Limiting effects in clusters of misaligned toroids orbiting static SMBHs

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ABSTRACT
We consider agglomerates of misaligned, pressure supported tori orbiting a Schwarzschild black hole. A leading function is introduced, regulating the toroids distribution around the central static attractor—it enables to model the misaligned tori aggregate as a single orbiting (macro-)configuration. We first analyze the leading function for purely hydrodynamical perfect fluid toroids. Later, the function is modified for presence of a toroidal magnetic field. We study the constraints on the tori collision emergence and the instability of the agglomerates of misaligned tori with general relative inclination angles. We discuss the possibility that the twin peak high-frequency quasi-periodic oscillations (HF-QPOs) could be related to the agglomerate inner ringed structure. The discrete geometry of the system is related to HF-QPOs considering several oscillation geodesic models associated to the toroids inner edges. We also study possible effect of the tori geometrical thickness on the oscillatory phenomena.

Key words: black hole physics – accretion, accretion discs – hydrodynamics– (magnetohydrodynamics) MHD— galaxies: active – galaxies: jets

1 INTRODUCTION
In various phases of the black hole (BH) accretion in active galactic nuclei (AGN), the angular momentum orientation of the infalling material related to various accretion periods, can be expected to be misaligned, leading to aggregates of misaligned toroidal structures. These and similar processes open up the prospect of investigation into the possibility that during more or less long periods of the attractor life there may be different orbiting toroidal structures, demonstrating substantially different inclinations of their symmetry planes relative to a distant observer. Misaligned tori could be created even around a BH accreting matter in a binary system, being raised from a warped accretion disk. This scenario is now widely focused in several studies and simulations evidencing how such complex orbiting structures are essentially regulated by initial data of their formation.

Above all these toroidal structures are governed by the geometry of their attractor. Particularly essential factor governing the tori formation and misalignment is the presence or absence of the central BH spin and its magnitude. Purely hydrodynamic (HD) models of these toroidal structures can provide effective descriptions of limiting effects associated with these systems and indicate observable situations constraining their formation and stability. The interest in these structures is then manifold, they can throw light on the processes of accretion disk formation and tori instability associated with the accretion phases. Furthermore, such complex configurations are of clear observational interest due to existence of different phenomena possibly caused by their non-homogeneous inner structure. A multi-disk model can be also a frame for the interpretation of the mass accretion rates of SMBHs in AGNs, and for the evolution of the attractor spin and jet emission. (We note that the Kerr BH warped torus can evolve together with its attractor changing its mass and spin—see Bardeen&Petterson (1975)– see also Martin et al. (2014); Nixon et al. (2012a,b); Nealon et al. (2015); Lodato & Pringle (2006); Scheuerl&Feiler (1996); Aly et al. (2015)).

In this article we consider some of these limiting effects for aggregates of misaligned (tilted) tori orbiting a central
spherically symmetric, Schwarzschild BH. In Pugliese & Stuchlík (2019) agglomerates of misaligned tori were framed within an adapted modified ringed accretion disks (RADs) model. These ringed structures were introduced in Pugliese & Montani (2015); Pugliese & Stuchlík (2015) as aggregates of axisymmetric accretion configurations, coplanar and centered on the equatorial plane of the central Kerr super massive (SM)BH. From now on we will refer to the original equatorial model considered in Pugliese & Stuchlík (2015, 2016, 2017a, 2018a, b, 2017c, 2018a) as equatorial-RAD or eRAD to distinguish it from the misaligned case, referred here for now on as RADs. The RAD and eRAD models are essentially "constrainting-models" from methodological and technical viewpoints. One of the main goals of these models was to provide constraints as initial data for dynamic situations, for example in simulations of complex structures such as GRMHD (general relativistic magnetohydrodynamic) supported tori orbiting a Kerr BH. A fundamental characteristic in these approaches is that the strong gravity of the SMBH attractor has a dominant role in determining the morphology and stability of the aggregate toroidal components– Pugliese & Stuchlík (2016, 2017a). An essential methodological aspect of the eRAD-approach was the adoption of a leading function capable of describing the distribution of toroids in the eRAD, and therefore its inner ringed structure– Pugliese & Stuchlík (2015); Pugliese & Stuchlík (2019). Here we use the leading function for the RAD model of the misaligned structures in the case of HD toroids–we elaborate in addition an energy function linked to the RAD tori densities and energetics. (The leading function can be modified to consider the contribution of a toroidal magnetic field for some of the aggregate toroids).

A further relevant aspect of the eRAD investigation consisted in the discussion of significant observational tracks of a ringed structure which we expect to appear particularly through optical effects (Karask & Sochora 2010; Sochora et al. 2011; Schae & Stuchlík 2009, 2013) see also discussion in Marchesi et al. (2016); Gilli et al. (2007); Marchesi et al. (2017); Masini et al. (2016); DeGraf et al. (2017); Storchi-Bergmann et al. (2017). Here we return to this aspect showing typical structures of RAD tori that should be evident as signature of their inner discrete inhomogeneous structure.

In these first adaptations of the eRAD into a RAD model, the advantage of considering the spherically symmetric case is essentially methodological. Then for the description of the situation for a rotating and axially symmetric Kerr BH, a perturbative approach can be considered having the limiting case of central Schwarzschild BH. In the special case of a static, Schwarzschild black hole each plane crossing the center of the attractor can serve as central plane of the toroid; only one of them can be related as the possible symmetry plane for a given orbiting toroid. A RAD aggregate on static spacetime has a number of similarities with the eRAD on a Kerr spacetime, specifically with the case of the "corotating" tori sequences which are sequences of orbiting coplanar tori orbiting on the equatorial plane of a Kerr BH being all corotating or counterrotating with respect to the central Kerr attractor. Nevertheless, as discussed in Pugliese & Stuchlík (2019), for very slowly rotating attractors, tori misalignment allows to reconsider in some extent the possibility of the presence of multi accreting tori on different planes, enlightening interesting situations and phenomenology which were considered not possible for the eRAD. Geometry of the RAD accreting or equilibrium tori, stability and collision emergence, were studied in Pugliese & Stuchlík (2019) where evaluations of quantities related to tori energetics such as the mass-flux, the enthalpy-flux (evaluating also the temperature parameter), and the flux thickness, in dependence on the model parameters were provided for polytropic fluids. Some notes on the RAD models including proto-jets, which are open and cusped solutions associated to geometrically thick tori were also reported. In the present investigation we also consider the possibility of oscillating tori causing quasi-periodic oscillations (QPOs) oscillations emission.

We focus in particular on the fluid specific angular momentum ℓ(r) in the HD-torus, as "leading-RAD function", i.e. as a "reference" function constraining the location of the maximum and minimum points of the hydrostatic pressure and density in the RAD. Therefore the function represents the distribution of tori centers and inner and outer edges of the toroidal aggregate components. The idea behind the elaboration of a leading function is to represent the possible tori "distribution" of the aggregate in the RAD. In the case focused in this work, considered as reference for more complex situations, the distribution is described by the specific angular momentum ℓ(r), this function of the radius r (radial distance from the central Schwarzschild black hole), has mainly a geometrical and centrifugal origin. The analysis via the leading function frame has to be developed along three main steps.

Firstly, the effective existence of such leading function has to be established– it has to assess the possibility of representing the macro-structure through one function which can provide effective constraints to the RAD tori. In this investigation we have discussed this issue through the analysis of the force balance equation (the relativistic Euler equation) of the hydrodynamical model, where the existence of a leading function is guaranteed by an integral condition on an ordinary differential equation due to the application of the von Zeipel theorem.

A second step of the investigation concerns the identification of a proper leading function which in general will depend on the specificity of the accretion models. In Sec. 4 we consider a deviation from the hydrodynamic case, considered as a (geometrical) reference case, including the contribution of a toroidal magnetic field in the force-balance equation.

Thirdly, the constraints obtained in the leading-function approach are applied in different models which can consider this setting as reference. Indeed, together with the conceptual relevance of constraining the orbiting tori system with one distribution leading function, this approach has the advantage to provide general constraints for the macro-structure which, in some extent, avoids the explicit solution of the GRHD equations or, eventually, the study of a series of tori characteristic...
effective potentials. In this last case the effective potentials, recovered from some integrability conditions on the Euler equations, would require a fine tuning of some model parameters related to each toroidal component of the RAD and the management of the particular boundary conditions adapted to the specific case in examination arising from the ringed structure of the misaligned orbiting configurations. This especially holds when the RAD tori are considered as set of initial data in complex models which have to be treated through numerical integration. (We report, with regard to this aspect of the applicability of this method that these toroidal analytic models are commonly used as starting condition for numerical studies of black hole accretion. In these studies, the simulations of accretion flows largely verify the agreement with the model predictions.)

Since we consider geometrically thick tori as RAD constituents it is convenient to describe here the accreting toroids in comparison with different models. There are several accreting disks models which differently consider the variety of processes of very diversified nature characterizing the accretion disk physics. It turns useful to distinguish these models according to the following three aspects. (1) The disk geometry is a first significant aspect of disk physics which, distinguishing the geometrically thin from the geometrically thick disks (tori), is essentially defined by the disk vertical thickness (on the disk symmetry plane). (2) A second characterizing element is the matter accretion rate of the accretion disk (correlated to sub or super-Eddington luminosity). (3) Third featuring element for the accretion disk is its optical depth (for transparent or opaque disks)–see (Abramowicz & Fragile 2013). Typically, thick disks, for example the Polish doughnuts tori (P-D), have very high, super-Eddington, accretion rates, and high optical depth, while the ion tori, the ADAF (Advection-Dominated Accretion Flow) disks have low optical depth and relatively low accretion rates (i.e. sub-Eddington). Importantly for our analysis, in thin disks the dissipative viscosity processes, which are usually framed with the local magnetic fields in the magnetorotational instability (MRI), are relevant for accretion.

In the toroidal disks, as those considered here, pressure gradients are crucial for the accretion mechanism as well as the disk morphology. Given the relevance of the issue, we studied in section (3) the tori geometrical thickness in RAD frame (for geometrically thick disks including viscous effects see also Lahiri & Lämmerzahl (2019)). (Recently astronomers of observatory Karl G. Jansky Very Large Array (VLA) provided a picture of the dusty, thick torus and associated emitted jets of material ejected by the disk orbiting the SMBH at the core Cygnus A (Carilli, et al., 2019). Tori in this analysis are therefore a case of radiation pressure supported thick disks with Super-Eddington luminosity, generally having very small accretion efficiency and consequentially highly super-Eddington accretion rates, often with strong outflows, and advection cooled (for some $q \approx 1$, i.e. energy flux per advection-energy flux per radiation). In the case considered here the accreting flow "starts" across a "Lagrangian point" (Roche lobe overflow, due to Paczyński accretion mechanics (Paczyński 1980; Paczyński & Wiita 1980)). This is the cusp of the orbiting toroidal surface, which is an important aspect of thick disks since its presence also stabilizes the tori against several instabilities (thermal and viscous local instabilities, and globally against the Papaloizou-Pringle instability- PPI and it could be possibly connected to QPOs emission.)

This fully general relativistic model of an opaque, pressure supported and super-Eddington torus, traces back to the Boyer theory of the equilibrium and rigidity in general relativity, i.e. the analytic theory of equilibrium configurations of rotating perfect fluids (Boyer 1965). Within the so called “Boyer’s condition”, we can determine the boundary of stationary, barotropic, perfect fluid body as the surfaces of constant pressure (eventually also equipotential surfaces $\partial_r V_{eff} = 0$). This occurs in many models, essentially thanks to the condition $\Omega = \Omega(\ell)$ on the fluid relativistic frequency $\Omega$ that has to be a function of $\ell$ (fluid specific angular momentum), a result known as von Zeipel condition (Zanotti & Pugliese 2015; Kozłowski et al., 1979; Abramowicz 1971; Chakrabarti 1991; Chakrabarti 1990). The advantage of this model turns to be both conceptual and technical: from a technical view-point, essential features of the disk morphology like the thickness, the elongation on its symmetric (equatorial) plane, the distance from the attractor are predominantly regulated by the geometric properties of spacetime via the pressure gradients in the relativistic Euler equation, reducible to an ordinary differential equation (ODE), often integrable with the introduction of an effective potential. In this context also the torus inner (outer) edge, in the different torus topological phases related to the stable phases of the disk (emergency of the Roche lobe and the cusp formation) are well defined and constrained.

The effective potential function, $V_{eff}(r; a, \ell)$, expressible here for each RAD component in the purely hydrodynamic model (without the presence of additional factors such as magnetic fields) contains two essential features. 1. It encodes a geometrical factor related to the properties of the spacetime background. This property constitutes essentially the conceptual advantage of having such a model as a reference model, even for decidedly more complex systems. In fact it allows to extrapolate geometrical basic information, when the "curvature" effects due to the very strong gravity of the central SMBH attractor are relevant, for example where these models are used as initial data of GRMHD simulations. 2. The second factor expressed in the effective potential $V_{eff}(r; a, \ell)$, which we can consider as a dynamical factor, is related to the orbiting matter, and represents the centrifugal component of the forces balance in the toroid. This is expressed here by the fluid specific angular momentum $\ell$ on which we will dwell long in this analysis. The function $f(\ell)$ is generally assumed, in many applications of these toroidal models, constant in the torus (see also Lei et al. 2008 for a more general discussion on the functional form of the specific angular momentum). In this model the entropy is constant along the flow (also in the magnetized case) and the rotation law $\ell = \ell(\Omega)$ is independent of the equation of state (Lei et al. 2008; Abramowicz 2009). In these tori in fact the functional form of the angular momentum and entropy distribution during the evolution of dynamical processes, could be considered as
dependent only on the initial conditions of the system and on the details of the dissipative processes— for the relevance of this assumption see Abramowicz & Fragile (2013). However, in the present paper the specific angular momentum $\ell(r)$ assumes a broader meaning by adapting it to the macro-structure, and we also discuss the fundamental of this assumption. We use this function as leading function for the RAD, representing the distribution of the pressure gradient points in the RAD. Thus, we assume in the hydrodynamical model, the radial profile of the Keplerian fluid specific angular momentum $\ell(r) = \ell_K(r)$ as the leading function.

Finally, regarding the possible associated phenomenology, a composite model of accretion tori obviously opens a scenario to numerous phenomena derived from the ringed inner structure and particularly related to instabilities, tori oscillations and their possible traces due to the QPOs emission. The accretion disks (tori) are characterized by natural oscillation modes depending mainly on the disk geometrical thickness and the location of the inner edges. These modes arise consequently to disks internal processes whose traces would appear in the typical light curves of the disk which is the mechanism at base of the QPOs models from accretion disks Stuchlík et al. (2013).

Thick tori are dynamically unstable for the non-axisymmetric oscillation modes, Papaloizou and Pringle instability (PPI), affecting the non acreting torus, particularly for the $\ell$ = constant tori considered here. However, in these tori the proper accretion process across the cusps of the closed configurations (modulated by global oscillations) regulates the accretion rate (due to the mass loss) stabilizing the torus for (local) thermal and viscous instabilities and globally against the PPI—Blaauw (1943); Abramowicz (2009); Paczyński (1980); Paczyński & Wiita (1980); Kozlowski et al. (1979). Thus, thick torus can have global instabilities although the flow can be locally stable. Eventually, it has been shown that such tori turn to be (marginally) stable for local axisymmetric perturbations and unstable to non-axisymmetric modes. In this respect the PPI is the tori characteristic global instability, mainly regulated by the boundary conditions which are extensively considered here in the aggregate model of misaligned tori. These modes are stabilized (suppressed) by the overflows across the cusp (for a discussion on the PPI in the presence of a toroidal magnetic field see for example Bugli et al. (2018) and discussion in Pugliese & Montani (2018)) Tori oscillations and perturbations are strongly dependent on the geometrical torus thickness (i.e. the "vertical" direction on its symmetric plane). Nevertheless, so far a complete spectrum of modes for dynamical oscillations of geometrically thick tori is still an open (technical) problem. For this reason, in this paper we extensively consider the evaluation of the torus geometric thickness and the related $\beta$ parameter to guarantee a validity regime of the approximations considered in the analysis. In the hydrodynamic models of thick disks incompressible and axisymmetric modes of global oscillations are associated with typical characteristic frequencies. We consider as the frequencies relevant in the thick torus, the Keplerian frequency and the two epicyclic (geodesic) frequencies (radial and vertical); their applicability is discussed in Stuchlík et al. (2013). Other modes (surface gravity, acoustic and internal inertial modes) can be studied in the so called relativistic Papaloizou-Pringle equation. To complete the discussion on the tori instabilities we mention the Runaway instability (RI) affecting thick tori and their BH attractors. In this article we have considered a frozen-background spacetime (the black hole mass and zero spin do not change following matter accretion). Runaway instability follows the large accretion rates typical of these tori, the BH mass increases changing the spacetime properties and in turn affecting the orbiting accreting disk, the inner edge moves inwards, this can leads to a a stable situation, or the cusp moves inside the disk inducing an increases of mass transfer. In Pugliese & Stuchlík (2017a) we have considered the possibility of Runaway-Runaway instability (RRI) when the Runaway instability affecting the BH and the inner accreting torus of the agglomeration is accompanied by the consequences of the background modification on the outer tori of the RAD which can collide, accrete or stabilize depending on initial conditions.

Article overview: To make more clear and simple the description and explanation of the RAD model, we provided Table 1 with a list of the main symbols and relevant notation used throughout this article. The article is structured in two parts: in first part, section 2, we develop the model introducing its essential quantities and discussion of its main aspects that will be used in the second part. In details, we introduce the agglomerate of misaligned perfect fluid tori orbiting a central Schwarzschild black hole including a description of the thick disk model. In this analysis we use also results of Pugliese & Stuchlík (2019), reported for convenience in Appendix A, which particularly contains a review of the main characteristics of misaligned (accreting) tori morphology. The second part of the article, section 3, develops application of the results of our analysis to oscillation models often considered in the modeling of high-frequency QPOs. Specifically, we discuss the possibility that the twin peak high-frequency QPOs could be related to the RAD inner structure, relating the RAD discrete geometry to the QPOs emission. In Sec. 4 we include some comments on the RAD structures and a brief discussion on the outcomes of our analysis. Finally, as sideline of this analysis, in Sec. 5 we discuss the case when the leading RAD function, defining the distribution of the tori in the RAD, has been modified to an alternative definition, considering the case of RADs where some of their components can be magnetized tori with the toroidal magnetic field introduced in Komissarov (2006), in the approach considered in Pugliese & Montani (2013); Pugliese & Montani (2018). Concluding remarks follow in section 6.
The fluid four-velocity satisfies the conditions 
\( \nabla \cdot u = 0 \) on the symmetry plane, at an inclination angle \( \theta \) of each toroid of the orbiting agglomeration, where \( p \) and \( p \) are the total energy density and pressure, respectively, as measured by an observer moving with fluid with fluid four-velocity \( u \). The projection tensor and \( g_{ab} \) the Schwarzschild metric tensor (where \( \nabla_{ab} = 0 \)), \( M \) is the BH mass. In Figures 1, we show a solution of the Euler equations with appropriate boundary conditions for misaligned barotropic tori, governed by the integral of Eq. (1) that states the form of definition of an effective potential due to relation

\[
\int \frac{dp}{\rho + p} = -\ln V_{eff} = -\ln \left[ \frac{(r - r_\ast)r^2}{\sqrt{r^3 - \ell^2(r - r_\ast)}} \right]
\]

where \( r_\ast = 2M \) is the BH horizon, and \( \ell(r) \) denotes radial profile of the specific angular momentum of the orbiting fluids in the symmetry plane, here and in the following we shall use dimensionless quantities if not otherwise specified.

1 The continuity equation, \( \nabla \cdot u + (p + p)\nabla u = 0 \) is identically satisfied because of the symmetries. The choice of a perfect fluid stress-energy tensor is closely correlated with the typical relation between the (fictitious) time-scales of the main physical processes assumed relevant for geometrically thick disks, which in turns is linked to the forces balance inside the disk. The disk physical processes are generally conveniently considered having three main origins: (1) a dynamical origin, (2) a thermal and (3) a viscous origin. Specifically, the relation between the related timescales assumed for these processes is determined by \( \ell_{dyn} \ll \ell_{therm} < \ell_{visc} \). The dynamic part is represented by the time reached by the pressure forces to balance the centrifugal and gravitation component. The thermal time scale concerns the entropy redistribution, dissipative heating and the cooling processes. The part most directly related to our choice of the rotation law (i.e \( \ell(r) \) interpreted for each torus, and here considered as RAD leading function) is the time scale involved when the angular momentum changes because of torque and dissipative effects—time scale of the viscous effects.

2 The fluid four-velocity satisfies \( u^\mu u_\mu = -1 \). We adopt the geometrical units \( c = 1 \equiv G \) and the \((-+,+,+,+)\) signature. The radius \( r \) that has unit of mass \( [M] \), and the angular momentum units of \([M]^2 \), the velocities \([u^r] = [u^\phi] = 1 \) and \([u^\theta] = [u^\phi] = [M]^{-1} \) with \([u^\theta]/[u^r] = [M]^{-1} \) and \([u^\phi]/[u^r] = [M] \). For the seek of convenience, we always consider the dimensionless energy and effective potential \( [V_{eff}] = 1 \) and an angular momentum per unit of mass \([L]/[M] = [M] \).
The model we adopt here for each component of the RAD aggregate provides the the (rigid) boundary of a stationary, barotropic, perfect fluid toroid constituted by the equipotential and equipressure surface given by the conditions \( V_\text{eff}(\ell, \theta) = \mathcal{K} \) =constant. The main features of the equipotential surfaces for a generic rotation law \( \Omega = \Omega(\ell) \) are described here by the equipotential surface for uniform distribution of the angular momentum density \( \ell \equiv L/E \) (also the fluid specific angular momentum where \( E \) and \( L \) are the particle energy and angular momentum per unit of mass as seen by infinity). The choice \( \ell \) =constant for each torus is a well known assumption, widely used in several contexts where geometrically thick tori are considered. Conditions \( \ell = \) constant, and \( \Omega = \) constant for the fluid relativistic angular velocity \( \Omega \), define the surfaces known as von Zeipel’s cylinders. From the series of results related to the von Zeipel theorem it follows that the equipotential surfaces of the marginally stable configurations orbiting in a Schwarzschild spacetime correspond to constant definition of \( \ell \). In the static spacetimes, the family of von Zeipel's surfaces depend only on the background spacetime, therefore they do not depend on the particular fluid rotation law \( \Omega = \Omega(\ell) \). If the fluid is barotropic, as we are considering here, then von Zeipel theorem guarantees that the surfaces \( \Omega = \) constant coincide with the surfaces \( \ell = \) constant. Solutions of equation \( \ell \) with appropriate boundary conditions lead to four classes of configurations corresponding to closed and open surfaces, and surfaces with or without a cusp, i.e. self-crossing open or closed configurations. The closed, not cusped, surfaces are associated to stationary equilibrium (quiescent) toroidal configurations. For the cusped and closed equipotential surfaces, the accretion onto the central black hole can occur through the cusp of the equipotential surface: the torus surface exceeds the critical equipotential surface (having a cusp), leading to a mechanical non-equilibrium process where matter inflows into the central black hole (a violation of the hydrostatic equilibrium known as Paczyński mechanism). Therefore, in this accretion model we shortly indicate the cusp of the self-crossed closed toroidal surface as the "inner edge of accreting torus". Finally, the open equipotential surfaces, which we do not consider explicitly here, have been associated to the formation of proto-jets.

**Tori agglomerate in the RAD framework**

We describe the aggregate of misaligned tori adopting the RAD framework developed in [Pugliese & Stuchlik (2015, 2016, 2017a)] in the case of an agglomerate composed by tori orbiting on the equatorial plane of a Kerr attractor (the eRAD). As in [Pugliese & Stuchlik (2019)], it is convenient to introduce a RAD rotational law \( \ell(r) \) as the distribution of specific angular momentum of its toroidal components. This function plays the part of leading RAD function proving also the toroids RAD distribution in its ringed structure. Therefore \( \ell(r) \) is intended to be the absolute magnitude of the fluid specific angular momentum distribution of each toroid of the aggregate orbiting at distance \( r \) from the central static BH on its general symmetry plane (while we stress the specific angular momentum is constant inside each toroid). The Keplerian, geodesic distribution of the specific angular momentum \( \ell_K(r) \) then parameterizes each torus in the RAD, due to RAD rotational law

\[
\ell_K(r) \equiv \frac{3/2}{(r - r_\gamma)}.
\]

We introduce also the RAD energy function parameterizing each torus of the agglomerate with further \( K \)-parameter governing extension of the torus [Pugliese & Stuchlik (2019)]

\[
K(r) \equiv \frac{(r - r_\gamma)}{\sqrt{(r - r_\gamma)r}}.
\]

\( r_\gamma = 3M \) is the location of the last photon circular orbit.

**Origin of \( \ell(r) \) and \( K(r) \)**

The specific angular momentum adopted in Eq. (3) and energy function \( K(r) \) of Eq. (4), have a clear geometric origin related to the symmetries of the Schwarzschild background. It is convenient to explicit the following quantities as follows

\[
E = -g_{ab} \xi_b^a p^b = -\xi_t p^t, \quad L = g_{ab} \xi_b^a p^b = g_{\phi\phi} p^\phi.
\]

We note that the von Zeipel condition guarantees an integrability condition on the Euler ODE, and it is therefore essential in many models where the reduction of ODE considered here (effective potential approach, leading function) is to be extended to more complex objects, for example in the extension to poloidal magnetic field a la Komissarov which is now present in integral form only as a toroidal magnetic field [Komissarov (2006), Zanotti & Pugliese (2015)]. The von Zeipel conditions \( \ell(\Omega) \) is closely related to the barotropic EoS of the fluid. In other flows, where dissipative effects must be considered, \( \ell(\Omega) \) depends on these in a way that is still unclear and often given on assumptions on various parameters, usually related to viscosity factors to be taken ad hoc, for example in the so called alpha-preservation in MRI or GRMRI models [Balbus (2011)]. Compared to these assumptions, the Paczyński prescription has the notable advantage of having a narrow geometric sense, which in fact makes it the bottom boundary condition on practically any accreting disk, with the respect to the quasi-spherical Bondi accretion [Abramowicz & Fragile (2013)]. On the function \( \ell_K \) generalization (that can have also different regimes in regions discriminated by \( r_{\text{exto}} \)) see [Lei et al. (2008)].
Pugliese&Montani (2013), Pugliese et al. (2013), where the metric components are $g_{tt} = -e^{-\phi(r)}$ and $g_{\phi\phi} = 4\pi^2 \sin^2 \theta$, written in standard spherical coordinates $(t, r, \theta, \phi)$, where $e^{\phi(r)} \equiv (1 - 2M/r)$. Expressed in terms of the four momentum components and $(E, L)$ are constants of motion for test particle geodesics with four-velocity $u^a$, related to the Schwarzschild geometry Killing vectors $\xi_t$ and $\xi_\phi$. We can define the quantity $V_{\text{eff}}$, considering the normalization condition on the fluid four-velocity (assuming $p^\phi = 0$ for the circular configurations), we introduce also the relativistic angular frequency $\Omega$ and the fluid specific angular momentum $\ell$:

$$u^t \equiv \sqrt{E^2 - V^2_{\text{eff}}}, \quad \Omega \equiv \frac{u^\phi}{u^t} = \frac{g_{tt}L}{E g_{\phi\phi}} = \frac{g_{tt}\ell}{g_{\phi\phi}}, \quad \ell \equiv \frac{L}{E} = \frac{g_{\phi\phi}\Omega}{8\pi},$$

and considering $u^t = 0$, we obtain $V_{\text{eff}} = E$ and

$$L_K = \pm \frac{(\sin \theta)^2 r^2}{(r - 3)} \quad \ell_K = \sqrt{\frac{(\sin \theta)^2 r^3}{(r - 2)^2}}, \quad V_{\text{eff}} = \sqrt{-\frac{g_{tt}\ell g_{\phi\phi}}{g_{\phi\phi} + \ell^2 g_{tt}}}.$$  

where $L = L_K$ and $\ell = \ell_K$ are the test particles and fluids angular momentum respectively, correspondent to the extremes for the $V_{\text{eff}}$ when expressed in terms of the particle momentum $L$ or the fluid specific momentum $\ell$ respectively. Energy function $K(r)$ of Eq. (6) can be found considering $V_{\text{eff}}$ of Eq. (7) evaluated in $\ell = \ell_K(r)$.

We stress that the function $(\ell(r), \ell_K)$, adopted for the most part here as the RAD leading function, has a meaning for the single toroidal RAD component as discussed in the previous paragraph. Here, we reinterpret $(\ell(r), \ell_K)$ as a tori distribution in the RAD, this setting has been detailed discussed in Pugliese&Stuchlik [2015, 2017a, 2016, 2017c] for the eRAD. It follows that, because of its geometric and centrifugal origins, function $(\ell(r), \ell_K)$ is naturally considered as a possible reference distribution. (It is then the boundary level for the "Bondi condition" on the fluid angular momentum in an extended region of any ("fast" rotating) accretion disk). Similar argument applies for the energy function $K(r)$. Each value of the function $K(r)$, in fact, a role for a toroidal RAD disk, being directly related to the maxima and minima of the RAD tori effective potential $V_{\text{eff}}(a, \ell; r)$ and therefore to the notable points of the tori topology.

Critical density and pressure points in the tori

According to Eq. (1), the maxima of the effective potential correspond to the minima of the HD pressure (and density because of the barotropic EoS), viceversa the minima of the potential correspond to the maxima of the HD pressure and density (the tori centers). Specifically:

(a) The constant values $K(r) = K \in [K_{\text{mas}}, 1]$, are associated to

i. closed configurations, that is to quiescent tori, if the $K > \max V_{\text{eff}}(r) < 1$ i.e. $K$, constant for each torus, is mower then the torus effective potential maximum.

ii. cusped tori if the $K \equiv \max V_{\text{eff}}(r) < 1$ i.e. $K$, constant for each torus, corresponds to the maximum point of the torus effective potential. Here $K_{\text{mas}} \equiv K_{\text{max}}$ is the minimum point of the energy function $K(r)$—see Table 1. This maximum point will be located on the cusp $r_{\text{mas}} \equiv r_c$ of the torus Roche lobe. The torus center, $r_{\text{cent}}$, is a minimum of the effective potential, and corresponds to $K \equiv \min V_{\text{eff}}(r) < 1$.

(b) The condition $K(r) = K > 1$ does not correspond to the minima of the potential, but it can correspond to the maximum points. In this last case $K \equiv \max V_{\text{eff}}(r) > 1$ is associated to the critical point, $r_c$, of the open surfaces cusp, associated with proto-jets. This radius is located on $r_c \in [r_j, r_{\text{mas}}]$, where the boundary radii of this range are defined in Table 1. These configurations have very large specific angular momentum $\ell \in [\ell_{\text{mas}}, \ell_j]$.

(c) For $\ell > \ell_j$, the potential has no maximum points i.e. the torus cusp is suppressed by the centrifugal force of the extremely fast rotating fluid, and only quiescent toroidal configurations are possible. Typically, however, these toroids are located quite far from the central attractor i.e. $r_{\text{cent}} > r_j$.

4 Thick accretion disks are characterized by a significant contribution of the centrifugal force represented by "high" angular momentum of matter, that is superior or equal the Keplerian $\ell_K$ angular momentum. The slow rotation cases are often referred to as "Bondi flows" in accordance with the spherically-symmetric (non-rotating) accretion (Bondi 1952). Angular momentum distribution $\ell_K(r; a)$ turns thus to be a natural reference function of the accretion model. More precisely, in the reference accretion flow known as the (quasi-)spherical “Bondi” accretion, the angular momentum is not so relevant in the dynamical forces compared with others ingredients of the forces balance and the (specific) angular momentum in the disk turns smaller than the Keplerian one. This frame is generally expanded by assuming the condition that an accretion disk must have an extended region where matter has a large (fast) enough centrifugal component $(\ell \geq \ell_K)$. Abramowicz&Pringle [1983]. "Bondi accretion" in this sense can be associated to small accretion rates (for example in the limit free fall accretion disks-slow rotation), in this sense fast rotation disks are for example thick disks (Bondi 1952) see also Mach, Piróg & Font (2018). Thus in these thick tori, where the strong gravitational field is assumed dominant with respect to the dissipative forces, and a perfect fluid energy-momentum tensor is assumed, the specific angular momentum $\ell_K$ turns to be an upper boundary conditions with the respect to the "Bondi condition". These tori represent therefore a reference model for more complex systems.
(Quantities introduced here are given in Table 1 and discussed below). The effective potential is widely used in literature for the description of these configurations, however we mention for analogy with notation and conventions on signs used here for example [Pugliese & Stuchlik 2015, 2018a, 2017c].

We should note that, although the energy function $K(r)$ is related to the fluid effective potentials in the RAD $V_{\text{eff}}(a, \ell; r)$, which is a function of the radius $r$ and depends on the parameters $(a, \ell)$, function $K(r)$ does not depend on $\ell$, but it provides, point by point $r$, the $K$-parameter values correspondent to the extremes of the effective potential for each possible toroidal in the RAD agglomeration that is, as explained below, it relates configurations with different $\ell$. The energy function $K(r)$ can be recovered directly from the effective potential $V_{\text{eff}}$ of Eq. (2) evaluated on the curve $\ell_k(r)$ in Eq. (3). Function $K(r)$ provides therefore the distribution of the effective potential values corresponding to the maximum and minimum density (and HD pressure) points. The dependence from $\ell^2$, of the tori characteristics, including morphological and stability properties introduced in Appendix A, is expression of the spacetime spherical symmetry. For fixed $\ell = \text{constant}$ of a torus, the solution of $\ell = \ell_k(r)$ gives the torus center $r_{\text{cent}}$ which is also the maximum density and HD pressure points; RAD tori have also equal minimum pressure (and density)-points, but not in general equal geometric center which depends also on $K$. The function $K(r)$ of Eq. (3) is related instead to an independent tori parameter $K$ which regulates the torus elongation $\lambda$ on its symmetry plane and the emergence of hydro-dynamical instability. Furthermore, this is also associated to the torus density, the torus thickness and other characteristics related to the torus energetics, as cusp luminosity and accretion rates [Pugliese & Stuchlik 2019].

Below we use the leading function $\ell_k(r)$ and the energy function $K(r)$ of the RAD, constraining its inner structure. The idea is to consider limiting points representing constraints on the toroidal components and therefore constraints on the RAD as the entire system (inner structure, inner and outer boundaries, limits on instabilities, as accretion or tori collision—Pugliese & Stuchlik 2015). Since $\ell_k(r)$ and $K(r)$ correlate different points, we study them by introducing some functions (radii) obtained from the conditions on the leading function. In brief, these represent the boundary conditions for the existence and location of the inner and outer edges, clearly regulated by the Euler equations, or maxima and minima of the pressure (and density in barotropic models like tori considered here where the pressure is the hydrostatic pressure).

The phases of the torus evolution towards accretion are supposed to be associated generally to a decrease of the momentum magnitude $\ell$ and possibly an increase of the $K$-parameter (Pugliese & Montani 2015). Moreover, the study of $\ell(r)$ and $K(r)$ functions is also important to set constrains on the tori collision. Equal $\ell$ tori, not possible in the eRAD frame, might be possible configurations in the RAD because of the different tori inclination angles. They present a doubled collisional region, minimized in case of torus maximum relative inclination $\theta_{ij} = \pi/2$ coinciding with the toroidal section with min($K(\ell)$) and having equal maximum density points. We introduce therefore the two functions $r_p^f(r)$ and $r_p^r(r)$.

The function $r_p^f(r)$ (from the leading function $\ell_k(r)$)

Condition $(\ell) = \text{constant}$ identifies the center $r_{\text{cent}} > r_{\text{mso}}$ (maximum pressure points), where $r_{\text{mso}}$ denotes radius of marginally stable orbits, and eventually, the instability point $r_x < r_{\text{mso}} < r_{\text{cent}}$ of an cusped torus with cusp $r_x$. We can relate these radii with a function $r_p^f(r)$

$$r_p^f(r) = \frac{2r}{(\ell - r_x)^2} \left( \sqrt{(2r - \gamma_x) + r - r_x^b} - \ell_k(r_p^f) \right), \text{ where } r \in [r_x, r_x^b], \text{ and } r_x^b = M$$

Function $r_p^f$ relates radii $r_{\text{cent}}$ or $r_{\text{crit}}$ (= $r_{\text{cusp}}$) for the configurations with equal $\ell$, for closed cusped tori the critical point $r_{\text{crit}}$ is the accretion point $r_x$ as function of the other radius of the couple. Note that the evaluation of $K(r = r_p^f)$ from equation 3 provides the $K$-parameter value of the maximum pressure $r_{\text{cent}}$ or minimum $r_{\text{crit}}$ (if this exists), turning in a RAD parametrization in terms of disks pressure gradients, where the disk center is expressed in terms of the instability point and vice versa.

The location of the accretion tori edges and center.

(Here we adopt notation $Q_\star = Q(r_\star)$ for any quantity $Q$ and for a general radius $r_\star$, in particular $Q_x$ refers generally to quantities evaluated at the cusp $r_x$ or related to cusped tori. There is $r_\circ = 3M$, the marginal circular orbit (photon circular orbit), $r_{\text{mbo}} = 4M$ (marginally bounded orbit), $r_{\text{mso}} = 6M$ (marginally stable circular orbit).)

- The proto-jet (open cusped configurations) cusp is at $r_{\text{crit}} = r_j \in [r_y, r_{\text{mbo}}]$ with specific angular momentum $\ell \in [\ell_{\text{mbo}}, \ell_y]$;
- The torus cusp is at $r_{\text{cent}} = r_x \in [r_{\text{mbo}}, r_{\text{mso}}]$ (specific angular momentum $\ell \in [\ell_{\text{mso}}, \ell_{\text{mbo}}]$);
- The center of a cusped closed configuration is at $r_{\text{cent}} \in [r_{\text{mso}}, \ell_{\text{mbo}}^b]$, where $\ell_{\text{mbo}}^b = 2\left(\sqrt{5} + 3\right)M \approx 10.4721M$ (solution of $\ell(r_{\text{mbo}}) = \ell(r_{\text{mbo}}^b)$).

5 From the point of view of RAD model considered in [Pugliese & Stuchlik 2019], independently of the relative direction of rotation and the angle of inclination of the toroids in the static spacetime all the toroids have characteristics similar to the $\ell$corotating pairs composing an eRAD orbiting a Kerr central SMBH as discussed in [Pugliese & Stuchlik 2017a].
Figure 1. Density profiles of a RAD orbiting misaligned tori obtained from the 3D HD integration of the perfect fluid Euler equation \( \frac{\rho}{m_{\text{b}o}} \). Black region is the central Schwarzschild BH. The RAD is of the order \( n = 3 \) (number of orbiting disks). Different viewing angles are shown, where the central black hole (outer horizon) is embedded into the orbiting RAD, the ringed structure also be recognized by the observation. Tori parameters \( \ell \), being the fluid specific angular momenta and \( K \), a parameter related to matter density and tori energetics are as follows: \( T_1 = (\ell_1 \approx 3.75, K_1 = 0.95) \), \( T_2 = (\ell_2 \approx 4.7958, K_2 = 0.976) \), \( (T_3 \approx 6, K_3 = 0.98487) \), where \( T_1 < T_2 < T_3 \), indicates the tori closest to the central attractor. Torus \( T_1 \) is at distance \( r_{in,1} = 5.75538M \) from the central BH (location of the inner edge of the orbiting torus), and elongation on the equatorial plane \( \lambda_1 \approx 4.65846M \), \( T_2 \) is at distance \( r_{in,2} = 14.16B \) from the central BH, with elongation \( \lambda_2 = 11.2734M \). \( T_3 \) is at distant \( r_{in,3} = 27.867M \) from the central Schwarzschild BH with elongation \( \lambda_3 = 8.57362M \). In Figures \( A2 \), schemes of RAD of order \( n = 5 \) of quiescent and non-interacting tori are represented.

- Proto-jet configuration centers are in \( \left[ r_{\text{mbo}}, r^b_\ell \right] \) where \( r^b_\ell = 6 \left( \sqrt{3} + 2 \right) M = 22.3923M \).
- Finally configurations with center in \( r > r^b_\ell \) are quiescent and closed, with specific angular momentum \( \ell_K > \ell_\gamma \). Radii \( r^{b}_{\text{mbo}} \) and \( r^b_\ell \) are solutions of \( \ell_K(r) = \ell_{\text{mbo}} \) and \( \ell_K(r) = \ell_\gamma \), respectively–Fig 2.

All tori with equal \( \ell \) have the same center and, eventually, same location of the cusp, therefore they orbit in the same spherical shell across the radii \( r_{\text{cent}} \) and limiting \( r_{\gamma} \). On different planes (different polar \( \theta \) angles), two tori, \( T_1 \) and \( T_2 \), having equal specific angular momentum \( \ell \) but with different inner and outer edges radii, \( r = r_{in} \) and \( r = r_{out} \), are in equal center \( r_{\text{cent}} \) spheres where \( r^1_{in} < r^2_{in} < r_{\text{cent}} < r^2_{out} < r^1_{out} \), that is, they are concentric. Location of \( r_{in}, r_{out} \) is determined by the \( K \) function. For quiescent (not cusped) \( T_1 < T_\alpha \) (i.e. \( T_1 \) is the torus closest to the central BH) there is \( \ell_1 < \ell_\alpha \) and \( r^1_{out} < r^\alpha_{in} \). Considering then that cusped configurations are fixed by the \( \ell \) parameter only, where for \( \ell_1 < \ell_\alpha \), there is \( \ell^2_{\gamma} < \ell^1_{\gamma} < r^2_{\text{cent}} < r^1_{\text{cent}} \) and it is \( r^1_{out} < r^\alpha_{out} \). Therefore the outer cusped torus "incorporates" the inner cusped torus, similarly to the eRAD case. Nevertheless, the isothermal compression can reduce, for high relative inclination angles, up to the limiting orthogonal case, the flow of impacting accreting material from an outer to the inner torus of the agglomerate. The tori misalignment can effectively reduce the collisional effect.

The function \( r^b_\ell (r) \) (from the energy function \( K(r) \))

Similarly to the considerations leading to the function \( r^b_\ell \) of equation \( 8 \), the condition \( K(r) = K(r^b_\ell) \), determines a radius
\( r = r_{\text{orbit}} \) for test particle and limit for inner edge of the torus (outer boundary) functions governing properties of the tori, discussed in Pugliese & Stuchlík (2019).

\( \lambda \) have larger magnitude of the specific angular momentum leading in general to a larger elongation.

Condition \( K(r) = K(p_r) \equiv K_p \).

\( r_p(r) = K_m \begin{cases} \frac{x + 1}{x} & \text{where } x = r - r_m \end{cases} \quad \text{K} = r - r_m \).

Function \( r_p \) identifies a pair of tori \( (T_1, T_2) \) with \( K_m \begin{cases} \frac{x + 1}{x} & \text{where } x = r - r_m \end{cases} \) orbit for test particle and limit for inner edge of the torus (outer boundary). 

\( K(r) = K(p_r) \equiv K_p \).

Correlating the notable points of the RAD constraints. It is clear that the points are symmetrically correlated, the fixed point of the transformation represented by the function \( r_p(r) \) are, as expected, the horizon \( r_+ \) and the marginally stable orbit \( r_m \) (limit of torus with center, maximum density point, \( r_{\text{cent}} \equiv r_m \), and negligible elongation and pressure being approximated to a free dust particles string). The couples \( r_{mbo}, r_m \) is particularly notable, considering that \( r_m \) is the marginally bounded orbit for test particle and limit for inner edge of the torus \( r_m \) to the open proto-jet configuration (with no closed outer boundary). 

Further notes on the morphological constraints on RADs stability can be found in Appendix (A), as well as exact expression of many characteristic functions governing properties of the tori, discussed in Pugliese & Stuchlík (2019).
3 RELATING TWIN PEAK QUASI-PERIODIC OSCILLATIONS WITH RADs STRUCTURE

The aim of this section is to investigate the possibility that the twin peak quasi-periodic oscillations (QPOs) could be related to the RAD inner structure, linking therefore the RAD discrete geometry to the QPOs emission. The RAD and the eRAD tori are characterized by a special and distinctive ringed structure that, as pointed out in Pugliese & Stuchlý (2015, 2018b, 2017a), could be evidenced in the X-ray emission spectrum, and as an imprint of the discrete inner RAD composition in the combined oscillatory phenomena associated to the tori models. Here we explore this conjecture, considering the eRAD and RAD case studying the expected epicyclic frequencies in the case of ringed structures. The idea is that the discrete structure of a RAD could be related to QPOs emission associated to the accretion torus structure, particularly with the respect to the inner edges \( r_{in} \) of (quiescent or) accreting tori. As for the eRAD discussed in Pugliese & Stuchlý (2015, 2016), the RAD is characterized by relation between \( r_{in}/r_{out} \) (or elongation \( \lambda = r_{out} - r_{in} \)) and height \( y_{max} \). Tori spacing \( \lambda = r_{in}^2 - r_{out}^2 \) (for inner \( T' \) and outer \( T'' \) tori) is strongly dependent on the background geometry. More specifically, the issues we address in the frame of the QPO-RAD hypothesis are (1) the interpretation of multiple combined signals as expression of the ringed structure, (2) the recognition of the role of each torus edges and of the RAD internal structure in the emission frequencies. However, in this analysis we propose a first comparative investigation on the problem of the QPO interpretation in the RAD frame, we also recognize that this hypothesis should be considered by properly disk-seismology effects for each toroid, which is in many aspects dependent on its geometrical features. Therefore, a part of this section is also dedicated to an evaluation of the impact of the disk geometry (specifically its thickness) in this analysis. Here, we mainly focus on the so called geodesis oscillation models (Stuchlý et al. 2013) analyzing the radial profiles and assuming specific oscillation models of the RADs constituents. In the test particle limit, the frequencies of the epicyclic oscillations in the Schwarzschild spacetime are

\[
\nu(r) = \nu_K(r) \left(1 - \frac{r_{mso}}{r}\right), \quad \nu_\beta(r) = \nu_K(r) \equiv \frac{1}{r^{3/2}},
\]

shown in Figs 1 as associated to different tori models. Furthermore, we also consider also the modifications of the geodesic oscillation models due to the tori structure.

The torus geometrical thickness and the "\( \beta \)-parameter" In the case of a thick disk, the role of the toroidal geometry and especially its geometrical thickness is predominant in the resolution of the oscillatory problem. For this purpose we evaluate the dimensionless \( \beta_{crit} \) parameter for cusped tori of our RAD models

\[
\beta_{crit} = \frac{(r_{cent} - 2\lambda)^2(r_{cent} - r_{cusp})\sqrt{r_{cent}r_{cusp} - 2(r_{cent} + 2r_{cusp})}}{r_{cent}r_{cent} - 3r_{cusp}\sqrt{r_{cusp} - 2}},
\]

introduced in Török et al. (2016), and derived in the more general form in Straub & Sramkova (2009) (see also Abramowicz et al. (2006)) that characterize the size of the torus. This quantity is roughly proportional to the flow Mach number at the torus center and to the ratio of the radial (or vertical) extension of the torus to its central radius (i.e., as pointed out in Török et al. 2016), in this situation the sound-crossing time and the dynamical timescale of the torus are similar. We consider the parameter \( \beta_{crit} \) within the constraints discussed in Appendix A: then there is \( r_{cent} \leq r_{mso}, r_{cusp} \geq r_{mbo} \) (note that \( \beta_{crit} \), specified in Török et al. 2016, should be considered for \( r_{cent} \leq 10.47M \approx \beta_{mbo} \) having \( \beta = \beta_{mbo} \). The frequencies have been therefore related to the epicyclic oscillations with the radial and vertical epicyclic modes describing eventually a global motion of the torus (Abramowicz et al. 2006). In fact it has been shown then that for slender tori having

8 QPOs have been also related to tilted disks as consequential to Lense–Thirring effect. In AGN accretion for example or in the binary (X-ray) systems. The mechanism in brief is as follows: a tilted accretion disks orbiting Kerr BHs undergoes a break in its central part for Lense–Thirring effect (Nixon et al. 2012c). As a consequence of this, the disk is splitted into different planes, giving raise to QPOs emission. Clearly the break would depend on the tilt angels (and the BH spin).

9 Eventually in the aggregate RAD model considered here we might consider a full thick tori model for the oscillation analysis including dependence on the outer edge \( r_{out} \) and the maximum point \( y_{max} \) of the torus surface. On regards of the particle model approximation of the oscillations considered here, we note that in this model the inner margins of the closed surfaces are related to constat and zero pressure point.

10 A non trivial topic in this discussion concerns the actual role of the disk morphology in the disk oscillation and emission, both in the most specific sense of the RAD morphology, which we address extensively, and more generally in each disk component. This issue regards the disk oscillations and the disk emission spectra. Here we have dealt with this argument through the evaluation of \( \beta \) geometric thickness parameter, which is a relevant aspect of the disk perturbation theory. However, a second aspect concerns the disk as extended object. More precisely, we could say that oscillations originate in the entire disk, but it may be reasonable to assume that their traces (for example as QPOs) would be evident in the emitted radiation that is usually associated to the disk more active part which is, in many senses, the disk boundary (and particularly the disk inner edge). It is clear that this specific issue turns to be extremely relevant in the ringed composite structure of RAD misaligned tori (which has lead also to the idea of luminous annuli), where each torus boundary is significant in the model. In this respect the accreting flow, especially in the case of cusp (Roche lobe) overflow, can have a great relevance.

11 Notice that the frequencies are given as dimensionless, also the radius is constructed dimensionless \( (r = r/M) \) being expressed in units of mass parameter \( M \). In order to obtain frequency in standard units of \( Hz = 1/s \) we have to use the correction factor \( c^3/2\pi GM \).
\( \beta \approx 0 \), frequencies of these tori modes, \( v_r \) and \( v_\theta \), as measured in the fluid reference frame are equal to the test particles geodesic, epicyclic frequencies. In a more realistic case, where there is \( \beta > 0 \), the relevant pressure gradients in the disk force balance are expected to induce a frequencies shift (i.e., for non-slender tori the epicyclic modes frequencies are modified by pressure). As perturbations are generally of the order of \( \beta^2 \), we will evaluate this parameter in the various tori models. Thus, according to Eq. (12) the QPO frequencies will depend on the location of center and cusp of the torus and its thickness (here the \( \beta_{\text{crit}} \) parameter). We also compare in Fig. 3 the \( \beta_{\text{crit}} \) Parameter for toroids considered here with thickness \( S_x = 2h_{\text{c}/}\lambda \) as defined in Appendix A. To start with we recall that the disk elongation and height are maximum for accreting configurations (cusped toroids) and in general these quantities grow with \( \ell \) and \( K \). The thickness \( S_x \) is higher than 1 only for in particular ranges of the model parameters highlighted in Appendix A. Moreover, the location of the disk center and cusp depend on the model parameters \( \ell \) and \( K \) and, as from the analysis in Appendix A and Sec. 2, there can not be two toroids with the same cusp, or with the same center, this fact constitutes a way to strongly relate QPOs and RAD structure. There can be however more toroids with equal geometrical thickness \( S_x \) (and eventually \( \beta_{\text{crit}} \)), thus relating not uniquely but to more tori models the \( \beta_{\text{crit}} \) regulating the oscillation modes. (We should then also consider that the eRAD is a geometrically thin disk even with geometrically thick components.) In Pugliese & Stuchlisky (2019) we have studied this possibility extensively, and in particular considering the limiting case of the plane (\( \ell, K \)) for toroids with thickness \( S_x = 1 \), discriminating geometrically thick disks from geometrically thin disks. Specifically, we have represented the situation for the cusped tori (thickness of the disk in accretion). The \( \beta_{\text{crit}} \) parameter in Eq. (12) has been studied in Fig. 3 considering \((r_{\text{cent}}, r_{\text{cusp}})\) in equations A3 and equations A7. Finally, we note that we take full advantage of RAD symmetry in the static case and consider toroids oscillation on each symmetry plane. We are actually considering the problem for misaligned RAD tori in static spacetime as eRAD, nevertheless we expect that the combination of the oscillator models for toroids could depend on the tori inclination angles \( \theta_{ij} \). For a discussion on QPO in titled accretion disks we mention Dexter & Fragile (2011); Banerjee et al. (2018).

Then, there is \( \beta_{\text{crit}}(\ell) = 1 \) for \( \ell = \ell_0^B \approx 3.7432 < \ell_1 \), where \( \ell_1 = 3.887 \in [m_{\text{soo}}, m_{\text{mbb}}] \) and \( K_1 = 0.975 \) define the \( S = 1 \) case. Note that curve \( \ell(r) = \ell_0^B \) provides also the two radii \( r^-_\beta = 4.87956M \) (a cusp) and \( r^+_\beta = 7.6259M < r_{\text{mbb}}^b \) (a center) for a cusped torus, corresponding to \( \ell = \ell_1^B \) such that \( \beta_{\text{crit}}(r^-_\beta) = \beta_{\text{crit}}(r^+_\beta) = 1 \) having expressed \( r_{\text{cent}} \) as function of \( r_{\text{cusp}} \) and, viceversa, \( r_{\text{cusp}} \) as function of \( r_{\text{cent}} \) in Eq. 12. This relation between the torus center and the cusp can be written in a compact form as

\[
\tilde{r}(r) = \frac{2}{(r_1 - 1)^2} \left( \frac{r_1^2 (2r_1 - 3) + (r_1 - 1)r_1}{(r_1 - 2)^2} \right), \quad \text{for} \ r_1 \in [m_{\text{bo}}, m_{\text{mbb}}] \quad \text{and} \quad \tilde{r}(r) \in [m_{\text{bo}}, m_{\text{mbb}}], \text{where} \ r = r_{\text{cusp}} \in [m_{\text{bo}}, m_{\text{soo}}]
\]

for \( \tilde{r}(r_1) = r_{\text{cent}} \in [m_{\text{soo}}, r_{\text{mbb}}^b] \) and \( r \in [m_{\text{soo}}, r_{\text{mbb}}^b] \) for \( \tilde{r}(r_1) = r_{\text{cusp}} \in [m_{\text{bo}}, m_{\text{soo}}] \) as shown in Figs 4. For convenience we also report to the following relation

\[
r_x^c \equiv \frac{r_{\text{cent}}}{(2r_{\text{cent}} - 3) + 1} \left( \frac{1}{(r_{\text{cent}} - 2)^2} \right)
\]

which is actually a specialization of Eq. 9 and \( \tilde{r}(r_1) \), Eq. 13. It is also clear from Figure 3 that RADs toroidal components have a prevalent \( \beta_{\text{crit}} > 1 \), and essentially it is \( \beta_{\text{crit}} \leq 1 \) for \( r \in [r_{\text{mbb}}^b, r_{\text{crit}}^B] \) where \( r_{\text{mbb}}^b \) correspond to \( \ell = \ell_0^B \). (Note that for a cusped torus, where \( (r_{\text{cent}}, r_{\text{cusp}}) \) are related by equation 13 it is then \( \beta_{\text{crit}} \equiv 0 \) only in the limiting case of \( r_{\text{cent}} = r_{\text{cusp}} = r_{\text{soo}} \).

**Frequency models-relations of the HF QPO models** We now focus on models that consider main epicyclical frequencies, especially we refer to the analysis of Stuchlisky et al. (2017); Stuchlisky & Kolos (2016); Stuchlisky et al. (2007); Török, & Stuchlisky (2005); Kotrlová et al. (2017); Török et al. (2016); Srámková et al. (2015); Stuchlisky et al. (2013); Török et al. (2011); Stuchlisky et al. (2011); Kotrlová et al. (2008); Stuchlisky et al. (2007). Here we want to test the RAD as a frame for QPO models assuming the geodesic frequencies governed by the background geometry but determined by the constraints imposed on the RAD. Each frequency model (TD, RP, RE, WD) we consider is borrowed from a specific context from which they are derived including slender tori and hot spot models (assumptions radiating spots in thin accretion disks)—we refer to the mentioned literature for further details on these models relevant for both accreting systems orbiting a BH or a neutron star. Here we provide, for each of the considered (geodesic) oscillation models, only the frequency relations corresponding to the twin high-frequency oscillations, giving the upper \( v_L \) and lower \( v_L \) frequency of the pair in terms of the radial and vertical oscillations and their combinations with the azimuthal frequency—for more details see Stuchlisky et al. (2013); Stuchlisky & Kolos (2016). Results of our analysis are shown in Figures 10-11 under the assumption \( \beta = 0 \) (slender tori) while we leave the case of thick torus with \( \beta_{\text{crit}} \neq 0 \) to future analysis. Particular attention is given to recognize the emergence of the twin HF QPOs with resonant frequency ratios \( v_L/v_L \approx 3 : 2, 4 : 3 \) and \( 5 : 4 \) or \( 2 : 1 \) and \( 3 : 1 \) assumed in BH systems. For the

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12 Note that this analysis is in fact independent by the fluid equation of state, nevertheless as noted in Török et al. (2016), when \( \beta > 0 \) (non zero thickness), there is the possibility that, in some cases, they also depend on the polytropic index.
Figure 3. Study of RAD βcrit in equation (12). Upper panels: left - βcrit as function of the torus center rcen and cusp rcusp, the limiting value βcrit = 1 is also shown. Upper right: curves βcrit = constant in the plane (ℓ, K) for rcen(l) in equation (A3) and rcusp = r(t, K) in equation (A1). This plot shows therefore sets of tori with equal βcrit. The curve βcrit = 1 distinguishes tori with equal βcrit > 1 from tori with βcrit < 1. Limiting ℓcrit (red curve) and also ℓcrit as functions of K are defined in equations (A7). ℓ and K are parameter values for torus thickness S = 1 and Sx = 1. Upper right panel: βcrit as function of the fluid specific angular momentum ℓ, where we made use of rcen(l) and rcusp(l) = r(x)(ℓ) of equation (A3). The discriminant value βcrit = 1 is also shown this occurs for the torus with fluid specific angular momentum ℓ = ℓcrit with cusp in ℓcrit and center in 1. The value of the fluid specific angular momentum for the cusped torus with geometrical thickness Sx = 1. Bottom left panel: red curve is the radius r = rcen (rcusp) as function of rcusp (rcen) defined in equation (A3). Radii rmo, rmb, and rmb are also shown. Curves are tori families in the plane (rcusp, rcen) with equal βcrit defined in the classes βcrit > 1 and βcrit < 1 with the discriminant curve βcrit = 1 (dark-yellow). Bottom right panel: βcrit as function of r, rc(1) (red curve) and rcen (blue curve). In βcrit of equation (13) we made use of r = rcen or rcusp respectively defined in equation (A3). The regions βcrit > 1 and βcrit < 1 are shown, the discriminant βcrit = 1 is shown with radii rmo. Role of rmo, rmb, and rmb is also clear.

The evaluation of the underlying circular orbit, we use in Figures 10, 11, the analysis of the inner edge and centers of the toroid in Appendix A. Figure 4 shows the radial profile of the radial and vertical epicyclic frequencies νr and νθ = νK(r) of Eqs 11 in different tori models.

One of the first QPO models is the so called “relativistic-precession model” (RP model or standard relativistic precession (RP) model (in our application it is coincident also with total precession models TP). This identifies the twin-peak kHz QPO frequencies νT (upper) and νL (lower) with the two fundamental frequencies of a nearly circular geodesic motion: the Keplerian orbital frequency and the periastron-precession frequency, νT = νK and νL = νR = νK − νr. This model is investigated in Figure 5 the radial profiles of the frequencies are shown in Figs 6, 7. Further HF QPOs model, RE model i.e. simple resonance epicyclic model, features resonances between epicyclic oscillation modes of the orbiting fluids. Among the most favored is the so-called “3:2 epicyclic resonance model”, identifying the resonant eigen-frequencies with frequencies (νK, νr) of radial and vertical epicyclic axisymmetric disk modes, with νT = νK and νL = νr, particularly of the ratio νT/νL = 3/2. This model is considered in Figure 6, while Figure 7 shows the radial profiles of the νL and νT frequencies. We also consider the following alternatives tidal distortion (TD) model where νL = νK and νR = (νK + νr), considered in Figure 2 and Figure 8 with radial profile of the frequencies given in Figure 6 and warped disk (WD) model featuring a combination of the orbital and epicyclic frequencies where νL = 2(νK − νr), and νR = (2νK − νr). This model is studied in Figures 11 and Figure 7. We consider the frequencies in different models introduced in Figures 4 in the (a)-model as functions of r/M. In the (b)-model frequencies are evaluated in r = rcent(ℓ) in equation (A3) as function of the fluids angular momentum ℓ in [rmo, rmb]. In this way the frequencies will be parameterized by the specific moment of the fluid setting the orbits. In the (c)-model (νL, νR) are evaluated for r = rout(ℓ) in equation (A15) as functions of ℓ. We use the outer edge of the torus as a test according to the analysis on the marginally collisional sphere considered in Paglietse & Stuchlík (2019). In the (d)-model (νL, νR) are evaluated in r = rmax(ℓ) (the geometric maximum of the torus) of Eq. (A12) as function of ℓ. The (e)-model considers (νL, νR) for r = rX of Eq. (A14) as function of r/M.
generally to the limiting frequency values. Also in component, describing their stability, as associated with a given frequency radial profile, by showing the limiting values $r_cusp$, for any quantity $Q_*$, evaluated in $r_s \in \{r_{msso},r_{mbo},r_{out}\}$. For the frequency models considered here, $Q_*$ corresponds generally to the limiting frequency values.

Figure 4. Radial profiles of the radial and vertical epicyclic frequencies $\nu_r(r)$ (orange curve) and $\nu_\theta(r) = \nu_K(r)$ (darker cyan curve) of equations [1] and (RE) frequencies for simple resonance epicyclic models where $\nu_L = \nu_\theta$ and $\nu_L = \nu_\theta$, see Figures [6] for different tori models. Radii $r_{msso} = 6M$, $r_{mbo} = 4M$, $r_s = 3M$ and $r_{out} = 10.4721M$ as the fluid specific angular momentum $\ell_{msso}$ and $\ell_{mso}$ are shown. Upper left panel (a)-model ($\nu_\phi,\nu_r$) as functions of $r/M$. Second panel: (b)-model ($\nu_\phi,\nu_r$) for $r = r_{cent}(r)$ in equation [A3] as function of the fluids angular momentum $\ell \in [\ell_{msso},\ell_{mbo}]$. Third panel: (c)-model ($\nu_L,\nu_\nu$) for $r = r_{out}(r)$ in equation [A15] as function of $r$. Fourth panel: (d)-model ($\nu_L,\nu_U$) for $x_{max}(r)$ in equation [A12] as function of $r$. Fifth panel: (e)-model ($\nu_L,\nu_U$) for $r_{cent}(r)$ in equation [A12] as function of $r/m/M$. Sixth panel: (f)-model ($\nu_L,\nu_U$) for $r_{out}(r)$ in equation [A12] as function of $r/m/M$. Seventh panel: (g)-model ($\nu_L,\nu_U$) for $r = r_{cent}(r)$ in equation [A8] as function of $K$.

center. The (f)-model features ($\nu_L,\nu_U$) as function of $r_{out}(r)$ in Eq. [3] therefore depending on the cusp or center if considered in the center or cusp respectively. In the (g)-model, ($\nu_L,\nu_U$) are considered for $r = r_{cent}(K)$ in Eq. [A3] as function of $K$. The profiles of the two-peaks frequencies ($\nu_L,\nu_U$) and their ratios in the oscillation models (TD,RP,RE,WD) of Figures [10] [9] [8] [11] can be interpreted, according to toroidal models (a,b,c,d,e,f,g) of Figures [4] taking into account the radial dependence of the epicyclic geodesic frequencies ($\nu_r,\nu_\theta$) of Eq. [11] and the radial profile of leading functions $\ell_K(r)$ in Eq. [3] and energy function $K(r)$ in equation [1]. The radial dependence of $\ell_K(r)$ and $K(r)$ are clearly reflected in the frequency analysis. In fact, the increase of $\ell_K(r)$ corresponds in general to an increase of the radius $r > r_{msso}$ and parameter $K = K(r)$, corresponding also to an increase of the torus outer edge $r_{out}$ and the torus center $r_{cent}$ and in general the torus height $x_{max}$. Viceversa, the increase of the specific angular momentum $\ell_K(r)$ corresponds to a decrease of radius $r < r_{msso}$, therefore it corresponds to a decrease of the cusp $r_{cusp}$ location, which corresponds particularly to the toroid cusp $r_{cusp} = r_\times \in [r_{mbo},r_{msso}]$ in accreting models, present also in $r_\times$ definition of equations [14]–[11] see also Figures [8] In Figures [2] [10] [9] [8] we pointed out the topological status of each RAD component, describing their stability, as associated with a given frequency radial profile, by showing the limiting values $Q_*$, for any quantity $Q_*$, evaluated in $r_s \in \{r_{msso},r_{mbo},r_{out}\}$. For the frequency models considered here, $Q_*$ corresponds generally to the limiting frequency values.
Figure 5. Frequencies $\nu_U = \nu_K$ (upper) and $\nu_L = \nu_{\text{per}} = \nu_K - \nu_r$ (lower) for the RP, relativistic-precession, model, in the (a,b,c,d,e,f,g) tori models of Figures 4. The two frequencies in dashed (orange and cyan) curves are increased by a factor present in parentheses.

3.1 Comments on the RAD structures and the outcomes of the analysis

In our investigation we focus on a clustered set of misaligned tori framed in the RAD context firstly developed for the case of equatorial tori in Kerr spacetime (eRAD) in Pugliese & Stuchlik (2015). By considering the tori agglomerate as one orbiting object we use a global approach to the characterization of the structure singling out a leading function for the distribution of tori in the RAD rather than focusing on the details of the physics of each specific toroid components hence enhancing a macro-structure approach. We are thus mapping the Schwarzschild geometry in relation to complex structures of arbitrarily inclined perfect fluid tori. The mapping is reflected by a leading function connected to the distribution of the specific angular momentum of the orbiting fluid; its generalization to the perfect fluid combined with internal toroidal magnetic field is also introduced. The mapping can be useful in establishing the starting point in studies of a variety of dynamical situations, including collisional effects, jets, etc. Here we have considered the case of small oscillations of slender toroidal configurations that can be related to HF QPOs. As a consequence of this analysis we provide a set of initial data for dynamical situation, and a procedure to select the initial configuration for dynamical simulations adapted also to more complex situations where toroidal components of the RAD have diversified nature. We provide constrains on morphology and stability on the RAD and first evaluation of multiple tori associate QPOs emission. We also discussed the possible correlated observational properties. We propose the RAD of clustered set of inclined tori as base for a different series of phenomena while we indicate possible observational outcomes related to different phases of activity of these structures. In fact, the RADs are therefore characterized by a typical ringed structure that could be evidenced primely for example in the X-ray emission spectrum and as an imprint of the discrete inner RAD composition, or in the combined oscillatory phenomena associated to the tori model observable for example by the X-ray observatory ATHENA.

In details we summarize below some key points of this analysis and the main results.

http://the-athena-x-ray-observatory.eu/
Figure 6. Frequencies $\nu_U = \nu_K$ (upper) and $\nu_L = (\nu_K + \nu_r)$ (lower) in the (TD) models for the (a,b,c,d,e,f,g) tori models of Figures 4. Frequencies are increased by a factor present in parentheses, in dashed (orange and cyan) curves.

**The clusters: an overview** In this article we provide the main characteristics of a set of accreting misaligned tori around the central BH. Within this analysis therefore we provided limitations on RAD existence and stability and an overview of the possible emissions associated with these structures as the QPOs. From methodological point of view we believe relevant the approach shift which characterizes the RAD frame, where the cluster is studied as one gravitating composite (macro-)structure orbiting around one central Schwarzschild attractor, hence the identification of a leading function for the tori distribution. This analysis also considers the misaligned tori collision emergence. Constrains on existence of such configurations are discussed in Sec. 2, particularly against tori collision and Paczynski instabilities (instability associated to installing of accretion phases, the cusped tori, or open cusped configurations, the so-called proto-jet) and developed more extensively in Pugliese & Stuchlík (2019). Many morphological properties of the torus are discussed in relation to the stability problem for the RAD and each of its gravitating components. We discuss as possible QPOs emerging from the RAD ringed structure in Sec. 3, proposing the RAD to be considering for entangled emission from each of its misaligned component. Part of our analysis was dedicated to the evaluation of the torus considered for its thickness which is crucially significant in many aspects of the accretion disk physics and phenomenology. In particular we focus on a more specific analysis of the role of geometrical thickness in relation to the disco-seismology effects for each toroid. In Sec. 3 we provided the conditions for which these can be considered geometrically thick---Figs 3. We have therefore characterized the model on the basis of the parameters determining the particular configurations according to their stability as related to the toroids morphology. The evaluation of the toroids geometrical thickness is indication of the predicted BH accretion rates correlation\footnote{Pugliese & Stuchlík (2019)}. The analysis identifies the sets of RAD inclined toroids having equal characteristics as the torus thickness. Therefore we provided the tori distribution in the RAD in Figs 3 by considering classes depending on the geometric thickness $S$ and an evaluation of the geometric thickness of the disks considered in the RAD frame establishing conditions under which disks are geometrically thick according to the model parameters.

The phenomenological outcomes: The RAD in any stage of its life, we believe can offer an interesting set of complex observational aftereffects. The RAD frame implies some relevant consequences from phenomenological view-point. We could
Limiting effects in misaligned tori clusters

Figure 7. Frequencies in the (WD) (warped disk) models where \( \nu_L = 2(\nu_K - \nu_r) \), and \( \nu_U = (2\nu_K - \nu_r) \) in the (a,b,c,d,e,f,g) tori models of Figures 4. Frequencies are represented increased by a factor present in parentheses, in dashed (orange and cyan) curves.

relate different phenomenological aspects to the presence of a RAD structure, we briefly discuss some of these here. (i.) Firstly we mention obscuration in the emission spectrum induced by an inner torus of the agglomerate, the analysis of this case requires also the understanding of main morphological characteristics of the torus closest to the central attractor. (ii.) RAD implies the possibility of an globulus depending on tori number thickness and inclination angle, the BH horizon in this case would be "covered" in a multipole embedding composed by different tori having different orientation rotation. (iii.) The RAD may be considered as a model to increasing accretion mass rates and to explain high masses for the SMBHs, due to multiple accreting tori emerging from the typical ringed structure, which also ensures the possibility of interrupted phases of accretion; (iv.) In Sec. (3) we investigated the possibility of QPOs emission enabled in complicated structure of the cluster described within different emission models and assumptions. (v.) The inner composite structure of RAD implies an inner activity which could end also in violent, catastrophic outburst with ejection of matter and possible destruction of the RAD and formation of a large SMBH. This situation implies the existence of periods characterized by different levels of activity for the BH and the RAD which would relate them ultimately also to the host characteristics. (vi.) The main interesting aspect of these clusters relies in their internal activity, particularly the internal exchanges of energy and matter between the tori as well as the tori and central BH. Tori accretion, in the case of a globular model, could end also into a relatively fast collapse of the entire structure into the central BH contributing therefore with a huge mass and spin and a great release of energy and matter. Instability in one point of the structure could initiate a sequence of associated, complicated RAD phenomenology. RAD would be recognizable by an articulated internal life triggered by each torus dynamics and empowered by its inner structure. Some of these phenomena are for example the tori collisions, internal jet emission and accretions, tori oscillation modes, eventually related to QPOs observed in non-thermal
X-ray emission from compact objects. Concerning the possible correlation with QPOs emission, which is considered Sec. (3) the oscillations of each component are added to others and are pulsations of the RAD, and possibly the globule creating eventually a rather distinct detectable emission spectra.

The model setup: In this work we propose also an adapted model setup. The novelty of our approach, resides on the methodological view point, proposing a conceptual global setup that constitutes the RAD frame pursuing the existence of a leading function to represent and constraint the tori distribution around the central attractor. We then identified also an energy function \( K(\ell) \) regulating the RAD stability (cusp emergence) and defining relevant quantities as the mass accretion rate and cusp luminosity. Our analysis places constraints on the existence and stability of misaligned tori which can be used in dynamical (time-dependent, evolutive) analysis of a similar system with these initial configurations. This analysis can be compared with similar studies in Martin et al. (2014); Nixon et al. (2012a,b); Nealon et al. (2015); Lodato & Pringle (2006); Scheuerl & Feiler (1996). The current literature considers similar objects within a numerical approach, fixing very specific initial data and physical setup for each torus. Our approach, with respect to other studies has the advantage to be an exact analysis of different morphological characteristics and emergence of tori collision conditions, in a non dynamical frame. The key element consists in the fact that we privileged a global approach focusing on the tori distribution around the RAD (where the general relativists effects are relevant) rather than on the analysis the details of physics and evolutions of each specific toroidal components, which would narrow the analysis to the selected system. Results therefore can be interpreted as initial configurations for dynamic simulations, focusing on the global issue the RAD structure, as the tori location in the cluster, the location of the maximum and minimum pressure points and other morphological characteristics, hence the choice \( \ell = \text{constant} \) for each torus. Therefore the torus parametrization with the \( \ell \) value is a very convenient choice for the HD RAD macrostructure scenario. In fact we provided the classes toroidal components and therefore RAD which can be
Limiting effects in misaligned tori clusters

Figure 9. (RE)-models (simple resonance epicyclic models). Plot of $v_U/v_L$ (dashed line) in the (a,b,c,d,e,f,g) models of Figures 4. There is $v_U = v_\theta$ and $v_L = v_r$. The two frequencies are also represented increased by a factor (nx) present in parentheses (where n is generally $\{5,10,30\}$), in dashed (orange and cyan) curves. Resonant frequency ratios $R_1 = 2:1$, $R_2 = 3:1$, $R_3 = 3:2$, $R_4 = 4:3$, $R_5 = 5:4$ (black lines) are also shown. See Figures 4 for the radial frequencies profiles.

Globuli as limiting cases and embedded BHs: Conceiving the RAD as a whole macro-structure leads also to focus on interesting limit situations, both limiting from the point of view of the conditions on the configurations, for example the thickness, and regarding conditions imposed on the activity that can be inferred from the very imposed constraints. In this sense a particular interesting limiting case of the RAD proposed in this work consists in the possibility of an embedded BH or a globulus. The formation of these objects would refer to periods of low activity, (cold-globuli), giving rise, when activated, to catastrophic outbursts distinguished by a huge release of energy and matter. The end of this process would ultimately go into a different SMBHs or with the formation of a different RAD configuration. This is a limiting case where the BH horizon is expected to be “covered” to an observer at infinity–Figs 1. Figure 2–right depicts this situation the emergence of luminous anuli and the complexity of the RAD case. However, a key aspect to focus on the observation of this situation remains the stability of the set of accreting RAD tori especially in case of attractor with spin. Primarily in this case it is necessary to evaluate the maximum torus distance from the central attractor considering the dependence from the model parameter of the outer edge of the torus, the geometrical thickness considered in Sec. 3 and the density of tori in the RAD here discussed in Sec. 2.

On the other hand, although we focus on a non-dynamical structure, we note that we could follow the evolution of a torus considering as a sequences of tori at differences stages having different values of the defining model parameters for example choosing the specific angular momentum as evolutive parameter \cite{Pugliese&Montani2015}.

\footnote{On the other hand, although we focus on a non-dynamical structure, we note that we could follow the evolution of a torus considering as a sequences of tori at differences stages having different values of the defining model parameters for example choosing the specific angular momentum as evolutive parameter \cite{Pugliese&Montani2015}.}
Morphology: A large part of our analysis concerns the investigation of the geometry of RAD of orbiting misaligned tori, which is in fact related to the emergence of unstable phases of RAD, such as topology collision and accretion or open cusped configurations which are variously related to proto-jet emission. Even in the simplest case of static background, the RAD toroidal components are characterized by boundary conditions dependent on the distance, in the clusters, from the central attractor. These systems have therefore several restrictions on the possibility of formation, their evolutions and related observational characteristics. The occurrence of accretion and collision are here regulated by the model parameters which in turn determines the disk morphology. We provide the conditions determining these cases. We mention the tori distance from the central attractor, here considered in Appendix A, especially the characteristics of the outer and the inner toroids. The analysis of outer edge of the outer tori of the clusters, which is an aspect deepened in Pugliese & Stuchlik (2019), results in constrains on the inner structure of the RAD and therefore the tori distribution in the clusters, which is relevant also for the formation, evolution and stability of the RAD. A further significant quantity for the RAD systems affecting both stability, including the BH tori energetics as the accretion rates or the cusp luminosity, observation and oscillation, for example in the evaluation of the effects of disc-seismology, is the torus and the RAD geometrical thickness. Two tasks of the RAD investigation was therefore to establish conditions of geometrical thickness by considering the \((S, \beta)\) parameters of the model and the limiting value \(S = 1\) and to characterize the tori distribution in the RAD considering the characteristic of the geometric thickness—Figs 3. This analysis is shown in Figs. 3 where we analyze also the geometrical thickness parameter \(S\) of the tori especially in the range of parameter values adapted to the onset of accretion phases (from cusped toroidal configurations) as well as the \(\beta_{crit}\) thickness parameters of Eq. (12) which is used to establish the approach for the oscillation analysis. A comparing between the classes of tori and RAD with equal \(S\) or \(\beta_{crit}\) is therefore shown in Figs. 3. We can see that the farthest the torus of the cluster is from the...
Figure 11. (WD) (warped disk) models, plot of \( \nu_U/\nu_L \) (dashed line) where \( \nu_L = 2(\nu_K - \nu_T) \), and \( \nu_U = (2\nu_K - \nu_T) \). The two frequencies are also represented increased by a factor present in parentheses (nx) (where \( n \) is generally \( \{5, 10, 30\} \)), in dashed (orange and cyan) curves. Resonant frequency ratios \( R_1 = 2 : 1 \), \( R_2 = 3 : 1 \), \( R_3 = 3 : 2 \), \( R_4 = 4 : 3 \), \( R_5 = 5 : 4 \) (black lines) are also shown. Frequencies are in Figures 7. Models (a,b,c,d,e,f,g) of Figures 4 have been considered.

central BH attractor, in a limiting spherical region of Figures (2)–left and the largest can be its thicknesses. However \( \beta_{\text{crit}} \)-parameter becomes relevant for smaller values of \( \ell \) and \( K \) then the values of \((K,\ell)\) for geometrically thick tori according to \( S \), therefore for thinner tori (according to \( S \) or \( S_x \) specifically for the cusped configurations) and for tori close to the BH, in other words for tori whose parameter of geometrical thickness would be significantly smaller than the reference values for thick tori. Figs (3)–Upper right and Bottom-right ultimately show the region where the QPOs models featuring an adapted particle oscillation model can be applied distinguishing also the main parameters \((\ell, K)\) and the relative location of the critical pressure points in the tori and their geometrical thickness \((S)\). Figs. (4)–Bottom-left and upper–center show the classes of tori having similar conditions, in terms of \( \beta_{\text{crit}} \)-parameter, for the QPOs emergence analysis, considering the torus cusp \( r_K \), torus center (maximum density point) and the parameters \( \ell \) and \( K \), essential for the identification of the toroidal components. A comparative analysis of these features would start from the assessment of one or more of the following elements: the \( \beta_{\text{crit}} \) (through QPOs detection) or \( K \) and \( \ell \) by means of one morphological characteristic reported in Sec. (A) and from the torus state, accreting or quiescent (closed, not cusped tori ). A more comprehensive view of \( \beta_{\text{crit}} \) in terms of cusp and center location is given by the three dimensional plot of the Figs. (3)–upper-center. Figures (4,5,6,7,8,9,10,11) show the frequencies, relevant in our approximation, for a set of the QPOs emission models, to be composed for each torus of the agglomerated, confronted with the observational data and compared with the data of different morphological properties of the torus, as the inner edge or center of maximum pressure, which can be translated into the model parameters according to the analysis of Sec. (2) and Appendix (A). Here we are interested particularly in the identification of the trend of the frequencies with the location of cusps orbits, clearly defining, with location of the torus inner edge (or the center), the torus and consequently the RAD inner structure and recognizing the possible status of the torus by considering for example Eq. (13) and Eq. (14) or, more explicitly, from the quantities listed in Sec. (A). Our results ultimately would serve as a guideline for possible observational
identification of a RAD. This morphological aspect is in fact a key element for the observation and recognition for the RAD and the establishments of its stability conditions. We investigate the RAD geometry and morphology considering the parameters \((\ell, K)\) (2n parameters for a RAD tori) introduced in Sec. (2), points on the curves \(\ell(r)\) and \(K(r)\). One or more of these characteristics may correspond to an entire class of objects, determining a class of components rather than one specific RAD. Therefore we considered a number of morphological characteristics, here listed in Appendix (A) and thoroughly considered in Pugliese & Stuchlík (2019), correlating in a comparative analysis the different RAD aspects, narrowing these classes through the combinations of the analysis on the other quantities.

**Stability:** A further outcome of this analysis is the characterization of the stability of these structures which however is firstly crucially related to the tori distance of the central BH and the eventual spin of central attractor.\(^{15}\) However, in the stationary (steady) frame developed in this work where the spacetime is spherical symmetric, RAD analysis of the instability emergence corresponds in the establishment of the occurrence of tori collisions, and conditions for tori accretion into BH, or a combinations of these, constraining the torus dimensions and the specific fluid angular momentum.\(^{16}\) Sec. (2)–Appendix (A)–Pugliese & Stuchlík (2019).

### 4 CONCLUDING REMARKS

The investigation of stationary axisymmetric toroids orbiting a central SMBH is a timely issue especially for the physics of super-Eddington accretion onto very compact objects, being related to several high energy phenomenological environments as Active Galactic Nuclei, X-ray sources and Gamma-Ray Bursts. In general, accretion tori are related to a huge variety of physical phenomena, and the analysis of these objects is important also for the identification of the central attractor features, being the disks directly involved, especially in the case of misaligned or tilted disks, in SMBHs dynamics (as for example the spin/mass evolution). In this view, however, several notable aspects of the disks structure and morphology are still unclear, for example the location of the innermost boundary of the disk (inner edge of accreting disk), or the disks dynamics as the accretion process and the QPOs associated to these structures. The existence and picture of a complete theoretical interpretation of the associated phenomenology in an unique satisfactory framework (for example covering both jets emission and accretion) remains still to be proved. In this work we considered a model of aggregated misaligned (inclined) tori orbiting one central Schwarzschild attractor, using the approach developed in Pugliese & Stuchlík (2015, 2016, 2017a) for the eRAD. Such orbiting aggregates were first considered in Pugliese & Stuchlík (2019) where constraints on the existence and stability of misaligned tori were discussed, also as possible initial data for dynamical (time-dependent) analysis of related systems. Then geometry of RAD accreting tori, stability and collision emergence were also constrained. Special sets of RAD misaligned toroids were identified having equal values of one or more of model characteristics \(P \equiv (r_{\text{out}}, r_{\text{in}}, r_{\text{cent}}, \lambda, S)\), where \((r_{\text{out}}, r_{\text{in}})\) are the outer and inner edges of the orbiting configurations, \(r_{\text{cent}}\) is the center of the toroid (point of maximum density and HD pressure), \(\lambda\) is the torus elongations of its symmetric plane, \(S\) is the torus geometrical thickness. Configurations of these classes might correspond to similar observational effects depending on the P-characteristics. We used these results in the present analysis. Particularly the evaluation of the toroids geometrical thickness \(S\) has an essential role in establishing the effects of disk-seismology discussed in Sec. (3). We analyzed the geometric thickness \(S\) discussing conditions under which these disks can be considered geometrical thick according to the model parameters, using the limiting value \(S \approx 1\). From the observational viewpoint, the special and distinctive ringed discrete structure of these aggregates could be evidenced in the X-ray emission spectrum and as a track of the inner RAD composition. In Sec. (3) we explored this possibility studying the expected epicyclic frequencies in the case of ringed structures with misaligned tori, for a first approximate description of the twin peak quasi-periodic oscillations (QPOs) in the context of the RAD tori structure. Particularly we showed the possibility that the twin peak quasi-periodic oscillations could reflect RAD inner structure, particularly with respect to the inner edges \(r_{\text{in}}\) of the cusped tori. However, this analysis is a first comparative investigation on the problem of the QPOs interpretation in the RAD frame, which focuses on geodesics oscillation models with the analysis of the radial profiles and assuming specific oscillation models of the tori. The geodesic frequencies are governed by the background geometry and additionally determined by the constraints imposed on the RAD. Nevertheless, the disko-seismology effect, for each toroid is in many aspects dependent on its geometrical features and particularly its geometrical thickness. Therefore, part of our

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15 In this case obviously the system instability would be dependent on the torus inclination angle and fluid rotation as well as the eventual presence of magnetic field. In this scenario the Lense-Thirring effect from the central spinning attractor, could induce even a torus break, with the consequent formation of two equatorial disks, and the emerging, especially for viscous disks, of the so called Bardeen & Peterson effect. Bardeen & Peterson (1973), Nealon et al. (2015).

16 There is a further mechanism of instability for the RAD which originates from a typical instability of geometrically thick tori orbiting a central SMBH. The Runaway–runaway instability, introduced first in Pugliese & Stuchlík (2017a), consists in the combination of runaway instability, involving the inner edge of the inner accreting torus of the RAD with the consequent destabilization of the aggregate. The accretion induces a change of the inner torus morphology, and the change of background geometry which has repercussions in the all RAD structure establishing a sequence of events having different possible outcomes. Font & Daigne (2002a).
analysis was also dedicated to an evaluation of the impact of the disk geometry (specifically its thickness) in this investigation. In future work we plan to investigate the influence of the tori thickness on the modifications of the oscillatory frequencies related to HFQPOs in the RAD framework. Another possibility is the study of the role of the toroidal magnetic fields and the relation to oscillating string loops (Stuchlik & Koles 2012).

As a sideline of this study, we explored in Appendix [14] the case of RADs where some of their components are magnetized tori with toroidal magnetic field (Komissarov 2006, Pugliese & Montani 2013, Pugliese & Montani 2018). In this Appendix we also discuss the case when leading RAD function, defining the distribution of tori in the RAD with misaligned disks has changed to an alternative definition.

In Pugliese & Stuchlik (2019) we also pointed out the possibility that such RAD systems could form accreting globuli of matter “embedding” the central static BH in a set of orbiting RAD tori where the BH horizon would be partially or totally “covered” to an observer at infinity [17]. The presence of combined oscillatory modes from different orbiting tori could possibly be understood as the “pulsation” of such globules and therefore give track for their existence. Luminous anuli were traces of the presence of a complex structure for inner edges of orbiting accretion disks that could be traced (for example as intertwined luminous profiles) from the observations of Event Horizon Telescope (Akiyama et al. 2019a, b, c, d, e; Akiyama et al. 2019f). Further notes on these issues are also discussed in Appendix A. In Fig. 1 we have shown different significant view angles for orbiting RAD found from the integration of the Euler equations for the misaligned tori. These orbiting configurations should be recognizable by their peculiar ringed structure, therefore in Fig. A2 we report also schemes for different RAD view angles. Eventually they should be also recognizable from absorption due to the presence of an inert or active (covered) inner orbiting torus. Further optical effects associated to geometrically thick tori that can be considered as the components of these orbiting structures are discussed for example in Karas & Sochora (2010); Sochora et al. (2011); Schee & Stuchlik (2009, 2013).

For the accreting globuli model, the issue of the stability of these static BHs which would be immersed in the set of misaligned thick tori becomes particularly significant. In this context the influence of the tori self-gravity should be also discussed [18]. In future works, we expect to use the constraints given in Pugliese & Stuchlik (2019) and the analysis considered here, as initial data for more complex dynamical situations. The analysis presented here will be extended to the case of Kerr attractors, where a co-evolution of central BH with the disks can occur and influence of magnetic fields could be more dominant in several aspects of tori and BH energetics.

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APPENDIX A: REVIEW OF MISALIGNED (ACCRETING) TORI MORPHOLOGY

In this Appendix we present explicitly some morphological characteristics of the misaligned (accreting) tori, used in Sec. 3, especially with regard to the geometric thickness and the toroids unstable phases (relative to the cusped closed configurations). The evaluation of the geometrical thickness for instance has an essential role in the evaluation of the effects of disk-seismology discussed in Sec. 3. Details on these quantities and their derivation are presented in Pugliese & Stuchlik (2019). The RAD geometry and morphology can be parameterized in terms of $(\ell, K)$, and only one of these parameters in the case of cusped tori. We consider especially the torus elongation $\lambda(\ell, K)$, the location of inner and outer edge $(r_{in}, r_{out})$, the location of the torus center $r_{cent}$ (point of maximum density and hydrostatic pressure in HD model), the location of the geometric maximum $r_{max}$ or $y_{max}$ of the RAD tori, and the torus thickness $S = 2h_{max}/\lambda$, where $h_{max}$ or $y_{max}$ is the torus height. In the particular case of cusped disk where these quantities depend on one parameter $(r_{crit}$ or $K_{crit}$ or $\ell_{crit})$ only. (In the following with the notation crit we indicate quantities calculated at the critical points of the torus potential, i.e., either the torus centers or the cusps.)

List of main toroids features

17 Note that in Kovar et al. (2011, 2016), Siani et al. (2013), Cremaschini et al. (2013), Kovar et al. (2014), Trova et al. (2018); Schroven et al. (2018) were considered electrically, off-equatorial, charged configurations, i.e. levitating tori or clouds, existing in addition to extended toroidal structures crossing the equatorial plane of the central massive attractor. In some circumstances these may be possibly interpreted as a sort of BH “horizon covering” (with respect to a very wide observational angle). In this sense similarly to the case of globules considered here, where the black hole could be “embedded”, for a period of its life, in such orbiting configuration having several maximum (and eventually minimum) points of pressure (and density). Such obscuring effects were also predicted by the case of inscribed Keplerian disk orbiting with different inclination angles (Pánis, Koles & Stuchlik 2019).

18 For example in Pugliese & Kroon (2016) the stability of Einstein-Maxwell perfect fluid configurations with a privileged radial direction (a self-gravitating spherically symmetric ideal plasma ball) has been constrained showing an important dependence on the sound velocity.
structures are associated with the quiescent disks with radius \( r \) of Figures A2–A4, example represented in Fig A2 in the case of cusped tori–(see equation 1 (with appropriate boundary conditions) typical of the geometrically thick disks considered here. These surfaces are for \( \lambda \approx 1 \) close to the horizon and coincident with the inner "Roche lobe" of cusped torus, which means that the distance

\[ r_{\text{out}} = \frac{2}{3} \left[ 1 + \frac{3}{2} \cos^{-1} \left( \frac{1}{6} \cos^{-1}(\alpha) \right) \right]. \]

It is possible to find the location of the outer radius of the innermost surface embracing the central BH with inner edge \( r_{\text{in}} \). This quantity \( \lambda \) is the elongation on each symmetry plane of a cusped or quiescent torus.

\[ \lambda = \frac{2r \cos \left( \frac{1}{6} \left[ 2 \cos^{-1}(\alpha) + \pi \right] \right)}{\sqrt{3}}. \]

where \( r \approx \sqrt{3} - \frac{Q}{K} + \frac{4}{3K^2} \), \( K \equiv \sqrt{1 - \frac{r}{M}} \), \( a \equiv \left[ 8 - 9Q(\kappa - 1)K(3K - 1) \right]/K^3\tau^3 \), \( Q \equiv \ell^2 \). The critical points of cusped tori:

The center of maximum density (and hydrostatic pressure):

\[ r_{\text{cent}}(\ell) \equiv \frac{1}{3} \left[ Q + 2L_\ell \cos \left( \frac{1}{3} \cos^{-1}(L_{\text{III}}) \right) \right], \]

where \( L_\ell = \sqrt{Q(2Q - 12)} \) and \( L_{\text{III}} = \frac{Q\left(\ell^4 - 18Q + 54\right)}{L_\ell^4} \).

The inner edge:

\[ r_{\text{in}}(\ell) \equiv \frac{1}{3} \left[ Q - 2L_\ell \cos \left( \frac{1}{3} \cos^{-1}(L_{\text{III}}) + \pi \right) \right]. \]

The critical \( K \)-parameter: At the center of maximum density (and hydrostatic pressure), and at the inner edge of accreting

\[ \lambda_{\text{in}} \equiv r_{\text{in}} - r_{\text{out}}, \]

close to the horizon and coincident with the inner "Roche lobe" of cusped torus, which means that the distance

\[ \lambda_{\text{in}} = \frac{2}{3} \left[ - \sin \left( \frac{1}{6} \left[ 2 \cos^{-1}(\alpha) + \pi \right] \right) \right]. \]

It is possible to find the location of the outer radius of the innermost surface embracing the central BH, a closed toroidal solution of equation 1 (with appropriate boundary conditions) typical of the geometrically thick disks considered here. These surfaces are for example represented in Fig A2 in the case of cusped tori (A¹, A²)-models. We have removed these structures in the case of quiescent tori of Figures A3–A5 (B, C)-models. They generally appear in three different classes of solutions whose role still needs to be clarified. Doubled structures are associated with the quiescent disks with radius \( r_{\text{in}}^B(K, \ell) \):

\[ \lambda_{\text{in}} \equiv r_{\text{in}} - r_{\text{in}}^B = \frac{2}{3} \left[ \sin \left( \frac{1}{6} \left[ 2 \cos^{-1}(\alpha) + \pi \right] \right) \right]. \]

It is possible to find the location of the outer radius of the innermost surface embracing the central BH, a closed toroidal solution of equation 1 (with appropriate boundary conditions) typical of the geometrically thick disks considered here. These surfaces are for example represented in Fig A2 in the case of cusped tori (A¹, A²)-models. We have removed these structures in the case of quiescent tori of Figures A3–A5 (B, C)-models. They generally appear in three different classes of solutions whose role still needs to be clarified. Doubled structures are associated with the quiescent disks with radius \( r_{\text{in}}^B(K, \ell) \):

\[ \lambda_{\text{in}} \equiv r_{\text{in}} - r_{\text{in}}^B = \frac{2}{3} \left[ \sin \left( \frac{1}{6} \left[ 2 \cos^{-1}(\alpha) + \pi \right] \right) \right]. \]
torus there is
\[
K_{\text{cen}}(\ell) \equiv \sqrt{\frac{Q + 2L_\ell \cos \left( \frac{1}{3} \cos^{-1}(L_{11}) \right) - 6 \left[ Q + 2L_\ell \cos \left( \frac{1}{3} \cos^{-1}(L_{11}) \right) \right]^2}{3Q \left( 3\ell^4 + 2(2Q - 15) L_\ell \cos \left( \frac{1}{3} \cos^{-1}(L_{11}) \right) - 39 \ell^2 \cos \left( \frac{2}{3} \cos^{-1}(L_{11}) \right) + 54 \right)},
\]
(A5)
\[
K_x(\ell) \equiv \sqrt{\frac{Q - 2L_\ell \sin \left( \frac{1}{3} \sin^{-1}(L_{11}) \right) - 6 \left[ Q - 2L_\ell \sin \left( \frac{1}{3} \sin^{-1}(L_{11}) \right) \right]^2}{3Q \left( 3\ell^4 + 2(15 - 2Q) L_\ell \sin \left( \frac{1}{3} \sin^{-1}(L_{11}) \right) - 39 \ell^2 \sin \left( \frac{2}{3} \sin^{-1}(L_{11}) \right) + 54 \right)}.
\]
(A6)

These expressions allow to explicit the distribution of $K$-parameters in the RAD in terms of the leading function $\ell(r)$ in equation 3 correlating $K$ and $\ell$ in the misaligned tori.

**The critical $\ell(K)$ and $r_K$ dependence**

Eliminating also the radial dependence there is
\[
\ell^\text{crit}(K) = \sqrt{\frac{-27K^4 + K (9K^2 - 8) \frac{3}{2} + 36K^2 - 8}{2K^2 (K^2 - 1)}}, \quad \ell^\text{crit}(K) = \sqrt{\frac{27K^4 + K (9K^2 - 8) \frac{3}{2} - 36K^2 + 8}{2K^2 (K^2 - 1)}},
\]
(A7)
\[
\ell^\text{crit}(K) > \ell^\text{crit}(K) > \ell_\text{msco}, \quad \ell^\text{crit}(K) \in [\ell_\text{msco}, \ell],
\]

note $K_\text{msco} = 8/9$. The function $\ell^\text{crit}(K)$, for $K = K_{\text{max}}$, provides the specific angular momentum of the torus whose instability accreting phase is associated to the occurrence of the value $K = K_{\text{max}}$, while $\ell^\text{crit}(K)$, for $K = K_{\text{cen}}$, provides the specific angular momentum of the torus whose center of maximum density corresponds to $K = K_{\text{cen}}$. For a quiescent torus $K \in [K_{\text{min}}, K_{\text{max}}]$ is a free parameter, where $K_{\text{msco}} < K_{\text{min}} < K < R$, and $R = K_{\text{max}}$ for torus in the accreting range of specific angular momentum values or $R = 1$ otherwise. The couple $(K_{\text{min}}, K_{\text{max}})$ is defined by the $K(r)$ RAD energy function of equation 3 at maximum $r_{\text{max}}$ or minimum $r_{\text{min}}$ of the hydrostatic pressure. Finally, considering $\ell(r) = \ell^\text{crit}(K)$ and $\ell(r) = \ell^\text{crit}(K)$, we find an expression for the critical radii $r^\text{crit}(K)$ of the torus as a function of $K^\text{crit}$ as
\[
r^\text{crit}(K) = \frac{8}{K \left( \sqrt{9K^2 - 8} + 3K \right)} - 4, \quad r^\text{crit}(K) = \frac{8}{K \left( \sqrt{9K^2 - 8} - 3K \right) + 4}.
\]
(A8)

There is $r^\text{crit}(K_x) = r^{\text{inner}}_\text{max}$ (inner edge for accreting torus), and $r^\text{crit}(K_{\text{cen}}) = r^{\text{cent}}_\text{max}$ (center of cusped configurations).

**The tori geometric thickness**

Geometric maximum radius $r^\text{max}(K, \ell)$ of the torus, and the innermost surface $r^\text{max}(K, \ell)$, and the maximum height $h^\text{max}(K, \ell)$ of the torus surface, given as functions of $K$ and $\ell$, where $r^\text{max}$ is the location of maximum point for the inner Roche lobe close to the central BH, $h^\text{max}$ is the semi height of the torus:
\[
r^\text{max}(K, \ell) \equiv \sqrt{\frac{K^2 Q}{K^2 - 1} + 4 \sqrt{\frac{1}{3} \psi \cos \left( \frac{1}{3} \cos^{-1}(\psi_\ell) \right)}} \quad \text{and} \quad r^\text{max}(K, \ell) \equiv \sqrt{\frac{K^2 Q}{K^2 - 1} - 4 \sqrt{\frac{1}{3} \psi \sin \left( \frac{1}{3} \sin^{-1}(\psi_\ell) \right)}}.
\]
(A9)
\[
h^\text{max}(K, \ell) \equiv \sqrt{\frac{K^2 Q}{K^2 - 1} + 4 \sqrt{\frac{1}{3} \psi \cos \left( \frac{1}{3} \cos^{-1}(\psi_\ell) \right) + 4 \sqrt{\frac{2}{3} (K^2 - 1) \psi^2}} - 4 \sqrt{\frac{2}{3} \psi \cos \left( \frac{1}{3} \cos^{-1}(\psi_\ell) \right)}}.
\]
(A10)
\[
\psi \equiv -\frac{K^2 Q}{(K^2 - 1)^{3/2}}, \quad \text{and} \quad \psi \equiv \frac{3}{4} \sqrt{\frac{2}{3} (K^2 - 1)^2} \psi.
\]
(A11)

RADs with cusped misaligned configurations constitute a particularly significant case. Therefore we evaluated also the torus height $h^\text{max}(r_\psi)$ and the locations $r^\text{max}(r_{\text{crit}})$ and $r^\text{max}(r_{\text{crit}})$ as functions of cusp location $r = r_\psi \in [r_{\text{msco}}, r_{\text{msco}}]$:
\[
r^\text{max}(r_{\text{crit}}) = \sqrt{4 \frac{\frac{1}{3} \sin^{-1} \left( \frac{3}{4} \frac{1}{2} \psi \frac{\psi_\ell}{\sqrt{2}} \right) + \frac{1}{r - r_r \psi_{\text{crit}}}^2}}
\]
(A12)
\[
r^\text{max}(r_{\text{crit}}) = \sqrt{\frac{r^2}{(r - r_r \psi_{\text{crit}}) - 4 \frac{1}{3} \psi \cos \left( \frac{1}{3} \cos^{-1} \left( \frac{3}{4} \frac{1}{2} \psi \frac{\psi_\ell}{\sqrt{2}} \right) + \pi \right)},
\]
and the torus height
\[
h^\text{max}(r_\psi) = \left( -2 \sqrt{6} \frac{(r_\psi - r_r)(r_\psi - r_s)^2 r_s^2}{(r_\psi - r_{\text{msco}})^2} \sec \left( \frac{1}{3} \cos^{-1}(\psi_\ell) \right) + \frac{9 r_\psi^2 - 2 r_s^2 \sec^2 \left( \frac{1}{3} \cos^{-1}(\psi_\ell) \right)}{8 (r_\psi - r_{\text{msco}})(r_\psi - r_r)} \right)^{1/2}.
\]
(A13)
where

\[ \psi_a \equiv \frac{r_{mbo} - r}{(r - r_g)\ell} \]

\[ \psi_I \equiv \sqrt{\frac{(r - r_+)^2}{(r - r_-)^2}} \psi_a \]

\[ \psi_p \equiv -\frac{3\sqrt{2}(r_X - r_{mbo})}{4(r_X - r_g)^2 r_X^2}, \tag{A14} \]

The outer and inner edges of an accreting torus as function of \( r_X \) are

\[ r_{\text{out}}(r_X) = \frac{2}{3} \left[ \frac{(r_X - r_{mbo})^2 r_X^2}{(r_X - r_{mbo})^2} \right] \]

\[ \cos \left[ \frac{1}{3} \cos^{-1} \left( \frac{r_X - r_{mbo}}{r_X - r_{mbo} + r} \right) + \frac{r_X - r_{mbo}}{r_X - r_{mbo} + r} \right] \]

\[ r_{\text{inner}}(r_X) = \frac{1}{3} \left[ \frac{r_X^3}{(r_X - r_+)^2} - 2 \sqrt{\frac{r_X^2}{(r_X - r_+)^2} - 12} \right] \]

\[ \cos \left[ \frac{1}{3} \cos^{-1} \left( \frac{r_X^3}{(r_X - r_+)^2} - \frac{18r_X^2}{(r_X - r_+)^2} + 54 \right) \right] + \pi \right] \tag{A16} \]

from which we can obtain the critical elongation \( \lambda_c \), and thickness \( S_X = 2h_X/(\lambda_X) \), of the cusped tori where \( (r_{\text{inner}}(r_X), r_{\text{out}}(r_X)) \) combine together solutions \( r = r_X \) and \( r = r_{\text{out}} \). Disk geometric thickness underlies different aspects of the physics of the accretion disks, including the disks oscillation modes and accretion rates. These conditions set also the asymptotic limits, and conditions on the quasi-sphericity of the torus discussed in Pugliese & Stuchlík (2019). The point of maximum density and (hydrostatic) pressure in the torus is not coincident with the geometric maximum of the torus surface, i.e., in general there is \( r_{\text{max}} \neq r_{\text{cent}} \). Nevertheless, there is a special class of toroids where \( r_{\text{cent}} = r_{\text{max}} \).

The torus thickness is \( S = 1 \) for \( K_I = 0.975 \) and \( l_I = 3.887 \epsilon/\ell_{mso}, \ell_{mbo} \). Schemes of different angle views of an orbiting RAD are in Figures [A2]. It is clear that for accreting configurations the farther from the central attractor the accreting torus the larger it is. For low magnitude of the specific angular momentum the thickness is essentially determined by the \( K \) parameter (i.e. fluid density), particularly for the accreting tori, viceversa, the main governing parameter for quiescent tori at \( \ell > \ell_{mbo} \) is the specific angular momentum.

Figure A2. Schematic representation of RAD of order (number of orbiting disk components) \( n = 5 \) all orthogonal, quiescent and non-interacting orbiting around a central attractor. The scheme represents different viewing angles where the central black hole (outer horizon) is embedded into the orbiting globule. The eRADs (on the order \( n = 2 \)) embedded in the RAD are differently colored, the coplanar ringed structure is evident and should also be recognized also in the observations—see Figures [1].
APPENDIX B: AN ALTERNATIVE RAD LEADING FUNCTION

In this section we show a possible alternative choice for the RAD leading function $\ell(r)$ in equation (3) (which was coincident with the RAD rotational law), considering a distribution function of alternative parameters that fix the toroids in the RAD, for the eRAD case this approach has been fully considered in Pugliese & Montani (2018) – see also Agol & Krolik (2000), Bugli et al. (2018), Das et al. (2017), Del Zanna et al. (2007), Grassi & Rubinstein (2001), Montero et al. (2007), Parker (1955), Porth et al. (2017), Safarzadeh (2018), Karas et al. (2014), Mach, et al. (2019), Gimeno-Soler, et al. (2019), Mach, Piróg & Font (2018). In order to do that we consider a properly adapted RAD model with toroids having toroidal magnetic field. The tori considered in this model are regulated, according to equations (1), by the balance of the gravitational and centrifugal effects with $p$ being the hydrostatic pressure. In Pugliese & Stuchlík (2019) we discussed some characteristics related to the tori and RAD energetics dependent on the polytropic index $\gamma$ different for each toroidal component. RAD components can be formed however in different periods of the BH life, having consequently different matter compositions. (On the other hand, the constraints on the inner edge variation range, as discussed in Pugliese & Stuchlík (2017a), guarantee the viability of this analysis for large part of models where the curvature effects of the background are significant). Here we consider the RADs where some of their components are magnetized tori with toroidal magnetic field introduced in Komissarov (2006) and also discussed in Abramowicz & Fragile (2013), Pugliese & Montani (2013), Adamek & Stuchlík (2013), Hamersky & Karas (2013), Montero et al. (2007), Fragile & Sadowski (2017), Gimeno-Soler & Font (2017), Zanotti & Pugliese (2015), using the approach considered in Pugliese & Montani (2018), Pugliese & Montani (2013) where the magnetic pressure is treated as a deformation of the potential term in the force balance equation. For an extensive discussion and comparison on different solutions of the Euler equation for this problem see for example Adamek & Stuchlík (2013), Hamersky & Karas (2013), (it should be noted that, as discussed in these references, the total pressure extremes are unchanged with respect to the hydrodynamic case). We use the analysis presented in Pugliese & Montani (2015), adapted to our case of a static attractor as done in Pugliese & Montani (2013). For such tori the entropy per particle is constant on the flow lines of each torus. The magnetic pressure is $p_B = \mu_0 q M \omega^{q-1} / (q - 1)$, where $(M, \omega, q)$ are constants related to the magnetic field and the enthalpy at the center of the configurations. Here we do not want to focus our investigation on the discussion of the properties of the magnetic field definition and role in the magnetized accretion disks models, but rather use this case to verify the situation related to the RAD function has changed to an alternative definition, and a modification of the force balance equation has to be considered at least for a RAD component. Thus our purpose is to derive the new leading function, and show how tori distribution varies according to the new parametrization, testing the validity of the study of the hydrodynamic case. As such we adopt here approach of Pugliese & Montani (2013) allowing to explicit these considerations. The toroidal magnetic field $B^\phi$ component is:

$$B^\phi = \sqrt{\frac{2p_B}{\ell^2 q \mu_0 + \mu_0 \omega}} = \sqrt{\frac{r M [(r - 2) r]^q - 1 \omega q}{r^3 - (r - 2) q^2}}. \quad (B1)$$

We introduce the following parameter $F \equiv q M \omega^{q-1} / (q - 1)$. Explicitly the deformed (modified) fluid-magnetic potential $V_B$ is

$$V_B = \frac{(r - 2) r^2 \sigma^2 F [r^2 - (2 - 2) r] \omega q}{r^3 - (r - 2) q^2} \quad (B2)$$

$$(\sigma \equiv \sin^2 \theta$$, where the deformed fluid magnetic-specific momentum $\ell_b$ is

$$\ell_b = \frac{2(q - 1) \omega (r^2 - 2) - 2}{2(q - 1) \omega (r^2 - 2) + (r - 2)^2} \quad (B3)$$

The radial profile of this distribution of fluid specific momentum is shown Figure (3) for different values of the model parameters, and as also noted in Pugliese & Montani (2018) it is clear that this distribution is strongly dependent on the magnetic parameters $q$ and $F$. It is more convenient to introduce the alternative critical parameter $F_{crit}$

$$F_{crit} \equiv \frac{q - 1 + \ell^2 - 2}{2(q - 1) \omega (r^2 - 2)} \quad (B4)$$

for the toroid distribution in the RAD – Figures (B2-B3). On the other hand, $F = 0$ implies $\ell_b = \ell_b(r)$ in equation (3). The $F_{crit}$ has a maximum as function of $r/M$ with fixes the $F$ upper boundary for the toroidal solution of the Euler equation, therefore we can express the maximum as

$$\ell_{max} = \sqrt{\frac{2(q - 1) \omega (r^2 - 2) - 2}{2(q - 1) \omega (r^2 - 2) + (r - 2)^2}} \quad (B5)$$

where $s_i \equiv (r - 2)(r - 1) r^3 [q(r - 1) - 2 + 4] - 4 r + 8 \quad \ell_i \equiv (r - 2)^3 [2q(r - 1) - r^3 - (r - 2) - 2]$, and

$$\ell \equiv \sqrt{(r - 2)^2 \sigma^2 F [q^2 (r - 2) (r - 1)^2 - 2q(r - 3)(r - 2) \omega (r^2 - 2) - (r - 2) r^2 (r^2 - 2) + (r - 2) (r^2 - 2) (r^2 - 2) + 76 | - 90 + 36)}. \quad (B5)$$

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Figure B1. Fluid specific angular momentum distribution $\ell_b$ in equation \(B3\) for magnetized tori in the \(RAD\) with a toroidal magnetic field. Left panel: $\ell_b$ as function of $r/M$ for different values of $q$ and $\mathcal{F}$; center and right panel $\ell_b$ for two values of $\mathcal{F}$.

Figure B2. Magnetized tori. Function $\mathcal{F}_{\text{crit}}$ defined in equation \(B4\) for fixed values of $q$ parameters. Note the different distributions in relation with the location of the marginally stable circular orbit $r_{\text{mso}}$, marginally bounded orbit $r_{\text{mbo}}$ and photon orbit $r_\gamma$. It is also clear that $q < 1$ represents a singular region in the space of the $q$ parameter, as discussed in Pugliese & Montani (2018), in the case of central Kerr attractor.

Analysis of these cases in show in Figures B1, B2, B3.

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Figure B3. First panel line: curves $\ell_{\text{max}} = \text{constant}$ defined in equation B5 as maximum of the $F_{\text{crit}}$ in the plane $(r/M, q)$. It is clear the difference between the regions $q > 1$ and $q < 1$. Second panel line: curves $F_{\text{max}} = F_{\text{crit}}(\ell_{\text{max}}) = \text{constant}$ in the plane $(r/M, q)$. Third and fourth panel lines: Classes of tori with equal $F_{\text{crit}} = \text{constant}$ (see also Pugliese & Montani (2018)) defined in equation B4 in the plane $(\ell, r)$ for different values of the magnetic $q$ parameter. Particularly there are shown the regions $F_{\text{crit}} > 0$ and $F_{\text{crit}} < 0$, associated to the regions $q > 1$ and $q < 1$. Curve $\ell_{\text{max}}$ as functions of $r/M$ are also shown.

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