Global Properties of Exact Solutions in Integrable Dilaton-Gravity Models

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Abstract

Global canonical transformations to free chiral fields are constructed for DG models minimally coupled to scalar fields. The boundary terms for such canonical transformations are shown to vanish in asymptotically static coordinates if there is no scalar field.

1 Introduction

Here we consider a special class of 1+1 dimensional dilaton - gravity - matter models (DGM) [1]. Gravitational variables in these models are the metric tensor $g_{ij}$ and the scalar dilaton field $\phi$. The matter is represented by a scalar field $\psi$, which is minimally coupled to the gravitational variables [2]. The Lagrangian of these models may be written in the form

$$L = \sqrt{-g} \left[ \phi R(g) + V(\phi) - g^{ij} \psi_i \psi_j / 2 \right],$$

(1)

where $R$ is the scalar curvature, the subscripts denote partial derivatives ($\phi_i = \partial_i \phi$, etc.), except when used in $g_{ij}$. Many models (describing black holes, strings, cosmologies, etc.) can be written in the form (1). The formal exact solutions for some classes of these models can be obtained by Bäcklund transformations to free fields [3]. Recipes to find canonical transformations to free field for the exactly integrable models were suggested in [3].

In what follows we study the model in the conformal flat, light-like metric $ds^2 = -4f(u,v)du dv$, in which $R = (\log f)_{uv}/f$. Then the general covariance is lost but there exists residual covariance under reparametrization of light-cone coordinates $u = a(\bar{u}), \ v = b(\bar{v})$, where $a(\bar{u})$ and $b(\bar{v})$ are arbitrary monotonic coordinate functions.

2 Equations of Motion

The equations of motion can be derived by varying $L$ with respect to the variables $g_{ij}, \phi$, and $\psi$,

$$f(\phi_i/f)_i = -(\psi_i)^2 / 2, \ (i = u, v),$$

(2)
\[
\phi_{uv} + fV(\phi) = 0, \\
(\log f)_{uv} + fV'(\phi) = 0, \\
\psi_{uv} = 0.
\] (3)
(4)
(5)

These equations are not independent. Thus, eqs. (2)-(4) imply eq. (5). A useful corollary is the following: we only need to solve eqs. (3) and (4), then the scalar field can be derived from eqs. (2) while eq. (5) is automatically satisfied.

If the scalar field is constant, we can find the general solution to eqs. (2)-(4) in terms of one free field \[1\]. Indeed, from the constraints (2) it follows that

\[
f/\phi_u = b'(v), \quad f/\phi_v = a'(u),
\]

where \(a(u)\) and \(b(v)\) are arbitrary (smooth) functions. Thus

\[
\phi = F(\tau), \quad f = F'(\tau) a'(u) b'(v), \quad \tau \equiv a(u) + b(v),
\]

where \(F(\tau)\) is an arbitrary function of the free field \(\tau(u,v)\). This means that the general dilaton gravity (coupled to Abelian gauge fields) is a topological theory and can be dimensionally reduced to a 1 + 0 dimensional theory \[1\], \[4\], \[5\], \[6\].

To obtain the exact form of the function \(F(\tau)\), we define a new function \(N(\phi)\) by

\[
N' = V(\phi).
\]

The function \(M = F'(\tau) + N(\phi) = \phi_u \phi_v / f(u,v) + N(\phi)\), is locally conserved, i.e. \(M_u = 0\) and \(M_v = 0\). Thus, we find

\[
\tau = \int (M - N(\phi))^{-1} d\phi,
\]

and this implicitly defines the general solution in terms of the free field \(\tau = a(u) + b(v)\). If \(N(\phi_0) = M\), \(f(a,b) = F'(\tau) = M - N(\phi_0) = 0\), the function \(\tau(\phi)\) is infinite at \(\phi = \phi_0\), and there is a horizon at \(\tau = \infty\).

If the functions \(a(u)\) and \(b(v)\) are monotonic, the transformation from variables \((u,v)\) to variables \((a,b)\) can be used. The solution is static in the coordinates \((a,b)\) (it depends on \(\tau = a + b\) only).

The general DGM \[1\] is not integrable. However, there exists a class of the potentials \(V(\phi)\) for which the general solution to eqs. (2)-(4) can be found in terms of two free fields, or equivalently, in terms of two pairs of chiral functions. In what follows they are denoted by \(a(u), b(v), A(u),\) and \(B(v)\).

### 3 Canonical transformations to free fields

To study quantum properties of the models, it is useful to find a canonical transformation to free fields. With this aim, we rewrite the model in the canonical form. Let us write the two-dimensional metric in the form

\[
ds^2 = e^{2\rho} \left[-\alpha^2 dt^2 + (\beta dt + dr)^2\right],
\]

where \(\alpha(u,v)\) and \(\beta(u,v)\) play the role of the lapse function and of the shift vector, respectively; \(\rho = (\log f)/2\). We denote the derivative with respect to \(t = u + v\) by the dot, and the derivative with respect to \(r = u - v\) by the prime,

\[
\partial_t = (\partial_u + \partial_v)/2, \quad \partial_r = (\partial_u - \partial_v)/2.
\]
When, using (8), the action (1) can be written in the Hamiltonian form (see e.g. [7])

\[ S = \int d^2x \left( \pi_\rho \dot{\rho} + \pi_\phi \dot{\phi} + \pi_\psi \dot{\psi} - \alpha H - \beta P \right), \] (10)

where \( \pi_\rho = -2\phi' \), \( \pi_\phi = -2\rho' \), and \( \pi_\psi = \dot{\psi} \) are the conjugate momenta to \( \rho, \phi, \) and \( \psi \); \( \alpha \) and \( \beta \) are the Lagrangian multipliers; \( H \) and \( P \) are the constraints:

\[ H = -\pi_\rho \pi_\phi / 2 + 2(\phi'' - \phi' \rho') - e^{2\phi} V(\phi) + (\pi_\phi^2 + \psi^2) / 2, \]

\[ P = \rho' \pi_\rho - \pi_\rho' + \phi' \pi_\phi + \pi_\psi \psi'. \]

To get a canonical transformation to free fields let us rewrite the symplectic 2-form

\[ \omega = \int dr (\delta \pi_\rho \wedge \delta \rho + \delta \pi_\phi \wedge \delta \phi + \delta \pi_\psi \wedge \delta \psi) \] (11)

in the new parametrization given by the exact solution of the Lagrange equations. In the exact solution, written in terms of free fields, we will substitute the free chiral functions by arbitrary functions and the derivatives with respect to \( u, v \) by

\[ a(u) \to a(u, v), \quad b(v) \to b(u, v), \]

\[ A(u) \to A(u, v), \quad B(v) \to B(u, v), \]

\[ \partial_u \to 2\partial_r, \quad \partial_v \to -2\partial_r, \]

since \( 2\partial_r = \partial_u - \partial_v \) (This recipe was proposed in [3]).

For the CGHS model [2], \( V = g = \text{const} \), the global solution can be rewritten as

\[ \rho = \log(4a'b')/2, \quad \phi = gab + A + B, \]

\[ \pi_\rho = -2g(a'b - ab') - 2A' + 2B', \quad \pi_\phi = -\log(a')' + \log(b')'. \] (12)

Thus, we get

\[ \omega = 2 \int dr [-\delta(\log a') \wedge \delta A' + \delta(\log b') \wedge \delta B'] + \omega_{b_0} = 2 \int dr [\delta(\log(A'/a')) \wedge \delta A' - \delta(\log(B'/b')) \wedge \delta B'] + \omega_{b_0}, \] (13)

Here \( \omega_{b_0} \) is the boundary term, which arises from integration by parts,

\[ \omega_{b_0} = \int d[-\delta\phi \wedge \delta \log(a'/b') - 2g\delta a \wedge \delta b], \] (14)

and \( \phi \) is defined in (12). To rewrite the constraints \( C^\pm = \pm(H \pm P)/2 \) in the simple form [4, 5],

\[ C^\pm = \pm X^\pm \Pi^\pm + (\pi_\psi \pm \psi')^2 / 4, \] (15)

we can choose the canonical coordinate \( X^+ \) and momentum \( \Pi^+ \) in one of the following four forms \( (i = 1, \ldots, 4) \):

\[ X^+_i = a, \quad (A'/a'), \quad A, \quad \log(A'/a'), \]

\[ \Pi^+_i = 2(A'/a')', \quad 2a', \quad 2(\log(A'/a'))', \quad 2A'. \] (16)
The canonical coordinate $X^-$ and momentum $\Pi^-$ can be chosen independently of $X^+, \Pi^+$. Altogether this gives $4 \times 4 = 16$ different canonical transformations to the chiral coordinates and momenta

$$\omega_{ij} = \int dr \left( \delta X^+_i \wedge \delta \Pi^+_j + \delta X^-_j \wedge \delta \Pi^-_i + \delta \psi \wedge \delta \pi \psi \right) + \omega_{b_0} + \omega_{b_1} + \omega_{b_2} - \omega_{b_3} ,$$

where $\omega_{b_0}$ is given by (14), $i,j = 1,\ldots,4$

$$\omega^+_{b_1} = -2 \int d[\delta a \wedge \delta (A'/a')] , \quad \omega^+_{b_2} = 0 , \quad \omega^+_{b_3} = -2 \int d[\delta A \wedge \delta \log(A'/a')] , \quad \omega^+_{b_4} = 0 ,$$

and $\omega^-_{b_i}$ are obtained from $\omega^+_{b_i}$ by substituting $a$ with $b$ and $A$ with $B$.

In what follows we choose $i = j = 1, X^+ = a, X^- = b$. This choice of canonical coordinates seems natural because $a$ and $b$ are the global coordinate functions on the maximally extended solution. Thus, the canonical transformation is global. Below we will write the global canonical transformations for other DG models.

Since the functions $X^\pm$ are monotonic coordinate functions, we can include $X^\pm'$ into the Lagrange multipliers [8], [9], and the new constraints will be

$$\bar{C}^\pm \equiv C^\pm/X^\pm = \pm \Pi^\pm + (\pi \psi \pm \psi')^2/(4X^\pm) .$$

We can simplify the form of the boundary term by choosing a special asymptotic behavior of the coordinate functions $a$ and $b$. If we choose, up to arbitrary linear transformations, the static form of the solution (with a constant scalar field) at spatial infinities, we will completely get rid of the boundary term. Moreover, in this way we can get rid of the boundary term for any DG model with constant scalar fields. Indeed, for any model of this kind it is easy to prove the following statement. For all the solutions in the static parametrization $(a, b)$ the boundary term is

$$\omega^+_{b} = -\int d[\delta a \wedge \delta \log(a'/b')] .$$

4 Conclusions

Starting from global solutions of exactly integrable models we have constructed global canonical transformations to free chiral fields. Monotonic chiral coordinate functions of the global solution (in the conformal metric) can be chosen as new canonical coordinates. This choice of canonical coordinates guarantees the global character of the canonical transformation. We can rewrite the constraints as linear functions of momenta canonically conjugate to coordinate functions. To get rid of the boundary term in the canonical transformation we demand the static form of the solution (with a constant scalar field) at spatial infinities.
Let us compare the constructions of the canonical transformations to free chiral fields for different DG models. The CGHS model is the simplest integrable model. It has only one static solution. The coordinate functions of the global solution were chosen as canonical coordinates. We get rid of the boundary term of the canonical transformation, if the global coordinate functions are chosen static at spatial infinities, and the scalar field is constant.

The generalized CGHS model \((V = g_1 + g_2\dot{\phi})\) has several static solutions. The global static solution can be chosen as a vacuum and may be used in the boundary conditions to get rid of the boundary term. To study the solutions obtained from the periodic static solutions with horizons, we have to restrict the manifold to a cylinder. Then we may completely get rid of the boundary term by choosing periodic boundary conditions for free chiral fields.

For the bi-Liouville model \((V = (g_+ e^{g\phi} - g_- e^{-g\phi})/g)\) minimally coupled to one scalar field we can construct canonical transformations to free chiral fields and choose the boundary conditions similarly to the CGHS (for \(\text{sign}\{g_+g_-\} = -1\)) or the generalized CGHS (for \(\text{sign}\{g_+g_-\} = 1\)) and get rid of the boundary term.

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References

[1] A.T. Filippov, Mod. Phys. Lett. A11 (1996) 1691, Int.J.Mod.Phys. A12 (1997) 13-22.

[2] C.G. Callan, S.B. Giddings, J.A. Harvey, A. Strominger, Phys. Rev. D45 (1992) 1005.

[3] J. Cruz, J. M. Izquierdo, D. J. Navarro, and J. Navarro-Salas, Phys. Rev. D58 (1998) 044010; J. Cruz and J. Navarro-Salas, Mod.Phys.Lett. A12 (1997) 2345-2352; J. Cruz, D. J. Navarro, and J. Navarro-Salas, hep-th/9712194.

[4] T. Banks and M. O'Loughlin, Nucl. Phys. B362 (1991) 649.

[5] M. Cavagliá, V. de Alfaro and A.T. Filippov, Int. J. Mod. Phys. D4 (1995) 661; Int.J.Mod.Phys. D5 (1996) 227-250; see also Int. J. Mod. Phys. A10 (1995) 611.

[6] T. Klösch and T. Strobl, Class.Quant.Grav. 13 (1996) 965-984, 2395-2422; Class.Quant.Grav. 14 (1997) 1689.

[7] D. Cangemi, R. Jackiw and B. Zwiebach, Annals Phys. 245 (1996) 408-444.

[8] K.V. Kuchar, J.D. Romano and M. Varadarajan, Phys. Rev. D55 (1997) 795.

[9] M. Cavagliá, V. de Alfaro and A.T. Filippov, Phys. Lett. B424 (1998) 265-270.