SHARP INEQUALITIES FOR THE NUMERICAL RADII OF BLOCK OPERATOR MATRICES

M. GHADERI AGHIDEH\textsuperscript{1,2}, M. S. MOSLEHIAN\textsuperscript{3,*} and J. ROOIN\textsuperscript{1}

\textsuperscript{1}Department of Mathematics, Institute for Advanced Studies in Basic Sciences (IASBS),
Zanjan 45137-66731, Iran
e-mails: m.ghaderiaghideh@iasbs.ac.ir, rooin@iasbs.ac.ir

\textsuperscript{2}Tusi Mathematical Research Group (TMRG), Mashhad, Iran

\textsuperscript{3}Department of Pure Mathematics,
Center of Excellence in Analysis on Algebraic Structures (CEAAS),
Ferdowsi University of Mashhad, P. O. Box 1159, Mashhad 91775, Iran
e-mails: moslehian@um.ac.ir, moslehian@member.ams.org

(Received May 7, 2018; revised September 13, 2018; accepted September 17, 2018)

Abstract. In this paper we present several sharp upper bounds for the numerical radii of the diagonal and off-diagonal parts of the $2 \times 2$ block operator matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$. Among extensions of some results of Kittaneh et al., it is shown that if $T = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$, and $f$ and $g$ are non-negative continuous functions on $[0, \infty)$ such that $f(t)g(t) = t$ ($t \geq 0$), then for all non-negative nondecreasing convex functions $h$ on $[0, \infty)$, we obtain that

$$h(w^r(T)) \leq \max \left( \left\| \frac{1}{p}h(f^p(|A|)) + \frac{1}{q}h(g^q(|A^*|)) \right\|, \left\| \frac{1}{p}h(f^p(|D|)) + \frac{1}{q}h(g^q(|D^*|)) \right\| \right),$$

where $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, and $r \min(p, q) \geq 2$.

1. Introduction

Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space, and $\mathbb{B}(\mathcal{H})$ denote the $C^*$-algebra of all bounded linear operators on $\mathcal{H}$. The spectral radius and the numerical radius of an operator $A \in \mathbb{B}(\mathcal{H})$ are defined by $\rho(A) = \sup \{|\lambda| : \lambda \in \text{sp}(A)\}$ and

$$w(A) = \sup \{|\langle Ax, x \rangle| : x \in \mathcal{H}, \|x\| = 1\},$$

\textsuperscript{*}Corresponding author.

Key words and phrases: numerical radius, convexity, mixed Cauchy–Schwarz inequality, polar decomposition.

Mathematics Subject Classification: 47A12, 47A63, 47A30.

0133-3852 © 2019 Akadémiai Kiadó, Budapest
respectively. It is well known that $\rho(A) \leq w(A)$ and $w(\cdot)$ defines a norm on $\mathcal{B}(\mathcal{H})$, which is equivalent to the usual operator norm $\| \cdot \|$; more precisely,

\begin{equation}
\frac{1}{2} \| A \| \leq w(A) \leq \| A \|
\end{equation}

for any $A \in \mathcal{B}(\mathcal{H})$. The inequalities in (1.1) are sharp; the second inequality becomes an equality, e.g., if $A$ is normal, while the first one becomes an equality, e.g., if $A^2 = 0$.

An important inequality for $w(A)$ is the power inequality stating that

$$w(A^n) \leq w(A)^n \quad (n = 1, 2, \ldots).$$

The quantity $w(A)$ is useful in the study of perturbation, convergence, and approximation problems. For more information see [1,6,7,17].

Let $A$, $B$, $C$, and $D$ be in $\mathcal{B}(\mathcal{H})$. We call $\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$ and $\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$ the diagonal and off-diagonal parts of the block matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$, respectively. Hirzallah, Kittaneh, and Shebrawi [6] proved that

\begin{equation}
w \left( \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right) \leq \frac{1}{2} (\| B \| + \| C \|),
\end{equation}

for $B, C \in \mathcal{B}(\mathcal{H})$. Kittaneh [8,9] showed the following precise estimates of $w(A)$:

\begin{equation}
w(A) \leq \frac{1}{2} \left( \| A \| + |A^*| \right)
\end{equation}

and

\begin{equation}
\frac{1}{4} \| |A|^2 + |A^*|^2 \| \leq w^2(A) \leq \frac{1}{2} \| |A|^2 + |A^*|^2 \|,
\end{equation}

where $|A| = (A^*A)^{1/2}$ denotes the absolute value of $A$.

Also, El-Haddad and Kittaneh [4] proved that if $A \in \mathcal{B}(\mathcal{H})$ and $A = B + iC$ is the Cartesian decomposition of $A$, then

\begin{equation}
2^{-\frac{r}{2}-1} \| |B + C|^r + |B - C|^r \| \leq w^r(A) \leq \frac{1}{2} \| |B + C|^r + |B - C|^r \|,
\end{equation}

for all $r \geq 2$.

The purpose of this paper is to present some general inequalities involving powers of the numerical radius for the diagonal and off-diagonal parts of $2 \times 2$ block operator matrices. As a consequence, we generalize inequalities (1.2), (1.3), and second inequalities in (1.4) and (1.5).
2. Inequalities for the off-diagonal part

To achieve our results, we need the functional calculus (see, e.g. [13]) and the following lemmas. The first lemma is a consequence of the classical Young and Hölder inequalities.

**Lemma 2.1** [12, pp. 100, 127]. For $a, b \geq 0$ and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, we have

(a) $ab \leq \left(\frac{a^p}{p} + \frac{b^q}{q}\right)^{\frac{1}{r}}$ for $r \geq 1$,

(b) $a_1b_1 + a_2b_2 + \cdots + a_nb_n \leq (a_1^p + a_2^p + \cdots + a_n^p)^{\frac{1}{r}} (b_1^q + b_2^q + \cdots + b_n^q)^{\frac{1}{r}}$.

The second lemma is an operator version of the classical Jensen inequality.

**Lemma 2.2** [14, Theorem 1.2]. Let $A \in \mathcal{B}(\mathcal{H})$ be a self-adjoint operator with $\text{sp}(A) \subseteq [m, M]$ for some scalars $m \leq M$, and let $x \in \mathcal{H}$ be a unit vector. If $f(t)$ is a convex function on $[m, M]$, then

$$f(\langle Ax, x \rangle) \leq \langle f(A)x, x \rangle.$$ 

In particular, if $A \geq 0$, then

$$\langle Ax, x \rangle^r \leq \langle A^r x, x \rangle$$ \quad ($r \geq 1$).

The third lemma is known as the generalized mixed Cauchy–Schwarz inequality.

**Lemma 2.3** [10]. Let $A \in \mathcal{B}(\mathcal{H})$, and let $x, y \in \mathcal{H}$ be any vectors. If $f$ and $g$ are non-negative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$ ($t \geq 0$), then

$$|\langle Ax, y \rangle| \leq \left( f^2(|A|)x, x \right)^{\frac{1}{2}} \left( g^2(|A^*|)y, y \right)^{\frac{1}{2}}.$$

The fourth lemma can be found in [8,16].

**Lemma 2.4.** Let $A, B$, and $D$ be operators in $\mathcal{B}(\mathcal{H})$. Then

(a) $w(A) = \max_{\theta \in \mathbb{R}} \| \text{Re}(e^{i\theta} A) \|$,

(b) $w(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}) = \max(w(A), w(D))$,

(c) $w(\begin{bmatrix} A & B \\ 0 & A \end{bmatrix}) = \max(w(A + B), w(A - B))$,

(d) $w(\begin{bmatrix} A & B \\ -B & A \end{bmatrix}) = \max(w(A + iB), w(A - iB))$.

The following result is a variant of a known result (see [11, Corollary 3.5]) but with a different proof.
Lemma 2.5. Let $h$ be a non-negative nondecreasing convex function on $[0, \infty)$ and $A, B \in \mathcal{B}(\mathcal{H})$ be positive operators. Then

$$h\left(\left\| \frac{A + B}{2} \right\| \right) \leq \frac{h(A) + h(B)}{2}.$$  

Proof. For each unit vector $x \in \mathcal{H}$ we have

$$h\left(\frac{\langle A + B, x, x \rangle}{2}\right) = h\left(\frac{\langle Ax, x \rangle + \langle Bx, x \rangle}{2}\right) \leq \frac{h(\langle Ax, x \rangle) + h(\langle Bx, x \rangle)}{2} \quad \text{(by the convexity of $h$)}$$

$$\leq \frac{\langle h(A)x, x \rangle + \langle h(B)x, x \rangle}{2} \quad \text{(by the operator Jensen inequality)}$$

$$= \left\langle \frac{h(A) + h(B)}{2}, x, x \right\rangle \leq \left\| \frac{h(A) + h(B)}{2} \right\|.$$  

Now, since $h$ is a non-negative, non-decreasing and convex (continuous) function, by considering (2.1) and taking the supremum from the left-hand side, we get

$$h\left(\left\| \frac{A + B}{2} \right\| \right) = h\left(\left\| w\left(\frac{A + B}{2}\right) \right\| \right) = h\left(\sup \left\langle \frac{A + B}{2}, x, x \right\rangle \right)$$

$$= \sup\left\{ h\left(\frac{\langle A + B, x, x \rangle}{2}\right) \right\} \leq \left\| \frac{h(A) + h(B)}{2} \right\|. \quad \square$$

We are in a position to demonstrate the main results of this section by adopting and extending some techniques of [2–5]. The following theorem gives a generalization of inequality (1.2). Recall the polarization identity: for any elements $x, y$ of an inner product space $\mathcal{H}$ we have

$$\langle x, y \rangle = \frac{1}{4} \sum_{k=0}^{3} i^k \| x + i^k y \|^2.$$  

Theorem 2.6. Let $S = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$, and let $f$ and $g$ be non-negative continuous functions on $[0, \infty)$ such that $f(t)g(t) = t \ (t \geq 0)$. Then for all non-negative nondecreasing convex functions $h$ on $[0, \infty)$,

$$h(w(S)) \leq \frac{1}{4} \left\| h(f^2(|B|)) + h(g^2(|B|)) \right\| + \frac{1}{4} \left\| h(f^2(|C|)) + h(g^2(|C|)) \right\|.$$  

Proof. Let $B = U|B|$, and let $C = V|C|$ be the polar decompositions of the operators $B$ and $C$. Then

$$S = W|S| = \begin{bmatrix} 0 & U \\ V & 0 \end{bmatrix} \begin{bmatrix} |C| & 0 \\ 0 & |B| \end{bmatrix}$$
is the polar decomposition of $S$. Let $x = (x_1, x_2)$ be any unit vector in $\mathcal{H} \oplus \mathcal{H}$; that is, $\|x_1\|^2 + \|x_2\|^2 = 1$. Then for all $\theta \in \mathbb{R}$, we obtain

$$\text{Re} \langle e^{i\theta} S x, x \rangle = \text{Re} \langle e^{i\theta} W |S| x, x \rangle = \text{Re} \langle e^{i\theta} W f(|S|)g(|S|) x, x \rangle$$

(by functional calculus)

$$= \text{Re} \langle e^{i\theta} g(|S|) x, f(|S|) W^* x \rangle$$

$$= \text{Re} \left( e^{i\theta} \begin{bmatrix} g(|C|) & 0 \\ 0 & g(|B|) \end{bmatrix} \begin{bmatrix} f(|C|) & 0 \\ 0 & f(|B|) \end{bmatrix} \begin{bmatrix} U^* & 0 \\ 0 & V^* \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right)$$

$$= \text{Re} \left( e^{i\theta} \left( g(|C|) x_1, g(|B|) x_2 \right), \left( f(|C|) V^* x_2, f(|B|) U^* x_1 \right) \right)$$

$$= \text{Re} \left( \langle e^{i\theta} g(|C|) x_1, f(|C|) V^* x_2 \rangle + \langle e^{i\theta} g(|B|) x_2, f(|B|) U^* x_1 \rangle \right)$$

$$= \frac{1}{4} \left( \|e^{i\theta} g(|C|) x_1 + f(|C|) V^* x_2 \|^2 - \|e^{i\theta} g(|C|) x_1 - f(|C|) V^* x_2 \|^2 \right)$$

$$+ \frac{1}{4} \left( \|e^{i\theta} g(|B|) x_2 + f(|B|) U^* x_1 \|^2 - \|e^{i\theta} g(|B|) x_2 - f(|B|) U^* x_1 \|^2 \right)$$

(by the polarization identity)

$$\leq \frac{1}{4} \|e^{i\theta} g(|C|) x_1 + f(|C|) V^* x_2 \|^2 + \frac{1}{4} \|e^{i\theta} g(|B|) x_2 + f(|B|) U^* x_1 \|^2$$

$$= \frac{1}{4} \left( \|e^{i\theta} g(|C|) f(|C|) V^* \| \left\| \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\|^2 + \frac{1}{4} \left( \|f(|B|) U^* e^{i\theta} g(|B|) \| \left\| \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\|^2 \right)$$

$$\leq \frac{1}{4} \left( \|e^{i\theta} g(|C|) f(|C|) V^* \|^2 + \frac{1}{4} \left( \|f(|B|) U^* e^{i\theta} g(|B|) \|^2 \right)$$

$$= \frac{1}{4} \left( \|e^{i\theta} g(|C|) f(|C|) V^* \| \|e^{-i\theta} g(|C|) V f(|C|) \| \right)$$

$$+ \frac{1}{4} \left( \|f(|B|) U^* e^{i\theta} g(|B|) \| \|U f(|B|) \| \right)$$

$$= \frac{1}{4} \left( g^2(|C|) + f(|C|) V^* V f(|C|) \right) + \frac{1}{4} \left( \|f(|B|) U^* U f(|B|) \| + g^2(|B|) \right)$$

$$= \frac{1}{4} \left( g^2(|C|) + g^2(|C|) \right) + \frac{1}{4} \left( f^2(|B|) + g^2(|B|) \right).$$

Taking the supremum over all unit vectors $x = (x_1, x_2)$ and utilizing Lemma 2.4(a), we get

$$w(S) \leq \frac{1}{4} \left( g^2(|C|) + g^2(|C|) \right) + \frac{1}{4} \left( f^2(|B|) + g^2(|B|) \right).$$
Therefore, since $h$ is nondecreasing and convex, by Lemma 2.5 we have
\[
h(w(S)) \leq \frac{1}{2} h\left(\left\| \frac{f^2(|C|)}{2} + g^2(|C|) \right\| \right) + \frac{1}{2} h\left(\left\| \frac{f^2(|B|)}{2} + g^2(|B|) \right\| \right)
\]
\[
\leq \frac{1}{4} \left( \left\| h\left(\frac{f^2(|C|)}{2}\right) + h\left(\frac{g^2(|C|)}{2}\right) \right\| + \frac{1}{4} \left\| h\left(\frac{f^2(|B|)}{2}\right) + h\left(\frac{g^2(|B|)}{2}\right) \right\| \right)
\]

The next corollary gives a generalization of inequality (1.2).

**Corollary 2.7.** Let $B, C \in \mathbb{B}(H)$. Then
\[(2.3) \quad w^r \left( \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right) \leq \frac{1}{4} \left\| B \right\|^{2\alpha} + |B|^{2r(1-\alpha)} \right\| + \frac{1}{4} \left\| C \right\|^{2\alpha} + |C|^{2r(1-\alpha)} \right\|,
\]
for all $\alpha \in [0, 1]$ and $r \geq 1$.

**Proof.** Inequality (2.3) follows from inequality (2.2) by putting $h(t) = t^r$, $f(t) = t^\alpha$, and $g(t) = t^{1-\alpha}$. □

**Remark 2.8.** Let $B, C \in \mathbb{B}(H)$. The lower bound
\[
w^{\frac{1}{2}}(BC) \leq w \left( \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right)
\]
was obtained in [2]. Also, in the same paper, it was shown that if $B, C \geq 0$, then
\[
\| B^{\frac{1}{2}} C^{\frac{1}{2}} \|^2 = \rho(BC) \quad (\leq w(BC)).
\]

**Corollary 2.9.** Let $B, C \in \mathbb{B}(H)$, and let $C$ be normal. Then
\[
\| B + C \|^r \leq 2^{r-2} \left( \left\| B \right\|^{2\alpha} + |B|^{2r(1-\alpha)} \right\| + \left\| C \right\|^{2\alpha} + |C|^{2r(1-\alpha)} \right\|
\]
for all $\alpha \in [0, 1]$ and $r \geq 1$.

**Proof.** We have
\[
\| B + C \|^r = \left\| \begin{bmatrix} 0 & B \\ C^* & 0 \end{bmatrix} \right\|^r = 2^r w^r \left( \begin{bmatrix} 0 & B \\ C^* & 0 \end{bmatrix} \right)
\]
\[
\leq 2^r \max_{\theta \in \mathbb{R}} \left\| \text{Re} \left( e^{i\theta} \begin{bmatrix} 0 & B \\ C^* & 0 \end{bmatrix} \right) \right\|^r = 2^r w^r \left( \begin{bmatrix} 0 & B \\ C^* & 0 \end{bmatrix} \right) \quad (\text{by Lemma 2.4(a)})
\]
\[
\leq 2^{r-2} \left( \left\| B \right\|^{2\alpha} + |B|^{2r(1-\alpha)} \right\| + \left\| C^* \right\|^{2\alpha} + |C^*|^{2r(1-\alpha)} \right\|
\]
by inequality (2.3). Since $C$ is normal, $|C| = |C^*|$, and the proof is completed. □
Theorem 2.10. Let $S = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \in \mathbb{B} (\mathcal{H} \oplus \mathcal{H})$, $r \geq 2$, and $p,q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $f_1, g_1, f_2,$ and $g_2$ are non-negative continuous functions on $[0, \infty)$ such that $f_1(t)g_1(t) = f_2(t)g_2(t) = t$ $(t \geq 0)$, then

$$w^r (S) \leq 2^{-\frac{r}{2} - 1} \max^\frac{1}{r}(\alpha, \beta) \max^\frac{1}{q}(\gamma, \delta),$$

and

$$w^r (S) \leq 2^{-\frac{r}{2} - 1} \max^\frac{1}{r}(\alpha', \beta') \max^\frac{1}{q}(\gamma', \delta'),$$

where

\[ \alpha = \| f_1^TP (|B^* - iC|) + f_2^TP (|B^* + iC|) \|, \]
\[ \beta = \| f_1^TP (|B + iC^*|) + f_2^TP (|B - iC^*|) \|, \]
\[ \gamma = \| g_1^qR (|B^* - iC|) + g_2^qR (|B^* + iC|) \|, \]
\[ \delta = \| g_1^qR (|B + iC^*|) + g_2^qR (|B - iC^*|) \|, \]
\[ \alpha' = \| f_1^TP (|B^* - iC|) + g_2^TP (|B^* + iC|) \|, \]
\[ \beta' = \| f_1^TP (|B + iC^*|) + g_2^TP (|B - iC^*|) \|, \]
\[ \gamma' = \| g_1^qR (|B^* - iC|) + f_2^qR (|B^* + iC|) \|, \]
\[ \delta' = \| g_1^qR (|B + iC^*|) + f_2^qR (|B - iC^*|) \|. \]

Proof. Assume that $S = S_1 + iS_2$ is the Cartesian decomposition of $S$, and that $x$ is any unit vector in $\mathcal{H} \oplus \mathcal{H}$. Then

\[ |\langle Sx, x \rangle|^r = |\langle (S_1 + iS_2)x, x \rangle|^r = (\langle S_1x, x \rangle^2 + \langle S_2x, x \rangle^2)^{\frac{r}{2}} \]
\[ = 2^{-\frac{r}{2}} (\langle (S_1 + S_2)x, x \rangle^2 + \langle (S_1 - S_2)x, x \rangle^2)^{\frac{r}{2}} \]
\[ \leq 2^{-\frac{r}{2}} 2^{\frac{r}{2} - 1} (\langle (S_1 + S_2)x, x \rangle^r + |\langle (S_1 - S_2)x, x \rangle|^r) \]

(by the convexity of $t^\frac{r}{2}$ for $r \geq 2$)

\[ \leq \frac{1}{2} (|\langle S_1 + S_2|x, x \rangle|^r + |\langle S_1 - S_2|x, x \rangle|^r), \]

by the convexity of $|t|$ and Lemma 2.2. A straightforward computation shows that

\[ 2 |\langle Sx, x \rangle|^r \leq \left[ \begin{array}{cc} \frac{1}{\sqrt{2}} |B^* - iC| & 0 \\ 0 & \frac{1}{\sqrt{2}} |B + iC^*| \end{array} \right] x, x \right]^r \]
\[ + \left[ \begin{array}{cc} \frac{1}{\sqrt{2}} |B^* + iC| & 0 \\ 0 & \frac{1}{\sqrt{2}} |B - iC^*| \end{array} \right] x, x \right]^r. \]
Hence,
\[
2^{r+1}|\langle Sx, x \rangle|^r \leq \left\langle \begin{bmatrix} |B^* - iC| & 0 \\ 0 & |B + iC^*| \end{bmatrix} x, x \right\rangle^r + \left\langle \begin{bmatrix} |B^* + iC| & 0 \\ 0 & |B - iC^*| \end{bmatrix} x, x \right\rangle^r
\]
\[
\leq \left\langle \begin{bmatrix} f_1^i (|B^* - iC|) & 0 \\ 0 & f_1^i (|B + iC^*|) \end{bmatrix} x, x \right\rangle^\frac{r}{2} + \left\langle \begin{bmatrix} g_1^i (|B^* - iC|) & 0 \\ 0 & g_1^i (|B + iC^*|) \end{bmatrix} x, x \right\rangle^\frac{r}{2}
\]
\[
+ \left\langle \begin{bmatrix} f_2^i (|B^* + iC|) & 0 \\ 0 & f_2^i (|B - iC^*|) \end{bmatrix} x, x \right\rangle^\frac{r}{2} + \left\langle \begin{bmatrix} g_2^i (|B^* + iC|) & 0 \\ 0 & g_2^i (|B - iC^*|) \end{bmatrix} x, x \right\rangle^\frac{r}{2}
\]
(by the mixed Cauchy–Schwarz inequality)
\[
\leq \left\langle \begin{bmatrix} f_1^p (|B^* - iC|) & 0 \\ 0 & f_1^p (|B + iC^*|) \end{bmatrix} x, x \right\rangle \left\langle \begin{bmatrix} g_1^p (|B^* - iC|) & 0 \\ 0 & g_1^p (|B + iC^*|) \end{bmatrix} x, x \right\rangle
\]
\[
+ \left\langle \begin{bmatrix} f_2^p (|B^* + iC|) & 0 \\ 0 & f_2^p (|B - iC^*|) \end{bmatrix} x, x \right\rangle \left\langle \begin{bmatrix} g_2^p (|B^* + iC|) & 0 \\ 0 & g_2^p (|B - iC^*|) \end{bmatrix} x, x \right\rangle
\]
(by Lemma 2.2)
\[
\leq \left\langle \begin{bmatrix} f_1^r (|B^* - iC|) + f_2^r (|B^* + iC|) & 0 \\ 0 & f_1^r (|B + iC^*|) + f_2^r (|B - iC^*|) \end{bmatrix} x, x \right\rangle^\frac{1}{2}
\]
\[
\times \left\langle \begin{bmatrix} g_1^r (|B^* - iC|) + g_2^r (|B^* + iC|) & 0 \\ 0 & g_1^r (|B + iC^*|) + g_2^r (|B - iC^*|) \end{bmatrix} x, x \right\rangle^\frac{1}{2}
\]
by the Hölder inequality and Lemma 2.2. Taking the supremum over all unit vectors we get inequality (2.4). Inequality (2.5) is achieved by a similar argument. □

3. Inequalities for the diagonal part

In this section we obtain some upper bounds for the numerical radius of diagonal operator matrices.

**Theorem 3.1.** Let \( T = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \in B(\mathcal{H} \oplus \mathcal{H}) \), and let \( f \) and \( g \) be non-negative continuous functions on \([0, \infty)\) such that \( f(t)g(t) = t \) \((t \geq 0)\). Then for all non-negative nondecreasing convex functions \( h \) on \([0, \infty)\), the following inequality holds:

\[
h(w(T)) \leq \frac{1}{2} \max \left( \|h(f^2(|A|))\| + h(g^2(|A|))\| , \|h(f^2(|D|))\| + h(g^2(|D|))\| \right).
\]

**Proof.** Let \( A = U|A| \) and \( D = V|D| \) be the polar decompositions of operators \( A \) and \( D \). Then

\[
T = W|T| = \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} |A| & 0 \\ 0 & |D| \end{bmatrix}
\]

Analysis Mathematica 45, 2019
is the polar decomposition of $T$. Let $x = (x_1, x_2)$ be any unit vector in $\mathcal{H} \oplus \mathcal{H}$; that is, $\|x_1\|^2 + \|x_2\|^2 = 1$. Then for all $\theta \in \mathbb{R}$, we obtain

$$\text{Re} \langle e^{i\theta} T x, x \rangle = \text{Re} \langle e^{i\theta} W |T|x, x \rangle = \text{Re} \langle e^{i\theta} W f(|T|) g(|T|) x, x \rangle$$

(by functional calculus)

$$= \text{Re} \langle e^{i\theta} g(|T|) x, f(|T|) W^* x \rangle$$

$$= \text{Re} \langle e^{i\theta} \left[ g(|A|) \begin{bmatrix} 0 & 0 \\ 0 & g(|D|) \end{bmatrix} \right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \left[ f(|A|) \begin{bmatrix} 0 & 0 \\ 0 & f(|D|) \end{bmatrix} \right] \begin{bmatrix} U^* & 0 \\ 0 & V^* \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rangle$$

$$= \text{Re} \langle e^{i\theta} (g(|A|) x_1, g(|D|) x_2), (f(|A|) U^* x_1, f(|D|) V^* x_2) \rangle$$

$$= \text{Re} \left( \langle e^{i\theta} g(|A|) x_1, f(|A|) U^* x_1 \rangle + \langle e^{i\theta} g(|D|) x_2, f(|D|) V^* x_2 \rangle \right)$$

$$= \frac{1}{4} \left( \|e^{i\theta} g(|A|) x_1 + f(|A|) U^* x_1\|^2 - \|e^{i\theta} g(|A|) x_1 - f(|A|) U^* x_1\|^2 \right)$$

$$+ \frac{1}{4} \left( \|e^{i\theta} g(|D|) x_2 + f(|D|) V^* x_2\|^2 - \|e^{i\theta} g(|D|) x_2 - f(|D|) V^* x_2\|^2 \right)$$

(by the polarization identity)

$$\leq \frac{1}{4} \|e^{i\theta} g(|A|) x_1 + f(|A|) U^* x_1\|^2 + \frac{1}{4} \|e^{i\theta} g(|D|) x_2 + f(|D|) V^* x_2\|^2$$

$$= \frac{1}{4} \left\| \left[ e^{i\theta} g(|A|) f(|A|) U^* \right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\|^2 + \frac{1}{4} \left\| \left[ e^{i\theta} g(|D|) f(|D|) V^* \right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\|^2$$

$$\leq \frac{1}{2} \left\| \left[ e^{i\theta} g(|A|) f(|A|) U^* \right]\right\| \|x_1\|^2 + \frac{1}{2} \left\| \left[ e^{i\theta} g(|D|) f(|D|) V^* \right]\right\| \|x_2\|^2.$$

Let $\alpha := \left\| \left[ e^{i\theta} g(|A|) f(|A|) U^* \right]\right\|$ and $\beta := \left\| \left[ e^{i\theta} g(|D|) f(|D|) V^* \right]\right\|$. Clearly,

$$\max_{\|x_1\|^2 + \|x_2\|^2 = 1} (\alpha^2 \|x_1\|^2 + \beta^2 \|x_2\|^2) = \max_{\theta \in [0, \frac{\pi}{2}]} (\alpha^2 \sin^2 \theta + \beta^2 \cos^2 \theta) = \max(\alpha^2, \beta^2).$$

Hence we have

$$\text{Re} \langle e^{i\theta} T x, x \rangle \leq \frac{1}{2} \max \left( \left\| \left[ e^{i\theta} g(|A|) f(|A|) U^* \right]\right\|^2, \left\| \left[ e^{i\theta} g(|D|) f(|D|) V^* \right]\right\|^2 \right)$$

$$= \frac{1}{2} \max \left( \left\| \left[ e^{i\theta} g(|A|) f(|A|) U^* \right]\left[ e^{-i\theta} g(|A|) \right] f(|A|) U f(|A|) \right\|, \left\| \left[ e^{i\theta} g(|D|) f(|D|) V^* \right]\left[ e^{-i\theta} g(|D|) \right] V f(|D|) \right\| \right)$$

$$= \frac{1}{2} \max \left( \|g^2(|A|) + f(|A|) U^* U f(|A|)\|, \|g^2(|D|) + f(|D|) V^* V f(|D|)\| \right)$$

*Analysis Mathematica* 45, 2019
\[ w(T) = \frac{1}{2} \max \left( \| f^2(|A|) + g^2(|A|) \|, \| f^2(|D|) + g^2(|D|) \| \right). \]

Taking the supremum over all unit vectors \( x = (x_1, x_2) \) and using Lemma 2.4(a) yields that

\[ w(T) \leq \frac{1}{2} \max \left( \| f^2(|A|) + g^2(|A|) \|, \| f^2(|D|) + g^2(|D|) \| \right). \]

Therefore, since \( h \) is nondecreasing, using Lemma 2.5 gives

\[ h(w(T)) \leq \max \left( h\left( \frac{1}{2} \| f^2(|A|) + g^2(|A|) \| \right), h\left( \frac{1}{2} \| f^2(|D|) + g^2(|D|) \| \right) \right) \]

\[ \leq \frac{1}{2} \max \left( \| h(f^2(|A|)) + h(g^2(|A|)) \|, \| h(f^2(|D|)) + h(g^2(|D|)) \| \right). \] □

**Corollary 3.2.** Let \( A, D \in \mathcal{B}(\mathcal{H}) \). Then we have

\[ w^r \left( \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \right) \leq \frac{1}{2} \max \left( \| |A|^{2r\alpha} + |A|^{2r(1-\alpha)} \|, \| |D|^{2r\alpha} + |D|^{2r(1-\alpha)} \| \right), \]

and, in particular,

\[ w^r(A) \leq \frac{1}{2} \| |A|^{2r\alpha} + |A|^{2r(1-\alpha)} \| \]

for all \( r \geq 1 \) and \( \alpha \in [0, 1] \).

**Proof.** Take \( h(t) = t^r \), \( f(t) = t^\alpha \), and \( g(t) = t^{1-\alpha} \) in inequality (3.1). Inequality (3.3) follows from (3.2) by putting \( A = D \) and considering Lemma 2.4(b). □

**Corollary 3.3.** Let \( A, B, C \) and \( D \in \mathcal{B}(\mathcal{H}) \). With the assumptions of Theorem 3.1, if \( Y = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \), then

\[ h\left( \frac{w(Y)}{2} \right) \leq \frac{1}{4} \max \left( \| h(f^2(|A|)) + h(g^2(|A|)) \|, \| h(f^2(|D|)) + h(g^2(|D|)) \| \right) \]

\[ + \frac{1}{8} \left( \| h(f^2(|B|)) + h(g^2(|B|)) \| + \| h(f^2(|C|)) + h(g^2(|C|)) \| \right). \]

**Proof.** Using the triangular inequality together with the nondecreasingness and the convexity of \( h \), and applying Theorems 2.6 and 3.1 give the result. □
Corollary 3.4. Let $A, B \in \mathcal{B}(\mathcal{H})$. Then
\[
\max\left(w^r(A \pm B), w^r(A \pm iB)\right) \\
\leq 2^{r-2} \left(\|A\|^{2r\alpha} + |A|^{2r(1-\alpha)}\right) + 2^{r-2} \left(\|B\|^{2r\alpha} + |B|^{2r(1-\alpha)}\right)
\]
holds for all $\alpha \in [0, 1]$ and $r \geq 1$.

Proof. Taking $Y = \begin{bmatrix} A & B \\ \pm B & A \end{bmatrix}$, $h(t) = t^r$, $f(t) = t^\alpha$, $g(t) = t^{1-\alpha}$ $(t \geq 0)$ in Corollary 3.3 we get
\[
\left(\frac{1}{2} w\left(\begin{bmatrix} A & B \\ \pm B & A \end{bmatrix}\right)\right)^r \leq \frac{1}{4} \left(\|A\|^{2r\alpha} + |A|^{2r(1-\alpha)}\right) \\
+ \frac{1}{8} \left(\|B\|^{2r\alpha} + |B|^{2r(1-\alpha)}\right) + \frac{1}{4} \left(\|B\|^{2r\alpha} + |B|^{2r(1-\alpha)}\right)
\]
Hence
\[
w^r\left(\begin{bmatrix} A & B \\ \pm B & A \end{bmatrix}\right) \leq 2^{r-2} \left(\|A\|^{2r\alpha} + |A|^{2r(1-\alpha)}\right) + 2^{r-2} \left(\|B\|^{2r\alpha} + |B|^{2r(1-\alpha)}\right).
\]
Now, applying Lemma 2.4(c) and (d) completes the proof.

The next result reads as follows.

Theorem 3.5. Let $T = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$, and let $f$ and $g$ be non-negative continuous functions on $[0, \infty)$ such that $f(t)g(t) = t$ $(t \geq 0)$. Then, for all non-negative nondecreasing convex functions $h$ on $[0, \infty)$, we have
\[
h(w(T)) \leq \frac{1}{2} \max\left(\left\|h(f^2(|A|)) + h(g^2(|A^*|))\right\|, \left\|h(f^2(|D|)) + h(g^2(|D^*|))\right\|\right).
\]

Proof. Let $x$ be any unit vector in $\mathcal{H} \oplus \mathcal{H}$. We observe that
\[
h(\langle Tx, x \rangle) \leq h(\langle f^2(|T|)x, x \rangle^{\frac{1}{2}} \langle g^2(|T^*|)x, x \rangle^{\frac{1}{2}})
\]
(by the mixed Cauchy–Schwarz inequality)
\[
\leq h\left(\frac{\langle f^2(|T|)x, x \rangle + \langle g^2(|T^*|)x, x \rangle}{2}\right) \quad \text{(by the Young inequality)}
\]
\[
\leq \frac{1}{2} \left(h(\langle f^2(|T|)x, x \rangle) + h(\langle g^2(|T^*|)x, x \rangle)\right) \quad \text{(by the convexity of $h$)}
\]
\[
\leq \frac{1}{2} \left(\langle h(f^2(|T|))x, x \rangle + \langle h(g^2(|T^*|))x, x \rangle\right) \quad \text{(by Lemma 2.2)}
\]
\[
\frac{1}{2} \begin{bmatrix} h(f^2(|A|)) + h(g^2(|A^*|)) & 0 \\ 0 & h(f^2(|D|)) + h(g^2(|D^*|)) \end{bmatrix} x, x.
\]

Taking the supremum over all unit vectors \(x\), we reach the required result.

\[\square\]

**Corollary 3.6.** Let \(A, D \in \mathbb{B}(\mathcal{H})\). Then for all \(r \geq 1\) and \(\alpha \in [0, 1]\) we have

\[
w^r \left( \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \right) \leq \frac{1}{2} \max \left( \| A^{2\alpha} + |A^*|^{2r(1-\alpha)} \|, \| D^{2\alpha} + |D^*|^{2r(1-\alpha)} \| \right).
\]

In particular (see [4, Theorem 1]),

\[
w^r(A) \leq \frac{1}{2} \| A^{2\alpha} + |A^*|^{2r(1-\alpha)} \|,
\]

and

\[
w^r(A) \leq \frac{1}{2} \| A^r + |A^*|^r \|.
\]

**Proof.** Inequality (3.6) follows from inequality (3.5) by putting \(h(t) = t^r\), \(f(t) = t^\alpha\) and \(g(t) = t^{1-\alpha}\).

**Remark 3.7.** Let \(A \in \mathbb{B}(\mathcal{H})\). For all \(r \geq 1\) and \(\alpha \in [0, 1]\), by using inequalities (3.3) and (3.7), we get

\[
w^r(A) \leq \frac{1}{2} \min \left( \| A^{2\alpha} + |A^*|^{2r(1-\alpha)} \|, \| A^{2\alpha} + |A|^{2r(1-\alpha)} \| \right).
\]

The following theorem presents a generalization of inequality (1.3) and the second inequality in (1.4).

**Theorem 3.8.** Let \(T = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \in \mathbb{B}(\mathcal{H} \oplus \mathcal{H})\) and \(f\) and \(g\) be non-negative continuous functions on \([0, \infty)\) such that \(f(t)g(t) = t\) \((t \geq 0)\). Then for all non-negative nondecreasing convex functions \(h\) on \([0, \infty)\) we have

\[
h(w^r(T)) \leq \max \left( \| \frac{1}{p} h(f^{pr}(|A|)) + \frac{1}{q} h(g^{qr}(|A^*|)) \|, \| \frac{1}{p} h(f^{pr}(|D|)) + \frac{1}{q} h(g^{qr}(|D^*|)) \| \right),
\]

where \(p, q > 1\) with \(\frac{1}{p} + \frac{1}{q} = 1\), and \(r \min(p, q) \geq 2\).

**Proof.** Without loss of generality, we can assume that \(p \geq q\). Let \(x\) be any unit vector in \(\mathcal{H} \oplus \mathcal{H}\). Then

\[
h(|\langle Tx, x \rangle|^r) \leq h(\langle f^2(|T|)x, x \rangle^\frac{r}{2} \langle g^2(|T^*|)x, x \rangle^\frac{r}{2})
\]

(by the mixed Cauchy–Schwarz inequality)
\[ \leq h \left( \frac{1}{p} \langle f^2(\|T\|)x, x \rangle^{\frac{p}{2}} + \frac{1}{q} \langle g^2(\|T^*\|)x, x \rangle^{\frac{q}{2}} \right) \text{ (by the Young inequality)} \]

\[ \leq h \left( \frac{1}{p} \langle f^{pr}(\|T\|)x, x \rangle + \frac{1}{q} \langle g^{qr}(\|T^*\|)x, x \rangle \right) \text{ (by Lemma 2.2)} \]

\[ \leq \frac{1}{p} \langle h(f^{pr}(\|T\|))x, x \rangle + \frac{1}{q} \langle h(g^{qr}(\|T^*\|))x, x \rangle \text{ (by Lemma 2.2)} \]

\[ = \left\langle \begin{bmatrix} \frac{1}{p} h(f^{pr}(\|A\|)) & \frac{1}{q} h(g^{qr}(\|A^*\|)) & 0 \\ 0 & \frac{1}{p} h(f^{pr}(\|D\|)) & \frac{1}{q} h(g^{qr}(\|D^*\|)) \end{bmatrix} x, x \right\rangle. \]

Taking the supremum over all unit vectors \( x \) gives the desired result. \( \square \)

In the next corollary, inequality (3.10) is a generalization of the second inequality in (1.4).

**Corollary 3.9.** Let \( A, D \in \mathbb{B}(\mathcal{H}) \), \( p, q > 1 \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \), and \( r \min(p, q) \geq 2 \). Then

\[ w^{2r} \left( \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \right) \leq \max \left( \left\| \frac{1}{p} |A|^{2pr\alpha} + \frac{1}{q} |A^*|^{2qr(1-\alpha)} \right\|, \left\| \frac{1}{p} |D|^{2pr\alpha} + \frac{1}{q} |D^*|^{2qr(1-\alpha)} \right\| \right), \]

and, in particular,

\[ w^{2r}(A) \leq \left\| \frac{1}{p} |A|^{2pr\alpha} + \frac{1}{q} |A^*|^{2qr(1-\alpha)} \right\| \quad (3.10) \]

for all \( \alpha \in [0, 1] \).

Note that if we take \( \alpha = \frac{1}{p} \) in Corollary 3.9, then

\[ w^{2r} \left( \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \right) \leq \max \left( \left\| \frac{1}{p} |A|^{2r} + \frac{1}{q} |A^*|^{2r} \right\|, \left\| \frac{1}{p} |D|^{2r} + \frac{1}{q} |D^*|^{2r} \right\| \right). \]

In particular (see [4, Theorem 2]),

\[ w^{2r}(A) \leq \left\| \frac{1}{p} |A|^{2r} + \frac{1}{q} |A^*|^{2r} \right\|. \]

In the next corollary, inequality (3.11) is a generalization of inequality (1.3).
Corollary 3.10. Let $A, D \in \mathbb{B}(H)$, $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, and $r \min(p, q) \geq 2$. Then

$$w^r \left( \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \right) \leq \max \left( \left\| \frac{1}{p} |A|^{pr\alpha} + \frac{1}{q} |A|^* |(1-\alpha)\right\|, \left\| \frac{1}{p} |D|^{pr\alpha} + \frac{1}{q} |D|^* |(1-\alpha)\right\| \right),$$

and, in particular (see [15, Corollary 3]),

$$w^r(A) \leq \left\| \frac{1}{p} |A|^{pr\alpha} + \frac{1}{q} |A|^* |(1-\alpha)\right\|,$$

for all $\alpha \in [0, 1]$.

Note that, if we take $\alpha = \frac{1}{p}$ in Corollary 3.10, then

$$w^r \left( \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \right) \leq \max \left( \left\| \frac{1}{p} |A|^r + \frac{1}{q} |A|^* |r\right\|, \left\| \frac{1}{p} |D|^r + \frac{1}{q} |D|^* |r\right\| \right).$$

In particular,

$$w^r(A) \leq \left\| \frac{1}{p} |A|^r + \frac{1}{q} |A|^* |r\right\|,$$

which is a generalization of inequality (3.8).

In the next theorem we give a generalization of the second inequality in (1.5).

Theorem 3.11. Let $T = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \in \mathbb{B}(H \oplus H)$, $r \geq 2$, and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $f_1$, $g_1$, $f_2$, and $g_2$ are non-negative continuous functions on $[0, \infty)$ such that $f_1(t)g_1(t) = f_2(t)g_2(t) = t$ ($t \geq 0$), then

$$w^r(T) \leq \frac{1}{2} \max_{\alpha, \beta} \left( \alpha, \beta \right) \max_{\gamma, \delta} \left( \gamma, \delta \right),$$

and

$$w^r(T) \leq \frac{1}{2} \max_{\alpha', \beta'} \left( \alpha', \beta' \right) \max_{\gamma', \delta'} \left( \gamma', \delta' \right),$$

where

$$\alpha = \left\| f_1^{rp}(|\text{Re} A + \text{Im} A|) + f_2^{rp}(|\text{Re} A - \text{Im} A|) \right\|,$$

$$\beta = \left\| f_1^{rp}(|\text{Re} D + \text{Im} D|) + f_2^{rp}(|\text{Re} D - \text{Im} D|) \right\|,$$

$$\gamma = \left\| g_1^{rq}(|\text{Re} A + \text{Im} A|) + g_2^{rq}(|\text{Re} A - \text{Im} A|) \right\|,$$

$$\delta = \left\| g_1^{rq}(|\text{Re} D + \text{Im} D|) + g_2^{rq}(|\text{Re} D - \text{Im} D|) \right\|,$$

Analysis Mathematica 45, 2019
Assume that $T = T_1 + iT_2$ is the Cartesian decomposition of $T$ and that $x$ is any unit vector in $H \oplus H$. Then

$$
\langle Tx, x \rangle^r = \langle (T_1 + iT_2)x, x \rangle^r = \langle (T_1x, x)^2 + (T_2x, x)^2 \rangle^{\frac{r}{2}} \leq 2^{-\frac{r}{2}} (\langle (T_1 + T_2)x, x \rangle^2 + \langle (T_1 - T_2)x, x \rangle^2)^{\frac{r}{2}} \leq 2^{-\frac{r}{2}} 2^{\frac{r}{2}} \langle (T_1 + T_2)x, x \rangle^{\frac{r}{2}} + \langle (T_1 - T_2)x, x \rangle^{\frac{r}{2}}
$$

(by the convexity of $t^\frac{r}{2}$ for $r \geq 2$)

$$
\leq \frac{1}{2} \left( \langle |T_1 + T_2|x, x \rangle^r + \langle |T_1 - T_2|x, x \rangle^r \right) \quad \text{(by the convexity of } |t| \text{)}
$$

$$
\leq \frac{1}{2} \left( \langle f_1^r(|T_1 + T_2|)x, x \rangle^{\frac{r}{2}} \langle g_1^r(|T_1 + T_2|)x, x \rangle^{\frac{r}{2}} \right) + \frac{1}{2} \left( \langle f_2^r(|T_1 - T_2|)x, x \rangle^{\frac{r}{2}} \langle g_2^r(|T_1 - T_2|)x, x \rangle^{\frac{r}{2}} \right)
$$

(by the mixed Cauchy–Schwarz inequality)

$$
\leq \frac{1}{2} \left( \langle f_1^r(|T_1 + T_2|)x, x \rangle \langle g_1^r(|T_1 + T_2|)x, x \rangle \right) + \frac{1}{2} \left( \langle f_2^r(|T_1 - T_2|)x, x \rangle \langle g_2^r(|T_1 - T_2|)x, x \rangle \right) \quad \text{(by Lemma 2.2)}
$$

$$
\leq \frac{1}{2} \left( \langle f_1^r(|T_1 + T_2|)x, x \rangle + \langle f_2^r(|T_1 - T_2|)x, x \rangle \right)^{\frac{1}{p}} \times \left( \langle g_1^r(|T_1 + T_2|)x, x \rangle + \langle g_2^r(|T_1 - T_2|)x, x \rangle \right)^{\frac{1}{q}},
$$

by the Hölder inequality and Lemma 2.2. Therefore,

$$
\langle Tx, x \rangle^r \leq \frac{1}{2} \left[ \begin{array}{cc} f_1^r(|\text{Re } A + \text{ Im } A|) & f_2^r(|\text{Re } A - \text{ Im } A|) \\ 0 & f_1^r(|\text{Re } D + \text{ Im } D|) + f_2^r(|\text{Re } D - \text{ Im } D|) \end{array} \right] \langle x, x \rangle^{\frac{1}{p}} \times \left[ \begin{array}{cc} g_1^r(|\text{Re } A + \text{ Im } A|) & g_2^r(|\text{Re } A - \text{ Im } A|) \\ 0 & g_1^r(|\text{Re } D + \text{ Im } D|) + g_2^r(|\text{Re } D - \text{ Im } D|) \end{array} \right] \langle x, x \rangle^{\frac{1}{q}}.
$$

\textit{Analysis Mathematica 45, 2019}
Take the supremum over all unit vectors $x$ to get inequality (3.12). Inequality (3.13) is obtained by a similar reasoning. □

**Corollary 3.12.** Let $A \in \mathbb{B}(\mathcal{H})$ with the Cartesian decomposition $A = B + iC$, and let $r \geq 2$. With the assumptions of Theorem 3.11,

$$w^r(A) \leq \frac{1}{2} \left\| f_1^{rp}(|B + C|) + f_2^{rp}(|B - C|) \right\| \frac{1}{p} \left\| g_1^{rq}(|B + C|) + g_2^{rq}(|B - C|) \right\|$$

and

$$w^r(A) \leq \frac{1}{2} \left\| f_1^{rp}(|B + C|) + f_2^{rp}(|B - C|) \right\| \frac{1}{p} \left\| g_1^{rq}(|B + C|) + g_2^{rq}(|B - C|) \right\|^\frac{1}{q},$$

where $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

**Corollary 3.13.** Let $A \in \mathbb{B}(\mathcal{H})$ with the Cartesian decomposition $A = B + iC$. Then for all $\alpha \in [0, 1]$, $r \geq 2$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$w^r(A) \leq \frac{1}{2} \left\| |B + C|^{rp\alpha} + |B - C|^{rp(1-\alpha)} \right\| \frac{1}{p} \left\| |B + C|^{rq\alpha} + |B - C|^{rq(1-\alpha)} \right\|$$

which is a generalization of the second inequality in (1.5).

**Proof.** Take $f_1(t) = f_2(t) = t^{\alpha}$ and $g_1(t) = g_2(t) = t^{1-\alpha}$ in inequality (3.14). The second inequality in (1.5) follows from inequality (3.15) by putting $p = q = 2$ and $\alpha = \frac{1}{2}$. □

**Remark 3.14.** We end our work by mentioning that all inequalities in this paper are sharp. This fact comes from the sharpness of the second inequality of (1.1). For example, if in Theorem 2.6, we take $h(t) = t$, $f(t) = g(t) = \sqrt{t}$ ($t \geq 0$) and $B = C$, then we get

$$(w(B) =) w\left( \begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix} \right) \leq \|B\|.$$  

Also, if in Theorem 2.10 we choose $f_1(t) = f_2(t) = g_1(t) = g_2(t) = \sqrt{t}$ ($t \geq 0$), $r = 2$, $p = q = 2$ and $C = B = B^*$, we obtain

$$(w^2(B) =) w^2\left( \begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix} \right) \leq \|B\|^2.$$  

The sharpness of the other inequalities is handled in the same manner.

**References**

[1] O. Axelsson, H. Lu and B. Polman, On the numerical radius of matrices and its application to iterative solution methods, Special issue: The numerical range and numerical radius, Linear Multilinear Algebra, 37 (1994), 225–238.
[2] M. Bakherad and K. Shebrawi, Some generalizations of the Aluthge transform of operators and their consequences, arXiv:1710.04893 (2017).

[3] M. Bakherad and K. Shebrawi, Upper bounds for numerical radius inequalities involving off-diagonal operator matrices, *Ann. Funct. Anal.*, 9 (2018), 297–309.

[4] M. El-Haddad and F. Kittaneh, Numerical radius inequalities for Hilbert space operators. II, *Studia Math.*, 182 (2007), 133–140.

[5] M. Hajmohamadi, R. Lashkaripour and M. Bakherad, Some generalizations of numerical radius on off-diagonal part of $2 \times 2$ operator matrices, *J. Math. Inequal.*, 12 (2018), 447–457.

[6] O. Hirzallah, F. Kittaneh and K. Shebrawi, Numerical radius inequalities for certain $2 \times 2$ operator matrices, *Integral Equations Operator Theory*, 71 (2011), 129–147.

[7] O. Hirzallah, F. Kittaneh and K. Shebrawi, Numerical radius inequalities for commutators of Hilbert space operators, *Numer. Funct. Anal. Optim.*, 32 (2011), 739–749.

[8] F. Kittaneh, Numerical radius inequalities for Hilbert space operators, *Studia Math.*, 168 (2005), 73–80.

[9] F. Kittaneh, A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix, *Studia Math.*, 158 (2003), 11–17.

[10] F. Kittaneh, Notes on some inequalities for Hilbert Space operators, *Publ. Res. Inst. Math. Sci.*, 24 (1988), 283–293.

[11] T. Kosem, Inequalities between $\|f(A + B)\|$ and $\|f(A) + f(B)\|$, *Linear Algebra Appl.*, 418 (2006), 153–160.

[12] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Classical and New Inequalities in Analysis*, Mathematics and its Applications (East European Series), vol. 61. Kluwer Academic Publishers Group (Dordrecht, 1993).

[13] G. J. Murphy, *$C^*$-Algebras and Operator Theory*, Academic Press (Boston, 1990).

[14] J. Pečarić, T. Furuta, J. Mičić Hot and Y. Seo, *Mond–Pečarić Method in Operator Inequalities, Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Monographs in inequalities, Element (Zagreb, 2005).

[15] A. Salemi and A. Sheikhhosseini, Matrix Young numerical radius inequalities, *Math. Inequal. Appl.*, 16 (2013), 783–791.

[16] T. Yamazaki, On upper and lower bounds of the numerical radius and an equality condition, *Studia Math.*, 178 (2007), 83–89.

[17] A. Zamani, Some lower bounds for the numerical radius of Hilbert space operators, *Adv. Oper. Theory*, 2 (2007), 98–107.