On the fattening of ACM arrangements of codimension 2 subspaces in $\mathbb{P}^N$

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Abstract

In the present note we study configurations of codimension 2 flats in projective spaces and classify those with the smallest rate of growth of the initial sequence. Our work extends those of Bocci, Chiantini in $\mathbb{P}^2$ and Janssen in $\mathbb{P}^3$ to projective spaces of arbitrary dimension.

Keywords ACM subschemes, Symbolic powers, Star configurations, Pseudo-star configurations.

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1 Introduction

A homogeneous ideal $I \subset R = \mathbb{K}[\mathbb{P}^N]$ in the ring of polynomials with coefficients in a field $\mathbb{K}$, decomposes as the direct sum of graded parts $I = \oplus_{t \geq 0} I_t$. For a nontrivial homogeneous ideal $I$ in $\mathbb{K}[\mathbb{P}^N]$, the initial degree $\alpha(I)$ of $I$ is the least integer $t$ such that $I_t \neq 0$.

For a positive integer $m$, the $m^{th}$ symbolic power $I^{(m)}$ of $I$ is defined as

$$I^{(m)} = \bigcap_{P \in \text{Ass}(I)} (I^m R_P \cap R),$$

where Ass($I$) is the set of associated primes of $I$ and the intersection takes place in the field of fractions of $R$.

We define the initial sequence of $I$ as the sequence of integers $\alpha_m = \alpha(I^{(m)})$. If $I$ is a radical ideal determined by the vanishing along a closed subscheme $Z \subset \mathbb{P}^N$, then the Nagata-Zariski Theorem [10, Theorem 3.14] provides a nice geometric interpretation of symbolic powers of $I$, namely $I^{(m)}$ is the ideal of polynomials vanishing to order at least $m$ along $Z$. This implies, in particular, that the initial sequence is strictly increasing.

The study of the relationship between the sequence of initial degrees of symbolic powers of homogeneous ideals and the geometry of the underlying algebraic sets in projective spaces has been initiated by Bocci and Chiantini in [3]. They proved that if $Z \subset \mathbb{P}^2$ is a finite set of points and $I = I(Z)$ is the vanishing ideal of $Z$, then the equality

$$\alpha(I^{(2)}) = \alpha(I) + 1$$

implies that either all points in $Z$ are collinear or they form a star configuration (see Definition 2.1).
This result has been considerably generalized in several directions. Dumnicki, Tutaj-Gasińska and the third author studied higher symbolic powers of ideals supported on points in [8] and [9]. Natural analogies of the problem have been studied on $\mathbb{P}^1 \times \mathbb{P}^1$ in [1] and on Hirzebruch surfaces in general in [5]. Bauer and the third author proposed in [2] the following conjecture for points in higher dimensional projective spaces and they proved it for $N = 3$.

**Conjecture.** [Bauer, Szemberg] Let $Z$ be a finite set of points in the projective space $\mathbb{P}^N$ and let $I$ be the radical ideal defining $Z$. If

$$d := \alpha(I^{(N)}) = \alpha(I) + N - 1,$$

then either $\alpha(I) = 1$ and the set $Z$ is contained in a single hyperplane or $Z$ is a star configuration of codimension $N$ associated to $d$ hyperplanes in $\mathbb{P}^N$.

In recent years many problems stated originally for points in projective spaces have been generalized to arrangements of flats, see e.g. [12, 7, 14, 15, 6]. In particular, Janssen in [13] generalized results of Bocci and Chiantini to configurations of lines in $\mathbb{P}^3$ defined by homogeneous Cohen-Macaulay ideals. Symbolic powers of codimension 2 Cohen-Macaulay ideals have been studied recently in [4]. Results of these two articles, especially Section 3 in [13], have motivated our research presented here. Our main result is the following, (see Definition 2.4, for the definition of a pseudo-star configuration).

**Theorem 1.1** (Main result). Let $\mathbb{L}$ be the union of a finite set of codimension 2 projective subspaces in $\mathbb{P}^N$ and let $J$ be its vanishing ideal. If $J$ is Cohen-Macaulay and

$$d := \alpha(J^{(2)}) = \alpha(J) + 1,$$

then $\mathbb{L}$ is either contained in a single hyperplane or it is a codimension 2 pseudo-star configuration determined by $d$ hypersurfaces.

Throughout this note we work over a field $K$ of characteristic zero.

## 2 Preliminaries

The term "star configuration" has been coined by Geramita. It is motivated by the observation that five general lines in $\mathbb{P}^2$ resemble a pentagram. The objects defined below have appeared in recent years in various guises in algebraic geometry, commutative algebra and combinatorics, see [11] for a throughout account.

**Definition 2.1** (Star configuration). Let $\mathcal{H} = \{H_1, \ldots, H_s\}$ be a collection of $s \geq 1$ mutually distinct hyperplanes in $\mathbb{P}^N$ defined by linear forms $\{h_1, \ldots, h_s\}$. We assume that the hyperplanes meet properly, i.e., the intersection of any $c$ of them is either empty or has codimension $c$, where $c$ is any integer in the range $1 \leq c \leq \min \{s, N\}$. The union

$$S(c, \mathcal{H}) = \bigcup_{1 \leq i_1 < \ldots < i_c \leq s} H_{i_1} \cap \ldots \cap H_{i_c}$$

is the codimension $c$ star configuration associated to $\mathcal{H}$. We have

$$I(c, \mathcal{H}) = I(S(c, \mathcal{H})) = \bigcap_{1 \leq i_1 < \ldots < i_c \leq s} (h_{i_1}, \ldots, h_{i_c}).$$

where $h_i, i = 1, \ldots, s$ are linear forms in $R$, defining the hyperplanes $H_i$. 

The condition of meeting properly is satisfied by a collection of general hyperplanes. If the collection \( \mathcal{H} \) is clear from the context or irrelevant, we write\[ S_N(c, s) \]
to denote a codimension \( c \) star configuration determined by \( s \) hyperplanes in \( \mathbb{P}^N \).

For the purpose of this note, it is convenient to use the following terminology: an \( r \)-flat in a projective space is a linear subspace of (projective) dimension \( r \). Thus a codimension \( c \) star configuration determined by \( s \) hyperplanes is the union of \( \binom{s}{c} \) distinguished \((N-c)\)-flats.

The following notion is essential for our arguments.

**Definition 2.2** (Cohen-Macaulay). A noetherian local ring \((R, m)\) is called Cohen-Macaulay, if\[ \text{depth}_m R = \dim(R). \]

A noetherian ring \( R \) is Cohen-Macaulay (CM) if all of its local rings at prime ideals are Cohen-Macaulay.

A closed subscheme \( Z \subseteq \mathbb{P}^N \) with defining ideal \( I(Z) \), is called arithmetically Cohen-Macaulay (ACM for short) if its coordinate ring \( \mathbb{k}[\mathbb{P}^N]/I(Z) \) is CM.

By [11, Proposition 2.9] every star configuration is ACM.

The following feature of ACM subschemes makes them particularly suited for inductive arguments.

**Proposition 2.3.** Let \( Z \subseteq \mathbb{P}^N \) be an ACM subscheme of dimension at least 1, and let \( H \subseteq \mathbb{P}^N \) be a general hyperplane. Then the intersection scheme \( Z \cap H \) is ACM and\[ \alpha(I(Z)) = \alpha(I(Z \cap H)). \]

**Proof.** A general hyperplane section of any curve is ACM, because all subschemes of dimension zero are ACM. For general hyperplane sections of higher dimensional subschemes see [16, Theorem 1.3.3]. The second claim follows from [16, Corollary 1.3.8] and some basic properties of postulation. \(\square\)

In [13] Janssen introduced the notion of a pseudo-star configuration for lines in \( \mathbb{P}^3 \). In the present note we extend this notion to higher dimensional flats in projective spaces of arbitrary dimension. To begin with, note that if \( \mathcal{H} = \{H_1, \ldots, H_s\} \) is a collection of \( s > N \) hyperplanes in \( \mathbb{P}^N \), then the assumption that they intersect properly (see Definition 2.1) is equivalent to assuming that any \((N+1)\) of them have an empty intersection. For our purposes, we need to weaken this condition, i.e.

**Definition 2.4** (Pseudo-star configuration). Let \( \mathcal{H} = \{H_1, \ldots, H_s\} \) be a collection of hyperplanes in \( \mathbb{P}^N \) and let \( 1 \leq c \leq N \) be a fixed integer. We assume that the intersection of any \( c+1 \) of hyperplanes in \( \mathcal{H} \) has codimension \( c+1 \) (equivalently: no \( c+1 \) hyperplanes in \( \mathcal{H} \) have the same intersection as any \( c \) of them). The union\[ P(c, \mathcal{H}) = \bigcup_{1 \leq i_1 < \ldots < i_c \leq s} H_{i_1} \cap \ldots \cap H_{i_c} \]
is called the codimension \( c \) pseudo-star configuration determined by \( \mathcal{H} \).
If \( H \) is clear from the context or irrelevant, we write \( P_N(c, s) \) for a codimension \( c \) pseudo-star configuration in \( \mathbb{P}^N \) determined by \( s \) hypersurfaces. Of course, any star configuration is a pseudo-star configuration. If \( N = c = 2 \), then also the converse holds, i.e., any pseudo-star configuration of points in \( \mathbb{P}^2 \) is a star configuration. In general the two notions go apart, see Section 3. Moreover, being a pseudo-star configuration is stable under taking cones over the configuration.

**Remark 2.5.** Let \( I \subset K[\mathbb{P}^N] \) be an ideal of a codimension \( c \) pseudo-star configuration \( P_N(c, s) \). Then the extension of the ideal \( I \) to \( K[\mathbb{P}^N+1] \) defines a \( P_{N+1}(c, s) \).

The construction known in the Liaison Theory as the *Basic Double Linkage* (see [16, chapter 4]) was used in [11] to prove some basic properties of star configuration. These properties are also satisfied for pseudo-star configurations as the following proposition shows.

**Proposition 2.6.** Let \( \mathcal{H} = \{H_1, \ldots, H_s\} \) be a collection of mutually distinct hyperplanes in \( \mathbb{P}^N \) such that any \( c+1 \) of them intersect in a subspace of codimension \( c+1 \). Let \( P(c, \mathcal{H}) \) be the associated codimension \( c \) pseudo-star configuration and let \( I \) be its vanishing ideal. Then:

1) \( \deg P(c, \mathcal{H}) = \binom{s}{c} \);
2) \( P(c, \mathcal{H}) \) is ACM;
3) \( I^{(m)} \) is CM for all \( 1 \leq m \leq c \);
4) \( \alpha(I) = s - c + 1 \) and all minimal generators of \( I \) occur in this degree.

**Proof.** According to the definition of a pseudo-star configuration 1) is obvious. Properties 2) and 4) were proved in [11, Proposition 2.9] (see also [11, Remark 2.13]). Symbolic powers of an ideal defining a pseudo-star configuration are Cohen-Macaulay by the first part of the proof of [11, Theorem 3.2]. Note that Proposition 2.9 and Theorem 3.2 in [11] are stated for star configuration but the assumption that the hyperplanes meet properly can be relaxed to the assumption in the mentioned Proposition. Hence the proof of that Proposition, works for pseudo-stars too. \( \square \)

**Remark 2.7.** Since the ideal of every linear subspace in a projective space is a complete intersection, by unmixedness theorem, we can describe the \( m^{th} \) symbolic power of a pseudo-star configuration in a straightforward manner. In fact, let \( \mathcal{H} = \{H_1, \ldots, H_s\} \) be a pseudo-star configuration in \( \mathbb{P}^N \), defined by the linear forms \( h_1, \ldots, h_s \in R \). Let \( c \geq 1 \) be a fixed integer. Then

\[
I = \bigcap_{1 \leq i_1 < \ldots < i_c \leq s} (h_{i_1}, \ldots, h_{i_c})
\]

is the defining ideal of codimension \( c \) pseudo-star configuration associated to \( \mathcal{H} \). Then by unmixedness theorem, for any positive integer \( m \), one has

\[
I^{(m)} = \bigcap_{1 \leq i_1 < \ldots < i_c \leq s} (h_{i_1}, \ldots, h_{i_c})^m.
\]

We use this description, to compute the second symbolic powers of the ideals in the following section.
3 Examples

In this section, we assume $R$ is $\mathbb{K}[\mathbb{P}^3]$.

**Example 3.1** (Star configurations). Let $\mathcal{H} = \{H_1, H_2, H_3, H_4\}$ be a collection of hyperplanes in $\mathbb{P}^3$ defined by the following linear forms

\[ h_1 = x + 2y + 3z, \quad h_2 = x + y + w, \quad h_3 = x + z + w, \quad h_4 = y + z + w. \]

These hyperplanes meet properly. Let $I$ be the ideal of the star configuration of lines $S(2, \mathcal{H})$ and let $J$ be the ideal of the associated star configuration of points $S(3, \mathcal{H})$. The minimal free resolutions of $I, I^{(2)}, J$ and $J^{(2)}$ respectively are as follows

\[
0 \to R^3(-4) \to R^4(-3) \to I \to 0,
\]

\[
0 \to R^4(-7) \to R(-4) \oplus R^4(-6) \to I^{(2)} \to 0,
\]

\[
0 \to R^3(-4) \to R^8(-3) \to R^6(-2) \to J \to 0,
\]

\[
0 \to R^6(-6) \to R^3(-4) \oplus R^{12}(-5) \to R^4(-3) \oplus R^6(-4) \to J^{(2)} \to 0.
\]

We see immediately that $\alpha(I) = 3$ and $\alpha(I^{(2)}) = 4$. Similarly for the ideal $J$ we have $\alpha(J) = 2$ and $\alpha(J^{(2)}) = 3$.

Our next example, is a pseudo-star configuration in $\mathbb{P}^3$.

**Example 3.2** (A pseudo-star configuration). Let $\mathcal{H} = \{H_1, H_2, H_3, H_4\}$ be a collection of hyperplanes in $\mathbb{P}^3$ defined by the following linear forms

\[ h_1 = x + 5z, \quad h_2 = 17x + 19y, \quad h_3 = 2x + 3y + 11z, \quad h_4 = 13x + 7z. \]

These hyperplanes do not meet properly but the intersection of any three of them has codimension 3 and they all intersect in the point $P = (0 : 0 : 0 : 1)$. Let $I$ be the ideal of the pseudo-star configuration of lines $P(2, \mathcal{H})$. The minimal free resolutions of $I$ and $I^{(2)}$ are

\[
0 \to R^3(-4) \to R^4(-3) \to I \to 0,
\]

\[
0 \to R^4(-7) \to R(-4) \oplus R^4(-6) \to I^{(2)} \to 0.
\]

Now we have also $\alpha(I) = 3$ and $\alpha(I^{(2)}) = 4$.

Note that the codimension 3 pseudo-star configuration defined by the ideal $J = (h_1, h_2, h_3) \cap (h_1, h_2, h_4) \cap (h_1, h_3, h_4) \cap (h_2, h_3, h_4)$ is now just a single point $\{P\}$, i.e. its defining ideal in $\mathbb{K}[x, y, z, w]$ is $J = (x, y, z)$. 
4 Proof of the Main Result

In the course of proving the main result of this note, the following lemma plays a crucial role. In fact, it is a higher dimensional analogue of [13, Proposition 2.10].

**Lemma 4.1.** If a collection of \((N - 2)\)-planes in \(\mathbb{P}^N\) with \(N \geq 4\) is not contained in a hyperplane in \(\mathbb{P}^N\) (so that there are in particular at least \(t \geq 2\) such planes), then its intersection with a general hyperplane \(H \subset \mathbb{P}^N\) is not contained in a hyperplane in \(H\).

**Proof.** It suffices to prove the statement for \(t = 2\). Let \(U, V\) be \((N - 2)\)-planes in \(\mathbb{P}^N\). By the dimension formula

\[
\dim \langle U, V \rangle = \dim U + \dim V - \dim (U \cap V)
\]

and by the assumption that \(U\) and \(V\) span \(\mathbb{P}^N\), we have

\[
N = N - 2 + N - 2 - \dim (U \cap V),
\]

so that \(\dim (U \cap V) = N - 4\). Since \(N \geq 4\) by assumption, the intersection \(U \cap V\) is non-empty. With the usual convention that the dimension of the empty set equals \(-1\), we have for a general hyperplane \(H\)

\[
\begin{align*}
\dim (U \cap H) &= \dim U - 1 = N - 3, \\
\dim (V \cap H) &= \dim V - 1 = N - 3, \\
\text{and } \quad \dim ((U \cap V) \cap H) &= \dim (U \cap V) - 1 = N - 5.
\end{align*}
\]

Hence

\[
\dim \langle U \cap H, V \cap H \rangle = N - 3 + N - 3 - N + 5 = N - 1.
\]

This means that \((U \cap H)\) and \((V \cap H)\) span \(H\) and we are done. \(\square\)

**Remark 4.2.** The above proof fails for two lines in \(\mathbb{P}^3\). This is the reason that the argument in [13] is somewhat more involved. In fact, the proof of \(N = 3\) seems the most difficult case, contrary to what one might naively expect.

We are now in the position to prove our main result.

**Proof of Theorem 1.1.** Let \(L = L_1 \cup \ldots \cup L_t\) be the union of \((N - 2)\)-flats in \(\mathbb{P}^N\) such that the initial sequence \(\alpha_m\) of the vanishing ideal \(J\) of \(L\) satisfies

\[
d := \alpha_2 = \alpha_1 + 1
\]

and \(J\) is CM.

To prove our claim, we proceed by induction on \(N\). For \(N = 2\) see [3, Theorem 1.1], and for \(N = 3\) see [13, Theorem 2.13]. Moreover, if \(t = 1\), then the claim is clear so we can assume \(N \geq 4\) and \(t \geq 2\). If \(L\) is contained in a hyperplane, then there is nothing to prove. So we assume that \(L\) spans the space \(\mathbb{P}^N\). Let \(H\) be a general hyperplane in \(\mathbb{P}^N\). Then, the intersection \(L_H = L \cap H\) can be represented as

\[
L_H = (L_1 \cap H) \cup \ldots \cup (L_t \cap H).
\]

Since \(H\) is general, \(\dim (L_i \cap H) = N - 3\) for all \(i = 1, \ldots, t\). By Proposition 2.3 the ideal \(J_H\) of \(L_H\) is CM and its initial sequence \(\beta_m = \alpha(J_H^{(m)})\) satisfies

\[
d = \beta_2 = \beta_1 + 1.
\]
By the induction assumption, \( L_H \) is a codimension two pseudo-star configuration determined by hypersurfaces \( F_1, \ldots, F_d \) in \( H \). Indeed, \( L_H \) cannot be contained in a hyperplane since otherwise, by Lemma \[4.1\] \( L \) would be contained in a hyperplane. The hypersurface \( F_1 \) contains its intersections with the remaining \((d - 1)\) hypersurfaces \( F_2, \ldots, F_d \). These intersections are traces of some of the \((N - 2)\)-flats \( L_1, \ldots, L_d \). There are exactly \( \binom{d}{2} \) intersections among \( F_i \)'s by 1) in Proposition [2.6]. There must be exactly as many traces so that \( t = \binom{d}{2} \). Since the intersections of \( F_1 \) with \( F_2, \ldots, F_d \) are by definition contained in \( F_1 \), a hyperplane in \( H \), and they are on the other hand intersections of some \( L_i \)'s with \( H \), the corresponding \( L_i \)'s must be themselves contained in a hyperplane, say \( H_1 \) in \( \mathbb{P}^N \). Permuting the indices we obtain hypersurfaces \( H_1, \ldots, H_d \) in \( \mathbb{P}^N \) such that

\[
F_i = H_i \cap H \quad \text{for} \quad i = 1, \ldots, d.
\]

Since every \((N - 3)\)-flat \((L_i \cap H)\) is contained in exactly two of \( F_i \)'s (by the definition of a codimension two pseudo-star configuration), every \( L_i \) must be contained in at least two of \( H_i \)'s. But there are \( \binom{d}{2} \) of \( L_i \)'s and at most that many pair intersections among the \( H_i \)'s. Hence every \( L_i \) is contained in exactly two of the \( H_i \)'s. This shows that \( L \) is the codimension two pseudo-star configuration determined by \( H_1, \ldots, H_d \) and we are done.

We complete the picture by showing that the converse statement holds for arbitrary codimension two pseudo-star configurations.

**Theorem 4.3.** Let \( L \) be the union of \((N - 2)\)-flats \( L_1, \ldots, L_t \) with the vanishing ideal \( J \). If \( L \) is

a) contained in a hyperplane, then the initial sequence of its vanishing ideal is

\[
1, 2, 3, 4, \ldots;
\]

b) a \( P_{N}(2, s) \), then the initial sequence of its vanishing ideal is

\[
s - 1, s, 2s - 1, 2s, 3s - 1, 3s, \ldots.
\]

**Proof.** In case a) there is nothing to prove since the initial sequence is strictly increasing.

In case b) we have, to begin with, \( t = \binom{s}{2} \) for some \( s \). Taking subsequent sections by general hyperplanes \( H_1, \ldots, H_{N - 2} \) we arrive to a pseudo-star (hence a star) configuration of \( \binom{s}{2} \) points in \( \mathbb{P}^2 \). In this case [3, Proposition 3.2] implies that \( \alpha(J, H_1, \ldots, H_{N - 2}) = s - 1 \). This implies

\[
\alpha(J) \geq s - 1. \tag{1}
\]

On the other hand the union of \( s \) hyperplanes in \( \mathbb{P}^N \) vanishes to order two along \( L \), so that \( \alpha(J^{(2)}) \leq s \). Combining this with (1) we obtain \( \alpha(J) = s - 1 \) and \( \alpha(J^{(2)}) = s \). The argument for higher symbolic powers is similar and we leave the details to the reader.

In the view of our results, it is natural to conclude this note with the following challenge.

**Problem 4.4.** Is there any codimension two pseudo-star configuration which is not ACM?
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References

[1] M. Baczynska, M. Dumnicki, A. Habura, G. Malara, P. Pokora, T. Szemberg, J. Szpond, and H. Tutaj-Gasińska. Points fattening on $\mathbb{P}^1 \times \mathbb{P}^1$ and symbolic powers of bi-homogeneous ideals. *J. Pure Appl. Algebra*, 218(8):1555–1562, 2014.

[2] T. Bauer and T. Szemberg. The effect of points fattening in dimension three. In *Recent advances in algebraic geometry*, volume 417 of *London Math. Soc. Lecture Note Ser.*, pages 1–12. Cambridge Univ. Press, Cambridge, 2015.

[3] C. Bocci and L. Chiantini. The effect of points fattening on postulation. *J. Pure Appl. Algebra*, 215(1):89–98, 2011.

[4] S. Cooper, G. Fatabbi, E. Guardo, A. Lorenzini, J. Migliore, U. Nagel, A. Sceceleanu, J. Szpond, and A. Van Tuyl. Symbolic powers of codimension two Cohen-Macaulay ideals, arXiv:1606.00935.

[5] S. Di Rocco, A. Lundman, and T. Szemberg. The effect of points fattening on Hirzebruch surfaces. *Math. Nachr.*, 288(5-6):577–583, 2015.

[6] M. Dumnicki, M. Fashami, J. Szpond, and H. Tutaj-Gasińska. Lower bounds for Waldschmidt constants of generic lines in $\mathbb{P}^3$ and a Chudnovsky-type theorem, arXiv:1803.02387.

[7] M. Dumnicki, B. Harbourne, T. Szemberg, and H. Tutaj-Gasińska. Linear subspaces, symbolic powers and Nagata type conjectures. *Adv. Math.*, 252:471–491, 2014.

[8] M. Dumnicki, T. Szemberg, and H. Tutaj-Gasińska. Symbolic powers of planar point configurations. *J. Pure Appl. Algebra*, 217(6):1026–1036, 2013.

[9] M. Dumnicki, T. Szemberg, and H. Tutaj-Gasińska. Symbolic powers of planar point configurations II. *J. Pure Appl. Algebra*, 220(5):2001–2016, 2016.

[10] D. Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. With a view toward algebraic geometry.

[11] A. V. Geramita, B. Harbourne, and J. Migliore. Star configurations in $\mathbb{P}^n$. *J. Algebra*, 376:279–299, 2013.

[12] E. Guardo, B. Harbourne, and A. Van Tuyl. Asymptotic resurgences for ideals of positive dimensional subschemes of projective space. *Adv. Math.*, 246:114–127, 2013.

[13] M. Janssen. On the fattening of lines in $\mathbb{P}^3$. *J. Pure Appl. Algebra*, 219(4):1055–1061, 2015.
[14] G. Malara and J. Szpond. Fermat-type configurations of lines in $\mathbb{P}^3$ and the containment problem. *J. Pure Appl. Algebra*, 222(8):2323–2329, 2018.

[15] G. Malara and J. Szpond. On codimension two flats in Fermat-type arrangements, arXiv:1705.00639.

[16] J. C. Migliore. *Introduction to liaison theory and deficiency modules*, volume 165 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1998.

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