Stored energies in electric and magnetic current densities for small antennas

B. L. G. Jonsson and Mats Gustafsson

1 KTH Royal Institute of Technology, School of Electric Engineering, Teknikringen 33, 100 44 Stockholm, Sweden
2 Department of Electrical and Information Technology, Lund University, Box 118, 221 00 Lund, Sweden

Electric and magnetic currents are essential to describe electromagnetic-stored energy, and the associated antenna Q and the partial directivity to antenna Q-ratio, $D/Q$, for arbitrarily shaped structures. The upper bound of previous $D/Q$ results for antennas modelled by electric currents is accurate enough to be predictive. This motivates us to develop the analysis required to determine upper and/or lower bounds for electromagnetic problems that include magnetic model currents. Here we derive new expressions for the stored energies, which are used to determine antenna Q bounds and $D/Q$ bounds for the combination of electric and magnetic currents, in the limit of electrically small antennas. In this investigation, we show both new analytical results and we illustrate numerical realizations of them.

We show that the lower bound of antenna Q is inversely proportional to the largest eigenvalue of certain combinations of the electric and magnetic polarizability tensors. These results are an extension of the electric only currents, which come as a special case. The proposed method to determine the minimum Q-value which is based on the new expressions for the stored energies, also yields a family of current-density minimizers for optimal electric and magnetic currents that can lend insight into antenna designs.

1. Introduction

Time harmonic electromagnetic radiating systems do not in general have a finite total energy associated with them. This is well known since the radiated electric and magnetic fields decay as $r^{-1}$ and the corresponding energy density hence decay as $r^{-2}$, which is not an
integrable quantity for exterior unbounded regions. This non-integrability differs from the
singularities of the electromagnetic energy for charged particles (e.g. [1,2]), where the challenge
is the finite mass of particles in coupling Maxwell’s equation to the dynamics of the
charged particles.

To consistently extract a finite stored energy from the energy densities associated with classical
time-harmonic energy has been investigated in [3–9]. These stored energies have been based
on spherical (and spheroidal) modes, circuit equivalents and on the input impedance for small
antennas. In 2010, Vandenbosch [10] proposed a current-density approach to stored energies
also applicable to larger antennas. This approach has generated new interest in electromagnetic
stored energy that is explored in [11–16]. This ‘stored energy’ is similar to the results of Collin
& Rothschild [5], and it also has similarities with the stored energies proposed in [17,18]. The
generalization of stored energy in [19] and in this paper allows for the first time the analysis of
both electric and magnetic current densities for arbitrary shapes. Antennas embedded in lossy or
dispersive material have been considered in [20–22].

The drive to find a well-defined stored energy stems partly from that it is closely related to
the antenna quality factor Q. Lower bounds on antenna Q are directly related to the electric
size of the antenna, and indirectly to the maximal matching bandwidth that can be obtained.
The relation between antenna Q and bandwidth is not trivial; for a discussion and examples, see
[9,14,23,24]. An alternative method to derive bandwidth bounds is sum-rules (e.g. [9,25–29]). The
approach given here is related to [9,12,14,19,27]. In this paper, we mathematically derive a direct
relation between antenna Q (also called the Q-factor) and the electric and magnetic polarizabilities
[30–32] as the leading order term for electrically small structures, using the here given expressions
of the stored energies. Realizations of antennas that use both electric and magnetic currents are
investigated in [33,34].

Our investigation is based on the asymptotic behaviour of stored energy in the electrically
small case for both electric and magnetic currents. This result is an extension of the stored energies
in [10] and connect to both antenna Q and partial directivity over antenna Q. That scattering
properties are related to the polarizabilities is known (e.g. [35,36]), but that polarizability tensors
appear directly as the essential factor in antenna Q estimates is a recent result [19,27,37,38].

The current-representation approach to stored energy enables the maximal partial directivity
over antenna Q problem to be reduced to a convex optimization problem [15]. It also enables
us to consider fundamental limitations for arbitrary geometries. Convex optimization problems
are efficiently solvable [39]. From a user perspective, it can be compared with solving a matrix
equation. To numerically find the physical bounds on antenna Q or partial directivity over
antenna Q, \( D/Q \) is here reduced to tractable problems, solvable with common electromagnetic
tools. In this paper, we illustrate how this can be applied to a range of shapes, both numerically
and analytically. The here considered minimization problems investigate how different current
and charge density combinations yield different lower bounds on antenna Q. For the electrical
dipole problem, we show that the minimizing currents result in Q and \( D/Q \) that agree
with [8,27,37]. For the case of a generalized electric dipole with both electric charges and magnetic
current densities as sources our result agrees with the sphere in [3]. When we allow dual-modes,
i.e. both electrically and magnetically radiation dipoles, we find that the result agree with [19,38].
The mathematical framework provided here for the first time easily account for all these different
cases with a generic approach. Another result of the here derived method is that we can show
that small antennas for a given shape have a family of current densities that realize the associated
optimal antenna Q [12].

This paper is based on the here given stored energies for both electric and magnetic current
densities and formulate upper and lower bounds for \( D/Q \) and Q, respectively, as mathematical
optimization problems, which we solve. Another approach to these energies and associated
bounds are given in [19] and provide a more heuristic approach. We investigate the small
antenna limit and illustrate how antenna Q and related optimization problems behave for electric
and magnetic currents for a range of antenna shapes. These results are based on the leading
order terms of the stored energies as the electric size of the domain approach zero. One of the
advantages here is that the bounds on \( Q \) and \( D/Q \) are known once the polarizability tensors are determined for a given shape. We use this knowledge to sweep shape parameters to illustrate how \( Q \) and \( D/Q \) depend on the shape of the antenna. Analytical expressions for the electrically small case provide physical insight into limiting factors for \( Q \) and \( D/Q \). These more general results are shown to reduce to the analytically known cases in \([19,37,40]\).

In §2, we recall the definitions of key antenna and energy quantities. Using an asymptotic expansion of the electric and magnetic currents in §3, we give the explicit leading order current-density representation of the radiated power, stored energies and the radiation intensities. Analytical and numerical examples for antenna \( Q \) and \( D/Q \) under different constraints are given in §4. In §5, we formulate the problem as a convex optimization problem and determine \( Q \) for some shapes. The conclusion and appendix A end the paper.

### 2. Antenna \( Q \) and partial directivity

Let \( V \subset \mathbb{R}^3 \) be the joint bound support of the electric and magnetic current densities \( I_e \) and \( I_m \), respectively (figure 1). The support \( V \) is here assumed to be bounded and connected. Through the continuity equations, we define the associated electric and magnetic charge densities \( \rho_e \) and \( \rho_m \). The time-harmonic Maxwell’s equations with electric and magnetic current densities in free-space take the form

\[
\nabla \times E + j\eta \omega H = -j I_m, \quad \nabla \cdot E = \frac{\rho_e}{\varepsilon} = -\frac{\eta}{jk} \nabla \cdot I_e
\]

and

\[
\nabla \times H - jk \frac{\eta}{\varepsilon} E = I_e, \quad \nabla \cdot H = \frac{\rho_m}{\mu} = -\frac{1}{j k \eta} \nabla \cdot I_m,
\]

where we use the time convention \( e^{j\omega t} \), which is suppressed. In this paper, we let \( \varepsilon = \varepsilon_0, \mu = \mu_0 \) and \( \eta = \eta_0 = \sqrt{\mu/\varepsilon} \) be the free space permittivity, permeability and impedance, respectively. \( E \) is the electric field and \( H \) is the magnetic field. The dispersion relation between the wavenumber, \( k \), and the angular frequency, \( \omega \), is \( k = \omega \sqrt{\varepsilon \mu} \) and \( t \) is time.

The field energy densities are \( \varepsilon |E|^2/4 \) and \( \mu |H|^2/4 \). Here we are interested in stored electric \( W_e \) and magnetic \( W_m \) energies, which are more challenging to define. We define stored electric and magnetic energies as

\[
W_e = \frac{\varepsilon}{4} \int_{\mathbb{R}^3} |E(r)|^2 \frac{dV}{r^2} \quad \text{and} \quad W_m = \frac{\mu}{4} \int_{\mathbb{R}^3} |H(r)|^2 \frac{dV}{r^2},
\]

where \( F_E, F_H \) are the far-fields, i.e. \( E \to F_E(e^{-jkr}/r) \) as \( r \to \infty \) and \( \eta F_H = \hat{r} \times F_E \). Let \( r \) denote a vector in \( \mathbb{R}^3 \), with length \( r = |r| \) and corresponding unit vector \( \hat{r} = r/r \). Here \( \int_{\mathbb{R}^3} \) is an abbreviation of the limit \( \lim_{r_0 \to \infty} \int_{|r| < r_0} \). Note that expressions (2.3) can for certain antennas become coordinate dependent, and for large structures (2.3) may become negative \([12]\), these artefacts do not appear in the small electrical limit, as shown later in this paper as all obtained minimal antenna \( Q \) are non-negative, see e.g. §4. A similar coordinate dependence were also observed in \([9]\). The choice of (2.3) as compared to far-field subtraction outside a given volume as in e.g. \([3]\) is that (2.3) is well defined independently of the shape of the antenna. Concepts like ‘inside’ and ‘outside’ for e.g. a spherical cap are not well defined, and the heuristics that energy should vanish ‘inside’ the volume become ambiguous, a numerical example of the spherical cap illustrate this in §5. The definition in (2.3) is similar to \([5,9,10,14,17,24,41]\).

Given these stored energies, we define the two main antenna parameters that appear in the physical bounds. The antenna quality factor \( Q = \max(Q_e, Q_m, 0) \), where

\[
Q_e = \frac{2\omega W_e}{P_{\text{rad}}} \quad \text{and} \quad Q_m = \frac{2\omega W_m}{P_{\text{rad}}},
\]

and
Here, $P_{rad}$ is the radiated power of the system described by (2.1) and (2.2). Defined as

$$P_{rad} = \frac{1}{2\eta} \int_\Omega |F_E(\hat{r})|^2 d\Omega,$$

(2.5)

where $\Omega$ is the unit sphere in $\mathbb{R}^3$.

The partial directivity $D(\hat{k}, \hat{e})$ in the direction $\hat{k}$ from an antenna with polarization $\hat{e}$ is [42]

$$D(\hat{k}, \hat{e}) = 4\pi \frac{P(\hat{k}, \hat{e})}{P_{rad}},$$

(2.6)

where $P(\hat{k}, \hat{e})$ is the partial radiation intensity $|\hat{e}^* \cdot F_E|^2/(2\eta)$. The other main antenna parameter here is the partial directivity over antenna $Q$, $D/Q$, which with the above notation is

$$\frac{D(\hat{k}, \hat{e})}{Q} = \frac{2\pi P(\hat{k}, \hat{e})}{\omega \max(W_e, W_m, 0)},$$

(2.7)

The goal in this paper is to optimize and investigate $Q$ and $D/Q$ in terms of the electric and magnetic current densities, in the small antenna limit. We hence express these quantities in terms of the current densities. The stored energies are given in appendix A as a quadratic form in terms of the electric and magnetic current densities. While these calculations are straightforward, they are also rather lengthy, see e.g. [10] for a similar effort, see also [38,43]. The stored energies are

$$W_e = \frac{\mu}{4k} \text{Im} \left[ \langle J_e, L_e J_e \rangle + \frac{1}{\eta^2} \langle J_m, L_m J_m \rangle \right] + W_{e,corr}$$

(2.8)

and

$$W_m = \frac{\mu}{4k} \text{Im} \left[ \langle J_e, L_m J_e \rangle + \frac{1}{\eta^2} \langle J_m, L_e J_m \rangle \right] + W_{m,corr}.$$  

(2.9)

The correction terms $W_{e,corr}$ and $W_{m,corr}$ are explicitly given in appendix A, where they also are shown to be perturbation terms for the small antenna case. Note that $W_e - W_{e,corr}$ and $W_m - W_{m,corr}$ are symmetric in the current densities and a natural extension of the electric only current case, $J_m = 0$. This current-density explicit representation of the stored energies given in (2.8) and (2.9) is new, and, as illustrated below, useful to increase our understanding of stored energies as exemplified in the below obtained bounds on $Q$ and $D/Q$. The associated operators in (2.8) and (2.9) are

$$\langle J, L_e J \rangle = -\frac{1}{jk} \int_V \int_V \nabla_1 \cdot J(r_1) \nabla_2 \cdot J^*(r_2) G(r_1 - r_2) dV_1 dV_2$$

(2.10)

and

$$\langle J, L_m J \rangle = jk \int_V \int_V J(r_1) \cdot J^*(r_2) G(r_1 - r_2) dV_1 dV_2.$$  

(2.11)

The operators are similar to the electric field integral equation (EFIE) operators $L = L_e - L_m$, when the currents are on a surface of an object, and for such currents there is a range of
implementations in the standard method-of-moment codes. Here and below we occasionally use the notation ‘current’, in place of ‘current density’, to shorten the notation. The kernel $G(r)$ is Green’s function, $e^{-ikr}/(4\pi r)$ and $*$ indicates the complex conjugate, see also figure 1.

The radiation intensity, $P(\hat{k})$ in the direction $\hat{k}$, have a representation in terms of the current densities [38]

$$P(\hat{k}) = \frac{\eta k^2}{32\pi^2} \left[ \int_V \left( \hat{e}^* \cdot J_e(r_1) + \frac{1}{\eta} \hat{k} \times \hat{e}^* \cdot J_m(r_1) \right) e^{ik \hat{k} \cdot r_1} dV_1 \right]^2$$

$$+ \left[ \int_V \left( \hat{h}^* \cdot J_e(r_1) + \frac{1}{\eta} \hat{k} \times \hat{h}^* \cdot J_m(r_1) \right) e^{ik \hat{k} \cdot r_1} dV_1 \right]^2 = P(\hat{k}, \hat{e}) + P(\hat{k}, \hat{h}), \quad (2.12)$$

where we use that $\hat{k}, \hat{e}, \hat{h}$ is an orthogonal triplet with $\hat{k} \times \hat{e} = \hat{h}$. We recognize the partial radiation intensity $P(\hat{k}, \hat{e})$ for the polarization $\hat{e}$. For electric currents only, i.e. $J_m = 0$, these expression agree with e.g. [10,12].

To find the total radiated power $P_{\text{rad}}$, in terms of its current-density representation, we can integrate (2.12) over the unit sphere. A more direct route to $P_{\text{rad}}$ is based on (2.5) and the observation that the electric far-field, $F_E$, have the representation

$$F_E(\hat{r}) = \frac{\eta k}{4\pi} \hat{r} \times \int_V \left( \hat{r} \times J_e(r_1) + \frac{1}{\eta} J_m(r_1) \right) e^{ik \hat{r} \cdot r_1} dV_1. \quad (2.13)$$

Somewhat lengthy calculations [38,43] show that the corresponding quadratic form in terms of the currents are

$$P_{\text{rad}} = \frac{\eta}{2} \text{Re}(J_e, \mathcal{L}J_e) + \frac{1}{2\eta} \text{Re}(J_m, \mathcal{L}J_m) - \text{Im}(J_e, \mathcal{K}_1 J_m), \quad (2.14)$$

where $\mathcal{K}_1$ is the operator defined by

$$\langle J_e, \mathcal{K}_1 J_m \rangle = \frac{k^2}{4\pi} \int V J_e^*(r_1) \cdot \hat{R} \times J_m(r_2) j_1(kR) dV_1 dV_2. \quad (2.15)$$

Here $R = r_1 - r_2$, $R = |R|$, $\hat{R} = R/R$ and $j_n(x)$ is the spherical Bessel function of order $n$ [44].

3. Electrically small volume approximation

The assumption of that the volume of the antenna is electrical small simplify the above energy and power-related expressions $W_e, W_m, P(\hat{k})$ and $P_{\text{rad}}$ and subsequently $Q$ and $D/Q$. An object is electrically small if its wavelength normalized size is small enough. A common method to quantify this is to enclose the antenna volume in a sphere of radius $a$ and let the object be electrically small if $ka < 1$. The physical size, here characterized by $a$, appears implicitly in the stored energies, as a consequence we adopt the ordo-notation $O(k)$ to indicate that $ka$ is bounded by $C a$, as $ka \to 0$, for some finite constant $C$. This abuse of the notation conforms to fact that for a given fixed $a$, we have that $k \to 0$ implies that $ka \to 0$.

To expand the above quantities like the stored energies in terms of small $ka$, we assume that the currents have the asymptotic behaviour

$$J_e = J_e^{(0)} + k j_e^{(1)} + O(k^2) \quad \text{with} \quad \nabla \cdot J_e^{(0)} = 0$$

$$J_m = J_m^{(0)} + k j_m^{(1)} + O(k^2) \quad \text{with} \quad \nabla \cdot J_m^{(0)} = 0. \quad (3.1)$$

This assumption is consistent with the continuity equations for the electric and magnetic current densities. Note that $j_e^{(0)}, j_m^{(0)}, j_e^{(1)}$ and $j_m^{(1)}$ are all $k$-independent and the two latter correspond to a lowest order static charge through the continuity equation. The expansion of the stored energies are given in appendix A.
We apply the small $ka$ approximation, and (3.1) and (3.2) to the partial radiation intensity and the far-field $F_E$ in the form of (2.13). We first note that
\[
\int_V J(r) e^{jkr} \, dV = \int_V J^{(0)}(r) + k f^{(1)}(r) + jk (\hat{k} \cdot r) f^{(0)}(r) + \mathcal{O}(k^2) \, dV \\
= -jk \int_V j f^{(1)}(r) + \frac{1}{2} \hat{k} \times (r \times f^{(0)}(r)) \, dV + \mathcal{O}(k^2),
\]
where we have used that $[45, p. 432]$
\[
\int_V J^{(n)}(r) \, dV = \begin{cases} 0, & n = 0, \\ -\int_V r \nabla \cdot J^{(n)}(r) \, dV, & n \neq 0, \end{cases}
\]
and $[45 (p. 433), 46 (p. 127)]$
\[
\int_V (\hat{k} \cdot r) f^{(0)}_{e,m}(r) \, dV = \frac{1}{2} \hat{k} \times \int_V r \times f^{(0)}_{e,m}(r) \, dV,
\]
since $\nabla \cdot J^{(0)} = 0$. Here $J^{(n)}_{e,m}$, indicate that the expression is valid for $J^{(n)}_e$ and $J^{(n)}_m$, $n = 0, 1$. It follows that the partial radiation intensity (2.12), for a wave with polarization $\hat{e}$ and propagating in direction $\hat{k}$ is $P(\hat{k}, \hat{e}) = P^{(0)}(\hat{k}, \hat{e}) + \mathcal{O}(k^5)$, where $P^{(0)}$ reduces to
\[
P^{(0)}(\hat{k}, \hat{e}) = \frac{\eta k^4}{32\pi^2} \left| \int_V e^* \cdot \left( j f^{(1)}_e + \frac{1}{2} r \times j f^{(0)}_m \right) + \hat{k} \times e^* \cdot \left( \frac{1}{\eta} f^{(1)}_m - \frac{1}{2} r \times j f^{(0)}_e \right) \, dV \right|^2
\]
\[
= \frac{\eta k^4}{32\pi^2} |\hat{e}^* \cdot \pi_e + \hat{k} \times \hat{e}^* \cdot \pi_m|^2.
\]
Here we used that the triplet $\hat{k}, \hat{e}^*, \hat{m}^*$ forms an orthogonal basis system. The
\[
\pi_e = \int_V j f^{(1)}_e + \frac{1}{2} r \times j f^{(0)}_m \, dV \quad \text{and} \quad \pi_m = \int_V \frac{j f^{(1)}_m}{\eta} - \frac{1}{2} r \times j f^{(0)}_e \, dV
\]
terms are generalized dipole-moments that account for both the electric and magnetic dipole radiating fields, respectively.

To find the total radiated power in (2.5), we start with inserting the expansion (3.3) into the far-field (2.13) to find the small $ka$ approximation of the far-field:
\[
F_E(\hat{k}) = \frac{\eta k^2}{4\pi} \hat{k} \times \int_V \hat{k} \times \left( j f^{(1)}_e + \frac{1}{2} r \times j f^{(0)}_m \right) + \left( \frac{j f^{(1)}_m}{\eta} - \frac{1}{2} r \times j f^{(0)}_e \right) \, dV + \mathcal{O}(k^3).
\]
We insert (3.8) into the expression for the total radiated power (2.5), to find that $P_{\text{rad}} = P^{(0)}_{\text{rad}} + \mathcal{O}(k^5)$ where
\[
P^{(0)}_{\text{rad}} = \frac{\eta k^4}{32\pi^2} \left\{ \left| \pi_e \right|^2 - |\hat{k} \cdot \pi_e|^2 + |\pi_m|^2 - |\hat{k} \cdot \pi_m|^2 - 2\hat{k} \cdot \text{Re}(\pi_m \times \pi_e^*) \right\} d\Omega
\]
\[
= \frac{\eta k^4}{12\pi} \left( |\pi_e|^2 + |\pi_m|^2 \right)
\]
\[
= \frac{\eta k^4}{12\pi} \left\{ \left| \int_V j f^{(1)}_e + \frac{1}{2} r \times j f^{(0)}_m \, dV \right|^2 + \left| \int_V \frac{j f^{(1)}_m}{\eta} - \frac{1}{2} r \times j f^{(0)}_e \, dV \right|^2 \right\} = P_e + P_m.
\]
Here we used the integration over the unit sphere $\Omega$ of the angular variables in $\hat{k}$ to find the relations $\int_\Omega \hat{k} d\Omega = 0$ and $\int_\Omega |\hat{k} \cdot \pi_e|^2 d\Omega = (4\pi/3)|\pi_e|^2$. The radiated power consists of two types of terms: terms that radiate as electric dipoles with power $P_e$ and terms that radiate as magnetic
dipoles with power $P_m$. The total radiated power can alternatively be expressed as

$$P_{\text{rad}} = \frac{k^4}{12\pi \sqrt{\varepsilon \mu}} \left[ \left\| \frac{1}{\sqrt{\varepsilon}} p_e + \sqrt{\mu} m_m \right\|^2 + \left\| \frac{1}{\sqrt{\mu}} p_m + \sqrt{\varepsilon} m_e \right\|^2 \right] + \mathcal{O}(k^5),$$

(3.10)

where $j_e p_e = \int_V J_e^{(1)}(0)\, dV$ and $m_e = \frac{1}{\pi} \int_V r \times J_e^{(0)}\, dV$ are the dipole moments from an electric current source and analogously for the magnetic currents and moments with subscript $m$, i.e. $m_m$. Here $c = 1/\sqrt{\varepsilon \mu}$ is the speed of light.

A check that the above expressions agree with what is known for small antennas that radiate as dipoles is obtained by comparing the maximal partial directivity, i.e. $P^{(0)}(\hat{k}, \hat{e})$ to the total radiated power $P^{(0)}_{\text{rad}}$. We consider two cases: fixed generalized electric dipole moments and no generalized magnetic dipole moment (3.7), i.e. $\pi_m = 0$ and $\pi_e \neq 0$ (or vice versa) and fixed non-zero $\pi_m, \pi_e$:

$$\max_{\hat{e}} \frac{4\pi P^{(0)}(\hat{k}, \hat{e})}{P^{(0)}_{\text{rad}}} = \frac{3}{2}, \quad \text{for } \pi_m = 0$$

(3.11)

and

$$\max_{\hat{e}, \hat{e} \perp \hat{k}} \frac{4\pi P^{(0)}(\hat{k}, \hat{e})}{P^{(0)}_{\text{rad}}} = \max_{\hat{e}, \hat{e} \perp \hat{k}} \frac{3 |\hat{e}^* \cdot (\hat{e}^* - \hat{k} \times \pi_m)|^2}{|\pi_e|^2 + |\pi_m|^2} \leq 3.$$  

(3.12)

Stating that a small antenna with electric dipole radiation from a generalized electric dipole moment have directivity $3/2$, but upon adding a magnetic generalized dipole $\pi_m$ we find that appropriately oriented combinations of $\pi_e$ and $\pi_m$ can have a directivity of 3, corresponding to a Huygens source, e.g. [47].

The small electric volume stored energies follow directly from their integral representation (2.8), we find that $W_e = W_e,0 + \mathcal{O}(k)$, where

$$W_{e,0} = \frac{\mu}{16\pi} \int_V \int_V \left[ \frac{1}{\eta^2} J_m^{(0)}(r_1) \cdot J_m^{(0)*}(r_2) + (\nabla_1 \cdot J_m^{(1)}(r_1))(\nabla_2 \cdot J_m^{(1)*}(r_2)) \right] \frac{1}{R_{12}} \, dV_1 \, dV_2$$

(3.13)

and similarly (2.9) yields $W_m = W_m,0 + \mathcal{O}(k)$, where

$$W_{m,0} = \frac{\mu}{16\pi} \int_V \int_V \left[ J_e^{(0)}(r_1) \cdot J_e^{(0)*}(r_2) + \frac{1}{\eta^2} (\nabla_1 \cdot J_e^{(1)}(r_1))(\nabla_2 \cdot J_e^{(1)*}(r_2)) \right] \frac{1}{R_{12}} \, dV_1 \, dV_2.$$  

(3.14)

These electrically small representation of stored energy is one of our results here, upon which the subsequent bounds on $Q$ and $D/Q$ are based.

### 4. Minimal antenna $Q$ and analytical and numerical illustrations

One of the goals with the expansion in the small antenna limit in §3 is that they should give us some insight into limitations of $Q$ and $D/Q$ and antenna design. It is reasonable to ask the question of what shapes that give low antenna $Q$. Similarly we investigate which charge and current densities that gives low antenna $Q$. Another goal with the expressions is to derive a priori bounds of antenna $Q$ and $D/Q$. Partial answers are given in this section, that extends the relation that a large charge-separation ability in the domain imply a small antenna $Q$ (e.g. [12,19,27,37,40]). Similarly we may think of a shape with low antenna $Q$, as a structure that supports a large ‘current loop area’ for a magnetic dipole-moment. One of the new results here is that the generic shape results in [12] for $D/Q$ is extended to lower bounds on antenna $Q$.

An often studied case is the electric-dipole case [10,15,27,37,40], here represented by the electric charges only and we illustrate below how an optimization problem is used to determine the minimal $Q$. We continue and show that the method and its associated eigenvalue problem extend to the more general case of both electric and magnetic currents that radiate as an electrical dipole. Here we also find that the magnetic polarizability enters in the lower bounds on $Q$. A short review of polarizability tensors are given in [48, App. B].
Consider the minimization problem for finding the lower bound on antenna $Q$:

$$\begin{align*}
Q &= \min_{\rho_e^{(1)}, \rho_m^{(1)}, J_0^{(0)}, J_m^{(0)}} \frac{2\omega \max\{W_{e,0}(\rho_e^{(1)}, J_0^{(0)}), W_{m,0}(\rho_m^{(1)}, J_m^{(0)})\}}{P_e(f_e^{(1)}, f_m^{(1)}) + P_m(\rho_m^{(1)}, J_m^{(0)})},
\end{align*}$$

(4.1)

with the two constraints $\int_V \rho_e^{(1)}(r) \, dV = 0$ and $\int_V \rho_m^{(1)}(r) \, dV = 0$. Here $j \omega \rho_e^{(1)} = -kV \cdot J_e^{(1)}$, and similarly for $\rho_m^{(1)}$. One of the interesting cases in antenna design is when the antenna radiates as an electric dipole, i.e. when $P_m$ is negligible and $W_{m,0} \leq W_{e,0}$. Once the optimal $(P_e, W_{e,0})$ is determined, we tune the antenna with a tuning circuit to make the antenna resonant, i.e. $W_{m,0} = W_{e,0}$. Thus, we start with the optimization problem for a pure $(W_{e,0}, P_e)$-case. The ‘dual-mode’ case, where both $P_e$ and $P_m$ are comparable is considered in §4d. Before we consider the general case, let us start with the easier case of an electric dipole when we have only $\rho_e^{(1)}$, i.e. $J_m^{(0)} = 0$.

(a) Antenna $Q$ for an electric dipole, i.e. $P_m = 0$

Different approaches to lower bounds of this antenna $Q$ case have also been investigated in e.g. [10–12,15,27,37,40]. However, one of the goals here is to arrive at a generic method that works for different cases of current-density sources, and the first step towards this goal is to verify that this method indeed gives the previously derived result on the lower bound (e.g. [8,12,19,27,37]). The electric dipole is here equivalent to the assumption $P_m = 0$ and $W_{e,0} \geq W_{m,0}$, which yields that $Q = Q_e$ and that we have an optimization problem that depend only on the electric charge-densities $\rho_e$. Once the design is determined we can tune the antenna with a tuning circuit to make $W_{e,0} = W_{m,0}$. This case is the classical electrical dipole case. Let $\rho_e = \rho_e^{(1)}$. The minimization problem (4.1) reduces to

$$Q_e = \min_{\rho_e^{(1)}} \frac{2\omega W_{e,0}(\rho_e)}{P_e(\rho_e)} = \frac{6\pi}{k^3} \min_{\rho_e^{(1)}} \frac{\int_V \int_V \rho_e^{*}(r_1) \rho_e(r_2)/(4\pi |r_1 - r_2|) \, dV_1 \, dV_2}{\int_V \rho_e(r) \, dV},$$

(4.2)

where we have used (3.4) to re-write the denominator. This minimization problem satisfies the constraint that no current flows through the surface $\partial V$, i.e. $0 = \int_{\partial V} \hat{n} \cdot J_{e}(r) \, d\mathcal{S} = -j\omega \int_V \rho_e(r) \, dV$, where $c$ is the speed of light. Hence, (4.2) is accompanied with the constraint of total zero charge, $\int_V \rho_e(r) \, dV = 0$.

The associate problem to maximize $D/Q$ in the small electric volume limit for arbitrary $\rho_e$, see [12] corresponds to

$$D = \max_{\rho_e^{(1)}} \frac{2\pi P(0)\hat{k} \cdot \hat{e}}{\kappa W_{e,0}(\rho_e)} = \frac{k^3}{4\pi} \frac{\int_V \int_V \hat{e}^{*} \cdot r \rho_e(r) \, dV^2}{\int_V \rho_e^{(1)}(r_1) \rho_e(r_2)/(4\pi |r_1 - r_2|) \, dV_1 \, dV_2},$$

(4.3)

with the same constraint of a total zero charge, $\int_V \rho_e(r) \, dV = 0$. These two problems are related but the $D/Q$ problem has the simplification in that the integrand in $P(0)\hat{k} \cdot \hat{e}$ see (3.6), is scalar-valued and the maximization has a convex optimization formulation, see [12,15].

The method that we apply below to (4.2) works on both problems (4.2) and (4.3) and yield the same result as in [12] where it is applied to (4.3). The final result is similar to the result in [19,37], but obtained with different methods. Note that both (4.2) and (4.3) remain unchanged under the scaling, $\rho_e \rightarrow \alpha \rho_e$. Thus, the solutions to (4.2) are a family of scaling invariant solutions. We determine the minimum by breaking the scaling-invariance by selecting a particular value of the amplitude of the dipole moment, $\rho_e$. We rewrite (4.2) as the minimization problem as

$$\min_{\rho_e^{(1)}} \int_V \int_V \rho_e^{*}(r_1) \rho_e(r_2) \, dV_1 \, dV_2,$$

subject to

$$\left| \int_V \rho_e(r) \, dV \right|^2 = \rho_e^2,$$

(4.5)

$$\int_V \rho_e(r) \, dV = 0.$$ 

(4.6)
To explicitly find the minimum, we use the method of Lagrange multipliers (e.g. [51, §4.14]) and define the Lagrangian $Q$ as

$$Q(\rho, \rho^*, \lambda_1, \lambda_2) = \int_{V} \int_{V} \frac{\rho^*(r_1)\rho(r_2)}{4\pi |r_1 - r_2|} \, dV_1 \, dV_2 - \lambda_1 \left( \int_{V} \rho \rho \, dV \right)^2 - \lambda_2 \int_{V} \rho^* \, dV. \quad (4.7)$$

Here $\lambda_1$ and $\lambda_2$ are Lagrange multipliers, and we use the short-hand notation $\rho = \rho_0$. Variation of $Q$ with respect to $\lambda_1$ and $\lambda_2$ gives the two constraints above. Taking the variation of $Q$ with respect to $\rho^*$, or equivalently, taking a Fréchet derivative of $Q$ yields the Euler–Lagrange equation for the critical points

$$\int_{V} \left( \frac{1}{4\pi |r_1 - r_2|} - \lambda_1 r_1 \cdot r_2 \right) \rho \, dV_1 = \lambda_2, \quad r_2 \in V. \quad (4.8)$$

Note that this is an integral equation with unknown $\rho$. Accompanied with the constraints we find three equations (4.8), (4.5) and (4.6) and three unknown $\rho, \lambda_1$, and $\lambda_2$.

Upon multiplying (4.8) with $\rho^*$ and integration over $V$, using the zero total charge constraint, we find that $Q_e$ in (4.2) is equivalent with

$$Q_e = \frac{6\pi}{k^3} \min_\rho \lambda_1. \quad (4.9)$$

The unknown Lagrange multiplier, $\lambda_1$, depends implicitly on $\rho$ and $\lambda_2$. The lower bound of the minimization problem (4.2) is hence determined by the unknown Lagrange multiplier $\lambda_1$ times a constant. Another property of the solution appears if we apply the Laplace operator on (4.8), for $r \notin \partial V$ we have that $\rho(r) = 0$. Thus, we reduce (4.8) to

$$\int_{\partial V} \left( \frac{1}{4\pi |r_1 - r_2|} - \lambda_1 r_1 \cdot r_2 \right) \rho_0 \, dS_1 = \lambda_2, \quad r_2 \in \partial V, \quad (4.10)$$

where $\rho_0$ is the surface charge density, i.e. we have formally the relation that $\rho \, dV = \rho_0 \, dS$. A similar result for $D/Q$ was shown in [12].

Using the constraint $|\int_{\partial V} \rho_0(r) \, dS| = p_0 > 0$, we re-write the critical equation (4.10) into

$$\int_{\partial V} \frac{\rho_0(r_1)}{4\pi |r_2 - r_1|} \, dS_1 = \lambda_1 p_0 \hat{\rho} \cdot r_2 + \lambda_2, \quad r_2 \in \partial V \quad (4.11)$$

for some unknown unit vector $\hat{\rho}$.

To solve equation (4.11), we make first a few observations: any solution $\rho_0$ of (4.11) for given right-hand sides, yields an associated potential that solves an electrostatic boundary-value problem (cf. [48, App. B]). Such solutions are restricted in their asymptotic behaviour by the electric polarizability tensor $\gamma_e$, which depends only on the shape of $V$. To make this restriction explicit, we note that the electric polarizability tensor $\gamma_e$ is defined through [48, eqn (88)] and $\gamma_e \cdot \hat{e}D_0 = p$. Comparing this with (4.11), we see that $D_0\hat{e} = \lambda_1 p_0 \hat{\rho}$, and the dipole-moment is by definition $p = \int_{\partial V} \rho_0(r) \, dS$. Since the electric polarizability tensor $\gamma_e$ is given, once the shape $V$ is known, we thus have a constraint on $(\lambda_1, \hat{\rho})$ in order for $\rho_0$ and its associated potential to comply with the polarizability tensor. The constraint is that

$$\gamma_e \cdot \hat{\rho} = \frac{1}{\lambda_1} \hat{\rho}, \quad (4.12)$$

which we recognize as an eigenvalue problem in $(\lambda_1, \hat{\rho})$ for $\gamma_e$. Here we have used that $p = p_0 \hat{\rho}$.

We conclude that critical points of (4.2) correspond to solutions $(\lambda_1, \hat{\rho})$ of the eigenvalue problem (4.12). Given such a solution $(\lambda_1, \hat{\rho})$, we determine the associated charge density through (4.11) with $(\lambda_1, \hat{\rho})$ given as solutions to (4.12). A charge density that solves (4.11) is hence the base for the family of current sources that supports the optimal radiation, which we obtain.
from the continuity equation. Through the re-writing of the optimal \( Q_e \) in (4.9), it follows that the largest eigenvalue, \( (\gamma_e)_3 \) of the polarizability matrix \( \gamma_e \) yields the minimum \( Q_e \), i.e.

\[
Q_e = \frac{6\pi}{k^3(\gamma_e)_3}.
\]  

(4.13)

We have hence reduced the variational problem of finding the minimum \( Q_e \) for the electric dipole to finding eigenvalues of \( \gamma_e \). This result have large similarities to [19,37], derived with different methods. We conclude that \( Q(ak)^3 \) in the small volume size only depend on the shape expressed through the normalized electric polarizability \( \gamma/\alpha^3 \). The physical interpretation connects large polarizability eigenvalues to the ability of the structure to separate charge under an external static field in a given direction. The polarizability \( \gamma_e \) is associated with the scalar Dirichlet problem of the Laplace operator and depends only on the shape of the object [30]. We note also that \( \gamma_e \) is identical to the high-contrast electric polarizability in e.g. [27].

We note that the low-frequency magnetic charge-density and electric charge-density antenna \( Q \) are dual-similar, and hence if we consider a case with either a \( \rho_e \)-term or a \( \rho_m \)-terms both of these problems result in identical minimization problems with a lower bound on antenna \( Q \) given by (4.13).

To compare with the \( D/Q \) problem, we note that the constraint \( \int \hat{\epsilon} \cdot r\rho_e^* (r) dS = \alpha^2 > 0 \), was in [12] reduced to \( \int \hat{\epsilon} \cdot r\rho_e(r) dS = \alpha \), yielding the critical equation corresponding to (4.11) as

\[
\int_{\partial V} \frac{\rho_e (r_1)}{4\pi |r_1 - r_2|} dS_1 = v_1 \hat{\epsilon} \cdot r_2 + v_2, \quad r_2 \in \partial V.
\]  

(4.14)

Similar to the \( Q_e \)-case above, we find that \( v_1 \) is connected to \( \gamma_e \) through the relation \( \hat{\epsilon}^* \cdot \gamma_e \cdot \hat{\epsilon} = \alpha/v_1 \). The corresponding maximum is \( D/Q = (k^3/4\pi)\hat{\epsilon}^* \cdot \gamma_e \cdot \hat{\epsilon} \). We thus see that the two problems are related, but that they describe different optimization problems. The antenna \( Q \) lower bound minimizes \( Q \) without concern of polarization direction of the antenna, whereas \( D/Q \) assume a fixed \( \hat{\epsilon} \) polarization direction throughout its optimization. With a priori knowledge about the optimal polarization direction of the structure or alternatively the principal eigenvalue of \( \gamma_e \) associated with a given structure, we select \( \hat{\epsilon} \) in this direction, to find the expected 3/2 difference between \( 1/Q_e \) and \( D/Q_e \). This \( D/Q \) result is similar to the sum-rule in [27] for electric sources. With the \( Q_e \) result and the observation of principal directions of \( \gamma_e \), we see that these three approaches illustrate closely connected results here with a common energy principle method to obtain them.

To illustrate the result, we begin with a sphere: \( \gamma_e = 4\pi a^3 I \), where \( I \) is the three times three unit tensor, and all eigenvalues of \( \gamma_e \) are identical. Note that if we added degenerate eigenvalues there are three orthogonal eigenvectors, and the corresponding charge densities in (4.10) for a given amplitude of the dipole-moment \( p_e \). This degeneracy is due to the geometrical symmetries of the shape. Thus even when we remove the scaling invariance, we may have multiple \( \rho \) that yield the same lower bound on \( Q \). Note also that for any arbitrary optimizer \( \rho_e = \rho_e^{(1)} \) that the associated electric current connected to \( \rho_e^{(1)} \), here \( f_e^{(1)} \), i.e. \( \partial \rho_e^{(1)} = -k\nabla \cdot f_e^{(1)} \), has an infinite dimensional subspace that all yields the same \( \rho_e^{(1)} \). It allows a potentially large design freedom that does not change \( Q_e \) in the quasi-static limit. This case is analogous to the case discussed in [12].

For the sphere, we find \( (ka)^3 Q_e = 3/2 \) and for a disc \( (ka)^3 Q_e = 9\pi/8 \) for the electrical dipole case ([48, App. D]). If we instead study \( \gamma_e \) of a rectangular plate of size \( \ell_2 \times \ell_1 \) and sweep the ratio \( \ell_1/\ell_2 \) we find that the two non-zero eigenvalues depicted as the two curves with highest value, marked with (E), in figure 2a, and corresponding \( Q \) in figure 2b marked with (E). Note that \( D/Q = k^3 \hat{\epsilon}^* \cdot \gamma_e \cdot \hat{\epsilon}^* /4\pi \), and hence proportional to the two electric polarizability curves given in figure 2a, for appropriate directions \( \hat{\epsilon} \). The electrical polarizability is a measure of how well a structure allows charge separation, in the sense that large eigenvalues in one direction corresponds to large static electric dipole-moment, or equivalently large ability to separate charges.

The corresponding, electric charge maximization problem of \( D/Q \) is solved in [12,27,40]. We have thus the solution to both the \( \min_{\rho} Q \) and the \( \max_{\rho} D/Q \) problems for small antennas that radiate as electric dipoles.
to find that $Q$ where $\hat{\mathbf{J}}$ is the boundary, i.e. $J$. By applying the operator $\nabla \times \nabla \times$, we take the scalar product of (4.17) with $W$ which reduces the problem (4.15) to an equivalent problem with Lagrange multipliers. The Lagrangian is analogous to how the electric dipole, $\rho_e^{(1)}$, and the magnetic dipole with $\rho_m^{(1)}$ yield the same optimization problem in the previous section, we see that an electric $J_e^{(0)}$ or a magnetic $J_m^{(0)}$ current density result in identical optimization problems. We associate a magnetic dipole moment $m = \hat{m}$ and electric current density, $J_e^{(0)}$ here denoted $\hat{J}$, to find the minimization problem:

$$Q_m = \min \frac{W_{m,0}(J)}{P_m(J)} = \min \frac{6\pi \int_V \hat{J}(r_1) \cdot J(r_2)/(4\pi |r_1 - r_2|) \, dV_1 \, dV_2}{|\int_V (1/2) r \times J(r) \, dV|^2}$$

(4.15)

with the constraint that $\hat{\mathbf{n}} \cdot \mathbf{J} = 0$ over the surface and $\mathbf{J} \in X_0$. Here $X_0 = \{ \mathbf{J}: W_{m,0}(J) < \infty, \nabla \cdot \mathbf{J} = 0 \}$. This problem is associated with an antenna that radiates as a magnetic dipole, i.e. $P_e = 0$, and $W_{e,0} \leq W_{m,0}$. Once the optimization is done, we can tune the antenna with a tuning circuit to reach resonance $W_{e,0} = W_{m,0}$.

We apply once again the method in (4.2)–(4.7) to the minimization of (4.15). Scaling invariance is broken by the assumption that $|1/2 \int_V \mathbf{r} \times J(r) \, dV| = m$, which reduces the problem (4.15) to an equivalent problem with Lagrange multipliers. The Lagrangian is

$$Q(J, J^*, \lambda_1) = \int_V \int_V \frac{J(r_1) \cdot J(r_2)}{4\pi |r_1 - r_2|} \, dV_1 \, dV_2 - \lambda_1 \left( \int_V \frac{1}{2} \mathbf{r} \times J(r) \, dV \right)^2$$

(4.16)

for $J \in X_0$. The associated critical point equation is

$$\int_V \frac{J(r_2)}{4\pi |r_1 - r_2|} \, dV_2 = -\frac{\lambda_1}{2} \mathbf{r}_1 \times \int_V \frac{1}{2} \mathbf{r}_2 \times J(r_2) \, dV_2 = -\frac{\lambda_1 m}{2} \mathbf{r}_1 \times \hat{m}.$$

(4.17)

Similar to the electric case (4.9), we take the scalar product of (4.17) with $J^*$ and integrate over $V$ to find that $Q_m$ is determined by $\lambda_1$.

$$Q_m = \frac{6\pi}{k^3} \min \lambda_1.$$  

(4.18)

By applying the operator $\nabla \times \nabla \times$ to (4.17), we realize that the currents have support only on the boundary, i.e. $J \, dV = J_s \, dS$, and (4.17) reduces to

$$\hat{n} \times \int_{\partial V} \frac{J_s(r_2)}{4\pi |r_1 - r_2|} \, dS_2 = \frac{\lambda_1 m}{2} \hat{n} \times (\hat{m} \times \mathbf{r}_1), \quad \text{for } r_1 \in \partial V,$$

(4.19)

where $\hat{n}$ is normal to $\partial V$. 

(b) Antenna $Q$ for an electric current magnetic dipole

Figure 2. (a) Eigenvalues for the electric and magnetic polarizability tensor for an infinitesimally thin plate normalized by $a^3$, where $a = (\sqrt{\ell_1^2 + \ell_2^2})/2$ is the radius of the smallest circumscribed sphere. The polarization directions are indicated by $\hat{\mathbf{e}}$ and $\hat{\mathbf{h}}$ for the $\mathbf{E}$ and $\mathbf{H}$-fields, respectively. The curves are marked with (E) for electric polarizability or (M) for magnetic polarizability. The curves are symmetric with respect to $\ell_1/\ell_2 = 1$, the lowest curve is the single non-zero eigenvalue of $\gamma_m$. (b) The corresponding $Q$-value from (4.13) and (4.21), once again the (E) corresponds to the electric and (M) to the magnetic case. (Online version in colour.)
Similar to the electric case (4.11), we compare this with the definition of the magnetic polarizability tensor, \( \gamma_m \) in [48, App. B] where \( \gamma_m \cdot \hat{h}H_0 = m \), for a given tangential boundary condition \( \mu_0 H_0 \hat{n} \times (\hat{h} \times r)/2 \) for the vector potential. The magnetic polarizability tensor is known, once the region \( V \) is given. The \((\lambda_1, \hat{m})\) in equation (4.19) is hence subject to the constraint

\[
\gamma_m \cdot \hat{m} = \frac{1}{\lambda_1} \hat{m}. \tag{4.20}
\]

The eigenvalue solution \((\lambda_1, \hat{m})\) of (4.20) yields the solution to the minimization problem

\[
Q_m = \frac{6\pi}{k^3} \min_{\lambda_1} \lambda_1 = \frac{6\pi}{k^3 (\gamma_m)_{3}}, \tag{4.21}
\]

where \((\gamma_m)_{3}\) is the largest eigenvalue of \( \gamma_m \). The analogous case for \( D/Q \) is given in [12]. The sphere has the magnetic polarizability tensor \( 2\pi a^3 I \), which yields \((ka)^3 Q_m = 3\) (cf. [8, 47]). Here \( I \) is a unit three times three tensor.

The electric and magnetic polarizabilities of a rectangular plate are depicted in figure 2a marked with (E) and (M), respectively. The polarizability tensors are diagonal for geometries with two orthogonal reflection symmetries and co-aligned coordinate systems [31, 52] and for planar structures we have only one eigenvalue of \( \gamma_m \), orthogonal to the plane. We can physically think of large \( \gamma_m\)-eigenvalues as that the region \( V \) supports a large loop current for the corresponding dipole-moment. Note that planar structures have one non-zero eigenvalue in \( \gamma_m \) which is associated with the normal-to-the-surface dipole-moment with the planar ‘current loop area’. We observe that the magnetic polarizability tensor is connected to the scalar Neumann problem of Laplace equation ([48, App. B]). Note that the magnetic polarizability corresponds to the permeable case of \( \mu \to 0 \). There are different sign conventions for \( \gamma_m \); however, we note that \( \lambda_1 \geq 0 \) in (4.21), see (4.15).

A similar current loop-area argument is illustrated in figure 3 for a flat ellipse and a thin ellipsoid. The eigenvalues of the polarizability tensor of an ellipsoid are known ([48, App. D]), and they are depicted in figure 3a,b. The two curves marked with (M) in figure 3c correspond to \( Q_m \), the upper one, marked (M), is for an ellipse of zero thickness and only one \( \gamma_m\)-eigenvalue corresponding to a current loop-area over the surface. The other marked (M, thick) corresponds to an ellipsoid identical to the flat one, but where the radius normal to the paper is \( h/100 \) where \( h \) is the height of the ellipse. The two transverse eigenvalues of \( \gamma_m \) are ignored by \( Q_m \) until the width, \( w \), is \( h/100 \), where equivalent current loop-area spanned by the height and normal (out of the paper) dominates the transverse current loop-area and \( Q_m \) changes slowly for \( w/h < 10^{-2} \) since this area is essentially preserved.

(c) Lower bound on antenna Q for both electric charge and magnetic currents

The common electric and magnetic dipoles cases above agree with previously derived results [14, 19]. We here extend these results to include both the electric charge density \( \rho_e \) and the magnetic current density \( J_m \), i.e. the components making up a generalized electric dipole-moment \( \pi_e \) (3.7). We once again consider the case where the antenna radiates as an electrical dipole, i.e. \( P_m = 0 \) and where the stored energy is mainly electric, \( W_{m,0} \leq W_{e,0} \). After the optimization, we tune the antenna to make the stored electric and magnetic energies equal. Optimizing for the \((P_m, W_{m,0})\)-case is identical to the \((P_e, W_{e,0})\)-case up to a sign and the free-space impedance normalization of the currents. Similar to the above discussion in §4a,b of electric and magnetic dipoles, we optimize

\[
Q = \frac{6\pi}{k^3} \min_{\rho J} \frac{\int_V \int_V (\rho(r_1) \rho(r_2) + J^a(r_1) \cdot J(r_2))/(4\pi |r_1 - r_2|) dV_1 dV_2}{\int_V r \rho - (1/2)r \times J(r) dV}^2. \tag{4.22}
\]

We used above the short-hand notation \( J = J_m^{(0)}/\eta \), and \( \rho = c \pi_e^{(1)} = j \nabla \cdot J_e^{(1)} \). Here we also have the constraints \( \int_V \rho dV = 0 \) and that \( \nabla \cdot J = 0 \) to account for the Gauge freedom of the associated vector potential. To include this Gauge freedom into the optimization problem, we restrict the current-density space to \( J \in X_0 = \{ J : W_{m,0}(J) < \infty, \nabla \cdot J = 0 \} \).
respect to the Lagrange parameters $\lambda$. Critical points of equivalently consider the problem

$$\int \rho(r) \lambda \, dV = 0,$$

subject to

$$\int_{V} r \rho(r) \, dV = \tilde{\pi}_e^2,$$

$$J \in X_0.$$  

Using the method of Lagrange multipliers $\lambda_1, \lambda_2$, we define the Lagrangian

$$Q = \int_{V} \int_{V} \frac{\rho^*(r_1) \rho(r_2) + J^*(r_1) \cdot J(r_2)}{4\pi |r_1 - r_2|} \, dV_1 \, dV_2$$

$$- \lambda_1 \left( \int_{V} r \rho(r) \, dV \right) - \lambda_2 \int_{V} \rho^*(r) \, dV.$$  

Critical points of $Q$ are determined by the variation (Fréchet derivative) of $Q$. Variation with respect to the Lagrange parameters $\lambda_1$ and $\lambda_2$ gives the constraints. The variation with respect to $\rho^*$ and $J^*$ yields

$$\int_{V} \frac{\rho(r_2)}{4\pi |r_1 - r_2|} \, dV_2 = \lambda_1 \left[ r_1 \cdot \int_{V} r_2 \rho(r_2) - \frac{1}{2} r_2 \times J(r_2) \, dV_2 \right] + \lambda_2$$

and

$$\int_{V} \frac{J(r_2)}{4\pi |r_1 - r_2|} \, dV_2 = \lambda_1 r_1 \times \int_{V} r_2 \rho(r_2) - \frac{1}{2} r_2 \times J(r_2) \, dV_2.$$  

Figure 3. (a) Eigenvalues to the electric polarizability tensor, $\gamma_e$. Solid lines are the flat case, dashed lines correspond to the case with normal (out of the paper) radius of the ellipsoidal is $h/100$. Polarization direction is indicated with an arrow. (b) Eigenvalues to the magnetic polarizability tensor $\gamma_m$. Solid line corresponds to the flat case, dashed lines are the case with normal radius $h/100$. Note that the x-axis is the same as in (a). (c) The antenna $Q$ for a flat ellipse indicated by (E), and (M) and (E + M) corresponding to $Q_m$ from magnetic sources (4.21), $Q_e$ from electric sources (4.13), $Q_m$ and $Q_e$ from combined dual-mode in (4.34), respectively. Two lines are also marked with ‘thick’, to indicate that the ellipsoidal radius normal to the ellipse-surface in the figure is $h/100$. Note in particular for $Q_m$ that as the width becomes smaller than $h/100$, the thickness become important, as is clear in (4.21), since it implies a switch of dominant eigenvalue. The reduction of $Q$ as compared to $Q_e$ due to the eigenvalue of $\gamma_m$ is absent for flat structures since the non-zero eigenvalues of $\gamma_e$ and $\gamma_m$ have orthogonal directions. It is a marginal reduction for thin structures. (Online version in colour.)

The minimization problem is scaling invariant under transformations $(\rho, J) \mapsto (\rho, J) \alpha$ for any complex valued scalar $\alpha$. By assuming that the denominator has a given value $\tilde{\pi}_e^2 > 0$, we may equivalently consider the problem
Here we used that \( \hat{\mathbf{n}} \cdot \mathbf{J} = 0 \) on \( \partial V \), and we recognize \( \lambda_2 \) as a way to ensure that the total charge is zero. To investigate the properties of these Euler–Lagrange equations, we first note that the inner product of these equations with \( \rho^e \) and \( \mathbf{J}^e \), respectively, and that their sum can be rewritten as the original problem

\[
Q = \frac{6\pi}{k^3} \min_{\rho, \mathbf{J}} \frac{\int_V (\rho^e(r_1)\rho(r_2) + J^e(r_1) \cdot J(r_2))/(4\pi |r_1 - r_2|) \, dV_1 \, dV_2}{\int_V \rho \rho(r) - (1/2) r \times J(r) \, dV^2} = \frac{6\pi}{k^3} \min_{\rho, \mathbf{J}} \lambda_1. \tag{4.30}
\]

The minimization problem is thus reduced to finding \( \lambda_1 \) for \( \rho, \mathbf{J} \) that solves (4.28) and (4.29).

Similar to the charge-density case (4.9), we note that \( \lambda_1 \) implicitly depend on \( \rho \) and \( \mathbf{J} \) through the Euler–Lagrange equations. Another property of the minimization problem appears if we for \( \rho \in \partial V \) operate with \( \Delta \) and with \( V \times \nabla \times \) on (4.28) and (4.29), respectively. We find that \( \rho \) and \( \mathbf{J} \) only have support on the boundary, and we use the notation \( \mathbf{J} \) the Euler–Lagrange equations.

\[
\rho_\delta(r_2) \int_{\partial V} \frac{\rho_\delta(r_2)}{4\pi |r_1 - r_2|} \, dS_2 = \lambda_1 \left( \int_{\partial V} r_2 \rho_\delta(r_2) - \frac{1}{2} r_2 \times J_\delta(r_2) \, dS_2 \right) + \lambda_2 = -\lambda_1 r_1 \cdot \mathbf{\pi}_e + \lambda_2 \tag{4.31}
\]

and

\[
\hat{\mathbf{n}}_1 \times \int_{\partial V} \frac{J_\delta(r_2)}{4\pi |r_1 - r_2|} \, dS_2 = \lambda_1 \hat{\mathbf{n}}_1 \times \left( \frac{1}{2} r_1 \times \int_{\partial V} r_2 \rho_\delta(r_2) - \frac{1}{2} r_2 \times J_\delta(r_2) \, dS_2 \right)
= -\lambda_1 \hat{\mathbf{n}}_1 \times \left( \frac{1}{2} r_1 \times \mathbf{\pi}_e \right), \tag{4.32}
\]

for \( r_1 \in \partial V \). We have here introduced the electric and magnetic dipole-moments for the current and charge-distribution that solve (4.31) and (4.32): \( \mathbf{p} = \int_{\partial V} r \rho_\delta \, dS, \quad m = \frac{1}{2} \int_{\partial V} r \times J_\delta \, dS \) and \( \mathbf{\pi}_e = m - \mathbf{p} \). However, both \( m \) and \( \mathbf{p} \) are presently unknown apart from the constraints that \( |\mathbf{\pi}_e| = |m - \mathbf{p}| = |\mathbf{\pi}_e| \).

To determine \( \lambda_1 \), we recall the definitions of the electric polarizability tensor \( \gamma_e \) and magnetic polarizability tensor \( \gamma_m \) in [48, App. B]. We compare (4.31) and (4.32) with the corresponding electric and magnetic boundary integrals [48, eqns (88), (92)]. The polarizability tensors \( \gamma_e \) and \( \gamma_m \) are known, once \( V \) is given, and they impose constraints on \( \lambda_1 \) and \( \mathbf{\pi}_e \)

\[
\gamma_e \cdot (\mathbf{p} - m) = \frac{1}{\lambda_1} \mathbf{p}, \quad \gamma_m \cdot (\mathbf{p} - m) = -\frac{1}{\lambda_1} m. \tag{4.33}
\]

Adding the two equations yields that \( \lambda_1^{-1} \) is an eigenvalue to the matrix \( \gamma_e + \gamma_m \). Furthermore, \( m - \mathbf{p} = \mathbf{\pi}_e \mathbf{\pi}_e \), where \( \mathbf{\pi}_e \) is an eigenvector of \( \gamma_e + \gamma_m \) of unit length. Thus, we have found that in this case the lower bound on \( Q \) is given by

\[
Q = \frac{6\pi}{k^3 \gamma_e + \gamma_m}, \tag{4.34}
\]

where \((\gamma_e + \gamma_m)_3\) is the largest eigenvalue of the \( \gamma_e + \gamma_m \) tensor. The corresponding \( \rho_\delta, J_\delta \) are hence the solution of (4.31) and (4.32), where \( m - \mathbf{p} = \mathbf{\pi}_e \mathbf{\pi}_e \), i.e. in the direction of the unit eigenvector corresponding to the largest eigenvalue. This result is similar to [19] but derived with a different method. Note that \( \gamma_e + \gamma_m \geq 0 \).

The minimization procedure also establishes that there exists a \( \lambda_1 \geq 0 \) such that

\[
\left| \int_V r \rho(r) - \frac{1}{2} r \times J(r) \, dV \right|^2 \leq \frac{1}{\lambda_1} \int_V \frac{\rho^e(r_1)\rho(r_2) + J^e(r_1) \cdot J(r_2)}{4\pi |r_1 - r_2|} \, dV_1 \, dV_2 \tag{4.35}
\]
for all \( \rho \) and \( J \) that satisfy the bi-condition \( J \in X_0 \) and \( \int_V \rho \, dV = 0 \). Equality is reached when \( \rho \) and \( J \) satisfy the Euler–Lagrange equations above, yielding \( 1/\lambda_1 = (\gamma_e + \gamma_m)/3 \). An equivalent formulation of this result is

\[
P_e \leq (\gamma_e + \gamma_m)^3 \frac{ck^4}{3\pi} W_{e,0} \quad \text{or} \quad P_m \leq (\gamma_e + \gamma_m)^3 \frac{ck^4}{3\pi} W_{m,0}.
\]

for the above described currents. The identity is achieved in either case for currents that realize the minimization of \( Q_e \) or \( Q_m \). The inequality for the \((P_m, W_{m,0})\)-case is obtained identically with the above described case starting from \( P_m \) and \( W_{m,0} \) with the substitution of \( J = -J^{(1)}_e \) and \( \rho = c\rho_m^{(1)}/\eta \) giving \ref{eq:4.22} with \( r\rho + r \times J/2 \) of the integrand in the denominator.

(i) Comparisons and numerical examples for the \( Q \)-lower bound for the dual-mode case \ref{eq:4.34}

We note that for a sphere where both electric and magnetic currents contribute to the generalized electric dipole-moment we find that \((ka)^3 Q_e = 1\) \cite{3}. The \( Q \)-lower bound for the flat ellipse and the thin ellipsoid are depicted in figure 3. For planar structures, we note that there is only one non-zero eigenvalue of \( \gamma_m \), in the direction normal to the surface and hence perpendicular to the non-zero direction of \( \gamma_e \). For a rectangular plate, this eigenvalue is depicted in figure 2. We conclude that in planar structures \( \gamma_e \) and \( \gamma_m \) do not couple to improve the antenna \( Q \). As is clear from the case where we add a small thickness of the domain as in figure 3c, we see that there is a rather small reduction of \( Q \) when compared with the flat case.

The polarizability tensors for spheroidal shapes are known \cite[48, App. D]{48}, figure 4a,b). We depict \( Q \) for spheroidal bodies as a function of the ratio between height and diameter in figure 4c. Here, the curves marked with \((+)\) correspond to \( Q \) given in \ref{eq:4.34} are shown for both the prolate (dashed lines) and oblate cases (solid lines).

The approach in \cite{37} provides an antenna \( Q, Q_V \), depending only on \( \gamma_e \) and volume \( V \). To compare \( Q_V \) with \ref{eq:4.34}, we use the inequality \cite[1.5.19]{30}

\[
(\hat{e} \cdot \gamma_e \cdot \hat{e} - V)(\hat{e} \cdot \gamma_m \cdot \hat{e} - V) \geq V^2 \iff (\hat{e} \cdot \gamma_e \cdot \hat{e} - V)(\gamma_e + \gamma_m) \cdot \hat{e} \geq (\hat{e} \cdot \gamma_e \cdot \hat{e})^2.
\]

Rewriting and comparing with the results, we find that

\[
Q = \frac{6\pi}{k^4 \hat{e} \cdot (\gamma_e + \gamma_m) \cdot \hat{e}} \leq \frac{6\pi}{k^4 \hat{e} \cdot \gamma_e \cdot \hat{e}} \left( 1 - V \hat{e} \cdot \gamma_e \cdot \hat{e} \right) = Q_V,
\]
Figure 5. The figure depicts the antenna $Q$, for energies that corresponds to currents that radiate as an electrical dipole aligned with the vertical axis. The dotted line correspond to $Q_V$ in (4.38), the $J_e$, $J_m$ and $J_e + J_m$ correspond to $Q_e$, $Q_m$ and $Q$ in, respectively, (4.13), (4.21) and (4.34). (Online version in colour.)

if we choose the $\hat{e}$ to be the unit eigenvectors corresponding to the largest eigenvalue of $\gamma_e + \gamma_m$. Equality holds for several cases in particular for ellipsoidal shapes. An alternative approach to antenna $Q$ is given in [19], see also [38]. To illustrate that there is a difference between $Q$ and $Q_V$, we calculate both antenna $Q$’s for a cylinder. We assume here that the currents radiate as an electrical dipole aligned with the cylinder axis, i.e. the vertical $\hat{x}_3$-axis, the resulting $Q$ from (4.34) and $Q_V$ are shown in figure 5. To demand that a small antenna radiates as an electric dipole in a given direction is equivalent to selecting the corresponding eigenvalue of the polarizability tensor. Such a choice of eigenvalue does not necessarily minimize antenna $Q$.

The above examples illustrate how the shape of a small antenna enters into the antenna $Q$-bound. The shape characterization in antenna $Q$ is encoded in the respective polarizability tensors. The electric polarizability is a measure on how easy it is to separate charge for a given volume $V$, i.e. to create a large electric dipole-moment. Similarly, the magnetic polarizability measure how easy it is to create a large magnetic dipole moment, i.e. finding a large ‘current-loop area’ in the domain.

If we similarly [19,33,34] associate the magnetic currents with layers/volumes of magnetization or synthesized Amperian current loops, we note that the associated volumes for the electric and magnetic currents do not necessary need to occupy identical volumes/surfaces. In such a case, there are a considerable design freedom for $\gamma_e$ and $\gamma_m$, with the performance bounded by the eigenvalues of $\gamma_e + \gamma_m$ for the total volume $V$.

(d) Dual-mode antennas

Self-resonant dual-mode antennas where both the electric $P_e$ and magnetic $P_m$ dipole radiation contribute significantly to the radiation and $W_{e,0} = W_{m,0}$ is considered here. Using that the problem decouples, we use the respective electric and magnetic case above with identities (4.35) where $\lambda_1 \geq 0$ for both $W_{e,0}$ and $W_{m,0}$. We hence find that the general case can be bounded by

$$Q \geq \frac{6\pi}{k^3} \max(W_{e,0}, W_{m,0}) \geq \frac{6\pi}{k^3} \frac{\lambda_1}{2} \gamma \frac{3\pi}{k^3(\gamma_e + \gamma_m)^3}$$

(4.39)

which follows directly from the Hölder inequality [53]. Equality follows when both electric and magnetic charges are optimized and the antenna is self-resonant. Clearly, we find that $Q$ is half the value of $Q_e$ or $Q_m$ when only electric or magnetic dipole radiation is allowed. The sphere yields $(ka)^3 Q = 1/2$, which agrees with the result of the sphere given in [3,54,55] see also [22]. A similar
result is given in [19], derived with a different method. The antenna $Q$ for this case is illustrated for spheroidal shapes in figure 4c, for curves marked with a (T).

5. Convex optimization for optimal currents

Bounds on $D/Q$ can be expressed as a convex optimization problems [15]. Here, these results are generalized to include electric and magnetic current densities. We consider a volume $V$ with electric $J_e$ and magnetic $J_m$ current densities. We expand the current densities in local basis-functions

\[ J_e(r) \approx \sum_{n=1}^{N} J_{e,n} \psi_n(r) \quad \text{and} \quad J_m(r) \approx \sum_{n=1}^{N} J_{m,n} \psi_n(r) \]  

(5.1)

and introduce the $1 \times 2N$ matrix $J_v$ with elements $\{J_{e,n}\}$ for $n = 1, \ldots, N$ and $\{\eta^{-1}J_{m,n-N}\}$ for $n = N+1, \ldots, 2N$ to simplify the notation. The basis functions are assumed to be real valued, divergence conforming, and having vanishing normal components at the boundary [57].

A standard method of moment implementation using the Galerkin procedure computes the stored energies given in appendix A as matrices. For simplicity, we here compute these stored energy matrices $X_e$ and $X_m$ only for the leading order term in (2.10) and (2.11), for $ka \ll 1$, i.e.

\[ X_{ij}^e = \frac{1}{k} \int_V \nabla_1 \cdot \psi_i(r_1) \nabla_2 \cdot \psi_j(r_2) \frac{\cos(kR_{12})}{4\pi R_{12}} \, dV_1 \, dV_2 \]  

(5.2)

and

\[ X_{ij}^m = k \int_V \psi_i(r_1) \cdot \psi_j(r_2) \frac{\cos(kR_{12})}{4\pi R_{12}} \, dV_1 \, dV_2. \]  

(5.3)

The quadratic forms for the stored energies (2.8) and (2.9) are then approximated as

\[ W_e \approx \frac{\eta}{4\omega} J_v^H X_e J_v = \frac{\eta}{4\omega} \sum_{i,j=1}^{N} J_{e,i}^* X_{ij}^e J_{e,j} + J_{m,i}^* X_{ij}^m J_{m,j} \]  

and

\[ W_m \approx \frac{\eta}{4\omega} J_v^H X_m J_v = \frac{\eta}{4\omega} \sum_{i,j=1}^{N} J_{e,i}^* X_{ij}^m J_{e,j} + J_{m,i}^* X_{ij}^e J_{m,j}. \]  

(5.5)

where the superscript, H, denotes the Hermitian transpose.

We also use the radiated far field, $F_\infty (\hat{r})$ see (3.8). The radiation vector projected on $\hat{e}$, cf. (3.8), defines the $2N \times 1$ matrix $E_\infty$ from

\[ \hat{e}^* \cdot F_\infty (\hat{k}) \approx E_\infty J_v = -j\eta k \sum_{n=1}^{N} \int_V [J_{e,n} \hat{e}^* \cdot \psi_n(r) + J_{m,n} \hat{k} \times \hat{e}^* \cdot \psi_n(r)] e^{jk \cdot r} \, dV. \]  

(5.6)

Using the scaling invariance of $D/Q$, we rewrite the maximization of $D/Q$ into the convex optimization problem of maximization of the far-field in one direction subject to a bounded stored energy [15], i.e.

\[ \max_{J_v} \quad \text{Re}(E_\infty J_v), \]  

subject to \[ J_v^H X_e J_v \leq 1, \]  

\[ J_v^H X_m J_v \leq 1. \]  

(5.7)

The formulation is easily generalized by adding additional convex constraints [15]. There are several efficient implementations that solve convex optimization problems, here we use CVX [58].

We consider planar geometries and bodies of revolution to illustrate the bound. The resulting $Q$ of (2.4) for a small spherical capped dipole antenna is depicted in figure 6a as a function of the angle $\theta$ for a maximized omnidirectional partial directivity in $\theta = 90^\circ$ and polarized in the

\[ \text{Optimization that use a fixed electric to magnetic dipole radiation ratio is discussed in [19,56].} \]
Figure 6. (a) The capped spherical dipole. The figure shows the optimized antenna $Q$ for different values of the cap-angle, see the figure in at top right. The purely electric and the purely magnetic cases are marked with $J_e$ and $J_m$, respectively. The joint case is marked $J_e + J_m$. Note that the constraint of only electrical energy $J_e$ approaches $Q_e(ka)^3 = \frac{3}{2}$, similarly $J_m$ yields $Q_m(ka)^3 = 3$ and the combined electric case $J_e$ and $J_m$ yield $Q_e(ka)^3 = 1$ as $\theta = 90^\circ$. (b, c) The figures compare the interior field without (b) and with (c) magnetic currents for dipoles that radiate as an electric dipole. (Online version in colour.)

Figure 7. Sweeping the two diameters of a spheroid, with purely electric and purely magnetic currents, as well as the combination are shown. Here the optimization is done under the assumption that the far-field radiates as an electric dipole aligned with the vertical axis. See also the discussion at the end of §5. (Online version in colour.)

$\hat{z}$-direction. The resulting radiation pattern is as from a $\hat{z}$-directed electric Hertzian dipole, i.e. $D = 1.5 \sin^2 \theta$. The three cases electric and magnetic currents $J_e + J_m$, only electric currents $J_e$, and only magnetic currents $J_m$ are analysed. The requirement of electric dipole-radiation implies $P_m = 0$, $P_e \neq 0$, and that we can use $\rho_e$ to represent the electric currents $J_e$. We observe that the $\theta = 90^\circ$ case corresponds to a spherical shell with the classical [3,8,34,37] bounds $Qk^3a^3 = \{1, 1.5, 3\}$ for the $J_e + J_m$, $J_e$ and $J_m$ cases, respectively. The reduced $Q$ of the combined $J_e + J_m$ case is understood from the suppression of the energy in the interior of the structure. This is also shown in figure 6b,c, where the resulting electric energy density is depicted for the cases to electric currents $J_e$ and combined electric and magnetic currents $J_e + J_m$. We also note that the potential improvement with combined electric and magnetic currents $J_e + J_m$ decreases as $\theta$ deceases. This can be understood from the increased internal energy as the magnetic current can only cancel
the internal field for closed structures. Moreover, the faster increase of $Qk^3\sigma^3$ as $\theta \to 0$ for the $J_m$ case than for the $J_e$ case is understood from the loop-type currents of $J_m$, whereas $J_e$ is due to charge separation.

The case of a spheroidal body with the additional radiation constraint corresponding to an electrical dipole along the vertical axis is given in figure 7. It is interesting to compare this constrained result with the minimal $Q$ as shown in figure 4c, the (+)-curve. Small $\ell_1/\ell_2$ in figure 7 corresponds to small $\xi$ as depicted with solid lines in figure 4c. We see that in the constrained case $Q$ approaches the pure magnetic current case marked $J_m$, whereas in figure 4c, $Q$ marked with (+), approaches the pure $Q_e$ case (solid line marked (E)), and it is a lower value than the result indicated in figure 7. The cause of this difference is the requirement of the radiation pattern, locking $Q$ to a disadvantageous eigenvalue, see figure 4a and the vertical polarization direction (solid line). The physical interpretation is clear: for the disc it is easier to excite an electrical dipoles aligned with the surface. The required vertical electric dipole is the cause of the higher $Q$ in figure 7. For $\ell_1/\ell_2$ large, we see that both results agree (dashed lines in figure 4c, as $\xi \to 0$).

6. Conclusion

This paper introduces a common mathematical framework for deriving lower bounds on antenna $Q$ to arbitrary shapes for electric and magnetic current densities. For the corresponding cases considered in [19,27], we get identical results for appropriate choices of the ratio of electric and magnetic dipole radiation $P_e$ and $P_m$. This is rather remarkable since the underlying physics and mathematical approaches use widely different ways to arrive to antenna $Q$ and $D/Q$. The result also verify that both electric and magnetic current densities are required to reach the classical results for a sphere in e.g. [3,54]. This method also provides a minimization method to determine the minimizing currents, which is attractive for optimization procedures, where antenna $Q$-related problems can be considered. A few of these minimization problems are demonstrated in the present paper, and extensions analogously to the convex optimization results in [15] follows directly from the explicit results shown here.

In this paper, we derive the antenna $Q$ lower bound for small electric antennas. The lower bound on antenna $Q$ depends symmetrically on both the electric and magnetic polarizabilities, which reflect the dual symmetry of the electromagnetic equations with electrical and magnetic current densities. The explicit lower bound enables $a$ priori estimates of antenna $Q$ given the shape of the object in terms of the static polarizability tensors $\gamma_e$ and $\gamma_m$. We also determine the antenna $Q$ for planar rectangles, ellipsoids and cylinders. Here we sweep a geometrical shape parameter, to illustrate how the antenna properties $Q$ and $D/Q$ depend on the shape. Low antenna $Q$ is associated with low fields inside closed domains, with the present technique we can study objects like the spherical cap to observe how the cancellation of the fields in the interior of an essentially open structures behave for optimal or constrained antenna $Q$.

We conclude that the presented new current-density representation of the stored energy yields explicit analytical expressions on antenna $Q$ and $D/Q$ in terms of the polarizability tensors. We also illustrate that the polarizabilities and different antenna $Q$-related optimization problems are straight forward to calculate, given standard software. This follows through the relation of the polarizabilities to the scalar Dirichlet and Neumann problems. The present results are applicable to a range of practical antenna problem, as $a$ priori limitations of their antenna $Q$-performance, and more subtle as explicit current minimizers that might give insight into antenna design problems.

Data accessibility. This manuscript does not contain primary data and as a result has no supporting material associated with the results presented.

Author contributions. Both authors contributed to the formulation, did numerical simulations and drafted the manuscript. Both authors gave final approval for publication.

Funding statement. The authors would like to acknowledge the support of the Swedish Strategic Research Agency (SSF) on the grant ‘Complex analysis and convex optimization for EM design’ AM13-0011 and the Swedish Research Council (Vetenskapsrådet). B.L.G.J. would also like to acknowledge the funding from the
Appendix A. Stored energy: general sources

The stored energies are derived from (2.3) using an approach with potentials. The result consists of a sum of terms each of a given leading \( k^n \)-behaviour for \( n = 0, 1, \ldots \) as \( k \to 0 \). We have that

\[
W_e = W^{(0)}_e + W^{(2)}_e, \quad W_m = W^{(0)}_m + W^{(2)}_m, \quad \text{where}
\]

\[
W^{(0)}_e, \quad W^{(2)}_e, \quad W^{(0)}_m, \quad W^{(2)}_m, \quad \text{and} \quad W^{\text{rest}}_e = W^{(1)}_e + W^{(3)}_e - W^{(2)}_e + W^{\text{rest}}_e.
\]

The EFIE operators \( \mathcal{L}_e \) and \( \mathcal{L}_m \) are given in (2.10) and (2.11), and we find the leading order electric and magnetic stored energy as

\[
W^{(0)}_e = \frac{\mu}{4k\eta} \text{Im} \left[ \langle J_e, \mathcal{L}_e J_e \rangle + \frac{1}{\eta^2} \langle J_m, \mathcal{L}_m J_m \rangle \right] \sim O(1), \quad k \to 0
\]

(A 2)

and

\[
W^{(0)}_m = \frac{\mu}{4k\eta} \text{Im} \left[ \langle J_e, \mathcal{L}_m J_e \rangle + \frac{1}{\eta^2} \langle J_m, \mathcal{L}_m J_m \rangle \right] \sim O(1), \quad k \to 0.
\]

(A 3)

Both terms are to leading order 1 for small \( k \), as is indicated by the \( O(1) \) above.

The second term contains the leading order cross-term

\[
W^{(1)}_e = -\frac{\mu}{4k\eta} \text{Im} \langle J_e, \mathcal{K}_2 J_m \rangle \sim O(k^1),
\]

(A 4)

where

\[
\langle J_e, \mathcal{K}_2 J_m \rangle = \frac{k^2}{4\pi} \int_V \int_V \mathcal{J}_3^e(r_1) \cdot \hat{R} \times J_m(r_2) \cos(k|r_1 - r_2|) \, dV_1 \, dV_2.
\]

(A 5)

Here \( R = r_1 - r_2 \), \( R = |R| \) and \( \hat{R} = R/R \). The next higher order term is

\[
W^{(2)}_e = \frac{\mu}{4k\eta} \text{Im} \left[ \langle J_e, \mathcal{L}_e J_e \rangle + \frac{1}{\eta^2} \langle J_m, \mathcal{L}_m J_m \rangle \right] \sim O(k^2),
\]

(A 6)

where

\[
\langle J_e, \mathcal{L}_e J_e \rangle = \frac{1}{8\pi} \int_V \left[ k^2 \mathcal{J}_1 \cdot \mathcal{J}_2 - (\nabla \cdot \mathcal{J}_e) (\nabla \cdot \mathcal{J}_e) \right] \frac{\sin(k|r_1 - r_2|)}{8\pi} \, dV_1 \, dV_2.
\]

(A 7)

The \( W^{(3)}_e \) term is

\[
W^{(3)}_e = \frac{\mu}{4k\eta} \text{Re} \langle J_e, \mathcal{K}_1 J_m \rangle \sim O(k^3),
\]

(A 8)

where

\[
\langle J_e, \mathcal{K}_1 J_m \rangle = \frac{k^2}{4\pi} \int_V \int_V \mathcal{J}_3^e(r_1) \cdot \hat{R} \times J_m(r_2) \mathcal{J}_1(kR) \, dV_1 \, dV_2.
\]

(A 9)

The last term \( W^{\text{rest}}_e \) is \( O(k^3) \) for small \( k \) and it is coordinate dependent in certain cases [24]

\[
W^{\text{rest}}_e = \frac{\mu}{4} \left[ K_3(J_e) + \frac{1}{\eta^2} K_3(J_m) + K_4(J_e, J_m) \right],
\]

(A 10)

where

\[
K_3(J) = -\int_V \int_V \text{Im}[k^2 \mathcal{J}_{e,1} \cdot \mathcal{J}_{e,2} - (\nabla \cdot J_e)(\nabla \cdot J_e)] \frac{(r_1^2 - r_2^2)}{8\pi R} \mathcal{J}_1(kR) \, dV_1 \, dV_2 \sim O(k^4)
\]

and

\[
K_4(J_e, J_m) = \frac{k}{\eta} \int_V \int_V \text{Re} \left[ \mathcal{J}_{m,2} \times J_{e,1} \right] \frac{r_2 + r_1}{4\pi R} \mathcal{J}_1(kR) + k \hat{R} \frac{r_1^2 - r_2^2}{4\pi R} \mathcal{J}_2(kR) \, dV_1 \, dV_2 \sim O(k^3).
\]
Note that both $W_{\text{rest}}$ and $W_{\text{em}}^{(3)}$ are of the same asymptotic order in $k$. We keep the terms separate due to the sign-change of $W_{\text{em}}^{(3)}$ in (A 1) and since $W_{\text{rest}}$ can depend on the coordinate system. We consider the coordinate independent part of these energies as the essential physical quantity of the stored energy.

References

1. Dirac PAM. 1938 Classical theory of radiating electrons. *Proc. R. Soc. Lond. A* 167, 148–169. (doi:10.1098/rspa.1938.0124)
2. Yaremko Y. 2003 On the regularization procedure in classical electrodynamics. *J. Phys. A Math. Gen.* 36, 5149–5156. (doi:10.1088/0305-4470/36/18/318)
3. Chu LJ. 1948 Physical limitations of omni-directional antennas. *J. Appl. Phys.* 19, 1163–1175. (doi:10.1063/1.1715038)
4. Harrington RF. 1961 *Time harmonic electromagnetic fields*. New York, NY: McGraw-Hill.
5. Collin RE, Rothschild S. 1964 Evaluation of antenna Q. *IEEE Trans. Antennas Propagat.* 12, 23–27. (doi:10.1109/TAP.1964.1138151)
6. Foltz H, McLean J. 1999 Limits on the radiation Q of electrically small antennas restricted to oblong bounding regions. In *IEEE Antennas and Propagation Society International Symposium, Orlando FL, 11–16 July*, vol. 4, pp. 2702–2705. Piscataway, NJ: IEEE.
7. Sten JCE, Koivisto PK, Hujanen A. 2001 Limitations for the radiation Q of a small antenna enclosed in a spheroidal volume: axial polarisation. *AEÜ Int. J. Electron. Commun.* 55, 198–204. (doi:10.1078/1434-8411-00030)
8. Vandenbosch GAE. 2010 Reactive energies, impedance, and Q factor of radiating structures. *IEEE Trans. Antennas Propagat.* 58, 13–27. (doi:10.1109/TAP.2010.2041166)
9. Thal HL. 2006 New radiation Q limits for spherical wire antennas. *IEEE Trans. Antennas Propagat.* 54, 2757–2763. (doi:10.1109/TAP.2006.882184)
10. Yaghjian AD, Best SR. 2005 Impedance, bandwidth, and Q of antennas. *IEEE Trans. Antennas Propagat.* 53, 1298–1324. (doi:10.1109/TAP.2005.844443)
11. Gustafsson M, Nordebo S. 2006 Bandwidth, Q-factor, and resonance models of antennas. *Progr. Electromagnetic Res.* 163, 1–20. (doi:10.2528/PIER06033003)
24. Gustafsson M, Jonsson BLG. 2015 Antenna Q and stored energy expressed in the fields, currents, and input impedance. *IEEE Trans. Antennas Propagat.* 63, 240–249. (doi:10.1109/TAP.2014.2368111)

25. Fano RM. 1950 Theoretical limitations on the broadband matching of arbitrary impedances. *J. Franklin Inst.* 249, 57–83. (doi:10.1016/0016-0032(50)90006-8)

26. Rozanov KN. 2000 Ultimate thickness to bandwidth ratio of radar absorbers. *IEEE Trans. Antennas Propagat.* 48, 1230–1234. (doi:10.1109/8.884491)

27. Gustafsson M, Sohl C, Kristensson G. 2007 Physical limitations on antennas of arbitrary shape. *Proc. R. Soc. A* 463, 2589–2607. (doi:10.1098/rspa.2007.1893)

28. Jonsson BLG, Kolitsidas CI, Hussain N. 2013 Array antenna limitations. *IEEE Antenn. Wireless Propag. Lett.* 12, 1539–1542. (doi:10.1109/LAWP.2013.2291362)

29. Jonsson BLG 2014 Bandwidth limitations and trade-off relations for wide- and multi-band array antennas over a ground plane. In *Progress In Electromagnetics Research Symposium Proceedings (PIERS)*, Guangzhou, China, 25–28 August 2014, pp. 419–423. Cambridge, MA: Electromagnetics Academy.

30. Schiffer M, Szegö G. 1949 Virtual mass and polarization. *Trans. Amer. Math. Soc.* 67, 130–205. (doi:10.1090/S0002-9947-1949-0033922-9)

31. Kleinman RE, Senior TBA. 1986 Rayleigh scattering. In *Low and high frequency asymptotics* (eds VV Varadan, VK Varadan). Handbook on Acoustic, Electromagnetic and Elastic Wave Scattering, vol. 2, ch. 1, pp. 1–70. Amsterdam, The Netherlands: Elsevier Science Publishers.

32. Sihvola A, Ylä-Oijala P, Järvenpää S, Avelin J. 2004 Polarizabilities of Platonic solids. *IEEE Trans. Antennas Propagat.* 52, 2226–2233. (doi:10.1109/TAP.2004.834081)

33. Hansen TV, Kim OS, Breinbjerg O. 2012 Stored energy and quality factor of spherical wave functions – in relation to spherical antennas with material cores. *IEEE Trans. Antennas Propagat.* 60, 1281–1290. (doi:10.1109/TAP.2011.2180330)

34. Stuart HR, Yaghjian AD. 2010 Approaching the lower bound on Q for electrically small electric-dipole antennas using high permeability shells. *IEEE Trans. Antennas Propagat.* 58, 3865–3872. (doi:10.1109/TAP.2010.2078466)

35. Strutt JW. 1871 On the light from the sky, its polarization and colour. *Phil. Mag.* 41, 107–120 (274–279, also published in Lord Rayleigh, Scientific Papers, volume I, Cambridge University Press, Cambridge, 1899).

36. Bethe HA. 1944 Theory of diffraction by small holes. *Phys. Rev.* 66, 163–182. (doi:10.1103/PhysRev.66.163)

37. Yaghjian AD, Stuart HR. 2010 Lower bounds on the Q of electrically small dipole antennas. *IEEE Trans. Antennas Propagat.* 58, 3114–3121. (doi:10.1109/TAP.2010.2055790)

38. Jonsson BLG, Gustafsson M. 2013 Stored energies for electric and magnetic currents with applications to Q for small antennas. In *Proc. Int. Symp. on Electromagnetic Theory, Hiroshima, Japan, 20-24 May 2013*, pp. 1050–1053. Piscataway, NJ: IEEE.

39. Boyd SP, Vandenberghe L. 2004 *Convex optimization*. Cambridge, UK: Cambridge University Press.

40. Gustafsson M, Sohl C, Kristensson G. 2009 Illustrations of new physical bounds on linearly polarized antennas. *IEEE Trans. Antennas Propagat.* 57, 1319–1327. (doi:10.1109/TAP.2009.2016683)

41. Fante RL. 1969 Quality factor of general antennas. *IEEE Trans. Antennas Propagat.* 17, 151–155. (doi:10.1109/TAP.1969.1139411)

42. Antenna Standards Committee of the IEEE Antenna and Propagation Society, The Institute of Electrical and Electronics Engineers Inc., USA, 1993, *IEEE Standard Definition of Terms for Antennas*. IEEE Std 145–1993.

43. Jonsson BLG, Gustafsson M. 2015 Stored energies for electric and magnetic currents densities. In preparation.

44. Abramowitz M, Stegun IA (eds). 1970 *Handbook of mathematical functions*. Applied Mathematics Series, no. 55. Washington DC: National Bureau of Standards.

45. Stratton JA. 1941 *Electromagnetic theory*. New York, NY: McGraw-Hill.

46. Kristensson G 1999 *Spridningsteori med antenntillämpningar*. Lund, Sweden: Studentlitteratur. [In Swedish.]

47. Pozar DM 2009 New results for minimum Q, maximum gain, and polarization properties of electrically small antennas. In *Proc. EuCAP 3rd Eur. Conf. on Antennas and Propagation, Berlin, Germany, 23–27 March*, pp. 1993–1996. Piscataway, NJ: IEEE.
48. Jonsson BLG, Gustafsson M 2014 Stored energies in electric and magnetic current densities for small antennas. (http://arxiv.org/abs/1410.8704)
49. Landkof NS. 1972 *Foundations of modern potential theory*. Berlin, Germany: Springer.
50. Helsing J, Perfekt KM. 2013 On the polarizability and capacitance of the cube. *Appl. Comput. Harmonic Anal.* 34, 445–468. (doi:10.1016/j.acha.2012.07.006)
51. Zeidler E. 1995 *Applied functional analysis: main principles and their applications*. New York, NY: Springer.
52. Payne LE. 1956 New isoperimetric inequalities for eigenvalues and other physical quantities. *Commun. Pure Appl. Math.* 9, 531–542. (doi:10.1002/cpa.3160090323)
53. Lieb EH, Loss M 1997 *Analysis*. Graduate studies in mathematics, vol. 14. Providence, RI: American Mathematical Society.
54. Harrington RF. 1960 Effect of antenna size on gain, bandwidth and efficiency. *J. Res. Natl Bureau Standards* 64D, 1–12. (doi:10.6028/jres.064D.003)
55. McLean JS. 1996 A re-examination of the fundamental limits on the radiation Q of electrically small antennas. *IEEE Trans. Antennas Propagat.* 44, 672–676. (doi:10.1109/8.496253)
56. Jonsson BLG, Gustafsson M. In press. Antenna Q for small antennas with radiation constraints and perturbations. In *Proc. of the 9th Eur. Conf. on Antennas and Propagation (EuCap)*, 2–17 April, Lisbon, Portugal. Piscataway, NJ: IEEE.
57. Peterson AF, Ray SL, Mittra R. 1998 *Computational methods for electromagnetics*. New York, NY: IEEE Press.
58. Grant M, Boyd S 2014 CVX: Matlab software for disciplined convex programming, version 2.1. cvxr.com/cvx.