Bose-Einstein condensation for a self-interacting theory in curved spacetime

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Abstract

The effective action is derived for a self-interacting theory with a finite fixed $O(2)$ charge at finite temperature in curved spacetime. We obtain the high temperature expansion of the effective action in the weak coupling limit. In the relativistic temperature, we discuss about the phase transition in a homogeneous spacetime.

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I. INTRODUCTION

Since the discovery of the Bose-Einstein condensation (BEC) much effort has been made to understand the effect of the non-vanishing chemical potential on thermodynamical quantities in quantum field theories in curved spacetime, where the spatial section may have a boundary. The high temperature behaviors of the grand thermodynamic potential in an arbitrary static spacetime have been investigated by utilizing the heat kernel and zeta-function regularization technique \cite{1}. Especially, for an Einstein universe which is a homogeneous space and have no boundary, the exact results for the thermodynamical quantities are derived in the non-relativistic limit and in the relativistic limit \cite{2}. Recently D. J. Toms studied the BEC in a static spacetime with a possible spatial boundary and derived the universal character—the portion of the charge in the ground state to the total charge is identical to the result in the flat spacetime \cite{3}. But to our knowledge all of them does not treat the interaction term.

For the system having the interaction terms, the effective actions (potential) have been only calculated in the flat spacetime. Kapusta, by studying the effective equation of motion for fields, showed that charge conservation can affect the phase transition of a theory \cite{4}. Haber and Weldon studied the effects of a net background charge on ideal and interacting relativistic Bose gases in the large $N$ approximation \cite{5}. More recently Benson, Bernstein, and Dodelson exactly calculated the one-loop finite temperature effective potential for a self-interacting theory with a fixed $O(2)$ charge \cite{6}.

In this paper we calculate the effective action for a model with an interaction term and a fixed charge at finite temperature in curved spacetime, and study the effects of the interaction on the phase transition. The simple
model under investigation, is described by the matter Lagrangian density

\[ -\mathcal{L} = \frac{1}{2} \left( \partial_\mu \phi_1 \partial_\mu \phi_1 + \partial_\mu \phi_2 \partial_\mu \phi_2 + m^2 \phi^2 + \xi R \phi^2 + \frac{\lambda}{4!} \phi^4 \right), \]

\[ \text{(1)} \]

where \( \xi \) is a numerical factor and \( R \) is the Ricci scalar curvature. This model is \( O(2) \) invariant, and there is a conserved charge \( Q \) associated with \( O(2) \) global symmetry.

In Sec. II we set up the formalism for evaluating the effective action at finite temperature and charge. To evaluate the effective action, we introduce the Riemann normal coordinates and use the heat-kernel. In Sec. III by using the dimensional regularization method, we derive the high temperature expansion of the effective action in the case that the coupling constant \( \lambda \) is small, and briefly discuss about the renormalization. In Sec. IV using the result of Sec. III, we discuss about the phase transition of the theory in the relativistic temperature limit in a homogeneous spacetime with no boundary. Finally we summary the results.

The spacetime we discuss is the ultrastatic spacetime \( \mathcal{M} = R \times \Sigma \).

\[ ds^2 = d\tau^2 + g_{ij}(x)dx^i dx^j, \]

\[ \text{(2)} \]

where the Wick rotation \( \tau = ix^0 \) has to be understood.

**II. ONE-LOOP EFFECTIVE ACTION**

In this section we compute the effective action at finite temperature and finite charge. According to Ref.\[1, 3\], the grand partition function \( Z[\beta, \mu] = e^{-\beta \Omega(\beta, \mu)} \), \( \Omega(\beta, \mu) \) being the thermodynamic potential, can be expressed as a path integral:

\[ Z = N \int_{\phi_1(\tau=0,x)=\phi_1(\tau=\beta,x)} [d\phi_1][d\phi_2] \exp(-S[\phi_i]), \]

\[ \text{(3)} \]
where \( N \) is a constant and the integration has to be taken over all fields \( \phi(\tau, x) \) with periodicity \( \beta = \frac{1}{T} \) with respect to \( \tau \). The action \( S \) is

\[
S = \int_0^\beta d\tau \int \Sigma d^3x \sqrt{g} \left[ \frac{1}{2} \dot{\phi}_1^2 + \dot{\phi}_2^2 + (\nabla \phi_1)^2 + (\nabla \phi_2)^2 + m^2 \phi_2^2 \right.
\]
\[
+ \xi R \phi_2^2] + \frac{\lambda}{4!} \phi^4 + i\mu(\phi_2 \dot{\phi}_1 - \dot{\phi}_2 \phi_1) - \frac{\mu^2}{2} \phi^2 \right]
\]

\( \phi_2 = \phi_1 + \phi_2, \dot{\phi}_i = \frac{\partial \phi_i}{\partial \tau}, \) and \( \mu \) is the Lagrangian multiplier related to the conserved charge \( Q \). It is the same form as the flat spacetime.

In order to calculate the effective action, some perturbative approach must be adopted. The usual one is the loop expansion \([7]\). We now expand the action around a background field configuration, \( \phi_i(x) = \bar{\phi}_i(x) \). \( \bar{\phi}_i \) are determined by the equation \( \frac{\delta \Gamma}{\delta \phi_i} \bigg|_{\phi_i = \bar{\phi}_i} = 0 \), where \( \Gamma \) is the effective action. In general \( \bar{\phi}_i(x) \) is not a constant field in curved spacetime. Then the action in Eq.(4) is expanded in powers of \( \phi'_i(x) = \phi_i(x) - \bar{\phi}_i(x) \):

\[
S = S^{(0)} + S^{(1)} + S^{(2)} + \cdots
\]

The zeroth order term is just the action evaluated at \( \bar{\phi}_i(x) \):

\[
S^{(0)} = \beta \int \Sigma d^3x \sqrt{g} \left[ \frac{1}{2} (\nabla \bar{\phi}_1)^2 + (\nabla \bar{\phi}_2)^2 + m^2 \bar{\phi}_2^2 + \xi R \bar{\phi}_2^2] + \frac{\lambda}{4!} \bar{\phi}^4 - \frac{\mu^2}{2} \bar{\phi}^2 \right]
\]

where \( \bar{\phi}^2 = \bar{\phi}_1^2 + \bar{\phi}_2^2. \) The first order in Eq.(4), which is linear in \( \phi'_i(x) \), can be neglected. The second order term in the action is

\[
S^{(2)} = \frac{1}{2} \int_M d^4x \sqrt{g} \int_M d^4y \sqrt{g} \phi'_i(x) M_{ij} \phi'_j(y),
\]

where the \( 2 \times 2 \) matrix \( M_{ij} \) is the second functional derivative of the action with respect to the field evaluated at \( \bar{\phi}_i \):

\[
M_{ij} = \frac{\delta^2 S}{\delta \phi_i(x) \delta \phi_j(y)} \bigg|_{\phi_i = \bar{\phi}_i}.
\]
Since the boundary condition at finite temperature is periodic for a bosonic field, we may expand $\phi_i'(\tau, x)$ in Fourier modes:

$$
\phi_i'(\tau, x) = \frac{1}{\sqrt{\beta}} \sum_{n=-\infty}^{\infty} \phi_{i,n}(x)e^{i\omega_n\tau}
$$

with $\omega_n = \frac{2\pi n}{\beta}$. Substituting Eq. (9) into Eq. (7), we obtain

$$
Z^{(1)} = \int [d\phi_{1,n}][d\phi_{2,n}] \exp \left[-\frac{1}{2} \sum_n \int_\Sigma d^3x \sqrt{g} \phi_{i,n}(x)\mathcal{M}_{i,j}' \phi_{j,n}(x) \right],
$$

where $\mathcal{M}_{i,j}' = (-\Delta_3 + V_{ij} + \omega_n^2 - \mu^2 - 2\mu\omega_n\epsilon_{ij})$. Here $\Delta_3$ is the Laplacian on the spatial section $\Sigma$. $V_{ij}$ and $\epsilon_{ij}$ are given by

$$
V_{ij} = \xi R\delta_{ij} + \bar{m}_1^2(\bar{\phi})\hat{\phi}_i\hat{\phi}_j + \bar{m}_2^2(\bar{\phi})(\delta_{ij} - \hat{\phi}_i\hat{\phi}_j), \quad \epsilon_{ij} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},
$$

where

$$
\bar{m}_1^2(\bar{\phi}) = m^2 + \frac{\lambda}{2}\bar{\phi}^2, \quad \bar{m}_2^2(\bar{\phi}) = m^2 + \frac{\lambda}{6}\bar{\phi}^2,
$$

$$
\hat{\phi}_i = \frac{\phi_i}{\sqrt{\phi^2}}.
$$

In terms of a loop expansion the effective action is then given by

$$
\Gamma[\bar{\phi}] = S^{(0)}[\bar{\phi}] + \Gamma_{\text{one-loop}} + \cdots,
$$

where the one-loop effective action $\Gamma_{\text{one-loop}}$ is given by

$$
\Gamma_{\text{one-loop}} = -\ln Z^{(1)}.
$$

It is noted that the functional $Z^{(1)}$ at finite temperature in $d = 4$ dimensions is completely reduced to 3–dimensional expression. Using this fact Eq. (16) can be calculated by the proper time formalism in 3-dimensions.
Following DeWitt [8], let us write
\[
\ln Z^{(1)} = \frac{1}{2} \sum_{n=-\infty}^{\infty} \int_{\Sigma} d^{3} x \sqrt{g} \int_{0}^{\infty} \frac{ds}{s} \langle x | e^{-sH_{n}} | x \rangle
\]
\[
\equiv \frac{1}{2} \beta \int_{\Sigma} d^{3} x \sqrt{g} \Gamma
\]
with
\[
H_{n} = -\Delta_{3} + V_{ij} + \omega_{n}^{2} - \mu^{2} - 2\mu\omega_{n}e_{ij}.
\] (18)

Since the eigenvalues of the operator $H_{n}$ are not exactly known for a general background metric $g$, it is necessary to make use of some approximation scheme. The useful approximations are the Riemann normal coordinates [10] and the weak field expansion. We will use the result of Ref. [11]. Following the ansatz for the heat kernel suggested by Jack and Parker, we use the nonlocal form
\[
e^{-s(\tilde{V} - \frac{1}{6} R)}
\]
in the heat kernel [12]. The explicit form of $\Gamma$, up to adiabatic order four, is given by (see Appendix A)
\[
\Gamma = \text{tr} \frac{1}{\beta} \sum_{n=\infty}^{\infty} \int_{0}^{\infty} \frac{ds}{s} \int \frac{d^{N} p}{(2\pi)^{N}} (I_{0} + I_{1} + I_{2} + I_{3} + I_{4} + \cdots)
\]
\[
\equiv \Gamma_{0} + \Gamma_{1} + \Gamma_{2} + \Gamma_{3} + \Gamma_{4} + \cdots,
\]
(19)

where
\[
I_{0} = e^{-s(p^{2} + \tilde{V} - \frac{1}{6} R)},
\]
(20)
\[
I_{1} = 0,
\]
(21)
\[
I_{2} = -se^{-s(p^{2} + \tilde{V} - \frac{1}{6} R)} \left[ \frac{1}{6} R - 2A_{\alpha\beta}(\delta^{\alpha\beta} - sp^{\alpha} p^{\beta}) \right],
\]
(22)
\[
I_{3} = -ise^{-s(p^{2} + \tilde{V} - \frac{1}{6} R)} \left[ s\tilde{V}_{\alpha\beta} p^{\alpha} - 3A_{3\alpha\beta}(-s\delta^{\alpha\beta} p^{\gamma} + \frac{4}{3} s^{2} p^{\alpha} p^{\beta} p^{\gamma}) \right],
\]
(23)
\[
I_{4} = -se^{-s(p^{2} + \tilde{V} - \frac{1}{6} R)} \left[ \frac{1}{2} \tilde{V}_{\alpha\beta}(s\delta^{\alpha\beta} - \frac{4}{3} s^{2} p^{\alpha} p^{\beta}) \right. \\
+ \left. (4A_{4\alpha\beta\gamma\delta} - 2A_{2\alpha\beta A_{2\gamma\delta}}) \left[-s(\delta^{\alpha\beta} \delta^{\gamma\delta} + \delta^{\alpha\gamma} \delta^{\beta\delta} + \delta^{\alpha\delta} \delta^{\beta\gamma}) - 2s^{3} p^{\alpha} p^{\beta} p^{\gamma} p^{\delta} + \frac{4}{3} s^{2} (\delta^{\alpha\beta} \delta^{\gamma\delta} p^{\delta} + \delta^{\alpha\gamma} p^{\beta} p^{\delta}) \right] \right],
\]
\[
+\delta^{\alpha\delta}p^{\gamma}p^{\beta} + \delta^{\beta\gamma}p^{\alpha}p^{\delta} + \delta^{\beta\delta}p^{\gamma}p^{\alpha} + \delta^{\gamma\delta}p^{\alpha}p^{\beta})
\]

\[
+ s^2 e^{-s(p^2 + \tilde{V} - \frac{1}{6}R)} \left[ \frac{1}{12} R - 2A_{2\alpha\beta}(\frac{1}{2} \delta^{\alpha\beta} - \frac{2}{3} p^\alpha p^\beta) \right]
\]

\[
\cdot \left[ \frac{1}{6} R - 2A_{2\alpha\beta} \delta^{\alpha\beta} \right] + 2s A_{2\alpha\beta} \left[ -\frac{1}{18} R p^\alpha p^\beta \right]
\]

\[
+ \frac{1}{6} s p^\alpha p^\beta p^\gamma p^\delta R_{\gamma\delta} - \frac{2}{9} p^\gamma p^\delta R_{\gamma\delta} \right].
\]

(24)

Here \( \tilde{V} = V_{ij} + \omega_n^2 - \mu^2 - 2\mu\omega_n\epsilon_{ij} \). After integrating about the momentum \( p \), we get

\[
\Gamma_0 = \text{tr} \int_0^\infty ds \frac{1}{s} \sum_n \frac{1}{(4\pi s)^2} e^{-s(\tilde{V} - \frac{1}{6}R)}, \quad (25)
\]

\[
\Gamma_1 = \Gamma_2 = \Gamma_3 = 0, \quad (26)
\]

\[
\Gamma_4 = \text{tr} \frac{1}{\beta} \sum_n \int_0^\infty ds \frac{s^2}{(4\pi s)^2} e^{-s(\tilde{V} - \frac{1}{6}R)} \left[ -\frac{1}{6} \Box \tilde{V} \right.
\]

\[
\left. + \frac{1}{30} \Box R + \frac{1}{180} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} - \frac{1}{180} R_{\alpha\beta} R^{\alpha\beta} \right], \quad (27)
\]

where \( R, R_{\alpha\beta\gamma\delta} \) and \( R_{\alpha\beta} \) are curvatures on the 3-dimensional hypersurface \( \Sigma \). The evaluation of the \( \Gamma_0 \) and \( \Gamma_4 \) is not easy because the potential \( \tilde{V} \) is not a diagonal matrix. However if we use the property of the trace, we can diagonalized \( e^{-s(\tilde{V} - \frac{1}{6}R)} \) (see Appendix B). If we choose the matrix \( S \) as

\[
S = \left( \begin{array}{cc} (\bar{m}_1^2 - \bar{m}_2^2) \hat{\phi}_1 \hat{\phi}_2 - 2\mu\omega_n & (\bar{m}_1^2 - \bar{m}_2^2) \hat{\phi}_1 \hat{\phi}_2 - 2\mu\omega_n \\ \lambda_+ - C & \lambda_- - C \end{array} \right), \quad (28)
\]

where

\[
C = (\xi - \frac{1}{6})R + \bar{m}_1^2 \hat{\phi}_1^2 + \bar{m}_2^2 \hat{\phi}_2^2 + \omega_n^2 - \mu^2, \quad (29)
\]

and \( \lambda_\pm \) are the eigenvalues of the matrix \( \tilde{V} - \frac{1}{6}R \):

\[
\lambda_\pm = (\xi - \frac{1}{6})R + m^2 + \frac{\lambda}{3} \hat{\phi}^2 + \omega_n^2 - \mu^2 \pm \frac{1}{2} \left[ \frac{\lambda^2}{9} \hat{\phi}^4 - 16\mu^2\omega_n^2 \right]^{\frac{1}{2}}. \quad (30)
\]
Then we can rewritten the $\Gamma_0$ and $\Gamma_4$ as a more tractable form:

$$
\Gamma_0 = \frac{1}{\beta} \sum_n (4\pi)^{-N/2} \int_0^\infty \frac{ds}{s} s^{-N/2} \text{tr} \left( e^{-s\lambda_+} \begin{pmatrix} 0 & 0 \\ 0 & e^{-s\lambda_-} \end{pmatrix} \right),$$  
(31)

$$
\Gamma_4 = \frac{1}{\beta} \sum_n (4\pi)^{-N/2} \int_0^\infty \frac{ds}{s} s^{-N/2+2} \text{tr} \left( e^{-s\lambda_+} \begin{pmatrix} 0 & 0 \\ 0 & e^{-s\lambda_-} \end{pmatrix} \right) 
\cdot \left[ -\frac{1}{6} S^{-1} \Box V S + \tilde{a}_2 \right] 
- \frac{\lambda}{18} \left[ \frac{\lambda^2}{9} \sigma^4 - 16\mu^2 \omega^2 \right]^{-\frac{1}{2}} \left[ \frac{\lambda}{6} (\sigma^2 - \sigma_0^2) \Box (\sigma^2 - \sigma_0^2) \right] 
+ \frac{2}{3} \lambda \sigma_0 \sigma_2 \sigma \Box (\sigma_0 \sigma_2) \right] \left( e^{-s\lambda_+} - e^{-s\lambda_-} \right),$$  
(32)

where $\tilde{a}_2 = \frac{1}{30} \Box R + \frac{1}{180} R_{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} R_{\alpha\beta} R_{\alpha\beta} - \frac{1}{180} R_{\alpha\beta} R_{\alpha\beta}$.

Now, let us consider limiting case. First of all, we consider the case that the metric is flat. In flat spacetime the background fields $\phi_i$ is a constant because the spacetime is homogeneous. Then only $\Gamma_0$ survive. Using the formula

$$
G_0 = \frac{\partial \Gamma_0}{\partial m^2},
$$

$$
= -\frac{1}{\beta} \sum_n \int \frac{d^N p}{(2\pi)^N} \int_0^\infty ds (e^{-s(p^2 + \lambda)} + e^{-s(p^2 + \lambda)}) 
- \frac{1}{\beta} \sum_n \int \frac{d^N p}{(2\pi)^N} \left( \frac{1}{p^2 + \lambda} + \frac{1}{p^2 + \lambda} \right) 
= -\frac{1}{\beta} \sum_n \int \frac{d^N p}{(2\pi)^N} \frac{\partial}{\partial m^2} \ln \left[ p^2 + (\xi - \frac{1}{6}) R + m^2 + \lambda \sigma^2 + \omega^2 - \mu^2 \right] \sigma^2 - \frac{1}{4} \lambda^2 \sigma^4 - 16\mu^2 \omega^2 \]
+ \left( e^{-s\lambda_+} - e^{-s\lambda_-} \right),$$  
(33)

we obtain, after sum over $n$,

$$
\Gamma_0 = -\frac{1}{\beta} \int \frac{d^N p}{(2\pi)^N} \left[ \beta E_+ + 2 \ln(1 + e^{-\beta E_+}) \right]$$
\[ E_{\pm}^2 = p^2 + M_f^2 + \mu^2 \pm [4\mu^2(p^2 + M_f^2)^{1/2} + \frac{\lambda^2}{36}\bar{\phi}^4]^{1/2} \]

with \( M_f^2 = m^2(1 + \frac{\lambda}{3m^2}\bar{\phi}^2) \). In this case, the effective action (potential) is given by

\[
\Gamma[\bar{\phi}] = S^{(0)}[\bar{\phi}] + \beta \int d^3x \sqrt{g} \frac{1}{\beta} \int \frac{d^N p}{(2\pi)^N} \left[ \frac{\beta(E_+ + E_-)}{2} + \ln(1 + e^{-\beta E_+}) + \ln(1 + e^{-\beta E_-}) \right].
\]

This is exactly the same form as the previous result in the flat spacetime \cite{1}. Second, for the case \( \lambda \to 0, \lambda_{\pm} \to (\xi - \frac{1}{6}) R + m^2 + (\omega_n \pm i\mu)^2 \). Then \( \Gamma_0 \) and \( \Gamma_4 \) go to the previous result because \( \Box \tilde{V} \to 0 \). For \( \mu = 0, \lambda_+ = (\xi - \frac{1}{6}) R + m^2 + \omega_n^2 + \frac{1}{2} \xi \lambda \bar{\phi}^2 \) and \( \lambda_- = (\xi - \frac{1}{6}) R + m^2 + \omega_n^2 + \frac{1}{6} \xi \lambda \bar{\phi}^2 \). In this case the effective action goes to the previously known result \cite{1}.

### III. HIGH-TEMPERATURE EXPANSION

In this section we will calculate \( \Gamma_0 \) and \( \Gamma_4 \) in the high temperature limit and the coupling constant \( \lambda \) is small. The regularization method is the dimensional one \cite{13}. After the regularization, we take \( N \to 3 \). First of all, let us calculate \( \Gamma_0 \). For small \( \lambda \), \( \Gamma_0 \) can be written as

\[
\Gamma_0 = \frac{1}{\beta} \sum_{n=-\infty}^{\infty} (4\pi)^{-N/2} \Gamma(-\frac{N}{2}) \left\{ \left[ M^2 + \omega_n^2 - \mu^2 + \frac{1}{2} \left( \frac{\lambda^2}{9}\bar{\phi}^4 - 16\mu^2\omega_n^2 \right) \right]^{\frac{N}{2}} \right. \\
+ \left. \left[ M^2 + \omega_n^2 - \mu^2 - \frac{1}{2} \left( \frac{\lambda^2}{9}\bar{\phi}^4 - 16\mu^2\omega_n^2 \right) \right]^{\frac{N}{2}} \right\} = \Gamma_0^{(n=0)} + \Gamma_0^{(1)} + \Gamma_0^{(2)} + \cdots,
\]

where

\[
\Gamma_0^{(n=0)} = \frac{1}{\beta} (4\pi)^{-N/2} \Gamma(-\frac{N}{2}) \left\{ \left[ M^2 - \mu^2 + \frac{\lambda^2}{6}\bar{\phi}^2 \right]^{\frac{N}{2}} \right.
\]


\[ M^2 - \mu^2 - \frac{\lambda}{6} \phi^2 \] \]

\[ + \left( M^2 - \mu^2 \right) \left( \omega_n + i\mu \right)^2 \right]^{\frac{N}{2}} \]

\( \Gamma_0^{(1)} = \frac{1}{\beta} (4\pi)^{-N/2} \Gamma \left( \frac{N}{2} \right) 2 \sum_{n=1}^{\infty} \left\{ \left[ M^2 + (\omega_n - i\mu)^2 \right]^{\frac{N}{2}} + \left[ M^2 + (\omega_n + i\mu)^2 \right]^{\frac{N}{2}} \right\}, \] \( \Gamma_0^{(2)} = \frac{1}{\beta} (4\pi)^{-N/2} \Gamma \left( \frac{N}{2} \right) 2 \sum_{n=1}^{\infty} \left\{ i \frac{N}{2} \frac{\lambda}{144 \mu \omega_n} \left[ M^2 + (\omega_n - i\mu)^2 \right]^{\frac{N}{2}-1} - i \frac{N}{2} \frac{\lambda}{144 \mu \omega_n} \left[ M^2 + (\omega_n + i\mu)^2 \right]^{\frac{N}{2}-1} \right\}, \)

where \( M^2 = m^2 - \mu^2 + \frac{\lambda}{6} \phi^2 \). The evaluations of the Matsubara sums in Eqs. (37,38,39) are not difficult [14]. By using the dimensional regularization, we obtain

\[ \Gamma_0^{(0)} = \frac{1}{\beta} \frac{1}{6\pi} \left( M^2 - \mu^2 + \frac{\lambda}{6} \phi^2 \right)^{3/2} + \left( M^2 - \mu^2 - \frac{\lambda}{6} \phi^2 \right)^{3/2}, \]

\[ \Gamma_0^{(1)} = \frac{\pi^2}{45 \beta^4} - \frac{1}{12 \beta^2} \left( M^2 - 2\mu^2 \right) + \left[ - \frac{1}{\epsilon} + \frac{1}{2} \gamma + \ln \left( \frac{\beta^2 M^2}{4\pi} \right) \right] \frac{M^4}{(4\pi)^2} \]

\[ + \frac{\mu^2}{8 \pi^2} \left( M^2 - \frac{1}{3} \mu^2 \right) + \frac{\pi^2}{\beta^4} \sum_{k,r \geq 1, k+r \neq 0, 2} \frac{8}{3} C \left( \frac{-3}{2}, k, 2r \right) (-1)^r \]

\[ \left( \frac{\mu \beta}{2\pi} \right)^{2r} \left( \frac{\beta M}{2\pi} \right)^{2k} \zeta \left( -3 + 2k + 2r \right) \]

\[ \Gamma_0^{(2)} = \frac{\lambda^2 \phi^4}{288 \pi^2} \left[ - \frac{1}{\epsilon} + \frac{1}{2} \gamma + \ln \left( \frac{\beta^2 M^2}{4\pi} \right) \right] + \sum_{k,r \geq 0, k+r \neq 0} C \left( -\frac{1}{2}, k, 2r + 1 \right) \]

\[ (-1)^r \left( \frac{\mu \beta}{2\pi} \right)^{2r} \left( \frac{\beta M}{2\pi} \right)^{2k} \zeta \left( 2k + 2r + 1 \right), \]

where \( \epsilon = 4 - d \) and \( \gamma \) is the Euler constant 0.577..., and

\[ C \left( s - \frac{N}{2}, k, r \right) = (-1)^{k+r} \frac{\Gamma \left( s - \frac{N}{2} + k \right) \Gamma \left( 2s - N + 2k + 2r \right)}{k! r! \Gamma \left( s - \frac{N}{2} \right) \Gamma \left( 2s - N + 2k \right)} \]
Now let us calculate the $\Gamma_4$. After some algebra, we get

$$\Gamma_4 = \frac{1}{4\pi^2} \left\{ \left[ -\frac{1}{\epsilon} + \frac{1}{2} \gamma + \ln\left(\frac{\beta^2 M^2}{4\pi}\right) \right] + \sum_{k,r \geq 0, k+r \neq 0} C \left( \frac{1}{2}, k, 2r \right) \right\} \cdot \left( \frac{i\mu \beta}{2\pi} \right)^{2r} (\frac{\beta M^{2k}}{2\pi} \zeta(1 + 2k + 2r)) \left( \tilde{a}_2 - \frac{\lambda}{18} \square \tilde{\phi}^2 \right)$$

$$+ \frac{1}{8\pi^2} \left\{ \left[ \left[ M^2 - \mu^2 + \frac{\lambda}{6} \tilde{\phi}^2 \right]^{-\frac{1}{2}} + \left[ M^2 - \mu^2 - \frac{\lambda}{6} \tilde{\phi}^2 \right]^{-\frac{1}{2}} \right] (\tilde{a}_2 - \frac{\lambda}{18} \square \tilde{\phi}^2) \right\}$$

$$- \frac{\lambda}{6\tilde{\phi}^2} \left( \frac{1}{6} (\tilde{\phi}_1^2 - \tilde{\phi}_2^2) \Box (\tilde{\phi}_1^2 - \tilde{\phi}_2^2) + \frac{2}{3} \tilde{\phi}_1 \tilde{\phi}_2 \Box (\tilde{\phi}_1 \tilde{\phi}_2) \right)$$

$$\cdot \left[ \left[ M^2 - \mu^2 + \frac{\lambda}{6} \tilde{\phi}^2 \right]^{-\frac{1}{2}} - \left[ M^2 - \mu^2 - \frac{\lambda}{6} \tilde{\phi}^2 \right]^{-\frac{1}{2}} \right] \right\}$$

$$+ \frac{\lambda^2}{288\pi^2} \left[ \frac{1}{6} (\tilde{\phi}_1^2 - \tilde{\phi}_2^2) \Box (\tilde{\phi}_1^2 - \tilde{\phi}_2^2) + \frac{2}{3} \tilde{\phi}_1 \tilde{\phi}_2 \Box (\tilde{\phi}_1 \tilde{\phi}_2) \right]$$

$$\cdot \sum_{k,r \geq 0} C \left( \frac{1}{2}, k, 2r + 1 \right) (-1)^{2r} \left( \frac{\mu \beta}{2\pi} \right)^{2r} (\frac{\beta M}{2\pi})^{2k} \zeta(3 + 2k + 2r), \quad (44)$$

where we have calculated up to $\lambda^2$ order except $n = 0$ term. The final result for the effective action is given by

$$\Gamma[\tilde{\phi}] = S[\tilde{\phi}] - \frac{1}{2} \beta \int_\Sigma d^3x \sqrt{g} \left( \Gamma_0^{(n=0)} + \Gamma_0^{(1)} + \Gamma_0^{(2)} + \Gamma_4 + \cdots \right) \quad (45)$$

up to $\lambda^2$ order.

Let us consider the infinite part of the effective action. It is given by

$$\Gamma_{\text{infinite}}[\tilde{\phi}] = -\int_\Sigma d^3x \sqrt{g} \left( -\frac{1}{\epsilon} + \frac{1}{2} \gamma + \ln\left(\frac{M^2}{4\pi \bar{\mu}^2}\right) \right)$$

$$\left[ \frac{M^2}{2} + \frac{\lambda^2}{72} \tilde{\phi}^4 + (\tilde{a}_2 - \frac{\lambda}{18} \Box \tilde{\phi}^2) \right], \quad (46)$$

where $\bar{\mu}$ is a arbitrary mass scale. It is note that the value of the pole part differs from one that calculated by the zeta function regularization method [1], [14]. In the case $\lambda \to 0$, this reduce to the known result. These infinite terms
can be removed by absorbing it into the gravitational Lagrangian and by introducing the counter terms \[15\]. Up to terms which are total divergence, the coupling constant and mass counter terms are only needed. This is coincident with the fact that at one loop level, there is no wave functional counter term \[16\].

**IV. BOSE-EINSTEIN CONDENSATION**

In quantum field theory, BEC is interpreted as a symmetry breaking effect in flat as well as in curved spacetime \[3, 5, 6\]. In order to discuss the BEC in the case of a relativistic Bose gas, it is important to clarify in what sense we mean that the temperature is relativistic. In flat spacetime, it means that \( T \gg M \). In curved spacetime, in addition to \( T \gg M \), we require \( T \gg |R|^1/2 \), where \( |R| \) is the magnitude of a typical curvature of the spacetime \[3\].

The high temperature effective action, from Eq.(45), has the form in the relativistic temperature,

\[
\Gamma[\bar{\phi}] = S[\bar{\phi}] + \Gamma_{\text{one-loop}} = \beta \int_\Sigma d^3x \sqrt{g} \left[ \frac{1}{2} \nabla \bar{\phi} \cdot \nabla \bar{\phi} + \frac{1}{2} m^2 \bar{\phi}^2 + \frac{1}{2} \xi R \bar{\phi}^2 + \frac{\lambda}{4!} - \frac{\mu^2}{2} \bar{\phi}^2 \right.
\]

\[
- \frac{\pi^2}{45} \frac{1}{\beta^4} + \frac{1}{12 \beta^2} (M^2 - 2 \mu^2) + \cdots \right].
\]

At first we discuss about the case that the spacetime is homogeneous. The translational invariance implies that the background field is a constant (This is always not true, For \( \lambda = 0 \), see Ref. \[3, 17\]). In this case, the equation of motion for \( \bar{\phi}_i \) implies

\[
(m^2 - \mu^2 + \xi R + \frac{\lambda}{6} \bar{\phi}^2 + \frac{\lambda}{18} T^2) \bar{\phi}_i = 0.
\]
Thus the effective action have two minima: \( \bar{\phi}_i = 0 \), with unbroken symmetry,

\[
\bar{\phi}^2 = \frac{6}{\lambda} \left[ \mu^2 - \left( m^2 + \xi R + \frac{\lambda}{18} T^2 \right) \right] \\
= \frac{6}{\lambda} \left[ \mu^2 - m^2(T) \right]
\]

(49)

with broken symmetry. This is the same form as the result in the flat space-time except for changing \( m^2 \rightarrow m^2 + \xi R \) [6]. The critical temperature when phase transition occurs is

\[
T_c^2 = \frac{18}{\lambda} \left[ \mu^2(T) - (m^2 + \xi R) \right]
\]

(50)

In general, \( \mu^2 \) has the temperature dependence due to the charge conservation.

Now consider the expectation value of the charge operator \( Q \) which is given in terms of the effective action by

\[
Q = -\frac{1}{\beta} \frac{\partial \Gamma}{\partial \mu} \equiv Q_0 + Q_{th} \\
= \int_{\Sigma} d^3 x \sqrt{g}(\mu \bar{\phi}^2 + \frac{\mu}{3} T^2)
\]

(51)

in the high temperature approximation. \( Q_0 \) is the charge in the ground state and \( Q_{th} \) is the charge in the thermal excited state. If \( T \) is high, \( \bar{\phi}_i = 0 \). Then \( Q = \frac{\mu}{3} T^2 V \), where \( V \) is the volume of the spatial section \( \Sigma \). As \( T \) decreases, \( \mu \) must increase until the temperature becomes \( T_c \). From Eq.(49) and Eq.(51), the critical temperature is rewritten as

\[
T_c = \left( \frac{3Q}{V} \right)^{1/2} (m^2(T))^{-1/2}.
\]

(52)

For \( T \leq T_c \), it is easily seen that

\[
Q_0 = Q \left[ 1 - \left( \frac{T}{T_c} \right)^2 \right],
\]

(53)
which is identical form to the result in flat spacetime [6].

Up to date we have restricted the value of the fields $\bar{\phi}_i$ to be a constant. But in curved spacetime or in the case that there are boundaries, as others showed, the value of the fields $\bar{\phi}_i$ is not a constant even for $\lambda = 0$. In addition, the critical temperature differs from our result because there is a non-zero lowest eigenvalue for the Laplacian operator [3, 17].

In an inhomogeneous spacetime, the equation of motion for the fields $\bar{\phi}_i$ is given by

$$\Box \bar{\phi}_i - (m^2 - \mu^2 + \xi R + \frac{1}{6} \bar{\phi}^2 + \frac{\lambda}{18} T^2) \bar{\phi}_i = 0.$$  \hspace{1cm} (54)

This is a nonlinear equation, so it is not easy to find the analytic solution. Therefore it is difficult to discuss about the phase transition of the system.

**V. DISCUSSION**

In this paper we have calculated the effective action for the $O(2)$ invariant model with quartic self-interaction and global $O(2)$ fixed charge at finite temperature in curved spacetime. We calculated the effective action by using the modified heat kernel technique suggested by Jack and Parker and introducing the normal coordinates expansion method. We also used the fact that a $2 \times 2$ matrix can be diagonalized in general. We showed that the coefficient $a_1$ - the Minakshiuundaram-Seeley-DeWitt coefficient- does not appear similarly to the result at zero temperature case.

Using the effective action, we discussed about the phase transition in the homogeneous spacetime. It is shown that the charge condensation property is the same as the flat spacetime case, except for the changing of the mass. The form of the condensed charge fraction to the total charge is the same as the non-interacting case (Eq.(53)). For the inhomogeneous spacetime we have not discussed about the phase transition because the equation of motion is nonlinear.
Even if we have studied the simple $O(N = 2)$ model, our result can be generalized to an arbitrary potential type and to the $N > 2$ model. But for $N > 2$ one will be confronted with the diagonalization problem of the matrix.

For conformally ultrastatic spacetime the effective action may also be derived by using the conformal transformation technique.

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**APPENDIX A**

In normal coordinates, an operator $\mathcal{O}(x, x')$ may be expanded in powers of $y$ about the point $x'$, where $y^\mu = \frac{dx^\mu(\tau)}{d\tau} |_{\tau = 0}, x(\tau)$ being a geodesic connecting $x$ and $x'$ [10]. Then, up to adiabatic order four [11],

\[
(-\Box + \tilde{V}) = C_1 + C_2 + C_3 + C_4 + \cdots, \tag{55}
\]

where

\[
C_1 = (-\partial^2 + \tilde{V} - \frac{1}{6} R), \tag{56}
\]
\[
C_2 = -(A_{2\alpha\beta} y^\alpha y^\beta + B^2_{\alpha\beta\gamma\delta} \partial\mu y^\alpha y^\beta y^\gamma \partial\nu - \frac{1}{6} R), \tag{57}
\]
\[
C_3 = -(A_{3\alpha\beta\gamma} y^\alpha y^\beta y^\gamma \partial^2 + B^{\mu\nu}_{\alpha\beta\gamma} \partial\mu y^\alpha y^\beta y^\gamma \partial\nu - \tilde{V}_\alpha y^\alpha), \tag{58}
\]
\[
C_4 = -(A_{2\alpha\beta} B^2_{\gamma\delta} y^\alpha y^\beta \partial\mu y^\gamma \partial\nu + A_{4\alpha\beta\gamma\delta} y^\alpha y^\beta y^\gamma y^\delta \partial^2 + B^4_{\alpha\beta\gamma\delta} \partial\mu y^\alpha y^\beta y^\gamma \partial\nu - \frac{1}{2} \tilde{V}_\alpha y^\alpha y^\beta), \tag{59}
\]

with

\[
A_{2\alpha\beta} = \frac{1}{6} R_{\alpha\beta}, \tag{60}
\]

15
\[ A_{3\alpha\beta\gamma} = \frac{1}{12} R_{\alpha\beta\gamma} \] (61)

\[ A_{4\alpha\beta\gamma\delta} = \frac{1}{40} R_{\alpha\beta\gamma\delta} + \frac{1}{180} R_{\mu\alpha\beta}\Sigma_{\gamma\delta} + \frac{1}{72} R_{\alpha\beta} R_{\gamma\delta} \] (62)

\[ B_{2}^{\mu\nu}_{\alpha\beta} = \frac{1}{3} R^\lambda_{\alpha\beta} \delta^{\mu\nu} - \frac{1}{6} \delta^{\mu\nu} R_{\alpha\beta} \] (63)

\[ B_{3}^{\mu\nu}_{\alpha\beta\gamma} = \frac{1}{6} R^\lambda_{\alpha\beta\gamma} - \frac{1}{12} \delta^{\mu\nu} R_{\alpha\beta\gamma} \] (64)

\[ B_{4}^{\mu\nu}_{\alpha\beta\gamma\delta} = \delta^{\mu\nu} \left( -\frac{1}{40} R_{\alpha\beta\gamma\delta} + \frac{1}{180} R_{\lambda\alpha\beta\delta} + \frac{1}{72} R_{\alpha\beta} R_{\gamma\delta} \right) + \left( \frac{1}{20} R^\lambda_{\alpha\beta\gamma\delta} + \frac{1}{15} R^\mu_{\alpha\beta\delta} R^\lambda_{\gamma\delta} - \frac{1}{18} R_{\alpha\beta} R^\mu_{\gamma\delta} \right) \] (65)

By using the Schwinger expansions [9]

\[ e^{-(H_0 + H_I)s} = e^{-H_0 s} + (-s) \int_0^1 du e^{-(1-u)H_0 s} H_I e^{-uH_0 s} \] (66)

\[ + (-s)^2 \int_0^1 du \int_0^1 dv e^{-(1-u)H_0 s} H_I e^{-u(1-v)H_0 s} H_I e^{-uvH_0 s} + \ldots, \]

one may evaluate the heat kernel

\[ K = e^{(-\Box + V)s} \]

\[ = e^{-(\Box + \tilde{V} - \frac{1}{6} R)s} + (-s) \int_0^1 du e^{-(1-u)(\Box + \tilde{V} - \frac{1}{6} R)s} \]

\[ \cdot \left( C_2 + C_3 + C_4 \right) e^{-u(\Box + \tilde{V} - \frac{1}{6} R)s} \]

\[ + (-s)^2 \int_0^1 du \int_0^1 dv e^{-(1-u)(\Box + \tilde{V} - \frac{1}{6} R)s} C_2 e^{-u(1-v)(\Box + \tilde{V} - \frac{1}{6} R)s} \]

\[ \cdot C_2 e^{-uv(\Box + \tilde{V} - \frac{1}{6} R)s} + \ldots. \] (67)

In local momentum space, for any operator \( O(x, x') \)

\[ \langle x | O(x, x') | x' \rangle = \langle y | O(y, x') | 0 \rangle \]

\[ = \int dp dq \langle y | p \rangle \langle p | O(i \frac{\partial}{\partial p}, x') | q \rangle \langle q | y \rangle, \] (68)

where \( y = x - x' \). Then, using the trace property \( \text{tr}(ABC) = \text{tr}(CAB) \), one can obtain

\[ \text{tr} \langle x | K | x \rangle = \text{tr} \int \frac{d^N p}{(2\pi)^N} \left( I_0 + I_1 + I_2 + I_3 + I_4 + \ldots \right). \] (69)
For any real $2 \times 2$ matrix $A$, a matrix $S$ can be chosen so that $S^{-1}AS$ has the diagonal form. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad A \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} = \lambda_+ \begin{pmatrix} \alpha \\ \gamma \end{pmatrix}, \quad A \begin{pmatrix} \beta \\ \delta \end{pmatrix} = \lambda_- \begin{pmatrix} \beta \\ \delta \end{pmatrix},$$

(70)

where $\lambda_\pm$ are eigenvalues, and $\begin{pmatrix} \alpha \\ \gamma \end{pmatrix}$ and $\begin{pmatrix} \beta \\ \delta \end{pmatrix}$ are the corresponding eigenvectors. They are given by

$$\lambda_\pm = \frac{1}{2} \left[ (a + d) \pm \left( (a + d)^2 - 4(ad - bc) \right)^{1/2} \right],$$

(71)

$$\begin{pmatrix} \alpha \\ \gamma \end{pmatrix} = \begin{pmatrix} b \\ \lambda_+ - a \end{pmatrix}, \quad \begin{pmatrix} \beta \\ \delta \end{pmatrix} + \begin{pmatrix} b \\ \lambda_- - a \end{pmatrix}.$$  \hspace{1cm} (72)

If we choose the matrix $S$ as

$$S = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

(73)

then

$$S^{-1}AS = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}. \hspace{1cm} (74)$$

From this we can show that

$$S^{-1} e^{tA}S = e^{S^{-1}AS} = \exp \begin{pmatrix} \lambda_+ t & 0 \\ 0 & \lambda_- \end{pmatrix} = \begin{pmatrix} e^{\lambda_+ t} & 0 \\ 0 & e^{\lambda_- t} \end{pmatrix},$$

(75)

$$\text{tr} e^{At} = \text{tr}(S^{-1}e^{A}S) = (e^{\lambda_+ t} + e^{\lambda_- t}),$$

(76)

$$\text{tr} e^{At}B = \text{tr}(S^{-1}e^{A}SS^{-1}BS)$$

$$= \text{tr} \left[ \begin{pmatrix} e^{\lambda_+ t} & 0 \\ 0 & e^{\lambda_- t} \end{pmatrix} S^{-1}BS \right].$$  \hspace{1cm} (77)
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