Degree of nearly comonotone approximation of periodic functions

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Abstract. Let a $2\pi$-periodic function $f \in C$ changes its monotonicity at a finitely even number of points $y_i$ of the period. The degree of approximation of this $f$ by trigonometric polynomials which are comonotone with it, i.e. that change their monotonicity exactly at the points $y_i$ where $f$ does, is restricted by $\omega_2(f, \pi/n)$ (with a constant depending on the location of these $y_i$). Recently, we proved that relaxing the comonotonicity requirement in intervals of length proportional to $\pi/n$ about the points $y_i$ (so called nearly comonotone approximation) allows the polynomials to achieve the approximation rate of $\omega_3$. By constructing a counterexample, we show here that even with the relaxation of the requirement of comonotonicity for the polynomials on sets with measures approaching 0 (no matter how slowly or how fast) $\omega_4$ is not reachable.

1 Introduction

1. Let $C := C_\mathbb{R}$ and $C^r$ denote, respectively, the space of continuous $2\pi$-periodic functions $f : \mathbb{R} \to \mathbb{R}$, and that of $r$-times continuously differentiable functions on the real axis $\mathbb{R}$, equipped with the uniform norm

$$\|f\| := \|f\|_\mathbb{R} := \max_{x \in \mathbb{R}} |f(x)|;$$

$\mathbb{T}_n$, $n \in \mathbb{N}$, is the space of trigonometric polynomials $P_n(x) = a_0 + \sum_{j=1}^n (a_j \cos jx + b_j \sin jx)$ of degree $\leq n$ (of order $\leq 2n + 1$) with $a_j, b_j \in \mathbb{R}$. By

$$E_n(f) := \inf_{P_n \in \mathbb{T}_n} \|f - P_n\|$$

we denote the value (error) of the best uniform approximation of the function $f$ by polynomials $P_n \in \mathbb{T}_n$. For any bounded $2\pi$-periodic function $f$, and $k \in \mathbb{N}$, the $k$-th symmetric difference of $f$ at the point $x$ with the step $h \geq 0$ is defined as

$$\Delta_h^k f(x) := \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} f(x - \frac{k}{2} h + ih), \quad x \in [-\pi, \pi],$$

and the (ordinary) $k$-th (order) modulus of continuity (or smoothness) of $f \in C$ is defined as

$$\omega_k(f, t) := \sup_{0<h\leq t} \sup_{x \in [-\pi, \pi]} |\Delta_h^k f(x)|.$$

We recall the classical Jackson-Zygmund-Akhiezer-Stechkin estimate (obtained by Jackson for $k = 1$ [10, 12], Zigmund for $k = 2$, $\omega_2(f, t) \leq t$, [25], Akhiezer for $k = 2$ [1] and Stechkin for $k \geq 3$ [23]): If a function $f \in C$, then

$$E_n(f) \leq c(k) \omega_k(f, \pi/n), \quad n \in \mathbb{N},$$

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where \( c(k) \) is a constant that depends only on \( k \). for details see, for example, [3, Section 4]. And hence, in particular, if \( f \in \mathbb{C}^r, \ r \in \mathbb{N} \), then

\[
E_n(f) \leq \frac{c(r, k)}{n^r} \omega_k \left( f^{(r)}; \pi/n \right), \quad n \in \mathbb{N}.
\] (1.2)

In 1968 Lorentz and Zeller [16, 17] proved a bell-shaped analogue of the estimate (1.1) with \( k = 1 \), that is, for approximation of bell-shaped (i.e. even and nonincreasing on \([0, \pi]\)) functions from \( \mathbb{C} \) by bell-shaped polynomials from \( \mathbb{T}_n \), and thus gave rise to the search for its other analogues, i.e. with other restrictions on the shape of the function and polynomials such as piecewise positivity, monotonicity, convexity (now this is called Shape Preserving Approximation, see, for example, the surveys of Kopotun, Leviatan, Prymak, Shevchuk [11, 12]).

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2'. In the paper of Pleshakov [19], and in [6] a comonotone analogue of (1.1) is proved with \( k = 1 \) and \( k = 2 \), respectively. To write it we need some notations. Let on \([-\pi, \pi]\) there are \( 2s, \ s \in \mathbb{N} \), fixed points

\[
y_i : -\pi \leq y_{2s} < y_{2s-1} < \ldots < y_1 < \pi,
\]

and for the rest \( i \in \mathbb{Z} \), the points \( y_i \) are defined by the equality \( y_i = y_{i+2s} + 2\pi \) (i.e. \( y_0 = y_{2s} + 2\pi, \ldots, y_{2s+1} = y_1 - 2\pi, \ldots \)). Let \( Y_s := \{y_i\}_{i \in \mathbb{Z}} \). We denote by \( \Delta^{(1)}(Y_s) \) the collection of all functions \( f \in \mathbb{C} \) that are nondecreasing on \([y_1, y_0]\), nonincreasing on \([y_2, y_1]\), nondecreasing on \([y_3, y_2]\) and so on. Thus, if \( f \in \mathbb{C}^1 \) then \( f \in \Delta^{(1)}(Y_s) \Leftrightarrow f'(x)\Pi(x) \geq 0, \ x \in \mathbb{R} \), where

\[
\Pi(x) := \Pi(x, Y_s) := \prod_{i=1}^{2s} \sin \frac{1}{2}(x - y_i)
\]

(\( \Pi(x) > 0, \ x \in (y_1, y_0), \ \Pi \in \mathbb{T}_s \)). The functions from \( \Delta^{(1)}(Y_s) \) are called piecewise monotone or comonotone (each other or between themselves), and the approximation of them by polynomials also from \( \Delta^{(1)}(Y_s) \) is called comonotone approximation.

Let

\[
E_n^{(1)}(f, Y_s) := \inf_{P_n \in \mathbb{T}_n \cap \Delta^{(1)}(Y_s)} \|f - P_n\|
\]

is the value (error) of the best uniform approximation of the function \( f \) by polynomials \( P_n \in \mathbb{T}_n \cap \Delta^{(1)}(Y_s) \).

So, it is proved in [19] and [6] that: If a function \( f \in \Delta^{(1)}(Y_s) \), then there exists a constant \( N(Y_s) \), which depends only on \( \min_{i=1, \ldots, 2s} \{y_i - y_{i+1}\} \), such that

\[
E_n^{(1)}(f, Y_s) \leq c(s) \omega_k(f; \pi/n), \quad k = 1, 2, \ n \geq N(Y_s),
\] (1.3)

where \( c(s) := \max_{k=1, 2} c(s, k) \) is a constant depending only on \( s \).

Note that the estimate (1.3) below is a simple consequence of (1.3) and the Whitney inequality \( [21] \|f - f(0)\| \leq c \omega_k(f; 2\pi) \) with an absolute constant \( c \) and \( k \in \mathbb{N} \), (since \( f(0) \in \Delta^{(1)}(Y_s) \cap \mathbb{T}_0 \) and at least twice interpolates \( f \) on the period): If \( f \in \Delta^{(1)}(Y_s) \), then

\[
E_n^{(1)}(f, Y_s) \leq C(Y_s) \omega_2(f; \pi/n), \quad n \in \mathbb{N},
\] (1.4)
where \( C(Y_s) \) is a constant depending only on \( \min_{i=1,\ldots,2s} \{y_i - y_{i+1}\} \).

Moreover, Pleshakov and Tyshkevich [18] using considerations of the papers of Shvedov [22] and DeVore, Leviatan, Shevchuk [2], for each \( n \in \mathbb{N} \), constructed a function \( g_n(x) = g_n(x, Y_s) \in \Delta^{(1)}(Y_s) \) such that

\[
E_n^{(1)}(g_n, Y_s) \geq C(Y_s) \sqrt{n} \omega_3(g_n, \pi/n)
\]

for some \( C(Y_s) > 0 \). In other words, for each \( n \in \mathbb{N} \), they found a function from \( \Delta^{(1)}(Y_2) \), for which the inequality (1.3) is invalid with \( \omega_k, \ k > 2 \), i.e. it is impossible to improve it in the order of the modulus of smoothness (unlike (1.1) and (1.2) of approximation without restrictions, which hold for all \( k \in \mathbb{N} \)). In [4], one such function is constructed (for all \( n \in \mathbb{N} \)), i.e. it is proved that in the set \( \Delta^{(1)}(Y_s) \) there exists a function \( g(x) = g(x, Y_s) \) such that

\[
\limsup_{n \to \infty} \frac{E_n^{(1)}(g, Y_s)}{\omega_3(g, \pi/n)} = \infty.
\]

3°. However, from the result of Leviatan and Shevchuk [14] on approximation on a segment by algebraic polynomials, we knew that relaxing the comonotonicity requirement in small intervals about the points \( y_i \) (so called nearly comonotone approximation) allows the polynomials to achieve one additional approximation rate, and it was proved in [5] that: If \( f \in \Delta^{(1)}(Y_s) \) then there exists a constant \( N(Y_s) \), which depends only on \( \min_{i=1,\ldots,2s} \{y_i - y_{i+1}\} \), and for every \( n \geq N(Y_s) \) there exists a polynomial \( P_n \in T_{cn} \) such that

\[
P_n'(x) \Pi(x) \geq 0, \quad x \in \mathbb{R} \setminus \bigcup_{i \in \mathbb{Z}} (y_i - \pi/n, y_i + \pi/n),
\]

where \( c \) and \( c(s) \) are constants depending only on \( s \).

Like (1.4) the estimate (1.6) below follows from (1.5) and the Whitney inequality [24]: If \( f \in \Delta^{(1)}(Y_s) \) then for every \( n \in \mathbb{N} \) there exists \( P_n \in T_n \) such that

\[
P_n'(x) \Pi(x) \geq 0, \quad x \in \mathbb{R} \setminus \bigcup_{i \in \mathbb{Z}} (y_i - c/n, y_i + c/n),
\]

where \( c \) and \( C(Y_s) \) are constants depending only on \( s \) and \( \min_{i=1,\ldots,2s} \{y_i - y_{i+1}\} \).

4°. In this paper we prove Theorem 1 below. For any \( \epsilon > 0 \) and \( f \in \Delta^{(1)}(Y_s) \) let

\[
E_n^{(1)}(f, \epsilon, Y_s) := \inf_{P_n} \|f - P_n\|
\]

where the infimum is taken over all polynomials \( P_n \in T_n \) satisfying

\[
\text{meas}(\{x : P_n'(x) \Pi(x, Y_s) \geq 0\} \cap [-\pi, \pi]) \geq 2\pi - \epsilon.
\]

(Obviously, \( E_n^{(1)}(f, Y_s) = E_n^{(1)}(f, 0, Y_s) \).)

**Theorem 1** For each \( s \in \mathbb{N} \) and each sequence \( \bar{\epsilon} = \{\epsilon_n\}_{n=1}^\infty \) of nonnegative numbers tending to 0, there exist a set \( Y_s \) and a function \( f = f_{\bar{\epsilon}} \in \Delta^{(1)}(Y_s) \) such that

\[
\limsup_{n \to \infty} \frac{E_n^{(1)}(f, \epsilon_n, Y_s)}{\omega_4(f, \pi/n)} = \infty.
\]
Remark 1 The considerations in the proof of Theorem 1 are inspired by their algebraic analogue, namely, by the paper of Leviatan and Shevchuk [15], and their joint paper with DeVore [2]. Partially these considerations were also used in [7, Example 3.2], [4] and [8, Theorem 4.1].

Remark 2 In the algebraic case, i.e. in [15], a stronger result is obtained, namely, the algebraic analogue of Theorem 1 is stated there for an arbitrary $Y_s \subset (-1, 1)$, and this is natural, because no need to regard the periodicity. It seems that for an arbitrary $Y_s$ in the trigonometric case, it is necessary to construct a different counterexample than in the proof of Theorem 1.

2 Construction of a counterexample
(Proof of Theorem 1)

As we said, this paper is inspired by the work of Leviatan and Shevchuk [15] and their joint work with DeVore [2], so now we will follow along the lines of [15] and [2] with the following main differences: instead of the Chebyshev and truncated Chebyshev (algebraic) polynomials we will use (have to use) others (trigonometric), to preserve the periodicity we will use not an arbitrary $Y_s$ rather a specific $Y_s^*$ (with equidistant $y_i$’s) and we will change the definition of the function $g_{j,M}$ in [15] for a more precise description of the continuity of the function $g$ (see [15] after (2.7)).

1°. Let $Y^* := Y_s^* = \{y_i := \pi + \frac{\pi}{2s} - i\frac{\pi}{s}, \ i = 1, ..., 2s\}, \ s \in \mathbb{N}$ is even for definiteness, $[y_{s+1}, y_s]$ is the interval containing 0 and $I_0 := [-\frac{\pi}{3s}, \frac{\pi}{3s}]$ is its middle two thirds (central four parts of six), so that, in particular, $\Pi(0) > 0$, $\|\Pi\| = \|\Pi\|_{[-\pi, \pi]} = \frac{1}{2^{2s-1}}$, and

$$\min_{x \in I_0} |\Pi(x)| = \Pi(\pm \frac{\pi}{3s}) = \frac{1}{2^{2s}}, \quad (2.1)$$

here and in the sequel $\Pi = \Pi(x) = \Pi(x, Y^*)$ (one can verify the both equalities using math software).

For $\nu > 10s$ and even for definiteness, let

$$t_\nu(x) := \cos \nu x,$$

with the extrema $x_j := \pi - \frac{j\pi}{\nu}, \ -\infty < j < \infty$. Given $0 < b < \frac{1}{\nu}$, we take two points on both side of $x_j$, namely, set $x_{j,l} := \pi - \frac{(j+b)\pi}{\nu}$ and $x_{j,r} := \pi - \frac{(j-b)\pi}{\nu}$. Then

$$|t_\nu(x_{j,l})| = |t_\nu(x_{j,r})| = \cos \pi b,$$

and

$$x_{j,r} - x_{j,l} = 2\pi \frac{b}{\nu}. \quad (2.2)$$

We also need the truncated $t_\nu$, namely,

$$t^*_\nu(x) := t^*_{\nu,b}(x) := \begin{cases} -\cos \pi b, & \text{if } t_\nu(x) < -\cos \pi b, \\ t_\nu(x), & \text{otherwise.} \end{cases}$$

Put

$$t_{\nu,b} := 2^{2s}(t_\nu + \cos \pi b)\Pi, \quad \text{and} \quad \tilde{t}_{\nu,b} := 2^{2s}(t^*_{\nu,b} + \cos \pi b)\Pi.$$
It is readily seen that
\[
\| \tilde{t}_{\nu,b} \| \leq \| t_{\nu,b} \| = \| t_{\nu,b} \|_{[-\pi,\pi]} \leq 2 \cdot 2^s \| \Pi \|_{[-\pi,\pi]} = 4. \tag{2.3}
\]

Now, we define a function \( g = g_{\nu,b} \) on \([-\pi, \pi]\) by defining it on each interval \([x_{2j+1}, x_{2j-1}] \cap [-\pi, \pi], \ j = 0, ..., \nu, \) (containing one positive extrema of \( t_{\nu} \) at its center and at an end-point for \( j = 0, \nu \)). Note that there are at most 4s intervals \([x_{2j+1}, x_{2j-1}] \) such that \( Y^* \cap (x_{2j+2}, x_{2j-2}) = \emptyset \) (roughly speaking either on both sides of each \( y_i \) or directly on it) we call them type I and put
\[
g(x) := \tilde{t}_{\nu,b}(x), \quad x \in [x_{2j+1}, x_{2j-1}].
\]
All other intervals \([x_{2j+1}, x_{2j-1}] \cap [-\pi, \pi], \ j = 0, ..., \nu, \) are of type II and satisfy \( Y^* \cap [x_{2j+1}, x_{2j-1}] = \emptyset. \) For these intervals the definition of \( g \) requires some preliminaries. Note that for each \( 1 \leq i \leq 2s \) and \( x \in [x_{2j+1}, x_{2j-1}], \) in view of \( \sin t > \frac{3}{\pi} t, \) for \( 0 < t < \frac{\pi}{6}, \) we have
\[
\frac{1}{\pi} \leq \frac{\sin \frac{x-y_i}{2}}{\sin \frac{x+y_i}{2}} \leq \frac{2\pi}{3}, \tag{2.4}
\]
which in turn implies
\[
(\frac{1}{\pi})^{2s} \leq \frac{\Pi(x)}{\Pi(x_{2j})} \leq (\frac{2\pi}{3})^{2s}. \tag{2.5}
\]
Indeed, without loss of generality assume that \( y_i \leq x_{2j+2}. \) Then if \( x > x_{2j}, \) then
\[
\frac{2}{\pi} \leq \frac{\sin \frac{\pi}{\nu}}{\sin \frac{3\pi}{2\nu}} \leq \frac{\sin \frac{x-y_i}{2}}{\sin \frac{x+y_i}{2}} \leq \frac{\sin \frac{2\pi}{\nu}}{\sin \frac{\pi}{\nu}} \leq \frac{2\pi}{3},
\]
while if \( x < x_{2j}, \) then
\[
\frac{\pi}{2} \geq \frac{\sin \frac{3\pi}{2\nu}}{\sin \frac{\pi}{\nu}} \geq \frac{\sin \frac{x-y_i}{2}}{\sin \frac{x+y_i}{2}} \geq \frac{\sin \frac{\pi}{2\nu}}{\sin \frac{3\pi}{2\nu}} \geq \frac{1}{\pi}.
\]
Thus (2.4) is proved (arithmetic with math tools gives \( \frac{2}{\pi} \leq \frac{\sin \frac{x-y_i}{2}}{\sin \frac{x+y_i}{2}} \leq \frac{6}{\pi} \)). Now, in view of (2.5), it follows that there is a constant \( B < 1/2, \) depending only on \( s, \) such that if \( b < B, \) then
\[
\Pi(x_{2j}) \int_{x_{2j+1}}^{x_{2j-1}} t_{\nu,b}(x) \, dx > 0. \tag{2.6}
\]
Let \( m_j := \| \tilde{t}_{\nu,b} \|_{[x_{2j+1}, x_{2j-1}]} \). For each \( 0 \leq M \leq 4, \) where 4 is from (2.3), and \( x \in [x_{2j+1}, x_{2j-1}], \) we set
\[
g_{j,M}(x) := \begin{cases} \frac{M}{m_j} \tilde{t}_{\nu,b}(x), & \text{if } m_j > M, \\ \tilde{t}_{\nu,b}(x), & \text{otherwise.} \end{cases}
\]
We now define the function \( g \) in intervals \([x_{2j+1}, x_{2j-1}] \) of type II where \( \Pi(x_{2j}) > 0. \) The case where \( \Pi(x_{2j}) < 0 \) requires obvious modifications in its proof and is left without specification. Since
\[
\int_{x_{2j+1}}^{x_{2j-1}} (t_{\nu,b}(x) - g_{j,M}(x)) \, dx = \int_{x_{2j+1}}^{x_{2j-1}} (t_{\nu,b}(x) - \tilde{t}_{\nu,b}(x)) \, dx < 0,
\]

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and by (2.6),
\[ \int_{x_{2j+1}}^{x_{2j-1}} (t_{\nu,b}(x) - g_{j,0}(x))dx = \int_{x_{2j+1}}^{x_{2j-1}} t_{\nu,b}(x)dx > 0, \]
hence a constant \( 0 < M_j < 4 \) exists such that
\[ \int_{x_{2j+1}}^{x_{2j-1}} t_{\nu,b}(x)dx = \int_{x_{2j+1}}^{x_{2j-1}} g_{j,M_j}(x)dx. \]  
(2.7)

Thus we define
\[ g(x) := g_{j,M_j}(x), \quad x \in [x_{2j+1}, x_{2j-1}]. \]

This completes the definition of \( g = g_{\nu,b} \), and in view of the fact that \( g(x_{2j+1}) = t_{\nu,b}(x_{2j+1}) = 0 \), for all \( x_{2j} \in [-\pi, \pi] \), \( j = 0, \ldots, \nu \), it follows that \( g \) is continuous in \( [-\pi, \pi] \) and hence in \( \mathbb{R} \), and \( 2\pi \)-periodic. We also note that
\[ g(x)\Pi(x) \geq 0, \quad x \in \mathbb{R}. \]  
(2.8)

Next, for any interval \( [x_{2j+1}, x_{2j-1}] \) of either type I or II, from (2.2) and (2.6) it follows that
\[ \int_{x_{2j+1}}^{x_{2j-1}} |t_{\nu,b} - g(x)|dx \]
\[ \leq 2 \left( \int_{x_{2j+1}}^{x_{2j+1}+\epsilon} |t_{\nu,b}|dx + \int_{x_{2j-1}+\epsilon}^{x_{2j-1}} |t_{\nu,b}|dx \right) \]
\[ \leq 2((x_{2j+1} - x_{2j+1}+\epsilon) + (x_{2j-1}+\epsilon - x_{2j-1}))2^{2s}(1 - \cos \pi b)||\Pi|| \]
\[ < 16\pi \frac{b}{\nu} 5b^2 =: C_0 \frac{b^3}{\nu}, \]
where we used the inequality \( 1 - \cos \pi b = 2\sin^2 \frac{\pi b}{2} < 5b^2 \). Here and in the sequel we denote by \( C_0, C_1, \ldots, C_9 \), different constants which may depend only on \( s \).

Given \( n \geq 1 \) and \( 0 < b < B \), let \( \nu = [b^{1/4}n] + 13 \), where \([\alpha]\) denotes the largest odd integer not exceeding \( \alpha \). Put
\[ T_{\nu,b} := \int_0^x t_{\nu,b}(u)du \quad \text{and} \quad f_{\nu,b} := \int_0^x g_{\nu,b}(u)du. \]

Since \( t_{\nu,b}, g_{\nu,b} \) II and \( t_{\nu} \) are even periodic functions and besides II and \( t_{\nu} \) have equidistant zeros, then it follows from (2.8) that \( f_{\nu,b} \in \Delta^{(1)}(Y^*) \). Also, in view of (2.2), the following estimate holds
\[ \|T_{\nu,b} - f_{\nu,b}\| = \|T_{\nu,b} - f_{\nu,b}\|_{[-\pi, \pi]} \leq C_0(4s + 2)\frac{b^3}{\nu} =: C_1 \frac{b^3}{\nu} \leq C_1 \frac{b^{9/4}}{n}, \]  
(2.10)

where we have counted the \( 4s \) intervals of type I, and possibly two intervals of type II, namely, those containing 0 and \( x \), respectively.

Set \( \overline{\pi}_{j,t} := \pi - \frac{(j+b/2)\pi}{\nu} \) and \( \overline{\pi}_{j,r} := \pi - \frac{(j-b/2)\pi}{\nu} \), and note
\[ \overline{\pi}_{j,r} - x_j = x_j - \overline{\pi}_{j,t} = \frac{\pi b}{2\nu}, \quad j \in \mathbb{Z}. \]  
(2.11)
Let $j$ be odd. Since $\sin \frac{b\pi}{4} > \frac{3b}{4}$ for $b$ satisfying $b\pi/4 < \pi/6$, then for $x \in [\overline{x}_{j,l}, \overline{x}_{j,r}]$, with (2.11), the following inequality holds

$$T'_{\nu,b}(x) = t_{\nu,b}(x) < -\cos \frac{b\pi}{2} + \cos \frac{b\pi}{4} = -2\sin \frac{b\pi}{4} \sin 3\frac{b\pi}{4} < -\frac{9b^2}{4}. \quad (2.12)$$

It follows by the Bernstein inequality that

$$\|T^{(4)}_{\nu,b}\| = \|t^{(3)}_{\nu,b}\|_{[-\pi,\pi]} \leq \|t_{\nu,b}\|_{[-\pi,\pi]} (\nu + 2s)^3 \leq 4(1 + 2s)^3 \nu^3 =: C_2 \nu^3. \quad (2.13)$$

Hence, by (2.10),

$$\omega_4(f_{n,b}, \pi/n) \leq \omega_4(f_{n,b} - T_{\nu,b}, \pi/n) + \omega_4(T_{\nu,b}, \pi/n) \leq (2\pi)^4 \|f_{\nu,b} - T_{\nu,b}\|_{[-\pi,\pi]} + \frac{\pi^4}{n^4} \|T^{(4)}_{\nu,b}\| \leq (2\pi)^4 C_1 \frac{b^{9/4}}{n} + \frac{\pi^4 C_2 \nu^3}{n^4} \leq C_3 \frac{b^{9/4}}{n}, \quad (2.14)$$

where the value of $\nu$ was also used.

**Lemma 2.1** For any interval $J \subseteq I_0$ there exists an absolute constant $C_4$ such that, if a polynomial $P_n$ satisfies

$$P_n'(x) \geq 0, \quad x \in J \setminus E, \quad (2.15)$$

where $E \subseteq I_0$ is any measurable set, then

$$\|f_{n,b} - P_n\|_J \geq \frac{b^2 |J|}{\pi n} - \frac{C_4}{n} \left( b^{9/4} + b|E| \right) + \frac{b^{5/4}}{n} \quad (2.16)$$

**Proof.** Let $J_0$ denotes the middle third of $J$. We consider two cases. First assume that $J_0$ contains at most one of the $x_j$'s such that $f'_{n,b}(x_j) = 0$, then by the definition of $\nu$ the following inequality holds

$$|J| < C_5 \frac{\pi}{\nu} < C_6 \frac{b^{-3/4}}{n},$$

and hence

$$\|f_{n,b} - P_n\|_J \geq 0 > \frac{b^2 |J|}{n} - C_6 \frac{b^{5/4}}{n^2}. \quad (2.17)$$

On the other hand, if $J_0$ contains at least two such extrema, then it contains at least $\frac{4}{\pi} C_7 \nu |J|$ of them for some constant $C_7$. These extrema satisfy (2.11), and about half of them (and at least one) have odd indices, then together with (2.12) we conclude that

$$\text{meas}(J_0 \cap \{ x : T'_{\nu,b}(x) < -\frac{9b^2}{4} \}) \geq \frac{1}{2} \frac{\pi b}{2 \nu} C_7 \nu |J| = C_7 b |J|. \quad (2.18)$$

Now, if $C_7 b |J| \leq |E|$, then

$$\|f_{n,b} - P_n\|_J \geq 0 \geq \frac{b^2 |J|}{n} - \frac{b|E|}{nC_7}. \quad (2.19)$$
Otherwise, by (2.18), there is a point \( x^* \in J_0 \setminus E \) such that
\[
T'_{\nu,b}(x^*) < -\frac{9b^2}{4}.
\]
Hence, (2.15) yields
\[
\frac{9b^2}{4} \leq P_n'(x^*) - T'_{\nu,b}(x^*) \leq \frac{2\pi}{|J|} n\|P_n - T_{\nu,b}\|_J,
\]
where we used a special case of I.I.Privalov’s Theorem (see in [21, p.96-98]) with sharp estimates proved in [13, Lemma 3.2]: for every \( R_n \in \mathbb{T}_n \) and each \( 0 < h \leq \pi \) the following inequality holds
\[
\|R_n'\|_{[-h/2,h/2]} \leq \frac{n}{\sin \frac{h}{2}} \|R_n\|_{[-h,h]} \leq \frac{\pi n}{h} \|R_n\|_{[-h,h]}.
\]
Therefore it follows from (2.10) that
\[
\frac{b^2 |J|}{\pi n} \leq \frac{9b^2}{4n} \frac{|J|}{2\pi} \leq \|P_n - T_{\nu,b}\|_J
\]
\[
\leq \|P_n - f_{n,b}\|_J + \|f_{n,b} - T_{\nu,b}\|_J
\]
\[
\leq \|P_n - f_{n,b}\|_J + C_1 \frac{b^{9/4}}{n}.
\]
Taking \( C_4 := \max\{C_6, 1/C_7, C_1\} \), (2.16) follows from combining (2.17), (2.19) and (2.20). Lemma 2.1 is proved.

\(2\). For a given sequence \( \bar{\varepsilon} = \{\varepsilon_n\} \) we define \( f = f_{\bar{\varepsilon}} \). Let \( b_n := (\max\{\varepsilon_n^2, \frac{1}{n}\})^{2/5} \), set \( d_0 := 1 \), and let
\[
d_j := \frac{b^{9/4}_n}{n_j} d_{j-1} = \prod_{\mu=1}^{j} \frac{b^{9/4}_{n_\mu}}{n_\mu}, \quad j \geq 1,
\]
where the sequence \( \{n_\mu\} \) is defined by induction as follows. First, we choose \( n_1 \) so large that \( b_{n_1} < \min\{B, |J_0|^8\} \) so that it satisfies (2.22) below and let \( J_0 := I_0 \). Suppose that \( \{n_1, ..., n_{\sigma-1}\} \) and \( J_{\sigma-2} \subseteq J_{\sigma-3} \subseteq ... \subseteq J_0 \), \( \sigma \geq 2 \), have been defined. Then put
\[
F_{\sigma-1} := \sum_{j=1}^{\sigma-1} d_{j-1} f_{n_j,b_{n_j}},
\]
and let \( J_{\sigma-1} \) be an interval such that \( J_{\sigma-1} \subseteq J_{\sigma-2} \) and
\[
F'_{\sigma-1}(x) = 0, \quad x \in J_{\sigma-1}.
\]
(The induction process will guarantee the existence of such intervals.) Let \( N_{1,\sigma} \) be such that
\[
\min\{B, |J_{\sigma-1}|^8\} \geq b_n, \quad n \geq N_{1,\sigma},
\]
and let
\[
N_{2,\sigma} := \left( \frac{\|F_{\sigma-1}^{(2)}\|}{d_{\sigma-1}} \right)^{10}.
\]
(2.23)
Finally, we take
\[ n_\sigma > \max\{n_{\sigma-1}, N_{1,\sigma}, N_{2,\sigma}\} \]
so big that the function \( f_{n_\sigma,b_{n_\sigma}}' \) oscillates a few times inside the interval \( J_{\sigma-1} \) and so it vanishes on some interval in each oscillation, that is, inside \( J_{\sigma-1} \) there exists an interval \( J_{\sigma} \subset J_{\sigma-1} \) as required in (2.21).

Now denote
\[ \Phi_\sigma := \sum_{j=\sigma}^{\infty} d_{j-1} f_{n_j,b_{n_j}}, \]
where the convergence of the series is justified by the definition of the \( d_j \)'s and the fact that \( \|f_{n,b_n}\| \leq 4\pi \), for all \( n \). Indeed,
\[
\|\Phi_\sigma\| \leq 4\pi d_{\sigma-1} \left( 1 + \frac{b_{n_\sigma}^{9/4}}{n_\sigma} + \frac{b_{n_\sigma}^{9/4} b_{n_{\sigma+1}}^{9/4}}{n_\sigma n_{\sigma+1}} + \ldots \right) \leq 4\pi d_{\sigma-1} \sum_{j=0}^{\infty} 2^{-j} = 8\pi d_{\sigma-1}. \tag{2.24}
\]

Also, since \( \|f_{n,b_n}'\| \leq 4 \), for all \( n \), then the series is differentiable term by term, which in turn implies that \( \Phi_\sigma \in \Delta^{(1)}(Y^*) \) for all \( \sigma \geq 1 \). So we define
\[ f = f_\epsilon := \sum_{j=1}^{\infty} d_{j-1} f_{n_j,b_{n_j}}, \]
then \( f \in \Delta^{(1)}(Y^*) \) and in particular \( f \) is monotone in \( I_0 \). Without loss of generality and according with our assumption of even \( s \) at the beginning of the section \( f \) is nondecreasing in \( I_0 \).

Lemma 2.2 For each \( \sigma \geq 1 \) we have
\[ \omega_4(f, \pi/n_\sigma) \leq C_8 d_\sigma. \tag{2.25} \]

Proof. By (2.24),
\[ \omega_4(\Phi_{\sigma+1}, \pi/n_\sigma) \leq \pi^4 \|\Phi_{\sigma+1}\| \leq 8\pi^5 d_\sigma. \tag{2.26} \]

Also, by (2.14),
\[ \omega_4(d_{\sigma-1} f_{n_\sigma,b_{n_\sigma}}, \pi/n_\sigma) \leq d_{\sigma-1} C_3 \frac{b_{n_\sigma}^{9/4}}{n_\sigma} = C_3 d_\sigma. \tag{2.27} \]

Finally,
\[ \omega_4(F_{\sigma-1}, \pi/n_\sigma) \leq \pi^2 \omega_2(F_{\sigma-1}, \pi/n_\sigma) \leq \pi^4 \frac{n_\sigma^2}{\|F_{\sigma-1}\|} \leq \pi^4 \frac{\|F_{\sigma-1}\|}{d_{\sigma-1}} \frac{1}{n_\sigma^{1/10}} \left( \frac{1}{n_\sigma^{2/5} b_{n_\sigma}} \right)^{9/4} d_\sigma \leq \pi^4 d_\sigma, \tag{2.28} \]
where we used (2.23) and the definitions of \( b_{n_\sigma}, d_\sigma \) and \( n_\sigma \). So (2.25) follows from combining (2.26), (2.27) and (2.28). Lemma 2.2 is proved.
Lemma 2.3  For any measurable $E \subset [-\pi, \pi]$ satisfying
\[ |E| \leq \epsilon_{n\sigma}, \quad (2.29) \]
and any polynomial $P_{n\sigma}$ satisfying
\[ P'_{n\sigma}(x) \geq 0, \quad x \in I_0 \setminus E, \quad (2.30) \]
there exists an absolute constant $C_9$ such that
\[ \| f - P_{n\sigma} \| \geq \left( \frac{b_{n\sigma}^{-1/8}}{\pi} - C_9 \right) d_{\sigma}. \quad (2.31) \]

Proof. Since $F_{\sigma-1}$ is constant on $J_{\sigma-1}$, we may write
\[ f(x) = d_{\sigma-1} f_{n\sigma, b_{n\sigma}}(x) + \Phi_{\sigma+1}(x) + M, \quad x \in J_{\sigma-1}. \quad (2.32) \]
Let
\[ Q_{n\sigma} := \frac{1}{d_{\sigma-1}} (P_{n\sigma} - M). \]
Then it follows from (2.30) that
\[ Q'_{n\sigma}(x) \geq 0, \quad x \in J_{\sigma-1} \setminus E. \]
Thus by virtue of Lemma 2.1,
\[ \| Q_{n\sigma} - f_{n\sigma, b_{n\sigma}} \|_{J_{\sigma-1}} \geq \frac{b_{n\sigma}^2 |J_{\sigma-1}|}{\pi n_{\sigma}} - \frac{C_4}{n_{\sigma}} \left( \frac{b_{n\sigma}^{9/4}}{n_{\sigma}} + b_{n\sigma} |E| + \frac{b_{n\sigma}^{5/4}}{n_{\sigma}} \right). \quad (2.33) \]
The definition of $n_{\sigma}$ and (2.22) yield
\[ b_{n\sigma}^2 |J_{\sigma-1}| = b_{n\sigma}^{17/8} \frac{|J_{\sigma-1}|}{b_{n\sigma}^{1/8}} \geq b_{n\sigma}^{17/8} \]
On the other hand, (2.29) and the definition of $b_{n\sigma}$ imply
\[ b_{n\sigma} |E| \leq b_{n\sigma} \epsilon_{n\sigma} \leq b_{n\sigma}^{9/4}, \]
and
\[ \frac{b_{n\sigma}^{5/4}}{n_{\sigma}} \leq b_{n\sigma}^{15/4} \leq b_{n\sigma}^{9/4}. \]
Hence (2.33) implies
\[ \| Q_{n\sigma} - f_{n\sigma, b_{n\sigma}} \|_{J_{\sigma-1}} \geq \frac{1}{n_{\sigma}} \left( \frac{b_{n\sigma}^{17/8}}{\pi - 3C_4 b_{n\sigma} b_{n\sigma}^{9/4}} \right) = \frac{b_{n\sigma}^{9/4}}{n_{\sigma}} \left( \frac{b_{n\sigma}^{-1/8}}{\pi - 3C_4} \right). \]
In other words,
\[ \| P_{n\sigma} - M - d_{\sigma-1} f_{n\sigma, b_{n\sigma}} \|_{J_{\sigma-1}} \geq \frac{b_{n\sigma}^{9/4}}{n_{\sigma}} \left( \frac{b_{n\sigma}^{-1/8}}{\pi - 3C_4} \right) = d_{\sigma} \left( \frac{b_{n\sigma}^{-1/8}}{\pi - 3C_4} \right). \]
In view of (2.32), it follows from (2.24) that,
\[
\| f - P_{n_0} \| \geq \| f - P_{n_0} \|_{J_{\sigma-1}} \geq \| P_{n_0} - M - d_{\sigma-1} f_{n_0, b_{n_0}} \|_{J_{\sigma-1}} - \| \Phi_{\sigma+1} \| \\
\geq (b_{n_0}^{-1/8}/\pi - (3C_4 + 8\pi)) d_{\sigma},
\]
and Lemma 2.3 is proved with \( C_9 := 3C_4 + 8\pi \).

3°. The proof of (1.7) now follows from Lemmas 2.2 and 2.3, namely,
\[
\limsup_{n \to \infty} \frac{E_{n}^{(1)}(f, \epsilon_n, Y^*)}{\omega_4(f, \pi/n)} \geq \limsup_{\sigma \to \infty} \frac{E_{n_\sigma}^{(1)}(f, \epsilon_{n_\sigma}, Y^*)}{\omega_4(f, \pi/n_\sigma)} \\
\geq \limsup_{\sigma \to \infty} \frac{1}{C_8} \left( b_{n_\sigma}^{-1/8}/\pi - C_9 \right) = \infty.
\]
Theorem \( \Box \) is proved.

References

[1] Akhiezer N. I. Lectures on Approximation Theory. — Moscow: Nauka, 1965. (in Russian)

[2] DeVore R. A., Leviatan D., Shevchuk I. A. Approximation of monotone functions: A counter example, Proceedings Curves and surfaces with applications in CAGD (Chamonix-Mont-Blanc, 1996), Nashville, TN: Vanderbilt Univ. Press, 1997, 95-102.

[3] Dzyadyk V. K. Introduction to the theory of uniform approximation of functions by polynomials. — Moscow: Nauka, 1977, 512 pp. (in Russian)

[4] Dzyubenko G. A. Contreexample in comonotone approximation of periodic functions, Transactions of Institute of Mathematics, the NAS of Ukraine, 5 (2008), No. 1, 113-123. (in Ukrainian)

[5] Dzyubenko G. A., Nearly comonotone approximation of periodic functions, Anal. Theory Appl., 33 (2017), 1, 74-92.

[6] Dzyubenko G. A., Pleshakov M. G. Comonotone Approximation of Periodic Functions, Mat. Zametki 83 (2008), no. 2, 199-209; Engl. transl. in Math. Notes 83 (2008), 180-189.

[7] Dzyubenko G. A., Gilewicz J., Shevchuk I. A., Piecewise monotone pointwise approximation, Constr. Approx., 14 (1998), 311-348.

[8] Dzyubenko G., Voloshyna V., Yushchenko L., Negative results in coconvex approximation of periodic functions, J. Approx. Theory, to appear.

[9] Jackson D. Über die Genauigkeit der Annäherung stetiger Funktionen durch ganze rationale Funktionen gegebenen Grades und trigonometrische Summen gegebener Ordnung, Göttingen (1911) (Thesis)

[10] Jackson D. On approximation by trigonometric sums and polynomials, Trans. Amer. Math. Soc., 13 (1912), 491-515.

[11] Kopotun K. A., Leviatan D., Prymak A., Shevchuk I. A. Uniform and pointwise shape preserving approximation by algebraic polynomials, Surveys in Approximation Theory, 6 (2011), 24-74.

[12] Kopotun K. A., Leviatan D., Shevchuk I. A. Uniform and pointwise shape preserving approximation (SPA) by algebraic polynomials: an update, SMAI Journal of Computational Mathematics S5 (2019), 99-108.

[13] Leviatan D., Motorna O. V., Shevchuk I. A. No Jackson-type estimates for piecewise \( q \)-monotone, \( q \geq 3 \), trigonometric approximation, will be published in Ukr. Math. J. Vol 74 No 5 (2022) (see also in Ukrainskyi Matematychnyi Zhurnal, Vol 74 No 5 (2022), 662-675; and on www.researchgate.net).
[14] Leviatan D., Shevchuk I. A., Nearly Comonotone Approximation, J. Approx. Theory, 95 (1998), 53-81.
[15] Leviatan D., Shevchuk I. A. Nearly comonotone approximation II, Acta Sci. Math. (Szeged), 66 (2000), 1151-135.
[16] Lorentz G. G., Zeller K. L. Degree of Approximation by Monotone Polynomials I, J. Approx. Theory, 1 (1968), 501-504.
[17] Lorentz G. G., Zeller K. L. Degree of Approximation by Monotone Polynomials II, J. Approx. Theory, 2 (1969), 265-269.
[18] Pleshakov M. G. Comonotone approximation of periodic functions of Sobolev classes. Candidate’s Dissertation. Saratov: Saratov State University, 1997. (in Russian)
[19] Pleshakov M. G., Comonotone Jackson’s Inequality, J. Approx. Theory, 99 (1999), 409-421.
[20] Pleshakov M. G., Tyshkevich S. V. One negative example of shape preserving approximation, Bulletin of the Saratov University. New series. Series Mathematics. Mechanics. Informatics, 14 (2014), 2, 144-150. (in Russian)
[21] Privalov A. A. Teoriya interpolirovaniya funktsei [Function Interpolation Theory]. Book 1. – Saratov: Saratov State University, 1990. – 230. (in Russian)
[22] Shvedov A. S. Orders of coapproximation of functions by algebraic polynomials, Mat. Zametki, 29 (1981), 1, 117-130. English transl. in Math. Notes 29 (1981), 63-70.
[23] Stechkin S. B. On the order of best approximations of continuous functions, Izv. USSR Academy of Sciences. Ser. mat., 15 (1951), No. 3, 219-242. (in Russian)
[24] Whitney H. On Functions with Bounded n-th Differences, J. Math. Pures Appl. 36 (1957), 9, 67-95.
[25] Zygmund A. Smooth functions, Duke Math. Journal, 12 (1945), 1, 47-76.