Lie symmetries of generalized Burgers equations: application to boundary-value problems

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The goal of the paper is twofold. The first target is to present an enhanced group classification for variable coefficient generalized Burgers equations $u_t + a(u^n)_x = g(t)u_{xx}$, which corrects and completes the similar results existing in the literature. The second target is to show that the direct procedure of solving boundary-value problems using Lie symmetries is more general and straightforward than the technique suggested by Moran and Gaggioli in [J. Engrg. Math. 3 (1969), 151–162].

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1 Introduction

The Burgers equation, $u_t + uu_x + u_{xx} = 0$, is one of the simplest nonlinear (1 + 1) evolution equations. Nevertheless it has a long history as it was already known to Forsyth [11] and discussed by Bateman not many years later [2]. However, it was a serious contribution made by Burgers which led to its present name [8]. Burgers equation has been used to describe many processes in fluid mechanics and a variety of other fields which do seem to be rather disparate. It’s remarkable feature is that it can be transformed to the standard heat equation by means of the Hopf–Cole transformation [9, 13].

In 1989 Hammerton and Crighton [12] derived the generalized Burgers equation describing the propagation of weakly nonlinear acoustic waves under the influence of geometrical spreading and thermoviscous diffusion, which in non-dimensional variables can be reduced to the form

$$u_t + uu_x = g(t)u_{xx}.$$

In this paper we investigate Lie symmetries of a generalized Burgers equation of the form

$$u_t + a(u^n)_x = g(t)u_{xx},$$

where $a$ is a nonzero constant, $g$ is an arbitrary smooth nonvanishing function and $n \neq 0,1$. The enhanced group classification for this class is achieved and comes to complete the results that exist in the literature [10, 31]. This generalization of Burgers equation was recently treated by Abd-el-Malek and Helal [1] who used what can only be described as a very ingenious method to obtain the solution. Here we present an alternate approach to solving the equation with associated boundary conditions and we are of the opinion that this approach is easier to implement and somewhat more transparent.

The paper is organized as follows. In Section 2 we determine the equivalence group. In Section 3 we calculate the Lie point symmetries of (1) and in Section 4 we demonstrate the solution of the boundary-value problem.
2 Equivalence transformations

The first step of solving a group classification problem is to find the equivalence group admitted by a given class \([22]\). Notions of different kinds of equivalence group can be found, e.g., in \([30]\). We were able to study all admissible \([27]\) (form-preserving \([15]\)) transformations in class \([1]\), in other words we described all point transformations that link equations from the class. The results of the study are given in the following statements.

**Theorem 1.** The usual equivalence group \(G^\sim\) of class \([1]\) comprises the transformations

\[
\tilde{t} = \delta_1 t + \delta_2, \quad \tilde{x} = \delta_3 x + \delta_4, \quad \tilde{u} = \delta_5 u, \quad \tilde{a} = \frac{\delta_3}{\delta_5} a, \quad \tilde{g} = \frac{\delta_3^2}{\delta_1} g, \quad \tilde{n} = n,
\]

where \(\delta_j, j = 1, \ldots, 5,\) are arbitrary constants with \(\delta_1 \delta_3 \delta_5 \neq 0\).

It appears that for \(n = 2\) class \([1]\) admits a nontrivial conditional equivalence group which is wider than \(G^\sim\).

**Theorem 2.** The generalized equivalence group \(G_2^\sim\) of the class,

\[
u_t + a(u^2)_x = g(t) u_{xx}, \tag{2}
\]

comprises the transformations

\[
\tilde{t} = \frac{\alpha t + \beta}{\gamma t + \delta}, \quad \tilde{x} = \frac{\kappa x + \mu_1 t + \mu_0}{\gamma t + \delta}, \quad \tilde{u} = \frac{\sigma}{2a(\alpha \delta - \beta \gamma)} (2a\kappa(\gamma t + \delta)u - \kappa \gamma x + \mu_1 \delta - \mu_0 \gamma), \\
\tilde{a} = \frac{a}{\sigma} \quad \text{and} \quad \tilde{g} = \frac{\kappa^2}{\alpha \delta - \beta \gamma} g,
\]

where \(\alpha, \beta, \gamma, \delta, \kappa, \mu_1, \mu_0, \sigma\) are constants defined up to a nonzero multiplier, \(\alpha \delta - \beta \gamma \neq 0\) and \(\kappa \sigma \neq 0\).

**Theorem 3.** Let the equations \(u_t + a(u^n)_x = g(t) u_{xx}\) and \(\tilde{u}_t + \tilde{a}(\tilde{u}^\tilde{n})_{\tilde{x}} = \tilde{g}(\tilde{t}) \tilde{u}_{\tilde{x} \tilde{x}}\) be connected by a point transformation \(T\) in the variables \(t, x\) and \(u\). Then \(\tilde{n} = n\), and the transformation \(T\) is the projection of a transformation from \(G^\sim\) or \(G_2^\sim\) on the space \((t, x, u)\), if \(n \neq 2\) or \(n = 2\), respectively.

**Proof.** Suppose that an equation from class \([1]\) is connected with an equation

\[
\tilde{u}_t + \tilde{a}(\tilde{u}^\tilde{n})_{\tilde{x}} = \tilde{g}(\tilde{t}) \tilde{u}_{\tilde{x} \tilde{x}} \tag{3}
\]

from the same class by a point transformation \(\tilde{t} = T(t, x, u), \tilde{x} = X(t, x, u), \tilde{u} = U(t, x, u)\), where \(\frac{\partial(T, X, U)}{\partial(t, x, u)} \neq 0\). It is known that for evolution equations we have the restrictions, \(T_x = T_u = 0\), on the general form of admissible transformations \([15]\) and moreover for equations of the form \(u_t = F(t, x, u) u_{xx} + G(t, x, u, u_x)\) we necessarily have the condition \(X_u = 0\) \([26]\). Therefore it is enough to consider a transformation, \(T\), of the form

\[
\tilde{t} = T(t), \quad \tilde{x} = X(t, x), \quad \tilde{u} = U(t, x, u),
\]

where \(T_x X_u U_u \neq 0\). After we change the variables in \((3)\), we obtain an equation in the variables without tildes. It should be an identity on the manifold \(L\) determined by \([1]\) in the second-order jet space \(J^2\) with the independent variables \((t, x)\) and the dependent variable \(u\). To involve the constraint between variables of \(J^2\) on the manifold \(L\), we substitute the expression of \(u_t\) implied
by equation (1). The splitting of this identity with respect to the derivatives \(u_{xx}\) and \(u_x\) results in the determining equations for the functions \(T, X\) and \(U\)

\[
U_{uu} = 0, \quad \ddot{g} T_t - g X_x^2 = 0, \\
X_t U_x - X_x U_t + \ddot{g} T_t \left( \frac{U_x}{X_x} \right)_x - \ddot{a} n T_t U_{\bar{n}-1} U_x = 0 \quad \text{and} \\
anu^{\bar{n}-1} - \ddot{a} n T_t U_{\bar{n}-1} + 2\dot{g} T_t U_{xx} - \ddot{g} T_t \frac{X_{xx}}{X_x^3} + X_t X_x = 0.
\]

Equations (1) imply that

\[
U = \eta^1(t, x) u + \eta^0(t, x), \quad X = \varphi(t) x + \psi(t) \quad \text{and} \quad \ddot{g} = \frac{\varphi^2}{T_t} g.
\]

Here the functions \(\eta^i(t, x), i = 1, 2, \varphi(t), \psi(t)\) are arbitrary smooth functions of their arguments and \(\eta^1 \varphi \neq 0\).

When we use the differential consequences of the fourth equation with respect to \(u\), we get that the arbitrary element \(n\) is invariant under the action of a point transformation, i.e., \(\ddot{n} = n\). Also we obtain that \(\eta^0 = 0 \quad \forall \quad n \neq 2\). After we substitute the expressions for \(U, X\) and \(\ddot{g}\) into the third and fourth determining equations, we can split them with respect to \(u\). Further consideration varies depending upon whether \(n \neq 2\) or \(n = 2\).

**I.** If \(n \neq 2\), then splitting of (5) and (6) results in the equations

\[
\eta_x^1 = 0, \quad \eta^1 (\varphi_t x + \psi_t) + 2\varphi g \eta_x^1 = 0, \\
\varphi \eta^1_t = \eta_x^1 (\varphi_t x + \psi_t) + \varphi g \eta_{xx}^1, \quad \text{and} \\
\varphi a \eta^1_t = \ddot{a} (\eta^1)^2 T_t.
\]

The general solution of this system is given by

\[
T = \delta_1 t + \delta_2, \quad \varphi = \delta_3, \quad \psi = \delta_4, \quad \eta^1 = \delta_5,
\]

where \(\delta_j, j = 1, \ldots, 5\), are arbitrary constants with \(\delta_1 \delta_3 \delta_5 \neq 0\). Then \(\ddot{a} = \frac{\delta_3 \delta_5^{1-n}}{\delta_1} a\) and \(\ddot{g} = \frac{\delta_5^2}{\delta_1} g\).

The statement of Theorem 1 is proved.

**II.** If \(n = 2\), then splitting of equations (5) and (6) leads to the system

\[
\eta_x^1 = 0, \quad \eta^1 (\varphi_t x + \psi_t) + 2\varphi g \eta_x^1 - 2\ddot{a} T_t \eta^0 \eta^1 = 0, \\
\varphi \eta^1_t = \eta_x^1 (\varphi_t x + \psi_t) + \varphi g \eta_{xx}^1 - 2\ddot{a} T_t (\eta^0 \eta^1)_x, \\
\varphi \eta^0_t = \eta_x^0 (\varphi_t x + \psi_t) + \varphi g \eta_{xx}^0 - 2\ddot{a} T_t \eta^0 \eta^0_x, \quad \text{and} \\
\varphi a = \ddot{a} \eta^1 T_t.
\]

From this system we initially obtain forms of \(\eta^1\) and \(\eta^0\) as

\[
\eta^1 = \sigma \frac{\varphi}{T_t}, \quad \eta^0 = \frac{\sigma}{2aT_t} (\varphi_t x + \psi_t), \quad \ddot{a} = \frac{a}{\sigma},
\]

where \(\sigma\) is a nonzero constant. The remaining equations for the functions \(T, \varphi\) and \(\psi\) are

\[
\left( \frac{\varphi^2}{T_t} \right)_t = 0, \quad \left( \frac{\varphi_t}{T_t} \right)_t = 0, \quad \left( \frac{\psi_t}{T_t} \right)_t = 0,
\]

the general solution of which can be written as

\[
T = \frac{\alpha t + \beta}{\gamma t + \delta}, \quad \varphi = \frac{\kappa}{\gamma t + \delta}, \quad \psi = \frac{\mu_1 t + \mu_0}{\gamma t + \delta}.
\]
where $\alpha, \beta, \gamma, \delta, \kappa, \mu_1$ and $\mu_0$ are constants defined up to a nonzero multiplier, $\alpha \delta - \beta \gamma \neq 0$ and $\kappa \neq 0$. When we substitute the functions $T, \varphi$ and $\psi$ into the formulas for $\eta^1$ and $\eta^0$, we get exactly the statement of Theorem 2. The group $\hat{G}_2$ is called generalized since transformation component for $u$ depends on arbitrary element $a$ of the class.

We have found that all admissible transformations in class (1) are exhausted by those presented in Theorems 1 and 2. Therefore, Theorem 3 is proved. \hfill \Box

Note that if $a = 1/n$ the group $\hat{G}_2$ was found previously in [14] (see also [23,24]) in the course of study of form-preserving (admissible) transformations for the class of generalized Burgers equations, $u_t + uu_x + f(t, x)u_{xx} = 0$.

3 Lie symmetries

We perform the group classification of class (1) within the framework of the classical Lie approach [21,22]. Naturally one performs the group classification for class (1) up to $G^*$-equivalence and for its subclass (2) up to $\hat{G}_2^*$-equivalence.

We search for operators of the form $\Gamma = \tau(t, x, u)\partial_t + \xi(t, x, u)\partial_x + \theta(t, x, u)\partial_u$ which generate one-parameter groups of point-symmetry transformations of an equation from class (1). Any such vector field, $\Gamma$, satisfies the infinitesimal invariance criterion, i.e., the action of the second prolongation, $\Gamma^{(2)}$, of the operator $\Gamma$ on equation (1) results in the conditions being an identity for all solutions of this equation. Namely, we require that

$$\Gamma^{(2)}\{u_t + anu^{n-1}u_x - g(t)u_{xx}\} = 0$$

identically, modulo equation (1).

The criterion of infinitesimal invariance implies that

$$\tau = \tau(t), \quad \xi = \xi(t, x), \quad \theta = \theta^1(t, x)u + \theta^0(t, x),$$

where $\tau, \xi, \theta^1$ and $\theta^0$ are arbitrary smooth functions of their variables. The remaining determining equations have the form

$$2g\xi_x = (g\tau)_t,$$  

$$an\theta^2_x u^{n+1} + an\theta^0_x u^n + (\theta^1_x - g\theta^1_{xx})u^2 + (\theta^0_t - g\theta^0_{xx})u = 0,$$  

$$an(\tau_t - \xi_x + (n-1)\theta^1_t)u^{n+1} + an(n-1)\theta^0_t u^n + (g\xi_x - 2g\theta^1_x - \xi_t)u^2 = 0.$$  

It is easy to see from (8) that $\xi_{xx} = 0$. The second and the third equations can be split with respect to different powers of $u$. Special cases of splitting arise if $n = 0, 1, 2$. If $n = 0$ or $n = 1$, equations (1) are linear and are excluded from consideration (Lie symmetries of second-order linear differential equations in two dimensions were studied over the century ago by S. Lie [18].) Therefore we investigate two cases, $n \neq 2$ and $n = 2$, separately.

I. If $n \neq 2$, then we immediately get that $\theta^1 = c_0$, where $c_0$ is an arbitrary constant, $\theta^0 = 0$.

$$\tau = c_1 t + c_2, \quad \xi = (c_1 + (n-1)c_0)x + c_3, \quad \theta = c_0 u.$$  

The classifying equation on $g$ has the form

$$(c_1 t + c_2)g_t = (c_1 + 2(n-1)c_0)g.$$  

Further consideration is performed using the method of furcate split suggested in [26]. For any operator $\Gamma$ from maximal Lie invariance algebra $A^{\text{max}}$ equation (11) gives some equations on $g$ of the general form

$$(pt + q)g_t = sg.$$
where \( p, q, s = \text{const} \). In general for all operators from \( A^{\text{max}} \) the number \( k \) of such independent equations is no greater than 2 otherwise they form an incompatible system on \( g \). There exist three inequivalent cases for the value of \( k \) given by \( k = 0, k = 1 \) and \( k = 2 \).

If \( k = 0 \), then \( \mathfrak{G} \) is identically zero and \( c_1 = c_2 = c_0 = 0 \). So, if \( g \) is arbitrary, we obtain that the kernel of maximal Lie invariance algebras of equations from (1) is the one-dimensional algebra \( \langle \partial_\xi \rangle \).

If \( k = 1 \), then \( g \in \{ \varepsilon e^t, \varepsilon t^\rho \} \mod G^\sim \), where \( \varepsilon = \pm 1 \) and \( \rho \neq 0 \). In the exponential case, \( g = \varepsilon e^t \), we have

\[
\Gamma = 2(n - 1)c_0 \partial_t + ((n - 1)c_0 x + c_3) \partial_x + c_0 u \partial_u.
\]

If \( g = \varepsilon t^\rho \) and \( \rho \neq 0 \), then

\[
\Gamma = c_1 t \partial_t + \left( \frac{1}{2}(\rho + 1)c_1 x + c_3 \right) \partial_x + \frac{1}{2(\rho - 1)}c_1 u \partial_u.
\]

In both these cases the maximal Lie-invariance algebras are two-dimensional with basis operators presented in Cases 2 and 3 of Table 1.

If \( k = 2, g = 1 \) mod \( G^\sim \). The infinitesimal operator takes the form

\[
\Gamma = (c_1 t + c_2) \partial_t + \left( \frac{1}{2}c_1 x + c_3 \right) \partial_x - \frac{1}{2(n - 1)}c_1 u \partial_u.
\]

Therefore we have proven that, if \( g \) is a constant, the maximal Lie invariance algebra of \( \mathfrak{G} \) with \( n \neq 2 \) is three-dimensional spanned by operators presented in Case 4 of Table 1.

II. If \( n = 2 \), then splitting of (9) and (10) results in the system

\[
\begin{align*}
\theta_x^0 &= 0, & 2a \theta_x^1 + \theta^1 &= 0, & \theta_t^0 - g \theta_{xx}^0 &= 0, \\
\tau_t - \xi_x + \theta^1 &= 0, & 2a \theta^0 - \xi_t &= 0.
\end{align*}
\]

The general solution of this system is

\[
\begin{align*}
\tau &= c_2 t^2 + c_1 t + c_0, & \xi &= \left( c_2 t + \frac{1}{2}c_1 + c_5 \right) x + 2ac_3 t + c_4, \\
\theta^1 &= -c_2 t - \frac{1}{2}c_1 + c_5, & \theta^0 &= \frac{1}{2a}c_2 x + c_3,
\end{align*}
\]

where \( c_i, i = 0, \ldots, 5 \), are arbitrary constants. The classifying equation (8) takes the form

\[
(c_2 t^2 + c_1 t + c_0)g_t = 2c_5 g.
\] (12)

For any operator \( \Gamma \) from the maximal Lie invariance algebra \( A^{\text{max}} \) equation (12) gives some equations for \( g \) of the general form

\[
(p t^2 + q t + r)g_t = s g,
\] (13)

where \( p, q, r, s = \text{const} \). As in the previous case the number, \( k \), of such independent equations is no greater than 2 otherwise they form an incompatible system on \( g(t) \). So three inequivalent cases for the value of \( k \) should be considered, namely \( k = 0, k = 1 \) and \( k = 2 \).

If \( k = 0 \), then (12) is identically zero and \( c_0 = c_1 = c_2 = c_5 = 0 \). So, if \( g(t) \) is arbitrary, we obtain that the kernel of the maximal Lie invariance algebras of equations from (2) is the two-dimensional algebra \( \langle \partial_x, 2at \partial_x + \partial_u \rangle \).

If \( k = 1 \), the following statement is true.
Table 1. Group classification of the class $u_t + a(u^n)_x = g(t)u_{xx}, n \neq 0, 1$.

| no. | $n$ | $g$ | Basis operators of $A^\text{max}$ |
|-----|-----|-----|----------------------------------|
| 1   | $\neq 2$ | $\forall$ | $\partial_x$ |
| 2   | $\neq 2$ | $\varepsilon t^\rho$ | $\partial_x, \ 2t\partial_t + (\rho + 1)x\partial_x + \frac{\rho - 1}{n - 1}u\partial_u$ |
| 3   | $\neq 2$ | $\varepsilon \varepsilon t$ | $\partial_x, \ 2\partial_t + x\partial_x + \frac{1}{n - 1}u\partial_u$ |
| 4   | $\neq 2$ | $1$ | $\partial_x, \partial_t, \ 2t\partial_t + x\partial_x - \frac{1}{n - 1}u\partial_u$ |
| 5   | $2$ | $\forall$ | $\partial_x, \ t\partial_x + \partial_u$ |
| 6   | $2$ | $\varepsilon t^\rho$ | $\partial_x, \ t\partial_x + \partial_u, \ 2t\partial_t + (\rho + 1)x\partial_x + (\rho - 1)u\partial_u$ |
| 7   | $2$ | $\varepsilon \varepsilon t$ | $\partial_x, \ t\partial_x + \partial_u, \ 2\partial_t + x\partial_x + u\partial_u$ |
| 8   | $2$ | $\varepsilon^2 \rho \arctan t$ | $\partial_x, \ t\partial_x + \partial_u, \ (t^2 + 1)\partial_t + (t + \rho)x\partial_x + (x + (\rho - t)u)\partial_u$ |
| 9   | $2$ | $1$ | $\partial_x, \ t\partial_x + \partial_u, \ \partial_t, \ 2t\partial_t + x\partial_x - u\partial_u, \ t^2\partial_t + tx\partial_x + (x - tu)\partial_u$ |

Here $\varepsilon = \pm 1 \mod G^\sim$ and $\rho$ is a nonzero constant. In all cases $a = 1/n \mod G^\sim$. In Case 6 we can set, modulo $G^\sim$, either $\rho > 0$ or $\rho < 0$.

**Lemma 1.** Up to $\hat{G}^\sim$-equivalence the parameter quadruple $(p, q, r, s)$ can be assumed to belong to the set

$$\{(0, 1, 0, \tilde{s}), (0, 0, 1, 1), (1, 0, 1, s')\},$$

where $\tilde{s}$, $s'$ are nonzero constants, $\tilde{s} > 0$.

**Proof.** Combined with multiplication by a nonzero constant, each transformation from the equivalence group $\hat{G}^\sim$ can be extended to the coefficient quadruple of equation (13) as

$$\begin{align*}
\tilde{p} &= \nu(p^2 - q\gamma\delta + r\gamma^2), \\
\tilde{q} &= \nu(-2p\beta\delta + q(\alpha\delta + \beta\gamma) - 2r\alpha\gamma), \\
\tilde{r} &= \nu(p^2 - q\alpha\beta + r\alpha^2), \\
\tilde{s} &= \nu s\Delta.
\end{align*}$$

Here $\Delta = \beta\gamma - \alpha\delta$ and $\nu$ is an arbitrary nonzero constant.

There are only three $\hat{G}^\sim$-inequivalent values of the triple $(p, q, r)$ depending upon the sign of $D = q^2 - 4pr$,

$$\begin{align*}
(0, 1, 0) & \text{ if } D > 0, & (0, 0, 1) & \text{ if } D = 0 \text{ and } (1, 0, 1) & \text{ if } D < 0.
\end{align*}$$

Indeed, if $D > 0$, then there exist two linearly independent pairs $(\delta, \gamma)$ and $(\alpha, \beta)$ such that $p^2 - q\gamma\delta + r\gamma^2 = 0$ and $p^2 - q\alpha\beta + r\alpha^2 = 0$. For these values of the constants, $\alpha$, $\beta$, $\gamma$ and $\delta$, we have $\tilde{p} = \tilde{r} = 0$. The coefficient $\tilde{q}$ is necessarily nonzero and it can be scaled to 1 using multiplication by an appropriate value of $\nu$. Certain freedom in varying group parameters is preserved even after fixing of the form of the triple $(p, q, r)$ and $(\tilde{p}, \tilde{q}, \tilde{r})$. This allows us to set constraint for the coefficient $\tilde{s}$. Thus, transformation $t = 1/t$ alternates the sign of $\tilde{s}$. So, it can be assumed positive one.

In the case $D = 0$ we choose values of $\alpha$, $\beta$, $\gamma$ and $\delta$ for which $p^2 - q\gamma\delta + r\gamma^2$ and the pair $(\delta, \gamma)$ is not proportional to the pair $(\alpha, \beta)$. Then we obtain that $\tilde{p} = 0$ and $\tilde{q} = \nu\beta(q\gamma - 2p\delta) + \nu\alpha(\delta\gamma - 2r\gamma) = 0$. Appropriate choice of the pair $(\alpha, \beta)$ allows us to set $\tilde{r} = 1$. Then the residual constant $\tilde{s}$ can be scaled to one by choice of $\nu$. 
If $D < 0$, we have $pr \neq 0$ and can set $p > 0$. We always can make $\tilde{p} = \tilde{r} = 1$ and $\tilde{q} = 0$, e.g., this gauge can be set by the transformation $\tilde{t} = \frac{2pt+q-\sqrt{4pr-q^2}}{2pt+q+\sqrt{4pr-q^2}}$. In this case the constant $\tilde{s}$ cannot be scaled.

Therefore, up to $\tilde{G}_2$-equivalence, we have three cases of $g$ which admit extension of Lie symmetry algebra by one basis operator. These are Cases 6–8 of Table 1.

If $k = 2$, and $g = 1 \text{ mod } \tilde{G}_2$ we get a five-dimensional Lie symmetry algebra which is $\mathfrak{s}(2, \mathbb{R}) \oplus 2A_1$.

The arbitrary constant element $a$ does not affect the results of the group classification problem and can be scaled to any fixed nonzero value by the transformations from the usual equivalence group $G^\sim$. It is convenient to perform the gauge $a = 1/n$.

Note that the group classification of subclass (2) was carried out independently in [10] and [31]. The results were presented there without gauging of the obtained forms of $g$ by equivalence transformations.

4 Solution of a boundary-value problem using Lie symmetries

There exist several approaches exploiting Lie symmetries in reduction of boundary-value problems (BVPs) for PDEs to those for ODEs. The classical technique is to require that both equation and boundary conditions are left invariant under the one-parameter Lie group of infinitesimal transformations. Of course the infinitesimal approach is usually applied (see, e.g., [5, Section 4.4]). The first works in this direction appeared in the late sixties (see, e.g., [4, 6, 19, 20]). The authors of [4, 6] used the classical approach, namely at first symmetries of a PDE were derived and then boundary conditions were checked whether they are also invariant under the action of the generators of symmetry found. In the case of a positive answer the BVP for the PDE was reduced to a BVP for an ODE. Using this technique a number of boundary-value problems were solved (see, e.g., [7, 28, 29]).

In [3, 25] it was mentioned that for most applications it is not natural to require that appropriate symmetries of a system of DEs preserve a particular BVP for this system but it is enough to impose that these symmetries map BVPs from a certain class of such problems to each other. Hence the induction of well-defined equivalence transformations on a properly chosen class of physically relevant BVPs can serve as a criterion for selecting symmetries to be taken into account, e.g., in the course of invariant parameterization or discretization of the system under consideration.

The method suggested in [20] uses specific one-parameter Lie groups of transformations of the independent and dependent variables of the PDE system as well as of all arbitrary elements which appear in the equations under study and in initial and boundary conditions. Namely, only the groups of scalings and translations are considered which can lead to self-similar or travelling-wave solutions only. After the admitted Lie group of scalings and/or translations is specified, the complete set of absolute invariants has to be found. Then a boundary-value problem for the PDE system is reduced to similar but simpler problem for the ODE system. Such an approach was applied to a number of engineering problems (see, e.g., [1] and references therein). There is also the approach in which the group classification of the PDE system and associated initial and boundary conditions is carried out simultaneously (see, e.g., [16, 17] and references therein).

Initial- and boundary-value problems for certain classes of nonlinear PDEs were solved in [1] using the method proposed in [20]. In this Section we demonstrate that the classical approach is much easier, especially taking into account that problems of group classification are solved already for wide classes of nonlinear PDEs.
We look for nonzero solution of the initial- and boundary-value problem
\[ u_t + a(u^n)_x = g(t)u_{xx}, \quad x \in [0, +\infty), \ t > 0, \]
\[ \lim_{t \to +0} u(t, x) = 0, \quad x \in (0, +\infty), \]
\[ u(t, 0) = q(t), \quad t > 0, \]
\[ \lim_{x \to +\infty} u(t, x) = 0, \quad t > 0, \]
where \( a \) is a nonzero constant, \( g \) and \( q \) are arbitrary smooth nonvanishing functions and \( n \neq 0, 1 \).

We have derived the Lie symmetries for the variable coefficient equation (11) and now we examine which of these symmetries leave the initial and boundary conditions of the problem (14) invariant. The procedure starts by assuming a general symmetry of the form
\[ \Gamma = \sum_{i=1}^{n} \alpha_i \Gamma_i, \tag{15} \]
where \( n \) is the number of Lie point symmetries of the given partial differential equation and \( \alpha_i, \ i = 1, \ldots, n, \) are constants to be determined.

Lie symmetries for equation (11) appear in Table 1. In Case 2, for which \( g(t) = \varepsilon t^\rho \), the generator (15) takes the form
\[ \Gamma = \alpha_1 \partial_x + \alpha_2 \left( 2t \partial_t + (\rho + 1)x \partial_x + \frac{\rho - 1}{n - 1} u \partial_u \right). \]

Application of \( \Gamma \) to the first boundary condition which is written as \( x = 0 \) and \( u(t, 0) = q(t) \) gives \( \alpha_1 = 0 \) and \( \alpha_2 \left( -2t \frac{\partial q}{\partial t} + \frac{\rho - 1}{n - 1} q \right) = 0 \). For nonzero \( \alpha_2 \), we have
\[ q(t) = \gamma t^{\frac{\rho - 1}{2n - 2}}, \]
where \( \gamma > 0 \) is a constant. It can be shown that the symmetry \( \Gamma \) with \( \alpha_1 = 0 \) leaves invariant the second boundary condition and the initial condition. Hence the admitted Lie symmetry can be used to reduce the initial- and boundary-value problem (14) to a problem with the governing equation being an ordinary differential equations. In fact the Lie symmetry \( 2t \partial_t + (\rho + 1)x \partial_x + \frac{\rho - 1}{n - 1} u \partial_u \) produces the transformation (called usually Ansatz)
\[ u = t^{\frac{\rho - 1}{2n - 2}} \phi(\eta), \quad \text{where} \quad \eta = xt^{\frac{\rho + 1}{2}}, \tag{16} \]
that reduces (14) into the BVP for ODE
\[ 2\varepsilon \phi'' + (\rho + 1)\eta \phi' - 2a(\phi^n)' - \frac{\rho - 1}{n - 1} \phi = 0, \quad \eta \in [0, +\infty), \tag{17} \]
\[ \phi(0) = \gamma, \tag{18} \]
\[ \lim_{\eta \to +\infty} \phi(\eta) = 0. \tag{19} \]

Let \( \rho = (2 - n)/n \). Then (17) takes the form
\[ \varepsilon \phi'' + \frac{1}{n} (\eta \phi' + \phi) - a(\phi^n)' = 0 \]
and can be integrated once to get \( \varepsilon \phi' + \frac{1}{n} \eta \phi - a \phi^n + c = 0 \). When we set \( c = 0 \) this equation becomes the Bernoulli equation that is linearizable to the form \( \frac{\varepsilon}{1 - n} z' + \frac{1}{n} \eta z - a = 0 \) by the substitution \( \phi^{1-n} = z \). The general solution of this equation is of the form
\[ z = e^{-\frac{1-n}{2\varepsilon \eta^2}} \left( C + \frac{a(1-n)}{\varepsilon} \int_{0}^{\eta} e^{\frac{1-n}{2\varepsilon \theta^2}} d\theta \right), \]

\(^1\)Similar BVP was solved in [1], but in our opinion, by the use of a more complicated method.
where $C$ is an arbitrary constant. If $\varepsilon n(n-1) > 0$ it can be written in terms of the error function as $z = e^{\frac{-x^2}{2\sigma^2}} \left( C + \frac{a(1-n)\sqrt{\pi}}{2\varepsilon\sigma} \operatorname{erf}(\sigma x) \right)$, where $\sigma = \sqrt{\frac{n-1}{2\varepsilon n}}$, $\operatorname{erf}(\theta) = \frac{2}{\sqrt{\pi}} \int_0^\theta e^{-s^2} \mathrm{d}s$ is the error function. Therefore, a particular solution of the second-order ODE on the function $\phi$ is

$$
\phi = \begin{cases} 
  e^{-\frac{1}{2\varepsilon n} \eta^2} \left( C + \frac{a(1-n)}{\varepsilon} \int_0^\eta e^{\frac{1-n}{2\varepsilon n} \theta^2} \mathrm{d}\theta \right)^{\frac{1}{1-n}}, & \text{if } \varepsilon n(n-1) < 0, \\
  e^{-\frac{1}{2\varepsilon n} \eta^2} \left( C + \frac{a(1-n)\sqrt{\pi}}{2\varepsilon\sigma} \operatorname{erf}(\sigma x) \right)^{\frac{1}{1-n}}, & \text{if } \varepsilon n(n-1) > 0,
\end{cases}
$$

(20)

where $\sigma = \sqrt{\frac{n-1}{2\varepsilon n}}$. This is the solution of BVP (17)–(19) with $\rho = (2-n)/n$, when $C = \gamma^{1-n}$ and $\varepsilon n > 0$. Its typical behavior is shown on Figure 1. Using (16) we can now obtain the solution of the following BVP

$$
\begin{aligned}
  u_t + a(u^n)_x &= \varepsilon t^{\frac{2-n}{n}} u_{xx}, \quad x \in [0, +\infty), \quad t > 0, \\
  \lim_{t \to +0} u(t, x) &= 0, \quad x \in (0, +\infty), \\
  u(t, 0) &= \gamma t^{-\frac{1}{n}}, \quad t > 0, \\
  \lim_{x \to +\infty} u(t, x) &= 0, \quad t > 0.
\end{aligned}
$$

(21)

For $\varepsilon > 0$ and $n > 1$ it is of the form

$$
\begin{aligned}
  u &= t^{-\frac{1}{n}} \exp \left( -\frac{1}{2\varepsilon n} x^2 t^{-\frac{2}{n}} \right) \left( \gamma^{1-n} + \frac{a(1-n)\sqrt{\pi}}{2\varepsilon\sigma} \operatorname{erf}(\sigma x t^{-\frac{1}{n}}) \right)^{\frac{1}{1-n}}, \\
  \sigma &= \sqrt{\frac{n-1}{2\varepsilon n}}.
\end{aligned}
$$

(22)

Note that it satisfies BVP (21) for all values of positive $\gamma$ if $a < 0$. If $a > 0$, then the parameters have to satisfy inequality $\gamma^{1-n} > \frac{a(n-1)\sqrt{\pi}}{2\varepsilon\sigma}$. The typical behavior of the latter solution is shown on Figure 2.

![Figure 1: Solution (20) for $\varepsilon = 1, \gamma = 0.5, a = 1$ and $n = 3$.](image1)

![Figure 2: Solution (22) for $\varepsilon = 1, \gamma = 0.5, a = 1$ and $n = 3$ (evolution in time).](image2)
Conclusion

The group classification problem for the class of variable-coefficient generalized Burgers equations (1) is firstly performed in the framework of modern group analysis. Namely, as preliminary step we investigated equivalence transformations within the class. It was shown that equivalence group of the subclass of class (1) singled out by the condition \( n = 2 \) is wider than the equivalence group of the whole class. Therefore, the group classification list is presented in Table 1 up to \( G \sim \)-equivalence for equations (1) with \( n \neq 2 \) and up to \( \hat{G}_2 \sim \)-equivalence for those with \( n = 2 \). The list of Lie symmetries for the class (1) comes to complete the existing results that appear in the literature. Then we have shown that direct application of Lie symmetries for finding solution of an initial and boundary-value problem is more general and easier procedure than that one used, e.g., in [1].

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