EKEDAHL-OORT STRATA OF CURVES OF GENUS FOUR IN CHARACTERISTIC THREE

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ABSTRACT. We study the induced Ekedahl-Oort stratification on the moduli space \( \mathcal{M}_4 \) of curves of genus 4 in characteristic 3. By constructing families of curves with given Ekedahl-Oort type and by computing the dimension of the intersection of the induced Ekedahl-Oort strata and the boundary divisor classes, we show that for certain induced Ekedahl-Oort strata in \( \mathcal{M}_4 \), they have the same codimension in \( \mathcal{M}_4 \) as the corresponding Ekedahl-Oort strata in moduli space \( \mathcal{A}_4 \) of principally polarized abelian varieties of dimension 4.

1. INTRODUCTION

Let \( k \) be an algebraically closed field of characteristic \( p > 0 \). Let \( \mathcal{A}_g \) be the moduli space (stack) of principally polarized abelian varieties of dimension \( g \) defined over \( k \) and let \( \mathcal{M}_g \) be the moduli space of (smooth projective) curves of genus \( g \) defined over \( k \). Ekedahl and Oort introduced a stratification on \( \mathcal{A}_g \) consisting of \( 2^g \) strata, cf. [7, 14]. It characterizes the \( p \)-torsion group scheme of abelian varieties. Later Ekedahl and van der Geer [7, 18] realized this stratification can be defined using the de Rham cohomology. These strata are indexed by \( n \)-tuples \( \mu = [\mu_1, \ldots, \mu_n] \) with \( 0 \leq n \leq g \) and \( \mu_1 > \mu_2 > \cdots > \mu_n > 0 \). The largest stratum is the locus of ordinary abelian varieties corresponding to the empty \( n \)-tuple \( \mu = \emptyset \). There are many results about the Ekedahl-Oort stratification on \( \mathcal{A}_g \). For example, Oort [14] showed that any positive dimensional stratum is connected and the Ekedahl-Oort stratum index by \( \mu \) is of codimension \( \sum_{i=1}^n \mu_i \) in \( \mathcal{A}_g \). Ekedahl and van der Geer [18] studied the irreducibility of certain Ekedahl-Oort strata and computed the Chern classes defined by Ekedahl-Oort strata.

Via the Torelli map \( \tau : \mathcal{M}_g \rightarrow \mathcal{A}_g \) this stratification can be pulled back to \( \mathcal{M}_g \) and one can ask what this stratification is. The curves of genus 4 in characteristic 3 are here of special interest. For hyperelliptic curves with \( p = 2 \), Elkin and Pries [8] describe their Ekedahl-Oort types completely. In [19], we determine the dimension and reducibility of Ekedahl-Oort strata of hyperelliptic curves in characteristic 3.

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In this paper, we study the dimension of Ekedahl-Oort strata in $\mathcal{M}_4$. For certain Ekedahl-Oort strata, we determine the dimension by looking at the intersection of the Ekedahl-Oort strata and the boundary divisor class. We also show the codimension of Ekedahl-Oort strata satisfying certain bounds by constructing families of curves lies in given Ekedahl-Oort strata. One can find related results in [4].

To describe the results precisely, we need the following notation.

**Notation 1.1.** For the Ekedahl-Oort stratum indexed by $\mu$, we denote the induced strata on $\mathcal{M}_g$ by $Z_\mu$. Note that the indices $\mu$ of the Ekedahl-Oort strata are partially ordered by $\mu = [\mu_1, \ldots, \mu_n] \preceq \nu = [\nu_1, \ldots, \nu_m]$ if $n \leq m$ and $\mu_i \leq \nu_i$ for $i = 1, \ldots, n$. We say a (smooth) curve has Ekedahl-Oort type $\mu$ if the corresponding point in $\mathcal{M}_g$ lies in $Z_\mu$.

**Theorem 1.2.** Let $p = 3$. The locus $Z_\mu$ is non-empty of codimension $\sum_{i=1}^n \mu_i$ in $\mathcal{M}_4$ if $\mu \preceq [4, 1]$. Moreover, the locus $Z_\mu$ is non-empty of codimension at most $\sum_{i=1}^n \mu_i$ for $\mu = [3, 2], [3, 2, 1]$ and $Z_{[4, 3]}$ is non-empty of dimension 3.

Part of Theorem 1.2 was known. Pries [16, Theorem 4.3] showed that there is a family (with dimension 6) of smooth curves of genus 4 with $p$-rank 1 and $a$-number 1 (that is, curve with Ekedahl-Oort type $\mu = [3]$). Moreover, if $p \geq 5$, then by [16, Corollary 4.5] there is a family (with dimension 6) of smooth curves of genus 4 with $p$-rank 2 and $a$-number 2 (Ekedahl-Oort type $\mu = [2, 1]$).

2. **Background**

From now on, let $k$ be an algebraically closed field of characteristic 3. All objects are defined on the category of $k$-schemes.

2.1. **The first de Rham cohomology.** Suppose $Y$ is a scheme. Here we recall the first de Rham cohomology of $Y$ using the Čech cohomology. Denote by $(\Omega^\cdot_Y, d)$ the complex of sheaves of Kähler differential forms on $Y$. The de Rham cohomology of $Y$ is defined as the hypercohomology of the functor $H^0(Y, \cdot)$ with respect to $(\Omega^\cdot_Y, d)$, cf. [5, Section 11.4]. Since we mainly focus on curves and abelian varieties, we are only interested in the first de Rham cohomology, which by [13] can also be described in terms of Čech cocycles as follows. Let $\mathcal{U} = \{ \mathcal{U}_i \mid i \in I \}$ be an open affine cover of $Y$. For a sheaf $\mathcal{F}$ of abelian groups, we denote the Čech complex of abelian groups associated to $\mathcal{F}$ by $C^\cdot(\mathcal{U}, \mathcal{F})$
Chapter III, Section 4. We define the $H^1_{dR}(U)$ with respect to the covering $U$ to be

$$H^1_{dR}(U) = Z^1_{dR}(U)/B^1_{dR}(U), \tag{1}$$

where $Z^1_{dR}(U)$ is defined as

$$\{(f, \omega) \in C^1(U, \mathcal{O}_U) \times C^0(U, \Omega^1_Y) \mid f_{i,k} = f_{i,j} + f_{j,k}, df_{i,j} = \omega_i - \omega_j, d\omega_i = 0\}$$

and $B^1_{dR}(U)$ is defined as

$$\{(f, \omega) \in C^1(U, \mathcal{O}_U) \times C^0(U, \Omega^1_Y) \mid \exists g \in C^0(U, \mathcal{O}_U), f_{i,j} = g_i - g_j, \omega_i = dg_i \}.$$

The first de Rham cohomology $H^1_{dR}(Y)$ is independent of the choice of the open affine cover and we have

$$H^1_{dR}(Y) \cong H^1_{dR}(U).$$

In particular for the case of a curve $X$, we can take an open affine cover $U = \{U_1, U_2\}$ consisting of only two open parts. Then the de Rham cohomology $H^1_{dR}(Y)$ can be described using (1) with

$$Z^1_{dR}(U) = \{(t, \omega_1, \omega_2) \mid t \in \mathcal{O}_Y(U_1 \cap U_2), \omega_i \in \Omega^1_Y(U_i), dt = \omega_1 - \omega_2\}$$

and

$$B^1_{dR}(U) = \{(t_1 - t_2, dt_1, dt_2) \mid t_i \in \mathcal{O}_Y(U_i)\}.$$  

2.2. The Verschiebung operator and the Cartier operator. The Cartier operator $\mathcal{C}$ is the operator on the rational differential forms on a curve $X$ defined in [3]. If $x$ is a separating variable of $k(X)$, any $f \in k(X)$ can be written as

$$f = f_0^p + f_1^p x + \cdots + f_{p-1}^p x^{p-1},$$

with $f_i \in k(X)$, then for a rational differential form $\omega = f \, dx$ with $f$ as above, we have $\mathcal{C}(\omega) = f_{p-1} \, dx$.

We define the rank of the Cartier operator to be the rank of the Cartier operator restricted to $H^0(X, \Omega^1_X)$ and the Cartier-Manin matrix is the matrix of the Cartier operator with respect to a given basis.

Suppose $A$ is a principally polarized abelian variety of dimension $g$ defined over $k$, e.g. $A = \text{Jac}(X)$ for some curve $X$ of genus $g$. Let $F_{A/k} : A \to A^{(p)}$ be the relative Frobenius morphism of $A$ defined by the universal property of $A^{(p)}$, which is the pullback of $A \to \text{Spec}(k)$ via the absolute Frobenius map $\sigma : \text{Spec}(k) \to \text{Spec}(k)$. The relative Frobenius morphism of $A$ is purely inseparable of degree $p^g$. Note that the multiplication by $p$ on $A$ is of degree $p^2 g$ and it factors through the relative Frobenius morphism, i.e. $\left[ p \right]_A = V_{A/k} \circ F_{A/k}$ with $V_{A/k} : A^{(p)} \to A$ the Verschiebung morphism.
Note that $V_{A/k}$ induces a $\sigma^{-1}$-linear map of $k$-vector spaces
$$V : H^1_{dR}(A) \hookrightarrow H^1_{dR}(A)$$
by considering the fact $H^1_{dR}(A^{(p)}) \cong (k; \sigma) \otimes_k H^1_{dR}(A)$. Moreover, for a curve $X$ and $(f, \omega) \in H^1_{dR}(X)$, one has $V(f, \omega) = (0, V(\omega)) = (0, C(\omega))$. Similarly one have the Frobenius operator $F : H^1_{dR}(A) \hookrightarrow H^1_{dR}(A)$ induced by the relative Frobenius of $A$.

2.3. The Ekedahl-Oort stratification. The Ekedahl-Oort stratification is defined by a final filtration on the first de Rham cohomology of abelian varieties, which we define as follows.

**Definition 2.1.** Let $(A, \lambda)$ be a principally polarized abelian variety with $G = H^1_{dR}(A)$. A final filtration is a filtration
$$0 = G_0 \subset G_1 \subset \cdots \subset G_g = V(G) \subset \cdots \subset G_{2g} = G$$
which is stable under $V$ and $F^{-1}$ with the following properties being satisfied:
$$\dim(G_i) = i, \ G_i^\perp = G_{2g-i}, \ i = 1, \ldots, 2g.$$  

The associated final type $v$ is the increasing and surjective map
$$v : \{0, 1, 2, \ldots, 2g\} \mapsto \{0, 1, \ldots, g\}$$
such that $V(G_i) = G_{v(i)}$.

By the properties of $V$ and $F$, we have $v(2g - i) = v(i) - i + g$ for $0 \leq i \leq g$. To find a final filtration of $(A, \lambda)$, we first construct a weak version of it, namely the canonical filtration, and refine it to a final one. Now to construct the canonical filtration of $A$, we start with $0 \subset G$ and add the images of $V$ and the orthogonal complements of the images. Repeating this process for a finite number of times, we obtain a filtration
$$0 = C_0 \subset C_1 \subset \cdots \subset C_r = V(G) \subset \cdots \subset C_{2r} = G,$$
which is stable under $V$ and $\perp$. Then it is also stable under $F^{-1}$ and it is unique. This filtration is called canonical filtration and can be refined to a final one by choosing a filtration of length $2g$ which is stable under $V$ and $\perp$. In general a final filtration is not unique but the final type is unique, cf. [13].

Given a final type $v$, we associate to it a Young diagram, or equivalently a $n$-tuple $\mu = [\mu_1, \ldots, \mu_n]$ with $0 \leq n \leq g$ and $\mu_1 > \mu_2 > \cdots > \mu_n > 0$ such that
$$\mu_j = \#\{i : 1 \leq i \leq j, v(i) + j \leq i\}.$$
Denote by $Z_\mu$ the set of geometric points of $A_g$ with given final type $\nu$ with Young diagram $\mu$. We say an abelian variety (resp. a curve) has Ekedahl-Oort type $\mu$ if the corresponding point in $A_g$ (resp. $M_g$) lies in $Z_\mu$ (resp. $Z_\nu$).

2.4. The $p$-rank and $a$-number. Let $X$ be a curve of genus $g$ defined over $k$. Such curve has several invariants, e.g. the $a$-number and the $p$-rank. The $a$-number of the curve $X$ is defined as $a_X = \dim_k(\Hom(\alpha_p, \Jac(X)))$ with $\alpha_p$ the group scheme which is the kernel of Frobenius on the additive group scheme $\mathbb{G}_a$. The $a$-number of $X$ is equal to $g - r$ where $g$ is the genus of $X$ and $r$ is the rank of the Cartier-Manin matrix, cf. [3, 17]. The $p$-rank of a curve $X$ is the number $f_X$ such that $\#\Jac(X)[p](k) = p^{f_X}$. One see that $1 \leq a_X + f_X \leq g$. Moreover, if $X$ has Ekedahl-Oort type $\mu = [\mu_1, \ldots, \mu_n]$, the $f_X = g - \mu_1$ and $a_X = n$.

3. Proof of Theorem 1.2

3.1. Set up. Recall that we denote by $M_g$ the moduli space (stack) of (smooth projective) curves of genus $g$ defined over $k$ and by $\overline{M}_g$ its Deligne-Mumford compactification. Write $\overline{Z}_\mu$ for the Zariski closure of $Z_\mu$ in $A_g$. Put $\tilde{M}_g := \overline{M}_g - \Delta_0$. It is an open substack of $\overline{M}_g$ parametrising stable curves of compact type of genus $g$. Note that a stable curve is said to be of compact type if the dual graph of the curve is a tree, we refer the definition of the dual graph of a curve to [2]. For $g = 1$, a stable curve of genus 1 of compact type is equivalent to an irreducible smooth elliptic curve. For $g \geq 2$, the Jacobian of a stable curve of genus $g \geq 2$ with a tree as its dual graph is an abelian variety.

The Torelli morphism $\tau : M_g \mapsto A_g$ can be extended to a regular map:

$$\tilde{\tau} : \tilde{M}_g \mapsto A_g.$$ 

Since curve with different marked point can have same Jacobian, the map $\tilde{\tau}$ has positive dimensional fibres. It is known that this morphism is proper and its image $\tilde{\tau}(\tilde{M}_g)$ is a reduced closed subscheme of $A_g$, cf. [12]. For $g = 4$, we have that $\dim(M_g) = 9$ and $\dim(A_4) = 10$. Moreover, by a work of Igusa [11], the image $\tilde{\tau}(\tilde{M}_4)$ is an ample divisor on $A_4$. Hence $\tilde{\tau}(\tilde{M}_4) \cap Z_\mu$ is of codimension at most $\sum_{i=1}^n \mu_i$ with $\mu = [\mu_1, \ldots, \mu_n]$. Recall that there is a partial order $\preceq$ on the set of Ekedahl-Oort strata. We have $Z_\mu \supset Z_\nu$ if $\mu \preceq \nu$.

Now we can give a proof of Theorem 1.2.
Proof of Theorem 1.2. The $p$-rank part were known by [9, Theorem 2.3], saying that $Z_\mu$ is of codimension $\mu_1$ in $M_g$ for $\mu = [\mu_1]$. For $\mu = [2,1]$, we compute the dimension of $\Delta_i \cap Z_{[2,1]}$ in $A_1$ for $i = 1, 2$. A curve corresponding to a point in the pull back of a generic point in $\Delta_i \cap Z_{[2,1]}$ is formed from two smooth pointed curves $X_1$ and $X_3$ of genus 1 and 3 respectively. Then $X_1$ is ordinary and $X_3$ has Ekedahl-Oort type $\mu = [2,1]$ or $X_1$ is supersingular and $X_3$ has Ekedahl-Oort type $\mu = [1]$. In the first case, we consider the action of Verschiebung operator $V$ on the de Rham cohomology. Let $A_2 = H^1_{dR}(X_1), B_6 = H^1_{dR}(X_3)$ and $C_8 = H^1_{dR}(X_1 \times X_3)$. We now compute the final filtration on $H^1_{dR}(X_1 \times X_3)$ as explained in Section 2.3. Recall that we have a pairing $\langle \cdot, \cdot \rangle$ on $H^1_{dR}(X)$ for any smooth irreducible projective curve $X$. We have

$$V(A_2) = A_1, \quad V(A_1) = A_1, \quad A^+_1 = A_1,$$

where $A^+_1$ is the orthogonal complement of $A_1$ in $A$ with respect to the pairing $\langle \cdot, \cdot \rangle$. Similarly, $V(B_6) = B_3, V(B_3) = B_1, V(B_1) = B_1, B^+_1 = B_5$. For a final filtration $0 \subset G_1 \subset G_2 \subset \cdots \subset G_g \subset \cdots \subset G_{2g}$, we have

$$\dim V(G_{2g-n}) = \dim V(G_n) + g - n$$

(2)

with $0 \leq n \leq g$. Then $\dim V(B_5) = \dim V(B_1) + 3 - 1 = 3$. It follows that $V(B_5) = B_3$ and

$$V(C_8) = V(A_2 + B_6) = A_1 + B_3 = C_4,$$

$$V(C_4) = V(A_1 + B_3) = A_1 + B_1 = C_2,$$

$$V(C_2) = V(A_1 + B_1) = A_1 + B_1.$$

Furthermore,

$$C^+_2 = A_1 + B_5 = C_6, \quad V(C_6) = V(A_1 + B_5) = A_1 + B_3 = C_4.$$

This gives the canonical filtration $0 \subset C_2 \subset C_4 \subset C_6 \subset C_8$ for $C_8 = H^1_{dR}(X_1 \times X_3)$ and the action of $V$. So the Ekedahl-Oort type is $[2,1]$. Note that the moduli space of pointed elliptic curves is irreducible of dimension 1 and $Z_{[2,1]} \subset M_3$ is irreducible of dimension 3 by [7, Theorem 11.3]. Hence in this case the component in $\Delta_i \cap Z_{[2,1]}$ is of dimension 4. By a similar argument, in the other case, a generic point also has Ekedahl-Oort type $[2,1]$ and the component of $\Delta_i \cap Z_{[2,1]}$ is non-empty of dimension 5. Hence $\Delta_i \cap Z_{[2,1]}$ is of dimension at most 5.

For $\Delta_2 \cap Z_{[2,1]}$, a generic point in $\Delta_2 \cap Z_{[2,1]}$ is the union of two pointed curves $X_2$ and $\tilde{X}_2$, both of genus 2, where $X_2$ and $\tilde{X}_2$ both have 3-rank 1 or $X_2$ is ordinary and $\tilde{X}_2$ is superspecial. In the first
case, let \( A_4 = H^1_{dR}(X_2) \), \( B_4 = H^1_{dR}(\tilde{X}_2) \) and \( C_8 = H^1_{dR}(X_2 \times \tilde{X}_2) \), then we have
\[
V(A_4) = A_2, \ V(A_2) = A_1, \ A_1^+ = A_3.
\]
By equation (2), we have \( V(A_3) = A_2 \). We also have \( V \) acts similarly on \( B_4 \). Furthermore,
\[
V(C_8) = V(A_4 + B_4) = A_2 + B_2 = C_4, \\
V(C_4) = V(A_2 + B_2) = A_1 + B_1 = C_2, \\
V(C_2) = V(A_1 + B_1) = A_1 + B_1.
\]
Combined with \( C_2^+ = A_3 + B_3 = C_6 \) and \( V(C_6) = A_2 + B_2 = C_4 \), we have the canonical filtration \( 0 \subset C_2 \subset C_4 \subset C_6 \subset C_8 \) and the action of \( V \). Hence the generic point has Ekedahl-Oort type \([2,1]\) and the component is of dimension 4. In a similar fashion we can prove in the other case that the component is of dimension 3.

On the other hand, \( \tilde{\tau}(\tilde{M}_4) \) is ample in \( A_4 \) and hence \( \tilde{\tau}(\tilde{M}_4) \cap Z_{[2,1]} \) is of dimension at least 6. Furthermore, \( Z_{[2,1]} \subset A_4 \) is irreducible of dimension 7 by [7, Theorem 11.5]. Suppose \( Z_{[2,1]} \subset M_4 \) is of dimension \( 7 \). By taking the closure in \( A_4 \) and by the irreducibility of \( Z_{[2,1]} \), we have \( \overline{Z_{[2,1]}} = \overline{\tilde{\tau}(Z_{[2,1]})} \). Since \( \tilde{\tau}(\tilde{M}_4) \) is closed in \( A_4 \), we have \( \overline{Z_{[2,1]}} \subset \tilde{\tau}(\tilde{M}_4) \).
This implies \( \overline{Z_{[4,1]}} \subset \overline{Z_{[2,1]}} \subset \tilde{\tau}(\tilde{M}_4) \) with \( \overline{Z_{[4,1]}} \) irreducible of dimension 5 by [7, Theorem 11.3]. Note that we also have \( \overline{Z_{[4,1]}} \subset \overline{Z_{[4]}} \). Then \( \overline{Z_{[4,1]}} \subset \overline{Z_{[4]}} \cap \tilde{\tau}(\tilde{M}_4) \) with \( \dim(\overline{Z_{[4,1]}}) = \dim(\overline{Z_{[4]}} \cap \tilde{\tau}(\tilde{M}_4)) = 5 \) by Theorem ???. Then \( \overline{Z_{[4,1]}} \) is an irreducible component of \( \overline{Z_{[4]}} \cap \tilde{\tau}(\tilde{M}_4) \) and every abelian variety corresponds to a point of this component of \( \overline{Z_{[4]}} \cap \tilde{\tau}(\tilde{M}_4) \) has \( a \)-number \( \geq 2 \). Write \( V_0(\tilde{M}_4) \) for the locus of curves with \( p \)-rank 0 in \( \tilde{M}_4 \). By the proof of Theorem 2.3 in [9], a generic point of any irreducible component of the locus \( \tilde{\tau}(V_0(\tilde{M}_4)) \) has \( a \)-number 1. This contradiction implies \( \overline{Z_{[2,1]}} \not\subset \tilde{\tau}(\tilde{M}_4) \) and \( Z_{[2,1]} \) is non-empty of dimension 6.

For \( \mu = [3,1] \), similarly we can compute that \( \tilde{\tau}(\Delta_i) \cap Z_{[3,1]} \) has dimension at most 3 for \( i = 1, 2 \). On the other hand, \( \tilde{\tau}(\tilde{M}_4) \cap Z_{[3,1]} \) is of dimension at least 5 as \( \tilde{\tau}(\tilde{M}_4) \) is ample. Moreover, \( Z_{[3,1]} \subset A_4 \) is irreducible of dimension 7 by [7, Theorem 11.5]. Suppose \( Z_{[3,1]} \) is of dimension 6. By taking the closure in \( A_4 \) and by the irreducibility of \( Z_{[3,1]} \), we have \( \overline{Z_{[3,1]}} = \overline{\tilde{\tau}(Z_{[3,1]})} \). Note that \( \overline{Z_{[4,1]}} \subset \overline{Z_{[4]}} \). By the similar argument in the case \( Z_{[2,1]} \), we have \( \overline{Z_{[3,1]}} \not\subset \tilde{\tau}(\tilde{M}_4) \) and the locus \( Z_{[3,1]} \) is non-empty of dimension 5.
For $\mu = [3, 2]$, we show that a smooth curve with affine equation
\[ y^3 + y^2 + by = x^5 + a_3 x^3 + a_2 x^2 + a_0, \tag{3} \]
where $a_i, b \in k$ and $a_2 \neq 0$ has Ekedahl-Oort type $[3, 2]$.

We first show that the map from the parameter space with coordinates $(a_3, a_2, a_0, b)$ to $\mathcal{M}_4$ has finite fibres. Denote by $\sigma$ an isomorphism between two smooth Artin-Schreier curves given by $y^3 + y^2 + by = x^5 + a_3 x^3 + a_2 x^2 + a_0$ and $y^3 + y^2 + b_1 y = x^5 + c_3 x^3 + c_2 x^2 + c_0$ as in (4). After possibly composing with an inversion $x \mapsto 1/x$, we may assume $\sigma(x) = ax + \beta$. Also since $\sigma$ is invertible, we have $\sigma(y) = zy + \delta$ with $z$ a unit in $k$ and $\delta \in k(x)$. Hence
\[ z^3 y^3 + \delta^3 + (zy + \delta)^2 + b(zy + \delta) = (ax + \beta)^5 + a_3 (ax + \beta)^3 + a_2(ax + \beta)^2 + a_0. \]
Then we have $\delta = \beta = 0$ and $y^3 + y^2/z + by/z^2 = 1/z^3 (\alpha^5 x^5 + a_3 \alpha^3 x^3 + a_2 \alpha^2 x^2 + a_0)$. This implies $\beta = \delta = 0, \alpha^5 = z = 1$.

Now we show that a curve $X$ with equation (3) has Ekedahl-Oort type $[3, 2]$. A basis of $H^0(X, \Omega^1_X)$ is given by $\omega_1 = 1/(y-b) \, dx, \omega_2 = x/(y-b) \, dx, \omega_3 = x^2/(y-b) \, dx, \omega_4 = 1 \, dx$. Then the Cartier-Manin matrix is given by
\[
\begin{pmatrix}
 a_2 & 0 & a_0 - b^3 & 0 \\
 1 & 0 & a_3 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 1 & 0
\end{pmatrix}^{1/3}
\]
and it has rank 2 and semisimple rank 1. One can compute that curve $X$ has final filtration
\[
0 \subset G_1 \subset G_2 \subset G_3 \subset G_4 = H^0(X, \Omega^1_X) \subset \cdots \subset G_8 = H^1_{dR}(X),
\]
where $G_1 = \langle a_2^{1/3} \omega_1 + \omega_2 \rangle, G_2 = \langle a_2 \omega_1 + \omega_2, (a_0 - b^3)^{1/3} \omega_1 + a_3^{1/3} \omega_2 + \omega_4 \rangle$ and $G_3 = \langle \omega_1, \omega_2, \omega_4 \rangle$. Hence the curve has Ekedahl-Oort type $[3, 2]$ and $Z_{[3,2]}$ is non-empty of dimension at least 4.

For $\mu = [3, 2, 1]$, we show that the family of curves given by equation
\[ y^3 - bx^3(y^2 + y) = x^5 + cx^3 + dx^2 + 1, \ b, c, d \in k, \ bd \neq 0 \tag{4} \]
has Ekedahl-Oort type $[3, 2, 1]$.

In a similar fashion, we first show that the map from the parameter space with coordinates $(b, c, d)$ to $\mathcal{M}_4$ has finite fibres. Denote by $\sigma$ an isomorphism between two smooth Artin-Schreier curves given by $y^3 - bx^3(y^2 + y) = x^5 + cx^3 + dx^2 + 1$ and $y^3 - b_1 x^3(y^2 + y) = x^5 + c_1 x^3 + d_1 x^2 + 1$ as in (4). After possibly composing with an inversion...
$x \mapsto 1/x$, we may assume $\sigma(x) = \alpha x + \beta$. Also since $\sigma$ is invertible, we have $\sigma(y) = zy + \delta$ with $z$ a unit in $k$ and $\delta \in k(x)$. Hence
\begin{equation}
z^3y^3 + \delta^3 + \beta(\alpha x + \beta)^3(z^2y^2 + (2\delta + z)y + \delta^2 + \delta) = \sigma(x)^3 + c\sigma(x)^3 + d\sigma(x)^3 + 1.
\end{equation}
By comparing the coefficients of $x^i$ for $i = 0, 1, 2, 3, 4, 5$, we have $\beta = \delta = 0$ and $z = 1$. Furthermore, we have $y^3 - b\alpha^3x^3(y^2 + y + \delta^2 + \delta) = \alpha^5x^5 + \delta^5x^3 + dx^2 + 1$ and hence $\alpha^5 = 1$. This implies $\beta = \delta = 0, \alpha^5 = z = 1$.

Now we compute the Ekedahl-Oort type of a curve $X$ given by equation (4). For a basis of $H^0(X, \Omega_X^1)$ we choose
\begin{equation}
1/s(x, y) \, dx, x/s(x, y) \, dx, x^2/s(x, y) \, dx, y/s(x, y) \, dx
\end{equation}
with $s(x, y) = x^3(y - 1)$. Then the Cartier-Manin matrix has rank 1 and semisimple rank 1. Hence $Z_{[3, 2, 1]}$ is non-empty with dimension at least 3.

For $\mu = [4, 1]$, we have $Z_{[4, 1]}$ is irreducible by [7, Theorem 11.5]. For $\tilde{\tau}(\Delta_1) \cap Z_{[4, 1]}$, a curve corresponding to a point in the pull back of a generic point in $\tilde{\tau}(\Delta_1) \cap Z_{[4, 1]}$ is formed from two pointed curves $X_1$ and $X_3$ of genus 1 and 3 respectively. Moreover, $X_1$ is superspecial and $X_3$ has Ekedahl-Oort type $\mu = [3]$. One can easily compute that in this case the curve formed from pointed curves $X_1$ and $X_3$ has Ekedahl-Oort type $[4, 2]$ and $\tilde{\tau}(\Delta_1) \cap Z_{[4, 1]}$ is of dimension 3. For $\tilde{\tau}(\Delta_2) \cap Z_{[4, 1]}$, a point in the pull back of a generic point in $\tilde{\tau}(\Delta_2) \cap Z_{[4, 1]}$ is formed from two pointed curves with Ekedahl-Oort type $\mu = [2]$. Then one can compute that a generic point of $\tilde{\tau}(\Delta_2) \cap Z_{[4, 1]}$ has Ekedahl-Oort type $[4, 3]$ and $\tilde{\tau}(\Delta_2) \cap Z_{[4, 1]}$ is of dimension 2.

On the other hand, $\tilde{\tau}(\bar{M}_4) \cap Z_{[4, 1]}$ is of dimension at least 4. Hence $Z_{[4, 1]}$ is of dimension at least 4. Note that if $Z_{[4, 1]}$ is of dimension 5. Then by taking the closure in $A_4$ and by the irreducibility of $Z_{[4, 1]}$, we have $\overline{Z_{[4, 1]}} \subset \tilde{\tau}(\bar{M}_4)$, which is a contradiction by the proof of the case $Z_{[2, 1]}$. Hence $Z_{[4, 1]}$ is non-empty of dimension 4.

For $\mu = [4, 2]$ and [4, 3], we have $\mathcal{H}_4 \cap Z_\mu$ non-empty hence $Z_\mu$ is non-empty by [19, Theorem 1.1].

Denote $\theta(x) = x^3 + x^2 + bx_1 x + b_2$ and $h(x) = x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1 x$. We consider the family of curves parametrized by equation
\begin{equation}
y^3 + y^2 + \theta(x)y = h(x),
\end{equation}
where $a_i, b_j \in k$ such that $b_2 = b_1^2 + a_2, a_4 = b_1 - 1$ and $a_3 = b_1^2 + b_2^2 + b_1 + a_2$.

Note that the map from the parameter space to the $\mathcal{M}_4$ is finite and hence it gives a 3-dimensional sublocus in $\mathcal{M}_4$. Indeed, denote by $\sigma$ an isomorphism between two smooth Artin-Schreier curves given by $\theta_1(x), h_1(x)$ and $\theta_2(x), h_2(x)$ as in (5). After possibly composing
with an inversion \( x \mapsto 1/x \), we may assume \( \sigma(x) = \alpha x + \beta \). Also since \( \sigma \) is invertible, we have \( \sigma(y) = zy + \delta \) with \( z \) a unit in \( k \) and \( \delta \in k(x) \). Then one can easily show that \( z = \alpha = 1, \beta^3 + \beta^2 = 0 \) and 
\[-(\delta^3 + \delta^2 + b_2 \delta) + h_1(\beta) = 0 \text{ with } \theta_1(x) = x^3 + x^2 + b_1 x + b_2 \text{ and } \]
\[h(x) = x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x.\]

Indeed, denote by \( \sigma \) an isomorphism between two smooth Artin-Schreier curves given by \( y^3 - y = f_1(x) = \sum_{i=0} a_i x^i \) and \( y^3 - y = f_2(x) = \sum_{i=0} b_i x^i \) as in (6). By [15, Lemma 2.1.5], two Artin-Schreier curves are isomorphic if and only if \( f_2(x) = z f_1(x) + \delta^3 - \delta \) with \( z \in \mathbb{F}_3^* \) and \( \delta \in k(x) \). Moreover, the \( \sigma \) is defined by \( \sigma(x) = \alpha x + \beta \) and \( \sigma(y) = zy + \delta \). Hence we have \( \beta = 0, \alpha^5 = z \in \mathbb{F}_3^* \) and \( \delta \in \mathbb{F}_3 \).

Now we show that any smooth curve \( X \) given by equation (5) above has Ekedahl-Oort type \([4, 3]\). For a basis of \( H^0(X, \Omega^1_X) \), we choose

\[
\frac{1}{s(x, y)} \, dx, \frac{x}{s(x, y)} \, dx, \frac{x^2}{s(x, y)} \, dx, \frac{y}{s(x, y)} \, dx
\]

with \( s(x, y) = y - (x^3 + x^2 + b_1 x + b_2) \). For the Cartier operator \( C \) we have

\[
C\left( \frac{1}{s(x, y)} \, dx \right) = C\left( \frac{1}{y - \theta(x)} \right) = C\left( \frac{(y - \theta(x))^2}{(y - \theta(x))^3} \right)
\]

\[
= \frac{1}{y - \theta(x)} C\left( y^2 + \theta^2(x) + \theta(x)y \right).
\]

Note that \( y^2 + \theta(x)y = -y^3 + h(x) \) and \( \theta^2(x) = x^6 + 2x^5 + (2b_1+1)x^4 + (2b_1+2b_2)x^3 + (b_1^2 + 2b_2)x^2 + 2b_1 b_2 x + b_2^2. \) Hence

\[
y^2 + \theta^2(x) + \theta(x)y = -y^3 + x^6 + (2b_1+1+a_4)x^5 + (2b_1 + 2b_2 + a_3)x^3
\]

\[+(b_1^2 + 2b_2 + a_2)x^2 + (2b_1 b_2 + a_1)x + b_2^2,
\]

and we have

\[
C\left( \frac{1}{s(x, y)} \, dx \right) = \frac{1}{y - \theta(x)} C\left( y^2 + \theta^2(x) + \theta(x)y \right)
\]

\[
= \frac{(2b_2 + b_1^2 + a_2)^{1/3}}{y - \theta(x)} = \frac{(2b_2 + b_1^2 + a_2)^{1/3}}{s(x, y)} \, dx.
\]
Similarly one can compute $C(x/s(x, y)), C(x^2/s(x, y))$ and $C(y/s(x, y))$. Then we have the Cartier-Manin matrix which equals to

$$
\begin{pmatrix}
2b_2 + b_1^2 + a_2 & 2b_1b_2 + a_1 & b_2^2 & b_2^2 + b_1(2b_1b_2 + a_1) \\
0 & 2b_1 + 1 + a_4 & 2b_1 + 2b_2 + a_3 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & -1 & -1
\end{pmatrix}^{1/3}
$$

We obtain that rank($C$) = 2 and $C^2 = 0$ on $H^0(X, \Omega_X^1)$. Hence $X$ with equation (5) has Ekedahl-Oort type $[4, 3]$ and $Z_{[4,3]}$ is of dimension 3.

For $\mu = [4, 3, 2, 1]$, a curve corresponding to a point in $Z_{[4,3,2,1]}$ is superspecial. Then $Z_{[4,3,2,1]}$ is empty as Ekedahl [6] showed that any superspecial curve of genus $g$ satisfies:

$$g \leq 3(3 - 1)/2 = 3.
$$

\[\square\]

**Remark 3.1.** As an example, a family of curves given by the equation

$$y^3 + bx^2y = x^5 + ax^4 + x, \ a, b \in k, b \neq 0$$

has Ekedahl-Oort type $[2, 1]$. Moreover, the family of curves parametrized by equation

$$y^3 - y = x^5 + a_2x^2 + a_1x, \ a_1, a_2 \in k$$

(6)

gives a 2-dimensional locus in $Z_{[4,3]}$. On the other hand, the family of curves parametrized by equation

$$y^3 + y^2 + (x^3 + x^2)y = x^5 + 2x^4 + a_1x + a_2,$$

(7)

where $a_1, a_2 \in k$ and $a_1 \neq 0$, gives another 2-dimensional locus in $Z_{[4,3]}$.

Consider a curve with equation (7) above, with respect to the basis given by

$$1/s(x, y) \ dx, x/s(x, y) \ dx, x^2/s(x, y) \ dx, y/s(x, y) \ dx$$

with $s(x, y) = y + 2x^3 + 2x^2$, one can easily compute the Cartier-Manin matrix which equals to

$$
\begin{pmatrix}
0 & a_1 & a_2 & a_2 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & -1 & -1
\end{pmatrix}^{1/3}
$$
Then the rank and semisimple rank of the Cartier-Manin matrix is 2 and 0 respectively and the curve with equation (7) has Ekedahl-Oort type $[4,3]$.

**Remark 3.2.** Note that Achter and Pries [1] have some result about generic Newton polygon of curves of given genus and $p$-rank.

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