ON CARLESON MEASURES OF BELTRAMI COEFFICIENTS BEING COMPATIBLE WITH INFINITELY GENERATED FUCHSIAN GROUPS RELATED TO DENJOY DOMIAN.

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Abstract. Let $\Omega$ be a Carleson-Denjoy domain and $G$ be its covering group. Let $\mu$ be a Beltrami coefficient on the unit disk which is compatible with the group $G$. In this paper we show that if \( \frac{|\mu|^2}{1-|z|^2} \, dx \, dy \) satisfies the Carleson condition on the infinite boundary of the Dirichlet fundamental domain of $G$, then \( \frac{|\mu|^2}{1-|z|^2} \, dx \, dy \) is a Carleson measure on the unit disk. We also show that the above property does not hold for Denjoy domain.

1. Introduction

This is a continuous work of our previous paper [15] which deal with the Beltrami coefficients being compatible with the finitely generated Fuchsian groups of the second kind. In this paper, we mainly focus our attention on the Beltrami coefficients being compatible with the infinitely generated Fuchsian groups.

We start by reviewing some basic definitions about Fuchsian groups. In this paper we call a Möbius group $G$ Fuchsian group if it acts on the unit disk $\Delta$ of the complex plane $\mathbb{C}$ properly discontinuously and freely. The limit set of a Fuchsian group $G$, denoted by $\Lambda(G)$, is the set of accumulation points of the $G$-orbit of any point $z \in \Delta$. A Fuchsian group is said to be of the first kind if its limit set is the entire circle and of the second kind otherwise. A Fuchsian group $G$ is of divergence type if \( \Sigma_{g \in G} (1 - |g(0)|) = \infty \) or \( \Sigma_{g \in G} \exp(-\rho(0, g(0))) = \infty \), where $\rho(0, g(0))$ is the hyperbolic distance between 0 and $g(0)$. Otherwise, we say that it is of convergence type. All second kind groups are of convergence type. The readers are suggested to see [5,11,18] for more details about Fuchsian groups.

In this paper, we still use $\mathcal{F}_z(G)$ to denote the Dirichlet fundamental domain of $G$ centered at $z$. For simplicity, we use the notation $\mathcal{F}$ to express the Dirichlet...
fundamental domain $\mathcal{F}_z(G)$ of $G$ centered at $z = 0$. We call the intersection of $\mathcal{F}$ with the unit circle $\partial \Delta$ the boundary at infinity of $\mathcal{F}$, denoted by $\mathcal{F}(\infty)$.

Recall that a positive measure $\lambda$ defined in a simply connected domain $\Omega$ is called a Carleson measure if there exists some constant $C$ which is independent of $r$ such that, for all $0 < r < \text{diameter}(\partial \Omega)$ and $z \in \partial \Omega$,

$$\lambda(\Omega \cap D(z, r)) \leq Cr.$$ 

The infimum of all such $C$ is called the Carleson norm of $\lambda$, denoted by $\| \lambda \|_*$. For more detail about Carleson measure, see [14, 25].

Let $G$ be a Fuchsian group and $\mu(z)$ a bounded measurable function on $\Delta$, then we say $\mu$ is a $G$-compatible Beltrami coefficient (or complex dilatation) if it satisfies

$$||\mu(z)||_\infty < 1 \text{ and } \mu(z) = \mu(g(z))\overline{g'(z)}/g'(z),$$

(1.1)

for every $g \in G$. We use $M(G)$ to denote the set of all $G$-compatible Beltrami coefficients. For a $G$-compatible Beltrami coefficient $\mu$, if the measure

$$\frac{||\mu||^2}{1 - |z|^2}dxdy$$

is a Carleson measure on $\Delta$ and when the Carleson norm is small, then $f_\mu(\partial \Delta)$ is a rectifiable (chord-arc) curve, where $f_\mu$ is the quasiconformal mapping of the complex plane $C$ with $i, 1$ and $-i$ fixed, whose Beltrami coefficient equals to $\mu$ a.e. on the unit disk and equals to zero on the outside of the unit disk. This is essential for the proof of the convergence-type first-kind Fuchsian groups failing to have Bowen’s property, see [3]. It is also the method to prove that some convergence-type Fuchsian groups fail to have Ruelle’s property, see [16, 17].

We say that a measurable function $\mu(z)$ belongs to $CM^*(\Delta)$ if the measure

$$\frac{||\mu||^2}{1 - |z|^2}dxdy \in CM(\Delta).$$

(1.2)

The importance of the class $CM^*(\Delta)$ lies on the fact that it has wide applications in BMOA-Teichmüller space and Weil-Petersson Teichmüller space, see [2, 7, 10, 21] etc. Hence it is important to investigate under which condition the $G$-compatible Beltrami coefficients belong to $CM^*(\Delta)$.

It is natural to ask whether or not can we determine a $G$ compatible Beltrami coefficient $\mu$ belonging to the class $CM^*(\Delta)$ by its value on the Dirichlet domains? Quite recently, we have proved

**Theorem 1.1.** ([15]) Let $G$ be a convex cocompact Fuchsian group of the second kind and $\mathcal{F}$ the Dirichlet fundamental domain of $G$ centered at 0. Let $\mu \in M(G)$: if there exists a constant $C$ such that, for any $\xi \in \mathcal{F}(\infty)$ (i.e. $\xi$ is in the free edges

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of $\mathcal{F}$) and for any $0 < r < 1$,
\[
\int \int_{B(\xi, r)} \frac{|\mu|^{2} \chi_{\mathcal{F}}}{1 - |z|^{2}} \, dx \, dy \leq C r,
\tag{1.3}
\]
then $\mu$ is in $CM^{\ast}(\Delta)$, where $\chi_{\mathcal{F}}$ is the characteristic function of the Dirichlet fundamental domain $\mathcal{F}$.

Furthermore, Theorem 1.1 can be generalized to the finitely generated Fuchsian group of the second kind with some parabolic elements. We have

**Theorem 1.2.** (15) Let $G$ be a finitely generated Fuchsian group of the second kind with some parabolic elements and $\mathcal{F}$ the Dirichlet fundamental domain of $G$ centered at 0. Let $\mu \in M(G)$: if there exists a constant $C$ such that, for any $\xi \in F(\infty)$ and for any $0 < r < 1$,
\[
\int \int_{B(\xi, r)} \frac{|\mu|^{2} \chi_{\mathcal{F}}}{1 - |z|^{2}} \, dx \, dy \leq C r,
\tag{1.4}
\]
then $\mu$ is in $CM^{\ast}(\Delta)$.

This theorem means that the Carleson property of the measures which are compatible with the finitely generated Fuchsian groups can be checked from the points in the set $\mathcal{F}(\infty)$ i.e., the boundary at infinity of the Dirichlet domain $\mathcal{F}$.

Notice that Theorem 1.1 fails for the case of the finitely generated Fuchsian groups of the first kind (i.e. cocompact groups), since Bowen [8] showed that cocompact groups hold a rigidity property, now called Bowen’s property, i.e. the image of the unit circle under any quasiconformal map whose Beltrami coefficient compatible with a cocompact group, is either a circle or has Hausdorff dimension bigger than 1. Hence for any $\mu$ being compatible with geometry finite groups, the measure $\frac{|\mu|^{2}}{1 - |z|^{2}} \, dx \, dy$ is not a Carleson measure.

Recall that a Denjoy domain is a connected open subset $\Omega$ of the extended complex plane $\mathcal{C}$ such that the complement $E = \mathcal{C} \setminus \Omega$ is a subset of the real axis $\mathbb{R}$. In addition, $\Omega$ is called a Carleson-Denjoy domain if there exists a positive constant $C$ such that
\[
|E \cap [x - t, x + t]| \geq C t
\tag{1.5}
\]
for all $x \in \mathbb{R}$ and $0, t < \text{diam}(E)$, where $| \cdot |$ denotes the Lebesque measure on $\mathbb{R}$. Let $G$ be a covering group of the unit disk $\Delta$ over $\Omega$, i.e., a Fuchsian group of $\Delta$ with quotient $\Delta/G$ conformally equivalent to $\Omega$. Any two covering groups of $\Omega$ are conjugate, and conversely. It is easy to see that when the boundary of $\Omega$ is totally disconnected, the covering group $G$ is of infinitely generated Fuchsian group of the first kind and of convergence type. For such groups $G$, we have
Theorem 1.3. Let $\Omega$ be any Carleson-Denjoy domain with a totally disconnected boundary. Let $G$ be the covering group of the unit disk $\Delta$ over $\Omega$ and $F$ the Dirichlet fundamental domain of $G$ centered at 0. Let $\mu \in M(G)$: if there exists a constant $C$ such that, for any $\xi \in F(\infty)$ (i.e. $\xi$ is in the free edges of $F$) and for any $0 < r < 1$,

$$\int \int_{B(\xi, r)} \frac{|\mu|^2 \chi_F}{1 - |z|^2} \, dxdy \leq Cr,$$  

(1.6)

then $\mu$ is in $CM^*(\Delta)$, where $\chi_F$ is the characteristic function of the Dirichlet fundamental domain $F$.

I thank professor Michel Zinsmeister for pointing out this result to me.

We will show that if a Denjoy domain $\Omega$ does not satisfy the homogeneous property (1.5), i.e. $\Omega$ is not a Carleson-Denjoy domain, Theorem 1.3 will not hold. We shall use the rich knowledge about the Ruelle’s property about Fuchsian groups to prove the following result.

Theorem 1.4. Suppose $(s_n)$ be a sequence of real numbers increasing to infinity and $G$ be the covering group of the surface $S = \mathbb{C}\{s_n, n \geq 0\}$. There exists a sequence $(s_n)$ and a constant $C$ such that, for any $\xi \in F(\infty)$ (i.e. $\xi$ is in the infinity boundary of $F$) and for any $0 < r < 1$,

$$\int \int_{B(\xi, r)} \frac{|\mu|^2 \chi_F}{1 - |z|^2} \, dxdy \leq Cr,$$  

(1.7)

however $\mu$ is not in $CM^*(\Delta)$, where $F$ is the Dirichlet fundamental domain of the group $G$.

The structure of the rest of the paper is as follows: Section 2, we recall some basic definitions and results on Bowen’s property and Ruelle’s property which will be used in the proof of Theorem 1.4.

2. Bowen’s property and Ruelle’s property revisited

Let $G$ be a Fuchsian group and $\mu \in M(G)$. By the measurable Riemann mapping theorem, there exists a unique quasiconformal self-mapping $f^\mu$ of $\Delta$ fixing 1,-1 and i, and satisfying

$$\bar{\partial} f^\mu = \mu \partial f^\mu \, a.e. \, z \in \Delta.$$

Similarly, there exists a unique quasiconformal homeomorphism $f^\mu$ of the plane $\mathbb{C}$ which is holomorphic outside the unit disk $\Delta$, fixing 1,-1 and i, and satisfying

$$\bar{\partial} f^\mu = \mu \partial f^\mu \, a.e. \, z \in \Delta.$$

Then $f^\mu(\partial \Delta)$ is a quasi-circle, i.e. the image of $\partial \Delta$ under a quasiconformal mapping of the plane.
For any $\mu \in M(G)$, $f^\mu \circ g \circ f^{-\mu}$ is a Möbius transformation for every $g \in G$. In this case, we call $G' = f^\mu \circ G \circ f^{-\mu}$ is a quasiconformal deformation of $G$. The group $G' = f^\mu \circ G \circ f^{-\mu}$ is called a quasi-Fuchsian group. We say that a Fuchsian group $G$ has Bowen’s property if the limit set of any quasiconformal deformation of $G$ is either a circle or has Hausdorff dimension $> 1$. In 1979, Bowen [8] proved that if $G$ is a finitely generated Fuchsian group of first kind without parabolic elements, then the limit set of any quasiconformal deformation of $G$ is either a circle or has Hausdorff dimension $> 1$. Soon, Sullivan [23, 24] extended Bowen’s property to all cofinite groups. In 1990, K. Astala and M. Zinsmeister [4] showed that Bowen’s property fails for all convergence groups of the first kind. At last, in 2001, C.J. Bishop [6] showed a excellent result for all divergence groups as follows:

**Lemma 2.1.** [6] Suppose $G$ is a divergence type Fuchsian group and $G' = f^\mu \circ G \circ f^{-\mu}$ is a quasiconformal deformation of $G$. Then either $f^\mu(\partial \Delta)$ is a circle or has Hausdorff dimension $> 1$.

Let us recall that a Fuchsian group $G$ has Ruelle’s property if, for any family of Beltrami coefficients $(\mu_t) \in M(G)$ which is analytic in $t \in \Delta$, the map $t \mapsto HD(\Lambda(G, \mu_t))$ is real-analytic in $\Delta$. In 1982, Ruelle [11] showed that all cocompact groups have this property. In 1997, J.W. Anderson and A.C. Rocha [11] extended this result to finitely-generated Fuchsian groups without parabolic elements. In [2, 3], Astala and Zinsmeister showed that for Fuchsian groups corresponding to Denjoy-Carleson domains or infinite $d$-dimensional "jungle gym" with $d \geq 3$, Ruelle’s property fails. It is easy to see that the Fuchsian groups studied in [3] and [6] are of the first kind. Very recently, the author and Zinsmeister showed that

**Lemma 2.2.** [17] All convergence type Fuchsian groups of the first kind fail to have Ruelle’s property.

In [17], we constructed an infinitely generated Fuchsian group which has Ruelle’s property. As the author’s known, this is the first concrete example of infinitely generated Fuchsian group which has Ruelle’s property.

**Lemma 2.3.** [17] There exists a sequence $(s_n)$ of real numbers increasing to infinity such that the Fuchsian group uniformizing $S = \mathbb{C} \setminus \{s_n, n \geq 0\}$ has Ruelle’s property.

Now we give the proofs of Theorem 1.3 and 1.4.
3. Proof of Theorem 3.3

For all $\xi \in \partial \Delta$ and all $0 < r < 2$, we need to show that there is a constant $C$ which is independent on $\xi$ and $r$ such that

$$\int \int_{B(\xi,r) \cap \partial \Delta} \frac{|\mu(w)|^2}{1 - |w|^2} dudv \leq Cr,$$

where $B(\xi, r)$ is the disk with center $\xi$ and radius $r$.

Note that

$$\int \int_{B(\xi,r) \cap \partial \Delta} \frac{|\mu(w)|^2}{1 - |w|^2} dudv = \int \int_{g^{-1}(B(\xi,r) \cap \partial \Delta)} \frac{\sum_{g \in G} |\mu(g(z))|^2 \chi_{g(F)} dx dy}{1 - |g(z)|^2} (3.1)$$

$$= \sum_{g \in G} \int \int_{g^{-1}(B(\xi,r) \cap \partial \Delta)} \frac{|\mu(g(z))|^2 |g'(z)|^2}{1 - |g(z)|^2} dx dy \quad (3.2)$$

$$= \sum_{g \in G} \int \int_{g^{-1}(B(\xi,r) \cap \partial \Delta)} \frac{|\mu(z)|^2 |g'(z)|}{1 - |z|^2} dx dy \quad (3.3)$$

The equality (3.3) holds since $\mu$ is compatible with the group $G$. By the statement of the theorem, suppose $C$ is the constant such that for any $\zeta \in F(\infty)$,

$$\int \int_{B(\zeta,r)} \frac{|\mu(u)|^2}{1 - |u|^2} dudy \leq Cr. \quad (3.4)$$

By some calculation or see (Lemma 2.1, [15]), we know that for any $\xi \in F$ and any $0 < r < 2$, the inequality (3.4) still holds.

Hence the measure

$$\int \int_{B(\zeta,r)} \frac{|\mu|^2 \chi_F}{1 - |z|^2} dxdy$$

is a Carleson on the domain $g^{-1}(B(\xi,r) \cap F)$. Combine with (3.3) we have

$$\int \int_{B(\zeta,r)} \frac{|\mu(w)|^2}{1 - |w|^2} dudv \leq C_1 \sum_{g \in G} \int_{\partial(g^{-1}(B(\xi,r) \cap F))} |g'(z)| ds \quad (3.5)$$

$$= C_1 \sum_{g \in G} \int_{\partial(B(\xi,r) \cap g(F))} ds \quad (3.6)$$

In [13] Fernandez and Hamilton showed that for Carleson-Denjoy domain $\Omega^*$, $\Gamma$ its covering group and $F_0$ the Dirichlet domain of $\Gamma$ with center 0,

$$\sum_{\gamma \in \Gamma} \text{length}(\partial(\gamma(F))) < \infty, \quad (3.7)$$

or see [12]. In fact, Carleson [9] for Carleson-Denjoy domain (in [9] it is called homogeneous), the harmonic measure is absolutely continuous.

Note that

$$\text{length} \partial(B(\xi,r) \cap g(F)) < 2 \text{length} (B(\xi,r) \cap \partial g(F)),$$

combine with (3.6) we prove the theorem.
4. Proof of Theorem 1.4

Since hyperbolic area is unchanged under conformal mapping, for convenience, we will first use the upper half plane $\mathbb{H} = \{ z : \text{Im}(z) > 0 \}$ as the covering group of the surface $S$. Let $D_n^*$ be the closed disk with diameter $[0, 2n]$ and $D_n^*, n \geq 2$, the closed disk with diameter $[2^{n-1}, 2^n]$. We consider the domain

$$\Omega = \mathbb{H} \setminus ((\cup_{n \geq 1} D_n^*) \cup (\cup_{n \geq 1} (-D_n^*)).$$

Let $\phi$ be the conformal mapping from $\Omega$ onto $\mathbb{H}$ fixing $0$, $2$ and $\infty$. We put $z_0 = 0$, and $z_n = \phi(2^n)$, $n \geq 1$ and $z_n = \phi(-2^{-n})$, $n \leq -1$. Let $\sigma_n$ be the reflection with respect to $\partial D_n^*$ and $\tau(z) = -\bar{z}$. By Rubel and Ryff’s construction [20] of the covering group of Riemann surface $S = \mathbb{C} \setminus \{ z_n \}$, the Fuchsian group $\Gamma$ generated by $\{ \tau \circ \sigma_n \}_{n=1}^\infty$ uniformities the surface $S$, in the sense that $S \simeq \mathbb{H}/\Gamma$. By the construction of $\Omega$ we can see that $\Gamma$ is of infinitely generated and of first kind. It is easy to see that $\Gamma$ contains infinitely many parabolic elements. Hence the Dirichlet domain $F$ of $\Gamma$ contains countably many cusps and the set $F(\infty)$, i.e., the infinity boundary of $\Gamma$, contains countably many points, denoted $F(\infty)$ by $\{ \zeta_n \}_{n=-\infty}^{\infty}$.

Let $B_n^* = B(\zeta_n, 1) \cap F$ and $B^* = \cup B_n^*$. In the following we will show that the hyperbolic area of $\cup B_n^*$ is finite. We first give the hyperbolic area of $B_0^*$. Without loss of generality, we may suppose $\zeta_0 = 0$ and let $C_{-1}$ and $C_1$ be the two infinity sides of the Dirichlet domain of the group $\Gamma$ with $0$ as a vertex, respectively.

Suppose $C_{-1}$ be the circle

$$(x - r_{-1})^2 + y^2 = r_{-1}^2$$

and $C_1$ the circle

$$(x + r_1)^2 + y^2 = r_1^2.$$

Then we have

$$\text{Area}(B_0^*) = \int\int_{B_0^*} \frac{1}{4v^2} dudv \quad (4.1)$$

$$= \int_0^1 dr \int_{\arccos \left( \frac{r}{2r_{-1}} \right)}^{\arccos \left( \frac{r}{2r_1} \right)} \frac{1}{4r \sin^2 \theta} d\theta. \quad (4.2)$$

$$= \int_0^1 (-\cot \theta) \frac{4r}{r_{-1} - 1} dr \quad (4.3)$$

$$\leq C \left( \frac{1}{r_{-1}} + \frac{1}{r_1} \right), \quad (4.4)$$

where $C$ is a universal constant. For any integer $n \in \mathbb{Z}$, by some calculation as above, we can get the hyperbolic area of $B_n^*$. We have, for any $n \in \mathbb{Z}$,
\[
\text{Area}(B_n^*) = \int_{B_n^*} \frac{1}{4r} dudv 
\]
(4.5)

\[
= \int_0^1 dr \int_{\arccos\left(\frac{2r}{2r_n} \right)}^{\arccos\left(\frac{-r}{2r_n} \right)} \frac{1}{4r \sin^2 \theta} d\theta. 
\]
(4.6)

\[
\leq C\left(\frac{1}{r_{n-1}} + \frac{1}{r_n}\right), 
\]
(4.7)

where the equality (4.6) is from the hyperbolic area of the domain in the upper half plane \(\mathbb{H}\) being unchanged under the translation \(f(z) = az + b\) along the horizontal direction, \(a, b \in \mathbb{R}\).

Note that the conformal mapping \(\phi\) from \(\Omega\) onto \(\mathbb{H}\) fixing \(\infty\), we have \(z_n = \phi(2^n)\) is comparable to \(2^n\) for \(n > 0\) and \(z_n = \phi(-2^{-n})\) is comparable to \(-2^{-n}\) for \(n < 0\). Furthermore, the radius \(r_n\) is comparable to \(z_n\). Combine with (4.5) and by some easy calculation, we know the hyperbolic area of \(B^*\) is finite.

In the following, we will show that for any \(\zeta\) in \(\mathcal{F}(\infty) = \{\zeta_m\}_{m=\infty}^{\infty}\) and \(0 < r < 1\),

\[
\lim_{r \to 0} \int_{\arccos\left(\frac{2r}{2r_n} \right)}^{\arccos\left(\frac{-r}{2r_n} \right)} \frac{1}{4r \sin^2 \theta} d\theta = \frac{1}{8} \left(\frac{1}{r_{n-1}} + \frac{1}{r_n}\right) 
\]
and the sequence \((r_n)\) is increasing to infinity as \(n\) tending to infinity.

Suppose \(\mu\) is a measurable function on \(\mathbb{H}\) with \(\|\mu\|_{\infty} < 1\). by some calculation as (4.5) to (4.7), for any \(0 < r < 1\), for any \(\xi \in \mathcal{F}(\infty)\)(i.e. \(\xi\) is in the infinity boundary of \(\mathcal{F}\)) and for any \(0 < r < 1\),

\[
\iint_{B(\xi, r) \cap \mathcal{F}} \frac{|\mu|^2}{(\text{Im}(z))^2} dxdy \leq \int_0^r dr \int_{\arccos\left(\frac{2r}{2r_n} \right)}^{\arccos\left(\frac{-r}{2r_n} \right)} \frac{1}{4r \sin^2 \theta} d\theta \leq Cr. 
\]
(4.9)

Note that any \(0 < r < 1\),

\[
\iint_{B(\xi, r) \cap \mathcal{F}} \frac{|\mu|^2}{\text{Im}(z))^2} dxdy \leq \iint_{B(\xi, r) \cap \mathcal{F}} \frac{|\mu|^2}{\text{Im}(z))^2} dxdy. 
\]
(4.10)

Define

\[
\mu_\mathcal{F}(z) = \begin{cases} 
\mu(z), & z \in B^* \cap \mathcal{F}; \\
0, & z \in \mathcal{F}\setminus B^*,
\end{cases} 
\]
(4.11)

and

\[
\mu^*(z) = \sum_{\gamma \in \Gamma} \mu_\mathcal{F} \circ \gamma^{-1}(z) \chi_\mathcal{F} \circ \gamma^{-1}(z). 
\]
(4.12)

It is easy to see that \(\mu^*(z) \in M(\Gamma)\).
Now it’s time to replace the upper half plane $\mathbb{H}$ by the unit disk $\Delta$ as the covering surface. Consider the Cayley transformation $\kappa(z) = \frac{z - i}{z + i}$ from the upper half plane $\mathbb{H}$ onto the unit disk $\Delta$.

By the transformation $\kappa$, we can conjugate the Fuchsian group $\Gamma$ to a Fuchsian group $G = \kappa \circ \Gamma \circ \kappa^{-1}$. Combine (4.8), (4.9), (4.10) and the construction of $\mu_\infty$, we know that there exists a constant $C$ such that, for any $\xi \in F_0^\infty$ (i.e. $\xi$ is in the free edges of $F_0$) and for any $0 < r < 1$,

$$\int \int_{B(\xi, r)} \frac{|\mu_0|^2 \chi F_0}{1 - |z|^2} \, dx \, dy \leq C r,$$

(4.13)

where $F_0$ denotes the Dirichlet domain of the group $G$ with center 0.

In the following we show that $\mu_0 \in CM^*(\Delta)$. It is easy to see that $\mu_0 \in M(G)$. By the construction of the Denjoy and Lemma 2.2, we know the group $G$ has Ruelle’s property. By Lemma 2.2 we have $\mu$ is divergence type. Suppose $\mu_0 \in CM^*(\Delta)$. It is known that in this case $\log(f^\mu_0)$ belongs to the space $BMOA(\Delta)$ with a norm controlled by the above Carleson measure norm. In particular, when the Carleson norm is small then $\partial f^\mu_0(\Delta)$ is a rectifiable (chord-arc) curve (19 or 22). Hence this is contradict with Lemma 2.1 which said that for all divergence type Fuchsian groups, if $G' = f^\mu \circ G \circ f^{-\mu}$ is a quasiconformal deformation of $G$, the Hausdorff dimension of $f^\mu(\partial \Delta)$ is bigger than one.

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