Spatio-temporal heterogeneities of entanglement in the many-body localized phase

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We propose a spatio-temporal characterization of the entanglement dynamics in many-body localized (MBL) systems, which exhibits a striking resemblance with dynamical heterogeneities in classical glasses. Specifically, we find that the relaxation times of local entanglement, as measured by the concurrence, are spatially correlated giving rise to a dynamical correlation length for quantum entanglement. Our work provides a yet unrecognized connection between the behavior of classical glasses and the genuine quantum dynamics of MBL systems.

Introduction.— The assumption of local equilibrium is at the core of statistical mechanics: even if isolated from the rest of the universe, a generic many-body system is expected to act as a thermal bath for itself, quickly driving the statistics of local observables to the Gibbs ensemble, by means of classical [1–3] or quantum [4, 5] chaos. The situations in which ergodization fails, and the system persists in non-thermal states for all relevant times, are therefore of paramount interest, both at the classical and the quantum level [5–7].

Glasses are a prototypical example of classical systems that remain trapped in metastable states for all experimentally accessible time scales [8–12]. In the quantum realm, after the seminal work [13], it has become clear that isolated, disordered many-body systems can elude thermal equilibrium even at infinite time. The existence of this nonergodic phase of matter, coined many-body localized (MBL), was found analytically [13–15] and numerically in a vast set of microscopic models [17–24], and observed in ultracold-atom experiments [25–27]. The lack of ergodicity in the MBL phase has been linked to the existence of an extensive number of local integrals of motion (LIOMs) [28–34], by which one can construct a phenomenological model of the l-bit model. The l-bit model qualitatively captures the features of the MBL phase, such as area-law entanglement for eigenstates, slow spreading of the entanglement after a quantum quench, and slow decay of correlation functions [35–39].

While entanglement can be completely characterized for two qubits [40–42], in many-body systems it has been proved difficult to quantify it in a definitive way, and only a handful of general properties are known. Several different measures of the entanglement have been proposed [43–45], which capture only specific properties of the quantity. The numerous studies on the entanglement growth in the MBL phase [18, 33, 35, 36, 46, 47] are mainly focused on global properties, employing measures such as the entanglement entropy, purity, quantum Fisher information [27, 48], or total correlations [40–49].

The purpose of this Letter is to go beyond those approaches, focusing on local properties of the entanglement, therefore characterizing its spreading in a more detailed way. We focus on the time and space behavior of the concurrence [41, 45, 50], which has been used to characterize entanglement in MBL before [51], and which can be measured experimentally [52, 53]. We quantitatively study the distribution of relaxation times of the concurrence in the l-bit model and describe its properties in a wide range of parameters and initial-state energies. We show that the relaxation times are spatially correlated and that they present a long-tailed, power-law-like distribution. Both the spatial correlation and the width of the distribution grow as disorder increases or energy decreases.

These properties provide a clear connection with dynamical heterogeneities, i.e. spatio-temporal fluctuations, observed in classical amorphous materials and spin glasses [10, 11, 54–58], adding one more point of contact between the physics of glasses and MBL systems [59–62]. It has already been argued that such heterogeneities have a quantum counterpart and, at ultra-low temperatures, can be induced by quantum fluctuations [63–66]. Previous studies have approached the topic modeling quantum glass systems based on a classical counterpart. For the first time, we focus on the MBL phase, which is a novel, non-ergodic state of matter existing only in the quantum realm. Moreover, we study the entanglement, which is a genuine quantum object with no classical analog: thus, the observed similarity to the classical case is even more remarkable.

Model.— We aim at studying general principles of the spatio-temporal entanglement dynamics of MBL systems. For that purpose, we focus on an effective description in terms of LIOMs, which allows us to access the nonequilibrium real-time dynamics of MBL systems for long times and large system sizes. Deep in the MBL phase, Hamiltonians of short-range interacting quantum spin-1/2 degrees of freedom can be diagonalized through a quasi-local unitary transformation [15, 51] yielding a
representation of the model in so-called l-bit form:

\[
H_{l\text{-bit}} = \sum_{i=1}^{L} h_i \sigma_i^z + \sum_{i,j=1}^{L} J_{ij} \sigma_i^x \sigma_j^x + \ldots
\]  

(1)

where \(\{\sigma_i^x, \sigma_i^y, \sigma_i^z\}\) are the localized spin-1/2 operators associated with the LIOMs. We neglect further terms in the Hamiltonian which comprise n-body interactions and \(n \geq 3\), which is a controlled approximation for weakly interacting spins in the original microscopic model. From the analytical results of Refs. [36, 37, 39, 41], it is known that the interactions \(J_{ij}\) are exponentially suppressed with the distance \(r_{ij}\) between localization centers. To achieve a model-independent effective description, we parametrize the l-bit model as follows. We assume that the \(h_i\) are independent identically distributed random fields, with a uniform distribution over \([-h, h]\), and that the \(J_{ij}\) are uncorrelated Gaussian variables of zero average and standard deviation \(J_0 e^{-r_{ij}/\kappa}\). For numerical purposes, we set \(h = J_0 = 1\).

The particular advantage of the l-bit model [1] is that it allows us to perform analytical estimates of few-body observables, and to efficiently compute them numerically, reaching system sizes up to \(L = 140\) spins for long times [36, 37, 39, 62]. It is worth stressing that the l-bits become more and more similar to the physical spins as the disorder increases, ultimately coinciding asymptotically at infinite disorder [33, 34]. Thus, at small values of \(\kappa\) (i.e., large disorder strength), one can safely consider the l-bits as uniformly spaced on a chain, and compute the distances between them as \(r_{ij} = |i - j|\), \(i, j = 1, 2, \ldots, L\). Our numerical results are obtained exactly in this strongly localized regime, deep in the MBL phase. It is important to note that the effective model allows us to tune: i) the interaction decay length \(\kappa\) (equivalent to varying the disorder strength); and ii), the initial condition, i.e. the energy density at which we prove the system’s properties. Concerning the latter case, we choose as initial state of the dynamics a product state in the effective spin basis:

\[
|\psi_0\rangle = \bigotimes_{i=1}^{L} (A_i |\uparrow_i\rangle + B_i |\downarrow_i\rangle),
\]  

(2)

where \(|\uparrow_i\rangle, |\downarrow_i\rangle\) are the eigenstates of \(\sigma_i^z\), and \(|A_i|^2 + |B_i|^2 = 1\). Employing Eq. [2], the system is initially prepared in a superposition of eigenstates, as it would be the case when deriving the l-bit model from a microscopic Hamiltonian. Moreover, Eq. (2) provides us with the flexibility to tune the coefficients \(A_i\) and \(B_i\) such that we can vary the initial-state energy: \(E := \langle \psi_0 | H_{l\text{-bit}} | \psi_0 \rangle\). This tuning can be achieved using a classical simulated annealing algorithm (see [67] for details), and allows us to explore different regions of the energy spectrum. We measure \(E\) in units of the standard deviations of \(h_i\) and \(\sum_{j} J_{jj}\), defining the dimensionless energy density \(\varepsilon := (E/N)/\sqrt{h^2/3 + 2J_0^2/(c^2/\kappa - 1)}\). Notice that \(\varepsilon = 0\) corresponds to the center of the spectrum, while \(\varepsilon \approx -1\) to the ground state [67, 68]. We also recall that the localization properties of MBL systems depend on the energy of the considered state, and are stronger near the edges of the spectrum [69].

**Methods.**— For the purpose of exploring dynamic entanglement heterogeneities in MBL systems, we concentrate on the two-site concurrence, which quantifies the pairwise entanglement [41, 42, 45, 51]. For two spins-1/2 located at lattice sites \(i\) and \(j\), the concurrence is defined as [41, 45]

\[
C_{i,j} := \max \{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}.
\]  

(3)

\(\lambda^2\) are the eigenvalues of the matrix \(R_{ij} = \rho_{ij}(\sigma_y \otimes \sigma_y)\rho_{ij}^\dagger(\sigma_y \otimes \sigma_y)\) sorted in descending order, where \(\rho_{ij}\) is the two-site reduced density matrix, and the complex conjugation is done in the standard computational basis. In MBL systems the mean value of the concurrence decays in time as a power law, in contrast to the exponential decay of ergodic systems [51]. Our key goal is to establish a more detailed spatio-temporal analysis of the concurrence beyond its mean value. For that purpose we define a local onsite concurrence via quantifying the total amount of entanglement of \(i\) with all the other lattice sites. For large systems and for a single disorder realization, we find that this local concurrence typically decays on a certain time scale, which motivates us to define a local relaxation time as

\[
\tau_i := t_0 e^{(\ln(t/t_0))/\kappa} = t_0 \exp \frac{\int_{t_0}^{t_{\text{fin}}} \ln(t/t_0) C_i(t)dt}{\int_{t_0}^{t_{\text{fin}}} C_i(t)dt},
\]  

(5)

where \(t_0 = J_0^{-1}\). Notice that \(C_i \geq 0\), i.e. the averages above are well-defined and independent of \(t_0\) in the thermodynamic limit. The definition [5] employs the logarithm \(\ln(t/t_0)\) to ensure that \(\tau_i\) is a good estimator of the typical time scale of the decay time of the concurrence, even if \(C_i(t)\) decays very slowly [70]. For finite systems we have that \(\text{typ}C_i(\infty) = O(2^{-L})\) [62, 67]. Thus, the function \(C_i(t)\) might be interpreted as a probability distribution in \(\mathbb{R}\) only in the thermodynamic limit. Therefore, when considering below the distribution functions of \(\tau_i\), the thermodynamic limit can be approached by fixing the maximum simulation time of the dynamics \(t_{\text{fin}}\) upon increasing \(L\), until convergence is reached. We find from our numerics that this typically happens for \(L \gtrsim 30\) [67], which is achievable in the l-bit but not for microscopic Hamiltonians by means of exact diagonalization. This represents a key argument for the use of the effective l-bit model.

While the \(\tau_i\)’s provide us with temporal information of the entanglement dynamics, we are further interested in
the spatial component. For that purpose we quantify the spatial correlations of the local relaxation time via

$$G_r(r) := \frac{\langle (\tau_i - \tau_j)^{\text{Is}} \rangle_{\text{Is}} - \langle \tau_i \rangle_{\text{Is}} \langle \tau_j \rangle_{\text{Is}}}{\langle (\tau_i^2)_{\text{Is}} - \langle \tau_i \rangle_{\text{Is}}^2 \rangle_{\text{Is}}},$$

where $\langle \bullet \rangle_{\text{Is}}$ denotes the average over different initial states, $\langle \bullet \rangle_{\text{Is}}^{\text{Is}}$ the average over all sites $i, j$ separated by a distance $r$, and $\langle \bullet \rangle$ the average over different disorder realizations. In [67] we show that $G_r(r)$ as defined in Eq. (6) is very robust to finite-size effects and disorder fluctuations: it is a self-averaging quantity. From our numerical simulations, we find that $G_r(r)$ experiences in general a stretched exponential decay as a function of $r$.

This allows us to define a length scale $\eta_r$ by performing a fit of the form $\log G_r(r) \sim a + (r/\eta_r)^{\beta}$ for some suitable $a$ and $b$. The length $\eta_r$ quantifies the distance over which the local entanglement relaxation is spatially correlated, i.e. it gives the size of the typical clusters of fast or slow entangling spins.

For small system sizes, we will also compare the results of the effective model with a full microscopic calculation for the spin-1/2 XXZ chain with random fields:

$$H_{\text{XXZ}} = \sum_{i=1}^{L-1} \left[ J (S_i^+ S_{i+1}^+ + \text{h.c.}) + V S_i^z S_{i+1}^z \right] + \sum_{i=1}^{L} \Delta_i S_i^z,$$

where $J = V = 1$ and $\Delta_i$ are random variables uniformly distributed over $[-W/2, W/2]$. For $W_c \simeq 7 \pm 2$ this model exhibits a MBL transition [19] (see Refs. 33, 34 about the relation between $W$ and the effective model parameters $h$, $\kappa$, and $J_0$). We probe the centre of the spectrum initializing the system in a Néel state $|\psi_0\rangle = |\uparrow\downarrow\uparrow\downarrow\ldots\rangle$, and average the results over different disorder realizations. We find that $C_{i,i} = 0$ for $|i-j| > 1$ so that we restrict the sum in Eq. (4) to the nearest-neighbour term, $C_i(t) = C_{i,i+1}(t)$. We compute the local relaxation times by Eq. (6) with $J_0 = 1$.

Results. — We show in Fig. 1 the probability distribution function (pdf) of $\log_{10} \tau_i$, obtained within both the XXZ and the l-bit model. We see that within the XXZ model (Fig. 1a) the pdf’s show a peak at the largest relaxation time, corresponding to the final simulation time $t_{\text{fin}}$. In [67] we argue that this feature is due to the asymptotic value $\text{typ}(C_i(\infty)) = O(2^{-L})$ (see also the discussion below Eq. (5)). If the time spent in such asymptotic region is too large, the decay time is heavily influenced by the final time of the dynamics. This is a finite-size effect, and it does disappear upon considering larger system sizes, as we show for the l-bit model in [67] (larger system sizes for the XXZ model cannot be presently considered).

The pdf’s obtained considering the l-bit model for $L = 80$, and for different values of $\kappa$ and $\varepsilon$, are shown in Figs. 1b, 1c. Thanks to the large system size, these plots do not present any peak at large times, and clearly show that the pdf of $\log_{10} \tau_i$ has a power-law tail: thus, in turn, also $\tau_i$ is power-law distributed. We see that the pdf’s become wider as the disorder is increased (both in the XXZ and the l-bit model), or the energy is lowered (in the l-bit model).

We define the typical value of $\tau_i$ as $\text{typ}[\tau_i] := t_0 \exp[\ln(\tau_i)/t_0]$, where $\langle \bullet \rangle_{\tau_i}$ is the average over the pdf of $\tau_i$. In Fig. 2a we show the behavior of $\text{typ}[\tau_i]$ as a function of the parameters $\kappa$ and $\varepsilon$. Following usual arguments for the l-bit model [30, 37], in [67] we derive the rough estimate: $\ln(\text{typ}[\tau_i]/t_0) \approx (2\kappa \ln 2 - 1)^{-1}$. Fig. 2a depicts the fits of $\text{typ}[\tau_i]$ with this functional relation with respect to $\kappa$, showing that our numerical results are in good agreement with this prediction. In Fig. 2b we plot the power-law exponents $\beta$ obtained from the fit of $P(\log_{10} \tau_i) \sim (\log_{10} \tau_i)^{-\beta}$ shown in Fig. 1b. We see that
FIG. 2. (a) The typical value of $\tau_i$, defined in the main text, is shown as a function of $\kappa$ for different $\epsilon$ (dots). The dashed lines are fits with the function $\exp[a + (bc + c)^{-1}]$. From $a$, we performed the linear fits depicted in the inset: $\ln(\text{typ}[\tau_i - a])^{-1}$ as a function of $\kappa$ is found to be linear, as expected. (b) Slope $\beta$ as a function of $\kappa$, obtained from the linear fits of the tails of $\log_{10}(P(\log_{10} \tau_i))$ in Fig. [11] $\beta(\kappa)$ is consistent with a linear behavior; with a linear fit we obtain: $\beta = 2.8(5)\kappa - 1.2(2)$.

We plot the spatial correlations between the $\tau_i$’s in Fig. [3]. Due to the strong finite-size effects for the XXZ model, we restrict ourselves to the $l$-bit model. In Fig. [3a] we show the spatial distribution of the $\tau_i$’s for a disorder realization. As $\kappa$ decreases, i.e. the disorder increases, the relaxation times of the local entanglement become spatially correlated over longer distances. The correlation function $G(\tau)$, defined in Eq. [4], is shown in Fig. [3b]. $G(\tau)$ decays more slowly upon decreasing $\kappa$ and $\epsilon$, confirming the pattern observed in Fig. [3a]. The same result is also supported by the (qualitative) behavior of the dynamical correlation length $\tau_{\eta}$, from stretched exponential fits of $G(\tau)$, as a function of $\kappa$. $\tau_{\eta}$ decreases as $\kappa$ increases: For larger disorder, larger clusters of dynamically correlated spins emerge.

The implications are twofold. The local entanglement spreading slows down when $\kappa$ decreases, i.e. the disorder increases, or the energy decreases (Fig. [1]), and increasingly larger clusters of spins emerge, in which the entanglement relaxation is correlated (Fig. [3]). As disorder increases, the distribution of relaxation times becomes wider, implying that more clusters are likely to assume an extreme value of the relaxation time in the slow, as well as in the fast tail. These findings might seem surprising in the quantum case, as a more localized structure might be expected when disorder increases. However, they are in agreement with the behavior of classical amorphous materials and spin glasses. In particular, they resemble the results in Ref. [54] in the case of the classical Ising spin glass in $d = 2, 3$, where the authors found that heterogeneities amplify when the temperature of the bath is lowered, and the glass transition is approached.

**Conclusions.**—We studied the spreading of entanglement in MBL systems by monitoring the onsite concurrence. We showed that in the MBL phase the relaxation times of the onsite concurrence increase upon increasing the disorder, or upon lowering the energy of the initial states. Specifically, the local relaxation times become more spatially correlated, and their distribution broadens. Thus, as disorder increases or energy decreases, entanglement heterogeneities arise: we observe the formation of increasingly larger dynamically correlated clusters, in which the entanglement relaxation times are likely to assume an extremely small or large value. Remarkably, the presence of such heterogeneities spotlights the connection between the local behavior of entanglement in quantum MBL systems and the dynamical properties of classical glasses.

Our analysis applies to the deep, many-body localized phase, where the $l$-bits are close to the physical spins. Entanglement heterogeneities are expected in all systems that present a long, localized transient, before they reach a thermal state. Such systems include MBL systems coupled to a bath [60, 73, 79], MBL systems in $d \geq 2$ [32, 48, 80], and two-level systems in structural glasses [62].
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We verified that, upon changing the definition of $\tau$, e.g. with $\tau := \langle t \rangle_C = \int_0^{\text{fin}} t C_i(t)dt / \int_0^{\text{fin}} C_i(t)dt$ or $\tau := \max \{ t | C_i(t) > 0 \}$, our findings do not qualitatively change.

Notice that the averages have to be taken in the proper order: first $\bullet_{\alpha \in [0, \ldots, \tau]}$, second $\bullet_{1 \sim j \sim r}$, finally $\bullet$.

We can compute the (annealed) density of states of the spin glass, if one substitutes $\sigma_i^z \rightarrow m_i \in [-1, 1]$ and $\sigma_i^x \rightarrow \pm 1$. This type of correlation functions were used in computing the “one-site concurrence” $C_i(t) = \sum_j C_{i,j}(t)$, Eq. (4), to see how it decays in time. The concurrence is a complicated nonlinear function of the two-site reduced density matrix $\rho_{i,j}$, therefore it is really hard to make analytical predictions for it. However, one can hope to get a rough estimate of its behavior by considering instead the correlation function $\langle \sigma_i^z(t) \sigma_j^z(t) \rangle$. This type of correlation functions were already considered in previous works (see e.g. Ref. 37).

Supplementary material

Energy of the initial states

In the $t$-bit model, given a disorder realization $\{J_{ij}\}$, we sample the local magnetization configurations $\{m_i\} = \{\sigma_i^z\}$ with probability $\propto e^{-E/T}$, $T$ being a fictitious temperature to be gradually lowered. Since $m_i \in [-1, 1]$ are continuous variables, the annealing procedure has easy access to states down to the edge of the spectrum. From $\{m_i\}$, we fix the coefficients in Eq. (2) as $A_1 = \sqrt{(1 + m_i)/2}$ and $B_1 = e^{\phi_m} \sqrt{(1 - m_i)/2}$. This choice guarantees that $\langle \psi_0 | H_{t,\text{bit}} | \psi_0 \rangle = E$, i.e., the quantum initial state has an energy expectation value equal to the desired one.

For what concerns our choice of the energy scale, namely

$$\varepsilon := \frac{E}{N \sqrt{h^2/3 + 2 J_0^2/(e^{2/\kappa} - 1)}} \quad \text{(8)}$$

(see also the the main text), the reasoning goes as follows. The Hamiltonian (1) can be interpreted as a classical spin glass, if one substitutes $\sigma_i^z \rightarrow s_i = \pm 1$. Then one can compute the (annealed) density of states of the model, finding that with high probability the ground state is at $E = -N \sqrt{h^2/4 + 4 J_0^2/(e^{2/\kappa} - 1)}$ (see also Ref. 68). Changing the spins to continuous variables $\sigma_i^z \rightarrow m_i \in [-1, 1]$ will just modify the prefactors of $h^2$ and $J_0^2/(e^{2/\kappa} - 1)$, without changing much the scale. For this reason, we have chosen to put in Eq. (8) simply the sum of the variances of $h_i$ and $\sum_{j} J_{i,j}$. The ground state will not be exactly at $\varepsilon = -1$, but close to it.

Analytical estimates of local time scales

Let us consider the $t$-bit model. Throughout this study, we were concerned in computing the “one-site concurrence” $C_i(t) = \sum_j C_{i,j}(t)$, Eq. (4), to see how it decayed in time. The concurrence is a complicated nonlinear function of the two-site reduced density matrix $\rho_{i,j}$, therefore it is really hard to make analytical predictions for it. However, one can hope to get a rough estimate of its behavior by considering instead the correlation function $\langle \sigma_i^z(t) \sigma_j^z(t) \rangle$. This type of correlation functions were already considered in previous works (see e.g. Ref. 37),
and are easy to access. Choosing \( i = 0 \) and \( j = 1 \) without loss of generality, it explicitly reads

\[
\langle \sigma_0^z(t) \sigma_1^z(t) \rangle = \sum_{s_0, s_1 = \pm 1} \langle \rho_{0,0} \rangle_{s_0} \langle \rho_{0,1} \rangle_{s_1, -s_1} e^{-2i\epsilon_1 s_0 t - 2i\epsilon_2 s_1 t + 8iJ_0 s_0 s_1} \\
\times \prod_{j \neq 0, 1} \left[ e^{-4iJ_0 s_0 t - 4iJ_1 s_1 t} \cos^2 \frac{\theta_j}{2} + e^{4iJ_0 s_0 t + 4iJ_1 s_1 t} \sin^2 \frac{\theta_j}{2} \right],
\]

where \( \rho_{0,0} \) and \( \rho_{0,1} \) are the initial density matrices of sites 0 and 1, and \( \theta_j \) is the azimuthal angle on the Bloch sphere for the initial state of site \( j \). We take a further step, and also simplify \( \theta_j = \pi/2 \), i.e. we choose a particular initial condition at infinite temperature. As a result, we find that \( \langle \sigma_j^z(t) \sigma_1^z(t) \rangle \) is an oscillating function, modulated by envelopes of the form

\[
A_j(t) := \prod_{j \neq 0, 1} \left| \cos (4J_0 j t \pm 4J_1 j t) \right|. 
\]

It is clear that, if we want to understand the leading-order behaviour in time, we can reduce to study the simpler function

\[
A(t) := \prod_{j \neq 0} \left| \cos (J_0 j t) \right|, 
\]

where, we recall, \( J_{0j} \) are Gaussian variables of zero average and standard deviation \( w_j := J_0 e^{-|j|^\alpha} \).

We can estimate the typical value of \( A(t) \) by means of \( \text{typ}[A(t)] := \exp \ln A(t) \) (we need to average the logarithm of \( A \) because, with hindsight, there is a power-law tail in the decay-time distribution). Since

\[
\ln A(t) = \sum_j \int dJ_{0j} \frac{e^{-J_{0j}^2/2w_j^2}}{\sqrt{2\pi w_j^2}} \ln \left| \cos (J_0 j t) \right|, 
\]

we just need to compute the integral

\[
\int dJ_{0j} \frac{e^{-J_{0j}^2/2w_j^2}}{\sqrt{2\pi w_j^2}} \ln \left| \cos (J_0 j t) \right| \\
= \int dJ_{0j} \frac{e^{-J_{0j}^2/2w_j^2}}{\sqrt{2\pi w_j^2}} \left[ \ln \left| 1 + e^{2J_0 j t} \right| - \ln 2 \right] \\
= \int dJ_{0j} \frac{e^{-J_{0j}^2/2w_j^2}}{\sqrt{2\pi w_j^2}} \sum_{n \geq 1} \frac{(-1)^{n+1}}{n} e^{2J_0 j t} - \ln 2 \\
= \sum_{n \geq 1} \frac{(-1)^{n+1}}{n} e^{-2n^2 w_j^2 t^2} - \ln 2. 
\]

Let us first focus on the asymptotic value in time of \( \text{typ}[A(t)] \). Substituting Eq. (13) in Eq. (12), for finite system size \( L \), and applying the dominated convergence theorem when performing the limit \( t \to \infty \), one can show that \( \text{typ}[A(\infty)] := \lim_{t \to \infty} \exp \ln A(t) \approx 2^{-L} \). This result is also related to the asymptotic value of the concurrence \( \text{typ}[C_i(\infty)] \), being exponentially small in the system size, as discussed in the main text.

Now, we focus on finite time \( t \), and we consider \( L \gg 1 \) so that \( \text{typ}[A(\infty)] \approx 0 \). We further proceed by approximating

\[
\sum_{n \geq 1} \frac{(-1)^{n+1}}{n} e^{-2n^2 w_j^2 t^2} \approx \begin{cases} 0 & \text{if } w_j^2 t^2 > 1, \\ \ln 2 & \text{if } w_j^2 t^2 \lesssim 1, \end{cases}
\]

which implies

\[
\sum_j \left\{ \sum_{n \geq 1} \frac{(-1)^{n+1}}{n} e^{-2n^2 w_j^2 t^2} - \ln 2 \right\} \approx -N(t) \ln 2 
\]

with \( N(t) \) given by

\[
N(t) = \# \{ j \mid w_j^2 t^2 > 1 \} = \begin{cases} 2 \kappa \ln (J_0 t) & t > 1/J_0, \\ 0 & t < 1/J_0. \end{cases}
\]

Finally, we find

\[
\text{typ}[A(t)] = \begin{cases} (J_0 t)^{-\kappa \ln 4} & t > 1/J_0, \\ 1 & t < 1/J_0. \end{cases}
\]

Substituting this typical value in the definition of \( \tau \), Eq. (5), we get

\[
\text{typ}[\tau] = J_0^{-1} \exp \left( \int_0^\infty dt \frac{\text{typ}[A(t)] \ln (J_0 t)}{\int_0^\infty dt \text{typ}[A(t)]} \right) \\
= (eJ_0)^{-1} \exp \left( \frac{1}{\kappa \ln 4 - 1} \right).
\]

**Finite size and finite disorder effects**

In view of the strong finite-size effects (see Fig. 1a in the main text), let us first consider the probability distribution function (pdf) of \( \tau \). 


In Fig. 4 we show the distribution of the local relaxation times of the concurrence, computed within the XXZ model. The tail of the distribution is cut away according to the following procedure. We observe that in some instances the nearest-neighbor concurrence $C_{i,i+1}(t)$ becomes numerically indistinguishable from 0 at a time $t^*$, and then stays equal to 0 definitively. We perform an evolution lasting only a finite time $t_{\text{fin}}$, so for the finite size we consider there will be many sites and realizations with $t^* > t_{\text{fin}}$. This is the reason why the full distributions in Figs. 4(a) and 4(b) show such a huge peak at large times: it is formed by the contributions of $C_{i,i+1}(t)$ which have not yet vanished on the finite-time window $t_{\text{fin}}$ of our evolution, for the finite size system we consider. In order to get rid of this peak, we choose a certain truncation time $t_{\text{tr}} \leq t_{\text{fin}}$, and select only the sites and the realizations for which $t^* < t_{\text{tr}}$. As we can see in Fig. 4(c) the huge peak disappears and there is a large-time tail which depends on the chosen value of $t_{\text{tr}}$. The small-time structure is, on the opposite, quite independent of the truncation, so we expect that it has a physical meaning. There is a peak around $\log_{10}(t_{\text{tr}}) \simeq -1$, which resembles the one appearing in the l-bit distributions; however, another peak is present around $\log_{10}(t_{\text{fin}}) \simeq 1$. The two-peak structure has no equal in the l-bit model results; we argue that this might be due to the $n$-body interactions with $n \geq 3$ missing in the l-bit model.

Fig. 5a shows the pdf’s of $\log_{10}(\tau_i)$, obtained within the l-bit model, for different values of $L$. We see that, when the system size is too small, the pdf’s present a spurious peak at large values of $\tau_i$. Indeed, entanglement cannot spread over many sites, and the concurrence of some strongly interacting couples remains finite even at infinite times ($C_i(\infty) = O(2^{-L})$). We collected data from 20 initial states for, at least, 300 disorder realizations. (b) Results for $L = 16$, $\varepsilon = 0$, and different values of $\kappa$. Data collected from 21000 disorder realizations. (c) Results for $L = 16$, $\kappa = 1$, $\varepsilon = 0$, and different values of the final time of the time evolution, $t_{\text{fin}}$. Data collected from 21000 disorder realizations.

FIG. 4. Results for the XXZ model. We show the pdf’s of $\log_{10} t_{\text{fin}}$, truncated as described in the text. The simulations were performed with a chain of length $L = 16$, final time $t_{\text{fin}} = 1000$, and disorder strength $W = 9$ (panel (a)), $W = 15$ (panel (b)), $W = 25$ (panel (c)). We compare the truncated distributions with the corresponding full distribution. Data from at least 8000 disorder realizations.

FIG. 5. Pdf’s of $\log_{10}(\tau_i)$, obtained within the l-bit model. (a) Results for $\kappa = 1$, $\varepsilon = 0$, and different values of $L$. We see that, when the system size is too small, the pdf’s present a spurious peak at large values of $\tau_i$. Indeed, entanglement cannot spread over many sites, and the concurrence of some strongly interacting couples remains finite even at infinite times ($C_i(\infty) = O(2^{-L})$). We collected data from 20 initial states for, at least, 300 disorder realizations. (b) Results for $L = 16$, $\varepsilon = 0$, and different values of $\kappa$. Data collected from 21000 disorder realizations. (c) Results for $L = 16$, $\kappa = 1$, $\varepsilon = 0$, and different values of the final time of the time evolution, $t_{\text{fin}}$. Data collected from 21000 disorder realizations.
FIG. 6. $G_\tau(r)$ defined in the main text, Eq. (6). Results for the $l$-bit model (a) for: (a) $\kappa = 1, \varepsilon = 0$, and various system sizes $L$, averaged over $D = 300$ disorder realizations and 20 initial states; (b) $\kappa = 1, \varepsilon = 0$, $L = 140$, averaged over a different number of disorder realizations $D$, and 20 initial states. We see that $G_\tau(r)$ converges quickly to its thermodynamic value (panel (a)), and is almost independent on the number of disorder realizations (panel (b)).

used in the numerical simulations for the time evolution. This is the same effect observed in Fig. 1a. In particular, we see that the shape of the pdf’s at $L = 10, 15, 20$ strongly resembles the behavior observed in the XXZ model, confirming that those results are strongly affected by finite-size effects.

In Fig. 5b–5c we reproduce the pdf’s at $L = 16, \varepsilon = 0$, and different values of $\kappa$ and $t_{\text{fin}}$. In the presence of finite-size effects we do not observe the decay of the pdf that is found in larger system sizes. We observe instead a peak in the distribution at $\tau_i = t_{\text{fin}}$.

As anticipated in the main text, the correlation function $G_\tau(r)$ defined in Eq. (6) is a self-averaging quantity, as depicted in Fig. 6. It is indeed only slightly sensitive to finite-size effects (see Fig. 6a), and to disorder fluctuations (see Fig. 6b).