BRAID MONODROMY OF UNIVARIATE FEWNOMIALS

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ABSTRACT. Let \( C_d \subset \mathbb{C}^{d+1} \) be the space of non-singular, univariate polynomials of degree \( d \). The Viète map \( \mathcal{V} : C_d \to \text{Sym}_d(\mathbb{C}) \) sends a polynomial to its unordered set of roots. It is a classical fact that the induced map \( \mathcal{V}_* \) at the level of fundamental groups realises an isomorphism between \( \pi_1(C_d) \) and the Artin braid group \( B_d \). For fewnomials, or equivalently for the intersection \( C \) of \( C_d \) with a collection of coordinate hyperplanes in \( \mathbb{C}^{d+1} \), the image of the map \( \mathcal{V}_* : \pi_1(C) \to B_d \) is not known in general.

In the present paper, we show that the map \( \mathcal{V}_* \) is surjective provided that the support of the corresponding polynomials spans \( \mathbb{Z} \) as an affine lattice. If the support spans a strict sublattice of index \( b \), we show that the image of \( \mathcal{V}_* \) is the expected wreath product of \( \mathbb{Z}/b\mathbb{Z} \) with \( B_d/b \). From these results, we derive an application to the computation of the braid monodromy for collections of univariate polynomials depending on a common set of parameters.

1. Introduction

1.1. Braid monodromy of univariate polynomials. Let \( C_d \subset \mathbb{C}^{d+1} \) denote the space of non-singular, unmonic univariate polynomials of degree \( d \). The Viète map \( \mathcal{V} \) that associates to a polynomial \( p(x) \in C_d \) its unordered set of roots realises an isomorphism between \( C_d \) and the configuration space \( C(\mathbb{C},d) \) of \( d \) distinct points in \( \mathbb{C} \). In turn, the induced map \( \mathcal{V}_* : \pi_1(C_d) \to \pi_1(C(\mathbb{C},d)) \) is an isomorphism onto the Artin braid group \( B_d := \pi_1(C(\mathbb{C},d)) \). Moreover, the higher homotopy groups of \( C_d \) vanish so that \( C_d \) is the Eilenberg-MacLane space \( K(B_d,1) \), see \[FN62\]. For instance, this fact was successfully used by V.I. Arnol’d in \[Arn71\] to compute the cohomology groups of \( B_d \).

From the perspective of fewnomial theory, it is natural to consider the related problem of determining the fundamental group (as well as higher homotopy groups) of the space \( C_A \subset \mathbb{C}^{d+1} \) consisting of polynomials \( p(x) = \sum_{a \in A} c_a x^a \) with a given set of exponents \( A \subset \{0, \cdots, d\} \). In particular, this study fits into the general program of determining the fundamental group of the complement to discriminant varieties, see \[DL81\]. The particular instance of the space \( C_A \) was considered in \[Lib90\] in relation to Smale-Vassiliev’s complexity for algorithms. In the latter, A. Libgober asked for the computation of \( \pi_1(C_A) \) and whether \( C_A \) is a \( K(G,1) \)-space. He answered both questions in the case of trinomials, i.e. \#\( A = 3 \). For general supports \( A \), it seems that not much is known about \( \pi_1(C_A) \).

As a first approximation, we suggest to investigate the image of \( \pi_1(C_A) \) under the map \( \mathcal{V}_* \). This study generalises the determination of the Galois group of the universal polynomial with support \( A \) (see for instance \[Coh80\]) whose multivariate analogues were studied in \[Est19\] and \[EL18\].

1.2. Main results. Since multiplying polynomials with monomials does not affect the set of roots in \( \mathbb{C}^* \), there is no loss of generality in restricting to supports \( A \) such that \( \{0,d\} \subset A \subset \{0, \cdots, d\} \).

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We define the space of conditions $C_A \subset \mathbb{C}^A$ to be the space of polynomial $p(x)$ with $d$ distinct roots in $\mathbb{C}^*$. Thus, the Viète map defined above restricts to a map $\mathcal{V}: C_A \to C(\mathbb{C}^*, d)$ into the configuration space of $d$ points in $\mathbb{C}^*$. We denote by
\[ \mu_A^* : \pi_1(C_A) \to \pi_1(C(\mathbb{C}^*, d)) \]
the induced map between fundamental groups and refer to it as the braid monodromy map relative to $A$. The fundamental group of the configuration space $C(\mathbb{C}^*, d)$ is the $d$-stranded surface braid group on $\mathbb{C}^*$ that we denote $B^*_d$. We will also consider the map $\mu_A$ obtained by composition of the map $\mu_A^*$ with the surjection $B^*_d \to B_d$ induced by the inclusion $\mathbb{C}^* \hookrightarrow \mathbb{C}$.

Following [Est19], the support set $A \subset \mathbb{N}$ is said to be reduced if the smallest sublattice of $\mathbb{Z}$ containing $A$ is $\mathbb{Z}$ itself. The set $A$ is said to be non-reduced otherwise.

**Theorem 1.** For any reduced set $A \subset \mathbb{N}$, the map $\mu_A^* : \pi_1(C_A) \to B^*_d$ is surjective.

For a non-reduced support set $B$ with largest element $\delta$, the map $\mu_B^*$ is not surjective. Indeed, consider the reduced support $A := B/b$ with largest element $d := \delta/b$, where $b := \gcd(B)$. Since every polynomial $q(x) \in C_B$ can be written as $p(x^b)$ for a unique $p(x) \in C_A$, we have the isomorphisms $C_A \cong C_B$ and $\pi_1(C_A) \cong \pi_1(C_B)$. The covering $f : x \mapsto x^b$ from $\mathbb{C}^*$ to itself induces the pullback $f^* : B^*_d \to B^*_d$ satisfying $\mu_B^*(\gamma) = f^*(\mu_A^*(\gamma))$ for any element $\gamma \in \pi_1(C_B) \cong \pi_1(C_A)$. Together with Theorem 1, the latter property implies the following.

**Corollary 2.** Let $B \subset \mathbb{N}$ be a non-reduced support and let $d$, $\delta$ and $f$ be defined as above. Then, the image of the map $\mu_B^* : \pi_1(C_B) \to B^*_d$ is the subgroup $f^*(B^*_d) \subset B^*_d$.

The image of $\mu_B^*$ is isomorphic to $B^*_d$ since $f^*$ is injective. However, the image of $\mu_B$, that is the projection of $f^*(B^*_d) \subset B^*_d$ to $B^*_d$, is isomorphic to the wreath product of $\mathbb{Z}/b\mathbb{Z}$ with $B_d$, see the end of Section 2.1.

**Remark 1.1.** 1. Corollary 2 justifies the consideration of braids in $\mathbb{C}^*$ rather than in $\mathbb{C}$. Indeed, the computation of $\text{im}(\mu_B)$ does not follow from the surjectivity of $\mu_A$, unlike for the case of $\mu_B^*$ and $\mu_A^*$, because there is no natural map $B_d \to B_d$ similar to $B^*_d \to B^*_d$. This phenomenon falls under the principle that the monodromy of an enumerative problem $P_B$ that is a covering of another enumerative problem $P_A$ can not be computed from the monodromy of the problem $P_A$ a priori. In general, more information is required on $P_A$, as illustrated for instance in [ELIS].

2. This in particular proves the theorem from [ELIS] that the Galois group of the indeterminate polynomial with the non-reduced support $B$ equals the wreath product of $\mathbb{Z}/b\mathbb{Z}$ with $\text{Sym}_d$. It would be interesting to extend results of this paper to multivariate polynomials so that they cover the respective results of [ELIS] on Galois groups of systems of polynomial equations of several variables.

Let us come back to the case of a reduced support $A := \{a_0, \ldots, a_n\}$ with $a_0 = 0$ and $a_n = d$. The set $C_A$ is the complement to a Zariski-closed subset of $\mathbb{C}^{d+1}$ that consists of the $A$-discriminant (see [GKZ08]) together with the first and last coordinate hyperplanes $\{c_0=0\}$ and $\{c_n=0\}$ in $\mathbb{C}^{d+1}$. While the monodromy obtained from small loops around the latter coordinate hyperplanes is easy to identify, there might be situations when one is interested in the contribution of the remaining component of $\mathbb{C}^{d+1} \setminus C_A$. To this regard, we prove the following.

**Theorem 3.** For a reduced support $A \subset \mathbb{N}$, the composition of $\pi_1(C_A \cap \{c_0=c_n=1\}) \to \pi_1(C_A)$ with $\mu_A$ is surjective. In particular, if $K \subset \pi_1(C_A)$ denotes the kernel of the map induced by the inclusion $C_A \hookrightarrow \mathbb{C}^A \setminus \{c_0c_n=0\}$, then we have that $\mu_A(K) = B_d$.
Theorem\textsuperscript{3} has useful application to collections of fewnomials depending on a common set of parameters. Such situations appear naturally in the computation of the monodromy of Severi varieties of toric surfaces, see [Lan19]. Consider a finite collection $A_1, \ldots, A_m \in \mathbb{N}$, of reduced supports and an affine linear map

$$L = (L_1, \ldots, L_m) : \mathbb{C}^k \rightarrow \mathbb{C}^{A_1} \times \cdots \times \mathbb{C}^{A_m}.$$  

For each index $j$, we denote by $D_j \subset \mathbb{C}^k$ (respectively $H_j \subset \mathbb{C}^k$) the pullback under $L_j$ of the $A_j$-discriminant (respectively of $\{c_0c_{n_j}=0\} \subset \mathbb{C}^{A_j}$, where $n_j$ is the index of the largest element $d_j$ in $A_j$).

**Definition 1.2.** The affine linear map $L$ is said to be generic with respect to the collection $A_1, \ldots, A_m$ if the following holds:

- the $m$ subsets $D_j \subset \mathbb{C}^k$ are reduced hypersurfaces,
- the $m$ subsets $H_j \subset \mathbb{C}^k$ have positive codimension,
- no $D_j$ has a common irreducible component with another $D_i$ or with any $H_i$.

Observe that $D_j$ is always reduced, irreducible and not contained in any of the $H_i$ whenever $k \geq 2$. Observe also that we do not impose any restriction on the relative position of the $H_j$-s and that the $H_j$-s are allowed to be non-reduced.

If we denote $U := L^{-1}(C_{A_1} \times \cdots \times C_{A_m})$, then we can define the map

$$\mu : \pi_1(U) \rightarrow B_{d_1} \times \cdots \times B_{d_m}$$

where the $j^{th}$ factor of $\mu$ is the composition of the map $L_j^* : \pi_1(U) \rightarrow \pi_1(C_{A_j})$ with the map $\mu_{A_j}$.

**Corollary 4.** If the linear map $L$ is generic in the sense of Definition\textsuperscript{1.2}, then $\mu$ is surjective.

**Proof.** Observe first that since $L$ is generic with respect to $A_1, \ldots, A_m$, there always exists an affine linear map $\iota : \mathbb{C} \rightarrow \mathbb{C}^k$ such that $\tilde{L} := L \circ \iota$ is generic. If we denote $\tilde{U} := \iota^{-1}(U)$ and $\tilde{\mu} : \pi_1(\tilde{U}) \rightarrow B_{d_1} \times \cdots \times B_{d_m}$ the corresponding monodromy map, it suffices to show that $\tilde{\mu}$ is surjective since it factorises through $\mu$. In other words, we can assume that $k = 1$.

Let us assume further that $L_j(C_{A_j}) \subset \{c_0=c_{n_j}=1\} \subset \mathbb{C}^{A_j}$ for all $j$. In particular, all the $H_j$-s are empty, the line $L_j(\mathbb{C})$ intersects transversely the $A_j$-discriminant for any $j$ and the $D_j$-s are pairwise disjoint. By [Che73 Théorème], the latter transversality property implies that $\pi_1(C \setminus D_j) \rightarrow \pi_1(C_{A_j} \cap \{c_0=c_{n_j}=1\})$ is surjective. It follows from Theorem\textsuperscript{3} that $\pi_1(C \setminus D_j) \rightarrow B_{d_j}$ is also surjective. Since $U = C \setminus (\cup_j D_j)$ and the fact that the $D_j$-s are pairwise disjoint, the natural map $\pi_1(U) \rightarrow \pi_1(C \setminus D_1) \times \cdots \times \pi_1(C \setminus D_m)$ is also surjective. Therefore, the map $\mu : \pi_1(U) \rightarrow B_{d_1} \times \cdots \times B_{d_m}$ is surjective too.

Let now $L : \mathbb{C} \rightarrow \mathbb{C}^{A_1} \times \cdots \times \mathbb{C}^{A_m}$ be any generic map. Then, there is a continuous one-parameter family of such maps joining $L = L_0$ to a map $L_1$ as in the above paragraph. Let us put an index $t$ to all piece of notation corresponding to the map $L_t$. Then, there exists a continuous one-parameter family of discs $V_t \subset \mathbb{C}$ such that $V_t$ contains $\cup_j D_{j,t}$ and is disjoint from $\cup_j H_{j,t}$. In particular, the disc $V_t$ determines the subgroup $G_t$ of $\pi_1(U_t)$ of all loops in $U_t$ supported in $V_t$. Clearly, the group $\mu_t(G_t)$ is independent of $t$. Since $\cup_j H_{j,1}$ is empty, we have that $G_1 = \pi_1(U_1)$ and therefore that $\mu_t(G_t) = \mu_1(G_1) = \text{im}(\mu_1) = B_{d_1} \times \cdots \times B_{d_m}$ by the previous paragraph. \hfill $\square$
1.3. Techniques and perspectives. Let us now comment on the proofs of Theorems 1 and 3. A possible approach is to use Zariski theorems such as [Che75, Théorème]. Indeed, the configuration space $C_A$ is an iteration of hyperplane sections of the space $C_d$. However, the latter hyperplane sections are not generic and the fundamental groups $\pi_1(C_A)$ and $\pi_1(C_d)$ are certainly not isomorphic in general. In particular, we cannot apply [Che75, Théorème]. At the current stage of our investigations, we are only interested in the surjectivity of maps between fundamental groups induced by restrictions to hyperplane sections. Therefore, the statement [Bes01, Theorem 2.5] could be used, provided that its assumptions are satisfied in the present situation.

Instead, we use elementary considerations from tropical geometry to construct explicit elements in the image of $\mu_\star A$ in the particular case $#A = 3$. We work out the general case by specialisation to the case of trinomials and by working out handy relations in the relevant braid groups, see Lemma 4.3. These relations lead to an analogue of the Euclidean algorithm which may be interesting in its own, see Proposition 4.1.

In addition to providing explicit elements in the image of the braid monodromy, the tropical techniques developed in this paper have the advantage of generalising to families of univariate polynomials $p(z, w) = c_0(w) + c_1(w)z^{a_1} + \cdots + c_n(w)z^{a_n}$ whose coefficients are polynomials in $w := (w_1, \ldots, w_k)$ of arbitrary degrees, situation in which the Zariski theorems mentioned above do not apply. In particular, these techniques can be used to determine the Galois groups of square systems $f_1 = f_2 = 0$ of bivariate polynomials supported on reducible tuples $A_1, A_2 \subset \mathbb{Z}^2$, see [Est19, Definition 1.3].

1.4. Organisation of the paper. In Section 2 we recall standard facts about braid groups and the basics of tropical geometry that we use in Section 3 to deal with the particular case of reduced trinomials. In Section 4 we use the case of trinomials and the aforementioned Euclidean algorithm to prove Theorems 1 and 3.

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2. Settings

2.1. Surface braid groups. Recall that $C(\mathbb{C}^*, d)$ denotes the configuration space of $d$ distinct points in $\mathbb{C}^*$. Its fundamental group is denoted by $B_d^\star$ and referred to as the $d$-stranded surface braid group on $\mathbb{C}^*$. We refer to [FM11, Ch.9] for the theory on surface braid groups.

Define a $d$-stranded geometric braid on $\mathbb{C}^*$ to be a topological 1-fold $b \subset S^1 \times \mathbb{C}^*$ such that the projection $\pi: b \to S^1$ is a degree $d$ covering. The map $S^1 \to C(\mathbb{C}^*, d)$ defined by $\theta \mapsto \pi^{-1}(\theta)$ defines in turn an element in $B_d^\star$. Two geometric braids define the same element in $B_d^\star$ if and only if there exists an isotopy of $d$-stranded geometric braids between them.

In order to perform computations in $B_d^\star$, we will represent braids by diagrams. To that purpose, observe that any braid $[b] \in B_d^\star$ can be represented by a geometric braid $b \subset S^1 \times \mathbb{C}^*$ with projection $\pi: b \to S^1$ such that:

- the set $\pi^{-1}(1)$ is evenly distributed on the circle $S^1 \subset \mathbb{C}^*$,
the only singularities of the projection of \( b \) under \( \pi \times \arg \) are transverse self-intersection points (here, the function \( \arg : \mathbb{C}^* \to \mathbb{S}^1 \) is defined by \( x \mapsto \frac{x}{|x|} \)).

We represent geometric braids with the above properties using the projection \( \rho \) given by the composition of \( \pi \times \arg \) with the projection of \( \mathbb{S}^1 \times \mathbb{S}^1 \) to a square fundamental domain \([a, a + 2\pi] \times [0, 2\pi]\) for any choice \( a \in \mathbb{R} \). When the projection of two strands of \( b \) under \( \rho \) cross each other, we picture the strand with the largest modulus in \( \mathbb{C}^* \) as passing underneath the other strand. Eventually, we use blue-shaded areas to represent a sequence of neighbouring strands that are parallel to each other.

![Figure 1](image)

**Figure 1.** The elements \([b_1],[\tau]\) and \([r_1]\) in \( B_d^* \).

Let us now introduce useful elements of the surface braid group \( B_d^* \). First, we define the elements \([b_1],[\tau]\) and \([r_1]\) according to Figure 1. From these elements, we define
\[
[b_{j+1}] := [\tau]^j \circ [b_1] \circ [\tau]^{-j} \quad \text{and} \quad [r_{j+1}] := [\tau]^j \circ [r_1] \circ [\tau]^{-j}
\]
for \( j \in \{1, \ldots, d - 1\} \). We have the following classical result.

**Lemma 2.1.** The projection of the elements \([b_1], \ldots, [b_{d-1}] \in B_d^* \) to \( B_d \) generate \( B_d \).

**Proof.** The projection of these elements are the elements \( \sigma_i \) used by E. Artin to give the presentation of the pure braid group \( PB_d^* \) in \( B_d^* \). We order the connected components of a pure braid by the ordering their starting points from left to right in the corresponding diagram (this ordering depends on the choice of the fundamental domain \([a, a + 2\pi] \times [0, 2\pi]\)). This ordering allows to define the map
\[
\text{ind} : PB_d^* \to \mathbb{Z}^d \quad [\beta] \mapsto \left( \frac{1}{2\pi} \int_{\beta_1} \frac{dx}{x}, \ldots, \frac{1}{2\pi} \int_{\beta_d} \frac{dx}{x} \right)
\]
where \( \beta_j \) is the \( j \)th component of any representative of \([\beta]\).

**Lemma 2.2.** The surface braid group \( B_d^* \) is generated by any of the following sets:
- the set \([\tau],[b_1], \ldots, [b_d],[r_1], \ldots, [r_d]\),
- the set \([\tau],[b_1],[r_1]\),
- the set \([\tau],[b_1]\).

**Proof.** By definition, the first set is generated by the second. The relation \([r_1] = [\tau] \circ [b_{d-1}] \circ \cdots \circ [b_1] \) that is easily check using the diagrams of Figure 1 implies that the third set generates the second one. It remains to show that the first set is a generating set.

To see this, observe first that for any element \([b] \in B_d^* \), there is an element \([b']\) in the group generated by \([b_1], \ldots, [b_{d-1}]\) such that \([b'] \circ [b] \) is a pure braid. Thus, there is no loss of generality
in assuming that \([b]\) is pure. Now, for any pure braid \([b]\), there is an element \([b']\) in the group generated by \([r_1], \ldots, [r_d]\) such that \(\text{ind}([b'] \circ [b]) = (0, \ldots, 0)\). Thus, there is no loss of generality in assuming that \([b]\) is pure and such that \(\text{ind}([b]) = (0, \ldots, 0)\). We now want to argue that this is possible to unravel such a braid with the only elements \([b_1], \ldots, [b_d]\). To see this, observe that there exists an integer \(m\) such that every string in the diagram of the pullback \([b']\) of the braid \([b]\) by the covering \(x \mapsto x^m\) fits inside a single fundamental domain. This follows from the fact that \(\text{ind}([b]) = (0, \ldots, 0)\). We know from Lemma 2.1 that the braids \([b_1], \ldots, [b_{md-1}]\) suffice to unravel the braid \([b'] \in B_{md}\), that is we can write \([b'] = [b_{j_1}] \circ \cdots \circ [b_{j_n}]\). Clearly, we can write \([b] = [b_{j_1}] \circ \cdots \circ [b_{j_n}]\) where \(j_i \in \{1, \ldots, d\}\) is the reduction of \(j_i^d\) modulo \(d\).

In the course of the paper, we will construct braids by means of tropical geometry. Unfortunately, the objects we will obtain are not as nice as geometric braids. However, we argue below that these objects lead to well defined elements in \(B_d\) in a rather canonical way.

**Definition 2.3.** A \(d\)-stranded coarse braid in \(\mathbb{C}^*\) is a topological 1-fold \(b' \subset S^1 \times \mathbb{C}^*\) with associated projection \(\pi' : b' \to S^1\) such that \(\pi'\) is a degree \(d\) covering outside of a finite set \(T \subset S^1\) and such that the preimage by \(\pi'\) of any connected arc in \(S^1\) with endpoints \(\alpha, \beta \in S^1 \setminus T\) consists of \(d\) connected arcs in \(b'\) connecting \(\pi'^{-1}(\alpha)\) to \(\pi'^{-1}(\beta)\).

**Lemma 2.4.** For any coarse braid \(b' \subset S^1 \times \mathbb{C}^*\), there exists a family of geometric braids \(b_\varepsilon\) continuous in \(\varepsilon > 0\) whose limit in the Hausdorff distance is \(b'\). Moreover, the class \([b_\varepsilon] \in B_d^*\) depends neither on \(\varepsilon\) nor on the choice of the family \(b_\varepsilon\).

**Definition 2.5.** For any \(d\)-stranded coarse braid \(b' \subset S^1 \times \mathbb{C}^*\), we define the class of the coarse braid \(b'\) to be the element \([b_\varepsilon] \in B_d^*\) for any family \(b_\varepsilon\) as in Lemma 2.4. We denote this class by \([b'] \in B_d^*\).

**Proof of Lemma 2.4.** Let us first show the existence of a family \(b_\varepsilon\) as in the statement. Let us fix \(\delta > 0\) arbitrarily small. We prescribe that \(b_\varepsilon\) is constant and coincides with \(b'\) on the preimage by \(\pi'\) of the complement of the \(\delta\)-neighbourhood of \(T \subset S^1\). Now, for any \(\theta \in T\), the preimage \((\pi')^{-1}(\theta - \delta, \theta + \delta)\) consists of \(d\) disjoints connected arcs, some of which are not transverse to the fiber \((\pi')^{-1}(\theta)\). Obviously, there is exists a deformation of each such arc satisfying the following:

\(-\) the deformation is continuous in the parameter \(\varepsilon\);
\(-\) the arcs have fixed endpoints;
\(-\) the arcs are transverse to the projection \(\pi'\) for any \(\varepsilon > 0\).

The latter deformation provides us the sought family \(b_\varepsilon\).

Finally, if \(b_\varepsilon\) is another family as in the statement, it is clear that \(b_\varepsilon\) is homotopic to \(b_\varepsilon\) along a family of geometric braids for small \(\varepsilon\). In particular, the class \([b_\varepsilon]\) does not depend on the choice of the family \(b_\varepsilon\) and obviously, it does not depend on \(\varepsilon\) either.

We conclude this section with the description of the isomorphism between the projection of \(f^*(B_d^*)\) in \(B_d\) and the wreath product of \(\mathbb{Z}/b\mathbb{Z}\) with \(B_d\) in the context of Corollary 2. Recall briefly the definition of the wreath product \(\mathbb{Z}/b\mathbb{Z} \wr B_d\). The group \(B_d\) acts on \(\{1, \ldots, d\}\) by permutation. The latter action extends naturally to the group \(K := \{1, \ldots, d\}_{\mathbb{Z}/b\mathbb{Z}}\). The group \(\mathbb{Z}/b\mathbb{Z} \wr B_d\) is defined as the semidirect product \(K \rtimes B_d\) with respect to the latter action. In other words, elements in \(\mathbb{Z}/b\mathbb{Z} \wr B_d\) are braids in \(B_d\) each strand of which is decorated with an element in \(\mathbb{Z}/b\mathbb{Z}\). The multiplication of two such elements is the one of \(B_d\) at the level of braids and to a strand of the product we associate the sum of the elements in \(\mathbb{Z}/b\mathbb{Z}\) decorating the two concatenated strands.
Recall now that \( \delta = bd \) and that the elements of \( f^*(B^*_d) \subset B^*_d \) are the braids globally invariant under multiplication by the \( b^\text{th} \) roots of unity. There is not loss of generality in assuming that the braids of \( B^*_d \) are based at \{1, \cdots, d\} and therefore that the braids of \( f^*(B^*_d) \) are based at the \( b^\text{th} \) roots of \( 1, \cdots, d \). In particular, the \( \delta \) strands of a braid \([b] \in f^*(B^*_d)\) are divided into \( d \) tuples of \( b \) strands, namely the strands based at the \( b^\text{th} \) root of \( j \) for any \( j \in \{1, \cdots, d\} \). If the strand based at \( j \in \mathbb{C} \) ends at \( j' e^{2\pi i k/b} \), then the strand based at \( j e^{2\pi i \ell/b} \) ends at \( j' e^{2\pi i (\ell+k)/b} \), since each tuple is globally invariant under multiplication by \( e^{2\pi i/b} \). Therefore, each of the \( d \) tuples defines an element in \( \mathbb{Z}/b\mathbb{Z} \). If \([b] = f^*([b'])\) then, each of the \( d \) tuples of \([b]\) contains exactly one strand of \([b']\).

The map \( \varphi \) is obviously surjective. It remains to show that the kernel of \( \varphi \) coincide with the kernel of the projection \( \pi : f^*(B^*_d) \rightarrow B_d \). It is an easy exercise to check that these kernels in \( f^*(B^*_d) \cong B^*_d \) both coincide with \( \langle r^1_d, \cdots, r^b_d \rangle \subset B^*_d \).

2.2. The phase-tropical line. Let \( z \) and \( w \) be the coordinates functions on \( \mathbb{C}^2 \) and denote by \( \mathcal{L} \) the line

\[
\mathcal{L} := \{(z, w) \in \mathbb{C}^2 | 1 + z + w = 0\}.
\]

The phase-tropical line \( \mathcal{L} \subset (\mathbb{C}^*)^2 \) is the Hausdorff limit of the sets \( H_t(\mathcal{L} \cap (\mathbb{C}^*)^2) \) where

\[
H_t : (\mathbb{C}^*)^2 \rightarrow (\mathbb{C}^*)^2
\]

\[
(z, w) \mapsto \left( \frac{z}{|z| \sqrt{1 + t}}, \frac{w}{|w| \sqrt{1 + t}} \right)
\]

as \( t \) goes to \( +\infty \). The set \( \mathcal{L} \) turns out to be homeomorphic to \( \mathcal{L} \cong \mathbb{C}P^1 \setminus \{0, 1, \infty\} \) and can be described through its projections under the maps

\[
\text{Arg} : (\mathbb{C}^*)^2 \rightarrow (S^1)^2
\]

\[
(z, w) \mapsto \left( \frac{z}{|z|}, \frac{w}{|w|} \right)
\]

and

\[
\text{Log} : (\mathbb{C}^*)^2 \rightarrow \mathbb{R}^2
\]

\[
(z, w) \mapsto (\log(|z|), \log(|w|))
\]

The set \( \text{Log}(\mathcal{L}) \) is the union of the three rays \( r_z := (-1, 0) \cdot \mathbb{R}_{\geq 0} \), \( r_w := (0, -1) \cdot \mathbb{R}_{\geq 0} \) and \( r_\infty := (1, 1) \cdot \mathbb{R}_{\geq 0} \) merging at the origin \( 0 \in \mathbb{R}^2 \). For any point \( p \) in the relative interior of \( r_z \) (respectively \( r_w \), respectively \( r_\infty \)), the set \( \text{Arg} \left( \mathcal{L} \cap \text{Log}^{-1}(p) \right) \subset (S^1)^2 \) is the geodesic with the same slope as \( r_z \) (respectively \( r_w \), respectively \( r_\infty \)) passing through the point \((-1, -1)\) (respectively \((-1, -1)\), respectively \((-1, 1)\)). Eventually, the set \( \text{Arg} \left( \mathcal{L} \cap \text{Log}^{-1}(0) \right) \subset (S^1)^2 \) is the closure of \( \text{Arg}(\mathcal{L}) \). The latter set is the union of the two blue triangles bounded by the three geodesics \( g_z, g_w \) and \( g_\infty \) pictured in Figure 2 (left). In the same figure, we illustrate the description of \( \mathcal{L} \) in terms of the projections \( \text{Log} \) and \( \text{Arg} \).

It will also be convenient to observe that the phase-tropical line \( \mathcal{L} \) is piecewise algebraic. Indeed, the restriction of \( \mathcal{L} \) to \( \text{Log}^{-1}(r_z \setminus 0) \) (respectively \( \text{Log}^{-1}(r_w \setminus 0) \), respectively \( \text{Log}^{-1}(r_\infty \setminus 0) \)) is the algebraic curve \( z = 1 \) (respectively \( w = 1 \), respectively \( z = -w \)).

3. Trinomials

Throughout this section, the support \( A \) will be of the form \( A := \{0, p, d\} \) where \( p \) and \( d \) are coprime.
3.1. Tropicalisation. In order to determine the image of \( \mu^* \), we will construct braids using tropical methods. To do so, we first reformulate our problem in a way that is suitable for tropicalisation.

In this section, we restrict to trinomials of the form
\[
p(x) = 1 + c_1 x^p + c_2 x^d
\]
in \( \mathbb{C}^A \), where \( c_0, c_1, c_2 \) are the coordinates on \( \mathbb{C}^A \). We denote simply by \( c := (c_1, c_2) \in \mathbb{C}^2 \) the corresponding coefficient vector and by \( S_c := \{ x \in \mathbb{C}^* \mid p(x) = 0 \} \) the corresponding set of solutions. The coefficient vector \( c \) also defines a map
\[
\phi_c : \mathbb{C}^* \to \mathbb{C}^2
\]
relative to \( A \). Thus, the set \( S_c \) equals \( \phi_c^{-1}(L) \) where \( L \) is the line defined in Section 2.2. Recall from the same section that \( L \subset (\mathbb{C}^*)^2 \) denotes the phase-tropical line. By analogy, we define the set of tropical solutions \( S_c := \phi_c^{-1}(L) \subset \mathbb{C}^* \) and the space of tropical conditions \( C_A \subset \mathbb{C}^2 \) to be the set of vectors \( c \in \mathbb{C}^2 \) such that \( S_c \) consists of \( d \) connected components. The following observation will be useful.

**Lemma 3.1.** A vector \( c \in \mathbb{C}^2 \) is in the complement of \( C_A \) if and only if the following conditions are satisfied:

- the line \( \text{Log}(\phi_c(\mathbb{C}^*)) \) passes through the origin \( 0 \in \mathbb{R}^2 \);
- the geodesic \( \text{Arg}(\phi_c(\mathbb{C}^*)) \) passes through the point \( (-1, 1) \in (S^1)^2 \).

Concisely, the set \( \mathbb{C}^2 \setminus C_A \) is the algebraic curve \( (-c_1)^d = (c_2)^p \).

**Proof.** If the line \( \text{Log}(\phi_c(\mathbb{C}^*)) \) does not pass through \( 0 \in \mathbb{R}^2 \), then the latter line intersects \( \text{Log}(L) \) transversely in either one or two points. The respective fibers in \( \phi_c(\mathbb{C}^*) \) and \( L \) over these points consist of geodesics intersecting transversely in the corresponding argument tori in \( d \) points in total. We deduce that the first condition is necessary.

Assume therefore that \( \text{Log}(\phi_c(\mathbb{C}^*)) \) passes through 0. Observe that the origin is the only intersection point between \( \text{Log}(\phi_c(\mathbb{C}^*)) \) and \( \text{Log}(L) \) in that case. The connected components of the set of tropical solutions \( S_c \) are then in correspondence with the connected components of
Arg(φ_c(C^*)) ∩ Arg(Ł). The geodesic Arg(φ_c(C^*)) has slope (p, d), hence it intersects the geodesic g_w of Figure 2 in d distinct points x_1, ..., x_d (recall that p and d are coprime). Each of the latter point is the endpoint of a connected component of Arg(φ_c(C^*)) ∩ Arg(Ł). If Arg(φ_c(C^*)) does not pass through (−1, 1), the other endpoint lies on exactly one of the two remaining geodesics g_w and g_∞. In particular, the other endpoint is away from the yellow geodesic and each of the x_j-s defines a distinct connect component of Arg(φ_c(C^*)) ∩ Arg(Ł). Thus, the second condition in the statement is also necessary.

By the same argument, we observe that Arg(φ_c(C^*)) ∩ Arg(Ł) consists of exactly d − 1 connected components when Arg(φ_c(C^*)) passes through (−1, 1). It follows that the two conditions are sufficient.

To conclude, the two conditions are equivalent to the fact that φ_c(C^*) passes through the point (−1, 1) ∈ C^2, which in turn is equivalent to the possibility of writing c = (−x^p, x^d) for some x ∈ C^*. We deduce that the set C^2 \ C_A is the algebraic curve (−c_1)^d = (c_2)^p.

**Remark 3.2.** 1. The above lemma is the tropical counterpart to [La00] Theorem B. Here, we recover the fact that the fundamental group of C_A is the group of the (p, d)-torus knot.

2. The algebraic curve (−c_1)^d = (c_2)^p is globally fixed by H_t for any t > 0, and so is its complement C_A. Indeed, the self-diffeomorphism H_t corresponds to the rescaling by 1/Log(t) on the first factor of the logarithmic coordinates Log × Arg and the projection of (−c_1)^d = (c_2)^p under Log is a line passing through 0 ∈ R^2 and is therefore preserved under the latter rescaling.

We are now ready to show that coarse braids obtained by tropical construction lead to braids in the image of μ_A.

**Proposition 3.3.** Let γ : S^1 → C_A be a loop such that the set b' := {(θ, p) ∈ S^1 × C^* | p ∈ S_θ(θ)} is a coarse braid. Then, there exists ℓ ∈ π_1(C_A) such that μ_A(ℓ) = [b'] ∈ B_d (see Definitions 2.3 and 2.5). Moreover, we can choose ℓ so that im(ℓ) is contained in {c_0=c_2=1} (respectively {c_0=c_1=1}) if im(γ) is.

**Proof.** The strategy is to construct an explicit one-parameter family of geometric braids as in Lemma 2.4. This family will be chosen in two different ways in order to fit with the extra assumption that im(γ) might be contained in {c_0=c_2=1} (respectively in {c_0=c_1=1}).

For any c = (c_1, c_2) ∈ C^2 and any t > 0, we define the element c^t ∈ C^2 by

\[ c^t := h_t\left(\frac{d}{dp/d} \cdot c_1, c_2\right) \quad \text{(respectively } c^t := (c_1, h_t\left(\frac{p}{d^p/d} \cdot c_2\right)) \text{)} \]

where h_t is the self-diffeomorphism h_t(x) = −x/|x| Log(t) on C^*. For both definitions of c^t, we claim first that H_t^{-1}(c^t) ∈ C_A for any t > 0 if c ∈ C_A and second that h_t(S_{H_t^{-1}(c^t)}) converges in Hausdorff distance to S_c when t becomes arbitrarily large. If we denote c_θ := γ(θ) and c^t_θ constructed from c_θ as above, we deduce that for any t > 0, the set

\[ b_t := \left\{ (θ, p) ∈ S^1 × C^* | p ∈ S_{H_t^{-1}(c^t_θ)} \right\} \]

is a geometric braid that can be made arbitrarily close to b' for sufficiently large t. Now, observe that the geometric braids b_t are isotopic to each other for any t ≥ e and that b_e is the braid

\[ b_e = \{ (θ, q) ∈ S^1 × C^* | q ∈ S_c \} . \]
Defining \( \ell : S^1 \to \mathcal{C}_A \) by \( \ell(\theta) = c^\theta \), we deduce that that \([b_c] = \mu^*_A(\gamma)\). Setting \( \varepsilon := 1/t \), we conclude that the one-parameter family \( b_c \) fulfils the assumptions of Lemma 2.4 and therefore that \( \mu^*_A(\ell) = [\theta'] \). Moreover, it is clear from the construction that \( \text{im}(\ell) \) is contained in \( \{c_0 = c_2 = 1\} \) (respectively \( \{c_0 = c_1 = 1\} \)) if \( \text{im}(\gamma) \) is.

It remains to prove the claims. We give the proof for the first definition of \( c^\ell \) (the proof for the second definition is similar). For the first claim, observe that \( \mathcal{C}_A \) is the complement of the curve \((-c_1/d)^d = (c_2/p)^p\). Indeed, we can parametrise the pairs \((c, x)\) such that the trinomial corresponding to \( c \in \mathbb{C}^A \) is singular at \( x \) by \( x \mapsto \left( -\frac{d}{x^{p(d-p)}}, \frac{p}{x^{p(d-p)}} \right) \). Then, the vector \( H_t^{-1}(c^\ell) \) is not in \( \mathcal{C}_A \) if and only if

\[
\left( -\frac{h_t^{-1}(c_1)}{d}, \frac{h_t^{-1}(c_2)}{p/d} \right)^d = \left( \frac{h_t^{-1}(c_2)}{p} \right)^p \iff \left( -h_t^{-1}(c_1) \right)^d = \left( h_t^{-1}(c_2) \right)^p
\]

if and only if \( H_t^{-1}(c) \) is not in \( \mathcal{C}_A \), by Lemma 3.1. By Remark 3.2.2, the latter is equivalent to \( c \) not lying in \( \mathcal{C}_A \). For the second claim, we have for any \( c = (c_1, c_2) \in \mathbb{C}^2 \)

\[
S_c := \phi_c^{-1} \left( \lim_{t \to \infty} H_t(\mathcal{L}) \right) \supset \lim_{t \to \infty} \phi_c^{-1} \left( H_t(\mathcal{L}) \right)
\]

\[
= \lim_{t \to \infty} \phi_c^{-1} \left\{ \{z, w\} \in (\mathbb{C}^*)^2 \mid 1 + h_t^{-1}(z) + h_t^{-1}(w) = 0 \right\}
\]

\[
= \lim_{t \to \infty} \left\{ x \in \mathbb{C}^* \mid 1 + h_t^{-1}(c_1 x^p) + h_t^{-1}(c_2 x^d) = 0 \right\}
\]

\[
= \lim_{t \to \infty} \left\{ x \in \mathbb{C}^* \mid 1 + h_t^{-1}(c_1) (h_t^{-1}(x))^p + h_t^{-1}(c_2) (h_t^{-1}(x))^d = 0 \right\}
\]

\[
= \lim_{t \to \infty} h_t(\phi_{H_t^{-1}(c)}(\mathcal{L})) := \lim_{t \to \infty} h_t(\mathcal{L}_{H_t^{-1}(c)})
\]

It implies that the set \( h_t(\mathcal{L}_{H_t^{-1}(c)}) \) can be made arbitrarily close to \( S_c \) for \( t \) sufficiently large. Since \( c^\ell \) converges to \( c \) for large \( t \), the claim follows.

\[ \square \]

### 3.2. Tropical construction of coarse braids.

Motivated by Proposition 3.3, we will construct explicit loops in the space of tropical conditions \( \mathcal{C}_A \) leading to coarse braids.

We will now construct the loop \( \gamma : S^1 \to \mathcal{C}_A \) giving rise to the sought coarse braid. First, we fix the base-point \( c(0) \) of \( \gamma \) to be of the form \( c(0) := (e^{i(\pi - \varepsilon)} \cdot 1, 1) \) where \( \varepsilon > 0 \) is arbitrarily small. In particular, the geodesic \( \text{Arg} \left( \phi_{c(0)}(\mathbb{C}^*) \right) \) of slope \((p, d)\) passes arbitrarily close to the point \((-1, 1)\). The line \( \text{Log} \left( \phi_{c(0)}(\mathbb{C}^*) \right) \) passes through \((-1, 0)\). We deduce from Lemma 3.1 that \( c(0) \) belongs to \( \mathcal{C}_A \). The itinerary of \( \gamma \) is as follows.

**Day 1:** we start from \( c(0) \) and follow the path \( x \mapsto c := (e^{x+i(\pi-\varepsilon)}, 1) \) from \(-1\) to \( 1\). While doing so, the geodesic \( \text{Arg} \left( \phi_{c}(\mathbb{C}^*) \right) \) remains unchanged. In particular, we stay inside \( \mathcal{C}_A \). The line \( \text{Log} \left( \phi_{c}(\mathbb{C}^*) \right) \) passes through \((x, 0)\) and moves to the right as \( x \) increases. In particular, the latter line passes through \( 0 \in \mathbb{R}^2 \) when \( x = 0 \). We denote the corresponding point \( c(x) := (e^{i(\pi-\varepsilon)}, 1) \). We take our first break at the point \( c(1) := (e^{1+i(\pi-\varepsilon)}, 1) \).

**Day 2:** we start from \( c(1) \) and follow the path \( \theta \mapsto c := (e^{1+i\theta}, 1) \) from \( \pi - \varepsilon \) to \( \pi + \varepsilon \). In particular, we stay inside \( \mathcal{C}_A \). The geodesic \( \text{Arg} \left( \phi_{c}(\mathbb{C}^*) \right) \) passes through \((e^{i\theta}, 1)\) and moves to the right as \( \theta \) increases, crossing \((-1, 1)\) on the way. We take our second break at the point \( c(2) := (e^{i(\pi+\varepsilon)+1}, 1) \).
Day 3: we start from $c^{(2)}$ and follow the path $x \mapsto c := (e^{x+i(\pi+\varepsilon)}, 1)$ from 1 to $-1$. While doing so, the geodesic $\text{Arg} \left( \phi_c(C^*) \right)$ remains unchanged. In particular, we stay inside $C_A$. Following this path, the geodesic $\text{Log} \left( \phi_c(C^*) \right)$ moves to the left as $x$ decreases. In particular, the latter line passes through $0 \in \mathbb{R}^2$ when $x = 0$. We denote the corresponding point $c^{(+)} := (e^{i(\pi+\varepsilon)}, 1)$. We take our third break at the point $c^{(3)} := (e^{i(\pi+\varepsilon)-1}, 1)$.

Day 4: we start from $c^{(3)}$ and follow the path $\theta \mapsto c := (e^{i\theta-1}, 1)$ from $\pi + \varepsilon$ to $\pi - \varepsilon$. We recover the point $c^{(0)}$ which is our final destination. During this last day, we stayed inside $C_A$ since the line $\text{Log} \left( \phi_c(C^*) \right)$ remained unchanged.

Proposition 3.4. For the loop $\gamma : S^1 \to C_A$ described above, the subset $b' := \{ (\theta, p) \mid p \in S_{\gamma(\theta)} \}$ of $S^1 \times \mathbb{C}^*$ is a coarse braid. Moreover, the class $[b'] \in B_d^*$ is the element $[b_1]^{-1}$.

The statement above depends on the choice of fundamental domain we use to represent braids, see Section 2.1. We will make this choice precise in the course of the proof. The arguments in the proof below are rather geometric and fairly elementary. It might be helpful though to consider the concrete example pictured in Figure 3.

Proof. Denote the projection $\pi' : b' \to S^1$ as at the end of Section 2.1. Travelling along $\gamma$, the line $\text{Log} \left( \phi_c(C^*) \right)$ passes through $0 \in \mathbb{R}^2$ exactly twice, namely when $c \in \{ c^{(x)}, c^{(+)} \}$. For such $c$, the geodesic $\text{Log} \left( \phi_c(C^*) \right)$ avoids the point $(-1, 1)$ so that the set $S_c$ consists of $d$ path connected components. Similarly, we observe that the preimage $\pi^{-1}(U)$ of a small neighbourhood $U$ around any of $c \in \{ c^{(x)}, c^{(+)} \}$ consists of $d$ path connected components. For any other $c$ in $\text{im}(\gamma)$, the set $S_c$ consists of $d$ distinct points varying continuously. It follows then that $b'$ is a coarse braid.

To show that $[b'] = [b_1]^{-1}$, we have the liberty to choose the real number $a$ determining the fundamental domain $[a, a + 2\pi] \times [0, 2\pi]$ which we use to represent braids. We choose $a = -\frac{2\pi}{d}$. Then, we have the $d$ points $\{ e^{i2\pi k/d} \mid k \in \mathbb{Z} \}$ (to which $e^{ia}$ belongs) are mapped under $\text{Arg} \circ \phi_{c(0)}$ bijectively to the intersection of $\text{Arg} \left( \phi_{c(0)}(C^*) \right)$ with the geodesic $\{ \text{arg}(w) = 1 \}$. The image under $\text{Arg}$ of the $d$ points of intersection $\phi_{c(0)}(C^*) \cap L$ are evenly distributed on the geodesic $\text{Arg} \left( \phi_{c(0)}(C^*) \right)$ and the geodesic $g := \{ \text{arg}(w) = -1 \}$. It follows that there is exactly one point of $\text{arg}(S_{c(0)})$ in each component of the complement to the collection of evenly distributed points $\{ e^{i2\pi k/d} \mid k \in \mathbb{Z} \}$ on $S^1$. While travelling along $\gamma$ between $c^{(0)}$ and $c^{(x)}$, the set $\text{arg}(S_{c})$ is constant until we reach $c^{(x)}$. By the above arguments, there is exactly one component of $\text{arg}(S_{c^{(x)}})$ in each component of $S^1 \setminus \{ e^{i2\pi k/d} \mid k \in \mathbb{Z} \}$. By construction of $\gamma$ and the choice of $a$, the two components of $\text{arg}(S_{c^{(x)}})$ in the arcs $[e^{ia}, e^{i(a+2\pi/d)}]$ and $[e^{ia+2\pi/d}, e^{i(a+4\pi/d)}]$ are arbitrarily close to each other while the distance between any other pair of adjacent components of $\text{arg}(S_{c^{(x)}})$ is arbitrarily large relative to $\varepsilon$. Equivalently, the projection under arg of the first two strands of $b'$ between $c^{(0)}$ and $c^{(x)}$ are arbitrarily close to each other while the projection of other strands are far apart. The projection of the latter strands is unchanged between $c^{(x)}$ and $c^{(2)}$. Between $c^{(2)}$ and $c^{(3)}$, the projection under arg of the first two strands of $b'$ meet when $\theta = \pi$. The projection of the first strand under Log sits on the ray $r_{\infty} \subset \text{Log}(L)$ while the projection of the second strand sits on the ray $r_{w} \subset \text{Log}(L)$. This implies that locally, the first strand has larger modulus than the second. In the corresponding diagram of $b'$, the first strand passes under the second. The projection under arg of the remaining strands of $b'$ do not meet between $c^{(2)}$ and $c^{(3)}$ and it should be clear from now that there will not meet between $c^{(3)}$ and our return to $c^{(0)}$. Since the same is true for the first two strands, the result follows. \qed
Recall that we have coordinates $c_0, c_1, c_2$ on $\mathbb{C}^A$ corresponding respectively to the monomials $1, x^p$ and $x^d$.

**Lemma 3.5.** For any $j \in \{1, \ldots, d\}$, there exists a loop $\ell_j : S^1 \to C_A$ such that the subset $b'_j := \{(\theta, p) \mid p \in S_{\ell_j(\theta)}\}$ of $S^1 \times \mathbb{C}^*$ is a coarse braid of class $[b_j] \in B^*_d$ and such that $\text{im}(\ell_j) \subset \{c_0=c_2=1\}$. Moreover, the loop $\ell : S^1 \to C_A$ given by $\ell(e^{i\theta}) = (e^{1+i(\pi-\varepsilon)}, e^{-i\theta})$ defines a geometric braid $b$ with class $[b] = [\tau]$.

**Proof.** For $j = 1$, it suffices to consider the loop $\ell_1 := \gamma^{-1}$ of Proposition 3.4. For any other $j$, observe that the set of tropical solutions $S_{c(0)}$ coincides with $S_{c(j)}$ where $c(j) := (e^{i(\pi-\varepsilon)} - 1, e^{2i\pi \frac{j-1}{d}}, 1)$. Therefore, we can also construct the loop $\gamma_j$ defined as $\gamma$ except that the base-point of $\gamma_j$ is $c(j)$.
instead of \( e^{(\theta)} \). Consider the path \( \rho_j : [0, 1] \to C_A \) given by
\[
\rho_j(\theta) = (e^{i(\pi-\epsilon)}-1)e^{2i\pi\theta\frac{(j-1)}{a}}, 1)
\]
connecting \( e^{(\theta)} \) to \( c_j \). Then, the sought loop \( \ell_j \) for \( j \in \{2, \ldots, d\} \) is defined by \( \ell_j := (\rho_j)^{-1} \circ (\gamma_j)^{-1} \circ \rho_j \). It follows from the construction that \( \text{im}(\ell_j) \subset \{ c_0 = c_2 = 1 \} \) for any \( j \).

Eventually, the set \( b := \{ (\theta, p) \mid p \in S_\ell(\theta) \} \) defined by \( \ell \) is easily seen to be a geometric braid since \( \text{Log}(\phi_\ell(\theta)(\mathbb{C}^*) \) never passes through 0 \( \in \mathbb{R}^2 \). Each of the points in the set of tropical solutions \( S_\ell(\theta) \) travels clockwise along \( S^1 \subset \mathbb{C} \) with constant velocity \( \frac{2\pi}{a} \) so that the diagram of \( b \) is the same as the one of \( \tau \).

**Corollary 5.** The map \( \mu_A : \pi_1(C_A) \to B_A^* \) is surjective. Moreover, the composition of the map \( \pi_1(C_A \cap \{c_0 = c_2 = 1\}) \to \pi_1(C_A) \) with \( \mu_A \) is surjective.

**Proof.** This follows from Proposition 3.3 and Lemmas 2.1, 2.2 and 3.5.

**Remark 3.6.** Using the tropical construction above, we could prove Theorems 2 and 3 for trinomials without having to work out any relations in the braid group \( B_A^* \). Unfortunately, these tropical techniques cannot be applied to arbitrary support sets. For supports \( A \) with four elements, the set \( L \) is to be replaced with the phase-tropical plane \( H \subset \mathbb{R}^3 \). For particular supports \( A \), the geodesic \( \text{Arg}(\phi_c(\mathbb{C}^*)) \) intersects \( \text{Arg}(H) \) in less than \( d \) components for any choice of \( c \). Therefore, there is no generalisation of the loop \( \gamma \) constructed above.

## 4. Proofs of Theorem 1 and 3

In this section, the support \( A \subset \mathbb{N} \) is reduced and with extremal elements 0 and \( d \). In order to prove the main theorems, we specialise to the case of all trinomials \( \{0, p, d\} \subset A \) and implement an analogue of the Euclidean algorithm in \( B_A^* \) in the name of Proposition 4.1.

For any divisor \( k \geq 2 \) of \( d \) and any \( j \in \{1, \ldots, d\} \), define \( J_{k,j} := \{ \ell \in \{1, \ldots, d\} : \ell \equiv j \mod k \} \) and the braid
\[
[b_{k,j}] := \prod_{\ell \in J_{k,j}} [b_{\ell}]
\]
Observe that the above elements \([b_{\ell}]\) commute so that \([b_{k,j}]\) is well defined.

**Proposition 4.1.** Let \( k, \ell \geq 2 \) be two distinct divisors of \( d \) and denote \( q := \text{lcm}(k, \ell) \). The subgroup of \( B_A^* \) generated by \( \cup_{1 \leq j \leq d} \{[b_{k,j}], [b_{\ell,j}]\} \) contains \( \cup_{1 \leq j \leq d} \{[b_{q,j}]\} \).

**Proof of Theorems 2 and 3.** Choose \( p \in A \setminus \{0, d\} \). Then, the space of conditions \( C_A \) contains the space of conditions \( \tilde{C}_A \) for \( A := \{0, p, d\} \). In particular, the image of \( \mu_A^* \) contains the image of \( \mu_{\tilde{A}}^* \). If we denote \( a := \text{gcd}(p, d) \), \( p' := \frac{p}{a} \) and \( d' := \frac{d}{a} \), then im \( (\mu_{\tilde{A}}^* \) contains the pullback by the covering \( x \mapsto x^a \) of every braid in \( \text{im} \mu_{\tilde{A}}^* \) since every polynomial in \( \tilde{C}_A \) is the composition of a polynomial in \( C_{\{0, p', d'\}} \) with the latter covering. In particular, the image of \( \mu_{\tilde{A}}^* \) contains the pullback \([b_{q,j}] \in B_{\tilde{A}}^* \) of \([b_j] \in B_A^* \) for all possible \( j \), according to Corollary 5. It follows now from Proposition 4.1 that im \( (\mu_A^* \) contains \([b_{q,j}] \) where \( q := \text{lcm}(\{\frac{d}{\text{gcd}(p, d)} \mid p \in A \setminus \{0, d\}\} \)). Since \( A \) is reduced, the integer \( q \) is equal to \( d \) and \([b_{q,j}] = [b_j] \in B_A^* \). Observe moreover that we can choose the loops in \( \pi_1(C_{\{0, p', d'\}}) \) giving rise to the \([b_j]\)-s in \( B_{\tilde{A}}^* \) to be in \( \{c_0 = c_2 = 1\} \subset \mathbb{C}^{(0, p', d')} \) by Corollary 5 and then the loops giving rise to \([b_{\ell,j}]\) and \([b_{q,j}] = [b_j] \in B_A^* \) are in \( \{c_0 = c_2 = 1\} \subset \mathbb{C}^\tilde{A} \) and \( \{c_0 = c_2 = 1\} \subset \mathbb{C}^A \).
respectively. This fact together with Lemma 2.1 imply Theorem 3. Theorem 1 follows from Lemma 2.2 and the fact that $[\tau]$ is always contained in $\text{im}(\mu^*_A)$. Indeed, the element $[\tau]$ is the image under $\mu^*_A$ of the loop $\theta \mapsto p_\theta(x) = x^d - e^{i\theta}$, $\theta \in [0, 2\pi]$.

We will now proceed to the proof of Proposition 4.1. In order to do so, we will need the following terminology.

**Definition 4.2.** A braid $[b] \in B_d^*$ is simple (respectively sparse) if it can be written as a composition $[b] = [b_{j_1}] \circ [b_{j_2}] \circ \cdots \circ [b_{j_k}]$ for a sequence of integers $1 \leq j_1 < j_2 < \cdots < j_k \leq d$ such that the difference between consecutive integers is at least 2 (respectively 3). In particular, all the elements in the above decomposition of $[b]$ commute. Any simple braid $[b]$ is determined by the collection $\{j_1, \cdots, j_k\}$ that we refer to as the support of $[b]$. Denote by $[b, J]$ the simple braid with support $J$ and define the support $J_n = \{k \in 2\mathbb{Z} + 1 \mid 1 \leq k \leq 2n - 1\}$ of cardinality $n$, for any integer $n \leq \lfloor d/2 \rfloor$.

Observe that the braid $[b_{k,j}]$ is always simple and that it is sparse provided that $k \geq 3$. For the sake of brevity, let us introduce the notation $g \ast g' := g \circ g' \circ g^{-1}$.

**Lemma 4.3.** Let $[b]$ and $[\beta]$ be two sparse braids in $B_d^*$ with respective supports $J$ and $J'$. Then, the braid $[b] \bullet [\beta] := [b]^{-1} \circ [\beta]^{-1} \circ [b] \circ [\beta] \circ [b]$ is the simple braid with support $J \bullet J'$ consisting of the union of $J \cap J'$ with the elements in $J$ at distance at least 2 from $J'$ with the elements in $J'$ at distance exactly 1 from $J$.

**Proof.** First, we observe that the set $J \bullet J'$ is the support of a simple braid, that is, two consecutive indices are at distance at least 2 from each other. This is a straightforward consequence of the fact that $[b]$ and $[\beta]$ are sparse.

If $j \in \{1, \cdots, d\}$ is such that $\{j - 1, j\}$ is disjoint from $J \cup J'$, then the $j$th strand of $[b] \bullet [\beta]$ is straight. Therefore, there are at most 15 configurations to check: each of the indices $j$ and $j + 1$ are in one of the sets $(J \cup J')^c$, $J \setminus J'$, $J' \setminus J$ or $J \cap J'$ and we should discard the case when both indices belong to $(J \cup J')^c$ by the previous argument. Among these configurations, some are prevented by the fact that $[b]$ and $[\beta]$ sparse and some configurations are redundant. All the relevant configurations are depicted in Figure 4.

![Figure 4](image-url)

**Figure 4.** Local pictures for the braid $[b] \bullet [\beta]$. The first two configurations on the left correspond to indices in $J'$ at distance 1 from $J$, the third to indices in $J \cap J'$, the fourth to indices in $J$ at distance at least 2 from $J'$ and the last one to indices in $J'$ at distance at least 2 from $J$. 
Lemma 4.4. Let $G \subset B_d^+$ be a subgroup that is invariant under conjugation by $[\tau]$. Assume further that $G$ contains either $[b_j] \circ [b_j + 1]^{-1}$ or $[b_j] \circ [b_j + 2]^{-1}$ for some $j$. If $G$ contains a simple braid $[b_j]$, then $G$ contains the simple braid $[b_{\tau}]$ for any support $J'$ such that $\#J = \#J'$.

Proof. Denote $n : = \#J$. We will show that $G$ contains $[b_j]$ if and only if it contains the simple braid $[b_{J_n}]$, see Definition 4.2.

Assume first that $G$ contains $[b_j] \circ [b_{j+1}]^{-1}$ for some $j$. Since $G$ is invariant under $[\tau]$, then it contains $[b_j] \circ [b_{j+1}]^{-1}$ for any $j$. In particular, if $j \in J$ and $j - 2 \notin J$ (recall that $j - 1 \notin J$ since $[b_j]$ is simple), then $[b_j] \circ [b_{j+1}] \circ [b_j]^{-1} = [b_{\tau}]$ where $J' := (J \setminus \{j\}) \cup \{j - 1\}$. Repeating this procedure as many times as necessary, we can shift the smallest index in $J$ to 1, then the second smallest index to 3 and so on, until we obtain the sought simple braid $[b_{J_n}]$. The result follows.

Assume now that $G$ contains $[b_j] \circ [b_{j+2}]^{-1}$ for some $j$, and thus for any $j$. Since $G$ is invariant under $[\tau]$, there is no loss of generality in assuming that $1 \in J$. As in the previous paragraph, we want to shift the indices of $J$ one by one to the left using elements in $G$, until we obtain the support $J_n$. For any $j \in J \setminus \{1\}$, there are two cases to consider: either $\{j - 3, j - 2\} \cap J = \emptyset$ or $j - 3 \in J$ and $j - 2 \notin J$ (the remaining case $j - 2 \in J$ corresponds to the situation where we cannot shift $j$ any further to the left). If we are in the first case, then $[b_j] \circ [b_{j-2}] \circ [b_j]^{-1} = [b_{J'}]$ where $J' := (J \setminus \{j\}) \cup \{j - 2\}$. In the second case, we have that

$$
\sigma_0 := \big(([[b_{j-3}]^{-1} \circ [b_{j-1}] \ast [b_j]^{-1}] \ast [b_j]\big)
$$

is equal to $[b_{J'}]$ with $J' = (J \setminus \{j\}) \cup \{j - 1\}$, see Figure 5. In both cases, the element $[b_{J'}]$ is in $G$ if and only if $[b_j]$ is. This proves inductively that $[b_{J_n}] \in G$ if and only if $[b_j] \in G$. The result follows.

![Figure 5](image_url)

**Figure 5.** The braid $[\sigma_0]$. We only picture the strands from $j - 3$ to $j + 1$ since $[b_{j-3}] \circ [b_{j-1}]^{-1}$ restricts to the identity on the remaining strands. The whole products restricts therefore to $[b_j]^{-1} \circ [b_j]^2 = [b_j]$ on the latter strands.

Proof of Proposition 4.1. Denote by $G$ the group generated by $\cup_{1 \leq j < d} \{[b_{k,j}], [b_{\ell,j}]\}$. The group $G$ is invariant by conjugation by $[\tau]$, since its generating set is. Therefore, it suffices to show that $G$ contains $[b_{\eta,j}]$ for a single $j$. 


Denote $a := \text{gcd}(k, \ell)$ and define $k', \ell'$, and $b$ such that $k = ak', \ell = a\ell'$ and $d = abk'\ell'$. First, observe that if $k$ divides $\ell$ or $\ell$ divides $k$, there is nothing to prove. Thus, we can assume that $k' > \ell' \geq 2$ are coprime. Second, observe that if the statement is true for the triple $k$, $\ell$ and $d' := abk'\ell'$, then the statement is also true for $k$, $\ell$ and $d$. This is easily seen by using the covering $x \mapsto x^b$ from $\mathbb{C}^*$ to itself. Therefore, there is not loss of generality in assuming that $d = ak'\ell'$.

In that case $q = d$ and $[b_{q,j}] = [b_j]$. Thus, we have to show that $[b_j]$ is an element in $G$ for some $j$.

Assume first that $a \geq 3$. Under the present assumptions, the braid $[b_{k,1}]$ is the product of the $\ell'$ elements $[b_i]$ for $i \equiv 1 \mod k$ and $[b_{k,1}']$ is the product of the $k'$ elements $[b_j]$ for $j \equiv 1 \mod k$. Observe that $J_{k,1} \cap J_{\ell,1} = \{1\}$ and that the distance $|j' - j''|$ between elements $j' \in J_{k,1}$ and $j'' \in J_{\ell,1}$ is always divisible by $a \geq 3$. By Lemma 4.3, we obtain that $[b] := [b_{k,1}] \bullet [b_{\ell,2}] \in G$ has support $(J_{k,1} \setminus \{1\}) \cup \{2\}$. In turn, we deduce that $[b_{k,1}] \circ [b] = [b_1] \circ [b_2]^{-1}$ is an element in $G$. Thus, we are in position to apply Lemma 4.4 and obtain that both $[b_{\ell,\sigma}]$ and $[b_{\ell,\tau}]$ are in $G$. The result follows now from the Euclidean algorithm. Indeed, we obtain that $[b_{\ell,\tau}] \in G$ as the conjugation of $[b_{\ell,\sigma}] \circ [b_{\ell,\tau}]^{-1} \in G$ by the appropriate power of $[\tau]$. Inductively, we obtain that $[b_{k,1}] \in G$ where $\delta := \text{gcd}(\ell', k')$. Since the latter $\text{gcd}$ is 1 by assumption, we conclude that $[b_{k,1}] \in G$. The result follows.

Assume now that $a = 2$. We will show that $G$ contains $[b_j] \circ [b_{j+2}]^{-1}$ for some $j$, apply Lemma 4.4 and conclude with the Euclidean algorithm as above. Assume first that $\ell' \geq 3$. Since $k'$ and $\ell'$ are coprime, the $\ell'$ elements of $J_{k,1}$ achieve the $\ell'$ classes modulo $\ell$ that are divisible by $a$. In particular, there are exactly two elements $j, j \in J_{k,1}$ such that $j + 2 \in J_{k,1}$ and $j - 2 \in J_{\ell,1}$. All elements in $J_{k,1} \setminus \{1; j; j\}$ are at distance at least 4 from $J_{k,1}$. Since $a = 3$ and $\ell' \geq 3$, two consecutive element in $J_{k,1}$ are at distance at least 6 from each other. Therefore, the index $j$ is at distance at least 2 from $J_{k,2}$ while 1 and $j$ are at distance 1. The remaining indices of $J_{k,1}$ are at distance at least 3 from $J_{\ell,2}$. It follows that $[b_{k,1}] \bullet [b_{\ell,2}] \in G$ is equal to $[b_{j'}]$ with $J' := J_{k,1} \setminus \{1, j\} \cup \{2, j - 1\}$. In turn, we have that $([\tau] \star [b_{j'}]) \bullet [b_{k,1}] \in G$ is equal to $[b_{j''}]$ with $J'' := (J_{k,1} \setminus \{1\}) \cup \{3\}$. We conclude that $G$ contains $[b_{k,1}] \circ [b_{j''}]^{-1} = [b_1] \circ [b_3]^{-1}$ and the result follows. If now $\ell' = 2$ (implying that $\ell = 4$), then $J_{k,1} = \{1, k + 1\}$ and $\{1, k - 1, k + 3\} \subset J_{k,1}$ since $k$ is divisible by 2 but not by 4. It follows that $[\tau] \star ([b_{k,1}] \bullet [b_{\ell,2}]) \in G$ is equal to $[b_{k+1}] \circ [b_{k+1}]$. Dividing $[b_{k,1}]$ by the latter element gives $[b_1] \circ [b_3]^{-1} \in G$. The result follows.

Assume now that $a = 1$. We postpone the case $\ell = 2$ and assume for now that $\ell \geq 3$. Since $k$ and $\ell$ are coprime, there are exactly two elements $j$ and $j$ in $J_{k,1}$ that are at distance 1 from $J_{\ell,1}$, namely the two elements with respective reduction 0 and 2 modulo $\ell$. Since $J_{k,1} \cap J_{\ell,1} = \{1\}$, any element in $J_{k,1} \setminus \{1; j; j\}$ is at distance at least 2 from $J_{\ell,1}$. We deduce that $[b_{k,1}] \bullet [b_{\ell,1}] \in G$ is equal to $[b_{j')]$ with $J'' := (J_{k,1} \setminus \{j, j\}) \cup \{j + 1, j - 1\}$ and then that $([\tau] \bullet [b_{\ell,1}]) \bullet [b_{k,1}] \in G$ is the simple braid with support $(J_{k,1} \setminus \{j\}) \cup \{j + 2\}$. Dividing $[b_{k,1}]$ by the latter element, we obtain that $[b_{j} \circ [b_{j+2}]^{-1} \in G$. The result follows again from Lemma 4.4 and the Euclidean algorithm.

It remains to treat the case $\ell = 2$ and $k$ odd. For $k = 3$, define the following elements of $G$

$$[\sigma_1] := [\tau]^{-1} \star ([b_{2,2}]^{-1} \circ [b_{3,1}] \circ [b_{2,1}]^{-1}) \circ [b_{3,2}] \circ [b_{3,1}]$$

and $[\sigma_2] := [b_{2,2}]^{-1} \circ ([b_{3,1}] \bullet [b_{2,1}])$, see Figure 6. Then, we have that $[\sigma_2]^{-1} \circ [\sigma_1] = [b_3] \in G$ and the result follows.
For \( k \geq 5 \), define the following elements of \( G \)

\[
[\sigma_3] := ([b_{k,2}]^{-1} \circ [b_{k,1}]^{-1} \circ [b_{k,2}]) \star [b_{k,1}], \quad [\sigma_4] := [b_{k,1}] \circ [b_{k,2}] \circ [b_{k,3}] \circ [b_{k,4}]
\]

and \([\sigma_5] := ([\sigma_4] \circ ([\tau]^{k+1} \circ [\sigma_3]) \circ [b_{k,1}])^{-1} \circ ([\tau] \circ [\sigma_3]).\)

The latter elements are depicted in Figure 7. In particular, we obtain that \([b_1] \circ [b_{k+3}] \in G\). In turn, we obtain \([b_{k,1}] \circ ([b_1] \circ [b_{k+3}])^{-1} = [b_{k+1}] \circ [b_{k+3}]^{-1} \in G\). The result follows from Lemma 4.4 and the Euclidean algorithm. \(\Box\)

**Figure 6.** The braids \([\sigma_1]\) and \([\sigma_2]\).

**Figure 7.** The braids \([\sigma_3]\), \([\sigma_4]\) and \([\sigma_5]\) (some of the blue areas are not present for \( k = 5 \)).
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