Maximum principle for state constrained optimal control problems governed by multisolution p-Laplacian elliptic equations in the absence of convexity

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Abstract

A state constrained optimal control problem governed by a class of multisolution p-Laplacian elliptic equations is studied in this paper. Both the control domain and cost functional considered may be non-convex. Combining the multiplicity and degeneracy of the state equation with the non-convex assumptions is the main difficulty we will overcome. By transforming the initial problem to a well-posed and non-degenerate problem with a point-point mixed constraint and then using Ekeland’s variational principle, the Pontryagin’s maximum principle for the initial problem is obtained by passing to the limits twice.

Keywords: optimal control, p-Laplacian equation, multiplicity, nonconvexity, Pontryagin’s maximum principle

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1. Introduction

2. Formulation of the control problem and the main result

Let $\Omega$ be a bounded open subset of $\mathbb{R}^n (n \geq 3)$ with $C^{1,1}$ boundary $\Gamma$. Consider the following $p$-Laplacian elliptic equation

$$
\begin{cases}
-\text{div}(|\nabla y|^{p-2}\nabla y) = f(x, y, u) & \text{in } \Omega, \\
y = 0 & \text{on } \Gamma,
\end{cases}
$$

(2.1)

where $1 < p < 2$ and $F(y) \in W$.

(2.2)

Let $U$ be a separate metric space and the set of controls $\mathcal{U}_{ad}$ is defined by $\mathcal{U}_{ad} \equiv \{ u : \Omega \to U | u \text{ is measurable} \}$. Define $d(u, v) = | \{ x \in \Omega | u(x) \neq v(x) \} |$, where $| \cdot |$ denotes the Lebesgue measure in $\mathbb{R}^n$. Then $(\mathcal{U}_{ad}, d)$ is a complete metric space (see Chapter 5 of [1]). Denote

$$
\mathcal{A} = \{(y, u) \mid y \in W^{1,p}_0(\Omega), u \in \mathcal{U}_{ad}, (y, u) \text{ satisfies (2.1) and (2.2)}\},
$$

a pair $(y, u) \in \mathcal{A}$ will be called an admissible pair. The cost functional we considered is as follows:

$$
J(y, u) = \int_{\Omega} f^0(x, y(x), u(x)) \, dx.
$$

We make the following assumptions.

(S1) The function $f : \Omega \times \mathbb{R} \times U \to \mathbb{R}$ is continuous and $f_y(x, \cdot , \cdot)$ is continuous in $\mathbb{R} \times U$. Moreover, $f$ satisfies the following conditions:

$$
|f(x, y, u)| \leq C(1 + |y|^{r_1}) \quad \text{for any } (x, y, u) \in \Omega \times \mathbb{R} \times U
$$

(2.3)

and

$$
|f_y(x, y, u)| \leq \tilde{C}(1 + |y|^{r_1 - 1}) \quad \text{for any } (x, y, u) \in \Omega \times \mathbb{R} \times U,
$$

(2.4)

and there exists a constant $L > 0$ such that for all $x \in \Omega$, $u \in U$, $y_1, y_2 \in \mathbb{R}$,

$$
|f_y(x, y_1, u) - f_y(x, y_2, u)| \\
\leq \begin{cases}
L|y_1 - y_2|(1 + |y_1|^{r_1 - 2} + |y_2|^{r_1 - 2}) & \text{if } r_1 - 2 > 0, \\
L|y_1 - y_2|^{r_1 - 1} & \text{if } r_1 - 2 \leq 0,
\end{cases}
$$

(2.5)
where $C, \tilde{C} \geq 0$ and $1 \leq r_1 \leq n/(n - 2)$.

(S2) Let $X$ be a Banach space with dual $X^*$ strictly convex and let $F : L^2(\Omega) \to X$ be of class $C^1$. $W \subset X$ is a closed and convex subset.

(S3) $f^0 : \Omega \times \mathbb{R} \times U \to \mathbb{R}$ satisfies that $f^0(\cdot, y, u)$ is measurable in $\Omega$, $f^0(x, \cdot, \cdot)$, $f^0_y(x, \cdot, \cdot)$ are continuous in $\mathbb{R} \times U$. Moreover,

$$|f^0(x, y, u)| + |f^0_y(x, y, u)| \leq a(x) + b|y|^{r_2} \quad \text{for any} \quad (x, y, u) \in \Omega \times \mathbb{R} \times U, \quad (2.6)$$

where $0 \leq r_2 \leq (n + 2)/(n - 2)$, $a(\cdot) \in L^{2n/(n+2)}(\Omega)$, $a(x) \geq 0$ a.e. in $\Omega$, $b \geq 0$.

We note that the following facts.

**Remark 2.1** Since we make no monotonicity assumption $f'(x, y, u) \leq 0$ for all $(x, y, u) \in \Omega \times \mathbb{R} \times U$ on $f$, the state equation (2.1) may admit more than one solution for any $u \in U_{ad}$. Hence, (2.1) is non-well-posed.

**Remark 2.2** Due to (S1), every solution of (2.1) belongs to $L^\infty(\Omega)$ (see [2]).

The optimal control problem we considered can be stated as follows.

**Problem (P).** Find a pair $(\bar{y}, \bar{u}) \in A$ such that

$$J(\bar{y}, \bar{u}) = \inf \{ J(y, u) | (y, u) \in A \}.$$

A solution of **Problem (P)** is said to be an optimal pair, $\bar{u}$ is called an optimal control, and $\bar{y}$ is called an optimal state.

Our purpose is to give an optimality condition for an optimal pair $(\bar{y}, \bar{u})$. To do this, we need one more assumption. Before stating it, we define, for $r_1, r_2 > 0$,

$$B_{r_1, r_2} = \{(q, w) \in X \times L^\infty(\Omega) : \forall (y, u) \in W_0^{1,2}(\Omega) \times U_{ad} \text{ with}$$

$$\|y - \bar{y}\|_{W_0^{1,2}(\Omega)} \leq r_1, \exists z \in W_0^{1,2}(\Omega) \text{ with} \|z\|_{W_0^{1,2}(\Omega)} \leq r_2 \text{ such that}$$

$$F'(\bar{y})z = q \text{ and } -\text{div}(A\nabla z) - f_y(x, y, \bar{u})z$$

$$-f(x, y, u) + f(x, y, \bar{u}) = w \text{ in } \Omega \} \quad (2.7)$$

where

$$A = |\nabla \bar{y}|^{p-2} \left( I + (p - 2)\frac{\nabla \bar{y}(\nabla \bar{y})^T}{|\nabla \bar{y}|^2} \right). \quad (2.8)$$
There exist \( r_1, r_2 > 0 \) such that \( B_{r_1, r_2} - W \times \{0\} \) has finite codimensionality in \( X \times L^\infty(\Omega) \).

The main result in this paper is as follows.

**Theorem 2.1** Suppose that (S1)-(S4) hold. Let \((\bar{y}, \bar{u}) \in W_0^{1,p}(\Omega) \times U_{ad}\) be an optimal pair of Problem (P) and \( A \) be given by (2.8). Then there exists a triplet \((\lambda_0, \varphi_0, \tilde{\psi}) \in \mathbb{R} \times X^* \times W_0^{1,2}(\Omega)\) with \((\lambda_0, \varphi_0, \tilde{\psi}) \neq 0\), such that

\[
- \text{div} \left( A \nabla \tilde{\psi} \right) = f(y(x, \bar{y}, \bar{u})), \quad \text{in } \{\nabla \bar{y} \neq 0\},
\]

\[
\varphi_0(x) = 0, \quad \text{a.e. } x \in \{\nabla \bar{y} = 0\},
\]

\[
\langle \varphi_0, \phi - F(\bar{y}) \rangle_{X^*, X} \leq 0 \quad \text{for all } \phi \in W,
\]

and for almost all \( x \in \Omega \)

\[
H_{\lambda_0}(x, \bar{y}(x), \bar{u}(x), \tilde{\psi}(x)) = \min_{v \in U} H_{\lambda_0}(x, \bar{y}(x), v, \tilde{\psi}(x)),
\]

where

\[
H_{\lambda}(x, y, v, \psi) = \lambda f^0(x, y, v) - \psi f(x, y, v) \quad \forall (x, y, v, \psi) \in \Omega \times \mathbb{R} \times U \times \mathbb{R}.
\]

3. Transformation of the initial problem and the preliminary lemmas

In this section, we will transform Problem (P) to a well-posed problem with a point-point mixed constraint and give some preliminary lemmas. For any \( \tau \in (0, 1) \), let us consider a new state equation

\[
\begin{cases}
- \text{div} \left( \sqrt{\tau^2 + |\nabla y_\tau|^2}^{p-2} \nabla y_\tau \right) = v & \text{in } \Omega, \\
y_\tau = 0 & \text{on } \Gamma,
\end{cases}
\]

with

\[
v(x) = f(x, y_\tau(x), u(x)) \quad \text{a.e. } x \in \Omega,
\]

and \( F(y_\tau) \in W\), where the control \((u, v) \in U_{ad} \times \mathcal{K}, \mathcal{K} \equiv \{v \in L^\infty(\Omega) : \|v - \tilde{v}\|_{L^\infty(\Omega)} \leq K, \tilde{v}(x) = f(x, \bar{y}(x), \bar{u}(x)) \text{ a.e. } x \in \Omega\}\) for some constant
$K > 0$ large enough, and $y_{\tau} \in W_{0}^{1,p}(\Omega)$ is a solution of (3.1) corresponding to $(u,v) \in U_{ad} \times K$.

The following two lemmas show that the existence, uniqueness and regularity of the solution of (3.1) which can be deduced from that Lemmas 3.1 and 3.2 in [3] and Remark 2.2.

**Lemma 3.1** Let (S1) hold. Then for any $\tau \in (0,1)$, $(u,v) \in U_{ad} \times K$, (3.1) admits a unique solution $y_{\tau} \in W_{0}^{1,p}(\Omega) \cap L^\infty(\Omega)$, moreover, there exists a constant $C > 0$, independent of $\tau$ and $(u,v)$, such that

$$\|y_{\tau}\|_{L^\infty(\Omega)} \leq C.$$

**Lemma 3.2** Let (S1) hold. Then for any $\tau \in (0,1)$, $(u,v) \in U_{ad} \times K$, there exist constants $C > 0$ and $\alpha \in (0,1)$, independent of $\tau$ and $(u,v)$, such that

$$\|y_{\tau}\|_{C^{1,\alpha}(\bar{\Omega})} \leq C,$$

where $y_{\tau}$ is the solution of (3.1) corresponding to $(u,v)$.

Next, we give some other important preliminary results which are proved using the same arguments as in [4].

**Lemma 3.3** Suppose that (S3) holds. Let $u_{k}, \tilde{u} \in U_{ad}$ and $v_{k}, \tilde{v} \in K(k > 0)$ be such that $u_{k} \to \tilde{u}$ in $U_{ad}$ and $v_{k} \to \tilde{v}$ strongly in $L^2(\Omega)$ as $k \to \infty$. Then

$$\int_{\Omega} |f^0(x, y_{\tau,k}, u_{k}) - f^0(x, \tilde{y}, \tilde{u})| dx \to 0$$

as $k \to \infty$ and then $\tau \to 0$, where $y_{\tau,k}$ is the solution of (3.1) corresponding to $v_{k}$, and $\tilde{y}$ is the solution of (2.1) corresponding to $\tilde{v}$.

**Proof.** For any $\tau \in (0,1)$, let $\tilde{y}_{\tau}$ be the solution of (3.1) corresponding
to \( \tilde{v} \), then it follows from (2.6) that

\[
\int_{\Omega} |f^0(x, y_{\tau,k}, u_k) - f^0(x, \tilde{y}_\tau, \tilde{u})| \, dx \\
\leq \int_{\Omega} |f^0(x, y_{\tau,k}, u_k) - f^0(x, \tilde{y}_\tau, u_k)| \, dx \\
+ \int_{\Omega} |f^0(x, \tilde{y}_\tau, u_k) - f^0(x, \tilde{y}_\tau, \tilde{u})| \, dx \\
\leq \int_{\Omega} |y_{\tau,k} - \tilde{y}_\tau| \int_0^1 |f^0_1(x, \tilde{y}_\tau + \theta(y_{\tau,k} - \tilde{y}_\tau), u_k)| \, d\theta \, dx \\
+ \int_{\Omega_k} |f^0(x, \tilde{y}_\tau, u_k) - f^0(x, \tilde{y}_\tau, \tilde{u})| \, dx \\
\leq \int_{\Omega} |y_{\tau,k} - \tilde{y}_\tau| \left( a + b \int_0^1 |\tilde{y}_\tau + \theta(y_{\tau,k} - \tilde{y}_\tau)| r^2 \, d\theta \right) \, dx \\
+ 2 \int_{\Omega_k} (a + b |\tilde{y}_\tau| r^2) \, dx \\
=: I_{1k} + I_{2k},
\]

where \( \Omega_k = \{ x \in \Omega : u_k(x) \neq \tilde{u}(x) \} \).

Since \( \| v_k - \tilde{v} \|_{L^2(\Omega)} \to 0 \) as \( k \to \infty \) and \( y_{\tau,k}, \tilde{y}_\tau \) is the solution of (3.1) corresponding to \( v_k \) and \( \tilde{v} \), respectively, then the standard energy estimate method implies that \( \| y_{\tau,k} - \tilde{y}_\tau \|_{W^{1,2}_0(\Omega)} \to 0 \) as \( k \to \infty \). Noting \( n \geq 3 \), thus Sobolev’s Imbedding Theorem implies that

\[
\| y_{\tau,k} - \tilde{y}_\tau \|_{L^{2n/(n-2)}(\Omega)} \to 0
\]

as \( k \to \infty \). In addition, one can check that for \( k > 0 \) large enough,

\[
\left\| a + b \int_0^1 |\tilde{y}_\tau + \theta(y_{\tau,k} - \tilde{y}_\tau)| r^2 \, d\theta \right\|_{L^{2n/(n+2)}(\Omega)} \leq C, 
\]

where \( C > 0 \) is independent of \( k \). Furthermore, using Hölder inequality, (3.4) and (3.5), we have that

\[
I_{1k} \leq \| y_{\tau,k} - \tilde{y}_\tau \|_{L^{2n/(n-2)}(\Omega)} \cdot \left\| a + b \int_0^1 |\tilde{y}_\tau + \theta(y_{\tau,k} - \tilde{y}_\tau)| r^2 \, d\theta \right\|_{L^{2n/(n+2)}(\Omega)} \\
\leq C \| y_{\tau,k} - \tilde{y}_\tau \|_{L^{2n/(n-2)}(\Omega)} \to 0
\]

as \( k \to \infty \).
Furthermore, (S3) and Sobolev's Imbedding Theorem imply that 
\[ (a + b|\tilde{y}_\tau|^2) \in L^{2n/(n+2)}(\Omega) \subset L^1(\Omega). \] Moreover, since \( u_k \to \tilde{u} \) in \( U_{ad} \), we have that \( |\Omega_k| \to 0 \) as \( k \to \infty \). Thus we deduce that
\[ I_{2k} \to 0 \quad \text{as} \quad k \to \infty. \] (3.7)

On the other hand, by (2.1), (3.1) and Lemma 3.2, we have that
\[ \tilde{y}_\tau \to \tilde{y} \quad \text{strongly in} \quad W^{1,2}_0(\Omega), \quad \text{uniformly in} \quad C^1(\bar{\Omega}). \] (3.8)
as \( \tau \to 0 \). This together with (S3), we have that
\[ \int_\Omega |f^0(x, y_{\tau,k}, u_k) - f^0(x, \tilde{y}, \tilde{u})|^2 dx \to 0 \quad \text{as} \quad \tau \to 0. \] (3.9)
Finally, by (3.3), (3.6), (3.7) and (3.9), we complete the proof. \( \square \)

**Lemma 3.4** Suppose that (S1) holds. Let \( u_k, \tilde{u} \in U_{ad} \) and \( v_k, \tilde{v} \in K(k > 0) \) be such that \( u_k \to \tilde{u} \) in \( U_{ad} \) and \( v_k \to \tilde{v} \) strongly in \( L^2(\Omega) \) as \( k \to \infty \). Then
\[ \int_\Omega |f(x, y_{\tau,k}, u_k) - f(x, \tilde{y}, \tilde{u})|^2 dx \to 0 \]
as \( k \to \infty \) and then \( \tau \to 0 \), where \( y_{\tau,k} \) is the solution of (3.1) corresponding to \( v_k \), and \( \tilde{y} \) is the solution of (2.1) corresponding to \( \tilde{v} \).

**Proof.** For any \( \tau \in (0, 1) \), let \( \tilde{y}_\tau \) be given as above, then we have that
\[ \int_\Omega |f(x, y_{\tau,k}, u_k) - f(x, \tilde{y}, \tilde{u})|^2 dx \]
\[ \leq 2 \int_\Omega |f(x, y_{\tau,k}, u_k) - f(x, \tilde{y}, u_k)|^2 dx \]
\[ + 2 \int_\Omega |f(x, \tilde{y}, u_k) - f(x, \tilde{y}, \tilde{u})|^2 dx \]
\[ \leq 2 \int_\Omega |y_{\tau,k} - \tilde{y}_\tau|^2 \left| \int_0^1 f_y(x, \tilde{y}_\tau + \theta(y_{\tau,k} - \tilde{y}_\tau), u_k)d\theta \right|^2 dx \]
\[ + 2 \int_{\Omega_k} |f(x, \tilde{y}, u_k) - f(x, \tilde{y}, \tilde{u})|^2 dx \]
\[ =: I_{3k} + I_{4k}. \] (3.10)
By (2.4) and (3.4), we have that
\[
I_{3k} \leq 2 \|y_{\tau,k} - \tilde{y}_\tau\|_{L^{2n/(n-2)}(\Omega)} \cdot \left\| \tilde{C} \left( 1 + \int_0^1 |\tilde{y}_\tau + \theta(y_{\tau,k} - \tilde{y}_\tau)|^{r_1-1} d\theta \right) \right\|_{L^{n/2}(\Omega)}
\]
\[
\leq C \|y_{\tau,k} - \tilde{y}_\tau\|_{L^{2n/(n-2)}(\Omega)}^2 \rightarrow 0
\]
as \( k \rightarrow \infty \).

On the other hand, since \( \tilde{y}_\tau \in L^{2n/(n-2)}(\Omega) \) and \( 1 \leq r_1 \leq n/(n-2) \), it is easy to see that \( (1 + |\tilde{y}_\tau|^{r_1})^2 \in L^1(\Omega) \). In addition, \( \Omega_k \rightarrow 0 \) as \( k \rightarrow \infty \). Therefore,
\[
I_{4k} \leq C \int_{\Omega_k} (1 + |\tilde{y}_\tau|^{r_1})^2 dx \rightarrow 0
\]
as \( k \rightarrow \infty \). Applying (3.8) and the continuity of \( f \) and by (3.10)-(3.12), the lemma is proved. □

**Lemma 3.5** Suppose that (S3) holds. Let \( u_k, \tilde{u} \in U_{ad} \) and \( v_k, \tilde{v} \in K \) \((k > 0)\) be such that \( u_k \rightarrow \tilde{u} \) in \( U_{ad} \) and \( v_k \rightarrow \tilde{v} \) strongly in \( L^2(\Omega) \) as \( k \rightarrow \infty \). In addition, let \( z_{\tau,k}, z \in W^{1,2}_0(\Omega) \) be such that \( z_{\tau,k} \rightarrow z \) strongly in \( L^2(\Omega) \) as \( k \rightarrow \infty \) and then \( \tau \rightarrow 0 \). Then there exist generalized subsequences of \( k \) and \( \tau \), denoted in the same way, such that
\[
\int_{\Omega} f_y^0(x, y_{\tau,k}, u_k) z_{\tau,k} dx \rightarrow \int_{\Omega} f_y^0(x, \tilde{y}, \tilde{u}) z dx
\]
as \( k \rightarrow \infty \) and then \( \tau \rightarrow 0 \), where \( y_{\tau,k} \) is the solution of (3.1) corresponding to \( v_k \), and \( \tilde{y} \) is the solution of (2.1) corresponding to \( \tilde{v} \).

**Proof.** First, we shall show that on a generalized subsequence of \( k \), denoted in the same way,
\[
f_y^0(x, y_{\tau,k}, u_k) \rightarrow f_y^0(x, \tilde{y}, \tilde{u}) \quad \text{weakly in } L^{2n/(n+2)}(\Omega)
\]
as \( k \rightarrow \infty \) and then \( \tau \rightarrow 0 \) because \( z \in W^{1,2}_0(\Omega) \subset L^{2n/(n-2)}(\Omega) \). For any \( \tau \in (0,1) \), firstly we note that there exists a generalized subsequence of \( k \), denoted in the same way, such that
\[
y_{\tau,k}(x) \rightarrow \tilde{y}_\tau(x), \quad u_k(x) \rightarrow \tilde{u} \quad \text{a.e. in } \Omega
\]
as \( k \to \infty \) since (3.4) holds and \( u_k \to \bar{u} \) in \( \mathcal{U}_{ad} \). Since \( f^0_y : \Omega \times \mathbb{R} \times U \to \mathbb{R} \) is continuous, it follows from (3.15) that

\[
f^0_y(x, y_{\tau,k}, u_k) \to f^0_y(x, \bar{y}, \bar{u}) \quad \text{a.e. in } \Omega \tag{3.16}
\]
as \( k \to \infty \).

On the other hand, by (2.6) and by Sobolev Imbedding Theorem, we have that

\[
\|f^0_y(x, y_{\tau,k}, u_k)\|_{L^{2n/(n+2)}(\Omega)} \leq C, \tag{3.17}
\]

where \( C \) is a positive constant independent of \( k \). Consequently, (3.14) can be deduced by (3.16), (3.17) and (3.8). Obviously, by (3.14), for any \( z \in W^{1,2}_0(\Omega) \), we have that

\[
\int_{\Omega} f^0_y(x, y_{\tau,k}, u_k)z \, dx \to \int_{\Omega} f^0_y(x, \bar{y}, \bar{u})z \, dx \tag{3.19}
\]
as \( k \to \infty \) and then \( \tau \to 0 \).

On the other hand, by (2.6) and Lemma 3.2, we have that

\[
\left| \int_{\Omega} f^0_y(x, y_{\tau,k}, u_k)(z_{\tau,k} - z) \, dx \right| \leq C\|z_{\tau,k} - z\|_{L^2(\Omega)}. \tag{3.20}
\]

Consequently, (3.19) and (3.20) imply (3.13) hold and thus the proof is over.

\[\Box\]

**Lemma 3.6** Suppose that (S1) holds. Let \( u_k, \bar{u} \in \mathcal{U}_{ad} \) and \( v_k, \bar{v} \in \mathcal{K}(k > 0) \) be such that \( u_k \to \bar{u} \) in \( \mathcal{U}_{ad} \) and \( v_k \to \bar{v} \) strongly in \( L^2(\Omega) \) as \( k \to \infty \). Suppose that \( \psi_{\tau,k}, \psi \in L^2(\Omega) \) with \( \psi_{\tau,k} \to \psi \) weakly in \( L^2(\Omega) \) and \( z_{\tau,k}, z \in W^{1,2}_0(\Omega) \) with \( z_{\tau,k} \to z \) strongly in \( L^2(\Omega) \) as \( k \to \infty \) and then \( \tau \to 0 \). Then there exist generalized subsequences of \( k \) and \( \tau \), respectively, denoted in the same way, such that

\[
\int_{\Omega} \psi_{\tau,k}f_y(x, y_{\tau,k}, u_k)z_{\tau,k} \, dx \to \int_{\Omega} \psi f_y(x, \bar{y}, \bar{u})z \, dx \tag{3.21}
\]
as \( k \to \infty \) and then \( \tau \to 0 \), where \( y_{\tau,k} \) is the solution of (3.1) corresponding to \( v_k \), and \( \bar{y} \) is the solution of (2.1) corresponding to \( \bar{v} \).

**Proof.** **Step 1** Firstly, for any \( z \in W^{1,2}_0(\Omega) \), we would like to prove that

\[
\int_{\Omega} \psi_{\tau,k}f_y(x, y_{\tau,k}, u_k)z \, dx \to \int_{\Omega} \psi f_y(x, \bar{y}, \bar{u})z \, dx \tag{3.22}
\]
as $k \to \infty$ and then $\tau \to 0$.

For any $\tau \in (0, 1)$. It is clear that

$$\left| \int_{\Omega} \left[ \psi_{\tau,k} f_y(x, y_{\tau,k}, u_k) - \psi f_y(x, \bar{y}_{\tau}, \bar{u}) \right] z dx \right| \leq I_{5k} + I_{6k}, \quad (3.23)$$

where

$$I_{5k} = \left| \int_{\Omega} \psi_{\tau,k} [f_y(x, y_{\tau,k}, u_k) - f_y(x, \bar{y}_{\tau}, u_k)] z dx \right| \quad (3.24)$$

and

$$I_{6k} = \left| \int_{\Omega} \psi_{\tau,k} [f_y(x, \bar{y}_{\tau}, u_k) - f_y(x, \bar{y}_{\tau}, \bar{u})] z dx \right|. \quad (3.25)$$

First, we estimate $I_{5k}$. We claim that on generalized subsequence of $k$, denoted in the same way,

$$[f_y(x, y_{\tau,k}, u_k) - f_y(x, \bar{y}_{\tau}, u_k)] z \to 0 \text{ strongly in } L^2(\Omega) \quad (3.26)$$

for any $z \in W^{1,2}_0(\Omega)$ as $k \to \infty$.

**Case 1**: $1 < r_1 \leq 2$.

In this case, by (2.5) and Sobolev’s Imbedding Theorem and Hölder’s inequality, we obtain that

$$\int_{\Omega} |f_y(x, y_{\tau,k}, u_k) - f(x, \bar{y}_{\tau}, u_k)|^2 z^2 dx$$

$$\leq C \int_{\Omega} |y_{\tau,k} - \bar{y}_{\tau}|^{2(r_1-1)} z^2 dx$$

$$\leq C \|y_{\tau,k} - \bar{y}_{\tau}\|^2_{L^{2n/(n-2)}(\Omega)} \|z\|^2_{L^{2n/(n-2)}(\Omega)}$$

$$\leq C \|y_{\tau,k} - \bar{y}_{\tau}\|^2_{L^{2n/(n-2)}(\Omega)} \|z\|^2_{W^{1,2}_0(\Omega)} \quad (3.27)$$

By (3.4) and (3.27), we get that (3.26) in this case.

**Case 2**: $r_1 = 1$.

In this case, by (2.5) again, we have that

$$|f_y(x, y_{\tau,k}, u_k) - f_y(x, \bar{y}_{\tau}, u_k)|^n \leq C \quad \forall x \in \Omega. \quad (3.28)$$

On the other hand, by the same argument as in the proof of Case 1, there exists a generalized subsequence of $k$, denoted in the same way, such that

$$|f_y(x, y_{\tau,k}(x), u_k(x)) - f_y(x, \bar{y}_{\tau}(x), u_k(x))|^n \to 0 \quad \text{a.e. in } \Omega \quad (3.29)$$
as $k \to \infty$. By (3.28) and (3.29) and Lebesgue’s Dominated Convergence Theorem, we get that

$$f_y(x, y_{r,k}, u_k) - f_y(x, \bar{y}_r, u_k) \to 0 \quad \text{strongly in } L^n(\Omega) \quad \text{as } k \to \infty. \quad (3.30)$$

Using Hölder’s inequality and Sobolev’s Imbedding Theorem again, we deduce that

$$\int_{\Omega} |f_y(x, y_{r,k}, u_k) - f(x, \bar{y}_r, u_k)|^2 \, dx \leq C \|f_y(x, y_{r,k}, u_k) - f_y(x, \bar{y}_r, u_k)\|_{L^n(\Omega)}^2 \|z\|^2_{W_0^{1,2}(\Omega)}. \quad (3.31)$$

Thus (3.26) follows from (3.30) and (3.31) immediately in this case.

**Case 3:** $r_1 > 2$.

In this case, we must have that $n = 3$ because $n \geq 3$. Thus $3 \geq r_1 > 2$. Since $\{y_{r,k}\}$ is bounded in $W_0^{1,2}(\Omega) \subset L^6(\Omega)$, $\{|y_{r,k}|^{r_1-2}\}$ is bounded in $L^{6/(r_1-2)}(\Omega)$. This implies that $\{1 + |y_{r,k}|^{r_1-2} + |ar{y}_r|^{r_1-2}\}$ is bounded in $L^{3/(r_1-2)}(\Omega)$ and therefore is bounded in $L^3(\Omega)$. Hence by (2.5) and by Hölder’s inequality and Sobolev’s Imbedding Theorem, we obtain that

$$\int_{\Omega} |f_y(x, y_{r,k}, u_k) - f(x, \bar{y}_r, u_k)|^2 \, dx \leq C \left( \int_{\Omega} |y_{r,k} - \bar{y}_r|^6 \, dx \right)^{1/3} \left[ \int_{\Omega} (1 + |y_{r,k}|^{r_1-2} + |ar{y}_r|^{r_1-2})^6 \, dx \right]^{1/3}$$

$$\leq C \|y_{r,k} - \bar{y}_r\|^2_{L^6(\Omega)} \|z\|^2_{W_0^{1,2}(\Omega)},$$

which shows that (3.26) holds in this case. Thus, we prove (3.26) in all cases.

Since $\psi_{r,k} \to \psi$ weakly in $L^2(\Omega)$ as $k \to \infty$ and then $\tau \to 0$. We have that $\|\psi_{r,k}\|_{L^2(\Omega)} \leq C_\tau$ for some constant $C_\tau > 0$. Thus it follows from (3.26) that

$$I_{5k} \to 0 \quad \text{as } k \to \infty. \quad (3.32)$$

Next, we claim that

$$f_y(x, \bar{y}_r, u_k)z \to f_y(x, \bar{y}_r, \bar{u})z \quad \text{strongly in } L^2(\Omega) \quad \text{as } k \to \infty. \quad (3.33)$$

Since $u_k \to \bar{u}$ in $U_{ad}$ and $f_y$ is continuous on $\Omega \times \mathbb{R} \times U$, it follows that

$$f_y(x, \bar{y}_r(x), u_k(x)) - f_y(x, \bar{y}_r(x), \bar{u}(x)) \to 0 \quad \text{a.e. in } \Omega \quad \text{as } k \to \infty. \quad (3.34)$$
On the other hand, by (2.4) and by Sobolev’s Imbedding Theorem, we have that
\[ |f_y(x, \tilde{y}_\tau, u^k) - f_y(x, \tilde{y}_\tau, \tilde{u})|^n \leq 2^n(1 + |\tilde{y}_\tau|^{r-1})^n \in L^1(\Omega). \]  
(3.35)
as \( k \to \infty \). By (3.34) and (3.35) and Lebesgue’s Dominated Convergence Theorem, we get that
\[ f_y(x, \tilde{y}_\tau, u^k) - f_y(x, \tilde{y}_\tau, \tilde{u}) \to 0 \text{ strongly in } L^n(\Omega) \text{ as } k \to \infty. \]  
(3.36)
Using Hölder’s inequality and Sobolev’s Imbedding Theorem again, we deduce that
\[ \int_\Omega |f_y(x, \tilde{y}_\tau, u^k) - f_y(x, \tilde{y}_\tau, \tilde{u})|^2 z^2 dx \]
\[ \leq C \|f_y(x, \tilde{y}_\tau, u^k) - f_y(x, \tilde{y}_\tau, \tilde{u})\|_{L^n(\Omega)}^2 \|z\|^2_{W^{1,2}_0(\Omega)}. \]  
(3.37)
Now (3.33) follows immediately from (3.36) and (3.37). Since \( \psi_{\tau,k} \to \psi \) weakly in \( L^2(\Omega) \) as \( k \to \infty \) and then \( \tau \to 0 \). We have that \( \|\psi_{\tau,k}\|_{L^2(\Omega)} \leq C_\tau \) for some constant \( C_\tau > 0 \). Thus it follows from (3.33) that
\[ I_{6k} \to 0 \text{ as } k \to \infty. \]  
(3.38)
On the other hand, similarly, by (3.8) and using the same arguments as above, we can get that
\[ \int_\Omega \psi f_y(x, \tilde{y}_\tau, \tilde{u}) z dx \to \int_\Omega \psi f_y(x, \tilde{y}, \tilde{u}) z dx \quad \forall z \in W^{1,2}_0(\Omega) \]  
(3.39)
as \( \tau \to 0 \). It is clear that (3.22) can be deduced from (3.23), (3.32), (3.38) and (3.39).

**Step 2** By (2.4) and Lemma 3.2, we have that
\[ \left| \int_\Omega \psi_{\tau,k} f_y(x, y_{\tau,k}, u^k)(z_{\tau,k} - z) dx \right| \leq C \|z_{\tau,k} - z\|_{L^2(\Omega)} \to 0 \]
as \( k \to \infty \) and then \( \tau \to \infty \). Combing this with (3.22), we see that (3.21) holds and thus we complete the proof of Lemma 3.6. \( \square \)

**Lemma 3.7** Suppose that (S) holds. Let \( u^k, \tilde{u} \in U_{ad} \) and \( v^k, \tilde{v} \in K(k > 0) \) be such that \( u^k \to \tilde{u} \) in \( U_{ad} \) and \( v^k \to \tilde{v} \) strongly in \( L^2(\Omega) \) as \( k \to \infty \).
Suppose that $\psi_{\tau,k}$ is bounded in $L^2(\Omega)$ for all $k > 0$ and $\tau \in (0,1)$. Then for any $\tau \in (0,1)$ and $M > 0$, there exist a generalized subsequence of $k$, denoted in the same way, such that

$$
\int_{\Omega} \psi_{\tau,k} [f(x, y_{\tau,k}, u_k) - f(x, y_{\tau,k}, \tilde{u})] \, dx \to 0 \quad (3.40)
$$

uniformly in $z \in W^{1,2}_0(\Omega)$ with $\|z\|_{W^{1,2}_0(\Omega)} \leq M$ as $k \to \infty$, and

$$
\int_{\Omega} \psi_{\tau,k} [f(x, y_{\tau,k}, u_k) - f(x, y_{\tau,k}, \tilde{u})] \, dx \to 0 \quad (3.41)
$$

as $k \to \infty$, where $y_{\tau,k}$ is the solution of (3.1) corresponding to $v_k$.

**Proof.** We fix $\tau \in (0,1)$. Let $\Omega_k = \{x \in \Omega : u_k(x) \neq \tilde{u}(x)\}$. It is clear that

$$
|\Omega_k| \to 0 \quad \text{as} \quad k \to \infty. \quad (3.42)
$$

Since $\psi_{\tau,k}$ is bounded in $L^2(\Omega)$, there exists a positive constant $C_\tau > 0$ in dependent of $k$, such that $\|\psi_{\tau,k}\|_{L^2(\Omega)} \leq C_\tau$. Let $M > 0$ be fixed and $z \in \{z \in W^{1,2}_0(\Omega) : \|z\|_{W^{1,2}_0(\Omega)} \leq M\}$. By H"older’s inequality, we have that

$$
\left| \int_{\Omega} \psi_{\tau,k} [f(y, x, y_{\tau,k}, u_k) - f(y, x, y_{\tau,k}, \tilde{u})] \, z \, dx \right|
\leq \left[ \int_{\Omega} \psi_{\tau,k}^2 \, dx \right]^{1/2} \cdot \left[ \int_{\Omega} |z|^{2n/(n-2)} \, dx \right]^{(n-2)/2n} \cdot \left[ \int_{\Omega} |f(y, x, y_{\tau,k}, u_k) - f(y, x, y_{\tau,k}, \tilde{u})|^{n} \, dx \right]^{1/n}
\leq C_\tau M \|f(y, x, y_{\tau,k}, u_k) - f(y, x, y_{\tau,k}, \tilde{u})\|_{L^2(\Omega)} \quad \text{as} \quad k \to \infty. \quad (3.43)
$$

We claim that on a generalized subsequence of $k$, denoted in the same way, such that

$$
\|f(y, x, y_{\tau,k}, u_k) - f(y, x, y_{\tau}, u_k)\|_{L^2(\Omega)} \to 0 \quad \text{as} \quad k \to \infty. \quad (3.44)
$$

**Case 1:** $1 < r_1 \leq 2$.

Since $r_1 \leq n/(n-2)$, we have that $(r_1 - 1)n \leq 2n/(n-2)$. Thus, by Sobolev’s Imbedding Theorem and by (2.5), (3.4), we have that

$$
\int_{\Omega} |f_g(x, y_{\tau,k}, u_k) - f_g(x, y_{\tau}, u_k)|^{(r_1-1)n} \, dx \to 0
$$
as \( k \to \infty \). Therefore (3.44) holds in this case.

Case 2: \( r_1 = 1 \). This case was proved in the proof of Lemma 3.6 (see (3.30)).

In this case, we must have that \( n = 3 \) and \( 3 \geq r_1 > 2 \). By Sobolev’s Imbedding Theorem, we infer that \( \| y_{r,k} \|^{3(r_1-2)} \) is bounded in \( L^2(\Omega) \) Hence by (2.5) and by Hölder’s inequality, we obtain that

\[
\int_{\Omega} |f_y(x, y_{r,k}, u_k) - f(x, \tilde{y}_r, u_k)|^3 \, dx \\
\leq C \int_{\Omega} |y_{r,k} - \tilde{y}_r|^3 \left[ 1 + |y_{r,k}|^{3(r_1-2)} + |\tilde{y}_r|^{3(r_1-2)} \right]^{1/3} \, dx \\
\leq C \| y_{r,k} - \tilde{y}_r \|_{L^6(\Omega)}^3 \| 1 + |y_{r,k}|^{3(r_1-2)} + |\tilde{y}_r|^{3(r_1-2)} \|_{L^2(\Omega)}
\]

as \( k \to \infty \). So by (3.4), we have that (3.44) holds in this case. Thus, we prove (3.44) in all cases.

Similarly, we can deduce that on a generalized subsequence of \( k \), denoted in the same way, such that

\[
\| f_y(x, \tilde{y}_{r,k}, \tilde{u}_k) - f_y(x, \tilde{y}_r, \tilde{u}) \|_{L^\infty(\Omega)} \to 0 \quad \text{as} \quad k \to \infty. \quad (3.45)
\]

Immediately, (3.40) can be deduced from (3.43)-(3.45) and (3.36).

Finally, using the same arguments in the proof of Lemma 3.4, we can prove that (3.41) holds because \( \psi_{r,k} \) is bounded in \( L^2(\Omega) \). Thus the proof of Lemma 3.7 is completed. \( \square \)

4. The proof of the main result

In this section, we will begin to prove Theorem 2.1.

**Proof of Theorem 2.1.** For any \( \tau \in (0, 1) \) fixed and \((u, v) \in U_{ad} \times K\), let

\[
J_{\tau}(u, v) = \int_{\Omega} f^0(x, y_{\tau}, u) \, dx
\]

and

\[
\bar{J}_{\tau} = \int_{\Omega} f^0(x, \bar{y}_{\tau}, \bar{u}) \, dx = J_{\tau}(\bar{u}, \bar{v}),
\]

where \( \bar{y}_{\tau} \) is the solution of (3.1) corresponding to \( \bar{v} \) and \( (\bar{u}, \bar{v}) \) satisfies (3.2).

For each \( 0 < \tau, \varepsilon < 1 \), we define a penalty functional on \( U_{ad} \times K \) by

\[
J_{\tau,\varepsilon}(u, v) = \left\{ \left[ (J_{\tau}(u, v) - \bar{J}_{\tau} + \varepsilon)^+ \right]^2 + \int_{\Omega} \| \nu - f(x, y_{\tau}, u) \|^2 \, dx + d_{W}(F(y_{\tau})) \right\}^{1/2}, \quad (4.1)
\]

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where \( d_W(\cdot) \) denotes the distance of \( \cdot \) to \( W \) in \( X \). By virtue of (4.1), Lemmas 3.1 and 3.2, we have that

1. \( J_{\tau,\varepsilon}(\cdot,\cdot) \) is continuous on \( U_{ad} \times K \);
2. \( J_{\tau,\varepsilon}(u,v) > 0 \) for all \( (u,v) \in U_{ad} \times K \);
3. \( J_{\tau,\varepsilon}(\bar{u},\bar{v}) = \varepsilon \leq \inf_{(u,v)\in U_{ad} \times K} J_{\tau,\varepsilon}(u,v) + \varepsilon \).

Thanks to Ekeland’s variational principle (see Chapter 4 of [1]), we conclude that there exists \( (\bar{u}_{\tau,\varepsilon}, \bar{v}_{\tau,\varepsilon}) \in U_{ad} \times K \) such that

\[
J_{\tau,\varepsilon}(\bar{u}_{\tau,\varepsilon}, \bar{v}_{\tau,\varepsilon}) \leq J_{\tau,\varepsilon}(u,v),
\]

and

\[
d^2(\bar{u}_{\tau,\varepsilon}, \bar{v}_{\tau,\varepsilon}) + \|\bar{v}_{\tau,\varepsilon} - \bar{v}\|_{L^2(\Omega)}^2 \leq \varepsilon
\]

and

\[
J_{\tau,\varepsilon}(\bar{u}_{\tau,\varepsilon}, \bar{v}_{\tau,\varepsilon}) \leq J_{\tau,\varepsilon}(u,v) + J_{\tau,\varepsilon}(u_{\delta}, \bar{v}_{\tau,\varepsilon}) \leq J_{\tau,\varepsilon}(u,v) + \sqrt{\varepsilon} \left[ d^2(u_{\delta}, u) + \|v - \bar{v}\|_{L^2(\Omega)}^2 \right]^{1/2}
\]

for all \( (u,v) \in U_{ad} \times K \).

Let \( (u,v) \in U_{ad} \times K \) be arbitrary but fixed, for each \( 0 < \delta < 1 \), by referring to the proofs of Theorem 2.2 of [1, Chapter 5], we deduce there exist subsets \( E^\delta \subset \Omega \) with \( |E^\delta| = \delta |\Omega| \), we set

\[
(u^\delta_{\tau,\varepsilon}(x), v^\delta_{\tau,\varepsilon}(x)) = \begin{cases} (u(x), v(x)) & \text{on } E^\delta, \\ (\bar{u}_{\tau,\varepsilon}(x), \bar{v}_{\tau,\varepsilon}(x)) & \text{on } \Omega \setminus E^\delta, \end{cases}
\]

and let \( y^\delta_{\tau,\varepsilon}, \bar{y}_{\tau,\varepsilon} \) be the solution of (3.1) corresponding to \( v^\delta_{\tau,\varepsilon}, \bar{v}_{\tau,\varepsilon} \), respectively. Denoting

\[
n^\delta_{\tau,\varepsilon}(\cdot) = \nabla \bar{y}_{\tau,\varepsilon}(\cdot) + t(\nabla y^\delta_{\tau,\varepsilon}(\cdot) - \nabla \bar{y}_{\tau,\varepsilon}(\cdot)), \quad t \in [0,1],
\]

\[
A^\delta_{\tau,\varepsilon} = \int_0^1 \left( \sqrt{\tau^2 + |n^\delta_{\tau,\varepsilon}|^2} \right)^{p-2} \left( I + (p-2) \frac{n^\delta_{\tau,\varepsilon} (n^\delta_{\tau,\varepsilon})^T}{\tau^2 + |n^\delta_{\tau,\varepsilon}|^2} \right) dt,
\]

and

\[
Y^\delta_{\tau,\varepsilon} = \frac{y^\delta_{\tau,\varepsilon} - \bar{y}_{\tau,\varepsilon}}{\delta},
\]

then we have that

\[
\begin{cases}
-\text{div}(A^\delta_{\tau,\varepsilon} \nabla Y^\delta_{\tau,\varepsilon}) = v - \bar{v}_{\tau,\varepsilon} & \text{in } \Omega, \\
Y^\delta_{\tau,\varepsilon} = 0 & \text{on } \Gamma.
\end{cases}
\]

(4.5)
On the other hand, by Lemma 3.2,
\[\|y_{\tau,\epsilon}\|_{C^{1,\alpha}(\tilde{\Omega})}, \|\bar{y}_{\tau,\epsilon}\|_{C^{1,\alpha}(\tilde{\Omega})} \leq C\] (4.6)
for some \(C > 0\) independent of \(\delta \in (0, 1)\). Consequently,
\[\|y_{\tau,\epsilon}\|_{C^{\alpha}(\tilde{\Omega})} \leq C \quad \forall t \in [0, 1].\]
Therefore, for some \(\beta > 0\)
\[\|A_{\tau,\epsilon}\|_{C^{\beta}(\tilde{\Omega};\mathbb{R}^{n \times n})} \leq C.\] (4.7)
Then it follows from (4.5) that
\[\|Y_{\tau,\epsilon}\|_{W^{1,2}_0(\Omega)} \leq C_{\tau,\epsilon}\]
for some \(C_{\tau,\epsilon} > 0\), independent of \(\delta\). Thus (as \(\delta \to 0\)),
\[y_{\tau,\epsilon}^\delta = \bar{y}_{\tau,\epsilon} + \delta Y_{\tau,\epsilon}^\delta \to \bar{y}_{\tau,\epsilon}\quad \text{strongly in } W^{1,2}_0(\Omega).\] (4.8)
Combing (4.6) with (4.8), we get that
\[y_{\tau,\epsilon}^\delta \to \bar{y}_{\tau,\epsilon}\quad \text{uniformly in } C^1(\tilde{\Omega}) \quad \text{as } \delta \to 0.\] (4.9)
Therefore, (4.7) implies that (at least in the sense of a subsequence)
\[A_{\tau,\epsilon}^\delta \to A_{\tau,\epsilon} \equiv \left(\sqrt{\tau^2 + |\nabla \bar{y}_{\tau,\epsilon}|^2}\right)^{p-2} \left(I + (p-2)\frac{\nabla \bar{y}_{\tau,\epsilon}(\nabla \bar{y}_{\tau,\epsilon})^T}{\tau^2 + |\nabla \bar{y}_{\tau,\epsilon}|^2}\right)\]
normally in \(C(\tilde{\Omega};\mathbb{R}^{n \times n})\) as \(\delta \to 0\). Consequently,
\[Y_{\tau,\epsilon}^\delta \to Y_{\tau,\epsilon}\quad \text{strongly in } W^{1,2}_0(\Omega)\]
with \(Y_{\tau,\epsilon}\) being the solution of the following equation
\[
\begin{cases}
-\text{div}(A_{\tau,\epsilon} \nabla Y_{\tau,\epsilon}) = v - \bar{v}_{\tau,\epsilon} & \text{in } \Omega, \\
Y_{\tau,\epsilon} = 0 & \text{on } \Gamma.
\end{cases}
\] (4.10)
Then by (4.9), (S1) and (S3), we have that as \(\delta \to 0\),
\[J_{\tau}(u_{\tau,\epsilon}^\delta, v_{\tau,\epsilon}^\delta) \to J_{\tau}(\bar{u}_{\tau,\epsilon}, \bar{v}_{\tau,\epsilon}),\] (4.11)
\begin{align*}
\frac{1}{\delta} \left\{ \left( J_\tau (u_{\tau, \varepsilon}^\delta, v_{\tau, \varepsilon}^\delta) - \bar{J}_\tau + \varepsilon \right)^+ \right\}^2 - \left\{ \left( J_\tau (\bar{u}_{\tau, \varepsilon}, \bar{v}_{\tau, \varepsilon}) - \bar{J}_\tau + \varepsilon \right)^+ \right\}^2 \\
\to 2 \left[ J_\tau (\bar{u}_{\tau, \varepsilon}, \bar{v}_{\tau, \varepsilon}) - \bar{J}_\tau + \varepsilon \right]^+ \left\{ \int_\Omega f^0 (x, \bar{y}_{\tau, \varepsilon}, \bar{u}_{\tau, \varepsilon}) Y_{\tau, \varepsilon} \, dx \right\} \\
+ \int_\Omega \left[ f^0 (x, \bar{y}_{\tau, \varepsilon}, u) - f^0 (x, \bar{y}_{\tau, \varepsilon}, \bar{u}_{\tau, \varepsilon}) \right] \, dx.
\end{align*}

(4.12)

\begin{align*}
\frac{1}{\delta} \left\{ \int_\Omega \left[ v_{\tau, \varepsilon}^\delta - f (x, y_{\tau, \varepsilon}^\delta, u_{\tau, \varepsilon}^\delta) \right]^2 \, dx \right\} - \int_\Omega \left[ \bar{v}_{\tau, \varepsilon} - f (x, \bar{y}_{\tau, \varepsilon}, \bar{u}_{\tau, \varepsilon}) \right]^2 \, dx \\
\to 2 \int_\Omega \left[ \bar{v}_{\tau, \varepsilon} - f (x, \bar{y}_{\tau, \varepsilon}, \bar{u}_{\tau, \varepsilon}) \right] \left[ v - \bar{v}_{\tau, \varepsilon} - f_y (x, \bar{y}_{\tau, \varepsilon}, \bar{u}_{\tau, \varepsilon}) Y_{\tau, \varepsilon} \right. \\
- f (x, \bar{y}_{\tau, \varepsilon}, u) + f (x, \bar{y}_{\tau, \varepsilon}, \bar{u}_{\tau, \varepsilon}) \] \, dx.
\end{align*}

(4.13)

By (S2), we imply that
\begin{equation}
\text{d}_W (F (y_{\tau, \varepsilon}^\delta)) \to \text{d}_W (F (\bar{y}_{\tau, \varepsilon})) \quad \text{as} \quad \delta \to 0.
\end{equation}

(4.14)

Using the same arguments in [Li, Chapter 5], we obtain that
\begin{equation}
\frac{1}{\delta} \left[ \text{d}_W^2 (F (y_{\tau, \varepsilon}^\delta)) - \text{d}_W^2 (F (\bar{y}_{\tau, \varepsilon})) \right] \to 2 \text{d}_W (F (\bar{y}_{\tau, \varepsilon})) \langle \xi_{\tau, \varepsilon}, F' (\bar{y}_{\tau, \varepsilon}) Y_{\tau, \varepsilon} \rangle x^*, x,
\end{equation}

(4.15)

where \( \xi_{\tau, \varepsilon} \in \partial \text{d}_W (F (\bar{y}_{\tau, \varepsilon})) \), and
\begin{equation}
\| \xi_{\tau, \varepsilon} \|_{x^*} = \begin{cases} 1 & \text{if } F (\bar{y}_{\tau, \varepsilon}) \notin W, \\
0 & \text{if } F (\bar{y}_{\tau, \varepsilon}) \in W.
\end{cases}
\end{equation}

By (4.4), we have that
\begin{equation}
- \sqrt{\varepsilon} \left( \| v - \bar{v}_{\tau, \varepsilon} \|_{L^2 (\Omega)}^2 + |\Omega| \right)^{1/2} \leq \frac{J_{\tau, \varepsilon}^2 (u_{\tau, \varepsilon}^\delta, v_{\tau, \varepsilon}^\delta) - J_{\tau, \varepsilon}^2 (\bar{u}_{\tau, \varepsilon}, \bar{v}_{\tau, \varepsilon})}{\delta \left[ J_{\tau, \varepsilon} (u_{\tau, \varepsilon}^\delta, v_{\tau, \varepsilon}^\delta) + J_{\tau, \varepsilon} (\bar{u}_{\tau, \varepsilon}, \bar{v}_{\tau, \varepsilon}) \right]}
\end{equation}

(4.16)

By taking the limit for \( \delta \to 0 \) in (4.16), applying (4.11)-(4.15), we obtain that
\begin{equation}
- \sqrt{\varepsilon} \left( \| v - \bar{v}_{\tau, \varepsilon} \|_{L^2 (\Omega)}^2 + |\Omega| \right)^{1/2} \leq \lambda_{\tau, \varepsilon} z_{\tau, \varepsilon} + \langle \varphi_{\tau, \varepsilon}, F' (\bar{y}_{\tau, \varepsilon}) Y_{\tau, \varepsilon} \rangle x^*, x \\
+ \int_\Omega \psi_{\tau, \varepsilon} \left[ (v - \bar{v}_{\tau, \varepsilon} - f_y (x, \bar{y}_{\tau, \varepsilon}, \bar{u}_{\tau, \varepsilon}) Y_{\tau, \varepsilon} \\
- f (x, \bar{y}_{\tau, \varepsilon}, u) + f (x, \bar{y}_{\tau, \varepsilon}, \bar{u}_{\tau, \varepsilon}) \] \, dx
\end{equation}

(4.17)
for all \((u, v) \in \mathcal{U}_\text{ad} \times \mathcal{K}\), where
\[
\lambda_{\tau, \varepsilon} = \left[ J_{\tau}(\bar{u}_{\tau, \varepsilon}, \bar{v}_{\tau, \varepsilon}) - \bar{J}_{\tau} + \varepsilon \right]/J_{\tau, \varepsilon}(\bar{u}_{\tau, \varepsilon}, \bar{v}_{\tau, \varepsilon}),
\]
\[
z^0_{\tau, \varepsilon} = \int_{\Omega} \left[ f^0_y(x, \bar{y}_{\tau, \varepsilon}, \bar{u}_{\tau, \varepsilon}) Y_{\tau, \varepsilon} + f^0(x, \bar{y}_{\tau, \varepsilon}, u) - f^0(x, \bar{y}_{\tau, \varepsilon}, \bar{u}_{\tau, \varepsilon}) \right] dx,
\]
and
\[
\varphi_{\tau, \varepsilon} = \frac{d\nu(F(\bar{y}_{\tau, \varepsilon})))\xi_{\tau, \varepsilon}/J_{\tau, \varepsilon}(\bar{u}_{\tau, \varepsilon}, \bar{v}_{\tau, \varepsilon}),
\]
and
\[
\psi_{\tau, \varepsilon} = \frac{[\bar{v}_{\tau, \varepsilon} - f(x, \bar{y}_{\tau, \varepsilon}, \bar{u}_{\tau, \varepsilon})]}{J_{\tau, \varepsilon}(\bar{u}_{\tau, \varepsilon}, \bar{v}_{\tau, \varepsilon})}.
\]

One can easily check that
\[
\lambda^2_{\tau, \varepsilon} + \|\varphi_{\tau, \varepsilon}\|^2_{X^*} + \|\psi_{\tau, \varepsilon}\|^2_{L^2(\Omega)} = 1 \quad \text{for any } 0 < \tau, \varepsilon < 1. \tag{4.18}
\]

By taking \(u = \bar{u}_{\tau, \varepsilon}\) in (4.17), we obtain that
\[
-\sqrt{\varepsilon} \left( \|v - \bar{v}_{\tau, \varepsilon}\|^2_{L^2(\Omega)} + |\Omega|^2 \right)^{1/2} \leq \lambda_{\tau, \varepsilon} \int_{\Omega} f^0_y(x, \bar{y}_{\tau, \varepsilon}, \bar{u}_{\tau, \varepsilon}) Y_{\tau, \varepsilon} dx
\]
\[
+ \int_{\Omega} \psi_{\tau, \varepsilon} [(v - \bar{v}_{\tau, \varepsilon} - f_y(x, \bar{y}_{\tau, \varepsilon}, \bar{u}_{\tau, \varepsilon}) Y_{\tau, \varepsilon})] dx
\]
\[
+ \int_{\Omega} F'(\bar{y}_{\tau, \varepsilon})^* \varphi_{\tau, \varepsilon} Y_{\tau, \varepsilon} dx \tag{4.19}
\]
for all \(v \in \mathcal{K}\).

By taking \(v = \bar{v}_{\tau, \varepsilon}\) in (4.17), we get that
\[
-\sqrt{\varepsilon} |\Omega| \leq \lambda_{\tau, \varepsilon} \int_{\Omega} \left[ f^0(x, \bar{y}_{\tau, \varepsilon}, u) - f^0(x, \bar{y}_{\tau, \varepsilon}, \bar{u}_{\tau, \varepsilon}) \right] dx
\]
\[
- \int_{\Omega} \psi_{\tau, \varepsilon} [f(x, \bar{y}_{\tau, \varepsilon}, u) - f(x, \bar{y}_{\tau, \varepsilon}, \bar{u}_{\tau, \varepsilon})] dx \tag{4.20}
\]
for all \(u \in \mathcal{U}_\text{ad}\).

In fact, (4.19) and (4.20) can be regarded as necessary conditions for \((\bar{u}_{\tau, \varepsilon}, \bar{v}_{\tau, \varepsilon})\). Next, we shall pass to the limits for \(\varepsilon \to 0\) and then \(\tau \to 0\) to derive necessary conditions for \((\bar{y}, \bar{u})\).

By (4.18), there exist generalized subsequences of \(\varepsilon\) and \(\tau\), respectively, denoted in the same way, such that for \(\varepsilon \to 0\) and then \(\tau \to 0\),
\[
\lambda_{\tau, \varepsilon} \to \lambda_0, \tag{4.21}
\]

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\[ \varphi_{\tau,\varepsilon} \rightarrow \varphi_0 \quad \text{weakly star in } X^*, \quad (4.22) \]

and

\[ \psi_{\tau,\varepsilon} \rightarrow \bar{\psi} \quad \text{weakly in } L^2(\Omega). \quad (4.23) \]

Furthermore, it follows from (4.3) that

\[ \bar{u}_{\tau,\varepsilon} \rightarrow \bar{u} \quad \text{in } \mathcal{U}_{ad}, \quad \bar{v}_{\tau,\varepsilon} \rightarrow \bar{v} \quad \text{strongly in } L^2(\Omega) \quad \text{as } \varepsilon \to 0. \quad (4.24) \]

Thus, combining (4.24) with (3.1), (3.2) and (2.1), we can deduce that

\[ \bar{y}_{\tau,\varepsilon} \rightarrow \bar{y} \quad \text{strongly in } W^{1,2}_0(\Omega), \quad \text{uniformly in } C^1(\bar{\Omega}) \quad (4.25) \]

by letting \( \varepsilon \to 0 \) and then \( \tau \to 0. \)

In addition, for any \( 0 < \tau, \varepsilon < 1 \), we have that

\[
\begin{align*}
\left( \sqrt{\tau^2 + |\nabla \bar{y}_{\tau,\varepsilon}|^2} \right)^{p-2} & \cdot |\nabla Y_{\tau,\varepsilon}|^2 \\
+ (p-2) \left( \sqrt{\tau^2 + |\nabla \bar{y}_{\tau,\varepsilon}|^2} \right)^{p-4} & \cdot (\nabla Y_{\tau,\varepsilon})^T \nabla \bar{y}_{\tau,\varepsilon} (\nabla \bar{y}_{\tau,\varepsilon})^T \nabla Y_{\tau,\varepsilon} \\
\end{align*}
\]

\[
= \left( \sqrt{\tau^2 + |\nabla \bar{y}_{\tau,\varepsilon}|^2} \right)^{p-2} |\nabla Y_{\tau,\varepsilon}|^2 \\
+ (p-2) \left( \sqrt{\tau^2 + |\nabla \bar{y}_{\tau,\varepsilon}|^2} \right)^{p-1} \cdot \frac{(\nabla Y_{\tau,\varepsilon})^T \nabla \bar{y}_{\tau,\varepsilon} (\nabla \bar{y}_{\tau,\varepsilon})^T \nabla Y_{\tau,\varepsilon}}{\left( \tau^2 + |\nabla \bar{y}_{\tau,\varepsilon}|^2 \right)^{\frac{1}{2}}} \\
= \left( \sqrt{\tau^2 + |\nabla \bar{y}_{\tau,\varepsilon}|^2} \right)^{p-2} |\nabla Y_{\tau,\varepsilon}|^2 + h_{\tau,\varepsilon} \cdot I_{\tau,\varepsilon},
\]

where

\[ h_{\tau,\varepsilon} = (p-2) \left( \sqrt{\tau^2 + |\nabla \bar{y}_{\tau,\varepsilon}|^2} \right)^{p-1}, \]

and

\[ I_{\tau,\varepsilon} = \frac{(\nabla Y_{\tau,\varepsilon})^T \nabla \bar{y}_{\tau,\varepsilon} (\nabla \bar{y}_{\tau,\varepsilon})^T \nabla Y_{\tau,\varepsilon}}{\left( \tau^2 + |\nabla \bar{y}_{\tau,\varepsilon}|^2 \right)^{\frac{1}{2}}}. \]

Next, we deal with the term of \( I_{\tau,\varepsilon} \). In fact, in \( \Omega \), it holds that

\[ I_{\tau,\varepsilon} \leq |I_{\tau,\varepsilon}| \leq \frac{|\nabla Y_{\tau,\varepsilon}|^2}{\tau^2 + |\nabla \bar{y}_{\tau,\varepsilon}|^2} \cdot \frac{|\nabla \bar{y}_{\tau,\varepsilon}|^2}{\tau^2 + |\nabla \bar{y}_{\tau,\varepsilon}|^2} \leq \frac{|\nabla Y_{\tau,\varepsilon}|^2}{\sqrt{\tau^2 + |\nabla \bar{y}_{\tau,\varepsilon}|^2}}. \]

Moreover, \( 1 < p < 2 \) implies that \( h_{\tau,\varepsilon} < 0 \). Thus we have that

\[
\begin{align*}
\left( \sqrt{\tau^2 + |\nabla \bar{y}_{\tau,\varepsilon}|^2} \right)^{p-2} & \cdot |\nabla Y_{\tau,\varepsilon}|^2 \\
+ (p-2) \left( \sqrt{\tau^2 + |\nabla \bar{y}_{\tau,\varepsilon}|^2} \right)^{p-4} & \cdot (\nabla Y_{\tau,\varepsilon})^T \nabla \bar{y}_{\tau,\varepsilon} (\nabla \bar{y}_{\tau,\varepsilon})^T \nabla Y_{\tau,\varepsilon} \\
\geq (p-1) \left( \tau^2 + |\nabla \bar{y}_{\tau,\varepsilon}|^2 \right)^{\frac{p-2}{2}} & \cdot |\nabla Y_{\tau,\varepsilon}|^2. \n\end{align*}
\]
It follows from this and (4.10), we get that
\[(p - 1) \int_{\Omega} (\tau^2 + |\nabla \bar{y}_{\tau,\varepsilon}|^2)^{\frac{p-2}{2}} |\nabla Y_{\tau,\varepsilon}|^2 \, dx \]
\[\leq \int_{\Omega} |v - \bar{v}_{\tau,\varepsilon}| \cdot |Y_{\tau,\varepsilon}| \, dx \leq C \int_{\Omega} |Y_{\tau,\varepsilon}| \, dx. \tag{4.26}\]

Since $1 < p < 2$, by (4.26) and Lemma 3.2, we get that
\[(p - 1) \int_{\Omega} |\nabla Y_{\tau,\varepsilon}|^2 \, dx \]
\[\leq (p - 1) \|\tau^2 + |\nabla \bar{y}_{\tau,\varepsilon}|^2\|_{L^\infty(\Omega)}^{\frac{2-p}{2}} \int_{\Omega} (\tau^2 + |\nabla \bar{y}_{\tau,\varepsilon}|^2)^{\frac{p-2}{2}} |\nabla Y_{\tau,\varepsilon}|^2 \, dx \]
\[\leq C \int_{\Omega} |Y_{\tau,\varepsilon}| \, dx. \]

Consequently, we have that
\[\|Y_{\tau,\varepsilon}\|_{W^{1,2}_0(\Omega)} \leq C \tag{4.27}\]
for some $C > 0$ independent of $\varepsilon$. Moreover, it follows from (4.25) and the definitions of $A_{\tau,\varepsilon}$, $A$ that
\[A_{\tau,\varepsilon} \to A \text{ uniformly in } C(\bar{\Omega}; \mathbb{R}^{n \times n}) \tag{4.28}\]
as $\varepsilon \to 0$ and then $\tau \to 0$. Thus, by (4.24), (4.25), (4.27) and (4.28), we can deduce that
\[Y_{\tau,\varepsilon} \to Y \text{ weakly in } W^{1,2}_0(\Omega), \text{ strongly in } L^2(\Omega) \tag{4.29}\]
as $\varepsilon \to 0$ and then $\tau \to 0$, where $Y$ satisfies that
\[\begin{cases} -\text{div}(A\nabla Y) = v - \bar{v} & \text{in } \Omega, \\ Y = 0 & \text{on } \Gamma \end{cases}\]
with
\[A = |\nabla \bar{y}|^{p-2}\left(I + (p - 2) \frac{\nabla \bar{y}(\nabla \bar{y})^T}{|\nabla \bar{y}|^2}\right).\]

By (4.21)-(4.25), (4.29) and by applying Lemmas 3.5 and 3.6, we may pass to the limits for $\varepsilon \to 0$ and then $\tau \to 0$ in (4.19) to obtain that
\[0 = \lambda_0 \int_{\Omega} f_y^0(x, \bar{y}, \bar{u})Y \, dx + \int_{\Omega} \bar{\psi} [v - \bar{v} - f_y(x, \bar{y}, \bar{u})Y] \, dx \]
\[+ \int_{\Omega} F'(\bar{y})^* \varphi_0 Y \, dx \tag{4.30}\]
for all \( v \in K \). This implies that (2.9) holds. Similarly, by (4.21)-(4.25) and by Lemmas 3.3 and 3.4, we pass to the limits for \( \varepsilon \to 0 \) and then \( \tau \to 0 \) in (4.20) to derive that
\[
\int_{\Omega} \lambda_0 \left[ f^0(x, \bar{y}, u) - f^0(x, \bar{y}, \bar{u}) \right] dx - \int_{\Omega} \bar{\psi} \left[ f(x, \bar{y}, u) - f(x, \bar{y}, \bar{u}) \right] dx \geq 0
\]
for all \( u \in \mathcal{U}_{ad} \). This implies that
\[
\int_{\Omega} \left[ H_{\lambda_0}(x, \bar{y}, u, \bar{\psi}) - H_{\lambda_0}(x, \bar{y}, \bar{u}, \bar{\psi}) \right] dx \geq 0 \quad \text{for all } u \in \mathcal{U}_{ad}. \tag{4.31}
\]
Then using the same arguments in [5], we get (2.12) from (4.31).

Since \( \varphi_{\tau, \varepsilon} \in \partial d_{W}(F(\bar{y}_{\tau, \varepsilon})) \), we must have that
\[
\langle \varphi_{\tau, \varepsilon}, \phi - F(\bar{y}_{\tau, \varepsilon}) \rangle_{X^*, X} \leq 0 \quad \text{for any } \phi \in W. \tag{4.32}
\]
By (4.25) and (S2), \( F(\bar{y}_{\tau, \varepsilon}) \to F(\bar{y}) \) strongly in \( X \) as \( \varepsilon \to 0 \) and then \( \tau \to 0 \). This together with (4.32) gives that
\[
\langle \varphi_{\tau, \varepsilon}, \phi - F(\bar{y}) \rangle_{X^*, X} \leq \langle \varphi_{\tau, \varepsilon}, F(\bar{y}_{\tau, \varepsilon}) - F(\bar{y}) \rangle_{X^*, X} \to 0 \quad \text{as } \varepsilon \to 0 \text{ and then } \tau \to 0. \tag{4.33}
\]

Now we turn to prove (2.10). By (4.25), we have that
\[
\lim_{\tau \to 0} \lim_{\varepsilon \to 0} \| \bar{y}_{\tau, \varepsilon} - \bar{y} \|_{C^1(\Omega)} = 0.
\]
For any \( \gamma > 0 \), there exists \( \rho \in (0, 1) \), when \( 0 < \tau < \rho \),
\[
\lim_{\varepsilon \to 0} \| \bar{y}_{\tau, \varepsilon} - \bar{y} \|_{C^1(\Omega)} \leq \gamma.
\]
Setting \( \lim_{\varepsilon \to 0} \| \bar{y}_{\tau, \varepsilon} - \bar{y} \|_{C^1(\Omega)} = h_{\tau} \), similarly, for \( 0 < \tau < \rho \), there exists an \( \varepsilon_{\tau} \in (0, 1) \), such that
\[
\| \bar{y}_{\tau, \varepsilon} - \bar{y} \|_{C^1(\Omega)} - h_{\tau} \leq \gamma.
\]
Thus,
\[
\| \bar{y}_{\tau, \varepsilon} - \bar{y} \|_{C^1(\Omega)} \leq 2\gamma.
\]
This implies that
\[
|\nabla \bar{y}_{\tau, \varepsilon}(x)| \leq 2\gamma \quad \forall x \in \{ \nabla \bar{y} = 0 \}, \quad \tau \in (0, \rho), \quad \varepsilon \in (0, \varepsilon_{\tau}).
\]
Since $1 < p < 2$, using (4.26) and (4.27), we have that 
\[(p - 1)(\tau^2 + 4\gamma^2)^{\frac{p-2}{p}} \int_{\{\bar{y} = 0\}} |\nabla Y_{\tau,\epsilon}|^2 \, dx \leq C.\]
That is 
\[\int_{\{\bar{y} = 0\}} |\nabla Y_{\tau,\epsilon}|^2 \, dx \leq \frac{C(\tau^2 + 4\gamma^2)^{\frac{2-p}{2}}}{p-1}.\]
Consequently, 
\[\int_{\{\bar{y} = 0\}} |\nabla Y|^2 \, dx \leq \frac{C(2\gamma)^{2-p}}{p-1}.\]
This also implies that $Y = 0$ a.e. on $\{\nabla \bar{y} = 0\}$ and then it follows from (4.30) that $\bar{\psi} = 0$ a.e. on $\{\nabla \bar{y} = 0\}$. That is to say that (2.10) holds.

It remains to show that $(\lambda_0, \varphi_0, \bar{\psi}) \neq 0$. To this end, we suppose that $\lambda_0 = 0$. Then it follows from (4.21) that $\lambda_{\tau,\epsilon} \to 0$ as $\epsilon \to 0$ and then $\tau \to 0$.

By (4.17) and (4.32), we obtain that
\[-\eta_{\tau,\epsilon}(u, v) \leq \langle \varphi_{\tau,\epsilon}, F'(\bar{y})Y_{\tau,\epsilon} - \phi + F(\bar{y}) \rangle_{X^*, X} + \int_\Omega \bar{\psi}_{\tau,\epsilon} [v - \bar{v}_{\tau,\epsilon} - f_y(x, \bar{y}_{\tau,\epsilon}, \bar{u})Y_{\tau,\epsilon}]
- f(x, \bar{y}_{\tau,\epsilon}, u) + f(x, \bar{y}_{\tau,\epsilon}, \bar{u})] \, dx \quad (4.34)\]
for all $\phi \in W$ and $(u, v) \in U_{ad} \times K$, where
\[\eta_{\tau,\epsilon}(u, v) = \sqrt{\epsilon} \left( \|v - \bar{v}_{\tau,\epsilon}\|_{L^2(\Omega)}^2 + |\Omega|^2 \right)^{1/2} + \lambda_{\tau,\epsilon} z_{\tau,\epsilon}^0 + \int_\Omega \bar{\psi}_{\tau,\epsilon} [f_y(x, \bar{y}_{\tau,\epsilon}, \bar{u}) - f_y(x, \bar{y}_{\tau,\epsilon}, \bar{u}_{\tau,\epsilon})] \, dx
+ \int_\Omega \bar{\psi}_{\tau,\epsilon} [f(x, \bar{y}_{\tau,\epsilon}, \bar{u}_{\tau,\epsilon}) - f(x, \bar{y}_{\tau,\epsilon}, \bar{u})] \, dx
+ \langle \varphi_{\tau,\epsilon}, [F'(\bar{y}_{\tau,\epsilon}) - F'(\bar{y})]Y_{\tau,\epsilon} + F(\bar{y}_{\tau,\epsilon}) - F(\bar{y}) \rangle_{X^*, X}.\]
Note that $z_{\tau,\epsilon}^0$ depends on $(u, v)$. By (4.29), Lemmas 3.3 and (3.5), we get that for $\epsilon, \tau > 0$ small enough, $|z_{\tau,\epsilon}^0| \leq C$, where $C$ is independent of $\epsilon, \tau$ and $(u, v) \in U_{ad} \times K$.

Since $\lambda_{\tau,\epsilon} \to 0$, we conclude that $\lambda_{\tau,\epsilon} z_{\tau,\epsilon}^0 \to 0$ uniformly in $U_{ad} \times K$. Then by Lemma 3.7, (4.18), (4.25) and the definition of $\eta_{\tau,\epsilon}$, we deduce that
\[\eta_{\tau,\epsilon}(u, v) \to 0 \quad \text{uniformly in } U_{ad} \times K. \quad (4.35)\]
By (4.25), there exists $\varepsilon_1$, $\tau_1 > 0$ such that
\[
\|\bar{y}_{\tau,\varepsilon} - \bar{y}\|_{W^{1,2}_0(\Omega)} \leq r_1 \tag{4.36}
\]
as $0 < \varepsilon < \varepsilon_1$, $0 < \tau < \tau_1$, where $r_1$ is given in (S4). Thus, by (2.7), for any $(q, w) \in B_{r_1, r_2}$ and for every $0 < \varepsilon < \varepsilon_1$, $0 < \tau < \tau_1$, there exists $Z_{\tau,\varepsilon} \in W^{1,2}_0(\Omega)$ with
\[
\|Z_{\tau,\varepsilon}\|_{W^{1,2}_0(\Omega)} \leq r_2, \tag{4.37}
\]
such that
\[
-\text{div}(A \nabla Z_{\tau,\varepsilon}) = f_y(x, \bar{y}_{\tau,\varepsilon}, \bar{u})Z_{\tau,\varepsilon} + f(x, \bar{y}_{\tau,\varepsilon}, u) - f(x, \bar{y}_{\tau,\varepsilon}, \bar{u}) + w \text{ in } \Omega, \tag{4.38}
\]
and
\[
F'(\bar{y})Z_{\tau,\varepsilon} = q, \tag{4.39}
\]
where $A$ is given by (2.8).

Let $\tilde{v}_{\tau,\varepsilon} \equiv \tilde{v}_{\tau,\varepsilon} + f_y(x, \bar{y}_{\tau,\varepsilon}, \bar{u})Z_{\tau,\varepsilon} + f(x, \bar{y}_{\tau,\varepsilon}, u) - f(x, \bar{y}_{\tau,\varepsilon}, \bar{u}) + w$, and then by (S1), (4.36) and (4.37), we have that $\tilde{v}_{\tau,\varepsilon} \in K$. Now, we take $v = \tilde{v}_{\tau,\varepsilon}$ and recall $Y_{\tau,\varepsilon}$ satisfies
\[
-\text{div}(A_{\tau,\varepsilon} \nabla Y_{\tau,\varepsilon}) = v - \bar{v}_{\tau,\varepsilon} \text{ in } \Omega. \tag{4.40}
\]
Letting
\[
Y_{\tau,\varepsilon} - Z_{\tau,\varepsilon} = \gamma_{\tau,\varepsilon} \text{ in } \Omega,
\]
then by (4.38), (4.40) and (4.28), we have that
\[
\|\gamma_{\tau,\varepsilon}\|_{W^{1,2}_0(\Omega)} \to 0 \text{ as } \varepsilon \to 0 \text{ and then } \tau \to 0. \tag{4.41}
\]
In (4.34), we take $v = \tilde{v}_{\tau,\varepsilon}$ and have that
\[
-\eta_{\tau,\varepsilon}(u, \tilde{v}_{\tau,\varepsilon}) - \langle \varphi_{\tau,\varepsilon}, F'(\bar{y})\gamma_{\tau,\varepsilon} \rangle_{X^*, X} + \int_{\Omega} \bar{\psi}_{\tau,\varepsilon} f_y(x, \bar{y}_{\tau,\varepsilon}, \bar{u}) \gamma_{\tau,\varepsilon} dx \leq \langle \varphi_{\tau,\varepsilon}, q - \phi + F'(\bar{y}) \rangle_{X^*, X} + \int_{\Omega} \bar{\psi}_{\tau,\varepsilon} wd x \tag{4.42}
\]
for all $\phi \in W$ and $(q, w) \in B_{r_1, r_2}$. Define the left terms of (4.42) to be $\rho_{\tau,\varepsilon}$, that is
\[
\rho_{\tau,\varepsilon} = -\eta_{\tau,\varepsilon}(u, \tilde{v}_{\tau,\varepsilon}) - \langle \varphi_{\tau,\varepsilon}, F'(\bar{y})\gamma_{\tau,\varepsilon} \rangle_{X^*, X} + \int_{\Omega} \bar{\psi}_{\tau,\varepsilon} f_y(x, \bar{y}_{\tau,\varepsilon}, \bar{u}) \gamma_{\tau,\varepsilon} dx.
\]
then by (4.35) and (4.41), we deduce that

$$\rho_{\tau,\varepsilon} \to 0 \quad \text{as} \quad \varepsilon \to 0 \quad \text{and then} \quad \tau \to 0.$$  \hfill (4.43)

On the other hand, since \(\lambda_{\tau,\varepsilon} \to 0\), it follows from (4.18) that there exists \(\sigma > 0\) such that

$$1 \geq \|\varphi_{\tau,\varepsilon}\|_{X^*}^2 + \|\bar{\psi}_{\tau,\varepsilon}\|_{L^2(\Omega)}^2 \geq \sigma > 0$$  \hfill (4.44)

for all \(\varepsilon, \tau\) small enough. By (S4), \(B_{r_1, r_2} - W \times \{0\} + (F(\bar{y}), 0)\) has finite codimensionality in \(X \times L^\infty(\Omega)\). Thanks to Lemma 3.6 of Chapter 4 in [1], we conclude by (4.42)-(4.44), (4.22) and (4.23) that \((\varphi_0, \bar{\psi}) \neq 0\). This gives that

$$(\lambda_0, \varphi_0, \bar{\psi}) \neq 0.$$  \hfill (4.45)

Finally, in the case that \(F'(\bar{y})\) is injective, suppose that \((\lambda_0, \bar{\psi}) = 0\), then by (2.9), \(F'(\bar{y})\varphi_0 = 0\), which shows that \(\varphi_0 = 0\). This contradicts to (4.45). Therefore \((\lambda_0, \bar{\psi}) \neq 0\) in this case.

Thus we complete the proof the Theorem 2.1. \(\square\)
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