Vector coherent states for nanoparticle Hamiltonians

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Abstract

The first part of this work deals with an extension of the formalism of vector coherent states developed for degenerate Hamiltonian systems by Ali and Bagarello [J. Math. Phys. 46, 053518 (2005)] to the study of $M$ fermionic level system associated with $N$ bosonic modes. Then follows a generalization leading to a Hamiltonian describing the translational motion of the center of mass of a nanoparticle, giving rise to a new mechanism for the electronic energy relaxation in nanocrystals which is intensively studied today in condensed matter physics. Finite and infinite degeneracies of these Hamiltonian systems are also investigated. Vector coherent states are defined and satisfy relevant mathematical properties of continuity, resolution of identity, temporal stability and action identity as stated by Gazeau and Klauder.
1 Introduction

Nanoscience has exploded in the last decade, primarily as the result of the development of new tools that have made the characterization and manipulation of nanostructures practical, and also as a result of new methods for preparation of these structures. Many technological applications including miniaturization in microelectronics devices yield research activities on electronic properties of new nanoscale structure. In at least one dimension, when the size of an object becomes comparable to the characteristic length of an electronic quantum effect, its physical manifestation is modified: interference and coherence effects, quantum tunnelling, discrete energy level, collective effects, electronic transport, role of interactions, coupling to the environment must be taken into account. All these phenomena occurring in quantum electronics lie at the heart of today scientific activities to which intense efforts are devoted due to their numerous applications in various domains ([1], [2] and references therein). Indeed the research activities of the quantum nanostructures are related to the fields of quantum information (quantum computers), spin electronic (spintronics), molecular electronics and nanophotonics. In the field of spintronics, the understanding of spin-related phenomena in magnetic or non-magnetic doped nanostructures is of paramount importance for using spin as information vector. Spin dynamics is studied in quantum confined nanostructures and ferromagnetic semiconductor layers by pump-probe techniques. Magnetic domain structure and domain wall motion are investigated in ferromagnetic semiconductors (also in superconductors) using magneto-optical Kerr microscopy. Electronic excitations in two-dimensional (2D) and one-dimensional nanostructures are investigated by means of electronic Raman spectroscopy: collective and individual spin polarized excitations of a (2D) electron gas, electronic excitations in quantum wells and wires. Moreover, a new development in near-field imaging [3] now makes it possible to map vector fields on the nanoscale as never before. Lee and co-workers [4] recently reported an experimental technique that can capture and map the vectoriel nature of the electric fields down to the nanoscale. This could lead to important applications in physics and biochemistry. Indeed, in-depth knowledge of the electric field vector on the nanoscale could help in the design of miniaturized optical components that may replace their electronic counterparts in the future. In addition, near-field vector imaging is also important in biosensing applications because the interaction between light and biological molecules strongly depends on the orientation of the electric field [3]. By extracting this information we can uncover new effects at play.

In the context of the theory of electron-phonon dynamics in insulating nanoparticles, the electron-phonon dynamics in an ensemble of nearly isolated nanoparticles, the vibrational dynamics of nanometer-scale semiconducting and insulating are probed by localized impurity states ([5]). Nanoparticles are modeled there as electronic two-level systems coupled to single vibrational modes.

Despite all these exciting explorations, to the best of our knowledge, not much has been done in the investigation of vector coherent states for nanoparticle systems. This work aims at filling this gap.

Coherent states originally introduced by Schrödinger ([6]) in the context of a harmonic oscillator and later popularized by Glauber and Klauder for the description of coherent light are generally acknowledged to provide a close connection between classical and quantum formulations for a given system. Indeed, they are useful for the phenomenological descrip-
tion of a nonlinear optical environment and nonlinear interacting systems in quantum optics. The concept of vector coherent states (VCS) has been introduced in mathematical literature in fifties (see [7] and references therein) in the study of induced representations of groups, constructed using vector bundles over homogeneous spaces and, while in the physical literature the idea was born with their formulation in connection with the use of the symplectic group for describing collective models of nuclei ([8] and references therein). Ever since, they have been widely used in a variety of symmetry problems in quantum mechanics in general, and in particular for quantum optical models ([9], [10], [8]), as well as for the study of spectra of two-level atomic systems in electromagnetic fields, for instance in the famous Jaynes-Cummings model ([11]). In previous works, in the context of quantum optics, a new formulation of VCS for nonlinear spin-orbit Hamiltonian model [1] in terms of the matrix eigenvalue problem for generalized annihilation operators has been provided. Let us cite also the VCS of Gazeau-Klauder type [12], constructed for a Hamiltonian describing the interaction between a single mode, \((a, a^\dagger)\) of the radiation field with the frequency \(\omega\) and 2 fermionic levels with energies \(\epsilon_1, \epsilon_2\), and creation and annihilation operators \(c_j^\dagger, c_j, j = 1, 2\).

This paper addresses investigations on general construction of VCS for nanoparticle systems which are today broadly used in the domain of condensed matter physics and include the physical system studied in [12] as a particular case. In section 2, VCS for the electron-phonon dynamics are built. Then, in section 3, we provide with a generalization of examined model and construct corresponding VCS. Finally we end with some concluding remarks in section 4.

2 Vector coherent states for electron-phonon dynamics

In this section, we deal with the construction of vector coherent states for the model describing the electron-phonon dynamics. This model consists of the following Hamiltonian given (with \(\hbar = 1\)) by

\[
H = \sum_{i=1}^{N} \omega_i a_i^\dagger a_i + \sum_{j=1}^{M} \epsilon_j c_j^\dagger c_j + \sum_{i=1}^{N} \sum_{l=1}^{M} g_{lk} c_l^\dagger c_i (a_i^\dagger + a_i),
\]

where the following commutation rules hold:

\[
\begin{align*}
[a_i, a_k^\dagger] &= \delta_{ik} 1, \quad \{c_j^\dagger, c_k\} = \delta_{jk} 1, \\
[a_i, a_i] &= 0 = [a_i^\dagger, a_i^\dagger], \quad \{c_j, c_j\} = 0 = \{c_j^\dagger, c_j^\dagger\}, \quad 1 \leq i, k \leq N, 1 \leq j, k \leq M,
\end{align*}
\]

where \(\{a_i, a_i^\dagger, N_i = a_i^\dagger a_i\}\) for each \(1 \leq i \leq N\) span the ordinary Fock-Heisenberg oscillator algebra.

The first term of this Hamiltonian describes the internal vibrational dynamics of a nanoparticle: the \(\omega_i\) are frequencies of the internal modes, and \(a_i^\dagger\) and \(a_i\) are the corresponding phonon creation and annihilation operators; the second term is the Hamiltonian for a non-interacting \(N\)-level system; the third term is the ordinary leading-order interaction between the \(N\)-level system and the internal vibrational modes. The constants \(g_{lk}(l = 1, 2, \cdots, M)\) are the electron-phonon interaction energy. This system generalizes a previous work [12] on a specific physical Hamiltonian of a single photon mode with the frequency \(\omega\) interacting with a pair of fermions, for which vector coherent states of Gazeau-Klauder type have been explicitly constructed, satisfying the four requirements of continuity, temporal stability, resolution
of the identity and action identity \cite{13}. The same model, describing an interaction between a single mode, \((a, a^\dagger)\), of the radiation field with two Fermi type modes, has been also object of investigation by Simon and Geller who outlined some physical aspects of the ensemble-averaged excited-state population dynamics of this model \cite{5} and showed its relevance in the study of electron-phonon dynamics in an ensemble of nearly isolated nanoparticles (\cite{5}), in the context of quantum effects in condensed matter systems. The vibrational spectrum of the nanoparticle is here provided by the localized electronic impurity states in doped nanocrystal (\cite{14}, \cite{15}). The impurity states can be used to probe the energy relaxation by phonon emission.

The generalized Hamiltonian (\cite{11}) considered in this work describes a nanoparticle modeled as a system of \(M\) Fermi types modes interacting with \(N\) vibrational modes of radiation (phonons) with frequencies \(\omega_i, 1 \leq i \leq N\). The eigenvector of the Hamiltonian \(H\) can be written as

\[
\varphi = \Phi_1 \otimes \Phi_2 \otimes \cdots \otimes \Phi_N \otimes \Psi_{[k]}, \quad [k] = k_1k_2\ldots k_M,
\]

where \(\Phi_l, l = 1, 2, \ldots, N\) and \(\Psi_{[k]}\) are respectively the bosonic and fermionic states with \(k_1, k_2, \ldots, k_M = 0, 1\). For all \(k_j, 1 \leq j \leq M\), the eigenvalue problem can be then defined as

\[
H \varphi = \left[ \sum_{i=1}^{N} \omega_i a_i^\dagger a_i + \sum_{i=1}^{N} \sum_{j=1}^{M} \delta_{k_j1} (\epsilon_j + g_j (a_i^\dagger + a_i)) \right] \varphi := E \varphi,
\]

\(E\) being the eigenenergy of the system. As a matter of appropriate analysis \cite{12}, let us consider the following self-adjoint operator \(B_{[k]12\ldots N}\) given by

\[
B_{[k]12\ldots N} = \sum_{l=1}^{N} B_{[k]l}, \quad B_{[k]l} = \omega_l A_{[k]l}^\dagger A_{[k]l} + \frac{\epsilon_{[k]}+g_{[k]}^2}{\omega_l},
\]

with

\[
[A_{[k]l}, A_{[k]l}^\dagger] = I, \quad A_{[k]l} = a_l + \frac{g_{[k]}}{\omega_l}, \quad l = 1, 2, \ldots, N,
\]

where we have introduced the quantities \(g_{[k]}\) and \(\epsilon_{[k]}\) with the dimension of energy (taking \(\hbar = 1\)), defined by

\[
g_{[k]} := \sum_{[k]=0}^{1} k_j g_j, \quad \epsilon_{[k]} := \sum_{[k]=0}^{1} k_j \epsilon_j, \quad 1 \leq j \leq M.
\]

The operators \(A_{[k]l}\) are such that

\[
A_{[k]l} = e^{i \sqrt{2} \frac{\epsilon_{[k]}+g_{[k]}^2}{\omega_l} P_l} a_l e^{-i \sqrt{2} \frac{\epsilon_{[k]}+g_{[k]}^2}{\omega_l} P_l}, \quad P_l = \frac{a_l - a_l^\dagger}{i \sqrt{2}}, \quad l = 1, 2, \ldots, N,
\]

where for all \(l\) the commutation relation

\[
[a_l, \sqrt{2} P_l] = i
\]

is satisfied. Thus, for all \(l, l = 1, 2, \ldots, N\), the eigenstates \(|\Phi_{[k]}^{[n_l]}\rangle\) of \(B_{[k]l}\) is of the type

\[
|\Phi_{[k]}^{[n_l]}\rangle = e^{i \sqrt{2} \frac{\epsilon_{[k]}+g_{[k]}^2}{\omega_l} P_l} |n_l\rangle = \frac{(A_{[k]l}^\dagger)^{n_l}}{\sqrt{n_l!}} |\Phi_{[k]}\rangle_l,
\]
where \( |\Phi^{[k]}_0\rangle = e^{i\sqrt{\frac{\pi}{2l}}|n_l|} |0\rangle \) with the corresponding eigenvalues

\[
E^{[k]}_{n_l} = \omega l n_l + \frac{\epsilon^{[k]}_l}{N} - \frac{g^{[k]}_l}{\omega_l}.
\]

Therefore, the eigenstates of \( B_{[k]_{12}...N} \) are given by

\[
|\Phi^{[k]}_{n_1 n_2 ... n_N}\rangle = |\Phi^{[k]}_{n_1}\rangle \otimes |\Phi^{[k]}_{n_2}\rangle \otimes \cdots \otimes |\Phi^{[k]}_{n_N}\rangle
= \left( \bigotimes_{l=1}^N \left( \frac{(A^\dagger_{[k]})^N}{\sqrt{n_l!}} e^{i\sqrt{\frac{\pi}{2l}g_{l}}|n_l|} \right) \right) (|0\rangle \otimes |0\rangle \otimes \cdots \otimes |0\rangle),
\]

where \( n_1, n_2, \ldots, n_N = 0, 1, 2, \ldots, \) associated with the eigenvalues

\[
E^{[k]}_{n_1 n_2 ... n_N} = \sum_{l=1}^N \omega l n_l + \epsilon^{[k]}_l - \frac{g^{[k]}_l}{\omega_l} \sum_{l=1}^N \frac{1}{\omega_l}, l = 1, 2, \ldots, N, n_l = 0, 1, 2, \ldots.
\]

Physically, \([k], k_1, k_2, \ldots, k_M = 0, 1\) can be interpreted as the two possible arrangements of fermionic particles corresponding to ferromagnetism or antiferromagnetism order, i.e. \([k] = [00 \ldots 0 \ldots 0]\) corresponds to the situation where all the \( M \) spins are down while \([k] = [11 \ldots 1 \ldots 1]\) in the opposite case when they are all up; the fermionic states \( \Psi^{[k]} \) correspond to spin waves, also called magnons by analogy to phonons ([10]) which constitute elementary excitations of the \( M \) spin system generated by the interaction of vibrational modes (or bosonic modes) of frequencies \( \omega_i, i = 1, 2, \ldots, N, \) with the \( M \) fermionic levels. The eigenstates of this Hamiltonian are defined as the tensor product of the bosonic and fermionic states and can be thus written as

\[
\varphi^{[k]}_{n_1 n_2 ... n_N} = \Phi^{[k]}_{n_1 n_2 ... n_N} \otimes \Psi^{[k]}, \Psi^{[k]} = (c_1^{\dagger})^{k_1} (c_2^{\dagger})^{k_2} \cdots (c_j^{\dagger})^{k_j} \cdots (c_M^{\dagger})^{k_M} \Psi_{00...0},
\]

where \( \Psi_{00...0} \) is the fermionic vacuum, with \( k_1, k_2, \ldots, k_M = 0, 1 \) and the associated eigenvalues are produced by \([13]\).

### 2.1 Construction of the Gazeau-Klauder vector coherent states for the nondegenerate Hamiltonian

We state now that the Hamiltonian \( H \) has a discrete positive spectrum and that its eigenvectors span a Hilbert space \( \mathcal{F} \). With \([k] = k_1 k_2 \cdots k_M \) where \( k_1, k_2, \ldots, k_M = 0, 1 \), these vectors can be grouped into \( 2^M \) families. Then \( \mathcal{F} = \bigoplus_{[k]=0} \mathcal{F}_{[k]} \), where \( \mathcal{F}_{[k]} \) is the subspace of \( \mathcal{F} \) spanned by the vectors \( \varphi^{[k]}_{n_1 n_2 ... n_N} \) for every \([k]\). This subspace remains stable under the action of the projection operator denoted by \( \mathbb{P}_{[k]} \).

Furthermore, assume that all the \( \omega_l \) are different, for all \( l, 1 \leq l \leq N, \) and \( E^{[k]}_{n_l} - E^{[k]}_{0} = \omega l n_l \) and that setting

\[
\epsilon_{n_l} = n_l = \frac{E^{[k]}_{n_l} - E^{[k]}_{0}}{\omega_l},
\]

leads to a sequence of dimensionless quantities \( (\epsilon_{n_l})_{n_l \in \mathbb{N}} \) satisfying the inequalities \( 0 = \epsilon_0 < \epsilon_1 < \epsilon_2 < \ldots \). Given \( l, 1 \leq l \leq N, \) we define the Gazeau-Klauder coherent states \( |J^{[k]}_l, \gamma^{[k]}_l\rangle \)
where \( J_{[k]} > 0 \) and \( \gamma_{[k]} \in \mathbb{R} \) (the variable \( J_{[k]} \) being generally identified with the classical action and \( \gamma_{[k]} \) with the conjugate angle), as follows

\[
|J_{[k]}, \gamma_{[k]}\rangle = e^{-J_{[k]}/2} \sum_{n_l=0}^{\infty} \frac{J_{[k]}^{n_l/2} e^{-\gamma_{[k]} n_l}}{\sqrt{n_l!}} |\Psi_{[k]}\rangle \otimes |\Phi_{[n_l]}^{[k]}\rangle,
\]

where the vectors \( \Psi_{[k]} \) form the canonical basis of \( \mathbb{C}^{2^M} \),

\[
\begin{align*}
\Psi_{00...00} &= \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix}, &
\Psi_{00...01} &= \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix}, &
\Psi_{[k]} &= \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ \vdots \end{pmatrix},
\end{align*}
\]

It is easily checked that each family of coherent states \( |J_{[k]}, \gamma_{[k]}\rangle, l = 1, 2, \ldots, N \) satisfies the properties of continuity, temporal stability, action identity and resolution of the identity.

Then, the space \( \mathcal{S}_{[k]} \) splits into subspaces \( \mathcal{S}_{[k]}, l = 1, 2, \ldots, N \) such that \( \mathcal{S}_{[k]} = \bigoplus_{l=1}^{N} \mathcal{S}_{[k][l]} \), which are associated for each bosonic mode with the frequency \( \omega_l \).

Set by definition

\[
J_{[k]}^{n/2} := \prod_{l=1}^{N} J_{[k][l]}^{n_l/2}, \quad \varepsilon_{n_1 n_2 \cdots n_N} := \begin{pmatrix} n_1 \\ n_2 \\ \vdots \\ n_N \end{pmatrix}, \quad \gamma_{[k]} := \begin{pmatrix} \gamma_{[k]_1} \\ \gamma_{[k]_2} \\ \cdots \\ \gamma_{[k]_N} \end{pmatrix},
\]

such that

\[
\gamma_{[k]_1} \varepsilon_{n_1 n_2 \cdots n_N} = n_1 \gamma_{[k]_1} + n_2 \gamma_{[k]_2} + \cdots + n_N \gamma_{[k]_N} = \sum_{l=1}^{N} n_l \gamma_{[k][l]}.
\]

For all \( l \), the normalization constant \( \mathcal{N}(J_{[k][l]}) \) is defined as

\[
\mathcal{N}(J_{[k][l]}) = \sum_{n_l=0}^{\infty} \frac{J_{[k][l]}^{n_l}}{n_l!} = e^{J_{[k][l]}},
\]

then \( \mathcal{N}(J_{[k][l]})^{-1/2} = e^{-J_{[k][l]}/2} \). Relatively to the vectors

\[
|\Phi_{[n_1 n_2 \cdots n_N]}^{[k]}\rangle = |\Phi_{n_1}^{[k]}\rangle \otimes |\Phi_{n_2}^{[k]}\rangle \otimes \cdots \otimes |\Phi_{n_l}^{[k]}\rangle \cdots \otimes |\Phi_{n_N}^{[k]}\rangle,
\]
the multidimensional Gazeau-Klauder coherent states ([17]) \(|\mathbf{J}_k, \gamma[k]\rangle\) are given by

\[
|\mathbf{J}_k, \gamma[k]\rangle = \mathcal{N}(J_{k_1})^{-1/2} \sum_{n_{l_1}=0}^{\infty} J_{k_1}^{n_{l_1}/2} e^{-m_1 \gamma[k_1]} \sqrt{n_{l_1}!} \cdots \mathcal{N}(J_{k_l})^{-1/2} \sum_{n_{l_1}=0}^{\infty} J_{k_l}^{n_{l_1}/2} e^{-m_l \gamma[k_l]} \sqrt{n_{l_1}!} \cdots
\]

\[
\mathcal{N}(J_{k_N})^{-1/2} \sum_{n_N=0}^{\infty} J_{k_N}^{n_N/2} e^{-m_N \gamma[k_N]} \sqrt{n_N!} |\Psi[k]\rangle \otimes |\Phi[k]_{n_1 n_2 \cdots n_N}\rangle,
\]

that is

\[
|\mathbf{J}_k, \gamma[k]\rangle = e^{-\frac{1}{2} \sum_{l=1}^{N} J_{k_l}} \prod_{l=1}^{N} \sum_{n_{l_1}=0}^{\infty} J_{k_l}^{n_{l_1}/2} e^{-m_l \gamma[k_l]} \sqrt{n_{l_1}!} |\Psi[k]\rangle \otimes |\Phi[k]_{n_1 n_2 \cdots n_N}\rangle.
\]

By setting

\[
\mathcal{N}(\mathbf{J}_k) = \prod_{l=1}^{N} \mathcal{N}(J_{k_l}) = e^{\sum_{l=1}^{N} J_{k_l}},
\]

the above relation becomes

\[
|\mathbf{J}_k, \gamma[k]\rangle = \mathcal{N}(\mathbf{J}_k)^{-1/2} \sum_{[n]=0}^{\infty} J_{k}^{n/2} e^{-\gamma[n] \varepsilon[n]} \sqrt{n!} |\Psi[k]\rangle \otimes |\Phi[k]_{[n]}\rangle,
\]

where \(\varepsilon[n] := \varepsilon_{n_1 n_2 \cdots n_N}; n! := n_1! n_2! \cdots n_N!\) and \([n] := n_1 n_2 \cdots n_N\).

**Remark:** Note that in this definition, for each \(l, l = 1, 2, \ldots, N\) the variables \(J_{k_l}\) on the one hand and the variables \(\gamma[k_l]\) on the other hand are assumed to be mutually independent.

The vector coherent states related to the coherent states ([25]) are

\[
|\mathbf{J}, \gamma; [k]\rangle = \mathcal{N}(\mathbf{J}_k)^{-1/2} \sum_{[n]=0}^{\infty} \frac{J_{k}^{n/2} e^{-\gamma[n] \varepsilon[n]}}{\sqrt{n!}} |\Psi[k]\rangle \otimes |\Phi[k]_{[n]}\rangle
\]

\[
= \left(\begin{array}{c}
0 \\
\vdots \\
0 \\
|\mathbf{J}_k, \gamma[k]\rangle \\
0 \\
\vdots \\
0
\end{array}\right),
\]

with \(\mathbf{J} = \text{diag}(\mathbf{J}_{00 \cdots 0}, \mathbf{J}_{00 \cdots 0}, \ldots, \mathbf{J}_{11 \cdots 1}), \gamma = \text{diag}(\gamma_{00 \cdots 0}, \gamma_{00 \cdots 0}, \ldots, \gamma_{11 \cdots 1})\) and where \(\varepsilon = \text{diag}(\varepsilon_{[00 \cdots 0]}, \varepsilon_{[00 \cdots 0]}, \ldots, \varepsilon_{[11 \cdots 1]}).\)

We must notice that in this representation the Hamiltonian \(H\) is a diagonal operator, \(H = \text{diag}(H_{00 \cdots 00}, H_{00 \cdots 00}, \ldots, H_{[k]}, \ldots, H_{11 \cdots 11})\), each \(H_{[k]}\) being an infinite diagonal matrix with eigenvalues \(E_{n_1 n_2 \cdots n_N} (n_1, n_2, \ldots, n_N = 0, 1, 2, \ldots)\), which acts on the Hilbert subspace \(\mathcal{H}_{[k]}\).

By denoting by a single indice, that is \([k] \equiv k, k = 1, 2, \ldots, 2^M\), we rewrite these vector coherent states as

\[
|\mathbf{J}, \gamma; k\rangle = \mathcal{N}(\mathbf{J}_k)^{-1/2} \sum_{[n]=0}^{\infty} \frac{J_{k}^{n/2} e^{-\gamma[n] \varepsilon[n]}}{\sqrt{n!}} |\Psi[k]\rangle \otimes |\Phi[k]_{[n]}\rangle.
\]
The Hamiltonian $H$ splits into $2^M$ orthogonal parts $H_{[k]}$. Then, we can write the following relation

$$H = \bigoplus_{k_1,k_2,\ldots,k_M=0}^1 H_{[k]}, \quad H_{[k]} = \sum_{|n|=0}^{\infty} E_{[n]}^{[k]} |\varphi_{[n]}^{[k]}\rangle \langle \varphi_{[n]}^{[k]}|,$$

(28)

with $|\varphi_{[n]}^{[k]}\rangle = |\Phi_{[n]}^{[k]}\rangle \otimes |\Psi_{[k]}\rangle$.

Since the lowest eigenvalue $E_0^{[k]}$ for the component $H_{[k]}$ is zero for $k_1 = k_2 = \ldots = k_M = 0$, we work with the Hamiltonian $H'$ such that

$$H' = \bigoplus_{k_1,k_2,\ldots,k_M=0}^1 H'_{[k]}, \quad H'_{[k]} = \sum_{|n|=0}^{\infty} (E_{[n]}^{[k]} - E_{[0]}^{[k]}) |\varphi_{[n]}^{[k]}\rangle \langle \varphi_{[n]}^{[k]}|.$$

(29)

In (13), we set, by definition, $\Omega_N$, the average frequency of the $N$ bosonic modes, as

$$\Omega_N := \frac{\sum_{l=1}^{N} \omega_l n_l}{\sum_{l=1}^{N} n_l}.$$

(30)

Then (13) becomes

$$E_{[n]}^{[k]} = \Omega_N \left[ \sum_{l=1}^{N} n_l + \frac{\epsilon_{[k]}}{\Omega_N} - \frac{g_{[k]}^2}{\Omega_N} \sum_{l=1}^{N} \frac{1}{\omega_l} \right].$$

(31)

The vector coherent states given in (26) or (27) satisfy temporal stability, action identity and the resolution of the identity given by the following equations:

$$e^{-iH'|J,\gamma;k} = |J,\gamma + \Omega_N t \beta_n d_k; k\rangle,$$

(32)

where $d_k$ is the diagonal matrix with one, 1, in the $kk$-position and zeroes, 0, elsewhere and $\beta_n = (1 \ldots 1 \ldots 1)$ stands for a line matrix with $N$ columns.

$$\langle J,\gamma;k|H'|J,\gamma;k \rangle = \Omega_N \sum_{l=1}^{N} J_{k_l}.$$

(33)

$$\sum_{k_1,k_2,\ldots,k_M=0}^{1} \int_{(\mathbb{R}^N)^2}^{2^M} \int_{(\mathbb{R}^N)^2}^{2^M} |J_{[k]},\gamma_{[k]}\rangle \langle J_{[k]},\gamma_{[k]}| \mathcal{N}(J_{[k]}) d\mu_B(\gamma) d\nu(J) = I_{\mathcal{S}},$$

(34)

with $\mathcal{S} = \bigoplus_{k_1,k_2,\ldots,k_M=0}^{1} \mathcal{S}_{[k]}$. Recall that, $d\mu_B$ which is really a functional (not a measure), is referred to as the **Bohr measure** defined through the relations below

$$\langle \mu_B|f \rangle_{ns} = \lim_{T \to -\infty} \frac{1}{2T} \int_{-T}^{T} f(\gamma) d(\gamma) = \int_{\mathbb{R}} f(\gamma) d\mu_B(\gamma),$$

$$\langle f|g \rangle_{ns} = \lim_{T \to -\infty} \frac{1}{2T} \int_{-T}^{T} \overline{f(\gamma)} g(\gamma) d\gamma := \int_{\mathbb{R}} \overline{f(\gamma)} g(\gamma) d\mu_B(\gamma),$$

(35)

where $\langle | \rangle_{ns}$ denotes the scalar product written by abuse of notation on the nonseparable Hilbert space $\mathcal{H}_{ns}$ (12), which is not an $L^2$ space, spanned by the vectors $f_x(\gamma) = e^{ix\gamma}, x \in \mathbb{R}$. 

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In particular, if \( f(x) = 1 \) for all \( x \), then \( \langle \mu_B|f \rangle_{ns} = 1 \), so that \( \mu_B \) resembles a probability measure.

The measures \( d\nu(J_{[k]}) = d\nu(J_{[k]_1})d\nu(J_{[k]_2}) \cdots d\nu(J_{[k]_N}) \) are such that for all \( l \), \( d\nu(J_{[k]_1}), d\nu(J_{[k]_2}), \ldots, d\nu(J_{[k]_l}), \ldots, d\nu(J_{[k]_N}) \) are defined through the moment problems given by

\[
\frac{1}{n_1!} \int_0^\infty J_{[k]_1}^{n_1} d\nu(J_{[k]_1}) = 1, \quad \frac{1}{n_2!} \int_0^\infty J_{[k]_2}^{n_2} d\nu(J_{[k]_2}) = 1, \ldots, \quad \frac{1}{n_N!} \int_0^\infty J_{[k]_N}^{n_N} d\nu(J_{[k]_N}) = 1,
\] (36)

and

\[
\int_0^\infty d\nu(J_{[k]_1}) = 1, \quad \int_0^\infty d\nu(J_{[k]_2}) = 1, \ldots, \quad \int_0^\infty d\nu(J_{[k]_l}) = 1, \ldots, \quad \int_0^\infty d\nu(J_{[k]_N}) = 1,
\] (37)

whose solutions \( d\nu(J_{[k]_l}) = e^{-J_{[k]_l}dJ_{[k]_l}} \), \( l = 1, 2, \ldots, N \).

On each Hilbert subspace \( \mathcal{H}_{[k]} \), one recursively gets that the following “partial resolution of the identity”

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |J_{[k]}, \gamma_{[k]} \rangle \langle J_{[k]}, \gamma_{[k]}| \mathcal{N}(J_{[k]}) d\mu_B(\gamma_{[k]}) d\nu(J_{[k]}) = P_{[k]}
\] (38)

is satisfied with \( P_{[k]} = \bigoplus_{l=1}^N P_{[k]_l} \), which means that the vectors \( |J_{[k]}, \gamma_{[k]} \rangle \) constitute an overcomplete set of each subspace \( \mathcal{H}_{[k]} \), where \( P_{[k]} \) denotes the projection operator onto \( \mathcal{H}_{[k]} \). Therefore, one obtains on the Hilbert space \( \mathcal{H} \) by summation over the projectors \( P_{[k]} \) that

\[
\sum_{k_1, k_2, \ldots, k_M = 0}^1 \int_{\mathbb{R}^N} d\nu(J_{[k]}) \int_{\mathbb{R}^N} d\mu_B(\gamma_{[k]}) |J_{[k]}, \gamma_{[k]} \rangle \langle J_{[k]}, \gamma_{[k]}| \mathcal{N}(J_{[k]}) d\nu(J_{[k]}) = \sum_{k_1, k_2, \ldots, k_M = 0}^1 P_{[k]} = I_{\mathcal{H}}.
\] (39)

### 2.2 Construction of vector coherent states for the degenerate Hamiltonian

Here we extend the preceding construction to the situation in which the eigenvalues of the given Hamiltonian \( H \) with discrete positive spectrum, are degenerate, these degeneracies being finite. As done in [12], all the required properties of the Gazeau-Klauder type coherent states are recovered by introducing a third parameter into the definition of the coherent states. Then \( |J, \gamma \rangle \) is replaced by \( |J, \gamma, \theta \rangle \), that is in the present case \( |J_{[k]}, \gamma_{[k]} \rangle \) by \( |J_{[k]}, \gamma_{[k]}, \theta \rangle \).

For \( \omega_l = \omega \), with \( l = 1, 2, \ldots, N \), the eigenvalues \( E_{n_1, n_2, \ldots, n_N}^{[k]} \) are degenerate. Set \( d(n) = C_{n+N-1}^n = \frac{(n+N-1)!}{n!(N-1)!} \) the degree of degeneracy of the \( n \)th energy level. The equations (13) and (14) with \( n = n_1 + n_2 + \ldots + n_N \) become

\[
E_n^{[k]} = \omega n + \epsilon_{[k]} + \frac{N\gamma_{[k]}^2}{\omega},
\] (40)

\[
\varphi_n^{[k]} = \varphi_{n,j}^{[k]} = \Phi_{n,j}^{[k]} \otimes \Psi^{[k]}, \quad j = 1, 2, \ldots, d(n).
\] (41)
We have $E_n^{[k]} - E_0^{[k]} = \omega n$. By setting $\epsilon_n = n$, $\forall n \geq 0$, it comes
\[
\epsilon_n = \frac{E_n^{[k]} - E_0^{[k]}}{\omega}.
\] (42)

Then we obtain a sequence of dimensionless quantities $(\epsilon_n)_{n\in\mathbb{N}}$ such that $0 = \epsilon_0 < \epsilon_1 < \epsilon_2 < \epsilon_3 < \ldots$. Then, the Hamiltonian $H$ is given by $H = \omega \sum_{n=0}^{\infty} \sum_{j=1}^{d(n)} \epsilon_n \phi_n^{[k]} \langle \phi_n^{[k]} | \phi_n^{[k]} \rangle \hbar = 1$. We introduce the parameter $\theta \in [0, 2\pi)$ such that the coherent states related to the Hamiltonian $H$ are defined (12) by
\[
\langle J[k], \gamma[k], \theta \rangle = e^{-J[k]/2} \sum_{n=0}^{\infty} \sum_{j=1}^{d(n)} \frac{J_n^{[k]}/2 e^{-m \gamma[k]} e^{-ij \theta}}{\sqrt{n!d(n)}} \Psi[k] \otimes \Phi[n, j].
\] (43)

The normalization constant $e^{-J[k]/2}$ is defined here such that
\[
\mathcal{N}(J[k]) = \sum_{n=0}^{\infty} \frac{n!d(n)}{\rho_n} = e^{J[k]},
\] (44)

with $\rho_n = n!d(n)$. Then, we notice that the radius of convergence of the above series also depends on the degree of degeneracies $d(n)$ of the eigenvalues. In this representation, the vectors $\Psi[k]$ form the canonical basis of $\mathbb{C}^2$. As noticed in (17),

With $\Phi[n, j] \equiv \Phi[j, n]$, $j = 1, 2, \ldots, d(n)$, we write
\[
\langle J[k], \gamma[k], \theta \rangle = e^{-J[k]/2} \sum_{n=0}^{\infty} \sum_{j=1}^{d(n)} \frac{J_n^{[k]}/2 e^{-m \gamma[k]} e^{-ij \theta}}{\sqrt{n!d(n)}} \Psi[k] \otimes \Phi[j, n].
\] (45)

The bosonic states $\Phi[j, n]$, $j = 1, 2, \ldots, d(n)$, are given for some values of the number of vibrational modes $N$ as below:

- For $N = 2$, for all $n \geq 1$,
\[
\Phi[j, n] = \frac{1}{\sqrt{(j - 1)!(n - j + 1)!}} \langle (A_{[k]_1}^\dagger)^{j-1} e^{\sqrt{2}g[k]_{\omega} P_1} \otimes (A_{[k]_2}^\dagger)^{n-j+1} e^{\sqrt{2}g[k]_{\omega} P_2} \rangle (0) \otimes (0).
\] (46)

- For $N = 3$, for all $n \geq 1$, the bosonic states $\Phi[j, 0, n]$, $\Phi[j, 1, n]$, and $\Phi[j, 1, n+1]$ are respectively given by
\[
\Phi[j, 0, n] = \frac{1}{\sqrt{(j - 1)!(n - j + 1)!}} \langle (A_{[k]_1}^\dagger)^{j-1} e^{\sqrt{2}g[k]_{\omega} P_1} \otimes I \otimes (A_{[k]_3}^\dagger)^{n-j+1} e^{\sqrt{2}g[k]_{\omega} P_3} \rangle (0) \otimes (0) \otimes (0)),
\] (47)

with $1 \leq j \leq n$,
\[
\Phi[j, 1, n+1] = \frac{1}{\sqrt{(j - 1)!(n - j + 1)!}} \langle I \otimes (A_{[k]_2}^\dagger)^{j-1} e^{\sqrt{2}g[k]_{\omega} P_2} \otimes (A_{[k]_3}^\dagger)^{n-j+1} e^{\sqrt{2}g[k]_{\omega} P_3} \rangle (0) \otimes (0) \otimes (0)),
\] (48)
with 1 ≤ j ≤ n + 1,

\[ |\Phi_{j-1,n-j+1}^{[k]}\rangle = \frac{1}{\sqrt{(j-1)!(n-j+1)!}} \times \left( A_{[k],1}^{\dagger} e^{i / 2^{n-j+1} P_1} \otimes (A_{[k],2}^{\dagger} e^{i / 2^{n-j+1} P_2} \otimes I) \right) (|0\rangle \otimes |0\rangle \otimes |0\rangle), \]

with 2 ≤ j ≤ n + 1.

In the sequel, as a matter of the notation simplification, \( |\Phi_{j-1,n-j+1}^{[k]}\rangle \) will denote a bosonic state for all values of \( N \).

The vector coherent states related to the coherent states (45) are defined (12) on the \( k \)-resolution of the identity properties defined through the equations below

\[ \langle J, \gamma, \theta, k | = e^{i / 2} \sum_{n=0}^{\infty} \sum_{j=1}^{d(n)} \frac{J^{n/2} e^{-m_j} e^{-ij\theta}}{n! d(n)} |\Psi_k^{[n]}\rangle \otimes |\Phi_{j-1,n-j+1}^{[k]}\rangle, \]

with \( J = \text{diag}(J_{00...0}, J_{00...01}, \ldots, J_{11...1}) \), \( \gamma = \text{diag}(\gamma_{00...0}, \gamma_{00...01}, \ldots, \gamma_{11...1}) \).

By denoting \( |k| = k \) (by a single index), we get

\[ |J, \gamma, \theta, k| = e^{-j} \sum_{n=0}^{\infty} \sum_{j=1}^{d(n)} \frac{J^{n/2} e^{-m_j} e^{-ij\theta}}{n! d(n)} |\Psi_k\rangle \otimes |\Phi_{j-1,n-j+1}^{[k]}\rangle, \]

where \( k = 0, 1, 2, \ldots, 2^M \).

The Hamiltonian \( H \) splits into \( 2^M \) orthogonal parts \( H_{[k]} \) such that

\[ H = \bigoplus_{k_1, k_2, \ldots, k_M = 0}^{1} H_{[k]}, \quad H_{[k]} = \sum_{n=0}^{\infty} \sum_{j=1}^{d(n)} E_n^{[k]} |\varphi_{j-1,n-j+1}^{[k]}\rangle \langle \varphi_{j-1,n-j+1}^{[k]}|, \]

with \( |\varphi_{j-1,n-j+1}^{[k]}\rangle = |\Phi_{j-1,n-j+1}^{[k]}\rangle \otimes |\Psi_{[n]}\rangle \).

Since the lowest eigenvalue \( E_0^{[k]} \) for the component \( H_{[k]} \) of the Hamiltonian \( H \) is zero for \( k_1 = k_2 = \cdots = k_j = \cdots = k_M = 0 \), we work with the Hamiltonian \( H' \) given by

\[ H' = \bigoplus_{k_1, k_2, \ldots, k_M = 0}^{1} H'_{[k]}, \quad H'_{[k]} = \sum_{n=0}^{\infty} \sum_{j=1}^{d(n)} (E_n^{[k]} - E_0^{[k]}) |\varphi_{j-1,n-j+1}^{[k]}\rangle \langle \varphi_{j-1,n-j+1}^{[k]}|, \]

The vector coherent states (50) and (51) fulfill the temporal stability, action identity and resolution of the identity properties defined through the equations below

\[ e^{-iH't} |J, \gamma, \theta; k\rangle = |J, \gamma + \omega t d_k, \theta; k\rangle, \]

where \( d_k \) is the diagonal matrix with one in the \( kk \)-position and zeroes elsewhere.

\[ \langle J, \gamma, \theta; [k]|H'|J, \gamma, \theta; [k]\rangle = \omega J_{[k]}, \]

\[ \frac{1}{2\pi} \sum_{k_1, k_2, \ldots, k_M = 0}^{1} \int_0^\infty \cdots \int_0^\infty \int_0^{2\pi} \int_{\mathbb{R}^{2^M}} \Phi_{[k]}(J, \gamma, \theta) \langle \Phi_{[k]}(J, \gamma, \theta)|N(J_{[k]}''\mu_B(\gamma) \times \times d\theta d\nu(J) = I_{\delta}, \]

\[ 56 \]
with $\mathfrak{H} = \bigoplus_{k_1,k_2,\ldots,k_M = 0} \mathfrak{H}[k]$, $d\nu(J) = d\nu_{00\ldots0}(J_{00\ldots0}) \cdots d\nu_{k_1,k_2,\ldots,k_M}(J_{k_1,k_2,\ldots,k_M}) \cdots d\nu_{11\ldots1}(J_{11\ldots1})$ and $d\mu_B(\gamma) = d\mu_B(\gamma_{00\ldots0}) \cdots d\mu_B(\gamma_{k_1,k_2,\ldots,k_M}) \cdots d\mu_B(\gamma_{11\ldots1})$, with $d\mu_B$ defined as in (55).

For every $[k]$, the measure $d\nu(J_{[k]})$ is defined through the moment problem

$$\int_0^\infty J_{[k]}^n d\nu[J_{[k]}](J_{[k]}) = \begin{cases} 1, & \text{if } n = 0, \\ n!d(n), & \text{if } n \geq 1, \end{cases}$$

(57)

whose solutions $d\nu[J_{[k]}](J_{[k]}) = [d(n)e^{-J_{[k]}} - \delta(J_{[k]})]dJ_{[k]}$ are such that the “partial resolution of the identity”

$$\frac{1}{2\pi} \int_0^\infty \int_0^{2\pi} \int_\mathbb{R} |J_{[k]}, \gamma_{[k]}, \theta\rangle \langle J_{[k]}, \gamma_{[k]}, \theta| \mathcal{N}(J_{[k]})d\mu_B(\gamma_{[k]})d\theta d\nu[J_{[k]}](J_{[k]}) = \mathbb{P}[k]$$

(58)

is satisfied, i.e. the overcompleteness of the vectors $|J_{[k]}, \gamma_{[k]}, \theta\rangle$ is guaranteed on each Hilbert subspace $\mathfrak{H}[k]$ and then on $\mathfrak{H}$. Indeed, one gets the relation

$$\frac{1}{2\pi} \int_0^\infty \int_0^{2\pi} \int_\mathbb{R} |J_{[k]}, \gamma_{[k]}, \theta\rangle \langle J_{[k]}, \gamma_{[k]}, \theta| \mathcal{N}(J_{[k]})d\mu_B(\gamma_{[k]})d\theta d\nu[J_{[k]}](J_{[k]}) = \sum_{n=0}^{\infty} \sum_{j=1}^n |\Psi_k \otimes \Phi^k_{j-1,n-j+1} \rangle \langle \Psi_k \otimes \Phi^k_{j-1,n-j+1}| = \mathbb{P}[k],$$

(59)

which yields

$$\frac{1}{2\pi} \sum_{k_1,k_2,\ldots,k_M = 0}^1 \int_0^\infty \cdots \int_0^\infty \int_\mathbb{R}^{2M} |J_{[k]}, \gamma_{[k]}, \theta\rangle \langle J_{[k]}, \gamma_{[k]}, \theta| \mathcal{N}(J_{[k]})d\mu_B(\gamma_{[k]}) \times d\theta d\nu[J_{[k]}](J_{[k]}) = \sum_{k_1,k_2,\ldots,k_M = 0}^1 \mathbb{P}[k] = I_{\mathfrak{H}}.$$ 

(60)

3 Generalized model and related vector coherent states

This section addresses a model of building vector coherent states for a Hamiltonian model describing a nonradiative relaxation mechanism caused by the inertial coupling of an electron to the nanoparticle’s translational center-of-mass. This interaction is present because an electron bound to an impurity center in an oscillating nanoparticle is in an accelerating reference frame, and, in accordance with the Einstein’s equivalence principle, it feels a fictitious time-dependent force. Such a relaxation mechanism is operative even at zero temperature, owing to the fact that quantum zero-point motion of the c.m. is sufficient to produce the fictitious force. See [18] for more details. This study is done in the context of low-energy decay of a localized electronic impurity state in a macroscopic semiconductor or insulator in condensed matter physics. The dissipation is following by a phonon emission. The model consists of a single nanoparticle of mass $M$ connected to a bulk substrate by a few atomic bonds. The effect of the substrate is to subject the nanoparticle to a one-dimensional harmonic oscillator potential $V = \frac{1}{2}M\Omega^2X^2$ with frequency $\Omega$, the $X$ direction being perpendicular to the plane of the substrate. The center of mass motion is that of a macroscopic harmonic oscillator interacting with many other degrees of freedom, such as the
phonons of the bulk substrate characterized by the creation and annihilation operators $b^\dagger, b$. After a series of gauge transformations, the general Hamiltonian describing such a system, can be given (where $\hbar = 1$) by

$$H_{CM} = \Omega b^\dagger b + \sum_{n=1}^{N} \omega_n a_n^\dagger a_n + \sum_{\alpha=1}^{M} \epsilon_\alpha c_\alpha^\dagger c_\alpha + \sum_{n=1}^{N} \sum_{\alpha,\alpha'=1}^{M} g_{n\alpha\alpha'} c_\alpha^\dagger c_{\alpha'} (a_n + a_n^\dagger) - g \sum_{\alpha,\alpha'=1}^{M} x_{\alpha\alpha'} c_\alpha^\dagger c_{\alpha'} (b + b^\dagger).$$

(61)

The first term describes the harmonic dynamics of the center of mass (c.m) of the nanoparticle; the nanoparticle is assumed to be constrained to move in the $x$ direction only. Hence, the c.m. translational motion is described by a single bosonic degree-of-freedom

$$b = \sqrt{\frac{M\Omega}{2}} \left( X + \frac{i}{M\Omega} P \right)$$

where $X$ and $P$ are the $x$ components of the c.m. position and momentum. The second term describes the nanoparticle’s internal vibrational dynamics: the $\omega_n$ are the frequencies of the internal modes, and $a_n^\dagger$ and $a_n$ are the corresponding phonon creation and annihilation operators. These internal vibrational modes have been observed by low-frequency Raman scattering [19]-[20] and by femtosecond pump-probe spectroscopy [21]. The third term is the Hamiltonian for a noninteracting $N$-level system. The fourth term is the ordinary leading-order interaction between the $N$-level system and the internal vibrational modes. Here $g_{n\alpha\alpha'}$ is the electron-phonon coupling constant; it depends on the detailed microscopic structure of the nanoparticle, the nature and position of the impurity, and the spatial dependence of the internal vibrational modes. The fifth term describes the inertial coupling between the $N$-level system and the center of mass motion; here $x_{\alpha\alpha'} \equiv \langle \phi_\alpha | x | \phi_{\alpha'} \rangle$ are dipole-moment matrix elements, which, of course, depend on the form of the impurity states.

Before all things, let us immediately clarify the terminology problems to avoid any confusion. By diagonal (resp. extra-diagonal) Hamiltonian, we always mean the Hamiltonian encompassing all free bosonic and fermionic contributions with only diagonal elements in the interaction coupling terms (resp. with only extra-diagonal elements in the interaction coupling terms).

3.1 The diagonal case

Supplementing the commutation rules given in (2) by the relations

$$[b, b^\dagger] = 1, \ [b, b] = 0 = [b^\dagger, b^\dagger], \ [a_i^\dagger, b] = 0 = [a_i, b^\dagger], \ [a_i^\dagger, b^\dagger] = 0 = [a_i, b], \ 1 \leq i \leq N.$$  

(62)

and setting $x_{\alpha\alpha'} = \delta_{\alpha\alpha'}$ for $\alpha, \alpha' = 1, 2, \ldots, M$, the Hamiltonian (61) can be transformed into a diagonal form

$$H = \Omega b^\dagger b + \sum_{i=1}^{N} \omega_i a_i^\dagger a_i + \sum_{\alpha=1}^{M} \epsilon_\alpha c_\alpha^\dagger c_\alpha + \sum_{i=1}^{N} \sum_{\alpha=1}^{M} g_{\alpha\alpha'} c_\alpha^\dagger c_{\alpha'} (a_i + a_i^\dagger)$$

12
\[-g' \sum_{\alpha=1}^{M} c_{\alpha}^\dagger c_{\alpha} (b^\dagger + b). \quad (64)\]

Let us denote by \( \varphi = \chi \otimes \Phi_1 \otimes \Phi_2 \otimes \cdots \otimes \Phi_N \otimes \Psi_{[k]} \) the eigenvector of \( H \), where \([k] = k_1 k_2 \ldots k_M\) with \( k_1, k_2, \ldots, k_M = 0, 1 \). So, for all \( M \geq 2 \), if there is not any zero in \([k]\), then

\[
H \varphi = \left[ \sum_{i=1}^{N} \omega_i a_i^\dagger a_i + \sum_{i=1}^{N} \sum_{j=1}^{M} \delta_{k_j 1} (\epsilon_j + g_j (a_i^\dagger + a_i)) + \Omega b^\dagger b - Mg' \right] \varphi. \quad (65)
\]

In the contrary, if there exists at least one zero in \([k]\), then

\[
H \varphi = \left[ \sum_{i=1}^{N} \omega_i a_i^\dagger a_i + \sum_{i=1}^{N} \sum_{j=1}^{M} \delta_{k_j 1} (\epsilon_j + g_j (a_i^\dagger + a_i)) + \Omega b^\dagger b - \frac{g'}{2} \sum_{j=1}^{M} (1 - \delta_{k_j 0} + \delta_{k_j 1}) \right] \varphi. \quad (66)
\]

As a matter of shortness of mathematical expression, let us also define the quantity \( \kappa_{[k]} \) such that

\[
\kappa_{[k]} = \begin{cases} 
M, & \text{if } [k] \text{ doesn't contain any zero}, \\
\frac{1}{2} \sum_{j=1}^{M} (1 - \delta_{k_j 0} + \delta_{k_j 1}), & \text{if } [k] \text{ contains at least one zero}.
\end{cases} \quad (67)
\]

By setting \( g' \kappa_{[k]} = g'_{[k]} \), it follows that, for all \([k]\), we readily find

\[
H \varphi = \left[ \sum_{i=1}^{N} \omega_i a_i^\dagger a_i + \sum_{i=1}^{N} \sum_{j=1}^{M} \delta_{k_j 1} (\epsilon_j + g_j (a_i^\dagger + a_i)) + \Omega b^\dagger b - g'_{[k]} \right] \varphi. \quad (68)
\]

Now, let us consider the self-adjoint operator \( B_{[k]_l} \), given by

\[
B_{[k]_l} = \left( \omega_l a_{[k]_l}^\dagger a_{[k]_l} + \frac{\epsilon_{[k]_l}}{\omega_l} \right) + \frac{1}{N} \left[ \Omega \left( b - \frac{g'_{[k]}}{\Omega} \right)^\dagger \left( b - \frac{g'_{[k]}}{\Omega} \right) - \frac{g'^2_{[k]}}{\Omega} \right], \quad (69)
\]

with the operator \( B_{[k]_{12\ldots N}} \) defined such that

\[
B_{[k]_{12\ldots N}} = \sum_{l=1}^{N} B_{[k]_l}, \quad (70)
\]

where

\[
[A_{[k]_l}, A_{[k]_l}^\dagger] = 1, A_{[k]_l} = a_l + \frac{g_{[k]}}{\omega_l}, \quad l = 1, 2, \ldots, N, \quad j = 1, 2, \ldots, M. \quad (71)
\]

The operator \( B_{[k]_{12\ldots N}} \) can be then written in the following manner

\[
B_{[k]_{12\ldots N}} = \left( \sum_{l=1}^{N} \omega_l a_l^\dagger a_l + \sum_{l=1}^{N} \sum_{j=1}^{M} \delta_{k_j 1} \epsilon_j + \sum_{l=1}^{N} \sum_{j=1}^{M} \delta_{k_j 1} \right) + \Omega \left( b - \frac{g'_{[k]}}{\Omega} \right)^\dagger \left( b - \frac{g'_{[k]}}{\Omega} \right) - \frac{g'^2_{[k]}}{\Omega}. \quad (72)
\]

It evidently possesses the same eigenvector and associated eigenvalue as the Hamiltonian \( H \). The latter will be deduced below according to the development performed in section 2.
3.1.1 The Gazeau-Klauder vector coherent states for eigenvalues with both nondegenerate and infinite degenerate degrees of freedom

We are now in the situation where the given Hamiltonian $H$ acts on a Hilbert space spanned by tensor products of different vectors which are eigenvectors of two Hamiltonians describing, respectively, the center-of-mass of the nanoparticle and the $N$ bosonic modes interacting with the fermionic levels. The vector coherent states associated to these Hamiltonians can be defined by including infinite degeneracies accordingly to [12] and furthermore satisfy as well the Gazeau-Klauder properties.

According to previous computations done in Section 1, the eigenstates of the Hamiltonian

$$H_1 = \sum_{l=1}^{N} \omega_l a_l^\dagger a_l + \sum_{l=1}^{N} \sum_{j=1}^{M} \delta_{kj} \left[ \epsilon_j + g_j (a_l^\dagger + a_l) \right],$$  \hfill (73)

are defined by [12]. The eigenstates of the operator $H_2$ given by

$$H_2 = \Omega \left( b - \frac{g_{K[k]}}{\Omega} \right) \dagger \left( b - \frac{g_{K[k]}}{\Omega} \right) - \frac{g_{K[k]}^2}{\Omega},$$  \hfill (74)

with

$$C^\dagger = e^{-\im \sqrt{\Theta_{K[k]}^L K[k]} \frac{p}{\Theta}} b^\dagger e^{\im \sqrt{\Theta_{K[k]}^L K[k]} \frac{p}{\Theta}} = b^\dagger - \frac{g’_{K[k]}}{\Omega}, \quad [b, \sqrt{2}p] = i,$$  \hfill (75)

can be given by

$$|\chi^{[k]}_m\rangle = \frac{(C^\dagger)^m}{\sqrt{m!}} e^{-\im \sqrt{\Theta_{K[k]}^L K[k]} \frac{p}{\Theta}} |0\rangle, $$  \hfill (76)

where $m = 0, 1, 2, \ldots$. The eigenvectors of the operator $B_{[k]}_{12\ldots N}$ can be then written as $|\chi^{[k]}_m\rangle \otimes |\Phi^{[k]}_{n_1 n_2\ldots n_N}\rangle$ leading to the following expression for the eigenstates of the Hamiltonian $H$:

$$|\xi^{[k]}_{m,n_1 n_2\ldots n_N}\rangle = |\chi^{[k]}_m\rangle \otimes |\Phi^{[k]}_{n_1 n_2\ldots n_N}\rangle \otimes |\Psi^{[k]}_{[k]}\rangle, $$  \hfill (77)

where $\Psi^{[k]}_{[k]} = (c_1^\dagger)^{k_1} (c_2^\dagger)^{k_2} \ldots (c_j^\dagger)^{k_j} \ldots (c_M^\dagger)^{k_M} \Psi_{00\ldots 0}$, with the corresponding eigenvalues

$$E^{[k]}_{m,n_1 n_2\ldots n_N} = \left( \Omega m - \frac{g_{K[k]}^2}{\Omega} \right) + \sum_{l=1}^{N} \omega_l n_l + \epsilon_{[k]} - \frac{g_{[k]}^2}{\omega} \sum_{l=1}^{N} \frac{1}{\omega_l}. $$  \hfill (78)

In the nondegenerate case, the eigenvalues of the operator $H_1$ are such that, for all $l$, they engender the sequence $(\epsilon_{n_l})_{n_l \in \mathbb{N}}$ given in (15) allowing the construction of the coherent states $|J_{[k]_l}, \gamma_{[k]_l}\rangle$, $1 \leq l \leq N$. The multidimensional coherent states $|J_{[k]}, \gamma_{[k]}\rangle$, defined in (25), can be therefore constructed.

The coherent states corresponding to the eigenvectors $|\chi^{[k]}_m\rangle$ of the self-adjoint operator $H_2$ are obtained as

$$|J'_{[k]}, \gamma'\rangle = \mathcal{N}(J'_{[k]})^{-1/2} \sum_{m=0}^{\infty} \frac{J'_{[k]}^m e^{-i m \gamma'}}{\sqrt{m!}} |\chi^{[k]}_m\rangle. $$  \hfill (79)
The eigenvalues of the operators $H_1$ and $H_2$, in the case of nondegenerate eigenvalues of $H_1$, are

$$E^{[k]}_{[n]} = \sum_{l=1}^{N} \omega_l n_l + \epsilon_{[k]} - g_{[k]}^2 \sum_{l=1}^{N} \frac{1}{\omega_l},$$  \hspace{1cm} (80)$$

$$E^{[k]} = \Omega \left( m - \frac{g_{[k]}^2}{\Omega^2} \right),$$ \hspace{1cm} (81)

respectively. The eigenvalues $E^{[k]}_m$ are such that

$$E^{[k]}_m = \Omega \mathcal{E}^{[k]}_m,$$ \hspace{1cm} (82)

with $\mathcal{E}^{[k]}_m$ given by

$$\mathcal{E}^{[k]}_m = m - \frac{g_{[k]}^2}{\Omega^2}.$$ \hspace{1cm} (83)

From the equation (30), the relation (80) becomes

$$E^{[k]}_{[n]} = \Omega_N \mathcal{E}^{[k]}_{[n]},$$ \hspace{1cm} (84)

where

$$\mathcal{E}^{[k]}_{[n]} = \sum_{l=1}^{N} \n_l + \frac{\epsilon_{[k]}}{\Omega_N} - \frac{g_{[k]}^2}{\Omega_N} \sum_{l=1}^{N} \frac{1}{\omega_l}.$$ \hspace{1cm} (85)

The Hamiltonians $H_1$ and $H_2$ are defined in the separable Hilbert space spanned by the vectors $\{ |\xi^{[k]}_{m,[n]}\rangle \}_{m,[n]=0}^{\infty}$ so that

$$H_1 = \sum_{m,[n]=0}^{\infty} \Omega_N \mathcal{E}^{[k]}_{[n]} |\xi^{[k]}_{m,[n]}\rangle \langle \xi^{[k]}_{m,[n]}|.$$ \hspace{1cm} (86)$$

$$H_2 = \sum_{m,[n]=0}^{\infty} \Omega \mathcal{E}^{[k]}_m |\xi^{[k]}_{m,[n]}\rangle \langle \xi^{[k]}_{m,[n]}|.$$ \hspace{1cm} (87)

In this case, the eigenvalues $\Omega_N \mathcal{E}^{[k]}_{[n]}$ and $\Omega \mathcal{E}^{[k]}_m$ are assumed to be infinitely degenerate, with $m = 0, 1, 2, \ldots, \infty$ and $[n] = 0, 1, 2, \ldots, \infty$ counting their degeneracies, respectively.

As the lowest eigenvalues $\mathcal{E}^{[k]}_0$ are zero only for $k_i = 0$, $i = 1, M$, we simply introduce the following shifted Hamiltonians, obtained from $H_1$ and $H_2$, respectively,

$$H_1' = \sum_{m,[n]=0}^{\infty} \Omega_N (\mathcal{E}^{[k]}_{[n]} - \mathcal{E}^{[k]}_0) |\xi^{[k]}_{m,[n]}\rangle \langle \xi^{[k]}_{m,[n]}|,$$ \hspace{1cm} (88)$$

$$H_2' = \sum_{m,[n]=0}^{\infty} \Omega (\mathcal{E}^{[k]}_m - \mathcal{E}^{[k]}_0) |\xi^{[k]}_{m,[n]}\rangle \langle \xi^{[k]}_{m,[n]}|,$$ \hspace{1cm} (89)

Let us now consider the separable Hilbert space $\mathcal{H}'$ spanned by $\{ |\chi^{[k]}_m\rangle \otimes |\psi^{[k]}_{[n]}\rangle \}_{m,[n]=0}^{\infty}$ and set $\mathcal{H} = \mathcal{H}' \otimes \mathbb{C}^2^M$, the Hilbert space spanned by the vectors

$$|\xi^{[k]}_{m,[n]}\rangle = |\chi^{[k]}_m\rangle \otimes |\Phi^{[k]}_{[n]}\rangle \otimes |\Psi^{[k]}\rangle.$$ \hspace{1cm} (90)
As stated in [12], the vector coherent states related to $H_1$ and $H_2$ can be deduced on $\mathcal{F}_t$ as

infinite component vector coherent states given by

$$\langle \mathbf{J}_{[k]}, \gamma_{[k]}, J'_{[k]}, \gamma'_{[n]} | \mathbf{J}_{[k]}, \gamma_{[k]}, J'_{[k]}, \gamma'_{[n]} \rangle = \frac{J_{[k]}^{n/2} e^{-i\gamma_{[k]}^\varepsilon_{[n]}_{[k]}}}{\mathcal{N}(\mathbf{J}_{[k]}), \mathcal{N}(J'_{[k]})} \sum_{m=0}^{\infty} \frac{J_{[k]}^{m/2} e^{im\gamma'}}{\sqrt{n! m!}} |\xi_{m,[n]}\rangle,$$  

(91)

respectively. As a matter of clarity, let us separately investigate the properties of these states.

**Analysis of the vector coherent states (91)**

The coherent states (91) satisfy the global normalization condition

$$\sum_{[n]=0}^{\infty} \langle \mathbf{J}_{[k]}, \gamma_{[k]}, J'_{[k]}, \gamma'_{[n]} | \mathbf{J}_{[k]}, \gamma_{[k]}, J'_{[k]}, \gamma'_{[n]} \rangle = 1. \quad (93)$$

Note that the vectors are not individually normalized. Furthermore, they fulfill the temporal stability, action identity and resolution of the identity properties stated, respectively, through the following relations

$$e^{-iH_2 t} |\mathbf{J}_{[k]}, \gamma_{[k]}, J'_{[k]}, \gamma'_{[n]}\rangle = |\mathbf{J}_{[k]}, \gamma_{[k]}, J'_{[k]}, \gamma'_{[n]}, \Omega t_{[n]}\rangle, \quad (94)$$

$$\sum_{[n]=0}^{\infty} \langle \mathbf{J}_{[k]}, \gamma_{[k]}, J'_{[k]}, \gamma'_{[n]} | H_2 | \mathbf{J}_{[k]}, \gamma_{[k]}, J'_{[k]}, \gamma'_{[n]} \rangle = \Omega J, \quad (95)$$

$$\sum_{[n]=0}^{\infty} \langle \mathbf{J}_{[k]}, \gamma_{[k]}, J'_{[k]}, \gamma'_{[n]} | H_2 | \mathbf{J}_{[k]}, \gamma_{[k]}, J'_{[k]}, \gamma'_{[n]} \rangle = \int_{[n]=0}^{\infty} \langle \mathbf{J}_{[k]}, \gamma_{[k]}, J'_{[k]}, \gamma'_{[n]} | H_2 | \mathbf{J}_{[k]}, \gamma_{[k]}, J'_{[k]}, \gamma'_{[n]} \rangle \int_{[n]=0}^{\infty} \langle \mathbf{J}_{[k]}, \gamma_{[k]}, J'_{[k]}, \gamma'_{[n]} | H_2 | \mathbf{J}_{[k]}, \gamma_{[k]}, J'_{[k]}, \gamma'_{[n]} \rangle = I_{\mathcal{F}_t}, \quad (96)$$

with $\mathcal{F}_t = \bigoplus_{[k]=0}^{1} \mathcal{F}_{[k]}$, $d\mu_B(\gamma) = d\mu_B(\gamma_{00}..0) d\mu_B(\gamma_{00}..01) \cdots d\mu_B(\gamma_{11}..1)$ and $d\nu(\mathbf{J}) = d\nu(\mathbf{J}_{00}..0) d\nu(\mathbf{J}_{00}..01) \cdots d\nu(\mathbf{J}_{11}..1)$. The measures $d\nu(\mathbf{J}_{[k]}) = d\nu(J_{[k]1}) d\nu(J_{[k]2}) \cdots d\nu(J_{[k]n})$ and $d\nu(J'_{[k]})$ stem from the moment problems given by

$$\int_{\mathbb{R}_+^N} J_{[k]}^n d\nu(\mathbf{J}_{[k]}) = \int_{0}^{\infty} J_{[k]1}^{n_1} d\nu(J_{[k]1}) \int_{0}^{\infty} J_{[k]2}^{n_2} d\nu(J_{[k]2}) \cdots \int_{0}^{\infty} J_{[k]n}^{n_n} d\nu(J_{[k]n}) = n!, \quad (97)$$

i.e.,

$$\frac{1}{n_1!} \int_{0}^{\infty} J_{[k]1}^{n_1} d\nu(J_{[k]1}) = 1, \quad \frac{1}{n_2!} \int_{0}^{\infty} J_{[k]2}^{n_2} d\nu(J_{[k]2}) = 1, \cdots, \frac{1}{n_l!} \int_{0}^{\infty} J_{[k]l}^{n_l} d\nu(J_{[k]l}) = 1,$$
\[
\cdots \frac{1}{n_N!} \int_0^\infty J_{[k]}^{n_N} d\nu(J_{[k]}^{n_N}) = 1, \tag{98}
\]

with
\[
\int_0^\infty d\nu(J_{[k]}^{l}) = 1, \quad l = 1, 2, \ldots, N, \tag{99}
\]

and
\[
\int_0^\infty J_{[k]}^{m} d\nu(J_{[k]}^{m}) = m!, \tag{100}
\]

whose solutions are \(d\nu(J_{[k]}^{l}) = e^{-J_{[k]}^{l} dJ_{[k]}^{l}}\) and \(d\nu(J_{[k]}^{m}) = e^{-J_{[k]}^{m} dJ_{[k]}^{m}}\), respectively; these provide the overcompleteness of the vectors \(|J_{[k]}, \gamma_{[k]}, J'_{[k]}, \gamma'_{[n]}\rangle\) on the subspace \(\mathcal{H}_{[k]}\). Indeed, one gets
\[
\sum_{[n]=0}^{\infty} \int_0^\infty \int_{\mathbb{R}_+^N} \int_{\mathbb{R}_+^N} \int_{\mathbb{R}_+^N} \int_{\mathbb{R}_+^N} \int_{\mathbb{R}_+^N} |J_{[k]}, \gamma_{[k]}, J'_{[k]}, \gamma'; [n]\rangle \langle J_{[k]}, \gamma_{[k]}, J'_{[k]}, \gamma'; [n]| \times \mathcal{N}(J_{[k]}^{l}) \mathcal{N}(J_{[k]}^{m}) d\mu_B(\gamma') d\mu_B(\gamma_{[k]}) d\nu(J_{[k]}^{l}) d\nu(J_{[k]}^{m}) = \sum_{[n]=0}^{\infty} \xi_{m,[n]}^{[k]} \xi_{m,[n]}^{[k]} = \mathbb{P}_{[k]}, \tag{101}
\]

yielding the relation
\[
\sum_{k_1, k_2, \ldots, k_M = 0}^{\infty} \int_0^\infty \int_{\mathbb{R}_+^N} \int_{\mathbb{R}_+^N} \int_{\mathbb{R}_+^N} \int_{\mathbb{R}_+^N} \int_{\mathbb{R}_+^N} |J_{[k]}, \gamma_{[k]}, J'_{[k]}, \gamma'; [n]\rangle \langle J_{[k]}, \gamma_{[k]}, J'_{[k]}, \gamma'; [n]| \times \mathcal{N}(J_{[k]}^{l}) \mathcal{N}(J_{[k]}^{m}) d\mu_B(\gamma') d\mu_B(\gamma_{[k]}) d\nu(J_{[k]}^{l}) d\nu(J_{[k]}^{m}) = \sum_{k_1, k_2, \ldots, k_M = 0}^{\infty} \mathbb{P}_{[k]} = I_{\mathcal{H}}, \tag{102}
\]
satisfied on \(\mathcal{H}\).

**Analysis of the vector coherent states (92)**

The vector coherent states (92) lead to the global normalization condition
\[
\sum_{m=0}^{\infty} \langle J_{[k]}, \gamma_{[k]}, J'_{[k]}, \gamma'; m|J_{[k]}, \gamma_{[k]}, J'_{[k]}, \gamma'; m\rangle = 1 \tag{103}
\]

and satisfy the following properties
\[
e^{-iH^1_{[k]}|J_{[k]}, \gamma_{[k]}, J'_{[k]}, \gamma'; m\rangle = |J_{[k]}, \gamma_{[k]} + \Omega_N t\beta_{[n]}, J'_{[k]}, \gamma'; m\rangle}, \tag{104}
\]

where \(\beta_{[n]} = (1 \ldots 1 \ldots 1)\) stands for a line matrix with \(N\) columns,

\[
\sum_{m=0}^{\infty} \langle J_{[k]}^{[l]}, \gamma_{[k]}^{[l]}, J'_{[k]}^{[l]}, \gamma'; J_{[k]}^{[l]}, \gamma_{[k]}^{[l]}, J'_{[k]}^{[l]}, \gamma'; m\rangle H^1_{[k]} = \sum_{q, n_p = 0}^{\infty} \omega_p(e^{[k]}_{p} - E^{[k]}_p)|\xi_{q, n_p}^{[k]}\rangle \langle \xi_{q, n_p}^{[k]}|, \tag{105}
\]
or
\[
\sum_{m=0}^{\infty} \langle J[k], \gamma[k], J'[k], \gamma'; m | H'_1 | J[k], \gamma[k], J'[k], \gamma'; m \rangle = \Omega_N \sum_{l=1}^{N} J[k],
\]

\[
H'_1 = \sum_{m,[n]=0}^{\infty} \Omega_N (\mathcal{E}_m^{[k]} - \mathcal{E}_0^{[k]}) |\xi_m,[n]\rangle \langle \xi_m,[n]|,
\]

and
\[
\sum_{k_1,k_2,\ldots,k_M=0}^{1} \sum_{[m]=0}^{\infty} \int_{(\mathbb{R}_N)^{2M}} \int_{(\mathbb{R}_N)^{2M}} \int_{R} |J[k], \gamma[k], J'[k], \gamma'; n\rangle \langle J[k], \gamma[k], J'[k], \gamma'; [n]| \times N(J[k])N(J'_1)d\mu_B(\gamma')d\mu_B(\gamma)d\nu(J)d\nu(J'_1) = I_S.
\]

Finally, the multidimensional vector coherent states [17], generated by the resulting Hamiltonian \(H = H_1 + H_2\), can be expressed by
\[
|J[k], \gamma'; J[k], \gamma[k]\rangle = N(J'_1)^{-1/2} \sum_{m=0}^{\infty} \frac{J_m^{m/2} e^{-m\gamma'}}{\sqrt{m!}} N(J_1)^{-1/2} \sum_{[n]=0}^{\infty} \frac{J_n^{n/2} e^{-i\gamma[k]C[n][k]}{\sqrt{n!}} \langle \xi_m,[n]|, (108)
\]

with the required properties, i.e,
\[
\langle J[k], \gamma'; J[k], \gamma[k]| J[k], \gamma'[k], \gamma'; J[k], \gamma[k]\rangle = 1, (109)
\]

\[
e^{-iH't}|J[k], \gamma'; J[k], \gamma[k]\rangle = |J[k], \gamma'+ \Omega t; J[k], \gamma[k] + \Omega_N t\beta_n\rangle, (110)
\]

\[
\langle J[k], \gamma'; J[k], \gamma[k]| H'| J[k], \gamma'; J[k], \gamma[k]\rangle = \Omega J' + \Omega_N \sum_{l=1}^{N} J[k], (111)
\]

\[
\sum_{k_1,k_2,\ldots,k_M=0}^{1} \int_{0}^{\infty} \int_{(\mathbb{R}_N)^{2M}} \int_{(\mathbb{R}_N)^{2M}} \int_{R} |J'[k], \gamma'; J[k], \gamma[k]\rangle \langle J'[k], \gamma'; J[k], \gamma[k]| \times N(J[k])N(J'[k])d\mu_B(\gamma')d\mu_B(\gamma)d\nu(J)d\nu(J'_1) = I_S. (112)
\]

### 3.1.2 The Gazeau-Klauder vector coherent states for eigenvalues with both finitely and infinitely degenerate degrees of freedom

The vector coherent states are now defined for the Hamiltonian \(H\) with eigenvalues having both finitely and infinitely degenerated degrees of freedom. The study will be conducted following the method developed in the preceding section.

When \(\omega_l = \omega\), for all \(l\), the eigenvalues \(E_m^{[k]}\) become
\[
E_m^{[k]} = \left(\Omega m - \frac{g_{[k]}^2}{\Omega}\right) + \omega n + \epsilon[k] - N \frac{g_{[k]}^2}{\omega}, n = n_1 + n_2 + \ldots + n_N, (113)
\]

and the associated eigenvectors are given by
\[
|\xi_m^{[k]}\rangle = |\chi_m\rangle \otimes |\Phi_{j-1,n-j+1}^{[k]}\rangle \otimes |\Psi[k]\rangle, j = 1, 2, \ldots, d(n), (114)
\]
where the vectors $|\Phi_{j-1,n-j+1}^{[k]}\rangle$ are defined as in (40), (47), (48) and (49). As the eigenvalues $E_{m,n}^{[k]}$, $n = n_1 + n_2 + \cdots + n_N$, are degenerate ($\omega_I = \omega$ for all $I$), the degree of degeneracy $d_{m,n}$ being the same as for $E_{j-1}^{[k]}$, $\omega(n) = \omega(n) - N\frac{g_{[k]}^2}{\omega}$, i.e., $d(n) = C_{n+N-1}^m$, it follows that the eigenvalues of $H_1$ generate the sequence of the quantities $(\epsilon_n)_{n \in \mathbb{N}}$ given in (42) and the corresponding coherent states $|J_{[k]}, \gamma_{[k]}, \theta\rangle$ are defined by (45). These eigenvalues (of $H_1$) are then built as

$$E_n^{[k]} = \omega n + \epsilon_{[k]} - N\frac{g_{[k]}^2}{\omega},$$

or equivalently

$$E_n^{[k]} = \omega \mathcal{E}_n^{[k]},$$

by setting

$$\mathcal{E}_n^{[k]} = n + \frac{\epsilon_{[k]}}{\omega} - N\frac{g_{[k]}^2}{\omega^2}.\quad (117)$$

The Hamiltonians $H_1$ and $H_2$ are given in the separable Hilbert space $\tilde{\mathcal{H}}$ spanned by the vectors $|\xi_{m,j-1,n-j+1}^{[k]}\rangle$ as below

$$H_1 = \sum_{m,n=0}^{\infty} \sum_{j=1}^{d(n)} \omega \mathcal{E}_n^{[k]} |\xi_{m,j-1,n-j+1}^{[k]}\rangle \langle \xi_{m,j-1,n-j+1}^{[k]}|,$$

$$H_2 = \sum_{m,n=0}^{\infty} \sum_{j=1}^{d(n)} \mathcal{O} \mathcal{E}_m^{[k]} |\xi_{m,j-1,n-j+1}^{[k]}\rangle \langle \xi_{m,j-1,n-j+1}^{[k]}|.$$  

Consequently, by similar arguments as in the previous section, we deduce the shifted Hamiltonians $H'_1$ and $H'_2$ as follows:

$$H'_1 = \sum_{m,n=0}^{\infty} \sum_{j=1}^{d(n)} \omega (\mathcal{E}_n^{[k]} - \mathcal{E}_0^{[k]}) |\xi_{m,j-1,n-j+1}^{[k]}\rangle \langle \xi_{m,j-1,n-j+1}^{[k]}|, \quad \mathcal{E}_0^{[k]} = \frac{\epsilon_{[k]}}{\omega} - N\frac{g_{[k]}^2}{\omega^2}.$$  

$$H'_2 = \sum_{m,n=0}^{\infty} \sum_{j=1}^{d(n)} \mathcal{O} (\mathcal{E}_m^{[k]} - \mathcal{E}_0^{[k]}) |\xi_{m,j-1,n-j+1}^{[k]}\rangle \langle \xi_{m,j-1,n-j+1}^{[k]}|, \quad \mathcal{E}_0^{[k]} = -\frac{g_{[k]}^2}{\mathcal{O}^2}.$$  

Then, as the eigenvalues of $H_1$ are degenerate with finite degeneracies, the vector coherent states can be defined on $\tilde{\mathcal{H}}$ in a similar way as in (91) and (92), but with $m = 0, 1, 2, \ldots, \infty$ and $n = 0, 1, 2, \ldots, \infty$, counting the infinite degeneracies of the eigenvalues $\mathcal{E}_n^{[k]}$ and $\mathcal{E}_m^{[k]}$. They are explicitly expressed as follows:

$$|J_{[k]}, \gamma_{[k]}, \theta, J'_{[k]}, \gamma', n\rangle = \frac{J_n^{m/2} e^{-m\gamma_{[k]} \theta}}{[\mathcal{N}(J_{[k]}), \mathcal{N}(J'_{[k]})]^{1/2}} \sum_{m=0}^{\infty} \frac{J_n^{m/2} e^{m\gamma'}}{\sqrt{n!d(n)m!}} |\xi_{m,j-1,n-j+1}^{[k]}\rangle,$$  

$$|J_{[k]}, \gamma_{[k]}, \theta, J'_{[k]}, \gamma', m\rangle = \frac{J_n^{m/2} e^{-m\gamma'}}{[\mathcal{N}(J_{[k]}), \mathcal{N}(J'_{[k]})]^{1/2}} \sum_{n=0}^{\infty} \frac{d(n) J_n^{m/2} e^{-m\gamma_{[k]} \theta}}{\sqrt{n!d(n)m!}} |\xi_{m,j-1,n-j+1}^{[k]}\rangle. \quad (123)$$

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As expected, the vector coherent states \[122\] fulfill the good mathematical properties such as the normalization condition, the temporal stability, the action identity and the resolution of the identity, respectively, through the following relations:

\[
\sum_{n=0}^{\infty} \langle J_{[k]}, \gamma_{[k]}, \theta, J'_{[k]}, \gamma'; n | J_{[k]}, \gamma_{[k]}, \theta, J'_{[k]}, \gamma'; n \rangle = 1, \tag{124}
\]

\[
e^{-iH_2 t} | J_{[k]}, \gamma_{[k]}, \theta, J'_{[k]}, \gamma'; n \rangle = | J_{[k]}, \gamma_{[k]}, \theta, J'_{[k]}, \gamma' + \Omega t; n \rangle, \tag{125}
\]

\[
\sum_{n=0}^{\infty} \langle J_{[k]}, \gamma_{[k]}, \theta, J'_{[k]}, \gamma'; n | H_2' | J_{[k]}, \gamma_{[k]}, \theta, J'_{[k]}, \gamma'; n \rangle = \Omega J'_{[k]}, \tag{126}
\]

\[
\frac{1}{2\pi} \sum_{k_1, k_2, \ldots, k_M} \int_0^{2\pi} \int_{\mathbb{R}^{2M}} \int_0^{2\pi} \int_{\mathbb{R}^{2M}} | J_{[k]}, \gamma_{[k]}, \theta, J'_{[k]}, \gamma'; n \rangle \langle J_{[k]}, \gamma_{[k]}, \theta, J'_{[k]}, \gamma'; n | N(J_{[k]}) N(J_{[k]}') d\mu_B(\gamma') d\theta d\nu(J) d\nu(J') | = I_{\tilde{\mathcal{S}}}, \tag{127}
\]

with \( \tilde{\mathcal{S}} = \Theta^{1}_{[k]=0} \delta_{[k]} \), \( d\nu(J) = d\nu_{00...0} d\nu_{00...0} \cdots d\nu_{11...1} d\nu_{11...1} \) and \( d\mu_B(\gamma) = d\mu_B(\gamma_{00...0}) \cdots d\mu_B(\gamma_{11...1}) \). The measures \( d\nu(J_{[k]}) \) are defined through the moment problem given by the equation \(100\).

By analogy, the vectors \[123\] verify the appropriate relations

\[
\sum_{m=0}^{\infty} \langle J_{[k]}, \gamma_{[k]}, \theta, J'_{[k]}, \gamma'; m | J_{[k]}, \gamma_{[k]}, \theta, J'_{[k]}, \gamma'; m \rangle = 1, \tag{128}
\]

\[
e^{-iH_1' t} | J_{[k]}, \gamma_{[k]}, \theta, J'_{[k]}, \gamma'; m \rangle = | J_{[k]}, \gamma_{[k]} + \omega t, \theta, J'_{[k]}, \gamma'; m \rangle, \tag{129}
\]

\[
\sum_{m=0}^{\infty} \langle J_{[k]}, \gamma_{[k]}, \theta, J'_{[k]}, \gamma'; m | H_1' | J_{[k]}, \gamma_{[k]}, \theta, J'_{[k]}, \gamma'; m \rangle = \omega J_{[k]}, \tag{130}
\]

\[
\frac{1}{2\pi} \sum_{k_1, k_2, \ldots, k_M} \int_0^{2\pi} \int_{\mathbb{R}^{2M}} \int_0^{2\pi} \int_{\mathbb{R}^{2M}} | J_{[k]}, \gamma_{[k]}, \theta, J'_{[k]}, \gamma'; n \rangle \langle J_{[k]}, \gamma_{[k]}, \theta, J'_{[k]}, \gamma'; n | N(J_{[k]}) N(J_{[k]})' d\mu_B(\gamma') d\mu_B(\gamma) d\theta d\nu(J) d\nu(J') | = I_{\tilde{\mathcal{S}}}, \tag{131}
\]

where the measure \( d\nu(J'_{[k]}) \) is solution of the moment problem \[100\].

Here again, the Hamiltonian \( H = H_1 + H_2 \) engenders the multidimensional vector coherent states

\[
| J'_{[k]}, \gamma', J_{[k]}, \gamma_{[k]}, \theta \rangle = \mathcal{N}(J'_{[k]})^{-1/2} \sum_{m=0}^{\infty} \frac{J_{[k]}^{m/2}}{\sqrt{m!}} N(J_{[k]})^{-1/2} \sum_{n=0}^{\infty} \frac{d(n)}{n!} \frac{J_{[k]}^{n/2} e^{-m\gamma'}}{\sqrt{n!d(n)}} | \xi_{m,j-1,n-j+1}^{[k]} \rangle \tag{132}
\]
satisfying all useful mathematical properties, namely

$$\langle J'_k \mid \gamma' + J_k \mid \gamma' \rangle = 1,$$

(133)

$$e^{-iH\prime t} | J'_k \rangle = | J'_k \rangle, \quad \langle J'_k \mid \gamma' + \Omega t \mid J'_k \rangle = \langle J'_k \mid \gamma' + \Omega t \mid J'_k \rangle,$$

(134)

$$\langle J'_k \mid \gamma' \rangle | J'_k \rangle = \Omega J'_k,$$

(135)

$$\frac{1}{2\pi} \sum_{k_1, k_2, \ldots, k_M} \int_0^\infty \int_0^{2\pi} \int_0^\infty \int_0^{2\pi} \langle J'_k \mid \gamma' \rangle | J'_k \rangle = \Omega J'_k,$$

(136)

3.2 The extradiagonal case

The model is now reduced to the Hamiltonian $H$ given by

$$H = \Omega b^\dagger b + \sum_{i=1}^N \omega_i a_i^\dagger a_i + \sum_{\alpha=1}^M c_\alpha^\dagger c_\alpha + \sum_{n=1}^N \sum_{\alpha, \alpha'=1}^M g_{n\alpha\alpha'} c_\alpha^\dagger c_{\alpha'} (a_n^\dagger + a_n)$$

$$-g' \sum_{\alpha, \alpha'=1}^M x_{\alpha\alpha'} c_\alpha^\dagger c_{\alpha'} (b^\dagger + b),$$

(137)

where

$$x_{\alpha\alpha'} = 1, \quad \text{if} \alpha \neq \alpha' \quad \text{and} \quad x_{\alpha\alpha'} = 0, \quad \text{if} \alpha = \alpha'.$$

(138)

This Hamiltonian is almost analogous to (61), except for the fourth term which describes the interaction between the $M$ fermionic levels of the system; the internal vibrational modes encompass the electron-phonon coupling constants defined for fermionic operators with different indices (extradiagonal case). The same remark holds for the fifth term which describes the inertial coupling between the $M$ fermionic levels of the system and the center of mass of the nanoparticle.

Taking into account the relations (138), and rearranging

$$\sum_{n=1}^N \sum_{\alpha, \alpha'=1}^M g_{n\alpha\alpha'} c_\alpha^\dagger c_{\alpha'} (a_n^\dagger + a_n) - g' \sum_{\alpha, \alpha'=1}^M x_{\alpha\alpha'} c_\alpha^\dagger c_{\alpha'} (b^\dagger + b)$$

$$= \sum_{\alpha, \alpha'=1}^M \left[ \sum_{i=1}^N g_{i\alpha\alpha'} (a_i^\dagger + a_i) - g' (b^\dagger + b) \right] c_\alpha^\dagger c_{\alpha'},$$

(139)

the Hamiltonian $H$ can be rewritten in the following form

$$H = \sum_{\alpha, \alpha'=1}^M \left[ \sum_{i=1}^N \left( g_{i\alpha\alpha'} (a_i^\dagger + a_i) + \omega_i a_i^\dagger a_i \right) + \Omega b^\dagger b - g' (b^\dagger + b) \right] c_\alpha^\dagger c_{\alpha'}$$

(139)
\[
- \left( \sum_{i=1}^{N} \omega_i a_i^\dagger a_i + \Omega b^\dagger b \right) + \sum_{\alpha, \alpha' = 1}^{M} c_{\alpha}^\dagger c_{\alpha'} + \sum_{\alpha = 1}^{M} \epsilon_{\alpha} c_{\alpha}^\dagger c_{\alpha} + \sum_{i=1}^{N} \omega_i a_i^\dagger a_i + \Omega b^\dagger b.
\]

Let us set
\[
H_1 := \sum_{\alpha, \alpha' = 1}^{M} \sum_{\alpha \neq \alpha'}^{\alpha, \alpha'}^{N} \left[ \sum_{i=1}^{N} \left( g_{i \alpha \alpha'} (a_i^\dagger + a_i) + \omega_i a_i^\dagger a_i \right) + \Omega b^\dagger b - g' (b^\dagger + b) \right] c_{\alpha}^\dagger c_{\alpha'},
\]

or, in a more expressive form in order to evidence the contributions of various interaction terms,
\[
H_1 := \sum_{i=1}^{N} \left( \sum_{j,l=1}^{M} \left[ g_{ijl} \kappa_{jil} (a_i^\dagger + a_i) + \omega_i a_i^\dagger a_i \right] c_{j}^\dagger c_{l} \right) + \left[ \Omega \left( b - g' \Omega \right)^\dagger \left( b - g' \Omega \right) - g'^2 \Omega \right] \sum_{j,l=1}^{M} \kappa_{jil} c_{j}^\dagger c_{l},
\]

where \( \kappa_{jil} = (1 - \delta_{k_j,1})(\delta_{k_j,0} - \delta_{k_l,0}) \), the \( k_j \) being the components of the multi index \( [k] \). Furthermore, let \( \varepsilon_{N,M} \) be the number of coupling constants \( g_{ijl}(j \neq l) \) for a fixed index \( i \) of a bosonic mode, and \( g_{i[k]} \) be the coupling constants such that
\[
g_{i[k]} := \sum_{j,l=1}^{M} g_{ijl} \kappa_{jil}, \quad 1 \leq i \leq N,
\]

where \( i \) is the index of a bosonic mode, \( [k] \) the multi index of a fermionic state. Then, the following relation can be worked out:
\[
g_{i[k]}^2 = \left( \sum_{j,l=1}^{M} g_{ijl} \kappa_{jil} \right)^2, \quad 1 \leq i \leq N.
\]
\[ B_i^{[k]} = \varepsilon_{N,M} \omega_i \left( a_i + \frac{g_i^{[k]}}{\varepsilon_{N,M} \omega_i} \right)^\dagger \left( a_i + \frac{g_i^{[k]}}{\varepsilon_{N,M} \omega_i} \right) - g_i^{2 [k]} \varepsilon_{N,M} \omega_i, \]  

with the operator \( B_i^{[k]} \) eigenvectors \( \Phi^{[k]}_{\alpha_i} \) given by

\[ |\Phi^{[k]}_{\alpha_i}\rangle = e^{i\sqrt{\frac{\varepsilon_{N,M} \omega_i}{\varepsilon_{N,M} \omega_i}} p_i} |\alpha_i\rangle \]

corresponding to the eigenvalues

\[ E^{[k]}_{\alpha_i} = \varepsilon_{N,M} \omega_i \alpha_i - \frac{g_i^{2 [k]}}{\varepsilon_{N,M} \omega_i}. \]

The eigenvectors of \( B^{[k]} \) are readily obtained

\[ |\Phi^{[k]}_{\alpha_1 \alpha_2 \cdots \alpha_N}\rangle = e^{i\sqrt{\frac{\varepsilon_{N,M} \omega_i}{\varepsilon_{N,M} \omega_i}} p_1} |\alpha_1\rangle \otimes e^{i\sqrt{\frac{\varepsilon_{N,M} \omega_i}{\varepsilon_{N,M} \omega_i}} p_2} |\alpha_2\rangle \otimes \cdots \otimes e^{i\sqrt{\frac{\varepsilon_{N,M} \omega_i}{\varepsilon_{N,M} \omega_i}} p_N} |\alpha_N\rangle, \]

associated to the eigenvalues

\[ E^{[k]}_{\alpha_1 \alpha_2 \cdots \alpha_N} = \sum_{i=1}^{N} \left( \frac{\varepsilon_{N,M} \omega_i \alpha_i - \frac{g_i^{2 [k]}}{\varepsilon_{N,M} \omega_i}}{\varepsilon_{N,M} \omega_i} \right). \]

As the fourth and fifth terms of the interaction parts of the Hamiltonian \( H \) are not diagonal on a given fermionic state \( \Psi^{[k]} \) because of the presence of fermionic creation and annihilation operators \( c_{\alpha}^\dagger c_{\alpha'} (\alpha \neq \alpha') \) in \((137)\) and \((140)\), in order to solve the eigenvalue problem for \( H \), we define a fermionic state \( \Psi \) such that

\[ \Psi := \Psi^{[k]} + \sum_{[k']=0}^{1} \Psi^{[k]}, \]

where the \([k']\) are the images of \([k]\) under the action of the fermionic operator

\[ \left( \sum_{j,l=1}^{M} \kappa_{jl} c_j^\dagger c_l \right) \]  

A straightforward computation, using the fact that the fermionic operator acts, through \((150)\), on each given index \([k]\) yielding a vector multiple of \( \Psi \) up to a multiplicative integer factor, allows the following identification

\[ \left( \sum_{j,l=1}^{M} \kappa_{jl} c_j^\dagger c_l \right) \Psi \equiv \Psi. \]

Then, the resolution of the eigenvalue problem for \( H_1 \) provides the eigenvectors given by

\[ |\xi_1\rangle = |\lambda^{[k]}_m\rangle \otimes |\Phi^{[k]}_{\alpha_1 \alpha_2 \cdots \alpha_N}\rangle \otimes |\Psi\rangle \]

\[ = e^{-i\sqrt{\frac{\varepsilon_{N,M} \omega_i}{\varepsilon_{N,M} \omega_i}} p} |\alpha_1\rangle \otimes |\Phi^{[k]}_{\alpha_1 \alpha_2 \cdots \alpha_N}\rangle \otimes |\Psi\rangle, \]

with the associated eigenvalues defined as

\[ E^{[k]}_{m,\alpha_1 \alpha_2 \cdots \alpha_N} = \left( \Omega m - \frac{g_i^{2 [k]}}{\Omega} \right) + \sum_{i=1}^{N} \left( \varepsilon_{N,M} \omega_i \alpha_i - \frac{g_i^{2 [k]}}{\varepsilon_{N,M} \omega_i} \right). \]
3.2.1 Construction of Gazeau-Klauder vector coherent states in the nondegenerate case

With the definition (150), the diagonalization of the Hamiltonian $H$ provides the eigenvectors defined as

$$\langle \xi | = f(P) | m \rangle \otimes g(p) | [n]^{[k]} \rangle \otimes h([k]) | \psi_{[k]} \rangle,$$

$$| \chi_{[k]} \rangle = f(P) | m \rangle \otimes | \phi_{[k]} \rangle \otimes h([k]) | \psi_{[k]} \rangle,$$  \hspace{1cm} (154)

where the operators $f(P), g(P)$ and $h([k])$ acting on the states $| m \rangle \otimes | [n]^{[k]} \rangle \otimes | \psi_{[k]} \rangle$ are determined as follows:

$$f(P) = F_{[k]} e^{-i \sqrt{\frac{2}{\hbar}} p_i \sum_{i=1}^{N} \Phi_{[k]}^{i}} + \Lambda_{[k]} ,$$

$$g(p) = F_{[k]} \left( \prod_{i=1}^{N} e^{i \sqrt{\frac{2}{\hbar}} g_i^{[k]} \Phi_{[k]}^{i}} \right) + \Lambda_{[k]} ,$$

$$h([k]) = F_{[k]} \left( \sum_{j \neq l}^{M} \kappa_{j,l} c_{j}^{\dagger} c_{l} + \mathbf{1} \right) + \Lambda_{[k]} .$$  \hspace{1cm} (155)

The operators $F_{[k]}$ and $\Lambda_{[k]}$ are given by

$$F_{[k]} = 1 \otimes \Lambda_{[k]} , \quad \Lambda_{[k]} = \prod_{j=1}^{M-1} \delta_{k_{j} k_{j+1}} ,$$  \hspace{1cm} (156)

the $k_{j}$ being the components of the multi index $[k]$. We have

$$\Lambda_{[k]} = \begin{cases} 1, & \text{if } [k] \text{ contains the same indices}, \\ 0, & \text{if not}, \end{cases}$$  \hspace{1cm} (157)

and

$$F_{[k]} = \begin{cases} 0, & \text{if } [k] \text{ contains the same indices}, \\ 1, & \text{if not}. \end{cases}$$  \hspace{1cm} (158)

The associated eigenvalues are

$$E_{m,n_{1}n_{2} \cdots n_{N}}^{[k]} = \Omega m + \sum_{i=1}^{N} \omega_{i} n_{i} + \epsilon_{[k]} + \alpha_{[k]} \lambda_{\varepsilon_{N},M} ,$$  \hspace{1cm} (159)

where

$$\alpha_{[k]} = 1 - \prod_{j=1}^{M-1} \delta_{k_{j} k_{j+1}} ,$$

$$\lambda_{\varepsilon_{N},M} = \sum_{i=1}^{N} \left( (\varepsilon_{N,M} - 1) \omega_{i} n_{i} - \frac{g_{i}^{2} \epsilon_{[k]}}{\varepsilon_{N,M} \omega_{i}} \right) - \frac{g^{2} \Omega}{\varepsilon_{N,M}} .$$  \hspace{1cm} (160)

In the nondegenerate case, for all $l, 1 \leq l \leq N$ :
- if $\alpha[k] = 1$, then

$$E_{n_l}^{[k]} = \varepsilon_{N,M} \omega_{l, n_l} + \frac{\varepsilon[k]}{N} - \frac{g_{l[k]}^2}{\varepsilon_{N,M} \omega_l}, \quad E_0^{[k]} = \frac{\varepsilon[k]}{N} - \frac{g_{l[k]}^2}{\varepsilon_{N,M} \omega_l}. \quad (161)$$

It follows that $E_{n_l}^{[k]} - E_0^{[k]} = \varepsilon_{N,M} \omega_{l, n_l}$. By setting

$$\varepsilon_{n_l} = n_l = \frac{E_{n_l}^{[k]} - E_0^{[k]}}{\varepsilon_{N,M} \omega_l}, \quad (162)$$

we obtain a sequence of dimensionless quantities $(\varepsilon_{n_l})_{n_l \in \mathbb{N}}$ such that $0 = \varepsilon_0 < \varepsilon_1 < \ldots.$

- if $\alpha[k] = 0$, we get

$$E_{n_l}^{[k]} = \omega_{l, n_l} + \frac{\varepsilon[k]}{N}, \quad E_0^{[k]} = \frac{\varepsilon[k]}{N}. \quad (163)$$

We have $E_{n_l}^{[k]} - E_0^{[k]} = \omega_{l, n_l}$. With

$$\varepsilon_{n_l} = n_l = \frac{E_{n_l}^{[k]} - E_0^{[k]}}{\omega_l}, \quad (164)$$

we obtain a sequence analogous to the previous.

For each $l, l = 1, 2, \ldots, N$, the coherent states are defined by

- if $h([k]) = 1$,

$$|J[k], \gamma[k]\rangle = e^{-\frac{J[k]}{2}} \sum_{n_l=0}^{\infty} \sqrt{n_l!} |\Psi[k]\rangle \otimes |\Phi_{n_l}^{[k]}\rangle. \quad (165)$$

- if $h([k]) \neq 1$,

$$|J[k], \gamma[k]\rangle = e^{-\frac{J[k]}{2}} \sum_{n_l=0}^{\infty} \sqrt{n_l!} |\Psi\rangle \otimes |\Phi_{n_l}^{[k]}\rangle. \quad (166)$$

By using the same methods as in (18) and (19), with

$$\mu[n][k] := \mu_{1n_2 \ldots n_N[k]} = \begin{pmatrix} n_1 \\ n_2 \\ \vdots \\ n_N \end{pmatrix}, \quad (167)$$

we obtain analogous equations to (25), i.e.,

- if $h([k]) = 1$,

$$|J[k], \gamma[k]\rangle = \mathcal{N}(J[k])^{-1/2} \sum_{[n]=0}^{\infty} \sqrt{\mathcal{N}[k]} \sum_{[\mu]} J[k]^{[\mu]} e^{-\gamma[k]\mu[n][k]} |\Psi[k]\rangle \otimes |\Phi_{[n]}^{[k]}\rangle. \quad (168)$$
- if \( h([k]) \neq 1 \),

\[
|J_{[k]}, \gamma_{[k]}\rangle = \mathcal{N}(J_{[k]})^{-1/2} \sum_{\alpha=0}^{\infty} J_{[k]}^{\alpha/2} e^{-\gamma_{[k]}/2} \sqrt{\frac{1}{\alpha!}} |\Psi \rangle \otimes |\Phi_{[k]}^{[\alpha]}\rangle. \tag{169}
\]

The coherent states related to the center of mass part Hamiltonian are identified as

\[
|J'_{[k]}, \gamma'\rangle = \mathcal{N}(J'_{[k]})^{-1/2} \sum_{m=0}^{\infty} J'_{[k]}^{m/2} e^{-m\gamma'} \sqrt{\frac{1}{m!}} |\chi_{m}^{[k]}\rangle. \tag{170}
\]

Let us denote by \( H^{b-f} \) and \( H^{c-m-f} \) the Hamiltonians describing the bosonic vibrational modes and the center of mass of the nanoparticle which interact with the fermionic levels, respectively:

\[
H^{b-f} = \sum_{i=1}^{N} \sum_{j=1}^{M} \left( \sum_{l \neq j} g_{ijl} \kappa_{jl} (a_{l}^{\dagger} + a_{l}) + \omega_{i} a_{i}^{\dagger} a_{i} \kappa_{jl} \right) c_{j}^{\dagger} c_{l} + \sum_{i=1}^{N} \omega_{i} a_{i}^{\dagger} a_{i} + \sum_{\alpha=1}^{M} \epsilon_{\alpha} c_{\alpha}^{\dagger} c_{\alpha}, \tag{171}
\]

\[
H^{c-m-f} = \sum_{j=1}^{M} \sum_{l \neq j} \left[ \Omega \left( b - \frac{g'}{\Omega} \right) \left( b - \frac{g'}{\Omega} \right) - \frac{g'}{\Omega} \right] \kappa_{jl} c_{j}^{\dagger} c_{l} + \Omega b^{\dagger} b. \tag{172}
\]

Let \( \mathcal{H}' \) be the Hilbert space spanned by \( \{|\chi_{m}^{[k]}\rangle \otimes |\Phi_{[n]}^{[\alpha]}\rangle\}_{m,n=0}^{\infty} \), and \( \tilde{\mathcal{H}} = \mathcal{H}' \otimes \mathbb{C}^{2M} \) and \( \tilde{\mathcal{H}} = \mathcal{H}' \otimes \{|\Psi\rangle\} \) be the separable Hilbert spaces spanned by the vectors

\[
|\gamma_{m,n}^{[k]}\rangle = |\chi_{m}^{[k]}\rangle \otimes |\Phi_{[n]}^{[\alpha]}\rangle \otimes |\Psi_{[k]}\rangle, \tag{173}
\]

\[
|\gamma_{m,n}^{[k]}\rangle = |\chi_{m}^{[k]}\rangle \otimes |\Phi_{[n]}^{[\alpha]}\rangle \otimes |\Psi\rangle, \tag{174}
\]

respectively. The eigenvalues are given in the nondegenerate case by

\[
E_{n}^{[k]} = \Omega N \mathcal{E}_{[n]}^{[k]}, \tag{175}
\]

\[
E_{m}^{[k]} = \Omega \mathcal{E}_{m}^{[k]}, \tag{176}
\]

with

\[
\mathcal{E}_{[n]}^{[k]} = \sum_{i=1}^{N} n_{i} + \frac{\epsilon_{[n]}^{[k]}}{\Omega N} + \alpha_{[n]}^{[k]} \left( \varepsilon_{N,M} - 1 \right) \sum_{i=1}^{N} n_{i} - \frac{1}{\varepsilon_{N,M} \Omega N} \sum_{i=1}^{N} g_{i}^{2} \frac{\Omega}{\omega_{i}}, \tag{177}
\]

\[
\mathcal{E}_{m}^{[k]} = m - \alpha_{[n]}^{[k]} \frac{g^{2}}{\Omega^{2}} \tag{178}
\]
for the Hamiltonians $H^{b-f}$ and $H^{cm-f}$, respectively, written in the separable Hilbert space $\mathcal{H}$ spanned by the vectors $|\xi_m\rangle$ such that

$$H^{b-f} = \sum_{m[n]=0}^{\infty} \Omega_N \xi_m^{[k]} \langle \xi_{m[n]} | \xi_m^{[k]} | \rangle,$$  
(179)

$$H^{cm-f} = \sum_{m[n]=0}^{\infty} \Omega_m \xi_m^{[k]} \langle \xi_{m[n]} | \xi_m^{[k]} | \rangle.$$  
(180)

The corresponding shifted Hamiltonians are given by

$$H^{b-f}_\theta = \sum_{m[n]=0}^{\infty} \Omega_N (\xi_m^{[k]} - \xi_0^{[k]}) \langle \xi_{m[n]} | \xi_m^{[k]} | \rangle,$$  
(181)

$$H^{cm-f}_\theta = \sum_{m[n]=0}^{\infty} \Omega_m (\xi_m^{[k]} - \xi_0^{[k]}) \langle \xi_{m[n]} | \xi_m^{[k]} | \rangle.$$  
(182)

The vector coherent states related to the Hamiltonians $H^{b-f}_\theta$ and $H^{cm-f}_\theta$ on $\mathcal{H}$, for the nondegenerate eigenvalues of $B^{[k]}$, can be defined as in (91) and (92) by

$$|J^{[k]}, \gamma^{[k]}, J_1^{[k]}, \gamma^{[k]}; [n]\rangle = \frac{J^{[k]}^{m/2} e^{-\gamma^{[k]}|\mu^{[n]}|_{[k]}}}{[\mathcal{N}(J^{[k]}), \mathcal{N}(J_1^{[k]})]^{1/2}} \sum_{m[n]=0}^{\infty} \frac{J^{m/2} e^{-\gamma^{[k]}|\mu^{[n]}|_{[k]}}}{\sqrt{n!m!}} |\xi_{m[n]}^{[k]}\rangle,$$  
(183)

$$|J^{[k]}, \gamma^{[k]}, J_1^{[k]}, \gamma^{[k]}; m\rangle = \frac{J^{m/2} e^{-\gamma^{[k]}|\mu^{[n]}|_{[k]}}}{[\mathcal{N}(J^{[k]}), \mathcal{N}(J_1^{[k]})]^{1/2}} \sum_{n[n]=0}^{\infty} \frac{J^{n/2} e^{-\gamma^{[k]}|\mu^{[n]}|_{[k]}}}{\sqrt{n!m!}} |\xi_{m[n]}^{[k]}\rangle.$$  
(184)

By analogy, the vector coherent states can be constructed on $\tilde{\mathcal{H}}$ in the nondegenerate eigenvalues, replacing, in the equations (183) and (184), the vectors $|\xi_{m[n]}^{[k]}\rangle$ by $|\xi_{m[n]}^{[k]}\rangle$.

The vector coherent states (183) fulfill the relevant properties of the normalization, temporal stability, action identity and resolution of the identity, respectively:

$$\sum_{m[n]=0}^{\infty} \langle J^{[k]}, \gamma^{[k]}, J_1^{[k]}, \gamma^{[k]}; [n]| J^{[k]}, \gamma^{[k]}, J_1^{[k]}, \gamma^{[k]}; [n]\rangle = 1,$$  
(185)

$$e^{-iH^{cm-f}_\theta} |J^{[k]}, \gamma^{[k]}, J_1^{[k]}, \gamma^{[k]}; [n]\rangle = |J^{[k]}, \gamma^{[k]}, J_1^{[k]}, \gamma^{[k]}; \gamma^{[k]} + \Omega t; [n]\rangle,$$  
(186)

$$\sum_{m[n]=0}^{\infty} \langle J^{[k]}, \gamma^{[k]}, J_1^{[k]}, \gamma^{[k]}; [n]| H^{cm-f}_\theta |J^{[k]}, \gamma^{[k]}, J_1^{[k]}, \gamma^{[k]}; [n]\rangle = \Omega J_1^{[k]}.$$  
(187)

$$\sum_{m[n]=0}^{\infty} \int_{[0,\infty)^2} \int_{\mathbb{R}^2} |J^{[k]}, \gamma^{[k]}, J_1^{[k]}, \gamma^{[k]}; [n]\rangle \langle J^{[k]}, \gamma^{[k]}, J_1^{[k]}, \gamma^{[k]}; [n]| H^{cm-f}_\theta |J^{[k]}, \gamma^{[k]}, J_1^{[k]}, \gamma^{[k]}; [n]\rangle = \Omega J_1^{[k]}.$$  
(188)

with $\mathcal{H} = \bigoplus_{k_1, k_2, \ldots, k_M=0}^{1} \mathcal{H}_k$, $d\mu_B(\gamma) = d\mu_B(\gamma_{00...0})d\mu_B(\gamma_{00...01}) \cdots d\mu_B(\gamma_{11...1})$ and
\[ dv(J) = dv(J_{00\ldots0})dv(J_{00\ldots01}) \cdots dv(J_{11\ldots1}). \]

These statements can be easily proved as in the case of relations (93), (94), (95) and (96).

The vector coherent states (184) also satisfy the same properties as required, i.e.,

\[ \sum_{m=0}^{\infty} \langle J[k], \gamma[k], J'[k], \gamma'; m | J[k], \gamma[k], J'[k], \gamma'; m \rangle = 1, \quad (189) \]

\[ e^{-iH^{th-f}} | J[k], \gamma[k], J'[k], \gamma'; m \rangle = | J[k], \gamma[k] + \Omega_N t \beta_n, J'[k], \gamma'; m \rangle, \quad (190) \]

\[ \sum_{m=0}^{\infty} \langle J[k], \gamma[k], J'[k], \gamma'; m | H^{th-f} | J[k], \gamma[k], J'[k], \gamma'; m \rangle = \omega_l J[k], \quad (191) \]

or

\[ \sum_{m=0}^{\infty} \langle J[k], \gamma[k], J'[k], \gamma'; m | H^{th-f} | J[k], \gamma[k], J'[k], \gamma'; m \rangle = \Omega_N \sum_{l=1}^{N} J[k], \quad (192) \]

and

\[ \frac{1}{k_1 k_2 \ldots k_M} \sum_{m=0}^{\infty} \int_{[R^{2M}]} \int_{[R^{2M}]} \int_{[R]} | J[k], \gamma[k], J'[k], \gamma'; m \rangle \langle J[k], \gamma[k], J'[k], \gamma'; m | \times \mathcal{N}(J[k]) \mathcal{N}(J'[k]) d\mu_B(\gamma') d\mu_B(\gamma) dv(J) dv(J') = I_S. \quad (193) \]

Note that the vector coherent states in the Hilbert space \( \tilde{J} \) can be constructed following step by step the above development.

Given the above development, the multidimensional vector coherent states read through the general expression (108) established in section 3.1.

### 3.2.2 Construction of Gazeau-Klauder vector coherent states in the degenerate case

Let us now use the formalism explicated in the section 2.1.2 to define vector coherent states for the Hamiltonian \( H \) when the bosonic mode term induces finite and infinite degeneracies.

We thus deal with the case when for all \( l, \omega_l = \omega \), leading to the degenerate eigenvalues of \( B^{|k|} \), with the degree of degeneracy \( d(n) \), corresponding to the eigenvectors given by

\[ |\xi_{m,j-1,n-j+1}^{k}| = f(P)|m \rangle \otimes |\Phi_{j-1,n-j+1}^{[k]}| \otimes h([k])|\Psi_{[k]}| \]

\[ = |\chi_{m}^{[k]} \rangle \otimes |\Phi_{j-1,n-j+1}^{[k]}| \otimes h([k])|\Psi_{[k]}|. \quad (194) \]

In this relation, the bosonic states \( |\Phi_{j-1,n-j+1}^{[k]} \rangle \) are defined in the same way as (16), (17), (48) and (49) where the operators \( (A_{[k]}^\dagger)^{j-1}e^{\sqrt{\tau_{[k]}} p_{l}} (A_{[k]}^\dagger)^{n-j+1}e^{\sqrt{\tau_{[k]}} p_{l}} \) are replaced by the operators \( (B_{[k]}^\dagger)^{j-1}e^{\sqrt{\tau_{[k]}} g_{[k]} p_{l}} \) and \( (B_{[k]}^\dagger)^{n-j+1}e^{\sqrt{\tau_{[k]}} g_{[k]} p_{l}} \), respectively.
The associated eigenvalues are determined as

\[ E_{m,j-1,n-j+1}^{[k]} = \Omega m + \omega n + \epsilon_{[k]} + \alpha_{[k]} \lambda_{\varepsilon_{N,M}}, \quad (195) \]

with

\[ \lambda_{\varepsilon_{N,M}} = (\varepsilon_{N,M} - 1) \omega n - \frac{1}{\omega} \sum_{i=1}^{N} \frac{g_{i[k]}^2}{\varepsilon_{N,M}} - \frac{g_{[k]}^2}{\Omega}, \quad (196) \]

where \( j = 1, 2, \ldots, d(n), n = n_1 + n_2 + \cdots + n_N. \)

The vectors \( |\xi_{m,j-1,n-j+1}^{[k]}\rangle \) and \( |\xi'_{m,j-1,n-j+1}^{[k]}\rangle \) are such that

\[ |\xi_{m,j-1,n-j+1}^{[k]}\rangle = |\lambda_{m}^{[k]}\rangle \otimes |\Phi_{j-1,n-j+1}^{[k]}\rangle \otimes |\Psi_{[k]}\rangle, \]

\[ |\xi'_{m,j-1,n-j+1}^{[k]}\rangle = |\lambda_{m}^{[k]}\rangle \otimes |\Phi_{j-1,n-j+1}^{[k]}\rangle \otimes |\Psi\rangle. \quad (197) \]

The eigenvalues \( E_{n}^{[k]} \) and \( E_{m}^{[k]} \) can be shortly written as

\[ E_{n}^{[k]} = \omega \mathcal{E}_{n}^{[k]}, \quad (199) \]

\[ E_{m}^{[k]} = \Omega \mathcal{E}_{m}^{[k]}, \quad (200) \]

with

\[ \mathcal{E}_{n}^{[k]} = n + \frac{\epsilon_{[k]}}{\omega} + \alpha_{[k]} \left( (\varepsilon_{N,M} - 1) n - \frac{1}{\varepsilon_{N,M} \omega^2} \sum_{i=1}^{N} g_{i[k]}^2 \right). \quad (201) \]

The suitable Hamiltonians \( H^{b-f} \) and \( H^{cm-f} \) are then given on the separable Hilbert space \( \mathcal{H} \), spanned by the vectors \( |\xi_{m,j-1,n-j+1}^{[k]}\rangle \), by

\[ H^{b-f} = \sum_{m,n=0}^{\infty} \sum_{j=1}^{d_{m,n}} \omega \mathcal{E}_{n}^{[k]} |\xi_{m,j-1,n-j+1}^{[k]}\rangle \langle \xi_{m,j-1,n-j+1}^{[k]}|, \quad (202) \]

\[ H^{cm-f} = \sum_{m,n=0}^{\infty} \sum_{j=1}^{d_{m,n}} \Omega \mathcal{E}_{m}^{[k]} |\xi_{m,j-1,n-j+1}^{[k]}\rangle \langle \xi_{m,j-1,n-j+1}^{[k]}|, \quad (203) \]

engendering the shifted Hamiltonians \( H^{b-f} \) and \( H^{cm-f} \) expressed as follows:

\[ H^{b-f} = \sum_{m,n=0}^{\infty} \sum_{j=1}^{d_{m,n}} \omega (\mathcal{E}_{n}^{[k]} - \mathcal{E}_{0}^{[k]}) |\xi_{m,j-1,n-j+1}^{[k]}\rangle \langle \xi_{m,j-1,n-j+1}^{[k]}|, \quad (204) \]

\[ H^{cm-f} = \sum_{m,n=0}^{\infty} \sum_{j=1}^{d_{m,n}} \Omega (\mathcal{E}_{m}^{[k]} - \mathcal{E}_{0}^{[k]}) |\xi_{m,j-1,n-j+1}^{[k]}\rangle \langle \xi_{m,j-1,n-j+1}^{[k]}|. \quad (205) \]

It then turns out that the vector coherent states corresponding to the degenerate eigenvalues of the operator \( B^{[k]} \), defined as in (122) and (123), can be evaluated as

\[ |J_{[k]}, \gamma_{[k]}, \theta, J'_{[k]}, \gamma'; n\rangle = \frac{J_{[k]}^{n/2} e^{-\frac{m_{n} \gamma_{[k]} J'}{2}} e^{-ij\theta}}{[N(J_{[k]}), N(J'_{[k]})]^{1/2}} \sum_{m=0}^{\infty} \frac{J_{[k]}^{m/2} e^{im\gamma'}}{\sqrt{n!d(n)m!}} |\xi_{m,j-1,n-j+1}^{[k]}\rangle, \quad (206) \]
The vector coherent states on an analogous way, simply replacing, in the equations (206) and (207), the vectors computed for the extra-diagonal Hamiltonian $H$ satisfying the subsequent analytic properties:

$$\sum_{n=0}^{\infty} \langle J[k], \gamma[k], \theta, J'[k], \gamma'; n | J[k], \gamma[k], \theta, J'[k], \gamma'; n \rangle = 1,$$

$$e^{-iH^{cm-f}t} | J[k], \gamma[k], \theta, J'[k], \gamma'; n \rangle = | J[k], \gamma[k], \theta, J'[k], \gamma' + \Omega t; n \rangle,$$

$$\sum_{n=0}^{\infty} \langle J[k], \gamma[k], \theta, J'[k], \gamma'; n | H^{cm-f} | J[k], \gamma[k], \theta, J'[k], \gamma'; n \rangle = \Omega J'[k],$$

and

$$\sum_{m=0}^{\infty} \langle J[k], \gamma[k], \theta, J'[k], \gamma'; m | J[k], \gamma[k], \theta, J'[k], \gamma'; m \rangle = 1,$$

$$e^{-iH^{nb-f}t} | J[k], \gamma[k], \theta, J'[k], \gamma'; m \rangle = | J[k], \gamma[k] + \omega t, \theta, J'[k], \gamma'; m \rangle,$$

$$\sum_{m=0}^{\infty} \langle J[k], \gamma[k], \theta, J'[k], \gamma'; m | H^{nb-f} | J[k], \gamma[k], \theta, J'[k], \gamma'; m \rangle = \omega J[k],$$

and

$$\sum_{m=0}^{\infty} \langle J[k], \gamma[k], \theta, J'[k], \gamma'; m | J[k], \gamma[k], \theta, J'[k], \gamma'; m \rangle = \Omega J'[k].$$

The vector coherent states on $\tilde{H}$, spanned by the vectors $|\xi_{m,j-1,n-j+1}^{[k]}\rangle$, can be constructed in an analogous way, simply replacing, in the equations (206) and (207), the vectors $|\xi_{m,j-1,n-j+1}^{[k]}\rangle$ by $|\xi_{m,j-1,n-j+1}^{[k]}\rangle$.

The expression (132) remains valid for the multidimensional vector coherent states computed for the extra-diagonal Hamiltonian $H$. 

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3.3 The general case of the total interaction Hamiltonian encompassing both diagonal and extra-diagonal terms

We now generalize our study to the full interaction Hamiltonian $H$ given by

$$H = H_1 + H_2,$$  \hfill (216)

where $H_1$ and $H_2$ are such that

$$H_1 = \frac{\Omega}{2} b^\dagger b + \sum_{i=1}^{N} \frac{\omega_i}{2} a_i^\dagger a_i + \frac{1}{2} \sum_{\alpha=1}^{M} \epsilon_{\alpha} c_{\alpha}^\dagger c_{\alpha} + \sum_{i=1}^{N} \sum_{\alpha=1}^{M} g_{\alpha} c_{\alpha}^\dagger c_{\alpha} (a_i^\dagger + a_i) - g' \sum_{\alpha=1}^{M} c_{\alpha}^\dagger c_{\alpha} (b^\dagger + b),$$

$$H_2 = \frac{\Omega}{2} b^\dagger b + \sum_{i=1}^{N} \frac{\omega_i}{2} a_i^\dagger a_i + \frac{1}{2} \sum_{\alpha=1}^{M} \epsilon_{\alpha} c_{\alpha}^\dagger c_{\alpha} + \sum_{\alpha,\alpha' = 1}^{M} \alpha \neq \alpha' \left[ \sum_{i=1}^{N} g_{\alpha\alpha'} (a_i^\dagger + a_i) - g' (b^\dagger + b) \right] c_{\alpha}^\dagger c_{\alpha'},$$ \hfill (217)

The Hamiltonian $H$ thus combines two different contributions describing the dynamics of the translational motion of the center of mass in the diagonal case (identified by same indices for the fermionic operators investigated in section 2.1) for the $H_1$ term and in the extra-diagonal case (referred to different indices for the fermionic operators examined in 2.2) for the $H_2$ part, respectively.

As now well performed from section 2.2, the Hamiltonian $H_2$ can be written in the following form:

$$H_2 = \sum_{\alpha,\alpha' = 1}^{M} \alpha \neq \alpha' \left[ \sum_{i=1}^{N} g_{\alpha\alpha'} (a_i^\dagger + a_i) + \frac{\omega_i}{2} a_i^\dagger a_i + \frac{\Omega}{2} b^\dagger b - g' (b^\dagger + b) \right] c_{\alpha}^\dagger c_{\alpha'}$$

$$- \left( \sum_{i=1}^{N} \frac{\omega_i}{2} a_i^\dagger a_i + \frac{\Omega}{2} b^\dagger b \right) \sum_{\alpha,\alpha' = 1}^{M} \alpha \neq \alpha' \left[ \sum_{i=1}^{N} g_{\alpha\alpha'} (a_i^\dagger + a_i) + \frac{\omega_i}{2} a_i^\dagger a_i + \frac{\Omega}{2} b^\dagger b \right] c_{\alpha}^\dagger c_{\alpha'},$$ \hfill (218)

from which we can deduce the boson-fermion and center of mass-fermion interaction contribution terms

$$H_{int} = \sum_{i=1}^{N} \left( \sum_{j,l=1}^{M} \sum_{l \neq j}^{M} g_{ijl} c_{jl}^\dagger (a_i^\dagger + a_i) + \frac{\omega_i}{2} a_i^\dagger a_i \kappa_{ijl} \right) c_{jl}^\dagger c_{jl}$$

$$+ \left[ \frac{\Omega}{2} \left( b - \frac{2g'}{\Omega} \right)^\dagger \left( b - \frac{2g'}{\Omega} \right) - \frac{2g'^2}{\Omega} \right] \sum_{j,l=1}^{M} \sum_{l \neq j}^{M} \kappa_{jl} c_{jl}^\dagger c_{jl}. \hfill (219)$$
The first term block contains the operator $B^{[k]}$ given by

$$B^{[k]} = \sum_{i=1}^{M} B_i^{[k]},$$

$$B_i^{[k]} = \frac{\varepsilon_{N,M} \omega_i}{2} \left( a_i + \frac{2g_{i[k]}}{\varepsilon_{N,M} \omega_i} \right)^\dagger \left( a_i + \frac{2g_{i[k]}}{\varepsilon_{N,M} \omega_i} \right) - \frac{2g_{i[k]}^2}{\varepsilon_{N,M} \omega_i},$$  \hspace{1cm} (220)

with the previous defined quantities $\kappa_{jl}, \varepsilon_{N,M}$ and constants of coupling $g_{i[k]}$ and $g_{i[k]}^2$.

The operator $B_i^{[k]}$ eigenvalues $E_i^{[k]}$ defined as

$$E_i^{[k]} = \frac{\varepsilon_{N,M} \omega_i n_i}{2} - \frac{2g_{i[k]}^2}{\varepsilon_{N,M} \omega_i}$$  \hspace{1cm} (221)

correspond to the eigenvectors

$$|\Phi_{n_i}^{[k]}\rangle = e^{i\sqrt{2}g_{i[k]}^2 n_i} |n_i\rangle.$$  \hspace{1cm} (222)

### 3.3.1 Gazeau-Klauder vector coherent states in the nondegenerate case

According to the computations performed in the section 2.2, we infer that the nondegenerate eigenvalues of the Hamiltonian $H_1$ are

$$E_{m,|n_1\rangle}^{[k]} = \left( \frac{\Omega m}{2} - \frac{2g_{N|k|}}{\Omega} \right) + \sum_{i=1}^{N} \left( \frac{\omega_i n_i}{2} - \frac{2g_{i[k]}^2}{\omega_i} \right) + \frac{\epsilon_{[k]}}{2},$$  \hspace{1cm} (223)

associated with eigenvectors

$$|\xi_{m,|n_1\rangle}^{[k]}\rangle = |\chi_{m}^{[k]}\rangle \otimes |\Phi_{|n_1\rangle}^{[k]}\rangle \otimes |\Psi_{[k]}\rangle = e^{-i\sqrt{2}g_{N|k|}^2 m} |m\rangle \otimes \left( \bigotimes_{i=1}^{N} e^{i\sqrt{2}g_{i[k]}^2 n_i} \right) (|n_1\rangle \otimes \cdots \otimes |n_N\rangle) \otimes |\Psi_{[k]}\rangle.$$  \hspace{1cm} (224)

In the same way, the nondegenerate eigenvalues of $H_2$ can be defined as

$$E_{m,|n_2\rangle}^{[k]} = \frac{\Omega m}{2} + \sum_{i=1}^{N} \frac{\omega_i n_i}{2} + \frac{\epsilon_{[k]}}{2} + \alpha_{[k]} \lambda_{\varepsilon_{N,M}},$$  \hspace{1cm} (225)

where

$$\alpha_{[k]} = 1 - \prod_{j=1}^{M-1} \delta_{k_j k_{j+1}},$$

$$\lambda_{\varepsilon_{N,M}} = \sum_{i=1}^{N} \left( \varepsilon_{N,M} - 1 \right) \frac{\omega_i n_i}{2} - \frac{2g_{i[k]}^2}{\varepsilon_{N,M} \omega_i} - \frac{2g_{[k]}^2}{\Omega}.$$  \hspace{1cm} (226)

Then, the $H_2$ eigenvectors defined with similar operators as defined in (155) acting on the states $|m\rangle \otimes |[n_1]^{[k]}\rangle_2 \otimes |\Psi_{[k]}\rangle$, are given by

$$|\xi_{m,|n_2\rangle}^{[k]}\rangle = f(P) |m\rangle \otimes g(p) |[n]^{[k]}\rangle_2 \otimes h([k]) |\Psi_{[k]}\rangle = |\chi_{m}^{[k]}\rangle \otimes |\Phi_{[n_1]}^{[k]}\rangle_2 \otimes h([k]) |\Psi_{[k]}\rangle,$$  \hspace{1cm} (227)
with

\[
\begin{align*}
    f(P) &= F[k] e^{-i\mathcal{S}_{P,k}^f / \hbar} + \Lambda[k], \\
    g(p) &= F[k] \left( \prod_{i=1}^{N} e^{i\mathcal{S}_{p_i,M}^f / \hbar} \right) + \Lambda[k], \\
    h([k]) &= F[k] \left( \sum_{j,l=1}^{M} \kappa_{jl} c_j^\dagger c_l + 1 \right) + \Lambda[k],
\end{align*}
\]

(228)

where the operators \( F[k] \) and \( \Lambda[k] \) are defined in (156).

Thus, the eigenvectors of \( H \), denoted by \( |\xi_{m,n}^{[k]}\rangle \), are easily found to be

\[
|\xi_{m,n}^{[k]}\rangle = |\xi_{m,n}^{[k]}\rangle_1 \otimes |\xi_{m,n}^{[k]}\rangle_2,
\]

(229)

with the associated eigenvalues

\[
E_{m,n}^{[k]} = E_{m,n}^{[k]}_1 + E_{m,n}^{[k]}_2.
\]

(230)

The bosonic modes and the center of mass of the nanoparticle interact with the fermionic levels through two Hamiltonians defined by

\[
\begin{align*}
    H_{b-f} &= \sum_{i=1}^{N} \omega_i a_i^\dagger a_i + \sum_{i=1}^{N} \sum_{j=1}^{M} \left( \delta_{k,j} (e_j + g_j (a_i^\dagger + a_i)) \right) \\
    &\quad + \sum_{i=1}^{N} \sum_{j,l=1}^{M} \left( g_{ijl} \kappa_{jl} (a_i^\dagger + a_i) + \frac{\omega_i}{2} a_i^\dagger a_i \kappa_{jl} \right) c_j^\dagger c_l, \\
    H_{cm-f} &= \frac{\Omega}{2} b^\dagger b + \left[ \frac{\Omega}{2} \left( b - \frac{2g'}{\Omega} \right)^\dagger \left( b - \frac{2g'}{\Omega} \right) - \frac{2g'^2}{\Omega} \right] \\
    &\quad + \frac{\Omega}{2} \left( b - \frac{2g_{k[k]}'}{\Omega} \right)^\dagger \left( b - \frac{2g_{k[k]}}{\Omega} \right) - \frac{2g_{k[k]}'^2}{\Omega},
\end{align*}
\]

(231)

(232)

respectively.

Let \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) be the separable Hilbert spaces, spanned by the vectors \( \{ |\xi_{m,n}^{[k]}\rangle_1 \}_{m,n=0}^{\infty} \) and \( \{ |\xi_{m,n}^{[k]}\rangle_2 \}_{m,n=0}^{\infty} \), respectively. Set \( \mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2 \). On \( \mathcal{F} \), the Hamiltonians \( H_{b-f} \) and \( H_{cm-f} \) are written as

\[
H_{b-f} = \sum_{m,n=0}^{\infty} \Omega_N \hat{c}_{b-f}^{[k]} |\xi_{m,n}^{[k]}\rangle \langle \xi_{m,n}^{[k]}|,
\]

33
\[ H_{cm-f} = \sum_{m,[n]=0}^{\infty} \Omega \mathcal{E}^{[k]}_{cm-f} |\xi_{m,[n]}^{[k]}\rangle \langle \xi_{m,[n]}^{[k]}|, \] (233)

with the quantities \( \mathcal{E}^{[k]}_{b-f} \) and \( \mathcal{E}^{[k]}_{cm-f} \) given by

\[
\mathcal{E}^{[k]}_{b-f} = \sum_{i=1}^{N} \frac{n_i}{2} + \frac{\epsilon^{[k]}}{\Omega_N} + \frac{\alpha^{[k]}}{\Omega_N} \left( \sum_{i=1}^{N} \left( (\varepsilon_{N,M} - 1) \frac{\omega_i n_i}{2} - \frac{2g_i^2}{\varepsilon_{N,M} \omega_i} \right) \right),
\]
\[
\mathcal{E}^{[k]}_{cm-f} = \mathcal{E}^{[k]}_{cm-f} = m - \frac{2g^2}{\Omega^2} - \alpha^{[k]} \frac{2g^2}{\Omega^2}. \] (234)

Since the lowest energy levels \( \mathcal{E}^{[k]}_{b-f_0} \) and \( \mathcal{E}^{[k]}_{cm-f_0} \) are zero for \( k_i = 0 \) for \( i = 1,M \), we deduce the shifted Hamiltonians

\[
H'_{b-f} = \sum_{m,[n]=0}^{\infty} \Omega_N (\mathcal{E}^{[k]}_{b-f} - \mathcal{E}^{[k]}_{b-f_0}) |\xi_{m,[n]}^{[k]}\rangle \langle \xi_{m,[n]}^{[k]}|,
\]
\[
H'_{cm-f} = \sum_{m,[n]=0}^{\infty} \Omega (\mathcal{E}^{[k]}_{cm-f} - \mathcal{E}^{[k]}_{cm-f_0}) |\xi_{m,[n]}^{[k]}\rangle \langle \xi_{m,[n]}^{[k]}| \] (235)

useful for constructing vector coherent states whose general expressions are given in section 2.2.1.

### 3.3.2 Gazéa-Klauuder vector coherent states in the degenerate case

The eigenvalues of the Hamiltonian \( H_1 \) are defined by

\[
E_{m,n_1}^{[k]} = \left( \Omega m - \frac{2g^2}{\Omega} \right) + \left( \frac{\omega n}{2} - N \frac{2g^2}{\Omega} \right) + \frac{\epsilon^{[k]}}{2}, \] (236)

corresponding to the eigenvectors

\[
|\xi_{m,j-1,n-j+1}^{[k]}\rangle_1 = |\chi_m^{[k]}\rangle \otimes |\Phi_{j-1,n-j+1}^{[k]}\rangle_1 \otimes |\Psi_k^{[k]}\rangle, \] (237)

while for the Hamiltonian \( H_2 \) they take the following form

\[
E_{m,n_2}^{[k]} = \frac{1}{2} (\Omega m + \omega n + \epsilon^{[k]} + \alpha^{[k]} \varepsilon_{N,M}) \] (238)

where

\[
\lambda_{\varepsilon_{N,M}} = (\varepsilon_{N,M} - 1) \frac{\omega n}{2} - \frac{2}{\varepsilon_{N,M} \omega} \sum_{i=1}^{N} g_i^2 - \frac{2g^2}{\Omega} \] (239)

with the associated eigenvectors

\[
|\xi_{m,j-1,n-j+1}^{[k]}\rangle_2 = |\chi_m^{[k]}\rangle \otimes |\Phi_{j-1,n-j+1}^{[k]}\rangle_2 \otimes h([k]) |\Psi_k^{[k]}\rangle. \] (240)

Therefore, the eigenvectors of the Hamiltonian \( H \) can be expressed as \( |\xi_{m,j-1,n-j+1}^{[k]}\rangle \) given by

\[
|\xi_{m,j-1,n-j+1}^{[k]}\rangle = |\xi_{m,j-1,n-j+1}^{[k]}\rangle_1 \otimes |\xi_{m,j-1,n-j+1}^{[k]}\rangle_2, \] (241)
associated with the eigenvalues

\[ E_{m,n}^{[k]} = E_{m,n_1}^{[k]} + E_{m,n_2}^{[k]} \]  

(242)

The Hamiltonians \( H_{b-f} \) and \( H_{cm-f} \) are given in the Hilbert space, spanned by the vectors \( |\xi_{m,j-1,n-j+1}\rangle \), by the equations

\[
H_{b-f} = \sum_{m,n=0}^{\infty} \sum_{j=1}^{d_{n,m}} \omega \mathcal{E}_{b-f}^{[k]} |\xi_{m,j-1,n-j+1}\rangle \langle |\xi_{m,j-1,n-j+1}\rangle,
\]

\[
H_{cm-f} = \sum_{m,n=0}^{\infty} \sum_{j=1}^{d_{n,m}} \Omega \mathcal{E}_{cm-f}^{[k]} |\xi_{m,j-1,n-j+1}\rangle \langle |\xi_{m,j-1,n-j+1}\rangle,
\]

where the quantities \( \mathcal{E}_{cm-f}^{[k]} \) and \( \mathcal{E}_{b-f}^{[k]} \) are defined by the relations

\[
\mathcal{E}_{cm-f}^{[k]} = \mathcal{E}_{b-f}^{[k]} = m - \frac{2g_{[k]}^2}{\Omega^2} - \frac{2g_{[k]}^2}{2},
\]

\[
= \frac{n}{2} + \frac{\epsilon_{[k]}}{\omega} + \alpha_{[k]} \left( (\varepsilon_{N,M} - 1) \frac{n}{2} - \frac{2}{\varepsilon_{N,M} \omega^2} \sum_{i=1}^{N} g_{[k]}^2 \right).
\]

(244)

The physically adequate shifted Hamiltonians \( H'_{b-f} \) and \( H'_{cm-f} \) then take the form

\[
H'_{b-f} = \sum_{m,n=0}^{\infty} \sum_{j=1}^{d_{n,m}} \omega (\mathcal{E}_{b-f}^{[k]} - \mathcal{E}_{b-f}^{[k]}) |\xi_{m,j-1,n-j+1}\rangle \langle |\xi_{m,j-1,n-j+1}\rangle,
\]

\[
H'_{cm-f} = \sum_{m,n=0}^{\infty} \sum_{j=1}^{d_{n,m}} \Omega (\mathcal{E}_{cm-f}^{[k]} - \mathcal{E}_{cm-f}^{[k]}) |\xi_{m,j-1,n-j+1}\rangle \langle |\xi_{m,j-1,n-j+1}\rangle,
\]

(245)

allowing to build corresponding vector coherent states. See their general expressions and properties in section 3.1.

### 4 Concluding remarks

In this work, we have studied in detail all relevant physical Hamiltonians considered in quantum optics and condensed matter physics to describe the nanoparticle dynamics in terms of a system of interacting bosons and fermions. For these systems, we have built vector coherent states that well satisfy required mathematical properties of continuity, temporal stability, resolution of the identity and action identity as postulated by Gazeau and Klauder. A generalization to multidimensional vector coherent states for the nondegenerate Hamiltonians has been also introduced. A more general Hamiltonian which is relevant for the study of a new mechanism for electronic energy relaxation in nanocrystals introduced by Yang et al. [13], manifesting finite and infinite degeneracies, has been also treated.

Particular features are expected from similar analysis as one done here when the ordinary Fock-Heisenberg oscillator algebra is replaced by the Jannussis et al. [22] and Man’ko et al. [23] f-deformed oscillator algebra. The latter is spanned by the generators \( \{a, a^\dagger, N\} \) coupled to a well defined function \( f \) of the number operator \( N \), such that

\[
A^- = af(N), \quad A^\dagger = f(N)a^\dagger, \quad N = A^\dagger A^- = Nf^2(N), \quad [A^-, A^\dagger] = \{N + 1\} - \{N\}.
\]

(246)

This will be in the core of a forthcoming paper.
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