Superselection Sectors and General Covariance. I

Romeo Brunetti(1) and Giuseppe Ruzzi(2)

(1) II Institute für Theoretische Physik, Universität Hamburg, Luruper Chaussee 149, D-22761 Hamburg, Germany
(2) Dipartimento di Matematica, Università di Roma “Tor Vergata,” Via della Ricerca Scientifica, I-00133 Roma, Italy
romeo.brunetti@desy.de, ruzzi@mat.uniroma2.it

July 6, 2018

Abstract

This paper is devoted to the analysis of charged superselection sectors in the framework of the locally covariant quantum field theories. We shall analyze sharply localizable charges, and use net-cohomology of J.E. Roberts as a main tool. We show that to any 4-dimensional globally hyperbolic spacetime it is attached a unique, up to equivalence, symmetric tensor C*-category with conjugates (in case of finite statistics): to any embedding between different spacetimes, the corresponding categories can be embedded, contravariantly, in such a way that all the charged quantum numbers of sectors are preserved. This entails that to any spacetime is associated a unique gauge group, up to isomorphisms, and that to any embedding between two spacetimes there corresponds a group morphism between the related gauge groups. This form of covariance between sectors also brings to light the issue whether local and global sectors are the same. We conjecture this holds that at least on simply connected spacetimes. It is argued that the possible failure might be related to the presence of topological charges. Our analysis seems to describe theories which have a well defined short-distance asymptotic behaviour.
## Contents

1. **Introduction**  
   - 3

2. **Spacetime geometry**  
   - 2.1 Lorentzian spacetimes  
     - 11
   - 2.2 Stable families of indices  
     - 13

3. **Locally covariant quantum field theory**  
   - 3.1 States and representations of nets  
     - 18
   - 3.2 State Space  
     - 19

4. **Homotopy of posets and net cohomology**  
   - 4.1 Homotopy of posets  
     - 23
   - 4.2 Net cohomology  
     - 27

5. **Charged superselection sectors**  
   - 5.1 Fixed spacetime background  
     - 5.1.1 A preferred reference states  
       - 33
     - 5.1.2 Independence of the choice of states I  
       - 34
     - 5.1.3 Independence of the choice of states II  
       - 36
     - 5.1.4 Restriction to subregions  
       - 39
   - 5.2 Net cohomology and sharply localized sectors  
     - 41
   - 5.3 Locally covariant structure of sectors  
     - 5.3.1 The Embedding functor  
       - 42
     - 5.3.2 The Superselection Functor  
       - 45
     - 5.3.3 The Gauge Weak Functor  
       - 46

6. **Local completeness**  
   - 49

7. **Conclusions and Outlook**  
   - 53

A. **Tensor C∗—categories**  
   - 55
1 Introduction

The present paper represents a first step toward a new investigation of superselection sectors based on the general framework of locally covariant quantum field theory [13].

One of the milestones of quantum field theory is the investigation of the superselection rules. They are, in essence, constraints to the nature and scope of possible measurements, proving that in quantum physics the superposition principle does not hold unrestrictedly. Stimulated by the preliminary investigation about the existence of superselection rules in quantum field theory by Wick, Wightman and Wigner [60], Haag and Kastler [34] found that the correct interpretation resides in the inequivalent representations of abstract C*-algebras describing local observables, i.e. equivalence classes of local measurements. Some years later, Doplicher, Haag and Roberts [21, 22] undertook an awesome analysis which culminated at the beginning of the nineties with a long series of remarkable papers (Doplicher and Roberts, e.g. see [24, 25]). The main result that has been proved is that besides any algebra of observables there is associated, canonically, a field algebra with normal Bose-Fermi commutation relations, on which a compact gauge group of the first kind acts. This group determines the superselection structure of the theory, in the sense that its irreducible representations are in one-to-one correspondence with the superselection sectors describing sharply localized charges. Well beyond the importance for the physical interpretation, they also obtained outstanding mathematical results, e.g. by founding an abstract theory of group duals extending that of Tannaka and Krein.

Let us describe in more specific terms the basic ideas of superselection sectors in the algebraic approach of Haag and Kastler [34, 33]. One assumes that all the physical information is contained in the a priori correspondence between (open, bounded) regions of Minkowski spacetime and algebras of observables that can be measured inside it, i.e., we have an abstract net of local observables, namely the correspondence

$$
\mathcal{O} \rightarrow \mathcal{A}(\mathcal{O})
$$

which associates, for instance, to every element $\mathcal{O}$ of the family $\mathcal{K}^{dc}$ of double cones of Minkowski spacetime the (unital) C*-algebra $\mathcal{A}(\mathcal{O})$. The C*-algebra generated by all local ones is termed the quasi-local algebra of observables and denoted by $\mathcal{A}_{\mathcal{K}^{dc}}$.

A few basic principles can be adopted, among which, causality and Poincaré covariance. The claim is that this is all one needs for investigating
(structural) properties of a physical theory. This claim is substantiated by many results, among which the theory of superselection sectors is the leading example [33], as far as the intrinsic perspective is concerned, although one should not forget to cite that also the time-honored perturbation theory can be fruitfully formulated in the algebraic framework [10, 26, 35].

An important point, where the physical interpretation enters heavily, is in the choice of the correct physical states for the theory at hand. As far as applications to elementary particle physics is concerned, one assumes the existence of a canonical state, that of the (pure) vacuum state $\omega_0$. Doplicher, Haag and Roberts invoked the intuitive ideas that any particle (at least in the case in which no massless excitations are present in the physical theory) arises as an elementary and localized excitation of the vacuum state. Hence, they put forward that, if $\pi_0$ is the representation of the algebra $\mathcal{A}_{K_{dc}}$ associated with the vacuum state $\omega_0$, any representation $\pi$ which describes an elementary and localized excitation of the vacuum should satisfy

$$\pi\lfloor \mathcal{A}(O_{\perp}) \cong \pi_0\lfloor \mathcal{A}(O_{\perp}) , \quad O \in K_{dc} ,$$

(1)

where the symbol $\cong$ means unitary equivalence, and $\perp$ denotes the causal complement.

This put the previous intuitive remark of particles as localized excitations of the vacuum on rigorous footing by declaring those representations to be “physically relevant” that are unitarily equivalent to the vacuum one in the causal complement of any double cone.

Superselection sectors of the quasi-local algebra $\mathcal{A}_{K_{dc}}$ are then defined to be the unitary equivalence classes of those irreducible representations, satisfying the selection condition, and where the labels distinguishing them are called charged quantum numbers. One then considers the representations satisfying condition (1) as referring to “localizable charges.” The domain of validity of such intrinsic development is, for instance, that of the massive sector of hadronic theories.

Borchers [8], in a previous attempt, based the selection of physical sectors on the requirement of positivity of the energy, since also in this case one can exclude states for which the matter density does not vanish at infinity. On the base of his remarks, the analysis of Doplicher, Haag and Roberts was further refined by Buchholz and Fredenhagen [17], at least for the case of gauge theories without massless excitations, by requiring the selection condition on unbounded regions called spacelike cones. These are regions extending to spacelike infinity. Their criterion is essentially that shown before, where one changes the double cones with the new regions. The so
selected representations bear charges termed “topological charges,” and the results are similar to those of Doplicher, Haag and Roberts.

However, there are important theories for which the above requirements are not fulfilled, for instance in abelian gauge theories like QED. There the above localization is not pertinent since any gauge charge has attached, via Gauss’ Law, its own flux line or string which forces a poorer localization. QED itself is still awaiting a complete determination of its sectors, but see e.g. [16] and [43, 44].

In both localizable and topological cases, the analysis leads to the following remarkable results shedding light on deep characteristic aspects of elementary particle physics, namely:

1. **Particle-Antiparticle symmetry**: To each charge corresponds a unique conjugated charge which entails the particle-antiparticle symmetry;

2. **Particle Statistics**: Each charge has a permutation symmetry to which there corresponds uniquely a sign and an integer $d$, its statistical dimension; the corresponding particles, if any, then satisfy parastatistics of order $d$, the sign distinguishes between para-Bose and para-Fermi statistics. Usual Bose and Fermi statistics correspond to $d = 1$.

3. **Composition of Charges**: Two charges can be composed because they can be created in spacelike separated regions.

The analysis developed so far has been fruitfully applied in other contexts, for instance in (chiral) conformal quantum field theory by Fredenhagen, Kawahigashi, Longo, Rehren, Schroer and various others collaborators [20, 37, 36], for quantum field theory on curved spacetime in a paper by Guido, Longo, Roberts and Verch [32], and initially for soliton physics and then towards more mathematical aims by Müger [40, 41]. In a more mathematical direction one should cite the work done by Baumgärtel and Lledó [3] and by Vasselli [54].

On the previous investigation of the superselection structure in the context of curved backgrounds Guido, Longo, Roberts and Verch [32] used as a major tool the results by Roberts on the connection between net cohomology and superselection sectors [15, 47].

Net cohomology has made clearer that it is the causal and topological structure of spacetime that are the basic relevant features when studying superselection sectors. In particular, we now understand that the main point is how these two properties are encoded in the structure of the index set of the net (e.g. the set of double cones $K^{dc}$ for Minkowski spacetime) as a
Let us then call $\mathcal{K}^\bullet$ the preferred choice of index set on the spacetime $M$, and call $\mathcal{A}_{\mathcal{K}^\bullet}$ the corresponding net of local observables. One proves, under some conditions, that representations satisfying the selection criterion (11) are, up to equivalence, in one-to-one correspondence with 1-cocycles of the poset with values in the (vacuum) representation of the net of local algebras, $\mathcal{B}_{\mathcal{K}^\bullet}: \mathcal{O} \to \mathcal{B}(\mathcal{O})$ where $\mathcal{B}(\mathcal{O}) \equiv \pi_0(\mathcal{A}(\mathcal{O}))''$, for any $\mathcal{O} \in \mathcal{K}^\bullet$. They define a $C^\ast$-category $Z_1^t(\mathcal{B}_{\mathcal{K}^\bullet})$. In order to be a good starting point for the analysis of superselection structure, it should possess good features like being a tensor category, having a permutation symmetry and a conjugation (in case of finite statistics).

Recently, in [51], one of us completed the investigation of [32] in the case where the spacetime is globally hyperbolic, relying on a refined form of net cohomology where a more precise investigation of the properties of indices was made clear, much stimulated by some previous results in [48]. Among several things, it is discussed why the choice made in [32] of basing the net structure on the so-called “regular diamonds” index set $\mathcal{K}^{rd}$ is unfortunate. There are two main problems with such a choice; the first is that this index set is not directed, causing problems with the definition of the tensor product structure of the category of 1-cocycles; the second is that it has elements with non-arcwise-connected causal complements, a topological feature that seems, at least, to forbid a straightforward application of Haag duality, if possible at all. In [51] a better choice was made, that of using as a basic index set that of “diamonds,” noted $\mathcal{K}^d$, having only elements with arcwise-connected causal complements. On this basis the analysis was easier and more powerful, and one arrives at a category of 1-cocycles $Z_1^t(\mathcal{B}_{\mathcal{K}^d})$ that indeed possesses all the previously stated properties.

Our aims, as announced at the beginning, is to initiate a study of superselection sectors for the case of locally covariant quantum field theories [13], stepping forward from the previous developments [32, 51, 58]. This new framework deals with a non-trivial blend of two principles, locality and general covariance, in quantum physics. We refer to the review paper [11] for further discussion. It contains the Haag-Kastler setting as a subcase, it found deep application to perturbation theory on curved spacetime [35] and appears to be well suited to developing a new look at perturbative quantum gravity [28]. Especially, it gives a new perspective in quantum field theory, for instance see [57, 50]. Hence, it is an important point to determine and study the features of the superselection structure. The basic features of the approach will be recalled in Section 3, here we just notice that it is a categorical approach in which a quantum field theory is considered as a covariant functor from a category $\textbf{Loc}$ whose objects are 4-dimensional
globally hyperbolic spacetimes, with isometric embeddings as arrows, to a category of \( \mathbb{C}^* \)-algebras of observables, with injective \(^*\)-homomorphisms as arrows.

Aiming at founding superselection theory for this more general setting, we are then faced with the following list of issues:

(a) The good choice of the index sets for any globally hyperbolic spacetime and their stability under isometric embeddings.

(b) A pertinent choice of a reference state (space), with a minimal set of assumptions for nets of observable algebras.

(c) The choice of what kind of 1-cocycles one wants to investigate, i.e., regarding their localization properties and the construction of their \( \mathbb{C}^* \)-categories, with the sought for properties, and especially their behaviour under isometric embeddings.

For the first problem, a “canonical” choice was made in [13]. Namely, a family of regions of each \( M \) with the properties of being relatively compact, causally convex, i.e., such that all causal curves starting and ending inside such a region would always remain inside it, and with non-empty causal complement. We denote it by \( \mathcal{K}^h(M) \). One immediately faces a problem with such a family. Indeed, it shares with the regular diamonds the same annoying topological feature of having non arcwise-connected causal complements. However, we learned in [51] how to overcome this by passing to the diamonds’ family \( \mathcal{K}^d(M) \). Although for the first family stability under isometric embeddings is essentially obvious, our proof for the second relies on a nice recent result due to Bernal and Sánchez [7] on the extension of compact hypersurfaces.

A choice of a reference state is more involved and delicate. Notice that, in the Minkowskian analysis of sectors à la Doplicher-Haag-Roberts, a choice is made in terms of a single pure reference state, typically the vacuum. In the locally covariant setting one can prove [13] that there is no choice of single states on each \( M \), pure or not, which is covariant under local diffeomorphisms. However, the covariance works well [13], in examples, if instead one chooses folia of states for each manifold. This entails that it is appropriate [13] to consider a functorial description of state space that takes into account the covariance under local diffeomorphisms. It is then rather natural to choose, for our present purposes, a locally quasi-equivalent state space \( \mathcal{S}_o(M) \) for any \( M \), that satisfies some further technical conditions as Borchers property and such that there is at least one state for which the
associated net of von Neumann algebras satisfies irreducibility and punctured Haag duality. Since no preferred choice of states is at hand, one of our main tasks would be therefore to exploit the relation between nets defined in terms of different states belonging to the reference state space. It is perhaps one of the virtues of our approach that such a relation can be fully discussed, and brings, under certain technical conditions, an isomorphism between the nets, which moreover behaves well under isometric embeddings. At the level of categories of 1-cocycles, we have been able to prove that they do not depend on the choice of the states inside the reference state space. As far as assumptions for the net are concerned, we have chosen mainly to require irreducibility and punctured Haag duality. This is really a minimal requirement since a form of Haag duality seems to be really necessary for exploiting properties of superselection sectors. Innocent as it looks, this choice will turn out to be crucial for our construction.

However, the real crux of the matter comes with the last issue. Here, there are several possibilities, depending on the choice of the index set and the topological features of the spacetimes. We recall that according to the results in general relativity [61], Cauchy surfaces can have any topology as (closed) 3-dimensional topological spaces. For instance, they can be compact and/or non simply connected, e.g. as in de Sitter space and for the $\mathbb{RP}^3$ geon (see, e.g., [30]), respectively. Hence, the problem lies in the large number of possibilities that we have for choosing the localization properties of the 1-cocycles, namely what kind of index set we would like to use, and accordingly, what features of the charges do we want to highlight. This is additionally complicated by the fact that some of the index sets have the ability to recover the topology of the spacetime while others do not [51].

It is in order to clarify another point, namely, although we are working within the locally covariant setting, this does not mean that the charges have to be necessarily localized in relatively compact regions. Local covariance only requires that the charges are local functionals of the metric, i.e., they depend only on local geometrical data, nothing forbids the charges testing the topological structure of spacetime. Something like the Buchholz and Fredenhagen analysis of topological charges can be envisaged.

In this first paper, however, we restrict our attention to localizable charges, namely we fix as a main index set that of diamonds $\mathcal{K}^d(M)$ on any globally hyperbolic spacetime $M$, and we choose as 1-cocycles those having the property of being path-independent, in the precise sense of paths in the poset $\mathcal{K}^d(M)$. In this case the 1-cocycles provide trivial representations of the first fundamental group of the manifold [51], hence they do not probe the topology of the spacetime. This choice corresponds to that of localizable
charges in the Doplicher, Haag and Roberts sense on Minkowski spacetime. In the forthcoming paper [14] we will discuss how different choices of the posets are related to each other, still in the localizable case. We are working on the more complicated analysis of “charges carrying topological structure” i.e., on path-dependent 1-cocycles.

Even in the easier case of localizable charges there is a subtlety. Although diamonds form a well behaved family for several of the questions underlying the mathematical development of the structure of sectors, one needs the family $\mathcal{K}^h(M)$ for some crucial results, since it contains nonsimply connected regions. For instance, under the isomorphism between nets for different choices of states, a 1-cocycle of $\mathcal{K}^d(M)$ does not remain necessarily path-independent. The stability under net isomorphisms seems to hold only when the nets are extended to $\mathcal{K}^h(M)$ (see Section 5.1.2). Nonetheless, these results involve only investigating the category of 1-cocycles, not the tensor structure. Hence, one of the main points of the analysis to prove that the categories of 1-cocycles associated to the different families are actually equivalent. Granted that, one can proceed to the analysis of the tensor structure of the categories. It is not clear to us, at the moment, whether the presence of both families underlies some further subtle point of more physical origin. A similar situation has been studied by Ciolli [18] for massless scalar fields in 1+1 dimensions.

The main result of the paper is the following: Calling $\mathcal{R}$ the restriction operation of 1-cocycles defined on spacetime $M$ to a subspace $N \subset M$, we have

**Main Technical Theorem.** The restriction $\mathcal{R}$ lifts to a full and faithful covariant $^\ast$-functor between $\mathcal{Z}_1^1(\omega, \mathcal{K}^d(M))$ and $\mathcal{Z}_1^1(\omega, \mathcal{K}^d(N))$, for any choice of the state $\omega$ in the reference state space $\mathcal{S}_0$.

This result entails the good behaviour under isometric embeddings of spacetimes. Furthermore, it shows that the kind of theories we study behave well under the scaling limit. Indeed, the Main Technical Theorem seems to be the cohomological counterpart of the “equivalence between local and global intertwiners” that “good” theories posses in the scaling limit [20].

Furthermore, the main result of the paper is that, according to the expectations coming out the locally covariant approach, the suitably defined superselection structure is also functorial, namely, one can define a map $\mathcal{S}$ from the category of spacetimes $\text{Loc}$ to the category $\text{Sym}$ of symmetric tensor $C^\ast$—categories, with full and faithful symmetric tensor $^\ast$—functors as arrows, for which there holds
**Main Theorem.** The map \( S : \text{Loc} \rightarrow \text{Sym} \) is a contravariant functor.

Roughly speaking, the message coming out the Theorem is that the physical content of superselection sectors based on different spacetimes can be faithfully transported provided the spacetimes can be embedded. Namely, all charged quantum numbers are preserved in this embedding.

An important byproduct of the Main Theorem is that to any spacetime there is associated a unique gauge group, up to isomorphism, and that to any embedding between spacetimes there corresponds a group morphism between the respective gauge groups.

Another consequence of the Main Theorem is that it makes possible a clear investigation of the possible relation between local and global superselection sectors of a spacetime. Because of the sharp localization, one expects an equivalence between local and global superselection sectors. However there is no general evidence of this equivalence, that we call *local completeness of superselection sectors*. Indeed, one can prove it only in models derived from free quantum fields. We point out that a possible violation of local completeness might be related to the nontrivial topology of spacetimes and, in particular, to the existence of path-dependent 1-cocycles.

We now pass on to outline the content of the paper. In Section 2 we start by elaborating geometrical features which are crucial for the following parts. We discuss some background results on Lorentzian spacetime mainly for fixing our notation. Then we pass to the discussion of families of subsets for each globally hyperbolic spacetime in 4-dimensions and their stability properties under isometric embeddings. Some categorical notions are briefly outlined. In Section 3 we start by recalling the basic definitions of the locally covariant quantum field theory as a covariant functor. There we introduce some crucial definitions and assumptions. Moreover we prove some results on state spaces. Section 4 contains only a brief discussion of the basic definitions and properties of net cohomology. The heart of the paper is in Section 5. There we prove our Main Theorem and several related results. Eventually, these last results are applied, in subsection 5.3, to prove the generally covariant behaviour of the superselection structures. In Section 6 we point out the difficulties in proving the equivalence between local and global sectors. Conclusions and Outlook follow. An Appendix is provided where some categorical notions and some results about Doplicher-Roberts Reconstruction Theorem are briefly recalled.
2 Spacetime geometry

In the first part of this section we review some basics notions of Lorentzian geometry, introduce the category of spacetimes and define some families of sets of technical importance. We prove some results of independent interests.

2.1 Lorentzian spacetimes

We recall some basics on the causal structure of spacetimes and establish our notation. Standard references for this topic are [4, 12, 27, 59].

**Spacetimes:** A spacetime \( M \), in our framework, consists of a Hausdorff, paracompact, connected, without boundary, smooth, oriented 4-dimensional manifold \( M \) endowed with a smooth metric \( g \) with signature \((- , + , + , + )\), and with a time-orientation, that is a smooth vector field \( v \) satisfying the equation \( g(p, v) < 0 \) for each \( p \in M \). (Throughout this paper smooth means \( C^\infty \)).

A curve \( \gamma \) in \( M \) is a continuous, piecewise smooth, regular function \( \gamma : I \rightarrow M \), where \( I \) is a connected subset of \( \mathbb{R} \) with nonempty interior. It is called timelike, lightlike, spacelike if respectively \( g(\dot{\gamma}, \dot{\gamma}) < 0, = 0, > 0 \) all along \( \gamma \), where \( \dot{\gamma} = \frac{d\gamma}{dt} \). Assume now that \( \gamma \) is causal, i.e. a non-spacelike curve; we can classify it according to the time-orientation \( v \) as future-directed or past-directed if respectively \( g(\dot{\gamma}, v) < 0, > 0 \) all along \( \gamma \). When \( \gamma \) is future-directed and there exists \( \lim_{t \rightarrow \sup I} \gamma(t) \) (\( \lim_{t \rightarrow \inf I} \gamma(t) \)), then it is said to have a future (past) endpoint. In the negative case, it is said to be future (past) endless; \( \gamma \) is said to be endless if none of them exist. Analogous definitions are assumed for past-directed causal curves.

The **chronological future** \( I^+(S) \), the **causal future** \( J^+(S) \) and the **future domain of dependence** \( D^+(S) \) of a subset \( S \subset M \) are defined as:

\[
I^+(S) \equiv \{ x \in M \mid \text{exists a future-directed timelike curve from } S \text{ to } x \} ; \\
J^+(S) \equiv S \cup \{ x \in M \mid \text{exists a future-directed causal curve from } S \text{ to } x \} ; \\
D^+(S) \equiv \{ x \in M \mid \text{any past-directed} \\
\text{endless causal curve through } x \text{ meets } S \} .
\]

These definitions have duals in which "future" is replaced by "past" and the \(+\) by \(-\). By this, we define \( I(S) \equiv I^+(S) \cup I^-(S) \), \( J(S) \equiv J^+(S) \cup J^-(S) \) and \( D(S) \equiv D^+(S) \cup D^-(S) \).
Remark 2.1. It is worth recalling the following properties of causal sets\(^1\).

(a) Let \( S \subseteq M \). Then \( I^+(S) \) is an open set and \( I^+(cl(S)) = I^+(S) \); \( cl(J^+(S)) = cl(I^+(S)) \) and \( int(J^+(S)) = I^+(S) \).

(b) Let \( S_1, S_2, S_3 \subseteq M \). If \( S_1 \subseteq J^+(S_2) \) and \( S_2 \subseteq I^+(S_3) \), then \( S_1 \subseteq I^+(S_3) \).

By (a), we have that \( cl(J^+(cl(S))) = cl(J^+(S)) \) for any \( S \subseteq M \).

The causal disjointness relation is a symmetric binary relation \( \perp \) on the subsets of \( M \) defined as follows

\[
S \perp V \iff V \subseteq M \setminus J(S) .
\] (2)

The causal complement of a set \( S \) is the open set \( S^\perp \) defined as

\[
S^\perp \equiv M \setminus cl(J(S)) .
\] (3)

A set \( S \) is acausal if \( \{p\} \perp \{q\} \) for each pair \( p, q \in S \). A set \( S \) is achronal if \( I^+(S) \cap S = \emptyset \). A (causal) Cauchy surface \( \mathcal{C} \) of \( M \) is an achronal (causal) set verifying \( D(\mathcal{C}) = M \). Any Cauchy surface is a closed, connected, Lipschitz hypersurface of \( M \). A spacelike Cauchy surface is a smooth Cauchy surface whose tangent space is everywhere spacelike. Any spacelike Cauchy surface is acausal.

Global hyperbolicity and the category \( \text{Loc} \): A spacetime \( M \) is globally hyperbolic if it admits a smooth foliation by spacelike Cauchy surfaces, namely, there is a 3-dimensional smooth manifold \( \Sigma \) and a diffeomorphism \( F : \mathbb{R} \times \Sigma \longrightarrow M \) such that: for each \( t \in \mathbb{R} \) the set \( \mathcal{C}_t \equiv \{F(t, y) \mid y \in \Sigma\} \) is a spacelike Cauchy surface of \( M \); the curve \( t \in \mathbb{R} \rightarrow F(t, y) \in M \) is a future-directed (by convention) endless timelike curve for any \( y \in \Sigma \). We recall some causality properties of a globally hyperbolic spacetimes.

Remark 2.2. Assume that \( M \) is a globally hyperbolic spacetime.

(a) Let \( K \) be a relatively compact subset of \( M \). Then, \( J^+(cl(K)) \) is closed, and \( J^+(cl(K)) = cl(J^+(K)) \); \( D^+(cl(K)) \) is compact. If \( \mathcal{C} \) is a spacelike Cauchy surface, then \( J(cl(K)) \cap \mathcal{C} \) is a compact subset of \( \mathcal{C} \).

(b) The family of the open subsets \( I^+(x) \cap I^-(y) \), for any \( x, y \in M \), is a basis for the topology of \( M \). Hence, if \( S \) is open set and \( x \in S \), then there are \( y_1, y_2 \in S \) such that \( x \in I^+(y_1) \cap I^-(y_2) \subseteq S \).

As an easy consequence of the Remark 2.2, we have the following

\(^1\)\( cl(S) \) and \( int(S) \) denote respectively the closure and the internal part of the set \( S \).
Lemma 2.3. Let $M$ be a globally hyperbolic spacetime. If $S$ is an open subset of $M$, then the sets $J^+(S)$, $J^-(S)$ and $J(S)$ are open, hence $J^+(S) = I^+(S)$, $J^-(S) = I^-(S)$ and $J(S) = I(S)$.

Proof. It is clear that $I^+(S) \subseteq J^+(S)$. Now, let $x \in J^+(S)$. Take a point $x_0$ in $J^-(x) \cap S \neq \emptyset$. Since $S$ is an open set, there is $y \in S$ such that $x_0 \in I^+(y)$ (see Remark 2.2(b)). By Remark 2.1(b), we have that $x \in J^+(S)$. Analogously, $J^-(S) = I^-(S)$. Hence $J(S) = I(S)$.

Lemma 2.4. Let $K, S$ be two subsets of a globally hyperbolic spacetime $M$. Assume that $K$ is relatively compact with $K^\perp \neq \emptyset$, and that $S$ is open. If $cl(K) \subseteq S$, then $K^\perp \cap S \neq \emptyset$.

Proof. Assume that $K^\perp \cap S = \emptyset$. This is equivalent to say that $S \subseteq (M \setminus K^\perp) = cl(J(K)) = J(cl(K))$ (see Remark 2.2(a)). Since $cl(K) \subseteq S$, we have that $J(cl(K)) \subseteq J(S) \subseteq J(cl(K))$. Hence $J(S) = J(cl(K))$ and this leads to a contradiction. Indeed, by Lemma 2.3, $J(S)$ is an open set while $J(cl(K))$ is closed and different from $M$ since $K^\perp \neq \emptyset$. Therefore $J(S)$ is an open and closed proper subset of $M$; this is not possible because $M$ is arcwise connected.

Let $M$ and $M_1$ be globally hyperbolic spacetimes with metrics $g$ and $g_1$ respectively. A smooth function $\psi$ from $M_1$ into $M$ is called an isometric embedding if $\psi : M_1 \to \psi(M_1) \subseteq M$ is a diffeomorphism and $\psi^*g_1 = g|_{\psi(M_1)}$. The category $\text{Loc}$ is the category whose objects are the 4-dimensional globally hyperbolic spacetimes; the arrows $(M_1, M)$ are the isometric embeddings $\psi : M_1 \to M$ preserving the orientation and the time-orientation of the embedded spacetime, and that satisfy the property

$$\forall x, x_1 \in \psi(M_1), \ J^+(x) \cap J^-(x_1) \text{ is contained in } \psi(M_1).$$

The composition law between two arrows $\psi$ and $\phi$, denoted by $\psi\phi$, is given by the usual composition between smooth functions; the identity arrow $id_M$ is the identity function of $M$.

2.2 Stable families of indices

Given $M \in \text{Loc}$, the aim is to introduce families of open subsets of $M$ which will be used as index set for nets of local algebras. We prove their stability under isometric embeddings.
Globally hyperbolic regions: Let $\mathcal{K}^h(M)$ be the collection of subsets $\mathcal{O} \subseteq M$ satisfying the following properties:

(i) $\mathcal{O}$ is open, arcwise connected, relatively compact set, and $\mathcal{O} \cap \mathcal{O}^\perp \neq \emptyset$;

(ii) if $x_1, x_2 \in \mathcal{O}$, then $J^+(x_1) \cap J^-(x_2)$ contained in $\mathcal{O}$.

It turns out by this definition that $\mathcal{K}^h(M)$ is a basis for the topology of $M$ and that any element $\mathcal{O}$ of $\mathcal{K}^h(M)$ – with the metric $g|_\mathcal{O}$ and with the induced orientation and time orientation – is a globally hyperbolic spacetime, hence $\mathcal{O} \in \text{Loc}$. We now show some straightforward geometrical results.

**Lemma 2.5**. Let $M \in \text{Loc}$, then the following assertions hold:

(a) Let $\mathcal{O}, \mathcal{O}_1 \in \mathcal{K}^h(M)$ be such that $\text{cl}(\mathcal{O}_1) \perp \mathcal{O}$. Then there exists $\mathcal{O}_2 \in \mathcal{K}^h(M)$ such that $\text{cl}(\mathcal{O}) \cup \text{cl}(\mathcal{O}_1) \subset \mathcal{O}_2$;

(b) For any $\mathcal{O} \in \mathcal{K}^h(M)$ there are $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{K}^h(M)$ such that $\text{cl}(\mathcal{O}_1) \perp \mathcal{O}$ and $\text{cl}(\mathcal{O}) \cup \text{cl}(\mathcal{O}_1) \subseteq \mathcal{O}_2$;

(c) Let $\mathcal{O} \in \mathcal{K}^h(M)$ and let $S \subset M$ be an open set, with $\text{cl}(\mathcal{O}) \subset S$, then $\mathcal{O} \cap S \neq \emptyset$.

**Proof.** (a) Since $\text{cl}(\mathcal{O}_1) \perp \mathcal{O}$, then $\text{cl}(\mathcal{O}_1) \subset M \setminus J(\text{cl}(\mathcal{O}))$. By Lemma 2.4 $\mathcal{O}_1 \cap \mathcal{O}^\perp = \mathcal{O}_1 \cap M \setminus J(\text{cl}(\mathcal{O})) \cup J(\text{cl}(\mathcal{O}_1)))$ is open and nonempty. Fix $x \in \mathcal{O}_1 \cap \mathcal{O}^\perp$ and let $\mathcal{C}$ be a spacelike Cauchy surface that meets $x$. Consider the sets $K \equiv J(\text{cl}(\mathcal{O})) \cap \mathcal{C}$ and $K_1 \equiv J(\text{cl}(\mathcal{O}_1)) \cap \mathcal{C}$. In the relative topology of $\mathcal{C}$, both $K$ and $K_1$ are compact because $\text{cl}(\mathcal{O})$ and $\text{cl}(\mathcal{O}_1)$ are compact (see Remark 2.2(a)), and they are arcwise connected because so are $\text{cl}(\mathcal{O})$ and $\text{cl}(\mathcal{O}_1)$.

Moreover $x \notin K \cup K_1$. It is clear that we can find two arcwise connected and relatively compact open sets $W$ and $W_1$ of $\mathcal{C}$ such that: $K \subset W$, $K_1 \subset W_1$ and $x \notin \text{cl}(W) \cup \text{cl}(W_1)$. We now consider the following cases. First, assume that $W \cap W_1 \neq \emptyset$. In this case we define $\mathcal{O}_2 \equiv D(W \cup W_1)$. $\mathcal{O}_2$ verifies the properties written in the statement. In fact $W \cup W_1$ is an open and arcwise connected subset of $\mathcal{C}$, then $\mathcal{O}_2$ is open and globally hyperbolic (Lemma 43). Moreover, since $W \cap W_1$ is relatively compact, $\mathcal{O}_2$ is relatively compact (see Remark 2.2(a)). Finally, since $\mathcal{C} \setminus (\text{cl}(W) \cup \text{cl}(W_1)) \neq \emptyset$, we have that $\mathcal{O}_2^\perp \neq \emptyset$. Hence $\mathcal{O}_2 \in \mathcal{K}^h(M)$. It is also clear that $\text{cl}(\mathcal{O}) \cup \text{cl}(\mathcal{O}_1) \subset \mathcal{O}_2$.

Secondly, assume now that $\text{cl}(W) \cap (W_1) = \emptyset$. Let $\gamma : [0, 1] \to \mathcal{C}$ be a curve that meets $x$ and such that $\gamma(0) \in W$ and $\gamma(1) \in W_1$. We can find a family $U_i, i = 1, \ldots, n$ of open, relatively compact and arcwise connected sets of $\mathcal{C}$ which cover the curve $\gamma$ and such that, if $G \equiv W \cup W_1 \cup U_n \cup \cdots \cup U_1$, then $\mathcal{C} \setminus \text{cl}(G) \neq \emptyset$. By setting $\mathcal{O}_2 \equiv D(G)$, as before we have that $\mathcal{O}_2$ verifies the properties written in the statement. (b) follows from (a): it is enough to observe that, since $\mathcal{O} \cap \mathcal{O}_1 \neq \emptyset$ and since $\mathcal{K}^h(M)$ is a basis for the topology of $M$, there is $\mathcal{O}_1 \in \mathcal{K}^h(M)$, with $\text{cl}(\mathcal{O}_1) \perp \mathcal{O}$. (c) follows from Lemma 2.4. \[\square\]
We now use the previous Lemma to show that compactness of Cauchy surfaces is the only obstruction to the directedness of the family $\mathcal{K}^h(M)$, namely that for any $O_1, O_2 \in \mathcal{K}^h(M)$ there is $O \in \mathcal{K}^h(M)$ such that $O_1 \cup O_2 \subseteq O$.

**Lemma 2.6.** Let $M \in \text{Loc}$ have noncompact Cauchy surfaces, then $\mathcal{K}^h(M)$ is directed.

**Proof.** Let $\mathcal{C}$ be a spacelike Cauchy surface of $M$. Note that the set $K \equiv (J(\text{cl}(O_1)) \cap \mathcal{C}) \cup (J(\text{cl}(O_2)) \cap \mathcal{C})$ is compact. Since $\mathcal{C}$ is noncompact, then $\mathcal{C} \setminus K$ is open and nonempty. From this point on the proof proceeds as in Lemma 2.5a. \hfill $\square$

Consider $M_1, M \in \text{Loc}$ and let $\psi \in (M_1, M)$. Define

\[
\psi(\mathcal{K}^h(M_1)) \equiv \{ \psi(O) \subseteq M \mid O \in \mathcal{K}^h(M_1) \},
\]

\[
\mathcal{K}^h(M)|_{\psi(M_1)} \equiv \{ O \in \mathcal{K}^h(M) \mid \text{cl}(O) \subseteq \psi(M_1) \}. \tag{4}
\]

**Lemma 2.7.** Given $M_1, M \in \text{Loc}$ and $\psi \in (M_1, M)$, then

\[
\mathcal{K}^h(M)|_{\psi(M_1)} = \psi(\mathcal{K}^h(M_1)).
\]

**Proof.** By the definition of $\text{Loc}$ it is clear that if $O_1 \in \mathcal{K}^h(M_1)$, then $\psi(O_1) \in \mathcal{K}^h(M)|_{\psi(M_1)}$. Conversely, let $O \in \mathcal{K}^h(M)|_{\psi(M_1)}$. By Lemma 2.5(a), $O^\perp \cap \psi(M_1) \neq \emptyset$. Then the causal complement of $\psi^{-1}(O)$ in $M_1$ is nonempty. By this and the definition of $\text{Loc}$ we have that $\psi^{-1}(O) \in \mathcal{K}^h(M_1)$. \hfill $\square$

Finally, we want to stress that $\mathcal{K}^h(M)$ have elements which are non-simply connected subsets of $M$ and elements whose causal complement is nonarcwise connected. This fact creates some problems in studying superselection sectors of a net indexed by $\mathcal{K}^h(M)$. However, as we shall see in Section 4 these problems will be overcome by basing the theory on a particular subfamily of $\mathcal{K}^h(M)$ whose elements do not present the above topological features. To this end, define

\[\text{Sub}(\mathcal{K}^h(M)) \equiv \{ \mathcal{K}_1 \subseteq \mathcal{K}^h(M) \mid \mathcal{K}_1 \text{ is a basis for the topology of } M \}.\]

It is clear that $\mathcal{K}^h(M) \in \text{Sub}(\mathcal{K}^h(M))$. Further properties are discussed in Section 4.
Define \( \psi \) and \( \mathcal{O} \) exists in \( M \) of \( \psi \phi \) Observe that \( \mathcal{O} \) is a spacelike Cauchy surface \( \mathcal{O}_1 \) such that there exists a spacelike Cauchy surface \( \mathcal{O} \) that \( \mathcal{O}_1 \) is an open, relatively compact, arcwise and simply connected subset of \( O \). Any diamond \( \mathcal{O} \) is based \( \mathcal{O} \) and \( \phi^{-1}(B) \) is a spacelike Cauchy surface \( \mathcal{O} \). The set of diamonds \( \mathcal{O} \) is a basis for the topology of \( \mathcal{O} \) which may be of a certain interest. For instance, one such is the family of regular diamonds used in \( \mathcal{O} \). It contains the family of diamonds as a subset, and has the same stability property as the others. We refrain from giving details here, since this family does not play a rôle hereafter. More discussion will be found in \( \mathcal{O} \).

**Diamonds regions:** A relevant element of \( \text{Sub}(\mathcal{K}^h(M)) \) is the set \( \mathcal{K}^d(M) \) of diamonds of \( M \) \[51\]. An open subset \( \mathcal{O} \) of \( M \) is called a diamond if there is a spacelike Cauchy surface \( \mathcal{O} \), a chart \( (U, \phi) \) of \( \mathcal{O} \), and an open ball \( B \) of \( \mathbb{R}^3 \) such that
\[ \mathcal{O} = D(\phi^{-1}(B)), \quad \mathcal{O} \subseteq \phi(U) \subset \mathbb{R}^3. \]
We will say that \( \mathcal{O} \) is based on \( \mathcal{O} \) and call \( \phi^{-1}(B) \) the base of \( \mathcal{O} \). Any diamond \( \mathcal{O} \) is an open, relatively compact, arcwise and simply connected subset of \( M \), and its causal complement \( \mathcal{O} \) is arcwise connected. Furthermore, for any diamond \( \mathcal{O} \) there exists a pair of diamonds \( \mathcal{O}_1, \mathcal{O}_2 \) such that
\[ \mathcal{O} \equiv \{ \mathcal{O} \subseteq M \mid \mathcal{O} \in \mathcal{K}^d(M) \} \]
\[ \mathcal{K}^d(M) \equiv \{ \mathcal{O} \in \mathcal{K}^d(M) \mid \mathcal{O} \subseteq \mathcal{O}_1 \}. \]

**Lemma 2.8.** Given \( M_1, M \in \text{Loc} \) and \( \psi \in (M_1, M) \), then
\[ \mathcal{K}^d(M_1) = \psi(\mathcal{K}^d(M_1)). \]

**Proof.** We prove the inclusion \((\supseteq)\). Let \( \mathcal{O} \) be a diamond of \( M_1 \). This means that there exists a spacelike Cauchy surface \( \mathcal{C}_1 \) of \( M_1 \), a chart \( (U, \phi_1) \) such that
\[ \mathcal{O}_1 = D(\phi_1^{-1}(B)), \quad \text{where} \quad B \text{ is a ball of } \mathbb{R}^3 \text{ such that } cl(\phi_1^{-1}(B)) \subseteq U. \]
Let \( B_1 \) be a ball of \( \mathbb{R}^3 \) such that \( cl(B) \subseteq cl(B_1) \) and \( cl(\phi_1^{-1}(B_1)) \subseteq U. \) Observe that \( \psi \phi_1^{-1}(B_1) \) is a relatively compact, spacelike acausal open set of \( M_1 \) with boundaries and with a nonempty complement. By \[7\], there exists in \( M \) a spacelike Cauchy surface \( \mathcal{C} \) such that \( cl(\psi \phi_1^{-1}(B_1)) \subseteq \mathcal{C}. \) Define \( V \equiv \psi \phi_1^{-1}(B_1) \) and \( \phi \equiv \phi_1 \psi^{-1}. \) The pair \( (V, \phi) \) is a chart of \( \mathcal{C} \) and \( cl(\phi^{-1}(B)) \subseteq V. \) Finally observe that by the properties of \( \psi \) we have that \( \psi(D(\phi_1^{-1}(B)) = D(\psi \phi_1^{-1}(B)) = D(\phi_1^{-1}(B)), \) namely \( \mathcal{K}^d(M_1) \supseteq \psi(\mathcal{K}^d(M_1)). \) The proof of the reverse inclusion is very similar to the previous one and we omit it.

There are many other elements of \( \text{Sub}(\mathcal{K}^h(M)) \) which may be of a certain interest. For instance, one such is the family of regular diamonds used in \[32\] (see it for the exact definition, or \[56\]). It contains the family of diamonds as a subset, and has the same stability property as the others. We refrain from giving details here, since this family does not play a rôle hereafter. More discussion will be found in \[14\].
3 Locally covariant quantum field theory

Locally covariant quantum field theory is a categorical approach to the quantum theory of fields which incorporates the locality principle of classical field theory in a generally covariant manner [13, 11]. In order to introduce the axioms of the theory, we give a preliminary definition. Let us denote by \( \text{Obs} \) the category whose objects \( A \) are unital \( C^* \)-algebras and whose arrows \((A_1, A_2)\) are the unit-preserving injective \( C^* \)-morphisms from \( A_1 \) into \( A_2 \).

The composition law between the arrows \( \alpha_1 \) and \( \alpha_2 \), denoted by \( \alpha_1 \alpha_2 \), is given by the usual composition between \( C^* \)-morphisms; the unit arrow \( \text{id}_A \) of \((A, A)\) is the identity morphism of \( A \).

A locally covariant quantum field theory is a covariant functor \( \mathcal{A} \) from the category \( \text{Loc} \) (see Section 2) into the category \( \text{Obs} \), that is, a diagram

\[
\begin{array}{ccc}
M_1 & \xrightarrow{\psi} & M_2 \\
\downarrow \mathcal{A} & & \downarrow \mathcal{A} \\
\mathcal{A}(M_1) & \xrightarrow{\alpha_\psi} & \mathcal{A}(M_2)
\end{array}
\]

where \( \alpha_\psi \equiv \mathcal{A}(\psi) \), such that \( \alpha_{\text{id}_M} = \text{id}_{\mathcal{A}(M)} \), and \( \alpha_\phi \alpha_\psi = \alpha_{\phi \psi} \) for each \( \psi \in (M_1, M) \) and \( \phi \in (M, M_2) \). The functor \( \mathcal{A} \) is said to be causal if, given \( \psi_1 \in (M_1, M) \) for \( i = 1, 2 \),

\[
\psi_1(M_1) \perp \psi_2(M_2) \Rightarrow [\alpha_{\psi_1}(\mathcal{A}(M_1)), \alpha_{\psi_2}(\mathcal{A}(M_2))] = 0 ,
\]

where \( \psi_1(M_1) \perp \psi_2(M_2) \) means that \( \psi_1(M_1) \) and \( \psi_2(M_2) \) are causally disjoint in \( M \). From now on \( \mathcal{A} \) will denote a causal locally covariant quantum field theory.

In conclusion, let us see how a net of local algebras over \( M \in \text{Loc} \) can be recovered from a locally covariant quantum field theory \( \mathcal{A} \). To this end, recall that any \( \emptyset \in \mathcal{K}^h(M) \), considered as a spacetime with the metric \( g|_\emptyset \), belongs to \( \text{Loc} \). The injection \( \iota_{M, \emptyset} \) of \( \emptyset \) into \( M \) is an element of \( (\emptyset, M) \) because of the definition of \( \mathcal{K}^h(M) \). Then, using \( \alpha_{\iota_{M, \emptyset}} \in (\mathcal{A}(\emptyset), \mathcal{A}(M)) \) to define \( \mathcal{A}(\emptyset) \equiv \alpha_{\iota_{M, \emptyset}}(\mathcal{A}(\emptyset)) \), it turns out [13] that the correspondence

\[
\mathcal{A}_{\mathcal{K}^h(M)} : \mathcal{K}^h(M) \ni \emptyset \longrightarrow \mathcal{A}(\emptyset) \subset \mathcal{A}(M) , \tag{7}
\]

is a net of local algebras satisfying the Haag-Kastler axioms:

\[
\emptyset_1 \subseteq \emptyset_2 \Rightarrow \mathcal{A}(\emptyset_1) \subseteq \mathcal{A}(\emptyset_2) , \quad \text{isotony},
\]

\[
\emptyset_1 \perp \emptyset_2 \Rightarrow [\mathcal{A}(\emptyset_1), \mathcal{A}(\emptyset_2)] = 0 , \quad \text{causality}.
\]
As for the local covariance of the theory, let $M_1 \in \text{Loc}$ with the metric $g_1$, and let $\psi \in (M_1, M)$. Because of Lemma 2.7, $\psi(\emptyset) \in \mathcal{K}_h(M)$ for each $\emptyset \in \mathcal{K}_h(M_1)$. Since $\iota_{M_1, \psi(\emptyset)}$ is an isometric embedding of the spacetime $\emptyset$ onto the spacetime $\psi(\emptyset)$ — the latter equipped with the metric $g|_{\psi(\emptyset)}$ — one has that

$$\alpha_\psi : A(\emptyset) \subset \mathcal{A}(M_1) \to A(\psi(\emptyset)) \subset \mathcal{A}(M).$$

is a $C^*$-isomorphism. As $\psi(\mathcal{K}_h(M_1)) \in \text{Sub}(\mathcal{K}_h(M))$, denote by $A_{\psi(\mathcal{K}_h(M_1))}$ the net index by $\psi(\mathcal{K}_h(M_1))$ obtained by restricting $A_{\mathcal{K}_h(M)}$ to $\psi(\mathcal{K}_h(M_1))$. Then the relation (8) says that

$$\alpha_\psi : A_{\mathcal{K}_h(M_1)} \to A_{\psi(\mathcal{K}_h(M_1))},$$

is a net-isomorphism.

### 3.1 States and representations of nets

Fix $M \in \text{Loc}$, and consider the net $A_{\mathcal{K}_h(M)}$. A state $\omega$ of the algebra $A(M)$, is defined to be a positive ($\omega(A^*A) \geq 0, A \in A(M)$), and normalized ($\omega(1) = 1$) linear functional on it. For any state $\omega$ of the algebra $A(M)$ we will denote by $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$ the corresponding GNS-construction; by $\omega^* A_{\mathcal{K}_h(M)}$ we will denote the net of von Neumann algebras defined as the correspondence

$$\omega^* A_{\mathcal{K}_h(M)} : \mathcal{K}_h(M) \ni \emptyset \to A_\omega(\emptyset) \subseteq \mathcal{B}(\mathcal{H}_\omega),$$

where $A_\omega(\emptyset) \equiv (\pi_\omega(A(\emptyset)))''$ for any $\emptyset \in \mathcal{K}_h(M)$, and $\mathcal{B}(\mathcal{H}_\omega)$ is the $C^*$-algebra of linear bounded operators of $\mathcal{H}_\omega$. Furthermore, we set $A_\omega(M) \equiv \pi_\omega(A(M))$.

Once $\omega^* A_{\mathcal{K}_h(M)}$ is given, to any element $\mathcal{K}_1 \in \text{Sub}(\mathcal{K}_h(M))$ there corresponds a net of von Neumann algebras $\omega^* A_{\mathcal{K}_1}$ defined as

$$\omega^* A_{\mathcal{K}_1} \equiv \{ A_\omega(\emptyset) \mid \emptyset \in \mathcal{K}_1 \}.$$  

Properties of such a net of local algebras, which will be important for our purposes, are the following:

**P1.** $\omega^* A_{\mathcal{K}_1}$ is said to be irreducible whenever, given $T \in \mathcal{B}(\mathcal{H})$ such that $T \in A_\omega(\emptyset)'$ for any $\emptyset \in \mathcal{K}_1$, then $T = c \cdot 1$.

**P2.** $\omega^* A_{\mathcal{K}_1}$ satisfies the Borchers property if given $\emptyset \in \mathcal{K}_1$ for any $\emptyset_1 \in \mathcal{K}_1$ with $\text{cl}(\emptyset) \subseteq \emptyset_1$ any nonzero orthogonal projection $E \in A_\omega(\emptyset)$ is equivalent to $1$ in $A_\omega(\emptyset_1)$.
P3. \( \omega^* \mathcal{A}_{\mathcal{K}_1} \) is locally definite if \( C \cdot 1 = \cap \{ A_\omega(O) \mid O \in \mathcal{K}_1, \ x \in O \} \), for any point \( x \) of \( M \).

P4. \( \omega^* \mathcal{A}_{\mathcal{K}_1} \) satisfies punctured Haag duality if given \( O \in \mathcal{K}_1 \), with \( cl(O) \perp \{ x \} \), then

\[
A_\omega(O) = \cap \{ A_\omega(O_1)' \mid O_1 \in \mathcal{K}_1, \ O_1 \perp O, \ cl(O_1) \perp \{ x \} \},
\]

for any point \( x \) of \( M \).

Some observations on these definitions are in order. First, given \( \mathcal{K}_1 \in \text{Sub}(\mathbb{K}_h(M)) \), the irreducibility of the net \( \omega^* \mathcal{A}_{\mathcal{K}_1} \) is, in general, stronger than irreducibility of the representation \( \pi_\omega \) of \( \mathcal{A}(M) \). This is because the collection \( \cup_{O \in \mathcal{K}_1} A_\omega(O) \) needs not to be dense in \( A_\omega(M) \) (see also [13]). Moreover, it is clear that if \( \omega^* \mathcal{A}_{\mathcal{K}_1} \) satisfies punctured Haag duality, then it satisfies Haag duality; given \( O \in \mathcal{K}_1 \), then \( A_\omega(O) = \cap \{ A_\omega(O_1)' \mid O_1 \in \mathcal{K}_1, \ O_1 \perp O \} \).

If \( \omega^* \mathcal{A}_{\mathcal{K}_1} \) satisfies punctured Haag duality and is irreducible, then it is locally definite [50].

From now on we will say that a state \( \omega \in \mathcal{S}(M) \) satisfies punctured Haag duality if \( \omega^* \mathcal{A}_{\mathcal{K}_h(M)} \) is irreducible and satisfies punctured Haag duality. We note the following straightforward results:

**Lemma 3.1.** The following assertions hold:

(a) If \( \mathcal{A}_{\mathcal{K}_1} \) is irreducible, then \( \mathcal{A}_{\mathcal{K}_2} \) is irreducible, for any \( \mathcal{K}_2 \in \text{Sub}(\mathbb{K}_h(M)) \) such that \( \mathcal{K}_1 \subseteq \mathcal{K}_2 \);

(b) If \( \mathcal{A}_{\mathcal{K}_1} \) satisfies the Borchers property, then \( \mathcal{A}_{\mathcal{K}_2} \) satisfies the Borchers property for any \( \mathcal{K}_2 \in \text{Sub}(\mathbb{K}_h(M)) \) such that \( \mathcal{K}_2 \subseteq \mathcal{K}_1 \);

(c) If there is \( \mathcal{K}_1 \in \text{Sub}(\mathbb{K}_h(M)) \) such that \( \mathcal{A}_{\mathcal{K}_1} \) is locally definite, then \( \mathcal{A}_{\mathcal{K}_2} \) is locally definite for any \( \mathcal{K}_2 \in \text{Sub}(\mathbb{K}_h(M)) \).

**Proof.** (a) and (b) are obvious. (c) derives from the fact that any element of \( \text{Sub}(\mathbb{K}_h) \) is a basis for the topology of \( M \). \( \square \)

### 3.2 State Space

We now turn to the notion of a state space of \( \mathcal{A} \). A state space of a unital C*-algebra \( A \) is a family of states \( \mathcal{S}(A) \) of \( A \) which is closed under finite convex combinations and operations \( \omega(\cdot) \rightarrow \omega(A^* \cdot A)/\omega(A^* A) \) for \( A \in \mathcal{A} \). We denote by \( \text{Sts} \) be the category whose objects are the state spaces \( \mathcal{S}(A) \) of unital C*-algebras \( A \) whose arrows are the positive maps \( \gamma^* : \mathcal{S}(A) \rightarrow \mathcal{S}(A') \), arising as dual maps of injective morphisms of C*-algebras \( \gamma : A' \rightarrow A \), by \( \gamma^* \omega(A) \equiv \omega(\gamma(A)) \) for each \( A \in \mathcal{A} ' \). The composition law between
two arrows, as the definition of the identity arrow of an object, are obvious.

A state space for $\mathcal{A}$ is a contravariant functor $\mathcal{S}$ between $\text{Loc}$ and $\text{Sts}$, that is, a diagram

$$
\begin{array}{ccc}
M_1 & \xrightarrow{\psi} & M_2 \\
\downarrow \mathcal{S} & & \downarrow \mathcal{S} \\
\mathcal{S}(M_1) & \xrightarrow{\alpha^*_\psi} & \mathcal{S}(M_2)
\end{array}
$$

where $\mathcal{S}(M_1)$ is a state space of the algebra $\mathcal{A}(M_1)$, such that $\alpha^*_\text{id}_M = \text{id}_{\mathcal{S}(M)}$, and $\alpha^*_\psi \alpha^*_\phi = \alpha^*_\phi \psi$ for each $\psi \in (M_1, M)$ and $\phi \in (M, M_2)$.

Some of the properties, involving nets and states, can be generalized to state spaces. First we recall, that a state space $\mathcal{S}$ is said to be locally quasi-equivalent if for any $M \in \text{Loc}$ and for any pair $\omega, \sigma \in \mathcal{S}(M)$ we have that

$$
F(\pi_\omega | A(\mathcal{O})) = F(\pi_\sigma | A(\mathcal{O})) , \quad \forall \mathcal{O} \in \mathcal{K}^h(M) ,
$$

where $F(\pi_\omega | A(\mathcal{O}))$ is the local folium, i.e. the collection of normal states of the algebras $\pi_\omega(A(\mathcal{O}))$.

**Definition 3.2.** We will say that a locally quasi-equivalent $\mathcal{S}$ satisfies the Borchers property (resp. local definiteness) if for any $M \in \text{Loc}$ and for any $\omega \in \mathcal{S}(M)$ the net $\omega^* \mathcal{A}(\mathcal{K}^h(M))$ satisfies the Borchers property (resp. local definiteness).

**Lemma 3.3.** Let $\mathcal{S}$ be a locally quasi-equivalent state space. Given $M \in \text{Loc}$, for any pair $\sigma, \omega \in \mathcal{S}(M)$ the nets $\omega^* \mathcal{A}(\mathcal{K}^h(M))$ and $\sigma^* \mathcal{A}(\mathcal{K}^h(M))$ are isomorphic, namely, there is a collection

$$
\rho_{\omega, \sigma} \equiv \{ \rho_\mathcal{O} : \mathcal{A}_\omega(\mathcal{O}) \rightarrow \mathcal{A}_\sigma(\mathcal{O}) \mid \mathcal{O} \in \mathcal{K}^h(M) \}
$$

made of $^*$-isomorphisms of von Neumann algebras such that $\rho_\mathcal{O} | \mathcal{A}_\omega(\mathcal{O}_1) = \rho_{\mathcal{O}_1}$ if $\mathcal{O}_1 \subseteq \mathcal{O}$.

**Proof.** Local quasi-equivalence means that for any $\mathcal{O} \in \mathcal{K}^h(M)$ there exists a unique isomorphism $\rho_\mathcal{O} : \mathcal{A}_\omega(\mathcal{O}) \rightarrow \mathcal{A}_\sigma(\mathcal{O})$ such that

$$
\rho_\mathcal{O} \pi_\omega(A) = \pi_\sigma(A) \quad A \in \mathcal{A}(\mathcal{O}) .
$$

The collection $\rho_{\omega, \sigma} \equiv \{ \rho_\mathcal{O} \mid \mathcal{O} \in \mathcal{K}^h(M) \}$ is compatible with the net structure. Indeed, note that given $\mathcal{O}_1 \subseteq \mathcal{O}$, then $\rho_\mathcal{O} \pi_\omega(A) = \pi_\sigma(A) = \rho_{\mathcal{O}_1} \pi_\omega(A)$ for any $A \in \mathcal{A}_\omega(\mathcal{O}_1)$. By uniqueness we have $\rho_\mathcal{O} | \mathcal{A}_\omega(\mathcal{O}_1) = \rho_{\mathcal{O}_1}$ if $\mathcal{O}_1 \subseteq \mathcal{O}$. 

\[ \Box \]
As an easy consequence of this lemma we have

Corollary 3.4. Let $\mathcal{S}$ be a locally quasi-equivalent state space. Assume that for any $M \in \text{Loc}$ there is $\omega \in \mathcal{S}(M)$ such that $\omega^* \mathcal{A}_{\mathcal{H}_M} \omega$ satisfies the Borchers property (resp. local definiteness). Then $\mathcal{S}$ fulfills the Borchers property (resp. local definiteness).

Now, consider $\psi \in (M_1, M)$ and the associated injective $C^*$-morphism $\alpha_\psi : \mathcal{A}(M_1) \rightarrow \mathcal{A}(M)$. By means of $\alpha_\psi$, a state $\omega \in \mathcal{S}(M)$ induces two different representations on $\mathcal{A}(M_1)$. On the one hand, if $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$ is the GNS construction associated with $\omega$, then $\pi_\omega \alpha_\psi$ is a representation of $\mathcal{A}(M_1)$. On the other hand, by local covariance $\omega_\alpha \psi \in \mathcal{S}(M_1)$. If $(\pi_{\omega_\alpha \psi}, \mathcal{H}_{\omega_\alpha \psi}, \Omega_{\omega_\alpha \psi})$ is the GNS construction associated with $\omega_\alpha \psi$, then $\pi_{\omega_\alpha \psi}$ is a representation of $\mathcal{A}(M_1)$. In order to understand the relation between $\pi_\omega \alpha_\psi$ and $\pi_{\omega_\alpha \psi}$, define

$$V \pi_\omega \alpha_\psi(A) \Omega_\omega \equiv \pi_{\omega_\alpha \psi}(A) \Omega_{\omega_\alpha \psi}, \quad A \in \mathcal{A}(M_1).$$

Since $\Omega_{\omega_\alpha \psi}$ is cyclic, $V$ is a partial isometry from $\mathcal{H}_\omega$ onto $\mathcal{H}_{\omega_\alpha \psi}$ such that $VV^* = 1_{\mathcal{H}_{\omega_\alpha \psi}}$ and $V^*V \in \pi_\omega(\alpha_\psi(\mathcal{A}(M_1)))'$. Let

$$\tau_\psi^\omega(\pi_\omega \alpha_\psi(A)) \equiv \pi_{\omega_\alpha \psi}(A), \quad A \in \mathcal{A}(M_1).$$

Observe that $\tau_\psi^\omega : \pi_\omega \alpha_\psi(\mathcal{A}(M_1)) \rightarrow \pi_{\omega_\alpha \psi}(\mathcal{A}(M_1))$ is a $C^*$-morphism. Furthermore, $\tau_\psi^\omega$ defines a net-morphism from the net $\omega^* \mathcal{A}_{\mathcal{H}(M_1)} \omega$ into the net $(\omega_\alpha \psi)^* \mathcal{A}_{\mathcal{H}(M_1)} \omega_\alpha \psi$ (see also [13]). Collecting the definitions and properties just stated we have that

Proposition 3.5. Assume that $\omega^* \mathcal{A}_{\mathcal{H}(M_1)}$ satisfies the Borchers property.

(a) $\pi_\omega \alpha_\psi$ and $\pi_{\omega_\alpha \psi}$ are locally quasi-equivalent representations of $\mathcal{A}(M_1)$.

(b) $\tau_\psi^\omega$ is a net-isomorphism.

Proof. (a) Since $V^*V \in (\pi_\omega \alpha_\psi(\mathcal{A}(M_1)))'$, we have

$$V^*V \in \mathcal{A}_\omega(0)' = \pi_\omega(\mathcal{A}(0))', \quad \forall \omega \in \psi(\mathcal{H}(M_1)).$$

We show that $V^*V$ has central support equal to $1$ on $\mathcal{A}_\omega(0)'$ for $0 \in \psi(\mathcal{H}(M_1))$. This is equivalent to showing that

$$E \cdot V^*V = 0 \iff E = 0,$$

for any orthogonal projection $E$ of $\mathcal{A}_\omega(0)$. Assume that $E \neq 0$. By Lemma [16](b), there is $0_1 \in \psi(\mathcal{H}(M_1))$ such that $cl(0) \subseteq 0_1$. By the Borchers property there is an isometry $W \in \mathcal{A}_\omega(0_1)$ such that $WW^* = E$. Then

$$E \cdot V^*V = 0 \iff WW^* \cdot V^*V = 0 \iff V^*V \cdot W^* = 0 \iff V^*V = 0.$$
This leads to a contradiction. Hence $V^*V$ has central support $1$ on $A_\omega(0)'$.

(b) The proof follows from (a) because by 12, $\tau_\psi^\omega(\pi_\omega \alpha_\psi(A)) = V\pi_\omega \alpha_\psi(A)V^*$ for any $A \in \mathcal{A}(M_1)$. 

We now study the relations between the net-isomorphisms introduced in this section.

**Lemma 3.6.** Let $\mathcal{I}$ be a locally quasi-equivalent state space satisfying the Borchers property. Given $\psi \in (M_1, M)$, then $\rho_{\omega \alpha_\psi, \sigma \alpha_\psi}^\omega \tau_\psi^\omega = \tau_\sigma^\omega \rho_{\omega, \sigma}$, for any pair $\sigma, \omega \in \mathcal{I}(M)$.

**Proof.** Let $0 \in K^h(M_1)$ and $A \in A(0)$. By the definition of $\rho_{\omega, \sigma}$ (see Lemma 3.3) and by (13), we have that

\[
\tau_\psi^\omega(\pi_\omega(\alpha_\psi(A))) = \pi_{\sigma \alpha_\psi}(A) = \rho_{\omega \alpha_\psi, \sigma \alpha_\psi}(\pi_{\omega \alpha_\psi}(A)) = \rho_{\omega \alpha_\psi, \sigma \alpha_\psi}^\omega(\pi_\omega(\alpha_\psi(A))) = \rho_{\omega \alpha_\psi, \sigma \alpha_\psi}^\omega \rho_{\sigma, \omega}(\pi_\sigma(\alpha_\psi(A))) .
\]

**Lemma 3.7.** Let $\mathcal{I}$ be a locally quasi-equivalent state space satisfying the Borchers property. Given $\psi \in (M_1, M)$, $\phi \in (M_2, M_1)$, then $\tau_\psi^\omega = \tau_\phi^\omega \tau_\psi^\omega$, for any $\omega \in \mathcal{I}(M)$.

**Proof.** First observe that $\omega \alpha_\phi \in \mathcal{I}(M_2)$. By the definition (13) we have that

\[
\tau_\psi^\omega(\pi_\omega \alpha_\phi(A)) = \pi_{\omega \alpha_\phi}(A) , \quad A \in \mathcal{A}(M_2) ,
\]

\[
\tau_\psi^\omega(\pi_\omega(\alpha_\phi(A))) = \pi_{\omega \alpha_\phi}(A) , \quad A \in \mathcal{A}(M_1) ,
\]

\[
\tau_\phi^\omega(\pi_{\omega \alpha_\phi}(\alpha_\phi(A))) = \pi_{\omega \alpha_\phi}(A) , \quad A \in \mathcal{A}(M_2) .
\]

By using these relations we have that

\[
\tau_\phi^\omega \tau_\psi^\omega(\pi_\omega \alpha_\phi(A)) = \tau_\phi^\omega(\pi_{\omega \alpha_\phi}(\alpha_\phi(A))) = \pi_{\omega \alpha_\phi}(A) = \tau_\phi^\omega(\pi_\omega \alpha_\phi(A)) ,
\]

for any $A \in \mathcal{A}(M_2)$. 

**4 Homotopy of posets and net cohomology**

The mathematical framework under which we will study superselection sectors is that of the net cohomology of posets. In the present section we give some notions and results referring to [47, 48, 51]. Throughout this section, we fix $M \in \mathbf{Loc}$, and denote by $\mathcal{P}$ a basis for the topology of $M$ whose elements are open and arcwise connected subsets of $M$, and have a nonempty
causal complement. Furthermore, by the same symbol $\mathcal{P}$, we denote the partially ordered set (poset) formed by the basis $\mathcal{P}$ ordered under inclusion $\subseteq$.

### 4.1 Homotopy of posets

**The simplicial set of $\mathcal{P}$**: We need a few observations on standard n-simplices. Consider the standard n-simplex $\Delta_n$, namely the subset of $\mathbb{R}^{n+1}$, defined as

$$\Delta_n \equiv \{(\lambda_0, \ldots, \lambda_n) \in \mathbb{R}^{n+1} \mid \lambda_0 + \cdots + \lambda_n = 1, \ \lambda_i \in [0, 1]\}.$$ 

Observe that $\Delta_n$ can be regarded as a partially ordered set with respect to the inclusion of its subsimplices, and that for any $n \geq 1$ there is a family $d^n_i : \Delta_{n-1} \to \Delta_n$, with $0 \leq i \leq n$, of inclusion preserving maps, defined as

$$d^n_i(\lambda_0, \ldots, \lambda_{n-1}) = (\lambda_0, \lambda_1, \ldots, \lambda_{i-1}, 0, \lambda_i, \ldots \lambda_{n-1}).$$

Now, a singular n-simplex of the poset $\mathcal{P}$ is an inclusion preserving map $f : \Delta_n \to \mathcal{P}$. We denote by $\Sigma^*_n(\mathcal{P})$ the set of singular n-simplices. The collection $\Sigma^*(\mathcal{P})$ of all singular simplices is called the simplicial set of $\mathcal{P}$.

The inclusion maps $d^n_i$ between standard simplices are extended to maps $\partial^n_i : \Sigma^*_n(\mathcal{P}) \to \Sigma^*_{n-1}(\mathcal{P})$, called boundaries, between singular simplices by setting $\partial^n_i f \equiv f \circ d^n_i$. One can easily check, by the definition of $d^n_i$, that the following relations

$$\partial^n_{i-1} \circ \partial^n_j = \partial^n_{j+1} \circ \partial^n_i, \quad i \geq j,$$  

hold. From now on, we will omit the superscripts from the symbol $\partial^n_i$, and will denote: the composition $\partial_i \circ \partial_j$ by the symbol $\partial_{ij}$; 0-simplices by the letter $a$; 1-simplices by $b$ and 2-simplices by $c$. Notice that a 0-simplex $a$ is nothing but an element of $\mathcal{P}$. A 1-simplex $b$ is formed by two 0-simplices $\partial_0 b$, $\partial_1 b$ and an element $|b|$ of $\mathcal{P}$, called the support of $b$, such that $\partial_0 b, \partial_1 b \subseteq |b|$; a 2-simplex $c$ is formed by three 1-simplices $\partial_0 c, \partial_1 c, \partial_2 c$, whose 0-boundaries are chained between them according to (14), and by a 0-simplex $|c|$, the support, such that $\partial_0 c, \partial_1 c, \partial_2 c \subseteq |c|$. Given a 1-simplex $b$, the reverse of $b$ is the 1-simplex $\overline{b}$ defined as

$$\partial_0 \overline{b} = \partial_1 b, \quad \partial_1 \overline{b} = \partial_0 b, \quad |\overline{b}| = |b|.$$ 

Finally, a 1-simplex $b$ is said to be degenerate to a 0-simplex $a_0$ whenever

$$\partial_0 b = a_0 = \partial_1 b, \quad a_0 = |b|.$$ 

We will denote by $b(a_0)$ the 1-simplex degenerate to $a_0$. 

23
Paths: Given \(a_0, a_1 \in \Sigma_0(P)\), a path from \(a_0\) to \(a_1\) is a finite ordered sequence \(p = \{b_n, \ldots, b_1\}\) of 1-simplices enjoying the relations

\[
\partial_1 b_1 = a_0, \quad \partial_0 b_i = \partial_1 b_{i+1} \text{ with } i \in \{1, \ldots, n-1\}, \quad \partial_0 b_n = a_1.
\]

The starting point of \(p\), written \(\partial_1 p\), is the 0-simplex \(a_0\), while the end point of \(p\), written \(\partial_0 p\), is the 0-simplex \(a_1\). The support \(|p|\) of the path \(p\) is the open set

\[
|p| \equiv \bigcup_{i=1}^{n} |b_i|.
\]

We will denote by \(P(a_0, a_1)\) the set of paths from \(a_0\) to \(a_1\), and by \(P(a_0)\) the set of closed paths whose end point is \(a_0\).

The set of paths is equipped with the following operations. Consider a path \(p = \{b_n, \ldots, b_1\} \in P(a_0, a_1)\). The reverse \(p\) of \(p\) is the path

\[
\overline{p} \equiv \{\overline{b}_1, \ldots, \overline{b}_n\} \in P(a_1, a_0).
\]

The composition of \(p\) with a path \(q = \{b'_k, \ldots, b'_1\}\) of \(P(a_1, a_2)\), is defined as

\[
q \ast p \equiv \{b'_k, \ldots, b'_1, b_n, \ldots, b_1\} \in P(a_0, a_2).
\]

Note that the reverse \(-\) is involutive, while the composition \(*\) is associative.

An elementary deformation of a path \(p\) consists in replacing a 1-simplex \(\partial_1 c\) of the path by a pair \(\partial_0 c, \partial_2 c\), where \(c \in \Sigma_2(P)\), or, conversely in replacing a consecutive pair \(\partial_0 c, \partial_2 c\) of 1-simplices of \(p\) by a single 1-simplex \(\partial_1 c\). Two paths with the same endpoints are homotopic if they can be obtained from one other by a finite set of elementary deformations. Homotopy defines an equivalence relation \(\sim\) on the set of paths with the same end points. The reverse and the composition are compatible with the homotopy equivalence relation, namely

\[
\begin{align*}
p \sim q & \iff \overline{p} \sim \overline{q}, \\
p \sim q, \ p_1 \sim q_1 & \Rightarrow p_1 \ast p \sim q_1 \ast q, \quad p, q \in P(a_0, a_1), \quad p_1, q_1 \in P(a_1, a_2).
\end{align*}
\]

Furthermore, for any \(p \in P(a_0, a_1)\), the following relations hold:

\[
\begin{align*}
p \ast b(a_0) & \sim p & \text{and} & \quad p \sim b(a_1) \ast p, \\
\overline{p} \ast p & \sim b(a_0) & \text{and} & \quad b(a_1) \sim p \ast \overline{p},
\end{align*}
\]

where \(b(a_0)\) is the 1-simplex degenerate to \(a_0\). The following lemma will be useful for later purposes.
Lemma 4.1. Let \( p = \{ b_n, \ldots, b_1 \} \) be a path, and let \( q = q_n \ast \cdots \ast q_1 \) be a path such that
\[
|q_i| \subseteq |b_i|, \quad \partial_0 q_i \subseteq \partial_0 b_i, \quad \partial_1 q_i \subseteq \partial_1 b_i,
\]
for \( i = 1, \ldots, n \). The following assertions hold:
(a) there exist a pair of 1-simplices \( \tilde{b}, \hat{b} \) such that \( p \sim \tilde{b} \ast q \ast \hat{b} \);
(b) if \( p \) and \( q \) are closed paths, there is a 1-simplex \( b \) such that \( p \sim b \ast q \ast b \), where \( b \) is the reverse of \( b \).

Proof. (a) By (15) we can assume, without loss of generality, \( p \) to be a 1-simplex \( b \). So, let \( q \) be a path such that \( |q| \subseteq |b|, \partial_0 q \subseteq \partial_0 b, \) and \( \partial_1 q \subseteq \partial_1 b \).

Let \( \hat{b} \) be the 1-simplex defined as
\[
|\hat{b}| = |b|, \quad \partial_1 \hat{b} = \partial_1 b, \quad \partial_0 \hat{b} = \partial_1 q,
\]
and let \( \tilde{b} \) be the 1-simplex defined as
\[
|\tilde{b}| = |b|, \quad \partial_1 \tilde{b} = \partial_0 q, \quad \partial_0 \tilde{b} = \partial_0 b.
\]

Note that \( b \) and \( \tilde{b} \ast q \ast \hat{b} \) have the same endpoints. Since the poset formed by the collection \( \emptyset \in P \) such that \( \emptyset \subseteq |b| \) is directed under inclusion, these two paths are homotopic [51] (see also [58]). (b) follows from (a).

Poset approximation and pathwise connectedness: Let \( S \) be an open subset of \( M \). Let \( P_S \equiv \{ O \in P \mid cl(O) \subseteq S \} \). Since the elements of \( P \) are arcwise connected subsets of \( M \), it turns out that \( S \) is arcwise connected if, and only if, \( P_S \) is pathwise connected, namely
\[
\forall O_1, O_2 \in P_S \text{ there exists a path } p \text{ from } O_1 \text{ to } O_2 \text{ with } cl(|p|) \subseteq S.
\]
In particular, since \( M \) is arcwise connected, then \( P \) is pathwise connected.

Given a curve \( \gamma : [0, 1] \to M \), a path \( p = \{ b_n, \ldots, b_1 \} \) is said to be an approximation of \( \gamma \) if there is a partition \( 0 = s_0 < s_1 < \ldots < s_n = 1 \) of the interval \([0, 1]\) such that
\[
\gamma([s_{i-1}, s_i]) \subseteq |b_i|, \quad \gamma(s_{i-1}) \in \partial_1 b_i, \quad \gamma(s_i) \in \partial_0 b_i,
\]
for \( i = 1, \ldots, n \). We denote by \( \text{App}(\gamma) \) the set of approximations of \( \gamma \). It turns out that \( \text{App}(\gamma) \neq \emptyset \) for any curve \( \gamma \).

Lemma 4.2 (poset-approximation [51]). Assume that the elements of \( P \) are simply connected. Let \( p, q \in P(a_0, a_1) \) be respectively two approximations of a pair of curve \( \gamma \) and \( \beta \) with the same endpoints. Then \( \gamma \) and \( \beta \) are homotopic if, and only if, \( p \) and \( q \) are homotopic.
**Fundamental group of a poset:** Fix a base 0-simplex \( a_0 \), and define

\[ \pi_1(\mathcal{P}, a_0) \equiv \mathcal{P}(a_0)/\sim, \]

i.e. the quotient of the set \( \mathcal{P}(a_0) \) of closed paths with end point \( a_0 \) with respect to homotopy equivalence relation. Denote by \([p]\) the equivalence class of \( p \in \mathcal{P}(a_0) \) with respect to homotopy equivalence relation, and define

\[ [p] \cdot [q] \equiv [p \ast q], \quad [p], [q] \in \pi_1(\mathcal{P}, a_0). \]

By the relations (15), (16), it turns out that \( \pi_1(\mathcal{P}, a_0) \) is, with respect to this composition law, a group. The identity of the group is the equivalence class \([b(a_0)]\) associated with the 1-simplex degenerate to \( a_0 \); the inverse \([p]^{-1}\) of \([p]\) is the equivalence class \([\overline{p}]\) associated with the reverse \(\overline{p}\) of the path \( p \). This group is the *first homotopy group of \( \mathcal{P} \) with base 0-simplex \( a_0 \).* Moreover, as \( \mathcal{P} \) is pathwise connected the first homotopy group does not depend, up to isomorphism, on the chosen base 0-simplex \( a_0 \). This equivalence class of groups, denoted by \( \pi_1(\mathcal{P}) \), is called the *fundamental group of \( \mathcal{P} \).* In the case that \( \pi_1(\mathcal{P}) \) is trivial we will say that \( \mathcal{P} \) is simply connected. We have

\[ \mathcal{P} \text{ is directed } \Rightarrow \mathcal{P} \text{ is simply connected}. \quad (18) \]

The link with the corresponding topological notions of the underlying spacetime is obtained by means of the poset approximation lemma.

**Theorem 4.3 ([51]).** Assume that all the elements of \( \mathcal{P} \) are simply connected. Then the fundamental group \( \pi_1(\mathcal{P}) \) of the poset and the fundamental group \( \pi_1(M) \) of the spacetime \( M \) are isomorphic.

Since the set \( \mathcal{K}^d(M) \) of diamonds of \( M \) verifies the property written in the statement, we have that \( \pi_1(\mathcal{K}^d(M)) \simeq \pi_1(M) \). The statement fails if \( \mathcal{P} \) admits nonsimply connected elements. A counterexample is provided by \( \mathcal{K}^h(M) \) which is simply connected irrespectively of the topology of \( M \). To show this a preliminary result is necessary.

**Lemma 4.4.** Let \( p \) be a closed path of \( \mathcal{K}^d(M) \) with end point \( a_0 \). There exists a closed path \( q \) of \( \mathcal{K}^d(M) \), with end point \( a_0 \), and an element \( \mathcal{O} \in \mathcal{K}^h(M) \) such that \( p \) is homotopic to \( q \) and \( |q| \subseteq \mathcal{O} \).

**Proof.** Let \( \gamma \) be a closed curve with \( p \in \text{App}(\gamma) \). Furthermore if \( a_0 \) is a diamond of the form \( D(G) \), with \( G \) contained in a spacelike Cauchy surface \( \mathcal{C}_0 \), then we can assume that \( \gamma \) meets \( G \) in a point \( x_0 \). By [51] Lemma 3.1 \( \gamma \) is homotopic to a closed curve \( \beta \) lying on \( \mathcal{C}_0 \), and with endpoint \( x_0 \). Let
Let \( q = \{b_n, \ldots b_1\} \) be a closed path, with endpoint \( a_0 \), formed by diamonds \( b_i = D(G_i) \), with \( G_i \subseteq C_0 \) for any \( i \), and such that \( q \in \text{App}(\beta) \). The poset approximation Lemma 4.2 entails that \( p \sim q \). Let \( K \equiv \cup_i G_i \). \( K \subset C_0 \) is open, relatively compact and arcwise connected. Let \( \emptyset \equiv D(K) \) and note that \( |q| \subseteq \emptyset \). \( \emptyset \) is open relatively compact, arcwise connected and, since it is contained in the Cauchy surface \( C_0 \), by a standard argument (see, e.g. [2]) \( \emptyset \) is globally hyperbolic. It is also clear that we can shrink the sets \( G_i \) in such a way that \( \emptyset \neq \emptyset \). Hence \( \emptyset \in \mathcal{K}(M) \).

**Proposition 4.5.** The poset \( \mathcal{K}(M) \) is simply connected for any \( M \in \text{Loc} \).

**Proof.** Let \( p \) be a closed path of \( \mathcal{K}(M) \), and let \( \gamma \) be a closed curve with \( p \in \text{App}(\gamma) \). Since \( \mathcal{K}(M) \) is a basis for the topology of \( M \) we can find a closed path \( q \) in \( \mathcal{K}(M) \), with endpoint \( a_0 \), such that: \( q \in \text{App}(\gamma) \); \( q \) satisfies (17) with respect to \( p \). By Lemma 4.1.(b) there is \( b \in \Sigma_1(\mathcal{K}(M)) \), with \( \partial_1 b = \partial_1 p \), and \( \partial_0 b = a_0 \), such that \( p \sim b * q * b \), where \( b \) is the reverse of \( b \). By using the previous lemma, there is a closed path \( q_1 \) in \( \mathcal{K}(M) \), with endpoint \( a_0 \), and \( \emptyset \in \mathcal{K}(M) \) such that \( q_1 \sim q \) and \( |q_1| \subseteq \emptyset \). As the subposet of \( \mathcal{K}(M) \) formed by \( \emptyset \) and \( \{\emptyset_1 \in \mathcal{K}(M) \mid \emptyset_1 \subset \emptyset \} \) is directed under inclusion, by (18) we have that \( q_1 \sim b(a_0) \). Finally, by (15) and (16) we have
\[
\begin{align*}
p & \sim b * q * b \sim b * b(a_0) * b \sim b(a_0),
\end{align*}
\]
and this completes the proof. \( \square \)

### 4.2 Net cohomology

**The category of 1-cocycles:** Consider a net of von Neumann algebras over a Hilbert space \( \mathcal{H} \), \( \mathcal{A}_P : P \ni O \rightarrow \mathcal{A}(O) \subseteq \mathcal{B}(\mathcal{H}) \) indexed by \( P \). A 1-cocycle \( z \) of the poset \( P \), with values in \( \mathcal{A}_P \), is a field
\[
z : \Sigma_1(P) \ni b \rightarrow z(b) \in \mathcal{B}(\mathcal{H})
\]
of unitary operators of \( \mathcal{B}(\mathcal{H}) \) satisfying the following properties:
\[
\begin{align*}
(i) \quad & z(\partial_0 c) \cdot z(\partial_2 c) = z(\partial_1 c), \quad c \in \Sigma_2(P) ; \\
(ii) \quad & z(b) \in \mathcal{A}([b]), \quad b \in \Sigma_1(P).
\end{align*}
\]
The property \((i)\) is called the 1-cocycle identity, while \((ii)\) is called the locality condition for cocycles. Given a pair \( z, z_1 \) of 1-cocycles an intertwiner \( t \) between \( z, z_1 \) is a field
\[
t : \Sigma_0(P) \ni a \rightarrow t_a \in \mathcal{B}(\mathcal{H})
\]

27
satisfying the following properties:

(iii) \( t_{\partial_0 b} \cdot z(b) = z_1(b) \cdot t_{\partial_1 b} \), \( b \in \Sigma_1(\mathcal{P}) \);
(iv) \( t_a \in \mathcal{A}(a) \), \( a \in \Sigma_0(\mathcal{P}) \).

(iii) is the intertwining property while (iv) is the locality condition for intertwiners. We will denote by \((z, z_1)\) the set of the intertwiners between \(z\) and \(z_1\).

The category of 1-cocycles \(Z_1(\mathcal{P})\) is the category whose objects are 1-cocycles and whose arrows are the corresponding set of intertwiners. It turns out that this is a C*–category, details can be found in [51]. Two 1-cocycles \(z, z_1\) are equivalent (or cohomologous) if there exists a unitary arrow \(t \in (z, z_1)\). A 1-cocycle \(z\) is trivial if it is equivalent to the identity cocycle \(\iota\) defined as \(\iota(b) = 1\) for any 1-simplex \(b\).

**Lemma 4.6.** If \(\mathcal{A}_\mathcal{P}\) is either irreducible or locally definite, then the identity cocycle \(\iota\) is irreducible, that is \((\iota, \iota) = C \cdot 1\).

**Proof.** If \(t \in (\iota, \iota)\), then \(t_{\partial_0 b} = t_{\partial_1 b}\) for any 1-simplex \(b \in \Sigma_1(\mathcal{P})\). This entails that \(t\) is a constant field, that is, \(t_a = t_{a_1}\) for any pair \(a, a_1 \in \Sigma_0(\mathcal{P})\), because \(\mathcal{P}\) is pathwise connected. Hence \(t_a \in \mathcal{A}(\emptyset)\) for any \(\emptyset \in \mathcal{P}\). This in turn entails that \(t_a\) belongs both to \(\bigcap \{\mathcal{A}(\emptyset) \mid \emptyset \in \mathcal{K}^h(M)\}\) and to \(\bigcap \{\mathcal{A}(\emptyset) \mid \emptyset \in \mathcal{K}^h(M), x \in \emptyset\}\) for any \(x \in M\). Hence, the proof follows from the definition of irreducibility and local definiteness (see Section 3.1). \(\square\)

From now on we will assume that \(\mathcal{A}_\mathcal{P}\) is irreducible; hence, by the above lemma, \(\iota\) is an irreducible object of \(Z_1(\mathcal{P})\).

The evaluation of a 1-cocycle \(z \in Z_1(\mathcal{P})\) on a path \(p = \{b_n, \ldots, b_1\}\) in \(\mathcal{P}\) is defined as

\[ z(p) \equiv z(b_n) \cdots z(b_2) \cdot z(b_1). \]

A 1-cocycle \(z\) is said to be path-independent if for any \(a_0, a_1 \in \Sigma_0(\mathcal{P})\) we have that

\[ z(p) = z(q) \quad \text{for any pair of paths} \ p, q \in \mathcal{P}(a_0, a_1). \]

It turns out that path-independence is equivalent to triviality in \(\mathcal{B}(\mathcal{H})\), namely, there exists a field \(V : \Sigma_0(\mathcal{P}) \ni a \rightarrow V_a \in \mathcal{B}(\mathcal{H})\) of unitary operators such that

\[ z(b) = V_{\partial_0 b} \cdot V_{\partial_1 b}^*, \quad b \in \Sigma_1(\mathcal{P}). \]

We denote by \(Z_1^t(\mathcal{P})\) the set of path-independent 1-cocycles \(z\) of \(\mathcal{P}\), and with the same symbol the full C*–subcategory of \(Z_1(\mathcal{P})\) whose objects belong to \(Z_1^t(\mathcal{P})\). We will refer to \(Z_1^t(\mathcal{P})\) as the category of path-independent 1-cocycles.
Changing the index set: We now focus on path-independent 1-cocycles and show how they behave under a change of the index set. Given the poset \( P \), consider a subfamily \( P_1 \subseteq P \) which forms a basis for the topology of \( M \). As we have already done in Section 3.1, we obtain a net of von Neumann algebras \( A_P \), indexed by \( P \), by restricting the \( A_P \) to \( P_1 \). We want to understand the relation between \( Z^1_t(P_1) \) and the category \( Z^1_t(P) \) of the path-independent 1-cocycles of \( P_1 \) with values in \( A_{P_1} \). This topic has been analyzed in great generality in [51] in terms of abstract posets. According to this paper, for our aim, it is enough to observe that the following two properties are verified.

1. For any \( O \in P \) there exists \( O_1 \in P_1 \) such that \( O_1 \subseteq O \).
2. Given \( O \in P \), for any pair \( O_1, O_2 \in P_1 \) with \( O_1, O_2 \subseteq O \) there exists a path \( p \in P_1 \) from \( O_1 \) to \( O_2 \) such that \( |p| \subseteq O \).

Both of them derive from the fact that \( P_1 \) is a basis for the topology of \( M \) (the second, in particular, is also a consequence of the fact that the elements of \( P \) are, by assumption, arcwise connected). The properties 1 e 2 imply that \( P_1 \) is a locally relatively connected refinement of \( P \) ([51, Def. 2.9]). This entails, first, that the identity cocycle \( \iota \) of \( Z^1_t(P_1) \) is irreducible ([51, Lemma 2.11]). Secondly, for any \( z, z_1 \in Z^1_t(P) \) and for any \( t \in (z, z_1) \), define

\[
\mathcal{R}(z)(b) \equiv z(b), \quad b \in \Sigma_1(P_1),
\]

\[
\mathcal{R}(t)_a \equiv t_a, \quad a \in \Sigma_0(P_1).
\]

(19)

It can be easily seen that \( \mathcal{R} : Z^1_t(P) \to Z^1_t(P_1) \) is a covariant functor (the restriction to \( P_1 \)). By [51, Def. 2.11] we have the following

**Lemma 4.7.** The categories \( Z^1_t(P_1) \) and \( Z^1_t(P) \) are equivalent, in particular there exists a covariant functor \( E : Z^1_t(P_1) \to Z^1_t(P) \) such that

\[
E \circ \mathcal{R} \simeq \text{id}_{Z^1_t(P)}, \quad \mathcal{R} \circ E = \text{id}_{Z^1_t(P_1)},
\]

where the symbol \( \simeq \) stands for natural equivalence.

Connection between homotopy and net cohomology: To begin with we recall some properties of 1-cocycles which will be useful also for later applications. First, any 1-cocycle \( z \) is invariant for homotopic paths, i.e.

\[
p \sim q \implies z(p) = z(q).
\]

(20)
Secondly,
\[
(a) \quad z(p) = z(p)^* \quad \text{for any path } p ,
\]
\[
(b) \quad z(b(a_0)) = 1 \quad \text{for any 0-simplex } a_0 .
\]  

Following [51], by using homotopic invariance of 1-cocycles and the relations (21), one can easily deduce the following result.

**Theorem 4.8 ([51])**. To any \(z \in Z^1(P)\) there corresponds a unitary representation \(\sigma_z\) of the fundamental group \(\pi_1(P)\) such that \(z\) is path-independent if, and only if, \(\sigma_z\) is trivial.

As a consequence of this result we have that if \(P\) is simply connected then any 1-cocycle of \(P\) is path-independent. Hence by Lemma 4.5 we have
\[
Z^1(K^h(M)) = Z^1_t(K^h(M)), \quad M \in \text{Loc} .
\]  

So, any 1-cocycle of the poset \(K^h(M)\) is path-independent.

The relation between the net cohomology of \(P\) and the topology of the underlying spacetime is obtained by means of the theorems 4.3 and 4.8.

**Theorem 4.9 ([51])**. Assume that all the elements of \(P\) are simply connected. To any 1-cocycle \(z \in Z^1(P)\), there corresponds a unitary representation \(\tilde{\sigma}_z\) of \(\pi_1(M)\) such that \(z\) is path-independent if, and only if, \(\tilde{\sigma}_z\) is trivial.

The set \(K^d(M)\) of diamonds of \(M\) verifies the hypotheses of Theorem 4.9. This allows us to give a topological characterization of the set \(Z^1(K^d(M))\) of 1-cocycles of \(K^d(M)\).

1. The 1-cocycles of \(K^d(M)\) which are path-independent do not carry any information about the topology of the spacetime since they provide trivial representations of the fundamental group. On the other hand, this type of 1-cocycles has a direct physical interpretation; they represent sharply localized sectors of the net of local observables (see Section 5.2). The inspection of their charge structure and of their local covariance is the subject of the present paper, and it will be carried out throughout Section 6.

2. The 1-cocycles of \(K^d(M)\) which are path-dependent are of a topological nature since they provide nontrivial representations of the fundamental group of the spacetime \(M\) (clearly, if \(M\) is nonsimply connected). In our opinion, path-dependent 1-cocycles might be charged sectors induced by the nontrivial topology of the spacetime, a phenomenon predicted and studied in the literature [31, 2, 53]. However, until now, there is no interpretation of these 1-cocycles in terms of superselection sectors, namely representations of the net of local observables. We investigate in Section 6 a notion that possibly points towards the correct interpretation.
5 Charged superselection sectors

By charged superselection sectors in the Minkowski space $\mathbb{M}^4$, it is meant the unitary equivalence classes of the irreducible representations of a net of local observables which are local excitations of the vacuum representation. We can distinguish two types of charged sectors according to the regions of the spacetime used as index set of the net of local observables. Charged sectors of Doplicher-Haag-Roberts type, when one considers double cones of $\mathbb{M}^4$, and the charges of Buchholz-Fredenhagen type associated with a particular class of nonrelatively compact regions like spacelike cones. In both cases, sectors define a $C^*$-category in which the charge structure manifests itself by the existence of a tensor product, a permutation symmetry, and a conjugation (Doplicher-Haag-Roberts analysis [21, 22], and Buchholz-Fredenhagen analysis [17]). Furthermore, it is possible to reconstruct the (unobservable) fields and the gauge group underlying the theory [25].

Our purpose is to study superselection sectors of Doplicher-Haag-Roberts type in the framework of a locally covariant quantum field theory $\mathcal{A}$. The first step is to introduce the notion of reference state space, which will play for the theory the same rôle played by the vacuum representation. To this aim, it is worth observing that the vacuum representation plays the rôle of a reference representation that singles out charged sectors, and that in both cited analysis it is enough to take as vacuum representation one satisfying the Borchers property and Haag duality [25].

**Definition 5.1.** We call a reference state space for $\mathcal{A}$ a locally quasi-equivalent state space $\mathcal{S}_0$ satisfying the Borchers property, and such that for any $M \in \text{Loc}$ there is at least one state $\omega \in \mathcal{S}_0(M)$ satisfying punctured Haag duality.

An example of a locally covariant quantum field theory with a state space verifying the properties of the Definition 5.1 is provided by the Klein-Gordon scalar field and by the space of quasi-free states satisfying the microlocal spectrum condition [13, 50] (see also in Section 6). We stress that we require the existence for any $M \in \text{Loc}$ of at least one state $\omega \in \mathcal{S}_0(M)$ satisfying punctured Haag duality. This property seems to be the right generalization of Haag duality, which apparently seems to deal well with the nontrivial topology of arbitrary globally hyperbolic spacetimes [51]. The reason why punctured Haag duality is assumed on the net indexed by $\mathcal{K}^d(M)$,

---

2It is possible to make do with less than the Borchers property, [19]. On the contrary, up to now, Haag duality or a weaker form of it [17], seems to be an essential requirement on the vacuum representation for the theory of superselection sectors.
is that $\mathcal{K}^h(M)$ contains elements which are not simply connected, and elements whose causal complement is not arcwise connected. So, punctured Haag duality (and also Haag duality) might not hold for a net indexed by $\mathcal{K}^h(M)$ (see [50]). Finally, recall that with our convention a state $\omega$ satisfying punctured Haag duality means that the net $\omega^*\mathcal{A}_{\mathcal{K}^d(M)}$ is irreducible and satisfies punctured Haag duality. These two properties entail local definiteness (see Section 3.1); Then, by Corollary 3.4, $\mathcal{S}_\omega$ is locally definite.

**Definition 5.2.** The charged superselection sectors of $\mathcal{A}$, with respect to the reference state space $\mathcal{S}_\omega$, are the unitary equivalence classes of the irreducible elements of the categories $Z_1^t(\omega, \mathcal{K}^d(M))$ of path-independent 1-cocycles of $\mathcal{K}^d(M)$ with values in $\omega^*\mathcal{A}_{\mathcal{K}^d(M)}$, as $\omega$ varies in $\mathcal{S}_\omega(M)$ and as $M$ varies in Loc.

Our program for investigating superselection sectors is divided in two parts. Our first aim is to understand the charge structure of the categories $Z_1^t(\omega, \mathcal{K}^d(M))$ on a fixed spacetime background $M$ and how these categories are related as $\omega$ varies in $\mathcal{S}_\omega(M)$. Secondly, we will inspect the locally covariant structure of sectors. This means that we will study the connection of sectors associated with different isometrically embedded spacetimes backgrounds.

Now, some observations concerning the definition of superselection sectors in a locally covariant quantum field theory are in order.

1. Our definition of superselection sectors in terms of net cohomology is equivalent to the usual one given in terms of representations of the net of local observables which are sharp excitations of a reference representation. In particular, we will show in Section 5.2 that for any spacetime $M$ and for any $\sigma \in \mathcal{S}_\omega(M)$, to any 1-cocycle $z \in Z_1^t(\sigma, \mathcal{K}^d(M))$ there corresponds, up to equivalence, a unique representation $\pi^z$ of the net of local observables which is a sharp excitation of a representation $\pi_\omega$ associated with a state $\omega \in \mathcal{S}_\omega(M)$ satisfying punctured Haag duality.

2. There are several reasons why we choose to study 1-cocycles of the poset $\mathcal{K}^d(M)$ instead of 1-cocycles of $\mathcal{K}^h(M)$. On the one hand, $\mathcal{K}^d(M)$ reflects the topological and causal properties of $M$ better than $\mathcal{K}^h(M)$: The fundamental group of $\mathcal{K}^d(M)$ is the same as that of $M$ and any diamond has an arcwise connected causal complement. These two properties have been one of the keys of the paper [51] where, in the Haag-Kastler framework, the charge structure of sharply localized sectors in a fixed background spacetime has been investigated. On the contrary, $\mathcal{K}^h(M)$ is simply connected irrespective of the topology of $M$ (Proposition 4.5), and it has elements with
a nonarcwise connected causal complement. On the other hand, Lemma 4.7 states that there is no loss of generality in studying path-independent 1-cocycles of $K^d(M)$ instead of those of $K^h(M)$, because the corresponding categories are equivalent. We have to say, however, that the cited result provides an equivalence of $C^*$-categories, but ignores the tensorial structure of the categories. This topic will be analyzed in [14] where we will provide a symmetric tensor equivalence between the categories associated with $K^d(M)$ and $K^h(M)$.

(3) Since $S_o$ satisfies the Borchers property, by a routine calculation (see [48]), it turns out that the category $Z^1_t(\omega, K^d(M))$ is closed under direct sums and subobjects for any $\omega \in S_o(M)$ and any $M \in \text{Loc}$. As observed above $S_o$ is locally definite. Then, by Lemma 4.6, the identity cocycle of $Z^1_t(\omega, K^d(M))$ is irreducible for any $\omega \in S_o(M)$ and any $M \in \text{Loc}$.

5.1 Fixed spacetime background

In the present section we investigate the charge structure of superselection sectors in a fixed spacetime background $M \in \text{Loc}$. We will start by noticing that for a state $\omega \in S_o(M)$ satisfying punctured Haag duality, the corresponding category has a tensor product, a permutation symmetry, and the objects with a finite statistics have conjugates. Afterwards, we will show that all the constructions can be coherently extended to the categories $Z^1_t(\sigma, K^d(M))$ for any $\sigma \in S_o(M)$. We conclude this section by studying the behaviour of these categories under restriction to subregions of $M$.

5.1.1 A preferred reference states

As a starting point of our investigation we apply to our framework the results of the analysis [51] of sharply localized sectors on a fixed spacetime background $M$, carried out in the Haag-Kastler framework.

Given $M \in \text{Loc}$, let $\omega \in S_o(M)$. Consider the category $Z^1_t(\omega, K^d(M))$, and for any tuple $z, z_1, z_2, z_3 \in Z^1_t(\omega, K^d(M))$ and $t \in (z, z_1)$, $s \in (z_2, z_3)$, define

$$
(z \otimes z_1)(b) \equiv z(b) \cdot \text{ad}_{z(p)}(z_1(b)), \quad b \in \Sigma_1(K^d(M)),
$$

$$
(t \otimes s)_a \equiv t_a \cdot \text{ad}_{z(q)}(s_a), \quad a \in \Sigma_0(K^d(M)),
$$

(23)

where $p$ is a path with $\partial_1 p \perp |b|$ and $\partial_0 p = \partial_1 b$; $q$ is a path with $\partial_1 q \perp a$ and $\partial_0 q = a$; $\text{ad}_{z(p)}$ denotes the adjoint action. Since the elements of $K^d(M)$ have arcwise connected causal complements, path-independence of
1-cocycles implies that these definitions do not depend on the choice of the paths $p$ and $q$. Now, given $z, z_1 \in \mathcal{Z}_1^t(\omega, \mathcal{K}^d(M))$, for any 0-simplex $a$ define
\[
\varepsilon_\omega(z, z_1)_a \equiv z_1(b)^* \cdot \text{ad}_{z_1(p)}(z(b_1)^*) \cdot z(b_1) \cdot \text{ad}_{z(p_1)}(z_1(b)) ,
\] (24)
where $b_1, b$ are 1-simplices such that $\partial_0 b_1 \perp \partial_0 b$ and $\partial_1 b_1 = \partial_1 b = a$; $p$ is a path from the causal complement of $|b_1|$ to $\partial_1 b$; $p_1$ is a path from the causal complement of $|b|$ to $\partial_0 b_1$. This expression is independent of the choices of $b_1, b, p, p_1$. We will refer to the pair $(\otimes_\omega, \varepsilon_\omega)$ as the tensor structure of $\mathcal{Z}_1^t(\omega, \mathcal{K}^d(M))$, although, up until now, we cannot affirm either that $\otimes_\omega$ is a tensor product or that $\varepsilon_\omega$ is a permutation symmetry. One can easily see, for instance, that $z \otimes_\omega z_1$ satisfies the 1-cocycle identity, but it is not clear if it satisfies the locality condition nor if it is path-independent. However, for a particular choice of $\omega$ we have the following result

**Theorem 5.3 ([51]).** Let $\omega \in \mathcal{S}_0(M)$ satisfy punctured Haag duality. Then relations (23) and (24) define, respectively, a tensor product $\otimes_\omega$ and a permutation symmetry $\varepsilon_\omega$ of $\mathcal{Z}_1^t(\omega, \mathcal{K}^d(M))$; the category has left inverses and a notion of statistics of objects; the objects with finite statistics have conjugates.

By using this result, in the next two sections we will be able to prove that $(\otimes_\omega, \varepsilon_\omega)$ define a tensor product and a permutation symmetry of $\mathcal{Z}_1^t(\omega, \mathcal{K}^d(M))$ for any $\sigma \in \mathcal{S}_0(M)$. Thus, the existence, for any $M \in \text{Loc}$, of at least one state $\omega \in \mathcal{S}_0(M)$ satisfying punctured Haag duality is a corner stone for our analysis.

**Remark 5.4.** It is worth observing that it is possible to choose the paths $b, b_1, p, p_1$ involved in the definition (24) in such a way that all their supports are contained in a unique diamond $\mathcal{O}$. In particular it is possible to replace the paths $p$ and $p_1$ by two 1-simplices $\tilde{b}$ and $\tilde{b}_1$ which have the same endpoints of $p$ and $p_1$, respectively, and whose support is $\mathcal{O}$. This is an easy consequence of the definition of diamonds and of the property (5). The same holds true for the paths involved in the definition (23).

### 5.1.2 Independence of the choice of states I

The aim of this section is to show that for any pair $\sigma, \omega \in \mathcal{S}_0(M)$ the corresponding categories $\mathcal{Z}_1^t(\omega, \mathcal{K}^d(M))$ and $\mathcal{Z}_1^t(\sigma, \mathcal{K}^d(M))$ are $^*$–isomorphic. Let
\[
\rho_{\omega, \sigma} : \omega^* \mathcal{A}^{kh}(M) \longrightarrow \sigma^* \mathcal{A}^{kh}(M),
\]
be the net isomorphism defined in Section 3.2 (see Lemma 3.3). We stress here that despite considering the categories associated with the set $\mathcal{K}^d(M)$, the fact that $\rho_{\omega,\sigma}$ is a net isomorphism of the nets indexed by $\mathcal{K}^h(M)$ is of a crucial importance when stating the claimed isomorphism (see proof of the Lemma 5.5). Now, for any pair $z, z_1 \in \mathcal{Z}^1_t(\omega,\mathcal{K}^d(M))$ and $t \in (z, z_1)$ define

\[
\mathcal{F}_{\omega,\sigma}(z)(b) \equiv \rho_{|b|}(z(b)) , \quad b \in \Sigma_1(\mathcal{K}^d(M)) , \\
\mathcal{F}_{\omega,\sigma}(t)_a \equiv \rho_a(t_a) , \quad a \in \Sigma_0(\mathcal{K}^d(M)) .
\]

Clearly, $\mathcal{F}_{\omega,\sigma}(z)(b) \in A_\sigma(|b|)$ and $\mathcal{F}_{\omega,\sigma}(t)_a \in A_\sigma(a)$. For any $c \in \Sigma_2(\mathcal{K}^d(M))$ we have that

\[
\mathcal{F}_{\omega,\sigma}(z)(\partial_0 c)\cdot\mathcal{F}_{\omega,\sigma}(z)(\partial_2 c) = \\
= \rho_{|\partial_0 c|}(z(\partial_0 c)) \cdot \rho_{|\partial_2 c|}(z(\partial_2 c)) = \rho_{|c|}(z(\partial_0 c)) \cdot \rho_{|c|}(z(\partial_2 c)) \\
= \rho_{|c|}(z(\partial_0 c)) - \rho_{|c|}(z(\partial_2 c)) = \rho_{|\partial_1 c|}(z(\partial_1 c)) \\
= \mathcal{F}_{\omega,\sigma}(z)(\partial_1 c) .
\]

Hence $\mathcal{F}_{\omega,\sigma}(z)$ is a 1-cocycle of $\mathcal{K}^d(M)$. Observe that if $M$ is simply connected, then, by Theorem 4.9 $\mathcal{F}_{\omega,\sigma}(z)$ is a path-independent 1-cocycle. For the general case we have the following

**Lemma 5.5.** For any $M \in \mathbf{Loc}$ then $\mathcal{F}_{\omega,\sigma}(z)$ is a path-independent 1-cocycle of $\mathcal{Z}^1_t(\sigma,\mathcal{K}^d(M))$.

**Proof.** Let $p$ be a closed path of $\mathcal{K}^d(M)$ with endpoint $a_0$. By Lemma 4.4 there exists a closed path $q$ of $\mathcal{K}^d(M)$, with endpoint $a_0$, and an element $\partial \in \mathcal{K}^h(M)$ such that $p \sim q$ and $|q| \subseteq \partial$. Assume $q = \{b_n, \ldots, b_1\}$. By homotopic invariance of 1-cocycles we have

\[
\mathcal{F}_{\omega,\sigma}(z)(p) = \mathcal{F}_{\omega,\sigma}(z)(q) = \rho_{|b_n|}(z(b_n)) \cdots \rho_{|b_1|}(z(b_1)) \\
= \rho_\partial(z(b_n)) \cdots \rho_\partial(z(b_1)) = \rho_\partial(z(b_n)) \cdots z(b_1) \\
= \rho_\partial(z(q)) = \rho_\partial(1) = 1 ,
\]

where the path-independence of $z$ has been used. \qed

It is clear that $\mathcal{F}_{\omega,\sigma}(t) \in (\mathcal{F}_{\omega,\sigma}(z), \mathcal{F}_{\omega,\sigma}(z_1))$. Moreover, since $\rho_{\omega,\sigma}$ is net isomorphism, $\mathcal{F}_{\omega,\sigma}$ is an isomorphism. Indeed, given the functor $\mathcal{F}_{\sigma,\omega}$ which is associated with the net isomorphism $\rho_{\sigma,\omega}$, one can easily see that

\[
\mathcal{F}_{\sigma,\omega} \circ \mathcal{F}_{\omega,\sigma} = \text{id}_{\mathcal{Z}^1_t(\omega,\mathcal{K}^d(M))} , \quad \mathcal{F}_{\omega,\sigma} \circ \mathcal{F}_{\sigma,\omega} = \text{id}_{\mathcal{Z}^1_t(\sigma,\mathcal{K}^d(M))} .
\]

In conclusion we have the following
Proposition 5.6. \( \mathcal{F}_{\omega,\sigma} : \mathcal{Z}^1_1(\omega, \mathcal{K}^d(M)) \rightarrow \mathcal{Z}^1_1(\sigma, \mathcal{K}^d(M)) \) is an isomorphism of \( C^* \)-categories for any pair \( \omega, \sigma \in \mathcal{I}_0(M) \).

We will refer to the functor \( \mathcal{F}_{\omega,\sigma} \) as the flip functor.

5.1.3 Independence of the choice of states II

The aim of this section is to show that the superselection sectors of the category \( \mathcal{Z}^1_1(\sigma, \mathcal{K}^d(M)) \), for any \( \sigma \in \mathcal{I}_0(M) \), have a charge structure, and that all these categories carry the same physical information. In other words we want to show that for any \( \sigma \in \mathcal{I}_0(M) \) the category \( \mathcal{Z}^1_1(\sigma, \mathcal{K}^d(M)) \) has a tensor product, a permutation symmetry, and that the objects with finite statistics have conjugates; furthermore, that all these categories are symmetric tensor \( \ast \)-isomorphic. To this end, we will first note that, thanks to the flip functor and Theorem 5.3, the category associated with a state satisfying punctured Haag duality, induces a tensor structure on \( \mathcal{Z}^1_1(\sigma, \mathcal{K}^d(M)) \) for any \( \sigma \in \mathcal{I}_0(M) \). Secondly, we will show that the induced structure coincide with the ambient one, defined by (23) and (24). This, in turn, will entail that the flip functor is a symmetric tensor \( \ast \)-isomorphism.

Consider a state \( \omega \in \mathcal{I}_0(M) \) satisfying punctured Haag duality. Recall that by Theorem 5.3, \( \otimes_\omega \) and \( \varepsilon_\omega \) are, respectively, a tensor product and a permutation symmetry of \( \mathcal{Z}^1_1(\omega, \mathcal{K}^d(M)) \). Now, given \( \sigma \in \mathcal{I}_0(M) \), for any \( z,z \in \mathcal{Z}^1_1(\sigma, \mathcal{K}^d(M)) \) and for any pair of arrows \( t,s \) of the category \( \mathcal{Z}^1_1(\sigma, \mathcal{K}^d(M)) \), define

\[
\otimes_\sigma \equiv \mathcal{F}_{\omega,\sigma}(\mathcal{F}_{\sigma,\omega}(z) \otimes_\omega \mathcal{F}_{\sigma,\omega}(z_1)) \quad \otimes_\sigma \quad (26)
\]

and

\[
\varepsilon_\sigma(z,z_1) \equiv \mathcal{F}_{\omega,\sigma}(\varepsilon_\omega(\mathcal{F}_{\sigma,\omega}(z), \mathcal{F}_{\sigma,\omega}(z_1))) \quad (27)
\]

Since the flip functor is an \( \ast \)-isomorphism the above formulas define, respectively, a tensor product and a permutation symmetry of \( \mathcal{Z}^1_1(\sigma, \mathcal{K}^d(M)) \). We will refer to the pair \( (\otimes_\sigma, \varepsilon_\sigma) \) as the tensor structure of \( \mathcal{Z}^1_1(\sigma, \mathcal{K}^d(M)) \) induced by \( \omega \).

Lemma 5.7. Let \( \omega \in \mathcal{I}_0(M) \) satisfy punctured Haag duality. Then, the following assertions hold for any \( \sigma \in \mathcal{I}_0(M) \).

(a) \( \otimes_\sigma = \otimes_\sigma \) and \( \varepsilon_\sigma = \varepsilon_\sigma \);  
(b) The pair \( (\otimes_\sigma, \varepsilon_\sigma) \) define a tensor product and a permutation symmetry of the category \( \mathcal{Z}^1_1(\sigma, \mathcal{K}^d(M)) \).
Proof. (a) Given a 1-simplex $b \in \Sigma_1(\mathcal{X}^d(M))$, by Remark 3.4 there is a 1-simplex $b_1$ and a diamond $\emptyset$ such that $\partial_0 b = \partial_1 b$, $\partial_0 b_1 \perp |b|$, and $|b|, |b_1| \subseteq \emptyset$. Now recalling the definition (23), we have that

\[
\mathcal{F}_{\sigma,\omega}(z) \otimes_{\omega} \mathcal{F}_{\sigma,\omega}(z_1)(b) = \mathcal{F}_{\sigma,\omega}(z)(b) \cdot \text{ad}_{\mathcal{F}_{\sigma,\omega}(z)(b)}(\mathcal{F}_{\sigma,\omega}(z_1)(b)) \\
= \rho_{|b|}(z(b)) \cdot \text{ad}_{\rho_{|b|}(z_1(b))}((\rho_{|b|}(z_1(b))) \\
= \rho_{\emptyset}(z(b)) \cdot \text{ad}_{\rho_{\emptyset}(z_1(b))}(\rho_{\emptyset}(z_1(b)) \\
= \rho_{\emptyset}(z(b) \cdot \text{ad}_{z_1(b)}(z_1(b))) .
\]

Using this equation and the fact that $\mathcal{F}_{\sigma,\omega}(z) \otimes_{\omega} \mathcal{F}_{\sigma,\omega}(z_1)(b) \in \mathcal{A}_\omega(|b|)$, we have that

\[
(z \otimes_{\sigma}^\omega z_1)(b) = \\
\mathcal{F}_{\sigma,\omega}(z)(b) \cdot \text{ad}_{\mathcal{F}_{\sigma,\omega}(z)(b)}(\mathcal{F}_{\sigma,\omega}(z_1)(b)) \\
= \rho_{|b|}(z(b)) \cdot \text{ad}_{\rho_{|b|}(z_1(b))}((\rho_{|b|}(z_1(b))) \\
= \rho_{\emptyset}(z(b)) \cdot \text{ad}_{\rho_{\emptyset}(z_1(b))}(\rho_{\emptyset}(z_1(b)) \\
= \rho_{\emptyset}(z(b) \cdot \text{ad}_{z_1(b)}(z_1(b))).
\]

The same argument leads to the identity $t \otimes_{\sigma}^\omega s = t \otimes_{\sigma} s$ for any pair of arrows $t, s$ of $\mathcal{Z}_1^\omega(\sigma, \mathcal{X}^d(M))$. So we have $\otimes_{\sigma}^\omega = \otimes_{\sigma}$. We now apply a similar argument to the permutation symmetry $\varepsilon^\omega_{\sigma}$ (see the definition (24)). Consider a 0-simplex $a$. By the Remark 3.4 there are four 1-simplices $b, b_1, \bar{b}, \bar{b}_1$ which fulfil, with respect to $a$, the properties of the definition (24), and such that all the supports $|b|, |b_1|, |\bar{b}|, |\bar{b}_1|$ are contained in a diamond $\emptyset$. So

\[
\varepsilon^\omega_{\sigma}(\mathcal{F}_{\sigma,\omega}(z), \mathcal{F}_{\sigma,\omega}(z_1))_a = \\
= \mathcal{F}_{\sigma,\omega}(z_1)(b_1)^* \cdot \text{ad}_{\mathcal{F}_{\sigma,\omega}(z_1)(\bar{b}_1)}(\mathcal{F}_{\sigma,\omega}(z)(b)^*) \cdot \\
\cdot \mathcal{F}_{\sigma,\omega}(z)(b) \cdot \text{ad}_{\mathcal{F}_{\sigma,\omega}(z)(\bar{b})}(\mathcal{F}_{\sigma,\omega}(z_1)(b_1)) \\
= \rho_{|b_1|}(z_1(b_1)^*) \cdot \text{ad}_{\rho_{|b_1|}(z_1(\bar{b}_1))}(\rho_{|b_1|}(z(b)^*)) \cdot \\
\cdot \rho_{|\bar{b}|}(z(b)) \cdot \text{ad}_{\rho_{|\bar{b}|}(z(\bar{b}))}(\rho_{|\bar{b}_1|}(z_1(b_1)) \\
= \rho_{\emptyset}(z_1(b_1)^* \cdot \text{ad}_{z_1(\bar{b}_1)}(z(b)^*)) \cdot \rho_{\emptyset}(z(b) \cdot \text{ad}_{z_1(b)}(z_1(b_1)) \\
= \rho_{\emptyset}(z_1(b_1)^* \cdot \text{ad}_{z_1(\bar{b}_1)}(z(b)^*) \cdot z(b) \cdot \text{ad}_{z_1(b)}(z_1(b_1)) \\
= \rho_{\emptyset}(\varepsilon_{\sigma}(z, z_1))_a .
\]

Using this equation and the fact that $\varepsilon^\omega_{\sigma}(\mathcal{F}_{\sigma,\omega}(z), \mathcal{F}_{\sigma,\omega}(z_1))_a \in \mathcal{A}_\omega(a)$ we
have

\[ \varepsilon_\omega^\omega(z, z_1)_a = F_{\omega, \sigma}(\varepsilon_\omega(F_{\sigma, \omega}(z), F_{\sigma, \omega}(z_1)))_a = \rho_\alpha^{-1}(\varepsilon_\omega(F_{\sigma, \omega}(z), F_{\sigma, \omega}(z_1)))_a \]
\[ = \rho_\beta^{-1}(\varepsilon_\omega(F_{\sigma, \omega}(z), F_{\sigma, \omega}(z_1)))_a = \rho_\beta^{-1}(\rho_\beta(\varepsilon_\omega(z, z_1)))_a \]
\[ = \varepsilon_\sigma(z, z_1)_a , \]

which completes the proof. (b) follows from (a) \qed

The following relations are a consequence of the previous lemma and of the invertibility of the flip functor. Consider \( \omega, \sigma \in \mathcal{O}(M) \) such that \( \omega \) satisfies punctured Haag duality. For any \( z, z_1 \in \mathcal{Z}^1_1(\sigma, \mathcal{K}^d(M)) \) and for any pair \( t, s \) of arrows of \( \mathcal{Z}^1_1(\sigma, \mathcal{K}^d(M)) \), we have

\[ F_{\sigma, \omega}(z) \otimes_\omega F_{\sigma, \omega}(z_1) = F_{\sigma, \omega}(z \otimes_\omega z_1) , \]
\[ F_{\sigma, \omega}(t) \otimes_\omega F_{\sigma, \omega}(s) = F_{\sigma, \omega}(t \otimes_\omega s) , \] (28)

and

\[ \varepsilon_\omega(F_{\sigma, \omega}(z), F_{\sigma, \omega}(z_1)) = F_{\sigma, \omega}(\varepsilon_\omega(z, z_1)) . \] (29)

In conclusion we have the following

**Theorem 5.8.** For any \( \omega \in \mathcal{O}(M) \) the category \( \mathcal{Z}^1_1(\omega, \mathcal{K}^d(M)) \) equipped, with \( \otimes_\omega \) and \( \varepsilon_\omega \), is a symmetric tensor \( C^* \)-category with left inverses. Any object with finite statistics have conjugates. For any \( \sigma \in \mathcal{O}(M) \),

\[ F_{\omega, \sigma} : \mathcal{Z}^1_1(\omega, \mathcal{K}^d(M)) \to \mathcal{Z}^1_1(\sigma, \mathcal{K}^d(M)) \]

is a covariant symmetric tensor \( \ast \)-isomorphism.

**Proof.** Consider a pair of states \( \omega, \sigma \in \mathcal{O}(M) \). In order to prove that \( F_{\omega, \sigma} : \mathcal{Z}^1_1(\omega, \mathcal{K}^d(M)) \to \mathcal{Z}^1_1(\sigma, \mathcal{K}^d(M)) \) is a symmetric tensor \( \ast \)-isomorphism, we consider a state \( \omega_0 \in \mathcal{O}(M) \) satisfying punctured Haag duality and use the fact that the tensor structures of \( \mathcal{Z}^1_1(\omega, \mathcal{K}^d(M)) \) and \( \mathcal{Z}^1_1(\sigma, \mathcal{K}^d(M)) \) are equal to the tensor structure induced by \( \omega_0 \). Given \( z, z_1 \in \mathcal{Z}^1_1(\omega, \mathcal{K}^d(M)) \), we have

\[ F_{\omega, \sigma}(z \otimes_\omega z_1) = F_{\omega, \sigma}(\varepsilon_\omega^\omega(z \otimes_\omega^\omega z_1)) \]
\[ = F_{\omega, \sigma} \circ F_{\omega_0, \sigma}(F_{\omega_0, \sigma}(z) \otimes_{\omega_0, \sigma} F_{\omega_0, \sigma}(z_1)) \]
\[ = F_{\omega_0, \sigma}(F_{\omega_0, \sigma}(z) \otimes_{\omega_0, \sigma} F_{\omega_0, \sigma}(z_1)) \]
\[ = F_{\omega_0, \sigma}(F_{\omega_0, \sigma}(z) \otimes_{\omega_0, \sigma} F_{\omega_0, \sigma}(z_1)) \]
\[ = F_{\omega_0, \sigma}(z) \otimes_{\omega_0, \sigma} F_{\omega_0, \sigma}(z_1) \] by (28)
\[ = F_{\omega, \sigma}(z) \otimes_\sigma F_{\omega, \sigma}(z_1) . \]
The same reasoning applied to arrows and to the permutation symmetry shows that the flip functor is a covariant symmetric tensor \(\ast\)–isomorphism. As the category associated with a state satisfying punctured Haag duality, has left inverses and the object with finite statistics have conjugates, the same holds for the category associated with any state of \(\mathcal{I}_o(M)\). This is because the flip functor is a covariant symmetric tensor \(\ast\)–isomorphism (see in Appendix).

5.1.4 Restriction to subregions

Let \(N \subset M\) be an open arcwise connected subset of \(M\), such that for any pair \(x_1, x_2 \in N\) then \(J^+(x_1) \cap J^-(x_2)\) is contained in \(N\). This property says that \(N\) is a globally hyperbolic spacetime. As \(N\) is isometrically embedded in \(M\) and as diamonds are stable under isometric embeddings (Lemma 2.18) we have

\[
\mathcal{K}^d(M) |_{N} \equiv \{ \mathcal{O} \in \mathcal{K}^d(M) \mid \overline{O} \subset N \} = \mathcal{K}^d(N).
\]

Let \(\mathcal{A}_{\mathcal{K}^d(M)}\) be the net of local algebras indexed by \(\mathcal{K}^d(N)\), obtained by restricting \(\mathcal{A}_{\mathcal{K}^d(M)}\) to \(\mathcal{K}^d(N)\). Let \(\omega \in \mathcal{I}_o(M)\), \(\omega^*\mathcal{A}_{\mathcal{K}^d(M)}\) inherits from \(\omega^*\mathcal{A}_{\mathcal{K}^d(M)}\) the Borchers property and local definiteness, as follows from Lemma 3.1. However, it needs not be irreducible. Let \(Z^1_\omega(\omega, \mathcal{K}^d(N))\) be the category of path-independent 1-cocycles of \(\mathcal{K}^d(N)\) with values in \(\omega^*\mathcal{A}_{\mathcal{K}^d(N)}\). This is a \(\mathcal{C}^\ast\)–category closed under direct sums and subobjects, and, by local definiteness, the identity cocycle is irreducible. The aim now is to show that the restriction functor \(\mathcal{R} : Z^1_\omega(\omega, \mathcal{K}^d(M)) \rightarrow Z^1_\omega(\omega, \mathcal{K}^d(N))\), defined by (19), is a full and faithful covariant \(\ast\)–functor.

Now, let \(\omega \in \mathcal{I}_o(M)\) and recall that the restriction functor (19) is defined, for any \(z, z_1 \in Z^1_\omega(\omega, \mathcal{K}^d(M))\) and \(t \in \langle z, z_1 \rangle\), by

\[
\mathcal{R}(z)(b) \equiv z(b), \quad b \in \Sigma_1(\mathcal{K}^d(N)),
\]

\[
\mathcal{R}(t)(a) \equiv t_a, \quad a \in \Sigma_0(\mathcal{K}^d(N)).
\]

\(\mathcal{R}\) is a covariant \(\ast\)–functor from \(Z^1_\omega(\omega, \mathcal{K}^d(M))\) into \(Z^1_\omega(\omega, \mathcal{K}^d(N))\). Note that if we take \(\sigma \in \mathcal{I}_o(M)\), then it can be easily shown that the following diagram is commutative

\[
\begin{array}{ccc}
Z^1_\omega(\omega, \mathcal{K}^d(M)) & \xrightarrow{\mathcal{R}_{\omega,\sigma}} & Z^1_\omega(\sigma, \mathcal{K}^d(M)) \\
\mathcal{R} \downarrow & & \mathcal{R} \\
Z^1_\omega(\omega, \mathcal{K}^d(N)) & \xrightarrow{\mathcal{R}_{\omega,\sigma}} & Z^1_\omega(\sigma, \mathcal{K}^d(N))
\end{array}
\]
Therefore if we prove that $\mathcal{R}$ is full an faithful for a particular choice of $\omega$ then it would be full and faithful for any other element $\sigma \in \mathcal{F}_0(M)$.

**Theorem 5.9.** $\mathcal{R}$ is a full and faithful $^*$-functor.

**Proof.** In the first part of the proof we follow [IS Theorem 30.2] As observed above it is enough to prove the assertion when $\omega$ satisfies punctured Haag duality. Given $z, z_1 \in Z^1_\mathcal{R}(\omega, \mathcal{K}^d(M))$, let $t \in (\mathcal{R}(z), \mathcal{R}(z_1))$. This means that

$$t_{\partial b} \cdot z(b) = z_1(b) \cdot t_{\partial b} \quad \forall b \in \Sigma_1(\mathcal{K}^d(N)) .$$

We want prove that there exists $\hat{t} \in (z, z_1)$ such that $\hat{t}_a = t_a$ whenever $a \in \Sigma_0(\mathcal{K}^d(M))$. Fix $a_0 \in \mathcal{K}^d(N)$, define

$$\hat{t}_a \equiv z_1(p_a) \cdot t_{a_0} \cdot z(p_a)^* \quad a \in \Sigma_0(\mathcal{K}^d(M)) ,$$

where $p_a$ is a path in $\mathcal{K}^d(M)$ from $a_0$ to $a$. This definition does not depend on the chosen $a_0 \in \Sigma_0(\mathcal{K}^d(M))$ nor on the chosen path $p_a$. Moreover $\hat{t}_{\partial b} \cdot z(b) = z_1(b) \cdot \hat{t}_{\partial b}$ for any $b \in \Sigma_1(\mathcal{K}^d(M))$, and

$$\hat{t}_a = t_a \quad a \in \Sigma_0(\mathcal{K}^d(N)) .$$

What remains to be shown is that $\hat{t}$ satisfies the locality condition, i.e. $\hat{t}_a \in \mathcal{A}_\omega(a)$ for any $a \in \Sigma_0(\mathcal{K}^d(M))$. From now on the proof is very similar to the proof of [51 Proposition 4.19]. Let $x_0 \in N$, we prove that $\hat{t}_a \in \mathcal{A}_\omega(a)$ for any $a \in \Sigma_0(\mathcal{K}^d(M))$ whose closure $cl(a)$ is causally disjoint from $\{x_0\}$. Fix a 0-simplex $a_1$ of $\mathcal{K}^d(M)$ to be such that $cl(a_1) \perp \{x_0\}$ and $a_1 \perp a$. Recall that the definition of $\hat{t}$ does not depend both on the choice of $a_0$ and on the choice of the path. First observe that we can always find $a_0 \in \Sigma_0(\mathcal{K}^d(N))$ such that $a_0 \perp a_1$ and $cl(a_0) \perp \{x_0\}$. Furthermore, since the causal complement of $a_1$ is arcwise connected, there is a path $p_a$ which lies in the causal complement of $a_1$. Therefore:

$$\hat{t}_a \cdot A = z_1(p_a) \cdot t_{a_0} \cdot z(p_a)^* \cdot A = z_1(p_a) \cdot t_{a_0} \cdot A \cdot z(p_a)^* = A \cdot \hat{t}_a,$$

for any $A \in \mathcal{A}_\omega(a_1)$. Hence $\hat{t}_a \in \mathcal{A}_\omega(a_1)'$ for any $a_1 \perp a$ and $cl(a_1) \perp x$. By punctured Haag duality $\hat{t}_a \in \mathcal{A}_\omega(a)$. Thus, we have shown that $\hat{t}_a \in \mathcal{A}_\omega(a)$ for any 0-simplex $a$ such that $cl(a) \perp \{x_0\}$. By [51 Proposition 4.19] the proof follows. \[\square\]

Two comments about Theorem 5.9 are in order.

1. This is a key result for our aims. It will entail that the embedding of a sector into a different spacetime preserves the statistical properties (see
Remark 5.16, this being the genesis of the local covariance of gauge groups (see Section 5.3.3).

(2) Theorem 5.9 is nothing but the cohomological version of the “equivalence between local and global intertwiners,” a property that the superselection sectors which are preserved in the scaling limit fulfills [20] (see also [45]). We emphasize that in the present paper this equivalence arises as a natural consequence of punctured Haag duality.

(3) It can be easily shown that the restriction functor \( R \) is a symmetric tensor functor. The proof is contained, implicitly, in the proof of Proposition 5.12.

5.2 Net cohomology and sharply localized sectors

The purpose of this section is to provide the interpretation of our definition of superselection sectors in terms of representations of the net of local observables which are sharply localized with respect to a reference representation.

A representation \( \pi \) on a Hilbert space \( \mathcal{H}_\pi \) of the net \( \mathcal{A}_{\mathcal{K}^d(M)} \) is a collection \( \{ \pi_O \}_{O \in \mathcal{K}^d(M)} \) of representations \( \pi_O \) of the algebras \( \mathcal{A}(O) \) on \( \mathcal{H}_\pi \), which is compatible with the net structure, i.e. \( \pi_O \big|_{\mathcal{A}(O)} = \pi_O \) if \( O \subseteq O_1 \). Given \( \omega \in \mathcal{S}_\sigma(M) \) let \( \pi_\omega \) be the GNS representation of the algebra \( \mathcal{A}(M) \), on the Hilbert space \( \mathcal{H}_\omega \), which is associated with \( \omega \). A representation \( \pi = \{ \pi_O \}_{O \in \mathcal{K}^d(M)} \) is a sharp excitation of \( \pi_\omega \) if there exists a family \( \{ V_O \}_{O \in \mathcal{K}^d(M)} \) of unitary operators from \( \mathcal{H}_\pi \) onto \( \mathcal{H}_\omega \) such that

\[
V_{O_1} \pi_O(A) = \pi_\omega(A) V_{O_1}, \quad A \in \mathcal{A}(O), \ O \perp O_1. \tag{30}
\]

Let \( (\pi, \pi_1) \) be the collection of linear bounded operators \( T : \mathcal{H}_\pi \to \mathcal{H}_{\pi_1} \) such that \( T \pi_O(A) = \pi_1 O(A) T \) for \( A \in \mathcal{A}(O) \) and for \( O \in \mathcal{K}^d(M) \). We denote by \( \text{Rep}(\omega) \) the category whose objects are those representations of \( \mathcal{A}_{\mathcal{K}^d(M)} \) which are a local excitation of \( \pi_\omega \) and with arrows the corresponding intertwiner operators.

**Proposition 5.10.** For any \( \sigma \in \mathcal{S}_\sigma(M) \), the category \( \mathcal{Z}_1^1(\sigma, \mathcal{K}^d(M)) \) is equivalent to the category \( \text{Rep}(\omega) \) for any \( \omega \in \mathcal{S}_\sigma(M) \) satisfying punctured Haag duality.

**Proof.** We first prove that \( \mathcal{Z}_1^1(\omega, \mathcal{K}^d(M)) \) is equivalent to the category \( \text{Rep}(\omega) \) for a state \( \omega \) as in the statement. We give only a sketch of the the proof because is very similar to the proof of [22, Lemma 3A.6]. Consider \( z \in \)}
\[ Z^1_t(\omega, \mathcal{K}^d(M)). \] Fix \( \mathcal{O}_1 \in \mathcal{K}^d(M) \) and define

\[ \pi^z_{\mathcal{O}}(A) \equiv z(p_{\mathcal{O}_1}) \cdot \pi_{\omega}(A) \cdot z(p_{\mathcal{O}_1})^* \quad A \in \mathcal{A}(\mathcal{O}), \]

where \( p_{\mathcal{O}_1} \) is a path with \( \partial_0 p_{\mathcal{O}_1} = \mathcal{O}_1 \) and \( \partial_1 p_{\mathcal{O}_1} \perp \mathcal{O} \). This definition is well posed as diamonds have arcwise connected causal complements, and it can be easily shown that \( \pi^z \) is a representation of \( \mathcal{A}_{\mathcal{K}^d(M)} \). For any \( \mathcal{O} \) let \( q_{\mathcal{O}} \) be a path from \( \mathcal{O}_1 \) to \( \mathcal{O} \), and let \( V_{\mathcal{O}} \equiv z(q_{\mathcal{O}}) \). One can easily check that \( V_{\mathcal{O}} \pi_{\mathcal{O}_2}(A) = \pi_{\omega}(A) V_{\mathcal{O}}, \) for any \( A \in \mathcal{A}(\mathcal{O}_2) \) with \( \mathcal{O}_2 \perp \mathcal{O} \). Hence \( \pi^z \in \text{Rep}(\omega) \).

Conversely, given \( \pi \in \text{Rep}(\omega) \), let \( V_{\mathcal{O}}, \mathcal{O} \in \mathcal{K}^d(M) \), be the collection of unitary operators associated with \( \pi \) by (30). Define

\[ z^\pi(b) \equiv V_{\partial_b} \cdot V_{\partial_b}^* \quad b \in \Sigma_1(\mathcal{K}^d(M)). \]

\( z^\pi \) satisfies the 1-cocycle identity and is path-independent. Punctured Haag duality (hence, Haag duality) entails that \( z^\pi \) fulfills the locality condition, hence \( z^\pi \in Z^1_t(\omega, \mathcal{K}^d(M)) \). Following the cited reference, one arrives to the categorical equivalence between \( Z^1_t(\omega, \mathcal{K}^d(M)) \) and \( \text{Rep}(\omega) \). Now the proof follows from the Proposition 5.6.

It is now clear that, in a fixed spacetime background \( M \) all the categories \( Z^1_t(\sigma, \mathcal{K}^d(M)) \) carry the same physical information for any choice of \( \sigma \in \mathcal{J}_\sigma(M) \). Indeed, they are associated with representations of the net of local observables which are a sharp excitation of the representation \( \pi_{\omega} \) associated with a state \( \omega \in \mathcal{J}_\sigma(M) \) satisfying punctured Haag duality.

### 5.3 Locally covariant structure of sectors

We show how the locally covariant structure of superselection sectors arises. We introduce the embedding functor which gives a first important information on the covariant structure of sectors. Such a structure is encoded in the superselection functor analyzed in the subsequent section. Finally, by applying the Doplicher-Roberts duality theory of compact groups to the superselection functor, we investigate the locally covariant properties of the associated gauge groups.

#### 5.3.1 The Embedding functor

Consider \( M_1, M \in \text{Loc} \), with \( \psi \in (M_1, M) \), let \( \alpha_\psi : \mathcal{A}(M_1) \rightarrow \mathcal{A}(M) \) be the \( C^* \)-morphism associated with \( \psi \). Given \( \omega \in \mathcal{J}_\sigma(M) \), let \( \tau_{\psi}^\omega : \omega^* \mathcal{A}_\psi(\mathcal{K}^d(M_1)) \rightarrow (\omega \alpha_\psi)^* \mathcal{A}_\psi(\mathcal{K}^d(M_1)) \) the corresponding net-isomorphism introduced in Section 3.2.
Definition 5.11. Given \( \psi \in (M_1, M) \) and \( \omega \in \mathcal{S}(M) \). We call embedding the map of categories \( \mathcal{C}^\omega_\psi : \mathcal{Z}_1^1(\omega, \mathcal{K}^d(M)) \to \mathcal{Z}_1^1(\omega \alpha \psi, \mathcal{K}^d(M_1)) \) defined, for \( z, z_1 \in \mathcal{Z}_1^1(\omega, \mathcal{K}^d(M)) \) and \( t \in (z, z_1) \), as
\[
\mathcal{C}^\omega_\psi(z)(b) \equiv \tau^\omega_\psi(z(\psi(b))) \quad b \in \Sigma_1(\mathcal{K}^d(M_1)) , \\
\mathcal{C}^\omega_\psi(t)_a \equiv \tau^\omega_\psi(t_\psi(a)) \quad a \in \Sigma_0(\mathcal{K}^d(M_1)) .
\]

In this definition \( \psi(b) \) is the 1-simplex of \( \mathcal{K}^d(M) \) defined as
\[
|\psi(b)| = \psi(|b|) , \quad \partial_0 \psi(b) = \psi(\partial_0 b) , \quad \partial_1 \psi(b) = \psi(\partial_1 b) .
\]

Proposition 5.12. Given \( \psi \in (M_1, M) \) and \( \omega \in \mathcal{S}(M) \). The embedding
\[
\mathcal{C}^\omega_\psi : \mathcal{Z}_1^1(\omega, \mathcal{K}^d(M)) \to \mathcal{Z}_1^1(\omega \alpha \psi, \mathcal{K}^d(M_1))
\]
is a covariant symmetric tensor \(*\)-functor which is full and faithful.

Proof. That \( \mathcal{C}^\omega_\psi \) is a covariant \(*\)-functor is obvious from the fact that \( \tau^\omega_\psi \) is a net-isomorphism. Given \( z, z_1 \in \mathcal{Z}_1^1(\omega, \mathcal{K}^d(M)) \) and \( t \in (\mathcal{C}^\omega_\psi(z), \mathcal{C}^\omega_\psi(z_1)) \), define
\[
s_a \equiv (\tau^\omega_\psi)^{-1}(t_{\psi^{-1}(a)}) , \quad a \in \psi(\mathcal{K}^d(M_1)) ,
\]
and observe that \( s \in (\mathcal{R}(z), \mathcal{R}(z_1)) \) where \( \mathcal{R} \) is the restriction functor from \( \mathcal{Z}_1^1(\omega, \mathcal{K}^d(M)) \) into \( \mathcal{Z}_1^1(\omega, \mathcal{K}^d(\psi(M_1))) \). Since \( \mathcal{R} \) is full, there is \( t' \in (z, z_1) \) such that
\[
(\tau^\omega_\psi)^{-1}(t_{\psi^{-1}(a)}) = \mathcal{R}(t')_a = t'_a ,
\]
for any \( a \in \Sigma_0(\psi(\mathcal{K}^d(M_1))) \) which is equivalent to \( t_a = \tau^\omega_\psi(t'_{\psi(a)}) = \mathcal{C}^\omega_\psi(t'_a) \) for any \( a \in \Sigma_0(\mathcal{K}^d(M_1)) \). This proves that \( \mathcal{C}^\omega_\psi \) is full. If \( \mathcal{C}^\omega_\psi \) were not-faithful, there would be \( t_1, t_2 \in (z, z_1) \) such that \( \mathcal{C}^\omega_\psi(t_1) = \mathcal{C}^\omega_\psi(t_2) \). Therefore
\[
\tau^\omega_\psi(t_{1\psi(a)}) = \tau^\omega_\psi(t_{2\psi(a)}) \iff t_{1\psi(a)} = t_{2\psi(a)}
\]
for any \( a \in \Sigma_0(\mathcal{K}^d(M_1)) \). This, in turns, is equivalent to the identity
\[
\mathcal{R}(t_1)_a = \mathcal{R}(t_2)_a , \quad a \in \Sigma_0(\psi(\mathcal{K}^d(M_1))) ,
\]
and this leads to a contradiction because \( \mathcal{R} \) is faithful. What remains to be shown is that the embedding is a symmetric and tensor \(*\)-functor. To this end let \( z, z_1 \in \mathcal{Z}_1^1(\omega, \mathcal{K}^d(M)) \). Recalling the definition (23), for any \( b \in \Sigma_1(\mathcal{K}^d(M_1)) \) we have
\[
\mathcal{C}^\omega_\psi(z \otimes_\omega z_1)(b) = \tau^\omega_\psi(z \otimes_\omega z_1)(\psi(b)) = \tau^\omega_\psi(z(\psi(b)) \cdot \text{ad}_{z(b)}(z_1(\psi(b)))) = \tau^\omega_\psi(z(\psi(b))) \cdot \text{ad}_{\tau^\omega_\psi(z(b))}(\tau^\omega_\psi(z_1(\psi(b)))) = (\mathcal{C}^\omega_\psi(z) \otimes_{\omega \alpha \psi} \mathcal{C}^\omega_\psi(z_1))(b) ,
\]
where the fact that $\tau^\omega_\psi$ is a morphism of C*-algebras has been used (see Section 3.2). The same reasoning leads to the identity

$$\mathcal{E}^\omega_\psi(t \otimes s)_a = (\mathcal{E}^\omega_\psi(t) \otimes \omega_\psi, \mathcal{E}^\omega_\psi(s))_a, \quad a \in \Sigma_0(\mathcal{K}^d(M_1))$$

for any pair $t, s$ of arrows of $\mathcal{Z}^1_\psi(\omega, \mathcal{K}^d(M))$. Now, let $z, z_1 \in \mathcal{Z}^1_\psi(\omega, \mathcal{K}^d(M))$.

By recalling definition (24), for any $a \in \Sigma_0(\mathcal{K}^d(M_1))$ we have

$$\mathcal{E}^\omega_\psi(\varepsilon(z, z)_1)_a = \tau^\omega_\psi(z_1(b_1)^* \cdot \text{ad}_{z_1(\tilde{q})}(z(b))^*) \cdot \tau^\omega_\psi(z(b) \cdot \text{ad}_{z(\tilde{q})}(z_1(b_1))),$$

where $b_1, b$ are 1-simplices in $\mathcal{K}^d(M)$ such that $\partial_1 b = \partial_1 b_1 = \psi(a)$ and $\partial_0 b \perp \partial_0 b_1$. Let us consider the first term of the product. We can take in $\mathcal{K}^d(M_1)$ two 1-simplices $b'_1, b'$ of the following form:

$$\partial_1 b'_1 = \partial_1 b' = a, \quad \partial_0 b'_1 \perp \partial_0 b'.$$

so the 1-simplices in $\mathcal{K}^d(M)$ defined as $\psi(b'_1)$ and $\psi(b')$ fulfill the hypotheses of the definition of a permutation symmetry. Furthermore, let $q'$ be a path in $\mathcal{K}^d(M_1)$ such that $\partial_0 q' = \partial_1 b'_1$ and $\partial_1 q' \perp |b'|$. Then $\psi(q')$ has the same properties of $q'$. Therefore

$$\tau^\omega_\psi(z_1(b_1)^* \cdot \text{ad}_{z_1(\tilde{q})}(z(b))^*) =$$

$$\tau^\omega_\psi(z_1(\psi(b'_1))^* \cdot \text{ad}_{z_1(\psi(q'))}(z(\psi(b'_1))^*))$$

$$= \tau^\omega_\psi(z_1(\psi(b'_1)) \cdot \text{ad}_{\tau^\omega_\psi(z_1(\psi(q')))}(\tau^\omega_\psi(z(\psi(b'_1)))^*))$$

$$= \mathcal{E}^\omega_\psi(z_1(b'_1)^* \cdot \text{ad}_{\mathcal{E}^\omega_\psi(z_1(q'))}(\mathcal{E}^\omega_\psi(z(b')))^*.$$

Applying the same reasoning to the other term of the product we arrive at $\mathcal{E}^\omega_\psi(\varepsilon(z, z)_1)_a = \varepsilon(\mathcal{E}^\omega_\psi(z), \mathcal{E}^\omega_\psi(z_1)_a)$ for any $a \in \Sigma_0(\mathcal{K}^d(M_1))$. \hfill \square

**Lemma 5.13.** Given $\psi \in (M_1, M)$ let $\omega, \sigma \in \mathcal{S}(M)$. Then

$$\mathcal{E}^\omega_\psi \circ \mathcal{F}_{\sigma, \omega} = \mathcal{F}_{\sigma \cdot \omega, \omega} \circ \mathcal{E}^\sigma_\psi.$$

**Proof.** Given $z \in \mathcal{Z}^1_\psi(\sigma, \mathcal{K}^d(M))$ and $b \in \Sigma_1(\mathcal{K}^d(M_1))$, by Lemma 3.6 we have

$$\mathcal{F}_{\sigma \cdot \omega, \omega} \circ \mathcal{E}^\sigma_\psi(z)(b) = \rho_{\sigma \cdot \omega, \omega}(\tau^\sigma_\psi(z(\psi(b))))$$

$$= \tau^\omega_\psi(\rho_{\sigma, \omega}(z(\psi(b)))) = \mathcal{E}^\omega_\psi(\mathcal{F}_{\sigma, \omega}(z)(b)).$$

\hfill \square

44
5.3.2 The Superselection Functor

We now are in the position to show the covariant structure of superselection sectors. Let $\text{Sym}$ be the category whose objects are symmetric tensor $C^*$–categories, and whose arrows are the full and faithful, symmetric tensor $\ast$–functors. According to the philosophy of the locally covariant quantum field theories, we expect that the superselection sectors can exhibit a structure of functor from the category $\text{Loc}$ into the category $\text{Sym}$. We know that the superselection sectors of any spacetime $M \in \text{Loc}$ identify a family of categories within the same isomorphism class, any such category $Z^1_\omega(\omega, \mathcal{K}^d(M))$ is labeled by an element $\omega \in \mathcal{H}_\omega(M)$. Since there is no natural way to associate an element of this isomorphism class to the spacetime $M$, as $M$ varies in $\text{Loc}$, we will be forced to make a choice.

Given a locally covariant quantum field theory $\mathcal{A}$ and a reference state space $\mathcal{H}_\omega$, let

$$\omega \equiv \{\omega_M \in \mathcal{H}_\omega(M) \mid M \in \text{Loc}\} \quad (31)$$

be a choice of states.

We call the superselection functor associated with the choice $\omega$, the categories map $S_\omega : \text{Loc} \to \text{Sym}$ defined as

$$S_\omega(M) \equiv Z^1_\omega(\omega_M, \mathcal{K}^d(M)), \quad M \in \text{Loc},$$

$$S_\omega(\psi) \equiv F_{\omega_M \alpha \psi_M} \circ E_\omega^M, \quad \psi \in (M_1, M). \quad (32)$$

Theorem 5.14. $S_\omega : \text{Loc} \to \text{Sym}$ is a contravariant functor.

Proof. Let $\psi \in (M_1, M)$. Since $S_\omega(\psi)$ is defined as the composition of the flip and of the embedding functor, Proposition 5.12 and Theorem 5.8 imply that $S_\omega(\psi) : S_\omega(M) \to S_\omega(M_1)$ is a full, faithful symmetric tensor $\ast$–functor. Given $\phi \in (M_2, M_1)$, by Lemma 5.13 we have

$$S_\omega(\phi) \circ S_\omega(\psi) = \mathcal{F}_{\omega_M \alpha \psi_M} \circ E_{\phi}^{M_1} \circ \mathcal{F}_{\omega_M \alpha \psi_M} \circ E_{\psi}^M = \mathcal{F}_{\omega_M \alpha \psi_M} \circ E_{\phi}^{M_1} \circ \mathcal{F}_{\omega_M \alpha \psi_M} \circ E_{\psi}^M = \mathcal{F}_{\omega_M \alpha \psi_M} \circ E_{\psi \phi}^M = S_\omega(\psi \phi).$$

Finally, by the definitions of the flip and of the embedding functors we have

$$S_\omega(\text{id}_M) = \mathcal{F}_{\omega_M \omega_M} \circ E_{\text{id}_M}^M = \text{id}_{Z^1_\omega(\omega_M, \mathcal{K}^d(M))} = \text{id}_{S_\omega(M)},$$

and the proof is now completed. \qed
Proposition 5.15. If $\omega$ and $\sigma$ is a pair of choice of states, then the functors $S_\omega$ and $S_\sigma$ are isomorphic.

Proof. Define $u_{\omega,\sigma}(M) = F_{\omega,\sigma}M$, $M \in \text{Loc}$. By Theorem 5.8 it follows that $u_{\omega,\sigma}(M) : S_\omega(M) \to S_\sigma(M)$ is an symmetric tensor isomorphism. Given $\psi \in (M_1, M)$. By Lemma 5.13 we have that $u_{\omega,\sigma}(M_1) \circ S_\omega(\psi) = F_{\omega,\sigma}M_1 \circ F_{\omega,\sigma,\alpha,\psi,M_1} \circ E_{\psi} = F_{\sigma,\omega,\alpha,\psi,M_1} \circ E_{\psi} = S_\sigma(\psi) \circ u_{\omega,\sigma}(M)$.

Hence, $u_{\omega,\sigma} : S_\omega \to S_\sigma$ is a natural isomorphism. □

Remark 5.16. This theorem is the main result of this paper because it shows the covariance of charged superselection sectors: if $\psi \in (M_1, M)$, then to any sector of $M$ there correspond a unique sector of $M_1$ with the same charged quantum numbers. To be precise, let $z \in S_\omega(M)$ be an irreducible object with statistical parameter $\lambda([z]) = \chi([z]) \cdot d([z])$, where $[z]$ denotes the equivalence class of $z$. Let $\overline{\psi}$ be the conjugate of $\psi$ (see Appendix A). Then $S_\omega(\psi)(z)$ is an irreducible object of $S_\omega(M_1)$ such that

$$ [S_\omega(\psi)(z)] = S_\omega(\psi)([z]). $$

Furthermore $z$ and $S_\omega(\psi)(z)$ have the same statistics, i.e.

$$ \chi([z]) = \chi([S_\omega(\psi)(z)]), \quad d([z]) = d([S_\omega(\psi)(z)]). $$

Moreover

$$ S_\omega(\psi)([\overline{\psi}]) = [S_\omega(\psi)(z)], $$

that means that $S_\omega(\psi)([\overline{\psi}])$ is the conjugate sector of $S_\omega(\psi)(z)$.

5.3.3 The Gauge Weak Functor

The next natural step of the present investigation should be the application of Doplicher-Roberts reconstruction theorem \[25\] to the pair of functors $(\omega, S_\omega)$ in order to analyze the local covariance of fields and of the gauge groups underlying the theory. In the present section we will follow partially this investigation line by showing the covariant structure of gauge groups
associated with the superselection sectors. Because of some technical problems, that will be exposed below, we will not deal with the reconstruction of fields. Conversely, gauge groups can be easily reconstructed by using the Doplicher-Roberts duality theorem (hereafter, DR-theorem) for compact groups \[24\] (note that in the reconstruction theorem \[25\] the gauge group of fields is the dual of the category of superselection sectors).

Before beginning the analysis we need a preliminary observation on the DR-theorem whose functorial properties are described in some detail in the Appendix \[A\]. The DR-theorem states that any tensor $C^*$—category with a permutation symmetry, and conjugates, is the abstract dual of a compact group which is uniquely associated with the category only up to isomorphism. To any full and faithful, symmetric tensor $^*-$functor between categories there corresponds, in a contravariant fashion, a group morphism. Also in this case the correspondence between functors and group morphisms is not injective. These degrees of freedom will reflect in the weakening of the local covariance of the gauge groups (see below).

Consider a choice of states $\omega$. For any spacetime $M$, we choose a compact group $G_\omega(M)$ among the isomorphism class of compact groups associated with the full subcategory of $S_\omega(M)$ whose objects have conjugates. Furthermore, for any $\psi \in (M_1, M)$, we choose a group morphism

$$\alpha_\omega(\psi) : G_\omega(M_1) \to G_\omega(M),$$

among the set of group morphisms associated with the functor

$$S_\omega(\psi) : S_\omega(M) \to S_\omega(M_1).$$

Let $\text{Grp}$ be the category whose objects are compact groups and whose arrows are the corresponding set of group morphisms. Now, for any choice of states $\omega$ we denote by $\mathcal{G}_\omega$ the categories map $\mathcal{G}_\omega : \text{Loc} \to \text{Grp}$ defined as

$$\begin{cases} 
G_\omega(M) \equiv G_\omega(M), & M \in \text{Loc}, \\
S_\omega(\psi) \equiv \alpha_\omega(\psi), & \psi \in (M_1, M),
\end{cases} \quad (33)$$

We have the following

**Theorem 5.17.** The map $\mathcal{G}_\omega$ satisfies the following properties:

(a) given $M \in \text{Loc}$, there exists $g_\omega(M) \in \mathcal{G}_\omega(M)$ such that

$$\mathcal{G}_\omega(\text{id}_M) = \text{ad}_{g_\omega(M)};$$

47
(b) given \( \phi \in (M_2, M_1) \) and \( \psi \in (M_1, M) \), there exists \( g_{\omega}(\phi, \psi) \in S_{\omega}(M_2) \) such that
\[
S_{\omega}(\phi) \circ S_{\omega}(\psi) = \text{ad}_{g_{\omega}(\phi, \psi)} \circ S_{\omega}(\phi \psi) .
\]

Proof. The proof is an easy application of the DR-theorem to the Theorem 5.14.

Theorem 5.17b says that the map \( S_{\omega} \) is not, in general, a covariant functor because the defining properties of functors are verified in a weak sense, namely up to isomorphisms of the set of arrows. We will refer to the map \( S_{\omega} : \text{Loc} \to \text{Grp} \) as the gauge weak functor\(^3\).

Theorem 5.18. Given a pair \( \omega, \sigma \) of choices of states. Then for any \( M \in \text{Loc} \) there exists a group isomorphism
\[
\alpha_{\omega, \sigma}(M) : S_{\sigma}(M) \to S_{\omega}(M) ,
\]
such that: if \( \psi \in (M_1, M) \) there exists \( g_{\omega, \sigma}(\psi) \in S_{\omega}(M) \) such that
\[
\alpha_{\omega, \sigma}(M) \circ S_{\sigma}(\psi) = \text{ad}_{g_{\omega, \sigma}(\psi)} \circ S_{\omega}(\psi) \circ \alpha_{\omega, \sigma}(M_1) .
\]

Proof. Consider the natural isomorphism \( u_{\omega, \sigma} : S_{\omega} \to S_{\sigma} \) defined in the proof of the Theorem 5.15. Recall that \( u_{\omega, \sigma} \) satisfies the following properties: for any \( M \in \text{Loc} \) we have that
\[
u_{\omega, \sigma}(M) : S_{\omega}(M) \to S_{\sigma}(M) , \quad (\ast)
\]
is a covariant symmetric tensor functor which satisfies, for any \( \psi \in (M_1, M) \) the following equation:
\[
u_{\omega, \sigma}(M) \circ S_{\omega}(\psi) = S_{\sigma}(\psi) \circ u_{\omega, \sigma}(M_1) . \quad (**)
\]

Consider the equation (\( \ast \)). Since \( u_{\omega, \sigma}(M) \) is an isomorphism, by the DR-theorem there corresponds group isomorphism \( \alpha_{\omega, \sigma}(M) : S_{\sigma}(M) \to S_{\omega}(M) \). Consider now the equation (\( ** \)), and note that \( u_{\omega, \sigma}(M_1) \circ S_{\omega}(\psi) \) is full faithful symmetric tensor functor from \( S_{\omega}(M) \) into \( S_{\sigma}(M_1) \). Therefore we can find an \( g \in S_{\omega}(M) \) such that
\[
u_{\omega, \sigma}(M_1) \circ \alpha_{\omega, \sigma}(M_1) = \text{ad}_g(\alpha_{u_{\omega, \sigma}(M_1) \circ S_{\omega}(\psi)}). \]

\(^3\)Note that the notion of a weak covariant (or contravariant) functor can be given in terms of a 2-category, see for instance [38].
In an analog fashion, there exists $g_1 \in G_\omega(M)$ such that 
\[
\alpha_{\omega,\sigma}(M) \circ \mathcal{S}_\omega(\psi) = \text{ad}_{g_1}(\alpha_{\omega,\sigma}(\psi) \circ u_{\omega,\sigma}(M_1))
\]
therefore by (***) we have 
\[
\text{ad}_{g_1^{-1}} \circ \alpha_{\omega,\sigma}(M) \circ \mathcal{S}_\omega(\psi) = \text{ad}_{g_1^{-1}} \circ \mathcal{S}_\omega(\psi) \circ \alpha_{\omega,\sigma}(M_1)
\]
Namely 
\[
\alpha_{\omega,\sigma}(M) \circ \mathcal{S}_\omega(\psi) = \text{ad}_{g_1^{-1}} \circ \mathcal{S}_\omega(\psi) \circ \alpha_{\omega,\sigma}(M_1),
\]
and this completes the proof. \qed

In conclusion, this theorem shows that the gauge weak functor does not depend on the choice of states $\omega$. In particular the groups $G_\omega(M)$ belong to the same isomorphism class for any possible choice $\omega$. Therefore we call $G_\omega(M)$, for some choice $\omega$, the \textbf{gauge group associated with} $M \in \text{Loc}$.

\textbf{Remark 5.19.} We point out two problems connected to the study of the local covariance of fields. \textit{First}, in the case that the set $\mathcal{K}^h(M)$ is nondirected, the Doplicher-Roberts reconstruction theorem does not apply straightforwardly. This happen for instance when either $M$ is nonsimply connected or $M$ has compact Cauchy surfaces. \textit{Secondly}, consider $M, M_1 \in \text{Loc}$ such that there is $\psi \in (M_1, M)$. Assume that $\mathcal{K}^h(M)$ and $\mathcal{K}^h(M_1)$ are directed.\footnote{For instance we can take as $M$ the Minkowski space, and as $M_1$ a diamond of the Minkowski space. One can easily check that the set of diamonds of these two spaces are directed.} In this case one can apply the reconstruction theorem and obtain the algebras of fields, say $\mathcal{F}(M)$ and $\mathcal{F}(M_1)$. One expects that $\mathcal{F}(M_1)$ is isomorphic to the subalgebra $\mathcal{F}(\psi(M_1))$ of $\mathcal{F}(M)$. It is not clear how to show this because in the definition of $\mathcal{F}(M_1)$ intervenes the category $S_{\omega}(M_1)$, while in that of $\mathcal{F}(\psi(M_1))$ intervenes $S_{\omega}(M)$, and we do not know whether these categories are equivalent.

\section{Local completeness}

The local covariance of the theory makes possible the analysis of the relation between local and global superselection sectors. This section is devoted to a preliminary analysis of this topic. In particular we will discuss how the possible nonequivalence between local and global superselection sectors might be related to the nontrivial topology of spacetimes and in particular
to the existence of path-dependent 1-cocycles.

Fix a spacetime $M \in \text{Loc}$ and consider the set of diamonds $\mathcal{D}(M)$. Any diamond $O$ is a globally hyperbolic spacetime, and the injection $\iota_{M,O} : O \to M$ provides an embedding of $O$ into $M$. Given a choice $\omega$ of states, we focus on the superselection sectors $S_\omega(M)$ associated with $M$ and to those associated with any diamond $O$, that is $S_\omega(O)$. We know that

$$S_\omega(\iota_{M,O}) : S_\omega(M) \to S_\omega(O)$$

is a covariant symmetric tensor $\ast$-functor which is full and faithful. From the physical point of view, we expect that this functor is an equivalence. The sectors under investigation are sharply localized, hence they should not be affected by the nontrivial topology of the spacetime and there should be no difference between local and global behaviour. However, up until now, we have no argument to establish this equivalence. We explain what is the technical problem, to which we will refer as the extension problem.

Given $O \in \mathcal{D}(M)$, let $\mathcal{D}^d(O)$ be the set of diamonds of $O$ considered as a globally hyperbolic spacetime. Given $\omega \in \mathcal{S}_o(M)$, let $Z_1^d(\omega,\mathcal{D}^d(O))$ be the category of path-independent 1-cocycles of $\mathcal{D}^d(O)$ with values in the net $\omega^* \mathcal{A}^d(O)$. This category is a tensor C$^*$-category with a permutation symmetry and its objects with finite statistics have conjugates. One can easily see that the functor (34) is an equivalence if, and only if, for any diamond $O$ and for any 1-cocycle $z_O$ of $Z_1^d(\omega,\mathcal{D}^d(O))$, having finite statistics, there exists $z \in Z_1^d(\omega,\mathcal{D}(M))$ such that

$$z|_{\mathcal{D}^d(O)} = z_O.$$  

(35)

Now, it is convenient to introduce a new terminology.

**Definition 6.1.** Given $M \in \text{Loc}$. We will say that the superselection sectors of $(\mathcal{A}(M), \mathcal{S}_o(M))$ are **locally complete** whenever $S_\omega(M)$ is equivalent to $S_\omega(O)$ for any $O \in \mathcal{D}(M)$; conversely, we will say that they are **locally incomplete**.

Examples of theories with locally complete sectors can be easily provided (see below), while it seems to be a very hard task to prove the converse. On the other hand, we have no argument to prove that this holds true in general, and the only attempts, known by the authors, to solve this problem in the Haag-Kastler framework are due to Roberts [45] and Longo$^5$. However,

$^5$Private communication.
only partial results have been achieved. Concerning the failure of local completeness, it seems to be related in a subtle way to the nontrivial topology of the spacetime: in particular to the nonsimply connectedness, as we explain by means of the following reasonings.

(1) The first example of a locally covariant quantum field theory with a state space has been provided in [13]. The correspondence which associates to any \( M \in \text{Loc} \) the CCR algebra \( \mathcal{F}(M) \) of the scalar Klein-Gordon field over \( M \) is a locally covariant quantum field theory \( \mathcal{F} \). The correspondence which associates the space \( \mathcal{S}_\mu(M) \) of the quasi-free states of \( \mathcal{F}(M) \) satisfying the microlocal spectrum condition [12] is a locally quasi-equivalent states space \( \mathcal{S}_\mu \) of \( \mathcal{F} \). The pure states of \( \mathcal{S}_\mu(M) \) fulfill the split property, and satisfies punctured Haag duality (see [50]). So the pair \( (\mathcal{F}, \mathcal{S}_\mu) \) satisfies all the properties that we have assumed in this analysis. Fix \( M \in \text{Loc} \) and consider a pure state \( \omega \) of \( \mathcal{S}_\mu(M) \). By means of the same reasoning used in [15] it turns out that any 1-cocycle of the category \( \mathcal{Z}_1(\omega, \mathcal{K}^d(M)) \) is a finite direct sums of the trivial 1-cocycle (the proof is relies on [46, Theorem 3.5], which provides sufficient conditions for the absence of nontrivial superselection sectors, and on the properties of the nets of free fields [1]). The Proposition 5.6 entails that this holds for any \( \sigma \in \mathcal{S}_\mu(M) \). So for any choice \( \omega \) and for any \( M \) the category \( \mathcal{S}_\omega(M) \) is trivial. Hence the functor \( \mathcal{S}_\omega(\psi) : \mathcal{S}_\omega(M_1) \to \mathcal{S}_\omega(M) \) is an equivalence for any \( \psi \in (M_1, M) \) and the theory is locally complete. This easy\(^6\) example says that the spacetime topology, in particular the nonsimply connectedness, does not represent an obstruction to local completeness.

(2) The existence of a possible relation between topology and local incompleteness, can be seen by analyzing a situation easier than the extension problem. Consider a spacetime \( M \), and let \( \omega \) be a state of \( \mathcal{S}(M) \). Assume that there is a family \( \{z_O\} \) with \( z_O \in \mathcal{Z}_1(\omega, \mathcal{K}^d(\mathcal{O})) \) for any diamond \( \mathcal{O} \), satisfying the following property: given \( b \in \Sigma_1(\mathcal{K}^d(M)) \), then

\[
z_O(b) = z_{O_1}(b) = z_{O_2}(b).
\]

for any pair \( \mathcal{O}, \mathcal{O}_1 \) of diamonds of \( M \) such that \( cl(|b|) \subseteq \mathcal{O} \cap \mathcal{O}_1 \). Now define

\[
z(b) \equiv z_O(b), \quad b \in \Sigma_1(\mathcal{K}^d(M)),
\]

where \( \mathcal{O} \) is a diamond which contains \( cl(|b|) \). By (36), this definition does not depend on the chosen diamond \( \mathcal{O} \). It is clear that for any 1-simplex \( b \)

\(^6\)Modifying this example one can construct examples of locally complete theories having a nontrivial superselection content. It is enough to enlarge the symplectic space associated with any \( M \in \text{Loc} \) in a such way that it is possible to introduce an action of a compact group.
we have that $z(b) \in A_\omega(|b|)$ because so does $z_\O(b)$. Furthermore, as for any $c \in \Sigma_2(\K^d(M))$, by (5), there is $\O_1 \in \K^d(M)$, with $cl(|c|) \subseteq \O_1$, we have that

$$z(\partial_0 c) \cdot z(\partial_2 c) = z_{\O_1}(\partial_0 c) \cdot z_{\O_1}(\partial_2 c) = z_{\O_1}(\partial_1 c) = z(\partial_1 c).$$

Therefore, $z$ is a 1-cocycle of $\K^d(M)$ which extends the family $\{z_\O\}$. Notice, however, that in the case that $M$ is nonsimply connected, then $z$ might be a path-dependent 1-cocycle. Hence the local incompleteness might be related to the existence of path-dependent 1-cocycles. Recall that this type of 1-cocycles are of a topological nature as they provide nontrivial representations of the fundamental group of the manifold (see Section 4.2).

(3) In order to strengthen the idea in (2), consider a nonsimply connected spacetime $M$, and let $\omega$ be a state of $\mathcal{S}_\rho(M)$. Assume that there exists a path-dependent 1-cocycle $z$ of $\K^d(M)$ with values in $\omega^* \mathcal{A}_{\K^d(M)}$. For any $\O \in \K^d(M)$ define

$$z_\O(b) \equiv z(b), \quad b \in \Sigma_1(\K^d(\O)).$$

Since $\O$ is simply connected, $z_\O$ is a path-independent 1-cocycle of $\K^d(\O)$. Hence, the family $\{z_\O\}$ satisfies the condition (6.1) and its extension to $\K^d(M)$ is a path-dependent 1-cocycle. One should be cautious at this point. The above does not imply the violation of local completeness because one should first ensure that $z_\O$ has finite statistics. Unfortunately, at the moment we are unable to prove this.

**Remark 6.2.** According to the above discussion, the obstruction to local completeness of sectors seems to be the presence of topological 1-cocycles. So, we expect that, in a simply connected spacetime, the sectors are locally complete.

**Remark 6.3.** To avoid confusion, we want to stress that there is no relation between path-dependent 1-cocycle of a poset and the topological charged sectors discovered by Buchholz and Fredenhagen in [17]. As observed the formers are of a topological nature because they provide nontrivial representation of the fundamental group of the poset. The latter are called topological because of their localization properties. In this case the poset underlying the theory is that formed by the set of spacelike cones of the Minkowski space. One can easily see that the 1-cocycles associated with this type of charges are path-independent. Thus, they provide only trivial representations of the fundamental group of the poset.
7 Conclusions and Outlook

The paper concerned with the analysis of superselection sectors in the framework of a locally covariant quantum field theory $\mathcal{A}$. The main purpose was the understanding of the covariance behaviour of those sectors which describe sharply localized charges, namely those type of sectors that on a fixed spacetime background are a sharp excitation of a reference representation, the vacuum in the Minkowski space, of the observable net.

As the present paper is the first investigation on this topic, it is worth recalling in some detail the basic assumptions and results. The first, very useful result, is of geometrical nature:

1. The set of diamonds $\mathcal{K}^d(M)$ of globally hyperbolic spacetimes $M$ is stable under isometric embeddings (Lemma 2.8).

This allows to express the locally covariant principle in terms of nets of local observables $\mathcal{A}_{\mathcal{K}^d(M)}$ indexed by the set of diamonds $\mathcal{K}^d(M)$ of $M$. The set $\mathcal{K}^h(M)$ introduced in [13] does not fit the topological and causal properties of the spacetime $M$, compared to $\mathcal{K}^d(M)$.

The first needed step to introduce the superselection sectors of the quantum field theory functor $\mathcal{A}$ has been the definition of a reference state space $\mathcal{S}_o$ for the theory. We required that $\mathcal{S}_o$ is locally quasi equivalent, that it satisfies the Borchers property and punctured Haag duality (see Definition 5.1). In particular, we emphasize that the local quasi equivalence is required not only on diamonds but also on the larger family $\mathcal{K}^h(M)$. This was an important assumption, as we are requiring that the elements $\mathcal{S}_o(M)$ behave locally in the same way also on non simply connected regions. Given $\mathcal{S}_o$ we have defined the sectors as follows: The superselection sectors associated with a spacetime $M$ are the families $Z^1_t(\omega,\mathcal{K}^d(M))$, as $\omega$ varies in $\mathcal{S}_o(M)$, of path-independent 1-cocycles of the poset obtained by ordering under inclusion $\mathcal{K}^d(M)$, which take their values in the net $\omega^*\mathcal{A}_{\mathcal{K}^d(M)}$ (see Definition 5.2).

2. For any spacetime $M$ the sets $Z^1_t(\sigma,\mathcal{K}^d(M))$, as $\sigma$ varies in $\mathcal{S}_o(M)$, carry the same physical information: 1-cocycles of $Z^1_t(\sigma,\mathcal{K}^d(M))$ are, up to equivalence, in bijective correspondence with representations of the observable net which are sharp excitations of the representation $\pi_\omega$ induced by a state $\omega$ in $\mathcal{S}_o(M)$ satisfying punctured Haag duality (see Proposition 5.10).

3. Sectors manifest a charge structure: Their quantum numbers have a composition law, a statistics and a charge conjugation symmetry. This
structure is independent from the choice of the state $\omega$ in $\mathcal{S}_0(M)$. This is expressed by the fact that $Z^1_t(\omega, \mathcal{K}^d(M))$ is a symmetric tensor $C^*$-category whose objects with finite statistics have conjugates; all the categories $Z^1_t(\omega, \mathcal{K}^d(M))$ belong to the same isomorphism class (see Theorem 5.8).

4. The charge structure is contravariant: If a spacetime $M_1$ can be isometrically embedded in $M$, then to any sector of $M$ there corresponds a unique sector of $M_1$ with the same charged quantum numbers; the correspondence associating to any spacetime $M$ the category $Z^1_t(\omega_M, \mathcal{K}^d(M))$, with $\omega_M \in \mathcal{S}_0(M)$, is a contravariant functor. (see Theorem 5.14 and Remark 5.16). \footnote{We point out that a similar functorial structure arises in the theory of subsystems [19].}

These results imply that the physical content of superselection sectors carried by each spacetime remains stable when the spacetimes can be coherently embedded.

Notice that in this first paper we took a conservative point of view, namely, we work at a level that is as faithful as possible to the tradition in superselection theory. Indeed, one might have even started in a more abstract way by defining representations in the generally covariant sense from the beginning. This amounts to view them as natural transformations. However, at the moment it is not clear how to generalize some of the important technical features that we used.

We now pass to briefly outline a prospect for future works.

An important point that will be tackled in a forthcoming contribution is how various indices are related to each other, in the sense of tensor categories. Here, one make use of more machinery from algebraic topology, in the sense of injections and rejections, always tailored to the poset situation.

New directions of research can be envisaged. For instance, an important step would be to clarify whether the reconstruction procedure of Doplicher and Roberts can be fully implemented, in general, by reconstructing the field nets, and proving them to be locally covariant. In case the theory is locally complete and the family index is directed, the reconstruction can be done, and it will appear in the third paper of the series.

A most important direction would be the incorporation of the case when the 1-cocycles are \textit{not} path-independent. Here the full topological structure of the spacetimes enters the game. This is strongly related also to the problem of proving, or disproving, local completeness in the sense of Section...
It looks like a difficult task but worth exploring. We hope to come back to this point in the near future.

We conclude with a couple of possible further directions of research. The first deals with the gauge groups. One may imagine that the gauge groups associated with any local region act as local gauge groups, thus opening a fresh look at this problem in the algebraic setting. Here, a more geometrical setting might prove helpful. The second direction deals with the fact that the theories described in this paper seems to have from the outset a well-defined ultraviolet behaviour, a fact which is exemplified by the properties of the functor of restriction $\mathcal{R}$ (see Theorem 5.9). A closer connection with the work done by D’Antoni, Morsella and Verch [20], possibly within the framework outlined in [9], would be desirable.

Acknowledgements

We are particularly thankful to J.E. Roberts and K. Fredenhagen for their kind remarks during this investigation. We thank as well E. Vasselli for his precious help with the intricacies of the Doplicher-Roberts reconstruction Theorem. We are grateful to the DFG and to the European Network “Quantum Spaces – Noncommutative Geometry” for financial support.

A Tensor $C^*$–categories

For easy of notation we will denote the set of objects of a category $\mathcal{C}$ by the same symbol $\mathcal{C}$. We denote by $z, z_1, z_2, \ldots$ the objects of the category and the set of the arrows between $z, z_1$ by $(z, z_1)$. The composition of arrows is indicated by “$\cdot$” and the identity arrow of $(z, z)$ by $1_z$. Recall that an arrow $t \in (z_1, z_2)$ is an isomorphism if there exists an arrow $s \in (z_2, z_1)$ such that

$$s \cdot t = 1_{z_1}, \quad t \cdot s = 1_{z_2}.$$ 

The objects $z_1$ and $z_2$ are said to be isomorphic, written $z_1 \sim z_2$. References for this appendix are [39] [21] [38].

Functors and natural transformations: Consider two categories $\mathcal{C}_1$ and $\mathcal{C}_2$. A covariant functor $F : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ is said to be:

- faithful, if $s, t \in (z_1, z_2)$ with $t \neq s$, then $F(s) \neq F(t)$;
- full, if $F(z, z_1) = (F(z), F(z_1))$;
- dense, if for any $z_2 \in \mathcal{C}_2$ there exists $z_1 \in \mathcal{C}_1$ such that $F(z_1) \sim z_2$;
involutive, if \( \mathcal{C}_2 = \mathcal{C}_1 \) and \( F \circ F = \text{id}_{\mathcal{C}_1} \) of \( \mathcal{C}_1 \),

where \( \text{id}_{\mathcal{C}_1} \) is the identity functor of \( \mathcal{C}_1 \).

A natural transformation \( u : F \to G \), between a pair of functors \( F, G : \mathcal{C}_1 \to \mathcal{C}_2 \), is a map which assigns an arrow \( u(z) \) of \( \mathcal{C}_2 \) to any object \( z \) of \( \mathcal{C}_1 \) such that

\[
(i) \quad u(z) \in (F(z), G(z)) \quad z \in \mathcal{C}_1 \\
(ii) \quad F(f) \cdot u(z) = u(z_1) \cdot G(f) \quad f \in (z, z_1).
\]

\( u \) is a natural isomorphism whenever \( u(z) \) is an isomorphism for any \( z \in \mathcal{C}_1 \). In this case we will say that \( F \) and \( G \) are isomorphic, written as \( F \sim G \).

An isomorphism of categories, is functor \( F : \mathcal{C}_1 \to \mathcal{C}_2 \) for which there exists another functor \( G : \mathcal{C}_2 \to \mathcal{C}_1 \) such that

\[
G \circ F = 1_{\mathcal{C}_1}, \quad F \circ G = 1_{\mathcal{C}_2}.
\]

Whenever

\[
G \circ F \sim 1_{\mathcal{C}_1}, \quad F \circ G \sim 1_{\mathcal{C}_2},
\]

then \( F \) is said to be an equivalence. If \( F \) is an isomorphism (equivalence), then \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) are said to be isomorphic (equivalent). It turns out that \( F \) is equivalence if, and only if, it is a full, faithful and dense.

**C*-categories:** A category \( \mathcal{C} \) is said to be a C*-category if the set of the arrows \((z, z_1)\) between two objects \( z, z_1 \) is a complex Banach space and the composition between arrows is bilinear a bilinear map \( t, s \to t \cdot s \) with \( \|t \cdot s\| \leq \|t\| \cdot \|s\| \); there should be an adjoint, that is an involutive contravariant functor \( * \) acting as the identity on the objects and the norm should satisfy the C*-property, namely \( \|r^* \cdot r\| = \|r\|^2 \) for each \( r \in (z, z_1) \). Notice, that if \( \mathcal{C} \) is a C*-category then \((z, z)\) is a C*-algebra for each \( z \).

Assume that \( \mathcal{C} \) is a C*-category. An arrow \( v \in (z, z_1) \) is said to be an isometry if \( v^* \cdot v = 1_z \); a unitary, if it is an isometry and \( v \cdot v^* = 1_{z_1} \). The property of admitting a unitary arrow, defines an equivalence relation on the set of the objects of the category. We denote by the symbol \([z]\) the unitary equivalence class of the object \( z \). An object \( z \) is said to be irreducible if \((z, z) = \mathbb{C} \cdot 1_z \). \( \mathcal{C} \) is said to be closed under subobjects if for each orthogonal projection \( e \in (z, z) \), \( e \neq 0 \) there exists an isometry \( v \in (z_1, z) \) such that \( v \cdot v^* = e \). \( \mathcal{C} \) is said to be closed under direct sums, if given \( z_i i = 1, 2 \) there exists an object \( z \) and two isometries \( w_i \in (z_i, z) \) such that \( w_1 \cdot w_1^* + w_2 \cdot w_2^* = 1_z \).
Given two C*-categories \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) a \(*\)-functor \( F : \mathcal{C}_1 \to \mathcal{C}_2 \) is a functor which commute with the adjoint, and preserve the linear structure of arrows:

\[
F(\alpha \cdot t + \beta \cdot s) = \alpha \cdot F(t) + \beta \cdot F(s)
\]

\( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) are said to be equivalent (isomorphic) if there exists a \(*\)-functor \( F : \mathcal{C}_1 \to \mathcal{C}_2 \) which is an equivalence (isomorphism). A natural isomorphism \( u : F \to G \), with \( F,G : \mathcal{C}_1 \to \mathcal{C}_2 \) \(*\)-functors, is said to be natural unitary whenever \( u(z) \) is a unitary arrow of \((F(z),G(z))\) for any object \( z \) of \( \mathcal{C}_1 \).

### Symmetric tensor C*-categories

A tensor C*-category \( \mathcal{C} \) is a C*-category which is equipped with a tensor product \( \otimes \). This means that to each pair of objects \( z,z_1 \) there is a product object \( z \otimes z_1 \), and \( \mathcal{C} \) has a unit object \( t \) such that \( z \otimes t = z = t \otimes z \). Given two arrows \( t \in (z,z_1) \) and \( s \in (z_2,z_3) \) there is an arrow \( t \otimes s \in (z \otimes z_2,z_1 \otimes z_3) \). The mapping \( t,s \mapsto t \otimes s \) is associative and bilinear, and

\[
1_t \otimes t = t = t \otimes 1_t, \quad (t \otimes s)^* = t^* \otimes s^*,
\]

and the interchange law

\[
(t \otimes s) \cdot (t_1 \otimes s_1) = t \cdot t_1 \otimes s \cdot s_1
\]

holds whenever the right hand side is defined.

From now on we will consider only tensor C*-categories, with irreducible unit \( \iota \), which are closed under direct sums and subobjects.

\( \mathcal{C} \) is said to be a symmetric if it has a permutation symmetry. This means that there is a map \( \varepsilon : \mathcal{C} \ni z_1,z_2 \to \varepsilon(z_1,z_2) \in (z_1 \otimes z_2,z_2 \otimes z_1) \) satisfying the relations:

\[
\begin{align*}
(i) \quad & \varepsilon(z_3,z_4) \cdot t \otimes s = s \otimes t \cdot \varepsilon(z_1,z_2) \\
(ii) \quad & \varepsilon(z_1,z_2)^* = \varepsilon(z_2,z_1) \\
(iii) \quad & \varepsilon(z_1,z_2 \otimes z) = 1_{z_2} \otimes \varepsilon(z_1,z) \cdot \varepsilon(z_1,z_2) \otimes 1_z \\
(iv) \quad & \varepsilon(z_1,z_2) \cdot \varepsilon(z_2,z_1) = 1_{z_2 \otimes z_1}
\end{align*}
\]

where \( t \in (z_2,z_4), s \in (z_1,z_3). \) By \( ii \)–\( iv \) it follows that \( \varepsilon(z,t) = \varepsilon(t,z) = 1_z \) for any \( z \).

A left inverse of an object \( z \) of \( \mathcal{C} \) is a set of nonzero linear maps \( \phi^z = \{ \phi^z_{x_2,z_2} : (z \otimes z_1,z \otimes z_2) \to (z_1,z_2) \} \) satisfying

\[
\begin{align*}
(i) \quad & \phi^z_{x_3,z_4} \cdot 1_z \otimes t \cdot r \cdot 1_z = t \cdot \phi^z_{x_1,z_2} (r) \cdot s^*, \\
(ii) \quad & \phi^z_{x_1 \otimes z_2 \otimes z_3} (r \otimes 1_{x_3}) = \phi^z_{x_1,z_2} (r) \otimes 1_{x_3}, \\
(iii) \quad & \phi^z_{x_1,z_2} (s^*_1 \cdot s_1) \geq 0, \\
(iv) \quad & \phi^z_{x_1} (1_z) = 1,
\end{align*}
\]
where $s \in (z_1, z_3)$, $t \in (z_2, z_4)$, $r \in (z \otimes z_1, z \otimes z_2)$ and $s_1 \in (z \otimes z_1, z \otimes z_1)$. $\mathcal{C}$ is said to have left inverses if any object of $\mathcal{C}$ has left inverses.

Consider two symmetric tensor $\mathcal{C}^*$-categories $\mathcal{C}_1$, $\mathcal{C}_2$. Let $(\otimes_1, \varepsilon_1)$ and $(\otimes_2, \varepsilon_2)$ be the corresponding tensor products and permutation symmetries. A $\ast$-functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ is said to be a symmetric tensor $\ast$-functor, if for any pair of objects $z_1, z_2$ of $\mathcal{C}_1$ and for any pair of arrows of $\mathcal{C}_1$, we have that

$$F(z \otimes_1 z_1) = F(z) \otimes_2 F(z_1),$$
$$F(t \otimes_1 s) = F(t) \otimes_2 (F(s),$$
$$F(\varepsilon_1(z, z_1)) = \varepsilon_2(F(z), F(z_1)).$$

Two symmetric tensor $\mathcal{C}^*$-category $\mathcal{C}_1$ and $\mathcal{C}_2$ are said to be equivalent (isomorphic) if there exists a symmetric tensor $\ast$-functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ which is an equivalence (isomorphism). A tensor natural transformation $u : F \to G$ between two symmetric tensor $\ast$-functors $G, F : \mathcal{C}_1 \to \mathcal{C}_2$ is a natural transformation such that $u(z \otimes_1 z_1) = u(z) \otimes_2 u(z_1)$. It will be said to be a tensor natural isomorphism (tensor natural unitary) if it is a natural isomorphism (a natural unitary).

**Statistics and conjugation** : Let $\mathcal{C}$ be a symmetric tensor $\mathcal{C}^*$-category with left inverses. An object $z$ of $\mathcal{C}$ is said to have finite statistics if it admits a standard left inverse, that is a left inverse $\phi^z$ satisfying the relation

$$\phi^z_{z, z}(\varepsilon(z, z)) \cdot \phi^z_{z, z}(\varepsilon(z, z)) = c \cdot 1_z \text{ with } c > 0$$

The full subcategory $\mathcal{C}_f$ of $\mathcal{C}$ whose objects have finite statistics, is closed under direct sum, subobjects, tensor products, and equivalence. Any object of $\mathcal{C}_f$ is direct sums of irreducible objects. Given an irreducible object $z$ of $\mathcal{C}_f$ and a left inverse $\phi^z$ of $z$, then

$$\phi^z_{z, z}(\varepsilon(z, z)) = \lambda(z) \cdot 1_z.$$

The number $\lambda(z)$ is an invariant of the equivalence class of $z$, called the statistics parameter. It is the product of two invariants:

$$\lambda(z) = \chi(z) \cdot d(z)^{-1} \text{ where } \chi(z) \in \{1, -1\}, \ d(z) \in \mathbb{N}$$

The possible statistics of $z$ are classified by the statistical phase $\chi(z)$ distinguishing para-Bose (1) and para-Fermi (−1) statistics and by the statistical dimension $d(z)$ giving the order of the parastatistics. Ordinary Bose and Fermi statistics correspond to $d(z) = 1$. 

58
An object $z$ of $C$ has conjugates if there exists an object $\tau$ and a pair of arrows $r \in (\iota, \tau \otimes z), \tau \in (\iota, z \otimes \tau)$ satisfying the conjugate equations

$$\tau^* \otimes 1_z \cdot 1_z \otimes r = 1_z, \quad r^* \otimes 1_\tau \cdot 1_\tau \otimes \tau = 1_\tau.$$  

Conjugation is a property stable under subobjects, direct sums, tensor products and, furthermore, it is stable under equivalence. It turns out that if $z$ has conjugates, then $z$ has finite statistics.

Consider two symmetric tensor $C^*$-categories $C_1$ and $C_2$, and let $F : C_1 \to C_2$ be a full and faithful symmetric tensor $^*-$functor. Let $r \in (\iota, \tau \otimes z)$ and $\tau \in (\iota, z \otimes \tau)$ solve the conjugate equations with respect to $z$ and $\tau$. Then it turns out that the pair $F(r) \in (\iota, F(\tau) \otimes z)$ and $F(\tau) \in (\iota, z \otimes F(\tau))$ solve the conjugate equations in $C_2$ with respect to $F(z)$ and $F(\tau)$. In particular, $z$ is irreducible, if and only, $F(z)$ is irreducible and has statistics as $z$.

Finally, a symmetric tensor $C^*$-category is said to have conjugates if any object of the category has conjugates. Note that given a symmetric tensor $C^*$-category $C$, let $C_1$ the full subcategory of $C$ whose objects have conjugates. Then $C_1$ has conjugates.

**Doplicher-Roberts duality theorem:** We recall some basic facts on the Doplicher-Roberts duality theorem of compact groups [24] (DR-theorem) focusing on its functorial properties.

Denote by $\text{Sym}$ the full subcategory of $\text{Sym}$ (see Section 5.3.2) whose objects are symmetric tensor $C^*$-categories having conjugates. The main technical result of the DR-theorem is the embedding theorem: Any $\mathcal{C} \in \text{Sym}$ admits an embedding into a category of finite dimensional Hilbert spaces. To be precise, an embedding of $\mathcal{C}$ is a pair $(H, \mathcal{H})$, where $\mathcal{H} \in \text{Sym}$ is a category of finite dimensional Hilbert spaces, while $H : \mathcal{C} \to \mathcal{H}$ is a symmetric tensor $^*-$functor which is full and faithful. Given an embedding $(H, \mathcal{H})$ of $\mathcal{C}$, the set

$$\text{End}_\otimes(H) \equiv \{\text{tensor natural unitaries } u : H \to H\}$$

equipped with the composition $(u_1 \circ u)(z) \equiv u_1(z) \cdot u(z)$ is, with a suitable topology, a compact group. It turns out that $\mathcal{C}$ is the abstract dual of $\text{End}_\otimes(H)$. This group is uniquely associated with the category $\mathcal{C}$ only up to isomorphism. This is because the embedding of the category $\mathcal{C}$ is not uniquely determined, so the compact group associated with $\mathcal{C}$ depends on
the choice of the embedding. However for any other embedding \((H',\mathcal{H}')\) of \(\mathcal{C}\) there exists a tensor natural unitary equivalence
\[
w : H \to H'
\] (A.1)
which associates to any object \(z \in \mathcal{C}\) a unitary operator \(w(z)\) from the Hilbert space \(H(z)\) onto the Hilbert space \(H'(z)\). \(w\) preserves the tensor products
\[
w(z \otimes z_1) = w(z) \otimes' w(z_1), \quad z, z_1 \in \mathcal{C},
\]
\(\otimes'\) is the tensor product of \(\mathcal{H}'\), and \(w(z_1) \cdot H(t) = H'(t) \cdot w(z)\) for any \(z, z_1 \in \mathcal{C}\) and \(t \in (z, z_1)\). It turns out that the mapping
\[
\text{End}_{\otimes}(H) \ni u \to w \circ u \circ w^* \in \text{End}_{\otimes}(H')
\]
is a group isomorphism, where
\[
(w \circ u \circ w^*)(z) \equiv w(z) \cdot u(z) \cdot w^*(z), \quad z \in \mathcal{C}.
\]
We now want to see the functorial properties of the DR-theorem. Given two categories \(\mathcal{C}_1, \mathcal{C}\), let \((H, \mathcal{H})\) and \((H_1, \mathcal{H})\) be a choice of embeddings. Consider a full and faithful, symmetric tensor \(^*\)-functor \(F_1 : \mathcal{C}_1 \to \mathcal{C}\). Then the pair \((H \circ F_1, \mathcal{H})\) provides another embedding of \(\mathcal{C}_1\). Let us denote by \(w_{F_1}\) the tensor natural unitary equivalence \(w_{F_1} : H_1 \to H \circ F_1\) associated, by (A.1) with the embeddings \((H_1, \mathcal{H})\) and \((H \circ F_1, \mathcal{H})\). Given \(u \in \text{End}_{\otimes}(H)\), define
\[
\alpha_{F_1}(u)(z_1) \equiv w_{F_1}^*(z_1) \circ u(F(z_1)) \circ w_{F_1}(z_1), \quad z_1 \in \mathcal{C}_1
\]
It turns out that
\[
\alpha_{F_1} : \text{End}_{\otimes}(H) \to \text{End}_{\otimes}(H_1)
\]
is a group morphism. Consider now a second category \(\mathcal{C}_2\) and let \((H_2, \mathcal{H}_2)\) be an embedding of this category. If \(F_2 : \mathcal{C}_2 \to \mathcal{C}_1\) is a functor, then by proceeding as above there is tensor natural unitary equivalence \(w_{F_2} : H_2 \to H_1 \circ F_2\) and a group morphism
\[
\alpha_{F_2} : \text{End}_{\otimes}(H_1) \to \text{End}_{\otimes}(H_2)
\]
defined, for any \(u_1 \in \text{End}_{\otimes}(H_1)\), as
\[
\alpha_{F_2}(u_1)(z_2) = w_{F_2}^*(z_2) \cdot u_1(F_2(z_2)) \cdot w_{F_2}(z_2), \quad z_2 \in \mathcal{C}_2
\]
Now, observe that $\alpha_{F_2} \circ \alpha_{F_1} : \text{End}_\otimes(H) \to \text{End}_\otimes(H_2)$. In particular given $u \in \text{End}_\otimes(H)$ and $z_2 \in \mathcal{C}_2$ we have that
\[
(\alpha_{F_2} \circ \alpha_{F_1})(u)(z_2) = \\
= w_{F_2}^*(z_2) \cdot \alpha_{F_1}(u)(F_2(z_2)) \cdot w_{F_2}(z_2) \\
= w_{F_2}^*(z_2) \cdot w_{F_1}^*(F_2(z_2)) \cdot u(F_1 F_2(z_2)) \cdot w_{F_1}(F_2(z_2)) \cdot w_{F_2}(z_2).
\]

Consider the tensor natural unitary equivalence $w_{F_1 F_2} : H_2 \to H \circ F_1 \circ F_2$, and the corresponding group morphism $\alpha_{F_1 F_2} : \text{End}_\otimes(H) \to \text{End}_\otimes(H_2)$ defined, for any $u \in \text{End}_\otimes(H)$, as
\[
\alpha_{F_1 F_2}(u)(z_2) = w_{F_1 F_2}^*(z_2) \cdot u(F_1 F_2(z_2)) \cdot w_{F_1 F_2}(z_2), \quad z_2 \in \mathcal{C}_2.
\]

Now, define
\[
w_{F_1, F_2; F_1 F_2}(z_2) \equiv w_{F_2}^*(z_2) \cdot w_{F_1}^*(F_2(z_2)) \cdot w_{F_1 F_2}(z_2), \quad z_2 \in \mathcal{C}_2.
\]

Then it turns out that $w_{F_1, F_2; F_1 F_2}$ is an element of the group $\text{End}_\otimes(H_2)$ and that
\[
(\alpha_{F_2} \circ \alpha_{F_1})(u)(z_2) = \text{ad}_{w_{F_1, F_2; F_1 F_2}(z_2)}(\alpha_{F_1 F_2}(u)(z_2)).
\]

**References**

[1] H. Araki. *von Neumann algebras of local observables for free scalar field.* J. Math. Phys. **5**, (1964), 1–13.

[2] A. Ashtekar, A. Sen. *On the role of space-time topology in quantum phenomena: superselection of charge and emergence of nontrivial vacua.* J. Math. Phys. **21**, (1980), 526–533.

[3] H. Baumgärtel, F. Lledó. *Duality of compact groups and Hilbert $C^\ast$–systems for $C^\ast$–algebras with nontrivial center.* Internat. J. Math. **15**, (2004), 759–812.

[4] J.K. Beem, P.E. Ehrlich, K.L. Easley. *Global Lorentzian geometry.* Marcel Dekker, Inc. New York. 1996, 2nd. edition.

[5] A.N. Bernal, M. Sánchez. *On smooth Cauchy hypersurfaces and Geroch’s splitting theorem.* Commun. Math. Phys. **243**, (2003), 461–470.
[6] A.N. Bernal, M. Sánchez. Smoothness of time functions and the metric splitting of globally hyperbolic spacetimes. Commun. Math. Phys. 257, (2005), 43–50.

[7] A.N. Bernal, M. Sánchez. A note on the extendability of compact hypersurfaces to smooth Cauchy hypersurfaces preprint, [gr-qc/0507018].

[8] H.J. Borchers. Local rings and the connection of spin with statistics. Commun. Math. Phys. 1, (1965), 281–307.

[9] R. Brunetti. Locally covariant quantum field theories. To be published on the Proceedings of the Conference in honour of J. Bros, 2004.

[10] R. Brunetti, K. Fredenhagen. Microlocal analysis and quantum field theory: Renormalization on physical backgrounds. Commun. Math. Phys. 208, (2000), 623–661.

[11] R. Brunetti, K. Fredenhagen. Algebraic Quantum Field Theory. Encyclopedia of Mathematical Physics, Elsevier, to be published.

[12] R. Brunetti, K. Fredenhagen, M. Köhler. The microlocal spectrum condition and Wick polynomials of free fields on curved spacetimes. Commun. Math. Phys. 180, (1996), 633–652.

[13] R. Brunetti, K. Fredenhagen, R. Verch. The generally covariant locality principle – A new paradigm for local quantum physics. Commun. Math. Phys. 237, (2003), 31–68.

[14] R. Brunetti, G. Ruzzi. Superselection sectors and general covariance II. preprint in preparation.

[15] D. Buchholz, S. Doplicher, R. Longo, J.E. Roberts, A new look at Goldstone’s theorem. Rev. Math. Phys. Special Issue, (1992), 49–83.

[16] D. Buchholz. The physical state space of quantum electrodynamics. Commun. Math. Phys. 85, (1982), 49–71.

[17] D. Buchholz, K. Fredenhagen. Locality and the structure of particle states. Commun. Math. Phys. 84, (1982), 1–34.
[18] F. Ciolli. *Massless scalar free Field in 1+1 dimensions II: Net Cohomology and Completeness of Superselection Sectors*. preprint in preparation.

[19] R. Conti, S. Doplicher, J.E. Roberts. *Superselection theory for subsystems*. Comm. Math. Phys. 218, (2001), no. 2, 263–281.

[20] C. D’Antoni, G. Morsella, R. Verch. *Scaling algebras for charged fields and short-distance analysis for localizable and topological charges*. Annales Henri Poincaré 5, (2005), 809–870.

[21] S. Doplicher, R. Haag, J.E. Roberts. *Local observables and particle statistics I*. Commun. Math Phys. 23, (1971), 199–230.

[22] S. Doplicher, R. Haag, J.E. Roberts. *Local observables and particle statistics II*. Commun. Math Phys. 35, (1974), 49–85.

[23] S. Doplicher, R. Longo. *Standard and split inclusions of von Neumann algebras*. Invent. Math. 35, (1984), 493–536.

[24] S. Doplicher, J.E. Roberts. *A new duality theory for compact groups*. Invent. Math. 98, no.1, (1989), 157–218.

[25] S. Doplicher, J.E. Roberts. *Why there is a field algebra with a compact gauge group describing the superselection sectors in particle physics*. Commun. Math. Phys. 131, no.1, (1990), 51–107.

[26] M. Dütsch, K. Fredenhagen. *Causal perturbation theory in terms of retarded products and, and a proof of the action Ward identity*. Rev. Math. Phys. 16, no.10, (2004), 1291–1348.

[27] G.F.R. Ellis, S.W. Hawking. *The large scale structure of spacetime*. Cambridge University Press, 1973.

[28] K. Fredenhagen. *A background independent formulation of perturbative quantum gravity*, talk given at Blaubeuren (2005).

[29] K. Fredenhagen, K.-H. Rehren, B. Schroer. *Superselection sectors with braid group statistics and exchange algebras. II: Geometric aspects and conformal invariance*. Rev. Math Phys. Special Issue, (1992), 113–157.

[30] J.L. Friedman, K. Schleich, D.M. Witt. *Topological censorship*. Phys. Rev. Lett. 71, (1993), 1486–1489.
[31] R.W. Fuller, J.A. Wheeler. *Causality and multiply connected space-time.* Phys. Rev. (2) **128**, (1962), 919–929.

[32] D. Guido, R. Longo, J.E. Roberts, R. Verch. *Charged sectors, spin and statistics in quantum field theory on curved spacetimes.* Rev. Math. Phys. **13**, no. 2, (2001), 125–198.

[33] R. Haag: *Local Quantum Physics.* Springer Texts and Monographs in Physics, 1996, 2nd edition.

[34] R. Haag, D. Kastler. *An algebraic approach to quantum field theory.* J. Math. Phys. **5**, (1964), 848–861.

[35] S. Hollands, R.M. Wald. *Local Wick polynomials and time ordered products of quantum fields in curved spacetime.* Commun. Math. Phys. **223**, (2001), 289–326.

[36] D. Kastler, M. Mebkhout, K-H. Rehren. *Introduction to the algebraic theory of superselection sectors. Space-time dimension = 2 – Strictly localizable morphisms.* In: *The algebraic theory of superselection sectors.* (Palermo 1989), D. Kastler ed., 113–214, World Sci. Publishing, River Edge, NJ, 1990.

[37] Y. Kawahigashi, R. Longo. *Classification of local conformal nets. Case c < 1.* Ann. of Math. (2) **160**, (2004), 493–522.

[38] R. Longo, J.E. Roberts. *A theory of dimension.* K-Theory **11**, no.2, (1997), 103–159.

[39] S. Mac Lane. *Categories for the working mathematician* Springer Verlag, New York-Heidelberg-Berlin, 1971.

[40] M. Müger. *The superselection structure of massive quantum field theories in 1+1 dimensions.* Rev. Math. Phys. **10**, (1998), 1147–1170.

[41] M. Müger. *Frobenius algebras in and Morita equivalence of tensor categories I.* J. Pure Appl. Alg. **180**, (2003), 81–157.

[42] B. O’Neill. *Semi–Riemannian geometry.* Academic Press, New York, 1983.

[43] M. Porrmann. *Particle weights and their disintegration I.* Commun. Math. Phys. **248**, (2004), 269–304.
[44] M. Porrmann. Particle weights and their disintegration II. Commun. Math. Phys. 248, (2004), 305–333.

[45] J.E. Roberts. Local cohomology and superselection structure. Commun. Math. Phys 51, no. 2, (1976) 107–119.

[46] J.E. Roberts. Net cohomology and its applications to field theory. “Quantum Fields—Algebras, Processes” (L.Streit ed.) Springer, Wien, New York, 1980.

[47] J.E. Roberts. Lectures on algebraic quantum field theory. In: The algebraic theory of superselection sectors. (Palermo 1989), D. Kastler ed., 1–112, World Sci. Publishing, River Edge, NJ, 1990.

[48] J.E. Roberts. More lectures in algebraic quantum field theory. In: Noncommutative geometry C.I.M.E. Lectures, Martina Franca, Italy, 2000. Editors: S. Doplicher, R. Longo, Springer (2003).

[49] G. Ruzzi. Essential properties of the vacuum sector for a theory of superselection sectors. Rev. Math. Phys. 15, no.10, (2003), 1255–1283.

[50] G. Ruzzi. Punctured Haag duality in locally covariant quantum field theories. Commun. Math. Phys. 256, (2005), 621–634.

[51] G. Ruzzi. Homotopy of posets, net-cohomology, and theory of superselection sectors in globally hyperbolic spacetimes. Rev. Math. Phys. 17, no.9, (2005), 1021–1070.

[52] I.M. Singer, J.A. Thorpe Lecture Notes on Elementary Topology and Geometry. Springer-Verlag, New York, 1967.

[53] R. Sorkin. The quantum electromagnetic field in multiply connected space. J. Phys. A 12, (1979), 403–421.

[54] E. Vasselli. Continuous Fields of C*-algebras arising from extensions of tensor C*-categories. J. Funct. Anal. 199, (2003), 123–153.

[55] R. Verch. Continuity of symplectically adjoint maps and the algebraic structure of Hadamard vacuum representations for quantum fields in curved spacetime. Rev. Math. Phys. 9, no.5, (1997), 635–674.
[56] R. Verch. Notes on regular diamonds., preprint, available as ps-file at [http://www/lqp.uni-goettingen.de/lqp/papers/](http://www/lqp.uni-goettingen.de/lqp/papers/)

[57] R. Verch. A spin-statistics theorem for quantum fields on curved spacetimes manifolds in a generally covariant framework. Commun. Math. Phys. 223, 261, (2001).

[58] R. Verch Stability Properties of Quantum Fields on Curved Spacetimes. Habilitation Thesis, University of Göttingen, March 2003.

[59] R.M. Wald. General Relativity. University of Chicago Press, 1984.

[60] G.C. Wick, A.S. Wightman, E.P. Wigner. The intrinsic parity of elementary particles. Phys. Rev. 88, (1952), 101–105.

[61] D.M. Witt Vacuum space-times that admit no maximal slice Phys. Rev. Lett. 57, (1986), 1386–1389.