OPTIMAL GEVREY REGULARITY FOR CERTAIN SUMS OF SQUARES IN TWO VARIABLES

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ABSTRACT. For $q, a$ integers such that $a \geq 1, 1 < q; (x, y) \in U, U$ a neighborhood of the origin in $\mathbb{R}^2$, we consider the operator

$$D_x^2 + x^{2(q-1)}D_x^a + y^aD_y^2.$$

Slightly modifying the method of proof of [9] we can see that it is Gevrey $s_0$ hypoelliptic, where $s_0^{-1} = 1 - a^{-1}(q - 1)q^{-1}$. Here we show that this value is optimal, i.e. that there are solutions to $Pu = f$ with $f$ more regular than $G^{s_0}$ that are not better than Gevrey $s_0$.

The above operator reduces to the Métivier operator ([24]) when $a = 1, q = 2$. We give a description of the characteristic manifold of the operator and of its relation with the Treves conjecture on the real analytic regularity for sums of squares.

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1. Introduction and Statement of the Result

For $q, a$ integers such that $a \geq 1, 1 < q$, $(x, y) \in \mathbb{R}^2$, we consider the operator

\[ P(x, y, D_x, D_y) = D_x^2 + x^{2(q-1)}D_y^2 + y^{2a}D_y^2. \]

Here $D_x = 1/\sqrt{-1} \partial_x$.

The operator $P$ generalizes the operator $P_M(x, y, D_x, D_y) = D_x^2 + x^{2}D_y^2 + y^{2}D_y^2$, defined by Métivier and for which Métivier, in an article in the Comptes Rendus de l'Académie des Sciences in 1981, stated that it is not real analytic hypoelliptic [24]. Actually in [24] it was stated that $P_M$ is Gevrey 2 hypoelliptic and not better, meaning that it is not Gevrey-$s$ hypoelliptic for any $s, 1 \leq s < 2$.

For the sake of completeness we give the definition of $G^s(\Omega)$—the class of all Gevrey-$s$ functions in the open set $\Omega$:

**Definition 1.1.** If $\Omega \subset \mathbb{R}^n$ is an open set we say that the function $u$ belongs to the Gevrey class of order $s \geq 1$, $G^s(\Omega)$, if $u \in C^\infty(\Omega)$ and for every compact set $K \subset \Omega$ there exists a positive constant $C_K$ such that

\[ \sup_K |\partial^\alpha u(x)| \leq C_K |\alpha|^{s+1}. \]

Note that $G^1(\Omega)$ is the class of the real analytic functions in $\Omega$.

We point out explicitly that the characteristic set of $P_M$ is the non symplectic submanifold of $T^*\mathbb{R}^2 \setminus \{0\}$ given by

\[ \text{Char}(P_M) = \{(x, y; \xi, \eta) \mid x = y = 0, \xi = 0, \eta \neq 0\}. \]

We have

\[ \text{Char}(P) = \{(x, y; \xi, \eta) \mid x = y = 0, \xi = 0, \eta \neq 0\} = \text{Char}(P_M). \]

The problem of the analytic or Gevrey regularity of the operator (1.1) is part of the broader problem of the analytic regularity for sums of squares of vector fields with analytic coefficients. Technically speaking the operator (1.1) is not a sum of squares, but it is easy to see that it shares the same properties with the sum of squares

\[ D_x^2 + (x^{q-1}D_y)^2 + (y^{a}D_y)^2. \]

There has been a fair amount of literature on the problem in the eighties and nineties and we refer to the paper [10] for a survey of the main results.

We focus on the conjecture stated by Treves (see [32], [11] and [33] for its statement) in 1999, characterizing the analytic hypoellipticity of sums of squares.

The Treves conjecture defines a stratification of $\text{Char}(P)$ such that the strata satisfy the following properties (everything is meant to be microlocal near a fixed point in $\text{Char}(P)$)

(i) Real analytic submanifolds of $\text{Char}(P)$.

(ii) The symplectic form $\sigma = d\xi \wedge dx$ restricted to each stratum has constant rank.

(iii) For each stratum there exists an integer $\nu \in \mathbb{N}$ such that all the Poisson brackets of the symbols of the vector fields of length $\leq \nu$ are identically zero, but there is a Poisson bracket of length $\nu + 1$ which is not zero. By
length of a Poisson bracket we mean the number of vector fields of which we take an iterated bracket.

According to the conjecture an operator is analytic hypoelliptic if and only if every stratum of its characteristic variety is a symplectic manifold.

For the operator $P$ the stratification is made up of two half lines, $\Sigma_+ \cup \Sigma_-$, where

$$\Sigma_{\pm} = \{(x, y; \xi, \eta) \mid x = y = 0, \xi = 0, \pm \eta > 0]\.$$ 

$\Sigma_{\pm}$ is an isotropic submanifold of $T^*\mathbb{R}^2 \setminus \{0\}$ and, as such, it coincides with its Hamilton leaf along the half-fiber $\pm \eta > 0$.

Thus the Treves stratification is not symplectic, which suggests that $P$ is not analytic hypoelliptic. We recall that if a manifold is not symplectic there is always a foliation, called the Hamilton foliation, whose leaves can or cannot be transverse to the fibers of the cotangent bundle.

We point out that the operator in (1.1) actually is one of the simplest model operator exhibiting a Hamilton leaf non transverse to the fibers.

By adapting the technique of [9] one can prove that $P$ is Gevrey $s_0$ hypoelliptic, with

$$s_0^{-1} = 1 - \frac{1}{a} \frac{q - 1}{q}.$$ 

In the present paper we want to show that $s_0$ is an optimal index, i.e.

**Theorem 1.1.** The operator $P$ is not Gevrey-$s$ hypoelliptic for any $s$ such that $1 \leq s < s_0$. Hence $P$ is not analytic hypoelliptic.

We point out that the optimality proof in the non transverse case is technically much more involved than that in the transverse case and, to our knowledge, in the non transverse case only M´etivier paper [24] is available.

In the remaining part of this section we want to give a brief exposition about our motivation to study the operator $P$ and set it in a more general context in the framework of the problem of the real analytic regularity of the solutions to sums of squares operators, of which $P$ is a two variable example.

An essential tool in M´etivier’s paper, [24], is the expansion of a solution as a linear combination of the eigenfunctions of the corresponding harmonic oscillator, $D^2_x + x^2$. It is known that the eigenfunctions of the harmonic oscillator are a basis in $L^2(\mathbb{R})$, are rapidly decreasing at infinity and satisfy finite recurrence relations allowing us to express e.g. the derivative, or the multiplication by $x$, of an eigenfunction as a combination of two different eigenfunctions. These recurrence relations are essential in the M´etivier approach.

When the degree of the potential is larger than two, i.e. for the anharmonic oscillator $D^2_x + x^{2(q-1)}$, $q > 2$, Gundersen in [15] proved the

**Theorem 1.2 ([15]).** Consider the equation in $\mathbb{C}$

$$w''(z) + (\lambda - p(z))w(z) = 0, \quad \lambda \in \mathbb{R}, \; z \in \mathbb{C}.$$ 

Here $p$ denotes the polynomial $p(z) = a_{2m}z^{2m} + a_{2m-2}z^{2m-2} + \ldots + a_2z^2$. Assume that $a_i \geq 0$ for every $i$ and $a_{2m} > 0$.

Denote by $(\psi_n(z))_{n \in \mathbb{N}}$ a set of solutions that is a complete orthonormal basis in $L^2(\mathbb{R})$.

Then if $m$ is even and $g$ is a non zero polynomial we have that $g(z) \cdot \psi^{(k)}_n(z)$ is not a finite linear combination of the $\psi_n$, for $k \in \mathbb{N} \cup \{0\}$, $\ell \in \mathbb{N}$ fixed and $\deg(g) \geq 1$ if $k = 0$. 

Thus for $q$ odd there is no finite recurrence relation for the anharmonic oscillator. For an even $q$, Bender and Wang, [2], showed that the eigenfunctions for the operator
\begin{equation}
-u''(x) + x^{2N+2}u(x) = Eu^N(x), \quad N = -1, 0, 1, 2, \ldots,
\end{equation}
do have finite recurrence relations using confluent hypergeometric functions. We refer to Chinni, [12], for a result in this sense.

Our proof of Theorem 1.1, as well as Mézivier’s approach, shares the general pathway of constructing an asymptotic formal solution with all the optimality proofs in the transverse case. This is done first at the formal level and then substantiated by introducing suitable cutoff functions to turn a formal solution into a true solution of an equation with a better right hand side.

We shall show that the Gevrey regularity of $P$ is strictly related to the functions satisfying the equation
\begin{equation}
-u''(x) + x^{2(q-1)}u(x) = \lambda u(x),
\end{equation}
i.e. the eigenfunctions of the operator
\begin{equation}
Q = D^2 + x^{2(q-1)}.
\end{equation}
However we do not require any recurrence relations among the eigenfunctions.

From a more geometric point of view we point out that even though the Treves conjecture has been shown not to hold for $n \geq 4$ (we refer to [1], [7] for a proof and to [6] for a case that might suggest that strata are still the object to be defined) there are no counterexamples when $n = 2, 3$.

We observe that in the proofs of [1], [7] there is a non symplectic submanifold of the characteristic manifold that is not identified by the Treves conjecture stratification procedure and plays a crucial role in carrying the non real analytic wave front set. This submanifold has a foliation on the base of the cotangent bundle, i.e. there is a Gevrey (or analytic) wave front set propagation in the space variables.

When $n = 3$ this may no longer occur due to a dimensional constraint, but we think that e.g. the following example
\begin{equation}
D^2_1 + x_1^{2(r-1)}D^2_y + D^2_2 + x_2^{2(q-1)}D^2_y + x_2^{2(p-1)} y^{2a}D^2_y,
\end{equation}
where $a, r, p, q \in \mathbb{N}$, $1 < r < p < q$, should be a candidate to violate the Treves conjecture in dimension 3, since the characteristic manifold has a foliation whose leaves are the $\eta$ fibers of the cotangent bundle that, in our opinion, should carry analytic wave front set.

To our knowledge no proof is known either of its analytic regularity or of its non-analytic hypoellipticity. We hope that the same techniques we are using for the operator $P$ above can be suitably modified to prove that the above example is not analytic hypoelliptic.

While for $n \geq 3$ there is no conjecture about the stratification to use in order to characterize analytic hypoellipticity, when $n = 2$ the Treves conjecture seems to describe accurately the geometry of the characteristic variety and we think that it might actually be true (see [9] again). Furthermore the operator studied in this paper is a microlocal model for a class of operators in two variables.

In this perspective, proving Theorem 1.1 seems a necessary step to accomplish a proof of the conjecture when $n = 2$. However no proof is available up to now.
Finally a few words about the proof of the theorem. We just sketch the idea of the proof with a simplified notation.

First we construct a formal solution of $P(A(u)) = 0$, of the form

$$A(u)(x, y) = \int_{0}^{+\infty} e^{iy\rho^{s_0}} u(x\rho^{\frac{a}{2}}, \rho) d\rho,$$

where

$$u(x, \rho) = \sum_{j \geq 0} u_j(x, \rho).$$

This is accomplished by observing that if $P(A(u)) = 0$, then $u$ must satisfy

$$\sum_{j=0}^{2a} \frac{1}{\rho^j} P_j(x, D_x, D_\rho) u(x, \rho) = 0,$$

where the $P_j$ are differential operators of order $2a$. This is done in section 2. In section 3 we compute the $u_j$ as solutions to countably many PDEs of the form

$$P_0 u_j = -\sum_{k=1}^{\min(j, 2a)} \frac{1}{\rho^k} P_k u_{j-k},$$

where, essentially, $P_0 = D_x^2 + x^{2(q-1)} + a^{2a}$ on the half plane $\mathbb{R}_x \times ]0, +\infty[$. In order to compute $u_j$ we have to compute an inverse of $P_0$ in the half plane $\rho > 0$. This is accomplished by separating variables, using the eigenfunctions of the anharmonic oscillator $D_x^2 + x^{2(q-1)}$, and solving an ODE in $\rho$ for $\rho > 0$. It can be seen that when solving the above mentioned ODE corresponding to higher eigenstates of the anharmonic oscillator we gain a decreasing rate of the form $O(\rho^{-j}e^{-c\rho})$. When it comes to the fundamental eigenstate the situation is more involved and, to obtain a similar decreasing rate, one has to decouple the equation for $u_j$, using the projector onto the fundamental eigenstate, $\pi$, and its complement $1 - \pi$ (see (3.18a), (3.18b)).

In sections 4, 5, 6, 7 we derive weighted $L^2$-type estimates for $\pi u_j$, $(1 - \pi) u_j$ and hence for $u_j$ (Theorem 4.1). The reason why we resorted to $L^2$, and not $L^\infty$, estimates is that we exploit the orthonormality of the eigenfunctions of the anharmonic oscillator, since there are no finite recurrence relations.

In section 8 the pointwise estimates for the $u_j$ are derived via the Sobolev immersion theorem (Theorem 8.1).

In order to turn the formal solution $A(u)$ of the equation $PA(u) = 0$ into a true solution, in section 9 we replace $u$ with $v$ where

$$v(x, \rho) = \sum_{j \geq 0} \psi_j(\rho) u_j(x, \rho),$$

where $\psi_j$ denotes a suitable cutoff function whose precise definition is given in Lemma 9.1. Then $A(v)$ solves an equation of the form $PA(v) = f$, for a function $f$ in a Beurling class of order $s_0$ (see Definition 9.1 for the definition of such classes). From Theorem 3.1 of M´etivier, [22], arguing by contradiction, if $P$ is Gevrey hypoelliptic of order less than $s_0$, then $A(v)$ belongs to the Beurling class of order $s_0$.

The purpose of section 10 is to obtain a contradiction to the fact that $A(v)$ is in a Beurling class of order $s_0$ by using the structure of $A(v)$ given above. This is done by comparing the growth rate of its $y$-Fourier transform as a member of the Beurling class of order $s_0$ (see Lemma 10.3) as well as a function given by the above
expression (see Lemma 10.4), when $x$ is frozen at the origin. As a technical detail we mention that the use of the Fourier transform forces us to replace the Beurling class of order $s_0$ with $L^2(\mathbb{R}_y)$ intersected with the global Beurling class of order $s_0$ (see definition 9.1). To show that actually $A(v)$ belongs to the latter class we use Lemma 10.2.

Finally we gathered in the Appendixes some results needed in the main body of the paper. Appendixes A and B present some basically known facts and fix the notation in our setting. Appendix C and D present some estimates that are essential for sections 5 and 6. We postponed them in order to allow the reader to better follow the deployment of the proof in those sections.

2. A Formal Solution

First we look for a formal solution to $Pv = 0$ by taking $v$ of the form

$$A(u)(x, y) = \int_{0}^{+\infty} e^{iy\rho s_0} \rho^r u(x\rho^{s_0}, \rho) d\rho,$$

where

$$\frac{1}{s_0} = 1 - \frac{1}{a} \frac{q - 1}{q},$$

$r$ is a complex number to be chosen later and $u$ denotes a smooth function defined in $\mathbb{R} \times \mathbb{R}$, with $\text{supp } u \subset \{ \rho > 0 \}$, and rapidly decreasing for $\rho \to +\infty$.

We have

$$D_x^2 A(u)(x, y) = \int_{0}^{+\infty} e^{iy\rho s_0} \rho^{r+2s_0} (D_x^2 u)(\rho x, \rho) \, d\rho.$$  

(2.3)

$$x^{2(q-1)} D_y^2 A(u)(x, y) = \int_{0}^{+\infty} e^{iy\rho s_0} \rho^{r+2s_0} \left[ x^{2(q-1)} u(x, \rho) \right]_{x=x\rho^{s_0/q}} \, d\rho.$$  

(2.4)

Here the notation $x = x\rho^{s_0/q}$ means that the variable $x$ inside the square brackets has to be replaced by the r.h.s. of the equation.

Finally

$$y^{2a} D_y^2 A(u)(x, y) = \int_{0}^{+\infty} e^{iy\rho s_0} \rho^{r+2s_0} y^{2a} \left[ u(x, \rho) \right]_{x=x\rho^{s_0/q}} \, d\rho.$$  

(2.5)

Since

$$ye^{iy\rho s_0} = \frac{1}{i s_0 \rho^{s_0-1}} \partial_{\rho} e^{iy\rho s_0},$$

integrating by parts we may rewrite the above expression as

$$y^{2a} D_y^2 A(u)(x, y) = \int_{0}^{+\infty} e^{iy\rho s_0} \left( -\partial_{\rho} \frac{1}{i s_0 \rho^{s_0-1}} \right)^{2a} \rho^{r+2s_0(\frac{q-1}{2} + \frac{1}{2})} \left[ u(x, \rho) \right]_{x=x\rho^{s_0/q}} \, d\rho.$$  

(2.6)

Remark that

$$\left( \frac{\partial_{\rho}}{i s_0 \rho^{s_0-1}} \right)^{2a} = \sum_{h=0}^{2a} \gamma_{2a, h} \frac{1}{\rho^{2s_0-h}} \partial_{\rho}^h, $$
where the $\gamma_{2a,h}$ are constants satisfying
\begin{equation}
|\gamma_{2a,h}| \leq C^2 a^h (2a - h)!,
\end{equation}

$$\gamma_{2a,2a} = \left( \frac{i}{s_0} \right)^{2a}, \quad \gamma_{2a,0} = 1.$$  

Hence, writing for the sake of simplicity, $\gamma_h$ instead of $\gamma_{2a,h}$, we obtain
\begin{align*}
y^{2a} D^2_y A(u)(x, y) &= \int_0^{+\infty} e^{iyx} \sum_{h=0}^{2a} \sum_{h=0}^{2a} h \gamma_h \rho^{-2a s_0 - h} \partial^h \rho^{-2a s_0 + h} \left( \frac{q - 1}{q} \right) \left[ u(x, \rho) \right]_{x=x \rho^{s_0}/q} \rho^{h} e^{iyx} \sum_{h=0}^{2a} \sum_{h=0}^{2a} h \gamma_h \left( \rho^{-2a s_0 + h} + 2 \frac{q - 1}{q} \right) \left[ u(x, \rho) \right]_{x=x \rho^{s_0}/q} \rho^{h},
\end{align*}

where $(\lambda)_\beta$ denotes the Pochhammer symbol, defined by
\begin{equation}
(\lambda)_\beta = \lambda(\lambda - 1) \cdots (\lambda - \beta + 1), \quad (\lambda)_0 = 1, \quad \lambda \in \mathbb{C}.
\end{equation}

Moreover, since
\begin{equation}
a(1 - s_0) + s_0 \frac{q - 1}{q} = a \left( 1 - s_0 \left( 1 - \frac{1}{a} \frac{q - 1}{q} \right) \right) = 0,
\end{equation}

we obtain that
\begin{align*}
y^{2a} D^2_y A(u)(x, y) &= \int_0^{+\infty} e^{iyx} \sum_{h=0}^{2a} \sum_{h=0}^{2a} h \gamma_h \rho^{-2a s_0 + h} \left( \frac{q - 1}{q} \right) \left[ u(x, \rho) \right]_{x=x \rho^{s_0}/q} \rho^{h},
\end{align*}

Since
\begin{equation}
\partial^\alpha \rho v(\rho^x, \rho) = \left[ \left( \frac{\partial}{\rho^x} \partial_{x} + \partial_\rho \right)^\alpha v(x, \rho) \right]_{x=x \rho^{s_0}/q},
\end{equation}

we have
\begin{align*}
y^{2a} D^2_y A(u)(x, y) &= \sum_{h=0}^{2a} \sum_{h=0}^{2a} h \gamma_h \left( \rho^{-2a s_0 + h} + 2 \frac{q - 1}{q} \right) \left[ u(x, \rho) \right]_{x=x \rho^{s_0}/q} \rho^{h},
\end{align*}

Because of the identity
\begin{equation}
\frac{a}{\rho} + \partial_\rho = \rho^{-a} \partial^a \rho^a = \sum_{k=0}^{a} \binom{a}{k} (\alpha)_k \rho^{-k} \partial^a \rho^{-k},
\end{equation}

we deduce that
\begin{equation}
\begin{align*}
y^{2a} D^2_y A(u)(x, y) &= \sum_{h=0}^{2a} \sum_{h=0}^{2a} h \gamma_h \left( \rho^{-2a s_0 + h} + 2 \frac{q - 1}{q} \right) \left[ u(x, \rho) \right]_{x=x \rho^{s_0}/q} \rho^{h},
\end{align*}
\end{equation}
\[ \int_0^{+\infty} e^{iy\rho^\alpha} \rho^{r+2a\alpha+2a} \left[ \sum_{k=0}^{m} \binom{\alpha}{k} \left( \frac{s_0}{q} x \partial_x \right)^k \rho^{-k} \partial_\rho^{\alpha-k} u(x, \rho) \right] d\rho, \]

where, in analogy with (2.8), we used the notation
\[ \left( \frac{s_0}{q} x \partial_x \right)_k = \frac{s_0}{q} x \partial_x \left( \frac{s_0}{q} x \partial_x - 1 \right) \cdots \left( \frac{s_0}{q} x \partial_x - k + 1 \right). \]

An inspection of (2.10) readily gives that the differential operator above is a polynomial of degree \( k \) in \( x \partial_x \) with uniformly bounded coefficients. We therefore may write it in the form
\[ \left( \frac{s_0}{q} x \partial_x \right)_k = p_k(x \partial_x) = \sum_{j=1}^{k} b_{k,j}(x \partial_x)^j, \]

where
\[ b_{k,k} = \left( \frac{s_0}{q} \right)^k, \quad b_{k,0} = \frac{s_0}{q} (-1)^{k-1} (k-1)! \]

Hence
\[ y^{2\alpha} D_x^2 A(u)(x, y) = \sum_{h=0}^{2a} \sum_{\alpha=0}^{h} \binom{\alpha}{k} \gamma_h \left( r + 2 \frac{s_0}{q} + 2s_0 q - 1 \right) \frac{q}{q} h - \alpha \]
\[ \cdot \int_0^{+\infty} e^{iy\rho^\alpha} \rho^{r+2a\alpha+2a} \left[ \sum_{k=0}^{m} \binom{\alpha}{k} p_k(x \partial_x) \rho^{-k} \partial_\rho^{\alpha-k} u(x, \rho) \right] d\rho, \]

where \( p_0(x \partial_x) = 1 \) by convention.

From (2.3), (2.4) and the above we may then write
\[ PA(u)(x, y) = \int_0^{+\infty} e^{iy\rho^\alpha} \rho^{r+2a\alpha} \left[ D_x^2 + x^2(q-1) \right] \]
\[ + \sum_{h=0}^{2a} \sum_{\alpha=0}^{h} \sum_{k=0}^{2a} \binom{\alpha}{k} \gamma_h \left( r + 2 \frac{s_0}{q} + 2s_0 q - 1 \right) \frac{q}{q} h - \alpha \left( \frac{\alpha}{2a-j} \right) \]
\[ p_{\alpha+j-2a}(x \partial_x) \rho^{-j} \partial_\rho^{\alpha-j} u(x, \rho) \mid_{x=x^p^{\alpha/q}} d\rho. \]

Let us consider the sums on the r.h.s. of the above expression. First we call \( \ell = \alpha - k \) and then rewrite the sums setting \( \ell = 2a - j \). We obtain
\[ PA(u)(x, y) = \int_0^{+\infty} e^{iy\rho^\alpha} \rho^{r+2a\alpha} \left[ D_x^2 + x^2(q-1) \right] \]
\[ + \sum_{h=0}^{2a} \sum_{\alpha=0}^{h} \sum_{j=2a-\alpha}^{2a} \binom{h}{\alpha} \gamma_h \left( r + 2 \frac{s_0}{q} + 2s_0 q - 1 \right) \frac{q}{q} h - \alpha \left( \frac{\alpha}{2a-j} \right) \]
\[ p_{\alpha+j-2a}(x \partial_x) \rho^{-j} \partial_\rho^{\alpha-j} u(x, \rho) \mid_{x=x^p^{\alpha/q}} d\rho. \]

We may then interchange the sums in \( \alpha \) and \( j \) and after that interchange the sums in \( h \) and \( j \) to get
\[ PA(u)(x, y) = \int_0^{+\infty} e^{iy\rho^\alpha} \rho^{r+2a\alpha} \left[ D_x^2 + x^2(q-1) \right] \]
+ \sum_{j=0}^{2a} \rho^{-j} \left( \sum_{h=2a-j}^{2a} \sum_{\alpha=2a-j}^{h} \binom{h}{\alpha} \gamma_{h} \left( r + \frac{2s_{0}}{q} + 2s_{0} \frac{q-1}{q} \right)_{h-\alpha} \right)
\left( \binom{\alpha}{2a-j} p_{\alpha+j-2a}(x\partial_{x}) \partial_{\rho}^{2a-j} u(x, \rho) \right)
\bigg|_{x=x\rho^{\alpha/q}} d\rho.

Denote by $P_{0}(x, D_{x}, D_{\rho})$ the differential operator corresponding to $j = 0$:

$$P_{0}(x, D_{x}, D_{\rho}) = D_{x}^{2} + x^{2(q-1)} + s_{0}^{-2a} D_{\rho}^{2a},$$
the differential operator induced by $P$—modulo some normalization—on the $\eta$ fibers. Furthermore set, for $j \geq 1$,

$$P_{j}(x, \partial_{x}, \partial_{\rho}) = \sum_{h=2a-j}^{2a} \sum_{\alpha=2a-j}^{h} \binom{h}{\alpha} \gamma_{h} \left( r + \frac{2s_{0}}{q} + 2s_{0} \frac{q-1}{q} \right)_{h-\alpha} \cdot \binom{\alpha}{2a-j} p_{\alpha+j-2a}(x\partial_{x}) \partial_{\rho}^{2a-j}$$

$$= \tilde{P}_{j}(x\partial_{x}) \partial_{\rho}^{2a-j},$$

with $\text{ord}(\tilde{P}_{j}) = j$. We also have, see also (2.6),

$$P_{1}(x, \partial_{x}, \partial_{\rho}) = \left( 2a\gamma_{2a} \left( p_{l}(x\partial_{x}) + r + \frac{2s_{0}}{q} + 2s_{0} \frac{q-1}{q} \right) + \gamma_{2a-1} \right) \partial_{\rho}^{2a-1}.$$

To keep the notation simple we write

$$\tilde{P}_{j}(x\partial_{x}) = \sum_{\ell=0}^{j} p_{j\ell}(x\partial_{x})^{\ell},$$

for $j = 1, \ldots, 2a$ and suitable numbers $p_{j\ell}$.

Equation (2.13) can thus be written as

$$PA(u)(x, y) = \int_{0}^{+\infty} e^{iy\rho^{\alpha/q}} \rho^{r+2a+2} \left[ \sum_{j=0}^{2a} \frac{1}{\rho^{j}} P_{j}(x, D_{x}, D_{\rho}) u(x, \rho) \right]_{x=x\rho^{\alpha/q}} d\rho.$$

**Remark 2.1.** The whole computation has been carried out for the model in (1.1). We might have allowed variable coefficients in (1.1), at least to a certain extent, which would have resulted in an infinite sum in (2.18). The rest of the proof should proceed essentially with minor modifications. We however preferred to stick to the proposed model for a greater clarity in the exposition, with a very tiny degree of generality lost.

Our next step is to formally solve the equation

$$\sum_{j=0}^{2a} \frac{1}{\rho^{j}} P_{j}(x, D_{x}, D_{\rho}) u(x, \rho) = 0.$$
3. Computing the Formal Solution

We start this section by discussing how to solve the prototype equation

\[(3.1) \quad P_0(x, D_x, D_\rho)u(x, \rho) = f(x, \rho),\]

where \(P_0\) is given by (2.14).

As a preliminary result we consider the kernel of the operator

\[(3.2) \quad Q_\mu(x, D) = D^2 + x^{2(q-1)} - \mu,\]

\(\mu\) being a real parameter.

The following proposition is well known (see e.g. [3].)

**Proposition 3.1.** There exist countably many positive numbers, \(\mu_j, \mu_{j+1} > \mu_j, j \geq 0\), such that

\[
\ker Q_{\mu_j} \neq \{0\},
\]

and actually \(\dim \ker Q_{\mu_j} = 1\). Here when we write \(\ker Q_{\mu_j}\) we mean the kernel of the operator \(Q_{\mu_j}\) as an unbounded operator in \(L^2(\mathbb{R})\).

Let us denote by \(\mu_j, \varphi_j(x), j \geq 0\), the eigenvalues and the eigenfunctions of the operator \(Q\), defined in (1.4), constructed in the above proposition.

They are an orthonormal basis in \(L^2(\mathbb{R})\).

Consider now the equation

\[
\left(D^2 + x^{2(q-1)} + s_0^{-2a} D_\rho^{2a}\right)u = f,
\]

with \(f \in L^2\). We want to find an expression for \(u \in L^2\) intersected with the natural domain of \(P_0\). We note explicitly that the operator \(P_0\) has no tempered distributions in its kernel if we consider it in the \((x, \rho)\)-plane. We are considering it in a half plane, and hence we can find non trivial distributions in its kernel.

Write

\[
u(x, \rho) = \sum_{k \geq 0} u_{(k)}(\rho) \varphi_k(x),
\]

where \(u_{(k)}(\rho) = \langle u, \varphi_k \rangle\). Here \(\langle \cdot, \cdot \rangle\) denotes the (complex) scalar product in \(L^2(\mathbb{R})\), in the only variable \(x\).

Hence, with an analogous expansion of \(f\),

\[
P_0(x, D_x, D_\rho)u = \sum_{k \geq 0} \left(u_{(k)} Q \varphi_k + \varphi_k s_0^{-2a} D_\rho^{2a} u_{(k)}\right) = \sum_{k \geq 0} \varphi_k \left(u_{(k)} \mu_k + s_0^{-2a} D_\rho^{2a} u_{(k)}\right) = \sum_{k \geq 0} f_k \varphi_k,
\]

where \(f_k = \langle f, \varphi_k \rangle\).

Identifying the coefficients of \(\varphi_k\) in the last equality above we may find the \(u_{(k)}\) as the solution of the differential equations

\[(3.3) \quad \left(\partial_\rho^{2a} + (-1)^a s_0^{2a} \mu_k\right) u_{(k)} = (-1)^a s_0^{2a} f_k.
\]

Let us denote again by \(f_k\) the r.h.s. of (3.3), the difference being just a multiplicative constant.

For \(j = 1, \ldots, 2a\), denote by \(\mu_{kj}\) the \(2a\)-roots of \((-1)^{a+1}s_0^{2a} \mu_k\).

Assume now that \(k \geq 1\).
Applying the result of Appendix A, we find that

\[ u(k)(\rho) = \sum_{j=1}^{2a} A_{kj} I_{kj}(f_k)(\rho), \]

where the \( A_{kj} \) are defined as in (A.4) and

\[ I_{kj}(f_k)(\rho) = -\text{sign}(\text{Re} \mu_{kj}) \int_{\mathbb{R}} e^{\mu_{kj}(\rho - \sigma)} H(-\text{sign}(\text{Re} \mu_{kj})(\rho - \sigma)) \cdot H(\sigma - R)f_k(\sigma)d\sigma, \]

where \( H \) denotes the Heaviside function, \( R \) is a positive number that can be chosen depending on some parameter to be precised. We note that (3.5) defines a function solving (3.3) for \( \rho > R \).

Consider the case \( k = 0 \).

We define \( u(0) \) in a slightly different way. The motivation for such a distinction will be clear in the subsequent sections.

Consider \( \mu_0 \) the smallest of the values \( \mu_0 < \mu_1 < \cdots \) defined in Proposition 3.1 and denote by \( \mu_{0i}, i = 1, \ldots, 2a \), the \( 2a \)-roots of

\[ (-1)^{a+1}s_0^a \mu_0. \]

Then define \( \tilde{\mu}_0 \) by

\[ \tilde{\mu}_0 = \text{Re} \mu_{0i*}, \]

where \( \mu_{0i*} \) is a \( 2a \)-root of \((-1)^{a+1}s_0^a \mu_0 \) with maximum negative real part.

We remark explicitly that, if \( a > 1 \), there are always two (complex conjugate) roots, \( \mu_{0i1}, \mu_{0i2} \) of maximal negative real part, since we are taking even roots. Of course the definition of \( \tilde{\mu}_0 \) is independent of the choice of the root.

The reason of the above choice for \( \tilde{\mu}_0 \), which will have important implications in the sequel, is that we have better decay rates for the components along higher eigenfunctions than that of the fundamental eigenfunction.

Then we define

\[ u(0)(\rho) = -\sum_{i\in\{1,\ldots,2a\}} A_{0i} \int_{\rho}^{+\infty} e^{\mu_{0i}(\rho - \sigma)} H(\sigma - R)f_0(\sigma)d\sigma \]

\[ + \sum_{i\in\{1,\ldots,2a\}} A_{0i} \int_{R}^{\rho} e^{\mu_{0i}(\rho - \sigma)} H(\sigma - R)f_0(\sigma)d\sigma. \]

We point out that if \( f_0(\sigma) = O(\sigma^{-1-\delta}e^{\tilde{\mu}_0\sigma}) \), with \( \delta > 0 \), the integrals where \( \text{Re} \mu_{0i} \neq \tilde{\mu}_0 \) are well defined.

It is convenient to harmonize the notation for (3.4), (3.5) and (3.7). To this end we point out that, for \( k > 0 \), \( \text{Re} \mu_{ki} > 0 \) implies that \( \text{Re} \mu_{ki} - \tilde{\mu}_0 > 0 \), and that if \( \text{Re} \mu_{ki} < 0 \) then \( \text{Re} \mu_{ki} - \tilde{\mu}_0 < \text{Re} \mu_{0i} - \tilde{\mu}_0 \leq 0 \), due to the fact that the sequence \( (\mu_k)_{k\in\mathbb{N}\cup\{0\}} \) is strictly increasing. Since these inequalities for \( k > 0 \) can...
be perturbed it is clear that there exists a small positive number, say \( \varepsilon_\mu \), such that for any \( k \geq 0 \) we may write that

\[
u_{(k)}(\rho) = \sum_{j=1}^{2a} A_{kj} I_{kj}(f_k)(\rho),
\]

with

\[
I_{kj}(f_k)(\rho) = - \text{sign} \left( \text{Re} \mu_{kj} - \tilde{\mu}_0 + \varepsilon_\mu \right) \int \text{Re} \mu_{kj} (\rho - \sigma) \cdot H(-\text{sign} \left( \text{Re} \mu_{kj} - \tilde{\mu}_0 + \varepsilon_\mu \right) (\rho - \sigma)) \cdot H(\sigma - R) f_k(\sigma) d\sigma,
\]

where the \( A_{0j} \) are chosen according to the above prescription.

Thus we obtain an expression for \( u \):

\[
u(x, \rho) = \sum_{k \geq 0} \left( \sum_{j=1}^{2a} A_{kj} I_{kj}(f_k)(\rho) \right) \varphi_k(x)
\]

\[
= \sum_{k \geq 0} E_k(f_k)(\rho) \varphi_k(x),
\]

where for the sake of brevity we used the notation

\[
E_k(f_k)(\rho) = \sum_{j=1}^{2a} A_{kj} I_{kj}(f_k)(\rho).
\]

In order to find a formal solution to

\[
P(x, y, D_x, D_y) A(u) = 0,
\]

we look for a function \( u \) of the form

\[
u(x, \rho) = \sum_{j \geq 0} u_j(x, \rho).
\]

such that, when we plug it into (2.19), it gives

\[
\sum_{k=0}^{2a} \frac{1}{\rho^k} P_k(x, D_x, D_\rho) u(x, \rho) = \sum_{k=0}^{2a} \frac{1}{\rho^k} P_k(x, D_x, D_\rho) \sum_{j \geq 0} u_j(x, \rho) = 0.
\]

It seems natural to split (3.12) according to

\[
P_0 u_j = - \sum_{k=1}^{\min\{j, 2a\}} \frac{1}{\rho^k} P_{j-k} u_j = 0, \quad j \geq 0.
\]

(The sum is understood to be zero if its upper index is zero).

Consider the first equation to be solved:

\[
P_0(x, D, D_\rho) u_0(x, \rho) = 0.
\]

Arguing as we just did, we may take, see (3.6),

\[
u_0(x, \rho) = \varphi_0(x)e^{\mu_{0^*} \rho}.
\]
We would like to compute the \( u_j, j \geq 1 \), in such a way that (3.13) is satisfied and that \( u_j = \mathcal{O}(\rho^{-j}e^{\bar{\mu}_0\rho}) \) (actually, for technical reasons, we prove in Theorem 8.1 a slightly weaker estimate).

Solving (3.13), using the spectral decomposition \( \varphi_k(x) \), by (3.3), boils down to solving an ode of the form
\[
\left( \partial^2_{\rho} + (-1)^a s_0^{-2a} \mu_k \right) w = g_k,
\]
where \( g_k = \mathcal{O}(\rho^{-j}e^{\bar{\mu}_0\rho}) \) and \( \bar{\mu}_0 \) is defined in (3.6). If \( k \geq 1 \) it is easy to see that \( w = \mathcal{O}(\rho^{-j}e^{\bar{\mu}_0\rho}) \), since \( \bar{\mu}_0 \) differs from the real part of the roots of the characteristic equation.

On the other hand if \( k = 0 \) this is not true anymore and we get that \( w = \mathcal{O}(\rho^{-j+1}e^{\bar{\mu}_0\rho}) \), since \( \bar{\mu}_0 \) is the real part of two roots of the characteristic equation. This prevents us from proving that \( u_j = \mathcal{O}(\rho^{-j}e^{\bar{\mu}_0\rho}) \).

A way around this is to reshuffle the transport equations using the projection onto the ground state function defined as

**Definition 3.1.** Denote by \( \pi \) the orthogonal projection onto
\[
L^2(\mathbb{R}_\rho) \otimes [\varphi_0],
\]
whose action is described by
\[
\pi(f)(x, \rho) = \langle f(\cdot, \rho), \varphi_0 \rangle_{L^2(\mathbb{R}_\rho)} \varphi_0(x).
\]

**Proposition 3.2.** We have that
\[
(1 - \pi) P_0 = P_0 (1 - \pi), \quad \pi P_0 = P_0 \pi,
\]
where \( P_0 \) is given by (2.14).

**Proof.** Let us prove the first relation. The second has the same proof. Compute first the l.h.s.
\[
(1 - \pi) P_0 v = (1 - \pi) P_0 \sum_{k \geq 0} v_k \varphi_k
\]
\[
= (1 - \pi) \sum_{k \geq 0} (v_k Q \varphi_k + s_0^{-2a} D_{\rho}^{2a} v_k \varphi_k)
\]
\[
= (1 - \pi) \sum_{k \geq 0} (\mu_k v_k + s_0^{-2a} D_{\rho}^{2a} v_k) \varphi_k
\]
\[
= \sum_{k \geq 1} (\mu_k v_k + s_0^{-2a} D_{\rho}^{2a} v_k) \varphi_k.
\]

On the other hand
\[
P_0 (1 - \pi) v = P_0 (1 - \pi) \sum_{k \geq 0} v_k \varphi_k
\]
\[
= P_0 \sum_{k \geq 1} v_k \varphi_k
\]
\[
= \sum_{k \geq 1} (v_k Q \varphi_k + \varphi_k \mu_k \varphi_k)
\]
\[
= \sum_{k \geq 1} (\mu_k v_k + s_0^{-2a} D_{\rho}^{2a} v_k) \varphi_k.
\]

This ends the proof. \( \square \)
Next we are going to choose $r$ in the definition (2.1) of $A(u)$. This is necessary in order to bootstrap a formal recursive calculation of all the $u_j$.

**Proposition 3.3.** It is possible to choose $r$ in (2.1) so that

\[(3.17) \quad \pi P_1 u_0 = 0.\]

**Proof.** It is just a computation. By (2.16) the above condition can be written as

\[
\pi P_1 u_0 = \pi \left( [ax \partial_x + \beta + 2a\gamma_2 r] \partial_{\rho}^{2a-1} e^{\mu_0, \rho} \varphi_0(x) \right) = (\partial_{\rho}^{2a-1} e^{\mu_0, \rho}) \left( [ax \partial_x \varphi_0, \varphi_0] + \beta + 2a\gamma_2 r \right) \varphi_0.
\]

Here we used the fact that a function of $\rho$ only commutes with the projectors and that $\| \varphi_0 \| = 1$.

Finally it is clear that the quantity in square brackets can be made zero by suitably choosing $r$, since $\gamma_2 \neq 0$ by (2.16), (2.7). $\square$

In order to get the optimal decreasing rate $u_j = O(\rho^{-j} e^{\mu_0, \rho})$, we split the equation (3.12) into two sets of equations using the projection $\pi$.

First we define $u_0$ by (3.7) and then for $j \geq 1$ we solve recursively the equations

\[
(1 - \pi) P_0 u_j = P_0 (1 - \pi) u_j = - \sum_{k=1}^{\min\{j, 2a\}} \frac{1}{\rho^k} (1 - \pi) \pi P_k u_{j-k}
\]

\[
\pi P_0 u_j = P_0 \pi u_j = - \frac{1}{\rho} \pi P_1 (1 - \pi) u_j - \frac{1}{\rho} \pi P_1 \pi u_{j-1} - \sum_{k=1}^{\min\{j, 2a-1\}} \frac{1}{\rho^{k+1}} \pi P_{k+1} u_{j-k}.
\]

We point out that the idea of the above splitting of the set of the transport equations is due to Métilvier, who first used it in [24]. The motivation for the splitting is that one has to match the decreasing rate of the $u_j$ for $\rho$ large.

Moreover the decay rate for large $\rho$ of the $(1 - \pi) u_j$ will turn out to be slightly better than that of the $\pi u_j$, due to the choice of $\tilde{\mu}_0$ in (3.6).

**Proposition 3.4.** The set of equations (3.18a), (3.18b) for $j \geq 1$ as well as the equation $P_0 u_0 = 0$ are formally equivalent to (3.12).

**Proof.** We have, for $j \geq 1$,

\[
P_0 u_j = P_0 (1 - \pi) u_j + P_0 \pi u_j
\]

\[
= - \sum_{\ell=1}^{\min\{j, 2a\}} \frac{1}{\rho^\ell} (1 - \pi) P_\ell u_{j-\ell} - \frac{1}{\rho} \pi P_1 (1 - \pi) u_j
\]

\[
- \frac{1}{\rho} \pi P_1 \pi u_{j-1} - \sum_{\ell=1}^{\min\{j, 2a-1\}} \frac{1}{\rho^{\ell+1}} \pi P_{\ell+1} u_{j-\ell}
\]

Since

\[
\sum_{k=0}^{2a} \frac{1}{\rho^k} P_k \sum_{j \geq 0} u_j = 0
\]
iff
\[ \sum_{j \geq 1} P_0 u_j + \sum_{j > 0, k \geq 1} \frac{1}{\rho^k} P_k u_j = 0, \]
from (3.19) we get
\[ \sum_{j \geq 1} P_0 u_j = -\sum_{j \geq 1} \min\{j, 2a\} \sum_{\ell=1}^{\min\{j, 2a-1\}} \frac{1}{\rho^{\ell+1}} \pi P_{\ell+1} u_{j-\ell}. \]

The third term on the r.h.s. above is written as
\[ \frac{1}{\rho} \sum_{j \geq 1} \pi P_1 \pi u_{j-1} = \frac{1}{\rho} \pi P_1 u_0 + \frac{1}{\rho} \sum_{j \geq 2} \pi P_1 \pi u_{j-1} \]
\[ = \frac{1}{\rho} \pi P_1 u_0 + \frac{1}{\rho} \sum_{j \geq 1} \pi P_1 \pi u_j. \]

Hence
\[ \sum_{j \geq 1} P_0 u_j = -\frac{1}{\rho} (1 - \pi) P_1 u_0 - \sum_{j \geq 1} \sum_{\ell=1}^{\min\{j, 2a\}} \frac{1}{\rho^{\ell+1}} (1 - \pi) P_{\ell+1} u_{j-\ell} \]
\[ -\frac{1}{\rho} \sum_{j \geq 1} \pi P_1 (1 - \pi) u_j - \frac{1}{\rho} \pi P_1 u_0 \]
\[ -\frac{1}{\rho} \sum_{j \geq 1} \pi P_1 \pi u_j - \sum_{j \geq 1} \sum_{\ell=1}^{\min\{j, 2a-1\}} \frac{1}{\rho^{\ell+1}} \pi P_{\ell+1} u_{j-\ell} \]
\[ = -\frac{1}{\rho} \pi P_1 u_0 - \frac{1}{\rho} \pi P_1 u_j - \sum_{j \geq 2} \sum_{\ell=1}^{\min\{j, 2a-1\}} \frac{1}{\rho^{\ell+1}} (1 - \pi) P_{\ell+1} u_{j-\ell} \]
\[ -\sum_{j \geq 1} \sum_{\ell=1}^{\min\{j, 2a-1\}} \frac{1}{\rho^{\ell+1}} \pi P_{\ell+1} u_{j-\ell}. \]

The third term in that last expression above when \( \ell = 1 \) is
\[ \sum_{j \geq 2} \frac{1}{\rho} (1 - \pi) P_1 u_{j-1} = \frac{1}{\rho} \sum_{j \geq 1} (1 - \pi) P_1 u_j, \]
so that
\[ \sum_{j \geq 1} P_0 u_j = -\frac{1}{\rho} P_1 u_0 - \frac{1}{\rho} \sum_{j \geq 1} \pi P_1 u_j - \frac{1}{\rho} \sum_{j \geq 1} (1 - \pi) P_1 u_j \]
\[ -\sum_{j \geq 2} \sum_{\ell=2}^{\min\{j, 2a\}} \frac{1}{\rho^{\ell+1}} (1 - \pi) P_{\ell+1} u_{j-\ell} \]
\[ -\sum_{j \geq 1} \sum_{\ell=1}^{\min\{j, 2a-1\}} \frac{1}{\rho^{\ell+1}} \pi P_{\ell+1} u_{j-\ell}. \]
Then

\[
\sum_{j \geq 1} P_0 u_j = -\frac{1}{\rho} P_1 u_0 - \frac{1}{\rho} \sum_{j \geq 1} P_1 u_j - \sum_{\ell=1}^{\min\{j,2a-1\}} \frac{1}{\rho^{\ell+1}} \pi P_{\ell+1} u_{j-\ell}
\]

\[-\sum_{j \geq 1} \sum_{\ell=1}^{\min\{j+1,2a\}-1} \frac{1}{\rho^{\ell+1}} (1-\pi) P_{\ell+1} u_{j-\ell}.\]

And finally, since \(\min\{j+1,2a\}-1 = \min\{j,2a-1\}\),

\[
\sum_{j \geq 1} P_0 u_j = -\frac{1}{\rho} \sum_{j \geq 0} P_1 u_j - \sum_{\ell=1}^{\min\{j,2a-1\}} \frac{1}{\rho^{\ell+1}} P_{\ell+1} u_{j-\ell}
\]

\[
= -\frac{1}{\rho} P_1 u_0 - \sum_{j \geq 1} \sum_{\ell=0}^{\min\{j,2a-1\}} \frac{1}{\rho^{\ell+1}} P_{\ell+1} u_{j-\ell}
\]

\[= -\sum_{j \geq 0} \sum_{\ell=0}^{\min\{j,2a-1\}} \frac{1}{\rho^{\ell+1}} P_{\ell+1} u_{j-\ell}.\]

Since \(P_0 u_0 = 0\) we have

\[
\sum_{j \geq 0} P_0 u_j + \sum_{j \geq 0} \sum_{\ell=0}^{\min\{j+1,2a\}-1} \frac{1}{\rho^{\ell+1}} P_{\ell+1} u_{j-\ell} = 0.
\]

Changing the indices from \((j,\ell)\) to \((t,\ell)\), \(\ell \geq 0\), \(j - \ell = t\), we conclude

\[
\sum_{\ell \geq 0} \sum_{\ell=0}^{2a} \frac{1}{\rho^{\ell+1}} P_\ell u_t = 0.
\]

\[\square\]

The equations in (3.18a) and (3.18b) can be solved, in principle. We must however make sure that the so obtained solutions have estimates allowing us to turn our formal solution into a real one.

Since the degeneration of the operator coefficients is higher than quadratic, there is no possibility of using the three terms recurrence relations valid for the quadratic case or in some special case, like in [24] and [12]. This leads us to getting estimates directly by inspecting the form of the \(u_j\).

This technique, although more involved than the classical one, seems however promising for more general (and generic) cases in two variables. This will be the object of the next section.

4. Sobolev Type Estimates of the \(u_j\)

In this section we deduce estimates of the functions \(u_j\) solving (3.18a), (3.18b) in the region \(\rho > \text{Const} \, j\).

As it can be seen from (3.9) the functions \(u_j\), or rather their projections, are computed as infinite expansions in the eigenfunctions of an anharmonic oscillator. It is then natural to use the \(L^2\) norms of the \(u_j\) and of their derivatives, to take advantage of the orthonormal system of the \(\varphi_k\).

We point out that in Métivier’s case each \(u_j\) is given by a finite expansion in the eigenfunctions and furthermore their derivatives are a finite linear combination
of the same eigenfunctions and allow to get the \( L^\infty \) estimates of the \( u_j \) in a direct way.

In the present setting the pointwise estimates will be deduced using the Sobolev embedding theorems in Section 8.

First we define the weight function (see (3.6))
\[
w_j(\rho) = e^{[|\tilde{\mu}_0| \rho (j-\delta)\kappa],}
\]
where
\[
0 < \delta < 1, \quad \frac{1}{s_0} < \kappa < 1, \quad \kappa\delta > \frac{1}{2}.
\]
We point out that the role of \( \delta, \kappa \) is exquisitely technical and due to the use of \( L^2 \) norms, i.e. making certain integrals in Lemmas 6.4, 6.5 absolutely convergent.

We define the Sobolev spaces \( B^k(\mathbb{R}_x) \) as the space of all \( L^2 \) functions such that
\[
\|u\|_k = \max_{\alpha + \beta \leq k} \|x^\beta D^\alpha x u\|_0.
\]
For the estimates we shall use the norms
\[
\|w_j(\rho)f(x, \rho)\|_{k,A}^2 = \int_A^{+\infty} w_j^2(\rho)\|f(\cdot, \rho)\|_k^2 d\rho,
\]
where \( A \) is a suitable positive constant to be chosen later.

We want to prove the

**Theorem 4.1.** There exist positive constants \( C_u, R_0 \) such that
\[
\|w_j(\rho)\partial^\gamma \rho^\nu Q^s u_j(x, \rho)\|_{0,A} \leq C_1 + j + \sigma + \gamma + \frac{j}{s_0} + \frac{\nu}{\kappa} + \frac{\gamma}{\alpha} + \frac{1}{q},
\]
where
\[
\gamma \leq j + \gamma^# \quad \text{and} \quad \kappa = A(j) = R_0(j+1),
\]
and \( R_0 > 0, \gamma^# \) a positive constant.

Here we understand that for a positive number \( x, x! = \Gamma(x+1) \).

Using the Sobolev immersion theorem, from Theorem 4.1 we deduce the desired pointwise estimates 8.3.

We are going to prove a slightly different statement. Actually in the theorem below both space derivatives and multiplication by \( x \) have been replaced by the corresponding powers of the operator \( Q \). This has the advantage that the action of \( Q \) onto an eigenfunction expansion keeps the orthonormality of the basis, while this is not true for both derivatives and multiplication by \( x \).

The transition from \( Q \) to the derivatives and multiplication by \( x \) is done in Proposition C.1.

**Theorem 4.2.** Let \( \nu \) denote a rational number \( \geq -1 \) and \( \gamma \in \mathbb{N} \) such that \( \nu + \gamma(2a)^{-1} \geq 0 \). There exist positive constants \( C_u, R_0, \sigma, \sigma' \) such that
\[
\|w_j(\rho)\partial^\nu Q^s u_j(x, \rho)\|_{0,A} \leq C_0^{1 + \sigma + \sigma'(\nu + \gamma)} \lambda(j, \nu, \gamma) + 1)^{\lambda(j, \nu, \gamma)},
\]
where
\[
\lambda(j, \nu, \gamma) = \frac{j}{s_0} + (2\nu + \frac{\gamma}{a}) \frac{q - 1}{q},
\]
and $A = A(j) = R_0(j + 1)$, $R_0 > 0$, $\gamma \leq j + \gamma^\#$.

**Remark 4.1.** We point out that the constants $R_0$, $C_0$ may be enlarged independently from each other.

We shall prove in the following that Theorem 4.2 actually implies Theorem 4.1. First we prove Theorem 4.2.

The proof proceeds by induction on $j$. Consider first $u_0$ defined in (3.7). We have for $A = A(0, \gamma)$,

\[
\|w_0(\rho)\partial^\gamma_\rho Q^\nu u_0(x, \rho)\|_{0,A}^2 = \int_0^{\infty} \frac{\rho^{-2\delta} e^{2|\bar{\mu}| \rho}}{\rho^2} |\partial^\gamma_\rho e^{i\mu_0 \cdot \rho}|^2 \|Q^\nu \varphi_0(x)\|^2_0 d\rho
\leq C^\gamma\|Q^\nu \varphi_0(x)\|_0^2 \leq C^\gamma \mu_0^0 \leq C^\gamma_0 \lambda(0, \nu, \gamma)^{\lambda(0, \nu, \gamma)}.
\]

Assume now that (4.7) holds for any $k$, $0 \leq k < j$. We recall that

\[
u_j = (1 - \pi)u_j + \pi u_j,
\]

where the summands in the r.h.s. above are defined by (3.18a) and (3.18b) respectively.

We have

\[
w_j(\rho)\partial^\gamma_\rho Q^\nu u_j(x, \rho) \leq \|w_j(\rho)\partial^\gamma_\rho Q^\nu (1 - \pi)u_j(x, \rho)\|_{0,A} + \|w_j(\rho)\partial^\gamma_\rho Q^\nu \pi u_j(x, \rho)\|_{0,A}.
\]

We are going to prove estimates for each of the two summands on the right hand side of (4.8).

5. **Estimate of $(1 - \pi)u_j$**

This section is devoted to proving the estimates (4.7) for $(1 - \pi)u_j$. Actually we shall prove a slightly better kind of inequality only for $(1 - \pi)u_j$. This improvement will be crucial in inductively proving the estimates (4.7) for $\pi u_j$.

**Theorem 5.1.** Let $\nu$ denote a rational number $\geq -1$ and $\gamma \in \mathbb{N}$ such that $\nu + \gamma(2\nu)^{-1} \geq 0$. There exist positive constants $C_\nu$, $R_0$, $\sigma$, $\sigma'$ such that

\[
w_j(\rho)\rho^{1-\kappa} \partial^\gamma_\rho Q^\nu (1 - \pi)u_j(x, \rho) \leq C_0^{\lambda(0, \nu, \gamma)} (\lambda(j, \nu, \gamma) + 1)\lambda(j, \nu, \gamma),
\]

where

\[
\lambda(j, \nu, \gamma) = \frac{j}{s_0} + \left(2\nu + \frac{\gamma}{\nu} \right) \frac{q - 1}{q},
\]

and $A = A(j) = R_0(j + 1)$, $R_0 > 0$, $\gamma \leq j + \gamma^\#$.

**Proof.** We have, by (3.18a),

\[
(1 - \pi)u_j = - \sum_{k=1}^{\min(j, 2\nu)} P_0^{-1} \left( \frac{1}{\rho^j} (1 - \pi) P_k u_{j-k} \right),
\]

where $P_0^{-1}$ has been defined in (3.19). Hence

\[
\|w_j(\rho)\rho^{1-\kappa} \partial^\gamma_\rho Q^\nu (1 - \pi)u_j(x, \rho)\|_{0,A}
\]
The proof of the second relation is straightforward.

Lemma 5.1. We have
\[ Q'' P_0^{-1}(f) = P_0^{-1}(Q'' f), \quad Q''(1 - \pi) = (1 - \pi)Q''. \]

Proof of Lemma 5.1. From (3.9) we get
\[ Q'' P_0^{-1}(f) = Q'' \sum_{k \geq 0} \left( \sum_{j=1}^{2a} A_{kj} I_{kj}(f_k)(\rho) \right) \varphi_k(x) \]
\[ = \sum_{k \geq 0} \left( \sum_{j=1}^{2a} A_{kj} I_{kj}(f_k)(\rho) \right) \mu_k^r \varphi_k(x) \]
\[ = \sum_{k \geq 0} \left( \sum_{j=1}^{2a} A_{kj} I_{kj}(\mu_k^r f_k)(\rho) \right) \varphi_k(x) \]
\[ = \sum_{k \geq 0} \left( \sum_{j=1}^{2a} A_{kj} I_{kj}((Q'' f)_k)(\rho) \right) \varphi_k(x). \]

The proof of the second relation is straightforward. \( \square \)

Using the above lemma we get
\[ \|w_j(\rho)\rho^{1-x} \partial_\rho^2 Q''(1 - \pi)u_j(x, \rho)\|_{0,A} \\leq \sum_{k=1}^{\min\{j,2a\}} \|w_j(\rho)\rho^{1-x} \partial_\rho^2 P_0^{-1} \left( \frac{1}{\rho^k}(1 - \pi)P_k u_{j-k} \right)\|_{0,A}. \]

Lemma 5.2. Let \( f \in \mathcal{S}(\mathbb{R}) \) and \( \gamma \in \mathbb{N} \). Write \( \gamma = 2ar + \gamma_1 \), with \( 0 \leq \gamma_1 < 2a \). Then for \( k \geq 0 \) we have
\[ \left( \frac{1}{s_0} D_\rho \right)^\gamma E_k(f_k)(\rho) = \sum_{s=0}^{r-1} (-\mu_k)^s \left( \frac{1}{s_0} D_\rho \right)^{\gamma - 2a(1+s)} f_k \]
\[ + (-\mu_k)^r \left( \frac{1}{s_0} D_\rho \right)^{\gamma_1} E_k(f_k)(\rho), \]
where \( E_k(f_k) \) is defined in (3.10) and \( f_k = \langle f, \varphi_k \rangle \), \( \varphi_k \) denoting the \( k \)-th eigenfunction of \( Q \).

The proof is just a computation using the fact that \( E_k(f_k) \) is a solution of (3.3) rapidly decreasing at infinity.

Corollary 5.1. Let \( f \in \mathcal{S}(\mathbb{R}) \) and \( \gamma \in \mathbb{N} \). Write \( \gamma = 2ar + \gamma_1 \), with \( 0 \leq \gamma_1 < 2a \). Then
\[ \left( \frac{1}{s_0} D_\rho \right)^\gamma P_0^{-1}(f) = \sum_{s=0}^{r-1} (-1)^s \left( \frac{1}{s_0} D_\rho \right)^{\gamma - 2a(1+s)} \]
\[ + (-1)^r \left( \frac{1}{s_0} D_\rho \right)^{\gamma_1} P_0^{-1}(Q'' f). \]
Using Corollary 6.1 inequality (5.2) becomes

\[
\|w_j(\rho)\rho^{1-\kappa}D_\rho^{\gamma_1}P_0^{\gamma}(1-\pi)u_j(x,\rho)\|_{0,A} \\
\leq \sum_{k=1}^{\min\{j,2a\}} \sum_{s=0}^{r-1} \|w_j(\rho)\rho^{1-\kappa}D_\rho^{\gamma_1-2a(1+s)}\left(\frac{1}{\rho^k}(1-\pi)Q^{\nu+s}P_ku_j\right)\|_{0,A} \\
+ \sum_{k=1}^{\min\{j,2a\}} \|w_j(\rho)\rho^{1-\kappa}D_\rho^{\gamma_1}P_0^{\gamma}(1-\pi)Q^{\nu+r}P_ku_j\|_{0,A},
\]

where we used the notation

\[
D_\rho = \frac{1}{s_0}D_\rho = \frac{1}{i s_0} \partial_\rho.
\]

First consider the norm in the third line of (5.5). We denote by \(g_{jk} = Q^{\nu+r}P_ku_j\). Then for \(k \in \{1, \ldots, \min\{j, 2a\}\},

\[
\|w_j(\rho)\rho^{1-\kappa}D_\rho^{\gamma_1}P_0^{\gamma}(1-\pi)g_{jk}\|_{0,A} = \|w_j(\rho)\rho^{1-\kappa}D_\rho^{\gamma_1} \sum_{i=1}^{2a} A_{\ell_i} I_{\ell_i}(\rho^{-k}(g_{jk}, \varphi_\ell))\|_{0,A}
\]

\[
= \left(\sum_{i=1}^{2a} \|w_j(\rho)\rho^{1-\kappa}D_\rho^{\gamma_1} \sum_{i=1}^{2a} A_{\ell_i} I_{\ell_i}(\rho^{-k}(g_{jk}, \varphi_\ell))\|_{A}^2\right)^{\frac{1}{2}},
\]

where

\[
\|f(\rho)\|^2_A = \int_A^\infty |f(\rho)|^2 d\rho.
\]

Remark that, by (3.5), (4.10), since \(0 \leq \gamma_1 < 2a,

\[
(5.7) \quad D_\rho^{\gamma_1} \sum_{i=1}^{2a} A_{\ell_i} I_{\ell_i}(f_k) = \frac{1}{(i s_0)^{\gamma_1}} \sum_{i=1}^{2a} A_{\ell_i} \mu_{\ell_i}^{\gamma_1} I_{\ell_i}(f_k).
\]

Hence

\[
\|w_j(\rho)\rho^{1-\kappa}D_\rho^{\gamma_1}P_0^{\gamma}(1-\pi)g_{jk}\|_{0,A} = \left(\sum_{i=1}^{2a} \|w_j(\rho)\rho^{1-\kappa} \sum_{i=1}^{2a} A_{\ell_i} I_{\ell_i}(\rho^{-k}(g_{jk}, \varphi_\ell))\|_{A}^2\right)^{\frac{1}{2}}
\]

\[
= s_0^{-\gamma_1} \left(\sum_{i=1}^{2a} \|w_j(\rho)\rho^{1-\kappa} \sum_{i=1}^{2a} A_{\ell_i} \mu_{\ell_i}^{\gamma_1} I_{\ell_i}(\rho^{-k}(g_{jk}, \varphi_\ell))\|_{A}^2\right)^{\frac{1}{2}}
\]

Now

\[
A_{\ell_i} = \prod_{r=1}^{2a} \frac{1}{\mu_{\ell_i} - \mu_{r}}.
\]

Since the roots \(\mu_{\ell_i}\) are the vertices of a \(2a\)-regular polygon whose circumscribed circle has radius \(\mu_{\ell_i}^{\frac{1}{1+1-2a}}\) (arithmetic root), we have that

\[
\frac{|A_{\ell_i}\mu_{\ell_i}^{\gamma_1}|}{\mu_{\ell_i}^{\frac{1}{1+1-2a}}} = c,
\]
Remark 5.2. In (5.10) side of the inequality because the right hand side may contain both \( C \) for a suitable constant \( C \) independent of \( \ell, i, \gamma_1 \). Thus

\[
\|w_j(\rho)\rho^{1-\kappa}D^\gamma P_0^{-1} \left( \frac{1}{\rho^k}(1-\pi)g_{jk} \right) \|_{0,A} \\
\leq C_1 \sum_{\ell \geq 1} \left( \sum_{i=1}^{2\alpha} \|w_j(\rho)\rho^{1-\kappa}I_{\ell i}(\rho^{-k}(g_{jk}, \mu_{\ell}^{\frac{2\alpha-1}{2\alpha}} \varphi_{\ell i}))\|_{0,A}^2 \right)^{\frac{1}{2}}
\]

Lemma 5.3. Assume that \( \ell \geq 1 \), \( \text{Re} \mu_{\ell i} > 0 \). Then

\[
\|w_j(\rho)\rho^{1-\kappa}I_{\ell i}(\rho^{-k}f(\rho))\|_{0,A}^2 \leq C_0 \|w_{j-k}(\rho)\mu_{\ell}^{-\frac{k}{\kappa}}f(\rho)\|_{0,A}^2,
\]

for a suitable constant \( C_0 > 0 \) independent of \( \ell, j, k, i \).

Remark 5.2. In (5.9) the factor \( \rho^{1-\kappa} \) appears only in the norms on the left hand side of the inequality because the right hand side may contain both \((1-\pi)u_{j-k}\) and \(\pi u_{j-k}\).

Proof. By (3.3), (3.6), (4.1) we have

\[
\|w_j(\rho)\rho^{1-\kappa}I_{\ell i}(\rho^{-k}f(\rho))\|_{0,A}^2 = \int_0^{\infty} \rho^{2(\delta-\kappa) + 1} \rho^{2|\tilde{\mu}_{\ell i}||\rho|} \int_\mathbb{R} e^{\mu_{\ell i}(\rho - \sigma)} \cdot H(\text{sign}(\text{Re} \mu_{\ell i})(\rho - \sigma)) \frac{1}{|\sigma|^k} H(\sigma - R)f(\sigma) d\sigma d\rho
\]

where, since \( \sigma \geq \rho \), we bounded \( \rho \) by \( \sigma \). Moreover for the same reason \( \sigma \geq A \) so that we may replace \( H(\sigma - R) \) by \( H(\sigma - A) \).

We estimate the integral on the right hand side of the above expression with an integral over the whole real line and apply Young inequality to obtain

\[
\|w_j(\rho)\rho^{1-\kappa}I_{\ell i}(\rho^{-k}f(\rho))\|_{0,A}^2 \leq \int_{-\infty}^{\infty} \left( \int_\mathbb{R} e^{(\text{Re} \mu_{\ell i} + |\tilde{\mu}_{\ell i}|)(\rho - \sigma)} H(\text{sign}(\text{Re} \mu_{\ell i})(\rho - \sigma))
\cdot w_{j-k}(\sigma)H(\sigma - A)\sigma^{-(k-1)(1-\kappa)}|f(\sigma)| d\sigma d\rho \right)^2
\]

where\( \sigma \geq \rho \), we bounded \( \rho \) by \( \sigma \). Moreover for the same reason \( \sigma \geq A \) so that we may replace \( H(\sigma - R) \) by \( H(\sigma - A) \).

This concludes the proof of the lemma.

Lemma 5.4. Assume that \( \ell \geq 1 \), \( \text{Re} \mu_{\ell i} < 0 \). Then

\[
\|w_j(\rho)\rho^{1-\kappa}I_{\ell i}(\rho^{-k}f(\rho))\|_{0,A}^2 \leq C_0 \|w_{j-k}(\rho)\mu_{\ell}^{-\frac{k}{\kappa}}f(\rho)\|_{0,A}^2,
\]

for a suitable constant \( C_0 > 0 \) independent of \( \ell, j, k, i \), and \( R = A(j) = R_0(j + 1) \).
Proof. As before by (3.3), (3.6), (4.1) we have
\[
\|w_j(\rho)\rho^{1-x}I_{\ell_1}(\rho^{-k} f(\rho))\|^2_A = \int_A^{+\infty} \rho^2(j-\delta)x+2(1-x) \leq 2|\tilde{\mu}_0|\rho \int_\mathbb{R} e^{\mu_1(\rho-\sigma)} \\
\cdot H(-\text{sign}(\text{Re} \mu_\ell)(\rho-\sigma)) \frac{1}{\sigma^x} H(\sigma-A)f(\sigma) d\sigma \bigg) d\rho \\
\leq \int_A^{+\infty} \left( \rho^{(j-\delta)x+1-x} \int_\mathbb{R} e^{(\text{Re} \mu_\ell+|\tilde{\mu}_0|)(\rho-\sigma)} \\
\cdot H(-\text{sign}(\text{Re} \mu_\ell)(\rho-\sigma)) e^{[\tilde{\mu}_0] \sigma^{-k} H(\sigma-A) f(\sigma) |d\sigma|} \right)^2 d\rho \\
\cdot \left( 1 + \frac{\rho - \sigma}{\sigma} \right)^{(j-\delta)x+1-x} e^{[\tilde{\mu}_0] \sigma^{(j-k-\delta)x} H(\sigma-A)\sigma^{-k(1-x)} f(\sigma) |d\sigma|} \bigg) d\rho
\]
Now, since \((j-\delta)x + 1 - x \leq j,\)
\[
\left( 1 + \frac{\rho - \sigma}{\sigma} \right)^{(j-\delta)x+1-x} \leq \left( 1 + \frac{\rho - \sigma}{\sigma} \right)^j,
\]
keeping in mind that \(A \leq \sigma \leq \rho,\) we have
\[
\|w_j(\rho)\rho^{1-x}I_{\ell_1}(\rho^{-k} f(\rho))\|^2_A \leq \int_A^{+\infty} \left( \int_\mathbb{R} e^{(\text{Re} \mu_\ell+|\tilde{\mu}_0|)H(-\text{sign}(\text{Re} \mu_\ell)(\rho-\sigma))} \right) \\
\cdot \left( \sum_{r=0}^{j} \left( \begin{array}{c} j \\ r \end{array} \right) \frac{(\rho - \sigma)^r}{\sigma^r} \right) e^{[\tilde{\mu}_0] \sigma^{(j-k-\delta)x} H(\sigma-A)\sigma^{-k(1-x)} f(\sigma) |d\sigma|} \bigg) d\rho \\
= \int_A^{+\infty} \left( \sum_{r=0}^{j} \left( \begin{array}{c} j \\ r \end{array} \right) \int_\mathbb{R} e^{(\text{Re} \mu_\ell+|\tilde{\mu}_0|)(\rho-\sigma)} \right) \\
\cdot \left( \frac{\rho - \sigma}{\sigma} \right)^r e^{[\tilde{\mu}_0] \sigma^{(j-k-\delta)x} H(\sigma-A)\sigma^{-k(1-x)} f(\sigma) |d\sigma|} \bigg) d\rho.
\]
Using the inequality
\[
(5.11) \quad \left( \sum_{i=1}^{n} a_i \right)^2 \leq \sum_{i=1}^{n-1} 2^i a_i^2 + 2^{n-1} a_n^2 \leq \sum_{i=1}^{n} 2^i a_i^2,
\]
we get
\[
\|w_j(\rho)\rho^{1-x}I_{\ell_1}(\rho^{-k} f(\rho))\|^2_A \leq \sum_{r=0}^{j} 2^{r+1} \left( \begin{array}{c} j \\ r \end{array} \right)^2 \int_A^{+\infty} \left( \int_\mathbb{R} e^{(\text{Re} \mu_\ell+|\tilde{\mu}_0|)(\rho-\sigma)} \right) \\
\cdot \left( \frac{\rho - \sigma}{\sigma} \right)^r e^{[\tilde{\mu}_0] \sigma^{(j-k-\delta)x} H(\sigma-A)\sigma^{-k(1-x)} f(\sigma) |d\sigma|} \bigg) d\rho.
\]
Now \(\sigma^{-r-k(1-x)} \leq R_0^{-r-j-r}\) and
\[
\|e^{(\text{Re} \mu_\ell+|\tilde{\mu}_0|)\rho^r} \|_{L^1(\mathbb{R}^+)} = \frac{r!}{|\text{Re} \mu_\ell + |\tilde{\mu}_0||^{r+1}}.
\]
Hence, using Young inequality, we obtain
\[
\|w_j(\rho)^{1-\nu}I_{\ell_1}(\rho^{-k}f(\rho))\|_{A}^2 \\
\leq C_5 \sum_{r=0}^{j} 2^{r+1} \frac{1}{|\Re \mu_{\ell} + |\mu_0|^{2(r+1)}} \left( \frac{j}{r} \right)^{r!} |R_0^{j-r} \left( w_j-k(\rho)f(\rho) \right)_{A}^2.
\]

Remark that, for \( \nu > 0 \),
\[
\left( \frac{j}{r} \right)^{r!} \frac{r!}{R_0^{j-r}} = \frac{1}{R_0^r} \left( 1 - \frac{1}{j} \right) \cdots \left( 1 - \frac{r-1}{j} \right) \leq \frac{1}{R_0^r}
\]
and that
\[
\frac{1}{|\Re \mu_{\ell} + |\mu_0|^{r+1}} \leq \left( \frac{C_4}{\Re \mu_{\ell}} \right)^{r+1} \leq C_5^{r+1} \mu_{\ell}^{-\frac{r+1}{2\alpha}},
\]
we obtain the conclusion of the lemma recalling (3.6), provided
\[
R_0 \geq \max\{4C_5\mu_{\ell}^{-\frac{1}{2\alpha}}, 2\}.
\]

Using the above lemmas (5.12) can be bound as
\[
\|w_j(\rho)^{1-\nu}D_\rho^{\gamma_1} P_{0}^{-1} \left( \frac{1}{\rho^{\alpha}}(1 - \pi)g_{jk} \right) \|_{0,A}
\leq 2aC_1 C_0 \left( \sum_{j \geq 1} \|w_j-k(\rho)\|_{A}^{2\nu} \mu_{\ell}^{-1} (g_{jk}, \varphi_{\ell})_{A}^2 \right)^{1/2},
\]
where \( g_{jk} = Q^{\nu+r} P_k u_{j-k} \).

We have \( \mu_{\ell}^{-\frac{1}{2\alpha}} (g_{jk}, \varphi_{\ell}) = (Q^{\nu+r} \varphi_{\ell} - P_k u_{j-k} - \varphi_{\ell}) \), see Lemma 5.11.

Since the \( \varphi_{\ell} \) are an orthonormal basis we obtain
\[
\|w_j(\rho)^{1-\nu}D_\rho^{\gamma_2} P_{0}^{-1} \left( \frac{1}{\rho^{\alpha}}(1 - \pi)Q^{\nu+r} P_k u_{j-k} \right) \|_{0,A}
\leq C_6 \|w_j-k(\rho)Q^{\nu+r} P_k u_{j-k}\|_{0,A},
\]
As a consequence the last term on the r.h.s. of (5.5) is estimated by
\[
A_r = C_6 \sum_{k=1}^{\min\{j,2a\}} \|w_j-k(\rho)Q^{\nu+r} P_k u_{j-k}\|_{0,A}.
\]
Recalling (2.15), (2.17) we write
\[
A_r \leq C_6 \sum_{k=1}^{\min\{j,2a\}} \sum_{m=0}^{k} \sum_{\ell=1}^{p_{km}} \|w_j-k(\rho)\partial_{\rho}^{2a-k} Q^{\nu+r} P_k u_{j-k}\|_{0,A}.
\]
Applying Proposition 3.3 and keeping in mind that the \( \rho \)-derivative is conserved, we have to estimate
\[
A_r \leq C_7 \sum_{k=1}^{\min\{j,2a\}} \sum_{m=0}^{k} \left( \|w_j-k(\rho)\partial_{\rho}^{2a-k} Q^{\nu+r} P_k u_{j-k}\|_{0,A}
+ \left( \nu + \frac{\gamma}{2a} - 1 \right) \|w_j-k(\rho)\partial_{\rho}^{2a-k} u_j-k\|_{0,A} \right)
\]
integration we have norms over the half line \( \gamma \leq \gamma \) for \( 0 < \gamma \leq \gamma \) for \( 0 < \gamma \leq \gamma \). As for the constant \( \gamma \), we estimated the coefficients of the polynomials \( \tilde{P}_k \), \( p_{km} \), with a uniform constant and used the estimate

\[
\|Q^{(m-k)}\|_{\infty} \leq C \|v\|_0,
\]

for \( 0 \leq m < k \leq 2a \).

We also remark that, since \( A = A(j) = R_0(j + 1) \), enlarging the domain of integration we have norms over the half line \( A(j - k), + \infty \), to which we may apply our inductive hypothesis. In fact the exponent of \( Q \) is evidently \( \geq -1 \) and

\[
\nu + \frac{\gamma}{2a} - 1 + k \left( \frac{q}{2(q-1)} - \frac{2a-k}{2} \right) \geq k \left( \frac{q}{2} - 1 \right) s_0 > 0.
\]

Thus, choosing \( \gamma \# \geq 2a \),

\[
\|w_{j-k}(\rho)\rho^{2a-k}Q^{\nu + \frac{\gamma}{2a} - 1 + k \left( \frac{q}{2(q-1)} - \frac{2a-k}{2} \right)} u_{j-k}\|_0,A
\]

\[
\leq C_0^{1+\sigma(j-k)+\sigma'(\nu + \frac{\gamma}{2a} - 1 + k \left( \frac{q}{2(q-1)} - \frac{2a-k}{2} \right) + 2a-k)} (\lambda + 1)^{\lambda},
\]

where

\[
\lambda = \frac{j-k}{s_0} + \left( 2\nu + \frac{\gamma}{a} - 2 + k \left( \frac{q}{2(q-1)} - \frac{2a-k}{2} \right) \right) \frac{q-1}{q}
\]

\[
= \frac{j}{s_0} + \left( 2\nu + \frac{\gamma}{a} \right) \frac{q-1}{q} = \lambda(j,\nu,\gamma).
\]

As for the constant \( C_0 \) we have

\[
\sigma(j-k) + \sigma'(\nu + \frac{\gamma}{2a} - 1 + k \left( \frac{q}{2(q-1)} - \frac{2a-k}{2} \right) + 2a-k)
\]

\[
= \sigma j + \sigma'(\nu + \frac{\gamma}{2a}) - \sigma k + \sigma'(2a-1) - \sigma' k \frac{q-2}{2(q-1)}
\]

\[
\leq \sigma j + \sigma'(\nu + \gamma) - \sigma',
\]

if we choose

\[
(5.13) \quad \sigma = 2a\sigma'.
\]

An analogous computation can be made for the summands of the second type in \( (5.12) \).

This completes the estimate of the second line on the right hand side of \( (5.5) \).

Next we are going to estimate the first line of the right hand side of \( (5.5) \):

\[
\sum_{k=1}^{\min\{j,2a\}} \Lambda_k = \sum_{k=1}^{\min\{j,2a\}} \sum_{r=0}^{\nu-1} \sum_{s=0}^{\nu-1} \|w_j(\rho)\rho^{1-s}D_\rho^{2a(1+s)} \left( \frac{1}{\rho^k(1-\pi)Q^{\nu+s}} P_k u_{j-k} \right) \|_{0,A}.
\]

Since the sum over \( k \) is finite it is enough to consider just one summand. Observe that
\[ \Lambda_k \leq C_9 \sum_{s=0}^{r-1} \sum_{m=0}^{\gamma - 2a(1+s)} \sum_{t=0}^{k} \frac{(\gamma - 2a(1+s))(k + m - 1)!}{m} \left( \frac{1}{(k-1)!} \right) \| w_j(\rho) \left( 1 - \frac{1}{\rho^{\gamma - 2a(1+s)-m+2a-k}} \right) Q^{\nu+s}(x) u_{j-k} \|_{0,A} \]

\[ \leq C_{10} \sum_{s=0}^{r-1} \sum_{m=0}^{\gamma - 2a(1+s)} \sum_{t=0}^{k} \frac{(\gamma - 2a(1+s))(k + m - 1)!}{m} \left( \frac{1}{(k-1)!} \right) \| w_{j-k}(\rho) D_{\rho}^{-2as-m-k} Q^{\nu+s}(x) u_{j-k} \|_{0,A} \]

We apply Proposition D.3 we obtain

\[ \sum_{s=0}^{r-1} \sum_{m=0}^{\gamma - 2a(1+s)} \sum_{t=0}^{k} \frac{(\gamma - 2a(1+s))(k + m - 1)!}{m} \left( \frac{1}{(k-1)!} \right) \| w_{j-k}(\rho) D_{\rho}^{-2as-m-k} Q^{\nu+s}(x) u_{j-k} \|_{0,A} \]

\[ \leq C_{11} \sum_{s=0}^{r-1} \sum_{m=0}^{\gamma - 2a(1+s)} \sum_{t=0}^{k} \frac{(\gamma - 2a(1+s))(k + m - 1)!}{m} \left( \frac{1}{(k-1)!} \right) \| w_{j-k}(\rho) D_{\rho}^{-2as-m-k} Q^{\nu+s+k+1}(x) u_{j-k} \|_{0,A} \]

\[ \leq C_{12} \sum_{s=0}^{r-1} \sum_{m=0}^{\gamma - 2a(1+s)} \sum_{t=0}^{k} \frac{(\gamma - 2a(1+s))(k + m - 1)!}{m} \left( \frac{1}{(k-1)!} \right) \| w_{j-k}(\rho) D_{\rho}^{-2as-m-k} Q^{\nu+s+k+1}(x) u_{j-k} \|_{0,A} \]

First we remark that, by (4.4),

\[ \frac{(\gamma - 2a(1+s))(k + m - 1)!}{m} \left( \frac{1}{(k-1)!} \right) \]

\[ = \frac{(k + m - 1)(\gamma - 2a(1+s))!}{m} \left( \frac{1}{(k-1)!} \right) \]

\[ \leq \frac{(k + m - 1)(\gamma - 2a(1+s))!}{m} \left( \frac{1}{(k-1)!} \right) \]

\[ \leq \frac{(k + m - 1)(\gamma - 2a(1+s))!}{m} \left( \frac{1}{(k-1)!} \right) \]

\[ \leq \frac{(k + m - 1)(\gamma - 2a(1+s))!}{m} \left( \frac{1}{(k-1)!} \right) \]

\[ \leq \frac{(k + m - 1)(\gamma - 2a(1+s))!}{m} \left( \frac{1}{(k-1)!} \right) \]

\[ \leq \left( \frac{2a}{R_0} \right)^{m \gamma} \]

if \( R_0 \) is suitably chosen. Moreover the quantities in the norms above verify the assumptions of Theorem 1.2 i.e. \( \nu \geq -1 \) and \( \nu + \gamma(2a)^{-1} \geq 0 \). In fact the first is trivially true. As for the second we have

\[ \nu + s + k \frac{q}{2(q-1)} + \gamma \frac{2a}{2a} - s - \frac{m}{2a} - \frac{k}{2a} \]

\[ = \nu + k \left( \frac{q}{q-1} - \frac{1}{a} \right) + \gamma - \frac{m}{2a} \]
\[ \geq \frac{k}{2} \frac{q - 1}{q - 1} s_0 + \nu + 1 > 0, \]

since \( m \leq \gamma - 2a \) and \( \nu \geq -1 \). Moreover \( \gamma - k \leq j - k + \gamma^# \) if \( \gamma \leq j + \gamma^# \).

We may thus apply the inductive hypothesis to both norms in the right hand side of (5.14). Starting with the first on the next to last line we have

\[
\|w_{j-k}(\rho)\partial_{\rho}^{\gamma-2as-m-k}Q^{\nu+s+k}u_{j-k}\|_{0,A} \leq C_0^{1+\sigma(j-k)+\sigma'(\nu+s+k)\frac{q}{m\alpha-1}+\gamma-2as-m-k}(\lambda+1)^{\lambda},
\]

where

\[
\lambda = \frac{j-k}{s_0} + \left(2\nu + 2s + k\frac{q}{q-1} + \frac{\gamma}{a} - 2s - \frac{m}{a} - \frac{k}{a}\right)\frac{q-1}{q}.
\]

As for the exponent in the constant \( C_0 \), we have

\[
\sigma(j-k) + \sigma'\left(\nu + s + k\frac{q}{2(q-1)} + \gamma - 2as - m - k\right) = \sigma j + \sigma'(\nu + \gamma) - \sigma k + \sigma'k\frac{q}{2(q-1)} - \sigma's(2a+1) - \sigma'm \leq \sigma j + \sigma'(\nu + \gamma) - \sigma'(2a+1) - \sigma's(2a+1),
\]

since \( k \geq 1 \) and \( \sigma = 2a\sigma' \).

Consider now the second term in the right hand side of (5.14): by the inductive hypothesis

\[
(\nu + s)^{2+\frac{1}{\alpha}}(\nu+s+k)\|w_{j-k}(\rho)\partial_{\rho}^{\gamma-2as-m-k}u_{j-k}\|_{0,A} \leq C_0^{1+\sigma(j-k)+\sigma'(\gamma-2as-m-k)}(\nu+s)^{2+\frac{1}{\alpha}}(\nu+s+k)(\lambda_1+1)^{\lambda_1},
\]

where

\[
\lambda_1 = \frac{j-k}{s_0} + \left(\frac{\gamma}{a} - 2s - \frac{m}{a} - \frac{k}{a}\right)\frac{q-1}{q}.
\]

Both \( \nu + s \) and \( \lambda_1 \) can be enlarged to be \( \lambda(j,\nu,\gamma) \), so that it is enough to check that the sum of the exponents has the right value. As before this is

\[
\frac{j-k}{s_0} + \left(\frac{\gamma}{a} - 2s - \frac{m}{a} - \frac{k}{a}\right)\frac{q-1}{q} + 2\frac{q-1}{q}(\nu+s) + k
\]

\[
= \frac{j}{s_0} + \left(2\nu + \frac{\gamma}{a}\right)\frac{q-1}{q} - \frac{m}{a}\frac{q-1}{q} < \lambda(j,\nu,\gamma).
\]

Hence

\[
\Lambda_k \leq C_0 \sum_{s=0}^{r-1} \gamma^{-2a(1+s)}(\gamma - 2a(1+s))(k+m-1)!(\nu+s+k)^{2+\frac{1}{\alpha}}(\nu+s+k)(\lambda_1+1)^{\lambda_1}
\]

\[
+ (\nu + s)^{2+\frac{1}{\alpha}}(\nu+s+k)\|w_{j-k}(\rho)\partial_{\rho}^{\gamma-2as-m-k}u_{j-k}\|_{0,A}
\]

\[
+ (\nu + s)^{2+\frac{1}{\alpha}}(\nu+s+k)\|w_{j-k}(\rho)\partial_{\rho}^{\gamma-2as-m-k}u_{j-k}\|_{0,A}
\]
where

Recall that from (3.18b)

Proof. A and γ

right hand side of (6.2) is

Keeping into account that

due to the choice of

in Proposition 3.3.

6. Estimate of πuj

This section is devoted to proving estimates, analogous to those in [5,1], for πuj. We use the same notation of the preceding section.

Theorem 6.1. Let ν denote a rational number ≥ −1 and γ ∈ ℤ such that ν + γ(2a)−1 ≥ 0. There exist positive constants C0, R0, σ, σ′ such that

\[ \|w_j(\rho)\partial^\sigma Q^\nu \pi u_j(x, \rho)\|_{0,A} \leq C_0^{1+\sigma j+\sigma'(\nu+\gamma)}(\lambda(j, \nu, \gamma) + 1)^{\lambda(j, \nu, \gamma)} \]

where

\[ \lambda(j, \nu, \gamma) = j \left( \frac{\beta}{\alpha} + \frac{\gamma}{\alpha} \right) \]

and A = A(j) = R_0(j + 1), γ ≤ j + γ#, 2γ# < R_0.

Proof. Recall that from (6.185) πuj is obtained as a solution of

\[ \pi P_0 u_j = P_0 \pi u_j = -\frac{1}{\rho} \pi P_1 (1 - \pi) u_j - \frac{1}{\rho} \pi P_1 \pi u_j - \sum_{k=1}^{\min\{j, 2a-1\}} \frac{1}{\rho^{k+1}} \pi P_{k+1} u_j - k. \]

First of all we observe that the decay rate in ρ of the first and third terms on the right hand side of (6.2) is

\[ \rho^{-[j-\delta]s+(1-\xi)+1}, \quad \rho^{-[j-\delta]s+(1-\xi)k+1} \]

respectively, while for the second term we have \( \rho^{-[j-\delta]s+1-k} \) which would not allow us to make an inductive argument.

Lemma 6.1. We have

\[ \pi P_1 \pi v = 0. \]

Proof. We just remark that \( \pi P_1 \pi v = \pi P_1 \left( \langle v, \phi_0 \rangle \phi_0 \right) \). Using the notation of Proposition 3.3 we may write

\[ \pi P_1 \left( \langle v, \phi_0 \rangle \phi_0 \right) = \pi \left( \alpha x \partial_x + \beta + 2a\gamma_2 r \right) \partial^2_{\rho^{2a-1}} \left( \langle v, \phi_0 \rangle \phi_0 \right) \]

due to the choice of r in Proposition 3.3.

□
As a consequence (6.2) becomes

\[
(6.4) \quad P_0 \pi u_j = -\frac{1}{\rho} \pi P_{1} (1 - \pi) u_j - \sum_{k=1}^{\min\{j, 2a-1\}} \frac{1}{\rho^{k+1}} \pi P_{k+1} u_{j-k}.
\]

Let us write

\[
(6.5) \quad \tilde{u}_j^{(j)} = \begin{cases} 
(1 - \pi) u_j & \text{if } \ell = j \\
u_{\ell} & \text{if } \ell < j.
\end{cases}
\]

Hence (6.4) becomes

\[
(6.6) \quad P_0 \pi u_j = - \sum_{k=0}^{\min\{j, 2a-1\}} \frac{1}{\rho^{k+1}} \pi P_{k+1} \tilde{u}_j^{(j)}.
\]

By the inductive hypothesis and by Theorem 5.1 the functions \(\tilde{u}_j^{(j)}\) verify the estimates (4.7) and (5.1) for \(k = 0\).

To start with, by Lemma 5.1

\[
(6.7) \quad \|w_j(\rho) \partial^{\gamma}_\rho Q^{\nu} \pi u_j(x, \rho)\|_{0, A}
\]

\[
\leq \sum_{k=0}^{\min\{j, 2a-1\}} \|w_j(\rho) \partial^{\gamma}_\rho P_{0}^{-1} \left( \frac{1}{\rho^{k+1}} \pi Q^{\nu+1} P_{k+1} \tilde{u}_j^{(j)} \right)\|_{0, A}.
\]

By Corollary 5.1 we have

\[
(6.8) \quad \|w_j(\rho) \partial^{\gamma}_\rho Q^{\nu} \pi u_j(x, \rho)\|_{0, A}
\]

\[
\leq \sum_{k=0}^{\min\{j, 2a-1\}} \sum_{s=0}^{r-1} \|w_j(\rho) D^{\gamma}_\rho P_{0}^{-1} \left( \frac{1}{\rho^{k+1}} \pi Q^{\nu+s} P_{k+1} \tilde{u}_j^{(j)} \right)\|_{0, A}
\]

\[
+ \sum_{k=0}^{\min\{j, 2a-1\}} \|w_j(\rho) D^{\gamma}_\rho P_{0}^{-1} \left( \frac{1}{\rho^{k+1}} \pi Q^{\nu+1} P_{k+1} \tilde{u}_j^{(j)} \right)\|_{0, A}.
\]

Consider the terms on the third line of the above inequality. Denote by \(\tilde{g}_j = Q^{\nu+1} P_{k+1} \tilde{u}_j^{(j)}\). Then for \(k \in \{0, \ldots, \min\{j, 2a-1\}\},

\[
(6.9) \quad \|w_j(\rho) D^{\gamma}_\rho P_{0}^{-1} \left( \frac{1}{\rho^{k+1}} \pi \tilde{g}_j \right)\|_{0, A}
\]

\[
= \|w_j(\rho) D^{\gamma}_\rho \sum_{i=1}^{2a} A_{0i} J_{0i} (\rho^{-(-k+1)} \langle \tilde{g}_j, \varphi_0 \rangle) \varphi_0\|_{0, A}
\]

\[
= \|w_j(\rho) D^{\gamma}_\rho \sum_{i=1}^{2a} A_{0i} J_{0i} (\rho^{-(-k+1)} \langle \tilde{g}_j, \varphi_0 \rangle)\|_{A}.
\]

Preliminarily we need the analogous of lemmas 5.3 5.4 for the fundamental eigenfunction, corresponding to the projection \(\pi\).

**Lemma 6.2.** Assume that \(\text{Re } \mu_{0} > 0\). Then

\[
(6.10) \quad \|w_j(\rho) J_{0i} (\rho^{-k} f(\rho))\|_{A} \leq \tilde{C}_0 \mu_{0}^{-\frac{1}{2}} \|w_{j-k}(\rho) f(\rho)\|_{A}^{2},
\]

for a suitable constant \(\tilde{C}_0 > 0\) independent of \(j, k, i\), where we choose \(R = A(j)\), see (5.8).
Lemma 6.3. Assume that \( \text{Re} \mu_{0i} < \tilde{\mu}_0 < 0 \). Then

\[
\|w_j(\rho)I_{0i}(\rho^{-k}f(\rho))\|_A \leq \tilde{C}_0\mu_0^{-\frac{k}{\kappa}}\|w_{j-k}(\rho)f(\rho)\|^2_A,
\]

for a suitable constant \( \tilde{C}_0 > 0 \) independent of \( j, k, i, \) and \( R = A(j) \).

Under the assumptions of lemmas 6.3, 6.2 \( \text{Re} \mu_{0i} - \tilde{\mu}_0 \neq 0 \) and, as a consequence, their proofs are completely analogous to those of lemmas 6.1, 6.3 since the factor \( \rho^{1-\kappa} \) plays no role.

Lemma 6.4. Assume that \( \text{Re} \mu_{0i} = \tilde{\mu}_0 < 0 \) and \( 1 \leq k \leq \min\{j, 2a - 1\} \). Then

\[
\|w_j(\rho)I_{0i}(\rho^{-(k+1)}f(\rho))\|_A \leq \tilde{C}'_0\frac{1}{jA(j)^{2(1-\kappa)}}\|w_{j-k}(\rho)f(\rho)\|^2_A,
\]

for a suitable constant \( \tilde{C}'_0 > 0 \) independent of \( j, k, \) and \( R = A(j) \).

Proof of Lemma 6.4. We have to estimate, using (3.8),

\[
\left\| w_j(\rho)I_{0i}(\rho^{-(k+1)}f(\rho)) \right\|^2_A = \int_A^{+\infty} \rho^{2(j-\delta)\kappa}e^{-2\tilde{\mu}_0\rho} \int_R \epsilon^{\mu_{0i}(\rho-\sigma)}
\]

\[
\cdot \sigma H(\sigma - A)f(\sigma) \frac{H(\sigma - A)}{\sigma^{k+1}} d\sigma d\rho.
\]

Since \( \text{Re} \mu_{0i} - \tilde{\mu}_0 = 0 \) and \( \epsilon_{\mu} > 0 \), we obtain

\[
\int_A^{+\infty} \rho^{2(j-\delta)\kappa}e^{-2\tilde{\mu}_0\rho} \int_R \epsilon^{\mu_{0i}(\rho-\sigma)}
\]

\[
\cdot H(\sigma - A)f(\sigma) \frac{H(\sigma - A)}{\sigma^{k+1}} d\sigma d\rho.
\]

Using Hölder inequality on the inner integral above we get

\[
\|w_j(\rho)I_{0i}(\rho^{-(k+1)}f(\rho))\|_A^2 \leq \int_A^{+\infty} \rho^{2(j-\delta)\kappa} \int_R \epsilon^{\mu_{0i}(\rho-\sigma)}
\]

\[
\cdot H(\sigma - A)f(\sigma) \frac{H(\sigma - A)}{\sigma^{k+1}} d\sigma d\rho = L_{jk} \|w_{j-k}(\rho)f(\rho)\|^2_A \int_A^{+\infty} \frac{1}{\rho^{2k(1-\kappa)+1}} d\rho
\]

where we set

\[
\int_R^{+\infty} \frac{1}{\sigma^{2(j-\delta)\kappa+2k(1-\kappa)+2}} d\sigma = \frac{L_{jk}}{\rho^{2(j-\delta)\kappa+2k(1-\kappa)+1}}.
\]

The fact that \( k \geq 1 \) allows us to conclude the proof of the lemma. \( \square \)

Finally we need one more result to estimate the first term in (6.9), i.e. the term corresponding to \( k = 0 \) when \( \text{Re} \mu_{0i} = \tilde{\mu}_0 \).
Lemma 6.5. Assume that \( \text{Re} \mu_0 = \tilde{\mu}_0 < 0 \).

\[
\|w_j(\rho)I_{0_1}(\rho^{-1}f(\rho))\|_A^2 \leq \frac{C_0}{f(A(j))^{2(1-\omega)}}\|w_j(\rho)^{1-\omega}f(\rho)\|_A^2, \tag{6.13}
\]

for a suitable constant \( C_0 > 0 \) independent of \( j \), and \( R = A(j) \).

Proof of Lemma 6.5. Arguing as in the beginning of the proof of Lemma 6.4 we have

\[
\|w_j(\rho)I_{0_1}(\rho^{-1}f(\rho))\|_A^2 = \int_\mathbb{R} e^{\mu_0(\rho-\sigma)} \cdot H(-\text{sign}(\text{Re} \mu_0 - \tilde{\mu}_0 + \varepsilon_\mu)(\rho - \sigma)) \frac{H(\sigma - A)}{\sigma}f(\sigma)d\sigma d\rho \leq \int_\mathbb{R} e^{\mu_0(\rho-\sigma)} H(-\text{sign}(\text{Re} \mu_0 - \tilde{\mu}_0 + \varepsilon_\mu)(\rho - \sigma)) \frac{H(\sigma - A)}{\sigma}f(\sigma)d\sigma d\rho \leq \int_\mathbb{R} e^{\mu_0(\rho-\sigma)} \frac{H(\sigma - A)}{\sigma(\sigma+2(1-\omega))}f(\sigma)d\sigma d\rho = \int_\mathbb{R} e^{\mu_0(\rho-\sigma)} \frac{H(\sigma - A)}{\sigma(\sigma+2(1-\omega))}f(\sigma)d\sigma d\rho
\]

Using Hölder inequality on the inner integral above we get

\[
\|w_j(\rho)I_{0_1}(\rho^{-1}f(\rho))\|_A^2 \leq \|w_j(\rho)\rho^{1-\omega}f(\rho)\|_A^2 \int_\mathbb{R} e^{\mu_0(\rho-\sigma)} \frac{1}{\sigma^{2(1-\omega)}}d\sigma \int_\mathbb{R} e^{\mu_0(\rho-\sigma)} \frac{1}{\sigma(\sigma+2(1-\omega))}d\sigma \leq L_j \|w_j(\rho)\rho^{1-\omega}f(\rho)\|_A^2 \int_\mathbb{R} e^{\mu_0(\rho-\sigma)} \frac{1}{\sigma^{2(1-\omega)}}d\sigma
\]

where we set

\[
\int_\mathbb{R} e^{\mu_0(\rho-\sigma)} \frac{1}{\sigma^{2(1-\omega)}}d\sigma = \frac{L_j}{\rho^{2(1-\omega)}/2(1-\omega)}.
\]

This concludes the proof of the lemma.

Going back to (5.9) and using (5.7) we have

\[
(6.14) \quad \|w_j(\rho)D_{\rho}^{\gamma_1}P_{0}^{-1}\left(\frac{1}{\rho^{k+1}}\pi \tilde{g}_{jk}\right)\|_0,A
\]

\[
= \|w_j(\rho)D_{\rho}^{\gamma_1} \sum_{i=1}^{2a} A_{0i}I_{0_1}(\rho^{-(k+1)}(\tilde{g}_{jk},\varphi_0))\|_A
\]

\[
= s_0^{-\gamma_1} \|w_j(\rho) \sum_{i=1}^{2a} A_{0i} \mu_0^{\gamma_1} I_{0_1}(\rho^{-(k+1)}(\tilde{g}_{jk},\varphi_0))\|_A
\]

\[
\leq C_1 \sum_{i=1}^{2a} \|w_j(\rho)I_{0_1}(\rho^{-(k+1)}(\tilde{g}_{jk},\mu_0^{\gamma_1} \varphi_0))\|_A.
\]

To the last quantity we apply Lemmas 6.2, 6.3 when \( \text{Re} \mu_{0i} \neq \tilde{\mu}_0 \), Lemma 6.4 when \( \text{Re} \mu_{0i} = \tilde{\mu}_0 \) and \( 1 \leq k \leq \min\{j,2a\} \) and Lemma 6.5 when \( \text{Re} \mu_{0i} = \tilde{\mu}_0 \) and \( k = 0 \). In the first case, i.e. when \( \text{Re} \mu_{0i} \neq \tilde{\mu}_0 \), we obtain a decay rate better than that we get when \( \text{Re} \mu_{0i} = \tilde{\mu}_0 \) so that the gain must be neglected.

We also remark that there is no need to keep track of the precise powers of \( \mu_0 \) in the formula above because they may always be absorbed by a constant. We did that only to have a more symmetric formula.
Consider first the terms where $k = 0, 1, \ldots, \min\{j, 2a - 1\}$ and Re $\mu_0 \neq \bar{\mu}_0$. Applying Lemmas 6.2, 6.3 in (6.14), we get (see (2.15))

$$\sum_{\text{Re} \mu_0 \neq \bar{\mu}_0} \|w_j(\rho) I_{0i}(\rho^{-(k+1)}(\tilde{g}_{jk}, \mu_0^{\frac{2a-k-1}{2a}} \varphi_0))\|_A$$

$$\leq C_2 \|w_{j-k-1}(\rho)(\tilde{g}_{jk}, \mu_0^{\frac{2a-k-1}{2a}} \varphi_0)\|_A$$

$$= C_2 \|w_{j-k-1}(\rho)(Q^{\nu+i\lambda} P_{k+1}(j, \mu_0^{\frac{2a-k-1}{2a}} \varphi_0))\|_A$$

$$= C_2 \|w_{j-k-1}(\rho)\partial_p^{2a-k-1} (\bar{u}_{j-k}^{(j)}(j, \mu_0^{\frac{2a-k-1}{2a}} \varphi_0))\|_A$$

$$= C_2 \mu_0^{\frac{2a-k-1}{2a}} \|w_{j-k-1}(\rho)\partial_p^{2a-k-1} (Q^{-\frac{2a-k-1}{2a}} \bar{u}_{j-k}^{(j)}(j, \mu_0^{\frac{2a-k-1}{2a}} \varphi_0))\|_A$$

$$\leq \frac{C_2 \mu_0^{\frac{2a-k-1}{2a}}}{A^{(j)\kappa}} \|Q^{-\frac{2a-k-1}{2a}} \bar{P}_{k+1}(j, \varphi_0)\|_A \|w_{j-k}(\rho)\partial_p^{2a-k-1} Q^{-\frac{2a-k-1}{2a}} \bar{u}_{j-k}(j, \varphi_0)\|_A$$

We remark that the norm $\|Q^{-\frac{2a-k-1}{2a}} \bar{P}_{k+1}(j, \varphi_0)\|_A$ is an absolute constant since $0 \leq k \leq 2a - 1$. Furthermore we see that the conditions $\nu \geq -1$ and $\nu + \frac{2a}{2a} \geq 0$, when $\nu, \gamma$ are the exponents of $Q$, $\partial_p$, respectively, are satisfied; moreover $2a - k - 1 \leq j - k + \gamma^1$ and hence we may apply the inductive hypothesis when $k \geq 1$ and the result of the preceding section for $(1 - \pi) u_j$ when $k = 0$, thus obtaining the bound

$$\sum_{k=0}^{\min\{j, 2a-1\}} \sum_{\text{Re} \mu_0 \neq \bar{\mu}_0} \|w_j(\rho) I_{0i}(\rho^{-(k+1)}(\tilde{g}_{jk}, \mu_0^{\frac{2a-k-1}{2a}} \varphi_0))\|_A$$

$$\leq \frac{C_3 \mu_0^{\frac{2a-k-1}{2a}}}{A^{(j)\kappa}} \sum_{k=0}^{\min\{j, 2a-1\}} \|w_{j-k}(\rho)\partial_p^{2a-k-1} Q^{-\frac{2a-k-1}{2a}} \bar{u}_{j-k}(j, \varphi_0)\|_A$$

$$\leq \frac{C_3 \mu_0^{\frac{2a-k-1}{2a}}}{A^{(j)\kappa}} \min\{j, 2a-1\} \|w_{j-k}(\rho)\partial_p^{2a-k-1} Q^{-\frac{2a-k-1}{2a}} u_{j-k}(j, \varphi_0)\|_A$$

$$\leq \frac{C_3 \mu_0^{\frac{2a-k-1}{2a}}}{A^{(j)\kappa}} \sum_{k=0}^{\min\{j, 2a-1\}} \|w_{j-k}(\rho)\partial_p^{2a-k-1} Q^{-\frac{2a-k-1}{2a}} u_{j-k}(j, \varphi_0)\|_A$$

$$\leq \frac{C_3 \mu_0^{\frac{2a-k-1}{2a}}}{A^{(j)\kappa}} \sum_{k=0}^{\min\{j, 2a-1\}} C_0^{1+\sigma(j-k)+\sigma'(2a-k-1)}$$

$$\cdot (\lambda(j-k, 0, 0) + 1) \lambda(j-k, 0, 0).$$

Hence we get the estimate

$$\sum_{k=1}^{\min\{j, 2a-1\}} \|w_j(\rho) D_{\rho}^{\gamma_1} \sum_{\text{Re} \mu_0 \neq \bar{\mu}_0} A_{0i} I_{0i}(\rho^{-(k+1)}(\tilde{g}_{jk}, \varphi_0))\|_A$$

$$\leq \frac{1}{2} C_1^{1+\sigma j + \sigma'(\nu + \gamma) - \sigma'(\lambda(j, \nu, \gamma) + 1) \lambda(j, \nu, \gamma)},$$

when $k \geq 1$ and Re $\mu_0 \neq \bar{\mu}_0$, provided we choose $C_0$ large enough.

Let us now consider the case $k = 0$ and Re $\mu_0 \neq \bar{\mu}_0$. We have to estimate

$$\|w_j(\rho) D_{\rho}^{\gamma_1} \sum_{\text{Re} \mu_0 \neq \bar{\mu}_0} A_{0i} I_{0i}(\rho^{-1}(1 - \pi) u_j, \varphi_0)\|_A$$
\[ \frac{C_3 \mu_0^\nu + \gamma}{A(j)^\nu} \rho_{j}^{1+\sigma_j + \gamma'(2a-1)} (\lambda(j, 0, 0) + 1)^{\lambda(j, 0, 0)} \]

\[ \leq \frac{C_3 \rho_{j}^{\nu + \gamma}}{R_0^{\nu + \gamma}} C_0^{1+\sigma_j + \gamma' \nu + \gamma} C_0^{1+\sigma_j + \gamma' (\nu + \gamma) - \sigma'} (\lambda(j, \nu, \gamma) + 1)^{\lambda(j, \nu, \gamma)} \]

We make the choice \( C_0^{\sigma'} > \mu_0 \) and \( R_0^{\nu} > 2C_3\rho_{j}^{2\nu} \), so that

\[ \sum_{k=0}^{\min\{j, 2a-1\}} \| w_j(\rho) D_{\rho}^{\nu+1} \sum_{\text{Re} \mu_0 \neq \mu_0} A_{\mu_0} I_{0i}(\rho^{-1}(k+1) (\bar{g}_{jk}, \varphi_0)) \|_A \leq C_0^{1+\sigma_j + \gamma' (\nu + \gamma) - \sigma'} (\lambda(j, \nu, \gamma) + 1)^{\lambda(j, \nu, \gamma)} \]

So far we treated \((6.9)\) in the case when \( \text{Re} \mu_0 \neq \mu_0 \) for any value of \( k \).

Consider now \((6.9)\); we have to estimate \((6.14)\) when \( \text{Re} \mu_0 = \mu_0 \).

\[ C_1 \sum_{k=0}^{\min\{j, 2a-1\}} \| w_j(\rho) I_{0i}(\rho^{-1}(k+1) (\bar{g}_{jk}, \mu_0^{\sigma_j + \gamma' \nu + \gamma} \varphi_0)) \|_A \]

Let us start considering the case when \( k = 0 \):

\[ C_1 \| w_j(\rho) I_{0i}(\rho^{-1}(\bar{g}_{jk}, \mu_0^{\sigma_j + \gamma' \nu + \gamma} \varphi_0)) \|_A \]

Applying Lemma \((6.5)\) we get

\[ C_1 \| w_j(\rho) I_{0i}(\rho^{-1}(\bar{g}_{jk}, \mu_0^{\sigma_j + \gamma' \nu + \gamma} \varphi_0)) \|_A \leq \frac{C_0}{j^{\frac{1}{2}} A(j)^{1-\nu}} \| w_j(\rho) \rho^{1-\nu} (\bar{g}_{jk}, \mu_0^{\sigma_j + \gamma' \nu + \gamma} \varphi_0) \|_A \]

\[ = C_1 \frac{\tilde{C}_0}{j^{\frac{1}{2}} A(j)^{1-\nu}} \| w_j(\rho) \rho^{1-\nu} (Q^{\nu + \gamma} \bar{P}_{\gamma} (1-\pi) u_j, \mu_0^{\sigma_j + \gamma' \nu + \gamma} \varphi_0) \|_A \]

\[ = C_1 \frac{\tilde{C}_0}{j^{\frac{1}{2}} A(j)^{1-\nu}} \| w_j(\rho) \rho^{1-\nu} (Q^{\nu + \gamma} \bar{P}_{\gamma} (1-\pi) u_j, Q^{\nu + \gamma} \mu_0^{\sigma_j + \gamma' \nu + \gamma} \varphi_0) \|_A \]

\[ \leq C_1 \frac{\tilde{C}_0}{j^{\frac{1}{2}} A(j)^{1-\nu}} \mu_0^{\sigma_j + \gamma' \nu + \gamma} \| Q^{\nu + \gamma} \bar{P}_{\gamma} \varphi_0 \|_0 \cdot \| w_j(\rho) \rho^{1-\nu} Q^{\nu + \gamma} \bar{P}_{\gamma} (1-\pi) u_j \|_0 \]

We may apply the estimate obtained in the preceding section for \((1-\pi) u_j\), since \(-\frac{2a-1}{2a} \geq -1\) and \(-\frac{2a-1}{2a} + \frac{2a-1}{2a} = 0\), \(2a-1 \leq j + \gamma'\). We thus get

\[ \| w_j(\rho) I_{0i}(\rho^{-1}(\bar{g}_{jk}, \mu_0^{\sigma_j + \gamma' \nu + \gamma} \varphi_0)) \|_A \leq \frac{C_3}{j^{\frac{1}{2}} A(j)^{1-\nu}} \mu_0^{\sigma_j + \gamma' \nu + \gamma} \| Q^{\nu + \gamma} \bar{P}_{\gamma} (1-\pi) u_j \|_0 \]

\[ \leq \frac{C_3}{j^{\frac{1}{2}} A(j)^{1-\nu}} \mu_0^{\sigma_j + \gamma' \nu + \gamma} C_0^{1+\sigma_j + \gamma' \nu + \gamma} (\lambda(j, 0, 0) + 1)^{\lambda(j, 0, 0)}, \]

where

\[ \lambda(j, 0, 0) = \frac{j}{s_0} \]
Hence from (6.18) we have
\[
(6.19) \quad C_1 \| w_j(r) I_{00}(r^{-1} \langle \tilde{\varphi}(r), \mu_0 \rangle \frac{\gamma + 2a + 1}{2a} \varphi_0) \|_A \leq \frac{C_0 C_0^{2a}}{j^2 A(j)^{1-\sigma}} \mu_0^{\gamma + 2a + 1} + \nu \left( C_0^{1+\sigma(j-1)^2} - \sigma'(\nu+\gamma) C_0^{1+\sigma(j)^2} - \sigma' \gamma \right) \left( 1 + \lambda(j, \nu, \gamma) \right) 
\]
\[
\leq C_0^{1+\sigma(j-1)^2} - \sigma' \gamma \left( 1 + \lambda(j, \nu, \gamma) \right),
\]
if we make the choice \( C_0^{q'} > \mu_0, R_0^{1-\sigma} > C_3 C_0^{2a} \).

To conclude the analysis of the last term on the right hand side of (6.18) we must examine (6.17) when \( \Re \mu_{0i} = \tilde{\mu}_0 \) for \( k \in \{1, \ldots, \min\{j, 2a - 1\} \} \).

It is enough to estimate one out of the two terms occurring:
\[
\min\{j, 2a - 1\} \sum_{k=1}^{\min\{j, 2a - 1\}} \| w_j(r) I_{00}(r^{-1} \langle \tilde{\varphi}(r), \mu_0 \rangle \frac{\gamma + 2a + 1}{2a} \varphi_0) \|_A.
\]

Applying Lemma 6.4 we have
\[
\min\{j, 2a - 1\} \sum_{k=1}^{\min\{j, 2a - 1\}} \| w_j(r) I_{00}(r^{-1} \langle \tilde{\varphi}(r), \mu_0 \rangle \frac{\gamma + 2a + 1}{2a} \varphi_0) \|_A \leq \tilde{C}_0 C_1 \sum_{k=1}^{\min\{j, 2a - 1\}} \| w_j(r) I_{00}(r^{-1} \langle \tilde{\varphi}(r), \mu_0 \rangle \frac{\gamma + 2a + 1}{2a} \varphi_0) \|_A
\]
\[
= \tilde{C}_0 C_1 \sum_{k=1}^{\min\{j, 2a - 1\}} \| w_j(r) I_{00}(r^{-1} \langle \tilde{\varphi}(r), \mu_0 \rangle \frac{\gamma + 2a + 1}{2a} \varphi_0) \|_A
\]
\[
\leq \tilde{C}_0 C_1 \sum_{k=1}^{\min\{j, 2a - 1\}} \| w_j(r) I_{00}(r^{-1} \langle \tilde{\varphi}(r), \mu_0 \rangle \frac{\gamma + 2a + 1}{2a} \varphi_0) \|_A
\]
\[
\leq \tilde{C}_0 C_2 \frac{\mu_0^{\gamma + 2a + 1}}{j^2 A(j)^{1-\sigma}} \sum_{k=1}^{\min\{j, 2a - 1\}} \| w_j(r) I_{00}(r^{-1} \langle \tilde{\varphi}(r), \mu_0 \rangle \frac{\gamma + 2a + 1}{2a} \varphi_0) \|_A.
\]

Here we absorbed the norm \( \| Q^{2a-k-1} \varphi_0 \|_A \) into an absolute constant \( C_2 \) since \( 1 \leq k \leq 2a - 1 \).

We point out that the conditions \( \nu \geq -1 \) and \( \nu + \frac{\gamma}{2a} \geq 0 \), when \( \nu, \gamma \) are the exponents of \( Q, \partial_\rho \) respectively, are satisfied and hence we may apply the inductive hypothesis thus obtaining the bound
\[
\tilde{C}_0 C_2 \frac{\mu_0^{\gamma + 2a + 1}}{j^2 A(j)^{1-\sigma}} \sum_{k=1}^{\min\{j, 2a - 1\}} C_0^{1+\sigma(j-k)^2} \frac{\gamma}{2a} \varphi_0 \|_A.
\]
where
\[ \lambda = \frac{j - k}{s_0} < \lambda(j, \nu, \gamma). \]

Since \( \sigma = 2a\sigma' \) (see (6.13)) we may bound the above quantity by

\[
(6.20) \quad C_3 C_0^{-\sigma'} C_0^{\frac{\nu}{\sigma' - \nu}} C_0^{1 + \sigma j + \sigma' (\nu + \gamma) - \sigma'} (\lambda(j, \nu, \gamma) + 1)^{\lambda(j, \nu, \gamma)}
\]

\[
\leq C_0^{1 + \sigma j + \sigma' (\nu + \gamma) - \sigma'} (\lambda(j, \nu, \gamma) + 1)^{\lambda(j, \nu, \gamma)}.
\]

This concludes the estimation of the second term on the right hand side of (6.8). We are left with the estimate of the first term in (6.8).

We split the sum as

\[
(6.21) \quad \sum_{s=0}^{r-1} \|w_j(\rho)D_\rho^{-2a(1+s)} \left( \frac{1}{\rho} \pi Q^{\nu+s} P_1(1-\pi)u_j \right) \|_{0,A}
\]

\[
= B_0 + \sum_{k=1}^{\min\{j, 2a-1\}} B_k.
\]

Let us examine \( B_k, k > 0 \), first.

\[
\sum_{s=0}^{r-1} \|w_j(\rho)D_\rho^{-2a(1+s)} \left( \frac{1}{\rho^{k+1}} \pi Q^{\nu+s} P_{k+1} u_{j-k} \right) \|_{0,A}
\]

\[
\leq C_5 \sum_{s=0}^{r-1} \sum_{m=0}^{s+1} \sum_{\ell=0}^{m} \left( \gamma - 2a(1+s) \right) \left( k + m \right) (k+m)! \frac{1}{A(j)^m}
\]

\[
\cdot \|w_j(\rho)\frac{1}{\rho^{k+1}} \partial_\rho^{-2a_\nu + m - k - 1} \pi Q^{\nu+s}(x\partial_x)^{\ell} u_{j-k} \|_{0,A}
\]

\[
\leq C_5 \sum_{s=0}^{r-1} \sum_{m=0}^{s+1} \sum_{\ell=0}^{m} \left( k + m \right) \left( \gamma - 2a(1+s) \right)! (\gamma - 2a(1+s) - m)!
\]

\[
\cdot \frac{1}{A(j)^{m+1+k(1-\xi)}} \|w_{j-k}(\rho)\partial_\rho^{-2a_\nu + m - k - 1} \pi Q^{\nu+s}(x\partial_x)^{\ell} u_{j-k} \|_{0,A}.
\]

Now \( \pi Q^{\nu+s}(x\partial_x)^{\ell} u_{j-k} = \langle Q^{\nu+s}(x\partial_x)^{\ell} u_{j-k}, \varphi_0 \rangle \varphi_0 \), so that we get

\[
\langle (x\partial_x)^{\ell} u_{j-k}, \mu_0^{\nu+s} \varphi_0 \rangle \varphi_0 = \langle u_{j-k}, \mu_0^{\nu+s} (x\partial_x)^{\ell} \varphi_0 \rangle \varphi_0.
\]

Hence the above expression is bounded by

\[
C_6 \sum_{s=0}^{r-1} \mu_0^{\nu+s} \sum_{m=0}^{s+1} \sum_{\ell=0}^{m} \left( k + m \right) (k+m)! (\gamma - 2a(1+s))! (\gamma - 2a(1+s) - m)!
\]

\[
\cdot \frac{1}{A(j)^{m+1+k(1-\xi)}} \|w_{j-k}(\rho)\partial_\rho^{-2a_\nu + m - k - 1} u_{j-k} \|_{0,A},
\]

where we used the fact that since \( \ell \) takes up a finite number of values, we can bound \( \| (x\partial_x)^{\ell} \varphi_0 \|_0 \) by an absolute constant. Now
As for the sums above we remark that, choosing $R(6.23)$

Taking the $\rho$

Next we have to estimate in (6.20)

We see immediately that $C$

Using the inductive hypothesis we have that the above sum is bounded by

Thus the above quantity is estimated by

Using the inductive hypothesis we have that the above sum is bounded by

where

We see immediately that

As for the sums above we remark that, choosing $R_0 > 2R_1$, we may estimate them by

so that the choice $C_0^{s'} > \max\{\mu_0, \frac{\mu_0}{\nu}\}$, $C_0^{s'} > 2aC_9$ allows us to obtain from (6.21)

Next we have to estimate in (6.20)

Taking the $\rho$-derivative as above and recalling (5.6) we have

$$B_0 \leq C_9 \sum_{s=0}^{r-1} \sum_{m=0}^{\gamma - 2a(1+s)} \left( \gamma - 2a(1+s) \right)_m!$$

$$\cdot \left\| w_j(\rho) \frac{1}{\rho^{1+m}} \frac{\partial^{\gamma - 2a(1+s) - m} \pi Q^{s'} P_1(1 - \pi) u_j}{\partial \rho^{s'}} \right\|_{0,A}$$

$$= C_9 \sum_{s=0}^{r-1} \sum_{m=0}^{\gamma - 2a(1+s)} \left( \gamma - 2a(1+s) \right)_m!$$
Taking (6.24) we see immediately that

\[ \sum_{m=0}^{\gamma-2a(1+s)} \frac{(\gamma - 2a(1+s))!}{(\gamma - 2a(1+s) - m)! A(j)^2 - \kappa + m} \leq C_{10} \sum_{s=0}^{r-1} \mu_0^{\nu+s} \sum_{m=0}^{\gamma-2a(1+s)} \rho^{1-x} \partial^m \varphi U_j \varphi_0 \|_{0,A} \]

where we used (2.15) and included in C_{11} the norm \| \tilde{P}_1 \varphi_0 \|_0, which is an absolute constant.

Arguing as we did in (6.22) we get

\[ B_0 \leq C_{12} \frac{1}{R_0} \sum_{s=0}^{r-1} \mu_0^{\nu+s} \sum_{m=0}^{\gamma-2a(1+s)} \left( \frac{2\gamma^m}{R_0} \right) \cdot \| w_j(p) \rho^{1-x} \partial^m \varphi U_j \varphi_0 \|_{0,A} \]

Using the inductive hypothesis we may bound the above quantity by

\[ \lambda = \frac{s_0}{j} + \left( \frac{\gamma - 2as - m - 1}{a} \right) \frac{q - 1}{q} \]

We see immediately that

\[ \lambda < \lambda(j, \nu, \gamma) \]

Taking \( R_0 \) large enough and choosing \( C_0 \) as above we conclude that

\[ B_0 \leq C_0^{1+\sigma(j, \nu, \gamma)} + 1 \]

Summing up, when we plug (6.24), (6.23), (6.20), (6.19), (6.16) into (6.8) and choose \( C_0 \) sufficiently large we get the desired estimate.

This ends the proof of Theorem 6.1. \( \square \)

7. END OF THE PROOF OF THEOREM 4.1

Using Theorem 5.1 and renaming \( C_0 \) we obtain Theorem 4.2

Next we want to prove Theorem 4.1

To this end we are going to use Proposition C.1 with \( \theta = 0 \):

\[ \| w_j(p) \partial^\lambda \varphi U_j \varphi \|_{0,A} \]

\[ \leq \sum_{\ell=0}^{m(0, I)} C_{\ell+1} \left( \frac{m(0, I)}{\ell} \right) \langle I \rangle^{\ell} \| w_j(p) \partial^\lambda \varphi U_j \varphi \|_{0,A} \]

where

\[ \langle I \rangle = \alpha + \frac{\beta}{q - 1} \]
and
\[ m(0, I) = \left\lceil \frac{\langle I \rangle q - 1}{q} \right\rceil, \]
where the square brackets denote the integer part.

By Theorem 4.2, we have
\[ \| w_j(\rho) \partial_x^\alpha \partial_\rho^\beta u_j(x, \rho) \|_{0, A} \]
\[ \leq \sum_{\ell=0}^{m(0, I)} C_1^{\ell+1} \left( m(0, I) \right) \| I \|^{1+\sigma_j+\sigma'(\gamma + I - \ell \frac{\rho}{q-1})} \left( 1 + \lambda(j, \frac{1}{2} \langle I \rangle, \gamma) \right) \]
\[ \cdot \left( 1 + \lambda(j, \frac{1}{2} \left( \langle I \rangle - \ell \frac{q}{q-1} \right), \gamma) \right) \]
\[ \lambda(j, \frac{1}{2} \left( \langle I \rangle - \ell \frac{q}{q-1} \right), \gamma) = \frac{j}{s_0} + \frac{\langle I \rangle q - 1}{a q} + \frac{\gamma q - 1}{a} - \ell, \]
and, since
\[ \langle I \rangle \leq \frac{q}{q-1} \lambda(j, \frac{1}{2} \left( \langle I \rangle - \ell \frac{q}{q-1} \right), \gamma), \]
we obtain that
\[ \| w_j(\rho) \partial_x^\alpha \partial_\rho^\beta u_j(x, \rho) \|_{0, A} \]
\[ \leq \sum_{\ell=0}^{m(0, I)} C_1^{\ell+1} \left( m(0, I) \right) \| I \|^{1+\sigma_j+\sigma'(\gamma + I - \ell \frac{\rho}{q-1})} \left( 1 + \lambda(j, \frac{1}{2} \langle I \rangle, \gamma) \right) \]
\[ \leq C_1^{1+\sigma_j+\sigma'(\gamma + I)} \left( 1 + \lambda(j, \frac{1}{2} \langle I \rangle, \gamma) \right) \]
\[ \sum_{\ell=0}^{m(0, I)} C_1^{\ell+1} \left( m(0, I) \right) \| I \|^{1+\sigma_j+\sigma'(\gamma + I) - \ell \frac{\rho}{q-1}} \]
\[ \leq C_0^{1+\sigma_j+\sigma'(\gamma + I)} \left( 1 + \lambda(j, \frac{1}{2} \langle I \rangle, \gamma) \right) \]
\[ \cdot C_1 \left( 1 + \frac{C_1}{C_0^{\alpha+\beta+\gamma}} \right)^{m(0, I)} \]
Choosing \( C_0^{\alpha+\beta+\gamma} > C_1 \) and recalling that \( m(0, I) < \langle I \rangle \leq \alpha + \beta \) we have that
\[ \| w_j(\rho) \partial_x^\alpha \partial_\rho^\beta u_j(x, \rho) \|_{0, A} \]
\[ \leq 2^{\alpha+\beta} C_1 C_0^{1+\sigma_j+\sigma'(\gamma + I)} \left( 1 + \lambda(j, \frac{1}{2} \langle I \rangle, \gamma) \right) \]
Moreover
\[ \lambda(j, \frac{1}{2} \langle I \rangle, \gamma) = \frac{j}{s_0} + \frac{\alpha q - 1}{q} \frac{\beta}{q} + \frac{\gamma q - 1}{a}, \]
so that
\[ \langle 7.1 \rangle \quad \| w_j(\rho) \partial_x^\alpha \partial_\rho^\beta u_j(x, \rho) \|_{0, A} \leq C_0^{1+\sigma_j+\alpha+\beta+\gamma} \]
\[ \cdot \left( 1 + \frac{j}{s_0} + \frac{\alpha q - 1}{q} \frac{\beta}{q} + \frac{\gamma q - 1}{a} \right)^{\frac{1}{s_0}} + \frac{\alpha q - 1}{q} \frac{\beta}{q} + \frac{\gamma q - 1}{a} \].
Renaming the constant $C_u$, we finish the proof of Theorem 4.1.

8. Pointwise Estimates of the $u_j$

First of all we point out that from the estimates of Theorem 4.1 it is straightforward to deduce the same type of estimates for

\[(8.1) \quad \| \partial_{\rho}^{\alpha'} \partial_x^\gamma w_j(\rho) \partial_{\rho}^{\alpha} \partial_x^\beta u_j(x, \rho) \|_{0, \Lambda} \leq C_u^{1+j+\alpha + \beta + \gamma} \cdot \left( \frac{j}{s_0} + a \frac{q-1}{q} + \frac{\beta}{q} + \frac{\gamma q-1}{aq} \right)! \]

with $\alpha' + \gamma' \leq 2$, with a different meaning of the constant $C_u$.

The purpose of adding two extra derivatives with respect to $(x, \rho)$ is to apply the Sobolev immersion theorem to deduce pointwise estimates.

To this end we use cutoff functions in order to apply the Sobolev immersion theorem on the whole plane.

**Lemma 8.1.** There exist smooth functions $\chi_j(\rho)$ such that

(i) $\text{supp } \chi_j \subset \{ \rho \geq R_0(j + 1) \}$.

(ii) $\chi_j \equiv 1$ for $\rho \geq 3R_0(j + 1)$.

(iii) $|D_j^\gamma \chi_j(\rho)| \leq C_\gamma^{(j)}$, for $0 \leq \gamma \leq R_0(j + 1)$.

**Proof.** Let $\chi$ denote the characteristic function of the half line $[2R_0(j + 1), +\infty[$, and denote by $\psi$ a function in $C_0^\infty(\mathbb{R})$, supp $\psi \subset \{ |x| \leq 1 \}$, and such that $\int_\mathbb{R} \psi(x)dx = 1$.

Define, assuming that $R_0$ is an integer,

$\chi_j = \chi * \underbrace{\psi * \cdots * \psi}_{R_0(j+1) \text{ times}}$.

The support of $\chi_j$ is evidently contained in $\{ \rho \geq R_0(j + 1) \}$ and $\chi_j \equiv 1$ for $\rho \geq 3R_0(j + 1)$.

Moreover

$D^\gamma \chi_j(\rho) = \chi * D^\gamma \underbrace{\psi * \cdots * \psi}_{\gamma \text{ times}}$,

for $\gamma \leq R_0(j + 1)$. Hence, by Young inequality for the convolution,

$|D^\gamma \chi_j(\rho)| \leq (\| D^\gamma \psi \|_{L^1(\mathbb{R})})^\gamma$.

This completes the proof of the lemma.

Consider now

$\| \partial_{\rho}^{\gamma'} \partial_x^\alpha w_j(\rho) \partial_{\rho}^{\gamma} x^\beta \partial_x^\gamma \chi_j(\rho) u_j(x, \rho) \|_0$,

where $\| \cdot \|_0$ denotes the norm in $L^2(\mathbb{R}_\rho \times \mathbb{R}_x)$. We have

$\| \partial_{\rho}^{\gamma'} \partial_x^\alpha w_j(\rho) \partial_{\rho}^{\gamma} x^\beta \partial_x^\gamma \chi_j(\rho) u_j(x, \rho) \|_0$

$\leq \| \chi_j(\rho) \partial_{\rho}^{\gamma'} \partial_x^\alpha w_j(\rho) \partial_{\rho}^{\gamma} x^\beta \partial_x^\gamma u_j(x, \rho) \|_0$

$+ 2^{\gamma + \gamma'} \sum_{\gamma_1' + \gamma_1 = \gamma' + \gamma - k} C_{\gamma_1'}^{(j)} \cdot \| D_{\rho}^{\gamma_{1'}} \chi_j(\rho) \cdot \partial_{\rho}^{\gamma_1} \left( \partial_x^\alpha w_j(\rho) \partial_{\rho}^{\gamma} x^\beta \partial_x^\gamma u_j(x, \rho) \right) \|_0$
Next we also need a lemma to estimate $x$-derivatives of the projections $\pi$ and $1 - \pi$ applied to $\partial^{a'}_\rho x^\beta \partial^a_x u_j(x, \rho)$. This will be used in the proof of Theorem 9.1.

**Lemma 8.2.** Under the same assumptions of Theorem 8.1 we have the estimates

\[
|\partial^a_x \pi (\partial^{a'}_\rho x^\beta \partial^a_x u_j(x, \rho))| \leq M_1^{1+j+a+a'+\beta+\gamma}
\]
Our purpose is to turn \((9.2)\)

\[
\phi \rho^{-\gamma^2(j-\delta)x} \left( \frac{j}{s_0} + (\alpha + \alpha') \frac{q-1}{q} + \frac{\beta + \gamma q - 1}{q} \right)!
\]

and

\[
(8.5) \quad \left| \partial^{\gamma'}_{x}(1 - \pi) \left( \partial^\alpha_{\rho} \partial^\beta_{x} \partial^\gamma_{x} u_j(x, \rho) \right) \right| \leq M_{\alpha}^{1+j+\alpha+\beta+\gamma}
\]

\[
. \phi \rho^{-\gamma^2(j-\delta)x} \left( \frac{j}{s_0} + (\alpha + \alpha') \frac{q-1}{q} + \frac{\beta + \gamma q - 1}{q} \right)!
\]

for \( \rho \geq 3R_0(j + 1) \) and \( \gamma \leq j + \gamma^2 \) and for any \( \alpha, \alpha' \).

Proof. Let us start by proving (8.4). Since the term including the projection has been already proved. that the bound of the term with the identity follows from (8.3), while the bound of Lemma 9.1. Lemma 8.1.

In the preceding sections we constructed functions \( \psi_j \), \( j \geq 0 \), such that the formal sum \( u = \sum_{j \geq 0} u_j \) formally verifies the equation (2.19)

\[
(9.1) \quad P(x, y, D_x, D_y)A(u(x, y)) = \int_0^{+\infty} e^{i\rho^\alpha x} \rho^{j + 2a} \left[ \sum_{j=0}^{2a} \frac{1}{\rho^j} P_j(x, D_x, D_y)u(x, \rho) \right] d\rho = 0.
\]

Our purpose is to turn \( u \) into a true solution of an equation of the form

\[
(9.2) \quad P(x, y, D_x, D_y)A(u(x, y)) = f(x, y),
\]

for some smooth function \( f \) defined in \( \mathbb{R}_{\gamma}^2(x, y) \).

In order to do so we need a set of cutoff functions similar to those discussed in Lemma 8.1.

Lemma 9.1. There exist smooth functions \( \psi_j(\rho), j \geq 0 \), such that

(i) \( \mathsf{supp} \psi_j \subset \{ \rho \geq 3R_0(j + 1) \} \).

(ii) \( \psi_j \equiv 1 \) for \( \rho \geq 6R_0(j + 1) \).
Theorem 9.1. We have that
\[ \text{By (9.1) we have} \]
\[ \text{Proof.} \]
\[ \text{Definition 9.1.} \]
\[ \text{Remark 9.3.} \]
\[ \text{Theorem 9.1 below,} \]
\[ \text{We define the class} \]
\[ \text{for every} \]
\[ \text{It is a consequence of the above definition that a function} \]
\[ \text{Obviously} \]
\[ \text{First we recall the definition of the Beurling classes:} \]
\[ \text{Remark 9.3.} \]
\[ \text{We point out that} \]
\[ \text{We define the global class} \]
\[ \text{The remaining part of the present section is devoted to computing} \]
\[ \text{The proof is the same as that of Lemma 8.1 and we omit it.} \]
\[ \text{The idea behind the next theorem is the following: applying} \]
\[ \text{We define the class} \]
\[ \text{We define the class} \]
\[ \text{Theorem 9.1.} \]
\[ \text{Proof.} \]
\[ \text{Since the sum} \]
\[ \text{We have} \]
In view of (2.15), the above sum has a typical summand of the form
\[ \frac{\alpha}{\alpha'} |D^\gamma \psi_j| \left| \partial^{2a-k-\gamma} x^{m-\alpha'} \partial^m x^{m+\alpha-\alpha'} u_j \right|, \]
if \( k > 0 \), for \( m = 0, 1, \ldots, k \), with the proviso that the term is zero if \( 2a - k - \gamma < 0 \) or \( m - \alpha' < 0 \), and of the form
\[ \frac{\alpha}{\alpha'} \left| D^\gamma \psi_j \right| \left| \partial^{2a-\gamma} \partial^\alpha u_j \right|, \]
if \( k = 0 \). We have
\[ \sum_{k=0}^{2a} \frac{1}{\rho^k} P_k(x, D_x, D_\rho)v(x, \rho) = \sum_{k=0}^{2a} \frac{1}{\rho^k} P_k(x, D_x, D_\rho) \sum_{j=0}^{2a} \psi_j u_j \]
where \( P_k^{(\gamma)} \) denotes the symbol \( \partial^\gamma P_k(x, \xi, \sigma) \). Hence
\[ \sum_{k=0}^{2a} \frac{1}{\rho^k} P_k(x, D_x, D_\rho)v(x, \rho) = \sum_{j=0}^{2a} \frac{1}{\rho^k} \psi_j P_k u_j \]
\[ + \sum_{j=0}^{2a} \frac{1}{\rho^k} \sum_{\gamma=0}^{2a} \partial^\gamma \psi_j \frac{1}{\gamma!} P_k^{(\gamma)}(x, D_x, D_\rho) u_j \]
\[ = S_1(x, \rho) + S_2(x, \rho). \]
We remark that the \( \rho \)-derivatives appearing in the above expression have order at most \( 2a \).

For the first summand we are going to organize the terms using the transport equations (3.18a) and (3.18b), while for the second summand we use the estimates (8.3) and the fact that the derivatives of \( \psi_j \) have compact support.

Define
\[ f_i(x, y) = \int_0^{+\infty} e^{iy\rho'^\alpha - (2+\alpha)\frac{\rho^\gamma}{\rho}} |S_i(x, \rho)|_{x=x, \rho=x, \rho} d\rho. \]
Let us start with \( f_2 \). We have
\[ \partial^\alpha x \partial^\beta y f_2(x, y) = i^\beta \int_0^{+\infty} e^{iy\rho'^\alpha - (2+\alpha)\frac{\rho^\gamma}{\rho}} |S_2(x, \rho)|_{x=x, \rho=x, \rho} d\rho. \]
In view of (2.15), the \( x \)-derivatives of \( S_2 \) are given by
\[ \partial^\alpha x S_2(x, \rho) = \sum_{j=0}^{2a} \sum_{k=0}^{2a} \sum_{\gamma=0}^{2a} \sum_{\alpha'=0}^{2a} \left( \frac{\alpha}{\alpha'} \frac{1}{\rho^k} \partial^\gamma \psi_j \frac{1}{\gamma!} P_{k, (\alpha')}^{(\gamma)}(x, D_x, D_\rho) \partial^{\alpha-\alpha'} u_j, \right. \]
where \( P_{k, (\alpha')}^{(\gamma)} \) denotes the symbol \( \partial^\gamma \partial^\alpha \partial^\alpha P_k(x, \xi, \sigma) \) and we can assume without loss of generality that \( \alpha > 2a \). Let us estimate \( |\partial^\alpha x S_2(x, \rho)| \). We have
\[ |\partial^\alpha x S_2(x, \rho)| \]
\[ \leq \sum_{j=0}^{2a} \sum_{k=0}^{2a} \sum_{\gamma=0}^{2a} \left( \frac{\alpha}{\alpha'} \frac{1}{\rho^k} \partial^\gamma \psi_j \frac{1}{\gamma!} \left| P_{k, (\alpha')}^{(\gamma)}(x, D_x, D_\rho) \partial^{\alpha-\alpha'} u_j \right| \right). \]
Recalling the definitions (2.12), (2.19), (2.15) and (2.17), we see that the above sum has a typical summand of the form
\[ \left( \frac{\alpha}{\alpha'} \frac{1}{\gamma!} |D^\gamma \psi_j| \left| \partial^{2a-k-\gamma} x^{m-\alpha'} \partial^m x^{m+\alpha-\alpha'} u_j \right| \right), \]
if \( k > 0 \), for \( m = 0, 1, \ldots, k \), with the proviso that the term is zero if \( 2a - k - \gamma < 0 \) or \( m - \alpha' < 0 \), and of the form
\[ \left( \frac{\alpha}{\alpha'} \frac{1}{\gamma!} |D^\gamma \psi_j| \left| \partial^{2a-\gamma} \partial^\alpha u_j \right| \right), \]
if \( k = 0 \).
if \( k = 0 \). Let us start by examining the first type. First observe that, since \( \gamma > 0 \), the \( \rho \)-support of every term is contained in the interval \( 3R_0(j + 1) \leq \rho \leq 6R_0(j + 1) \) and that \( |D_{\rho}^j \varphi| \) is uniformly bounded since \( \gamma \) runs over a finite set of indices. As for the function \( u_j \) we may apply (8.3), thus obtaining
\[
\left| \partial_{\rho}^{2a-k-\gamma m-\alpha'} \partial_x^{m+\alpha-\alpha'} u_j \right| \\
\leq K_1^{1+j+m+\alpha-\alpha'+m-\alpha'}e^{\tilde{\alpha}_0 \rho - (j-\delta)\rho} \\
\cdot \left( \frac{j}{s_0} + (m+\alpha-\alpha')\frac{q-1}{q} + \frac{m-\alpha'}{q} + \frac{2a-k-\gamma q-1}{q} \right)!
\leq K_1^{1+j+k+\alpha-2a-\gamma \tilde{\alpha}_0 \rho - (j-\delta)\rho} \\
\cdot \left( \frac{j}{s_0} + \frac{k}{s_0} + (\alpha-\alpha')\frac{q-1}{q} - \frac{\alpha'}{q} + \frac{2a-\gamma q-1}{q} \right)!
\]
where we bound \( \rho \) by \( \tilde{\alpha}_0 \).

We can then see that there is a suitable positive constant, \( M_1 \), independent of \( j, \alpha, \alpha' \), such that
\[
\left| \partial_{\rho}^{2a-\gamma} \partial_x^{\alpha} u_j \right| \leq M_1^{1+j+\alpha}e^{\tilde{\alpha}_0 \rho - (j-\delta)\rho} \frac{1}{\alpha^{\frac{a+1}{\gamma}}}.
\]
Furthermore we can make the same argument to estimate the term with \( k = 0 \), i.e.
\[
\left| \partial_{\rho}^{2a-\gamma} \partial_x^{\alpha} u_j \right| \leq M_1^{1+j+\alpha}e^{\tilde{\alpha}_0 \rho - (j-\delta)\rho} \frac{1}{\alpha^{\frac{a+1}{\gamma}}}.
\]
Hence, when \( k > 0 \) we get
\[
\left( \frac{\alpha}{\alpha'} \right)^\frac{1}{\gamma!} |D_{\rho}^j \varphi| \left| \partial_{\rho}^{2a-k-\gamma m-\alpha'} \partial_x^{m+\alpha-\alpha'} u_j \right| \\
\leq M_1^{1+j+\alpha} \left( \frac{1}{3R_0} \right)^\frac{1}{\alpha'} \left| D_{\rho}^j \varphi \right| e^{\tilde{\alpha}_0 \rho - (j-\delta)\rho} \frac{1}{\alpha^{\frac{a+1}{\gamma}}} \\
\leq M_1^{1+j+\alpha} C_1 \left( \frac{1}{3R_0} \right)^\frac{1}{\alpha'} \tilde{\alpha}_0 \rho - (\frac{1}{\alpha'} - \frac{1}{\alpha}) \frac{1}{\alpha^{\frac{a+1}{\gamma}}} \\
\leq M_1^{1+j+\alpha} C_2 \left( \frac{1}{3R_0} \right)^\frac{1}{\alpha'} e - \left( \frac{1}{\alpha' - \frac{1}{\alpha}} \right)^{\rho} \frac{\rho + \frac{\alpha'}{\alpha}}{\alpha^{\frac{a+1}{\gamma}}},
\]
and an analogous estimate for the term containing \( 9.10 \).

Plugging the above estimate into (9.8) and keeping into account that the summations in \( k, \gamma, \alpha' \) run over a finite range of indices, we obtain with a new positive constant \( M_3 \)
\[
\left| \partial_{\rho}^{\alpha} S_2(x, \rho) \right| \leq \sum_{j \geq 0} M_1^{1+j+\alpha} \left( \frac{1}{3R_0} \right)^\frac{1}{\alpha'} e - \left( \frac{1}{\alpha' - \frac{1}{\alpha}} \right)^{\rho} \frac{\rho + \frac{\alpha'}{\alpha}}{\alpha^{\frac{a+1}{\gamma}}},
\]
Thus choosing \((3R_0)^\frac{1}{\alpha'} > M_3\) we get
\[
\left| \partial_{\rho}^{\alpha} S_2(x, \rho) \right| \leq M_4^{1+\alpha} e - \left( \frac{1}{\alpha' - \frac{1}{\alpha}} \right)^{\rho} \frac{\rho + \frac{\alpha'}{\alpha}}{\alpha^{\frac{a+1}{\gamma}}},
\]
We need the following
Lemma 9.2. Let $\mu > 0$, $s > 1$. For any $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that

\begin{equation}
\int_0^{\pm \infty} e^{-\mu \log \rho} \rho^{s\alpha} d\rho \leq C_\varepsilon \varepsilon^{\alpha} \alpha^{s}. \tag{9.14}
\end{equation}

Proof of Lemma 9.2. Pick a positive $M$ to be chosen later and write

\begin{equation}
\int_0^{\pm \infty} e^{-\mu \log \rho} \rho^{s\alpha} d\rho = \int_0^M e^{-\mu \log \rho} \rho^{s\alpha} d\rho + \int_M^{\pm \infty} e^{-\mu \log \rho} \rho^{s\alpha} d\rho = I_1 + I_2.
\end{equation}

Consider $I_2$. Because $e^{-\mu \log \rho} \leq e^{-\mu \log M}$, we get

\begin{equation}
I_2 \leq \int_0^{\pm \infty} e^{-\mu \log M} \rho^{s\alpha} d\rho = \left( \frac{1}{\mu \log M} \right)^{s\alpha+1} \alpha^{s}. \tag{9.15}
\end{equation}

Choosing $\mu^{-\varepsilon}(\log M)^{-s} \leq \varepsilon$ we prove the assertion for $I_2$.

Consider $I_1$.

\begin{equation}
I_1 \leq e^{\mu/\varepsilon} M^{s\alpha+1} \alpha^{s} \leq e^{\mu/\varepsilon} M^{\left( \frac{\alpha}{\alpha s} \right) +} \varepsilon^{\alpha} \alpha^{s}.
\end{equation}

and this implies the assertion also for $I_1$. \hfill \Box

Going back to the derivative of $f_2$ we have

\begin{equation}
|\partial_x^{\alpha_1} \partial_y^{\alpha_2} f_2(x, y)| \leq \int_0^{\pm \infty} \rho^{\Re (2+\alpha) \frac{\alpha_1}{q} + \beta \frac{\alpha_2}{q}} d\rho \leq \int_0^{\pm \infty} \rho^{\Re (2+\alpha) \frac{\alpha_1}{q} + \beta \frac{\alpha_2}{q}} e^{-\left( \frac{\alpha_1}{\alpha_0 \rho} \right) \rho^{\log \rho} + \frac{\alpha_2}{2} \rho} d\rho
\end{equation}

Applying Lemma 9.2 we obtain that

\begin{equation}
|\partial_x^{\alpha_1} \partial_y^{\alpha_2} f_2(x, y)| \leq M_5^{1+\alpha} \alpha^{\frac{1}{q-1}} \varepsilon_1 \varepsilon_1^{\frac{1}{q-1}} + \beta \left( \frac{\alpha}{q} + \beta \right)^{s_0} \leq \tilde{C}_\varepsilon \varepsilon_1^{\alpha + \beta} \alpha^{\frac{1}{q-1} + \frac{m_0}{q}} \beta^{s_0}, \tag{9.16}
\end{equation}

since the inequality

\begin{equation}
\frac{q - 1}{q} + \frac{s_0}{q} < s_0
\end{equation}

is obviously true, being equivalent to $s_0 > 1$. This proves that the assertion is true for $f_2$.

Consider now $f_1$:

\begin{equation}
f_1(x, y) = \int_0^{\pm \infty} e^{i\gamma \rho^{s_0}} \rho^{r+2 \frac{\alpha}{q}} \left[ \sum_{j \geq 0} \sum_{k=0}^{2a} \rho^k \psi_0 P_k \psi_j(x, \rho) \right] \rho^s d\rho.
\end{equation}

Using Proposition 3.3 or rather its proof, we may regroup the terms in the above summation according to the scheme of (3.18a), (3.18b).
In view of Lemma 8.2, arguing as we did before for the derivatives of (9.17)

\[ f_1(x, y) = \int_0^{+\infty} e^{iy\rho\alpha} \rho^{r+2a} \left[ \psi_0 P_0 u_0 + \sum_{j \geq 1} \left( \sum_{k=0}^{\min\{j, 2a\}} \frac{1}{\rho^k} \psi_{j-k}(1 - \pi) P_k u_{j-k} \right) \right. \]

\[ + \left. \sum_{j \geq 1} \left( \psi_j \pi P_0 u_j + \frac{1}{\rho} \psi_j \pi P_1(1 - \pi) u_j + \frac{1}{\rho} \psi_{j-1} \pi P_1 \pi u_{j-1} \right) \right] d\rho \]

\[ = f_{10}(x, y) + f_{11}(x, y) + f_{12}(x, y). \]

Observe that \( f_{10} = 0 \) since \( P_0 u_0 = 0 \) because of (3.15).

Consider \( f_{11} \). For a fixed \( j \), the support of

\[ \sum_{k=0}^{\min\{j, 2a\}} \frac{1}{\rho^k} \psi_{j-k}(1 - \pi) P_k u_{j-k} \]

is contained in the interval \([3R_0((j - 2a)_+ + 1), 6R_0(j + 1)]\). In fact for \( \rho \geq 6R_0(j + 1) \) the functions \( \psi_{j-k} \equiv 1 \) and (3.18a) is satisfied. On the other hand if \( \rho \leq 3R_0((j - 2a)_+ + 1) \), then \( \psi_{j-k} \equiv 0 \).

Moreover

\[ \partial_x^\alpha \partial_y^\beta f_{11}(x, y) = i^\beta \int_0^{+\infty} e^{iy\rho\alpha} \rho^{r+2a} \left[ \psi_j \pi P_0 u_j + \frac{1}{\rho} \psi_j \pi P_1(1 - \pi) u_j + \frac{1}{\rho} \psi_{j-1} \pi P_1 \pi u_{j-1} \right. \]

\[ + \left. \sum_{k=1}^{\min\{j, 2a\}-1} \frac{1}{\rho^k} \psi_{j-k} \pi P_{k+1} u_{j-k} \right] d\rho. \]

In view of Lemma 8.2, arguing as we did before for the derivatives of \( f_2 \), we conclude that \( f_{11} \in \mathcal{B}_\rho^{\alpha\beta}(\mathbb{R}^2) \).

Finally consider \( f_{12} \). We first remark that since \( \pi P_1 \pi = 0 \), due to Lemma 6.21

\[ \psi_j \pi P_0 u_j + \frac{1}{\rho} \psi_j \pi P_1(1 - \pi) u_j + \frac{1}{\rho} \psi_{j-1} \pi P_1 \pi u_{j-1} \]

\[ + \sum_{k=1}^{\min\{j, 2a\}-1} \frac{1}{\rho^k} \psi_{j-k} \pi P_{k+1} u_{j-k} \]

\[ = \psi_j \pi P_0 u_j + \frac{1}{\rho} \psi_j \pi P_1 u_j + \sum_{k=1}^{\min\{j, 2a\}-1} \frac{1}{\rho^k} \psi_{j-k} \pi P_{k+1} u_{j-k} \]

Again as above we have that, for a fixed \( j \), the \( \rho \)-support of the above quantity is contained in the interval \([3R_0((j - 2a)_+ + 1), 6R_0(j + 1)]\).

Arguing as above we achieve the proof of the theorem.

We remark that, as a consequence of Theorem 9.1, we found a function \( f(x, y) \in \mathcal{B}_\rho^{\alpha\beta}(\mathbb{R}^2) \) such that

\[ PA(v) = f, \quad \text{in } \mathbb{R}^2. \]
We also need a technical variant of Theorem 9.1 adding a finite order vanishing rate at infinity to the property of being in $B^s_v(\mathbb{R}^2)$.

**Corollary 9.1.** We use the same notation of Theorem 9.1. Let $b > 1$ denote a fixed positive integer and $\varepsilon > 0$. Then for every $k, 1 \leq k \leq b$, there is a constant $C_{\varepsilon} > 0$ such that

\[(9.20) \quad |(y^k P_x^\alpha \partial_y^\beta P A(v)(x,y))| \leq C_{\varepsilon} \varepsilon^{\alpha + \beta} \alpha^{\mu_0} \beta^{\mu_0},\]

for every $\alpha, \beta \in \mathbb{N} \cup \{0\}$ and for any $(x,y) \in \mathbb{R}^2$.

**Proof.** The proof is made along the same lines as the proof of Theorem 9.1 so we are going to give a sketch of it detailing those parts where it is different from that of the above theorem.

We have, using (2.6) with $2a$ replaced by $k$,

\[y^k \partial_x^\alpha \partial_y^\beta (P(x,y,D_x,D_y)A(v)(x,y)) = \int_0^{+\infty} e^{iy\rho + \alpha} \sum_{h=0}^{k} \frac{(-1)^h}{\rho^{k_{s_0} - h}} \partial_y^h \rho^{(2 + \alpha) + \beta} \partial_x^\alpha \sum_{j=0}^{2a} \frac{1}{\rho^j} P_j(x,D_x,D_y) v(x,\rho) \bigg|_{x=x_\rho} \frac{\rho^b}{\varepsilon^{\mu_0}} d\rho.\]

Proceeding as we did for (2.9), (2.13), (2.15) we obtain

\[y^k \partial_x^\alpha \partial_y^\beta (P(x,y,D_x,D_y)A(v)(x,y)) = \int_0^{+\infty} e^{iy\rho + \alpha} \rho^{k(1-s_0) + N_{\alpha,\beta}} \cdot \sum_{j=0}^{k} \frac{1}{\rho^j} P_j^\#(x,\partial_x) \partial_x^{k-j} \partial_y^\alpha \sum_{j=0}^{2a} \frac{1}{\rho^j} P_j(x,D_x,D_y) v(x,\rho) \bigg|_{x=x_\rho} \frac{\rho^b}{\varepsilon^{\mu_0}} d\rho,\]

where $P_j^\#$ are polynomials of degree $j'$ in $x \partial_x$ whose coefficients are $O(C^{\alpha + \beta})$ with $C$ a positive absolute constant.

Using (9.7) we are reduced to estimating, for $i = 1, 2$, the expressions

\[\int_0^{+\infty} e^{iy\rho + \alpha} \rho^{k(1-s_0) + N_{\alpha,\beta}} \sum_{j=0}^{k} \frac{1}{\rho^j} P_j^\#(x,\partial_x) \partial_x^{k-j} \partial_y^\alpha \partial_y^\alpha S_i(x,\rho) \bigg|_{x=x_\rho} \frac{\rho^b}{\varepsilon^{\mu_0}} d\rho.\]

Consider first the quantity involving $S_2$. The typical summand in the quantity in square brackets has the form

\[O(C^{\alpha + \beta}) \frac{1}{\rho^\gamma} x^\gamma \partial_x^\gamma \partial_y^{k-j} S_2(x,\rho),\]

where $\gamma, j' \leq k$.

Arguing as for the derivation of (9.13), choosing $\gamma^\#$ large enough depending on $k$, we obtain

\[(9.21) \quad |O(C^{\alpha + \beta}) x^\gamma \partial_x^\gamma \partial_y^{k-j} S_2(x,\rho)|\]
Consider derivatives of $A$ for any $\delta > 0$ to show that for any $C$ for a suitable $\lambda > 0$.

Here is a brief description of what we are going to do in what follows. First we proceed analogously to the proof of Theorem 9.1, from (8.3) we get

\[ \text{Then the conclusion follows as in the proof of Theorem 9.1.} \]

Then the proof goes along the same lines as that of Theorem 9.1.

This completes the proof of the corollary.

\[ \square \]

10. END OF THE PROOF

To finish the proof of Theorem 1.1 we argue by contradiction. Assume that $P$ is Gevrey-$s$ hypoelliptic for an $s$, with $1 \leq s < s_0$.

By Theorem 3.1 in [22] it follows from (9.19) that

\[ A(v) \in B^{s_0}(\mathbb{R}^2). \]

Here is a brief description of what we are going to do in what follows. First we show that any $y$-derivative of $A(v)$ is integrable over $\mathbb{R}$ for $x = 0$. Then we prove that $A(v)(0, y) \in B^{s_0}(\mathbb{R})$ with some decreasing rate at infinity, and this allows us to show that for any $\delta > 0$

\[ |\mathcal{F}(A(v))(0, \eta)| \leq C_\delta e^{-\delta^{-1}|\eta|^{a_0}}, \]

for a suitable $C_\delta > 0$.

On the other hand the construction of $A(v)$ implies that its Fourier transform satisfies a bound from below of the form

\[ |\mathcal{F}(A(v))(0, \eta)| \geq C_0 \eta^{\lambda} e^{\hat{\mu}_0 \eta^{-\delta}}, \quad \eta \geq (6R_0)^{\lambda_0}, \]

for a suitable $C_0 > 0$ and $\lambda \in \mathbb{R}$, where $\hat{\mu}_0$ is defined in (5.6). This is the desired contradiction.

**Lemma 10.1.** For any $\alpha, \beta$ there exists a positive constant $C_{\alpha, \beta}$ such that

\[ (1 + x^{2k} + y^{2k}) \left| \partial_x^\alpha \partial_y^\beta A(v)(x, y) \right| \leq C_{\alpha, \beta}, \]

for $k \leq b$, $b \in \mathbb{N}$ a fixed integer. In particular, if we choose $b$ suitably, the $(\alpha, \beta)$-derivatives of $A(v)$ are in $L^2(\mathbb{R}^2)$.

**Proof.** Consider

\[ x^{2k} \partial_x^\alpha D_y^\beta A(v)(x, y) = \int_0^{+\infty} e^{iy\rho^{\alpha_0} \rho^{r+(\alpha-2k)\alpha_q + \beta s_0}} \left( x^{2k} \partial_x^\alpha v \right)(x, x^{\alpha_q} \rho) \rho d\rho. \]

Proceeding analogously to the proof of Theorem 9.1 from (8.3) we get

\[ |x^{2k} \partial_x^\alpha v(x, \rho)| \leq \sum_{j \geq 0} \psi_j(\rho) \left| x^{2k} \partial_x^\alpha u_j(x, \rho) \right| \leq \sum_{j \geq 0} \psi_j(\rho) e^{	ilde{\mu}_0 \rho} K_1^{1+j+\alpha+2k} \rho^{-(j-\delta)\alpha} \left( \frac{j}{s_0} + \alpha \frac{q-1}{q} + \frac{2k}{q} \right)! \]
\[
\leq C_\alpha \varepsilon \rho \sum_{j \geq 0} K_{\alpha + j} (3R_0(j + 1))^{-(j - \delta)\kappa} (j + 1)^{\frac{a}{\delta}} \leq C_\alpha \varepsilon \rho \sum_{j \geq 0} K_{\alpha + j} (j + 1)^{\delta^{-\kappa}} = C_2 \alpha \varepsilon \rho,
\]
whence we conclude.

Consider then \( y^{2k} \partial_x^\alpha \partial_y^\beta A(v)(x, y) \). Arguing in the same way as we did when deducing (2.18), and disregarding the behaviour of the coefficients, we may write

\[
y^{2k} \partial_x^\alpha \partial_y^\beta A(v)(x, y) = \int_0^{\infty} e^{iy\rho^\alpha} \rho^{N_{\alpha, \beta}} \sum_{\ell = 0}^{2k} \frac{1}{\rho^\ell} [L_\ell(x \partial_x) \partial_x^\ell \psi(x, \rho)] \rho^\ell \partial_x^\ell \psi(x, \rho) d\rho.
\]

where \( N_{\alpha, \beta} \) is a suitable complex number independent of \( \ell \) and \( L_\ell \) is a polynomial of degree \( \ell \) with respect to \( x \partial_x \).

In order to estimate the above integral we proceed as before using (8.3). Because of (9.3) the sum inside the above integral reads

\[
\sum_{j \geq 0} \sum_{\ell = 0}^{2k} \frac{1}{\rho^\ell} \partial_x^\ell L_\ell(x \partial_x) \partial_x^\ell \psi(x, \rho) u_j(x, \rho).
\]

Writing \( L_\ell(x \partial_x) = \sum_{m = 0}^\ell c_{\ell m} (x \partial_x)^m \) we have, for fixed \( j \), a finite sum \( (\ell \leq 2k \leq b \) and \( b \) is a fixed positive integer) whose typical summand has the form

\[
c_{\ell m} \rho^{2k - \ell} x^m \partial_x^{m + \alpha} \psi(x, \rho) u_j(x, \rho).
\]

Choosing \( R_0, \gamma \# \) so that \( 4k \leq 2\gamma \# \leq R_0 \), we obtain by (8.3) and Lemma 9.1 that the above quantity is bounded by

\[
C_2 1^{\alpha + j} \varepsilon \rho^{-(j - \delta)\kappa} \left( \frac{j}{s_0} + a \frac{q - 1}{q} \right)!.
\]

We may then argue as in (10.3) and conclude that (10.4) holds. \( \square \)

To prove (10.2) we follow the proving lemma, from which (10.2) will be deduced.

**Lemma 10.2.** For any \( \varepsilon > 0 \), there exists a \( C_\varepsilon > 0 \), such that for any index \( \alpha \) we have

\[
|\langle y \rangle^\alpha \partial_y^\alpha \partial_x^\beta A(v)(0, y)| \leq C_\varepsilon \varepsilon \alpha!^\alpha,
\]

where \( \beta = 0, 1 \) and \( \langle y \rangle = 1 + y^2 \frac{1}{\delta} \).

**Proof.** Let \( \chi \) denote a smooth function in \( G^{s'}(\mathbb{R}) \), \( s' < s_0 \), such that \( \chi(t) = 0 \) for \( |t| \leq 1 \) and \( \chi(t) = 1 \) for \( |t| \geq 2 \).

Observe that, by (9.19),

\[
P \chi(y) A(v) = \chi(y) f - [P, \chi] A(v).
\]

By Corollary 9.1 and formula (10.1), if we denote by \( g \) the right hand side of the above equation and \( \psi(x) \in G^0_0(\mathbb{R}) \) we have that

\[
|\langle y \rangle^\alpha \partial_y^\alpha \partial_x^\beta (\psi g)(x, y)| \leq C_\delta \delta^{\alpha + \alpha} \alpha!^\alpha \beta!^\beta,
\]
for every \((x, y) \in \mathbb{R}^2\) and \(\delta > 0\). From the above inequality it readily follows that, possibly renaming \(C_\delta\),

\[
\|\partial_y^\alpha \partial_x^\beta (\psi g)(x, y)\|_0 \leq C_\delta \delta^{\alpha + \beta} \alpha! \beta! \epsilon!
\]

We are going to use the maximal estimate for the above equation:

\[
\sum_{i=1}^{3} \|X_i u\|^2_0 \leq C \left( \|Pu, u\| + \|y^{\alpha-1} u\|^2_0 \right),
\]

where \(u \in L^2(\mathbb{R}^2)\), \(Pu \in L^2(\mathbb{R}^2)\) and \(y^{\alpha-1} u \in L^2(\mathbb{R}^2)\), and \(X_1 = D_x\), \(X_2 = x^{q-1} D_y\), \(X_3 = y^{q} D_y\).

Let \(\varphi = \varphi(x) \in \mathcal{G}_0' (\mathbb{R})\) denote a cutoff function near the origin.

We want to show that for any \(\epsilon > 0\), for any positive integers \(\alpha, \beta\), with \(0 \leq \beta \leq N\), \(N\) arbitrarily fixed natural number, there exists a positive constant \(C_\epsilon\) such that

\[
\|X_i \partial_y^\alpha \partial_x^\beta \varphi A(v)\|_0 \leq C_\epsilon \epsilon^{\alpha} \alpha! \epsilon!,
\]

\(i = 1, 2, 3\). Since \(A(v) \in \mathcal{B}^\gamma(\mathbb{R}^2)\), it is enough to prove \((10.10)\) for \(A'(v) = \chi A(v)\).

Actually we are going to prove that for any \(\epsilon_1 > 0\), \(\epsilon_2 > 0\), for any positive integers \(\alpha\) and \(\beta\), \(0 \leq \beta \leq N\), there exists a positive constant \(C_{\epsilon_1 \epsilon_2}\) such that

\[
\sum_{i=1}^{3} \|X_i \partial_y^\alpha \partial_x^\beta \varphi^{(k)} A'(v)\|_0 \leq C_{\epsilon_1 \epsilon_2} \epsilon_1^{\alpha + \beta} \epsilon_2^{k} (\alpha + \beta + k)! \epsilon!.
\]

The estimate \((10.10)\) then follows keeping into account that \(\beta\) takes a finite number of values and choosing \(\epsilon_1, \epsilon_2\) suitably.

We proceed by induction on \(\alpha + \beta\). Actually \((10.11)\) is true when \(\alpha + \beta = 0\). In fact we use Lemma \((10.1)\) as well as the fact that \(\varphi \in \mathcal{G}_0' (\mathbb{R}) \subset \mathcal{B}^\gamma_0 (\mathbb{R})\).

Consider

\[
\sum_{i=1}^{3} \|X_i \partial_y^\alpha \partial_x^\beta \varphi^{(k)} A'(v)\|^2_0.
\]

By \((10.9)\) we have

\[
\sum_{i=1}^{3} \|X_i \partial_y^\alpha \partial_x^\beta \varphi^{(k)} A'(v)\|^2_0 \\
\leq C \left( \|\partial_y^\alpha \partial_x^\beta \varphi^{(k)} A'(v)\|_0 \right) \\
+ \|y^{\alpha-1} \partial_y^\alpha \partial_x^\beta \varphi^{(k)} A'(v)\|^2_0 \\
\leq C_1 \frac{\|\partial_y^\alpha \partial_x^\beta \varphi^{(k)} [P(A'(v)), \partial_y^\alpha \partial_x^\beta \varphi^{(k)} A'(v)]\|_0}{B_1} \\
+ 2 \sum_{i=1}^{3} \frac{\|X_i \partial_y^\alpha \partial_x^\beta \varphi^{(k)} A'(v)\|^2_0}{B_{2i}} \\
+ \|X_i \partial_y^\alpha \partial_x^\beta \varphi^{(k)} A'(v)\|^2_0 \frac{\|X_i \partial_y^\alpha \partial_x^\beta \varphi^{(k)} A'(v)\|^2_0}{B_{3i}}
\]

\(i = 1, 2, 3\).
Let us start with $B_1$. Let $\psi \in \mathcal{G}_0^{'}(\mathbb{R})$ be such that $\varphi \psi = \varphi$. Then,

$$|B_1| \leq \left( \|\partial_y^\alpha \partial_x^\beta \varphi^{(k)} P(A'(v))\|_0 + \|\partial_y^\alpha \partial_x^\beta \varphi^{(k)} A'(v)\|_0 \right)^2$$

$$\leq C_2 \left( \sum_{\ell=0}^\beta \left( \begin{array}{c} \beta \\ \ell \end{array} \right) \|\varphi^{(k+\ell)} \partial_y^\alpha \partial_x^{\beta-\ell} \psi P(A'(v))\|_0 + \|X \partial_y^\alpha \partial_x^{\beta'} \varphi^{(k)} A'(v)\|_0 \right)^2,$$

where $X$ denotes the vector field $X_1$ or $X_4$ according to the fact that $\beta > 0$ or $\alpha > 0$ respectively, since on the support of $A'(v)$, $|y| \geq c > 0$ and $\alpha' + \beta' = \alpha + \beta - 1$.

Consider the first term in the sum above. By assumption (10.8) we have

$$\sum_{\ell=0}^\beta \left( \begin{array}{c} \beta \\ \ell \end{array} \right) \|\varphi^{(k+\ell)} \partial_y^\alpha \partial_x^{\beta-\ell} \psi P(A'(v))\|_0$$

$$\leq \sum_{\ell=0}^\beta \left( \begin{array}{c} \beta \\ \ell \end{array} \right) C_\delta_2 \delta_2^{k+\ell} (k+\ell)!^s \delta_2^\alpha \delta_2^{\alpha-\ell} (\alpha + \beta - \ell)!^s$$

$$\leq \frac{1}{\nu C_2} C_{\varepsilon_1 \epsilon_2 \varepsilon_1^\alpha \varepsilon_2^\beta} (\alpha + \beta + k)!^s,$$

choosing $\delta_1 < \frac{\beta}{\nu}$, $\delta_2 < \min\{\varepsilon_2, \delta_1\}$ and $C_{\varepsilon_1 \epsilon_2} > \nu C_\delta_2 C_{\delta_2} C_2$.

As for the second term we have, by applying the inductive assumption (10.11), that

$$\|X \partial_y^\alpha \partial_x^{\beta'} \varphi^{(k)} A'(v)\|_0 \leq C_{\varepsilon_1 \epsilon_2} \varepsilon_1^{\alpha+\beta} \varepsilon_2^{k} (\alpha' + \beta' + k)!^s$$

$$= C_{\varepsilon_1 \epsilon_2} \frac{1}{(\alpha + \beta + k)!^s} \varepsilon_1^{\alpha+\beta} \varepsilon_2^{k} (\alpha + \beta + k)!^s$$

$$\leq \frac{1}{\nu C_{\varepsilon_1 \epsilon_2} \varepsilon_1^{\alpha+\beta} \varepsilon_2^k (\alpha + \beta + k)!^s},$$

provided

$$\alpha + \beta + k > (\nu \varepsilon_1^{-1})^\frac{1}{\alpha}.$$  

By Lemma [10.1] we obtain the same estimate for any $\alpha$, $\beta$, $0 \leq \beta \leq N$, $k$, if $C_{\varepsilon_1 \epsilon_2}$ is chosen large enough.

Consider now $B_{2i}$. Remark that if $i = 1$, $[X_1, \partial_y^\alpha \partial_x^{\beta} \varphi^{(k)}] = \partial_y^\alpha \partial_x^{\beta} \frac{1}{i} \varphi^{(k+1)}$. Hence for $i = 1$ we have

$$\langle [X_1, \partial_y^\alpha \partial_x^{\beta} \varphi^{(k)}] A'(v), X_1^* \partial_y^\alpha \partial_x^{\beta} \varphi^{(k)} A'(v) \rangle$$

$$= \langle \partial_y^\alpha \partial_x^{\beta} \frac{1}{i} \varphi^{(k+1)} A'(v), X_1 \partial_y^\alpha \partial_x^{\beta} \varphi^{(k)} A'(v) \rangle.$$
Thus
\[
\left| \langle [X_1, \partial_y^\alpha \partial_x^\beta \varphi^{(k)}] A'(v), X_1^* \partial_y^\alpha \partial_x^\beta \varphi^{(k)} A'(v) \rangle \right|
\leq \| \partial_y^\alpha \partial_x^\beta \varphi^{(k+1)} A'(v) \|_0 \cdot \| X_1 \partial_y^\alpha \partial_x^\beta \varphi^{(k)} A'(v) \|_0
\leq \mu \| X_1 \partial_y^\alpha \partial_x^\beta \varphi^{(k)} A'(v) \|_0^2 + \frac{1}{\mu} \| \partial_y^\alpha \partial_x^\beta \varphi^{(k+1)} A'(v) \|_0^2.
\]

The first term is absorbed on the left hand side of (10.13) if \( \mu \) is small enough. We are left with the second term. As before it can be bounded by
\[
C_3 \frac{1}{\mu} \| X \partial_y^\alpha \partial_x^\beta \varphi^{(k+1)} A'(v) \|_0^2,
\]
where \( \alpha' + \beta' = \alpha + \beta - 1 \), and \( X \) denotes either \( X_1 \) or \( X_3 \) according to the fact that \( \beta > 0 \) or \( \alpha > 0 \) respectively, since, on the support of \( \varphi^{(k+1)} A'(v) \), \( y \) is bounded away from zero.

The above norm is bounded by
\[
C_3 \frac{1}{\mu} C_{\varepsilon_1 \varepsilon_2} \varepsilon_1^{\alpha + \beta} \varepsilon_2^{k+1} (\alpha' + \beta' + k + 1)! \varepsilon_0,
\]
which becomes
\[
\left( C_3 \frac{1}{\mu} \frac{\varepsilon_2}{\varepsilon_1} \right) C_{\varepsilon_1 \varepsilon_2} \varepsilon_1^{\alpha + \beta} \varepsilon_2^{k} (\alpha + \beta + k)! \varepsilon_0,
\]
so that, choosing
\[
(10.15) \quad \frac{\varepsilon_2}{\varepsilon_1} < \frac{1}{\nu C_3},
\]
we obtain, modulo terms absorbed on the left,
\[
(10.16) \quad \left| \langle [X_1, \partial_y^\alpha \partial_x^\beta \varphi^{(k)}] A'(v), X_1^* \partial_y^\alpha \partial_x^\beta \varphi^{(k)} A'(v) \rangle \right|
\leq \frac{1}{\nu} C_{\varepsilon_1 \varepsilon_2} \varepsilon_1^{\alpha + \beta} \varepsilon_2^{k} (\alpha + \beta + k)! \varepsilon_0.
\]

Consider now \( B_{22} \). Since \([X_2, \partial_y^\alpha \partial_x^\beta \varphi^{(k)}] = \partial_y^\alpha [x^{q-1}, \partial_x^\beta \varphi^{(k)}] D_y\), we may write, for suitable positive constants \( C_q, C_{1q} \), depending only on \( q \),
\[
\left| \langle [X_2, \partial_y^\alpha \partial_x^\beta \varphi^{(k)}] A'(v), X_2 \partial_y^\alpha \partial_x^\beta \varphi^{(k)} A'(v) \rangle \right|
\leq C_q \min \{ \beta, q-1 \} \sum_{\ell=1}^{\min \{ \beta, q-1 \}} \left( \begin{array}{c} \beta \\ \ell \end{array} \right) \| x^{q-1-\ell} \partial_x^{\beta-\ell} \varphi^{(k)} \partial_y^{\alpha+1} A'(v), X_2 \partial_y^\alpha \partial_x^\beta \varphi^{(k)} A'(v) \|_0 \cdot \| X_2 \partial_y^\alpha \partial_x^\beta \varphi^{(k)} A'(v) \|_0
\leq C_{1q} \sum_{\ell=1}^{\min \{ \beta, q-1 \}} \frac{\beta!}{(\beta-\ell)!} \| x^{q-1-\ell} \partial_x^{\beta-\ell} \partial_y^{\alpha+1} A'(v) \|_0 \cdot \| X_2 \partial_y^\alpha \partial_x^\beta \varphi^{(k)} A'(v) \|_0
\leq C_{1q} \left( \mu \| X_2 \partial_y^\alpha \partial_x^\beta \varphi^{(k)} A'(v) \|_0^2 + \frac{1}{\mu} \sum_{\ell=1}^{\min \{ \beta, q-1 \}} \frac{\beta!}{(\beta-\ell)!} \| x^{q-1-\ell} \partial_x^{\beta-\ell} \partial_y^{\alpha+1} \varphi^{(k)} A'(v) \|_0 \right)^2.
\]
The first term is absorbed on the left hand side of (10.13) provided \( \mu \) is small but otherwise fixed, so that we are left with the estimate of the square of the sum. The latter can be bounded as

\[
\min\{\beta,q-1\} C_4 \sum_{\ell=1} \beta(\beta - 1) \cdots (\beta - \ell + 1) \| x_3 \partial_x^{\beta-\ell} \partial_y^{\alpha} \varphi(k) A'(v) \|_0
\]

\[
\leq C_4 \sum_{\ell=1} \beta(\beta - 1) \cdots (\beta - \ell + 1) C_{\varepsilon_1 \varepsilon_2} \varepsilon_1^{\beta-\ell+\alpha} \varepsilon_2^k (\alpha + \beta - \ell + k)!^{\kappa_0}
\]

\[
\leq C_{\varepsilon_1 \varepsilon_2} \varepsilon_1^{\beta+\alpha} \varepsilon_2^k (\alpha + \beta + k)!^{\kappa_0}, C_4 \sum_{\ell=1} \min\{\beta,q-1\} \left( \frac{1}{(\alpha + \beta + k - (q-1))^{\kappa_0-1} \varepsilon_1} \right) ^\ell
\]

\[
\leq \frac{1}{\mu} \sqrt{\mu} C_{\varepsilon_1 \varepsilon_2} \varepsilon_1^{\beta+\alpha} \varepsilon_2^k (\alpha + \beta + k)!^{\kappa_0},
\]

provided

\[
\alpha + \beta + k - (q-1) \geq \left( \frac{2\nu C_4}{\sqrt{\mu \varepsilon_1}} \right)^{\frac{1}{\kappa_0}}.
\]

By Lemma [10.4] we obtain the same estimate of \( B_{22} \) for any \( \alpha, \beta, k \), if \( C_{\varepsilon_1 \varepsilon_2} \) is chosen large enough.

Consider now \( B_{23} \) in (10.13). Since \([X_3, \partial_0^a \partial_x^\beta \varphi(k)] = [X_3, \partial_0^a] \partial_x^\beta \varphi(k) = [y^a, \partial_0^a] D_y \partial_x^\beta \varphi(k)\). Hence

\[
\left| \langle [X_3, \partial_0^a \partial_x^\beta \varphi(k)] A'(v), X_3 \partial_0^a \partial_x^\beta \varphi(k) A'(v) \rangle \right|
\]

\[
\leq C_{1a} \sum_{\ell=1} \min\{\alpha,a\} \left( \frac{\alpha}{\ell} \right) \left| \langle y^{a-\ell} \partial_y^{a-\ell+1} \partial_x^\beta \varphi(k) A'(v), X_3 \partial_0^a \partial_x^\beta \varphi(k) A'(v) \rangle \right|
\]

\[
\leq C_{1a} \sum_{\ell=1} \min\{\alpha,a\} \left( \frac{\alpha}{\ell} \right) \| y^{a-\ell} \partial_y^{a-\ell+1} \partial_x^\beta \varphi(k) A'(v) \|_0 \| X_3 \partial_0^a \partial_x^\beta \varphi(k) A'(v) \|_0
\]

\[
\leq C_{2a} \left( \mu \| X_3 \partial_0^a \partial_x^\beta \varphi(k) A'(v) \|_0^2 + \mu \| y^{a-\ell} \partial_y^{a-\ell+1} \partial_x^\beta \varphi(k) A'(v) \|_0^2
\]

\[
+ \frac{1}{\mu} \left( \sum_{\ell=1} \min\{\alpha,a\} \left( \frac{\alpha}{\ell} \right) \| y^{a-\ell} \partial_y^{a-\ell+1} \partial_x^\beta \varphi(k) A'(v) \|_0 \right)^2 \right)
\]

The first summand is absorbed on the left of (10.13), so that we have to treat the second as well as the sum. Consider the second term above.

If \( \alpha = 0 \) we use Lemma [10.4] and the fact that \( \beta \) takes a finite number of values to conclude that the second term verifies the desired estimate. Assume \( \alpha > 0 \). Then

\[
C_{2a} \mu \| y^{a-1} \partial_y^a \partial_x^\beta \varphi(k) A'(v) \|_0^2 \leq C_5 \| X_3 \partial_0^a \partial_x^\beta \varphi(k) A'(v) \|_0^2
\]

\[
\leq C_5 \left( C_{\varepsilon_1 \varepsilon_2} \varepsilon_1^{\alpha+\beta-1} \varepsilon_2^k (\alpha + \beta + k - 1)!^{\kappa_0} \right)^2
\]

\[
= C_5 \frac{1}{(\alpha + \beta + k)^{2\kappa_0} \varepsilon_1} \left( C_{\varepsilon_1 \varepsilon_2} \varepsilon_1^{\alpha+\beta} \varepsilon_2^k (\alpha + \beta + k)!^{\kappa_0} \right)^2
\]
If (10.19) is not satisfied we choose a larger \( \alpha \) provided

\[
\alpha + \beta + k \geq \left( \frac{\nu \sqrt{C_2}}{\varepsilon_1} \right)^{s_{\gamma_0}}.
\]

If (10.19) is not satisfied we choose a larger \( C_{\varepsilon_1, \varepsilon_2} \) as we did above, in view of Lemma 10.4.

The same argument treats also \( B_4 \) and \( B_2' \). Consider now

\[
\frac{\sqrt{C_{2\alpha}}}{\mu} \sum_{\ell = 1}^{\text{min\{\alpha, a\}}} \left( \frac{\alpha}{\ell} \right) \| y^\alpha - \ell \delta_x^\alpha \partial_y^\beta \varphi^{(k)} A'(v) \|_0.
\]

The above sum is bounded as

\[
\frac{\sqrt{C_{2\alpha}}}{\mu} \sum_{\ell = 1}^{\text{min\{\alpha, a\}}} \left( \frac{\alpha}{\ell} \right) \| X_3 \delta_x^\alpha \partial_y^\beta \varphi^{(k)} A'(v) \|_0
\]

\[
\leq \frac{\sqrt{C_{2\alpha}}}{\mu} \sum_{\ell = 1}^{\text{min\{\alpha, a\}}} \left( \frac{\alpha}{\ell} \right) C_{\varepsilon_1, \varepsilon_2} \varepsilon_1^{\alpha + \beta - \ell} \varepsilon_2^{k (\alpha + \beta + k - \ell)}
\]

\[
\leq C_{\varepsilon_1, \varepsilon_2} \varepsilon_1^{\alpha + \beta} \varepsilon_2^{k} (\alpha + \beta + k)!
\]

whence we conclude as for \( B_{22} \) (see (10.18)).

We are left with the estimate of \( B_{31} \) in (10.13). Let us start with \( B_{31} \). Since

\[
[X_1, [X_1, \partial_y^\alpha \partial_x^\beta \varphi^{(k)}]] = [X_1, \partial_y^\alpha \partial_x^\beta \varphi^{(k+1)}] = \partial_y^\alpha \partial_x^\beta \varphi^{(k+2)},
\]

we have

\[
\left| \langle [X_1, [X_1, \partial_y^\alpha \partial_x^\beta \varphi^{(k)}]] A'(v), \partial_y^\alpha \partial_x^\beta \varphi^{(k)} A'(v) \rangle \right|
\]

\[
= \left| \langle \partial_y^\alpha \partial_x^\beta \varphi^{(k+2)} A'(v), \partial_y^\alpha \partial_x^\beta \varphi^{(k)} A'(v) \rangle \right|.
\]

We may assume that \( \alpha + \beta \geq 2 \), since otherwise bounding the original norm (10.12) is easily done by choosing \( C_{\varepsilon_1, \varepsilon_2} \) sufficiently large. Thus taking one derivative to the right hand side the above scalar product becomes

\[
\left| \langle \partial_y^\alpha \partial_x^\beta \varphi^{(k+2)} A'(v), \partial_y^\alpha \partial_x^\beta \varphi^{(k)} A'(v) \rangle \right|,
\]

where \( \alpha' + \beta' = \alpha + \beta - 1 \) and \( \alpha'' + \beta'' = \alpha + \beta + 1 \). Next we may write

\[
\left| \langle \partial_y^\alpha \partial_x^\beta \varphi^{(k+2)} A'(v), \partial_y^\alpha \partial_x^\beta \varphi^{(k)} A'(v) \rangle \right|
\]

\[
\leq \| \partial_y^\alpha \partial_x^\beta \varphi^{(k+2)} A'(v) \|_0 \cdot \| \partial_y^\alpha \partial_x^\beta \varphi^{(k)} A'(v) \|_0
\]

\[
\leq M_1 \left( \frac{1}{\mu} \| X \partial_y^\alpha \partial_x^\beta \varphi^{(k+2)} A'(v) \|_0^2 + \mu \| X \partial_y^\alpha \partial_x^\beta \varphi^{(k)} A'(v) \|_0^2 \right),
\]
where $\mu$ is a small fixed positive number, $\alpha_+ + \beta_- = \alpha + \beta - 2$. Moreover in the above expression $X$ denotes either the vector field $X_1$ or the field $X_3$, since the $y$-support of the integrated function is bounded away from zero. The second term above can be absorbed on the left of (10.13) provided $\mu$ is chosen small but finite, so that we have to estimate the first, to which we may apply the inductive hypothesis:

$$\frac{M_1^\frac{\nu}{\mu^2}}{\mu^2 \epsilon_1^2} \|X \partial_y^\alpha \partial_x^{\beta} \varphi^{(k+2)} A'(v)\|_0 \leq \frac{M_1^\frac{\nu}{\mu^2}}{\mu^2} C_{\epsilon_1 \epsilon_2} \epsilon_1^{\alpha - \beta} \epsilon_2^{k+2} (\alpha_+ + \beta_- + k + 2)^{\nu_0}$$

$$= \frac{1}{\nu} C_{\epsilon_1 \epsilon_2} \epsilon_1^{\alpha - \beta} \epsilon_2^k (\alpha + \beta + k)^{\nu_0} \cdot \frac{M_1^\frac{\nu}{\mu^2}}{\mu^2 \epsilon_1^2},$$

which gives the desired estimate provided $\epsilon_1, \epsilon_2$ are such that

$$\frac{M_1^\frac{\nu}{\mu^2}}{\mu^2 \epsilon_1^2} < 1.$$

Consider now $B_32$. Since

$$[X_2, [X_2, \partial_y^\alpha \partial_x^{\beta} \varphi^{(k)}]] = [X_2, \partial_y^\alpha [x^{q-1}, \partial_x^{\beta}] \varphi^{(k)} D_y]$$

$$= \partial_y^\alpha \sum_{\ell_1 = 1}^{\min\{\beta, q-1\}} \left( \begin{array}{c} \beta \\ \ell_1 \end{array} \right) (\text{ad } \partial_x)^{\ell_1} (x^{q-1}) [\partial_x^{\beta-\ell_1}, x^{q-1}] \varphi^{(k)} D_y^2$$

$$= - \sum_{\ell_1 = 1}^{\min\{\beta, q-1\}} \sum_{\ell_2 = 1}^{\min\{\beta - \ell_1, q-1\}} \left( \begin{array}{c} \beta \\ \ell_1 \end{array} \right) \left( \begin{array}{c} \beta - \ell_1 \\ \ell_2 \end{array} \right)$$

$$\cdot (\text{ad } \partial_x)^{\ell_1} (x^{q-1}) (\text{ad } \partial_x)^{\ell_2} (x^{q-1}) \partial_x^{\beta-\ell_1-\ell_2} \varphi^{(k)} \partial_y^{\alpha+2}.$$  

Hence, $\beta$ being bound by $N$, 

$$\left| \langle [X_2, [X_2, \partial_y^\alpha \partial_x^{\beta} \varphi^{(k)}]] A'(v), \partial_y^\alpha \partial_x^{\beta} \varphi^{(k)} A'(v) \rangle \right|$$

$$\leq M_2 \sum_{\ell_2 = 2}^{\min\{\beta, 2(q-1)\}} \left| \langle \partial_x^{\beta-\ell} \partial_y^{\alpha+2} \varphi^{(k)} A'(v), \partial_y^\alpha \partial_x^{\beta} \varphi^{(k)} A'(v) \rangle \right|$$

$$\leq M_3 \sum_{\ell_2 = 2}^{\min\{\beta, 2(q-1)\}} \left( \|X_3 \partial_x^{\beta-\ell} \partial_y^{\alpha+1} \varphi^{(k)} A'(v)\|_0^2 + \|X_3 \partial_y^{\alpha+1} \partial_x^{\beta} \varphi^{(k)} A'(v)\|_0^2 \right).$$

We may apply the inductive hypothesis to both summands above and treat each of them as we did in (10.17).

Finally consider $B_{33}$. We may suppose that $\alpha > 0$, otherwise the double commutator is identically zero. We have

$$[X_3, [X_3, \partial_y^\alpha \partial_x^{\beta} \varphi^{(k)}]] = -[y^\alpha \partial_y, [y^\alpha, \partial_y^\alpha] \partial_y] \partial_x^{\beta} \varphi^{(k)}$$

$$= \sum_{\ell_2 = 2}^{\min\{\alpha, 2q\}} c_{\alpha, \ell_2} \ell_2 y^{2\alpha-\ell_2} \partial_y^{\alpha+2-\ell_2} \partial_x^{\beta} \varphi^{(k)},$$
where the $c_{a\ell}$ are absolute constants. Hence
\[
\left|\langle [X_3, [X_3, \partial_y^\alpha \partial_x^\beta \varphi^{(k)}], A'(v), \partial_y^a \partial_x^b \varphi^{(k)} A'(v)]\right|
\leq \sum_{\ell=2}^{\min\{\alpha, 2\alpha\}} |c_{a\ell}| \|g^{2\alpha-\ell \partial_y^\alpha + 2-\ell \partial_y^\beta \varphi^{(k)} A'(v), \partial_y^a \partial_x^b \varphi^{(k)} A'(v)}\|
\leq M_4 \sum_{\ell=2}^{\min\{\alpha, 2\alpha\}} \alpha^{\ell-1} \|X_3 \partial_y^\alpha + 1-\ell \partial_y^b \varphi^{(k)} A'(v)\|_0
\cdot \alpha \|X_3 \partial_y^\alpha - 1 \partial_x^b \varphi^{(k)} A'(v)\|_0.
\]
The inductive hypothesis can be applied to both factor in the sum above and we proceed as for (10.20).
Choosing $\nu$ sufficiently large achieves the proof of (10.11) and hence of (10.10).
Via the Sobolev immersion theorem as well as Lemma 10.1, we get a pointwise estimate of the same type, from which setting $x = 0$ we deduce the conclusion of the lemma.

Let us now prove (10.2).

**Lemma 10.3.** For any $\varepsilon > 0$ there exist positive constants $K_\varepsilon$, $M$, such that
\[
(10.22) \quad |\mathcal{F}(\partial_x^\beta A(v))(0, \eta)| \leq K_\varepsilon e^{-M (\frac{|\eta|}{\varepsilon})}^\frac{1}{\alpha},
\]
for $\beta = 0, 1$.

**Proof.** Set $\beta = 0$. The argument is exactly the same when $\beta = 1$. We have,
\[
|D_y^a A(v)(0, y)| \leq C_\varepsilon (y)^{-\alpha \varepsilon |\alpha|},
\]
by Lemma 10.2. Then we obtain
\[
(10.23) \quad |\mathcal{F}(A(v))(0, \eta)| \leq \frac{1}{|\eta|^{\alpha}} C_\varepsilon \varepsilon^\alpha |\alpha| \int_{\mathbb{R}} (y)^{-\alpha} dy \leq C_1 \frac{1}{|\eta|^{\alpha}} \varepsilon^\alpha |\alpha|.
\]
Then
\[
|\mathcal{F}(A(v))(0, \eta)|^{\frac{1}{\alpha}} \frac{\left(\frac{|\eta|}{\varepsilon}\right)^\frac{1}{\alpha}}{\alpha!} \leq C_2 \left(\frac{1}{2 |\sigma|}\right)^\alpha,
\]
whence, summing on $\alpha$, we deduce the assertion of the lemma. \qed

Next we prove an estimate from below of $|\mathcal{F}(\partial_x^\beta A(v))(0, \eta)|$, where $\beta$ takes either the value 0 or 1, depending on the ground state, $\varphi_0$, of the operator $Q$.

**Lemma 10.4.** There exist positive constants $\lambda$, $C_\lambda$ and a real constant $\lambda'$ such that for $\eta \geq (6R_0)^{\alpha_0}$ and $R_0$ large enough, we have
\[
(10.24) \quad |\mathcal{F}(\partial_x^\beta A(v))(0, \eta)| \geq C_\lambda \varepsilon^{1-\lambda |\eta|}^{\frac{1}{\alpha}}.
\]

**Proof.** We have
\[
\partial_x^\beta A(v)(0, y) = \int_0^\infty e^{iy \rho^{\alpha_0}} \rho^{r+\beta \eta} \varphi^{(k)} \partial_x^\beta v(0, \rho) d\rho.
\]
where \( v \) is given by (9.3). By Lemma 9.1 we see that the \( \rho \)-support of \( v \) is contained in \([3R_0, +\infty]\), hence changing variables according to \( \eta = \rho \varphi \) we obtain

\[
\partial_x^\beta A(v)(0, y) = \frac{1}{s_0} \int_{-\infty}^{+\infty} e^{iy\eta} \eta^{\frac{\beta}{q} - 1} \partial_x^\beta v(0, \eta \varphi) d\eta.
\]

By (9.5) we have that

\[
\mathcal{F}(\partial_x^\beta A(v))(0, \eta) = \frac{2\pi}{s_0} \eta^{\frac{\beta}{q} - 1} \sum_{j \geq 0} \psi_j(\eta \varphi) \partial_x^\beta u_j(0, \eta \varphi).
\]

Moreover by (8.8), (8.6) we have

\[
\left| \sum_{j \geq 0} \psi_j(\eta \varphi) \partial_x^\beta u_j(0, \eta \varphi) \right| \geq \psi_0(\eta \varphi) \left| e^{\mu_0 \eta \varphi - \xi} \partial_x^\beta \varphi_0(0) \right| - \sum_{j > 0} \psi_j(\eta \varphi) \left| \partial_x^\beta u_j(0, \eta \varphi) \right| \geq \psi_0(\eta \varphi) e^{\mu_0 \eta \varphi} |\partial_x^\beta \varphi_0(0)| - \sum_{j > 0} \psi_j(\eta \varphi) e^{\mu_0 \eta \varphi} \left| \partial_x^\beta u_j(0, \eta \varphi) \right|.
\]

In view of the estimates (8.3) and Lemma 9.1 we examine the sum on the right hand side of the above inequality.

\[
\sum_{j > 0} \psi_j(\eta \varphi) e^{\mu_0 \eta \varphi} \left| \partial_x^\beta u_j(0, \eta \varphi) \right| \leq \sum_{j > 0} \psi_j(\eta \varphi) e^{\mu_0 \eta \varphi} K_u^{1 + j + \beta} \eta^{-\frac{(j+\delta)\kappa}{\alpha}} \left( \frac{j}{s_0} + \beta \frac{q - 1}{q} \right)!
\]

\[
\leq C \psi_0 e^{\mu_0 \eta \varphi} \sum_{j > 0} K_u^{1 + j} (3R_0(j+1))^{-(j+\delta)\kappa} (j+1)^\beta
\]

\[
\leq C \psi_0 e^{\mu_0 \eta \varphi} (3R_0)^{-(1-\delta)\kappa} \sum_{j > 0} K_u^{1 + j} (j+1)^{(\frac{1}{s_0} - \kappa) + \delta \kappa}
\]

\[
= C_1 (3R_0)^{-(1-\delta)\kappa} e^{\mu_0 \eta \varphi}.
\]

In the last line above we used the fact that \( \delta < 1, \kappa > \frac{1}{s_0} \) (see the statement of Theorem 8.1).

As a consequence, taking \( \eta \varphi \geq 6R_0 \), we have the following lower bound

\[
(10.26) \left| \sum_{j \geq 0} \psi_j(\eta \varphi) \partial_x^\beta u_j(0, \eta \varphi) \right| \geq e^{\mu_0 \eta \varphi} \left| \partial_x^\beta \varphi_0(0) \right| - C_1 (3R_0)^{-(1-\delta)\kappa}
\]

Here \( \varphi_0 \) denotes the ground state of the operator \( Q \) defined in (1.3). We point out that either \( \varphi_0(0) \) or \( \partial_x \varphi_0(0) \) are non zero. This is easily seen by remarking that
if both $\varphi_0(0)$ and $\partial_x \varphi_0(0)$ are zero then all derivatives of $\varphi_0$ vanish at the origin, which is false due to the analyticity of $\varphi_0$.

Choosing $R_0$ large enough we deduce that there is a $\beta \in \{0, 1\}$ and there is a constant $C_2 > 0$, such that, for $\eta \geq 6 R_0$,

$$\left| \sum_{j \geq 0} \psi_j(\eta^{1/\alpha}) \partial_x^j u_j(0, \eta^{1/\alpha}) \right| \geq C_2 e^{\tilde{\mu} \eta^{1/\alpha}}.$$  

Then, plugging this into (10.25) we get

$$|\mathcal{F}(\partial_x^j A(v))(0, \eta)| \geq C_3 \eta^{\frac{\Re \mu + 1}{\alpha} + \frac{\beta}{2} - 1} e^{\tilde{\mu} \eta^{1/\alpha}},$$

for $\eta \geq 6 R_0$, and a suitable positive constant $C_3$.

This proves the lemma.

Now Lemma 10.3 as well as Lemma 10.4 yield a contradiction, thus proving Theorem 1.1.

A. Appendix: Solution of an ODE

Let $\mu > 0$ be a positive number. We want to find the solution of the ordinary differential equation

$$(D_{\rho}^{2a} + \mu) u = f,$$

where $f \in C^\infty(\mathbb{R})$, $\text{supp} f \subset \mathbb{R}^+$, which is rapidly decreasing at $+\infty$.

Let us denote by $\mu_i$, $i = 1, \ldots, 2a$, the $2a$-roots of $(-1)^{a+1} \mu$. We observe that $\Re \mu_i \neq 0$ for every $i$. In fact if $a$ is an even integer, in order to have $\Re \mu_{k+1} = 0$, for some $k = 0, 1, \ldots, 2a - 1$, we should have

$$\frac{\pi}{2a} + k \frac{\pi}{a} = \ell \frac{\pi}{2},$$

for some odd integer $\ell$. This would imply $2k + 1 = a \ell$, which is impossible. Analogously, assume $a$ is an odd integer. Then if $\Re \mu_{k+1} = 0$ we must have

$$k \frac{\pi}{a} = \ell \frac{\pi}{2}.$$

This would imply $2k = a \ell$, which is also impossible, since $a \ell$ is odd.

For any $j = 1, \ldots, 2a$, define

$$I_j(f)(\rho) = - \text{sign} (\Re \mu_j) \int_{\mathbb{R}} e^{\mu_j (\rho - \sigma)} H (- \text{sign} (\Re \mu_j) (\rho - \sigma)) f(\sigma) d\sigma,$$

where $H$ denotes the Heaviside function.

Since

$$\frac{1}{\sigma^{2a} + (-1)^{a} \mu} = \prod_{j=1}^{2a} \frac{1}{\sigma - \mu_j},$$

define the positive numbers $A_j$ by the relation

$$\sum_{j=1}^{2a} A_j \frac{1}{\sigma - \mu_j} = \prod_{j=1}^{2a} \frac{1}{\sigma - \mu_j}.$$
Multiplying both sides of (A.3) by $\sigma - \mu_{\ell}$, $\ell \in \{1, \ldots, 2a\}$, we get

$$A_{\ell} + \sum_{j=1 \atop j \neq \ell}^{2a} A_j \frac{\sigma - \mu_{\ell}}{\sigma - \mu_j} = \prod_{j=1 \atop j \neq \ell}^{2a} \frac{1}{\sigma - \mu_j}.$$  

Computing the above identity for $\sigma = \mu_{\ell}$ we obtain an expression for the $A_j$:

(A.4) $$A_{\ell} = \prod_{j=1 \atop j \neq \ell}^{2a} \frac{1}{\mu_{\ell} - \mu_j}.$$  

Define now

(A.5) $$u(\rho) = \sum_{j=1}^{2a} A_j I_j(f)(\rho).$$  

We want to show that $u$ is the desired solution.

First observe that

\[ \partial_\rho I_j(f) = \mu_j I_j(f) + f. \]

Hence we deduce

(A.6) $$\partial_\rho u = \sum_{j=1}^{2a} A_j \left( f + \mu_j I_j(f) \right)$$

\[ \vdots \]

\[ \partial^{2a}_\rho u = \sum_{j=1}^{2a} A_j \left( f^{(2a-1)} + \mu_j f^{(2a-2)} + \cdots + \mu_j^{2a-1} f + \mu_j^{2a} I_j(f) \right). \]

Now, because of (A.3), we have

(A.7) $$\sum_{i=1}^{2a} A_i \prod_{j=1 \atop j \neq i}^{2a} (\sigma - \mu_j) = 1.$$  

Moreover the polynomial multiplying a single $A_i$ above is written as

(A.8) $$\prod_{j=1 \atop j \neq i}^{2a} (\sigma - \mu_j) = \sigma^{2a-1} - \sigma^{2a-2} \sum_{j=1 \atop j \neq i}^{2a} \mu_j + \sigma^{2a-3} \sum_{j_1 < j_2 = 1 \atop j_1, j_2 \neq i}^{2a} \mu_{j_1} \mu_{j_2}$$

$$+ \cdots + (-1)^k \sum_{j_1 < \cdots < j_k = 1 \atop j_1, \ldots, j_k \neq i}^{2a} \mu_{j_1} \cdots \mu_{j_k} \sigma^{2a-k-1} + \cdots + (-1)^{2a-1} \prod_{j=1 \atop j \neq i}^{2a} \mu_j.$$  

Defining as $s_k^{(i)} = s_k^{(i)}(\mu_1, \ldots, \mu_{2a})$ the symmetric function of degree $k$ of the $2a - 1$ arguments $\mu_1, \ldots, \mu_{i-1}, \mu_{i+1}, \ldots, \mu_{2a}$, with $s_0^{(i)} = 1$, the above identity can be
rewritten as
\[ \prod_{j=1, j \neq i}^{2a} (\sigma - \mu_j) = \sigma^{2a-1} + \sum_{k=1}^{2a-1} (-1)^k s_k^{(i)} \sigma^{2a-1-k}. \]

Let us show inductively that for \( 1 \leq k \leq 2a \) we have
\[ s_k^{(i)} = s_k - \mu_i s_{k-1} + \mu_i^2 s_{k-2} + \cdots + (-1)^k \mu_i^k, \]
where \( s_k \) denotes the symmetric function of \( k \) of the \( 2a \) arguments \( \mu_1, \ldots, \mu_{2a} \), and we make the convention that \( s_0 = 1 \). For \( k = 1 \) it is obviously true. Consider
\[ s_k^{(i)} = \sum_{j_1 < \cdots < j_k = 1, j_1, \ldots, j_k \neq i}^{2a} \mu_{j_1} \cdots \mu_{j_k} \]
\[ = \sum_{j_1 < \cdots < j_k = 1, j_2 < \cdots < j_{k-1} = i+1}^{2a} \mu_{j_1} \cdots \mu_{j_k} - \mu_i \sum_{j_2 < \cdots < j_{k-1} = i+1}^{2a} \mu_{j_2} \cdots \mu_{j_k} \]
\[ = \cdots = \sum_{j_1 < \cdots < j_k} \mu_{j_1} \cdots \mu_{j_k} - \mu_i \left( \sum_{j_1 < \cdots < j_k} \mu_{j_1} \cdots \mu_{j_k} \right) \]
\[ + \sum_{j_1 < j_2 < \cdots < j_{k-1} < j_i} \mu_{j_1} \mu_{j_2} \cdots \mu_{j_{k-1}} + \cdots + \sum_{j_1 < \cdots < j_{k-1} < i} \mu_{j_1} \cdots \mu_{j_{k-1}} \]
\[ = s_k - \mu_i s_{k-1}^{(i)}. \]

Applying the inductive hypothesis
\[ s_k^{(i)} = s_k - \mu_i \left( s_{k-1} - \mu_i s_{k-2} + \cdots + (-1)^k \mu_i^{k-1} \right) \]
\[ = \sum_{\ell=0}^{k} (-1)^\ell \mu_i^\ell s_{k-\ell}, \]
which is the desired conclusion.

Going back to \((A.7)\), \((A.8)\) we see that \((A.8)\) can be written as
\[ \prod_{j=1, j \neq i}^{2a} (\sigma - \mu_j) = \sum_{\ell=0}^{2a-1} (-1)^\ell \sigma^{2a-1-\ell} s_\ell^{(i)}, \]
so that, identifying the polynomials on both sides of \((A.7)\), we obtain
\[ \sum_{i=1}^{2a} A_i s_k^{(i)} = 0, \quad \text{for } k = 0, \ldots, 2a - 2. \]
When \( k = 0 \) we immediately get that \( \sum_{i=1}^{2a} A_i = 0 \). Assume \( k = 1 \). Since \( s_1 = s_1 - \mu_i \) we have that \( \sum_{i=1}^{2a} A_i s_1 = s_1 \sum_{i=1}^{2a} A_i - \sum_{i=1}^{2a} A_i \mu_i = 0 \), giving that \( \sum_{i=1}^{2a} A_i \mu_i = 0 \).

Iterating this kind of argument we obtain that

\[
(A.10) \quad \sum_{i=1}^{2a} A_i \mu_i^k = 0, \quad \text{for } k = 0, \ldots, 2a - 2.
\]

Finally, again from (A.7), (A.8), the identity

\[
- \sum_{i=1}^{2a} A_i \mu_1 \cdots \mu_{i-1} \mu_{i+1} \cdots \mu_{2a} = 1
\]

implies that

\[
(A.11) \quad 1 = - \sum_{i=1}^{2a} A_i \frac{1}{\mu_i} \mu_1 \cdots \mu_{2a} = - \sum_{i=1}^{2a} A_i \frac{1}{\mu_i} (-1)^a \mu
\]

\[
= \sum_{i=1}^{2a} A_i \frac{1}{\mu_i} (-1)^{a+1} \mu = \sum_{i=1}^{2a} A_i \frac{1}{\mu_i} 2a = \sum_{i=1}^{2a} A_i \mu_i 2^{a-1}.
\]

Thus \( u \), as defined in (A.5), is a solution of (A.1).

**B. Appendix: Estimate of the Ground Level Eigenfunction**

This section contains the proof of the estimates of the function \( \varphi_0 \in \ker Q_{\lambda_0} \), where \( Q_{\lambda_0} = D_x^2 + x^{2(q-1)} - \lambda_0 \). The method for obtaining such estimates has been introduced by Métivier in [23] in a homogeneous, i.e. quadratic, case.

Let us start by defining

\[
(B.1) \quad \begin{cases} X_+ = \partial_x \\ X_- = x \end{cases}
\]

For \( k \in \mathbb{N} \) we denote by \( I \) the multiindex \( I = (i_1, \ldots, i_k) \), where \( i_j \in \{\pm\} \) for \( j = 1, \ldots, k \). We also write \( k = |I| \). Define

\[
(B.2) \quad X_I = X_{i_1} \cdots X_{i_k}.
\]

Set

\[
I_+ = (i_1^+, \ldots, i_k^+),
\]

where

\[
(i_\nu)^+ = \begin{cases} +, & \text{if } i_\nu = + \\ 0, & \text{if } i_\nu = - \end{cases}
\]

and analogously for \( I_- \) and \( i_\nu^- \). Define \( |I_+| \) as the number of the non-zero components of \( I_+ \) and similarly for \( |I_-| \).

We finally set

\[
(B.4) \quad (I) = |I_+| + \frac{|I_-|}{q-1}.
\]
We are going to need the spaces \( H^k_q(\mathbb{R}) \), which for suitable \( k \) are natural domains of the operator \( Q_{\lambda_0} \) in \( L^2(\mathbb{R}) \):

\[
H^k_q(\mathbb{R}) = \{ u \in L^2(\mathbb{R}) \mid X_I u \in L^2(\mathbb{R}), \text{ for every } I, \langle I \rangle \leq k \},
\]

for \( k \in \mathbb{N} \cup \{0\} \).

We equip \( H^k_q(\mathbb{R}) \) with the norm

\[
\| u \|_k = \max_{0 \leq \ell \leq k} |u|_\ell,
\]

where

\[
|u|_\ell = \max_{\langle I \rangle = \ell} \| X_I u \|_{L^2(\mathbb{R})}.
\]

For the sake of simplicity we write \( \| u \|_0 \) for the \( L^2(\mathbb{R}) \) norm of \( u \).

Due to Grušin, [14], or rather Lemma C.2 below for the anisotropic case, we know that the following a priori estimate is satisfied

\[
\| u \|_2 \leq C_0 (\| Q_{\lambda_0} u \|_0 + \| u \|_0),
\]

for a suitable positive constant \( C_0 \).

We want to prove the following

**Proposition B.1.** Assume that \( u \in \ker Q_{\lambda_0} \). Then there exist positive constants \( C, R \), depending only on the operator \( Q_{\lambda_0} \), such that, for every multiindex \( I \), we have the inequality

\[
\| X_I u \| \leq CR^{\langle I \rangle} (\| Q_{\lambda_0} u \|_0 + \| u \|_0),
\]

where, for \( x > 0 \), \( x! \) means \( \Gamma(x + 1) \) and \( 0! = 1 \).

**Corollary B.1.** We have for any \( u \in \ker Q_{\lambda_0} \)

\[
\| x^\beta \partial_x^\alpha u \|_0 \leq C^{\alpha + \beta + 1} (\alpha q - 1 q + \beta q - 1 q + 1 q - 1 q + 1 q + 1 q)!.
\]

Before proving the proposition we state a couple of lemmas that are used in its proof.

**Lemma B.1.** Let \( I \) be a multiindex. Then

\[
\| [Q_{\lambda_0}, X_I] u \|_0 \leq C |I| \| u \|_{2 + \langle I \rangle} - \frac{q}{q - 1},
\]

for any \( u \) in the \( L^2 \) domain of the operator on the left hand side.

**Proof of Lemma B.1.** The assertion is proved by remarking that

\[
[Q_{\lambda_0}, X_I] = \sum_{I_1, I_2} b_{I_1, I_2} X_{I_1} X_{I_2},
\]

where \( b_{I_1, I_2} \in \mathbb{C} \) and are bounded by a quantity depending only on the problem and \( \langle I_1 \rangle = \langle I \rangle + 2 - \frac{q}{q - 1} \). Moreover the number of terms in the summation above is bounded by \( \varkappa |I| \), \( \varkappa \) denoting a positive constant depending only on the problem data.

Inequality (B.10) then immediately follows. \( \square \)

**Lemma B.2.** Let \( I \) a multiindex such that \( \langle I \rangle > 2 \). Then we may decompose \( X_I \) as

\[
X_I = X_I^{\nu} X_I^{\mu} + A,
\]
where $\langle I'' \rangle = 2$, $\langle I' \rangle = \langle I \rangle - 2$ and $A$ is a finite sum of the form

$$A = \sum_J c_J X_J,$$

with $\langle J \rangle = \langle I \rangle - \frac{q-1}{q-1}$. Here both the coefficients $c_J$ and the number of summands are bounded by an absolute constant.

The proof of the lemma is straightforward and we skip it.

**Proof of Proposition B.7.** Instead of proving (B.8) directly we are going to show that the following inequality holds:

\[
\|X_J u\| \leq C_1 R_1^{(I)} \|u\|_0 \langle I \rangle^{\frac{k}{q-1}},
\]

for certain positive constants $C_1$, $R_1$, for every multiindex $I$. It is then obvious that (B.11) implies (B.8) slightly modifying the constants, because of the Stirling formula for the Euler Gamma function.

We argue by induction on $k$ where $\langle I \rangle = \frac{k}{q-1}$.

First of all we observe that when $\langle I \rangle \leq 2$ we have by (B.7) that

$$\|X_J u\| \leq \|Q_{\lambda_0} X_J u\|_0 \langle I \rangle_0 + \|Au\|_0 = C_0 \|u\|_0.$$

Assume now that the assertion holds for any $I$ with $\langle I \rangle = \frac{k-\ell}{q-1}$, $\ell = 0, 1, \ldots, k$, $k > 2(q-1)$. We want to show that the assertion is true for $I$ with $\langle I \rangle = \frac{k}{q-1}$.

Let $I$ be a multiindex with $\langle I \rangle = \frac{k}{q-1}$. Using Lemma B.2 we write $X_I = X_J X_J + A$, with $\langle I \rangle = 2 \langle J \rangle = \langle I \rangle - 2$ and $A$ of the form specified in Lemma B.2. By (B.7) we have

$$\|X_I u\| \leq \|X_J X_J u\|_0 + \|Au\|_0 \leq C_0 \left( \|Q_{\lambda_0} X_J u\|_0 + \|X_J u\|_0 + \|Au\|_0 \right).$$

Since $u \in \ker Q_{\lambda_0}$,

$$Q_{\lambda_0} X_J u = X_J Q_{\lambda_0} u + [Q_{\lambda_0}, X_J] u = [Q_{\lambda_0}, X_J]\langle I \rangle_0$$

Hence

\[
\|X_I u\| \leq C_1 R_1^{(J)} \|u\|_0 \langle J \rangle^{\frac{k}{q-1}} = C_1 R_1^{(J)} \left( \|Q_{\lambda_0} X_J u\|_0 + \|X_J u\|_0 + \|Au\|_0 \right).
\]

Since $\langle J \rangle = \langle I \rangle - 2 = \frac{k-\ell}{q-1} - 2 \leq \frac{k}{q-1}$ and $A$ is a sum of terms involving $X_J$, with $\langle J' \rangle = \langle I \rangle - \frac{q-1}{q-1} = \frac{k-\ell}{q-1} - 1 < \frac{k}{q-1}$, we see that both terms $\|X_J u\|$ and $\|Au\|$ satisfy the inductive hypothesis:

$$\|X_J u\| \leq C_1 R_1^{(J)} \|u\|_0 \langle J \rangle^{\frac{k}{q-1}} \leq \left( C_1 R_1^{(I)} \right)^{\frac{k}{q-1}} \langle I \rangle^{\frac{k}{q-1}} \|u\|_0 \langle I \rangle^{\frac{k}{q-1}}.$$

Furthermore writing $A = \sum_{J'} c_{J'} X_{J'}$, with $\langle J' \rangle = \frac{k}{q-1} - 1$ and where both the number of summands and the constants $c_{J'}$ are bounded by a universal constant, say $M$, we have

$$\|Au\|_0 \leq \sum_{J'} |c_{J'}| \|X_{J'} u\|_0 \leq \sum_{J'} |c_{J'}| C_1 R_1^{(J')} \|u\|_0 \langle J' \rangle^{\frac{k}{q-1}} \leq \left( C_1 M^2 R_1^{(I)} \right)^{\frac{k}{q-1}} \|u\|_0 \langle I \rangle^{\frac{k}{q-1}}.$$
Note that there is always a gain of a negative power of $R_1$ in the above terms.

Consider now the norm with the commutator in the right hand side of (B.12). By Lemma B.1 we have

$$\| [Q_{\lambda_{0}}, X_J]u \|_0 \leq C|J| \|u\|_{2(\mathcal{J})} \max_{0 \leq \ell \leq 2(\mathcal{J})} \max_{(J') = \ell} \|X_{(J')}u\|_0.$$  

(B.13)

Observe that

$$2 + \langle J \rangle - \frac{q}{q-1} = \langle I \rangle - \frac{k}{q-1} = \frac{k}{q-1} - 1.$$  

Thus the norms in the right hand side of (B.13) satisfy the inductive hypothesis. We deduce that

$$\| [Q_{\lambda_{0}}, X_J]u \|_0 \leq CkC_1 R_1^{\frac{k}{q-1} - 1} \|u\|_0 \left( \frac{k}{q-1} - 1 \right) \left( \frac{k}{q-1} - 1 \right)^{\frac{k+1-q}{q-1} + 1} \left( \frac{k+1}{q-1} \right)^{\frac{k+1-q}{q-1} + 1} \left( \frac{k+1}{q-1} \right)^{\frac{k+1}{q-1}}.$$  

Choosing $R_1$ in such a way that the constant in parentheses is bounded by $C_1$, achieves the proof of the proposition. \qed

Taking $|I_-| = \beta$, $|I_+| = \alpha$ and using the Sobolev embedding theorem (in one dimension,) we obtain a bound for the derivatives of the functions in ker $Q_{\lambda_0}$:

**Corollary B.2.** Let $\varphi \in \mathcal{S} (\mathbb{R})$ be such that $Q_{\lambda_0} \varphi = 0$. Then for every $\alpha, \beta \in \mathbb{N} \cup \{0\}$ there exists a positive constant $C_\varphi$ such that

$$|x^\beta \partial_x^\alpha \varphi(x)| \leq C_\varphi^{\alpha+\beta+1} \alpha! \beta! \left( \frac{q}{q-1} \right)^{\gamma},$$  

for every $x \in \mathbb{R}$.

**C. Appendix: An Inequality Involving Powers of Q**

We prove here the following inequality

**Proposition C.1.** Let $u \in \mathcal{S} (\mathbb{R})$, and $Q$ the operator $D_x^2 + x^2(q^{-1})$, $\theta \in \mathbb{Q}^+ \cup \{0\}$, and $X_I$ an operator of the type defined in (B.2).

Set

$$m(\theta, I) = \left[ (2\theta + \langle I \rangle) \frac{q-1}{q} \right],$$  

where the square brackets denote the integer part. Then there exists a positive constant $C$ such that

$$\|Q^\theta X_I u\|_0 \leq \sum_{\ell=0}^{m(\theta, I)} C^{\ell+1} \binom{m(\theta, I)}{\ell} p^\ell \|Q^{\frac{1}{\ell} (p-\ell \theta)} u\|_0,$$  

(C.1)

where

$$p = 2\theta + \langle I \rangle.$$  

(C.2)
Here we used the fact that $Q$ is a positive globally elliptic operator with discrete spectrum and its rational powers are defined via the spectral mapping theorem, see Helffer [17].

Proof. The proof is carried out via a number of lemmas.

**Lemma C.1.** Let $\mu \in \mathbb{Q}^+$, $\mu \geq 1$, $I$ a multiindex, then

\[
\|Q^\mu X_I u\|_0 \leq \|Q^{\mu-1} X_I Q u\|_0 + \sum_{I_1} c_{I_1} \|Q^{\mu-1} X_{I_1} u\|_0,
\]

where $I_1$ is a multiindex such that

\[
\langle I_1 \rangle = \langle I \rangle + 2 - \frac{q}{q-1},
\]

the constants $c_{I_1}$ are uniformly bounded by a constant and the number of summands is bounded by $M(I)$, $M > 0$ independent of $I$.

**Proof of Lemma C.1.** We have $Q^\mu X_I = Q^{\mu-1} X_I Q + Q^{\mu-1} [Q, X_I]$. We then remark that

\[
\{Q, X_I\} = \sum_{I_1} c_{I_1} X_{I_1},
\]

where $\langle I_1 \rangle = \langle I \rangle + 2 - \frac{q}{q-1}$ and the bounds in the above statement hold. \qed

**Lemma C.2.** Let $I$ be such that $\langle I \rangle = 2$. Then for any $u \in \mathcal{S}(\mathbb{R})$ we have the estimate

\[
\|X_I u\|_0 \leq \|Qu\|_0 + C_q \|x^{q-2} u\|_0,
\]

where $C_q$ denotes a positive constant.

**Proof.** It is a very simple computation. First remark that

\[
\|Qu\|_0^2 = \|D_x^2 u\|_0^2 + \|x^{2(q-1)} u\|_0^2 + \langle x^{2(q-1)} D_x^2 + D_x^2 x^{2(q-1)}, u, u \rangle.
\]

Now

\[
x^{2(q-1)} D_x^2 + D_x^2 x^{2(q-1)} = 2 D_x x^{2(q-1)} D_x + [D_x, [D_x, x^{2(q-1)}]],
\]

so that we get the identity

\[
\|D_x^2 u\|_0^2 + \|x^{2(q-1)} u\|_0^2 + 2 \|x^{q-1} D_x u\|_0^2 = \|Qu\|_0^2 + (2q-2)(2q-3) \|x^{q-2} u\|_0^2,
\]

and (C.3) immediately follows remarking that

\[
\|X_I u\|_0 \leq \|D_x^2 u\|_0 + \|x^{2(q-1)} u\|_0 + \|x^{q-1} D_x u\|_0 + C \|x^{q-2} u\|_0.
\]

\qed

**Lemma C.3.** Let $\mu \in \mathbb{Q}^+$, $\mu < 1$, $I$ a multiindex, then

\[
\|Q^\mu X_I u\|_0 \leq \|Q^{\mu-1} X_I Q u\|_0 + \sum_{I_1} c_{I_1} \|Q^{\mu} X_{I_1} u\|_0 + C \|Q^{\mu+1} \frac{q}{q-1} X_I u\|_0,
\]

where $I_1$ is a multiindex such that $\langle I_1 \rangle = \langle I \rangle - 2$, $\langle I \rangle = \langle I \rangle - \frac{q}{q-1}$, and for the sum we have bounds analogous to those in Lemma C.1.
Proof. We may write

\[ X_I = X_I Q_I + \sum_{I_i} c_{I_i} X_{I_i}, \]

where \( \langle I \rangle = 2, \langle \tilde{I} \rangle = \langle I \rangle - 2, \langle I_1 \rangle = \langle I \rangle - \frac{1}{q-1}. \) Moreover both the constants \( c_{I_i} \) and the number of the summands are bounded by a universal constant.

Now \( Q^\mu X_I X_I = X_I Q^\mu X_I + [Q^\mu, X_I] X_I. \) By lemma C.2 we have

\[
\|X_I Q^\mu X_I u\|_0 \leq \|Q^{\mu + 1} X_I u\|_0 + C_q \|x^{\gamma-2} Q^\mu X_I u\|_0 \\
\leq \|Q^{\mu + 1} X_I u\|_0 + C_q \|Q^\mu X_I u\|_0 \\
+ C_q \|Q^\mu, x^{\gamma-2} X_I u\|_0,
\]

where \( \langle I_3 \rangle = \langle I \rangle - \frac{1}{q-1} \) so that the second term in the last line above has the same weight as the terms containing \( X_{I_i} \) above. Adapting to the anisotropic case the calculus of globally elliptic pseudodifferential operators we get—see Helffer [17], Theorem 1.11.2 and Proposition 1.6.11—that

\[
\|Q^\mu, x^{\gamma-2} X_I u\|_0 \leq C_1 \|Q^{\mu + 1} x^{\gamma-1} X_I u\|_0.
\]

The conclusion follows applying Lemma C.4 to the term \( \|Q^{\mu + 1} X_I u\|_0. \)

Let us now go back to the proof of inequality (C.1). We proceed by induction with respect to \( p = 2\theta + \langle I \rangle. \) Since \( \theta \in \mathbb{Q} \) and \( \langle I \rangle \) is a rational number whose denominator is \( q-1, \) we may write \( p \) as a fraction \( p = p_1 / d(\theta, I), \) so that, proceeding by induction actually means inducing with respect to \( p_1. \)

If \( p_1 = 0 \) there is nothing to prove. Assume thus that (C.1) holds for any \( q_1 / d(\theta, I), \) \( q_1 < p_1 \) and let us show that it is true when \( q_1 = p_1. \)

Assume \( \theta < 1. \) Applying Lemma C.3 we obtain

\[
\|Q^\theta X_I u\|_0 \leq \|Q^\theta X_I Q u\|_0 + \sum_{I_i} c_{I_i} \|Q^\theta X_{I_i} u\|_0 \\
+ C \|Q^{\theta + 1} x^{\gamma-1} X_I u\|_0 \\
= A_1 + A_2 + A_3,
\]

where \( \langle I_1 \rangle = \langle I \rangle - \frac{1}{q-1} \) and \( \langle \tilde{I} \rangle = \langle I \rangle - 2. \) Let us start by examining \( A_1. \) Write

\[
(2\theta + \langle \tilde{I} \rangle) \frac{q - 1}{q} = m(\theta, \tilde{I}) + \bar{\sigma}, \quad 0 \leq \bar{\sigma} < 1,
\]

with \( m(\theta, \tilde{I}) \in \mathbb{N} \cup \{0\}. \) Since

\[
(2\theta + \langle \tilde{I} \rangle) \frac{q - 1}{q} = (2\theta + \langle I \rangle) \frac{q - 1}{q} - 1 - \left(1 - \frac{2}{q}\right),
\]

we deduce that

\[
m(\theta, I) - 2 \leq m(\theta, \tilde{I}) \leq m(\theta, I) - 1.
\]

Hence applying the induction to \( A_1 \) we obtain

\[
\|Q^\theta X_I (Qu)\|_0 \leq \sum_{\ell=0}^{m(\theta, \tilde{I})} C^{\theta + 1} \left(m(\theta, \tilde{I})\right)^{\ell} \|Q^\theta X_I (Qu)\|_0,
\]

where \( \bar{\theta} = 2\theta + \langle \tilde{I} \rangle. \)
Since 
\[ \tilde{p} + 2 = 2 \theta + \langle I \rangle + 2 = 2 \theta + \langle I \rangle = p, \]
we may write
\[
\| Q^\theta X_I(Qu) \|_0 \leq \sum_{\ell=0}^{m(\theta, I)} C^{\ell+1} \left( \frac{m(\theta, I)}{\ell} \right) p^\ell \| (Q^{\frac{1}{2}})^{p - \ell \frac{q}{p-1}} u \|_0
\]
\[
(C.7) \leq \sum_{\ell=0}^{m(\theta, I) - 1} C^{\ell+1} \left( \frac{m(\theta, I) - 1}{\ell} \right) p^\ell \| (Q^{\frac{1}{2}})^{p - \ell \frac{q}{p-1}} u \|_0
\]
Next let us examine \( A_2 \). Applying the inductive hypothesis we may write
\[
(C.8) \sum_{I_1} C_{I_1} \| Q^\theta X_{I_1} u \|_0
\]
\[
\leq \sum_{I_1} C_1 \sum_{\ell=0}^{m(\theta, I_1)} C^{\ell+1} \left( \frac{m(\theta, I_1)}{\ell} \right) p^\ell \| (Q^{\frac{1}{2}})^{2\theta + \langle I_1 \rangle - \ell \frac{q}{p-1}} u \|_0
\]
Since
\[
2 \theta + \langle I \rangle = 2 \theta + \langle I \rangle - \frac{q}{q-1} = p - \frac{q}{q-1},
\]
we have that
\[
m(\theta, I_1) = \left\lfloor \frac{q - 1}{q} \left( 2 \theta + \langle I \rangle - \frac{q}{q-1} \right) \right\rfloor = \left\lfloor \frac{q - 1}{q} p \right\rfloor - 1 = m(\theta, I) - 1.
\]
Hence we may conclude that
\[
\sum_{I_1} c_{I_1} \| Q^\theta X_{I_1} u \|_0
\]
\[
\leq MC_1 \sum_{\ell=0}^{m(\theta, I) - 1} C^{\ell+1} \left( \frac{m(\theta, I) - 1}{\ell} \right) p^\ell \| (Q^{\frac{1}{2}})^{2\theta + \langle I \rangle - \ell \frac{q}{p-1}} u \|_0
\]
\[
= MC_1 \sum_{\ell=1}^{m(\theta, I)} C^\ell \left( \frac{m(\theta, I) - 1}{\ell - 1} \right) p^\ell \| (Q^{\frac{1}{2}})^{2\theta + \langle I \rangle - \ell \frac{q}{p-1}} u \|_0
\]
\[
= MC_1 C^{-1} \sum_{\ell=1}^{m(\theta, I)} C^{\ell+1} \left( \frac{m(\theta, I) - 1}{\ell - 1} \right) p^\ell \| (Q^{\frac{1}{2}})^{2\theta + \langle I \rangle - \ell \frac{q}{p-1}} u \|_0
\]
\[
\leq \frac{1}{2} \sum_{\ell=1}^{m(\theta, I)} C^{\ell+1} \left( \frac{m(\theta, I) - 1}{\ell - 1} \right) p^\ell \| (Q^{\frac{1}{2}})^{2\theta + \langle I \rangle - \ell \frac{q}{p-1}} u \|_0
\]
provided \( C \) is chosen so that \( MC_1 C^{-1} \leq \frac{1}{2} \).
Finally consider \( A_3 \). Since
\[
2 \theta + 2 - \frac{q}{q-1} + \langle I \rangle = 2 \theta + 2 - \frac{q}{q-1} + \langle I \rangle - 2 = 2 \theta + \langle I \rangle - \frac{q}{q-1} < p,
\]
we may apply the inductive hypothesis. Moreover as above
\[
\left\lfloor \frac{q - 1}{q} \left( p - \frac{q}{q-1} \right) \right\rfloor = \left\lfloor \frac{q - 1}{q} p \right\rfloor - 1 = m(\theta, I) - 1,
\]
so that, renaming \( \tilde{C} \) the constant \( C \) in the definition of \( A_3 \), we have
\[
\tilde{C} \|Q^{\theta+1} - \frac{8}{\pi^2} X \|_0 \\
\leq \tilde{C} \sum_{\ell=0}^{m(\theta, I)-1} C^{\ell+1} \left( \frac{m(\theta, I) - 1}{\ell} \right) p^{\ell} \| (Q^{2\frac{1}{p}})^{p-\frac{\theta}{p} + \frac{1}{\pi^2}} u \|_0 \\
= \tilde{C} \sum_{\ell=1}^{m(\theta, I)} C^{\ell} \left( \frac{m(\theta, I) - 1}{\ell - 1} \right) p^{\ell-1} \| (Q^{2\frac{1}{p}})^{p-\frac{\theta}{p} + \frac{1}{\pi^2}} u \|_0 \\
\leq \frac{1}{2} \sum_{\ell=1}^{m(\theta, I)} C^{\ell+1} \left( \frac{m(\theta, I) - 1}{\ell - 1} \right) p^{\ell} \| (Q^{2\frac{1}{p}})^{p-\frac{\theta}{p} + \frac{1}{\pi^2}} u \|_0,
\]
provided \(C\) is chosen so that \(\tilde{C} C^{-1} \leq \frac{1}{2}\). Hence we conclude that
\[
\|Q^n X u\|_0 \leq \sum_{\ell=0}^{m(\theta, I)-1} C^{\ell+1} \left( \frac{m(\theta, I) - 1}{\ell} \right) p^{\ell} \| (Q^{2\frac{1}{p}})^{p-\frac{\theta}{p} + \frac{1}{\pi^2}} u \|_0 \\
+ \sum_{\ell=1}^{m(\theta, I)} C^{\ell+1} \left( \frac{m(\theta, I) - 1}{\ell - 1} \right) p^{\ell-1} \| (Q^{2\frac{1}{p}})^{p-\frac{\theta}{p} + \frac{1}{\pi^2}} u \|_0 \\
= \sum_{\ell=0}^{m(\theta, I)} C^{\ell+1} \left( \frac{m(\theta, I)}{\ell} \right) p^{\ell} \| (Q^{2\frac{1}{p}})^{p-\frac{\theta}{p} + \frac{1}{\pi^2}} u \|_0,
\]
thus concluding the proof of Proposition \(C.1\).

If \(\theta > 1\) the proof is completely analogous, using Lemma \(C.1\). \(\square\)

D. APPENDIX: REDUCING VECTOR FIELDS TO POWERS OF \(Q\)

We first prove the following

**Proposition D.1.** Let \(\mu \in Q^+\). Then there exists a positive constant, \(C\), independent of \(\mu\), such that

\((D.1)\) \[
\|Q^n x \partial_x u\|_0 \leq C \left( \|Q^{\mu+\frac{n}{\pi^2 \tau^2}} u\|_0 + \mu^{2\frac{n}{\pi^2 \tau^2} + 1} \|u\|_0 \right).
\]

**Proof.** Let \([\mu] = k\), so that \(\mu = k + \theta, 0 \leq \theta < 1\). Then

\[
Q^n x \partial_x = x \partial_x Q^n + [Q^n, x \partial_x] = x \partial_x Q^n + Q^n [x \partial_x] + [Q^n, x \partial_x] Q^n.
\]

Since \(x \partial_x\) has weight \(\frac{n}{\pi^2 \tau^2 - 1}\) we have

\[
\|x \partial_x v\|_0 \leq C_1 \|Q^{\frac{n}{\pi^2 \tau^2}} v\|_0,
\]
for a suitable constant \(C_1\). Here we used the pseudodifferential calculus adapted to the anharmonic oscillator \(Q\) (see \([17]\) and \([4]\), Definition 2.1.) Hence

\[
\|x \partial_x Q^n u\|_0 \leq C_1 \|Q^{\mu+\frac{n}{\pi^2 \tau^2}} u\|_0.
\]

Analogously for the third term we have the estimate

\[
\|Q^n, x \partial_x] Q^n u\|_0 \leq C_2 \|Q^n u\|_0 \leq C_3 \|Q^{\mu+\frac{n}{\pi^2 \tau^2}} u\|_0,
\]
where \(C_2, C_3\) are independent of \(k\). Here we also used the fact that if \(\sigma, \tau\) are rational numbers such that \(0 < \sigma < \tau\), then \(\|Q^\sigma u\|_0 \leq C \|Q^\tau u\|_0\), for a suitable constant \(C > 0\), independent of \(\sigma, \tau\).
Let us consider the term $Q^\theta[Q^k, x\partial_x]$. We have

$$Q^\theta[Q^k, x\partial_x] = \sum_{i=1}^k \binom{k}{i} Q^\theta(\text{ad} Q)^i (x\partial_x) Q^{k-i}.$$  

The iterated commutator above is a sum of products of $X_-, X_+$ (see equation (B.1))

$$(\text{ad} Q)^i (x\partial_x) = \sum_I c_I X_I,$$

where the $|c_I| \leq C^i_4$, the number of summands is bounded by $C^i_4$ and

$$(I) = \frac{q-2}{q-1} + \frac{q}{q-1}.$$  

Applying Proposition C.1 to $Q^\theta X_I$ we obtain the estimate

$$\|Q^\theta(\text{ad} Q)^i (x\partial_x)v\|_0 \leq C^i_4 \sum_{\ell=0}^{m(i)} C^{\ell+1} \binom{m(i)}{\ell} p(i)^\ell \|Q^{\frac{q}{q-1}}\|_0 \|v\|_0,$$

where

$$p(i) = 2\theta + \frac{q-2}{q-1} + \frac{q}{q-1}, \quad m(i) = \left\lfloor 2\theta \frac{q-1}{q} + \frac{q-2}{q} \right\rfloor + 1.$$  

It will be useful to simplify the sums of the above type by, roughly, taking the terms with the maximum and the minimum power of $Q$, and, correspondingly, with the minimum and maximum power of $p(i)$. This is done by a sort of convexity estimate of the following type

**Proposition D.2.** Let $1 \leq p, p' < +\infty$, $p^{-1} + p'^{-1} = 1$, and let $\lambda > 0$ be a real number. Then

$$\|\lambda Qv\|_0 \leq \frac{1}{p} \|Q^p v\|_0 + \frac{1}{p'} \|Q^\lambda v\|_0.$$  

The proof is straightforward, by using the spectral mapping theorem and we omit it.

Going back to (D.2) we may write

$$\|Q^\theta(\text{ad} Q)^i (x\partial_x)v\|_0 \leq C^i_4 \sum_{\ell=0}^{m(i)} C^{\ell+1} \binom{m(i)}{\ell} \left( \|Q^{\frac{q}{q-1}}\|_0 \|v\|_0 + p(i)^{\frac{q-2}{q}} \right) \|v\|_0 \leq C^{i+1}_6 \left( \|Q^{\frac{q}{q-1}}\|_0 \|v\|_0 + p(i)^{\frac{q-2}{q}} \right),$$

due to the bound

$$C^i_4 \sum_{\ell=0}^{m(i)} C^{\ell+1} \binom{m(i)}{\ell} = CC^i_4 (C + 1)^{m(i)} \leq C^{i+1}_6.$$  

As a consequence

$$\|Q^\theta x\partial_x u\|_0 \leq C_7 \|Q^{\frac{q}{q-1}}\|_0 \|v\|_0 + \sum_{i=1}^k C^{i+1}_6 \binom{k}{i} \left( \|Q^{\frac{q}{q-1}}\|_0 \|v\|_0 + p(i)^{\frac{q-2}{q}} \right),$$

$$+ \sum_{i=1}^k C^{i+1}_6 \binom{k}{i} \left( \|Q^{\frac{q}{q-1}}\|_0 \|v\|_0 + p(i)^{\frac{q-2}{q}} \right) \|Q^{k-i} u\|_0$$.
$$\begin{align*}
&= C_7 \|Q^{\mu + \frac{\sigma}{2(\sigma-1)}} u\|_0 \\
&\quad + \sum_{i=1}^{k} C_i^{i+1} \binom{k}{i} \left( \|Q^{\mu - (i-1) \frac{\sigma}{2(\sigma-1)}} u\|_0 + p(i) \frac{\sigma-1}{\sigma} i + 2 \frac{\sigma-1}{\sigma} \|Q^{k-i} u\|_0 \right)
\end{align*}$$

Observe now that

$$p(i) \frac{\sigma-1}{\sigma} i + 2 \frac{\sigma-1}{\sigma} \leq C_0 \frac{\sigma-1}{\sigma} i + 1 + 2 \frac{\sigma-1}{\sigma} \leq C_0 \mu \frac{\sigma-1}{\sigma},$$

since \( i \geq 1 \), for a suitable constant \( C_0 \). Moreover \( \binom{k}{i} \leq \mu! \leq 1 \), since \( k = \lceil \mu \rceil \).

Applying Proposition D.2 to both terms under the sum sign above, we obtain

$$\mu^i \left\| Q^{\mu + \frac{\sigma}{2(\sigma-1)}} - i \frac{\sigma}{2(\sigma-1)} u \right\|_0 \leq \left\| Q^{\mu + \frac{\sigma}{2(\sigma-1)}} u \right\|_0 + \mu^2 \frac{\sigma-1}{\sigma} u \| 0 \|_0,$$

and

$$\mu^{i-1} \left\| Q^{\mu + \frac{\sigma}{2(\sigma-1)} - i} u \right\|_0 \leq \left\| Q^{\mu + \frac{\sigma}{2(\sigma-1)}} u \right\|_0 + \mu^2 \frac{\sigma-1}{\sigma} u \| 0 \|_0.$$

Plugging the above estimates into (D.4), we find

$$\|Q^\mu x \partial_x u\|_0 \leq C_0 \sum_{i=0}^{\lceil \mu \rceil} C_i^{i+1} \left( \left\| Q^{\mu + \frac{\sigma}{2(\sigma-1)}} u \right\|_0 + \mu^2 \frac{\sigma-1}{\sigma} u \| 0 \|_0 \right) \leq C_0 \left( \left\| Q^{\mu + \frac{\sigma}{2(\sigma-1)}} u \right\|_0 + \mu^2 \frac{\sigma-1}{\sigma} u \| 0 \|_0 \right).$$

This completes the proof of the proposition.

Proposition D.1 takes care of the action of (rational) powers of \( Q \) on the principal part of the transport operator \( P_1 \). We need a similar result for the other transport operators, \( P_k \), \( k = 2, \ldots, 2a \), in (2.14).

**Proposition D.3.** Let \( \mu \in \mathbb{Q}^+ \). Then there exists a positive constant, \( C \), independent of \( \mu \), such that

$$\left\| Q^\mu (x \partial_x)^j u \right\|_0 \leq C_1 \left( \left\| Q^{\mu + \frac{j(\sigma)}{2(\sigma-1)}} u \right\|_0 + \mu^2 \frac{\sigma-1}{\sigma} u \| 0 \|_0 \right),$$

\( j = 1, \ldots, 2a \).

**Proof.** We proceed by induction with respect to \( j \). When \( j = 1 \) the assertion is just Proposition D.1. Assume that the assertion is valid for \( j \) holds and let us prove the assertion for \( j + 1 \). We have

$$\left\| Q^\mu (x \partial_x)^{j+1} u \right\|_0 = \left\| Q^\mu (x \partial_x)^j (x \partial_x) u \right\|_0.$$

Applying the above estimate for \( j \) we obtain

$$\left\| Q^\mu (x \partial_x)^{j+1} u \right\|_0 \leq C_1 \left( \left\| Q^{\mu + \frac{j(\sigma)}{2(\sigma-1)}} (x \partial_x) u \right\|_0 + \mu^2 \frac{\sigma-1}{\sigma} (x \partial_x) u \| 0 \|_0 \right).$$

For the first term we use Proposition D.1, while for the second we apply Propositions C.1 and D.2. Now

$$\|Q^\mu (x \partial_x)^{j+1} u\|_0 \leq C_1 \left( C_1 \left( \left\| Q^{\mu + \frac{\sigma}{2(\sigma-1)}} u \right\|_0 + \mu^2 \frac{\sigma-1}{\sigma} u \| 0 \|_0 \right) \right) + C_1 \left( \left\| Q^{\mu + \frac{\sigma}{2(\sigma-1)}} u \right\|_0 + \mu^2 \frac{\sigma-1}{\sigma} u \| 0 \|_0 \right),$$
so that choosing $C_1 \geq 2 \max\{C_*, C_{**}\}$ achieves the proof. □

References

[1] P. Albano, A. Bove and M. Mughetti, Analytic Hypoellipticity for Sums of Squares and the Treves Conjecture, J. Funct. Anal. 274 (2018), no. 10, 2725–2753.

[2] C. M. Bender and Q. Wang, A class of exactly solvable eigenvalue problems, J. Physics A 34 (2001), 9835–9847.

[3] F.A. Berezin and M.A. Shubin, The Schrödinger equation, Mathematics and its Applications (Soviet Series), 66. Kluwer Academic Publishers Group, Dordrecht, 1991.

[4] A. Bove, M. Mughetti and D.S. Tartakoff, Hypoellipticity and nonhypoellipticity for sums of squares of complex vector fields, Analysis and PDE 6 (2) (2013), 371–446.

[5] A. Bove, M. Mughetti, On a new method of proving Gevrey hypoellipticity for certain sums of squares, Advances in Mathematics 293 (2016), 146–220.

[6] A. Bove and M. Mughetti, Analytic hypoellipticity for sums of squares in the presence of symplectic non Treves strata, J. Inst. Math. Jussieu 19 (6) (2020), 1877–1888.

[7] A. Bove and M. Mughetti, Analytic and Gevrey hypoellipticity for a class of pseudodifferential operators in one variable, J. Differential Equations 255 (2013), no. 4, 728-758.

[8] A. Bove and M. Mughetti, Gevrey Regularity for a Class of Sums of Squares of Monomial Vector Fields, Advances in Math. 373 (2020), 35 pages.

[9] A. Bove and M. Mughetti, Analytic regularity for solutions to sums of squares: an assessment, Complex Analysis and its Synergies 6 (2020), article no. 18.

[10] A. Bove and F. Treves, On the Gevrey hypo-ellipticity of sums of squares of vector fields, Ann. Inst. Fourier (Grenoble) 54 (2004), 1443-1475.

[11] L. Ehrenpreis, Solutions of some problems of division IV, Amer. J. Math 82 (1960), 522-588.

[12] V. V. Grušin, On a class of Hypoelliptic operators, Math. USSR Sb. 12 (1970), 458–476.

[13] G. G. Gundersen, A class of anharmonic oscillators whose eigenfunctions have no recurrence relations, Proc. Amer. Math. Soc. 58 (1976), 109–113.

[14] B. Helffer, Sur l’hypoellipticité des opérateurs pseudodifférentiels à caractéristiques multiples (perte de 3/2 dérivées), Mémoires de la S. M. F. 51–52 (1977), 13–61.

[15] B. Helffer, Théorie spectrale pour des opérateurs globalement elliptiques, Astérisque, 112 (1984), ix+197 pp.

[16] L. Hörmander, Hypoelliptic second order differential equations, Acta Math. 119 (1967), 147–171.

[17] L. Hörmander, Uniqueness Theorems and Wave Front Sets for Solutions of Linear Differential Equations with Analytic Coefficients, Communications Pure Appl. Math. 24 (1971), 671–704.

[18] L. Hörmander, The Analysis of Partial Differential Operators, I, Springer Verlag, 1985.

[19] L. Hörmander, The Analysis of Partial Differential Operators, III, Springer Verlag, 1985.

[20] M. Métivier, Une Classe d’Opérateurs Non Hypoelliptiques Analytiques, Indiana Univ. Math. J. 29 (1980), 823–860.

[21] M. Métivier, Hypoellipticité analytique sur des groupes nilpotents de rang 2, Duke Math. J. 47 (1980), 195–221.

[22] M. Métivier, Non-hypoelliptic Analytique pour $D^2 + (x^2 + y^2)D_y$, C. R. Acad. Sci. Paris Sér. I Math. 292 (1981), no. 7, 401–404.

[23] M. Mughetti, Regularity properties of a double characteristics differential operator with complex lower order terms, J. Pseudo-Differ. Oper. Appl. 5 (2014), no. 3, 343-358.

[24] M. Mughetti and F. Nicola, Hypoellipticity for a class of operators with multiple characteristics, J. Analyse Math. 103 (2007), 377–396.

[25] O. A. Oleinik and E. V. Radkevič, The analyticity of the solutions of linear partial differential equations, (Russian) Mat. Sb. (N.S.), 90(132) (1973), 592–606.

[26] L. Rothschild and E. M. Stein, Hypoelliptic differential operators and nilpotent groups, Acta Math. 137 (3 - 4) (1976), 247–320.
[29] D.S. Tartakoff, *On the Local Real Analyticity of Solutions to $\Box_b$ and the $\bar{\partial}$-Neumann Problem*, Acta Math. 145 (1980), 117–204.

[30] F. Trèves, *Analytic hypo-ellipticity of a class of pseudodifferential operators with double characteristics and applications to the $\bar{\partial}$-Neumann problem*, Comm. Partial Differential Equations 3 (1978), no. 6-7, 475–642.

[31] F. Treves, *Introduction to pseudodifferential and fourier integral operators*, Vol. 1, Plenum Press, New York-London, 1980.

[32] F. Treves, *Symplectic geometry and analytic hypo-ellipticity, in Differential equations, La Pietra 1996 (Florence)*, Proc. Sympos. Pure Math. 65, Amer. Math. Soc., Providence, RI, 1999, 201-219.

[33] F. Treves, *On the analyticity of solutions of sums of squares of vector fields*, Phase space analysis of partial differential equations, Bove, Colombini, Del Santo ed.’s, 315-329, Progr. Nonlinear Differential Equations Appl., 69, Birkhäuser Boston, Boston, MA, 2006.

[34] F. Treves, *Aspects of Analytic PDE*, book in preparation.

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