FIXED POINT THEOREMS OF VARIOUS NONEXPANSIVE ACTIONS OF SEMITOPOLOGICAL SEMIGROUPS ON WEAKLY/WEAK* COMPACT CONVEX SETS

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Abstract. Let $S$ be a right reversible semitopological semigroup, and let $\text{LUC}(S)$ be the space of left uniformly continuous functions on $S$. Suppose that $\text{LUC}(S)$ has a left invariant mean. Let $K$ be a weakly compact convex subset of a Banach space not necessarily with normal structure. We show that there always exists a common fixed point for any jointly weakly continuous and super asymptotically nonexpansive action of $S$ on $K$. Several variances involving the weak* compactness, the RNP, the distality of $K$ and/or the left reversibility of $S$ are also provided.

1. Introduction

Let $K$ be a non-empty convex subset of a Banach space $E$. Let $T: K \to K$ be a nonexpansive map, namely $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y$ in $K$. Schauder [29] shows that $T$ has a fixed point if $K$ is norm compact. Kirk [14] shows that $T$ has a fixed point if $E$ is reflexive and $K$ is weakly compact with normal structure. Nevertheless, Alspach [1] gives an example in which $K$ is weakly compact without normal structure, and $T$ has no fixed point.

The fixed point theorems for actions of a left amenable semigroup were first investigated by Day [9] as an extension of the results for commutative family of maps in [8, 13]. In [17, Problem 4] Lau raised a question about whether the left amenability of a semitopological semigroup $S$ implies the following fixed point property.

($F_{w^*}$) Every jointly weak* continuous nonexpansive action of $S$ on a weak* compact convex subset $K$ of a dual Banach space has a common fixed point.

By embedding a Banach space into its double dual space, the fixed point property ($F_{w^*}$) implies that

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(F_w) every jointly weakly continuous nonexpansive action of S on a weakly compact convex subset K of a Banach space has a common fixed point.

Some partial answers for this question can be found in [19, 20, 22, 28, 30]. In particular, it is stated in [27] that if S is σ–ELA, that is, S is n-extremely left amenable for some positive integer n, then (F_{w*}) and (F_w) hold without any nonexpansiveness assumption.

*Question 1.1.* What can we say if the action of a left amenable semitopological semigroup is not nonexpansive?

We can construct an action of a commutative (and thus amenable) discrete semigroup on a compact convex set without common fixed point. Indeed, Boyce [7] showed that there are two commutative continuous functions f, g : [0, 1] → [0, 1], i.e., f(g(x)) = g(f(x)) for all x ∈ [0, 1], with no common fixed point. Therefore, in general, the assumption of nonexpansiveness can not be totally dropped. For a resolve, Holmes and Lau [10] considered asymptotically nonexpansive actions (see section 2 for definitions) on norm-compact convex sets, while Amini, Medghalchi and Naderi [3] considered pointwise eventually nonexpansive actions on weakly compact convex sets with normal structure. Under some conditions, these actions have common fixed points.

*Question 1.2.* Without normal structure, what can we say for asymptotically nonexpansive like actions on weakly/weak* compact convex sets?

In this paper, we provide a positive answer for the super asymptotically nonexpansive actions (see Section 2 for definitions). We show in Theorem 3.1 that if S is right reversible and left amenable then it holds a fixed point property similar to (F_w) for super asymptotically nonexpansive actions. When we assume further that the domain K of the action is norm-separable, we show in Theorem 3.6 that it holds a fixed point property similar to (F_{w*}). We also provide some results about the existence of a common fixed point for an action of any reversible semigroup. In Section 4, motivated by recent results of Wiśnicki [32], we establish fixed point theorems involving the Radon-Nikodým property or the distality. In Section 5, applying the results in previous sections, we establish fixed point theorems for commutative pointwise eventually nonexpansive mappings on weakly/weak* compact convex sets.

2. Preliminaries

A semitopological semigroup S is a semigroup with a Hausdorff topology such that the product is separately continuous, i.e., for each fixed t ∈ S, both the maps s → ts and s → st from S into S are continuous. Let CB(S) be the Banach space of bounded and continuous real-valued functions on S equipped with the supremum norm.
For each \( s \in S \) and \( f \in \text{CB}(S) \), we denote by \( l_s f \) the left translation of \( f \) by \( s \), where 
\[
l_s f(t) = f(st) \quad \text{for all} \quad t \in S.
\]
Let \( \text{LUC}(S) \) be the space of left uniformly continuous functions on \( S \), namely those \( f \in \text{CB}(S) \) for which the map \( s \mapsto l_s f \) from \( S \) into \( \text{CB}(S) \) is norm continuous. A bounded linear functional \( m \) on \( \text{LUC}(S) \) is called a mean if \( \|m\| = m(1) = 1 \). A mean \( m \) is called a left invariant mean, or LIM in short, if \( m(l_s f) = m(f) \) for all \( s \in S \) and all \( f \in \text{LUC}(S) \). We call \( S \) left amenable if \( \text{LUC}(S) \) has a LIM.

An action of a semitopological semigroup \( S \) on a Hausdorff topological space \( K \) is a mapping of \( S \times K \) into \( K \), denoted by \((s,x) \mapsto s.x \) (or simply \( s \cdot x \)), such that \((st)x = s(tx)\) for all \( s,t \in S \) and \( x \in K \). We call the action separately (resp. jointly) continuous if the mapping \((s,x) \mapsto s.x \) is separately (resp. jointly) continuous. A point \( x_0 \in K \) called a common fixed point for \( S \) if \( s.x_0 = x_0 \) for all \( s \in S \).

Recall that a left (resp. right) ideal \( I \) of a semigroup \( S \) is a nonempty subset of \( S \) such that \( SI \subseteq I \) (resp. \( IS \subseteq I \)). We say that a left ideal \( I \) is supported by an element \( t \) in \( S \) if \( I \subseteq St \). In this case, we can set the left ideal \( I^t = \{ s \in S : st \in I \} \) and write \( I = I^t t \).

**Definition 2.1.** An action \( S \times K \to K \) of a semitopological semigroup \( S \) on a subset \( K \) of a Banach space is called

1. **nonexpansive** if \( \|s.x - s.y\| \leq \|x - y\| \) for all \( s \in S \) and \( x, y \in K \).
2. **asymptotically nonexpansive** (see [10]) if for each given \( x, y \in K \), there exists a left ideal \( I_{xy} \) of \( S \) such that \( \|s.x - s.y\| \leq \|x - y\| \) for all \( s \in I_{xy} \);
3. **pointwise eventually nonexpansive** (see [3]) if for each given \( x \in K \), there exists a left ideal \( I_x \) of \( S \) such that \( \|s.x - s.y\| \leq \|x - y\| \) for all \( s \in I_x \) and all \( y \in K \);
4. **super asymptotically nonexpansive** if for each given \( x \in K \) and \( t \in S \), there exists a left ideal \( I^t_x \) of \( S \) supported by \( t \) such that \( \|s.x - s.y\| \leq \|x - y\| \) for all \( s \in I^t_x \) and all \( y \in K \).

By taking \( I^t_x = \{ s \in S : st \in I_x \} \), the definition of a super asymptotically nonexpansive action of \( S \) on \( K \) can be restated that for each given \( x \in K \) and \( t \in S \), there exists a left ideal \( I^t_x \) of \( S \) such that
\[
\|st.x - st.y\| \leq \|x - y\| \quad \text{for all} \quad s \in I^t_x \text{ and all} \quad y \in K.
\]

**Remark 2.2.** (a) Pointwise eventually nonexpansive actions are also called semi-asymptotically nonexpansive actions in [2], and strongly asymptotically nonexpansive actions in [4].

(b) The notion of pointwise eventually nonexpansive action extends the one introduced by Kirk and Xu in [15], in which a map \( T : K \to K \) is called pointwise eventually nonexpansive if for each \( x \in K \) there exists \( N(x) \in \mathbb{N} \) such that
\[
\|T^n x - T^n y\| \leq \|x - y\|, \quad \forall y \in K \text{ and} \quad n \geq N(x).
\] (2.1)
(c) Let \( \{T_1, \ldots, T_k\} \) be a commutative family of pointwise eventually nonexpansive maps on \( K \). Let \( S \) be the discrete semigroup generated by this family. Then the action of \( S \) on \( K \) is super asymptotically nonexpansive. Indeed, for each \( x \in K \), there exist \( N_1(x), \ldots, N_k(x) \in \mathbb{N} \) satisfying (2.1) for \( T_1, \ldots, T_k \), respectively. For any \( T_0 = T_1^{n_1} \cdots T_k^{n_k} \in S \), set \( q_i^{T_0}(x) = \max\{p_i, N_i(x)\} \) for \( i = 1, \ldots, k \). Consider the left ideal \( I_{x,T_0} = \{T_1^{n_1} \cdots T_k^{n_k} : n_i \geq q_i^{T_0}(x), i = 1, \ldots, k\} \) of \( S \) supported by \( T_0 \). Then for each \( T \in I_{x,T_0} \) and \( y \in K \), we have \( \|Tx - Ty\| \leq \|x - y\| \).

We have the following implications for the above mentioned actions of semitopological semigroups:

\[
\text{nonexpansiveness} \implies \text{super asymptotic nonexpansiveness} \implies \text{point-wise eventual nonexpansiveness} \implies \text{asymptotic nonexpansiveness}.
\]

As seen in Examples 2.3 and 2.4 below, these implications can be strict. However, as seen in Proposition 2.7, when the semigroup \( S \) is compact and right reversible, all three asymptotic nonexpansiveness coincide.

**Example 2.3** (based on [2, Example 3.3(ii)]). Let \( K \) be the closed unit ball in \( \mathbb{R}^2 \) and \( f : [-1, 1] \to [-1, 1] \) be given by \( f(x) = x^3 \). Consider two maps \( T_1 \) and \( T_2 \) from \( K \) into \( K \) defined by

\[
T_1(x_1, x_2) = (f(x_2), 0) \quad \text{and} \quad T_2(x_1, x_2) = (0, f(x_1)).
\]

Clearly, \( T_1^n = 0 \) and \( T_2^n = 0 \) for all \( n \geq 2 \). Consider the non-commutative discrete semigroup \( S \) generated by \( T_1 \) and \( T_2 \), that is

\[
S = \{0, T_1, T_2, (T_1T_2)^n, (T_2T_1)^n, T_2(T_1T_2)^n, T_1(T_2T_1)^n : n \in \mathbb{N}\}.
\]

Since \( f \) is not nonexpansive, so are \( T_1 \) and \( T_2 \). Hence the action of \( S \) on \( K \), defined by \( (T, (x_1, x_2)) \mapsto T(x_1, x_2) \), is not nonexpansive.

However, for each \( T \in S \) we can find an element \( T' \in S \) such that \( T'T = 0 \). Thus the left ideal \( \{0\} = (ST')T \) is supported by any \( T \) in \( S \). Therefore, the action of \( S \) on \( K \) is super asymptotically nonexpansive. We also see that \( (0, 0) \) is a common fixed point of the action.

**Example 2.4** (based on [10, Example]). Let \( K = \{(r, \theta) : 0 \leq r \leq 1, 0 \leq \theta < 2\pi\} \) be the closed unit disk in \( \mathbb{R}^2 \) in polar coordinates and the usual Euclidean norm. Define two continuous mappings \( f, g \) from \( K \) into \( K \) by

\[
f(r, \theta) = (r/2, \theta) \quad \text{and} \quad g(r, \theta) = (r, 2\theta \mod 2\pi).\]

Let \( S \) be the discrete semigroup generated by \( f \) and \( g \) under composition. Any left ideal \( I \) of \( S \) must have the form \( I = \{f^ng^m : n \geq n_0, m \geq m_0\} \), for some \( n_0, m_0 \in \mathbb{N}_0 \), where \( \mathbb{N}_0 = \{0, 1, 2, \ldots\} \).
An action of $S$ on $K$ is given by
\[ f^n g^m(r, \theta) = \left( \frac{r}{2^n}, 2^m \theta \mod 2\pi \right). \]
As seen in [10], the action is asymptotically nonexpansive. However, it is not super asymptotically nonexpansive. To see this, choose $x = (1, 0) \in K$, $t = f \in S$, and any left ideal $I_0 = \{ f^n g^m : n \geq n_0, m \geq m_0 \}$, together with the left ideal $I = I_0t$ supported by $t$. For each $n \geq n_0, m \geq m_0$, consider $s = f^n g^m \in I$ and $y_m = (1, \frac{\pi}{2^m}) \in K$. We have
\[ \| f^{n+1} g^m(x) - f^{n+1} g^m(y_m) \| = \| \left( \frac{1}{2^{n+1}}, 0 \right) - \left( \frac{1}{2^{n+1}}, \pi \right) \| = \frac{1}{2^n}, \]
and $\| x - y_m \| = \left[ 2(1 - \cos \frac{\pi}{2^m}) \right]^{1/2} \to 0$ as $m \to \infty$. Thus, the inequality
\[ \| s \cdot t_0 x - s \cdot t_0 y_m \| \leq \| x - y_m \| \]
fails to hold for all $m \geq m_0$ when $n$ is fixed.

Remark 2.5. (a) The semigroup $S$ in Example 2.3 is left amenable because the point evaluation at 0, defined by $\delta_0(f) = f(0)$, is a LIM on $\text{CB}(S)$.

(b) Theorem 3.1 in [10] does not apply to Example 2.3, since the action in Example 2.3 does not satisfy the property (B) in its assumptions. The property (B) says that for each $x \in K$ whenever a net $\{ s_\alpha x \}$ converges to $x$, the net $\{ s_\alpha sx \}$ converges to $sx$ for any $s \in S$. However, consider $x = (1, 0) \in K$, $s = T_2$ and the sequence $s_n = (T_1 T_2)^n$ in $S$. We see that $(T_1 T_2)^n(1, 0) = (1, 0)$ and $T_2(1, 0) = (0, 1)$ while $(T_1 T_2)^n T_2(1, 0) = (0, 0)$ for all $n \in \mathbb{N}$.

A semitopological semigroup $S$ is called right (resp. left) reversible if any two closed left (resp. right) ideals of $S$ always intersect. We call $S$ reversible if it is both left and right reversible. For example, the semigroup $S$ in Example 2.3 is reversible.

Lemma 2.6 (see [20, Lemma 3.4]). For every separately continuous action of a left reversible semitopological semigroup $S$ on a compact Hausdorff space $K$, there is a nonempty closed subset $A$ of $K$ such that $A \subset sA$ for all $s \in S$.

Recall that a Hausdorff topological space $X$ is countably compact if every countable open covering of $X$ has a finite subcovering; in other words, every countable collection of closed sets in $X$ with the finite intersection property has a nonempty intersection. A topological space $Y$ is called $C$-closed if every countably compact set in $Y$ is closed (see [11]). For example, first countable and countably compact spaces are regular, and thus they are $C$-closed spaces by [11, Proposition 1.4].

Proposition 2.7. Let $S$ be a countably compact right reversible semitopological semigroup. Consider a separately weakly continuous and asymptotically nonexpansive action of $S$ on a set $K$ in a Banach space.
(a) If $K$ is norm separable, then the action is pointwise eventually nonexpansive.

(b) If $S$ is $C$-closed and $K$ is norm separable, then the action is super asymptotically nonexpansive.

(c) If $S$ is compact, then the action is super asymptotically nonexpansive.

Proof. (a) Let $\{y_n\}_{n=1}^\infty$ be a norm dense subset of $K$. Suppose an action of $S$ on $K$ is asymptotically nonexpansive. For each $x \in K$ and $n \in \mathbb{N}$, there is a left ideal $I_n$ such that
\[ \|a.x - a.y_n\| \leq \|x - y_n\|, \quad \forall a \in I_n. \]
By the separately weak continuity of the action, we can assume that $I_n$ is closed.

By the right reversibility and the countable compactness of $S$, there is $r \in \bigcap_n I_n$. Consider the left ideal $I_x := S r \subseteq \bigcap_n I_n$. It follows
\[ \|s.x - s.y_n\| \leq \|x - y_n\|, \quad \forall s \in I_x, \forall n = 1, 2, \ldots. \]
By the norm denseness of $\{y_n\}_{n=1}^\infty$ in $K$ and the separately weak continuity of the action, we see that
\[ \|s.x - s.y\| \leq \|x - y\|, \quad \forall s \in I_x, \forall y \in K. \]
In other words, the action is pointwise eventually nonexpansive.

(b) Continuing with the arguments in (a), we see that the action is already pointwise eventually nonexpansive. Thus, for any $n = 1, 2, \ldots$, there is a left ideal $J_n$ of $S$ such that
\[ \|s.y_n - s.y\| \leq \|x - y\|, \quad \forall s \in J_n, \forall y \in K. \]
By the separate weakly continuity of the action, we can assume that $J_n$ is closed. By the right reversibility and the countable compactness of $S$, we have a (nonempty) closed left ideal $J := \bigcap_n J_n$. For any $t \in S$, the left ideal $St$ is countably compact, and thus closed since $S$ is $C$-closed. By the right reversibility of $S$, we have $(St) \cap J$ is nonempty. Consequently, $J^t = \{s \in S : st \in J\}$ is a (nonempty) left ideal of $S$. We have
\[ \|s.(t.y_n) - s.(t.y)\| = \|(st).y_n - (st).y\| \leq \|y_n - y\|, \quad \forall s \in J^t, \forall y \in K. \]
Since $\{y_n\}_{n=1}^\infty$ is norm dense in $K$ and the action is separately weakly continuous,
\[ \|s.(t.x) - s.(t.y)\| \leq \|x - y\|, \quad \forall s \in J^t, \forall y \in K. \]
Therefore, the action is super asymptotically nonexpansive.

(c) Suppose that $S$ is compact but $K$ is not necessarily norm separable. Note that the left ideal $St$ is compact and thus closed for every $t$ in $S$. We can go through the same arguments as in (a) and (b), but by considering the whole set $K$ rather than the sequence $\{y_n\}$, to get the desired conclusion. \qed
Let $A$ be a bounded subset of a Banach space $E$. A point $x \in A$ is called **diametral** if $\sup_{y \in A} \|x - y\| = \text{diam } A$, where $\text{diam } A = \sup_{a,b \in A} \|a - b\|$. A convex set $K$ of $E$ is said to have **normal structure** if each bounded, convex subset $A$ of $K$ with $\text{diam } A > 0$ contains a non-diametral point. Any norm compact convex set has normal structure, see [8, Lemma 1], while a weakly (resp. weak*) compact convex set might not be so, see [1, 21].

A closed convex subset $K$ of a Banach space $E$ is said to have the **Radon-Nikodým property** (RNP for short) if the following condition is satisfied: for any probability space $(\Omega, \mathcal{F}, \mu)$ and any $E$-valued measure $F : \mathcal{F} \to E$ which is absolutely continuous with respect to $\mu$ such that

$$\{F(B) / \mu(B) : B \in \mathcal{F}, \mu(B) > 0\} \subset K,$$

there is a Bochner integral function $f \in L^1_B(\Omega, \mathcal{F}, \mu)$ such that

$$F(B) = \int_B f \, d\mu \quad \text{for every } B \in \mathcal{F}.$$

See, e.g., [6, Definition 2.1.1].

**Lemma 2.8** (see [6, Theorem 4.2.13]). Let $K$ be a weak* compact convex subset of a dual Banach space $E^*$. Then $K$ has the RNP if and only if for any weak* compact subset $Y$ of $K$, the identity map $\text{id} : (Y, \text{wk}^*) \to (Y, \| \cdot \|)$ has a point of continuity.

**Lemma 2.9** (based on [24, Proposition 3.5]). Let $K$ be a weakly compact subset of a locally convex space $(E, Q)$, where $Q$ is a family of seminorms of $E$ determining the topology. Let $q \in Q$ and $\varepsilon > 0$. Then there is a weakly open subset $U$ of $E$, and a point $x \in K \cap U$ such that $q(x - y) < \varepsilon$ for every $y \in K \cap U$.

An action $S \times K \to K$ of a semigroup $S$ on a subset $K$ of the locally convex space $(E, \tau)$ is called **$\tau$-distal** if $0 \notin \{s.x - s.y : s \in S\}^\tau$ for every pair of distinct points $x, y \in K$. We call the action **affine** if $s.(\lambda x + (1 - \lambda)y) = \lambda s.x + (1 - \lambda)s.y$ for all $s \in S$, all $x, y \in K$ and all $\lambda \in (0, 1)$.

**Theorem 2.10** (see [25, Theorem 4.1]). Let $S \times K \to K$ be a continuous affine action of a semigroup $S$ on a compact convex subset $K$ of a Hausdorff locally convex space $(E, \tau)$. Suppose there exists a nonempty compact subset $A$ of $K$ such that $s.A \subset A$ for all $s \in S$ and the action of $S$ on $A$ is $\tau$-distal. Then there is a common fixed point of $S$ in the convex hull of $A$.

We also consider other classical function spaces on $S$ instead of $\text{LUC}(S)$. Let $\text{AP}(S)$ (resp. $\text{WAP}(S)$) be the subspace of $\text{CB}(S)$ consisting of **almost periodic** (resp. **weakly periodic**...
almost periodic) functions; namely those \( f \) for which the set \( \{l_s f : s \in S\} \) is relatively compact in the norm (resp. weak) topology of \( \text{CB}(S) \). In general, we have
\[
\text{AP}(S) \subseteq \text{LUC}(S) \subseteq \text{CB}(S) \quad \text{and} \quad \text{AP}(S) \subseteq \text{WAP}(S) \subseteq \text{CB}(S).
\]

For the existence of \( \text{LIM} \) on these spaces and the associated fixed point properties, the reader can see [16,19,31].

3. Fixed point theorems assured by the amenability and the reversibility

**Theorem 3.1.** Let \( S \) be a right reversible and left amenable semitopological semigroup. Let \( K \) be a weakly compact convex subset of a Banach space. Then every jointly weakly continuous and super asymptotically nonexpansive action of \( S \) on \( K \) has a common fixed point.

The proof of Theorem 3.1 needs several lemmas. The first one arises from the proof of [10, Theorem 3.1].

**Lemma 3.2.** Let \( S \) be a right reversible semitopological semigroup. Assume \( S \times K \to K \) is a separately continuous action of \( S \) on a compact convex subset \( K \) of a locally convex space. Then there exists a subset \( L_0 \) of \( K \) which is minimal with respect to being nonempty, compact, convex and satisfying the following conditions (\( \ast 1 \)) and (\( \ast 2 \)).

(\( \ast 1 \)) there exists a collection \( \Lambda = \{\Lambda_i : i \in I\} \) of closed subsets of \( K \) such that \( L_0 = \bigcap \Lambda \), and

(\( \ast 2 \)) for each \( x \in L_0 \) there is a left ideal \( J_i \subset S \) such that \( J_i . x \subset \Lambda_i \) for each \( i \in I \).

Furthermore, \( L_0 \) contains a subset \( Y \) that is minimal with respect to being nonempty, compact, and \( S \)-invariant, i.e., \( s . Y \subset Y \) for all \( s \in S \).

The following is a variant of [18, Lemma 5.1], we sketch a proof here since we need to consult its argument later.

**Lemma 3.3.** Let \( S \times Y \to Y \) be a jointly weakly continuous action of a semitopological semigroup \( S \) on a weakly compact subset \( Y \) of a normed space. Assume that \( S \) is left amenable. Assume further that \( Y \) is minimal with respect to being an \( S \)-invariant, nonempty and weakly compact subset of \( Y \). Then \( Y \) is norm separable, and \( S \)-preserving, i.e., \( s . Y = Y \) for all \( s \in S \).

**Proof.** For each pair of \( y \in Y \) and \( f \in C(Y) (= \text{CB}(Y)) \) while \( Y \) is the compact space equipped with the weak topology, define a function \( R_y f \) by \( R_y f(s) = f(s,y) \). It can be shown that \( R_y f \in \text{LUC}(S) \).
Let $m$ be a LIM on $\text{LUC}(S)$. Define a left invariant linear functional $\psi$ on $C(Y)$ by $\psi(f) = m(R_y f)$. Let $\mu$ be the Radon probability measure on $Y$ defining $\psi$, and let $Y_0$ be the support of $\mu$, i.e.,

$$Y_0 = \text{supp}(\mu) = \bigcap \{F \subseteq Y : F \text{ is weakly closed and } \mu(F) = 1\}.$$

It can be shown that $Y_0$ is $S$-preserving, hence $Y = Y_0$ by the minimality of $Y$. Since every finite Radon measure on a weakly compact set in a Banach space has a norm separable support (see, e.g., [12, Theorem 4.3, page 256]), $Y$ is norm separable. □

**Lemma 3.4.** Let $Y$ be a norm separable and weakly compact set in a Banach space $E$. For a super asymptotically nonexpansive action of a right reversible semitopological semigroup $S$ on $Y$, suppose that $Y$ is minimal with respect to being weakly compact and $S$-invariant. Let $F$ be any nonempty weakly closed subset of $Y$ such that $F \subseteq s.F$ for all $s \in S$. Then $F$ is norm compact. In particular, $Y$ is norm compact if $s.Y = Y$ for all $s \in S$.

**Proof.** We follow the idea in [18, Lemma 5.2] in which a nonexpansive action of $S$ is considered instead.

Define $N_{\varepsilon} = \{x \in E : \|x\| \leq \varepsilon\}$ for any given $\varepsilon > 0$. Since $Y$ is norm separable, there exists $\{x_i : i \in \mathbb{N}\} \subseteq Y$ such that $Y \subseteq \bigcup \{x_i + N_{\varepsilon} : i \in \mathbb{N}\}$. By the Baire category theorem, there exist $\bar{x} \in \{x_i : i \in \mathbb{N}\}$ such that $(\bar{x} + N_{\varepsilon}) \cap Y$ has nonempty interior in $Y$ in the relative weak topology. Hence, there exist a $z \in Y$ and a weakly open neighborhood $V$ of 0 such that $(z + V) \cap Y \subseteq (\bar{x} + N_{\varepsilon}) \cap Y$. We can choose a weak neighborhood $U$ of 0 such that $U + U \subseteq V$. Since $U$ also contains a norm open neighborhood of 0, there exists $\delta > 0$ such that $N_{\delta} \subseteq U$. By the norm separability of $Y$ again, we can assume, with a new sequence $\{x_i : i \in \mathbb{N}\}$, that

$$Y = \bigcup \{(x_i + N_{\delta}) \cap Y : i \in \mathbb{N}\}.$$  \hspace{1cm} (3.2)

By the definition of the super asymptotic nonexpansiveness, for each given $r_0 \in S$, there exists a left ideal $I_1 = I_1^{r_0}$ of $S$ such that $\|sr_0.x_1 - sr_0.y\| \leq \|x_1 - y\|$ for all $s \in I_1, y \in Y$. Since $I_1r_0.x_1$ is $S$-invariant in $Y$, by the minimality of $Y$, its weak closure must be exactly $Y$. Thus, there exists an $s_1 \in I_1$ such that $s_1r_0.x_1 \in (z + U) \cap Y$. Let $r_1 = s_1r_0$, we have $r_1.x_1 \in (z + U) \cap Y$ and $\|r_1.x_1 - r_1.y\| \leq \|x_1 - y\|$ for all $y \in Y$.

Similarly, there exists a left ideal $I_2 = I_2^{r_1} \subseteq S$ such that $\|sr_1.x_2 - sr_1.y\| \leq \|x_2 - y\|$ for all $s \in I_2, y \in Y$. There exists an $s_2 \in I_2$ such that $s_2r_1.x_2 \in (z + U) \cap Y$. Let $r_2 := s_2r_1 = s_2s_1r_0$, we have $r_2.x_2 \in (z + U) \cap Y$ and $\|r_2.x_2 - r_2.y\| \leq \|x_2 - y\|$ for all $y \in Y$. By induction, we can choose a sequence $\{s_i : i \in \mathbb{N}\}$ in $S$ such that for

$$r_i = s_is_{i-1} \cdots s_1r_0.$$
we have

\[ r_i x_i \in (z + U) \cap Y, \quad \text{and} \]

\[ \|r_i x_i - r_i y\| \leq \|x_i - y\|, \quad \forall y \in Y, \; i \geq 1. \]

For each \( y \in (x_i + N_\delta) \cap Y \), we can write

\[ r_i y = (r_i y - r_i x_i) + r_i x_i \]

where \( r_i x_i \in (z + U) \cap Y \) and \( \|r_i x_i - r_i y\| \leq \|x_i - y\| < \delta \). Thus,

\[ r_i ((x_i + N_\delta) \cap Y) \subseteq (z + U + N_\delta) \cap Y \subseteq (z + V) \cap Y. \]

We rewrite the action \( r x \) in the form of \( L_r x \). Then \( (x_i + N_\delta) \cap Y \subseteq L_{r_i}^{-1}((z + V) \cap Y) \), where \( L_{r_i}^{-1}((z + V) \cap Y) \) is weakly open by the weak continuity of the action. By the weak compactness and (3.2), we can cover \( Y \) by finitely many such weakly open sets. Let

\[ Y = \bigcup_{i=1}^{n} L_{r_i}^{-1}((z + V) \cap Y). \]

It follows from the super asymptotic nonexpansiveness of the action that there exist left ideals \( J_i = J_{x_i}^i \), where

\[ t_i = s_{n+1}s_n \cdots s_{i+1}, \quad \text{for } i = 1, \ldots, n, \]

such that \( \|s_{t_i} \bar{x} - s_{t_i} y\| \leq \|\bar{x} - y\| \) for all \( y \in Y, s \in J_i \). Since \( S \) is right reversible, there exists \( t_0 \in \cap_{i=1}^{n} J_i \).

For each \( i \), there exists a net \( \{s_\lambda\} \subseteq J_i \) converging to \( t_0 \). Hence \( s_{\lambda}(t_i \bar{x}) - s_{\lambda}(t_i y) \) converges to \( t_0 t_i \bar{x} - t_0 t_i y \) weakly. Therefore, from the lower continuity of the norm function in the weak topology,

\[ \|t_0 t_i \bar{x} - t_0 t_i y\| \leq \|\bar{x} - y\| \quad \text{for all } y \in Y, \; i = 1, \ldots, n. \]  

(3.3)

Since \( F \subseteq s_i F \) for all \( s \in S \), we have

\[
F \subseteq L_{t_0} L_{r_{n+1}} F \subseteq L_{t_0} L_{r_{n+1}} Y' = L_{t_0} L_{r_{n+1}} \left\{ \bigcup_{i=1}^{n} L_{r_i}^{-1}((z + V) \cap Y) \right\} \\
\subseteq \bigcup_{i=1}^{n} \left\{ L_{t_0} L_{s_{n+1} \cdots s_{i+1}} ((\bar{x} + N_\varepsilon) \cap Y) \right\} \\
= \bigcup_{i=1}^{n} \left\{ L_{t_0 t_i} ((\bar{x} + N_\varepsilon) \cap Y) \right\} \\
\subseteq \bigcup_{i=1}^{n} \left\{ ((t_0 t_i \bar{x} + N_\varepsilon) \cap Y) \right\} \\
\subseteq \bigcup_{i=1}^{n} \left\{ L_{t_0 t_i} \bar{x} + N_\varepsilon \right\} .
\]

The second last inclusion above follows from (3.3). This proves that the norm closed set \( F \) can be covered by a finite \( \varepsilon \)-net for any \( \varepsilon > 0 \). Hence, \( F \) is norm compact.  \( \square \)
Together with above lemmas and motivated by [10, Theorem 3.1] and [4, Theorem 4.2], we are ready to prove Theorem 3.1. Note that, if the subset $Y$ in Lemma 3.4 is known to be convex then we can apply [26, Lemmas 2.5] for a shorter proof. However, at the current stage, we do not have the convexity of $Y$.

Proof of Theorem 3.1. By Lemmas 3.3 and 3.4, the nonempty $S$-preserving subset $Y$ given in Lemma 3.2 is separable and norm compact in the Banach space $E$. Consequently, the norm topology and the weak topology agree on $Y$. If $Y$ contains exactly one point then we are done. Otherwise, let

$$r = \text{diam}(Y) = \sup \{ \|x - y\| : x, y \in Y \}.$$

By DeMarr’s Lemma [8, Lemma 1], there is an element $u \in \overline{\text{conv}}(Y)$ such that $r_0 = \sup \{ \|u - y\| : y \in Y \} < r$.

Let $0 < \varepsilon < r - r_0$. Let $L_0$ and $\{\Lambda_i : i \in I\}$ be given in Lemma 3.2. For each $\Lambda \in \{\Lambda_i : i \in I\}$, set

$$N_{\varepsilon, \Lambda} = \bigcap_{y \in Y} \{x \in \Lambda : \|x - y\| \leq r_0 + \varepsilon\}$$

and

$$N_0 = \bigcap \{N_{\varepsilon, \Lambda_i} : i \in I\} = L_0 \cap \bigcap_{y \in Y} \overline{B}[y, r_0 + \varepsilon],$$

where $\overline{B}[y, \delta]$ stands for the norm closed ball centered at $y$ of radius $\delta$.

We show that $N_0$ satisfies ($\star_1$) and ($\star_2$). Indeed, every $N_{\varepsilon, \Lambda_i}$ is weakly compact. Thus $N_0$ is a weakly compact subset of $L_0$, and contains $u$. For each $x \in N_0$ and $i \in I$, there exists a left ideal $I \subseteq S$ such that $I.x \subseteq \Lambda_i$. By the super asymptotic nonexpansiveness of the action, for each $t \in S$ there exists a left ideal $I^t_s$ such that $\|st.y - st.x\| \leq \|y - x\|$ for all $y \in K$ and $s \in I^t_s$. By the right reversibility of $S$, there exists a $t_0 \in T \cap I^t_s$. Since $\Lambda_i$ is weakly closed, $St_0.x \subseteq \Lambda_i$. Consider a net $s_\lambda \in I^t_s$ such that $s_\lambda t \to t_0$. From $\|ss_\lambda t.y - ss_\lambda t.x\| \leq \|y - x\| \leq r_0 + \varepsilon$ for all $\lambda$, $y \in Y$ and $s \in S$, we have $\|st_0.y - st_0.x\| \leq \|y - x\| \leq r_0 + \varepsilon$. Since $Y \subseteq st_0.Y$, we have $\|y' - st_0.x\| \leq r_0 + \varepsilon$ for all $y' \in Y$. In other words, there exists a left ideal $J = St_0$ of $S$ such that $J.x \subseteq N_{\varepsilon, \Lambda_i}$. Consequently, the nonempty weakly compact convex subset $N_0$ of $L_0$ also satisfies conditions ($\star_1$) and ($\star_2$).

By the minimality of $L_0$, we have $Y \subseteq L_0 = N_0 \subseteq \bigcap_{y \in Y} \overline{B}[y, r_0 + \varepsilon]$. This gives us a contradiction that $\text{diam}(Y) \leq r_0 + \varepsilon < r$. Therefore, $Y$ contains a unique point and it is the common fixed point for the action of $S$ on $K$. □
Remark 3.5. The converse of Theorem 3.1 is not true in general. In fact, by Proposition 3.12 below, we will see that for any separable and reversible semitopological semigroup $S$, any jointly weakly continuous and super asymptotically nonexpansive action of $S$ on a weakly compact convex set $K$ has a common fixed point. However, $S$ is not necessarily left amenable. For example, take $S$ to be the free group of two generators.

Lau and Zhang [20, Theorem 6.2] established that a left amenable semitopological semigroup $S$ has the following fixed point property.

(F\textsubscript{wr,sep}) Let $K$ be a weak* compact convex and norm-separable subset of a dual Banach space. Then every jointly weak* continuous and nonexpansive action of $S$ on $K$ has a common fixed point.

Replacing the assumption of nonexpansiveness with the weaker one of super asymptotically nonexpansiveness, but together with the right reversibility of the semigroup, we obtain the following result.

Theorem 3.6. Let $S$ be a right reversible and left amenable semitopological semigroup. Let $K$ be a weak* compact convex and norm-separable subset of a dual Banach space. Then every jointly weak* continuous and super asymptotically nonexpansive action of $S$ on $K$ has a common fixed point.

Proof. It follows similarly as in proving Theorem 3.1, but with the norm-separability coming from the assumption, and noticing that Lemma 3.4 is also valid for the weak* compact case. \hfill \Box

Remark 3.7. Since the support of a finite Radon measure on a weak* compact set in a dual Banach space may not be norm separable, the conclusion of Lemma 3.3 about the norm separability of $Y$ may not hold for the weak* compact case.

The following result supplements [19, Theorem 3.4].

Theorem 3.8. Let $S$ be a right reversible semitopological semigroup.

(i) Assume $\text{AP}(S)$ has a LIM. Let $K$ be a weakly compact (resp. weak* compact and norm-separable) convex subset of a Banach (resp. dual Banach) space. Then every separately weakly (resp. weak*) continuous, equicontinuous and super asymptotically nonexpansive action of $S$ on $K$ has a common fixed point.

(ii) Assume $\text{WAP}(S)$ has a LIM. Let $K$ be a weakly compact (resp. weak* compact and norm-separable) convex subset of a Banach (resp. dual Banach) space. Then every separately weakly (resp. weak*) continuous, quasi-equicontinuous and super asymptotically nonexpansive action of $S$ on $K$ has a common fixed point.
Proof. These are direct consequences of [16, Lemma 3.1], [19, Lemma 3.2], and the proofs of Theorems 3.1 and 3.6.

Corollary 3.9. Let $S$ be a semitopological semigroup as well as a normal topological space. Assume that $\text{CB}(S)$ has an invariant mean. Let $K$ be a weakly compact (resp. weak* compact and norm-separable) convex subset of a Banach (resp. dual Banach) space. Then every separately weakly (resp. weak*) continuous and super asymptotically nonexpansive action of $S$ on $K$ has a common fixed point.

Proof. It is known that if $S$ is normal and $\text{CB}(S)$ has a right invariant mean then $S$ is right reversible. The assertion now follows from Theorems 3.1 and 3.6.

Remark 3.10. The only place we need the joint continuity of the action in Theorems 3.1 and 3.6 is where we derive that $R_yf$ belongs to $LUC(S)$ for each $f \in C(Y)$ and $y \in Y$. Then we can construct a LIM $\psi$ of $C(Y)$. For Theorem 3.8 and Corollary 3.9, we need only separate continuity since other assumptions there suffice to ensure that such a LIM $\psi$ exists for the stated function spaces on $S$.

Without any amenability assumption, we consider in the following fixed point properties of reversible semitopological semigroups. Borzdyński and Wiśnicki [5, Theorem 3.5] established that any commutative (and thus left amenable and reversible) semigroup has the fixed point property $(F_{w^*})$. For a discrete semigroup, the left amenability implies the left reversibility (see [19, page 2549]), while in general it might not be the case. The following two results supplement Theorems 3.1 and 3.6. The key point in their proofs is to bypass Lemma 3.3.

Proposition 3.11. Let $S$ be a reversible semitopological semigroup. Let $K$ be a weak* compact convex and norm-separable subset of a dual Banach space. Then every separately weak* continuous and super asymptotically nonexpansive action of $S$ on $K$ has a common fixed point.

Proof. By Lemma 3.2, there is a subset $L_0$ of $K$ which is minimal with respect to being nonempty, weak* compact, convex, and satisfying conditions $(\star 1)$ and $(\star 2)$ where the weak* topology is involved. Moreover, $L_0$ contains a subset $Y$ that is minimal with respect to being nonempty, weak* compact and $S$-invariant.

By Lemma 2.6, there is a non-empty weak* closed subset $F$ of $Y$ such that $F \subset s.F$ for all $s \in S$. Applying Lemma 3.4, we see that $F$ is compact. The remaining parts now follow similarly as in proving Theorem 3.1 where the set $Y$ is replaced by its norm compact subset $F$. 

$\square$
Proposition 3.12. Let $S$ be a separable and reversible semitopological semigroup. Let $K$ be a weakly compact convex subset of a Banach space. Then every separately weakly continuous and super asymptotically nonexpansive action of $S$ on $K$ has a common fixed point.

Proof. By Lemma 3.2, we establish a subset $Y$ of $L_0 \subset K$ that is minimal with respect to being nonempty, weakly compact and $S$-invariant. Following an idea in [19, Lemma 3.3], we show that $Y$ is norm separable. Indeed, for any fixed $y \in Y$, we have $Sy = \{s.y : s \in S\}$ is an $S$-invariant subset of $Y$. Thus, its weak closure $\overline{Sy}^{wk}$ must be exactly $Y$. Assume $S$ contains a countable dense subset $S_c$. Since the action is separately weakly continuous, $\overline{Sy}^{wk} = \overline{S_c y}^{wk}$. Moreover, $\overline{\text{conv}^{wk}(Sy)} = \overline{\text{conv}^{wk}(S_c y)} = \overline{\text{conv}^{\|\|}(S_c y)}$ by Mazur’s Theorem. It follows that $Y = \overline{\text{conv}^{wk}(Sy)}$ is norm separable. From Lemma 2.6, the left reversibility of $S$ ensures that there is a weakly compact subset $F$ of $Y$ satisfying $F \subset s.F$ for all $s \in S$. The remaining parts follow similarly as in the proof of Theorem 3.1.

Remark 3.13. In Example 2.3, all assumptions of Theorem 3.1 and Proposition 3.12 are satisfied. Thus we can conclude that the action there has a fixed point. Indeed, the action has a common fixed point $(0, 0)$.

Motivated by [19, Page 2550], we call a semitopological semigroup $S$ strongly reversible if there is a family of separable and reversible subsemigroups $\{S_\alpha : \alpha \in I\}$ of $S$ such that $S = \bigcup_{\alpha \in I} S_\alpha$ and for each $\alpha_1, \alpha_2 \in I$ there is an $\alpha_3 \in I$ such that $S_{\alpha_1} \cup S_{\alpha_2} \subset S_{\alpha_3}$. Here, the topology on each $S_\alpha$ is the subspace topology inherited from $S$. Obviously, if $S$ is strongly reversible then it is reversible. Follow [19, Lemma 5.2], if $S$ is metrizable and reversible then it is strongly reversible.

Corollary 3.14. Let $S$ be a strongly reversible semitopological semigroup. Let $K$ be a weakly compact convex subset of a Banach space. Then every separately weakly continuous and super asymptotically nonexpansive action of $S$ on $K$ has a common fixed point.

Proof. Suppose $S = \bigcup_{\alpha \in I} S_\alpha$ where $S_\alpha$ is separable and reversible. For each $\alpha \in I$, consider the sub-action $\overline{S_\alpha} \times K \to K$ of $\overline{S_\alpha}$ on $K$. It follows from the proof of Proposition 3.12 that there is a norm separable subset $Y_\alpha$ of $K$ that is minimal with respect to being nonempty, weakly compact and $\overline{S_\alpha}$-invariant. Follow the arguments in proving [19, Lemma 5.3] and Lemma 2.6, we can show that there is a nonempty weakly compact subset $F_\alpha$ of $Y_\alpha$ such that $F_\alpha \subset s.F_\alpha$ for all $s \in \overline{S_\alpha}$. By Lemma 3.4, $F_\alpha$ is norm compact. As that verified in the proof of Theorem 3.1, the common fixed points set $F_\alpha = \{x \in K : s.x = x \text{ for all } s \in \overline{S_\alpha}\}$ is nonempty. Since the action is weakly continuous, $F_\alpha$ is weakly closed. Indeed, for any net $\{x_\alpha\}$ in $F_\alpha$ weakly converging to $x$
in $K$, we have $s.x = \lim_{\alpha} s.x_\alpha = \lim_{\alpha} x_\alpha = x$ and thus $x \in F_\alpha$. Since for each $\alpha_1, \alpha_2 \in I$ there is an $\alpha_3 \in I$ such that $S_{\alpha_1} \cup S_{\alpha_2} \subset S_{\alpha_3}$, the family $\{F_\alpha : \alpha \in I\}$ has the finite intersection property. By the weak compactness of $K$, the set of common fixed points $F(S) = \bigcap_{\alpha \in I} F_\alpha$ of $S$ is nonempty.

\[ \square \]

4. Fixed point theorems involving the Radon-Nikodým property or the distality

Recently, Wiśnicki [32] provided an extension for the Ryll-Nardzewski’s Theorem. It is about the existence of a common fixed point for a nonlinear action of a semigroup on a weakly compact convex set in a locally convex space. Following his idea, we establish in this section some extensions of the results in Section 3 as well as the results in [32], for the asymptotically nonexpansive type actions of right reversible semigroups.

The main idea is to replace the norm-separability assumption in [19, Theorem 6.2] and in Theorem 3.6 with the Radon-Nikodým property or the norm-distality, and to derive the norm compactness of an $S$-invariant subset of $K$ in the action.

**Theorem 4.1.** Let $S$ be a right reversible and left amenable semitopological semigroup. Let $K$ be a weak* compact convex subset of a dual Banach space with the RNP. Then every jointly weak* continuous and pointwise eventually nonexpansive action of $S$ on $K$ has a common fixed point.

**Proof.** By Lemmas 3.2 and 3.3, there is an $S$-invariant Radon probability measure $\mu$ such that the support $Y = \text{supp} \mu$ is minimal with respect to being a nonempty $S$-invariant weak* compact subset of $K$. Below, we follow the approach in [32, Theorem 3.1], see also [33, Theorem 4.2], in which nonexpansive actions are considered instead.

By Lemma 2.8, there exists an $x \in Y$ such that for each $\varepsilon > 0$, there is a weak* open neighbourhood $U$ of $x$ such that $\|x - y\| < \varepsilon$ for all $y \in U \cap Y$. Hence

\[
\delta := \mu(\{y \in Y : \|x - y\| < \varepsilon\}) \geq \mu(U \cap Y) > 0. \tag{4.4}
\]

Since the action is pointwise eventually nonexpansive, there is a left ideal $I$ of $S$ such that

\[
\|s.x - s.y\| \leq \|x - y\|, \quad \forall s \in I, \forall y \in Y.
\]

Therefore, for each $s \in I$, we have

\[
\{y \in Y : \|x - y\| < \varepsilon\} \subset L_s^{-1}\{z \in Y : \|s.x - z\| < \varepsilon\},
\]

where $L_s(y) := s.y$ for $y \in K$. Since $\mu$ is invariant,

\[
\mu(\{y \in Y : \|s.x - y\| < \varepsilon\}) = \mu(L_s^{-1}\{y \in Y : \|s.x - y\| < \varepsilon\}) \geq \mu(\{y \in Y : \|x - y\| < \varepsilon\}) = \delta > 0.
\]
Since \( Ix \) is \( S \)-invariant, by the minimality of \( Y \), we have \( Y = \overline{Ix}^{wk^*} \). We shall see that there are only finitely many elements \( s_1, \ldots, s_k \) in \( I \) such that \( \|s_ix - s_jx\| \geq 2\varepsilon \) for any \( i \neq j \). In fact, all \( Y_i = \{y \in Y : \|y - s_i x\| < \varepsilon\} \) are pairwise disjoint subsets of \( Y \), and \( \mu(Y_i) \geq \delta \) for all \( i \). Now the fact \( \mu(Y) = 1 \) ensures that at most finitely many of such elements exist. In other words, \( Ix \) is totally bounded in norm, and hence \( Ix \|\cdot\| \) is norm-compact. Since the identity map from \( (Y, \|\cdot\|) \) into \( (Y, wk^*) \) is continuous, \( Y = \overline{Ix}^{wk^*} = \overline{Ix}^{\|\cdot\|} \) is norm-compact. The remaining follows similarly as the proof of Theorem 3.1.

The following result supplements Theorem 4.1, and applies to the case when we do not have the left amenability in stock.

**Theorem 4.2.** Let \( S \) be a right reversible semitopological semigroup. Let \( K \) be a weak* compact convex subset of a dual Banach space with the RNP. Then every separately weak* continuous, pointwise eventually nonexpansive and norm-distal action of \( S \) on \( K \) has a common fixed point.

**Proof.** By Lemma 3.2, there is a subset \( L_0 \) of \( K \) which is minimal with respect to being nonempty, weak* compact, convex, and satisfying conditions (\( \ast \)1) and (\( \ast \)2) where the weak* topology is involved. Moreover, \( L_0 \) contains a subset \( Y \) that is minimal with respect to being nonempty, weak* compact and \( S \)-invariant.

We are going to construct an \( S \)-invariant Radon probability measure \( \mu \) on \( K \) with respect to the weak* topology. As in the proof of [32, Theorem 3.1], the norm-distality of the action together with the minimality of \( Y \) implies that the action is weak*\( ^* \)-distal.

Let \( C(Y) \) be the space of continuous functions on \( Y \). Let \( P(Y) \) be the weak* compact convex set of all means on \( C(Y) \). Consider an action of \( S \) on \( P(Y) \) given by \( s.\mu = l_s^*\mu \), where \( l_s \) is the left translation operator by \( s \), and \( \langle l_s^*\mu, f \rangle = \langle \mu, l_s f \rangle \) for all \( s \in S \) and \( f \in C(Y) \). Let \( \phi : Y \to P(Y) \) be the isometric natural embedding, \( x \mapsto \hat{x} \), defined by \( \hat{x}(f) = f(x) \) for all \( f \in C(Y) \). Then, we obtain an action of \( S \) on \( \phi(Y) \) by defining \( s.\phi(x) = \phi(s.x) \) for all \( s \in S \) and \( x \in Y \).

Since the action of \( S \) on \( Y \) is weak*\( ^* \)-distal, so is the action of \( S \) on \( \phi(Y) \). It follows from Theorem 2.10 that there is a common fixed point \( \mu \) of \( S \) in \( P(Y) \). In other words, \( \mu \) is an \( S \)-invariant Radon probability measure on \( Y \) with respect to the weak* topology. As in the proof of Theorem 4.1, we see that \( S \) has a common fixed point in \( K \).

For the super asymptotically (resp. pointwise eventually) nonexpansive actions on a weakly compact convex subset of a locally convex space, we obtain the following fixed point properties without assuming neither RNP nor norm-separability.
Let \((E, Q)\) be a locally convex space where \(Q\) is a family of seminorms determining the topology. An action \(S \times K \to K\) is said to be

1. \textbf{pointwise eventually \(Q\)-nonexpansive} if for each \(x \in K\) and \(q \in Q\), there exists a left ideal \(I = I(x, q)\) of \(S\) such that \(q(s.x - s.y) \leq q(x - y)\) for all \(s \in I\) and all \(y \in K\);
2. \textbf{super asymptotically \(Q\)-nonexpansive} if for each \(x \in K\), \(t \in S\) and \(q \in Q\), there exists a left ideal \(I = I(x, t, q)\) of \(S\) supported by \(t\) such that \(q(s.x - s.y) \leq q(x - y)\) for all \(s \in I\) and all \(y \in K\).

See [23] for more discussions.

The following result is an extension of Theorem 3.1 for pointwise eventually nonexpansive actions.

\textbf{Proposition 4.3.} Let \(S\) be a right reversible and left amenable semitopological semigroup. Let \(K\) be a weakly compact convex subset of a locally convex space \((E, Q)\). Then every jointly weakly continuous and pointwise eventually \(Q\)-nonexpansive action of \(S\) on \(K\) has a common fixed point.

\textbf{Proof.} Using Lemma 2.9 and arguing as in proving Theorem 4.1, we obtain a similar inequality as (4.4) in which the norm is replaced by a seminorm in \(Q\). We can then derive that \(Y\) is \(Q\)-compact. Finally, with an argument similar to the one proving Theorem 3.1, see also [23, Theorem 2.14], we will arrive at the conclusion. \(\square\)

\textbf{Theorem 4.4.} Let \(S\) be a separable and right reversible semitopological semigroup. Let \(K\) be a weakly compact convex subset of a locally convex space \((E, Q)\). Then every separately weakly continuous, super asymptotically \(Q\)-nonexpansive and \(Q\)-distal action of \(S\) on \(K\) has a common fixed point.

\textbf{Proof.} By Lemma 3.2, there is a subset \(Y\) of \(K\), that is minimal with respect to being nonempty, weakly compact and \(S\)-invariant. As in proving Proposition 3.12, we can show that \(Y\) is \(Q\)-separable.

As in proving [32, Theorem 4.1], the \(Q\)-distality of the action together with the minimality of \(Y\) ensures that the action is indeed weakly-distal. Following the proof of Theorem 4.2, we can show the existence of an \(S\)-invariant Radon probability measure \(\mu\) defined on \(Y\) such that \(Y = \supp(\mu)\) and \(sY = Y\) for all \(s \in S\). Arguing as in proving Lemma 3.4, but with the seminorms in \(Q\) replacing the norm, we see that \(Y\) is \(Q\)-compact. The remaining now follows as in the proof of Theorem 3.1. \(\square\)
5. **Fixed point theorems for pointwise eventually nonexpansive mappings**

In the following, applying the result in previous sections, we establish fixed point theorems for a finite commutative family of continuous maps on weakly/weak* compact convex sets which are pointwise eventually nonexpansive.

**Corollary 5.1.** Let $K$ be a non-empty weakly compact convex subset of a Banach space. Let $\{T_1, \ldots, T_k\}$ be a commutative family of weakly continuous and pointwise eventually nonexpansive maps on $K$. Then they have a common fixed point.

**Proof.** Let $S$ be the discrete semigroup generated by this family. As seen in Remark 2.2(c), the action of $S$ on $K$ is super asymptotically nonexpansive. Moreover, since $S$ is commutative, it is reversible and left amenable. From Theorem 3.1, as well as Proposition 3.12, $S$ has a common fixed point. Hence $\{T_1, \ldots, T_k\}$ has a common fixed point in $K$. \qed

**Corollary 5.2.** Let $K$ be a weak* compact convex subset of a dual Banach space with the RNP. Let $\{T_1, \ldots, T_k\}$ be a commutative family of weak* continuous and pointwise eventually nonexpansive maps on $K$. Then they have a common fixed point.

**Proof.** Let $S$ be the semigroup generated by this finite family. From Theorem 4.1, the canonical action of $S$ on $K$ has a common fixed point. Therefore, the finite family has a common fixed point in $K$. \qed

Since every weak* compact convex and norm-separable subset has the RNP, we have the following result.

**Corollary 5.3.** Let $K$ be a weak* compact convex and norm-separable subset of a dual Banach space. Let $\{T_1, \ldots, T_k\}$ be a commutative family of weak* continuous and pointwise eventually nonexpansive maps on $K$. Then they have a common fixed point.

**Proof.** This is a consequence of Theorem 3.6, as well as Remark 2.2(c) and Proposition 3.11. \qed

A map $T : K \to K$ on a subset $K$ of a locally convex space $(E, Q)$ is called **eventually nonexpansive** if for each $x, y \in K$ and each seminorm $q \in Q$, there exists an $n(x, y, q) \in \mathbb{N}$ such that $q(T^n x - T^n y) \leq q(x - y)$ for all $n \geq n(x, y, q)$.

**Corollary 5.4.** Let $K$ be a compact convex subset of a locally convex space $(E, Q)$. Let $\{T_1, \ldots, T_k\}$ be a commutative family of continuous and eventually nonexpansive maps on $K$. Then they have a common fixed point.

**Proof.** This follows Theorem 3.1 in [10]. Noting that for commutative semigroups, the property (B) is always satisfied. \qed
We end this paper with an open problem about possible extensions of our results. Under some conditions, we establish that a super asymptotically nonexpansive or pointwise eventually nonexpansive action of a semitopological semigroup $S$ on weakly/weak* compact convex sets has a common fixed point.

**Question 5.5.** Do we have similar results as Theorems 3.1, 3.6, 4.1, 4.2, 4.4, and Propositions 3.11, 3.12 for asymptotically nonexpansive actions?

From Proposition 2.7, if $S$ is compact and right reversible then the asymptotic nonexpansiveness coincides with the super asymptotic nonexpansiveness. Hence the question has an affirmative answer in this case.

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