On the Complexity of Computing Maximum and Minimum Min-Cost-Flows

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Abstract
Consider a flow network, i.e., a directed graph where each arc has a non-negative capacity value and an associated length, together with nonempty supply intervals for the sources and nonempty demand intervals for the sinks. The Maximum Min-Cost-Flow Problem (MaxMCF) is to find fixed supply and demand values within these intervals such that the optimal objective value of the induced Min-Cost-Flow Problem (MCF) is maximized. In this paper, we show that MaxMCF as well as its uncapacitated variant, the Maximum Transportation Problem (MaxTP), are NP-hard. Further, we prove that MaxMCF is APX-hard if a connectedness-condition regarding the sources and the sinks of the flow network is dropped. Finally, we show how the Minimum Min-Cost-Flow Problem (MinMCF) can be solved in polynomial time.

KEYWORDS
approximation hardness, attacker-defender problem, bilevel programming, minimum cost flow, network interdiction, parametrized flow problems

1 INTRODUCTION

The Minimum Cost Flow Problem (MCF), which is commonly called Min-Cost-Flow Problem, is one of the classic network flow problems in operations research. Fixed amounts of supply have to be transported through a flow network from a set of sources towards a set of sinks in order to satisfy their demands while respecting capacity restrictions on the arcs. The goal is to find a feasible flow with minimum cost. Due to its applicability to a broad variety of real-world problems, including telecommunication, network design, transportation, routing, scheduling, resource planning, and manufacturing, MCF has been investigated thoroughly during the last decades. Several concrete examples are for example discussed in the book of Ahuja et al. [1].

However, in many real-world applications the supplies and demands are not necessarily fixed and can only be narrowed down to a certain range. An example for this is given by Hoppmann and Schwarz [9], who model an optimization problem arising in the context of natural gas transport as MCF. Here the supplies and demands can take on values within predefined intervals, which are given for each source and each sink individually. Due to this setup, there is a natural interest in the worst case and in the best case with respect to (w.r.t.) the potential transportation cost. This is the motivation for the introduction of the Maximum Min-Cost-Flow Problem (MaxMCF) and the Minimum Min-Cost-Flow Problem (MinMCF), respectively.

The remainder of this paper is structured as follows. First of all, we introduce MaxMCF and MinMCF in Section 2 and discuss some related work in Section 3. Next, we show that MinMCF can be solved in polynomial time using a purpose-built...
MCF instance in Section 4. After introducing a linear bilevel optimization model for MaxMCF in Section 5, we prove that it and its uncapacitated variant, the Maximum Transportation Problem (MaxTP), are NP-hard in Section 6. Further, in Section 7 we additionally demonstrate that MaxMCF is APX-hard if a connectedness-condition regarding the sources and sinks of the flow network is dropped. We conclude with an outlook on our future research in Section 8.

2 PROBLEM DEFINITIONS

First, we revisit the definition of the Min-Cost-Flow Problem. Consider a flow network, i.e., a directed graph \((V, A)\) with vertex set \(V\) and arc set \(A \subseteq V \times V\) where each arc \(a \in A\) has an associated nonnegative capacity value \(c_a \in \mathbb{R}_{\geq 0}\) and a length value \(\ell_a \in \mathbb{R}\). Further, \(V^+ \subseteq V\) and \(V^- \subseteq V\) denote the sources and the sinks of the flow network, respectively, and we assume w.l.o.g. that \(V^+ \cap V^- = \emptyset\). The remaining nodes \(V^0 := V \setminus (V^+ \cup V^-)\) are called inner nodes. Additionally, for each source \(u \in V^+\) we are given a nonpositive demand value \(b_u \in \mathbb{R}_{\leq 0}\) and for each sink \(w \in V^-\) we are given a nonpositive demand value \(b_v \in \mathbb{R}_{\leq 0}\). Hence, we denote an instance of MCF as a triple \((V, A, b)\). Further, it can be stated as the following linear program (LP)

\[
\begin{align*}
\min & \sum_{a \in A} \ell_a f_a \\
\text{s.t.} & \sum_{a \in \delta^+(v)} f_a - \sum_{a \in \delta^-(v)} f_a = b_v \quad \forall v \in V^+ \cup V^- \\
& \sum_{a \in \delta^+(v)} f_a - \sum_{a \in \delta^-(v)} f_a = 0 \quad \forall v \in V^0 \\
& f_a \in [0, c_a] \quad \forall a \in A,
\end{align*}
\]

where \(\delta^+(v)\) and \(\delta^-(v)\) denote the set of outgoing and incoming arcs of \(v \in V\), respectively. Here, the nonnegative variable \(f_a\) describes the amount of flow on arc \(a \in A\), which is bounded by its capacity \(c_a\), see (4). Furthermore, constraints (2) guarantee that the supplies and the demands are satisfied while constraints (3) ensure that flow conservation holds at all inner nodes. A flow vector \(f \in \mathbb{R}_{\geq 0}^{|A|}\) is a feasible solution for MCF if it satisfies (2)–(4). The goal of MCF is to find an optimal solution, i.e., a feasible flow vector \(f\) minimizing the cost function (1).

For MaxMCF and MinMCF we consider the same setup as for MCF, but instead of fixed supplies and demands we are given intervals. In particular, for each source \(u \in V^+\) we have a nonempty supply interval \([\bar{b}_u, \overline{b}_u] \subseteq \mathbb{R}_{\geq 0}\), where \(\underline{b}_u \in \mathbb{R}_{\geq 0}\) is a lower bound and \(\overline{b}_u \in \mathbb{R}_{\geq 0}\) is an upper bound on \(u\)'s supply. Analogously, for each sink \(w \in V^-\) we have a nonempty demand interval \([\underline{b}_w, \overline{b}_w] \subseteq \mathbb{R}_{\leq 0}\) with \(\underline{b}_w \in \mathbb{R}_{\leq 0}\). Furthermore, we call a supply and demand vector \(b \in \mathbb{R}^{|V|+|V^-|}\) feasible if \(b_v \in [\underline{b}_v, \overline{b}_v]\) for all \(v \in V^+ \cup V^-\) and if \(\sum_{v \in V^0} b_v = 0\) holds, i.e., if the supplies and the demands respect the interval bounds and if they are balanced. Thus, we are about to denote an instance of MaxMCF or MinMCF as quadruple \((V, A, b, \bar{b})\).

In the following, we denote solutions for MaxMCF and MinMCF as tuples \((b, f)\), where \(b\) is a feasible supply and demand vector and \(f\) is a feasible solution for the MCF instance induced by \(b\). The cost of a solution \(c(b, f)\) is equal to the cost of \(f\). Next, a solution \((b, f)\) is called feasible if \(f\) is optimal for the MCF instance induced by \(b\). The goal of MaxMCF is to find supply and demand values within the given intervals such that the optimal objective value of the induced MCF instance is maximized. Hence, a solution \((b, f)\) is optimal for MaxMCF if it is feasible and if \(c(b, f) \geq c(\bar{b}, \bar{f})\) holds for all feasible solutions \((\bar{b}, \bar{f})\). On the other hand, \((b, f)\) is optimal for MinMCF if it is feasible and if \(c(b, f) \leq c(\bar{b}, \bar{f})\) holds for all feasible solutions \((\bar{b}, \bar{f})\), since it is MinMCF's goal to find supply and demand values within the given intervals such that the optimal objective value of the induced MCF instance is minimized. An example illustrating the previous definitions is shown in Figure 1.

Similar to Ahuja et al. [1], we assume that there exists an uncapacitated directed path from each source towards each sink to ensure the existence of a feasible solution for all MCF instances induced by feasible supply and demand vectors. We impose this connectedness-condition, if necessary, by adding direct arcs with large cost and infinite capacity. No such arc appears in any feasible solution for MaxMCF or MinMCF unless there exists a feasible supply and demand vector inducing an originally infeasible MCF instance.

3 RELATED WORK

To the best of our knowledge, MaxMCF and MinMCF have not been investigated, yet. In this section we discuss optimization problems and corresponding complexity results that are closely related to it. Since MCF has been used in many different contexts, we focus on a selection of corresponding applications, which motivated the introduction of MaxMCF and MinMCF.

First of all, the variant of MaxMCF where the capacity restrictions on the arcs are dropped has previously been considered by Hennig and Schwarz [8] and Hoppmann and Schwarz [9]. This so-called Maximum Transportation Problem (MaxTP) and its complexity are further discussed in Section 6. However, in their work MaxTP is used to determine severe scenarios for natural
In (A) an example flow network with sources \( S = \{ u_1, u_2, u_3 \} \) and sinks \( T = \{ w_1, w_2, w_3 \} \) is shown. The supply and demand intervals are stated above or below the corresponding nodes. Furthermore, while all arcs have capacity \( c_a = 2 \), the vertical dashed arcs have length \( \ell_a = 1 \) and the solid horizontal arcs have length \( \ell_a = 3 \). In (B,C,D) feasible solutions for MaxMCF and MinMCF are depicted. The chosen supply and demand values are stated below or above the corresponding nodes. In all three figures the blue arcs represent a flow value of \( f_a = 2 \) while the flow on all non-visible arcs is \( f_a = 0 \). (B) shows a solution having maximum total supply of six units. On the other hand, (C) and (D) represent optimal solutions for MaxMCF and MinMCF, respectively, and both feature a total supply of four units.

In gas transport, in particular, the network operators sell bookings for individual sources and sinks to traders, which allow them to insert or withdraw gas up to a certain amount at the corresponding nodes. Thereby, the network operator guarantees that any balanced supplies and demands within these bounds can be transported, i.e., the feasibility of the booking. Determining an optimal booking is NP-hard, see Schewe et al. [15]. On the other hand, the complexity of checking whether a booking is feasible or not for different network types and different linear and nonlinear gas flow models is for example discussed by Labbe et al. [10,11].

Furthermore, as we are going to see in Section 5, MaxMCF can be modeled as a linear bilevel optimization problem with interdicting objective functions. Here, the leader chooses a feasible supply and demand vector anticipating the response of the follower, who determines a feasible flow with minimum cost. For a general introduction and more details on bilevel optimization we refer to the books of Bard [3] and Dempe [5]. Network interdiction problems, which are similar to MaxMCF, are discussed by Smith and Lim [16], Smith and Song [17], and Wood [18]. For example, for the Minimum Cost Flow Interdiction Problem the supplies and demands of the sources and sinks are fixed, but the leader has a budget to decrease arc capacities and thereby tries to maximize the optimal objective value of the corresponding MCF problem. However, general linear bilevel optimization is NP-hard, see for example the book of Bard [3] for a proof.

Finally, another class of optimization problems related to MaxMCF are Capacitated or Robust Network Design Problems. Here, capacities have to be installed on the arcs in order to accommodate all scenarios which are for example given by a HOSE-model type demand polytope. The HOSE-model, which was first introduced by Duffield et al. [6] in the context of virtual private networks, is similar to the set of feasible supply and demand vectors that we consider in this paper. We recommend the PhD thesis of Raack [14] for an overview. Furthermore, complexity results regarding this kind of problem and concerning the feasibility of potential solutions can be found in the work of Minoux [12].
4 COMPLEXITY OF MINMCF

First of all, we consider MinMCF, whose goal is to find a supply and demand vector such that the optimal objective value of the induced MCF instance is minimized.

**Theorem 1.** MinMCF can be solved in polynomial time.

In order to prove Theorem 1, we define a corresponding MCF instance $I_{MCF} = (V_{MCF}, A_{MCF}, b)$ for each MinMCF instance $I = (V, A, \tilde{b}, \tilde{b})$. Let

$$B^+_\text{min} := \sum_{v \in V^+} \tilde{b}_v, \quad B^+_\text{max} := \sum_{v \in V^+} \tilde{b}_v, \quad B^-_\text{min} := \sum_{v \in V^-} |\tilde{b}_v|, \quad B^-_\text{max} := \sum_{v \in V^-} |\tilde{b}_v|$$

$$B_{\text{min}} := \min\{B^+_\text{min}, B^-_\text{min}\}, \quad B_{\text{max}} := \min\{B^+_\text{max}, B^-_\text{max}\}.$$

$B_{\text{min}}^+$ denotes the minimum and $B_{\text{max}}^+$ denotes the maximum possible total amount of supply w.r.t. the corresponding bounds. Analogously, $B_{\text{min}}^-$ and $B_{\text{max}}^-$ are the minimum and maximum possible absolute demand. Consequently, $B_{\text{min}}$ and $B_{\text{max}}$ represent the minimum and maximum possible amount of flow to enter and leave the network in any feasible solution.

The vertex set $V_{MCF}$ is equal to $V$ together with four additional vertices, i.e., we define $V_{MCF} := V \cup \{s, s', t', t\}$. Here, $s$ serves as the only source while $t$ represents the only sink, i.e., $V_{MCF}^+ := \{s\}$ and $V_{MCF}^- := \{t\}$. Further, we set $b_s := B_{\text{max}}^+$ and $b_t := -B_{\text{max}}^-$ as corresponding supply and demand value.

Next, we describe the composition of the arc set $A_{MCF}$. First, we add a copy of each $a \in A$ together with the corresponding capacity and length values. These copied arcs we denote by $A^1$ in the following. Additionally, we add an arc $a = (s, u)$ from $s$ towards each vertex corresponding to a source $u \in V^+$ of $I$ having capacity $c_a := \tilde{b}_u$. This set of arcs we denote by $A^2 := \{(s, u) \mid u \in V^+\}$. Analogously, we add an arc $a = (s', u)$ from $s'$ towards each vertex corresponding to a source $u \in V^+$ of $I$ with capacity $c_a := \tilde{b}_u - \tilde{b}_u$. This set of arcs we denote by $A^3 := \{(s', u) \mid u \in V^+\}$. Similarly, we add an arc $a = (w, t)$ from each vertex corresponding to a sink $w \in V^-$ of $I$ towards $t$ with capacity $c_a := |\tilde{b}_w|$. This set of arcs we denote by $A^4 := \{(w, t) \mid w \in V^-\}$. Additionally, we add an arc $a = (w, t)$ from each vertex corresponding to a sink $w \in V^-$ of $I$ towards $t'$ with capacity $c_a := |\tilde{b}_w| - |\tilde{b}_w|$. This arc set we denote by $A^5 := \{(w, t') \mid w \in V^-\}$. Finally, we add three more arcs to the graph. First, an arc $a_1 = (s, s')$ having capacity $c_{a_1} := B_{\text{max}}^+ - B_{\text{min}}^+$. Second, an arc $a_2 = (t', t)$ having a capacity value of $c_{a_2} := B_{\text{max}}^+ - B_{\text{min}}^-$. And third, an arc $a_3 = (s', t')$ with capacity $c_{a_3} := B_{\text{max}}^- - B_{\text{min}}^-$. Finally, for all $a \in A_{MCF} \setminus A^1$ we set $c_{a} := 0$. As an example, the MCF instance $I_{MCF}$ corresponding to the example MinMCF instance $I$ based on Figure 1(A) together with an optimal solution for it are shown in Figure 2.

**Lemma 1.** There exists a one-to-one correspondence between the solutions of MinMCF instance $I$ and the feasible solutions of the corresponding MCF instance $I_{MCF}$ preserving the objective function value.

**Proof.** Let $(b, f)$ be a solution for $I$. Then

$$\tilde{f}_a := \begin{cases} f_a & \text{if } a \in A_1 \\ b_u & \text{if } a = (s, u) \in A_2 \\ b_u - \tilde{b}_u & \text{if } a = (s', u) \in A_3 \\ |\tilde{b}_w| & \text{if } a = (w, t) \in A_4 \\ |\tilde{b}_w| - |\tilde{b}_w| & \text{if } a = (w, t') \in A_5 \\ B_{\text{max}}^+ - B_{\text{min}}^+ & \text{if } a = a_1 \\ B_{\text{max}}^+ - B_{\text{min}}^- & \text{if } a = a_2 \\ B_{\text{max}}^- - \sum_{u \in V} b_u & \text{if } a = a_3 \end{cases}$$

is a feasible solution for MCF instance $I_{MCF}$. The flow values for the arcs in $A^2$ and $A^4$ as well as for $a_1$ and $a_2$ do not depend on $(b, f)$. In fact, these values have to be the same for all feasible solutions of $I_{MCF}$ in order to satisfy the supply and demand of $s$ and $t$, respectively. Further, the flow values of $A_1$ directly correspond to the flow values $f$ and the flow values for the arcs in $A^3$ and $A^5$ depend on the supply and demand of the corresponding source $u \in V^+$ or sink $w \in V^-$, respectively. Similarly, the flow on arc $a_3$ depends on the total supply. By construction, all flow values respect the capacities and flow conservation is ensured. Thus, there are no two pairwise different feasible solutions for $I$ that map towards the same feasible solution for $I_{MCF}$.  


(A) MCF instance $\mathcal{I}_{MCF}$ for the MinMCF instance $\mathcal{I}$ based on Figure 1(A).

(B) An Optimal solution for $\mathcal{I}_{MCF}$ having objective value 8.

**FIGURE 2** (A) shows the Min-Cost-Flow Problem (MCF) instance $\mathcal{I}_{MCF}$ corresponding to the example MinMCF instance $\mathcal{I}$, which is based on the flow network and the supply and demand intervals in Figure 1(A). All newly added arcs that are not shown here have capacity $c_a = 0$. On the other hand, all arcs shown here have capacity $c_a = 2$, except for $(s, s')$, $(s', r)$, and $(r, t)$, which have capacity 4. Further, all newly added and visible arcs are dotted and have length $l_a = 0$. (B) shows an optimal solution for $\mathcal{I}_{MCF}$, which coincides with the example solution from Figure 1(D). The two thick blue arcs $(s, s')$ and $(t', t)$ represent a flow value of $f_a = 4$, while all other blue arcs have $f_a = 2$. Finally, the flow on all nonvisible arcs is $f_a = 0$. [Color figure can be viewed at wileyonlinelibrary.com]

On the other hand, let $\tilde{f}_a$ be a feasible solution for MCF instance $\mathcal{I}_{MCF}$:

$$b_v := \begin{cases} \tilde{f}_{vu} + \tilde{f}_{vu} & \text{for } v = u \in V^+ \\ -\tilde{f}_{wt} - \tilde{f}_{wt} & \text{for } v = w \in V^- \end{cases}$$

and

$$f_a := \tilde{f}_a \text{ for } a \in A.$$ 

denotes a solution for MinMCF instance $\mathcal{I}$. By construction, flow conservation is ensured at the inner nodes, the supplies and demands of all sources and sinks are satisfied, respectively, and the corresponding interval bounds are respected. Additionally, as argued above, only the flow values of the arcs in $A^1, A^2,$ and $A^3$ are not fixed and any change in one of the corresponding variable values leads to a different solution $(b, f)$. Thus, there are no two pairwise different feasible solutions for $\mathcal{I}_{MCF}$ that map towards a common solution for $\mathcal{I}$.

Finally, only the arcs in $A^1$ in $\mathcal{I}_{MCF}$ can have nonzero length. Thus, since the flow values on these arcs are preserved by the bijection given by the two mappings above, the objective value is preserved.

**Corollary 1.** There exists a one-to-one mapping between the optimal solutions of MinMCF instance $\mathcal{I}$ and the optimal solutions of MCF instance $\mathcal{I}_{MCF}$.

**Theorem 1.** MinMCF can be solved in polynomial time.

**Proof.** Creating MCF instance $\mathcal{I}_{MCF}$, which has $|V_{MCF}| = |V| + 4$ vertices and $|A_{MCF}| = | \bigcup_{i=1}^{5} A_i \cup \{a_1, a_2, a_3\}| = |A| + 2|V^+| + 2|V^-| + 3$ arcs, solving it using any polynomial-time algorithm for MCF, and applying the mapping defined in the proof of Lemma 1 yields an optimal solution for $\mathcal{I}$ in polynomial time.

### 5 A BILEVEL MODEL FOR MAXMCF

Before discussing the complexity of MaxMCF, we first state a mathematical programming formulation for it, which is adapted from Hoppmann and Schwarz [9]. In particular, we show that MaxMCF can be modelled as a linear bilevel optimization program
with interdicting objective functions.

\[
\begin{align*}
\max_b & \quad \sum_{a \in \mathcal{A}} \ell_a f_a \\
\text{s.t.} & \quad b_v \in [\bar{b}_v, \overline{b}_v] \\
\min & \quad \sum_{a \in \mathcal{A}} \ell_a f_a \\
\text{s.t.} & \quad \sum_{a \in \delta^+(v)} f_a - \sum_{a \in \delta^-(v)} f_a = b_v \quad \forall v \in \mathcal{V}^+ \cup \mathcal{V}^- \\
& \quad \sum_{a \in \delta^+(v)} f_a - \sum_{a \in \delta^-(v)} f_a = 0 \quad \forall v \in \mathcal{V}^0 \\
& \quad f_a \in [0, c_a] \quad \forall a \in \mathcal{A}.
\end{align*}
\] (5) (6) (1) (2) (3) (4)

For each source or sink \( v \in \mathcal{V}^+ \cup \mathcal{V}^- \) the variable \( b_v \) represents the supply or demand value, respectively. These values are chosen by the leader w.r.t. the corresponding bounds, see (6). Note that the leader has to balance supply and demand, because otherwise the MCF problem, which the follower solves subsequently, does not admit any feasible solution. Thus, the \( b \)-variables form a feasible supply and demand vector for the corresponding MaxMCF instance.

The follower solves the MCF problem induced by the selected \( b \)-variables, see constraints (1)–(4), which were already explained in Section 2. However, while the follower routes the flow such that the cost is minimized, it is the leaders goal to choose a feasible supply and demand vector maximizing it, compare (1) and (5), respectively.

Note that in order to adapt the model above for MinMCF, we change the upper level’s objective sense in (5) from max to min. Since the objective functions of both levels are now identical, the formulation can be transformed into an LP. This is in line with Theorem 1, as LPs can be solved in polynomial time.

\section{Complexity of MaxMCF}

The goal of this section is to prove the following theorem:

\textbf{Theorem 2.} \textit{MaxMCF is NP-hard.}

Actually, we show an even more extensive statement as we are going to consider the special case of MCF in the following, where we drop the capacity restrictions on the arcs, i.e., we replace (4) with

\[ f_a \geq 0 \quad \forall a \in \mathcal{A}. \] (7)

This problem is equivalent to the Transportation Problem (TP) when considering shortest paths and their lengths between all source and sink pairs, see Ahuja et al. [1]. Hence, we define the MaxTP analogous to MaxMCF but without the arc capacity restrictions. Note that MaxTP was first introduced and discussed by Hennig and Schwarz [8]. However, since MaxMCF is a generalization of MaxTP, proving the following statement directly implies Theorem 2.

\textbf{Theorem 3.} \textit{MaxTP is NP-hard.}

We reduce from PARTITION. Its definition is adapted from SP12 in the book of Garey and Johnson [7].

\textbf{Definition 1.} Given a finite set \( Z := \{z_1, \ldots, z_n\} \) and a size \( s(z) \in \mathbb{Z}^+ \) for each \( z \in Z \). Is there a feasible partition of \( Z \), i.e., a set \( Z' \subseteq Z \) such that \( \sum_{z \in Z'} s(z) = \sum_{z \in Z \setminus Z'} s(z) \).

For an instance \( Z \) of PARTITION, we construct a corresponding MaxTP instance \( I_Z = (\mathcal{V}, \mathcal{A}, \overline{b}, \overline{b}) \) as follows. For each \( z \in Z \) we add a source \( u_i \in \mathcal{V}^+ \) and a sink \( w_i \in \mathcal{V}^- \). Further, for each source \( u_i \in \mathcal{V}^+ \) we define \( \overline{b}_{u_i} := 0 \) and \( \overline{b}_{u_i} := s(z_i) \), while for each sink \( w_i \in \mathcal{V}^- \) we set \( \overline{b}_{w_i} = -s(z_i) \) and \( \overline{b}_{w_i} := 0 \). Additionally, we add a single inner node \( v \in \mathcal{V}^0 \).

The arc set \( \mathcal{A} \) consists of three different arc types, i.e., \( \mathcal{A} := \mathcal{A}^1 \cup \mathcal{A}^2 \cup \mathcal{A}^3 \). First, for each source \( u_i \in \mathcal{V}^+ \) an arc towards the corresponding sink \( w_i \in \mathcal{V}^- \) is added, i.e., \( \mathcal{A}^1 := \{(u_i, w_i) \mid i \in \{1, \ldots, n\}\} \), and we define \( \ell_{u_i} := 0 \) for each \( a \in \mathcal{A}^1 \). Further, an arc between each source \( u_i \in \mathcal{V}^+ \) and the inner node \( v \) is added, i.e., \( \mathcal{A}^2 := \{(u_i, v) \mid i \in \{1, \ldots, n\}\} \) and we define \( \ell_{u_i} := 1 \) for each \( a \in \mathcal{A}^2 \). Finally, an arc between the inner node \( v \) and each sink \( w_i \in \mathcal{V}^- \) is added, i.e., \( \mathcal{A}^3 := \{(v, w_i) \mid i \in \{1, \ldots, n\}\} \) and we define \( \ell_{u_i} := 1 \) for each \( a \in \mathcal{A}^3 \). This concludes the construction of instance \( I_Z \), which features \(|\mathcal{V}| = 2|Z| + 1 \) vertices and \(|\mathcal{A}| = 3|Z| \) arcs. Figure 3 shows the MaxTP instance \( I_Z \) corresponding to the example PARTITION instance \( Z \) described in its caption.
are stated above and below the corresponding nodes, respectively.

Next, a supply and demand vector \( b \) for \( I_Z \) with the property that \( b_{u_i} = 0 \) or \( b_{w_i} = 0 \) for all \( i \in \{1, \ldots, n\} \) is called complementary.

**Lemma 2.** For each feasible solution \((b, f)\) of \( I_Z \) there exists a feasible solution \((\tilde{b}, \tilde{f})\) with complementary \( \tilde{b} \) such that \( c(b, f) = c(\tilde{b}, \tilde{f}) \).

**Proof.** For each \( i \in \{1, \ldots, n\} \) let \( \gamma_i := \min\{b_{u_i}, |b_{w_i}|\} \geq 0 \). Since \( f \) is an optimal solution for the TP instance induced by \( b \), we have \( f_{u_iw_i} = \gamma_i \). Thus,

\[
\tilde{b}_v := \begin{cases} 
  b_{u_i} - \gamma_i & \text{if } v = u_i \in V^+ \\
  b_{w_i} + \gamma_i & \text{if } v = w_i \in V^- 
\end{cases}
\]

and

\[
\tilde{f}_a := \begin{cases} 
  f_{u_iw_i} - \gamma_i & \text{if } a = (u_i, w_i) \in A^1 \\
  f_a & \text{otherwise}
\end{cases}
\]

is a feasible solution and \( \tilde{b} \) is complementary. Additionally, since \( \ell_a = 0 \) for all \( a \in A^1 \), we have \( c(b, f) = c(\tilde{b}, \tilde{f}) \).

**Corollary 2.** There exists an optimal solution \((b, f)\) for \( I_Z \) with complementary supply and demand vector \( b \).

**Lemma 3.** There exists a feasible partition \( Z' \) of \( Z \) if and only if there exists a feasible solution \((b, f)\) for \( I_Z \) such that \( c(b, f) \geq \sum_{z \in Z} s(z) \).

**Proof.** Let \( Z' \subseteq Z \) be a feasible partition. In other words, it holds that \( \sum_{z \in Z'} s(z) \geq \sum_{z \in Z \setminus Z'} s(z) \). Consider solution \((b, f)\) defined as

\[
b_v := \begin{cases} 
  s(z) & \text{if } v = u_i \in V^+, z_i \in Z' \\
  - s(z) & \text{if } v = w_i \in V^-, z_i \in Z \setminus Z' \\
  0 & \text{otherwise}
\end{cases}
\]

and

\[
f_a := \begin{cases} 
  s(z) & \text{if } a = (u_i, v) \in A^2, z_i \in Z' \\
  |s(z)| & \text{if } a = (v, w_i) \in A^3, z_i \in Z \setminus Z' \\
  0 & \text{otherwise}
\end{cases}
\]

By construction, \( b \) is a feasible supply and demand vector and complementary, \( f \) represents an optimal solution for the TP instance induced by \( b \), and

\[
c(b, f) = \sum_{(u_i, v) \in A^2} f_{u_i} + \sum_{(v, w_i) \in A^3} f_{w_i} = \sum_{z \in Z'} f_{u_i} + \sum_{z \in Z \setminus Z'} f_{w_i} = \sum_{z \in Z'} s(z) + \sum_{z \in Z \setminus Z'} s(z)
\]

\[
= \sum_{z \in Z} \frac{s(z)}{2} + \sum_{z \in Z} \frac{s(z)}{2} = \sum_{z \in Z} s(z).
\]
Conversely, by Lemma 2 there exists a feasible and complementary solution \((b, f)\) for \(I_Z\) such that \(c(b, f) = \sum_{z \in Z} s(z)\).

Due to the complementarity of \(b\), we have \(f_a = 0\) for all \(a \in A^1\). Let \(Z' := \{z_i \in Z \mid b_i > 0\} \subseteq Z\), then

\[
\sum_{z \in Z} s(z) \leq c(b, f) = \sum_{(u, v) \in A} f_{uv} + \sum_{(v, w) \in A^1} f_{vw} = \sum_{z \in Z'} b_z + \sum_{z \in Z \setminus Z'} |b_z| = \sum_{z \in Z'} s(z) + \sum_{z \in Z \setminus Z'} s(z) = \sum_{z \in Z} s(z).
\]

This shows that \(c(b, f) = \sum_{z \in Z} s(z)\) as well as \(\sum_{z \in Z} b_z = \sum_{z \in Z'} s(z)\) and \(\sum_{z \in Z \setminus Z'} |b_z| = \sum_{z \in Z \setminus Z'} s(z)\) hold with equality. Furthermore, since \(b\) must be balanced, it follows that

\[
\sum_{z \in Z'} b_z = \sum_{z \in Z'} s(z) = \sum_{z \in Z \setminus Z'} s(z) = \sum_{z \in Z \setminus Z'} |b_z|,
\]

showing that \(Z'\) is a feasible partition.

**Theorem 3.** \(\text{MaxTP}\) is NP-hard.

**Proof.** Deciding whether there exists a feasible partition or not is an NP-complete problem, see for example SP12 in Garey and Johnson [7]. Therefore, \(\text{MaxTP}\) is NP-hard, since any polynomial-time algorithm applied to \(I_Z\) deciding whether it admits a feasible solution \((b, f)\) with \(c(b, f) \geq \sum_{z \in Z} s(z)\) or not could be used to decide whether or not \(Z\) contains a feasible partition by Lemma 3.

## 7 MAXMCF WITHOUT CONNECTEDNESS-CONDITION

So far we assumed that there exists an uncapacitated directed path from each source towards each sink in the flow network, see Section 2. This connectedness-condition guarantees the existence of a feasible and therefore also of an optimal solution for all MCF instances induced by feasible supply and demand vectors. However, in this section we investigate the complexity of MaxMCF when this connectedness-condition is dropped, which we denote as MaxMCF-CC in the following. Hence, the goal of MaxMCF-CC is to find a feasible supply and demand vector \(b\) for which the induced MCF instance admits a feasible solution and for which the objective value of an optimal solution is maximized. Note that the bilevel optimization model presented in Section 5 remains valid, since the leader must ensure that the lower level, i.e., the MCF instance, admits a feasible solution by definition. Nevertheless, the described additional necessity to ensure feasibility for the induced MCF instances allows us to prove the following result.

**Theorem 4.** \(\text{MaxMCF-CC}\) is APX-hard.

We reduce from the Maximum Independent Set with Bounded Degree Problem (MIS). Its definition is adapted from GT23 in Ausiello et al. [2]:

**Definition 2.** Let \(H = (W, E)\) be an undirected graph such that the degree of each node is bounded by some constant \(B \geq 3\), i.e., we have \(\Delta(v) \leq B\) for each \(v \in W\), where \(\Delta(v)\) denotes \(v\)'s degree. The goal of the MIS is to find a maximum subset \(W' \subseteq W\) w.r.t. the cardinality such that no two vertices in \(W'\) are joined by an edge.

**Theorem 5.** MIS is APX-complete.

**Proof.** A proof can be found in the papers of Berman and Fujito [4] or Papadimitriou and Yannakakis [13].

Given an undirected graph \(H = (W, E)\) as instance of MIS, we create a corresponding MaxMCF-CC instance \(I_H\) as follows. Note that we assume w.l.o.g. that \(H\) contains no isolated vertices, since these are obviously contained in every maximum independent set.

First of all, for each vertex \(v \in W\) we add a source node \(v^+ \in V^+\) as well as two inner nodes \(v^0, v^1 \in V^0\). Furthermore, for each edge \(e \in E\) we add a sink node \(e^- \in V^-\). For each source \(v^+ \in V^+\) we define \(b_{v^+} := 0\) as the lower supply bound while
the upper supply bound is set to $\overline{b}_v := \Delta(v)$, i.e., the degree of the corresponding vertex $v \in W$. Additionally, for each sink $e^- \in \mathcal{V}^-$ we define $\overline{b}_{e^-} := -1$ and $\overline{b}_{b^-} := 0$.

Next, we add four different types of arcs to $I_H$, i.e., $A := A^1 \cup A^2 \cup A^3 \cup A^4$. First, for each vertex $v \in W$ we add an arc from $v^+ \in \mathcal{V}^+$ to $v^0 \in \mathcal{V}^0$, i.e., $A^1 := \{(v^+, v^0) \mid v \in W\}$. Second, for $v \in W$ we add an arc from $v^0 \in \mathcal{V}^0$ to each sink $e^- \in \mathcal{V}^-$ whose corresponding edge $e \in E$ is incident to $v$, i.e., $A^2 := \{(v^0, e^-) \mid v \in W, e \in \delta(v)\}$. Third, for $v \in W$ we add an arc from $v^+ \in \mathcal{V}^+$ to $v^1 \in \mathcal{V}^1$, i.e., $A^3 := \{(v^+, v^1) \mid v \in W\}$. And fourth, for each $v \in W$ we add an arc from $v^1 \in \mathcal{V}^1$ to each sink $e^- \in \mathcal{V}^-$ whose corresponding edge $e \in E$ is incident to $v$, i.e., $A^4 := \{(v^1, e^-) \mid v \in W, e \in \delta(v)\}$. Additionally, we define $\ell_a := 0$ for each $a \in A^1 \cup A^2 \cup A^4$ and $\ell_a := 1$ for each $a \in A^3$. Finally, we set $c_a := \Delta(v) - 1$ for all $a = (v^+, v^0) \in A^1$, and $c_a := 1$ for all $a \in A^2 \cup A^3 \cup A^4$. This concludes the construction of $I_H$. Note that $I_H$ is of linear size w.r.t. $H$, since the number of nodes is equal to $|\mathcal{V}| = 3|W| + |E|$ and the number of arcs is equal to $|A| = 2|W| + 4|E|$. The MaxMCF-CC instance $I_H$ corresponding to example MIS instance $H$ in Figure 4 can be found in Figure 5.

Next, we introduce some definitions regarding a feasible solution $(b, f)$ for $I_H$. For each source $v^+ \in \mathcal{V}^+$ the flow towards a sink $e^- \in \mathcal{V}^-$, whose corresponding edge $e \in E$ is incident to the corresponding node $v \in W$, can be uniquely partitioned into flows on two paths: Flow on the short path $v^+ \rightarrow v^0 \rightarrow e^-$ with total length zero and flow on the long path $v^+ \rightarrow v^1 \rightarrow e^-$ with total length one, which can directly be read from $f_{v^0 e^-}$ and $f_{v^1 e^-}$, respectively.

Furthermore, we call a solution $(b, f)$ assigning if it is feasible and for each sink $e^- \in \mathcal{V}^-$ with $b_{e^-} < 0$, which corresponds to an edge $e = \{x, y\} \in E$, we have that either $f_{v^0 e^-} + f_{v^1 e^-} > 0$ or $f_{v^0 e^-} + f_{v^1 e^-} > 0$ but not both.

**Lemma 4.** Let $(b, f)$ be a feasible solution for $I_H$. There exists an assigning solution $(\tilde{b}, \tilde{f})$, which can be determined in $O(|E|)$, such that $c(\tilde{b}, \tilde{f}) \geq c(b, f)$.

**Proof.** Let $(b, f)$ be a feasible solution and assume there exists a sink $e^- \in \mathcal{V}^-$ corresponding to $e = \{x, y\} \in E$ such that we have $f_{v^0 e^-} + f_{v^1 e^-} > 0$ and $f_{v^0 e^-} + f_{v^1 e^-} > 0$. W.l.o.g. we assume that $\overline{b}_{e^-} - b_{e^-} \leq \overline{b}_{v^0} - b_{v^0}$. Next, we shift the supply routed from $y^+$ towards $e^-$ over to $x^+$. Therefore, let $\omega := c_{v^0 e^0} - f_{v^0 e^-}$ denote the remaining capacity on the short path from $x^+$ towards $e^-$. We define

$$\tilde{b}_v := \begin{cases} b_{v^+} + f_{v^0} & \text{for } v = x^+ \in \mathcal{V}^+ \\ b_{v^+} - f_{v^0} & \text{for } v = y^+ \in \mathcal{V}^+ \\ b_v & \text{otherwise} \end{cases}$$

and

$$\tilde{f}_a := \begin{cases} f_{v^0 e^-} + \min(\omega, f_{v^0}) & \text{if } a = (x^+, x^0) \\ f_{v^0 e^-} + \min(\omega, f_{v^0}) & \text{if } a = (x^0, e^-) \\ f_{v^0 e^-} + f_{v^0} & \text{if } a = (x^0, e^-) \\ f_{v^0 e^-} + f_{v^1 e^-} & \text{if } a = (y^+, e^-) \\ f_{v^0 e^-} - f_{v^1 e^-} & \text{if } a = (y^+, y^0) \\ 0 & \text{if } a = (y^0, e^-) \\ f_{v^0 e^-} - f_{v^1 e^-} & \text{if } a = (y^+, y^1) \\ 0 & \text{if } a = (y^1, e^-) \\ f_a & \text{otherwise.} \end{cases}$$
Algorithm 1. Shift flow from long to short paths to ensure that $\tilde{f}$ is optimal

\begin{algorithm}
\caption{Shift flow from long to short paths to ensure that $\tilde{f}$ is optimal}
\begin{algorithmic}
\FORALL{$e \in \delta(y)$}
\STATE $r \leftarrow \min\{c_{y^+y^0} - \tilde{f}_{y^+y^0}, \tilde{f}_{y^-y^-}\}$
\STATE $\tilde{f}_{y^+y^0} \leftarrow \tilde{f}_{y^+y^0} + r$
\STATE $\tilde{f}_{y^-v} \leftarrow \tilde{f}_{y^-v} + r$
\STATE $\tilde{f}_{y^+y^-} \leftarrow \tilde{f}_{y^+y^-} - r$
\STATE $\tilde{f}_{y^-v} \leftarrow \tilde{f}_{y^-v} - r$
\ENDFOR
\end{algorithmic}
\end{algorithm}

$x^+$ can only supply sinks, whose corresponding edges are incident to $x \in W$. Therefore, we have

\[
\sum_{v \in \delta(x)} b_{x^+} = f_{x^+x^0} + f_{x^+y^+} = \sum_{v \in \delta(x)} \tilde{f}_{v^+v^-} + \sum_{v \in \delta(x)} f_{x^+v} = \sum_{v \in \delta(x)} (\tilde{f}_{v^+v^-} + f_{x^+v})
\]

\[
\sum_{v \in \delta(x)} (b_{x^+v}) = (\sum_{v \in \delta(x)} f_{v^+v^-} + f_{x^+v}) \leq \Delta(x) - f_{x^-} = \overline{b}_x - f_{x^-}
\]

showing $\overline{b}_x = b_{x^+} + f_{x^-} \leq \overline{b}_x$, which makes $\overline{b}$ a feasible supply and demand vector. Furthermore, while the flows on the short and long path from $y^+$ towards $e^-$ are set to 0, we route up to $\omega$ units from $x^+$ towards $e^-$ on the short path and the remaining supply on the long path.

If there was flow on the short path from $y^+$ towards $e^-$ in $\tilde{f}$ and if there is still flow on some long paths starting at $y^+$ in $\tilde{f}$, we need to shift it onto the short paths in order to make $\tilde{f}$ an optimal solution for the induced MCF. This is done by Algorithm 1.

Finally, it remains to show that $c(\overline{b}, \tilde{f}) \geq c(b, f)$. Recall that we assumed that $\overline{b}_x = b_{x^+} - b_{x^-} \leq b_{x^+} - b_{x^+}$. There are two cases that we have to consider: First, if $\overline{b}_x - b_{x^-} \leq 1$ then $\overline{b}_x - b_{x^+} \leq 1$ and all the shifted supply is routed along the long path from $x^+$ towards $e^-$. Second, if we have $\overline{b}_x - b_{x^-} \geq 1$, all the supply from $y^+$ towards $e^-$ was routed along the short path in $\tilde{f}$. In both cases we do not decrease the objective value and therefore $c(\overline{b}, \tilde{f}) \geq c(b, f)$.

Iteratively applying the procedure above to all sinks $e^-$ results in an assigning solution $(\overline{b}, \tilde{f})$ such that $c(\overline{b}, \tilde{f}) \geq c(b, f)$ in $O(|E|)$. This is because for each sink $e^- \in V^-$ the construction of $(\overline{b}, \tilde{f})$ can be done in constant time. Further, instead of applying Algorithm 1 after each sink $e^-$ individually, it suffices to run it once for each node $y \in W$ after all the supply shifts.

Next, let us call a solution $(b, f)$ bound-tight if it is feasible and if $b_{x^+} = 0$ or $b_{x^-} = \Delta(v)$ holds for all $v^+ \in V^+$.

Lemma 5. Let $(b, f)$ be a feasible solution for $I_H$. There exists a bound-tight solution $(\overline{b}, \tilde{f})$, which can be determined in $O(|V| + |E|)$, with $c(\overline{b}, \tilde{f}) \geq c(b, f)$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{MaxMCF-CC instance $I_H$ corresponding to MIS instance $H$ from Figure 4. For a dashed arc $a = (x^+, y^0) \in A^1$ we have $c_a = \Delta(v) - 1$ and $c_a = 0$, while for a dotted arc $a \in A^1$ we have $c_a = 1$ and $c_a = 1$. Finally, for a solid arc $a \in A^1 \cup A^2$ we have $c_a = 1$ and $c_a = 0$. Obviously, the connectedness-condition described in Section 2 does not hold for this flow network.}
\end{figure}
Proof. Let \((b,f)\) be a feasible solution. Using Lemma 4, we furthermore assume that \((b,f)\) is assigning. We define

\[
\tilde{b}_v := \begin{cases} 
\Delta(x) - 1 & \text{if } a = x^+x^0 \text{ and } b_{x^+} > \Delta(x) - 1 \\
1 & \text{if } a = x^+x^1 \text{ and } b_{x^+} > \Delta(x) - 1 \\
-1 & \text{if } v = e^- \in V^- \text{ and } e = \{x,y\} \text{ if } b_{x^+} > \Delta(x) - 1 \text{ or } b_{y^+} > \Delta(y) - 1 \\
0 & \text{for } v = e^+ \in V^+ \text{ with } b_{x^+} \leq \Delta(x) - 1 \\
0 & \text{for } v = e^- \in V^- \text{ with } e = \{x,y\} \text{ if } b_{x^+} \leq \Delta(x) - 1 \text{ and } b_{y^+} \leq \Delta(y) - 1
\end{cases}
\]

and

\[
\tilde{f}_a := \begin{cases} 
\Delta(x) - 1 & \text{if } a = x^+x^0 \text{ and } b_{x^+} > \Delta(x) - 1 \\
1 & \text{if } a = x^+x^1 \text{ and } b_{x^+} > \Delta(x) - 1 \\
f_a & \text{if } a = x^0e^- \text{ and } b_{x^+} > \Delta(x) - 1 \\
\frac{1}{\Delta(x)} (f_a + (1 - |b_e|)) & \text{if } a = x^1e^- \text{ and } b_{x^+} > \Delta(x) - 1 \\
0 & \text{otherwise}.
\end{cases}
\]

First, the supply of each source \(x^+ \in V^+\) with \(b_{x^+} > \Delta(x) - 1\) is increased up to its upper bound, i.e., \(\tilde{b}_{x^+} := b_{x^+} = \Delta(x)\), and the demand of each sink \(e^-\) with \(e \in \delta(x)\) up to \(-1\), i.e., \(\tilde{b}_{e^-} := -1\). The additional supply of \(1 - |b_e|\) is then routed along the corresponding long path.

Second, the supply of all sources \(x^+ \leq \Delta(x) - 1\), the demands of their assigned sinks, as well as the flows on the corresponding short paths are set to 0. Since \(\tilde{b}_{x^+} \leq \Delta(x) - 1\), there is no flow on any of the corresponding long paths.

In both cases, the objective value does not decrease and we determined a bound-tight solution \((\tilde{b},\tilde{f})\) with the property that \(c(\tilde{b},\tilde{f}) \geq c(b,f)\) in \(\Theta(|V| + |E|)\).

Lemma 6. Let \((b,f)\) be a bound-tight solution for MaxMCF-CC instance \(\mathcal{I}_H\). Then \(W' := \{v \in W : b_v = \tilde{b}_v\}\) is an independent set in \(H\). Furthermore, we have that \(c(b,f) = |W'|\).

Proof. Assume that \(W'\) is not independent. Then there exist \(x,y \in W'\) such that \(b_{x^+} = \tilde{b}_{x^+}, b_{y^+} = \tilde{b}_{y^+}\), and \(\{x,y\} = e \in E\).

Since only sinks whose corresponding edges are contained in \(\delta(x) \cup \delta(y)\) can be supplied by \(x^+\) and \(y^+\) and because \(e \in \delta(x) \cap \delta(y)\), we have

\[
\tilde{b}_{x^+} + \tilde{b}_{y^+} = \sum_{e \in \delta(x) \cup \delta(y)} |b_e| \leq |\delta(x) \cup \delta(y)| \leq \Delta(x) + \Delta(y) - 1 < \tilde{b}_{x^+} + \tilde{b}_{y^+}
\]

which is a contradiction. Thus, \(W'\) is an independent set and

\[
c(b,f) = \sum_{a \in A^1} \ell_{a^+} + \sum_{a \in A^2} \ell_{a^-} + \sum_{a \in A^3} \ell_{a^+} + \sum_{a \in A^4} \ell_{a^-} + \sum_{a \in A^5} \ell_{a^+} = \sum_{a \in A^1} f_{a^+} = \sum_{a \in A^2} f_{a^-} = \sum_{a \in A^3} f_{a^+} + \sum_{a \in A^4} f_{a^-} = 1 = |W'|.
\]

Lemma 7. Let \(W'\) be an independent set in \(H\). Then there exists a bound-tight solution \((b,f)\) for \(\mathcal{I}_H\) with \(|W'| = c(b,f)\).

Proof. Consider \((b,f)\) defined as

\[
b_v := \begin{cases} 
\Delta(x) & \text{if } v = x^+ \in V^+ \text{ and } x \in W' \\
1 & \text{if } v = e^- \in V^- \text{ and } e \in \delta(W') \\
0 & \text{otherwise}
\end{cases}
\]

and

\[
f_a := \begin{cases} 
\Delta(x) & \text{if } a = x^+x^0 \text{ and } x \in W' \\
\frac{1}{\Delta(x)} & \text{if } a = x^1e^- \text{ and } x \in W' \\
\frac{\Delta(x) - 1}{\Delta(x)} & \text{if } a = x^0e^- \text{ and } x \in W' \\
1 & \text{if } a = x^1x^0 \text{ and } x \in W' \\
0 & \text{otherwise}.
\end{cases}
\]

By construction, \((b,f)\) is feasible, bound-tight and we have \(c(b,f) = |W'|\).
Lemma 8. There exists an independent set $W'$ of $H$ with size $|W'|=k$ if and only if there exists a bound-tight solution $(b,f)$ for $I_H$ with $c(b,f)=k$.

Proof. Given a feasible and bound-tight solution $(b,f)$ for $I_H$ with $c(b,f)=k$, the induced independent set $W'$ from Lemma 6 has size $|W'|=k$. Conversely, if there exists an independent set $W'$ of $H$ with size $k$, by Lemma 7 there exists a bound-tight solution $(b,f)$ for $I_H$ with $c(b,f)=k$. □

Corollary 3. The size of a maximum independent set $W'$ of $H$ is $|W'|=k$ if and only if an optimal solution $(b,f)$ for $I_H$ has objective value $c(b,f)=k$.

Using the previous results, we can now prove Theorem 4.

Theorem 4. MaxMCF-CC is APX-hard.

Proof. Suppose there exists a PTAS for MaxMCF-CC yielding a $(1-\epsilon)$-factor approximate solution. Let $I_H=(\mathcal{V},A)$ be the corresponding MaxMCF-CC instance for MIS instance $H=(W,E)$ and let $k$ denote the optimal objective value of $I_H$, which is equal to the size of a maximum independent set in $H$ by Corollary 3. A PTAS would give us a feasible solution $(b,f)$ for $I_H$ with solution value $c(b,f) \geq (1-\epsilon)k$. Using Lemma 4 and Lemma 5, we can determine a bound-tight feasible solution $(\tilde{b},\tilde{f})$ in polynomial time w.r.t. the number of vertices and edges with $c(\tilde{b},\tilde{f}) \geq c(b,f)$. Next, by Lemma 6 we can extract an independent set in $\mathcal{G}(|W|)$, which has size at least $(1-\epsilon)k$. Hence, a $(1-\epsilon)$-factor PTAS for MaxMCF-CC together with the algorithms from Lemma 4, Lemma 5, and Lemma 6 would yield a $(1-\epsilon)$-factor PTAS for MIS. Thus, no PTAS for MaxMCF-CC can exist unless P = NP. □

8 | CONCLUSION AND OUTLOOK

In this paper, we showed that MinMCF can be solved in polynomial time using a purpose-built MCF instance. On the other hand, we proved that MaxMCF and MaxTP are NP-hard using a reduction from PARTITION. Furthermore, if we drop the connectedness-condition and therefore have to additionally ensure the existence of a feasible solution for the induced MCF instance, we are able to show that MaxMCF is an APX-hard problem using a reduction from MIS.

There are two open questions regarding the complexity of MaxMCF and MaxTP that we would like to answer next. First, does there exist a PTAS for MaxTP? And second, can the APX-hardness result be extended to MaxMCF as defined in Section 2, i.e., with the connectedness-condition? Additionally, we are currently developing algorithmic approaches to efficiently solve MaxMCF and MaxTP instances originating in practice, for example, instances from the real-world application in natural gas transport as described by Hoppmann and Schwarz [9].

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DATA AVAILABILITY STATEMENT

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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