Twisted Quantum Affine Algebras
and Solutions to the Yang-Baxter Equation

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Abstract

We construct spectral parameter dependent R-matrices for the quantized enveloping algebras of twisted affine Lie algebras. These give new solutions to the spectral parameter dependent quantum Yang-Baxter equation.


1 Introduction

The solutions to the Yang-Baxter equation play a central role in the theory of quantum integrable models \[28, 23\]. In statistical mechanics they are the Boltzman weights of exactly solvable lattice models \[3\]. In quantum field theory they give the exact factorizable scattering matrices \[29\]. For an introduction to the mathematical aspects of the Yang-Baxter equation see e.g. \[24\].

The Yang-Baxter equation with spectral parameter has the form

$$R_{12}^1(u)R_{13}^1(uv)R_{23}^2(v) = R_{23}^2(v)R_{13}^1(uv)R_{12}^1(u).$$

(1.1)

The $R^{ab}(u)$ are matrices which depend on a spectral parameter $u$ and which act on the tensor product of two vector spaces $V_a$ and $V_b$

$$R^{ab}(u) : V_a \otimes V_b \rightarrow V_a \otimes V_b.$$  

(1.2)

The products of $R$’s in (1.1) act on the space $V_1 \otimes V_2 \otimes V_3$.

The mathematical framework for the construction of trigonometric solutions of the quantum Yang-Baxter equation (1.1) is given by the quantum affine algebras $U_q(\hat{L})$ introduced by Jimbo \[18\] and Drinfeld \[11\]. These are deformations of the enveloping algebras $U(\hat{L})$ of affine Lie algebras \[22\]. Associated to any two finite-dimensional irreducible $U_q(\hat{L})$-modules $V(\lambda)$ and $V(\mu)$ there exists a trigonometric R-matrix $R^{\lambda\mu}(u)$. Given three modules, the R-matrices for all pairs of these three modules are a solution of (1.1). Many R-matrices of untwisted quantum affine algebras have since been determined (see references in \[7\]), leading to a large number of new quantum integrable models, quantum spin chains, exactly solvable lattice models and exact scattering matrices. The method has also been extended to quantum affine superalgebras \[8\].

While it was clear from the beginning \[11\] that all (twisted and untwisted) affine Lie algebras can be quantized and give rise to trigonometric R-matrices, the twisted algebras have hardly been treated in the literature. The only R-matrices associated to twisted quantum affine algebras which we have found in the literature are those associated to the vector representation of $U_q(A_1^{(2)})$ and $U_q(D_{l+1}^{(2)})$ \[14, 19\]. (The $U_q(A_2^{(2)})$ R-matrix was found already in \[17\]; another R-matrix for $U_q(D_{l+1}^{(2)})$ has been found in \[14\] by “Baxterizing” the so-called dilute BWM algebra \[\dagger\].) The knowledge of these R-matrices has had many physical applications. They have for example been used to obtain transfer matrices of solvable lattice models \[25\] or to diagonalize quantum spin chain Hamiltonians on the periodic chain \[27\] and on the open chain \[1, 2\]. To generalize these works to the models associated to higher dimensional representations it is necessary to know the corresponding R-matrices and that is the topic of this paper.

R-matrices are also needed to construct the S-matrices of quantum field theories with quantum affine symmetries \[10\]. In particular the R-matrices associated to all

\*We thank Ole Warnaar for drawing our attention to the ref. \[14\]
fundamental representations of $A_{2n}^{(2)}$ which we construct in this paper will give the scattering matrices for the solitons in $A_{2n}^{(2)}$ affine Toda theory.

The paper is organized as follows: In section 2 we review the necessary facts about twisted Lie affine algebras [22]. In section 3 we discuss the quantized affine algebras and give the equations which uniquely determine the R-matrices (Jimbo’s equations). In section 4 we explain how to solve these equations. Our technique is an extension of the tensor product graph method introduced in [30] and generalized in [7]. We have obtained the R-matrices for the following twisted algebras and tensor products:

- $U_q(A_{2l}^{(2)})$, $l \geq 2$, section 4.1
- $U_q(A_{2l-1}^{(2)})$, $l \geq 3$, section 4.2
- $U_q(D_{l+1}^{(2)})$, $l \geq 2$, section 4.3

Here $\lambda_i$ denotes the i-th fundamental weight. We give some of the technical details in appendices. In particular in appendix B we derive the tensor product decompositions and branching rules which we need in the paper.

## 2 Twisted affine Lie algebras

We recall the relevant information about twisted affine Lie algebras [22]. Let $L$ be a finite dimensional simple Lie algebra and $\sigma$ a diagram automorphism of $L$ of order $k$. Associated to these one constructs the twisted affine Lie algebra $\hat{L}^{(k)}$. In this paper we will assume $k = 2$. Let $L_0$ be the fixed point subalgebra under the diagram automorphism $\sigma$. We recall that

$$L = L_0 \oplus L_1, \quad [L_i, L_j] = L_{(i+j)\text{mod}2}. \quad (2.1)$$

$L_1$ gives rise to an irreducible $L_0$-module under the adjoint action of $L_0$. Let $\theta_0$ be its highest weight. In table 1 we list all the cases with $k = 2$. Below we restrict ourselves to the three families and leave out the exceptional case for technical reasons which will become apparent later.

We recall that $L$ admits generators $E_i, F_i, H_i$, $0 \leq i \leq l$, satisfying the defining relations

$$[H_i, E_j] = (\bar{\alpha}_i, \bar{\alpha}_j)E_j, \quad [H_i, F_j] = -(\bar{\alpha}_i, \bar{\alpha}_j)F_j, \quad [E_i, F_j] = \delta_{ij}H_i,$$

$$(\text{ad}_{E_i})^{1-a_{ij}}E_j = (\text{ad}_{F_i})^{1-a_{ij}}F_j = 0, \quad i \neq j, \quad (2.2)$$

where $a_{ij} = 2(\bar{\alpha}_i, \bar{\alpha}_j)/(\bar{\alpha}_i, \bar{\alpha}_i)$ are the entries of the corresponding (twisted) Cartan matrix of $\hat{L}^{(2)}$. Here the $E_i, F_i, H_i$, $1 \leq i \leq l$, form the Chevalley generators for

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*The only diagram automorphism which is not of order $k = 2$ is the triality of the Lie algebra $D_4$ and we will not treat this case in this paper.

*The rescaled generators $E_i' = \sqrt{2/\bar{\alpha}_i^2}E_i$, $F_i' = \sqrt{2/\bar{\alpha}_i^2}E_i$, $H_i' = 2/\bar{\alpha}_iH_i$ satisfy the more usual commutation relations with the structure constants given by the Cartan matrix.
| $L$  | $L_0$  | $\theta_0$ |
|------|--------|------------|
| $A_{2l}$,  $l \geq 1$ | $B_l$ | $2\lambda_1 = 2\epsilon_1$ |
| $A_{2l-1}$,  $l \geq 3$ | $C_l$ | $\lambda_2 = \epsilon_1 + \epsilon_2$ |
| $D_{l+1}$,  $l \geq 2$ | $B_l$ | $\lambda_1 = \epsilon_1$ |
| $E_6$ | $F_4$ | $\lambda_4$ |

Table 1: Table of the finite dimensional simple Lie algebras $L$ which possess a diagram automorphism of order $k = 2$, their fixed point subalgebras $L_0$ and the highest weight $\theta_0$ of the adjoint $L_0$-module $L_1$. Here and in the rest of the paper we give weights either as integer combinations of the fundamental weights $\lambda_i$ or alternatively we give them in terms of the $\epsilon_i$ which form a basis of the root space of $gl(n)$ into which we embed the other algebras, see Appendix A.

$L_0$ and the $\bar{\alpha}_i$, $(1 \leq i \leq l)$ are the simple roots of $L_0$. $E_0 \in L_1$ corresponds to the minimal weight vector and thus has weight $-\theta_0$. It follows that $\bar{\alpha}_0 = -\theta_0$ and that $H_0 = -\sum_{i=1}^{l} a_i H_i$ lies in the Cartan subalgebra $H$ of $L_0$. The integers $a_i$ are known as the Kac labels of $\hat{L}^{(2)}$.

Throughout we let $(\ , \ )$ be a fixed invariant bilinear form on $L$ which induces a corresponding invariant form $(\ , \ )$ on $H^*$. A suitable choice for the invariant form on $L$ together with a realization of the simple generators is given in Appendix A for completeness. With our choice we have

$$(E_i, F_j) = \delta_{ij}, \quad (H_i, H_j) = (\bar{\alpha}_i, \bar{\alpha}_j). \quad (2.3)$$

We now introduce the corresponding twisted affine Lie algebra $\hat{L}^{(2)'}$ which admits the decomposition

$$\hat{L}^{(2)'} = \bigoplus_{m \in \frac{1}{2} \mathbb{Z}} \hat{L}_m \oplus \mathbb{C} c_0, \quad \hat{L}_m = \begin{cases} L_0(m), & m \in \mathbb{Z} \\ L_1(m), & m \in \mathbb{Z} + \frac{1}{2} \end{cases} \quad (2.4)$$

with $L_a(m) = \{x(m)|x \in L_a\}$, $a = 0, 1$ and $c_0$ a central charge. The Lie bracket is given by

$$[x(m), y(n)] = [x, y](m+n) + m c_0 \delta_{m+n,0} (x, y), \quad [c_0, x(m)] = 0. \quad (2.5)$$

Here $(\ , \ )$ is the fixed invariant bilinear form on $L$. Note that $\hat{L}_0 = L_0$. A suitable set of generators for $\hat{L}^{(2)'}$ is given by

$$e_i = E_i(0), \quad h_i = H_i(0), \quad f_i = F_i(0), \quad 1 \leq i \leq l,$$

$$e_0 = E_0(1/2), \quad h_0 = H_0(0) + 1/2c_0, \quad f_0 = F_0(-1/2). \quad (2.6)$$

This algebra is extended to $\hat{L}^{(2)} = \hat{L}^{(2)'} \oplus \mathbb{C} d_0$ by the introduction of the level operator $d_0$ satisfying

$$[d_0, x(m)] = m x(m), \quad [d_0, c_0] = 0. \quad (2.7)$$

3
As a Cartan subalgebra of \( \hat{L}^{(2)} \) we take
\[
\hat{H} = H(0) \oplus \mathbb{C}c_0 \oplus \mathbb{C}d_0.
\] (2.8)

The weights for \( \hat{L}^{(2)} \) are of the form \( \lambda = (\bar{\lambda}, c_\lambda, d_\lambda) \) where \( \bar{\lambda} \in H^* \) and \( c_\lambda, d_\lambda \) are the eigenvalues of the central extension \( c_0 \) and the level operator \( d_0 \) respectively. The simple roots corresponding to the set of simple generators in (2.6) are
\[
\alpha_i = (\bar{\alpha}_i, 0, 0), \quad 1 \leq i \leq l, \quad \alpha_0 = (-\theta_0, 0, \frac{1}{2}).
\] (2.9)

The invariant bilinear form on \( \hat{H}^* \) is given by
\[
(\lambda, \mu) = (\bar{\lambda}, \bar{\mu}) + c_\lambda d_\mu + d_\lambda c_\mu.
\] (2.10)

With this convention we have \( (\alpha_i, \alpha_j) = (\bar{\alpha}_i, \bar{\alpha}_j) \), \( 0 \leq i, j \leq l \) and our simple generators satisfy the defining relations
\[
[h_i, e_j] = (\alpha_i, \alpha_j)e_j, \quad [h_i, f_j] = -(\alpha_i, \alpha_j)f_j, \quad [e_i, f_j] = \delta_{ij}h_i, \quad (\text{ad}_{e_i})^{1-a_{ij}}e_j = (\text{ad}_{f_i})^{1-a_{ij}}f_j = 0, \quad i \neq j.
\] (2.11)

Following Kac [22] it is useful to introduce the weights \( \gamma = (0, 1, 0) \) and \( \delta = (0, 0, 1) \) so that \( (\gamma, \delta) = 1, (\gamma, \gamma) = (\delta, \delta) = 0 \). Our simple roots are then given by \( \alpha_i = \bar{\alpha}_i, \quad 1 \leq i \leq l, \quad \alpha_0 = -\theta_0 + 1/2\delta \).

We have an algebra homomorphism, called the evaluation map, \( \text{ev}_t : U(\hat{L}^{(2)}) \to \mathbb{C}[t, t^{-1}] \otimes U(L) \), with \( U(\hat{L}^{(2)}), U(L) \) the enveloping algebras of \( \hat{L}^{(2)}, L \) respectively, given by
\[
\text{ev}_t(x(m)) = t^mx, \quad \text{ev}_t(c_0) = 0, \quad \text{ev}_t(d_0) = \frac{1}{2}t\frac{d}{dt},
\] (2.12)

and extended to all of \( U(\hat{L}^{(2)}) \) in the natural way. Thus given a finite dimensional \( L \)-module \( V \) carrying a representation \( \pi \) we have a corresponding \( \hat{L}^{(2)} \) module \( V(t) = \mathbb{C}[t, t^{-1}] \otimes V \) carrying the loop representation \( \hat{\pi} \) given by
\[
\hat{\pi} = (1 \otimes \pi)\text{ev}_t.
\] (2.13)

Below we consider the problem of quantizing such representations to give solutions of the Yang-Baxter equation.

An important role will be played below by those irreducible \( L \)-modules which are also irreducible under the \( L_0 \) subalgebra. We call these \( L_0 \)-irreducible modules. We will see below that the loop representations built on \( L_0 \)-irreducible modules can all be quantized. Such \( L_0 \)-irreducible modules appear to exist for the first three cases in table 4 only and this is the reason why we are restricting to these cases. In table 3 we list for each of the three families the highest weights of all the \( L_0 \)-irreducible irreps of \( L \) together with their highest weight with respect to \( L_0 \).

In appendix 3 we show that the tensor product of any two such \( L_0 \)-irreducible \( L \)-modules decomposes into a multiplicity free direct sum of irreducible \( L_0 \)-modules. This is important because it implies that a solution to Jimbo’s equations will always exist for such tensor products (see below).
| $L$  | $L_0$ | $\Lambda$  | $\Lambda_0$ |
|------|------|-----------|-----------|
| $A_{2l}$ | $\lambda_k, \lambda_{2l+1-k}$ | $\lambda_k, \ 1 \leq k \leq l$ | $\lambda_{2l+1-k}$ |
| $A_{2l-1}$ | $a\lambda_1$ | $a\lambda_1, \ a \in \mathbb{Z}_+$ | $a\lambda_1$ |
| $D_{l+1}$ | $a\lambda_1, a\lambda_{l+1}$ | $a\lambda_l, \ a \in \mathbb{Z}_+$ | $a\lambda_l$ |

Table 2: $L_0$-irreducible irreps. $\Lambda$ are the highest weights of the $L_0$-irreducible irreps of $L$ and $\Lambda_0$ are the corresponding highest weights under $L_0$.

### 3 Twisted quantum affine algebras

Corresponding to the twisted affine algebra $\hat{L}^{(2)}$ we have the twisted quantum affine algebra $U_q(\hat{L}^{(2)})$ with generators $q^{\pm h_i/2}, e_i, f_i, d_0, \ (0 \leq i \leq l)$ and defining relations

$$[h_i, e_j] = (\alpha_i, \alpha_j)e_j, \quad [h_i, f_j] = -(\alpha_i, \alpha_j)f_j, \quad [h_i, h_j] = 0,$$

$$[d_0, e_i] = \frac{1}{2}\delta_{i,0}e_i, \quad [d_0, f_i] = -\frac{1}{2}\delta_{i,0}f_i, \quad [d_0, h_i] = 0,$$

$$[e_i, f_j] = \delta_{ij}q^{h_i} - q^{-h_i},$$

$$\sum_{k=0}^{1-a_{ij}}(-1)^ke_i^{(1-a_{ij}-k)}e_je_i^{(k)} = 0, \quad i \neq j,$$

$$\sum_{k=0}^{1-a_{ij}}(-1)^kf_i^{(1-a_{ij}-k)}f_jf_i^{(k)} = 0, \quad i \neq j, \quad (3.1)$$

where

$$e_i^{(k)} = \frac{e_i^k}{[k]_{q_i}!}, \quad f_i^{(k)} = \frac{f_i^k}{[k]_{q_i}!},$$

$$[k]_q = q^k - q^{-k}, \quad [k]_q! = \prod_{n=1}^{k} [n]_q \quad q_i = q^{\frac{1}{2}(\alpha_i, \alpha_i)}. \quad (3.2)$$

$U_q(\hat{L}^{(2)})$ is a quasi-triangular Hopf algebra with coproduct $\Delta$ and antipode $S$ given by

$$\Delta(e_i) = q^{-h_i/2} \otimes e_i + e_i \otimes q^{h_i/2}, \quad S(e_i) = -q_i e_i,$$

$$\Delta(f_i) = q^{-h_i/2} \otimes f_i + f_i \otimes q^{h_i/2}, \quad S(f_i) = -q_i^{-1} f_i,$$

$$\Delta(q^{\pm h_i/2}) = q^{\pm h_i/2} \otimes q^{\pm h_i/2}, \quad S(q^{\pm h_i/2}) = q^{\mp h_i/2},$$

$$\Delta(d_0) = 1 \otimes d_0 + d_0 \otimes 1, \quad S(d_0) = -d_0. \quad (3.3)$$

Throughout $\check{R}$ denotes the universal R-matrix of $U_q(\hat{L}^{(2)})$ which by definition satisfies

$$\check{R}\Delta(a) = \Delta^T(a)\check{R}, \quad \forall a \in U_q(\hat{L}^{(2)}),$$

$$(1 \otimes \Delta)\check{R} = \check{R}_{13}\check{R}_{12}, \quad (\Delta \otimes 1)\check{R} = \check{R}_{13}\check{R}_{23} \quad (3.4)$$
where $\Delta^T(a)$ is the opposite coproduct. A direct consequence of the above relations is that $R$ satisfies the quantum Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$  \hspace{1cm} (3.5)$$

Note that the generators $q^{\pm h_i/2}, e_i, f_i, \ (1 \leq i \leq l)$ generate the quantum algebra $U_q(L_0)$ which is a quasi-triangular Hopf subalgebra of $U_q(\hat{L}^{(2)})$. We denote its universal R-matrix by $R$.

We shall see below that any minimal irrep $V_0(\lambda)$ of $U_q(L_0)$ can be affinized to give rise to an irrep of $U_q(\hat{L}^{(2)})$. To perform such an affinization it is necessary and sufficient to find operators $\pi_\lambda(e_0)$ and $\pi_\lambda(f_0)$ acting on $V_0(\lambda)$ which satisfy the required defining relations (3.1) of $V_0(\lambda)$.

We define an automorphism $D_t$ of $U_q(\hat{L}^{(2)})$ by

$$D_t(e_i) = t^{\delta_{i0}} e_i, \quad D_t(f_i) = t^{-\delta_{i0}} f_i, \quad D_t(h_i) = h_i.$$  \hspace{1cm} (3.6)$$

Given any two minimal irreps $\pi_\lambda$ and $\pi_\mu$ of $U_q(L_0)$ and their affinizations to irreps of $U_q(\hat{L}^{(2)})$, we obtain a one-parameter family of representations $\Delta^u_{\lambda\mu}$ of $U_q(\hat{L}^{(2)})$ on $V_0(\lambda) \otimes V_0(\mu)$ defined by

$$\Delta^u_{\lambda\mu}(a) = \pi_\lambda \otimes \pi_\mu \left( (D_a \otimes 1) \Delta(a) \right), \quad \forall a \in U_q(\hat{L}^{(2)}),$$  \hspace{1cm} (3.7)$$

where $u$ is the spectral parameter. We define the spectral parameter dependent R-matrix

$$R^{\lambda\mu}(u) = (\pi_\lambda \otimes \pi_\mu) \left( (D_u \otimes 1) \hat{R} \right).$$  \hspace{1cm} (3.8)$$

It follows from (3.3) that this R-matrix gives a solution to the spectral parameter dependent Yang-Baxter equation (1.1). From the defining property (3.4) of the universal R-matrix one derives the equations

$$R^{\lambda\mu}(u) \Delta^u_{\lambda\mu}(a) = (\Delta^T(a))^{u}_{\lambda\mu} R^{\lambda\mu}(u)$$  \hspace{1cm} (3.9)$$

which, because the representations $\Delta^u_{\lambda\mu}$ are irreducible for generic $u$, uniquely determine $R^{\lambda\mu}(u)$ up to a scalar function of $u$. These are the Jimbo equations for twisted affine algebras.

As in [7] we normalize $R^{\lambda\mu}(u)$ such that

$$\check{R}^{\lambda\mu}(u) R^{\mu\nu}(u^{-1}) = I \quad \text{and} \quad R(0) = \pi_\lambda \otimes \pi_\mu(R),$$  \hspace{1cm} (3.10)$$

where $R$ is the R-matrix of $U_q(L_0)$ and $\check{R}^{\lambda\mu}(u) = P R^{\lambda\mu}(u)$ with $P : V_0(\lambda) \otimes V_0(\mu) \rightarrow V_0(\mu) \otimes V_0(\lambda)$ the usual permutation operator.

In order for the equation (3.9) to hold for all $a \in U_q(\hat{L}^{(2)})$ it is sufficient that it holds for all $a \in U_q(L_0)$ and in addition for the extra generator $e_0$. The relation for $e_0$ reads explicitly

$$R^{\lambda\mu}(u) \left( u \pi_\lambda(e_0) \otimes \pi_\mu(q^{h_{0/2}}) + \pi_\lambda(q^{-h_{0/2}}) \otimes \pi_\mu(e_0) \right) = \left( u \pi_\lambda(e_0) \otimes \pi_\mu(q^{h_{0/2}}) + \pi_\lambda(q^{-h_{0/2}}) \otimes \pi_\mu(e_0) \right) R^{\lambda\mu}(u),$$  \hspace{1cm} (3.11)$$
or equivalently
\[
\tilde{R}^{\lambda\mu}(u) \left( u \pi_\lambda(e_0) \otimes \pi_\mu(q^{h_0/2}) + \pi_\lambda(q^{-h_0/2}) \otimes \pi_\mu(e_0) \right) \\
= \left( \pi_\mu(e_0) \otimes \pi_\lambda(q^{h_0/2}) + u \pi_\mu(q^{-h_0/2}) \otimes \pi_\lambda(e_0) \right) \tilde{R}^{\lambda\mu}(u).
\] (3.12)

### 4 Solutions to Jimbo’s equations

With \(V_0(\lambda)\) and \(V_0(\mu)\) denoting two minimal irreps of \(U_q(\hat{L}^{(2)})\) we write the tensor product decomposition into irreducible \(U_q(L_0)\)-modules as
\[
V_0(\lambda) \otimes V_0(\mu) = \bigoplus_{\nu} V_0(\nu)
\] (4.1)
and note that there are no multiplicities in this decomposition for the cases which we are considering (c.f. Appendix B). We let \(P^{\lambda\mu}_\nu\) be the projection operator of \(V_0(\lambda) \otimes V_0(\mu)\) onto \(V_0(\nu)\) and set
\[
\tilde{P}^{\lambda\mu}_\nu = \tilde{R}^{\lambda\mu}(1) P^{\lambda\mu}_\nu = P^{\mu\lambda}_\nu \tilde{R}^{\lambda\mu}(1).
\] (4.2)

We may thus write
\[
\tilde{R}^{\lambda\mu}(u) = \sum_\nu \rho_\nu(u) \tilde{P}^{\lambda\mu}_\nu, \quad \rho_\nu(1) = 1.
\] (4.3)

Following our previous approach [7], the coefficients \(\rho_\nu(u)\) may be determined according to the recursion relation
\[
\rho_\nu(u) = \frac{q^{C(\nu)/2} + \epsilon_\nu \epsilon_{\nu'} u q^{C(\nu')/2}}{u q^{C(\nu)/2} + \epsilon_\nu \epsilon_{\nu'} q^{C(\nu')/2}} \rho_{\nu'}(u),
\] (4.4)
which holds for any \(\nu \neq \nu'\) for which
\[
P^{\lambda\mu}_\nu \left( \pi_\lambda(e_0) \otimes \pi_\mu(q^{h_0/2}) \right) P^{\lambda\mu}_{\nu'} \neq 0.
\] (4.5)

Here \(C(\nu)\) is the eigenvalue of the universal Casimir element of \(L_0\) on \(V_0(\nu)\) and \(\epsilon_\nu\) denotes the parity of \(V_0(\nu) \subseteq V_0(\lambda) \otimes V_0(\mu)\), (c.f. Appendix C).

To graphically encode the recursive relations between the different \(\rho_\nu\) we introduce the **Twisted Tensor Product Graph** \(\tilde{G}^{\lambda\mu}\) associated to the tensor product module \(V_0(\lambda) \otimes V_0(\mu)\). The nodes of this graph are given by the highest weights \(\nu\) of the \(U_q(L_0)\)-modules occuring in the decomposition (4.1) of the tensor product module. There is an edge between two nodes \(\nu \neq \nu'\) iff (4.5) holds.

Given a tensor product module and its decomposition, it is not in general an easy task to determine the twisted tensor product graph because in order to determine between which nodes of the graph relation (4.5) holds requires detailed calculations. We therefore introduce the **Extended Twisted Tensor Product Graph** \(\tilde{\Gamma}^{\lambda\mu}\)
which has the same set of nodes as the twisted tensor product graph (TPG) but has an edge between two vertices $\nu \neq \nu'$ whenever

$$V_0(\nu') \subseteq V_0(\theta_0) \otimes V_0(\nu)$$

and

$$\epsilon_\nu \epsilon_{\nu'} = \begin{cases} +1 & \text{if } V_0(\nu) \text{ and } V_0(\nu') \text{ are in the same irrep of } L \\ -1 & \text{if } V_0(\nu) \text{ and } V_0(\nu') \text{ are in different irreps of } L. \end{cases}$$

The conditions (4.6) and (4.7) are necessary conditions for (4.5) to hold and therefore the twisted TPG is contained in the extended twisted TPG. To see why (4.6) is a necessary condition for (4.5) one must realize that $e_0 \otimes q^{h_0/2}$ is the lowest component of a tensor operator corresponding to $V_0(\theta_0)$, see \[30\] for details. The necessity of (4.7) follows from the following fact derived in Appendix C: Two vertices $\nu \neq \nu'$ connected by an edge in the twisted TPG (i.e., for which (4.5) is satisfied) must have the same parity if $V_0(\nu)$ and $V_0(\nu')$ belong to the same irreducible $L$-module while they must have opposite parities if they belong to different irreducible $L$-modules.

While the extended twisted TPG will always include the twisted TPG, it will in general have more edges. Only if the extended twisted TPG is a tree are we guaranteed that it coincides with the twisted TPG.

**Note:** Unlike the untwisted case [7], we may now get an edge between $\nu$ and $\nu'$ of the same parity. This gives rise to a twisted TPG which may be topologically quite different to the untwisted TPG.

We will impose a relation (4.4) for every edge in the extended twisted TPG. Because the extended TPG will in general have more edges than the unextended twisted TPG, we will be imposing too many relations. These relations may be inconsistent and we are therefore not guaranteed a solution. If however a solution exists, then it must be the unique correct solution to Jimbo’s equations.

As seen below, for the minimal cases we are considering, the extended twisted TPG is always consistent and thus will always give rise to a solution of the quantum Yang-Baxter equation.

Throughout we adopt the convenient notation

$$\langle a \rangle_{\pm} = \frac{1 \pm x q^a}{x \pm q^a},$$

so that the relation (4.4) may be expressed as

$$\rho(\nu)(u) = \left\langle \frac{C(\nu') - C(\nu)}{2} \right\rangle_{\epsilon_\nu \epsilon_{\nu'}} \rho(\nu')(u).$$

We will now determine the R-matrices for any tensor product of any two $L_0$-irreducible representations for all the three families of twisted quantum affine algebras $U_q(A^{(2)}_{2l}), U_q(A^{(2)}_{2l-1})$ and $U_q(D^{(2)}_{l+1})$. 
4.1 R-matrices for $U_q(A_{2l}^{(2)})$

This is the case of the first line in table [1], i.e. $L = A_{2l} = sl(2l+1)$, $L_0 = B_l = so(2l+1)$ and $\theta_0 = 2\epsilon_1 = 2\lambda_1$.

The defining (vector) irrep $V(\lambda_1)$ of $U_q(L)$ is undeformed. By this we mean that the representation matrices for the fundamental generators are the same as those in the classical case, i.e. they are independent of $q$. It is also a minimal irrep, i.e. $V(\lambda_1) = V_0(\lambda_1)$ is also irreducible as a module of $U_q(L_0)$. Furthermore it is affinizable, i.e. it carries a representation of $U_q(A_{2l}^{(2)})$. Also this affinized representation is undeformed, i.e. $\pi(e_0)$ and $\pi(f_0)$ are given by the classical expressions.

We have the corresponding twisted TPG for $V(\lambda_1) \otimes V(\lambda_1)$

\[
\begin{array}{c c c}
+ & + & - \\
0 & 2\lambda_1 & \lambda_2
\end{array}
\] (4.10)

where $\pm$ indicate the parities. This is quite different to the untwisted TPG

\[
\begin{array}{c c c}
+ & - & + \\
0 & \lambda_2 & 2\lambda_1
\end{array}
\] (4.11)

Since $\lambda_2$ is an extremal node on the twisted TPG it follows that $V_0(\lambda_2)$ is affinizable, i.e. it too carries an irrep of $U_q(A_{2l}^{(2)})$ (for a discussion of the relation between TPGs and finite dimensional irreps of quantum affine algebras see [13]). More generally we have the following twisted TPG for for $V(\lambda_1) \otimes V(\lambda_k)$, $k < l$

\[
\begin{array}{c c c c}
+ & + & - \\
\lambda_{k-1} & \lambda_1 + \lambda_k & \lambda_{k+1}
\end{array}
\] (4.12)

so that $\lambda_{k+1}$ is an extremal node and hence, by recursion, each of the fundamental irreps $V_0(\lambda_k)$, $1 \leq k \leq l$, is affinizable. Again the above twisted TPG (4.12) is different to the untwisted one which is

\[
\begin{array}{c c c c}
+ & - & + \\
\lambda_{k-1} & \lambda_{k+1} & \lambda_1 + \lambda_k
\end{array}
\] (4.13)

Note: In the untwisted case $V_0(\lambda_k)$, $k > 1$ does not occur on extremal nodes of any TPG and can therefore not be shown to be affinizable to a representation of $U_q(\hat{L}_0^{(1)})$. In fact it is generally not affinizable [13, 14] in the untwisted sense. But, as seen above, it is nevertheless affinizable to a representation of the twisted algebra $U_q(\hat{L}^{(2)})$.

Now for $1 \leq k \leq r \leq l$ we have the tensor product decomposition

\[
V_0(\lambda_k) \otimes V_0(\lambda_r) = \bigoplus_{a=0}^{k} \bigoplus_{c=0}^{a} V_0(\lambda_c + \lambda_d)
\] (4.14)
where
\[
d = \begin{cases} 
  k + r - 2a + c & \text{for } 2a - c \geq r + k - l \\
  2l + 1 - (k + r - 2a + c) & \text{for } 2a - c < r + k - l 
\end{cases}
\] (4.15)

which is multiplicity free (c.f. Appendix [3]). The corresponding extended twisted tensor product graph is consistent and quite different in topology to the the extended untwisted TPG (which is inconsistent). We illustrate this below with the case \( r + k \leq l, \ k \leq r \). In this case \( d = k + r - 2a + c \) and we have the extended twisted TPG depicted in figure 1.

Figure 1: The extended twisted TPG for \( U_q(A^{(2)}_2) \) for the product \( V_0(\lambda_k) \otimes V_0(\lambda_r) \) \((k \leq r, \ r + k \leq l)\). The nodes correspond to representations whose highest weight is given by the sum of the weight labeling the column and the weight labeling the row. The \( \pm \) indicate the parity. The parities are equal along the northwest-southeast diagonals and they alternate along the northeast-southwest diagonals.

To see that the extended twisted TPG in figure 1 is consistent, consider a typical
where we have indicated the relative parities of the vertices. Using the fact that on $V_0(\lambda_c + \lambda_d)$ the universal Casimir element of $L_0$ takes the eigenvalue

$$C_{c,d} = (c + d)(2l + 2 - c) - (d + 1)(d - c),$$

(4.17)

it is easily seen that

$$C_{c,d} - C_{c-1,d-1} = C_{c-1,d+1} - C_{c-2,d} = 2(2l + 3 - c - d),$$

$$C_{c,d} - C_{c-1,d+1} = C_{c-1,d-1} - C_{c-2,d} = 2(d - c + 2).$$

(4.18)

This implies that the extended twisted TPG is consistent, i.e. that the recursion relations (4.4) give the same result independent of the path along which one recurses.

We are now in a position to write down our solution to Jimbo’s equation and thus to the quantum Yang-Baxter equation arising from the above extended twisted TPG:

$$\tilde{R}^{\lambda_k,\lambda_r}(u) = \sum_{a=0}^{k} \sum_{c=0}^{a} \prod_{i=a}^{k-1} \langle k + r - 2i \rangle - \prod_{j=1}^{a-c} \langle n - r - k + 2j \rangle \cdot \tilde{R}^{\lambda_k,\lambda_r}_{\lambda_c + \lambda_k + r - 2a + c}$$

(4.19)

### 4.2 R-matrices for $U_q(A_{2l-1}^{(2)})$

This is the case of the second line in table [4], i.e. $L = A_{2l-1} = sl(2l)$, $L_0 = C_l = sp(2l)$ and $\theta_0 = \epsilon_1 + \epsilon_2 = \lambda_2$.

Starting with the vector irrep $V_0(\lambda_1)$ of $U_q(L_0)$ (and also of $U_q(L)$) we have the following twisted TPG for $V(\lambda_1) \otimes V(\lambda_1)$

$$\begin{array}{ccc}
0 & \lambda_2 & 2\lambda_1 \\
\hline
- & - & +
\end{array}$$

(4.20)

which has quite a different topology to the untwisted TPG

$$\begin{array}{ccc}
0 & 2\lambda_1 & \lambda_2 \\
\hline
- & + & -
\end{array}$$

(4.21)
Because $V_0(2\lambda_1)$ appears as an extremal node on the twisted TPG $[4.20]$, it is affinizable. Continuing in this way it is easily seen that $V_0(a\lambda_1)$ is affinizable for any positive integer $a$. We have the following $U_q(L_0)$-module decomposition of the tensor product of any two such representations

$$V_0(k\lambda_1) \otimes V_0(r\lambda_1) = \bigoplus_{a=0}^{k} \bigoplus_{b=0}^{a} V_0((k + r - 2a)\lambda_1 + b\lambda_2), \quad k \leq r. \quad (4.22)$$

Figure 2: The extended twisted TPG for $U_q(A_{2l-1}^{(2)})$ for the tensor product $V(k\lambda_1) \otimes V(r\lambda_1)$. The nodes correspond to modules whose highest weight is the sum of the weight labeling the column and the weight labeling the row. Some of the parities are indicated below the nodes. The parities are the same along the horizontal and they alternate along the vertical.

The corresponding extended twisted TPG is shown in figure 2. To see that this graph is consistent we have to consider the closed loops of the form

$$c\lambda_1 + (d - 1)\lambda_2 \quad (c - 2)\lambda_1 + d\lambda_2 \quad c\lambda_1 + d\lambda_2 \quad (c - 2)\lambda_1 + (d + 1)\lambda_2$$

(4.23)
where we have indicated the relative parities of the vertices. Using the fact that on $V_0(c\lambda_1 + d\lambda_2)$ the quadratic Casimir element of $L_0$ takes the eigenvalue
\[ C_{c,d} = (c + d)(n + c + d) + (n - 2 + d)d, \] (4.24)
it is easily seen that the above loop is consistent:
\[
\begin{align*}
C_{c,d} - C_{c,d+1} &= C_{c-2,d+1} - C_{c-2,d} = 2(n - 2 + c + 2d), \\
C_{c,d} - C_{c-2,d+1} &= C_{c,d-1} - C_{c-2,d} = 2c.
\end{align*}
\] (4.25)

We can now read off the R-matrix from the extended twisted TPG
\[
\tilde{R}^{k\lambda_1,r\lambda_1}(u) = \sum_{a=0}^{k} \sum_{b=0}^{a} \prod_{i=1}^{a} \langle n + k + r - 2i \rangle_+ \prod_{j=1}^{a} \langle k + r + 2 - 2j \rangle_- \\
\times \tilde{P}^{k\lambda_1,r\lambda_1}_{(k+r-2a)\lambda_1+b\lambda_2}.
\] (4.26)

Again it can be seen that the above eigenvalues of $\tilde{R}(u)$ are quite different to those arising from the untwisted case and thus give rise to new solutions to the quantum Yang-Baxter equation.

### 4.3 R-matrices for $U_q(D_{l+1}^{(2)})$

This is the case of $L = D_{l+1} = so(2l + 2)$, $L_0 = B_l = so(2l + 1)$ and $\theta_0 = \epsilon_1 = \lambda_1$. We set $n = 2l + 1$.

The fundamental spinor irrep $V_0(\lambda_l)$, $\lambda_l = \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_l)$ of $U_q(L_0)$ is undeformed and affinizable and also carries the spinor irreps $V(\lambda_l)$ and $V(\lambda_{l+1})$ of $U_q(L)$. We have the tensor product decomposition, for $a \in \mathbb{Z}_+$,
\[
V_0(\lambda_l) \otimes V_0(a\lambda_l) = \bigoplus_{k=0}^{l-1} V_0(\lambda_k + (a - 1)\lambda_l) \oplus V_0((a + 1)\lambda_l) \] (4.27)

with the corresponding twisted TPG
\[
\begin{array}{cccccccc}
\bullet & - & \bullet & + & \bullet & + & \cdots & \bullet \\
(a+1)\lambda_l & \mu_1 & - & \mu_2 & \cdots & \mu_l & (a-1)\lambda_l
\end{array}
\] (4.28)

where we have indicated the parity of the vertices and introduced $\mu_i = (a-1)\lambda_l + \lambda_{l-i}$. Note that his graph has a very different topology from the corresponding untwisted TPG given in Fig. 1 of [7]. Because $(a + 1)\lambda_l$ is on an extremal node of this twisted TPG it then follows by induction that $V_0(a\lambda_l)$ is affinizable for any positive integer $a$. The eigenvalue of the universal Casimir element of $L_0$ on $V_0(\lambda_k + (a - 1)\lambda_l)$ is given by
\[
C(\lambda_k + (a - 1)\lambda_l) = k(n + a - k - 1) + \frac{1}{4}l(a - 1)(n + a - 2), \quad 1 \leq k \leq l - 1,
\]
\[
C((a + 1)\lambda_l) = l(n + a - l - 1) + \frac{1}{4}l(a - 1)(n + a - 2)
\] (4.29)
Using this we can read off the R-matrix from the twisted TPG (4.28):

\[
\tilde{R}_{\lambda, a\lambda l}^{\lambda_l, a\lambda l}(u) = \sum_{k=0}^{l-1} \rho_k(u) \tilde{P}_{\lambda_k, (a-1)\lambda_l}^{\lambda_l, a\lambda_l} + \tilde{P}_{(a+1)\lambda_l}^{\lambda_l, a\lambda_l}
\]  

(4.30)

with the eigenvalues \( \rho_k(u) \) given by

\[
\rho_k(u) = \prod_{i=1}^{l-k} \left\langle \frac{1}{2}(a - 1) + i \right\rangle (-1)^i.
\]

(4.31)

We now proceed to the general case \( V_0(a\lambda_l) \otimes V_0(b\lambda_l), \ 0 \le a \le b \in \mathbb{Z} \). In view of appendix B we now have the multiplicity-free tensor product decomposition

\[
V_0(a\lambda_l) \otimes V_0(b\lambda_l) = \bigoplus_{\Lambda} V_0(\Lambda + (b-a)\lambda_l),
\]

(4.32)

where the sum is over all dominant weights \( \Lambda = (\Lambda_1, \Lambda_2, \ldots, \Lambda_l) \) satisfying

\[
a \ge \Lambda_1 \ge \Lambda_2 \ge \cdots \ge \Lambda_l \ge 0, \quad \Lambda_i \in \mathbb{Z},
\]

(4.33)

each such weight appearing exactly once.

The extended twisted TPG will typically be \( l \)-dimensional and we can therefore not draw a diagram for it. However to determine whether it is consistent, it is sufficient to look at closed loops of the form (labeling the vertices by the weight \( \Lambda \) since \( a \) and \( b \) are here fixed and thus redundant labels):

\[
\begin{align*}
&\Lambda - \epsilon_j \\
&\Lambda - \epsilon_i \\
&\Lambda - \epsilon_i - \epsilon_j
\end{align*}
\]

(4.34)

To show that all the edges in this loop really exist, i.e. that (4.33) is satisfied, one uses the following theorem proven in [12]:

**Theorem 1** Suppose \( \lambda, \mu \) are dominant weights and \( \nu \) is a weight Weyl group conjugate to \( \lambda \). If \( \mu + \nu \) is dominant then \( V(\mu + \nu) \) occurs exactly once in \( V(\lambda) \otimes V(\mu) \).

To obtain the relative (to the top vertex) parities of the vertices of the closed loops it is necessary to determine which irreps of \( so(2l + 1) \) belong to the same irrep of \( so(2l + 2) \) (which will all have the same parity). As seen in appendix B, two such
irreps $V_0(\Lambda + (b - a)\lambda_l)$ and $V_0(\Lambda' + (b - a)\lambda_l)$ belong to the same irrep of $so(2l + 2)$ iff $\Lambda_i = \Lambda'_i$ for $l + i$ odd. From this it follows that the difference in parity along any edge in (4.34) is equal to the difference in parity along the opposite edge. Using the eigenvalue formula for the universal Casimir element of $L_0$ on $V_0(\Lambda + (b - a)\lambda_l)$

$$C_\Lambda = \sum_{i=1}^l \Lambda_i (\Lambda_i + b - a + n - 2i) + \frac{1}{4} l(b - a)(b - a + n - 1), \quad (4.35)$$

we see that also the difference of the Casimirs are equal along opposite edges:

$$C_\Lambda - C_{\Lambda - \epsilon_i} = C_{\Lambda - \epsilon_j} - C_{\Lambda - \epsilon_i - \epsilon_j} = 2(\Lambda_i - i) + b - a + n - 1, \quad i \neq j. \quad (4.36)$$

From these facts it follows that the extended twisted TPG is consistent.

We are now in a position to determine the eigenvalues $\rho_\Lambda(u)$ in the expression for the R-matrix

$$\check{R}^{a\lambda_i, b\lambda_i}(u) = \sum_\Lambda \rho_\Lambda(u) \check{R}^{a\lambda_i, b\lambda_i}_{\Lambda + (b - a)\lambda_i}. \quad (4.37)$$

We start with the first component $\Lambda_1$ and proceed along the following path:

$$\Lambda \quad \Lambda + \epsilon_1 \quad \ldots \quad \Lambda + (a - \Lambda_1)\epsilon_1 \quad (4.38)$$

We then proceed in this way component by component. By this means we arrive at the formula

$$\rho_\Lambda(u) = \prod_{i=1}^l \prod_{k=\Lambda_i}^{a-1} \left\langle k - i + l + 1 + \frac{1}{2} (b - a) \right\rangle_{(-1)^{i+l}}. \quad (4.39)$$

If $\Lambda_i = a$ for some $i$ then the $i$th term is understood to contribute 1 to the product. It is easily seen that this formula reduces to our previous one when $a = b = 1$.

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A Kac generators

In this appendix we give detailed expressions for the generators $E_i, F_i$ and $H_i$ of (2.2). We will write them in terms of the familiar basis elements $e_{ij}, i \leq i, j \leq n$ of $gl(n)$ which satisfy the Lie bracket

$$[e_{ij}, e_{kl}] = \delta_{kj} e_{il} - \delta_{il} e_{kj}. \quad (A.1)$$
As our invariant bilinear form on \( gl(n) \) we take
\[
(e_{ij}, e_{kl}) = \frac{1}{2} \delta_{jk} \delta_{il}.
\] (A.2)
We choose a basis \( \{ \epsilon_i | 1 \leq i \leq n \} \) for the root space, i.e. the dual space to the Cartan subalgebra of \( gl(n) \) such that \( \sqrt{2} \epsilon_{ii} \) is paired with \( \epsilon_i \). Then the bilinear form (A.2) induces the scalar product \( (\epsilon_i, \epsilon_j) = \delta_{ij} \).

The diagram automorphism \( \sigma \) of order \( k = 2 \) by which we will twist is given by
\[
\sigma(e_{ij}) = (-1)^{i+j+1} e_{\bar{j}\bar{i}}, \quad \bar{i} = n + 1 - i.
\] (A.3)

The fixed point subalgebra \( L_0 \) is generated by the linear combinations
\[
a_{ij} = e_{ij} - (-1)^{i+j} e_{\bar{j}\bar{i}}.
\] (A.4)

Its Cartan subalgebra \( H \) is spanned by the elements
\[
b_{ij} = e_{ij} + (-1)^{i+j} e_{\bar{j}\bar{i}} - \frac{2}{n} \delta_{ij} I_1, \quad I_1 = \sum_{i=1}^{n} e_{ii}.
\] (A.5)

**A.1** \( L = A_{2l} = sl(2l + 1), \ L_0 = B_l = so(2l + 1) \)

In this case we perform the above construction with \( n = 2l + 1 \). Our simple generators are given by
\[
E_i = a_{i,i+1}, \quad F_i = a_{i+1,i}, \quad H_i = a_{i,i} - a_{i+1,i+1}, \quad 1 \leq i < l,
\]
\[
E_l = a_{l,l+1}, \quad F_l = a_{l+1,l}, \quad H_l = a_{l,l},
\]
\[
E_0 = \frac{1}{\sqrt{2}} b_{11} = \sqrt{2} a_{2l+1,1}, \quad F_0 = \sqrt{2} a_{1,2l+1}, \quad H_0 = -2a_{11},
\] (A.6)

with the corresponding simple roots
\[
\bar{\alpha}_i = \epsilon_i - \epsilon_{i+1}, \quad (1 \leq i < l), \quad \bar{\alpha}_l = \epsilon_l, \quad \bar{\alpha}_0 = -2\epsilon_1.
\] (A.7)

They can be checked to satisfy the defining relations (2.2). The invariant bilinear form (A.2) induces the form (2.3).

**Note:** Our notation here differs from that of Kac [22] who interchanges indices \( i = 0 \) and \( i = l \). We prefer our notation since it conforms to the usual notation in the untwisted case.

**A.2** \( L = A_{2l-1} = sl(2l), \ L_0 = C_l = sp(2l) \)

In this case \( n = 2l \) and our simple generators are
\[
E_i = a_{i,i+1}, \quad F_i = a_{i+1,i}, \quad H_i = a_{i,i} - a_{i+1,i+1}, \quad 1 \leq i < l,
\]
\[
E_l = \frac{1}{\sqrt{2}} a_{l,l+1}, \quad F_l = \frac{1}{\sqrt{2}} a_{l+1,l}, \quad H_l = 2a_{l,l},
\]
\[
E_0 = b_{2l-1,1}, \quad F_0 = b_{1,2l-1}, \quad H_0 = -(a_{11} + a_{22}),
\] (A.8)
with the corresponding simple roots

\[ \tilde{\alpha}_i = \epsilon_i - \epsilon_{i+1}, \quad (1 \leq i < l), \quad \tilde{\alpha}_l = 2\epsilon_l, \quad \tilde{\alpha}_0 = -(\epsilon_1 + \epsilon_2). \]  

(A.9)

They can be checked to satisfy the defining relations (2.2). The invariant bilinear form (A.2) induces the form (2.3).

A.3 \( L = D_{l+1} = so(2l + 2), \ L_0 = B_l = so(2l + 1) \)

In this case we embedd \( so(2l + 1) \) and \( so(2l + 2) \) into \( gl(2l + 2) \). The \( L_0 = so(2l + 1) \) generators are

\[ a_{i,j} = e_{ij} - (-1)^{i+j} e_{ji}, \quad 1 \leq i,j \leq 2l + 1, \]  

(A.10)

where \( i = 2l + 2 - j \). The extra generators which span \( L_1 \) and complete \( so(2l + 1) \) to \( so(2l + 2) \) are

\[ a_{i,2l+2} = e_{i,2l+2} - (-1)^{i} e_{2l+2,i}, \quad 1 \leq i \leq 2l + 1 \]  

(A.11)

As the set of simple generators for \( L = so(2l + 2) \) we take

\[ E_i = a_{i,i+1}, \quad F_i = a_{i+1,i}, \quad H_i = a_{i,i} - a_{i+1,i+1}, \quad 1 \leq i \leq l, \]  

\[ E_l = a_{l,l+1}, \quad f_l = a_{l+1,l}, \quad H_l = a_{l,l}, \]  

\[ E_0 = b_{2l+1,2l+2}, \quad F_0 = b_{2l+2,2l+1}, \quad H_0 = -a_{11}, \]  

(A.12)

with the corresponding simple roots

\[ \tilde{\alpha}_i = \epsilon_i - \epsilon_{i+1}, \quad (1 \leq i < l), \quad \tilde{\alpha}_l = \epsilon_l, \quad \tilde{\alpha}_0 = -\epsilon_1. \]  

(A.13)

They can be checked to satisfy the defining relations (2.2). The invariant bilinear form (A.2) induces the form (2.3).

B \ Decompositions and Branching rules

If \( \pi \) is any finite dimensional irrep of \( L \) on a space \( V \) then we obtain a representation \( \hat{\pi} \) of \( \hat{L}^{(2)} \) on the loop space \( V(t) \) with the prescription

\[ \hat{\pi}(e_i) = \pi(E_i), \quad \hat{\pi}(f_i) = \pi(F_i), \quad \hat{\pi}(h_i) = \pi(H_i), \quad i \leq i \leq l, \]  

\[ \hat{\pi}(e_0) = t\pi(E_0), \quad \hat{\pi}(f_0) = t^{-1}\pi(F_0), \quad \hat{\pi}(h_0) = \pi(H_0). \]  

(B.1)

Here we wish to consider the \( L_0 \)-irreducible irreps of \( L \) which, by definition, are also irreps of \( L_0 \). From the known \( L \downarrow L_0 \) branching rules this restricts us to the irreps of \( L \) with highest weights given in table 3. These \( L_0 \)-irreducible irreps are of interest because they can always be quantized to give rise to irreps of the twisted quantum affine algebra \( U_q(\hat{L}^{(2)}) \), as we have seen in the paper. Our aim here is to show that in the decomposition of the tensor product of any two \( L_0 \)-irreducible irreps all irreps of \( L_0 \) occur with at most unit multiplicity. At the same time we shall deduce the tensor product branching rules used in the paper. As before it is convenient to treat each of the three families separately.
\section*{B.1 $A_{2l}^{(2)}$}

This is the case with $L = A_{2l} = sl(n = 2l + 1)$, $L_0 = B_l = so(2l + 1)$. For $k \leq l$ we have seen that $V_0(\lambda_k) = V(\lambda_k)$, where $\lambda_k = \sum_{i=1}^{k} \epsilon_i$, is $L_0$-irreducible. We first consider the decomposition of the tensor product of two such irreps into irreps of $L$

$$V(\lambda_k) \otimes V(\lambda_r) = \bigoplus_{a=0}^{k} V(\lambda_a + \lambda_b), \quad b = k + r - a, \quad 1 \leq k \leq r \leq l. \quad (B.2)$$

We have

$$\dim V(\lambda_a + \lambda_b) = \frac{b - a + 1}{n + 1} \binom{n + 1}{a} \binom{n + 1}{b + 1}. \quad (B.3)$$

From well established techniques (e.g. Young diagrams \cite{young} or the quasi-spin formalism \cite{quasi-spin1, quasi-spin2, quasi-spin3}) we deduce the $L \downarrow L_0$ branching rules (here and below $a \wedge b = \min(a, b)$)

$$V(\lambda_a + \lambda_b) = \bigoplus_{c=0}^{a \wedge (n-b)} V_0(\lambda_c + \lambda_d), \quad d = (b - a + c) \wedge (n - (b - a + c)). \quad (B.4)$$

This decomposition is consistent with the dimension formula

$$\dim V_0(\lambda_c + \lambda_d) = \frac{(1 + d - c)(n + 1 - c - d)}{(n + 1)(n + 2)} \binom{n + 2}{c} \binom{n + 2}{d + 1}. \quad (B.5)$$

For the case at hand $1 \leq a \leq k \leq r \leq l$ from which it follows that $a \wedge (n - b) = a$ and we obtain the $L_0$-module decomposition

$$V_0(\lambda_k) \otimes V_0(\lambda_r) = \bigoplus_{a=0}^{k} \bigoplus_{c=0}^{a} V_0(\lambda_c + \lambda_d), \quad d = (k + r - 2a + c) \wedge (n - (k + r - 2a + c)). \quad (B.6)$$

To see that this decomposition is indeed multiplicity free suppose that the $L_0$-module $V_0(\lambda_c + \lambda_d)$ occurred twice. This could only happen if there existed $a, a'$ such that $(k + r - 2a + c) = n - (k + r - 2a' + c)$ which is however impossible because $n = 2l + 1$ is odd.

\section*{B.2 $A_{2l-1}^{(2)}$}

This is the case of $L = A_{2l-1} = sl(n = 2l)$, $L_0 = C_l = sp(2l)$. In this case $V(k\lambda_1) = V_0(k\lambda_1)$ is minimal and irreducible as both an $L$-module and an $L_0$-module. For $k \leq r$ we have the $L$-module decomposition

$$V(k\lambda_1) \otimes V(r\lambda_1) = \bigoplus_{a=0}^{k} V(b\lambda_1 + a\lambda_2), \quad b = k + r - 2a. \quad (B.7)$$
The $L \downarrow L_0$ branching rules for such modules is

$$V(b\lambda_1 + a\lambda_2) = \bigoplus_{c=0}^{a} V_0(b\lambda_1 + c\lambda_2),$$  \hspace{1cm} (B.8)

which again follows from Young diagrams or the quasi-spin formalism [21, 8, 16].

The decomposition is consistent with the dimension formulae

$$\dim V(b\lambda_1 + a\lambda_2) = \frac{b+1}{n-1} \left( \begin{array}{c} a+b+n-1 \\ n-2 \end{array} \right) \left( \begin{array}{c} a+n-2 \\ n-2 \end{array} \right),$$

$$\dim V_0(d\lambda_1 + c\lambda_2) = \frac{(1+d)(2c+d+n-1)}{(n-1)(n-2)} \left( \begin{array}{c} c+d+n-2 \\ n-3 \end{array} \right) \left( \begin{array}{c} c+n-3 \\ n-3 \end{array} \right).$$  \hspace{1cm} (B.9)

We thus arrive at the irreducible $L_0$-module decomposition

$$V_0(k\lambda_1) \otimes V_0(r\lambda_1) = \bigoplus_{a=0}^{k} \bigoplus_{c=0}^{a} V_0((k+r-2a)\lambda_1 + c\lambda_2), \hspace{1cm} k \leq r, \hspace{1cm} (B.10)$$

which is easily seen to be multiplicity free.

### B.3 $D_{l+1}^{(2)}$

This is the case of $L = D_{l+1} = so(n = 2l + 2), \hspace{0.5cm} L_0 = B_l = so(2l+1)$.

We first recall the PRV theorem [20]. Let $L$ be a finite dimensional simple Lie algebra with simple roots $\alpha_i, 1 \leq i \leq l$ and corresponding Chevalley generators $e_i, f_i, h_i, 1 \leq i \leq l$. We denote the set of dominant weights of $L$ by $D_+$ and for $\lambda \in D_+$ we let $\Pi(\lambda)$ denote the set of distinct weights in the finite dimensional irreducible module $V(\lambda)$.

**Theorem 2** For $\lambda, \mu \in D_+$ we have the tensor product decomposition

$$V(\lambda) \otimes V(\mu) = \bigoplus_{\nu \in \Pi(\lambda)} \dim(V_{\nu, \mu}(\lambda)) V(\mu + \nu).$$ \hspace{1cm} (B.11)

Here

$$V_{\nu, \mu}(\lambda) = \left\{ v \in V(\lambda) | e_i^{(\mu, \alpha_i)+1} v = 0, 1 \leq i \leq l \right\}$$ \hspace{1cm} (B.12)

where $V(\lambda)$ is the weight space consisting of weight vectors of $V(\lambda)$ of weight $\nu \in \Pi(\lambda)$ and

$$\langle \mu, \alpha_i \rangle = \frac{2(\mu, \alpha_i)}{\langle \alpha_i, \alpha_i \rangle}. \hspace{1cm} (B.13)$$

To determine the $so(2l+1)$ branching rules for $V_0(a\lambda_l) \otimes V_0(b\lambda_l), \hspace{1cm} a \leq b, \hspace{0.5cm} \lambda_l = (\frac{1}{2}, \ldots, \frac{1}{2}, \lambda_l)$, we use the fact that $V_0(a\lambda_l)$ is the carrier space for parafermistatistics
of order \( a \mid 3 \). From this it is known that \( V_0(a\lambda) \) decomposes into a direct sum of irreps of the \( gl(l) \) subalgebra with highest weight of the form \( \Lambda - a\lambda \) where \( \Lambda = (\Lambda_1, \ldots, \Lambda_l) \) satisfies
\[
a \geq \Lambda_1 \geq \Lambda_2 \geq \cdots \geq \Lambda_l \geq 0, \quad \Lambda_i \in \mathbb{Z},
\]
each occurring exactly once. We deduce from this, together with the PRV theorem, the tensor product decomposition
\[
V_0(a\lambda) \otimes V_0(b\lambda) = \bigoplus_{\Lambda} V_0(\Lambda + (b-a)\lambda),
\]
where the sum is over all \( \Lambda \) satisfying (B.14).

To obtain the correct parity pattern for the extended twisted tensor product graph we need to investigate the \( so(2l+2) \) branching law. It is convenient to consider the so-called statistical operator \( \tilde{q} \) which takes a constant value on the irreducible \( gl(l) \) submodule with highest weight \( \Lambda - a\lambda \) (with \( \Lambda \) as in (B.14)) given by
\[
\tilde{q} = -\sum_{i=1}^{l} (-1)^i \Lambda_i.
\]

Then \( \epsilon_{l+1}^\vee \equiv a/2 - \tilde{q} \) is the additional Cartan generator of \( so(2l+2) \) so that, in this representation, \( V_0(a\lambda) \) gives rise to an irreducible \( so(2l+2) \) module \( V(\hat{\lambda}_s) \) with highest weight \( \hat{\lambda}_s \),
\[
\hat{\lambda}_s = \left( \frac{1}{2}, \ldots, \frac{1}{2}, (-1)^{l+1} \frac{1}{2} \right) \quad (l + 1 \text{ components}).
\]

Again applying the PRV theorem we deduce the following \( so(2l+2) \) branching law:
\[
V_0(a\lambda) \otimes V_0(b\lambda) = V(a\hat{\lambda}_s) \otimes V(b\hat{\lambda}_s) = \bigoplus_{\hat{\Lambda}} V(\hat{\Lambda} + (b-a)\hat{\lambda}_s),
\]
where the sum is over all \( so(2l+2) \) weights \( \hat{\Lambda} \in D_+ \) of the form
\[
\hat{\Lambda} = \begin{cases} 
(\hat{\Lambda}_1, \hat{\Lambda}_1, \hat{\Lambda}_2, \cdots, \hat{\Lambda}_{l/2}, \hat{\Lambda}_{l/2}, 0), & l \text{ even} \\
(a, \hat{\Lambda}_1, \hat{\Lambda}_1, \hat{\Lambda}_2, \cdots, \hat{\Lambda}_{(l-1)/2}, \hat{\Lambda}_{(l-1)/2}, 0), & l \text{ odd}
\end{cases}
\]
with \( a \geq \Lambda_i \geq \hat{\Lambda}_{i+1}, \ \forall i \).

**Note:** The above may be uniquely characterized as those weights \( \hat{\Lambda} = (\Lambda, 0) \), where \( \Lambda \in D_0 \) are those \( so(2l+1) \) weights of the form (B.14), for which the corresponding statistical quantum number \( \tilde{q} \) takes its minimal (resp. maximal) value \( \tilde{q} = 0 \) (resp. \( \tilde{q} = a \)) when \( l \) is even (resp. odd).

It follows from this that an irreducible \( so(2l+1) \) module \( V_0(\Lambda + (b-a)\lambda) \) belongs to the irreducible \( so(2l+2) \) module \( V(\hat{\Lambda} + (b-a)\hat{\lambda}_s) \) where
\[
\hat{\Lambda}_i = \begin{cases} 
\Lambda_i, & i + l \text{ odd} \\
\Lambda_{i-1}, & i + l \text{ even}
\end{cases}
\]
with \( \Lambda_0 \equiv a \). This leads to the parity pattern used in the paper.
C  Note on Parities

Here we derive the following result:

Two vertices \( \nu \neq \nu' \) connected by an edge in the twisted TPG (i.e., for which (4.5) is satisfied) must have the same parity if \( V_0(\nu) \) and \( V_0(\nu') \) belong to the same irreducible \( L \)-module while they must have opposite parities if they belong to different irreducible \( L \)-modules.

We used this in section 4 to show that the twisted tensor product graph is contained in the extended twisted tensor product graph. The following considerations are a generalization of those for the untwisted case contained in Appendix B of [7] to which we refer the reader for the details.

It is sufficient to consider the case \( q = 1 \). We introduce the twisted parity operator

\[
\tilde{P} \equiv R^{\lambda \mu}(1)|_{q=1},
\]

which satisfies \( \tilde{P}^2 = 1 \). As in eqs. (B.12) and (B.13) of [7] we obtain the equations (omitting \( \pi_\lambda \) and \( \pi_\mu \) below)

\[
\tilde{P} P^{(0)\lambda \mu}_\nu (e_0 \otimes 1) P^{(0)\lambda \mu}_\nu = P^{(0)\lambda \mu}_\nu (1 \otimes e_0) P^{(0)\lambda \mu}_\nu \tilde{P},
\]

\[
\tilde{P} P^{(0)\lambda \mu}_\nu (1 \otimes e_0) P^{(0)\lambda \mu}_\nu = P^{(0)\lambda \mu}_\nu (e_0 \otimes 1) P^{(0)\lambda \mu}_\nu \tilde{P}, \quad \nu \neq \nu',
\]

where \( P^{(0)\lambda \mu}_\nu = P^{\lambda \mu}_{\nu}|_{q=1} \). We will now show that \( \tilde{P} \) coincides with the normal (untwisted) parity operator \( P \) on \( L_0 \) defined in our previous work. From (C.2) we deduce, since \( \tilde{P} \) and \( P^{(0)\lambda \mu}_\nu \) commute with the diagonal action of \( L_0 \), that, for \( \nu \neq \nu' \)

\[
\tilde{P} P^{(0)\lambda \mu}_\nu (a \otimes 1) P^{(0)\lambda \mu}_\nu = P^{(0)\lambda \mu}_\nu (1 \otimes a) P^{(0)\lambda \mu}_\nu \tilde{P},
\]

\[
\tilde{P} P^{(0)\lambda \mu}_\nu (1 \otimes a) P^{(0)\lambda \mu}_\nu = P^{(0)\lambda \mu}_\nu (a \otimes 1) P^{(0)\lambda \mu}_\nu \tilde{P}, \quad \forall a \in L_1
\]

since \( L_1 \) is irreducible under the adjoint action of \( L_0 \). It follows that, for all \( a \in L_1 \), \( 1 \otimes a - a \otimes 1 \) reverses the twisted parity while \( 1 \otimes a + a \otimes 1 \) preserves it. Then also, for \( a, b \in L_1 \),

\[
[1 \otimes a + a \otimes 1, 1 \otimes b - b \otimes 1] = 1 \otimes [a, b] - [a, b] \otimes 1
\]

must reverse the twisted parity and since \( [L_1, L_1] = L_0 \) this implies that \( 1 \otimes a - a \otimes 1 \) reverses the parity for any \( a \in L_0 \). But this is the defining property of the normal (untwisted) parity operator \( P \) so that both \( P \) and \( \tilde{P} \) have the same eigenvalues on the irreducible \( L_0 \) modules \( V_0(\nu) \subseteq V_0(\lambda) \otimes V_0(\mu) \). Thus we must have \( \tilde{P} = P \).

It follows that \( 1 \otimes a + a \otimes 1 \) preserves the usual parity for all \( a \in L \) while \( 1 \otimes a - a \otimes 1 \) reverses it. In particular, since all irreducible \( L_0 \) modules contained in a given irreducible \( L \) module are connected to one another by repeated application of the generators \( \Delta(a) = a \otimes 1 + 1 \otimes a, \ a \in L_1 \), we deduce that they must all have
the same parity. Thus two vertices $\nu \neq \nu'$ connected by an edge in the twisted tensor product graph must have the same parity if $V_0(\nu)$ and $V_0(\nu')$ belong to the same irreducible $L$ module while they must have opposite parities if they belong to different irreducible $L$ modules since then

$$0 \neq P^{(0)\lambda \mu}_\nu (e_0 \otimes 1) P^{(0)\lambda \mu}_{\nu'} = \frac{1}{2} P^{(0)\lambda \mu}_\nu (e_0 \otimes 1 - 1 \otimes e_0) P^{(0)\lambda \mu}_{\nu'}.$$

(C.5)

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