Quantum convolutional codes: fundamentals

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Abstract

We describe the theory of quantum convolutional error correcting codes. These codes are aimed at protecting a flow of quantum information over long distance communication. They are largely inspired by their classical analogs which are used in similar circumstances in classical communication. In this article, we provide an efficient polynomial formalism for describing their stabilizer group, derive an on-line encoding circuit with linear gate complexity and study error propagation together with the existence of on-line decoding. Finally, we provide a maximum likelihood error estimation algorithm with linear classical complexity for any memoryless channel.

1 Introduction

Quantum information science has been developed in the past two decades as a way to process information more efficiently than with classical means. It lead to great theoretical advances and to impressive experimental realizations (see [14, 12] for a review). The main results motivating the interest for quantum computation concern integer factorization [15] and unsorted database search [7]. Both contribute to the widely accepted idea that quantum computers are intrinsically more powerful than their classical analogs, and justify the ever increasing interest for this new model of computation.

In parallel to these developments, the difficulty of building quantum information processing devices has been thoroughly pointed out: the quantum world is extremely sensitive to interactions with its surrounding environment [19, 20, 21]. This process, called decoherence, is responsible for the instability of the fragile quantum superpositions necessary to obtain a speedup over classical computation [9]. In absence of any control over the decoherence process, these quantum devices would be turned into—at best—classical computers. Fortunately, the discovery of quantum error correction schemes [16], together with their fault-tolerant implementation [6] cleared the future of quantum computation: quantum codes protect from unwanted evolutions and noise, whereas their fault-tolerant implementation guarantees that, below a certain error rate, quantum information processing can be done without loss of coherence [4, 18, 6, 1].

However, generic encoded quantum computation requires a large overhead in costly quantum resources: up to now, only a single encoded qubit has been produced and manipulated.

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successfully on an experimental quantum processing unit [10]. On the other hand, quantum communication protocols—e.g. quantum key distribution—achieve the production of large numbers of qubits often represented by some degrees of freedom of light modes. In most of the protocols, the manipulation of quantum bits is very limited and errors occur mainly during the transmission—loss of photons, noise, etc. In the perspective of quantum communication, we develop a theory of quantum convolutional error correcting codes. These codes are largely inspired by their classical analogs [11, 8] and share many of their properties: efficient encoding and decoding circuits and an efficient maximum likelihood error estimation procedure for any memoryless channel. As in the classical context, these codes can deal with infinitely long streams of “to-be-protected” information without introducing unacceptable delays in the transmission.

The article is organized as follows: sec. 2 describes the structure of quantum convolutional codes and introduces an appropriate formalism; sec. 3 provides an encoding circuit for this class of codes; sec. 4 studies error propagation properties, and sec. 5 details the efficient maximum likelihood error estimation algorithm. Throughout the text, abstract concepts are readily applied to a previously introduced example of quantum convolutional code [13].

2 Structure of quantum convolutional codes

All error protection strategies share many common ingredients. First, they must define the structure in which quantum information will be stored and, as second step, explain how information can be manipulated within this structure. Quantum error correcting codes impose the information to be stored in a subspace of the total Hilbert space of the physical qubits. This subspace, \( \mathcal{C} \), is called the code subspace. \( \mathcal{C} \) is usually further decomposed into—e.g. single qubit—subspaces for which elementary operations are then provided.

However, to arrive at a practical definition of a quantum error correction scheme, it is usually necessary to further restrict the possibilities offered by the above general program. One such restriction leads to stabilizer codes. Those are often compared to classical linear codes: they are defined by a set of linear equations—called syndromes—which allow an efficient description of \( \mathcal{C} \) together with a great flexibility in their design. To facilitate the introduction of quantum convolutional codes, we will use abundantly the stabilizer formalism, even though convolutional codes can be generalized to a wider framework\(^1\).

More precisely, the code subspace \( \mathcal{C} \) of any stabilizer code is defined as the largest subspace stabilized by an Abelian group \( S \) acting on the \( N \) physical qubits of the code. In practice, \( S \) is a subgroup of the multiplicative Pauli group \( G_N = sp\{I, X, Y, Z\}^\otimes N \), where \( I, X, Y, Z \) are the well known Pauli matrices\(^2\). The description of \( \mathcal{C} \) is further simplified by the introduction of a set of independent generators \( \{M_i\} \) of \( S \). This leads to the definition of \( \mathcal{C} \) in terms of syndromes:

\[
\forall i, \ |\psi\rangle = M_i |\psi\rangle \Leftrightarrow |\psi\rangle \in \mathcal{C}.
\]

\(^1\)In particular, our main theorem concerning error propagation in sec. 4 does not rely on the stabilizer formalism.

\(^2\)\( I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), \( X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), \( Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \), \( Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \).
2.1 Definition

The particularity of convolutional codes is to impose a specific form to the generators of the stabilizer group such that on-line encoding, decoding and error correction become possible even in the presence of an infinitely long to-be-protected stream of information.

However, convolutional codes do not consider groups of qubits independently of each other: the encoding operation cannot be decomposed into a tensor product of encoding operations acting on a small number of qubits. By contrast, an \((N,K)\)-block code can protect such stream only by cutting it into successive \(K\)-qubit blocks. As a result, the code subspace defined by these independent applications can be decomposed as a tensor product of the \(N\)-qubit subspaces of each output block. Furthermore, increasing the parameter \(K\) is usually not an option as it requires, in most cases, a quadratic overhead in the complexity of the encoding circuit \cite{5} and, more dramatically, an exponentially growing complexity of the error estimation algorithm\(^3\).

Quantum convolutional codes are especially designed to offer an alternative to small block codes in counteracting the effect of decoherence and noise over long-distance communications while using a limited overhead of costly quantum resources.

**Definition 2.1 \(((n,k,m))\)-convolutional code** The stabilizer group, \(S\), for an \((n,k,m)\)-convolutional code is given by:

\[
S = \text{sp} \{ M_{j,i} = I^\otimes j \times n \otimes M_{0,i}, \ 1 \leq i \leq n - k, \ 0 \leq j \},
\]

where \(M_{0,i} \in G_{n+m}\). Above \(M_{j,i}\)’s are required to be independent and to commute with each other.

**Remark 2.1** As expected, the length of the code (i.e. the number \(N\) of physical qubits of the code) as well as the number of logical qubits are left unspecified. In fact, the maximum value of the integer \(j\) controls this length implicitly. However, and contrarily to block codes, this maximum value does not need to be known in advance for encoding and decoding qubits. Instead, it will be fixed a posteriori when the transmission ends. This specific issue will be addressed in sec. 3. Hence, in most situations the length of the code can simply be set to infinity. The only associated restriction is to consider operators whose support\(^4\) has size of order 1. This also explains why in Eq. (2) the \(M_{j,i}\)’s seem to have different length: in the rest of the article we simply assume that the operators are “padded” by identities on the right-most physical qubits to adjust them to the appropriate length.

With this remark in mind, the structure of the stabilizer group generators can be sum-

\(^3\)This holds for random codes without particular structure—not belonging to a restricted class—and with constant rate as \(K\) increases.

\(^4\)In this article the definition of support of an element \(A\) of the Pauli group is—rather unconventionally—the smallest block of consecutive qubits on which \(A\) acts non-trivially.
marized easily with the help of a semi-infinite matrix $M$:

$$M = \begin{pmatrix}
M_{0,1} & \cdots & M_{0,n-k} \\
\vdots & \ddots & \vdots \\
M_{1,1} & \cdots & M_{1,n-k} \\
\vdots & \ddots & \vdots \\
m & \cdots & n
\end{pmatrix} \quad (3)$$

Each line of the matrix represents one of the $M_{j,i}$ and each column a different qubit. A given entry in $M$ is thus the Pauli matrix for the corresponding qubit and generator. The rectangles represent graphically which qubits are potentially affected by the action of the generators. The form of Eq. (3) visually emphasizes the structure of convolutional codes:

- $M$ has a block-band structure;
- the overlap of $m$ qubits between two neighboring sets of generators forces to consider the code subspace as a whole.

By contrast, for a block code used repeatedly to protect an infinitely long stream of qubits, the above parameter $m$ would be equal to 0.

**Remark 2.2** In addition to the above generators, and in order to properly account for the finiteness of real-life communications, a few other generators will be added to the matrix $M$. This will however not interfere with the rest of this section.

### 2.2 Polynomial representation

Although, it is in principle possible to carry out a complete the analysis of the code with the matrix $M$ only, we will introduce a polynomial formalism which greatly simplifies this task. Such formalism is the exact translation of the polynomial formalism for classical convolutional codes. Its advantage is to capture in a convenient and efficient way the fact that the generators in $M$ are $n$-qubit shifted versions of the $M_{0,i}$’s.

More precisely, for a $(n,k,m)$-convolutional code, we define the delay operator $D$ acting on any element $A$ of the Pauli group of the physical qubits with bounded support by:

$$D[A] = I^\otimes n \otimes A,$$

with the same “padding rule” as before. Naturally, one can consider powers of $D$ as repeated applications of the delay operator. For instance, the generators of the code can now be written as:

$$M_{j,i} = D^j[M_{0,i}], \quad 0 \leq j, \quad 1 \leq i \leq n - k. \quad (5)$$

Therefore, and to further continue with simplifications, it is obviously not necessary to keep more than the first $n - k$ lines of the matrix $M$ defined in Eq. (3). All the omitted ones can be easily recovered by applying $D$ the appropriate number of times.
In addition to applying a single $D^j$ to an element of the Pauli group, it is, under certain conditions, possible to consider more complex operations—for instance, these will be necessary for deriving the encoding circuit. Namely, consider $A$, an element of the Pauli group with bounded support, such that $A$ and $D^j[A]$ commute for any value of $j$. Then, the full polynomial ring $GF_2[D]$ can act on $A$. For $P(D) = \sum_j \alpha_j D^j$, the action of $P(D)$ on $A$ is naturally defined as:

$$P(D)[A] = \prod_j \alpha_j D^j[A].$$

Above, the commutation relation is crucial: the sum operation in $GF_2[D]$ is commutative and must therefore be translated into another commutative operation—here the product—on the multiplicative group spanned by $\{D^j[A]\}$.

Finally, we will sometimes use a short hand in our notation and, instead of restricting ourselves only to polynomials in $D$, consider formal Laurent series acting on $A$. In such case, we do not really need to define the action of negative powers of $D$, but we impose that, at the end of the calculation—possibly concerning several operators—, all the negative powers of $D$ are removed by globally applying the smallest possible positive power of $D$. For instance, if we end with

$$L(D)[A] = \left( \sum_{j=-p}^{q} \alpha_j D^j \right) [A],$$

it will be turned into

$$P(D)[A] = \left(D^p \sum_{j=-p}^{q} \alpha_j D^j \right) [A] = \left( \sum_{j=-p}^{q} \alpha_j D^{j+p} \right) [A].$$

In practice, the representation of the code generators as a matrix $M$ with entries $I$, $X$, $Y$, $Z$ is often replaced by the one of [3]. In this representation, the first $n-k$ generators of an $(n,k,m)$-convolutional code would be written as a pair of $(n+m) \times (n-k)$ binary matrices arranged side by side. Each line corresponds to a generator and each column to a qubit. A 1 for the left matrix indicates the presence of an $X$ or $Y$ and, similarly, a 1 for the right matrix indicates the presence of a $Y$ or $Z$. Within this framework, it is easy to realize that the polynomial formalism can be fruitfully extended to lead an even more compact notation for the generators of the stabilizer group.

First, recall that the addition of two pairs of binary vectors simply results in the multiplication of the corresponding generators provided that these commute. For instance, suppose $A$ and $B$ are two elements of $G_n$, and $(A_X|A_Z)$, $(B_X|B_Z)$ their respective representations as pair of binary vectors. In such case, the operator $A \otimes B$ is represented by $(A_X : B_X|A_Z : B_Z)$ where "::" indicates the concatenation of the vectors. With the polynomial formalism, we also have $A \otimes B = A \times D[B]$, which leads to\(^7\) $(A_X : B_X|A_Z : B_Z) = (A_X|A_Z) + D[(B_X|B_Z)]$.

\(^5\)This means on all the operators involved in the calculation.

\(^6\)This representation as a pair of binary vectors or matrices is not restricted to elements of the stabilizer group, and can indeed be used for any element of $G_{n+m}$.

\(^7\)Here again we apply the implicit "padding rule" to adjust the length of the vectors.
Here, the commutation of $A$ and $D[B]$ is trivially verified since their supports do not intersect. This last equality suggests the following modification of the representation. A generic element $P$ of the Pauli group of the physical qubits with bounded support is represented by a pair of length $n$ vectors with coefficients in $GF_2[D]$ such that, by definition,

\[
(P_X|P_Z) = (P_X^{(0)} : P_X^{(1)} : P_X^{(2)} : \ldots | P_Z^{(0)} : P_Z^{(1)} : \ldots) = (P_X^{(0)} + D \times P_X^{(1)} + D^2 \times P_X^{(2)} + \ldots | P_Z^{(0)} + D \times P_Z^{(1)} + \ldots),
\]

where the $P_X^{(j)}$'s and $P_Z^{(j)}$'s are length $n$ binary vectors.

All these new concepts are best illustrated by applying them to a particular convolutional code. The simplest one with non-trivial behavior is the $(5, 1, 2)$-convolutional code given in [13]:

\[
M_{0,1} = Z X X Z I I I, \\
M_{0,2} = I Z X X Z I I, \\
M_{0,3} = I I Z X X Z I, \\
M_{0,4} = I I I Z X X Z, \\
M_{j,i} = D^j[M_{0,i}], 0 \leq j.
\]

Thus, using the pair of polynomial matrices representation, the generators of the stabilizer group can be written:

\[
M = \begin{pmatrix}
0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & D & 0 & 1 & 0 & 0 & 0 \\
D & 0 & 0 & 0 & 1 & 0 & D & 0 & 1 & 0 & 0 \\
\end{pmatrix}.
\]

### 2.3 Generalized commutator

In the context of block codes, the main reason justifying the introduction of the representation of the elements of the Pauli group as pairs of binary vectors [3, 5] is the existence of an easy way to compute the group commutator. We shall see below that the same kind of advantage holds for the representation as pair of polynomial vectors.

First, consider two elements $A = (A_X|A_Z)$ and $B = (B_X|B_Z)$ of $G_N$. It is then easy to check on their representation as pair of binary vectors that,

\[
AB = BA \iff A_X B_Z + A_Z B_X = 0,
\]

where we use the standard inner product of two vectors of length $N$ and addition modulo 2.

Now suppose that $P$ and $Q$ are two elements of the Pauli group of the physical qubits of an $(n, k, m)$-convolutional code, and $(P_X(D)|P_Z(D))$, $(Q_X(D)|Q_Z(D))$ their representation as pair of polynomial vectors. Using the above method, one can conclude that the commutation of $P$ and $Q$ is simply expressed by:

\[
PQ = QP \iff \sum_i P_X^{(i)} Q_Z^{(i)} + P_Z^{(i)} Q_X^{(i)} = 0,
\]
where \( P_X(D) = \sum_j P_X^{(j)} D^j \) with \( P_X^{(j)} \) a binary vector of length \( n \) and similarly for \( P_Z(D), Q_X(D), Q_Z(D) \). This also leads to,

\[
D^r[P]D^s[Q] = D^s[Q]D^r[P] \Leftrightarrow \sum_i P_X^{(i+s)} Q_Z^{(i+r)} + P_Z^{(i+s)} Q_X^{(i+r)} = 0.
\]

(14)

The last equation is particularly interesting since its right hand side is the coefficient of \( D^{s−r} \) in \( P_X(D)Q_Z(1/D) + P_Z(D)Q_X(1/D) \). Therefore, one can readily conclude that the representation as pair of polynomial vectors allows an easy computation of the “generalized commutation relation”—i.e. the commutation of any \( n \)-qubit shifted version of \( P \) with any \( n \)-qubit shifted version of \( Q \)—:

\[
\forall r, s, D^r[P]D^s[Q] = D^s[Q]D^r[P] \Leftrightarrow P_X(D)Q_Z(1/D) + P_Z(D)Q_X(1/D) = 0.
\]

(15)

We will see below that this property of the polynomial representation is crucial as it allows the derivation of almost all the encoded Pauli operators by considering only the first \( n − k \) generators \( M_0, i \)’s of the stabilizer group.

2.4 Encoded Pauli operators

The encoded Pauli operators for a quantum error correcting code are some operators of the Pauli group of the physical qubits which allow the manipulation of the information without requiring any decoding. More precisely, these are operators that leave the code subspace \( \mathcal{C} \) globally invariant, but which have a non-trivial action on it. Indeed, it is possible to require such operators to reproduce exactly the commutation relations of the Pauli group for the encoded qubits. This is mathematically expressed by [5]:

\[
[X_i, Z_i] \in N(S)/S,
\]

(16)

\[
[X_i, X_j] = 0,
\]

(17)

\[
[Z_i, Z_j] = 0,
\]

(18)

\[
[X_i, Z_j] = 0, \quad i \neq j,
\]

(19)

\[
\{X_i, Z_i\} = 0,
\]

(20)

where the index \( i \) in \( \overline{X}_i \) and \( \overline{Z}_i \) denotes the \( i \)-th logical qubit.

In the rest of this paragraph we exploit Eq. (15) to find an algorithmic procedure for deriving the \( \overline{X}_i \)’s and \( \overline{Z}_i \)’s. First we define the standard polynomial form of \( M \) and, as a second step, we translate Eqs. (16–20) into a set of equations for polynomial vectors which can be solved easily.

To obtain the standard polynomial form for the generators of the stabilizer group one can perform two Gaussian eliminations\(^9\) on \( M \) written in its representation as pair of polynomial matrices over \( GF_2[D] \). This can be done by using line additions, column swaps and multiplication of a line by a power of \( D \):

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\(^8\)In all this article, and following the notation of [5], the encoded Pauli operators are denoted by, e.g. \( \overline{X} \) and \( \overline{Z} \).

\(^9\)See also [5] for a similar procedure for block codes.
where \( A(D) \) and \( K(D) \) are diagonal matrices with polynomial coefficients, and where \( r \) is the rank of the \( X \)-part of \( M \).

By definition, \( A(D) \) has full rank. In fact, this holds for \( K(D) \) as well: if it was not the case, then there would exist a line with zeroes everywhere except for at least one position in the first \( r \) columns of the \( Z \)-part. Then, the operator corresponding to this line cannot commute in the generalized sense with all the other generators, which would contradict the assumption that the stabilizer group \( S \) generated by \( M_{\text{std}} \) is Abelian.

We now turn to the determination of the encoded Pauli operators. Here, we restrict our search to operators that preserve the convolutional nature of the code: we want to find a finite set of independent operators with bounded support which generate through \( n \)-qubit shifts—almost all—the encoded Pauli operators\(^{10} \). This can be accomplished by considering a \( k \)-line matrix,

\[
\overline{X} = (U_1(D), U_2(D), U_3(D)|V_1(D), V_2(D), V_3(D)),
\]

representing the encoded \( \overline{X} \) operators—the rest of the discussion shows that such encoded Pauli operators exist. Since these operators can be multiplied by any element of the stabilizer group, \( U_1(D) \) and \( V_2(D) \) can be set to 0. The generalized commutation with the lines of \( M \) imposed by Eq. (16) can be simply written:

\[
\begin{pmatrix}
A(D) & B(D) & C(D) \\
0 & 0 & 0 \\
E(D) & F(D) & G(D) \\
J(D) & K(D) & L(D)
\end{pmatrix}
\begin{pmatrix}
V_1^T(1/D) \\
0 \\
V_2^T(1/D) \\
0 \\
U_3^T(1/D)
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0 \\
u_3^T(1/D)
\end{pmatrix}.
\]

On the other hand, Eq. (17) is expressed by

\[
U_3(D)V_3^T(1/D) + V_3(D)U_3^T(1/D) = 0,
\]

which can be trivially satisfied with \( V_3(D) = 0 \) and \( U_3(D) = \Lambda(D) \times I \), where \( \Lambda(D) \) is a non-zero polynomial of \( GF_2[D] \). This choice guarantees that the operators in \( \overline{X} \) together with their \( n \)-qubit shifted versions are independent of each other and from the generators of \( S \). In this case, Eq. (23) becomes,

\[
\begin{pmatrix}
A(D)V_1(1/D)^T + F(D)U_2(1/D)^T + G(D)U_3(1/D)^T \\
K(D)U_2(1/D)^T + L(D)U_3(1/D)^T
\end{pmatrix} =
\begin{pmatrix}
0 \\
0
\end{pmatrix}.
\]

Then we can write the encoded \( \overline{X} \) operators:

\[
U_1(D) = 0.
\]

\(^{10}\)For the purpose of introducing the theory of quantum convolutional codes, it is not necessary to consider encoded Pauli operators that do not respect the convolutional structure of the code. However, in more elaborated error correction scheme, this might prove to be useful.
\[ U_2(D) = L^T(1/D)K^{-1}(1/D)\Lambda(D) \] (27)
\[ U_3(D) = \Lambda(D) \times I \] (28)
\[ V_1(D) = (U_2(D)F(1/D)^T + \Lambda(D)G(1/D)^T)A^{-1}(1/D) \] (29)
\[ V_2(D) = 0 \] (30)
\[ V_3(D) = 0. \] (31)

One must realize that the encoded Pauli operators \( \overline{X} \) are not yet properly defined as the division by polynomials is in general problematic. The reason is that generic polynomial fractions cannot be written as finite formal Laurent series. Thus, the operators that they describe have an unbounded support. In such case, and without further modifications, the formalism introduced earlier imposes transmissions of infinite length. However, when the result of the division can be written with a finite Laurent series, such operation is permitted.

**Definition 2.2 (Conditioning polynomial)** The conditioning polynomial \( \Lambda(D) \) of a convolutional code is the non-zero polynomial with minimum degree such that the equations (26–31) only involve finite Laurent series.

As it can be seen easily, the conditional polynomial always exists, and the \( \overline{X} \)'s operators are well defined. They correspond to operators with a finite support, respecting the convolutional structure of the code.

We now turn to the derivation of some \( \overline{Z} \)'s by applying the same tools. First note that once the \( \overline{X} \)'s are fixed, there is a unique set of valid \( \overline{Z} \)'s. Quite surprisingly, we will also see here that it is not always possible to impose to the \( \overline{Z} \)'s the convolutional structure—the invariance by \( n \)-qubit shifts.

For instance, first define the \( k \)-line matrix

\[ \overline{Z} = (0, U'_2(D), U'_3(D)|V'_1(D), 0, V'_3(D)). \] (32)

Above, the zeroes have been set for the same reason as in the derivation of the \( \overline{X} \)'s. In addition to satisfying an equation similar to Eq. (23), the matrix \( \overline{Z} \) must anti-commute in the generalized sense with \( \overline{X} \), Eq. (20). Equivalently, this can be expressed as \( V'_3(D)U'_2(1/D)^T = I \), which can be fulfilled if and only if \( V'_3 = I/\Lambda(D) \). As discussed above, only when \( \Lambda(D) \) is a monomial in \( D \) does \( V'_3(D) \) correspond to a valid polynomial vector (i.e. \( 1/\Lambda(D) \) is a bounded Laurent series). In this latter case, we obtain \( \overline{Z} \):

\[ U'_1(D) = 0 \] (33)
\[ U'_2(D) = 0 \] (34)
\[ U'_3(D) = 0 \] (35)
\[ V'_1(D) = C^T(1/D)A(1/D)^{-1}/\Lambda(1/D) \] (36)
\[ V'_2(D) = 0 \] (37)
\[ V'_3(D) = I/\Lambda(1/D). \] (38)

Note that for \( \Lambda(D) \) to be a monomial, all the \( A_{i,i}(D) \)'s must be monomials as well, so that Eq. (36) is automatically well defined.
Remark 2.3 The obvious question raised by this derivation concerns the case where \( \Lambda(D) \) is not a monomial. The rigorous answer will be given in sec. 4 where it will be shown that if such code were to be used, it would have bad error propagation properties.\(^{11}\) One can also consider the following hand-waving argument: when \( \Lambda(D) \) is not a monomial, and for a finite length communication, the \( \bar{Z} \)'s have a support with a size of the order of the length of the code. Thus, if one implements an encoded phase flip by applying individual \( Z \)'s on the physical qubits with finite precision, then for long streams of to-be-protected information this will result in an error with probability close to 1.

Finally, and to conclude this section on the structure of convolutional codes, we should count how many logical qubits are described by our construction in the case of a finite transmission. To simplify this discussion, we define the integer \( \lambda \) as the highest degree in the polynomial matrices \( \bar{X} \) and \( \bar{Z} \). This sets an upper bound on the size of the support of any of the \( \bar{X} \)'s and \( \bar{Z} \)'s: they extend on at most \( \lambda + 1 \) consecutive \( n \)-qubit blocks. Further consider the stabilizer group \( S \) generated by the \( \{ M_{j,i} \} \) for \( 0 < i \leq n - k \) and \( 0 \leq j < p \) with \( p > \lambda \). In this case, the above derivation leads to at least\(^{12}\) \( k \times (p + \lceil m/n \rceil - \lambda) \) logical qubits while we used \((n - k) \times p \) generators and \((p + \lceil m/n \rceil) \times n \) physical qubits. Therefore, only \([m/n] \times (n - k) + \lambda k \) logical qubits—a number independent of \( p \)—do not follow the convolutional structure of the code. These will simply be discarded in the encoding process as this does not change the asymptotic rate of the code. This can be done consistently with the stabilizer formalism by adding their encoded \( Z \) operators to the generators of \( S \).

By working out the example of Eq. (11), one easily finds the standard form of \( M \),

\[
M_{\text{std}} = \begin{pmatrix}
D & 0 & 0 & 0 & 1 & 0 & D & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & D & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & D & 0 & 1 & 0 \\
\end{pmatrix}.
\tag{39}
\]

The \( \bar{X} \) operators are obtained from a single 5-dimensional vector, with the polynomial \( \Lambda(D) \) equal to 1:

\[
\bar{X} = (0, 0, 0, 0, 1, 0, 1, 0, 0),
\]

\[
\bar{Z} = (0, 0, 0, 0, 0, D, 1, 1, 1, 1).
\tag{40}
\]

3 Encoding

This section provides an operational method to arrive at an encoding circuit which respects the convolutional structure of the code: a simple unitary operation—indeed of the length of the to-be-protected stream—and its \( n \)-qubit shifted versions will be applied successively to arrive at the protected state. Therefore, the complexity of this scheme in terms of number of gates in the encoding circuit only grows linearly with the number of encoded qubits. This is of particular relevance since dealing with convolutional codes as if they were

\(^{11}\)Only the \( \bar{X} \) operators are used to derive the encoding circuit. Then, if one renounces to manipulate information in its encoded form, the code can be, in principle, successfully used to protect quantum information.

\(^{12}\)Here, we consider an integer number of physical \( n \)-qubit blocks. I wrote “at least” because it is possible that the support of some of the \( \bar{X} \) and \( \bar{Z} \) is smaller than \( n \times (\lambda + 1) \).
generic block codes would lead to an encoding circuit with quadratic gate complexity. It would also require increasing precision in the applications of the encoding gates and would cause severe delays in the transmission of the information.

The derivation of the encoding circuit will nonetheless be very similar to the one for block codes [5]. Here, instead of the usual standard form for the generators, we use the standard polynomial form. The circuit that will be obtained is relative to the encoding of $q \times k$ logical qubits. The encoded Pauli operators corresponding to these qubits will be denoted $X_{j,i}$ and $Z_{j,i}$. For instance, $X_{0,i}$ is defined as the $i$-th line of the $X$ matrix derived in the previous section, $X_{j,i} = D_j[X_{0,j}]$, and similarly for the $Z$'s.

The encoding circuit maps the to-be-protected qubits $c_{j,i}$ onto the code subspace. Its action on the computational basis can be written as:

$$|c_{0,1}, \ldots, c_{q-1,k}\rangle \rightarrow \left(\prod_{i,j} \frac{1 + M_{j,i}}{\sqrt{2}}\right) \prod_{r,s} X^{s,r}_{j,i} |0,\ldots,0\rangle,$$  \hspace{1cm} (41)

for $c_{s,r} \in \{0, 1\}$, $0 < i \leq n - k$, $0 \leq j < q + \lambda$, $0 \leq s < q$ and $1 \leq r \leq k$.\footnote{Here $\lambda$ is defined as the in the previous section. With this definition, the operators $X$ have support on at most $\lambda + 1$ consecutive n-qubit blocks. The choice $j < q + \lambda$ then ensures that the support of each logical qubit is covered by the same number of generators of the stabilizer group.} This operation can be decomposed in two steps. The first one, $\prod_{r,s} X^{s,r}_{j,i}$, applies the different flip operators depending on the value of the to-be-protected qubits in the computational basis. The second, $\prod_{i,j}(1 + M_{j,i})/\sqrt{2}$, projects this state onto the code subspace.\footnote{The way of writing this projection follows from the realization that any element of the stabilizer group is a product where each generator appears at most once—any element of the Pauli group is its own inverse.}

We first focus on the conditional application of the $X$'s:

$$|c_{0,1}, \ldots, c_{q-1,k}\rangle \rightarrow \prod_{r,s} X^{s,r}_{j,i} |0,\ldots,0\rangle.$$ \hspace{1cm} (42)

The number of $n$-qubit blocks involved in the right hand side of Eq. (41) is equal to $q + \lambda + \lceil m/n \rceil$. Hence, the first requirement is to supplement the to-be-protected stream of information with some ancillary qubits prepared in the $|0\rangle$ state. Both are arranged in the following way:

\begin{align*}
|c_{0,1}, \ldots, c_{q-1,k}\rangle \rightarrow & \quad \underbrace{|0\ldots0, \underbrace{c_{0,1}, \ldots, c_{0,k}}_{\text{n-k blocks}}, \underbrace{c_{1,0}, \ldots, c_{1,k}}_{\text{n-k blocks}}, \ldots, \underbrace{c_{q-1,0}, \ldots, c_{q-1,k}}_{\text{n-k blocks}}, \ldots, 0\ldots0\rangle}_{\text{n-k \times m/n}}. \hspace{1cm} (43)
\end{align*}

The notation $X^{s,r}_{j,i}$ means that $X_{s,r}$ needs to be applied on the all-zeroes state if and only if $c_{s,r} = 1$. Now, in the standard polynomial form, $X_{s,r}$ has a factor $X$ exactly at the position of $c_{s,r}$ in the state of the right hand side of Eq. (43). Therefore, if all the other logical qubits are set to zero, the output state of Eq. (42) can be obtained from the right hand side of Eq. (43) by applying $X_{s,r}$—without the above mentioned $X$—conditioned on qubit $c_{s,r}$. Unlike for quantum block codes, these conditional operations can confuse each other when the conditioning polynomial $\Lambda(D)$ (see sec. 2) is not a monomial\footnote{Here, for sake of generality, we describe the encoding circuit without imposing $\Lambda(D)$ to be a monomial even though in this case, the encoding shows bad error propagation properties.}. In this situation,
applying $X_{s,r}$ might flip some control qubits $c_{s',r'}$ for $s' < s$. Therefore, these modified qubits $c_{s',r'}$ cannot be used anymore to condition the application of $X_{s',r'}$. This also indicates the way-out of this problem: when the $X$'s are applied by increasing successively the index $s$ by one, there is no risk that one application flips a qubit later used to condition another $X$.

For the example given in Eq. (11), this part of the encoding circuit is illustrated in Fig. 1.

The rest of the encoding circuit must implement the effect of the projection onto the code subspace for this partially encoded state:

$$\prod_{r,s} X_{s,r}^{c_{s,r}} |0,\ldots,0\rangle \rightarrow \left( \prod_{i,j} \frac{1 + M_{j,i}}{\sqrt{2}} \right) \prod_{r,s} X_{s,r}^{c_{s,r}} |0,\ldots,0\rangle.$$ (44)

There are two classes of $M_{j,i}$'s. Either $M_{j,i}$ is a tensor product of $I$'s and $Z$'s only, or there is a polynomial $A_{i,i}$ on the $i$-th column of the $X$ part when it is expressed in the standard polynomial form (see sec. 2). In the first case, nothing needs to be done. In the latter, consider the $i$-th qubit of the $(j + \deg A_{i,i}(D))$-th $n$-qubit block in Eq. (44). The resulting state is an equal weight superposition of a state with a $|0\rangle$ and a state with a $|1\rangle$ on the previously mentioned qubit. This can be created by first applying a Hadamard gate for this qubit, which later controls the application of $M_{j,i}$—ignoring the $X$ factor for the control. If there is a $Z$ factor for the control qubit, it does not need to be conditioned on anything and can be applied right after the Hadamard gate. Once again, since $A_{i,i}$'s are not required to be monomials, the above operations might confuse each other when a control qubit, supposedly still in its initial $|0\rangle$ state, has indeed already been modified. As before, this can be overcome by applying the conditional gates and increasing the index $j$ one by one successively.

The method described here details how to obtain the encoding circuit when the generators $M_{j,i}$ have a positive sign. When this is not the case, the procedure described here must be modified so that a $Z$ gate is applied to the qubit conditioning the application of those particular $M_{j,i}$ with a negative sign.

---

Figure 1: Circuit for generating the state $\prod_{r,s} (X_{s,r})^{c_{s,r}} |0,\ldots,0\rangle$ for the 5-qubit convolutional code. For obvious reasons, the control-$Z$ operations have been kept even though they act on $|0\rangle$ and should be simplified: this part of the encoding circuit would be reduced to no circuit at all!
Remark 3.1 For sake of simplicity in the presentation of the whole encoding circuit, the usual simplifications corresponding to the removal of control-Z gates acting on a target in state |0⟩ have not been described. Of course, these should be performed to obtain a simpler circuit.

Remark 3.2 Note also that the circuit described in this section encodes the qubits on-line:

- the second step rotating the partially encoded state into the code subspace can start before all the X’s are applied;
- sending the qubits can be done before all the stream has been encoded.

This is a simple consequence of the fact that each conditional gate in the circuit acts only on the last λ + 1 n-qubit blocks.

For the 5-qubit convolutional code, the full encoding circuit is presented in Fig. 2, where all the simplifications have been implemented. Here, the existence of sacrificed logical qubits is clearly apparent: the first qubit is never involved in any gate and does not contain any quantum information. This comes from the finiteness of the to-be-protected sequence: at the beginning and at the end of the stream, there are less commutation constraints for the encoded Pauli operators imposed by the generators in M. Thus, it is not surprising that there exist a finite number of encoded Pauli operators that do not follow the convolutional structure. It is also important to remark that there is no need to determine these operators explicitly for deriving the encoding circuit. Setting the sacrificed qubits to the logical |0⟩ state is taken care of by setting the first λ n-qubit blocks to the all-zeroes state.
4 Error propagation and on-line decoding

The previous section was devoted to the derivation of the encoding circuit for quantum convolutional codes. It showed how the standard polynomial form for the generators of the code leads to an automated procedure for finding an on-line encoding circuit. In this section, the focus shifts to decoding quantum convolutional codes. The need for a clear discussion on this issue comes from the specificity of convolutional codes: usual decoding circuits—obtained by running the encoding one in reverse direction—require to wait for the last logical qubit before running them. This is not a practical option as it would cause long transmission delays.

Here, we show that the existence of an on-line decoding circuit is implied by a more fundamental property of the encoding operation: the absence of catastrophic errors. These errors will be defined carefully below, but we can already mention that they are not specific to quantum codes. Rather they, and more generally all the error propagation problems considered in this section, are also encountered in the theory of classical convolutional codes [11, 8].

To build our intuition on the error propagation problems that might arise when using convolutional codes, consider a generic encoding circuit as derived in the previous section (see also Fig. 3). Because of the overlap between the generators on \( m \) qubits as defined in Eq. (3), quantum information is propagated from one \( n \)-qubit block to another. As a consequence, even though the to-be-protected stream of information is in a separable state, say \( |0, \ldots, 0\rangle \), the encoded state is not, in general, separable with respect to any bipartite cut. In spite of their relatively simple form—invariant by shifts of \( n \)-qubit—encoding circuits apply global unitary transformations that cannot be cast in tensor products of smaller unitary operations.

The good spreading of quantum information induced by the particular structure of convolutional codes might in some cases have a bad consequences: nothing prevents an error affecting a finite number of qubits before the complete decoding of the stream to propagate infinitely through the decoding circuit. Such error is called catastrophic.

Definition 4.1 (Catastrophic error) Consider an \((n, k, m)\)-convolutional encoding scheme for protecting \( q \times k \) logical qubits. A catastrophic error is an error that affects \( O(1) \) qubits before the end of the decoding operation and that can only be corrected by a unitary transformation whose size of support grows with \( q \), for large \( q \).

Remark 4.1 The theory of classical convolutional codes explicitly shows the existence of catastrophic errors for some convolutional encoders. As these are a special case of quantum codes—their generators are tensor products of \( I \)'s and \( Z \)'s—it proves the existence of catastrophic errors for some quantum encoding circuits.

4.1 Catastrophicity condition

In this paragraph, we will find a catastrophicity condition for convolutional encoders without relying on the stabilizer description of the code. Instead, we simply assume a generic form for the encoding operation of \( q \times k \) to-be-protected qubits:

\[
C(q) = T_{\text{erm}} \times D^{q-1}[U] \times \ldots \times D[U] \times U \times I_{\text{init}},
\]  

(45)
Figure 3: Typical encoding circuit for a convolutional code. The circuit is run from left to right. Horizontal lines of a given type (i.e. with single or double vertical bar) always represent the same number of qubits. The unitary operation $U$ is implemented as a series of elementary gates acting only on the qubits with which it intersects.

where $I_{\text{init}}$ and $T_{\text{term}}$ are two fixed unitary transformations, respectively the initialization—acting at the beginning of the to-be-protected stream of information—, and the termination—acting on the last qubits of the stream.\footnote{The delay operator, initially introduced only for elements of the Pauli group with finite support, is easily generalized to handle unitary matrices with finite support.} The unitary $U$ has a finite support independent of $q$. In the standard encoding presented in the previous section, $U$ corresponds to the encoding of $k$ consecutive qubits containing information—i.e. it corresponds to applying some $X$'s and some $M_{j,i}$'s. The presence of $I_{\text{init}}$ and $T_{\text{term}}$ is due to the sacrificed logical qubits at the beginning and at the end of the encoded stream. The typical arrangement of the unitary operations $D_i[U]$ far from the beginning and the end of the stream of information is depicted in Fig. 3.

\textbf{Proposition 4.1} A quantum convolutional encoder is non-catastrophic if and only if the encoding operation $C(q)$ can be decomposed in the following way for large $q$:

$$C(q) = \tilde{T}_{\text{term}}(q) \times \left( \prod_{i=0}^{[q/l_2]} D_i^{l_2}[U_1] \right) \times \ldots \times \left( \prod_{i=0}^{[q/l_1]} D_i^{l_1}[U_1] \right) \times \tilde{I}_{\text{init}}(q),$$

where $\tilde{I}_{\text{init}}(q)$ and $\tilde{T}_{\text{term}}(q)$ are modified initialization and termination steps which can vary with $q$, but whose support is bounded; $\{U_j\}_{j}$ is a finite set of unitary operators independent of $q$—thus with bounded support—such that $D_i^j[U_j]$ and $D_i^{j'}[U_j]$ commute; and $l_j$'s are integers independent of $q$.

Even though this condition might seem at first sight quite complicated, it corresponds to a reordering of the unitaries—or gates—in the quantum circuit which is easy to understand.
Figure 4: Example of pearl-necklace structure for the encoding circuit. We have depicted four layers of unitaries, $U_1$ through $U_4$. Here, the condition of commutation inside a layer is guaranteed by the disjointness of the support of the different unitaries $\{D^i[U_i]\}_i$.

The new circuit must have the following form: first an initialization step is performed; then, there are $t$ layers of unitaries (each of them made out of a single unitary, e.g. $U_i$, and its $n$-qubit shifted versions) such that the gates inside a layer commute with each other; finally it is followed by a termination step, $\tilde{T}_{\text{term}}(q)$ with bounded support. This structure resembles a pearl-necklace as it can be seen on Fig. 4.

**Proof 4.1 (Sufficiency)** To simplify the discussion, we will consider the case where the error $E$ occurs before the beginning of the decoding operation. This is not general, since the definition of non-catastrophicity also imposes to consider errors occurring on a partially decoded stream. Nonetheless, the proof presented here can be easily adapted for this other case.

Here, we have to show that for $q$ large, whenever $E$ has bounded support, $C(q)^\dagger EC(q)$ has a bounded support as well. Since $\tilde{T}_{\text{term}}(q)$ has a bounded support at the end of the stream, it is always possible to increase $q$ such that $E$ and $\tilde{T}_{\text{term}}(q)$ commute. Therefore, after simplifying $C(q)^\dagger EC(q)$ by $\tilde{T}_{\text{term}}$, we have:

$$C(q)^\dagger EC(q) = \tilde{I}_{\text{nit}}(q)^\dagger \times \left( \prod_{i=0}^{\lfloor q/l_1 \rfloor} D^{i_1}[U_{i_1}^1] \right) \times \ldots \times \left( \prod_{i=0}^{\lfloor q/l_t \rfloor} D^{i_t}[U_{i_t}^1] \right) \times \times \prod_{i=0}^{\lfloor q/l_i \rfloor} D^{i_i}[U_i] \times E \times \prod_{i=0}^{\lfloor q/l_i \rfloor} D^{i_i}[U_i] \times \ldots \times \prod_{i=0}^{\lfloor q/l_i \rfloor} D^{i_i}[U_i] \times \tilde{I}_{\text{nit}}(q).$$ (47)

Similarly, in the above equation all the $D^{i_t}[U_t]$ whose support does not intersect the one of $E$ commute with it and can be simplified (recall also that the $D^{i_t}[U_t]$ also commute with

\[18\] Because of possible side effects $\tilde{I}_{\text{nit}}(q)$ can depend on $q$ but the size of its support must be of order 1 and it can act non-trivially only on the first few qubits
each other). Only a finite number of the $D^{i_l}[U_i]$’s remain, say $\{D^{i_l}[U_i]\}_{i \in I}$. Note that for $q$ large, this number is independent of $q$. We thus have

$$C(q)\dagger EC(q) = \tilde{I}_{\text{mit}}(q)\dagger \times \left( \prod_{i=0}^{\lfloor q/l \rfloor} D^{i_1}[U_i^\dagger] \right) \times \ldots \times \left( \prod_{i=0}^{\lfloor q/l-1 \rfloor} D^{i_{l-1}}[U_{l-1}^\dagger] \right) \times$$

$$\times E_1 \times \left( \prod_{i=0}^{\lfloor q/l \rfloor} D^{i_{l-1}}[U_{l-1}] \right) \times \ldots \times \left( \prod_{i=0}^{\lfloor q/l-1 \rfloor} D^{i_1}[U_1] \right) \times \tilde{I}_{\text{mit}}(q) \quad (48)$$

where $E_1 = \left( \prod_{i \in I} D^{i_l}[U_i^\dagger] \right) \times E \times \left( \prod_{i \in I} D^{i_l}[U_i] \right)$ has a bounded support, independent of $q$. The rest of the proof follows immediately by applying the same technique to the remaining layers: another step generates $E_2$, by considering $E_1$ instead of $E$ and $U_{l-1}$ instead of $U_l$. Following the same arguments, $E_2$ has a bounded support independent of $q$ and so will $E_3, \ldots, E_t$. Thus it proves that $C(q)\dagger EC(q) = \tilde{I}_{\text{mit}}(q)\dagger E_1 \tilde{I}_{\text{mit}}(q)$ has bounded support.

**Proof 4.2 (Necessity)** To prove that this condition is necessary, we will show that a non-catastrophic encoding operation $C(q)$ can be put in the special form of Eq. (46), for $q$ large. The outline of the proof is the following: we will work on the circuit of the decoding operation $C(q)\dagger$, obtained by running the encoding circuit in the reverse direction (see Fig. 5). Our goal is to convert this decoding circuit into an equivalent one which displays the pearl-necklace structure. To do so, we will consider a possible—but yet very particular—error which could occur on the physical qubits during the transmission. The chosen error indeed corresponds to a local reordering of the unitaries in $C(q)\dagger$. Since the encoding is supposed to have no catastrophic errors, this local reordering can be compensated by applying a unitary operation with finite support after complete decoding. This will give us an identity between two decoding circuits, which we can apply as many times as required to arrive at the pearl-necklace structure.

More specifically, consider the decoding unitary operation,

$$C(q)\dagger = \tilde{I}_{\text{mit}}\dagger \times U^\dagger \times D[U^\dagger] \times \ldots \times D^{q-1}[U^\dagger] \times T_{\text{erm}}^\dagger. \quad (49)$$

We define the integer $l$ such that $U$ and $D^l[U]$ have disjoint support for $|i| > l$.\(^{19}\) The circuit identity that will be derived is:

$$C(q)\dagger = D^{q-l'}[V^\dagger] \times \tilde{C}(q)\dagger, \quad (50)$$

where $V$ has finite support extending on $l'$ n-qubit blocks, and where $\tilde{C}(q)\dagger$ is obtained from $C(q)\dagger$ by locally reordering its last $2l + 1$ unitaries $U$:

$$\tilde{C}(q)\dagger = \tilde{I}_{\text{mit}}\dagger \times U^\dagger \times \ldots \times D^{q-2l-3}[U^\dagger] \times$$

$$\times D^{q-2l-2}[D^{l+1}[U^\dagger] \times U^\dagger] \times D^{q-2l-1}[D^{l+1}[U^\dagger] \times U^\dagger] \times \ldots \times D^{q-l-2}(D^{l+1}[U^\dagger] \times U^\dagger) \times T_{\text{erm}}^\dagger. \quad (51)$$

\(^{19}\)This integer exists because $U$ has finite support.
Figure 5: Typical decoding circuit for a convolutional code. The circuit is obtained by running the encoding circuit in reverse direction and with appropriate Hermitian conjugates.

Consider $E$, a unitary operation, defined by:

$$E = (D^{l+1}[U] \times U^\dagger) \times D[D^{l+1}[U] \times U^\dagger] \times \ldots \times D'[D^{l+1}[U] \times U^\dagger] \times$$

$$\times D^{2l+1}[U] \times D^{2l}[U] \times \ldots \times D[U] \times U. \quad (52)$$

An illustration of the arrangement of the unitaries in $E$ is presented on Fig. 6 for $l=1$. By construction, $E$ satisfies:

$$\tilde{C}(q)^\dagger \times \ldots \times D^{q-2l-3}[U] \times \ldots$$

$$\times D^{q-2l-2}[E] \times$$

$$\times D^{q-2l-2}[U] \times D^{q-2l-1}[U] \times \ldots \times D^{q-1}[U] \times T_{\text{erm}}, \quad (53)$$

which simply corresponds to the initial decoding operation $C(q)^\dagger$ with an error $E$ happening between the unitaries $D^{q-2l-3}[U]$ and $D^{q-2l-2}[U]$. Since, the encoding is non-catastrophic, there exists a unitary $V^\dagger$ with finite support—also obviously independent of $q$—such that $C(q)^\dagger = D^{q-l'}[V] \times \tilde{C}(q)$, where $l'$ is the size of the support of $V$ counted in number of $n$-qubit blocks, which gives the circuit identity (see Figs. 7 & 8 for the local reordering implied by Eqs. (49–53).

Moreover, this identity concerns only the unitary operations around the position where $E$ is applied. It is then possible to apply it at repeated intervals—e.g. separated from $\max(l,l') + 1$ n-qubit blocks—in the decoding circuit. It is then straightforward to show that $\tilde{C}(q)^\dagger$—and similarly $\tilde{C}(q)$—has the form of Eq. (46), and to conclude the proof (see Figs. 9 & 10).

Remark 4.2 Note also, that this demonstrates the possibility of on-line decoding for non-catastrophic quantum convolutional codes: in this form, the “directionality” of the quantum circuit which imposed to begin the decoding at the end of the received stream disappeared.
Figure 6: Error operation $E$ as defined in Eq. (52). Here, $l = 1$ because $D^i[U]$ commutes with $U$ for $i > 1$. When introduced in the decoding circuit, such operation induces a local reordering of the unitaries $U^\dagger$.

Figure 7: Derivation of a circuit identity for decoding. Because there is no catastrophic error, the effect of applying $E$ as defined in Eq. (6) in the decoding circuit can be corrected by a unitary operation $V$ with finite support: this circuit induces the same unitary transformation on the received stream of information.
Figure 8: Local reordering in the decoding circuit. By using the specific form of $E$, this circuit is equivalent to the ones given in Figs. 5 & 8.

Figure 9: Global reordering of the decoding circuit. Exploiting the circuit identity described in Fig. 8, the fact that it corresponds to a local reordering only (i.e. only a finite number of unitaries with bounded support are involved in this identity), and the invariance of the initial decoding circuit by $n$-qubit shifts, it is possible to induce local reorderings at regular intervals in the decoding circuit.
The pearl-necklace structure of the encoding circuit for the 5-qubit convolutional code is presented in Fig. 11.

4.2 Catastrophicity condition for standard encoders

Proposition 4.2 Encoders derived from the standard polynomial form are non-catastrophic if and only if $\Lambda(D)$ is a monomial.

Proof 4.3 Simple commutations rules between controlled gates can be used to show that when $\Lambda(D)$ is a monomial, the quantum circuit can be put in the form of Eq. (46). To prove the necessity, suppose $\Lambda(D)$ is not a monomial and consider the decoding circuit for this code. More precisely, focus on the qubits that control the application of $X_{0,1}, \ldots, X_{q-1,1}$. If the decoding circuit is restricted to those qubits only, the only two-qubit gates that are used are controlled-NOT’s. Thus, this part of the quantum circuit in fact implements a rate 1 classical convolutional encoder with feedback. This encoder links its output stream $y(D)$ with its input $x(D)$ through (see [11] for a rapid introduction to classical convolutional codes and their polynomial formalism),

$$y(D) = x(D) + (\Lambda(1/D) - 1)y(D).$$

(54)

Thus, an error affecting the input stream—corresponding to a bit flip in the quantum case—propagates to an infinite number of output bits when $\Lambda(D)$ is not a monomial:

$$y(D) = \frac{x(D)}{\Lambda(1/D)}.$$
Similarly, in the quantum case, a single bit flip could propagate to an infinite number of qubits. Thus non-catastrophic standard encoders have a monomial $\Lambda(D)$. 

**Remark 4.3** Note also that the condition “$\Lambda(D)$ is a monomial” is equivalent to having the $Z$ operators efficiently described with the polynomial formalism. These two questions are in fact intimately related. The application of a $Z$ can be done before encoding by applying the corresponding $Z$ to the physical unprotected qubit. It is well known that phase flips propagate through controlled-NOT gates from the target to the control. Here, this phase flip propagates in the same way the bit flip of the proof propagates in the decoding circuit. The number of qubits affected by this $Z$ operation after running the encoding increases linearly with $q$, the number of $k$-qubit blocks to be protected. More generally, the non-catastrophicity condition shows that contrarily to classical convolutional codes, an operation with finite support acting before encoding cannot propagate to an infinite number of qubits after encoding. 

### 5 Error estimation algorithm

The last subject that must be addressed to arrive at a theory of quantum convolutional codes is the error estimation algorithm. A naive attempt at finding the most likely error could be to search among all the possible errors. In turn, this usually implies an exponential complexity in the number of encoded qubits, thus making this scheme impractical for large amounts of to-be-protected information. In this section, a maximum likelihood estimation algorithm with a linear complexity is provided. This algorithm is similar to its classical analog, known as the Viterbi algorithm [17, 11, 8].
5.1 Notation

To simplify the description of the algorithm, some additional notation will be useful. Recall Eq. (3) which defines the generators of the stabilizer group $M_{j,i}$. The expression “block $j$” will refer to the qubits involved in $M_{j,i}$ for $i = 1, \ldots, n - k$. The qubits are numbered in increasing number from left to right, so that the first $m$ qubits and the last $n$ qubits of the second block are those separated on Eq. (3) by a dashed line. Note also that due to the convolutional nature of the code and because of the definition of $m$, the last $m$ qubits of block $j$ are the same as the first $m$ qubits of block $j + 1$. The syndrome $s_{j,i}$ for a received stream of information is the result of the projective measurement associated to the $M_{j,i}$. It is equal to $+1$ (resp. $-1$) if the measured state belongs to the $+1$ (resp. $-1$) eigenspace of $M_{j,i}$. An element of the Pauli group of the transmitted qubits is said to be compatible with the syndrome $s_{j,i}$ if it commutes (resp. anti-commutes) with $M_{j,i}$ when $s_{j,i} = 1$ (resp. $-1$). An error candidate up to block $j$ is an operator of the Pauli group defined on all the qubits up to block $j$ and which satisfies all the syndromes up to block $j$. The likelihood of an error candidate is the logarithm of the probability of getting this particular error pattern according to the channel model. Since we consider memoryless channels, the likelihood is the sum of the logarithms of single-qubit-error probabilities.

5.2 Quantum Viterbi algorithm

The algorithm examines the syndromes block by block and updates a list of error candidates among which one of them coincides with the most likely error. All this algorithm is classical except the syndrome extraction procedure.

The value of the syndrome is obtained by the usual phase estimation circuit: an ancillary qubit is prepared in the $|0\rangle$ state; undergoes a Hadamard gate; conditionally applies $M_{j,i}$; is once again modified by a Hadamard gate; and is measured according to the $Z$ observable. The result of this measure is the value of the syndrome $s_{j,i}$.

Algorithm 5.1 (Quantum Viterbi algorithm)

**Inputs:** (i) The list of syndromes $\{s_{j+1,i}\}$ for $i = 1, \ldots, n - k$; (ii) a list $\{E_{j}^{(e)}\}$ with $e \in \{I, X, Y, Z\} \otimes m$ of error candidates up to block $j$ such that the element $E_{j}^{(e)}$ corresponding to the index $e$ has a tensor product decomposition ending by $e$ for its last $m$ qubits, and such that it maximizes the likelihood given the previous constraint. The list $\{E_{j}^{(e)}\}$ is constructed recursively.

**Step $j + 1$:** For a given value of $e' \in \{I, X, Y, Z\} \otimes m$, consider all the possible $n$-qubit extensions of the elements of $E_{j}^{(e)}$ such that:

- they satisfy the syndromes $s_{j+1,i}$ for $i = 1, \ldots, n - k$;
- they have the prescribed tensor product decomposition $e'$ on their last $m$ positions.

By construction, these extensions are error candidates up to block $j + 1$. For each $e' \in \{I, X, Y, Z\} \otimes m$ select one such extension with maximum likelihood—take one at random among them in case of tie. This constitutes the new list of error candidates $\{E_{j+1}^{(e')}\}_{e'}$. When all the syndromes have been taken care of in this way, select the most likely error candidate of the list. This error candidate is one of the most likely errors compatible with all the syndromes.
Proof 5.1 Consider a most likely error $E_p$ for the whole $p$ blocks of syndromes. The truncation of this error to the first $p-1$ blocks, $E_{p-1}$, is by construction an error candidate up to block $p-1$. This error candidate has maximum likelihood given its decomposition on the last $m$ qubits. If it was not the case, another error candidate, $E_{p-1}$, with the same decomposition on the last $m$ qubits could be extended up to block $p$ by concatenation with the last $n$ Pauli operators of $E$. It would therefore have a strictly greater likelihood than $E$. Recursively, this property holds for $E_j$: it has maximum likelihood given its tensor product decomposition on the last $m$ positions. Thus, at each step $j$ of the algorithm, one element of the list coincides with the most likely error up to block $j$. 

Remark 5.1 Note that in the encoding of quantum convolutional codes, we chose to set to $|0\rangle$ some logical qubits that were not described by the polynomial formalism. This was done formally by adding their $Z$ operators to the stabilizer group of the code. Hence either the first and last steps of the algorithm should be modified to take into account these extra syndromes.

Remark 5.2 It is also important to understand that in the error estimation algorithm presented above, the most likely error is known only at the end of the algorithm. However, in practice the error candidates considered at step $j$ all coincide except on the last few blocks. Hence, the most likely error is known except on the last few blocks. Some simulations for a depolarizing channel with error probability less than 0.05 showed that keeping two blocks in the 5-qubit convolutional code was enough to estimate the most likely error with high probability.

6 Conclusion

This article showed the basis of quantum convolutional coding. An appropriate polynomial formalism has been introduced to handle the codes efficiently and to make calculations consistently with their specific structure. A procedure for deriving an encoding circuit with linear gate complexity has been given together with a condition which warrants the good behavior of this circuit with respect to error propagation effects. Finally, the quantum Viterbi algorithm has been given explicitly. This algorithm finds the most likely error with a complexity growing linearly with the number of encoded qubits.

More importantly, as the reader familiar with classical convolutional codes can notice, other error estimation algorithms, such as Bahl’s [2] algorithm—a stepping stone toward turbo-decoding—, can readily be employed with the codes described here. Hence, quantum convolutional codes open a new range of efficient error correction strategies.

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