Weighted averages of $n$-convex functions via extension of Montgomery’s identity

Abstract

Using an extension of Montgomery’s identity and the Green’s function, we obtain new identities and related inequalities for weighted averages of $n$-convex functions, i.e., the sum $\sum_{i=1}^{m} \rho_i h(\lambda_i)$ and the integral $\int_{a}^{b} \rho(\lambda) h(\gamma(\lambda)) d\lambda$ where $h$ is an $n$-convex function.

Mathematics Subject Classification

26A51 · 26D15 · 26D20

1 Introduction

In this paper we will give some identities and related inequalities involving weighted discrete or integral averages of functions, i.e., containing sums $\sum_{i=1}^{m} \rho_i h(\lambda_i)$ or integrals $\int_{a}^{b} \rho(\lambda) h(\gamma(\lambda)) d\lambda$. As a consequence we will give conditions on numbers $\lambda_1, \ldots, \lambda_m, \rho_1, \ldots, \rho_m$ under which the inequality $\sum_{i=1}^{m} \rho_i h(\lambda_i) \geq 0$ holds for every function $h$ from a particular class of functions. For example, for the class of convex functions such results were studied in [5], while Popoviciu [7–9] gave results for the class of $n$-convex functions (see [6, Chapter 9] also). We will extend the results of Popoviciu and we would start first with some basic definitions and properties of $n$-convex functions.

Definition 1.1

The $n$th order divided difference of a function $h : [a, b] \to \mathbb{R}$ at distinct points $\lambda_i, \lambda_{i+1}, \ldots, \lambda_{i+n} \in [a, b] \subset \mathbb{R}$ for some $i \in \mathbb{N}$ is defined recursively by:

$$\left[\lambda; h\right] = h\left(\lambda\right), \quad j \in \{i, \ldots, i+n\}$$

$$\left[\lambda_{i}, \ldots, \lambda_{i+n}; h\right] = \frac{\left[\lambda_{i+1}, \ldots, \lambda_{i+n}; h\right] - \left[\lambda_{i}, \ldots, \lambda_{i+n-1}; h\right]}{\lambda_{i+n} - \lambda_{i}}.$$
A function \( h \) is said to be convex of order \( n \) or \( n \)-convex if for all choices of distinct points \( \lambda_1, \ldots, \lambda_{i+n} \), we have \( \left[ \lambda_i, \ldots, \lambda_{i+n}; h \right] \geq 0 \).

From the definition, it is easy to see that 1-convex functions are nondecreasing functions, while 2-convex functions are the classical convex functions, so \( n \)-convex functions are a generalization of the notion of convexity. If the \( n \)th order derivative \( h^{(n)} \) exists, then \( h \) is \( n \)-convex iff \( h^{(n)} \geq 0 \). For \( 1 \leq k \leq n-2 \), a function \( h \) is \( n \)-convex iff \( h^{(k)} \) exists and is \((n-k)\)-convex.

The following result is due to Popoviciu [7, 8] (see [6] also).

**Proposition 1.2** The inequality

\[
\sum_{i=1}^{m} \rho_i h(\lambda_i) \geq 0, \tag{1.1}
\]

holds for all \( n \)-convex functions \( h : [a, b] \to \mathbb{R}, n \in \mathbb{N} \), iff the \( m \)-tuples \( \lambda \in [a, b]^m, \rho \in \mathbb{R}^m \) satisfy

\[
\sum_{i=1}^{m} \rho_i \lambda_i^k = 0, \quad \text{for all } k = 0, 1, \ldots, n-1, \tag{1.2}
\]

\[
\sum_{i=1}^{m} \rho_i (\lambda_i - t)^{n-1}_+ \geq 0, \quad \text{for every } t \in [a, b], \tag{1.3}
\]

where \( y_+ = \max(y, 0) \).

In fact, Popoviciu proved a stronger result that it is enough to assume that the inequality in (1.3) holds for every \( t \in [\lambda_{(1)}, \lambda_{(m-n+1)}] \), where \( \lambda_{(1)} \leq \cdots \leq \lambda_{(m)} \) is the ordered \( m \)-tuple \( \lambda \), since this, together with (1.2), implies that it holds for every \( t \in [a, b] \) (see [9]). In the case of convex functions, i.e., \( n = 2 \), Pečarić [5] proved the result with the conditions (1.2) and (1.3) replaced with

\[
\sum_{i=1}^{m} \rho_i = 0 \quad \text{and} \quad \sum_{i=1}^{m} \rho_i |\lambda_i - \lambda_k| \geq 0 \quad \text{for } k \in \{1, \ldots, m\}. \tag{1.4}
\]

The integral analogue of Proposition 1.2 is given in the next proposition.

**Proposition 1.3** Let \( n \geq 2, \rho : [a, \beta] \to \mathbb{R} \) and \( \gamma : [a, \beta] \to [a, b] \). The inequality

\[
\int_{a}^{\beta} \rho(\lambda) h(\gamma(\lambda)) \, d\lambda \geq 0, \tag{1.5}
\]

holds for all \( n \)-convex functions \( h : [a, b] \to \mathbb{R} \) iff

\[
\int_{a}^{\beta} \rho(\lambda) \gamma(\lambda)^k \, d\lambda = 0, \quad \text{for all } k = 0, 1, \ldots, n-1, \tag{1.6}
\]

\[
\int_{a}^{\beta} \rho(\lambda) (\gamma(\lambda) - t)^{n-1}_+ \, d\lambda \geq 0, \quad \text{for every } t \in [a, b].
\]

We will also need the following extension of Montgomery’s identity given in [1] which was derived using Taylor’s formula.

**Proposition 1.4** Let \( n \in \mathbb{N}, h : I \to \mathbb{R} \) be such that \( h^{(n-1)} \) is absolutely continuous, \( I \subset \mathbb{R} \) an open interval, \( a, b \in I, a < b \). Then the following identity holds

\[
\begin{align*}
\int_{a}^{b} h(\lambda) = & \int_{a}^{b} h(s) \, ds + \sum_{k=0}^{n-2} \frac{h^{(k+1)}(a)(\lambda - a)^{k+2}}{k!(k+2)} - \frac{1}{b-a} \\
& - \sum_{k=0}^{n-2} \frac{h^{(k+1)}(b)(\lambda - b)^{k+2}}{k!(k+2)} + \frac{1}{b-a} \sum_{k=0}^{n-1} T_n(\lambda, t) h^{(k)}(t) \, dt. \tag{1.7}
\end{align*}
\]
where
\[ T_n(\lambda, t) = \begin{cases} \frac{(\lambda-t)^{n}}{n(b-a)} + \frac{\lambda-a}{b-a} (\lambda-t)^{n-1}, & a \leq t \leq \lambda, \\ \frac{(\lambda-t)^{n}}{n(b-a)} + \frac{\lambda-b}{b-a} (\lambda-t)^{n-1}, & \lambda < t \leq b. \end{cases} \] (1.8)

In case \( n = 1 \) the sum \( \sum_{k=0}^{n-2} \) is empty, so identity (1.7) reduces to the well-known Montgomery identity (see for instance [4])

\[ h(\lambda) = \frac{1}{b-a} \int_{a}^{b} h(t) \, dt + \int_{a}^{b} P(\lambda, s) h'(s) \, ds, \]

where \( P(\lambda, s) \) is the Peano kernel defined by

\[ P(\lambda, s) = \begin{cases} \frac{s-a}{b-a}, & a \leq s \leq \lambda, \\ \frac{s-b}{b-a}, & \lambda < s \leq b. \end{cases} \]

Let us denote by \( G : [a, b] \times [a, b] \rightarrow \mathbb{R} \) the Green’s function of the boundary value problem

\[ z''(\lambda) = 0, \quad z(a) = z(b) = 0. \]

The function \( G \) is given by

\[ G(t, s) = \begin{cases} \frac{(t-b)(s-a)}{b-a}, & \text{for } a \leq s \leq t, \\ \frac{(s-b)(t-a)}{b-a}, & \text{for } t \leq s \leq b \end{cases} \] (1.9)

and integration by parts easily yields that for any function \( h \in C^2[a, b] \) the following identity holds

\[ h(\lambda) = \frac{b-\lambda}{b-a} h(a) + \frac{\lambda-a}{b-a} h(b) + \int_{a}^{b} G(\lambda, s) h''(s) \, ds. \] (1.10)

The function \( G \) is continuous, symmetric and convex with respect to both variables \( t \) and \( s \).

The paper is organized as follows: in Sect. 2, we obtain new identities involving discrete and integral weighted averages of \( n \)-convex functions by appropriate use of the extended Montgomery’s identity and the Green’s function. Related Popoviciu type inequalities are also derived. In Sect. 3, we obtain new Grüss- and Ostrowski-type inequalities by obtaining bounds for the remainders of the identities from Sect. 2.

2 Identities and related Popoviciu-type inequalities for \( n \)-convex functions

We will first prove couple of identities which will have a key role in the rest of the paper.

**Theorem 2.1** Let \( n \in \mathbb{N}, n \geq 3, h : I \rightarrow \mathbb{R} \) be a function such that \( h^{(n-1)} \) is absolutely continuous, \( I \subset \mathbb{R} \) an open interval, \( a, b \in I, a < b \). Furthermore, let \( \lambda \in [a, b] \) and \( \rho \in \mathbb{R}^{m} \) satisfy \( \sum_{i=1}^{m} \rho_{i} \lambda_{i} = 0 \) and \( \sum_{i=1}^{m} \rho_{i} \lambda_{i} = 0 \) and let \( G \) be given by (1.9). Then

\[
\sum_{i=1}^{m} \rho_{i} h(\lambda_{i}) = \frac{h'(a) - h'(b)}{b-a} \int_{a}^{b} \sum_{i=1}^{m} \rho_{i} G(\lambda_{i}, s) \, ds \\
+ \sum_{k=2}^{n-1} \frac{k}{(k-1)!} \int_{a}^{b} \sum_{i=1}^{m} \rho_{i} G(\lambda_{i}, s) \frac{h^{(k)}(s-a)(s-a)^{k-1} - h^{(k)}(b)(s-b)^{k-1}}{b-a} \, ds \\
+ \frac{1}{(n-3)!} \int_{a}^{b} h^{(n)}(t) \left( \int_{a}^{b} \sum_{i=1}^{m} \rho_{i} G(\lambda_{i}, s) \tilde{T}_{n-2}(s, t) \, ds \right) \, dt.
\] (2.1)
Moreover, the following identity holds
\[\sum_{i=1}^{m} \rho_i h(\lambda_i) = \frac{h'(b) - h'(a)}{b-a} \int_{a}^{b} \sum_{i=1}^{m} \rho_i G(\lambda_i, s) ds
+ \sum_{k=3}^{n-1} \frac{k-2}{(k-1)!} \int_{a}^{b} \sum_{i=1}^{m} \rho_i G(\lambda_i, s) \frac{h^{(k)}(a)(s-a)^{k-1} - h^{(k)}(b)(s-b)^{k-1}}{b-a} ds
+ \frac{1}{(n-3)!} \int_{a}^{b} h^{(n)}(t) \left( \int_{a}^{b} \sum_{i=1}^{m} \rho_i G(\lambda_i, s) T_{n-2}(s, t) ds \right) dt, \tag{2.3}\]
where \(T_n\) is as defined in (1.8).

Proof Using (1.10) in \(\sum_{i=1}^{m} \rho_i h(\lambda_i)\) and the fact that \(\sum_{i=1}^{m} \rho_i = 0\) and \(\sum_{i=1}^{m} \rho_i \lambda_i = 0\), we get
\[\sum_{i=1}^{m} \rho_i h(\lambda_i) = \int_{a}^{b} \sum_{i=1}^{m} \rho_i G(\lambda_i, s) h''(s) ds. \tag{2.4}\]

Differentiating the function \(h\) in (1.7) twice gives
\[h''(s) = \frac{h'(a) - h'(b)}{b-a} + \sum_{k=2}^{n-1} \frac{k}{(k-1)!} \frac{h^{(k)}(a)(s-a)^{k-1} - h^{(k)}(b)(s-b)^{k-1}}{b-a}
+ \frac{1}{(n-3)!} \int_{a}^{b} \tilde{T}_{n-2}(s, t) h^{(n)}(t) dt. \tag{2.5}\]

Inserting (2.5) in (2.4) yields
\[\sum_{i=1}^{m} \rho_i h(\lambda_i) = \frac{h'(a) - h'(b)}{b-a} \int_{a}^{b} \sum_{i=1}^{m} \rho_i G(\lambda_i, s) ds
+ \sum_{k=2}^{n-1} \frac{k}{(k-1)!} \int_{a}^{b} \sum_{i=1}^{m} \rho_i G(\lambda_i, s) \frac{h^{(k)}(a)(s-a)^{k-1} - h^{(k)}(b)(s-b)^{k-1}}{b-a} ds
+ \frac{1}{(n-3)!} \int_{a}^{b} \sum_{i=1}^{m} \rho_i G(\lambda_i, s) \left( \int_{a}^{b} \tilde{T}_{n-2}(s, t) h^{(n)}(t) dt \right) ds, \tag{2.6}\]
and then using Fubini's theorem in the last term we get (2.1).

Moreover, by applying formula (1.7) with \(h\) and \(n\) replaced by \(h''\) and \(n-2\), respectively, and rearranging the indices, we get
\[h''(s) = \frac{h'(b) - h'(a)}{b-a} + \sum_{k=3}^{n-1} \frac{k-2}{(k-1)!} \frac{h^{(k)}(a)(s-a)^{k-1} - h^{(k)}(b)(s-b)^{k-1}}{b-a}
+ \frac{1}{(n-3)!} \int_{a}^{b} T_{n-2}(s, t) h^{(n)}(t) dt. \tag{2.6}\]

Similarly, using (2.6) in (2.4) and applying Fubini's Theorem, we get (2.3). \qed

Next we will state some inequalities that can be derived from the obtained identities.
Theorem 2.2 Let all the assumptions of Theorem 2.1 hold with the additional condition
\[ \int_a^b \sum_{i=1}^m \rho_i G(\lambda_i, s) \tilde{T}_{n-2}(s, t) \, ds \geq 0, \quad \forall t \in [a, b], \]  
(2.7)
where G and \( \tilde{T}_{n-2} \) are defined in (1.9) and (2.2). If h is n-convex, then the following inequality holds
\[ \sum_{i=1}^m \rho_i h(\lambda_i) - \frac{h'(a) - h'(b)}{b - a} \int_a^b \sum_{i=1}^m \rho_i G(\lambda_i, s) \, ds \]

\[ - \sum_{k=2}^{n-1} \frac{k}{(k-1)!} \int_a^b \sum_{i=1}^m \rho_i G(\lambda_i, s) \frac{h^{(k)}(a)(s-a)^{k-1} - h^{(k)}(b)(s-b)^{k-1}}{b - a} \, ds \geq 0. \]

(2.8)

Proof Since the function h is n-convex, we have \( h^{(n)} \geq 0 \). Using this fact and (2.7) in (2.1) we easily arrive at our required result. \( \square \)

Theorem 2.3 Let all the assumptions of Theorem 2.1 hold with the additional condition
\[ \int_a^b \sum_{i=1}^m \rho_i G(\lambda_i, s) T_{n-2}(s, t) \, ds \geq 0, \quad \forall t \in [a, b], \]  
(2.9)
where G and \( T_n \) are defined in (1.9) and (1.8). If h is n-convex, then the following inequality holds
\[ \sum_{i=1}^m \rho_i h(\lambda_i) - \frac{h'(a) - h'(b)}{b - a} \int_a^b \sum_{i=1}^m \rho_i G(\lambda_i, s) \, ds \]

\[ - \sum_{k=2}^{n-1} \frac{k}{(k-1)!} \int_a^b \sum_{i=1}^m \rho_i G(\lambda_i, s) \frac{h^{(k)}(a)(s-a)^{k-1} - h^{(k)}(b)(s-b)^{k-1}}{b - a} \, ds \geq 0. \]

(2.10)

Proof Since the function h is n-convex we have \( h^{(n)} \geq 0 \). Using this fact and (2.9) in (2.3) we easily arrive at our required result. \( \square \)

Now we state an important consequence.

Theorem 2.4 Let all the assumptions from Theorem 2.1 hold with the additional assumptions \( \sum_{i=1}^m \rho_i = 0 \) and \( \sum_{i=1}^m \rho_i |\lambda_i - \lambda_k| \geq 0 \) for \( k \in \{1, \ldots, m\} \). If h is n-convex and n is even, then inequalities (2.8) and (2.10) hold.

Proof The Green’s function G(s, t) is convex with respect to t for every s \( \in [a, b] \). Therefore, from Proposition 1.2, with conditions (1.2) and (1.3) replaced by (1.4) as in [5], we have
\[ \sum_{i=1}^m \rho_i G(\lambda_i, s) \geq 0 \quad \text{for} \quad s \in [a, b]. \]

(2.11)

Also note that for even n both \( \tilde{T}_{n-2}(s, t) \geq 0 \) and \( T_{n-2}(s, t) \geq 0 \). Therefore, combining this fact with (2.11) we get inequalities (2.7) and (2.9). As h is n-convex, the results follow from Theorems 2.2 and 2.3. \( \square \)

We will next state the integral versions of our main results. Since the proofs are of similar nature we will omit the details.

Theorem 2.5 Let \( n \in \mathbb{N}, n \geq 3, h : I \rightarrow \mathbb{R} \) be a function such that \( h^{(n-1)} \) is absolutely continuous, \( I \subset \mathbb{R} \) an open interval, \( a, b \in I, a < b \). Furthermore, let \( \gamma : [\alpha, \beta] \rightarrow [a, b] \) and \( \rho : [\alpha, \beta] \rightarrow \mathbb{R} \) satisfy
Theorem 2.7 Let all the assumptions from Theorem 2.5 hold with the additional assumption that \( \beta \) following two identities hold:

\[
\int_a^b p(\lambda) h(\gamma(\lambda)) d\lambda = 0 \quad \text{and} \quad \int_a^b \frac{\rho(\lambda) \gamma(\lambda)}{\rho(\lambda) \gamma(\lambda)} d\lambda = 0,
\]

and let \( G \) be defined by (1.9), \( T_n \) be defined by (1.8). Then the following two identities hold:

\[
\int_a^b p(\lambda) h(\gamma(\lambda)) d\lambda = \frac{h'(a) - h'(b)}{b - a} \int_a^b p(\lambda) G(\gamma(\lambda), s) d\lambda ds + \sum_{k=2}^{n-1} \frac{k}{(k-1)!} \int_a^b \left( \int_a^b p(\lambda) G(\gamma(\lambda), s) d\lambda \right) \frac{h^{(k)}(a)(s-a)^{k-1} - h^{(k)}(b)(s-b)^{k-1}}{b - a} ds
\]

\[
+ \frac{1}{(n-3)!} \int_a^b h^{(n)}(t) \left( \int_a^b \left( \int_a^b p(\lambda) G(\gamma(\lambda), s) d\lambda \right) \tilde{T}_{n-2}(s,t) ds \right) dt.
\]

\[
\int_a^b p(\lambda) h(\gamma(\lambda)) d\lambda = \frac{h'(b) - h'(a)}{b - a} \int_a^b p(\lambda) G(\gamma(\lambda), s) d\lambda ds + \sum_{k=2}^{n-1} \frac{k - 2}{(k-1)!} \int_a^b \left( \int_a^b p(\lambda) G(\gamma(\lambda), s) d\lambda \right) \frac{h^{(k)}(a)(s-a)^{k-1} - h^{(k)}(b)(s-b)^{k-1}}{b - a} ds
\]

\[
+ \frac{1}{(n-3)!} \int_a^b h^{(n)}(t) \left( \int_a^b \left( \int_a^b p(\lambda) G(\gamma(\lambda), s) d\lambda \right) T_{n-2}(s,t) ds \right) dt.
\]

Theorem 2.6 Let all the assumptions of Theorem 2.5 hold with the additional condition

\[
\int_a^b \int_a^b p(\lambda) G(\gamma(\lambda), s) \tilde{T}_{n-2}(s,t) d\lambda ds \geq 0, \quad \forall \ t \in [a, b],
\]

where \( G \) is defined in (1.9) and \( \tilde{T}_n \) is defined in (2.2). If \( h \) is \( n \)-convex, then the following inequality holds:

\[
\int_a^b p(\lambda) h(\gamma(\lambda)) d\lambda = \frac{h'(a) - h'(b)}{b - a} \int_a^b p(\lambda) G(\gamma(\lambda), s) d\lambda ds + \sum_{k=2}^{n-1} \frac{k}{(k-1)!} \int_a^b \left( \int_a^b p(\lambda) G(\gamma(\lambda), s) d\lambda \right) \frac{h^{(k)}(a)(s-a)^{k-1} - h^{(k)}(b)(s-b)^{k-1}}{b - a} ds \geq 0.
\]

Theorem 2.7 Let all the assumptions of Theorem 2.5 hold with the additional condition

\[
\int_a^b \int_a^b p(\lambda) G(\gamma(\lambda), s) T_{n-2}(s,t) d\lambda ds \geq 0, \quad \forall \ t \in [a, b],
\]

where \( G \) is defined in (1.9) and \( T_n \) is defined in (1.8). If \( h \) is \( n \)-convex, then the following inequality holds:

\[
\int_a^b p(\lambda) h(\gamma(\lambda)) d\lambda = \frac{h'(b) - h'(a)}{b - a} \int_a^b p(\lambda) G(\gamma(\lambda), s) d\lambda ds + \sum_{k=2}^{n-1} \frac{k - 2}{(k-1)!} \int_a^b \left( \int_a^b p(\lambda) G(\gamma(\lambda), s) d\lambda \right) \frac{h^{(k)}(a)(s-a)^{k-1} - h^{(k)}(b)(s-b)^{k-1}}{b - a} ds \geq 0.
\]

Theorem 2.8 Let all the assumptions from Theorem 2.5 hold with the additional assumption that \( \gamma : [\alpha, \beta] \rightarrow [a, b] \) and \( \rho : [\alpha, \beta] \rightarrow \mathbb{R} \) satisfy (1.6). If \( h \) is \( n \)-convex and \( n \) is even, then inequalities (2.15) and (2.18) hold.
3 Bounds for the remainders

In these sections, we will give bounds for the remainders which occur in certain representations of the sum \( \sum_{i=1}^{m} \rho_i h(\lambda_i) \) and the integral \( \int_{a}^{b} \rho(\lambda) h(\gamma(\lambda)) \, d\lambda \). Namely, we will give some Grüss- and Ostrowski-type inequalities.

Let \( h, \gamma : [a, b] \to \mathbb{R} \) be two Lebesgue integrable functions. We consider the Čebyshev functional

\[
T(h, g) = \frac{1}{b - a} \int_{a}^{b} h(\lambda) g(\lambda) \, d\lambda - \left( \frac{1}{b - a} \int_{a}^{b} h(\lambda) \, d\lambda \right) \left( \frac{1}{b - a} \int_{a}^{b} g(\lambda) \, d\lambda \right) \tag{3.1}
\]

\( L_\infty [a, b] \) denotes the space of essentially bounded functions on \([a, b]\) with the norm

\[
\|h\|_\infty = \text{ess sup}_{t \in [a, b]} |h(t)|.
\]

The following results can be found in [3]:

**Proposition 3.1** Let \( h : [a, b] \to \mathbb{R} \) be a Lebesgue integrable function and let \( g : [a, b] \to \mathbb{R} \) be an absolutely continuous function with \((\cdot - a)(b - \cdot)|g'|^2 \in L[a, b] \). Then we have the inequality

\[
|T(h, g)| \leq \frac{1}{\sqrt{2}} \left( \frac{1}{b - a} |T(h, h)| \int_{a}^{b} (\lambda - a)(b - \lambda)|g'|^2 \, d\lambda \right)^{1/2}.
\tag{3.2}
\]

The constant \( \frac{1}{\sqrt{2}} \) in (3.2) is the best possible.

**Proposition 3.2** Let \( g : [a, b] \to \mathbb{R} \) be a monotonic nondecreasing function and let \( h : [a, b] \to \mathbb{R} \) be an absolutely continuous function such that \( h' \in L_\infty[a, b] \). Then we have the inequality

\[
|T(h, g)| \leq \frac{1}{2(b - a)} \|h'|_\infty \int_{a}^{b} (\lambda - a)(b - \lambda) \, d\lambda.
\tag{3.3}
\]

The constant \( \frac{1}{2} \) in (3.3) is the best possible.

For the ease of notation, throughout this section \( \Omega_j, \ j \in \{1, 2, 3, 4\} \), will denote the following functions: under the assumption of Theorems 2.1 and 2.5 we define

\[
\Omega_1(t) = \int_{a}^{b} \sum_{i=1}^{m} \rho_i G(\lambda_i, s) \tilde{T}_{n-2}(s, t) \, ds, \quad t \in [a, b],
\]

\[
\Omega_2(t) = \int_{a}^{b} \sum_{i=1}^{m} \rho_i G(\lambda_i, s) T_{n-2}(s, t) \, ds \geq 0, \quad t \in [a, b],
\]

\[
\Omega_3(t) = \int_{a}^{b} \int_{a}^{b} p(\lambda) G(\gamma(\lambda), s) \tilde{T}_{n-2}(s, t) \, d\lambda \, ds, \quad t \in [a, b],
\]

\[
\Omega_4(t) = \int_{a}^{b} \int_{a}^{b} p(\lambda) G(\gamma(\lambda), s) T_{n-2}(s, t) \, d\lambda \, ds, \quad t \in [a, b].
\]

**Theorem 3.3** Let \( n \in \mathbb{N}, n \geq 3, h : [a, b] \to \mathbb{R} \) be such that \( h^{(m)} \) is an absolutely continuous function with \((\cdot - a)(b - \cdot)|h^{(n+1)}|^2 \in L[a, b] \) and let \( \lambda \in [a, b]^m \) and \( \rho \in \mathbb{R}^m \) satisfy \( \sum_{i=1}^{m} \rho_i = 0 \) and \( \sum_{i=1}^{m} \rho_i \lambda_i = 0 \). Then

\[
\sum_{i=1}^{m} \rho_i h(\lambda_i) = \frac{h'(a) - h'(b)}{b - a} \int_{a}^{b} \sum_{i=1}^{m} \rho_i G(\lambda_i, s) \, ds
\]

\[
+ \sum_{k=2}^{n-1} \frac{k}{(k-1)!} \int_{a}^{b} \sum_{i=1}^{m} \rho_i G(\lambda_i, s) \frac{h^{(k)}(a)(s - a)^{k-1} - h^{(k)}(b)(s - b)^{k-1}}{b - a} \, ds
\]

\[
+ \frac{h^{(n-1)}(b) - h^{(n-1)}(a)}{(n-3)! (b - a)} \int_{a}^{b} \Omega(s) \, ds + R_n^1(h; a, b),
\tag{3.4}
\]

\( \Omega(s) \) is an absolutely continuous function with \( \int_{a}^{b} \Omega(s) \, ds = 0 \) and \( \int_{a}^{b} s \, \Omega(s) \, ds = 0 \).
and

\[
\sum_{i=1}^{m} \rho_i h(\lambda_i) = \frac{h'(b) - h'(a)}{b - a} \int_{a}^{b} \sum_{i=1}^{m} \rho_i G(\lambda_i, s)\,ds \\
= \frac{-k}{2} \int_{a}^{b} \sum_{i=1}^{m} \rho_i G(\lambda_i, s)\,ds \\
= \frac{h^{(n-1)}(b) - h^{(n-1)}(a)}{(n-3)!(b - a)} \int_{a}^{b} \Omega_2(s)\,ds + R_n^{1}(h; a, b),
\]

(3.5)

where the remainders \( R_n^{j}(h; a, b), j = 1, 2, \) satisfy the bounds

\[
|R_n^{j}(h; a, b)| \leq \frac{1}{(n-3)!} \left( \frac{b - a}{2} \right) \left( \frac{b - a}{2} \right) \left( \frac{(s - a)(b - s)}{[h^{(n+1)}(s)]^2} \right) \right)^{1/2}.
\]

(3.6)

**Proof** We will prove the claim for \( j = 1, \) while the proof for \( j = 2 \) is analogous. Proposition 3.1 with \( h \rightarrow \Omega_1 \) and \( g \rightarrow h^{(n)} \) yields

\[
\left| \frac{1}{b - a} \int_{a}^{b} \Omega_1(t)h^{(n)}(t)\,dt - \left( \frac{1}{b - a} \int_{a}^{b} \Omega_1(t)\,dt \right) \left( \frac{1}{b - a} \int_{a}^{b} h^{(n)}(t)\,dt \right) \right| \\
\leq \frac{1}{\sqrt{2}} \left( \frac{1}{b - a} \left| \int_{a}^{b} (t - a)(b - t)[h^{(n+1)}(t)]^2\,dt \right| \right)^{1/2}.
\]

(3.7)

By identity (2.1) from Theorem 2.1

\[
\sum_{i=1}^{m} \rho_i h(\lambda_i) = \frac{h'(a) - h'(b)}{b - a} \int_{a}^{b} \sum_{i=1}^{m} \rho_i G(\lambda_i, s)\,ds \\
- \sum_{k=2}^{n-1} \frac{k}{(k-1)!} \int_{a}^{b} \sum_{i=1}^{m} \rho_i G(\lambda_i, s)\,ds \frac{h^{(k)}(a)(s-a)^{k-1} - h^{(k)}(b)(s-b)^{k-1}}{b - a} \\
= \frac{1}{(n-3)!} \int_{a}^{b} \Omega_1(t)h^{(n)}(t)\,dt
\]

and since

\[
\frac{1}{(n-3)!} \int_{a}^{b} \Omega_1(t)h^{(n)}(t)\,dt = \frac{h^{(n-1)}(b) - h^{(n-1)}(a)}{(n-3)!(b - a)} \int_{a}^{b} \Omega_1(t)\,dt + R_n^{1}(h; a, b),
\]

the bound (3.6) for the remainder \( R_n^{1}(h; a, b) \) follows from (3.7).

Using Proposition 3.2, we obtain the following Grüss-type inequality.

**Theorem 3.4** Let \( n \in \mathbb{N}, n \geq 3, h : [a, b] \rightarrow \mathbb{R} \) be such that \( h^{(n)} \) is an absolutely continuous function with \( h^{(n+1)} \geq 0 \) and let \( k \in [a, b]^m \) and \( \rho \in \mathbb{R}^m \) satisfy \( \sum_{i=1}^{m} \rho_i = 0 \) and \( \sum_{i=1}^{m} \rho_i \lambda_i = 0. \) Then representations (3.4) and (3.5) hold and the remainders \( R_n^{j}(h; a, b), j = 1, 2, \) satisfy the bounds

\[
|R_n^{j}(h; a, b)| \leq \frac{1}{(n-3)!} \| \Omega_j' \|_{\infty} \left\{ \frac{b - a}{2} \left[ h^{(n-1)}(b) + h^{(n-1)}(a) \right] \\
- \left[ h^{(n-2)}(b) - h^{(n-2)}(a) \right] \right\}.
\]

(3.8)
Proof Proposition 3.2 with \( h \rightarrow \Omega_j \) and \( g \rightarrow h^{(n)} \) yields
\[
\left| \frac{1}{b-a} \int_a^b \Omega_j(t)h^{(n)}(t)dt - \left( \frac{1}{b-a} \int_a^b \Omega_j(t)dt \right) \left( \frac{1}{b-a} \int_a^b h^{(n)}(t)dt \right) \right| \\
\leq \frac{1}{2(b-a)} \| \Omega_j' \|_{\infty} \int_a^b (t-a)(b-t)h^{(n+1)}(t)dt.
\]
Since
\[
\int_a^b (t-a)(b-t)h^{(n+1)}(t)dt = \int_a^b (2t-a-b)h^{(n)}(t)dt \\
= (b-a) \left[ h^{(n-1)}(b) + h^{(n-1)}(a) \right] - 2 \left[ h^{(n-2)}(b) - h^{(n-2)}(a) \right], \tag{3.9}
\]
using (3.9) and the identities from Theorem 2.1, we deduce (3.8).

Here, the symbol \( L_p[a,b] \) \((1 \leq p < \infty)\) denotes the space of \( p \)-power integrable functions on the interval \([a,b]\) equipped with the norm
\[
\| h \|_p = \left( \int_a^b |h(t)|^p dt \right)^{1/p}.
\]

Now we state some Ostrowski-type inequalities related to the generalized linear inequalities.

**Theorem 3.5** Let \( n \in \mathbb{N}, n \geq 3, 1 \leq q, r \leq \infty, \quad \frac{1}{q} + \frac{1}{r} = 1, \) \( h^{(n)} \in L_q[a,b] \) and let \( \lambda \in [a,b]^m \) and \( \rho \in \mathbb{R}^m \) satisfy \( \sum_{i=1}^m \rho_i = 0 \) and \( \sum_{i=1}^m \rho_i \lambda_i = 0. \) Then
\[
\left| \sum_{i=1}^m \rho_i h(\lambda_i) - \frac{h'(a) - h'(b)}{b-a} \int_a^b \sum_{i=1}^m \rho_i G(\lambda_i, s)ds \\
- \sum_{k=2}^{n-1} \frac{k}{(k-1)!} \int_a^b \sum_{i=1}^m \rho_i G(\lambda_i, s) \frac{h^{(k)}(a)(s-a)^{k-1} - h^{(k)}(b)(s-b)^{k-1}}{b-a}ds \right| \\
\leq \frac{1}{(n-3)!} \| h^{(n)} \|_q \| \Omega_1 \|_r. \tag{3.10}
\]

and
\[
\left| \sum_{i=1}^m \rho_i h(\lambda_i) - \frac{h'(b) - h'(a)}{b-a} \int_a^b \sum_{i=1}^m \rho_i G(\lambda_i, s)ds \\
- \sum_{k=2}^{n-1} \frac{k-2}{(k-1)!} \int_a^b \sum_{i=1}^m \rho_i G(\lambda_i, s) \frac{h^{(k)}(a)(s-a)^{k-1} - h^{(k)}(b)(s-b)^{k-1}}{b-a}ds \right| \\
\leq \frac{1}{(n-3)!} \| h^{(n)} \|_q \| \Omega_2 \|_r. \tag{3.11}
\]

The constant on the right-hand sides of (3.10) and (3.11) is sharp for \( 1 < q \leq \infty \) and the best possible for \( q = 1. \)

**Proof** Let us denote
\[
\mu_j(t) = \frac{1}{(n-3)!} \Omega_j(t), \quad j = 1, 2.
\]

Using identities (2.1) and (2.3) from Theorem 2.1 and Hölder’s inequality, we obtain inequalities (3.10) and (3.11), i.e. that the left-hand sides of these inequalities are less than or equal to
\[
\text{L.H.S.} \leq \| h^{(n)} \|_q \| \mu_j \|_r. \tag{3.12}
\]
For the proof of the sharpness of the constant \( \left( \int_a^b |\mu_j(t)|^r \, dt \right)^{1/r} \), let us find a function \( h \) for which the equality in (3.12) is obtained.

For \( 1 < q < \infty \) take \( h \) to be such that

\[
h^{(n)}(t) = sgn \mu_j(t) \cdot |\mu_j(t)|^{1/(q-1)}.
\]

For \( q = \infty \), take \( h \) such that

\[
h^{(n)}(t) = sgn \mu_j(t).
\]

Finally, for \( q = 1 \), we prove that

\[
\left| \int_a^b \mu_j(t) h^{(n)}(t) \, dt \right| \leq \max_{t \in [a, b]} |\mu_j(t)| \int_a^b h^{(n)}(t) \, dt
\]

is the best possible inequality.

Suppose that \(|\mu_j(t)|\) attains its maximum at \( t_0 \in [a, b] \). First we consider the case \( \mu_j(t_0) > 0 \). For \( \delta \) small enough we define \( h_\delta(t) \) by

\[
h_\delta(t) = \begin{cases} 
0, & a \leq t \leq t_0, \\
\frac{1}{\delta^n} (t - t_0)^n, & t_0 \leq t \leq t_0 + \delta, \\
\frac{b}{(a-\delta)^n} (t - t_0)^{n-1}, & t_0 + \delta \leq t \leq b.
\end{cases}
\]

Therefore, we have

\[
\left| \int_a^b \mu_j(t) h_\delta^{(n)}(t) \, dt \right| = \left| \int_{t_0}^{t_0+\delta} \mu_j(t) \frac{1}{\delta} \, dt \right| = \frac{1}{\delta} \int_{t_0}^{t_0+\delta} \mu_j(t) \, dt
\]

Now from inequality (3.13), we have

\[
\frac{1}{\delta} \int_{t_0}^{t_0+\delta} \mu_j(t) \, dt \leq \mu_j(t_0) \frac{1}{\delta} \int_{t_0}^{t_0+\delta} \, dt = \mu_j(t_0)
\]

Since

\[
\lim_{\delta \to 0} \frac{1}{\delta} \int_{t_0}^{t_0+\delta} \mu_j(t) \, dt = \mu_j(t_0)
\]

the statement follows.

In the case \( \mu_j(t_0) < 0 \), we define \( h_\delta(t) \) by

\[
h_\delta(t) = \begin{cases} 
\frac{1}{\delta^n} (t - t_0 - \delta)^n, & a \leq t \leq t_0, \\
\frac{-b}{(a-\delta)^n} (t - t_0 - \delta)^{n-1}, & t_0 \leq t \leq t_0 + \delta, \\
0, & t_0 + \delta \leq t \leq b.
\end{cases}
\]

and the rest of the proof is the same as above. \( \square \)

We will end this section with the integral versions of the results, the proofs of which are analogous to the discrete case and are omitted.

**Theorem 3.6** Let \( n \in \mathbb{N}, n \geq 3, h : [a, b] \to \mathbb{R} \) be such that \( h^{(n)} \) is an absolutely continuous function with \((e - a)(b - c)[h^{(n+1)}]^2 \in L[a, b] \) and let \( \gamma : [\alpha, \beta] \to [a, b] \) and \( \rho : [\alpha, \beta] \to \mathbb{R} \) satisfy \( \int_a^\beta \rho(\lambda) \, d\lambda = 0 \) and \( \int_a^\beta \rho(\lambda) \gamma(\lambda) \, d\lambda = 0 \). Then

\[
\int_a^\beta p(\lambda) h(\gamma(\lambda)) \, d\lambda = \frac{h'(a) - h'(b)}{b - a} \int_a^b p(\lambda) G(\gamma(\lambda), s) \, d\lambda \, ds
\]

\[
+ \sum_{k=2}^{n-1} \frac{k}{(k-1)!} \int_a^b \left( \int_a^b p(\lambda) G(\gamma(\lambda), s) \, d\lambda \right) \frac{h^{(k)}(a)(s - a)^{k-1} - h^{(k)}(b)(s - b)^{k-1}}{b - a} \, ds
\]

\[
+ \frac{h^{(n-1)}(b) - h^{(n-1)}(a)}{(n-3)!(b - a)} \int_a^b \Omega_3(s) \, ds + R_n(h; a, b), \tag{3.14}
\]

\( \square \) Springer
and
\[
\int_\alpha^\beta p(\lambda) h(\gamma(\lambda)) \, d\lambda = \frac{h'(b) - h'(a)}{b - a} \int_\alpha^b p(\lambda) G(\gamma(\lambda), s) \, d\lambda \, ds \\
+ \sum_{k=3}^{n-1} \frac{k - 2}{(k - 1)!} \left( \int_\alpha^b p(\lambda) G(\gamma(\lambda), s) \, d\lambda \right) \frac{h^{(k)}(a)(s - a)^{k-1} - h^{(k)}(b)(s - b)^{k-1}}{b - a} \, ds \\
+ \frac{h^{(n-1)}(b) - h^{(n-1)}(a)}{(n - 3)!(b - a)} \int_\alpha^b \Omega_4(s) \, ds + R_n^j(h; a, b),
\]
(3.15)

where the remainders \( R_n^j(h; a, b), j = 3, 4, \) satisfy the bounds
\[
|R_n^j(h; a, b)| \leq \frac{1}{(n - 3)!} \|\Omega_j'\|_\infty \left\{ \frac{b - a}{2} \left[ h^{(n-1)}(b) + h^{(n-1)}(a) \right] \\
- \left[ h^{(n-2)}(b) - h^{(n-2)}(a) \right] \right\}.
\]

**Theorem 3.7** Let \( n \in \mathbb{N}, n \geq 3, h : [a, b] \to \mathbb{R} \) be such that \( h^{(n)} \) is an absolutely continuous function with \( h^{(n+1)} \geq 0 \) and let \( \gamma : [a, \beta] \to [a, b] \) and \( \rho : [a, \beta] \to \mathbb{R} \) satisfy \( \int_a^\beta \rho(\lambda) \, d\lambda = 0 \) and \( \int_a^\beta \rho(\lambda) \gamma(\lambda) \, d\lambda = 0 \). Then representations (3.14) and (3.15) hold and the remainders \( R_n^j(h; a, b), j = 3, 4, \) satisfy the bounds
\[
|R_n^j(h; a, b)| \leq \frac{1}{(n - 3)!} \|\Omega_j'\|_\infty \left\{ \frac{b - a}{2} \left[ h^{(n-1)}(b) + h^{(n-1)}(a) \right] \\
- \left[ h^{(n-2)}(b) - h^{(n-2)}(a) \right] \right\}.
\]

**Theorem 3.8** Let \( n \in \mathbb{N}, n \geq 3, 1 \leq q, r \leq \infty, \frac{1}{q} + \frac{1}{r} = 1, h^{(n)} \in L_q[a, b] \) and let \( \gamma : [a, \beta] \to [a, b] \) and \( \rho : [a, \beta] \to \mathbb{R} \) satisfy \( \int_a^\beta \rho(\lambda) \, d\lambda = 0 \) and \( \int_a^\beta \rho(\lambda) \gamma(\lambda) \, d\lambda = 0 \). Then
\[
\left| \int_\alpha^\beta p(\lambda) h(\gamma(\lambda)) \, d\lambda - \frac{h'(a) - h'(b)}{b - a} \int_\alpha^b p(\lambda) G(\gamma(\lambda), s) \, d\lambda \, ds \\
- \sum_{k=2}^{n-1} \frac{k}{(k - 1)!} \left( \int_\alpha^b p(\lambda) G(\gamma(\lambda), s) \, d\lambda \right) \frac{h^{(k)}(a)(s - a)^{k-1} - h^{(k)}(b)(s - b)^{k-1}}{b - a} \, ds \right| \\
\leq \frac{1}{(n - 3)!} \|h^{(n)}\|_q \|\Omega_4\|_r,
\]
(3.16)

and
\[
\left| \int_\alpha^\beta p(\lambda) h(\gamma(\lambda)) \, d\lambda - \frac{h'(b) - h'(a)}{b - a} \int_\alpha^b p(\lambda) G(\gamma(\lambda), s) \, d\lambda \, ds \\
- \sum_{k=3}^{n-1} \frac{k - 2}{(k - 1)!} \left( \int_\alpha^b p(\lambda) G(\gamma(\lambda), s) \, d\lambda \right) \frac{h^{(k)}(a)(s - a)^{k-1} - h^{(k)}(b)(s - b)^{k-1}}{b - a} \, ds \right| \\
\leq \frac{1}{(n - 3)!} \|h^{(n)}\|_q \|\Omega_4\|_r.
\]
(3.17)

The constant on the right hand sides of (3.16) and (3.17) is sharp for \( 1 < q \leq \infty \) and the best possible for \( q = 1 \).

**Remark 3.9** Left-hand sides of the inequalities (2.8), (2.10), (2.15) and (2.18) can be defined as linear functionals in \( h \). Using similar methods as in [2] we can prove mean value results for these functionals, as well as construct new families of exponentially convex functions and Cauchy-type means. Then, using some known properties of exponentially convex functions, we can derive new inequalities and prove monotonicity of the obtained Cauchy-type means analogously as in [2].
Acknowledgements. The publication was supported by the Ministry of Education and Science of the Russian Federation (the Agreement number No. 02.a03.21.0008).

Open Access. This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

References

1. Aljinović, A. Aglić; Pečarić, J.; Vukelić, A.: On some Ostrowski type inequalities via Montgomery identity and Taylor’s formula. II. Tamkang Jour. Math 36(4), 279–301 (2005)
2. Butt, S.I.; Pečarić, J.; Praljak, M.: Reversed Hardy inequality for C-monotone functions. J. Math. Inequal. 10(3), 603–622 (2016)
3. Cerone, P.; Dragomir, S.S.: Some new Ostrowski-type bounds for the Čebyšev functional and applications. J. Math. Inequal. 8(1), 159–170 (2014)
4. Mitrinović, D.S.; Pečarić, J.E.; Fink, A.M.: Inequalities for Functions and Their Integrals and Derivatives. Kluwer Academic Publishers, Dordrecht (1994)
5. Pečarić, J.: On Jessens inequality for convex functions. III. J. Math. Anal. Appl. 156, 231–239 (1991)
6. Pečarić, J.E.; Proschan, F.; Tong, Y.L.: Convex Functions, Partial Orderings and Statistical Applications. Academic Press, New York (1992)
7. Popoviciu, T.: Notes sur les fonctions convexes d’orde superieur III. Mathematica (Cluj) 16, 74–86 (1940)
8. Popoviciu, T.: Notes sur les fonctions convexes d’orde superieur IV. Disquisitiones Math. 1, 163–171 (1940)
9. Popoviciu, T.: Les Fonctions Convexes. Herman and Cie, Editeurs (1944)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.