Random resistor network model of minimal conductivity in graphene

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Transport in undoped graphene is related to percolating current patterns in the networks of N- and P-type regions reflecting the strong bipolar charge density fluctuations. Transmissions of the P-N junctions, though small, are vital in establishing the macroscopic conductivity. We propose a random resistor network model to analyze scaling dependences of the conductance on the doping and disorder, the quantum magnetoresistance and the corresponding dephasing rate.

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Graphene – an atomic monolayer of graphite [1] - is a gapless semiconductor with linear [2] electron spectrum. The carrier density in a graphene-based field effect transistor (GraFET) can be varied continuously, from P-type to N-type. Experimental results [1] are surprising: the resistance per square never exceeds several kΩ’s in contrast to the pinch-off in conventional (gapful) semiconductors. Such a behaviour of GraFET persists over a broad temperature range 10−2K ≲ T ≲ 102K. The conductivity reaches its minima when the gate-controlled carrier density, n_d = 0, – “neutrality point”. Similar observations were reported for the bilayer graphene [3].

Attempts to understand the finite value of the minimal conductivity in graphene have so far addressed the role of “chirality” of the Dirac-type quasiparticles and influence of various short-range defects on the quantum transport [4] in a homogeneous graphene sheet. However, there is an emerging evidence [5, 6, 7] that charge density in a “charge-neutral” graphene sheet, is an emerging evidence [5, 6, 7] that charge density in graphene has so far addressed the role of various short-range defects on the quantum transport [4] in a homogeneous graphene sheet. However, there is an emerging evidence [5, 6, 7] that charge density in a “charge-neutral” graphene sheet, n_d = 0 implies that it can be viewed as a checkerboard of P-type and N-type doped regions separated by weakly conducting P-N junctions. In this Letter we propose a random resistor network model for such a system and use it to describe classical and quantum transport.

Model of the random resistor network (RRN) is formulated on the square lattice with the lattice constant a and sites labelled by integers (i, j). Sites (i, j), (i’, j’) with |i’ - i| > 1, |j’ - j| > 1 are not connected directly. Each pair of sites with |i’ - i| ≤ 1, and |j’ - j| ≤ 1, is connected by a link with the conductance G_i,j = G_{i’,j’}

\[ G_{i,j}^{(i,j)} = G_{i,j}^{(i’,j’)} = g \left[ 1 + (-1)^{i+j}\eta_{i,j} \right]/2; \]

\[ G_{i,j}^{(i+1,j+1)} = g \left[ 1 + (-1)^{i+j+1}\eta_{i,j} \right]/2; \]

\[ G_{i,j}^{(i,j+1)} = g \left[ 1 - (-1)^{i+j}\eta_{i,j} \right]/2; \]

\[ G_{i,j}^{(i+1,j)} = G_{i,j}^{(i,j+1)} = \gamma g, \; \gamma \ll 1. \]

Here \( \eta_{i,j} \) is a random variable,

\[ \eta_{i,j} = \pm 1, \; \langle \eta_{i,j} \rangle = p, \; \langle \eta_{i,j}\eta_{j,k} \rangle = \delta_{ik}\delta_{j,l}, \]

where \( \langle \ldots \rangle \) means the ensemble averaging.

The relation of model [1] to graphene with inhomogeneous charge density, \( n(r) = n_d + \delta n(r) \) is illustrated on Fig. 1. Assume that the density fluctuations \( \delta n(r) \) are characterised by the length scale \( a \) so that \( \langle \delta n(r)\delta n(r + x) \rangle = \delta n^2 f(x/a); \) a may be determined by, e.g., the thickness of the insulating substrate. As long as \( a^2\delta n \gg 1, \) one can view the system as a combination of electron (N) and hole (P) puddles of the size of the order of \( a. \) Each puddle contains a large number of carriers, which are characterised by the Fermi wavevector \( k_F \sim \sqrt{\pi \delta n} \gg a^{-1}. \) If \( i + j \) is even (odd), the site of the RRN corresponds to the N(P)-puddle and is marked by red (blue) colour on Fig. 1. The correspondence between the puddles and the lattice sites encodes the fact that the observable conductivity is determined by the random links between the puddles rather than the local conductivity of a puddle.

Each plaquette of the RRN, see the inset for Fig. 1 has one and only one diagonal connection – either P-P or N-N link. This is described by Eqs. (1a) and (1b). If parameter \( p = 0, \) then P-P and N-N connections appear with equal probability, and \( p > 0 \) (\( p < 0 \)) describe the

FIG. 1: Random resistor network (RRN) representation of a graphene sheet with nominally zero doping, \( n_d = 0. \)
electron (hole) doping. The boundary between the puddles has a finite, though small, transparency \([\varepsilon]\). This transparency is characterised by \(\gamma \ll 1\) in Eq. (1c) and depicted by the square-lattice grid on Fig. 1.

**Scaling analysis** – First, let us consider a RRN with 
\(p = 0, \gamma = 0\) in Eq. (1), when two-colour random networks at a bipartite lattice have peculiar geometrical features illustrated in Fig. 1 by a computer-generated sample. Emerging patterns are typical for the percolation theory \([\varepsilon]\). The RRN is critical, i.e. the geometry of the RRN is self-similar on all length scales: \(L \times L\) network contains, typically, a larger cluster of one polarity (such as the red cluster in the middle part of Fig. 1) which separates a pair of smaller blue clusters. Each of those clusters is in turn a shell for several smaller red clusters, etc. Such alternating cluster-embedding represents a scale-invariant property of an infinite network. As a result, larger and larger parts of the network become excluded from the mono-colour percolation upon the increase of the size \(L\). Therefore, the observable conductance \(G(L)\) decreases with the increase of \(L\). The corresponding critical behaviour of its mean value is [3]

\[
\langle G(L) \rangle \sim (a/L)^x g; \quad x \approx 0.97. \tag{2}
\]

This means that the conductivity is not defined.

Finite \(p\) and \(\gamma\) are relevant perturbations for the percolation leading to a finite correlation length \(\xi(p, \gamma)\). At \(L \gg \xi\), the RRN is not critical and consists of independent patches of size \(\xi\). Thus, the conductivity is finite:

\[
\sigma(\gamma, p) \equiv \langle G(L \to \infty) \rangle \sim [a/\xi(p, \gamma)]^x g. \tag{3}
\]

Because of the scale invariance, \(\xi\) depends on \(p, \gamma\) as

\[
\xi(p, \gamma \to 0) \sim a|p|^{-\nu}; \quad \xi(p \to 0, \gamma) \sim a\gamma^{-\mu}. \tag{4a}
\]

Its \(p\)-dependence is the dependence of the correlation length on deviations from the percolation threshold [3]:

\[
\nu = 4/3. \tag{4b}
\]

To the best of our knowledge, the scaling of \(\xi\) with \(\gamma\) was not considered in literature, yet. We found

\[
\mu = 1/(h + x) \approx 0.37, \tag{4c}
\]

where \(x\) is the conductance exponent, Eq. (2), and \(h = 7/4\) (see Ref. [4]) is the exponent of the cluster outer perimeter \(P(L) \simeq a(L/a)^h\) for the cluster of the size \(L\).

To derive Eq. (1c), consider the red cluster of the size \(L\) embedded into the conducting blue cluster. Small but finite \(\gamma\) does not affect the criticality if the leak through the perimeter of this cluster \(P\) is much smaller than the conductance of the blue cluster, the latter is also of the order of \(G(L)\). One can use Eqs. (2) and (1c) to write this condition as \((a/L)^x \gtrsim (L/a)^h \gamma\). This yields \(L \lesssim \xi = a/\gamma^\mu\), with \(\mu\) given by Eq. (4c). For \(L > \xi\), the leakage through the cluster boundary is efficient enough for the clusters to become distinguishable, i.e. RRN is uniform.

The dependence of the conductivity on \(p, \gamma \neq 0\), is

\[
\xi(p, \gamma) \sim a^\gamma min / f(p/p^*) , \quad p^* = \gamma^{\mu/\nu}, \tag{5}
\]

where \(f\) is a universal scaling function. The form of Eq. (5) is protected by the scale invariance. The numerics described below are well fit by the interpolation formula

\[
f(z) = (1 + z^2)^{\nu/2}, \tag{6}
\]

reproducing both \(z \ll 1\) and \(z \gg 1\) asymptotic behaviour. Substituting Eqs. (5), (6) into Eq. (3) we find

\[
\sigma(\gamma, p) = u a^\gamma \xi f(p/p_\ast), \quad \alpha = \mu \approx 0.36, \tag{7}
\]

where \(u \approx 1.3\) is a coefficient found from the fit of the numerical data. Equation (7) completely describes non-analytic dependence of the conductivity on both the leakage parameter \(\gamma \ll 1\) [11] and adjustable doping \(p\). Recently, an attempt was made in Ref. [11] to explain the graphene minimal conductivity using a mean field theory. We believe that Ref. [11] correctly describes the limit of high carrier density, where the fluctuations of local conductivity are small, but fails near the neutrality point where the percolation physics start to dominate.

**Numerical simulations** of model (1) were performed on square lattices with \(L \leq 500a\). Numerical evaluation of the network conductance has been done using the bus-bar boundary conditions at the two opposite edges along the direction of the current flow and the periodic boundary conditions in the orthogonal direction. Realizations of the RRN were generated in families spanning the interval \(p \in [-1; 1]\) of the network parameter, in steps of \(\Delta p = 0.0125\). Each family was obtained from the regular network at \(p = 1\) (containing only blue links) by sequentially replacing \(L^2/\Delta p\) blue links by red ones in randomly chosen plaquettes. Averaging of the data has been performed over 300 families. To minimise the finite-size effects we (i) identified \(1/L\) corrections by finite size scaling from \(L = 50a\) to \(L = 500a\); and (ii) subtracted those corrections by means of numerical extrapolation. Numerical results summarised on Fig. 2 are in excellent agreement with the scaling formulas (6), (7).

**Sample-to-sample fluctuations** – Geometry of the percolation cluster is non-trivial. In particular, its linear size depends on the particular realization and reconnecting only few bottleneck links may change it substantially. Therefore, the conductance \(G(L)\) is a random quantity, which is characterised by its distribution function \(P(G)\) rather than only by its average. Scaling invariance of the critical cluster at \(L \ll \xi(p, \gamma)\) constrains the functional form of this distribution to \(\langle P(G) = (G)^{-1} P(G/L)\rangle, \int P(x)dx = 1\), where \(P(x)\) is a universal function. It implies the variance

\[
\langle \delta G^2(L \ll \xi) \rangle = u_2 \langle G(L \ll \xi) \rangle^2 , \tag{8}
\]
where \( u_2 \simeq 1 \) is a universal numerical coefficient. For \( L \gg \xi \), the central limit theorem is restored, and we find

\[
\langle \delta G^2 (L \gg \xi) \rangle = \langle \delta G^2 (\xi) \rangle (\xi/L)^2 = u_2 \sigma^2 * (\xi/L)^2. \tag{9}
\]

Figure 3 (a) illustrates the fluctuations of RRN conductance. One can see that not only value of the minimal conductance but also its position fluctuates from sample to sample. Figure 3 (b) shows an excellent agreement with the scaling form (9) for \( p = 0 \).

The fluctuations (9) are self-averaging. Notice, however, that because \( \xi \) is large, those fluctuations can exceed the universal conductance fluctuations \( \simeq e^2/h \) \[12\] which are not self-averaging as long as the phase coherence is preserved. The scale for the energy dependence of fluctuations (9) is determined by the properties of the individual bottle-neck links in the cluster and significantly exceeds the Thouless energy for the UCF. This observation is consistent with numerical findings of Ref. \[13\].

**Quantum magnetoresistance** – Up to now we ignored the quantum interference and interaction effects. Some, and most importantly those that give rise to the quantum magnetoresistance \[14\], can be analysed within the percolation cluster framework \[15\]. The role of the weak localisation (WL) is to renormalize the conductance of each link in Eq. (1) due to the interference of the random walk paths that go through the same link twice.

We can write the conductance fluctuations as

\[
\delta \sigma = \xi / \min (L, \xi)^2 D_\xi. \tag{10}
\]

Our treatment of the WL differs from that of Ref. \[16\], only by the scale dependence of the diffusion constant. The magnetoconductance \( \delta \sigma (B) = \sigma (B) - \sigma (0) \) in not-so-low magnetic field \( B \) can be estimated as

\[
\delta \sigma \simeq \frac{e^2}{\pi^2 \hbar} \sum_{J=0}^{1} (-1)^J \sum_{M=-J}^{J} \sqrt{eB/\hbar c} \frac{qdq D_\xi}{D(1/4)q^2 + \Gamma M^2}, \tag{11}
\]

where \( \Gamma_0 = \frac{1}{\tau_0} \) and \( \Gamma_1 = \frac{1}{\tau_0} + \frac{1}{1+|M|} \left( \frac{1}{\tau_\perp} + \frac{|M|}{\tau_\parallel} \right) \).
Here $\tau_z$ is the inter-valley scattering time and $1/\tau_\parallel$ is the rate for intra-valley scatterings breaking certain symmetries of the system, see Ref. [10] for more details. The phase relaxation time $\tau_\phi$ is estimated below.

If $\tau_\phi \gg \xi^2/D_z$, Eq. (11) reduces to the results of Ref. [10]. The specifics of the scale dependent diffusion [10] are revealed in the opposite limit $\tau_\phi \ll \xi^2/D_z$:

$$\delta \sigma \approx \frac{e^2}{2\pi \hbar} \sum_{J=0}^{1} (-1)^J \sum_{M=-J}^{J} \left( \mathcal{L}_{J,M} / \xi \right)^x \tilde{Y} \left( \frac{eB\mathcal{L}_{J,M}^2}{\mathcal{e}h} \right);$$

$$\mathcal{L}_{J,M} = \xi \left[ D_z / (\xi^2 \mathcal{G}^{(1)}) \right]^{1/2},$$

where $\tilde{Y}(z)$ is a universal function [17] with the asymptotic behaviour $\tilde{Y}(z \ll 1) \approx z^2$, and $\tilde{Y}(z \gg 1) \approx 1 - z^{-1/2}$. The overall magnetoresistance [12] is strongly suppressed in comparison with that for the homogeneous graphene, $|\delta \sigma| \ll e^2/\pi^2 \hbar$.

The phase relaxation in ordinary disordered conductors is dominated by the electron-electron interaction [18]. The time defining the magnetoresistance curvature $\partial^2 \sigma / \partial B^2|_{B=0}$ is controlled by the dimensionless conductance $G(L) = 2\pi \hbar G(L)/e^2$:

$$\hbar/\tau_\phi \approx T/G(\phi); \quad L_\phi = (D\tau_\phi)^{1/2},$$

where $L_\phi \equiv L_{0,0}$ is the dephasing length. As the all the geometric properties of the system in Eq. (13) are encoded into the scale dependences of $G(L)$ and $D(L)$, Eq. (13) is valid even for diffusion along the critical cluster. We obtain from Eqs. (12), (13), (14), and (10)

$$L_\phi = \xi \Phi (T_\xi / T); \quad T_\xi = h\mathcal{G}(\xi)D_\xi / \xi^2 \propto \xi^{2(1+\gamma)},$$

where $\Phi(x)$ is a scaling function with the asymptotic behaviour $\Phi(z \gg 1) \approx \sqrt{z}$, $\Phi(z \ll 1) \approx z^{1/2(2+2\gamma)}$. Condition $T > T_\xi$ determines the crossover from the critical to the normal diffusion: $L_\phi (T > T_\xi) > \xi$, $L_\phi (T < T_\xi) < \xi$.

Equations (11) and (12) predict interesting doping behaviour of the magnetoresistance at fixed temperature. At $p \ll 1$, $\xi(p)$ is small, $T_\xi > T$ and the usual logarithmic magnetoresistance occurs. For $p \rightarrow 0$, $T_\xi < T$, and the WL becomes suppressed as $1/\xi^2$. Such prediction seems to be consistent with the experiment of Ref. [19].

Relation to the experimental parameters − To relate Eq. (7) to the properties of graphene, we have to connect the parameters $\gamma$ and $g$ with the physical parameters of the sample. If the charge inhomogeneity is the dominant disorder in the graphene monolayer, we can estimate the conductance of $N$-$N$ ($P$-$P$) connections between puddles as $g \sim \frac{e^2}{\pi} ak_F$ and conductance of the $P$-$N$ junction separating puddles of the opposite polarity as $\gamma g \sim \frac{e^2}{\pi} (ak_F)^{1/2}$, see Eq. (2) of Ref. [18]. $\gamma \sim (ak_F)^{-1/2} \sim (a^2 \delta n)^{-1/4}$. According to Eq. (17) we then estimate $\sigma_{\min} = \sigma(\gamma, p = 0) \sim \frac{e^2}{\pi} (a^2 \delta n)^{1/2} \sim \frac{e^2}{\pi} (a^2 \delta n)^{0.41}$.

In conclusion, we constructed a random resistor network model, which adequately takes into account strong fluctuations of the local charge density and, thus, of the local conductivity of the mono- and bilayer graphene near neutrality point. This model describes inhomogeneous current percolating through the system, giving rise to the scaling dependencies of the observable conductivity on the doping and disorder. Quantum magnetoresistance and the sample-to-sample fluctuations are analysed within the model.

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