A MINIMAL MODEL OF LORENTZ GAUGE GRAVITY WITH DYNAMIC TORSION

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A new Lorentz gauge gravity model with $R^2$-type Lagrangian is proposed. In the absence of classical torsion the model admits a topological phase with an arbitrary metric. We analyze the equations of motion in constant curvature space-time background using the Lagrange formalism and demonstrate that the model possesses a minimal set of dynamic degrees of freedom for the torsion. Surprisingly, the number of torsion dynamic degrees of freedom equals the number of physical degrees of freedom for the metric tensor. An interesting feature of the model is that the spin two mode of torsion becomes dynamical essentially due to the non-linear structure of the theory. We perform covariant one-loop quantization of the model for a special case of constant curvature space-time background. We treat the contortion as a quantum field variable whereas the metric tensor is kept as a classical object. We discuss a possible mechanism of an emergent Einstein gravity as a part of the effective theory induced due to quantum dynamics of torsion.

Keywords: quantum gravity; torsion; Lorentz gauge symmetry.

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1. Introduction

The gauge approach to gravity based on gauging Lorentz and Poincare group was developed as a powerful tool in attempts to construct a consistent quantum theory of gravity in the framework of field theory formalism. The extension of gauge gravity models to the case of Riemann-Cartan space-time geometry reveals wide possibilities towards formulation of numerous models of quantum gravity with torsion. Recently, a Lorentz gauge model of gravity with Yang-Mills type Lagrangian including contortion (torsion) has been developed further. It has been proposed that the Einstein gravity with a cosmological term can be induced as an effective theory due to quantum corrections of contortion. In that model the space-time metric is treated as a fixed classical field while the contortion supposed to be a quantum field. Such a treatment of the metric is not satisfactory from the conceptual point of view since one has to assume the existence of a classical space-time with a metric given a priori. In other words, we encounter the problem of space-time background dependence which is similar to the space-time dependence problem in superstrings where the string lives in a pre-defined fiducial space-time. One possible way to resolve this problem is to generalize the Lorentz gauge model by extending the gauge group to Poincare one. In that case the gauge potential of the Poincare group, the vielbein, becomes dynamical on equal footing with torsion. Another interesting possibility is to look for a quantum gravity model which has only one basic quantum field, contortion, whereas the metric describes a topological phase in the absence of contortion at classical level. In the topological phase the torsion can be unobservable quantity at classical level (this assumption may have physical sense if there is some mechanism of hiding the classical torsion like the Meissner effect) but manifests itself through the quantum dynamics which triggers the phase transition to the phase of an effective theory where the metric obtains its dynamic content.

In the present paper we propose a special $R^2$-type model of Lorentz gauge gravity which admits a topological phase at classical level and has non-trivial quantum dynamics of torsion. The proposed model is minimal in a sense that only the contortion possesses dynamic degrees of freedom whereas the metric does not. We will demonstrate that the contortion has six propagating modes with spins $J = (2; 1; 0; 0)$, exactly the same number of physical degrees of freedom the metric tensor has in general.

Notice, the structure of Poincare gauge theory with a most general $R + R^2$ type Lagrangian was studied in Refs. A special case with vanishing linear in curvature term was considered partially in Ref. What we consider in the present paper represents a strongly degenerate case of the theory, and its analysis was missed in previous studies. Moreover, in our analysis we take into account the effects conditioned by the curved space-time. We show that these effects are important and lead to appearance of the propagating torsion mode with spin two. In a fact, introducing the non-flat space-time background we estimate indirectly
the effect of the non-linear structure of the full theory. So that, we expect that our model even with flat metric has very non-trivial dynamic content for the contortion due to non-linearity of the equations of motion. The non-linearities of constraints in Poincare gauge theory are very important and can change the number of constraints and degrees of freedom.\cite{27,29}

In section II, we consider a general Lagrangian quadratic in Riemann-Cartan curvature which admits the topological phase in the limit of vanishing classical torsion. In section III, we study the dynamic content of the model in the framework of Lagrange formalism. By explicit solving all constraints among the equations of motion we show that the contortion possesses propagating modes. A covariant quantization scheme and the analysis of the structure of the one-loop effective Lagrangian is presented in Section IV. The last section contains discussion on possibility of inducing the Einstein-Hilbert term and cosmological constant by quantum torsion corrections.

2. Lorentz gauge gravity model with a topological phase

Let us start first with the main outlines of Riemann-Cartan geometry. The basic geometric objects in Poincare gauge models of gravity\cite{1,2,3,9,10} are the vielbein $e^i_a$ and the general Lorentz affine connection $A_{i}^{cd}$. The infinitesimal Lorentz transformation of the vielbein $e^i_a$ is given by

$$\delta e^i_a = [A_i, e^i_a] = \Lambda_{a}^{b}e^i_{b}, \quad (1)$$

where $\Lambda \equiv \Lambda_{cd}^{\Omega}^{cd}$ is a Lie algebra valued gauge parameter, and $\Omega^{cd}$ is a generator of the Lorentz Lie algebra. We use $(i, j, k, ... = 0, 1, 2, 3)$ to denote coordinate indices, and $(a, b, c, ... = 0, 1, 2, 3)$ for Lorentz frame indices. We assume that the vielbein is invertible and the signature of the flat metric $\eta_{ab}$ is Minkowskian, $\eta_{ab} = \text{diag}(+-\cdots)$.

The covariant derivative with respect to the Lorentz group transformation is defined in a standard manner

$$D_a = e^i_a(\partial_i + gA_i), \quad \text{(2)}$$

where $A_i \equiv A_{icd}^{\Omega}^{cd}$ is a general affine connection taking values in the Lorentz Lie algebra, and $g$ is a new gravitational gauge coupling constant. For brevity of notation we will use a redefined connection which absorbs the coupling constant. The original Lorentz gauge transformation of the connection $A_i$ has the form

$$\delta A_i = -\partial_i \Lambda - [A_i, \Lambda]. \quad \text{(3)}$$

The affine connection $A_{icd}$ can be rewritten as a sum of the Levi-Civita spin connection $\varphi_{ic}^{d}(e)$ and the contortion $K_{ic}^{d}$

$$A_{ic}^{d} = \varphi_{ic}^{d}(e) + K_{ic}^{d}. \quad \text{(4)}$$

$$\varphi_{ia}^{b}(e) = -\frac{1}{2}(e^{jb}\partial_i e_{ja} - e^{ja}_i e_{jb}^{d} e_{jc}^{d} + \partial_i e^{ja}_j - e^{ja}_i e^{jb}_j e_{jc}^{d} + e^{jb}_i e^{ja}_j e_{jc}^{d} - \partial^b e_{ia}).$$
The torsion and curvature tensors are defined in a standard way

$$ [D_a, D_b] = -T_{ab}^c D_c - R_{ab}, $$

$$ T_{ab}^c = K_{ba}^c - K_{ab}^c, $$

(5)

where, $R_{ab} \equiv R_{abcd} \Omega^{cd}$. Under the decomposition (4) the Riemann-Cartan curvature is splitted into two parts

$$ R_{abcd} = \hat{R}_{abcd} + \tilde{R}_{abcd}, $$

$$ \hat{R}_{abcd} = \hat{D}_b \varphi_{ac} + \varphi_{bc} \varphi_{ad} - (a \leftrightarrow b), $$

$$ \tilde{R}_{abcd} = \tilde{D}_b K_{ac} + K_{bc} K_{ad} - (a \leftrightarrow b), $$

(6)

where, $\hat{D}_a$ is a restricted covariant derivative containing only the Levi-Civita connection, and the underlined indices stand for indices over which the covariantization is performed.

We exclude from our consideration any quadratic invariants with a dual Riemann-Cartan curvature which breaks the parity. The most general quadratic in Riemann-Cartan curvature Lagrangian reads

$$ L = c_1 R_{abcd} R^{abcd} + c_2 R_{abcd} R^{cdab} + c_3 R_{ab} R^{ab} + c_4 R_{ab} R^{ba} + c_5 R^2 + c_6 A_{abcd}^2, $$

(7)

where the last term is an additional invariant which appears in Riemann-Cartan space-time. The tensor $A_{abcd}$ is defined as follows

$$ A_{abcd} = \frac{1}{6} (R_{abcd} + R_{acdb} + R_{adbc} + R_{bca} + R_{cdab}). $$

(8)

In Riemannian space-time the tensor $A_{abcd}$ vanishes due to Jacobi cyclic identity

$$ R_{abcd} + R_{acdb} + R_{adbc} \equiv 0. $$

(9)

In Riemann-Cartan geometry the proper generalization of the topological Gauss-Bonnet invariant is given by the Bach-Lanczos density

$$ I_{BL} = R_{abcd} R^{cdab} - 4 R_{ab} R^{ba} + R^2. $$

(10)

The properties of the Bach-Lanchos invariant are described in a detail in Ref. [32].

We are interested in such a Lagrangian in Riemann-Cartan space-time which is reduced to the topological Gauss-Bonnet density in the limit of Riemannian geometry. A proper Lagrangian can be derived from the general expression (7) by fitting the parameters $c_i$ as follows

$$ L = -\frac{1}{4} \left( \alpha R_{abcd}^2 + (1 - \alpha) R_{abcd} R^{cdab} - 4 \beta R_{bd}^2 - 4 (1 - \beta) R_{bd} R^{db} - R^2 + 6 \gamma A_{abcd}^2 \right), $$

(11)

where the parameters $\alpha, \beta, \gamma$ define the remaining arbitrariness. One can check that the Lagrangian reduces to the Gauss-Bonnet density in the limit of Riemannian space-time geometry. One can rewrite the Lagrangian in a more simple form

$$ L = -\frac{1}{4} \left[ (\alpha + \gamma) R_{abcd}^2 - (\alpha - \gamma) R_{abcd} R^{cdab} - 4 \beta (R_{bd}^2 - R_{bd} R^{db}) + 4 \gamma R_{abcd} R^{adcb} + I_{BL} \right]. $$

(12)
We will demonstrate that the model described by the Lagrangian (12) admits dynamic degrees of freedom for the contortion only for special values of the parameters $\alpha, \beta, \gamma$.

3. Equations of motion, constraints

A detailed analysis of the equations of motion of the general $R + T^2 + R^2$ type Poincare gauge gravity for most of non-degenerate cases was performed in Refs. 23, 25. A Hamiltonian formalism of the Poincare gauge gravity was considered in Refs. 33, 34. For our model, which is strongly degenerate and has a non-trivial space-time background, it is much more easier to use the Lagrange formalism instead of the canonical Hamiltonian one 35. We start with arbitrary parameters $\alpha, \beta, \gamma$ in the initial Lagrangian (12), and then we will find constraints on the parameters under which the contortion obtains propagating modes. Let us consider linearized equations of motion corresponding to the Lagrangian (12)

$$\frac{\delta L}{\delta K_{bcd}} = \frac{1}{2}(\alpha + \gamma)(\bar{D}^a \bar{D}_a K_{bcd} - \bar{D}^a \bar{D}_b K_{a0d}) - (\alpha - \gamma)\bar{D}^a \bar{D}_c K_{dab} +$$

$$\gamma \left( \bar{D}^a \bar{D}_a K_{c0d} - \bar{D}^a \bar{D}_c K_{ad0} - \bar{D}^a \bar{D}_b K_{cd0} - \bar{D}^a \bar{D}_e K_{bda} + \right)$$

$$\beta \left( \bar{D}_c \bar{D}^a K_{dab} - \bar{D}_e \bar{D}^a K_{bad} + \bar{D}_d \bar{D}_b K_d - \bar{D}_e \bar{D}_d K_b + \right)$$

$$\eta_{bc} \left( \bar{D}^a \bar{D}^e K_{a0d} - \bar{D}^a \bar{D}^e K_{d0a} - \bar{D}^a \bar{D}_a K_d + \bar{D}^a \bar{D}_d K_a \right) - (c \leftrightarrow d) = 0, \quad (13)$$

where, $K_{d} \equiv K^c_{cd}$. It is convenient to impose gauge fixing conditions which are compatible with equations of motion. To fix the Lorentz gauge symmetry we choose the following constraints (indices denoted by Greek letters refer to spatial coordinates)

$$\partial^\beta (K_{\beta0\delta} - K_{\delta0\beta}) = 0,$$

$$\gamma (\partial^\beta K^\gamma_{\beta\delta} - \partial^\delta K^\gamma_{\beta\gamma}),$$

$$\partial^\beta \partial^\delta K_{\beta0\delta} = 0. \quad (14)$$

The consistence of these gauge conditions with the Lagrange equations will be verified during solving all equations of motion.

The equation of motion $\delta L/\delta K_{000} = 0$ with a flat metric and non-vanishing parameter $\beta$ represents a most stringent constraint which implies that the transverse part of the vector field $K_{\mu0\delta}$ vanishes identically. Since we are interested in finding propagating modes for the spin one vector field even in the flat space-time we will consider in the following only the case $\beta = 0$. It should be stressed, that the presence of curved space-time and non-linearity of the theory is an essential factor which can not be ignored. The perturbative analysis of linearized equations of motion in the flat space-time may lead to incorrect results in degenerate cases. For that reason we will study the equations of motion in the curved space-time, with a non-flat metric, while keeping a linearized approximation in contortion field. Such a treatment allows us to describe the main features of our model since the non-trivial Levi-Civita connection $\varphi_{ia}^{\ b}(e)$ can be treated as a background part of...
the general Lorentz connection. For simplicity we choose the background space-time as a Riemannian space-time of covariant constant curvature

\[ \hat{R}_{abcd} = \frac{1}{12} \hat{R}(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc}). \] (15)

In constant curvature background space-time one can rewrite the equations of motion in normal coordinates keeping only linear terms in Ricci scalar \( \hat{R} \). The metric tensor, vielbein and Levi-Civita connection have the following decomposition in normal coordinates

\[
\begin{align*}
g_{mn} &= \eta_{mn} + \frac{2\rho}{3} (\eta_{mn}x^kx_k - x_mx_n), \\
e^a_m &= \delta^a_m + \frac{\rho}{3} (\delta^a_mx^kx_k - x^a), \\
\varphi^b_m &= \rho(\eta_{ma}x^b - \delta^b_mx_a),
\end{align*}
\] (16)

where \( \rho = \frac{\hat{R}}{24} \). Using these relations one can write down the equations of motion as follows

\[
\frac{1}{2}(\alpha + \gamma)(\partial^a \partial_a K_{bcd} - \partial^a \partial_b K_{acd}) - (\alpha - \gamma)\partial^a \partial_a K_{dab} + \gamma(\partial^a \partial_a K_{cde} - \partial^a \partial_c K_{ade})
\]

\[
+ \gamma\rho K_{bcd} + (3\alpha - 2\gamma)\rho K_{cde} - \alpha\rho \eta_{bc} K_{da} - (c \leftrightarrow d) = 0.
\] (17)

Using the gauge conditions (14) one can rewrite the equations of motion in component form

\[
\frac{\delta \mathcal{L}}{\delta K_{00\delta}} = \Delta K_{00\delta} - \partial^3 \partial_3 K_{00\beta} - \partial^3 \partial_0 K_{\beta0\delta} + 2\rho K_{00\delta} - \rho K^{\nu}_{\nu\delta} = 0,
\] (18)

\[
\frac{\delta \mathcal{L}}{\delta K_{0\gamma\delta}} = (\alpha + \gamma)\Delta K_{0\gamma\delta} + \alpha(\partial^3 \partial_3 K_{0\beta\delta} - \partial^3 \partial_0 K_{\beta0\delta}) - (\alpha + \gamma)\partial_0 \partial^3 K_{\beta\gamma\delta} + \\
\gamma(\partial_0 \partial^3 K_{0\beta\delta} - \Delta K_{0\gamma\delta} - \partial_0 \partial^3 K_{\gamma\beta\delta} - (\gamma \leftrightarrow \delta)) + 2\gamma\rho K_{0\gamma\delta} + (3\alpha - 2\gamma)\rho (K_{0\gamma\delta} - K_{\delta0\gamma}),
\] (19)

\[
\frac{\delta \mathcal{L}}{\delta K^{\nu}_{\nu\delta}} = \Delta K^{\nu}_{\nu\delta} + \partial_0 \partial^3 K^{\nu}_{\nu\beta} = 0,
\] (20)

\[
\frac{\delta \mathcal{L}}{\delta K^{\nu}_{\nu\delta}} = \square K^{\nu}_{\nu\delta} - \partial^3 \partial_3 K_{\beta\nu\delta} - \partial_0 \partial^3 K^{\nu}_{\nu\beta} - \partial_0 \partial^3 K_{\beta0\delta} - \partial_0 \partial^3 K_{0\beta\delta} - \partial_0 \partial^3 K_{0\beta\delta} \\
- \partial_0 \partial_3 K_{\delta} + \rho K^{\nu}_{\nu\delta} - 2\rho K_{00\delta} = 0,
\] (21)
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\[
\frac{\delta L}{\delta K^{0\beta\delta}} \equiv \alpha [\Delta K_{0\beta\delta} - \partial_\beta \partial^\alpha K_{\alpha 0\delta} + \partial_0 \partial_\beta (K_{0\beta\delta} - K_{\delta 0\beta}) + \partial^\alpha \partial_\beta K_{0\alpha\beta} - \partial_\beta \partial^\alpha K_{\alpha 0\delta} + \partial_0 \partial_\beta (K_{0\beta\delta} - K_{\delta 0\beta})
+ \gamma [\Delta (K_{0\beta\delta} - K_{\delta 0\beta}) + \partial_0 \partial^\alpha (K_{\delta\alpha\beta} - K_{\alpha\delta\beta}) + \partial_\beta \partial^\alpha K_{\beta\delta\alpha} - \partial_\beta \partial^\alpha K_{\beta\delta\alpha} + \partial_0 \partial_\beta (K_{0\beta\delta} - K_{\delta 0\beta})] - 2\gamma \rho K_{0\beta\delta} + 2\gamma \rho (K_{30\delta} - K_{\delta 30}),
\]  

(22)

\[
\frac{\delta L}{\delta K_{\beta\gamma\delta}} \equiv \alpha [\square K_{\beta\gamma\delta} - \partial_\beta \partial^\alpha K_{\alpha\gamma\delta} - \partial_\gamma \partial^\alpha K_{\alpha\beta\delta} + \partial_\gamma \partial_\delta K_{\alpha\beta\gamma} + \partial_0 \partial_\beta K_{0\gamma\delta} + \partial_0 \partial_\gamma K_{0\beta\delta} + \partial_\delta \partial^\alpha K_{\alpha\beta\gamma} + 3\rho (K_{\gamma\beta\delta} - K_{\delta\beta\gamma}) - \rho (\eta_{\beta\gamma} K_{\delta} - \eta_{\beta\delta} K_{\gamma})]
+ \gamma [\square + 2\rho] K_{(\beta\gamma\delta)} + \partial_\beta \partial^\alpha K_{(\alpha\beta\delta)} - \partial_\beta \partial^\alpha K_{(\alpha\beta\gamma)} - \partial_\delta \partial^\alpha K_{(\alpha\gamma\delta)} + \partial_\delta \partial_\gamma (K_{0\beta\delta} - K_{0\beta\delta})] = 0,
\]  

(23)

where, \[\square = \partial^\alpha \partial_\alpha, \quad \Delta = \partial^\alpha \partial_\alpha, \] and a cyclic combination over indices enclosed in brackets is assumed, for instance,

\[K_{(\beta\gamma\delta)} = K_{\beta\gamma\delta} + K_{\gamma\delta\beta} + K_{\delta\beta\gamma}.
\]  

(24)

We will use the following decomposition of the contortion \(K_{\mu cd}\) into irreducible parts,

\[K_{\mu\gamma\delta} = \epsilon_{\gamma\delta}^\rho K^*_{\mu\rho},
\]

\[K^*_{\mu\rho} = \mathcal{S}_{\mu\rho} + \frac{1}{2} (\delta_{\mu\rho} \Delta - \delta_{\mu\rho} \partial_\rho) \mathcal{S} + (\partial_\mu \mathcal{S}_\rho + \partial_\rho \mathcal{S}_\mu) + \epsilon_{\mu\rho}^\sigma A_\sigma,
\]

\[K_{\mu 0\rho} = \mathcal{R}_{\mu\rho} + \frac{1}{2} (\delta_{\mu\rho} \Delta - \delta_{\mu\rho} \partial_\rho) \mathcal{R} + (\partial_\mu \mathcal{R}_\rho + \partial_\rho \mathcal{R}_\mu) + \epsilon_{\mu\rho}^\sigma Q_\sigma,
\]  

(25)

where the superscript "T" stands for traceless components and the superscript "TT" denotes irreducible parts which are traceless and transverse.

The equations of motion \([18, 29]\) and gauge fixing conditions \([14]\) represent both constraints and dynamic equations. Below we solve all constraints and derive remaining independent dynamic equations.

The first equation of motion, Eq. \([18]\), is a constraint which allows to express the longitudinal and transverse components of the pure gauge field \(K_{0\beta\delta}\) in terms of corresponding components of \(K_{\nu\rho\delta}\)

\[K^l_{0\beta\delta} = -\frac{1}{2} K^l_{\nu\rho\delta},
\]  

(26)

\[K^{tr}_{0\beta\delta} = -\frac{\rho}{\Delta + 2\rho} K^{tr}_{\nu\rho\delta}.
\]  

(27)

To solve the Eqn. \([19]\) it is convenient to find first the longitudinal part of \(K_{0\gamma\delta}\)

\[\partial^\gamma K^{l}_{0\gamma\delta} = -\frac{\alpha \Delta}{\alpha \Delta + 2\gamma \rho} \partial^\delta K^{l}_{0\alpha\beta}.
\]  

(28)
With this one can resolve completely the constraint \( (29) \)

\[
K_{\gamma\delta} = \frac{1}{(\alpha + \gamma)\Delta + 2\gamma\rho} \left[ -\left( \alpha + \frac{\alpha\gamma\Delta}{\alpha\Delta + 2\gamma\rho} \right) \left( \partial_{\gamma} \partial^{\alpha} K_{\delta0} - \partial_{\delta} \partial^{\alpha} K_{\alpha0} \right) 
- \left( (3\alpha - 2\gamma)\rho - \gamma\Delta \right) (K_{\gamma0} - K_{0\gamma}) \right].
\]

The constraint \( (20) \) has a simple solution \( K_0 = \frac{1}{\Delta} \partial_0 \partial^{\alpha} K_{\nu\alpha}. \) (30)

This solution can be rewritten in terms of irreducible component fields \( \Delta R^\top = \frac{2\partial_0 \partial^{\alpha}}{\Delta} A_{\alpha}. \) \( (31) \)

The longitudinal projection of Eqn. \( (21) \) represents a constraint which has the following solution \( S_{\alpha} = -\frac{\gamma}{2(\alpha + \gamma)} \partial_\alpha \Delta R^\top. \) \( (32) \)

Using the above solutions one can check that the transverse projection of the Eqn. \( (21) \) produces a dynamic equation for the transverse field \( K_{\nu\nu\delta} \)

\[
\Box K_{\nu\nu\delta} + \rho K_{\nu\nu\delta} + 2\rho K_{0\delta0} + 2\frac{\gamma\rho}{\alpha\Delta} \partial_\delta R^0_{\alpha} = 0.
\] \( (33) \)

In the next section we will show that the original quadratic in \( K_{mod} \) Lagrangian possesses additional local \( U(1) \) and \( \chi \) symmetries. One can check that the field \( K_{\nu\nu\delta}^{tr} \) is invariant under \( \chi \)-transformations and represents two transverse modes corresponding to \( U(1) \) gauge boson. Notice, one should check the consistence of the Eq. \( (33) \) with all remaining equations of motion which can constrain strongly the field dynamics of \( K_{\nu\nu\delta}^{tr}. \)

Solving the Eqn. \( (22) \) is a little tedious but straightforward. Notice, that we consider all equations including \( \rho \)-dependence, i.e., we take into account next linear order terms in \( \rho. \) One can verify that in a case of constant space-time background, \( \rho = const, \) there is no higher order corrections to the equations of motion. The antisymmetric part of the Eqn. \( (22) \) implies two constraints on the irreducible field components

\[
\left( C_1 - \frac{4C_2^2}{A} \right) \partial^{\alpha} Q_{\alpha} + (2\gamma - \alpha) \partial_\delta \Delta \Delta R^\top + 4\gamma \partial_\delta \Delta \partial^{\alpha} S_{\alpha} = 0,
\]

\[
\left[ \frac{2C_2}{(\alpha + \gamma)\Delta + 2\gamma\rho} \left( - (\alpha + \gamma)\Delta + \frac{2\gamma^2\rho}{\alpha} \right) + (-\alpha + \alpha\xi - \frac{2\gamma\Delta}{\Delta - 6\rho}) \right] R^t_{\alpha} = -4\alpha \rho \partial_\delta A^{tr}_{\alpha},
\]

where \( C_1, C_2 \) are operator quantities

\[
C_1 = 2(\alpha + 2\gamma)\Delta + 4\alpha \partial_\delta \partial_\delta + 2(4\gamma - 3\alpha)\rho,
C_2 = -\gamma \Delta + (3\alpha - 2\gamma)\rho.
\] \( (36) \)
Notice, that the relationship (35) between the transverse components $A^{tr}_\alpha$ and $R^{tr}_{\alpha}$ appears due to taking into account the presence of the curved space-time with a non-zero curvature $\rho$.

Taking the divergence of the symmetrized part of the Eqn. (22) one can obtain another two constraints

$$2 \partial^\rho R_\alpha = \partial_\alpha \partial^\rho A_\alpha,$$

$$\rho(1 + \frac{\gamma}{3\alpha}) \Delta R^{tr}_{\delta} = 0.$$  

Due to the imposed gauge condition (14) the Eqs. (37) and (38) imply that $R^l_{\alpha} = A^l_{\alpha} = 0$. One should stress that the equation (35) is non-trivial, and it appears in linear order in curvature $\rho$. An important consequence of the equations (38) and (35) is that the field $A^{tr}_\alpha$ vanishes unless $\gamma = -3\alpha$. This is the only case, $\gamma = -3\alpha$, when the field $A^{tr}_\alpha$ gains propagating modes. Finally, taking into account the above solutions, the symmetric traceless part of the Eqn. (22) leads to a relationship between spin two fields $\mathbb{T} \mathbb{T} R^{\beta\delta} \bigg( \Delta + 3\rho \bigg)$ and $\mathbb{T} \mathbb{T} S^{\beta\delta}$

$$\bigg( \Delta + 3\rho \bigg) R^{\beta\delta} - \frac{1}{2} \epsilon_{\beta\delta} \rho \partial^\sigma \partial^\gamma S^{\sigma\gamma} + (\beta \leftrightarrow \delta) = 0.$$  

Surprisingly, the general Lagrangian (12) admits dynamic equations for the contortion only with a unique choice of the parameters, ($\beta = 0, \gamma = -3, \alpha = 1$), ($\alpha$ is unimportant overall number factor which can be set to one).

Let us rewrite the gauge conditions and solutions to the equations of motion for the special case of chosen parameters $\alpha = 1, \beta = 0, \gamma = -3$ and small values of $\rho$. Notice, the approximation $\rho \simeq 0$ corresponds to small deviation of the space-time geometry from the flat limit which is enough for our purpose to analyze the dynamic properties of the fields. The gauge conditions (14) can be written in terms of independent fields as follows

$$Q^{tr}_\gamma = 0,$$

$$S^{tr}_\alpha = \epsilon_{\alpha\gamma\delta} \mathbb{T} A^{tr}_\delta,$$

$$\mathcal{S}_\gamma = \frac{3}{4} \partial_\gamma S,$$

$$R^l_{\alpha} = 0.$$  

The solution (29) is simplified as follows

$$K_{\gamma\delta} = -\frac{\Delta}{\Delta - 6\rho} \big( \partial_\gamma R_\delta - \partial_\delta R_\gamma \big) + 3\epsilon_{\gamma\delta} \rho Q_\sigma.$$  

Now we can write down the dynamic equation (33) for $A^{tr}_\alpha$ in a final form

$$(\Box + 2\rho) A^{tr}_\alpha = 0.$$  

Since the fields $A^{tr}_\alpha$ and $S^{tr}_\alpha$ are related by the Eqn. (11), the same equation holds for the field $S^{tr}_\alpha$. 
The Eqns. (34,35) can be rewritten in a simple form

$$2\partial_0\partial_0 Q_\alpha + 4(\Delta + 3\rho)Q_\alpha + \partial_0\partial_\alpha \Delta S - \epsilon_{\alpha\beta\delta}(\partial_0\partial^\beta K_{00\delta} + 6\rho\partial^\beta R_\delta) = 0,$$

$$R_{\delta}^{tr} = \frac{\partial_0}{3\Delta} A_{\delta}^{tr}.$$

(46)

Using the identity for the irreducible field $S^{tr}_{\alpha\beta}$

$$S^{tr}_{\alpha\beta} + \epsilon_{\alpha\gamma\epsilon} \epsilon_{\beta\nu} \frac{\partial^\gamma}{\Delta} S^{tr}_{\delta\rho} = 0$$

(48)

one can solve the last equation of motion, (23), which reproduces the eqn. (45) and leads to additional equations

$$(\Box + 3\rho)(\partial^\alpha Q_\alpha - \frac{2}{3}\Delta\partial_0\partial^\alpha S_\alpha) = 0,$$

$$\rho \Box S_{\alpha\beta} = 0.$$

(49)

(50)

Notice, that there is no equation for spin two field $S^{tr}_{\alpha\beta}$ at zero order in $\rho$. The non-trivial dynamics for the spin two field $S^{tr}_{\alpha\beta}$ appears only in linear order in $\rho$, i.e., in the presence of curved space-time background. The Eqns. (46, 49) implies that the longitudinal components of the vector fields $S_i, Q_i$ become propagating. Defining scalar fields corresponding to the longitudinal components of $S_i, Q_i$

$$\varphi = \partial^\alpha Q_\alpha,$$

$$\psi = -\frac{2}{3}\partial^\alpha S_\alpha,$$

one can easily derive the following dynamic equations

$$(\Box + 6\rho)\varphi + \Delta \sigma = j_1,$$

$$(\Box + 6\rho)\psi + \partial_0 \sigma = j_2,$$

$$\sigma = \varphi + \partial_0 \psi,$$

(52)

where we introduce explicitly sources $j_1, j_2$ for the fields $\varphi, \psi$. The linear dependent field $\sigma$ satisfies the standard hyperbolic equation

$$(\Box + 3\rho)\sigma = \frac{1}{2}(j_1 + \partial_0 j_2).$$

(53)

So that, our model has six dynamic degrees of freedom corresponding to the irreducible field components $(S^{tr}_{\alpha\beta}, A^{tr}_{i\alpha}, \varphi, \psi)$ with spins $(2; 1; 0; 0)$ respectively. One can treat the field $S^{tr}_{\alpha\beta}$ as an independent one, since the fields $A^{tr}_{i\alpha}, S^{tr}_{\alpha}$ are related by Eq. (41). Then, the scalar $\psi = -2/3\partial^\alpha S_\alpha$ represents a longitudinal component of the vector $S_\alpha$.

Notice, that the kinetic terms for the fields $A^{tr}_{i\alpha}$ and $\varphi$ are positively defined, whereas the terms for the fields $S^{tr}_{\alpha\beta}$ and $\psi$ have negative contribution to the classical Hamiltonian. However, one should stress that the action might have still
positive definiteness since the theory has highly non-linear structure. For instance, the kinetic term for the field $S_{\alpha,\beta}$ contains the curvature $\rho$ as a multiplier, so that, it represents actually the interaction term between the vielbein and contortion.

4. Covariant quantization and the effective Lagrangian

One-loop effective action with a constant curvature space-time background and quantum torsion has been calculated recently in the model with Yang-Mills type Lagrangian quadratic in Riemann-Cartan curvature \[21\]. In the previous section we have demonstrated that the model given by the Lagrangian \[12\] \((\alpha = 1, \beta = 0, \gamma = -3)\) has dynamic contortion. In this section we perform covariant quantization of the model which is more suitable for practical calculation of a quantum effective action.

We apply the functional integral formalism to derive the quantum effective Lagrangian in one-loop approximation. In background field formalism one starts with splitting the general gauge connection $A_{mcd}$ into background (classical) and quantum parts

$$A_{mcd} = A_{mcd}^{(cl)} + A_{mcd}^{(q)}.$$ \[54\]

We identify the classical field $A_{mcd}^{(cl)}$ with the Levi-Civita connection $\varphi_{mcd}(e)$ corresponding to the Riemannian space-time geometry and the quantum field $A_{mcd}^{(q)}$ with contortion $K_{mcd}$.

Let us define two types of Lorentz gauge transformations consistent with the original Lorentz gauge transformation \[3\] and the decomposition \[54\]:

(I) the classical, or background, gauge transformation

$$\delta e^m_a = \Lambda^b_a e^m_b,$$
$$\delta \varphi_m(e) = -\partial_m \Lambda - [\varphi_m, \Lambda],$$
$$\delta K_m = -[K_m, \Lambda].$$ \[55\]

(II) the quantum gauge transformation

$$\delta e^m_a = \delta \varphi_m(e) = 0,$$
$$\delta K_m = -\hat{D}_m \Lambda - [K_m, \Lambda],$$ \[56\]

where $\varphi_m \equiv \varphi_{mcd} \Omega^{cd}$, and the restricted covariant derivative $\hat{D}_m$ is defined by means of the Levi-Civita connection only

$$\hat{D}_m \Lambda = \partial_m \Lambda + [\varphi_m, \Lambda].$$ \[57\]

Notice that the restricted derivative $\hat{D}_m$ is covariant under the classical Lorentz gauge transformation. An interesting feature of the Lagrangian \[12\] is its invariance under above two types of gauge transformations. In general, this invariance can be broken by interaction of contortion with matter fields.
In one-loop approximation it is sufficient to keep only quadratic in contortion terms in the Lagrangian. After integration by parts and neglecting surface terms the quadratic Lagrangian (12) can be reduced to the form

\[ L^{(2)} = \hat{D}_a K_{bcd} (\hat{D}^a K^{bcd} - \hat{D}^b K^{acd}) + 4 \hat{D}_a K_{bcd} \hat{D}^c K^{dab} + 3 \hat{D}_a K_{bcd} (\hat{D}^a K^{cde} - \hat{D}^c K^{ade} - \hat{D}^b K^{cda} + \hat{D}^c K^{bda}). \] (58)

The Riemann curvature is supposed to be covariant constant, i.e., \( \hat{D}_a \hat{R}^{bcde} = 0 \).

An interesting property of the quadratic Lagrangian (58) is the presence of an additional local \( U(1) \) symmetry

\[ \delta_{U(1)} K_{bcd} = \frac{1}{3} (\eta_{bc} \hat{D}_d \lambda - \eta_{bd} \hat{D}_c \lambda), \]

\[ \delta_{U(1)} K_d = \hat{D}_d \lambda. \] (59)

As it was shown in the previous section using the Lagrange formalism, the vector field \( K_d \) contains two transverse dynamic modes, \( A^{tr}_d \). Notice, that a quadratic part of the Yang-Mills type Lagrangian

\[ L_{YM} = -\frac{1}{4} R_{abcd}^2 \] (60)

does not have such a local \( U(1) \) symmetry.

It is convenient to decompose the contortion into irreducible parts

\( K_{bcd} = Q_{bcd} + \frac{1}{3} (\eta_{bc} K_d - \eta_{bd} K_c) + \frac{1}{6} \epsilon_{bcde} S^e, \)

\( Q^{ce}_{cd} = 0, \)

\( \epsilon^{abcd} Q_{bcd} = 0. \) (61)

Another unexpected feature of the quadratic Lagrangian \( L^{(2)}, \) is that it admits another local symmetry. The symmetry is provided by the following transformations with a new constrained parameter \( \chi_{bc} \)

\[ \delta_\chi Q_{bcd} = \hat{D}_c \chi_{db} - \hat{D}_d \chi_{cb}, \]

\[ \delta_\chi K_d = 0, \]

\[ \delta_\chi S^a = 0, \]

\[ \chi_{bc} = \chi_{cb}, \quad \chi_c^e = 0, \quad \hat{D}_c \chi_{cd} = 0. \] (62)

The presence of the additional local gauge symmetries implies that the model is degenerate and has additional constraints. The field \( Q_{bcd} \) has sixteen field components in general. After subtracting six pure gauge degrees of freedom due to Lorentz gauge symmetry and five degrees due to \( \chi \)-symmetry one has exactly five physical degrees of freedom for the spin two field. The fact that we have only one physical spin two field is unexpected, and it does not occur in the gauge gravity model with Yang-Mills type Lagrangian \( L_{YM}, \) where the contortion \( Q_{bcd} \) contains a pair of spin two fields.
To perform one-loop quantization one has to fix the gauges corresponding to the local Lorentz, \( \lambda \) and \( \chi \) symmetries. The simplest gauge fixing function we have chosen is the following

\[ F_{1cd} = \hat{D}^b Q_{bcd}. \]  

One has a simple Lorentz gauge transformation rule for the function

\[ \delta F_{1cd} = -\frac{2}{3}(\hat{D} \hat{D} + \frac{\hat{R}}{6})\Lambda_{cd}. \]  

With this one can write down the corresponding gauge fixing term and Faddeev-Popov ghost Lagrangian

\[ L_{gf}^{(1)} = -\frac{1}{2\xi_1}(\hat{D}^b Q_{bcd})^2, \]
\[ L_{FP}^{(1)} = \bar{c}_1^d (\hat{D} \hat{D} + \frac{\hat{R}}{6}) c_{1cd}, \]  

where \( \bar{c}_1^d, c_{1cd} \) are ghost fields. For simplicity we choose the gauge parameter \( \xi_1 = 1 \). Notice, that the gauge function \( F_{1cd} \) is invariant under \( U(1) \) and \( \chi \)-transformations. To fix the gauge for the local \( U(1) \) symmetry one has to introduce a second gauge fixing function which can be chosen as

\[ F_2 = \hat{D}^b K_b. \]  

The corresponding gauge fixing term and Faddeev-Popov Lagrangian have the following form

\[ L_{gf}^{(2)} = -\frac{1}{\xi_2}(\hat{D}^b K_b)^2, \]
\[ L_{FP}^{(2)} = \bar{c}_2 \hat{D} \hat{D} c_2. \]

We choose a Feynman gauge (\( \xi_2 = 1 \)) for simplicity. Finally, to fix the gauge for the \( \chi \)-transformations one can choose a constrained gauge fixing function

\[ F_{3bd} = \frac{1}{2}(\hat{D}^a Q_{bad} + \hat{D}^a Q_{dab}), \]
\[ \eta^{bd} F_{3bd} = 0, \]
\[ \hat{D}^b F_{3bd} = \frac{1}{2} \hat{D}^a \hat{D}^b Q_{bad} \simeq 0, \]  

where the last equality takes place on the hypersurface \( \hat{D}^b Q_{bcd} = 0 \) in the configuration space of functions \( \{Q_{bcd}\} \). One can easily find the corresponding gauge fixing and ghost terms

\[ L_{gf}^{(3)} = -\frac{1}{2\xi_3} F_{3bd}^2, \]
\[ L_{FP}^{(3)} = \bar{\psi}^{cd} (\hat{D} \hat{D} - \frac{\hat{R}}{3}) \psi_{cd}. \]  

\[ \]
We set condition $\xi_3 = \frac{1}{2}$ which corresponds to a symmetric gauge. In calculation of the one-loop effective action after functional integration over the ghost fields $\psi_{cd}$ the last equation (69) leads to a functional determinant of the operator $\hat{D}^2 - \frac{\hat{R}}{3}$ which has an additional pole (for a constant positive curvature $\hat{R}$). This implies the presence of the tachyon mode which has the same origin as the known Savvidy-Nielsen unstable mode in quantum chromodynamics\textsuperscript{37,38}.

The final expression for a total one-loop effective Lagrangian is given by the sum of all gauge fixing and ghost terms

$$L_{\text{eff}}^{(2)} = L_{0}^{(2)} + \sum_{i=1,2,3} (L_{gf}^{(i)} + L_{FP}^{(i)}).$$ \hspace{1cm} (70)

The expression for the effective Lagrangian is ready for calculation of the one-loop effective action. The contributions of ghosts are given by scalar functional determinants which can be easily calculated in analytic form\textsuperscript{21}. Unfortunately, the calculation of the functional determinants obtained after integration over contortion is more complicated due to the tensorial structure of the corresponding propagator.

5. Discussion

In our previous paper\textsuperscript{21} we have proposed a mechanism of dynamical generation of Einstein gravity through the quantum corrections of torsion in the Yang-Mills type Lorentz gauge gravity model. The main ingredient of such a mechanism is the formation of torsion vacuum condensate which we assumed to be covariant constant

$$<\hat{R}_{abcd}> = \frac{1}{2} M^2 (\eta_{ac} \eta_{bd} - \eta_{ad} \eta_{bc}),$$ \hspace{1cm} (71)

where $M^2$ is a mass scale characterizing the torsion condensate. We suppose that the model proposed in the present paper admits the generation of the Einstein-Hilbert and cosmological terms as well.

In conclusion, we propose a simple $R^2$ model of Lorentz gauge quantum gravity with torsion. Our model has a number of advantages to compare with Yang-Mills type Lorentz gauge gravity. In the absence of classical torsion the model reduces to a pure topological gravity, i.e., one has a topological phase where the metric is not specified a priori. The metric can obtain its dynamical content after dynamical symmetry breaking in the phase of the effective gravity which includes the Einstein gravity as its part. An unexpected feature of our model is that the contortion has the same number of dynamic degrees of freedom as the number of physical components of the metric tensor in Einstein gravity. The difference is that in Einstein gravity the metric has only two propagating spin 2 modes, other polarization modes corresponding to spin states $(1; 0; 0)$ are not dynamical. Whereas in our model the contortion has six propagating modes with spin polarizations $(2; 1; 0; 0)$, that means the torsion might have a dynamical mass like a gluon in quantum chromodynamics. This fact can be considered as an additional hint that the contortion could be a quantum counter-part to the classical graviton.
We have analyzed the dynamic structure of our model considering linearized equations of motion in a constant curvature space-time. It should be noted, that our results can reflect the principal dynamic properties of the full non-linear theory since the Levi-Civita connection can be treated as a background part of the Lorentz gauge connection. A more detailed analysis of the non-linear structure of our model and interaction with matter will be considered in a separate forthcoming paper.

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