Abstract

Many definitions of chaos have appeared in the last decades and with them the question if they are equivalent in some more specific spaces. Our focus will be distributional chaos, first defined in 1994 and later subdivided into three major types (and even more subtypes). These versions of chaos are equivalent on a closed interval, but distinct in more complicated spaces. Since dendrites have much in common with the interval, we explore whether or not we can distinguish these kinds of chaos already on dendrites. In the end of the paper we will also briefly look at the correlation with other types of chaos.

Keywords: Chaos, Dendrites, Distributional Chaos, Horseshoes.

1 Introduction

When we look in the literature for some examples showing that (in general) various types of Distributional Chaos (DC) are not equivalent ([13],[2]), we find systems with more complicated structure, spaces with subsets homeomorphic to the circle or at least some triangular (skew product) maps on unconnected spaces. On the other hand, we know that all types of DC are equivalent on the interval, trees and graphs ([10], [7], [5], [8]). Since dendrites, trees and intervals are so much alike, can it be that all types of DC are equivalent on dendrites? We will answer this question in section 3.

Also if we look into the literature we can find different relationships between the original definition of distributional chaos (which required the existence of DC-pairs) and other types of chaos. How will the situation on dendrites change, if we will require an uncountable DC-scrambled set? In the end of section 3 you will find the relation between DC-pairs and uncountable DC-scrambled sets, and in section 4 the relation with other types of chaos.

2 Terminology

We will use the following notation through the whole paper, if not indicated otherwise. Let $(X, d)$ be a non-empty compact metric space. A pair $(X, f)$, where $f$ is a continuous
self-map acting on $X$, is called a (topological) dynamical system. The orbit of a point $x \in X$ is the set $\{f^n(x) : n \geq 0\}$.

A pair of two different points $(x, y) \in X^2$ is scrambled or Li-Yorke if

$$\liminf_{k \to \infty} d(f^k(x), f^k(y)) = 0$$  \hspace{1cm} (1)

and

$$\limsup_{k \to \infty} d(f^k(x), f^k(y)) > 0.$$  \hspace{1cm} (2)

A subset $S \subset X$ is LY-scrambled if it contains at least 2 distinct points and every pair of distinct points in $S$ is scrambled. According to the size of $S$ we say that $f$ is LY$_2$, if $S$ contains a scrambled pair, LY$_\infty$, if $S$ is an infinite LY-scrambled set, or LY$_u$, if $S$ is an uncountable LY-scrambled set. The system $(X, f)$ is usually called Li-Yorke chaotic if there exists an uncountable LY-scrambled set.

As we already mentioned, at the beginning, there was a definition of one kind of DC (see [12]), this type is called DC1 in these days, later ([11]) that type was divided into 3 different types DC1, DC2 and DC3, different in general, but the same in the interval. It also turned out, that DC3 can be a really weak and unstable type of chaos, so in [2] appeared a new kind of DC, namely DC2$\frac{1}{2}$ which as was shown, fixed those problems, but in general it is essentially weaker than DC2. (There is also DC1$\frac{1}{2}$ see [3], but we will not discuss this kind in this paper.)

For a pair $(x, y)$ of points in $X$, define the lower distribution function generated by $f$ as

$$\Phi_{(f,x,y)}(\delta) = \liminf_{n \to \infty} \frac{1}{n} \#\{0 \leq k \leq n; d(f^k(x), f^k(y)) < \delta\},$$  \hspace{1cm} (3)

and the upper distribution function as

$$\Phi^*_{(f,x,y)}(\delta) = \limsup_{n \to \infty} \frac{1}{n} \#\{0 \leq k \leq n; d(f^k(x), f^k(y)) < \delta\},$$  \hspace{1cm} (4)

where $\#A$ denotes the cardinality of the set $A$.

A pair $(x, y) \in X^2$ is called distributionally scrambled of type 1 (or a DC1 pair) if

$$\Phi^*_{(f,x,y)}(\delta) = 1, \text{ for every } 0 < \delta \leq \text{diam } X$$

and

$$\Phi_{(f,x,y)}(\epsilon) = 0, \text{ for some } 0 < \epsilon \leq \text{diam } X,$$

distributionally scrambled of type 2 (or a DC2 pair) if

$$\Phi^*_{(f,x,y)}(\delta) = 1, \text{ for every } 0 < \delta \leq \text{diam } X$$

and

$$\Phi_{(f,x,y)}(\epsilon) < 1, \text{ for some } 0 < \epsilon \leq \text{diam } X,$$

distributionally scrambled of type 3 (or a DC3 pair) if

$$\Phi_{(f,x,y)}(\delta) < \Phi^*_{(f,x,y)}(\delta), \text{ for every } \delta \text{ in some interval } (a, b), \text{ where } 0 \leq a < b \leq \text{diam } X.$$

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A subset $S$ of $X$ is distributionally scrambled of type $i$ (or a DC$i$ set), where $i = 1, 2, 3$, if every pair of distinct points in $S$ is a DC$i$ pair. Originally, the dynamical system $(X, f)$ was called distributionally chaotic of type $i$ (a DC$i$ system), where $i = 1, 2, 3$, if there was a DC$i$ pair (DC$i_2$), later the focus was moved to an uncountable $i$-distributionally scrambled sets (DC$i_u$).

We can also define both distribution functions at 0 as limits:

$$
\Phi_{\delta}(f,x,y)(0) = \lim_{\delta \to 0^+} \Phi_{\delta}(f,x,y)(\delta)
$$

and

$$
\Phi^\ast_{\delta}(f,x,y)(0) = \lim_{\delta \to 0^+} \Phi^\ast_{\delta}(f,x,y)(\delta).
$$

Then $(x,y) \in X^2$ is called distributionally scrambled of type $2\frac{1}{2}$ (or a DC$2\frac{1}{2}$ pair) if

$$
\Phi_{\delta}(f,x,y)(0) < \Phi^\ast_{\delta}(f,x,y)(0).
$$

We define DC$2\frac{1}{2}$ sets and DC$2\frac{1}{2}$ systems in the same way as for the other 3 versions of distributional chaos.

**Note 1.** Notice that

$$
\Phi_{\delta}(f,x,y)(\delta) = 1 - \limsup_{n \to \infty} \frac{1}{n} \#\{0 \leq k \leq n; d(f^k(x), f^k(y)) \geq \delta\}.
$$

Suppose that there are disjoint compact sets $A, B \subset X$ such that

$$
f(A) \cap f(B) \supset A \cup B. \tag{5}
$$

Then we say that $f$ has a horseshoe or that $A$ and $B$ form a horseshoe for $f$. (Since several definitions of horseshoes have appeared over time, this general case is in other literature also known as a *strict general horseshoe*). If $X$ is a graph or dendrite and there are arcs $A, B \subset X$ which intersect at most in their endpoints, satisfying (5), then we say that $A, B$ form an arc horseshoe for $f$. Moreover if the sets $A, B$ are disjoint, we say that $A, B$ form a *strict arc horseshoe* for $f$. (For other types of horseshoes see [6] or [7].)

The set of limit points of the sequence $(f^n(x))_{n \in \mathbb{N}}$ is called the $\omega$-limit set of the point $x$ under $f$ and denoted by $\omega_f(x)$. A set $S \subset X$ is called $\omega$-scrambled for $f$ if it contains at least two points and for any 2 distinct points $x, y \in S$ the following conditions hold:

1) $\omega_f(x) \setminus \omega_f(y)$ is uncountable,
2) $\omega_f(x) \cap \omega_f(y)$ is nonempty,
3) $\omega_f(x)$ is not contained in the set of periodic points.

A *dendrite* is a locally connected continuum (compact connected metric space) containing no subset homeomorphic to the circle (no simple closed curve). A point of a dendrite is called a *branch point* if it is the endpoint of three arcs with disjoint interiors. A point of a dendrite is called an *end point* if for every one of its neighborhoods $U$ there exists a neighborhood $V \subset U$ with a one-point boundary.
3   Equivalence of different types of DC on dendrites

3.1   DC1 and DC2

(Un)fortunately, there are already several articles showing that we can associate the Gehman dendrite (the topologically unique dendrite whose set of end points is homeomorphic to the Cantor set) and a map \( g \) on it with a shift space and the shift map \( (\{0, 1\}^\mathbb{N}, \sigma) \). Moreover, we can also build a subdendrite of the Gehman dendrite associated with any subshift of the full 2-shift (see e.g. [6], [7], [4]). That means we can use not just general results but also results known from shift spaces.

Lemma 1 ([4] Lem 4.1). If \( X \subset \{0, 1\}^\mathbb{N} \) is a subshift then there is a subdendrite \( G_X \) of the Gehman dendrite \( G \) invariant under \( g \). Let \( E_X \) be the set of end points of \( G_X \), then \( (E_X, g|_{E_X}) \) is topologically conjugate to \( (X, \sigma) \).

Note 2. Notice, that all \( x \in G_X \setminus E_X \) are eventually fixed and so all the interesting dynamics happen exclusively in \( E_X \) and so if we are looking for DC-pairs/sets we need only look in \( E_X \) (for more details see the proof of Lem. 4 in [6]).

Theorem 1 ([3] Th 1.1). Assume that a topological dynamical system \( (X, T) \) has positive topological entropy. Then the system possesses an uncountable DC2-scrambled set.

Theorem 2 ([11] Ex 4.1). There exists a subshift with positive topological entropy without DC1 pairs.

As a combination of the previous theorems and lemma we obtain the following result:

Theorem 3. DC2 does not imply DC1 on dendrites.

3.2   DC2 and DC3

While for DC1 and DC2 we could go directly to known results on subshifts of the full 2-shift, for DC3 and DC2 we will start with a less complicated structure. The dendrite \( D \) will be a comb-style dendrite and we will use a level-inductive approach for the construction of our dendrite.
In the following construction $I$ denotes the unit interval $\langle 0, 1 \rangle \times \{0\}$, $(s, t) = \{s\} \times \{t\}$ and if we will need a metric, $d$ will be the max metric in $\mathbb{R}^2$.

**Construction of the dendrite $\mathcal{D}$:**

$$\mathcal{D} = I \cup \left( \bigcup_{n \in \mathbb{N}} (Z^{(n)} \times \langle 0, h_n \rangle) \right)$$

where $Z^{(n)} = \left\{ z^{(n)}_j = \frac{j}{3^n} : j \in J_n \right\}$, $J_n = \{j : 1 \leq j \leq 3^n \land 3 \nmid j\}$, $h_n = \frac{1}{3^n}$.

The set $\left( \bigcup_{n \in \mathbb{N}} (Z^{(n)} \times \{h_n\}) \right) \cup \{(0, 0), (1, 0)\}$ is the set of endpoints of $\mathcal{D}$. Moreover let’s denote $l_n = \# \left( \bigcup_{i=1}^{n} Z^{(i)} \right) = 3^n - 1$.

We can easily imagine the dendrite construction by an inductive approach: we start with 1 horizontal line (the interval $I$). Then we add levels of “spikes:” in every next level we add 2 equally distributed spikes between every 2 “old” spikes. Spikes in the $n$-th level emanate from the points of $Z^{(n)}$ and have height $h_n = 1/3^n$. So the first level has 2 “spikes” at $\frac{1}{3}$ and $\frac{2}{3}$ with height $\frac{1}{3}$, in the second level we add 6 spikes of height $\frac{1}{9}$ at $\frac{1}{9}, \frac{4}{9}, \frac{7}{9}, \frac{2}{9}, \frac{5}{9}, \frac{8}{9}$ (all $i/9$ between 0 and 1 except for $i$-s which are multiples of 3, those are already in the previous level, so after finishing the second level we have 8 spikes total) and so on.

**Construction of the map $f : \mathcal{D} \to \mathcal{D}$:**

Let’s first construct helping maps. For $n \in \mathbb{N}$ and $j \in J_n$ denote by $\phi(n,j)$, $\psi(n,j)$ the increasing linear functions such that:

![Figure 2: Construction of the dendrite $\mathcal{D}$ - first 4 levels.](image)
\[
\psi_{(n,j)} \left( \left< \frac{2}{3} h_n \right> \right) = \begin{cases} 
\left< z_j^{(n)} , z_{j+1}^{(n)} \right> , & \text{if } j = 3a + 1 \\
\left< z_j^{(n)} , z_{j+2}^{(n)} \right> , & \text{if } j = 3a + 2 \neq l_n \\
\left< z_l^{(n)} , z_{l+1}^{(n+1)} \right> , & \text{if } j = l_n = 3^n - 1 
\end{cases}
\]

for odd \( n \)

\[
\phi_{(n,j)} \left( \left< \frac{2}{3} h_n , h_n \right> \right) = \begin{cases} 
\left< 0, h_n \right> , & \text{if } j \neq 3^n - 1 \\
\left< 0, h_n+1 \right> , & \text{if } j = 3^n - 1 
\end{cases}
\]

for odd \( n \)

\[
\phi_{(n,j)} \left( \left< \frac{2}{3} h_n , h_n \right> \right) = \begin{cases} 
\left< 0, h_n \right> , & \text{if } j \neq 1 \\
\left< 0, h_n+1 \right> , & \text{if } j = 1 
\end{cases}
\]

for even \( n \)

Now we use the functions \( \psi_{(n,j)} \) and \( \phi_{(n,j)} \) to construct \( f \) in three parts:

1. \( f|_I = id. \)

2. for \((x, y) \in \left( \{ z_j^{(n)} \} \times \left< \frac{2}{3} h_n \right> \right) : \)
\[
f(x, y) = f \left( z_j^{(n)} , y \right) = (\psi_{(n,j)}(y), 0) \]

3. for \((x, y) \in \left( \{ z_j^{(n)} \} \times \left< \frac{2}{3} h_n , h_n \right> \right) : \)
\[
f(x, y) = f \left( z_j^{(n)} , y \right) = (\psi_{(n,j)}(\frac{2}{3} h_n), \phi_{(n,j)}(y)) \]
Lemma 2.  

a) The set \( \mathcal{D} \) is a dendrite.

b) The map \( f \) is continuous.

c) If \( x \in \mathcal{D} \setminus \left( \bigcup_{n \in \mathbb{N}} \left( Z^{(n)} \times \{ h_n \} \right) \right) \) then \( x \) is an eventually fixed point.

d) If \( x,y \in \bigcup_{n \in \mathbb{N}} \left( Z^{(n)} \times \{ h_n \} \right) \), then \( \lim_{n \to \infty} d(f^n(x), f^n(y)) = 0 \).

Proof. Proofs of a), b) and d) are very similar as in [4] section 7, so we will not repeat them. The proof of c) is obvious since for every such \( x \) it is not so hard to find \( m \) such that \( f^m(x) \in I \).

Lemma 3. The system \( (\mathcal{D}, f) \) has a DC3 pair.

Proof. We will prove that the pair \((x, y), y = \left( \frac{1}{3}, \frac{1}{3} \right), x = (1, 0)\) is a DC3 pair. For showing that it is a DC3 pair we need to find an interval such that \( \Phi_{(f,x,y)}(\delta) < \Phi_{(f,x,y)}^*(\delta) \),
for every $\delta$ in that interval. Since distribution functions are nondecreasing, it is enough to find 2 values $a, b$ such that $a < b$ and $\Phi_{(f,x,y)}(b) < \Phi^*_{(f,x,y)}(a)$.

Let’s consider $b = 1/2$.

Since the metric is maximal, the height will be irrelevant, it is enough to count the total number of spikes and the number of spikes on the right side of $D$ at some concrete times - for half way through the levels. For $\Phi^*_{(f,x,y)}$ we will need an odd level + half of the next even level (because in even levels points move to the left), whereas for $\Phi_{(f,x,y)}$ the counting will go from the “left” and we will “stop” just before entering the $b$—neighborhood of $x$.

$$\Phi^*_{(f,x,y)}(0.5) = \lim_{k \to \infty} \left( \frac{\frac{1}{2}l_{2k+2}}{l_{2k+2} + l_{2k+1}} \right) = \lim_{k \to \infty} \left( \frac{\frac{1}{2}(3^{(2k+2)} - 1)}{(3^{2k+2} - 1) + 3^{3^{2k+1} - 1}} \right) = \frac{3}{4}$$

$$\Phi_{(f,x,y)}(0.5) = \lim_{k \to \infty} \left( \frac{\frac{1}{2}l_{2k}}{l_{2k+1} + l_{2k}} \right) = \frac{1}{4}.$$

Let $a = 1/4$.

$$\Phi^*_{(f,x,y)}(0.25) = \lim_{k \to \infty} \left( \frac{\frac{1}{4}l_{2k+2}}{\frac{1}{4}l_{2k+2} + \frac{1}{4}l_{2k+1}} \right) = \lim_{k \to \infty} \left( \frac{(3^{(2k+2)} - 1)}{(3^{2k+2} - 1) + 3^{(3^{2k+1} - 1)}} \right) = \frac{1}{2}$$

$$\left[ \Phi_{(f,x,y)}(0.5) < \Phi^*_{(f,x,y)}(0.25) \right] \Rightarrow \Phi_{(f,x,y)}(\delta) < \Phi^*_{(f,x,y)}(\delta) \text{ for every } \delta \in (0.25; 0.5) \text{ and so the pair } (x, y) \text{ is DC3.} \qed$$

**Lemma 4.** The system $(\mathcal{D}, f)$ has no DC2-pairs.

**Proof.** By Lemma 2 it is enough to check pairs with 1 fixed point and 1 point of type \(\left( \frac{(n)}{3}, \frac{1}{3^n} \right)\) (endpoints), moreover each such point has a preimage in \(\left( \frac{1}{3}, \frac{1}{3} \right)\). So we need to prove that for every $x \in I$ and $y = \left( \frac{1}{3}, \frac{1}{3} \right)$ there is a $\delta \in (0, 1)$ such that $\Phi^*_{(f,x,y)}(\delta) \neq 1$.

It is also clear that for $\delta \geq 1/2$ there are points $x$ for which $\Phi_{(f,x,y)}(\delta) = \Phi^*_{(f,x,y)}(\delta) = 1$, so we will search $\delta \in (0, 1/2)$.

![Figure 5: Illustration of a $\delta$-neighborhood for $\Phi^*_{(f,x,y)}(\delta)$.](image)

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For $\delta \in (0, \frac{1}{2})$, $y = (\frac{1}{3}, \frac{1}{3})$ and $x = (x_1, 0)$, where $x_1 \in (0, 1)$, there is an exact formula for $\Phi^*(f,x,y)(\delta)$:

\[
\Phi^*(f,x,y)(\delta) = \lim_{k \to \infty} \frac{a(x,\delta) \cdot l_{2k+1} + c(x,\delta,k)}{b(x,\delta) \cdot l_{2k+1} + \left[1 - b(x,\delta)\right] \cdot l_{2k}}
\]

where

\[
a(x,\delta) = |\min\{(x_1 + \delta), 1\} - \max\{(x_1 - \delta), 0\}|,
\]

\[-3^ma(x,\delta) - 2 \leq c(x,\delta,k) \leq 2 \text{ where } m = \frac{\log \delta^{-1}}{\log 3},\]

and

\[
b(x,\delta) = \left\{ \begin{array}{ll} x_1 + \delta, & \text{if } x_1 \in (0, \frac{1}{2}), \\ 1 - x_1 + \delta, & \text{if } x_1 \in (\frac{1}{2}, 1). \end{array} \right.
\]

Moreover, if we fix $\delta$, the function $u(x_1) = \Phi^*(f,(x_1,0),\left(\frac{1}{3}, \frac{1}{3}\right))(\delta)$ has 2 local maxima at $x_{11} = \delta$, $x_{12} = 1 - \delta$. Now, let $\delta = \frac{1}{4}$. Then $\Phi^*(f,(x,y))(0.25) \leq \frac{3}{4} < 1$ for every $(x,y)$ defined as above and so there are no DC2-pairs in $D$. \hfill \Box

### 3.3 DC3 and DC2$^{\frac{1}{2}}$

Since the new type of DC (DC2$^{\frac{1}{2}}$) defined in [2] was supposed to fix some problems of DC3, (e.g. DC2$^{\frac{1}{2}} \Rightarrow$ LY) and the above example has also LY-pairs, there was a hope that DC3 might imply at least DC2$^{\frac{1}{2}}$. Unfortunately, as the next lemma shows, even this weaker hope is not fulfilled on dendrites.

**Lemma 5.** The system $(D, f)$ has no DC2$^{\frac{1}{2}}$-pairs.

**Proof.** The system $(D, f)$ has no DC2$^{\frac{1}{2}}$-pairs if $\Phi^*(f,x,y)(0) = \Phi(f,x,y)(0)$ for every pair $x, y \in D$. As in Lemma 4, it is enough to check pairs such that $x \in I$ and $y = (\frac{1}{3}, \frac{1}{3})$. Since both distribution functions are nonnegative and nondecreasing, it is sufficient to show that for such a pair $(x, y)$, $\lim_{\delta \to 0^+} \Phi^*(f,x,y)(\delta) = 0$.

Let $x \in I$ and $y = (\frac{1}{3}, \frac{1}{3})$. $\Phi^*(f,x,y)$ is defined by (6) in Lemma 4 then

\[
\lim_{\delta \to 0^+} \Phi^*(f,x,y)(\delta) = \lim_{\delta \to 0^+} \frac{3a(x,\delta)}{1 + 2b(x,\delta)} = \left\{ \begin{array}{ll} \lim_{\delta \to 0^+} \frac{3(x_1 + \delta)}{1 + 2(x_1 + \delta)} = \lim_{\delta \to 0^+} \frac{3\delta}{1 + 2\delta}, & \text{if } x_1 = 0 \\ \lim_{\delta \to 0^+} \frac{6\delta}{1 + 2b(x,\delta)}, & \text{if } x_1 \in (0, 1) \end{array} \right\} = 0.
\]

Then by the previous discussion and lemmas, $\Phi^*(f,x,y)(0) = \Phi(f,x,y)(0)$ for every $x, y \in D$ and so there is no DC2$^{\frac{1}{2}}$ pair. \hfill \Box
By combining the results of Lemmas 3, 4 and 5 we obtain the following theorem:

**Theorem 4.** DC3 implies neither DC2 nor DC2\(\frac{1}{2}\) on dendrites.

### 3.4 DC-pairs and uncountable DC-sets

It is easy to see that the example in the above subsection also shows that existence of a DC3-pair implies neither existence of an uncountable DC3-set nor an infinite DC3-set (as a simple corollary of Lemma 2 we get that the system does not have any DC3-triples). However the whole construction above was inspired by the example constructed in [4] proving the next theorem and as a simple corollary of that theorem, and the fact that every DC1-pair is also a LY-pair, we obtain a stronger result:

**Theorem 5** ([4] Th 7.6). There exists a continuous self-map \(f\) of a dendrite such that: \(f\) has a DC1-pair but does not have any infinite LY-scrambled set.

**Corollary 6.** Existence of a DC\(i\) pair does not imply existence of an infinite DC\(i\)-set for any known \(i\)-type of distributional chaos \((i \in \{1, 1.5, 2, 2.5, 3\})\).

**Note 3.** Notice, that the above construction of the dendrite \(\mathcal{D}\) can be changed to have DC2 or DC1-pairs (one example for the DC1 case can be seen in [4]). All that we need to change is how many spikes we will add in each level between each 2 “old” spikes. If we use any bounded number of “new spikes” between every 2 “old spikes”, we will get DC3, if the number of added spikes between every 2 “old spikes” will grow without bound we will get at least DC2. But for this construction, there is a clear jump between DC3 and DC2, so that leaves us with the question whether it can be that DC2\(\frac{1}{2}\) implies DC2.

### 3.5 Uncountable DC3-set

In subsection 3.1 we just simply used known results on the 2-shift. One would think we can do the same for the case DC3 and DC2. Unfortunately even though many tried to find such a 2-shift, to our knowledge there is no such result. We tried too, but even though we were able to construct an infinite DC3-set, it was not bigger than countable. Fortunately, just when we were ready to give up, we found out that there is an example of a subshift of the full 5-shift in [14], which has the required properties. And so we just need to construct a Gehman-like dendrite for the full 5-shift and its subshifts. We will denote this dendrite \(G_5\) and the corresponding map \(g_5\).

We will not prove Lemma 3, since the proof would be just a small modification of previous proofs for the classic Gehman dendrite. But let us briefly recall how the map \(g_5\) works:

We denote the branching points as follows: \(c\) is the top point, then from left to right in the next level: \(c_0, c_1, c_2, c_3, c_4\), in the next level down: \(c_{00}, c_{01}, c_{02}, c_{03}, c_{04}, c_{05}, \ldots, c_{44}\), and so on. Similarly for the branches we start with \(B_0 = [c, c_0], \ldots, B_4 = [c, c_4]\), and for every \(n \in \mathbb{N}\) the branch \(B_{i_1 \ldots i_{n+1}} = [c_{i_1 \ldots i_n}, c_{i_1 \ldots i_{n+1}}]\), where each \(i_j \in \{0, 1, 2, 3, 4\}\). The map \(g_5\) is defined such that: \(c\) is a fixed point, \(B_i \rightarrow c\) and every \(B_{i_1 \ldots i_n} \rightarrow B_{i_2 \ldots i_n}\), in such a way that \(c_{i_1 \ldots i_n} \rightarrow c_{i_2 \ldots i_n}\), and \(c_i \rightarrow c\), where all \(i\)’s are from the set \(\{0, 1, 2, 3, 4\}\). The map on the limit set is the full 5-shift.
Lemma 6. If \( Y \subset \{0, 1, 2, 3, 4\}^\mathbb{N}_0 \) is a subshift then there is a subdendrite \( \mathcal{G}_{5Y} \) of the dendrite \( \mathcal{G}_5 \) invariant under \( g_5 \). Let \( \mathcal{E}_{5Y} \) be the set of end points of \( \mathcal{G}_{5Y} \), then \( (\mathcal{E}_{5Y}, g|_{\mathcal{E}_{5Y}}) \) is topologically conjugate to \( (Y, \sigma) \) and all DC-pairs of \( \mathcal{G}_{5Y} \) are contained in \( \mathcal{E}_{5Y} \).

Theorem 7 ([14] sec. 5). There exists a system \((Y, \sigma)\), such that \( Y \subset \{0, 1, 2, 3, 4\}^\mathbb{N}_0 \) and has an uncountable DC3-set, but has no DC2-pair.

As a combination of the previous theorems and lemma we obtain the following result:

Theorem 8. DC3 does not imply DC2 on dendrites in the sense of uncountable sets or pairs.

4 The strongest type of DC and other types of chaos

In [6] and other literature are shown different relationships between DC1 and other types of chaos. How will the situation change, if we want that some kind of chaos will imply DC1_u (the strongest possible DC) instead of just pairs? The most important example is the horseshoe, because in [6] it is the only studied property which implies DC1, but it was shown just for DC1_u.

Lemma 7. The dynamical system \( \left( \{0, 1\}^\mathbb{N}_0, \sigma \right) \), where \( \sigma \) is the shift map has an uncountable DC1 set.

Proof. Definition of the metric: To discuss DC chaos we have to specify a metric. Let \( d \) be the metric on \( \{0, 1\}^\mathbb{N}_0 \) given by \( d(x, y) = 2^{-i} \), where \( i = \inf \{j : x_j \neq y_j\} \) and \( x \in \{0, 1\}^\mathbb{N}_0 \) represents an infinite sequence \( x = x_0x_1x_2x_3 \cdots \) of 0s and 1s.

Construction of the uncountable scrambled set (\( \Lambda \)):
We can either use a tail equivalence relation which involves the axiom of choice or we may follow a more constructive approach and take \( \Lambda = \nu(\lambda(\{0, 1\}^\mathbb{N}_0)) \), where \( \lambda : \{0, 1\}^\mathbb{N}_0 \to \{0, 1\}^\mathbb{N}_0 \) provides tail inequivalence \( \lambda(x_0x_1x_2\cdots) = x_0x_0x_1x_0x_1x_2\cdots \) so if \( x \neq y \) then \( \lambda(x)_i \neq \lambda(y)_i \), infinitely many times \( [\lambda(x) \not\sim \lambda(y)] \) and then \( \nu : \{0, 1\}^\mathbb{N}_0 \to \)
\(\{0, 1\}^{N_0}\), provides blocks (of \(x_i\)’s and 0’s) of sufficient length: 
\[
\nu(x_0 x_1 x_2 x_3 x_4 x_5 \cdots) = 0^{a_1} x_0^{a_2} 0^{a_3} x_1^{a_4} x_2^{a_5} 0^{a_6} x_3^{a_7} x_4^{a_8} x_5^{a_9} 0^{a_{10}} \cdots,
\]
(where \(0^a\) denotes \(0\) \(a\) times). Then by composition we will get:
\[
\nu(\lambda(x_0 x_1 x_2 x_3 \cdots)) = 0^{a_1} x_0^{a_2} 0^{a_3} x_1^{a_4} x_2^{a_5} 0^{a_6} x_3^{a_7} x_4^{a_8} x_5^{a_9} 0^{a_{10}} \cdots,
\]
(7)
where \(\{a_i\}_{i \in \mathbb{N}}\) is an increasing sequence (eg: \(i^2\)) such that
\[
\frac{b_n + n}{a_{n+1}} \to 0 \text{ for } n \to \infty, \text{ where } b_n = \sum_{i=1}^{n} a_i.
\]
(8)
Since both \(\lambda\) and \(\nu\) are injective maps, \(\Lambda = \nu(\lambda(\{0, 1\}^{N_0}))\) is an uncountable set, and we claim that \(\Lambda\) is also a DC1 set for the shift map \(\sigma\).
Moreover the construction of \(\Lambda\) motivates us to think about elements \(\hat{x} \in \Lambda\) in blocks, where the \(i\)-th block \((B_i = (\hat{x}_{b_i-1+1} \ldots \hat{x}_{b_i}))\) has length \(a_i\) and for any element \(\hat{x} \in \Lambda\) there exists exactly one \(x \in \{0, 1\}^{N_0}\) such that \(\hat{x} = \nu \circ \lambda(x)\) (\(\hat{x} \neq \hat{y}\) implies \(x \neq y\)).
So for any two \(\hat{x} \neq \hat{y} \in \Lambda\) there are not just infinitely many \(i \in \mathbb{N}\) such that \(\hat{x}_i \neq \hat{y}_i\) but infinitely many blocks \(B_i\) such that \(\hat{x}_i \neq \hat{y}_i\) for every \(i\) in those blocks. Then for the calculation of \(\Phi_{(\sigma, \hat{x}, \hat{y})}(\delta)\), the worst scenario is when we use \(x, y\) which are different just in one coordinate or are very close (if there are more \(j\)-s such that \(x_j \neq y_j\) or the first such \(j\) is smaller, then \(\Phi_{(\sigma, \hat{x}, \hat{y})}(\delta)\) will be smaller/converge faster). For \(\Phi^*_{(\sigma, \hat{x}, \hat{y})}(\delta)\), the worst scenario is when we use \(x, y\) which are different in every coordinate.
Calculation of \(\Phi_{(\sigma, \hat{x}, \hat{y})}(\delta)\) for \(\delta \in (0, 1)\) (Instead of the definition we will use Note 1):
For any two \(x \neq y \in \{0, 1\}^{N_0}\) there exists at least one \(j\) such that \(x_j \neq y_j\) (it does not matter which such \(j\) will we use in the sequel, but for simplicity, let’s take the smallest and denote it \(j_0\)). Then for \(\hat{x}, \hat{y}\) there exists a subsequence \(\{B_{i(j_0)}\}_{i \in \mathbb{N}} \subset \{B_i\}_{i \in \mathbb{N}}\) of blocks with sequence of lengths \(\{a_{i(j)}\}_{i \in \mathbb{N}}\) such that \(\hat{x}_j \neq \hat{y}_j\) for every \(\hat{x}_j, \hat{y}_j \in B_{i(j_0)}\) and so
d\((\sigma^j(\hat{x}), \sigma^j(\hat{y})) = 1\) for those \(j\)-s.
By (8) \(a_i/b_i \to 1\) as \(i \to \infty\) and so
\[
\lim_{i(j_0) \to \infty} \frac{1}{b_{i(j_0)}} \#\{b_{i(j_0)}-1 \leq k \leq b_{i(j_0)}; d(\sigma^k(\hat{x}), \sigma^k(\hat{y})) \geq \delta\} = \lim_{i(j_0) \to \infty} \frac{a_{i(j_0)}}{b_{i(j_0)}} = 1.
\]
This shows that
\[
\Phi_{(\sigma, \hat{x}, \hat{y})}(\delta) \leq 1 - \lim_{i(j_0) \to \infty} \frac{1}{b_{i(j_0)}} \#\{b_{i(j_0)}-1 \leq k \leq b_{i(j_0)}; d(\sigma^k(\hat{x}), \sigma^k(\hat{y})) \geq \delta\} = 0.
\]
Calculation of \(\Phi^*_{(\sigma, \hat{x}, \hat{y})}(\delta)\) for \(\delta \in (0, 1)\) (We get to use the blocks of 0’s.)
For every \(\hat{x} \in \Lambda\) there is a subsequence \(\{B_i\}_{i \in \mathbb{N}} \subset \{B_i\}_{i \in \mathbb{N}}\) of blocks such that \(\hat{x}_j = 0\) for every \(\hat{x}_j \in B_i\), where \(i = \sum_{j=1}^{l} j\).
Since \(d(\hat{x}, \hat{y}) \in \{2^{-m}\}_{m \in \mathbb{N}}\), the function \(\Phi^*_{(\sigma, \hat{x}, \hat{y})}\) is constant on intervals \((2^{-m}, 2^{-m+1})\), where \(m \in \mathbb{N}\), so it is enough to check \(\Phi^*_{(\sigma, \hat{x}, \hat{y})}(\delta)\) for \(\delta \in 2^{-m}\). But for every \(m \in \mathbb{N}\) there exists an \(N \in \mathbb{N}\) such that for every \(l \geq N\), \(a_i > m\) and since \(\lim_{i \to \infty} \frac{a_{i+1}}{b_i + 1} = 1\),
\[
\Phi^*_{(\sigma, \hat{x}, \hat{y})}(\delta) \geq \lim_{l \to \infty} \frac{1}{b_i} \#\{b_{i(l)}-1 \leq k \leq b_{i(l)} - m; d(\sigma^k(\hat{x}), \sigma^k(\hat{y})) < \delta\} = 1
\]
That shows \(\Phi_{(\sigma, \hat{x}, \hat{y})}(\delta) = 0\) and \(\Phi^*_{(\sigma, \hat{x}, \hat{y})}(\delta) = 1\) for any \(\hat{x}, \hat{y} \in \Lambda\) and \(\delta \in (0, 1)\).
As a corollary of Lemma 7 we can strengthen the statement of Lemma 2 from [6].

**Theorem 9.** Let $f$ be a continuous self-map of a dendrite. If an iterate of $f$ has an arc horseshoe then $f$ is DC$_1$ and $\omega$-chaotic.

**Proof.** In [6] this lemma was proved for DC$_2$. The author used the fact that if an iterate of the map $f$ has an arc horseshoe, then there is a set $D \subset X$ such that some iterate of $f$ restricted to $D$ is 2-to-1 semiconjugate to the shift map $\sigma$ on the space $\{0, 1\}^{\mathbb{N}_0}$ which by [5] and [9] is DC$_2$ and $\omega$-chaotic and thus $f$ is as well DC$_1$ and $\omega$-chaotic. Moreover by Lemma 7, $(\{0, 1\}^{\mathbb{N}_0}, \sigma)$ is also DC$_1$.

5 Open questions and remarks

While we showed that already on dendrites DC$_3$ $\not\Rightarrow$ DC$_2$ $\not\Rightarrow$ DC$_1$ for any size of DC-scrambled set, nor do DC-pairs imply infinite DC-scrambled sets, there are at least 2 more important questions for DC on dendrites:

**Question.** Do DC$_2$ and DC$^1$ coincide on dendrites?

Or we can single out an even more obvious question (since DC$^3$ $\not\Rightarrow$ DC$_2$).

**Question.** Does DC$^3$ imply at least LY on dendrites?

To be clear, we are asking about the implications DC$_3$ $\Rightarrow$ LY$_2$ and DC$_3$ $\Rightarrow$ LY$_u$, since we know from [4] that DC$_2$ $\not\Rightarrow$ LY$_\infty$.

Moreover we can ask if any of the theorems in this paper would change, if we required the functions on dendrites to be monotone (ours are not).

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