Abstract

The octonion X-product changes the octonion multiplication table, but does not change the role of the identity. The XY-product is very similar, but shifts the identity as well. This will be of interest to those applying the octonions to string theory.
1. Moufang Identities.

The Moufang identities are listed here for future reference. I list them in their conventional form and in the form I introduced in [1].

\[(xy)(ax) = x(ay)x; \quad (xy)LxR = xLxRyL;\]
\[(xa)(yx) = x(ya)x; \quad (yx)RxL = xLxRyR;\]
\[(xax)y = x(a(xy)); \quad yRxLxR = xL(xy)R;\]
\[y(xax) = ((yx)a)x; \quad yLxLxR = xR(yx)L.\]  

(1)

2. \(O_{X,Y}\).

The X-product \([1][2]\) changes \(O\) to \(O_X\), which is isomorphic to \(O\). Furthermore, the identities of \(O\) and \(O_X\) are both \(e_0 = 1\). However, it is possible to modify the octonion product in such a way that \(e_0\) is not the identity of the result. In particular, define

\[A \circ_{X,Y} B = (AX)(Y\dagger B)\]
\[= A \circ_X ((XY\dagger) \circ_X B)\]
\[= (A \circ_Y (XY\dagger)) \circ_Y B,\]  

(2)

where as usual we assume that both \(X, Y \in S^7\) (the X-product is obtained by setting \(X = Y\)). Let \(O_{X,Y}\) be \(O\) with this modified product.

The question is, is this still isomorphic \(O\) itself? Let \(O\) be the copy of the octonions employing the cyclic multiplication table introduced in [1], and let

\[E_a, \quad a = 0, ..., 7,\]

be a basis for \(O_{X,Y}\). We will attempt to make assignments for the \(E_a\) so that they satisfy the same table, and consequently \(e_a \rightarrow E_a, \quad a = 0, ..., 7,\) will be an isomorphism from \(O\) to \(O_{X,Y}\).

IDENTITY

As a start, it is not difficult to prove that in general,

\[E_0 = YX\dagger\]  

(3)
Hermitian conjugation is also altered, and it will help to determine this before proceeding. In particular, let $A^*$ denote the $O_{X,Y}$ conjugate of $A$. It must satisfy

\[(AX)(Y^\dagger A^*) = (A^*X)(Y^\dagger A) = \|A\|^2 E_0 = AA^\dagger E_0,\] (4)

where $\|A\|^2$ is the square of the norm of $A$. Therefore, (4) implies the two results,

\[
\begin{align*}
A^* &= \|A\|^2 Y((AX)^{-1}E_0) = Y((X^\dagger A^\dagger)(YX^\dagger)) = Y(X^\dagger(A^\dagger Y)X^\dagger), \\
A^* &= \|A\|^2 (E_0(Y^\dagger A)^{-1})X^\dagger = ((YX^\dagger)(A^\dagger Y))X^\dagger = (Y(X^\dagger A^\dagger)Y)X^\dagger
\end{align*}
\] (5)

(the Moufang identities were used in (5), along with $A^{-1} = A^\dagger/\|A\|^2$). Proving the two versions of $A^*$ in (5) are the same is equivalent to proving

\[Y_LX_L^\dagger X_R^\dagger Y_R = X_R^\dagger Y_L Y_R X_L^\dagger.\]

I leave it to the reader to prove this using the righthand identities in (1).

**ISOMORPHISM**

In order to prove the general isomorphism of $O$ and $O_{X,Y}$, we will start with the simpler case of $O_{1,Z}$, the product of which is

\[A_{O_{1,Z}} B = A(Z^\dagger B).\] (6)

Therefore,

\[E_0 = Z,\] (7)

and

\[A^* = ZA^\dagger Z.\] (8)

Without any loss in generality we can set

\[Z = Z^0 + Z^7 e_7, \quad ZZ^\dagger = (Z^0)^2 + (Z^7)^2 = 1.\] (9)
After playing around a bit, I came up with the following assignments:

\[ E_1 = Ze_1 = e_1Z; \quad E_5 = Ze_5 = e_5Z; \]
\[ E_2 = Ze_2 = e_2Z; \quad E_3 = Ze_3 = e_3Z; \]
\[ E_4 = Ze_4 = e_4Z; \quad E_6 = Ze_6 = e_6Z; \]
\[ E_7 = Ze_7 = e_7Z. \]  

(10)

Observe that

\[ (Ze_a)^* = -Ze_aZ = -Ze_a, \]
\[ (e_aZ)^* = -ZZ^*e_aZ = -e_aZ \]  

(11)

(no parentheses needed), so each of the \( E_a, a = 1, \ldots, 7, \) is perpendicular to \( E_0 = Z. \)

To make life easier, we’ll first check that these elements anticommut e. That is, if \( a, b \in \{1, \ldots, 7\} \) are distinct, then

\[ E_a \circ_{1,Z} E_b = -E_b \circ_{1,Z} E_a. \]  

(12)

PROOF OF ANTICOMMUTATION

CASE I: \( E_a = e_aZ, \quad E_b = e_bZ. \)

Therefore \( a, b \in \{4, 6, 7\}, \) a quaternionic triple. Since \( Z \) is also linear in 1 and \( e_7, \) the parentheses below can be dropped.

\[
E_a \circ_{1,Z} E_b = (e_aZ)(Z^*e_bZ)
\]
\[
= e_aZZ^*e_bZ
\]
\[
= e_ae_bZ
\]
\[
= -e_be_aZ
\]
\[
= -E_b \circ_{1,Z} E_a. \]  

(13)

CASE II: \( E_a = Ze_a, \quad E_b = e_bZ. \)

In this case,

\[ a \in \{1, 2, 3, 5\}, \quad b \in \{4, 6, 7\}. \]  

(14)
This is used twice below.

\[ E_{a \circ_{1,Z} b} = (Ze_a)(Z^\dagger(e_bZ)) \]
\[ = (Ze_a)(Z^\dagger e_bZ) \quad \text{Associate} \]
\[ = Z[e_a(Z^\dagger e_b)]Z \quad \text{Moufang} \]
\[ = [e_a(Z^\dagger e_b)]Z^\dagger Z \quad \text{From 9,14} \]
\[ = e_a(Z^\dagger e_b) \]  \hspace{1cm} (15)  
\[ = [(e_bZ)e_a]^\dagger \]
\[ = -(e_bZ)e_a \quad \text{From 9,14} \]
\[ = (e_bZ)(Z^\dagger(Ze_a)) \]
\[ = -E_{b \circ_{1,Z} a}. \]

\[ (16) \]

CASE III: \( E_a = Ze_a, \ E_b = Ze_b. \)

In this final case, \( a, b \in \{1, 2, 3, 5\}. \)

\[ E_{a \circ_{1,Z} b} = (Ze_a)(Z^\dagger(Ze_b)) \]
\[ = (Ze_a)e_b \]
\[ = -(Ze_b)e_a \quad \text{See [1]} \]  \hspace{1cm} (16)  
\[ = -(Ze_b)(Z^\dagger(Ze_a)) \]
\[ = -E_{b \circ_{1,Z} a}. \]

So (12) is proven.

MULTIPLICATION TABLE

Because of (12), to complete the multiplication table (and prove the isomorphism) we need merely check that

if \( e_a e_b = e_c, \) then \( E_{a \circ_{1,Z} b} = E_c, \) \hspace{1cm} (17)  

and

\[ E_{a \circ_{1,Z} a} = -E_0, \quad a = 1, ..., 7. \]  \hspace{1cm} (18)
Prove (18) first. Note that parentheses may be dropped in this case. There are two possibilities:
\[
(Ze_a)(Z\dagger(Ze_a)) = Ze_a Z\dagger Ze_a = Ze_a e_a = -Z,
\]
\[
(e_aZ)(Z\dagger(e_aZ)) = e_a ZZ\dagger e_aZ = e_a e_aZ = -Z.
\]
This proves (18).

We'll prove (17) by cases.

CASE I: \(E_1 \circ_{1,Z} E_2\).
\[
E_1 \circ_{1,Z} E_2 = (Ze_1)(Z\dagger(Ze_2))
= (Ze_1)e_2
= Z\dagger(e_1e_2) \quad \text{Nonassociativity}
= Z\dagger e_6
= e_6Z
= E_6.
\]
This example covers the four products,
\[
E_1 \circ_{1,Z} E_2 = E_6, \quad E_3 \circ_{1,Z} E_5 = E_6,
E_5 \circ_{1,Z} E_2 = E_4, \quad E_1 \circ_{1,Z} E_3 = E_4.
\]
(19)

CASE II: \(E_a \circ_{1,Z} E_7\).

In this case again, parentheses may be deleted.
\[
E_a \circ_{1,Z} E_7 = E_a(Z\dagger(Ze_7))
= E_a e_7
= e_aZe_7 \text{ or } Ze_ae_7
= (e_ae_7)Z \text{ or } Z(e_ae_7).
\]
This example covers the six products,
\[
E_7 \circ_{1,Z} E_1 = E_5, \quad E_7 \circ_{1,Z} E_2 = E_3, \quad E_7 \circ_{1,Z} E_4 = E_6,
E_5 \circ_{1,Z} E_7 = E_1, \quad E_3 \circ_{1,Z} E_7 = E_2, \quad E_6 \circ_{1,Z} E_7 = E_4.
\]
(20)
CASE III: $E_a \circ_1 Z E_b$, where $e_a e_b = e_7$.

Parentheses may be deleted in this case too, as all products associate. Note that if $E_a = Z e_a$ (or $e_a Z$), then $E_b = Z e_b$ (or $e_b Z$). So

$$E_a \circ_1 Z E_b = (e_a Z)(Z^\dagger (e_b Z)) \quad \text{or} \quad (Ze_a)(Z^\dagger (Ze_b))$$

$$= e_a Z Z^\dagger e_b Z \quad \text{or} \quad Ze_a Z^\dagger Ze_b$$

$$= e_a e_b Z \quad \text{or} \quad Ze_a e_b$$

$$= e_7 Z \quad = Z_7$$

$$= E_7.$$ 

This example covers the three products,

$$E_1 \circ_1 Z E_6 = E_2 \circ_1 Z E_3 = E_4 \circ_1 Z E_6 = E_7.$$ (21)

CASE IV: $E_4 \circ_1 Z E_1$.

$$E_4 \circ_1 Z E_1 = (e_4 Z)(Z^\dagger (Ze_1))$$

$$= (e_4 Z)e_1$$

$$= (Z^\dagger e_4)e_1$$

$$= Z(e_4 e_1) \quad \text{Nonassociativity}$$

$$= Ze_3$$

$$= E_3.$$ 

This example covers the four cases,

$$E_4 \circ_1 Z E_1 = E_3, \quad E_4 \circ_1 Z E_5 = E_2, \quad E_6 \circ_1 Z E_1 = E_2, \quad E_6 \circ_1 Z E_3 = E_5.$$ (22)
CASE V: $E_2 \circ_{1,Z} E_4$.

Lastly,

$$E_2 \circ_{1,Z} E_4 = (Ze_2)(Z^\dagger (e_4 Z)) = (Ze_2)((Z^\dagger e_4)Z) = Z[e_2(Z^\dagger e_4)]Z \quad \text{Moufang}$$

$$= [e_2(Z^\dagger e_4)]Z^\dagger Z \quad \text{Think about it}$$

$$= e_2(e_4 Z)$$

$$= (e_2 e_4)Z^\dagger \quad \text{Nonassociativity}$$

$$= e_5 Z^\dagger$$

$$= Ze_5$$

$$= E_5.$$ 

This example covers the four products,

$$E_2 \circ_{1,Z} E_4 = E_5, \quad E_3 \circ_{1,Z} E_4 = E_1, \quad (23)$$

$$E_2 \circ_{1,Z} E_6 = E_1, \quad E_5 \circ_{1,Z} E_6 = E_3.$$ 

These five cases, covering all $4 + 6 + 3 + 4 + 4 = 21$ ordered quaternionic triples, prove (17). Together with (12) and (18), they prove

$$O_{1,Z} \simeq O. \quad (24)$$

So $O_{1,Z}$ is in fact a copy of the octonions.

GENERAL $O_{X,Y}$.

In general, starting from any copy of the octonions, the $O_{1,Z}$ modification will be another copy of the octonions. In particular,

$$(O_X)_{1,Z} \simeq O. \quad (25)$$

The product of $O_X$ is

$$A \circ_X B = (AX)(X^\dagger B).$$
Now modify this to the product of $(O_X)_{1,Z}$:

\[
A_{O_X}(Z^{\dagger}o_X B) = (AX)[X^{\dagger}((Z^{\dagger}X)(X^{\dagger}B))] \\
= (AX)[X^{\dagger}(X((X^{\dagger}Z^{\dagger})B))] \\
= (AX)[(X^{\dagger}Z^{\dagger})B] \\
= (AX)(Y^{\dagger}B),
\]

(see [1][2][3]) where we define

\[
Y = ZX.
\]

Therefore, for all $X, Y \in S^7$,

\[
O_{X,Y} \simeq O.
\]

Note that by virtue of (27),

\[
YX^{\dagger} = Z
\]

is the identity of both $O_{1,Z}$ and $O_{X,Y}$.

Finally, in [3] it was shown how the X-product could be used to generate all the 480 renumberings of the $e_a, a = 1, ..., 7$, which leave $e_0 = 1$ fixed as the identity. There are 7680 renumberings of the entire collection, $e_a, a = 0, ..., 7$, and the XY-product plays exactly the same role in this context. In addition, in the X-product case the 480 renumberings arose from a pair of octonion $E_8$ lattices. The XY-product renumberings are related in a similar fashion to the pair of octonion $\Lambda_{16}$ lattices developed in [4]. I’ll leave it to the reader to prove this, or the reader can wait for a complete development in the monograph which is in preparation.

References

[1] G.M. Dixon, *Division Algebras: Octonions, Quaternions, Complex Numbers, and the Algebraic Design of Physics*, (Kluwer, 1994).

[2] M. Cederwall, C.R. Preitschopf, $S^7$ and $\hat{S}^7$, hep-th-9309030.

[3] G.M. Dixon, Octonion X-Product Orbits, [hep-th 9410202](http://arxiv.org/abs/hep-th/9410202).

[4] G.M. Dixon, Octonions: $E_8$ Lattice to $\Lambda_{16}$, [hep-th 9501007](http://arxiv.org/abs/hep-th/9501007).