On a General Theorem of Number Theory Leading to the Gibbs, Bose–Einstein, and Pareto Distributions as well as to the Zipf–Mandelbrot Law for the Stock Market

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Abstract

The notion of density of a finite set is introduced. We prove a general theorem of set theory which refines the Gibbs, Bose–Einstein, and Pareto distributions as well as the Zipf law.

Suppose that $M(n)$ is a sequence of finite sets tending as $n \to \infty$ to an infinite set. Suppose that $N(M(n))$ is the number of elements in the set $M(n)$.

The set $M(n)$ is said to be $\rho(\cdot)$-measurable if there exists a smooth convex function $\rho(\cdot)$, called a density function, such that the limit

$$\lim_{n \to \infty} \frac{\rho(N(M(n)))}{\rho(n)}$$

is finite. This limit is called the $\rho(\cdot)$-density of the sequence $M(n)$ of sets.

Let us present a few examples.

Example 1 Consider the eigenvalues of the $k$-dimensional oscillator with potential

$$U(x) = \sum_{i=1}^{k} (\omega_{i}x_{i}^{2}), \quad x \in \mathbb{R}^{k},$$

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where the \( \omega_i \) are commensurable:

\[
-\Delta \Psi_i + U(x) \Psi_i = \lambda_i \Psi_i, \quad \Psi_i(x) \in L^2(\mathbb{R}^k).
\]

Suppose that \( N_\lambda(\lambda_i) \) is the number of its eigenvalues not exceeding a given positive number \( \lambda \). If \( \lambda \to \infty \), then the limit

\[
\lim_{\lambda \to \infty} \frac{\ln N_\lambda(\lambda_i)}{\ln \lambda} = k
\]

coincides with the dimension of the oscillator.

**Example 2** Suppose that \( F \) is a compact set and \( N_F(\varepsilon) \) is the minimal number of sets of diameter at most \( \varepsilon \) needed to cover \( F \). Then \( N_F(\varepsilon) \) is \( \ln(\cdot) \)-measurable and its density coincides with the metric order of the compact set \( F \) (see [1]).

Consider the set \( \{M_1\} \) of nonnegative numbers \( \lambda_1, \lambda_2, \ldots, \lambda_n \) and the set \( \{M_2\} \) of integers \( 1, 2, \ldots, N; \ N = N(n) \). Suppose that the set \( \{M_2\} \) is \( \ln(\cdot)@\)-commensurable and \( s \) is its density:

\[
\lim_{n \to \infty} \frac{\ln N}{\ln n} = s.
\]

Besides, let \( \overline{\lambda}(n) \) be the arithmetic mean of the ensemble of \( \lambda_i \):

\[
\overline{\lambda}(n) = \frac{1}{n} \sum_{i=1}^{n} \lambda_i.
\]

Suppose that we are given a number \( E(n) \). Consider the following cases:

1. \( \varepsilon \leq E(n) \leq \overline{\lambda}(n)N, \ \varepsilon > 0; \)
2. \( E(n) \geq \overline{\lambda}(n)N. \)

Consider the set of mappings of \( \{M_2\} \) onto \( \{M_1\} \). Two mappings are said to be equivalent if their images are identical. Further, we shall only consider nonequivalent mappings and denote them by \( \{M_3\} \).

Suppose that the sum of elements in \( \{M_2\} \) is equal to \( N = \sum_{i=1}^{n} N_i \) and the bilinear form of the pair of sets \( \{M_1\} \) and \( \{M_2\} \) satisfies the condition

\[
\left| \sum_{i=1}^{n} N_i \lambda_i \right| \leq E(n)
\]
in case (1) and
\[ \left| \sum_{i=1}^{n} N_i \lambda_i \right| \geq E(n) \]
in case (2).

Note that the set \( \{ \mathcal{M}_3 \} \) is \( \ln \ln(\cdot) \)-measurable.

Without loss of generality, we assume that the real numbers \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are naturally ordered, i.e., \( 0 \leq \lambda_i \leq \lambda_{i+1} \), and we split the interval \( 1, 2, \ldots, n \) into \( k \) intervals (to within 1), where \( k \) is independent of \( n \):

\[ 1, 2, \ldots, n, \quad n_1 + 1, n_1 + 2 \ldots, n_2, \quad n_2 + 1, n_2 + 2, \ldots, n_3, \ldots, \]
\[ n_{k-1} + 1, n_{k-1} + 2, \ldots, n_k, \quad \sum_{l=1}^{k} n_l = n; \]

here \( l = 1, \ldots, k \) is the number of the interval.

Denote by \( \overline{\lambda}_l \), \( l = 1, 2, \ldots, k \), the nonlinear average of \( \lambda_i \) over each interval:

\[ \overline{\lambda}_l = \Phi_{\alpha \beta} \left( \sum_{n_{l-1}}^{n_l} \psi_{\alpha \beta}(\lambda_i) \right), \]
where \( \psi_{\alpha \beta}(x) \) is the two-parameter family of functions and \( \Phi_{\alpha \beta} \) is its inverse:
\( \Phi_{\alpha \beta}(\psi_{\alpha \beta}(x)) = 1 \); namely,

(a) \[ \psi_{\alpha \beta} = \alpha e^{-\beta x} \quad \text{for} \quad s > 1; \quad (1) \]
(b) \[ \psi_{\alpha \beta} = \frac{1}{\alpha e^{\beta x} - 1} \quad \text{for} \quad s = 1; \]
(c) \[ \psi_{\alpha \beta} = \frac{1}{\beta x + \ln \alpha} \quad \text{for} \quad 0 < s < 1. \]

The parameters \( \alpha \) and \( \beta \) are related to \( N(n) \) and \( E(n) \) by the conditions

\[ \sum_{i=1}^{n} \psi_{\alpha \beta}(\lambda_i) = N(n), \quad \sum_{i=1}^{n} \lambda_i \psi_{\alpha \beta}(\lambda_i) = E(n). \quad (2) \]

Consider the subset \( \mathcal{A} \subset \mathcal{M}_3 \):

\[ \mathcal{A} = \left\{ \sum_{l=1}^{k} \left( \sum_{n_{l-1}}^{n_l} N_i - \psi_{\alpha \beta}(\overline{\lambda}_l) \right) \leq \Delta \right\}, \quad (3) \]
where
\[ \Delta = \begin{cases} 
\sqrt{N \ln^{1/2+\epsilon} N} & \text{for } N \ll n, \\
\sqrt{n \ln^{1/2+\epsilon} n} & \text{for } N \sim n, \\
\frac{N}{\sqrt{n}} \ln^{1/2+\epsilon} n & \text{for } N \gg n
\end{cases} \] (4)
is called the resolving power.

**Theorem 1** The following inequality holds:
\[ \frac{N(M_3 \setminus A)}{N(M_3)} \leq C n^k + \frac{C}{N^k}, \]
where \( k \) is arbitrary and \( C \) is a constant independent of \( n \) and \( N \).

**Proof.** Obviously,
\[ N(M_3 \setminus A) = \sum_{\{N_i\}} \Theta \left\{ N(n) E(n) - \sum_{i=1}^n N_i \lambda_i \right\} \delta_{\sum_{i=1}^n N_i, N(n)} \]
(5)
\[ \sum_{\{N_i\}} \left( \times \Theta \left\{ \left| \sum_{l=1}^k \left( \sum_{i=n_{l-1}}^{n_l} N_i - \psi_{\alpha\beta}(\lambda_l) \right) \right| - \Delta \right\} \right). \]
Here the sum is taken over all integers \( N_i \), \( \Theta(x) \) is the Heaviside function,
\[ \Theta(x) = \begin{cases} 
1 & \text{for } x \geq 0 \\
0 & \text{for } x < 0,
\end{cases} \]
and \( \delta_{k_1, k_2} \) is the Kronecker delta,
\[ \delta_{k_1, k_2} = \begin{cases} 
1 & \text{for } k_1 = k_2, \\
0 & \text{for } k_1 \neq k_2.
\end{cases} \]

Let us use the integral representations
\[ \delta_{N, N'} = \frac{e^{-\nu N}}{2\pi} \int_{-\pi}^{\pi} d\phi e^{-iN\phi} e^{\nu N'} e^{iN'\phi}, \]
(6)
\[ \Theta(y) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dx \frac{1}{x - i} e^{\beta y(1+ix)}. \] (7)
We have
\[ \int_0^\infty dE \Theta \left( E - \sum_{i=1}^n N_i \lambda_i \right) e^{-\beta E} = \int_0^\infty dE e^{-\beta E} = \frac{e^{-\beta \sum_{i=1}^n N_i \lambda_i}}{\beta}. \tag{8} \]

Denote
\[ Z(\beta, N) = \sum_{\{N_i\}} e^{-\beta \sum_{i=1}^n N_i \lambda_i}, \quad \zeta(\nu, \beta) = \prod_{l=1}^k \zeta_l(\nu, \beta), \]
\[ \zeta_l(\nu, \beta) = \prod_{i=n_l-1}^{n_l} \xi_i(\nu, \beta), \quad \xi_i(\nu, \beta) = \frac{1}{1 - e^{\nu - \beta \lambda_i}}, \quad i = 1, \ldots, n, \]
and
\[ \Gamma(E, N) = \mathcal{N}\{\mathcal{M}_3\}. \]

Since \( \mathcal{N}\{\mathcal{M}_3\}(E) < \mathcal{N}\{\mathcal{M}_3\}(E + \epsilon) \) for \( \epsilon > 0 \), we have
\[ Z(\beta, N) \geq \beta \int_E^\infty dE' \Gamma(E', N)e^{-\beta E'} = \Gamma(E, N)e^{-\beta E}. \tag{9} \]

Therefore,
\[ \mathcal{N}\{\mathcal{M}_3\} \leq Z(\beta, N)e^{\beta E}. \tag{10} \]

But, by (6),
\[ Z(\beta, N) = \frac{e^{-\nu N}}{2\pi} \int_{-\pi}^{\pi} d\phi \ e^{-iN\alpha} \zeta(\beta, \nu + i\alpha); \tag{11} \]

hence
\[ \mathcal{N}\{\mathcal{M}_3 \setminus \mathcal{A}\} \tag{12} \]
\[ \leq \left| \frac{e^{-\nu N + \beta E}}{2\pi} \int_{-\pi}^{\pi} \left[ \exp(-iN\phi) \sum_{\{N_j\}} \left( \exp \left\{ \left( -\beta \sum_{j=1}^n N_j \lambda_j + (i\phi + \nu)N_j \right) \right\} \right) \right] d\phi \]
\[ \times \Theta \left( \left| \sum_{l=1}^k \left( \sum_{j=n_l-1}^{n_l} N_j - \psi_\alpha(\lambda_l) \right) \right| - \Delta \right) \right|, \]

where \( \beta \) and \( \nu = -\ln \alpha \) are real parameters for which the series is convergent.
Estimating the right-hand side, carrying the modulus through the integral sign and then through the sign of the sum, and integrating over \( \phi \), we obtain

\[
\mathcal{N}\{M_3^n \setminus \mathcal{A}\} \leq e^{-\nu N} \exp \beta E(n) \sum_{\{N_i\}} \exp \left\{-\beta \sum_{i=1}^{n} N_i \lambda_i + \nu N_i \right\} 
\times \Theta \left\{ \left| \sum_{l=1}^{k} \left( \sum_{i=n_l-1}^{n_l} N_i - \psi_{\alpha\beta}(\lambda_l) \right) \right| - \Delta \right\}. 
\]

Let us use the following inequality for the hyperbolic cosine \( \cosh(x) = (e^x + e^{-x})/2 \):

\[
\prod_{l=1}^{k} \cosh(x_l) \geq 2^{-k} e^\delta \quad \text{for all} \quad x_l, \quad \sum_{l=1}^{k} |x_l| \geq \delta \geq 0.
\]

Hence, for all positive \( c \) and \( \Delta \), we have the inequality (cf. [2, 3])

\[
\Theta \left\{ \left| \sum_{l=1}^{k} \left( \sum_{i=n_l-1}^{n_l} N_i - \psi_{\alpha\beta}(\lambda_l) \right) \right| - \Delta \right\} \leq 2^k e^{-\nu \Delta} \prod_{l=1}^{k} \cosh \left( c \sum_{i=n_l-1}^{n_l} N_i - c \psi_{\alpha\beta}(\lambda_l) \right).
\]

We obtain

\[
\mathcal{N}\{M_3 \setminus \mathcal{A}\} \leq 2^k e^{-\nu \Delta} \exp(\beta E(n) - \nu N) \exp \left\{-\beta \sum_{i=1}^{n} N_i \lambda_i + \nu N_i \right\} \prod_{l=1}^{k} \cosh \left( \sum_{i=n_l-1}^{n_l} cN_i - c \psi_{\alpha\beta}(\lambda_l) \right)
\]

\[
= e^{\beta E(n)} e^{-\nu N} e^{-c\Delta} \prod_{l=1}^{k} \left( \zeta_l(\nu + c, \beta) \exp(-c \psi_{\alpha\beta}(\lambda_l)) + \zeta_l(\nu - c, \beta) \exp(c \psi_{\alpha\beta}(\lambda_l)) \right).
\]

Let us apply Taylor’s formula to \( \zeta_l(\nu + c, \beta) \). Namely, there exists a \( \gamma < 1 \) such that

\[
\ln(\zeta_l(\nu + c, \beta)) = \ln(\zeta_l(\nu, \beta)) + c(\ln(\zeta_l)'(\nu, \beta) + \frac{c^2}{2}(\ln(\zeta_l)''(\nu + c, \beta)).
\]

Obviously,

\[
\frac{\partial}{\partial \nu} \ln(\zeta_l) \equiv \psi_{\alpha\beta}(\lambda_l).
\]
Let $c = \Delta / D(\nu, \beta)$, where $D(\nu, \beta) = (\ln \zeta(\nu, \beta))''$.

The right-hand side of relation (16) is equal to

$$2^k e^{\beta E(n)} e^{-\nu N} \prod_{i=1}^{k} \zeta_i(\nu, \beta) \exp \left\{ -\frac{\Delta^2}{D(\nu, \beta)} + \frac{\Delta^2 D(\nu + \gamma \Delta / D(\nu, \beta), \beta)}{2(D(\nu, \beta))^2} \right\}.$$ 

Imposing the following constraint on $\Delta$:

$$D\left(\nu + \frac{\Delta}{D(\nu, \beta)}, \beta\right) \leq (2 - \epsilon) D(\nu, \beta),$$

where $\epsilon > 0$, and taking into account the fact that $D(\nu, \beta)$ is monotone increasing in $\nu$, we finally obtain

$$N(\mathcal{M}_3 \setminus A) \leq 2^k e^{\beta E(n)} e^{-\nu N} \zeta(\nu, \beta) e^{-\epsilon \Delta^2 / D(\nu, \beta)}.$$ 

Next, let us estimate $\zeta(\nu, \beta)$.

The following lower bound for $Z(\beta, N)$ was obtained in [2], relation (95):

$$\zeta(\nu', \beta) \leq \sqrt{27D(\nu', \beta)} Z(\beta, N) e^{\nu' N},$$

where $\nu' = \nu'(\beta, N)$ is determined from the condition

$$\sum_{i=1}^{n} \xi_i(\nu, \beta) = N. \quad (18)$$

Suppose that $\beta = \beta'$ is determined from the condition

$$\sum_{i=1}^{n} \lambda_i \xi_i(\nu, \beta) = E(n). \quad (19)$$

Since $Z(\beta, N)$ is determined by the integral (11), its asymptotics given by the saddle-point method (the stationary phase due to Laplace) yields a unique saddle point for $\alpha = 0$.

The square root of the second derivatives with respect to $\alpha$ will appear in the denominator. As a result, we obtain

$$\zeta(\nu', \beta') \leq C e^{N \nu'} e^{-\beta' E(\mu)} D(\nu', \beta') N\{\mathcal{M}_3\},$$

$$\text{7}$$
where $C$ is a constant. Therefore, we finally obtain

$$\frac{\mathcal{N}(\mathcal{M}_3 \setminus \mathcal{A})}{\mathcal{N}(\mathcal{M}_3)} \leq 2^k C D(\nu', \beta') e^{-\epsilon \Delta^2 / D(\nu', \beta')}.$$  \hspace{1cm} (20)

Further, it is easy to estimate $D(\nu, \beta)$ as a function of $N, n$: $D \sim N$ for $N < n$, while, for $n \gg N$, the estimate for $D$ yields the relation $D \sim N^2 / n$. Hence we obtain the estimate for $\mathcal{N}(\mathcal{M}_3 \setminus \mathcal{A}) / \mathcal{N}(\mathcal{M}_3)$, given in the theorem.

**Example 3** For the case in which $s > 0$ is sufficiently small (and hence, $\sum_{i=1}^k N_i = N$ not very large), the Bose–Einstein distribution is of the form

$$\frac{N_i}{N_{i+1}} \sim \sum_{i=n_i-1}^{n_i} e^{-\lambda_i \beta} \sum_{i=n_i-1}^{n_i} e^{-\lambda_i \beta},$$  \hspace{1cm} (21)

where $\beta = 1/(kT)$, $T$ is the temperature, and $k$ is the Boltzmann constant.

In the case of a Bose gas, for $s < 1$, we have a distribution of Gibbs type, i.e., the ratio of the number of particles on the $l$th interval to the number of particles on the $(l+1)$th interval obeys formula (21).

**Example 4** In the case $s > 1$, we obtain a refinement of the Zipf–Mandelbrot law [5], namely,

$$\frac{N_i}{n} \sim \left\{ \sum_{i=n_i-1}^{n_i} \frac{1}{\lambda_i + \nu} \right\}.$$  \hspace{1cm} (22)

However, if $s$ is close to 1, then it is better to use relation (b) in (1), which uniformly passes into relation (c) and relation (a).

Note that if all the $\lambda_i$ on the $l$th interval are identical and equal to $\lambda^{(l)}$, then $N_i / n_l \sim 1 / \lambda^{(l)}$, and since $n_l \sim N_l^{1/s}$, it follows that, in this case, we obtain the Zipf–Mandelbrot formula.

**Example 5** (relation between the sales volume and the prices on the stock market) Let us now consider the relation between the prices and the number of sold (bought) shares of some particular company on the stock market.

Since the number $n_i$ of sold shares of that company during the $i$th day is equal to the number of bought shares and $\lambda_i$ is the price of the shares at the

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1 A lower bound for $\mathcal{N}(\mathcal{M}_3)$ was obtained by G. V. Koval’ and the author in [4] without recourse to the saddle-point method.
end of the day, we set, averaging over $n_l$ days, the nonlinear average price as

$$
\bar{\lambda} = \Phi \left( \frac{\sum_{i=n_l}^{n_{l+1}} \phi(\lambda_i)}{n_{l+1} - n_l} \right),
$$

where $\phi(x) = 1/(x + \nu)$, $\nu = \text{const}$, and $\Phi(x)$ is the function inverse to $\phi(x)$. Then, by Theorem 1, we have

$$
n_l \simeq A \phi(\bar{\lambda}_l),
$$

where $A$ is a constant.

Thus, the stock market obeys the refined Zipf–Mandelbrot law if all the types of transactions are equiprobable (see [6, 7]).

In conclusion, note that although the theorem is stated in terms of set theory owing to the fact that we have introduced the notion of equivalent mappings, it belongs, most likely, to number theory. Under the same conditions, considering the set of mappings of the set $M_1^{(n)}$ onto the set $M_2^{(n)}$ without the condition for the equivalence of mappings, i.e., considering all mappings, we can obtain a similar theorem that will only be relevant to the refinement of the Gibbs distribution. At the same time, such a theorem is related to information theory and a generalization of Shannon’s entropy. Here the estimate has special features, and the corresponding article will be published jointly with G. V. Koval’.

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