Natural Vibrations of Micropolar Elastic Flexible Plates and Shallow Shells

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Abstract—A mathematical model representing the dynamics of geometrically nonlinear (flexible) micropolar elastic thin plates in Cartesian and curvilinear coordinates is constructed (the approach is generalized to the case of micropolar flexible shallow shells as well). The model is developed under the assumption that the elastic deflection of a plate is comparable with the plate thickness but is small compared to the characteristic plate size in plan. Based on the given model of micropolar elastic flexible plates, the problem on free vibrations is solved for rectangular and circular plates and shallow shells. Effective manifestations of characteristic features of a micropolar material are considered in comparison with the corresponding classical material.

Keywords: micropolar elasticity theory, applied nonlinear dynamic model, flexible plates and shallow shells, nonlinear natural vibrations, characteristic features of a micropolar material

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INTRODUCTION

The problem of nonlinear dynamic deformation of thin-walled elastic structures is one of the fundamental problems in the mechanics of solids under deformation. Within classical elasticity theory, the Föppl–Karman equations, which pertain to the quadratic version of the dynamic nonlinear applied theory of plate bending [1], and Marguerre equations, which refer to the theory of shallow shell bending [2], have become popular. In terms of these theories (see, e.g., [3, 4]), numerous problems on nonlinear dynamic deformation of thin elastic plates and shallow shells of different shapes had been solved for different types of dynamic external actions and boundary conditions.

At present, a topical problem is the development and investigation of applied mathematical theories for describing nonlinear dynamic deformation of micropolar thin elastic plates and shallow shells. A review of publications devoted to the theory of micropolar thin elastic shells and plates was given in [5]. We note that theoretical foundations of the dynamic micropolar elasticity theory and its applications to different problems had been developed in [6–10] and other works. Construction of the general nonlinear theory of micropolar thin shells was the subject of [11].

In [12–16], based on the hypothesis method, which was justified by analyzing a three-dimensional theory with the use of the asymptotic integration method applied to the corresponding boundary-value problem [17], a linear theory of the dynamics of micropolar elastic shells and plates was developed and, in terms of this theory, various applied problems on natural and forced vibrations were investigated.

In this paper, we develop an applied nonlinear theory of dynamic bending of micropolar thin elastic plates and shallow shells, which extends the Föppl–Karman–Marguerre theory to micropolar elasticity theory.

THREE-DIMENSIONAL GEOMETRICALLY NONLINEAR THEORY OF MICROPOLAR ELASTIC PLATES WITH INDEPENDENT DISPLACEMENT AND ROTATION FIELDS

Let us consider a rectangular thin plate with constant thickness $2h$ and assume that it represents a three-dimensional micropolar elastic isotropic body. We consider the plate in $(x_1, x_2, z)$ Cartesian coordinate system. Let the $x_1, x_2$ coordinate plane coincide with the median plane of the plate, and the $Oz$ axis be normal to the median plane. We use the following notation: $(V_1, V_2, V_3)$ is the displacement vector, and $(\omega_1, \omega_2, \omega_3)$ is the free rotation vector.

Let $M(x_1, x_2, z)$ be an arbitrary point of the plate and $N(x_1 + dx_1, x_2 + dx_2, z + dz)$ be a neighboring point of the plate; the distance between them being
infinitely small. Before deformation, we preset vector \( MN \) with projections \((dx_1, dx_2, dz)\). After deformation, point \( M \) shifts to position \( M^* (\xi, \eta, \zeta) \) and point \( N \) to position \( N^* (\xi + d\xi, \eta + d\eta, \zeta + d\zeta) \). New vector \( M^* N^* \) has projections \( d\xi, d\eta, d\zeta \). In terms of micropolar elasticity theory, the displacement of element \( d\mathbf{r} = (dx_1, dx_2, dz) \) is determined not only by displacement vector \([18]\)

\[
\begin{align*}
  d\xi &= \left(1 + \frac{\partial V_1}{\partial x_1}\right) dx_1 + \frac{\partial V_1}{\partial x_2} dx_2 + \frac{\partial V_1}{\partial z} dz, \\
  d\eta &= \frac{\partial V_2}{\partial x_1} dx_1 + \left(1 + \frac{\partial V_2}{\partial x_2}\right) dx_2 + \frac{\partial V_2}{\partial z} dz, \\
  d\zeta &= \frac{\partial V_3}{\partial x_1} dx_1 + \frac{\partial V_3}{\partial x_2} dx_2 + \left(1 + \frac{\partial V_3}{\partial z}\right) dz,
\end{align*}
\]

but also by the free rotation vector according to formula \( d\mathbf{r} \times \omega \) (because, within the micropolar theory, differential element \( MN \) under rotation can be considered as a perfectly rigid body). As a result, we obtain

\[
\begin{align*}
  d\xi &= \left(1 + \frac{\partial V_1}{\partial x_1}\right) dx_1 + \left(1 + \frac{\partial V_2}{\partial x_2}\right) dx_2 + \left(1 + \frac{\partial V_3}{\partial z}\right) dz, \\
  d\eta &= \frac{\partial V_1}{\partial x_1} dx_1 + \left(1 + \frac{\partial V_2}{\partial x_2}\right) dx_2 + \left(1 + \frac{\partial V_3}{\partial z}\right) dz, \\
  d\zeta &= \frac{\partial V_3}{\partial x_1} dx_1 + \frac{\partial V_3}{\partial x_2} dx_2 + \left(1 + \frac{\partial V_3}{\partial z}\right) dz,
\end{align*}
\]

The distances between neighboring points before and after deformation are as follows:

\[
\begin{align*}
  (MN)^2 &= ds^2 = dx_1^2 + dx_2^2 + dz^2, \\
  (M^* N^*)^2 &= ds^* = d\xi^2 + d\eta^2 + d\zeta^2.
\end{align*}
\]

Based on Eqs. (1)–(3) we obtain

\[
\begin{align*}
  ds^2 - ds^* &= \left(1 + \frac{\partial V_1}{\partial x_1}\right)^2 dx_1^2 + \frac{\partial V_1}{\partial x_2}^2 dx_2^2 + \left(1 + \frac{\partial V_1}{\partial z}\right)^2 dz^2 \\
  &+ \left(1 + \frac{\partial V_2}{\partial x_1}\right)^2 dx_1^2 + \left(1 + \frac{\partial V_2}{\partial x_2}\right)^2 dx_2^2 + \left(1 + \frac{\partial V_2}{\partial z}\right)^2 dz^2 \\
  &+ \left(1 + \frac{\partial V_3}{\partial x_1}\right)^2 dx_1^2 + \left(1 + \frac{\partial V_3}{\partial x_2}\right)^2 dx_2^2 + \left(1 + \frac{\partial V_3}{\partial z}\right)^2 dz^2 \\
  &+ 2 \left(1 + \frac{\partial V_1}{\partial x_1}\right) \frac{\partial V_1}{\partial x_2} dx_1 dx_2 + 2 \left(1 + \frac{\partial V_2}{\partial x_1}\right) \frac{\partial V_2}{\partial x_2} dx_1 dx_2 \\
  &+ 2 \left(1 + \frac{\partial V_3}{\partial x_1}\right) \frac{\partial V_3}{\partial x_2} dx_1 dx_2 + 2 \frac{\partial V_1}{\partial x_1} dx_1 dx_2 + 2 \frac{\partial V_2}{\partial x_2} dx_1 dx_2 \\
  &+ 2 \frac{\partial V_3}{\partial x_1} dx_1 dx_2 + 2 \frac{\partial V_3}{\partial x_2} dx_1 dx_2 \\
  &= 2 \left( \frac{\partial V_1}{\partial x_1} - \omega_3 \right) \left( \frac{\partial V_1}{\partial z} + \omega_1 \right) dx_1 dz \\
  &+ 2 \left( \frac{\partial V_2}{\partial x_2} - \omega_3 \right) \left( \frac{\partial V_2}{\partial z} + \omega_1 \right) dx_2 dz \\
  &+ 2 \left( \frac{\partial V_3}{\partial x_1} - \omega_1 \right) \frac{\partial V_3}{\partial x_2} dx_1 dx_2.
\end{align*}
\]

Using strain tensor components \( \gamma_{ii}, \gamma_{ij}, \gamma_{33}, \gamma_{3i} \), difference \( ds^2 - ds^* \) can be represented as

\[
\begin{align*}
  ds^2 - ds^* &= 2\gamma_{11} dx_1^2 + 2\gamma_{22} dx_2^2 + 2\gamma_{33} dz^2 \\
  &+ 2(\gamma_{12} + \gamma_{21}) dx_1 dx_2 \\
  &+ 2(\gamma_{13} + \gamma_{31}) dx_1 dz + 2(\gamma_{23} + \gamma_{32}) dx_2 dz.
\end{align*}
\]
In a similar way, we obtain bending-torsion tensor components \( \chi_{ii}, \chi_{ij}, \chi_{33}, \chi_{3i}, \chi_{3r} \):

\[
\chi_{ii} = \frac{\partial \sigma_{ii}}{\partial x_i} + \frac{1}{2} \left( \left( \frac{\partial \omega_i}{\partial x_i} \right)^2 + \left( \frac{\partial \omega_i}{\partial x_j} \right)^2 + \left( \frac{\partial \omega_i}{\partial z} \right)^2 \right),
\]

\[
\chi_{33} = \frac{\partial \sigma_{33}}{\partial z} + \frac{1}{2} \left( \left( \frac{\partial \omega_3}{\partial x_i} \right)^2 + \left( \frac{\partial \omega_3}{\partial x_j} \right)^2 + \left( \frac{\partial \omega_3}{\partial z} \right)^2 \right),
\]

\[
\chi_{ij} = \frac{\partial \sigma_{ij}}{\partial x_i} + \frac{\partial \omega_i}{\partial x_j} + \frac{1}{2} \frac{\partial \sigma_{ij}}{\partial x_j},
\]

\[
\chi_{3i} = \frac{\partial \sigma_{3i}}{\partial z} + \frac{\partial \omega_3}{\partial x_i} + \frac{1}{2} \frac{\partial \sigma_{3i}}{\partial x_i},
\]

\[
\chi_{3r} = \frac{\partial \sigma_{3r}}{\partial z} + \frac{\partial \omega_3}{\partial x_r} + \frac{1}{2} \frac{\partial \sigma_{3r}}{\partial x_r}.
\]  

Expressions (6) and (7) determine a three-dimensional geometric model of a micropolar flexible plate in Cartesian coordinates.

Now, we generalize the geometric model to \( \alpha_1, \alpha_2, z \) orthogonal curvilinear coordinates (where the \( \alpha_1, \alpha_2 \) curvilinear axes lie in the median plane of the plate and the \( z \) rectilinear axis is perpendicular to this plane) and write down the equations of motion and the elasticity relations of a micropolar material (with independent displacement and rotation fields).

The equations of motion have the form [6]

\[
\frac{1}{H_i} \frac{\partial \sigma_{ii}}{\partial x_i} + \frac{1}{H_j} \frac{\partial H_j}{\partial x_j} (\sigma_{ii} - \sigma_{jj}) + \frac{1}{H_j} \frac{\partial \sigma_{ji}}{\partial x_j} + \frac{1}{H_j} \frac{\partial \sigma_{ji}}{\partial x_j} \frac{\partial H_j}{\partial \alpha_j} = \rho \frac{\partial^2 V_j}{\partial t^2},
\]

\[
\frac{1}{H_i} \frac{\partial \sigma_{13}}{\partial x_1} + \frac{1}{H_1} \frac{\partial H_1}{\partial \alpha_1} \sigma_{13} + \frac{1}{H_2} \frac{\partial \sigma_{33}}{\partial z} = \frac{\partial^2 V_j}{\partial z^2} + \frac{\partial V_j}{\partial \alpha_1},
\]

\[
\frac{1}{H_1} \frac{\partial H_1}{\partial \alpha_1} \sigma_{23} + \frac{\partial \sigma_{23}}{\partial z} + \frac{\sigma_{23}}{H_1} \frac{\partial^2 V_j}{\partial z^2} + \frac{\partial V_j}{\partial \alpha_2} = \sigma_{12} + \sigma_{21} + \frac{\partial V_j}{\partial \alpha_2}.
\]  

As in the Foepp–Karman theory, the above equations of motion are written for a deformed state of the plate.

The physical elasticity relations are as follows:

\[
\gamma_{ii} = \frac{1}{E} \left( \sigma_{ii} - \nu (\sigma_{jj} + \sigma_{33}) \right),
\]

\[
\gamma_{33} = \frac{1}{E} \left( \sigma_{33} - \nu (\sigma_{11} + \sigma_{22}) \right),
\]

\[
\gamma_{ij} = \frac{\mu + \alpha}{4\mu \alpha} \sigma_{ij} - \frac{\mu - \alpha}{4\mu \alpha} \sigma_{ji},
\]

\[
\gamma_{13} = \frac{\mu + \alpha}{4\mu \alpha} \sigma_{13} - \frac{\mu - \alpha}{4\mu \alpha} \sigma_{31},
\]

\[
\gamma_{3i} = \frac{\mu + \alpha}{4\mu \alpha} \sigma_{3i} - \frac{\mu - \alpha}{4\mu \alpha} \sigma_{3i}.
\]  

Note that physical relations (9) are represented in the form of linear dependences.

The geometric relations have the form

\[
\gamma_{ii} = \frac{1}{H_i} \frac{\partial V_i}{\partial x_i} + \frac{1}{H_j} \frac{\partial H_j}{\partial x_j} V_j + \frac{1}{2} \frac{\partial^2 V_j}{\partial t^2} \left[ \left( \frac{\partial V_j}{\partial x_i} \right)^2 + \left( \frac{\partial V_j}{\partial \alpha_i} \right)^2 + \left( \frac{\partial V_j}{\partial \alpha_j} \right)^2 \right] + \omega_1^2 \omega_2^2 - 2 \omega_1 \omega_2 \omega_j \frac{\partial^2 V_j}{\partial t^2} \left[ \left( \frac{\partial V_j}{\partial x_i} \right)^2 + \left( \frac{\partial V_j}{\partial \alpha_i} \right)^2 + \left( \frac{\partial V_j}{\partial \alpha_j} \right)^2 \right]
\]

\[
\gamma_{33} = \frac{1}{H_i} \frac{\partial V_i}{\partial x_i} + \frac{1}{H_j} \frac{\partial H_j}{\partial x_j} V_j + \frac{1}{2} \frac{\partial^2 V_j}{\partial t^2} \left[ \left( \frac{\partial V_j}{\partial x_i} \right)^2 + \left( \frac{\partial V_j}{\partial \alpha_i} \right)^2 + \left( \frac{\partial V_j}{\partial \alpha_j} \right)^2 \right] + \omega_1^2 \omega_2^2 - 2 \omega_1 \omega_2 \omega_j \frac{\partial^2 V_j}{\partial t^2} \left[ \left( \frac{\partial V_j}{\partial x_i} \right)^2 + \left( \frac{\partial V_j}{\partial \alpha_i} \right)^2 + \left( \frac{\partial V_j}{\partial \alpha_j} \right)^2 \right].
\]
\begin{equation}
\gamma_y = \frac{1}{H_i} \frac{\partial V_i}{\partial \alpha_j} - \frac{1}{H_i} \frac{\partial H_i}{\partial \alpha_j} V_i - (-1)^y \omega_i,
\end{equation}

\begin{equation}
+ \frac{1}{H_i H_j} \frac{\partial V_j}{\partial \alpha_i} \frac{\partial V_i}{\partial \alpha_j} - (-1)^y \frac{1}{H_i} \frac{\partial V_j}{\partial \alpha_j} \omega_i,
\end{equation}

\begin{equation}
+ \left( \frac{1}{2} \frac{\partial V_j}{\partial \alpha_i} \frac{\partial V_i}{\partial \alpha_j} - \frac{1}{2} \frac{\partial V_j}{\partial \alpha_j} \omega_i \right),
\end{equation}

\begin{equation}
\gamma_3 = \frac{1}{H_i} \frac{\partial V_i}{\partial \alpha_j} + (-1)^y \omega_j + \frac{1}{2} \frac{\partial V_j}{\partial \alpha_i} \omega_i,
\end{equation}

\begin{equation}
- (-1)^y \frac{1}{H_i} \frac{\partial V_j}{\partial \alpha_i} \omega_j + \frac{1}{2} \frac{\partial V_j}{\partial \alpha_i} \omega_i,
\end{equation}

\begin{equation}
+ (-1)^y \frac{\partial V_j}{\partial \alpha_i} \omega_i - (-1)^y \frac{\partial V_j}{\partial \alpha_i} \omega_i,
\end{equation}

\begin{equation}
\chi_{ii} = \frac{1}{H_i} \frac{\partial \omega_i}{\partial \alpha_i} + \frac{1}{H_i H_j} \frac{\partial H_i}{\partial \alpha_j} \omega_i + \frac{1}{2H_i} \left( \frac{\partial \omega_i}{\partial \alpha_i} + \frac{\partial \omega_i}{\partial \alpha_i} \right) \omega_i,
\end{equation}

\begin{equation}
\chi_{33} = \frac{\partial \omega_i}{\partial \alpha_i} + \frac{1}{2} \left( \frac{\partial \omega_i}{\partial \alpha_i} + \frac{\partial \omega_i}{\partial \alpha_i} \right) \omega_i,
\end{equation}

\begin{equation}
\chi_{ij} = \frac{1}{H_i} \frac{\partial \omega_i}{\partial \alpha_j} - \frac{1}{H_i} \frac{\partial \omega_i}{\partial \alpha_i} V_i - (-1)^y \omega_i,
\end{equation}

\begin{equation}
+ \frac{1}{H_i H_j} \frac{\partial \omega_i}{\partial \alpha_j} \frac{\partial V_j}{\partial \alpha_i} + \frac{1}{H_i H_j} \frac{\partial \omega_i}{\partial \alpha_i} \frac{\partial V_j}{\partial \alpha_j},
\end{equation}

\begin{equation}
\chi_3 = \frac{1}{H_i} \frac{\partial \omega_i}{\partial \alpha_i} + \frac{1}{H_i} \frac{\partial \omega_i}{\partial \alpha_i} \omega_i + \frac{1}{2H_i} \frac{\partial \omega_i}{\partial \alpha_i} \omega_i,
\end{equation}

\begin{equation}
\chi_3 = \frac{1}{H_i} \frac{\partial \omega_i}{\partial \alpha_i} + \frac{1}{H_i} \frac{\partial \omega_i}{\partial \alpha_i} \omega_i + \frac{1}{2H_i} \frac{\partial \omega_i}{\partial \alpha_i} \omega_i.
\end{equation}

For boundary conditions at the face surfaces of the plate, we take the conditions of the first boundary-value problem of micropolar elasticity theory with independent displacement and rotation fields:

\begin{equation}
\sigma_{3n} = p_n^*, \quad \mu_{3n} = m_n^* \quad \text{for} \quad \Sigma = \pm h, \quad n = 1, 2, 3,
\end{equation}

where \( p_n^*, m_n^* \) are the components of external forces and moments preset at the face surfaces of the plate.

The boundary conditions at the lateral plate surface \( \Sigma = \Sigma_1 \cup \Sigma_2 \) are formulated depending on the type of external load application or fastening: either in terms of force and moment stresses or in terms of displacements and rotations, or, else, in a mixed form:

\begin{equation}
\sigma_{mn} n_m = p_n^*, \quad \mu_{mn} m_n = m_n^* \quad \text{on} \quad \Sigma_1;
\end{equation}

\begin{equation}
V_n = V_n^*, \quad \omega_n = \omega_n^* \quad \text{on} \quad \Sigma_2, \quad m, n = 1, 2, 3,
\end{equation}

where \( p_n^*, m_n^* \) are components of the preset external forces and moments at \( \Sigma_1 \); \( V_n^*, \omega_n^* \) are the preset components of displacement and free rotation vectors at \( \Sigma_2 \).

Using the initial conditions at \( t = 0 \), we preset the components of the displacement and independent rotation vectors and the linear and rotational velocities for different points of the body; i.e., \( V_n^*, \omega_n^*, \frac{\partial V_n^*}{\partial t}, \frac{\partial \omega_n^*}{\partial t} \).

Note that, generalizing the aforementioned approach, it is possible to obtain in a similar way the basic equations of the geometrically nonlinear three-dimensional theory of micropolar elastic shallow shells with independent displacement and rotation fields.

**GEOMETRICALLY NONLINEAR APPLIED MODEL OF A MICROPOLAR ELASTIC PLATE UNDER LARGE DEFORMATION**

To reduce a three-dimensional geometrically nonlinear problem of micropolar elasticity theory to a two-dimensional problem, we use the following basis for the proposed theory of micropolar elastic thin flexible plates: (1) the basic hypotheses of the applied linear theory of thin plates [12–17]; (2) the assumptions of the Karman nonlinear classical theory of flexible plates [1]. Below, this basis is considered in more detail.

1. After deformation, a normal element initially perpendicular to the median plane of the plate remains rectilinear and is freely rotated through a certain angle without changing its length. The tangential components of the free rotation vector represent constant functions throughout the plate thickness, whereas the normal component is a linear function.

Then, the displacements and free rotations vary along the plate thickness according to the following laws:
\[ V_i = u_i(\alpha_1, \alpha_2, t) + z\psi_i(\alpha_1, \alpha_2, t), \]
\[ \psi_i = \Omega_i(\alpha_1, \alpha_2, t), \quad (i = 1, 2), \]
\[ V_3 = w(\alpha_1, \alpha_2, t), \quad w_3 = \Omega_3(\alpha_1, \alpha_2, t) + z(\alpha_1, \alpha_2, t), \] (13)
where \( u_i, u_2 \) are displacements of the points of the median plane along the \( z \) axis (i.e., the deflection of the plate); \( \psi_1, \psi_2 \) are total rotation angles and \( \Omega_1, \Omega_2, \Omega_3 \) are free rotations of the initially normal element about the \( \alpha_1, \alpha_2, z \) lines; and \( t \) is the intensity of free rotation along the \( z \) axis.

2. We assume that the plate experiences large deflections \( w \), but, at the same time, we assume that displacements \( u_i, u_2 \) in the median plane are small. We make similar assumptions for derivatives \( \frac{\partial u_i}{\partial \alpha_1}, \frac{\partial u_2}{\partial \alpha_2}, \frac{\partial w}{\partial \alpha_1}, \frac{\partial w}{\partial \alpha_2} \) by assuming that they are small compared to derivatives \( \frac{\partial w}{\partial \alpha_1}, \frac{\partial w}{\partial \alpha_2} \). We also assume that the squares of derivatives \( \frac{\partial w}{\partial \alpha_1}, \frac{\partial w}{\partial \alpha_2} \) are of the same order of smallness as that characterizing the first degree of derivatives of displacements \( u_i, u_2 \) with respect to \( \alpha_1 \) and \( \alpha_2 \).

We also assume that both the rotation angles of normals to the median plane before deformation and their free rotations are small; in addition, the strain normals to the median plane due to moment stresses.

3. In physical formulas for \( \gamma_1 \) and \( \gamma_2 \), it is possible to ignore force stress \( \sigma_{33} \) in comparison with force stresses \( \sigma_{ii} \).

4. In determining deformations, bending torsions, and force and moment stresses, we initially accept the following expressions for force stresses \( \sigma_{3i} \) and moment stress \( \mu_{33} \):
\[ \sigma_{3i} = \mu_{3i}(\alpha_1, \alpha_2, t), \quad \mu_{33} = \mu_{33}(\alpha_1, \alpha_2, t). \] (15)

After determining the aforementioned quantities, we refine expressions (15) for \( \sigma_{3i} \) and \( \mu_{33} \) by combining Eqs. (15) with the results of integration of equations of motion for \( \sigma_{ii} \) and \( \mu_{33} \) over \( z \) under the condition that their values averaged over the plate thickness be zero.

It can be easily shown that components of the strain tensor and bending torsion tensor are expressed by the formulas
\[ \gamma_{ii} = \Gamma_{ii} + zK_{ii}, \quad \gamma_{ij} = \Gamma_{ij} + zK_{ij}, \]
\[ \gamma_{i3} = \Gamma_{i3}, \quad \gamma_{33} = \Gamma_{33}, \quad \chi_{ii} = \chi_{ii}, \]
\[ \chi_{ij} = \chi_{ij}, \quad \chi_{i3} = \chi_{i3} + zl_{i3}. \] (16)

Here, \( \Gamma_{ii} \) are extensional strains in \( \chi_1, \chi_2 \) directions; \( \Gamma_{ij}, \Gamma_{i3}, \Gamma_{33} \) are shear strains in the corresponding planes; \( K_{ii} \) are bends of the median plane due to force stresses; \( K_{ij} \) are torsions of the median plane due to force stresses; \( \chi_{ii}, \chi_{33} \) are bends of the median plane due to moment stresses; \( \chi_{ij} \) are torsions of the median plane due to moment stresses; and \( l_{i3} \) are hypershears of the median plane due to moment stresses.

As it is usually done in applied theories of thin plates, we replace the components of force and moment stress tensors by integral characteristics statically equivalent to them, namely, forces \( (T_{ii}, S_{ii}, N_{ii}, N_{ii}) \), moments \( (M_{ii}, M_{ij}, L_{ii}, L_{ij}, L_{i3}, L_{33}) \), and hypermoments \( (A_{ij}) \) [12–17]:
\[ T_{ii} = \int_{-h}^{h} \sigma_{ii} dz, \quad S_{ii} = \int_{-h}^{h} \sigma_{ii} dz, \]
\[ N_{ii} = \int_{-h}^{h} \sigma_{3i} dz \quad (i = 3), \quad M_{ii} = \int_{-h}^{h} \sigma_{i3} dz, \]
\[ M_{ij} = \int_{-h}^{h} \sigma_{ij} dz, \quad L_{ii} = \int_{-h}^{h} \mu_{ii} dz, \quad L_{i3} = \int_{-h}^{h} \mu_{i3} dz. \] (17)

With allowance for all the aforementioned assumptions concerning torsional shear deformations, the basic equations describing the dynamics of geometrically nonlinear micropolar elastic thin plates with free rotation of the median plane in curvilinear coordinates \( (H_i = A_i \text{ for thin plates}) \) form the set of equations given below.

Equations of motion:
\[ \frac{1}{A_i} \frac{\partial T_{ii}}{\partial \alpha_i} + \frac{1}{A_i A_j} \frac{\partial A_j}{\partial \alpha_i} (T_{ii} - T_{ij}) + \frac{1}{A_j} \frac{\partial S_{ii}}{\partial \alpha_j} \]
\[ + \frac{1}{A_i} \frac{\partial A_j}{\partial \alpha_j} (S_{ii} + S_{ij}) = 2 \rho h^2 \frac{\partial^2 u_i}{\partial t^2} (p_i^* - p_i^-), \]
\[ \frac{1}{A_i} \frac{\partial M_{ii}}{\partial \alpha_i} + \frac{1}{A_i A_j} \frac{\partial A_j}{\partial \alpha_i} (M_{ii} - M_{ij}) + \frac{1}{A_j} \frac{\partial M_{ij}}{\partial \alpha_j} \]
\[ + \frac{1}{A_i A_j} \frac{\partial A_j}{\partial \alpha_j} (M_{ii} + M_{ij}) - N_{ii} \]
\[ = \frac{2 \rho h^3}{3} \frac{\partial^2 \psi_i}{\partial t^2} - h (p_i^* + p_i^-), \]
\[ \frac{1}{A_i A_j} \left[ \left( \frac{\partial (A_i N_{ij})}{\partial \alpha_i} + \frac{\partial (A_i N_{ij})}{\partial \alpha_j} \right) + A_i \frac{\partial^2 w}{\partial \alpha_i^2} T_{ii} \right] \]
\[ + A_j \frac{\partial^2 w}{\partial \alpha_i^2} T_{ij} + \frac{\partial^2 w}{\partial \alpha_i^2} \left( S_{ij} + S_{ji} \right) + \frac{\partial w}{\partial \alpha_i} \]
\[ \times \left[ \frac{A_i}{A_i} \frac{\partial T_{ii}}{\partial \alpha_i} + \frac{1}{A_i} \frac{\partial A_j}{\partial \alpha_i} T_{ij} + \frac{1}{A_i} \frac{\partial (S_{ij} + S_{ji})}{\partial \alpha_i} \right] \]
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NATURAL VIBRATIONS OF MICROPOLAR

Physical relations of elasticity theory:

\[
\begin{aligned}
\frac{\partial w}{\partial \alpha_i} + \frac{1}{A_i} \frac{\partial A_i}{\partial \alpha_i} T_{22} + \frac{1}{2} \left( \phi_i \frac{\partial \phi_i}{\partial \alpha_i} \right)^2 & = 2\rho h \frac{\partial^2 w}{\partial t^2} - (p_i^+ - p_i^-), \\
\frac{1}{A_i} \frac{\partial A_i}{\partial \alpha_i} (L_{ii} - L_{jj}) & = 2Jh \frac{\partial^2 \Omega_i}{\partial t^2} - (m_i^+ - m_i^-), \\
\frac{1}{A_i A_j} \left[ \frac{\partial A_i}{\partial \alpha_i} (L_{ij} + L_{ij}) \right] + (S_{12} - S_{21}) & = 2Jh \frac{\partial^2 \Omega_i}{\partial t^2} - (m_i^+ - m_i^-), \\
\frac{1}{A_i A_j} \left[ \frac{\partial A_i}{\partial \alpha_i} (L_{ij} + L_{ij}) \right] + (M_{12} - M_{21}) - L_{33} & = 2Jh \frac{\partial^2 \Omega_i}{\partial t^2} - h(m_i^+ + m_i^-) - h(m_i^+ + m_i^-) - h(m_i^+ + m_i^-).
\end{aligned}
\]

Physical relations of elasticity theory:

\[
T_{ij} = \frac{2Eh}{1 - \nu^2} \left( \Gamma_{ii} + \nu \Gamma_{ij} \right), \quad M_{ij} = \frac{2Eh^3}{3(1 - \nu^2)} \left[ K_{ii} + \nu K_{ij} \right],
\]

\[
\begin{aligned}
S_{ij} & = 2\beta ((\mu + \alpha)K_{ij} + (\mu - \alpha)K_{ij}), \\
N_{ij} & = 2\beta ((\mu + \alpha)\Gamma_{ij} + (\mu - \alpha)\Gamma_{ij}), \\
\psi_{ij} & = 2\beta ((\mu + \alpha)\psi_{ij} + (\mu - \alpha)\psi_{ij}), \\
\Gamma_{ij} & = 2\beta ((\mu + \alpha)\psi_{ij} + (\mu - \alpha)\psi_{ij}), \\
\psi_{ij} & = 2\beta ((\mu + \alpha)\psi_{ij} + (\mu - \alpha)\psi_{ij}), \\
\psi_{ij} & = 2\beta ((\mu + \alpha)\psi_{ij} + (\mu - \alpha)\psi_{ij}), \\
\psi_{ij} & = 2\beta ((\mu + \alpha)\psi_{ij} + (\mu - \alpha)\psi_{ij}).
\end{aligned}
\]

Geometric relations:

\[
\begin{aligned}
\Gamma_{ii} & = \frac{1}{A_i} \frac{\partial w}{\partial \alpha_i} + \frac{1}{A_i} \frac{\partial A_i}{\partial \alpha_i} u_i + \frac{1}{2} \left( \frac{1}{A_i} \frac{\partial w}{\partial \alpha_i} \right)^2, \\
\Gamma_{ij} & = \frac{1}{A_i} \frac{\partial u_i}{\partial \alpha_j} - \frac{1}{A_j} \frac{\partial u_i}{\partial \alpha_j} - \frac{1}{2} \frac{\partial w}{\partial \alpha_i} - \frac{1}{2} \frac{\partial w}{\partial \alpha_j},
\end{aligned}
\]

\[
\begin{aligned}
\psi_{ij} & = \psi_i u_j + \psi_j u_i, \\
\psi_{ij} & = \psi_i u_j + \psi_j u_i, \\
\psi_{ij} & = \psi_i u_j + \psi_j u_i, \\
\psi_{ij} & = \psi_i u_j + \psi_j u_i.
\end{aligned}
\]

Boundary conditions, for example, in the case of a hinge-type support, have the form

\[
T_{ii} = 0, \quad u_j = 0, \quad M_{ii} = 0, \quad \psi_{ij} = 0, \quad w = 0, \quad L_{ii} = 0, \quad \Omega_i = 0, \quad \Omega_i = 0.
\]

The basic system of equations of the dynamics of micropolar plates with free rotation (18)–(20) and boundary conditions (21) should be complemented with the corresponding initial conditions for \(w, \partial w/\partial t, \psi_i, \partial \psi_i/\partial t, \Omega_i, \partial \Omega_i/\partial t, \iota, \partial \iota/\partial t\).

Note that a similar approach is serves to construct a geometrically nonlinear applied model of thin elastic shallow shells.

NATURAL VIBRATIONS OF MICROPOLAR ELASTIC RECTANGULAR PLATES UNDER LARGE DEFORMATIONS

In the case of a rectangular plate, for the basic equations, we set \(A_i = A_j = 1\). In subsequent calculations, we ignore all the inertial terms in equations of motion, except for \(2\rho h \frac{\partial^2 w}{\partial t^2}\).

Considering natural vibrations, we represent the solution to boundary-value problem (18)–(21) in the form

\[
\begin{aligned}
\psi_{ij}(x_1, x_2, t) & = \psi_i(t) \cos \frac{\pi x_1}{a} \cos \frac{\pi x_2}{b}, \\
\psi_{ij}(x_1, x_2, t) & = \psi_i(t) \sin \frac{\pi x_1}{a} \sin \frac{\pi x_2}{b}, \\
\psi_{ij}(x_1, x_2, t) & = \psi_i(t) \cos \frac{\pi x_1}{a} \cos \frac{\pi x_2}{b}, \\
\psi_{ij}(x_1, x_2, t) & = \psi_i(t) \sin \frac{\pi x_1}{a} \sin \frac{\pi x_2}{b}.
\end{aligned}
\]

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\[
\begin{aligned}
\Gamma_{ij} & = \frac{1}{A_i} \frac{\partial w}{\partial \alpha_i} + (-1)^i \Omega_j, \quad \Gamma_{ij} = \psi_i - (-1)^i \Omega_j, \\
K_{ii} & = \frac{1}{A_i} \frac{\partial u_i}{\partial \alpha_i} + \frac{1}{A_i} \frac{\partial \psi_i}{\partial \alpha_i}, \\
K_{ij} & = \frac{1}{A_i} \frac{\partial \psi_i}{\partial \alpha_i} - \frac{1}{A_i} \frac{\partial \psi_j}{\partial \alpha_i}, \quad \psi_j - (-1)^i \Omega_j, \\
\kappa_{ij} & = \frac{1}{A_i} \frac{\partial \Omega_i}{\partial \alpha_i} - \frac{1}{A_i} \frac{\partial \Omega_i}{\partial \alpha_j}, \quad \Omega_i, \partial \Omega_i/\partial t, \iota, \partial \iota/\partial t.
\end{aligned}
\]
\[ \Omega_2(x_1, x_2, t) = \Omega_2(t) \cos \frac{\pi x_1}{a} \sin \frac{\pi x_2}{b}, \]
\[ \Omega_3(x_1, x_2, t) = \Omega_3(t) \cos \frac{\pi x_1}{a} \cos \frac{\pi x_2}{b}, \]
\[ u(x_1, x_2, t) = u(t) \cos \frac{\pi x_1}{a} \cos \frac{\pi x_2}{b}. \]

Solution (22) satisfies boundary conditions (21). We substitute these representations in geometric relations (20). Then, substituting the resulting expressions for strains and bending torsions in physical relations (19), we obtain equations for forces, moments, and hypermoments. Applying Galerkin’s method to systems of equations of motion (18), we obtain

\[
\int_0^b \int_0^b \left( \frac{\partial T_1}{\partial x_1} + \frac{\partial S_{11}}{\partial x_1} \right) \cos \frac{\pi x_1}{a} \sin \frac{\pi x_2}{b} \, dx_1 \, dx_2 = 0, \\
\int_0^b \int_0^b \left( \frac{\partial T_2}{\partial x_2} + \frac{\partial S_{12}}{\partial x_1} \right) \sin \frac{\pi x_1}{a} \cos \frac{\pi x_2}{b} \, dx_1 \, dx_2 = 0, \\
\int_0^b \int_0^b \left( \frac{\partial N_{13}}{\partial x_1} + \frac{\partial N_{23}}{\partial x_2} + \frac{\partial \left[ T_1 \frac{\partial w}{\partial x_1} + S_{12} + S_{21} \frac{\partial w}{\partial x_2} \right]}{2 \frac{\partial x_2}{\partial x_1}} \right) \times \sin \frac{\pi x_1}{a} \sin \frac{\pi x_2}{b} \, dx_1 \, dx_2 = 0, \\
\int_0^b \int_0^b \left( \frac{\partial N_{31}}{\partial x_1} + \frac{\partial N_{32}}{\partial x_2} \right) \cos \frac{\pi x_1}{a} \sin \frac{\pi x_2}{b} \, dx_1 \, dx_2 = 0, \\
\int_0^b \int_0^b \left( \frac{\partial L_{11}}{\partial x_1} + \frac{\partial L_{21}}{\partial x_2} + (N_{23} - N_{32}) \right) \times \cos \frac{\pi x_1}{a} \cos \frac{\pi x_2}{b} \, dx_1 \, dx_2 = 0, \\
\int_0^b \int_0^b \left( \frac{\partial L_{12}}{\partial x_1} + \frac{\partial L_{22}}{\partial x_2} - (N_{13} - N_{31}) \right) \times \cos \frac{\pi x_1}{a} \sin \frac{\pi x_2}{b} \, dx_1 \, dx_2 = 0, \\
\int_0^b \int_0^b \left( \frac{\partial L_{13}}{\partial x_1} + \frac{\partial L_{23}}{\partial x_2} + (S_{12} - S_{21}) \right) \times \cos \frac{\pi x_1}{a} \cos \frac{\pi x_2}{b} \, dx_1 \, dx_2 = 0, \\
\int_0^b \int_0^b \left( \frac{\partial L_{31}}{\partial x_1} + \frac{\partial L_{32}}{\partial x_2} + (M_{12} - M_{21}) \right) \times \cos \frac{\pi x_1}{a} \cos \frac{\pi x_2}{b} \, dx_1 \, dx_2 = 0. 
\]

Performing integration, we represent functions \( w(t), u_1(t), u_2(t), \psi_1(t), \psi_2(t), \Omega_1(t), \Omega_2(t), \Omega_3(t), u(t) \) in the form

\[ w(t) = W \cos(pt), \quad u_1(t) = U, \cos(pt), \quad \psi_1(t) = \Psi_1, \cos(pt), \quad \Omega_1(t) = \Omega_1, \cos(pt), \]
\[ \Omega_2(t) = \Omega_2, \cos(pt), \quad u(t) = I \cos(pt). \]

We substitute these functions in the equations obtained above, multiply them by \( \cos(pt) \), and integrate over \( t \) from 0 to \( \frac{\pi}{2p} \). As a result, we obtain a system of algebraic equations in coefficients \( W, U_1, U_2, \Psi_1, \Psi_2, \Omega_1, \Omega_2, \Omega_3, I \). From this system, it is possible to derive the \( W-p \) dependence.

This problem is also solved in the framework of the corresponding linear model of micropolar elastic thin plates. We introduce dimensionless deflection \( A = W/h, \) and quantity \( \eta = p/\rho_0, \) i.e., the ratio of quantity \( p \) to the corresponding linear vibration frequency \( \rho_0 \).

We perform numerical calculations for a square plate with dimensions \( b = a = 0.005 \) m; the relative thickness is taken to be \( \delta = \frac{h}{a} = \frac{1}{100} \). For physical constants, we use the following values [19]: \( \alpha = 0.115 \times 10^9 \) Pa, \( \mu = 1.033 \times 10^9 \) Pa, \( \lambda = 2.1951 \times 10^9 \) Pa, \( \gamma = 4.1 \) H, \( \varepsilon = 0.13 \) H, \( \beta = -2.34 \) H, \( \rho = 590 \) kg/m^3, \( J = 5.31 \times 10^{-6} \) kg/m. Figure 1 shows the \( (\eta, A) \) dependences; this line is called the skeleton curve, which reflects the main properties of the deformed system [3]. Curve \( (\eta, A) \) represents a rigid-type line; i.e., as the amplitude grows, the frequency increases. At very small amplitudes, we have \( \eta \to 1 \). As the amplitude grows, the vibration frequency increases with progressively growing steepness.

Figure 2 shows the dependence of deflection \( W \) of a rectangular plate on frequency \( p \). The dashed line corresponds to the classical model, and the continuous line to the micropolar model. In the presence of \( W-p \) dependence (in dimensional quantities), Fig. 2 can be used to compare the micropolar and classical cases. One can see that, for frequencies within 3709 to 5287 s^{-1}, in the classical case, vibrations are present, whereas, in the micropolar case, they are absent. When vibrations occur in both cases (for a frequency of 5500 s^{-1}), the displacement in the micropolar case is 2.7 times smaller, as compared to the classical case. This means that, all other conditions being identical, the plate is more rigid in the micropolar case, as compared to the classical case.
NATURAL VIBRATIONS OF MICROPOLAR ELASTIC CIRCULAR PLATES UNDER LARGE DEFLECTIONS

In the case of a circular plate, for basic equations (18)–(20) we set \( A_1 = 1, \ A_2 = r \). Then, we obtain the following system of equations.

Equations of motion:

\[
\begin{align*}
\frac{\partial T_{11}}{\partial r} + \frac{1}{r} (T_{11} - T_{22}) + \frac{1}{r} \frac{\partial S_{11}}{\partial \theta} &= 2\rho h \frac{\partial^2 u_1}{\partial t^2}, \\
\frac{1}{r} \frac{\partial T_{22}}{\partial r} + \frac{\partial S_{12}}{\partial r} + \frac{1}{r} (S_{12} + S_{21}) &= 2\rho h \frac{\partial^2 u_2}{\partial r^2}, \\
\frac{\partial N_{13}}{\partial r} + \frac{1}{r} N_{13} + \frac{1}{r} \frac{\partial N_{23}}{\partial \theta} + \frac{\partial^2 w}{\partial r^2} T_{11} &= \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} T_{22} + \frac{1}{r} \frac{\partial^2 w}{\partial \theta^2} (S_{12} + S_{21}) \\
+ \frac{\partial w}{\partial \theta} \left[ \frac{\partial T_{11}}{\partial r} + \frac{1}{r} T_{11} + \frac{1}{2} \frac{\partial (S_{12} + S_{21})}{\partial \theta} \right]
\end{align*}
\]

\[
\begin{align*}
&+ \frac{\partial w}{\partial \theta} \left[ \frac{1}{2} \frac{\partial (S_{12} + S_{21})}{\partial \theta} \right] = 2\rho h \frac{\partial^2 w}{\partial t^2}, \\
&\frac{\partial L_{11}}{\partial r} + \frac{1}{r} (L_{11} - L_{22}) + \frac{1}{r} \frac{\partial L_{21}}{\partial \theta} + \frac{\partial L_{22}}{\partial \theta} = 2Jh \frac{\partial^2 \Omega_1}{\partial t^2}, \\
&\frac{\partial L_{12}}{\partial r} + \frac{1}{r} (L_{12} + L_{21}) + \frac{1}{r} \frac{\partial L_{23}}{\partial \theta} + \frac{\partial L_{23}}{\partial \theta} = 2Jh \frac{\partial^2 \Omega_2}{\partial t^2}, \\
&N_{23} - N_{32} = 2Jh \frac{\partial^2 \Omega_3}{\partial t^2}, \\
&+ \frac{1}{r} (L_{12} + L_{21}) + \frac{1}{r} \left( \frac{\partial S_{12}}{\partial r} + \frac{1}{r} (S_{12} - S_{21}) \right) = 2Jh \frac{\partial^2 \Omega_3}{\partial t^2}
\end{align*}
\]

(25)

\[
\begin{align*}
N_{31} &= \left( \frac{\partial M_{11}}{\partial r} + \frac{1}{r} (M_{11} - M_{22}) + \frac{1}{r} \frac{\partial M_{21}}{\partial \theta} \right) \\
&+ \frac{2\rho h^3}{3} \frac{\partial^2 \psi_1}{\partial t^2} = 0, \\
N_{32} &= \left( \frac{1}{r} \frac{\partial M_{12}}{\partial r} + \frac{1}{r} \frac{\partial M_{21}}{\partial r} + \frac{1}{r} (M_{12} + M_{21}) \right) \\
&+ \frac{2\rho h^3}{3} \frac{\partial^2 \psi_3}{\partial t^2} = 0, \\
L_{33} &= \frac{\partial^2 L_{11}}{\partial r^2} - \frac{1}{r} \frac{\partial L_{11}}{\partial \theta} - \frac{1}{r} \frac{\partial L_{13}}{\partial \theta} + \frac{1}{r} \frac{\partial L_{23}}{\partial \theta} \\
&- (M_{12} - M_{21}) + \frac{2\rho h^3}{3} \frac{\partial^2 \lambda}{\partial t^2} = 0.
\end{align*}
\]

Physical-geometric relations:

\[
\begin{align*}
T_{11} &= \frac{2Eh}{1 - \nu^2} \left[ \frac{\partial u_1}{\partial r} + \frac{1}{2} \left( \frac{\partial w}{\partial r} \right)^2 \right], \\
&+ \nu \left[ \frac{1}{r} \frac{\partial u_2}{\partial \theta} + \frac{1}{r} u_2 + \frac{1}{2r^2} \left( \frac{\partial w}{\partial \theta} \right)^2 \right], \\
T_{22} &= \frac{2Eh}{1 - \nu^2} \left[ \frac{1}{r} \frac{\partial u_1}{\partial \theta} + \frac{1}{r} u_1 + \frac{1}{2r^2} \left( \frac{\partial w}{\partial \theta} \right)^2 \right], \\
&+ \nu \left[ \frac{1}{r} \frac{\partial u_2}{\partial r} + \frac{1}{2r^2} \left( \frac{\partial w}{\partial r} \right)^2 \right], \\
S_{12} &= 2h \left[ (\mu + \alpha) \frac{\partial u_1}{\partial r} + (\mu - \alpha) \left( \frac{1}{r} \frac{\partial u_1}{\partial \theta} - \frac{1}{r} u_2 \right) \right] \\
&- 2\alpha \Omega_3 + \mu \frac{1}{r} \left( \frac{\partial w}{\partial r} \right)^2, \\
S_{21} &= 2h \left[ (\mu + \alpha) \left( \frac{1}{r} \frac{\partial u_1}{\partial \theta} - \frac{1}{r} u_2 \right) \right] \\
&+ (\mu - \alpha) \frac{\partial u_2}{\partial r} + 2\alpha \Omega_1 + \mu \frac{1}{r} \left( \frac{\partial w}{\partial r} \right)^2, \\
N_{13} &= 2h \left[ (\mu + \alpha) \frac{1}{r} \frac{\partial w}{\partial \theta} + (\mu - \alpha) \psi_1 + 2\alpha \Omega_2 \right], \\
N_{23} &= 2h \left[ (\mu + \alpha) \frac{1}{r} \frac{\partial w}{\partial \theta} + (\mu - \alpha) \psi_2 - 2\alpha \Omega_1 \right],
\end{align*}
\]
A. A. SARGSYAN, C. O. SARGSYAN

Equations of motion:
\[
\begin{align*}
\frac{\partial N_{13}}{\partial r} + \frac{1}{r} N_{13} + \frac{1}{r} \frac{\partial}{\partial r} \left( r T_{11} \frac{\partial w}{\partial r} \right) &= 2 \rho h \frac{\partial^2 u_1}{\partial t^2}, \\
N_{31} - \left( \frac{\partial M_{11}}{\partial r} + \frac{1}{r} \left( M_{11} - M_{21} \right) \right) + 2 \rho h^3 \frac{\partial^3 \psi_1}{\partial r^3} &= 0, \\
\frac{\partial L_{12}}{\partial r} + \frac{1}{r} \left( L_{12} - L_{21} \right) + N_{31} - N_{13} &= 2 J h \frac{\partial^2 \Omega_2}{\partial t^2}, \\
\frac{\partial T_{11}}{\partial r} + \frac{1}{r} \left( T_{11} - T_{22} \right) &= 2 \rho h \frac{\partial^2 u_i}{\partial r^2},
\end{align*}
\]

Physical-geometric relations:
\[
\begin{align*}
T_{11} &= \frac{2 E h}{1 - \nu^2} \left[ \frac{\partial u_i}{\partial \rho} + \frac{1}{2} \left( \frac{\partial w}{\partial \rho} \right)^2 + \nu \frac{1}{r} u_i \right], \\
T_{22} &= \frac{2 E h}{1 - \nu^2} \left[ \frac{1}{r} u_i + \nu \frac{1}{r} \left( \frac{\partial w}{\partial r} \right)^2 \right], \\
M_{11} &= \frac{2 E h^3}{3 (1 - \nu^2)} \left[ \frac{\partial \psi_1}{\partial \rho} + \nu \frac{1}{r} \frac{\partial \psi_1}{\partial r} \right], \\
M_{22} &= \frac{2 E h^3}{3 (1 - \nu^2)} \left[ \frac{1}{r} \frac{\partial \psi_1}{\partial r} + \nu \frac{1}{r} \frac{\partial \psi_1}{\partial \rho} \right], \\
L_{12} &= 2 h \left[ (\gamma + \varepsilon) \frac{\partial \Omega_2}{\partial \rho} + (\gamma - \varepsilon) \frac{1}{r} \left( \frac{\partial \Omega_1}{\partial \rho} - \frac{1}{r} \frac{\partial \Omega_2}{\partial r} \right) \right], \\
L_{21} &= 2 h \left[ (\gamma + \varepsilon) \left( \frac{1}{r} \frac{\partial \Omega_1}{\partial \rho} - \frac{1}{r} \frac{\partial \Omega_2}{\partial r} \right) + (\gamma - \varepsilon) \frac{\partial \Omega_2}{\partial r} \right], \\
L_{31} &= 2 h \left[ (\beta + \gamma) \frac{1}{r} \left( \frac{\partial \Omega_1}{\partial \rho} + \frac{1}{r} \frac{\partial \Omega_2}{\partial r} \right) + \beta \frac{\partial \Omega_2}{\partial r} \right], \\
\Lambda_{13} &= 2 h^3 \frac{4 \gamma \varepsilon \frac{\partial \Omega_3}{\partial \rho}}{3 \gamma + \varepsilon r \frac{\partial \Omega_3}{\partial \rho}}, \\
\Lambda_{23} &= 2 h^3 \frac{4 \gamma \varepsilon \frac{1}{r} \frac{\partial \Omega_3}{\partial r}}{3 \gamma + \varepsilon r \frac{\partial \Omega_3}{\partial \rho}}.
\end{align*}
\]

Boundary conditions are set in the form:
\[
\begin{align*}
T_{11} = T_{11}^0, & \quad S_{12} = S_{12}^0, \quad M_{11} = M_{11}^0, \quad M_{12} = M_{12}^0, \\
T_{11} \frac{\partial w}{\partial r} + S_{12} + \frac{1}{2} \frac{\partial w}{\partial \rho} + N_{13} &= N_{13}^0, \\
L_{12} = L_{12}^0, & \quad L_{13} = L_{13}^0, \quad \Lambda_{13} = \Lambda_{13}^0.
\end{align*}
\]

We consider an axisymmetric problem. In this case, the equations fall in two separate systems of equations: the problem of bending and the problem of torsion of a circular plate.

In the case of the bending problem, we obtain the following system of equations:

We consider an axisymmetric problem. In this case, the equations fall in two separate systems of equations: the problem of bending and the problem of torsion of a circular plate.

In the case of the bending problem, we obtain the following system of equations.
\[M_{12} = \frac{2h^3}{3} \left[ (\mu + \alpha) \frac{\partial \psi_2}{\partial r} - (\mu - \alpha) \frac{1}{r} \psi_2 - 2\alpha \Omega_1 \right],\]
\[N_{21} = 2h\left[ (\mu + \alpha) \psi_2 - (\mu - \alpha) \frac{1}{r} \psi_2 + 2\alpha \Omega_1 \right],\]
\[M_{21} = \frac{2h^3}{3} \left[ -(\mu + \alpha) \frac{1}{r} \psi_2 + (\mu - \alpha) \frac{\partial \psi_2}{\partial r} + 2\alpha \right],\]
\[N_{32} = 2h\left[ (\mu + \alpha) \psi_2 + 2\alpha \Omega_1 \right],\]
\[L_{41} = 2h \left[ (\beta + 2\gamma) \frac{\partial \Omega_1}{\partial r} + \beta \frac{1}{r} \Omega_1 + \beta \right],\]
\[L_{22} = 2h \left[ (\beta + 2\gamma) \frac{1}{r} \Omega_1 + \frac{\partial \Omega_1}{\partial r} - \beta \right],\]
\[L_{33} = 2h \left[ (\beta + 2\gamma) \Omega_1 + \frac{\partial \Omega_1}{\partial r} + \frac{1}{r} \Omega_1 \right].\]  

(32)

Boundary conditions corresponding to a hinged support:

\[\Omega_1 = 0, \quad \psi_2 = 0, \quad \Lambda_{13} = 0, \quad \text{when} \quad r = R. \quad (33)\]

Now, we consider bending problem (28)–(30). We ignore all the inertial terms in the equations of motion, except for \(2\rho h \frac{\partial^2 w}{\partial t^2}\).

In studying natural vibrations, we represent the solution to boundary-value problem (28)–(30) in the form

\[u_i(r, t) = u_i(t) \left(1 - \left(\frac{r}{R}\right)^2\right) \frac{r}{R},\]
\[w(r, t) = w(t) \left(1 - \left(\frac{r}{R}\right)^2\right),\]
\[\psi_i(r, t) = \psi_i(t) \left(1 - \left(\frac{r}{R}\right)^2\right) \frac{r}{R},\]
\[\Omega_2(r, t) = \Omega_2(t) \left(1 - \frac{11 + \nu}{5 + \nu} \left(\frac{r}{R}\right)^2\right) \frac{r}{R},\]
\[r(\partial T_{ii} + \frac{1}{r} (T_{ii} - T_{21})) \left(1 - \left(\frac{r}{R}\right)^2\right) \frac{r}{R} \, dr = 0,\]
\[r(\partial \psi_1 + \frac{1}{r} (\psi_1 - \psi_2)) \left(1 - \left(\frac{r}{R}\right)^2\right) \frac{r}{R} \, dr = 0,\]
\[r(\partial L_{12} + \frac{1}{r} (L_{12} + L_{21}) + N_{31} - N_{13}) \left(1 - \left(\frac{r}{R}\right)^2\right) \frac{r}{R} \, dr = 0,\]
\[\int \left(1 - \frac{1}{\gamma + 2\epsilon} \left(\frac{r}{R}\right)^2\right) \frac{r}{R} \, dr = 0.\]

We perform integration and then represent functions \(w(t), u_i(t), \psi_1(t), \Omega_2(t)\) in the form

\[w(t) = W \cos(pt), \quad u_i(t) = U_i \cos(pt),\]
\[\psi_1(t) = \Psi_1 \cos(pt), \quad \Omega_2(t) = \Omega_2 \cos(pt).\]  

(36)

Substituting these functions in the above equations, we multiply them by \(\cos(pt)\) and perform integration over \(t\) from 0 to \(\pi / 2p\). As a result, we arrive at a system of algebraic equations in coefficients \(W, U_i, \Psi_1, \Omega_2\), which allows us to obtain the \(W-p\) dependence. Then, we introduce notations similar to those introduced in the case of a rectangular plate.

Now, we perform a numerical analysis for the results obtained above. For geometric dimensions, we use the following values: plate radius \(R = 0.005\) m and relative thickness \(\delta = h / R = 1 / 100\). Figure 3 shows the \((\eta, A)\) dependence. Curve \((\eta, A)\) is a rigid-type line, as in the previous case; i.e., as the amplitude grows, the frequency increases. At very small amplitudes, we have \(\eta \to 1\). As the amplitude grows, the vibration frequency increases with progressively increasing rate. Comparing \(W-p\) dependences obtained for the micropolar and classical cases, we arrive at the same conclusions as those formulated for the rectangular plate.
terms in the equations of motion, except for $2\rho h \frac{\partial^2 w}{\partial t^2}$.

The solution to the boundary-value problem for natural vibrations can also be represented in the form of Eqs. (22).

Solution (22) satisfies the hinged support boundary conditions. Applying Galerkin’s method to the systems of equations of motion, we obtain

$$\int_0^b \left( \frac{\partial T_{22}}{\partial x_2} + \frac{\partial S_{12}}{\partial x_1} \sin \frac{\pi x_1}{a} \cos \frac{\pi x_2}{b} dx_1 dx_2 \right) \cos \frac{\pi x_1}{a} \sin \frac{\pi x_2}{b} dx_1 dx_2 = 0,$$

$$\int_0^b \left( \frac{\partial T_{11}}{\partial x_1} + \frac{\partial S_{21}}{\partial x_2} \right) \cos \frac{\pi x_1}{a} \sin \frac{\pi x_2}{b} dx_1 dx_2 = 0,$$

$$\int_0^b \left( \frac{\partial N_{13}}{\partial x_1} + \frac{\partial N_{23}}{\partial x_2} + \frac{\partial S_{12}}{\partial x_1} \left[ \frac{T_{11}}{2} \frac{\partial w}{\partial x_1} + \frac{S_{12}}{2} \frac{\partial w}{\partial x_2} \right] + \frac{\partial}{\partial x_2} \left[ \frac{S_{12} + S_{21}}{2} \frac{\partial w}{\partial x_1} + T_{22} \frac{\partial w}{\partial x_2} \right] \right. $$

$$+ \left. \frac{T_{11}}{R_1} - 2\rho h \frac{\partial^2 w}{\partial t^2} \right) \sin \frac{\pi x_1}{a} \cos \frac{\pi x_2}{b} dx_1 dx_2 = 0,$$

$$\int_0^b \left( N_{31} - \left( \frac{\partial M_{11}}{\partial x_1} + \frac{\partial M_{21}}{\partial x_2} \right) \right) \cos \frac{\pi x_1}{a} \sin \frac{\pi x_2}{b} dx_1 dx_2 = 0,$$

$$\int_0^b \left( N_{32} - \left( \frac{\partial M_{22}}{\partial x_2} + \frac{\partial M_{12}}{\partial x_1} \right) \right) \sin \frac{\pi x_1}{a} \cos \frac{\pi x_2}{b} dx_1 dx_2 = 0,$$

$$\int_0^b \left( \frac{\partial L_{11}}{\partial x_1} + \frac{\partial L_{21}}{\partial x_2} + (N_{23} - N_{32}) \right) \cos \frac{\pi x_1}{a} \sin \frac{\pi x_2}{b} dx_1 dx_2 = 0,$$

$$\int_0^b \left( \frac{\partial L_{22}}{\partial x_2} + \frac{\partial L_{12}}{\partial x_1} - (N_{13} - N_{31}) \right) \sin \frac{\pi x_1}{a} \cos \frac{\pi x_2}{b} dx_1 dx_2 = 0,$$

$$\int_0^b \left( \frac{\partial L_{13}}{\partial x_1} + \frac{\partial L_{23}}{\partial x_2} + (S_{12} - S_{21}) \right) \cos \frac{\pi x_1}{a} \sin \frac{\pi x_2}{b} dx_1 dx_2 = 0,$$

$$\int_0^b \left( L_{33} - \left( \frac{\partial N_{13}}{\partial x_1} + \frac{\partial N_{23}}{\partial x_2} \right) \right) \left. - \left( \frac{\partial M_{12}}{\partial x_1} + \frac{\partial M_{21}}{\partial x_2} \right) \right) \cos \frac{\pi x_1}{a} \sin \frac{\pi x_2}{b} dx_1 dx_2 = 0.$$

Here, it is necessary to substitute the expressions for physical and geometric relations and the solution given by Eqs. (22).

We perform integration and represent functions $w(t), u_i(t), u_i(t), \psi_i(t), \psi_i(t), \Omega_i(t), \Omega_i(t), \Omega_i(t), \Omega_i(t)$ in the form of Eqs. (24). We substitute them in the equations obtained as a result of integration, perform multiplication by $\cos(pt)$, and integrate over $t$ from 0 to $\frac{\pi}{2p}$. As a result, we obtain a system of algebraic equations in coefficients $W, U_1, U_2, \Psi_1, \Psi_2, O_1, O_2, O_3, I$, which yields the $W-p$ dependence. Numerical analysis was performed for the same material and geometric dimensions as those used in the case of the plate; in addition, $R = 0.005$ m and the relative thickness was $\delta = \frac{h}{R} = \frac{1}{100}$. Note that $k_2 = \frac{1}{R_2} = 0, k_1 = \frac{1}{R}$ (i.e., we consider a cylindrical shallow shell, where $R_1, R_2$ are the curvature radii along the $\alpha_1, \alpha_2$ directions), and dimensionless curvature $k^* = \frac{a^2}{R_1 h}$ of the shallow shell is taken to be $k^* = 20$. 

Fig. 3. Dimensionless deflection $A$ of circular plate versus quantity $\eta$.

Fig. 4. Dimensionless deflection $A$ of shallow shell versus quantity $\eta$. 

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Figure 4 shows the $(\eta, A)$ skeleton curve for a shallow shell. The $(\eta, A)$ curve represents a soft-type line; i.e., its initial part deviates toward the ordinate axis; after this, as $A$ increases, the amplitude grows steeper and steeper. Similar qualitative results are obtained for the case of a shallow shell that is circular in plan.

CONCLUSIONS

We constructed mathematical models of micropolar flexible plates and shallow shells. The models represent generalizations of the well-known Foeppl–Karman–Marguerre models corresponding to the classical case. Using the proposed models, we numerically solved the problems on free vibrations of rectangular and circular plates and shallow shells. We revealed the characteristic features of effective properties of a micropolar material in comparison with the corresponding classical material.

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