NONCOMMUTATIVE GEOMETRY AND CONFORMAL GEOMETRY, II.
CONNES-CHERN CHARACTER AND AND THE LOCAL EQUIVARIANT
INDEX THEOREM.

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Abstract. This paper is the second part of a series of papers on noncommutative geometry
and conformal geometry. In this paper, we compute explicitly the Connes-Chern character
of an equivariant Dirac spectral triple. The formula that we obtain for which was used in the first
paper of the series. The computation has two main steps. The first step is the justification
that the CM cocycle represents the Connes-Chern character. The second step is the computation
of the CM cocycle as a byproduct of a new proof of the local equivariant index theorem of
Donnelly-Patodi, Gilkey and Kawasaki. The proof combines the rescaling method of Getzler
with an equivariant version of the Greiner-Hadamard approach to the heat kernel asymptotics.
Finally, as a further application of this approach, we compute the short-time limit of the JLO
cocycle of an equivariant Dirac spectral triple.

1. Introduction

The paper is the second part of a series of papers whose aim is to use tools of noncommutative
geometry to study conformal geometry and noncommutative versions of conformal geometry. In
the prequel [PW1] (referred throughout this paper as Part I) we derived a local index formula in
conformal-diffeomorphism invariant geometry and exhibited a new class of conformal invariants
taking into account the action by a group of conformal-diffeomorphisms (i.e., the conformal gauge
group). These results make use of the conformal Dirac spectral triple of Connes-Moscovici [CM4]
and two key features of its Connes-Chern character, namely, its conformal invariance and its
explicit computation in terms of equivariant characteristic forms. The former feature is established
in Part I. The latter is one of the main goal of this paper.

It should be stressed out that the conformal Dirac spectral triple is not an ordinary spectral
triple, but a twisted spectral triple in the sense of [CM4]. As shown by Connes-Moscovici [CM3],
under suitable conditions, the Connes-Chern is represented by a cocycle, called CM cocycle, which
is given by formulas that are “local” in the sense of noncommutative geometry. Except for a small
class of twisted spectral triples, which don’t include the conformal Dirac spectral triple, there is
no analogue of the CM cocycle for twisted spectral triples (see [Mo] and the related discussion
in Part I). However, in the case of the conformal Dirac spectral triple, the conformal invariance
of the Connes-Chern character allows us to choose any metric in the given conformal class. In
particular, when the conformal structure is non-flat, thanks to Ferrand-Obata theorem we may
choose a metric invariant under the whole conformal-diffeomorphism group. Moreover, in this case
the conformal Dirac spectral triple becomes an ordinary spectral triple, namely, an equivariant
Dirac spectral triple $(C^\infty(M) \rtimes G, L^2_g(M, \mathcal{S}), \mathcal{D}_g)$. We are thus reduced to computing the Connes-
Chern character of such a Dirac spectral triple.

A main goal of this paper is the explicit computation of the Connes-Chern character of an
equivariant Dirac spectral triple $(C^\infty(M) \rtimes G, L^2_g(M, \mathcal{S}), \mathcal{D}_g)$, where $(M^n, g)$ is a compact Rie-
mannian oriented spin manifold of even dimension, $\mathcal{D}_g$ is the Dirac operator acting on the spinor
bundle $\mathcal{S}$, and $C^\infty(M) \rtimes G$ is the (discrete) crossed product of the algebra $C^\infty(M)$ with a group

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Chern character in the periodic cyclic cohomology of continuous cochains on the locally convex algebra \( C^\infty(M) \times G \), rather than the usual representation in the periodic cyclic cohomology of arbitrary cochains. Therefore, we actually aim at computing the Connes-Chern character of \((C^\infty(M) \times G, L^2_0(M, \mathcal{S}), \mathcal{D}_g)\) in the cyclic cohomology \( HP^0(C^\infty(M) \times G) \) of continuous cochains. We refer to Theorem \( \text{CM} \) for the explicit formula for this Connes-Chern character. The computation has two main steps. The first step is the justification that the CM cocycle makes sense and represents the Connes-Chern character in \( HP^0(C^\infty(M) \times G) \) (and not just in the ordinary periodic cyclic cohomology \( HP^0(C^\infty(M) \times G) \)). The second step is the actual computation of the CM cocycle.

In the original paper of Connes-Moscovici [CM3], the existence of the CM cocycle and the representation of the Connes-Chern character by this cocycle was proved under several assumptions, which were subsequently relaxed (see, e.g., [HI, CPRS]). In Part I, we introduced a natural class of spectral triples over locally convex algebras, called smooth spectral triples. Given a smooth spectral triple \((A, H, D)\), it was shown that the Connes-Chern character descends to a class \( \text{Ch}(D) \) in \( HP^0(A) \). It is natural to ask for further conditions ensuring that the CM cocycle represents the Connes-Chern character in \( HP^0(A) \). Such conditions can be figured out by a careful look at the construction of the CM cocycle in \( \text{CM} \) and \( \text{HI} \) (see [Pôh] and Section \( \text{B} \)). For our purpose, it is useful to re-express these conditions in terms of heat-trace asymptotics (see Proposition \( \text{D.7} \)). Using the differentiable equivariant heat kernel asymptotics that we establish in Section \( \text{B} \), it is fairly straightforward to check these conditions. Therefore, the Connes-Chern character of \((C^\infty(M) \times G, L^2_0(M, \mathcal{S}), \mathcal{D}_g)\) is represented in \( HP^0(C^\infty(M) \times G) \) by the CM cocycle (Proposition \( \text{D.5} \)).

We then are left with computing the CM cocycle of the equivariant Dirac spectral triple \((C^\infty(M) \times G, L^2_0(M, \mathcal{S}), \mathcal{D}_g)\). As it turns out, Azmi [Az1] computed the CM cocycle when \( G \) is a finite group. However, she did not check the assumptions of [CM3] hold in her setting, and so she did not show that the CM cocycle represents the Connes-Chern character. Likewise, Chern-Hu [CH] computed the CM cocycle defined as an equivariant periodic cyclic cochain, but they didn’t verify the conditions of [CM3]. Note also that in these papers the computations are carried out by elaborating on the approach of Lafferty-Yu-Zhang [LYZ] to the local equivariant index theorem, but some additional work is required to derive the heat-kernel asymptotics that are needed in the computation of the CM cocycle.

Another goal of this paper is to give a new proof of the local equivariant index theorem of Donnelly-Patodi [DP], Gilkey [Gi] and Kawasaki [Ka] (see also [Bi1, BV1, LR, LM]). More precisely, we produce a proof which, as an immediate byproduct, yields a differentiable version of the local equivariant index theorem. Once it is recasted in terms of heat kernel asymptotics (see Proposition \( \text{D.7} \)), the computation of the CM cocycle then becomes a simple corollary of this differentiable version of the local equivariant index theorem. Thus, hardly any additional work is required to compute the CM cocycle.

Recall that the local equivariant index theorem provides us with a heat kernel proof of the equivariant index theorem of Atiyah-Singer [AS, AS2], which is a fundamental extension of both the index theorem of Atiyah-Singer [AS1, AS2] and the fixed-point formula of Atiyah-Bott [AB1, AB2]. Given a \( G \)-equivariant Hermitian vector bundle \( E \), any \( \phi \in G \) gives rise to a unitary operator \( U_\phi \) of \( L^2(M, \mathcal{S} \otimes E) \). If we let \( \mathcal{D}_{\nabla E} \) be the Dirac operator associated to a \( G \)-equivariant connection \( \nabla^E \) on \( E \), then the local equivariant index theorem for Dirac operators states that, for all \( \phi \in G \), we have

\[
\lim_{t \to 0^+} \text{Str} \left[ e^{-t \mathcal{D}_{\nabla E}^2} U_\phi \right] = (-i)^{\frac{n}{2}} \sum_{0 \leq n < \infty} (2\pi)^{-\frac{n}{2}} \int_{M^0_\alpha} A(R^{TM^0}) \wedge \nu_\phi \left(R^{N^0}\right) \wedge \text{Ch}_\phi \left(F^E\right),
\]

where \( M^0_\alpha \) is the \( \alpha \)-th dimensional submanifold component of the fixed-point set \( M^0 \) of \( \phi \), and \( A(R^{TM^0}), \nu_\phi \left(R^{N^0}\right), \text{Ch}_\phi \left(F^E\right) \) are polynomials in \( \phi' \) and the respective curvatures \( R^{TM^0}, R^{N^0}, \)
The equivariant asymptotics for kernels of Volterra ΨDOs follow from an application of Taylor’s formula. Differentiable pseudodifferential calculus reduces the derivation of the short-time asymptotics for kernels of Volterra ΨDOs to a mere application of Taylor’s formula. This approach yields a short-time asymptotic for the heat kernel $L$ by exploiting the well known fact that the heat semi-group enables us to invert the heat operator $L + \partial_t$ (see Section 2). As it turns out, the inverse $(L + \partial_t)^{-1}$ lies in a natural class of ΨDOs, called Volterra ΨDOs (see [Ge2, P2]). These operators have the Volterra property, in the sense that they are causal and time-translation invariant. The short-time asymptotics for the heat kernel then follows from the short-time asymptotics for kernels of Volterra ΨDOs. Observing that, given any differential operator $P$, the operator $P(L + \partial_t)^{-1}$ is a Volterra ΨDO, the Greiner-Hadamard approach yields for free a short-time asymptotic for the kernel of $Pe^{-tL}$. That is, it provides us for free with differentiable heat kernel asymptotics. It should also be mentioned that the Volterra pseudodifferential calculus reduces the derivation of the the short-time asymptotics for kernels of Volterra ΨDOs to a mere application of Taylor’s formula.

There is no difficulty to extend the Greiner-Hadamard approach to the equivariant setting by working in tubular coordinates (see Proposition 3.7). Like in the non-equivariant setting, the equivariant asymptotics for kernels of Volterra ΨDOs follow from an application of Taylor’s formula (cf. Lemma 5.3). Once these equivariant asymptotics are established, we can extend the approach of [Po1] to prove the local equivariant index theorem. More precisely, as observed in [Po1], the rescaling of Getzler [Ge2, Ge3] defines a natural filtration on Volterra ΨDOs. Incidentally, this gives rise to a finer notion of order for Volterra ΨDOs (called Getzler order) and a notion of model operator (see Section 4). The existence of a limit in (1.1) and its computation by means of the inverse kernel of the model heat operator then follow from elementary considerations on Getzler orders and model operators of Volterra ΨDOs (see Lemmas 4.1 and 4.10). The proof of the local equivariant index theorem is then completed by the computation of this inverse kernel. This is done by using Melcher’s formula for the heat kernel of an harmonic oscillator (see [LM] and Section 4).

As it is based on considerations on Volterra ΨDOs, this approach to the proof of the local equivariant index theorem actually yields a version of the local equivariant index theorem for Volterra ΨDOs (Theorem 4.22). In particular, specializing this theorem to operators of the form $P(L^2 + \partial_t)^{-1}$ then provides us with the differentiable version of the local equivariant index theorem (Theorem 4.22) which we need in the computation of the CM cocycle. This enables us to complete the computation of the Connes-Chern character of the equivariant Dirac spectral triple $(\mathcal{C}^\infty(M) \rtimes G, L^2(M, \mathcal{E}), \mathcal{D}_g)$.

As mentioned the existence of the CM cocycle, as defined in [CM1], requires various assumptions. One of these assumptions is the regularity condition which can be regarded as some kind of operator theoretic version of the scalarness of the principal symbol of the square of the Dirac operator (cf. Definition 5.10). However, there are natural geometric examples of spectral triples associated to hypoelliptic operators on contact or more generally Carnot manifolds which are not regular. Nevertheless, for these spectral triples the Connes-Chern character is represented by the short-time partie finie (finite part) of the JLO cocycle. In the case of the spectral triple is regular the short-time partie finie of the JLO cocycle agrees with that of the CM cocycle.

In the case of a Dirac spectral triple the JLO cocycle actually has a short-time limit, and all previous computations of this limit implicitly use the regularity of the Dirac spectral triple as they involve commuting the heat semi-group with Clifford elements. As a further application of our version of the local equivariant index theorem alluded to above we give a computation of the short-time limit of the JLO cocycle of an equivariant Dirac spectral triple that does not assume the regularity of this spectral triple (Theorem 4.1). To our knowledge this seems to be the first
computation of the short-time limit of the JLO cocycle of a Dirac spectral triple that does not use the regularity of this spectral triple. In some sense, this paves the way for the computation of the Connes-Chern character of spectral triples associated to hypoelliptic operators on contact or Carnot manifolds.

It is believed that the approach of this paper to the proof of the local equivariant index theorem and computation of the Connes-Chern character is fairly general and can be applied to numerous geometric situations. In particular, it can be extended to spin manifolds with boundary equipped with h-metric (cf. Remark 8.8 see also [10]) and to various (non-equivariant or equivariant) family settings or even bivariant settings (cf. Remarks 13.20 and 8.6 see also [22, 33]).

In this paper, the computation of the Connes-Chern character and short-time limit of the JLO cocycle is carried out in even dimension only. We refer to [14] for an extension of these results to odd dimension (see also Remark 4.25 and 7.10). Note also that [14] contains applications to the construction of the eta cochain for a Dirac spectral triple (see also Remark 8.7 on this point).

The paper is organized as follows. In Section 2, we review the Volterra pseudodifferential calculus and its relationship with the heat kernel. In Section 3, we use the Volterra pseudodifferential calculus to derive equivariant heat kernel asymptotics. In Section 3, we present our proof of the local equivariant index theorem, which leads us to a version of the local equivariant index theorem for Volterra ΨDOs. In Section 5, we review the construction of the Connes-Chern character and CM cocycle in the framework of smooth spectral triples. In Section 6, we re-interpret the CM cocycle in terms of heat kernel asymptotics. In Section 7, we determine the Connes-Chern character of an equivariant Dirac spectral triple by computing its CM cocycle. In Section 8, we use the version of the local equivariant index theorem for Volterra ΨDOs to compute the short-time limit of the JLO cocycle of an equivariant Dirac spectral triple.

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2. Volterra Pseudodifferential Calculus and Heat Kernels

In this section, we recall the main definitions and properties of the Volterra pseudodifferential calculus and its relationship with the heat kernel of an elliptic operator. The pseudodifferential representation of the heat kernel appeared in [14], but some of the ideas can be traced back to Hadamard [12]. The presentation here follows closely that of [14].

Let $(M^n, g)$ be a compact Riemannian manifold and $E$ a Hermitian vector bundle over $M$. The metrics of $M$ and $E$ naturally define a continuous inner product on the space $L^2(M, E)$ of the $L^2$-sections of $E$. In addition, we let $L : C^\infty(M, E) \to C^\infty(M, E)$ be a selfadjoint 2nd order differential operator whose principal symbol is positive-definite. In particular, $L$ is elliptic. The operator $L$ then generates a smooth semigroup $(0, \infty) \ni t \mapsto e^{-tL} \in \mathcal{L}(L^2(M, E))$ (called heat semigroup) such that, for all $u \in L^2(M, E)$, we have

\[
\lim_{t \to 0^+} e^{-tL}u = u \quad \text{ and } \quad \frac{d}{dt} e^{-tL}u = -Le^{-tL}u \quad \forall t > 0.
\]

In what follows, we shall make some abuse of notation by denoting by $E$ the vector bundle over $M \times \mathbb{R}$ obtained as the pullback of $E$ by the projection the projection $(x, t) \mapsto x$ (so that the fiber at $(x, t)$ is $E_x$). The heat operator $L + \partial_t$ then acts on the sections of this vector bundle. As is well known, the heat semigroup enables us to invert this operator. More precisely, let us denote by $C^\infty_c(\mathbb{R}, L^2(M, E))$ the space of continuous functions $u : \mathbb{R} \to L^2(M, E)$ that are supported on some interval $I_c = [c, \infty)$, $c \in \mathbb{R}$. We also denote by $C^\infty_c(M \times \mathbb{R}, E)$ the subspace of $C^\infty_c(M \times \mathbb{R}, E)$ consisting of sections supported on $M \times I_c$ for some $c \in \mathbb{R}$. We
shall regard $C_c^\infty (M \times \mathbb{R}, E)$ as a subspace of $C^0_+ (\mathbb{R}, L^2(M, E))$. We then define a linear operator 
\[(L + \partial_t)^{-1} : C^0_+ (\mathbb{R}, L^2(M, E)) \rightarrow C^0_+ (\mathbb{R}, L^2(M, E))\]
by
\[(L + \partial_t)^{-1} u(s) := \int_0^\infty e^{-tL} u(s - t) dt \quad \forall u \in C^0_+ (M \times \mathbb{R}, E).\]

The following result shows that $(L + \partial_t)^{-1}$ is really an inverse for the heat operator.

**Proposition 2.1** ([6],[8]). For all $u \in C_c^\infty (M \times \mathbb{R}, E)$, we have
\[(L + \partial_t)^{-1} (L + \partial_t) u = (L + \partial_t)(L + \partial_t)^{-1} u = u.\]

**Remark 2.2.** The space $C^0_+ (\mathbb{R}, L^2(M, E))$ carries a natural locally convex topology defined as follows. For $I_c = [c, \infty)$, $c \in \mathbb{R}$, denote by $C^0_+(I_c, L^2(M, E))$ the subspace of $C^0 (\mathbb{R}, L^2(M, E))$ consisting of functions supported on $I_c$. We equip $C^0_+(\mathbb{R}, L^2(M, E))$ with the induced topology. The topology of $C^0_+(\mathbb{R}, L^2(M, E))$ then is the coarsest locally convex topology that makes continuous the inclusion of $C^0_+(\mathbb{R}, L^2(M, E))$ into $C^0 (\mathbb{R}, L^2(M, E))$ for all $c \in \mathbb{R}$. We observe that with respect to this topology, the inclusion of $C_c^\infty (M \times \mathbb{R}, E)$ into $C^0_+ (\mathbb{R}, L^2(M, E))$ is continuous.

**Remark 2.3.** Let $c \in \mathbb{R}$ and $u \in C^0_+(\mathbb{R}, L^2(M, E))$. Then, for all $s \in \mathbb{R},$
\[(L + \partial_t)^{-1} u(s) = \int_{\{0 \leq t \leq s - c\}} e^{-tL} u(s - t) dt.\]
Therefore, we see that $(L + \partial_t)^{-1} u \in C^0_+(\mathbb{R}, L^2(M, E))$ and, for $s \geq c$, we have
\[\| (L + \partial_t)^{-1} u(s) \| \leq \int_0^{s-c} \| e^{-tL} u(s - t) \| dt \leq (s - c) \sup \{ \| u(s') \| : c \leq s' \leq s \}.\]
We then deduce that $(L + \partial_t)^{-1}$ induces a continuous endomorphism of $C^0_+(\mathbb{R}, L^2(M, E))$ for all $c \in \mathbb{R}$, and hence is a continuous operator from $C^0_+(\mathbb{R}, L^2(M, E))$ to itself.

**Remark 2.4.** The space $C_c^\infty (M \times \mathbb{R}, E)$ carries a natural locally convex topology defined in a similar fashion as the topology of $C^0_+(\mathbb{R}, L^2(M, E))$ described in Remark 2.2. Furthermore, by using the ellipticity of $L$ it can be further shown that with respect to this topology the operator $(L + \partial_t)^{-1}$ induces a continuous endomorphism of $C_c^\infty (M \times \mathbb{R}, E)$ (see also Remark 2.22 below on this point).

Let us denote by $E \boxtimes E^*$ the vector bundle over $M \times \mathbb{R}$ whose fiber at $(x, y, t) \in M \times \mathbb{R}$ is $E_n|_{E_n}$. We define the heat kernel $k_t(x, y), t > 0,$ as the smooth section of $E \boxtimes E^*$ over $M \times \mathbb{R}$ such that
\[e^{-tL} u(x) = \int_M k_t(x, y) u(y) dy \quad \forall u \in L^2(M, E),\]
where $|dy|$ is the Riemannian density defined by $g$ on $M$. That is, $k_t(x, y)|dy|$ is the Schwartz kernel of $e^{-tL}$. In addition, $(L + \partial_t)^{-1}$ is a continuous linear operator from $C^0_+(\mathbb{R}, L^2(M, E))$ to itself, we may regard it as a continuous linear operator from $C_c^\infty (M \times \mathbb{R}, E)$ to $C^0_0 (M \times \mathbb{R}, E)$. As such, it has a kernel $k_{(L+\partial_t)^{-1}}(x, y, t) \in C^0_0 (M \times \mathbb{R}, E) \boxtimes \mathcal{D}'(M_n \times \mathbb{R}, E)$ such that
\[(L + \partial_t)^{-1} u(x, s) = \{ k_{(L+\partial_t)^{-1}}(x, s, y, t), u(y, t) \} \quad \forall u \in C_c^\infty (M \times \mathbb{R}, E).\]
We then observe that, at the level of Schwartz kernels, the definition (2.3) means that
\[k_{(L+\partial_t)^{-1}}(x, s, y, t) = \begin{cases} k_{s-t}(x, y) & \text{for } s - t > 0, \\ 0 & \text{for } s - t < 0. \end{cases}\]

It then follows that $(L + \partial_t)^{-1}$ has the Volterra property in the sense of the following definition.

**Definition 2.5** ([2]). A linear operator $Q : C^\infty_c (M \times \mathbb{R}, E) \rightarrow C^0_0 (M \times \mathbb{R}, E)$ has the Volterra property when it satisfies the following properties:

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In addition, we denote by $C_{2.9}$ (See [BGS]) for the inverse of the heat operator. The idea is to modify the classical ΨDO calculus in order to
with an asymptotic expansion

In particular, we see that $\tilde{\cdot}$

Then there is a unique $G$ such that $k_Q(x,s,y,t) = k_Q(x,y,s-t)$,

for some $K_Q(x,y,t)$ in $C^\infty(M,E) \otimes D'(M \times \mathbb{R},E)$ such that $K_Q(x,y,t) = 0$ for $t < 0$. The distribution $K_Q(x,y,t)$ is then called the Volterra kernel of $Q$. We also observe that in this case $Q$ uniquely extends to a continuous linear operator from $C^\infty(M \times \mathbb{R},E)$ to itself.

The Volterra ΨDO calculus aims at constructing a class of ΨDOs which is a natural receptacle for the inverse of the heat operator. The idea is to modify the classical ΨDO calculus in order to take into account the following properties:

(i) The aforementioned Volterra property.

(ii) The parabolic homogeneity of the heat operator $L + \partial_t$, i.e., the homogeneity with respect to the dilations,

\[ \lambda (\xi, \tau) := (\lambda \xi, \lambda^2 \tau) \quad \forall (\xi, \tau) \in \mathbb{R}^{n+1} \forall \lambda \in \mathbb{R}^*. \]

In what follows, for $G \in S'(\mathbb{R}^{n+1})$ and $\lambda \neq 0$, we denote by $G_\lambda$ the distribution in $S'(\mathbb{R}^{n+1})$ defined by

\[ \langle G_\lambda(\xi, \tau), u(\xi, \tau) \rangle := |\lambda|^{-(n+2)} \langle G(\xi, \tau), u(\lambda^{-1} \xi, \lambda^{-2} \tau) \rangle \quad \forall u \in S(\mathbb{R}^{n+1}). \]

In addition, we denote by $\mathbb{C}_-$ the complex halfplane $\{ \Im \tau < 0 \}$ with closure $\overline{\mathbb{C}}_-$.

**Definition 2.7.** A distribution $G \in S'(\mathbb{R}^{n+1})$ is (parabolic) homogeneous of degree $m$, $m \in \mathbb{Z}$, when

\[ G_\lambda = \lambda^m G \quad \forall \lambda \in \mathbb{R} \setminus 0. \]

We mention the following version of Paley-Wiener-Schwartz Theorem.

**Lemma 2.8 ([BGS], Prop. 1.9]).** Let $q(\xi, \tau) \in C^\infty((\mathbb{R}^n \times \mathbb{R}) \setminus 0)$ be a parabolic homogeneous function of degree $m$, $m \in \mathbb{Z}$, such that

(i) $q(\xi, \tau)$ extends to a continuous function on $(\mathbb{R}^n \times \overline{\mathbb{C}}_-) \setminus 0$ in such way to be holomorphic with respect to the variable $\tau$ on $\mathbb{R}^n \times \mathbb{C}_-$.

Then there is a unique $G \in S'(\mathbb{R}^{n+1})$ agreeing with $q$ on $\mathbb{R}^{n+1} \setminus 0$ and such that

(ii) $G$ is homogeneous of degree $m$.

(iii) The inverse Fourier transform $\hat{G}(x,t)$ vanishes for $t < 0$.

**Remark 2.9** (See [BGS]). The homogeneity of $G$ implies that

\[ \hat{G}_\lambda = |\lambda|^{-(n+2)} \lambda^{-m} \hat{G} \quad \forall \lambda \in \mathbb{R}^*. \]

In particular, we see that $\hat{G}$ is positively homogeneous of degree $-(m + n + 2)$.

Let $U$ be an open subset of $\mathbb{R}^n$. We define Volterra symbols and Volterra ΨDOs on $U \times \mathbb{R}^{n+1} \setminus 0$ as follows.

**Definition 2.10.** $S^m_\infty(\mathbb{R}^{n+1})$, $m \in \mathbb{Z}$, consists of smooth functions $q(x,\xi,\tau)$ on $U \times \mathbb{R}^n \times \mathbb{R}$ with an asymptotic expansion $q(x,\xi,\tau) \sim \sum_{j \geq 0} q_{m-j}(x,\xi,\tau)$, where

- $q(x,\xi,\tau) \in C^\infty((\mathbb{R}^n \times \mathbb{R}) \setminus 0)$ is a homogeneous Volterra symbol of degree $m$, i.e., $q_i$ is parabolic homogeneous of degree $l$ and satisfies the property (i) in Lemma 2.8 with respect to the variables $\xi$ and $\tau$. 

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- The asymptotic expansion is meant in the sense that, for all compact sets $K \subset U$, integers $N$ and $k$ and multi-orders $\alpha$ and $\beta$, there is a constant $C_{N,K,\alpha,\beta,k} > 0$ such that, for all $(x, \xi, \tau) \in K \times \mathbb{R}^n \times \mathbb{R}$ with $|\xi| + |\tau|^\frac{1}{2} \geq 1$, we have

$$
(2.8) \quad \sum_{j \in \mathbb{N}} q_{m-j} \partial_x^\alpha \partial_\xi^\beta \partial_\tau^k (q - \sum_{j \in \mathbb{N}} q_{m-j}) (x, \xi, \tau) \leq C_{N,K,\alpha,\beta,k} (|\xi| + |\tau|^{1/2})^{m-N-|\beta|-2k}.
$$

In what follows, for a symbol $q(x, \xi, \tau) \in S^0(U \times \mathbb{R})$ we shall denote by $q(x, D_x, D_t)$ the operator from $C^\infty_c(U \times \mathbb{R})$ to $C^\infty(U \times \mathbb{R})$ defined by

$$
q(x, D_x, D_t)u(x, t) = (2\pi)^{-(n+1)} \int e^{i(x \cdot \xi + t \cdot \tau)} q(x, \xi, \tau) \hat{u}(\xi, \tau) d\xi d\tau \quad \forall u \in C^\infty_c(U \times \mathbb{R}).
$$

**Definition 2.11.** $\Psi^m(U \times \mathbb{R})$, $m \in \mathbb{Z}$, consists of continuous linear operators $Q$ from $C^\infty_c(U \times \mathbb{R})$ to $C^\infty_c(U \times \mathbb{R})$ such that

(i) $Q$ has the Volterra property in the sense of Definition 2.5

(ii) $Q$ can be put in the form

$$
(2.9) \quad Q = q(x, D_x, D_t) + R,
$$

for some symbol $q(x, \xi, \tau) \in S^m(U \times \mathbb{R})$ and some smoothing operator $R$.

**Remark 2.12.** For any operator $Q \in \Psi^m(U \times \mathbb{R})$ there is a unique Volterra kernel $K_Q(x, y, t)$ in $C^\infty(U, \mathcal{D}'(U \times \mathbb{R}))$ such that $K_Q(x, y, t) = 0$ for $t < 0$ and

$$
Qu(x, s) = (K_Q(x, y, s - t), u(y, t)) \quad \forall u \in C^\infty_c(U \times \mathbb{R}).
$$

In fact, if we put $Q$ in the form (2.9) and we denote by $k_R(x, s, y, t)$ the Schwartz kernel of the smoothing operator $R$ as defined in (2.7), then

$$
K_Q(x, y, t) = \delta(x, y - t) + k_R(x, 0, y, -t).
$$

**Example 2.13.** Let $P$ be a differential operator of order 2 on $U$ with principal symbol $p_2(x, \xi)$. Then the operator $P + \partial_\tau$ is a Volterra $\Psi$DO of order 2 with principal symbol $p_2(x, \xi) + i\tau$. In particular, if $p_2(x, \xi) > 0$ for all $(x, \xi) \in U \times (\mathbb{R}^n \setminus 0)$, then $p_2(x, \xi) + i\tau \neq 0$ for all $(x, \xi, \tau) \in U \times [(\mathbb{R}^n \setminus 0) \times \mathbb{R}]$.

The following definition provides us with further examples of Volterra $\Psi$DOs.

**Definition 2.14.** Let $q_m(x, \xi, \tau) \in C^\infty(U \times (\mathbb{R}^{n+1} \setminus 0))$ be a homogeneous Volterra symbol of order $m$ and let $G_m(x, \xi, \tau) \in C^\infty(U, S'(\mathbb{R}^{n+1}))$ denote its unique homogeneous extension given by Lemma 2.8. Then

- $q_m(x, y, t)$ is the inverse Fourier transform of $G_m(x, \xi, \tau)$ w.r.t. the last $n + 1$ variables.

- The operator $q_m(x, D_x, D_t) : C^\infty_c(U \times \mathbb{R}) \rightarrow C^\infty_c(U \times \mathbb{R})$ is defined by

$$
(2.10) \quad q_m(x, D_x, D_t)u(x, s) := \langle \hat{q}_m(x, \xi, \tau), u(x, t) \rangle \quad \forall u \in C^\infty_c(U \times \mathbb{R}).
$$

**Remark 2.15.** It follows from the proof of [BG, Prop. 1.9] that the homogeneous extension $G_m(x, \xi, \tau)$ depends smoothly on $x$, i.e., it belongs to $C^\infty(U, S'(\mathbb{R}^{n+1}))$.

**Lemma 2.16.** The operator $q_m(x, D_x, D_t)$ is a Volterra $\Psi$DO of order $m$ with symbol $q \sim q_m$.

**Sketch of Proof.** Let $Q = q_m(x, D_x, D_t)$. Since $q_m(x, y, t)$ belongs to $C^\infty(U, S'(\mathbb{R}^{n+1}))$, it follows from (2.10) that the operator $q_m(x, D_x, D_t)$ is continuous and satisfies the Volterra property. Denote by $G_m(x, \xi, \tau)$ the unique homogeneous extension of $q_m(x, \xi, \tau)$ given by Lemma 2.8. In addition, let $\varphi \in C^\infty_c(\mathbb{R}^{n+1})$ be such that $\varphi(\xi, \tau) = 1$ near $(\xi, \tau) = (0, 0)$. Then the symbol $\tilde{q}_m(x, \xi, \tau) := (1 - \varphi(\xi, \tau)) q_m(x, \xi, \tau)$ lies in $S^m(U \times \mathbb{R}^{n+1})$ and we have

$$
K_Q(x, y, t) = \langle \tilde{q}_m(x, \xi, \tau), u(x, t) \rangle + \langle \varphi G_m(x, \xi, \tau), u(x, t) \rangle.
$$

Observe that $(\varphi G_m)(x, y, t)$ is smooth since this is the inverse Fourier transform of a compactly supported function. Thus $Q$ agrees with $\tilde{q}_m(x, D_x, D_t)$ up to a smoothing operator, and hence is a Volterra $\Psi$DO of order $m$. Furthermore, it has symbol $\tilde{q}_m \sim q_m$. The proof is complete. □

We gather the main properties of Volterra $\Psi$DOs in the following statement.
Proposition 2.17 ([Gr] [Pi] [BCS]). The following properties hold.

1. Pseudolocality. For any $Q \in \Psi^m(U \times \mathbb{R})$, the Volterra kernel $K_Q(x, y, t)$ is smooth on the open subset $\{(x, y, t) \in M \times M \times \mathbb{R}; x \neq y \text{ or } t \neq 0\}$.

2. Proper Support. For any $Q \in \Psi^m(U \times \mathbb{R})$ there exists $Q' \in \Psi^m(U \times \mathbb{R})$ such that $Q'$ is properly supported and $Q - Q'$ is a smoothing operator.

3. Composition. Let $Q_j \in \Psi^m(U \times \mathbb{R})$, $j = 1, 2$, have symbol $q_j$ and assume that $Q_1$ or $Q_2$ is properly supported. Then $Q_1Q_2$ lies in $\Psi^{m_1+m_2}(U \times \mathbb{R})$ and has symbol $q_1q_2$.

4. Parametrices. Any $Q \in \Psi^m(U \times \mathbb{R})$ admits a parametrix in $\Psi^{-m}(U \times \mathbb{R})$ if and only if its principal symbol is nowhere vanishing on $U \times \{(\mathbb{R}^n \times \mathbb{C}) \setminus 0\}$.

5. Diffeomorphism Invariance. Let $\phi$ be a diffeomorphism from $U$ onto an open subset $V$ of $\mathbb{R}^n$. Then for any $Q \in \Psi^m(U \times \mathbb{R})$ the operator $(\phi \oplus \text{id}_{\mathbb{R}})Q$ is contained in $\Psi^m(V \times \mathbb{R})$.

Remark 2.18. Most properties of Volterra $\Psi$DOs can be proved in the same way as with classical $\Psi$DOs or by observing that Volterra $\Psi$DOs are $\Psi$DOs of type $(\frac{1}{2}, 0)$ in the sense of [He].

As usual with $\Psi$DOs, the asymptotic expansion (2.8) for the symbol of a given $\Psi$DO can be translated in terms of an asymptotic expansion for the Schwartz kernel in terms of distributions that are smoother and smoother. For Volterra $\Psi$DOs we obtain the following result.

Proposition 2.19 ([Gr] [Pi] [BCS]). Let $Q \in \Psi^m(U \times \mathbb{R})$ have symbol $q \sim \sum q_{m-j}$. Then, for all $N \in \mathbb{N}_0$, there is $J \in \mathbb{N}$ such that

$$K_Q(x, y, t) = \sum_{j \leq J} q_{m-j}(x, x - y, t) \text{ mod } C^N(U \times U \times \mathbb{R}).$$

Sketch of Proof. As Volterra $\Psi$DOs are $\Psi$DOs of type $(\frac{1}{2}, 0)$, the kernel of a Volterra $\Psi$DO of order $\leq -(n + 2 + 2N)$ is $C^N$ (see [He]). Let us choose $J$ so that $m - J \leq -(n + 2 + 2N)$, then $Q - \sum q_{m-j}(x, D_x, D_t)$ is a Volterra $\Psi$DOs with symbol $q^J \sim \sum q_{m-j}$, and hence it has order $m - J - 1 \leq -(n + 2 + 2N)$. Therefore, its kernel is $C^N$. This proves the result. \qed

The invariance property in Proposition 2.17 enables us to define Volterra $\Psi$DOs on products of manifolds with $\mathbb{R}$ and acting on sections of vector bundles. In particular, we can define Volterra $\Psi$DOs on the manifold $M \times \mathbb{R}$ and acting on the sections of our vector bundle $E$ (seen as a vector bundle over $M \times \mathbb{R}$). All the aforementioned properties of Volterra $\Psi$DOs hold verbatim in this context. We shall denote by $\Psi^m(M \times \mathbb{R}, E)$ the space of Volterra $\Psi$DOs of order $m$ on $M \times \mathbb{R}$ acting on sections of $E$.

Remark 2.20. By Remark 2.18 Volterra $\Psi$DOs on $M \times \mathbb{R}$ uniquely extend to continuous operators $C^\infty_+(M \times \mathbb{R}, E)$ to itself. Therefore, the composition of such Volterra $\Psi$DOs always make sense as a continuous operator from $C^\infty_+(M \times \mathbb{R}, E)$ to itself. This enables us to drop the proper support assumption in part (3) of Proposition 2.17.

If $Q \in \Psi^m(M \times \mathbb{R}, E)$, then there is a unique $K_Q(x, y, t) \in C^\infty_+(M \times \mathbb{R}) \oplus D'(M, E)$ such that $K_Q(x, y, t) = 0$ for $t < 0$ and

$$Qu(x, s) = (K_Q(x, y, s - t)u(y, t)) \quad \forall u \in C^\infty_+(M \times \mathbb{R}, E).$$

We shall refer to $K_Q(x, y, t)$ as the Volterra kernel of $Q$. Proposition 2.17 ensures us that $K_Q(x, y, t)$ is smooth for $t \neq 0$. Therefore, on $M \times M \times \mathbb{R}^*$ we may regard $K_Q(x, y, t)$ as a smooth section of $E \otimes E^*$ over $M \times M \times \mathbb{R}^*$ such that

$$\langle K_Q(x, y, t), u(y, t) \rangle = \int_{M \times \mathbb{R}} K_Q(x, y, t)u(y, t)dydt \quad \forall u \in C^\infty_+(M \times \mathbb{R}^*, E),$$

where $\langle \cdot, \cdot \rangle$ is the pairing of sections of $E$ with sections of $E^*$. We shall denote this pairing by $\int_{M \times \mathbb{R}}$.
where in the l.h.s. \( K_Q(x, y, t) \) is an element of \( C^\infty(M \times \mathbb{R}) \hat{\otimes} \mathcal{D}'(M, E) \) and in the r.h.s. it is a smooth section of \( E \boxtimes E^* \).

It follows from Example 2.13 and Proposition 2.17 that the heat operator \( L + \partial_t \) admits a parametrix in \( \Psi^{-2}(M \times \mathbb{R}, E) \). Comparing such a parametrix with the inverse \((L + \partial_t)^{-1}\) defined by (2.22) and using (2.23) we arrive at the following result.

**Proposition 2.21** (Gr [9], BCS pp. 363-362). The operator \((L + \partial_t)^{-1}\) defined by (2.23) is a Volterra \( \Psi \)DO of order \(-2\). Moreover, we have

\[
(2.12) \quad k_t(x, y) = K_{(L + \partial_t)^{-1}}(x, y, t) \quad \forall t > 0.
\]

**Remark 2.22.** The fact that \((L + \partial_t)^{-1}\) is Volterra \( \Psi \)DO implies that it induces a continuous linear operator from \( C^\infty_0(M \times \mathbb{R}, E) \) to itself (cf. Remark 2.4).

Proposition 2.21 provides us with a representation of the heat kernel as the (Volterra) kernel of a Volterra \( \Psi \)DO. Combining it with (2.11) enables us to describe the asymptotic behaviour of \( k_t(x, x) \) as \( t \to 0^+ \) (see Gr [9] BCS and next section). More generally, we have the following result.

**Proposition 2.22.** Let \( P : C^\infty(M, E) \to C^\infty(M, E) \) be a differential operator of order \( m \). For \( t > 0 \) denote by \( h_t(x, y) \) the kernel of \( Pe^{-tL} \) defined as in (2.14). Then, \( P(L + \partial_t)^{-1} \) is a Volterra \( \Psi \)DO of order \( m - 2 \), and we have

\[
(3.1) \quad h_t(x, y) = K_{P(L + \partial_t)^{-1}}(x, y, t) \quad \forall t > 0.
\]

**Proof.** As the order of \( P \) as a Volterra \( \Psi \)DO is \( m \), it follows from Proposition 2.17 that \((L + \partial_t)^{-1}\) is a Volterra \( \Psi \)DO of order \( m - 2 \). Moreover, we have

\[
(3.2) \quad Tr [Pe^{-tL}U_\phi] = \int_M \text{tr}_E [h_t(x, \phi(x)) \phi^E(x)] \, dx = \int_M \text{tr}_E [\phi^E(x)h_t(x, \phi(x))] \, dx.
\]

We are thus led to understand the short-time behavior of \( h_t(x, \phi(x)) \). Since by Proposition 2.22 we can represent \( h_t(x, y) \) as the kernel of a Volterra \( \Psi \)DO, we shall more generally study the short-time behavior of \( K_Q(x, \phi(x), t) \), where \( Q \in \Psi^m_0(M \times \mathbb{R}, E), m \in \mathbb{Z} \).

In what follows, we denote by \( M^\phi \) the fixed-point set of \( \phi \), and for \( a = 0, \ldots, n \), we let \( M^\phi_a \) be the subset of \( M^\phi \) consisting of fixed-points \( x \) at which \( \phi^a(x) - 1 \) has rank \( n - a \), i.e., the eigenvalue 1 of \( \phi^a(x) \) has multiplicity \( a \). Therefore, we have the disjoint-sum decomposition,

\[
M^\phi = \bigsqcup_{0 \leq a \leq n} M^\phi_a.
\]

In addition, we pick some \( \epsilon_0 \in (0, \rho_0) \), where \( \rho_0 \) is the injectivity radius of \((M, g)\).
Let $x_0$ be a point in some component $M^\phi_\epsilon$. Denote by $B_{x_0}(\epsilon_0)$ the ball of radius $\epsilon_0$ around the origin in $T_{x_0}M$. Then $\exp_{x_0}$ induces a diffeomorphism from $B_{x_0}(\epsilon_0)$ onto an open neighborhood $U_{\epsilon_0}$ of $x_0$ in $M$. Moreover, as $\phi$ is an isometry, for all $X \in B_{x_0}(\epsilon_0)$, we have

\[(3.3) \quad \phi(\exp_{x_0}(X)) = \exp_{\phi(x_0)}(\phi'(x_0)X) = \exp_{x_0}(\phi'(x_0)X).\]

Thus under $\exp_{x_0}|_{B_{x_0}(\epsilon_0)}$ the diffeomorphism $\phi$ corresponds to $\phi'(x_0)$, and hence $M^\phi \cap U_{\epsilon_0}$ is identified with $B_{\epsilon_0}^{\phi}(\epsilon_0) : = B_{x_0}(\epsilon_0) \cap \ker(\phi'(x_0) - 1)$. Incidentally, the tangent bundle $TM^\phi_{|M^\phi \cap U_{\epsilon_0}}$ is identified with $B_{\epsilon_0}^{\phi}(\epsilon_0) \times \ker(\phi'(x_0) - 1)^\perp$. Note also that when $\epsilon = 0$ this shows that $x_0$ is an isolated fixed-point. It follows from this that each component $M^\phi_\epsilon$ is a (closed) submanifold of dimension $a$ of $M$ and over $M^\phi_\epsilon$ the set $\mathcal{N}^\phi := \cup_{x \in M^\phi} \ker(\phi'(x) - 1)^\perp$ can be organized as a smooth vector bundle. We denote by $\pi : \mathcal{N}^\phi \to M^\phi$ the corresponding canonical map. We shall refer to $\mathcal{N}^\phi$ as the normal bundle of $M^\phi$. Note that $\phi'$ induces (over each component $M^\phi_\epsilon$) an isometric vector bundle isomorphism of $\mathcal{N}^\phi$ onto itself.

As is well known, using the normal bundle $\mathcal{N}^\phi$ we can construct a tubular neighborhood of $M^\phi$ as follows. Let $\mathcal{N}^\phi(\epsilon_0)$ be the ball bundle of $\mathcal{N}^\phi$ of radius $\epsilon_0$ around the zero-section. Then the map $\mathcal{N}^\phi(\epsilon_0) \ni X \mapsto \exp_{x_0}(X)$ is a homeomorphism from $\mathcal{N}^\phi(\epsilon_0)$ onto an open tubular neighborhood $V_{\epsilon_0}$ of $M^\phi$ in $M$. Moreover, over each submanifold $M^\phi_\epsilon, a = 0, 1, \ldots, n$, this maps induces a diffeomorphism from $\mathcal{N}^\phi(\epsilon_0)_{|M^\phi_\epsilon}$ onto its image. Let us fix some $\epsilon \in (0, \epsilon_0)$ and let $(x, t) \in M^\phi \times (0, \infty)$. Observe that, in view of (3.3), for all $x \in \mathcal{N}^\phi_\epsilon(\epsilon)$, we have

\[K_Q(\exp_x v, \exp_x (\phi'(x)v), t) = K_Q(\exp_x v, \phi(\exp_x v), t).\]

For $x \in M^\phi$ and $t > 0$ set

\[(3.4) \quad I_Q(x, t) := \phi^E(x)^{-1} \int_{\mathcal{N}^\phi_\epsilon(\epsilon)} \phi^E(\exp_x v) K_Q(\exp_x v, \exp_x (\phi'(x)v), t) \, dv.\]

This defines a smooth section of $\text{End} \, E$ over $M^\phi \times (0, \infty)$, since $\phi^E(x) \in \text{End} \, E_x$ for all $x \in M^\phi$.

In what follows, we shall say that a function $f(t)$ is $O(t^\infty)$ as $t \to 0^+$ when $f(t)$ is $O(t^N)$ for all $N \in \mathbb{N}$.

**Lemma 3.1.** As $t \to 0^+$, we have

\[\int_M \text{tr}_E [\phi^E(x) K_Q(x, \phi(x), t)] \, dx = \int_{M^\phi} \text{tr}_E [\phi^E(x) I_Q(x, t)] \, dx + O(t^\infty).\]

**Proof.** If we regard $K_Q(x, y, t)$ as a distributional section of $E \boxtimes E^*$ over $M \times M \times \mathbb{R}$, then Proposition 2.17 tells us that $K_Q(x, y, t)$ is smooth on $\{(x, y, t) \in M \times M \times \mathbb{R}; x \neq y\}$. Incidentally, $K_Q(x, \phi(x), t)$ is smooth on $(M \setminus M^\phi) \times \mathbb{R}$. Let $N \in \mathbb{N}$. Since $K_Q(x, y, t) = 0$ for $t < 0$, we see that $\partial_t^K K_Q(x, \phi(x), 0) = 0$ for all $x \in M \setminus M^\phi$. The Taylor formula at $t = 0$ then implies that, uniformly on compact subsets of $M \setminus M^\phi$, we have

\[K_Q(x, \phi(x), t) = O(t^N) \quad \text{as } t \to 0^+.\]

As $M$ is compact and $V_\epsilon$ is an open neighborhood of $M^\phi$, the complement $M \setminus V_\epsilon$ is a compact subset of $M \setminus M^\phi$. Thus,

\[\int_M \text{tr}_E [\phi^E(x) K_Q(x, \phi(x), t)] \, dx = \int_{V_\epsilon} \text{tr}_E [\phi^E(x) K_Q(x, \phi(x), t)] \, dx + O(t^N)\]

\[= \int_{M^\phi} \left( \int_{\mathcal{N}^\phi_\epsilon(\epsilon)} \text{tr}_E [\phi^E(\exp_x v) K_Q(\exp_x v, \phi(\exp_x v), t)] \, dv \right) \, dx + O(t^N)\]

\[= \int_{M^\phi} \text{tr}_E [\phi^E(x) I_Q(x, t)] \, dx + O(t^N).\]

This proves the lemma. \(\square\)
Thanks to Lemma 3.1, we are reduced to study the short-time behavior of \( I_Q(x, t) \). Note this is a purely local issue and \( I_Q(x, t) \) depends on \( \epsilon \) only up to \( O(t^\infty) \) near \( t = 0 \). Therefore, upon choosing \( \epsilon_0 \) small enough so that there is a local trivialization of \( E \) over the tubular neighborhood \( V_{\epsilon_0} \), we may assume that \( E \) is a trivial vector bundle.

Given a fixed-point \( x_0 \) in a submanifold component \( M^\phi_0 \), consider some local coordinates \( x = (x^1, \ldots, x^a) \) around \( x_0 \). Setting \( b = n - a \), we may further assume that over the range of the domain of the local coordinates there is an orthonormal frame \( e_1(x), \ldots, e_b(x) \) of \( N^\phi \). This defines fiber coordinates \( v = (v^1, \ldots, v^b) \). Composing with the map \( N^\phi(\epsilon_0) \ni (x, v) \mapsto \exp_x v \) we then get local coordinates \( x^1, \ldots, x^a, v^1, \ldots, v^b \) for \( M \) near the fixed-point \( x_0 \). We shall refer to this type of coordinates as \textit{tubular coordinates}. Let \( q(x, v; \xi, \nu; \tau) \sim \sum_{j \geq 0} q_{m-j}(x, v; \xi, \nu; \tau) \) be the symbol \( Q \) in these tubular coordinates. We denote by \( K_Q(x, v; y, w; t) \) the kernel of \( Q \) in these coordinates. In the local coordinates \( x^1, \ldots, x^a \) we have

\[
I_Q(x, t) = \int_{|v|<\epsilon} \phi^E(x,0)^{-1} \phi^E(x,v)K_Q(x,v,\phi'(v);t)dv,
\]

where \( \phi^E(x,v) \) is defined in the local coordinates \( x, v \).

In what follows, we let \( U \) be the open subset of \( \mathbb{R}^a \) over which the coordinates \( x = (x^1, \ldots, x^a) \) range. Moreover, we denote by \( B(\epsilon_0) \) (resp., \( B(\epsilon) \)) the open ball about the origin in \( \mathbb{R}^b \) with radius \( \epsilon_0 \) (resp., \( \epsilon \)). Note that the range of \( v = (v^1, \ldots, v^b) \) is \( B(\epsilon_0) \). In addition, for \( j = 0, 1, \ldots \) we set

\[
q_{m-j}(x, v; \xi, \nu; \tau) := \phi^E(x,0)^{-1} \phi^E(x,v)q_{m-j}(x, v; \xi, \nu; \tau).
\]

**Lemma 3.2.** As \( t \to 0^+ \) and uniformly on compact subsets of \( U \), we have

\[
I_Q(x, t) \sim \sum_{j \geq 0} \int_{|v|<\epsilon} (q_{m-j}^E)^{\nu}(x, v; 0, (1 - \phi'(x))v; t)dv.
\]

**Proof.** Let \( N \in \mathbb{N}_0 \). By Proposition 2.19 there is \( J \in \mathbb{N} \) such that \( K_Q = \sum_{j \leq J} q_{m-j} \) is \( C^N \). Set

\[
R_N(x, v, t) := K_Q(x, v, \phi'(v); t) - \sum_{j \leq J} q_{m-j}(x, v; 0, (1 - \phi'(x))v; t).
\]

Then \( R_N(x, v, t) \) is \( C^N \) on \( U \times B(\epsilon_0) \times (0, 1) \). Moreover \( R_N(x, v, t) = 0 \) for \( t < 0 \), since \( K_Q(x, v, y, w; t) \) and all the \( q_{m-j}(x, y, w; t) \) vanish for \( t < 0 \). This implies that \( \partial^t R_N(x, v, 0) = 0 \) for all \( j \leq N \). Applying Taylor’s formula at \( t = 0 \) to \( R_N(x, v, t) \) then shows that, as \( t \to 0^+ \) and uniformly on compact subsets of \( U \times B(\epsilon_0) \), the function \( R_N(x, v, t) \) is \( O(t^N) \), that is,

\[
K_Q(x, v, \phi'(v); t) = \sum_{j \leq J} q_{m-j}(x, v; 0, (1 - \phi'(x))v; t) + O(t^N).
\]

Therefore, uniformly on compact subsets of \( U \),

\[
I_Q(x, t) = \sum_{j \leq J} \int_{|v|<\epsilon} (q_{m-j}^E)^{\nu}(x, v; 0, (1 - \phi'(x))v; t)dv + O(t^N).
\]

This proves the lemma.

**Lemma 3.3.** As \( t \to 0^+ \) and uniformly on compact subsets of \( U \), we have

\[
\int_{|v|<\epsilon} (q_{m-j}^E)^{\nu}(x, v; 0, (1 - \phi'(x))v; t)dv \sim \sum_{a \in \mathbb{Z}^+} \sum_{j \leq m-n} t^{j+(m-j+2)\nu} \int_{\mathbb{R}^b} \frac{v^a}{\alpha!} (\partial^a \phi^E_{m-j})^{\nu}(x, 0; 0, (1 - \phi'(x))v; 1)dv.
\]

**Proof.** Let \( h(x, v, w, t) \) be the function on \( U \times B(\epsilon_0) \times [(\mathbb{R}^b \times \mathbb{R}) \setminus 0] \) defined by

\[
h(x, v, w, t) := (q_{m-j}^E)^{\nu}(x, v; 0, (1 - \phi'(x))w; t)dv.
\]
We observe that \( h(x, v, w, t) \) is smooth on \( U \times B(\epsilon_0) \times (\mathbb{R}^k \setminus \{0\}) \times \mathbb{R} \) and vanishes for \( t < 0 \). Moreover, the homogeneity of \( q_{m-j} \) in the sense of (3.4) implies that
\[
(3.8) \quad h(x, v, \lambda w, \lambda^2 t) = |\lambda|^{-(n+2)} \lambda^{j-m} h(x, v, w, t) \quad \forall \lambda \in \mathbb{R}^*.
\]
Setting \( k = j - (m + n + 2) \), we see that, for all \( t > 0 \), we have
\[
(3.9) \quad \int_{|v| < \epsilon} h(x, v, w, t) \, dv = t^{\frac{b}{a}} \int_{B(\epsilon)} h(x, \sqrt{t} v, \sqrt{t} w, t) \, dv = t^{\frac{2}{a}} \int_{B(\epsilon)} h(x, \sqrt{t} v, 1) \, dv.
\]
Let \( N \in \mathbb{N} \). By Taylor’s formula,
\[
(3.10) \quad h(x, \sqrt{t} v, 1) = \sum_{|\alpha| < N} \frac{t^{\frac{|\alpha|}{a}}}{\alpha!} \partial_v^\alpha h(x, 0, v, 1) + t^{\frac{2}{a}} R_N(x, \sqrt{t} v, v),
\]
where \( R_N(x, v, w) \) is the function on \( U \times B(\epsilon_0) \times \mathbb{R}^b \) given by
\[
R_N(x, v, w) = \sum_{|\alpha| = N} \int_0^1 (1-s)^{N-1} w^\alpha \partial_v^\alpha h(x, sv, w, 1) \, ds.
\]
Let \( K \) be a compact subset of \( U \). As \( w^\alpha \partial_v^\alpha h(x, v, w, t) \) is smooth on \( U \times B(\epsilon_0) \times (\mathbb{R}^k \setminus \{0\}) \times \mathbb{R} \) and vanishes for \( t < 0 \), we see that \( w^\alpha \partial_v^\alpha h(x, v, w, 0) = 0 \) for all \( l \in \mathbb{N}_0 \). Therefore, using once more Taylor’s formula at \( t = 0 \) shows that, for all \( l \in \mathbb{N}_0 \), there is a constant \( C_{K \alpha} > 0 \) such that
\[
|w^\alpha \partial_v^\alpha h(x, v, w, t)| \leq C_{K \alpha} |t|^l \quad \forall (x, v, w, t) \in K \times B(\epsilon) \times \mathbb{R}^b \times (0, 1).
\]
In addition, the homogeneity of \( h(x, v, w, t) \) implies that, when \( w \neq 0 \), we have
\[
0^\alpha \partial_v^\alpha h(x, v, w, 1) = w^\alpha |w| \partial_v^\alpha h(x, v, |w|^{-1} w, 1).
\]
Thus,
\[
|w^\alpha \partial_v^\alpha h(x, v, w, t)| \leq C_{K \alpha} |w|^{\alpha + |\alpha|^{-2}t} \quad \forall (x, v, w) \in K \times B(\epsilon) \times (\mathbb{R}^k \setminus \{0\}).
\]
The above estimate shows that \( w^\alpha \partial_v^\alpha h(x, v, w, 1) \) has rapid decay in \( w \) uniformly with respect to \( x \) and \( v \), as \( x \) ranges over \( K \) and \( v \) ranges over \( B(\epsilon) \). Incidentally, both \( w^\alpha \partial_v^\alpha h(x, v, w, 1) \) and \( R_N(x, v, w) \) are uniformly bounded on \( K \times B(\epsilon) \times \mathbb{R}^b \). It then follows that there is a constant \( C_{K N} > 0 \) such that
\[
|R_N(x, \sqrt{t} v, v)| \leq C_{K N} \quad \forall (x, v, w) \in K \times B(\epsilon).
\]
Therefore, integrating both sides of (3.10) with respect to \( v \) over \( B(\epsilon) \) we see that, as \( t \to 0^+ \) and uniformly on \( K \), we have
\[
\int_{B(\epsilon)} h(x, \sqrt{t} v, 1) \, dv = \sum_{|\alpha| < N} \frac{t^{\frac{|\alpha|}{a}}}{\alpha!} \int_{B(\epsilon)} \partial_v^\alpha h(x, 0, v, 1) \, dv + O \left( t^{\frac{2}{a}} \right).
\]
Together with (3.9) this proves that, as \( t \to 0^+ \) and uniformly on \( K \), we have
\[
(3.11) \quad \int_{|v| < \epsilon} h(x, v, w, t) \, dv \sim \sum_{|\alpha| < N} \frac{t^{\frac{k+b}{a}+|\alpha|}}{\alpha!} \int_{B(\epsilon)} \partial_v^\alpha h(x, 0, v, 1) \, dv.
\]
We observe that \( k+b = j - (m+a+2) \). Moreover, as mentioned above, the function \( w^\alpha \partial_v^\alpha h(x, 0, w, 1) \) has rapid decay uniformly with respect to \( x \), as \( x \) ranges over \( K \). Therefore, as \( t \to 0^+ \) and uniformly on \( K \), we have
\[
(3.12) \quad \int_{B(\epsilon)} w^\alpha \partial_v^\alpha h(x, 0, v, 1) \, dv = \int_{\mathbb{R}^b} w^\alpha \partial_v^\alpha h(x, 0, v, 1) \, dv + O(t^\infty).
\]
In addition, the homogeneity property (3.3) for \( \lambda = -1 \) gives
\[
\int_{\mathbb{R}^b} w^\alpha \partial_v^\alpha h(x, 0, v, 1) \, dv = \int_{\mathbb{R}^b} (-v)^\alpha \partial_v^\alpha h(x, 0, -v, (1)^2) \, dv = (-1)^{|\alpha|} \int_{\mathbb{R}^b} v^\alpha \partial_v^\alpha h(x, 0, v, 1) \, dv.
\]
Thus \( \int_{\mathbb{R}} v^n \partial^n h(x,0,v,1)dv = 0 \) whenever \(|\alpha| + j - m\) is odd. Combining this with \(3.11\) and \(3.12\) shows that, as \( t \to 0^+ \) and uniformly on \( K \), we have

\[
\int_{|x|<r} h(x,v,t)dv \sim \sum_{m=0}^{\infty} t^{-(m+a+2)+|\alpha|} \int_{\mathbb{R}^b} \frac{v^n}{\alpha!} \partial^n h(x,0,v,1)dv.
\]

This proves the lemma. \( \square \)

Combining Lemmas \(3.2\) and \(3.3\) we see that, as \( t \to 0^+ \) and uniformly on compact subsets of \( U \), we have

\[
I_Q(x,t) \sim \sum_{j \geq 0} \int_{\mathbb{R}^b} \frac{v^n}{\alpha!} \left( \partial^\alpha q_m \right)^j (x,0;0,(1-\phi'(x))v;1) dv.
\]

If \(|\alpha| + j - m\) is even, then \( -\frac{m+\alpha+2+j+|\alpha|}{2} \) is an integer and is greater than \(-\frac{m+a}{2} - 1\), i.e., it is greater than or equal to \(-\left[ \frac{m+a}{2} \right] - 1\). We actually have an equality when \( j = |\alpha| = 0 \) and \( m \) is even and when \(|\alpha| + j = 1 \) and \( m \) is odd. Thus, grouping together all the terms with same powers of \( t \), we can rewrite the above asymptotic in the form,

\[
I_Q(x,t) \sim \sum_{j \geq 0} t^{-(\frac{\alpha}{2}+1)+j} I_Q^{(j)}(x),
\]

where we have set

\[
I_Q^{(j)}(x) := \sum_{|\alpha| \leq m-2\left[ \frac{m+a}{2} \right] + 2j} \int_{\mathbb{R}^b} \frac{v^n}{\alpha!} \left( \partial^\alpha q_m \right)^j (x,0;0,(1-\phi'(x))v;1) dv.
\]

Therefore, we arrive at the following result.

**Proposition 3.4.** Let \( Q \in \Psi_v^m(M \times \mathbb{R}, E), m \in \mathbb{Z} \). Uniformly on each fixed-point submanifold \( M^\phi_a, a = 0,2,...,n \), we have

\[
I_Q(x,t) \sim \sum_{j \geq 0} t^{-(\frac{\alpha}{2}+1)+j} I_Q^{(j)}(x) \quad \text{as } t \to 0^+,
\]

where \( I_Q^{(j)}(x) \) is the section of \( \text{End}E \) over \( M^\phi_a \) defined by \(3.13\) in terms of the symbol of \( Q \) in local tubular coordinates over which \( E \) is trivialized.

**Remark 3.5.** On \( M^\phi_a \) the leading term in \(3.16\) is \( t^{-(\frac{\alpha}{2}+1)+j} I_Q^{(0)}(x) \), where \( I_Q^{(0)}(x) \) depends only on the principal symbol of \( Q \). Namely,

\[
I_Q^{(0)}(x) = \int_{\mathbb{R}^b} (q_m)\nu(x,0;0,(1-\phi'(x))v;1)dv = |1-\phi'(x)|^{-1} \int_{\mathbb{R}^b} \xi_m(x,0;0,v;1)dv.
\]

**Remark 3.6.** The asymptotic \(3.16\) is expressed in terms of the symbol of \( Q \) in tubular coordinates. However, we usually start with a symbol in some local coordinates before passing to tubular coordinates. We determine the symbol in the tubular coordinates by using the change of variable formula for symbols (as given, e.g., in \(16\)) in the case of the change of variables \( \psi(x,v) = \exp_{+}(v) \).

We are now in a position to state and prove the main result of this section.

**Proposition 3.7.** Let \( P : C^\infty(M, E) \to C^\infty(M, E) \) be a differential operator of order \( m \).

(1) Uniformly on each fixed-point submanifold \( M^\phi_a, a = 0,2,...,n \), we have

\[
I_{P(L+\partial_+)}(x,t) \sim \sum_{j \geq 0} t^{-(\frac{\alpha}{2}+1)+j} I_{P(L+\partial_+)^{-1}}^{(j)}(x) \quad \text{as } t \to 0^+,
\]

where \( I_{P(L+\partial_+)^{-1}}^{(j)}(x) \) is the section of \( \text{End}E \) over \( M^\phi_a \) defined by \(3.13\) in terms of the symbol of \( P(L+\partial_+)^{-1} \) in any tubular coordinates over which \( E \) is trivial.
(2) As $t \to 0^+$, we have

$$\text{Tr} \left[ P e^{-tL} U_\phi \right] = \int_{M^0} \text{tr}_E \left[ \phi^E(x) I_{P_1} \left( x, t \right) \right] |dx| + O(t^\infty)$$

$$\sim \sum_{0 \leq a \leq n \geq 0} t^{-\left( \frac{a}{2} + \left[ \frac{a}{2} \right] \right)} \int_{M^0} \text{tr}_E \left[ \phi^E(x) I^{(j)}_{P_1} \left( x \right) \right] |dx|.$$  

Proof. The first part is an immediate consequence of Proposition 3.3, since $P(L + \partial_t)^{-1}$ is a Volterra $\Psi$DO of order $m - 2$. Combining Proposition 3.23 and 3.29 shows that

$$\text{Tr} \left[ P e^{-tL} U_\phi \right] = \int_{M} \text{tr}_E \left[ \phi^E(x) K_{P_1} \left( x, \phi(x), t \right) \right].$$

The 2nd part then follows from Lemma 3.1 and the first part. The proof is complete. \(\square\)

Remark 3.8. When $P = 1$, the asymptotic (3.19) was originally established by Gilkey \cite{Gilkey} and Shih-Chi Lee \cite{Lee}.

Remark 3.9. The formula (3.15) expresses the coefficients $I^{(j)}_{P_1}(x)$ in terms of the homogeneous components of the symbol of $P(L + \partial_t)^{-1}$. Therefore, in order to compute them at a given point $x_0 \in M^0$, we may replace $P(L + \partial_t)^{-1}$ by $PQ$, where $Q$ is Volterra $\Psi$DO parametrix of $L + \partial_t$ defined near $x_0$. In particular, we have

$$I_{P_1}(x_0, t) = I_{PQ}(x_0, t) + O(t^\infty).$$

4. The Local Equivariant Index Theorem

In this section, we shall use the Volterra pseudodifferential calculus and the equivariant heat kernel asymptotics from the previous section to give a new proof of the local equivariant index theorem for Dirac operators.

Let $(M^\infty, g)$ be an even dimensional compact spin oriented Riemannian manifold with the spinor bundle $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$. We let $G$ be a subgroup of the connected component of the group of smooth orientation-preserving isometries of $(M, g)$ preserving the spin structure. Then any $\phi \in G$ uniquely lifts to a unitary vector bundle isomorphism $\phi^E : \mathcal{S} \rightarrow \phi_* \mathcal{S}$ (see \cite{BC}). In addition, we allow $E$ to be a $G$-equivariant Hermitian vector bundle over $M$ of rank $p$ equipped with a $G$-equivariant Hermitian connection $\nabla^E$. We also form the $G$-equivariant $\mathbb{Z}_2$-graded Hermitian vector bundle $\mathcal{S} \otimes E = (\mathcal{S}^+ \otimes E) \oplus (\mathcal{S}^- \otimes E)$, and let $\mathcal{D}_{\nabla^E} : C^\infty(M, \mathcal{S} \otimes E) \rightarrow C^\infty(M, \mathcal{S} \otimes E)$ be the Dirac operator associated to these data. Namely,

$$\mathcal{D}_{\nabla^E} = \mathcal{D}_g \otimes 1_E + (c \otimes 1_E) \circ \nabla^E,$$

where $\mathcal{D}_g : C^\infty(M, \mathcal{S}) \rightarrow C^\infty(M, \mathcal{S})$ is the Dirac operator of $M$, and $c : T^*M \times \mathcal{S} \rightarrow \mathcal{S}$ is the Clifford action of $T^*M$ on $\mathcal{S}$. Note that with respect to the splitting $\mathcal{S} \otimes E = (\mathcal{S}^+ \otimes E) \oplus (\mathcal{S}^- \otimes E)$ the Dirac operator $\mathcal{D}_{\nabla^E}$ takes the form,

$$\mathcal{D}_{\nabla^E} = \begin{pmatrix} 0 & \mathcal{D}_{\nabla^E}^+ \\ \mathcal{D}_{\nabla^E}^- & 0 \end{pmatrix}, \quad \mathcal{D}_{\nabla^E}^\pm : C^\infty(M, \mathcal{S}^\pm \otimes E) \rightarrow C^\infty(M, \mathcal{S}^\mp \otimes E).$$

Given $\phi \in G$, as in the previous section, we let $\phi^E$ be the unitary vector bundle isomorphism from $E$ onto $\phi_* E$ induced by the action of $\phi$ on $E$. Then $\phi^E \circ \phi^E$ is a lift of $\phi$ to a unitary vector bundle isomorphism from $\mathcal{S} \otimes E$ onto $\phi_* \mathcal{S} \otimes \phi_* E = \phi_*(\mathcal{S} \otimes E)$. As in the previous section, this defines a unitary operator $U_\phi : L^2(M, \mathcal{S} \otimes E) \rightarrow L^2(M, \mathcal{S} \otimes E)$. Note that the $G$-equivariant setup implies that $U_\phi$ commutes with the Dirac operator $\mathcal{D}_{\nabla^E}$. The equivariant index $\text{ind} \mathcal{D}_{\nabla^E} : G \rightarrow \mathbb{Z}$ is defined by

$$\text{ind} \mathcal{D}_{\nabla^E}(\phi) = \text{Tr} \left[ U_\phi |_{\ker \mathcal{D}_{\nabla^E}^+} \right] - \text{Tr} \left[ U_\phi |_{\ker \mathcal{D}_{\nabla^E}^-} \right] \quad \forall \phi \in G.$$

When $\phi = \text{id}_M$ we recover the Fredholm index of $\mathcal{D}_{\nabla^E}$. When $\phi$ has only isolated fixed-points, the equivariant index of $\phi$ agrees with the Lefschetz number associated to the spin complex with coefficient in $E$ (see \cite{AB1}). The equivariant index theorem of Atiyah-Segal-Singer (ASS) expresses the equivariant index in terms of universal curvature polynomials defined as follows.
Let $\phi \in G$. As in the previous section, we shall denote by $M^\phi$ the fixed-point set of $\phi$ and by $N^\phi$ the normal bundle of $M^\phi$. We observe that, as $\phi$ preserves the orientation, the fixed-point submanifolds $M^\phi_a$ have even dimensions. Moreover, we shall orient each fixed-point submanifold $M^\phi_a$, $a = 0, 2, \ldots, n$, as in [BGV] Prop. 6.14, so that $\phi^* \hat{\omega}$ gives rise to a section of $\Lambda^{n-a}(N^\phi)^*|_{M^\phi_a}$ which is positive with respect to the orientation of $N^\phi$ defined by the orientations of $M$ and $M^\phi_a$.

Let $R^{TM}$ be the curvature of $(M, g)$, regarded as a section of $\Lambda^2 T^* M \otimes \text{End}(TM)$. The $G$-equivariance of Levi-Civita connection $\nabla^{TM}$ implies that, its restriction to each fixed-point submanifold $M^\phi_a$, $a = 0, 2, \ldots, n$, preserves the splitting $TM|_{M^\phi_a} = TM^\phi_a \oplus N^\phi$ over $M^\phi_a$. Therefore, it induces connections $\nabla^{TM^\phi}$ and $\nabla^{N^\phi}$ on $TM^\phi_a$ and $N^\phi$, respectively, so that we have

$$\nabla^{TM^\phi}_{|M^\phi_a} = \nabla^{TM^\phi} \oplus \nabla^{N^\phi} \quad \text{on } M^\phi_a.$$

Note that $\nabla^{TM^\phi}$ is the Levi-Civita connection of $TM^\phi_a$. Let $R^{TM^\phi}$ and $R^{N^\phi}$ be the respective curvatures of $\nabla^{TM^\phi}$ and $\nabla^{N^\phi}$. We observe that

$$R_{|M^2,TM^\phi} = R^{TM^\phi} \oplus R^{N^\phi}.$$  

Define

$$\hat{A}(R^{TM^\phi}) = \det \left( \frac{R^{TM^\phi}/2}{\sinh(R^{TM^\phi}/2)} \right) \quad \text{and} \quad \nu_\phi \left( R^{N^\phi} \right) = \det -\frac{1}{2} \left( 1 - \phi^N e^{-R^{N^\phi}} \right),$$

where $\det -\frac{1}{2} \left( 1 - \phi^N e^{-R^{N^\phi}} \right)$ is defined in the same way as in [BGV] Section 6.3.

In addition, let $F^E$ be the curvature of the $G$-equivariant connection $\nabla^E$ and denote by $F^E_0$ its restriction to $M^\phi_a$. Note that $F^E_0$ is a smooth section of $(\Lambda^2 T^* M^\phi_a) \otimes \text{End}(E)$. We then define

$$\text{Ch}_\phi \left( F^E \right) := \text{Tr} \left[ \phi^E \exp \left( -F^E_0 \right) \right] \in C^\infty \left( M^\phi_a, \Lambda^2 T^* M^\phi_a \right).$$

We can now state the equivariant index theorem in the following form.

**Theorem 4.1 (Equivariant Index Theorem [AS, AS2]).** For all $\phi \in G$, we have

$$\text{ind} \mathcal{D}_{\nabla^E}(\phi) = (-i)^{\frac{n}{2}} \sum_{\substack{0 \leq a \leq n \geq \text{even}}} (2\pi)^{-\frac{n}{2}} \int_{M^\phi_a} \hat{A}(R^{TM^\phi}) \wedge \nu_\phi \left( R^{N^\phi} \right) \wedge \text{Ch}_\phi \left( F^E \right).$$

**Remark 4.2.** When $\phi = \text{id}_M$ we recover the index theorem of Atiyah-Singer [AS1, AS2] for Dirac operators. In case $\phi$ has only isolated fixed-points, the formula (4.3) reduces to the fixed-point formula of Atiyah-Bott [AB1, AB2].

Let us recall how the equivariant index theorem can be proved by heat kernel techniques. In what follows we let $\text{str}_{\otimes E} = \text{tr}_{\otimes E} - \text{tr}_{\otimes E}$ be the supertrace on the fibers of the $\mathbb{Z}_2$-graded bundle $\mathcal{S} \otimes E = (\mathcal{S}^+ \otimes E) \oplus (\mathcal{S}^- \otimes E)$. We also denote by $\text{Str}$ the corresponding supertrace on $L^1(L^2(M, \mathcal{S} \otimes E))$. By the equivariant McKean-Singer formula (see, e.g., [BGV] Prop. 6.3), for any given $\phi \in G$, we have

$$\text{ind} \mathcal{D}_{\nabla^E}(\phi) = \text{Str} \left[ e^{-\mathcal{D}_{\nabla^E}^2 U_\phi} \right] \quad \text{for all } t > 0.$$

Therefore, the equivariant index theorem is a consequence of the following result.

**Theorem 4.3 (Local Equivariant Index Theorem [DP, CH, Ka]).** For all $\phi \in G$, we have

$$\lim_{t \to 0^+} \text{Str} \left[ e^{-\mathcal{D}_{\nabla^E}^2 U_\phi} \right] = (-i)^{\frac{n}{2}} \sum_{\substack{0 \leq a \leq n \geq \text{even}}} (2\pi)^{-\frac{n}{2}} \int_{M^\phi_a} \hat{A}(R^{TM^\phi}) \wedge \nu_\phi \left( R^{N^\phi} \right) \wedge \text{Ch}_\phi \left( F^E \right).$$

**Remark 4.4.** The local equivariant index theorem for the Dirac complex with coefficients in a vector bundle is originally due to Gilkey [CH], who also obtained versions of this theorem for several other elliptic complexes. Versions for the signature complex were also given by Donnelly-Postot [DP] and Kawasaki [Ka] around the same time. All these proofs partly rely on Riemannian invariant theory. Purely analytical proofs were subsequently given by Bismut [BH], Berline-Vergne [BV1, BV2].
and Lafferty-Yu-Zhang \[LYZ\]. In addition, Liu-Ma \[LM\] proved a version of the local equivariant
index theorem for families of Dirac operators.

In what follows, given a differential form \(\omega\) on \(M\) (resp., \(M^\phi_a\)) we shall denote by \(|\omega|^{(n)}\) (resp.,
\(|\omega|^{(a,0)}\)) its Berezin integral, i.e., its inner product with the volume form of \(M\) (resp., the induced
volume form of \(M^\phi_a\)). We note that if \(\omega\) is differential form on \(M^\phi_a\), then
(4.5) \[
\int_{M^\phi_a} |\omega(x)|^{(a,0)}|dx| = \int_{M^\phi_a} \omega.
\]

Bearing this in mind, let \(\phi \in G\). Applying the 2nd part of Proposition \[LYZ\] to \(L = D^2_{\nabla_E}\) and \(P = \gamma\),
where \(\gamma = \text{id}_{E^+\otimes E} - \text{id}_{E^-\otimes E}\) is the grading operator, we see that, as \(t \to 0^+\), we have
\[
\text{Str}\left[e^{-t\Phi^2_{\nabla_E}^a}U_\phi\right] = \int_{M^\phi_a} \text{str}_{\nabla_E}\left[\phi^{\otimes E}(x)I_{(\Phi^2_{\nabla_E} + \partial_t)^{-1}}(x,t)\right]|dx| + O(t^\infty).
\]

Therefore, we see that the local equivariant index theorem is a consequence of the following result.

**Theorem 4.5.** Let \(\phi \in G\). Then, as \(t \to 0^+\) and uniformly on each fixed-point submanifold \(M^\phi_a\),
we have
(4.6) \[
\text{Str}_{\nabla_E}\left[\phi^{\otimes E}(x_0)I_{(\Phi^2_{\nabla_E} + \partial_t)^{-1}}(x_0,t)\right] = \left(-i\right)^{\frac{2}{2}}(2\pi)^{-\frac{n}{2}} \hat{A}(R^{TM^\phi_a}) \wedge \nu_\phi \left(R^{\nabla^\phi}\right) \wedge \text{Ch}_\phi \left(F^E\right)(x)|^{(a,0)} + O(t).
\]

We shall now prove Theorem 4.5. By the 1st part Proposition \[LYZ\] as \(t \to 0^+\) and uniformly on
each fixed-point submanifold \(M^\phi_a\), we have
(4.7) \[
I_{(\Phi^2_{\nabla_E} + \partial_t)^{-1}}(x_0,t) \sim \sum_{j \geq 0} t^{-\frac{n}{2} + j} I^{(j)}_{(\Phi^2_{\nabla_E} + \partial_t)^{-1}}(x).
\]

Comparing the asymptotics \[4.6\] and \[4.7\] we deduce that in order to prove Theorem 4.5 it is
enough to show that the asymptotics \[4.6\] hold pointwise at any fixed-point of \(\phi\).

Let \(x_0 \in \bar{M}^\phi_a\), \(a = 0, 2, \ldots, n\). By Remark \[LYZ\] given any Volterra \(\Psi DO\) parametrix \(Q\) defined
near \(x_0\), we have
(4.8) \[
I_{(\Phi^2_{\nabla_E} + \partial_t)^{-1}}(x_0,t) = I_Q(x_0,t) + O(t^\infty).
\]

As a result we may replace the Dirac operator \(\Phi_{\nabla_E}\) by any differential operator that agrees with
\(\Phi_{\nabla_E}\) in a given local chart near \(x_0\). In other words, this enables us to localize the problem and
replace \(\Phi_{\nabla_E}\) by a Dirac operator on \(\mathbb{R}^n\) and acting on the trivial bundle with fiber \(S^a \otimes \mathbb{C}^p\), where
\(\mathbb{R}^n\) is the spinor space of \(\mathbb{R}^n\).

To wit let \(e_1, \ldots, e_n\) be an oriented orthonormal basis of \(T_{x_0}M\) such that \(e_1, \ldots, e_a\) span \(T_{x_0}M^\phi\) and \(e_{a+1}, \ldots, e_n\) span \(N^\phi_{x_0}\). This provides us with normal coordinates \((x^1, \ldots, x^a) \to \exp_{x_0}(x_1 e_1 + \cdots + x^a e_a)\). Moreover, using parallel transport enables us to construct a synchronous
local oriented tangent frame \(e_1(x), \ldots, e_a(x)\) such that \(e_1(x), \ldots, e_a(x)\) form an oriented
frame of \(TM^\phi_a\) and \(e_{a+1}(x), \ldots, e_n(x)\) form an (oriented) frame \(N^\phi\) (when both frames are re-
stricted to \(M^\phi_a\)). This gives rise to trivializations of the tangent and spinor bundles. We also
trivialize \(E\) near \(x_0\). Using these coordinates and trivializations, we let \(\Phi\) be a Dirac operator on
\(\mathbb{R}^n\) acting on the trivial bundle with fiber \(S^a \otimes \mathbb{C}^p\) associated to a metric on \(\mathbb{R}^n\) and a connection
on \(\mathbb{C}^p\) that agree near \(x = 0\) with the metric \(g\) and connection \(\nabla^E\), respectively. Incidentally, \(\Phi_{\nabla_E}\)
agrees with \(\Phi\) near \(x = 0\). Note that \(e_1(x) = \partial_j\) at \(x = 0\). Moreover, the coefficients \(g_{ij}(x)\) of the
metric and the coefficients \(\omega_{ijkl} := \langle \nabla^1 \nabla \partial_i e_k, e_j \rangle\) of the Levi-Civita connection satisfy
(4.9) \[
g_{ij}(x) = \delta_{ij} + O(|x|^2), \quad \omega_{ijkl}(x) = -\frac{1}{2} R_{ijkl} x^j + O(|x|^2),
\]
where \(R_{ijkl} := \langle R^{TM}(0)\partial_i \partial_j \partial_k \partial_l \rangle\) are the coefficients of the curvature tensor at \(x = 0\) (see, e.g., \[BG\] Chap. 1). Moreover, in order to simplify notations we shall denote by \(\phi'\) the endomorphism \(\phi'(0)\) of \(\mathbb{R}^n\). We shall use a similar notation for \(\phi''(0), \phi''(0),\) and \(\phi''(0)\). In particular,
\( \phi^N \) is the element of \( \text{SO}(n-a) \) such that
\[
\phi' = \begin{pmatrix} 1 & 0 \\ 0 & \phi^N \end{pmatrix}.
\]

Let \( Q \in \Psi_0^{-2}(\mathbb{R}^n \times \mathbb{R}, \mathcal{S}_n \otimes \mathbb{C}^p) \) be a parametrix for \( \mathcal{D}^2 + \partial \). Using (4.3) we obtain
\[
(4.10) \quad \text{str}_{\mathfrak{g} \otimes E} \left[ \phi^E \phi^E(x_0)I_{\mathcal{D}^{\infty} + \partial}^{-1}(x_0, t) \right] = (\text{str}_{\mathfrak{g} \otimes \mathbb{C}^p}) \left[ (\phi^E \phi^E)IQ(0, t) \right] + O(t^\infty).
\]

The supertrace of an endomorphisms on spinors is computed as follows. Let \( \Lambda(n) = \Lambda^\phi \mathbb{R}^n \) be the complexified exterior algebra of \( \mathbb{R}^n \). We shall use the following gradings on \( \Lambda(n) \),
\[
\Lambda(n) = \bigoplus_{1 \leq j \leq n} \Lambda^j(n) = \bigoplus_{1 \leq j \leq n-a} \Lambda^{k,l}(n),
\]
where \( \Lambda^j(n) \) is the space of forms of degree \( j \) and \( \Lambda^{k,l}(n) \) is the span of forms \( dx^{i_1} \wedge \cdots \wedge dx^{i_k} \) with \( 1 \leq i_1 < \cdots < i_k \leq a \) and \( a+1 \leq i_{k+1} < \cdots < i_{k+l} \leq n \). Given a form \( \omega \in \Lambda(n) \) we shall denote by \( \omega^{(j)} \) (resp., \( \omega^{(k,l)} \)) its component in \( \Lambda^j(n) \) (resp., \( \Lambda^{k,l}(n) \)). Let \( \text{Cl}(n) \) be the complexified Clifford algebra of \( \mathbb{R}^n \) (seen as a subalgebra of \( \text{End} \Lambda(n) \)) and let \( c : \Lambda(n) \to \text{Cl}(n) \) be the linear isomorphism given by the Clifford action of \( \Lambda(n) \) on itself. Composing \( c \) with the spinor representation \( \text{Cl}(n) \to \text{End} \Sigma_n^\phi \) (which is an algebra isomorphism since \( n \) is even), we get a linear isomorphism \( \Lambda(n) \to \text{End} \Sigma_n^\phi \) which we shall also denote by \( c \). We denote by \( \sigma : \text{End} \Sigma_n^\phi \to \Lambda(n) \) its inverse. We note that, although \( c \) and \( \sigma \) are not isomorphisms of algebras, for \( \omega_j \in \Lambda^{k_j,l_j}(n) \), \( j = 1, 2 \), we have
\[
(4.11) \quad \sigma [c(\omega_1)c(\omega_2)] = \omega_1 \wedge \omega_2 \mod \bigoplus_{(k,l) \in \mathcal{K}} \Lambda^{k,l}(n),
\]
where \( \mathcal{K} \) consists of all pairs \((k,l)\) such that, either \( k \leq k_1 + k_2 - 2 \) and \( l \leq l_1 + l_2 \), or \( k \leq k_1 + k_2 \) and \( l \leq l_1 - l_2 - 2 \).

**Lemma 4.6** ([Gei Thm. 1.8]; see also [BGV Prop. 3.21]). Let \( a \in \text{End} \Sigma_n^\phi \). Then
\[
\text{str}_{\Sigma_n^\phi}^a = (-2i)^e \sigma(a)^{(n)}.
\]

In the case of \( \phi^E \) we have the following result.

**Lemma 4.7** ([BGV LYZ]). The form \( \sigma \phi^E \) belongs to \( \Lambda^{0,2*}(n) \) and we have
\[
(4.12) \quad \sigma \left[ \phi^E \right]^{(0,n-a)} = 2^{-\frac{n-a}{2}} \det^\phi \left( 1 - \phi^N \right) dx^{a+1} \wedge \cdots \wedge dx^n.
\]

**Proof.** As \( \phi^N \) is an element of \( \text{SO}(n-a) \), there is an oriented orthonormal basis \( \{ v_{a+1}, \ldots, v_n \} \) of \( \{0\}^a \times \mathbb{R}^{n-a} \) such for \( j = 2, +1, \ldots, n \) the subspace \( \text{Span}\{v_{2j-1}, v_{2j}\} \) is invariant under \( \phi^N \) and the matrix of \( \phi^N \) with respect to the basis \( \{ v_{2j-1}, v_{2j}\} \) is a rotation matrix of the form,
\[
\begin{pmatrix}
\cos \theta_j & -\sin \theta_j \\
\sin \theta_j & \cos \theta_j
\end{pmatrix}
\]
\[
= \exp \left( \begin{pmatrix} 0 & -\theta_j \\ \theta_j & 0 \end{pmatrix} \right), \quad 0 < \theta_j \leq \pi.
\]

Using [BGV Eqs. (3.4)–(3.5)] we then see that
\[
(4.13) \quad \phi^E \left( v_{i_{j+1}}, \ldots, v_n \right) = \prod_{\frac{\pi}{2} < j \leq \frac{\pi}{2}} \left( \cos \left( \frac{\theta_j}{2} \right) + \sin \left( \frac{\theta_j}{2} \right) c(v_{2j-1})c(v_{2j}) \right),
\]
where \( \{ v_{a+1}, \ldots, v_n \} \) is the basis of \( \Lambda^{0,1}(n) \) that is dual to \( \{ v_{a+1}, \ldots, v_n \} \). It then follows that \( \sigma(\phi^E) \) is an element of \( \Lambda^{0,2*}(n) \) and we have
\[
\sigma[\phi^E]^{(0,n-a)} = \prod_{\frac{\pi}{2} < j \leq \frac{\pi}{2}} \sin \left( \frac{\theta_j}{2} \right) v^{a+1} \wedge \cdots \wedge v^n = 2^{-\frac{n-a}{2}} \det^\phi \left( 1 - \phi^N \right) dx^{a+1} \wedge \cdots \wedge dx^n,
\]
where we have used the equality \( 4 \sin^2 \frac{\theta}{2} = \frac{1 - \cos \theta}{2} = \frac{\sin \theta}{1 - \cos \theta} \). The proof is complete. \( \square \)
We shall determine the short-time behavior of $\sigma[\phi^E \text{tr}_{C^\infty} \phi^E I_Q(0,t)]$ in (4.11) by means of elementary considerations on Getzler orders of Volterra ΨDOs as in [Po1]. This notion of order is intimately related to the rescaling of Getzler [Ge2], which is motivated by the following assignment of degrees:

$$\text{deg} \, \partial_j = \text{deg} \, c(dx^j) = 1, \quad \text{deg} \, \partial_0 = 2, \quad \text{deg} \, dx^j = -1.$$

As observed in [Po1] this defines another filtration on Volterra ΨDOs as follows.

Let $Q \in \Psi^m_v(\mathbb{R}^n \times \mathbb{R}, \mathcal{S}_0 \otimes \mathbb{C})$ have symbol $q(x,\xi,\tau) \sim \sum_{r \geq m} q_{m-r}(x,\xi,\tau)$. Taking components in each subspace $\Lambda^j T^\ast_{C^\infty} \mathbb{R}^n$ and using Taylor expansions at $x = 0$ we get asymptotic expansions of symbols,

$$\sigma[q(x,\xi,\tau)] \sim \sum_{j,r} \sigma[q_{m-r}(x,\xi,\tau)]^{(j)} \sim \sum_{j,r,\alpha} \frac{x^\alpha}{\alpha!} \sigma[\partial^\alpha_x \partial^\beta_{\xi} q_{m-r}(0,\xi,\tau)]^{(j)},$$

The last asymptotic is meant in the following sense: for $j = 0, \ldots, n$ and all $N \in \mathbb{N}$, as $x \to 0$ and $|\xi| + |\tau|^{1/2} \to \infty$, we have

$$\sigma[q(x,\xi,\tau)]^{(j)} - \sum_{r+|\alpha|=N+j} \frac{x^\alpha}{\alpha!} \sigma[\partial^\alpha_x \partial^\beta_{\xi} q_{m-r}(0,\xi,\tau)]^{(j)} = O \left( ||\xi,\tau||^{m'} ||x|| + ||\xi,\tau||^{-1} N \right),$$

where we have set $||\xi,\tau|| := ||\xi|| + ||\tau||^{1/2}$, and there are similar asymptotics for all $\partial^\alpha_x \partial^\beta_{\xi}$-derivatives (upon replacing the exponent $m'$ by $m' - |\beta| - k$). Moreover, the degree assignment (4.14) leads us to define (Getzler)-rescaling operators $\delta^\lambda_{\tau}$, $\lambda \in \mathbb{R}$, on Volterra symbols with coefficients in $\Lambda(n) \otimes M_p(\mathbb{C})$ by letting

$$\delta^\lambda_{\tau} q(x,\xi,\tau) := \lambda^j q(\lambda^{-1} x, \lambda^k \xi, \lambda^{2} \tau) \quad \forall q \in \mathcal{S}_0^v(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}) \otimes \Lambda^j \otimes M_p(\mathbb{C}).$$

Note that in (4.15) each symbol $x^\alpha \partial^\alpha_{\xi} \sigma[q_{m-r}(0,\xi,\tau)]^{(j)}$ is homogeneous of degree $\mu := m' - r + j - |\alpha|$ with respect to this rescaling. We shall say that such a symbol is Getzler homogeneous of degree $\mu$. The asymptotic expansion (4.16) then imply that, in the sense of (4.17), we have

$$\sigma[q(x,\xi,\tau)] \sim \sum_{\mu \leq m} q_{(\mu)}(x,\xi,\tau),$$

where $q_{(\mu)}(x,\xi,\tau)$ is the Getzler-homogeneous symbol of degree $\mu$ given by

$$q_{(\mu)}(x,\xi,\tau) := \sum_{m' - r + j - |\alpha| = \mu} \frac{x^\alpha}{\alpha!} \sigma[\partial^\alpha_x \partial^\beta_{\xi} q_{m-r}(0,\xi,\tau)]^{(j)},$$

and $m$ is the greatest integer $\mu$ such that $q_{(\mu)} \neq 0$.

Alternatively, in terms of the rescaling operators $\delta^\lambda_{\tau}$, for all $(x,\xi,\tau) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, $(\xi,\tau) \neq 0$, we have

$$\delta^\lambda_{\tau} \sigma[q(x,\xi,\tau)] \sim \sum_{\mu \leq m} \lambda^\mu q_{(\mu)}(x,\xi,\tau) \quad \text{as} \, \lambda \to 0.$$

We observe that the homogeneous symbols $q_{(\mu)}(x,\xi,\tau)$ are uniquely determined by the above asymptotic. In particular, the leading Getzler-homogeneous symbol $q_{(m)}(x,\xi,\tau)$ is uniquely determined by

$$\delta^\lambda_{\tau} \sigma[q(x,\xi,\tau)] = \lambda^m q_{(m)}(x,\xi,\tau) + O(\lambda^{m-1}).$$

Remark 4.8. Let $\mathcal{S}(\mathbb{R}^n \times \mathbb{R})$ be the Fréchet space of Schwartz-class functions on $\mathbb{R}^n \times \mathbb{R}$. As each symbol $q_{(\mu)}(x,\xi,\tau)$ is a polynomial with respect to the variable $x$, it defines an element $q_{(\mu)}(x,D_x,D_t)$ of $\mathcal{L}(\mathcal{S}(\mathbb{R}^n \times \mathbb{R}, \mathbb{C}^p)) \otimes \Lambda(n)$ by

$$q_{(\mu)}(x,D_x,D_t) u(x,s) = \langle \hat{q}(x,x-y,s-t), u(y,t) \rangle \quad \forall u \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}, \mathbb{C}^p).$$

Note that by using the action of $\Lambda(n)$ on itself by left-multiplication we may regard $q_{(\mu)}(x,D_x,D_t)$ as an actual operator of $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}, \Lambda(n) \otimes \mathbb{C}^p)$ to itself.

Definition 4.9 (Po1). Bearing in mind the notation (4.17) and (4.18), we make the following definitions:
Lemma 4.13. Let $Q_{(m)} = q_{(m)}(x, D_x, D_t)$ be the model operator of $Q$.

Remark 4.10. As the symbol $\sigma[\partial^\alpha_x \partial^\beta_x q_{uv-\ell}(0, \xi, \tau)]^0$ is Getzler-homogeneous of degree $m' - r + j - |\alpha| \leq m' + n$, we see that the Getzler order of $Q$ is always $\leq m + n$.

Example 4.14. Let $A = A_I dx^I$ be the connection 1-form on $\mathbb{C}^n$. Then it follows from (4.13) that the covariant derivative $\nabla_i = \partial_i + \frac{1}{2} \omega_{ikl}(x)c(e^k)c(e^l) + A_i$ on $\mathcal{S}_m \otimes \mathbb{C}^p$ has Getzler order 1 and its model operator is

$$\nabla_i(1) = \partial_i - \frac{1}{4} R_{ij} x^j,$$

where $R_{ij} = \sum_{k<l} R^T_{ijkl}(0) dx^k \wedge dx^l$.}

Remark 4.12. In what follows, we shall often look at symbols or operators up to terms that have lower Getzler orders. For this reason, it is convenient to use the notation $O_G(m)$ to denote any remainder term (symbol or operator) of Getzler order $\leq m$. Note that in view of (4.13) a Volterra symbol $q \in S^2((\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}, M_p(\mathbb{C})) \otimes \Lambda(n))$ is $O_G(m)$ if and only if, for all $(x, \xi, \tau) \in (\mathbb{R}^n \times \mathbb{R} \times \mathbb{R})$, $(\xi, \tau) \neq 0$,

$$\delta q(x, \xi, \tau) = O(\lambda^m) \quad \text{as } \lambda \to 0.$$ 

Moreover, two Volterra symbols have Getzler order $m$ and same leading Getzler-homogeneous symbol if and only if their difference is $O_G(m - 1)$.

Lemma 4.15. Let $Q_j \in \Psi^*(\mathbb{R}^n \times \mathbb{R}, \mathcal{S}_m \otimes \mathbb{C}^p)$ have Getzler order $m_j$ and model operator $Q_{(m_j)}$, and assume that either $Q_1$ or $Q_2$ is properly supported. Then

$$\sigma[Q_1 Q_2] = Q_{(m_1)} Q_{(m_2)} + O_G(m_1 + m_2 - 1).$$

Example 4.14. By Lichnerowicz’s formula, near $x = 0$, we have

$$\mathcal{D}^2 = \mathcal{D}_{\mathcal{X}}^2 = -g^{ij} (\nabla_i \otimes E \nabla_j \otimes E - \Gamma^k_{ij} \nabla_k \otimes E) + \frac{\kappa}{4} + \frac{1}{2} \epsilon^i \epsilon^j \nabla_{ij},$$

where the $\Gamma^k_{ij}$ are the Christoffel symbols of the metric, $\kappa$ is the scalar curvature, and $\{\epsilon^i\}$ is the coframe dual to the synchronous frame $\{e_i\}$. Therefore, combining Lemma 4.13 with (4.20) shows that $\mathcal{D}_{\mathcal{X}}^2 E$ has Getzler order 2 and its model operator is

$$\mathcal{D}_{(2)}^2 = H_R + F^E(0),$$

where we have set

$$H_R = -\sum_{i=1}^{n} (\partial_i - \frac{1}{4} R_{ij} x^j)^2 \quad \text{and} \quad F^E(0) = \sum_{i<j} F^E(0)(\partial_i, \partial_j) dx^j \wedge dx^j.$$

In what follows, it would be convenient to introduce the variables $x' = (x^1, ..., x^n)$ and $x'' = (x^{n+1}, ..., x^n)$, so that $x = (x', x'')$. When using these variables we shall denote by $q(x', x'', \xi', \xi'', \tau)$ and $K_Q(x', x''; y', y''; t)$ the respective symbol and kernel of any given ”operator” $Q \in \Psi^*((\mathbb{R}^n \times \mathbb{R}, C^p) \otimes \Lambda(n))$. We then define

$$I_Q(x', t) := \int_{\mathbb{R}^{n+m}} K_Q(x', x''; 0, (1 - \phi^N)x''; t) dx''', \quad x' \in \mathbb{R}^n.$$

Lemma 4.15. Let $Q \in \Psi^*((\mathbb{R}^n \times \mathbb{R}, \mathcal{S}_m \otimes \mathbb{C}^p)$ have Getzler order $m$ and model operator $Q_{(m)}$. In addition, let $j$ be an integer $\leq n$.

1. If $m - j$ is an odd integer, then

$$\sigma[I_Q(0, t)]^{(j)} = O(t^{\frac{m-j-n}{2}}) \quad \text{as } t \to 0^+.$$

2. If $m - j$ is an even integer, then

$$\sigma[I_Q(0, t)]^{(j)} = t^{\frac{m-j-n}{2}} I_Q_{(m)}(0, 1)^{(j)} + O(t^{\frac{m-j-n}{2}}) \quad \text{as } t \to 0^+.$$
Proof. Let \( q(x, \xi, \tau) \sim \sum_{k \leq m} q_k(x, \xi, \tau) \) be the symbol of \( Q \) and denote by \( q_{(m)}(x, \xi, \tau) \) its principal Getzler homogeneous symbol. Recall that Proposition 3.4 provides us with an asymptotic for 
\[
\sigma[I_Q(t, 0)] = \sum_{l \leq m} a_{l} q_{l}(0, 0, (1 - \Phi^N(0))v; 1) dv.
\]
where \( a_{l} q_{l}(0, 0, (1 - \Phi^N(0))v; 1) \) is the symbol of \( Q \). As in the tubular coordinates, the derivative \( \phi^\beta \) is constant along the fibers of \( N^0 \); we see that \( \phi^\beta \) too is fiberwise constant. Incidentally, in the notation of (3.3) the symbols \( \tilde{q}_k^\beta \) and \( \tilde{q}_k \) agree for all \( k \leq m' \). Bearing this mind, Proposition 3.4 shows that, as \( t \to 0^+ \),
\[
\sigma[I_Q(0, t)] \sim \sum_{\vert \alpha \vert - k \text{ even}} t^{(\frac{2}{4} + 1 + m - \vert \alpha \vert)} \int_{\mathbb{R}^n} a_{\alpha\beta}(0, v)^{\vert \alpha \vert} \partial^\alpha_{\eta} \sigma[\tilde{q}_k^{\beta}]^{(\frac{2}{4})} (0, 0; 0, (1 - \Phi^N(0))v; 1) dv.
\]
Using (4.27), the change of variable formula for symbols ([H, Thm. 18.1.17]) gives
\[
\tilde{q}_k(0, v, \xi', v, \tau) = \sum_{l \leq \vert \beta \vert + \vert \gamma \vert \leq \vert \alpha \vert} a_{\alpha\beta}(0, v)^{\vert \beta \vert} D^\beta_{\eta} q_k(0, 0, \xi', v, \tau)
\]
where the \( a_{\alpha\beta}(x', v) \) are some smooth functions such that \( a_{\alpha\beta}(x) = 1 \) when \( \beta = \gamma = 0 \). Thus,
\[
\sigma[I_Q(0, t)] \sim \sum_{\vert \alpha \vert - l + \vert \beta \vert - \vert \gamma \vert \text{ even}} t^{(\frac{2}{4} + 1 + m - \vert \alpha \vert)} I^{(j)}_{l, \alpha\beta\gamma},
\]
where we have set
\[
I^{(j)}_{l, \alpha\beta\gamma} := \int_{\mathbb{R}^n} a_{\alpha\beta}(0, v)^{\vert \alpha \vert} \partial^\alpha_{\eta} \sigma[\tilde{q}_k^{\beta}]^{(\frac{2}{4})} (0, 0; 0, (1 - \Phi^N(0))v; 1) dv.
\]
Note that the symbol \( v^\alpha \partial^\alpha_{\eta} \sigma[\tilde{q}_k^{\beta}]^{(\frac{2}{4})} (0, 0, \xi', v, \tau) \) is Getzler homogeneous of degree \( l + j - \vert \alpha \vert \). Therefore, it must be zero if \( l + j - \vert \alpha \vert > m \), since otherwise \( Q \) would have Getzler order \( m \). This implies that in (4.29) all the coefficients \( I^{(j)}_{l, \alpha\beta\gamma} \) with \( l + j - \vert \alpha \vert > m \) must be zero. Furthermore, the condition \( 2 \vert \gamma \vert \leq \vert \beta \vert \) and implies that \( \vert \gamma \vert - \vert \beta \vert \leq -\frac{1}{2} \vert \beta \vert \), and hence \( \vert \gamma \vert - \vert \beta \vert \leq -1 \) unless \( \beta = \gamma = 0 \). Therefore, if \( l + j - \vert \alpha \vert \leq m \) and \( 2 \vert \gamma \vert \leq \vert \beta \vert \), then \( t^{(\frac{2}{4} + 1 + m - \vert \alpha \vert)} I^{(j)}_{l, \alpha\beta\gamma} \) is \( O(t^{\frac{2}{4} + 1 + m - l}) \) and even is \( O(t^{\frac{2}{4} + 1 + m - l}) \) if we further have \( l + j - \vert \alpha \vert < m \) or \( \beta, \gamma \neq (0, 0) \). Observe that the asymptotic (4.29) contains only integer powers of \( t \) (non-negative or negative). Therefore, from the above observations we deduce that if \( m - j \) is odd, then all the (non-zero) terms in (4.29) are \( O(t^{\frac{2}{4} + 1 + m - l}) \), and hence
\[
\sigma[I_Q(0, t)] = O(t^{\frac{2}{4} + 1 + m - l})).
\]
Likewise, if \( m - j \) is even, then all the terms in (4.29) with \( l + j - \vert \alpha \vert \neq m \) or with \( l - \vert \alpha \vert = m - j \) and \( \beta, \gamma \neq (0, 0) \) are \( O(t^{\frac{2}{4} + 1 + m - l}) \). Thus,
\[
\sigma[I_Q(0, t)] = t^{\frac{2}{4} + 1 + m - l} \sum_{l - \vert \alpha \vert = m - j} I^{(j)}_{l, \alpha\beta\gamma} + O(t^{\frac{2}{4} + 1 + m - l})).
\]
To complete the proof it remains to identify the coefficient of \( t^{- (m+2)} \) in \((4.31)\) with \( I_{Q(m)}(0,1)^{(j)} \).
To this end observe that the formula \((4.1)\) for \( q_{(m)} \) at \( x' = 0 \) gives
\[
q_{(m)}(0,v;\xi,\nu;\tau)^{(j)} = \sum_{k+j-\lvert a\rvert = m} \frac{v^\alpha}{\alpha!} \partial^\alpha_{\xi^a} \left( \sigma[q_k]^{(j)} \right) (0;0,\xi,\nu;\tau).
\]
Thus,
\[
I_{Q(m)}(0,1)^{(j)} = \sum_{k-\lvert a\rvert = m-j} \int_{\mathbb{R}^{n-a}} \frac{v^\alpha}{\alpha!} \left( \partial^\alpha_{\xi^a} \sigma[q_{m-j+\lvert a\rvert}]^{(j)} \right) (0,0;0,1-\phi^N(0);v;1) \, dv
= \sum_{t-\lvert a\rvert = m-j} I_{t(0)}^{(j)}
\]
This completes the proof.

We are now in a position to prove the key lemma of the proof of Theorem \ref{thm:main_result}.

**Lemma 4.16.** Let \( Q \in \Psi^\tau_n(\mathbb{R}^n \times \mathbb{R}, \mathcal{S}_m \otimes \mathbb{C}^p) \) have Getzler order \( m \) and model operator \( Q_{(m)} \).

1. If \( m \) is an odd integer, then
   \[
   \text{str}_{g \otimes \mathbb{C}^p} \left[ \phi^\eta \otimes E I_Q(0,t) \right] = O(t^{-m+1}) \quad \text{as } t \to 0^+.
   \]
2. If \( m \) is an even integer, then, as \( t \to 0^+ \), we have
   \[
   \text{str}_{g \otimes \mathbb{C}^p} \left[ \phi^\eta \otimes E I_Q(0,t) \right] = \left( -i \right)^{\frac{\ell}{2}} t^{-\left(\frac{\ell}{2}+1\right)} 2^\frac{\ell}{2} \det \phi^\eta \left( 1 - \phi^N \right) \left[ \text{tr}_{\mathbb{C}^p} [\phi^E I_{Q(m)}(0,1)] \right]^{(a,0)} + O(t^{-\frac{\ell}{2}}).
   \]

Proof. For \( t > 0 \) set \( a(t) = (1g_n \otimes \text{tr}_{\mathbb{C}^p})[\phi^E I_Q(0,t)] \). Then by Lemma \ref{lem:integral_representation} we have
\[
\text{str}_{g \otimes \mathbb{C}^p} \left[ \phi^\eta \otimes E I_Q(0,t) \right] = \left( -2i \right)^{\frac{\ell}{2}} \left[ \sigma \left[ \phi^\eta a(t) \right] \right]^{(n,0)} \quad \text{for all } t > 0.
\]
As Lemma \ref{lem:coefficient_identification} ensures us that \( \sigma \left[ \phi^\eta \right] \) is an element of \( \Lambda^{(0,2\ell)}(n) \), using \ref{eq:coefficient_identification} we deduce that
\[
\sigma \left[ \phi^\eta a(t) \right]^{(n,0)} = \sigma \left[ \phi^\eta \right]^{(0,n-a)} \wedge \sigma \left[ a(t) \right]^{(a,0)} + \sum_{1 \leq \ell \leq \frac{\ell}{2}(n-a)} \varphi_{\ell} \left( \sigma \left[ a(t) \right] \right)^{(a,2\ell)},
\]
where \( \varphi_{\ell} : \Lambda^{(0,2\ell)}(n) \to \Lambda^{(0,\ell)}(n) \) is a linear map which does not depend on \( a(t) \). Moreover, for \( \ell = 0,1,\ldots,\frac{\ell}{2}(n-a) \), Lemma \ref{lem:coefficient_identification} ensures that, as \( t \to 0^+ \), we have
- \( \sigma \left[ a(t) \right]^{(a,2\ell)} = O \left( t^{-\frac{\ell}{2}+\frac{\ell}{2}a} \right) \) when \( m \) is odd.
- \( \sigma \left[ a(t) \right]^{(a,2\ell)} = t^{\ell-\frac{\ell}{2}} \text{tr}_{\mathbb{C}^p} [\phi^E I_{Q(m)}(0,1)]^{(a,2\ell)} + O \left( t^{\ell-\frac{\ell}{2}} \right) \) when \( m \) is even.

Therefore, when \( m \) is odd, we obtain
\[
\sigma \left[ \phi^\eta a(t) \right]^{(n,0)} = O \left( t^{-\frac{\ell}{2}+\frac{\ell}{2}a} \right) \quad \text{as } t \to 0^+.
\]
Combining this with \ref{eq:coefficient_identification} we get the asymptotic \ref{eq:integral_representation_odd} when \( m \) is odd. When \( m \) is even, we see that, as \( t \to 0^+ \), we have
\[
\sigma \left[ \phi^\eta a(t) \right]^{(n,0)} = t^{\ell-\frac{\ell}{2}} \sigma \left[ \phi^\eta \right]^{(0,n-a)} \wedge \text{tr}_{\mathbb{C}^p} [\phi^E I_{Q(m)}(0,1)]^{(a,0)} + O(t^{\ell-\frac{\ell}{2}}).
\]
Combining this with \ref{eq:coefficient_identification} and the formula \ref{eq:coefficient_identification_even} for \( \sigma \left[ \phi^\eta \right]^{(0,n-a)} \) yields the asymptotic \ref{eq:integral_representation_even} when \( m \) is even. The proof is complete.
(1) Q has Getzler order −2 and its model operator is
\[ Q_{(-2)} = (H_R + \partial_t)^{-1} \wedge \exp (-tF^E(0)). \]

(2) For all \( t > 0 \), we have
\[ I_{Q_{(-2)}}(0, t) = I_{(H_R + \partial_t)^{-1}}(0, t) \wedge \exp (-tF^E(0)), \]
\[ I_{(H_R + \partial_t)^{-1}}(0, t) = \frac{(4\pi t)^{-\frac{n}{2}}}{\det (1 - \phi^N)} \det^{\frac{1}{2}} \left( \frac{tR^2/2}{\sinh(tR^2/2)} \right) \det^{-\frac{1}{2}} \left( 1 - \phi^N e^{-tR^*} \right). \]

**Proof.** The first part is contained in [Pol1 Lemma 5]. This immediately gives the formula (4.34). The formula for \( I_{(H_R + \partial_t)^{-1}}(0, t) \) is obtained exactly like in [LM] p. 459. For reader’s convenience we mention the main details of this computation.

The kernel of \((H_R + \partial_t)^{-1}\) can be determined from the arguments of [Ge2]. More precisely, let \( A \in \mathfrak{so}_n(\mathbb{R}) \) and set \( B = A^t A \). Consider the harmonic oscillators,
\[ H_A := - \sum_{1 \leq i \leq n} (\partial_i + \sqrt{-1} A_{ij} x^j)^2 \quad \text{and} \quad H_B := - \sum_{1 \leq i \leq n} \partial_i^2 + \frac{1}{4} (Bx, x). \]
In particular substituting \( A = \frac{1}{2} \sqrt{-1} R \) in the formula for \( H_A \) gives \( H_R \). In addition, define
\[ X := -\sqrt{-1} \sum_{i,j} A_{ij} x^i \partial_j = \sqrt{-1} \sum_{i < j} A_{ij} (x^i \partial_j - x^j \partial_i). \]
Note that \( H_A = H_B + X \). Observe also that, as \( X \) is a linear combination of the infinitesimal rotations \( x^i \partial_i - x^j \partial_j \), the \( O(n) \)-invariance of \( H_B \) implies that \([H_B, X] = 0\). Thus,
\[ e^{-tH_A} = e^{-tX} e^{-tH_B} \quad \forall t \geq 0. \]

The heat kernel of \( H_B \) is determined by using Melcher’s formula in its version of [Ge2]. We get
\[ K_{(H_R + \partial_t)^{-1}}(x, y, t) = (4\pi t)^{-\frac{n}{2}} \det^{\frac{1}{2}} \left( \frac{t\sqrt{B}}{\sinh(t \sqrt{B})} \right) \exp \left( -\frac{1}{4t} \Theta_B(x, y, t) \right), \quad t > 0, \]
where we have set
\[ \Theta_B(x, y, t) := \left< \frac{t\sqrt{B}}{\tanh(t \sqrt{B})}, x, x \right> + \left< \frac{t\sqrt{B}}{\tanh(t \sqrt{B})}, y, y \right> - 2\left< \frac{t\sqrt{B}}{\sinh(t \sqrt{B})}, x, y \right>. \]
Here \( \sqrt{B} \) is any square root of \( B \) (e.g., \( \sqrt{B} = \sqrt{-1} A \)). Note that the right-hand side of (4.37) is actually an analytic function of \( \sqrt{B} \).

We also observe that for \( t \in \mathbb{R} \) the matrix \( e^{-t\sqrt{-1}A} \) is an element of \( O(n) \), since in a suitable orthonormal basis it can be written as a block diagonal of 2 × 2 rotation matrices with purely imaginary angles. Moreover, the family of operators \( u \to u(e^{-t\sqrt{-1}A}), t \in \mathbb{R} \), is a one-parameter group of operators on \( L^2(\mathbb{R}^n) \) with infinitesimal generator \( X \), so it agrees with \( e^{-tX} \) for \( t > 0 \). Combining this with (4.36) and (4.37) then gives
\[ K_{(H_A + \partial_t)^{-1}}(x, y, t) = (4\pi t)^{-\frac{n}{2}} \det^{\frac{1}{2}} \left( \frac{t\sqrt{B}}{\sinh(t \sqrt{B})} \right) \exp \left( -\frac{1}{4t} \Theta_A(x, y, t) \right), \]
\[ \Theta_A(x, y, t) := \left< \frac{t\sqrt{B}}{\tanh(t \sqrt{B})}, x, x \right> + \left< \frac{t\sqrt{B}}{\tanh(t \sqrt{B})}, y, y \right> - 2\left< \frac{t\sqrt{B}}{\sinh(t \sqrt{B})}, e^{-t\sqrt{-1}A}, x, y \right>, \]
where we have used the fact that \( e^{-t\sqrt{-1}A} \) is an orthogonal matrix. Substituting \( A = \frac{1}{2} \sqrt{-1} R \) and \( \sqrt{B} = \frac{1}{2} R \) then gives the kernel of \((H_R + \partial_t)^{-1}\). We obtain
\[ K_{(H_R + \partial_t)^{-1}}(x, y, t) = (4\pi t)^{-\frac{n}{2}} \det^{\frac{1}{2}} \left( \frac{tR^2/2}{\sinh(tR^2/2)} \right) \exp \left( -\frac{1}{4t} \Theta_R(x, y, t) \right), \quad t > 0, \]
where we have set
\[ \Theta_R(x, y, t) := \left< \frac{tR^2/2}{\tanh(tR^2/2)}, x, x \right> + \left< \frac{tR^2/2}{\tanh(tR^2/2)}, y, y \right> - 2\left< \frac{tR^2/2}{\sinh(tR^2/2)}, e^{tR^2/2}, x, y \right>. \]
We are ready to compute \( I_{(H_n + \hat{\partial}_t)^{-1}}(0, t) \). From (4.25) and (4.38) we get

\[
I_{(H_n + \hat{\partial}_t)^{-1}}(0, t) = (4\pi t)^{-\frac{n}{2}} \det^{\frac{2}{n}} \left( \frac{tR/2}{\sinh(tR/2)} \right) \int_{\mathbb{R}^{n-1}} \exp \left( -\frac{1}{4t} \Theta(v, t) \right) dv,
\]

where \( \Theta(v, t) := \Theta_t(v, \phi^N, v). \) Set \( S = \frac{1}{2} tR'' \). As \([\phi^N, S] = 0\), we see that

\[
\Theta(v, t) = \left( \frac{\partial v}{\tanh S} \right) + \left( \frac{\partial v}{\sinh S} \phi^N v, \phi^N v \right) - 2 \left( \frac{\partial v}{\sinh S} e^{Sv} v, \phi^N v \right).
\]

Note that

\[
\left( \cosh S - (\phi^N)^{-1} e^{Sv} \right) + \left( \cosh S - (\phi^N)^{-1} e^{Sv} \right)^T = e^{Sv} + e^{-Sv} - (\phi^N)^{-1} e^{Sv} - \phi^N e^{-Sv} = e^{Sv} \left( 1 - (\phi^N)^{-1} \right) (1 - \phi^N e^{-2Sv}).
\]

Therefore, using the formula for the integral of a Gaussian function and its extension to Gaussian functions associated to form-valued symmetric matrices, we get

\[
\int_{\mathbb{R}^{n-1}} \exp \left( -\frac{1}{4t} \Theta(v, t) \right) dv = \int_{\mathbb{R}^{n-1}} \exp \left( -\frac{1}{4t} \left( \frac{\partial v}{\sinh S} \right) e^{Sv} \left( 1 - (\phi^N)^{-1} \right) (1 - \phi^N e^{-2Sv}) v, v \right) dv
\]

\[
= (4\pi)^{-\frac{n}{2}} \det^{\frac{1}{2}} \left( \frac{\partial v}{\sinh S} \right) \det^{-\frac{1}{2}} \left[ e^{Sv} \left( 1 - (\phi^N)^{-1} \right) \right] \det^{-\frac{1}{2}} \left( 1 - \phi^N e^{-2Sv} \right).
\]

We observe that \( \det^{-\frac{1}{2}} \left[ e^{Sv} \left( 1 - (\phi^N)^{-1} \right) \right] = \det^{-\frac{1}{2}} (1 - \phi^N) \), and so using (4.39) we get

\[
I_{(H_n + \hat{\partial}_t)^{-1}}(0, t) = (4\pi)^{-\frac{n}{2}} \det^{-\frac{1}{2}} \left[ 1 - \phi^N \right] \det^{\frac{1}{2}} \left( \frac{tR/2}{\sinh(tR/2)} \right) \det^{-\frac{1}{2}} \left( 1 - \phi^N e^{-tR''} \right).
\]

This proves (4.36) and completes the proof.

Let us go back to the proof of Theorem 4.5. Let \( Q \in \Psi^{-2}(\mathbb{R}^n \times \mathbb{R}, \mathcal{S}_n \otimes \mathbb{C}^p) \) be a parametrix for \( \mathcal{D}^2 + \hat{\partial}_t \). Thanks to Lemma 4.17 we know that \( Q \) has Getzler order \(-2\). Therefore, using (4.10) and (4.32) we see that, as \( t \to 0^+ \),

\[
\text{str}_{\mathbb{R}^n E} \left[ (\phi^S \otimes \phi^F)_{I_{(H_n + \hat{\partial}_t)^{-1}}(x_0, t)} \right] = \left( \text{str}_{\mathbb{R}^n \otimes \mathbb{C}^p} \right) \left[ (\phi^S \otimes \phi^F)_{I_{Q}(0, t)} \right] + O(t^\infty)
\]

\[
= - (i)^{\frac{p}{2} + 2} \det^{\frac{1}{2}} (1 - \phi^N) \left[ \text{tr}_{\mathbb{C}^p} (\phi^F_{I_{Q(0,1)}} (0, 1)) \right] (a_0) + O(t).
\]

As noted above, the components in \( \Lambda^{*,0}(n) \) of the curvatures \( R' \) and \( R'' \) are \( R^{TM^o}(0) \) and \( R^{N^o}(0) \), respectively. Likewise, the component in \( \Lambda^{*,0}(n) \) of the curvature \( F^E(0) \) is \( F^E_0(0) \). Therefore, using (4.34)–(4.35) we see that the component in \( \Lambda^{*,0}(n) \) of \( I_{Q(-1)}(0, 1) \) is

\[
I_{Q(-1)}(0, 1) = (4\pi)^{-\frac{n}{2}} \det^{\frac{1}{2}} (1 - \phi^N) \hat{A}(R^{TM^o}(0)) \wedge \nu(a_0) \wedge \text{Ch}(F^E(0)).
\]

Combining this with (4.40) we deduce that, as \( t \to 0^+ \), we have

\[
\text{str}_{\mathbb{R}^n E} \left[ (\phi^S(x_0) I_{(H_n + \hat{\partial}_t)^{-1}}(x_0, t)) \right]
\]

\[
= - (i)^{\frac{p}{2} + 2} \left( (4\pi)^{-\frac{n}{2}} \det^{\frac{1}{2}} (1 - \phi^N) \hat{A}(R^{TM^o}(0)) \wedge \nu(a_0) \wedge \text{Ch}(F^E(0)) \right) + O(t).
\]

This gives the asymptotic (4.5) at the point \( x_0 \). This completes the proof of Theorem 4.5 and the local equivariant index theorem.
Remark 4.18. As Theorem 4.5 is a purely local statement, it also allows us to obtain the local equivariant index theorem for Dirac operators acting on sections of any $G$-equivariant Clifford module $E$, where $G$ is any compact group of orientation-preserving isometries. This only amounts to replace $Ch_\phi(F^E)$ by $Ch_\phi(F^E/\mathcal{S})$, where $F^E/\mathcal{S}$ is the twisted curvature in the sense of HGV. In particular, this enables us to recover the local equivariant index theorem for the de Rham and signature complexes with coefficients in any $G$-equivariant Hermitian vector bundle.

It should be stressed out that the above proof of the local equivariant index theorem actually gives a more general result. The key lemma in the proof is Lemma 4.10 which was specialized to $Q = (\mathcal{D}^2 + \partial_0)^{-1}$. This lemma actually holds for general Volterra $\Psi$DOs. As a result, this allows us to obtain a version of Theorem 4.22 for Volterra $\Psi$DOs as follows.

Following [DG], we shall call synchronous normal coordinates centered at a point $x_0 \in M$ the data of normal coordinates centered at $x_0$ and trivialization of the spinor bundle via synchronous tangent frame associated to an oriented orthonormal basis $e_1, \ldots, e_n$ of $T_{x_0}M$. If $x_0 \in M^\phi_a$ for a given $\phi \in G$, we shall call such a basis admissible if $e_1, \ldots, e_n$ is an oriented orthonormal basis of $T_{x_0}M^\phi_a$ (which implies that $e_{a+1}, \ldots, e_n$ is an oriented orthonormal basis of $N^\phi_{x_0}$).

Definition 4.19. Let $Q = \Psi^*(M \times \mathbb{R}, \mathcal{S} \otimes E)$.

1. We shall say that $Q$ has Getzler order $m$ at a given point $x_0 \in M$, when, for any synchronous normal coordinates centered at $x_0$ over which $E$ is trivialized, the operator $Q$ agrees up near $x = 0$ with an operator $\tilde{Q} \in \Psi^*_t(\mathbb{R}^n \times \mathbb{R}, \mathcal{S}_a \otimes \mathbb{C}^p)$ that has Getzler order $m$.

2. Given a subset $S \subset M$, we shall say that $Q$ has Getzler order $m$ along $S$ when it has Getzler order $\leq m$ at every point of $S$.

Example 4.20. The operator $(\mathcal{D}^2_{\mathcal{F}R} + \partial_0)^{-1}$ has Getzler order $-2$ at every point of $M$.

Remark 4.21. It can be shown that the condition in (1) holds for some synchronous normal coordinates centered at $x_0$ and some trivialization of $E$ over these coordinates, then it holds for any such data. That is, the notion of Getzler of order at a given point of $M$ is independent of the choice of the synchronous normal coordinates and trivialization of $E$ near that point.

Theorem 4.22. Given $\phi \in G$, let $Q = \Psi^*(M \times \mathbb{R}, \mathcal{S} \otimes E)$ have Getzler order $m$ along the fixed-point set $M^\phi$.

1. If $m$ is odd, then, uniformly on $M^\phi$,

\begin{equation}
\text{str}_{\mathcal{S} \otimes E} \left[ \phi^\mathcal{S}E(x)I_Q(x,t) \right] = O \left( t^{-\frac{m+1}{2}} \right) \quad \text{as } t \to 0^+.
\end{equation}

2. If $m$ is even, then, as $t \to 0^+$ and uniformly on each fixed-point submanifold $M^\phi_a$, we have

\begin{equation}
\text{str}_{\mathcal{S} \otimes E} \left[ \phi^\mathcal{S}E(x)I_Q(x,t) \right] = \gamma_\phi(Q)(x) t^{-\frac{m}{2}} + O \left( t^{-\frac{m}{2}} \right),
\end{equation}

where $\gamma_\phi(Q)(x)$ is a function on $M^\phi_a$ such that, for all $x_0 \in M^\phi_a$, in any synchronous normal coordinates centered at $x_0$ over which $E$ is trivialized, we have

$$
\gamma_\phi(Q)(0) = (-i)^{\frac{n}{2}} 2^\frac{n}{2} \det \frac{1}{2} (1 - \phi^N(0)) \left| \text{tr}_E \left[ \phi^{E'}(0)I_{Q(m)}(0,1) \right] \right|^{(a,0)},
$$

where $Q_{(m)}$ is the model operator of $Q$.

Proof. The proof follows the outline of the proof of Theorem 4.5. More precisely, as in the proof of Theorem 4.5 it is enough to prove the asymptotics (4.42) - (4.43) pointwise, since we are already provided by a uniform short-time asymptotic for $I_Q(x,t)$ thanks to Proposition 4.3. However, these asymptotics are nothing but the contents of Lemma 4.10. The proof is complete. □

Remark 4.23. With some additional work it can be shown that, if $Q \in \Psi^*(M \times \mathbb{R}, \mathcal{S} \otimes E)$ have Getzler order $m$ along $M^\phi$, then, for all $a = 0, 2, \ldots, n$, there is a unique section $\Upsilon_\phi(Q)(x)$ of $(\Lambda^a T^a M^\phi_a) \otimes \text{End}(E)$ over $M^\phi_a$ such that, for every point $x_0 \in M^\phi_a$, in any admissible synchronous normal coordinates centered at $x_0$ over which $E$ is trivialized, we have

$$
\Upsilon_\phi(Q)(0) = (-i)^{\frac{n}{2}} 2^\frac{n}{2} \det \frac{1}{2} (1 - \phi^N(0)) I_{Q(m)}(0,1)^{\bullet,0}.
$$
Lemma 4.15 holds verbatim observed in [BF], the trace on $C$ when $m/S$ the infinitesimal equivariant index theorem (a.k.a. Kirillov Formula) of Berline-Vergne [BV] (see Theorem 4.22 and some of their applications. As the operator $(4.44) Str \left[ Pe^{-t \Phi^E} U_\phi \right] = O \left( t^{-\frac{m}{2} + 1} \right)$ as $t \to 0^+$. 

(2) If $m$ is even, then, as $t \to 0^+$, we have $(4.45) Str \left[ Pe^{-t \Phi^E} U_\phi \right] = t^{-\frac{n}{2}} \int_{M^\phi} \gamma_\phi(P; \Phi^E)(x) dx + O \left( t^{-\frac{m}{2} + 1} \right),$ where $\gamma_\phi(P; \Phi^E)(x)$ is a function on $M^\phi$ such that, for all $x_0 \in M^\phi_a$, $a = 0, 2, \ldots, n$, in any synchronous normal coordinates centered at $x_0$ over which $E$ is trivialized, we have

\[
\gamma_\phi(P; \Phi^E)(x)(0) = (-i)^{\frac{n}{2}} 2^{\frac{n}{2}} \det \partial \left( 1 - \phi^N(0) \right) \left| tr_E \left[ \phi^E(0) I_{P_{(m)}}(H_R + \partial) \right] \right|^{(a, 0)},
\]

where $P_{(m)}$ is the model operator of $P$.

Proof. By Proposition (4.24) as $t \to 0^+$, we have

\[
Str \left[ Pe^{-t \Phi^E} U_\phi \right] = \int_{M^\phi} \nabla_{\phi^E} \left[ \phi^E(x) I_{P_{(m)}}(H_R + \partial) \right] dx + O(t^{\infty}).
\]

As the operator $Q = P(\Phi^E + \partial)^{-1}$ has Getzler order $m - 2$, using Theorem (4.22) we see that, when $m$ is odd the asymptotics (4.44) holds. Moreover, when $m$ is even, as $t \to 0^+$, we obtain

\[
Str \left[ Pe^{-t \Phi^E} U_\phi \right] = t^{-\frac{n}{2}} \int_{M^\phi} \gamma_\phi(P; \Phi^E)(x) dx + O \left( t^{-\frac{m}{2} + 1} \right),
\]

where we have set $\gamma_\phi(P; \Phi^E)(x) = \gamma(Q)(x)$.

Let $x_0 \in M^\phi_a$, $a = 0, 2, \ldots, n$, and let us consider admissible normal coordinates centered at $x_0$. Then the operator $Q = P(\Phi^E + \partial)^{-1}$ has model operator $Q_{(m-2)} = P_{(m)}(H_R + \partial)^{-1} \exp(-tF^E(0))$.

Therefore, setting $\mu = (-i)^{\frac{n}{2}} 2^{\frac{n}{2}} \det \partial \left( 1 - \phi^N(0) \right)$, we get

\[
\gamma_\phi(Q)(0) = \mu \left| tr_E \left[ \phi^E(0) I_{Q_{(m-2)}}(0, 1) \right] \right|^{(a, 0)} = \mu \left| tr_E \left[ \phi^E(0) I_{P_{(m)}}(H_R + \partial) \right] \right|^{(a, 0)}.
\]

This proves (4.45) and completes the proof. □

Remark 4.25. The considerations of this section on Getzler order and model operators of operators in $\Psi^*_c(\mathbb{R}^n \times \mathbb{R}, \mathcal{S}_n \otimes \mathbb{C}^p)$ are restricted to even dimension $n$ in order to use the isomorphism $\text{End} \mathcal{S}_n \simeq \text{Cl}(n)$. As pointed out in [Po1] the notions of Getzler order and model operators make sense for elements of $\Psi^*_c(\mathbb{R}^n \times \mathbb{R}, \mathbb{C}^p) \otimes H_S(n)$ independently of the parity of $n$. In particular, Lemma (4.15) holds verbatim in this context. When $n$ is odd, $\text{Cl}(n)$ is a 2-cover to $\text{End} \mathcal{S}_n$, but, as observed in [BF], the trace on $\text{Cl}(n)$ behaves essentially like the supertrace on $\text{End} \mathcal{S}_n$, in even dimension. We refer to [Po1] for odd-dimensional analogues of Lemma (4.16) and Theorem (4.22) and some of their applications.

Remark 4.26. It is not difficult to various family settings the considerations of this sections on Getzler orders and model operators on Volterra $\Psi$DOs (see, e.g., [Wa2, Wa3]). In particular, this provides us with proofs of the local equivariant family index theorem of Liu-Ma [LM] and the infinitesimal equivariant index theorem (a.k.a. Kirillov Formula) of Berline-Vergne [BV] (see also [B2]). We refer to [Wa2] for a proof of the former result and to [Wa3] of the latter. We
note that the proofs of [Wa2, Wa3] rely on Lemma 9.5 of [PW0] which are not correct. This can fixed by using the appropriate extensions of Lemma 4.16 and Theorem 4.22 to the respective family settings at stake in [Wa2] and [Wa3].

5. Connes-Chern Character and CM Cocycle

In this section, we briefly recall the framework for the local index formula and explain how to extend it to the setup of spectral triples over locally convex algebras. In what follows we assume the notation, definitions and results of the prequel [PW1] (which we shall refer as Part I).

5.1. The Connes-Chern character. The role of manifolds in noncommutative geometry is played by spectral triples. More specifically, a spectral triple is a triple \((A, \mathcal{H}, D)\), where

- \(\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-\) is a \(\mathbb{Z}_2\)-graded Hilbert space.
- \(A\) is an \(\ast\)-algebra represented by bounded operators on \(\mathcal{H}\) preserving its \(\mathbb{Z}_2\)-grading.
- \(D\) is a selfadjoint unbounded operator on \(\mathcal{H}\) such that
  - \(D\) maps \(\text{dom}(D)\) to \(\mathcal{H}^\pm\).
  - The resolvent \((D + i)^{-1}\) is a compact operator.
  - \(a\text{dom}(D) \subset \text{dom}(D)\) is bounded for all \(a \in A\).

In particular, with respect to the splitting \(\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-\) the operator \(D\) takes the form,

\[
D = \begin{pmatrix}
0 & D^\perp \\
D^+ & 0
\end{pmatrix}, \quad D^\pm : \text{dom}(D) \cap \mathcal{H}^\pm \to \mathcal{H}^\mp.
\]

The paradigm of a spectral triple is given by a Dirac spectral triple,

\[
(C^\infty(M), L^2_g(M, \mathcal{S}), \mathcal{D}_g),
\]

where \((M^n, g)\) is a compact spin Riemannian manifold \((n\text{ even})\) and \(\mathcal{D}_g\) is its Dirac operator acting on the spinor bundle \(\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-\). Given any Hermitian vector bundle \(E\) over \(M\), the datum of any Hermitian connection on \(E\) enables us to form the coupled operator \(\mathcal{D}_{\nabla_E}\) acting on the sections of \(\mathcal{S} \otimes E\). Its Fredholm index depends only on the \(K\)-theory class of \(E\) and is given by the local index formula for Atiyah-Singer [AS1, AS2], i.e., the equivariant index formula \((4.1)\) in the case \(\phi = \text{id}_M\).

Likewise, given any spectral triple \((A, \mathcal{H}, D)\) and a Hermitian finitely generated projective bundle \(\mathcal{E}\) over \(A\), the datum of any Hermitian connection \(\nabla^\mathcal{E}\) on \(\mathcal{E}\) gives rise to an unbounded operator \(D_{\nabla^\mathcal{E}}\) on the Hilbert space \(\mathcal{H}(\mathcal{E}) = \mathcal{H} \otimes_A \mathcal{E}\). Furthermore, with respect to the splitting \(\mathcal{H}(\mathcal{E}) = \mathcal{H}(\mathcal{E}^+) \oplus \mathcal{H}(\mathcal{E}^-)\), where \(\mathcal{H}(\mathcal{E}^\pm) := \mathcal{H}^\mp \otimes_A \mathcal{E}\), the coupled operator takes the form,

\[
D_{\nabla^\mathcal{E}} = \begin{pmatrix}
0 & D_{\nabla^\mathcal{E}^+} \\
D_{\nabla^\mathcal{E}^-} & 0
\end{pmatrix}, \quad D_{\nabla^\mathcal{E}}^\pm : \text{dom}(D^\pm) \otimes_A \mathcal{E} \to \mathcal{H}^\mp(\mathcal{E}).
\]

The operator \(D_{\nabla^\mathcal{E}}\) is selfadjoint and Fredholm, and its Fredholm index is then defined by

\[
\text{ind } D_{\nabla^\mathcal{E}} = \text{ind } D_{\nabla^\mathcal{E}^+} = \dim \ker D_{\nabla^\mathcal{E}^+} - \dim \ker D_{\nabla^\mathcal{E}^-}.
\]

This index depends only on the class of \(\mathcal{E}\) in the \(K\)-theory \(K_0(A)\), and hence gives rise to an additive index map \(\text{ind}_D : K_0(A) \to \mathbb{Z}\).

Let us further assume that the spectral triple is \(p\)-summable for some \(p \geq 1\), i.e, the operator \(|D|^{-p}\) is trace-class. In this case, Connes [C02] constructed an even periodic cyclic cohomology class \(\text{Ch}(D) \in \text{HP}(A)\), called Connes-Chern character, whose pairing with \(K\)-theory computes the index map. This construction is described in Part I in the more general setting of twisted spectral triples. For reader’s convenience, we recall this construction in the special case of ordinary spectral triples.

**Proposition 5.1** (Connes [C02]). Assume that \((A, \mathcal{H}, D)\) is \(p\)-summable and \(D\) is invertible.

1. For any \(q \geq \frac{1}{2}(p - 1)\) the following formula defines a \(2q\)-cyclic cocycle on the algebra \(A\):

\[
\tau D_q(a^0, \ldots, a^{2q}) := \frac{1}{2(2q)!} \text{Str} \left\{ D^{-1}[D, a^0] \cdots D^{-1}[D, a^{2q}] \right\}, \quad a^j \in A,
\]

where \(\text{Str} = \text{Tr}_{\mathcal{H}^+} - \text{Tr}_{\mathcal{H}^-}\) is the supertrace on \(\mathcal{L}^1(\mathcal{H})\).
(2) The class of the cocycle $\tau_{2q}^D$ in the periodic cyclic cohomology $\text{HP}^0(\mathcal{A})$ is independent of the value of $q$.

The case where $D$ is non-invertible is dealt with by passing to the invertible double $(\tilde{\mathcal{A}}, \tilde{\mathcal{H}}, \tilde{D})$. Here $\tilde{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}$ is equipped with the $\mathbb{Z}_2$-grading $\tilde{\gamma} = \gamma \oplus \gamma$ and $\tilde{\mathcal{A}} = \mathcal{A} \oplus \mathbb{C}$ is the unitalization of $\mathcal{A}$ represented in $\tilde{\mathcal{H}}$ by $(a, \lambda) \rightarrow \begin{pmatrix} a + \lambda & 0 \\ 0 & \lambda \end{pmatrix}$. The operator $\tilde{D}$ is given by

\begin{equation}
\tilde{D} = \begin{pmatrix} D & \Pi_0 \\ \Pi_0 & -D \end{pmatrix},
\end{equation}

where $\Pi_0$ is the orthogonal projection onto $\ker D$. This provides us with a $p$-summable spectral triple $(\tilde{\mathcal{A}}, \tilde{\mathcal{H}}, \tilde{D})$ and it can be checked that the operator $\tilde{D}$ is invertible. We thus have well defined cyclic cocycles $\tau_{2q}^D$, $q \geq \frac{1}{2}(p-1)$, on $\tilde{\mathcal{A}}$.

**Proposition 5.2** (Connes [Co2]). Assume that $(\mathcal{A}, \mathcal{H}, D)$ is $p$-summable. For $q \geq \frac{1}{2}(p-1)$, denote by $\tau_{2q}^D$ the restriction to $\mathcal{A}$ of the cyclic cocycle $\tau_{2q}^D$.

1. The cochain $\tau_{2q}^D$ is a cyclic cocycle on $\mathcal{A}$ whose class in $\text{HP}^0(\mathcal{A})$ is independent of the value of $q$.

2. When $D$ is invertible, the cyclic cocycles $\tau_{2q}^D$ and $\tau_{2q}^D$ are cohomologous in $\text{HC}^{2q}(\mathcal{A})$, and hence define the same class in $\text{HP}^0(\mathcal{A})$.

**Remark 5.3.** The homotopy invariance of the cohomology class of $\tau_{2q}^D$ shows that, for $q \geq \frac{1}{2}(p+1)$, the class of $\tau_{2q}^D$ in $\text{HC}^{2k}(\mathcal{A})$ remains unchanged if in the formula for $\tilde{D}$ we replace the projection $\Pi_0$ by any operators of the form $t_1 + t_2 \Pi_0$, with $t_j \geq 0$, $t_1 + t_2 > 0$ (cf. [Co2]).

**Definition 5.4** (Connes [Co2]). Assume that $(\mathcal{A}, \mathcal{H}, D)$ is $p$-summable. The Connes-Chern character of $(\mathcal{A}, \mathcal{H}, D)$, denoted by $\text{Ch}(D)$, is defined as follows:

- If $D$ is invertible, then $\text{Ch}(D)$ is the common class in $\text{HP}^0(\mathcal{A})$ of the cyclic cocycles $\tau_{2q}^D$ and $\tau_{2q}^D$, with $q \geq \frac{1}{2}(p-1)$.
- If $D$ is not invertible, then $\text{Ch}(D)$ is the common class in $\text{HP}^0(\mathcal{A})$ of the cyclic cocycles $\tau_{2q}^D$, $q \geq \frac{1}{2}(p-1)$.

The above definition is motivated by the following index formula.

**Proposition 5.5** (Connes [Co2]). For any Hermitian finitely generated projective module $\mathcal{E}$ over $\mathcal{A}$ and any Hermitian connection $\nabla^\mathcal{E}$ on $\mathcal{E}$, we have

\begin{equation}
\text{ind } D_{\nabla^\mathcal{E}} = \langle \text{Ch}(D), \text{Ch}(\mathcal{E}) \rangle,
\end{equation}

where $\langle \cdot, \cdot \rangle$ is the duality pairing between cyclic cohomology and cyclic homology and $\text{Ch}(\mathcal{E})$ is the Connes-Chern character of $\mathcal{E}$ in the periodic cyclic homology $\text{HP}^0(\mathcal{A})$ (cf. Part I).

**Remark 5.6.** All the above results continue to hold if in the definition of the cocycles $\tau_{2q}^D$ and $\tau_{2q}^D$ we replace the operator $D$ by any operator $D[|D|^{-1}$ with $t \in [0,1]$. In particular, for $t = 1$ we obtain the sign operator $F = |D|^{-1}$, which is often convenient to use for defining the Connes-Chern character.

5.2. The CM cocycle. The cocycles $\tau_{2q}^D$ and $\tau_{2q}^D$ used in the definition of the Connes-Chern character are difficult to compute in practice, even in the case of a Dirac spectral triple (see [Co3, BM3]). Therefore, it was sought for an alternative representative of the Connes-Chern character which would be easier to compute. Such a representative is provided by the CM cocycle $\tau_{2q}^D$, which is constructed as follows.

In what follows we shall assume that $(\mathcal{A}, \mathcal{H}, D)$ is $p$-summable for some $p \geq 1$, that is,

$$
\mu_j(D^{-1}) = O(j^{-\frac{1}{2}}) \quad \text{as } j \to \infty,
$$

where $\mu_j(D^{-1})$ is the $(j + 1)$-th eigenvalue of $|D^{-1}| = |D|^{-1}$. This implies that $(\mathcal{A}, \mathcal{H}, D)$ is $p$-summable for all $q > p$. In addition, we set

\begin{equation}
\mathcal{D}_{2q}(\mathcal{A}) = \mathcal{A} + [D, \mathcal{A}],
\end{equation}

where $\mathcal{D}_{2q}(\mathcal{A})$ is the $p$-summable ideal of $\mathcal{A}$.
where $\gamma = 1_{H^+} - 1_{H^-}$ is the grading operator. Note that the very definition of a spectral triple implies that the operators in $\mathcal{D}^D_0(A)$ are bounded.

**Definition 5.7.** We shall say that $(A, \mathcal{H}, D)$ is hypo-regular when, for any $X \in \mathcal{D}^D_0(A)$, all the operators $D^mXD^{-m}$, $m \in \mathbb{N}_0$, are bounded.

From now on we further assume that $(A, \mathcal{H}, D)$ is hypo-regular. Set $\mathcal{H}^\infty = \cap_{m \geq 0} \text{dom}(D^{-m})$ and equip $\mathcal{H}^\infty$ with the Fréchet space topology defined by the seminorms $\xi \rightarrow \langle (1+D^{2m})\xi, \xi \rangle$, $m \geq 0$. Then the hyporegularity condition implies that any $X \in \mathcal{D}^D_0(A)$ induces a continuous linear operator from $\mathcal{H}^\infty$ to itself. Note also that $\mathcal{H}^\infty$ is a dense subspace of $\mathcal{H}$. In addition, for $m \geq 0$, we denote by $\mathcal{D}^D_0(A)$ the class of unbounded operators on $\mathcal{H}$, where $\mathcal{D}^D_0(A)$ is defined as in Definition 5.8.

and

\[
\mathcal{D}^1(A) = D \mathcal{D}^D_0(A) + \mathcal{D}^D_0(A)D,
\]

\[
\mathcal{D}^D_0(A) = \mathcal{D}^1(A)\mathcal{D}^D_0(A) + \mathcal{D}^2(A)\mathcal{D}^D_0(A) + \cdots + \mathcal{D}^m\mathcal{D}^D_0(A)\mathcal{D}^1(A), \quad m \geq 2.
\]

Alternatively, we may regard each class $\mathcal{D}^D_0(A)$ as a subspace of $\mathcal{L}(\mathcal{H}^\infty)$. Aspanning set of $\mathcal{D}^D_0(A)$ is then obtained as follows. For $m \in \mathbb{N}_0$, define

\[
\mathcal{P}(m) = \{(\alpha, \beta) \in \mathbb{N}_0^{m+2} \times \{0, 1\}^{m+1} : |\alpha| = \alpha_0 + \cdots + \alpha_{m+1} = m\}.
\]

Then $\mathcal{D}^D_0(A)$ is spanned by operators of the form

\[
P_{\alpha, \beta}(0^\alpha, \ldots, a^\ell) = D^{\alpha_0}(a^\ell)\beta_0[D, a_0^{-1}\beta_0] \cdots D^{\alpha_j}(a^\ell)\beta_j[D, a_0^{-1}\beta_j]D^{\alpha_{j+1}}
\]

where the pair $(\alpha, \beta)$ ranges over $\mathcal{P}(m)$ and $a_0^\alpha, \ldots, a_\ell^\alpha$ range over $A$. Setting $X^\alpha = (a^\ell)^\beta[D, a_0^{-1}\beta_j]d^\alpha$ such an operator can be rewritten as

\[
(D^{\alpha_0}X^\alpha D^{-\alpha_0})(D^{a_0 + \cdots + a_m}X^D D^{-\alpha_0 - \cdots - a_m})D^m.
\]

This and the hypo-regularity condition imply that the operator $P_{\alpha, \beta}(0^\alpha, \ldots, a^m)$ is bounded. Combining this with the $p^\gamma$-summability of $D$ we then deduce that, for all $X \in \mathcal{D}^D_0(A)$, the operator $X[D]^{-\gamma}$ is trace-class for $\mathbb{R} > p$, and so by taking its super-trace we then obtain a function $z \rightarrow \text{Str} [X[D]^{-\gamma}]$ which is holomorphic on the half-plane $\mathbb{R} > p$.

**Definition 5.8.** We say that $(A, \mathcal{H}, D)$ has discrete and simple dimension spectrum when $(A, \mathcal{H}, D)$ is hypo-regular and there is a discrete subset $\Sigma \subset \mathbb{C}$ such that, for all $m \in \mathbb{N}_0$ and $X \in \mathcal{D}^D_0(A)$, the function $\text{Str} [X[D]^{-\gamma}]$ has an analytic continuation to $\mathbb{C} \setminus \Sigma$ with at worst simple pole singularities on $\Sigma$. The dimension spectrum of $(A, \mathcal{H}, D)$ is then defined as the smallest such set.

**Remark 5.9.** The $p^\gamma$-summability of $D$ implies that the dimension spectrum of $(A, \mathcal{H}, D)$ is contained in the half-plane $\mathbb{R} \leq p$.

In what follows, given $X \in \mathcal{D}^D_0(A)$, for $j = 0, 1, 2, \ldots$ we denote by $X^{[j]}$ the $j$-th iterated commutator of $D^2$ with $X$, i.e.,

\[
X^{[j]} = X, \quad X^{[j]} = [D^2, [D^2, \ldots, [D^2, X], \ldots]], \quad j \geq 1.
\]

Note that $X^{[j]}$ is an element of $\mathcal{D}^D_0(A)$.

**Definition 5.10.** We say that $(A, \mathcal{H}, D)$ is regular when, for any $X \in \mathcal{D}^D_0(A)$, all the operators $X^{[m]}D^{-m}$, $m \in \mathbb{N}_0$, are bounded.

As shown by Connes-Moscovici [CMR], assuming that $(A, \mathcal{H}, D)$ is regular and has discrete and simple dimension spectrum enables us to construct a class of $\Psi$DOs and an analogue of Guillemin-Wodzicki’s noncommutative residue trace as follows.

In the following we denote by $\mathcal{B}$ the class of unbounded operators on $\mathcal{H}$ that are linearly combination of operators of the form $X^{[m]}D^{-m}$ with $X \in \mathcal{D}^D_0(A)$ and $m \in \mathbb{N}_0$. The regularity assumption implies that all the operators in $\mathcal{B}$ are bounded. Moreover, for $r \in \mathbb{R}$ we denote by $\text{OP}^r$ the class of unbounded operators $T$ on $\mathcal{H}$ such that $\text{dom}(T) \supset \mathcal{H}(\infty)$ and $|T|^r$ is bounded.
**Definition 5.11.** \( \Psi^j_D(A) \), \( q \in \mathbb{C} \), consists of unbounded operators \( P \) on \( \mathcal{H} \) such that the domain of \( P \) contains \( \mathcal{H}^\infty \) and there is an asymptotic expansion of the form,

\[
P \sim \sum_{j \geq 0} b_j |D|^{q-j} \quad b_j \in \mathcal{B},
\]

in the sense that

\[
\left( P - \sum_{j < N} b_j |D|^{q-j} \right) \in O \mathbb{P}^{-N} \quad \forall N \in \mathbb{N}.
\]

In particular, the above definition implies that any operator in \( \Psi^j_D(A) \) induces a continuous linear operator from \( \mathcal{H}^\infty \) to itself. Moreover, the operators in \( \Psi^j_D(A) \) with \( \Re q \leq 0 \) extend to bounded operators of \( \mathcal{H} \) to itself. Those operators are compact (resp., trace-class) when \( \Re q < 0 \) (resp., \( \Re q < -p \)). Furthermore, it can be shown (see \[CM3\]) that if \( X \in \mathcal{D}^j_D(A) \), then, for all \( j \in \mathbb{N} \) and \( z \in \mathbb{C} \),

\[
|D|^{2z} X[j] \sim \sum_{k \geq 0} \binom{z}{j} X[j+k]|D|^{2z-2k},
\]

and hence \( |D|^{2z} X[j] \) is contained in \( \Psi^{2z+j}_D(A) \). This implies that

\[
\Psi^q_D(A) \Psi^q_D(A) \subset \Psi^{q+q}_D(A) \quad \forall q_j \in \mathbb{C}.
\]

In addition, the existence of a discrete and simple dimension spectrum implies that, for any \( P \in \Psi^j_D(A) \), the function \( z \to \text{Str} \left[ |P| |D|^{-2z} \right] \) has a meromorphic extension to the entire complex plane with at worst simple pole singularities. Note that the poles are contained in the half-plane \( \Re z \leq \frac{1}{2} (p + \Re q) \). We then set

\[
\int P := \text{Res}_{z=0} \text{Str} \left[ |P| |D|^{-2z} \right] \quad \forall P \in \Psi^j_D(A).
\]

As it turns out (cf. \[CM3\]), this defines a trace on \( \Psi^j_D(A) \), i.e.,

\[
\int P_1 P_2 = \int P_2 P_1 \quad \text{for all } P_j \in \Psi^j_D(A).
\]

This residual functional is the analogue of the noncommutative residue trace of Guillemin \[Gu\] and Wodzicki \[Wo1\]. Note also that it vanishes on all operators \( P \in \Psi^j_D(A) \) with \( \Re q < -p \), and so this is a local functional.

In what follows for \( q \geq 1 \) and \( \alpha \in \mathbb{N}_0^{2q} \) we set

\[
c_{q, \alpha} = (-1)^{\alpha_1} (q-1)! (\alpha_1 + \cdots + \alpha_{2q})! \quad \alpha_1 (\alpha_1 + 1) \cdots (\alpha_{2q} + 2q)
\]

**Theorem 5.12** (Connes-Moscovici \[CM3\]). Assume that \( (\mathcal{A}, \mathcal{H}, D) \) is \( p^+ \)-summable, regular and has a discrete and simple dimension spectrum. Then

1. The following formulas define an even periodic cyclic cocycle \( \varphi^{\text{CM}} \) (cf. \[CM3\]) on \( \mathcal{A} \):

\[
\varphi^{\text{CM}}_0(a^0) = \text{Res} \left\{ \Gamma(z) \text{Str} \left[ a^0 |D|^{-2z} \right] \right\} + \text{Str} \left[ a^0 \Pi_0 \right] , \quad a^0 \in \mathcal{A},
\]

\[
\varphi^{\text{CM}}_{2q}(a^0, \ldots, a^{2q}) = \sum_{\alpha \in \mathbb{N}_0^{2q}} c_{q, \alpha} \int \gamma a^{0}[D, a^{1}]^{[\alpha_1]} \cdots [D, a^{2q}]^{[\alpha_{2q}]} |D|^{-2[(\alpha_1 + \cdots + \alpha_{2q})+q]}, \quad a^j \in \mathcal{A}, \quad q \geq 1.
\]

2. The cocycle \( \varphi^{\text{CM}} \) represents the Connes-Chern character \( \text{Ch}(D) \) in \( \mathcal{H}^0(D) \).

3. For any Hermitian finitely projective module \( \mathcal{E} \) over \( \mathcal{A} \) and any Hermitian connection \( \nabla^\mathcal{E} \) over \( \mathcal{E} \), we have

\[
\text{ind} D_{\nabla^\mathcal{E}} = \langle \varphi^{\text{CM}}, \text{Ch}(\mathcal{E}) \rangle.
\]

**Remark 5.13.** The formulas \[5.7 \text{-- } 5.8\] provide us with the local index formula in noncommutative geometry \[CM3\]. The cocycle \( \varphi^{\text{CM}} \) is called the CM cocycle of the spectral triple \( (\mathcal{A}, \mathcal{H}, D) \).

**Remark 5.14.** Connes-Moscovici \[CM3\] proved Theorem 5.12 under the additional assumption that the functions \( \Gamma(z) \text{Str}[X |D|^{-z}] \), \( X \in \mathcal{D}^0_D(A) \), have rapid decay along vertical lines in the complex plane. This technical assumption is removed in \[Hi\].
Example 5.15. A Dirac spectral triple \((C^\infty(M), L^2(M, \mathcal{S}), \mathcal{D})\) is \(n^+\)-summable with \(n = \dim M\). It is also regular and has a discrete dimension spectrum contained in \(\{k \in \mathbb{N} : k \leq n\}\). Each-space \(\mathcal{D}q\), \(q \in \mathbb{C}\), is contained in the space of classical \(\Psi DOs\) of order \(q\). In addition, the residual trace \(\hat{\tau}\) agrees with the noncommutative residue trace of Guillemin and Wodzicki. Finally (see [CM3, Remark II.1], [Po1]), the CM cocycle of a smooth spectral triple.

We observe that the hypo-regularity assumption enables us to endowed each space \(\mathcal{D}\alpha,\beta\) with the norm, \(\|X\|_\alpha = \|X(1 + D^2)^{-m/2}\|, \quad X \in \mathcal{D}^\alpha_m(A), \quad m \geq 0\).

This gives rise to a natural normed topology on each space \(\mathcal{D}^\alpha_m(A), \quad m \geq 0\).

In what follows, given any open \(\Omega \subset \mathbb{C}\), we shall denote by \(\text{Hol}(\Omega)\) the Fréchet space of holomorphic functions on \(\Omega\).

Remark 5.17. If \((A, \mathcal{H}, D)\) is uniformly hypo-regular, then, for any pair \((\alpha, \beta) \in \mathcal{P}(m), \ m \in \mathbb{N}_0\), the map \((a^0, \ldots, a^m) \mapsto P_{\alpha,\beta}(a^0, \ldots, a^m)\) is a continuous \((m+1)\)-linear map from \(A^{m+1}\) to \(\mathcal{D}^\alpha_m(A)\), where the operator \(P_{\alpha,\beta}(a^0, \ldots, a^m)\) is given by \([5.4]\). It then follows that we obtain a continuous \((m+1)\)-linear map,

\[A^{m+1} \ni (a^0, \ldots, a^m) \mapsto \text{Str}[P_{\alpha,\beta}(a^0, \ldots, a^m) D^{-z-m}] \in \text{Hol}(\mathbb{R}z > p).\]

The above remark leads us to the following notion of uniform dimension spectrum.

Definition 5.18. We say that \((A, \mathcal{H}, D)\) has a simple and discrete uniform dimension spectrum when it is uniformly hypo-regular and has a simple and discrete dimension spectrum in such way that, for any \(m \in \mathbb{N}_0\) and pair \((\alpha, \beta) \in \mathcal{P}(m)\), the following conditions are satisfied:

(i) The \((m+1)\)-linear map \((a^0, \ldots, a^m) \mapsto \text{Str}[P_{\alpha,\beta}(a^0, \ldots, a^m) D^{-z-m}]\) is continuous from \(A^{m+1}\) to \(\text{Hol}(\mathbb{C} \setminus \Sigma)\).

(ii) Any \(\sigma \in \Sigma\) has an open neighbourhood \(\Omega \subset \mathbb{C}\) such that \(\Sigma \cap \Omega = \{\sigma\}\) and the \((m+1)\)-linear map \((a^0, \ldots, a^m) \mapsto (z - \sigma) \text{Str}[P_{\alpha,\beta}(a^0, \ldots, a^m) D^{-z-m}]\) is continuous from \(A^{m+1}\) to \(\text{Hol}(\Omega)\).

Definition 5.19. We say that \((A, \mathcal{H}, D)\) is uniformly regular when it is regular and, for all \(m \in \mathbb{N}_0\), the linear maps \(a \mapsto a^{[m]} D^{-m}\) and \(a \mapsto [D, a]^{[m]} D^{-m}\) are continuous from \(A\) to \(\mathcal{L}(\mathcal{H})\).
We are now in a position to answer the question about the representation of the Connes-Chern character \( \text{Ch}(D) \in H^0_p(A) \) by means of the CM cocycle.

**Proposition 5.20** ([P3]). Suppose that \((A, \mathcal{H}, D)\) is smooth, \(p^+\)-summable, uniformly regular and has a discrete and simple uniform dimension spectrum. Then

1. The components \((X_1^\varphi, X_2^\varphi, X_3^\varphi)\) of the CM cocycle \(\varphi^\text{CM}\) are continuous cochains on \(\text{CM}\).
2. The class of \(\varphi^\text{CM}\) in \(H^0_p(A)\) agrees with the Connes-Chern character \(\text{Ch}(D)\).

6. **CM Cocycle and Heat-Trace Asymptotics**

In this section, we shall now re-interpret the CM cocycle and its representation of the Connes-Chern character in terms of heat-trace asymptotics. As we shall see, this is especially convenient for smooth spectral triples over barrelled locally convex algebras.

In what follows we let \((A, \mathcal{H}, D)\) be a hypo-regular \(p^+\)-summable spectral triple. We denote by \(L^1(\mathcal{H})\) the Banach ideal of trace-class operators on \(\mathcal{H}\) equipped with the norm \(\|T\| := \text{Tr}|T|\), \(T \in L^1(\mathcal{H})\). In addition, we denote by \(\Pi_0\) the orthogonal projection onto \(\text{ker} \, D\). Note that \(\Pi_0\) is a finite-rank operator whose range is contained in \(\mathcal{H}^\infty\).

**Lemma 6.1.** Assume that \((A, \mathcal{H}, D)\) is hypo-regular, and let \(m \in \mathbb{N}_0\).

1. For all \(X \in D^n_D(A)\), the operators \(X e^{-tD^2}\), \(t > 0\), form a continuous family of trace-class operators.
2. For all \(q > p\), there is a constant \(C_{mq} > 0\) such that
   \[
   \|X e^{-tD^2}\| \leq C_{mq} \|X\|_{(m)} t^{-\frac{mq}{2}} \quad \text{for all} \, \, t > 0 \, \, \text{and} \, \, X \in D^n_D(A).
   \]
3. Let \(\lambda_0\) be the smallest eigenvalue of \(D^2\). For all \(q > p\), there is a time \(t_q > 0\) and a constant \(C_{mq} > 0\), such that
   \[
   \|X(1 - \Pi_0) e^{-tD^2}\| \leq C_{mq} \|X\|_{(m)} e^{-t\lambda_0} \quad \text{for all} \, \, t \geq t_q \, \, \text{and} \, \, X \in D^n_D(A).
   \]

**Proof.** The proof relies on the fact that \(L^1(\mathcal{H})\) is a two-sided ideal and its norm is symmetric, i.e.,
\[
\|A_1 TA_2\| \leq \|A_1\|\|T\|\|A_2\| \quad \forall T \in L^1(\mathcal{H}) \forall A_j \in \mathcal{L}(\mathcal{H}).
\]

Let \(q > p\) and \(X \in D^n_D(A)\). For all \(t > 0\), we have
\[
X e^{-tD^2} = X(1 - \Pi_0) e^{-tD^2} + X\Pi_0.
\]
\[
= t^{-\frac{mq}{2}} (XD^{-m}) D^{-q} (tD^2)^{\frac{mq}{2}} (1 - \Pi_0) e^{-tD^2} + X\Pi_0.
\]

We note that \(X\Pi_0\) has finite-rank, and hence is trace-class. In addition, the operators \(XD^{-m}\), \((tD^2)^{\frac{mq}{2}} e^{-tD^2}\) and \(D^{-q}\) are trace-class. Therefore, we see that \(X e^{-tD^2}\) is trace-class for all \(t > 0\).

Using (5.3) and the fact that \(\Pi_0 = (1 + D^2)^{-\frac{m}{2}} \Pi_0\) we get
\[
\|X\Pi_0\| = \|X(1 + D^2)^{-\frac{m}{2}} \Pi_0\| \leq \|X\|_{(m)} \|\Pi_0\|_1.
\]

Using (5.3) and (5.4) we also obtain
\[
\|X(1 - \Pi_0) e^{-tD^2}\| \leq t^{-\frac{mq}{2}} \|XD^{-m}\| \|D^{-q}\| \|tD^2\|^{\frac{mq}{2}} (1 - \Pi_0) e^{-tD^2}.
\]

We observe that
\[
\|XD^{-m}\| = \|X(1 + D^2)^{-\frac{m}{2}} (1 + D^2)^{\frac{m}{2}} D^{-m}\| \leq \|X\|_{(m)} \|1 + D^2\| \|D^{-m}\|,
\]
\[
\|tD^2\|^{\frac{mq}{2}} (1 - \Pi_0) e^{-tD^2} \leq \sup\{\mu^{\frac{mq}{2}} e^{-\mu}; \mu \geq t\lambda_0\} \leq \sup\{\mu^{\frac{mq}{2}} e^{-\mu}; \mu \geq 0\}.
\]

Combining this with (5.3) and (5.4) and the fact that \(X e^{-tD^2} = X(1 - \Pi_0) e^{-tD^2} + X\Pi_0\) we deduce there is a constant \(C_{mq} > 0\) such that, for all \(t > 0\) and \(X \in D^n_D(A)\), we have
\[
\|X e^{-tD^2}\| \leq \|X(1 - \Pi_0) e^{-tD^2}\| + \|X\Pi_0\| \leq C_{mq} (t^{-\frac{mq}{2}} + 1) \|X\|_{(m)}.
\]

We further observe that the function \(\mu \to \mu^{\frac{mq}{2}} e^{-\mu}\) has a single critical point \(\mu_0\) on \((0, \infty)\) and is decreasing on \([\mu_0, \infty)\). Therefore, if we set \(t_q = \mu_0^{\frac{1}{mq}}\), then, for all \(t \geq t_q\), the function
\[ \mu \to \mu_{\geq \alpha} e^{-\mu} \text{ is decreasing on } [\alpha, \infty), \text{ and hence } \mu_{\geq \alpha} e^{-\mu} \leq (\mu_{\geq \alpha}) e^{-\mu_{\geq \alpha}} \text{ for all } \mu \geq \alpha. \]

Combining this with (6.10), we deduce there is a constant \( C'_{\text{mq}} > 0 \) such that
\[ \|X(1 - \Pi_0) e^{-tD^2}\|_1 \leq C'_{\text{mq}} \|X\|_{\infty} e^{-t\lambda_0} \text{ for all } t \geq t_0 \text{ and } X \in D^*_0(A). \]

To complete the proof it remains to show that, for any \( X \in D^*_0(A) \), the family \( \left(X e^{-tD^2}\right)_{t \geq 0} \) is a continuous family in \( L^1(\mathcal{H}) \). To this end let \( c > 0 \) and \( t_j \in (c, \infty), \) \( j = 1, 2 \). Set \( t = \min\{t_1, t_2\} \) and \( h = |t_1 - t_2| \). In addition, let \( f(x) \) be the function on \([0, \infty)\) defined by \( f(x) = e^{-x(1 - e^{-x})} \) for \( x > 0 \) and \( f(0) = 1 \). Note that \( f(x) \) is a continuous function on \([0, \infty)\) with values in \((0, 1] \).

We then have
\[ (6.9) \quad X e^{-t_1D^2} - X e^{-t_2D^2} = \pm X e^{-tD^2} (1 - e^{-hD^2}) = h XD^2 e^{-cD^2} e^{-(t-c)D^2} f(hD^2). \]

As \( XD^2 \) is contained in \( D^{m+2}_0(A) \), the first part of the proof shows that \( XD^2 e^{-cD^2} \) is a trace-class operator.

We also note that \( \|e^{-(t-c)D^2} f(hD^2)\| \leq \|e^{-(t-c)D^2}\| \|f(hD^2)\| \leq \max f \leq 1 \). Therefore, combining (6.3) and (6.9) we obtain
\[ \|X e^{-t_1D^2} - X e^{-t_2D^2}\|_1 \leq h \|XD^2 e^{-cD^2}\|_1 \quad \forall t_j \in (c, \infty). \]

This proves that \( \left(X e^{-tD^2}\right)_{t \geq 0} \) is a continuous family in \( L^1(\mathcal{H}) \). The proof is complete. \( \square \)

**Remark 6.2.** It follows from Lemma 6.1 that, for any \( X \in D^*_0(A) \), the supertrace \( \text{Str} \left[X e^{-tD^2}\right] \) is well defined for all \( t > 0 \). Moreover, the inequalities (6.1)–(6.2) imply that, for all \( m \in \mathbb{N}_0 \), we actually have continuous linear maps,
\[ D^{m}_0(A) \ni X \to e^{-tD^2} \text{Str} \left[X e^{-tD^2}\right] \in C^0_b(0, 1], \quad q > p, \]
\[ D^{m}_0(A) \ni X \to e^{t\lambda} \text{Str} \left[X(1 - \Pi_0) e^{-tD^2}\right] \in C^0_b(1, \infty), \]
where \( C^0_b(0, 1] \) (resp., \( C^0_b[1, \infty) \)) is the Banach space of bounded continuous functions on \((0, 1]\) (resp., \([1, \infty)\)).

**Definition 6.3.** Given a discrete subset \( \Sigma \) of \((-\infty, p] \), we shall say that \((A, \mathcal{H}, D)\) has the asymptotic expansion property relatively to \( \Sigma \) when it is hypo-regular and, for any \( X \in D^*_0(A), \) \( m \geq 0 \), there is an asymptotic expansion of the form,
\[ (6.10) \quad \text{Str} \left[X e^{-tD^2}\right] \sim \sum_{\sigma \in \Sigma} \alpha^{(m)}_\sigma(X) t^{-\frac{\sigma}{2}(\sigma+m)} \quad \text{as } t \to 0^+. \]

**Remark 6.4.** Let \( \theta(t) \) be a function on \((0, \infty)\) which as \( t \to 0^+ \) admits an asymptotic expansion of the form (6.11). We shall call \( \theta^\text{finie} \) (i.e., finite part), and denote by \( \text{Pf} X(t) \), the constant coefficient in this asymptotic expansion. For instance, in the notation of (6.10), for any \( X \in D^*_0(A), \) \( m \geq 0, \) and \( \sigma \in \Sigma \), we have
\[ (6.11) \quad \alpha^{(m)}_\sigma(X) = \text{Pf}_{t \to 0^+} \left\{ \theta^\text{finie} (\sigma+m) \text{Str} \left[X e^{-tD^2}\right] \right\} \]
\[ (6.12) \quad = \lim_{t \to 0^+} \theta^\text{finie} (\sigma+m) \left\{ \text{Str} \left[X e^{-tD^2}\right] - \sum_{\sigma' \in \Sigma, \sigma' > \sigma} \alpha^{(m)}_{\sigma'}(X) t^{-\frac{\sigma'}{2}(\sigma'+m)} \right\}. \]

Note also that both right-hand sides make sense when \( \sigma \notin \Sigma \).

**Remark 6.5.** The asymptotic expansion property implies that, for any \( m \in \mathbb{N}_0 \), we define a linear map \( R^*_N : D^*_0(A, D) \to C^0_b[0, 1] \) by
\[ (6.13) \quad R^*_N(X)(t) = t^{-N} \left\{ \text{Str} \left[X e^{-tD^2}\right] - \sum_{\sigma \in \Sigma_N^{(m)}} \alpha^{(m)}_\sigma(X) t^{-\frac{\sigma}{2}(\sigma+m)} \right\}, \quad X \in D^*_0(A), \quad t \in (0, 1], \]
where we have set \( \Sigma_N^{(m)} := \{ \sigma \in \Sigma; \sigma + m > 2N \} \).
Lemma 6.6. Assume that \((A, H, D)\) has the asymptotic expansion property relatively to \(\Sigma\). Then

1. \((A, H, D)\) has a discrete and simple dimension spectrum contained in \(\Sigma\).
2. For any \(X \in \mathcal{D}_0^q(A)\) and \(q \geq 0\), we have

\[
\text{Res}_{z=0} \left\{ \Gamma(z) \text{ Str} \left[ X|D|^{-2(z+q)} \right] \right\} = \begin{cases} \text{Pf} \ t^q \text{ Str} \left[ X e^{-tD^2} \right] & \text{if } q > 0, \\ \text{Pf} \ t^q \text{ Str} \left[ X e^{-tD^2} \right] - \text{Str} [X\Pi_0] & \text{if } q = 0. \end{cases}
\]

Proof. Let \(X \in \mathcal{D}_0^q(A)\), \(m \geq 0\). In the following, given any \(s \in \mathbb{C}\) we set \(\frac{s}{2} = \frac{1}{2} (s + m)\). Using the Mellin formula \(\Gamma(\hat{z}) |D|^{-(z+m)} = \int_0^\infty t^{z-1} (1 - \Pi_0) e^{-tD^2} dt\), \(\Re \hat{z} > 0\), and the boundedness of \(X|D|^{-m}\), it can be shown that, for \(\Re z > 0\), we have

\[
\Gamma(\hat{z}) X|D|^{-(z+m)} = \int_0^\infty t^{z-1} X (1 - \Pi_0) e^{-tD^2} dt,
\]

where the integral converges in \(L(\mathcal{H})\). In fact, Lemma 6.1 further ensures us that, for \(\Re z > p\) the operators \(t^{\frac{s}{2}} X (1 - \Pi_0) e^{-tD^2}\), \(t > 0\), form a continuous family in \(L^1(\mathcal{H})\) which remains bounded as \(t \to 0^+\) and decay exponentially fast as \(t \to \infty\). Therefore, for \(\Re z > p\), we have

\[
\Gamma(\hat{z}) \text{ Str} \left[ X|D|^{-(z+m)} \right] = \int_0^\infty t^{z-1} \text{Str} \left[ X (1 - \Pi_0) e^{-tD^2} \right] dt.
\]

That is, when regarded as a function of the variable \(\hat{z}\), the function \(\Gamma(\hat{z}) \text{ Str} \left[ X|D|^{-(z+m)} \right]\) is the Mellin transform of the function,

\[
\theta_X(t) = \text{Str} \left[ X (1 - \Pi_0) e^{-tD^2} \right] = \text{Str} \left[ X e^{-tD^2} \right] - \text{Str} [X\Pi_0], \quad t > 0.
\]

It is well known how to relate the short-time behavior of \(\theta_X(t)\) to the meromorphic singularities of its Mellin transform (see, e.g., [BGV, GS]). First, the fact that \(t^{\frac{s}{2}} X (1 - \Pi_0) e^{-tD^2}\), \(t \geq 1\), is a continuous family with rapid decay in \(L^1(\mathcal{H})\) enables us to define an entire function \(F(X)(z) := \int_1^\infty t^{z-1} \theta_X(t) dt\), \(z \in \mathbb{C}\). We then may rewrite (6.15) as

\[
\Gamma(\hat{z}) \text{ Str} \left[ X|D|^{-(z+m)} \right] = \int_1^\infty t^{z-1} \text{Str} \left[ X (1 - \Pi_0) e^{-tD^2} \right] dt,
\]

\(\Re z > p\).

Second, the asymptotic expansion (6.14) implies that, for all \(N \in \mathbb{N}\), we can write

\[
\theta_X(t) = \sum_{\sigma \in \Sigma_N^{(m)}} t^{-\sigma} a^{(m)}_{\sigma}(X) - \text{Str} [X\Pi_0] + t^N R_N(X)(t), \quad 0 < t \leq 1,
\]

where \(R_N(X)(t)\) is the bounded continuous function on \((0, 1]\) defined by (6.12). The boundedness of \(R_N(X)(t)\) enables us to define a holomorphic function \(H_N(X)(z) := \int_0^\infty t^{z+N-1} R_N(X)(t) dt\) on the half-plane \(\Re z > -N\) (i.e., the half-plane \(\Re z > -(2N + m)\)). Combining this with (6.16) and (6.17) we see that on the half-plane \(\Re z > p\) the function \(\Gamma(\hat{z}) \text{ Str} \left[ X|D|^{-(z+m)} \right]\) is equal to

\[
\sum_{\sigma \in \Sigma_N^{(m)}} \int_1^\infty t^{\frac{z}{2} - \sigma - 1} a^{(m)}_{\sigma}(X) dt - \int_1^\infty t^{\frac{z}{2} + (m+1)} \text{Str} [X\Pi_0] dt + H_N(X)(z) + F(X)(z)
\]

\[
= \sum_{\sigma \in \Sigma_N^{(m)}} \frac{2}{z - \sigma} a_{\sigma}(X) - \frac{2}{z - m} \text{Str} [X\Pi_0] + H_N(X)(z) + F(X)(z).
\]

This shows that, for all \(N \in \mathbb{N}\), the function \(\Gamma(\hat{z}) \text{ Str} \left[ X|D|^{-(z+m)} \right]\) has a meromorphic continuation to the half-plane \(\Re z > -(2N + m)\) with at worst simple pole singularities on \(\Sigma_N^{(m)} \cup \{-m\}\). Furthermore, for all \(\sigma \in \Sigma_N^{(m)}\) with \(\sigma \neq -m\), we have

\[
\text{Res}_{z=\sigma} \left\{ \Gamma(\hat{z}) \text{ Str} \left[ X|D|^{-(z+m)} \right] \right\} = 2 a^{(m)}_{\sigma}(X) = 2 \text{Pf} \left\{ \hat{t}^{\sigma} \text{ Str} \left[ X (1 - \Pi_0) e^{-tD^2} \right] \right\},
\]

while for \(\sigma = -m\) we obtain

\[
\text{Res}_{z=-m} \left\{ \Gamma(\hat{z}) \text{ Str} \left[ X|D|^{-(z+m)} \right] \right\} = 2 \text{Pf} \text{ Str} \left[ X (1 - \Pi_0) e^{-tD^2} \right] - 2 \text{Str} [X\Pi_0].
\]
As $\Gamma(z)^{-1} = \Gamma(z)^{-1}$ is an entire function that vanishes on $-m - 2\mathbb{N}_0$, we then deduce that the function $\text{Str} \left[ X[D]^{-q(z+m)} \right]$ has a meromorphic extension to $\mathbb{C}$ with at worst simple pole singularities on $\bigcup_{N \geq 1} \Sigma_N^{(m)} = \Sigma$. This proves that $(\mathcal{A}, \mathcal{H}, D)$ has a simple and discrete dimension spectrum contained in $\Sigma$. Finally, the formula $(6.14)$ follows from $(6.18) - (6.19)$ and the fact that, for $q > 0$ both sides of $(6.14)$ vanish when $q \leq \frac{N}{2} (\Sigma + m)$. The proof is complete. 

Combining Lemma 6.6 with Theorem 5.12 we then arrive at the following result.

**Proposition 6.7** (P6.3). Assume that $(\mathcal{A}, \mathcal{H}, D)$ is $p^+$-summable, regular and has the asymptotic property. Then

1. $(\mathcal{A}, \mathcal{H}, D)$ has a discrete and simple dimension spectrum, and hence the Connes-Chern character $\text{Ch}(D)$ is represented by the CM cocycle $(7.7)$–$(7.8)$.
2. We have the following formulas for the components $\varphi_{\Sigma q}$, $q \geq 0$, of the CM cocycle:

\begin{equation}
\varphi_{\Sigma q} (a^0) = \text{Pf} \text{Str} \left[ a^0 e^{-tD^2} \right], \quad a^0 \in \mathcal{A},
\end{equation}

\begin{equation}
\varphi_{\Sigma q} (a^0, \ldots, a^{2q}) = \sum_{a \in \mathbb{N}_0^{2q}} c_{q,a} \Gamma(|q| + q)^{-1} \text{Pf} \left\{ \ell^{|a|+q} \text{Str} \left[ T_{q,a} e^{-tD^2} \right] \right\}, \quad a^j \in \mathcal{A}, \quad q \geq 1,
\end{equation}

where $c_{q,a}$ is given by (5.8) and we have set $T_{q,a} = a^0[D,a^1]^{|a_1|} \cdots [D,a^{2q}]^{|a_{2q}|}$.

In the rest of this section we further assume that $(\mathcal{A}, \mathcal{H}, D)$ is smooth (in addition to be $p^+$-summable and hypo-regular).

**Definition 6.8.** Given a discrete subset $\Sigma \subset \mathbb{R}$, we say that $(\mathcal{A}, \mathcal{H}, D)$ has the uniform asymptotic expansion property relatively to $\Sigma$ when $(\mathcal{A}, \mathcal{H}, D)$ is uniformly hypo-regular and it has the asymptotic expansion property relatively to $\Sigma$ in such way that, for any $m \in \mathbb{N}_0$ and pair $(\alpha, \beta) \in \mathcal{P}(m)$, the following properties are satisfied:

(i) For all $\sigma \in \Sigma$, the $(m+1)$-linear form $(a^0, \ldots, a^m) \mapsto a^{(m)}_{\sigma} \left( P_{\alpha,\beta}(a^0, \ldots, a^m) \right)$ is continuous on $\mathcal{A}^{m+1}$.

(ii) For all $N \in \mathbb{N}_0$, the $(m+1)$-linear map $(a^0, \ldots, a^m) \mapsto R_N \left( P_{\alpha,\beta}(a^0, \ldots, a^m) \right)$ is continuous from $\mathcal{A}^{m+1}$ to $C_0^N(\mathbb{R})$ (where $R_N$ is defined as in (6.13)).

**Remark 6.9.** The condition (i) means there is a continuous seminorm $N_{\alpha,\beta}^{(m)}$ on $\mathcal{A}$ such that

\begin{equation}
\left| a^{(m)}_{\sigma} \left( P_{\alpha,\beta}(a^0, \ldots, a^m) \right) \right| \leq N_{\alpha,\beta}^{(m)}(a^0) \cdots N_{\alpha,\beta}^{(m)}(a^m) \quad \text{for all } a^j \in \mathcal{A}.
\end{equation}

Likewise, the condition (ii) means there is a continuous semi-norm $N_{\alpha,\beta}^{(m)}$ on $\mathcal{A}$ such that

\begin{equation}
\left| R_N \left( P_{\alpha,\beta}(a^0, \ldots, a^m) \right)(t) \right| \leq N_{\alpha,\beta}^{(m)}(a^0) \cdots N_{\alpha,\beta}^{(m)}(a^m) \quad \text{for all } a^j \in \mathcal{A} \text{ and } t \in (0,1].
\end{equation}

As the following lemma shows, requiring uniformness in the heat trace asymptotics (6.10) turns out to be irrelevant when the $\mathcal{A}$ has a barelled locally convex topology. We refer to [SW] for the precise definition of a barelled locally convex topology. For our purpose it is enough to know that Banach-Steinhaus theorem continues to hold for barelled topological vector spaces and main examples of barelled topological spaces include Baire topological vector spaces, as well as inductive limits of such spaces (cf. [SW]). In particular, Fréchet spaces and inductive limits of Fréchet spaces are barelled locally convex spaces.

**Lemma 6.10.** Assume that the topology of $\mathcal{A}$ is barelled, and let $\Sigma$ be a discrete subset of $(-\infty, p]$. Then the following are equivalent:

1. $(\mathcal{A}, \mathcal{H}, D)$ has the uniform asymptotic expansion property relatively to $\Sigma$.
2. $(\mathcal{A}, \mathcal{H}, D)$ is uniformly hypo-regular and has the asymptotic expansion property relatively to $\Sigma$.

**Proof.** It is immediate that (1) implies (2), so we only need to prove the converse. Assume that $(\mathcal{A}, \mathcal{H}, D)$ is uniformly hypo-regular and has the asymptotic expansion property relatively to $\Sigma$. Let $m \in \mathbb{N}_0$ and $(\alpha, \beta) \in \mathcal{P}(m)$. By Remark 5.17 the $(m+1)$-linear map $(a^0, \ldots, a^m) \mapsto a^{(m)}_{\sigma} \left( P_{\alpha,\beta}(a^0, \ldots, a^m) \right)$ is continuous on $\mathcal{A}^{m+1}$.


Lemma 6.11. \( P_{\alpha,\beta}(a^0, \ldots, a^m) \) is continuous from \( A^{m+1} \) to \( D_B^0(A) \). Let \( t > 0 \). As the estimate (6.11) implies that the linear form \( X \to \text{Str}[X e^{-tD^2}] \) is continuous on \( D_B^0(A) \), we deduce that the \((m+1)\)-linear form \( (a^0, \ldots, a^m) \to \text{Str}[P_{\alpha,\beta}(a^0, \ldots, a^m)e^{-tD^2}] \) is continuous on \( A^{m+1} \).

Bearing this in mind, let us enumerate \( \Sigma \) as a decreasing sequence \( \sigma_0 > \sigma_1 > \cdots \). Then by (6.11) we have

\[
\sigma_0 (P_{\alpha,\beta}(a^0, \ldots, a^m)) = \lim_{t \to 0^+} t^2 (a^0 + m) \text{Str}[P_{\alpha,\beta}(a^0, \ldots, a^m)e^{-tD^2}] \quad \forall a^1 \in A.
\]

Therefore, we see that \( \sigma_0 (P_{\alpha,\beta}(a^0, \ldots, a^m)) \) is the pointwise limit of continuous \((m+1)\)-linear forms on \( A \). As the topology of \( A \) is barrelled, the Banach-Steinhaus theorem holds, and hence ensures us that \((m+1)\)-linear form \( (a^0, \ldots, a^m) \to \sigma_0 (P_{\alpha,\beta}(a^0, \ldots, a^m)) \) is continuous on \( A^{m+1} \).

More generally, using (6.12), an induction on \( j \) and repeated use of the Banach-Steinhaus theorem show that, for all \( j = 0, 1, 2, \ldots \), the \((m+1)\)-linear form \( (a^0, \ldots, a^m) \to \sigma_j (P_{\alpha,\beta}(a^0, \ldots, a^m)) \) is continuous on \( A^{m+1} \). That is, condition (i) of Definition 6.8 is satisfied.

Bearing this in mind, let \( N \in \mathbb{N}_0 \). An examination of (6.13) shows that, for all \( t \in (0, 1) \), the \((m+1)\)-linear form \( (a^0, \ldots, a^m) \to R_N (P_{\alpha,\beta}(a^0, \ldots, a^m)) \) is continuous on \( A^{m+1} \). Moreover, the asymptotic expansion property implies that \( \lim_{t \to 0^+} R_N (P_{\alpha,\beta}(a^0, \ldots, a^m)) \) exists for all \( a^1 \in A \). The Banach-Steinhaus theorem then implies that we obtain an equicontinuous family of \((m+1)\)-linear forms on \( A \) parametrized by \( t \in (0, 1] \). That is, the estimate (6.23) holds.

Therefore, using Remark 6.9 we see that condition (ii) of Definition 6.8 holds as well. This shows that \((A, H, D)\) has the uniform asymptotic expansion property. The proof is complete.

The following lemma provides us with a relationship between uniform asymptotic property and uniform dimension spectrum.

**Lemma 6.11.** If \((A, H, D)\) has the uniform asymptotic expansion property, then it has a simple and discrete uniform dimension spectrum.

**Proof.** We know by Lemma 6.6 that \((A, H, D)\) has a discrete and simple dimension spectrum contained in \( \Sigma \). Therefore, we only need to show that the dimension spectrum is uniform. Given \( m \in \mathbb{N}_0 \), the proof of Lemma 6.6 shows that, given any \( X \in D_B^0(A) \) and \( N \in \mathbb{N} \), for \( \Re z > p \) we have

\[
(6.24) \quad \Gamma(z) \text{Str}[X D^{-(z+m)}] = \sum_{\sigma \in \Sigma_N} \frac{2}{z - \sigma} a_\sigma(X) - \frac{2}{z - m} \text{Str}[X P_0] + H_N(X)(z) + F(X)(z),
\]

where \( F(X)(z) \) and \( H_N(X)(z) \) are given by

\[
F(X)(z) = \int_1^\infty t^{-1} \text{Str}[X (1 - P_0) e^{-tD^2}] dt \quad \text{and} \quad H_N(X)(z) = \int_0^1 t^{z+N} R_N(X)(z) dt.
\]

Recall also that \( F(X)(z) \) is an entire function and the function \( H_N(X)(z) \) is holomorphic on the half-plane \( \Re z > -(2N + m) \).

Let \( \alpha \in \mathbb{N}_0^{m+2}, |\alpha| = m, \) and \( \beta \in \{0, 1\}^m \). By assumption, for all \( \sigma \in \Sigma \), the \((m+1)\)-linear map \((a^0, \ldots, a^m) \to a^{(m)}_\sigma(P_{\alpha,\beta}(a^0, \ldots, a^m)) \) is continuous on \( A^{m+1} \). Moreover, it follows from the seminorm estimate (6.25) that \( H_N(P_{\alpha,\beta}(a^0, \ldots, a^m))(z) \) satisfies a uniform estimate of the form (6.23) on any closed halfspace \( \Re z \geq -(2N + m) + \epsilon, \epsilon > 0 \). In addition, as pointed out in Remark 6.14 the \((m+1)\)-linear map \((a^0, \ldots, a^m) \to P_{\alpha,\beta}(a^0, \ldots, a^m) \) is continuous from \( A^{m+1} \) to \( D_B^0(A) \). Combining this with the estimate (6.22) we deduce that \( F(P_{\alpha,\beta}(a^0, \ldots, a^m))(z) \) satisfies a uniform estimate of the form (6.23) on any closed vertical stripe \( c_1 \leq \Re z \leq c_2, c_1, c_2 \in \mathbb{R} \). Combining these observations with (6.24) we then deduce that the conditions (i)–(ii) of Definition 6.13 are satisfied on any halfspace \( \Re z \geq -(2N + m), N \in \mathbb{N} \). This shows that \((A, H, D)\) has a simple and discrete uniform dimension spectrum. The proof is complete.

Combining Lemmas 6.11 and 6.11 with Proposition 5.24 we then arrive at the final result of this section.
Proposition 6.12 ([P03]). Assume that \((A, H, D)\) is smooth, \(p^+\)-summable, uniformly regular, and one of the following conditions holds:

(i) \((A, H, D)\) has the uniform asymptotic expansion property.

(ii) The topology of \(A\) is barrelled and \((A, H, D)\) has the asymptotic expansion property.

Then the CM cocycle represents the Connes-Chern character \(\text{Ch}(D)\) in \(\text{HP}^0(A)\) and its components are computed by the formulas (6.30)–(6.31).

7. The Connes-Chern Character of an Equivariant Dirac Spectral Triple

The aim of this section is to compute the Connes-Chern character of an equivariant Dirac spectral triple by means of its representation by the CM cocycle. By Connes-Chern character we shall mean its version as a class in the periodic cyclic cohomology of continuous cochains.

Throughout this section we let \((M^\infty, g)\) be an even dimensional compact spin oriented Riemannian manifold. We denote by \(D_g\) its Dirac operator acting on the spinor bundle \(\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-\). In addition, we let \(G\) be a subgroup of the connected component of the group of orientation-preserving smooth isometries preserving the spin structure. For \(\phi \in G\) we denote by \(U_\phi\) the unitary operator of \(L^2_0(M, \mathcal{S})\) defined by (3.1) using the unique lift of \(\phi\) to a unitary vector bundle isomorphism \(\phi^S : \mathcal{S} \to \phi_* \mathcal{S}\). The map \(\phi \to U_\phi\) then provides us with a unitary representation of \(G\) in the Hilbert space \(L^2_0(M, \mathcal{S})\).

Equipping \(G\) with its discrete topology, the crossed-product algebra \(C^\infty(M) \rtimes G\) is the space \(C^\infty_c(M \times G)\) with the product and involution given by

\[F_1 \ast F_2(x, \phi) = \sum_{\phi_1 \circ \phi_2 = \phi} F_1(x, \phi_1)F_2(\phi_1^{-1}(x), \phi_2), \quad F^*(x, \phi) = F(x, \phi^{-1}).\]

Alternatively, if we denote by \(u_\phi\) the characteristic function of \(M \times \{\phi\}\). Then \(u_\phi \in C^\infty_c(M \times G)\) and any \(F \in C^\infty_c(M \times G)\) is uniquely written as a finitely supported sum,

\[F = \sum_{\phi \in G} f_\phi u_\phi,\]

where \(f_\phi(x) := F(x, \phi) \in C^\infty(M)\). Moreover, we have the relations,

\[u_{\phi_1} u_{\phi_2} = u_{\phi_1 \circ \phi_2}, \quad u_{\phi_j}^* = u_{\phi_j^{-1}} = u_{\phi_j^{-1}}, \quad \phi_j \in G,\]

(7.1)

\[u_\phi f = (f \circ \phi^{-1})u_\phi, \quad f \in C^\infty_c(M), \quad \phi \in G.\]

(7.2)

In addition we shall endow \(C^\infty(M) \rtimes G\) with its standard locally convex \(*\)-algebra topology. As mentioned in Part I, this topology is obtained as the inductive limit of the topologies of the Fréchet spaces \(C^\infty(M) \rtimes F\), where \(F\) ranges over finite subsets of \(G\). In particular, the topology of \(C^\infty_c(M) \rtimes G\) is barrelled. Moreover, given any topological vector space \(X\), a linear map \(\Phi : C^\infty(M) \rtimes G \to X\) is continuous if and only if, for all \(\phi \in G\), the map \(f \to \Phi(f u_\phi)\) is a continuous linear map from \(C^\infty_c(M)\) to \(X\).

We also observe that the relations (7.1)–(7.2) are satisfied by the operators \(U_\phi, \phi \in G\), and the functions \(f \in C^\infty_c(M)\) are represented as multiplication operators on \(L^2_0(M, \mathcal{S})\). Therefore, we have a natural representation \(f u_\phi \to f U_\phi\) of the crossed-product algebra \(C^\infty(M) \rtimes G\) as bounded operators on \(L^2_0(M, \mathcal{S})\).

Proposition 7.1. \((C^\infty(M) \rtimes G, L^2_0(M, \mathcal{S}), \mathcal{D}_g)\) is an \(n^+\)-summable smooth spectral triple.

Proof. We know that \((C^\infty(M), L^2_0(M, \mathcal{S}), \mathcal{D}_g)\) is an \(n^+\)-summable spectral triple. As the Dirac operator \(\mathcal{D}_g\) commutes with the unitary operators \(U_\phi, \phi \in G\), we see that \((C^\infty(M) \rtimes G, L^2_0(M, \mathcal{S}), \mathcal{D}_g)\) is a spectral triple as well. Obviously, this spectral triple is \(n^+\)-summable.

In order to show that the spectral triple \((C^\infty(M) \rtimes G, L^2_0(M, \mathcal{S}), \mathcal{D}_g)\) is smooth we only need to show that, given any \(\phi \in G\), the linear maps \(f \to f U_\phi\) and \(f \to [\mathcal{D}_g, f U_\phi]\) are continuous from \(C^\infty(M)\) to \(L(L^2_0(M, \mathcal{S}))\). The continuity of the former map is immediate and that of the latter is a consequence of the identities \([\mathcal{D}_g, f U_\phi] = [\mathcal{D}_g, f U_\phi] = c(df)U_\phi\). The proof is complete. □
As \((C^\infty(M) \times G, L^2_g(M, S), \mathcal{D}_g)\) is an \(n^\ast\)-summable smooth spectral triple it has a well defined Chern-Connes character \(\text{Ch}(\mathcal{D}_g)\) in \(\mathbb{HP}^0(C^\infty(M) \times G)\). The first step is showing that this Connes-Chern character is represented in \(\mathbb{HP}^0(C^\infty(M) \times G)\) by the CM cocycle.

In what follows, as in Section 5, for \(m \in \mathbb{N}_0\), we let \(\mathcal{D}^m(M, S)\) be the Fréchet space of \(m\)-th order differential operators on \(M\) acting on the sections of \(S\). We then have the following result.

**Lemma 7.2.** Let \(m \in \mathbb{N}_0\) and \(\phi \in G\). Then the linear map \(P \to \mathbb{P}\_g^m U\_\phi\) from \(\mathcal{D}^m(M, S)\) to \(\mathcal{L}(L^2_g(M, S))\) is continuous.

**Proof.** Let \(\Psi^0(M, S)\) be the space of zero-th order \(\Psi\)DOs on \(M\) acting on the sections of \(S\). We equip it with its standard Fréchet space topology (see, e.g., [LMP1] Appendix A) for a description of this topology. We note that with respect to this topology the following properties are satisfied:

- The inclusion of \(\Psi^0(M, S)\) to \(\mathcal{L}(L^2_g(M, S))\) is continuous.
- For all \(m \in \mathbb{N}_0\), the linear map \(P \to \mathbb{P}\_g^m\) from \(\mathcal{D}^m(M, S)\) to \(\Psi^0(M, S)\) is continuous.

Using these two properties we deduce that, for all \(m \in \mathbb{N}_0\) and \(\phi \in G\), the linear map \(P \to \mathbb{P}\_g^m U\_\phi\) is continuous from \(\mathcal{D}^m(M, S)\) to \(\mathcal{L}(L^2_g(M, S))\), proving the lemma.

**Lemma 7.3.** The spectral triple \((C^\infty(M) \times G, L^2_g(M, S), \mathcal{D}_g)\) is uniformly hypo-regular and uniformly regular.

**Proof.** Let \(m \in \mathbb{N}_0\) and \(\phi \in G\). As the operators \(U\_\phi\) and \(\mathcal{D}_g\) commute with each other we see that, for any \(f \in C^\infty(M)\), we have

\[
(\mathcal{D}_g f U\_\phi) \mathcal{D}_g^m = (\mathcal{D}_g f) \mathcal{D}_g^m U\_\phi \quad \text{and} \quad \mathcal{D}_g^m f U\_\phi = (\mathcal{D}_g^m f U\_\phi) \mathcal{D}_g^m.
\]

As the linear maps \(f \to \mathcal{D}_g^m f\) and \(f \to \mathcal{D}_g^m U\_\phi f\) are continuous from \(C^\infty(M)\) to \(\mathcal{D}^m(M, S)\), using Lemma 7.2 we then deduce that, for all \(m \in \mathbb{N}_0\) and \(\phi \in G\), the linear maps \(f \to \mathcal{D}_g^m f\) and \(f \to \mathcal{D}_g^m U\_\phi f\) are continuous from \(C^\infty(M)\) to \(\mathcal{L}(L^2_g(M, S))\). This proves that the spectral triple \((C^\infty(M) \times G, L^2_g(M, S), \mathcal{D}_g)\) is uniformly hypo-regular.

Similarly, given \(m \in \mathbb{N}_0\) and \(\phi \in G\), for all \(f \in C^\infty(M)\), we have

\[
(f U\_\phi)^{[m]} \mathcal{D}_g^m = f^{[m]} \mathcal{D}_g^m U\_\phi \quad \text{and} \quad \mathcal{D}_g f U\_\phi = (c(df))^{[m]} \mathcal{D}_g^m.
\]

As the principal symbol of \(\mathcal{D}_g^m\) is scalar, we see that \(f^{[m]}\) and \((c(df))^{[m]}\) are \(m\)-th order differential operators. Incidentally, the linear maps \(f \to f U\_\phi^{[m]}\) and \(f \to (c(df)) U\_\phi^{[m]}\) are continuous from \(C^\infty(M)\) to \(\mathcal{D}^m(M, S)\). Combining this with the equalities (7.3) and using Lemma 7.2 we deduce that, for all \(m \in \mathbb{N}_0\) and \(\phi \in G\), the linear maps \(f \to (f U\_\phi)^{[m]} \mathcal{D}_g^m f\) and \(f \to (\mathcal{D}_g f U\_\phi)^{[m]} \mathcal{D}_g^m f\) are continuous from \(C^\infty(M)\) to \(\mathcal{L}(L^2_g(M, S))\). It then follows that \((C^\infty(M) \times G, L^2_g(M, S), \mathcal{D}_g)\) is uniformly regular. The proof is complete.

**Lemma 7.4.** Set \(\Sigma = \{\frac{1}{2}(m - \ell); \ell \in \mathbb{N}_0\}\). Then \((C^\infty(M) \times G, L^2_g(M, S), \mathcal{D}_g)\) has the asymptotic expansion property relatively to \(\Sigma\).

**Proof.** Set \(A = C^\infty(M)\) and \(A_G = C^\infty(M) \times G\). In addition, for all \(m \in \mathbb{N}_0\), we shall denote by \(\mathcal{D}^m(M, S) \times G\) the unbounded operators on \(\mathcal{H}\) that are linear combinations of operators of the form \(P U\_\phi\), where \(P\) ranges over \(\mathcal{D}^m(M, S)\) and \(\phi\) ranges over \(G\). Note that \(\mathcal{D}_g^{[m]} (A_G)\) is spanned by operators of the form \(f U\_\phi\) and \([\mathcal{D}_g f U\_\phi] = c(df) U\_\phi\), with \(f \in A\) and \(\phi \in G\). Therefore, the space \(\mathcal{D}_g^{[m]} (A_G)\) is spanned by \(\mathcal{D}_g^{[m]} (A_G)\) and operators of the form

\[
\mathcal{D}_g f U\_\phi, 
\mathcal{D}_g^m f U\_\phi, 
\mathcal{D}_g f (c(df)) U\_\phi, 
\mathcal{D}_g^m (c(df)) U\_\phi,
\]

where \(f\) ranges over \(A\) and \(\phi\) ranges over \(G\). We thus see that \(\mathcal{D}_g^{[m]} (A_G)\) (resp., \(\mathcal{D}_g^{[2]} (A_G)\)) is contained in \(\mathcal{D}^0(M, S) \times G\) (resp., \(\mathcal{D}^1(M, S) \times G\)). An induction on \(m\) then shows that

\[
\mathcal{D}_g^{[m]} (A_G) \subset \mathcal{D}^m(M, S) \times G 
\]

for all \(m \in \mathbb{N}_0\).
Bearing this in mind, let $\phi \in G$ and $P \in \mathcal{D}(M, S)$, $m \in \mathbb{N}_0$. As $\mathcal{D}_g$ commutes with the action of $G$, we see that the unitary operator $U_\phi$ commutes with the heat semigroup $e^{-t\mathcal{D}_g^2}$. Thus,

\begin{equation}
\text{Str} \left[ PU_\phi e^{-t\mathcal{D}_g^2} \right] = \text{Str} \left[ Pe^{-t\mathcal{D}_g^2} U_\phi \right] \quad \text{for all } t > 0.
\end{equation}

Using Proposition 3.7 we then see that, as $t \to 0^+$, we have

\[ \text{Str} \left[ PU_\phi e^{-t\mathcal{D}_g^2} \right] \sim \sum_{0 \leq a \leq n} \sum_{j \geq 0} t^{-\left(J + \left(\frac{n}{2}\right)\right) + j} \int_{M^g_{\mathbb{R}}} \text{Str} \left[ \phi^{\mathcal{D}}(x) I_{P(D^g_{\mathbb{R}} + \alpha)^{-1}}(x) \right] |dx| . \]

Combining this with (6.10) shows that $(C^\infty(M) \times G, L^2(M, S), \mathcal{D}_g)$ has the asymptotic expansion property relatively to $\Sigma = \left\{ \frac{1}{2}(n - \ell); \; \ell \in \mathbb{N}_0 \right\}$. The lemma is proved. \hfill $\Box$

As the topology of $C^\infty(M) \times G$ is bared, combining Lemma 7.3 and Lemma 7.4 with Proposition 6.12 and Proposition 7.1 we then obtain the following statement.

**Proposition 7.5.** The Connes-Chern character of $(C^\infty(M) \times G, L^2(M, S), \mathcal{D}_g)$ is represented in $\text{HF}^0(C^\infty(M) \times G)$ by the CM cocycle. Moreover, the formulas (6.20)–(6.21) compute this cocycle.

It then remains to compute the CM cocycle by using the formulas (6.20)–(6.21). As we shall see the computation will follow from the differentiable version of the local equivariant index theorem provided by Theorem 1.21.

In the following, given a differential form $\omega$ on $M$, for $a = 0, 2, \ldots, n$, we shall denote by $\int_{M^g_{\mathbb{R}}} \omega$ the integral of top component of $(t_{M^g_{\mathbb{R}}})^* \omega$ over $M^g_{\mathbb{R}}$, where $t_{M^g_{\mathbb{R}}}$ is the inclusion of $M^g_{\mathbb{R}}$ into $M$. We shall also denote by $|\omega|^{(a, 0)}$ the Berezin integral $|(t_{M^g_{\mathbb{R}}})^* \omega|^{(a, 0)}$. We note that

\begin{equation}
\int_{M^g_{\mathbb{R}}} \omega = \int_{M^g_{\mathbb{R}}} (t_{M^g_{\mathbb{R}}})^* \omega = \int_{M^g_{\mathbb{R}}} |\omega|^{(a, 0)} |dx| .
\end{equation}

**Proposition 7.6.** Let $\phi \in G$ and $f^0, \ldots, f^{2q}$ in $C^\infty(M)$. When $q \geq 1$ and $\alpha \in \mathbb{N}^{2q}$ set

\[ P_{q, \alpha} = f^0(\mathcal{D}_g, f^1|^{(\alpha_1)} \ldots |(\mathcal{D}_g, f^{2q}|^{(\alpha_{2q})}) . \]

In addition, set $P_{0, 0} = f^0$ when $q = 0$.

1. If $q \geq 1$ and $\alpha = 0$, then

\begin{equation}
\text{Str} \left[ P_{q, \alpha} e^{-t\mathcal{D}_g^2} U_\phi \right] = O \left( t^{-((|\alpha|+q)+1)} \right) \quad \text{as } t \to 0^+. \end{equation}

2. If $\alpha = 0$, then, as $t \to 0^+$, we have

\begin{equation}
\text{Str} \left[ P_{q, 0} e^{-t\mathcal{D}_g^2} U_\phi \right] = t^{-q} \sum_{0 \leq a \leq n} (2\pi)^{-\frac{n}{2}} \int_{M^g_{\mathbb{R}}} \Upsilon_q + O \left( t^{-(q+1)} \right) ,
\end{equation}

where we have set $\Upsilon_q = \left( -i \frac{q}{2} \right) \int_0^1 df^1 \wedge \cdots \wedge df^{2q} \wedge \hat{A}(R^TM^g) \wedge \nu_\phi \left( R^{N^g} \right)$.

**Remark 7.7.** The above result is proved in [CH] (see Theorem 2 of [CH] for (7.6) and Corollary 3.16 of [CH] for (7.6); see also [AZ]). Our aim here is to show that these asymptotics are simple consequences of the differentiable version of the local equivariant index theorem provided by Theorem 1.21.

**Proof of Proposition 7.6.** Let $x_0 \in M^g_{\mathbb{R}}$, $a = 0, 2, \ldots, n$, and let us work in admissible normal coordinates centered at $x_0$. The results of Section 3 show that

- The multiplication operator $f^0$ has Getzler order 0 and model operator $f^0(0)$.
- Each Clifford multiplication operator $c(df^j)$ has Getzler order 1 and model operator $df^j(0)$.
- The operator $\mathcal{D}_g^{-1}$ has Getzler order 2 and model operator $H_R$. 

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Therefore, using Lemma \ref{lem:phi_0}, we deduce that, at \( x = 0 \), we have

\begin{equation}
\sigma \left[ P_{q, \alpha} \right] = f^0(0) df^1(0) |^\alpha_1 \cdots df^{2q}(0) |^{\alpha_{2q}} + O_G(2(|\alpha| + q) - 1),
\end{equation}

where \( df^j(0)^{|\alpha_j|} \) is the \( \alpha_j \)-th iterated commutator with \( H_R \). In fact, as \( H_R \) takes coefficients in forms of degree 0 and degree 2, it commutes with the forms \( df^j(0) \), and hence \( df^j(0)^{|\alpha_j|} = 0 \) whenever \( \alpha_j \neq 0 \). Therefore, if \( \alpha \neq 0 \), then we see that \( P_{q, \alpha} \) has Getzler order \( 2(|\alpha| + q) - 1 \). As \( 2(|\alpha| + q) - 1 \) is an odd integer, Theorem \ref{thm:main} shows that, when \( \alpha \neq 0 \), we have

\[ \text{Str} \left[ P_{q, \alpha} e^{-\hat{\phi}^2 U_\phi} \right] = O \left( t^{(|\alpha|+q)+1} \right) \quad \text{as} \quad t \to 0^+. \]

Suppose now that \( \alpha = 0 \) and set \( \omega = f^0 df^1 \wedge \cdots \wedge df^{2q} \). Then (7.8) shows that \( P_{q, 0} \) has Getzler order \( 2q \) and model operator \( (P_{q, 0})(2q) = \omega(0) \). It then follows that

\begin{equation}
K_{(P_{q, 0})(2q)}(H_R + \partial_t)^{-1} (x, y, t) = K_{\omega(0)}(H_R + \partial_t)^{-1} (x, y, t) = \omega(0) \wedge K_{(H_R + \partial_t)^{-1}}(x, y, t).
\end{equation}

Thus,

\begin{equation}
I_{(P_{q, 0})(2q)}(H_R + \partial_t)^{-1} (x, t) = \omega(0) \wedge I_{(H_R + \partial_t)^{-1}} (x, t).
\end{equation}

Combining this with (4.11) we deduce that

\[ I_{(P_{q, 0})(2q)}(H_R + \partial_t)^{-1} (0, 1) = \frac{(4\pi)^{-\frac{q}{2}}}{\det (1 - \phi \nu)} \omega(0) \wedge \hat{A}(R^{TM^0}) \wedge \nu_{\phi} \left( R^{N^0} \wedge \nu_{\phi} \right) \].

Therefore, we obtain

\begin{equation}
\gamma_{\phi}(P_{q, 0}; \mathcal{D}_g)(0) = (-i)^{\frac{q}{2}} 2\pi \det \frac{\omega(0) \wedge \hat{A}(R^{TM^0}) \wedge \nu_{\phi} \left( R^{N^0} \wedge \nu_{\phi} \right)}{(a, 0)}.
\end{equation}

As \( P_{q, 0} \) has Getzler order \( 2q \), combining this with Theorem \ref{thm:main} and using (7.5) gives the asymptotic (7.4). The proof is complete.

We are now in a position to prove the main result of this section.

**Theorem 7.8.** The Connes-Chern character of \( (C^\infty(M) \times G, L^2_g(M, \mathcal{S}), \mathcal{D}_g) \) is represented in \( H^0(C^\infty(M) \times G) \) by the CM cocycle \( \phi_{\text{CM}} = (\varphi_{2q}^{\text{CM}}) \). Moreover, for all \( f_1, \ldots, f_{2q} \) in \( C^\infty(M) \) and \( \phi_0, \ldots, \phi_{2q} \) in \( G \), we have

\begin{equation}
\varphi_{2q}^{\text{CM}}(f_0 U_{\phi_0}, \ldots, f_{2q} U_{\phi_{2q}}) = (-i)^{\frac{q}{2}} \sum_{0 \leq a \leq n, a \text{ even}} (2\pi)^{-\frac{q}{2}} \int_{M^0_{g}} f_0 df^1 \wedge \cdots \wedge df_{2q} \wedge \hat{A}(R^{TM^0}) \wedge \nu_{\phi(2q)} \left( R^{N^0(2q)} \right),
\end{equation}

where we have set \( \phi(j) := \phi_0 \circ \cdots \circ \phi_j \) and \( f(j) := f^0 \circ \phi_{j-1}^{-1} \).

**Proof.** Thanks to Proposition \ref{prop:phi_0} we know that the Connes-Chern character \( \text{Ch}(\mathcal{D}_g) \) is represented in \( H^0(C^\infty(M) \times G) \) by the CM cocycle \( \phi_{\text{CM}} = (\varphi_{2q}^{\text{CM}}) \). Moreover, this CM cocycle is given by the formulas \( (7.6) \) - \( (7.7) \). It then remains to compute the CM cocycle \( \varphi_{\text{CM}}(\cdot) = (\varphi_{2q}^{\text{CM}}) \) by using these formulas. Note that \( \varphi_{2q}^{\text{CM}} = 0 \) for \( q \geq \frac{1}{2} n + 1 \), since \( (C^\infty(M) \times G, L^2_g(M, \mathcal{S}), \mathcal{D}_g) \) is \( n^+ \)-summable, so we only have to compute \( \varphi_{2q}^{\text{CM}} \) for \( q = 0, \ldots, \frac{1}{2} n \).

Let \( \phi_0 \in G \) and \( f^0 \in C^\infty(M) \). Using \( (7.6) \) and \( (7.7) \) we get

\[ \varphi_{0}^{\text{CM}}(f^0 U_{\phi_0}) = \text{Pf} \text{ Str} \left[ f^0 U_{\phi_0} e^{-\hat{\phi}^2 U_\phi} \right] = \text{Pf} \text{ Str} \left[ f^0 e^{-\hat{\phi}^2 U_\phi} U_{\phi_0} \right]. \]

Combining this with \( (4.1) \) then gives

\[ \varphi_{0}^{\text{CM}}(f^0 U_{\phi_0}) = (-i)^{\frac{q}{2}} \sum_{0 \leq a \leq n, a \text{ even}} (2\pi)^{-\frac{q}{2}} \int_{M^0_{g}} f^0 \hat{A}(R^{TM^0}) \wedge \nu_{\phi_0} \left( R^{N^0} \right), \]

which is the formula \( (7.12) \) for \( q = 0 \).
Given \( q \in \{1, \ldots, \frac{1}{2} n\} \), let \( \phi_j \in G \) and \( f^j \in C^\infty(M) \), \( j = 0, \ldots, q \). In addition, for \( \alpha \in \mathbb{N}_0^{2q} \), set \( T_{q,\alpha} = f^0 U_{\phi_0} \left[ \mathbb{P}_g, f^1 U_{\phi_1} \right]^{[\alpha_1]} \cdots \left[ \mathbb{P}_g, f^{2q} U_{\phi_{2q}} \right]^{[\alpha_{2q}]} \). We observe that, as \( \mathbb{P}_g \) commutes with the action of \( G \), given any \( \psi_1 \) and \( \psi_2 \) in \( G \) and \( f \in C^\infty(M) \), for all \( j \in \mathbb{N} \), we have

\[
U_{\psi_1} \mathbb{P}_g U_{\psi_2}^{[q]} = \mathbb{P}_g U_{\psi_1} f U_{\psi_2}^{-1}[q] U_{\psi_1} U_{\psi_2} = \mathbb{P}_g f \circ \psi_2^{-1}[q] U_{\psi_1} U_{\psi_2}.
\]

Repeated use of these equalities enables us to rewrite \( T_{q,\alpha} \) as

\[
T_{q,\alpha} = P_{q,\alpha} U_{(\varphi_{2q}, \ldots, \varphi_0)} , \quad \mathcal{P}_{q,\alpha} := f^0 \left[ \mathbb{P}_g, f^1 \right]^{[\alpha_1]} \cdots \left[ \mathbb{P}_g, f^{2q} \right]^{[\alpha_{2q}]},
\]

where we have set \( \varphi_{(j)} := \phi_0 \circ \cdots \circ \phi_j \) and \( \hat{f}^j := f^j \circ \phi_{(j-1)}^{-1} \). Therefore, using (7.11) and (7.3) we obtain

\[
\varphi_{2q}^\text{CM} (f^0 U_{\phi_0}, \ldots, f^{2q} U_{\phi_{2q}}) = \sum_{\alpha \in \mathbb{N}_0^{2q}} c_{q,\alpha} \gamma(|\alpha| + q)^{-1} \sum_{a \text{ even}} (2\pi)^{-\frac{n}{2}} \int_{M^{a}(\mathbb{Z})} \mathcal{P}_{q,\alpha} U_{(\varphi_{(a)}, \ldots, \varphi_{(1)})}.
\]

Combining this with Proposition 7.6 then gives

\[
\varphi_{2q}^\text{CM} (f^0 U_{\phi_0}, \ldots, f^{2q} U_{\phi_{2q}}) = c_{q,0} \gamma(q)^{-1} \sum_{a \text{ even}} (2\pi)^{-\frac{n}{2}} \int_{M^{a}(\mathbb{Z})} \mathcal{P}_{q,0} U_{(\varphi_{(a)}, \ldots, \varphi_{(1)})}.
\]

where we have set \( \Upsilon_q = (-i)^{\frac{q}{2}} f^0 d\hat{f}^1 \wedge \cdots \wedge d\hat{f}^q \wedge \Delta(R^M(\phi_{(2q)})) \wedge \nu_{\phi_{(2q)}} \left( R^{\mathcal{N}(\phi_{(2q)})} \right. \left. \wedge \nu_{\phi_{(2q)}} \right) \). As \( c_{q,0} \gamma(q)^{-1} \) is equal to \((2q)!)^{-1}\) this gives the formula (7.13) for \( q = 1, \ldots, \frac{1}{2} n \). The proof is complete.

\[\text{Remark 7.9.}\] To understand the formula (7.13) it is worth looking at the top-degree component \( \varphi_{n}^\text{CM} \). For \( q = \frac{1}{2} n \) the r.h.s. of (7.13) reduces to an integral over \( M_{\phi_{(n)}} \) and this submanifold is is empty unless \( \phi_{(n)} = \text{id} \). Thus,

\[
\varphi_{n}^\text{CM} (f^0 U_{\phi_0}, \ldots, f^n U_{\phi_n}) = \begin{cases} (2\pi)^{-\frac{n}{2}} \int_M f^0 d\hat{f}^1 \wedge \cdots \wedge d\hat{f}^n & \text{if } \phi_0 \circ \cdots \circ \phi_n = \text{id}, \\ 0 & \text{otherwise}, \end{cases}
\]

That is, \( \varphi_{n}^\text{CM} \) agrees with the transversal fundamental class cocycle of Connes [Co1]. This implies that the Hochschild class of the Connes-Chern character agrees with Connes’ transversal fundamental class (see also [Mo] Proposition 3.7).

\[\text{Remark 7.10.}\] We refer to [Pe4] for the computation of the Connes-Chern character of a crossed-product Dirac spectral triple \((C^\infty(M) \times G, L^2(M, H), \mathcal{P}_g)\) in odd dimensions. In this case the spectral triple is an odd spectral triple. The Connes-Chern character then is defined as a class \( \text{Ch}^L(D) \in \text{HP}^1(C^\infty(M) \times G) \) and is represented by a CM cocycle. The formula for the CM cocycle is similar to that in the even dimensional case given by (7.13).

8. The JLO Cocycle of a Dirac Spectral Triple

In this section, as a further application of Theorem 7.22, we shall compute the short-time limit of the JLO cocycle of an equivariant Dirac spectral triple. As we shall see, with our approach, the computation of JLO-type cochains associated to a Dirac spectral triple is not more difficult than the computation of the CM cocycle.

8.1. The JLO cocycle of a spectral triple. Let \((\mathcal{A}, \mathcal{H}, D)\) be a \( p^+\)-summable spectral triple. In what follows, given operators \( X^0, \ldots, X^n \) in \( D^p_+(\mathcal{A}) \), for all \( t > 0 \), we define

\[
H_t(X^0, \ldots, X^n) = \int_{\Delta_m} X^0 e^{-s_0 t D^2} X^1 e^{-s_1 t D^2} \cdots X^n e^{-s_n t D^2} d\delta,
\]

where \( \Delta_m = \{ s = (s_0, \ldots, s_m) \in [0,1]^{m+1} : s_0 + \cdots + s_m = 1 \} \). The JLO cocycle \( \varphi_{t}^\text{JLO} \) is actually a family of infinite-supported even cochains \( \varphi_{t}^\text{JLO} = (\varphi_{2q,t}^\text{JLO})_{q \geq 0} \), where \( \varphi_{2q,t}^\text{JLO} \) is the \( 2q \)-cochain on \( \mathcal{A} \) given by

\[
\varphi_{t}^\text{JLO} (a^0, \ldots, a^{2q}) = t^q \text{Str} \left[ H_t \left( (a^0, [D,a^1], \ldots, [D,a^{2q}]) \right) \right], \quad a^i \in \mathcal{A}, \ t > 0.
\]
Let $A$ be the Banach $*$-algebra obtained as the completion of $A$ with respect to the norm $a \rightarrow \|a\| + \|[D, a]\|$. Then it can be shown that $\varphi^{\text{JLO}}_t$ gives rise to a cocycle in the entire cyclic cohomology of $A$ whose class is independent of $t$ (see [JLO, GS]). Moreover, this class agrees with the Connes-Chern character in entire cyclic cohomology $K_0$ (see [C]). In fact, as pointed out by Quillen [Q], the JLO cocycle can be naturally interpreted as the Chern character of a suitable superconnection with values in cochains. It should also be mentioned that the definition of the JLO cocycle and its aforementioned properties only require the spectral triple $(A, \mathcal{H}, D)$ to be $\theta$-summable, which is a weaker condition than $p^1$-summability or $p$-summability.

When $(A, \mathcal{H}, D)$ is $p$-summable, Connes-Moscovici [CM2] showed that the JLO cocycle retracts to a periodic cyclic cocycle representing the Connes-Chern character in $HP^0(A)$. They also proved that, under suitable short-time asymptotic properties for the supertraces $\operatorname{Str} [H_t \{X^0, X^1, \ldots, X^m\}]$, $X^j \in \mathcal{D}_j(A)$, we can define partie finies $\operatorname{Pf}_{t=0^+} \varphi^{\text{JLO}}_t(a^0, \ldots, a^{2q})$, $a^j \in A$, in such a way to obtain a periodic cyclic cocycle $\operatorname{Pf}_{t=0^+} \varphi^{\text{JLO}}_t$ representing the Connes-Chern character in $HP^0(A)$. In fact, this cocycle is naturally identified with the CM cocycle (see [CM3]).

We stress out that the results of [CM2] do not require the spectral triple $(A, \mathcal{H}, D)$ to be regular. Note that there are natural geometric examples of spectral triples that are not regular. See Section 7 (or in [Po1] in the non-equivariant setting).

8.2. Equivariant Dirac spectral triple. As a further example of application of the local equivariant index theorem for Volterra $Ψ$DOs provided by Theorem [122], we shall now show how this result enables us to establish the short-time limit of the JLO cocycle $\varphi^{\text{JLO}}_t$ of an equivariant Dirac spectral triple $(C^\infty(M) \rtimes G, L^2_g(M, \mathcal{S}), D_g)$. As in previous sections, $(M^n, \mathcal{S})$ is an even dimensional compact spin oriented Riemannian manifold, $D_g$ is its Dirac operators acting on spinors, and $G$ is a subgroup of the connected component of the identity component of the group of orientation-preserving isometries preserving the spin structure. More precisely, our goal is to prove the following result.

**Theorem 8.1.** For all $f^0, \ldots, f^{2q}$ in $C^\infty(M)$ and $\phi_0, \ldots, \phi_{2q}$ in $G$, as $t \rightarrow 0^+$, we have

$$
(8.1) \quad \lim_{t \rightarrow 0^+} \varphi^{\text{JLO}}_t(f^0 U_{\phi_0}, \ldots, f^{2q} U_{\phi_{2q}}) = \left(\frac{(-i)^q}{(2q)!}\right) \sum_{0 \leq \alpha \leq n, \tau \text{ even}} (2\pi)^{-\frac{\tau}{2}} \int_{M^\alpha} f^0 d\tilde{f}^1 \wedge \cdots \wedge d\tilde{f}^{2q} \wedge \tilde{A} R^{T M^\alpha} \wedge \nu_{\phi} \left(R^{N^\alpha}\right),
$$

where we have set $\phi = \phi_0 \circ \cdots \circ \phi_{2q}$ and $\tilde{f}^j := f^j \circ (\phi_0 \circ \cdots \circ \phi_{j-1})^{-1}$, $j = 1, \ldots, 2q$.

**Remark 8.2.** When $G = \{\text{id}\}$ we recover the result of Block-Fox [BF0] on the short-time limit of the JLO cocycle of a non-equivariant Dirac spectral triple $(C^\infty(M), L^2_g(M, \mathcal{S}), D_g)$.

In what follows, given differential operators $X^0, \ldots, X^m$ on $M$ acting on spinors we set

$$
Q(X^0, \ldots, X^m) = X^0 (\partial^2_g + \partial_t)^{-1} \cdots X^m (\partial^2_g + \partial_t)^{-1}.
$$

Note that $Q(X^0, \ldots, X^m)$ is a Volterra $Ψ$DO of order $\operatorname{ord} X^0 + \cdots + \operatorname{ord} X^m - 2m - 2$. We will deduce Theorem 8.1 from the following result.
Proposition 8.3. Given $\phi \in G$, let $f \in C^\infty(M)$ and $\omega^j \in C^\infty(M,T^*_xM)$, $j = 1, \ldots, 2q$. In addition, set $Q = Q\{f,c(\omega^1),\ldots,c(\omega^{2q})\}$. Then, as $t \to 0^+$ and uniformly on each fixed-point submanifold $M^j$, we have

\[
(8.2) \quad \text{str}_g \left[ \phi^s(x) U_Q(x,t) \right] = \frac{1}{(2\pi)^q} \mathcal{L} \left[ f \wedge \omega^1 \wedge \cdots \wedge \omega^{2q} \wedge \hat{A}(R^{TM^n}) \wedge \nu_{\phi}(R^{TM^n}) \right]^{(0)} + O(t^{q+1}).
\]

The proof of Proposition 8.3 is a direct application of Theorem 8.1. Before getting to this let us explain how Proposition 8.3 enables us to prove Theorem 8.1. The key observation is the following elementary lemma.

Lemma 8.4. Let $X^0, \ldots, X^m$ be differential operators on $M$ acting on spinors. For $t > 0$ denote by $h_t(X^0, \ldots, X^m)(x,y)$ the kernel of $H_t(X^0, \ldots, X^m)$ in the sense of $\left[2.6\right]$. Then

\[
h_t(X^0, \ldots, X^m)(x,y) = t^{-m} K_{Q(X^0, \ldots, X^m)}(x,y,t) \quad \text{for all } t > 0.
\]

Proof. Let $u \in C^\infty_c([0, L^2(M, S)])$. Then

\[
Q(X^0, \ldots, X^m)u(t) = \int_0^\infty X^0 e^{-t_0 \Delta} Q(X^1, \ldots, X^m)u(t-t_0)dt_0
\]

An induction then shows that

\[
Q(X^0, \ldots, X^m)(x,y,t) = t^{-m} K_{Q(X^0, \ldots, X^m)}(x,y,t) \quad \text{for all } t > 0.
\]

The change of variables $\sigma = t_0 + \cdots + t_m$ and $s_j = \sigma^{-1} t_j$, $j = 0, \ldots, m$, gives

\[
Q(X^0, \ldots, X^m)(x,y,t) = \int_0^\infty \int_{\Delta_m} X^0 e^{-s_0 \sigma \Delta} \cdots X^m e^{-s_m \sigma \Delta} u(t - \sigma) \sigma^m ds_0 d\sigma.
\]

In the same way as we obtained (2.5), this shows that

\[
K_{Q(X^0, \ldots, X^m)}(x,y,t) = t^m h_t(X^0, \ldots, X^m)(x,y) \quad \text{for all } t > 0.
\]

The lemma is proved.

Remark 8.5. The formula (8.3) is reminiscent of the resolvent formula for the JLO cocycle given by Connes [55, Eq. (17)]. In fact, at least at a formal level, we go from one formula to the other by a conjugation by the Laplace transform with respect to the variable $t$, since this transforms the inverse heat operator $(\mathcal{D}^2_\gamma + \partial_t)^{-1}$ into the resolvent $(\mathcal{D}^2_\gamma - \lambda)^{-1}$, $\lambda > 0$.

Proof of Theorem 8.1. Let $f^j \in C^\infty(M)$ and $\phi_j \in G$, $j = 0, 1, \ldots, 2q$. By definition, for all $t > 0$, we have

\[
\varphi_{\gamma t}^{(j)}(f^0 U_{\phi_0}, \ldots, f^{2q} U_{\phi_{2q}}) = t^q \text{str} \left[ H_t \left( f^0 U_{\phi_0}, [\mathcal{D}_\gamma,f^1 U_{\phi_1}], \ldots, [\mathcal{D}_\gamma,f^{2q} U_{\phi_{2q}}] \right) \right].
\]

As the unitary operators $U_{\phi_j}$ commute with the heat semigroup $e^{-t \mathcal{D}^2_\gamma}$, $t > 0$, by arguing as in [7.30] it can be shown that, for all $t > 0$, we have

\[
H_t \left( f^0 U_{\phi_0}, [\mathcal{D}_\gamma,f^1 U_{\phi_1}], \ldots, [\mathcal{D}_\gamma,f^{2q} U_{\phi_{2q}}] \right) = H_t \left( f^0, [\mathcal{D}_\gamma,f^1], \ldots, [\mathcal{D}_\gamma,f^{2q}] \right) U_{\phi}
\]

\[
= H_t \left( f^0, c(df^1), \ldots, c(df^{2q}) \right) U_{\phi},
\]

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where we have set \( \phi = \phi_0 \circ \cdots \circ \phi_2 \) and \( \hat{f} := f^j \circ (\phi_0 \circ \cdots \circ \phi_j)^{-1} \), \( j = 1, \ldots, 2q \). Therefore, using Lemma 5.1 and its notation, we see that, for all \( t > 0 \), we have

\[
\varphi_{2q,t}^\{H\} (f^0 U_{\phi_0}, \ldots, f^q U_{\phi_2}) = t^q \text{Str} \left[ H_t \left( f^0, c(d\hat{f}^1), \ldots, c(d\hat{f}^q) \right) U_{\phi} \right]
\]

where \( \varphi_{2q,t}^\{H\} \) denotes the \( H \)-kernel of \( \varphi_{2q,t} \). The proof is based on the following observation: for any \( t > 0 \), we have

\[
\varphi_{2q,t}^\{H\} (f^0 U_{\phi_0}, \ldots, f^q U_{\phi_2}) = t^q \int_M \text{Str}_g \left[ \phi^g (x) h_t \left( f^0, c(d\hat{f}^1), \ldots, c(d\hat{f}^q) \right) (x, \phi(x)) \right] |dx|
\]

where \( \phi^g (x) \) denotes the \( g \)-kernel of \( \phi \). Moreover, Lemma 5.1 and Proposition 8.3 shows that, as \( t \to 0^+ \), we have

\[
\int_M \text{Str}_g \left[ \phi^g (x) K_Q (x, \phi(x), t) \right] |dx| = \frac{1}{(2q)!} t^q \sum_{0 \leq a \leq n} (2\pi)^{-\frac{n}{2}} \int_{M^{a}} \gamma_t + O(t^{q+1}),
\]

where \( \gamma_t = (-t)^{q} f^0 d\hat{f}^1 \cdots d\hat{f}^q \). Moreover, \( \gamma_t \) has Getzler order 0, each Clifford multiplication operator \( c(\omega^j) \) has Getzler order 1, and uniformly on each fixed-point submanifold \( M^a \), we have

\[
\int_M \text{Str}_g \left[ \phi^g (x) I_Q (x, t) \right] |dx| = t^q \gamma_t (Q)(x) + O(t^{q+1}).
\]

To complete the proof it then remains to identify \( \gamma_t (Q)(x) \).

Let \( x_0 \) be a point in some fixed-point submanifold \( M^a \), \( a = 0, 2, \ldots, n \), and let us work in admissible normal coordinates centered at \( x_0 \). At \( x = 0 \) the respective model operators of \( f, c(\omega^j) \) and \( (D^2_g + \partial_t)^{-1} \) are \( f(0), \omega^j(0) \) and \( (H_R + \partial_t)^{-1} \). Therefore, using Lemma 10.2 we see that \( Q \) has model operator

\[
Q_{(-2q-2)} = f(0) (H_R + \partial_t)^{-1} \omega^1(0) (H_R + \partial_t)^{-1} \cdots \omega^q(0) (H_R + \partial_t)^{-1}.
\]

As pointed out in the proof of Theorem 4.3 in Section 4, the operator \( (H_R + \partial_t)^{-1} \) commutes with the forms \( \omega^j(0) \). Thus, setting \( \omega = f \omega^1 \wedge \cdots \wedge \omega^{2q} \), we can rewrite \( Q_{(-2q-2)} \) as

\[
Q_{(-2q-2)} = f(0) \omega^1(0) \wedge \cdots \wedge \omega^{2q}(0) \wedge (H_R + \partial_t)^{-2q} = \omega(0) \wedge (H_R + \partial_t)^{-2q+1}.
\]

Therefore, arguing as in 10.9, 10.10 shows that

\[
\sum_{0 \leq a \leq n} (2\pi)^{-\frac{n}{2}} \int_{M^{a}} \gamma_t + O(t^{q+1}).
\]

To complete the proof it then remains to identify \( \gamma_t (Q)(x) \).

Let \( x_0 \) be a point in some fixed-point submanifold \( M^a \), \( a = 0, 2, \ldots, n \), and let us work in admissible normal coordinates centered at \( x_0 \). At \( x = 0 \) the respective model operators of \( f, c(\omega^j) \) and \( (D^2_g + \partial_t)^{-1} \) are \( f(0), \omega^j(0) \) and \( (H_R + \partial_t)^{-1} \). Therefore, using Lemma 10.2 we see that \( Q \) has model operator

\[
Q_{(-2q-2)} = f(0) (H_R + \partial_t)^{-1} \omega^1(0) (H_R + \partial_t)^{-1} \cdots \omega^q(0) (H_R + \partial_t)^{-1}.
\]

As pointed out in the proof of Theorem 4.3 in Section 4, the operator \( (H_R + \partial_t)^{-1} \) commutes with the forms \( \omega^j(0) \). Thus, setting \( \omega = f \omega^1 \wedge \cdots \wedge \omega^{2q} \), we can rewrite \( Q_{(-2q-2)} \) as

\[
Q_{(-2q-2)} = f(0) \omega^1(0) \wedge \cdots \wedge \omega^{2q}(0) \wedge (H_R + \partial_t)^{-2q} = \omega(0) \wedge (H_R + \partial_t)^{-2q}.
\]

Therefore, arguing as in 10.9, 10.10 shows that

\[
\sum_{0 \leq a \leq n} (2\pi)^{-\frac{n}{2}} \int_{M^{a}} \gamma_t + O(t^{q+1}).
\]
Bearing this in mind, we shall prove (8.7) by induction on \( m \). It is immediate that (8.7) is true for \( m = 0 \). Assume it is true for \( m \geq 0 \). As \([H_R + \partial_t, t] = [\partial_t, t] = 1\), we have
\[
[t, (H_R + \partial_t)^{-1}] = (H_R + \partial_t)^{-1}[H_R + \partial_t, t][H_R + \partial_t]^{-1} = (H_R + \partial_t)^{-2}.
\]
This implies that \([t, (H_R + \partial_t)^{-m}]\) is equal to
\[
\sum_{0 \leq j \leq m-1} (H_R + \partial_t)^{-j}[t, (H_R + \partial_t)^{-1}](H_R + \partial_t)^{-m+j+1} = m(H_R + \partial_t)^{-(m+1)}.
\]
Combining this with (8.8) we then get
\[
I_{(H_R + \partial_t)^{-(m+1)}}(x, t) = \frac{1}{m} I_{[t, (H_R + \partial_t)^{-(m)}]}(x, t) = \frac{1}{m} I_{(H_R + \partial_t)^{-m}}(x, t).
\]
As formula (8.7) is true for \( m \), we deduce that
\[
I_{(H_R + \partial_t)^{-(m+1)}}(x, t) = \frac{1}{(m + 1)!} I_{(H_R + \partial_t)^{-1}}(x, t).
\]
This shows that formula (8.7) is true for \( m + 1 \). The proof of the claim is thus complete. \( \square \)

Let us go back to the proof of Proposition 8.3. Combining (8.6) with (8.7) shows that
\[
I_{Q_{(-2q-2)}}(x, t) = \frac{1}{(2q)!} \omega(0) \wedge I_{(H_R + \partial_t)^{-1}}(x, t).
\]
Therefore, by arguing as in (7.11), we obtain
\[
\gamma_\phi(Q)(0) = \frac{1}{(2q)!} (-i)^{\frac{m}{2}} (2\pi)^{-n} \left[ \omega(0) \wedge \tilde{A}(R^{TM^g}(0)) \wedge \nu_\phi \left( R^{N^g}(0) \right) \right]_{(a,0)}.
\]
Combining this with (8.5) gives the asymptotic (8.3). The proof of Proposition 8.3 is complete. \( \square \)

**Remark 8.6.** There is no major difficulty to extend this approach to the computation of the JLO cocycle of a Dirac spectral triple to various equivariant and non-equivariant family settings, as those discussed in Remark 7.26. In particular, this approach can be used to compute the bivariant JLO cocycle of an equivariant Dirac spectral triple with coefficients in suitable algebras (compare [AZ2]). As pointed out by Wu [Wu2] (who introduced the bivariant JLO cocycle), this enables us to recover the higher-index theorem of Connes-Moscovici [CM1] (in the formulation of Lott [Lo4]). This approach can also be used to simplify the computations of the infinitesimal equivariant JLO cocycle and proof of the integrability of the transgressed infinitesimal equivariant JLO cocycle in [Wa3].

**Remark 8.7.** The eta cochain of Wu [Wu1] implements the explicit homotopy between the large-time limit of the JLO cocycle and its short-time finite part. It also naturally appears in the description of the Connes-Chern character of a spin manifold with boundary equipped with a \( b \)-metric (see [Ge3, Wu1, LMP]). In particular, in the odd dimensional case, its first degree component agrees with the eta invariant of Atiyah-Patodi-Singer [APS]. The eta cochain is formally defined as the integral over \([0, \infty)\) of a transgressed version of the JLO cocycle. The main issue at stake in this definition is the integrability near \( t = 0 \) of the transgressed JLO cocycle [Wu1].

We refer to [Po1] for further applications of Lemma 8.1 and Theorem 4.22 to a new proof of the integrability at \( t = 0 \) of the transgressed JLO cocycle of an equivariant Dirac spectral triple. In particular, this bypasses the crossing with \( S^1 \) and the use of a Grassmannian variable from the previous approaches of Wu [Wu1] in the non-equivariant case and Yong Wang [Wa1] in the equivariant case.

**Remark 8.8.** By combining the heat \( b \)-calculus of Melrose [Mel] with a version of Theorem 4.22 for the \( b \)-differential operators we also can compute the short-time limit of the relative JLO cocycle of Lesch-Moscovici-Pflaum [LMP] associated to a Dirac operator on a spin manifold with boundary equipped with a \( b \)-metric. It would be interesting to extend the results [LMP] to the equivariant setting.

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