Modeling and study of properties of surfaces equidistant to a sphere and a plane

V I Vyshnepolsky¹, A V Efremov¹ and E V Zavariikhina²

¹MIREA – Russian Technological University, Moscow, Russia
²Moscow Aviation Institute (National Research University), Moscow, Russia
vyshnep@mail.ru

Abstract. In the present paper geometric locus of points (GLP) equidistant to a sphere and a plane is considered; the properties of the acquired surfaces are studied. Four possible cases of mutual location of a sphere and a plane are considered: the plane passing through the center of the sphere, the plane intersecting the sphere, the plane tangent to the sphere and the plane passing outside the sphere. GLP equidistant to a sphere and a plane constitutes two co-axial co-focused paraboloids of revolution. General properties of the acquired paraboloids were studied: the location of foci, vertices, axis and directing planes, distance between the sphere center and the vertices, the distance between the vertices. GLP for each case of mutual location of a plane and a sphere constitutes: in case one passes through the center of other, two co-axial co-focused oppositely directed paraboloids of revolution symmetrical with respect to the given plane; in case they intersect each other, two co-axial co-focused oppositely directed non-symmetrical paraboloids; in case they are tangent to each other, a paraboloid and a straight line passing through the tangency point; in case they have no common points, a pair of co-axial co-focused mutually directed paraboloids of revolution.

1. Introduction

Numerous geometers dedicated their work to the problems of Geometric Locus of Points (GLP). Among them the name of Dmitry Ivanovich Kargin (1880-1949) is considered the most outstanding. The problems of GLP were the subject of research by A.D. Posvyanskiy (1909-1992), I.I. Aleksandrov (1856-1919), M. Ya. Vygodskyi (1898 – 1965) and N.A. Salkov [1]. V.V. Glogovskiy (1920 – 1998) has published a number of papers on equidistant GLP. The present paper extends the series of articles on the topic of GLP previously published by the authors [2, 3, 4].

Solution to numerous tasks of analytic and descriptive geometry requires knowing geometric locus of points and curves. Surfaces generated as geometric locus of points equidistant to various geometric shapes offer practical value in architecture, industrial construction and cosmonautics [5]. A frame designed of GLP surfaces can be optimized with methods documented in paper [6]. At the moment, various studies on modeling and application of complex surfaces are being held [7, 8]. The present paper describes GLP equidistant to a sphere and a plane.

| Geometric objects | No. | Point | Line | Circle | Plane | Sphere | Cylindrical surface | Conical surface | Torus |
|-------------------|-----|-------|------|--------|-------|--------|---------------------|----------------|-------|
|                   | 0   | 1.1   | 2.1  | 2.2    | 2.3   | 2.4    | 2.5                 | 2.6            | 2.7   |
| Point             | 1   | 1.1   | 2.1  | 3.1    | 4.1   | 5.1    | 6.1                 | 7.1            | 8.1   |
| Line              | 2   | 1.2   | 2.2  | 3.2    | 4.2   | 5.2    | 6.2                 | 7.2            | 8.2   |
| Circle            | 3   | 1.3   | 2.3  | 3.3    | 4.3   | 5.3    | 6.3                 | 7.3            | 8.3   |
Table 1 [2] proposed above provides GLP classification. As one can see from Table 1, the GLP equidistant to a sphere and a plane are designated 5.4.

2. Problem definition
The present study is aimed at the problem of construction of geometric locus of points equidistant to a sphere and a plane. The problem is posed: to construct projections and 3D models of GLP surfaces, to derive equations for the constructed surfaces, to investigate regularities in shape and location of the surfaces constituting the geometric locus of points equidistant to a sphere and a plane.

3. Theory
Let us consider the methods of geometric locus construction. A number of geometric objects are selected as given, for example, a point and a cylindrical surface, a torus and a sphere, or a circle and a sphere. There can be two or more geometric objects. In order to acquire the geometric locus of points equidistant to the given objects, surfaces equidistant to the given objects on a certain distances \( t \) are constructed; the lines of intersection of those surfaces are found. The procedure above is repeated a number of times with step-by-step incrementation \( t_i \). For the sake of convenience, the step of incrementation should remain fixed. The multitude of the acquired curves of intersection of surfaces equidistant to the given objects on distance \( t_i \) comprises the sought surface equidistant to the given objects.

A geometric locus of points equidistant on \( t \) to a sphere of radius \( R \) constitutes a pair of spheres of radiiuses \( R \pm t \). A geometric locus of points equidistant on \( t \) to a plane constitutes a pair of planes located on distance \( t \) to the given plane.

GLP 3D modeling in CAD software involves constructing of the mentioned surfaces on distance \( t \) to the given objects and acquiring of the curves of their intersection. Then, through use of the “lofted surface” tool, the sough surface of GLP equidistant to the given objects is constructed. Surface modeling can also be performed in CAD software through methods documented in paper [9]. There are two known approaches to GLP studies: the analytic approach and the graphic approach (see Figure 1):
1. The projectional method allows one to construct projections of GLP equidistant to the considered objects, determine foci, directrices, vertices of the acquired curves and to estimate their type.
2. The 3D models of GLP created by means of CAD software allow one to visually analyze their shape.
3. In order to determine the types of the studied GLP surfaces, it is required to derive equations for these surfaces.
4. 3D models created in computer algebra systems allow one to observe variation in shape of the constructed GLP upon variation of mutual location of the given objects.

4. Results of the study
It has been established that GLP equidistant to a sphere and a plane constitutes a pair of co-axial co-focused paraboloids of revolution. These paraboloids have common properties described below. 1. They are both focused at the center of the given sphere. 2. Their vertices belong to a perpendicular raised from
the center of the sphere to the given plane. 3. The direction of the perpendicular matches the axes of revolution of the paraboloids. 4. The distance between the vertices of the paraboloids is equal to the sphere radius \( R \). 5. The distance \( n \) between the center of the sphere and a vertex of any paraboloid is equal to \( 0.5(R - a) \), with a single exception: if the given plane is tangent to the given sphere, the vertices of the paraboloids match the tangency point. 6. Directing planes of the paraboloids are located at distance \( R \) to the given plane.

Let us consider the geometric locus of points in the four possible cases of mutual location of the given plane and the given sphere:

1. The GLP equidistant to a plane and a sphere, one passing through the center of the other, constitutes a pair of co-axial co-focused oppositely directed paraboloids of revolution symmetrical with respect to the given plane (Figure 2).
2. The GLP equidistant to a plane and a sphere intersecting each other constitutes two co-axial co-focused oppositely directed, but not symmetrical, paraboloids of revolution (Figure 3).
3. The GLP equidistant to a plane and a sphere tangent to each other constitutes a paraboloid and a straight line passing through the point of tangency (Figure 4).
4. The GLP equidistant to a plane and a sphere that have no common points constitutes a pair of co-axial co-focused mutually directed paraboloids of revolution (Figure 5).

5. Consideration of the results

Before we begin, it is worth mentioning that the equation defining a sphere allows for imaginary values of radius. Spheres of complex radiiues are also capable of generating surfaces of equidistant points; however, the scope of the present paper is restricted to real solutions.

Let us consider the following four cases of mutual location of the given sphere and plane and study the number and the shape of the GLP for each case:

– case 5.4.1: the plane passes through the center of the sphere, \( a = 0 \);
– case 5.4.2: the plane intersects the sphere, \( a < R \);
– case 5.4.3: the plane is tangent to the sphere, \( a = R \);
– case 5.4.4: the plane is outside the sphere, \( a > R \);

In the expressions above \( a \) represents the distance between the center of the given sphere and the given plane; \( R \) represents the radius of the given sphere.

Note that mutual location of other objects was studied in previous papers by the authors: case 5.1 a point and a sphere [1]; case 6.1 a point and a cylindrical surface [1], case 5.2 a sphere and a straight line [3].

The given plane passes through the center of the given sphere: \( a = 0 \) (5.4.1)

Every sphere of radius \( R + t \) intersects a plane \( \Delta \pm t \). These intersections generate the parallel of the larger circle \( AB \) (the intersection between the given sphere \( \Sigma \) and the given plane \( \Delta \)) as well as parallels 14, 25, 36, i.e. \( \Delta^1 \cap \Sigma^1 = 14; \Delta^2 \cap \Sigma^2 = 25; \Delta^3 \cap \Sigma^3 = 36. \)

Figure 2. GLP equidistant to a sphere and a plane. The plane passes through the center of the sphere, \( a = 0 \): (a) front projection, (b) 3D model
These parallels grow in diameter. Intersections of the same spheres with planes constructed to the left of the given plane \( \Delta \) result in parallels \( \Delta'' \cap \Sigma'' = 710; \Delta'' \cap \Sigma'' = 811; \Delta'' \cap \Sigma'' = 912 \) (Figure 2 (a)) equal to the circles 14, 25, 36, namely, 14=710, 25=811, 36=912. The mentioned parallels are located symmetrically with respect to the given plane \( \Delta \).

These parallels along with an infinite number of others generate two symmetrical (with respect to plane \( \Delta \)) surfaces of revolution focused at the center \( O \) of the sphere with axis \( i \) of revolution that constitute two co-focused and co-axial paraboloids of revolution. The vertices of the paraboloids \( K \) and \( N \) are located midway between the plane \( \Delta \) and the sphere \( \Sigma \), i.e. at distance \( R/2 \) to the sphere center \( O \). The distance between the vertices is therefore \( KN = R \). Directing planes are located at distance \( R/2 \) to the vertices, therefore, they are tangent to the given sphere and are located at distance \( R \) to the given plane \( \Delta \).

Conclusion: In the case 5.4.1, when the plane \( \Delta \) passes through the center of the sphere \( \Sigma \), as shown on Figure 2 (a), (b), the GLP equidistant to the sphere \( \Sigma \) and the plane \( \Delta \) constitutes two symmetrically located co-focused and co-axial paraboloids of revolution, their vertices located at distance \( R/2 \) to the sphere center \( O \) and at distance \( R \) to each other, and their directing planes located at distance \( R \) to the given plane \( \Delta \).

The given plane intersects the given sphere: \( a < R \) (5.4.2)

It is safe to assume that the surface \( \Gamma^{5.4.2} \) has to be similar to the surface \( \Gamma^{5.4.1} \) acquired in the previous case, yet asymmetrical, since the circle \( AB \) of intersection of the given plane \( \Delta \) and the given sphere \( \Sigma \) does not match the larger circle of \( \Sigma \), hence the diameter of circle \( AB \) is less than the diameter of the sphere \( \Sigma \) (Figure 3 (a), (b)).

**Figure 3.** GLP equidistant to a sphere and a plane. The plane intersects the sphere, \( a < R \): (a) front projection, (b) 3D model

Spheres of radius \( R + ti \) intersect planes \( \Delta = t, \Delta', \Delta^2, \Delta^3 \) (to the right of the plane \( \Delta \)) and \( \Delta''^1, \Delta''^2, \Delta''^3 \) (to the left of the plane \( \Delta \)) generating parallels 16 and 16'', 27 and 27'', 38 and 38'' etc,
Spheres of radiuses $R_{t_i}$: $\Sigma^1$, $\Sigma^2$, $\Sigma^3$ intersect planes $\Delta^{+t_i}$: $\Delta^1$, $\Delta^2$, $\Delta^3$: $\Delta^1 \cap \Sigma^1 = 1'6'$; $\Delta^2 \cap \Sigma^2 = 2'5'$; $\Delta^3 \cap \Sigma^3 = 3'4'$. Since the spheres $\Sigma_{t_i}$ then degenerate into a point, there are no other intersections.

Surface $\Gamma^{4.5.2}$ consists of two hyperboloids, their vertices are at distance $R$, as in the case 4.5.1: $CD = R$. The directing planes $H$ and $\Phi$ are located at distance $R$ to the given plane $\Delta$ as in the case 4.5.1. The vertex $C$ is at distance $n$ to the center of the sphere:

$$n = \frac{R - a}{2},$$

where $n$ represents the distance between the center of the sphere and the vertex; $R$ represents the radius of the sphere $\Sigma$; $a$ represents the distance between the given plane $\Delta$ and the center $O$ of the sphere $\Sigma$.

The expression (1) is true for surfaces $\Gamma^{4.5.2}$ and $\Gamma^{4.5.1}$. As one may notice, at $a = 0$ $n = R/2$, as in the previous case 4.5.1.

Conclusion: In the case 5.4.2, when the plane $\Delta$ intersects the sphere $\Sigma$, the GLP equidistant to the sphere $\Sigma$ and the plane $\Delta$ constitutes two asymmetrical co-focused and co-axial oppositely directed paraboloids of revolution (Figure 3 (a), (b)), their vertices $C$ and $D$ located at distance $R$ to each other, and their directing planes located at distance $R$ to the given plane $\Delta$.

**The given plane is tangent to the given sphere:** $a = R$ (5.4.3)

In general, GLP equidistant to a plane and a sphere constitutes two surfaces. However, if the given surfaces are tangent to each other, one of the GLP surfaces degenerates into a straight line. In the current case one of the paraboloids transforms into line $CA$ as shown on Figure 4 (a), (b). This happens since each sphere $\Sigma_{t_t}$ is not intersecting each respective plane $\Delta^{\pm t_t}$, but is tangent to it in points $C$, $A$, $B$, $O$ etc., since the common step incrementation is started at the point of tangency $A$ between the given plane $\Delta$ and the given sphere $\Sigma$.

![Figure 4. GLP equidistant to a sphere and a plane. The plane is tangent to the sphere, $a = R$. (a) front projection, (b) 3D model](image-url)
distance $R$ to the given plane $\Delta$. The latter is obvious in the present case, since $AO$ is the radius of the given sphere $\Sigma$, and the vertex is located midway between the focus and the directing plane.

Conclusion: In the case 5.4.3, when the plane $\Delta$ is tangent to the sphere $\Sigma$, the GLP equidistant to the sphere $\Sigma$ and the plane $\Delta$ constitutes a paraboloid of revolution $\Gamma$ and a straight line $CA$ passing through the center $O$ of the sphere $\Sigma$ and the point of tangency $A$. The center $O$ divides the line $CA$ into two rays, a positive one and a negative one. The vertex of the paraboloid $\Gamma$ is the point of tangency $A$, its focus is the center $O$ of the sphere, its directing plane is located at distance $R$ to the given plane $\Delta$, its axis matches the line $AO$.

The given plane passes outside the given sphere: $a > R$ (5.4.4)

Figure 5. GLP equidistant to a sphere and a plane. The plane is outside to the sphere, $a > R$: (a) front projection, (b) 3D model

Spheres $\Sigma_i$ of radius $R + t$: $\Sigma^1$, $\Sigma^6$, $\Sigma^7$, $\Sigma^8$ etc. intersect the corresponding planes $\Delta^1$, $\Delta^6$, $\Delta^7$, $\Delta^8$ constructed to the right of the given plane $\Delta$. Let us consider this direction as positive. The intersections generate a point $\Sigma^1 \cap \Delta^8 = 1$ and parallels $\Sigma^i \cap \Delta^6 = 23$; $\Sigma^j \cap \Delta^7 = 45$; $\Sigma^k \cap \Delta^8 = 67$ etc. forming paraboloid $\Gamma$ with vertex 1, and focus $O$ matching the center of the sphere.

Spheres $\Sigma_i$ of negative radius, starting with $\Sigma^9$, intersect the corresponding planes generating the vertex $\Sigma^9 \cap \Delta^8 = 1'$ of the second paraboloid of revolution $T$ and parallels $\Sigma^{10} \cap \Delta^6 = 2'3'$; $\Sigma^{11} \cap \Delta^7 = 4'5'$; $\Sigma^{12} \cap \Delta^8 = 6'7'$ etc. forming this paraboloid. The paraboloid $T$ is focused, as in any other case of $\Gamma^{5,4}$, at the center $O$ of the given sphere $\Sigma$.

As mentioned previously, both directing planes are located at distance $R$ to the given plane $\Delta$. The vertex $1'$ is located at distance $n$ to the center $O$ of the given sphere $\Sigma$, according to the expression (1), in the present example $n = -5t$, where $t$ represents incrementation step. The distance between the vertices of the paraboloids is $R$. The common axis of the paraboloids passes through the vertices 1, 1', and through the center $O$ of the given sphere $\Sigma$.

Conclusion: In the case 5.4.4, when the plane $\Delta$ passes outside the sphere $\Sigma$, the GLP equidistant to the sphere $\Sigma$ and the plane $\Delta$ constitutes two co-axial, co-focused mutually directed paraboloids of revolution; their directing planes are located at distance equal to sphere radius $R$ to the given plane $\Delta$; their vertices are located at distance $R$ to each other, while one of the vertices is located at distance $n$ (1) to the center of the given sphere.

Analytic representation of GLP equidistant to the given sphere and the given plane

Let us apply canonical equations for second-degree surfaces.
Expressions (2) and (3) are the equations for a sphere and a plane located as depicted to Figure 6.

\[ x^2 + y^2 + z^2 = (R \pm t)^2 , \]  
\[ x = a \pm t . \]  

Let us raise (3) to the second power and acquire the expression (4).

\[ x^2 = (a \pm t)^2 . \]  

Figure 6. A drawing on derivation of equations for GLP surfaces equidistant to a sphere and a plain

Substitution of expression (4) from the equation (2) with subsequent transformation and simplification results in the equation for GLP equidistant to a sphere and a plane (5).

\[ y^2 + z^2 = R^2 - a^2 \pm 2t(R \pm a) . \]  

The equation (5), with consideration for plus and minus signs, constitutes four equations

\[ y^2 + z^2 = R^2 - a^2 + 2t(R - a) , \]  
\[ y^2 + z^2 = R^2 - a^2 + 2t(R + a) , \]  
\[ y^2 + z^2 = R^2 - a^2 - 2t(R - a) , \]  
\[ y^2 + z^2 = R^2 - a^2 - 2t(R + a) . \]  

The equations (6), (7), (8), (9) presented in canonical form are (10), (11), (12), (13) respectively.

\[ x = \frac{y^2}{2(R - a)} + \frac{z^2}{2(R - a)} - \frac{R - a}{2} , \]  
\[ x = \frac{y^2}{2(R + a)} + \frac{z^2}{2(R + a)} - \frac{R - a}{2} , \]  
\[ -x = \frac{y^2}{2(R - a)} + \frac{z^2}{2(R - a)} - \frac{R - 3a}{2} , \]  
\[ -x = \frac{y^2}{2(R + a)} + \frac{z^2}{2(R + a)} - \frac{R - 3a}{2} . \]  

As one may observe, the equations (10), (11), (12), (13) are the canonical equations for four paraboloids of revolution, or, more precisely, two pairs of oppositely directed paraboloids (10), (13) and (11), (12). The axis of revolution of the paraboloids (10), (11), (12), (13) is either parallel, or matching the \( x \) axis.

The negative sign of \( x \) in the equations (12), (13) indicates that the paraboloids (10), (13) and (11), (12) are oppositely directed.

The absolute terms \( 0.5(R - a) \) and \( 0.5(R - 3a) \) in equations (10), (11), (12), (13) indicate the position of vertices of the paraboloids on the \( x \) axis.

For example, at \( a = 0 \), when the plane passes through the center of the sphere, the equation (10) is reduced to

\[ x = \frac{y^2}{2R} + \frac{z^2}{2R} - \frac{R}{2} . \]  

while the equation (13) is reduced to

\[ -x = \frac{y^2}{2R} + \frac{z^2}{2R} - \frac{R}{2} . \]  

The equations (14), (15) define two symmetrical co-focused co-axial paraboloids of revolution with vertices at distance \( R/2 \) to the center of the sphere and at distance \( R \) to each other conforming to the conclusions drawn earlier and the Figure 2.
Consider another example: at $a = R$, when the plane is tangent to the sphere, the equation (10) at $x = 0$ is reduced to the equation of a straight line matching the $x$ axis.

As follows from the equation (13),

$$-x = \frac{y^2}{4R} + \frac{z^2}{4R} + R.$$  

The equation (16) is the equation for a paraboloid of revolution with vertex belonging to the outline of the sphere, see Figure 4.

Let us summarize the results of the study.

The GLP equidistant to a sphere and a plane constitutes two paraboloids of revolution possessing the following common properties:

1. The paraboloids are always focused at the center of the given sphere $\Sigma$.
2. The vertices of the paraboloids belong to a perpendicular raised from the center of the given sphere $\Sigma$ to the given plane $\Delta$.
3. The direction of the perpendicular matches the axes or revolution of the paraboloids.
4. The distance between the vertices of the paraboloids is equal to the sphere radius $R$, as one can see from Figures 2, 3, 5.
5. The distance $n$ between the center of the sphere and the vertex of one of the paraboloids $n = 0.5(R - a)$ with one exception: in case 5.4.3, when the plane is tangent to the sphere $a = R$, the vertex of the paraboloid matches the point of tangency, see Figure 4.
6. The directing planes are located on distance $R$ to the given plane $\Delta$, as shown on Figures 2 – 5.

Let us list the GLP in the four cases of mutual location of the sphere and the plane (see Table 2).

1. Case 5.4.1, $a = 0$, the given plane $\Delta$ passes through the center of the sphere, illustrated on Figure 2. The GLP $\Gamma^{5.4.1}$ constitutes a pair of co-axial co-focused oppositely directed paraboloids of revolution symmetrical with respect to the given plane $\Delta$.
2. Case 5.4.2, $a < R$, the given plane $\Delta$ intersects the given sphere $\Sigma$, illustrated on Figure 3. The GLP $\Gamma^{5.4.2}$ constitutes two co-axial co-focused oppositely directed, but not symmetrical, paraboloids of revolution.
3. Case 5.4.3, $a = R$, the given plane $\Delta$ is tangent to the given sphere $\Sigma$, illustrated on Figure 4. The GLP $\Gamma^{5.4.3}$ constitutes a paraboloid and a straight line passing through the point of tangency $A$ and the center $O$ of the given sphere $\Sigma$.
4. Case 5.4.4, $a > R$, the given plane $\Delta$ passes outside the given sphere $\Sigma$, illustrated on Figure 5. The GLP $\Gamma^{5.4.4}$ constitutes two co-axial co-focused mutually directed paraboloids of revolution.

The results of the study are summarized in Table 2. Note that the first GLP in every case of mutual location of a sphere and a plane is a paraboloid of revolution.

| No. | Designation | Distance $a$ between the plane and the center of the sphere | First GLP | Second GLP | Figure |
|-----|-------------|-----------------------------------------------|----------|-----------|-------|
| 1   | 5.4.1       | $a = 0$                                      | A paraboloid of revolution | A paraboloid of revolution symmetrical and oppositely directed to the first GLP | Figure 2 |
| 2   | 5.4.2       | $a < R$                                      | A paraboloid of revolution | A paraboloid of revolution oppositely directed, but not symmetrical to the first GLP | Figure 3 |
| 3   | 5.4.3       | $a = R$                                      | A paraboloid of revolution | A straight line | Figure 4 |
| 4   | 5.4.4       | $a > R$                                      | A paraboloid of revolution | A paraboloid of revolution directed mutually to the first GLP | Figure 5 |
6. Conclusion
In the present paper the GLP equidistant to a sphere and a plane were constructed. The equations for the surfaces of GLP equidistant to a sphere and a plane, namely various paraboloids of revolution, were derived. The regularities in shape and location of these surfaces were studied; the formulas for location of vertices and directing planes of paraboloids of revolution were acquired.

The GLP surfaces have numerous applications. One of the surfaces of GLP equidistant to a point and a conic surface [3] can have wide application in architecture and construction. Surfaces equidistant to two spheres (planets) should find application in cosmonautics [5]. The tasks featuring the geometric locus of points should be (and already are) included in the curriculum for descriptive geometry, graphics, and computer graphics. For example, numerous student competitions on descriptive geometry, engineering and computer graphics held in RTU MIREA, Moscow city and nation-wide Russia featured the challenges requiring the student to be able to understand and construct the geometric locus of points.

The near-term prospect of this line of research is to study the GLP equidistant to each pair of geometric objects presented in Table 1.

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