Extremal Metrics on del Pezzo Threefolds

I. A. Cheltsov\textsuperscript{a} and K. A. Shramov\textsuperscript{a}

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Abstract—We prove the existence of Kähler–Einstein metrics on a nonsingular section of the Grassmannian $\text{Gr}(2,5) \subset \mathbb{P}^9$ by a linear subspace of codimension 3 and on the Fermat hypersurface of degree 6 in $\mathbb{P}(1,1,1,2,3)$. We also show that a global log canonical threshold of the Mukai–Umemura variety is equal to $1/2$.

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1. INTRODUCTION

Let $X$ be a variety\textsuperscript{1} with at most log canonical singularities (see [20]), and let $D$ be an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on the variety $X$. Then the number

$$\text{lct}(X,D) = \sup\{ \lambda \in \mathbb{Q} \mid \text{the log pair} (X, \lambda D) \text{ is log canonical} \} \in \mathbb{Q} \cup \{ +\infty \}$$

is called the log canonical threshold of the divisor $D$ (see [8]).

Suppose that $X$ is a Fano variety with at most log terminal singularities (see [19]).

Definition 1.1. The global log canonical threshold of the Fano variety $X$ is the number

$$\text{lct}(X) = \inf\{ \text{lct}(X,D) \mid D \text{ is an effective } \mathbb{Q}\text{-divisor on } X \text{ such that } D \sim_{\mathbb{Q}} -K_X \} \geq 0.$$  

Recall that every Fano variety $X$ is rationally connected (see [27]). Thus, the group $\text{Pic}(X)$ is torsion free. Hence

$$\text{lct}(X) = \sup\left\{ \lambda \in \mathbb{Q} \mid \text{the log pair} (X, \lambda D) \text{ is log canonical for every effective } \mathbb{Q}\text{-divisor } D \sim_{\mathbb{Q}} -K_X \right\}.$$  

Example 1.2. Let $X$ be a smooth hypersurface in $\mathbb{P}^n$ of degree $m$, where $2 \leq m \leq n$. Then

$$\text{lct}(X) = \frac{1}{n + 1 - m}$$

if $m < n$ (see [5]). Thus, we have $\text{lct}(\mathbb{P}^n) = 1/(n + 1)$. Suppose that $n = m$. By [5]

$$1 \geq \text{lct}(X) \geq \frac{n - 1}{n}.$$  

It follows from [4] and [12] that if $X$ is general, then

$$\text{lct}(X) \geq \begin{cases} 1 & \text{if } n \geq 6, \\ 22/25 & \text{if } n = 5, \\ 16/21 & \text{if } n = 4, \\ 3/4 & \text{if } n = 3. \end{cases}$$

One has $\text{lct}(X) = 1 - 1/n$ if $X$ contains a cone of dimension $n - 2$.

\textsuperscript{a}School of Mathematics, University of Edinburgh, King’s Buildings, Mayfield Road, Edinburgh, EH9 3JZ, UK.

E-mail address: I.Cheltsov@ed.ac.uk (I.A. Cheltsov).

\textsuperscript{1}All varieties are assumed to be complex, algebraic, projective, and normal.
Example 1.3. Let $X$ be a rational homogeneous space such that $-K_X \sim rD$ and

$$\text{Pic}(X) = \mathbb{Z}[D],$$

where $D$ is an ample divisor and $r \in \mathbb{Z}_{>0}$. Then $\lct(X) = 1/r$ (see [17]).

Example 1.4. Let $X$ be a quasismooth hypersurface in $\mathbb{P}(1, a_1, \ldots, a_4)$ of degree $\sum_{i=1}^{4} a_i$ such that $X$ has at most terminal singularities, where $a_1 \leq a_2 \leq a_3 \leq a_4$. Then

$$-K_X \sim \mathcal{O}_{\mathbb{P}(1, a_1, \ldots, a_4)}(1)|_X$$

and there are 95 possibilities for the quadruple $(a_1, a_2, a_3, a_4)$ (see [18]). One has

$$1 \geq \lct(X) \geq \begin{cases} 
16/21 & \text{if } a_1 = a_2 = a_3 = a_4 = 1, \\
7/9 & \text{if } (a_1, a_2, a_3, a_4) = (1, 1, 1, 2), \\
4/5 & \text{if } (a_1, a_2, a_3, a_4) = (1, 1, 2, 2), \\
6/7 & \text{if } (a_1, a_2, a_3, a_4) = (1, 1, 2, 3), \\
1 & \text{in the remaining cases}
\end{cases}$$

if $X$ is general (see [10, 12, 6]).

Example 1.5. Let $X$ be smooth del Pezzo surface. It follows from [11] that

$$\lct(X) = \begin{cases} 
1 & \text{if } K_X^2 = 1 \text{ and } |-K_X| \text{ contains no cuspidal curves,} \\
5/6 & \text{if } K_X^2 = 1 \text{ and } |-K_X| \text{ contains a cuspidal curve,} \\
5/6 & \text{if } K_X^2 = 2 \text{ and } |-K_X| \text{ contains no tacnodal curves,} \\
3/4 & \text{if } K_X^2 = 2 \text{ and } |-K_X| \text{ contains a tacnodal curve,} \\
3/4 & \text{if } X \text{ is a cubic in } \mathbb{P}^3 \text{ with no Eckardt points,} \\
2/3 & \text{if } \text{either } X \text{ is a cubic in } \mathbb{P}^3 \text{ with an Eckardt point or } K_X^2 = 4, \\
1/2 & \text{if } X \cong \mathbb{P}^1 \times \mathbb{P}^1 \text{ or } K_X^2 \in \{5, 6\}, \\
1/3 & \text{in the remaining cases.}
\end{cases}$$

Let $G \subset \text{Aut}(X)$ be an arbitrary subgroup.

Definition 1.6. The global $G$-invariant log canonical threshold $\lct(X, G)$ of the Fano variety $X$ is the number

$$\sup \left\{ \epsilon \in \mathbb{Q} \mid \text{the log pair } \left( X, \frac{\epsilon}{n}D \right) \text{ has log canonical singularities for every } G\text{-invariant linear system } D \subset |-nK_X| \text{ and every } n \in \mathbb{Z}_{>0} \right\}.$$  

If the Fano variety $X$ is smooth and $G$ is compact, then it follows from [7, Appendix A] that

$$\lct(X, G) = \alpha_G(X),$$

where $\alpha_G(X)$ is the invariant introduced in [25]. We have $\lct(X) \leq \lct(X, G) \in \mathbb{R} \cup \{+\infty\}.$

Remark 1.7. Suppose that the subgroup $G$ is finite. Then

$$\lct(X, G) = \sup \left\{ \lambda \in \mathbb{Q} \mid \text{the log pair } (X, \lambda D) \text{ is log canonical for every effective } G\text{-invariant } \mathbb{Q}\text{-divisor } D \sim_{\mathbb{Q}} -K_X \right\}.$$
Indeed, it is enough to show that if $D \subset |-mK_X|$ is a $G$-invariant linear system such that the log pair $(X, cD)$ is not log canonical for some $c \in \mathbb{Q}_{\geq 0}$, then there is a $G$-invariant effective $\mathbb{Q}$-divisor $B \sim_{\mathbb{Q}} -mK_X$ such that the log pair $(X, cB)$ is not log canonical. Put $k = |G|$. Suppose that the log pair $(X, cD)$ is not log canonical. Let $D \in D$ be a general divisor. Then the log pair

$$(X, \frac{c}{k} \sum_{g \in G} g(D))$$

is not log canonical either (see the proof of [20, Theorem 4.8]), which implies the required assertion.

**Example 1.8.** The simple group $\text{PGL}(2, F_7)$ is a group of automorphisms of the quartic

$$x^3y + y^3z + z^3x = 0 \subset \mathbb{P}^2 \cong \text{Proj}(\mathbb{C}[x, y, z]),$$

which gives an embedding $\text{PGL}(2, F_7) \subset \text{Aut}(\mathbb{P}^2)$. One has $\text{lct}(\mathbb{P}^2, \text{PGL}(2, F_7)) = 4/3$ (see [24, 11]).

**Example 1.9.** Let $X$ be the cubic surface in $\mathbb{P}^3$ given by the equation

$$x^3 + y^3 + z^3 + t^3 = 0 \subset \mathbb{P}^3 \cong \text{Proj}(\mathbb{C}[x, y, z, t]),$$

and let $G = \text{Aut}(X) \cong \mathbb{Z}_3^3 \rtimes S_4$. Then $\text{lct}(X, G) = 4$ by [11].

The following result is proved in [25, 23, 13] (cf. [7, Appendix A]).

**Theorem 1.10.** Suppose that $X$ has at most quotient singularities, the group $G$ is compact, and the inequality

$$\text{lct}(X, G) > \frac{\dim(X)}{\dim(X) + 1}$$

holds. Then $X$ admits an orbifold Kähler–Einstein metric.

**Remark 1.11.** Let $G \subset \text{Aut}(X)$ be a reductive subgroup and $G' \subset G$ the maximal compact subgroup of $G$. Then a restriction to $G'$ of any irreducible representation of $G$ remains irreducible as a complex representation of $G'$. This implies that all linear systems on $X$ that are invariant with respect to $G$ are also invariant with respect to $G'$ (the converse holds by obvious reasons). In particular, $\text{lct}(X, G) = \text{lct}(X, G')$.

Theorem 1.10 has many applications (see Examples 1.2, 1.4, and 1.9).

**Example 1.12.** Let $X$ be one of the following smooth Fano varieties:

- a Fermat hypersurface in $\mathbb{P}^n$ of degree $n/2 \leq d \leq n$ (cf. Example 1.9);
- a smooth complete intersection of two quadrics in $\mathbb{P}^5$ that is given by

$$\sum_{i=0}^{5} x_i^2 = \sum_{i=0}^{5} \zeta^i x_i^2 = 0 \subset \mathbb{P}^5 \cong \text{Proj}(\mathbb{C}[x_0, \ldots, x_5]),$$

where $\zeta$ is a primitive sixth root of unity;
- a hypersurface in $\mathbb{P}(1^{n+1}, q)$ of degree $pq$ that is given by the equation

$$w^p = \sum_{i=0}^{5} x_i^{pq} \subset \mathbb{P}(1^{n+1}, q) \cong \text{Proj}(\mathbb{C}[x_0, \ldots, x_n, w])$$

such that $pq - q \leq n$, where $\text{wt}(x_0) = \ldots = \text{wt}(x_n) = 1$, $\text{wt}(w) = q \in \mathbb{Z}_{>0}$, and $p \in \mathbb{Z}_{>0}$.

Let $G = \text{Aut}(X)$. Then $G$ is finite and the inequality $\text{lct}(X, G) \geq 1$ holds (see [25, 23]).
The numbers $\lct(X)$ and $\lct(X, G)$ also play an important role in birational geometry. For instance, the following result holds (see [11]).

**Theorem 1.13.** Let $X_i$ be a Fano variety, and let $G_i \subset \text{Aut}(X_i)$ be a finite subgroup such that the variety $X_i$ is $G_i$-birationally superrigid (see [7]) and the inequality $\lct(X_i, G_i) \geq 1$ holds, where $i = 1, \ldots, r$. Then the following assertions hold:

- there is no $(G_1 \times \ldots \times G_r)$-equivariant birational map $\rho: X_1 \times \ldots \times X_r \rightarrow \mathbb{P}^n$;
- every $(G_1 \times \ldots \times G_r)$-equivariant birational automorphism of $X_1 \times \ldots \times X_r$ is biregular;
- for every $(G_1 \times \ldots \times G_r)$-equivariant rational dominant map $\rho: X_1 \times \ldots \times X_r \rightarrow Y$ whose general fiber is a rationally connected variety, there a commutative diagram

$$
\begin{array}{ccc}
X_1 \times \ldots \times X_r & \longrightarrow & Y \\
\downarrow{\pi} & & \\
X_{i_1} \times \ldots \times X_{i_k} \end{array}
$$

where $\xi$ is a birational map, $\pi$ is a natural projection, and $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, r\}$.

Varieties satisfying all hypotheses of Theorem 1.13 do exist.

**Example 1.14.** The simple group $A_6$ is a group of automorphisms of the sextic

$$
10x^3y^3 + 9zx^5 + 9zy^5 + 27z^6 = 45x^2y^2z^2 + 135xyz^4 \subset \mathbb{P}^2 \cong \text{Proj}(\mathbb{C}[x, y, z]),
$$

which induces an embedding $A_6 \subset \text{Aut}(\mathbb{P}^2)$. Then $\mathbb{P}^2$ is $A_6$-birationally superrigid and the equality $\lct(\mathbb{P}^2, A_6) = 2$ holds (see [24, 11]). Thus, there is an induced embedding $A_6 \times A_6 \cong \Omega \subset \text{Bir}(\mathbb{P}^4)$ such that $\Omega$ is not conjugate to any subgroup in $\text{Aut}(\mathbb{P}^4)$ by Theorem 1.13.

Let $V$ be a smooth Fano threefold (see [19]) such that $-K_V \sim 2H$, where $H$ is an ample Cartier divisor that is not divisible in $\text{Pic}(V)$.

**Remark 1.15.** The variety $V$ is called a del Pezzo variety, since a general element in the linear system $|H|$ is a smooth del Pezzo surface.

It is well-known that $V$ is one of the following varieties:

- $V_1$, i.e., a hypersurface in $\mathbb{P}(1, 1, 1, 2, 3)$ of degree 6;
- $V_2$, i.e., a hypersurface in $\mathbb{P}(1, 1, 1, 1, 2)$ of degree 4;
- $V_3$, i.e., a cubic surface in $\mathbb{P}^3$;
- $V_4$, i.e., a complete intersection of two quadrics in $\mathbb{P}^5$;
- $V_5$, i.e., a section of the Grassmannian $\text{Gr}(2, 5) \subset \mathbb{P}^9$ by a linear subspace of codimension 3 (all such sections are isomorphic);
- $W$, a divisor in $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(1, 1)$;
- $V_7$, i.e., a blow-up of $\mathbb{P}^3$ at a point;
- the product $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

**Remark 1.16.** In [7] the values of the global log canonical thresholds of smooth del Pezzo threefolds were found:

$$
\lct(V) = \begin{cases} 
1/4 & \text{if } V \text{ is a blow-up of } \mathbb{P}^3 \text{ at a point,} \\
1/2 & \text{in the remaining cases.}
\end{cases}
$$
Concerning Kähler–Einstein metrics on $V$, the following is known:

- $V_7$ does not admit a Kähler–Einstein metric (see [26]);
- $V_4$ admits a Kähler–Einstein metric (see [9], cf. Example 1.12);
- $W$ and $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ admit Kähler–Einstein metrics, since their automorphism groups are reductive and act on them transitively (see Theorem 1.10 and Remark 1.11);
- there are examples of varieties $V_2 \subset \mathbb{P}(1,1,1,2)$ and $V_3 \subset \mathbb{P}^4$ with large automorphism groups (see Example 1.12) that admit Kähler–Einstein metrics.

The question of existence of Kähler–Einstein metrics on the varieties $V_1$ and $V_5$ has not been studied in the literature yet (cf. a remark before [9, Theorem 3.2]).

The main purpose of this paper is to prove the following assertions.

**Theorem 1.17.** Let $G$ be a maximal compact subgroup in $\text{Aut}(V_5)$. Then

$$\text{lct}(V_5, G) = \text{lct}(V_5, \text{Aut}(V_5)) = 5/6.$$ 

**Theorem 1.18.** Let $V_1$ be a hypersurface in $\mathbb{P}(1,1,1,2,3)$ given by the equation

$$w^2 = t^3 + x^6 + y^6 + z^6 \subset \mathbb{P}(1,1,1,2,3) \cong \text{Proj}(\mathbb{C}[x,y,z,t,w]),$$

where $\text{wt}(x) = \text{wt}(y) = \text{wt}(z) = 1$, $\text{wt}(t) = 2$, and $\text{wt}(w) = 3$. Then $\text{lct}(V_1, \text{Aut}(V_1)) \geq 1$.

Note that the latter results combined with Theorem 1.10 imply the existence of Kähler–Einstein metrics on the variety $V_5$ and on the Fermat hypersurface of degree 6 in $\mathbb{P}(1,1,1,2,3)$.

**Remark 1.19.** Let $V_1$ be a smooth hypersurface in $\mathbb{P}(1,1,1,2,3)$ of degree 6. Assume that $\text{lct}(V_1, G) \geq 1$, where $G$ is a subgroup in $\text{Aut}(V_1)$. Then

- the linear system $|H|$ does not contain $G$-invariant surfaces,
- the linear system $|H|$ does not contain $G$-invariant pencils (cf. the proof of [24, Theorem 1.2]),
- the variety $V_1$ is $G$-birationally superrigid (see [1, 2]).

**Remark 1.20.** The methods we use to prove Theorem 3.2 (see below) are similar to those of [23]. Nevertheless, some statements of [23] (say, [23, Corollary 4.2] or a standard method for excluding zero-dimensional components of a subscheme of log canonical singularities) cannot be directly applied in our case, since the group $\text{Aut}(V_1)$ never acts on $V_1$ without fixed points.

The structure of the paper is as follows. Section 2 contains some auxiliary statements. In Section 3 we prove Theorem 1.18. In Section 4 we prove Theorem 1.17. The methods of Section 4 can be applied without significant changes to one more interesting Fano threefold, the so-called Mukai–Umemura variety (see [14] and Remark 5.2). To complete the picture, in Section 5 we calculate the global log canonical threshold of the Mukai–Umemura variety without any group action.

2. PRELIMINARIES

Let $X$ be a variety with log terminal singularities. Let us consider a $\mathbb{Q}$-divisor $B_X = \sum_{i=1}^{r} a_i B_i$, where $B_i$ is a prime Weil divisor on the variety $X$ and $a_i$ is an arbitrary nonnegative rational number. Suppose that $B_X$ is a $\mathbb{Q}$-Cartier divisor such that $B_i \neq B_j$ for $i \neq j$.

Let $\pi: \overline{X} \to X$ be a birational morphism such that $\overline{X}$ is smooth. Put

$$B_{\overline{X}} = \sum_{i=1}^{r} a_i \overline{B_i},$$
where \( \overline{B_i} \) is a proper transform of the divisor \( B_i \) on the variety \( \overline{X} \). Then

\[
K_{\overline{X}} + B_{\overline{X}} \sim_\mathbb{Q} \pi^*(K_X + B_X) + \sum_{i=1}^n c_i E_i,
\]

where \( c_i \in \mathbb{Q} \) and \( E_i \) is an exceptional divisor of the morphism \( \pi \). Suppose that

\[
\left( \bigcup_{i=1}^r B_i \right) \cup \left( \bigcup_{i=1}^n E_i \right)
\]

is a divisor with simple normal crossings. Put

\[
B_{\overline{X}} = B_{\overline{X}} - \sum_{i=1}^n c_i E_i.
\]

**Definition 2.1.** The singularities of \((X, B_X)\) are log canonical (respectively, log terminal) if

- the inequality \( a_i \leq 1 \) holds (respectively, the inequality \( a_i < 1 \) holds),
- the inequality \( c_j \geq -1 \) holds (respectively, the inequality \( c_j > -1 \) holds)

for every \( i = 1, \ldots, r \) and \( j = 1, \ldots, n \).

One can show that Definition 2.1 does not depend on the choice of the morphism \( \pi \). Put

\[
\text{LCS}(X, B_X) = \left( \bigcup_{a_i \geq 1} B_i \right) \cup \left( \bigcup_{c_i \leq -1} \pi(E_i) \right) \subset X
\]

and let us call \( \text{LCS}(X, B_X) \) the locus of log canonical singularities of the log pair \((X, B_X)\).

**Definition 2.2.** A proper irreducible subvariety \( Y \subset X \) is said to be a center of log canonical singularities of the log pair \((X, B_X)\) if one of the following conditions is satisfied:

- either the inequality \( a_i \geq 1 \) holds and \( Y = B_i \),
- or the inequality \( c_i \leq -1 \) holds and \( Y = \pi(E_i) \)

for some choice of the birational morphism \( \pi : \overline{X} \to X \).

Let \( \text{LCS}(X, B_X) \) be the set of all centers of log canonical singularities of \((X, B_X)\). Then

\[
Y \in \text{LCS}(X, B_X) \quad \Rightarrow \quad Y \subset \text{LCS}(X, B_X)
\]

and let \( \text{LCS}(X, B_X) = \emptyset \iff \text{LCS}(X, B_X) = \emptyset \iff \text{the log pair } (X, B_X) \text{ is log terminal} \).

**Remark 2.3.** We can use similar constructions and notation for any log pair \((X, \lambda D)\), where \( D \) is a linear system and \( \lambda \) is a nonnegative rational number.

The set \( \text{LCS}(X, B_X) \) can be naturally equipped with a scheme structure (see [23, 8]). Put

\[
\mathcal{I}(X, B_X) = \pi_* \left( \sum_{i=1}^n [c_i] E_i - \sum_{i=1}^r [a_i] \overline{B_i} \right),
\]

and let \( \mathcal{L}(X, B_X) \) be a subscheme that corresponds to the ideal sheaf \( \mathcal{I}(X, B_X) \).

**Remark 2.4.** The scheme \( \mathcal{L}(X, B_X) \) is usually called the subscheme of log canonical singularities of the log pair \((X, B_X)\), and the ideal sheaf \( \mathcal{I}(X, B_X) \) is usually called the multiplier ideal sheaf of the log pair \((X, B_X)\).

It follows from the construction of the subscheme \( \mathcal{L}(X, B_X) \) that

\[
\text{Supp}(\mathcal{L}(X, B_X)) = \text{LCS}(X, B_X) \subset X.
\]
The following result is known as the Shokurov vanishing theorem (see [8]) or the Nadel vanishing theorem (see [21, Theorem 9.4.8]).

**Theorem 2.5.** Let $H$ be a nef and big $\mathbb{Q}$-divisor on $X$ such that $K_X + B_X + H \sim_\mathbb{Q} D$ for some Cartier divisor $D$ on the variety $X$. Then for every $i \geq 1$

$$H^i(X, \mathcal{I}(X, B_X) \otimes D) = 0.$$ 

The following result is known as the Shokurov connectedness theorem.

**Theorem 2.6.** Suppose that $-(K_X + B_X)$ is nef and big. Then LCS$(X, B_X)$ is connected.

**Proof.** It follows from Theorem 2.5 that the sequence

$$\mathbb{C} = H^0(O_X) \to H^0(O_{\mathcal{L}(X, B_X)}) \to H^1(\mathcal{I}(X, B_X)) = 0$$

is exact. Thus, the locus

$$\text{LCS}(X, B_X) = \text{Supp}(\mathcal{L}(X, B_X))$$

is connected. $\square$

One can generalize Theorem 2.6 in the following way (see [8, Lemma 5.7]).

**Theorem 2.7.** Let $\psi: X \to Z$ be a morphism. Then the set

$$\text{LCS}(X, B_X)$$

is connected in a neighborhood of every fiber of the morphism $\psi \circ \pi: X \to Z$ in the case when

- the morphism $\psi$ is surjective and has connected fibers,
- the divisor $-(K_X + B_X)$ is nef and big with respect to $\psi$.

The following result is a corollary of Theorem 2.5 (see [23, Theorem 4.1]).

**Lemma 2.8.** Suppose that $-(K_X + B_X)$ is nef and big and $\dim(\text{LCS}(X, B_X)) = 1$. Then

- the locus LCS$(X, B_X)$ is a connected union of smooth rational curves,
- the locus LCS$(X, B_X)$ does not contain a cycle of smooth rational curves,
- any intersecting irreducible components of the locus LCS$(X, B_X)$ meet transversally.

Let $P$ be a point in $X$. Let us consider an effective divisor

$$\Delta = \sum_{i=1}^r \varepsilon_i B_i \sim_\mathbb{Q} B_X,$$

where $\varepsilon_i$ is a nonnegative rational number. Suppose that

- the divisor $\Delta$ is a $\mathbb{Q}$-Cartier divisor,
- the equivalence $\Delta \sim_\mathbb{Q} B_X$ holds,
- the log pair $(X, \Delta)$ is log canonical at the point $P \in X$.

**Remark 2.9.** Suppose that $(X, B_X)$ is not log canonical at the point $P \in X$. Put

$$\alpha = \min\left\{ \frac{a_i}{\varepsilon_i} \mid \varepsilon_i \neq 0 \right\}.$$ 

Note that $\alpha$ is well defined, because there is $\varepsilon_i \neq 0$. Then $\alpha < 1$, the log pair

$$\left( X, \sum_{i=1}^r \frac{a_i - \alpha \varepsilon_i}{1 - \alpha} B_i \right)$$
is not log canonical at the point $P \in X$, the equivalence
\[ \sum_{i=1}^{r} \frac{a_i - \alpha \varepsilon_i}{1 - \alpha} B_i \sim_{Q} B_X \sim_{Q} \Delta \]
holds, and at least one irreducible component of the divisor $\text{Supp}(\Delta)$ is not contained in
\[ \text{Supp} \left( \sum_{i=1}^{r} \frac{a_i - \alpha \varepsilon_i}{1 - \alpha} B_i \right) \).

The following result is an easy corollary of Remark 2.9.

**Lemma 2.10.** Let $X$ be a smooth Fano variety such that $\text{Pic}(X) = \mathbb{Z}[H]$ for some divisor $H \in \text{Pic}(X)$, and let $G \subset \text{Aut}(X)$ be a subgroup. Let $\lambda$ be a rational number such that
- $\text{lct}(X, D) \geq \lambda / n$ for every $G$-invariant divisor $D \in \{ nH \}$,
- $\text{lct}(X, D) \geq \lambda / n$ for every $G$-invariant linear subsystem $\mathcal{D} \subset \{ nH \}$ that has no fixed components.

Then
\[ \text{lct}(X, G) \geq \lambda. \]

**Proof.** Suppose that $\text{lct}(X, G) < \lambda$. Then there are a natural number $n$ and a $G$-invariant linear subsystem $\mathcal{D} \subset \{ nH \}$ such that the log pair
\[ \left( X, \frac{\lambda}{n} \mathcal{D} \right) \]
is not log canonical. Put $\mathcal{D} = F + \mathcal{M}$, where $F$ is a fixed part of the linear system $\mathcal{D}$ and $\mathcal{M}$ is a $G$-invariant linear system that has no fixed components.

Let $M_1, \ldots, M_r$ be general divisors in $\mathcal{M}$, where $r \gg 0$. Then
\[ \left( X, \frac{\lambda}{n} \left( F + \frac{\sum_{i=1}^{r} M_i}{r} \right) \right) \]
is not log canonical by [20, Theorem 4.8].

Since $\text{Pic}(X) = \mathbb{Z}[H]$, we have $F \sim n_1 H$ and $\mathcal{M} \sim n_2 H$ for some $n_1, n_2 \in \mathbb{Z}_{>0}$ such that $n_1 + n_2 = n$. By Remark 2.9, we see that the log pair
\[ \left( X, \frac{\lambda}{n_2} \sum_{i=1}^{r} M_i \right) \]
is not log canonical, because $F$ is $G$-invariant. Then the log pair
\[ \left( X, \frac{\lambda}{n_2} \mathcal{M} \right) \]
is not log canonical by [20, Theorem 4.8], which is a contradiction. \qed

The following simple calculation will be useful in Section 4.

**Lemma 2.11.** Let $\dim(X) = 3$; let $C \subset X$ be an irreducible reduced curve and $P \in C$ a point such that
\[ \text{Sing}(C) \not\ni P \not\in \text{Sing}(X). \]
Let $L \subset X$ be a curve such that $P \not\in \text{Sing}(L)$ and $D$ a $\mathbb{Q}$-divisor on $X$ such that $C \subset \text{Supp}(D) \not\ni L$. Assume that $L$ and $C$ are tangent at $P$. Then
\[ \text{mult}_{P}(D \cdot L) \geq 2 \text{mult}_{C}(D). \]
Proof. Let \( \pi : \tilde{X} \to X \) be a blow-up of the point \( P \), and let \( E \) be an exceptional divisor of \( \pi \). Denote by \( \tilde{L} \), \( \tilde{C} \), and \( \tilde{D} \) the proper transforms on \( \tilde{X} \) of the curves \( L \) and \( C \) and the divisor \( D \), respectively. Then the intersection
\[
\tilde{L} \cap \text{Supp}(\tilde{D})
\]
contains some point \( \tilde{P} \in E \), since \( L \) and \( C \) are tangent at \( P \). Hence
\[
\text{mult}_P(D \cdot L) = \text{mult}_P(D) + \text{mult}_{\tilde{P}}(\tilde{D} \cdot \tilde{L}) \geq \text{mult}_C(D) + \text{mult}_{\tilde{C}}(\tilde{D}) = 2 \text{mult}_C(D). \quad \square
\]

3. VERONESE DOUBLE CONE

We will use the following notation: if \( D \) is a (nonempty) linear system on the variety \( X \), then \( \varphi_D \) denotes the rational map defined by \( D \).

Let \( V \) be a smooth Fano threefold such that \((-K_V)^3 = 8\) and
\[
\text{Pic}(V) = \mathbb{Z}[H]
\]
for some \( H \in \text{Pic}(V) \). Then \( V \) is a hypersurface in \( \mathbb{P}(1,1,1,2,3) \) of degree 6.

The linear system \( |H| \) has the only base point \( O \in V \) and defines a rational map
\[
\varphi_{|H|} : V \dashrightarrow \mathbb{P}^2
\]
with irreducible fibers; a general fiber of \( \varphi_{|H|} \) is an elliptic curve.

Remark 3.1. We will refer to the subvarieties of \( V \) that are swept out by the fibers of \( \varphi_{|H|} \) as vertical subvarieties.

Let \( G \subset \text{Aut}(V) \) be a subgroup. Note that \( G \) is finite, its action on \( V \) extends to \( \mathbb{P}(1,1,1,2,3) \), and \( G \) naturally acts on \( \mathbb{P}(|H|) \cong \mathbb{P}^2 \). Moreover, the following conditions are equivalent:

- \( G \) has no fixed points on \( \mathbb{P}(|H|) \cong \mathbb{P}^2 \);
- \( G \) has no invariant lines on \( \mathbb{P}(|H|) \cong \mathbb{P}^2 \);
- \( |H| \) contains no \( G \)-invariant surfaces;
- \( |H| \) contains no \( G \)-invariant pencils (cf. the proof of [24, Theorem 1.2]);
- \( V \) is \( G \)-birationally superrigid (see [1, 2]).

Let \( \mathcal{B} \) be a linear subsystem in \( |-K_X| \) generated by divisors of the form
\[
\lambda_0x^2 + \lambda_1y^2 + \lambda_2z^2 + \lambda_3xy + \lambda_4xz + \lambda_5yz = 0,
\]
where \( x, y, \) and \( z \) are coordinates of weight 1 on \( \mathbb{P}(1,1,1,2,3) \). The statement of Theorem 1.18 is implied by the following result.

Theorem 3.2. Suppose that the linear system \( \mathcal{B} \) contains no \( G \)-invariant divisors. Then \( \text{lct}(V, G) \geq 1 \).

Proof. Assume that \( \text{lct}(V, G) < 1 \). Then the linear system \( |H| \) does not contain \( G \)-invariant divisors, but there exists an effective \( G \)-invariant \( \mathbb{Q} \)-divisor \( D \sim_{\mathbb{Q}} -K_V \) such that
\[
\text{LCS}(V, \lambda D) \neq \emptyset
\]
for some \( 1 > \lambda \in \mathbb{Q} \). The set \( \text{LCS}(V, \lambda D) \) is \( G \)-invariant.

Lemma 3.3. The set \( \text{LCS}(V, \lambda D) \) does not contain divisors.

Proof. Easy. \quad \square
Lemma 3.4. The set $\mathcal{LCS}(V, \lambda D)$ does not contain curves.

Proof. Let $C \subset \mathcal{LCS}(V, \lambda D)$ be a $G$-invariant curve. Then for any point $P \in C$ one has $\operatorname{mult}_P D > 1/\lambda$. 

Lemma 2.8 implies that $C$ has a nonvertical component. Then $\deg(\phi|_H(C)) \geq 3$, since the linear system $\mathcal{B}$ does not contain $G$-invariant surfaces.

Let $S$ be a general surface in $[H]$. Put 

$$S \cap C = \{P_1, \ldots, P_s\},$$

where $P_1, \ldots, P_s$ are distinct points. Then $s \geq 3$. Moreover, one has $s > 3$ if $O \in C$. So it is easy to see that one may assume the following:

- $O \notin \{P_1, P_2, P_3\}$,
- $\phi|_H(P_1), \phi|_H(P_2),$ and $\phi|_H(P_3)$ are distinct points.

The surface $S$ is a del Pezzo surface. One has 

$$D|_S \sim \mathbb{Q} -2K_S$$

and $-K_S^2 = 1$. The log pair $(S, \lambda D|_S)$ is not log terminal at $P_1, P_2,$ and $P_3$. Theorem 2.5 implies that the sequence 

$$\mathbb{C}^2 = H^0(\mathcal{O}_S(-K_S)) \rightarrow H^0(\mathcal{O}_{\mathcal{L}(S, \lambda D|_S)}) \rightarrow H^1(I(S, \lambda D|_S) \otimes \mathcal{O}_S(-K_S)) = 0$$

is exact, since the scheme $\mathcal{L}(S, \lambda D|_S)$ is zero-dimensional by Lemma 3.3. In particular, the support of the subscheme $\mathcal{L}(S, \lambda D|_S)$ contains at most two points, which is a contradiction. 

So $\mathcal{LCS}(V, \lambda D)$ is zero-dimensional. Theorem 2.6 implies that $\mathcal{LCS}(V, \lambda D)$ consists of a single point $P \in V$.

Lemma 3.5. $P = O$.

Proof. Assume that $P \neq O$. Then the $G$-orbit of $P$ is nontrivial, since so is the $G$-orbit of $\varphi_{|H|(P)}$, which is a contradiction. 

Let $\pi: \overline{V} \rightarrow V$ be a blow-up of the point $O$ with an exceptional divisor $E$; let $\overline{D}$ be a proper transform of $D$ on $\overline{V}$. Then the log pair 

$$(\overline{V}, \lambda \overline{D} + (\lambda \operatorname{mult}_O(D) - 2)E)$$

is not log canonical in the neighborhood of $E$. On the other hand, one has $\operatorname{mult}_O(D) \leq 2$, since otherwise $\operatorname{Supp}(\overline{D})$ would contain all fibers of the elliptic fibration $\varphi|_{\pi^*(H) - E}$. Hence the set 

$$\mathcal{LCS}(\overline{V}, \lambda \overline{D} + (\lambda \operatorname{mult}_O(D) - 2)E)$$

contains some $G$-invariant subvariety $Z \subset E$ and is contained in $E \cong \mathbb{P}^2$.

Lemma 3.6. One has $\dim(Z) = 0$.

Proof. Suppose that $\dim(Z) = 1$. Let $L$ be a general line in $E \cong \mathbb{P}^2$. Then 

$$2 \geq \operatorname{mult}_O(D) = L \cdot \overline{D} \geq \deg(Z) \operatorname{mult}(\overline{D}) \frac{\deg(Z)}{\lambda} > \deg(Z).$$

Hence $Z$ contains a $G$-invariant line. But $|H|$ does not contain $G$-invariant surfaces, which gives a contradiction. 

So we see that the $G$-invariant set $\mathcal{LCS}(\overline{V}, \lambda \overline{D})$ consists of a finite number of points. By Theorem 2.7 the set $\mathcal{LCS}(Y, \lambda \overline{D})$ consists of a single point, since the divisor $-(K_Y + \lambda \overline{D})$ is $\pi$-ample. But $G$ acts on $E$ without fixed points, since $|H|$ contains no $G$-invariant pencils. The contradiction concludes the proof of Theorem 3.2. 

□
4. QUINTIC DEL PEZZO THREEFOLD

Let $V_5$ be a smooth Fano variety such that
\[ \text{Pic}(V_5) = \mathbb{Z}[H] \]
and $H^3 = 5$. One has $-K_{V_5} \sim 2H$ (see, for example, [19]). Let $W \cong \mathbb{C}^3$ be a vector space endowed with a nondegenerate quadratic form $q$. Then the variety $V_5$ is isomorphic to the variety of triples of pairwise orthogonal (with respect to $q$) lines in $W$ (see [19]). In particular, there is a natural action of the group $\text{SO}_3(\mathbb{C})$ (or $\text{SL}_2(\mathbb{C})$) on the variety $V_5$.

**Remark 4.1.** One can show that $\text{Aut}(V_5) = \text{PSL}_2(\mathbb{C})$. By Remark 1.11, to prove Theorem 1.17 it suffices to check that $\text{lct}(V_5, \text{PSL}_2(\mathbb{C})) = 5/6$.

The variety $V_5$ has a natural $\text{PSL}_2(\mathbb{C})$-equivariant stratification:
\[ V_5 = U \cup \Delta \cup C, \]
where $U$ is an open orbit that consists of triples of pairwise distinct lines, $\Delta$ is a two-dimensional orbit that consists of the triples $(l_1, l_1, l_2)$, where $q(l_1, l_1) = 0$ and $q(l_1, l_2) = 0$, and $C$ is a one-dimensional orbit that consists of the triples $(l, l, l)$, where $q(l, l) = 0$.

The linear system $|H|$ defines an embedding $V_5 \subset \mathbb{P}^6$. Under this embedding the curve $C$ is a rational normal curve of degree 6, and $\Delta$ is swept out by the lines that are tangent to $C$.

**Lemma 4.2.** One has $\text{lct}(V_5, \Delta) = 5/6$.

**Proof.** The surface $\Delta$ is smooth outside $C$ and has a singularity along $C$ that is locally isomorphic to $T \times \Delta^1$, where $T$ is a germ of a cuspidal curve. □

In particular, $\text{lct}(V_5, \text{PSL}_2(\mathbb{C})) \leq 5/6$.

**Lemma 4.3.** Let $D \subset |nH|$ be a $\text{PSL}_2(\mathbb{C})$-invariant linear system on $V_5$ such that $\Delta \not\subset \text{Bs}(D)$. Then $\text{lct}(X, D) \geq 1/n$.

**Proof.** Suppose that $\text{lct}(X, D) < 1/n$. Then there exists a $\text{PSL}_2(\mathbb{C})$-invariant subvariety $Z \subset X$ such that
\[ \text{mult}_Z(D) > n, \]
where $D$ is a general divisor in $D$. Since $\Delta \not\subset \text{Bs}(D)$, the subvariety $Z$ is the curve $C$. Let $P$ be a general point of $C$ and $L$ be the tangent line to $C$ at $P$. Then $L \not\subset \text{Supp}(D)$. By Lemma 2.11 one has
\[ 2n = D \cdot L \geq \text{mult}_P(D \cdot L) > 2n, \]
which is a contradiction. □

Lemmas 2.10, 4.2, and 4.3 imply that $\text{lct}(V_5, \text{PSL}_2(\mathbb{C})) \geq 5/6$, and hence $\text{lct}(V_5, \text{PSL}_2(\mathbb{C})) = 5/6$.

5. THE MUKAI–UMEMURA THREEFOLD

Let $X$ be a smooth Fano threefold such that
\[ \text{Pic}(X) = \mathbb{Z}[-K_X], \]
the equality $-K_X^3 = 22$ holds, and $\text{Aut}(X) \cong \text{PSL}_2(\mathbb{C})$. It is well known that the variety having such properties is unique (see [22, 3]).

**Proposition 5.1.** The equality $\text{lct}(X) = 1/2$ holds.

**Proof.** Let $U \subset \mathbb{C}[x, y]$ be a subspace of forms of degree 12. Consider $U \cong \mathbb{C}^{13}$ as the affine part of $\mathbb{P}(U \oplus \mathbb{C}) \cong \mathbb{P}^{15}$, and let us identify $\mathbb{P}(U)$ with the hyperplane at infinity.
The natural action of $\text{SL}(2, \mathbb{C})$ on $\mathbb{C}[x, y]$ induces an action on $\mathbb{P}(U \oplus \mathbb{C})$. Put
\[
\varphi = xy(x^{10} - 11x^5y^5 - y^{10}) \in U
\]
and consider the closure $\overline{\text{SL}(2, \mathbb{C}) \cdot [\varphi + 1]} \subset \mathbb{P}(U \oplus \mathbb{C})$. It follows from [22] that
\[
X \cong \text{SL}(2, \mathbb{C}) \cdot [\varphi + 1],
\]
and the natural embedding $X \subset \mathbb{P}(U \oplus \mathbb{C}) \cong \mathbb{P}^{13}$ is induced by $-K_X$.

It is well known (see [19, Theorem 5.2.13]) that the action of $\text{SL}(2, \mathbb{C})$ on $X$ has the following orbits:
- the three-dimensional orbit $\Sigma_3 = \text{SL}(2, \mathbb{C}) \cdot [\varphi + 1]$;
- the two-dimensional orbit $\Sigma_2 = \text{SL}(2, \mathbb{C}) \cdot [xy^{11}]$;
- the one-dimensional orbit $\Sigma_1 = \text{SL}(2, \mathbb{C}) \cdot [y^{12}]$.

The orbit $\Sigma_3$ is open. The orbit $\Sigma_1 \cong \mathbb{P}^1$ is closed. One has $\Sigma_2 = \Sigma_1 \cup \Sigma_2$, and
\[
X \cap \mathbb{P}(U) = \Sigma_1 \cup \Sigma_2.
\]

Put $R = X \cap \mathbb{P}(U)$. It follows from [22] that
- the surface $R$ is swept out by lines on $X \subset \mathbb{P}^{13}$,
- the surface $R$ contains all lines on $X \subset \mathbb{P}^{13}$,
- for any lines $L_1 \subset R \supset L_2$ such that $L_1 \neq L_2$, one has $L_1 \cap L_2 = \emptyset$,
- the surface $R$ is singular along the orbit $\Sigma_1 \cong \mathbb{P}^1$,
- the normalization of the surface $R$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$,
- for every point $P \in \Sigma_1$, the surface $R$ is locally isomorphic to
\[
x^2 = y^3 \subset \mathbb{C}^3 \cong \text{Spec}(\mathbb{C}[x, y, z]),
\]
which implies that $\text{lc}(X, R) = 5/6$.

The structure of the surface $R$ can be described as follows. We see that
\[
\Sigma_2 = \{(ax + by)(cx + dy)^{11} \mid ad - bc = 1\} \subset \mathbb{P}(U),
\]
which implies that there is a birational morphism $\nu: \mathbb{P}^1 \times \mathbb{P}^1 \to R$ that is defined by
\[
\nu: [a : b] \times [c : d] \mapsto [(ax + by)(cx + dy)^{11}] \in R,
\]
which is a normalization of the surface $R$.

Let $V_5$ be a smooth Fano threefold such that
\[
-K_{V_5} \sim 2H
\]
and $H^3 = 5$, where $H$ is a Cartier divisor on $V_5$. Then $|H|$ induces an embedding $V_5 \subset \mathbb{P}^6$ (see Section 4).

Let $L \cong \mathbb{P}^1$ be a line on $X$. Then
\[
\mathcal{N}_{L/X} \cong \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(1).
\]
Let $\alpha_L: U_L \to X$ be a blow-up of the line $L$, and let $E_L$ be the exceptional divisor of $\alpha_L$. Then it follows from Theorem 4.3.3 in [19] that there is a commutative diagram

\[
\begin{array}{ccc}
U_L & \xrightarrow{\rho_L} & W_L \\
\downarrow{\alpha_L} & & \downarrow{\beta_L} \\
X & \xrightarrow{\psi_L} & V_5
\end{array}
\]

where $\rho_L$ is a flop in the exceptional section of $E \cong \mathbb{P}_3$, the morphism $\beta_L$ contracts a surface $D_L \subset W_L$ to a smooth rational curve of degree 5, and $\psi_L$ is a double projection from the line $L$.

Let $\overline{D}_L \subset X$ be the proper transform of the surface $D_L$. Then $\text{mult}_L(\overline{D}_L) = 3$ and $\overline{D}_L \sim -K_X$.

It follows from [15] that $X \setminus \overline{D}_L \cong \mathbb{C}^3$.

It follows from [16] that there is an open subset $\tilde{D}_L \subset \overline{D}_L$ that is given by

\[
\mu_0 x^4 + (\mu_1 y z + \mu_2 z^3) x^3 + (\mu_3 y^3 + \mu_4 y^2 z^2 + \mu_5 y z^4) x^2 + (\mu_6 y^4 z + \mu_7 y^3 z^3) x + \mu_8 y^6 + \mu_9 y^5 z^2 = 0
\]

in $\mathbb{C}^3 \cong \text{Spec}(\mathbb{C}[x, y, z])$, where the point $L \cap \Sigma_1 \in \tilde{D}_L$ is given by the equations $x = y = z = 0$ and $\mu_0 = -2^8 \times 5^2$, $\mu_1 = 2^9 \times 3^3 \times 5$, $\mu_2 = -2^6 \times 3^4 \times 5$, $\mu_3 = -2^8 \times 3^3 \times 7$, $\mu_4 = -2^4 \times 3^4 \times 127$, $\mu_5 = 2^9 \times 3^5$, $\mu_6 = 2^2 \times 3^6 \times 89$, $\mu_7 = -2^8 \times 3^6$, $\mu_8 = -3^6 \times 5^3$, $\mu_9 = 2^5 \times 3^7$.

Put $O_L = \Sigma_1 \cap L$. Then $\text{mult}_{O_L}(\overline{D}_L) = 4$, and it follows from [20, Proposition 8.14] that

\[
\text{LCS}(X, \frac{1}{2} \overline{D}_L) = O_L
\]

and let $(X, \overline{D}_L) = 1/2$. Thus, we see that $\text{let}(X) \leq 1/2$.

Suppose that $\text{let}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor

\[
D \sim_{\mathbb{Q}} -K_X
\]

such that the log pair $(X, \lambda D)$ is not log canonical for some positive rational number $\lambda < 1/2$. By Remark 2.9, we may assume that $R \not\subset \text{Supp}(D)$, because $\text{let}(X, R) = 5/6$.

Let $C$ be a line in $X$ such that $C \not\subset \text{Supp}(D)$. Then

\[
1 = D \cdot C \geq \text{mult}_{O_C}(D) \text{mult}_{O_C}(C) = \text{mult}_{O_C}(D),
\]

which implies that $O_C \not\in \text{LCS}(X, \lambda D)$. In particular, we see that $\Sigma_1 \not\in \text{LCS}(X, \lambda D)$.

Let $\Gamma$ be an irreducible curve in $\text{Supp}(D)$ such that $O_C \in \Gamma$. Then

\[
\text{mult}_{\Gamma} \left( \frac{1}{2} \overline{D}_C + \lambda D \right) = \frac{\text{mult}_{\Gamma}(\overline{D}_C)}{2} + \lambda \text{mult}_{\Gamma}(D) \leq \frac{\text{mult}_{\Gamma}(\overline{D}_C)}{2} + \lambda \text{mult}_{O_C}(D) < 1,
\]

because $\lambda < 1/2$ and $\text{Sing}(\overline{D}_C) = C$, because $\overline{D}_C \neq R$. Thus, we see that

\[
\Gamma \not\subset \text{LCS}(X, \frac{1}{2} \overline{D}_C + \lambda D) \supset \text{LCS}(X, \lambda D) \cup O_C,
\]

which is impossible by Theorem 2.6, because $O_C \not\in \text{LCS}(X, \lambda D)$ and $\lambda < 1/2$. \[\square\]
Remark 5.2. It follows from [14] that
\[
lct(X, SO_3(\mathbb{R})) = \frac{5}{6},
\]
which implies, in particular, that \(X\) has a Kähler–Einstein metric. This equality can be obtained by arguing as in the proof of Theorem 1.17 (the only difference is that we do not need to use Lemma 2.11 here).

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REFERENCES

1. M. M. Grinenko, “On a Double Cone over a Veronese Surface,” Izv. Ross. Akad. Nauk, Ser. Mat. 67 (3), 5–22 (2003) [Izv. Math. 67, 421–438 (2003)].
2. M. M. Grinenko, “Mori Structures on a Fano Threefold of Index 2 and Degree 1,” Tr. Mat. Inst. im. V.A. Steklova, Ross. Akad. Nauk 246, 116–141 (2004) [Proc. Steklov Inst. Math. 246, 103–128 (2004)].
3. Yu. G. Prokhorov, “Automorphism Groups of Fano Manifolds,” Usp. Mat. Nauk 45 (3), 195–196 (1990) [Russ. Math. Surv. 45, 222–223 (1990)].
4. A. V. Pukhlikov, “Birational Geometry of Fano Direct Products,” Izv. Ross. Akad. Nauk, Ser. Mat. 69 (6), 153–186 (2005) [Izv. Math. 69, 1225–1255 (2005)].
5. I. A. Cheltsov, “Log Canonical Thresholds on Hypersurfaces,” Mat. Sb. 192 (8), 155–172 (2001) [Sb. Math. 192, 1241–1285 (2001)].
6. I. Cheltsov, “Extremal Metrics on Two Fano Varieties,” Mat. Sb. 200 (1), 97–136 (2009).
7. I. A. Cheltsov and K. A. Shramov, “Log Canonical Thresholds of Smooth Fano Threefolds,” Usp. Mat. Nauk 63 (5), 73–138 (2008) [Russ. Math. Surv. 63, 859–958 (2008)].
8. V. V. Shokurov, “3-fold Log Flips,” Izv. Ross. Akad. Nauk, Ser. Mat. 56 (1), 105–203 (1992) [Russ. Acad. Sci., Izv. Math. 40, 95–202 (1993)].
9. C. Arezzo, A. Ghigi, and G. P. Pirola, “Symmetries, Quotients and Kähler–Einstein Metrics,” J. Reine Angew. Math. 591, 177–200 (2006).
10. I. Cheltsov, “Fano Varieties with Many Selfmaps,” Adv. Math. 217, 97–124 (2008).
11. I. Cheltsov, “Log Canonical Thresholds of del Pezzo Surfaces,” Geom. Funct. Anal. 18, 1118–1144 (2008).
12. I. Cheltsov, J. Park, and J. Won, “Log Canonical Thresholds of Certain Fano Hypersurfaces,” arXiv:0706.0751.
13. J.-P. Demailly and J. Kollár, “Semi-continuity of Complex Singularity Exponents and Kähler–Einstein Metrics on Fano Orbifolds,” Ann. Sci. Éc. Norm. Supér. 34, 525–556 (2001).
14. S. K. Donaldson, “A Note on the \(\alpha\)-Invariant of the Mukai–Umemura 3-fold,” arXiv:0711.4357.
15. M. Furushima, “Complex Analytic Compactifications of \(\mathbb{C}^3\),” Compos. Math. 76, 163–196 (1990).
16. M. Furushima, “Mukai–Umemura’s Example of the Fano Threefold with Genus 12 as a Compactification of \(\mathbb{C}^3\),” Nagoya Math. J. 127, 145–165 (1992).
17. J.-M. Hwang, “Log Canonical Thresholds of Divisors on Fano Manifolds of Picard Number 1,” Compos. Math. 143, 89–94 (2007).
18. A. R. Iano-Fletcher, “Working with Weighted Complete Intersections,” in Explicit Birational Geometry of 3-folds (Cambridge Univ. Press, Cambridge, 2000), LMS Lect. Note Ser. 281, pp. 101–173.
19. V. A. Iskovskikh and Yu. G. Prokhorov, Fano Varieties (Springer, Berlin, 1999), Encycl. Math. Sci. 47.
20. J. Kollár, “Singularities of Pairs,” Proc. Symp. Pure Math. 62, 221–287 (1997).
21. R. Lazarsfeld, Positivity in Algebraic Geometry (Springer, Berlin, 2004), Vol. 2.
22. S. Mukai and H. Umemura, “Minimal Rational Threefolds,” in Algebraic Geometry (Springer, Berlin, 1983), Lect. Notes Math. 1016, pp. 490–518.

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23. A. Nadel, “Multiplier Ideal Sheaves and Kähler–Einstein Metrics of Positive Scalar Curvature,” Ann. Math. 132, 549–596 (1990).

24. Yu. G. Prokhorov and D. Markushevich, “Exceptional Quotient Singularities,” Am. J. Math. 121, 1179–1189 (1999).

25. G. Tian, “On Kähler–Einstein Metrics on Certain Kähler Manifolds with $C_1(M) > 0$,” Invent. Math. 89, 225–246 (1987).

26. X.-J. Wang and X. Zhu, “Kähler–Ricci Solitons on Toric Manifolds with Positive First Chern Class,” Adv. Math. 188, 87–103 (2004).

27. Q. Zhang, “Rational Connectedness of Log $Q$-Fano Varieties,” J. Reine Angew. Math. 590, 131–142 (2006).

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