On the Regularization of On-Shell Diagrams

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In this letter we discuss a regularization scheme for the integration of generic on-shell forms. The basic idea is to extend the three-particle amplitudes to the space of unphysical helicities keeping the dimension of the related coupling constant fixed, and construct on-shell forms out of them. We briefly discuss the analytic structure of the extended on-shell diagrams, both at tree level and one loop. Furthermore, we propose an integration contour which, applied to the relevant on-shell forms, allows to extract the four-particle amplitudes in Lorentz signature at one loop. With this contour at hand, we explicitly apply our procedure to this case obtaining the IR divergencies as poles in the deformation parameter space, as well as the correct functional form for the finite term. This procedure provides a natural regularization for generic on-shell diagrams.

INTRODUCTION

In recent years the development of on-shell methods to compute scattering amplitudes has led to a new way of thinking of perturbation theory. In the regime in which asymptotic states can be defined, it is possible to formulate a theory in terms of on-shell processes from first principles, with no reference to a pre-existing Lagrangian. This program has been extensively developed for planar $\mathcal{N} = 4$ Super Yang-Mills theory in four dimensions and $\mathcal{N} = 6$ ABJM [1], leading to the formulation of scattering processes first in terms of a(n auxiliary) Grassmannian $\mathbb{G}[4]$ as well as, just in the case of planar $\mathcal{N} = 4$ SYM, of a geometrical object, the amplituhedron, from which the amplitudes can be read off as volumes [2].

In a nutshell, the general idea underlying these constructions is to build-up on-shell processes by suitably gluing the three-particle amplitudes: one imposes momentum conservation on the internal states keeping them on-shell, and integrating over their phase-space [3]. Depending on how many legs are glued together and how many three-particle amplitudes are involved, the gluing procedure can fix all the degrees of freedom on the internal legs, or some of them may stay unfixed, or also constraints can be imposed on the external momenta.

In this language, the perturbative expansion of a scattering amplitude in four-dimensions is represented as a series of $4L$-forms

$$M_n((p_i, h_i)) = \sum_{L=0}^{\infty} M_n^{(L)}((p_i, h_i), \{z_l\}) \Lambda_{l=1}^{4L} dz_l,$$

where $h_i$'s are the helicities of the external states, $z_l$ ($l = 1, \ldots, L$) are the unfixed degrees of freedom left over by the gluing procedure which generates the on-shell diagrams at a given order $L$, and the coefficients $M_n^{(L)}((p_i, h_i), \{z_l\})$ are rational functions of both the momenta of the external states $p_i$ and of $z_l$. Notice that $M_n^{(L)}$ can, in principle, contain terms with different powers of the coupling constant — for example, the $L = 0$ term contains both the tree-level amplitude and further rational terms coming from higher loops. At a given order $L$, the parameters $z_l$ form a $4L$-dimensional phase space which turns out to be equivalent to the loop momenta phase-space at $L$-loops in the standard Feynman diagram representation. Thus, the on-shell diagrammatic involves the integrands and the rational terms characterizing an amplitude rather than the integrated amplitude. In order to obtain an actual physical amplitude at a given order $L$, the relevant on-shell diagrams need to be integrated over a suitable contour, but it is currently not known how to identify the one which provides the Lorentz-signature amplitude. This issue comes necessarily with the question of how to regularize the IR-divergencies which plague massless theories.

A first approach to this issue for $\mathcal{N} = 4$ SYM amplitudes was taken in [2], where the regulator was introduced by making the external states slightly off-shell, and in [3, 4], where using integrability arguments the on-shell diagrams were deformed by introducing spectral parameters which are supposed to play the role of regulators. In the latter case, the integration of the relevant on-shell diagram was performed by going back to the off-shell loop momentum and choosing the parameters in such a way that the resulting integrand was simply the original loop integrand with the powers of the propagators analytically continued to complex values. As proposed,
this approach breaks Yangian symmetry, while demanding Yangian symmetry to be preserved beyond MHV level makes the deformation become trivial [9]. A way out has been argued to be the implementation of the spectral deformation directly on the Grassmannian formulation of the theory [10]: while at undeformed level the two representations are equivalent, the deformed Grassmannian does not have a direct on-shell diagrammatic interpretation.

In this paper, we consider the on-shell diagrammatics as a more general framework and introduce a regularization prescription which can be performed for a general theory, treating directly the integration without any need of referring to any off-shell loop momenta.

CONTINUATION OF THE HELICITY SPACE AND LOCALITY

An S-matrix theory can be defined through the fundamental symmetries which one wants to attribute to it, and its fundamental physical objects which are determined by such symmetries. For asymptotically Minkowski space-times in the regime in which asymptotic states can be defined, Poincaré symmetry fixes through the space-time translations and the Lorentz little group the three-particle amplitudes for massless theories [11]

\[
M_3 = \begin{cases} 
\kappa (1,2)^d_3 (2,3)^d_1 (3,1)^d_2, & \sum_{i=1}^3 h_i < 0 \\
\kappa [1,2]^{-d_3} [2,3]^{-d_1} [3,1]^{-d_2}, & \sum_{i=1}^3 h_i > 0 
\end{cases}
\] (2)

with the momenta represented as direct products of SL(2) spinors \( p_{\alpha a} = \lambda_{\alpha a} \) (\( \cdot, \cdot \)) being the Lorentz invariant internal product for (un)dotted spinors, \( h_i = h_i - h_{i+1} - h_{i-1} \) and the dimension of the coupling constant \( \kappa \) being \( |\kappa| = 1 - |h_2 + h_3 - h_3| \). The latter can be used to classify theories [12]: a class is identified by fixing the dimension of the coupling constant, which constrains the possible helicity configurations allowed. The expression (2) is non-trivial either if we consider the complexified Lorentz group \( SO(3,1; \mathbb{C}) \) and, thus, complex momenta, or if the Lorentz group is taken to be \( SO(2,2) \). Diagrammatically, the three-particle amplitudes are represented by a black (white) vertex for \( h_1 + h_2 + h_3 < 0 \) (\( > 0 \)) with incoming (outgoing) arrows for negative (positive) helicity states:

Let us now consider the three-particle amplitudes (2), fix the dimension of the coupling constant, and perform an extension of the helicity space to complex values

\[
h_i \rightarrow \tilde{h}_i \equiv h_i + \sigma_i \varepsilon_i, \quad \sigma_i = -\frac{h_i}{|h_i|} \] (3)

in such a way that the dimension of the coupling constant is unchanged (notice that the choice of \( \sigma_i \) is just a matter of convention and it does not imply any assumption about the real part of \( \varepsilon_i \)). This requirement constrains the sum of the parameters \( \varepsilon_i \) (weighted by \( \sigma_i \)) to vanish and allows to extend a given class of four-dimensional theories to the unphysical helicity space. The extended three-particle amplitudes acquire the following form

\[
M_3(\varepsilon) = M_3 \times \begin{cases} 
\frac{(2,3)^{2\varepsilon_1} (3,1)^{2\varepsilon_2}}{(1,2)^{2\varepsilon_1+1}} \cdot \sum_{i=1}^3 h_i < 0 \\
\frac{(1,2)^{2\varepsilon_1+1}}{(2,3)^{2\varepsilon_1} (3,1)^{2\varepsilon_2}} \cdot \sum_{i=1}^3 h_i > 0
\end{cases}
\] (4)

where the constraint on the parameters \( \varepsilon_i \) has been used to solve \( \varepsilon_3 \) as a function of the other two parameters.

With this extension at hand, we can build generalized on-shell diagrams by gluing the extended three-particle amplitudes (4) as prescribed for the undeformed case. First of all, the constraint on the parameters \( \varepsilon_i \) of the three-particle amplitudes generalizes to the parameters associated to the external states of any \( n \)-particle on-shell diagram

\[
\sum_{i=1}^n \sigma_i \varepsilon_i = 0. \] (5)

Secondly, the equivalence relation named merger [3], which connects two on-shell diagrams made up by two three-particle amplitudes of the same type but glued along different channels, still holds because it is just a consequence of the proportionality of all the spinors of the same type.

Let us now consider the following extended on-shell diagram

\[
\begin{align*}
M_4^{\text{tree}} & = M_4^{\varepsilon} \left( \frac{(2,3)}{(1,3)} \right)^{2\varepsilon_{12}} \left( \frac{(1,2)}{(1,3)} \right)^{2\varepsilon_{23}} \\
& \times \left( \frac{(4,1)}{(3,1)} \right)^{2\varepsilon_{34}} \left( \frac{(3,4)}{(3,1)} \right)^{2\varepsilon_{41}},
\end{align*}
\] (6)

where \( \varepsilon_{i,i+1} \) is the deformation parameter related to the internal states between the particles labelled by \( i \) and \( i+1 \), \( M_4^{\text{tree}} \) is, generically, a contribution to the tree-level 4-particle amplitude (or the full amplitude in presence of color ordering) with the helicity configuration for the external states given on the l.h.s of (6). Furthermore, the constraints on the deformation parameters have been used to express the external ones in terms of the internal \( \varepsilon_{i,i+1} \)’s. One comment is now in order. Even at tree level, our procedure breaks locality either enhancing already existing singularities or generating new ones. For configurations which at \( \varepsilon = 0 \) admit several representations, and thus further equivalence relations, our deformation
may in general break such equivalences\textsuperscript{12}. To which extent is this a big issue? For tree-level results this is not really relevant for two reasons: a naive one is that they are reproduced setting the parameters $\varepsilon_{i,i+1}$ exactly to zero. A more subtle one is related to the analytic structure of (6): taking properly a collinear limit on it, the related singularity appears as a simple pole $1/\varepsilon$ whose residue reproduces the correct factorization. A neat way to see this is to consider a BCFW deformation \textsuperscript{14} of (6). Taking for simplicity the case of pure Yang-Mills, let us perform the BCFW shift $\lambda_1(w) = \lambda_1 - w\lambda_2$, $\lambda_2(w) = \lambda_2 + w\lambda_1$, and consider the following integration

$$\int_7 \frac{dw}{w} M^{\text{tree}}(\varepsilon; w) = \mathcal{F}(\varepsilon) \frac{M_3^H M_3^A}{(2,3)\{1,3\}} \int_7 \frac{dw}{w} (w-w_{23})^{2\varepsilon_{12}-1},$$

where $M^{\text{tree}}(\varepsilon; w)$ represents the rhs of (6), $\mathcal{F}(\varepsilon)$ is all the $\varepsilon$-dependent factor of (6) which stays $w$-independent, $w_{23} \equiv -(2,3)/(1,3)$, the remaining $w$-independent factor comes just from the BCFW representation of $M^{\text{tree}}_4$ (multiplied by $w_{23}$) of the r.h.s. of (6), and $\gamma$ is a closed path encircling $w_{23}$. This is equivalent to analyzing one of the complex collinear limits in the $t$-channel. Expanding the integrand of (7) in a neighborhood of $w_{23}$, taking its primitive and taking the limit $\varepsilon_{i,i+1} \longrightarrow 0 \ (\forall i)$ in such a way that all the parameters go to zero in the same way, one gets

$$\lim_{(2,3) \longrightarrow 0} P_{23}^2 \int_7 \frac{dw}{w} M^{\text{tree}}(\varepsilon; w) = \frac{1}{2\varepsilon_{12}} M_3^H(4,1,P_{23}) M_3^A(-P_{23},2,3) + \mathcal{O}(\varepsilon^0).$$

In this specific case, it is actually the full tree-level amplitude which appears as the residue of the pole in the $\varepsilon$-parameter space (before taking the limit $\langle 2,3 \rangle \longrightarrow 0$).

At loop level the physical singularities get enhanced but also new (unphysical) ones can appear. The first effect is just an analytic continuation of the powers of the loop propagators, while the new singularities just provide scheme-dependent quantities. For a deeper and more extensive discussion about the helicity continuation and the effect of locality breaking, we refer to \textsuperscript{13}.

HELICITY CONTINUATION AND IR REGULARIZATION

Let now explore the possibility that the parameters $\varepsilon_{i,i+1}$ can actually work as IR regulators. For this purpose, let us consider a contribution to a one-loop amplitude which contains just IR singularities (i.e. we consider the on-shell representative of a contribution to a scalar box integral):

$$M^{\text{tree}}_4 = M^{\text{tree}}_4 \mathcal{F}(\varepsilon, \bar{\varepsilon}) \Omega_4(\zeta), \quad \Omega_4(\zeta) \equiv \int_{i=1}^4 d\zeta_{i,i+1} \zeta_{i,i+1}^{2\varepsilon_{i,i+1}-1}(1 - \zeta_{i,i+1})^{-2(\varepsilon_{i,i+1} + \varepsilon_{i,i+1})},$$

Some comments are now in order. Firstly, setting all the parameters to zero, one obtains the $\zeta_{i,i+1}^4 \log \zeta_{i,i+1}$ term for the above integrand, as it should be for UV-finite contributions. Secondly, the $\Omega_4(\zeta)$ in (9) can be seen as a four-form with branch points at $\zeta_{i,i+1} = 0, \infty, 1$. Thus, the introduction of our helicity deformations opens up the poles naturally present in the integrand producing branch cuts as well as new singularities ($\zeta_{i,i+1} = 1$) not present in the undeformed integrand. The 4-form $\Omega_4$ can be thought of as a holomorphic multi-valued 4-form on the complement of a branch locus in $C^4$. In order to extract physical information from (9), one has to identify the correct integration path and how to correctly treat
the branch locus. We propose that the contour of integration which allows to compute (a UV-finite contribution to) the four-particle amplitude in Lorentz signature is given by

\[ \Gamma = \left\{ \zeta \in \mathbb{C}^4 \Big| -\frac{8}{u} \zeta_{12} \zeta_{34} - \frac{t}{u} \zeta_{23} \zeta_{41} = \frac{\zeta_{i,i+1}^*}{\zeta_{i,i+1}}, \forall i \right\}, \tag{10} \]

\( \zeta_{i,i+1}^* \) being the complex conjugate of \( \zeta_{i,i+1} \). Notice that \( \Gamma \) contains the right symmetries and it can be checked bottom-up by relating the \( \zeta \)-parametrization of the phase-space to the off-shell loop momentum. The path (11) implies that all the \( \zeta \)'s have the same phase (up to \( k \pi \)), so one can adopt a convenient parametrization of the path in terms of a phase and three real variables. Since our path crosses branch points, as it is usually done in complex analysis, we split the integral into several ones, keeping track of the \( e^{\mp \pi} \) factors generated by the hidden logarithms in (9) (\( z^w = e^{w \log(z)} \)) when moving from one region to the other. After some manipulations, the result can be obtained by computing integrals of the form

\[ I_4 = \oint d\omega \frac{\omega^2}{\omega^3} \prod_{i=1}^4 dx_{i,i+1} \frac{2^{\epsilon_{i,i+1}-1}}{(1 - \omega x_{i,i+1})^{-2(\epsilon_{i,i+1} + \epsilon_{i,i+1})}} \delta \left( 1 + \frac{s}{u} x_{12} x_{34} + \frac{t}{u} x_{23} x_{41} \right), \tag{11} \]

where the range for the \( x_{i,i+1} \) is from 0 to \( \infty \) or from \( -\infty \) to 0 depending on the region. After using the \( \delta \)-function to solve for, again depending on the region, either \( x_{34} \) or \( x_{41} \), one is left with an integral whose divergent part can be extracted using two Mellin-Barnes transforms on the terms involving the variable we solved for. In spirit it is the same calculation one can use when dimensionally regularizing the box integral, but the details are a bit more involved and we postpone a detailed discussion to [13].

\[ \frac{1}{\epsilon^2} - 3 \frac{\log \left( -\frac{s}{u} \right) + \log \left( -\frac{t}{u} \right)}{2\epsilon} + \frac{1}{2} \log \left( -\frac{s}{u} \right)^2 + \frac{1}{2} \log \left( -\frac{t}{u} \right)^2 + 2 \log \left( -\frac{s}{u} \right) \log \left( -\frac{t}{u} \right) - \frac{56\pi^2}{3}, \tag{12} \]

where we are working in the kinematical region \( s < 0, t < 0 \). The result above is symmetric under the label exchange \( s \leftrightarrow t \), but this is just an artifact of our choice of regularization parameters: \( I_4 \) possesses the same symmetries as the l.h.s. of (9), and as such is invariant under relabelings \( 2 \leftrightarrow 4 \) and \( 1 \leftrightarrow 3 \). For generic values of the regularization parameters, \( I_4 \) will not be invariant under \( s \leftrightarrow t \), as it happens for dimensional regularization. This is consistent with the fact that introducing different helicity deformations for the scattering particles breaks some of these discrete symmetries. Another difference with dimensional regularization that we can observe in [12] is that the third Mandelstam variable \( u \) is playing the role of “renormalization scale”. Of course our regularization procedure remains four-dimensional all along and we do not introduce any extra dimensionful scale in the problem.

\[ \text{CONCLUSION AND OUTLOOK} \]

In this paper we propose a general (theory-independent) scheme for regularizing on-shell forms, which allows to have well-defined integrals on the Lorentz sheet. Our idea comes from two basic considerations. First, the fundamental objects for the on-shell representation of a scattering amplitude are the three-particle ones, whose coupling constant dimensionality \( [\kappa] \) allows to classify theories. The dimension of the three-particle coupling constant depends on the helicities, so that fixing it restricts the possible states which can interact in a given three-particle amplitude. Secondly, the on-shell construction breaks locality at intermediate steps, \( i.e. \) a single on-shell diagram can show poles which are not in the amplitude it contributes to. With these considera-
tions in mind, we propose to perform a deformation of the helicity space on the three-particle amplitudes, in such a way that its dimensionality is not changed. This constraint allows us to keep a four-dimensional framework as well as to remain in a given class of theories at fixed $|\kappa|$ but extending it to unphysical values of the helicities. As a consequence of this extension, locality gets broken. A regularization scheme typically breaks some features of a theory, which is the price one has to pay to be able to have well-defined quantities. Given that, at least at intermediate steps, the on-shell construction breaks locality, it can be more suitable in such a framework to introduce a regularization scheme which performs such a breaking (in a controlled way), rather than giving up any other feature or symmetry. We discuss the locality breaking in our scheme (even if a more detailed discussion will appear in [13]), arguing that it allows to associate the relevant physical quantities to poles in the parameter space. More precisely, even at tree level, one can read off the collinear behavior of an amplitude as well as the underformed on-shell diagrams from poles in the deformation parameter space. At loop level, the IR singularities are reflected also as poles in the parameter space, in a similar fashion to what happens in dimensional regularization. Our computation of a UV-finite contribution to a one-loop four-particle amplitude reveals both the correct IR structure and the correct functional structure at one-loop four-particle amplitude reveals both the correct IR structure and the correct functional structure at order $O(1)$. It is interesting that at integrand level, our deformation maps the four-form $\Lambda_i^{4} d(\log \zeta_{i,i+1})$ into another four-form which is nothing but the integrand of the Euler beta-function, even if then the integration contour $\Gamma$ complicates the branch cut structure. As already said in the text, our computation has been performed in a sort of brute force way, but it would be interesting to exploit the power of the hypergeometric function theory to have a cleaner and somehow more natural way to treat the integration and the branch locus. In order to perform the integration directly in the on-shell variables we needed to find a contour of integration implementing the Lorentz sheet. We proceeded by inspection but still we do not have a general, first principle criterium to define such contours. Finding such a criterium, with the hope of extending it to higher loops, still remains an open question, even if we consider instructive to have shown both the shape of the Lorentz sheet from the on-shell perspective and have performed directly the integration.

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