Solving the Puzzle of \( \frac{M_{D^*-D}}{M_{D^*_{s}-D_{s}}} \simeq \frac{M_{B^*-B}}{M_{B^*_{s}-B_{s}}} \simeq 1 \)

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Abstract

The commonly used Hamiltonian of the chromomagnetic hyperfine splitting is inversely proportional to the product of the masses of two constituent quarks composing the meson. So it is expected to have \( \frac{(M_{D^*-D})}{(M_{D^*_{s}-D_{s}})} \simeq \frac{(M_{B^*-B})}{(M_{B^*_{s}-B_{s}})} \simeq 1.6 \), when the constituent quark masses \( m_{u,d} = 0.33 \) GeV and \( m_{s} = 0.53 \) GeV are used. However, the experimental results show that the above ratios are very close to 1. We solve this puzzle by employing the Hamiltonian recently proposed by Scora and Isgur.

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Review of Particle Physics [1] presents the masses of the $D$ mesons (MeV):

$$M_{D^\pm} = 1869.3 \pm 0.5, \quad M_{D^0} = 1864.5 \pm 0.5, \quad M_{D^*_\pm} = 1968.5 \pm 0.6,$$

$$M_{D^{*\pm}} = 2010.0 \pm 0.5, \quad M_{D^{*0}} = 2006.7 \pm 0.5, \quad M_{D_{s*}^\pm} = 2112.4 \pm 0.7, \quad (1)$$

$$M_{D^*} - M_{D} = 140.64 \pm 0.09, \quad M_{D^{*0}} - M_{D^0} = 142.12 \pm 0.07,$$

$$M_{D_{s*}^*} - M_{D_{s*}} = 143.8 \pm 0.4,$$

and the masses of the $B$ mesons (MeV):

$$M_{B^\pm} = 5278.9 \pm 1.8, \quad M_{B^0} = 5279.2 \pm 1.8, \quad M_{B^*} = 5369.3 \pm 2.0,$$

$$M_{B^*} = 5324.8 \pm 1.8, \quad M_{B^{*0}} = 5416.3 \pm 3.3, \quad (2)$$

$$M_{B^*} - M_B = 45.7 \pm 0.4, \quad M_{B^{*0}} - M_{B^0} = 47.0 \pm 2.6,$$

where $M_B \equiv (M_{B^\pm} + M_{B^0})/2$. We note that $M_{D^*} - M_{D*}$ is very close to $M_{D^*} - M_{D}$ and $M_{D^{*0}} - M_{D^0}$ in (1), and that $M_{B^{*0}} - M_{B^0}$ to $M_{B^*} - M_B$ in (2).

The commonly used Hamiltonian for the chromomagnetic hyperfine splitting of vector and pseudoscalar mesons is given by [2]

$$H_{hf}(r) = \frac{32\pi\alpha_s}{9m_Qm_{\bar{q}}}|s_Q \cdot s_{\bar{q}} \delta(r)|, \quad (3)$$

where $\alpha_s = \bar{g}^2/4\pi$ of the QCD running coupling constant $\bar{g}$, and $m_Q (s_Q)$ and $m_{\bar{q}} (s_{\bar{q}})$ are the masses (spins) of the heavy quark $Q$ and the light antiquark $\bar{q}$ inside a heavy-light meson, respectively. The Hamiltonian in (3) corresponds to the potential due to one-gluon exchange for $s$-wave bound states. Treating the chromomagnetic hyperfine splitting as a perturbation, one obtains

$$M_{M^*} - M_M = \frac{32\pi\alpha_s}{9m_Qm_{\bar{q}}}|\Psi_{Q\bar{q}}(0)|^2, \quad (4)$$

where $\Psi_{Q\bar{q}}(0)$ is the two-body bound state wave function at origin. When it is assumed that $D_{u,d}$ and $D_s$ (or $B_{u,d}$ and $B_s$) have similar values of $\alpha_s$ and $|\Psi_{Q\bar{q}}(0)|$, $M_{M^*} - M_M$ is proportional to $1/m_{\bar{q}} (\bar{q} = \bar{u}, \bar{d}, \bar{s})$ for $D$ (or $B$) mesons. Then it is expected to have $(M_{D^*} - M_D)/(M_{D_{s*}} - M_{D_s}) \simeq (M_{B^*} - M_B)/(M_{B_{s*}} - M_{B_s}) \simeq$
1.6, when the constituent quark masses $m_{u,d} = 0.33 \text{GeV}$ and $m_s = 0.53 \text{GeV}$ are used. However, the experimental results in (1) and (2) give the values

$$\frac{M_{D^*} - M_{D^0}}{M_{D^*} - M_D} = 0.978 \pm 0.003, \quad \frac{M_{B^*} - M_B}{M_{B^*} - M_{B^0}} = 0.972 \pm 0.054,$$

which are very close to 1. Goity and Hou [3] noticed this peculiar property of the heavy meson masses. Randall and Sather [4] emphasized this discrepancy and called it a puzzle. They suggested that these data might be an interesting probe of heavy mesons. The purpose of this letter is to solve this puzzle by employing the Hamiltonian recently proposed by Scora and Isgur.

Scora and Isgur [5] proposed the following $\bar{H}_{hf}(r)$ as a modification of $H_{hf}(r)$ in (3):

$$\bar{H}_{hf}(r) = \left[ \frac{m_{Q\bar{q}}}{E_Q E_{\bar{q}}} \right]^\frac{1}{2} \left( \frac{32\pi a\alpha_s}{9m_Qm_{\bar{q}}} s_Q \cdot s_{\bar{q}} \delta(r) \right) \left[ \frac{m_{Q\bar{q}}}{E_Q E_{\bar{q}}} \right]^\frac{1}{2},$$

where the term in the parentheses would be the ordinary Fermi contact term in (3) if the anomalous coupling coefficient $a$ were unity, and where $E_i = (m_i^2 + p^2)^{1/2}$. For s-wave bound states, the expectation value of the Hamiltonian $\bar{H}_{hf}(r)$ in (6) is given by [5]

$$\langle \bar{H}_{hf}(r) \rangle = \left[ \frac{2S(S + 1) - 3}{4} \right] \left( \frac{32\pi a\alpha_s}{9m_Qm_{\bar{q}}} \right) |\bar{\Psi}_{Q\bar{q}}(0)|^2,$$

where $S = 0$ or 1 is the total spin of the meson.

In order to obtain the heavy meson wave function $\Phi_{Q\bar{q}}(p)$ in (8), Scora and Isgur [5] applied the variational method to the Hamiltonian with the nonrelativistic kinetic energy terms. They adopted the Gaussian wave function as a trial wave function for the heavy meson ground state. We follow the same procedure as theirs in obtaining the heavy meson ground state wave function, except for that we use the relativistic kinetic energy terms since the velocity of the light quark inside heavy meson is large. We apply the variational method to the relativistic
Hamiltonian

\[ H = \sqrt{p^2 + m_q^2} + \sqrt{p^2 + m_{\bar{q}}^2} + V(r), \]  

(9)

where \( r \) and \( p \) are the relative coordinate and its conjugate momentum. We take the following potential energy for \( V(r) \) in (9),

\[ V(r) = -\frac{\alpha_c}{r} + Kr + \bar{H}_{hf}(r), \]  

(10)

where \( \bar{H}_{hf}(r) \) is the modified chromomagnetic hyperfine Hamiltonian in (8) proposed by Scora and Isgur. We take the Gaussian wave function as a variational wave function for the heavy meson ground state,

\[ \Psi(r) = \left( \frac{\beta}{\sqrt{\pi}} \right)^{3/2} e^{-\beta^2 r^2/2}, \quad \Phi(p) = \frac{1}{(\sqrt{\pi} \beta)^{3/2}} e^{-p^2/2\beta^2}. \]  

(11)

The ground state wave function is then given by

\[ \langle H \rangle = \langle \psi | H | \psi \rangle = E(\beta), \quad \frac{dE(\beta)}{d\beta} = 0 \text{ at } \beta = \bar{\beta}, \]  

(12)

where \( \bar{\beta} \) represents the inverse size of the meson \( \langle r^2 \rangle^{1/2} = 3/(2 \bar{\beta}) \), and \( E \equiv E(\bar{\beta}) \) the meson mass. In explicit calculations, we used the following four different potential models which have the values of \( \alpha_c, K \) and quark masses given in Table 1: (A) Scora and Isgur \( \bar{\xi} \), (B) Eichten \( \bar{\xi} \) et al. \( \bar{\xi} \), (C) Hagiwara \( \bar{\xi} \) et al. \( \bar{\xi} \), and (D) Model D which is the same as the Hagiwara et al.’s model (C) except for the values \( \alpha_c(D) = 0.48 \) and \( \alpha_c(B) = 0.32 \) which are given from the running coupling constants at the energy scales of \( M_D \) and \( M_B \), respectively.

For each of the four models of (A), (B), (C), and (D), we obtained the function \( E(\beta) \) in (12). As a representative, we present \( E(\beta) \equiv M \) for the model (C) of Hagiwara et al. in Fig 1, where (a), (b), (c), and (d) correspond to \( D_d, D_s, B_d, \) and \( B_s \) mesons, respectively. In Fig 1, the graphs with the mark \( \Box \) are what we obtain when we take the anomalous coupling coefficient \( a \) in (9) as zero, that is, when we do not include the \( \bar{H}_{hf}(r) \) term in (10). The values of the variational parameter \( \beta \) which give the minimum of \( E(\beta) \) are presented in the \( D_d(0), D_s(0), B_d(0), \) and \( B_s(0) \) columns of Table 2.
The graphs with the marks ◊ and + in Fig. 1 correspond to those for the vector and pseudoscalar mesons, respectively, which we obtain when we use desirable values of $a$, as we explain in the following. By including of the $H_{nf}(r)$ term in \[ (10) \] with a fixed value of the anomalous coupling coefficient $a$ in \[ (3) \], we obtain $E(\beta) \equiv M$ and its minimum from the condition in \[ (12) \], for the vector and pseudoscalar mesons separately. Then, by performing the same procedure with different values of $a$, we obtain the vector and pseudoscalar meson masses $M_{M^*}$ and $M_M$ which are given by the minimum of $E(\beta)$, and their difference $\Delta M \equiv M_{M^*} - M_M$ as functions of the coefficient $a$. We obtained these funtions for each of the four models of (A), (B), (C), and (D). In Fig. 2, as a representative, we present $\Delta M \equiv M_{M^*} - M_M$ as functions of $a$ for $D_d$, $D_s$, $B_d$, and $B_s$ mesons obtained for the model (C).

The nice feature of Fig. 2 is that the graph of $D_d$ is very close to that of $D_s$, also for $B_d$ and $B_s$, and this interesting character is the clue for solving the puzzle in this letter. The dotted vertical lines indicate the values of $a$ which give rise to the experimental values of $\Delta M$. In Table 3, we present these values of $a$ and $\Delta M \equiv M_{M^*} - M_M$ obtained by these $a$ values. Table 3 shows that the ratio $R \equiv \Delta M(M_d)/\Delta M(M_s)$ are obtained as the values which are very close to 1 for both $D$ and $B$ mesons, and the puzzle is solved. The graphs with the marks ◊ and + in Fig. 1 are $E(\beta) \equiv M$ obtained with the values of $a$ in Table 3. Fig. 3 shows that the minimum of $E(\beta)$ for the vector meson is bigger than that for the pseudoscalar meson as it should be, and the value of the corresponding $\bar{\beta}$ (denoted by dotted vertical lines) for the pseudoscalar meson is bigger than that for the vector meson. This means that the size of the pseudoscalar meson is smaller than that of the vector meson, since $\langle r^2 \rangle^{1/2} = 3/(2\bar{\beta})$. We present the sizes of the vector and pseudoscalar mesons and their ratios in Table 4.

In conclusion, we have solved the puzzle of $\frac{(M_{D^*} - M_D)}{(M_{D^*} - M_{D_s})} \approx \frac{(M_{B^*} - M_B)}{(M_{B^*} - M_{B_s})} \approx 1$ by adopting the modified chromomagnetic hyperfine Hamiltonian in \[ (3) \] proposed by Scora and Isgur [5] instead of the commonly
used chromomagnetic hyperfine Hamiltonian in (3).
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\[ \alpha_c \quad K \quad m_u = m_d \quad m_s \quad m_c \quad m_b \]

|       | \alpha_c | \( K \) | \( m_u = m_d \) | \( m_s \) | \( m_c \) | \( m_b \) |
|-------|----------|--------|----------------|--------|--------|--------|
| Scora | 0.67     | 0.18   | 330           | 550    | 1820   | 5200   |
| Eichten | 0.52     | 0.18   | 330           | 530    | 1840   | 5180   |
| Hagiwara | 0.47     | 0.19   | 330           | 530    | 1320   | 4750   |
| Model D | (D)0.48, (B)0.32 | 0.19 | 330           | 530    | 1320   | 4750   |

Table 1: The values of the parameters in the potential Eq. (10) of the models used in this letter.

|       | \( D_d(0) \) | \( D_d^* \) | \( D_d \) | \( D_s(0) \) | \( D_s^* \) | \( B_d(0) \) | \( B_d^* \) | \( B_d \) | \( B_s(0) \) | \( B_s^* \) | \( B_s \) |
|-------|--------------|--------------|----------|--------------|--------------|--------------|--------------|----------|--------------|--------------|----------|
| Scora | 556          | 526          | 715      | 592          | 561          | 767          | 663          | 646      | 729          | 716          | 697      |
| Eichten | 515         | 488          | 662      | 547          | 518          | 709          | 591          | 577      | 647          | 636          | 620      |
| Hagiwara | 484         | 460          | 605      | 512          | 487          | 642          | 574          | 561      | 625          | 616          | 601      |
| Model D | 486          | 462          | 608      | 514          | 490          | 639          | 523          | 511      | 567          | 558          | 545      |

Table 2: The values of the variational parameter \( \beta \) (MeV) of the Gaussian wave function which minimize \( \langle H \rangle \). Here, the values in the column of \( D_d(0) \) are \( \beta \)'s obtained without taking \( \bar{H}_{hf}(r) \) into account, and those in the columns of \( D_d^* \) and \( D_d \) are \( \beta \)'s obtained with taking \( \bar{H}_{hf}(r) \) into account for vector and pseudoscalar mesons, respectively, and the same for other mesons.
Table 3: The obtained values of anomalous coupling constant $a$, the mass differences $\Delta M \equiv M_{M^*} - M_M$ (MeV) of the vector and pseudoscalar mesons, and the ratio $R \equiv \Delta M(M_d)/\Delta M(M_s)$.

|       | $a(D)$ | $\Delta M(D_d)$ | $\Delta M(D_s)$ | $R(D)$ | $a(B)$ | $\Delta M(B_d)$ | $\Delta M(B_s)$ | $R(B)$ |
|-------|--------|-----------------|-----------------|--------|--------|-----------------|-----------------|--------|
| Scora | 1.21   | 141             | 144             | 0.98   | 0.78   | 46.0           | 49.5           | 0.93   |
| Eichten | 1.85  | 143             | 144             | 0.99   | 1.27   | 45.7           | 48.2           | 0.95   |
| Hagiwara | 1.86  | 147             | 143             | 1.03   | 1.37   | 45.3           | 47.2           | 0.96   |
| Model D | 1.80  | 146             | 143             | 1.02   | 2.49   | 45.6           | 46.5           | 0.98   |
| Expts. | —     | 141 ± 0.1       | 144 ± 0.4       | 0.98±0.00 | —     | 45.7 ± 0.4     | 47.0 ± 2.6     | 0.97±0.05 |

Table 4: The sizes of mesons $\langle r^2 \rangle^{1/2} (= 3/(2 \bar{\beta}))$ in the unit of GeV$^{-1}$ (1 GeV$^{-1}$ = 0.197 fm) obtained from the values of $\bar{\beta}$’s given in Table 2, and the ratios of the vector and pseudoscalar meson sizes.

|       | $D_d^*$ | $D_s^*$ | $D_d^*/D_s$ | $D_s^*$ | $D_s^*/D_s$ | $B_d^*$ | $B_d^*/B_d$ | $B_s^*$ | $B_s^*/B_s$ | $\bar{\beta}_d^*$ | $\bar{\beta}_s^*$ |
|-------|---------|---------|-------------|---------|-------------|---------|------------|---------|------------|-----------------|-----------------|
| Scora | 2.85    | 2.10    | 1.36        | 2.67    | 1.96        | 1.36    | 2.32       | 2.06    | 1.13       | 2.15           | 1.89            |
| Eichten | 3.07   | 2.27    | 1.35        | 2.90    | 2.12        | 1.37    | 2.60       | 2.32    | 1.12       | 2.42           | 2.14            |
| Hagiwara | 3.26   | 2.48    | 1.31        | 3.08    | 2.34        | 1.32    | 2.67       | 2.40    | 1.11       | 2.50           | 2.23            |
| Model D | 3.25   | 2.47    | 1.32        | 3.06    | 2.35        | 1.30    | 2.94       | 2.65    | 1.11       | 2.75           | 2.47            |
| Average | 3.11   | 2.33    | 1.34        | 2.93    | 2.19        | 1.34    | 2.63       | 2.36    | 1.12       | 2.46           | 2.18            |
Figure Captions

Fig. 1. The meson masses $E(\beta) \equiv M$ as functions of the variational parameter $\beta$ for (a) $D_d$, (b) $D_s$, (c) $B_d$, and (d) $B_s$ mesons.

Fig. 2. The difference of the vector and pseudoscalar meson masses $\Delta M \equiv M_{M^*} - M_M$ as functions of the anomalous coupling coefficient $a$ for $D_d$, $D_s$, $B_d$, and $B_s$ mesons.
