Accelerating Stochastic Probabilistic Inference

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Amazon

Abstract

Recently, Stochastic Variational Inference (SVI) has been increasingly attractive thanks to its ability to find good posterior approximations of probabilistic models. It optimizes the variational objective with stochastic optimization, following noisy estimates of the natural gradient. However, almost all the state-of-the-art SVI algorithms are based on first-order optimization algorithm and often suffer from poor convergence rate. In this paper, we bridge the gap between second-order methods and stochastic variational inference by proposing a second-order based stochastic variational inference approach. In particular, firstly we derive the Hessian matrix of the variational objective. Then we devise two numerical schemes to implement second-order SVI efficiently. Thorough empirical evaluations are investigated on both synthetic and real dataset to backup both the effectiveness and efficiency of the proposed approach.

1 Introduction

Performing large scale inference for complex models is a fundamental task in modern machine learning and statistical applications. Bayesian learning provides a probabilistic framework for inference that combines prior knowledge with observed data in a principled manner. However, Bayesian computations are intractable in general case. Thus one might resort to either Markov Chain Monte Carlo (MCMC) [Robert and Casella, 2013] or variational Bayesian inference [Jordan et al., 1999]. They are widely applied in probabilistic modelling [Chen et al., 2015, 2014], and molecule generation [Li et al., 2020, Qian et al., 2021, Huang et al., 2020]. While MCMC provides unbiased estimates of Bayesian expectation, in practice designing MCMC algorithms that reliably converge to the desired posterior distribution is a notoriously difficult task especially in complex model. On the other hand, variational Bayesian approach approximates the full posterior by attempting to minimize the Kullback-Leibler (KL) divergence between the true posterior and a predefined distribution from a simple class of distribution on the same variables. Minimizing the KL divergence is equivalent to maximizing the familiar variational objective function. Specifically, let $\Theta = \{\theta_1, \ldots, \theta_d\}$ denotes the parameter that we are interested and $X$ the observed data. Variational methods approximate the intractable posterior distribution $p(\Theta | X)$ with a $q$-distribution from a simple family of distribution. The $q$-distribution is characterized by variational parameter, denoted $\Gamma$. For computational convenience, it is always assumed that the $q$-distribution is factorized. That is,

$$q(\Theta | \Gamma) = \prod_{i=1}^{d} q_i(\theta_i | \gamma_i). \tag{1}$$

Accordingly, the variational parameter $\Gamma$ can be factorized as $\Gamma = \{\gamma_1, \ldots, \gamma_d\}$. Factorized variational distribution has been proved to be efficient and effective. Thus, in this paper, we pay our main attention to factorized variational distribution.

The variational objective function (sometimes called variational lower bound) arises by bounding the marginal likelihood using the $q$-distribution, i.e.,

$$\mathcal{L}(\Gamma) = \int_{\Theta} q(\Theta | \Gamma) \ln \frac{p(X, \Theta)}{q(\Theta | \Gamma)} d\Theta$$

$$\leq \ln \int_{\Theta} p(X, \Theta) d\Theta = \ln p(X), \tag{2}$$

where the inequality follows from the Jensen Inequality and $p(X, \Theta) = p(\Theta) p(X | \Theta)$. Then the optimization problem can be formulated as

$$\arg \max_{\Gamma} \mathcal{L}(\Gamma) = \mathbb{E}_{q(\Theta | \Gamma)}[\ln p(X | \Theta)] + \mathbb{H}(q(\Theta | \Gamma)) + \ln p(X), \tag{3}$$

where $\mathbb{H}(q(\Theta | \Gamma)) = -\int_{\Theta} q(\Theta | \Gamma) \ln q(\Theta | \Gamma) d\Theta$, is a function of $\Gamma$ ($\Theta$ is integrated out). Since $q$ comes from a simple family of distribution, say, exponential family, $\mathbb{H}(q(\Theta | \Gamma))$ usually exhibits tractable form and differentiable w.r.t. the
variational parameter. The last term (i.e., \( \ln p(X) \), logarithm of normalizing constant) on RHS, independent of \( \Gamma \), can be omitted. It is observed that \( \mathcal{L}(\Gamma) = -\ln p(X) = -\text{KL}(q(\Theta|\Gamma)||p(\Theta|X)) \). Thus, maximizing \( \mathcal{L} \) with respect to \( \Gamma \) is equivalent to minimizing the KL divergence between \( q(\Theta|\Gamma) \) and \( p(\Theta|X) \).

Unfortunately, Problem (3) is not a typical optimization problem owing to the fact that both the variational objective and its gradient are intractable in general case, apart from simple cases involving some conjugate models. To address this issue, [Faisley et al. 2012, Ranganath et al. 2014] presented a method to approximate the gradient of the variational lower bound. In particular, they showed that the gradient of variational objective \( \mathcal{L} \) can be written into the form

\[
\nabla_\Gamma \mathcal{L}(\Gamma) = \mathbb{E}_{q(\Theta|\Gamma)}[\nabla_\Theta \log q(\Theta|\Gamma) \ln p(\Theta, X)] + \nabla_\Gamma \mathbb{E}(q(\Theta|\Gamma)),
\]

(4)

where the second term on RHS owns closed form while Monte Carlo integration is adopted to approximate the first term, given as

\[
\nabla_\Gamma \mathcal{L}(\Gamma) = \sum_{i=1}^{T} \nabla_\Gamma \log q(\Theta^{(i)}|\Gamma) \ln p(\Theta^{(i)}|x) + \nabla_\Gamma \mathbb{E}(q(\Theta|\Gamma)),
\]

(5)

where \( \Theta^{(1)}, \ldots, \Theta^{(T)} \) are i.i.d. \( q(\cdot|\Gamma) \).

Since the estimator is unbiased for the true gradient, a series of off-the-shelf tools in stochastic optimization can be adapted in this scenario [Ranganath et al. 2013, Wang et al. 2013]. These methods are all based on first-order optimization algorithms.

On the other hand, second-order methods, exploiting curvature information of the objective function, are acknowledged to enjoy faster per-iteration convergence and are well-studies in numerical optimization literature [Nocedal and Wright 2006]. However, they have been much less explored in the context of stochastic variational inference owing to the absence of second-order information. In this paper, we fill this blank and bridge the gap between second-order methods and stochastic variational inference. In particular, we first derive the Hessian matrix of variational lower bound. Then we provide two numerical schemes to implement second-order SVI efficiently. Furthermore, empirical results are also satisfactory.

The remainder of the paper is organized as follows: Section 2 briefly introduces Newton’s methods and its variants as preliminaries. In Section 3, second-order SVI is demonstrated elaborately.

## 2 Preliminaries—Newton’s Method

In this section, we provide a concise description for Newton’s method. In the context of numerical optimization, given some differentiable function \( f(x) : \mathbb{R}^d \to \mathbb{R} \), the task is either minimize or maximize it by altering \( x \). Among numerous optimization methods, the most commonly used method is steepest descent or gradient descent. Given current point \( x \), it proposes a new point \( x' \) via

\[
x' = x - \epsilon \nabla_x f(x),
\]

(6)

where \( \epsilon \) represents the learning rate, a positive scalar determining the size of the step. Numerous variants are raised based on the updating rule in Equation (6). Optimization algorithms that use only gradients are called first-order optimization algorithms.

Accordingly, algorithms that use the Hessian matrix (defined later) are called second-order algorithms. Newton’s method, the mainstream second-order method, is also known as the Newton-Raphson method. It is based on using a second-order Taylor series expansion to approximate \( f(x) \) near some point \( x^{(0)} \):

\[
f(x) \approx f(x^{(0)}) + (x - x^{(0)})^T \nabla_x f(x^{(0)}) + \frac{1}{2}(x - x^{(0)})^T \mathbf{H}(f)(x^{(0)})^{-1}(x - x^{(0)}),
\]

where \( \mathbf{H}(f)(x) \in \mathbb{R}^{d \times d} \), referred to as the Hessian matrix of \( f \), is defined such that

\[
\mathbf{H}(f)(x)_{i,j} = \frac{\partial^2}{\partial x_i \partial x_j} f(x) = \frac{\partial^2}{\partial x_j \partial x_i} f(x).
\]

(7)

Equivalently, the Hessian is the Jacobian of the gradient. Optimization algorithms such as Newton’s method that use the Hessian matrix are called second-order optimization algorithms. If we solve for the critical point of this function, we obtain:

\[
x' = x^{(0)} - \mathbf{H}(f)(x^{(0)})^{-1} \nabla_x f(x^{(0)}).
\]

(8)

It can be shown that the update rule described in Equation (8) eventually reaches quadratic convergence under reasonable assumptions. But it usually suffers from the prohibitively high per-iteration cost owing to the computation of the second-order information, especially the inversion of the Hessian matrix \( \mathbf{H}(f)(x^{(0)}) \).

To handle this problem, a number of quasi-Newton methods were raised [Nocedal and Wright 2006], which require only the gradient of the objective function to approximate Hessian matrix. The most popular quasi-Newton method is the BFGS and L-BFGS (limited-memory variant). Worth to mention that [Fan et al. 2013, Glass et al. 2020] argued that L-BFGS works in the context of stochastic variational inference.
3 Second-Order Stochastic Variational Inference

In this section, we explore the possibility of marrying the second-order method with stochastic variational inference. First, we derive the Hessian matrix of \( L(\Gamma) \). Then the efficient implementation of second-order SGVI is studied and two schemes are developed.

3.1 Hessian matrix

The Hessian matrix of objective function plays the critical role in second-order methods. To derive the Hessian matrix of \( \nabla^2_L L(\Gamma) \), we return to the gradient of variational objective \( \nabla_T L(\Gamma) \) governed in Equation (4), whose derivation is given as

\[
\nabla_T L(\Gamma) = \nabla_T \mathbb{E}_q[\ln p(\Theta | X)] + \nabla_T \mathbb{H}(q(\Theta | \Gamma))
\]

\[
= \nabla_T \int q(\Theta | \Gamma) \ln p(\Theta | X) d\Theta + \nabla_T \mathbb{H}(q(\Theta | \Gamma))
\]

\[
= \int q(\Theta | \Gamma) \nabla_T \ln p(\Theta | X) d\Theta + \nabla_T \mathbb{H}(q(\Theta | \Gamma))
\]

\[
= \int q(\Theta | \Gamma) \nabla_T \log q(\Theta | \Gamma) \ln p(\Theta | X) d\Theta
\]

\[
+ \nabla_T \mathbb{H}(q(\Theta | \Gamma))
\]

\[
= \mathbb{E}_{q(\Theta | \Gamma)}[\nabla_T \log q(\Theta | \Gamma)] + \nabla_T \mathbb{H}(q(\Theta | \Gamma))
\]

(9)

where the fourth equality follows from the identity \( \nabla_T q(\Theta | \Gamma) = q(\Theta | \Gamma) \nabla_T \log q(\Theta | \Gamma) \), which is also known as log-derivative trick. In the above equation, the integration \( \int q(\Theta | \Gamma) \nabla_T \ln p(\Theta | X) d\Theta \) (the first term in third line) is computationally intractable generally while the integration \( \int q(\Theta | \Gamma) \nabla_T \log q(\Theta | \Gamma) \ln p(\Theta | X) d\Theta \) can be estimated by Monte Carlo method, as mentioned in Equation (4).

Based on this, we can derive the Hessian matrix \( \nabla^2_T L(\Theta) \) via taking derivative to \( \nabla_T L(\Gamma) \)

\[
\nabla^2_T L(\Gamma) = \nabla_T \mathbb{E}_{q(\Theta | \Gamma)}[\nabla_T \log q(\Theta | \Gamma) \ln p(\Theta, X)]
\]

\[
+ \nabla_T \mathbb{H}(q(\Theta | \Gamma))
\]

\[
= \nabla_T \left[ \int q(\Theta | \Gamma) \nabla_T \log q(\Theta | \Gamma) \ln p(\Theta, X) d\Theta \right]
\]

\[
+ \nabla_T \mathbb{H}(q(\Theta | \Gamma))
\]

\[
= \int q(\Theta | \Gamma) \nabla_T \log q(\Theta | \Gamma) [\nabla_T \log q(\Theta | \Gamma)]^T \ln p(\Theta, X) d\Theta
\]

\[
+ \int q(\Theta | \Gamma) \nabla_T \log q(\Theta | \Gamma) \ln p(\Theta, X) d\Theta
\]

\[
+ \nabla_T \mathbb{H}(q(\Theta | \Gamma))
\]

\[
= \mathbb{E}_{q(\Theta | \Gamma)} \left[ \left[ \nabla_T \log q(\Theta | \Gamma) \right] \left[ \nabla_T \log q(\Theta | \Gamma) \right]^T \ln p(\Theta, X) \right]
\]

\[
+ \nabla_T \mathbb{H}(q(\Theta | \Gamma))
\]

(10)

where the fourth equality (the eighth line) employs the log-derivative trick again, borrowing the idea from [Paisley et al. 2012]. Worth to mention that for factorized variational distribution \( q(\Theta | \Gamma) \) satisfying that \( q(\Theta | \Gamma) = \prod_{i=1}^{d} q_i(\theta_i | \gamma_i) \), the matrix \( \nabla^2_T \log q(\Theta | \Gamma) \) is diagonal or block diagonal.

That is,

\[
\nabla^2_T \log q(\Theta | \Gamma) = \nabla^2_{\gamma_1, \ldots, \gamma_d} \sum_{i=1}^{d} \log q_i(\theta_i | \gamma_i)
\]

\[
= \begin{pmatrix}
\nabla^2_{\gamma_1} \log q_1(\theta_1 | \gamma_1) & \cdots & \nabla^2_{\gamma_d} \log q_d(\theta_d | \gamma_d)
\end{pmatrix}
\]

(11)

Similarly, we know that the last term \( \nabla^2_T \mathbb{H}(q(\Theta | \Gamma)) \) in RHS of Equation (10) can be simplified as

\[
\nabla^2_T \mathbb{H}(q(\Theta | \Gamma)) = \nabla^2_T \left( \sum_{i=1}^{d} \mathbb{H}(q_i(\theta_i | \gamma_i)) \right)
\]

\[
= \begin{pmatrix}
\nabla^2_{\gamma_1} \mathbb{H}(q_1(\theta_1 | \gamma_1)) & \cdots & \nabla^2_{\gamma_d} \mathbb{H}(q_d(\theta_d | \gamma_d))
\end{pmatrix}
\]

(12)

where \( \mathbb{H}(q_i(\theta_i | \gamma_i)) \) is a function of \( \gamma_i \). Thus \( \nabla^2_T \mathbb{H}(q(\Theta | \Gamma)) \) is also diagonal or block diagonal. These structures would be exploited to cut down the complexity later. Then we discuss the implementation of second-order SVI.

\[1\] The size of block is equal to the dimension of \( \gamma_i \), if \( \gamma_1, \ldots, \gamma_d \in \mathbb{R} \), then the matrix reduces to diagonal.
The most straightforward approach is to stochastically approximate the Hessian matrix $\nabla^2 L(\Theta)$ using Monte Carlo integration, i.e.,

$$\nabla^2 L(\Theta) = \frac{1}{S} \sum_{i=1}^{S} \left[ \nabla^2 \log q(\Theta^{(i)}|\Gamma) \left\{ \nabla_{\Theta} \log q(\Theta^{(i)}|\Gamma) \right\}^T \right]$$

$$\quad \ln p(\Theta^{(i)}|\Gamma) + \nabla^2 \log q(\Theta^{(i)}|\Gamma) \ln p(\Theta^{(i)}|\Gamma) + \nabla^2 \mathbb{E}(g(\Theta|\Gamma)),$$

where $\Theta^{(1)}, \ldots, \Theta^{(S)} \sim q(\Theta|\Gamma)$.

We can therefore replace $\nabla^2 L(\Theta)$ with the unbiased stochastic approximation of the Hessian matrix in Equation (10). Combining the update rule in Newton’s method described in Equation (8), the algorithm is shown in Algorithm 1.

**Algorithm 1** A straightforward approach to implement second-order SGVI

**Input:** initial value $\Gamma_0$, $S$, $T$.

**Output:** final value of variational parameter $\Gamma_t$

1: for $t = 1, 2, \ldots$ do
2: Estimate gradient $\nabla^2 L(\Gamma_{t-1})$ according to Equation (5). # $O(d)$
3: Estimate Hessian matrix $\nabla^2 L(\Theta)$ according to Equation (13). # $O(d^2)$
4: Update the variational parameter via $\Gamma_t = \Gamma_{t-1} - \left(\nabla^2 L(\Theta)\right)^{-1} \nabla^2 L(\Gamma_{t-1})$. # $O(d^3)$
5: if Convergence condition is met then
6: break.
7: end if
8: end for

The computational complexity for each step is listed. We find that per-iteration cost is prohibitively large due to the matrix inversion operation to the estimated Hessian matrix $\nabla^2 L(\Theta)$, which costs $O(d^3)$ computation, as shown in Step 4 of Algorithm 1. Thus we attempt to circumvent this operation in the follows.

### 3.2 Scheme I

To do this, we first rearrange Equation (13) as

$$\nabla^2 L(\Theta) = \frac{1}{S} \sum_{i=1}^{S} \left[ \nabla^2 \log q(\Theta^{(i)}|\Gamma) \ln p(\Theta^{(i)}, X) \right] + \nabla^2 \mathbb{E}(\Gamma)$$

$$+ \sum_{i=1}^{S} \left\{ [c_i \nabla^2 \log q(\Theta^{(i)}|\Gamma)] [c_i \nabla^2 \log q(\Theta^{(i)}|\Gamma)]^T \right\},$$

where $c_i = \sqrt{\ln p(\Theta^{(i)}, X) / S}$.

We find that $\nabla^2 L(\Theta)$, the estimated Hessian matrix, can be expressed as the sum of a diagonal (or block diagonal) matrix $A$ and rank-$S$ correction $B$. Then we demonstrate the celebrated Sherman–Morrison formula as follows.

**Lemma 3.1.** Suppose $A \in \mathbb{R}^{d \times d}$ is an invertible square matrix and $u, v \in \mathbb{R}^d$ are $d$-dimensional vectors. Suppose furthermore that $1 + v^T A^{-1} u \neq 0$. Then the Sherman–Morrison formula states that

$$(A + uv^T)^{-1} = A^{-1} - \frac{A^{-1} uv^T A^{-1}}{1 + v^T A^{-1} u}. \tag{15}$$

Note that the computation of Equation (15) only involves matrix-vector product and requires $O(d^2)$ computation. Thus we can iteratively use the Sherman–Morrison formula to compute the inversion of $\nabla^2 L(\Theta)$ described in Equation (13). Each iteration requires $O(d^2)$ computations. However, this strategy will be useful only in the case where $S \ll d$.

Then we focus on a more general case. In particular, our target is to compute $\nabla^2 L(\Theta)^{-1}$ efficiently. The target is just the solution of the following line system

$$(\nabla^2 L(\Theta)) y = \nabla^2 L(\Gamma), \tag{16}$$

where $y \in \mathbb{R}^d$.

Conjugate gradient algorithm [Nocedal and Wright, 2006], a well-studies algorithm in the context of numerical optimization, is suited to solve it. Note that in conjugate gradient algorithm, matrix-vector product $\nabla^2 L(\Theta)x$ (where $x \in \mathbb{R}^d$) is frequently calculated, thus when $S \ll d$, via making use of the special structure described in Equation (14), the computational complexity can be reduced to $O(Sd)$, instead of $O(d^2)$.

\(^2\)corresponding to the term $A$ in above equation, according to Equation (11) and (12).

\(^3\)corresponding to the term $B$. 
Then a sequence of matrix \( \{ B_i \} \) is defined as
\[
B_i = \sum_{j=0}^{i} (I - A)^j,
\]
i.e., the first \( i \) terms of Taylor expansion described Equation (17). It is easy to find the limiting properties
\[
\lim_{i \to \infty} B_i = A^{-1}
\]
as long as the assumptions in Lemma 3.2 holds. Additionally, we have the recursion
\[
B_i = I + (I - A)B_{i-1}.
\]

An unbiased estimator of Hessian matrix is devised based on this recursive formulation, given as
\[
\begin{align*}
\hat{B}_0 &= I, \\
\hat{B}_1 &= I + (I - X_i)\hat{B}_{i-1}.
\end{align*}
\]
where \( \{ X_i \} \) are unbiased samples of the Hessian matrix \( \nabla_\Gamma^2 \mathcal{L}(\Gamma) \). Concretely,
\[
X_i = \nabla_\Gamma \log q(\Theta^{(i)}|\Gamma) | \nabla_\Gamma \log q(\Theta^{(i)}|\Gamma)^T \ln p(\Theta^{(i)}, X) + \nabla_\Gamma^2 \log q(\Theta^{(i)}|\Gamma) \ln p(\Theta^{(i)}, X),
\]

Lemmas 3.2, 3.3. \( \hat{B}_i \) is an unbiased estimator for \( B_i \), i.e., \( \mathbb{E}[\hat{B}_i] = B_i \). Furthermore, we have
\[
\mathbb{E}[\hat{B}_i] \to (\nabla_\Gamma^2 \mathcal{L}(\Gamma))^{-1} \text{ as } i \to \infty.
\]

In practice, we attempt to avoid matrix-matrix\(^2\) or matrix-vector product for computational efficiency. Additionally, it is observed that \( X_i \) described in Equation (22) exhibits a special structure ((block) diagonal plus a rank-one correction) so that the computational complexity can be significantly reduced. The resulting algorithm is computationally light. Because per-iteration computation involves only vector-vector product (\( O(d) \) complexity). The computational complexity of each step is also listed. Details are listed in Algorithm 3.

Note that in Lemma 3.2, the assumption that \( \| A \| < 1 \) is too restrictive for Hessian matrix, thus we circumvent this case via estimating the inverse of \( \frac{1}{C_0} A \) (corresponding to Step 6, 7, 12 in Algorithm 3), where constant \( C_0 \) is pre-specified and satisfies that \( C_0 > \max_{\Gamma} \| \nabla_\Gamma \mathcal{L}(\Gamma) \| \).

In practice, \( C_0 \) usually takes a large value satisfying that \( C_0 \gg \| \nabla_\Gamma \mathcal{L}(\Gamma) \| \).

3.3 Scheme II

Agarwal et al. [2016] devised a novel estimator for the Hessian matrix when optimizing the finite sums. Here we adapt this strategy to variational inference setting. The motivation is from the well known fact about the Taylor series expansion of the matrix inverse as follows.

**Lemma 3.2.** For a semi-positive definite matrix \( A \in \mathbb{R}^{d \times d} \) satisfying that \( \| A \| < 1 \) we have that
\[
A^{-1} = \sum_{i=0}^{\infty} (I - A)^i, \tag{17}
\]
where \( I \) denotes the identity matrix.

The resulting algorithm is simple and listed in Algorithm 2, referred to as Second-Order Stochastic Gradient Variational Inference Scheme-I (SO-SGVI-I).

**Algorithm 2** Scheme I

**Input:** initial value \( \Gamma_0 \),

**Output:** final value of variational parameter \( \Gamma_t \).

1: for \( t = 1, 2, \ldots \) do
2: Estimate gradient \( \nabla_\Gamma \mathcal{L}(\Gamma_{t-1}) \) according to Equation (5).
3: Estimate Hessian matrix \( \nabla_\Gamma^2 \mathcal{L}(\Theta) \) according to Equation (13).
4: Option I: Compute \( (\nabla_\Gamma^2 \mathcal{L}(\Theta))^{-1} \) via iteratively using Sherman–Morrison formula, then update the variational parameter as \( \Gamma_t = \Gamma_{t-1} - \epsilon (\nabla_\Gamma^2 \mathcal{L}(\Theta))^{-1} \nabla_\Gamma \mathcal{L}(\Gamma_{t-1}). \)
5: Option II: Solve the linear system \( (\nabla_\Gamma^2 \mathcal{L}(\Theta)) \hat{y} = \nabla_\Gamma \mathcal{L}(\Gamma) \) using conjugate algorithm. Then update the variational parameter as \( \Gamma_t = \Gamma_{t-1} - \hat{y}. \)
6: if Convergence condition is met then
7: break.
8: end if
9: end for

[1] In this paper, \( \| \cdot \| \) represents the spectral norm for a matrix and \( l_2 \) norm for a vector.

[2] Like direct computation of recursion in Equation (21).

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Algorithm 3 Scheme II of Second-Order SVI

Input: initial value $\Gamma_0$, maximal iteration $T_{\text{max}}$, tolerance $\eta$, constant $C_0 \gg \|\nabla_2^2 L(\Gamma)\|

Output: final value of variational parameter $\Gamma_t$. 

1: for $t = 1, 2, \ldots$ do
2: Estimate gradient $\nabla_1 L(\Gamma_{t-1})$ according to Equation (5). # $O(d)$
3: $y_0 = \nabla_1 L(\Gamma_{t-1})$.
4: for $j = 1, \ldots, T_{\text{max}}$ do
5: Sample $\Theta^{(j)}$ from the variational distribution $q(\cdot|\Gamma)$ and compute the Hessian matrix for $\Theta^{(j)}$, denoted $X_i$. # $O(d)$: Note that $X_i$ is not computed explicitly here.
6: $X_i = X_i / C_0$.
7: $y_j = \nabla_1 L(\Gamma_{t-1}) + \frac{1}{C_0} y_{j-1} - \dot{X}_i y_{j-1}$. # $O(d)$: $\dot{X}_i y_{j-1}$ can be computed in linear time.
8: if $\|y_j - y_{j-1}\| \leq \eta$ then
9: break.
10: end if
11: end for
12: Then update the variational parameter as $\Gamma_t = \Gamma_{t-1} - \frac{1}{C_0} y_j$.
13: if Convergence condition is met then
14: break.
15: end if
16: end for

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