ is a spindle, or a clam maybe with an antenna, or we find a contradiction.

**Case 1:** $\partial e_1 \cup \partial e_2$ is four points. In this case $G_1 \cap G_2$ is a relative core graph, because each of the four points $\partial e_1 \cup \partial e_2$ has valence 2 in $G_1 \cap G_2$ or valence 1 in $T/\Gamma$. But $G_1 \cap G_2$ is trivial, and so all four endpoints must have valence 1 in $T/\Gamma$. It follows that $T/\Gamma$ is the disjoint union of $e_1$ and $e_2$ and so is disconnected, a contradiction.

**Case 2:** $\partial e_1 \cup \partial e_2$ is three points. Let $\partial e_1 \cap \partial e_2 = \{p\}$, a single point. In this case $G_1 \cap G_2$ is not a core graph, because $p$ has valence 1 in $G_1 \cap G_2$ but valence 3 in $T/\Gamma$. Letting $e_3$ be the edge of $G_1 \cap G_2$ incident to $p$, and letting $H = (G_1 \cap G_2) - (\{p\} \cup \text{int}(e_3))$, it follows that $\text{core}(H) = \text{core}(G_1 \cap G_2)$, and so $H$ is trivial but we cannot yet conclude that $H$ itself is a relative core graph. Let $q_i \neq p$ be the endpoint of $e_i$ opposite $p$. There are two subcases, depending on whether $q_3$ equals one of $q_1$, $q_2$.

If $q_3$ is distinct from both $q_1$ and $q_2$ then each of $q_1$, $q_2$, $q_3$ has valence 2 in $H$ or valence 1 in $T/\Gamma$ and so $H$ is a relative core graph. But $H$ is trivial and so $H = V_{\text{int}} = \{q_1, q_2, q_3\}$ implying that $|A| = 3$, and implying that $T/\Gamma$ is a tree, more specifically a triod, and so $\text{corank}(A) = 0$. But then $D_{\text{GF}}(A) = 2$, a contradiction.

Suppose that $q_3$ equals one of $q_1$ or $q_2$, say $q_3 = q_1$, a point of valence 3 in $T/\Gamma$ and of valence 1 in $H$, and so $H$ is not a relative core graph. Let $e'$ be the edge of $H$ incident to $q_3$ and let $H' = H - (\{q_3\} \cup \text{int}(e'))$, so $\text{core}(H') = \text{core}(H) = \text{core}(G_1 \cap G_2)$ and $H'$ is trivial, but again $H'$ need not be a relative core graph. Let $q'$ be the endpoint of $e'$ opposite $q_3$. Depending on whether $q_2 = q'$ we will see that $T/\Gamma$ is either a spindle or a clam with an antenna. If $q_2 \neq q'$ then each has valence 1 in $T/\Gamma$ or valence 2 in $H'$ and so $H'$ is a relative core graph, but $H'$ is trivial and so both $q_2$, $q'$ have valence 1 in $T/\Gamma$, and in this case $T/\Gamma$ is a spindle. If $q_2 = q'$ then that point has valence 1 in $H'$ and valence 3 in $T/\Gamma$, and so $H'$ is not a relative core graph. Letting $e''$ be the edge of $H'$ with endpoint $q''$ opposite $q'$ it follows that $H'' = H' - (\{q'\} \cup \text{int}(e''))$ satisfies $\text{core}(H'') = \text{core}(H') = \text{core}(G_1 \cap G_2)$ and so $H''$ is trivial. Also, $q''$ has either valence 2 in $H''$ or valence 1 in $T/\Gamma$ so $H''$ is, at last, a relative core graph. By triviality it follows that $q''$ has valence 1 in $T/\Gamma$ and that $T/\Gamma$ is a clam with an antenna.

**Case 3:** $\partial e_1 \cup \partial e_2 = \{p, q\}$ is two points. These two points each have valence 1 in $G_1 \cap G_2$ and valence 3 in $T/\Gamma$. Let $e_p, e_q \subset T/\Gamma$ be the natural edges incident to $p, q$ respectively. If $e_p = e_q$ then $G_1 \cap G_2 = e_p$, and so $\text{core}(G_1 \cap G_2) = \emptyset$ and $T/\Gamma$ is a clam.

We may therefore assume that $e_p \neq e_q$. Let $H' = (G_1 \cap G_2) - (\{p, q\} \cup \text{int}(e_p) \cup \text{int}(e_q))$, so $\text{core}(H') = \text{core}(G_1 \cap G_2)$, implying that $H'$ is trivial. Let $p', q'$ be the endpoints of $e_p, e_q$ opposite $p, q$ respectively. Depending on whether $p' = q'$ the graph $T/\Gamma$ is either a spindle or a clam with an antenna, which is proved exactly as in Case 2 but with the notation changed to replace $q_2$ in Case 2 with $p'$ in Case 3.

This completes Step 1.
Step 2. Knowing that \( T/\Gamma \) is the clam, spindle, or clam with an antenna, we now consider these graphs one-at-a-time, and we consider the possibilities for the subgraphs \( G_1, G_2 \). In each case we will obtain a path in \( \mathcal{F}(\Gamma; A) \) of length \( \leq 4 \) connecting \( G_1 \) and \( G_2 \).

Notational alert: The symbols \( e_1, e_2 \) no longer assume their earlier meanings and are freed up for new use.

\( T/\Gamma \) is a clam. We may denote its edges \( e_1, e_2, e' \) so that \( G_1 = e_1 \cup e' \) and \( G_2 = e_2 \cup e' \). There is a collapse map \( T \searrow T' \) whose effect on \( T/\Gamma \) is to collapse \( e' \) to a point, and then there is an expansion \( T' \searrow T'' \) whose effect is to pull the two loops apart, so \( T''/\Gamma \) is a barbell graph with disjoint circles \( C_1, C_2 \) connected by an edge, and \( |G_1| = |C_1| \). We obtain a length 2 path \([G_1] = [C_1] \sqcup [C_1 \cup C_2] \sqcup [C_2] = [G_2]\) in \( \mathcal{F}(\Gamma; A) \).

\( T/\Gamma \) is a spindle. Let the circle edges be denoted \( e_1, e_2 \) with \( \partial e_1 = \partial e_2 = \{p, q\} \), and let the edges \( e_p, e_q \) be attached to \( p, q \) with opposite endpoints \( P, Q \) respectively. Consider the proper, nontrivial relative core graph \( C = e_1 \cup e_2 \cup \{P, Q\} \), representing the free factor system \([C]\). The graph \( G_1 \cap G_2 \), being trivial, cannot contain \( e_1 \cup e_2 \). By symmetry we may therefore suppose that \( G_1 = T/\Gamma - \text{int}(e_1) = e_p \cup e_2 \cup e_q \).

Consider the case that \( G_2 = T/\Gamma - \text{int}(e_2) = e_p \cup e_1 \cup e_q \). For each choice of \( i \neq j \in \{1, 2\} \) we may carry out the following operations. First collapse \( T \searrow T' \) with the effect on \( T/\Gamma \) of collapsing \( e_j \) to a point and taking \( e_i, e_p, e_q, P, Q \subset T/\Gamma \) to \( e'_i, e'_p, e'_q, P', Q' \subset T'/\Gamma \) respectively. The collapsed image of \( G_i \) is \( G'_i = e'_p \cup e'_q \) and the collapsed image of \( C \) is \( C' = e'_i \cup \{P', Q'\} \). Next, there is an expansion \( T''/\Gamma \) whose effect on \( T'/\Gamma \) is to pull apart the arc \( G'_1 \) and the circle \( e'_j \), so that in the quotient graph \( T''/\Gamma \) the free factor systems \([G_i]\) and \([C]\) are represented by relative core graphs \( G''_i, C'' \) having proper union \( G''_i \cup C'' \) which is also a relative core graph. We obtain a length 2 path \([G_i] = [G''_i] \sqcup [G''_i \cup C''] \sqcup [C''] = [C]\). Putting these together for \( i = 1, 2 \) we get a length 4 path connecting \([G_1]\) and \([G_2]\).

By symmetry of notation it remains to consider the case that \( G_2 = T/\Gamma - \text{int}(e_2) \). From the argument of the previous paragraph we get a length 2 path connecting \([G_1]\) to \([C]\), and since \( C \subset G_2 \) we get a length 1 path \([C] \sqcup [G_2]\), which together give a length 3 path connecting \([G_1]\) and \([G_2]\).

\( T/\Gamma \) is a clam with an antenna. Let the valence 1 vertex be \( R \), let its incident edge be \( e_R \) with opposite vertex \( r \), let other two incident edges to \( r \) be \( e_1, e_2 \) with opposite vertices \( p_1, p_2 \) respectively, and let the two edges with endpoints \( p_1, p_2 \) be \( e_3, e_4 \). Since \( G_1 \cap G_2 \) is trivial, it cannot contain a circle, and therefore it contains at most one of \( e_3, e_4 \). We assume \( e_4 \not\subset G_1 \cap G_2 \), and by symmetry we may assume \( e_4 \not\subset G_1 \), and so \( G_1 = e_R \cup e_1 \cup e_2 \cup e_3 \). We have \( e_4 \subset G_2 \). Exactly one of the edges \( e_R, e_1, e_2, e_3 \) is not in \( G_2 \), and by symmetry we may assume \( e_2 \subset G_2 \). If \( e_R \not\subset G_2 \) then \( G_1 \cap G_2 \) contains the circle \( e_1 \cup e_2 \cup e_3 \), contradicting that \( G_1 \cap G_2 \) is trivial. This shows that the edge not in \( G_2 \) is either \( e_1 \) or \( e_3 \). It follows that \( G_1 \cap G_2 = \tau \) is a tree containing \( R \): if
\( e_1 \not\subset G_2 \) then \( \tau = e_R \cup e_2 \cup e_3 \); whereas if \( e_3 \not\subset G_2 \) then \( \tau = e_R \cup e_1 \cup e_2 \). Let \( T > T' \) be the collapse map whose effect on \( T/\Gamma \) is to collapse the tree \( \tau \) to a point. The graph \( T'/\Gamma \) is a rank 2 rose whose rose point \( R' \) is the unique vertex with nontrivial vertex group, and whose petals \( C'_1, C'_2 \) satisfy \([G_1] = [C'_1]\) and \([G_2] = [C'_2]\). Let \( T' \prec T'' \) be the expansion which pulls the petals \( C'_1, C'_2 \) apart into the disjoint circles \( C''_1, C''_2 \) of the barbell graph \( T''/\Gamma \), so that the arc connecting \( C''_1 \) to \( C''_2 \) is subdivided at its midpoint \( R'' \), having nontrivial vertex group, into edges \( E''_i \) connecting \( C''_i \) to \( R'' \) \((i = 1, 2)\). We have \([G_i] = [C''_i] = [C''_i \cup E''_i]\). We then have a chain of proper, nontrivial relative core graphs

\[
C''_1 \cup E''_1 \supset C''_1 \cup R'' \subset C''_1 \cup R'' \cup C''_2 \supset R'' \cup C''_2 \subset E''_2 \cup C''_2
\]

producing a length 4 path

\[
[G_1] = [C''_1 \cup E''_1] \supset [C''_1 \cup R''] \subset [C''_1 \cup R'' \cup C''_2] \supset [R'' \cup C''_2] \subset [E''_2 \cup C''_2] = [G_2]
\]

\( \diamond \)

### 6.3 Proof of hyperbolicity of \( \mathcal{FF}(\Gamma; A) \)

We shall apply the following theorem of I. Kapovich and K. Rafi:

**Theorem 6.6** ([KR14] Proposition 2.5). Let \( X \) be a connected simplicial complex which is \( \delta \)-hyperbolic with respect to the simplicial metric. Let \( Y \) be a connected simplicial complex. Suppose that there exists a map of 0-skeleta \( \pi: X^{(0)} \to Y^{(0)} \) with the following properties:

1. \( \pi \) is surjective
2. \( \pi \) is \( K \)-Lipschitz
3. There exists a constant \( D \) such that for all \( v, w \in X^{(0)} \), if \( d_Y(v, w) \leq 1 \), and if \( v = v_0, v_1, \ldots, v_L = w \) are the vertices along a geodesic in the 1-skeleton \( X^{(1)} \) between \( v \) and \( w \), then \( \text{diam}_Y\{\pi(v_0), \ldots, \pi(v_L)\} \leq D \).

It follows that \( Y \) is \( \delta_1 \)-hyperbolic with respect to the simplicial metric. Furthermore,

4. If \( v_0, v_1, \ldots, v_L \) are the vertices along a geodesic in \( X^{(1)} \) then \( \{\pi(v_0), \pi(v_1), \ldots, \pi(v_L)\} \) is \( C \)-Hausdorff close to a geodesic in \( Y \).

The constants \( \delta_1 \) and \( C \) depend only on \( \delta, K, \) and \( D \). \( \diamond \)

By incorporating item (4) of the previous theorem into the proof of Theorem 1.5 we get the following:
Theorem 6.7 (Enhanced version of Theorem 1.5). For any group $\Gamma$ and any nonexceptional free factor system $A$ of $\Gamma$, the complex $\mathcal{FF}(\Gamma; A)$ is nonempty, connected, and hyperbolic. Furthermore, the image under $\pi: \mathcal{FS}(\Gamma; A) \to \mathcal{FF}(\Gamma; A)$ of any geodesic in $\mathcal{FS}(\Gamma; A)$ is uniformly Hausdorff close to a geodesic in $\mathcal{FF}(\Gamma; A)$.

Proof. Choose a special projection map $\pi: \mathcal{FS}(\Gamma; A) \to \mathcal{FF}(\Gamma; A)$ having the property that for each 0-simplex $[T] \in \mathcal{FS}(\Gamma; A)$, if $\mathcal{F}(T) \neq A$ then $\pi[T] = \mathcal{F}(T)$; such a map exists since clearly $\mathcal{F}(T) \in \Pi[T]$. Surjectivity of this special $\pi$ follows from Lemma 3.1, and $\pi$ is Lipschitz by Proposition 6.5 (2a). These are hypotheses (1) and (2) of the Kapovich–Rafi Theorem 6.6 above. It remains to verify hypothesis (3).

Consider 0-simplices $[S], [T] \in \mathcal{FS}(\Gamma; A)$ such that $d(\pi(S), \pi(T)) \leq 1$.

We first reduce to the case that $\mathcal{F}(S) = \pi(S)$ and that $\mathcal{F}(T) = \pi(T)$; by the special choice of $\pi$ this is equivalent to reducing to the case $\mathcal{F}(S) \neq A$ and $\mathcal{F}(T) \neq A$. From the requirement that $\pi[S] \in \Pi[S]$ and $\pi[T] \in \Pi[T]$ it follows that there exist collapse maps $S \succ S'$ and $T \succ T'$ such that $\pi(S) = \mathcal{F}(S') \neq A$ and $\pi(T) = \mathcal{F}(T') \neq A$, and from the special choice of $\pi$ it follows that $\mathcal{F}(S') = \pi(S')$ and $\mathcal{F}(T') = \pi(T')$. Note that in $\mathcal{FS}(\Gamma; A)$ we have $d(S, S') \leq 1$ and $d(T, T') \leq 1$. By hyperbolicity of $\mathcal{FS}(\Gamma; A)$, any geodesic connecting $S$ to $T$ stays uniformly Hausdorff close to any geodesic connecting $S'$ to $T'$, and since $\pi$ is Lipschitz it follows that the $\pi$-images of these geodesics are uniformly Hausdorff close in $\mathcal{FF}(\Gamma; A)$. Once we have verified that hypothesis (3) holds for an $S', T'$ geodesic, it holds as well for an $S, T$ geodesic, completing the reduction.

Henceforth we assume $\mathcal{F}(S) = \pi(S)$ and that $\mathcal{F}(T) = \pi(T)$. Since $d(\mathcal{F}(S), \mathcal{F}(T)) \leq 1$, up to transposing notation we may assume $\mathcal{F}(S) \subset \mathcal{F}(T)$. Combining Lemma 4.2 and Lemma 4.3, there exists a collapse map $S \succ S''$ and a fold sequence from $S''$ to $T$. Since $d(S, S'') \leq 1$ it follows, just as in the previous paragraph, that once we have verified the desired conclusions for an $S'', T$ geodesic, the conclusions for an $S, T$ geodesic follow.

Henceforth we may assume that there exists a fold sequence from $S$ to $T$, denoted

$$S = S_0 \xrightarrow{f_1} S_1 \xrightarrow{f_2} \cdots \xrightarrow{f_L} S_L = T$$

By Theorem 5.4 the sequence $S_0, S_1, \ldots, S_L$ can be reparameterized as a uniform quasigeodesic. By hyperbolicity of $\mathcal{FS}(\Gamma; A)$ this quasigeodesic is uniformly Hausdorff close in $\mathcal{FS}(\Gamma; A)$ to any $S, T$ geodesic. And by the Lipschitz property for $\pi$ the images of the quasigeodesic and the geodesics are uniformly Hausdorff close in $\mathcal{FF}(\Gamma; A)$. It therefore suffices to bound the diameter of the set $\{\pi(S_0), \pi(S_1), \ldots, \pi(S_L)\}$. By Lemma 3.2(3) we have $\mathcal{F}(S_0) \subset \mathcal{F}(S_1) \subset \cdots \subset \mathcal{F}(S_L)$ and so the set $\{\mathcal{F}(S_0), \mathcal{F}(S_1), \ldots, \mathcal{F}(S_L)\}$ has diameter $\leq 1$ in $\mathcal{FF}(\Gamma; A)$. Since $A$ is properly contained in $\mathcal{F}(S) = \mathcal{F}(S_0)$, it follows that $A$ is properly contained in each of $\mathcal{F}(S_0), \mathcal{F}(S_1), \ldots, \mathcal{F}(S_L)$, and so $\pi(S_i) = \mathcal{F}(S_i)$ for $0 \leq i \leq L$. The set $\{\pi(S_0), \pi(S_1), \ldots, \pi(S_L)\}$ therefore has diameter $\leq 1$. \hfill \Box
References

[BBF10] M. Bestvina, K. Bromberg, and K. Fujiwara, *Constructing group actions on quasi-trees and applications to mapping class groups*, arXiv:1006.1939, 2010.

[BF91] M. Bestvina and M. Feighn, *Bounding the complexity of simplicial group actions on trees*, Invent. Math. **103** (1991), no. 3, 449–469.

[BF02] M. Bestvina and K. Fujiwara, *Bounded cohomology of subgroups of mapping class groups*, Geom. Topol. **6** (2002), 69–89 (electronic).

[BF12] M. Bestvina and M. Feighn, *Subfactor projections*, arXiv:1211.1730, 2012.

[BF14] ———, *Hyperbolicity of the complex of free factors*, Adv. Math. **256** (2014), 104–155, arXiv:1107.3308.

[BFH00] M. Bestvina, M. Feighn, and M. Handel, *The Tits alternative for Out(F_n)*. I. *Dynamics of exponentially-growing automorphisms*, Ann. of Math. **151** (2000), no. 2, 517–623.

[BKMM12] J. Behrstock, B. Kleiner, Y. Minsky, and L. Mosher, *Geometry and rigidity of mapping class groups*, Geom. Topol. **16** (2012), no. 2, 781–888, Preprint arXiv:0801.2006.

[BM08] J. Behrstock and Y. Minsky, *Dimension and rank for mapping class groups*, Ann. of Math. **167** (2008), no. 3, 1055–1077.

[Bow08] B. Bowditch, *Tight geodesics in the curve complex*, Invent. Math. **171** (2008), no. 2, 281–300.

[Chi76] I. Chiswell, *The Grushko-Neumann theorem*, Proc. London Math. Soc. (3) **33** (1976), no. 3, 385–400.

[Coh89] D. E. Cohen, *Combinatorial group theory: a topological approach*, LMS Student Texts, no. 14, Cambridge Univ. Press, 1989.

[CT94] D. Collins and E. Turner, *Efficient representatives for automorphisms of free products*, Michigan Math. J. **41** (1994), no. 3, 443–464.

[FM14] S. Francaviglia and A. Martino, *Stretching factors, metrics and train tracks for free products*, arXiv:1312.4172v2, February 2014.
[GL07] V. Guirardel and G. Levitt, The outer space of a free product, Proc. London Math. Soc. (3) 94 (2007), no. 3, 695–714, preprint, arXiv:math.GR/0501288.

[Hat95] A. Hatcher, Homological stability for automorphism groups of free groups, Comment. Math. Helv. 70 (1995), no. 1, 39–62.

[HM] M. Handel and L. Mosher, Relative free splitting and free factor complexes II: Loxodromic outer automorphisms, In preparation.

[HM13] ———, The free splitting complex of a free group I: Hyperbolicity, Geom. Topol. 17 (2013), 1581–1670, arXiv:1111.1994.

[HM14] ———, The free splitting complex of a free group II: Loxodromic outer automorphisms, arXiv:1402.1886, 2014.

[HV98] A. Hatcher and K. Vogtmann, The complex of free factors of a free group, Quart. J. Math. Oxford Ser. (2) 49 (1998), no. 196, 459–468.

[KR14] I. Kapovich and K. Rafi, On hyperbolicity of free splitting and free factor complexes, Groups Geom. Dyn. 8 (2014), no. 2, 391–414, arXiv:1206.3626.

[Man10] J. Mangahas, Uniform uniform exponential growth of subgroups of the mapping class group, Geom. Funct. Anal. 19 (2010), no. 5, 1468–1480.

[Mar99] A. Martino, An index theorem for automorphisms of free products, J. Group Theory 2 (1999), no. 2, 199–211.

[McM96] D. McCullough and A. Miller, Symmetric automorphisms of free products, Mem. Amer. Math. Soc. 122 (1996), no. 582, viii+97.

[MM99] H. Masur and Y. Minsky, Geometry of the complex of curves, I. Hyperbolicity, Invent. Math. 138 (1999), no. 1, 103–149.

[MM00] ———, Geometry of the complex of curves II: Hierarchical structure, Geom. Funct. Anal. 10 (2000), no. 4, 902–974.

[SW79] P. Scott and C. T. C. Wall, Topological methods in group theory, Homological group theory, Proceedings of Durham symposium, Sept. 1977, London Math. Soc. Lecture Notes, vol. 36, 1979, pp. 137–203.

[Wei] Z. Weiner, Saturday Morning Breakfast Cereal, {http://www.smbc-comics.com/?id=2595}.