Isospin symmetry-breaking corrections for superallowed $\beta$ decay in relativistic RPA approaches

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Abstract. The isospin symmetry-breaking corrections $\delta_c$ obtained with the self-consistent relativistic RPA approaches are presented. It is shown that the proper treatment of the Coulomb mean field is very important for extracting these corrections. The nucleus-independent $Ft$ value, the $V_{ud}$ matrix element and the unitarity of the Cabibbo-Kobayashi-Maskawa matrix are discussed. The effects of neutron-proton mass difference on the isospin symmetry-breaking corrections are also investigated.

1. Introduction

The Cabibbo-Kobayashi-Maskawa (CKM) matrix [1, 2] relates the quark eigenstates of the weak interaction with the quark mass eigenstates. Its unitarity is one of the most important properties of the Standard Model of particle physics. Measuring its matrix elements independently and verifying its unitarity condition provide a rigorous test for the Standard Model. For the leading matrix element $V_{ud}$, there are four experimental methods to determine its value: nuclear $0^+ \rightarrow 0^+$ superallowed $\beta$ decays [3], neutron decay [4], pion $\beta$ decay [5] and nuclear mirror transitions [6]. In particular, the first method provides the most precise determination of the $|V_{ud}|$ value [7].

In order to determine the $|V_{ud}|$ value with these nuclear superallowed $\beta$ transitions, the radiative corrections $\Delta V_R$, $\delta'_{NS}$ and the isospin symmetry-breaking corrections $\delta_c$ for the experimental $ft$ values have to be taken into account [3], i.e.,

$$V_{ud}^2 = \frac{K}{2G_F^2(1 + \Delta V_R)Ft},$$

and

$$Ft = ft(1 + \delta'_{NS})(1 + \delta_{NS} - \delta_c),$$

where $G_F$ is the Fermi coupling constant for purely leptonic decays and $K$ is a physical constant related to electron mass [7]. The $f$ and $t$ factors represent the statistical rate function and partial half-life, respectively, and they are obtained through measurements of the $Q$ values, branching ratios, and half-lives of the superallowed decays. With the radiative and isospin symmetry-breaking corrections, the quantity $Ft$ should be nucleus independent.
Since the isospin symmetry is not an exact symmetry mainly due to the presence of the Coulomb forces in nuclei, the superallowed transition strength $|M_F|^2$ from initial state $|i\rangle$ to final state $|f\rangle$ slightly differs from its ideal value $|M_0|^2$ as

$$|M_F|^2 = |\langle f | T_\pm |i\rangle|^2 = |M_0|^2 (1 - \delta_c),$$

where $M_0 = \sqrt{2}$ for $T = 1$ states having exact isospin symmetry.

Shell model calculations are generally used to evaluate the isospin symmetry-breaking corrections $\delta_c$ [8]. Alternatively, self-consistent Random Phase Approximation (RPA) based on microscopic mean field theories is another reliable approach for evaluating the superallowed transition strength $M_F$. Such calculations have been performed for a few nuclei with the non-relativistic Skyrme Hartree-Fock approach in Ref. [9].

During the last decade, great efforts have been dedicated to developing the charge-exchange (Q)RPA within the relativistic framework [10–12]. In particular, a fully self-consistent charge-exchange RPA based on the relativistic Hartree-Fock (RHF) approach [13] was established recently [12]. A very satisfactory description of spin-isospin resonances was obtained without any readjustment of the energy functional. Therefore, it is appropriate now to re-investigate the isospin symmetry-breaking corrections $\delta_c$ for superallowed $\beta$ decay with the relativistic approaches.

In this paper, the calculations of the isospin symmetry-breaking corrections $\delta_c$ with the self-consistent relativistic RPA approaches will be presented. The nucleus-independent $Ft$ value, the $V_{ud}$ matrix element and the unitarity of the CKM matrix will then be discussed. The effects of neutron-proton mass difference on the isospin symmetry-breaking corrections will be also investigated.

2. Self-consistent Relativistic RPA

The basic ansatz of the relativistic Hartree (RH), also known as relativistic mean field (RMF), and relativistic Hartree-Fock (RHF) theories is a Lagrangian density $\mathcal{L}$, where nucleons are described as Dirac spinors that interact via the exchange of $\sigma$, $\omega$, $\rho$, $\pi$-mesons and the photon [14–16]. In order to give a satisfactory description of nuclear matter and finite nuclei, the nonlinear self-coupling of mesons [17, 18] or density-dependent meson-nucleon couplings [13, 19] are introduced.

In the Hartree and Hartree-Fock approximations, the total energy $E$ of the system is the trial ground state (Slater Determinant) expectation value of the effective Hamiltonian operator $\hat{H}$, which can be obtained with the general Legendre transformation from the Lagrangian density $\mathcal{L}$. This total energy contains the kinetic energy and the direct (Hartree) and exchange (Fock) potential energies. In the Hartree approximation, the Fock terms are neglected for simplicity, and the parameters of $\mathcal{L}$ are readjusted accordingly.

In the self-consistent RPA calculations, the particle-hole residual interaction $V_{\text{ph}}$ in the RPA equations is derived from the second derivative of the same energy functional as that used in the ground-state description. In general, there are additional rearrangement terms in the particle-hole residual interaction when the explicit density dependence of the meson-nucleon couplings are included [20]. However, since the rearrangement terms are due to the dependence on isoscalar ground-state densities, it is easy to see that they are absent in the charge-exchange channels, e.g., in the description of superallowed $\beta$ decays. The RPA equations can be written as

$$\left( \begin{array}{cc} A^J_{p\bar{p}'n'n} & B^J_{p\bar{p}'n'n} \\ -B^J_{\bar{p}p'n'n} & -A^J_{\bar{p}p'n'n} \end{array} \right) \left( \begin{array}{c} X^J_{p'n'n'} \\ Y^J_{p'n'n'} \end{array} \right) = \omega_{J'+J} \left( \begin{array}{c} X^J_{p'n'n'} \\ Y^J_{p'n'n'} \end{array} \right),$$

where $p$ and $\bar{p}$ ($n$ and $\bar{n}$) denote unoccupied and occupied proton (neutron) states. It should be emphasized that within the no-sea approximation the unoccupied states include not only
the states above the Fermi surface, but also the states in the Dirac sea. With the $X$ and $Y$ amplitudes, the transition probabilities $B_{\nu}^{\pm}$ between the $0^+$ parent ground-state and the $0^+$ excited states $\nu$ in the daughter nuclei ($N \pm 1, Z \mp 1$) can be obtained [21].

3. Results and Discussion

For all the calculations presented in this paper, the spherical symmetry is assumed and the filling approximation is applied to the last partially occupied orbital. The radial Dirac equations are solved in coordinate space by the Runge-Kutta method within a spherical box with a box radius $R = 15$ fm and a mesh size $dr = 0.1$ fm. The single-particle wave functions thus obtained are used to construct the RPA matrices $A$ and $B$ in Eq. (4) with the single-particle energy truncation $[-M, M + 120$ MeV], i.e., the occupied states are the positive energy states below the Fermi surface, whereas the unoccupied states can be either positive energy states above the Fermi surface or bound negative energy states [12]. With these numerical inputs, the model-independent sum rule,

$$\sum_{\nu} B_{\nu}^{-} - \sum_{\nu} B_{\nu}^{+} = N - Z,$$

where $B_{\nu}^{\pm}$ corresponds to the transition operator $T_{\pm}$, can be fulfilled up to $10^{-5}$ accuracy, and the isospin symmetry-breaking corrections $\delta_c$ are stable with respect to these numerical inputs at the same level of accuracy.

|          | PKO1  | PKO1* | DD-ME2 | T&H [8] |
|----------|-------|-------|--------|---------|
| $^{10}$C $\rightarrow$ $^{10}$B | 0.082 | 0.148 | 0.150  | 0.175(18) |
| $^{14}$O $\rightarrow$ $^{14}$N | 0.114 | 0.178 | 0.197  | 0.330(25) |
| $^{18}$Ne $\rightarrow$ $^{18}$F | 0.270 | 0.357 | 0.430  | 0.565(39) |
| $^{26}$Si $\rightarrow$ $^{26}$Al | 0.176 | 0.246 | 0.252  | 0.435(27) |
| $^{30}$S $\rightarrow$ $^{30}$P | 0.497 | 0.625 | 0.633  | 0.855(28) |
| $^{34}$Ar $\rightarrow$ $^{34}$Cl | 0.268 | 0.359 | 0.376  | 0.665(56) |
| $^{38}$Ca $\rightarrow$ $^{38}$K | 0.313 | 0.406 | 0.441  | 0.765(71) |
| $^{42}$Ti $\rightarrow$ $^{42}$Sc | 0.384 | 0.460 | 0.523  | 0.935(75) |
| $^{26}$Al $\rightarrow$ $^{26}$Mg | 0.139 | 0.193 | 0.198  | 0.310(18) |
| $^{34}$Cl $\rightarrow$ $^{34}$S | 0.234 | 0.298 | 0.307  | 0.650(46) |
| $^{38}$K $\rightarrow$ $^{38}$Ar | 0.278 | 0.344 | 0.371  | 0.655(59) |
| $^{42}$Sc $\rightarrow$ $^{42}$Ca | 0.333 | 0.395 | 0.448  | 0.665(56) |
| $^{54}$Co $\rightarrow$ $^{54}$Fe | 0.319 | 0.392 | 0.393  | 0.770(67) |
| $^{68}$As $\rightarrow$ $^{68}$Ge | 0.475 | 0.571 | 0.572  | 1.56(40)  |
| $^{70}$Br $\rightarrow$ $^{70}$Se | 1.140 | 1.234 | 1.268  | 1.60(25)  |
| $^{74}$Rb $\rightarrow$ $^{74}$Kr | 1.088 | 1.230 | 1.258  | 1.63(31)  |

Table 1. Isospin symmetry-breaking corrections $\delta_c$ for the $0^+ \rightarrow 0^+$ superallowed transitions obtained by self-consistent RHF+RPA calculations with PKO1 [13] as well as RH+RPA calculations with DD-ME2 [19]. The column PKO1* presents the results obtained with PKO1 without the Coulomb exchange (Fock) term. The results obtained by shell model calculations [8] are listed in the column T&H for comparison. All values are expressed in %.

In Table 1, the isospin symmetry-breaking corrections $\delta_c$ in Eq. (3) for the $0^+ \rightarrow 0^+$ superallowed transitions are shown. The results are obtained by self-consistent RHF+RPA...
calculations with PKO1 [13] model as well as by self-consistent RH+RPA calculations with DD-ME2 [19] model, and the results obtained by shell model calculations (T&H) [8] are also listed for comparison. The present corrections $\delta_c$ range from about 0.1% for the lightest nucleus $^{10}$C to about 1.2% for the heaviest nucleus $^{74}$Rb, which are two or three times smaller than the T&H results. It is noticed that even smaller estimates of $\delta_c$ compared to the shell model calculations have been recently obtained in Ref. [22] using perturbation theory.

In Table 1, it is found that the corrections $\delta_c$ of RHF+RPA are systematically smaller than those of RH+RPA. To understand this systematic discrepancy between RHF+RPA and RH+RPA, it must be kept in mind that in RHF+RPA the exchange (Fock) terms of mesons and photon are kept in both the mean field and RPA levels, whereas they are neglected altogether in RH+RPA. Among all the Fock terms, we expect, in particular, the exchange terms of the Coulomb field to play an important role due to the following reason. The isobaric analog state would be degenerate with its isobaric multiplet partners, and it would contain 100% of the model-independent sum rule (5), i.e., $\delta_c = 0$, if the nuclear Hamiltonian commutes with the isospin raising and lowering operators $T_{\pm}$. This would be true when the Coulomb field is switched off. While this degeneracy is broken by the mean field approximation, no matter the exchange terms of mesons are included or not, it can be restored by the RPA calculations as long as the RPA calculations are self-consistent [23]. Therefore, the Coulomb field is essential for the $0^+ \rightarrow 0^+$ superallowed transitions and the Coulomb exchange (Fock) term should be responsible for the different isospin symmetry-breaking corrections $\delta_c$ in RHF+RPA and RH+RPA approaches.

In order to verify the above argument, we have performed the following calculations. Using PKO1, the Hartree-Fock calculations are performed by switching off the exchange contributions of the Coulomb field. From the single-particle spectra thus obtained, self-consistent RPA calculations are then performed. The isospin symmetry-breaking corrections $\delta_c$ thus obtained are listed in the column denoted as PKO1* in Table 1. It is seen that these results are almost the same as those of RH+RPA calculations with DD-ME2, i.e., by switching off the exchange contributions of the Coulomb field, the corrections $\delta_c$ in RHF+RPA calculations recover the results in RH+RPA calculations. Therefore, one can conclude that the proper treatment of the Coulomb field is very important to extract the isospin symmetry-breaking corrections $\delta_c$.

![Figure 1](image_url)

**Figure 1.** Nucleus-independent $\mathcal{F}t$ values as a function of the charge $Z$ for the daughter nucleus. The values of $\delta_c$ are respectively obtained by RHF+RPA calculations with PKO1 (left panel) and by RH+RPA calculations with DD-ME2 (right panel). The shaded horizontal band gives one standard deviation around the average $\mathcal{F}t$ value.

Combining the $\delta'_{R}$ and $\delta_{NS}$ values from recent calculations [8], and $\delta_c$ in Table 1, as well as measured $\mathcal{F}t$ values [3], the nucleus-independent $\mathcal{F}t$ values for superallowed $\beta$ decays are obtained and plotted as a function of the charge $Z$ for the daughter nucleus in Fig. 1, where the
uncertainty of $\delta_c$ is taken as zero and the shaded horizontal band gives one standard deviation around the average $\mathcal{F}t$ value. It is found that the $\chi^2/\nu$ is 1.0 $\sim$ 1.1 s, which indicates that the constancy of the nucleus-independent $\mathcal{F}t$ values is satisfied, even though not as well as the shell model calculations in Ref. [3]. It is also found that the $\mathcal{F}t$ value of RHF+RPA is about 2 s larger than that of RH+RPA, which is larger than the difference due to the different effective interactions in either RHF or RH approximations [21]. It again demonstrates that the proper treatment of the Coulomb field plays an important role in this specific subject, and one cannot effectively take care of the Coulomb exchange term by adjusting the parameters in the direct terms of mesons as done in the usual RH approximation.

With the nucleus-independent $\mathcal{F}t$ value, the element $V_{ud}$ of the CKM matrix can be calculated by Eq. (1) with $K/(\bar{h}c)^{6} = 8120.2787(11) \times 10^{-10}$ GeV$^{-4}$s, $G_F/(\bar{h}c)^{3} = 1.16637(1) \times 10^{-5}$ GeV$^{-2}$ [7], and $\Delta^{\nu}_{R} = 2.361(38)\%$ [8]. Then in combination with the other two CKM matrix elements $|V_{us}| = 0.2255(19)$ and $|V_{ub}| = 0.00393(36)$ [7], one can test the unitarity of the matrix.

![Figure 2](image-url)  

**Figure 2.** The sum of squared top row elements of the CKM matrix obtained by RHF+RPA calculations with PKO1 and by RH+RPA calculations with DD-ME2 in comparison with those in shell model (H&T) [3] as well as in neutron decay [7], pion $\beta$ decay [5] and nuclear mirror transitions [6].

In Fig. 2, the sum of squared top row elements of the CKM matrix obtained by RHF+RPA calculations with PKO1 and by RH+RPA calculations with DD-ME2 is shown in comparison with those in shell model (H&T) [3] as well as in neutron decay [7], pion $\beta$ decay [5] and nuclear mirror transitions [6]. It is interesting to note that the present results agree well with those obtained in neutron decay, pion $\beta$ decay and nuclear mirror transitions. However, they deviate somewhat from the unitarity condition, which is in contradiction with the conclusion of shell model calculations (H&T) [3]. However, it should be noticed that the uncertainties of the present results are underestimated to some extent as the uncertainty of $\delta_c$ is assumed to be zero and the systematic errors are not taken into account. These systematic deviations would come from extracting $\delta_c$ with different effective interactions, with the proper neutron-proton mass difference, with deformation, or with pairing correlations.

Finally, we examine the effects of the neutron-proton mass difference on the $V_{ud}$ value. We repeat the self-consistent relativistic RPA calculations with effective interactions PKO1 and DD-
Table 2. Isospin symmetry-breaking corrections $\delta_c$ for the $0^+ \rightarrow 0^+$ superallowed transitions. The columns PKO1$^\dagger$ and DD-ME2$^\dagger$ present the results obtained with effective interactions PKO1 and DD-ME2 but adopting experimental values of neutron and proton masses. The columns $\Delta \delta_c$ show the difference between the results of PKO1$^\dagger$ and PKO1, as well as DD-ME2$^\dagger$ and DD-ME2. All values are expressed in %.

|       | PKO1$^\dagger$ $\Delta \delta_c$ | DD-ME2$^\dagger$ $\Delta \delta_c$ |
|-------|-------------------|-------------------|
| $^{10}$C $\rightarrow$ $^{10}$B | 0.089 0.007 | 0.161 0.011 |
| $^{14}$O $\rightarrow$ $^{14}$N | 0.127 0.013 | 0.214 0.017 |
| $^{18}$Ne $\rightarrow$ $^{18}$F | 0.288 0.017 | 0.454 0.024 |
| $^{26}$Si $\rightarrow$ $^{26}$Al | 0.188 0.012 | 0.268 0.016 |
| $^{30}$S $\rightarrow$ $^{30}$P | 0.531 0.034 | 0.672 0.039 |
| $^{34}$Ar $\rightarrow$ $^{34}$Cl | 0.287 0.019 | 0.400 0.024 |
| $^{38}$Ca $\rightarrow$ $^{38}$K | 0.334 0.021 | 0.467 0.026 |
| $^{42}$Ti $\rightarrow$ $^{42}$Sc | 0.405 0.020 | 0.549 0.026 |
| $^{26}$Al $\rightarrow$ $^{26}$Mg | 0.150 0.011 | 0.212 0.014 |
| $^{34}$Cl $\rightarrow$ $^{34}$S | 0.252 0.018 | 0.329 0.022 |
| $^{38}$K $\rightarrow$ $^{38}$Ar | 0.299 0.020 | 0.395 0.024 |
| $^{42}$Sc $\rightarrow$ $^{42}$Ca | 0.352 0.019 | 0.472 0.024 |
| $^{54}$Co $\rightarrow$ $^{54}$Fe | 0.336 0.017 | 0.414 0.020 |
| $^{66}$As $\rightarrow$ $^{66}$Ge | 0.500 0.026 | 0.601 0.028 |
| $^{70}$Br $\rightarrow$ $^{70}$Se | 1.188 0.048 | 1.320 0.051 |
| $^{74}$Rb $\rightarrow$ $^{74}$Kr | 1.132 0.044 | 1.308 0.050 |

ME2, but adopting experimental values of neutron and proton masses. The isospin symmetry-breaking corrections $\delta_c$ thus obtained are listed in the columns PKO1$^\dagger$ and DD-ME2$^\dagger$ of Table 2. The difference between the results of PKO1$^\dagger$ and PKO1, as well as DD-ME2$^\dagger$ and DD-ME2 are presented in the columns $\Delta \delta_c$ to show the net effects of the neutron-proton mass difference.

It can be seen that the effects of the neutron-proton mass difference on the corrections $\delta_c$ range from about 0.01% to about 0.05%, and roughly speaking, larger $\delta_c$ leads to larger $\Delta \delta_c$. With these effects, the $|V_{ud}|$ value predicted by RHF+RPA with PKO1 is changed from $|V_{ud}| = 0.97273(27)$ to $|V_{ud}| = 0.97280(27)$, and the value predicted by RH+RPA with DD-ME2 is changed from $|V_{ud}| = 0.97311(26)$ to $|V_{ud}| = 0.97321(26)$. This indicates the effects of the neutron-proton mass difference are small compared to other kinds of uncertainties, and it is not the main reason why the present results of the sum of squared top row elements of the CKM matrix deviate from the unitarity condition.

4. Summary
In summary, self-consistent relativistic RPA approaches are applied to calculate the isospin symmetry-breaking corrections $\delta_c$ for the superallowed $\beta$ transitions. It is found that the proper treatments of the Coulomb field is very important to extract the isospin symmetry-breaking corrections $\delta_c$. By switching off the exchange contributions of the Coulomb field, the corrections $\delta_c$ in RHF+RPA calculations recover the results in RH+RPA calculations. In other words, one cannot effectively take care of the Coulomb exchange term by adjusting the parameters in the direct terms of mesons as done in the usual RH approximation.

With the isospin symmetry-breaking corrections $\delta_c$ calculated by relativistic RPA approaches, the values of $|V_{ud}|$ thus obtained agree well with those obtained in neutron decay, pion $\beta$ decay and nuclear mirror transitions. However, the sum of squared top-row elements seems to deviate
from the unitarity condition.

The effects of the neutron-proton mass difference on the isospin symmetry-breaking corrections $\delta_c$ have been investigated. It is shown that these effects are small compared to other kinds of uncertainties. It is not the main reason why the present results of the sum of squared top row elements of the CKM matrix deviate from the unitarity condition.

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