Towards Relativistic Atomic Physics. I. The Rest-Frame Instant Form of Dynamics and a Canonical Transformation for a System of Charged Particles plus the Electro-Magnetic Field

David Alba

Dipartimento di Fisica
Universita’ di Firenze
Polo Scientifico, via Sansone 1
50019 Sesto Fiorentino, Italy
E-mail ALBA@FI.INFN.IT

Horace W. Crater

The University of Tennessee Space Institute
Tullahoma, TN 37388 USA
E-mail: hcrater@utsi.edu

Luca Lusanna

Sezione INFN di Firenze
Polo Scientifico
Via Sansone 1
50019 Sesto Fiorentino (FI), Italy
E-mail: lusanna@fi.infn.it
Abstract

A complete exposition of the rest-frame instant form of dynamics for arbitrary isolated systems (particles, fields, strings, fluids) admitting a Lagrangian description is given. The starting point is the parametrized Minkowski theory describing the system in arbitrary admissible non-inertial frames in Minkowski space-time, which allows one to define the energy-momentum tensor of the system and to show the independence of the description from the clock synchronization convention and from the choice of the 3-coordinates. The restriction to the inertial rest frame, centered on the inertial observer having the Fokker-Pryce center-of-inertia world-line, and the study of relativistic collective variables replacing the non-relativistic center of mass lead to the description of the isolated system as a decoupled globally-defined non-covariant canonical external center of mass carrying a pole-dipole structure (the invariant mass $M$ and the rest spin $\vec{S}$ of the system) and an external realization of the Poincare' group. $Mc$ and $\vec{S}$ are the energy and angular momentum of a unfaithful internal realization of the Poincare' group built with the energy-momentum tensor of the system and acting inside the instantaneous Wigner 3-spaces where all the 3-vectors are Wigner covariant. The vanishing of the internal 3-momentum and of the internal Lorentz boosts eliminate the internal 3-center of mass inside the Wigner 3-spaces, so that at the end the isolated system is described only by Wigner-covariant canonical internal relative variables.

Then an isolated system of positive-energy charged scalar articles with mutual Coulomb interaction plus a transverse electro-magnetic field in the radiation gauge is investigated as a classical background for defining relativistic atomic physics. The electric charges of the particles are Grassmann-valued to regularize the self-energies. The external and internal realizations of the Poincare' algebra in the rest-frame instant form of dynamics are found. This allows one to define explicitly the rest-frame conditions and their gauge-fixings (needed for the elimination of the internal 3-center of mass) for this isolated system.

It is shown that there is a canonical transformation which allows one to describe the isolated system as a set of Coulomb-dressed charged particles interacting through a Coulomb plus Darwin potential plus a free transverse radiation field: these two subsystems are not mutually interacting (the internal Poincare' generators are a direct sum of the two components) and are interconnected only by the rest-frame conditions and the elimination of the internal 3-center of mass. Therefore in this framework with a fixed number of particles there is a way out from the Haag theorem, at least at the classical level.
I. INTRODUCTION

As shown in Refs.[1, 2, 3], standard atomic physics is a semi-relativistic treatment of a sector of QED in which:

a) the matter fields are approximated by scalar or spinning particles at the classical level and by their first quantization at the quantum level;
b) the relevant energies are below the threshold $2m c^2$ of pair production so that there is a fixed number of particles;
c) the electro-magnetic field is described in the Coulomb gauge: it is often approximated with an external radiation field and often one works in the long wavelength approximation ($\lambda_{em} >> size\ of\ atom$);
d) the atom is described in the electric dipole representation (Göppert - Mayer or Power - Zieman - Woolley unitary transformations [1, 3]) as a monopole (non-relativistic center of mass) carrying an electric dipole and a magnetic spin dipole; often the atom is approximated with a two-level system carried by the monopole (Jaynes - Cummings - Paul model [3]).

As a consequence, even if the theory is formulated in Minkowski space-time, there is no consistent realization of the Poincare’ group available. As shown in Ref.[4], due to the relation $\varepsilon_0 \mu_0 = 1/c^2$ ($\varepsilon_0$ electric permittivity, $\mu_0$ magnetic permeability), there are two non-relativistic limits with two different realizations of the Galilei group: a) the electric one without Faraday’s law of induction (a time-varying magnetic field does not produce an electric field); b) the magnetic one without the displacement current (a time-varying electric field does not produce a magnetic field).

The open problems are whether

a) it is possible to develop a consistent relativistic version of atomic physics starting from a classical system of relativistic charged particles plus the electro-magnetic field followed by a quantization in which the number of particles is fixed;
b) there is a clock synchronization convention defining instantaneous 3-spaces in Minkowski space-time (possible Cauchy surfaces for Maxwell equations);
c) the Poincare’ algebra is correctly implemented.

The need of this formulation comes from at least two directions:
A) The emergence of a new generation of extremely precise and stable atomic clocks, to be put in space and synchronized with similar clocks on Earth so to be able to measure the gravitational redshift of the Earth in a post-Newtonian approximation of the geo-potential [5], requires the consideration of general relativistic corrections in a special relativistic setting (i.e. the post-Newtonian modification of null geodesics starting from the Minkowski ones; these notions do not exist in non-relativistic physics).
B) The absence of a relativistic theory of the entanglement (see Ref.[6] for preliminary steps in this direction) to be used, for instance, to formulate teleportation from the Earth to the Space Station.

In this paper and subsequent ones we bring to bear intertwined tools on the problem of relativistic atomic physics that do not appear in the semi-relativistic treatments of the past. They are Dirac’s constraint theory, the maintenance of covariance by construction of Poincare’ generators, and the use of the three existing notions of relativistic center-of-mass variables. These tools have been developed over the past fifty years as a systematic
response to problems connected with the foundations of relativistic mechanics as old as one hundred years. The result has been a full clarification and solution of all these problems, see Refs.[7, 8, 9, 10] and Subsection F of Section I. As shown in Ref.[7], containing a review of previous work [8, 9], we have now a description of every isolated system (particles, strings, fields, fluids) admitting a Lagrangian formulation in arbitrary global inertial or non-inertial frames in Minkowski space-time by means of parametrized Minkowski theories. They allow one to get a Hamiltonian description of the relativistic isolated systems, in which the transition from a non-inertial (or inertial) frame to another one is a gauge transformation generated by suitable first-class Dirac constraints. Therefore, all the admissible conventions for clock synchronization (identifying the instantaneous 3-spaces containing the system) turn out to be gauge equivalent.

The basic motivation of parametrized Minkowski theories has been the absence of an intrinsic notion of instantaneous 3-space (and of spatial distance and 1-way velocity of light) in Minkowski space-time, where to visualize the dynamics, due to the Lorentz signature, only the light-cone, i.e. the conformal structure of space-time identifying the allowed trajectories for incoming and outgoing rays of light, is intrinsically given. Usually physics is described in inertial frames centered on inertial observers, where the instantaneous Euclidean 3-spaces are identified by means of Einstein’s $\frac{1}{2}$ clock synchronization convention $^1$.

However, very little is known about non-inertial frames [9]. The only known way to have a global description of them is to choose an arbitrary time-like observer and a 3+1 splitting of Minkowski space-time with space-like hyper-surfaces (namely an arbitrary clock synchronization convention) with a set of 4-coordinates adapted to the foliation and having the observer as origin of the 3-coordinates on each instantaneous 3-space. Each such foliation defines a global non-inertial frame centered on the given observer if it satisfies the Møller admissibility conditions [11], [9], and if the instantaneous (in general non-Euclidean) 3-spaces, described by the functions giving their embedding in Minkowski space-time, tend to space-like hyper-planes at spatial infinity [9]. The 4-metric in the non-inertial frame is a function of the embedding functions, obtained from the flat metric with a general coordinate transformation from the inertial Cartesian 4-coordinates to curvilinear 4-coordinates adapted to the Møller-admissible foliation.

If we couple the Lagrangian of an arbitrary isolated system to an external gravitational field and we replace the external gravitational metric with the embedding-dependent 4-metric of a non-inertial frame we get the Lagrangian of parametrized Minkowski theories. It is a function of the matter of the isolated system (now described in a non-inertial frame with variables knowing the instantaneous 3-spaces) and of the embedding of the instantaneous 3-spaces of the non-inertial frame in Minkowski space-time. However, the associated action is invariant [8], [7, 9], under frame-preserving diffeomorphisms $^2$: this implies that the embeddings are gauge variables, so that all Møller-admissible clock synchronization conventions

---

$^1$ An inertial observer sends rays of light to another arbitrary time-like observer, who reflects them back towards the inertial observer. Given the emission $(x^o_i)$ and adsorption $(x^o_f)$ times on the inertial world-line, the point $P$ of reflection on the other world-line is assumed to be simultaneous with the mid-point $Q$ between emission and reabsorption where $x^o_P \overset{def}{=} x^o_Q = x^o_i + \frac{1}{2}(x^o_f - x^o_i) = \frac{1}{2}(x^o_i + x^o_f)$. Only with this convention the 1-way velocity of light is constant and isotropic and coincides with the 2-way (or round-trip) velocity $c$.

$^2$ Schmutzer and Plebanski [12] were the only ones emphasizing the relevance of this subgroup of diffeomor-
are gauge equivalent. As expected special relativistic physics does not depend on how we conventionally define the instantaneous 3-spaces!

As a particular case, it is now possible to get the rest-frame instant form of dynamics of such isolated systems: every configuration of the system is defined in an inertial frame whose instantaneous 3-spaces are the Wigner hyper-planes orthogonal to the conserved 4-momentum of the configuration (intrinsic definition of the rest frame).

In this instant form there are two realizations of the Poincare' algebra:
1) an external one in which the extended isolated system is described as a point particle (by means of the canonical non-covariant external 3-center of mass of the system) carrying an internal space of Wigner-covariant relative variables determining the invariant mass $M$ of the system and its overall spin $\vec{S}$;
2) an unfaithful internal one inside the Wigner hyper-planes, whose only non-vanishing generators are $M$ and $\vec{S}$; the internal space is defined by the vanishing of the internal 3-momentum, i.e. by the rest-frame condition, and by the vanishing of the internal Lorentz boosts, implying the elimination of the internal 3-center of mass inside the Wigner hyper-planes which avoids a double counting of the center of mass.

As said in Refs. [7, 8], we have now a complete characterization of the relativistic collective variables (all of them tend to the Newtonian center of mass in the non-relativistic limit), which can be built only in terms of the Poincare’ generators 4 and, as shown in Ref.[10], it is now possible to reconstruct the relativistic orbits of interacting particles.

In Ref.[14] there is the rest-frame description of $N$ relativistic positive-energy charged scalar particles plus the electro-magnetic field (described in the radiation gauge, a special case of Lorentz gauge). The particles have Grassmann-valued electric charges to regularize the Coulomb self-energies. It was shown that the use of the Lienard-Wiechert solution with no incoming radiation field allows one to arrive at a description of $N$ charged particles dressed with a Coulomb cloud and mutually interacting through the Coulomb plus the full relativistic Darwin potential. This happens independently from the choice of the Green function (retarded, advanced, symmetric,...) due to the Grassmann regularization. In this way one can build the potential in the rest frame describing all the static and non-static effects of the one-photon exchange in QED. The quantization allows one to recover the standard instantaneous approximation for relativistic bound states, which till now had only been obtained starting from QFT (either in the instantaneous approximations of the Bethe-Salpeter equation or in the quasi-potential approach). In Ref.[15] the same scheme was

---

3 It is the classical counterpart of the Newton-Wigner position operator: its non-covariance is the way out from the no-interaction theorem in relativistic mechanics [10]. This breaking of Lorentz covariance is universal (independent from the isolated system), but has no dynamical effect being associated with the decoupled relativistic canonical center of mass.

4 Besides the canonical non-covariant center of mass there are only the covariant non-canonical Fokker-Pryce center of inertia and the non-covariant non-canonical Møller center of energy. See Refs.[7, 8, 9, 10] based on Pauri-Prosperi formulation [13] of theory of canonical realizations of (rotation, Galilei and Poincare’) Lie groups in phase space and on Dirac’s theory of Hamiltonian constraints.
applied to spinning particles (with a pseudo-classical Grassmann-valued spin generating the Dirac matrices after quantization) and the Salpeter potential was identified after a pseudo-classical Foldy-Wouthuysen transformation.

In this paper we will consider the isolated system of N relativistic positive-energy charged scalar particles plus the electro-magnetic field as a candidate for a description of relativistic atomic physics, in which the atoms will turn out to be relativistic bound states of subsets of the scalar (or spinning) particles. After defining its rest-frame conditions, we will identify the external and internal Poincare’ generators of relativistic atomic physics. Then we study whether it is meaningful to put the arbitrary transverse electro-magnetic field as the sum of a transverse radiation field plus the particle-dependent Lienard-Wiechert field found in Ref.[14].

The main result will be to show the existence of a canonical transformation in the rest-frame instant form, which sends the N charged scalar particles, interacting only through Coulomb potentials, plus an arbitrary transverse electro-magnetic field into a set of N charged Coulomb-dressed particles mutually interacting with the Coulomb + Darwin potential plus a decoupled transverse radiation field.

Our surprising result shows that, in the rest-frame framework for the description of relativistic atomic physics, at least at the classical level, at every finite time (and not only asymptotically) there is a canonical transformation from the radiation field to the interpolating electro-magnetic one appearing in the phenomena described by atomic physics (laser beams in a cavity interacting with beams of atoms comes to mind).

This has to be contrasted with QED, where we must consider only the S matrix between IN and OUT free fields and not the interpolating fields due to the Haag theorem, saying that the interaction picture does not exist [16, 17] and that there is no unitary transformation from the asymptotic IN and OUT states of the radiation field to interpolating states of the general (non radiation) electro-magnetic field. See the end of Section III.

After the canonical transformation each internal Poincare’ generator is the sum of the one of the radiation field plus the one of the dressed particles interacting with the Coulomb + Darwin potential. However the two subsystems are connected by the rest-frame condition (vanishing of the internal 3-momentum) and by the condition eliminating the overall internal center of mass (vanishing of the internal Lorentz boosts).

In a second paper we will study the problems connected with the collective variables of the isolated system ”atoms plus electro-magnetic field” and with the multipolar expansions of the open subsystem formed by the atoms. We will show how to get relativistic orbits for the atoms, how to define a relativistic electric dipole representation and how to define a pseudo-classical relativistic two-level atom.

Then, in a third paper, we will delineate how to make a canonical quantization taking into account the relativistic need of clock synchronization implying that the instantaneous 3-spaces are the Wigner hyper-planes. If the canonical transformation of this paper will turn out to be unitarily implementable, there will be a sound definition of interpolating electro-magnetic fields starting from asymptotic free radiation ones.

Moreover the absence of relative times among the particles imposed by the clock synchronization, the non-locality of the Poincare’ generators and the non-covariance of the
relativistic canonical center of mass will be shown to introduce a spatial non-separability of composite systems, whose description will be the basic problem in the development of a theory about relativistic entanglement independently from the adopted point of view about quantum non-locality.

In Section II we give a concise updated review of the rest-frame description of the isolated system "N positive-energy charged scalar particles plus the electro-magnetic field" and of its external and internal Poincare’ generators (Subsections A, B, C and D). In Subsection F there is a comparison with other approaches to relativistic mechanics, while in Subsection G there is the study of the non-relativistic limit. We give also the rest-frame description of the radiation field (Subsections E and H) and of the particle-dependent Lienard-Wiechert fields of Ref.[14] (Subsection L) and a discussion of the relation between the Fourier transform of an arbitrary electro-magnetic field versus a radiation field (Subsection I).

In Section III there is the definition of the canonical transformation. In Section IV there is the form of the internal Poincare’ generators (Subsection A) and of the Hamilton equations (Subsection B) after the canonical transformation. In Section V there are some concluding remarks.

In Appendix A there is the definition of the transverse polarization vectors for the radiation field in the radiation gauge and their transformation properties under Poincare’ transformations. In Appendix B there are some properties of the Lienard-Wiechert electromagnetic fields and their Fourier transform. In Appendix C there is the explicit evaluation of the internal Poincare’ generators after the canonical transformation. Finally in Appendix D there are the conventions for the dimensions of the various quantities adopted in this paper.
II. N CHARGED PARTICLES, WITH GRASSMANN-VALUED CHARGES, PLUS THE ELECTROMAGNETIC FIELD IN THE REST-FRAME INSTANT FORM OF DYNAMICS

In this Section we shall review the results of Ref.[14], which make use of the radiation gauge for the electro-magnetic field, after recalling the main steps of relativistic kinematics [8, 10] leading to the rest-frame instant form of dynamics and emphasizing the differences with previous approaches to relativistic mechanics.

As said in the Introduction, the main tool is the description of isolated relativistic systems (particles, fluids, strings, fields) admitting a Lagrangian formulation by means of parametrized Minkowski theories. After having done a (Møller-admissible [9, 11]) 3+1 splitting of Minkowski space-time (i.e. the choice of a global non-inertial frame centered on an arbitrary time-like observer) and defined observer-dependent radar 4-coordinates \( \sigma^A = (\tau, \sigma^r) \), we use the embeddings \( z^\mu(\tau, \vec{\sigma}) \), identifying the instantaneous 3-spaces \( \Sigma_\tau \) as space-like surfaces in Minkowski space-time, to find the metric \( g_{AB}[z(\tau, \vec{\sigma})] \) induced in the non-inertial frame. The action of parametrized Minkowski theories is obtained by coupling the Lagrangian of the isolated system to an external gravitational field and by replacing the gravitational metric tensor \( g_{\mu\nu} \) with \( g_{AB}[z] \). Therefore the resulting Lagrangian depends on both the matter variables and on the embeddings \( z^\mu(\tau, \vec{\sigma}) \). However the invariance of this action under frame-preserving diffeomorphisms implies that the embeddings \( z^\mu(\tau, \vec{\sigma}) \) are gauge variables. As a consequence the description of the isolated system does not depend on the choice of the non-inertial frame with its clock synchronization convention (the choice of the proper time \( \tau \)) and its choice of 3-coordinates \( \sigma^r \) in the (in general non-Euclidean) instantaneous 3-space.

An aspect of this approach that is distinctly new and not seen in other approaches is the use of this embedding and its conjugate momentum as an additional canonical pair of fields playing an equal role with canonical particle and field variables. In essence, the constraint formalism is applied not only to the particle and field degrees of freedom, but also to the foliation of space-time identifying the instantaneous 3-spaces.

Subsections A-G of this Section will emphasize the particle aspect of the rest-frame instant form with only Subsection E devoted to the electro-magnetic field and to its reduction to the radiation gauge.

In contrast, in Subsections I,H and G we will give the rest-frame description of the radiation field in the radiation gauge, the connection of it with a generic electro-magnetic

---

5 For a review on relativistic mechanics see Refs.[7, 10]. For the details concerning parametrized Minkowski theories and the definition of the rest-frame instant form of dynamics see the first paper in Ref.[8].
6 We use the metric \( \eta_{\mu\nu} = (+, -, -, -) \). See Appendix D for the dimensions adopted in this paper.
7 To simplify the notation we will use \( \vec{\sigma} \) to denote the curvilinear 3-coordinates \( \{\sigma^r\} \).
8 For an elementary description see the Appendix of Ref.[7]. The basic steps specified there which we repeatedly use are a) identifying local symmetries of a singular Lagrangian and associated first class constraints; b) this also allows us to identify related gauge variables; c) one can then explicitly break the gauge freedom by the introduction of gauge fixing conditions on these gauge variable pairing up the first class constraints of b): this is equivalent to have pairs of second class constraints for the elimination of redundant pairs of canonical variables.
field in presence of charges (adapting the Coulomb gauge formulation of ch.1 of Ref.[1] to the radiation gauge) and the expression of the rest-frame radiation-gauge Lienard-Wiechert solutions of Ref.[14].

A. Parametrized Minkowski Theories

As shown in Ref.[14] the description of N scalar positive energy particles with Grassmann-valued electric charges plus the electro-magnetic field is done in parametrized Minkowski theories with the action

\[
S = \int d\tau d^3\sigma \mathcal{L}(\tau, \vec{\sigma}) = \int d\tau L(\tau),
\]

\[
\mathcal{L}(\tau, \vec{\sigma}) = \frac{i}{2} \sum_{i=1}^{N} \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)) \left[ \theta_i^\tau(\tau) \dot{\theta}_i(\tau) - \dot{\theta}_i^\tau(\tau) \theta_i(\tau) \right] - 
\]

\[
- \sum_{i=1}^{N} \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)) \left[ m_i c \sqrt{g_{\tau\tau}(\tau, \vec{\sigma}) + 2 g_{\tau\sigma}(\tau, \vec{\sigma}) \dot{\eta}_i^\tau(\tau) + g_{\sigma\sigma}(\tau, \vec{\sigma}) \dot{\eta}_i^\sigma(\tau) + g_{\tau \sigma}(\tau, \vec{\sigma})} + 
\]

\[
+ \frac{Q_i(\tau)}{c} \left( A_\tau(\tau, \vec{\sigma}) + A_\sigma(\tau, \vec{\sigma}) \dot{\eta}_i^\sigma(\tau) \right) \right] - 
\]

\[
- \frac{1}{4} \frac{\sqrt{g(\tau, \vec{\sigma})} g^{AC}(\tau, \vec{\sigma}) g^{BD}(\tau, \vec{\sigma}) F_{AB}(\tau, \vec{\sigma}) F_{CD}(\tau, \vec{\sigma})}{c^2},
\]

\[
Q_i(\tau) = e \theta_i^\tau(\tau) \theta_i(\tau).
\]

In this action the configuration variables are \( z^\mu(\tau, \vec{\sigma}), \vec{\eta}_i(\tau), \theta_i(\tau) \) and \( A_\mu(\tau, \vec{\sigma}) \).

Here \( z^\mu(\tau, \vec{\sigma}) \) are the embeddings of the instantaneous 3-spaces \( \Sigma_\tau \), leaves of an arbitrary 3+1 splitting of Minkowski space-time. Instead of the standard Cartesian 4-coordinates, observer-dependent Lorentz-scalar radar 4-coordinates \( \sigma^A = (\tau; \vec{\sigma}) \) are used, where \( \tau \) is a monotonically increasing function of the proper time of an arbitrary time-like observer and \( \sigma^r \) are 3-coordinates on each \( \Sigma_\tau \) having the observer as origin. The induced metric is \( g_{AB}(\tau, \vec{\sigma}) = z_A^\nu(\tau, \vec{\sigma}) \eta_{\mu\nu} z_B^\nu(\tau, \vec{\sigma}) \) with \( z_A^\nu = \frac{\partial z^\nu}{\partial \sigma^A} \). \( g_{AB} \) is a functional \( g_{AB}[z] \) of \( z^\mu(\tau, \vec{\sigma}) \).

The scalar positive-energy particles are described by the Lorentz-scalar 3-coordinates \( \vec{\eta}_i(\tau) \) defined by \( x_i^\mu(\tau) = z^\mu(\tau, \vec{\eta}_i(\tau)) \), where \( x_i^\mu(\tau) \) are their world-lines. \( Q_i \) are the Grassmann-valued electric charges satisfying \( Q_i^2 = 0, Q_i Q_j \neq 0 \) for \( i \neq j \): they are described in terms of the complex Grassmann variables \( \theta_i(\tau), \theta_i^*(\tau) \).

We also use Lorentz-scalar electro-magnetic potentials \( A_\mu(\tau, \vec{\sigma}) = \frac{\partial z^\mu(\tau, \vec{\sigma})}{\partial \sigma^A} A_\mu(z(\tau, \vec{\sigma})) \) adapted to the foliation, i.e. knowing the clock synchronization convention. For the field

---

\( ^9 \text{Since we identify the instantaneous 3-space } \Sigma_\tau \text{ with a global clock synchronization convention, the particles are identified by the 3-coordinates } \vec{\eta}_i(\tau) \text{ giving the intersection of their world-lines with } \Sigma_\tau. \text{ There are no relative times. This implies an independent description for the positive-energy and negative energy sectors of the particle mass shell } p_i^2(\tau) = m_i^2 c^2. \) Like the world-lines \( x_i^\mu(\tau) \), also the particle 4-momenta \( p_i^\mu(\tau) \) are derived quantities functions of the canonical variables of parametrized Minkowski theories.
Due to this Poisson bracket we will use the following vector notation:

\[ \{ z^\mu_\tau (\tau, \vec{\sigma}), \rho_\mu (\tau, \vec{\sigma}), \vec{\eta}_i (\tau), \vec{\kappa}_j (\tau) \} = \delta_{ij} \delta^z_s. \]

By two first-class constraints for the electro-magnetic field (see Ref. [14])

\[ z^\mu_\tau (\tau, \vec{\sigma}) \approx 0, \]

and by two first-class constraints for the electro-magnetic field (see Ref. [14])

\[ \Gamma(\tau, \vec{\sigma}) = \partial_\tau \pi^\tau (\tau, \vec{\sigma}) - \sum_{i=1}^N Q_i \delta^\sigma (\vec{\sigma} - \vec{\eta}_i (\tau)) \approx 0. \]

In Eqs. (2.2) \( T^{AB}(\tau, \vec{\sigma}) = -\left( \frac{1}{\sqrt{g}} \frac{\delta S}{\delta g_{\alpha\beta}(\tau, \vec{\sigma})} \right) (\tau, \vec{\sigma}) \) is the energy-momentum of the isolated system of particles plus the electro-magnetic field. See Ref. [14] for the second class constraints eliminating the momenta conjugate to the Grassmann variables \( \theta_i (\tau) \): here we will only use the Grassmann character of the electric charges \( Q_i \), which are constants of the motion.

The constraints (2.3) arise due to the invariance of the action (2.1) under frame-preserving diffeomorphisms [7] and electro-magnetic gauge transformations.

As a consequence of the constraints (2.2), \( z^\mu_\tau (\tau, \vec{\sigma}) \) are gauge variables and this implies the gauge equivalence of the clock synchronization conventions defining the instantaneous 3-spaces \( \Sigma_\tau \) and of the choice of the 3-coordinates on each \( \Sigma_\tau \).

The Dirac Hamiltonian is

\[ H_D = \int d^3 \sigma \left[ \lambda^\mu \mathcal{H}_\mu + \lambda_\tau \pi^\tau - A_\tau \Gamma \right] (\tau, \vec{\sigma}), \]

where the \( \lambda \)'s are the arbitrary Dirac multipliers associated with the primary first-class constraints. In Eq. (2.4) \( H_c = -\int d^3 \sigma A_\tau (\tau, \vec{\sigma}) \Gamma (\tau, \vec{\sigma}) \) is the canonical Hamiltonian determined by the Legendre transformation: it is weakly zero due to the secondary first-class constraint \( \Gamma (\tau, \vec{\sigma}) \approx 0 \) (the Gauss law).

---

10 Due to this Poisson bracket we will use the following vector notation: \( \vec{\eta}_i (\tau) = (\eta_i^r (\tau)), \vec{\kappa}_i (\tau) = (\kappa_{ir} (\tau) = -\kappa^r_i (\tau)). \)
Since only the embedding variables have Minkowski indices, the Poincaré' generators are

\[ P^\mu = \int d^3 \sigma \, \rho^\mu(\tau, \vec{\sigma}), \]
\[ J^{\mu\nu} = \int d^3 \sigma \left( z^\mu \rho^\nu - z^\nu \rho^\mu \right)(\tau, \vec{\sigma}). \]  

(2.5)

they satisfy the Poincaré' algebra: \( \{ P^\mu, P^\nu \} = 0, \{ P^\mu, J^{\alpha\beta} \} = \eta^{\mu\alpha} P^{\beta} - \eta^{\mu\beta} P^{\alpha}, \{ J^{\mu\nu}, J^{\alpha\beta} \} = C^{\mu\nu\alpha\beta}_\gamma J^{\gamma\delta}, C^{\mu\nu\alpha\beta}_\gamma = \delta^\nu_\gamma \delta^\beta_\delta \eta^{\mu\alpha} + \delta^\mu_\gamma \delta^\beta_\delta \eta^{\nu\alpha} - \delta^\nu_\gamma \delta^\alpha_\delta \eta^{\mu\beta} - \delta^\mu_\gamma \delta^\alpha_\delta \eta^{\nu\beta}. \)

See Ref.[7] for the properties of the isolated system in non-inertial frames. Here we will only describe the properties of the inertial foliation corresponding to its intrinsic rest frame.

To this end we have to restrict the 3+1 splittings of Minkowski space-time to inertial frames, whose instantaneous 3-spaces \( \Sigma_\tau \) are described by the following embeddings (\( b^\mu_r(\tau) \) are three orthonormal space-like vectors)

\[ z^\mu_F(\tau, \vec{\sigma}) = x^\mu(\tau) + b^\mu_r(\tau) \sigma^r. \]  

(2.6)

These space-like hyper-planes still depend on 10 residual gauge degrees of freedom:

a) the world-line \( x^\mu(\tau) \) of the inertial observer chosen as origin of the 3-coordinates \( \sigma^r; \)

b) 6 variables parametrizing an orthonormal tetrad \( b^\mu_A(\tau) \) such that the constant (future-pointing) unit normal to the hyper-planes is \( l^\mu = b^\mu_r = \epsilon^\mu_{\alpha\beta\gamma} b^\beta_A(\tau) b^\gamma_B(\tau). \)

If we impose the gauge fixings \( z^\mu(\tau, \vec{\sigma}) - z^\mu_F(\tau, \vec{\sigma}) \approx 0 \) to the first-class constraints (2.2), only 10 degrees of freedom associated with the momenta \( \rho^\mu(\tau, \vec{\sigma}) \) survive:

a) the total 4-momentum \( P^\mu \) canonically conjugate to \( x^\mu(\tau), \{ x^\mu, P^\nu \} = -\eta^{\mu\nu}; \)

b) 6 momentum variables canonically conjugate to the tetrads \( b^\mu_A(\tau). \)

After this gauge fixing the Dirac Hamiltonian depends only on the 10 surviving first class constraints

\[ H_D = \tilde{\lambda}^\mu(\tau) \tilde{\mathcal{H}}^\mu(\tau) - \frac{1}{2} \tilde{\lambda}^{\mu\nu}(\tau) \tilde{\mathcal{H}}_{\mu\nu}(\tau) + (electromagnetic \ constraints), \]
\[ \tilde{\mathcal{H}}^\mu(\tau) = \int d^3 \sigma \, \mathcal{H}^\mu(\tau, \vec{\sigma}) \approx 0, \]
\[ \tilde{\mathcal{H}}^{\mu\nu}(\tau) = \int d^3 \sigma \, \sigma^r \left[ b^\mu_r(\tau) \mathcal{H}^{\nu}(\tau, \vec{\sigma}) - b^\nu_r(\tau) \mathcal{H}^\mu(\tau, \vec{\sigma}) \right] \approx 0. \]  

(2.7)

The Lorentz generators become

\[ J^{\mu\nu} = x^\mu P^\nu - x^\nu P^\mu + S^{\mu\nu}, \]
\[ S^{\mu\nu} = \int d^3 \sigma \, \sigma^r \left[ b^\mu_r(\tau) \rho^\nu(\tau, \vec{\sigma}) - b^\nu_r(\tau) \rho^\mu(\tau, \vec{\sigma}) \right]. \]  

(2.8)
From now on we will use the notation $P^\mu = (P^\alpha = E/c; \vec{P}) = M c u^\mu(P) = M c (\sqrt{1 + \vec{h}^2}; \vec{h}) \overset{\text{def}}{=} M c h^\mu$, where $\vec{h} = \vec{v}/c$ is an a-dimensional 3-velocity and with $M c = \sqrt{P^2}$.

B. The Rest-Frame Instant Form

To get the rest-frame instant form we add the gauge fixings (only 6 of them are independent)

$$b^\mu_A(\tau) - \epsilon^\mu_A(u(P)) \approx 0,$$

(2.9)

where $\epsilon^\mu_A(u(P)) = \epsilon^\mu_A(\vec{h}) = L^\mu_A(P, \vec{P})$ are the column of the standard Wigner boost sending the 4-momentum $P^\mu$ (assumed time-like) to its rest-frame form $P^\mu = L^\mu_A(P, \vec{P}) = M c \sqrt{1 + \vec{h}^2}$; $\vec{h}$ is an a-dimensional 3-velocity and with $M c = \sqrt{P^2}$.

In this way the 3+1 splittings of Minkowski space-time are restricted to the inertial frames centered on the inertial observer $x^\mu(\tau)$ and whose instantaneous 3-spaces $\Sigma^\tau$ are orthogonal to $P^\mu$: they are named Wigner hyper-planes, because by construction the 3-vectors inside them are Wigner spin-1 3-vectors.

At this stage only 4 gauge degrees of freedom, $x^\mu(\tau)$, of the original embedding $z^\mu(\tau, \vec{r})$ survive. However, due to the dependence of the gauge fixings (2.9) upon $P^\mu$, the final canonical gauge variables are not $x^\mu$, $P^\mu$, but a non-covariant $\tilde{x}^\mu(\tau)$ and $P^\mu$ (see the first paper in Ref.[8])

$$\tilde{x}^\mu = x^\mu - \frac{1}{M c (P^\alpha + M c)} \left[ P_\nu S^{\nu\mu} + M c \left( S^{\alpha\nu} - S^{\alpha\nu} \frac{P_\nu P^\mu}{M^2 c^2} \right) \right],$$

$$u(p) \cdot \tilde{x} = u(P) \cdot x, \quad \{ \tilde{x}^\mu(\tau), P^\nu \} = -\eta^{\mu\nu}.$$  (2.10)

After the gauge fixing (2.9) the Lorentz generators become

$$J^{\mu\nu} = \tilde{x}^\mu P^\nu - \tilde{x}^\nu P^\mu + \tilde{S}^{\mu\nu},$$

$$\tilde{S}^r = \tilde{S}^r = \frac{1}{2} \epsilon^{ruv} S^{uv}, \quad \tilde{S}^{0r} = \frac{\epsilon^{rsu} P^s \tilde{S}^u}{M c + P^\alpha},$$  (2.11)
with \( \tilde{S}^{\mu\nu} \) function only of the 3-spin \( \tilde{S}^r \) of the rest spin tensor \( \tilde{S}_{AB} = \epsilon^\mu_A(u(P)) \epsilon^\nu_B(u(P)) S_{\mu\nu} \) (both the spin tensors \( \tilde{S}^{\mu\nu} \) and \( \tilde{S}^{AB} \) satisfy the Lorentz algebra). Since we assume \( P^2 > 0 \), the Pauli-Lubanski invariant is \( W^2 = -P^2 \tilde{S}^2 \).

The main result is that the \( P^\mu \)-dependent gauge fixing (2.9) change the interpretation of the residual gauge variables, because the remaining 4 first class constraints can be written in the following form

\[
\mathcal{H}^\mu(\tau) = \int d^3 \sigma \mathcal{H}^\mu(\tau, \bar{\sigma}) = P^\mu - u^\mu(P) \int d^3 \sigma T^{\tau\tau}(\tau, \bar{\sigma}) - \epsilon^\mu_r(u(P)) \int d^3 \sigma T^{\tau\tau}(\tau, \bar{\sigma}) \approx 0,
\]

\[
M_c = \sqrt{P^2} \approx \int d^3 \sigma T^{\tau\tau}(\tau, \bar{\sigma}),
\]

\[
\mathcal{P}_{(int)}^r = \int d^3 \sigma T^{\tau\tau}(\tau, \bar{\sigma}) \approx 0.
\]

The Dirac Hamiltonian is now \( H_D = \lambda(\tau) \left( M_c - \int d^3 \sigma T^{\tau\tau}(\tau, \bar{\sigma}) \right) + \bar{\lambda}(\tau) \cdot \bar{P}_{(int)} + \) (electromagnetic constraints) and the embedding of Wigner hyper-planes is

\[
z^\mu_W(\tau, \bar{\sigma}) = x^\mu(\tau) + \epsilon^\mu_r(u(P)) \sigma^r,
\]

with \( x^\mu \) function of \( \tilde{x}^\mu \), \( P^\mu \) and \( S^{\mu\nu} \) (or \( \tilde{S}^{AB} \)) according to Eq.(2.10). The gauge freedom in the choice of \( x^\mu \) is connected with the arbitrariness of the spin boosts \( \tilde{S}^{\tau\tau} \) (or \( S^{\alpha} \)).

Eqs.(2.12) imply that the 8 variables \( \tilde{x}^\mu(\tau) \) and \( P^\mu \) are restricted only by a first class constraint identifying \( M_c = \sqrt{P^2} \) with the invariant mass of the isolated system, evaluated by using its energy-momentum tensor: therefore these 8 variables are to be reduced to 6 physical variables describing the external decoupled relativistic center of mass of the isolated system (\( \sqrt{P^2} \) is determined by the constraint and its conjugate variable, the rest time \( u(P) \cdot \tilde{x}(\tau) \), is a gauge variable). Therefore these 3 collective degrees of freedom hidden in the embedding field \( z^\mu(\tau, \bar{\sigma}) \) become physical variables. However, there are 3 first class constraints \( \bar{P}_{(int)}^r \approx 0 \) (the rest-frame conditions) implying that the Wigner hyper-plane \( \Sigma_{\tau} \) is the rest frame of the isolated system. Therefore the remaining 3 gauge degrees of freedom are shifted from the embeddings to the internal variables inside the Wigner hyper-planes: now the final 3 gauge variables are the 3-coordinates of the internal 3-center of mass, which have to be eliminated with a gauge fixing to the the rest-frame conditions. In this way we avoid a double counting of the center of mass and the dynamics inside the Wigner hyper-planes is described only by relative variables.

C. The External Center of Mass and the External Poincare’ Group

Let us now add the \( \tau \)-dependent gauge fixing \( c T_s - \tau \approx 0 \), where \( c T_s = u(P) \cdot \tilde{x} = u(P) \cdot x \) is the Lorentz-scalar rest time, to the first class constraint \( M_c - \int d^3 \sigma T^{\tau\tau}(\tau, \bar{\sigma}) \approx 0 \). After
In non-relativistic physics, the center of mass variable take on two essential roles. First, it is a locally observable 3-vector with the necessary transformation properties under the Galilei group. Secondly it, together with the total momentum, form a canonical pair. In relativity the first two properties are split respectively between the Fokker-Pryce 4-center of inertia $Y^\mu(\tau)$ and $\tilde{\dot{x}}^\mu(\tau)$. Both as well as $\dot{R}^\mu(\tau)$ move with constant velocity $\tilde{h}^\mu$ for an isolated system in analogy to what happens in non-relativistic physics.

1) The 4-momentum $P^\mu$ becomes the 4-momentum of the isolated "particles plus electromagnetic field" system: $P^\mu = Mc(\sqrt{1 + \tilde{h}^2}; \tilde{h}) = Mc \tilde{h}^\mu$ with $\tilde{h}$ arbitrary a-dimensional 3-velocity and $Mc \equiv \int d^3\sigma T^{\tau \tau}(\tau, \tilde{\sigma})$.

2) The 6 physical degrees of freedom describing the external decoupled relativistic center of mass are the non-evolving Jacobi data

$$\vec{z} = Mc(\vec{x} - \vec{P}, \vec{P}^\alpha \vec{x}^\alpha), \quad \vec{h} = \vec{P} / Mc, \quad \{z^i, h^j\} = \delta^{ij}. \quad (2.14)$$

The 3-vector $\vec{x}_{NW} = \vec{z}/Mc$ is the external non-covariant canonical 3-center of mass (the classical counterpart of the ordinary Newton-Wigner position operator) and is canonically conjugate to the 3-momentum $Mc \vec{h}$. We have $\tilde{x}^\mu(\tau) = (\vec{x}^\alpha(\tau); \vec{x}_{NW} + \vec{P}/Mc \vec{x}^\alpha(\tau))$. From Appendix B of Ref.[18] we have the following transformation properties under Poincare' transformations $(a, \Lambda)$:

$$h^\mu = u^\mu(P) = (\sqrt{1 + \tilde{h}^2}; \tilde{h}) \mapsto h'^\mu = \Lambda^\mu_\nu h^\nu,$$
$$z^i \mapsto z'^i = (\Lambda^i_j - \frac{\Lambda^i_\mu \tilde{h}^\mu}{\Lambda^\alpha_\nu \tilde{h}^\nu} \Lambda^\alpha_j) z^j + \left(\frac{\Lambda^i_\mu \tilde{h}^\nu}{\Lambda^\alpha_\nu \tilde{h}^\rho} \Lambda^\alpha_\mu \Lambda^\rho_j\right) (\Lambda^{-1} a)^\mu,$$
$$\tau \mapsto \tau' + h_\mu (\Lambda^{-1} a)^\mu. \quad (2.15)$$

As a consequence, under Lorentz transformations we have $\vec{h} \cdot \vec{z}' = h \cdot \vec{z} + \frac{\Lambda^\alpha_\mu \tilde{x}^\alpha}{\Lambda^\nu_\mu \tilde{h}^\nu}$.

3) As shown in Refs.[7, 8, 10], for every isolated relativistic system there are only three collective variables (replacing the unique non-relativistic 3-center of mass), which can be constructed by using only the generators $P^\mu, J^{\mu\nu}$ of the associated realization of the Poincare' group. They are the external covariant non-canonical Fokker-Pryce 4-center of inertia $Y^\mu(\tau)$, the external non-covariant canonical 4-center of mass (also called center of spin) $\tilde{x}^\mu(\tau)$ and the external non-covariant non-canonical Møller 4-center of energy $\dot{R}^\mu(\tau)$. All of them have unit 4-velocity $h^\mu = u^\mu(P)$, but only $Y^\mu(\tau) = Y^\mu(0) + u^\mu(P) \tau = Y^\mu(0) + \left(\sqrt{1 + \tilde{h}^2}; \tilde{h}\right) \tau$ is a 4-vector, whose world-line can be used as an inertial observer. As we shall see these collective variables are functions of $\vec{z}, \vec{h}, Mc, \vec{S}$ and $\tau$.

Let us remark that, since the Poincare' generators $P^\mu, J^{\mu\nu}$ are global quantities (they know the whole instantaneous 3-space), these collective variables, being defined in terms of them (as shown in Ref.[8, 13]), are also global quantities. They cannot be locally determined: this is a fundamental difference from the non-relativistic 3-center of mass.

4) Due to the $\tau$-dependence of the gauge fixing $c T_s - \tau \approx 0$, the Dirac Hamiltonian (2.7) becomes $H_D = Mc + \vec{\Lambda}(\tau) \cdot \vec{P} + (\text{electromagnetic constraints})$. 

---

11 In non-relativistic physics, the center of mass variable take on two essential roles. First, it is a locally observable 3-vector with the necessary transformation properties under the Galilei group. Secondly it, together with the total momentum, form a canonical pair. In relativity the first two properties are split respectively between the Fokker-Pryce 4-center of inertia $Y^\mu(\tau)$ and $\tilde{x}^\mu(\tau)$. Both as well as $\dot{R}^\mu(\tau)$ move with constant velocity $\tilde{h}^\mu$ for an isolated system in analogy to what happens in non-relativistic physics.
5) As shown in Refs. [8, 10] the final form of the external Poincare’ generators (2.5) of an arbitrary isolated system in the rest-frame instant form is

\[ P^\mu , \quad J^{\mu \nu} = \tilde{x}^\mu P^\nu - \tilde{x}^\nu P^\mu + \tilde{S}^{\mu \nu}, \]

\[ P^o = \sqrt{M^2 c^2 + \vec{P}^2} = M c \sqrt{1 + \vec{h}^2}, \quad \vec{P} = M c \vec{h}, \]

\[ J^{ij} = \tilde{x}^i P^j - \tilde{x}^j P^i + \epsilon^{iju} \tilde{S}^u = z^i h^j - z^j h^i + \epsilon^{iju} \tilde{S}^u, \]

\[ K^i = J^{oi} = \tilde{x}^o P^i - \tilde{x}^i \sqrt{M^2 c^2 + \vec{P}^2} - \frac{\epsilon^{isu} P^s \tilde{S}^u}{M c + \sqrt{M^2 c^2 + \vec{P}^2}} = \]

\[ = -\sqrt{1 + \vec{h}^2} z^i + \frac{(\tilde{S} \times \vec{h})^i}{1 + \sqrt{1 + \vec{h}^2}} \quad \text{(2.16)} \]

Note that both \( \tilde{L}^{\mu \nu} = \tilde{x}^\mu P^\nu - \tilde{x}^\nu P^\mu \) and \( \tilde{S}^{\mu \nu} = J^{\mu \nu} - \tilde{L}^{\mu \nu} \) are conserved.

It is this external realization which implements the Wigner rotations on the Wigner hyper-planes through the last term in the Lorentz boosts.

Let us remark that this realization is universal in the sense that it depends on the nature of the isolated system only through an \( U(2) \) algebra [19], whose generators, which will be defined in Eqs.(2.23), are the invariant mass \( M \) (which in turn depends on the relative variables and on the type of interaction) and the internal spin \( \vec{S} \), which is interaction-independent being in an instant form of dynamics.

Therefore the isolated system of particles plus fields is described by a canonical external non-covariant 3-center of mass \( \vec{z} \) (canonically conjugate to \( \vec{h} \)), i.e. a decoupled pseudo-particle of mass \( M \) and spin \( \vec{S} \) \(^{12}\) (point particle clock), and an internal space spanned by internal relative 3-variables living on the Wigner hyper-planes (they identify the particular isolated system and are Wigner spin-1 3-vectors) and with the internal 3-center of mass eliminated to avoid double counting. \( M \) and \( \vec{S} \) are (Lorentz scalar and Wigner spin-1 3-vector) functions of the internal relative variables. As a consequence we have a non-covariant pseudo-particle carrying a pole-dipole structure described by the internal relative degrees of freedom. The natural inertial observer for this description is the external Fokker-Pryce center of inertia (the only covariant collective variable), which is also a function of \( \tau \), \( \vec{z} \) and \( \vec{h} \).

6) To eliminate the residual gauge freedom in the choice of \( x^\mu(\tau) \), i.e. of the inertial observer origin of the 3-coordinates on the Wigner hyper-planes, we add the gauge fixings \( K^r(\text{int}) = \vec{S}^{rr} = -\vec{S}^{rr} \approx 0 \). Its preservation in \( \tau \) using the previous \( H_D \) implies \( \vec{\lambda}(\tau) = 0 \), since \( K^r(\text{int}) \) has non-vanishing Poisson bracket with \( \vec{P}^s(\text{int}) \). Thus the final Dirac Hamiltonian becomes \( H_D = M c + (\text{electromagnetic constraints}) \), i.e. the invariant mass of the isolated

\(^{12}\) For \( N \) free positive-energy scalar particles we have \( M c = \sum_{i=1}^{N} \sqrt{m_i^2 c^2 + \vec{k}_i^2}, \quad \vec{S} = \sum_{i=1}^{N} \vec{\eta}_i \times \vec{k}_i \) with the rest-frame conditions \( \sum_{i=1}^{N} \vec{k}_i \approx 0, \vec{S}^{rr} = -\sum_{i=1}^{N} \vec{\eta}_i \sqrt{m_i^2 c^2 + \vec{k}_i^2} \approx 0. \)
In this way $x^\mu(\tau)$ is identified with the Fokker-Pryce 4-center of inertia $Y^\mu(\tau)$ and the embedding (2.13) of the Wigner hyper-planes becomes

$$z^\mu_W(\tau, \vec{\sigma}) = Y^\mu(\tau) + \epsilon^\mu_r(\vec{h}) \sigma^r.$$  (2.17)

A check of the consistency of this identification can be easily done by putting Eq. (2.17) into Eqs. (2.2) and (2.5): Eqs. (2.16) will be recovered if the rest-frame conditions $\vec{P}_{(int)} \approx 0$ and $\vec{K}_{(int)} \approx 0$, of Eqs. (2.23) hold.

The world-lines of the positive-energy particles are by definition the derived quantities

$$x^\mu_i(\tau) = Y^\mu(\tau) + \epsilon^\mu_r(\vec{h}) \eta^r_i(\tau),$$  (2.18)
i.e. they are described by the Wigner spin-1 3-vectors $\vec{\eta}_i(\tau)$.

As shown in Ref. [10] for particle systems and as it will be clarified in paper II for the system of particles plus electro-magnetic field, the rest-frame conditions $\vec{P}_{(int)} \approx 0$ and their gauge fixings $\vec{K}_{(int)} \approx 0$ eliminate the internal 3-center of mass of the whole isolated system. This implies that the 3-positions $\vec{\eta}_i(\tau)$ (and also their conjugate momenta $\vec{\kappa}_i(\tau)$) become functionals only of a set of relative coordinates and momenta satisfying Hamilton equations governed by the invariant mass $M_c$. Once these equations are solved and the orbits $\vec{\eta}_i(\tau)$ are reconstructed, Eqs. (2.18) lead to world-lines functions of $\tau$, of the non-evolving Jacobi data $\vec{z}$, $\vec{\eta}$, and of the solutions for the relative variables (Eq. (2.20) has to be used for $Y^\mu(\tau)$).

If $\vec{\kappa}_i(\tau)$ are the conjugate momenta, with Poisson brackets $\{\eta^r_i(\tau), \kappa^j_s(\tau)\} = \delta_{ij} \delta^r_s$, then the particle 4-momenta $p^\mu_i(\tau)$ are the derived quantities $p^\mu_i(\tau) = \sqrt{m^2_i c^2 + \vec{\kappa}^2_i(\tau) \vec{h}^\mu - \epsilon^\mu_r(\vec{h}) \kappa^{ir}(\tau)}$ satisfying $p^2_i(\tau) = m^2_i c^2$.

7) In Refs. [8] it is shown that the three relativistic collective variables, originally defined in terms of the Poincare’ generators in Refs. [13], [8], can be expressed in terms of the the variables $\tau$, $\vec{z}$, $\vec{\eta}$, $M$ and $\vec{S}$ (the spin or total baricentric angular momentum of the isolated system) in the following way:

a) the pseudo-world-line of the canonical non-covariant 4-center of mass (or center of spin) is

$$\tilde{x}^\mu(\tau) = \left(\tilde{x}^\mu(\tau); \tilde{x}(\tau)\right) = \left(\sqrt{1 + \vec{h}^2} (\tau + \frac{\vec{h} \cdot \vec{z}}{M_c}); \frac{\vec{z}}{M_c} + (\tau + \frac{\vec{h} \cdot \vec{z}}{M_c}) \vec{h}\right) =$$

$$z^\mu_W(\tau, \vec{\sigma}) = Y^\mu(\tau) + \left(0; \frac{-\vec{S} \times \vec{h}}{M_c (1 + \sqrt{1 + \vec{h}^2})}\right),$$  (2.19)

so that we get $Y^\mu(0) = \left(\sqrt{1 + \vec{h}^2} \frac{\vec{h} \cdot \vec{z}}{M_c}; \frac{\vec{h} \cdot \vec{z}}{M_c} \vec{h} + \frac{-\vec{S} \times \vec{h}}{M_c (1 + \sqrt{1 + \vec{h}^2})}\right)$ (we have used [8] $\vec{\sigma} = \frac{-\vec{S} \times \vec{h}}{M_c (1 + \sqrt{1 + \vec{h}^2})}$);

b) the world-line of the non-canonical covariant Fokker-Pryce 4-center of inertia is
\[ Y^\mu(\tau) = \left( \tilde{x}^\mu(\tau); \tilde{Y}(\tau) \right) = \]
\[ = \left( \sqrt{1 + \vec{h}^2 (\tau + \frac{\vec{h} \cdot \vec{z}}{Mc})}; \frac{\vec{z}}{Mc} + (\tau + \frac{\vec{h} \cdot \vec{z}}{Mc}) \vec{h} + \frac{\vec{S} \times \vec{h}}{Mc (1 + \sqrt{1 + \vec{h}^2})} \right) = z^\mu_W(\tau, \tilde{0}); \]
\[ (2.20) \]

c) the pseudo-world-line of the non-canonical non-covariant Møller 4-center of energy is
\[ R^\mu(\tau) = \left( \tilde{x}^\mu(\tau); \tilde{R}(\tau) \right) = \]
\[ = \left( \sqrt{1 + \vec{h}^2 (\tau + \frac{\vec{h} \cdot \vec{z}}{Mc})}; \frac{\vec{z}}{Mc} + (\tau + \frac{\vec{h} \cdot \vec{z}}{Mc}) \vec{h} - \frac{\vec{S} \times \vec{h}}{Mc \sqrt{1 + \vec{h}^2 (1 + \sqrt{1 + \vec{h}^2})}} \right) = \]
\[ = z^\mu_W(\tau, \tilde{\sigma}_R) = Y^\mu(\tau) + \left( 0; -\frac{\vec{S} \times \vec{h}}{Mc \sqrt{1 + \vec{h}^2}} \right), \]
\[ (2.21) \]
(we have used [8] \[ \tilde{\sigma}_R = \frac{-\vec{S} \times \vec{h}}{Mc \sqrt{1 + \vec{h}^2}} \].

D. The Internal Poincare’ Generators

Eqs.(2.2) show that all the dependence on the isolated system is hidden in the energy-momentum tensor \( T^{AB}(\tau, \tilde{\sigma}) \) (see also Ref.[10]), like it happens in general relativity.

For the system "N charged particles plus electro-magnetic field" in the inertial rest frame it has the following form [14] (the Grassmann-valued electric charges eliminate diverging self-energies) \(^{13}\) in the radiation gauge (see Subsection E)

\[ T^{\tau \tau}(\tau, \tilde{\sigma}) = \sum_{i=1}^{N} \delta^3(\tilde{\sigma} - \tilde{\eta}_i(\tau)) \sqrt{m_i^2 c^2 + \left[ \vec{h}_i(\tau) - \frac{Q_i}{c} \vec{p}_\perp(\tau, \tilde{\eta}_i(\tau)) \right]^2} + \]
\[ + \frac{1}{2c} \left( \tilde{\pi}_\perp + \sum_{i=1}^{N} Q_i \frac{\delta^3(\tilde{\sigma} - \tilde{\eta}_i(\tau))}{\Delta} \right)^2 + \vec{B}^2 \right)(\tau, \tilde{\sigma}) = \]
\[ = T^{\tau \tau}_{\text{matter}}(\tau, \tilde{\sigma}) + T^{\tau \tau}_{\text{em}}(\tau, \tilde{\sigma}), \]
\[ T^{\tau \tau}_{\text{em}}(\tau, \tilde{\sigma}) = \frac{1}{2c} \left( \tilde{\pi}_{\perp}^2 + \vec{B}^2 \right)(\tau, \tilde{\sigma}), \]

\(^{13}\) The internal Lorentz generators are determined by the rest-frame spin tensor \( \vec{S}^{AB} \). See Ref.[14] for the final form of the internal generators: \( M \) is given in Eq. (6.19) [Eq. (6.35) with use of the rest-frame condition], \( \vec{P}_{(\text{int})} \) in Eq. (5.49), \( \vec{J}_{(\text{int})} \) in Eq. (6.39), \( \vec{K}_{(\text{int})} \) in Eq. (6.46).
\[ T^{\tau}(\tau, \bar{\sigma}) = \sum_{i=1}^{N} \delta^3(\bar{\sigma} - \bar{\eta}_i(\tau)) \left[ \kappa_i^\tau(\tau) - \frac{Q_i}{c} A_i^\tau(\tau, \bar{\eta}_i(\tau)) \right] + \]
\[ + \frac{1}{c} \left[ \bar{\pi}_\perp + \sum_{i=1}^{N} Q_i \bar{\Delta} \delta^3(\bar{\sigma} - \bar{\eta}_i(\tau)) \right] \times \bar{B} \right](\tau, \bar{\sigma}) = \]
\[ = T^{\tau}_{\text{matter}}(\tau, \bar{\sigma}) + T^{\tau}_{\text{em}}(\tau, \bar{\sigma}), \]
\[ T^{\tau}_{\text{em}}(\tau, \bar{\sigma}) = \frac{1}{c} \left( \bar{\pi}_\perp \times \bar{B} \right)(\tau, \bar{\sigma}), \]
\[ T^{\tau}(\tau, \bar{\sigma}) = \sum_{i=1}^{N} \delta^3(\bar{\sigma} - \bar{\eta}_i(\tau)) \left[ \kappa_i^\tau(\tau) - \frac{Q_i}{c} A_i^\tau(\tau, \bar{\eta}_i(\tau)) \right] - \]
\[ - \frac{1}{c} \left[ \frac{1}{2} \delta^s \left[ \left( \bar{\pi}_\perp + \sum_{i=1}^{N} Q_i \bar{\Delta} \delta^3(\bar{\sigma} - \bar{\eta}_i(\tau)) \right)^2 + \bar{B}^2 \right] - \right. \]
\[ - \left. \left[ \left( \bar{\pi}_\perp + \sum_{i=1}^{N} Q_i \bar{\Delta} \delta^3(\bar{\sigma} - \bar{\eta}_i(\tau)) \right)^{\tau} \left( \bar{\pi}_\perp + \sum_{i=1}^{N} Q_i \bar{\Delta} \delta^3(\bar{\sigma} - \bar{\eta}_i(\tau)) \right)^s + \right. \right. \]
\[ + B^\tau B^s \right](\tau, \bar{\sigma}) = \]
\[ = T^{\tau}_{\text{matter}}(\tau, \bar{\sigma}) + T^{\tau}_{\text{em}}(\tau, \bar{\sigma}), \]
\[ T^{\tau}_{\text{em}}(\tau, \bar{\sigma}) = -\frac{1}{c} \left[ \frac{1}{2} \delta^s \left( \bar{\pi}_\perp^2 + \bar{B}^2 \right) - \left( \bar{\pi}_\perp^\tau \bar{\pi}_\perp^s + B^\tau B^s \right) \right](\tau, \bar{\sigma}). \quad (2.22) \]

By using \( T^{A\bar{B}}(\tau, \bar{\sigma}) \) we can define an internal realization of the Poincare’ group acting inside the Wigner hyper-planes. The resulting internal Poincare’ generators, acting in the internal space, are

\[ \mathcal{E}_{\text{(int)}} = \mathcal{P}_{\text{(int)}} c = M c^2 = c \int d^3 \sigma T^{\tau}(\tau, \bar{\sigma}) = \]
\[ = c \sum_{i=1}^{N} \sqrt{m_i^2 c^2 + \left( \bar{\kappa}_i(\tau) - \frac{Q_i}{c} \bar{A}_i(\tau, \bar{\eta}_i(\tau)) \right)^2} + \]
\[ + \sum_{i \neq j} \frac{Q_i Q_j}{4\pi \left| \bar{\eta}_i(\tau) - \bar{\eta}_j(\tau) \right|} + \frac{1}{2} \int d^3 \sigma \left[ \bar{\pi}_\perp^2 + \bar{B}^2 \right](\tau, \bar{\sigma}) = \]
\[ = c \sum_{i=1}^{N} \left( \sqrt{m_i^2 c^2 + \bar{\kappa}_i^2(\tau)} - \frac{Q_i \bar{\kappa}_i(\tau) \cdot \bar{A}_i(\tau, \bar{\eta}_i(\tau))}{\sqrt{m_i^2 c^2 + \bar{\kappa}_i^2(\tau)}} \right) + \]
\[ + \sum_{i \neq j} \frac{Q_i Q_j}{4\pi \left| \bar{\eta}_i(\tau) - \bar{\eta}_j(\tau) \right|} + \frac{1}{2} \int d^3 \sigma \left[ \bar{\pi}_\perp^2 + \bar{B}^2 \right](\tau, \bar{\sigma}), \]
\[ \rightarrow_{c \to \infty} \left( \sum_i m_i c^2 + \sum_i \frac{\bar{\kappa}_i^2(\tau)}{2m_i} + \sum_{i \neq j} \frac{Q_i Q_j}{4\pi |\bar{\eta}_i(\tau) - \bar{\eta}_j(\tau)|} \right) - \]
\[ - \frac{1}{c} \sum_i Q_i \frac{\bar{\kappa}_i(\tau)}{m_i} \cdot \bar{A}_\perp(\tau, \bar{\eta}_i(\tau)) - \frac{1}{c^2} \sum_i \frac{\bar{\kappa}_i^4(\tau)}{8m_i^3} + \]
\[ + \frac{1}{c^3} \sum_i Q_i \frac{\bar{\kappa}_i^2(\tau)}{2m_i^2} \frac{\bar{\kappa}_i(\tau)}{m_i} \cdot \bar{A}_\perp(\tau, \bar{\eta}_i(\tau)) + \]
\[ + O(c^{-4}) + \frac{1}{2} \int d^3\sigma [\bar{\pi}_\perp^2 + \bar{B}^2](\tau, \bar{\sigma}), \]

\[ \bar{P}_{(int)} = \int d^3\sigma T^{\tau\tau}(\tau, \bar{\sigma}) = \sum_{i=1}^N \bar{\kappa}_i(\tau) + \frac{1}{c} \int d^3\sigma [\bar{\pi}_\perp \times \bar{B}](\tau, \bar{\sigma}) \approx 0, \]

\[ J^r_{(int)} = S^r = \frac{1}{2} \epsilon^{ruv} \int d^3\sigma \sigma^u T^{v\tau}(\tau, \bar{\sigma}) = \]
\[ = \sum_{i=1}^N \left( \bar{\eta}_i(\tau) \times \bar{\kappa}_i(\tau) \right)^r + \frac{1}{c} \int d^3\sigma (\bar{\sigma} \times [\bar{\pi}_\perp \times \bar{B}])^r(\tau, \bar{\sigma}), \]

\[ K^r_{(int)} = \bar{S}^{\tau\tau} = -\bar{S}^{\tau\tau} = -\int d^3\sigma \sigma^r T^{\tau\tau}(\tau, \bar{\sigma}) = \]
\[ = -\sum_{i=1}^N \eta_i^r(\tau) \sqrt{m_i^2 c^2 + \left( \bar{\kappa}_i(\tau) - \frac{Q_i}{c} \bar{A}_\perp(\tau, \bar{\eta}_i(\tau)) \right)^2} + \]
\[ + \frac{1}{c} \sum_{i=1}^N \sum_{j \neq i}^{1..N} Q_i Q_j \left[ \frac{1}{\Delta_{ij} \eta_j} \frac{\partial}{\partial \eta_j} c(\bar{\eta}_i(\tau) - \bar{\eta}_j(\tau)) - \eta_j^r(\tau) c(\bar{\eta}_i(\tau) - \bar{\eta}_j(\tau)) \right] + \]
\[ + Q_i \int d^3\sigma \pi^r_\perp(\tau, \bar{\sigma}) c(\bar{\sigma} - \bar{\eta}_i(\tau)) \right] - \frac{1}{2c} \int d^3\sigma \sigma^r (\bar{\pi}_\perp^2 + \bar{B}^2)(\tau, \bar{\sigma}) = \]

20
\[ \begin{align*}
&= - \sum_{i=1}^{N} \eta_i^r(\tau) \left( \sqrt{m_i^2 c^2 + \kappa_i^2(\tau)} - \frac{Q_i}{c} \frac{\vec{\kappa}_i(\tau) \cdot \vec{A}_\perp(\tau, \eta_i(\tau))}{\sqrt{m_i^2 c^2 + \kappa_i^2(\tau)}} \right) + \\
&\quad + \frac{1}{c} \sum_{i=1}^{N} \sum_{j \neq i}^{1,N} Q_i Q_j \left[ \int d^3\sigma \frac{1}{4\pi |\vec{\sigma} - \eta_j(\tau)|} \frac{\partial}{\partial \sigma^r} \frac{1}{4\pi |\vec{\sigma} - \eta_i(\tau)|} \right] + \\
&\quad - \frac{\eta_j^r(\tau)}{4\pi |\eta_j(\tau) - \eta_i(\tau)|} - \\
&\quad - \frac{1}{c} \sum_{i=1}^{N} Q_i \int d^3\sigma \frac{\pi_i^r(\tau, \vec{\sigma})}{4\pi |\vec{\sigma} - \eta_i(\tau)|} - \frac{1}{2c} \int d^3\sigma \sigma^r \left( \frac{\pi_i^2}{4\pi} + \vec{B}^2 \right)(\tau, \vec{\sigma}) \approx 0,
\end{align*} \]

\[ K^r_{(int)} \to c \to \infty c K^r_{Galilei} + O(c^{-1}) \approx 0, \]

\[ \vec{K}_{Galilei} = - \sum_{i=1}^{N} m_i \vec{\eta}_i(\tau) = - \left( \sum_{i=1}^{N} m_i \right) \vec{\eta}_{12} \approx 0, \quad \text{with} \quad c(\vec{\eta}_i - \vec{\eta}_j) := -1/(4\pi |\vec{\eta}_j - \vec{\eta}_i|). \]

Note that, as required by the Poincare’ algebra in an instant form of dynamics \cite{10}, there are interaction terms both in the internal energy and in the internal Lorentz boosts.

This realization of the Poincare’ group is \textit{unfaithful} due to the rest-frame conditions \( \vec{P}_{(int)} \approx 0 \). Moreover also the internal boosts have been put equal to zero as gauge fixings to the rest-frame conditions \( \vec{P}_{(int)} \approx 0 \). Since we have \( \vec{K}_{(int)} = - M c \vec{R}_+ \approx 0 \), the internal 3-center of mass \( (\vec{q}_+ \approx \vec{R}_+ \approx \vec{y}_+ \approx 0) \) is eliminated \cite{10}, avoiding a double counting.

The only surviving internal generators are \( M \) and \( \vec{J}_{(int)} \), which appear in the external Poincare’ generators.

In Eqs.(2.23) there is also the non-relativistic limit, identifying the Galilei generators plus \( 1/c \) electro-magnetic corrections in the rest frame and with the non-relativistic center of mass \( \vec{x}_{com} \) put in the origin of the coordinates (only relative variables survive). From Eqs.(2.16) one can get the Galilei generators associated with the description of the non-relativistic center of mass of the particles.

### E. The Electro-Magnetic Field in the Radiation Gauge

In the previous Subsection we used the radiation gauge for the electro-magnetic field, because it is the one naturally selected by the Shanmugadhasan canonical transformation identifying Darboux bases of phase space adapted to Dirac first class constraints \cite{7, 8, 10}.
As already said (see also Ref.[14]), the electromagnetic potential in the adapted radar 4-coordinates is \( A_A(\tau, \vec{\sigma}) = \frac{\partial z^\nu(\tau, \vec{\sigma})}{\partial x^\nu} A_\mu(z(\tau, \vec{\sigma})) \), so that \( A_\mu(z(\tau, \vec{\sigma})) = \frac{\partial z^\nu(\tau)}{\partial x^\nu} A_A(\tau, \vec{\sigma}) \). In the rest-frame instant form for the electromagnetic potential we have (see Eq.(3.39) of Ref.[14])

\[
A^\mu \left( Y^\alpha(\tau) + \epsilon^\alpha_r(\vec{h}) \sigma^r \right) = h^\mu A^\tau(\tau, \vec{\sigma}) - \epsilon^\mu_r(\vec{h}) A^\tau(\tau, \vec{\sigma}) = h^\mu A^\tau(\tau, \vec{\sigma}) + \epsilon^\mu_r(\vec{h}) \frac{\partial}{\partial A^\tau(\tau, \vec{\sigma})} - \epsilon^\mu_r(\vec{h}) A^\tau_{\perp}(\tau, \vec{\sigma}),
\]

\[
A^\tau(\tau, \vec{\sigma}) = h^\mu A_\mu \left( h^\alpha \tau + \epsilon^\alpha_r(\vec{h}) \sigma^r \right),
\]

where \( \triangle = -\vec{\partial}^2 \) and \( \vec{\partial} \cdot \vec{A}_\perp(\tau, \vec{\sigma}) \equiv 0 \).

In the Dirac Hamiltonian (2.4) the term \( \int d^3\sigma \left[ \lambda_\tau \pi^r - A_r \Gamma \right](\tau, \vec{\sigma}) \) is the generator of the electro-magnetic gauge transformations. The two first class constraints (2.3) imply that \( A_r(\tau, \vec{\sigma}) \) and one component of \( A_\perp(\tau, \vec{\sigma}) \) are gauge variables. The second natural gauge variable at the Hamiltonian level aside from \( A_r \) is \( \eta(\tau, \vec{\sigma}) = -\frac{1}{\lambda} \vec{\partial} \cdot \vec{A}(\tau, \vec{\sigma}) \), since it is canonically conjugate to the Gauss law: \( \{ \eta(\tau, \vec{\sigma}), \Gamma(\tau, \vec{\sigma}_1) \} = \delta^3(\vec{\sigma} - \vec{\sigma}_1) \).

Therefore in the rest-frame instant form the natural gauge fixing is \( \eta(\tau, \vec{\sigma}) \approx 0 \), namely the Coulomb gauge \( \vec{\partial} \cdot \vec{A}(\tau, \vec{\sigma}) \approx 0 \). Its preservation in \( \tau \) implies the gauge fixing determining the scalar potential: \( A_\perp(\tau, \vec{\sigma}) \approx \sum_{i=1}^N \frac{Q_i}{4\pi |\vec{\sigma} - \vec{\eta}_i(\tau)|} \). The \( \tau \)-preservation of this secondary gauge fixing implies \( \lambda_\tau(\tau, \vec{\sigma}) = 0 \).

After this elimination of the electro-magnetic gauge degrees of freedom we remain only with transverse electromagnetic degrees of freedom (they are a canonical basis of Dirac observables of the electro-magnetic field). This is a rest-frame radiation gauge where the final form of the Dirac Hamiltonian is \( H_D = M_\tau c \) and where we have

\[
A^\tau(\tau, \vec{\sigma}) = \sum_{i=1}^N \frac{Q_i}{4\pi |\vec{\sigma} - \vec{\eta}_i(\tau)|},
\]

\[
\Rightarrow h^\mu A_\mu \left( h^\alpha \tau + \epsilon^\alpha_r(\vec{h}) \sigma^r \right) = \sum_{i=1}^N \frac{Q_i}{4\pi |\vec{\sigma} - \vec{\eta}_i(\tau)|},
\]

\[
\vec{\partial} \cdot \vec{A}(\tau, \vec{\sigma}) = 0, \Rightarrow -\epsilon^\mu_r(\vec{h}) \frac{\partial}{\partial A^\tau(\tau, \vec{\sigma})} A_\mu \left( h^\alpha \tau + \epsilon^\alpha_r(\vec{h}) \sigma^r \right) =
\]

\[
= -\sum_{r} \epsilon_r^\mu(\vec{h}) \epsilon_r^\alpha(\vec{h}) \frac{\partial A_\mu(x)}{\partial x^\mu} = \left( \eta^{\mu\nu} - h^\mu h^\nu \right) \frac{\partial A_\mu(x)}{\partial x^\mu} =
\]

\[
= \partial_\tau A^\mu(x)|_{x=z(\tau, \vec{\sigma})} = 0.
\]

Therefore this radiation gauge is a particular case of Lorentz gauge where \( A^\mu \left( Y^\alpha(\tau) + \epsilon^\alpha_r(\vec{h}) \sigma^r \right) = h^\mu \sum_{i=1}^N \frac{Q_i}{4\pi |\vec{\sigma} - \vec{\eta}_i(\tau)|} - \epsilon^\mu_r(\vec{h}) A^\perp_{\perp}(\tau, \vec{\sigma}). \)

If \( \vec{\sigma}_\perp(\tau, \vec{\sigma}) = \vec{E}_\perp(\tau, \vec{\sigma}) \) (\( \frac{\partial}{\partial \tau} = -\vec{\partial} \cdot \vec{A}_\perp(\tau, \vec{\sigma}) \) due to the first half of Hamilton equations) is the momentum conjugate to \( \vec{A}_\perp(\tau, \vec{\sigma}) \), we have the Poisson brackets
\{A^s_\perp(\tau, \vec{\sigma}), \pi^{s, \perp}_\perp(\tau, \vec{\sigma}_1)\} = -c P^s_\perp(\vec{\sigma}) \delta^3(\vec{\sigma} - \vec{\sigma}_1),

\text{(2.26)}

where \(P^s_\perp(\vec{\sigma}) = \delta^r s + \frac{\partial^s}{\Delta^s} \), \(\Delta = -\partial^2, \partial^r = -\frac{\partial}{\partial r} \).

F. The Møller Radius and the Comparison with other Approaches to Relativistic Mechanics

As said in Refs.[7, 8], if in an arbitrary fixed Lorentz frame we draw the pseudo-world-lines corresponding to the position of \(\tilde{x}^u(\tau)\) and \(R^u(\tau)\) in all possible inertial frames, it turns out that they fill a world-tube [11], the Møller world-tube, around the world-line \(Y^u(\tau)\) of the covariant non-canonical Fokker-Pryce 4-center of inertia \(Y^u(\tau)\). The invariant radius of the tube is \(\rho = \sqrt{-W^2}/p^2 = |\vec{S}|/\sqrt{P^2}\) where \((W^2 = -P^2 \vec{S}^2)\) is the Pauli-Lubanski invariant when \(P^2 > 0\). This classical intrinsic radius delimitates the non-covariance effects (the pseudo-world-lines) of the canonical 4-center of mass \(\tilde{x}^u(\tau)\).

The existence of the Møller world-tube for rotating systems is a consequence of the Lorentz signature of Minkowski space-time. It identifies a region which cannot be explored without a breaking of manifest Lorentz covariance: this implies a limitation on the localization of the canonical center of mass due to its frame-dependence. Moreover it leads to the identification of a fundamental length, the Møller radius, associated with every configuration of an isolated system and built with its global Poincare’ Casimirs. This unit of length is really remarkable for the following two reasons:

A) At the quantum level \(\rho\) becomes the Compton wavelength of the isolated system times its spin eigenvalue \(\sqrt{s(s + 1)}\), \(\rho \rightarrow \hat{\rho} = \sqrt{s(s + 1)}h/M = \sqrt{s(s + 1)}\lambda_M\) with \(M = \sqrt{P^2}\) the invariant mass and \(\lambda_M = h/M\) its associated Compton wavelength. Therefore the region of frame-dependent localization of the canonical center of mass is also the region where classical relativistic physics is no longer valid, because any attempt to make a localization more precise than the Compton wavelength at the quantum level leads to pair production. The interior of the classical Møller world-tube must be described by using quantum mechanics!

In string theory extended Heisenberg relations \(\Delta x = \frac{h}{\lambda_\perp p} + \frac{\lambda_\perp p}{\lambda_\perp b} \frac{h}{\lambda_\perp b} + \frac{\lambda_\perp p}{\lambda_\perp b} (l_s = \sqrt{h/\lambda_\perp b})\) is the fundamental string length) have been proposed [20] to get the lower bound \(\Delta x > l_s\) (due to the \(y + 1/y\) form) forbidding the exploration of distances below \(l_s\). By replacing \(l_s\) with the Møller radius of an isolated system and \(x\) with its Newton-Wigner 3-center of mass \(\tilde{x}_{NW} = \tilde{z}/Mc\) in the modified Heisenberg relations, \(\Delta x^2_{NW} = \frac{h}{\lambda_\perp p} + \frac{\lambda_\perp p}{\lambda_\perp b} (l_s = \sqrt{h/\lambda_\perp b})\), we could obtain the impossibility \((\Delta x^2 > \rho)\) to explore the interior of the Møller world-tube of the isolated system [7]. This would be compatible with a non-self-adjoint Newton-Wigner position operator after quantization.

---

15 In each Lorentz frame one has different pseudo-world-lines describing \(R^u(\tau)\) and \(\tilde{x}^u(\tau)\): the canonical 4-center of mass \(\tilde{x}^u(\tau)\) lies in between \(Y^u(\tau)\) and \(R^u(\tau)\) in every (non rest)-frame.

16 In the rest-frame the world-tube is a cylinder: in each instantaneous 3-space there is a disk of possible positions of the canonical 3-center of mass orthogonal to the spin. In the non-relativistic limit the radius \(\rho\) of the disk tends to zero and we recover the non-relativistic center of mass.
Moreover, the Møller radius of a field configuration (think of the radiation field discussed in the next Subsection H) could be a candidate for a physical (configuration-dependent) ultraviolet cutoff in QFT [7].

B) As shown in Refs. [11], where the Møller world-tube was introduced in connection with the Møller center of energy $R^\mu(\tau)$, an extended rotating relativistic isolated system with the material radius smaller than its intrinsic radius $\rho$ one has: i) its peripheral rotation velocity can exceed the velocity of light; ii) its classical energy density cannot be positive definite everywhere in every frame 17. Therefore the Møller radius $\rho$ is also a remnant of the energy conditions of general relativity in flat Minkowski space-time [7].

Let us add some comments clarifying the connection of the rest-frame instant form of dynamics with the other existing forms of relativistic mechanics.

In Refs.[10, 22] there are reviews of the various approaches to classical relativistic mechanics induced by the No-Interaction theorem [23], which shows the existence of a conflict among the following three requirements: i) Hamiltonian dynamics; ii) relativistic invariance; iii) manifestly covariant world-lines; except in the case of free particles 18. The way out from this theorem turns out to be the non-covariance of the relativistic canonical (Newton-Wigner) 3-center of mass.

The existing approaches to relativistic mechanics besides the present one may be classified in three main groups (see Refs.[10, 22] for more bibliography):

A) The models in which each particle is described in phase space by 8 canonical variables $x_i^\mu(\tau), p_i^\nu(\tau)$ (with $x_i^\mu(\tau)$ assumed to describe the world-line of particle $i$) and by a modified mass-shell first-class constraint $\chi_i = p_i^\mu - m_i^2 c^2 + V_i \approx 0$. These constraints eliminate the variables $p_i^\mu$ and imply that the time variables $x_i^0$ are gauge variables (absence of physical relative times) to be fixed with gauge fixings (the clock synchronization problem). The non-linear first-class property $\{\chi_i, \chi_j\} \approx 0$ determines the arguments of the allowed potentials $V_i$. Only for the two-body case solutions are known. The simplest solution [25], [26], is $V_1 = V_2 = V(r_1^2)$ with $r_1^\mu = (\eta^\mu\nu - P^\mu P^\nu / p_2^\nu) (x_{1\nu} - x_{2\nu}), P^\mu = p_1^\mu + p_2^\mu$: therefore the interaction is instantaneous in the rest frame (see Ref.[18] for its quantization). However, since we have $\{x_i^\mu, \chi_j\} \neq 0$ for all $i, j$, the canonical variables $x_i^\mu(\tau)$ (and therefore also the world-lines) are not Dirac observables. As a consequence, each gauge fixing on the times $x_0^0$’s generates different world-lines, whose totality spans a so-called world-sheet (see Komar

---

17 Classically, energy density is always positive and the stress-energy tensor for all classical fields satisfies the weak energy condition $T_{\mu\nu} u^\mu u^\nu \geq 0$, where $u^\mu$ is any time-like or null vector. In a sense the Møller world-tube is a classical version of the Epstein, Glaser, Jaffe theorem [21] in QFT: if a field $Q(x)$ satisfies $\langle \Psi | Q(x) | \Psi \rangle \geq 0$ for all states and if $\langle \Omega | Q(x) | \Omega \rangle = 0$ for the vacuum state, then $Q(x) = 0$. Therefore in QFT the weak energy condition does not hold for the renormalized stress-energy tensor. Since it has by definition a null vacuum expectation value, there are states $|Y>$ such that $\langle Y | T_{\mu\nu} u^\mu u^\nu | Y \rangle < 0$. This holds both for the scalar field and for the squeezed state of the electromagnetic field (see also Ref.[6]).

18 This theorem appears also in non-relativistic mechanics [24], if one reformulates it as a many-time theory, namely with an independent time variable for each particle plus suitable first-class constraints. The gauge fixings of these constraints, identifying of the particle times with the Newton time, imply the recovering of the standard one-time Newton mechanics. The problem in special relativity is the absence of an absolute time and of an absolute instantaneaus 3-space.
in Ref.[25]). Moreover the gauge fixings must satisfy the so-called world-line conditions (WLC) [27], according to which the manifest covariance of the world-lines after a good gauge fixing is saved by correcting the Lorentz transformations with suitable gauge transformations generated by the first-class constraints $\chi \approx 0$.

Let us remark that this class of models can be connected to the rest-frame instant form by leaving the derived world-lines as in Eq.(2.18) but modifying the derived momenta from the form given after Eqs.(2.18) to the new form $\tilde{p}_i^\mu(\tau) = \left(\sqrt{m_i^2 c^2 + \left[\bar{r}_i(\tau) - Q_i \bar{A}_\perp(\tau, \bar{\eta}_i(\tau))\right]^2} - Q_i V(\tau, \bar{\eta}_i(\tau))\right) h^\mu - e^\mu_r(\bar{h}) \kappa_{ir}(\tau)$, where $V(\tau, \bar{\sigma}) = \sum_j \frac{Q_j}{4\pi|\bar{\sigma} - \bar{\eta}_i(\tau)|}$ is the radiation gauge term along $h^\mu$ of the gauge potential $A^\mu(Y^\alpha(\tau) + e^\mu_r(\bar{h}) \sigma^r) = V(\tau, \bar{\sigma}) h^\mu - e^\mu_r(\bar{h}) A^\perp(\tau, \bar{\sigma})$ given after Eq.(2.25), because in this case we have $\left(\tilde{P}_i(\tau) - Q_i A(x_i(\tau))\right)^2 = m_i^2 c^2$.

B) If we add $N-1$ gauge fixings to the N first-class constraints in A), implying that the N particles are simultaneously interacting in a particular inertial or non-inertial frame, then the world-lines are well defined and a Lagrangian description becomes possible [28]. Since the natural, intrinsically defined, frame is the rest frame, these models are the ancestors of the rest-frame instant form of dynamics.

Both the approaches A) and B) have to face the problem of separability or cluster decomposition property [29] (see also Todorov in Ref.[25]) essential for scattering theory at the quantum level: with non-confining potentials falling off at big inter-particle separations we must recover the world-lines of free particles. The models B) are natural for confining potentials.

C) A third, non-Hamiltonian approach started with the Currie-Hill WLC [30] and led to predictive mechanics [31], where it is emphasized that the world-line 4-coordinates $q^\mu_i(\tau)$ are each one labeled by the proper time of particle $i$ also in presence of interactions, so that a many-time formulation of relativistic mechanics with a non-linear realization of the Poincare’ algebra can be formulated. Droz Vincent [26] found the Hamiltonian reformulation of predictive mechanics and showed that the covariant predictive 4-coordinates $q^\mu_i(\tau)$ are not canonical, $\{q^\mu_i, q^\nu_j\} \neq 0$ for any pair $i, j$, (except in the free case), so that they cannot coincide with the canonical 4-coordinates $x^\mu_i(\tau)$ of the approach A).

The rest-frame instant form has the independent canonical variables $\vec{z}, \vec{h}$ (with $\vec{z}$ non-covariant, thus avoiding the No-Interaction theorem), $\bar{p}_a(\tau), \bar{\pi}_a(\tau), a = 1, \ldots, N-1$ (the relative canonical variables), and rebuilds the particle world-lines as derived quantities, $x^\mu_i(\tau) = z^\mu(\tau, \bar{\eta}_i(\tau))$ with the $\bar{\eta}_i(\tau)$ function of the relative variables due to the rest-frame conditions. It turns out that what we denoted with $x^\mu_i(\tau)$ are not the canonical 4-coordinates of approach A), but must be interpreted as a realization of the covariant non-canonical predictive 4-coordinates $q^\mu_i(\tau)$. Moreover, our derived world-lines satisfy the cluster decomposition property when the potentials in the internal mass $M$ are non-confining 19, even if our independent canonical variables give a spatially non-separable description of the isolated system (induced by the notion of relativistic center of mass, by the structure of the Poincare’ group and by the clock synchronization problem).

19 Since they depend upon the relative variables, on the solution of the Hamilton equations with Hamiltonian the internal mass $M$ they know the type of potential. For non-confining potentials, modulo tails the on-shell relative variables tend to their value for free particles.
Let us also remark that, as shown in Ref.[8], the 4-vector $Y^\mu(\tau)$ is non canonical since \{\(Y^\mu, Y^\nu\)\} \neq 0 (it is a function of \(P^\mu\) and \(\vec{S}\)): as a consequence there is a non-commutative structure associated to it already at the classical level. The same happens for the 4-vectors \(x^\mu_i(\tau)\). Are these non-commutative structures playing any role at the quantum level?

The advantage of the rest-frame instant form of dynamics is to allow the explicit treatment of N-body problems also in presence of interactions, while in all the previously quoted approaches it was possible to study in detail only the \(N = 2\) case. Moreover only with this approach has it been possible to make contact with the treatment of relativistic bound states in such a way to recover the Darwin and Salpeter potentials starting from the classical theory [14, 15].

G. The Non-Relativistic Limit of the Rest-Frame Instant Form

Let us consider the non-relativistic limit of two positive-energy scalar free particles, disregarding the electro-magnetic field with its two electric and magnetic limits [4].

The particles are described by the Newtonian canonical variables \(\vec{x}_{(n)}\), \(\vec{p}_{(n)}\); \(i = 1, 2\), or by the canonically equivalent center-of-mass and relative variables \(\vec{x}_{(n)}\), \(\vec{p}_{(n)}\), \(\vec{r}_{(n)}\), \(\vec{q}_{(n)}\) (see Ref.[32] for the case of \(N\) particles)

\[
\vec{x}_{(n)} = \frac{1}{m} \sum_{i=1}^{2} m_i \vec{x}_{(n)}^i, \quad \vec{p}_{(n)} = \sum_{i=1}^{2} \vec{p}_{(n)}^i, \quad m = m_1 + m_2,
\]

\[
\vec{r}_{(n)} = \vec{x}_{(n)}^1 - \vec{x}_{(n)}^2, \quad \vec{q}_{(n)} = \frac{1}{m} \left( m_2 \vec{p}_{(n)}^1 - m_1 \vec{p}_{(n)}^2 \right),
\]

\[
\vec{x}_{(n)}^1 = \vec{x}_{(n)} + \frac{m_2}{m} \vec{r}_{(n)}, \quad \vec{x}_{(n)}^2 = \vec{x}_{(n)} - \frac{m_1}{m} \vec{r}_{(n)},
\]

\[
\vec{p}_{(n)}^1 = \frac{m_1}{m} \vec{p}_{(n)} + \vec{q}_{(n)}, \quad \vec{p}_{(n)}^2 = \frac{m_2}{m} \vec{p}_{(n)} - \vec{q}_{(n)}.
\]  \hspace{1cm} (2.27)

The generators of the centrally extended Galilei algebra are (we have changed the sign of the Galilei boosts with respect to Refs.[33])

\[
E_{\text{Galilei}} = \sum_{i=1}^{2} \frac{\vec{p}^2_{(n)}^i}{2m_i} = \frac{\vec{p}^2_{(n)}}{2m}, \quad \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2},
\]

\[
\vec{P}_{\text{Galilei}} = \vec{p}_{(n)} = \sum_{i=1}^{2} \vec{p}_{(n)}^i,
\]

\[
\vec{J}_{\text{Galilei}} = \sum_{i=1}^{2} \vec{x}_{(n)}^i \times \vec{p}_{(n)}^i = \vec{x}_{(n)} \times \vec{p}_{(n)} + \vec{S}_{(n)}, \quad \vec{S}_{(n)} = \vec{r}_{(n)} \times \vec{q}_{(n)};
\]

\[
\vec{K}_{\text{Galilei}} = t \vec{p}_{(n)} - m \vec{x}_{(n)},
\]

\[
\{E_{\text{Galilei}}, \vec{K}_{\text{Galilei}}\} = \vec{P}_{\text{Galilei}}, \quad \{P_{\text{Galilei}}^i, K_{\text{Galilei}}^j\} = m \delta^{ij}, \quad \{K_{\text{Galilei}}^i, K_{\text{Galilei}}^j\} = 0,
\]

\[
\{A^i, J_{\text{Galilei}}^j\} = \epsilon^{ijk} A^k, \quad \vec{A} = \vec{P}_{\text{Galilei}}, \vec{J}_{\text{Galilei}}, \vec{K}_{\text{Galilei}}.
\]  \hspace{1cm} (2.28)
The main property of the Galilei algebra is that the presence of interactions changes the energy, \( E_{\text{Galilei}} \rightarrow E^\prime_{\text{Galilei}} = E_{\text{Galilei}} + V(\vec{r}(n)) \) but not the Galilei boosts \(^{20}\).

Also at the non-relativistic level the 2-body system can be presented as a decoupled particle, the external center of mass \( \vec{x}_{(n)}(t) \) with momentum \( \vec{p}_{(n)} \), of mass \( m \) in the absolute Euclidean 3-space carrying an internal space of relative variables \( (\vec{r}_{(n)}(t), \vec{q}_{(n)}(t)) \) with Hamiltonian \( H_{\text{rel}} = \frac{\vec{p}_{(n)}^2}{2m} \) and rest spin \( \vec{S}_{(n)} \). The external center of mass is associated with an external realization of the Galilei group with generators \( E_{\text{Galilei}} = \frac{\vec{p}_{(n)}^2}{2m} + H_{\text{rel}}, \vec{P}_{\text{Galilei}} = \vec{p}_{(n)} \), \( \vec{J}_{\text{Galilei}} = \vec{x}_{(n)} \times \vec{p}_{(n)} + \vec{S}_{(n)} \), \( \vec{K}_{\text{Galilei}} = t \vec{p}_{(n)} - m \vec{x}_{(n)}(t) \). The internal space can be identified with the rest frame \( (\vec{p}_{(n)} \approx 0) \) if we choose the origin of 3-coordinates in the external center of mass \( (\vec{x}_{(n)}(t) \approx 0) \): in it the particles variables are \( \vec{\eta}_{(n)i}(t) = \vec{x}_{(n)i}(t)|_{\vec{x}_{(n)}=\vec{p}_{(n)}=0}, \vec{\kappa}_{(n)i}(t) = \vec{p}_{(n)i}(t)|_{\vec{x}_{(n)}=\vec{p}_{(n)}=0} \) (they are the non-relativistic counterpart of the variables \( \vec{\eta}(\tau), \vec{\kappa}(\tau) \) on the instantaneous Wigner 3-spaces). With this identification we get a unfaithful internal realization of the Galilei group with generators \( E_{\text{Galilei}} = H_{\text{rel}}, \vec{P}_{\text{Galilei}} = \vec{p}_{(n)} \approx 0 \) (the rest-frame conditions), \( \vec{J}_{\text{Galilei}} = \vec{S}_{(n)}, \vec{K}_{\text{Galilei}} = t \vec{p}_{(n)} - m \vec{x}_{(n)}(t) \approx 0 \) (the gauge fixings to the rest-frame conditions implying \( \vec{x}_{(n)}(t) \approx 0 \)). Inside the internal space we have \( \vec{x}_{(n)1} \approx \vec{\eta}_{(n)1} + \frac{m}{m_p} \vec{p}_{(n)}, \vec{x}_{(n)2} \approx \vec{\eta}_{(n)2} - \frac{m}{m_p} \vec{p}_{(n)}, \vec{p}_{(n)1} \approx \vec{\kappa}_{(n)1} = \vec{q}_{(n)}, \vec{p}_{(n)2} \approx \vec{\kappa}_{(n)2} = -\vec{q}_{(n)} \) and we can introduce the following auxiliary variables (having an obvious relativistic counterpart) \( \vec{\rho}_{(n)12} = \vec{\eta}_{(n)1} - \vec{\eta}_{(n)2} = \vec{r}_{(n)}, \vec{p}_{(n)12} = \frac{m}{m_p} \vec{\kappa}_{(n)1} - \frac{m}{m_p} \vec{\kappa}_{(n)2} = \vec{q}_{(n)}, \vec{\eta}_{(n)12} = \frac{m}{m_p} \vec{\eta}_{(n)1} + \frac{m}{m_p} \vec{\eta}_{(n)2} \approx 0, \vec{\kappa}_{(n)12} = \vec{\kappa}_{(n)1} + \vec{\kappa}_{(n)2} \approx 0 \).

In the relativistic rest-frame instant form the two-particle system is described by

1) the external center-of-mass frozen Jacobi data \( \vec{z}, \vec{h}, \), carrying the internal mass \( M c = \sum_{i=1}^{2} \sqrt{m_i c^2 + \vec{h}_i^2} \) and the spin \( \vec{S} = \sum_{i=1}^{2} \vec{\eta}_i \times \vec{\kappa}_i \);

2) the two pairs of Wigner 3-vectors \( \vec{\eta}_i, \vec{\kappa}_i, i = 1, 2 \), or by the canonically equivalent variables

\[
\vec{\eta}_{12} = \frac{1}{m} \sum_{i=1}^{2} m_i \vec{\eta}_i, \quad \vec{\kappa}_{12} = \sum_{i=1}^{2} \vec{\kappa}_i, \\
\vec{p}_{12} = \vec{\eta}_1 - \vec{\eta}_2, \quad \vec{\pi}_{12} = \frac{1}{m} \left( m_2 \vec{\kappa}_1 - m_1 \vec{\kappa}_2 \right),
\]

restricted by the rest-frame conditions \( \vec{\kappa}_{12} \approx 0 \) (so that \( \vec{\pi}_{12} \approx \vec{\kappa}_1 \approx -\vec{\kappa}_2 \)) and \( -\sum_{i=1}^{2} \vec{\eta}_i \sqrt{m_i c^2 + \vec{h}_i^2} \approx 0 \) (elimination of the internal 3-center of mass).

In terms of these variables we can rebuild the world-lines \( x^\mu_i \) of Eq.(2.18) and the 4-momenta \( p^\mu_i \).

Since in the non-relativistic limit we have \( \vec{P} = \vec{p}_{(n)}, \vec{h} = \frac{\vec{p}_{(n)}}{4\pi c} \rightarrow c_{-\infty} 0 \), implying \( u^\mu(P) \rightarrow c_{-\infty} \left(1; \vec{0}\right) \) and \( e^\mu_\tau(u(P)) \rightarrow c_{-\infty} \left(0; \delta^\mu_\tau\right) \), it turns out that \( \tau/c, \vec{x}/c, Y^\tau/c, R^\tau/c \) and \( x^\rho/c \) all become the absolute Newton time \( t \).

\(^{20}\) This is the reason why there is no ”No-Interaction Theorem” in Newtonian mechanics, so that Newtonian kinematics is trivial. However, as already said, this theorem reappears when we make a many-time reformulation of Newtonian mechanics \([24]\).
Moreover from Subsection C we have the following results:

A) In the reference inertial system we get \( \vec{x}(\tau), \vec{Y}(\tau), \vec{P}(\tau) \rightarrow_{\tau \rightarrow \infty} \vec{x}_{(n)}(t), \vec{x}_{NW} = \frac{\vec{z}}{mc} \rightarrow_{\tau \rightarrow \infty} \vec{x}_{(n)}(0) \) because Eq.(2.14) implies \( z \rightarrow_{\tau \rightarrow \infty} \infty \) and \( \vec{h} \cdot \vec{z} \rightarrow_{\tau \rightarrow \infty} \vec{p}_{(n)} \cdot \left( \vec{x}_{(n)}(t) - \frac{\vec{p}_{(n)}(t)}{m} t \right) = \vec{p}_{(n)} \cdot \vec{x}_{(n)}(0) \) (it is a Jacobi data of the non-relativistic theory).

B) In the inertial rest frame, \( \vec{p}_{(n)} \approx 0 \), we get \( \vec{x}_{(n)}(t) \rightarrow_{\tau \rightarrow \infty} \vec{x}_{(n)}(t), \vec{p}_{(n)}(t), \vec{p}_{(n)}^{\theta} \rightarrow_{\tau \rightarrow \infty} \vec{p}_{(n)}(t), p_{i}^{\theta} \rightarrow_{\tau \rightarrow \infty} m_{i} c + \frac{\vec{p}_{(n)}^{2}(t)}{2m_{i}}. \)

The matter part of the internal Poincare’ generators (2.23) has the limit

\[
M c \rightarrow_{\tau \rightarrow \infty} m c + \sum_{i=1}^{2} \frac{\vec{p}^{2}_{(n)}(t)}{2m_{i}} \approx m c + \frac{\vec{p}^{2}_{(n)}(t)}{2} = m c + H_{rel},
\]

\[
\vec{p}_{(int)} \rightarrow_{\tau \rightarrow \infty} \vec{p}_{(n)}(t) \approx 0,
\]

\[
\vec{S} \rightarrow_{\tau \rightarrow \infty} \sum_{i=1}^{2} \vec{p}_{(n)}(t) \times \vec{x}_{(n)}(t) \approx \vec{p}_{(n)}(t) \times \vec{p}_{(n)}(t) = \vec{S}_{(n)},
\]

\[
\vec{K}_{(int)} \rightarrow_{\tau \rightarrow \infty} - \sum_{i=1}^{2} m_{i} \vec{x}_{(n)}(t) - m \vec{p}_{(n)}(t) \approx 0,
\]

while the limit of the external Poincare’ generators (2.16) is

\[
\vec{P} = \vec{p}_{(n)} = \vec{P}_{Galilei},
\]

\[
P^{\theta} \rightarrow_{\tau \rightarrow \infty} m c + \frac{\vec{p}^{\theta}_{(n)}}{2m} + \sum_{i=1}^{2} \frac{\vec{p}^{2}_{(n)}(t)}{2m_{i}} \approx m c + \frac{\vec{p}^{2}_{(n)}}{2m} + \frac{\vec{p}^{2}_{(n)}(t)}{2m_{i}} = m c + E_{Galilei},
\]

\[
\vec{J} \rightarrow_{\tau \rightarrow \infty} \vec{x}_{(n)}(t) \times \vec{p}_{(n)}(t) + \vec{S}_{(n)} = \vec{J}_{Galilei},
\]

\[
\vec{K}/c \rightarrow_{\tau \rightarrow \infty} t \vec{p}_{(n)}(t) - m \vec{x}_{(n)}(t) = \vec{K}_{Galilei}.
\]

Therefore the non-relativistic limit of the rest-frame instant form leads to the following presentation of the Newton 2-body problem:

1) we have a decoupled external center of mass described by the canonical variables \( \vec{x}_{(n)}, \vec{p}_{(n)} \) and carrying an internal space of relative variables coinciding with the non-relativistic rest frame centered on the center of mass, \( \vec{p}_{(n)} \approx 0 \) and \( \vec{x}_{(n)}(t) \approx 0 \) with the Hamiltonian \( H_{rel} \) and the rest spin \( \vec{S}_{(n)} \);

2) in the internal space we have two pairs of variables \( \vec{x}_{(n)}(t), \vec{p}_{(n)}(t), \) or the canonically equivalent \( \vec{n}_{(n)}(t), \vec{r}_{(n)}(t), \) and, as a consequence from Eqs. (2.18) and (2.27) we have the following identifications
\[ \bar{\rho}_{12}(\tau) = \bar{\eta}_1(\tau) - \bar{\eta}_2(\tau) \to_{c \to \infty} \bar{\rho}_{(n)12}(t) = \bar{\eta}_{(n)1}(t) - \bar{\eta}_{(n)2}(t) = \bar{r}_{(n)}(t), \]

\[ \bar{\pi}_{12}(\tau) = \frac{m_2}{m} \bar{\kappa}_1(\tau) - \frac{m_1}{m} \bar{\kappa}_2(\tau) \to_{c \to \infty} \bar{\pi}_{(n)12}(t) = \frac{m_2}{m} \bar{\kappa}_{(n)1}(\tau) - \frac{m_1}{m} \bar{\kappa}_{(n)2}(\tau) = \bar{q}_{(n)}(t), \]

\[ \bar{x}_1(\tau) \to_{c \to \infty} \bar{x}_{(n)1}(t) + \bar{\eta}_{(n)1}(t) = \bar{x}_{(n)1}(t), \]

\[ \bar{x}_2(\tau) \to_{c \to \infty} \bar{x}_{(n)2}(t) + \bar{\eta}_{(n)2}(t) \bar{x}_{(n)2}(t). \]  

(2.32)

Let us remark that, while at the relativistic level the rest-frame world-lines (2.18) depend upon the 4-momentum \( P^\mu \) of the external 4-center of mass (because it identifies the instantaneous Wigner 3-space in every inertial frame, being orthogonal to it), the non-relativistic trajectories \( \bar{x}_{(n)1}(t) \) do not depend upon \( \bar{\rho}_{(n)} \), but only on \( \bar{x}_{(n)} \) (the non-relativistic definition of center of mass and relative variables does not mix coordinates and momenta).

H. The Radiation Field in the Radiation Gauge

Till now we have emphasized the description of particles. Only in Subsection E have we given the description of the electro-magnetic field in the radiation gauge. If we eliminate the particles we get the rest-frame description of a transverse radiation field in the radiation gauge, solution of \( \Box \bar{A}_\perp^{rad}(\tau \sigma) = 0 \).

In this Subsection we give the rest-frame parametrization (Fourier coefficients and their Poisson brackets) of the radiation field in the radiation gauge by using the results of Ref.[34]. The needed transverse polarization vectors are given in Appendix A. This material will be needed in Section III together with the Lienard-Wiechert solution in the radiation gauge of the rest-frame Hamilton equations, reviewed in the Subsection J.

Instead in Subsection I we will study the connection between the electro-magnetic and the radiation field in presence of charges by adapting to the radiation gauge the Coulomb gauge treatment of atomic physics [1].

On the Wigner hyperplane we have \(^{21}\)

\[ \sigma^A = (\sigma^\tau = \tau; \sigma^\sigma), \quad k^A = (k^\tau = |\vec{k}| = \omega(\vec{k}); k^\sigma), \quad k^2 = 0, \quad \text{with } \vec{k} \text{ Wigner spin-1 3-vector and } k^\tau \text{ Lorentz scalar; } d\vec{k} = \frac{d^3k}{2\omega(\vec{k})(2\pi)^3}, \quad \Omega(\vec{k}) = 2\omega(\vec{k})(2\pi)^3, \quad [d\vec{k} = [l^{-2}]. \]
\[
\vec{A}_{\text{rad}}(\tau, \vec{\sigma}) = \int d\vec{k} \sum_{\lambda=1,2} \vec{e}_\lambda(\vec{k}) \left[ a_\lambda(\vec{k}) e^{-i[\omega(\vec{k}) \tau - \vec{k} \cdot \vec{\sigma}]} + a_\lambda^*(\vec{k}) e^{i[\omega(\vec{k}) \tau - \vec{k} \cdot \vec{\sigma}]} \right],
\]

\[
\vec{\pi}_{\text{rad}}(\tau, \vec{\sigma}) = \vec{E}_{\text{rad}}(\tau, \vec{\sigma}) - \frac{\partial}{\partial \tau} \vec{A}_{\text{rad}}(\tau, \vec{\sigma}) = i \int d\vec{k} \omega(\vec{k}) \sum_{\lambda=1,2} \vec{e}_\lambda(\vec{k}) \left[ a_\lambda(\vec{k}) e^{-i[\omega(\vec{k}) \tau - \vec{k} \cdot \vec{\sigma}]} - a_\lambda^*(\vec{k}) e^{i[\omega(\vec{k}) \tau - \vec{k} \cdot \vec{\sigma}]} \right],
\]

\[
\vec{B}_{\text{rad}}(\tau, \vec{\sigma}) = \vec{\delta} \times \vec{A}_{\text{rad}}(\tau, \vec{\sigma}) = i \int d\vec{k} \sum_{\lambda} \vec{k} \times \vec{e}_\lambda(\vec{k}) \left[ a_\lambda(\vec{k}) e^{-i[\omega(\vec{k}) \tau - \vec{k} \cdot \vec{\sigma}]} - a_\lambda^*(\vec{k}) e^{i[\omega(\vec{k}) \tau - \vec{k} \cdot \vec{\sigma}]} \right],
\]

\[
\vec{A}_{\text{rad}}(\tau, \vec{k}) = \int d^3\sigma \vec{A}_{\text{rad}}(\tau, \vec{\sigma}) e^{-i\vec{k} \cdot \vec{\sigma}} = \frac{1}{2\omega(\vec{k})} \sum_{\lambda=1,2} \left[ \vec{e}_\lambda(\vec{k}) a_\lambda(\vec{k}) e^{-i\omega(\vec{k}) \tau} + \vec{e}_\lambda(-\vec{k}) a_\lambda^*(-\vec{k}) e^{i\omega(\vec{k}) \tau} \right],
\]

\[
\vec{\pi}_{\text{rad}}(\tau, \vec{k}) = \int d^3\sigma \vec{\pi}_{\text{rad}}(\tau, \vec{\sigma}) e^{-i\vec{k} \cdot \vec{\sigma}} = \frac{i}{2} \sum_{\lambda=1,2} \left[ \vec{e}_\lambda(\vec{k}) a_\lambda(\vec{k}) e^{-i\omega(\vec{k}) \tau} - \vec{e}_\lambda(-\vec{k}) a_\lambda^*(-\vec{k}) e^{i\omega(\vec{k}) \tau} \right],
\]

\[
\vec{B}_{\text{rad}}(\tau, \vec{k}) = \int d^3\sigma \vec{B}_{\text{rad}}(\tau, \vec{\sigma}) e^{-i\vec{k} \cdot \vec{\sigma}} = \frac{i}{2\omega(\vec{k})} \vec{k} \times \sum_{\lambda=1,2} \left[ \vec{e}_\lambda(\vec{k}) a_\lambda(\vec{k}) e^{-i\omega(\vec{k}) \tau} + \vec{e}_\lambda(-\vec{k}) a_\lambda^*(-\vec{k}) e^{i\omega(\vec{k}) \tau} \right],
\]

\[
a_\lambda(\vec{k}) = \int d^3\sigma \vec{e}_\lambda(\vec{k}) \cdot \left[ \omega(\vec{k}) \vec{A}_{\text{rad}}(\tau, \vec{\sigma}) - i \vec{\pi}_{\text{rad}}(\tau, \vec{\sigma}) \right] e^{-i\vec{k} \cdot \vec{\sigma}},
\]

\[
\{ A_{\text{rad}}^r(\tau, \vec{\sigma}), \pi_{\text{rad}}^a(\tau, \vec{\sigma}_1) \} = -c P_{\text{rad}}(\vec{\sigma}) \delta^a(\vec{\sigma} - \vec{\sigma}_1),
\]

\[
\{ a_\lambda(\vec{k}), a_\lambda^*(\vec{k}') \} = -i 2 \omega(\vec{k}) c (2\pi)^3 \delta_{\lambda\lambda'} \delta^3(\vec{k} - \vec{k}') \equiv -i \Omega(\vec{k}) c \delta_{\lambda\lambda'} \delta^3(\vec{k} - \vec{k}'),
\]

\[
\{ a_\lambda(\vec{k}), a_{\lambda'}^*(\vec{k}') \} = \{ a_\lambda^*(\vec{k}), a_{\lambda'}^*(\vec{k}') \} = 0,
\]

\[
\delta^{ij} = \sum_{\lambda=1,2} \vec{e}_\lambda(\vec{k}) \vec{e}_\lambda(\vec{k}) + \frac{k_i k_j}{|k|^2}, \quad \vec{k} \cdot \vec{e}_\lambda(\vec{k}) = 0,
\]

\[
\vec{e}_\lambda(\vec{k}) \cdot \vec{e}_{\lambda'}(\vec{k}) = \delta_{\lambda\lambda'}, \quad \frac{\vec{k}}{|k|} \cdot [\vec{e}_1(\vec{k}) \times \vec{e}_2(\vec{k})] = 1. \quad (2.33)
\]
In Ref.[34] it is shown the following result

See Appendix A for a choice of the polarization 3-vectors $\bar{\epsilon}_\lambda(\vec{k})$. In the circular basis (A6) with $\bar{\epsilon}_\pm(\vec{k})$ we have

$$\vec{A}_{\perp rad}(\tau, \vec{\sigma}) = \int d\vec{k} \sum_{\sigma = \pm} \left[ \bar{\epsilon}_\sigma(\vec{k}) a_\sigma(\vec{k}) e^{-i k^A \sigma_A} + \bar{\epsilon}_\sigma^*(\vec{k}) a_\sigma^*(\vec{k}) e^{i k^A \sigma_A} \right],$$

$$\vec{\pi}_{\perp rad}(\tau, \vec{\sigma}) = i \int d\vec{k} \omega(\vec{k}) \sum_{\sigma = \pm} \left[ \bar{\epsilon}_\sigma(\vec{k}) a_\sigma(\vec{k}) e^{-i k^A \sigma_A} - \bar{\epsilon}_\sigma^*(\vec{k}) a_\sigma^*(\vec{k}) e^{i k^A \sigma_A} \right],$$

$$\vec{B}_{rad}(\tau, \vec{\sigma}) = i \int d\vec{k} \sum_{\sigma = \pm} \vec{k} \times \left[ \bar{\epsilon}_\sigma(\vec{k}) a_\sigma(\vec{k}) e^{-i k^A \sigma_A} - \bar{\epsilon}_\sigma^*(\vec{k}) a_\sigma^*(\vec{k}) e^{i k^A \sigma_A} \right],$$

$$a_\pm(\vec{k}) = \frac{1}{\sqrt{2}} \left[ a_1(\vec{k}) \mp i a_2(\vec{k}) \right],$$

$$a_1(\vec{k}) = \frac{1}{\sqrt{2}} \left[ a_+(\vec{k}) + a_-(\vec{k}) \right], \quad a_2(\vec{k}) = \frac{i}{\sqrt{2}} \left[ a_+(\vec{k}) - a_-(\vec{k}) \right],$$

$$\{a_\sigma(\vec{k}), a_\sigma^*(\vec{k})\} = -i \Omega(\vec{k}) c \delta_{\sigma\sigma'} \delta^3(\vec{k} - \vec{k'}),$$

$$\{a_\sigma^*(\vec{k}), a_\sigma^*(\vec{k'})\} = \{a_\sigma(\vec{k}), a_\sigma(\vec{k'})\} = 0,$$

(2.34)

By eliminating the particles in Eq.(2.23) we get the following expression for the internal Poincare' generators of the radiation field in rest-frame instant form $[\mathcal{P}_{rad}^A = (\mathcal{P}_{rad}^\tau = \mathcal{E}_{rad}/c = M_{rad} c; \mathcal{J}_{rad}^u = \frac{1}{2} e^{urs} \mathcal{J}_{rad}^{rs})$

$$M_{rad} c^2 = \mathcal{E}_{rad} = c \mathcal{P}_{rad}^\tau = \frac{1}{2} \int d^3 \sigma \left[ \vec{\pi}_{\parallel rad}^2 + \vec{B}_{rad}^2 \right](\tau, \vec{\sigma}) = \sum_{\lambda = 1,2} \int d\vec{k} \omega(\vec{k}) a^*_\lambda(\vec{k}) a_\lambda(\vec{k}),$$

$$\vec{\mathcal{P}}_{rad} = \frac{1}{c} \int d^3 \sigma \left[ \vec{\pi}_{\perp rad} \times \vec{B}_{rad} \right](\tau, \vec{\sigma}) = \frac{1}{c} \sum_{\lambda = 1,2} \int d\vec{k} \vec{a}^*_\lambda(\vec{k}) a_\lambda(\vec{k}) \approx 0,$$

$$\vec{\mathcal{J}}_{rad} = \vec{\mathcal{S}}_{rad} = \frac{1}{c} \int d^3 \sigma \vec{\sigma} \times \left( \vec{\pi}_{\perp rad} \times \vec{B}_{rad} \right)(\tau, \vec{\sigma}) =$$

$$= \frac{i}{c} \sum_{\lambda} \int d\vec{k} a^*_\lambda(\vec{k}) \vec{k} \times \frac{\partial}{\partial \vec{k}} a_\lambda(\vec{k}) +$$

$$+ \frac{i}{2c} \sum_{\lambda\lambda'} \int d\vec{k} \left[ a_\lambda(\vec{k}) a^*_{\lambda'}(\vec{k}) - a^*_\lambda(\vec{k}) a_{\lambda'}(\vec{k}) \right] \bar{\epsilon}_\lambda(\vec{k}) \cdot \left( \vec{k} \times \frac{\partial}{\partial \vec{k}} \right) \bar{\epsilon}_{\lambda'}(\vec{k}) -$$

$$- \frac{i}{c} \sum_{\lambda\lambda'} \int d\vec{k} \bar{\epsilon}_\lambda(\vec{k}) \times \bar{\epsilon}_{\lambda'}(\vec{k}) a^*_\lambda(\vec{k}) a_{\lambda'}(\vec{k}),$$

---

22 In Ref.[34] it is shown the following result $|a_\sigma(\vec{k})| = |\vec{k}|^{-\gamma} |\vec{k}|^{-1+\gamma}$ with $\gamma > 0$. 31
\[ \mathcal{K}^r_{\text{rad}} = \mathcal{J}^r_{\text{rad}} = -\frac{1}{2c} \int d^3 \sigma \sigma^r \left[ \mathcal{\vec{\pi}}^2_{\text{rad}} + \vec{B}^2_{\text{rad}} \right] (\tau, \vec{\sigma}) = \]
\[ = \frac{i}{c} \int d\vec{k} a^*_\lambda(\vec{k}) \omega(\vec{k}) \frac{\partial}{\partial k^r} a^{\dagger}_\lambda(\vec{k}) + \]
\[ + \frac{i}{2c} \sum_{\lambda, \lambda'} \int d\vec{k} \left[ a^{\dagger}_{\lambda'}(\vec{k}) a^*_\lambda(\vec{k}) - a_{\lambda'}^*(\vec{k}) a^{\dagger}_\lambda(\vec{k}) \right] \vec{\epsilon}_\lambda(\vec{k}) \cdot \omega(\vec{k}) \frac{\partial \vec{\epsilon}_\lambda(\vec{k})}{\partial k^r} \approx 0, \]
\[ h^r_{\text{rad}} = \frac{\mathcal{J}^r_{\text{rad}} \cdot \vec{k}}{|\vec{k}|} = \frac{i}{c} \int d\vec{k} \left[ a^*_2(\vec{k}) a_1(\vec{k}) - a^*_1(\vec{k}) a_2(\vec{k}) \right] = \]
\[ = \frac{1}{c} \int d\vec{k} \left[ a_+^*(\vec{k}) a_+(\vec{k}) - a_-^*(\vec{k}) a_-(\vec{k}) \right], \quad (2.35) \]
where in the last line we defined the helicity.

One needs [34] \( a_\lambda(\vec{k}), \tilde{\partial}_\lambda(\vec{k}) \in L_2(R^3, d^3 k) \) for the existence of the previous ten integrals (and of the occupation number \( N_\lambda = \int d\vec{k} a^*_\lambda(\vec{k}) a_\lambda(\vec{k}) \)) as finite quantities. Moreover one can show [34] the existence of the following behavior: i) \( |a_\lambda(\vec{k})| \to |\vec{k}|^{-\frac{d}{2} - \rho} \) with \( \rho > 0 \); ii) \( |a_\lambda(\vec{k})| \to |\vec{k}|^{-\frac{d}{2} + \epsilon} \) with \( \epsilon > 0 \).

I. The Connection between the Electro-Magnetic and Radiation Fields in the Radiation Gauge

In this Subsection we discuss the treatment of the electromagnetic field in presence of dynamical charges used by atomic physicists in the Coulomb gauge (we follow chapter I of Ref.[1]).

Let us remark that in our rest-frame radiation gauge (a special case of Lorentz gauge) we have Wigner covariance, so that quantities like \( \vec{k} \cdot \vec{\sigma} \) and \( d^3 k \) and \( \vec{k}^2 \) are Lorentz scalars, since the 3-vectors are Wigner spin-1 3-vectors.

In the radiation gauge we have \( A^r(\tau, \vec{\sigma}) = \sum_{i=1}^N \frac{Q_i}{4\pi |\vec{\sigma} - \eta(\tau)|} \) and \( \vec{\partial} \cdot \vec{A}(\tau, \vec{\sigma}) = 0 \) (Coulomb gauge), so that we have the following fields: \( \vec{A}_{\perp}(\tau, \vec{\sigma}), \vec{B}(\tau, \vec{\sigma}) = \vec{\partial} \times \vec{A}_{\perp}(\tau, \vec{\sigma}), \vec{\pi}_{\perp}(\tau, \vec{\sigma}) = \vec{E}_{\perp}(\tau, \vec{\sigma}) \equiv -\frac{\partial \vec{A}_{\perp}(\tau, \vec{\sigma})}{\partial \tau} \). We used \( \vec{\partial} \) in the last equation to make explicit that this result is equivalent to the kinematical first half of Hamilton equations, whose complete set in the radiation gauge is obtained from Eqs. (4.11), (4.15) of Section IV and from the transverse matter current \( \vec{J}_{\perp}(\tau, \vec{\sigma}) \) of Eq.(B7)

\[ \partial_\tau \vec{A}_{\perp}(\tau, \vec{\sigma}) \equiv -\vec{\pi}_{\perp}(\tau, \vec{\sigma}), \quad (\text{kinematical}), \]
\[ \partial_\tau \vec{\pi}_{\perp}(\tau, \vec{\sigma}) \equiv -\vec{\partial}^2 \vec{A}_{\perp}(\tau, \vec{\sigma}) - \vec{J}_{\perp}(\tau, \vec{\sigma}), \quad (\text{dynamical}), \]
\[ \Rightarrow \Box \vec{A}_{\perp}(\tau, \vec{\sigma}) \equiv \vec{J}_{\perp}(\tau, \vec{\sigma}). \quad (2.36) \]
From the point of view of Maxwell equation the use of the potentials in the radiation gauge automatically satisfies the two equations

\[ \tilde{\partial} \cdot \vec{B}(\tau, \vec{\sigma}) = 0, \quad \tilde{\partial} \times \vec{\pi}_\perp(\tau, \vec{\sigma}) = -\partial_\tau \vec{B}(\tau, \vec{\sigma}). \] (2.37)

In the radiation gauge the following equation is trivial and sourceless (the charge density, connected to the longitudinal electric field, has been reabsorbed due to the presence of the Coulomb potential among the charges)

\[ \vec{\partial} \cdot \vec{\pi}_\perp(\tau, \vec{\sigma}) = 0. \] (2.38)

The real dynamical Maxwell equation in the radiation gauge is

\[ \tilde{\partial} \times \vec{B}(\tau, \vec{\sigma}) = \partial_\tau \vec{\pi}_\perp(\tau, \vec{\sigma}) + \vec{j}_\perp(\tau, \vec{\sigma}), \quad \Rightarrow \quad \Box \vec{A}_\perp(\tau, \vec{\sigma}) = \vec{j}_\perp(\tau, \vec{\sigma}). \] (2.39)

In the last line we used \( \tilde{\partial} \times \vec{B} = \tilde{\partial} \times (\tilde{\partial} \times \vec{A}_\perp) = -\Box \vec{A}_\perp \) and the kinematical Hamilton equation \( \vec{\pi}_\perp = -\partial_\tau \vec{A}_\perp \).

Let us now define the following Fourier transforms of the fields

\[
\vec{A}_\perp(\tau, \vec{\sigma}) = \frac{1}{(2\pi)^3} \int d^3k \vec{\tilde{A}}_\perp(\tau, \vec{k}) e^{i\vec{k} \cdot \vec{\sigma}} = \\
= \frac{1}{(2\pi)^3} \int d^3k \sum_{\lambda=1,2} \vec{\epsilon}_\lambda(\vec{k}) \left[ b_{e m \lambda}(\tau, \vec{k}) e^{i\vec{k} \cdot \vec{\sigma}} + b^*_{e m \lambda}(\tau, \vec{k}) e^{-i\vec{k} \cdot \vec{\sigma}} \right],
\]

\[
\vec{\pi}_\perp(\tau, \vec{\sigma}) = \vec{E}_\perp(\tau, \vec{\sigma}) = \frac{1}{(2\pi)^3} \int d^3k \vec{\tilde{\pi}}_\perp(\tau, \vec{k}) e^{i\vec{k} \cdot \vec{\sigma}} = -\frac{\partial \vec{A}_\perp(\tau, \vec{\sigma})}{\partial \tau} = \\
= -\frac{1}{(2\pi)^3} \int d^3k \sum_{\lambda=1,2} \vec{\epsilon}_\lambda(\vec{k}) \left[ \frac{\partial b_{e m \lambda}(\tau, \vec{k})}{\partial \tau} e^{i\vec{k} \cdot \vec{\sigma}} + \frac{\partial b^*_{e m \lambda}(\tau, \vec{k})}{\partial \tau} e^{-i\vec{k} \cdot \vec{\sigma}} \right],
\]

\[
\vec{B}(\tau, \vec{\sigma}) = \frac{1}{(2\pi)^3} \int d^3k \vec{\tilde{B}}(\tau, \vec{k}) e^{i\vec{k} \cdot \vec{\sigma}} = \\
= \frac{i}{(2\pi)^3} \int d^3k \sum_{\lambda=1,2} \vec{k} \times \vec{\epsilon}_\lambda(\vec{k}) \left[ b_{e m \lambda}(\tau, \vec{k}) e^{i\vec{k} \cdot \vec{\sigma}} - b^*_{e m \lambda}(\tau, \vec{k}) e^{-i\vec{k} \cdot \vec{\sigma}} \right],
\]

\[
\tilde{\partial} \cdot \vec{A}_\perp(\tau, \vec{\sigma}) \Rightarrow \vec{k} \cdot \vec{\epsilon}_\lambda(\vec{k}) = \vec{k} \cdot \vec{\epsilon}_\lambda(-\vec{k}) = 0,
\]

\[
\vec{A}_\perp(\tau, \vec{k}) = \int d^3\sigma \vec{\tilde{A}}_\perp(\tau, \vec{\sigma}) e^{-i\vec{k} \cdot \vec{\sigma}} = \vec{A}_\perp^\dagger(\tau, -\vec{k}) = \\
= \sum_{\lambda=1,2} \left[ \vec{\epsilon}_\lambda(\vec{k}) b_{e m \lambda}(\tau, \vec{k}) + \vec{\epsilon}_\lambda(-\vec{k}) b^*_{e m \lambda}(\tau, -s\vec{k}) \right],
\]

\[
\vec{k} \cdot \vec{A}_\perp(\tau, \vec{k}) = 0,
\]
\[ \vec{\pi}_\perp(\tau, \vec{k}) = \int d^3\sigma \vec{\pi}_\perp(\tau, \vec{\sigma}) e^{-i \vec{k} \cdot \vec{\sigma}} = \vec{\pi}_\perp^*(\tau, -\vec{k}) = \]
\[ = - \sum_{\lambda=1,2} \left[ \vec{\varepsilon}_\lambda(\vec{k}) \left( |\vec{k}| b_{\text{em} \lambda}(\tau, \vec{k}) + i \frac{\partial b_{\text{em} \lambda}(\tau, \vec{k})}{\partial \tau} \right) + \vec{\varepsilon}_\lambda(-\vec{k}) \left( |\vec{k}| b^*_{\text{em} \lambda}(\tau, -\vec{k}) + i \frac{\partial b^*_{\text{em} \lambda}(\tau, -\vec{k})}{\partial \tau} \right) \right] = \]
\[ \overset{\text{def}}{=} \frac{i}{2} \left[ \vec{\alpha}(\tau, \vec{k}) - \vec{\alpha}^*(\tau, -\vec{k}) \right], \]
\[ \vec{k} \cdot \vec{\pi}_\perp(\tau, \vec{k}) = \frac{i}{2} \vec{k} \cdot \left[ \vec{\alpha}(\tau, \vec{k}) - \vec{\alpha}^*(\tau, -\vec{k}) \right], \]

\[ \vec{B}(\tau, \vec{k}) = \int d^3\sigma \vec{B}(\tau, \vec{\sigma}) e^{-i \vec{k} \cdot \vec{\sigma}} = \vec{B}^*(\tau, -\vec{k}) = \]
\[ = i \sum_{\lambda=1,2} \vec{k} \times \left[ \vec{\varepsilon}_\lambda(\vec{k}) b_{\text{em} \lambda}(\tau, \vec{k}) + \vec{\varepsilon}_\lambda(-\vec{k}) b^*_{\text{em} \lambda}(\tau, -\vec{k}) \right] = \]
\[ \overset{\text{def}}{=} i \frac{\vec{k}}{|\vec{k}|} \times \left[ \vec{\alpha}(\tau, \vec{k}) + \vec{\alpha}^*(\tau, -\vec{k}) \right], \]
\[ \vec{k} \cdot \vec{B}(\tau, \vec{k}) = 0, \quad \vec{k} \times \left[ \vec{\alpha}(\tau, \vec{k}) + \vec{\alpha}^*(\tau, -\vec{k}) \right] = -2i |\vec{k}| \vec{B}(\tau, \vec{k}). \quad (2.40) \]

We see that the Fourier coefficients \( b_{\text{em} \lambda}(\tau, \vec{k}) \) do not enjoy of the nice properties of the Fourier coefficients \( a_\lambda(\vec{k}) \) of the free radiation field (2.33). Now the electric field \( \vec{\pi}_\perp = \vec{E}_\perp \) depends upon \( \frac{\partial b_{\text{em} \lambda}(\tau, \vec{k})}{\partial \tau} \).

As a consequence, following Ref.[1], in Eqs.(2.40) we introduced the following function \( \vec{\alpha}(\tau, \vec{k}) \)

\[ \vec{\alpha}(\tau, \vec{k}) = -i \left[ \vec{\pi}_\perp - \frac{\vec{k}}{|\vec{k}|} \times \vec{B} \right](\tau, \vec{k}) = \]
\[ = \sum_{\lambda=1,2} \left[ \vec{\varepsilon}_\lambda(\vec{k}) \left( |\vec{k}| b_{\text{em} \lambda}(\tau, \vec{k}) + i \frac{\partial b_{\text{em} \lambda}(\tau, \vec{k})}{\partial \tau} \right) + \vec{\varepsilon}_\lambda(-\vec{k}) \left( |\vec{k}| b^*_{\text{em} \lambda}(\tau, -\vec{k}) + i \frac{\partial b^*_{\text{em} \lambda}(\tau, -\vec{k})}{\partial \tau} \right) \right], \]
\[ \overset{\text{def}}{=} i \left[ \vec{\pi}_\perp + \frac{\vec{k}}{|\vec{k}|} \times \vec{B}^* \right](\tau, \vec{k}) = i \left[ \vec{\pi}_\perp^* - \frac{(-\vec{k})}{|\vec{k}|} \times \vec{B}^* \right](\tau, -\vec{k}), \]
\[ \vec{k} \cdot \vec{\alpha}(\tau, \vec{k}) = 0, \]

34
\[ \tilde{\pi}(\tau, \vec{k}) = \frac{i}{2} \left[ \tilde{\alpha}(\tau, \vec{k}) - \tilde{\alpha}^*(\tau, -\vec{k}) \right], \]

\[ \tilde{B}(\tau, \vec{k}) = \frac{i}{2} \frac{\vec{k}}{|\vec{k}|} \times \left[ \tilde{\alpha}(\tau, \vec{k}) + \tilde{\alpha}^*(\tau, -\vec{k}) \right]. \quad (2.41) \]

The Fourier transform of Eqs.(2.37), (2.38) and (2.39) is

\[ i \vec{k} \cdot \tilde{\Pi}(\tau, \vec{k}) \overset{\circ}{=} 0, \quad i \vec{k} \times \tilde{\Pi}_\perp(\tau, \vec{k}) \overset{\circ}{=} -\partial_\tau \tilde{B}(\tau, \vec{k}), \]

\[ i \vec{k} \cdot \tilde{\pi}_\perp(\tau, \vec{k}) \overset{\circ}{=} 0, \quad i \vec{k} \times \tilde{B}(\tau, \vec{k}) \overset{\circ}{=} \partial_\tau \tilde{\pi}_\perp(\tau, \vec{k}) + \tilde{j}_\perp(\tau, \vec{k}), \]

\[ \Rightarrow -i \vec{k} \times \tilde{B}^*(\tau, -\vec{k}) - \partial_\tau \tilde{\pi}_\perp(\tau, -\vec{k}) \overset{\circ}{=} \tilde{j}_\perp(\tau, -\vec{k}) = \tilde{j}_\perp^*(\tau, -\vec{k}), \quad (2.42) \]

where \( \tilde{j}_\perp(\tau, \vec{k}) \) is the Fourier transform of \( \tilde{j}_\perp(\tau, \vec{\sigma}) \).

A consequence of Eqs.(2.42) is (\( \omega(\vec{k}) = |\vec{k}| \))

\[ \partial_\tau \left[ \tilde{\pi}_\perp \pm \frac{\vec{k}}{|\vec{k}|} \times \tilde{B} \right](\tau, \vec{k}) \overset{\circ}{=} \pm \omega(\vec{k}) \left[ \tilde{\pi}_\perp \pm \frac{\vec{k}}{|\vec{k}|} \times \tilde{B} \right](\tau, \vec{k}) - \tilde{j}_\perp(\tau, \vec{k}). \quad (2.43) \]

At the level of Maxwell equations in the radiation gauge one uses the dynamical equation and one of the two automatic ones.

The form of the function

\[ \tilde{\alpha}(\tau, \vec{k}) \overset{\text{def}}{=} \sum_{\lambda=1,2} \tilde{c}_\lambda(\vec{k}) a_{\text{em} \lambda}(\tau, \vec{k}), \quad (2.44) \]

is suggested by Eqs.(2.43). Eqs.(2.42) imply the following equations of motion for \( \tilde{\alpha}(\tau, \vec{k}) \)

\[ \partial_\tau \tilde{\alpha}(\tau, \vec{k}) \overset{\circ}{=} -i \omega(\vec{k}) \tilde{\alpha}(\tau, \vec{k}) + i \tilde{j}_\perp(\tau, \vec{k}), \]

\[ \partial_\tau \tilde{\alpha}^*(\tau, -\vec{k}) \overset{\circ}{=} i \omega(\vec{k}) \tilde{\alpha}^*(\tau, -\vec{k}) - i \tilde{j}_\perp(\tau, -\vec{k}), \]

\[ \Rightarrow \partial_\tau a_{\text{em} \lambda}(\tau, \vec{k}) \overset{\circ}{=} -i \omega(\vec{k}) a_{\text{em} \lambda}(\tau, \vec{k}) + i \tilde{c}_\lambda(\vec{k}) \cdot \tilde{j}_\perp(\tau, \vec{k}). \quad (2.45) \]

Instead Eqs.(2.45) imply the following equations of motion for \( b_{\text{em} \lambda}(\tau, \vec{k}) \) [we put \( \tilde{j}_\perp(\tau, \vec{k}) = \frac{1}{2} \left[ \tilde{j}_\perp(\tau, \vec{k}) + \tilde{j}_\perp^*(\tau, -\vec{k}) \right] \) due to Eqs.(2.42)]
\[ \frac{\partial^2 b_{em}(\tau, \vec{k})}{\partial \tau^2} = -\omega^2(\vec{k}) b_{em}(\tau, \vec{k}) + \frac{1}{2} \tilde{j}(\tau, \vec{k}). \] (2.46)

Since the relation \( \tilde{\partial} \times \tilde{B}(\tau, \vec{\sigma}) = \partial \times \left( \tilde{\partial} \times \tilde{A}_\perp(\tau, \vec{\sigma}) \right) = -\partial^2 \tilde{A}_\perp(\tau, \vec{\sigma}) \) implies \( i \vec{k} \times \tilde{B}(\tau, \vec{k}) = \vec{k}^2 \tilde{A}_\perp(\tau, \vec{k}) \), we can write the following representation of the function \( \tilde{a}(\tau, \vec{k}) \)

\[ \tilde{a}(\tau, \vec{k}) = \left[ |\vec{k}| \tilde{A}_\perp - i \vec{\pi}_\perp \right](\tau, \vec{k}). \] (2.47)

As a consequence, we arrive at the following representation of the fields formally identical to Eqs.(2.33) for the radiation field by sending \( a_\lambda(\vec{k}) \) into \( b_{em}(\tau, \vec{k}) \)

\[ \tilde{A}_\perp(\tau, \vec{k}) = \frac{i}{\vec{k}^2} \times \tilde{B}(\tau, \vec{k}) = \frac{1}{2 |\vec{k}|} \left[ \tilde{a}(\tau, \vec{k}) + \tilde{a}^*(\tau, -\vec{k}) \right], \]

\[ \tilde{A}_\perp(\tau, \vec{\sigma}) = \frac{1}{(2\pi)^3} \int \frac{d^3k}{2 \omega(\vec{k})} \sum_{\lambda=1,2} \tilde{e}_\lambda(\vec{k}) \left[ a_{em,\lambda}(\tau, \vec{k}) e^{i \vec{k} \cdot \vec{\sigma}} + a_{em,\lambda}^*(\tau, \vec{k}) e^{-i \vec{k} \cdot \vec{\sigma}} \right], \]

\[ \tilde{\pi}_\perp(\tau, \vec{\sigma}) = \frac{i}{2(2\pi)^3} \int d^3k \sum_{\lambda=1,2} \tilde{e}_\lambda(\vec{k}) \left[ a_{em,\lambda}(\tau, \vec{k}) e^{i \vec{k} \cdot \vec{\sigma}} - a_{em,\lambda}^*(\tau, \vec{k}) e^{-i \vec{k} \cdot \vec{\sigma}} \right] = \]

\[ \equiv_{dyn} - \frac{\partial}{\partial \tau} \tilde{A}_\perp(\tau, \vec{\sigma}), \]

\[ \tilde{B}(\tau, \vec{\sigma}) = \frac{i}{2(2\pi)^3} \int d^3k \frac{\vec{k} \times \sum_{\lambda=1,2} \tilde{e}_\lambda(\vec{k}) \left[ a_{em,\lambda}(\tau, \vec{k}) e^{i \vec{k} \cdot \vec{\sigma}} - a_{em,\lambda}^*(\tau, \vec{k}) e^{-i \vec{k} \cdot \vec{\sigma}} \right], \]

\[ a_{em,\lambda}(\tau, \vec{k}) = \int d^3\sigma \tilde{e}_\lambda(\vec{k}) \cdot \left[ \omega(\vec{k}) \tilde{A}_\perp(\tau, \vec{\sigma}) - i \tilde{\pi}_\perp(\tau, \vec{\sigma}) \right] e^{-i \vec{k} \cdot \vec{\sigma}}. \] (2.48)

The peculiarity of this new representation (in terms of \( a_{em,\lambda}(\tau, \vec{k}) \) and not of \( b_{em,\lambda}(\tau, \vec{k}) \)) is that it is only by using the dynamical equations (2.45) (i.e. the dynamical Hamilton equations) and the property \( \tilde{J}_\perp(\tau, \vec{k}) = \tilde{j}_\perp(\tau, -\vec{k}) \) of Eqs.(2.42) that we can show the validity of the kinematical Hamilton equations \( \pi_\perp(\tau, \vec{\sigma}) \equiv -\partial_\tau \tilde{A}_\perp(\tau, \vec{\sigma}) \), which were definitory of the old representation (2.40). With this representation the kinematical Hamilton equations hold only in the space of the solutions of the dynamical Hamilton equations.

However, due to the formal identity of Eqs.(2.48) and (2.33), the Poisson brackets \( \{ A_\perp^r(\tau, \vec{\sigma}_1), \pi_\perp^s(\tau, \vec{\sigma}_2) \} = -c P^r_{\perp \perp}^{s} \delta(\vec{\sigma}_1 - \vec{\sigma}_2) \) imply the Poisson brackets \( \{ a_{em,\lambda}(\tau, \vec{k}, \vec{k}), a_{em,\lambda}^*(\tau, \vec{k}) \} = -i \Omega(\vec{k}) c \delta_{\lambda\lambda'} \delta(\vec{k} - \vec{k}') \) like in Eqs.(2.33) for the radiated fields.

By using the last of Eqs.(2.48), Eqs. (2.23) and (B14) and (B15) of Appendix B we have from \( \tilde{A}_\perp(\tau, \vec{\sigma}) \equiv \tilde{A}_\perp(\tau, \vec{\sigma}) + \tilde{A}_{\perp rad}(\tau, \vec{\sigma}) \)
This already shows that on shell an arbitrary electro-magnetic field in the radiation gauge can be describe as a transverse radiation field plus particle terms coming from the Lienard-Wiechert fields.

Moreover we have the following expression for the terms of the internal Poincare’ generators (2.23) involving only the electromagnetic field

\[
\begin{align*}
c p_{em}(\tau) &= e_{em}(\tau) = M_{em}(\tau) c^2 = \frac{1}{2} \int d^3 \sigma \left( \pi_\perp + B^2 \right)(\tau, \sigma) = \frac{1}{2} \int d^3 k \left( \frac{2}{m_i^2 c^2 + k^2} \right)(\tau, k) = \\
&= \sum_{\lambda=1,2} \int d\tilde{k} \omega(\tilde{k}) a_{em,\lambda}(\tau, \tilde{k}) a_{em,\lambda}(\tau, \tilde{k}), \\
\tilde{p}_{em}(\tau) &= \frac{1}{c} \int d^3 \sigma \left( \pi_\perp \times \tilde{B} \right)(\tau, \tilde{\sigma}) = \frac{1}{c} \int d^3 k \left( \frac{2}{m_i^2 c^2 + k^2} \right)(\tau, k) = \\
&= \frac{1}{c} \sum_{\lambda=1,2} \int d\tilde{k} \tilde{k} a_{em,\lambda}^*(\tau, \tilde{k}) a_{em,\lambda}(\tau, \tilde{k}), \\
\tilde{j}_{em}(\tau) &= \frac{1}{c} \int d^3 \sigma \left( \pi_\perp \times \tilde{B} \right)(\tau, \tilde{\sigma}) = \\
&= i \frac{1}{c} \sum_{\lambda} \int d\tilde{k} a_{em,\lambda}^*(\tilde{k}) \tilde{k} \times \frac{\partial}{\partial \tilde{k}} a_{em,\lambda}(\tilde{k}) + \\
&+ \frac{i}{2c} \sum_{\lambda,\lambda'} \int d\tilde{k} \left[ a_{em,\lambda}(\tilde{k}) a_{em,\lambda'}^*(\tilde{k}) - a_{em,\lambda}(\tilde{k}) a_{em,\lambda'}(\tilde{k}) \right] \tilde{e}_\lambda(\tilde{k}) \cdot \left( \tilde{k} \times \frac{\partial}{\partial \tilde{k}} \tilde{e}_{\lambda'}(\tilde{k}) \right) - \\
&- \frac{i}{c} \sum_{\lambda,\lambda'} \int d\tilde{k} \tilde{e}_\lambda(\tilde{k}) \times \tilde{e}_{\lambda'}(\tilde{k}) a_{em,\lambda}^*(\tilde{k}) a_{em,\lambda'}(\tilde{k}),
\end{align*}
\]
\[ k_{\text{em}}^*(\tau) = -\frac{1}{2c} \int d^3\sigma \sigma^r \left[ \bar{\pi}^2_\perp + \vec{B}^2 \right](\tau, \vec{\sigma}) = \]
\[ = \frac{i}{c} \int d\tilde{k} a_{\text{em}0}(\tilde{k}) \omega(\tilde{k}) \frac{\partial}{\partial k^r} a_{\text{em}0}(\tilde{k}) + \]
\[ + \frac{i}{2c} \sum_{\lambda, \lambda'=1,2} \int d\tilde{k} \left[ a_{\text{em}0}^\lambda(\tilde{k}) a_{\text{em}0}^{\lambda'}(\tilde{k}) - a_{\text{em}0}^{\lambda'}(\tilde{k}) a_{\text{em}0}^\lambda(\tilde{k}) \right] \vec{e}_\lambda(\tilde{k}) \cdot \omega(\tilde{k}) \frac{\partial \vec{e}_{\lambda'}(\tilde{k})}{\partial k^r}, \]
\[ h_{\text{em}}(\tau) = \frac{\vec{j}_{\text{em}}(\tau) \cdot \vec{k}}{|\vec{k}|} = \frac{i}{c} \int d\tilde{k} \left[ a_{\text{em}2}^\lambda(\tilde{k}) a_{\text{em}1}(\tilde{k}) - a_{\text{em}1}(\tilde{k}) a_{\text{em}2}^\lambda(\tilde{k}) \right] = \]
\[ = \frac{1}{c} \int d\tilde{k} \left[ a_{\text{em}1}^\lambda(\tilde{k}) a_{\text{em}}(\tilde{k}) - a_{\text{em}}(\tilde{k}) a_{\text{em}1}^\lambda(\tilde{k}) \right], \quad (2.50) \]

Therefore at the end the internal Poincare- generators (2.23) \( M, \vec{S} = \vec{J}_{(\text{int})}, \vec{P}_{(\text{int})} \approx 0, \vec{K}_{(\text{int})} \approx 0 \), will depend on the particles and on \( a_{\text{em}}^\lambda(\tau, \tilde{k}) \).

J. The Lienard-Wiechert Electro-Magnetic Potential and Field associated with the Particles

In Ref.[14] we obtained the following expression for the Lienard-Wiechert transverse electromagnetic potential, electric field and magnetic field inhomogeneous solution of the equations \( \Box A_\perp^r(\tau, \vec{\sigma}) = j_\perp^r(\tau, \vec{\sigma}) = \sum_i Q_i P_\perp^r(\vec{\sigma}) \tilde{\eta}_i^r(\tau) \delta^3(\vec{\sigma} - \tilde{\eta}_i(\tau)) \) (see Eq.(B7) for the Hamiltonian expansion of \( \vec{j}_\perp^r(\tau, \vec{\sigma}) \))

\[ \vec{A}_\perp S(\tau, \vec{\sigma}) = \sum_{i=1}^N Q_i \vec{A}_\perp S_i(\vec{\sigma} - \tilde{\eta}_i(\tau), \vec{K}_i(\tau)), \]
\[ \vec{A}_\perp S_i(\vec{\sigma} - \tilde{\eta}_i, \vec{K}_i) = \frac{1}{4\pi |\vec{\sigma} - \tilde{\eta}_i|} \sqrt{m_i^2 \tau^2 + \vec{K}_i^2} \times \]
\[ \frac{1}{\sqrt{m_i^2 \tau^2 + \vec{K}_i^2}} \frac{\sqrt{m_i^2 \tau^2 + \vec{K}_i^2}}{\sqrt{m_i^2 \tau^2 + \vec{K}_i^2}} \]
\[ = \left( \frac{\alpha_1}{c} + \frac{\alpha_3}{c^3} + \sum_{k=2}^{\infty} \frac{\alpha_{k+1}}{c^{2k+1}} \right) (\tau, \vec{\sigma}). \quad (2.51) \]
\[ \vec{E}_\perp S(\tau, \vec{\sigma}) = \vec{\pi}_\perp S(\tau, \vec{\sigma}) = -\frac{\partial \vec{A}_\perp S(\tau, \vec{\sigma})}{\partial \tau} = \]
\[ = \sum_{i=1}^N Q_i \vec{\pi}_\perp S_i(\vec{\sigma} - \tilde{\eta}_i(\tau), \vec{K}_i(\tau)) = \]

38
\[= \sum_{i=1}^{N} Q_i \frac{\vec{\kappa}_i(\tau) \cdot \vec{\sigma}}{\sqrt{m_i^2 c^2 + \vec{\kappa}_i^2(\tau)}} \vec{A}_i \cdot (\vec{\sigma} - \vec{\eta}_i(\tau), \vec{\kappa}_i(\tau)) =\]

\[= - \sum_{i=1}^{N} Q_i \times \]

\[\frac{1}{4\pi |\vec{\sigma} - \vec{\eta}_i(\tau)|^2} \left[ \sec^2 \frac{\vec{\kappa}_i(\tau) \cdot \vec{\sigma} - \vec{\eta}_i(\tau)}{|\vec{\sigma} - \vec{\eta}_i(\tau)|} \right] \frac{\sqrt{m_i^2 c^2 + \vec{\kappa}_i^2(\tau)}}{[m_i^2 c^2 + (\vec{\kappa}_i(\tau) \cdot \vec{\sigma} - \vec{\eta}_i(\tau))^2]^{3/2}} + \]

\[+ \frac{\vec{\sigma} - \vec{\eta}_i(\tau)}{|\vec{\sigma} - \vec{\eta}_i(\tau)|} \left( \frac{\vec{\kappa}_i^2(\tau) + (\vec{\kappa}_i(\tau) \cdot \vec{\sigma} - \vec{\eta}_i(\tau))^2}{\sec^2 \frac{\vec{\kappa}_i(\tau) \cdot \vec{\sigma} - \vec{\eta}_i(\tau)}{|\vec{\sigma} - \vec{\eta}_i(\tau)|} \right) \frac{\sqrt{m_i^2 c^2 + \vec{\kappa}_i^2(\tau)}}{[m_i^2 c^2 + (\vec{\kappa}_i(\tau) \cdot \vec{\sigma} - \vec{\eta}_i(\tau))^2]^{3/2}} - 1) + \]

\[= \left( \frac{\beta_i}{c^2} + \sum_{k=2}^{\infty} \frac{\beta_i 2k}{c^{2k}} \right) (\tau, \vec{\sigma}). \quad (2.52)\]

\[\vec{B}_i(\tau, \vec{\sigma}) = (\vec{\sigma} \times \vec{A}_i) \cdot (\vec{\sigma} - \vec{\eta}_i(\tau), \vec{\kappa}_i(\tau)) = \]

\[= \sum_{i=1}^{N} Q_i \frac{1}{4\pi |\vec{\sigma} - \vec{\eta}_i(\tau)|^2} \frac{m_i^2 c^2 \vec{\kappa}_i(\tau) \times \vec{\sigma} - \vec{\eta}_i(\tau)}{[m_i^2 c^2 + (\vec{\kappa}_i(\tau) \cdot \vec{\sigma} - \vec{\eta}_i(\tau))^2]^{3/2}} = \]

\[= \left( \frac{\gamma_{i1}}{c} + \frac{\gamma_{i3}}{c^3} + \sum_{k=2}^{\infty} \frac{\gamma_{i2k+1}}{c^{2k+1}} \right) \frac{1}{c^2} (\tau, \vec{\sigma}). \quad (2.53)\]

See Appendix B, Subsection 1 for the coefficients of the expansions and other properties, and Subsection 2 for the Fourier transform of the Liéard-Wiechert quantities.

In Ref.[14] the final internal Poincare’ generators for \( N = 2 \) in absence of the radiation field were 23

\[\mathcal{E}_{(\text{int})} = M c^2 = c \sum_{i=1}^{2} \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2} + \frac{Q_1 Q_2}{4\pi |\vec{\eta}_1 - \vec{\eta}_2|} + V_{\text{DARWIN}}(\vec{\eta}_1(\tau) - \vec{\eta}_2(\tau); \vec{\kappa}_i(\tau)),\]

\[\vec{P}_{(\text{int})} = \vec{\kappa}_1 + \vec{\kappa}_2 \approx 0,\]

\[\vec{J}_{(\text{int})} = \sum_{i=1}^{2} \vec{\eta}_i \times \vec{\kappa}_i,\]

\[V_{\text{DARWIN}} = V_D(\vec{\eta}_1 - \vec{\eta}_2, \vec{\kappa}_1, \vec{\kappa}_2) \text{ is given in Eq.(6.19) of Ref.[14] and in Eq.(4.5).} \quad \vec{\eta}_i, \vec{\kappa}_i \text{ are dressed particle canonical variables given in Eqs.(5.51) of Ref.[14], where } \vec{J}_{(\text{int})} \text{ is given in Eq.(6.40) and } \vec{K}_{(\text{int})} \text{ in Eq.(6.46).} \quad \vec{K}_{ij} \text{ are a-dimensional quantities given in Eq.(5.35) of Ref. [14], given in Eq.(3.9).}\]
\[ \tilde{K}_{(int)} = - \sum_{i=1}^{2} \tilde{\eta}_i \left[ \sqrt{m_i^2 c^2 + \tilde{\kappa}_i^2} + \right. \\
+ \tilde{\kappa}_i \cdot \sum_{j \neq i} Q_i Q_j \left[ \partial_{\tilde{\eta}_i} \frac{1}{2} \tilde{K}_{ij} (\tilde{\kappa}_i, \tilde{\kappa}_j, \tilde{\eta}_i - \tilde{\eta}_j) - 2 \tilde{A}_{\perp Sj} (\tilde{\kappa}_j, \tilde{\eta}_i - \tilde{\eta}_j) \right] \right] - \\
- \frac{1}{2c} \sum_{i=1}^{2} \sum_{j \neq i} Q_i Q_j \sqrt{m_i^2 c^2 + \tilde{\kappa}_i^2} \partial_{\tilde{\eta}_i} \tilde{K}_{ij} (\tilde{\kappa}_i, \tilde{\kappa}_j, \tilde{\eta}_i - \tilde{\eta}_j) + \\
+ \sum_{i=1}^{2} \sum_{j \neq i} \frac{Q_i Q_j}{8\pi c} \frac{\tilde{\eta}_i - \tilde{\eta}_j}{|\tilde{\eta}_i - \tilde{\eta}_j|} - \sum_{i=1}^{2} \sum_{j \neq i} \frac{Q_i Q_j}{4\pi c} \int d^3 \sigma \tilde{\pi}_{\perp Sj} (\tilde{\sigma} - \tilde{\eta}_j, \tilde{\kappa}_j) - \\
- \frac{1}{2c} \sum_{i=1}^{2} \sum_{j \neq i} Q_i Q_j \int d^3 \sigma \tilde{\pi}_{\perp Si} (\tilde{\sigma} - \tilde{\eta}_i, \tilde{\kappa}_i) \cdot \tilde{\pi}_{\perp Sj} (\tilde{\sigma} - \tilde{\eta}_j, \tilde{\kappa}_j) + \\
+ \tilde{B}_{Si} (\tilde{\sigma} - \tilde{\eta}_i, \tilde{\kappa}_i) \cdot \tilde{B}_{Sj} (\tilde{\sigma} - \tilde{\eta}_j, \tilde{\kappa}_j) \approx 0. \] (2.54)
III. THE CANONICAL TRANSFORMATION.

In this Section we will define a canonical transformation allowing the identification of the radiation electro-magnetic field in the radiation gauge, as an alternative to the method of Ref.[1] (see also Eq.(4)-(5) of the Appendix of Ref.[2]) valid in the Coulomb gauge. The non-radiative part of the electro-magnetic field will become a dressing of charged particles and will imply the replacement of the Coulomb potential with the Darwin one (it is a modification beyond the $O(1/c)$ semi-relativistic standard atomic physics). The only interaction between the dressed particles and the radiation field is due to the rest-frame conditions and its gauge fixings (vanishing of the internal boosts), eliminating the internal 3-center of mass. This is possible due to $Q_i^2 = 0$.

Knowing the Lienard-Wiechert (LW) solution of Eqs. (2.51), (2.52), (2.53), let us look for a canonical transformation to a new canonical basis $\vec{A}_\perp^{\text{rad}}(\tau, \vec{\sigma}), \vec{\pi}_\perp^{\text{rad}}(\tau, \vec{\sigma}), \hat{\eta}_i(\tau), \hat{\kappa}_i(\tau)$ such that the new transverse electromagnetic field be the radiation field of Eqs.(2.33)

\[
\begin{align*}
\vec{A}_\perp^{\text{rad}}(\tau, \vec{\sigma}) &= \vec{A}_\perp(\tau, \vec{\sigma}) - \vec{A}_\perp^{S}(\tau, \vec{\sigma}), \\
\vec{\pi}_\perp^{\text{rad}}(\tau, \vec{\sigma}) &= \vec{\pi}_\perp(\tau, \vec{\sigma}) - \vec{\pi}_\perp^{S}(\tau, \vec{\sigma}), \\
\hat{\eta}_i(\tau) &= \eta_i(\tau) + Q_i K_i(\tau), \\
\hat{\kappa}_i(\tau) &= \kappa_i(\tau) + Q_i W_i(\tau),
\end{align*}
\]

\[
\{A_r^{\perp}(\tau, \vec{\sigma}), \pi_s^{\perp}(\tau, \vec{\sigma}_1)\} = -c P_{rs}^{\perp}(\vec{\sigma}) \delta^3(\vec{\sigma} - \vec{\sigma}_1), \quad \{\eta_i^r(\tau), \kappa_{js}(\tau)\} = \delta_{ij} \delta^r_s,
\]

\[
\downarrow
\]

\[
\{A_r^{\perp}(\tau, \vec{\sigma}), \pi_s^{\perp}(\tau, \vec{\sigma}_1)\} = -c P_{rs}^{\perp}(\vec{\sigma}) \delta^3(\vec{\sigma} - \vec{\sigma}_1), \quad \{\hat{\eta}_i(\tau), \hat{\kappa}_{js}(\tau)\} = \delta_{ij} \delta^r_s,
\]

or

\[
\{a_\lambda(\vec{k}), a_{\lambda'}^*(\vec{k}_1)\} = -i \Omega(\vec{k}) c \delta_{\lambda\lambda'} \delta^3(\vec{k} - \vec{k}_1),
\]

(3.1)

where Eqs.(2.33) have been used.

If this canonical transformation exists, then, consistently, Eqs.(4.17) of Ref.[14] imply [see also Eq.(B7)]

\[
\square \vec{A}_\perp(\tau, \vec{\sigma}) \sim \vec{J}_\perp(\tau, \vec{\sigma}), \quad \Rightarrow \quad \square \vec{A}_\perp^{\text{rad}}(\tau, \vec{\sigma}) \sim 0.
\]

(3.2)

Let us build the canonical transformation. For the field sector we have (the Poisson brackets of the LW fields vanish because only the terms proportional to $Q_i^2 = 0$ survive due to Eqs.(2.51) and (2.52))
\[
0 = \{ A_{\perp \text{rad}}^r(\tau, \vec{\sigma}), A_{\perp \text{rad}}^s(\tau, \vec{\sigma}_1) \} = \sum_{i \neq j}^1 \cdots N Q_i Q_j \{ A_{\perp \text{rad}}^r(\tau, \vec{\sigma}), A_{\perp \text{rad}}^s(\tau, \vec{\sigma}_1) \} = 0, \\
0 = \{ \pi_{\perp \text{rad}}^r(\tau, \vec{\sigma}), \pi_{\perp \text{rad}}^s(\tau, \vec{\sigma}_1) \} = \sum_{i \neq j}^1 \cdots N Q_i Q_j \{ \pi_{\perp \text{rad}}^r(\tau, \vec{\sigma}), \pi_{\perp \text{rad}}^s(\tau, \vec{\sigma}_1) \} = 0, \\
- c P_{\perp}^r(\vec{\sigma}) \delta^3(\vec{\sigma} - \vec{\sigma}_1) = \{ A_{\perp \text{rad}}^r(\tau, \vec{\sigma}), \pi_{\perp \text{rad}}^s(\tau, \vec{\sigma}_1) \} = - c P_{\perp}^s(\vec{\sigma}) \delta^3(\vec{\sigma} - \vec{\sigma}_1) + \sum_{i \neq j}^1 \cdots N Q_i Q_j \{ A_{\perp \text{rad}}^r(\tau, \vec{\sigma}), \pi_{\perp \text{rad}}^s(\tau, \vec{\sigma}_1) \} = - c P_{\perp}^s(\vec{\sigma}) \delta^3(\vec{\sigma} - \vec{\sigma}_1). \quad (3.3)
\]

Let us now determine \( \hat{\eta}_i(\tau) \) and \( \hat{\kappa}_i(\tau) \). To this end let introduce the following functionals

\[
T_i(\tau) = \int d^3\sigma \left[ \vec{\pi}_{\perp} \cdot \vec{A}_{\perp Si}(\tau, \vec{\sigma}) - \vec{A}_{\perp} \cdot \vec{\pi}_{\perp Si}(\tau, \vec{\sigma}) \right], \quad (3.4)
\]

which have the property (\( K_{ij} \) had been defined in Ref.[14] to find the particle canonical basis of the Dirac brackets)

\[
Q_i Q_j \{ T_i(\tau), T_j(\tau) \} = - c Q_i Q_j K_{ij}(\tau) = c Q_i Q_j K_{ji}(\tau), \\
K_{ij}(\tau) = \int d^3\sigma \left[ \vec{A}_{\perp Si}(\tau, \vec{\sigma}) \cdot \vec{\pi}_{\perp Sj}(\tau, \vec{\sigma}) - \vec{\pi}_{\perp Si}(\tau, \vec{\sigma}) \cdot \vec{A}_{\perp Sj}(\tau, \vec{\sigma}) \right], \quad (3.5)
\]

and let us make the following ansatz

\[
\hat{\eta}_i^r(\tau) = \eta_i^r(\tau) + \frac{Q_i}{c} \frac{\partial T_i(\tau)}{\partial \kappa_{ir}} - \frac{1}{2} \frac{Q_i}{c} \sum_{k \neq i}^1 \cdots N Q_k \frac{\partial K_{ik}(\tau)}{\partial \kappa_{ir}}, \\
\hat{\kappa}_{ir}(\tau) = \kappa_{ir}(\tau) - \frac{Q_i}{c} \frac{\partial T_i(\tau)}{\partial \eta_i^r} + \frac{1}{2} \frac{Q_i}{c} \sum_{k \neq i}^1 \cdots N Q_k \frac{\partial K_{ik}(\tau)}{\partial \eta_i^r}, \\
Q_i \hat{\eta}_i = Q_i \eta_i, \quad Q_i \hat{\kappa}_i = Q_i \kappa_i. \quad (3.6)
\]

Let us verify whether the ansatz defines a canonical transformation. We have immediately
0 = \{A^r_{\perp rad}(\tau, \vec{\sigma}), \hat{\eta}_i^s(\tau)\} = Q_i \left[ \frac{1}{c} \left\{ A^r_{\perp}(\tau, \vec{\sigma}), \frac{\partial T_i(\tau)}{\partial \kappa_{is}} \right\} + \frac{\partial A^r_{\perp Si}(\tau, \vec{\sigma})}{\partial \kappa_{is}} \right] = \\
= Q_i \left[ \frac{\partial}{\partial \kappa_{is}} \int d^3 \sigma_1 \left( - P^r_{\perp}(\vec{\sigma}) \delta^3(\vec{\sigma} - \vec{\sigma}_1) \right) A^u_{\perp Si}(\tau, \vec{\sigma}_1) + \frac{\partial A^r_{\perp Si}(\tau, \vec{\sigma})}{\partial \kappa_{is}} \right] = 0, \\
0 = \{\pi^r_{\perp rad}(\tau, \vec{\sigma}), \hat{\eta}_i^s(\tau)\} = Q_i \left[ \frac{1}{c} \left\{ \pi^r_{\perp}(\tau, \vec{\sigma}), \frac{\partial T_i(\tau)}{\partial \kappa_{is}} \right\} + \frac{\partial \pi^r_{\perp Si}(\tau, \vec{\sigma})}{\partial \kappa_{is}} \right] = 0, \\
0 = \{A^r_{\perp rad}(\tau, \vec{\sigma}), \hat{\kappa}_{is}\} = Q_i \left[ - \frac{1}{c} \left\{ A^r_{\perp}(\tau, \vec{\sigma}), \frac{\partial T_i(\tau)}{\partial \eta_i^s} \right\} - \frac{\partial A^r_{\perp Si}(\tau, \vec{\sigma})}{\partial \eta_i^s} \right] = 0, \\
0 = \{\pi^r_{\perp rad}(\tau, \vec{\sigma}), \hat{\kappa}_{is}\} = Q_i \left[ - \frac{1}{c} \left\{ \pi^r_{\perp}(\tau, \vec{\sigma}), \frac{\partial T_i(\tau)}{\partial \eta_i^s} \right\} - \frac{\partial \pi^r_{\perp Si}(\tau, \vec{\sigma})}{\partial \eta_i^s} \right] = 0. \quad (3.7)

Finally we have

\[ \delta_{ij} \phi_i = \{\hat{\eta}_i^s(\tau), \hat{\kappa}_{js}(\tau)\} = \delta_{ij} \phi_i^s + \]

\[ + \frac{\partial}{\partial \kappa_{ir}} \left( - \frac{Q_i}{c} \frac{\partial T_j(\tau)}{\partial \eta_j^s} \right) + \frac{1}{2} \frac{Q_i}{c} \sum_{k \neq j} Q_k \frac{\partial K_{ik}(\tau)}{\partial \eta_j^s} \]

\[ + \frac{\partial}{\partial \eta_j^s} \left( \frac{Q_i}{c} \frac{\partial T_i(\tau)}{\partial \kappa_{ir}} - \frac{1}{2} \frac{Q_i}{c} \sum_{k \neq i} Q_k \frac{\partial K_{ik}(\tau)}{\partial \kappa_{ir}} \right) - \frac{Q_i}{c} \left\{ \frac{\partial T_i(\tau)}{\partial \kappa_{ir}}, \frac{\partial T_j(\tau)}{\partial \eta_j^s} \right\} = \]

\[ = \delta_{ij} \phi_i^s - \frac{1}{2} \frac{Q_i}{c} \sum_{k \neq i} Q_k \left( \frac{\partial^2 K_{ik}(\tau)}{\partial \eta_j^s \partial \eta_k^s} - \frac{\partial^2 K_{ik}(\tau)}{\partial \kappa_{ir} \partial \eta_j^s} \right) \]

\[ + \frac{1}{2} \frac{Q_i}{c} \left( \frac{\partial^2 K_{ii}(\tau)}{\partial \eta_j^s \partial \kappa_{ir}} - \frac{\partial^2 K_{ij}(\tau)}{\partial \kappa_{ir} \partial \eta_j^s} \right) + \frac{Q_i}{c} \frac{\partial^2 K_{ij}(\tau)}{\partial \kappa_{ir} \partial \eta_j^s} = \delta_{ij} \phi_i^s. \quad (3.8) \]

Analogously we have that the conditions \(0 = \{\hat{\eta}_i^s, \hat{\eta}_j^s\} = \{\hat{\kappa}_{ir}, \hat{\kappa}_{js}\}\) are satisfied.

Since \(Q_i, \hat{\kappa}_{is}\) and \(Q_i, \hat{\pi}_{\perp Si}\) are the same functions (2.51) and (2.52) of \(\hat{\eta}_i, \hat{\kappa}_i\) as \(Q_i, \hat{\kappa}_{is}\) and \(Q_i, \hat{\pi}_{\perp Si}\) were of \(\hat{\eta}_i, \hat{\kappa}_i\) and since we have

\[ Q_i, Q_j K_{ij}(\tau) = Q_i, Q_j \hat{K}_{ij}(\tau) = Q_i, Q_j \int d^3 \sigma \left[ \hat{\kappa}_{\perp Si} \cdot \hat{\pi}_{\perp Sj} - \hat{\pi}_{\perp Si} \cdot \hat{\kappa}_{\perp Sj} \right](\tau, \vec{\sigma}), \]

\[ Q_i, T_i(\tau) = Q_i, \hat{T}_i(\tau) + Q_i \sum_{k \neq i} Q_k \hat{K}_{ik}(\tau), \quad \text{with} \]

\[ \hat{T}_i(\tau) = \int d^3 \sigma \left[ \hat{\pi}_{\perp rad} \cdot \hat{A}_{\perp Si} - \hat{A}_{\perp rad} \cdot \hat{\pi}_{\perp Si} \right](\tau, \vec{\sigma}), \quad (3.9) \]
the inverse of the canonical transformation (3.1), (3.6) is

\[ \tilde{A}_\perp(\tau, \vec{\sigma}) = \tilde{A}_{\perp rad} + \hat{A}_\perp S(\tau, \vec{\sigma}), \]

\[ \tilde{\pi}_\perp(\tau, \vec{\sigma}) = \tilde{\pi}_{\perp rad} + \hat{\pi}_\perp S(\tau, \vec{\sigma}), \]

\[ \eta_i^r(\tau) = \hat{\eta}_i^r(\tau) - \frac{Q_i}{c} \frac{\partial \hat{T}_i(\tau)}{\partial \hat{\eta}_i^r} - \frac{1}{2} \sum_{k \neq i}^{1..N} Q_k \frac{\partial \hat{K}_{ik}(\tau)}{\partial \hat{\eta}_i^r}, \]

\[ \kappa_{ir}(\tau) = \hat{\kappa}_{ir}(\tau) + \frac{Q_i}{c} \frac{\partial \hat{T}_i(\tau)}{\partial \hat{\eta}_i^r} + \frac{1}{2} \sum_{k \neq i}^{1..N} Q_k \frac{\partial \hat{K}_{ik}(\tau)}{\partial \hat{\eta}_i^r}. \] (3.10)

The generating function of this canonical transformation is

\[ S = \frac{1}{c} \sum_i^{1..N} Q_i T_i[\tilde{A}_\perp, \tilde{\pi}_\perp, \tilde{\eta}_i, \tilde{\kappa}_i]. \] (3.11)

Due to the Grassmann-valued charges for every phase space variable \( A = \tilde{A}_\perp(\tau, \vec{\sigma}), \tilde{\pi}_\perp(\tau, \vec{\sigma}), \tilde{\eta}_i(\tau), \tilde{\kappa}_i(\tau), \) we have the following truncation

\[ e^{\{-S\}} A = A + \{A, S\} + \frac{1}{2} \{ \{A, S\}, S\}. \] (3.12)

Eqs.(3.1) and (3.6) are reproduced by means of Eqs.(3.12)

\[ \hat{\eta}_i^r(\tau) = e^{\{-S\}} \eta_i^r(\tau), \]

\[ \hat{\kappa}_{ir}(\tau) = e^{\{-S\}} \kappa_{ir}(\tau), \]

\[ \tilde{A}_{\perp rad}(\tau, \vec{\sigma}) = e^{\{-S\}} \tilde{A}_\perp(\tau, \vec{\sigma}), \]

\[ \tilde{\pi}_{\perp rad}(\tau, \vec{\sigma}) = e^{\{-S\}} \tilde{\pi}_\perp(\tau, \vec{\sigma}). \] (3.13)

This result says that, at least at the classical level, the isolated system of \( N \) particles with Grassmann-valued electric charges and mutual Coulomb interaction plus the transverse electro-magnetic field” is canonically equivalent to a free radiation field plus a system of Coulomb-dressed charged particles mutually interacting through a Coulomb plus Darwin potential. This is possible because the Grassmann-valued charges imply the following properties:

a) A regularization of Coulomb self-energies, namely a ultraviolet cutoff.

b) The emergence of the Darwin potential, namely of the main ingredient for the theory of relativistic bound states in scalar QED.

c) Transverse Lienard-Wiechert potential and electric field depending only on the positions and momenta of the charged particles and not on the higher accelerations. Therefore they are the action-at-a-distance potentials hidden in all the possible (retarded, advanced, symmetric,..) solutions of Maxwell equations. The Coulomb plus Darwin potentials give the
Cauchy problem formulation of the interaction hidden in the one-photon exchange Feynman diagrams. All the loop diagrams and the soft and hard photon emission diagrams are eliminated. Therefore there is also an infrared regularization. Also the classical problems with causality violations (either runaway solutions or pre-accelerations) are absent. There is no Larmor emission of radiation from a single Grasmann-value d charge. However, as shown in Ref.[35], in the wave zone of a system of Grassmann-valued charges one can obtain the Larmor formula for the radiated energy as a result of he interference terms $Q_i Q_j$ with $i \neq j$

\[
\frac{dE}{d\tau} = \frac{2}{3} \sum_{i \neq j} \frac{Q_i Q_j}{(4\pi)^2} \vec{\eta}_i(\tau) \cdot \vec{\eta}_j(\tau).
\]

d) In the rest-frame instant form the isolated system of ”N particles with Grassmann-valued electric charges and with mutual Coulomb interaction plus the transverse electromagnetic field” is described as a non-covariant external decoupled center of mass carrying the internal mass and the spin of the system. The system lives inside the instantaneous Euclidean Wigner 3-spaces, where the canonical equivalence with a radiation field and a system of Coulomb-dressed Grassmann-valued charges with mutual Coulomb plus Darwin interaction holds. However, the two non-interacting subsystems are connected by the rest-frame condition $\vec{P}(\text{int}) \approx 0$ and by the elimination, $\vec{K}(f) \approx 0$, of the internal 3-center of mass. As a consequence, the radiation field knows the particle subsystem: strictly speaking it is not a free field. Therefore the existence of the canonical equivalence may be a classical implementation of the mechanism which allows the Haag-Ruelle scattering theory (see for instance Ref.[16]) to avoid the Haag theorem. As clearly said in Ref.[17], this happens when it is possible to define in the Hilbert space of the interacting fields ”surrogates” of the free field states in the asymptotic $t \to \pm \infty$ regimes (these surrogates are not unitarily equivalent to free fields). Moreover in our case there is the universal breaking of Lorentz covariance connected to the decoupled external center of mass, which is present in the rest-frame instant form even in absence of particles and even if the No-Interaction theorem does not hold in field theory [23]. This is the price for having this instant form emerging from the general description of isolated systems in non-inertial frames.
IV. THE INTERNAL POINCARE’ GENERATORS AND THE HAMILTON EQUATIONS AFTER THE CANONICAL TRANSFORMATION

A. The New Expression of the Internal Poincare’ Generators

For the internal 3-momentum of Eq.(2.23) in the new canonical basis Eqs. (C1)-(C3) imply the following expression

\[
\vec{P}_{(\text{int})} = \sum_{i=1}^{N} \hat{\kappa}_i(\tau) + \frac{1}{c} \int d^3\sigma \left( \vec{\pi}_{\text{rad}} \times \vec{B}_{\text{rad}} \right)(\tau, \vec{\sigma}) + \\
+ \sum_{i=1}^{N} \frac{Q_i}{c} \left[ \frac{\partial \hat{T}_i(\tau)}{\partial \hat{\eta}_i} + \int d^3\sigma \left( \vec{\pi}_{\text{rad}} \times \hat{\pi}_{\text{rad}} \cdot \vec{B}_{\text{rad}} \right)(\tau, \vec{\sigma}) \right] + \\
+ \sum_{i \neq j}^{1..N} \frac{Q_i Q_j}{c} \left[ \int d^3\sigma \left( \hat{\pi}_{\text{rad}} \times \hat{B}_{\text{rad}} \right)(\tau, \vec{\sigma}) + \frac{1}{2} \frac{\partial \hat{K}_{ij}}{\partial \hat{\eta}_i} \right] = \\
= \sum_{i=1}^{N} \hat{\kappa}_i(\tau) + \frac{1}{c} \int d^3\sigma \left( \vec{\pi}_{\text{rad}} \times \vec{B}_{\text{rad}} \right)(\tau, \vec{\sigma}) = \\
= \vec{P}_{\text{matter}} + \vec{P}_{\text{rad}} \approx 0. \quad (4.1)
\]

For the internal angular momentum of Eq.(2.23) Eq.(C4) implies

\[
\vec{J}_{(\text{int})} = \sum_{i=1}^{N} \hat{\eta}_i \times \hat{\kappa}_i + \frac{1}{c} \int d^3\sigma \vec{\sigma} \times \left( \vec{\pi}_{\text{rad}} \times \vec{B}_{\text{rad}} \right)(\tau, \vec{\sigma}) + \\
+ \sum_{i=1}^{N} \frac{Q_i}{c} \left[ \hat{\pi}_i(\tau) \times \frac{\partial \hat{T}_i(\tau)}{\partial \hat{\eta}_i} + \hat{\kappa}_i(\tau) \times \frac{\partial \hat{T}_i(\tau)}{\partial \hat{\kappa}_i} \right] \hat{T}_i(\tau) + \\
+ \int d^3\sigma \vec{\sigma} \times \left( \vec{\pi}_{\text{rad}} \times \hat{B}_{\text{rad}} \right)(\tau, \vec{\sigma}) \right] + \\
+ \sum_{i \neq j}^{1..N} \frac{Q_i Q_j}{c} \left[ \frac{1}{2} \left( \hat{\pi}_i \times \frac{\partial \hat{\sigma}}{\partial \hat{\eta}_i} + \hat{\kappa}_i \times \frac{\partial \hat{\sigma}}{\partial \hat{\kappa}_i} \right) \hat{K}_{ij}(\tau) + \\
+ \int d^3\sigma \vec{\sigma} \times \left( \vec{\pi}_{\text{rad}} \times \hat{B}_{\text{rad}} \right)(\tau, \vec{\sigma}) \right] = \\
= \sum_{i=1}^{N} \hat{\eta}_i \times \hat{\kappa}_i + \frac{1}{c} \int d^3\sigma \vec{\sigma} \times \left( \vec{\pi}_{\text{rad}} \times \vec{B}_{\text{rad}} \right)(\tau, \vec{\sigma}) = \\
= \vec{J}_{\text{matter}} + \vec{J}_{\text{rad}}. \quad (4.2)
\]

By adding the second class constraints \( \vec{A}_{\text{rad}}(\tau, \vec{\sigma}) \approx 0 \) and \( \vec{\pi}_{\text{rad}}(\tau, \vec{\sigma}) \approx 0 \) we recover the expression of Eqs.(2.54) for the internal 3-momentum and angular momentum.
Since we have
\[
c \sqrt{m_i^2 c^2 + \kappa_i^2} = c \sqrt{m_i^2 c^2 + \kappa_i^2 + \frac{2Q_i}{c} \hat{k}_i \cdot \left( \frac{\partial \hat{T}_i}{\partial \eta_i} + \frac{1}{2} \sum_{k \neq i}^N Q_k \frac{\partial \hat{K}_{ik}}{\partial \eta_i} \right)} = c \sqrt{m_i^2 c^2 + \kappa_i^2 + Q_i \frac{\hat{k}_i \cdot \left( \frac{\partial \hat{T}_i}{\partial \eta_i} + \frac{1}{2} \sum_{k \neq i}^N Q_k \frac{\partial \hat{K}_{ik}}{\partial \eta_i} \right)}{\sqrt{m_i^2 c^2 + \kappa_i^2}}},
\]
the internal energy of (2.23) takes the following form
\[
E_{(\text{int})} = M c^2 = c \sum_{i=1}^N \sqrt{m_i^2 c^2 + \kappa_i^2(\tau)} +
\left[ \left( \frac{\partial \hat{T}_i}{\partial \eta_i} + \frac{1}{2} \sum_{j \neq i}^N Q_j \frac{\partial \hat{K}_{ij}}{\partial \eta_i} \right) - \hat{A}_{\text{rad}}(\tau, \hat{\eta}_i(\tau)) - \sum_{j \neq i}^N Q_j \hat{A}_{\text{Sj}}(\tau, \hat{\eta}_j(\tau)) \right] + \sum_{i \neq j}^{1,N} \frac{Q_i Q_j}{4\pi |\hat{\eta}_i(\tau) - \hat{\eta}_j(\tau)|} +
\frac{1}{2} \int d^3\sigma \left( \hat{\pi}_{\text{rad}}^2 + \hat{B}_{\text{rad}}^2 \right)(\tau, \vec{\sigma}) + \sum_i Q_i \int d^3\sigma \left( \hat{\pi}_{\text{Si}} \cdot \hat{\pi}_{\text{Sj}} + \hat{B}_{\text{Si}} \cdot \hat{B}_{\text{Sj}} \right)(\tau, \vec{\sigma}) +
\frac{1}{2} \sum_{i \neq j}^{1,N} Q_i Q_j \int d^3\sigma \left( \hat{\pi}_{\text{Si}} \cdot \hat{\pi}_{\text{Sj}} + \hat{B}_{\text{Si}} \cdot \hat{B}_{\text{Sj}} \right)(\tau, \vec{\sigma}) =
\]
\[
= c \sum_{i=1}^N \sqrt{m_i^2 c^2 + \kappa_i^2(\tau)} + \sum_{i \neq j}^{1,N} \frac{Q_i Q_j}{4\pi |\hat{\eta}_i(\tau) - \hat{\eta}_j(\tau)|} + V_{\text{DARWIN}} +
\sum_{i=1}^N Q_i \int d^3\sigma \left[ \hat{\pi}_{\text{rad}} \cdot \left( \hat{\pi}_{\text{Si}} + \left( \frac{c \hat{k}_i}{\sqrt{m_i^2 c^2 + \kappa_i^2}} \frac{\partial}{\partial \eta_i} \right) \hat{A}_{\text{Si}} \right) - \hat{A}_{\text{rad}} \cdot \left( \hat{\pi}_{\text{Si}} + \left( \frac{c \hat{k}_i}{\sqrt{m_i^2 c^2 + \kappa_i^2}} \frac{\partial}{\partial \eta_i} \right) \hat{A}_{\text{Si}} \right) \right] -
\hat{k}_i \cdot \hat{A}_{\text{rad}}(\tau, \hat{\eta}_i(\tau)) + \frac{1}{2} \int d^3\sigma \left( \hat{\pi}_{\text{rad}}^2 + \hat{B}_{\text{rad}}^2 \right)(\tau, \vec{\sigma}) =
\]
47
\[ V_{\text{DARWIN}}(\hat{\eta}_1(\tau) - \hat{\eta}_2(\tau); \hat{\kappa}_i(\tau)) = \sum_{i \neq j}^{1,N} Q_i Q_j \left( \frac{\hat{\kappa}_i \cdot \hat{A}_{\perp S_j}(\tau, \hat{\eta}_i(\tau))}{\sqrt{m_i^2 c^2 + \hat{\kappa}_i^2}} \right) + \frac{1}{2} \int d^3 \sigma \left( \frac{1}{2} \left( \hat{\pi}_{\perp S_i} \cdot \hat{\pi}_{\perp S_j} + \hat{B}_{S_i} \cdot \hat{B}_{S_j} \right) + \left( \frac{\hat{\kappa}_i}{\sqrt{m_i^2 c^2 + \hat{\kappa}_i^2}} \cdot \frac{\partial}{\partial \hat{\eta}_i} \right) \left( \hat{A}_{\perp S_i} \cdot \hat{\pi}_{\perp S_j} - \hat{\pi}_{\perp S_i} \cdot \hat{A}_{\perp S_j} \right) \right)(\tau, \vec{\sigma}). \]

By adding the second class constraints \( \hat{A}_{\perp \text{rad}}(\tau, \vec{\sigma}) \approx 0 \) and \( \hat{\pi}_{\perp \text{rad}}(\tau, \vec{\sigma}) \approx 0 \) (implying \( \hat{T}_i(\tau) \approx 0 \)) we recover the internal energy of Eqs.(2.54).

Since the canonical transformation is not explicitly \( \tau \)-dependent, \( M c^2 \) is still the hamiltonian and both the energy \( P_{\text{matter}}^\tau \) of the dressed particles and the energy \( P_{\text{rad}}^\tau \) of the radiation field are constants of motion.

See Subsection 3 of Appendix B for the \( 1/c \) expansion of Eq.(4.4 ).

For the internal boosts, we substitute expressions for the new canonical variables given in Eqs.(3.10) into the boost expression given in Eq.(2.24) to obtain
\[
\vec{K}_{(\text{int})} = - \sum_{i=1}^{N} \vec{\eta}_i(\tau) \left( \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2(\tau)} + \frac{Q_i}{c \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2(\tau)}} \hat{\vec{K}}_i(\tau) \cdot \left[ \frac{\partial \hat{T}_i(\tau)}{\partial \vec{\eta}_i} + \frac{1}{2} \sum_{j \neq i}^{1..N} Q_j \frac{\partial \hat{K}_{ij}(\tau)}{\partial \vec{\eta}_i} \right] - 
\right.

\left. \vec{A}_{\perp \text{rad}}(\tau, \vec{\eta}_i(\tau)) - \sum_{j \neq i} Q_j \hat{\vec{A}}_{\perp S_j}(\tau, \vec{\eta}_i(\tau)) \right) + 

\left. \sum_{i=1}^{N} \frac{Q_i}{c} \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2(\tau)} \frac{\partial \hat{T}_i(\tau)}{\partial \vec{\kappa}_i} - \frac{1}{2} \sum_{i \neq j}^{1..N} \frac{Q_i Q_j}{c} \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2(\tau)} \frac{\partial \hat{K}_{ij}(\tau)}{\partial \vec{\kappa}_i} + 
\right.

\left. \frac{1}{c} \sum_{i=1}^{N} Q_i \left[ \sum_{j \neq i}^{1..N} Q_j \left[ \frac{1}{\Delta \vec{\eta}_j} \frac{\partial}{\partial \vec{\eta}_j} c(\vec{\eta}_i(\tau) - \vec{\eta}_j(\tau)) - \vec{\eta}_j(\tau) c(\vec{\eta}_i(\tau) - \vec{\eta}_j(\tau)) \right] + 
\right. 

\left. \int d^3\sigma \ c(\vec{\sigma} - \vec{\eta}_i(\tau)) \left( \vec{\pi}_{\perp \text{rad}} + \sum_{j \neq i} Q_j \hat{\vec{\pi}}_{\perp S_j}(\tau, \vec{\sigma}) \right) \right] - 

\left. - \frac{1}{2c} \int d^3\sigma \ \vec{\sigma} \left( \vec{\pi}_{\perp \text{rad}}^2 + \vec{B}_{\perp \text{rad}}^2 \right)(\tau, \vec{\sigma}) - 
\right.

\left. - \sum_{i=1}^{N} \frac{Q_i}{c} \int d^3\sigma \ \vec{\sigma} \left( \vec{\pi}_{\perp \text{rad}} \cdot \hat{\vec{\pi}}_{\perp S_i} + \vec{B}_{\perp \text{rad}} \cdot \hat{\vec{B}}_{S_i} \right)(\tau, \vec{\sigma}) - 
\right.

\left. - \frac{1}{2c} \sum_{i \neq j}^{1..N} Q_i Q_j \int d^3\sigma \ \vec{\sigma} \left( \hat{\vec{\pi}}_{\perp S_i} \cdot \hat{\vec{\pi}}_{\perp S_j} + \hat{\vec{B}}_{S_i} \cdot \hat{\vec{B}}_{S_j} \right)(\tau, \vec{\sigma}) \right) (4.6)
\]
Then using the results of Eqs. (C5)-(C18) we get

\[ \vec{K}_{(\text{int})} = -\sum_{i=1}^{N} \hat{\eta}_i(\tau) \left[ \sqrt{m_i^2 c^2 + \hat{\kappa}_i^2} + \frac{\hat{\kappa}_i}{2c \sqrt{m_i^2 c^2 + \hat{\kappa}_i^2}} \sum_{j \neq i}^{1..N} Q_i Q_j \left( \frac{1}{2} \frac{\partial \hat{K}_{ij}(\hat{\kappa}_i, \hat{\kappa}_j, \hat{\eta}_i - \hat{\eta}_j)}{\partial \hat{\eta}_i} \right) - 2 \hat{A}_{\perp S_j}(\hat{\kappa}_i, \hat{\eta}_i - \hat{\eta}_j) \right] - \frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i}^{1..N} \frac{Q_i Q_j}{c} \sqrt{m_i^2 c^2 + \hat{\kappa}_i^2} \frac{\partial \hat{K}_{ij}(\hat{\kappa}_i, \hat{\kappa}_j, \hat{\eta}_i - \hat{\eta}_j)}{\partial \hat{\kappa}_i} + \sum_{i=1}^{N} \sum_{j \neq i}^{1..N} \frac{Q_i Q_j}{8\pi c} \int d^3\sigma \frac{\hat{\pi}_{\perp S_j}(\hat{\sigma} - \hat{\eta}_i, \hat{\kappa}_i)}{|\hat{\sigma} - \hat{\eta}_i|} - \frac{1}{2c} \sum_{i=1}^{N} \sum_{j \neq i}^{1..N} Q_i Q_j \int d^3\sigma \hat{\sigma} \left[ \hat{\pi}_{\perp S_i}(\hat{\sigma} - \hat{\eta}_i, \hat{\kappa}_i) \cdot \hat{\pi}_{\perp S_j}(\hat{\sigma} - \hat{\eta}_j, \hat{\kappa}_j) + \hat{B}_S(\hat{\sigma} - \hat{\eta}_i, \hat{\kappa}_i) \cdot \hat{B}_S(\hat{\sigma} - \hat{\eta}_j, \hat{\kappa}_j) \right] - \frac{1}{2c} \int d^3\sigma \hat{\sigma} \left( \hat{\pi}_{\perp \text{rad}}^2 + \hat{B}_{\text{rad}}^2 \right)(\tau, \hat{\sigma}) = \vec{K}_{\text{matter}} + \vec{K}_{\text{rad}} = c \vec{K}_{\text{Galilei}} + O\left(\frac{1}{c}\right) = \]

\[ \equiv -\frac{1}{c} \mathcal{E}_{(\text{int})} \vec{R}_+ = -\frac{1}{c} \left( \mathcal{E}_{\text{matter}} + \mathcal{E}_{\text{rad}} \right) \vec{R}_+ = \]

\[ \overset{\text{def}}{=} -\frac{1}{c} \mathcal{E}_{(\text{int})} \vec{R}_+ = -\frac{1}{c} \left( \mathcal{E}_{\text{matter}} + \mathcal{E}_{\text{rad}} \right) \vec{R}_+ \approx 0. \] (4.7)

By adding the second class constraints \( \hat{A}_{\perp \text{rad}}(\tau, \hat{\sigma}) \approx 0 \) and \( \hat{\pi}_{\perp \text{rad}}(\tau, \hat{\sigma}) \approx 0 \) (implying \( \hat{T}_i(\tau) \approx 0 \)) we recover the internal boost of Eqs.(2.54) [of Eq.(6.46) of Ref.[14]].

Even if all the internal generators are the direct sum of the generators of the two subsystems, the two subsystems are connected by the rest-frame conditions and by the vanishing of the internal boosts, i.e. by the necessity of eliminating the position and momentum of the internal 3-center of mass.

**B. The New Hamilton Equations**

Let us consider the new Hamiltonian (4.4) after the canonical transformation.
with the Darwin potential given in Eq.(4.5).

The first half of Hamilton equations is (see Eqs.(3.1) for the Poisson brackets)

\[
\frac{\partial A_{i,\text{rad}}'(\tau, \vec{\sigma})}{\partial \tau} \overset{\circ}{=} \{A_{i,\text{rad}}'(\tau, \vec{\sigma}), \mathcal{E}_{\text{int}}\} = -\pi_{i,\text{rad}}'(\tau, \vec{\sigma}) - \sum_{i=1}^{N} Q_i \left[ \hat{\pi}_{\perp S_i}(\tau, \vec{\sigma}) + \left( \frac{\hat{\kappa}_i}{\sqrt{m_i^2 c^2 + \hat{\kappa}_i^2}} \frac{\partial}{\partial \hat{\eta}_i} \right) \hat{A}_{\perp S_i}(\tau, \vec{\sigma}) \right],
\]

\[\tag{4.9}\]

\[
\frac{d \hat{\eta}_i'(\tau)}{d \tau} \overset{\circ}{=} \{\hat{\eta}_i'(\tau), \mathcal{E}_{\text{int}}\} =
\]

\[
= \frac{\hat{\kappa}_i'(\tau)}{\sqrt{m_i^2 c^2 + \hat{\kappa}_i^2(\tau)}} + \frac{1}{c} \frac{\partial V_{\text{DARWIN}}}{\partial \hat{\kappa}_i} + \frac{Q_i}{c} \int d^3 \sigma \left( \hat{\pi}_{\perp \text{rad}}(\tau, \vec{\sigma}) \cdot \frac{\partial}{\partial \hat{\kappa}_i} \left[ \hat{\pi}_{\perp S_i}(\tau, \vec{\sigma}) + \left( \frac{\hat{\kappa}_i}{\sqrt{m_i^2 c^2 + \hat{\kappa}_i^2}} \frac{\partial}{\partial \hat{\eta}_i} \right) \hat{A}_{\perp S_i}(\tau, \vec{\sigma}) \right] \right) - \hat{A}_{\perp \text{rad}}(\tau, \vec{\sigma}) \cdot \frac{\partial}{\partial \hat{\kappa}_i} \left[ \hat{\pi}_{\perp S_i}(\tau, \vec{\sigma}) + \left( \frac{\hat{\kappa}_i}{\sqrt{m_i^2 c^2 + \hat{\kappa}_i^2}} \frac{\partial}{\partial \hat{\eta}_i} \right) \hat{A}_{\perp S_i}(\tau, \vec{\sigma}) \right] - \frac{Q_i}{c} \hat{A}_{\perp \text{rad}}(\tau, \hat{\eta}_i(\tau)) \frac{\hat{\kappa}_i'(\tau)}{\sqrt{m_i^2 c^2 + \hat{\kappa}_i^2(\tau)}}.
\]

\[\tag{4.10}\]

Therefore, like in Eq.(6.2) of Ref.[14], we get
\[
Q_i \frac{d \hat{\eta}^\tau_i(\tau)}{d \tau} = Q_i \frac{\hat{k}^\tau_i(\tau)}{\sqrt{m_i c^2 + \hat{k}^2_i(\tau)}} \\
\]

\[
\Rightarrow \quad Q_i \hat{\pi}_{\perp Si} = -Q_i \left( \frac{\hat{k}_i}{\sqrt{m_i c^2 + \hat{k}^2_i}} \cdot \frac{\partial}{\partial \hat{\eta}_i} \right) \hat{A}_{\perp Si} = -Q_i \frac{\partial \hat{A}_{\perp Si}}{\partial \tau},
\]

\[
\frac{\partial \hat{A}_{\perp rad}(\tau, \hat{\sigma})}{\partial \tau} = -\pi_{\perp rad}(\tau, \hat{\sigma}). \quad (4.11)
\]

Therefore the new Hamiltonian, i.e. the internal energy, takes the following simplified form

\[
\mathcal{E}_{(int)} = c \sum_{i=1}^{N} \sqrt{m_i^2 c^2 + \hat{k}^2_i(\tau)} + \frac{1}{4\pi} \sum_{i \neq j} Q_i Q_j \| \hat{\eta}_i(\tau) - \hat{\eta}_j(\tau) \| + V_{DARWIN} + \\
+ \sum_{i=1}^{N} \frac{Q_i}{c} \left( \int d^3 \sigma \left[ -\hat{A}_{\perp rad} \cdot \left( \partial^2 \hat{A}_{\perp Si} + \left( \frac{c \hat{k}_i}{\sqrt{m_i^2 c^2 + \hat{k}^2_i}} \cdot \frac{\partial}{\partial \hat{\eta}_i} \right) \hat{\pi}_{\perp Si} \right) \right] (\tau, \hat{\sigma}) - \\
\frac{\hat{k}_i \cdot \hat{A}_{\perp rad}(\tau, \hat{\eta}_i(\tau))}{\sqrt{m_i^2 c^2 + \hat{k}^2_i}} + \frac{1}{2} \int d^3 \sigma \left( \pi^2_{\perp rad} + B^2_{rad} \right)(\tau, \hat{\sigma}). \quad (4.12)
\]

By using Eq.(B7) and (4.11) we have

\[
\sum_{i=1}^{N} \frac{Q_i}{c} \left( \int d^3 \sigma \left[ -\hat{A}_{\perp rad} \cdot \left( \partial^2 \hat{A}_{\perp Si} + \left( \frac{c \hat{k}_i}{\sqrt{m_i^2 c^2 + \hat{k}^2_i}} \cdot \frac{\partial}{\partial \hat{\eta}_i} \right) \hat{\pi}_{\perp Si} \right) \right] (\tau, \hat{\sigma}) - \\
\frac{\hat{k}_i \cdot \hat{A}_{\perp rad}(\tau, \hat{\eta}_i(\tau))}{\sqrt{m_i^2 c^2 + \hat{k}^2_i}} \right) = \\
= \sum_{i=1}^{N} \frac{Q_i}{c} \left( \int d^3 \sigma \left[ -\hat{A}_{\perp rad}(\tau, \hat{\sigma}) \left( -\square \hat{A}_{\perp Si}(\tau, \hat{\sigma}) \right) \right] - \frac{\hat{k}_i \cdot \hat{A}_{\perp rad}(\tau, \hat{\eta}_i(\tau))}{\sqrt{m_i^2 c^2 + \hat{k}^2_i}} \right) = \\
= \sum_{i=1}^{N} \frac{Q_i}{c} \left( \int d^3 \sigma \hat{A}^\tau_{\perp rad}(\tau, \hat{\sigma}) P_{\perp}^s(\hat{\sigma}) \frac{c \hat{k}^\tau_i(\tau)}{\sqrt{m_i^2 c^2 + \hat{k}^2_i(\tau)}} \delta^2(\hat{\sigma} - \hat{\eta}_i(\tau)) - \frac{\hat{k}_i \cdot \hat{A}_{\perp rad}(\tau, \hat{\eta}_i(\tau))}{\sqrt{m_i^2 c^2 + \hat{k}^2_i}} \right) = \\
= 0. \quad (4.13)
\]

so that the final form of the new Hamiltonian and of the particle velocities are
\[ \mathcal{E}_{(int)} = c \sum_{i=1}^{N} \sqrt{m_i^2 c^2 + \mathbf{k}_i^2(\tau)} + \frac{1}{N} \sum_{i \neq j} Q_i Q_j \frac{4\pi}{|\tilde{\eta}_i(\tau) - \tilde{\eta}_j(\tau)|} + V_{DARWIN} + \]
\[ + \frac{1}{2} \int d^3\sigma \left( \tilde{\pi}_{\perp rad}^2 + \tilde{B}_{rad}^2 \right)(\tau, \bar{\sigma}) = \mathcal{P}^\tau_{\text{matter}} + \mathcal{P}^\tau_{\text{rad}}, \]
\[ \frac{d\tilde{\eta}_i^r(\tau)}{d\tau} = \frac{\tilde{k}_i^r(\tau)}{\sqrt{m_i^2 c^2 + \mathbf{k}_i^2(\tau)}} + \frac{1}{c} \frac{\partial V_{DARWIN}}{\partial \tilde{\kappa}_{ir}}. \tag{4.14} \]

As a consequence, the second half of Hamilton equation for the radiation field, i.e. for \( \tilde{\pi}_{\perp rad} \), and for the particle momenta are (\( \tilde{B}^2 = \sum_{rs} \left[ \partial^r A^s_\perp \partial^r A^s_\perp - \partial^r A^s_\perp \partial^r A^s_\perp \right] \))
\[ \frac{\partial \tilde{\pi}_{\perp rad}(\tau, \bar{\sigma})}{\partial \tau} \equiv \{ \tilde{\pi}_{\perp rad}, \mathcal{E}_{(int)} \} = -\partial^2 \tilde{A}_{\perp rad}(\tau, \bar{\sigma}), \]
\[ \Rightarrow \boxdot \tilde{A}_{\perp rad}(\tau, \bar{\sigma}) \equiv 0, \]
\[ \frac{d\tilde{k}_{i}(\tau)}{d\tau} = -\frac{1}{c} \frac{\partial}{\partial \tilde{\eta}_i} \left( \sum_{i \neq j} \frac{Q_i Q_j}{4\pi |\tilde{\eta}_i(\tau) - \tilde{\eta}_j(\tau)|} + V_{DARWIN} \right). \tag{4.15} \]

Therefore we get a decoupling of the radiation field from the the particles, which are mutually interacting not with the Coulomb potential but with the full Darwin potential. It is a kind of decoupling like the one assumed to exist to define the asymptotic \textit{IN} states after having given the asymptotic conditions.

However the vanishing of the internal boosts \textit{reintroduces a coupling between the particles and the radiation field} at the level of the reconstruction of the orbits.

Therefore the final form of the new equations of motion is given by Eqs.(4.11), (4.14) and (4.15)
\[ \frac{d\tilde{\eta}_i^r(\tau)}{d\tau} = \frac{\tilde{k}_i^r(\tau)}{\sqrt{m_i^2 c^2 + \mathbf{k}_i^2(\tau)}} + \frac{1}{c} \frac{\partial V_{DARWIN}}{\partial \tilde{\kappa}_{ir}}, \]
\[ \frac{d\tilde{k}_i(\tau)}{d\tau} = -\frac{1}{c} \frac{\partial}{\partial \tilde{\eta}_i} \left( \sum_{i \neq j} \frac{Q_i Q_j}{4\pi |\tilde{\eta}_i(\tau) - \tilde{\eta}_j(\tau)|} + V_{DARWIN} \right), \]
\[ \frac{\partial \tilde{A}_{\perp rad}(\tau, \bar{\sigma})}{\partial \tau} \equiv -\tilde{\pi}_{\perp rad}(\tau, \bar{\sigma}), \]
\[ \frac{\partial \tilde{\pi}_{\perp rad}(\tau, \bar{\sigma})}{\partial \tau} \equiv \{ \tilde{\pi}_{\perp rad}, \mathcal{P}^\tau = \frac{\mathcal{E}_{(int)}}{c} \} = -\partial^2 \tilde{A}_{\perp rad}(\tau, \bar{\sigma}), \]
\[ \Rightarrow \boxdot \tilde{A}_{\perp rad}(\tau, \bar{\sigma}) \equiv 0. \tag{4.16} \]
V. CONCLUSIONS

In this paper we have given a complete updated review of the rest-frame instant form of dynamics for isolated systems in Minkowski space-time starting from their description in non-inertial frames done by means of parametrized Minkowski theories. In them the nature of the isolated system is hidden in their energy-momentum tensor like in general relativity. We have completely clarified the problem of the existence of the many extensions of the notion of the Newtonian center of mass in special relativity. Only the non-canonical Fokker-Pryce center of inertia $Y^\mu(\tau)$ is a 4-vector, so that its world-line describes an inertial observer. The canonical center of mass $\tilde{x}^\mu(\tau)$ and the non-canonical Møller center of energy $R^\mu(\tau)$ are not 4-vectors. There is a Møller world-tube of non-covariance, centered on $Y^\mu(\tau)$ and with its Møller radius determined by the Poincare’ Casimirs, which contains all the pseudo-world-lines connected with these two non-covariant quantities. These three collective variables, all tending to the Newton center of mass in the non-relativistic limit, are determined by the Poincare’ generators, so that they are global non-locally determinable quantities. This complicated structure is due to the Lorentz signature of the metric of Minkowski space-time and to the structure of the Poincare’ group, implying that in the instant form of dynamics the interaction potentials are present also in the Lorentz boosts and not only in the energy generator.

In the rest-frame instant form the instantaneous 3-spaces are the Wigner hyper-planes orthogonal to the 4-momentum of the isolated system and the origin of the 3-coordinates is the Fokker-Pryce center of inertia. The isolated system is described by the non-covariant canonical center of mass (a decoupled particle described by six Jacobi data) carrying a pole-dipole structure, i.e. carrying the invariant mass $M$ (assumed non zero) and the rest spin $\vec{S}$ (barycentric angular momentum) of the isolated system. The external Poincare’ generators are built in terms of the six Jacobi data and of $M$ and $\vec{S}$. For each such system $M$ and $\vec{S}$ are functions only of Wigner-covariant relative canonical variables living in the instantaneous Wigner 3-space, because the rest-frame conditions $\vec{P}(\text{int}) \approx 0$ and their gauge fixing $\vec{K}(\text{int}) \approx 0$ eliminate the canonical variables of the internal 3-center of mass. $M$ is the Hamiltonian determining the evolution of the relative variables.

We showed explicitly how to describe massive charged positive-energy scalar particles, the transverse radiation field and an arbitrary transverse electro-magnetic field in the rest-frame instant form.

Before imposing the rest-frame conditions the basic canonical variables for describing the particles in the instantaneous Wigner 3-spaces are the Wigner spin-1 3-vectors $\vec{\eta}_i(\tau)$ and their conjugate 3-momenta $\vec{\kappa}_i(\tau)$. The particle world-lines are derived covariant quantities, $x^\mu_i(\tau) = Y^\mu(\tau) + \epsilon^\mu(\vec{h}) \eta^i(\tau)$, which are non-canonical, $\{x^\mu_i(\tau), x^\nu_j(\tau)\} \neq 0$. Therefore, they have to be identified with the covariant non-canonical predictive particle positions of predictive mechanics.

Let us remark that also massless positive-energy particles can be described in this way. There will be the extra condition $\vec{\eta}^2_i(\tau) = 1$ (so that $\dot{x}^2_i(\tau) = 0$), which will induce a constraint on the momenta $\vec{\eta}_i(\tau)$. This will be investigated elsewhere, being relevant for the description of rays of light (see the pseudo-classical photon of Ref.[36] and the problem of the spatial localization of photons in Refs.[37]).
Let us remark that it is possible to extend [38] the previous description of isolated systems to non-inertial rest frames, where the instantaneous 3-spaces are non-Euclidean but only asymptotically Euclidean: at spatial infinity they are orthogonal to the total 4-momentum of the isolated system.

In the second part of the paper we have considered the isolated system of N positive energy charged scalar particles with mutual Coulomb interaction plus the electromagnetic field in the radiation gauge as the classical basis of a relativistic formulation of atomic physics.

As shown in Ref.[14], in the rest-frame instant form of dynamics and by using Grassmann-valued electric charges to regularize the self-energies, we are able to find the explicit expression of the external and internal Poincare’ generators associated with this isolated system. In the limit \( c \to \infty \) we recover the Galilei generators of the particles plus the \( 1/c \) corrections connected with the electro-magnetic field. The same results could be obtained for spinning particles (with the spin described by Grassmann variables) by using the results of Ref.[15].

In Ref.[14], relying on the Grassmann regularization, we evaluated the effective particle-dependent electro-magnetic potential and fields coming from the Lienard-Wiechert solution in absence of homogeneous solutions corresponding to incoming radiation fields. As a consequence, we decided to investigate whether it was acceptable to put the transverse electro-magnetic potential and fields equal to the sum of a transverse radiation term plus the particle-dependent Lienard-Wiechert term.

To our surprise, it turned out that at this classical level, with a fixed number of particles, at every instant (and not only at asymptotic times \( \tau \to \pm \infty \)) we can define a canonical transformation sending the isolated system of a transverse electro-magnetic field plus N charged particles interacting through the Coulomb potential into a free transverse radiation field plus a set of N Coulomb-dressed charged particles interacting through a Coulomb + Darwin potential. Moreover, the internal Poincare’ generators (in particular the interaction-dependent internal mass and internal Lorentz boosts) become the direct sum of the corresponding generators of the two non-interacting subsystems. These subsystems know each other only due to the conditions \( \vec{P}_{(int)} \approx 0, \vec{K}_{(int)} \approx 0 \) defining the rest frame and eliminating the internal 3-center of mass.

This shows the limitation of the approximation, often made in atomic physics, of replacing the electro-magnetic field coupled to the atoms with a radiation field.

If this canonical transformation will turn out to be unitarily implementable after a canonical quantization compatible with the rest-frame instant form of dynamics, we will have an example of avoidance of the Haag theorem. In QED this theorem says that there is no unitary transformation sending the asymptotic IN and OUT radiation field into an interpolating non-radiation field when charged matter is present. We added some comments on why this canonical transformation exists: the conditions \( \vec{P}_{(int)} \approx 0, \vec{K}_{(int)} \approx 0 \), imply that the radiation field knows the particles, so that there is some analogy with the inner reasons allowing the Haag-Ruelle scattering theory to avoid the Haag theorem.

In the second paper we will investigate how to solve the equations \( \vec{P}_{(int)} \approx 0, \vec{K}_{(int)} \approx 0 \) for the elimination of the internal 3-center of mass for the following isolated systems: a) charged particles with a Coulomb plus Darwin mutual interaction; b) transverse radiation field; c)
charged particles with a mutual Coulomb interaction plus a transverse electro-magnetic field. Then we will study the multipolar expansion of the particle energy-momentum tensor and we will find the relativistic generalization of the dipole approximation and of the electric dipole representation used in atomic physics. Then in the third paper we will define the canonical quantization of the rest-frame instant form of isolated systems to get a formulation of relativistic atomic physics, to see whether the previous canonical transformation is unitarily implementable and to explore the implications of special relativity for the theory of entanglement.
APPENDIX A: WIGNER TETRADS FOR NULL POINCARÉ ORBITS AND THE TRANSVERSE POLARIZATION VECTORS FOR THE RADIATION FIELD.

1. Helicity Tetrad

For \( k^2 = 0 \), we have \( k^\mu = (\omega(\vec{k}) = |\vec{k}| = \sqrt{k^2}; \vec{k}) \). We have chosen positive energy \( k^0 > 0 \); more in general one should put \( |\vec{k}| \mapsto \eta_s|\vec{k}| \) with \( \eta_s = \pm 1 = \text{sign} k^0 \).

By choosing a reference vector \( \tilde{\epsilon}^\mu = \omega_s (1; 0, 0, 1) \), where \( \omega_s \) is a dimensional parameter, we can define the standard Wigner helicity boost \( H L^\mu_\nu(k, \vec{k}) \) such that \( k^\mu = H L^\mu_\nu (k, \vec{k}) \tilde{\epsilon}^\nu \). We have [36]

\[
H L^\mu_\nu(k, \vec{k}) = \begin{pmatrix}
\frac{1}{2} \left( \frac{\vec{k}}{|k|} + \frac{\eta_s}{|k|} \right) & 0 & \frac{1}{2} \left( \frac{\vec{k}}{|k|} - \frac{\eta_s}{|k|} \right) \\
\frac{1}{2} \left( \frac{\vec{k}}{|k|} - \frac{\eta_s}{|k|} \right) + \frac{k^\alpha k^\beta}{k^2 |k|^2} & \frac{k^\alpha}{|k|} \delta^\beta_\gamma & \frac{1}{2} \left( \frac{\vec{k}}{|k|} + \frac{\eta_s}{|k|} \right) \\
\frac{1}{2} \left( \frac{\vec{k}}{|k|} + \frac{\eta_s}{|k|} \right) + \frac{k^\alpha k^\beta}{k^2 |k|^2} & \frac{1}{2} \left( \frac{\vec{k}}{|k|} + \frac{\eta_s}{|k|} \right) & \frac{1}{2} \left( \frac{\vec{k}}{|k|} + \frac{\eta_s}{|k|} \right)
\end{pmatrix},
\]

\( H \epsilon^\mu_A(\vec{k}) = H L^\mu_\nu (k, \vec{k}) \tilde{\epsilon}^\nu_A = H L^\mu_A(k, \vec{k}), \quad A = (\tau, r), \)

\[
\epsilon^\mu_A : \quad (1; 0, 0, 0), \quad (0; 1, 0, 0), \quad (0; 0, 1, 0), \quad (0; 0, 0, 1),
\]

\( \eta^{\mu\nu} = H \epsilon^\mu_A(\vec{k}) \eta^{AB} H \epsilon^\nu_B(\vec{k}). \) (A1)

In this way we get a helicity tetrad \( H \epsilon^\mu_A(\vec{k}) \) and a null basis

\[
k^\mu = (|\vec{k}|; \vec{k}) = \omega_s \left[ H \epsilon^\mu_A(\vec{k}) + H \epsilon^\mu_A(\vec{k}) \right], \quad \text{when} \quad k^2 = 0,
\]

\[
\tilde{k}^\mu(\vec{k}) = \frac{1}{2|k|^2} (|k|; -\vec{k}) = \frac{1}{2 \omega_s} \left[ H \epsilon^\mu_A(\vec{k}) - H \epsilon^\mu_A(\vec{k}) \right],
\]

\[
H \epsilon^\mu_\lambda(\vec{k}) = (0; H \epsilon^\mu_\lambda(\vec{k})) = (0; \delta^\mu_\lambda + \frac{k^\alpha k_\lambda}{|k|(|k| + k^3)} \frac{k_\lambda}{|k|}), \quad a, \lambda = 1, 2,
\]

\[
k^2 = \vec{k}^2(\vec{k}) = 0, \quad k \cdot \vec{k}(\vec{k}) = 1,
\]

\[
k \cdot H \epsilon_\lambda(\vec{k}) = \vec{k}(\vec{k}) \cdot H \epsilon_\lambda(\vec{k}) = 0,
\]

\[
h \epsilon_\lambda(\vec{k}) \cdot H \epsilon_\lambda(\vec{k}) = -h \epsilon_\lambda(\vec{k}) \cdot H \epsilon_\lambda(\vec{k}) = -\delta_{\lambda\lambda'},
\]

\[
\eta^{\mu\nu} = k^\mu \tilde{k}^\nu(\vec{k}) + k^\nu \tilde{k}^\mu(\vec{k}) - \sum_{\lambda=1}^{2} h \epsilon^\mu_\lambda(\vec{k}) h \epsilon^\nu_\lambda(\vec{k}),
\]

\[
h \epsilon^\mu_\tau(\vec{k}) = \frac{k^\mu}{2 \omega_s} + \omega_s \tilde{k}^\mu(\vec{k}),
\]

\[
h \epsilon^\mu_3(\vec{k}) = \frac{k^\mu}{2 \omega_s} - \omega_s \tilde{k}^\mu(\vec{k}),
\]

\[
h \epsilon^\mu_3(-\vec{k}) = H \epsilon^\mu_3(\vec{k}), \quad H \epsilon^\mu_3(-\vec{k}) = -H \epsilon^\mu_3(\vec{k}),
\]

\[
\Rightarrow \quad H \epsilon_\lambda(-\vec{k}) \cdot H \epsilon_\lambda' = \delta_{\lambda\lambda'} - 2 \frac{k_\lambda k_{\lambda'}}{k^2},
\]

57
\[ H \varepsilon_{\lambda}(-k) = \left( \delta_{\lambda\lambda'} - 2 \frac{k_{\lambda} k_{\lambda'}}{k^2} \right) H \varepsilon'_{\lambda'}(k). \] (A2)

The quantities \( H \varepsilon_{\mu}(\vec{k}) \) are a possible set of transverse polarization vectors to be used in Eqs. (2.33) and (2.34) for the radiation field in the radiation gauge.

2. Transformation Properties of Polarization Vectors under Lorentz Transformations

We give the transformation properties of the polarization vectors defined in Eqs. (A1) and (A2). To this end one needs to find the Wigner matrix belonging to the little group \( E_2 \) of \( \vec{k} = \omega_s(1; 0, 0, 1) \). From Ref.[39] and from Appendix A of Ref. [36] we have

\[ H \varepsilon_{\mu}^B(\vec{k}) = \Lambda_{\mu\nu} H \varepsilon_{\nu}^B(\vec{k}) \mathcal{R}(\vec{k}, \Lambda)^{B A}, \]

where \( \Lambda \) is a Lorentz transformation in \( O(3,1) \). If \( \Lambda \) is obtained from the SL(2,C) matrix \( \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \) with \( \alpha \delta - \beta \gamma = 1 \), one obtains the following parametrization of the Wigner matrix

\[ \mathcal{R}(\vec{k}, \Lambda) = \left( \begin{array}{cccc}
1 + \frac{1}{2} u^2 & u^1 & u^2 & -\frac{1}{2} u^2 \\
u^1 \cos 2\theta + u^2 \sin 2\theta & \cos 2\theta & \sin 2\theta & -u^1 \cos 2\theta - u^2 \sin 2\theta \\
-u^1 \sin 2\theta + u^2 \cos 2\theta & -\sin 2\theta & \cos 2\theta & u^1 \sin 2\theta - u^2 \cos 2\theta \\
\frac{1}{2} u^2 & u^1 & u^2 & 1 - \frac{1}{2} u^2
\end{array} \right), \]

\( u^2 = (u^1)^2 + (u^2)^2 \),

\[ e^{i\theta} = \frac{d^r}{|d|}, \]

\[ u^1 \cos 2\theta + u^2 \sin 2\theta + i \left( u^1 \sin 2\theta - u^2 \cos 2\theta \right) = \frac{\omega_s}{|\vec{p}|} \frac{ac^* + bd^*}{|c|^2 + |d|^2}, \]

\[ a = \delta(|\vec{k}| + k^3) - \gamma(k^1 - i k^2), \]
\[ b = -\beta(|\vec{k}| + k^3) + \alpha(k^1 - i k^2), \]
\[ c = -\gamma(|\vec{k}| + k^3) - \delta(k^1 + i k^2), \]
\[ d = \alpha(|\vec{k}| + k^3) + \beta(k^1 + i k^2). \] (A4)

Therefore, we get
\((\Lambda \vec{k})^\mu = \Lambda^\mu_\nu k^\nu\),
\(\hbar \epsilon^\mu_{\lambda=1}(\Lambda \vec{k}) = \Lambda^\mu_\nu \left[ \cos 2\theta \hbar \epsilon^\nu_{\lambda=1}(\vec{k}) - \sin 2\theta \hbar \epsilon^\nu_{\lambda=2}(\vec{k}) + u^1 k^\nu \right] + \frac{1}{2} \hbar \epsilon^\nu_{\lambda=1}(\vec{k}) - \frac{1}{2} \hbar \epsilon^\nu_{\lambda=2}(\vec{k})\),
\(\hbar \epsilon^\mu_{\lambda=2}(\Lambda \vec{k}) = \Lambda^\mu_\nu \left[ \sin 2\theta \hbar \epsilon^\nu_{\lambda=1}(\vec{k}) + \cos 2\theta \hbar \epsilon^\nu_{\lambda=2}(\vec{k}) + u^2 k^\nu \right] - \frac{1}{2} \hbar \epsilon^\nu_{\lambda=1}(\vec{k}) - \frac{1}{2} \hbar \epsilon^\nu_{\lambda=2}(\vec{k})\),
\(\tilde{\vec{k}}^\mu(\Lambda \vec{k}) = \Lambda^\mu_\nu \left[ \tilde{k}^\nu(\vec{k}) + \frac{1}{2} u^2 k^\nu \right] + \frac{1}{2} \hbar \epsilon^\nu_{\lambda=1}(\vec{k}) - \frac{1}{2} \hbar \epsilon^\nu_{\lambda=2}(\vec{k})\),
\(\hbar \epsilon^\mu_{\lambda=1}(\Lambda \vec{k}) = \frac{1}{2 \omega_s} (\Lambda \vec{k})^\mu + \omega_s \tilde{\vec{k}}^\mu(\Lambda \vec{k}) = \Lambda^\mu_\nu \left[ \hbar \epsilon^\nu_{\lambda=1}(\vec{k}) + \frac{\omega_s}{2} u^2 k^\nu \right] + (u^1 \cos 2\theta + u^2 \sin 2\theta) \hbar \epsilon^\nu_{\lambda=1}(\vec{k}) - (u^1 \sin 2\theta - u^2 \cos 2\theta) \hbar \epsilon^\nu_{\lambda=2}(\vec{k})\),
\(\hbar \epsilon^\mu_{\lambda=2}(\Lambda \vec{k}) = \frac{1}{2 \omega_s} (\Lambda \vec{k})^\mu - \omega_s \tilde{\vec{k}}^\mu(\Lambda \vec{k}) = \Lambda^\mu_\nu \left[ \hbar \epsilon^\nu_{\lambda=1}(\vec{k}) - \frac{\omega_s}{2} u^2 k^\nu \right] - (u^1 \cos 2\theta + u^2 \sin 2\theta) \hbar \epsilon^\nu_{\lambda=1}(\vec{k}) + (u^1 \sin 2\theta - u^2 \cos 2\theta) \hbar \epsilon^\nu_{\lambda=2}(\vec{k})\). (A5)

For a circular basis we have
\(\hbar \epsilon^\mu_{(\pm)}(\vec{k}) = \frac{1}{\sqrt{2}} \left[ \hbar \epsilon^\mu_{\lambda=1}(\vec{k}) \pm i \hbar \epsilon^\mu_{\lambda=2}(\vec{k}) \right]\),
\(\hbar \epsilon^\mu_{(\pm)}(\Lambda \vec{k}) = \Lambda^\mu_\nu \left( e^{\pm 2i \theta} \hbar \epsilon^\nu_{(\pm)}(\vec{k}) + \frac{u^1 \pm i u^2}{\sqrt{2}} k^\nu \right)\). (A6)
APPENDIX B: THE LIENARD-WIECHERT ELECTROMAGNETIC POTENTIALS AND FIELDS.

1. Properties of the Lienard-Wiechert Fields

Let us study the properties of the Lienard-Wiechert fields of Eqs. (2.51)-(2.53) taken from Ref.[14]. For the vector potential we have

\[
\vec{A} \bot S(\tau, \vec{\sigma}) = \sum_{i=1}^{N} Q_i \vec{A} \bot S_i(\vec{\sigma} - \vec{\eta}_i(\tau), \vec{k}_i(\tau)),
\]

\[
\vec{A} \bot S_i(\vec{\sigma} - \vec{\eta}_i(\tau), \vec{k}_i(\tau)) = \frac{1}{4\pi|\vec{\sigma} - \vec{\eta}_i|} \sqrt{m^2 c^2 + \vec{k}_i^2 + \sqrt{m^2 c^2 + (\vec{k}_i \cdot \frac{\vec{\sigma} - \vec{\eta}_i}{|\vec{\sigma} - \vec{\eta}_i|})^2}} \times \]

\[
\left[ \vec{k}_i + \frac{[\vec{k}_i \cdot (\vec{\sigma} - \vec{\eta}_i)] (\vec{\sigma} - \vec{\eta}_i)}{|\vec{\sigma} - \vec{\eta}_i|^2} \frac{\sqrt{m^2 c^2 + \vec{k}_i^2}}{\sqrt{m^2 c^2 + (\vec{k}_i \cdot \frac{\vec{\sigma} - \vec{\eta}_i}{|\vec{\sigma} - \vec{\eta}_i|})^2}} \right] = \]

\[
= \left( \frac{\alpha_{i1}}{c} + \frac{\alpha_{i3}}{c^3} + \sum_{k=2}^{\infty} \frac{\alpha_{ik}}{c^{2k+1}} \right)(\tau, \vec{\sigma}), \tag{B1}
\]

in which

\[
\alpha_{i1}(\tau, \vec{\sigma}) = \frac{1}{8\pi m_i |\vec{\sigma} - \vec{\eta}_i|} \left( \vec{k}_i + \frac{[\vec{k}_i \cdot (\vec{\sigma} - \vec{\eta}_i)] (\vec{\sigma} - \vec{\eta}_i)}{|\vec{\sigma} - \vec{\eta}_i|^2} \right),
\]

\[
\alpha_{i3}(\tau, \vec{\sigma}) = \frac{1}{8\pi m_i^4 |\vec{\sigma} - \vec{\eta}_i|^2} \left[ -\frac{1}{4} (\vec{k}_i + \frac{[\vec{k}_i \cdot (\vec{\sigma} - \vec{\eta}_i)] (\vec{\sigma} - \vec{\eta}_i)}{|\vec{\sigma} - \vec{\eta}_i|^2}) (\vec{k}_i^2 + (\vec{k}_i \cdot \frac{\vec{\sigma} - \vec{\eta}_i}{|\vec{\sigma} - \vec{\eta}_i|^2})^2) + \right.
\]

\[
+ \left. \frac{[\vec{k}_i \cdot (\vec{\sigma} - \vec{\eta}_i)] (\vec{\sigma} - \vec{\eta}_i)}{2|\vec{\sigma} - \vec{\eta}_i|^2} (\vec{k}_i^2 - (\vec{k}_i \cdot \frac{\vec{\sigma} - \vec{\eta}_i}{|\vec{\sigma} - \vec{\eta}_i|^2})^2) \right]. \tag{B2}
\]

From Eqs.(6.2) of Ref.[14] [by using \(Q_i \dot{\vec{\eta}}_i(\tau) = Q_i \vec{k}_i(\tau) / \sqrt{m^2 c^2 + \vec{k}_i^2} \) (see Eqs.(4.5) and after Eq.(5.27) in Ref.[14]) and \(Q_i \vec{k}_i(\tau) \neq 0 \) (see Eqs.(4.5) of Ref.[14])] it follows that the electric field is minus the \(\tau\)-derivative of the vector potential.
\[ E_{\perp S}(\tau, \bar{\sigma}) = \sum_{i=1}^{N} Q_i \tilde{p}_{\perp S} = \sum_{i=1}^{N} Q_i \tilde{p}_{\perp S}(\bar{\sigma} - \tilde{\eta}_i(\tau), \tilde{\kappa}_i(\tau)) = \sum_{i=1}^{N} Q_i \frac{\tilde{\kappa}_i(\tau) \cdot \bar{\sigma}}{\sqrt{m_i^2 c^2 + \tilde{\kappa}_i^2(\tau)}} \tilde{A}_{\perp S}(\bar{\sigma} - \tilde{\eta}_i(\tau), \tilde{\kappa}_i(\tau)) = \]  
\[ = -\sum_{i=1}^{N} Q_i \frac{\tilde{\kappa}_i(\tau) \cdot \bar{\sigma}}{\sqrt{m_i^2 c^2 + \tilde{\kappa}_i^2(\tau)}} \tilde{A}_{\perp S}(\bar{\sigma} - \tilde{\eta}_i(\tau), \tilde{\kappa}_i(\tau)) = \]  
\[ = \sum_{i=1}^{N} Q_i \frac{\partial \tilde{A}_{\perp S}(\tau, \bar{\sigma})}{\partial \tau} |_{\tilde{\kappa}_i} = -\frac{\partial \tilde{A}_{\perp S}(\tau, \bar{\sigma})}{\partial \tau} = \]  
\[ = -\sum_{i=1}^{N} Q_i \times \]  
\[ \frac{1}{4\pi |\bar{\sigma} - \tilde{\eta}_i(\tau)|^2} \left[ \tilde{\kappa}_i(\tau) \left[ \tilde{\kappa}_i(\tau) \cdot \frac{\bar{\sigma} - \tilde{\eta}_i(\tau)}{|\bar{\sigma} - \tilde{\eta}_i(\tau)|} \right] \sqrt{m_i^2 c^2 + \tilde{\kappa}_i^2(\tau)} \right] + \]  
\[ + \frac{\tilde{\kappa}_i^2(\tau) + (\tilde{\kappa}_i(\tau) \cdot \frac{\bar{\sigma} - \tilde{\eta}_i(\tau)}{|\bar{\sigma} - \tilde{\eta}_i(\tau)|})^2}{\sqrt{m_i^2 c^2 + \tilde{\kappa}_i^2(\tau)}} \left( \sqrt{m_i^2 c^2 + \tilde{\kappa}_i^2(\tau)} \right) \]  
\[ = \left( \tilde{\beta}_{21} \frac{1}{c^2} + \sum_{k=2}^{\infty} \tilde{\beta}_{2k} \frac{1}{c^2k} \right)(\tau, \bar{\sigma}). \] (B3)  

with  
\[ \tilde{\beta}_{21}(\tau, \bar{\sigma}) = -\sum_{i=1}^{N} Q_i \frac{1}{4\pi m_i^2 |\bar{\sigma} - \tilde{\eta}_i(\tau)|^2} \left[ \tilde{\kappa}_i(\tau) \left[ \tilde{\kappa}_i(\tau) \cdot \frac{\bar{\sigma} - \tilde{\eta}_i(\tau)}{|\bar{\sigma} - \tilde{\eta}_i(\tau)|} \right] + \right. \]  
\[ + \frac{\tilde{\kappa}_i^2(\tau) + (\tilde{\kappa}_i(\tau) \cdot \frac{\bar{\sigma} - \tilde{\eta}_i(\tau)}{|\bar{\sigma} - \tilde{\eta}_i(\tau)|})^2}{\sqrt{m_i^2 c^2 + \tilde{\kappa}_i^2(\tau)}} \left( \sqrt{m_i^2 c^2 + \tilde{\kappa}_i^2(\tau)} \right) \]  
\[ + \left. \left( \tilde{\kappa}_i(\tau) \cdot \frac{\bar{\sigma} - \tilde{\eta}_i(\tau)}{|\bar{\sigma} - \tilde{\eta}_i(\tau)|} \right)^2 \right] \]  
\[ = \sum_{k=2}^{\infty} \tilde{\beta}_{2k} \frac{1}{c^2k} \left( \right). \] (B4)  

The magnetic field is
\[ \mathbf{B}_S (\tau, \vec{\sigma}) = \sum_{i=1}^{N} Q_i \mathbf{B}_{Si} (\vec{\sigma} - \vec{n}_i (\tau), \vec{k}_i (\tau)) = \sum_{i=1}^{N} Q_i \frac{1}{4\pi|\vec{\sigma} - \vec{n}_i (\tau)|^2} \mathbf{m}_i^2 c^2 \mathbf{k}_i (\tau) \times \frac{\vec{\sigma} - \vec{n}_i (\tau)}{|\vec{\sigma} - \vec{n}_i (\tau)|} \]

\[ = \left( \frac{\gamma_{11}}{c} + \frac{\gamma_{13}}{c^3} + \sum_{k=2}^{\infty} \frac{\gamma_{2k+1}}{c^{2k+1}} \right) (\tau, \vec{\sigma}), \]  \hspace{1cm} \text{(B5)}

with

\[ \gamma_{11} (\tau, \vec{\sigma}) = \sum_{i=1}^{N} Q_i \frac{\mathbf{k}_i (\tau) \times \frac{\vec{\sigma} - \vec{n}_i (\tau)}{|\vec{\sigma} - \vec{n}_i (\tau)|}}{4\pi \mathbf{m}_i |\vec{\sigma} - \vec{n}_i (\tau)|^2}, \]

\[ \gamma_{13} (\tau, \vec{\sigma}) = -\sum_{i=1}^{N} Q_i \frac{3\mathbf{k}_i (\tau) \times \frac{\vec{\sigma} - \vec{n}_i (\tau)}{|\vec{\sigma} - \vec{n}_i (\tau)|}}{8\pi \mathbf{m}_i^3 |\vec{\sigma} - \vec{n}_i (\tau)|^2} (\mathbf{k}_i (\tau) \cdot \frac{\vec{\sigma} - \vec{n}_i (\tau)}{|\vec{\sigma} - \vec{n}_i (\tau)|})^2. \]  \hspace{1cm} \text{(B6)}

From Eqs.(4.17) of Ref.[14] we have (the source term has \( \dot{\eta}_i / c \))

\[ \Box A_{\perp S}^r (\tau, \vec{\sigma}) = (\partial^2 - \partial^2) A_{\perp S}^r (\tau, \vec{\sigma}) = -\left( \frac{\partial}{\partial \tau} A_{\perp S}^r + \partial^2 A_{\perp S}^r \right) (\tau, \vec{\sigma}) = \sum_{i=1}^{N} Q_i \Box A_{\perp S}^r (\vec{\sigma} - \vec{n}_i (\tau); \vec{k}_i (\tau)) = \sum_{i=1}^{N} Q_i P_{\perp}^r (\vec{\sigma}) \frac{\mathbf{k}_i^2 (\tau)}{\sqrt{m_i^2 c^2 + \mathbf{k}_i^2 (\tau)}} \delta^3 (\vec{\sigma} - \vec{n}_i (\tau)) \overset{\text{def}}{=} \vec{j}_{\perp} (\tau, \vec{\sigma}). \]  \hspace{1cm} \text{(B7)}

2. Fourier Transforms of the Lienard-Wiechert Transverse Electromagnetic Potential, Electric Field and Magnetic Field

In Appendix A of [14] we find the following series form for the Lienard-Wiechert transverse electromagnetic potential

\[ \vec{A}_{\perp} (\tau, \vec{\sigma}) = \sum_{i=1}^{N} \frac{Q_i}{4\pi} \sum_{m=0}^{\infty} \left[ \frac{1}{(2m)!} \frac{\mathbf{k}_i}{\sqrt{m_i^2 c^2 + \mathbf{k}_i^2}} \left( \frac{\mathbf{k}_i}{\sqrt{m_i^2 c^2 + \mathbf{k}_i^2}} \cdot \vec{\sigma} \right)^{2m} \right] |\vec{\sigma} - \vec{n}_i|^{2m-1} \]

\[ - \frac{1}{(2m+2)} \vec{\sigma} \left( \frac{\mathbf{k}_i}{\sqrt{m_i^2 c^2 + \mathbf{k}_i^2}} \cdot \vec{\sigma} \right)^{2m+1} |\vec{\sigma} - \vec{n}_i|^{2m+1} \]  \hspace{1cm} \overset{\text{def}}{=} \vec{A}_{\perp 1} (\tau, \vec{\sigma}) + \vec{A}_{\perp 2} (\tau, \vec{\sigma}). \]  \hspace{1cm} \text{(B8)}
To obtain its Fourier transform we need

\[
I_1 = \int d^3\sigma e^{i\vec{k}\cdot\vec{\sigma}} \left( \frac{\vec{k}_i}{\sqrt{m_i^2 c^2 + \vec{k}_i^2}} \cdot \vec{\sigma} \right)^{2m} |\vec{\sigma} - \vec{\eta}|^{2m-1},
\]

\[
I_2 = \int d^3\sigma e^{i\vec{k}\cdot\vec{\sigma}} \left( \frac{\vec{k}_i}{\sqrt{m_i^2 c^2 + \vec{k}_i^2}} \cdot \vec{\sigma} \right)^{2m+1} |\vec{\sigma} - \vec{\eta}|^{2m+1}.
\]

Clearly each integral converges. Thus

\[
I_i = \lim_{\varepsilon \to 0} \int_0^\infty \sigma^2 d\sigma \int d\Omega_{\sigma}(\ )e^{-|\vec{\sigma} - \vec{\eta}|} := \lim_{\varepsilon \to 0} \int d^3\sigma(\ )e^{-\varepsilon|\vec{\sigma} - \vec{\eta}|} = \lim_{\varepsilon \to 0} I_i(\varepsilon).
\]

Change to \(\partial/\partial \eta\) from \(\partial/\partial \sigma\), so that we can bring out the derivatives, translate and we obtain

\[
I_1 = \left( -\frac{\vec{k}_i}{\sqrt{m_i^2 c^2 + \vec{k}_i^2}} \cdot \vec{\sigma} \right)^{2m} e^{i\vec{k}\cdot\vec{\eta}} \int d^3\sigma e^{i\vec{k}\cdot\vec{\sigma}} \sigma^{2m-1} e^{-\varepsilon \sigma} =
\]

\[
= \left( \frac{\vec{k}_i}{\sqrt{m_i^2 c^2 + \vec{k}_i^2}} \cdot \vec{\sigma} \right)^{2m} e^{i\vec{k}\cdot\vec{\sigma}} \frac{4\pi}{k} \int_0^\infty \sigma^{2m} \frac{e^{-\sigma(\varepsilon - ik)}}{2i} + c.c. =
\]

\[
= \left( \frac{\vec{k}_i}{\sqrt{m_i^2 c^2 + \vec{k}_i^2}} \cdot \vec{\sigma} \right)^{2m} e^{i\vec{k}\cdot\vec{\sigma}} \frac{4\pi}{k} \frac{d^{2m}}{d\varepsilon^{2m}} \int_0^\infty \sigma^{2m} \frac{e^{-\sigma(\varepsilon - ik)}}{2i} + c.c. =
\]

\[
= \left( \frac{\vec{k}_i}{\sqrt{m_i^2 c^2 + \vec{k}_i^2}} \cdot \vec{\sigma} \right)^{2m} e^{i\vec{k}\cdot\vec{\sigma}} \frac{4\pi}{k^{2m+2}} \varepsilon^{m} (2m)! = (2m)! \left( \frac{\vec{k}_i}{\sqrt{m_i^2 c^2 + \vec{k}_i^2}} \cdot \vec{k} \right)^{2m} \frac{4\pi}{k^{2m+2}}.
\]

(B11)

in which we have set \(\varepsilon = 0\). This integral now permits us to perform the summation explicitly. We find

\[
\sum_{m=0}^\infty \int d^3\sigma e^{i\vec{k}\cdot\vec{\sigma}} \frac{1}{(2m)!} \sqrt{m_i^2 c^2 + \vec{k}_i^2} \left( \frac{\vec{k}_i}{\sqrt{m_i^2 c^2 + \vec{k}_i^2}} \cdot \vec{\sigma} \right)^{2m} |\vec{\sigma} - \vec{\eta}|^{2m-1} =
\]

\[
= \left( \frac{\vec{k}_i}{\sqrt{m_i^2 c^2 + \vec{k}_i^2}} \cdot \vec{\sigma} \right)^{2m} \sum_{m=0}^\infty \left( \frac{\vec{k}_i}{\sqrt{m_i^2 c^2 + \vec{k}_i^2}} \cdot \vec{k} \right)^{2m} \frac{4\pi}{k^{2m+2}} =
\]

\[
= \frac{4\pi \vec{k}_i}{k^2 \sqrt{m_i^2 c^2 + \vec{k}_i^2}} e^{i\vec{k}\cdot\vec{\eta}} \frac{1}{1 - \left( \frac{\vec{k}_i}{\sqrt{m_i^2 c^2 + \vec{k}_i^2}} \right)^{2m}} = 4\pi \frac{\vec{k}_i}{k^2} e^{i\vec{k}\cdot\vec{\eta}} \sqrt{m_i^2 c^2 + \vec{k}_i^2} \left( \frac{m_i^2 c^2 + \vec{k}_i^2}{m_i^2 c^2 + \vec{k}_i^2 - \left( \vec{k}_i \cdot \vec{k} \right)^2} \right) \varepsilon^{m} (2m)!.
\]

(B12)

In a similar way we find

\[
\sum_{m=0}^\infty \int d^3\sigma e^{i\vec{k}\cdot\vec{\sigma}} \frac{1}{(2m+2)!} \varepsilon^{m+1} (2m+1) ! \frac{\vec{k}_i}{\sqrt{m_i^2 c^2 + \vec{k}_i^2}} \cdot \vec{\sigma} \left( \frac{\vec{k}_i}{\sqrt{m_i^2 c^2 + \vec{k}_i^2}} \cdot \vec{\sigma} \right)^{2m+1} |\vec{\sigma} - \vec{\eta}|^{2m+1} =
\]

\[
= \frac{4\pi \vec{k}_i}{k^4} \frac{\vec{k}_i}{m_i^2 c^2 + \vec{k}_i^2 - \left( \vec{k}_i \cdot \vec{k} \right)^2} e^{i\vec{k}\cdot\vec{\eta}} \sqrt{m_i^2 c^2 + \vec{k}_i^2} \left( \frac{m_i^2 c^2 + \vec{k}_i^2}{m_i^2 c^2 + \vec{k}_i^2 - \left( \vec{k}_i \cdot \vec{k} \right)^2} \right) \varepsilon^{m+1} (2m+1) !.
\]

(B13)
Combining we obtain

\[ \tilde{A}_{LS}(\tau, \vec{k}) \ \stackrel{def}{=} \ \int d^3 \sigma e^{-i \vec{k} \cdot \vec{\sigma}} \tilde{A}_{LS}(\tau, \vec{\sigma}) = \sum_{i=1}^{N} \frac{Q_i}{k^4} \frac{e^{-i \vec{k} \cdot \vec{\eta}_i}}{k} \frac{\vec{k} \times (\vec{k}_i \times \vec{k})}{m_i^2 c^2 + \vec{k}_i^2} \frac{\sqrt{m_i^2 c^2 + \vec{k}_i^2}}{m_i^2 c^2 + \vec{k}_i^2 - (\vec{k}_i \cdot \vec{k})^2}, \]

\( (B14) \)

From this and Eqs.(2.52) and (2.53) we obtain

\[ \tilde{\pi}_{LS}(\tau, \vec{k}) \ \stackrel{def}{=} \ \frac{1}{c} \int d^3 \sigma e^{-i \vec{k} \cdot \vec{\sigma}} \tilde{E}_{LS}(\tau, \vec{\sigma}) = \]

\[ = - \sum_{i=1}^{N} \frac{\vec{k}_i}{\sqrt{m_i^2 c^2 + \vec{k}_i^2}} \cdot \vec{\eta}_i \int d^3 \sigma e^{-i \vec{k} \cdot \vec{\sigma}} \tilde{A}_L(\tau, \vec{\sigma} - \vec{\eta}_i, \vec{k}_i) = \]

\[ = i \sum_{i=1}^{N} \frac{Q_i}{k^4} \frac{e^{-i \vec{k} \cdot \vec{\eta}_i}}{k} \frac{\vec{k}_i \cdot \vec{k}}{\sqrt{m_i^2 c^2 + \vec{k}_i^2}} \frac{\vec{k} \times (\vec{k}_i \times \vec{k})}{m_i^2 c^2 + \vec{k}_i^2} \frac{\sqrt{m_i^2 c^2 + \vec{k}_i^2}}{m_i^2 c^2 + \vec{k}_i^2 - (\vec{k}_i \cdot \vec{k})^2}, \]

\( (B15) \)

\[ \tilde{B}_{LS}(\tau, \vec{k}) \ \stackrel{def}{=} \ \int d^3 \sigma e^{-i \vec{k} \cdot \vec{\sigma}} \tilde{B}_{LS}(\tau, \vec{\sigma}) = \]

\[ = i \sum_{i=1}^{N} \frac{Q_i}{k^2} \frac{e^{-i \vec{k} \cdot \vec{\eta}_i}}{k} \frac{\vec{k} \times \vec{k}_i \sqrt{m_i^2 c^2 + \vec{k}_i^2}}{m_i^2 c^2 + \vec{k}_i^2 - (\vec{k}_i \cdot \vec{k})^2}. \]

3. The \( 1/c \) Expansion of \( \mathcal{E}_{(int)} \) after the Canonical Transformation (3.6).

By using Eqs. (B1) - (B4) the non-relativistic limit of the Darwin potential (4.5) is

\[ V_{Darwin}(\hat{\eta}_i(\tau) - \hat{\eta}_j(\tau); \hat{\eta}_i(\tau)) \rightarrow c \rightarrow \infty \ 1 \sum_{i \neq j}^{1,N} \frac{Q_i Q_j}{c^2} \left( \frac{\hat{\kappa}_i(\tau)}{m_i} \cdot \hat{\alpha}_{j1}(\tau, \hat{\eta}_i(\tau)) + \right. \]

\[ + \frac{1}{2} \int d^3 \sigma \left[ \hat{\gamma}_{i1} \cdot \hat{\gamma}_{j1} \right](\tau, \vec{\sigma}) + \]

\[ + \frac{1}{c^2} \sum_{i \neq j}^{1,N} \frac{Q_i Q_j}{m_i} \left( \hat{\alpha}_{j3} - \frac{\hat{\kappa}_i(\tau)}{2m_i^2} \hat{\alpha}_{j1}(\tau, \hat{\eta}_i(\tau)) + \right. \]

\[ + \int d^3 \sigma \left[ \frac{1}{2} \left( \hat{\beta}_{j2} \cdot \hat{\beta}_{j2} + \hat{\gamma}_{i1} \cdot \hat{\gamma}_{j3} + \hat{\gamma}_{i3} \cdot \hat{\gamma}_{j1} \right) + \right. \]

\[ + \left. \left( \frac{\hat{\kappa}_i(\tau)}{m_i} \frac{\partial}{\partial \hat{\eta}_i} \right) \left( \hat{\alpha}_{i1} \cdot \hat{\beta}_{j2} - \hat{\beta}_{i2} \cdot \hat{\alpha}_{j1} \right) \right] (\tau, \vec{\sigma}). \]

\( (B16) \)

Therefore the non-relativistic limit of \( \mathcal{E}_{(int)} \), given by Eq.( 4.4), is
\( \mathcal{E}_{(\text{int})} \to_{c \to \infty} \left( \sum_{i=1}^{N} m_i \right) c^2 + \sum_{i=1}^{N} \frac{\tilde{\eta}_i^2(\tau)}{2m_i} + \sum_{i \neq j}^{1..N} \frac{Q_i Q_j}{4\pi |\tilde{\eta}_i(\tau) - \tilde{\eta}_j(\tau)|} + \\
+ \frac{1}{2} \int d^3\sigma \left( \tilde{\pi}_{\text{rad}}^2 + \tilde{B}_{\text{rad}}^2 \right)(\tau, \vec{\sigma}) + \\
+ \frac{1}{c^2} \left( - \sum_i \frac{\tilde{\eta}_i^4(\tau)}{8m_i^3} + \sum_{i \neq j}^{1..N} \left[ Q_i Q_j \frac{\tilde{\eta}_i(\tau)}{m_i} \cdot \tilde{\alpha}_j(\tau, \tilde{\eta}_i(\tau)) \right] + \\
+ \frac{1}{2} \int d^3\sigma \left( \tilde{\gamma}_{i1} \cdot \tilde{\gamma}_{j1} \right)(\tau, \vec{\sigma}) \right) + \\
+ O(c^{-4}). \tag{B17}
APPENDIX C: THE INTERNAL POINCARE’ GENERATORS AFTER THE CANONICAL TRANSFORMATION

1. The New Internal Momentum and Angular Momentum.

To obtain the instant form of the 3-momentum (4.1) we used the following results [remember from Eqs. (2.51) and (2.52) that in the new canonical basis the fields of the Lienard-Wiechert solution depend upon $\vec{\sigma} - \vec{\eta}_i(\tau)$ and $\vec{\kappa}_i(\tau)$]

\[
\sum_{i \neq j}^{1..N} Q_i Q_j \int d^3\sigma \left( \vec{\pi}_{\perp Si} \times \vec{B}_{Sj} \right)^r(\tau, \vec{\sigma}) = \sum_{i \neq j}^{1..N} Q_i Q_j \int d^3\sigma \left( \vec{\pi}_{\perp Si} \frac{\partial}{\partial \sigma} \hat{A}^s_{\perp Sj} \right)(\tau, \vec{\sigma}), \tag{C1}
\]

where the last line was obtained by integration by parts and using the transversality condition; then we get

\[
\sum_{i \neq j}^{1..N} Q_i Q_j \int d^3\sigma \left( \vec{\pi}_{\perp Si} \frac{\partial}{\partial \sigma} \hat{A}^s_{\perp Sj} \right)(\tau, \vec{\sigma}) = \frac{1}{2} \sum_{i \neq j}^{1..N} Q_i Q_j \int d^3\sigma \left( - \frac{\partial}{\partial \eta_i} (\hat{\pi}_{\perp Sj}^s \hat{A}^s_{\perp Si}) + \frac{\partial}{\partial \eta_i^*} (\hat{A}^s_{\perp Sj} \hat{\pi}_{\perp Si}^s) \right)(\tau, \vec{\sigma}) = \frac{1}{2} \sum_{i \neq j}^{1..N} Q_i Q_j \frac{\partial}{\partial \eta_i^*} \hat{K}_{ij}. \tag{C2}
\]

Moreover we have

\[
\int d^3\sigma \left( \vec{\pi}_{\perp rad} \times \vec{B}_{Si} \right)^r(\tau, \vec{\sigma}) = \frac{\partial}{\partial \eta_i^*} \int d^3\sigma \left( - \pi_{\perp rad}^s \hat{A}^s_{\perp Si} + \hat{\pi}_{\perp Si}^s \hat{A}^s_{rad} \right)^r(\tau, \vec{\sigma}) = \frac{\partial}{\partial \eta_i^*} \int d^3\sigma \left( - \pi_{\perp rad} \cdot \hat{A}_{\perp Si} + \hat{A}_{\perp rad} \cdot \hat{\pi}_{\perp Si} \right)^r(\tau, \vec{\sigma}) = - \frac{\partial \hat{T}_i(\tau)}{\partial \eta_i}. \tag{C3}
\]

The instant form of the internal angular momentum has been obtained by using Eqs. (6.41)-(6.45) of Ref. [14], based on the explicit knowledge of the Lienard Wiechert fields (2.51) and (2.52), since they imply
\[
\int d^3\sigma \left[ \hat{\sigma} \times \left( \vec{\pi}_{\perp rad} \times \hat{B}_{Si} + \vec{\pi}_{\perp Si} \times \hat{B}_{rad} \right)(\tau, \hat{\sigma}) \right]
= -\left( \hat{\eta}_i \times \frac{\partial}{\partial \hat{\eta}_i} + \hat{\kappa}_i \times \frac{\partial}{\partial \hat{\kappa}_i} \right) \int d^3\sigma \left( \hat{A}_{\perp Si} \cdot \vec{\pi}_{\perp rad} - \vec{\pi}_{\perp Si} \cdot \hat{A}_{\perp rad} \right)(\tau, \hat{\sigma}) = 0.
\]

\[
\frac{1}{N} \sum_{i \neq j} Q_i Q_j \int d^3\sigma \hat{\sigma} \times \left( \vec{\pi}_{\perp Si} \times \hat{B}_{Sj} \right)(\tau, \hat{\sigma}) = -\frac{1}{2} \sum_{i \neq j} Q_i Q_j \left( \hat{\eta}_i \times \frac{\partial}{\partial \hat{\eta}_i} + \hat{\kappa}_i \times \frac{\partial}{\partial \hat{\kappa}_i} \right) \hat{K}_{ij}.
\]

2. The Internal Boosts.

By using Eqs.(3.4) the internal boost in Eq.(4.6) below can be shown to assume the following form

\[
\hat{K}_{(int)} = -\sum_{i=1}^{N} \hat{\eta}_i(\tau) \left[ \sqrt{m_i^2 c^2 + \hat{\kappa}_i^2} + \frac{\hat{\kappa}_i}{2 c \sqrt{m_i^2 c^2 + \hat{\kappa}_i^2}} \cdot \sum_{j \neq i}^{1..N} Q_i Q_j \left( \frac{1}{2} \frac{\partial \hat{K}_{ij}(\hat{\kappa}_i, \hat{\kappa}_j, \hat{\eta}_i - \hat{\eta}_j)}{\partial \hat{\kappa}_i} - 2 \hat{A}_{\perp Sj}(\hat{\kappa}_j, \hat{\eta}_i - \hat{\eta}_j) \right) \right] - \frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i}^{1..N} Q_i Q_j \sqrt{m_i^2 c^2 + \hat{\kappa}_i^2} \left( \frac{\partial \hat{K}_{ij}(\hat{\kappa}_i, \hat{\kappa}_j, \hat{\eta}_i - \hat{\eta}_j)}{\partial \hat{\eta}_i} \right) + \frac{1}{c} \sum_{i=1}^{N} \sum_{j \neq i}^{1..N} Q_i Q_j \frac{\hat{\eta}_i - \hat{\eta}_j}{8 \pi |\hat{\eta}_i - \hat{\eta}_j|} - \frac{1}{2 c} \sum_{i=1}^{N} \sum_{j \neq i}^{1..N} Q_i Q_j \int d^3\hat{\sigma} \frac{\vec{\pi}_{\perp Sj}(\hat{\sigma} - \hat{\eta}_j, \hat{\kappa}_j)}{|\hat{\sigma} - \hat{\eta}_i|} - \frac{1}{2 c} \sum_{i=1}^{N} \sum_{j \neq i}^{1..N} Q_i Q_j \int d^3\hat{\sigma} \left[ \vec{\pi}_{\perp Si}(\hat{\sigma} - \hat{\eta}_i, \hat{\kappa}_i) \cdot \vec{\pi}_{\perp Sj}(\hat{\sigma} - \hat{\eta}_j, \hat{\kappa}_j) + \hat{B}_{Si}(\hat{\sigma} - \hat{\eta}_i, \hat{\kappa}_i) \cdot \hat{B}_{Sj}(\hat{\sigma} - \hat{\eta}_j, \hat{\kappa}_j) \right] - \frac{1}{2 c} \int d^3\hat{\sigma} \left( \vec{\pi}_{\perp rad}^2 + \hat{B}_{rad}^2 \right)(\tau, \hat{\sigma}) -
\]

67
cancelations occur here as well. We first point out that Eq.(4.11) implies the Hamiltonian given in Eq.(4.14) that all such terms cancel. We investigate whether such involve both radiation and Lienard-Wiechert fields. We have seen that in the final form of the second part of the tenth line and this last term cancels the seventh line on the right hand side of Eq.(C5). Furthermore the final four lines of the long expression on the right hand side contain terms that involve both radiation and Lienard-Wiechert fields. We have seen that in the final form of the Hamiltonian given in Eq.(4.14) that all such terms cancel. We investigate whether such cancelations occur here as well. We first point out that Eq.(4.11) implies

\[ + \frac{1}{c} \sum_{i=1}^{N} Q_i \frac{\hat{\eta}_i(\tau) \cdot \hat{A}_\perp \cdot \hat{A}_\perp}{\sqrt{m_i^2 c^2 + \hat{k}_i^2(\tau)}} + \frac{1}{c} \sum_{i=1}^{N} Q_i \left[ \sqrt{m_i^2 c^2 + \hat{k}_i^2(\tau)} \frac{\partial}{\partial \hat{k}_i} - \hat{\eta}_i(\tau) \frac{\partial}{\partial \hat{\eta}_i} \right] \]

\[ \times \int d^3\sigma \left( \hat{\pi}_\perp \cdot \hat{A}_\perp - \hat{A}_\perp \cdot \hat{\pi}_\perp \right)(\tau, \vec{\sigma}) - \frac{1}{c} \sum_{i=1}^{N} Q_i \int d^3\sigma \frac{\hat{\pi}_\perp(\tau, \vec{\sigma})}{4\pi |\vec{\sigma} - \hat{\eta}_i(\tau)|} - \frac{1}{c} \sum_{i=1}^{N} Q_i \int d^3\sigma \left( \hat{\pi}_\perp \cdot \hat{A}_\perp + \hat{B}_\perp \cdot \hat{B}_\perp \right)(\tau, \vec{\sigma}). \] (C5)

The final four lines of the long expression on the right hand side contain terms that involve both radiation and Lienard-Wiechert fields. We have seen that in the final form of the Hamiltonian given in Eq.(4.14) that all such terms cancel. We investigate whether such cancelations occur here as well. We first point out that Eq.(4.11) implies

\[ + \frac{1}{c} \sum_{i=1}^{N} Q_i \left[ - \hat{\eta}_i(\tau) \frac{\hat{k}_i(\tau)}{\sqrt{m_i^2 c^2 + \hat{k}_i^2(\tau)}} \cdot \frac{\partial}{\partial \hat{\eta}_i} \right] \]

\[ \times \int d^3\sigma \left( \hat{\pi}_\perp \cdot \hat{A}_\perp - \hat{A}_\perp \cdot \hat{\pi}_\perp \right)(\tau, \vec{\sigma}) \]

\[ = \frac{1}{c} \sum_{i=1}^{N} Q_i \left[ \hat{\eta}_i(\tau) \int d^3\sigma \left( \hat{\pi}_\perp \cdot \hat{\pi}_\perp - \hat{A}_\perp \cdot \frac{\partial^2 \hat{A}_\perp}{\partial \tau^2} \right)(\tau, \vec{\sigma}) \right] \]

\[ = \frac{1}{c} \sum_{i=1}^{N} Q_i \left[ \hat{\eta}_i(\tau) \int d^3\sigma \left( \hat{\pi}_\perp \cdot \hat{\pi}_\perp - \hat{A}_\perp \cdot \left( \Box \hat{A}_\perp + \partial^2 \hat{A}_\perp \right) \right)(\tau, \vec{\sigma}) \right] \]

\[ = \frac{1}{c} \sum_{i=1}^{N} Q_i \left[ \hat{\eta}_i(\tau) \int d^3\sigma \left( \hat{\pi}_\perp \cdot \hat{\pi}_\perp - \hat{A}_\perp \cdot \partial^2 \hat{A}_\perp \right)(\tau, \vec{\sigma}) \right] \]

\[ - \sum_{i=1}^{N} \hat{\eta}_i(\tau) Q_i \frac{\hat{k}_i(\tau) \cdot \hat{A}_\perp \cdot \hat{A}_\perp}{c \sqrt{m_i^2 c^2 + \hat{k}_i^2(\tau)}}, \] (C6)

and this last term cancels the seventh line on the right hand side of Eq.(C5). Furthermore the second part of the tenth line

\[ - \frac{1}{c} \sum_{i=1}^{N} Q_i \int d^3\sigma \left( \vec{B}_\perp \cdot \vec{B}_\perp \right)(\tau, \vec{\sigma}) \]

\[ = - \frac{1}{c} \sum_{i=1}^{N} Q_i \int d^3\sigma \left( \left( \partial_\sigma \hat{A}_\perp \right) \hat{A}_\perp - \vec{\sigma} \left( \partial_\sigma \hat{A}_\perp \right) \cdot \hat{A}_\perp \right)(\tau, \vec{\sigma}). \] (C7)
Taking into account the canceling from Eqs. (C6) and (C7), the final four lines of the long expression on the right hand side of Eq. (C5) thus reduce to

\[
+ \frac{1}{c} \sum_{i=1}^{N} Q_i \left[ \sqrt{m_i^2 c^2 + \tilde{\kappa}_i^2} \frac{\partial}{\partial \tilde{\pi}_{i}} \right] \\
\times \int d^3 \sigma \left( \tilde{\pi}_{\perp rad} \cdot \tilde{A}_{\perp Si} - \tilde{A}_{\perp rad} \cdot \tilde{\pi}_{\perp Si} \right)(\tau, \tilde{\sigma}) \\
- \frac{1}{c} \sum_{i=1}^{N} Q_i \int d^3 \sigma \left( \frac{\tilde{\pi}_{\perp rad}(\tau, \tilde{\sigma})}{4\pi |\tilde{\sigma} - \tilde{\eta}_i(\tau)|} + (\tilde{\partial}_{\sigma} A_{\perp rad}^*) \hat{A}_{\perp Si}^* \right)(\tau, \tilde{\sigma}) \\
+ \frac{1}{c} \sum_{i=1}^{N} Q_i \int d^3 \sigma \left( \tilde{\eta}_i(\tau) - \tilde{\sigma} \right) \left( \tilde{\pi}_{\perp rad} \cdot \tilde{\pi}_{\perp Si} - (\tilde{\partial}_{\sigma}^2 \tilde{A}_{\perp rad}^*) \cdot \hat{A}_{\perp Si} \right)(\tau, \tilde{\sigma}).
\]  

(C8)

To further simplify this expression we make use of the Fourier transforms of the potentials and fields given in Appendix B. In particular the first part of the last integral in Eq. (C8) becomes

\[
- \frac{1}{c} \sum_{i=1}^{N} Q_i \int d^3 \sigma \tilde{\sigma} \left( \tilde{\pi}_{\perp rad} \cdot \tilde{\pi}_{\perp Si} \right)(\tau, \tilde{\sigma}) = \\
= - \frac{1}{c} \sum_{i=1}^{N} Q_i \int d^3 k \omega(\vec{k}) \sum_{\lambda=1,2} \tilde{\epsilon}_\lambda(\vec{k}) \left\{ \left[ i a_{\lambda}(\vec{k}) \int d^3 \sigma e^{-i[\omega(\vec{k})\tau - \vec{k} \cdot \vec{\sigma}]} \cdot \tilde{\pi}_{\perp Si} \right] + c.c. \right\} = \\
= \frac{1}{c} \sum_{i=1}^{N} Q_i \int d^3 k \omega(\vec{k}) \sum_{\lambda=1,2} \tilde{\epsilon}_\lambda(\vec{k}) \left\{ \left[ i a_{\lambda}(\vec{k}) e^{-i[\omega(\vec{k})\tau - \vec{k} \cdot \vec{\sigma}]} \frac{\partial}{\partial \vec{k}} (-i) \frac{e^{i \vec{k} \cdot \vec{\pi}_i} \frac{\vec{k} \times (\vec{\pi}_i \times \vec{k})}{4\pi m_i^2 c^2 + \vec{\kappa}_i^2}}{k^4} \right] + c.c. \right\} = \\
= \frac{1}{c} \sum_{i=1}^{N} Q_i \int d^3 k \omega(\vec{k}) \sum_{\lambda=1,2} \tilde{\epsilon}_\lambda(\vec{k}) \left\{ \left[ i a_{\lambda}(\vec{k}) e^{-i[\omega(\vec{k})\tau - \vec{k} \cdot \vec{\sigma}]} \frac{\partial}{\partial \vec{k}} \frac{1}{k^4} \frac{\vec{k} \times (\vec{\pi}_i \times \vec{k})}{4\pi m_i^2 c^2 + \vec{\kappa}_i^2} \right] + c.c. \right\} - \\
- \frac{1}{c} \sum_{i=1}^{N} Q_i \int d^3 \sigma \tilde{\eta}_i(\tau)[\tilde{\pi}_{\perp rad} \cdot \tilde{\pi}_{\perp Si}](\tau, \tilde{\sigma}),
\]  

(C9)

while the second part becomes

\[
\frac{1}{c} \sum_{i=1}^{N} Q_i \int d^3 \sigma \tilde{\sigma} \left( \tilde{\partial}_{\sigma}^2 \tilde{A}_{\perp rad}^* \cdot \hat{A}_{\perp Si} \right)(\tau, \tilde{\sigma}) = \\
= \frac{1}{c} \sum_{i=1}^{N} Q_i \int d^3 \sigma \tilde{\eta}_i(\tau) \left( \tilde{\partial}_{\sigma}^2 \tilde{A}_{\perp rad}^* \cdot \hat{A}_{\perp Si} \right)(\tau, \tilde{\sigma}) + \\
+ \frac{1}{c} \sum_{i=1}^{N} Q_i \int d^3 \vec{k} \sum_{\lambda=1,2} \tilde{\epsilon}_\lambda(\vec{k}) \left( i a_{\lambda}(\vec{k}) e^{-i[\omega(\vec{k})\tau - \vec{k} \cdot \vec{\eta}_i]} \frac{\partial}{\partial \vec{k}} \frac{1}{k^4} \frac{\vec{k} \times (\vec{\pi}_i \times \vec{k})}{4\pi m_i^2 c^2 + \vec{\kappa}_i^2} \right) + c.c.
\]  

(C10)
Substituting Eqs. (C9) and (C10) into Eq. (C8) gives

\[
+ \frac{1}{c} \sum_{i=1}^{N} Q_i \left[ \sqrt{m_i^2 c^2 + \kappa_i^2(\tau)} \frac{\partial}{\partial \kappa_i} \right] \\
\times \int d^3 \sigma \left( \vec{\pi}_{\text{rad}} \cdot \vec{A}_{\text{Sl}} - \vec{A}_{\text{rad}} \cdot \vec{\pi}_{\text{rad}} \right)(\tau, \vec{\sigma}) - \\
- \frac{1}{c} \sum_{i=1}^{N} Q_i \int d^3 \sigma \left( \frac{\vec{\pi}_{\text{rad}}(\tau, \vec{\sigma})}{4\pi |\vec{\sigma} - \vec{\eta}_i(\tau)|} + (\bar{\sigma}_A \vec{A}_{\text{rad}}) \vec{A}_{\text{Sl}} \right)(\tau, \vec{\sigma}) + \\
+ \frac{1}{c} \sum_{i=1}^{N} Q_i \int d^3 \vec{k} \sum_{\lambda=1,2} \bar{e}_\lambda(\vec{k}) \{ i a_\lambda(\vec{k}) e^{-i [\omega(\vec{k}) \tau - \vec{k} \cdot \vec{\eta}_i]} \\
\times [\vec{k} \times (\vec{k} \times \vec{\kappa}_i)]_r \sqrt{m_i^2 c^2 + \kappa_i^2} + \omega(\vec{k}) \frac{\partial}{\partial \vec{k}} \frac{1}{k^2} \left( \vec{\kappa}_i \cdot \vec{k} \times (\vec{\kappa}_i \times \vec{k}) \right)_r + \text{c.c}. \\
\] (C11)

Including all Fourier transforms the above becomes

\[
+ \frac{1}{c} \sum_{i=1}^{N} Q_i \int d^3 \vec{k} \sum_{\lambda=1,2} \left[ i a_\lambda(\vec{k}) e^{-i [\omega(\vec{k}) \tau - \vec{k} \cdot \vec{\eta}_i]} \\
(\omega(\vec{k}) \sqrt{m_i^2 c^2 + \kappa_i^2(\tau)}) \frac{\partial}{\partial \vec{k}} \frac{1}{k^2} \bar{e}_\lambda(\vec{k}) \cdot [\vec{k} \times (\vec{\kappa}_i \times \vec{k})] \sqrt{m_i^2 c^2 + \kappa_i^2} + \\
\sqrt{m_i^2 c^2 + \kappa_i^2(\tau)} \frac{\partial}{\partial \vec{k}} \frac{1}{k^2} \bar{e}_\lambda(\vec{k}) \cdot [\vec{k} \times (\vec{\kappa}_i \times \vec{k})] \vec{\kappa}_i \cdot \vec{k} \right]_r - \omega(\vec{k}) \frac{\partial}{\partial \vec{k}} \frac{1}{k^2} \bar{e}_\lambda(\vec{k}) + \\
+ \frac{1}{k^4} \bar{e}_\lambda(\vec{k}) \cdot [\vec{k} \times (\vec{\kappa}_i \times \vec{k})] \sqrt{m_i^2 c^2 + \kappa_i^2} + \\
+ \omega(\vec{k}) \frac{\partial}{\partial \vec{k}} \frac{1}{k^2} \bar{e}_\lambda(\vec{k}) \cdot [\vec{k} \times (\vec{\kappa}_i \times \vec{k})] \sqrt{m_i^2 c^2 + \kappa_i^2} + \text{c.c}. \\
\] (C12)

Expanding the triplet products in the above and using the transversality yields for Eq. (C8)
\[
+ \frac{1}{c} \sum_{i=1}^{N} Q_i \int d^3 \vec{k} \sum_{\lambda=1,2} \left[ i a(\vec{k}) e^{-i[\omega(\vec{k}) \tau - \vec{k} \cdot \vec{n}]} \right]

\left( \omega(\vec{k}) \sqrt{m_i^2 c^2 + \vec{k}_i^2(\tau)} \right) \frac{\partial}{\partial \vec{k}_i} \frac{1}{k^2} \vec{\epsilon}_\lambda(\vec{k}) \cdot \vec{\nu}_i \sqrt{m_i^2 c^2 + \vec{\nu}_i^2} + \frac{\omega(\vec{k})}{k^2} \vec{\epsilon}_\lambda(\vec{k}) + \frac{1}{k^2} \vec{\epsilon}_\lambda(\vec{k}) \cdot \vec{k} \sqrt{m_i^2 c^2 + \vec{k}_i^2} + \frac{\omega(\vec{k})}{k^2} \vec{\epsilon}_\lambda(\vec{k}) + \frac{\epsilon_\lambda(\vec{k})}{k^2} \frac{1}{\partial \vec{k} \partial \vec{k} \cdot \vec{k}_i \times (\vec{k}_i \times \vec{k})] \sqrt{m_i^2 c^2 + \vec{k}_i^2} + \frac{\omega(\vec{k})}{k^2} \frac{1}{m_i^2 c^2 + \vec{k}_i^2} \left[ \vec{k} \cdot \vec{\nu}_i \right] \right) + c.c.]
\]

(C13)

Using
\[
\frac{\partial}{\partial \vec{k}} \frac{1}{2 \vec{k}^2} = \frac{\omega(\vec{k})}{k^2} \frac{1}{m_i^2 c^2 + \vec{k}_i^2} \left( \frac{\vec{k}_i \cdot \vec{k}_i}{\vec{k}^2} \right),
\]
performing, the indicated operations and combining like terms yields

\[
= \frac{1}{c} \sum_{i=1}^{N} Q_i \int d^3 \vec{k} \sum_{\lambda=1,2} \left[ i a(\vec{k}) e^{-i[\omega(\vec{k}) \tau - \vec{k} \cdot \vec{n}]} \right]

\left( \vec{\epsilon}_\lambda(\vec{k}) \right) \frac{\omega(\vec{k})}{k^2} \frac{m_i^2 c^2 + \vec{k}_i^2}{m_i^2 c^2 + \vec{k}_i^2} - \frac{\omega(\vec{k})}{k^2} \frac{(\vec{k}_i \cdot \vec{k}) \omega(\vec{k})}{m_i^2 c^2 + \vec{k}_i^2} + \frac{\vec{k}_i \cdot \vec{k}}{k^2} \frac{\vec{k}_i \cdot \vec{k}}{m_i^2 c^2 + \vec{k}_i^2} - \frac{\vec{k}_i \cdot \vec{k}}{k^2} \left( \frac{\vec{k}_i \cdot \vec{k}}{m_i^2 c^2 + \vec{k}_i^2} \right)
\]

(C14)
\begin{align}
&+ \bar{e}_\lambda(\vec{k}) \cdot \vec{k} \cdot \vec{k} \cdot \vec{k} \left[ \frac{\omega(\vec{k})}{k^2} \left( \frac{1}{m_i^2 c^2 + \vec{k}_i^2 - (\vec{k}_i \cdot \vec{k})^2} - \frac{2(m_i^2 c^2 + \vec{k}_i^2)}{m_i^2 c^2 + \vec{k}_i^2 - (\vec{k}_i \cdot \vec{k})^2} \right) + \right. \\
&\quad \left. \frac{2 \left( \vec{k}_i \cdot \vec{k} \right)^2}{k^2 \left( m_i^2 c^2 + \vec{k}_i^2 - (\vec{k}_i \cdot \vec{k})^2 \right)^2} + \frac{1}{m_i^2 c^2 + \vec{k}_i^2 - (\vec{k}_i \cdot \vec{k})^2} \right] + \\
&\quad - \sqrt{m_i^2 c^2 + \vec{k}_i^2} \left[ \frac{2 \vec{k}_i \cdot \vec{k}}{k^2 \left( m_i^2 c^2 + \vec{k}_i^2 - (\vec{k}_i \cdot \vec{k})^2 \right)^2} + \frac{2 \sqrt{m_i^2 c^2 + \vec{k}_i^2} \vec{k}_i \cdot \vec{k}}{k^2 \left( m_i^2 c^2 + \vec{k}_i^2 - (\vec{k}_i \cdot \vec{k})^2 \right)^2} \right] + \\
&\quad + \bar{e}_\lambda(\vec{k}) \cdot \vec{k} \cdot \vec{k} \cdot \vec{k} \left[ \frac{\omega(\vec{k})}{k^2} \left( \frac{2(m_i^2 c^2 + \vec{k}_i^2) (\vec{k}_i \cdot \vec{k})}{k^2 \left( m_i^2 c^2 + \vec{k}_i^2 - (\vec{k}_i \cdot \vec{k})^2 \right)^2} - \frac{4 \vec{k}_i \cdot \vec{k}}{k^2 \left( m_i^2 c^2 + \vec{k}_i^2 - (\vec{k}_i \cdot \vec{k})^2 \right)^2} \right] - \\
&\quad - \frac{2 \left( \vec{k}_i \cdot \vec{k} \right)^2}{k^2 \left( m_i^2 c^2 + \vec{k}_i^2 - (\vec{k}_i \cdot \vec{k})^2 \right)^2} + \frac{2 \vec{k}_i \cdot \vec{k}}{k^2 \left( m_i^2 c^2 + \vec{k}_i^2 - (\vec{k}_i \cdot \vec{k})^2 \right)^2} \right] + \\
&\quad + \sqrt{m_i^2 c^2 + \vec{k}_i^2} \left[ \frac{1}{m_i^2 c^2 + \vec{k}_i^2 - (\vec{k}_i \cdot \vec{k})^2} + \frac{2 \vec{k}_i \cdot \vec{k}}{k^2 \left( m_i^2 c^2 + \vec{k}_i^2 - (\vec{k}_i \cdot \vec{k})^2 \right)^2} \right] + \\
&\quad - \frac{4 \left( \vec{k}_i \cdot \vec{k} \right)^2}{k^2 \left( m_i^2 c^2 + \vec{k}_i^2 - (\vec{k}_i \cdot \vec{k})^2 \right)^2} - \frac{2 \left( \vec{k}_i \cdot \vec{k} \right)^2}{k^2 \left( m_i^2 c^2 + \vec{k}_i^2 - (\vec{k}_i \cdot \vec{k})^2 \right)^2} + \\
&\quad + \frac{2}{\left( m_i^2 c^2 + \vec{k}_i^2 - (\vec{k}_i \cdot \vec{k})^2 \right) + c.c.} \right], \quad \text{(C15)}
\end{align}

or

\begin{align}
&= \frac{1}{c} \sum_{i=1}^{N} Q_i \int d^3 \vec{k} \sum_{\lambda=1,2} \left[ i a_{\lambda}(\vec{k}) e^{-i [\omega(\vec{k}) - \vec{k} \cdot \vec{p}]} \right] \\
&\quad \left( \bar{e}_\lambda(\vec{k})[0 + 0] + \bar{e}_\lambda(\vec{k}) \cdot \vec{k} \cdot \vec{k} \cdot \vec{k} \frac{\omega(\vec{k})}{k^2} (0) - \frac{\sqrt{m_i^2 c^2 + \vec{k}_i^2} (\tau)}{k^2} \right) + \\
&\quad + \frac{\bar{e}_\lambda(\vec{k}) \cdot \vec{k} \cdot \vec{k} \cdot \vec{k} \omega(\vec{k})}{k^2} \left[ \frac{2(m_i^2 c^2 + \vec{k}_i^2)}{\left( m_i^2 c^2 + \vec{k}_i^2 - (\vec{k}_i \cdot \vec{k})^2 \right)^2} - \right. \\
&\quad \left. \left( m_i^2 c^2 + \vec{k}_i^2 - (\vec{k}_i \cdot \vec{k})^2 \right) \right]
\end{align}
\[ 
- \frac{2 \left( \hat{\kappa}_i \cdot \hat{k} \right)^2}{\left( m_i^2 c^2 + \hat{\kappa}_i^2 - (\hat{\kappa}_i \cdot \hat{k})^2 \right)} + \sqrt{m_i^2 c^2 + \hat{\kappa}_i^2(0)} + c.c = 0. \quad (C16) 
\]

Thus all four lines of the long expression on the right hand side of Eq. (C5) cancel among one another.

Hence our boost reduces to

\[ 
\tilde{K}_{(\text{int})} = - \sum_{i=1}^{N} \hat{\eta}_i(\tau) \left[ \sqrt{m_i^2 c^2 + \hat{\kappa}_i^2} + \frac{\hat{\kappa}_i}{2c} \cdot \sum_{j \neq i}^{1..N} \frac{Q_i Q_j}{c} \sqrt{m_j^2 c^2 + \hat{\kappa}_j^2} \frac{1}{\hat{\kappa}_i} \frac{\partial \hat{K}_{ij}(\hat{\kappa}_i, \hat{\kappa}_j, \hat{\eta}_i - \hat{\eta}_j)}{\partial \hat{\eta}_i} - \frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i}^{1..N} \frac{Q_i Q_j}{c} \sqrt{m_j^2 c^2 + \hat{\kappa}_j^2} \frac{\partial \hat{K}_{ij}(\hat{\kappa}_i, \hat{\kappa}_j, \hat{\eta}_i - \hat{\eta}_j)}{\partial \hat{\kappa}_i} + \frac{1}{c} \sum_{i=1}^{N} \sum_{j \neq i}^{1..N} \frac{Q_i Q_j}{8\pi} \frac{\hat{\eta}_i - \hat{\eta}_j}{|\hat{\eta}_i - \hat{\eta}_j|} - \frac{1}{2c} \sum_{i=1}^{N} \sum_{j \neq i}^{1..N} Q_i Q_j \int d^3\sigma \frac{\hat{\pi}_{\perp Si}(\hat{\sigma} - \hat{\eta}_i, \hat{\kappa}_i) \cdot \hat{\pi}_{\perp Sj}(\hat{\sigma} - \hat{\eta}_j, \hat{\kappa}_j)}{|\hat{\sigma} - \hat{\eta}_i|} + \frac{1}{2c} \sum_{i=1}^{N} \sum_{j \neq i}^{1..N} Q_i Q_j \int d^3\sigma \frac{\hat{\kappa}_i^2 + \hat{\kappa}_j^2}{\hat{\kappa}_i^2 + \hat{\kappa}_j^2} (\tau, \hat{\sigma}) \right] 
\]

which consists only of the decoupled radiation fields portion at the end plus particle and Lienard-Wiechert potential induced terms.

3. Semi-relativistic Expansions of \( \mathcal{E}_{(\text{int})} \) and \( \tilde{K}_{(\text{int})} \) after the Canonical Transformation

By using Eqs. (B1) - (B4) we get \( \hat{K}_{12} = O(c^{-3}) \), so that all the terms after the first in Eq. (4.6) are of order \( O(c^2), O(c^{-3}), O(c^{-3}), O(c^{-5}) \), respectively. Therefore we get

\[ 
\tilde{K}_{(\text{int})} = - \sum_{i=1}^{2} \hat{\eta}_i(\tau) \left( m_i c + \frac{\hat{\kappa}_i^2}{2m_i c} \right) + O(c^{-2}) - \frac{1}{2c} \int d^3\sigma \hat{\sigma} \left( \hat{\pi}_{\perp \text{rad}}^2 + \hat{B}_{\text{rad}}^2 \right)(\tau, \hat{\sigma}) = \tilde{K}_{\text{matter}} + \tilde{K}_{\text{rad}} = -c \sum_{i=1}^{2} m_i \hat{\eta}_i \approx 0. \quad (C18) 
\]
APPENDIX D: DIMENSIONS

Let \([t], [l] \) and \([m] \) be the units of time, length and mass. Then for the proper time we have \([\tau] = [x^0 = ct] = [l] \).

For the 4-momentum \(P^\mu = (E^0 = \vec{E}; \vec{P}) \) with \(Mc = \sqrt{P^2}\) we have the following dimensions \([M] = [m], [E = c P^0] = [m l^2 t^{-2}], [\vec{P}] = [m l t^{-1}] \).

For the Lorentz generators \(J^i = \epsilon^{ijk} J^j k, K^r = J^{or} \) we have \([J^{\mu\nu}] = [\vec{J}] = [\vec{K}] = [m l^2 t^{-1}] \). For the non-relativistic Galilei boosts we have \([\vec{K}] = [\vec{K} / c] = [m l] \).

For the particles we have \([\vec{\eta}_i] = [l], [\vec{\kappa}_i] = [m l t^{-1}], [\hat{d} \vec{\eta}_i(\tau)] = [0] \) a-dimensional, \([\hat{d} \vec{\kappa}_i(\tau)] = [m t^{-1}] \). For the energy-momentum tensor we have \([T^{AB}] = [m l^{-2} t^{-1}] \).

For the electro-magnetism we adopt the Heaviside-Lorentz system of units, so that the Coulomb potential is \(\frac{Q_1 Q_2}{4\pi |\vec{r}_1(\tau) - \vec{r}_2(\tau)|} \) and \(\epsilon_o = \mu_o = 1 \) for the electric permittivity and magnetic permeability of the vacuum (this implies \(\vec{D} = \vec{E} \) and \(\vec{B} = \vec{H} \)).

Therefore the dimensions of the electric charge and of the electro-magnetic potentials and fields are \([Q_i] = [m^{1/2} l^{3/2} t^{-1}], [\vec{A}_\perp] = [Q l^{-1}] = [m^{1/2} l^{1/2} t^{-1}], [\vec{E}_\perp] = [\vec{B}] = [\vec{A}_\perp l^{-1}] = [m^{1/2} l^{-1/2} t^{-1}] \). Consistently we have \([Q^2 l^{-1}] = [Q] [\vec{A}_\perp] = [m e^2] \).

In Eqs.(2.33) we have \([k] = [l^{-1}], [\vec{A}_\perp] = [m^{1/2} l^{1/2} t^{-1}], [\hat{a}_\perp] = [m^{1/2} l^{5/2} t^{-1}] \), while for the helicity in Eqs.(2.35) we have \([\hat{h}] = [\vec{j}] = [m l^2 t^{-1}] \).

For the Lienard-Wiechert electro-magnetic potentials and fields of Eqs. (2.50)-(2.54) we have \([\vec{A}_{\perp Si}] = [l^{-1}], [\vec{E}_{\perp Si}] = [\vec{B}_{Si}] = [l^{-2}], [V_{Darwin}] = [m l^2 t^{-2}] \).

For the functionals (3.4) and (3.5) we have \([T_i] = [m^{1/2} l^{3/2} t^{-1}] = [Q_i], [\mathcal{K}_{ij}] = [0] \) a-dimensional.
[1] C.Cohen-Tannoudji, J. Dupont-Roc and G.Grynberg, *Photons and Atoms. Introduction to Quantum Electrodynamics* (Wiley, New York, 1989).

[2] C.Cohen-Tannoudji, J. Dupont-Roc and G.Grynberg, *Atom-Photon Interactions. Basic Processes and Applications* (Wiley, New York, 1992).

[3] W.P.Schleich, *Quantum Optics in Phase Space* (Wiley-VCH, Berlin, 2001).

[4] M. LeBellac and J.M.Levy-Leblond, *Galilean Electromagnetism*, Nuovo Cimento 14B, 217 (1973).

[5] L.Cacciapuoti and C.Salomon, *ACES: Mission Concept and Scientific Objective*, 28/03/2007, ESA document, Estec (ACES_science(printout.doc)).

[6] A.Peres and D.R.Terno, *Quantum Information and Relativity Theory*, Rev.Mod.Phys. 76, 93 (2004)(quant-ph/0212023).

[7] D.R.Terno, *Introduction to Relativistic Quantum Information* (quant-ph/0508049).

[8] L.Lusanna, *The Chrono-Geometrical Structure of Special and General Relativity: A Re-Visititation of Canonical Geometrodynamics*, lectures at 42nd Karpacz Winter School of Theoretical Physics: Current Mathematical Topics in Gravitation and Cosmology, Ladek, Poland, 6-11 Feb 2006, Int.J.Geom.Methods in Mod.Phys. 4, 79 (2007). (gr-qc/0604120).

[9] D.Alba, L.Lusanna and M.Pauri, *New Directions in Non-Relativistic and Relativistic Rotational and Multipole Kinematics for N-Body and Continuous Systems* (2005), in Atomic and Molecular Clusters: New Research, ed.Y.L.Ping (Nova Science, New York, 2006) (hep-th/0505005).

D.Alba, L.Lusanna and M.Pauri, *Centers of Mass and Rotational Kinematics for the Relativistic N-Body Problem in the Rest-Frame Instant Form*, J.Math.Phys. 43, 1677-1727 (2002) (hep-th/0102087).

D.Alba, L.Lusanna and M.Pauri, *Multipolar Expansions for Closed and Open Systems of Relativistic Particles*, J. Math.Phys. 46, 062505, 1-36 (2004) (hep-th/0402181).

[10] D.Alba, H.W.Crater and L.Lusanna, *Hamiltonian Relativistic Two-Body Problem: Center of Mass and Orbit Reconstruction*, J.Phys. A40, 9585 (2007) (gr-qc/0610200).
[11] C. Møller, *Sur la dynamique des systèmes ayant un moment angulaire interne*, Ann. Inst. H. Poincare’ **11**, 251 (1949).
C. Möller, *The Theory of Relativity* (Oxford Univ. Press, Oxford, 1957).

[12] E. Schmutzer and J. Plebanski, *Quantum Mechanics in Noninertial Frames of Reference*, Fortschr. Phys. **25**, 37 (1978).

[13] M. Pauri and G. Prosperi, *Canonical Realizations of Lie Symmetry Groups*, **7**, 366 (1966); *Canonical Realizations of the Rotation Group*, J. Math. Phys. **8**, 2256 (1967); *Canonical Realizations of the Galilei Group*, J. Math. Phys. **9**, 1146 (1968); *Canonical Realizations of the Poincaré group: I. General Theory*, J. Math. Phys. **16**, 1503 (1975); *Canonical Realizations of the Poincaré Group: II. Space-Time Description of Two Particles Interacting at a Distance, Newtonian-like Equations of Motion and Approximately Relativistic Lagrangian Formulation*, J. Math. Phys. **17**, 1468 (1976).

[14] H. W. Crater and L. Lusanna, *The Rest-Frame Darwin Potential from the Lienard-Wiechert Solution in the Radiation Gauge*, Ann. Phys. (N.Y.) **289**, 87 (2001).

[15] D. Alba, H. W. Crater and L. Lusanna, *The Semiclassical Relativistic Darwin Potential for Spinning Particles in the Rest Frame Instant Form: Two-Body Bound States with Spin 1/2 Constituents*, Int. J. Mod. Phys. **A16**, 3365-3478 (2001) (hep-th/0103109).
F. Bigazzi and L. Lusanna, *Spinning Particles on Spacelike Hypersurfaces and their Rest Frame Description*, Int. J. Mod. Phys. **A14**, 1429 (1999) (hep-th/9807052).

[16] N. N. Bogoliubov and D. V. Shirkov, *Introduction To The Theory of Quantized Fields*, (Wiley, New York, 1980).

[17] J. Earman and D. Fraser, *Haag’s Theorem and its Implications for the Foundations of Quantum Field Theory*, Erkenntnis **64**, 305 (2006) (philsci-archive.pitt.edu/archive/00002673/).

[18] G. Longhi and L. Lusanna, *Bound-State Solutions, Invariant Scalar Products and Conserved Currents for a Class of Two-Body Relativistic Systems*, Phys. Rev. **D34**, 3707 (1986).

[19] H. Leutwyler and J. Stern, *Relativistic Dynamics on a Null Plane*, Ann. Phys. (N.Y.) **112**, 94 (1978).

[20] G. Veneziano, *Quantum Strings and the Constants of Nature*, in *The Challenging Questions*, ed. A. Zichichi, the Subnuclear Series n. 27 (Plenum Press, New York, 1990).

[21] H. Epstein, V. Glaser and A. Jaffe, *Nonpositivity of the Energy Density in Quantized Field Theories*, Nuovo Cimento **36**, 1016 (1965).

[22] L. Lusanna, *Gauge Fixings, Evolution Generators and World-line Conditions in Relativistic Classical Mechanics with Constraints*, Nuovo Cimento **65B**, 135 (1981).

[23] D. G. Currie, T. F. Jordan and E. C. G. Sudarshan, *Relativistic Invariance and Hamiltonian Theories of Interacting Particles*, Rev. Mod. Phys. **35**, 350 (1965).
H. Leutwyler, *A no Interaction Theorem in Classical Relativistic Hamiltonian Particle Mechanics*, Nuovo Cimento **37**, 556 (1965).
S. Chelkowski, J. Nietendel and R. Suchanek, *The No-Interaction Theorem in Relativistic Particle Mechanics*, Acta Phys. Pol. **B11**, 809 (1980).

[24] G. Longhi, L. Lusanna and G. Longhi, *On the Many-Time Formulation of Classical Particle Dynamics*, J. Math. Phys. **30**, 1893 (1989).

[25] I. T. Todorov, *Dynamics of Relativistic Point Particles as a Problem with Constraints*, Dubna Joint Institute for Nuclear Research No. E2-10175, 1976; *On the Quantification of a Mechanical System with Second Class Constraints*, Ann. Inst. H. Poincare’ **A28**, 207 (1978).
A. Komar, *Constraint Formalism of Classical Mechanics*, Phys. Rev. **D18**, 1881 and *Interacting Relativistic Particles*, Phys. Rev. **D18**, 1887 (1978).
[26] Ph.Droz Vincent, Is Interaction Possible without Heredity?, Lett.Nuovo Cimento 1, 839 (1969); Relativistic Systems of Interacting Particles, Phys.Scr. 2, 129 (1970); Hamiltonian Systems of Relativistic Particles, Rep. Math. Phys., 8, 79 (1975); Two-Body Relativistic Systems, Ann.Inst.H.Poincaré 27, 407 (1977) and N-Body Relativistic Systems, 32A, 377 (1980); Action at a Distance and Relativistic Wave Equations for Spinless Quarks, Phys.Rev. D19, 702 (1979).

[27] N.Mukunda and E.C.G.Sudarshan, Form Of Relativistic Dynamics With World Lines, Phys.Rev. D23, 2210 (1981).
A.Kihlberg, R.Marnelius and N.Mukunda, Relativistic Potential Models As Systems With Constraints And Their Interpretation, Phys.Rev. D23, 2201 (1981).
J.N.Goldberg, Relativistically Interacting Particles and World-lines, Syracuse Univ. preprint (1980).

[28] M. Kalb and P. Van Alstine, Invariant Singular Actions for the Relativistic Two-Body Problem: a Hamiltonian Formulation, Yale Reports, C00-3075-146 (1976), C00-3075-156 (1976).
D.Domincici, J.Gomis and G.Longhi, A Lagrangian For Two Interacting Relativistic Particles, Nuovo Cimento B48, 152 (1978); A Lagrangian For Two Interacting Relativistic Particles: Canonical Formulation, Nuovo Cimento A48, 257 (1978); A Possible Approach To The Two-Body Relativistic Problem, Nuovo Cimento A56, 263 (1980).
J.Gomis, J.A.Lobo and J.M.Pons, A Singular Lagrangian Model For Two Interacting Relativistic Particles, Ann.Inst.H.Poincare’ A35, 17 (1981).
R.Giachetti and E.Sorace, Canonical Theory of Relativistic Interactions, Nuovo Cimento A43, 281 (1978); Relativistic Two-Body Interactions: A Hamiltonian Formulation, Nuovo Cimento A56, 263 (1980).
L.Lusanna, A Model for N Classical Relativistic Particles, Nuovo Cimento A64, 65 (1981).

[29] F.Rohrlich, Many-Body Forces and the Cluster Decomposition, Phys.Rev. D23, 1305 (1981).
L.P.Horwitz and F.Rohrlich, Constraint Relativistic Quantum Dynamics, Phys.Rev. D24, 1928 (1981).
H.Sazdjian, Separable Interactions In Classical Relativistic Hamiltonian Mechanics, Lett.Math.Phys. 5, 319 (1981); Position Variables in Classical Relativistic Hamiltonian Mechanics, Nucl.Phys. B161, 469 (1979).
S.N.Sokolov, Theory of Relativistic Direct Interaction, Serpukhov report IHEP, OTF 78-125 (1978).

[30] D.G.Currie, Poincar-Invariant Equations of Motion for Classical Particles, Phys.Rev. 142, 817 (1966).
R.H.Hill, Inatntaneous Action-at-a-Distance in Classical Relativistic Mechanic, J.Math.Phys. 8, 201 (1967).

[31] L.Bel, Mecanica Relativista Predictiva, curso impartido en el Departamento de Fisica Teorica de la Universidad Autonoma de Barcelona, UAB FT-34 (1977); Dynamique des systmes de N particules ponctuelles en relativit restreinte, Ann.Inst.Henry Poincare’ 12, 307 (1970); Predictive Relativistic Mechanics., Ann.Inst.Henry Poincare’ 14, 189 (1971).
L.Bel and F.X.Fustero, Predictive Relativistic Mechanics of n Particle Systems., Ann.Inst.Henry Poincare’ 25, 411 (1976).
L.Bel and J.Martin, Hamiltonians and Conservative Systems, Ann.Inst.Henry Poincare’ 22, 173 (1975) and Predictive Relativistic Mechanics of Systems of N Particles with Spin, 33, 409 (1980).

[32] D. Alba, L. Lusanna and M. Pauri, Dynamical Body Frames, Orientation-Shape Variables and
Canonical Spin Bases for the Nonrelativistic N-Body Problem , J. Math. Phys. 43, 373 (2002) (hep-th/0011014).

[33] M.Pauri and G.Prosperi, Canonical Realizations of the Galilei Group, J.Math.Phys. 9, 1146 (1968).
R.DePietri, L.Lusanna and M.Pauri, Gauging Kinematical and Internal Symmetry Groups for Extended Systems: the Galilean One-Time and Two-Times Harmonic Oscillators, Class.Quantum Grav. 11, 1417 (1996).

[34] L.Assenza and G.Longhi, Collective and Relative Variables for Massless Fields, Int.J.Mod.Phys. A15, 4575-4601 (2000).

[35] D.Alba and L.Lusanna, The Lienard-Wiechert Potential of Charged Scalar Particles and their Relation to Scalar Electrodynamics in the Rest-Frame Instant Form, Int.J.Mod.Phys. A13, 2791 (1998).

[36] A.Barducci and L.Lusanna, The Photon in Pseudo-Classical Mechanics, Nuovo Cimento 77A, 39 (1983).

[37] N.H.Lindner, A.Peres and D.R.Terno, Wigner’s Little Group and Berry’s Phase for Massless Particles, J.Phys. A36, L449 (2003) (hep-th/0304017).
J.A.Brooke and F.E.Schroeck, Localization of the Photon on Phase Space, J.Math.Phys. 37, 5958 (1996).
O.Keller, On the Theory of Spatial Photon Localization, Phys.Rep. 411, 1 (2005).

[38] D.Alba and L.Lusanna, Charged Particles and the Electro-Magnetic Field in Non-Inertial Frames, in preparation.

[39] S.Weinberg, Feynman Rules for Any Spin. II. Massless Particles, Phys.Rev. 134, B882 (1964); Feynman Rules for Any Spin , Phys. Rev. 133, B1318.