A general approach for lookback option pricing under Markov models

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We propose a computationally efficient method for pricing various types of lookback options under Markov models. We utilize the model-free representations of lookback option prices as integrals of first passage probabilities. We combine efficient numerical quadrature with continuous-time Markov chain approximation for the first passage problem to price lookbacks. Our method is applicable to a variety of models, including one-dimensional time-homogeneous and time-inhomogeneous Markov processes, regime-switching models and stochastic local volatility models. We demonstrate the efficiency of our method through various numerical examples.

Keywords: Lookback options; Drawdown; Markov chain approximation; Gauss quadrature

1. Introduction

Lookback options are an important class of path-dependent derivatives in financial markets, and they can be monitored continuously or discretely (the maximum price is calculated from a discrete set of dates). The pricing of lookback options has been extensively studied. Fusai (2010) provides a comprehensive review of the topic. Under the Black-Scholes model, analytical solutions for different types of continuous lookback options are derived by Goldman et al. (1979) and Conze (1991), while for discrete lookback options, several numerical methods are put forth, including binomial/trinomial trees (Cheuk and Vorst 1997, Babbs 2000, Tse et al. 2001), continuity correction (Broadie et al. 1999), a numerical integration scheme based on random walk duality (Aït-Sahalia and Lai 1998), double-exponential fast Gauss transform (Broadie and Yamamoto 2005) and z-transform (Atkinson and Fusai 2007, Green et al. 2010). For the CEV model, Davydov and Linetsky (2001) price continuous lookback options by numerically inverting Laplace transform and Boyle and Tian (1999) employ trinomial trees. Furthermore, Linetsky (2004) proposes a spectral expansion approach that is applicable to a class of one-dimensional diffusions. For one-dimensional exponential Lévy models that can have jumps, Boyarchenko and Levendorski (2004) devise a method based on Wiener-Hopf factorization for continuous lookback options, whereas Petrella and Kou (2004), Feng and Linetsky (2009) and Fusai et al. (2016a) develop methods based on Laplace transform, Hilbert transform, and a combination of z-transform and Hilbert transform for discretely monitored ones. Additionally, a numerical PDE approach based on finite element is pursued by Forsyth et al. (1999) for some stochastic volatility models. Last but not the least, Monte Carlo simulation techniques are discussed in Glasserman (2013) for discrete lookback options.

In this paper, we propose a new computational method for pricing lookback options based on continuous-time Markov chain (CTMC) approximation for general Markov models. CTMC approximation has become a popular method for solving various option pricing problems under Markov models in recent years. See Mijatović and Pistorius (2013) and Cui and Taylor (2021) for barrier options, Eriksson and Pistorius (2015) for American options, Cai et al. (2015), Song et al. (2018) and Cui et al. (2018b) for Asian options, Zhang and Li (2023a) for Parisian options, Zhang and Li (2023b) for maximum drawdown options, Zhang et al. (2021) for American drawdown options, Li et al. (2022) for speed and duration of drawdown options, and Meier et al. (2021, 2023) for pricing and simulation under financial models with sticky behavior. Boyarchenko and Levendorski (2007a, 2007b, 2007c, 2009, 2013) employed Markov chains to approximate stochastic interest rate and stochastic volatility processes. They imposed appropriate conditions at the localization level to reduce errors when the
maturity is large. In all these papers, the original Markov model is approximated by a CTMC, and then the option price under the CTMC model is derived. We can follow this approach to derive the lookback option price under a CTMC model. However, this algorithm is inferior in terms of computational efficiency to alternatives generated by our method (see Remark 3 for the explanation). In our approach, we combine CTMC approximation with numerical quadrature. We utilize the model-free representation that expresses the lookback option price as an integral of first passage probabilities. By using an efficient quadrature rule, our method can yield an efficient algorithm for pricing lookback options. Other applications of efficient quadrature rules for option pricing can be found in Andricopoulos et al. (2003), Andricopoulos et al. (2007), and Fusai and Recchioni (2007).

Our method has two nice features. First, it is applicable to very general models, including one-dimensional (1D) time-homogeneous and time-inhomogeneous Markov processes, regime-switching models and stochastic local volatility models. Second, it can generate very efficient algorithms by using efficient quadrature rules. In particular, using the Gauss-Legendre quadrature, we can obtain highly accurate results with a small or moderate number of quadrature points. In one example, we show that our method significantly outperforms the finite difference method for solving the partial differential equation for the lookback option price.

The rest of the paper is organized as follows. Section 2 first reviews the model-free representations for lookback options and CTMC approximation for the first-passage problem and then presents our algorithm. Section 3 develops error analysis of our algorithm. Section 4 provides various numerical examples to demonstrate the efficiency and convergence of our algorithm. Section 5 concludes. Appendix 1 provides two schemes for constructing CTMC approximation and appendix 2 gives the finite difference scheme that we compare our algorithm with.

2. Lookback option pricing

Let $X_t$ denote the underlying asset price at time $t$. Define $m_t = \inf_{0 \leq s \leq t} X_s$, and $M_t = \sup_{0 \leq s \leq t} X_s$, which are the running minimum and maximum of the price process starting from time 0, respectively. We also consider the seasonized running minimum and maximum $\bar{m}_t = m_t \wedge m_0$, $\bar{M}_t = M_t \vee M_0$, where $\bar{m}_0$ and $\bar{M}_0$ are the minimum and maximum before time 0.

We consider four types of standard lookback options and focus on continuous monitoring in this paper (see Remark 1 for how to treat discretely monitored ones in our algorithm). In the following, we consider general seasonal lookback options that mature at $T$ and price them at $t \in [0,T]$. Define $\tau := T - t$ and let $r, \sigma$ be the constant risk-free rate and dividend yield, respectively. Davydov and Linetsky (2001) show that their prices admit the following model-free representations:

- Floating-strike lookback put:
  \begin{align*}
  u^{\text{fp}}(t,x,M) & = e^{-\tau r} \mathbb{E}\left[(\bar{M}_T - X_T)^+ \mid X_0 = x, \bar{M}_t = M \right] \\
  & = e^{-\tau r} M - e^{-\sigma \tau} e^{r \tau} \int_0^\infty \mathbb{P}_x(M_T \geq y) \, dy.
  \end{align*}

  (1)

  It is worth noticing that the floating-strike lookback put is an option that compensates the option holder the drawdown of the asset at maturity. This type of options becomes particularly relevant in market turmoil.

- Floating-strike lookback call:
  \begin{align*}
  u^{\text{fc}}(t,x,m) & = e^{-\tau r} \mathbb{E}\left[(K - \bar{M}_T)^+ \mid X_0 = x, \bar{M}_t = m \right] \\
  & = e^{-\tau r} (K - m)^+ + e^{-\tau r} \int_0^m \mathbb{P}_x(m_T \leq y) \, dy.
  \end{align*}

  (2)

- Fixed-strike lookback put:
  \begin{align*}
  u^{\text{fpp}}(t,x,m) & = e^{-\tau r} \mathbb{E}\left[(K - \bar{M}_T)^+ \mid X_0 = x, \bar{M}_t = m \right] \\
  & = e^{-\tau r} (K - m)^+ + e^{-\tau r} \int_{m}^K \mathbb{P}_x(m_T \leq y) \, dy.
  \end{align*}

  (3)

- Fixed-strike lookback call:
  \begin{align*}
  u^{\text{fpc}}(t,x,M) & = e^{-\tau r} \mathbb{E}\left[(\bar{M}_T - K)^+ \mid X_0 = x, \bar{M}_t = M \right] \\
  & = e^{-\tau r} (M - K)^+ + e^{-\tau r} \int_{M}^\infty \mathbb{P}_x(M_T \geq y) \, dy.
  \end{align*}

  (4)

The distributions of $m_t$ and $M_t$ are essentially first passage probabilities. Using these representations, the lookback option pricing problem boils down to calculating the integral of first passage probabilities for different passage levels. It is important to note that these probabilities are for the unseasoned running minimum and maximum, thus the seasoned problem has been turned into an unseasoned one. A general and efficient method for the first passage probability calculation is CTMC approximation, which we review in the next subsection.

2.1. CTMC approximation for the first passage problem

Consider a 1D time-homogeneous Markov model for the asset price process $X_t$ with an infinitesimal generator given by

\begin{align*}
Gf(x) & = \mu(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x) \\
& + \int_{-\infty}^\infty \left(f(x + y) - f(x) - yf'(x)\right) \nu(x, dy).
\end{align*}

(5)
for \( f \in C^2_0(\mathbb{R}^+) \), where \( \mu(x), \sigma(x), \) and \( \nu(x, dy) \) are the drift, diffusion volatility, and jump measure of \( X \), respectively. In general, \( \nu \) is state-dependent and satisfies \( \int_{|y| \leq 1} \nu^y(x, dy) < \infty \) for all \( x \in \mathbb{R}^+ \).

In financial models, \( X_t \) lives on the continuous state space \( \mathbb{R}^+ \). We can approximate \( X_t \) by a CTMC and the construction procedure is reviewed in appendix 1. In particular, CTMC approximation can be constructed for jump-diffusion and pure-jump models with jumps of infinite activity by approximating small jumps around each grid point by a diffusion component. Let \( \{Y_t^{(n)}\} \) be a sequence of CTMCs that converges weakly to \( X_t \). For \( Y_t^{(n)} \), it lives on the state space \( S^{(n)} = \{s_0^{(n)}, \ldots, s_n^{(n)}\} \) with \( n + 1 \) grid points and \( \delta_n \) is the mesh size. Hereafter, all quantities with superscript \( (n) \) are defined for \( Y_t^{(n)} \) in the same way as those defined for \( X_t \).

For \( Y_t^{(n)} \), we denote its generator matrix by \( G_n \) with \( G_n(x, y) \) referring to the transition rate from state \( x \) to \( y \). Let \( P_n(t, x, y) := P(Y_t^{(n)} = y | Y_0^{(n)} = x) \). As \( t \to 0 \), \( P_n(t, x, y) \to 1_{[x, y]}(x) + G_n(x, y)t + o(t) \). It is a classical result that \( P_n(t, x, y) = (\exp(G_n t))(x, y) \) (Serfozo 2009, Section 4.4), where \( \exp(A) \) is the matrix exponential of matrix \( A \) defined as

\[
\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!}.
\]

Consider the first passage times of \( Y_t^{(n)} \) defined as

\[
T_y^{(n),+} = \inf\{t \geq 0 : Y_t^{(n)} > (y)\}.
\]

Closed-form formulas for first-passage probabilities under CTMCs with finite state spaces were derived in Mijatović and Pistorius (2013). Let \( G_{n,\geq 2} \) be the square sub-matrix of \( G_n \) by keeping only transition rates among states in the space \( \{z \in S^{(n)} : z \leq y\} \) and define \( G_{n,\leq 2} \) similarly. Then, we have

\[
P_x(T_y^{(n),+} > t) = (\exp(G_{n,\geq 2} t) I)(x), \tag{6}
\]

\[
P_x(T_y^{(n),-} > t) = (\exp(G_{n,\leq 2} t) I)(x). \tag{7}
\]

where \( I \) is a vector of ones. To calculate the matrix exponential in (6) and (7), a popular algorithm is the scaling and squaring algorithm (Higham 2005), which has a time complexity of \( O(n_1^2) \), where \( n_1 \) is the size of the matrix. When \( Y_t^{(n)} \) is a birth-and-death process, Li and Zhang (2016) propose an algorithm based on efficient matrix eigendecomposition which reduces the time complexity to \( O(n_1^2) \). Another very efficient algorithm for matrix exponentials can be found in Meier et al. (2021).

The construction of CTMC approximation for 1D time-inhomogenous Markov models, regime-switching models and stochastic local volatility models and the computation of first passage probabilities using CTMC approximation under these models can be found in Mijatović and Pistorius (2010), Cai et al. (2019) and Cui et al. (2018a), respectively. In appendix 1, we review the construction of CTMC approximation for 1D time-homogenous Markov models and present two schemes.

We introduce some notations. For any \( x \), let

\[
x^- := \max\{y \in S^{(n)} : y < x\}, \quad x^+ := \min\{y \in S^{(n)} : y > x\},
\]

which are the grid points next to \( x \) on the left and right, respectively.

### 2.2. Algorithm for lookback option pricing

We explain our ideas by way of the floating-strike lookback put option. Recall the model-free representation (1). We truncate the interval by replacing \( \infty \) with a large number \( A \) and obtain

\[
u^0(t, x, M) \approx u_A^0(t, x, M) \quad := e^{-rt}A - e^{-dr}x - e^{-r(t - \tau)} \int_M^A P_x (M_t < y) dy.
\]

(8)

We then apply a quadrature rule on \([M, A]\), which results in

\[
u^0(t, x, M) \approx u_{\lambda, n_1}^0(t, x, M)
\]

\[
:= e^{-rt}A - e^{-dr}x - e^{-r(t - \tau)} \sum_{i=0}^{n_1} \omega_i P_x (M_t < y_i),
\]

(9)

where \( \{y_0, \ldots, y_{n_1}\} \) are the quadrature points on \([M, A]\) and \( \omega_i \) is the weight at \( y_i \). In section 2.4, we will review several commonly used quadrature rules.

The first passage probability \( P_x (M_t < y_i) \) is unknown for a general Markov process \( X_t \) and we compute it by CTMC approximation. Recall that \( Y_t^{(n)} \) is a CTMC that approximates \( X_t \). This leads to the following approximation:

\[
P_x (M_t < y_i) \approx P_x (M_t^{(n)} < y_i) = P_x \left( T_{y_i}^{(n),+} > \tau \right),
\]

where \( y_i^- \) is left neighbor of \( y_i \) on the grid \( S^{(n)} \). Consequently, we obtain the following approximation to the option price:

- Floating-strike lookback put:

\[
u^0(t, x, M) \approx u_{\lambda, n_1}^{0,(n)}(t, x, M)
\]

\[
:= e^{-rt}A - e^{-dr}x - e^{-r(t - \tau)} \sum_{i=0}^{n_1} \omega_i P_x (T_{y_i}^{(n),+} > \tau).
\]

(10)

Applying the same ideas to the model-free representations (2)–(4), we can obtain the approximations for the prices of other types of lookback options as follows:

- Floating-strike lookback call:

\[
u^c(t, x, m) \approx u_{n_1}^{c,(n)}(t, x, m)
\]
\[ e^{-dr} x - e^{-rt} \sum_{i=0}^{n} \omega_i P_x \left( T_{y_i}^{(n)} > \tau \right) \]

where \( \{y_0, \ldots, y_{n}\} \) are the quadrature points on \([0, m]\) and \(\omega_i\) is the weight at \(y_i\).

- Fixed-strike lookback put:

\[ u^{bp}(t, x, m) \approx u^{bp}_{y_0, n}(t, x, m) = e^{-rt} K - e^{-rt} \sum_{i=0}^{n} \omega_i P_x \left( T_{y_i}^{(n)} > \tau \right) \]

where \( \{y_0, \ldots, y_{n}\} \) are the quadrature points on \([0, m \wedge K]\) and \(\omega_i\) is the weight at \(y_i\).

- Fixed-strike lookback call:

\[ u^{bc}(t, x, M) \approx u^{bc}_{y_0, n}(t, x, M) = e^{-rt} A - e^{-rt} K - e^{-rt} \sum_{i=0}^{n} \omega_i P_x \left( T_{y_i}^{(n)} > \tau \right) \]

where \(A\) is the truncation level of the integral, \(\{y_0, \ldots, y_{n}\}\) are the quadrature points on \([M \vee K, A]\) and \(\omega_i\) is the weight at \(y_i\).

The first passage probabilities \(P_x(T_{y_i}^{(n)} > \tau)\) and \(P_x(T_{y_i}^{(n)} > \tau)\) of the CTMC \(Y_t^{(n)}\) can be computed by (6) and (7) using an efficient algorithm for the matrix exponential.

**Remark 1** (discretely monitored lookback options) If the lookback option is monitored discretely, the model-free representation rates, which holds in scheme 1 presented in appendix 1, CTMC approximation error for pricing these lookback options is not affected by the truncation level \(U\) for the state space as long as we set \(U \geq A\).

For the floating-strike lookback put and fixed-strike lookback put, integration is done on a finite interval and hence truncation of the integral is not involved. We have

\[ P_x \left( T_{y_i}^{(n)} > \tau \right) = (\exp(G_{\rho \leq y} \tau)I)(x), \quad i = 0, 1, \ldots, n_q, \]

Now the rates of transitions to states outside the integral region are needed, which imply that the truncation level \(U\) affects the CTMC approximation error.

**Remark 2** For the floating-strike lookback put and fixed-strike lookback call, an integral of the form \(\int_0^\tau P_x(M_t < y) dy\) needs to be calculated, where \(C = M\) for the former and \(C = M \wedge K\) for the latter. It is obvious that \(P_x(M_t < y) = 0\) if \(x \geq y\). Hence, for the purpose of calculating such probability for \(y < A\), all the states greater than or equal to \(A\) behave as absorbing ones. As we choose the state space truncation level \(U \geq A\), using the absorbing boundary condition for \(U\) in the CTMC construction can match the absorbing behavior in the original process.

### 2.3. Grid design

Grid design is essential for the CTMC method to converge nicely. Zhang and Li (2019) propose grid design principles for pricing European and barrier options with call/put-type and digital-type payoffs. Here, the first passage probabilities \(P_x(M_t < y_i)\) and \(P_x(m_r > y_i)\) can be viewed as the price of an up-and-out and down-and-out barrier option, respectively, with barrier level \(y_i\) and unit payoff. From Zhang and Li (2019), it is essential to locate \(y_i\) on the grid of the Markov chain to achieve fast convergence. Thus, the grid for the CTMC should be designed according to the positions of the quadrature points. A grid design that satisfies the above requirement is as follows. A uniform grid is used on each \([y_i, y_{i+1}]\) with both end-points included on the grid as well as outside \([y_0, y_{n}]\). The piecewise uniform structure has the additional benefit of removing oscillations as shown in Zhang and Li (2019) so that Richardson extrapolation can be applied to speed up convergence.

### 2.4. The choice of quadrature rules

Clearly the efficiency of our algorithm hinges on the quadrature rule as well as CTMC approximation. Below we review several popular quadrature rules. Since any integral on a finite interval can be transformed to another integral on \([0, 1]\) by a change of variable, we present these rules for computing \(I = \int_0^1 f(x) \, dx\).

- **Rectangle rule:** \(I \approx \sum_{i=0}^{n-1} f(\bar{x})h\) or \(\sum_{i=1}^{n} f(\bar{x})h\).
- **Trapezoid rule:** \(I \approx \sum_{i=0}^{n-1} \frac{1}{2} f(\bar{x}) + f((i + 1)h))\).
- **Simpson’s rule:** \(I \approx \frac{h}{3} \sum_{i=1}^{n/2} (f(2 - 2h) + 4f((2j - 1)h) + f(2jh))\).
- **Gauss-Legendre quadrature:** \(I \approx \sum_{i=1}^{n} \omega_i f(x_i)\). The weights \(\omega_i\) and abscissas \(x_i\) are determined to optimize the convergence rate. For the formulas of \(\omega_i\) and \(x_i\), see e.g. Press et al. (2007), Section 4.5.
can be found in Fusai and Roncoroni (2008) Chapter 6. While a detailed discussion of these rules and their implementation is not sufficiently smooth, so we can utilize the Gauss–Legendre quadrature, which converges faster than any order of polynomial convergence. Simpson’s rule being the fastest, Gauss-Legendre quadrature shows that using a few quadrature points already suffices to achieve a high level of accuracy for integral discretization.

Table 1 displays their error bounds and convergence rates.

| Rule            | Error Bound       | Convergence Rate |
|-----------------|-------------------|-----------------|
| Rectangle       | $\frac{1}{2}h \max |f'(\xi)|$ | $O(n^{-1})$     |
| Trapezoid       | $\frac{h^2}{12} \max |f''(\xi)|$ | $O(n^{-2})$     |
| Simpson         | $\frac{h^4}{180} \max |f^{(4)}(\xi)|$ | $O(n^{-4})$     |
| Gauss-Legendre  | $\frac{(n+1)!}{(2n+1)!(2n+3)!} \max |f^{(2n)}(\xi)|$ | $O(n^{-2n})$    |

Equation (14) is identical to what we would obtain with the rectangle rule applied in (10).

Remark 4 Our method is applicable to a general setting where the payoff is additive in terms of the maximum or minimum and the terminal stock price, i.e. the payoff takes the form $f(M_T) + g(M_T^h) + h(X_T)$ for functions $f(\cdot)$, $g(\cdot)$ and $h(\cdot)$. This encompasses the floating/fixed strike call/put options considered in the paper as special cases. In the general case, we can compute the option price by calculating the following integrals:

$$\mathbb{E}[f(M_T)|X_t = x, M_t = M] = \int_M^\infty f(y) \, dP_x [M_t \leq y],$$

$$\mathbb{E}[g(M_T)|X_t = x, m_t = m] = \int_0^m g(y) \, dP_x [m_t \leq y].$$

The integrals can be discretized using Gauss quadrature, and $P_x[M_T \leq y]$ and $P_x[m_t \leq y]$ for any quadrature point $y$ can be calculated by CTMC approximation. The last part $\mathbb{E}[h(X_T)|X_t = x]$ is simply a European option price, which can also be calculated by CTMC approximation.

2.5. An alternative algorithm for exponential Lévy models

When the underlying asset price follows an exponential Lévy model, we can exploit the spatial homogeneity of the Lévy process to develop a more efficient algorithm with less time complexity. Recall that $[X_t, t \geq 0]$ is the asset price process. Then $\ln(X_t), t \geq 0$ is a Lévy process. Below we use the floating strike lookback put option as an example to illustrate the idea and the other types of lookback options can be dealt with similarly.

Let $\bar{M}_t = \sup_{0 \leq \tau \leq t} \ln(X_\tau), t \geq 0$. It’s easy to see that $\bar{M}_t = \ln(M_t)$. We start with (9) and replace the first passage probability for $X$ with those for $\bar{X}$. Then, we obtain

$$u^{\bar{p}}(t, x, M) \approx u^{\bar{p}}_{i,n}(t, x, M) = e^{-rt} A - e^{-dr} x - e^{-rt} \sum_{i=0}^n \omega_i P_x (M_t < y_i)$$

$$= e^{-rt} A - e^{-dr} x - e^{-rt} \int_M^\infty (1 - P_x (M_t^{(n)} < y)) \, dy.$$
where we use the spatial homogeneity of \( \ln X \) in (15). Comparing (16) with (10), we see that the first passage probabilities of the CTMC in the former are calculated at different starting points but for the same barrier level, whereas in the latter they are calculated at the same starting point but for different barrier levels. From (6),

\[
P_{A_i}(T^{(0), i} > \tau) = (\exp(G_{n, \leq 1})_i \leq 1)(x/y_i),
\]

so we only need to calculate one matrix exponential to get all the first passage probabilities in (16). In contrast, we have to calculate \( n_q + 1 \) matrix exponentials to obtain the first passage probabilities in (10). Since doing matrix exponentiation is the most time-consuming part of our method, using (16) can significantly reduce the computation time. For the grid design, we can construct a piecewise uniform grid to have \( x/y_i \) for all \( i = 0, 1, \ldots, n_q \) on the grid.

### 2.6. Complexity analysis

The pseudocodes for pricing a floating-strike lookback put option under general models are given in Algorithm 1. The most time-consuming step is calculating the first passage probabilities for various passage levels \( y_i, i = 0, 1, \ldots, n_q \), which involves computation of matrix exponentials multiplied by vectors. Following the discussions in section 2.1, in the general case, the complexity of calculating \( \exp(G_{n, \leq i} \tau) \) is \( O(n_i^3) \) using the scaling and squaring algorithm (Higham 2005) with \( n_i \) being the number of states less than \( y_i \). When the original model is a diffusion, the approximating CTMC is a birth-and-death process, and hence \( G_{n, \leq i} \) is tridiagonal. In this case, we can use the algorithm based on efficient matrix eigendecomposition that has complexity \( O(n_i^2) \) (Li and Zhang 2016). If the original model is an exponential Lévy process, we use Algorithm 2, which only requires computing one matrix exponential.

The total computational complexity is \( \sum_{i=0}^{n_q} O(n_i^3) = O(n_q n^3) \) for general jump-diffusion models, \( \sum_{i=0}^{n_q} O(n_i^2) = O(n_q n^2) \) for diffusion models, and \( O(n^3) \) for exponential Lévy models. Suppose the truncation error is \( O(\delta^3) \) and the discretization error is \( O(\delta A^3) \) (ignoring the quadrature error as it converges fast). To achieve an overall error level \( \varepsilon > 0 \), we need to take \( A = O(\varepsilon^{-1/3}) \) and \( \delta = O(\varepsilon^{-1/3} + 1/\rho) \). Correspondingly, the number of grid points is \( n = O(A/\delta) = O(\varepsilon^{-1/3} + 1/\rho) \). Thus the complexity w.r.t. error tolerance level is \( O(n_q \varepsilon^{-1/3} + 1/\rho) \) for general jump-diffusion models, \( O(n_q \varepsilon^{-2} + 1/\rho) \) for diffusion models, and \( O(n_q \varepsilon^{-3} + 1/\rho) \) for exponential Lévy models. The convergence orders \( \rho \) and \( \gamma \) can be estimated numerically or obtained by the theoretical analysis in section 3.

The algorithms for other types of lookback options and their complexity analysis are similar.

**Remark 5** From our numerical experiments, for typical ranges of model and contract parameters, setting \( A = 5\varepsilon \) for \( \tau \leq 1 \) and \( A = 30\varepsilon \) for \( \tau \leq 5 \) is sufficient for good accuracy in truncation, where \( x \) is the initial price.† Using \( n \) around 500 and 1000 with extrapolation can achieve a good accuracy in CTMC approximation for diffusion and jump-diffusion models, respectively. It is worth noting that these suggestions may be conservative in some cases, while in other cases with more extreme parameters, \( A \) and \( n \) may need to be greater than the suggested values.

### Algorithm 1 Pricing a floating-strike lookback put option under general models

**Require:** Model drift \( \mu(x) \), diffusion coefficient \( \sigma^2(x) \) and jump measure \( \nu(x, dy) \).

**Require:** \( \tau, d, r, \tau \times M \).

**Ensure:** The approximate floating-strike lookback put option price \( p_{flp} \).

1: Set the localization level \( A \).
2: Set the quadrature points \( y_i \) and weights \( \omega_i \) for \( i = 1, 2, \ldots, n_q \) over the interval \([M, A]\).
3: Construct the CTMC grid \( S^{(n)} = \{s^{(n)}_0, \ldots, s^{(n)}_{n_q}\} \).
4: Construct the generator matrix \( G_n \) of the CTMC using \( \mu(x), \sigma^2(x) \) and \( \nu(x, dy) \).
5: Set \( p_{flp} \leftarrow e^{-d\tau}A - e^{-d\tau}x \)
6: for \( i = 0 \) to \( n_q \) do
7: \( p_{flp} \leftarrow p_{flp} - \omega_i e^{d\tau} (\exp(G_{n, \leq i} \tau) \mathbf{1})(x) \).
8: end for
9: return \( p_{flp} \).

### Algorithm 2 Pricing a floating-strike lookback put option under exponential Lévy models

**Require:** Model drift \( \mu(x) \), diffusion coefficient \( \sigma^2(x) \) and jump measure \( \nu(x, dy) \).

**Require:** \( \tau, d, \tau \times x \) and \( M \).

**Ensure:** The approximate floating-strike lookback put option price \( p_{flp} \).

1: Set the localization level \( A \).
2: Set the quadrature points \( y_i \) and weights \( \omega_i \) for \( i = 1, 2, \ldots, n_q \) over the interval \([M, A]\).
3: Construct the CTMC grid \( S^{(n)} = \{s^{(n)}_0, \ldots, s^{(n)}_{n_q}\} \).
4: Construct the generator matrix \( G_n \) of the CTMC using \( \mu(x), \sigma^2(x) \) and \( \nu(x, dy) \).
5: Set \( G_{n, \leq i} \) as a square sub-matrix of \( G_n \) by keeping only transition rates among states less than \( y_i \).
6: Calculate \( p = \exp(G_{n, \leq i} \tau) \).
7: Set \( p_{flp} \leftarrow e^{-d\tau}A - e^{-d\tau}x \)
8: for \( i = 0 \) to \( n_q \) do
9: \( p_{flp} \leftarrow p_{flp} - \omega_i e^{-d\tau} p(x/y_i) \).
10: end for
11: return \( p_{flp} \).

† Similar values can be used for \( U \), the truncation level of the state space, for pricing floating-strike lookback calls.
3. Error analysis

We first perform error analysis for the floating-strike lookback put in section 3.1 and section 3.2. We then state the results for the floating-strike lookback call in section 3.3 without detailed proofs as they are similar to the put case. The analysis for the fixed-strike lookback put (call) is entirely analogous to the floating-strike lookback put (call) and hence omitted.

The error of our approximation for the floating-strike put can be decomposed as

\[ \begin{align*}
    u_{A,\nu}^{(\alpha)}(t, x, M) - u_{A}^{\nu}(t, x, M) \\
    = u_{A,\nu}^{(\alpha)}(t, x, M) - u_{A}^{\nu}(t, x, M) \\
    + u_{A}^{\nu}(t, x, M) - u_{A}^{\nu}(t, x, M) \\
    = u_{A,\nu}^{(\alpha)}(t, x, M) - u_{A}^{\nu}(t, x, M) \\
    - e^{-\gamma t} \int_{A}^{\infty} P_{\gamma}(M_{t} \geq y) \ dy, 
\end{align*} \]

where the last term is the error of truncating the infinite integral.

3.1. Analysis of integral truncation error

We provide an estimate of the truncation error of the infinite integral for diffusions and exponential Lévy models.

**Proposition 1** Assume that \( X \) is a diffusion process with \( \mu(x) + \sigma^2(x) \leq L(1 + x^2) \) for all \( x \geq 0 \) for some constant \( L > 0 \). Then for any integer \( p \geq 1 \), there exists a constant \( C > 0 \) independent of \( A \) such that the error of truncating the infinite integral is bounded as follows:

\[ e^{-\gamma t} \int_{A}^{\infty} P_{\gamma}(M_{t} \geq y) \ dy \leq C \frac{1 + x^2p}{A^{2p-1}}. \]

**Proof** By Karatzas and Shreve (2012) (Page 306, Problem 3.3.15), for any integer \( p \geq 1 \), \( \mathbb{E}[M_{T}^{2p}] \leq C(1 + x^{2p}) \) for some constant \( C > 0 \). Then by the Markov’s inequality, we have

\[ P_{\gamma}[M_{t} \geq y] \leq \frac{\mathbb{E}[M_{T}^{2p}]}{y^{2p}} \leq C \frac{1 + x^2p}{y^{2p}}. \]

Therefore,

\[ \int_{A}^{\infty} P_{\gamma}(M_{t} \geq y) \ dy \leq C(1 + x^{2p}) \int_{A}^{\infty} \frac{1}{y^{2p}} \ dy \]

\[ = \frac{C(1 + x^{2p})}{(2p - 1)A^{2p-1}}, \]

which concludes the proof.

Proposition 1 shows that the truncation error decays faster than any negative power of \( A \) if the drift and diffusion coefficient functions satisfy the linear growth condition.

**Proposition 2** Suppose \( X \) is an exponential Lévy process, and there exists a constant \( \beta > 0 \) such that

\[ \int_{|x| > 1} e^{(1+\beta)x} \nu(dx) < \infty, \]

where \( \nu(\cdot) \) is the Lévy measure of \( \ln X \). Then

\[ e^{-\gamma t} \int_{A}^{\infty} P_{\gamma}(M_{t} \geq y) \ dy \leq C \frac{A^{1+\beta}}{A^{2p}}. \]

for some constant \( C > 0 \) independent of \( x \) and \( A \).

**Proof** Using theorems 25.17 and 25.18 in Sato (1999), we obtain

\[ \mathbb{E}[e^{(1+\beta)\sup_{s \leq t} \ln X}] = \mathbb{E}[M_{t}^{1+\beta}] < \infty. \]

Applying the Markov’s inequality yields

\[ P_{\gamma}[M_{t} \geq y] \leq \frac{\mathbb{E}[M_{t}^{1+\beta}]}{y^{1+\beta}}. \]

Therefore, we have

\[ \int_{A}^{\infty} P_{\gamma}(M_{t} \geq y) \ dy \leq e^{-\gamma t} \int_{A}^{\infty} \frac{1}{y^{1+\beta}} \ dy \]

\[ = e^{-\gamma t} A^{1+\beta} \int_{A}^{\infty} \frac{1}{y^{1+\beta}} \ dy \]

\[ \leq C \frac{A^{1+\beta}}{A^{2p}}. \]

for some constant \( C > 0 \) independent of \( x \) and \( A \) because \( M_{t}/x \) is independent of \( x \).

Proposition 2 shows that the truncation error decays as a negative power of \( A \) under the integrability condition of the Lévy measure.

3.2. Analysis of quadrature and CTMC approximation errors

Sharp convergence rate estimates of CTMC approximation for European and barrier options under 1D time-homogeneous diffusion models are obtained in Li and Zhang (2018), Zhang and Li (2019) and Zhang and Li (2022) under different assumptions. For general Markov processes with jumps, sharp convergence rate estimates for these options are still an open problem. For this reason, here we only consider 1D time-homogeneous diffusions.

We analyze \( u_{A,\nu}^{(\alpha)}(t, x, M) - u_{A}^{\nu}(t, x, M) \). This error arises from the quadrature and CTMC approximation. To analyze its convergence rate, we impose the following conditions on the diffusion model.

**Assumption 1** Assume that \( X \) is a diffusion on \( I = [a, \infty) \) with drift \( \mu(x) \), diffusion coefficient \( \sigma(x) \) and \( \ell(x) \) is an absorbing boundary. Suppose \( \mu(x), \sigma(x) \in C^{\infty}([a, b]) \) and \( \min_{[a, b]} \sigma(x) > 0 \) for any \( [a, b] \subseteq I \).

In asset price models with \( X \in (0, \infty) \), assumption 1 does not hold. Our analysis requires an absorbing lower boundary because we need to use some results for the regular Sturm–Liouville eigenvalue problem with homogeneous boundary condition. In this case, in order to satisfy assumption 1 we
set $\ell = \varepsilon$ for some positive $\varepsilon$ very close to zero and introduce a new diffusion $X^\varepsilon$, which is obtained from the original diffusion $X$ by making it stay at $\varepsilon$ forever when it gets there. More precisely, let $\tau_\varepsilon := \inf\{\tau \geq 0 : X_\tau = \varepsilon\}$ be the first time $X$ hits $\varepsilon$, and $X^\varepsilon_\tau = X_\tau 1_{\tau < \tau_\varepsilon} + \varepsilon 1_{\tau \geq \tau_\varepsilon}$. We will estimate $u^{0}(t, x, M) - u^{0}_{A}(t, x, M)$ under $X^\varepsilon$. The error of localizing the diffusion to $\varepsilon$ is negligible if $\varepsilon$ is chosen very close to zero, and we can expect the estimated rate of convergence of CTMC approximation in the localized case applies to the model without localization. We emphasize that in our pricing algorithm the localization to $\varepsilon$ is not needed, as the implementation does not require assumption 1 and 0 is a finite model without localization. We emphasize that in our pricing algorithm the localization to $\varepsilon$ is not needed, as the implementation does not require assumption 1 and 0 is a finite model without localization. We do it here only to perform sharp error analysis.

The quadrature rule applied to integration on an interval $[a, b]$ in this paper satisfies

$$\sum_{i=0}^{n_q} a_i = b - a,$$

and $a_i > 0$ for $i = 0, 1, \ldots, n_q$. The equation holds because the quadrature rule is exact for integrating a constant function. The next assumption considers its convergence rate.

**Assumption 2.** For the quadrature rule, there exist positive integers $l$ and $p$ such that for any $f \in C^l[a, b]$, 

$$\left| \sum_{i=0}^{n_q} a_i f(y_i) - \int_a^b f(y) \, dy \right| \leq C n^{-l} \sup_{y \in [a, b]} |f^{(p)}(y)|,$$

where $C$ is a constant independent of $n_q, \{a_i\}_{i=0}^{n_q}$ and $f$.

To analyze the first part of the error in (17), we need to first establish the smoothness of the key quantity $P_{t}(M_{t}, < y)$ w.r.t. $y$. In previous works on barrier options (Li and Zhang 2018, Zhang and Li 2019, 2022), the barrier level is fixed, so the smoothness of the first passage probability w.r.t. the barrier level is not analyzed there.

To obtain the smoothness, we develop a representation for $g(t, x, y) := P_{t}(M_{t}, < y)$ based on eigenfunction expansions. Using Itô’s formula we can derive the PDE for $g(t, x, y)$ as

$$g_t = \mu(x)g_x + \frac{1}{2} \sigma^2(x)g_{xx}, \quad x \in (\ell, y), \ \tau > 0, $$

$$g_t(t, \ell, y) = 1, \quad g_t(t, x, y) = 0, \quad \tau \geq 0, $$

$$g(0, x, y) = 1, \quad x \in (\ell, y).$$

We decompose $g(t, x, y)$ as $g(t, x, y) = v(t, x, y) + w(x, y)$ with the two components satisfying

$$\mu(x)v'(x, y) + \frac{1}{2} \sigma^2(x)v''(x, y) = 0, \quad x \in (\ell, y), \quad w(\ell, y) = 1, \quad w(y, y) = 0, $$

and

$$v_t = \mu(x)v + \frac{1}{2} \sigma^2(x)v_x, \quad x \in (\ell, y), \quad v(\ell, y) = v(t, y, y) = 0, \quad \tau \geq 0, $$

$$v(0, x, y) = 1 - w(x, y), \quad x \in (\ell, y).$$

Let $T_0$ and $T_\varepsilon$ be the diffusion’s first hitting time of $\ell$ and $y$, respectively. The two quantities have a probabilistic meaning. We can show that $w(x, y) = P_x(T_0 < T_{\varepsilon})$ and $v(t, x, y) = P_x(t < T_{\varepsilon} < T_\varepsilon)$.

The ODE for $w(x, y)$ can be solved analytically with the solution given by

$$w(x, y) = 1 - \int_{x}^{y} \frac{d(z)}{d(z)} \, dz,$$

where

$$d(z) = \exp \left( -\int_{x}^{z} 2\mu(z) \frac{\sigma^2(z)}{\sigma^2(x)} \, dz \right)$$

is the scale density of the diffusion. The PDE for $v(t, x, y)$ can be solved by the separation of variables and we obtain the following representation as an eigenfunction expansion:

$$v(t, x, y) = \sum_{k=1}^{\infty} c_k(y) e^{-\lambda_k(t-x)} \varphi_k(x, y),$$

where $\{\varphi_k(x, y), \ k = 1, 2, \ldots\}$ are solutions to the Sturm–Liouville eigenvalue problem

$$\mu(x)\psi'(x) + \frac{1}{2} \sigma^2(x)\psi''(x) = \lambda \psi(x), \quad \ell < x < y,$$

$$\psi(\ell) = \psi(y) = 0,$$

and

$$c_k(y) = \int_{x}^{y} \varphi_k(z, y)(1 - w(z, y))m(z) \, dz, \quad m(z) = \frac{2}{\sigma^2(z)}.$$ 

Here, $c_k(y)$ is the $k$th expansion coefficient and $m(z)$ is the speed density of the diffusion.

**Lemma 1.** $\lambda_k(y), \varphi_k(x, y) \in C^\infty([M, A])$ as a function of $y$. For any nonnegative integer $m$, there exist constants $c_1, c_2 > 0$ and integer $p > 0$ independent of $y \in [M, A]$, $x \in [\ell, y]$ and $k \geq 1$ such that

$$\lambda_k(y) \geq c_1 k^2, \quad |\partial^m \lambda_k(y)| \leq c_2 k^p, \quad |\partial^m \varphi_k(x, y)| \leq c_2 k^p.$$

**Proof.** By Zhang and Li (2019) lemma 3, for any $y > \ell$, there exists constant $c_1(y) > 0$ such that $\lambda_k(y) \geq c_1(y) k^2$ for all $k \geq 1$. Extending the proof of Zhang and Li (2019) lemma 3, we can show that the coefficient $c_1(y)$ depends on $y$ continuously. Hence $c_1(y)$ has a lower bound $c_1 > 0$ for $y \in [M, A]$. It is proved in Zhang and Li (2023a) lemma 4.1 that the eigenvalues and eigenfunctions are three times continuously differentiable w.r.t. the boundary level $y$ by assuming that $\mu(x) \in C^3([a, b])$ and $\sigma(x) \in C^4([a, b])$ for any $[a, b] \subseteq [0, \infty)$. Further assuming that $\mu(x) \in C^{\infty}([a, b])$ and $\sigma(x) \in C^{\infty}([a, b])$ for any $[a, b] \subseteq [0, \infty)$ and extending the proof of Zhang and Li (2023a), we have that $\partial^m \lambda_k(y)$ and $\partial^m \varphi_k(x, y)$ are well defined for all nonnegative integer $m, k \geq 1, y \in [M, A]$ and $x \in [\ell, y]$, and they are bounded by $c_2 k^p$ for some constant $c_2 > 0$ independent of $k, y, x$.

**Lemma 2.** $P_x(M_{t} < y) \in C^\infty([M, A])$ as a function of $y$. For any nonnegative integer $m$, there exists a constant $C > 0$ independent of $y \in [M, A]$ and $x \in [\ell, y]$ such that $|\partial^m P_x(M_{t} < y)| \leq C$. 


Proof The smoothness of \( w(x;y) \) w.r.t. \( y \) can be seen from its analytical formula (18). For \( v(\tau,x,y) \), by lemma 1, for any nonnegative integer \( m \), we have that \( |\partial_y^m c_k(y)| \leq C_1 k^p \), \( |\partial_y^m e^{-\lambda(y)\tau}| \leq C_1 k^p e^{-C_1 k^2} \) for constant \( C_1 \), \( C_2 > 0 \) and integer \( p_1 > 0 \) independent of \( k \geq 1 \) and \( y \in [M,A] \). Hence, there exist constants \( C_3, C_4 \) and integer \( p_2 > 0 \) such that,

\[
\sum_{k=1}^\infty |\partial_y^m (c_k(y)e^{-\lambda(y)\tau} \varphi_k(x;y))| \\
\leq C_3 \sum_{k=1}^\infty k^{p_2} e^{-C_1 k^2} \leq C_4 < \infty.
\]

Then \( v(\tau,x,y) \) is \( m \)-times differentiable w.r.t. \( y \) and

\[
\partial_y^m v(\tau,x,y) = \sum_{k=1}^\infty \partial_y^m (c_k(y)e^{-\lambda(y)\tau} \varphi_k(x;y)).
\]

Hence, \( |\partial_y^m v(\tau,x,y)| \leq C_4 \). The claim follows by combining the smoothness of \( w(x;y) \) and \( v(\tau,x,y) \) and noting that \( P_t(M_t < y) = w(x;y) + v(\tau,x,y) \).

Now, we are ready to analyze the error caused by quadrature and CTMC approximation.

**THEOREM 1** Suppose assumptions 1 and 2 hold. Then we have that,

\[
|\tilde{u}_{t_0}^{\text{bp},(n)}(t,x,M) - u^{\text{bp}}(t,x,M)| \\
\leq C_1 \delta_n + C_2 \mathcal{E}_{n_t} (P_t(M_t < \cdot)),
\]

where \( C_1, C_2 > 0 \) are constants independent of \( x \in \mathbb{S}^{(n)} \), \( n \) and the quadrature scheme used, \( \gamma = 1 \) in general and \( \gamma = 2 \) if all of the points \( y_0, y_1, \ldots, y_n \in \mathbb{S}^{(n)} \), and

\[
\mathcal{E}_{n_t} (P_t(M_t < \cdot)) \\
= \sum_{i=0}^n \alpha_i P_x(M_t < y_i) - \int_A^M P_x(M_t < y) \, dy \\
\leq C_3 n_t^{-1} \sup_{y \in [M,A]} |\partial_y^0 P_x(M_t < y)|.
\]

**Proof** By (8) and (9), we have that,

\[
|\tilde{u}_{t_0}^{\text{bp},(n)}(t,x,M) - u^{\text{bp}}(t,x,M)| \\
= \sum_{i=0}^n \alpha_i P_x(M_t^{(n)} < y_i) - \int_A^M P_x(M_t < y) \, dy \\
\leq \sum_{i=0}^n \alpha_i \left| P_x(M_t^{(n)} < y_i) - P_x(M_t < y_i) \right| \\
+ \sum_{i=0}^n \alpha_i \max_{0 \leq s \leq t} |P_x(M_t^{(n)} < y_i) - P_x(M_t < y_i)| \\
\leq (A - M) \max_{0 \leq s \leq t} |P_x(M_t^{(n)} < y_i) - P_x(M_t < y_i)| \\
+ \mathcal{E}_{n_t} (P_x(M_t < \cdot)).
\]

It is proved that in Zhang and Li (2019) theorem 1 that

\[
|P_x(M_t^{(n)} < y) - P_t(M_t < y)| \leq \tilde{C}(y) \delta_n
\]

for some constant \( \tilde{C}(y) > 0 \) independent of \( n \) and \( x \). Furthermore, the theorem shows that if the barrier level \( y \in \mathbb{S}^{(n)} \), then there holds that

\[
|P_x(M_t^{(n)} < y) - P_t(M_t < y)| \leq \tilde{C}(y) \delta_n^2.
\]

Inspecting the proof of Zhang and Li (2019) theorem 1, we can show that the coefficient \( \tilde{C}(y) \) depends on \( y \) continuously and hence it has an upper bound \( \tilde{C} > 0 \) for \( y \in [y_0, y_1, \ldots, y_n] \subset [M,A] \). Therefore, the first part of error is bounded by \((A-M)C\delta_n^2\). For the second part of error, it suffices to recall that \( P_t(M_t < y) \) as a function of \( y \) is in \( C^{\infty}([M,A]) \) as proved in lemma 2. This concludes the proof.

**3.3. Results for the floating-strike lookback call**

The error of our approximation for the floating-strike call can be decomposed as

\[
\tilde{u}_{t_0}^{\text{fl},(n)}(t,x,m) - u^{\text{fl}}(t,x,m) \\
= u^{\text{fl},(n)}(t,x,m) - u^{\text{fl}}(t,x,m) + u^{\text{fl}}(t,x,m) - u^{\text{fl}}(t,x,m) \\
= u^{\text{fl},(n)}(t,x,m) - u^{\text{fl}}(t,x,m) + e^{-\tau t} \int_0^t P_x(m_t > y) \, dy \\
- e^{-\tau t} \int_0^t P_x(m_t^U > y) \, dy,
\]

where \( m_U \) is the running minimum of the localized version of \( X \) which is absorbed at \( U \), and

\[
e^{-\tau t} \int_0^t P_x(m_t > y) \, dy.
\]

The first difference in (19) is the truncation error for the floating-strike lookback call under the diffusion and exponential Lévy models.

**PROPOSITION 3** Assume that \( X \) is a diffusion process with \( \mu^2(x) + \sigma^2(x) \leq L(1 + |x|) \) for all \( x \geq 0 \) for some constant...
Let $x > 0$. Then for any integer $p \geq 1$, there exists a constant $C > 0$ independent of $x$ and $U$ such that

$$\left| e^{-rT} \int_{0}^{m} P_{x}(m_{\tau} > y) \, dy - e^{-rT} \int_{0}^{m} P_{x}(m_{U}^{U} > y) \, dy \right| \leq C \frac{1 + x^{2p}}{U^{2p}}.$$ 

**Proposition 4** Suppose $X$ is an exponential Lévy process, and there exists a constant $\beta > 0$ such that $\int_{|x| > 1} e^{(1+\beta)x} \nu(dx) < \infty$, where $\nu(\cdot)$ is the Lévy measure of $\ln X$. Then there exists a constant $C > 0$ independent of $x$ and $U$ such that

$$\left| e^{-rT} \int_{0}^{m} P_{x}(m_{\tau} > y) \, dy - e^{-rT} \int_{0}^{m} P_{x}(m_{U}^{U} > y) \, dy \right| \leq C \frac{x^{1+\beta}}{U^{1+\beta}}.$$ 

For 1D diffusion models, we can analyze the error of quadrature and CTMC approximation in the same way as theorem 1.

![Figure 1](image_url)

Figure 1. Convergence of a floating strike lookback put option in the Black-Scholes model, the regime-switching Black-Scholes model and the CEV model. Here ‘trap-11’ means the trapezoidal rule with 11 points is used in (10) and ‘gauss-11’ means the Gauss quadrature rule with 11 terms is applied. ‘trap-21’, ‘trap-31’, ‘gauss-21’ and ‘gauss-31’ have similar meanings. On the right panel, the ‘Original’ is the corresponding line of ‘gauss-11’ on the left panel except the CEV model for which ‘gauss-21’ is used. The ‘Extrapolation’ line is obtained from ‘Original’ by applying Richardson extrapolation based on second-order convergence.
Theorem 2. Suppose assumptions 1 and 2 hold. Then we have
\[ |u_{nq}^{ak}(t, x, m) - u_{nq}(t, x, m)| \leq C_1 \delta_n^\gamma + C_2 E_nq(P_x(m_t > \cdot)), \]
where \( C_1, C_2 > 0 \) are constants independent of \( x \in \mathbb{S}(n), n \) and the quadrature scheme used, \( \gamma = 1 \) in general and \( \gamma = 2 \) if all of the points \( y_0, y_1, \ldots, y_{nq} \in \mathbb{S}(n) \), and
\[ E_nq(P_x(m_t > \cdot)) = \left| \sum_{p=0}^{nq} \omega_p P_x(m_t > y_p) - \int_0^m P_x(m_t > y) \, dy \right| \leq C_3 n^{-l} \sup_{y \in (0, m)} \left| \partial_y \log P_x(m_t > y) \right| . \]

4. Numerical results

We consider four representative models to evaluate the performance of our method:

- The Black-Scholes (BS) model: \( dX_t = (r - d) X_t dt + \sigma X_t dW_t, \ t \geq 0 \). We set \( \sigma = 0.3 \).
- A regime-switching BS model with two regimes: \( dX_t = (r - d) X_t dt + \sigma_v t X_t dW_t, \ t \geq 0 \), where \( \sigma_v \) is a CTMC with state space \{1, 2\}, and its transition rate matrix is given by
\[ \Lambda = \begin{pmatrix} -0.75 & 0.75 \\ 0.25 & -0.25 \end{pmatrix} . \]

- The CEV model (Davydov and Linetsky 2001):
\[ dX_t = (r - d - \lambda \xi) X_t \, dt + \sigma X_t^{1+\beta} dW_t, \ t \geq 0 \]. We set \( \sigma = 0.25 \) and \( \beta = -0.5 \).

- Kou’s double-exponential jump-diffusion model (Kou 2002):
\[ dX_t = (r - d - \lambda \xi) X_t \, dt + \sigma dW_t + d \left( \sum_{i=1}^{N_t} (V_i - 1) \right) , \]
where \( N_t \) is a Poisson process with intensity \( \lambda \), \( \{V_i, i \geq 1\} \) is a sequence of i.i.d. random variables with the density of \( \ln V_i \) given by
\[ f_{\ln V_i}(y) = p^+ \eta^+ e^{\eta^+ y} 1_{y \geq 0} + p^- \eta^- e^{\eta^- y} 1_{y < 0} , \]
and \( \xi = \mathbb{E}[V_i] - 1 = p^+ \eta^+ + p^- \eta^- - 1 \). We set \( \sigma = 0.3, \lambda = 3.0, \eta^+ = \eta^- = 10, \) and \( p^+ = p^- = 0.5 \). The Kou model is a popular jump-diffusion model with finite jump activity.

Figure 2. Convergence of a floating strike lookback put option in the Kou model and the CGMY model with extrapolation results. Here ‘trap-11’, ‘gauss-11’, ‘trap-21’ and ‘gauss-21’ have the same meaning as in figure 1. On the right panel, the ‘Original’ is the corresponding line of ‘gauss-11’ on the left panel. The ‘Extrapolation’ line is obtained from ‘Original’ by three-point extrapolation.
The Carr-Geman-Madan-Yor (CGMY) model (Carr et al. 2002):

\[ X_t = X_0 e^{(r-d-o) t + Z_t}, \]

where \( \omega = C \Gamma (-Y) [(G + 1)^Y - G^Y + (M - 1)^Y - M^Y] \) and \( Z_t \) is a pure-jump Lévy process with

Lévy density

\[ \nu(x) = C \left( \frac{e^{-G|x|}}{|x|^2 + Y} I_{x < 0} + \frac{e^{-Mx}}{x^{1+Y}} I_{x > 0} \right). \]

We set the parameters as \( C = 1, G = 9, M = 8, \ Y = 0.5 \) or 1.2. The jumps of this process have finite variation if \( Y = 0.5 \) and infinite variation if

| Table 2. Results of CTMC approximation for floating-strike lookback put options with various \( M \) and \( T \) under the CGMY model with \( Y = 0.5 \). |
|---|---|---|---|---|
| \( T \) | \( n \) | CTMC | Error | Time (s) | Extra. |
|---|---|---|---|---|---|
| 0.5 | 111 | 0.124979 | 1.37E-02 | 0.11 |  
| 199 | 0.118876 | 7.57E-03 | 0.33 |  
| 397 | 0.115030 | 3.72E-03 | 1.31 | 0.111148 | 1.61E-04 |
| 793 | 0.113121 | 1.81E-03 | 5.07 | 0.111246 | 6.23E-05 |
| 1 | 111 | 0.185315 | 1.92E-02 | 0.11 |  
| 199 | 0.176797 | 1.07E-02 | 0.32 |  
| 397 | 0.171391 | 5.26E-03 | 1.37 | 0.165850 | 2.79E-04 |
| 793 | 0.168703 | 2.57E-03 | 5.63 | 0.166060 | 6.93E-05 |
| 5 | 127 | 0.452960 | 8.24E-02 | 0.17 |  
| 253 | 0.415163 | 4.46E-02 | 0.62 |  
| 505 | 0.393278 | 2.27E-02 | 2.66 | 0.363439 | 7.09E-03 |
| 967 | 0.382452 | 1.19E-02 | 9.86 | 0.369847 | 6.82E-04 |
| \( M = 1.02 \) | | | | |  
| 0.1 | 100 | 0.048534 | 4.70E-03 | 0.10 |  
| 199 | 0.046126 | 2.29E-03 | 0.33 |  
| 397 | 0.044965 | 1.13E-03 | 1.19 | 0.043895 | 5.90E-05 |
| 793 | 0.044395 | 5.59E-04 | 4.86 | 0.043847 | 1.05E-05 |
| 0.5 | 111 | 0.124979 | 1.37E-02 | 0.11 |  
| 199 | 0.118876 | 7.57E-03 | 0.33 |  
| 397 | 0.115030 | 3.72E-03 | 1.31 | 0.111148 | 1.61E-04 |
| 793 | 0.113121 | 1.81E-03 | 5.07 | 0.111246 | 6.23E-05 |
| 5 | 127 | 0.452960 | 8.24E-02 | 0.17 |  
| 253 | 0.415163 | 4.46E-02 | 0.62 |  
| 505 | 0.393278 | 2.27E-02 | 2.66 | 0.363439 | 7.09E-03 |
| 967 | 0.382452 | 1.19E-02 | 9.86 | 0.369847 | 6.82E-04 |
| \( M = 1.05 \) | | | | |  
| 0.1 | 100 | 0.066676 | 2.63E-03 | 0.11 |  
| 199 | 0.065324 | 1.27E-03 | 0.36 |  
| 397 | 0.064679 | 6.30E-04 | 1.35 | 0.064096 | 4.64E-05 |
| 793 | 0.064364 | 3.14E-04 | 5.41 | 0.064063 | 1.42E-05 |
| 0.5 | 111 | 0.132707 | 1.16E-02 | 0.12 |  
| 199 | 0.127487 | 6.38E-03 | 0.38 |  
| 397 | 0.124259 | 3.15E-03 | 1.32 | 0.121132 | 2.31E-05 |
| 793 | 0.122666 | 1.56E-03 | 5.38 | 0.121122 | 1.36E-05 |
| 5 | 127 | 0.452031 | 7.88E-02 | 0.16 |  
| 253 | 0.415741 | 4.25E-02 | 0.61 |  
| 505 | 0.394867 | 2.16E-02 | 2.37 | 0.366848 | 6.38E-03 |
| 967 | 0.384572 | 1.13E-02 | 10.40 | 0.372658 | 5.66E-04 |
| \( M = 1.1 \) | | | | |  
| 0.1 | 100 | 0.107002 | 1.21E-03 | 0.09 |  
| 199 | 0.106377 | 5.85E-04 | 0.33 |  
| 397 | 0.106081 | 2.89E-04 | 1.27 | 0.105819 | 2.71E-05 |
| 793 | 0.105396 | 1.45E-04 | 5.44 | 0.105798 | 6.19E-06 |
| 0.5 | 111 | 0.153991 | 8.68E-03 | 0.12 |  
| 199 | 0.150055 | 4.74E-03 | 0.38 |  
| 397 | 0.147646 | 2.33E-03 | 1.32 | 0.145363 | 4.96E-05 |
| 793 | 0.146468 | 1.15E-03 | 5.22 | 0.145347 | 3.42E-05 |
| 1 | 111 | 0.204592 | 1.44E-02 | 0.11 |  
| 199 | 0.198099 | 7.95E-03 | 0.33 |  
| 397 | 0.194059 | 3.91E-03 | 1.24 | 0.190093 | 5.35E-05 |
| 793 | 0.192076 | 1.93E-03 | 5.19 | 0.190176 | 2.88E-05 |
| 5 | 127 | 0.453184 | 7.33E-02 | 0.16 |  
| 253 | 0.419209 | 3.93E-02 | 0.59 |  
| 505 | 0.399876 | 1.99E-02 | 2.48 | 0.374564 | 5.37E-03 |
| 967 | 0.390386 | 1.05E-02 | 10.07 | 0.379516 | 4.17E-04 |

Note: The computation times are reported in seconds and the ‘Extra.’ columns present the approximate prices after extrapolation. In our experiment, we set \( A = 2 \) for \( T = 0.1, A = 5 \) for \( T = 0.5, 1, \) and \( A = 30 \) for \( T = 5 \).
Y = 1.2. The CGMY model belongs to the KoBoL family (Boyarchenko and Levendorskii 2000, 2002) whose Lévy density is in the following form:

$$\Pi(x) = c_+ e^{-\lambda_- |x|} \mathbb{1}_{|x| < 0} + c_+ e^{-\lambda_+ |x|} \mathbb{1}_{|x| > 0}.$$  

In our setting, $C = c_\pm$, $Y = \nu$, $\lambda_- = G$ and $\lambda_+ = M$.

The BS and CEV model are two popular 1D diffusion models, and the last two are well-known models with jumps. The Kou model is a jump-diffusion with finite jump activity and the CGMY model is a pure-jump process with infinite jump activity.

### Table 3. Results of CTMC approximation for floating-strike lookback put options with various $M$ and $T$ under the CGMY model with $Y = 1.2$.

| $T$ | $n$ | CTMC | Time (s) | Extra. | Error |
|-----|-----|------|----------|--------|-------|
|     |     |      |          |        |       |
| 0.1 | 100 | 1.167480 | 1.45E-02 | 0.11   |
| 199 | 0.161833 | 8.86E-03 | 0.40    |
| 397 | 0.158254 | 5.28E-03 | 1.55    |
| 793 | 0.156078 | 3.11E-03 | 6.21    |
| 0.5 | 111 | 0.421896 | 4.03E-02 | 0.15   |
| 199 | 0.409175 | 2.76E-02 | 0.38    |
| 397 | 0.398300 | 1.69E-02 | 1.51    |
| 793 | 0.391694 | 1.01E-02 | 6.87    |
| 1   | 111 | 0.619694 | 5.46E-02 | 0.15   |
| 199 | 0.597104 | 3.78E-02 | 0.38    |
| 397 | 0.582673 | 2.33E-02 | 1.44    |
| 793 | 0.573352 | 1.40E-02 | 6.69    |
| 5   | 127 | 1.737191 | 3.99E-01 | 0.21   |
| 253 | 1.529542 | 1.92E-01 | 0.77    |
| 505 | 1.437313 | 9.95E-02 | 2.97    |
| 967 | 1.395605 | 5.76E-02 | 13.19   |

Note: The computation times are reported in seconds and the ‘Extra.’ columns present the approximate prices after extrapolation. In our experiment, we set $A = 2$ for $T = 0.1$, $A = 5$ for $T = 0.5, 1$, and $A = 30$ for $T = 5$. 

4.1. Results for floating-strike lookback puts

We price a seasoned floating-strike lookback put option at \( t = 0 \). Except the CEV model, we set \( s = 1, M = 1.5, T = 1 \), the risk-free rate \( r = 0.05 \) and dividend yield \( d = 0.02 \). For the CEV model, we set \( s = M = 1, T = 0.5, r = 0.1 \) and \( d = 0 \), and these values are taken from Davydov and Linetsky (2001) so that we can use the price reported there computed by their analytical formula as a benchmark. In our implementation, we use scheme 1 (see appendix A.1) and the grid design in section 2.3 by placing all quadrature points on the grid of the CTMC.

The left panels of figures 1 and 2 display the convergence of the option price against the number of Markov chain grid points under these five models using the trapezoid rule and the Gauss-Legendre quadrature. From these plots, it is clear that using the grid design in section 2.3 attains smooth convergence in all cases. Gauss quadrature outperforms the trapezoid rule overwhelmingly in each case and should be the preferred choice. Under the trapezoid rule with a fixed number of quadrature points, the pricing error barely decays after the number of grid points for the CTMC passes some level. This is because the numerical integration error remains and it is quite significant even though the CTMC approximation error becomes very small. Except for the CEV model, the 11-point Gauss quadrature suffices for reaching a high level of accuracy for numerical integration. For the CEV model, 21 points are needed in the Gauss quadrature for highly accurate results, which is likely due to the exploding volatility near zero in this model.

Using Gauss quadrature, the numerical integration error becomes negligible. Thus, we can estimate the convergence rate of CTMC approximation numerically for each model by regressing the logarithmic error against the number of Markov chain states in the results from Gauss quadrature. To calculate the error, the benchmark is computed by the closed-form formula derived from Goldman et al. (1979) for the BS model and taken from Davydov and Linetsky (2001) for the CEV model. For the other three models, we use the result of our algorithm with a very large \( n \) as a benchmark. We find that the convergence order is 1.99 for the BS model, 2.01 for the CEV model and 2.02 for the regime-switching BS model, 1.92 for the Kou model and 1.06 for the CGMY model. The estimate convergence orders for BS and CEV are very close to the theoretical convergence order of two.

As there are no oscillations, we can apply extrapolation to accelerate convergence. For the Kou and CGMY model, we follow the extrapolation method used in section 5 of Zhang and Li (2019) with estimated convergence orders. Such method requires three points for extrapolation in contrast to the standard two-point extrapolation method when the convergence order is known. For the BS, CEV and regime-switching BS model, we apply the two-point extrapolation based on second-order convergence. The right panels of figures 1 and 2 clearly show the effectiveness of extrapolation. Using \( n \) below 100, one can attain a high level of accuracy that requires \( n \) to be several hundred or even more without extrapolation.

Tables 2 and 3 present the results for the floating strike lookback put option with maturities \( T = 0.1, 0.5, 1, 5 \) and maximum asset prices at time zero \( M = 1.02, 1.05, 1.1 \) under the CGMY models with \( Y = 0.5 \) and \( Y = 1.2 \). Other option contract and model parameters are set the same as before. The performance of our algorithm is similar for various values of \( M \) under consideration. When \( T \) increases and the other parameters remain fixed, a larger truncation level \( A \) is needed, which leads to a larger domain as the state space of the CTMC (\( A \) is the maximum state). Subsequently, we also need to use more grid points for the CTMC. Thus, our algorithm becomes less efficient as \( T \) increases. In the case of \( Y = 0.5 \) (the jumps of CGMY have finite variation), our algorithm still performs quite well after extrapolation is applied for \( T = 1, 5 \) years (see the last column of table 2). The estimated convergence order in this case is roughly first. When \( Y = 1.2 \) (the jumps of CGMY have infinite variation), extrapolation still reduces error, but the relative error after extrapolation remains around 0.4% for \( T = 1 \) year, and 1.1% for \( T = 5 \) years. In this case,

| \( U \) | \( n \) | Scheme 1 | Error | Extra.Err | Scheme 2 | Error | Extra.Err |
|-------|------|---------|-------|-----------|---------|-------|-----------|
| 20    | 201  | 0.472584| 7.53E - 02 | 0.475032 | 7.78E - 02 | 0.472584 | 7.53E - 02 |
| 401   | 0.438067 | 4.08E - 02 | 0.440442 | 4.32E - 02 | 0.438067 | 4.08E - 02 |
| 801   | 0.418930 | 2.17E - 02 | 0.421250 | 2.40E - 02 | 0.418930 | 2.17E - 02 |
| 1601  | 0.408574 | 1.13E - 02 | 0.410861 | 1.36E - 02 | 0.408574 | 1.13E - 02 |
| 25    | 251  | 0.473134 | 7.59E - 02 | 0.474935 | 7.77E - 02 | 0.473134 | 7.59E - 02 |
| 501   | 0.438600 | 4.13E - 02 | 0.440348 | 4.31E - 02 | 0.438600 | 4.13E - 02 |
| 1001  | 0.419452 | 2.22E - 02 | 0.421159 | 2.39E - 02 | 0.419452 | 2.22E - 02 |
| 2001  | 0.409088 | 1.18E - 02 | 0.410771 | 1.35E - 02 | 0.409088 | 1.18E - 02 |

| \( T = 5 \) |
| 20    | 201  | 0.757742 | 4.97E - 02 | 0.762118 | 5.40E - 02 | 0.757742 | 4.97E - 02 |
| 401   | 0.737508 | 2.94E - 02 | 0.741855 | 3.38E - 02 | 0.737508 | 2.94E - 02 |
| 801   | 0.723719 | 1.56E - 02 | 0.728024 | 2.00E - 02 | 0.723719 | 1.56E - 02 |
| 1601  | 0.715280 | 7.21E - 03 | 0.719553 | 1.15E - 02 | 0.715280 | 7.21E - 03 |
| 25    | 251  | 0.758843 | 5.08E - 02 | 0.762132 | 5.41E - 02 | 0.758843 | 5.08E - 02 |
| 501   | 0.738588 | 3.05E - 02 | 0.741850 | 3.38E - 02 | 0.738588 | 3.05E - 02 |
| 1001  | 0.724783 | 1.67E - 02 | 0.728011 | 1.99E - 02 | 0.724783 | 1.67E - 02 |
| 2001  | 0.716332 | 8.26E - 03 | 0.719536 | 1.15E - 02 | 0.716332 | 8.26E - 03 |
the estimated convergence rates range from 0.7 to 1.1 for different \( M \) and \( T \). Overall, our algorithm works best for short to medium maturities. For maturities greater than or equal to one year, it can become less efficient (in terms of computational time) and less accurate as the maturity increases.

### 4.2. Results for floating-strike lookback calls

We compare two CTMC construction schemes, where scheme 2 (see appendix A.2) can potentially reduce the error of truncating the state space compared with scheme 1, especially for models with jumps. Tables 4 and 5 show the results for floating-strike lookback call options with maturities \( T = 1, 5 \) and \( m = 1 \) under the CGMY models with parameters \( C = 1, G = 3, M = 2 \), and \( Y = 0.5, 1.2 \). Now the values of \( G \) and \( M \) are much smaller than those used for the lookback put, implying very slow decay of the intensity of large jumps. This is a computationally difficult case where the level of truncation can have a significant impact on accuracy. We consider two truncation levels for the state space: \( U = 20, 25 \). We use the same values for \( x, r, d \) as for the lookback put. We can see that with the same \( U \) the errors after extrapolation under scheme 2 are smaller than under scheme 1 in almost all the cases except when \( T = 1 \) and \( U = 20 \). Hence, we have confirmed that scheme 2 can reduce the truncation error.

### 4.3. Comparison with finite difference method

An alternative general approach to price a lookback option is numerically solving the PDE (for diffusions) or PIDE (for processes with jumps) it satisfies. We provide the detail of a finite difference scheme in appendix 2. Figure 3 compares the performance of our algorithm to the finite difference scheme under the CEV model for the same floating-strike lookback put. For a similar amount of time taken, our algorithm is significantly more accurate than finite difference. Table 6 reports the comparison under the BS model for various initial maximum asset prices \( M = 1.02, 1.05, 1.1 \) and maturities \( T = 0.1, 0.5, 1.5 \). Our algorithm performs very well and achieves much smaller error than the finite difference scheme for each combination of \( M \) and \( T \).

It should be pointed out that as our method is a general approach, it is not quite fair to compare it with an algorithm that only applies to specific models. These bespoke algorithms take advantage of some special properties of the specific models; see Linetsky (2004) for some diffusions, Boyarchenko and Levendorskii (2013), Feng and Linetsky (2009), and Fusai et al. (2016b) for Lévy processes. If one is only interested in a specific model, it is possible that a bespoke algorithm can be better than ours. The strength of our method lies in its general applicability and its competitive edge over some alternative general approaches in computational performance.

| \( U \) | \( n \) | Scheme 1 | Error | Extra.Err | Scheme 2 | Error | Extra.Err |
|---|---|---|---|---|---|---|---|
| 20 | 201 | 0.639290 | 4.9E−02 | | 0.639942 | 5.04E−02 |
| 401 | 0.621331 | 3.18E−02 | | 0.621986 | 3.24E−02 |
| 801 | 0.609688 | 2.01E−02 | 1.21E−03 | 0.610344 | 2.08E−02 | 5.35E−04 |
| 1601 | 0.602174 | 1.26E−02 | 1.02E−03 | 0.602831 | 1.33E−02 | 3.67E−04 |
| 25 | 251 | 0.639445 | 4.99E−02 | | 0.639903 | 5.03E−02 |
| 501 | 0.621487 | 3.19E−02 | | 0.621947 | 3.24E−02 |
| 1001 | 0.609845 | 2.03E−02 | 1.08E−03 | 0.610306 | 2.07E−02 | 6.14E−04 |
| 2001 | 0.602332 | 1.28E−02 | 8.76E−04 | 0.602793 | 1.32E−02 | 4.14E−04 |

\( T = 1 \)

| \( U \) | \( n \) | Scheme 1 | Error | Extra.Err | Scheme 2 | Error | Extra.Err |
|---|---|---|---|---|---|---|---|
| 20 | 201 | 0.845868 | 1.67E−02 | | 0.846879 | 1.77E−02 |
| 401 | 0.840638 | 1.15E−02 | | 0.841687 | 1.25E−02 |
| 801 | 0.836656 | 7.51E−03 | 5.09E−03 | 0.837726 | 8.58E−03 | 4.03E−03 |
| 1601 | 0.833715 | 4.57E−03 | 3.70E−03 | 0.834798 | 5.60E−03 | 2.62E−03 |
| 25 | 251 | 0.847341 | 1.82E−02 | | 0.848179 | 1.90E−02 |
| 501 | 0.842054 | 1.29E−02 | | 0.842915 | 1.38E−02 |
| 1001 | 0.838032 | 8.89E−03 | 3.81E−03 | 0.838906 | 9.76E−03 | 2.95E−03 |
| 2001 | 0.835066 | 5.92E−03 | 2.38E−03 | 0.835948 | 6.81E−03 | 1.51E−03 |

\( T = 5 \)

Figure 3. Comparison between a finite difference scheme and our algorithm for a floating-strike lookback put option under the CEV model. The benchmark price is obtained from Davydov and Linetsky (2001).
Table 6. Comparison between CTMC and the finite difference scheme for floating-strike lookback put options with various $M$ and $T$ under the BS model.

| $T$ | $n$     | CTMC       | Error | Time (s) | FD       | Error | Time (s) |
|-----|---------|------------|-------|----------|----------|-------|----------|
|     |         | $M = 1.02$ |       |          | $M = 1.02$ |       |          |
| 0.1 | 100     | 0.078206   | 3.02E-04 | 0.01     | 0.082615 | 4.71E-03 | 1.00 |
|     | 199     | 0.077981   | 7.67E-05 | 0.06     | 0.079124 | 2.12E-03 | 5.55 |
|     | 397     | 0.077923   | 1.93E-05 | 0.44     | 0.078135 | 2.31E-04 | 30.78 |
| 0.5 | 111     | 0.171658   | 4.41E-04 | 0.02     | 0.177360 | 6.14E-03 | 1.20 |
|     | 199     | 0.171356   | 1.39E-04 | 0.06     | 0.173202 | 1.98E-03 | 5.46 |
|     | 397     | 0.171252   | 3.46E-05 | 0.40     | 0.171537 | 3.19E-04 | 32.62 |
| 1   | 111     | 0.240523   | 3.51E-04 | 0.02     | 0.256155 | 1.60E-02 | 1.17 |
|     | 199     | 0.240282   | 1.10E-04 | 0.06     | 0.243979 | 3.81E-03 | 5.29 |
|     | 397     | 0.240200   | 2.76E-05 | 0.42     | 0.239758 | 4.14E-04 | 30.32 |
| 5   | 127     | 0.493870   | 8.82E-05 | 0.06     | 0.509701 | 1.59E-02 | 1.68 |
|     | 253     | 0.493801   | 1.95E-05 | 0.22     | 0.497445 | 3.66E-03 | 9.92 |
|     | 505     | 0.493786   | 4.22E-06 | 1.65     | 0.491666 | 2.12E-03 | 66.90 |
|     |         | $M = 1.05$ |       |          | $M = 1.05$ |       |          |
| 0.1 | 100     | 0.086653   | 2.17E-04 | 0.01     | 0.090869 | 4.43E-03 | 0.88 |
|     | 199     | 0.086491   | 5.52E-05 | 0.06     | 0.087499 | 1.06E-03 | 5.29 |
|     | 397     | 0.086450   | 1.38E-05 | 0.42     | 0.086647 | 2.12E-04 | 34.72 |
| 0.5 | 111     | 0.175521   | 3.94E-04 | 0.02     | 0.182100 | 6.97E-03 | 1.29 |
|     | 199     | 0.175247   | 1.21E-04 | 0.05     | 0.177128 | 2.00E-03 | 5.73 |
|     | 397     | 0.175156   | 2.93E-05 | 0.37     | 0.175458 | 3.31E-04 | 33.03 |
| 1   | 111     | 0.243255   | 3.31E-04 | 0.02     | 0.249379 | 6.45E-03 | 1.16 |
|     | 199     | 0.243026   | 1.02E-04 | 0.06     | 0.245794 | 2.87E-03 | 4.94 |
|     | 397     | 0.242949   | 2.55E-05 | 0.38     | 0.243315 | 3.92E-04 | 30.81 |
| 5   | 127     | 0.495014   | 1.45E-04 | 0.04     | 0.501965 | 7.10E-03 | 1.73 |
|     | 253     | 0.494903   | 3.40E-05 | 0.22     | 0.495970 | 1.10E-03 | 10.14 |
|     | 505     | 0.494877   | 7.87E-06 | 1.48     | 0.494322 | 5.47E-04 | 63.16 |
|     |         | $M = 1.1$  |       |          | $M = 1.1$  |       |          |
| 0.1 | 100     | 0.114152   | 1.40E-04 | 0.02     | 0.116933 | 2.92E-03 | 1.08 |
|     | 199     | 0.114047   | 3.52E-05 | 0.06     | 0.114781 | 7.69E-04 | 5.78 |
|     | 397     | 0.114020   | 8.80E-06 | 0.42     | 0.114171 | 1.59E-04 | 34.84 |
| 0.5 | 111     | 0.188838   | 3.12E-04 | 0.02     | 0.194812 | 6.29E-03 | 1.31 |
|     | 199     | 0.188622   | 9.54E-05 | 0.06     | 0.190471 | 1.94E-03 | 5.74 |
|     | 397     | 0.188549   | 2.28E-05 | 0.39     | 0.188860 | 3.34E-04 | 35.16 |
| 1   | 111     | 0.252716   | 2.87E-04 | 0.02     | 0.260785 | 8.36E-03 | 1.16 |
|     | 199     | 0.252518   | 8.87E-05 | 0.06     | 0.254786 | 2.36E-03 | 5.15 |
|     | 397     | 0.252452   | 2.22E-05 | 0.38     | 0.252747 | 3.18E-04 | 31.77 |
| 5   | 127     | 0.498873   | 2.23E-04 | 0.04     | 0.504826 | 6.18E-03 | 1.66 |
|     | 253     | 0.498704   | 5.38E-05 | 0.22     | 0.499702 | 1.05E-03 | 9.94 |
|     | 505     | 0.498663   | 1.28E-05 | 1.50     | 0.498081 | 5.69E-04 | 63.74 |

Note: The benchmark prices are obtained by the closed-form formula in Goldman et al. (1979). The columns ‘CTMC’ and ‘FD’ contain the results of our algorithm and the finite difference scheme, respectively. The computation times are reported in seconds.

5. Conclusion

This paper develops a general approach to price lookback options by combining numerical quadrature with continuous-time Markov chain approximation. CTMC approximation has proved to be a computationally efficient tool for pricing various types of options. Our analysis reveals that directly using CTMC approximation to price lookbacks is not efficient enough. However, when it is combined with an efficient numerical quadrature such as a Gauss quadrature, we can obtain a very efficient algorithm. Our method is applicable to a range of commonly used Markov models, including 1D time-homogeneous and time-inhomogeneous Markov processes, regime-switching models and stochastic local volatility models.

The lookback options considered in the paper do not allow early exercise. The pricing of American-style floating-strike lookback puts is treated in Zhang et al. (2021), where we convert it to the pricing of American drawdown options after a measure change. The problem can be efficiently solved by combining CTMC approximation with an efficient solver for variational inequalities. The other types of American-style lookback options can be priced using similar ideas. In future research, we can consider extending our approach to deal with other types of nonstandard lookback options (see Fusai 2010) as well as two-asset double lookback options proposed by He et al. (1998).

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Disclosure statement

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Here, \( d_x \) collects the distances of other states to state \( x \). Let \( \alpha(x, z_i) = (z_i + z_{i+1})/2 \) for \( 0 \leq i < n - 1 \). We also define \( \alpha(x, z_{-1}) := -x \) and \( \alpha(x, z_n) := \infty \). Let \( A_j(x) = (\alpha(x, z_{j-1}), \alpha(x, z_j)) \) for \( 0 \leq j \leq n \). For \( x = y_j \in S^p \), define
\[
\Lambda(x, A_j(x)) := \int_{A_j(x)} \nu(x, dz), \quad 0 \leq j \leq n, j \neq i.
\]
which is the jump intensity around \( y_j \) given the current state is \( x \), and
\[
\Delta^2(x) := \int_{A_j(x)} z_j^2 1_{\{|z| \leq 1\}} \nu(x, dz),
\]
\[
\mu(x) := \sum_{j \neq i} z_j \int_{A_j(x)} 1_{\{|z| \leq 1\}} \nu(x, dz).
\]
The first one is the variance of the diffusion approximation of small jumps around the current state \( x \), and the second one gives the average jump size for small jumps outside the neighborhood of \( x \). Then, the integral is approximated as
\[
\int_{-\infty}^{\infty} \left( f(x + z) - f(x) - z_1 1_{\{|z| \leq 1\}} f'(x) \right) \nu(x, dz)
\]
\[
\approx \int_{A_j(x)} \left( f(x + z) - f(x) - z_1 1_{\{|z| \leq 1\}} f'(x) \right) \nu(x, dz)
\]
\[
+ \sum_{j \neq i} \int_{A_j(x)} (f(x + z) - f(x)) \nu(x, dz)
\]
\[
- \sum_{j \neq i} z_1 1_{\{|z| \leq 1\}} f'(x) \nu(x, dz).
\]
\[
\approx \frac{1}{2} \Delta^2(x)f''(x) + \sum_{j \neq i} (f(x + z) - f(x)) \Lambda(x, A_j(x)) - \mu(x)f'(x).
\]

Hence, we have the following approximation:
\[
G_f(x) \approx \frac{1}{2} \left( \sigma^2(x) + \Delta^2(x)f''(x) + (\mu(x) - \mu(x))f'(x) \right)
\]
\[
+ \sum_{j \neq i} (f(x + z) - f(x)) \Lambda(x, A_j(x)).
\]

We further apply central difference to approximate \( f'(x) \) and \( f''(x) \) for \( x = y_j \in S^p \) and obtain
\[
f'(x) \approx \frac{\delta^+ x f(y_{j+1}) - f(y_j)}{2\delta x} + \frac{\delta^- x f(y_j) - f(y_{j-1})}{2\delta x}.
\]
\[
f''(x) \approx \frac{1}{\delta x} \left( \frac{f(y_{j+1}) - f(y_j)}{\delta x} - \frac{f(y_j) - f(y_{j-1})}{\delta x} \right),
\]
where \( \delta^+ x = |y_{j+1} - y_j| \) and \( \delta x = (\delta^+ x + \delta^- x)/2 \).

Finally, we obtain the CTMC transition rates as follows for \( x = y_j \in S^p \):
\[
G(x, y_{j+1}) = \frac{(\mu(x) - \mu(x))\delta^- x}{2\delta x} + \frac{\sigma^2(x) + \Delta^2(x)}{2\delta x} + \Lambda(x, A_{j+1}(x)).
\]
\[
G(x, y_{j-1}) = -\frac{(\mu(x) - \mu(x))\delta^+ x}{2\delta x} + \frac{\sigma^2(x) + \Delta^2(x)}{2\delta x} + \Lambda(x, A_{j-1}(x)),
\]
\[
G(x, y_j) = \Lambda(x, A_j(x)), \quad |j - i| > 1.
\]
\[
G(x, x) = -\sum_{j \neq i} G(x, y_j).
\]

We set \( y_0, y_n \) as absorbing states for the CTMC, i.e.
\[
G(x, y) = 0, \quad x \in \{y_0, y_n\}, y \in S.
\]

If \( X \) is a jump-diffusion with finite jump activity, the infinitesimal generator is in the form
\[
G_f(x) = \frac{1}{2} \Delta^2(x)f''(x) + \mu(x)f'(x)
\]
In this case, define \( \alpha(x, z_i), A_j(x) \), and \( \Lambda(x, A_j(x)) \) in the same way as the general case, and we approximate the generator as

\[
Gf(x) \approx \frac{1}{2} \sigma^2(x) f''(x) + \mu(x) f'(x) + \sum_{j \neq i} (f(x + z_j) - f(x)) \Lambda(x, A_j(x))
\]

Note that there is no need to approximate small jumps by a diffusion, possibility is ignored in the construction of CTMC in scheme 1.

Thus, the error of truncating the state space may be quite significant component in this case. For \( x = y_i \in S^p \), the transition rates are

\[
\begin{align*}
G(x, y_{i+1}) &= \frac{\nu(x) \delta^- x}{2\delta x} + \frac{\sigma^2(x)}{2\delta x} + \Lambda(x, A_{i+1}(x)), \\
G(x, y_{i-1}) &= -\frac{\nu(x) \delta^+ x}{2\delta x} + \frac{\sigma^2(x)}{2\delta x} + \Lambda(x, A_{i-1}(x)), \\
G(x, y_i) &= \Lambda(x, A_i(x)), \quad |i - j| > 1,
\end{align*}
\]

In this case, \( G \) is tridiagonal and \( Y \) is a birth-and-death process.

### A.2. Scheme 2

States greater than \( y_n \) have a positive probability to move to states smaller than \( y_n \) in finite time in the original model, but such a possibility is ignored in the construction of CTMC in scheme 1. Thus, the error of truncating the state space may be quite significant under this scheme. To reduce such errors, we adopt the idea in Boyarchenko and Levendorskii (2007a, 2007b, 2007c) to construct CTMC approximation by considering this possibility. In our setting, we let \( y_n = U \) be the largest CTMC state and extend \( S \) beyond \( U \) to have \( S = \cup \{2U - y : y \in S\} \). The points in \( S \) greater than \( y_n = U \) are denoted by \( y_{n+1}, y_{n+2}, \ldots, y_p \) \((n = 2n + 1)\) in ascending order. We adjust the definitions of \( \alpha(x, z_i) \) and \( \Lambda(x, A_j(x)) \) for \( x \in S^p \) as

\[
\begin{align*}
\{z_0, z_1, \ldots, z_i\}, & \quad z_j := y_j - x, \\
\alpha(x, z_i) &= (\alpha_i + \alpha_{i+1})/2, \quad 0 \leq i \leq n' - 1, \\
\alpha(x, z_{-1}) &= -x, \quad \alpha(x, z_p) := \infty, \\
\Lambda(x, A_i(x)) &= \int_{A_i(x)} v(x, dz), \quad 0 \leq j \leq n', j \neq i.
\end{align*}
\]

The integral part of \( Gf(x) \) for \( x = y_i \) in \( S^p \) is approximated as

\[
\int_{-\infty}^{\infty} \left( f(x + z) - f(x) - z \mathbf{1}_{|z| \leq 1} f'(x) \right) v(x, dz)
\]

\[
\approx \int_{A_i(x)} \left( f(x + z) - f(x) - z \mathbf{1}_{|z| \leq 1} f'(x) \right) v(x, dz)
\]

\[
+ \sum_{j \neq i, |j| \leq n - 1} \int_{A_j(x)} \left( f(x + z) - f(x) \right) v(x, dz)
\]

\[
- \sum_{j \neq i, |j| \leq n'} \int_{A_j(x)} z \mathbf{1}_{|z| \leq 1} f'(x) v(x, dz)
\]

\[
\approx \frac{1}{2} \sigma^2(x) f''(x) + \mu(x) f'(x)
\]

where

\[
\sigma^2(x) := \int_{\Lambda(x)} z^2 \mathbf{1}_{|z| \leq 1} v(x, dz),
\]

\[
\mu(x) := \sum_{j \neq i} z_j \int_{A_j(x)} \mathbf{1}_{|z| \leq 1} v(x, dz).
\]

We assume \( f(y) - 2f(y_n) + f(2y_n - y) \approx 0 \). Then

\[
\sum_{j \neq i, |j| \leq n - 1} \left( f(x + z_j) - f(x) \right) \Lambda(x, A_j(x))
\]

\[
\approx \mu(x) f'(x) + \sum_{j \neq i} \left( 2f(x + z_j) - f(x + z_{n-j}) - f(x) \right) \Lambda(x, A_{2n-j}(x))
\]

\[
= \mu(x) f'(x) + \sum_{j \neq i, |j| \leq n} \left( f(x + z_j) - f(x) \right) \Lambda(x, A_{2n-j}(x))
\]

Further applying central differences to approximate \( f'(x) \) and \( f''(x) \) yields transition rates as follows. For \( x = y_i \) in \( S^p \),

\[
G(x, y_{i+1}) = \frac{\nu(x) \delta^- x}{2\delta x} + \frac{\sigma^2(x) + \sigma^2(x)}{2\delta x} + \Lambda(x, A_{i+1}(x) - \Lambda(x, A_{2n-i+1}(x)),
\]

\[
G(x, y_{i-1}) = -\frac{\nu(x) \delta^+ x}{2\delta x} + \frac{\sigma^2(x) + \sigma^2(x)}{2\delta x} + \Lambda(x, A_{i-1}(x) - \Lambda(x, A_{2n-i+1}(x)),
\]

\[
G(x, y_i) = \Lambda(x, A_i(x)), \quad |i - j| > 1, 0 \leq j < i,
\]

\[
G(x, y_{n+1}) = \Lambda(x, A_n(x) - \Lambda(x, A_{2n-f}(x)), \quad |y - j| > 1, f \neq n,
\]

\[
G(x, y_n) = \Lambda(x, A_n(x) + 2 \sum_{i-j < n} \Lambda(x, A_{2n-f}(x)), \quad |y - j| > 1, f = n.
\]

For \( x = y_n \), we assume \( f''(x) \approx 0 \). We also assume

\[
\int_{-\infty}^{\infty} \left( f(x + z) - f(x) - z \mathbf{1}_{|z| \leq 1} f'(x) \right) v(x, dz) \approx 0 \quad \text{for } x = y_n
\]

as in Boyarchenko and Levendorskii (2007a) using these conditions in the above derivation and forward/backward difference to approximate the first-order derivative, we obtain the transition rates at the
boundary points as follows. For $x = y_0$,
\[
G(x, y_j) = \left(\frac{\mu(x) - \bar{\mu}(x)}{\delta x}\right)^+ + \Lambda(x, A_1(x)) - \Lambda(x, A_{2n-j}(x)), j = 1, \quad G(x, y_j) = \Lambda(x, A_j(x)) - \Lambda(x, A_{2n-j}(x)), \quad 1 < j < n,
\]
\[
G(x, y_j) = \Lambda(x, A_0(x)) + 2 \sum_{0 \leq j < n} \Lambda(x, A_{2n-j}(x)), \quad j = n,
\]
\[
G(x, x) = -\sum_{j \neq i} G(x, y_j).
\]

For $x = y_n$,
\[
G(x, y_j) = \left(\frac{\mu(x)}{\delta x}\right)^-, \quad j = n - 1,
\]
\[
G(x, y_j) = 0, \quad 0 \leq j < n,
\]
\[
G(x, x) = -\sum_{j \neq i} G(x, y_j).
\]

It is worth noting that using forward/backward difference can result in invalid transition rates which are negative. Hence, we take the positive/negative part in such case to obtain valid transition rates.

### Appendix 2. Finite difference scheme

For a diffusion model with drift $\mu(x)$ and volatility $\sigma(x)$, the PDE for the floating-strike lookback put price $u^\mathcal{B}(t, x, M)$ is given by

\[
\begin{align*}
\frac{\partial u^\mathcal{B}}{\partial t} + \mu(x) u^\mathcal{B} + \frac{1}{2} \sigma^2(x) u^\mathcal{B} - r u^\mathcal{B} &= 0, \quad 0 \leq t < T, \quad 0 < x < M, \\
n^\mathcal{B}(t, 0, M) &= 0, \quad 0 \leq t < T, \\
n^\mathcal{B}(T, x, M) &= e^{-r(T-t)} M, \quad 0 \leq t \leq T, \\
n^\mathcal{B}(T, x, M) &= M - x, \quad 0 \leq x \leq M.
\end{align*}
\]

We use second-order finite difference to approximate the derivatives in the PDE and use the Crank-Nicolson scheme to do time stepping. This is a standard finite difference scheme for such PDEs. We first discretize the $x$ and $M$ dimensions with a uniform grid as $\{(i \Delta x, j \Delta x) : i, j = 0, 1, \ldots, N_i, N_j \geq 1\}$ with $N_i \Delta x = M$ where $\Delta x$ and $\Delta M$ are the spatial step size and localization level, respectively. The time is discretized as $\{(i \Delta t : i = 0, 1, \ldots, N_t)\}$ with $N_t \Delta t = T$.

Let $u^\mathcal{B}_{ij}$ be the finite difference approximation to $u^\mathcal{B}(k \Delta t, i \Delta x, j \Delta x)$ for $k = 0, 1, \ldots, N_t, i, j = 0, 1, \ldots, N_i$, and $i < j$. The finite difference scheme proceeds as follows. First, applying the terminal condition, we obtain
\[
u^\mathcal{B}_{N_i} = (j - i) \Delta x, \quad 0 \leq i \leq j \leq N_i.
\]
At the upper localization level $j \Delta x = N_i \Delta x = M$, we apply the artificial boundary condition
\[
u^\mathcal{B}_{M} = (j - i) \Delta x, \quad 0 \leq i \leq j \leq N_i.
\]

For $k = N_t - 1, N_t - 2, \ldots, 0$, we do the following. Let $\mu = \mu(t)$ and $\sigma = \sigma(t)$. Applying central difference approximation and Crank-Nicolson time stepping, we have

\[
\begin{align*}
\frac{u^\mathcal{B}_{i+1,j} - u^\mathcal{B}_{i,j}}{\Delta t} &= \frac{1}{2} \left(\frac{u^\mathcal{B}_{i+1,j} - u^\mathcal{B}_{i-1,j}}{\Delta x^2} + \frac{u^\mathcal{B}_{i,j+1} - u^\mathcal{B}_{i,j-1}}{\Delta x^2}ight) \\
&\quad + \frac{1}{2} \left(\frac{u^\mathcal{B}_{i+1,j} - u^\mathcal{B}_{i,j}}{\Delta t} - r u^\mathcal{B}_{i,j}\right) \\
&\quad + \frac{1}{2} \left(\frac{u^\mathcal{B}_{i,j+1} - u^\mathcal{B}_{i,j}}{\Delta t} - r u^\mathcal{B}_{i,j}\right) \\
&\quad + \frac{1}{2} \left(\frac{u^\mathcal{B}_{i+1,j+1} - u^\mathcal{B}_{i-1,j-1}}{\Delta x^2} - r u^\mathcal{B}_{i,j}\right) + \frac{1}{2} \left(\frac{u^\mathcal{B}_{i,j+1} - u^\mathcal{B}_{i,j-1}}{\Delta x^2} - r u^\mathcal{B}_{i,j}\right)
\end{align*}
\]

\[
0 < i < j < N_i.
\]

The boundary condition at $x = 0$ yields
\[
u^\mathcal{B}_{0j} = e^{-r(T-k \Delta t)} j \Delta x, \quad 0 \leq j \leq N_i.
\]

At the boundary $x = M$, we discretize the boundary condition with a second-order and first-order one-sided finite difference approximation for $j < N_i - 1$ and $j = N_i - 1$, respectively, and obtain
\[
\begin{align*}
-3 u^\mathcal{B}_{i,j} + 4 u^\mathcal{B}_{i+1,j} - u^\mathcal{B}_{i-1,j} &= 0, \quad 0 < i \leq j < N_i - 1, \\
n^\mathcal{B}_{i,j+1} - n^\mathcal{B}_{i,j} &= 0, \quad i = j = N_i - 1.
\end{align*}
\]

The finite difference scheme proceeds as follows. Set $u^\mathcal{B}_{N_i} = (j - i) \Delta x$ for all $0 \leq i \leq j \leq N_i$. For each $k = N_t - 1, N_t - 2, \ldots, 0$, we do the following:

Step 1: Set $u^\mathcal{B}_{0j} = (N_i - i) \Delta x$ for $t = 0, 1, \ldots, N_t$ by (A1).

Step 2: For $j = N_t - 1, N_t - 2, \ldots, 1$,

- set $u^\mathcal{B}_{0j} = e^{-r(T-k \Delta t)} (j \Delta x)$;
- if $j = N_t - 1$, set $u^\mathcal{B}_{0j} = u^\mathcal{B}_{0j+1} = (j \Delta x)$ by (A1) and (A4), otherwise set $u^\mathcal{B}_{0j} = (4 u^\mathcal{B}_{0j+1} - u^\mathcal{B}_{0j-1})/3$ by (A3);
- obtain $\{u^\mathcal{B}_{ij} : 0 \leq i \leq j\}$ by solving the linear system (A2).