GENERATORS AND RELATIONS FOR LIE SUPERALGEBRAS
OF CARTAN TYPE

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Abstract. We give an analog of a Chevalley–Serre presentation for the Lie superalgebras $W(n)$ and $S(n)$ of Cartan type. These are part of a wider class of Lie superalgebras, the so-called tensor hierarchy algebras, denoted $W(g)$ and $S(g)$, where $g$ denotes the Kac–Moody algebra $A_r$, $D_r$ or $E_r$. Then $W(A_{n-1})$ and $S(A_{n-1})$ are the Lie superalgebras $W(n)$ and $S(n)$. The algebras $W(g)$ and $S(g)$ are constructed from the Dynkin diagram of the Borcherds–Kac–Moody superalgebras $B(g)$ obtained by adding a single grey node (representing an odd null root) to the Dynkin diagram of $g$. We redefine the algebras $W(A_r)$ and $S(A_r)$ in terms of Chevalley generators and defining relations. We prove that all relations follow from the defining ones at level $\geq -2$. The analogous definitions of the algebras in the $D$- and $E$-series are given. In the latter case the full set of defining relations is conjectured.
1. Introduction

In the classification of complex finite-dimensional simple Lie superalgebras, the classical ones are separated from those of Cartan type, and further divided into the two classes of basic and strange Lie superalgebras [1]. The simplest example of a Lie superalgebra of Cartan type (from which the other ones can be obtained) is $W(n)$, consisting of all derivations of the Grassmann algebra with $n$ generators. The basic Lie superalgebras are similar to finite-dimensional simple Lie algebras, in the sense that they can be constructed from Dynkin diagrams, where each node represents a simple root with corresponding Chevalley generators “$e$” and “$f$”. One example is $A(n - 1, 0) = \mathfrak{sl}(1|n)$. In the present paper, we will show how these two representatives of two important classes of finite-dimensional Lie superalgebras (Cartan type and classical) are in fact related to each other. As one of our main results (Theorem 4.5) we will show that $W(n)$ can be constructed from the same Dynkin diagram as $A(n - 1, 0)$, but with extended sets of generators (2.32) and relations (3.1)–(3.7), (4.16). The relationship extends to infinite-dimensional Lie superalgebras, with so-called tensor hierarchy algebras on the one hand side, and Borcherds–Kac–Moody (BKM) superalgebras on the other.

Any (possibly finite-dimensional) Kac–Moody algebra $\mathfrak{g}$ can be extended to a contragredient Lie superalgebra $\mathcal{B}$ by adding a node to the Dynkin diagram such that the corresponding Chevalley generators are odd elements [1]. When the simple root represented by the new node is a null root, the node is said to be grey, and drawn as $\otimes$. We are particularly interested in the case where $\mathfrak{g}$ belongs to the $A$-, $D$- or $E$-series of Kac–Moody algebras, say $\mathfrak{g} = X_r$ with $X$ being either $A$, $D$, or $E$, and the grey node replaces the usual white node added to the Dynkin diagram of $X_r$ in the extension to $X_{r+1}$. This allows us to identify the grey node uniquely, and write $\mathcal{B} = \mathcal{B}(\mathfrak{g})$. The Cartan matrices of $\mathcal{B}$ and $X_{r+1}$ only differ by the diagonal entry corresponding to the additional node. In this case $\mathcal{B}$ is a BKM superalgebra [2,3].

In [4] a Lie superalgebra closely related to $\mathcal{B}$, but not of BKM type, was defined from the same Dynkin diagram as $\mathcal{B}$ (for finite-dimensional $\mathfrak{g}$). It was called tensor hierarchy algebra, and here we denote it by $S(\mathfrak{g})$. We also introduce a third Lie superalgebra $W(\mathfrak{g})$ associated to the same Dynkin diagram. Our aim is to give a unified construction of these algebras. To this end, Chevalley-type generators for the tensor hierarchy algebras are defined from the same Dynkin diagram as $\mathcal{B}(\mathfrak{g})$, but with an asymmetry between generators “$e$” and “$f$” in the absence of a Cartan involution. Throughout the article, we will work over a field $\mathbb{K}$, which can be either the complex or real numbers.
We will focus mainly on the class of algebras obtained by taking $X_r = A_r$, and show that $W(A_{n-1})$ and $S(A_{n-1})$ are the well known finite-dimensional Lie superalgebras of Cartan type, denoted by $W(n)$ and $S(n)$, respectively. We will obtain presentations for $W(A_r)$ and $S(A_r)$ by giving an analog of Chevalley generators and defining relations. We will also comment on the cases $X_r = D_r$ and $X_r = E_r$. The latter case is interesting, and provides the main motivation from mathematical physics for this investigation, due to a deep, and to a large extent unexplored, connection to generalised and extended geometry (see [5–13]). The level decompositions of $W(E_r)$ and $S(E_r)$ correctly predict, for example, the embedding tensor used in the construction of gauged supergravities. Given the geometric character of the classical definition of $W(n)$ as operating on forms, one may envisage a similar rôle for $W(E_r)$ in exceptional geometry. Hopefully, the tensor hierarchy algebra can be shown to provide an underlying structure, on which the concept of exceptional geometry depends. The relationship between tensor hierarchies and Leibniz algebras recently studied in [14] might be useful in this respect. From a mathematical perspective, we expect that the present construction will be useful e.g. for addressing questions concerning the combinatorics and representation theory of Cartan type superalgebras.

Sections 1 and 2 contain a review of the BKM and Cartan type superalgebras used in the paper, and introduce some notation. In Section 3 we define the Lie superalgebras $\widehat{W}(g)$ and $\widehat{S}(g)$ in terms of Chevalley–Serre-like generators and relations, from which $W(g)$ and $S(g)$ are obtained by factoring out the maximal ideal intersecting the local part trivially. In Section 4 we construct this maximal ideal for the case $g = A_{n-1}$. We end in Section 5 with a discussion of and a conjecture for the $D$ and $E$ cases in Section 5 where the identification $S(D_r) \simeq H(2r)$ is made. Details about the root system of $W(A_{n-1}) = W(n)$ are given in an appendix.

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2. \(\mathbb{Z}\)-graded Lie superalgebras

In this section we review some basic definitions and results from Section 1.2 in [1]. We refer to this article for further details.

First we recall that a Lie superalgebra is a \(\mathbb{Z}_2\)-graded vector space \(G = G_0 \oplus G_1\) with a bilinear bracket
\[
G \times G \to G, \quad (x, y) \mapsto [x, y] \quad (2.1)
\]
satisfying \([G_i, G_j] \subseteq G_{(i+j) \pmod 2}\), and the identities
\[
[x, y] = (-1)^{|x||y|}[y, x], \quad (2.2)
\]
\[
[x, [y, z]] = [[x, y], z] - (-1)^{|x||y|}[y[x, z]], \quad (2.3)
\]
where \(|x| = 0\) if \(x \in G_0\) and \(|x| = 1\) if \(x \in G_1\).

A \(\mathbb{Z}\)-grading of the Lie superalgebra \(G\) is a decomposition of \(G\) into a direct sum of subspaces \(G_i\) for all integers \(i\), called levels, such that \([G_i, G_j] \subseteq G_{i+j}\). In all cases we consider in this paper, we have \([G_i, G_j] = G_{i+j}\).

Whenever we use the notation \(G_i\) for subspaces of an algebra \(G\), we assume a \(\mathbb{Z}\)-grading of \(G\). We will also use the notation \(G_\pm = \bigoplus_{i \in \mathbb{Z}_\pm} G_i\). The \(\mathbb{Z}\)-grading is said to be consistent if \(G_i \subseteq G_{(i \pmod 2)}\).

It follows from the relations \([G_i, G_j] \subseteq G_{i+j}\) that the subspace \(G_0\) of any \(\mathbb{Z}\)-graded Lie superalgebra \(G\) is a subalgebra, which is a Lie algebra if the \(\mathbb{Z}\)-grading is consistent, and all subspaces \(G_i\) can be considered as \(G_0\)-modules.

A \(\mathbb{Z}\)-graded Lie superalgebra can be constructed from a \(\mathbb{Z}_2\)-graded vector space \(g = g_0 \oplus g_1\) with a consistent decomposition into a direct sum of subspaces \(g = g_{-1} \oplus g_0 \oplus g_1\) and a bilinear bracket defined for all pairs of elements in \(G\) such that not both of them have nonzero components in the same subspace \(g_1\) or \(g_{-1}\). If \([g_i, g_j] \subseteq g_{i+j}\) and the identities (2.2)–(2.3) are satisfied whenever the brackets are defined, then \(g\) is a local Lie superalgebra.

Clearly, any \(\mathbb{Z}\)-graded Lie superalgebra \(G = \bigoplus_{i \in \mathbb{Z}} G_i\) gives rise to a local Lie superalgebra \(G_{-1} \oplus G_0 \oplus G_1\), which is called the local part of \(G\). If a subspace \(g'\) of a local Lie superalgebra \(g\) itself is a local Lie superalgebra with respect to the bracket and the decomposition \(g' = g'_{-1} \oplus g'_0 \oplus g'_1\) inherited from \(g\), then we call \(g'\) a local subalgebra of the local Lie superalgebra \(g\).
Given a local Lie superalgebra \( g = g_\pm \oplus g_0 \oplus g_1 \) we can construct maximal and minimal Lie superalgebras with \( g \) as the local part. The maximal one is defined as 
\[
\tilde{G} = F(g)/I
\]
where \( F(g) \) is the free Lie superalgebra generated by \( g \), and \( I \) the ideal generated by the commutation relations in \( g \). The maximal Lie superalgebra can then be decomposed as 
\[
\tilde{G} = \tilde{G}_- \oplus \tilde{G}_0 \oplus \tilde{G}_+,
\]
where \( \tilde{G}_\pm \) is the free Lie superalgebra generated by \( \tilde{G}_{\pm 1} = g_{\pm 1} \). The minimal Lie superalgebra with local part \( g \) is defined as \( G = \tilde{G}/J \), where \( J \) is the maximal homogeneous ideal of \( \tilde{G} \) intersecting the local part trivially. Thus \( G_{\pm 1} = \tilde{G}_{\pm 1} = g_{\pm 1} \). Maximality (respectively minimality) here means that any isomorphism between the local parts of \( \tilde{G} \) (respectively \( G \)) and any other \( \mathbb{Z}_2 \)-graded Lie superalgebra \( G' \) can be extended to a surjective homomorphism \( \tilde{G} \rightarrow G' \) (respectively \( G \rightarrow G' \)) \[1,15]\.

2.1. The local Lie superalgebra associated to a vector space. Any \( \mathbb{Z}_2 \)-graded vector space \( U_1 \) gives rise to a local Lie superalgebra in the following way. Set
\[
U_0 = \text{End} U_1, \quad U_{-1} = \text{Hom}(U_1, U_0).
\]
Then \( U_1 \oplus U_0 \oplus U_{-1} \) is a local Lie superalgebra, denoted \( u(U_1) \), with the bracket given by the following relations,
\[
\begin{align*}
[x_0, y_1] &= x_0(y_1), \\
[x_1, y_1] &= x_1(y_1), \\
[x_0, y_0] &= x_0 \circ y_0 - (-1)^{|x||y|} y_0 \circ x_0, \\
[x_0, y_{-1}] &= x_0 \circ y_{-1} - (-1)^{|x||y|} y_{-1} \circ x_0,
\end{align*}
\]
where \( x_i \) and \( y_i \) belong to \( U_i \) (i = 0, ±1), and have \( \mathbb{Z}_2 \)-degrees \( |x| \) and \( |y| \), respectively.

In particular, this means that \( U_0 \), as a Lie superalgebra, is \( \mathfrak{gl}(U_1) \). Let \( K_a \) be a basis of \( U_1 \), for \( a = 1, 2, \ldots, \dim U_1 \). Then we have bases \( K^a_b \) of \( U_0 \) (with \( a, b \) indexing rows and columns of a matrix in \( \text{End} U_1 \)) and \( K^{a,b,c} \) of \( U_{-1} \) (with \( a \) indexing the basis of the dual \( U_1^* \) and \( b, c \) indexing the basis of \( U_0 \)). When \( U_1 \) is an odd vector space, so that \( U_0 = \mathfrak{gl}(U_1) \) is a Lie algebra, the commutation relations for the basis elements that follow from (2.5) are
\[
\begin{align*}
[K_a, K^b_{\cdot c}] &= \delta_a^b K_c, \\
[K^a_b, K^c_{\cdot d}] &= \delta_b^c K^a_d - \delta_a^b K^c_d, \\
[K_a, K^{b,c}_{\cdot d}] &= \delta_a^b K^{c}_{\cdot d}, \\
[K^a_b, K^{c,d}_{\cdot e}] &= \delta_b^c K^{a,d}_{\cdot e} + \delta_a^d K^c_{\cdot e} - \delta_e^a K^{c,d}_{\cdot b},
\end{align*}
\]
where \( \delta_a^b \) is a Kronecker delta.

From this local Lie superalgebra, and its local subalgebras, we can construct the maximal and minimal Lie superalgebras as above. As we will see, the BKM algebras described in the next section, and the associated tensor hierarchy algebras, can be constructed in this way.
2.2. Borcherds–Kac–Moody superalgebras. We are particularly interested in Lie superalgebras \( \mathcal{B} \) that are minimal with respect to a consistent \( \mathbb{Z} \)-grading and with local part \( \mathcal{B}_1 \oplus \mathcal{B}_0 \oplus \mathcal{B}_1 \), where the subalgebra \( \mathcal{B}_0 \) is the direct sum of a simple and simply laced (possibly finite-dimensional) Kac–Moody algebra \( \mathfrak{g} \) and a one-dimensional center of \( \mathcal{B}_0 \), and the representation of \( \mathfrak{g} \) on \( \mathcal{B}_1 \) is fundamental and dual to the one on \( \mathcal{B}_{-1} \). Then in the Dynkin diagram of \( \mathcal{B} \), there is a grey node connected to one of the white nodes in the Dynkin diagram of \( \mathfrak{g} \) by a single line.

Let \( r \) be the rank of \( \mathfrak{g} \). We use a labelling \( 1, 2, \ldots, r \) of the nodes in the Dynkin diagram of \( \mathfrak{g} \) such that the grey node is connected to node 1, and can be included as node 0 in an extended labelling \( 0, 1, 2, \ldots, r \). Then \( \mathcal{B} \) has a corresponding Cartan matrix \( B_{ab} \) \((a, b = 0, 1, \ldots, r) \) such that

\[
B_{00} = 0, \quad B_{01} = B_{10} = -1,
\]

\[
B_{0i} = B_{i0} = 0 \quad (i = 2, \ldots, r).
\] (2.7)

Assuming \( B_{ab} \) to be non-degenerate, \( \mathcal{B} \) can be constructed as the Lie superalgebra generated by \( 2(r + 1) \) elements \( e_a, f_a \) (odd if \( a = 0 \), even otherwise) modulo the Chevalley–Serre relations

\[
[h_a, e_b] = B_{ab}e_b, \quad [h_a, f_b] = -B_{ab}f_b, \quad [e_a, f_b] = \delta_{ab}h_b,
\] (2.8)

\[
(ad e_a)^{1-B_{ab}}(e_b) = (ad f_a)^{1-B_{ab}}(f_b) = 0 \quad (a \neq b)
\] (2.9)

\((a, b = 0, 1, 2, \ldots, r)\), where the elements \( h_a = [e_a, f_a] \) span an abelian Cartan subalgebra. Thus the Chevalley–Serre relations for \( \mathcal{B} \) take the same form as those for \( \mathfrak{g} \) (keeping in mind that the index set is extended, and that the bracket \([e_0, f_0]\) is symmetric, but it is often relevant to consider \( \mathcal{B} \) as a special case of a BKM superalgebra with additional relations in the general case. We refer to [3] for details about general BKM superalgebras.

A comment on notation: We write \( \mathcal{B} \) as \( \mathcal{B}(\mathfrak{g}) \) when we want to emphasise the underlying Lie algebra \( \mathfrak{g} \). Strictly speaking, it is not sufficient to know the Lie algebra \( \mathfrak{g} \) in order to construct \( \mathcal{B}(\mathfrak{g}) \) from it; the data specifying \( \mathcal{B}(\mathfrak{g}) \) is a choice of \( \mathfrak{g} \) together with a choice of node 1 (the node connecting to the grey node 0). In the series of greatest interest to us, the A-, D- and E-series, our default choice of node 1 is the node connected to the additional white node added in order to obtain the next algebra in the series. However, other choices are possible. The same comment applies to the Lie superalgebras \( W(\mathfrak{g}) \) and \( S(\mathfrak{g}) \) that we will later apply to the same Dynkin diagram as \( \mathcal{B}(\mathfrak{g}) \).
Throughout the rest of the paper, the indices \( a, b, \ldots \) will take the \( r + 1 \) values 0, 1, \ldots, \( r \), where \( r \) is the rank of the Lie algebra \( g \). When \( g \) belongs to the \( A \)-series it is convenient to set \( n = r + 1 \), and thus the indices \( a, b, \ldots \) will take the \( n \) values 0, 1, \ldots, \( n - 1 \). The range of indices \( i, j, \ldots \) may vary, and will be specified explicitly whenever they appear (if not obvious from the context).

It follows from (2.7) that \( \det B = -\det A' \) where \( A' \) is the Cartan matrix obtained by removing the first two rows and columns (corresponding to removing nodes 0 and 1 from the Dynkin diagram), as is evident in (2.16) below. The subalgebra of \( g \) with the Cartan matrix \( A' \) obtained in this way will play an important role later, and we denote it by \( g' \). Since we assume \( B \) to be non-degenerate, this means that \( g' \) is simple.

As is the case for Kac–Moody algebras, the nodes in the Dynkin diagram of \( B \) correspond not only to generators \( e_a \) and \( f_a \), but also to simple roots \( \alpha_a \) that form a basis of the dual space of the Cartan subalgebra, and the Cartan matrix defines an inner product on this space up to an overall normalisation. Since we only consider simply laced Dynkin diagrams, we do not have to symmetrise the Cartan matrix, and we fix the normalisation as follows:

\[
(\alpha_a, \alpha_b) = B_{ab}. \tag{2.10}
\]

Thus \( (\alpha_0, \alpha_0) = 0 \), so \( \alpha_0 \) is a null root, whereas \( (\alpha_i, \alpha_i) = 2 \) for \( i = 1, 2, \ldots, r \).

The inner product also defines the basis \( \Lambda_1, \Lambda_2, \ldots, \Lambda_r \) of fundamental weights of \( g \), dual to the basis \( \alpha_1, \alpha_2, \ldots, \alpha_r \) of simple roots, with \( (\Lambda_i, \alpha_j) = \delta_{ij} \).

In the consistent \( \mathbb{Z} \)-grading of \( B \) mentioned above, the odd generators \( e_0 \) and \( f_0 \) belong to \( B_1 \) and \( B_{-1} \), respectively, and the even generators belong to \( B_0 \). As a \( g \)-module, \( B_1 \) has a lowest weight vector \( e_0 \) with weight \( -\Lambda_1 \), and \( B_{-1} \) has a highest weight vector \( f_0 \) with weight \( \Lambda_1 \). We will often use the following notation for weights where \( \Lambda_1 = (10\ldots 0) \), and the entries are the Dynkin labels, \( i.e., \) the coefficients of the weight in the basis \( \Lambda_1, \Lambda_2, \ldots, \Lambda_r \). We will occasionally use this notation not only for the weight itself, but also for the irreducible representation with that highest weight. Occasionally we will instead use the notation \( R(\lambda) \) for the irreducible module with highest weight \( \lambda \). Thus

\[
B_{-1} = R(\Lambda_1) = (10\ldots 0). \tag{2.11}
\]

2.3. The associated tensor hierarchy algebras. In this subsection, we recall the tensor hierarchy algebras of [4] and we consider variations on the construction of \( B \).
which allow us to modify the zero and negative levels of $B$ without affecting the positive levels. We assume a $\mathbb{Z}$-grading of $B$ as in the preceding subsection.

Consider the local Lie superalgebra $u(B_1) = U_{-1} \oplus U_0 \oplus U_1$ associated to the subspace $B_1$ of $B$, as defined in Section 2.1. Thus

$$U_{-1} = \text{Hom}(B_1, \text{End } B_1), \quad U_0 = \text{End } B_1, \quad U_1 = B_1.$$  \tag{2.12}

The adjoint action in $B$ of $B_0 = g \oplus \mathbb{K}$ on $B_1$ provides an embedding of $g \oplus \mathbb{K}$ into $\text{End } B_1$. By this embedding we can restrict $U_{-1}$ to the subspace

$$V_{-1} = \text{Hom}(B_1, g \oplus \mathbb{K}),$$  \tag{2.13}

and consider the local subalgebra

$$v(B_1, V_{-1}) = V_{-1} \oplus V_0 \oplus V_1$$  \tag{2.14}

of $u(B_1)$ generated by $V_1 = B_1$ and $V_{-1}$. We then have

$$V_0 = [V_1, V_{-1}] = g \oplus \mathbb{K},$$  \tag{2.15}

and as a $g$-module, $V_{-1}$ is isomorphic to $\overline{B}_1 \otimes (g \oplus \mathbb{K})$, the tensor product of the dual of $B_1$ and the adjoint of $g \oplus \mathbb{K}$.

We can restrict $U_{-1}$ further by varying our choice of $V_{-1}$. For example, we may choose $V_{-1}$ to be isomorphic to a submodule of the tensor product $\overline{B}_1 \otimes (g \oplus \mathbb{K})$, where $\overline{B}_1$ is the dual of $B_1$ (and thus isomorphic to $B_{-1}$). In particular, we may take $V_{-1}$ to be isomorphic to $\overline{B}_1$ as a $g$-module, noting that $\overline{B}_1$ has multiplicity 2 as a $g$-module in the tensor product. Taking $V_{-1}$ to be a particular linear combination of the two $\overline{B}_1$ modules in the tensor product, we get back the BKM superalgebra $B$ of Section 2.1 that is $B = V = V(B_1, V_{-1})$, where $V = V(B_1, V_{-1})$ is the minimal Lie superalgebra with local part $v(B_1, V_{-1})$. Hence there is a $\mathbb{Z}$-grading of $V$ such that the $i$-component indeed equals $V_i$ for $i = 0, \pm 1$.

For finite-dimensional $g$ it is also possible to choose $V_{-1}$ to be the direct sum of $\overline{B}_1$ and an additional module contained in the tensor product $\overline{B}_1 \otimes g$ such that $V_2 \subseteq B_2$. The maximal additional submodule of $\overline{B}_1 \otimes g$ for which this holds is known as the embedding tensor representation in the $g$-covariant formulation of gauged supergravity with broken global symmetry $g$ [3].

If instead of choosing $V_0 = g \oplus \mathbb{K}$, we have $V_0 = g$, and we require that $V_{-1}$ is the maximal subspace of $U_{-1}$ such that $V_2 \subseteq B_2$, then $V_{-1}$ consists only of the embedding tensor representation (not the direct sum with $\overline{B}_1$) and $V$ is precisely what was called the tensor hierarchy algebra of $g$ in [4]. Here we extend the definition of tensor
hierarchy algebra to include the case where $V_0 = g \oplus K$. In the application to gauged supergravity, the difference between the two algebras depends on whether or not the so-called trombone gauging is taken into account.

In Section 3 we will define Lie superalgebras $W = W(g)$ and $S = S(g)$ associated to $g$ and show that they agree (with some minor exceptions) with the tensor hierarchy algebras of $g$ in the case when $g$ is finite-dimensional ($S$ being the original one, and $W$ the extended version), although the definition of these algebras that we will give in Section 3 is very different from the construction above (and more general since it can be applied also to infinite-dimensional $g$). Thus we can write $W_0 = g' \oplus K$ and $S_0 = g$.

To understand the condition $V_2 \subseteq \mathcal{B}_2$, we recall that the minimal Lie superalgebra $V = V(\mathcal{B}_1, V_-)$ can be obtained from the maximal one $\tilde{V} = \tilde{V}(\mathcal{B}_1, V_-)$ by factoring out the maximal ideal that intersects the local part trivially, and that $\tilde{V}_+$ is the free Lie superalgebra generated by the odd subspace $V_1 = \mathcal{B}_1$. Thus $\tilde{V}_+^2$ is the full symmetric tensor product of two $\mathcal{B}_1$ modules, and decomposes into a direct sum of $\mathcal{B}_2$ and another submodule, which we denote by $\mathcal{B}_2^\xi$ (unless $\mathcal{B}_2 = 0$, in which case we set $\mathcal{B}_2^\xi = \tilde{V}_+^2$). The condition $V_2 \subseteq \mathcal{B}_2$ now means that $\mathcal{B}_2^\xi$ must be contained in the maximal ideal of $\tilde{V}$ that intersects the local part trivially, and is thus equivalent to $[V_{-1}, \mathcal{B}_2^\xi] = 0$.

Any irreducible submodule of $U_{-1}$ that is not contained in $V_{-1}$, and thus gives a nonzero bracket with $\mathcal{B}_2^\xi$, must be dual to a submodule of $\mathcal{B}_2^\xi \otimes \mathcal{B}_1$. Thus the modules $W_{-1}$ and $S_{-1}$ can be determined by decomposing $\mathcal{B}_1 \otimes (g \oplus K)$ and $\mathcal{B}_1 \otimes g$, respectively, into irreducible submodules, and subtracting the overlap with $\mathcal{B}_2^\xi \otimes \mathcal{B}_1$. For $W_{-1}$, this is precisely the computation that determines the torsion representation in exceptional geometry, defined as the part of the affine connection that transforms with the generalised Lie derivative under a generalised diffeomorphism [9]. In this context, the $\mathcal{B}_2$ representation is the one that appears in the so called section condition.

2.4. The BKM superalgebra $\mathcal{B}(A_{n-1}) = A(n - 1, 0)$. We now explicitly consider the case $g = A_{n-1} = \mathfrak{sl}(n)$, where the BKM algebra $\mathcal{B}(g)$ is the finite-dimensional Lie superalgebra $\mathcal{B} = A(n - 1, 0) = \mathfrak{sl}(1|n)$ with the Dynkin diagram given in Figure 1 and with Cartan matrix
The Dynkin diagram of $\mathcal{B}(A_{n-1}) = A(n - 1, 0)$. 

\[ B_{IJ} = \begin{pmatrix}
0 & -1 & 0 & \cdots & 0 & 0 \\
-1 & 2 & -1 & \cdots & 0 & 0 \\
0 & -1 & 2 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 2 & -1 \\
0 & 0 & 0 & \cdots & -1 & 2
\end{pmatrix}. \quad (2.16) \]

The consistent $\mathbb{Z}$-grading of $\mathcal{B}$ is in this case a 3-grading, which means that the full Lie superalgebra coincides with its local part,

\[ A(n - 1, 0) = \mathcal{B}_{-1} \oplus \mathcal{B}_0 \oplus \mathcal{B}_1. \quad (2.17) \]

The subalgebra $\mathcal{B}_0$ is $\mathfrak{sl}(n) \oplus \mathbb{K} = \mathfrak{gl}(n)$, and the basis elements can thus be written as $\mathfrak{gl}(n)$ tensors as in Table 1.

| level | basis | $A_{n-1}$ representation |
|-------|-------|--------------------------|
| 1     | $E_a$ | $(00 \cdots 01)$         |
| 0     | $G^a_b$ | $(10 \cdots 01) \oplus (00 \cdots 00)$ |
| $-1$  | $F^a$  | $(10 \cdots 00)$         |

Table 1. The $\mathbb{Z}$-grading of $A(n - 1, 0)$.

The commutation relations are

\[ [G^a_b, G^c_d] = \delta^c_b G^a_d - \delta^a_d G^c_b, \quad [E_a, F^b] = -G^b_a + \delta^b_a G, \]

\[ [G^a_b, F^c] = \delta^c_b F^a, \quad [G^a_b, E_c] = -\delta^a_c E_b, \quad [E_a, E_b] = [F^a, F^b] = 0, \quad (2.18) \]
where \( G = \sum_{a=0}^{n-1} G^a. \) Identifying the Chevalley generators,

\[
e_0 = E_0, \quad f_0 = F^0, \quad h_0 = G^1 + G^2 + \cdots + G^{n-1}, \quad G = G^0,
\]

(2.19)

\[
e_i = G^{i-1}, \quad f_i = G^i, \quad h_i = G^{i-1} - G^i, \quad (i = 1, 2, \ldots, n - 1)
\]

(2.20)

the commutation relations (2.18) follow from the Chevalley–Serre relations (2.8)–(2.9).

2.5. The Cartan type Lie superalgebra \( W. \) Let \( \Lambda \) be the Grassmann superalgebra with generators \( \xi_0, \xi_1, \ldots, \xi_{n}, \) that is, the associative superalgebra generated by these elements modulo the relations \( \xi_a \xi_b = -\xi_b \xi_a. \) The Cartan type superalgebra \( W \) is the derivation superalgebra of \( \Lambda, \) with basis elements

\[
K^{a_1 \cdots a_p} = \xi^{a_1} \cdots \xi^{a_p} \frac{\partial}{\partial \xi^b}
\]

(2.21)

acting on a monomial \( \xi^{c_1} \cdots \xi^{c_q} \) by a contraction,

\[
K^{a_1 \cdots a_p} = \xi^{c_1} \cdots \xi^{c_q} \mapsto q \delta_b^{[c_1} \xi^{a_1} \cdots \xi^{a_p]} \xi^{c_2} \cdots \xi^{c_q]},
\]

(2.22)

where the square brackets denote antisymmetrisation with total weight 1, excluding indices enclosed between vertical bars. It is easy to see that \( W \) has a consistent \( \mathbb{Z} \)-grading where \( W_{-p+1} \) has a basis of elements \( K^{a_1 \cdots a_p} \) which are antisymmetric in all upper indices, and thus \( W_{-p+1} = 0 \) for \( p > n \) (or \( p < 0 \)) as can be seen in Table 2.

| level | basis | \( A_{n-1} \) representation | dimension |
|-------|-------|-------------------------------|-----------|
| 1     | \( K_a \) | \((00 \ldots 01)\)           | \( n \)   |
| 0     | \( K^{a}_b \) | \((10 \ldots 01) \oplus (00 \ldots 00)\) | \( n^2 \) |
| -1    | \( K^{ab}_c \) | \((010 \ldots 01) \oplus (10 \ldots 00)\) | \( \binom{n}{2} \cdot n \) |
| -2    | \( K^{abc}_d \) | \((0010 \ldots 01) \oplus (010 \ldots 00)\) | \( \binom{n}{3} \cdot n \) |
| \vdots| \vdots | \vdots                       | \vdots    |
| \( -n + 2 \) | \( K^{a_1 \cdots a_{n-2}}_b \) | \((00 \ldots 02) \oplus (00 \ldots 010)\) | \( \binom{n-1}{n-2} \cdot n = n^2 \) |
| \( -n + 1 \) | \( K^{a_1 \cdots a_{n-1}}_b \) | \((000 \ldots 01)\)           | \( n \)   |

Table 2. The \( \mathbb{Z} \)-grading of \( W. \)
The commutation relations are
\[
[K^{a_1\ldots a_p}, K^{b_1\ldots b_q}] = q \delta_c^{[b_1} K^{[a_1\ldots a_p][b_2\ldots b_q]} - p \delta_d^{a_p} K^{a_1\ldots a_{p-1}]b_1\ldots b_q}_c.
\] (2.23)

The Lie superalgebra \(W(n)\) has a subalgebra \(S(n)\) spanned by the traceless linear combinations
\[
\hat{K}^{a_1\ldots a_p}_c = K^{a_1\ldots a_p} - \sum_{d=0}^{n-1} \frac{p}{n-p+1} \delta_c^{[a_p} K^{a_1\ldots a_{p-1}]b_1\ldots b_q}_d,
\] (2.24)
satisfying
\[
\sum_{c=0}^{n-1} \hat{K}^{a_1\ldots a_{p-1}c}_c = 0.
\] (2.25)

The commutation relations in \(S(n)\) are
\[
[\hat{K}^{a_1\ldots a_p}_c, \hat{K}^{b_1\ldots b_q}_d] = q \delta_c^{[b_1} \hat{K}^{[a_1\ldots a_p][b_2\ldots b_q]} - p \delta_d^{a_p} \hat{K}^{a_1\ldots a_{p-1}]b_1\ldots b_q}_c \\
\quad - \frac{p(q-1)}{n-p+1} \delta_c^{[a_p} \hat{K}^{a_1\ldots a_{p-1}]b_1\ldots b_q}_d \\
\quad + \frac{q(p-1)}{n-q+1} \delta_d^{[b_1} \hat{K}^{[a_1\ldots a_p][b_2\ldots b_q-1]}_c,
\] (2.26)
in particular
\[
[K_c, \hat{K}^{b_1\ldots b_q}_d] = q \delta_c^{[b_1} \hat{K}^{b_2\ldots b_q]}_d - \frac{q}{n-q+1} \delta_d^{[b_1} \hat{K}^{b_2\ldots b_q]}_c,
\] (2.27)
and the \(\mathbb{Z}\)-grading inherited from \(W(n)\) is given by Table 3.

| level | basis | \(A_{n-1}\) representation | dimension |
|-------|-------|----------------------------|-----------|
| 1     | \(K_a\) | (00 \ldots 01)           | \(n\)    |
| 0     | \(\hat{K}^a_b\) | (10 \ldots 01)           | \(n^2 - 1\) |
| -1    | \(\hat{K}^{ab}_c\) | (010 \ldots 01)         | \(\binom{n}{2} \cdot n - n\) |
| -2    | \(\hat{K}^{abc}_d\) | (0010 \ldots 01)      | \(\binom{n}{3} \cdot n - \binom{n}{2}\) |
| \vdots | \vdots | \vdots                   | \vdots    |
| -\(n + 2\) | \(\hat{K}^{a_1\ldots a_{n-1}}_b\) | (00 \ldots 02)          | \(\frac{1}{2}n(n + 1)\) |

Table 3. The \(\mathbb{Z}\)-grading of \(S(n)\).
2.5.1. Towards generators and relations for $W(n)$. We note that the subalgebra of $W(n)$ generated by $K_a$, $K^a b$ and $K^a = \sum_{b=0}^{n-1} K^{ab}$ is isomorphic to $A(n-1,0)$, with an injective homomorphism $\psi : A(n-1,0) \to W(n)$ given by

$$
\psi(E_a) = K_a, \quad \psi(G^a_b) = K^a_b, \quad \psi(F^a) = K^a.
$$

This follows from comparing the commutation relations (2.18) with those for $K_a$, $K^a b$ and $K^a$ coming from (2.23). It is therefore natural to ask whether the set of Chevalley generators of $A(n-1,0)$ can be extended by generators corresponding to the traceless part of $K^{ab}$, and whether $W(n)$ then can be constructed from the extended set of generators by some relations extending the Chevalley–Serre relations. In order to investigate this we redefine the Chevalley generators (2.19)–(2.20) as elements in $W(n)$ by the injective homomorphism $\psi$ above,

$$
e_0 = K_0, \quad f_0 = K^0, \quad h_0 = K_2 + K^2 + \cdots + K^{n-1}, = K - K^0, \quad e_i = K^{i-1}, \quad f_i = K^i, \quad h_i = K^{i-1} - K^i,
$$

for $i = 1, \ldots, n - 1$, where $K = \sum_{a=0}^{n-1} K_a$. Accordingly, we define the Cartan subalgebra of $W(n)$ as the subalgebra spanned by the elements $h_0, h_1, \ldots, h_{n-1}$. Usually the Cartan subalgebra of $W(n)$ is defined as the Cartan subalgebra of the $A_{n-1}$ subalgebra, which is spanned by $h_1, \ldots, h_{n-1}$, but from our point of view, it is natural to include also $h_0$. We can thus define roots in the usual way with respect to this Cartan subalgebra.

We describe the root system of $W(n)$ in the appendix, in general, and explicitly for $n = 3$ and $n = 4$. As can be seen for $W(3)$ in Table 6, there are not only positive and negative roots, but also a root $-\alpha_0 + \alpha_2$ mixing positive and negative coefficients for the simple roots. This is a general feature of $W(n)$ root systems. Another remarkable feature in which the Cartan type superalgebras differ from the BKM superalgebras is that the simple root $-\alpha_0$ has multiplicity greater than one (as usual, the multiplicity of a root is the dimension of its root space). We have

$$
[h_a, K^{0i}_i] = -B_{ab} K^{0i}_i, \quad (i = 1, 2, \ldots, n - 1)
$$

for all the $n - 1$ linearly independent elements $K^{0i}_i$, and thus they all have the same weight as $f_0 = K^0$. In other words, the negative $-\alpha_0$ of the simple root $\alpha_0$ has multiplicity $n - 1$ in $W(n)$, meaning that the corresponding root space is $(n-1)$-dimensional (whereas $\alpha_0$ itself has multiplicity one as usual). Furthermore, we note that the adjoint action of $e_0$ maps the root space of $-\alpha_0$, spanned by all $K^{0i}_i$, injectively
to the subspace of the Cartan subalgebra spanned by \( h_0, h_2, h_3, \ldots, h_{n-1} \), (that is, the Cartan generators corresponding to simple roots orthogonal to \( \alpha_0 \)). Defining generators

\[
f_{0i} = (\text{ad } e_0)^{-1}(h_i) = K_0^{0(i-1)}(i-1) - K_0^{0i}
\]  

(2.31)
corresponding to \(-\alpha_0\) for \( i = 2, 3, \ldots, n-1 \), we thus have \([e_0, f_{0i}] = h_i\). We set \( f_{00} \equiv f_0 \) to make this relation valid also for \( i = 0 \). Henceforth, whenever \( f_{0a} \) appears we assume \( a = 0, 2, 3, \ldots, r \) (with \( r = n-1 \) in the case considered here), and whenever \( f_{a} \) appears we assume \( a \neq 0 \). We thus have the \( \mathbb{Z}_2 \)-graded set of generators

\[
\mathcal{S} = \{e_a, f_{0a}, f_{a}\}
\]  

(2.32)
where \( e_0 \) and \( f_{0a} \) are odd, and the others are even, and we seek relations generating an ideal \( C \) of the free Lie superalgebra \( F \) generated by \( \mathcal{S} \) such that \( F / C \) is isomorphic to \( W(n) \).

Of course, one way of finding relations that generate the ideal \( C \) would be to write all the commutation relations (2.23) with the basis elements expressed in terms of the generators \( e_a, f_{0a}, f_{a} \). But there would be a great deal of redundancy in such a set of relations, and, most importantly, they would be applicable only to the case of \( A(n-1, 0) \), not to the other cases we are interested in. Thus we seek relations that are more fundamental in this sense. The obvious starting point is the Chevalley–Serre relations for \( B \) that do not involve \( f_0 \), and the relations

\[
[e_0, f_{0a}] = h_a, \quad [h_a, f_{0b}] = -B_{a0}f_{0b}
\]  

(2.33)
explained above. In addition we choose the relations (3.5)–(3.7) below, which can easily be verified in the \( A(n-1, 0) \) case, as fundamental ones. We will justify this choice for a general \( g \) in the next section, and present the final result for \( g = A_{n-1} \) in Section 4.

3. The Lie superalgebras \( W(g) \) and \( S(g) \)

3.1. Definition of \( \widehat{W}(g) \). We return to the general case where a grey node (node 0) is connected to node 1 in the Dynkin diagram of a simple and simply laced Kac–Moody algebra \( g \) of rank \( r \), and consider the corresponding \( \mathbb{Z}_2 \)-graded set of generators \( \mathcal{S} = \{e_a, f_{0a}, f_{a}\} \). Motivated by the observations made for \( W(n) \) at the end of the preceding section, we define \( \widehat{W} = \widehat{W}(g) \) as the Lie superalgebra generated by the set \( \mathcal{S} \) modulo the relations

\[
[h_a, e_b] = B_{ab}e_b, \quad [h_a, f_b] = -B_{ab}f_b, \quad [e_a, f_b] = \delta_{ab}h_b, \quad (\text{ad } e_a)^{1-B_{ab}}(e_b) = (\text{ad } f_a)^{1-B_{ab}}(f_b) = 0,
\]  

(3.1)
(3.2)
for $i, j = 2, 3, \ldots, r$, where we have defined $h_a$ by (3.3) for $a \neq 1$, and $h_1 = [e_1, f_1]$ (recall that $a \neq 1$ in $f_{0a}$, and $a \neq 0$ in $f_a$).

There is some redundancy in (3.1)–(3.7): If $a, b = 2, 3, \ldots, r$, then the last relation in (3.1) follows by acting with $e_0$ on (3.7), and acting with $e_1$ on (3.5) gives the relation $[e_a, [e_a, f_{0b}]] = 0$ in (3.6) for $a = 1$. Furthermore, for $a = 0$ this relation follows from $[e_0, [e_0, e_1]] = 0$ in (3.2) since

$$4[e_0, [e_0, f_{0b}]] = 2[[e_0, e_0], f_{0b}] = [[[f_1, [e_1, [e_0, e_0]]], f_{0b}] = -2[[f_1, e_0, [e_0, e_1]], f_{0b}].$$

The different relations (3.6) for different values of $b$ are not independent either. For example, if $B_{23} = -1$, then we have

$$[f_2, [f_2, f_{03}]] = -[f_2, [f_2, [e_3, [f_3, f_{02}]]]]$$

$$= -[e_3, [f_2, [f_2, [f_3, f_{02}]]]]$$

$$= -2[e_3, [f_2, [f_3, [f_2, f_{02}]]]] + [e_2, [f_3, [f_2, [f_2, f_{02}]]]]$$

$$= -2[f_2, [h_3, [f_2, f_{02}]]] + [e_2, [f_3, [f_2, [f_2, f_{02}]]]]$$

$$= -2[f_2, [f_2, f_{02}]] + [e_2, [f_3, [f_2, [f_2, f_{02}]]]],$$

where we have used $[f_2, [f_2, f_3]] = 0$, expanded as in (3.21). Thus (3.6) could be replaced by

$$[e_i, [e_i, f_{02}]] = [f_a, [f_a, f_{02}]] = 0,$$

$$[e_i, [e_i, f_{00}]] = [f_a, [f_a, f_{00}]] = 0$$

for $i = 2, 3, \ldots, r$. Also in (3.3)–(3.5) it is sufficient to consider $f_{02}$ and $f_{00}$.

Note the absence of relations setting $[e_i, f_{0a}]$ and $[f_i, f_{0a}]$ to zero for any $i = 2, 3, \ldots, r$.

However, if $B_{ai} = 0$, then it follows from (3.9) and (3.7) that $[e_i, f_{0a}] = [f_i, f_{0a}] = 0$.

We will proceed under the assumption that the algebra $\widehat{W}(g)$ is non-trivial, i.e., that the relations (3.1)–(3.7) do not generate the whole free Lie superalgebra $F$ generated
by \( S \). We have no proof that this assumption is true for arbitrary \( g \), only for finite-dimensional \( g \) (as we will see later in this section), and for \( g = E_r \) with \( r \geq 11 \) and the “outermost” node as node 1 (as we will show in Section 5.1). In the \( A \) case, we have already seen that the non-trivial Lie superalgebra \( \mathcal{W}(n) \) satisfies the relations.

3.2. The \( \mathbb{Z} \)-grading on \( \mathcal{W}(g) \) and definition of \( \mathcal{W}(g) \). The free Lie superalgebra \( F \) generated by \( S \) is spanned by all elements

\[
[x_1, [x_2, \ldots, [x_{p-1}, x_p], \ldots]] \quad (p \geq 1),
\]

where each \( x_i \) \((i = 1, 2, \ldots, p)\) is equal to \( e_a, f_a \) or \( f_0 a \). It follows that \( F \) has a consistent \( \mathbb{Z} \)-grading where \( F_k \) is spanned by all elements of the form \( (3.11) \) such that, among the generators \( x_1, \ldots, x_p \), the number of \( e_0 \) minus the number of \( f_0 a \) is equal to \( k \). Thus, if \( F_{(i,j)} \) is the subspace of \( F \) spanned by all elements of the form \( (3.11) \) where \( e_0 \) appears \( i \) times and \( f_0 a \) appears \( j \) times (in total, for possibly different \( a \)), then

\[
F_k = \bigoplus F_{(i,j)},
\]

where the sum ranges over all pairs \((i, j)\) of non-negative integers such that \( i - j = k \).

Let \( I \) be the ideal generated by the relations \( (3.1) - (3.7) \). Since these relations are homogeneous with respect to the \( \mathbb{Z} \)-grading \( (3.12) \) on \( F \), this \( \mathbb{Z} \)-grading is preserved in the quotient \( \mathcal{W} = F/I \), and we have

\[
\mathcal{W}_k = \sum \mathcal{W}_{(i,j)},
\]

where, as before, the sum ranges over all pairs \((i, j)\) of non-negative integers such that \( i - j = k \).

Note that the sum in \( (3.13) \) is not (necessarily) direct, since the Lie superalgebra \( \mathcal{W} \) is not free. In fact, we will show that \( \mathcal{W}_{(i,j)} = \mathcal{W}_{(i',j')} \) for all pairs \((i, j)\) and \((i', j')\) such that \( i - j = i' - j' \), or equivalently

\[
[\mathcal{W}_{(1,0)}, \mathcal{W}_{(0,1)}] = \mathcal{W}_{(0,0)}.
\]

In particular \( \mathcal{W}_k = \mathcal{W}_{(k,0)} \) and \( \mathcal{W}_{-k} = \mathcal{W}_{(0,k)} \) for all \( k \geq 0 \).

Since \( (3.13) \) is a \( \mathbb{Z} \)-grading, \( \mathcal{W}_{-1} \oplus \mathcal{W}_0 \oplus \mathcal{W}_1 \) is a local Lie superalgebra. We define \( \mathcal{W}(g) \) as the minimal Lie superalgebra with this local part. Thus we can write, as usual, \( \mathcal{W}_i = W_i \) for \( i = 0, \pm 1 \). Note however that \( \mathcal{W}(g) \) is not the maximal Lie superalgebra with local part \( \mathcal{W}_{-1} \oplus \mathcal{W}_0 \oplus \mathcal{W}_1 \), since we, for convenience, have included the relation \([e_0, [e_0, e_1]] = 0\) in \( (3.2) \), which occurs at level 2.
We define \( \tilde{S} = \tilde{S}(g) \) as the subalgebra of \( \tilde{W}(g) \) generated by the subset \( \mathcal{S} \setminus \{ f_{00} \} \), and \( S = S(g) \) as the minimal Lie superalgebra with local part \( \tilde{S}_{-1} \oplus \tilde{S}_0 \oplus \tilde{S}_1 \) with respect to the \( \mathbb{Z} \)-grading inherited from \( \tilde{W}(g) \). From our results for \( W(g) \) below, it is straightforward to deduce the corresponding results for \( S(g) \). We will not do that explicitly, but focus on \( W(g) \). We will come back to \( S(g) \) later when we relate the results to the construction of the tensor hierarchy algebra in [4].

3.3. The subalgebra \( \tilde{W}' \). Obviously, the generators associated to node 1 play a distinguished role in the relations (3.1)–(3.7). Therefore, it is convenient to first study the subalgebra of \( \tilde{W} \) generated by the subset \( \mathcal{S}' = \mathcal{S} \setminus \{ e_1, f_1 \} \) of \( \mathcal{S} \). We denote this subalgebra of \( \tilde{W} \) by \( \tilde{W}' \) and consider the \( \mathbb{Z} \)-grading inherited from \( \tilde{W} \), which in turn originates from the \( (\mathbb{N} \times \mathbb{N}) \)-grading of the free Lie superalgebra \( F \). Thus \( \tilde{W}'(0,0) \) is the subalgebra of \( \tilde{W} \) generated by \( e_i, f_i, h_0 \) for \( i = 2, 3, \ldots, r \), and since these generators commute with \( e_0 \), the subspace \( \tilde{W}'(1,0) \) is one-dimensional (spanned by \( e_0 \)). Since furthermore \([e_0, f_{0a}] = f_a \in \tilde{W}'(0,0)\), it follows that the condition for \( \tilde{W}' \) corresponding to (3.14) holds,

\[
[\tilde{W}'(1,0), \tilde{W}'(0,1)] = \tilde{W}'(0,0).
\]

Thus \( \tilde{W}'_k = \tilde{W}'(k,0) \) and \( \tilde{W}'_{-k} = \tilde{W}'(0,k) \) for all \( k \geq 0 \), in particular for \( k = 0, 1 \).

Having described \( \tilde{W}'_1 \) as the one-dimensional subspace of \( \tilde{W}' \) spanned by \( e_0 \), we now proceed to \( \tilde{W}'_0 \), and in the next subsection to \( \tilde{W}'_{-1} \).

**Proposition 3.1.** The subalgebra \( \tilde{W}'_0 = \tilde{W}'(0,0) \) of \( \tilde{W} \) is isomorphic to \( g' \oplus \mathbb{K} \).

**Proof.** It is clear that \( g' \oplus \mathbb{K} \) can be constructed as the Lie superalgebra generated by \( \mathcal{S}' = \{ e_0, f_{0a}, e_i, f_i \} \) for \( i = 2, 3, \ldots, r \) modulo the relations among (3.1)–(3.7) that only involve these generators. Thus there is a surjective homomorphism from \( g' \) to \( \tilde{W}'_0 \). The kernel of this homomorphism must be an ideal of \( g' \oplus \mathbb{K} \). Since we assume that \( g' \) is simple, the ideal is either \( g' \) or the one-dimensional center or zero. It is easy to see that in the first two cases the whole of \( \tilde{W} \) would be zero, contradicting the assumption above. We conclude that the homomorphism is injective, and thus an isomorphism.

Consider the map \( \mathcal{S}' \to \mathcal{S}' \) given by

\[
e_i \mapsto \pm f_i, \quad f_i \mapsto \pm e_i, \quad e_0 \mapsto e_0, \quad f_{0a} \mapsto -f_{0a}.
\]

Since those of the relations (3.1)–(3.7) that do not involve \( e_1 \) or \( f_1 \) are invariant under this map, it can be extended to an automorphism of the Lie superalgebra \( \tilde{W}' \).
We will now state and prove some additional relations that hold in $\tilde{W}(\mathfrak{g})'$, as consequences of the defining ones (3.1)-(3.7). Given the automorphism (3.16), these relations always come in pairs, and we will only prove half of them explicitly. The other half then follow by applying the automorphism.

**Proposition 3.2.** For $i, j = 2, 3, \ldots, r$ and $i \neq j$, the following relations hold in $\tilde{W}'$: \[ (\text{ad } e_i)^{1-B_{ij}}(\text{ad } e_j)(f_{0a}) = (\text{ad } f_i)^{1-B_{ij}}(\text{ad } f_j)(f_{0a}) = 0, \] \[ B_{ia}[e_i, f_{0a}] = B_{ib}[e_i, f_{0a}], \quad B_{ia}[f_i, f_{0a}] = B_{ib}[f_i, f_{0a}]. \] \[ (3.17) \]

**Proof.** If $B_{ij} = 0$, we have \[ 0 = \text{ad } [e_i, e_j] = [\text{ad } e_i, \text{ad } e_j] = \text{ad } e_i \text{ ad } e_j - \text{ad } e_j \text{ ad } e_i, \] and we get

\[
2[e_i, [e_j, f_{0a}]] = 2[e_i, [e_j, f_{0a}]] + [f_i, [e_j, [e_i, [e_i, f_{0a}]]]] \\
= [h_i, [e_i, [e_j, f_{0a}]]] + [f_i, [e_i, [e_i, f_{0a}]]]] \\
= [e_i, [f_i, [e_i, [e_j, f_{0a}]]]] \\
= [e_i, [h_i, [e_j, f_{0a}]]] + [e_i, [e_i, [f_i, [e_j, f_{0a}]]]] = 0. \] \[ (3.20) \]

Likewise, if $B_{ij} = -1$, we have \[ 0 = \text{ad } [e_i, [e_i, e_j]] = [\text{ad } e_i, \text{ad } [e_i, e_j]] = [\text{ad } e_i, [\text{ad } e_i, \text{ad } e_j]] \]
\[ = \text{ad } e_i \text{ ad } e_i \text{ ad } e_j - 2 \text{ ad } e_i \text{ ad } e_j \text{ ad } e_i + \text{ad } e_j \text{ ad } e_i \text{ ad } e_i, \] and we get

\[
3[e_i, [e_i, [e_j, f_{0a}]]] = 3[e_i, [e_i, [e_j, f_{0a}]]] + 4[f_i, [e_i, [e_j, [e_i, f_{0a}]]]] \\
- 2[f_i, [e_j, [e_i, [e_i, f_{0a}]]]] - [f_i, [e_j, [e_i, [e_i, f_{0a}]]]] \\
= [h_i, [e_i, [e_i, [e_j, f_{0a}]]]] + 2[f_i, [e_i, [e_j, [e_i, f_{0a}]]]] \\
- [f_i, [e_i, [e_i, [e_j, f_{0a}]]]] \\
= [e_i, [f_i, [e_i, [e_i, [e_j, f_{0a}]]]]] \\
= [h_i, [e_i, [e_i, [e_j, f_{0a}]]]] + [f_i, [e_i, [e_i, [e_j, f_{0a}]]]] \\
= [e_i, [f_i, [e_i, [e_j, f_{0a}]]]] \\
= - [e_i, [h_i, [e_i, [e_j, f_{0a}]]]] + [e_i, [f_i, [e_i, [e_j, f_{0a}]]]] \\
= - [e_i, [h_i, [e_i, [e_j, f_{0a}]]]] - [e_i, [h_i, [e_j, f_{0a}]]]] \\
+ [e_i, [e_i, [f_i, [e_j, f_{0a}]]]] = 0. \] \[ (3.22) \]
Finally we have
\[ B_{ia}[f_i, f_0] = [f_i, [e_i, [f_i, f_0]]] \]
\[ = -[h_i, [f_i, f_0]] - [e_i, [f_i, f_0]] = 2[f_i, f_0], \tag{3.23} \]
where \( a \) can be replaced by \( b \). Thus (3.18) follows directly if \( B_{ia} \) or \( B_{ib} \) is zero, and otherwise by
\[ [f_i, f_0] = \frac{2}{B_{ia}} [f_i, f_0] = \frac{2}{B_{ib}} [f_i, f_0]. \tag{3.24} \]

The meaning of this proposition is that we lose the \( r \)-fold multiplicity of the roots at level \( -1 \) when we apply \( \text{ad} \) \( e_i \) or \( \text{ad} \) \( f_i \) to \( f_0 \), for \( i = 2, 3, \ldots, r \). As we have seen, \( -\alpha_0 \) has multiplicity \( r \), whereas the proposition says that the roots \( -\alpha_0 + \alpha_i \) have multiplicity one, since for different \( a \), all root vectors \( [e_i, f_0] \) are proportional to each other (and correspondingly for \( -\alpha_0 - \alpha_i \) and \( [f_i, f_0] \)).

3.4. Determining \( \widehat{W}_{-1} \) when \( g' \) is finite-dimensional. From now on, we assume that \( g' \) is finite-dimensional (and still simply laced). Then we know that all roots \( \alpha \) of \( g' \) have length squared \( (\alpha, \alpha) = 2 \) with our normalization, and that the sum \( \alpha + \beta \) of two positive roots \( \alpha, \beta \) is a root if and only if \( (\alpha, \beta) = -1 \). We will use this repeatedly in the following proofs. At the end of this section we will comment on the case when \( g' \) is infinite-dimensional.

For all positive roots \( \alpha \) of \( g' \), and \( i = 2, 3, \ldots, r \), let \( \{e_\alpha, f_\alpha, h_i\} \) be a Chevalley basis of \( g \), so that
\[ [e_\alpha, f_\alpha] = h_\alpha, \quad [h_\alpha, e_\beta] = (\alpha, \beta)e_\beta, \quad [h_\alpha, f_\beta] = -(\alpha, \beta)f_\beta, \tag{3.25} \]
where \( h_\alpha = \sum_{i=2}^{r} a_i h_i \) if \( \alpha = \sum_{i=2}^{r} a_i \alpha_i \).

We introduce the following notation: for \( \alpha = \sum_{i=2}^{r} a_i \alpha_i \), set \( f_{0\alpha} = \sum_{i=2}^{r} a_i f_0 \). Then \( f_{0\alpha} = f_{0i} \), and the relation (3.7) can be written
\[ [e_i, [f_j, f_{0\alpha}]] = \delta_{ij}(\alpha, \alpha_j)f_{0j}. \tag{3.26} \]

Lemma 3.3. For any positive root \( \alpha \) of \( g' \) and for \( i = 2, 3, \ldots, r \), we have
\[ (\alpha_i, \alpha) \leq 0 \quad \Rightarrow \quad [f_i, [e_\alpha, f_{0\alpha}]] = 0, \tag{3.27} \]
\[ (\alpha_i, \alpha) \geq 0 \quad \Rightarrow \quad [e_i, [e_\alpha, f_{0\alpha}]] = 0, \tag{3.28} \]
\[ [e_\alpha, [e_i, f_{0\alpha}]] = B_{ia}[e_i, [e_\alpha, f_{0\alpha}]]. \tag{3.29} \]
**Proof.** First we note that (3.28) is a consequence of (3.29), since if \((\alpha_i, \alpha) \geq 0\), then \([\epsilon_i, \epsilon_\alpha] = 0\) and
\[
[\epsilon_\alpha, [\epsilon_i, f_{0a}]] = [\epsilon_i, [\epsilon_\alpha, f_{0i}]] .
\] (3.30)
Together with (3.29) this gives (3.28). Thus it suffices to prove (3.27) and (3.29), which can be done by induction over the height of \(\alpha\), denoted \(ht\alpha\). If \(ht\alpha = 1\), then \(\alpha\) is a simple root, say \(\alpha = \alpha_k\). Then (3.27) and (3.29) follow from (3.7) and
\[
[e_k, [\epsilon_i, f_{0a}]] = \frac{B_{ia}}{2}[e_k, [\epsilon_i, f_{0i}]] = \frac{1}{2}[e_k, [\epsilon_i, [f_i, f_{0a}]]])
\]
\[
= -\frac{1}{2}[\epsilon_i, [e_k, [f_i, f_{0a}]]] + [e_i, [\epsilon_k, [f_i, f_{0a}]]])
\]
\[
= [e_i, [e_k, [f_i, f_{0a}]]]) = B_{ia}[e_i, [\epsilon_k, f_{0i}]] ,
\] (3.31)
respectively.

Suppose now that the lemma holds for roots \(\alpha'\) such that \(ht\alpha' \leq p\) for some \(p \geq 1\). Thus the induction hypothesis consists of the three parts
\[
(\alpha_i, \alpha') \leq 0 \Rightarrow [f_i, [\epsilon_{\alpha'}, f_{0a}]] = 0 ,
\] (3.32)
\[
(\alpha_i, \alpha') \geq 0 \Rightarrow [\epsilon_i, [\epsilon_{\alpha'}, f_{0a}]] = 0 ,
\] (3.33)
\[
[e_{\alpha'}, [\epsilon_i, f_{0a}]] = B_{ia}[\epsilon_i, [\epsilon_{\alpha'}, f_{0i}]] ,
\] (3.34)
corresponding to (3.27)-(3.29).

Any root \(\alpha\) with \(ht\alpha = p + 1\) can be written \(\alpha = \alpha' + \alpha_j\), where \((\alpha, \alpha_j) = -1\), the condition \((\alpha_i, \alpha) \leq 0\) implies \(\alpha', \alpha_j \neq \alpha_i\), and with \(\epsilon_\alpha = [\epsilon_j, \epsilon_{\alpha'}]\) we get
\[
[f_i, [\epsilon_\alpha, f_{0a}]] = [f_i, [\epsilon_j, [\epsilon_{\alpha'}, f_{0a}]]] - [f_i, [\epsilon_{\alpha'}, [\epsilon_j, f_{0a}]]])
\]
\[
= [\epsilon_j, [f_i, [\epsilon_{\alpha'}, f_{0a}]]] - [f_i, [\epsilon_{\alpha'}, [\epsilon_j, f_{0a}]]]) .
\] (3.35)

If now \((\alpha_i, \alpha') \leq 0\), then the first term is zero by the induction hypothesis, and the second term is equal to \([\epsilon_{\alpha'}, [f_i, f_{0a}]]\), which is zero by (3.7). If \((\alpha_i, \alpha') = 1\), then we can write \(\alpha' = \alpha_i + \alpha''\) and \(\epsilon_{\alpha'} = [\epsilon_i, \epsilon_{\alpha''}]\), where \((\alpha_i, \alpha'') = -1\). Furthermore, we must have \((\alpha_i, \alpha_j) = -1\), since \((\alpha_i, \alpha_j) = 0\) would imply
\[
(\alpha_i, \alpha) = (\alpha_i, \alpha_j + \alpha_i + \alpha'') = 1 ,
\] (3.36)
contradicting \((\alpha_i, \alpha) \leq 0\). This means that
\[
(\alpha_j, \alpha'') = (\alpha_j, \alpha' - \alpha_i) = (\alpha_j, \alpha') - (\alpha_j, \alpha_i) = 0 ,
\] (3.37)
and we get

\[
[f_i, [e_{\alpha}, f_{0a}]] = [f_i, [e_j, [e_i, [e_{\alpha'}, f_{0a}]]]] - [f_i, [e_j, [e_{\alpha'}, [e_i, f_{0a}]]]] \\
- [f_i, [e_{\alpha'}, [e_i, f_{0a}]]] + [f_i, [e_{\alpha'}, [e_j, f_{0a}]]] \\
= -[e_j, [h_i, [e_{\alpha'}, f_{0a}]]] + [e_j, [e_i, [e_{\alpha'}, f_{0a}]]] \\
- [e_j, [e_{\alpha'}, [f_i, [e_i, f_{0a}]]]] + [h_i, [e_{\alpha'}, [e_j, f_{0a}]]] \\
- [e_{\alpha'}, [h_i, [e_j, f_{0a}]]].
\]

(3.38)

The second term is zero by (3.32), and the others are proportional to \([e_j, [e_{\alpha'}, f_{0a}]]\), which is zero by (3.33). Thus we have shown that (3.31) holds if \(\alpha = p + 1\). Now (3.33) for \(\alpha = p + 1\) and \((\alpha_i, \alpha) \leq 0\) can be shown in the same way as (3.31) with \(e_k\) replaced by \(e_{\alpha}\). If \((\alpha, \alpha_i) = 1\), then we can write \(e_{\alpha} = [e_i, e_{\alpha'}]\), where \((\alpha', \alpha_i) = -1\), and we get

\[
[e_i, [e_{\alpha}, f_{0a}]] = [e_{\alpha}, [e_i, f_{0a}]] = [e_i, [e_{\alpha'}, [e_i, f_{0a}]]] = \frac{1}{2} [e_i, [e_i, [e_{\alpha'}, f_{0a}]]],
\]

(3.39)

which can be shown to vanish in the same way as \([e_i, [e_i, [e_j, f_{0a}]]]\) in (3.32), using (3.32).

We will use (3.31) in the proof of Proposition 3.5 below, and also the corresponding statement

\[(\alpha_i, \alpha) \leq 0 \quad \Rightarrow \quad [e_i, [f_{\alpha}, f_{0a}]] = 0,\]

(3.40)

which follows by applying the automorphism given by (3.16).

**Theorem 3.4.** The \(\mathfrak{g}'\)-module \(W'_{-1}\) generated by the elements \(f_{0a}\) is isomorphic to \(\mathfrak{g}'\) itself (i.e., the adjoint module).

**Proof.** Consider the linear map \(\varphi : \mathfrak{g}' \to W'_{-1}\) given by

\[
\varphi : \quad e_{\alpha} \mapsto -\frac{1}{\mathrm{ht} \alpha} [e_{\alpha}, f_{0\alpha}], \quad f_{\alpha} \mapsto \frac{1}{\mathrm{ht} \alpha} [f_{\alpha}, f_{0\alpha}], \quad h_i \mapsto f_{0i},
\]

(3.41)
where $\rho$ is the Weyl vector of $g'$, satisfying $(\rho, \alpha_i) = 1$ for any $i = 2, 3, \ldots, r$. Then $\varphi$ is injective, with inverse $\text{ad} \ e_0$. Furthermore,

$$
\varphi([e_i, e_\alpha]) = -\frac{1}{\text{ht} \alpha + 1} \left( [e_i, [e_\alpha, f_0]] - [e_\alpha, [e_i, f_0]] \right)
$$

$$
= -\frac{1}{\text{ht} \alpha + 1} \left( [e_i, [e_\alpha, f_0]] - [e_i, [e_\alpha, f_0]] \right)
$$

$$
= -\frac{1}{\text{ht} \alpha + 1} [e_i, [e_\alpha, f_0]]
$$

$$
= -\frac{1}{\text{ht} \alpha + 1} \frac{(\alpha, \rho - \alpha_i)}{(\alpha, \rho)} [e_i, [e_\alpha, f_0]], \quad (3.42)
$$

which is equal to

$$
-\frac{1}{\text{ht} \alpha + 1} \left( 1 + \frac{1}{\text{ht} \alpha} \right) [e_i, [e_\alpha, f_0]] = -\frac{1}{\text{ht} \alpha} [e_i, [e_\alpha, f_0]] = [e_i, \varphi(e_\alpha)] \quad (3.43)
$$

if $(\alpha, \alpha_i) = -1$. If $(\alpha, \alpha_i) \geq 0$, then $(3.43)$ vanishes by $(3.28)$, and is thus equal to $\varphi([e_i, e_\alpha]) = 0$. Similarly, we get $\varphi([f_i, e_\alpha]) = [f_i, \varphi(e_\alpha)]$ (both if $\alpha$ is a simple root and otherwise), $\varphi([e_i, h_j]) = [e_i, \varphi(h_j)]$ and $\varphi([f_i, h_j]) = [f_i, \varphi(h_j)]$. Thus $\varphi$ is an isomorphism of $g'$-modules, and by repeated use of the homomorphism property $\varphi([x, y]) = [x, \varphi(y)]$, where $x$ is equal to $e_i$ or $f_i$, it follows that $\varphi$ is surjective. We conclude that $\varphi$ is an isomorphism.

**Proposition 3.5.** For any positive root $\alpha$ of $g'$, we have

$$
[e_\alpha, [f_\alpha, f_0]] = (\alpha, \alpha) f_0. \quad (3.44)
$$

**Proof.** As in Lemma 3.3, this can be proven by induction on the height of $\alpha$. If $\alpha$ is a simple root, the proposition follows from (3.7). Suppose now that the proposition holds for roots $\alpha'$ such that $\text{ht} \alpha' \leq p$ for some $p \geq 1$. Any root $\alpha$ with $\text{ht} \alpha = p + 1$ can then be written $\alpha = \alpha' + \alpha_i$, where $(\alpha', \alpha_i) = -1$. We set $e_\alpha = [e_i, e_{\alpha'}]$ and $f_\alpha = [f_{\alpha'}, f_i]$. 23
Then
\[
[e_{\alpha}, [f_\alpha, f_{0\beta}]] = [e_i, [e_{\alpha'}, [f_{i}, [f_i, f_{0\beta}]]]] - [e_{\alpha'}, [e_i, [f_\alpha', [f_i, f_{0\beta}]]]]
- [e_i, [e_{\alpha'}, [f_i, [f_\alpha', f_{0\beta}]]]] + [e_{\alpha'}, [e_i, [f_\alpha', [f_i, f_{0\beta}]]]]
\]
\[
= [e_i, [h_\alpha, [f_i, f_{0\beta}]]] + [e_i, [f_\alpha', [e_{\alpha'}, [f_i, f_{0\beta}]]]]
- [e_{\alpha'}, [f_\alpha', [e_{\alpha'}, [f_i, f_{0\beta}]]]] - [e_i, [f_i, [e_{\alpha'}, [f_\alpha', f_{0\beta}]]]]
\]
\[
+ [e_{\alpha'}, [h_i, [f_\alpha', f_{0\beta}]]] + [e_{\alpha'}, [f_i, [e_{\alpha'}, f_{0\beta}]]]
\]
\[
= - (\alpha', \alpha_i)[e_i, [f_i, f_{0\beta}]] - (\alpha_i, \beta)[e_{\alpha'}, [f_\alpha', f_{0\beta}]]
- (\alpha', \beta)[e_i, [f_i, f_{0\alpha'}]] - (\alpha_i, \alpha')[e_{\alpha'}, [f_\alpha', f_{0\beta}]]
\]
\[
= (\alpha, \beta)f_{0\beta} + (\alpha_i, \gamma)f_{0\alpha'}
+ (\alpha', \beta)f_{0\alpha} + (\alpha', \beta)f_{0\alpha'}
\]
\[
= (\alpha, \beta)(f_{0\beta} + f_{0\alpha'}) = (\alpha, \beta)f_{0\alpha} .
\] (3.45)

As a consequence of Proposition 3.5 we can replace not only \( e_k \) by \( e_\alpha \) in (3.31), but also \( e_i \) by \( e_\beta \). This gives the following proposition.

**Proposition 3.6.** For any positive roots \( \alpha, \beta, \gamma \) of \( g' \) we have
\[
(\alpha, \beta) \geq 0 \quad \Rightarrow \quad [e_\beta, [e_\alpha, f_{0\gamma}]] = 0 ,
\] (3.46)
and
\[
[e_\alpha, [e_\beta, f_{0\gamma}]] = (\beta, \gamma)[e_\beta, [e_\alpha, f_{0\gamma}]].
\] (3.47)

3.5. **Determining \( \tilde{W}_{(0, 1)} \) when \( g \) is finite-dimensional.** We now step out from \( g' \) to \( g \), and from \( \tilde{W}' \) to \( \tilde{W} \). Proposition 3.6 can be generalised in the following way.

**Proposition 3.7.** For any positive roots \( \alpha \) of \( g \) and \( \beta, \gamma \) of \( g' \) we have
\[
(\alpha, \beta) \geq 0 \quad \Rightarrow \quad [e_\beta, [e_\alpha, f_{0\gamma}]] = 0 ,
\] (3.48)
and
\[
[e_\alpha, [e_\beta, f_{0\gamma}]] = (\beta, \gamma)[e_\beta, [e_\alpha, f_{0\gamma}]].
\] (3.49)

In particular we can set \( e_\alpha = e_1 \), which gives \([e_1, [e_\beta, f_{0\gamma}]] = 0 \), since \([e_1, f_{0\gamma}] = 0 \). Thus the annihilation of \( f_{0\beta} = \varphi(h_\beta) \) upon the adjoint action of \( e_1 \) (which can be thought of as an arrow in the weight space of \( \tilde{W} \) pointing out of the hyperplane which is the weight space of \( g' \)) can be transported along any root of \( g' \), as shown for \( g = A_2 \) in Figure 2.
This can be generalised from $\alpha_1$ to all roots $\alpha_1 + \beta$, where $\beta$ is a root of $\mathfrak{g}'$.

**Proposition 3.8.** Let $\alpha$ be a positive root of $\mathfrak{g}$ such that $\langle \alpha_0, \alpha \rangle = -1$. Then $[e_\alpha, f_{0\beta}] = 0$.

**Proof.** We note that the hypothesis means that $\alpha$ appears at level 1 in a level decomposition of $\mathfrak{g}$ with respect to $\alpha_1$.

Any such root vector $e_\alpha$ can be written as $[e_{\beta_1}, [e_{\beta_2}, \ldots, [e_{\beta_p}, e_1]]]$ where $\beta_1, \ldots, \beta_p$ are roots of $\mathfrak{g}'$. Then the proposition can be proven by induction, using Proposition 3.7.

We would now like to show that the representation of $\mathfrak{g}$ on $W_{-1}$ is the direct sum of a module dual to $W_1$ and the irreducible representation of $\mathfrak{g}$ with highest weight $\Lambda_1 + \theta$, where $\theta$ is the highest root of $\mathfrak{g}'$. The only further piece of information used in the proof is the invariance of the relations (3.1)–(3.7) under the Weyl group of $\mathfrak{g}$, which we now proceed to prove.

**Lemma 3.9.** The relations (3.1)–(3.7) are preserved by the Weyl group of $\mathfrak{g}$, with the fundamental Weyl reflections $w_i$ ($i = 1, 2, \ldots, r$) mapping the generators $e_\alpha, f_\alpha, f_{0\alpha}, h_\alpha$
to their primed counterparts

\[
\begin{align*}
e'_a &= \begin{cases} -f_a & \text{if } i = a \\ e_i, e_a & \text{if } B_i a = -1 \\ e_a & \text{if } B_i a = 0 \end{cases}, \\
f'_a &= \begin{cases} -e_a & \text{if } i = a \\ -[f_i, f_a] & \text{if } B_i a = -1 \\ f_a & \text{if } B_i a = 0 \end{cases}, \\
f'_{0a} &= \begin{cases} f_{0a} - B_{ai} f_{0i} & \text{if } i \neq 1 \\ [f_1, f_{0a}] & \text{if } i = 1 \end{cases}, \\
h'_a &= h_a - B_{ai} h_i.
\end{align*}
\] (3.50)

**Proof.** The invariance under the Weyl group of $g'$ is quite obvious, and may easily be checked. The only additional generator is the Weyl reflection $w_1$ in the hyperplane orthogonal to $\alpha_1$. The proof proceeds by explicit evaluation of the identities for the transformed generators. The identities not containing $f_{00}$ are the same as in the BKM superalgebra $\mathcal{B}$, and are easily checked. The relation (3.3) transforms into

\[
[e'_0, f'_0] = -[[e_1, e_0], [f_1, f_0]] \\
= [e_0, [e_1, [f_1, f_0]]] - [e_1, [e_0, [f_1, f_0]]] \\
= [e_0, [h_1, f_0]] - [e_1, [f_1, h_i]] \\
= h_i - B_{1i} h_1 = h'_i,
\] (3.51)

and (3.4) into

\[
[h'_a, f'_0] = -([h_a - B_{ai} h_i], [f_1, f_0]) \\
= -(-B_{ai} - B_{a0} - B_{ai}(-2 + 1))[f_1, f_0] \\
= -B_{a0} f'_{0i}.
\] (3.52)

For (3.5), we obtain $[e'_1, f'_0] = [f_1, [f_1, f_0]] = 0$. We will not exhibit all the cases of (3.6) and (3.7), only give one example of each. The other cases are similar. One part of the transformed relation (3.6) is

\[
[e'_2, [e'_2, f'_0]] = -[[e_1, e_2], [[e_1, e_2], [f_1, f_0]]] \\
= -[e_1, [e_2, [e_1, [e_2, [f_1, f_0]]]]] + [e_1, [e_2, [e_1, [f_1, f_0]]]] \\
+ [e_2, [e_1, [e_2, [f_1, f_0]]]] - [e_2, [e_1, [e_2, [f_1, f_0]]]] \\
= -[e_1, [e_2, [h_1, [e_2, f_0]]]] + [e_1, [e_2, [e_2, [h_1, f_0]]]] \\
+ [e_2, [e_1, [h_1, [e_2, f_0]]]] - [e_2, [e_1, [e_2, [h_1, f_0]]]] \\
= [e_1, [e_2, [e_2, f_0]]] - [e_2, [e_1, [e_2, f_0]]] = 0,
\] (3.53)
where we have used \([e_1, [e_2, f_{0a}]] = 0\), see (3.39). An example of (3.7) is

\[
[e_2', [f_2', f_{0a}']] = [e_1, e_2], \left[[f_1, f_2], [f_1, f_{0a}]\right]]
\]

\[
= [e_1, [e_2, [f_1, [f_2, [f_1, f_{0a}]]]]] - [e_2, [e_1, [f_1, [f_2, [f_1, f_{0a}]]]]]
\]

\[
= [e_1, [f_1, [h_2, [f_1, f_{0a}]]]] + [e_1, [f_1, [f_2, [f_1, f_{0a}]]]]
\]

\[- [e_2, [h_1, [f_2, [f_1, f_{0a}]]]] - [e_2, [f_1, [f_2, [h_1, f_{0a}]]]]
\]

\[
= - [f_1, [e_2, [f_2, f_{0a}]]] = - B_{2a} [f_1, f_{02}] = B_{2a} f_{02}'.
\]

(3.54)

Part of the verification of Weyl invariance relies on the identity \([f_1, [e_2, f_{0a}]] = 0\), which may be derived from \([f_1, [f_1, [e_2, f_{0a}]]]] = 0\). □

Thus the map (3.50) gives an automorphism of \(\tilde{W}\). The invariance under the Weyl group could be used to prove Proposition 3.5, Proposition 3.6 and Proposition 3.7 directly from the corresponding relations for the simple roots since (in this case) all the positive roots are in one single orbit under the Weyl group \(\tilde{W}\). (On the other hand, we used a special case of Proposition 3.7 in the proof of the Weyl group invariance.) The Weyl group invariance also gives additional parts of Proposition 3.6 and Proposition 3.7 corresponding to (and generalising) the part (3.27) of Lemma 3.3.

**Theorem 3.10.** The \(\mathfrak{g}\)-module \(\tilde{W}_{0,1}\) is the direct sum of a module dual to \(W_1\) and the irreducible representation of \(\mathfrak{g}\) with highest weight \(\lambda = \Lambda_1 + \theta\), where \(\theta\) is the highest root of \(\mathfrak{g}'\).

**Proof.** The module \(R(\Lambda_1)\) dual to \(W_1\) is generated from its highest weight state \(f_{00}\) as in the BKM superalgebra \(\mathcal{B}\).

To prove that \(R(\Lambda_1 + \theta)\) is a submodule of \(\tilde{W}_{0,1}\), consider the element \(F_\lambda = [e_\theta, f_{0j}]\), where \(j = 2, 3, \ldots, r\) is such that \((\theta, \alpha_j) \neq 0\) (we can always find such a simple root \(\alpha_j\)). Then \(F_\lambda = [e_\theta, f_{0j}]\) is nonzero since \([f_\theta, [e_\theta, f_{0j}]] = (\theta, \alpha_j) f_{0\theta} \neq 0\). It is furthermore a highest weight vector in the adjoint representation of \(\mathfrak{g}'\), and carries \(\mathfrak{g}\)-weight \(\lambda = \Lambda_1 + \theta\). Therefore \(F_\lambda\) satisfies \((ad f_i)^{\lambda_i + 1} F_\lambda = 0\) for \(i = 2, \ldots, r\) [17]. It is also annihilated by \(e_1\), i.e., \([e_1, F_\lambda] = 0\). Consider the image of this relation under a Weyl reflection \(w_1\) in the hyperplane orthogonal to \(\alpha_1\). We have \(\lambda_1 = (\lambda, \alpha_1) = 1 - c_2\), where \(c_2\) is the Coxeter label of root 2 in \(\mathfrak{g}'\). The weight \(\lambda\) is thus orthogonal to \(\alpha_1\) if and only if \(c_2 = 1\), which is also a necessary condition for \(\mathfrak{g}\) to be finite-dimensional. This implies that \(w_1(F_\lambda) = F_\lambda\). We then have \(w_1([e_1, F_\lambda]) = - [f_1, F_\lambda] = 0\). This relation completes the set of null states for the irreducible representation of \(\mathfrak{g}\) with highest weight \(\lambda\), so that \((ad f_i)^{\lambda_i + 1} F_\lambda = 0\) for \(i = 1, \ldots, r\).
Finally, from $f_{0j}$ we can obtain any $f_{0i}$ with $i = 2, 3, \ldots, r$ such that $A_{ij} \neq 0$ by stepping up and down with generators $e_i$ and $f_i$, and by continuing the procedure we can reach all $f_{0i}$ with $i = 2, 3, \ldots, r$. Thus all such $f_{0i}$ belong to the same module $R(A_1 + \theta)$, whereas $f_{00}$ belongs to $R(A_1)$, and there cannot be any other submodules of $W_{(0,1)}$. \hfill \Box

3.6. Completing the local Lie superalgebra. We are now ready to show that the condition (3.14) indeed holds when $\mathfrak{g}'$ is finite-dimensional.

**Proposition 3.11.** We have

\[ [\tilde{W}_{(1,0)}, \tilde{W}_{(0,1)}] = \tilde{W}_{(0,0)}. \tag{3.55} \]

**Proof.** We write any element in $\tilde{W}_{(1,0)}$ as a sum of terms

\[ [e_{\beta_1}, [e_{\beta_2}, \ldots, [e_{\beta_p}, e_0] \cdots]] \tag{3.56} \]

where each $\beta_i$ ($i = 1, 2, \ldots, p$) is a root of $\mathfrak{g}$ such that $(\alpha_0, \beta_i) = -1$, and any element of $\tilde{W}_{(0,1)}$ can be written as a sum of terms

\[ [x_1, [x_2, \ldots, [x_q, f_{00}] \cdots]] \tag{3.57} \]

where each $x_j$ ($j = 1, 2, \ldots, q$) is equal to $e_k$ or $f_k$ for $k = 1, 2, \ldots, r$. For $q = 0$ we have

\[ [f_{00}, [e_{\beta_1}, [e_{\beta_2}, \ldots, [e_{\beta_p}, e_0] \cdots]]] = [e_{\beta_1}, [e_{\beta_2}, \ldots, [e_{\beta_p}, f_{00}] \cdots]] \]
\[ = [e_{\beta_1}, [e_{\beta_2}, \ldots, [e_{\beta_p}, h_0] \cdots]] \in \tilde{W}_{(0,0)} \tag{3.58} \]

by Proposition 3.8 and for $q = 1$ we get

\[ [[x_1, f_{00}], [e_{\beta_1}, [e_{\beta_2}, \ldots, [e_{\beta_p}, e_0] \cdots]]] = [x_1, [f_{00}, [e_{\beta_1}, \ldots, [e_{\beta_p}, e_0] \cdots]] \]
\[ - [f_{00}, [x_1, [e_{\beta_1}, \ldots, [e_{\beta_p}, e_0] \cdots]]]. \tag{3.59} \]

where $[x_1, [e_{\beta_1}, \ldots, [e_{\beta_p}, e_0] \cdots]]$ in the second term on the right hand side can be rewritten in the form (3.56) since this is an element of $\tilde{W}_{(1,0)}$. We can then as in (3.58) show that each of the two terms on the right hand side belongs to $\tilde{W}_{(0,0)}$, and, continuing in the same way, the proposition follows by induction on $q$. \hfill \Box

To summarise, when $\mathfrak{g}$ is finite-dimensional, the local part $W_{-1} \oplus W_0 \oplus W_1$ of $W(\mathfrak{g})$ consists of the $\mathfrak{g}$-modules

\[ W_{-1} = R(A_1 + \theta) \oplus R(A_1), \quad W_0 = \mathfrak{g} \oplus \mathbb{K}, \quad W_1 = \overline{R(A_1)}. \tag{3.60} \]
Correspondingly, the local part $S_{-1} \oplus S_0 \oplus S_1$ of $S(\mathfrak{g})$ consists of the $\mathfrak{g}$-modules

$$S_{-1} = R(\Lambda_1 + \theta), \quad S_0 = \mathfrak{g}, \quad S_1 = \overline{R(\Lambda_1)}.$$  \hspace{1cm} (3.61)

Since all the modules are irreducible in this case, the form of the commutator relations $[S_{-1}, S_1] = S_0$ is uniquely given by the projection of the tensor product $S_{-1} \otimes S_1$ onto the submodule $S_0$. Thus, to prove that the definition of $S(\mathfrak{g})$ here agrees with the definition of the tensor hierarchy algebra associated to $\mathfrak{g}$ given in \cite{4}, and reviewed in Section 2.3, it suffices to check that the $\mathfrak{g}$-modules in the local parts agree. The only cases where they do not agree are those where the module at level $-1$ in the tensor hierarchy algebra is not irreducible but contains a singlet in addition to $S_{-1}$. As explained in \cite{4}, this happens precisely when replacing the grey node in the Dynkin diagram with an ordinary white one gives the Dynkin diagram of an affine Lie algebra, for example $\mathfrak{g} = E_8$ with the default choice of “node 1”, where $S_{-1}$ is the module 3875, but the level $-1$ content of the tensor hierarchy algebra is 3875 + 1. In the application to gauged supergravity the additional singlet is important, but it does not fit naturally into the algebra from the point of view that we adopt here.

We conclude this section with a few words about the case when $\mathfrak{g}$ is infinite-dimensional. Then it is no longer true that all roots $\alpha$ of $\mathfrak{g}$ have length squared $\langle \alpha, \alpha \rangle = 2$. In particular there might be a root $\beta$ which satisfies $\langle \alpha_0, \beta \rangle = -1$ but has length squared $\langle \beta, \beta \rangle \leq 0$, and thus does not belong to the same Weyl orbit as $e_1$. Then a corresponding root vector $e_\beta$ does not need to commute with all $f_0a$, but $[f_0a, e_\beta]$ could be a root vector in $\widetilde{W}_{(0,1)}$ corresponding to a root $-\alpha_0 + \beta$. Taking the commutator with $e_0$ we get

$$[e_0, [f_0a, e_\beta]] = [h_0a, e_\beta] - [f_0a, [e_0, e_\beta]]$$  \hspace{1cm} (3.62)

where the right hand side now contains a second term in $\widetilde{W}_{(1,1)}$ which does not necessarily belong to $\mathfrak{g} = \widetilde{W}_{(0,0)}$ or vanish. This is in agreement with the results in \cite{11}, where a tensor hierarchy algebra associated to $E_{11}$ was constructed, and shown to contain elements at level zero beyond the original $E_{11}$ algebra. We will come back to this example in Section 5.1.

4. The ideal $J$ of $\widetilde{W}(A_{n-1})$

Now $W$ can be constructed from $\widetilde{W}$ as the minimal Lie superalgebra with local part (3.60), by factoring out the maximal homogeneous ideal intersecting the local part trivially. Let $J$ be this ideal of $\widetilde{W}$. Then $J$ is the direct sum of subspaces $J_k = J \cap \widetilde{W}_{-k}$ for $k \geq 2$. The intersection of $J$ and $\widetilde{W}_+$ must be empty since $\widetilde{W}_+ = \mathcal{B}_+$, and $\mathcal{B}$ is
simple. We conjecture that $J$ is generated by $J_2$, but we have only proven this for $\mathfrak{g} = A_{n-1}$, the proof of which is the goal of this section. Throughout the section we assume $\mathfrak{g} = A_{n-1}$ and thus $W = W(\mathfrak{g}) = W(n)$.

4.1. The intersection between $J$ and $\hat{W}_{-2}$. Before stating the identities needed to generate the ideal $J$, we examine $\hat{W}_-$, which is the free Lie superalgebra generated by the subspace $W_{-1}$ of $W(A_{n-1})$. The first observation is that the generators $f_{0a}$ anticommute among themselves in $W(\mathfrak{g})_{-1}$, the proof of which is the goal of this section. Throughout the section we assume $\mathfrak{g} = A_{n-1}$ and thus $W = W(\mathfrak{g}) = W(n)$.

All the anticommutators under consideration carry the $A_{n-1}$ weight $(20\ldots0)$. In the freely generated algebra, there are generators at level $-2$ from the anticommutators

$$[\hat{K}, \hat{K}]: (0010\ldots02) \oplus (0010\ldots01) \oplus (020\ldots02) \oplus (1010\ldots01) \oplus (110\ldots01) \oplus (20\ldots0),$$

$$[\hat{K}, \hat{K}']: (0010\ldots01) \oplus (010\ldots0) \oplus (110\ldots01),$$

$$[K', K'] : (20\ldots0). \quad (4.1)$$

Concerning $[\hat{K}, \hat{K}]$, the relation

$$[f_{0i}, f_{0j}] = 0 \quad (4.2)$$

provides $(n - 2)(n - 1)/2$ elements at $A_{n-1}$ weight $(20\ldots0)$. If on the other hand we count the multiplicity of this weight in the six representations in $[\hat{K}, \hat{K}]$, we obtain the numbers in Table 4.

The total multiplicity of the weight $(20\ldots0)$ in $[\hat{K}, \hat{K}]$ is $(n - 2)(n - 1)$. Note that $(0010\ldots01)$, which is the level $-2$ generator that should be outside the ideal, is not affected by relation (4.2), but also that $(00010\ldots02)$, which should be part of the ideal, remains untouched by relation (4.2).

It remains to verify that the remaining elements are eliminated by the relation (4.2). Before checking this, we note that the total multiplicity is larger than (twice) the number of relations.
The element representations not already eliminated, we get the numbers in Table 5.

| representation | multiplicity of $(20\ldots0)$ |
|----------------|-------------------------------|
| $(00010\ldots02)$ | 0                             |
| $(0010\ldots01)$  | 0                             |
| $(020\ldots02)$   | $\frac{(n-2)(n-1)}{2}$       |
| $(1010\ldots010)$ | $\frac{(n-3)(n-2)}{2} - 1$   |
| $(110\ldots01)$   | $n - 2$                       |
| $(20\ldots0)$     | 1                             |

Table 4. Multiplicities of the weight $20\ldots0$ in some $A_{n-1}$ modules.

Any element in $[\hat{K},\hat{K}]$ with $A_{n-1}$ weight $20\ldots0$ comes from the product of two elements in the adjoint representation of $A_{n-2}$ at opposite $A_{n-2}$ weights. In addition to the relation $[f_{0i},f_{0j}] = [\varphi(h_i),\varphi(h_j)]$, one may also consider $[\varphi(e_\alpha),\varphi(f_\alpha)]$, where $\alpha$ is a positive $A_{n-2}$ root, and the homomorphism $\varphi$ was defined in (3.41). The number of positive roots in $A_{n-2}$ is $\frac{(n-2)(n-1)}{2}$. Together with $[f_{0i},f_{0j}]$, the anticommutators $[\varphi(e_\alpha),\varphi(f_\alpha)]$ give all the $(n-2)(n-1)$ elements at $A_{n-1}$ weight $20\ldots0$. Now, consider

$$\text{(ad } e_k)(\text{ad } f_k)([f_{0i},f_{0j}]) = 2B_{i[k}f_{0j]},f_{0k}] - 2B_{ik}B_{jk}[,\varphi(e_k),\varphi(f_k)].$$

This shows that $[\varphi(e_k),\varphi(f_k)]$ are not needed as separate generators in the ideal. Continued action with $(\text{ad } e_k)(\text{ad } f_k)$ gives the full set of $[\varphi(e_\alpha),\varphi(f_\alpha)]$.

In $[\hat{K},\hat{K}']$, the representation $(110\ldots01)$ should be part of the ideal. It contains the weight $20\ldots0$ with multiplicity $n-2$, and is set to zero by the relations $[f_{00},f_{00}]=0$. In $[K',K']$ the relation $[f_{00},f_{00}]=0$ generates the whole irreducible $A_{n-1}$ representation $(20\ldots0)$, just as $[e_0,e_0]=0$ generates its dual at level 2.

At this stage, it remains to remove the representation $(00010\ldots02)$ in $[\hat{K},\hat{K}]$, and also to relate the $(0010\ldots01)$’s appearing in $[\hat{K},\hat{K}]$ and $[\hat{K},K']$. If, in addition to a pair of generators $f_{00}$ or $f_{0i}$, one $f_1$ is introduced ($f_{00}$ and $f_{0i}$ are annihilated by $\text{ad } e_1$), the result has $A_{n-1}$ weight $(010\ldots0)$. Counting the multiplicities of this weight in the representations not already eliminated, we get the numbers in Table 5.

The element $[f_{0i},[f_1,f_{0j}]]$ generates an ideal if $i, j = 3, \ldots, n - 1$, so that nodes $i, j$ are disconnected from node 1, which is easily seen by commuting with $e_0$. Its symmetric part vanishes modulo $[f_{0i},f_{0j}]$ by the Jacobi identity. The antisymmetric
Table 5. Multiplicities of the weight $010\ldots0$ in some $A_{n-1}$ modules.

| representation | multiplicity of $010\ldots0$ |
|----------------|-------------------------------|
| $(00010\ldots02)$ | $\frac{(n-4)(n-3)}{2}$ |
| $(0010\ldots01)$ | $n-3$ |
| $(010\ldots0)$ | $1$ |

part gives exactly the $(n - 4)(n - 3)/2$ relations needed to eliminate $(00010\ldots02)$. The $(0010\ldots01)$, which is not part of the ideal, contains the $n - 3$ elements $[f_{02}, [f_1, f_{0j}]]$, $j = 3, \ldots, n - 1$.

Finally, one can check that the elements $[(f_{02} - f_{00}), [f_1, f_{0j}]]$ for $i = 3, \ldots, n - 1$ are annihilated by $\text{ad} e_0$, and generate an ideal that intersects the local part trivially. This provides the relation between the $(0010\ldots01)$’s appearing in $\hat{K}, \hat{K}'$.

This completes the examination of the relations at level $-2$. We summarise the result.

**Theorem 4.1.** The intersection $J_2$ of the ideal $J$ and the subspace $\hat{W}_{-2}$ of $\hat{W}$ is generated by the relations

\[
[f_{0a}, f_{0b}] = 0, \\
[f_{0i}, [f_1, f_{0j}]] = 0, \quad i, j \geq 3, \\
[(f_{02} - f_{00}), [f_1, f_{0i}]] = 0, \quad i \geq 3.
\] (4.4)

Before being in a position to state that this is the full set of generators of the ideal $J$ at negative levels, we need to show that no new ideals appear at lower levels. This can be done in several ways. Below we will use the relations recursively and verify that $W(A_{n-1})$ arises by repeated use of the relations. This proof has the potential of generalization to the $D$- and $E$-series.

We thus consider $\hat{W}/J_2$. We will use repeatedly in the proofs of the two lemmas below that $[K_{ab}, K_{cd}] = 0$ in $\hat{W}/J_2$ if $b \neq c$ and $a \neq d$. (As elsewhere in the paper, repeated indices should not be summed over.)

For $p \geq 3$ and indices $a_1, \ldots, a_p, b$ such that either $b = a_p$ or $b \neq a_1, \ldots, a_p$, define $\check{K}^{a_1\ldots a_p}_b$ by $\check{K}^{a_1 a_2 a_3}_b = K^{a_1 a_2 a_3}_b$ for $p = 3$, and recursively by

\[
\check{K}^{a_1\ldots a_p}_b = [K^{a_1 a_2}_{a_2}, \check{K}^{a_2\ldots a_p}_b]
\] (4.5)
for $p \geq 4$. Let $V_{-p+1}$ be the subspace of $(\tilde{W}/J_2)_{-p+1}$ spanned by all such $\tilde{K}^{a_1 \cdots a_p} b$, and set

$$V = \left( \bigoplus_{k \geq 3} V_{-k} \right) \oplus (\tilde{W}/J_2)_{-2} \oplus W_{-1} \oplus W_0 \oplus W_1 .$$

Lemma 4.2. We have

$$\tilde{K}^{a_1 \cdots a_p} b = \begin{cases} \tilde{K}^{a_1 \cdots a_p} b & \text{if } a_p \neq b, \\ \tilde{K}^{a_1 \cdots a_{p-1} a_p} b & \text{if } a_p = b. \end{cases}$$

Proof. We prove this by induction. By the definition and the induction hypothesis we then already have

$$\tilde{K}^{a_1 \cdots a_p} b = \begin{cases} \tilde{K}^{a_1 a_2 [a_3 \cdots a_p]} b & \text{if } a_p \neq b, \\ \tilde{K}^{a_1 a_2 [a_3 \cdots a_{p-1} a_p]} b & \text{if } a_p = b, \end{cases}$$

and it remains to show that

$$\tilde{K}^{a_1 a_2 a_3 a_4 \cdots a_p} b = -\tilde{K}^{a_2 a_1 a_3 a_4 \cdots a_p} b = -\tilde{K}^{a_3 a_2 a_1 a_4 \cdots a_p} b .$$

Suppose $\tilde{K}^{a_1 \cdots a_p} b \neq 0$. By the antisymmetry in the upper indices of $K^{a_1 a_2 a_2}$ and the induction hypothesis we can assume that all indices $a_1, a_2, \ldots, a_p$ are distinct. It follows that $b \neq a_1, a_2, \ldots, a_{p-1}$. Now we have

$$\tilde{K}^{a_1 a_2 a_3 \cdots a_p} b = [K^{a_1 a_2 a_2}, [K^{a_2 a_3 a_3}, \tilde{K}^{a_3 a_4 \cdots a_p} b]]
= [[K^{a_1 a_2 a_2}, K^{a_2 a_3 a_3}], \tilde{K}^{a_3 a_4 \cdots a_p} b]
= -[[K^{a_2 a_1 a_1}, K^{a_1 a_3 a_3}], \tilde{K}^{a_3 a_4 \cdots a_p} b]
= -[K^{a_2 a_1 a_1}, [K^{a_1 a_3 a_3}, \tilde{K}^{a_3 a_4 \cdots a_p} b]] = -\tilde{K}^{a_2 a_1 a_3 a_4 \cdots a_p} b ,$$

and similarly

$$\tilde{K}^{a_1 a_2 a_3 \cdots a_p} b = [K^{a_1 a_2 a_2}, [K^{a_2 a_3 a_3}, K^{a_3 a_4 a_4}, \tilde{K}^{a_4 \cdots a_p} b]]
= [K^{a_1 a_2 a_2}, [[K^{a_2 a_3 a_3}, K^{a_3 a_4 a_4}], \tilde{K}^{a_4 \cdots a_p} b]]
= -[K^{a_1 a_2 a_2}, [[K^{a_3 a_2 a_2}, K^{a_2 a_4 a_4}], \tilde{K}^{a_4 \cdots a_p} b]]
= [K^{a_3 a_2 a_2}, [[K^{a_1 a_2 a_2}, K^{a_2 a_4 a_4}], \tilde{K}^{a_4 \cdots a_p} b]]
= -[K^{a_3 a_2 a_2}, [[K^{a_2 a_1 a_1}, K^{a_1 a_4 a_4}], \tilde{K}^{a_4 \cdots a_p} b]]
= -[K^{a_3 a_2 a_2}, [K^{a_2 a_1 a_1}, [K^{a_1 a_4 a_4}, \tilde{K}^{a_4 \cdots a_p} b]]] = -\tilde{K}^{a_3 a_2 a_1 a_4 \cdots a_p} b .$$
Lemma 4.3. The subspace \( V \) of \( \tilde{W}/J_2 \) is closed under the adjoint action of elements at level \(-1\) and \(0\), that is,

\[
[K^{cd}, \tilde{K}^{a_1\ldots a_p b}] \in V
\]  

(4.12)

and

\[
[K^{c_d}, \tilde{K}^{a_1\ldots a_p b}] \in V .
\]  

(4.13)

Proof. Thanks to Lemma 4.2, we can assume \( c \neq a_2, \ldots, a_p \) and \( d \neq a_1, \ldots, a_{p-2} \). Then

\[
[K^{cd}, \tilde{K}^{a_2\ldots a_p b}] = [K^{a_1a_2}, [K^{a_2a_3}, \ldots, [K^{a_{p-2}a_{p-1}}, [K^{cd}, K^{a_{p-1}a_p b}]]]] ,
\]

where \( [K^{cd}, K^{a_{p-1}a_p b}] \) can be written as a linear combination of terms \( K^{a_{p-1}ef g} \). Thus \( [K^{cd}, \tilde{K}^{a_1\ldots a_p b}] \) is equal to a corresponding linear combination of terms

\[
K^{a_1\ldots a_{p-1}ef g} ,
\]

and we have proven the first part (4.12) of the lemma. For the second part (4.13) we have

\[
[K^{c_d}, \tilde{K}^{a_1\ldots a_p b}] = [K^{c_d}, [K^{a_1a_2}, \tilde{K}^{a_2\ldots a_p b}]] = [K^{a_1a_2}, [K^{c_d}, \tilde{K}^{a_2\ldots a_p b}]] ,
\]

(4.15)

and the proof follows by induction, using (4.12).

Since the \( A_{n-1} \) module \( W_{-1} \) is generated by \( K^{cd} \) for any nonzero \( K^{cd} \) it follows that \( [W_{-1}, \tilde{W}_{p+1}] \in \tilde{V}_{-p} \), and thus \( \tilde{W}/J_2 = \tilde{V} \). On the other hand, there is an isomorphism \( \tilde{V} \leftrightarrow W \) given by \( \tilde{K}^{a_1\ldots a_p b} \leftrightarrow K^{a_1\ldots a_p b} \), and thus \( \tilde{W}/J_2 \) is isomorphic to \( W \).

We have proven the following theorem.

Theorem 4.4. The ideal \( J \) of \( \tilde{W}(A_{n-1}) \) is generated by the relations

\[
[f_{0a}, f_{0b}] = 0 ,
\]

\[
[f_{0i}, [f_1, f_{0j}]] = 0 , \quad i, j \geq 3 ,
\]

\[
[(f_{0i} - f_{00}), [f_1, f_{0i}]] = 0 , \quad i \geq 3 .
\]

(4.16)

This gives us the main result of the paper:

Theorem 4.5. The Lie superalgebra \( \tilde{W}(A_{n-1})/J \) is isomorphic to \( W(n) \). Thus \( W(n) \) has generators \( (2.32) \) and defining relations \( (3.7) \)–\( (3.7) \) and \( (4.16) \).
5. Comments on the $D$ and $E$ cases

The tensor hierarchy algebras $W(E_r)$ and $S(E_r)$ are relevant in exceptional geometry and $W(D_r)$ and $S(D_r)$ are their analogs in double geometry. The Dynkin diagrams of these algebras, i.e., those of the BKM superalgebras $\mathcal{B}(D_r)$ and $\mathcal{B}(E_r)$ are given in Figure 3.

The definition of $W(\mathfrak{g})$ given in Section 3 is formulated entirely in terms of generators, and holds for any $\mathfrak{g}$. This means that the local superalgebra $W_{-1} \oplus W_0 \oplus W_1$ of $W(\mathfrak{g})$ is obtained from the definition of $\hat{W}(\mathfrak{g})$. The algebra $W(\mathfrak{g})$ is then defined as $\hat{W}(\mathfrak{g})/J$, where as previously, $J$ is the maximal ideal intersecting the local superalgebra trivially. It is clear that the identities (4.16) generate such an ideal. The only instance when we specialised to $\mathfrak{g} = A_{n-1}$, and more specifically, the tensor structure, was in the proof that (4.16) indeed generates the ideal $J$. The corresponding statement for $W(D_r)$ and $W(E_r)$ remains a conjecture.

It is straightforward to check that in $W(D_r)$ for $r \geq 4$ and $W(E_r)$ for $r \geq 6$, the level $-2$ relation $[f_{0i}, [f_1, f_{0j}]] = 0$ for $i, j = 3, \ldots, r$, is superfluous, although it still generates an ideal intersecting the local part trivially. The corresponding weights are in the same $\mathfrak{g}$ representation as the one of $[f_{0i}, f_{0j}]$, and the full level $-2$ part of $J$ is then generated by the relations

$$[f_{0a}, f_{0b}] = 0,$$
$$[(f_{02} - f_{00}), [f_1, f_{0i}]] = 0, \quad i \geq 3. \tag{5.1}$$
The BKM algebra with the same Dynkin diagram as \(W(D_r)\) is finite-dimensional, \(\mathcal{B}(D_r) = \mathfrak{osp}(r, r|2)\). There are generators at level \(\ell = -2, -1, 0, 1, 2\), with \(D_r\) singlets at \(\ell = 0, \pm 2\). Therefore, \(W(D_r)\) will not have any generators at level \(\ell \geq 3\).

We conjecture that the level decomposition of \(W(D_r)\) consists of an infinitely repeating sequence of antisymmetric modules, such that there are scalars at levels \(2 - 2p\), vectors at levels \(1 - 2p\), antisymmetric 2-index tensors at levels \(-2p\) etc. for \(p = 0, 1, 2, \ldots\). The completeness of the ideal at level \(-2\) should be possible to check in this case. For \(S(D_r)\), however, the scalar at level 2 is part of the ideal \(J\), and there is (by definition) no singlet at level 0. There will be no recurrence of the antisymmetric tensors. Therefore, the Lie superalgebra \(S(D_r)\) is finite-dimensional, and it is isomorphic to \(H(2r)\) in the classification of Kac [1].

In the case \(g = E_r\), the BKM algebra \(\mathcal{B}(E_r)\) is infinite-dimensional (see e.g. [7]), and \(W(E_r)\) and \(S(E_r)\) contain generators at all integer levels. A list of \(E_r\) representations for \(4 \leq r \leq 8\) up to level \(12 - r\) can be found for example in [18]. We conjecture that the relations (5.1) generate the maximal ideal \(J\) also in this case, but a proof is so far lacking.

5.1. **Realization of** \(W(E_n)\). In Section 3 we assumed that the algebra \(W(\mathfrak{g})\) is non-trivial, i.e., that the ideal of the free Lie superalgebra \(F\) generated by the relations (3.1)–(3.7) is not equal to \(F\) itself, but a proper ideal. To verify the assumption it is sufficient to find a non-trivial algebra homomorphism from \(W(\mathfrak{g})\) to a non-trivial algebra. When \(\mathfrak{g}\) is finite-dimensional, it is straightforward to construct such a homomorphism from \(S(\mathfrak{g})\) to the (original) tensor hierarchy algebra associated to \(\mathfrak{g}\), given the structural details in Section 3. The homomorphism can then be extended to a homomorphism from \(W(\mathfrak{g})\) to the extended version of the tensor hierarchy algebra.

However, when \(\mathfrak{g}\) is infinite-dimensional, it is not obvious that the assumption is true. The infinite-dimensional cases that we are most interested in are \(\mathfrak{g} = E_r\) for \(r \geq 9\), with the default choice of “node 1”, which means \(\mathfrak{g}' = E_{r-1}\). The assumption that both \(\mathfrak{g}\) and \(\mathfrak{g}'\) are simple excludes the affine Lie algebra \(E_9\) and adjusts the range of \(r\) to \(r \geq 11\). In [11] a tensor hierarchy algebra associated to \(E_r\) was defined also for \(r \geq 9\), with particular focus on the case \(r = 11\). We will end this section (and the paper) by briefly giving a surjective homomorphism from \(W(E_r)\) to an extension of the algebra defined in [11] (which is the image of the \(S(E_r)\) subalgebra).

Consider the Grassmann superalgebra \(\Lambda = \Lambda(n)\). Since it is an associative algebra with identity element, there is an injective homomorphism \(\Lambda(n) \to \text{End} \Lambda(n)\) given by
left multiplication. It is common to use the same notation for the image of any element under this homomorphism as for the element itself, writing

\[ x : \Lambda(n) \to \Lambda(n), \quad y \mapsto xy. \]

(5.2)

We will employ this convention, but at the same time it will be important to distinguish between the two copies of \( \Lambda(n) \). Therefore, we denote the identity elements in \( \Lambda(n) \) and \( \text{End} \, \Lambda(n) \) by \( E \) and \( L \), respectively, and write out these elements explicitly in the expressions. For example, (5.2) then becomes

\[ xL : \Lambda(n) \to \Lambda(n), \quad yE \mapsto xyE. \]

(5.3)

For any triple of indices \( a, b, c = 0, 1, \ldots, n-1 \), we define a map \( F_{abc} : \Lambda \to \text{End} \, \Lambda \) by

\[ F_{abc}(xE) = 3(K_{[a}K_{b}x)K_{c]} + (-1)^{|x|}(K_{a}K_{b}K_{c}x)L, \]

(5.4)

where \( K_{b} \) is the contraction

\[ K_{b} : \xi^{c_1} \cdots \xi^{c_q} E \mapsto q\delta^{b}_i \xi^{c_2} \cdots \xi^{c_q} E \]

(5.5)
defined in \([2.22]\). Set \( K^{a_1 \cdots a_p}_{b} = \xi^{a_1} \cdots \xi^{a_p} K_{b} \) and \( K = \sum_{a=0}^{n-1} K^a \).

Consider now the local Lie superalgebra \( u(\Lambda) = U_{-1} \oplus U_0 \oplus U_1 \), as defined in Section \([2.1]\) (but note that the \( \mathbb{Z}_2 \)-grading is not consistent in this case, since the \( \mathbb{Z}_2 \)-graded vector space \( U_1 = \Lambda \) is not homogeneous). Thus

\[ U_1 = \Lambda, \quad U_0 = \text{End} \, \Lambda, \quad U_{-1} = \text{Hom} \, (\Lambda, \text{End} \, \Lambda). \]

(5.6)

Let \( w_E(n) \) be the local subalgebra of \( u(\Lambda) \) generated by all elements in \( U_1 \), the elements \( F_{abc} \) in \( U_{-1} \), and the elements \( K^{a_1 \cdots a_p}_{b} \) in \( U_0 \). Let \( W_E(n) \) be the minimal Lie superalgebra with local part \( w_E(n) \). Consider the map \( W(E_n) \to W_E(n) \) given by

\[ e_0 \mapsto K_0, \quad e_i \mapsto K^{-1}_{i}, \quad f_i \mapsto K_{i} \]

\[ e_n \mapsto \xi^{(n-3)}\xi^{(n-2)}\xi^{(n-1)}E, \quad f_n \mapsto F_{(n-3)(n-2)(n-1)}, \]

\[ h_0 \mapsto K - 3L - K^0_0, \quad h_i \mapsto K^{-1}_{i-1} - K^i_i, \]

\[ f_0 \mapsto K_0^0 - 3L^0, \quad f_{0i} \mapsto K^{0(i-1)}_{i-1} - K^{0i}_i, \]

\[ h_n \mapsto K^{n-3}_{n-3} + K^{n-2}_{n-2} + K^{n-1}_{n-1} - L, \quad f_{0n} \mapsto K^{0(n-3)}_{n-3} + K^{0(n-2)}_{n-2} + K^{0(n-1)}_{n-1} - L^0 \]

(5.7)

for the set of generators, where \( K^0 = \xi^0 K \) and \( L^0 = \xi^0 L \). It is straightforward to check that this map preserves all the relations \([3.1]-[3.7]\), and thus it is a homomorphism.
Appendix A. The root system of $W(A_{n-1}) = W(n)$

A weight $\lambda$ can be expressed in the form

$$\lambda = k\tilde{\Lambda}_0 + \sum_{i=1}^{n-1} \mu_i\tilde{\Lambda}_i. \quad (A.1)$$

Here $\tilde{\Lambda}_0$ is a weight which is orthogonal to all simple roots $\alpha_1, \alpha_2, \ldots, \alpha_r$, and has coefficient 1 for $\alpha_0$ when expressed in terms of simple roots. This implies that $\tilde{\Lambda}_0$ is proportional to $\Lambda_0$. The inverse of the Cartan matrix $B$ given in (2.16) is

$$B^{-1} = \begin{pmatrix}
-\frac{n}{n-1} & -1 & \frac{n-2}{n-1} & \cdots & -\frac{1}{n-1} \\
-1 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & A'^{-1} \\
-\frac{1}{n-1} & 0
\end{pmatrix}, \quad (A.2)$$

where $A'$ is the Cartan matrix for $A_{n-2}$. An analogous structure (with 0's in the second row and column, and the inverse Cartan matrix for the algebra with Dynkin diagram obtained by deleting nodes 0 and 1) arises also for other choices of $g$. From the upper left corner we get

$$\tilde{\Lambda}_0 = \frac{n-1}{n} \lambda_0. \quad (A.3)$$

This also implies that $(\tilde{\Lambda}_0, \tilde{\Lambda}_0) = -\frac{n-1}{n}$. The $\tilde{\Lambda}_i$'s satisfy $(\tilde{\Lambda}_i, \alpha_j) = \delta_{ij}$, and have vanishing $\alpha_0$ component when expressed in the basis of simple roots. Thus, $(\tilde{\Lambda}_0, \tilde{\Lambda}_i) = 0$. The length of the weight $\lambda$ becomes

$$(\lambda, \lambda) = -\frac{n-1}{n} k^2 + (\mu, \mu), \quad (A.4)$$

where the scalar product on the right hand side is calculated for weights $\mu$ of $A_{n-1}$.

We can use (A.4) together with the known $A_{n-1}$ representations to give the lengths of all roots in $W(n)$ or $S(n)$. The representation $(0 \ldots 010 \ldots 01)$, with the first 1 in position $k + 1$, occurs at level $-k$. It contains two Weyl orbits, represented by the dominant weights $\mu_{k+1} + \mu_{n-1}$ and $\mu_k$, where $\mu_i$ are simple $A_{n-1}$ weights. The other representation at level $-k$ has highest weight $\mu_k$. The lengths squared of these weights
are
\[(\mu_k, \mu_k) = \frac{k(n-k)}{n}, \quad (\mu_{k+1} + \mu_{n-1}, \mu_{k+1} + \mu_{n-1}) = \frac{k(n-k)}{n} + 2. \quad (A.5)\]

Insertion into (A.4) tells us that the root lengths at level \( -k \) are \( k - k^2 \) and \( 2 + k - k^2 \). Roots with \((\lambda, \lambda) > 0\) appear at level 0 and -1. Null roots appear at level 1, -1 and -2, the last case for \( n \geq 4 \).

The root system for \( W(3) \) is depicted in Figure 4 and the one for \( W(4) \) in Figure 5. The \( W(3) \) roots are listed in Table 6.

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| level | basis | $\mathfrak{s}(3)$ representation | roots $\alpha$ | mult $\alpha$ | $(\alpha, \alpha)$ |
|-------|-------|-------------------------------|---------------|--------------|------------------|
| 1     | $K_0$ | $\overline{3}$               | $\alpha_0$    | 1            | 0                |
|       | $K_1$ |                               | $\alpha_0 + \alpha_1$ | 1            | 0                |
|       | $K_2$ |                               | $\alpha_0 + \alpha_1 + \alpha_2$ | 1            | 0                |
| 0     | $K^0_1$ | $8 + 1$                        | $\alpha_1$    | 1            | 2                |
|       | $K^1_2$ |                               | $\alpha_2$    | 1            | 2                |
|       | $K^0_2$ |                               | $\alpha_1 + \alpha_2$ | 1            | 2                |
|       | $K^1_0$ |                               | $-\alpha_1$   | 1            | 2                |
|       | $K^2_1$ |                               | $-\alpha_2$   | 1            | 2                |
|       | $K^2_0$ |                               | $-\alpha_1 - \alpha_2$ | 1            | 2                |
| -1    | $K^{01}_{12}$ | $\overline{6} + 3$           | $-\alpha_0 + \alpha_2$ | 1            | 2                |
|       | $K^{20}_{1}$ |                               | $-\alpha_0 - \alpha_2$ | 1            | 2                |
|       | $K^{12}_{0}$ |                               | $-\alpha_0 - 2\alpha_1 - \alpha_2$ | 1            | 2                |
|       | $K^{01}_{1} K^{02}_{2}$ |           | $-\alpha_0$     | 2            | 0                |
|       | $K^{12}_{2} K^{10}_{0}$ |               | $-\alpha_0 - \alpha_1$ | 2            | 0                |
|       | $K^{20}_{0} K^{21}_{1}$ |                    | $-\alpha_0 - \alpha_1 - \alpha_2$ | 2            | 0                |
| -2    | $K^{012}_{2}$ | $\overline{3}$               | $-2\alpha_0 - \alpha_1$ | 1            | -2               |
|       | $K^{012}_{1}$ |                               | $-2\alpha_0 - \alpha_1 - \alpha_2$ | 1            | -2               |
|       | $K^{012}_{0}$ |                               | $-2\alpha_0 - 2\alpha_1 - \alpha_2$ | 1            | -2               |

Table 6. The $W(3)$ root system.
Figure 4. The root system of $W(3)$. Each ball corresponds to a root, and the sizes of the balls to the multiplicities of the roots. The roots at each plane are the weights of the $A_2$-representation that occurs at the corresponding level in $W(3)$. The circles are the intersections of the planes at the different levels with the “light cone” consisting of null weights.
Figure 5. The root system of $W(4)$, divided into levels, from 1 at the top to $-3$ at the bottom. The spheres indicate the intersections of the level planes with the cone of null weights. Note the presence of null roots at levels 1, $-1$ and $-2$. 

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