ON COUNTING PERMUTATIONS BY PAIRS
OF CONGRUENCE CLASSES OF MAJOR INDEX

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Abstract.
For a fixed positive integer $n$, let $S_n$ denote the symmetric group of $n!$ permutations on $n$ symbols, and let $\text{maj}(\sigma)$ denote the major index of a permutation $\sigma$. Fix positive integers $k < \ell \leq n$, and nonnegative integers $i, j$. Let $m_n(i; k; j; \ell)$ denote the cardinality of the set $\{ \sigma \in S_n : \text{maj}(\sigma) \equiv i \mod k, \text{maj}(\sigma^{-1}) \equiv j \mod \ell \}$. In this paper we give some enumerative formulas for these numbers. When $\ell$ divides $(n - 1)$ and $k$ divides $n$, we show that for all $i, j$,

$$m_n(i; k; j; \ell) = \frac{n!}{k \cdot \ell}.$$

1. Introduction

Denote by $S_n$ the symmetric group of all $n!$ permutations on the $n$ symbols $1, \ldots, n$. First recall some combinatorial definitions pertaining to permutations. See, e.g., [4].

Definition 1.1. Let $\sigma \in S_n$. For $1 \leq i \leq n - 1$, $i$ is said to be a descent of $\sigma$ if $\sigma(i) > \sigma(i + 1)$.

Definition 1.2. The major index of $\sigma$, denoted $\text{maj}(\sigma)$, is the sum of the descents of $\sigma$.

The values of the statistic $\text{maj}$ range from 0 (for the identity) to $\binom{n}{2}$.

In [1], the following result was discovered using certain representations of the symmetric group $S_n$, and then proved by means of a bijection as well.

Proposition 1.3. ([1, Theorem 2.6]) Fix an integer $0 \leq i \leq n - 1$.

$$(n - 1)! = |\{ \sigma \in S_n : \text{maj}(\sigma) \equiv i \mod n \}|.$$

This paper is similarly motivated by the algebraic discovery ([S. Sundaram, unpublished]) of the identity

(A) $$(n - 2)! = |\{ \sigma \in S_n : \text{maj}(\sigma) \equiv i \mod n, \text{maj}(\sigma^{-1}) \equiv j \mod (n - 1) \}|,$$

where $i, j$ are fixed nonnegative integers.

The paper is organised as follows. In Section 2 the main technical lemmas are presented. In Section 3 we derive the enumerative formulas, and in Section 4 we give purely bijective proofs of special cases of Theorem 3.1 and Proposition 2.5.

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2. Preliminaries

This section contains the main lemmas that are needed for the rest of the paper.

Let $\gamma \in S_n$ be the $n$-cycle which takes $i$ to $i + 1$ modulo $n$, for all $i$. We will sometimes write $\gamma_n$ for clarity. The circular class of $\sigma$ is the set of permutations $[\sigma] = \{\sigma^i, 0 \leq i \leq n - 1\}$. The following observation is due to Klaychko [3]. For our purposes it is more convenient to state the result in terms of the inverse permutation. This formulation also admits an easy proof, which we give below for the sake of completeness.

Lemma 2.1. ([3], [2, Lemma 4.1]) Let $\sigma \in S_n$. Then the function $\tau \mapsto \text{maj}(\tau^{-1})$ takes on all $n$ possible values modulo $n$ in the circular class of $\sigma$. More precisely, we have that $\text{maj}(\sigma^{-1}) = \text{maj}(\sigma^{-1}) + i \mod n, 0 \leq i \leq n - 1$.

Proof. Let $\tau = a_1 \ldots a_n$ (written as a word). Then $\tau \gamma = a_2 \ldots a_n a_1$. Note that $i$ is a descent of $\tau^{-1}$ if and only if $i$ appears to the right of $i + 1$ when $\tau$ is written as a word in $\{1, 2, \ldots, n\}$.

By looking at occurrences of $i$ to the right of $i + 1$, it is easy to see that $\text{maj}(\tau \gamma)^{-1} - \text{maj}(\tau^{-1}) = a_1 - (a_1 - 1) = +1$, if $a_1 \neq 1, a_1 \neq n$. If $a_1 = 1$, then clearly $\text{maj}(\tau \gamma)^{-1} - \text{maj}(\tau^{-1}) = +1$, while if $a_1 = n$, then the difference is $- (n - 1)$.

Hence in all cases the difference is $+1$ modulo $n$. □

Lemma 2.2. Let $\sigma \in S_{n-1}$, and let $\sigma_i$ denote the permutation in $S_n$ obtained by inserting $n$ in position $i$ of $\sigma$, $1 \leq i \leq n$. Then for each $k$ between $1$ and $n$, the values of the major index on the set $\{\sigma_1, \ldots, \sigma_k\}$ form a consecutive segment of integers $[m + 1, m + k]$, and the value of $\text{maj}(\sigma_{k+1})$ is either $m$ or $m + k + 1$ according as $k$ is a descent of $\sigma$ or not, respectively. Note that $\text{maj}(\sigma_n) = \text{maj}(\sigma)$.

In particular, on the set $\{\sigma_1, \ldots, \sigma_n\}$, the function $\text{maj}$ takes on each of the $n$ values in the interval $[\text{maj}(\sigma), \text{maj}(\sigma) + (n - 1)]$.

Proof:

Let $\sigma' = a_1 \ldots a_{n-1}$, with descents in positions $i_1, \ldots, i_d$. Let $\sigma$ be the permutation in $S_n$ obtained by appending $n$ to $\sigma'$. Hence $\text{maj}(\sigma) = \text{maj}(\sigma')$. We shall show that the value of $\text{maj}(\sigma')$ increases successively by 1 as $n$ is inserted into $\sigma'$ in the following order:

(1) first in the positions immediately following a descent, starting with the rightmost descent and moving to the left;

(2) then in the remaining positions, beginning with position 1, from left to right.

For instance, if $\sigma' = 14253$, then the resulting permutations, beginning with $\sigma$ and then in the order specified above, are

$$142536, 142563, 146253, 614253, 164253, 142653,$$

with respective major indices $6, 7, 8, 9, 10, 11$.

Let $\sigma_k$ denote the permutation in $S_n$ obtained from $\sigma'$ by inserting $n$ in position $k$. Thus $\sigma_k = a_1 \ldots a_{k-1} n a_k a_{k+1} \ldots a_n$ for $k = 2, \ldots, n-1$, and $\sigma_1 = n a_1 \ldots a_{n-1}$, $\sigma_n = a_1 \ldots a_{n-1} n$. Let $\Delta_k$ denote the difference $\text{maj}(\sigma_k) - \text{maj}(\sigma')$.

The following facts are easily verified:

(1) If $n$ is inserted immediately after a descent of $\sigma'$, i.e., if $k = i_j + 1, 1 \leq j \leq d$, then $n$ contributes a descent in position $i_j + 1$, but the $i_j$th element ceases to be
a descent. Also the \((d - j)\) descents to the right of \(n\) are shifted further to the right by one. Thus
\[
\Delta_k = (d - j) + (i_j + 1) - i_j = d - j + 1,
\]
and hence the difference \(\Delta_k\) ranges from 1 through \(d\).

(2) If \(1 \leq k \leq i_1\), then the \(d\) descents to the right are shifted over by 1, and thus
\[
\Delta_k = d + k,
\]
and hence \(\Delta_k\) ranges from \(d + 1\) through \(d + i_1\).

(3) If \(n\) is inserted in position \(k\) between two descents, but not immediately following a descent, i.e., if \(1 + i_j < k \leq i_{j+1}, j \leq d - 1\), then
\[
\Delta_k = (d - j) + k,
\]
and hence \(\Delta_k\) ranges from \((d - j + 2) + i_j\) through \(d - j + i_{j+1}\).

(4) Finally when \(i_d + 2 \leq k \leq n - 1\),
\[
\Delta_k = k,
\]
and hence \(\Delta_k\) ranges from \(i_d + 2\) through \(n - 1\).

This establishes the claim. It also shows that as \(n\) is inserted into \(\sigma'\) from left to right, the difference in major index goes up (from \(\text{maj}(\sigma')\)) first by \((d + 1)\), then up by one at each step, except when it is inserted immediately after the \(j\)th descent, in which case it goes down to \((d - j + 1)\). Since when \(n\) is in position \(n\), \(\text{maj}(\sigma')\) is unchanged, this establishes the statement of the lemma. \(\square\)

Remark 2.3. Note that in Lemma 2.2, it need not be true that the values of \(\text{maj}\) on an arbitrary set \(\{\sigma_j, \ldots, \sigma_{j+r}\}, j > 1\), form a consecutive set of integers.

Lemma 2.4. Let \(\sigma \in S_{n-1}\), and let \(\sigma_i\) denote the permutation in \(S_n\) obtained by inserting \(n\) in position \(i\) of \(\sigma\), for \(1 \leq i \leq n\). Then \(\text{maj}(\sigma^{-1}) = \text{maj}(\sigma) \mod (n-1)\).

Proof. Consider the effect of inserting \(n\) on the set of descents of \(\sigma^{-1}\). If \(n\) is inserted to the right of \((n - 1)\), there is no change; if \(n\) is inserted to the left of \((n - 1)\), then \((n - 1)\) becomes a descent of \(\sigma^{-1}\). In either case, the major index of the inverse permutation is unchanged modulo \((n - 1)\). \(\square\)

Finally we shall need the following result, which generalises Proposition 1.3. It is perhaps known, although we do not know of a precise reference. There is an easy generating function proof which we include for the sake of completeness. In Section 4 we will give a constructive proof of the equivalent statement for inverse permutations.

Proposition 2.5.
\[
\frac{n!}{k} = |\{\sigma \in S_n : \text{maj}(\sigma) \equiv j \mod k\}|.
\]

Proof. Recall the well-known formula (see [4])

\[
\sum_{\sigma \in S_n} q^{\text{maj}(\sigma)} = \prod_{i=1}^{n-1} (1 + q + \ldots + q^i)
\]
Note that Lemma 2.2 gives an immediate inductive proof of formula (B).

Now fix integers \( 1 \leq k \leq n \) and \( 0 \leq j \leq k-1 \). To show that the number of permutations in \( S_n \) with major index congruent to \( j \mod k \) is \( n!/k \), it suffices to show that, modulo the polynomial \((1 - q^k)\), the left-hand side of (B) equals
\[
(n!/k) \cdot (1 + q + \ldots + q^{k-1}).
\]

Since \( 1 + q + \ldots + q^t = (1 - q^{t+1})/(1 - q) \), it follows from the generating function that for fixed \( k \leq n \), the sum on the left-hand side vanishes at all \( k \)th roots of unity not equal to 1. Hence, modulo \((1 - q^k)\), there is a constant \( c \) such that
\[
\sum_{\sigma \in S_n} q^{\text{maj}(\sigma)} = c(1 + q + \ldots + q^{k-1}).
\]

Putting \( q = 1 \) yields \( c = n!/k \), as required. \( \square \)

3. Enumerative Results

Let \( m_n(i\backslash k; j\backslash \ell) \) denote the number of permutations \( \sigma \in S_n \) with \( \text{maj}(\sigma) \equiv i \mod k \) and \( \text{maj}(\sigma^{-1}) \equiv j \mod \ell \).

**Theorem 3.1.** Let \( \ell \) be a divisor of \( n - 1 \), \( \ell \neq 1 \), and let \( k \) be a divisor of \( n \), \( k \neq 1 \). Fix \( 0 \leq i \leq k-1 \), \( 0 \leq j \leq \ell - 1 \). Then
\[
m_n(i\backslash k; j\backslash \ell) = \frac{n!}{k \cdot \ell}.
\]

**Proof.** Let \( \sigma \in S_{n-1} \), and construct \( \sigma_i, i = 1, \ldots, n \) in \( S_n \) as in Lemma 2.2, by inserting \( n \) in position \( i \). Since \( \ell(n-1) \), we have by Lemma 2.4 that for all \( i \),
\[
\text{maj}(\sigma^{-1}) \equiv \text{maj}(\sigma_i^{-1}) \mod \ell.
\]

By Lemma 2.2, since the set \( \{\text{maj}(\sigma_i) : i = 1, \ldots, n\} \) consists of \( n \) consecutive integers, each congruence class modulo \( k \) appears exactly \( \frac{n}{k} \) times. Hence we have
\[
m_n(i\backslash k; j\backslash \ell) = \frac{n}{k} \cdot |\{\sigma \in S_{n-1} : \text{maj}(\sigma^{-1}) \equiv j \mod \ell\}|
\]
and the result now follows from Proposition 2.5. \( \square \)

By examining Lemma 2.2 more closely, we obtain the following recurrence on \( n \) for these numbers in the case when \( k \) and \( \ell \) are divisors of \( (n-1) \).

**Proposition 3.2.** Let \( k, \ell \) be divisors of \( n - 1 \), \( \ell \neq 1 \), \( k \neq 1 \). Then
\[
m_n(i\backslash k; j\backslash \ell) = (n-2)! \frac{(n-1)^2}{k \cdot \ell} + m_{n-1}(i\backslash k; j\backslash \ell).
\]

**Proof.** Let \( \sigma \in S_{n-1} \), and construct \( \sigma_i, i = 1, \ldots, n \) in \( S_n \) as in Lemma 2.2, by inserting \( n \) in position \( i \). Since \( \ell(n-1) \), we have by Lemma 2.4 that for all \( i \),
\[
\text{maj}(\sigma^{-1}) \equiv \text{maj}(\sigma_i^{-1}) \mod \ell.
\]

Now let \( k|(n-1) \). By Lemma 2.2, the major indices of the first \( (n-1) \) elements \( \sigma_i, i = 1, \ldots, n-1 \), form a segment of \( (n-1) \) consecutive integers, and hence the
residue class $i$ modulo $k$ appears exactly $\frac{n-1}{k}$ times among them. Also note that $\text{maj}(\sigma) = \text{maj}(\sigma')$.

Hence we have $m_n(i\backslash k; j\backslash \ell)$

$$= \frac{n-1}{k} |\{\sigma \in S_{n-1} : \text{maj}(\sigma^{-1}) \equiv j \mod \ell\}|$$

$$+ |\{\sigma \in S_{n-1} : \text{maj}(\sigma^{-1}) \equiv j \mod \ell, \text{maj}(\sigma) \equiv i \mod k\}|.$$ Collecting terms and using Proposition 2.5, we obtain

$$m_n(i\backslash k; j\backslash \ell) = \frac{n-1}{k} \frac{(n-1)!}{\ell} + m_{n-1}(i\backslash k; j\backslash \ell),$$

as required.

We note that while the above arguments are not symmetric in $k$ and $\ell$, the numbers $m_n(i\backslash k; j\backslash \ell)$ satisfy

$$(C) \quad m_n(i\backslash k; j\backslash \ell) = m_n(j\backslash \ell; i\backslash k).$$

This follows by applying the involution $\tau \mapsto \tau^{-1}$.

For arbitrary choices of $k, \ell$, these numbers usually depend on the values of $i$ and $j$. For example for $n = 4$, we have $m_4(0\backslash 2; 0\backslash 2) = 8 = m_4(1\backslash 2; 1\backslash 2)$, and $m_4(1\backslash 2; 0\backslash 2) = 4 = m_4(0\backslash 2; 1\backslash 2)$. When $k = \ell = 3$, we have $m_4(0\backslash 3; 0\backslash 3) = 4$, $m_4(0\backslash 3; 1\backslash 3) = 2 = m_4(0\backslash 3; 2\backslash 3)$; and $m_4(1\backslash 3; 1\backslash 3) = 3 = m_4(1\backslash 3; 2\backslash 3)$. The other values follow by symmetry from (C).

Note that in view of Proposition 2.5, we know that, for fixed $\ell$, the sum over $i = 0, 1, \ldots, k-1$ of the numbers $m_n(i\backslash k; j\backslash \ell)$ is $\frac{2^n}{\ell}.

4. Some bijections

In this section we present bijective proofs for some of the results derived in Sections 3 and 2. Recall that this paper was originally motivated by the algebraic discovery of the formula (A). We now give a bijective proof of (A), which is the special case $k = n, \ell = n-1$ of Theorem 3.1.

**Proposition 4.1.** (Bijection for the case $k = n, \ell = n-1$ of Theorem 3.1.) Fix integers $0 \leq i \leq n-1, 0 \leq j \leq n-2$. Then the number of permutations $\sigma$ in $S_n$ such that $\text{maj}(\sigma) \equiv i \mod n$ and $\text{maj}(\sigma^{-1}) \equiv j \mod (n-1)$, equals $(n-2)!$.

**Proof.** First note that $(n-2)!$ counts the number of permutations in $S_{n-1}$ having $(n-1)$ as a fixed point. Let $A_{n-1}$ be this set of permutations, and let $B_n$ be the subset of $S_n$ with major indices as prescribed in the statement of the theorem. Given $\sigma \in A_{n-1}$, by Lemma 2.1 there is a unique circular rearrangement $\sigma'$ in $S_{n-1}$ whose inverse has major index congruent to $j \mod (n-1)$. Lemma 2.2 then shows that, for each $i = 0, 1, \ldots, n-1$, there is a unique position in $\sigma'$ in which to insert $n$, in order to obtain a permutation $\sigma'' \in S_n$ such that $\text{maj}(\sigma'') \equiv i \mod n$. By Lemma 2.4, the passage from $\sigma'$ to $\sigma''$ does not change the major index of the inverses modulo $(n-1)$, and thus $\text{maj}(\sigma''^{-1}) = \text{maj}(\sigma'^{-1}) \equiv j \mod (n-1).$ Hence $\sigma \mapsto \sigma''$ gives a well-defined map from $A_{n-1}$ to $B_n$. To see that this is a bijection, given $\sigma'' \in B_n$, erase the $n$ to obtain $\sigma' \in S_{n-1}$, and let $\sigma$ be the unique circular rearrangement of $\sigma'$ such that $\sigma(n-1) = n-1$. Then $\sigma \in A_{n-1}$, and clearly the map is a bijection. $\square$
Example 4.1.1. Let \( n = 6, i = 2, j = 3 \). Take \( \sigma = 21345 \in A_5 \). Note that \( \text{maj}(\sigma^{-1}) = 1 \). The unique circular rearrangement whose inverse has major index equal to \( 3 \equiv 3 \pmod{5} \) is \( \sigma' = 34521 \). Now \( \text{maj}(\sigma') = 7 \), (descents in positions \( 3 \) and \( 4 \)). Now use (the proof of) Lemma 2.2. To obtain a permutation with major index \( 8 \equiv 2 \pmod{6} \), insert \( 6 \) into position \( 5 \) (immediately after the right-most descent). This gives \( \sigma'' = 345261 \in B_6 \).

The remainder of this section is devoted to giving a constructive proof of Proposition 2.5. A bijection for the case \( k = n \) was given in [1], using Lemma 2.1. We do not know of a bijection for arbitrary \( k \), but a bijection for the case \( k = n - 1 \) is given in the proof which follows.

Proposition 4.2. (Bijection for the case \( k = n - 1 \) of Proposition 2.5.) Fix an integer \( 0 \leq j \leq n - 2 \). The number of permutations in \( S_n \) with major index congruent to \( j \pmod{(n - 1)} \) is \( n(n - 2)! = n!/(n - 1) \).

Proof. Let \( B_n \) denote the set \{\( \sigma \in S_n : \text{maj}(\sigma^{-1}) \equiv j \pmod{(n - 1)} \}\}. It suffices to show that this set has cardinality \( n(n - 2)! \). Let \( C_n \) denote the set of permutations \( \tau \in S_n \) such that, when \( n \) is erased, \( (n - 1) \) is a fixed point of the resulting permutation \( \tau' \) \( \in S_{n-1} \). Observe that \( C_n \) has cardinality \( n(n - 2)! \), since the number of permutations in \( S_{n-1} \) which fix \( (n - 1) \) is \( (n - 2)! \), and there are \( n \) positions in which \( n \) can be inserted.

We describe a bijection between \( C_n \) and \( B_n \). If \( \tau \in C_n \), let \( \tau' \) be the permutation in \( S_{n-1} \) obtained by erasing \( n \). By definition of \( C_n \), \( \tau'(n - 1) = n - 1 \). By Lemma 2.1, there is a unique circular rearrangement \( \tau'' \in S_{n-1} \) of \( \tau' \) such that the major index of the inverse of \( \tau'' \) is congruent to \( j \pmod{(n - 1)} \). Now construct \( \tilde{\tau} \in S_n \) by inserting \( n \) into \( \tau'' \) in the same position that it occupied in \( \tau \), i.e., \( \tilde{\tau}^{-1}(n) = \tau^{-1}(n) \).

By Lemma 2.4, \( \text{maj}(\tilde{\tau}^{-1}) = \text{maj}(\tau''^{-1}) \equiv j \pmod{(n - 1)} \). Hence we have a map \( \tau \mapsto \tilde{\tau} \in B_n \). It is easy to see that this construction can be reversed exactly as in the proof of Proposition 4.1, and hence we have the desired bijection. □

Example 4.2.1. Let \( n = 5, j = 2 \). Take \( \tau = 32154 \). Then \( \tau \) belongs to the set \( C_5 \) of the preceding proof. Erasing 5 yields \( \tau' = 3214 \), whose inverse major index is 3. The third cyclic rearrangement \( \tau'' = 4321 \) then has inverse major index 6 \( \equiv 2 \pmod{4} \), and \( \tau \mapsto \tilde{\tau} = 43251 \).

Now we examine Klyachko’s Lemma 2.1 more closely. We obtain the following result, which specialises, in the case \( k = n \), to Proposition 1.3.

Lemma 4.3. Fix integers \( 1 \leq k \leq n \), \( 0 \leq j \leq k - 1 \) and \( 1 \leq a \leq n - k + 1 \).

1. Then

\[
(n-1)! = |\{ \sigma \in S_n : \text{maj}(\sigma^{-1}) \equiv j \pmod{k} \text{ and } n-a-k+2 \leq \sigma^{-1}(n) \leq n-a+1 \}| \]

2. Let \( n = qk + r \), \( 0 \leq r \leq k - 1 \). Fix an integer \( s \) between 1 and \( q \). Then

\[
s(n-1)! = \{ \sigma \in S_n : \text{maj}(\sigma^{-1}) \equiv j \pmod{k} \text{ and } \sigma^{-1}(n) \in [n-sk+1,n] \} \].

Proof. Let \( A_n \) denote the set of permutations in \( S_n \) which fix \( n \), and let \( B_n \) denote the subset of \( S_n \) subject to the conditions in the statement of Part (1). Let \( \tau \in A_n \). Consider the circular class of \( \tau \) consisting of the set \( \{ \tau, \tau\gamma, \ldots, \tau\gamma^{n-1} \} \). The proof of Lemma 2.1 shows that because \( \tau(n) = n \), we have the exact equality \( \text{maj}(\tau\gamma^i) = \)
maj(τ) + i, for 0 ≤ i ≤ n − 1. In particular, for any 1 ≤ k ≤ n, the first k circular
rearrangements τγi, 0 ≤ i ≤ k − 1, have the property that the major indices of their
inverses form a complete residue system modulo k. More generally, this observation
holds for any k consecutive circular rearrangements τγi, a ≤ i ≤ a + k − 1, where a
is any fixed integer 1 ≤ a ≤ n − k + 1.

Hence for every τ ∈ An, there is a unique i, a ≤ i ≤ a + k − 1 such σ = τγi has
maj(σ−1) ≡ j mod k. Since n is in position n − i in τγi, clearly n − a − k + 2 ≤
σ−1(n) ≤ n − a + 1. Thus τ → σ gives a well-defined map from An to Bn. Conversely
given σ ∈ Bn, with σ−1(n) = n − i + 1, a ≤ i ≤ a + k − 1, let τ ∈ Sn be defined
by τγi = σ. Then clearly τ(n) = n, and τ ∈ An. This shows that our map is a
bijection, and (1) is proved.

For (2), again we start with the set An of the (n−1)! permutations in Sn which
fix n. Let τ ∈ An. Then as in the preceding proof, for i = 0, 1, . . . , sk − 1, the
first sk circular rearrangements τγi have n in position (n − i), and maj((τγi)−1) =
maj(τ) + i. In particular, for each J = 1, . . . , s, the major index of the inverse
permutations in the subset {τγj−1 ≤ k : 0 ≤ i ≤ k − 1} is a complete residue
system modulo k. Hence the first sk rearrangements contain exactly s permutations
with inverse major index congruent to j mod k. This establishes (2).

We are now ready to give a constructive proof of an equivalent restatement of
Proposition 2.5, by looking at the circular classes of permutations τ ∈ Sn which fix
n. Note that the statement of Proposition 4.4 (or Proposition 2.5) is invariant with
respect to taking inverses, i.e., it says that \( \frac{n!}{k} \) is also the number of permutations
in Sn with constant major index modulo k. Our constructive proof, however, works
only for the inverse permutations.

**Proposition 4.4. (Constructive proof)**

\[
\frac{n!}{k} = |\{\sigma \in S_n : \text{maj}(\sigma^{-1}) \equiv j \mod k\}|
\]

**Proof.** We proceed inductively. We assume k ≤ n − 1, since the case k = n was
dealt with in Proposition 1.3. It is easy to verify directly that the statement holds
for n = 3. Assume we have constructed the permutations in Sn−1 with inverse
major index congruent to j mod k. Note that this means we can identify these
permutations in the subset An of Sn. Let τ ∈ An. We show how to pick out the
permutations in the circular class of τ with inverse major index congruent to j
mod k. Let n = qk + r. Taking s = q in Lemma 4.3 (2), the proof shows how to pick
out the q permutations in the first qk circular rearrangements τγi, 0 ≤ i ≤ qk − 1.
Now consider the remaining r (recall r < k) rearrangements τγi, qk ≤ i ≤ qk + r − 1.
These will contain a (necessarily unique) permutation with inverse major index
congruent to j mod k, iff maj(τ−1) ≡ j − i mod k, for qk ≤ i ≤ qk + r − 1, i.e.,
iff maj(τ−1) ≡ j − t mod k, for t = 0, . . . , r − 1. By induction hypothesis for each
t = 0, . . . , r − 1, there are exactly \( \frac{(n−1)!}{k} \) such permutations in An. Hence there
are \( r(n−1)!/k \) permutations in An whose circular class is such that, among the last
r rearrangements, there is a permutation with inverse maj congruent to j mod k.

We have thus accounted for exactly \( q(n−1)! + r(n−1)!/k = n!/k \) permutations
σ ∈ Sn with maj(σ−1) ≡ j mod k. □
REFERENCES

1. H. Barcelo and S. Sundaram, *On Some Submodules of the Action of the Symmetric Group on the Free Lie Algebra*, J. Alg. *154 No. 1* (1993), 12–26.

2. A. M. Garsia, *Combinatorics of the free Lie algebra and the symmetric group*, Analysis: Research papers Published in Honour of Jürgen Moser’s 60th Birthday, Paul H. Rabinowitz and Eduard Zehnder, eds., Academic Press, San Diego, CA, 1990.

3. A. A. Klaychko, *Lie elements in the tensor algebra*, Siberian Math. J. *15, No. 6* (1974), 1296–1304.

4. R. P. Stanley, *Enumerative Combinatorics, Vol. 1*, Wadsworth & Brooks Cole, Monterey, CA, 1986.

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