Generalization of Perfect Electromagnetic Conductor Boundary

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Abstract—Certain classes of electromagnetic boundaries satisfying linear and local boundary conditions can be defined in terms of the dispersion equation of waves matched to the boundary. A single plane wave is matched to the boundary when it satisfies the boundary conditions identically. The wave vector of a matched wave is a solution of a dispersion equation characteristic to the boundary. The equation is of the second order, in general. Conditions for the boundary are studied under which the dispersion equation is reduced to one of the first order or to an identity, whence it is satisfied for any wave vector of the plane wave. It is shown that boundaries associated with a dispersion equation of the first order, form a natural generalization of the class of perfect electromagnetic conductor (PEMC) boundaries. As a consequence, the novel class is labeled as that of generalized PEMC (GPEMC) boundaries. In another case, boundaries for which there is no dispersion equation (NDE) for the matched wave (because it is an identity) are labeled as NDE boundaries. They are shown to be special cases of GPEMC boundaries. Reflection of the general plane wave from the GPEMC boundary is considered and an analytic expression for the reflection dyadic is found. Some numerical examples on its application are presented for visualization.

Index Terms—Boundary conditions, electromagnetic theory.

I. INTRODUCTION

BOUNDARY conditions are known to play an essential role in defining electromagnetic field problems. It has been recently pointed out that the most general form of linear and local conditions for electromagnetic boundaries, valid at a surface with normal unit vector \( \mathbf{n} \), can be expressed in terms of four dimensionless vectors \( \mathbf{a}_1 \cdots \mathbf{a}_2 \) and \( \mathbf{b}_1 \cdots \mathbf{b}_2 \) as [1]–[3]

\[
\begin{align*}
\mathbf{a}_1 \cdot \mathbf{E} + \mathbf{b}_1 \cdot \eta_0 \mathbf{H} &= 0 \quad (1) \\
\mathbf{a}_2 \cdot \mathbf{E} + \mathbf{b}_2 \cdot \eta_0 \mathbf{H} &= 0. \quad (2)
\end{align*}
\]

Conditions (1) and (2) include a number of well-known boundaries as special cases. Denoting vectors tangential to the boundary surface by the subscript \( \alpha \), a few of them can be listed as follows [3].

1) The impedance boundary, defined by four vectors tangential to the boundary surface, \( \mathbf{a}_1 = \mathbf{a}_1, \mathbf{b}_1 = \mathbf{b}_1 \), \( i = 1, 2 \), or, more compactly as [4], [5]

\[
\mathbf{E}_r = \frac{\mathbf{Z}_r}{\mathbf{n} \times \mathbf{H}}
\]

with [3]

\[
\mathbf{Z}_r = \frac{-\eta_0}{\mathbf{n} \cdot \mathbf{a}_1 \times \mathbf{a}_2} (\mathbf{a}_1 \mathbf{b}_2 - \mathbf{a}_2 \mathbf{b}_1).
\]

For the dyadic notations and operational rules applied in this article, [4] and [6] may be consulted.

2) The soft-and-hard (SH) boundary [7], [8], defined by \( \mathbf{a}_1 = \mathbf{b}_2 = \mathbf{a}_1, \mathbf{b}_1 = \mathbf{a}_2 = 0 \), or

\[
\mathbf{a}_1 \cdot \mathbf{E} = \mathbf{a}_1 \cdot \mathbf{H} = 0.
\]

3) The DB boundary [9]–[11], defined by \( \mathbf{a}_1 = \mathbf{b}_2 = \mathbf{n}, \mathbf{b}_1 = \mathbf{a}_2 = 0 \), or

\[
\mathbf{n} \cdot \mathbf{E} = \mathbf{n} \cdot \mathbf{H} = 0.
\]

4) The SH/DB (SHDB) boundary [12], a generalization to the SH and DB boundaries, defined by \( \mathbf{a}_1 = \mathbf{a}_1, \mathbf{b}_1 = \alpha \mathbf{n} \) and \( \mathbf{a}_2 = \alpha \mathbf{n}, \mathbf{b}_2 = -\mathbf{a}_1 \), or

\[
\begin{align*}
\mathbf{a}_1 \cdot \mathbf{E} + \alpha \mathbf{n} \cdot \eta_0 \mathbf{H} &= 0 \quad (7) \\
\alpha \mathbf{n} \cdot \mathbf{E} - \mathbf{a}_1 \cdot \eta_0 \mathbf{H} &= 0. \quad (8)
\end{align*}
\]

5) The perfect electromagnetic conductor (PEMC) [13]–[15], defined by \( \mathbf{b}_{1,2} = (1/M \eta_0) \mathbf{a}_{1,2} \) and \( \mathbf{a}_1 \times \mathbf{a}_2 = \mathbf{n} \), or

\[
\mathbf{n} \times (\mathbf{H} + \mathbf{ME}) = 0. \quad (9)
\]

It has the special cases of PMC \( (M = 0) \) and PEC \( (|M| \to \infty) \) boundaries. Also, (9) is a special case of the impedance boundary condition (3) with \( \mathbf{Z}_r = (1/M)\mathbf{n} \times \mathbf{\hat{z}} \).

In [3], additional special cases of (1) and (2), have been discussed. In the past, many of the boundaries have been given realizations in terms of physical structures [16]–[25]. Also, many of the boundaries have recently found applications and generalizations [26]–[37].

The outline of this article is as follows. In Section II, the dispersion equation of matched waves, i.e., equation for
the wave vector $k$ of plane waves satisfying the boundary conditions identically, is found to be an algebraic equation of the second order, in general. In Section III, the class of special boundaries, for which the dispersion equation is reduced to one of the first order, is defined. In Section IV, the special boundaries of the previous Section are labeled as GPEMC boundaries, because they generalize the PEMC boundaries. Numerical examples of wave reflection and matched waves of such boundaries are computed. In Section V, the possibility of boundaries allowing matched waves for any wave vector $k$, is studied. Two Appendices containing details of the analysis, are included at the end of this article.

II. Matched Waves

A plane wave is called matched to a boundary when it satisfies the boundary conditions identically. Thus, there is no reflected wave when the incident wave is matched to the boundary. Surface waves associated with impedance boundaries serve as examples of matched waves.

Conditions for matched waves at the GBC boundary are obtained by writing the relation between fields of a plane wave

$$ k \times E = k_0\eta_o H $$

and requiring that the plane wave field $E$ satisfy the boundary conditions (1) and (2) as

$$ (k_o a_1 + b_1 \times k) \cdot E = 0 $$

$$ (k_o a_2 + b_2 \times k) \cdot E = 0 $$

$$ k \cdot E = 0. $$

Condition (13) is satisfied by any plane wave.

A. Dispersion Equation

For a solution $E \neq 0$, the three vectors in (11)–(13) must be coplanar, i.e., they must satisfy

$$ (k_o a_1 + b_1 \times k) \times (k_o a_2 + b_2 \times k) \cdot k = 0. \quad (14) $$

Equation (14) restricts the choice of the wave vector $k$ and it is called the dispersion equation for a matched wave [2]. After finding the solution $k$ of (14), the field of the matched wave can be expressed in the form

$$ E = E k \times (k_o a_1 + b_1 \times k) $$

in terms of some scalar factor $E$.

Despite its cubic form, the dispersion equation (14) can be expanded in a form which is actually quadratic in $k$ [3]

$$ (a_1 b_2 - b_1 a_2) : k k + k_o (a_1 \times a_2 + b_1 \times b_2) \cdot k $$

$$ + k_o^2 (a_2 \cdot b_1 - a_1 \cdot b_2) = 0. $$

Another form for the dispersion equation is [3]

$$ (a_1 \times k) \cdot (b_2 \times k) - (b_1 \times k) \cdot (a_2 \times k) $$

$$ - k_o (a_1 \times a_2 + b_1 \times b_2) \cdot k = 0. \quad (17) $$

Because the wave vector in the simple-isotropic medium is known to satisfy $k \cdot k = k_o^2$, we can write

$$ k = k_o u, \quad u \cdot u = 1 $$

whence (16) and (17) actually restrict the choice of the unit vector $u$. In the general case, $u$ is a complex vector, corresponding to exponential spatial dependence of the electric and magnetic fields.

B. Special Cases

The dispersion equation (16) depends on the four vectors defining the boundary. Defining

$$ k = n k_o + k_t, \quad n \cdot k_o = 0 $$

for the special cases listed in the Introduction, the dispersion equation takes the following simplified forms.

1) For the impedance boundary, the dispersion equation becomes

$$ (a_1 b_2 - b_1 a_2) : k, k_o + k_o (a_1 \times a_2 + b_1 \times b_2) \cdot n k_o $$

$$ + k_o^2 (a_2 \cdot b_1 - a_1 \cdot b_2) = 0 $$

or [3]

$$ k_o k_n \left( \eta_0^2 + \det \bar{Z} \right) + \eta_0 \left( \bar{Z} : k, k_1 + k_o^2 \bar{u} \bar{Z} \right) = 0 \quad (21) $$

where $\det \bar{Z}$ denotes the 2-D determinant. The equation can be solved for $k_o / k_n$ in terms of given vectors $k_t, k_o$, thus defining the locus of the $k$ vector of possible matched waves.

2) For the SH boundary, with $a_1 = b_2 = a_r$, $a_2 = b_1 = 0$, the dispersion equation (17) becomes

$$ (a_r \times k) \cdot (a_r \times k) = 0. \quad (22) $$

If $a_1$ is a real unit vector, matched waves propagate along the boundary as $k = \pm a_1 k_o$.

3) For the DB boundary, the dispersion equation is reduced to

$$ (n \cdot k)^2 = k_o^2 (n \cdot u)^2 = k_o^2. $$

Real solutions are $k = \pm n k_o$, which correspond to propagation normal to the DB boundary.

4) For the PEC boundary, the dispersion equation becomes

$$ (a_1 \times a_2) \cdot k = 0 $$

which is satisfied for any $k$ satisfying $n \cdot k = 0$, i.e., for lateral waves propagating along the boundary surface.

The form (16) of the dispersion equation suggests defining three classes of boundaries in terms of the order of the dispersion equation

1) Equation (16) is of the second order in $k$.

2) Equation (16) is of the first order in $k$.

3) Equation (16) is an identity, satisfied by any $k = k_o u$.

Actually, each class contains those below as special cases. Let us study restrictions to the boundary vectors $a_1 \cdots b_2$ corresponding to the cases 2) and 3).
III. FIRST-ORDER DISPERSION EQUATION

For the dispersion equation (16) to be of the first order, the quadratic term must vanish. Denoting

$$\overline{A} = a_1 b_2 - b_1 a_2$$

(25)

the dyadic \(\overline{A}\) must satisfy

$$\overline{A} : k k = \frac{1}{2}(\overline{A} + \overline{A}^T) : k k = 0$$

(26)

for any (possibly complex) vector \(k = k_o u\). Actually, (26) can be required to be valid for any vector \(k\) without restriction. Choosing \(k = k_1 + k_2\), (26) yields \((\overline{A} + \overline{A}^T) : k_1 k_2 = 0\) for any two vectors \(k_1, k_2\), which requires \(\overline{A} + \overline{A}^T = 0\). Thus, for the dispersion equation to be of the first order, the dyadic \(\overline{A}\) must be antisymmetric, whence the four vectors must satisfy the condition

$$a_1 b_2 + b_2 a_1 = b_1 a_2 + a_2 b_1.$$ (27)

Because this implies

$$a_1 \cdot b_2 = b_1 \cdot a_2$$

(28)

the first and last terms of (16) vanish simultaneously. The resulting first-order dispersion equation then becomes

$$(a_1 \times a_2 + b_1 \times b_2) \cdot k = 0$$

(29)

where the four vectors are restricted by the condition (27).

Dot-multiplying both sides of (27) by a vector from the left or from the right, leads to the conclusion that the two vector pairs \(a_1, b_2\) and \(a_2, b_1\) must be coplanar. Thus, there must exist relations of the form

$$a_2 = A_2 a_1 + B_2 b_2$$

(30)

$$b_1 = A_1 a_1 + B_1 b_2.$$ (31)

Inserting these, the condition (27) becomes

$$2 A_1 A_2 a_1 a_2 + 2 B_1 B_2 b_1 b_2$$

$$+ (A_1 B_2 + A_2 B_1 - 1) (a_1 b_2 + b_2 a_1) = 0.$$ (32)

Assuming \(a_1\) and \(b_2\) as linearly independent (otherwise all four vectors are multiples of the same vector, whence (29) is identically satisfied), (32) leads to the relations

$$A_1 A_2 = 0, \quad B_1 B_2 = 0, \quad A_1 B_2 + A_2 B_1 = 1$$

(33)

which have two possible solutions

$$A_1 = B_2 = 0, \quad A_2 = 1/B_1$$

(34)

$$a_2 = A_2 a_1, \quad b_1 = b_2/A_2$$

(35)

and

$$A_2 = B_1 = 0, \quad A_1 = 1/B_2$$

(36)

$$a_2 = B_2 b_2, \quad b_1 = a_1/B_2.$$ (37)

Corresponding to the case (35), the boundary conditions (1), (2) take the respective form

$$a_1 \cdot A_2 E + b_2 \cdot \eta_o H = 0$$

(38)

$$a_1 \cdot A_2 E + b_2 \cdot \eta_o H = 0$$

(39)

which are the same condition. Since they do not uniquely define a boundary, we can ignore this case.

For the case (37), the boundary conditions become

$$a_1 \cdot (B_2 E + \eta_o H) = 0$$

(40)

$$a_2 \cdot (B_2 E + \eta_o H) = 0.$$ (41)

To have two distinct conditions, we must assume

$$m = a_1 \times a_2 \neq 0.$$ (42)

IV. GENERALIZED PEMC BOUNDARY

The boundary conditions (40) and (41) can be written compactly as

$$m \times (H + ME) = 0, \quad M = B_2/\eta_o.$$ (43)

Because, for \(m = n\), (43) equals the PEMC boundary condition (9), we can call the boundary defined by (43) by the name generalized PEMC (GPEMC) boundary. Here we must note that \(m\) need not be a real vector.

The dispersion equation (29) restricting the \(k\) vectors for waves matched to the GPEMC boundary is reduced to

$$\left(\frac{B_2 + 1}{B_2}\right) m \cdot k = 0.$$ (44)

The case \(B_2^2 = -1\) will be considered in the following section. In the more general case, the linear dispersion equation must be of the simple form

$$m \cdot k = 0$$

(45)

whence the \(k\) vector can be expressed as

$$k = k_1 a_1 + k_2 a_2.$$ (46)

Because of the limitation \(k \cdot k = k_o^2\), there is one free (complex) parameter left in the representation (46). For a real vector \(m\), the real and imaginary parts of the \(k\) vectors of possible matched waves lie in the plane orthogonal to \(m\). For the special case of the PEMC boundary with \(m = n\), any lateral plane wave satisfying \(n \cdot k = 0\), is known to be a matched wave [3].

To interpret the boundary defined by (40) and (41), let us consider the duality transformation of fields defined by (91) and (92) in Appendix A, known to keep the isotropic medium invariant. Because the vectors defining the boundary conditions are transformed as (93), the dispersion equation (16) remains invariant, \(k_d = k\).

Excluding zero and infinite values of the parameter \(B_2\) and requiring the transformation parameters \(A, B, \varphi\) of (91) and (92) to satisfy

$$A/B = \cot \varphi = B_2.$$ (47)

Equations (40) and (41) can be expressed as conditions for the dual field

$$a_1 \cdot E_d = 0, \quad a_2 \cdot E_d = 0$$

(48)

$$m \times E_d = 0.$$ (49)
These conditions can be recognized as those of the E-boundary [2], [3], a generalization of the PEC boundary with \( m = n \).

On the other hand, if we define the parameters \( C, D, \varphi \) by
\[
C / D = - \tan \varphi = - B_2
\]
the conditions (40) and (41) can be expressed as
\[
a_1 \cdot H_d = 0, \quad a_2 \cdot H_d = 0
\]
or
\[
m \times H_d = 0
\]
which correspond to those of the H-boundary [2], [3], a generalization of the PMC boundary.

In conclusion, the dispersion equation (16) is reduced to one of the first order in \( k \) when the boundary belongs to the class of GPEMC boundaries, defined conditions of the form (40) and (41) or the form (43). In this case, the possible \( k \) vectors of a matched wave satisfy (45). The GPEMC boundary can be interpreted as a duality-transformed E-boundary or H-boundary.

A. Special Case

Let us consider the special GPEMC boundary defined by a real unit vector \( m \). Assuming a complex wave vector with real and imaginary parts
\[
k = k_{re} + j k_{im}
\]
for matched waves satisfying the dispersion condition (45), both \( k_{re} \) and \( k_{im} \) must lie in the plane orthogonal to \( m \), which is different from the plane of the boundary, in general. From \( k \cdot k = k_o^2 \), we obtain
\[
k_{re} \cdot k_{re} - k_{im} \cdot k_{im} = k_o^2
\]
(54)
\[
k_{re} \cdot k_{im} = 0.
\]
(55)

Assuming the \( x, y, z \) coordinate system with \( m = u_z \), we can assume \( k_z \) known in \( k = u_x k_x + u_y k_y + u_z k_z \), whence \( k_z \) is obtained from
\[
k_z = \sqrt{k_o^2 - k_x^2 - k_y^2}.
\]
(56)

This is visualized by Fig. 1.

B. Reflection From GPEMC Boundary

Assuming an incident wave with the electric field
\[
E'(r) = E' \exp(-jk' \cdot r)
\]
(57)
for a GPEMC boundary defined by (43) with a real vector \( m \)
\[
E'(r) = E' \exp(-jk' \cdot r)
\]
(58)
the reflected field can be found through the reflection dyadic \( \overline{R} \)
as
\[
E' = \overline{R} \cdot E'.
\]
(59)
The expression of the reflection dyadic can be written as (see Appendix B)
\[
\overline{R} = \frac{-1}{(1 + M^2 \eta_o^2) k_o^2 m \cdot k' \times \overline{R}} \cdot (m \times \overline{\pi}) \cdot \overline{R}
\]
(60)
with
\[
\overline{R} = k' \times i - k_o M \eta_o \overline{\pi}, \quad \overline{R} = k' \times i + k_o M \eta_o \overline{\pi}
\]
(61)
As a check of (60), let us assume \( |M| \to \infty \), which corresponds to the special case of E-boundary [3]. Expanding (60) yields
\[
\overline{R} = \frac{1}{m \cdot k'} k' \times \left( m \times \overline{\pi} \right) = - \overline{\pi} + \frac{mk'}{m \cdot k'}
\]
(62)
which coincides with [3, eq. (5.245)]. The total field satisfies the condition
\[
m \times (E' + E') = m \times (\overline{\pi} + \overline{R}) \cdot E' = 0.
\]
(63)
For \( m = n \), the E-boundary is reduced to the PEC boundary. Applying (5.66) from [3], we can write for the reflected magnetic field component the rule
\[
H' = \frac{1}{k_o^2} \left( k' k' \times \overline{R} \right) \cdot H'.
\]
(64)
As another check, let us consider the case \( M \to 0 \). Substituting (60), after some algebraic steps, we obtain
\[
H' \to \frac{1}{m \cdot k'} m (k' \cdot H') - H_i
\]
(65)
whence the total field satisfies the condition of the H-boundary,
\[
m \times (H' + H') = 0.
\]
(66)
For \( m = n \), this reduces to the condition of the PMC boundary.

C. Polarization of Matched Wave

The \( k \) vector of a wave matched to a GPEMC boundary is any solution of (45), \( m \cdot k = 0 \). Any incident plane wave with zero reflection is matched. The field \( E' \) of a matched wave corresponding to a solution \( k' \) of (45) satisfies
\[
k' \times \left( \overline{R} \cdot \left( m \times \overline{\pi} \right) \cdot \overline{R} \right) \cdot E' = 0.
\]
(67)
Applying the dyadic rule
\[
\overline{\mathbf{K}} \cdot \overline{\mathbf{K}}^{(2)T} = (\det(\overline{\mathbf{K}})) \mathbf{I}
\]  
(68)
where \((\cdot)^T\) denotes transpose of the dyadic, and the rules of Appendix D in [4], the double-cross square and the determinant of the dyadic \(\overline{\mathbf{K}}\) can be expanded as
\[
\overline{\mathbf{K}}^{(2)} = \mathbf{k}' \cdot \mathbf{k}' + k_o M \eta_o \mathbf{k}' \times \mathbf{\hat{I}} + k_o^2 M^2 \eta_o^2 \mathbf{\hat{I}}
\]  
(69)
and
\[
\det(\overline{\mathbf{K}}) = k_o^3 M \eta_o (1 + M^2 \eta_o^2).
\]  
(70)
The polarization for the field of a matched wave can now be expressed as
\[
\mathbf{E}' = E' \overline{\mathbf{K}}^{(2)T} \cdot \mathbf{m}
\]  
\[
= E' k_o M \eta_o (\mathbf{m} \times \mathbf{k}' + k_o M \eta_o \mathbf{m}).
\]  
(71)
(72)
To check this, because of (68), we can expand
\[
\mathbf{k'} \times \overline{\mathbf{K}} \cdot (\mathbf{m} \times \mathbf{\hat{I}}) \cdot \overline{\mathbf{K}}^{(2)T} \cdot \mathbf{m}
\]  
\[
= \det(\overline{\mathbf{K}}) \mathbf{k'} \times \overline{\mathbf{K}} \cdot (\mathbf{m} \times \mathbf{m}) = 0.
\]  
(73)
Because the field satisfies \(\overline{\mathbf{K}} \cdot \mathbf{E}' = 0\), there is no reflected wave and, consequently, the incident wave polarized as (72) is matched to the GPEMC boundary.

D. Normal Incidence

For a plane wave with normal incidence
\[
\mathbf{k'} = -\mathbf{k'} = k_o \mathbf{n}
\]  
(74)
we can substitute
\[
\overline{\mathbf{K}}' = -\overline{\mathbf{K}} = k_o (\mathbf{n} \times \mathbf{\hat{I}} - M \eta_o \mathbf{\hat{I}})
\]  
(75)
in the expression of the reflection dyadic (60), which is now reduced to
\[
\overline{\mathbf{K}} = \frac{\mathbf{n}}{A \mathbf{m} \cdot \mathbf{n}} \times \left( (\mathbf{n} \times \mathbf{\hat{I}} - M \eta_o \mathbf{\hat{I}}) \cdot (\mathbf{m} \times \mathbf{\hat{I}}) \cdot (\mathbf{n} \times \mathbf{\hat{I}} - M \eta_o \mathbf{\hat{I}}) \right)
\]  
(76)
with \(A = 1 + M^2 \eta_o^2\). Expanding this and noting that \(\mathbf{n} \cdot \mathbf{E}' = 0\), we obtain the relation
\[
\mathbf{E}' = \frac{1 - M^2 \eta_o^2}{1 + M^2 \eta_o^2} \mathbf{E} + \frac{2 M \eta_o}{1 + M^2 \eta_o^2} \mathbf{n} \times \mathbf{E}
\]  
(77)
\[
= \frac{1 - M^2 \eta_o^2}{1 + M^2 \eta_o^2} \mathbf{E}' - \frac{2 M \eta_o}{1 + M^2 \eta_o^2} \eta_o \mathbf{H}'.
\]  
(78)
It appears remarkable that the GPEMC vector \(\mathbf{m}\), real or complex, does not play any role in normal incidence. Actually, (78) reproduces the reflection rule for the PEMC boundary with \(\mathbf{m} = \mathbf{n}\) [3, eq. (2.36)].

E. Numerical Examples

As an example, let us consider a planar GPEMC boundary defined by \(\mathbf{m} = \mathbf{u}_s \sin(\pi/3) + \mathbf{u}_c \cos(\pi/3)\). The incident wave has unit amplitude and varying angle of incidence, \(\mathbf{k}'/k_o = \mathbf{u}_s \sin \theta - \mathbf{u}_c \cos \theta\). Fig. 2 illustrates the magnitude of the reflected wave for different polarizations. For \(M = 0\), the matched-wave condition can be seen to occur for the linear (perpendicular) polarization when \(\mathbf{m} \cdot \mathbf{k}' = 0\). However, for \(M \eta_o = 1\), the polarization of the matched wave is no longer linear, and the two reflection coefficients are equally strong for all incidences. As another example, the GPEMC surface is defined by randomly generated complex \(\mathbf{a}_1\) and \(\mathbf{a}_2\) vectors, yielding an \(\mathbf{m}\) vector with complex components as
\[
\mathbf{m} = (0.0682569 - 0.243121j) \mathbf{u}_s
\]  
\[
+ (-0.397047 + 0.364515j) \mathbf{u}_c
\]  
\[
+ (0.25906 + 0.0128787j) \mathbf{u}_z.
\]  
(79)
Fig. 3 displays the reflection characteristics when the angle of incidence is fixed as \((\theta = 5\pi/12 = 75^\circ)\) and the azimuth angle \(\phi\) varies over the \(2\pi\) range. The GPEMC parameter in this example is \(M \eta_o = 1.5\). There is no matched wave in this particular example.

V. NO Dispersion Equation

Let us finally consider the problem of defining conditions for the GBC boundary allowing matched waves for any vector \(\mathbf{k} = k_o \mathbf{u}\). Because (16) is now an identity, let us call such a boundary as NDE boundary. An example was found in the previous section as two special cases of the GPEMC boundary for \(M = B_2/\eta_o = \pm j/\eta_o\). Electromagnetic media with no dispersion equation (NDE) have been labeled in the past as NDE media [38, Ch. 10].

To find other possible solutions, let us start by requiring that the dispersion equation (16) written as
\[
(a_1 b_2 - b_1 a_2) : \left(\mathbf{uu} - \mathbf{\hat{I}}\right) + (\mathbf{a}_1 \times \mathbf{a}_2 + \mathbf{b}_1 \times \mathbf{b}_2) \cdot \mathbf{u} = 0
\]  
(80)
be valid for any unit vector \( \mathbf{u} \). Changing the sign of \( \mathbf{u} \), the sign of the last term of (80) is changed, hence the condition can be split in two parts as

\[
(a_1b_2 - b_1a_2) : (\mathbf{uu} - \mathbf{I}) = 0 \quad (81)
\]

\[
(a_1 \times a_2 + b_1 \times b_2) \cdot \mathbf{u} = 0 \quad (82)
\]

each of which must be valid for any unit vector \( \mathbf{u} \). Obviously, (82) requires

\[
a_1 \times a_2 + b_1 \times b_2 = 0. \quad (83)
\]

Choosing consecutively \( \mathbf{u} = u_1, u_2, u_3 \) as three vectors making an orthonormal basis, summing the corresponding three conditions (81) with \( \sum(u_i u_i - \mathbf{I}) = -2\mathbf{I} \), yields

\[
(a_1b_2 - b_1a_2) : \mathbf{I} = 0 \quad (84)
\]

whence (81) requires

\[
(a_1b_2 - b_1a_2) : \mathbf{uu} = 0 \quad (85)
\]

for any \( \mathbf{u} \). From reasons similar to those of the previous section, the symmetric part of the dyadic \( a_1b_2 - b_1a_2 \) must be zero, whence the previously obtained condition (27) must be valid. Thus, the relations of the form (37) must be valid between the four vectors defining the NDE boundary.

Substituting (37) to the condition (83), we arrive at

\[
a_1 \times a_2 + b_1 \times b_2 = \left(1 + \frac{1}{B_2^2}\right)a_1 \times a_2 = 0. \quad (86)
\]

The case \( a_1 \times a_2 = 0 \) applied to (40) and (41) would lead to an incomplete set of boundary conditions. Thus, the NDE boundary requires

\[
B_2 = \pm j. \quad (87)
\]

In conclusion, boundary conditions for which matched waves satisfy the dispersion equation for any \( \mathbf{k} = k_o \mathbf{u} \) must be of the form

\[
\mathbf{m} \times (\mathbf{H} \pm (j/\eta_o)\mathbf{E}) = 0 \quad (88)
\]

hence, there are no solutions beyond the two special cases \( M = \pm j/\eta_o \) of the GPEMC boundary.

Let us check this result. Assuming boundary conditions of either of the two forms in (88) and inserting

\[
\mathbf{b}_1 = \mathbf{a}_1/B_2 = \mp j\mathbf{a}_1, \quad \mathbf{b}_2 = \mathbf{a}_2/B_2 = \mp j\mathbf{a}_2 \quad (89)
\]

in the dispersion equation (14), and expanding

\[
(k_o\mathbf{a}_1 \mp j\mathbf{a}_1 \times \mathbf{k}) \times (k_o\mathbf{a}_2 \mp j\mathbf{a}_2 \times \mathbf{k}) \cdot \mathbf{k} = 0 \quad (90)
\]

term by term, it can be identified as being an identity.

The case, \( \mathbf{m} = \mathbf{n} \) of (88), corresponding to two special cases of the PEMC boundary, was previously noticed in [3] to define a boundary with NDE.

VI. CONCLUSION

The dispersion equation governing the matched plane waves associated with boundaries obeying GBC has been studied for its special cases. In general, the dispersion equation is of the second order in the wave vector \( \mathbf{k} = k_o \mathbf{u} \), defined by the unit vector \( \mathbf{u} \). Restrictions to the boundary conditions in the case when the dispersion equation is reduced to one of the first order were studied, and the boundaries were found to define a novel class for which the name GPEMC was suggested. The GPEMC boundary is defined by a vector \( \mathbf{m} \) with arbitrary magnitude. When \( \mathbf{m} \) is real and normal to the boundary, GPEMC equals the previously studied PEMC boundary. An expression for the reflection dyadic corresponding to plane-wave reflection from the GPEMC boundary was derived and a few numerical examples were considered. For normal incidence, the GPEMC boundary turns out to act as the PEMC boundary for any vector \( \mathbf{m} \). Finally, boundary conditions for which there is NDE (because it is identically satisfied by any \( \mathbf{k} \)), were studied to define the class of NDE boundaries. It was found to be a certain special case of the class of GPEMC boundaries.

Compared to the previously known PEMC boundary, GPEMC has additional design parameters and hence opens up new possibilities for polarization control and wave transformation in antenna applications. The existing PEMC realizations [16], [17] exploit ferrites, graphene layers, or periodic patches. Similar materialization approaches will inspire the fabrication of future GPEMC structures. These designs will have an additional boost by the fast advances in present metasurface engineering.

APPENDIX A

Duality Transformation

In its basic form, duality transformation, based on the symmetry of the Maxwell equations, swaps electric and magnetic quantities. More generally, it is based on the linear transformation [4]

\[
\begin{pmatrix}
\mathbf{E}_d \\
\eta_o \mathbf{H}_d
\end{pmatrix} = \begin{pmatrix}
A & B \\
C & D
\end{pmatrix} \begin{pmatrix}
\mathbf{E} \\
\eta_o \mathbf{H}
\end{pmatrix} \quad (91)
\]

with \( AD - BC \neq 0 \). The transformation changes fields, sources, and conditions of electromagnetic media and boundaries. Choosing

\[
A = D = \cos \varphi, \quad B = -C = \sin \varphi \quad (92)
\]
where \( \varphi \) is the transformation parameter, the simple isotropic medium is invariant [3], while the vectors defining the GBC boundary conditions (1), (2) are transformed as

\[
\begin{pmatrix}
    b_{1d} \\
    b_{2d}
\end{pmatrix}
= \begin{pmatrix}
    A & B \\
    -C & D
\end{pmatrix}
\begin{pmatrix}
    a_{1d} \\
    a_{2d}
\end{pmatrix}
= \begin{pmatrix}
    \cos \varphi \ b_1 + \sin \varphi \ b_2 \\
    -\sin \varphi \ a_1 + \cos \varphi \ b_1 \\
    \cos \varphi \ a_2 + \sin \varphi \ b_2 \\
    -\sin \varphi \ a_2 + \cos \varphi \ b_2
\end{pmatrix}
\]

(93)

Applying this, one can find the relations

\[
\begin{align*}
    a_{1d} b_{2d} - b_{1d} a_{2d} &= a_1 b_2 - b_1 a_2, \\
    a_{1d} \times a_{2d} + b_{1d} \times b_{2d} &= a_1 \times a_2 + b_1 \times b_2, \\
    a_{1d} \cdot b_{2d} - b_{1d} \cdot a_{2d} &= a_1 \cdot b_2 - b_1 \cdot a_2
\end{align*}
\]

(94) - (96)

whence the dispersion equation (16) is invariant in the duality transformation, \( k_j = k \). Thus, the wave vector of a matched wave does not change in the duality transformation (91), (92) of the boundary conditions.

**APPENDIX B**

**REFLECTION DYADIC FOR GPEMC BOUNDARY**

The reflection dyadic for the GPEMC boundary can be recovered from that of the more general GBC boundary by applying the expression from [3, eq. (5.63)]

\[
\mathbf{\Pi} = \frac{1}{j' \kappa} \mathbf{\tilde{\kappa}} \times \mathbf{\tilde{\Pi}}.
\]

(97)

Here we denote

\[
J' = c'_1 \times c'_2 \cdot \mathbf{k'}
\]

\[
\mathbf{\Pi} = c'^T c' - c' c'^T
\]

(98) - (99)

The vector functions are defined by

\[
\mathbf{c}'_j = \mathbf{k}' \times \mathbf{b}_j - k_0 \mathbf{a}_j
\]

(100)

\[
\mathbf{c}_j = \mathbf{k} \times \mathbf{b}_j - k_0 \mathbf{a}_j
\]

(101)

Substituting \( \mathbf{b}_j = \mathbf{a}_j / M_{\eta_0} \) for \( j = 1, 2 \), they become

\[
\mathbf{c}'_j = \frac{1}{M_{\eta_0}} \left( \mathbf{k}' \times \tilde{\mathbf{l}} - k_0 M_{\eta_0} \tilde{\mathbf{l}} \right) \cdot \mathbf{a}_j
\]

(102)

\[
\mathbf{c}_j = \frac{1}{M_{\eta_0}} \left( \mathbf{k} \times \tilde{\mathbf{l}} - k_0 M_{\eta_0} \tilde{\mathbf{l}} \right) \cdot \mathbf{a}_j
\]

(103)

Applying (40)–(43), we can expand after some algebraic steps

\[
J' = \frac{1 + M^2 \eta_0^2}{M^2 \eta_0^2} k_0^2 \mathbf{m} \cdot \mathbf{k'}
\]

(104)

and

\[
\mathbf{\Pi} = -\frac{1}{M^2 \eta_0^2} \left( \mathbf{k}' \times \tilde{\mathbf{l}} - k_0 M_{\eta_0} \tilde{\mathbf{l}} \right) \cdot \left( \mathbf{m} \times \tilde{\mathbf{l}} \right) \cdot \left( \mathbf{k} \times \tilde{\mathbf{l}} + k_0 M_{\eta_0} \tilde{\mathbf{l}} \right)
\]

(105)
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