DYNAMICS OF FRAGMENT FORMATION IN THE NUCLEAR SPINODAL REGION

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(July 12, 1994)

Abstract

The Vlasov-Nordheim equation is solved numerically on a lattice for nuclear matter in two dimensions. We discuss the reliability of the model at normal density and then study the response of the system to small perturbations. We find deterministic chaos inside the spinodal zone where fragment formation occurs. We discuss in detail the dynamical features of this phenomenon in order to clarify the mechanisms leading to nuclear disassembly in heavy–ion collisions.

Pacs: 25.70.Pq, 24.60.Lz, 21.65.+f
Many recent experiments in heavy-ion collisions at energies above 50 MeV/A have shown a very rapid break-up of the hot composite system formed in the reaction into several big fragments with \( Z \geq 3 \). This disassembly of the nuclear system, which can include a substantial fraction of the colliding masses, falls under the generic name of "nuclear multifragmentation". In parallel with experiments there has been an intense theoretical effort in the understanding of the mechanisms underlying the phenomenon. In fact a detailed and systematic study is expected to provide fundamental properties of nuclear matter. However, though many models are able to reproduce the data, the origin of multifragmentation has not been sufficiently clarified up to now.

It seems nowadays established that multifragmentation cannot be explained by the same scenarios valid at lower energies. The characteristic time ranges from one hundred to few hundreds \( fm/c \). Therefore it is a very fast process in comparison with phenomena like fission or compound nucleus formation whose typical time is of the order of thousands \( fm/c \). Mechanical instabilities play a fundamental role and they should be taken into due account when the excited nuclear system enters the spinodal zone of nuclear matter Equation of State (EOS). The latter is the region where the compressibility, i.e. the derivative of the pressure with respect to the density, becomes negative and the system is unstable to small perturbations.

The undoubted success of the statistical models in explaining some of the multifragmentation features, indicates that the phase space dominates the population of the different final channels. In such a fast process, the tendency of filling uniformly the phase space cannot occur in each collisional event. The system has not enough time to explore, during the reaction, the whole phase space, as can happen for compound nucleus collisions. It can be expected, therefore, that the reaction dynamics is dominated by the phase space only when the physical quantities are averaged over large
sets of events. This assumption implies that the nuclear dynamics in the multifrag-
mentation regime is irregular or chaotic enough to produce, at least approximately,
a uniform "a priori" probability to populate each region of the available phase space.
In particular, the formation of the final fragments must follow an irregular dynamics.
This conjecture can be also inferred by the large event-by-event fluctuations observed
experimentally on the charge and mass distributions.

Within this scenario, however, it is still not clear at which stage the dynamics is
dominated by a chaotic behaviour and which is the mechanism able to produce the
expected phase space mixing.

In the present work the problem of fragment formation is approached in a sim-
plified way. It is assumed that the nuclear system after an initial compression slowly
expands entering into the spinodal region. Some evidences of such a stage of the re-
action in central heavy ion collisions have been found in computer simulations based
on the BUU scheme. The fragments are supposed to emerge from this expanded
nuclear system through the unstable density fluctuations of the spinodal zone. We
study in details this phase of the nuclear dynamics and show that it exhibits a chaotic
behavior. In section 1 we discuss a simple example to illustrate the main features
of deterministic chaos. The theoretical background for the nuclear case is based on
the Vlasov equation, which is overviewed in section 2. In section 3 we discuss the
numerical method adopted to solve the Vlasov equation on a lattice and study the
corresponding two-dimensional Equation of State (EOS). The mean field dynamics is
investigated in section 4. The density profile of the initially almost uniform nuclear
system is followed until fragments can be identified. The dynamics of this process is
analyzed in order to investigate the possible non-linear and chaotic behaviour, with
methods analogous to the ones well known in the theory of dynamical systems. To
this purpose, a distance between "trajectories" is introduced, and the analysis in
terms of Lyapunov exponents is performed. In section 5 we discuss the consequences of a non-linear and erratic dynamics for nuclear multifragmentation. Finally in section 6 we draw our conclusions. The results reported in this paper are a complete and detailed review of an investigation started along these lines in refs.\textsuperscript{7,8}.

1. Deterministic chaos in a simple example.

In the last two decades the study of low-dimensional dynamical systems has given a fundamental contribution to the understanding of the onset of irregular motion. Due to the nonintegrability of the sets of deterministic equations an erratic and unpredictable behaviour, \textit{deterministic chaos}, can follow. What was previously considered as spurious noise has found rigorous mathematical and experimental foundations. This revolutionary concept has changed drastically our view of classical mechanics and is revealing important implications also in quantum physics. For a general review see ref.\textsuperscript{6} The consequences of this \textit{new paradigm} which is shading light into apparently unrelated different fields are difficult to overcast and will probably be an important guideline of research for the next decades. In this work, we explore the dynamics of nuclear matter from this new perspective. But before going to the discussion of the main topic, we will consider the dynamics of a very simple system. This example is used to illustrate the main features of a chaotic unstable system and the conditions under which deterministic chaos can occur.

Let us consider the two–dimensional problem described by the hamiltonian

$$\mathcal{H} = \frac{1}{2}(p_1^2 + p_2^2) + q_1^2 + q_2^2 - 2\mu q_1^2 q_2^2$$

which corresponds to two coupled harmonic oscillators with unit mass. It can be easily seen that, for a given total energy \(E\), the available space has a fourfold structure, with four “branches” extending to infinity. If \(\mu \neq 0\), the dynamics of the particle inside the
potential is chaotic. In other words, it is impossible to find an analytical solution for the equations of motion corresponding to the Hamiltonian (1). We have two degrees of freedom and only one constant of the motion, \textit{i.e.} the total energy. This implies an extreme sensitivity of the dynamics on the initial conditions and a very irregular motion.

One can consider a particle starting its motion inside this potential or impinging on it from outside. In the latter case the chaoticity of the interaction region manifests itself in strong irregularities, at all scales, of the final observables - final scattering angle, final internal excitation, etc. - as a function of the initial conditions. We have the well known phenomenon of \textit{chaotic scattering} \cite{10}. An incoming particle can be trapped for a long period inside the interaction region, where it bounces erratically to and fro before escaping. The extreme internal sensitivity influences drastically the final result.

Classical chaotic scattering is therefore intimately connected with the presence of trajectories which remain trapped in the interaction region for a time long enough that chaos can set in.

These considerations can be readily extended to the case of trajectories which starts directly from the interaction region and which therefore will describe systems which are unstable and decay. In particular, the Hamiltonian (1) does not describe a proper scattering situation, since the potential diverges at infinity. However it can describe a system which escapes from the interaction region after some trapping time. Since the system is chaotic, again the trapping time and the direction of emission depend in a very irregular way on the initial conditions. This point is illustrated in fig.1. For each trajectory starting from the internal region, one of the coordinates of the particle is recorded. The distance at which we take this value is so large to be considered asymptotic. The particle cannot return anymore to the interaction region.
In the plot, the value of the final coordinate $q_f$ is reported as a function of the initial one $q_i$. It is important to notice that the considered set of initial conditions lies on the same energy surface. The observed irregular structure is very similar to the one usual encountered for the deflection function in chaotic scattering systems. If chaotocity is strong enough, the distribution of the trapping time is actually exponential. More formally, if we call $\mathcal{T}$ the average trapping time and $\tau = 1/\lambda$ the average rate of divergence between trajectories, being $\lambda$ the average Lyapunov exponent, one can write, according to Tel,

$$\mathcal{T} = \tau/(1 - D)$$

(2)

where $D$ is the average linear Hausdorff dimension of the so-called strange repellor. If $D > 0$, which is the condition for a genuine chaotic scattering, then $\mathcal{T} > \tau$.

This condition can be used as a criterium to argue if a system presents a chaotic dynamics or not. It is useful in applications, because it is usually numerically difficult to assess the existence of positive $D$ by an extensive direct sampling of the whole phase space. Still it must be used with caution, due to the unavoidable uncertainties in the numerical values of the Lyapunov exponents, and therefore it can give only an indication of a possible chaotic dynamics.

The analysis can be extended to models with many degrees of freedom. In particular, for a fluid system the relevant degrees of freedom can be identified with its linear eigenmodes, and in the harmonic approximation a fluid is exactly equivalent to a set of harmonic oscillators. Such a system, which corresponds to the case $\mu = 0$ in eq. (1), is integrable, since the energy of each mode is conserved. Therefore, in this limit a fluid can undergo only a regular dynamical evolution. If the amplitude of the perturbation is large enough, the modes of the fluid can be coupled between each other, as in eq. (1) for $\mu \neq 0$. In this case the problem is, in general, non–integrable.
and chaotic dynamics can be present.

If a uniform homogeneous fluid is initially in its spinodal region, where the compressibility is negative, it will spontaneously "escape" towards a non-homogenous phase, where droplets are formed. However, before the final droplets are formed, the fluid can spend enough time in the spinodal region to allow non-linearity and chaotic motion to set in and dominate the time evolution. The situation is quite analogous to the example of eq. (1), where the particle escapes outside the interaction region but has enough time to experience the chaotic dynamics present in the inner sector of the potential, as illustrated in fig.1. If chaoticity is really present, then the average Lyapunov exponent $\bar{\lambda}$ must be positive and, as discussed above, the criterium $\mathcal{T} > \tau$ should be satisfied. Here $\mathcal{T}$ is the characteristic time for droplet formation and $\tau = 1/\bar{\lambda}$ is the average time of divergence between "trajectories" of the fluid. In the next sections we study - within the semiclassical approximation - the dynamics of nuclear matter inside the spinodal region. We will show, following a reasoning similar to the one discussed above, that nonlinearities and positive Lyapunov exponents are found inside the spinodal region. Therefore deterministic chaos characterizes the formation of fragments.

2. Theoretical background

2.1. The Vlasov equation

In the multifragmentation energy range the dynamics is characterized by an interplay between a purely mean field evolution, typical of a low-energy phenomenology (fusion, inelastic reactions), and two-body collisions due to the partial relaxation of the Pauli principle. A possible way of incorporating both of these effects is the inclusion of the residual interaction in models like TDHF, but this procedure is diffi-
cult to carry on numerically in realistic calculations. For this purpose semiclassical methods have been developed, whose starting point is the Wigner transform of the time-dependent Hartree-Fock (TDHF) equation for the one-body density matrix. By neglecting powers of $\hbar$ higher than two, one obtains the usual Vlasov equation which reads

$$\frac{\partial f}{\partial t} + \frac{p}{m} \cdot \nabla_r f - \nabla_r U[\rho(r)] \cdot \nabla_p f = 0$$

(3)

$f(r, p; t)$ is the Wigner phase-space distribution function, $(r, p)$ are the space and momentum coordinates and $U$ is a self-consistent single-particle potential depending on the density $\rho = \int d^3p \, f$. When eq.(3) includes a Nordheim-type collision integral on the right-hand side, $I[f]$, it is generally referred in literature as $BUU$ (Boltzmann-Uehling-Uhlenbeck) or $VUU$ (Vlasov-Uehling-Uhlenbeck) or $LV$ (Landau-Vlasov) or $VN$ (Vlasov-Nordheim) equation. For a complete review, see ref. The collision integral reads

$$I[f](r, p_1, t) = \int d^3p_2 \, d^3p_{1'} \, d^3p_{2'} \left( f_{1'} f_{2'} \bar{f}_1 \bar{f}_2 - f_1 f_2 \bar{f}_{1'} \bar{f}_{2'} \right) \omega_{121'2'}$$

(4)

where $f_j = f(s_j, t)$ is the phase-space occupation probability at the location $s_j = (r, p_j)$, and $\bar{f}_j = 1 - f_j$ is the Pauli blocking factor of the final states. $\omega_{121'2'}$ is the microscopic transition rate for the collision vertex $(p_1, p_2) \rightarrow (p_{1'}, p_{2'})$, and is related to the nucleon-nucleon scattering cross section. This collision term describes the momentum changes of two interacting particles during a collision with blocking factors forbidding transitions leading to occupied final states.

We notice the formal identity of eq.(3) with the Liouville equation of classical mechanics for a fluid of non-interacting particles. This underlines its intrinsic classical character and guarantees that quantum effects like the Pauli principle are conserved. It can be easily demonstrated that the Vlasov-Nordheim equation satisfies the conservation laws of mass, momentum and energy. The presence of an average potential $U$
dependent on the density introduces a non-linear constraint on the solution of eq.(3). This is a common point to all mean field theories.

Because of that, the Vlasov-Nordheim equation is difficult to solve numerically, in spite of its apparent simplicity. A large amount of literature has been devoted to its numerical solution techniques in the case of systems interacting through Coulomb fields like plasmas, to which the Vlasov equation was first applied.

Later on we will discuss the implications of non-linearity on physical processes like multifragmentation.

2.2. Linear response for the Vlasov equation

The linearized version of the Vlasov equation has been successfully applied to the study of small amplitude oscillations, see ref.\textsuperscript{17}, and it represents the starting point for a phase space approach to RPA. For small variations of the distribution function around the equilibrium solution \( f_0(r, p^2) \)

\[
f(r, p, t) = f_0(r, p^2) + g(r, p, t) \tag{5}
\]

we can expand eq.(3) to the first order in g. A Thomas-Fermi approximation for the ground state - which is obviously a solution of eq.(3) - can be used. That is

\[
f_0(r, p^2) = \frac{4}{(2\pi\hbar)^D} \Theta(E_F - \frac{p^2}{2m}) \tag{6}
\]

where D is the dimension of the physical space. Then one gets an equation for the Fourier transform of \( g(r, p, t) \) which can be analytically solved and produces a dispersion relation for the frequency \( \omega \) for each wave number \( k \). For stable systems the linear response theory yields real values for the energies \( \omega = \omega(k) \) of the normal modes, which can be represented by plane waves propagating in opposite directions. When the system is unstable, some energies are complex and the imaginary part
is related to the time growth of instabilities. In both cases, the dependence of the energy $\omega$ on $k$ is linear in this approximation, with a correction factor depending on the equation of state and the density of the system, see ref.16,17,18. We like to remind that the linear response approximation has a limited range of validity, since it can be applied only to small amplitude variations of the initial reference state. Therefore its applicability to processes like multifragmentation, where large fluctuations are involved, might be not completely correct, as it will be discussed later.

3. The model

3.1. The lattice calculation

As previously explained in the introduction, we study the behavior of a dilute nuclear system whose density fluctuations are growing inside the spinodal region. In order to get numerically robust results, we solved the Vlasov-Nordheim equation on a lattice, using the same code of ref.16, but neglecting the stochastic contribution to the collision integral. We divide the single particle phase space into several small cells, each of volume $V^D = \Delta r^D \cdot \Delta p^D$, being $\Delta r$ and $\Delta p$ the cell sidelongths respectively in coordinate and momentum space. Most numerical problems in calculating the Vlasov evolution of nuclear systems arise from the need to smooth the one-body density. For this purpose we must employ a lattice with a big number of small cells in order to get a nice paving of the phase space. Typically $\Delta r$ is 1 fm or less, while $\Delta p$ should be smaller than the Fermi momentum. Of course the discretization introduces some numerical error on the physical observables, but it gets smaller and smaller as we decrease the cell size and increase their number. Therefore the main limit of the lattice method is the memory resources of the computers and the huge computing time requested. For this reason, we performed all the calculations on a two-dimensional lattice.
We have studied a fermion gas situated on a large torus with periodic boundary conditions, and its size is kept constant during the evolution. The torus sidelengths are equal to $L_x = 51 \, fm$ and $L_y = 15 \, fm$. We employed in momentum space 51x51 small cells of size $\Delta p_x = \Delta p_y = 40 \, MeV/c$, while in coordinate space $\Delta x = 0.3333 \, fm$ and $\Delta y = 15 \, fm$, i.e. we have only one big cell on the $y$-direction.

The initial local momentum distribution was assumed to be the one of a Fermi gas at a fixed temperature $T$. We employ a local Skyrme interaction which generates a mean field $U[\rho] = t_0 (\rho/\rho_0) + t_3 (\rho/\rho_0)^2$. The density $\rho$ is folded along the $x$-direction with a gaussian of width $\mu = 0.61 \, fm$, in order to give a finite range to the interaction. The parameters of the force $t_0$ and $t_3$ have been chosen in order to reproduce correctly the binding energy of nuclear matter at zero temperature, and this gives $t_0 = -100.3 \, MeV$ and $t_3 = 48 \, MeV$. The resulting EOS gives a saturation density in two dimensions equal to $\rho_0 = 0.55 \, fm^{-2}$ which corresponds to the usual three-dimensional Fermi momentum equal to $P_F = 260 \, MeV/c$.

Then a complete dynamical evolution is performed by subdividing the total time in small time steps, each equal to $\Delta t = 0.5 \, fm/c$.

For more details concerning the mean field propagation on the lattice and the exact calculation of the collision integral, the reader is referred to ref.16.

3.2. Two-dimensional nuclear matter equation of state

We calculate the nuclear matter equation of state (EOS) for an homogeneous two dimensional system. We mainly follow the definitions given in ref.19 for the three dimensional case. For a gas of particles interacting through a local Skyrme single-particle potential, $U[\rho]$, we calculate the density of the free energy $F$ and the corresponding pressure $P$. They read

$$F = H - TS$$

(7)
\[ P = -F + \rho (\mu + U[\rho]) \] (8)

In eq.(7) \( H \) is the spatial energy density, and it is given by summing the kinetic energy density \( K \) and the potential energy density \( E = \int U[\rho]d\rho \). \( \mu \) denotes the chemical potential, \( T \) is the temperature and \( S \) is the density of entropy, for which we use the standard definition for noninteracting particles.\(^{19}\)

If we define the parameter

\[ \eta = \frac{\mu}{T} \] (9)

the single-particle density, the kinetic energy density and the entropy density can be expressed through the Fermi integrals \( J_0 \) and \( J_1 \)

\[ \rho = \frac{2m}{\pi\hbar^2} J_0(\eta) \] (10)

\[ K = \frac{2m}{\pi\hbar^2} J_1(\eta) \] (11)

\[ S = \frac{4m}{\pi\hbar^2 T} J_1(\eta) - \eta \rho \] (12)

where the Fermi integral is given by

\[ J_\nu(\eta) = \int \frac{e^\nu d\epsilon}{1 + e^{\frac{\mu T}{\hbar^2} - \eta}} \] (13)

Since the integral \( J_0 \) is analytical, the chemical potential \( \mu \) is readily calculated from the single-particle density \( \rho \)

\[ \mu = T \log[\exp(\frac{\pi\hbar^2 \rho}{2mT}) - 1] \] (14)
From the above definitions, the expressions for the density of the free energy and the pressure are

$$F = -\frac{2m}{\pi h^2} J_1(\eta) + E + \mu \rho$$  \hspace{1cm} (15)

$$P = \frac{2m}{\pi h^2} J_1(\eta) - E + \rho U[\rho]$$  \hspace{1cm} (16)

In Fig.2 we show the two-dimensional nuclear matter equation of state (EOS), calculated for the Skyrme interaction we discussed in the previous subsection. In the part a) of the figure, the free energy per particle is displayed along the isotherms for different values of the temperature $T$ as function of the density, while in part b) the corresponding thermostatic pressure is shown. This recalls the qualitative features of a classical Van der Waals gas. We notice that the general trend of a two-dimensional EOS does not differ appreciably from the realistic three-dimensional case.\[\text{[19]}\]

It has to be noticed that the pressure $P$ can be also calculated by the thermodynamical relationship $P = -\rho \frac{\partial F}{\partial \rho}$, being $F$ the free energy per particle. It can be readily verified that this procedure gives the same expression of eq.(16). The latter result indicates that in our particular simple model of nuclear matter, the thermodynamical relationships are exactly satisfied. This is a consequence of the simple form adopted for the interaction.

The spinodal zone can be identified in fig.2b) as the region where the compressibility, i.e. the slope of the pressure $P$ as a function of the density $\rho$, is negative. At increasing temperature the density interval where $\frac{\partial P}{\partial \rho}$ is negative becomes smaller and smaller. Finally at the critical temperature $T_c$ for the liquid-gas phase transition the spinodal zone reduces to a point where the corresponding isotherm has an inflection. For $T > T_c$ the pressure $P$ is a monotonic increasing function of $\rho$. From the figure one can deduce $T_c \sim 16 \text{ MeV}$.\[\text{[13]}\]
4. Mean field dynamics

In this section we want to study in detail the response of nuclear matter to small perturbations. We neglect for the moment the collision term I(f) and solve numerically only the Vlasov equation. If non-linear terms are negligible, linear response theory gives an accurate description of the time evolution. However we will see that this is not always the case.

Let us consider an initial sinusoidal perturbation along the $x-$direction in the density profile. That is

$$\rho(x) = (1 + \delta \sin(kx)) \bar{\rho}, \quad (17)$$

with

$$k = \frac{2\pi n_k}{L_x}, \quad (18)$$

being $L_x$ the size of the torus along the $x$-direction and $\bar{\rho} = \int dx \rho(x)/L_x$. We study the time evolution of this profile considering a small amplitude for the perturbation, i.e. $\delta = 0.01$. In fig.3 and 4 we show this evolution for $\bar{\rho}/\rho_0 = 0.8$ for the cases $n_k = 4, 5$. In these figures the mode initially excited is damped in a time range which is around $40 \text{ fm/c}$. This is the usual Landau damping of Fermi liquid theory which is a well known linear phenomenon. Actually, since we are working in a discrete lattice, we have also a small initial excitation of a subharmonic of higher order, i.e. $n_k = 12$ and $n_k = 25$, which is also damped very rapidly. In a previous paper we had shown a similar damping but at normal density. In that case the damping is slower and occurs within $70 \text{ fm/c}$. The Landau damping for nuclear matter in two dimensions is an important issue by itself and will be discussed in detail in a forthcoming paper.

The behavior shown in the two previous figures indicates that initial perturbations evolve linearly in time outside the spinodal zone. Moreover a small difference in the
initial conditions does not produce a different dynamical evolution, as we will discuss later.

The situation is completely different if one analyses the time evolution inside the spinodal zone, \( \frac{\bar{\rho}}{\rho_0} < \frac{2}{3} \). In fig. 5 we display the evolution of three initial perturbations at \( \frac{\bar{\rho}}{\rho_0} = 0.4 \). The small initial modulations evolve and form fragments with different shapes and sizes. In our case, due to the fact that we are considering a two-dimensional torus of nuclear matter, no real fragmentation occurs and we call “fragments” the macroscopic structures of the density profile. The behavior shown in fig. 5 is typical of a dynamical system in a chaotic region, where very small initial perturbations are rapidly amplified and distorted. The relevant characteristic is that the density profiles do not increase rapidly in their amplitude only, they also change their shape strongly during the time evolution. These features are strong indications of a non-linear regime.

In fig. 6 we show the power spectra corresponding to the modes involved in fig. 5. These spectra are different in each case and illustrate a population of more and more modes as the time evolution goes on. Hence the behaviour of the system for \( \frac{\bar{\rho}}{\rho_0} = 0.4 \) show a sensitive dependence on the initial conditions and an unpredictability of the evolution for a given initial perturbation. Small initial differences will produce large final deviations. A similar feature was illustrated in ref. 7 for \( \frac{\bar{\rho}}{\rho_0} = 0.5 \). As we will clarify in the following this behavior is typical inside the spinodal region and does not depend on the particular choice of varying the initial conditions. In the present case we have changed the initial \( n_k \), but we could have modified either the average density or the amplitude of the initial perturbation.

The sensitivity to the initial conditions occurs, in the dynamical evolution, together with a dominant role of the non-linear terms in the Vlasov equation. This is proved by the time evolution of the strength of the most important modes as illus-
trated in fig. 7. In the latter the numerical calculations are reported as open squares in steps of 5 \( fm/c \). Linear fits of the initial evolution are shown for comparison. It is evident that the growth is linear only in the first stages of the evolution while it strongly deviates from the fits after \( \sim 25 fm/c \). In general one cannot trust linear response theory after the very first dynamical stages. It is true that some modes follow a linear evolution for a longer time, however in general they do correspond to very small wavelengths. Therefore they do not affect the size of the main emerging fragments but only their surface modulations. One can consider \( \lambda/2 \) as the diameter of the nuclear fragment formed. In concluding this section one can say that when the system lies inside the spinodal region many modes are coupled and they interact strongly among each other. The final fragments are the result of this dynamics which goes beyond the linearized equations discussed in section 2.2.

### 4.1. Calculation of the largest Lyapunov exponent

The arguments discussed so far provide a strong indication that the dynamics is chaotic beyond the spinodal line, but they do not provide a real proof that this is so. Therefore in this section we present numerical calculations which define the degree of chaoticity according to the usual techniques adopted in the study of low-dimensional dynamical systems. The discussion follows the same lines of ref. but gives a more complete and detailed overview.

A way to characterize quantitatively the dynamics in a chaotic regime is by means of the rate of divergence of two nearby trajectories. The mean rate is usually called largest Lyapunov exponent and it is calculated by means of the expression

\[
\overline{\lambda} = \lim_{t \to \infty} \lim_{d_0 \to 0} \lambda(t)
\]  

(19)
where
\[ \lambda(t) = \frac{\log(d(t)/d_0)}{t} . \]  
(20)

In the latter \( d(t) \) is the distance between two phase space trajectories, along an unstable direction. In our notation \( d_0 = d(0) \). We have chosen as metric the following one

\[ d(t) = \sum_i |\rho^{(1)}(x_i, t) - \rho^{(2)}(x_i, t)|/N_c , \]  
(21)

where the index \( i \) runs over the \( N_c \) cells in ordinary space, and \( \rho^{(1)}(x_i) \), \( \rho^{(2)}(x_i) \) are the densities in the cell \( x_i \) for the trajectories 1 and 2 respectively. The definition of eq. (21) represents the difference between the two density profiles and is sufficient for our purpose, although possible differences in momentum space are averaged out. It should include the contribution of all the unstable modes which dynamically grow up during the evolution.

In fig. 8 we show the time evolution of \( \lambda(t) \) for various average densities. Inside the spinodal region \( \lambda(t) \) converges to a limiting value \( \lambda \), represented by a dashed line, which is just the largest Lyapunov exponent. This convergence occurs only for a limited time interval, which is the one needed for fragment formation. As it will be discussed below this is because we are describing a transient phenomenon. A different behavior is observed when the system evolves outside the spinodal region. In fact in this case the trajectories do not diverge exponentially and \( \lambda(t) \) goes slowly towards zero as time increases. This result is analogous to the general case occurring in simple dynamical systems, when \( \lambda \) finite and positive distinguishes a chaotic behavior from a regular one where \( \lambda = 0 \). The values of \( \lambda \) at a temperature \( T=3 \) MeV are reported on the second column of table 1. Some of the features here discussed were already presented in ref.\(^7\) where actually the evolution was followed for a shorter
time. For the calculations presented in this paper we improved the algorithm used and we could follow the time evolution for a longer period maintaining an error in the energy conservation smaller than 1%. Now one can see that, inside the spinodal zone, $\lambda(t)$ remains around the plateau - which indicates the value of the largest Lyapunov exponent - only for a limited time range. This fact has a simple explanation. In fact, the time at which the plateau stops corresponds to the primary fragment formation. Hence after fragments are formed, the different modes interact weaker than before among each other, because along the $x$-direction there are many points where the density is almost zero. These fragments stop increasing their size and start to move along the $x$-direction. They eventually collide against each other and therefore, in effect, the fragment configuration continues to change. However this dynamical regime is different from the previous one and we will not consider the time evolution following the plateau. It is important to notice that estimates of the expansion time in the BUU framework are surely larger than $100 \, fm/c$. Hence the expansion process is expected to hardly affect the dynamics of fragment formation, while it could have some effects on the successive stages of the reaction. A realistic investigation of this following regime can be approached only by a fully dynamical simulation and it is beyond the scope of the present work.

Now we discuss the reliability of the numerical estimates for $\bar{\lambda}$ in our calculations. In fig.9 we show the dependence of $\bar{\lambda}$ on the initial distance $d_0$ for average densities $\bar{\rho}/\rho_0=0.4, 0.5$. There is a strong dependence of $\bar{\lambda}$ on $d_0$ for large values, which however saturates for values smaller than $\sim 5 \cdot 10^{-3} \, fm^{-2}$. In the calculations displayed in fig.8 we actually used a value $d_0 \leq 10^{-4} \, fm^{-2}$, as in ref.\textsuperscript{7}, which lies in the saturation zone and therefore assures a stable numerical result. Last but not least, we also checked that $\bar{\lambda}$'s do not change varying the size of the torus or changing the time step of the integration\textsuperscript{8}. In fig.10 the dependence of $\bar{\lambda}$ on the initial $n_k$ is displayed for
\[ \bar{\rho}/\rho_0 = 0.4, 0.5. \] Even in this case the insensitivity to the variation of the wave number assures that the value of the Lyapunov exponent is numerically stable. It is important to notice that \( \bar{\lambda} \) is different from the growth rate calculated in the linear response theory, where one should expect a linear dependence with the largest mode having the fastest growth. Only if the system is linearly unstable the Lyapunov exponent and the growth rate do coincide. This is not true in our case where the latter is smaller by almost a factor of two.

Up to now the momentum distribution adopted was always that one of a Fermi gas at temperature \( T = 3 \) MeV. We varied also this temperature to see the effect on \( \bar{\lambda} \). In fig.11 the dependence of \( \bar{\lambda} \) on the temperature is shown. We found a small dependence on the temperature which indicates that thermal motion seems to reduce only slightly the degree of chaoticity. This is due to the shrinking of the spinodal zone at increasing temperatures, where vaporization processes start to compete with instabilities.

The calculations discussed up to now refer only to the Vlasov equation. One could wonder how much the inclusion of two-body collisions can modify the discussed scenario. To this purpose, we solved the Vlasov-Nordheim equation numerically and calculated the Lyapunov exponents also in this case. In fig.12 we show the comparison of \( \lambda(t) \) calculated with and without two-body collisions. The figure shows that the inclusion of the collision integral (4) in the r.h.s of eq. (3) influences only slightly \( \bar{\lambda} \) reducing its value. This reduction is of the order of 30\% for \( \bar{\rho}/\rho_0 = 0.5 \), but only of 10\% for \( \bar{\rho}/\rho_0 = 0.4 \). It is not difficult to understand this result by considering the diluteness of the fermion gas inside the spinodal region.
4.2. Time scales

In this section we discuss the interplay of the different time scales occurring in fragment formation.

In fig.13 we show the comparison of the Lyapunov time $\tau_{\text{chaos}} = 1/\lambda$ and the time needed to form the primary fragments $\tau_{\text{frag}}$, corresponding to the time end of the plateau. The former is always smaller than the latter. It is important to notice that $\tau_{\text{frag}}$ is analogous to the trapping time $T$ discussed in section 1. Hence according to the criterium there introduced for the simple example, the relation $\tau_{\text{frag}} > \tau_{\text{chaos}}$ implies that, during the interval of time in which fragments are formed, deterministic chaos can fully develop and characterizes the clusterization.

An additional evidence of that can be also seen in fig.7, where some modes, despite the overall growth of density fluctuation, can have amplitudes oscillating in time. This behaviour is strongly reminiscent of the erratic bouncing of the particles inside the potential of the example discussed in section 1. Of course the dynamics is followed in a finite time interval and therefore in our case chaoticity is only a transient phenomenon. Poincaré maps like in ref. 22 cannot be drawn. We believe however that the evidence found are strong enough to justify at least a strong analogy of the above discussed dynamics with the more familiar low-dimensional chaoticity in closed and scattering systems.

It can be instructive to compare the characteristic times $\tau_{\text{chaos}} = \hbar/\lambda$, which defines the time scale of the divergence between mean field trajectories, with the single particle characteristic time $\tau_{sp} = \hbar/E_F(\rho)$, being $E_F$ the Fermi energy at the given density. This is done in table 1 for a set of densities and for a temperature $T=3$ MeV. One can see that for densities $\rho \leq 0.4\rho_0$ the divergence time is smaller than the single particle time. In other words the motion associated with the mean field is faster than that of the particles moving inside. Therefore in this region the notion
itself of mean field ceases to have validity, being the gas too dilute.

5. Chaoticity implications for multifragmentation

In this section we discuss what are the consequences of a chaotic dynamics in the spinodal zone for the final fragments formation. In our schematic description we are neglecting the initial compression induced by the heavy ion collision which likely leads the composite system well inside the spinodal zone. However one could simulate the uncertainty in the initial conditions by considering in our model a small and random initial perturbation in the density profile. This is also another way of studying the response of our system inside the spinodal region. In particular we start the time evolution giving to each cell along the $x$-direction a random shift around the average density whose strength is 1% of the average. Also with this initialization the time evolution is non-linear and show the same features already discussed in the previous section for a sinusoidal shape. This behaviour is shown in figs.14 and 15 where the time evolution of two random initialized events are compared by displaying the density profiles and the power spectra at different times. In fig.14 the scale at $t=0$ is different to magnify the random initial shape. The power spectra of this shape is rather flat as fig.15 illustrates. During the first evolution a wide range of modes are privileged due to the finite interaction range and then among these an erratic and hardly predictable mixing occurs. The dynamics of the single modes exhibits the same non-linear trend illustrated in fig.7. We have checked that calculating the Lyapunov exponent with a random initialization one gets values identical to those shown previously.

The initial random shape is on one hand more realistic, simulating the missing dynamics, and on the other hand it gives the opportunity to sample many initial
conditions. In this perspective we performed 108 runs for an average density $\bar{\rho}/\rho_0=0.5$ and extracted the fragment size distribution out of each one. In our case we do not have real fragments but a natural criterium is to consider those contiguous cells whose density exceeds a threshold value as forming a cluster. We adopted a freeze-out time which is slightly larger than that shown in fig.13 and defined as $\tau_{\text{frag}}$ and is equal to 120 $fm/c$. This was necessary because the dynamics is somehow slower when many modes are coupled since the beginning. Therefore the time for fragment formation shown in fig.13 should be considered a minimum estimate. In fig.16 we show the average fragment size distribution $P(s)$ for three different threshold cuts equal to 5,7 and 10 % of the normal density $\rho_0$. As one can see the three distributions are rather large and do not differ significantly between each other. In the same figure two fits for each distribution are shown. The dashed curve is an exponential fit, $i.e. P(s) \sim e^{-\tau s}$, while the full curve corresponds to a power law fit, $i.e. P(s) \sim s^{-\tau}$. The corresponding $\tau$ and the relative errors are reported in the figure. The fits shown in the figure were performed up to a maximum value $s_{\text{max}} = 20$. In table 2 we report the values obtained, together with the respective $\chi^2$, by fitting the three distributions up to $s_{\text{max}} = 20, 30$. In all cases the power law fits have a smaller $\chi^2$ than the exponential ones. Though the difference is small, a power law fit always seems more appropriate. The value of the $\tau_{\text{power}}$ which one can extract from the power law fits oscillates around 2. In a preliminary analysis of this kind we took into account 100 events considering a freeze-out time smaller and a larger cut. However a power law fit gave also in that case a similar value for $\tau_{\text{power}}$. Thus the more refined present analysis does not change the previous results.

One could be tempted to compare these calculations to those published recently concerning experimental data and theoretical considerations. In fact, the large distributions shown in fig.16 remind those observed experimentally and the exponents
found are also very close to those of percolation at criticality. Actually, these similarities could be accidental. Our model is at the moment too schematic and the criteria adopted are rather arbitrary to draw final conclusions. However, we think it is very important the fact that a deterministic chaotic behavior allows for final large distributions which resemble those found in experimental data. On the contrary we note that within a linear unstable evolution the most unstable mode has the largest growth rate and wins always over the others. Therefore, in the linear regime, the path followed by the system in its dynamical evolution from a random small initialization towards a macroscopic fragmentation is always the same, apart from a very small spreading width. Hence one obtains a strongly peaked and gaussian size distribution. Such a kind of multifragmentation has not yet been observed experimentally.

In concluding this section we claim that a deterministic chaotic mechanism is the most natural explanation for the multifragmentation data. In a chaotic regime even selecting events with the same impact parameter, energy, temperature, etc. the small uncertainties which will always be present would lead inevitably to very large fluctuations with mass and charge distributions ranging from an exponential to a power law. Chaoticity does not exclude phase transitions or statistical hypotheses. On the other hand it seems likely to be a general feature underlying these scenarios.

6. Conclusions

The growth of density fluctuations, in the spinodal region of nuclear matter EOS, has been advocated as the mechanism of fragment formation in the multifragmentation process observed in heavy ion collisions. According to this scenario, the nuclear fragment formation is similar to the process of droplet formation in an over-saturated classical vapour. We have studied the dynamics of such a process by solving the
Vlasov–Nordheim equation numerically on a lattice. The nuclear system in the spinodal region was schematized by a two-dimensional fluid confined inside a fixed torus. The initially homogeneous system was perturbed by adding a small density fluctuation and the dynamics was followed up to the time of fragment formation, when several well separated density humps are apparent. Several characteristics, typical of chaotic dynamics, have been identified and studied.

i) Strong sensitivity to the initial conditions is apparent in the evolution of the system. Small variations of the initial conditions lead to large deviations in the final pattern of the density profile and therefore of the fragment distribution.

ii) The amplitudes of the modes increase, in general, non–exponentially, with possible oscillating behaviour, at least for the wavelengths relevant for the fragment formation (\( \lambda > 6 \text{fm} \)). This indicates that the modes are coupled and the dynamics is highly non–linear.

iii) The analysis in terms of Lyapunov exponents gives values different from zero and independent from the wavelength of the initial perturbation. This is in sharp contrast with the expectations for a linear dynamics, for which the inverse Lyapunov exponent coincides with the growth rate of the fluctuation and is proportional to the momentum of the mode.

iv) The growth rate of the density fluctuations is longer than the inverse average Lyapunov exponent, and therefore chaotic dynamics has time to develop during the process of fragment formation. This criteria of identifying chaotic dynamics has been illustrated with a simple example in Sec. 1.

Points i–iv give strong evidences of a chaotic behaviour of the dynamics in the fragment formation process. Of course such a process takes place in a finite time interval, of the order of 100 \( \text{fm/c} \), and therefore the traditional method of drawing Poincaré maps to identify possible chaotic dynamics, cannot be used here. The sit-
uation, however, is similar to the case of chaotic scattering, and we believe that the evidences are strong enough to justify the conclusion of a chaotic dynamics of the process.

This conclusion gives support to the statistical models used to analyze the multifragmentation data, since chaotic dynamics entails a tendency of the system to fill uniformly the available phase space. In fact, if the dynamics is chaotic, strong fluctuations are expected from one collision event to another, each one ending in a different region of the available phase space. On the average, therefore, if chaoticity is strong enough, the population of the final channels will be dominated by phase space.

Finally, a preliminary analysis of the fragment size distribution shows an approximate power law. It has to be pointed out that the presence of chaotic dynamics does not exclude the possibility that the system passes through a phase transition. On the other hand a power law does not necessarily imply a phase transition.

More detailed and realistic investigations have to be performed in order to confirm this appealing scenario. It should be also clear that we have not presented a new model, but rather a new perspective in nuclear dynamics which seems more appropriate and promising.

We would like to thank M. Ploszajczak, D.H.E. Gross, Ph. Chomaz, J. Randrup, X. Campi and M. Di Toro for stimulating discussions.

This work was partially supported by the Human Capital and Mobility Program of the European Community contract no. CHRX-CT92-0075.

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Table 1 For a temperature $T = 3$ MeV, the Lyapunov exponent $\lambda$, as calculated in the text, and the corresponding characteristic time $\tau_{\text{chaos}} = 1/\lambda$ are reported. For comparison the single particle time $\tau_{\text{sp}} = h/E_F(\rho)$ for various values of the average density outside and inside the spinodal region is shown.
Table 2  The values of the exponential and power law fits to the fragment distribution shown in fig.16 are shown. The fits were performed up to a maximum $s$ value $s_{\text{max}} = 20, 30$.  

| $s_{\text{max}}$ = 20 | $(\overline{p}/p_0)_{\text{cut}}$ | $\tau_{\text{power}}$ | $\chi^2_{\text{power}}$ | $\tau_{\text{expon}}$ | $\chi^2_{\text{expon}}$ |
|------------------------|----------------|-----------------|----------------|----------------|----------------|
|                        | 0.05           | 1.48±0.16       | 0.087          | 0.10±0.01      | 0.12          |
|                        | 0.07           | 2.17±0.49       | 0.80           | 0.15±0.04      | 0.96          |
|                        | 0.10           | 2.17±0.34       | 0.61           | 0.15±0.03      | 0.86          |

| $s_{\text{max}}$ = 30 | $(\overline{p}/p_0)_{\text{cut}}$ | $\tau_{\text{power}}$ | $\chi^2_{\text{power}}$ | $\tau_{\text{expon}}$ | $\chi^2_{\text{expon}}$ |
|------------------------|----------------|-----------------|----------------|----------------|----------------|
|                        | 0.05           | 0.96±0.13       | 0.62           | 0.05±0.01      | 0.73          |
|                        | 0.07           | 1.62±0.22       | 1.770          | 0.09±0.01      | 2.13          |
|                        | 0.10           | 2.10±0.15       | 1.010          | 0.12±0.01      | 1.32          |
Fig. 1 Final value $q_f$ of the coordinate $q_1$ for a given set of initial conditions, defined by the total energy $E = 3.9$, and the constraints $\frac{1}{2}p_1^2 + q_1^2 = 3.7$, $q_2 = 0$. The initial value $q_i$ of the coordinate $q_1$ is taken randomly in the corresponding allowed interval.

Fig. 2 The free energy per particle $F$ and the pressure $P$ are respectively shown in part (a) and (b). Both quantities are calculated as function of the density $\rho/\rho_0$ and for different temperatures $T$.

Fig. 3 Time evolution of the density profile (a) for an average density $\bar{\rho}/\rho_0 = 0.8$. The initial harmonic perturbation is characterized by a node number $n_k = 4$. The corresponding power spectrum is shown in (b).

Fig. 4 The same as fig.3 but for $n_k = 5$.

Fig. 5 Time evolution of the density profile for an average density $\bar{\rho}/\rho_0 = 0.4$. The examples displayed above correspond to three different initial conditions, obtained by changing the node number $n_k = 3, 4, 5$.

Fig. 6 Power spectra corresponding to the density profiles shown in fig.5.

Fig. 7 Time evolution of the strength of the main modes excited in the profile of fig.5 for $n_k = 5$ is shown as open diamonds in steps of $5 \text{ fm/c}$. The linear fits over the first four points are drawn as full lines. The corresponding wavelengths are also reported.

Fig. 8 The quantity $\lambda(t)$ is shown as a function of time for different values of the average density reported also in the figure. The dashed line correspond to the largest Lyapunov exponent $\lambda$. The full curve is only to guide the eye.
Fig. 9 The value of $\bar{\lambda}$ is shown as a function of the initial distance $d_0$ for different values of the initial density $\bar{\rho}/\rho_0 = 0.4, 0.5$. See text for further details.

Fig. 10 The value of $\bar{\lambda}$ is shown as a function of the node number $n_k$ characterizing the initial harmonic oscillation. The squares correspond to the initial average density $\bar{\rho}/\rho_0 = 0.4$, while the diamonds correspond to $\bar{\rho}/\rho_0 = 0.5$. The dashed lines represent the average value.

Fig. 11 The value of $\bar{\lambda}$ is shown as a function of the temperature $T$ for an initial average density $\bar{\rho}/\rho_0 = 0.5$.

Fig. 12 Time evolution of $\lambda(t)$ for an average initial density $\bar{\rho}/\rho_0 = 0.5$ respectively without (crosses) and with the collision integral $I[f]$ (squares).

Fig. 13 The Lyapunov time $\tau_{chaos} = 1/\bar{\lambda}$ (diamonds) is shown in comparison to the time requested by the system in order to form fragments $\tau_{frag}$ (squares) at the various average densities. The initial condition is an harmonic perturbation of the initial density profile.

Fig. 14 Time evolution of two random initial density profiles. The average density is $\bar{\rho}/\rho_0 = 0.5$. Please note the magnification of the initial scale.

Fig. 15 Power spectra corresponding to the density profiles displayed in fig.14. Also in this case the initial scale is magnified.
Fig. 16 Fragment size distributions $P(s)$ obtained by considering 108 events at an average density $\bar{\rho}/\rho_0 = 0.5$. $P(s)$ is normalized to the total number of events considered. Each event is generated through a random initialization of the kind shown in fig.14. The three panels correspond to the different density cuts chosen in order to define the fragments. In the figure are also drawn the power law fit (solid line) and the exponential one (dashed line). The maximum value for the fit was $s_{max} = 20$. The values of the fitted parameters with the relative errors are reported too. See text for further details.
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