Extremal Selections of Multifunctions
Generating a Continuous Flow

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1 - Introduction

Let $F : [0, T] \times \mathbb{R}^n \mapsto 2^{\mathbb{R}^n}$ be a continuous multifunction with compact, not necessarily convex values. If $F$ is Lipschitz continuous, it was shown in [4] that there exists a measurable selection $f$ of $F$ such that, for every $x_0$, the Cauchy problem

$$\dot{x}(t) = f(t, x(t)), \quad x(0) = x_0$$

has a unique Caratheodory solution, depending continuously on $x_0$.

In this paper, we prove that the above selection $f$ can be chosen so that $f(t, x) \in ext F(t, x)$ for all $t, x$. More generally, the result remains valid if $F$ satisfies the following Lipschitz Selection Property:

(LSP) For every $t, x$, every $y \in \overline{co} F(t, x)$ and $\varepsilon > 0$, there exists a Lipschitz selection $\phi$ of $\overline{co} F$, defined on a neighborhood of $(t, x)$, with $|\phi(t, x) - y| < \varepsilon$.

We remark that, by [7,9], every Lipschitz multifunction with compact values satisfies (LSP). Another interesting class, for which (LSP) holds, consists of those continuous multifunctions $F$ whose values are compact and have convex closure with nonempty interior.
Indeed, for any given $t, x, y, \varepsilon$, choosing $y' \in \text{int} \overline{co} F(t, x)$ with $|y' - y| < \varepsilon$, the constant function $\phi \equiv y'$ is a local selection from $\overline{co} F$ satisfying the requirements.

In the following, $\Omega \subseteq \mathbb{R}^n$ is an open set, $\overline{B}(0, M)$ is the closed ball centered at the origin with radius $M$, $\overline{B}(D; MT)$ is the closed neighborhood of radius $MT$ around the set $D$, while $AC$ the Sobolev space of all absolutely continuous functions $u : [0, T] \mapsto \mathbb{R}^n$, with norm $\|u\|_{AC} = \int_0^T (|u(t)| + |\dot{u}(t)|) \, dt$.

**Theorem 1.** Let $F : [0, T] \times \Omega \mapsto 2^{\mathbb{R}^n}$ be a bounded continuous multifunction with compact values, satisfying (LSP). Assume that $F(t, x) \subseteq \overline{B}(0, M)$ for all $t, x$ and let $D$ be a compact set such that $\overline{B}(D; MT) \subset \Omega$. Then there exists a measurable function $f$, with

$$f(t, x) \in \text{ext} F(t, x) \quad \forall t, x,$$

such that, for every $(t_0, x_0) \in [0, T] \times D$, the Cauchy problem

$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0$$

has a unique Caratheodory solution $x(\cdot) = x(\cdot, t_0, x_0)$ on $[0, T]$, depending continuously on $t_0, x_0$ in the norm of $AC$.

Moreover, if $\varepsilon_0 > 0$ and a Lipschitz continuous selection $f_0$ of $\overline{co} F$ are given, then one can construct $f$ with the following additional property. Denoting by $y(\cdot, t_0, x_0)$ the unique solution of

$$\dot{y}(t) = f_0(t, y(t)), \quad y(t_0) = x_0,$$

for every $(t_0, x_0) \in [0, T] \times D$ one has

$$|y(t, t_0, x_0) - x(t, t_0, x_0)| \leq \varepsilon_0 \quad \forall t \in [0, T].$$

The proof of the above theorem, given in section 3, starts with the construction of a sequence $f_n$ of selections from $\overline{co} F$, which are piecewise Lipschitz continuous in the $(t, x)$-space. For every $u : [0, T] \mapsto \mathbb{R}^n$ in a class of Lipschitz continuous functions, we then show that the composed maps $t \mapsto f_n(t, u(t))$ form a Cauchy sequence in $L^1([0, T]; \mathbb{R}^n)$, converging pointwise almost everywhere to a map of the form $f(\cdot, u(\cdot))$, taking values within the extreme points of $F$. This convergence is obtained through an argument which is considerably different from previous works. Indeed, it relies on a careful use of the
likelihood functional introduced in [3], interpreted here as a measure of “oscillatory non-
convergence” of a set of derivatives.

Among various corollaries, Theorem 1 yields an extension, valid for the wider class of
multifunctions with the property (LSP), of the following results, proved in [5], [4] and [6],
respectively.

(i) Existence of selections from the solution set of a differential inclusion, depending
continuously on the initial data.

(ii) Existence of selections from a multifunction, which generate a continuous flow.

(iii) Contractibility of the solution sets of \( \dot{x} \in F(t, x) \) and \( \dot{x} \in extF(t, x) \).

These consequences, together with an application to bang-bang feedback controls, are
described in section 4.

2 - Preliminaries

As customary, \( \bar{A} \) and \( \overline{A} \) denote here the closure and the closed convex hull of \( A \)
respectively, while \( A \backslash B \) indicates a set–theoretic difference. The Lebesgue measure of a
set \( J \subset IR \) is \( m(J) \). The characteristic function of a set \( A \) is written as \( \chi_A \).

In the following, \( K_n \) denotes the family of all nonempty compact convex subsets of
\( IR^n \), endowed with Hausdorff metric. A key technical tool used in our proofs will be the
function \( h: IR^n \times K_n \mapsto IR \cup \{-\infty\} \), defined by

\[
h(y, K) = \sup \left\{ \left( \int_0^1 |w(\xi) - y|^2 \, d\xi \right)^{\frac{1}{2}} \; : \; w: [0, 1] \mapsto K, \; \int_0^1 w(\xi) \, d\xi = y \right\}
\]  

(2.1)

with the understanding that \( h(y, K) = -\infty \) if \( y \notin K \). Observe that \( h^2(y, K) \) can be
interpreted as the maximum variance among all random variables supported inside \( K \),
whose mean value is \( y \). The following results were proved in [3]:

**Lemma 1.** The map \( (y, K) \mapsto h(y, K) \) is upper semicontinuous in both variables; for each
fixed \( K \in K_n \) the function \( y \mapsto h(y, K) \) is strictly concave down on \( K \). Moreover, one has

\[
h(y, K) = 0 \quad \text{if and only if} \quad y \in extK,
\]  

(2.2)
where $c(K)$ and $r(K)$ denote the Chebyshev center and the Chebyshev radius of $K$, respectively.

For the basic theory of multifunctions and differential inclusions we refer to [1]. As in [2], given a map $g : [0,T] \times \Omega \mapsto \mathbb{R}^n$, we say that $g$ is directionally continuous along the directions of the cone $\Gamma^N = \{ (s,y); |y| \leq Ns \}$ if

$$g(t,x) = \lim_{k \to \infty} g(t_k, x_k)$$

for every $(t,x)$ and every sequence $(t_k, x_k)$ in the domain of $g$ such that $t_k \to t$ and $|x_k - x| \leq N(t_k - t)$ for every $k$. Equivalently, $g$ is $\Gamma^N$-continuous iff it is continuous w.r.t. the topology generated by the family of all conical neighborhoods

$$\Gamma^N_{(\hat{t},\hat{x},\varepsilon)} = \{ (s,y); \hat{t} \leq s \leq \hat{t} + \varepsilon, |y - \hat{x}| \leq N(s - t) \}. \quad (2.4)$$

A set of the form (2.4) will be called an $N$-cone.

Under the assumptions on $\Omega$, $D$ made in Theorem 1, consider the set of Lipschitzian functions

$$Y = \{ u : [0,T] \mapsto \mathcal{B}(D, MT) ; \quad |u(t) - u(s)| \leq M|t-s| \quad \forall t,s \}. \quad (2.5)$$

The Picard operator of a map $g : [0,T] \times \Omega \mapsto \mathbb{R}^n$ is defined as

$$\mathcal{P}^g(u)(t) = \int_0^t g(s, u(s)) \, ds \quad u \in Y.$$ 

The distance between two Picard operators will be measured by

$$\|\mathcal{P}^f - \mathcal{P}^g\| = \sup \left\{ \left| \int_0^t [f(s, u(s)) - g(s, u(s))] \, ds \right| ; \quad t \in [0,T], \ u \in Y \right\}. \quad (2.6)$$

The next Lemma will be useful in order to prove the uniqueness of solutions of the Cauchy problems (1.2).

**Lemma 2.** Let $f$ be a measurable map from $[0,T] \times \Omega$ into $\mathcal{B}(0, M)$, with $\mathcal{P}^f$ continuous on $Y$. Let $D$ be compact, with $\mathcal{B}(D, MT) \subset \Omega$, and assume that the Cauchy problem

$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0, \quad t \in [0,T]$$

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has a unique solution, for each \((t_0, x_0) \in [0, T] \times D\).

Then, for every \(\epsilon > 0\), there exists \(\delta > 0\) with the following property. If \(g : [0, T] \times \Omega \to \overline{B}(0, M)\) satisfies \(\|P^g - P^f\| \leq \delta\), then for every \((t_0, x_0) \in [0, T] \times D\), any solution of the Cauchy problem

\[
\dot{y}(t) = g(t, y(t)) \quad y(t_0) = x_0 \quad t \in [0, T]
\]  

(2.7)

has distance \(< \epsilon\) from the corresponding solution of (2.6). In particular, the solution set of (2.7) has diameter \(\leq 2\epsilon\) in \(C^0([0, T]; \mathbb{R}^n)\).

**Proof.** If the conclusion fails, then there exist sequences of times \(t_\nu, t'_\nu\), maps \(g_\nu\) with \(\|P^{g_\nu} - P^f\| \to 0\), and couples of solutions \(x_\nu, y_\nu : [0, T] \to \overline{B}(D; MT)\) of

\[
\dot{x}_\nu(t) = f(t, x_\nu(t)), \quad \dot{y}_\nu(t) = g_\nu(t, y_\nu(t)) \quad t \in [0, T],
\]  

(2.8)

with

\[
x_\nu(t_\nu) = y_\nu(t_\nu) \in D, \quad |x_\nu(t'_\nu) - y_\nu(t'_\nu)| \geq \epsilon \quad \forall \nu.
\]  

(2.9)

By taking subsequences, we can assume that \(t_\nu \to t_0, t'_\nu \to \tau, x_\nu(t_0) \to x_0\), while \(x_\nu \to x\) and \(y_\nu \to y\) uniformly on \([0, T]\). From (2.8) it follows

\[
\left| y(t) - x_0 - \int_{t_0}^{t} f(s, y(s)) \, ds \right| \leq |y(t) - y_\nu(t)| + |x_0 - y_\nu(t_0)| + \int_{t_0}^{t} \left| f(s, y(s)) - f(s, y_\nu(s)) \right| \, ds + \int_{t_0}^{t} \left| g_\nu(s) - g(s, y_\nu(s)) \right| \, ds
\]  

(2.10)

As \(\nu \to \infty\), the right hand side of (2.10) tends to zero, showing that \(y(\cdot)\) is a solution of (2.6). By the continuity of \(P^f\), \(x(\cdot)\) is also a solution of (2.6), distinct from \(y(\cdot)\) because

\[
|x(\tau) - y(\tau)| = \lim_{\nu \to \infty} |x_\nu(\tau) - y_\nu(\tau)| = \lim_{\nu \to \infty} |x_\nu(t'_\nu) - y_\nu(t'_\nu)| \geq \epsilon.
\]

This contradicts the uniqueness assumption, proving the lemma.

### 3 - Proof of the main theorem

Observing that \(extF(t, x) = ext\overline{co}F(t, x)\) for every compact set \(F(t, x)\), it is clearly not restrictive to prove Theorem 1 under the additional assumption that all values of \(F\) are convex. Moreover, the bounds on \(F\) and \(D\) imply that no solution of the Cauchy problem

\[
\dot{x}(t) \in F(t, x(t)), \quad x(t_0) = x_0, \quad t \in [0, T],
\]  

(2.11)
with \( x_0 \in D \), can escape from the set \( \overline{B}(D, MT) \). Therefore, it suffices to construct the selection \( f \) on the compact set \( \Omega^+ = [0, T] \times \overline{B}(D, MT) \). Finally, since every convex valued multifunction satisfying (LSP) admits a globally defined Lipschitz selection, it suffices to prove the second part of the theorem, with \( f_0 \) and \( \varepsilon_0 > 0 \) assigned.

We shall define a sequence of directionally continuous selections of \( F \), converging a.e. to a selection from \( ext F \). The basic step of our constructive procedure will be provided by the next lemma.

**Lemma 3.** Fix any \( \varepsilon > 0 \). Let \( S \) be a compact subset of \([0, T] \times \Omega\) and let \( \phi : S \rightarrow \mathbb{R}^n \) be a continuous selection of \( F \) such that

\[
h(\phi(t, x), F(t, x)) < \eta \quad \forall (t, x) \in S,
\]

with \( h \) as in (2.1). Then there exists a piecewise Lipschitz selection \( g : S \rightarrow \mathbb{R}^n \) of \( F \) with the following properties:

(i) There exists a finite covering \( \{ \Gamma_i \}_{i=1}^{\nu} \), consisting of \( \Gamma^{M+1} \)-cones, such that, if we define the pairwise disjoint sets \( \Delta^i = \Gamma_i \setminus \bigcup_{\ell<i} \Gamma_\ell \), then on each \( \Delta^i \) the following holds:

(a) there exist Lipschitzian selections \( \psi_j^i : \Delta^i \rightarrow \mathbb{R}^n \), \( j = 0, \ldots, n \), such that

\[
g |_{\Delta^i} = \sum_{j=0}^n \psi_j^i \chi_{A_j^i},
\]

where each \( A_j^i \) is a finite union of strips of the form \( ([t', t''] \times \mathbb{R}^n) \cap \Delta^i \).

(b) For every \( j = 0, \ldots, n \) there exists an affine map \( \varphi_j^i(\cdot) = a_j^i + b_j^i \) such that

\[
\varphi_j^i(\psi_j^i(t, x)) \leq \varepsilon, \quad \varphi_j^i(z) \geq h(z, F(t, x)), \quad \forall (t, x) \in \Delta^i, \ z \in F(t, x).
\]

(ii) For every \( u \in Y \) and every interval \([\tau, \tau']\) such that \( (s, u(s)) \in S \) for \( \tau \leq s < \tau' \), the following estimates hold:

\[
\left| \int_{\tau}^{\tau'} [\phi(s, u(s)) - g(s, u(s))] \, ds \right| \leq \varepsilon,
\]

\[
\int_{\tau}^{\tau'} \left| \phi(s, u(s)) - g(s, u(s)) \right| \, ds \leq \varepsilon + \eta(\tau' - \tau).
\]
Remark 1. Thinking of $h(y, K)$ as a measure for the distance of $y$ from the extreme points of $K$, the above lemma can be interpreted as follows. Given any selection $\phi$ of $F$, one can find a $\Gamma^{M+1}$-continuous selection $g$ whose values lie close to the extreme points of $F$ and whose Picard operator $P^g$, by (3.4), is close to $P^\phi$. Moreover, if the values of $\phi$ are near the extreme points of $F$, i.e. if $\eta$ in (3.1) is small, then $g$ can be chosen close to $\phi$. The estimate (3.5) will be a direct consequence of the definition (2.1) of $h$ and of Hölder’s inequality.

Remark 2. Since $h$ is only upper semicontinuous, the two assumptions $y_\nu \to y$ and $h(y_\nu, K) \to 0$ do not necessarily imply $h(y, K) = 0$. As a consequence, the a.e. limit of a convergent sequence of approximately extremal selections $f_\nu$ of $F$ need not take values inside $\text{ext} F$. To overcome this difficulty, the estimates in (3.3) provide upper bounds for $h$ in terms of the affine maps $\varphi^t_{i_j}$. Since each $\varphi^t_{i_j}$ is continuous, limits of the form $\varphi^t_{i_j}(y_\nu) \to \varphi^t_{i_j}(y)$ will be straightforward.

Proof of Lemma 3. For every $(t, x) \in S$ there exist values $y_j(t, x) \in F(t, x)$ and coefficients $\theta_j(t, x) \geq 0$, with

$$
\phi(t, x) = \sum_{j=0}^n \theta_j(t, x)y_j(t, x), \quad \sum_{j=0}^n \theta_j(t, x) = 1,
$$

$$
h(y_j(t, x), F(t, x)) < \varepsilon/2.
$$

By the concavity and the upper semicontinuity of $h$, for every $j = 0, \ldots, n$ there exists an affine function $\varphi^t_{i_j}(\cdot) = \langle a^t_{i_j}(\cdot), \cdot \rangle + b_{i_j}^t$ such that

$$
\varphi^t_{i_j}(y_j(t, x)) < h(y_j(t, x), F(t, x)) + \frac{\varepsilon}{2} < \varepsilon,
$$

$$
\varphi^t_{i_j}(z) > h(z, F(t, x)) \quad \forall z \in F(t, x).
$$

By (LSP) and the continuity of each $\varphi^t_{i_j}$, there exists a neighborhood $U$ of $(t, x)$ together with Lipschitzean selections $\psi^t_{i_j} : U \to \mathbb{R}^n$, such that, for every $j$ and every $(s, y) \in U$,

$$
\left| \psi^t_{i_j}(s, y) - y_j(t, x) \right| < \frac{\varepsilon}{4T},
$$

$$
\varphi^t_{i_j}(\psi^t_{i_j}(s, y)) < \varepsilon.
$$
Using again the upper semicontinuity of \( h \), we can find a neighborhood \( U' \) of \((t,x)\) such that
\[
\varphi_j^{(t,x)}(z) \geq h(z, F(s, y)) \quad \forall z \in F(s, y), \quad (s, y) \in U', \quad j = 0, \ldots, n. \tag{3.8}
\]

Choose a neighborhood \( \Gamma_{t,x} \) of \((t,x)\), contained in \( U \cap U' \), such that, for every point \((s, y)\) in the closure \( \Gamma_{t,x} \), one has
\[
|\phi(s, y) - \phi(t, x)| < \frac{\varepsilon}{4T}, \tag{3.9}
\]

It is not restrictive to assume that \( \Gamma_{t,x} \) is a \((M + 1)\)-cone, i.e. it has the form \((2.4)\) with \( N = M + 1 \). By the compactness of \( S \) we can extract a finite subcovering \( \{ \Gamma_i; 1 \leq i \leq \nu \} \), with \( \Gamma_i \supseteq \Gamma_{t_i,x_i} \). Define \( \Delta^i = \Gamma_i \setminus \bigcup_{j<i} \Gamma_j \) and set \( \theta^i_j = \theta_j(t_i, x_i), \quad y^i_j = y_j(t_i, x_i), \quad \psi^i_j = \psi^i_j(t_i, x_i), \quad \varphi^i_j = \varphi_j(t_i, x_i) \). Choose an integer \( N \) such that
\[
N > \frac{8M\nu^2T}{\varepsilon} \tag{3.10}
\]
and divide \([0, T]\) into \( N \) equal subintervals \( J_1, \ldots, J_N \), with
\[
J_k = [t_{k-1}, t_k), \quad t_k = \frac{kT}{N}. \tag{3.11}
\]

For each \( i, k \) such that \((J_k \times \mathbb{R}^n) \cap \Delta^i \neq \emptyset\), we then split \( J_k \) into \( n + 1 \) subintervals \( J^i_{k,0}, \ldots, J^i_{k,n} \) with lengths proportional to \( \theta^i_0, \ldots, \theta^i_n \), by setting
\[
J^i_{k,j} = [t_{k,j-1}, t_{k,j}), \quad t_{k,j} = \frac{T}{N} \cdot \left( k + j \sum_{\ell=0}^{j} \theta^i_{\ell} \right), \quad t_{k,-1} = \frac{T}{N}. \tag{3.12}
\]

For any point \((t, x)\) in \( \overline{\Delta^i} \) we now set
\[
\begin{cases}
  g^i(t, x) = \psi^i_j(t, x) & \text{if } t \in \bigcup_{k=1}^{N} J^i_{k,j}.
  \\
  \bar{g}^i(t, x) = y^i_j
\end{cases} \tag{3.12}
\]

The piecewise Lipschitz selection \( g \) and a piecewise constant approximation \( \bar{g} \) of \( g \) can now be defined as
\[
g = \sum_{i=1}^{\nu} g^i \chi_{\Delta^i}, \quad \bar{g} = \sum_{i=1}^{\nu} \bar{g}^i \chi_{\Delta^i}. \tag{3.13}
\]

By construction, recalling (3.7) and (3.8), the conditions (a), (b) in (i) clearly hold.
It remains to show that the estimates in (ii) hold as well. Let \( \tau, \tau' \in [0, T] \) and \( u \in Y \) be such that \((t, u(t)) \in S \) for every \( t \in [\tau, \tau'] \), and define
\[
E^i = \{ t \in I ; (t, u(t)) \in \Delta^i \}, \quad i = 1, \ldots, \nu.
\]
From our previous definition \( \Delta^i = \Gamma^i \setminus \bigcup_{j < i} \Gamma^j \), where each \( \Gamma^j \) is a \((M + 1)\)-cone, it follows that every \( E^i \) is the union of at most \( i \) disjoint intervals. We can thus write
\[
E^i = \left( \bigcup_{J_k \subset E^i} J_k \right) \cup \hat{E}^i,
\]
with \( J_k \) given by (3.11) and
\[
m(\hat{E}^i) \leq \frac{2iT}{N} \leq \frac{2\nu T}{N}. \tag{3.14}
\]
Since
\[
\phi(t_i, x_i) = \sum_{j=0}^{n} \theta^i_j y^i_j,
\]
the definition of \( \bar{g} \) at (3.12), (3.13) implies
\[
\int_{J_k} \left[ \phi(t_i, x_i) - \bar{g}(s, u(s)) \right] ds = m(J_k) \cdot \left( \phi(t_i, x_i) - \sum_{j=0}^{n} \theta^i_j y^i_j \right) = 0.
\]
Therefore, from (3.9) and (3.6) it follows
\[
\left| \int_{J_k} \left[ \phi(s, u(s)) - g(s, u(s)) \right] ds \right| \leq \int_{J_k} \left| \phi(s, u(s)) - \phi(t_i, x_i) \right| ds \leq m(J_k) \cdot \left( \frac{\varepsilon}{4T} + 0 \right) = m(J_k) \cdot \frac{\varepsilon}{2T}.
\]
The choice of \( N \) at (3.10) and the bound (3.14) thus imply
\[
\left| \int_{\tau}^{\tau'} \left[ \phi(s, u(s)) - g(s, u(s)) \right] ds \right| \leq 2M \cdot m \left( \bigcup_{i=1}^{\nu} \hat{E}^i \right) + (\tau' - \tau) \frac{\varepsilon}{2T} \leq 2M \nu \frac{2\nu T}{N} + \frac{\varepsilon}{2} \leq \varepsilon,
\]
proving (3.4).

We next consider (3.5). For a fixed \( i \in \{1, \ldots, \nu\} \), let \( E^i \) be as before and define
\[
\xi_{-1} = 0, \quad \xi_j = \sum_{\ell=0}^{j} \theta^i_{\ell}, \quad w^i(\xi) = \sum_{j=0}^{n} y^i_j \chi_{[\xi_{j-1}, \xi_j]},
\]
Recalling (3.15), the definition of \( h \) at (2.1) and Hölder’s inequality together imply
\[
\begin{align*}
    h(\phi(t_i, x_i), F(t_i, x_i)) &\geq \left( \int_0^1 |\phi(t_i, x_i) - w^i(\xi)|^2 \, d\xi \right)^{\frac{1}{2}} \\
    &
    \geq \int_0^1 |\phi(t_i, x_i) - w^i(\xi)| \, d\xi = \sum_{j=0}^n \theta_j^i |\phi(t_i, x_i) - y_j^i|.
\end{align*}
\]

Using this inequality we obtain
\[
\int_{J_k} |\phi(t_i, x_i) - g(s, u(s))| \, ds = m(J_k) \cdot \sum_{j=0}^n \theta_j^i |\phi(t_i, x_i) - y_j^i|
\leq m(J_k) \cdot h(\phi(t_i, x_i), F(t_i, x_i)) \leq \eta \cdot m(J_k),
\]
and therefore, by (3.9) and (3.6),
\[
\int_{J_k} |\phi(s, u(s)) - g(s, u(s))| \, ds
\leq \int_{J_k} |\phi(s, u(s)) - \phi(t_i, x_i)| \, ds + \int_{J_k} |g(s, u(s)) - g(s, u(s))| \, ds
+ \int_{J_k} |\phi(t_i, x_i) - g(s, u(s))| \leq m(J_k) \cdot \left( \frac{\varepsilon}{4T} + \frac{\varepsilon}{4T} + \eta \right).
\]

Using again (3.14) and (3.10), we conclude
\[
\int_\tau^{\tau'} |\phi(s, u(s)) - g(s, u(s))| \, ds \leq (\tau' - \tau) \left( \frac{\varepsilon}{2T} + \eta \right) + 2M\nu \cdot \frac{2\nu T}{N} \leq \varepsilon + (\tau' - \tau)\eta.
\]

Q.E.D.

Using Lemma 3, given any continuous selection \( \tilde{f} \) of \( F \) on \( \Omega^\dagger \), and any sequence \( (\varepsilon_k)_{k \geq 1} \) of strictly positive numbers, we can generate a sequence \( (f_k)_{k \geq 1} \) of selections from \( F \) as follows.

To construct \( f_1 \), we apply the lemma with \( S = \Omega^\dagger, \phi = f_0, \varepsilon = \varepsilon_1 \). This yields a partition \( \{ A_1^i; \ i = 1, \ldots, \nu_1 \} \) of \( \Omega^\dagger \) and a piecewise Lipschitz selection \( f_1 \) of \( F \) of the form
\[
f_1 = \sum_{i=1}^{\nu_1} f_1^i \chi_{A_1^i}.
\]
In general, at the beginning of the \( k \)-th step we are given a partition of \( \Omega^1 \), say 
\[ \{ A^i_k; \ i = 1, \ldots, \nu_k \} \]
and a selection
\[ f_k = \sum_{i=1}^{\nu_k} f^i_k \chi_{A^i_k}, \]
where each \( f^i_k \) is Lipschitz continuous and satisfies
\[ h(f_k(t, x), F(t, x)) \leq \epsilon_k \quad \forall (t, x) \in \overline{A^i_k}. \]
We then apply Lemma 3 separately to each \( A^i_k \), choosing \( S = \overline{A^i_k}, \ \epsilon = \epsilon_k, \ \phi = f^i_k \). This yields a partition \( \{ A_{k+1}^i; \ i = 1, \ldots, \nu_{k+1} \} \) of \( \Omega^1 \) and functions of the form
\[ f_{k+1} = \sum_{i=1}^{\nu_{k+1}} f^i_{k+1} \chi_{A_{k+1}^i}, \quad \varphi^i_{k+1} = \langle a^i_{k+1}, \cdot \rangle + b^i_{k+1}, \]
where each \( f^i_{k+1} : \overline{A_{k+1}^i} \rightarrow \mathbb{R}^n \) is a Lipschitz continuous selection from \( F \), satisfying the following estimates:
\[ \varphi^i_{k+1}(z) > h(z, F(t, x)) \quad \forall (t, x) \in A^i_{k+1}, \quad (3.16) \]
\[ \varphi^i_{k+1}(f^i_{k+1}(t, x)) \leq \epsilon_{k+1} \quad \forall (t, x) \in A^i_{k+1}, \quad (3.17) \]
\[ \left| \int_\tau^{\tau'} [f_{k+1}(s, u(s)) - f_k(s, u(s))] \, ds \right| \leq \epsilon_{k+1}, \quad (3.18) \]
\[ \int_\tau^{\tau'} |f_{k+1}(s, u(s)) - f_k(s, u(s))| \, ds \leq \epsilon_{k+1} + \epsilon_k(\tau' - \tau), \quad (3.19) \]
for every \( u \in Y \) and every \( \tau, \tau' \), as long as the values \((s, u(s))\) remain inside a single set \( A^i_k \), for \( s \in [\tau, \tau') \).

Observe that, according to Lemma 3, each \( A^i_k \) is closed-open in the finer topology generated by all \((M + 1)\)-cones. Therefore, each \( f_k \) is \( \Gamma^{M+1} \)-continuous. By Theorem 2 in [2], the substitution operator \( S^{f_k} : u(\cdot) \mapsto f_k(\cdot, u(\cdot)) \) is continuous from the set \( Y \) defined at (2.5) into \( \mathcal{L}^1([0, T]; \mathbb{R}^n) \). The Picard map \( \mathcal{P}^{f_k} \) is thus continuous as well.

Furthermore, there exists an integer \( N_k \) with the following property. Given any \( u \in Y \), there exists a finite partition of \([0, T]\) with nodes \( 0 = \tau_0 < \tau_1 < \cdots < \tau_{n(u)} = T \), with \( n(u) \leq N_k \), such that, as \( t \) ranges in any \([\tau_{i-1}, \tau_i)\), the point \((t, u(t))\) remains inside one single set \( A^i_k \). Otherwise stated, the number of times in which the curve \( t \mapsto (t, u(t)) \)
crosses a boundary between two distinct sets \( A^i_k \), \( A^j_k \) is smaller that \( N_k \), for every \( u \in Y \).

The construction of the \( A^i_k \) in terms of \((M + 1)\)-cones implies that all these crossings are transversal. Since the restriction of \( f_k \) to each \( A^i_k \) is Lipschitz continuous, it is clear that every Cauchy problem

\[
\dot{x}(t) = f_k(t, x(t)), \quad x(t_0) = x_0
\]

has a unique solution, depending continuously on the initial data \((t_0, x_0) \in [0, T] \times D\).

From (3.18), (3.19) and the property of \( N_k \) it follows

\[
\left| \int_0^t \left[ f_{k+1}(s, u(s)) - f_k(s, u(s)) \right] ds \right| \leq \sum_{\ell=1}^{\hat{\ell}} \int_{\tau_{\ell-1}}^{\tau_\ell} \left[ f_{k+1}(s, u(s)) - f_k(s, u(s)) \right] ds \leq N_k \varepsilon_{k+1}, \tag{3.20}
\]

where \( 0 = \tau_0 < \tau_1 < \cdots < \tau_{\hat{\ell}} = t \) are the times at which the map \( s \to (s, u(s)) \) crosses a boundary between two distinct sets \( A^i_k, A^j_k \). Since (3.20) holds for every \( t \in [0, T] \), we conclude

\[
\left\| P^{f_{k+1}} - P^{f_k} \right\| \leq N_k \varepsilon_{k+1}. \tag{3.21}
\]

Similarly, for every \( u \in Y \) one has

\[
\left\| f_{k+1}(. , u(.)) - f_k(. , u(.)) \right\| \leq \sum_{\ell=1}^{n(u)} \int_{\tau_{\ell-1}}^{\tau_\ell} \left| f_{k+1}(s, u(s)) - f_k(s, u(s)) \right| ds \leq N_k \varepsilon_{k+1} + \varepsilon_k T.	ag{3.22}
\]

Now consider the functions \( \varphi_k : \mathbb{R}^n \times \Omega^\dagger \to \mathbb{R} \), with

\[
\varphi_k(y, t, x) = \langle a^i_k, y \rangle + b^i_k \quad \text{if} \quad (t, x) \in A^i_k. \tag{3.23}
\]

From (3.16), (3.17) it follows

\[
\varphi_k(y, t, x) \geq h(y, F(t, x)) \quad \forall (t, x) \in \Omega^\dagger, \quad y \in F(t, x), \tag{3.24}
\]

\[
\varphi_k(f_k(t, x), t, x) \leq \varepsilon_k \quad \forall (t, x) \in \Omega^\dagger. \tag{3.25}
\]
For every \( u \in Y \), (3.18) and the linearity of \( \varphi_k \) w.r.t. \( y \) imply

\[
\left| \int_0^T \left[ \varphi_k(f_{k+1}(s, u(s)), s, u(s)) - \varphi_k(f_k(s, u(s)), s, u(s)) \right] \, ds \right|
\leq \sum_{\ell=1}^{n(u)} \max \left\{ |a^1_k|, \ldots, |a^{\nu_k}_k| \right\} \cdot \left| \int_{t_{\ell-1}}^{t_\ell} \left[ f_{k+1}(s, u(s)) - f_k(s, u(s)) \right] \, ds \right|
\leq N_k \cdot \max \left\{ |a^1_k|, \ldots, |a^{\nu_k}_k| \right\} \cdot \varepsilon_{k+1}.
\]  

Moreover, for every \( \ell \geq k \), from (3.19) it follows

\[
\int_0^T \left| \varphi_k(f_{\ell+1}(s, u(s)), s, u(s)) - \varphi_k(f_\ell(s, u(s)), s, u(s)) \right| \, ds
\leq \max \left\{ |a^1_k|, \ldots, |a^{\nu_k}_k| \right\} \cdot \left| \int_0^T \left| f_{\ell+1}(s, u(s)) - f_\ell(s, u(s)) \right| \, ds \right|
\leq \max \left\{ |a^1_k|, \ldots, |a^{\nu_k}_k| \right\} \cdot (N_\ell \varepsilon_{\ell+1} + \varepsilon_\ell T).
\]  

Observe that all of the above estimates hold regardless of the choice of the \( \varepsilon_k \). We now introduce an inductive procedure for choosing the constants \( \varepsilon_k \), which will yield the convergence of the sequence \( f_k \) to a function \( f \) with the desired properties.

Given \( f_0 \) and \( \varepsilon_0 \), by Lemma 2 there exists \( \delta_0 > 0 \) such that, if \( g : \Omega^\dagger \rightarrow \mathcal{B}(0, M) \) and \( \|P^g - P^{f_0}\| \leq \delta_0 \), then, for each \( (t_0, x_0) \in [0, T] \times D \), every solution of (2.7) remains \( \varepsilon_0 \)-close to the unique solution of (1.3). We then choose \( \varepsilon_1 = \delta_0/2 \).

By induction on \( k \), assume that the functions \( f_1, \ldots, f_k \) have been constructed, together with the linear functions \( \varphi^i_k(\cdot) = \langle a^i_k, \cdot \rangle + b^i_k \) and the integers \( N_\ell, \quad \ell = 1, \ldots, k \). Let the values \( \delta_0, \delta_1, \ldots, \delta_k > 0 \) be inductively chosen, satisfying

\[
\delta_\ell \leq \frac{\delta_{\ell-1}}{2} \quad \ell = 1, \ldots, k,
\]  

and such that \( \|P^g - P^{f_\ell}\| \leq \delta_\ell \) implies that for every \( (t_0, x_0) \in [0, T] \times D \) the solution set of (2.7) has diameter \( \leq 2^{-\ell} \), for \( \ell = 1, \ldots, k \). This is possible again because of Lemma 2. For \( k \geq 1 \) we then choose

\[
\varepsilon_{k+1} = \min \left\{ \frac{\delta_k}{2N_k}, \frac{2^{-k}}{N_k \cdot \max \left\{ |a^i_k|; 1 \leq \ell \leq k, 1 \leq i \leq \nu_\ell \right\}} \right\}.
\]  

Using (3.28), (3.29) in (3.21), with \( N_0 = 1 \), we now obtain

\[
\sum_{k=p}^{\infty} \|P^{f_{k+1}} - P^{f_k}\| \leq \sum_{k=p}^{\infty} N_k \cdot \frac{\delta_k}{2N_k} \leq \sum_{k=p}^{\infty} \frac{2^{p-k}\delta_p}{2} \leq \delta_p
\]  

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for every \( p \geq 0 \). From (3.22) and (3.29) we further obtain
\[
\sum_{k=1}^{\infty} \| f_{k+1}(\cdot, u(\cdot)) - f_k(\cdot, u(\cdot)) \|_{L^1} \leq \sum_{k=1}^{\infty} \left( N_k \cdot \frac{2^{-k}}{N_k} + \frac{2^{1-k}T}{N_k} \right) \leq \sum_{k=1}^{\infty} (2^{-k} + 2^{1-k}T) \leq 1 + 2T.
\]
(3.31)

Define
\[
f(t, x) = \lim_{k \to \infty} f_k(t, x)
\]
for all \( (t, x) \in \Omega^t \) at which the sequence \( f_k \) converges. By (3.31), for every \( u \in Y \) the sequence \( f_k(\cdot, u(\cdot)) \) converges in \( L^1([0, T]; \mathbb{R}^n) \) and a.e. on \([0, T]\). In particular, considering the constant functions \( u \equiv x \in \overline{B(D, MT)} \), by Fubini’s theorem we conclude that \( f \) is defined a.e. on \( \Omega^t \). Moreover, the substitution operators \( S^{f_k} : u(\cdot) \mapsto f_k(\cdot, u(\cdot)) \) converge to the operator \( S^f : u(\cdot) \mapsto f(\cdot, u(\cdot)) \) uniformly on \( Y \). Since each \( S^{f_k} \) is continuous, \( S^f \) is also continuous. Clearly, the Picard map \( \mathcal{P}^f \) is continuous as well. By (3.30) we have
\[
\| \mathcal{P}^f - \mathcal{P}^{f_k} \| \leq \sum_{k=p}^{\infty} \| \mathcal{P}^{f_{k+1}} - \mathcal{P}^{f_k} \| \leq \delta_p \quad \forall p \geq 1.
\]
Recalling the property of \( \delta_p \), this implies that, for every \( p \), the solution set of (2.7) has diameter \( \leq 2^{-p} \). Since \( p \) is arbitrary, for every \( (t_0, x_0) \in [0, T] \times D \) the Cauchy problem can have at most one solution. On the other hand, the existence of such a solution is guaranteed by Schauder’s theorem. The continuous dependence of this solution on the initial data \( t_0, x_0 \), in the norm of \( \mathcal{A} \), is now an immediate consequence of uniqueness and of the continuity of the operators \( S^f, \mathcal{P}^f \). Furthermore, for \( p = 0 \), (3.30) yields \( \| \mathcal{P}^f - \mathcal{P}^{f_0} \| \leq \delta_0 \). The choice of \( \delta_0 \) thus implies (1.4).

It now remains to prove (1.1). Since every set \( F(t, x) \) is closed, it is clear that \( f(t, x) \in F(t, x) \). For every \( u \in Y \) and \( k \geq 1 \), by (3.24)–(3.27) the choices of \( \varepsilon_k \) at (2.29) yield
\[
\int_0^T h(f(s, u(s)), F(s, u(s))) \, ds \leq \int_0^T \varphi_k(f(s, u(s)), s, u(s)) \, ds
\]
\[
\leq \int_0^T \varphi_k(f_k(s, u(s)), s, u(s)) \, ds
\]
\[
+ \left| \int_0^T \left[ \varphi_k(f_{k+1}(s, u(s)), s, u(s)) - \varphi_k(f_k(s, u(s)), s, u(s)) \right] \, ds \right|
\]
\[
+ \sum_{\ell=k+1}^{\infty} \int_0^T \left| \varphi_k(f_{\ell+1}(s, u(s)), s, u(s)) - \varphi_k(f_{\ell}(s, u(s)), s, u(s)) \right| \, ds
\]
\[
\leq 2^{1-k}T + 2^{-k} + \sum_{\ell=k+1}^{\infty} (2^{-\ell} + 2^{1-\ell}T).
\]
\[14\]
Observing that the right hand side of (3.33) approaches zero as \( k \to \infty \), we conclude that
\[
\int_0^T h(f(t, u(t)), F(t, u(t))) \, dt = 0.
\]
By (2.2), given any \( u \in Y \), this implies \( f(t, u(t)) \in extF(t, u(t)) \) for almost every \( t \in [0, T] \).
By possibly redefining \( f \) on a set of measure zero, this yields (1.1).

4 - Applications

Throughout this section we make the following assumptions.

(H) \( F : [0, T] \times \Omega \mapsto \overline{B}(0, M) \) is a bounded continuous multifunction with compact values satisfying (LSP), while \( D \) is a compact set such that \( \overline{B}(D, MT) \subseteq \Omega \).

An immediate consequence of Theorem 1 is

**Corollary 1.** Let the hypotheses (H) hold. Then there exists a continuous map \( (t_0, x_0) \mapsto x(\cdot, t_0, x_0) \) from \( [0, T] \times D \) into \( \mathcal{AC} \), such that
\[
\begin{align*}
\dot{x}(t, t_0, x_0) &\in extF(t, x(t, t_0, x_0)) \quad \forall t \in [0, T], \\
x(t_0, t_0, x_0) & = x_0 \quad \forall t_0, x_0.
\end{align*}
\]

Another consequence of Theorem 1 is the contractibility of the sets of solutions of certain differential inclusions. We recall here that a metric space \( X \) is contractible if there exist a point \( \tilde{u} \in X \) and a continuous mapping \( \Phi : X \times [0, 1] \to X \) such that:
\[
\Phi(v, 0) = \tilde{u}, \quad \Phi(v, 1) = v, \quad \forall v \in X.
\]
The map \( \Phi \) is then called a null homotopy of \( X \).

**Corollary 2.** Let the assumptions (H) hold. Then, for any \( \bar{x} \in D \), the sets \( \mathcal{M}, \mathcal{M}^{ext} \) of solutions of
\[
\begin{align*}
x(0) & = \bar{x}, \quad \dot{x}(t) \in F(t, x(t)) \quad t \in [0, T], \\
x(0) & = \bar{x}, \quad \dot{x} \in extF(t, x(t)) \quad t \in [0, T],
\end{align*}
\]
are both contractible in \( \mathcal{AC} \).
Proof. Let \( f \) be a selection from \( extF \) with the properties stated in Theorem 1. As usual, we denote by \( x(\cdot, t_0, x_0) \) the unique solution of the Cauchy problem (1.2). Define the null homotopy \( \Phi : \mathcal{M} \times [0, 1] \to \mathcal{M} \) by setting

\[
\Phi(v, \lambda)(t) = \begin{cases} 
  v(t) & \text{if } t \in [0, \lambda T], \\
  x(t, \lambda T, v(\lambda T)) & \text{if } t \in [\lambda T, T]. 
\end{cases}
\]

By Theorem 1, \( \Phi \) is continuous. Moreover, setting \( \tilde{u}(\cdot) = u(\cdot, 0, \bar{x}) \), we obtain

\[
\Phi(v, 0) = \tilde{u}, \quad \Phi(v, 1) = v, \quad \Phi(v, \lambda) \in \mathcal{M} \quad \forall v \in \mathcal{M},
\]

proving that \( \mathcal{M} \) is contractible. We now observe that, if \( v \in \mathcal{M}^{ext} \), then \( \Phi(v, \lambda) \in \mathcal{M}^{ext} \) for every \( \lambda \). Therefore, \( \mathcal{M}^{ext} \) is contractible as well.

Our last application is concerned with feedback controls. Let \( \Omega \subseteq \mathbb{R}^n \) be open, \( U \subset \mathbb{R}^m \) compact, and let \( g : [0, T] \times \Omega \times U \to \mathbb{R}^n \) be a continuous function. By a well known theorem of Filippov [8], the solutions of the control system

\[
\dot{x} = g(t, x, u), \quad u \in U, \tag{4.1}
\]

correspond to the trajectories of the differential inclusion

\[
\dot{x} \in F(t, x) \doteq \{ g(t, x, \omega) : \omega \in U \}. \tag{4.2}
\]

In connection with (4.1), one can consider the “relaxed” system

\[
\dot{x} = g^\#(t, x, u^\#), \quad u^\# \in U^\#, \tag{4.3}
\]

whose trajectories are precisely those of the differential inclusion

\[
\dot{x} \in F^\#(t, x) \doteq \overline{coF}(t, x).
\]

The control system (4.3) is obtained defining the compact set

\[
U^\# \doteq U \times \cdots \times U \times \Delta_n = U^{n+1} \times \Delta_n,
\]

where

\[
\Delta_n \doteq \left\{ \theta = (\theta_0, \ldots, \theta_n) : \sum_{i=0}^{n} \theta_i = 1, \theta_i \geq 0 \quad \forall i \right\}
\]
is the standard simplex in $\mathbb{R}^{n+1}$, and setting
\[ g^\#(t, x, u^\#) = g^\#(t, x, (u_0, \ldots, u_n, (\theta_0, \ldots, \theta_n))) = \sum_{i=0}^{n} \theta_i f(t, x, u_i). \]

Generalized controls of the form $u^\# = (u_0, \ldots, u_n, \theta)$ taking values in the set $U^{n+1} \times \Delta_n$ are called chattering controls.

**Corollary 3.** Consider the control system (4.1), with $g : [0, T] \times \Omega \times U \mapsto \overline{B}(0, M)$ Lipschitz continuous. Let $D$ be a compact set with $\overline{B}(D; MT) \subset \Omega$. Let $u^\#(t, x) \in U^\#$ be a chattering feedback control such that the mapping
\[ (t, x) \mapsto g^\#(t, x, u^\#(t, x)) = f_0(t, x) \]
is Lipschitz continuous.

Then, for every $\varepsilon_0 > 0$ there exists a measurable feedback control $\bar{u} = \bar{u}(t, x)$ with the following properties:

(a) For every $(t, x)$, one has $g(t, x, \bar{u}(t, x)) \in \text{ext}F(t, x)$, with $F$ as in (4.2).

(b) for every $(t_0, x_0) \in [0, T] \times D$, the Cauchy problem
\[ \dot{x}(t) = g(t, x(t), \bar{u}(t, x(t))), \quad x(t_0) = x_0 \]
has a unique solution $x(\cdot, t_0, x_0)$,

(c) if $y(\cdot, t_0, x_0)$ denotes the (unique) solution of the Cauchy problem
\[ \dot{y} = f_0(t, y(t)), \quad y(t_0) = x_0, \]
then for every $(t_0, x_0)$ one has
\[ |x(t, t_0, x_0) - y(t, t_0, x_0)| < \varepsilon_0, \quad \forall t \in [0, T]. \]

**Proof.** The Lipschitz continuity of $g$ implies that the multifunction $F$ in (4.2) is Lipschitz continuous in the Hausdorff metric, hence it satisfies (LSP). We can thus apply Theorem
1, and obtain a suitable selection $f$ of $ext F$, in connection with $f_0$, $\varepsilon_0$. For every $(t, x)$, the set

$$W(t, x) = \{ \omega \in U ; \ g(t, x, \omega) = f(t, x) \} \subset \mathbb{R}^m$$

is a compact nonempty subset of $U$. Let $\bar{u}(t, x) \in W(t, x)$ be the lexicographic selection. Then the feedback control $\bar{u}$ is measurable, and it is trivial to check that $\bar{u}$ satisfies all required properties.

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