Analysis of instability of systems composed by dark and baryonic matter

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In this work the dynamics of self-gravitating systems composed by dark and baryonic matter was analyzed. Searching for a description of this dynamics, a system of Boltzmann equations for the two constituents and the Poisson equation for the gravitational field were employed. Through the solution of these equations the collapse criterion is determined from a dispersion relation. The collapse occurs in an unstable region where the solutions grow exponentially with time. Two cases were analyzed: (a) collisionless dark and baryonic matter and (b) collisionless baryons with self-interacting dark matter. For the former case it was shown that the unstable region becomes larger if the dispersion velocity of dark matter becomes larger than the one of the baryonic matter. For the later case it was shown that the unstable region becomes smaller by increasing the collision frequency of the self-interacting dark matter. The results obtained were also compared with the case where only the dark matter is present. The models of the present work have proven to have a higher limit instability and therefore, exhibited an advantage in the structure formation.

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I. INTRODUCTION

The problem of structure formation has been investigated for a long time, since the pioneer work of Jeans [1]. There are many models which explain the conditions for the beginning of the structure formation and when they occurred. The Jeans model describes the gravitational instability of a self-gravitating gas cloud and search for the conditions that small perturbations of a gas cloud grow exponentially, leading to the collapse of the cloud (see e.g. [2–10]). The simplest way to understand these criteria is to think in force balance arguments. The collapse occurs whenever the inwards directed gravitational force is larger than the outwards directed internal pressure of the gas. The parameters which are used to quantify this instability are the Jeans length and Jeans mass that will be discuss throughout this paper.

In this work the dark matter [11] is present together with baryonic matter. The accepted model of this unknown component is the model of cold dark matter. Such model postulates the dark matter as weakly interacting massive particles and whose typical velocity is much lower than that of the light. As a consequence, the cold dark matter is an auxiliary matter in a process of structure formation. In this scenario, the cold dark matter leads to a hierarchical formation process where small structures are formed first and massive structures later. Models with hot dark matter, on the other hand, show difficulties to explain the galaxies formation and other structures on a small scale. The higher velocities do not allow the agglomeration in the necessary scale. Thus, our model consist in cold dark matter and baryonic matter subjected to a gravitational field.

For the search of the collapse criteria for our model, we invite the system of Boltzmann equations for the constituents and the Poisson equation for the gravitational field. This set of equations leads to a dispersion relation which implies in a collapse criterion for infinity homogeneous fluid and stellar systems. Two systems are analyzed: (a) collisionless dark and baryonic matter and (b) collisionless baryons with self-interacting dark matter. Self-interacting dark matter is a subject of investigation in the literature (see e.g. [12–17]).

The aim of this work is to obtain the Jeans length and the Jeans mass for the two cases, to compare the results with the case when only one constituent is present and to discuss the role of the dispersion velocities and of the collision frequency in the case of a self-interaction dark matter.

Some works with dark and baryonic matter were analyzed in the references [3, 4, 6, 10] by taking into account the macroscopic balance equations, but to the best of our knowledge the use of the system of Boltzmann equations with and without interactions is new.

Another point of view without the inclusion of a dark matter constituent is to analyze the Jeans instability by using a f(R)- theory, which was the subject of investigation of the works [18, 19].

This paper is organized as follows. In section II, we present the set of equations, the dispersion relation and we explore the collapse limits through the Jeans mass and Jeans length for a system of collisionless baryonic and dark matter. The same analysis is done for the case of a system of collisionless baryonic matter and self-interacting dark matter in section III. The conclusions of the work are stated in section IV.

II. SYSTEMS COMPOSED OF COLLISIONLESS DARK AND BARYONIC MATTER

We start with the three equations that describe systems composed by dark and baryonic matter subjected to a gravitational field $\Phi$. Here we use the indices $d$ and $b$...
for the dark and baryonic matter, respectively. The evolution equations of the distributions functions of baryonic \( f_b \equiv f(x, v_b, t) \) and dark matter \( f_d \equiv f(x, v_d, t) \) are the collisionless Boltzmann equations (see Appendix)

\[
\frac{\partial f_b}{\partial t} + v_b \cdot \frac{\partial f_b}{\partial x} - \nabla \Phi \cdot \frac{\partial f_b}{\partial v_b} = 0,
\]

\[
\frac{\partial f_d}{\partial t} + v_d \cdot \frac{\partial f_d}{\partial x} - \nabla \Phi \cdot \frac{\partial f_d}{\partial v_d} = 0.
\]

The gravitational field must fulfill the Poisson equation:

\[
\nabla^2 \Phi = 4\pi G \left( \int f_b dv_b + \int f_d dv_d \right).
\]

where \( \rho_b \) and \( \rho_d \) are the mass densities of the baryonic and dark matter, respectively.

We consider that the equilibrium of the self-gravitating system is described by homogeneous and time-independent distribution functions \( f^0_b(v_b) \) and \( f^0_d(v_d) \) and a potential which is only a function of the space coordinates, i.e., \( \Phi_0(x) \). The equilibrium state is then subject to small perturbations represented by plane waves of frequency \( \omega \) and wavenumber vector \( k \), namely

\[
\begin{align*}
\tilde{f}_b &= f^0_b(v_b) + \int e^{i(k \cdot x - \omega t)} e^{i(k \cdot x - \omega t)} d^3v_b, \\
\tilde{f}_d &= f^0_d(v_d) + \int e^{i(k \cdot x - \omega t)} e^{i(k \cdot x - \omega t)} d^3v_d, \\
\Phi &= \Phi_0(x) + \Phi_1(x) \exp[i(k \cdot x - \omega t)],
\end{align*}
\]

where the overbarred quantities denote small amplitudes.

The equilibrium for a homogeneous system is achieved when

\[
\frac{\partial \tilde{f}_b}{\partial t} + v_b \cdot \frac{\partial \tilde{f}_b}{\partial x} - \nabla \Phi_0 \cdot \frac{\partial \tilde{f}_b}{\partial v_b} = 0,
\]

\[
\frac{\partial \tilde{f}_d}{\partial t} + v_d \cdot \frac{\partial \tilde{f}_d}{\partial x} - \nabla \Phi_0 \cdot \frac{\partial \tilde{f}_d}{\partial v_d} = 0.
\]

Here the modulus of the wavenumber vector was denoted by \( |k| \).

By eliminating the overbarred quantities from the system of equations (11) – (13) and linearize the resulting equations, we get

\[
\begin{align*}
- i\omega \tilde{J}_b + v_b \cdot (i k \tilde{F}_b) - (i k \tilde{F}_1) \cdot \frac{\partial f^0_b}{\partial v_b} &= 0, \\
- i\omega \tilde{J}_d + v_d \cdot (i k \tilde{F}_d) - (i k \tilde{F}_1) \cdot \frac{\partial f^0_d}{\partial v_d} &= 0,
\end{align*}
\]

\[
-k^2 \tilde{F}_1 = 4\pi G \left( \int \tilde{J}_b dv_b + \int \tilde{J}_d dv_d \right).
\]

where \( \rho^0_b \), \( \rho^0_d \) are constant mass densities of baryons and dark matter, respectively, and \( \sigma_b \) and \( \sigma_d \) their dispersion velocities.

Without loss of generality we can choose \( k = (k, 0, 0) \) so that the dispersion relation (11) together with (11) and (12) can be integrated with respect to the velocity components \( v_b \) and \( v_d \), yielding

\[
k^2 = 4\pi G \frac{2}{\sqrt{\pi}} \left[ \frac{\rho^0_b}{\sigma^2_b} \int_0^\infty \frac{x^2 e^{-x^2}}{x^2 - w^2/(2\sigma^2_b k^2)} dx + \frac{\rho^0_d}{\sigma^2_d} \int_0^\infty \frac{y^2 e^{-y^2}}{y^2 - w^2/(2\sigma^2_d k^2)} dy \right].
\]

Above, we have introduced new integration variables

\[
x = \frac{v_{b\perp}}{\sqrt{2} \sigma_b}, \quad y = \frac{v_{d\perp}}{\sqrt{2} \sigma_d}.
\]

Unstable solutions are such that \( \Re(\omega) = 0 \) and \( \omega_I = \Im(\omega) > 0 \), since in this case the solutions grow exponentially with time. When \( \omega = i\omega_I \) the integrals on the right-hand side of (13) can be evaluated by using the following relationship (see [21], eq. 3.466)

\[
\int_0^\infty \frac{x^2 e^{-x^2}}{x^2 - \beta^2 dx} = \frac{\sqrt{\pi}}{2\mu} \frac{\pi \beta}{2} - \frac{\pi \beta^2}{2} \exp(\beta \mu),
\]

where \( \text{erfc} \) is the complementary error function. By taking into account (13) the dispersion relation (13) with \( \omega = i\omega_I \) reduces to

\[
k^2 = 1 - \sqrt{\frac{\pi}{2 \kappa_s}} \exp \left( \frac{\omega_I^2}{2 \kappa_s^2} \right) \text{erfc} \left( \frac{\omega_I}{\sqrt{2 \kappa_s}} \right) + \frac{\rho^0_d}{\rho^0_b} \left[ \frac{\sigma^2_d}{\sigma^2_b} \text{erfc} \left( \frac{\sigma_d}{\sigma_b} \sqrt{2 \kappa_s} \right) \right].
\]

In the above equation we have introduced the dimensionless wavenumber and frequency defined by

\[
k_s = \frac{\sigma_d k}{\sqrt{4\pi G \rho^0_d}}, \quad \omega_s = \frac{\omega_I}{\sqrt{4\pi G \rho^0_d}}.
\]

Note that we have used the mass density \( \rho_d \) and the dispersion velocity \( \sigma_d \) of the dark matter in order to construct the dimensionless wavenumber and frequency, since the dark matter plays an important role in the cosmic structure formation.

From (16) we may determine the instability regions in the plane \((\omega_s, k_s)\), where disturbances grow with time. For that end we have to specify: (a) the mass densities ratio \( \rho^0_d/\rho^0_b \) and (b) the dispersion velocities ratio \( \sigma_d/\sigma_b \). The mass densities ratio can be taken as the ratio of the densities parameters \( \Omega_1/\Omega_b \) today which is approximately 5.5 (see the section 22 of [21]), i.e., \( \rho^0_d/\rho^0_b \approx 5.5 \). This ratio has not modified too much during the evolution of the Universe. For the dispersion velocities ratio we take the values given in the work [22] on the Milky-Way-like galaxy simulations including both dark matter and baryons, namely, \( \sigma_d/\sigma_b = 170/93 \approx 1.83 \).
FIG. 1. Dimensionless wavenumber versus dimensionless frequency for a model with baryonic and dark matter. Unstable solutions $\omega^2 > 0$ and stable solutions $\omega^2 < 0$.

In figures 1 and 2 the dimensionless wavenumber $k_*$ versus the dimensionless frequency $\omega_*$ are plotted by considering the dispersion relation (16). Figure 1 refers to the case where baryonic and dark matter are taken into account, while figure 2 refers only to the dark matter by requiring that the baryonic matter is absent ($\rho_0^b = 0$). By the comparison of the two figures, we infer that the model with baryonic and dark matter has a larger region of unstable solutions than the one with only the dark matter.

By considering $\omega_* = 0$ in the dispersion relation (16), we obtain that the dimensionless wavenumber is equal to $k_* = 1.2694$. This value can be interpreted as the ratio of the Jeans wavenumbers one referring to the dark matter-baryons system $k_{db}^J$ and the other only to dark matter $k_{d}^J = \sqrt{4\pi G \rho_0^d/\sigma_d^2}$. Hence, the ratio of the Jeans lengths $\lambda_{db}^J = 2\pi/k_{db}^J$ and $\lambda_{d}^J = 2\pi/k_{d}^J$ is $\lambda_{db}^J/\lambda_{d}^J = 0.7877$, showing that the Jeans length of the system dark matter-baryons is smaller than the one with only the dark matter. This reflects also in the value of the Jeans mass, which is defined as the mass contained in a sphere of diameter equal to the Jeans length. The ratio of the Jeans masses of the systems dark matter-baryons $M_{db}^J$ and only dark matter $M_d^J$ is

$$
\frac{M_{db}^J}{M_d^J} = \frac{\rho_0^d + \rho_0^b}{\rho_0^d} \left(\frac{\lambda_{db}^J}{\lambda_{d}^J}\right)^3
= \frac{\rho_0^d + \rho_0^b}{\rho_0^d} \sqrt{\left(\frac{\rho_0^d \sigma_b^2 + \rho_0^b \sigma_d^2}{\rho_0^d \sigma_b^2 + \rho_0^b \sigma_d^2}\right)}^3
= 0.5791. \quad (18)
$$

This shows that in the model where baryonic and dark matter are present, the structures began to form earlier, at the time that this dark component dominated, which reinforces the fact that a smaller Jeans mass was required to initiate the collapse.

FIG. 2. Dimensionless wavenumber versus dimensionless frequency for a model with only dark matter. Unstable solutions $\omega^2 > 0$ and stable solutions $\omega^2 < 0$.

FIG. 3. Dimensionless wavenumber versus dimensionless frequency showing the regions of unstable $\omega^2 > 0$ and stable solutions $\omega^2 < 0$. Models with baryonic and dark matter with: (a) $\sigma_d/\sigma_b = 1.83$ thick line; (b) $\sigma_d/\sigma_b = 1.20$ dotted line; (c) $\sigma_d/\sigma_b = 2.2$ dot-dashed line. Model with only dark matter dashed line.

We may ask about the role of the dispersion velocity ratio on the structure formation. This can be seen from Fig. 3 where three different values for $\sigma_d/\sigma_b$ were considered. From this figure we may infer that by increasing the dispersion velocity ratio the unstable region
also increases. As a consequence the ratios of the Jeans masses become: (a) $M_{Jb}^{db}/M_{Jb}^d = 0.8338$ for $\sigma_d/\sigma_b = 1.20$ and (b) $M_{Jb}^{db}/M_{Jb}^d = 0.4585$ for $\sigma_d/\sigma_b = 2.2$. Hence, the increase of the dispersion velocity ratio implies that a smaller mass is needed to begin the collapse.

### III. SYSTEM COMPOSED OF COLLISIONLESS BARYONIC AND SELF-INTERACTING DARK MATTER

In this section we shall analyze the same system consisted of baryonic and dark matter, but we shall consider that the baryons are collisionless while the particles of the dark matter are self-interacting. Within the frame-

work of Boltzmann equations the evolution equations of the one-particle distribution functions for the collisionless baryons $f_b$ and self-interacting dark matter $f_d$ read (see the Appendix)

$$\frac{\partial f_b}{\partial t} + v_b \cdot \frac{\partial f_b}{\partial x} - \nabla \Phi \cdot \frac{\partial f_b}{\partial \nabla b} = 0, \quad (19)$$

$$\frac{\partial f_d}{\partial t} + v_d \cdot \frac{\partial f_d}{\partial x} - \nabla \Phi \cdot \frac{\partial f_d}{\partial \nabla d} = -\nu \left(f_d - f_d^0\right). \quad (20)$$

In (20) $\nu$ is a frequency of order of the collision frequency.

As in the previous section the system of equations are the Boltzmann equations (19) and (20) and the Poisson equation (3). By following the same methodology equations (13) and (16), here become

$$k^2 = 4\pi G \frac{2}{\sqrt{\pi}} \left[ \frac{\rho_b^0}{\sigma_b^2} \int_0^\infty \frac{x^2 e^{-x^2}}{\sqrt{2\pi \sigma_b^2 k_s^2}} dx + \frac{\rho_d^0}{\sigma_d^2} \int_0^\infty \frac{y^2 e^{-y^2}}{\sqrt{2\pi \sigma_d^2 k_s^2}} dy \right], \quad (21)$$

and

$$\nu_s = \frac{\nu}{\sqrt{4\pi G \rho_b^0}}. \quad (23)$$

In Fig. 4 it is plotted – by taking into account (22) – the dimensionless wavenumber $k_s$ versus the dimensionless frequency $\nu_s$ for different systems: collisionless dark matter-baryons, collisionless baryons with self-interacting dark matter and collisionless dark matter.

We infer from Fig. 4 that the model of self-interacting dark matter with collisionless baryons has a smaller unstable region than the collisionless dark matter-baryons one. Furthermore, by increasing the dimensionless collision frequency the unstable region decreases. The Jeans mass ratios for the two cases plotted in Fig. 4 – with $\sigma_d/\sigma_b = 1.83$ – are: (a) $M_{Jb}^{db}/M_{Jb}^d = 0.9255$ for $\nu_s = 0.5$ and (b) $M_{Jb}^{db}/M_{Jb}^d = 0.8242$ for $\nu_s = 0.2$. Here the increase of the dimensionless collision frequency implicates that a larger mass is necessary in order to start the collapse.

### IV. CONCLUSIONS

In this work, we analyzed the dynamics and the collapse of collisionless self-gravitating system composed by dark and baryonic matter. This system is described by two Boltzmann equations, one for each constituent, and the Poisson equation for the gravitational field. The dispersion relation for this model was found by linearizing the equations about the equilibrium of the self-gravitating system, which is described by homogeneous

and time-independent Maxwellian distribution functions and a potential which by obeys Jeans “swindle”. The dispersion relation allows to obtain the collapse criterion of interstellar gas clouds composed by the mentioned constituents, resulting in the formation of structures.

The Jeans mass always has lower values in the models where the baryonic and dark matter are present. This phenomena occurs because the dark matter agglomerates more easily and the baryonic matter passes to aggregate due to the attraction of the gravitational field generated from this initial agglomerate. This result demonstrates that the structures began to form earlier when the dark matter dominated. We also conclude that when the dispersion velocity of the baryonic particles is relatively smaller than the one of the dark matter, they easily aggregate, since they hardly overcome the escape velocity of a given gravitational field. This behavior can be observed through the Fig. 5 when $\sigma_d/\sigma_b$ increases, the unstable region – where the solutions grow exponentially – increases also. Furthermore, for systems of collisionless baryons and self-interacting dark matter, the unstable region decreases by increasing the collision frequency according to Fig. 4.

Hence, the model with self-interaction dark matter and collisionless baryonic matter has always larger Jeans mass in comparison with the model of the collisionless dark and baryonic matter.

To sum up, this model with baryonic and dark matter proved to have a higher limit of instability and therefore, exhibited an advantage in the structure formation.
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APPENDIX: THE BOLTZMANN EQUATION

A state of a non-relativistic gas mixture of \( r \) constituents is characterized by the set of one-particle distribution functions \( f_a(x, v_a, t), (a = 1, \ldots, r) \) in the phase space spanned by \((x, v_a)\), such that \( f(x, v_a, t)\, d^3x\, d^3v_a \) gives the total mass of particles of constituent \( a \) in the volume element \( d^3x \) about the position \( x \) and with velocity in the range \( d^3v_a \) about \( v_a \). The space-time evolution of the one particle distribution function \( f_a \) in the phase space is governed by the Boltzmann equation, which in the presence of a gravitational potential \( \Phi \) and in the absence of collisions between the particles, reads (see e.g. \[23\])

\[
\frac{\partial f_a}{\partial t} + v_a \cdot \nabla f_a - \nabla \Phi \cdot \frac{\partial f_a}{\partial v_a} = 0, \quad a = 1, \ldots, r, \quad (24)
\]

where the force which acts on a particle of constituent \( a \) is only of gravitational nature \( F = -\nabla \Phi \).

The above equation follows also from a relativistic Boltzmann equation in the presence of gravitational fields, where a one-particle distribution function \( f_a \equiv f(x, p_a, t) \) of constituent \( a \) in the phase space spanned by \((x, p_a)\) satisfies the Boltzmann equation (see e.g. \[24\])

\[
p_a^\mu \frac{\partial f_a}{\partial x^\mu} - \Gamma_{\mu \nu}^i p_a^\mu p_a^\nu \frac{\partial f_a}{\partial p_a^\nu} = 0, \quad a = 1, \ldots, r. \quad (25)
\]

Here the mass-shell condition \( p_a^\mu p_a_\mu = m_a^2 c^2 \) - where \( m_a \) is the particle rest mass of constituent \( a \) - was taken into account. In the non-relativistic Newtonian limiting case \( p_a^0 \to m_a c, \ p_a \to m_a v_a \) and the Christoffel symbol \( \Gamma_{00}^i \to \nabla \Phi/c^2 \) so that \[24\] follows.

The equilibrium distribution function is a Maxwellian distribution of the velocities

\[
f_a^0(x, v_a, t) = \frac{\rho_a}{(2\pi \sigma_a^2)^{3/2}} e^{-v_a^2/2\sigma_a^2}, \quad (26)
\]

where \( \sigma_a = \sqrt{kT_a/\pi m_a} \) - with \( k \) denoting the Boltzmann constant and \( T_a \) the temperature of the constituent \( a \) - is the dispersion velocity and \( \rho_a \) the mass density of constituent \( a \), which is defined in terms of the one-particle distribution function by

\[
\rho_a = \int f_a d^3v_a. \quad (27)
\]

If the binary collisions between the constituents are included, we have to introduce the collision terms, which are integrals of products of two distribution functions. Hence, in the case of mixtures the Boltzmann equations become a system of integro-differential equations for the distribution functions \( f_a \). In several problems in kinetic theory of gases the collision term of the Boltzmann equation is replaced by a more simple model and the model most used in the literature is the so-called BGK-model (see e.g. \[23\]). The BGK collision model fulfills the same properties of the collision operator of the Boltzmann equation: it leads to the same conservation equations of mass, momentum and energy densities and the H-theorem holds.

In this work we are interested in the case where the baryons are treated as non-interacting particles, but we assume that the dark matter could be described by a system of self-interacting particles. In this case, the BGK-model of the Boltzmann equation for the dark matter reads

\[
\frac{\partial f_d}{\partial t} + v_d \cdot \nabla f_d - \nabla \Phi \cdot \frac{\partial f_d}{\partial v_d} = -\nu (f_d - f_d^0), \quad (28)
\]

where \( f_d^0 \) denotes a Maxwellian distribution function and \( \nu \) a frequency of order of the collision frequency.

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