POINTWISE GRADIENT ESTIMATES FOR SUBQUADRATIC ELLIPTIC SYSTEMS WITH DISCONTINUOUS COEFFICIENTS

FENG ZHOU*
School of Mathematics and Statistics
Shandong Normal University, Jinan, Shandong, 250358, China

ZHENQIU ZHANG
School of Mathematical Sciences and LPMC
Nankai University, Tianjin, 300071, China
(Communicated by Man Chun Leung)

Abstract. In this paper we study subquadratic elliptic systems in divergence form with VMO leading coefficients in \( \mathbb{R}^n \). We establish pointwise estimates for gradients of local weak solutions to the system by involving the sharp maximal operator. As a consequence, the nonlinear Calderón-Zygmund gradient estimates for \( L^q \) and \( \text{BMO} \) norms are derived.

1. Introduction. The objective of this paper is to study pointwise estimates for gradients of local weak solutions to the following subquadratic elliptic systems in divergence form

\[
\text{div } \mathbf{a}(x, \nabla u) = \text{div } \mathbf{F}(x) \quad \text{in } \mathbb{R}^n
\]

with a discontinuous nonlinearity \( \mathbf{a} \) and \( n \geq 3 \). Here \( u : \mathbb{R}^n \to \mathbb{R}^N \) is a vector-valued unknown function, \( \nabla u : \mathbb{R}^n \to \mathbb{R}^{N \times n} \) denotes its gradient, and \( \text{div} \) stands for the \( \mathbb{R}^N \)-valued divergence operator. Throughout this paper, the nonlinearity \( \mathbf{a}(x, \mathbf{z}) : \mathbb{R}^n \times \mathbb{R}^{N \times n} \to \mathbb{R}^{N \times n} \) is assumed to be

\[
\mathbf{a}(x, \mathbf{z}) := \langle b(x) \mathbf{z}, \mathbf{z} \rangle_{\frac{p-2}{2}} b(x) \mathbf{z},
\]

with \( 1 < p \leq 2 \). Moreover, we assume that \( b(x) = \{ b_{k\ell}^i(x) \} \) is measurable, uniformly bounded, and satisfies the strong ellipticity condition, i.e. there exist universal constants \( 0 < \nu \leq 1 \leq L \) such that for almost all \( x \in \mathbb{R}^n \) and every \( \mathbf{z} \in \mathbb{R}^{N \times n} \),

\[
\nu |\mathbf{z}|^2 \leq \langle b(x) \mathbf{z}, \mathbf{z} \rangle \leq L |\mathbf{z}|^2.
\]

Here we adopt the standard summation convention over the repeated upper and lower indices for \( 1 \leq k, \ell \leq n \) and \( 1 \leq i, j \leq N \). Note that a standard example of such a nonlinearity \( \mathbf{a}(x, \mathbf{z}) \) satisfying this condition is the \( p \)-Laplacian, if \( b(x) \) are unitary matrices. In this case, the system (1) can be converted to the \( p \)-Laplace elliptic system

\[
\text{div } (|\nabla u|^{p-2} \nabla u) = \text{div } \mathbf{F}(x).
\]
Such kind of equations and systems have been extensively studied in mathematical physics.

For the non-homogeneous term of (1), we suppose that \( F \in L^{p'}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^{N\times n}) \) is a given matrix-valued function with Hölder conjugate \( p' := \frac{p}{p - 1} \). As usual, a solution to the system (1) is understood in the standard weak sense, that is, \( u \in W^{1,p}_{\text{loc}}(\mathbb{R}^n) \) is a local weak solution to the system (1) if

\[
\int_{\Omega} a(x, \nabla u(x)) \nabla \psi(x) \, dx = \int_{\Omega} F(x) \nabla \psi(x) \, dx
\]

for any test function \( \psi \in W^{1,p}_0(\Omega) \) and every open set \( \Omega \circ \subset \subset \mathbb{R}^n \). Our goal is to establish pointwise gradient estimates for local weak solutions to the non-homogeneous problem, which enable us to obtain various integrability estimates. More precisely, under minimal conditions on the coefficients, we want to establish the nonlinear Calderón-Zygmund estimates.

The Calderón-Zygmund theory is a classical topic in the regularity theory of solutions to partial differential equations and systems. The pioneering work can be traced back to Calderón and Zygmund [5] in the 1950s. By establishing the standard linear Calderón-Zygmund theory of singular integrals, they proved the \( L^p \) estimate for the gradient of solutions to linear elliptic equations in the whole of \( \mathbb{R}^n \). For the nonlinear Calderón-Zygmund theory, Iwaniec [13] first established the foundations for the \( p \)-Laplacian equations via the sharp maximal operators. Based on Iwaniec’s original ideas, the non-trivial extension to the systems was subsequently obtained by DiBenedetto and Manfredi [6]. In the same paper, they discussed a borderline case and obtained the first BMO estimates for the \( p \)-Laplacian system in the superquadratic case. Furthermore, the BMO estimate was extended to the full range \( p \in (1, \infty) \) by Diening et al. in [8]. By using the Hardy-Littlewood maximal operators and the Calderón-Zygmund decomposition techniques, Caffarelli and Peral [4] obtained interior \( W^{1,p} \) estimates for solutions to fully nonlinear elliptic equations.

More recently, inspired by the works [10] and [17], Breit, et al. [2] developed the regularity of the solutions for the \( p \)-Laplacian system, and established interior pointwise estimates for the gradients of local weak solutions by using the sharp maximal operators. This result enables them to obtain the Calderón-Zygmund estimates in a wide range of function spaces in \( \mathbb{R}^n \), such as Lebesgue, Lorentz and Orlicz spaces. In addition, we would like to mention some important works [14, 15, 16] on the studies of pointwise elliptic gradient regularity by involving the nonlinear Wolff type potentials.

In this paper, we overcome the difficulties arising from the discontinuous coefficients, and apply the sharp maximal operator to derive pointwise estimates for gradients of local weak solutions to the system (1). As a consequence, the nonlinear Calderón-Zygmund estimates in \( L^q \) and BMO spaces are obtained.

In order to state our main results, we introduce the following definitions and assumptions. Firstly, the coefficients matrix \( b(x) \) is assumed to be in VMO space. We say a locally integrable function \( b \) has vanishing mean oscillations, i.e. belong to the space of VMO(\( \mathbb{R}^n \)), if

\[
\lim_{R \to 0} [b]_R := \lim_{R \to 0} \sup_{0 < r \leq R} \frac{1}{|B_r|} \int_{B_r} |b(y) - (b)_{B_r}| \, dy = 0.
\]  

(4)

Here \( (b)_{B_r} := \frac{1}{|B_r|} \int_{B_r} b(y) \, dy \).
Secondly, we introduce the following restricted sharp maximal operator which plays an important role to establish the pointwise estimates. For $f \in L^q_{\text{loc}}(\mathbb{R}^n)$, let
\[
M^q_{\mathbb{R}} f(x) := \sup_{B_r \ni x, r < R} \left( \frac{1}{|B_r|} \int_{B_r} |f(y) - (f)_{B_r}|^q \, dy \right)^{\frac{1}{q}}.
\]
Given $q \geq 1$, we also define the Hardy-Littlewood maximal operator as
\[
M^q f(x) := \sup_{B_r \ni x, r < R} \left( \frac{1}{|B_r|} \int_{B_r} |f(y)|^q \, dy \right)^{\frac{1}{q}}.
\]

To establish the Calderón-Zygmund estimates, we also need the definition of functions of bounded mean oscillation (BMO), which was introduced and studied by John and Nirenberg. We say that a function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ belongs to the space of functions with bounded mean oscillation $\text{BMO}(\mathbb{R}^n)$ if
\[
\|f\|_{\text{BMO}(\mathbb{R}^n)} := \sup_{B_r \subset \mathbb{R}^n} \int_{B_r} |f(y) - (f)_{B_r}| \, dy < \infty.
\]

The well known John-Nirenberg inequality tells us that if $f \in \text{BMO}(\mathbb{R}^n)$, then $f \in L^q_{\text{loc}}(\mathbb{R}^n)$ with $q \in [1, \infty)$.

By using the foregoing concepts and tools, we are in a position to generalize the results of [2, 3] to a subquadratic elliptic system with VMO coefficients in $\mathbb{R}^n$. We present our main result as follows.

**Theorem 1.1.** Let $p \in (1, 2]$, $n \geq 3$, and $N \geq 1$. Assume that $b(x)$ satisfies the strong ellipticity condition (3) and VMO condition (4), $F \in L^q_{\text{loc}}(\mathbb{R}^n)$ with $q \in [p', q_0]$ for some $q_0$. Let $u$ be a local weak solution to the system (1) with $\nabla u \in L^p(\mathbb{R}^n)$. Then for any $\delta > 0$, there exist positive constants $R_0$ and $C = C(n, N, p, q, \nu, L, b)$ such that
\[
M^p_{\mathbb{R}} \left( |\nabla u|^{p-2} \nabla u \right)(x) \leq C M^p_{\mathbb{R}} F(x) + \delta M^p_{\mathbb{R}} \left( |\nabla u|^{p-2} \nabla u \right)(x)
\]
for every $R \in (0, R_0)$, and for almost every $x \in \mathbb{R}^n$.

The key step towards the proof of Theorem 1.1 is the estimate (65) which not only presents the decay estimates of the gradients of weak solutions, but also reveals the relationship between the oscillations of solutions and the VMO coefficients. By applying Theorem 1.1, we establish the following results.

**Theorem 1.2.** Assume that $p \in (1, 2]$, $b(x) \in \text{VMO}(\mathbb{R}^n)$ satisfies (3), and $u$ is a local weak solution to the system (1) with $\nabla u \in L^p(\mathbb{R}^n)$.

\textbf{(L$^p$-estimate)} For $q \in [p', q_0]$, if $F \in L^q(\mathbb{R}^n)$ and $\nabla u \in L^{(p-1)q}(\mathbb{R}^n)$, then
\[
\| |\nabla u|^{p-2} \nabla u \|_{L^q(\mathbb{R}^n)} \leq C \| F \|_{L^q(\mathbb{R}^n)}.
\]

\textbf{(BMO-estimate)} If $F \in \text{BMO}_{\text{loc}}(\mathbb{R}^n)$ and $B_{2R} \subset \subset \mathbb{R}^n$, then
\[
\| |\nabla u|^{p-2} \nabla u \|_{\text{BMO}(B_R)} \leq C \| F \|_{\text{BMO}(B_{2R})} + C \left( \int_{B_{2R}} |\nabla u|^p \, dy \right)^{\frac{p-1}{p}}.
\]

Here the constants $C$ in (9) and (10) depend only on $n, N, p, q, \nu, L, b,$ and $\tilde{R}$.
Remark 1. The pointwise estimate in Theorem 1.1 provides an unified approach to gradient regularity of the nonlinear Calderón-Zygmund theory in not only the classical Lebesgue space, but also the BMO space. In addition, Theorem 1.2 could easily deduce the classical $L^p$ and BMO estimates.

From now on we adopt the following standard notations: we set $B_R(x_0) = \{ y \in \mathbb{R}^n \mid |y - x_0| < R \}$ for the open ball centered at the point $x_0 \in \mathbb{R}^n$ and with radius $R > 0$. We consider function spaces $X(\mathbb{R}^n, \mathbb{R}^{N \times n})$ of matrix valued measurable maps defined on $\mathbb{R}^n$, for example, the classical Lebesgue spaces $L^q(\mathbb{R}^n, \mathbb{R}^{N \times n})$ and Sobolev space $W^{1,q}(\mathbb{R}^n, \mathbb{R}^{N \times n})$. The symbol $A \approx B$ means that $A \leq CB$ and $B \leq CA$ hold for some constant $C > 0$ simultaneously. Finally, the constant in a chain of inequalities may change from line to line, as long as the dependence of such a constant does not interfere with the essence of the argument.

This paper is organized as follows. In the next section, we collect some notations and known preliminary results, while in section 3, we introduce the comparison system. In section 4, we deal with the degenerate case, and in section 5, under the non-degenerate assumption, we deduce the decay estimate. In the last section, we complete the proofs of Theorem 1.1 and Theorem 1.2.

2. Preliminaries. In this section, we give some auxiliary tools, concepts and no-
tations, and present some fundamental results that will be used in this paper. We first introduce the auxiliary maps $V$ and $A$ by

$$V(z) := |z|^\frac{p-2}{2} z \quad \text{and} \quad A(z) := |z|^{p-2} z \quad (11)$$

for any $z \in \mathbb{R}^{N \times n}$. Note that $V$ and $A$ are both locally bi-Lipschitz bijections on $\mathbb{R}^{N \times n}$. Obviously, $|V(z)|^2 = |A(z)|^p = |z|^p$. There is an intimate relationship between the auxiliary maps and the following inequality, see [1, Lemma A.2]. For any $z_1, z_2 \in \mathbb{R}^{N \times n}$, there exists a positive constant $C$ independent of the choices of $z_1$ and $z_2$ such that

$$\int_0^1 t z_1 + (1-t) z_2 \bigg|^{p-2} dt \approx C \left( |z_1|^2 - |z_2|^2 \right)^{\frac{p-2}{2}}. \quad (12)$$

To handle the subquadratic case, we next invoke the following definition which plays a key role in this paper. Indeed, these functions are standard examples of the $N$-functions in the literature, see [7, Section 2]. For $a \in \mathbb{R}$ and $1 < p \leq 2$, we define the shifted $p$-power function $\Phi_{p,a} : [0, \infty) \to [0, \infty)$ by

$$\Phi_{p,a}(t) := (a^2 + t^2)^{\frac{p-2}{2}} t^2. \quad (13)$$

We observe that $\Phi_{p,a}$ is a Young function and satisfies Young type inequality below. For any $\zeta > 0$ and any pair of non-negative real numbers $t$, $s$, there exists $c(\zeta, p) > 0$ such that

$$ts \leq \zeta \Phi_{p,a}(t) + c(\zeta, p) \Phi_{p', a^{p-1}}(s). \quad (14)$$

The next lemma from [2, (2.5)] will best reflect the connections between auxiliary maps and $\Phi_{p,a}$.

Lemma 2.1. For any $z_1, z_2 \in \mathbb{R}^{N \times n}$, there holds

$$\langle A(z_1) - A(z_2), z_1 - z_2 \rangle \approx |V(z_1) - V(z_2)|^2 \approx \left( |z_1|^2 + |z_2|^2 \right)^{\frac{p-2}{2}} |z_1 - z_2|^2$$

$$\approx \Phi_{p, |z_1|} \left( |z_1 - z_2| \right) \approx \Phi_{p', |A(z_1)|} \left( |A(z_1) - A(z_2)| \right) \quad (15)$$
up to equivalence constants depending on \(n, N, p\). Similarly, there holds
\[
|A(z_1) - A(z_2)| \approx \left( |z_1|^2 + |z_2|^2 \right)^{\frac{p-2}{2}} |z_1 - z_2|.
\] (16)

The different representations of (15) will be useful at each stage of our proofs. When we discuss the decay estimates, the expression involving \(A\) will appear. The representation with \(V\) is also helpful, as most of the information on \(u\) will be expressed in terms of \(V(\nabla u)\). The function \(\Phi_{p, a}\) will help us to simplify the proofs.

Lemma 2.2 enables us to transfer information among shifted \(p\)-power functions.

**Lemma 2.2.** For any \(b, d \in \mathbb{R}^{N \times n}\) and any \(\gamma \in (0, 1]\), there is \(C = C(n, N, p) > 0\) such that
\[
\Phi_{p', |b|}(t) \leq \gamma \Phi_{p', |b|}(|b - d|) + C \gamma^{\frac{p-2}{p-2}} \Phi_{p', |d|}(t).
\]

**Proof.** We divide our proof into two cases: \(0 \leq t \leq \gamma^{\frac{1}{2}}|b - d|\) and \(t > \gamma^{\frac{1}{2}}|b - d|\).

**Case 1.** Suppose that \(t \leq \gamma^{\frac{1}{2}}|b - d|\). Then we have
\[
\Phi_{p', |b|}(t) \leq \left( |b|^2 + \gamma |b - d|^2 \right)^{\frac{p-2}{2}} \gamma |b - d|^2 \leq \gamma \Phi_{p', |b|}(|b - d|).
\]

**Case 2.** We consider the case that \(t > \gamma^{\frac{1}{2}}|b - d|\). It is clear that \(t \geq \gamma^{\frac{1}{2}}|b| - \gamma^{\frac{1}{2}}|d|\). Let \(\omega = \frac{\gamma^{\frac{1}{2}} + 1}{\gamma^{\frac{1}{2}}}\). It follows that \(t + |b| \leq \omega (t + |d|)\). Hence
\[
\Phi_{p', |b|}(t) \leq \left( \omega (t + |d|) \right)^2 t^{\frac{p-2}{2}} \leq C \left( \frac{\gamma^{\frac{1}{2}} + 1}{\gamma^{\frac{1}{2}}} \right)^{p-2} \left( |d|^2 + t^2 \right)^{\frac{p-2}{2}} t^2 \leq C \gamma^{\frac{p-2}{p-2}} \Phi_{p', |d|}(t).
\]

Combining the above cases, we finally prove the lemma.

By taking \(b = A(z_1)\) and \(d = A(z_2)\) in Lemma 2.2, one has
\[
\Phi_{p', |A(z_1)|}(t) \leq \gamma \Phi_{p', |A(z_1)|}(|A(z_1) - A(z_2)|) + C \gamma^{\frac{p-2}{p-2}} \Phi_{p', |A(z_2)|}(t).
\]

Then by Lemma 2.1, we deduce the following inequality.

**Corollary 1.** For any \(\gamma \in (0, 1]\), and every \(z_1, z_2 \in \mathbb{R}^{N \times n}\), there exists a positive constant \(C = C(n, N, p)\) such that
\[
\Phi_{p', |A(z_1)|}(t) \leq \gamma \left| V(z_1) - V(z_2) \right|^2 + C \gamma^{\frac{p-2}{p-2}} \Phi_{p', |A(z_2)|}(t).\] (17)

In addition, we need the next lemma from [2, Lemma 2.3] which concerns self-improving properties of reverse Hölder inequality for shifted functions.

**Lemma 2.3.** Let \(\tau \geq 0, r > 0, g \in L^p_{\text{loc}}(\mathbb{R}^n)\) and \(h \in L^p_{\text{loc}}(\mathbb{R}^n)\). Assume that there exist \(0 < \sigma < 1\) and \(C_1 > 0\) such that
\[
\int_{B_r} \Phi_{p, \tau}(|g|) \, dx \leq C_1 \left[ \int_{B_{2r}} \Phi_{p, \tau}(|g|) \, dx \right]^\sigma + \int_{B_{2r}} |h| \, dx.
\]

Then there exists a number \(C_2(n, N, p, \sigma, C_1) > 0\) such that
\[
\int_{B_r} \Phi_{p, \tau}(|g|) \, dx \leq C_2 \Phi_{p, \tau} \left( \int_{B_{2r}} |g| \, dx \right) + \int_{B_{2r}} |h| \, dx.
\]
For the rest of this section, we provide some technical results which are also in integral form. In order to discuss the oscillation estimates, we will use the following estimates frequently. For any \( f \in L^2(E) \) with \( E \) a measurable subset of \( \mathbb{R}^n \), and for any \( h \in \mathbb{R}^{N \times n} \), we have

\[
\left( \int_E |f(x) - (f)_E|^2 \, dx \right)^{\frac{1}{2}} = \min_{h \in \mathbb{R}^{N \times n}} \left( \int_E |f(x) - h|^2 \, dx \right)^{\frac{1}{2}}.
\]  

(18)

For \( p \in [1, \infty) \) and every \( f \in L^p(E) \), there holds

\[
\left( \int_E |f(x) - (f)_E|^p \, dx \right)^{\frac{1}{p}} \leq 2 \min_{h \in \mathbb{R}^{N \times n}} \left( \int_E |f(x) - h|^p \, dx \right)^{\frac{1}{p}}.
\]  

(19)

One of the main techniques of this paper is expressing the integral mean in terms of \( A \) and \( V \) in the following sense. Since \( A \) and \( V \) are surjective, for any \( g \in L^p(B_r, \mathbb{R}^{N \times n}) \), there exist \( g_A, g_V \in \mathbb{R}^{N \times n} \) such that

\[
A(g_A) = (A(g))_{B_r}, \quad \text{and} \quad V(g_V) = (V(g))_{B_r}.
\]  

(20)

These integral means have the subsequent equivalence relations. We refer to [8, Lemma A.2].

**Lemma 2.4.** Let \( r > 0 \). For any \( g \in L^p(B_r, \mathbb{R}^{N \times n}) \) there are \( g_A \) and \( g_V \) satisfying (20). Then

\[
\int_{B_r} |V(g) - V(g_V)|^2 \, dx \approx \int_{B_r} |V(g) - V(g_A)|^2 \, dx
\]

\[
\approx \int_{B_r} |V(g) - V((g)_{B_r})|^2 \, dx.
\]

Via (20), for a local weak solution \( u \) to the system (1) with \( \nabla u \in L^p(\mathbb{R}^n) \), there are constant matrices \( Q, P \in \mathbb{R}^{N \times n} \) such that

\[
A(Q) = (A(\nabla u))_{B_{2R}} \quad \text{and} \quad A(P) = (A(\nabla u))_{B_{4R}}.
\]  

(21)

We end this section by discussing reverse Hölder inequality, self-improving properties, and higher integrability of gradients of local weak solutions. By [8, Lemma 3.2], there exist \( \varpi = \varpi(n, N, p, \nu, L) \in (0, 1) \) and \( C = C(n, N, p, \nu, L) > 0 \) such that

\[
\int_{B_R} |V(\nabla u) - V(g)|^2 \, dx \leq C \left( \int_{B_{2R}} |V(\nabla u) - V(g)|^{2\varpi} \, dx \right)^{\frac{1}{p}} + C \int_{B_{2R}} \Phi_{p', |A(g)|} \left( |F - (F)_{B_{2R}}| \right) \, dx
\]  

(22)

for any matrix \( g \). This reverse Hölder inequality can be proved by choosing the test function \( \varphi = \eta^p(u - w) \), where \( w \) is a linear function satisfying that \( \nabla w = g \) and \( (u - w)_{B_{2R}} = 0 \).

Next by using (22), (15), and Lemma 2.3, we find that the estimate of functional involving \( V \) can be converted to that of functional involving \( A \).

**Lemma 2.5.** Assume that \( u \) is a local weak solution to the system (1) with \( \nabla u \in L^p(\mathbb{R}^n) \), \( F \in L_{rad}^p(\mathbb{R}^n) \) and \( b \) satisfying (3). Let \( g \in \mathbb{R}^{N \times n} \) be an arbitrary matrix, and \( R > 0 \). Then there exists a constant \( c = c(n, N, p, \nu, L) > 0 \) such that

\[
\int_{B_R} |V(\nabla u) - V(g)|^2 \, dx \leq c \Phi_{p', |A(g)|} \left( \int_{B_{2R}} |A(\nabla u) - A(g)| \, dx \right)
\]
Furthermore, with the help of \( g = 0 \) in (22), we obtain the higher integrability of \( \nabla u \) by applying Gehring lemma.

**Lemma 2.6.** Let \( u \) be a local weak solution to the system (1) with \( \nabla u \in L^p(\mathbb{R}^n) \), \( F \in L^q_{loc}(\mathbb{R}^n) \) with \( q \in [p', q_0] \) for some \( q_0 \), and \( b \) satisfying (3). Then there exist positive constants \( s \in \left( p, \frac{pq}{n+p} \right) \) and \( C = C(n, N, p, q, \nu, L) \) such that \( \nabla u \in L^s(\mathbb{R}^n) \) with

\[
\left( \frac{\int_{B_R} |\nabla u|^s \, dx}{\int_{B_R} r^\frac{p}{2m} \, dr} \right) \leq C \left( \frac{\int_{B_R} |\nabla u|^p \, dx}{\int_{B_R} r^\frac{p}{2m} \, dr} \right)^\frac{s}{p} + C \left( \frac{\int_{B_R} |F|_B^q \, dx}{\int_{B_R} r^\frac{p}{2m} \, dr} \right)^\frac{s}{q},
\]

Proof. One has the following reverse Hölder inequality

\[
\left( \frac{\int_{B_r} |\nabla u|^{p-2} \nabla u \cdot \nabla u \, dx}{\int_{B_r} r^\frac{p}{2m} \, dr} \right) \leq C \left( \frac{\int_{B_r} |\nabla u|^p \, dx}{\int_{B_r} r^\frac{p}{2m} \, dr} \right)^\frac{1}{p} + C \left( \frac{\int_{B_r} |F - (F)_{B_r}|^q \, dx}{\int_{B_r} r^\frac{p}{2m} \, dr} \right)^\frac{1}{q},
\]

where \( p_* := \frac{np}{n+p} < p \). By Gehring lemma, there exist constants \( s \in (p, \frac{pq}{n+p}) \) with \( q \in [p', q_0] \) such that

\[
\left( \frac{\int_{B_r} |\nabla u|^s \, dx}{\int_{B_r} r^\frac{p}{2m} \, dr} \right) \leq C \left( \frac{\int_{B_r} |\nabla u|^p \, dx}{\int_{B_r} r^\frac{p}{2m} \, dr} \right)^\frac{s}{p} + C \left( \frac{\int_{B_r} |F - (F)_{B_r}|^q \, dx}{\int_{B_r} r^\frac{p}{2m} \, dr} \right)^\frac{s}{q},
\]

which yields the desired result.

**3. Comparison system.** In this section, our purpose is to construct a homogeneous system with constant coefficients in a ball for which we have known regularity results, and compare a weak solution of the original system to that of the comparison problem. To this end, we compare a local weak solution \( u \) of the system (1) to the unique weak solution \( v \in W^{1,p}(B_R) \) of the frozen Dirichlet problem

\[
\begin{align*}
\text{div} a_R (\nabla v) &= 0 \quad \text{in} \ B_R, \\
v &= u \quad \text{on} \ B_R \setminus B_R
\end{align*}
\]

with

\[
a_R (\nabla v) := \langle (b)_{B_R} \nabla v, \nabla v \rangle \frac{p-2}{2} (b)_{B_R} \nabla v.
\]

We first explore the properties of the vector field \( a_R \) and evaluate the difference between \( \nabla u \) and \( \nabla v \). Note that \( v \in W^{1,p}(B_R) \) is the weak solution to (24) if

\[
\int_{B_R} \langle (b)_{B_R} \nabla v, \nabla v \rangle \frac{p-2}{2} \langle (b)_{B_R} \nabla v, \nabla h \rangle \, dx = 0
\]

for every \( h \in W^{1,p}_0(B_R) \). It is clear that \( (b)_{B_R} \) is bounded and satisfies the strong ellipticity condition with the same constants \( \nu \) and \( L \) as in (3). For the vector field \( a_R \) with fixed coefficients, we need the following lemma.

**Lemma 3.1.** For any \( z_1, z_2 \in \mathbb{R}^{N \times n} \), there exists \( C = C(n, N, p, \nu, L) > 0 \) such that

\[
\langle a_R (z_1) - a_R (z_2), z_1 - z_2 \rangle \approx C |V(z_1) - V(z_2)|^2.
\]
Proof. We define a function $\varphi(z)$ by
\[\varphi(z) := \frac{1}{p} \langle (b)_{B_R} z, z \rangle^{\frac{p}{2}}\]
for $z \in \mathbb{R}^{N \times n}$. Then one has the Euclidean gradient by $\nabla \varphi(z) = a_R(z)$. Moreover,
\[\langle \nabla^2 \varphi(z), g \rangle = C_p \langle (b)_{B_R} z, z \rangle^{\frac{p-2}{2}} \langle (b)_{B_R} g, g \rangle \quad (27)\]
holds for any $g \in \mathbb{R}^{N \times n}$. For $t \in [0, 1]$, we set $z_t := t z_1 + (1-t) z_2$. Then by (27),
one has
\[
\Upsilon := \langle a_R(z_1) - a_R(z_2), z_1 - z_2 \rangle = \langle \nabla \varphi(z_1) - \nabla \varphi(z_2), z_1 - z_2 \rangle
\]
is
ip \frac{d}{dt} \nabla \varphi(t z_1 + (1-t) z_2), z_1 - z_2 \rangle dt
\]
is
C \int_0^1 \langle (b)_{B_R} t z_t, z_t \rangle^{\frac{p-2}{2}} \langle (b)_{B_R} (z_1 - z_2), z_1 - z_2 \rangle dt.
\]
Using the strong ellipticity condition and the inequalities (12), (15), we obtain
\[
\Upsilon \approx C \int_0^1 |t z_1 + (1-t) z_2|^{p-2} |z_1 - z_2|^2 dt
\]
is
C \left( |z_1|^2 + |z_2|^2 \right)^{\frac{p-2}{2}} |z_1 - z_2|^2 \approx C |\nabla(z_1) - \nabla(z_2)|^2.
\]
This completes the proof. $\square$

Using the definition (4) and Lemma 2.6, one derives the following approximation lemma.

\textbf{Lemma 3.2.} Let $F \in L^q_{\text{loc}}(\mathbb{R}^n)$ with $q \in [p', q_0]$, and $b \in \text{VMO}(\mathbb{R}^n)$ satisfy (3). Assume that $u$ and $v$ are weak solutions to the system (1) and (24) respectively. Let $s$ be the exponent appearing in Lemma 2.6. For any $\zeta > 0$, there exists a number $C = C(\zeta, s, N, \rho, q, p, q, s, v, L) > 0$ such that
\[
\left| \int_{B_R} \langle a(x, \nabla u) - a_R(\nabla u), \nabla u - \nabla v \rangle \, dx \right|
\]
is
\[\leq \zeta \int_{B_R} |\nabla(\nabla u) - \nabla(\nabla v)|^2 \, dx
\]
is
C [b]^{\frac{p-2}{2}} \left[ \int_{B_{2R}} |\nabla u|^p \, dx + \left( \int_{B_{2R}} |F - (F)_{B_{2R}}|^q \, dx \right)^{\frac{p}{q}} \right].
\]

\textbf{Proof.} In view of (2) and (25), we have
\[
I := \int_{B_R} \langle a(x, \nabla u) - a_R(\nabla u), \nabla u - \nabla v \rangle \, dx
\]
is
= \int_{B_R} \langle (b(x) \nabla u, \nabla u \rangle^{\frac{p-2}{2}} \langle (b(x) \nabla u, \nabla u - \nabla v \rangle \, dx
\]
is
- \int_{B_R} \langle (b)_{B_R} \nabla u, \nabla u \rangle^{\frac{p-2}{2}} \langle (b)_{B_R} \nabla u, \nabla u - \nabla v \rangle \, dx.
\]
To bound the above expression, we apply the inequality
\[
| I | \leq C_1 \int_{B_R} | b - (b)_{B_R} | |A(\nabla u)| |\nabla u - \nabla v| \, dx \\
\leq \zeta \int_{B_R} \Phi_{p',|A(\nabla u)|}(|\nabla u - \nabla v|) \, dx \\
+ C_2 \int_{B_R} \Phi_{p',|A(\nabla u)|}(b - (b)_{B_R} | |A(\nabla u)|) \, dx \\
\leq \zeta \int_{B_R} |V(\nabla u) - V(\nabla v)|^2 \, dx + C_3 \int_{B_R} |b - (b)_{B_R}|^2 |\nabla u|^p \, dx,
\]
where the last inequality follows from (13) with
\[
\Phi_{p',|A(\nabla u)|}(b - (b)_{B_R} | |A(\nabla u)|) \\
\leq \left( |A(\nabla u)|^2 + |b - (b)_{B_R}|^2 |A(\nabla u)|^2 \right)^{\frac{p-2}{2}} |b - (b)_{B_R}|^2 |A(\nabla u)|^p \\
\leq C |b - (b)_{B_R}|^2 |A(\nabla u)|^{p'} \leq C |b - (b)_{B_R}|^2 |\nabla u|^p.
\]
Let \( t := \frac{2s}{s-p} > 1 \), where \( s \) is given in Lemma 2.6. Using Hölder inequality, one gets
\[
| I | \leq \zeta \int_{B_R} |V(\nabla u) - V(\nabla v)|^2 \, dx + C_3 \left( \int_{B_R} |b - (b)_{B_R}|^t \, dx \right)^{\frac{p}{t}} \left( \int_{B_R} |\nabla u|^s \, dx \right)^{\frac{t}{s}}.
\]
Due to the definition (4) and the boundedness of \( b \), there exists \( C_4 = C_4(t, \nu, L) > 0 \) such that
\[
\int_{B_R} |b - (b)_{B_R}|^t \, dx \leq C_4 |b|_{B_R}.
\]
Thus we obtain
\[
| I | \leq \zeta \int_{B_R} |V(\nabla u) - V(\nabla v)|^2 \, dx + C_4 |b|_{B_R}^{\frac{t}{p}} \left( \int_{B_R} |\nabla u|^s \, dx \right)^{\frac{t}{s}}.
\]
By applying Lemma 2.6, we complete the proof. \( \Box \)

We are now in a position to estimate the difference between \( V(\nabla u) \) and \( V(\nabla v) \) as follows.

**Lemma 3.3.** Assume that \( F \in L^q_{\text{loc}}(\mathbb{R}^n) \) with \( q \in [p', q_0] \), and \( b \in VMO(\mathbb{R}^n) \) satisfies (3). Let \( u \) and \( v \) be weak solutions to the system (1) and (24) respectively, and let \( s \) be the exponent appearing in Lemma 2.6. For any \( \theta > 0 \), \( R > 0 \), there exist positive constants \( C_\theta = C(\theta, n, N, p, \nu, L) \) and \( C = C(n, N, p, q, s, \nu, L) \) such that
\[
\int_{B_R} |V(\nabla u) - V(\nabla v)|^2 \, dx \leq C_\theta R^s
\] (28)
\[ \gamma > (17), \text{we deduce that for any} \ 2.1, \text{we have} \]

\[
\int_{B_{2\gamma}} |A(\nabla u) - A(Q)| \, dx \\
+ C_\delta \int_{B_{2\gamma}} \Phi_{\gamma, A(\nabla u)} \left( |F - (F)_{B_{2\gamma}}| \right) \, dx \\
+ \gamma \int_{B_{2\gamma}} |\nabla u|^p \, dx + \left( \int_{B_{2\gamma}} |F - (F)_{B_{2\gamma}}|^q \, dx \right)^{\frac{p}{q}} 
\]

where \( Q \in \mathbb{R}^{N \times n} \) is given by (21).

**Proof.** We start by choosing \( u - v \in W_0^{1,p}(B_R) \) as a test function for the system (1) and (24), respectively. By the relation (26) and Lemma 3.2, for any \( \zeta > 0 \), we get

\[
\int_{B_R} |V(\nabla u) - V(\nabla v)|^2 \, dx \\
\leq C \int_{B_R} \langle a_R(\nabla u) - a_R(\nabla v), \nabla u - \nabla v \rangle \, dx \\
\leq C \int_{B_R} \langle a(x, \nabla u), \nabla u - \nabla v \rangle \, dx \\
+ \zeta \int_{B_R} \langle a_R(\nabla u) - a(x, \nabla u), \nabla u - \nabla v \rangle \, dx \\
\leq C \int_{B_R} \langle F - (F)_{B_R}, \nabla u - \nabla v \rangle \, dx + \zeta \int_{B_R} |V(\nabla u) - V(\nabla v)|^2 \, dx \\
+ C |b|_{R}^{\frac{p-2}{2}} \left[ \int_{B_{2\gamma}} |\nabla u|^p \, dx + \left( \int_{B_{2\gamma}} |F - (F)_{B_{2\gamma}}|^q \, dx \right)^{\frac{p}{q}} \right].
\]

For the first integral on the right side, by Young type inequality (14) and Lemma 2.1, we have

\[
C \int_{B_R} \langle F - (F)_{B_R}, \nabla u - \nabla v \rangle \, dx \\
\leq \zeta \int_{B_R} \Phi_{p, |\nabla u|} \left( |\nabla u - \nabla v| \right) \, dx + C \int_{B_R} \Phi_{p, |A(\nabla u)|} \left( |F - (F)_{B_R}| \right) \, dx \\
\leq \zeta \int_{B_R} |V(\nabla u) - V(\nabla v)|^2 \, dx + C \int_{B_R} \Phi_{p, |A(\nabla u)|} \left( |F - (F)_{B_R}| \right) \, dx
\]

with the same \( \zeta > 0 \) as in (29). Hence by taking \( \zeta < \frac{1}{2} \), and applying the inequality (17), we deduce that for any \( \gamma > 0 \), there is a constant \( C_\gamma > 0 \) such that

\[
\int_{B_R} |V(\nabla u) - V(\nabla v)|^2 \, dx \\
\leq C \int_{B_R} \Phi_{p, |A(\nabla u)|} \left( |F - (F)_{B_R}| \right) \, dx \\
+ C |b|_{R}^{\frac{p-2}{2}} \int_{B_{2\gamma}} |\nabla u|^p \, dx + C |b|_{R}^{\frac{p-2}{2}} \left( \int_{B_{2\gamma}} |F - (F)_{B_{2\gamma}}|^q \, dx \right)^{\frac{p}{q}} \\
\leq \gamma \int_{B_R} |V(\nabla u) - V(\nabla Q)|^2 \, dx + C_\gamma \int_{B_{2\gamma}} \Phi_{p, |A(\nabla u)|} \left( |F - (F)_{B_{2\gamma}}| \right) \, dx
\]
In order to finish the proof, it remains to apply Lemma 2.5, i.e. substitute (23) with \( g = Q \) into (30). This completes the proof of (28).

Before proceeding further, we make use of the decay estimate of gradients of the “\( a_R \)-harmonic” maps of the system (24) as follows. See [8, 9, 18].

**Lemma 3.4.** Let \( v \) be the unique weak solution to (24). There exist constants \( \alpha = \alpha(n,N,p) > 0 \) and \( c_1 = c_1(n,N,p) > 0 \) such that for every \( 0 < \sigma \leq \frac{1}{2} \),

\[
\sup_{y,z \in B_{\sigma R}} \left| V(\nabla v)(y) - V(\nabla v)(z) \right|^2 \leq c_1 \sigma^{2\alpha} \int_{B_R} \left| V(\nabla v) - (V(\nabla v))_{B_R} \right|^2 \, dx. \tag{31}
\]

Furthermore, for any \( 0 < \kappa < \min \left\{ 1, \frac{2\alpha}{p} \right\} \), there is a constant \( c_2 = c_2(n,N,p,\kappa) > 0 \) such that for every \( 0 < \sigma \leq \frac{1}{2} \),

\[
\sup_{y,z \in B_{\sigma R}} \left| A(\nabla v)(y) - A(\nabla v)(z) \right| \leq c_2 \sigma^\kappa \int_{B_R} \left| A(\nabla v) - (A(\nabla v))_{B_R} \right| \, dx. \tag{32}
\]

With the help of Lemma 3.3 and Lemma 3.4, we establish decay estimates for \( \nabla u \).

**Lemma 3.5.** Assume that \( u \) is a local weak solution to the system (1) with \( \nabla u \in L^p(\mathbb{R}^n) \), \( F \in L_{\text{loc}}^q(\mathbb{R}^n) \) with \( q \in [p', q_0] \), and \( b \in \text{VMO}(\mathbb{R}^n) \) satisfying (3), \( \alpha \) is defined in Lemma 3.4, \( s \) is the exponent appearing in Lemma 2.6, and \( Q \in \mathbb{R}^{N \times n} \) is given by (21). For any \( 0 < \sigma \leq \frac{1}{2} \), there exist positive constants \( c = c(n,N,p,\nu,L) \), \( c_\sigma = c_\sigma(n,N,p,\nu,L,\sigma) \), and \( C = C(n,N,p,q,s,\nu,L) \) such that

\[
\begin{align*}
\int_{B_{\sigma R}} & \left| V(\nabla u) - (V(\nabla u))_{B_{\sigma R}} \right|^2 \, dx \\
& \leq c \sigma^{2\alpha} \Phi_{p',|A(Q)|} \left( \int_{B_{2\sigma R}} |A(\nabla u) - A(Q)| \, dx \right) \\
& \quad + c_\sigma \int_{B_{2\sigma R}} \Phi_{p',|A(Q)|} \left( |F - (F)_{B_{2\sigma R}}| \right) \, dx \\
& \quad + C |b|^{\frac{s}{2}} \left[ \int_{B_{2\sigma R}} |\nabla u|^p \, dx + \left( \int_{B_{2\sigma R}} |F - (F)_{B_{2\sigma R}}|^q \, dx \right)^{\frac{p}{q}} \right].
\end{align*}
\]

**Proof.** Making use of the property (18) and the decay estimate (31) of \( \nabla v \), one obtains that

\[
\begin{align*}
\int_{B_{\sigma R}} & \left| V(\nabla u) - (V(\nabla u))_{B_{\sigma R}} \right|^2 \, dx \\
& \leq \frac{c}{\sigma^\alpha} \int_{B_{\sigma R}} |V(\nabla u) - V(\nabla v)|^2 \, dx + c \int_{B_{\sigma R}} |V(\nabla v) - (V(\nabla v))_{B_{\sigma R}}|^2 \, dx \\
& \leq \frac{c}{\sigma^\alpha} \int_{B_{\sigma R}} |V(\nabla u) - V(\nabla v)|^2 \, dx + c \sigma^{2\alpha} \int_{B_{\sigma R}} |V(\nabla v) - (V(\nabla v))_{B_{\sigma R}}|^2 \, dx.
\end{align*}
\]

Note that from (18), we also have

\[
\int_{B_R} |V(\nabla v) - (V(\nabla v))_{B_R}|^2 \, dx
\]
\[
\begin{align*}
&\leq \int_{B_R} |V(\nabla u) - V(\nabla v)|^2 \, dx + \int_{B_R} |V(\nabla u) - (V(\nabla u))_{B_R}|^2 \, dx \\
&\leq \int_{B_R} |V(\nabla u) - V(\nabla v)|^2 \, dx + \int_{B_R} |V(\nabla u) - V(Q)|^2 \, dx.
\end{align*}
\]

Hence we continue to obtain
\[
\int_{B_{2R}} |V(\nabla u) - (V(\nabla u))_{B_{2R}}|^2 \, dx \\
\leq c\sigma^{2n} \int_{B_R} |V(\nabla u) - V(Q)|^2 \, dx + c \left(\sigma^{2n} + \frac{1}{\sigma^m}\right) \int_{B_R} |V(\nabla u) - V(\nabla v)|^2 \, dx.
\]

Using Lemma 2.5 with \(g = Q\), and Lemma 3.3, we finally reach the conclusion. \(\square\)

**Remark 2.** Our decay estimate will be established depending on whether the term \(\int_{B_{2R}} |V(\nabla u)|^2 \, dx\) is “small” or “large” compared to the integral oscillation \(\int_{B_{2R}} |V(\nabla u) - (V(\nabla u))_{B_{2R}}|^2 \, dx\). We shall refer to the former case as the “degenerate case” appearing in the next section, and to the latter as the “non-degenerate case” which will be discussed in Section 5. Moreover, in the next two sections, we always assume that \(F \in L^q_{\text{loc}}(\mathbb{R}^n)\) with \(q \in [p', q_0]\), and \(b \in \text{VMO}(\mathbb{R}^n)\) satisfies the strong ellipticity condition (3).

4. **Degenerate case.** In this section, we consider the situation that \(u\) is a local weak solution to the system (1) and satisfies the following degenerate condition

\[
\int_{B_{2R}} |V(\nabla u)|^2 \, dx \leq \frac{1}{\lambda} \int_{B_{2R}} |V(\nabla u) - (V(\nabla u))_{B_{2R}}|^2 \, dx
\]

for some fixed number \(\lambda \in (0, \frac{1}{4})\). We recall that \(Q\) and \(P\) are given in (21). By the assumption (33), the property (18), and \(\lambda < \frac{1}{4}\), we have the boundedness of \(|V(Q)|\) as

\[
|V(Q)|^2 \leq 2 \int_{B_{2R}} |V(\nabla u)|^2 \, dx + 2 \int_{B_{2R}} |V(\nabla u) - V(Q)|^2 \, dx
\]

\[
\leq \frac{2}{\lambda} \int_{B_{2R}} |V(\nabla u) - (V(\nabla u))_{B_{2R}}|^2 \, dx + 2 \int_{B_{2R}} |V(\nabla u) - V(Q)|^2 \, dx \\
\leq \frac{3}{\lambda} \int_{B_{2R}} |V(\nabla u) - V(Q)|^2 \, dx.
\]

The next lemma tells us that \(V(\nabla u) - V(Q)\) can be “controlled” by \(A(\nabla u) - A(P)\).

**Lemma 4.1.** Let \(u\) be a local weak solution to the system (1) satisfying the degenerate condition (33) for some \(\lambda \in (0, \frac{1}{4})\). Then there exists \(c = c(n, N, p, \nu, L) > 0\) such that

\[
\int_{B_{2R}} |V(\nabla u) - V(Q)|^2 \, dx + \Phi_{p', |A(Q)|} \left(\int_{B_{2R}} |A(\nabla u) - A(Q)| \, dx\right)
\]

\[
\leq c\lambda^{\frac{p}{2p' - 2}} \left[\left(\int_{B_{4R}} |A(\nabla u) - A(P)| \, dx\right)^{p'} + \int_{B_{4R}} |F - (F)_{B_{4R}}|^{p'} \, dx\right].
\]

**Proof.** Recalling the inequality (17) and taking \(z_1 = Q, z_2 = 0\), we first find that for any \(\gamma \in (0, 1]\), there exists a constant \(C_1 > 0\) such that

\[
\Phi_{p', |A(Q)|}(t) \leq \gamma |V(Q)|^2 + C_1 \gamma^{\frac{2}{p' - 2}} |Q'|
\]

(35)
Then by applying Lemma 2.5 with $g = Q$, and using inequality (35), one has
\[
\int_{B_{2R}} |V(\nabla u) - V(Q)|^2 \, dx + \Phi'_{p'}(x)\int_{B_{2R}} |A(\nabla u) - A(Q)| \, dx \\
\leq C_2 \Phi'_{p'}(x)\int_{B_{4R}} |A(\nabla u) - A(Q)| \, dx \\
+ C_2 \int_{B_{4R}} \Phi'_{p'}(x)\left( |F - (F)_{B_{4R}}| \right) \, dx \\
\leq \gamma C_2 |V(Q)|^2 + C_1 C_2 \gamma \frac{p-2}{p} \left( \int_{B_{4R}} |A(\nabla u) - A(Q)| \, dx \right)^{p'} \\
+ \gamma C_2 |V(Q)|^2 + C_1 C_2 \gamma \frac{p-2}{p} \int_{B_{4R}} \left| F - (F)_{B_{4R}} \right|^{p'} \, dx,
\]
where $C_2$ is dependent on the constant appearing in Lemma 2.5, and the parameter $\gamma \in (0, 1]$ will be expressed in terms of $\lambda$ later. Next (34) implies that
\[
2 \gamma C_2 |V(Q)|^2 \leq \gamma \frac{6C_2}{\lambda} \int_{B_{2R}} |V(\nabla u) - V(Q)|^2 \, dx.
\]
By taking $\gamma := \frac{\sqrt{2}C_2}{12\sqrt{2}} \in (0, 1)$, one observes that $2 \gamma C_2 |V(Q)|^2$ can be absorbed by the left side of (36). Finally, one has the following computation
\[
\int_{B_{4R}} |A(\nabla u) - A(Q)| \, dx \leq \int_{B_{4R}} |A(\nabla u) - A(P)| \, dx + |A(Q) - A(P)| \\
\leq \int_{B_{4R}} |A(\nabla u) - A(P)| \, dx + \int_{B_{2R}} |A(\nabla u) \, dx - A(P)| \\
\leq C \int_{B_{4R}} |A(\nabla u) - A(P)| \, dx,
\]
which leads to the conclusion. \hfill \Box

Applying Lemma 4.1, we present the estimate of $A(\nabla u) - A(\nabla v)$ as follows.

**Lemma 4.2.** Let $u$ be a local weak solution to the system (1) satisfying the degenerate condition (33) for some $\lambda \in (0, \frac{1}{2})$, and $v \in W^{1, p}(B_R)$ the unique weak solution to (24). For any $\tau > 0$, there exists $C = C(n, N, p, q, \nu, L, b, \lambda, \tau) > 0$ such that
\[
\int_{B_R} |A(\nabla u) - A(\nabla v)|^{p'} \, dx \\
\leq \tau \left( \int_{B_{4R}} |A(\nabla u) - (A(\nabla u))_{B_{4R}}|^{p'} \, dx \right) + C \left( \int_{B_{4R}} |F - (F)_{B_{4R}}|^{q} \, dx \right)^{\frac{p'}{q}}.
\]

**Proof.** In view of the definition (13) with the inequality (17), by taking $z_1 = 0$, $z_2 = \nabla u$ and $t = |A(\nabla u) - A(\nabla v)|$, one has for any $\gamma > 0$, there exists a constant $C_1 > 0$ such that
\[
\int_{B_R} |A(\nabla u) - A(\nabla v)|^{p'} \, dx \\
\leq \gamma \int_{B_R} |V(\nabla u)|^2 \, dx + C_1 \gamma \frac{p-2}{p} \int_{B_R} \Phi'_{p'}(x)\left( |A(\nabla u) - A(\nabla v)| \right) \, dx.
\]
By (15) with \( z_1 = \nabla v \) and \( z_2 = \nabla u \), and the condition (33), we arrive at

\[
\int_{B_R} |A(\nabla u) - A(\nabla v)|^\prime \, dx
\leq \gamma \int_{B_R} |V(\nabla u)|^2 \, dx + C_1 \gamma \frac{p-2}{p} \int_{B_r} |V(\nabla u) - V(\nabla v)|^2 \, dx
\leq \frac{\gamma}{\lambda} \int_{B_{2R}} |V(\nabla u) - V(Q)|^2 \, dx + C_1 \gamma \frac{p-2}{p} \int_{B_R} |V(\nabla u) - V(\nabla v)|^2 \, dx.
\]

Applying (18) to the first integral, and Lemma 3.3 to the second integral on right side of the preceding inequality, we continue to deduce that for any \( \vartheta > 0 \) and \( \gamma > 0 \), there are \( C_1, C_\vartheta > 0 \) such that

\[
\int_{B_R} |A(\nabla u) - A(\nabla v)|^\prime \, dx 
\leq \frac{\gamma}{\lambda} \int_{B_{2R}} |V(\nabla u) - V(Q)|^2 \, dx
+ C_1 \gamma \frac{p-2}{p} \vartheta \int_{B_{2R}} |A(\nabla u) - (A(Q))_{B_{2R}}| \, dx
+ C_1 \gamma \frac{p-2}{p} C_\vartheta \int_{B_{2R}} \Phi_{p', |A(Q)|} (|F - (F)_{B_{2R}}|) \, dx
+ C_1 \gamma \frac{p-2}{p} \left[ b \int_{B_R} |V(\nabla u)|^2 \, dx + \left( \int_{B_{2R}} |F - (F)_{B_{2R}}|^q \, dx \right)^{\frac{p-2}{q}} \right].
\]

Here we recall that \( Q \) is the matrix defined in (21). We estimate the third term on the right side of (38). By applying (17) with \( z_1 = Q, z_2 = 0 \) and \( t = |F - (F)_{B_{2R}}| \), and using (34), we deduce that for any \( \rho > 0 \), there exists a constant \( C_2 > 0 \) such that

\[
\int_{B_{2R}} \Phi_{p', |A(Q)|} (|F - (F)_{B_{2R}}|) \, dx
\leq \rho |V(Q)|^2 + C_2 \rho \frac{p-2}{p} \int_{B_{2R}} |F - (F)_{B_{2R}}|^{p'} \, dx
\leq \frac{3\rho}{\lambda} \int_{B_{2R}} |V(\nabla u) - V(Q)|^2 \, dx + C_2 \rho \frac{p-2}{p} \left( \int_{B_{2R}} |F - (F)_{B_{2R}}|^q \, dx \right)^{\frac{p'}{q}}.
\]

Substituting (39) into (38), one finds that

\[
\int_{B_R} |A(\nabla u) - A(\nabla v)|^\prime \, dx
\leq \left[ \frac{\gamma}{\lambda} + 3C_1 C_\vartheta \gamma \frac{p-2}{p} \vartheta \right] \int_{B_{2R}} |V(\nabla u) - V(Q)|^2 \, dx
+ \left[ C_1 \gamma \frac{p-2}{p} \vartheta \right] \Phi_{p', |A(Q)|} (\int_{B_{2R}} |A(\nabla u) - (A(Q))_{B_{2R}}| \, dx)
+ \left[ C_1 C_2 C_\vartheta \gamma \frac{p-2}{p} \rho \frac{p-2}{p} + 2C_1 \gamma \frac{p-2}{p} \left[ b \int_{B_R} |V(\nabla u)|^2 \, dx \right] \right] \left( \int_{B_{2R}} |F - (F)_{B_{2R}}|^q \, dx \right)^{\frac{p'}{q}}.
\]
We make use of Lemma 4.1 and Hölder inequality to deduce that both
\[ \int_{B_{2R}} |V(\nabla u) - V(Q)|^2 \, dx \quad \text{and} \quad \Phi_{p'}(|A(Q)|) \left( \int_{B_{2R}} |A(\nabla u) - (A(\nabla u))_{B_{2R}}| \, dx \right) \]
are bounded from above by \( C \lambda^{\frac{\alpha}{2}} \Xi_1 + C \lambda^{\frac{\alpha}{2}} \Xi_2 \), where
\[ \Xi_1 := \left( \int_{B_{4R}} |A(\nabla u) - (A(\nabla u))_{B_{4R}}| \, dx \right)^{p'} \]
\[ \Xi_2 := \left( \int_{B_{4R}} |F - (F)_{B_{4R}}|^q \, dx \right)^{\frac{p'}{q}}. \]
With the above notations, we obtain that
\[ \int_{B_R} |A(\nabla u) - A(\nabla v)|^{p'} \, dx \]
\[ \leq \left[ \frac{\gamma}{\lambda} + 3 C_1 C_\sigma \sigma^{\frac{\alpha-2}{\alpha}} \right] \left[ C \lambda^{\frac{\alpha}{2}} \Xi_1 + C \lambda^{\frac{\alpha}{2}} \Xi_2 \right] \]
\[ + \left[ C_1 \gamma^{\frac{\alpha-2}{\alpha}} \right] \left[ C \lambda^{\frac{\alpha}{2}} \Xi_1 + C \lambda^{\frac{\alpha}{2}} \Xi_2 \right] \]
\[ + \left[ C_1 C_2 C_\sigma \gamma^{\frac{\alpha-2}{\alpha}} \right] \left[ 2 C_1 \gamma^{\frac{\alpha-2}{\alpha}} \left[ b \right]^{\frac{\alpha-2}{\alpha}} \Xi_2 \right] \]
\[ = C \left[ (\gamma + \gamma^{\frac{\alpha-2}{\alpha}} \rho) \lambda^{\frac{\alpha}{2}} - 1 + \gamma^{\frac{\alpha-2}{\alpha}} \vartheta \lambda^{\frac{\alpha}{2}} \right] \Xi_1 + C (\gamma, \vartheta, \rho; \lambda) \Xi_2. \]
By taking \( \gamma, \vartheta, \) and \( \rho \) small enough in turn, we finally conclude that, for any \( \tau > 0, \)
there exists \( C = C(n, N, p, q, \nu, L, \lambda, \tau) \) such that (37) holds. \( \square \)

We state the conclusion for this section as follows.

**Proposition 1.** Assume that \( u \) is a local weak solution to the system (1) satisfying
the degenerate assumption (33) for some \( \lambda \in \left( 0, \frac{1}{4} \right) \), \( \alpha \) is given in Lemma 3.4.
For any \( \sigma \in (0, 1) \), and \( \kappa < \min \left\{ 1, \frac{2a}{p} \right\} \), there exist positive constants \( C_1 = C_1(n, N, p, \nu, L) \) and \( C_2 = C_2(n, N, p, q, \nu, L, b, \lambda, \sigma, \kappa) \) such that
\[ \int_{B_{4R}} \left| A(\nabla u) - (A(\nabla u))_{B_{4R}} \right|^{p'} \, dx \]
\[ \leq C_1 \sigma^{\kappa p'} \left( \int_{B_{4R}} |A(\nabla u) - (A(\nabla u))_{B_{4R}}| \, dx \right)^{p'} + C_2 \left( \int_{B_{4R}} \left| F - (F)_{B_{4R}} \right|^q \, dx \right)^{\frac{p'}{q}}. \]

**Proof.** Using (19), we first have
\[ \int_{B_{4R}} \left| A(\nabla u) - (A(\nabla u))_{B_{4R}} \right|^{p'} \, dx \]
\[ \leq \frac{C}{\sigma^n} \int_{B_R} \left| A(\nabla u) - A(\nabla v) \right|^{p'} \, dx + C \int_{B_{4R}} \left| A(\nabla v) - (A(\nabla v))_{B_{4R}} \right|^{p'} \, dx \]
\[ =: I + II. \]
Applying Lemma 4.2, one derives that
\[ I \leq \frac{C}{\sigma^n} \left( \int_{B_{4R}} \left| A(\nabla u) - (A(\nabla u))_{B_{4R}} \right| \, dx \right)^{p'} + \frac{C}{\sigma^n} \left( \int_{B_{4R}} \left| F - (F)_{B_{4R}} \right|^q \, dx \right)^{\frac{p'}{q}}, \]
where $\tau > 0$ will be determined later. For the term $\Pi$, we make use of the decay estimate (32), property (19), and Lemma 4.2 to deduce that

$$
\Pi \leq C \left( \sigma^\alpha \int_{B_R} |A(\nabla v) - (A(\nabla u))_{B_R}| \, dx \right)^{p'}
$$

$$
\leq C \sigma^{\alpha p'} \left( \int_{B_R} |A(\nabla v) - A(\nabla u)| \, dx \right)^{p'}
$$

$$
+ C \sigma^{\alpha p'} \left( \int_{B_R} |A(\nabla u) - (A(\nabla u))_{B_R}| \, dx \right)^{p'}
$$

$$
\leq C \sigma^{\alpha p'} \left( \int_{B_{4R}} |A(\nabla u) - (A(\nabla u))_{B_{4R}}| \, dx \right)^{p'} + C \left( \int_{B_{4R}} |F - F_{B_{4R}}| \, dx \right)^{\frac{q'}{q'}}.
$$

By choosing $\tau := \sigma^{\alpha p'} + n$, we finally obtain the conclusion (40) for the degenerate case.

5. **Non-degenerate case.** In this section, we consider the counterpart of (33). That is, we assume that $u$ is a local weak solution to the system (1) and satisfies the non-degenerate condition

$$
\frac{1}{\lambda} \int_{B_{2R}} |V(\nabla u) - (V(\nabla u))_{B_{2R}}|^2 \, dx \leq \int_{B_{2R}} |V(\nabla u)|^2 \, dx
$$

(41)

for some fixed number $\lambda \in (0, \frac{1}{2})$. In the sequel we introduce $A_{y, \rho R}, V_{y, \rho R}$ and $U_{y, \rho R} \in \mathbb{R}^{N \times n}$ by

$$
A(A_{y, \rho R}) = (A(\nabla u))_{B_{\rho R}(y)},
$$

$$
V(V_{y, \rho R}) = (V(\nabla u))_{B_{\rho R}(y)},
$$

$$
U_{y, \rho R} = (\nabla u)_{B_{\rho R}(y)}.
$$

These notations are equivalent in the following sense.

**Lemma 5.1.** There is a constant $\tilde{c} = \tilde{c}(n, N, p)$ such that if

$$
\lambda \tilde{c} \leq \rho^{5n},
$$

(43)

then for every $\beta \in [\rho, 2]$ and any point $y \in B_R(x_0)$ with $B_{\beta R}(y) \subset B_{2R}(x_0)$, one has

$$
\max \{ |V(A_{y, \beta R})|, |V(U_{y, \beta R})| \} \leq 2 |V(V_{x_0, 2R})| \leq 4 \min \{ |V(A_{y, \beta R})|, |V(U_{y, \beta R})| \},
$$

(44)

$$
\max \{ |V(A_{x_0, 2R})|, |V(U_{x_0, 2R})| \} \leq 2 |V(V_{y, \beta R})| \leq 4 \min \{ |V(A_{x_0, 2R})|, |V(U_{x_0, 2R})| \}.
$$

Moreover, for every $\beta \in [\rho, 2]$,

$$
|V(A_{y, \beta R}) - V(V_{x_0, 2R})| + |V(V_{y, \beta R}) - V(V_{x_0, 2R})| + |V(U_{y, \beta R}) - V(V_{x_0, 2R})| \leq \frac{\rho^{2n}}{8} |V(V_{x_0, 2R})|.
$$

(45)

**Proof.** We divide the proof into several steps.

**Step 1.** By the definition (42), Cauchy-Schwarz inequality, and the condition (41), we have

$$
|V(V_{x_0, 2R})| \leq \int_{B_{2R}(x_0)} |V(\nabla u)|^2 \, dx.
$$


\[
\leq 2 \int_{B_{2R}(x_0)} |V(\nabla u) - (V(\nabla u))_{B_{2R}(x_0)}|^2 \, dx + 2 \int |V(\nabla u)_{B_{2R}(x_0)}|^2 \\
\leq 2 \lambda \int_{B_{2R}(x_0)} |V(\nabla u)|^2 \, dx + 2 |V(x_{0, 2R})|^2.
\]

Since \(0 < \lambda < \frac{1}{4}\), we deduce that
\[
|V(x_{0, 2R})|^2 \leq \int_{B_{2R}(x_0)} |V(\nabla u)|^2 \, dx \leq \frac{2}{1 - 2\lambda} |V(x_{0, 2R})|^2 < 4 |V(x_{0, 2R})|^2.
\]

From the leftmost and rightmost terms of the preceding inequalities, one finds that \(V_{x_0, 2R} \neq 0\).

**Step 2.** In this step, we shall prove that
\[
|V(A_{y, \beta R})| \leq 2 |V(x_{0, 2R})| \leq 4 |V(A_{y, \beta R})|.
\] (47)

Using Lemma 2.4, (18), (41), (46), and the fact that \(B_{\beta R}(y) \subset B_{2R}(x_0)\) with \(\beta \leq 2\), we obtain
\[
|V(x_{0, 2R}) - V(A_{y, \beta R})|^2 \\
\leq 2 \int_{B_{2R}(y)} |V(\nabla u) - V(x_{0, 2R})|^2 \, dx + 2 \int_{B_{2R}(y)} |V(\nabla u) - V(A_{y, \beta R})|^2 \, dx \\
\leq 2 \int_{B_{\beta R}(y)} |V(\nabla u) - V(x_{0, 2R})|^2 \, dx + C \int_{B_{2R}(y)} |V(\nabla u) - V(y, \beta R)|^2 \, dx \\
\leq \frac{C}{\beta^n} \int_{B_{2R}(x_0)} |V(\nabla u) - (V(\nabla u))_{B_{2R}(x_0)}|^2 \, dx \\
\leq \frac{C \lambda}{\beta^n} |V(x_{0, 2R})|^2.
\] (48)

Thus from (48), \(\beta \geq \rho\) and (43), one has
\[
|V(A_{y, \beta R})|^2 \leq 2 |V(x_{0, 2R}) - V(A_{y, \beta R})|^2 + 2 |V(x_{0, 2R})|^2 \\
\leq \left(\frac{2C \lambda}{\beta^n} + 2\right) |V(x_{0, 2R})|^2 \leq \left(\frac{2C \lambda}{\rho^n} + 2\right) |V(x_{0, 2R})|^2 \\
\leq \left(\frac{2C \rho^n}{\bar{c}} + 2\right) |V(x_{0, 2R})|^2 \leq 3 |V(x_{0, 2R})|^2.
\] (49)

The last inequality follows by taking \(\bar{c}\) large enough. On the other hand, via (48), we also get
\[
|V(x_{0, 2R})|^2 \leq 2 |V(x_{0, 2R}) - V(A_{y, \beta R})|^2 + 2 |V(A_{y, \beta R})|^2 \\
\leq \frac{1}{2} |V(x_{0, 2R})|^2 + 2 |V(A_{y, \beta R})|^2.
\] (50)

Combining (49) and (50), we obtain
\[
|V(A_{y, \beta R})|^2 \leq 3 |V(x_{0, 2R})|^2 \leq 12 |V(A_{y, \beta R})|^2.
\] (51)

We have similar estimates by replacing \(A_{y, \beta R}\) with \(V_{y, \beta R}\) and \(U_{y, \beta R}\), which enable us to establish (47) and so (44).
Step 3. Furthermore, making use of (48), $\beta \geq \rho$, and (43), we see there exists $C_n > 0$ such that
\[
|V(V_{x_0,2R}) - V(A_{x_0,R})|^2 \leq C_n \frac{\lambda}{\beta^n} |V(V_{x_0,2R})|^2
\]
\[
\leq \frac{C_n \lambda}{\rho^n} |V(V_{x_0,2R})|^2 \leq \frac{C_n \rho^4n}{c} |V(V_{x_0,2R})|^2.
\]
By choosing $\bar{c}$ sufficiently large, we obtain (45). \(\square\)

For the first sub-case of this section, we introduce the condition below. Assume that there is a constant $\bar{c}$ such that
\[
\left| (A(\nabla u))_{B_{2R}} \right| \leq \frac{\bar{c}_o}{\lambda^{\frac{1}{4}}} \left( \int_{B_{4R}} \left| F - (F)_{B_{4R}} \right|^q dx \right)^{\frac{1}{q}}.
\]
Under this condition, we have the following result:

**Proposition 2.** Let $u$ be a local weak solution to the system (1) satisfying (41) for some $\lambda \in (0, \frac{1}{4})$. Assume that (52) holds. Then there exists a constant $C_1 = C_1(n,N,p) > 0$ such that
\[
\left( \int_{B_{\rho R}} |A(\nabla u) - (A(\nabla u))_{B_{\rho R}}|^p dx \right)^{\frac{1}{p}} \leq C_1 \frac{\bar{c}_o}{\lambda^{\frac{1}{4}}} \rho^{\frac{n}{p}} \left( \int_{B_{4R}} |F - (F)_{B_{4R}}|^q dx \right)^{\frac{1}{q}}
\]
for any $\rho$ satisfying $\lambda \bar{c} \leq \rho^{5n}$ in (43).

**Proof.** By recalling (42), there exists $A_{x_0,\rho R}$ such that $A(A_{x_0,\rho R}) = (A(\nabla u))_{B_{\rho R}}$. Then by (19), Hölder inequality, (46), and (51), there holds
\[
\int_{B_{\rho R}} |A(\nabla u) - (A(\nabla u))_{B_{\rho R}}|^p dx
\]
\[
\leq \frac{C}{\rho^n} \int_{B_{2R}} |A(\nabla u) - A(A_{x_0,2R})|^p dx
\]
\[
\leq \frac{C}{\rho^n} \int_{B_{2R}} |A(\nabla u)|^p dx + \frac{C}{\rho^n} \int_{B_{2R}} |A(A_{x_0,2R})|^p dx
\]
\[
\leq \frac{C}{\rho^n} \int_{B_{2R}} |V(\nabla u)|^2 dx + \frac{C}{\rho^n} \int_{B_{2R}} |A(\nabla u)|^p dx
\]
\[
\leq \frac{C}{\rho^n} |V(V_{x_0,2R})|^2 \leq \frac{C}{\rho^n} \left| V(A_{x_0,2R}) \right|^2 \leq \frac{C}{\rho^n} \left| (A(\nabla u))_{B_{2R}} \right|^p.
\]
By (52), we complete the proof of Proposition 2. \(\square\)

Next we are in position to consider that
\[
\bar{c}_o \left( \int_{B_{4R}} \left| F - (F)_{B_{4R}} \right|^q dx \right)^{\frac{1}{q}} \leq \lambda^{\frac{1}{2}} \left| (A(\nabla u))_{B_{2R}} \right|
\]
for some positive parameter $\bar{c}_o$. By combining (41), (46), and (51), We note that
\[
\int_{B_{\rho R}} |V(V_{x_0}) - (V(V_{x_0}))_{B_{\rho R}}|^2 dx
\]
\[
\leq \lambda \int_{B_{2R}} |V(V_{x_0})|^2 dx \leq 4 \lambda |V(V_{x_0,2R})|^2
\]
\[
\leq c \lambda |V(A_{x_0,2R})|^2 \leq c \lambda \left| (A(\nabla u))_{B_{2R}} \right|^p.
\]
We are ready to state the decay estimate of the second sub-case of this section.

**Proposition 3.** Let \( u \) be a local weak solution to the system (1) satisfying conditions (41) and (53) for some \( \lambda \in (0, \frac{1}{4}) \) and \( \bar{c} = c_\rho \) for some \( \rho \) as in (43). Here \( c_\rho \) denotes the constant appearing in Lemma 3.5 with \( \sigma = \rho \). For any \( \delta > 0 \), there are positive numbers \( R_0, c = c(n, N, p, \nu, L) \) and \( \alpha = \alpha(n, N, p) \in (0, 1) \) such that

\[
\int_{B_{2R}} |A(\nabla u) - (A(\nabla u))_{B_{2R}}|^2 \, dx \\
\leq c \sigma^{\alpha p} \left[ \int_{B_{2R}} |A(\nabla u) - (A(\nabla u))_{B_{2R}}|^2 \, dx \right]^{\frac{1}{2}} + \delta \left[ M_{B_{2R}}(A(\nabla u)) \right]^2
\]

for every \( R \in (0, R_0) \), and any \( \sigma \) satisfying \( \rho \leq \sigma < \frac{1}{4} \).

**Remark 3.** For convenience’s sake, we define \( \text{osc}^1_{B_{2R}} A(\nabla u) \), \( \text{osc}^q_{B_{4R}} F \), and \( \mathcal{E} \) by

\[
\text{osc}^1_{B_{2R}} A(\nabla u) := \int_{B_{2R}} |A(\nabla u) - (A(\nabla u))_{B_{2R}}| \, dx, \\
\text{osc}^q_{B_{4R}} F := \int_{B_{4R}} |F - (F)_{B_{4R}}|^q \, dx, \\
\mathcal{E} := \left( \text{osc}^1_{B_{2R}} A(\nabla u) \right) + \frac{c_\rho}{\rho^q} \left( \text{osc}^q_{B_{4R}} F \right)^{\frac{1}{q}},
\]

where the constant \( c_\rho \) is given in Lemma 3.5 with \( \sigma = \rho \). It follows immediately that

\[
\left( \text{osc}^q_{B_{2R}} F \right)^2 \leq c \sigma^{\alpha p} \mathcal{E}^2 \quad \text{and} \quad \left( \text{osc}^q_{B_{2R}} F \right)^{\frac{\rho}{p}} \leq c \rho^{2\alpha} \mathcal{E}^p.
\]

Moreover, recalling (13), for any \( t > 0 \), one has

\[
t^p \leq \Phi_{p', \text{osc}^1_{B_{2R}} A(\nabla u)}(t).
\]

Then by taking \( t = |A(\nabla u) - (A(\nabla u))_{B_{2R}}| \) and using Hölder inequality, (15), and (54), we have

\[
\text{osc}^1_{B_{2R}} A(\nabla u) \leq \left( \int_{B_{2R}} |A(\nabla u) - (A(\nabla u))_{B_{2R}}|^{p'} \, dx \right)^{\frac{1}{p'}} \\
\leq \left[ \int_{B_{2R}} \Phi_{p', \text{osc}^1_{B_{2R}} A(\nabla u)} \left( |A(\nabla u) - (A(\nabla u))_{B_{2R}}| \right) \, dx \right]^{\frac{1}{p'}} \\
\leq c \left( \int_{B_{2R}} |V(\nabla u) - (V(\nabla u))_{B_{2R}}|^2 \, dx \right)^{\frac{1}{2}} \leq c \lambda^{\frac{3}{2}} \left( A(\nabla u) \right)_{B_{2R}}.
\]

Together with (53), we obtain the boundedness of \( \mathcal{E} \) by

\[
\mathcal{E} \leq c \left( 1 + \frac{c_\rho}{\rho^q} \right) \lambda^{\frac{3}{2}} \left( A(\nabla u) \right)_{B_{2R}}.
\]

With these notational conventions in place, we present a proof of Proposition 3.
Proof. Making use of the decay estimate for $\nabla u$, i.e. Lemma 3.5, and replacing $B_{kR}$ appearing in Lemma 3.5 with $B_{\frac{1}{2}R}(y)$, we deduce that for $\rho \leq \sigma < \frac{1}{4}$, and for any $y \in B_R(x_0)$,

$$
\int_{B_{1R}(y)} \left| \nabla u - (\nabla u)_{B_{1R}(y)} \right|^2 dx 
\leq c \sigma^{2n} \Phi_{p',|A(A_y,R)} \left( \frac{\text{osc}_1 A(\nabla u)}{B_{1R}(y)} \right) 
+ c_{\rho} \int_{B_{1R}(y)} \Phi_{p',|A(A_y,R)} \left( \frac{\text{osc}_1 A(\nabla u)}{B_{1R}(y)} \right) dx 
+ C \left[ b \frac{\alpha - p}{2} \int_{B_{1R}(y)} |\nabla u|^{p'} dx + C \left[ b \frac{\alpha - p}{2} \left( \frac{\text{osc}_1 A(\nabla u)}{B_{1R}(y)} \right)^{p'} \right] \right].
$$

$\Phi_{p',|A(A_y,R)}(t) \leq c |A(A_y,R)|^{p'-2} t^2 + c t^{p'}$. (59)

Via (59), by taking $t = |F - (F)_{B_{2R}}|$ and using Hölder inequality, (56), there holds

$$
\int_{B_{2R}} \Phi_{p',|A(A_{x_0},2R)} \left( \frac{\text{osc}_1 A(\nabla u)}{B_{2R}} \right) dx 
\leq c |A(A_{x_0},2R)|^{p'-2} \left( \frac{\text{osc}_1 A(\nabla u)}{B_{2R}} \right)^{p'} + c \left( \frac{\text{osc}_1 A(\nabla u)}{B_{2R}} \right)^{p'}.
$$

By using (57) and $\rho \leq \sigma < \frac{1}{4}$, one has

$$
c \rho^{2\sigma} \delta^{p'} \leq c \sigma^{\frac{4\rho}{p'}} \delta^{\frac{p'}{p'}} \left[ \rho^{\frac{2p'}{p'}} \left( 1 + \frac{c_{\rho}}{\rho^{p'q}} \right) \lambda^{\frac{2p'}{p'}} \right] \frac{\text{osc}_1 A(\nabla u)}{B_{2R}} \leq c |A(A_{x_0},2R)|^{p'-2} \sigma^{\frac{4\rho}{p'} \delta^{2}}.
$$

The last inequality holds since both $\lambda$ and $\rho$ are less than $\frac{1}{4}$. Thus we find

$$
\int_{B_{2R}} \Phi_{p',|A(A_{x_0},2R)} \left( \frac{\text{osc}_1 A(\nabla u)}{B_{2R}} \right) dx \leq c |A(A_{x_0},2R)|^{p'-2} \sigma^{\frac{4\rho}{p'} \delta^{2}}.
$$

(61)
Estimate of $I_1$ : Similar to (61), using (59) and (60), one obtains
\[
\sigma^{2\alpha} \Phi_{\rho'}, |A(A_{x_0,2R})| \left( \text{Osc} B_{2R} |\nabla u| \right) \leq c |A(A_{x_0,2R})|^{p'-2} \sigma^{2\alpha} g^2 + c \sigma^{2\alpha} g^{\rho'} 
\leq c |A(A_{x_0,2R})|^{p'-2} \sigma^2 g^{\rho'}.
\] (62)

Estimates of $I_3$ and $I_4$ : By Remark 3, and (60), we have
\[
C \left( \text{Osc} \frac{\eta}{B_{2R}} \right)^{\frac{p}{2}} \leq c \sigma^{2\alpha} g^{\rho'} \leq c |A(A_{x_0,2R})|^{p'-2} \sigma^2 g^{\rho'}.
\]

Applying the definition of VMO (4), one finds that for any $0 < R < R_0$, then
\[
I_3 + I_4 \leq \delta \int_{B_{2R}} |\nabla u|^p dx + C \left( \text{Osc} \frac{\eta}{B_{2R}} \right)^{\frac{p}{2}} 
\leq \delta \int_{B_{2R}} |A(\nabla u)|^{p'} dx + c |A(A_{x_0,2R})|^{p'-2} \sigma^2 g^{\rho'}
\leq \delta \left[ M_{2R}^{p'}(A(\nabla u)) \right]^{p'} + c |A(A_{x_0,2R})|^{p'-2} \sigma^2 g^{\rho'},
\] (63)

where we use the Hardy-Littlewood maximal operator introduced in (6).

Substituting (61) - (63) into (58), we conclude that
\[
\int_{B_{\sigma R}(y)} \left| \mathbf{V}(\nabla u) - (\mathbf{V}(\nabla u))_{B_{\sigma R}(y)} \right|^2 dx \leq c |A(A_{x_0,2R})|^{p'-2} \sigma^2 g^{\rho'} + \delta \left[ M_{2R}^{p'}(A(\nabla u)) \right]^{p'}.
\] (64)

By (18), (16), the assumption $1 < p \leq 2$, (11), and (15), we deduce that
\[
\int_{B_{\sigma R}(y)} \left| A(\nabla u) - (A(\nabla u))_{B_{\sigma R}(y)} \right|^2 dx \leq \int_{B_{\sigma R}(y)} |A(\nabla u) - A(V_{y,\sigma R})|^2 dx 
\leq \int_{B_{\sigma R}(y)} \left( |\nabla u|^2 + |V_{y,\sigma R}|^2 \right)^{p-2} |\nabla u - V_{y,\sigma R}|^2 dx 
\leq |V_{y,\sigma R}|^{\frac{2-p}{p-2}} \int_{B_{\sigma R}(y)} \left( |\nabla u|^2 + |V_{y,\sigma R}|^2 \right)^{\frac{p-2}{2}} |\nabla u - V_{y,\sigma R}|^2 dx 
\leq |A(V_{y,\sigma R})|^{\frac{p-2}{2}} \int_{B_{\sigma R}(y)} |\nabla (\nabla u) - V(V_{y,\sigma R})|^2 dx.
\]

Finally, from the relation $|A(V_{y,\sigma R})| \leq 2 |A(A_{x_0,2R})|$ and (64), one has
\[
\int_{B_{\sigma R}(y)} |A(\nabla u) - (A(\nabla u))_{B_{\sigma R}(y)}|^2 dx 
\leq c |A(A_{x_0,2R})|^{\frac{p-2}{p-2}} \int_{B_{\sigma R}(y)} |\nabla (\nabla u) - (\nabla(\nabla u))_{B_{\sigma R}(y)}|^2 dx 
\leq c |A(A_{x_0,2R})|^{\frac{p-2}{p-2}} |A(A_{x_0,2R})|^{p-2} \sigma^2 g^{\rho'} + c |A(A_{x_0,2R})|^{\frac{p-2}{p-2}} \delta |M_{2R}^{p'}(A(\nabla u))|^{p'}.
\]

Notice that
\[
|A(A_{x_0,2R})|^{\frac{p-2}{p-2}} \leq \left( \int_{B_{2R}(x_0)} |A(\nabla u)|^{p'} dx \right)^{\frac{1}{p}} \leq \left[ M_{2R}^{p'}(A(\nabla u)) \right]^{\frac{p-2}{p-2}},
\]
together with \( \frac{p-2}{p-1} + p' = 2 \), we obtain that
\[
\int_{B_{\sigma R}(y)} |A(\nabla u) - (A(\nabla u))_{B_{\sigma R}(y)}|^2 \, dx \leq c \sigma^{\frac{4n}{p'}} \delta^2 + \delta \left[ M_{2R}^p'(A(\nabla u)) \right]^2.
\]
This completes the proof of (55).

6. Proof of main results. In this section, we first prove the main result (8) in Theorem 1.1.

Proof of Theorem 1.1. We divide the proof into two steps.

In the first step, we present the following decay estimates for the gradients of local weak solutions, which is a key step to (8). For \( \beta \in (0, 1) \) and any \( \delta > 0 \), we show that there exist constants \( \sigma = \sigma(n, N, p, \nu, L, \beta) \in (0, 1) \) and \( C_\beta = C_\beta(n, N, p, \nu, L, b, \beta) > 0 \) such that
\[
\begin{align*}
&\left( \int_{B_R} |A(\nabla u) - (A(\nabla u))_{B_R}|^2 \, dx \right)^{\frac{1}{2}} \\
&\quad \leq \beta \left( \int_{B_R} |A(\nabla u) - (A(\nabla u))_{B_R}|^2 \, dx \right)^{\frac{1}{2}} + C_\beta \left( \int_{B_R} |F - (F)_{B_R}|^q \, dx \right)^{\frac{1}{q}} \\
&\qquad + \delta M_{2R}^p'(A(\nabla u))
\end{align*}
\]
for any \( B_R \subset \mathbb{R}^n \).

To prove (65), we let \( \alpha \) be given as in Lemma 3.4, and restate the results in Section 4 and Section 5 by using the notations introduced in Remark 3. By Proposition 1 and Hölder inequality, for every \( \sigma \in (0, 1) \) and \( \kappa \in \left( 0, \min \left\{ 1, \frac{2\alpha}{3p} \right\} \right) \), there exist two constants \( c_1 = c_1(n, N, p, \kappa, \nu, L) \) and \( c_2 \) such that
\[
\left( \operatorname{osc}^{p'}_{B_R} A(\nabla u) \right)^{\frac{1}{p'}} \leq c_1 \sigma^{\kappa} \left( \operatorname{osc}^{2}_B A(\nabla u) \right)^{\frac{1}{2}} + c_2 \left( \operatorname{osc}^{q}_B F \right)^{\frac{1}{q}}.
\]
Making use of Proposition 2, there exists a constant \( c_3 \) such that
\[
\left( \operatorname{osc}^{p'}_{B_R} A(\nabla u) \right)^{\frac{1}{p'}} \leq c_3 \left( \operatorname{osc}^{q}_B F \right)^{\frac{1}{q}}.
\]
Via Proposition 3, there are constants \( c_4 = c_4(n, N, p, \nu, L) \) and \( c_5 \) such that
\[
\left( \operatorname{osc}^{2}_B A(\nabla u) \right)^{\frac{1}{2}} \leq c_4 \sigma^{\frac{2\alpha}{3p}} \left( \operatorname{osc}^{2}_B A(\nabla u) \right)^{\frac{1}{2}} + c_5 \left( \operatorname{osc}^{q}_B F \right)^{\frac{1}{q}} + \delta M_{2R}^p(A(\nabla u)).
\]
Applying Hölder inequality and taking \( \sigma \) sufficiently small such that \( c_1 \sigma^{\kappa} + c_4 \sigma^{\frac{2\alpha}{3p}} \leq \beta \), we obtain the desired estimate (65).

In the second step, given any \( x \in \mathbb{R}^n \), let \( B_R \) be any ball such that \( x \in B_{\sigma R} \). Applying (65) in Step 1 and the definition of sharp maximal operator (5), one has
\[
M_{R}^{\sharp, 2}(A(\nabla u))(x) \leq C \beta M_{R}^{\sharp, 2}(A(\nabla u))(x) + C M_{R}^{\sharp, q}(F)(x) + \delta M_{2R}^p(A(\nabla u))(x).
\]
Since we assume that \( \nabla u \in L^p(\mathbb{R}^n) \), then \( M_{R}^{\sharp, 2}(A(\nabla u))(x) \) is finite for almost every \( x \in \mathbb{R}^n \). By taking \( \beta = \frac{1}{2\sigma} \), we conclude that
\[
M_{R}^{\sharp, 2}(A(\nabla u))(x) \leq C M_{R}^{\sharp, q}(F)(x) + \delta M_{2R}^p(A(\nabla u))(x),
\]
which completes the proof of Theorem 1.1.
With the help of Theorem 1.1, we are ready to prove the $L^q$ and BMO estimates.

**Proof of Theorem 1.2.** We first prove the $L^q$-estimate (9) for $q \in [p', q_0]$.

Note that since $F \in L^q(\mathbb{R}^n)$, then $M^{\sharp, q}_R F \in L^q(\mathbb{R}^n)$. We invoke the classical Fefferman-Stein inequality as follows. Suppose that $g \in L^1(\mathbb{R}^n)$ and $M^{\sharp, 1}_R g \in L^p(\mathbb{R}^n)$ for some $p > 1$. Then $g \in L^p(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} |g(y)|^p \, dy \leq C \int_{\mathbb{R}^n} |M^{\sharp, 1}_R g(y)|^p \, dy.$$  \hfill (66)

Applying Fefferman-Stein inequality (66), H"older inequality, and the pointwise estimate (8), we deduce that

$$\int_{\mathbb{R}^n} \left| \nabla u \right|^{p-2} \nabla u \, dy \leq C \int_{\mathbb{R}^n} \left| M^{\sharp, 1}_R \left( \left| \nabla u \right|^{p-2} \nabla u \right) \right|^q \, dy \leq C \int_{\mathbb{R}^n} \left| M^{\sharp, 2}_R \left( \left| \nabla u \right|^{p-2} \nabla u \right) \right|^q \, dy \leq C \int_{\mathbb{R}^n} \left| M^{\sharp, q}_R F \right|^q \, dy + \delta \int_{\mathbb{R}^n} \left| M^{p'}_{2R} \left( \left| \nabla u \right|^{p-2} \nabla u \right) \right|^q \, dy \leq C \int_{\mathbb{R}^n} \left| F \right|^q \, dy + \delta \int_{\mathbb{R}^n} \left| \nabla u \right|^{p-2} \nabla u \, dy.$$  \hfill (67)

Here $C$ is a positive constant depending on $n, N, p, q, \nu, L$, and $b$. We find that the expression involving the Hardy-Littlewood maximal operator on the right-hand side can be absorbed by the left-hand side.

We next prove the BMO estimate. From the definitions (5) and (7), it is clear that

$$\left\| M^{\sharp, 1}_R g \right\|_{L^\infty(B_R)} = \| g \|_{\text{BMO}(B_R)}.$$  \hfill (68)

Moreover, by using the well-known John-Nirenberg inequality, we infer that if $g \in \text{BMO}(B_R)$ and $p \in [1, \infty)$, then

$$\left\| M^{\sharp, p}_R g \right\|_{L^\infty(B_R)} \leq C \| g \|_{\text{BMO}(B_R)}.$$  \hfill (69)

Therefore, combining (67), H"older inequality, and (68), we obtain

$$\left\| \left| \nabla u \right|^{p-2} \nabla u \right\|_{\text{BMO}(B_R)} = \left\| M^{\sharp, 1}_R \left( \left| \nabla u \right|^{p-2} \nabla u \right) \right\|_{L^\infty(B_R)} \leq \left\| M^{\sharp, 2}_R \left( \left| \nabla u \right|^{p-2} \nabla u \right) \right\|_{L^\infty(B_R)} \leq C \| F \|_{\text{BMO}(B_R)} + C \left( \int_{B_{2R}} |\nabla u|^p \, dy \right)^{\frac{p-1}{p}}.$$  \hfill (70)

Therefore Theorem 1.2 is proved.

**Acknowledgments.** We thank the referees for their valuable and insightful comments.
REFERENCES

[1] L. Beck, *Boundary Regularity Results for Local Weak Solutions of Subquadratic Elliptic Systems*, Ph.D thesis, Friedrich-Alexander-Universität Erlangen-Nürnberg, 2008.

[2] D. Breit, A. Cianchi, L. Diening, T. Kuusi and S. Schwarzacher, *Pointwise Calderón-Zygmund gradient estimates for the p-Laplace system*, *Journal de Mathématiques Pures et Appliquées*, **114** (2018), 146–190.

[3] D. Breit, A. Cianchi, L. Diening, T. Kuusi and S. Schwarzacher, *The p-Laplace system with right-hand side in divergence form: Inner and up to the boundary pointwise estimates*, *Nonlinear Analysis: Theory, Methods and Applications*, **153** (2017), 200–212.

[4] L. Caffarelli and I. Peral, *On $W^{1,p}$ estimates for elliptic equations in divergence form*, *Communications on Pure and Applied Mathematics*, **51** (1998), 1–21.

[5] A. Calderón and A. Zygmund, *On the existence of certain singular integrals*, *Acta Mathematica*, **88** (1952), 85–139.

[6] E. DiBenedetto and J. Manfredi, *On the higher integrability of the gradient of weak solutions of certain degenerate elliptic systems*, *American Journal of Mathematics*, **115** (1993), 1107–1134.

[7] L. Diening and F. Ettwein, *Fractional estimates for non-differentiable elliptic systems with general growth*, *Forum Mathematicum*, **20** (2008), 523–556.

[8] L. Diening, P. Kaplický and S. Schwarzacher, *BMO estimates for the p-Laplacian*, *Nonlinear Analysis*, **75** (2012), 637–650.

[9] L. Diening, B. Stroffolini and A. Verde, *Everywhere regularity of functionals with $\varphi$-growth*, * Manuscripta Mathematica*, **129** (2009), 449–481.

[10] F. Duzaar and G. Mingione, *Gradient estimates via non-linear potentials*, *American Journal of Mathematics*, **133** (2011), 1093–1149.

[11] M. Giaquinta and L. Martinazzi, *An Introduction to the Regularity Theory for Elliptic Systems, Harmonic Maps and Minimal Graphs*, 2nd edition, Scuola Normale Superiore di Pisa, 2012.

[12] E. Giusti, *Direct Methods in the Calculus of Variations*, World Scientific Publishing Co. Pte. Ltd., Singapore, 2003.

[13] T. Iwaniec, *Projections onto gradient fields and $L^p$-estimates for degenerated elliptic operators*, *Studia Mathematica*, **75** (1983), 293–312.

[14] T. Kilpeläinen and J. Malý, *The Wiener test and potential estimates for quasilinear elliptic equations*, *Acta Mathematica*, **172** (1994), 137–161.

[15] T. Kuusi and G. Mingione, *Linear potentials in nonlinear potential theory*, *Archive for Rational Mechanics and Analysis*, **207** (2013), 215–246.

[16] T. Kuusi and G. Mingione, *A nonlinear Stein theorem*, *Calculus of Variations and Partial Differential Equations*, **51** (2014), 45–86.

[17] G. Mingione, *Gradient potential estimates*, *Journal of the European Mathematical Society*, **13** (2011), 459–486.

[18] S. Schwarzacher, *Hölder-Zygmund estimates for degenerate parabolic systems*, *Journal of Differential Equations*, **256** (2014), 2423–2448.

Received November 2018; revised November 2018.

E-mail address: zhoufeng@u.nus.edu
E-mail address: zqzhang@nankai.edu.cn