Boundary amenability of Out($F_N$)

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Kaum nennt man die Dinge beim richtigen Namen, so verlieren sie ihren gefährlichen Zauber.
Elias Canetti

What’s in a name? That which we call a rose by any other name would smell as sweet.
William Shakespeare

Mal nommer un objet, c’est ajouter au malheur de ce monde.
Albert Camus

Abstract

We prove that Out($F_N$) is boundary amenable. This also holds more generally for Out($G$), where $G$ is either a toral relatively hyperbolic group or a right-angled Artin group. As a consequence, all these groups satisfy the Novikov conjecture on higher signatures.

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1 Introduction

Definition 1.1 (Boundary amenability). A group \( \Gamma \) is boundary amenable (or exact) if there exist a compact Hausdorff space \( X \) equipped with a continuous \( \Gamma \)-action and a sequence of continuous maps

\[ \mu_n : X \rightarrow \text{Prob}(\Gamma) \]

such that for all \( \gamma \in \Gamma \), one has

\[ \sup_{x \in X} ||\mu_n(\gamma.x) - \gamma.\mu_n(x)||_1 \to 0 \]

as \( n \) goes to \( +\infty \).

The topology on the space \( \text{Prob}(\Gamma) \) of probability measures on \( \Gamma \) is that of pointwise convergence, or equivalently, subspace topology from \( \ell^1(\Gamma) \). An action \( \Gamma \curvearrowright X \) as in Definition 1.1 is called topologically amenable.

Boundary amenability has already been established for several important classes of groups. Guentner, Higson and Weinberger proved in [13] that all linear groups are exact. Campbell and Niblo proved in [7] that every group acting properly and cocompactly on a CAT(0) cube complex is exact. Boundary amenability is also known for many groups satisfying ‘hyperbolic-like’ properties: this was established by Adams [1] for hyperbolic groups and extended by Ozawa [41] to the case of relatively hyperbolic groups with exact parabolics. It was then established for mapping class groups of orientable surfaces of finite type by Kida [29] and Hamenstädter [21]. On the other hand, Gromov’s monster containing a properly embedded expander is not exact \([12, 38]\).

The goal of this paper is to establish exactness of \( \text{Out}(F_N) \), and more generally of various outer automorphism groups.

Theorem 1.2. Let \( G \) be either

1. a free group,
2. a torsion-free Gromov hyperbolic group,
3. a torsion-free toral relatively hyperbolic group,
4. a right-angled Artin group.
Then Out(G) is exact.

Since exactness passes to subgroups, this gives a new proof of exactness of mapping class groups of surfaces with non-empty boundary.

Applications. A key motivation behind the study of exactness comes from a theorem that follows from work of Yu [46], Higson–Roe [23] and Higson [22], stating that exactness of \( \Gamma \) implies the injectivity of the Baum–Connes assembly map, which in turn implies the Novikov conjecture on higher signatures for \( \Gamma \) (this theorem builds on the fact that exactness of \( \Gamma \) is equivalent to \( \Gamma \) satisfying Yu’s property A, which implies in turn that \( \Gamma \) admits a uniform embedding in a Hilbert space). Since exactness passes to subgroups [40], we get the following corollary to Theorem 1.2.

**Corollary 1.3.** Let \( G \) be either
1. a free group,
2. a torsion-free Gromov hyperbolic group,
3. a torsion-free toral relatively hyperbolic group,
4. a right-angled Artin group.

Then Out(G) and any of its subgroups satisfy the Novikov conjecture.

Another application of the exactness of a group \( \Gamma \) comes from the study of certain operator algebras associated to \( \Gamma \); for example, exactness of a group is equivalent to the exactness of its reduced \( C^* \)-algebra, see [2, 39]. We refer to [40] for a general survey of applications of boundary amenability.

**Boundary amenability of the automorphism group of a free product.** In order to establish Theorem 1.2, we actually work in the more general setting of groups that split as free products. Let \( \{G_1, \ldots, G_k\} \) be a finite collection of countable groups, and let

\[
G := G_1 \ast \cdots \ast G_k \ast F_N,
\]

where \( F_N \) is a free group of rank \( N \). We call \( \mathcal{F} := \{G_1, \ldots, G_k\} \) a free factor system of \( G \); subgroups that are conjugate into \( \mathcal{F} \) are called peripheral. We denote by Out\((G, \mathcal{F})\) the subgroup of Out\((G)\) made of all automorphisms which preserve the conjugacy classes of all the subgroups \( G_i \), and by Out\((G, \mathcal{F}^{(i)})\) the subgroup made of all automorphisms that act as the conjugation by an element \( g_i \in G \) on each subgroup \( G_i \). Our main theorem is the following.

**Theorem 1.4.** Let \( \{G_1, \ldots, G_k\} \) be a finite collection of countable groups, and let

\[
G := G_1 \ast \cdots \ast G_k \ast F_N,
\]

where \( F_N \) is a free group of rank \( N \).

Assume that for all \( i \in \{1, \ldots, k\} \), the group \( G_i \) is exact.

Then Out\((G, \{G_i\}^{(i)})\) is exact.
We would now like to make a few comments on the statement. First, the same statement also holds for the larger group Out(G, \{G_i\}) (instead of Out(G, \{G_i\}(t))) if we make the additional assumption that Out(G_i) is exact for each i \in \{1, \ldots, k\} (see Corollary 5.3). Second, we mention that (unless the given decomposition of G is trivial, i.e. G = G_1), our assumption that G_i be exact is necessary: indeed, the group G_i/Z(G_i) embeds into Out(G, \{G_i\}(t)) as the subgroup of all partial conjugations of the G_i factor, and exactness of G_i is equivalent to that of G_i/Z(G_i) (see Section 2.4).

We briefly explain the strategy to derive Theorem 1.2 from Theorem 1.4. The particular case where k = 0 shows that Out(F_N) is exact. If G is torsion-free and hyperbolic or toral relatively hyperbolic, then Theorem 1.4 basically reduces the proof of Theorem 1.2 to the case where G is one-ended, in which case JSJ theory implies that Out(G) is virtually built from mapping class groups of surfaces and free abelian groups [18] (see Section 5.2). The case where G is a right-angled Artin group is proved by induction on the number of vertices of the underlying graph, using work of Charney–Vogtmann [8] (see Section 5.3).

**General strategy of the proof: the inductive argument.** In the rest of the introduction, we sketch our proof of Theorem 1.4.

The group Out(G, F) has a natural action on a compact space, namely the compactified outer space of the free product (G, F). Points in this space correspond to certain actions of G on R-trees. Unfortunately, except in a few exceptional cases, the Out(G, F)-action on is not topologically amenable. Indeed, some points in have a non-amenable stabilizer which is a general obstruction for an action to be topologically amenable (see Corollary 2.12 for instance). For example, every tree in the boundary of Culler–Vogtmann’s outer space which is dual to a non-filling measured lamination on a surface with boundary is stabilized by every automorphism coming from a mapping class supported on the subsurface that avoids the lamination.

Because of this, we use the following inductive argument inspired from Kida’s proof of exactness of mapping class groups [29]. We use the decomposition of into arational and non-arational trees introduced by Reynolds [44]: a tree T in is arational if for every proper free factor A of G relative to F, the restriction of the A-action to its minimal subtree in T is relatively free and simplicial.

In the particular case where G = F_N and F = \emptyset, we prove that the action of Out(F_N) on the set of arational trees is amenable. On the other hand, to any tree T which is not arational, one can canonically associate a finite set of free factors [44]. In this situation, a theorem by Ozawa says that it is enough to prove that the stabilizers of free factors are exact ([11], [29], see Corollary 2.8). This allows to argue by induction on the complexity of the free product decomposition. But in order to carry out this inductive strategy, we need to understand the more difficult case where F \neq \emptyset. Here, we would like to notice that when F \neq \emptyset, the case G = F_N is as difficult as the general setting of Theorem 1.4.

When F \neq \emptyset, one can still associate a finite set of relative free factors to any non-arational tree [25]. We also construct sequences of probability measures indexed by
Rational trees, but instead of being supported on \( \text{Out}(G, F) \), they live on the (countable) set Simp of simplices of outer space.

**Theorem 1.5.** Let \( G \) be a countable group, and let \( F \) be a free factor system of \( G \). Then there exists a sequence of Borel maps

\[ \mu_n : \mathcal{P}\mathcal{A}T \rightarrow \text{Prob}(\text{Simp}) \]

such that for all \( \Phi \in \text{Out}(G, F) \) and all \( T \in \mathcal{P}\mathcal{A}T \), one has

\[ ||\Phi \mu_n(T) - \mu_n(\Phi T)||_1 \rightarrow 0 \]
as \( n \) goes to +\( \infty \).

Using an additional argument, we actually prove that the \( \text{Out}(G, F(t)) \)-action on \( \mathcal{A}T \) is Borel amenable (see Definition 2.7) under the (necessary) additional assumption that nontrivial peripheral elements have amenable centralizers (see Theorem 6.4).

Since stabilizers of simplices have smaller complexity, Ozawa’s theorem still allows for an inductive argument. The base cases of the induction correspond to either

- \( G = G_1 \ast G_2 \) and \( F = \{G_1, G_2\} \), in which case \( \text{Out}(G, F(t)) \) is isomorphic to \( G_1/Z(G_1) \times G_2/Z(G_2) \), or

- \( G = G_1 \ast \mathbb{Z} \) and \( F = \{G_1\} \), in which case \( \text{Out}(G, F(t)) \) has an index 2 subgroup isomorphic to \( (G_1 \times G_1)/Z(G_1) \).

**On the proof of Theorem 1.5** The key geometric feature used in the classical proof of boundary amenability of a free group \( F_N \) (more precisely, in the proof that the \( F_N \)-action on its boundary \( \partial_\infty F_N \) is topologically amenable) is that any two rays converging to a common point in \( \partial_\infty F_N \) have the same tail. The key observation in our proof of Theorem 1.5 is inspired by this phenomenon. Let \( T \in \mathcal{A}T \), and let \( S \) be a simplicial \((G, F)\)-tree coming with a morphism \( f : S \rightarrow T \) (see Section 2.1 for definitions). We define the turning class of \( f \) as the set of turns (i.e. pairs of directions) at branch points in \( T \) which lift to \( S \). Our key observation is the following.

**Lemma 1.6** (Factorization lemma). Let \( T \in \mathcal{O} \), let \( S, S' \in \mathcal{O} \), and let \( f : S \rightarrow T \) and \( f' : S' \rightarrow T \) be two optimal morphisms having the same turning class. Then there exists \( \varepsilon > 0 \) such that if \( U \in \mathcal{O} \) is a tree of covolume at most \( \varepsilon \) and \( f \) factors through \( U \), then \( f' \) also factors through \( U \).

The idea of turning classes and its consequences on factorization appear in an unpublished paper by Los–Lustig [35] (where these were called blowup classes).

It is noticeable that here, factorization occurs “on the nose”, while in the proofs of boundary amenability of hyperbolic groups [41] or mapping class groups [21, 29], one appeals to averaging arguments on finite collections of rays going to the same boundary point (in the case of mapping class groups, Kida uses tight geodesics to get this finiteness). Nevertheless, in our context, we still need a “finite width” statement about the collection.
of rays going to a given boundary point. This is given by the following important fact, established in Section 3.2: given \( t \in \mathbb{R} \), the set \( S_t(f) \) of all simplices in \( \text{Simp} \) that contain a tree of covolume \( e^{-t} \) through which the morphism \( f \) factors is finite.

Given \( t \in \mathbb{R} \), we define \( \nu_t(f) \) to be the uniform probability measure on the finite set \( S_t(f) \), and we then let

\[
\mu_n(f) := \frac{1}{n} \int_0^{2n} \nu_t(f) d\text{Leb}(t).
\]

The factorization lemma shows that as long as \( f \) and \( f' \) have the same turning class, we have \( \mu_n(f) = \mu_n(f') \) for all sufficiently large \( n \in \mathbb{N} \).

In the special case where \( G = F_N \) and \( F = \emptyset \), and \( T \) is an arational \( F_N \)-tree equipped with a free action of the free group, there are finitely many directions at branch points in \( T \), and therefore there are only finitely many possible turning classes for the morphisms with target \( T \). In this case, we are done by letting \( \mu_n(T) \) be the average of the measures \( \mu_n(f) \) taken over a finite set of morphisms representing all possible turning classes.

A main difficulty in the paper is that this averaging argument does not work in general because there are infinitely many turning classes. The rough idea to bypass this difficulty is to enumerate the possible turning classes in \( T \) by giving them “names” in an \( \text{Out}(G,F) \)-invariant way. Only finitely many turning classes should share a given name. We then define \( \mu_n(T) \) by averaging \( \mu_n(f) \) over a finite set of morphisms \( f \) whose turning classes coincide with the set of turning classes having the first possible name.

Our enumeration of turning classes is made differently for geometric and nongeometric trees in \( \mathcal{AT} \). If \( T \in \mathcal{AT} \) is nongeometric (in the sense of Levitt–Paulin [34]), then it can be strongly approximated by trees in \( \mathcal{O} \). It follows that there exists a morphism \( f : S \to T \) with \( S \in \mathcal{O} \), such that every turn in \( T \) lifts to \( S \). The turning class of \( f \) (which contains all turns) is then our preferred turning class, i.e. the first from the enumeration.

When \( T \) is geometric, the band complex that resolves \( T \) enables us to analyze a very particular set of turns that we call ubiquitous. These are defined as turns \( (d,d') \) such that for every nondegenerate segment \( I \subseteq T \), there exists \( g \in G \) such that \( gI \) contains \( (d,d') \). We show in particular that each direction is contained in only finitely many ubiquitous turns. These turns can therefore be used to define an angle between two directions \( d,d' \) at \( x \) by counting the number of overlapping ubiquitous turns at \( x \) needed to go from \( d \) to \( d' \) (see Definition 4.16). Using a Whitehead graph argument, we also prove that there are sufficiently many turns so that the angle between any pair of directions coming from the same minimal component of the band complex is finite. The “name” of a turning class is then defined from the angles of the turns it contains.

The above discussion is made formal in the following technical statement, which is key in our proof of Theorem 1.5. Here \( \mathcal{AT}^\text{turn} \) is the set of all turning classes on arational trees (this space is equipped with a \( \sigma \)-algebra). The symbol \( \bot \) is a special name for turning classes that we want to ignore.

**Proposition 1.7.** There exists a measurable map

\[
\text{Name} : \mathcal{AT}^\text{turn} \to \mathbb{N} \cup \{ \bot \}
\]

such that
• if $B$ is a turning class on a tree $T$, and $\lambda B$ is the turning class on the tree $\lambda T$ for some $\lambda > 0$ containing the same turns as $B$, then $\text{Name}(\lambda B) = \text{Name}(B)$,

• Name is $\text{Out}(G, \mathcal{F})$-invariant, i.e. $\text{Name}(\Phi.B) = \text{Name}(B)$ for all $B \in \mathcal{A}T^\text{turn}$ and all $\Phi \in \text{Out}(G, \mathcal{F})$,

• for all $T \in \mathcal{A}T$ and all $n \in \mathbb{N}$, there are only finitely many turning classes on $T$ whose name is $n$, and

• for all $T \in \mathcal{A}T$, there exists an optimal morphism with range $T$ whose turning class with respect to $T$ has a name different from $\perp$.

We finish by mentioning that Theorem 1.4 does not yield a concrete compact space equipped with a topologically amenable action of $\text{Out}(G, \mathcal{F}(t))$. However, in the case where $G$ is a free group, we describe an explicit such space by unraveling our inductive argument. This is a metrizable compact space obtained as a product space involving in particular all boundaries of the relative outer spaces associated to all free factors $A \subseteq F_N$ and all free factor systems of $A$, and all Gromov boundaries of free factors of $F_N$ (see Section 6.1).

**Structure of the paper.** The paper is organized as follows. In Section 2 we review necessary background on free products and their outer spaces, arational trees, geometric and nongeometric trees, and general facts concerning boundary amenability of groups.

In Section 3 we prove the factorization lemma (Lemma 1.6). We then define the probability measures associated to any arational tree, and explain the argument to derive Theorem 1.5 from Proposition 1.7.

In Section 4 we give names to turning classes on arational trees to prove Proposition 1.7 and we treat nongeometric and geometric trees separately.

In Section 5 we complete the proof of our main theorem (Theorem 1.4), and we explain how to derive boundary amenability of $\text{Out}(G)$ when $G$ is either a toral relatively hyperbolic group or a right-angled Artin group.

In Section 6 we establish two complementary results to our main theorem. First, we unravel the inductive argument to build an explicit compact metrizable space equipped with a topologically amenable action of $\text{Out}(F_N)$. Second, we prove that if the centralizer of every nontrivial peripheral element is amenable, then the $\text{Out}(G, \mathcal{F}(t))$-action on $\mathcal{A}T$ is Borel amenable.

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2 Background

2.1 General background on free products

Generalities on free products. Let $\mathcal{F} := \{G_1, \ldots, G_k\}$ be a finite collection of countable groups, and let

$$G := G_1 \ast \cdots \ast G_k \ast F_N,$$

where $F_N$ is a free group of rank $N$. Elements or subgroups of $G$ that are conjugate into one of the subgroups in $\mathcal{F}$ are called peripheral. We define the complexity of $(G, \mathcal{F})$ as the pair $\xi(G, \mathcal{F}) := (N + k - 1, N)$. We say that $(G, \mathcal{F})$ is sporadic if $\xi(G, \mathcal{F}) \leq (1, 1)$ which happens exactly if $G = G_1$, $G = Z$, $G = G_1 \ast G_2$, or $G = G_1 \ast Z$.

A $(G, \mathcal{F})$-tree is an $\mathbb{R}$-tree $T$ equipped with a $G$-action, such that every subgroup in $\mathcal{F}$ acts elliptically on $T$ (i.e. it fixes a point in $T$). A $(G, \mathcal{F})$-free splitting is a simplicial $(G, \mathcal{F})$-tree with trivial edge stabilizers. A $(G, \mathcal{F})$-free factor is a subgroup of $G$ that arises as a point stabilizer in some $(G, \mathcal{F})$-free splitting. A proper $(G, \mathcal{F})$-free factor is a free factor distinct from $G$ and non-peripheral. A $(G, \mathcal{F})$-free factor system is the collection of all conjugacy classes of point stabilizers in some $(G, \mathcal{F})$-free splitting.

A theorem of Kurosh [31] asserts that every subgroup $A \subseteq G$ decomposes as $\ast_j H_j \ast F$, where each $H_j$ is conjugate into one of the peripheral subgroups $G_i$, and $F$ is a free group. We denote by $\mathcal{F}|A$ the collection of all $A$-conjugacy classes of the subgroups $H_j$ from the above decomposition of $A$.

Directions and branch points in trees. A direction $d$ at $x$ in a $(G, \mathcal{F})$-tree $T$ is a germ of isometric maps $\eta : [0, \varepsilon]_{\mathbb{R}} \rightarrow T$ with $\eta(0) = x$. We say that a subtree $Y \subset T$ contains the direction $d$ if it is represented by a map $\eta$ whose image is contained in $Y$.

A turn at a point $x \in T$ is a pair $(d, d')$ where $d, d'$ are two distinct directions at $x$ (a turn is formally defined an ordered pair, but the ordering will not be important). The point $x$ is an inversion point if there are exactly two directions at $x$, and there exists $g \in G$ fixing $x$ and swapping the two directions at $x$. A branch point $x \in T$ is a point such that $T \setminus \{x\}$ has at least 3 connected components. A point that is either a branch point or an inversion point is called a generalized branch point. Note that in this paper, inversion points will occur only if one of the factors in $\mathcal{F}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. A branch direction in $T$ is a direction based at a branch point.

Outer space and its closure. A $(G, \mathcal{F})$-tree $T$ is relatively free if every element of $G$ elliptic in $T$ is conjugate into one of the subgroups in $\mathcal{F}$. A Grushko $(G, \mathcal{F})$-tree is a simplicial metric relatively free $(G, \mathcal{F})$-tree. We note that $(G, \mathcal{F})$ is non-sporadic if and only if every Grushko tree has at least 2 orbits of edges. The unprojectivized relative
outer space $\mathcal{O}$ is the space of all $G$-equivariant isometry classes of Grushko $(G,\mathcal{F})$-trees. We denote by $\mathbb{P}\mathcal{O}$ the projectivized outer space, where trees are considered up to homothety instead of isometry. Given a tree $S \in \mathcal{O}$, the set of all trees $S'$ obtained by keeping the same underlying simplicial structure but varying the metrics (keeping all edge lengths positive) projects to an open simplex in $\mathbb{P}\mathcal{O}$. We denote by $\text{Simp}$ the countable collection of all these open simplices. Given an open simplex $\Delta \in \text{Simp}$, we denote by $\tilde{\Delta}$ the preimage of $\Delta$ in $\mathcal{O}$.

The closure $\overline{\mathcal{O}}$ of $\mathcal{O}$ in the space of all $(G,\mathcal{F})$-trees (equipped with the Gromov–Hausdorff topology introduced in [42]) was identified in [24] with the space of all very small $(G,\mathcal{F})$-trees, i.e. those trees $T$ for which tripod stabilizers are trivial, and arc stabilizers are either trivial, or maximally cyclic and nonperipheral (see [9, 4] for free groups). We will let $\partial \mathcal{O} := \mathcal{O} \setminus \overline{\mathcal{O}}$.

**Levitt decomposition of a very small tree.** Very small trees with dense $G$-orbits have trivial arc stabilizers, see [24, Proposition 4.17]. By [33] (see also [24, Theorem 4.16] for free products), every tree $T \in \overline{\mathcal{O}}$ splits in a unique way as a graph of actions (in the sense of [33]), such that vertices of this decomposition correspond to orbits of connected components of the closure of the set of generalized branch points. Edges of this decomposition correspond to orbits of maximal arcs whose interiors contain no generalized branch points. In particular, vertex groups act with dense orbits on the corresponding subtree (maybe a point). The Bass–Serre tree of the underlying graph of groups is very small (maybe trivial). This decomposition is called the **Levitt decomposition** of $T$.

**Morphisms.** Given two trees $S, T \in \overline{\mathcal{O}}$, a morphism $f : S \to T$ is a $G$-equivariant map such that every segment in $S$ can be subdivided into finitely many subsegments, in such a way that $f$ is an isometry when restricted to any of these subsegments. We denote by Mor the space of all morphisms between trees in $\overline{\mathcal{O}}$, equipped with the Gromov–Hausdorff topology (see [16, Section 3.2] for a definition of the Gromov–Hausdorff topology on the set of morphisms). A morphism is optimal if every point $x \in S$ is contained in the interior of a nondegenerate segment $I_x$ such that $f|_{I_x}$ is injective. We denote by Opt the subspace of Mor consisting of all optimal morphisms. Given subsets $\sigma \subseteq \mathcal{O}$ and $\tau \subseteq \overline{\mathcal{O}}$, we will denote respectively by $\text{Opt}_\sigma$, $\text{Opt}_{\sigma \to \tau}$, and $\text{Opt}_{\sigma \to \tau}$ the set of optimal morphisms whose source tree lies in $\sigma$, or whose range tree lies in $\tau$, or both. Given $T \in \overline{\mathcal{O}}$, we will also use the notation $\text{Opt}_{\to T}$ for the set of optimal morphisms whose range tree is $T$.

**Lemma 2.1.** Let $\sigma \subseteq \mathcal{O}$ and $\tau \subseteq \overline{\mathcal{O}}$ be Borel subsets. Then $\text{Opt}_{\sigma \to \tau}$ is a Borel subset of Mor.

**Proof.** Since the source and range maps from Mor to $\overline{\mathcal{O}}$ are continuous, it is enough to check that Opt is a Borel subset of Mor. Since Simp is countable, it is enough to show that for all $\Delta \in \text{Simp}$, letting $F_\Delta \subseteq \mathcal{O}$ be the closure of the subset made of all trees projecting to $\Delta$, the space $\text{Opt}_{F_\Delta}$ is a closed subset of Mor. Let $S \in F_\Delta$, let $T \in \overline{\mathcal{O}}$, let $f : S \to T$ be a morphism, and let $(f_n)_{n \in \mathbb{N}} \in \text{Opt}^\mathbb{N}$ be a sequence of optimal morphisms converging to $f$, with sources in $F_\Delta$; we will prove that $f$ is optimal.
Assume not. Then there exists a vertex $v \in S$ such that any two segments $[v, v_1]$ and $[v, v_2]$ in $S$ with extremity $v$ have initial subsegments that are identified by $f$. Since arc stabilizers in $T$ are either trivial or nonperipheral, this implies that the stabilizer $G_v$ of $v$ is trivial because $S \in \mathcal{O}$. So $v$ has finite valence. In particular, there exists $\varepsilon > 0$ such that we can find subsegments $[v, x_i]$ of length $\varepsilon$ on each of the edges $e_i$ incident on $v$, which are all identified by $f$. For all $n \in \mathbb{N}$, the source $S_n$ of $f_n$ is in $F_\Delta$. Up to passing to a subsequence, we can assume that all trees $S_n$ have the same underlying simplicial tree, and one passes from $S_n$ to $S$ by slightly changing the edge lengths and collapsing a $G$-invariant forest. Let $F_n$ be the preimage of $v$ in $S_n$ under the collapse map. The edges $e^n_i$ incident on $F_n$ in $S_n$ are in natural bijection with the edges $e_i$ in $S$. For all $n \in \mathbb{N}$ sufficiently large, any approximation $x^n_i$ of the point $x_i$ in $S_n$ given by the equivariant Gromov–Hausdorff topology lies on the edge $e^n_i$, any approximation $v^n_n$ of $v$ lies at distance at most $\varepsilon/100$ from the forest $F$, and all images $f_n(x^n_i)$ lie in the same connected component of $T \setminus \{f_n(v_n)\}$. This implies that there exists a point $y_n$ in the $\varepsilon/100$-neighborhood of $F_n$ such that all directions at $y_n$ in $S_n$ are mapped to the same direction in $T$ by $f_n$. In other words $f_n$ is non-optimal, a contradiction.\ 

\subsection*{2.2 Arational trees}

A tree $T \in \partial \mathcal{O}$ is arational if for every proper $(G, F)$-free factor $A$, the $A$-action on its minimal subtree $T_A$ is equivariantly isometric to a Grushko $(A, F_A)$-tree ([14], [25]). We denote by $\mathcal{AT}$ the subspace of $\overline{\mathcal{O}}$ made of arational trees; we mention that it is a Borel subset of $\overline{\mathcal{O}}$, see [25, Lemma 5.5]. Notice that all arational trees have dense $G$-orbits. Indeed, a tree $T$ without dense orbits has a nontrivial Levitt decomposition $\Lambda$. Assume first that all edge stabilizers of $\Lambda$ are trivial. Then if all the vertex stabilizers of the Levitt decomposition are peripheral, then $T$ is a Grushko tree, contrary to the assumption $T \in \partial \mathcal{O}$. Otherwise, some vertex stabilizer of the Levitt decomposition is a proper free factor, and its action on its minimal subtree (maybe a point) has dense orbits, so $T$ is not arational. Finally, if some edge stabilizer of $\Lambda$ is non-trivial, then some edge stabilizer of $\Lambda$ is contained in a proper $(G, F)$-free factor by [24, Lemma 5.11], so $T$ is not arational.

Arational surface trees. One can construct examples of arational $(G, F)$-trees coming from trees that are dual to arational laminations on certain 2-orbifolds. In the context of free products, arational surface trees were introduced in [25, Section 4.1], and are defined in the following way (see Figure 1 for an illustration of the construction).

Let $\Sigma$ be a 2-orbifold with only conical singularities, having $s + 1$ boundary components $c_0, \ldots, c_s$ and $q$ conical points $c_{s+1}, \ldots, c_{s+q}$. Let $\Gamma$ be a graph of groups having a central vertex with vertex group isomorphic to $\pi_1(\Sigma)$, and $k$ other vertices with vertex groups isomorphic to the peripheral subgroups $G_i$. For all $j \in \{1, \ldots, s + q\}$, we choose a peripheral subgroup $G_{ij}$, and we amalgamate it with the cyclic subgroup of $\pi_1(\Sigma)$ generated by $c_j$, identifying $c_j$ with an element of $G_{ij}$ having the same order (notice that we allow the case where $c_j$ generates $G_{ij}$, in this case the Bass–Serre tree of $\Gamma$ may fail
to be minimal). The boundary curve $c_0$ is left unused. Choices are made so that the graph $\Gamma$ is connected, i.e. every peripheral subgroup is joined to the central vertex by an edge. We assume that the fundamental group of $\Gamma$ is isomorphic to $G$, and we choose such an isomorphism. We now build an action of $G$ on an $\mathbb{R}$-tree from a graph of actions on $\Gamma$ as defined in [32] (except that edges are given length 0): choose a $\pi_1(\Sigma)$-action on an $\mathbb{R}$-tree dual to an arational lamination on $\Sigma$; then take a copy $Y_v$ of this tree for each vertex $v$ of the Bass–Serre tree $S_\Gamma$ of $\Gamma$ that projects to the central vertex of $\Gamma$, and take a point $x_u$ for each vertex $u$ of $S_\Gamma$ that does not project to the central vertex; then for each edge $uv$ of $S_\Gamma$, glue the point $x_u$ to the unique point in $Y_v$ that is fixed by the stabilizer of $uv$. Every tree obtained by this construction is called an arational surface tree.

It was proved in [25, Section 4.1] that arational surface trees are indeed arational. In addition, every arational $(G,F)$-tree is either relatively free or arational surface: this was first proved by Reynolds [44] for free groups, and extended to free products in [25, Lemma 4.6].

Maps towards arational trees. Given $T,T' \in \mathcal{O}$, the bounded backtracking constant $BBT(f)$ of a Lipschitz map $f : T \to T'$ is the supremum of $d_T(f(y),[f(x),f(z)])$ over all points $x,y,z \in T$ aligned in this order. By [5, Lemma 3.1], [26, Proposition 3.12], if $T'$ is very small and $T$ is a Grushko tree, then for every Lipschitz map $f : T \to T'$, we have $BBT(f) \leq \text{Lip}(f)\text{covol}(T)$. The following fact extends [27, Corollary 3.9] to the case of free products.

**Lemma 2.2.** Let $T,T' \in \mathcal{O}$, and assume that $T'$ has dense orbits. Then every 1-Lipschitz map $f : T' \to T$ preserves alignment.

**Proof.** Assume by contradiction that it does not: there exist $x,y,z \in T'$ aligned in this order so that $f(y)$ is at positive distance (which we denote by $\varepsilon$) from $[f(x),f(z)]$. Since $T'$ has dense orbits, it has trivial arc stabilizers, and therefore we can find a
sequence of trees $S_n \in \mathcal{O}$ converging to $T'$ and coming with 1-Lipschitz morphisms $f_n : S_n \to T'$, see [24, Theorem 5.3]. By postcomposing $f_n$ by the map $f : T' \to T$, we get 1-Lipschitz maps $g_n : S_n \to T$. The covolumes of the trees $S_n$ converge to 0; since $BBT(g_n) \leq \text{covol}(S_n)$, we deduce that $BBT(g_n)$ converges to 0. For all $n \in \mathbb{N}$, let $x_n$ (resp. $z_n$) be an $f_n$-preimage of $x$ (resp. $z$) in $S_n$. Then there exists $y_n \in [x_n, z_n]$ such that $f_n(y_n) = y$. Now we have $d_T(g_n(y_n), [g_n(x_n), g_n(z_n)]) = d_T(f(y), [f(x), f(z)]) = \varepsilon$. Therefore $BBT(g_n) \geq \varepsilon$, which contradicts the fact that $BBT(g_n)$ converges to 0.

**Lemma 2.3.** Let $S \in \mathcal{O}$ be a tree without dense orbits, let $T \in \mathcal{AT}$. If there exists a 1-Lipschitz map from $S$ to $T$, then $S \in \mathcal{O}$.

**Proof.** Let $\Lambda$ be the Levitt decomposition of $S$. If $\Lambda$ contains an edge with nontrivial (whence nonperipheral) stabilizer, then there exists an edge $e \subseteq \Lambda$ whose stabilizer is a cyclic subgroup $G_e$ which is contained in a proper $(G, F)$-free factor $A$. Then $G_e$ is elliptic in $T$, so the minimal $A$-tree in $T$ is not Grushko, contradicting arationality of $T$. Therefore $\Lambda$ only has edges with trivial stabilizer. If $\Lambda$ has a nonperipheral vertex stabilizer $G_v$, then $G_v$ is a proper $(G, F)$-free factor and acts with dense orbits on its minimal subtree in $S$ (which can be a point). Since $f$ is 1-Lipschitz, the free factor $G_v$ acts with dense orbits on its minimal subtree in $T$ (which can be a point), again contradicting arationality of $T$. Therefore all vertex stabilizers of $\Lambda$ are peripheral, in other words $S$ is a Grushko tree.

**Corollary 2.4.** Let $T \in \mathcal{AT}$, and let $T' \in \partial \mathcal{O}$. If $T' \neq T$, then there is no morphism from $T'$ to $T$.

**Proof.** Assume that there exists a morphism $f : T' \to T$. Lemma 2.3 implies that $T'$ has dense orbits, and Lemma 2.2 then implies that $f$ preserves alignment. Since every alignment-preserving morphism is an isometry, we have $T' = T$.

### 2.3 Geometric and nongeometric trees

**Definition of geometric trees.** Geometric trees were first defined by Levitt–Paulin [34], see also [24] for the case of free products. Fix $R \in \mathcal{O}$ a Grushko tree. Let $T \in \mathcal{O}$, let $K \subseteq T$ be a finite subtree. We construct a band complex $\Sigma_K(T, R)$ in the following way. Let $K = (K_v)_{v \in R}$ be an equivariant family of subgroups $K_v \subset T$ (called base trees) indexed by vertices $v \in R$, such that for each vertex $v$, we have $K_v = G_v$. $K'$ for some finite subtree $K' \subset K_v$. We assume that for each edge $vv'$ of $R$, the intersection $K_v \cap K_{v'}$ is non-empty. Starting from the disjoint union of the base trees, the foliated band complex $\Sigma_K(T, R)$ is obtained by gluing for each edge $vv'$ of $R$ a band foliated by vertical segments, joining the copy of $K_v \cap K_{v'}$ in $K_v$ to its copy in $K_{v'}$. The space of leaves of $\Sigma_K(T, R)$ has a natural structure of an $\mathbb{R}$-tree $T_K$, see [34]. Notice that since $\Sigma_K(T, R)$ is modeled on the Grushko tree $R$, all leaves of $\Sigma_K(T, R)$ are trees. A tree $T \in \mathcal{O}$ is geometric if there exists a finite subtree $K \subseteq T$ such that $T = T_K$.

The quotient $\Sigma_K(T, R)/G$ is a compact foliated 2-complex. Indeed, each $K_v/G_v$ is a finite tree by assumption, there are only finitely many orbits of bands in $\Sigma_K(T, R)$,
and each band $B$ in $\Sigma_K(T, R)$ is a product $K_B \times [0,1]$ of a finite tree by an interval, and $K_B \times (0,1)$ embeds in the quotient because edges of $R$ have trivial stabilizer (the quotient map may however fold $K_B \times \{0\}$ or $K_B \times \{1\}$). One can then associate to this foliated 2-complex a system of partial isometries on a finite tree (see [24, Section 3.3]). This will allow us to apply standard results such as [11].

Characterization using strong limits. Geometric trees can be characterized as those that do not occur as nonstationary strong limits of trees in $\mathcal{O}$. Precisely, a sequence of trees $S_n \in \mathcal{O}$ is a direct system if it comes with a collection of morphisms $f_{nn'} : S_n \to S_{n'}$ (for all $n \leq n'$) and morphisms $f_n : S_n \to T$ (for all $n \in \mathbb{N}$) making the following diagram commute:

The direct system $(S_n)_{n \in \mathbb{N}}$ converges strongly to $T$ if for every finite subtree $X \subseteq S_n$, there exists $n' > n$ such that $f_{nn'}$ is isometric when restricted to $f_{nn'}(X)$.

It was proved in [34] that a tree $T \in \mathcal{O}$ is nongeometric if and only if it is a nonstationary strong limit of a direct system of trees in $\mathcal{O}$ (this is proved for finitely presented groups in [34], but this generalizes without difficulty to any finite free product when the factors are elliptic). If additionally $T$ is arational, then Corollary 2.4 enables us to add the requirement that $S_n \in \mathcal{O}$, as opposed to being in the boundary of outer space. We record this in form of the following lemma.

**Lemma 2.5.** For all $T \in \mathcal{A}T$, the following statements are equivalent.

- The tree $T$ is nongeometric.
- There exists a nonstationary direct system of trees $(S_n)_{n \in \mathbb{N}} \in \mathcal{O}^\mathbb{N}$ that converges strongly to $T$.
- For every finite subset $F \subseteq G$, there exists $S \in \mathcal{O} \setminus \{T\}$ with a morphism $S \to T$ such that $\|g\|_S = \|g\|_T$ for all $g \in F$.

**Proof.** The equivalence between the first two statements follows from [34] as mentioned above. The second assertion easily implies the third.

Assume that the third assertion holds. We are first going to construct trees with morphisms $S_1 \to S_2 \to S_3 \ldots$ and morphisms $f_j : S_j \to T$ without any commutation requirement. We choose an exhaustion of $G$ by finite subsets $B_1 \subset B_2 \subset \ldots$. We start with $S_1 \in \mathcal{O}$ with any morphism $f_1 : S_1 \to T$. Assuming that $S_i$ has already be constructed, consider $F_i$ a finite subset of $G$ containing $B_i$ and a finite set of candidates for $S_i$: this means that there is a morphism $S_i \to S'$ if and only if $\|g\|_{S_i} \geq \|g\|_{S'}$ for all
By assumption, there exist $S_{i+1}$ with a morphism $f_{i+1}: S_{i+1} \to T$ such that $||g||_{S_{i+1}} = ||g||_T$ for all $g \in F_i$. Since $||g||_{S_i} \geq ||g||_T = ||g||_{S_{i+1}}$ for all $g \in F_i$, there exists a morphism $S_i \to S_{i+1}$, and our inductive construction is complete.

In order to make the diagram commutative, define $f_j^i: S_i \to T$ for $i \leq j$ as the composition of the map $S_i \to S_j$ with $f_j: S_j \to T$. For every $j$, the following diagram commutes by construction:

By diagonal extraction, one can find a subsequence such that for every $i$, $f_j^i$ converges to $f_i^\infty$ (in the topology on the set of morphisms) and the following infinite diagram commutes:

This gives us a direct system of trees $(S_i)_{i \in \mathbb{N}}$ with morphisms to $T$ such that for every finite subset $F \subset G$, there exists $i$ such that $||g||_{S_i} = ||g||_T$ for all $g \in F$. It easily follows that $S_i$ converges strongly to $T$.

**Skeleton of a geometric tree.** Recall that a transverse family $Y$ in a tree $T \in \mathfrak{O}$ is a $G$-invariant collection of nondegenerate subtrees such that any two distinct subtrees in the collection intersect in at most one point. A tree $T \in \mathfrak{O}$ is indecomposable if it does not admit any transverse family of non-degenerate subtrees apart from the trivial one (i.e. $Y = \{T\}$). A transverse family in $T$ is a transverse covering if in addition, every segment in $T$ is covered by finitely many subtrees $Y_1, \ldots, Y_n$ in $Y$ with $Y_i \cap Y_{i+1} \neq \emptyset$ (see [14, Definition 4.6]). The skeleton of a transverse covering of $T$ is the bipartite simplicial $G$-tree $S$ having one vertex $v_Y$ for each subtree $Y$ in the family (the stabilizer of $v_Y$ in $S$ is the stabilizer of $Y$ in $T$), and one vertex $v_x$ for each point $x \in T$ belonging to at least two subtrees of the family (the stabilizer of $v_x$ in $S$ is the stabilizer of $x$ in $T$); the vertex $v_x$ is joined to $v_Y$ by an edge if and only if $x \in Y$.

Every geometric tree with dense orbits has a unique transverse covering by a family $\mathcal{Y}$ of non-degenerate indecomposable subtrees with finitely generated stabilizers, see e.g. [15, Proposition 1.25]. Moreover, each $Y \in \mathcal{Y}$ is itself dual to a band complex $\Sigma_Y$ such that $\Sigma_Y/G_Y$ is a finite band complex where every leaf is dense. The skeleton of $T$ is the skeleton of this transverse covering. The band complexes $\Sigma_Y$ are called the minimal components of $\Sigma$.

### 2.4 Boundary amenability

We now review a construction, due to Ozawa, that allows for an inductive argument to prove that a discrete countable group $\Gamma$ is boundary amenable. We first recall the defini-
tion of topological amenability of a group action on a compact space. The space $\text{Prob}(\Gamma)$ of probability measures on $\Gamma$ is equipped with the topology of pointwise convergence, or equivalently, the subspace topology from $\ell^1(\Gamma)$.

**Definition 2.6** (Topologically amenable action). Let $\Gamma$ be a group. A continuous $\Gamma$-action on a compact Hausdorff space $X$ is topologically amenable if there exists a sequence of continuous maps 

$$\mu_n : X \to \text{Prob}(\Gamma)$$

such that for all $\gamma \in \Gamma$, one has 

$$\sup_{x \in X} ||\mu_n(\gamma.x) - \gamma.\mu_n(x)||_1 \to 0$$

as $n$ goes to $+\infty$.

The group $\Gamma$ is boundary amenable (or exact) if it admits a topologically amenable action on a compact Hausdorff space.

In particular, the trivial action of $\Gamma$ on a point is topologically amenable if and only if $\Gamma$ is amenable. There is a similar notion in the Borel category, see [28, Definition 2.11].

**Definition 2.7** (Borel amenable action). A $\Gamma$-action on a Borel space $X$ is Borel amenable if there exists a sequence of Borel maps 

$$\mu_n : X \to \text{Prob}(\Gamma)$$

such that for all $\gamma \in \Gamma$ and all $x \in X$, one has 

$$||\mu_n(\gamma.x) - \gamma.\mu_n(x)||_1 \to 0$$

as $n$ goes to $+\infty$.

For a continuous action on a compact Hausdorff space, the two definitions are equivalent (see [43] or Remark 2.9 below).

The inductive procedure for proving that a group is exact.

**Proposition 2.8.** (Ozawa [41, Proposition 11]) Let $\Gamma$ be a countable group, let $X,Y$ be two compact Hausdorff spaces equipped with continuous $\Gamma$-actions, and let $K$ be a countable discrete space equipped with a $\Gamma$-action. Assume that there exists a sequence of Borel maps 

$$\mu_n : X \to \text{Prob}(K)$$

such that for all $\gamma \in \Gamma$ and all $x \in X$, one has 

$$||\mu_n(\gamma.x) - \gamma.\mu_n(x)||_1 \to 0$$

as $n$ goes to $+\infty$. Assume in addition that for all $k \in K$, the restriction of the $\Gamma$-action on $Y$ to the stabilizer $\text{Stab}(k) \subseteq \Gamma$ is a topologically amenable $\text{Stab}(k)$-action. Then the $\Gamma$-action on the compact space $X \times Y$ is topologically amenable, in particular $\Gamma$ is exact.
Remark 2.9. The proposition implies that if $X$ is compact Hausdorff, and $\Gamma \curvearrowright X$ is Borel amenable then it is topologically amenable by taking $K = \Gamma$ and $Y$ a point.

Proof of Proposition 2.8. The statement given in [41, Proposition 11] requires the convergence
\[
||\mu_n(\gamma.x) - \gamma.\mu_n(x)||_1 \to 0
\]
to be uniform in $x$, but the proof only requires that
\[
\int_X ||\mu_n(\gamma.x) - \gamma.\mu_n(x)||_1 dm(x) \to 0
\]
for all probability measures $m$ on $X$ and all $\gamma \in \Gamma$, see also [6, Proposition 5.2.1]. By noticing that $||\mu_n(\gamma.x) - \gamma.\mu_n(x)||_1 \leq 2$ and using Lebesgue’s dominated convergence theorem, this hypothesis can be replaced by pointwise convergence.

The following consequence of Proposition 2.8 is established in [29, Proposition C.1], by noticing that the restricted action of any exact subgroup of $\Gamma$ on the Stone–Čech compactification $\beta \Gamma$ is topologically amenable.

Corollary 2.10. (Ozawa, Kida [29, Proposition C.1]) Let $\Gamma$ be a countable group, let $X$ be a compact Hausdorff space equipped with a continuous $\Gamma$-action, and let $K$ be a countable space equipped with a $\Gamma$-action. Assume that there exists a sequence of Borel maps
\[
\mu_n : X \to \text{Prob}(K)
\]
such that for all $\gamma \in \Gamma$ and all $x \in X$, one has
\[
||\mu_n(\gamma.x) - \gamma.\mu_n(x)||_1 \to 0
\]
as $n$ goes to $+\infty$. Assume in addition that for all $k \in K$, the stabilizer $\text{Stab}(k) \subseteq \Gamma$ is exact.

Then $\Gamma$ is exact.

Arguing as in [41, Proposition 11] without the hypothesis that $X$ is compact, one also gets that the analogous statement holds in the context of Borel actions. In particular, we get the following.

Proposition 2.11. Let $\Gamma$ be a countable group, let $X$ be a space equipped with a continuous $\Gamma$-action, and let $K$ be a countable space equipped with a $\Gamma$-action. Assume that there exists a sequence of Borel maps
\[
\mu_n : X \to \text{Prob}(K)
\]
such that for all $\gamma \in \Gamma$ and all $x \in X$, one has
\[
||\mu_n(\gamma.x) - \gamma.\mu_n(x)||_1 \to 0
\]
as $n$ goes to $+\infty$. Assume in addition that for all $k \in K$, the stabilizer $\text{Stab}(k) \subseteq \Gamma$ is amenable.

Then the $\Gamma$-action on $X$ is Borel amenable.
Stability under subgroups. The following well-known fact says that boundary amenability is stable under passing to subgroups. It can be viewed as a consequence of Ozawa’s result (Proposition 2.8), applied with $K = \Gamma$ and $Y$ a point.

Corollary 2.12. Let $\Gamma$ be a countable group, let $\Delta \subseteq \Gamma$ be a subgroup, and let $X$ be a compact Hausdorff space equipped with a topologically amenable $\Gamma$-action. Then the $\Delta$-action on $X$ obtained by restriction of the $\Gamma$-action is topologically amenable, so in particular $\Delta$ is exact. In particular, every point stabilizer for the $\Gamma$-action on $X$ is amenable.

Remark 2.13. The same statement applies in the Borel category. In particular, if the $\Gamma$-action on $X$ is Borel amenable, then every point stabilizer in $\Gamma$ is amenable.

Stability under extensions. This was proved by Kirchberg–Wassermann in [30].

Proposition 2.14 (Kirchberg–Wassermann [30]). Any extension of two countable exact groups is exact.

If one wants to keep track of spaces, one can use Ozawa’s result to prove the following.

Corollary 2.15. Let

$$1 \to H \to \Gamma \to Q \to 1$$

be a short exact sequence of countable groups. Let $X$ be a compact Hausdorff space equipped with a continuous $Q$-action, and $Y$ be a compact Hausdorff space equipped with a continuous $\Gamma$-action. Assume that the action $Q \curvearrowright X$ and the restricted action $H \curvearrowright Y$ are topologically amenable. Then the $\Gamma$-action on $X \times Y$ is topologically amenable (where the $\Gamma$-action on $X$ is the one factoring through $Q$).

Proof. Since the $Q$-action on $X$ is topologically amenable, there exists a sequence of Borel maps

$$\mu_n : X \to \text{Prob}(Q)$$

such that for all $x \in X$ and all $\gamma \in \Gamma$, we have

$$||\mu_n(\gamma.x) - \gamma.\mu_n(x)||_1 \to 0$$

as $n$ goes to $+\infty$. The corollary then follows from Proposition 2.8 applied to $K = Q$.

Stability under finite-index overgroups. The following well-known easy fact implies that boundary amenability is a commensurability invariant; its proof is left to the reader.

Proposition 2.16. Let $\Gamma$ be a group, and let $X$ be a compact Hausdorff space equipped with a continuous $\Gamma$-action. Let $\Gamma^0$ be a finite index subgroup of $\Gamma$, such that the $\Gamma^0$-action on $X$ is topologically amenable. Then the $\Gamma$-action on $X$ is topologically amenable.
Quotient by an amenable subgroup.

**Proposition 2.17** (Nowak [37]). Let $G$ be a countable group, and $N \triangleleft G$ an amenable normal subgroup. Then $G$ is exact if and only if $G/N$ is exact. In particular, $G$ is exact if and only if $G/Z(G)$ is exact.

**Proof.** One implication follows from stability under extension. The other implication was proved in [37]. This is stated for $G$ finitely generated but extends to the general case since exactness is closed under increasing unions. \[\square\]

3 Scheme of the proof

3.1 Turning classes and the factorization lemma

The following definition, which is central in the present work, was inspired by a preprint of Los–Lustig [35] (where it is called a blow-up class). Recall that a turn at a point $x$ in a tree $T$ is a pair $(d,d')$ where $d,d'$ are two distinct directions at $x$.

**Definition 3.1** (Turning class). Let $T \in \mathcal{O}$. A turning class in $T$ is a $G$-invariant collection of turns at the branch points of $T$.

An important example of turning classes is the following. Given a morphism $f : S \to T$, recall that a subtree $Y \subseteq S$ is legal if $f$ is an isometry when restricted to $Y$. A turn $(d,d')$ in $S$ based at $x \in S$ is legal if there exist small intervals $[x,y],[x,y']$ representing respectively the directions $d,d'$ such that the interval $[y,x] \cup [x,y']$ is legal.

**Definition 3.2** (Turning class of a morphism). Let $S \in \mathcal{O}$, let $T \in \mathcal{O}$, and let $f : S \to T$ be an optimal morphism. The turning class of $f$ is the collection of all orbits of turns $(d,d')$ at branch points of $T$ such that there exists a legal segment $I \subseteq S$ whose $f$-image crosses the turn $(d,d')$.

The first important ingredient in our proof of boundary amenability of Out($G,F(t)$) is the following factorization lemma.

**Lemma 3.3** (Factorization lemma). Let $S,S' \in \mathcal{O}$, let $T \in \mathcal{O}$ with trivial arc stabilizers, and let $f : S \to T$ and $f' : S' \to T$ be optimal morphisms. Assume that $f$ and $f'$ have the same turning class. Then there exists $\varepsilon > 0$ such that whenever $U \in \mathcal{O}$ is a tree such that $f'$ factors through $U$, and such that the induced morphism $f'_U : U \to T$ has BBT smaller than $\varepsilon$, then $f$ also factors through $U$.

**Remark 3.4.** It is actually enough to assume that the turning class of $f$ is contained in the turning class of $f'$.

**Remark 3.5.** The hypothesis $BBT(f'_U) < \varepsilon$ can be replaced by the assumption that the volume of the quotient graph $U/G$ is at most $\varepsilon$: indeed, the Lipschitz constant of the morphism $f'_U$ is equal to 1, so we have $BBT(f'_U) \leq \text{vol}(U/G)$ by [5, Lemma 3.1] or [26, Proposition 3.12].
Proof. We say that a nondegenerate legal segment $I \subseteq S$ is $S'$-liftable if its image in $T$ lifts isometrically to $S'$, i.e. there exists an $f'$-legal segment $I' \subseteq S'$ such that $f(I) = f'(I')$.

Claim 1: There exists $\varepsilon_0 > 0$ such that

- every point in $S$ is the midpoint of an $S'$-liftable legal segment of length $2\varepsilon_0$,
- for every vertex $v$ of finite valence in $S$, every legal segment of length $2\varepsilon_0$ centered at $v$ is $S'$-liftable,
- for every vertex $v$ of infinite valence in $S$ and every legal interval $I$ of length $\varepsilon_0$ with endpoint $v$, there exists a nontrivial element $g \in G_v$ such that $I \cup gI$ is $S'$-liftable (here $G_v$ is the stabilizer of the vertex $v$).

Proof: We first observe that for every $x \in S$, every nondegenerate legal segment $I$ in $S$ centered at $x$ contains a nondegenerate $S'$-liftable subsegment centered at $x$. Indeed, the turn in $T$ defined by the image of the two directions at $x$ in $I$ is either based at a point of valence 2 in $T$, or else it is contained in the turning class of $f$, hence of $f'$. So there exists a neighborhood $V_x \subseteq I$ of $x$ in $I$ whose image in $T$ lifts isometrically to $S'$.

The second assertion of the claim follows immediately since there are only finitely many turns involved. To prove the third, let $e$ be an edge incident on a vertex $v$ of infinite valence in $S$. Let $g \neq 1$ be an arbitrary element of $G_v$. The turn $(e, g e)$ is legal because $T$ has trivial arc stabilizers. Applying the above observation shows that there exists $\varepsilon_0 > 0$ such that the $\varepsilon_0$-neighborhood of $v$ in the segment $e \cup g e$ is $S'$-liftable. The third assertion follows because there are only finitely many orbits of edges.

We now prove the first assertion. Let $l$ be the minimal length of an edge in $S$. Since $f$ is optimal, we can find legal segments $J_1, \ldots, J_k$ obtained by concatenation of two edges of $S$ such that the orbit of every point in $S$ intersects some $J_i$ in a point at distance at least $l/10$ from $\partial J_i$. We are going to use a compactness argument in the disjoint union $J_1 \sqcup \ldots \sqcup J_k$. For every $i \in \{1, \ldots, k\}$ and every $x \in J_i$, the observation made in the first paragraph of the proof provides a neighborhood $V_x \subseteq I$ of $x$ in $I$ whose image in $T$ lifts isometrically to $S'$.

Let now $U \in \mathcal{O}$ be a tree such that there exist morphisms $f'_{SU} : S' \to U$ and $f'_U : U \to T$, with $BBT(f'_U) \leq \varepsilon_0/100$, such that the following diagram commutes:

$$
\begin{array}{ccc}
S' & \xrightarrow{f'_{SU}} & U \\
\quad & \searrow \quad & \nearrow \\
& T & 
\end{array}
$$

We are going to define a map $f_{SU}$ from $S$ to $U$ in the following way. We first note that any $S'$-liftable segment is $U$-liftable. Now given $x \in S$, choose $I$ a $U$-liftable legal
segment of length $2\varepsilon_0$ centered at $x$, and choose a lift $\tilde{I}$ of $f(I)$ in $U$ (which naturally comes with an isometry $j : I \to \tilde{I}$); we want to map $x$ to the midpoint of $\tilde{I}$. We will prove that this definition is independent of choices, and that this defines an optimal morphism from $S$ to $U$. It will then be obvious that $f$ factors through this map.

**Claim 2:** If $I \subseteq S$ is an $S'$-liftable segment of length greater than $2\varepsilon_0/100$, and if $\tilde{I}_1$ and $\tilde{I}_2$ are two lifts of $f(I)$ in $U$, then the isometries $j_1 : I \to \tilde{I}_1$ and $j_2 : I \to \tilde{I}_2$ coincide on the complement of the $\varepsilon_0/100$-neighborhood of the endpoints of $I$.

**Proof:** Note that for all $x \in I$, the points $j_1(x)$ and $j_2(x)$ have the same $f_U'$-image in $T$. Write $I = [a, b]$. Let $x \in I$ be a point at distance greater than $\varepsilon_0/100$ from $\partial I$. For all $i \in \{1, 2\}$, we denote by $a_i, b_i, x_i$ the $j_i$-images of $a, b, x$. Let us prove that $x_1 = x_2$. If $x_1 \neq x_2$, then $x_1 \notin \tilde{I}_2$ and $x_2 \notin \tilde{I}_1$ because $x_1$ and $x_2$ have the same image in $T$, and both segments $\tilde{I}_i$ embed in $T$. It follows that $x_1$ lies in $[a_1, a_2]$ or $[b_1, b_2]$ (say $[a_1, a_2]$): indeed, if $[a_1, a_2] \cup [b_1, b_2]$ differs from $K = \text{Hull}(a_1, a_2, b_1, b_2)$, then the complement of $[a_1, a_2] \cup [b_1, b_2]$ in $K$ is contained in $[a_1, b_1] \cap [a_2, b_2]$, which cannot contain $x_1$. However $f_U'(a_1) = f_U'(a_2)$, while $d_T(f_U'(x_1), f_U'(a_1)) = d_U(x_1, a_1) > BBT(f_U')$ by assumption, contradicting the definition of the BBT.

**Claim 3:** Let $v$ be a vertex of infinite valence in $S$, and let $\tilde{v}$ be the point fixed by $G_v$ in $U$. Let $I \subseteq S$ be a legal segment that contains $v$, and such that $v$ is at distance at least $2\varepsilon_0/100$ from $\partial I$. Then any lift of $I$ to $U$ passes through $\tilde{v}$.

**Proof:** Write $I = I_1 \cup I_2$, where $I_1$ and $I_2$ are two segments of length at least $2\varepsilon_0/100$ with endpoint $v$. By Claim 1, there exists an element $g_1 \neq 1$ in $G_v$ such that $J_1 \cup g_1 I_1$ is $S'$-liftable (hence $U$-liftable); let $I_1' := g_1 I_1$, and $\tilde{J}_1$ be a lift of $I_1 \cup I_1'$ in $U$ with an isometry $j_1 : I_1 \cup I_1' \to \tilde{J}_1$, and let $\tilde{I}_1 = j_1(I_1), \tilde{I}_1' = j_1(I_1')$. Denote by $m$ the midpoint of $\tilde{I}_1$. Notice that $g_1 \tilde{I}_1$ is also a lift of $I_1' = g_1 I_1$ in $U$, so by Claim 2 it has the same midpoint as $I_1'$. In other words $g_1$ sends the midpoint of $I_1$ to the midpoint of $I_1'$. Therefore, the common endpoint of $\tilde{I}_1$ and $\tilde{I}_1'$ (which is also the midpoint of $[m, g_1 m]$) is equal to $\tilde{v}$. This shows that $I_1$ has a lift in $U$ with endpoint $\tilde{v}$, and by symmetry $I_2$ also has a lift in $U$ with endpoint $\tilde{v}$. Their concatenation gives a lift $\tilde{I}$ of $I$ in $U$ that contains $\tilde{v}$, and such that $\tilde{v}$ is at distance at least $2\varepsilon_0/100$ from $\partial I$. Claim 2 then implies that all lifts of $I$ in $U$ pass through $\tilde{v}$, so Claim 3 is proved.

Let now $x \in S$, and let $I_1$ and $I_2$ be two segments of length $\varepsilon_0$ centered at $x$. We are left showing that if $j_1 : I_1 \to \tilde{I}_1$ and $j_2 : I_2 \to \tilde{I}_2$ are lifts of $I_1$ and $I_2$ in $U$, then $j_1(x) = j_2(x)$. The fact that $f_{SU}$ is locally isometric when restricted to legal segments, hence an optimal morphism, will then follow from Claim 2. From now on, we will assume that $\varepsilon_0$ has been chosen smaller than the lengths of the edges of $S$ and small enough so that if two directions $d_1, d_2$ form an illegal turn at a vertex $v \in S$, then there are subsegments $[v, x_1] \subseteq d_1$ and $[v, x_2] \subseteq d_2$ of length at least $\varepsilon_0$ that have the same image in $T$: this is possible because there are only finitely many orbits of illegal turns in $S$, otherwise $T$ would have nontrivial arc stabilizers.

If $x$ is at distance at least $\varepsilon_0/100$ from all branch points of $S$, then $I_1$ and $I_2$ contain
a common subsegment of length $2\varepsilon_0/100$ centered at $x$, and it follows from Claim 2 that all lifts $j_1 : I_1 \rightarrow \hat{I}_1$ and $j_2 : I_2 \rightarrow \hat{I}_2$ in $U$ have the same midpoint.

We can thus assume that $x$ is at distance at most $\varepsilon_0/100$ from some branch point $v$ of $S$. We write $I_1 = J_1 \cup J'_1$ with $J_1 \cap J'_1 = \{v\}$, and $x \in J_1$. Similarly, we write $I_2 = J_2 \cup J'_2$ with $J_2 \cap J'_2 = \{v\}$, and $x \in J_2$ (in particular $J_1 = J_2$). If the tripod formed by $J_1, J'_1$ and $J'_2$ is illegal, then the hypothesis we made on $I_1$ and $I_2$ have the same image in $T$, and hence they have common lifts in $U$. The result then follows as above from Claim 2. We now assume that this tripod is legal.

For all $i \in \{1, 2\}$, denote by $m_i, m'_i$ the midpoints of $J_i, J'_i$ respectively, and let $\tilde{m}_i, \tilde{m}'_i$ be their images in $\hat{I}_i$. Since all three segments $J_1, J'_1$ and $J'_2$ have length at least $2\varepsilon_0/100$, Claim 2 implies that for all $i \in \{1, 2\}$, the lifts $\tilde{m}_i$ and $\tilde{m}'_i$ do not depend on the choice of a lift $\hat{I}_i$ of $I_i$, and in addition $\tilde{m}_1 = \tilde{m}_2$. Denote by $\hat{c}$ the center of the tripod $[\tilde{m}_1, \tilde{m}'_1, \tilde{m}'_2]$ in $U$. To complete the proof, it is enough to check that $d_U(\tilde{m}_1, \hat{c}) \geq d_S(m_1, v)$, because this implies that the images of $x$ in $\hat{I}_1$ and $\hat{I}_2$ are the same. If $v$ has finite valence, this follows from the second assertion of Claim 1 which implies that $J'_1 \cup J'_2$ is also $U$-liftable and contains $[m'_1, m'_2]$, so that $(\tilde{m}_1, \tilde{m}'_1, \tilde{m}'_2)$ is an isometric lift of $(m_1, m'_1, m'_2)$. If $v$ has infinite valence, then this is a consequence of Claim 3, showing that $\hat{I}_1$ and $\hat{I}_2$ both contain the point $\hat{v}$ of $U$ fixed by $G_v$ so $\hat{v} \in [\tilde{m}_1, \hat{c}]$. □

3.2 Probability measures associated to a morphism

Given a morphism $f : S \rightarrow T$ from a simplicial metric tree $S \in \mathcal{O}$ to an $\mathbb{R}$-tree $T \in \mathcal{AT}$, and a real number $t$, we let $S_t(f)$ be the collection of all simplices $\Delta \in \text{Simp}$ such that there exists a tree $U \in \mathcal{O}$ of covolume $e^{-t}$, whose image in $\mathcal{PO}$ is contained in $\Delta$, such that $f$ factors through $U$. The goal of the present section is to associate to the morphism $f$ a sequence of probability measures $\mu_n(f)$ on Simp, obtained by averaging uniform measures on the sets $S_t(f)$. The key proposition we need to establish is the following.

Proposition 3.6. Let $T \in \mathcal{AT}$, let $S \in \mathcal{O}$, and let $f : S \rightarrow T$ be a morphism. Then for all $t \in \mathbb{R}$, the set $S_t(f)$ is finite.

Before proving the proposition, we will prove two preliminary lemmas. Given a tree $S \in \mathcal{O}$, we define the systole of $S$ as the smallest translation length in $S$ of a nonperipheral element of $G$.

Lemma 3.7. Let $T \in \overline{\mathcal{O}}$ be a tree with trivial arc stabilizers. Let $f : S \rightarrow T$ be a morphism, and let $(U_i)_{i \in \mathbb{N}} \in \mathcal{O}^{\mathbb{N}}$ be a sequence of trees of systole at least $\lambda$ such that $f$ factors through all trees $U_i$. Then the set of simplices in Simp to which the trees $U_i$ project is finite.

Proof. Assume by contradiction that the set of simplices spanned by $\{U_i\}_{i \in \mathbb{N}}$ is infinite. Let $f_i : S \rightarrow U_i$ be such that $f$ factors through $f_i$. Consider the $f_i$-preimage $V_i \subset S$ of the set of vertices of $U_i$.

We claim that for each edge $e$ of $S$, $\#(e \cap V_i)$ is bounded. Otherwise, there exists oriented subedges $I, J \subset e$ (bounded by points in $V_i$) that are at distance at most $\lambda/2$
from one another and whose images in $U_i$ are two oriented edges in the same orbit under some element $g_i$. Since $f_i$ is an isometry when restricted to $e$, the translation length of $g_i$ in $U_i$ is positive and at most $\lambda/2$, contradicting our the hypothesis on the systole. This proves the claim.

Up to passing to a subsequence, we can therefore assume that the combinatorics of the subdivision defined by $V_i$ does not depend on $i$. We denote by $S^0$ this combinatorial tree.

For each $i \in \mathbb{N}$, choose a pair of adjacent edgelets folded by $f_i$, and let $S^1_i$ be the combinatorial tree obtained by folding these two edgelets together. We claim that only finitely many combinatorial trees $S^1_i$ appear. Otherwise, there exist two edgelets $e, e' \subset S$ sharing a vertex $v$, and infinitely many elements $g_i \in G_v$ such that $e$ is identified with $g_i e'$. Note that the precise (metric) subsegment of $S$ corresponding to $e$ depends on $i$. We denote by $E, E'$ the edges of $S$ (for its natural set of vertices) containing $e$ and $e'$ respectively ($E$ and $E'$ do not depend on $i$). Then $f(E)$ and $f(g_i E')$ share a common subsegment, so for all $i, j \in \mathbb{N}$, the element $g_i g_j^{-1}$ fixes an arc in $T$, a contradiction.

Thus, up to extracting a subsequence, we can assume that $S^1_i = S^1$ does not depend on $i$ (as a combinatorial tree). We now assume that we have constructed by induction combinatorial trees $S^1, S^2, \ldots, S^k$ such that for all $j < k$, the tree $S^{j+1}$ is obtained from $S^j$ by folding two edgelets, and such that for each $i$, $f_i$ factors through these folds. A coherent set of metrics on $S^0, S^1, \ldots, S^k$ is a metric for each $S^j$ such that $S^0$ becomes isometric to $S$, and each $S^j \to S^{j+1}$ is a morphism. By induction we will also have that for each $i$ there exists a coherent set of metrics such that $f_i : S \to U_i$ factors through $S^1, \ldots, S^k$, hence so does $f : S \to T$.

To construct $S^{k+1}$, for each $i \in \mathbb{N}$, choose a pair of adjacent edgelets of $S^k$ folded by $f_i$ (they exist for infinitely many $i$ because we assume that the set of simplices spanned by $\{U_i\}_{i \in \mathbb{N}}$ is infinite). For each $i$, let $S^{k+1}_i$ be the combinatorial tree obtained by folding these two edgelets together. As above, we claim that only finitely many combinatorial trees $S^{k+1}_i$ appear. Indeed, if not, there exist two edgelets $e, e' \subset S^k$ sharing a vertex $v$, and infinitely many elements $g_i \in G_v$ such that $e$ is identified with $g_i e'$. Let $\tilde{E}, \tilde{E}'$ be natural edges of $S$ containing a preimage of $e$ and $e'$ respectively. Then $f(\tilde{E})$ and $f(g_i \tilde{E}')$ share a common subsegment. Since $g_i \in G_v$ and arc stabilizers of $T$ are trivial, there are at most 4 indices $i$ such that $f(\tilde{E}) \cap g_i f(\tilde{E}')$ is non-degenerate, a contradiction. This proves the claim and allows to construct $S^{k+1}$.

This allows to construct an infinite sequence of trees $S^1, \ldots, S^k, \ldots$. Since $S^j$ has fewer orbits of edgelets than $S^{j-1}$, this is a contradiction. This completes the proof of the lemma.

Lemma 3.8. There exist $K, K' > 0$ such that for every $\lambda > 0$, the following holds.
Let $U \in \mathcal{O}$ be a tree of systole at least $\lambda > 0$, and let $f : U \to U'$ be a morphism with $\text{vol}(U'/G) \geq \text{vol}(U/G) - \frac{\lambda}{K'}$.
Then the systole of $U'$ is at least $\lambda/K'$.

Proof. Let $g \in G$ be an element which is hyperbolic in $U'$. Its axis $l$ in $U$ contains an edge of length at least $\lambda/4N$, where $N$ is the number of orbits of edges of $U$. Otherwise, $l$
We take for $v$ we let $S$ projecting to $\Delta$ numbers below by the length function of $i$. Therefore, up to diagonal extraction of a subsequence, we can assume that for each $i$, our inductive construction.

Lemma 3.8 implies that the systole of the trees in $\Delta$, real numbers $v_i > e^{-t}, \lambda_i > 0$, a set of trees $U_i \subseteq U_{i-1}$, and for each $U \in U_i$, a tree $S_{U,i}$ projecting to $\Delta_i$ and a morphism $f_{U,i} : S_{U,i} \to U$ such that the following holds:

- $\text{Simp}(U_i)$ is infinite,
- $\text{vol}(S_{U,i}/G) \leq v_i$ and $\text{systole}(S_{U,i}) \in [\lambda_i, 2\lambda_i]$ for all $U \in U_i$,
- $f_{U,i} : S_{U,i-1} \to U$ factors through $f_{U,i} : S_{U,i} \to U$.

We take for $v_0$ and $\lambda_0$ the covolume and the systole of $S$.

Let $v_{i+1} = v_i - \frac{\lambda_i}{K},$ where $K$ is the integer from Lemma 3.8. If $e^{-t} \geq v_{i+1}$, then Lemma 3.8 implies that the systole of the trees in $U_i$ is bounded from below. Thus Lemma 3.7 implies that $\text{Simp}(U_i)$ is finite, a contradiction.

Thus $e^{-t} < v_{i+1}$, so for each $U \in U_i$, one can choose a tree $S_{U,i+1}$ of covolume $v_{i+1}$ through which $f_{U,i}$ factors, and let $f_{U,i+1} : S_{U,i+1} \to U$ be the induced morphism. Lemma 3.8 implies that the systole of the trees $S_{U,i+1}$ is bounded from below as $U$ varies in $U_i$. Lemma 3.7 shows that $\text{Simp}(S_{U,i+1})$ is finite. Let $\Delta_{i+1}$ be a simplex such that $S_{U,i+1} \in \Delta_{i+1}$ for all trees $U$ in a set $U_{i+1}' \subset U_i$ with $\text{Simp}(U_{i+1}')$ infinite. By the pigeonhole principle, there exists $\lambda_{i+1} > 0$ such that the set $U_{i+1}'$ of trees $U \in U_{i+1}'$ such that $\text{systole}(S_{U,i+1}) \in [\lambda_{i+1}, 2\lambda_{i+1}]$ is such that $\text{Simp}(U_{i+1})$ is infinite. This concludes our inductive construction.

Since $v_{i+1} = v_i - \frac{\lambda_i}{K}$ and $v_i \geq e^{-t}$ for all $i$, we have $\lambda_i \to 0$. Consider for each $i \in \mathbb{N}, \ U_i \in U_i$. The tree $S_{U,i}$ is defined for $i \geq j$. Its length function is bounded from below by the length function of $T$, and bounded from above because $\text{vol}(S_{U,i}/G) \leq v_j$.

Therefore, up to diagonal extraction of a subsequence, we can assume that for each $j$, $S_{U_{i,j}}$ converges to a tree $S_{\infty,j} \in \overline{O}$. By continuity of the systole and the covolume on the closure of $\Delta_j$, we have $\text{systole}(S_{\infty,j}) \leq 2\lambda_j$, and $\text{vol}(S_{\infty,j}/G) \geq e^{-t}$. There are 1-Lipschitz maps $S_{\infty,j} \to S_{\infty,j+1}$ and $S_{\infty,j} \to T$. The trees $S_{\infty,j}$ converge to some
tree $S_{\infty,\infty}$ as $j \to \infty$ and there is a 1-Lipschitz map $S_{\infty,\infty} \to T$. By semi-continuity of the volume, we have $\text{vol}(S_{\infty,\infty}/G) \geq e^{-t}$, so Lemma 2.3 implies that $S_{\infty,\infty}$ is a Grushko tree. In particular, the systole of $S_{\infty,\infty}$ is positive and bounds $2\lambda_j$ from below, a contradiction. \hfill \Box

Remark 3.9. Notice that the same proof would also have worked if in the definition of $S_t(f)$, we had replaced the condition that the covolume of $U$ is equal to $e^{-t}$ by the condition that this covolume is at least $e^{-t}$.

Notice also that for all $t$ sufficiently large (so that the source of $f$ has covolume at least $e^{-t}$), the set $S_t(f)$ is nonempty.

Fix an arational tree $T \in \mathcal{AT}$ and $f \in \text{Opt}_{\to T}$ an optimal morphism with range $T$. For all $t > 0$ such that $S_t(f)$ is nonempty, we let $\nu_t(f)$ be the uniform probability measure on the finite set $S_t(f)$. For all $n \in \mathbb{N}$, we then define a probability measure $\mu_n(f)$ by letting

$$\mu_n(f) := \frac{1}{n} \int_n^{2n} \nu_t(f) d\text{Leb}(t)$$

if $S_{e^{-n}}(f) \neq \emptyset$, and otherwise we just let $\mu_n(f)$ be some fixed probability measure $\mu_0$ on $\text{Simp}$. As a consequence of the factorization lemma (Lemma 3.3, see also Remark 3.5), if $f, g \in \text{Opt}_{\to T}$ have the same turning class, then $S_t(f) = S_t(g)$ for all sufficiently large $t$, and therefore we obtain the following fact.
Corollary 3.10. Let $T \in \AT$ be a rational tree. Let $f, g \in \Opt_{\to T}$ be two optimal morphisms with range $T$ and having the same turning class. Then $\mu_n(f) = \mu_n(g)$ for all sufficiently large $n \in \mathbb{N}$. \hfill $\Box$

In the sequel, in order to have an action of $\Out(G, F)$ on a compact space, we will need to work with projective classes of trees instead of isometry classes of trees. The next lemma will ensure that the measures $\mu_n$ do not depend too strongly on a choice of representative in a projective class. Given a tree $T \in \mathcal{O}$ and $\lambda > 0$, we denote by $\lambda T$ the tree obtained from $T$ by dilating the metric by $\lambda$. Given an optimal morphism $f : S \to T$, we let $\lambda f : \lambda S \to \lambda T$ be the corresponding morphism.

Lemma 3.11. Let $T \in \AT$, $f \in \Opt_{\to T}$, and let $\lambda > 0$. Then $||\mu_n(f) - \mu_n(\lambda f)||_1 \to 0$ as $n$ goes to $+\infty$.

Proof. We have

$$\mu_n(\lambda f) = \frac{1}{n} \int_n^{2n} \nu_t(\lambda f) d\text{Leb}(t)$$

$$= \frac{1}{n} \int_n^{2n} \nu_{t+\log \lambda}(f) d\text{Leb}(t).$$

Therefore

$$\mu_n(\lambda f) - \mu_n(f) = \frac{1}{n} \int_{2n}^{2n+\log \lambda} \nu_t(f) d\text{Leb}(t') - \frac{1}{n} \int_n^{n+\log \lambda} \nu_t(f) d\text{Leb}(t'),$$

so

$$||\mu_n(f) - \mu_n(\lambda f)||_1 \leq \frac{2|\log \lambda|}{n}$$

and the conclusion follows. \hfill $\Box$

As a consequence of Corollary 3.10 and Lemma 3.11, we obtain the following fact.

Corollary 3.12. Let $T \in \AT$ and consider $f, g \in \Opt_{\to T}$ two morphisms having the same turning class, and let $\lambda > 0$. Then $||\mu_n(f) - \mu_n(\lambda g)||_1 \to 0$ as $n$ goes to $+\infty$. \hfill $\Box$

Lemma 3.13. For every $n \in \mathbb{N}$, the map $\mu_n : \Opt_{\to \AT} \to \Prob(\text{Simp})$ is Borel.

Proof. Given $\tau \in \text{Simp}$, let $Z_\tau \subset \mathbb{R} \times \Opt_{\to \AT}$ be defined by

$$Z_\tau = \{(t, f) \in \mathbb{R} \times \Opt_{\to \AT} \mid \tau \in \mathcal{S}_t(f)\}.$$

It suffices to prove that $Z_\tau$ is Borel since this implies that the map

$$\nu : \mathbb{R} \times (\Opt_{\to \AT}) \to \Prob(\text{Simp})$$

sending $(t, f)$ to $\nu_t(f)$ is Borel, and the lemma follows.
Given a simplex $\sigma \in \text{Simp}$ and $\varepsilon, v > 0$, denote by $\tilde{\sigma}_{v,\varepsilon}$ the subset of unprojectivized outer space $O$ defined as the closure of the set of trees projecting to $\sigma$, with covolume at most $v$, and systole at least $\varepsilon$. This is a compact set and the space $\text{Opt}_{\tilde{\sigma}_{v,\varepsilon}}$ of all optimal morphisms $f : S \to T$ with $S \in \tilde{\sigma}_{v,\varepsilon}$ and $T$ arbitrary is compact. Indeed, the bound on the covolume bounds the length functions of $S$ and $T$ from above, and it suffices to check that the length function of $T$ cannot accumulate to 0. Now there is a finite set $F \subset G$ of non-peripheral elements such that for all $f \in \text{Opt}_{\tilde{\sigma}_{v,\varepsilon}}$, there exists some element $g \in F$ whose axis in $S$ is legal (see the proof of [20, Theorem 4.7]), so $l_T(g) = l_S(g)$ is bounded from below by the systole of $S$ which is at least $\varepsilon$.

Let us now prove that $Z_\tau$ is Borel. Since there are countably many simplices, it suffices to prove that for each $n \in \mathbb{N}$ and every $v, \varepsilon > 0$, the set

$$Z' = Z_\tau \cap \left( [-n, n] \times \text{Opt}_{\tilde{\sigma}_{v,\varepsilon} \to AT} \right) = \left\{ (t, f) \in [-n, n] \times \text{Opt}_{\tilde{\sigma}_{v,\varepsilon} \to AT} \mid \tau \in \mathcal{S}_t(f) \right\}$$

is Borel. Let $\tilde{\tau} \subset O$ be the set of trees whose projective class lies in $\tau$. Given $\delta > 0$, let $X_{\tau,\delta}$ be the subset of $[-n, n] \times \text{Opt}_{\tilde{\sigma}_{v,\varepsilon}} \times \text{Opt}_{\tilde{\sigma}_{v,\varepsilon} \to \tilde{\tau}} \times \text{Opt}_{\tilde{\tau}}$ made of all tuples $(t, f, g, h)$ such that $f = h \circ g$, the covolume of the source of $h$ equals $\varepsilon^{-t}$, and the systole of the source of $h$ is equal to or larger than $\delta$. We note that $f, g, h$ belong to the compact spaces $\text{Opt}_{\tilde{\sigma}_{v,\varepsilon}}, \text{Opt}_{\tilde{\sigma}_{v,\varepsilon} \to \tilde{\tau}}$ and $\text{Opt}_{\tilde{\tau}_{v,\varepsilon} \to \tilde{\tau}}$, respectively. Since composition is continuous, and since the covolume and systole are continuous functions on $\tilde{\tau}$, the set $X_{\tau,\delta}$ is compact. Let $X'_{\tau,\delta}$ be the projection of $X_{\tau,\delta}$ to the first two coordinates. Then

$$Z' = \bigcup_{k \in \mathbb{N}} X'_{\tau,\delta} \cap [-n, n] \times \text{Opt}_{\tilde{\tau} \to AT},$$

and therefore $Z'$ is Borel. \qed

Remark 3.14. In the case where $G = F_N$ and $\mathcal{F} = \emptyset$, if $T$ is an arational tree equipped with a free action of $F_N$, then there are only finitely many possible turning classes in $T$. By averaging the measures $\mu_n$, associated to a finite set of morphisms representing all possible turning classes of $T$ achieved by morphisms, we get a sequence of measures showing that the action of $\text{Out}(F_N)$ on the space of free arational $F_N$-trees is amenable. The goal of the rest of the paper is to understand the general case where some vertices in the trees $T$ we consider have infinite valence, which implies that $T$ has infinitely many turning classes.

3.3 Measurability considerations

This technical section gives tools to prove some measurability properties. It can be omitted in a first reading if the reader wishes to ignore all measurability considerations.

3.3.1 Measurable enumeration of directions in trees

Let $\{(g, h)\}_{i \in \mathbb{N}}$ be an enumeration of $G^2$. Given $T \in \overline{O}$, we say that $(g, h) \in G^2$ is a disjoint pair for $T$ if $g$ and $h$ are both hyperbolic in $T$, and their axes are disjoint.
Given \( n \in \mathbb{N} \) and \( T \in \overline{\mathcal{O}} \), we let \((g_n(T), h_n(T))\) be the \( n\)th pair of elements in the above enumeration that is a disjoint pair for \( T \).

**Lemma 3.15.** For all \( n \in \mathbb{N} \) and all \((g, h) \in G^2\), the set

\[
B_{n,g,h} := \{ T \in \overline{\mathcal{O}} | (g_n(T), h_n(T)) = (g, h) \}
\]

is a Borel set of \( \overline{\mathcal{O}} \).

**Proof.** It is well-known (see [10, 1.8]) that \((g, h)\) is a disjoint pair for \( T \) if and only if \( ||g||_T, \|h\|_T > 0 \) and \( ||gh||_T > \|g\|_T + \|h\|_T \), an open condition. The fact that \((g, h)\) is the \( n\)th disjoint pair in the enumeration can be expressed as a boolean combination of such open sets. Therefore \( B_{n,g,h} \) is a Borel set. \( \square \)

Given \( g \in G \) and \( T \in \overline{\mathcal{O}} \) such that \( g \) is hyperbolic in \( T \), we denote by \( C_g(T) \) the axis of \( g \) in \( T \). Given a disjoint pair \((g, h) \in G^2\) for \( T \), we define \( v_{(g,h)}(T) \) as the endpoint in \( C_g(T) \) of the bridge joining \( C_g(T) \) to \( C_h(T) \). We also define \( d_{(g,h)}(T) \) as the branch direction at \( v_{(g,h)}(T) \) pointing towards \( C_h(T) \). Given \( n \in \mathbb{N} \), we then let \( v_n(T) := v_{(g_n(T), h_n(T))}(T) \) and \( d_n(T) := d_{(g_n(T), h_n(T))}(T) \), and we let \( v'_n(T) := v_{(h_n(T), g_n(T))}(T) \). In particular, \( d_n(T) \) is the direction based at \( v_n(T) \) pointing towards \( v'_n(T) \). Notice that for any branch direction \( d \) in \( T \), there exists \( n \in \mathbb{N} \) such that \( d = d_n(T) \). Similarly, any branch point arises as \( v_n(T) \) for some \( n \in \mathbb{N} \), see [42].

**Lemma 3.16.** For all \( n, m \in \mathbb{N} \), and \( g \in G \), the maps \( T \mapsto d_T(v_n(T), gv_m(T)) \) and \( T \mapsto d_T(v_n(T), C_g(T)) \) are Borel.

**Proof.** Let \( X_n(T) := \{g_n(T), g_n(T)h_n(T), h_n(T)g_n(T)\} \). The branch point \( v_n(T) \) can be defined as the unique point in the intersection of the axes of the elements in \( X_n(T) \). It follows that

\[
d_T(v_n(T), gv_m(T)) = \max \{ d_T(C_\alpha(T), C_\beta(T)) | \alpha \in X_n(T), \beta \in X_m(T)^g \}
\]

and that

\[
d_T(v_n(T), C_g(T)) = \max \{ d_T(C_\alpha(T), C_g(T)) | \alpha \in X_n(T) \}.
\]

Since \( d_T(C_\alpha(T), C_\beta(T)) = \frac{1}{2} \max \{0, ||\alpha||_T - ||\beta||_T - ||\alpha - \beta||_T \} \), the lemma follows. \( \square \)

**Lemma 3.17.** For all \( n, m, p, k, l \in \mathbb{N} \) and all \( g \in G \), the following sets are Borel subsets of \( \overline{\mathcal{O}} \):

1. \( \{ T \in \overline{\mathcal{O}} | v_n(T) = gv_m(T) \} \),
2. \( \{ T \in \overline{\mathcal{O}} | v_n(T), v_m(T), v_p(T) \text{ are pairwise distinct and aligned in this order} \} \),
3. \( \{ T \in \overline{\mathcal{O}} | d_n(T) \text{ points towards } g v_m(T) \} \),
4. \( \{ T \in \overline{\mathcal{O}} | d_n(T) = gd_m(T) \} \),
5. \( \{ T \in \overline{\mathcal{O}} | g(d_n(T), d_m(T)) \text{ is a turn lying in the segment } [v_k(T), v_l(T)] \} \).
Proof. The first two assertions follow from Lemma 3.16. For the third one, we note that $d_n(T)$ fails to point towards $gv_m(T)$ if and only if either $gv_m(T) = v_n(T)$, or else the points $v'_n(T), v_n(T), gv_m(T)$ are pairwise distinct and aligned in this order. The fourth assertion follows from the third because $d_n(T) = gd_m(T)$ if and only if $v_n(T) = gv_m(T)$ and $d_n(T)$ points towards $gv'_m(T)$. The last assertion also follows because $d_n(T) = gd_m(T)$ if and only if $v_n(T) = gv_m(T)$ and $d_n(T)$ points towards $g^{-1}v_k(T), v_l(T)$.

3.3.2 Enumerating morphisms

For every open simplex $\Delta \in \text{Simp}$, every tree in $\tilde{\Delta}$ is obtained by assigning positive lengths to the edges of some combinatorial tree $S_\Delta$. Choose a collection $v_\Delta^1, \ldots, v_\Delta^{k(\Delta)}$ of representatives of the orbits of vertices in $S_\Delta$ and view them as vertices in any $S$ in the closure of $\tilde{\Delta}$.

We define a decorated simplex as a pair $(\Delta, \theta)$, where $\Delta \in \text{Simp}$, and where $\theta : \{1, \ldots, k(\Delta)\} \to \mathbb{N}$ is a map. We denote by $\text{Simp}^*$ the countable collection of all decorated simplices. Given a tree $T \in \overline{\mathcal{O}}$ and a decorated simplex $(\Delta, \theta)$, there is a unique $S$ in the closure of $\tilde{\Delta}$, and a unique $G$-equivariant morphism $f : S \to T$ which is isometric on edges of $S$, and maps the vertex $v_\Delta^i \in S$ to the branch point $v_{\theta(i)}(T)$ in $T$ for all $i$. We denote this unique morphism $f$ by $f_{(\Delta, \theta), T}$.

Lemma 3.18. The map

$$\text{Simp}^* \times \overline{\mathcal{O}} \to \text{Mor}$$

$$(\Delta, \theta), T \mapsto f_{(\Delta, \theta), T}$$

is Borel. Its image consists of all morphisms that are isometric on edges, and send vertices to branch points.

Proof. The last statement in the lemma follows from the observation that every branch point in $T$ arises as $v_n(T)$ for some $n \in \mathbb{N}$, and a morphism which is isometric on edges is completely determined by the images of the vertices $v_\Delta^1, \ldots, v_\Delta^{k(\Delta)}$. To show that the map in the lemma is Borel, it is enough to show that for each decorated simplex $(\Delta, \theta)$, the map

$$\overline{\mathcal{O}} \to \text{Mor}$$

$$T \mapsto f_{(\Delta, \theta), T}$$

is Borel. This is because it is continuous when restricted to each of the countably many fibers of the map

$$\overline{\mathcal{O}} \to (G^2)^{k(\Delta)}$$

$$T \mapsto (g_{\theta(n)}(T), h_{\theta(n)}(T))_{1 \leq n \leq k(\Delta)}$$

and these fibers are Borel in view of Lemma 3.17. \qed
Remark 3.19. Given an enumeration of the set of decorated simplices, we obtain for each $T \in \mathcal{O}$ an enumeration of all morphisms $F_{n,T} : S_{n,T} \to T$ from all simplicial trees to $T$ that send vertices to branch points, and are isometric on edges. Moreover, the maps $T \mapsto S_{n,T}$ and $T \mapsto F_{n,T}$ are Borel for every $n$.

Given $S \in \mathcal{O}$ and $T \in \mathcal{O}$, we say that a morphism $f : S \to T$ is very optimal if $f$ is optimal and sends vertices of $S$ to branch points in $T$. We denote by $\text{VOpt}$ the subspace of $\text{Opt}$ made of very optimal morphisms.

Lemma 3.20. The preimage of $\text{VOpt}$ in $\text{Simp}^* \times \mathcal{O}$ is Borel.

Proof. It is a consequence of Lemma 3.17 that given a finite set $E$ of edges of the quotient graph $S_\Delta/G$, the collection of all $((\Delta, \theta), T) \in \text{Simp}^* \times \mathcal{O}$ such that the edges projecting to $E$ are collapsed in the source of $f_{((\Delta, \theta), T)}$ is Borel. By restricting to each of these finitely many subsets, we can assume that no edge in $S_\Delta$ is collapsed. Then $f_{((\Delta, \theta), T)}$ is very optimal if and only if for every $n \in \{1, \ldots, k(\Delta)\}$, there exist $p, q \in \{1, \ldots, k(\Delta)\}$ and $g, h \in G$ such that $gv_\Delta^p$ and $hv_\Delta^q$ are adjacent to $v_\Delta^n$ in $\Delta$, and the points $gv_{\theta(p)}(T), v_{\theta(n)}(T), hv_{\theta(q)}(T)$ are aligned in this order in $T$. The conclusion thus follows from Lemma 3.17.

3.3.3 A $\sigma$-algebra on the set of turning classes

We denote by $\mathcal{O}^\text{turn}$ the collection of all turning classes on trees in $\mathcal{O}$. There is an embedding $\Phi : \mathcal{O}^\text{turn} \rightarrow \mathcal{O} \times \{0, 1\}^{\mathbb{N}^2}$ sending $(T, T)$ to $\{(n, m) \in \mathbb{N}^2 | (d_n(T), d_m(T)) \in T\}$. This defines a $\sigma$-algebra on $\mathcal{O}^\text{turn}$ by pulling back the standard Borel $\sigma$-algebra.

The group $\text{Out}(G, \mathcal{F})$ has a natural action on $\mathcal{O}^\text{turn}$, defined as follows. Let $B$ be a turning class of finite type on a tree $T \in \mathcal{O}$, and let $\Phi \in \text{Out}(G, \mathcal{F})$. Then the tree $\Phi.T$ is the same metric space as $T$ (equipped with a twisted $G$-action), and we let $\Phi.B$ be the turning class on $\Phi.T$ consisting of the same turns as $B$. One easily checks that this action is measurable.

3.4 Namable turning classes

Recall that $\mathcal{O}^\text{turn}$ denotes the collection of all turning classes on trees in $\mathcal{O}$. More generally, given $X \subseteq \mathcal{O}$, we denote by $X^\text{turn}$ the collection of all turning classes on trees in $X$. Given a turning class $B$ on a tree $T \in \mathcal{O}$, and $\lambda > 0$, we denote by $\lambda B$ the turning class on $\lambda T$ consisting of the same turns as $B$.

Definition 3.21 (Having namable turning classes). An $\text{Out}(G, \mathcal{F})$-invariant Borel subset $X \subseteq \mathcal{O}$ has namable turning classes if there exists a map

$$\text{Name} : X^\text{turn} \rightarrow \mathbb{N} \cup \{\bot\}$$

such that

- $\text{Name}$ is measurable,
• for all \( \lambda \in \mathbb{R}_+^* \) and all turning classes \( B \in X^{\text{turn}} \), we have \( \text{Name}(\lambda B) = \text{Name}(B) \),
• Name is \( \text{Out}(G, F) \)-invariant, i.e. \( \text{Name}(\Phi B) = \text{Name}(B) \) for all \( B \in X^{\text{turn}} \) and all \( \Phi \in \text{Out}(G, F) \),
• for all \( T \in X \) and all \( n \in \mathbb{N} \), there are only finitely many turning classes on \( T \) whose name is \( n \), and
• for all \( T \in X \), there exists a very optimal morphism with range \( T \) whose turning class has a name different from \( \bot \).

\textbf{Proposition 3.22.} \( X \subseteq \mathcal{O} \) be an \( \text{Out}(G, F) \)-invariant Borel subset which has namable turning classes.
Then there exists a sequence of Borel maps
\[
\begin{align*}
P X & \to \text{Prob}(\text{Simp}) \\
T & \mapsto \mu_n^T
\end{align*}
\]
such that for all \( \Phi \in \text{Out}(G, F) \) and all \( T \in P X \), one has
\[
||\Phi \mu_n^T - \mu_n^{\Phi T}||_1 \to 0
\]
as \( n \) goes to \(+\infty\).

\textbf{Proof.} Fix a continuous section \( \alpha : \mathcal{P} \mathcal{O} \to \mathcal{O} \). To any \( T \in P X \), we are going to associate a finite set of very optimal morphisms \( f_i^T \) with range \( \alpha(T) \) as follows. Let \( i_0(T) \) be the smallest integer \( i \in \mathbb{N} \) for which there exists a very optimal morphism with range \( \alpha(T) \), whose turning class with respect to \( T \) has name \( i \): this exists in view of the last hypothesis from Definition 3.21. Let \( f_1^T, \ldots, f_{k(T)}^T \) be a collection of very optimal morphisms \( f_i^T : S_i^T \to \alpha(T) \) such that the turning classes of \( f_1^T, \ldots, f_{k(T)}^T \) are exactly the turning classes named \( i_0(T) \) (without repetition). We choose this collection that is smallest for the lexicographic order, relative to a measurable enumeration of morphisms with range \( T \) as in Remark 3.19. We claim that the probability measures
\[
\mu_n^T := \frac{1}{k(T)} \sum_{i=1}^{k(T)} \mu_n(f_i(T))
\]
satisfy the desired conclusion.

Notice that \( \text{Out}(G, F) \)-invariance of Name implies that for all \( T \in P X \) and all \( \Phi \in \text{Out}(G, F) \), we have \( i_0(\Phi T) = i_0(T) \) and \( k(\Phi T) = k(T) \). Applying \( \Phi^{-1} \) to the morphism \( f_i^{\Phi T} : S_i^{\Phi T} \to \alpha(\Phi T) \), we get a morphism \( f'_i : S'_i \to \lambda \alpha(T) \) with \( S'_i = \Phi^{-1} S_i^{\Phi T} \) and \( \lambda > 0 \) such that \( \Phi^{-1} \alpha(\Phi T) = \lambda \alpha(T) \). The morphism \( f'_i \) has the same turning class as \( f_{\sigma(i)}^T \) for some permutation \( \sigma \) of \( \{1, \ldots, k(T)\} \). Since the two target trees are homothetic, Corollary 3.12 implies that for all \( i \in \{1, \ldots, k(T)\} \), we have
\[
||\mu_n(f_i^T) - \mu_n(f'_i)||_1 \to 0
\]
as \( n \) goes to \(+\infty\). Averaging over \( i \in \{1, \ldots, k(T)\} \), it then follows that

\[
||\mu_n^T - \Phi^{-1}\mu_n^T||_1 \to 0
\]
as \( n \) goes to \(+\infty\).

We finally check that the maps \( T \mapsto \mu_n^T \) are Borel. Let \( F_{n,T} \) be a Borel enumeration of morphisms as in Remark 3.19. Then

\[
i_0(T) = \min\{i \in \mathbb{N} | \exists n \in \mathbb{N}, F_{n,a(T)} \text{ is optimal and its turning class has name } i\}.
\]

Since optimality is a Borel condition (Lemma 3.20) and the map \( \text{Name} \) is Borel, we deduce that the map \( T \mapsto i_0(T) \) is Borel. Similarly, the maps \( T \mapsto k(T) \) and \( T \mapsto f_1^T \) are Borel. In view of Lemma 3.13 we thus get that \( T \mapsto \mu_n^T \) is Borel. \( \square \)

## 4 Arational trees have namable turning classes.

We will prove separately that the sets \( \mathcal{G}_{\text{Geom}} \) and \( \mathcal{N}_{\text{Geom}} \) made of geometric and non-geometric arational trees have namable turning classes. We start with the following observation.

**Lemma 4.1.** The sets \( \mathcal{G}_{\text{Geom}} \) and \( \mathcal{N}_{\text{Geom}} \) are Borel subsets of \( \mathcal{AT} \).

**Proof.** It suffices to show that \( \mathcal{N}_{\text{Geom}} \) is a Borel subset of \( \mathcal{AT} \). Let \( \{g_i\}_{i \in \mathbb{N}} \) be an enumeration of \( G \).

Using the third characterization of non-geometric trees given in Lemma 2.3, a tree \( T \in \mathcal{AT} \) is non-geometric if and only if for every finite subset \( F \subset G \), there exists \( S \in \mathcal{O} \) with a morphism \( S \to T \) such that the length functions of \( T \) and \( S \) agree on \( F \). Given a simplex \( \Delta \in \text{Simp} \), denote by \( \Delta^\mathbb{Q} \) the set of trees in \( \Delta \) with rational edge lengths. It clearly follows that \( T \in \mathcal{AT} \) is non-geometric if and only if for all finite subsets \( F \subset G \), there exists a simplex \( \Delta \in \text{Simp} \) such that for all \( k \geq 1 \), there exists \( S \in \Delta^\mathbb{Q} \) such that

- \( \forall g \in G, ||g||_S \geq ||g||_T \) and
- \( \forall g \in F, ||g||_S \leq ||g||_T + \frac{1}{k} \).

This expresses \( \mathcal{N}_{\text{Geom}} \) as countable intersections and unions of Borel sets. \( \square \)

### 4.1 Non-geometric arational trees

Given a tree \( T \in \overline{O} \), the **full turning class** in \( T \) is the turning class consisting of all turns at all branch points in \( T \).

**Proposition 4.2.** For all \( T \in \mathcal{N}_{\text{Geom}} \), there exists a tree \( S \in \mathcal{O} \) and a very optimal morphism \( f : S \to T \) whose turning class with respect to \( T \) is the full turning class of \( T \).

**Proof.** Since \( T \) is non-geometric, it is not an arational surface tree, and therefore it is relatively free. Let \( X \subseteq T \) be a finite subtree that contains
• a set of representatives $V = \{v_1, \ldots, v_l\}$ of the orbits of branch points of $T$,
• for each branch point $v_i \in V$, a set of representatives of the orbits of directions at $v_i$.

Since $T$ is nongeometric, Lemma 2.5 ensures that there exists a nonstationary direct system of trees $(S_n)_{n \in \mathbb{N}} \in \mathcal{O}^\mathbb{N}$ that converges strongly to $T$, coming with morphisms $f_n : S_n \to T$ and $f_{nm} : S_n \to S_m$ for all $m \geq n$ (these morphisms can be chosen optimal). Let $X_0 \subseteq S_0$ be a finite subtree of $S_0$ whose $f_0$-image contains $X$, and such that for every $v_i \in V$ with nontrivial stabilizer in $T$, the vertex of $S_0$ with stabilizer $G_{v_i}$ is contained in $X_0$. By definition of strong convergence, there exists $n \in \mathbb{N}$ such that $f_n$ is isometric when restricted to $X_n := f_0^n(X_0)$. In particular $X_n$ contains an isometric copy of $X$, and the preimage in $X_n$ of every vertex $v$ of $X$ with non-trivial stabilizer in $T$, is the vertex of $S_n$ having the same peripheral point stabilizer $G_v$. Let now $(d, d')$ be a turn based at a vertex $v_i$ in $T$ (up to translating we can assume that $v_i \in V$). If $v$ has trivial stabilizer in $T$, then the assumptions made on $X$ ensure that $(d, d')$ lifts to $S_n$. If $v$ has non-trivial stabilizer in $T$, then there exist $g, g' \in G_v$ such that $(gd, g'd')$ lifts at the vertex of $S_n$ with stabilizer $G_v$, and therefore $(d, d')$ also lifts. Therefore, every turn in $T$ lifts to a turn in $S$, so the turning class of $f_n$ is the full turning class of $T$. Notice that by slightly folding if needed, we can always arrange that $f_n$ maps vertices of $S_n$ to branch points of $T$, i.e. $f_n$ is very optimal.

Corollary 4.3. The subset $\mathcal{N}_{\text{Geom}} \subseteq \mathcal{O}$ has namable turning classes.

Proof. In view of Proposition 4.2, the map $\text{Name}$ that sends the full turning class of any tree $T \in \mathcal{N}_{\text{Geom}}$ to 0, and any other turning class to $\perp$, satisfies the assumptions in Definition 3.21.

4.2 Geometric arational trees

The goal of the present section is to prove the following fact.

Proposition 4.4. The subset $\mathcal{G}_{\text{Geom}} \subseteq \mathcal{O}$ has namable turning classes.

4.2.1 Ubiquitous turns

Definition 4.5 (Ubiquitous turns). Let $T \in \mathcal{O}$. A turn $U$ in $T$ is ubiquitous if for every interval $I$, there exists $g \in G$ such that $gU$ is contained in $I$.

Let $T \in \mathcal{O}$ be a geometric tree, and let $G \curvearrowright \Sigma$ be a band complex as in Section 2.3 to which $T$ is dual. We denote by $V$ the set of vertices of $\Sigma$, i.e. points in $\Sigma$ belonging to a base tree $K_v$ which are either extremal in one of the bands they belong to, or are at least trivalent in $K_v$. Recall that all leaves of $\Sigma$ are trees. A leaf of $\Sigma$ is singular if it contains a vertex of $\Sigma$. Notice that if $x \in T$ is a branch point, then the leaf of $\Sigma$ that projects to $x$ in $T$ is singular. Given a singular leaf $l$ of $\Sigma$ projecting to a point $x \in T$, we denote by $l^0 \subseteq l$ the union of the convex hull of $l \cap V$ in $l$ together with all finite
connected components of $l \setminus V$. Notice that since the set $V$ is $G$-finite, the intersection $l \cap V$ is $G_l$-finite (where $G_l$ denotes the stabilizer of the leaf $l$). Since there are only finitely many $G_l$-orbits of finite components in $l \setminus V$, we deduce that $l^0/G_l$ is compact (it is a finite graph). We call a turn at $x$ in $T$ regular if it comes from a turn in a base tree at a point in $l \setminus l^0$, and singular otherwise.

**Lemma 4.6.** Let $T$ be a geometric tree which is mixing (i.e. dual to a band complex with a single orbit of minimal components), and let $x \in T$ be a branch point. A turn at $x$ is ubiquitous if and only if it is regular. Every segment in $T$ contains only finitely many singular turns based at a point in $G.x$.

**Proof.** Let $l \subset \Sigma$ be the leaf representing $x$. Since $l^0$ is $G_l$-cocompact, for each band $B$, the orbit $G.l^0$ intersects $B$ in a finite set of leaf segments (otherwise $B$ would have nontrivial stabilizer). This shows that singular turns at $x$ are non-ubiquitous. Moreover, it implies that for each segment $\bar{I} \subset K_v$ contained in a band, only finitely many turns in the projection $I$ of $\bar{I}$ to $T$ are singular turns based at a point in $G.x$. This obviously also holds if the interior of $\bar{I}$ intersects no band (which in fact cannot happen since $T$ is mixing). Since every arc in $T$ is contained in a finite union of such segments $I$, it also follows that every arc in $T$ contains only finitely many singular turns based at a point in $G.x$.

We will now prove that every regular turn $U$ at $x$ is ubiquitous. Since $U$ is regular, there exist a base tree $K_v$ intersecting $l \setminus l^0$ and a point $z \in K_v \cap (l \setminus l^0)$ having the following property: denoting by $l_z \subset l$ the connected component of $l \setminus l^0$ containing $z$, for every transverse interval $I \subset \Sigma$ intersecting $l_z$, the two directions at the point $I \cap l_z$ define the turn $U$. Since the band complex $\Sigma/G$ has a unique minimal component, by [11, Theorem 3.1] the orbit of $l_z$ intersects each transverse tree $K_v \subset \Sigma$ in a dense set. Since every arc in $T$ is a finite concatenation of arcs that are the image of intervals contained in base trees, it follows that $U$ is ubiquitous. This completes the proof of the lemma.

Using the fact that there are only finitely many $G$-orbits of singular leaves and $l^0/G_l$ is a finite graph, we get the following as a consequence of Lemma 4.6.

**Corollary 4.7.** Let $T$ be a geometric tree which is mixing (i.e. dual to a band complex with a single orbit of minimal components). Then there are only finitely many orbits of ubiquitous turns at branch points.
Every interval contains only finitely many non-ubiquitous turns.

Given a branch point $x \in T$ and a collection $\mathcal{T}$ of turns at $x$, the Whitehead graph at $x$ defined by $\mathcal{T}$ is the graph having one vertex for each direction at $x$, and an edge between two directions if the turn they form belongs to $\mathcal{T}$.

**Lemma 4.8.** Let $T \in Geom$. Let $x \in T$ be a branch point. If the Whitehead graph at $x$ defined by the collection of all ubiquitous turns is disconnected, then $T$ is not indecomposable.

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Proof. Since the Whitehead graph defined by ubiquitous turns at $x$ is not connected, we have an equivariant partition of the directions at $x$ such that every ubiquitous turn joins two equivalent directions. We say that a turn at a point in the orbit of $x$ is allowed if it is a translate of a turn joining two equivalent directions at $x$. We now construct a transverse covering of $T$: say that a subtree $Y$ is allowed if all its turns at points in the orbit of $x$ are allowed. Let $\mathcal{Y}$ be the set of all nondegenerate maximal allowed subtrees. The last assertion from Corollary 4.7 implies that $\mathcal{Y}$ is nonempty (and actually covers $T$): any arc $I$ can be covered by finitely many subarcs that only contain ubiquitous turns. In addition, is not the trivial covering (i.e. $\mathcal{Y} \neq \{T\}$) because there are at least two equivalence classes at $x$. 

We claim that the family $\mathcal{Y}$ is transverse; this will show that $T$ is not indecomposable and complete the proof of the lemma. Otherwise, we can find $Y,Y' \in \mathcal{Y}$ two distinct subtrees with nondegenerate intersection. By maximality, there exists a turn $(d,d')$ in $Y \cup Y'$ which is not allowed. Since $Y$ and $Y'$ are allowed, up to exchanging $d$ and $d'$ we can assume that $d \subset Y \setminus Y'$ and $d' \subset Y' \setminus Y$. Let $b \in Y \cap Y'$ be the common base point of these directions, and let $d''$ be a direction at $b$ in $Y \cap Y'$: this exists because $Y \cap Y'$ is nondegenerate. Then $(d,d'')$ and $(d',d'')$ are allowed, hence so is $(d,d')$, a contradiction. 

We finish this section by proving measurability of the ubiquity property (we recall the definition of the directions $d_n(T)$ from Section 3.3.1).

Lemma 4.9. For all $n,m \in \mathbb{N}$, the set

$$U_{nm} := \{T \in \mathcal{O}|(d_n(T),d_m(T)) \text{ is an ubiquitous turn}\}$$

is a Borel subset of $\mathcal{O}$.

Proof. By Lemma 3.17, the set

$$D_{k,l,m,n,g} := \{T \in \mathcal{O}|v_k(T) \neq v_l(T) \text{ and } g(d_n(T),d_m(T)) \text{ is a turn lying in the segment } [v_k(T),v_l(T)]\}$$

is a Borel subset of $\mathcal{O}$. Since

$$U_{nm} = \bigcap_{k,l \in \mathbb{N}} \bigcup_{g \in G} D_{k,l,m,n,g},$$

the result follows.

4.2.2 Surface arational trees

We denote by $\mathcal{SAT}$ the subspace of $\mathcal{AT}$ made of non relatively free trees: in other words, a tree $T \in \mathcal{AT}$ is in $\mathcal{SAT}$ if and only if there exists a nonperipheral element $g \in G$ such that $||g||T = 0$. This shows that $\mathcal{SAT}$ is a Borel subset of $\mathcal{AT}$. 

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Figure 3: The tree $T$ from the proof of Proposition 4.10 is the arational surface tree dual to the foliation on the orbifold. The tree $S_t$ is obtained by cutting a segment of size $t$ along each of the red slits, and considering the simplicial tree dual to the foliation obtained in the cut surface.

**Proposition 4.10.** The subset $\mathcal{SAT}$ has namable turning classes.

**Proof.** Let $T \in \mathcal{SAT}$. The tree $T$ splits as a graph of actions where the only nontrivial vertex action is dual to a foliation on a 2-orbifold with a single unused boundary component (see Section 2.2). Up performing Whitehead moves on the foliation (which does not change the dual tree), we can assume that half-leaves starting at the unused boundary curve do not contain singularities of the foliation, and that all singularities on the boundary are 3-pronged. By putting slits of sufficiently small length $t > 0$ between any two half-leaves of the foliation defining $T$ starting at the unused boundary component (see Figure 3), and cutting along these slits, all leaves of the obtained foliation are now compact and the dual tree provides a simplicial approximation $S_t$ of $T$, which naturally comes with a morphism $f_t : S_t \to T$. We observe that all morphisms $f_t$ with $t > 0$ sufficiently small have the same turning class $B_{\text{slit}}(T)$: indeed, any turn defined by two directions not based at the vertex corresponding to the unused boundary component is in $B_{\text{slit}}(T)$, and a turn based at the vertex corresponding to the boundary component is in $B_{\text{slit}}(T)$ if and only if it is made of two directions that are separated by only one singular leaf of the foliation. Using Lemma 4.6, we see that $B_{\text{slit}}(T)$ is nothing but the collection of all ubiquitous turns in $T$ based at a point corresponding to the unused boundary component. The map Name sending the turning class $B_{\text{slit}}(T)$ to 0, and any other turning class to $\perp$, satisfies the conditions from Definition 3.21 (the fact that it is measurable follows from Lemma 4.9).

4.2.3 The turning class is of finite type.

We are now left understanding relatively free geometric arational trees. We will now prove that turning classes of morphisms with such a tree as their target satisfy some finiteness properties. We make the following definition.
Definition 4.11 (The set $B_{[d],[d']}$). Let $T \in \overline{\mathcal{O}}$, and let $\mathcal{B}$ be a turning class in $T$. Let $x \in T$ be a branch point, and let $[d],[d']$ be two $G_x$-orbits of directions based at $x$. We denote by $B_{[d],[d']} \subseteq \mathcal{B}$ the set of turns $(\delta,\delta') \in \mathcal{B}$ with $\delta \in [d]$ and $\delta' \in [d']$. We say that $B_{[d],[d']}$ is full if it is equal to $[d] \times [d']$ minus the diagonal.

Notice that the set $B_{[d],[d']}$ is $G_x$-invariant.

Definition 4.12 (Turning class of finite type). A turning class $\mathcal{B}$ of a tree $T \in \overline{\mathcal{O}}$ is of finite type if for each branch point $x \in T$ and for each pair of $G_x$-orbits of directions $[d],[d']$ at $x$, the set $B_{[d],[d']}$ is either full or finite modulo $G_x$.

Notice that the set of all turning classes of finite type on trees in $\overline{\mathcal{O}}$ forms a measurable subset of the set $\overline{\mathcal{O}}^{\text{turn}}$ of all turning classes (see Section 3.3.3).

Lemma 4.13. Let $T \in \overline{\mathcal{O}}$ be geometric, with trivial arc stabilizers. Assume that $T$ is relatively free. Then for any $S \in \mathcal{O}$ and any morphism $f : S \to T$, the turning class of $f$ is of finite type.

Proof. Let $d,d'$ be a turn based at a branch point $x \in T$. We aim to show that $B_{[d],[d']}$ is either $G_x$-cofinite or full.

We first claim that all but finitely many $G_x$-orbits of turns in $B_{[d],[d']}$ lift at a vertex of $S$ with non-trivial stabilizer. To prove this claim, we first note that if two (ordered) turns $(\delta,\delta_1'), (\delta,\delta_2')$ at $x$ are in the same $G_x$-orbit, then they have to coincide since the stabilizer of any direction is trivial. It follows that $B_{[d],[d']}$ contains at most finitely many $G_x$-orbits of ubiquitous turns since there are only finitely many orbits of ubiquitous turns at branch points in $T$. Now for each edge $e$ of $S$, the segment $f(e)$ crosses only finitely many non-ubiquitous turns. It follows that turns taken by the images of the edges of $S$ contribute only finitely many orbits of turns in $B_{[d],[d']}$. Since $S$ has only finitely many $G$-orbits of turns at vertices with trivial stabilizer, the claim follows.

If no turn in $B_{[d],[d']}$ lifts at a vertex of $S$ with nontrivial stabilizer, then our claim shows that $B_{[d],[d']} = G_x$-cofinite. So consider a vertex $s \in f^{-1}(x)$ with nontrivial stabilizer, and a turn $\tilde{(\delta_1,\tilde{\delta}_2)}$ at $s$ that maps to some $(\delta_1,\delta_2) \in B_{[d],[d']}$. Since $T$ is relatively free, $s$ is the unique vertex with nontrivial stabilizer in $f^{-1}(x)$, and $G_s = G_x$. Then for all $g_1,g_2 \in G_s = G_x$, the turn $(g_1\delta_1,g_2\delta_2)$ is mapped to $(g_1\delta_1, g_2\delta_2)$ so $B_{[d],[d']} = \text{full}$. □

4.2.4 Definition of Name in the indecomposable case

We denote by $\mathcal{F} \text{Geom}$ the collection of all geometric, relatively free arational trees. This is a Borel subset of $\mathcal{AT}$, being the complement of the union $\mathcal{N} \text{Geom} \cup S \mathcal{AT}$. Every tree $T \in \mathcal{F} \text{Geom}$ has a natural decomposition given by a transverse covering by indecomposable trees [15]. Since $T$ is arational, it has to be mixing (see [14] Proposition 8.3 for free groups and [23] Lemma 4.9 in general), i.e. there is exactly one orbit of indecomposable trees in the transverse covering. The skeleton of this transverse covering is a bipartite simplicial tree which has one orbit of vertices corresponding to the indecomposable
pieces, whose stabilizer is finitely generated, while the stabilizer of every other vertex in $S$ is a peripheral group (because $T$ is relatively free). Edge stabilizers of this skeleton are finitely generated. In particular, the set $Sk$ of all possible skeleta of trees in $FG_{geom}$ is countable. We denote by $GI$ the subset of $FG_{geom}$ made of all indecomposable geometric relatively free arational trees in $\overline{O}$.

**Lemma 4.14.** Let $S$ be a simplicial $G$-tree. Then the set of all trees in $\overline{O}$ that are compatible with $S$ is a Borel subset of $\overline{O}$.

**Proof.** This is because a tree $T$ is compatible with $S$ if and only if the sum of the translation length functions of $T$ and $S$ is a translation length function of some $G$-tree $[19$, Theorem A.10$]$, and the space of translation length functions of $G$-trees is closed (hence a Borel subset) in $\mathbb{R}^G$, see $[10]$. $\square$

**Corollary 4.15.** The set $GI$ is a Borel subset of $Geom$.

**Proof.** This is a consequence of Lemma 4.14 because a geometric tree is indecomposable if and only if it is not compatible with any simplicial tree in $Sk$, and $Sk$ is countable. In addition, as already observed in the previous section, being relatively free is a Borel condition. $\square$

The map Name will be defined in terms of the following notion of angle.

**Definition 4.16.** Let $T \in GI$, and $d, d'$ be two directions in $T$ based at the same branch point $x \in T$.

The angle $\angle(d, d')$ between $d$ and $d'$ is the minimal number $k$ of ubiquitous turns $U_1, \ldots, U_k$ such that $d$ is a direction in $U_1$, $d'$ is a direction in $U_k$, and for all $i \in \{1, \ldots, k-1\}$, the turns $U_i$ and $U_{i+1}$ have a direction in common.

The angle can be interpreted as a distance in the Whithead graph at $x$ defined by ubiquitous turns. It is finite because this Whithead graph is connected (Lemma 4.8). Obviously, the angle is $G$-invariant. Moreover, since by Corollary 4.15 any direction is part of only finitely many ubiquitous turns, the Whithead graph is locally finite. In other words, for any direction $d$, and $k \in \mathbb{N}$, there are only finitely many directions $d'$ based at the same point as $d$ such that $\angle(d, d') \leq k$.

We now define the name of a turning class $B$ on $T$ as follows. If $B$ is not of finite type, we define its name as $\perp$. We now assume that $B$ is of finite type. Given a branch point $x \in T$ and a pair $([d], [d'])$ of $G_x$-orbits of directions based at $x$, we let $N_{[d], [d']} = 0$ if $B_{[d], [d']} = \emptyset$, and otherwise we let

$$N_{[d], [d']} = \max_{(\delta, \delta') \in B_{[d], [d']}} \angle(\delta, \delta').$$

This is well defined because $B$ is of finite type. We then let

$$Name(B) = \max_{([d], [d'])} (N_{[d], [d']}),$$

where the maximum is taken over all pairs of orbits of directions at branch points.
Proposition 4.17. The set $\mathcal{G}I$ has namable turning classes: the map 

$$\text{Name} : \mathcal{G}I^\text{turn} \to \mathbb{N} \cup \{\perp\}$$

satisfies

1. $\text{Name}$ is measurable,

2. for all $\mathcal{B} \in \mathcal{G}I^\text{turn}$ and all $\lambda > 0$, we have $\text{Name}(\lambda \mathcal{B}) = \text{Name}(\mathcal{B})$,

3. $\text{Name}$ is $\text{Out}(G, \mathcal{F})$-invariant, i.e. $\text{Name}(\Phi \cdot \mathcal{B}) = \text{Name}(\mathcal{B})$ for all $\mathcal{B} \in \mathcal{G}I^\text{turn}$ and all $\Phi \in \text{Out}(G, \mathcal{F})$,

4. for all $T \in \mathcal{G}I$, the set of turning classes on $T$ having a given name $n \neq \perp$ is finite,

5. for all $T \in \mathcal{G}I$, there exists a very optimal morphism with range $T$ whose turning class has a name different from $\perp$.

Proof. The fact that $\text{Name}$ is measurable can easily be checked using Lemmas 3.17 and 1.9 (representatives of the pairs of orbits of directions in $T$ can be chosen in a measurable way). The second and third assertions are clear from the definition of $\text{Name}$. The fourth assertion follows from the fact that there are only finitely many orbits of pairs of directions making a given angle, and there are finitely many possibilities for the pairs of orbits of directions for which $\mathcal{B}_{[d], [d']}$ is full. The fifth assertion follows from Lemma 4.13. \hfill $\square$

4.2.5 Definition of $\text{Name}$ in the decomposable case

The goal of the present section is to prove that the set of all geometric trees in $\mathcal{A}T$ has namable turning classes, without restricting to the indecomposable ones.

We start by defining a map $\text{Name}$ on the set $\mathcal{F}Geom^\text{turn}$ of turning classes on relatively free, arational, geometric trees. Let $T \in \mathcal{F}Geom$, and let $\mathcal{B}$ be a turning class on $T$. Let $\mathcal{Y}$ be the transverse covering of $T$ by its maximal indecomposable subtrees (recall that all trees in $\mathcal{Y}$ are in the same orbit because $T$ is mixing). We say that a turn $(d, d')$ in $T$ crosses $\mathcal{Y}$ if it is not contained in any $Y \in \mathcal{Y}$. We say that the turning class $\mathcal{B}$ is acceptable if given any turn $(d, d') \in \mathcal{B}$ based at $x \in T$ and $g \in G_x$ such that both $(d, d')$ and $(d, gd')$ cross $\mathcal{Y}$, then $(d, d') \in \mathcal{B}$ if and only if $(d, gd') \in \mathcal{B}$. We then say that $\mathcal{B}$ crosses $\mathcal{Y}$ along $([d], [d'])$ when there exists a turn $(d, d') \in \mathcal{B}$ with $d \in [d]$, and $d' \in [d']$ such that $(d, d')$ crosses $\mathcal{Y}$ (in which case, all turns in $[d] \times [d']$ crossing $\mathcal{Y}$ are in $\mathcal{B}$ if $\mathcal{B}$ is acceptable).

If $\mathcal{B}$ is not acceptable or is not of finite type, we define its name as $\perp$. Otherwise, choose $Y \in \mathcal{Y}$ and let $\mathcal{B}_{\mid Y}$ be the turning class of $Y$ obtained by restriction of $\mathcal{B}$. We then define $\text{Name}(\mathcal{B}) = \text{Name}(\mathcal{B}_{\mid Y})$. Since $\text{Name}(\mathcal{B}_{\mid Y})$ is defined as an angle when not full this name does not depend on the choice of $Y \in \mathcal{Y}$.

Proposition 4.18. The set $\mathcal{F}Geom$ has namable turning classes: the map 

$$\text{Name} : \mathcal{F}Geom^\text{turn} \to \mathbb{N} \cup \{\perp\}$$

satisfies
1. Name is measurable,
2. for all $B \in F\text{Geom}^\text{turn}$ and all $\lambda > 0$, we have $\text{Name}(\lambda B) = \text{Name}(B)$,
3. Name is $\text{Out}(G, \mathcal{F})$-invariant, i.e. $\text{Name}(\Phi B) = \text{Name}(B)$ for all $B \in F\text{Geom}^\text{turn}$ and all $\Phi \in \text{Out}(G, \mathcal{F})$,
4. for all $T \in F\text{Geom}$, the set of turning classes on $T$ having a given name $n \neq \perp$ is finite,
5. for all $T \in F\text{Geom}$, there exists a very optimal morphism with range $T$ whose turning class has a name different from $\perp$.

Proof. The map $\text{Name}$ clearly satisfies the second and third conditions.

Let us check that $\text{Name}$ satisfies Assertion 4. Given $n \neq \perp$, and a turning class $B$ on $T$ with name $n$, $B$ is acceptable, so $B$ is completely determined by $B_N$ and by the (finite) collection of all pairs $([d], [d'])$ of orbits of directions along which $B$ crosses $\mathcal{Y}$ (with the above notations). Finiteness thus follows from finiteness in the indecomposable case (Proposition 4.17).

We now prove Assertion 5. Given $T \in F\text{Geom}$, and $Y \in \mathcal{Y}$ as above, let $G_Y \subset R$ be any Grushko tree for $(G_Y, F_{[G_Y]})$ with a very optimal morphism $f_Y : R \to Y$. Let $S$ be the skeleton of $Y$. It has a vertex $v_Y$ whose stabilizer is $G_Y$. Blow-up in an equivariant way the vertex $v_Y$ of $S$ into $R$ and attach each edge $e$ incident on $v_Y$ to the unique point fixed by $G_e$ (recall that $G_e$ is non-trivial because $T$ is arational, and peripheral because $T$ is relatively free). Let $\hat{S}$ be the obtained blown-up tree, and let $S_0$ be the tree obtained from $\hat{S}$ by collapsing all the edges coming from $S$. The tree $S_0$ lies in $\mathcal{O}$, and we denote by $R$ the transverse covering of $S_0$ by the translates of $R$. The morphism $f_Y : R \to Y$ extends uniquely to a map $S \to T$ which is constant on all edges coming from $S$. This yields a morphism $f : S_0 \to T$ which is optimal and very optimal because $f_Y$ is. The turning class $B$ of $f$ is of finite type by Lemma 4.13.

To ensure that $\text{Name}(B) \neq \perp$, we now check that $B$ is acceptable. Let $(d_1, d_2)$ be a turn in $B$ that crosses $\mathcal{Y}$. Let $x$ be the base point of this turn, and $Y_1, Y_2 \in \mathcal{Y}$ the subtrees containing $d_1$ and $d_2$ respectively. Let $R_1, R_2 \subset S_0$ be the preimages of $Y_1, Y_2$ in $S_0$ (these are translates of $R \subset S_0$). These two subtrees intersect in a single vertex $v$ coming from the copy of the point $x$ in the skeleton, so $G_x = G_v$, and $f(v) = x$. Let $\hat{d}_1, \hat{d}_2$ be a lift of $(d_1, d_2)$ in $S_0$. Since $\hat{d}_i$ is contained in $R_i$, $\hat{d}_1$ and $\hat{d}_2$ are necessarily based at $v$. Then for all $g \in G_x = G_v$, the turn $(\hat{d}, gd')$ is a lift of $(d, gd')$. In particular, all turns $(d, gd')$ (and in particular all which cross $\mathcal{Y}$) are in $B$.

We finally establish that $\text{Name}$ is measurable. We claim that crossing is a Borel condition, i.e. for all $n, m \in \mathbb{N}$, the set

$$\{ T \in F\text{Geom}_S | (d_n(T), d_m(T)) \text{ defines a turn that crosses } \mathcal{Y} \}$$

is a Borel subset of $F\text{Geom}_S$. Indeed, if $d_n(T), d_m(T)$ are based at the same point, then $(d_n(T), d_m(T))$ does not cross if and only if for every arc $I \subset T$, there exists $g_1, \ldots, g_N \in G$ such that $d_n(T)$ is contained in $g_1.I$, $d_m(T)$ is contained in $g_N.I$, and
$g_i I \cap g_{i+1} I$ is non-degenerate for all $i < N$. This condition does not change if we impose that the endpoints of $I$ are branch points of $T$. The above characterization can therefore be expressed using the measurable enumeration of branch points, thus proving the claim. Since the name of a turning class is expressed in terms of angles of such turns, the measurability of Name easily follows.

5 Proof of the main theorem and applications

5.1 Proof of the main theorem

We start by establishing the following result.

**Theorem 5.1.** Let $G$ be a countable group, and let $\mathcal{F}$ be a free factor system of $G$. Then there exists a sequence of Borel maps

$$\mu_n : \mathbb{P}AT \to \text{Prob}(\text{Simp})$$

such that for all $\Phi \in \text{Out}(G, \mathcal{F})$ and all $T \in \mathbb{P}AT$, one has

$$||\Phi \cdot \mu_n(T) - \mu_n(\Phi \cdot T)||_1 \to 0$$

as $n$ goes to $+\infty$.

*Proof.* The subsets $\mathcal{S}AT$, $\mathcal{F}Geom$ and $\mathcal{N}Geom$ are $\text{Out}(G, \mathcal{F})$-invariant Borel subsets of $\mathcal{AT}$ which together partition $\mathcal{AT}$, and they have namable turning classes (Corollary 4.3, Proposition 4.10 and Proposition 4.18). Theorem 1.5 thus follows from Proposition 3.22.

We are now in position to complete the proof of our main theorem.

**Theorem 5.2.** Let $\{G_1, \ldots, G_k\}$ be a finite collection of countable groups, and let

$$G := G_1 \ast \cdots \ast G_k \ast F_N,$$

where $F_N$ is a free group of rank $N$.

Assume that for all $i \in \{1, \ldots, k\}$, the group $G_i$ is exact.

Then $\text{Out}(G, \{G_i\}^{(t)})$ is exact.

*Proof.* As usual, we denote by $\mathcal{F}$ the finite collection of the $G$-conjugacy classes of the subgroups $G_i$. The proof is by induction on the complexity $\xi(G, \mathcal{F}) = (N + k - 1, N)$, where complexities are ordered with respect to the lexicographic order. We start with the sporadic cases.

- The result is obvious if either $\mathcal{F} = \{[G]\}$ (i.e. $\xi(G, \mathcal{F}) = (0, 0)$) or $G = \mathbb{Z}$ (i.e. $\xi(G, \mathcal{F}) = (0, 1)$).

- If $G = G_1 \ast G_2$ and $\mathcal{F} = \{[G_1], [G_2]\}$ (i.e. $\xi(G, \mathcal{F}) = (1, 0)$), then the group $\text{Out}(G, \mathcal{F}^{(t)})$ is isomorphic to $G_1/Z(G_1) \times G_2/Z(G_2)$ (see [33]), which is exact, being a direct product of two groups that are exact by Proposition 2.17.
• If $G = G_1 \ast Z$ and $F = \{[G_1]\}$ (i.e. $\xi(G,F) = (1,1)$), then $\text{Out}(G,F^{(i)})$ has an index 2 subgroup which is isomorphic to $(G_1 \times G_1)/Z(G_1)$, where $Z(G_1)$ embeds diagonally in $G_1 \times G_1$ (see [33]). The group $(G_1 \times G_1)/Z(G_1)$ maps to $G_1/Z(G_1) \times G_1/Z(G_1)$ with central kernel, and is therefore exact by Propositions 2.14 and 2.17.

We now assume that $(G,F)$ is non-sporadic. It was proved in [44, 25] that to any (projective class of) non-arational tree, one can associate a canonical finite set of conjugacy classes of free factors: this means that there exists an $\text{Out}(G,F)$-equivariant map from $\mathbb{PO} \setminus \mathbb{PAT}$ to the countable collection $\mathcal{F}(FF)$ of finite sets of proper $(G,F)$-free factors.

Notice also that there exists an $\text{Out}(G,F)$-equivariant map $\pi : \text{Simp} \to \mathcal{F}(FF)$, sending a simplex $\Delta$ to the collection of all proper free factors that are elliptic in some tree obtained by collapsing some edges in the underlying tree of $\Delta$. Combining these facts with Theorem 5.1, we get a sequence of Borel maps

$$\mu_n : \mathbb{PO} \to \text{Prob}(FF)$$

such that for all $\Phi \in \text{Out}(G,F)$ and all $T \in \mathbb{PO}$, one has

$$||\Phi \cdot \mu_n(T) - \mu_n(\Phi \cdot T)||_1 \to 0$$

as $n$ goes to $+\infty$. This holds in particular for every $\Phi \in \text{Out}(G,F^{(i)})$. Since $\mathbb{PO}$ is compact and $FF$ is countable, using Corollary 2.10 it is enough to prove that the stabilizer in $\text{Out}(G,F^{(i)})$ of any proper $(G,F)$-free factor is exact. Let $A$ be a proper $(G,F)$-free factor, and let $\mathcal{F}'$ be the smallest $(G,F)$-free factor system that contains $A$. Then there is a morphism from the stabilizer of $A$ in $\text{Out}(G,F^{(i)})$ to $\text{Out}(A,\mathcal{F}'|A)$, whose kernel is contained in $\text{Out}(G,\mathcal{F}(A))$. An easy computation shows that $\xi(A,\mathcal{F}|A) < \xi(G,F)$ and $\xi(G,\mathcal{F}') < \xi(G,F)$, so arguing by induction on the complexity (and using the fact that extensions of countable exact groups are exact), we get that the stabilizer of $A$ in $\text{Out}(G,F^{(i)})$ is exact. \hfill \Box

**Corollary 5.3.** Let $\{G_1,\ldots,G_k\}$ be a finite collection of countable groups, and let

$$G := G_1 \ast \cdots \ast G_k \ast F_N,$$

where $F_N$ is a free group of rank $N$.

Assume that for all $i \in \{1,\ldots,k\}$, the groups $G_i$ and $\text{Out}(G_i)$ are exact. Then $\text{Out}(G,\{G_i\})$ is exact.

**Proof.** This follows from the fact that there is a short exact sequence

$$1 \to \text{Out}(G,F^{(i)}) \to \text{Out}(G,F) \to \prod_{i=1}^{k} \text{Out}(G_i) \to 1,$$

and exactness is stable under extensions (Proposition 2.14). \hfill \Box
5.2 Automorphisms of relatively hyperbolic groups

Corollary 5.4. Let $G$ be a torsion-free group which is hyperbolic relative to a finite collection of finitely generated groups $\mathcal{P} := \{P_1, \ldots, P_k\}$. Assume that for all $i \in \{1, \ldots, k\}$, the group $P_i$ is exact. Then $\text{Out}(G, \mathcal{P}(t))$ is exact.

Before proving Corollary 5.4 we establish the following consequence.

Corollary 5.5. Let $G$ be a torsion-free group which is hyperbolic relative to a finite collection of finitely generated groups $\mathcal{P} := \{P_1, \ldots, P_k\}$. Assume that for all $i \in \{1, \ldots, k\}$, the groups $P_i$ and $\text{Out}(P_i)$ are exact. Then $\text{Out}(G, \mathcal{P})$ is exact.

In particular, the outer automorphism group of any torsion-free Gromov hyperbolic group (or more generally any toral relatively hyperbolic group) is exact.

Proof. The first statement follows from Corollary 5.4 as in Corollary 5.3. The application to torsion-free hyperbolic groups is immediate. The case of toral relatively hyperbolic groups follows from the fact that if some parabolic subgroup is isomorphic to $\mathbb{Z}^n$, then its outer automorphism group is linear, whence exact by [13]; in addition, every automorphism of a toral relatively hyperbolic group permutes the conjugacy classes of all non-cyclic parabolic subgroups.

Proof of Corollary 5.4. First assume that $G$ is freely indecomposable relative to the parabolic subgroups. This case follows easily from exactness of mapping class groups [29, 21] and JSJ theory [45, 33, 18] as we now explain. By [18], the group $\text{Out}(G, \mathcal{P}(t))$ has a finite index subgroup that maps onto a product of mapping class groups of surfaces, with kernel contained in the group $\mathcal{T}$ of twists of the canonical elementary JSJ decomposition of $G$. The group $\mathcal{T}$ maps with abelian kernel to a direct product of copies of $P_i/Z(P_i)$ [20, Appendix, Lemma 5], so $\mathcal{T}$ is exact. By [29, 21], the mapping class group of an orientable surface $S$ is exact. This also holds if $S$ is non-orientable because $\text{MCG}(S)$ is commensurable to a subgroup of $\text{MCG}(\tilde{S})$ where $\tilde{S}$ is the orientation cover of $S$. Therefore, $\text{Out}(G, \mathcal{P}(t))$ is exact.

We now assume that $G$ splits as a free product of the form

$$G = G_1 \ast \cdots \ast G_k \ast F_N$$

relative to the parabolic subgroups, with $G_i$ freely indecomposable relative to $\mathcal{P}_{G_i}$. Then the group $\text{Out}(G, \mathcal{P}(t))$ has a finite index subgroup $\text{Out}^0(G, \mathcal{P}(t))$ which does not permute the conjugacy classes of the factors $G_i$. There is a morphism

$$\text{Out}^0(G, \mathcal{P}(t)) \to \prod_{i=1}^k \text{Out}(G_i, \mathcal{P}(t)_{|G_i}),$$

whose kernel is equal to $\text{Out}(G, \{G_i\}(t))$. The above paragraph shows that each group $\text{Out}(G_i, \mathcal{P}(t)_{|G_i})$ is exact, and therefore their direct product is exact. It is thus enough to
check that the kernel $\text{Out}(G, \{G_i\}^{(t)})$ is exact. Since each $G_i$ is hyperbolic relative to $\mathcal{P}_{|G_i|$ and relatively hyperbolic groups with exact parabolics are exact [111], we deduce that each $G_i$ is exact. Therefore, Theorem 5.2 shows that $\text{Out}(G, \{G_i\}^{(t)})$ is exact, which concludes the proof.

5.3 Automorphisms of right-angled Artin groups

We recall that given a finite simplicial graph $X$, the right-angled Artin group $A_X$ is the group whose generators are the vertices of $X$, in which the relations are given by commutation of any two generators corresponding to vertices of $X$ that are joined by an edge. Using Theorem 5.2 and previous work of Charney–Vogtmann [8], we will derive the following statement.

Corollary 5.6. For any right-angled Artin group $A$, the group $\text{Out}(A)$ is exact.

Proof. The proof is by induction on the number of vertices of the defining graph $X$ of $A$.

If $A$ has nontrivial center, then by [8, Proposition 4.4], there exists an equivalence class $[v]$ of vertices in $X$ (with the terminology from [8]), and a finite index subgroup $\text{Out}^0(A)$ of $\text{Out}(A)$, such that there is a morphism

$$\text{Out}^0(A) \to \text{Out}(A_{[v]}) \times \text{Out}(A_{\text{lk}([v])})$$

with abelian kernel. If $A$ is abelian, then $\text{Out}(A)$ is linear, whence exact [13]. Otherwise both $[v]$ and $\text{lk}([v])$ are proper subgraphs of $X$, so we can apply our induction hypothesis to deduce that $\text{Out}(A)$ is exact.

If $X$ is connected, and the center of $A$ is trivial, then by [8, Corollary 3.3], the group $\text{Out}(A)$ has a finite index subgroup $\text{Out}^0(A)$ that admits a morphism

$$\text{Out}^0(A) \to \prod \text{Out}(A_{\text{lk}([v])})$$

whose kernel is abelian [8, Theorem 4.2], where the product is taken over all maximal abelian equivalence classes of vertices in $X$. Again the result follows from our induction hypothesis.

We finally assume that $X$ is disconnected. Then $A$ splits as a free product

$$A = A_{X_1} \ast \cdots \ast A_{X_k} \ast F_N,$$

where $X_1, \ldots, X_k$ are the connected components of $X$ that are not reduced to a point, and $X$ has $N$ connected components reduced to a point. The group $A_{X_i}$ is exact [7], and $\text{Out}(A_{X_i})$ is exact by induction hypothesis. Corollary 5.3 implies that $\text{Out}(A, \{A_{X_i}\})$ is exact. Since it has finite index in $\text{Out}(A)$, the group $\text{Out}(A)$ is exact. \qed
6 Complements

6.1 An explicit compact space with an amenable action of $\text{Out}(F_N)$

We will now give an explicit compact space with a topologically amenable $\text{Out}(F_N)$-action. This could also be generalized to any free product given compact spaces on which the groups $G_i/Z(G_i)$ and $\text{Out}(G_i)$ act amenably.

Define a subfactor system in $F_N$ as the conjugacy class of a pair $(A, \mathcal{F})$ where $A \subseteq F_N$ is a non-trivial free factor (possibly $A = F_N$), and $\mathcal{F}$ is a proper free factor system of $A$ (possibly empty, but $\mathcal{F} \neq \{A\}$). We denote by $\text{SF}$ the set of all subfactor systems of $F_N$.

To each subfactor system $(A, \mathcal{F})$ we associate a compact space $\Omega_{(A, \mathcal{F})}$. These spaces will depend naturally on $(A, \mathcal{F})$ in the following sense: for each $\Phi \in \text{Out}(F_N)$, there will be a natural homeomorphism $h_\Phi : \Omega_{(A, \mathcal{F})} \to \Omega_{\Phi(A, \mathcal{F})}$ such that $h_{\Phi \circ \Psi} = h_\Phi \circ h_\Psi$ for all $\Phi, \Psi \in \text{Out}(F_N)$. The space we consider is then the product space

$$
\Omega := \prod_{(A, \mathcal{F}) \in \text{SF}} \Omega_{(A, \mathcal{F})},
$$
on which the $\text{Out}(F_N)$-action is defined by

$$
\Phi.(\omega_{(A, \mathcal{F})})_{(A, \mathcal{F}) \in \text{SF}} = (h_\Phi(\omega_{(A, \mathcal{F})}^{-1}))_{(A, \mathcal{F}) \in \text{SF}}.
$$

The spaces $\Omega_{(A, \mathcal{F})}$ are defined as follows. For non-sporadic subfactor systems $(A, \mathcal{F})$, we define $\Omega_{(A, \mathcal{F})}$ as the projectivization $\mathbb{P}\overline{\Omega}(A, \mathcal{F})$ of the closure of the associated outer space. When $A = \mathbb{Z}$, we let $\Omega_{(A, \mathcal{F})}$ be a point. If $A = A_1 * A_2$ and $\mathcal{F} = \{[A_1], [A_2]\}$, then we let $\Omega_{(A, \mathcal{F})} := \partial_\infty A_1 \times \partial_\infty A_2$. Finally, if $A = A_1 *$ and $\mathcal{F} = \{[A_1]\}$, we let $\Omega_{(A, \mathcal{F})} := (\partial_\infty A_1)^2$.

The naturality is obvious in the first two cases, let us make it explicit in the last two cases. We will only construct $h_\Phi$ when $\Phi$ belongs to the stabilizer of $(A, \mathcal{F})$, since this allows to compute $h_\Phi$ in general.

If $A = A_1 * A_2$ and $\mathcal{F} = \{[A_1], [A_2]\}$ then $\text{Out}(A, \mathcal{F})$ is isomorphic to $\text{Aut}(A_1) \times \text{Aut}(A_2)$: this is shown by observing that every element $\Phi \in \text{Out}(A, \mathcal{F})$ has a unique representative $\phi \in \text{Aut}(A)$ that fixes both subgroups $A_1$ and $A_2$ (as opposed to just fixing the conjugacy classes of these two subgroups); the isomorphism from $\text{Out}(A, \mathcal{F})$ to $\text{Aut}(A_1) \times \text{Aut}(A_2)$ then maps $\Phi$ to the pair $(\phi_1, \phi_2)$ made of the restrictions of $\phi$ to $A_1$ and $A_2$. Therefore, the group $\text{Out}(A, \mathcal{F})$ acts by homeomorphisms on $\partial_\infty A_1 \times \partial_\infty A_2 = \Omega(A, \mathcal{F})$.

If $A = A_1 *$ with $\mathcal{F} = \{[A_1]\}$, let $T$ be the Bass–Serre tree of the splitting $A = A_1 *$. We choose a base edge $e = uv$ where the stabilizer of $u$ is $A_1$, and we let $t$ be an element of $A$ sending $u$ to $v$ (thus $t$ is a stable letter of the HNN extension). The group $\text{Aut}(A, \mathcal{F})$ of (not outer) automorphisms of $A$ preserving the conjugacy class of $A_1$ acts on $T$. Any outer automorphism $\Phi \in \text{Out}(A, \mathcal{F})$ has a unique representative $\phi \in \text{Aut}(A, \mathcal{F})$ that preserves the edge $e$ (it might exchange its endpoints). Looking at the restriction of $\phi$ to $G_u = A_1$ and $G_v = tA_1t^{-1}$, we get a map $\text{Out}(A, \mathcal{F}) \to (\text{Aut}(A_1) \times \text{Aut}(A_1)) \rtimes \mathbb{Z}/2\mathbb{Z}$. This group acts by homeomorphisms on $\partial_\infty A_1 \times \partial_\infty A_1 = \Omega(A, \mathcal{F})$. 

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Theorem 6.1. The $\text{Out}(F_N)$-action on $\Omega$ is topologically amenable.

The proof is similar to the proof of Theorem 5.2, keeping track of the spaces when using Ozawa’s inductive procedure (Proposition 2.8 and Corollary 2.15). The base case of the induction follows from the fact that when $(A, F)$ is sporadic, the $\text{Out}(A, F^{(t)})$-action on $\Omega_{(A, F)}$ is topologically amenable, because the natural action of a hyperbolic group on its boundary is topologically amenable. We leave the details to the reader.

Remark 6.2. If one takes for $\Omega'$ the product of outer spaces of all non-elementary subgroups of $F_N$, the natural action on this space (and even on the closure of the diagonal embedding of the projectivized outer space) is not topologically amenable for $N \geq 3$.

Indeed, consider a decomposition $F_N = A \ast B$ with $A$ not cyclic. Let $T$ be the Bass–Serre tree of this action, and $T_A, T_B$ be two free simplicial actions of $A$ and $B$ respectively. This defines a point in $\Omega'$ as follows: let $H < F_N$ be a non-elementary subgroup; if the action of $H$ on $T$ is non-trivial, we take the minimal $H$-invariant subtree for this action as the $H$-coordinate. If not, then up to conjugating, we can assume that $H$ is contained in $A$ or $B$, and we take as $H$-coordinate the action of $H$ on the corresponding tree $T_A$ or $T_B$. We now note that the stabilizer of this point of $\Omega'$ contains a subgroup isomorphic to $A$, hence is non-amenable. Indeed, for all $a \in A$, the automorphism $\phi$ of $F_N$ that restricts to $ad_a$ on $A$ and to the identity on $B$ fixes this point.

Question 6.3. In [21], Hamenstädter describes a compact space equipped with a topologically amenable action of the mapping class group of a surface, in terms of complete geodesic laminations on the surface. A possible analogue for $\text{Out}(F_N)$ might be to consider free actions on general $\Lambda$-trees, instead of just actions on $\mathbb{R}$-trees; the space $\Omega'$ in the above remark can be viewed as a baby model for this space of $\Lambda$-trees. Is the $\text{Out}(F_N)$-action on this space topologically amenable?

6.2 On the amenability of the $\text{Out}(G, F^{(t)})$-action on $\mathcal{A}T$

In Remark 3.14, we noticed that the action of $\text{Out}(F_N)$ on the set of free arational trees is Borel amenable (this is the case where $F = \emptyset$). In fact Theorem 5.1 implies that the action of $\text{Out}(F_N)$ on the set of all arational trees in $\mathcal{P}O(F_N, \emptyset)$ is Borel amenable because the stabilizer of every simplex in $\mathcal{P}O(F_N, \emptyset)$ is finite.

We now explain how to refine the proof of Theorem 5.1 in order to show that the action of $\text{Out}(G, F^{(t)})$ on $\mathcal{A}T$ is amenable even though the stabilizer of a simplex may be non-amenable. It is important here to restrict to the subgroup $\text{Out}(G, F^{(t)})$ (whereas Theorem 1.5 applies to $\text{Out}(G, F)$): indeed, there are arational trees whose stabilizer in $\text{Out}(G, F)$ is non-amenable (one easily constructs examples where $T$ is an arational surface tree). For similar reasons, this will also require further assumptions on the peripheral groups $G_i$ (see Remark 6.5). The idea of the proof is to replace simplices in $\mathcal{O}$ by larger collections of simplices whose common stabilizer in $\text{Out}(G, F^{(t)})$ is amenable.

Theorem 6.4. Let $\{G_1, \ldots, G_k\}$ be a finite collection of countable groups, and let $G = G_1 \ast \cdots \ast G_k \ast F_N$ with $(G, \{G_i\})$ non-sporadic. Assume that in each group $G_i$, centralizers of non-trivial elements are amenable.

Then the $\text{Out}(G, \{G_i\}^{(t)})$-action on $\mathcal{A}T$ is Borel amenable.
Remark 6.5. The hypothesis on centralizers is necessary. Indeed, assume that one of the groups \( G_i \) and contains an element \( a \) whose centralizer \( Z_{G_i}(a) \) is non-amenable. Let \( T \) be an arational surface \( (G, F) \)-tree where \( G_i \) is amalgamated to one of the boundary curves (or conical points) along \( a \). Then the stabilizer of \( T \) contains a subgroup of twists isomorphic to \( Z_{G_i}(a)/Z(G_i) \). It is therefore non-amenable, which prevents the \( \text{Out}(G, F^{(l)}) \)-action on \( \mathcal{AT} \) from being Borel amenable, as noticed in Remark 2.13. This construction requires to write \( G \) as in Figure 1 in which the surface (or orbifold) holds an arational foliation. This can be achieved by taking a sphere with \( k \geq 4 \) punctures or conical points, or a projective plane with 3 punctures or conical points [3].

Stabilizers of pairs of Grushko trees. In the proof of Theorem 6.3, we will make use of the following proposition.

Proposition 6.6. Let \( G \) be a countable group, and let \( F \) be a free factor system of \( G \), such that \((G,F)\) is non-sporadic. Assume that in each peripheral group \( G_i \), centralizers of non-trivial elements are amenable.

Let \( S,S' \in \mathcal{O} \) be two trees, such that there is no \((G,F)\)-free splitting that is compatible with both \( S \) and \( S' \).

Then the common stabilizer of \( S \) and \( S' \) in \( \text{Out}(G, F^{(l)}) \) is amenable.

Notice that in the case where \( G = F_N \) and \( F = \emptyset \), the stabilizer of any point in Culler–Vogtmann’s outer space is finite, so the conclusion is obvious. From now on, we will assume that \( F \neq \emptyset \). We denote by \( \text{Aut}(G, F^{(l)}) \) the preimage in \( \text{Aut}(G) \) of \( \text{Out}(G, F^{(l)}) \), by \( \mathcal{A}_S, \mathcal{A}_{S'} \subseteq \text{Aut}(G, F^{(l)}) \) the preimage of the stabilizer of \( S \) and \( S' \) respectively, and we let \( \mathcal{A}_{S,S'} := \mathcal{A}_S \cap \mathcal{A}_{S'} \). Given \( g \in G \), we denote by \( \text{ad}_g \) the automorphism of \( G \) given by the conjugation by \( g \), i.e. \( \text{ad}_g(h) = ghg^{-1} \) for all \( h \in G \).

Lemma 6.7. For all \( \phi \in \mathcal{A}_S \), there exists a unique isometry \( H_\phi \) of \( S \) which is \( \phi \)-equivariant, i.e. such that \( H_\phi(gx) = \phi(g)H_\phi(x) \) for all \( g \in G \).

The map

\[
H : \mathcal{A}_S \to \text{Isom}(S) \\
\phi \mapsto H_\phi
\]

is an injective group morphism, which sends \( \text{ad}_g \) to the isometry \( x \mapsto gx \).

Proof. Let \( \phi \in \mathcal{A}_S \), and let \( H_\phi \) be a \( \phi \)-equivariant isometry of \( S \). Let \( v \in S \) be a vertex with nontrivial stabilizer. Then for all \( g \in G_v \), we have \( H_\phi(v) = H_\phi(gv) = \phi(g)H_\phi(v) \), so \( H_\phi(v) \) is the only vertex of \( S \) which is fixed by \( \phi(g) \). Therefore the \( H_\phi \)-images of all vertices of \( S \) with nontrivial stabilizer are completely determined. In addition, for every point \( x \in S \), there exist three vertices \( v_1, v_2, v_3 \) of \( S \) with nontrivial stabilizer so that \( x \) belongs to the tripod spanned by \( v_1, v_2, v_3 \). Therefore, the \( H_\phi \)-image of every point in \( S \) is determined. This shows that \( H_\phi \) is unique. The fact that \( H \) is a group morphism then follows from the observation that

\[
H_\phi \circ H_\psi(gv) = \phi \circ \psi(g)H_\phi \circ H_\psi(v)
\]

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for all \( \phi, \psi \in A_S \), all \( g \in G \) and all \( v \in S \). Injectivity follows from the observation that if \( H_\phi \) is the identity, then \( \phi(g)x = gx \) for all \( g \in G \) and all \( x \in S \), so \( \phi \) is the identity. The last statement of the lemma also follows because \( x \mapsto gx \) is \( \text{ad}_g \)-equivariant. \( \square \)

For \( \phi \in A_{S,S'} \), we will denote by \( H_\phi \) (resp. \( H'_\phi \)) the isometry of \( S \) (resp. \( S' \)) representing \( \phi \). Then \( A_{S,S'} \) acts on \( S \) (resp. \( S' \)) by \( \phi.x = H_\phi(x) \) (resp. \( \phi.x = H'_\phi(x) \)); these actions are denoted with a dot. We let \( A \subseteq A_{S,S'} \) be the finite index subgroup made of all automorphisms acting as the identity on the quotient graphs \( S/G \) and \( S'/G \). Given a vertex \( v \) of \( S \) or \( S' \), we denote by \( G_v \) its stabilizer in \( G \), and by \( A_v \) its stabilizer for the \( A \)-action. Similarly, we denote by \( A_v \) the stabilizer of an edge \( e \) for the \( A \)-action.

**Lemma 6.8.** The actions \( A \rhd S \) and \( A \rhd S' \) belong to the same deformation space, i.e. there exist \( A \)-equivariant maps from \( S \) to \( S' \) and from \( S' \) to \( S \).

**Proof.** By symmetry of roles of \( S \) and \( S' \), it is enough to show that every point stabilizer for the actions \( A \rhd S \) fixes a point in \( S' \). If \( v \in S \) is a vertex with \( G_v \not= \{1\} \), then \( A_v = \{ \phi \in A | \phi(G_v) = G_v \} \), and \( A_v \) fixes the unique point \( v' \) fixed by \( G_v \) in \( S' \). Let now \( w \in S \) be a vertex with \( G_w = \{1\} \). Let \( v \in S \) be a point with \( G_v \not= \{1\} \) such that the segment \([w,v]\) does not meet any other point with nontrivial \( G \)-stabilizer. Since every automorphism in \( A \) acts as the identity on the quotient graph \( S/G \), for every \( \phi \in A_w \), the isometry \( H_\phi \) fixes \([w,v]\) pointwise. Therefore \( A_w \subseteq A_v \), so \( A_w \) fixes a point in \( S' \). \( \square \)

**Lemma 6.9.** Let \( e, e' \subseteq S \) be two edges. If \( A_e \subseteq A_{e'} \), then \( A_e = A_{e'} \).

**Proof.** Let \( \phi \in A_{e'} \). Since \( \phi \) acts trivially on \( S/G \), there exists \( g \in G \) such that \( \phi.e = ge \), i.e. \( \phi.e = \text{ad}_g.e \). Then \( (\text{ad}_{g^{-1}} \circ \phi).e = e \), so by assumption \( (\text{ad}_{g^{-1}} \circ \phi).e' = e' \), so \( ge' = \phi.e' = e' \). Since \( S \) has trivial arc stabilizers for the \( G \)-action, it follows that \( g = 1 \), so \( \phi.e = e \), i.e. \( \phi \in A_e \).

**Corollary 6.10.** The actions \( A \rhd S \) and \( A \rhd S' \) have the same edge stabilizers.

**Proof.** By Lemma 6.8 there exists an \( A \)-equivariant map from \( S \) to \( S' \). Therefore, for every edge \( e' \subseteq S' \), there exists an edge \( e \subseteq S \) such that \( A_e \subseteq A_{e'} \). By symmetry, there also exists an edge \( e'_2 \subseteq S' \) such that \( A_{e'_2} \subseteq A_e \). We thus have \( A_{e'_2} \subseteq A_e \subseteq A_{e'} \), and Lemma 6.9 implies that these subgroups are all equal. \( \square \)

Let \( v \in S \) be a vertex, and \( e \subseteq S \) be an edge incident on \( v \). Then every automorphism in \( A_v \) preserves \( G_v \), and since \( A \subseteq \text{Aut}(G,F(v)) \), there is a restriction map \( \rho_{v,e} : A_v \to \text{Inn}(G_v) \). In the following statement, we denote by \( \overline{A} \) the image of \( A \) in \( \text{Out}(G,F(v)) \).

**Lemma 6.11.** Assume that \( \overline{A} \) is non-amenability. Then there exist a vertex \( v \in S \) and an edge \( e \) incident on \( v \) so that \( \rho_{v,e}(A_v) \) is non-amenability.

**Proof.** Given a vertex \( v \in S \), we let \( E(v) \) be a set of representatives of the \( G_v \)-orbits of edges incident on \( v \). We denote by \( Z(G_v) \) the center of \( G_v \), which we diagonally embed into \( G_v^{E(v)} \). There is an injective morphism

\[
\theta : \overline{A} \to \prod_v \left( \frac{G_v^{E(v)}}{Z(G_v)} \right),
\]

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where the product is taken over a set of representatives of the $G$-orbits of vertices of $S$, defined as follows: if $\Phi \in \mathcal{A}$ and $\phi \in \mathcal{A}$ is a representative fixing $v$ and acting as the identity on $G_v$, then to any incident edge $e$, one associates the element $g_{v,e} \in G_v$ such that $\phi.e = g_{v,e}e$; since $\phi$ is well defined modulo conjugation by an element of $Z(G_v)$, the tuple $(g_{v,e})_{e \in E(v)}$ is well defined $Z(G_v)$. Since $G_v^{E(v)}/Z(G_v)$ maps to $(G_v/Z(G_v))^{E(v)}$ with central kernel, we get a map

$$\theta' : \mathcal{A} \to \prod_v \text{Inn}(G_v)^{E(v)}$$

with central kernel. The image of this map is contained in $\prod_v \prod_{e \in E(v)} \rho_{v,e}(\mathcal{A}_e)$: indeed, if the representative $\phi$ of $\Phi$ acting as the identity on $G_v$ sends $e$ to $g_{v,e}e$, then $\Phi$ has a representative that fixes $e$ and acts by conjugation by $g_{v,e}$ on $G_v$. Since $\mathcal{A}$ is nonamenable, there exists a pair $(v, e)$ with $e \in E(v)$, such that $\rho_{v,e}(\mathcal{A}_e)$ is nonamenable. \hfill \Box

**Proof of Proposition 6.6** We assume that $\mathcal{A}$ is nonamenable, and we will prove that there exists a $(G, \mathcal{F})$-free splitting which is compatible with both $S$ and $S'$. Denote by $\mathcal{E}$ the collection of edge stabilizers for the action $A \curvearrowright S$ (equivalently $A \curvearrowright S'$, in view of Corollary 6.10). By Lemma 6.9, the trivial equivalence relation on $\mathcal{E}$ is admissible in the sense of [17, Definition 3.1]. Cylinders of $S$ are defined by

$$\text{Cyl}_e = \bigcup_{A' = A_e} e'$$

and the tree of cylinders $S^c$ is the bipartite simplicial tree having one vertex for each cylinder $C$, one vertex for each point $x \in S$ belonging to at least two cylinders, and an edge joining $x$ to $C$ whenever $x \in C$. Since $S$ and $S'$ belong to the same deformation space (Lemma 6.8), it follows from [17, Theorem 1] that $S^c = (S')^c$. From now on, we let $U := S^c$. By [17, Proposition 8.1], the tree $U$ is compatible with both $S$ and $S'$. Moreover, $U$ is a minimal $A$-tree by [14, Lemma 4.9], hence a minimal $G$-tree (the minimal $G$-tree is $A$-invariant because $\text{Inn}(G)$ is normal in $A$). Thus, it suffices to show that some edge of $U$ has trivial stabilizer in $G$.

Let $(v, e)$ be a pair as in Lemma 6.11 so that $\rho_{v,e}(\mathcal{A}_e)$ is non-amenable. We claim that $\text{Cyl}_e \cap G_{v,e} = \{e\}$. Indeed, let $e' = ge$ for some $g \in G_v$, with $A_{v'} = A_v$. Then for all $\phi \in A_e$, one has $ge = e' = \phi.e' = \phi(ge) = \phi(g)\phi.e = \phi(g)e$. Since edge stabilizers of the $G$-action on $S$ are trivial, we deduce that $\phi(g) = g$. Therefore $g$ is fixed by all the inner automorphisms in $\rho_{v,e}(\mathcal{A}_e)$. Since every non-trivial element of $G_v$ has an amenable centralizer, and $\rho_{v,e}(\mathcal{A}_e)$ is non-amenable, $g = 1$ and $e' = e$. This proves the claim.

Let now $e = (v, \text{Cyl}_e)$ be the corresponding edge of $S^c$. The above claim shows that any element $g \in G$ that fixes $e$ must fix $e$, so $g = 1$. This proves that $e$ has trivial stabilizer for the $G$-action on $U$, which concludes the proof of the proposition. \hfill \Box

**End of the proof of Theorem 6.4** From now on, we assume that $(G, \mathcal{F})$ satisfies the assumptions from Theorem 6.4.

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Recall that a $\mathcal{Z}$-splitting of $(G, \mathcal{F})$ is a minimal, simplicial $(G, \mathcal{F})$-tree whose edge stabilizers are either trivial, or cyclic and nonperipheral. The $\mathcal{Z}$-splitting graph $FZ$ is the simplicial graph whose vertices are the $\mathcal{Z}$-splittings of $(G, \mathcal{F})$, where two splittings are joined by an edge if they are compatible. Its hyperbolicity was proved by Mann \cite{Mann} for free groups, and extended to the context of free products in \cite{Popescu}. Since every optimal folding path projects to an unparameterized quasigeodesic in $FZ$, there exists $R > 0$ such that if $S_t$ is an optimal folding path and $t_1 < t_2 < t_3 < t_4$ and $d_{FZ}(S_{t_2}, S_{t_3}) > R$, then $S_{t_1}$ and $S_{t_4}$ are not compatible with any common free splitting.

Given an arational tree $T$, an optimal morphism $f : S \to T$ and positive real numbers $t < t'$, we let $[t, t']_f$ be the collection of all simplices $\Delta$ in $\text{Simp}$ such that there exists a tree $S' \in \hat{\Delta}$ through which $f$ factors, with $e^{-t'} \leq \text{vol}(S'/G) \leq e^{-t}$. By Proposition \ref{prop:volume_bound} and Remark \ref{rem:volume_bound}, the set $[t, t']_f$ is finite. Given $n \in \mathbb{N}$, we let $m_n(f)$ be the smallest integer such that the projection to $FZ$ of the set $[n, n + m_n(f)]_f$ has diameter at least $R$. Existence of $m_n(f)$ is justified by the following lemma.

**Lemma 6.12.** For all $n \in \mathbb{N}$, we have $m_n(f) < +\infty$.

**Proof.** Since the range of $f$ is arational, it follows from the description of the Gromov boundary of $FZ$ in terms of $\mathcal{Z}$-averse trees given in \cite{Popescu}, together with the fact that arational trees are $\mathcal{Z}$-averse \cite[Proposition 4.7]{Popescu}, that every folding path guided by $f$ projects to an infinite unparameterized quasigeodesic ray in $FZ$. Therefore, by choosing $m$ sufficiently large, we can ensure that $[k, k + m]_f$ contains two simplices whose projections to $FZ$ are at distance larger than $R$ from one another, as required. \hfill $\Box$

Let $M_n(f) := \max_{n \leq k \leq 2n} m_k(f)$. By definition of $R$ and Proposition \ref{prop:amenable_scope}, this implies that for all $t \in [n, 2n]$, the set $[t, t + M_n(f)]_f$ contains two simplices whose common stabilizer is amenable.

**Proof of Theorem \ref{thm:amenability}**. The same argument as in the proof of Lemma \ref{lem:amenable_scope} shows that for all $n, m$, the set $[n, n + m]_f$ depends measurably on $f \in \text{Opt}$. Therefore, for all $n \in \mathbb{N}$, the integers $m_n(f)$ and $M_n(f)$ depend measurably on $f$. Let $\mathcal{F}^{\text{amen}}(\text{Simp})$ be the countable collection of all finite sets in $\text{Simp}$ whose stabilizer in $\text{Out}(G, \mathcal{F}^{(t)})$ is amenable. For all $n \in \mathbb{N}$ and all $f \in \text{Opt}$, we then let $\mu_n(f)$ be the probability measure on $\mathcal{F}^{\text{amen}}(\text{Simp})$ defined as

$$
\mu_n(f) := \frac{1}{n} \int_0^{2n} \delta_{[t, t + M_n(f)]_f} d\text{Leb}(t),
$$

where $\delta_{[t, t + M_n(f)]_f}$ is the Dirac measure on the set $[t, t + M_n(f)]_f$. Then the map $f \mapsto \mu_n(f)$ is Borel. Arguing as in the proof of Proposition \ref{prop:amenability}, we can then associate to every tree $T \in AT$ a sequence of probability measures $\mu_n(T)$ on $\mathcal{F}^{\text{amen}}(\text{Simp})$, depending measurably on $T$, so that for all $n \in \mathbb{N}$ and all $\Phi \in \text{Out}(G, \mathcal{F}^{(t)})$, we have

$$
||\mu_n(\Phi(T)) - \Phi_\ast \mu_n(T)||_1 \to 0
$$

as $n$ goes to $+\infty$. Since the stabilizer of every point in $\mathcal{F}^{\text{amen}}(\text{Simp})$ is amenable, one can apply Proposition \ref{prop:amenability} with $K := \mathcal{F}^{\text{amen}}(\text{Simp})$, and deduce that the $\text{Out}(G, \mathcal{F}^{(t)})$-action on $AT$ is Borel amenable. \hfill $\Box$
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