The Entropic Dynamics of Spin

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Abstract
In the Entropic Dynamics (ED) approach the essence of quantum theory lies in its probabilistic nature while the Hilbert space structure plays a secondary and ultimately optional role. The dynamics of probability distributions is driven by the maximization of an entropy subject to constraints that carry the relevant physical information — directionality, correlations, gauge interactions, etc. The challenge is to identify those constraints and to establish a criterion for how the constraints themselves are updated. In this paper the ED framework is extended to describe a spin-1/2 point particle. In ED spin is neither modelled as a rotating body, nor through the motion of a point particle; it is an epistemic property of the wave function. The constraint that reflects the peculiar rotational properties of spin is most effectively expressed in the language of geometric algebra. The updating of all constraints is carried out in a way that stresses the central importance of symmetry principles. First we identify the appropriate symplectic and metric structures in the phase space of probabilities, their conjugate momenta, and the spin variables. This construction yields a derivation of the Fubini-Study metric for a spin-1/2 particle which highlights its deep connection to information geometry. Then we construct an ED that preserves both the symplectic structure (a Hamiltonian flow) and the metric structure (a Killing flow). We show that generic Hamiltonian-Killing flows are linear in the wave function. Imposing further that the Hamiltonian be the generator of an entropic evolution in time leads to an entropic dynamics described by the Pauli equation. We conclude with a discussion of dynamical variables such as energy, momentum, and spin; the magnetic dipole moment and Stern-Gerlach experiments; and the introduction of a Hilbert space.

1 Introduction
Quantum Mechanics (QM) has been commonly conceived as a generalization of classical mechanics with an added stochastic element. A Hilbert space structure was adopted as the foundation of the theory while the probabilistic structure was reluctantly included in order to handle those peculiar physical processes called measurements [1][2]. Unfortunately, the dynamical and the probabilistic
aspects of quantum theory turned out to be incompatible with each other causing all sorts of conceptual difficulties and endless controversy. Why a linear and unitary dynamical evolution for some processes and a wave function collapse for others? Why should some processes receive a privileged treatment? Are macroscopic superpositions at all possible? How can events actually happen? Prominent among these difficulties is the question of the interpretation of the quantum state itself: does the wave function represent the actual state of the system (its \textit{ontic} state) or does it represent our knowledge about the system (a merely \textit{epistemic} state)\textsuperscript{1}.

Entropic Dynamics (ED) resolves these difficulties by placing the probabilistic structure at the very foundation of the theory \textsuperscript{2}. Hilbert spaces are eventually introduced into the formalism but only as an optional calculational tool. In the ED approach QM is derived as an application of entropic methods of inference\textsuperscript{3}. The first concern is to achieve ontological clarity: What is real? What are the probabilities about? Once this issue is settled one can proceed to introduce probabilities following the standard practice of maximizing an entropy subject to constraints. The concern here is to identify the right constraints since it is through them that the information that is relevant to the particular physics problem is introduced.

In ED the central question is not how to obtain probabilities from a wave function, how to derive the Born rule, but the reverse: given a purely probabilistic foundation how do wave functions involving complex numbers arise? Eventually a wave function $\Psi$ is introduced and it turns out to be a fully epistemic concept. This has nontrivial implications: not only are “quantum” probabilities which obey alternative inference rules excluded from the start, but both the magnitude $|\Psi|$ and also the phase of $\Psi$ are given an epistemic interpretation. Furthermore, the notion that the change of probabilities by time evolution must be consistent with the change of probabilities by updating according to the rules of entropic inference places severe restrictions on the dynamics of $\Psi$.

The demand for ontological clarity places additional constraints on the theory. Once we decide to formulate a model in which the ontic state of the system

\textsuperscript{1}Excellent reviews with extended references to the literature are given in \textit{e.g.} \textsuperscript{3}-\textsuperscript{7}.

\textsuperscript{2}A note on terminology: a concept is referred to as ‘ontic’ when it describes something that is supposed to be real, to exist out there independently of any observer. A concept is referred to as ‘epistemic’ when it is related to the state of knowledge, opinion, or belief of an agent. (Here we are only concerned with idealized rational agents.) Some examples of epistemic quantities are probabilities and entropies. An important point is that the distinction ontic/epistemic is not the same as the distinction objective/subjective. For example, probabilities are fully epistemic — they are tools for reasoning with incomplete information — but they can lie anywhere in the spectrum from being completely subjective (two different agents can have different beliefs) to being completely objective. In QM, for example, probabilities are epistemic and objective. Indeed, at the non-relativistic level anyone who computes probabilities that disagree with QM will be led to experimental predictions that are demonstrably wrong. We will say that the wave function $\Psi$, which is fully epistemic and objective, represents a “physical” state when it represents information about an actual “physical” situation.

\textsuperscript{3}The principle of maximum entropy as a method for inference can be traced to the pioneering work of E. T. Jaynes \textsuperscript{10}-\textsuperscript{11}. For a pedagogical overview of Bayesian and entropic inference and further references see \textsuperscript{12}. 

2
is defined by the positions of the constituent particles then all other dynamical variables such as energy, momentum, and angular momentum are necessarily epistemic. These are not properties of the particles but properties of the wave function. Prior to a measurement these variables need not have definite values; their values are “created” by the process of measurement.

The ED approach has been applied to a variety of topics in quantum theory: the quantum measurement problem [13][14]; momentum and uncertainty relations [15][16]; the Bohmian limit [16][17] and the classical limit [18]; extensions to curved spaces [19] and to relativistic fields [20][21][22]. In this paper the ED framework is extended to describe the quantum mechanics of spin. Here we discuss a single spin-1/2 particle; in subsequent publications we will discuss multiparticle systems and the Dirac equation.

Ever since Pauli’s introduction of “a peculiar, not classically describable two-valuedness” [23] the nature of spin — an intrinsic, or internal, or inherent angular momentum — has been the subject of numerous studies. The interpretations range from the outright denial of the possibility of visualizable models to a wide variety of alternatives. One possibility is that the particle is a small spinning body described either by Bohmian mechanics [24][25], or by stochastic mechanics [26], or even by entropic dynamics [27]. There are also hydrodynamic models in which a particle-like inhomogeneity is carried along by a fluid composed of rigid spinning bodies [28][29]. Yet another approach is to attribute the spin to the motion of the point particle guided by a pilot wave represented either by a Pauli wave function [30], or by a Dirac wave function [24][31], or by a real spinor function expressed in the language of geometric algebra [32][33]. In all of these models spin is described as an ontic variable attributed either to the spinning particle, or to the particle’s motion, or to the ontic wave function. It goes without saying that all of these models have been met with some degree of success, but they share those characteristics of the hydrodynamic, Bohmian, or stochastic models, that have traditionally stood in the way of their broader acceptance. In particular, they provide no insights into the question of why does QM take the particular form it does.

Our purpose here is to formulate a nonrelativistic ED model in which spin is a purely epistemic property. We deal with a point particle the position of which is the only ontic variable. Spin is purely a property of the wave function.

The goal is to derive the Pauli equation. As mentioned above ED consists in the entropic updating of probabilities with information supplied by constraints. The challenge is to identify those constraints which correspond to spin and, in particular, to establish a clear criterion for how the constraints themselves are updated. The argument here generalizes our previous discussion for scalar particles [9]. The idea, which stresses the central importance of symmetry principles when implemented in conjunction with concepts of information geometry, is to identify the appropriate symplectic and metric structures in the phase space of probabilities, their conjugate momenta, and the spin variables. This construction yields a derivation of the Fubini-Study metric for a spin-1/2 particle which highlights its deep connection to information geometry. A welcome by-product is that the joint presence of a symplectic and a metric structure leads to a
complex structure which in this case receives a particularly elegant geometric algebra interpretation: the imaginary unit coincides with the pseudoscalar of Euclidean space.

Once these structures are in place we can formulate the desired geometric criterion for updating constraints: we construct an ED that preserves both the symplectic structure (a Hamiltonian flow) and the metric structure (a Killing flow). The result of these Hamiltonian-Killing flows is an Entropic Dynamics described by the linear Pauli equation.

The paper is structured as follows. In Section 2 we provide a review of the Geometric Algebra approach to spinors. This serves to emphasize that, while the spin of the electron is a quantum property, the spinors considered as mathematical objects have nothing to do with quantum theory. It also serves to collect in one place various equations that will be of later use and to establish the notation. In Section 3 we introduce the main ideas behind ED. These include the choice of constraints that allow us to assign the probability of a short step and the introduction of the notion of entropic time as a tool to keep track of how these short steps accumulate. The geometric criteria for updating the constraints so that the dynamics preserves the appropriate symplectic and metric structures are discussed in Sections 4 and 5 respectively. In Section 6 we discuss generic Hamilton-Killing flows and the Pauli equation is derived Section 7. In Section 8 we discuss the interpretation of the formalism; the definitions of energy, momentum, and spin; the Stern-Gerlach experiment; possible interactions; and the introduction of a Hilbert space as a convenient tool. Section 9 summarizes results and includes some final comments.

2 Brief review of geometric algebra

Geometric Algebra (GA) is Clifford algebra with an added geometric interpretation. It provides an efficient way to represent rotations and therefore it is particularly well suited for the study of spin. There is a vast literature devoted to the study of GA; an excellent introduction is the classic text by David Hestenes [34] (see also [35]).

2.1 Multivectors and the geometric product

The geometric algebra of Euclidean three-dimensional space — the Pauli algebra \( G_3 \) — is a graded algebra. Its elements, called multivectors, include real scalars (grade 0), directed lengths or vectors (grade 1), directed areas or bivectors (grade 2), and directed volumes or pseudoscalars (grade 3). A generic multivector consists of sums of elements of different grades. The central concept behind geometric algebra is the geometric product. The product of two vectors \( \vec{a} \) and \( \vec{b} \) is defined by

\[
\vec{a} \vec{b} = \vec{a} \cdot \vec{b} + \vec{a} \wedge \vec{b} .
\]  

4Specifically, it is a Clifford algebra in which the bilinear form of interest is a metric on the underlying vector space.
The inner product, $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} = \delta_{ab}$, is the familiar scalar dot product. The outer product or wedge product, $\vec{a} \wedge \vec{b} = -\vec{b} \wedge \vec{a}$, is a bivector and represents the directed area defined by $\vec{a}$ and $\vec{b}$. The notion of adding objects of different grades can at first appear strange but it turns out to be quite analogous to the addition of real and imaginary numbers: just as one manipulates complex numbers by keeping track of their real and imaginary parts one manipulates multivectors such as $\vec{a} \vec{b}$ by keeping track of their components of different grades. Since

$$\vec{a} \cdot \vec{b} = \frac{1}{2} \left( \vec{a} \vec{b} + \vec{b} \vec{a} \right) \quad \text{and} \quad \vec{a} \wedge \vec{b} = \frac{1}{2} \left( \vec{a} \vec{b} - \vec{b} \vec{a} \right),$$

(2)

the product of two vectors $\vec{a} \vec{b}$ can be considered to be the more basic operation in terms of which the dot and wedge products are defined.

The generalization of the geometric product to products of generic multivectors that turns out to be the most fruitful is achieved by requiring that the geometric product be associative — for example, $(\vec{a} \vec{b}) \vec{c} = \vec{a} (\vec{b} \vec{c})$. The corresponding wedge and dot products are introduced as follows. Let $B = \vec{b} \wedge \vec{c}$ be a bivector, then

$$\vec{a} B = \frac{1}{2} (\vec{a} B + B \vec{a}) + \frac{1}{2} (\vec{a} B - B \vec{a})$$

(3)

suggests the definitions

$$\vec{a} \wedge B = \frac{1}{2} (\vec{a} B + B \vec{a}) \quad \text{and} \quad \vec{a} \cdot B = \frac{1}{2} (\vec{a} B - B \vec{a})$$

(4)

which implies that the wedge product $\vec{a} \wedge \vec{b} \wedge \vec{c}$ is also associative,

$$\vec{a} \wedge (\vec{b} \wedge \vec{c}) = (\vec{a} \wedge \vec{b}) \wedge \vec{c} = \vec{a} \wedge \vec{b} \wedge \vec{c}.$$  

(5)

The grade three object $\vec{a} \wedge \vec{b} \wedge \vec{c}$ can then be interpreted as the directed volume defined by $\vec{a}$, $\vec{b}$ and $\vec{c}$. The elements of maximum grade, which in three dimensions is grade three, are called pseudoscalars.

Introducing an orthonormal basis $\{\vec{e}_a\}$,

$$\vec{e}_a \cdot \vec{e}_b = \frac{1}{2} (\vec{e}_a \vec{e}_b + \vec{e}_b \vec{e}_a) = \delta_{ab},$$

(6)

allows a generic vector $\vec{a}$ to be expressed as $\vec{a} = a^a \vec{e}_a$, where we sum over repeated indices.

\[\text{In a matrix representation of the Pauli algebra } G_3 \text{ the basis}\]

\[\text{indeed, this allows us to divide by vectors: if } \]

\[\vec{a} \vec{b} = \vec{c} \quad \text{then} \quad \vec{b} = (\vec{a})^{-1} \vec{c} \]

where $(\vec{a})^{-1} = \vec{a}/a^2$.

The inner product $\vec{a} \cdot B$, yields a grade one object

$$-\vec{a} \cdot (\vec{b} \wedge \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b}) = \vec{a} \times (\vec{b} \times \vec{c}).$$

In general the characteristic equation for a GA with respect to some metric $g_{ab}$ is

$$\vec{e}_a \vec{e}_b + \vec{e}_b \vec{e}_a = 2 g_{ab}.$$

In eq. (6) and throughout the rest of the paper we assume that the metric of interest is the three-dimensional Euclidean metric $g_{ab} = \delta_{ab}$. 

5
vectors $\vec{e}_a$ are represented by the Pauli matrices $\hat{\sigma}_a$, which we write as $\vec{e}_a \sim \hat{\sigma}_a$, and the geometric product is represented by the matrix product. Then (6) takes the familiar form
\[
\hat{\sigma}_a \hat{\sigma}_b + \hat{\sigma}_b \hat{\sigma}_a = 2\delta_{ab} \hat{1}.
\] (7)

In three dimensions all pseudoscalars are multiples of
\[
\vec{e}_1 \wedge \vec{e}_2 \wedge \vec{e}_3 = \vec{e}_1 \vec{e}_2 \vec{e}_3.
\] (8)

This quantity is of particular interest because its square is $-1$,
\[(\vec{e}_1 \vec{e}_2 \vec{e}_3)^2 = -1,
\] (9)
and commutes with all other elements in the algebra. Therefore, the pseudoscalar $\vec{e}_1 \vec{e}_2 \vec{e}_3$ is indistinguishable from the imaginary unit $i$ with the added advantage that it has a geometrical interpretation as a directed unit volume. Accordingly, we shall adopt the notation
\[
\vec{e}_1 \vec{e}_2 \vec{e}_3 = i \quad \text{or} \quad \vec{e}_a \vec{e}_b \vec{e}_c = i \varepsilon_{abc}.
\] (10)

In a matrix representation the pseudoscalar $\vec{e}_1 \vec{e}_2 \vec{e}_3$ is represented by $\hat{\sigma}_1 \hat{\sigma}_2 \hat{\sigma}_3 = i\hat{1}$. The basis bivectors will can be written as
\[
\vec{e}_1 \vec{e}_2 = i\vec{e}_3, \quad \vec{e}_2 \vec{e}_3 = i\vec{e}_1, \quad \text{and} \quad \vec{e}_3 \vec{e}_1 = i\vec{e}_2.
\] (11)

Therefore in three dimensions a generic bivector $B$ can always be written in the form of an “imaginary” vector,
\[
B = i\vec{b},
\] where $\vec{a}$ and $\vec{b}$ are (real) scalars and $\vec{a}$ and $\vec{b}$ are (real) vectors. We adopt the standard notation of denoting the grade $k$ part by $\langle \ldots \rangle_k$,
\[
\langle A \rangle_0 = \alpha, \quad \langle A \rangle_1 = \vec{a}, \quad \langle A \rangle_2 = i\vec{b}, \quad \langle A \rangle_3 = i\beta.
\] (14)

2.1.1 Involutions

There are three involutions — operations that when repeated yield the identity. One is the operation of reversion which consists of reversing the order of all products of vectors. In the Pauli algebra $\mathcal{G}_3$ this operation is denoted by a dagger $\dagger$ and is defined by
\[
(AB)^\dagger = B^\dagger A^\dagger \quad \text{and} \quad \vec{a}^\dagger = \vec{a}.
\] (15)

\[\text{This statement is valid in three dimensions; it does not generalize to all geometric algebras.}\]
In three dimensions we find that $i^\dagger = -i$ so that taking the reverse of (13),

$$A^\dagger = \alpha + \vec{a} - i\vec{b} - i\beta,$$

(16)

amounts to taking its Hermitian adjoint. The other involution is the operation of spatial inversion denoted by $*$ and defined by

$$(AB)^* = A^* B^* \quad \text{and} \quad \vec{a}^* = -\vec{a}.$$ (17)

In three dimensions this implies $i^* = -i$ and the spatial inverse of (13) is

$$A^* = \alpha - \vec{a} + i\vec{b} - i\beta.$$ (18)

Yet a third involution is obtained by taking both the reverse and the inverse, $A^{\dagger*}$.

2.1.2 The multivector derivative

We shall find it useful to take derivatives $\partial/\partial X = \partial_X$ with respect to a multivector $X$ [35][37]. To define $\partial_X$ we first introduce the scalar derivative $\langle A \partial_X \rangle_0$ of a function $F(X)$ in the “direction” of the multivector $A$,

$$\langle A \partial_X \rangle_0 F(X) \overset{\text{def}}{=} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[ F(X + \varepsilon A) - F(X) \right].$$ (19)

To ensure that the derivative $\partial_X$ has the correct transformation properties we define the reciprocal basis $\{\vec{e}^a\}$ as follows. Introduce the metric tensor $g_{ab}$ and its inverse $g^{ab}$,

$$\vec{e}_a \cdot \vec{e}_b = g_{ab} \quad \text{and} \quad g^{ab} g_{bc} = \delta^a_c.$$ (20)

Then the reciprocal basis vectors are given by

$$\vec{e}^a \overset{\text{def}}{=} g^{ab} \vec{e}_b \quad \text{so that} \quad \vec{e}^a \cdot \vec{e}_c = \delta^a_c.$$ (21)

In this paper we shall restrict ourselves to orthonormal bases, $\vec{e}_a \cdot \vec{e}_b = \delta_{ab}$, in which case the direct and reciprocal bases coincide, $\vec{e}^a = \vec{e}_a$.

Finally, just as the multivector $A$, eq.(13), can be expanded as

$$A = \langle A \rangle_0 + \vec{e}_a (e^{a\dagger} A)_{0} + i\vec{e}_a (i e^{a\dagger})_{0} + i i^0 A_{0},$$ (22)

then the multivector derivative $\partial_X$ with respect to

$$X = x_0 + \vec{e}_a x^a + i\vec{e}_a y^a + iy_0,$$ (23)

is defined through its expansion in the reciprocal basis,

$$\partial_X \overset{\text{def}}{=} \langle \partial_X \rangle_0 + e^a (\vec{e}_a \partial_X)_{0} + i e^a (i\vec{e}_a)_{0} \partial_X + i i^0 \partial_X_{0},$$ (24)

which, using (19), gives

$$\partial_X = \frac{\partial}{\partial x_0} + e^a \frac{\partial}{\partial x^a} - i e^a \frac{\partial}{\partial y^a} - i \frac{\partial}{\partial y_0}.$$ (25)
We can check that this includes the derivative with respect to a complex variable
\[ z = x_0 + iy_0 \] as a special case,
\[ \frac{\partial}{\partial z} = \frac{\partial}{\partial x_0} - i \frac{\partial}{\partial y_0}. \] (26)

Most of the derivatives we shall need below can be computed using the Leibniz rule for the derivative of a product and the formula
\[ \partial_X \langle XA \rangle_0 = \langle A \rangle_X X \] (27)
where \( \langle A \rangle_X \) denotes the projection of \( A \) onto those grades contained in \( X \).

2.2 Spinors

One effective way to define spinors in GA is as elements of a minimal left ideal — a subspace that remains invariant under multiplication from the left and does not itself contain invariant subspaces; see \[34\]. The decomposition into ideals is achieved by selecting an arbitrary unit vector, say \( \vec{e}_3 \), and rewriting any generic multivector \( \Psi \) in the form,
\[ \Psi = \Psi_1 + \frac{1}{2} \vec{e}_3. \] (28)

Multivectors such as \((1 + \vec{e})/2\) and \((1 - \vec{e})/2\) where \( \vec{e} \) is any unit vector are projectors — they are idempotent and orthogonal. The set \( \mathcal{I}_+ = \{ \Psi_+ \} \) of all multivectors \( \Psi_+ \) of the form
\[ \Psi_+ = \Psi_1 + \frac{1}{2} \vec{e}_3 \] (29)
is a left ideal; its elements will be called spinors. The set \( \mathcal{I}_- \) of elements of the form \( \Psi(1 - \vec{e}_3)/2 \) constitute an independent left ideal.

A basis for the spinors \( \Psi_+ \) is given by
\[ u_+ = \frac{1}{\sqrt{2}} (1 + \vec{e}_3) \quad \text{and} \quad u_- = \vec{e}_1 u_+ = \frac{1}{\sqrt{2}} \vec{e}_1 (1 + \vec{e}_3). \] (30)
The factors of \( \sqrt{2} \) are chosen so that these basis spinors are normalized according to
\[ \langle u_A^+ u_B \rangle_0 = \delta_{AB}. \] (31)

To show that \( \{ u_+, u_- \} \) form a basis note that using \[13\] the generic spinor \[29\] can be written as
\[ \Psi_+ = (c^0 + c^1 \vec{e}_1 + c^2 \vec{e}_2 + c^3 \vec{e}_3) u_+ \] (32)
where the \( c^\alpha \) are “complex” numbers, that is, \( \langle c^\alpha \rangle_{0+3} = c^\alpha \). Then, using \( \vec{e}_3 u_+ = u_+, \vec{e}_1 u_+ = u_- , \text{ and } \vec{e}_2 u_+ = i u_- \), we can further rewrite \( \Psi_+ \) in the alternative form,
\[ \Psi_+ = (c^0 + c^3) u_+ + (c^1 + i c^2) u_- \] (33)
In a matrix representation of $G_3$ the matrices of a minimal left ideal are characterized by a single non-vanishing column: using $\vec{e}_k \sim \hat{\sigma}_k$, we have

$$u_+ \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad u_- \sim \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$ \hspace{1cm} (34)

The action of $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ on $u_+$ and $u_-$ is precisely the action of the Pauli matrices $\hat{\sigma}_1$, $\hat{\sigma}_2$, and $\hat{\sigma}_3$ on the eigenvectors of $\hat{\sigma}_3$,

$$\vec{e}_1 u_+ = u_-; \quad \vec{e}_2 u_+ = i u_-; \quad \vec{e}_3 u_+ = u_+ \quad \vec{e}_1 u_- = u_+; \quad \vec{e}_2 u_- = -i u_+; \quad \vec{e}_3 u_- = -u_- .$$ \hspace{1cm} (35)

### 2.2.1 Rotations and rotors

Rotating a vector $\vec{r}$ by an angle $\theta$ about an axis described by the unit vector $\vec{n}$ gives a new vector $\vec{r}' = U \vec{r} U^\dagger$ where $U = \exp(-i\vec{n}\theta/2)$.

The proof is straightforward: just express $\vec{r}'$ in terms of its components parallel and perpendicular to $\vec{n}$ and use the identity

$$\exp(-i\vec{n}\theta/2) = \cos \frac{\theta}{2} - i\vec{n} \sin \frac{\theta}{2} .$$ \hspace{1cm} (36)

Thus unitary multivectors such as $U$ provide us with an elegant tool to represent rotations and are appropriately called rotors. In the usual matrix representation, $\vec{e}_k \sim \hat{\sigma}_k$, we have

$$U \sim \hat{U} = \exp(-i\vec{n}_k \hat{\sigma}_k \theta/2) .$$ \hspace{1cm} (37)

Most interesting is the appearance of half angles: since $U$ and $-U$ represent the same rotation the rotors provide a two-to-one representation of the rotation group, i.e., the rotors $U$ are elements of the corresponding spin group, $\text{Spin}(3)$. Furthermore, since the grade of $i\vec{n}$ is even, the rotor $\hat{U}$ belongs to the quaternion subalgebra. (The multivectors of even grade, i.e., $G_3^+ \overset{\text{def}}{=} (G_3)_{0+2}$, are also known as quaternions.)

There is a one-to-one correspondence between spinors in $G_3$ and quaternions. Indeed, using $\vec{e}_3 u_+ = u_+$, a generic spinor,

$$\Psi = (\alpha + \vec{a} + \vec{b} + i\beta)u_+ ,$$ \hspace{1cm} (38)

(from here on we drop the subscript $+$ and write $\Psi_+ = \Psi$) can be expressed as

$$\Psi = (\alpha + \vec{a} \vec{e}_3 + \vec{b} + i\vec{e}_3 \beta)u_+ .$$ \hspace{1cm} (39)

The four terms in the parenthesis belong to the even subalgebra and therefore, any spinor $\Psi$ can be written as

$$\Psi = \Upsilon u_+$$ \hspace{1cm} (40)
The orientation of an arbitrary frame \((x', y', z')\) relative to a standard frame \((x, y, z)\) is specified through Euler angles \(\{\theta, \phi, \chi\}\). The construction proceeds as follows. Rotate the original standard frame \((x, y, z)\) by an angle \(\theta\) about the \(\vec{e}_x = \vec{e}_2\) axis to a new frame \((x_1, y_1, z_1)\). Next rotate by an angle \(\phi\) about the original \(\vec{e}_y = \vec{e}_3\) axis to a second frame \((x_2, y_2, z_2)\). Finally rotate by an angle \(\chi\) about the \(\vec{e}_{z_2}\) axis to get the final primed frame \((x', y', z')\). (In all rotations the positive sense is given by the right hand rule or \textit{counterclockwise}.)

The composite rotation is

\[
U(\theta, \phi, \chi) = U_{z_2}(\chi)U_z(\phi)U_y(\theta) = e^{-i\vec{e}_z\chi/2}e^{-i\vec{e}_x\phi/2}e^{-i\vec{e}_y\theta/2}.
\]

It is convenient to express \(U(\theta, \phi, \chi)\) in terms of rotations about the original standard frame axes.

The result is

\[
U(\theta, \phi, \chi) = U_z(\phi)U_y(\theta)U_z(\chi) = e^{-i\vec{e}_z\phi/2}e^{-i\vec{e}_y\theta/2}e^{-i\vec{e}_x\chi/2}.
\]

A useful expression for \(U(\theta, \phi, \chi)\) is obtained by commuting the first factor past the second to get

\[
U(\theta, \phi, \chi) = e^{-i\vec{e}_z(\chi+\phi)/2} \cos \frac{\theta}{2} - i\vec{e}_2 e^{-i\vec{e}_3(\chi-\phi)/2} \sin \frac{\theta}{2},
\]

which, using \(-i\vec{e}_2 u_+ = u_-\), allows us to write a generic normalized spinor \(u\) as

\[
u = U(\theta, \phi, \chi)u_+ = u_+ e^{-i(\chi+\phi)/2} \cos \frac{\theta}{2} + u_- e^{-i(\chi-\phi)/2} \sin \frac{\theta}{2}.
\]

The orthogonal spinor \(u'\) is

\[
u' = U(\theta, \phi, \chi)u_- = -u_+ e^{i(\chi-\phi)/2} \sin \frac{\theta}{2} + u_- e^{i(\chi+\phi)/2} \cos \frac{\theta}{2}.
\]

\(\text{The convention we adopt for the Euler angles is common but by no means universal. Holland \cite{Holland}, for example, adopts a different convention.}\)
2.3 Rotors and the spin frame

Let the three orthonormal vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$ be the standard basis for Euclidean space — the laboratory basis. Rotate the standard frame with different rotors $U(\vec{x})$ at different points $\vec{x}$ in space. The result is a frame field $\{\vec{s}_1, \vec{s}_2, \vec{s}_3\}$ with

$$\vec{s}_k(\vec{x}) = U(\vec{x})\vec{e}_k U^\dagger(\vec{x}) ,$$  

which we will call the spin frame or spin triad. The orientation of the spin frame relative to the standard frame $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is specified through the local Euler angles $\{\theta(\vec{x}), \phi(\vec{x}), \chi(\vec{x})\}$. From (45) it is clear that the angles $\theta$ and $\phi$ are the polar angles of $\vec{s}_3$ while the angle $\chi$ describes the rotation of $\vec{s}_1$ and $\vec{s}_2$ about the $\vec{s}_3$ axis. From now on we drop the label 3 and the vector $\vec{s}_3 = \vec{s}$ will be called the spin vector,

$$\vec{s} = U\vec{e}_3 U^\dagger = \sin \theta \cos \phi \vec{e}_1 + \sin \theta \sin \phi \vec{e}_2 + \cos \theta \vec{e}_3 .$$  

(51)

2.3.1 Phase transformations

For future reference we consider the effect of multiplying the spinor $\Psi = \Upsilon u_+$ by a phase factor $e^{-i\alpha/2}$ where $\alpha$ is a real scalar. Specifically we write the modified spinor $\Psi' = e^{-i\alpha/2}\Psi$ in its polar form, that is, as an even multivector $\Upsilon' = |\Upsilon'| U'$ operating on the standard spinor $u_+$,

$$e^{-i\alpha/2}\Psi = \Upsilon \left( \cos \frac{\alpha}{2} - i \sin \frac{\alpha}{2} \right) u_+ = \Upsilon \left[ \cos \frac{\alpha}{2} - ie_3^r \sin \frac{\alpha}{2} \right] u_+ = \Upsilon e^{-i\vec{e}_3 \alpha/2} u_+ .$$

More explicitly, from (46), if

$$\Psi = |\Upsilon| U(\theta, \phi, \chi) u_+ , \quad \text{then} \quad e^{-i\alpha/2}\Psi = |\Upsilon| U(\theta, \phi, \chi + \alpha) u_+ .$$

(52)

In words, multiplication by a phase factor, $\Psi' = e^{-i\alpha/2}\Psi$, corresponds to an additional rotation of $\vec{s}_1$ and $\vec{s}_2$ about the spin vector $\vec{s}$ by the angle $\alpha$.

2.3.2 Infinitesimal rotations

The effect of a change $\delta U$ in the rotor $U$ is to induce an additional infinitesimal rotation by angles $\{\delta \theta, \delta \phi, \delta \chi\}$. To see this start from $UU^\dagger = 1$ so that

$$(\delta U)U^\dagger + U\delta U^\dagger = 0 \quad \text{or} \quad \delta U = -(U\delta U^\dagger)U .$$

(53)

Next we note that $U\delta U^\dagger$ is a bivector — its grade is even ($U$ is even) and its scalar part $(U\delta U^\dagger)_0$ vanishes because $(U\delta U^\dagger)^\dagger = -U\delta U^\dagger$. Therefore we can write $U\delta U^\dagger$ as an imaginary vector,

$$U\delta U^\dagger = i\delta \vec{\zeta} \quad \text{so that} \quad \delta U = -i\delta \vec{\zeta} U .$$

(54)

To find the meaning of $\delta \vec{\zeta}$ consider the rotor

$$U + \delta U = \left( 1 - \frac{i}{2} \delta \vec{\zeta} \right) U = e^{-\frac{i}{2}\delta \vec{\zeta} U} .$$

(55)
Thus, $\delta \vec{\zeta}$ represents an infinitesimal rotation by the angle $|\delta \vec{\zeta}|$ about the axis $\vec{n}_t = \delta \vec{\zeta} / |\delta \vec{\zeta}|$. Using eq. (46) to evaluate $(\delta U)U^\dagger$ gives

$$\delta \vec{\zeta} = \vec{e}_3 \delta \phi + \vec{e}_\phi \delta \theta + \vec{s} \delta \chi ,$$

(56)

where

$$\vec{e}_\phi = -\sin \phi \vec{e}_1 + \cos \phi \vec{e}_2$$

(57)

and $\vec{s}$ is given in (51).

Next we look at two special cases of $\delta \vec{\zeta}$ which lead to the concepts of angular velocity and of the spin connection.

**Angular velocity** — The rotor and the spin frame can evolve over time and this change is conveniently expressed in terms of an angular velocity vector $\vec{\omega}_t$. The infinitesimal rotation described by

$$\delta \vec{\zeta} = \vec{\omega}_t \delta t$$

where

$$\vec{\omega}_t = \vec{e}_3 \partial_t \phi + \vec{e}_\phi \partial_t \theta + \vec{s} \partial_t \chi$$

(58)

leads to

$$\partial_t U = -\frac{i}{2} \vec{\omega}_t U$$

(59)

and

$$\partial_t \vec{s}_k = \partial_t (U \vec{e}_k U^\dagger) = i \vec{s}_k \wedge \vec{\omega}_t = \vec{\omega}_t \times \vec{s}_k .$$

(60)

**Spatial derivatives** — The spatial derivatives of the rotor $U$ and of the spin frame are conveniently expressed by introducing three bivectors $\Omega_a$ and their duals, the three vectors $\vec{\omega}_a$,

$$\Omega_a = 2U \partial_a U^\dagger = -2(\partial_a U)U^\dagger = i \vec{\omega}_a ,$$

(61)

where $\partial_a = \partial / \partial x^a$. The rotor at $\vec{x} + \delta \vec{x}$ is

$$U(\vec{x} + \delta \vec{x}) = U(\vec{x}) + \delta x^a \partial_a U = \left(1 - \frac{i}{2} \vec{\omega}_a \delta x^a\right)U(\vec{x}) = e^{-\frac{i}{2} \vec{\omega}_a \delta x^a} U(\vec{x}) .$$

(62)

Thus, as one moves from $\vec{x}$ to $\vec{x} + \delta \vec{x}$ the spin frame is rotated by the infinitesimal angle $|\vec{\omega}_a \delta x^a|$. Therefore,

$$\delta \vec{\zeta} = \vec{\omega}_a \delta x^a$$

where

$$\vec{\omega}_a = \vec{e}_3 \partial_a \phi + \vec{e}_\phi \partial_a \theta + \vec{s} \partial_a \chi$$

(63)

which leads to

$$\partial_a U = -\frac{1}{2} \Omega_a U = -\frac{1}{2} i \vec{\omega}_a U$$

(64)

and to the analogue of (60),

$$\partial_a \vec{s}_k = \vec{\omega}_a \times \vec{s}_k .$$

(65)

The corresponding angles $\delta \zeta^k$ of rotation about the instantaneous axes $\vec{s}_k$ are

$$\delta \zeta^k = \delta \vec{\zeta} \cdot \vec{s}_k .$$

(66)
Therefore \( \delta \zeta^k = \delta \zeta^k \mathbf{s}_k \) and \( \mathbf{\omega}_a = (\partial_a \zeta^k) \mathbf{s}_k \).

The components of \( \mathbf{\omega}_t \) and \( \mathbf{\omega}_a \) along the spin vector \( \mathbf{s} \),
\[
\mathbf{\omega}_t \cdot \mathbf{s} = \partial_t \zeta^3 = \partial_t \chi + \cos \theta \partial_t \phi,
\]
\[
\mathbf{\omega}_a \cdot \mathbf{s} = \partial_a \zeta^3 = \partial_a \chi + \cos \theta \partial_a \phi,
\]
will play an important role; they capture information about energy and momentum. To see how this comes about we write \( \mathbf{\omega}_t \) and \( \mathbf{\omega}_a \) in terms of the spinor \( u = U u \)
\[
\hbar^2 \mathbf{\omega}_t \cdot \mathbf{s} = \langle u^+ i \hbar \partial_t u \rangle_0 \quad \text{and} \quad -\hbar^2 \mathbf{\omega}_a \cdot \mathbf{s} = \langle u^+ i \hbar \partial_a u \rangle_0.
\]
The proof is straightforward. For example, using (48), (63), and the cyclic identity \( \langle ABC \rangle_0 = \langle BCA \rangle_0 \), we have
\[
\langle u^+ i \partial_a u \rangle_0 = (u^+ U \partial_a U u)_+ = \langle (i \partial_a U)(1 + \epsilon_3)U^t \rangle_0 = \frac{1}{2} \langle \mathbf{\omega}_a U(1 + \epsilon_3)U^t \rangle_0 = \frac{1}{2} \langle \mathbf{\omega}_a \mathbf{s} \rangle_0 = \frac{1}{2} \langle \mathbf{\omega}_a \cdot \mathbf{s} \rangle.
\]

The spin connection and its curvature — The bivectors \( \Omega_a \) (or their duals \( \mathbf{\omega}_a \)) turn out to be what is called the spin connection. To justify the name write (65) as
\[
\partial_a \mathbf{s}_j = \omega^m_a \mathbf{s}_m \times \mathbf{s}_j = \omega^m_a \varepsilon_{mijn} \mathbf{s}_i \overset{\text{def}}{=} \Omega^i_{ja} \mathbf{s}_i
\]
where we introduced the (mixed index) spin connection coefficient \( \Omega^i_{ja} \),
\[
\Omega^i_{ja} = \varepsilon_{mijn} \omega^m_a \mathbf{s}_i \quad \text{or} \quad \Omega_{ija} = -\varepsilon_{ijk} \omega^k_a.
\]
The condition that partial derivatives commute, \( \partial_a \partial_b U = \partial_b \partial_a U \), implies that the derivatives of the \( \mathbf{\omega}_a \) are not independent. They obey certain compatibility conditions that can be expressed in terms of the six curvature bivectors \( R_{ab} \),
\[
R_{ab} = i (\partial_a \mathbf{\omega}_b - \partial_b \mathbf{\omega}_a + i \mathbf{\omega}_a \wedge \mathbf{\omega}_b) = 0.
\]
In other words: the curvature of Euclidean space vanishes. The analogous compatibility condition relating time and space derivatives is
\[
\partial_a \mathbf{\omega}_t - \partial_t \mathbf{\omega}_a + i \mathbf{\omega}_a \wedge \mathbf{\omega}_t = 0.
\]

Topological invariants — Additional evidence that the quantities (68) codify relevant rotational information is provided by the fact that since \( \delta \zeta^3 = \delta \zeta^j \mathbf{s}_j \) is the angle of rotation.

\(^{10}\)Quantities such as \( \delta \zeta^k \) and \( \partial_a \zeta^k \) are well defined functions of \( x \) but they cannot be integrated to yield functions \( \zeta^k(x) \). Nevertheless, when discussing non-holonomic constraints it is often useful to introduce the symbols \( \zeta^k \) which are sometimes called pseudo-coordinates.
about the spin vector $\vec{s}$ in a displacement $d\vec{\ell}$, the total angle of rotation accumulated in a closed loop $\Gamma$ gives rise to a topological invariant,

$$\oint_{\Gamma} \delta \xi^3 = 2\pi n_{\Gamma},$$

or

$$\oint_{\Gamma} d\ell^a \langle u^i \partial_a u \rangle_0 = \oint_{\Gamma} d\ell^a \frac{1}{2} (\partial_a \chi + \cos \theta \partial_a \phi) = \pi n_{\Gamma},$$

(77)

where $n_{\Gamma}$ is an integer that depends on the loop. This topological invariant will eventually allow us to conclude that the circulation integrals that have historically been proposed as "quantization" conditions are not of quantum origin but can instead be given a purely geometrical interpretation.

## 3 Entropic dynamics of a spin-1/2 particle

Our goal is to study the evolution of the probability $\rho(x)$ of the position $x = (x^1, x^2, x^3) \in \mathbf{X}$ of a point particle that lives in a three-dimensional Euclidean space $\mathbf{X}$. In contrast to the standard Copenhagen interpretation in our model the particle has a definite position at all times. Furthermore, in contrast to other approaches to spin, we deal with a point particle and not with a small rotating body. In the ED approach spin is a property of the particle’s motion and not a property of its internal structure.

The first step in defining the dynamics is to construct the probability $P(x'|x)$ that the particle moves from a point $x$ to a neighboring point $x'$. This is done by maximizing the entropy,

$$S[P,Q] = -\int d^3 x' P(x'|x) \log \frac{P(x'|x)}{Q(x'|x)},$$

(78)

of $P(x'|x)$ relative to a prior $Q(x'|x)$ and subject to the appropriate constraints that incorporate the relevant information about the motion. It is through the prior and the constraints that the “physical” information is introduced.

### 3.1 The prior

The main dynamical assumption is that the particle follows a continuous path. This allows us to analyze a generic motion as the accumulation of many infinitesimally short steps. We shall adopt a prior that forces the particle to take short steps but is otherwise maximally uninformative. Such a prior can itself be derived using the principle of maximum entropy. Indeed, maximize

$$S[Q] = -\int d^3 x' Q(x'|x) \log \frac{Q(x'|x)}{\mu(x'|x)},$$

(79)

relative to the uniform measure $\mu$ subject to normalization, and subject to a constraint that imposes short steps and implements the translational and

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11In Cartesian coordinates the uniform measure $\mu$ is a constant that may be ignored.
rotational invariance of space. The constraint is
\[ \langle \delta_{ab} \Delta x^a \Delta x^b \rangle = \kappa , \] (80)
where \( \Delta x^a = x'^a - x^a \) and \( \kappa \) is a small constant. The result is a Gaussian,
\[ Q(x'|x) \propto \exp \left( -\frac{1}{2} \alpha \delta_{ab} \Delta x^a \Delta x^b \right) , \] (81)
where \( \alpha \) is a Lagrange multiplier that will eventually be taken to infinity in order to enforce the fact that the steps are meant to be infinitesimally short.

3.2 The constraints

The distribution (81) leads to a diffusion process that is isotropic. We seek to generate a motion that exhibits directionality, the peculiar rotational properties attributed to spin, and also, that describes the appropriate response to electromagnetic fields. The relevant information is incorporated by imposing additional constraints. One can reasonably expect that the dynamics of particles with spin should, at least in those situations when the spin happens to be uniform and constant, resemble the dynamics of scalar particles. Accordingly, we shall construct the ED of a particle with spin in close analogy to the ED of scalar particles [9].

The drift potential constraint — Just as for scalar particles the physical information that the motion can be directional is introduced through a constraint expressed in terms of a drift potential \( \varphi(x) \). The latter is a multi-valued function on configuration space with the topological properties of an angle — \( \varphi \) and \( \varphi + 2\pi \) represent the same angle. The constraint consists of requiring that the displacement \( \Delta x^a \) be such that the expected change of the drift potential \( \langle \Delta \varphi \rangle \) be a small quantity \( \kappa' \) to be specified below,
\[ \langle \Delta \varphi \rangle = \langle \Delta x^a \rangle \frac{\partial \varphi}{\partial x^a} = \kappa' . \] (82)

The gauge constraint — Just as for scalar particles, to incorporate the effect of an external electromagnetic field we impose that the expected displacement \( \langle \Delta x^a \rangle \) satisfy
\[ \langle \Delta x^a \rangle A_a(x) = \kappa'' , \] (83)
where \( A_a(x) \) is the electromagnetic vector potential and \( \kappa'' \) is yet another small quantity to be specified below.

The spin constraint — The rotational properties of spin 1/2 are obtained by imposing an additional constraint involving “potentials” associated to a rotor field \( U(\theta, \phi, \chi) \) or equivalently to a spin frame field \( \{ \mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3 \} \). The relevant information is conveyed by the (pseudo-) gradient \( \partial_a \zeta^3 \) given by eq. (69) or (70).
\[ \langle \Delta x^a \rangle (u^i i \partial_a u)_0 = \langle \Delta x^a \rangle \frac{1}{2} (\partial_a \chi + \cos \theta \partial_a \phi) = \kappa''' . \] (84)

The quantities \( \kappa' \), \( \kappa'' \), and \( \kappa''' \) in (82), (83), and (84) can be specified directly — just state what \( \kappa' \), \( \kappa'' \), and \( \kappa''' \) are — or indirectly, by specifying the corresponding Lagrange multipliers. We shall adopt the latter alternative because it turns out to be much more convenient.

3.3 The probability of a short step

Maximizing the entropy (78) subject to the constraints (82-84) plus normalization leads to the transition probability

\[ P(x'|x) \propto \exp \left[ -\frac{\alpha}{2} \Delta x^a \Delta x^b \delta_{ab} + \alpha' \Delta x^a [\partial_a \varphi - \beta A_a - \gamma (\partial_a \chi + \cos \theta \partial_a \phi)] \right] , \] (85)

where \( \alpha' \), \( \beta \), and \( \gamma \) are Lagrange multipliers. Being Gaussian \( P(x'|x) \) is more conveniently rewritten as

\[ P(x'|x) \propto \exp \left[ -\frac{\alpha}{2} \delta_{ab} (\Delta x^a - \bar{\Delta} x^a)(\Delta x^b - \bar{\Delta} x^b) \right] . \] (86)

The generic displacement \( \Delta x^a = x'^a - x^a \), can be written as

\[ \Delta x^a = \bar{\Delta} x^a + \Delta w^a , \] (87)

where \( \bar{\Delta} x^a = \langle \Delta x^a \rangle \) is the drift

\[ \langle \Delta x^a \rangle = \frac{\alpha'}{\alpha} \delta^{ab} [\partial_b \varphi - \beta A_b - \gamma (\partial_b \chi + \cos \theta \partial_b \phi)] , \] (88)

and \( \Delta w^a \) is a fluctuation,

\[ \langle \Delta w^a \rangle = 0 \quad \text{and} \quad \langle \Delta w^a \Delta w^b \rangle = \frac{1}{\alpha} \delta^{ab} . \] (89)

Just as for scalar particles the fact that the constraints (82) and (83) are not independent — both involve the same displacements \( \langle \Delta x^a_n \rangle \) — leads to a gauge symmetry. As we can see in (85) and (88) where \( \varphi \) and \( A_a \) appear in the combination \( \partial_a \varphi - \beta A_a \) which is invariant under the gauge transformations,

\[ A_a(x) \rightarrow A'_a(x) = A_a(x) + \partial_a \xi(x) , \] (90)

\[ \varphi(x) \rightarrow \varphi'(x) = \varphi(x) + \beta \xi(x) . \] (91)

In the next section we address the issue of assigning values to the multipliers \( \alpha \) and \( \alpha' \). Later, in section 5.1 we shall argue that the value of \( \gamma \) that ensures the compatibility of spin with the superposition principle is \( \gamma = 1/2 \). Finally, in section 7 we shall see that the multiplier \( \beta \) will turn out to be related to the electric charge \( q \) through \( \beta = q/\hbar c \).
3.4 Entropic time: instants and duration

ED consists of a succession of short steps; in order to keep track of how these small changes accumulate one introduces the notion of time. As proposed in [8] and [9] the construction of entropic time involves the concepts of ‘instant’ and of ‘duration’. An instant is specified by the probability distribution \( \rho(x) \) and by the transition probability \( P(x'|x) \) given in (85). The distribution \( \rho(x) \) implicitly provides a criterion of simultaneity and the pair \( (\rho, P) \) conveys the information required for the construction of the next instant. More explicitly, if the distribution \( \rho_t(x) \) refers to one instant \( t \), then the distribution \( \rho_{t'}(x) \) generated by \( P(x'|x) \) through

\[
\rho_{t'}(x') = \int dx P(x'|x)\rho_t(x)
\]

(92)
defines what we mean by the “next” instant which we have arbitrarily labelled \( t' \). Iterating this process defines the entropic dynamics.\(^{12}\)

To complete the construction of the dynamics and of entropic time we must specify the Lagrange multipliers \( \alpha \) and \( \alpha' \) in \( P(x'|x) \), and also the duration or interval \( \Delta t = t' - t \) between successive instants. This is accomplished by specifying the relation between \( \alpha, \alpha' \), and \( \Delta t \). The basic criterion is convenience: \textit{duration is defined so that motion looks simple} \(^9\). For the non-relativistic regime we adopt a choice that reflects the translational symmetry of a Newtonian space and time: we choose \( \alpha' \) and \( \alpha \) to be independent of \( x \) and \( t \), and we choose the ratio \( \alpha'/\alpha \propto \Delta t \) so that eq. (88) leads to a well defined drift velocity. The proportionality constant is written as \( \hbar/m \),

\[
\frac{\alpha'}{\alpha} = \frac{\hbar}{m} \Delta t ,
\]

(93)

where \( m \) will eventually be identified as the particle’s mass and \( \hbar \) is a constant that fixes the units of \( m \) relative to the units of time.

Having specified the ratio \( \alpha'/\alpha \) it remains to specify \( \alpha' \). It turns out that different choices of \( \alpha' \) lead to different forms of dynamics at the sub-quantum level ranging from non-differentiable Brownian paths to smooth Bohmian trajectories. Remarkably, all these different sub-quantum dynamics lead to the same theory at the quantum level \(^9\)\(^{13}\). Here we explore the particular ED described by \( \alpha' \propto 1/\Delta t^2 \). We write the proportionality constant as \( \hbar/\eta \),

\[
\alpha' = \frac{\hbar}{\eta \Delta t^2} \quad \text{so that} \quad \alpha = \frac{m}{\eta \Delta t^3}.
\]

(94)

As we shall see below, unlike the non-differentiable trajectories typical of a Brownian motion, in this type of ED the particle follows a smooth trajectory.\(^{12}\)\(^\text{As noted in [8][9] the fact that } P(x'|x) \text{ is derived using the maximum entropy method introduces an arrow of time — entropic time only flows towards the future. Nevertheless, the resulting Schrödinger or Pauli equation exhibits the symmetry commonly referred to as time reversal or (more appropriately) motion reversal.}\(^{13}\)\(^\text{The choice } \alpha' = \text{const in [9]} \text{ leads to a sub-quantum dynamics that is a Brownian motion}\(^{12}\).
Then the transition probability becomes

\[ P(x'|x) = \frac{1}{Z} \exp \left[ -\frac{m}{2\eta \Delta t} \delta_{ab} \left( \frac{\Delta x^a}{\Delta t} - v^a \right) \left( \frac{\Delta x^b}{\Delta t} - v^b \right) \right], \]  

where we used to define the drift velocity,

\[ v_a = \frac{1}{m} \partial_a \Phi - \frac{q}{mc} A_a - \frac{\hbar}{m} \cos \theta \partial_a \phi. \]

In this expression the drift potential \( \varphi \) and the Euler angle \( \chi \) have been merged into the single variable

\[ \Phi = \hbar (\varphi - \gamma \chi) \]

which we will call the phase\(^{14}\) and we set \( \beta = q/\hbar c \). A generic displacement is then written as a drift plus a fluctuation,

\[ \Delta x^a = v^a \Delta t + \Delta w^a, \]

with the fluctuations \( \Delta w^a \) given by

\[ \langle \Delta w^a \rangle = 0 \quad \text{and} \quad \langle \Delta w^a \Delta w^b \rangle = \frac{\eta}{m} \delta^{ab} \Delta t^3, \]

or

\[ \left\langle \left( \frac{\Delta x^a}{\Delta t} - v^a \right) \left( \frac{\Delta x^b}{\Delta t} - v^b \right) \right\rangle = \frac{\eta}{m} \delta^{ab} \Delta t. \]

Since \( \langle \Delta x^a \rangle \approx O(\Delta t) \) and \( \Delta w^a \approx O(\Delta t^{3/2}) \) we see that for small \( \Delta t \) the fluctuations are negligible and the particle follows a smooth deterministic trajectory. Indeed, the limit

\[ \lim_{\Delta t \to 0} \frac{\Delta x^a}{\Delta t} = v^a \]

is well defined. Not only does the particle have a definite position, it also has a definite velocity given by the expected drift velocity \( \langle v_a \rangle \). This is a feature that this version of ED shares in common with Bohmian mechanics.

### 3.5 The evolution equation

The basic evolution equation can be written in differential form,

\[ \partial_t \rho = -\partial_a (\rho v^a). \]

The proof follows closely the analogous derivation given in for the scalar case and need not be repeated here. Equation states that the probability evolves according to a continuity equation with probability flux \( \rho v^a \), and the current velocity coincides with the drift velocity \( v^a \), eq.\((106)\)

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\(^{14}\)In the multiparticle case it will not be possible to combine the variable \( \varphi \) and the \( \chi \)s associated to different particles into a single phase variable \( \Phi \). They are independent degrees of freedom.

\(^{15}\)When \( \alpha' = \text{const} \) and the sub-quantum dynamics is a Brownian motion the evolution equation is a Fokker-Planck equation with a current velocity that differs from the drift velocity by an osmotic term \( \propto \log \rho^{1/2} \).
Definition of the current velocity— Since the current velocity \( \vec{v} \) is introduced through the continuity equation, eq. (102), there is some arbitrariness in its definition. The same physical predictions are obtained with any velocity field \( \vec{V} \) of the form,

\[
\rho \vec{V}_k = \rho \vec{v} + k \vec{\partial} \times (\rho \vec{s})
\]

where \( k \) is an arbitrary constant. Since \( \vec{v} \) can be interpreted as the particle velocity, eq. (96), from now on we shall set \( k = 0 \). In realistic interpretations of quantum mechanics the field \( \rho \) may refer to some sort of fluid or substance with \( \vec{v} \) being its actual velocity. In such interpretations it makes sense to ask which, among all the possible \( \vec{V}_k \), is the “true” velocity. In contrast, in ED \( \rho \) receives a purely epistemic interpretation, and the question of a “true” current velocity cannot and should not be raised. The significant quantity is the time derivative \( \partial_t \rho \) because probabilities can change; but probabilities are not substances and therefore cannot move. Strictly, any reference to a probability “flow” is an abuse of language that ought to be avoided.

Having said that, from eq. (96) we can immediately see one way that spin affects the “flow” of probability. The vorticity is given by

\[
\vec{\partial} \times \vec{v} = \frac{\hbar \gamma}{m} \vec{\partial} \cos \theta \times \vec{\partial} \phi - \frac{q}{mc} \vec{B} ,
\]

which means that the circulation \( \oint \vec{v} \cdot d\vec{\ell} \) does not in general vanish.\(^{16}\)

Introducing the ensemble Hamiltonian — For future use we note that the evolution equation (102) can be conveniently rewritten in the alternative form

\[
\partial_t \rho = -\partial_a (\rho v^a) = \frac{\delta \tilde{H}}{\delta \Phi} ,
\]

for some suitably chosen functional \( \tilde{H} = \tilde{H}[\rho, \Phi, \phi, \theta] \). It is easy to check that the appropriate functional \( \tilde{H} \) is

\[
\tilde{H} = \int d^3x \rho \left( \frac{1}{2} m \delta_{ab} v^a v^b + F[\rho, \phi, \theta] \right) ,
\]

where \( v^a \) is given by (96) and \( F[\rho, \phi, \theta] \) is some unspecified functional of \( \{\rho, \theta, \phi\} \). Indeed, from (96) the variation of \( \tilde{H} \) with respect to \( \delta \Phi \) with \( \{\rho, \theta, \phi\} \) fixed is

\[
\delta \tilde{H} = \int dx \rho v^a \partial_a \delta \Phi = - \int dx \partial_a (\rho v^a) \delta \Phi .
\]

With these results the ED of spin reaches a certain level of completion. Even though the multipliers \( \gamma \) and \( \beta \) remain unspecified we have figured out what small changes to expect — they are given by the transition probability \( P(x'|x) \) in eq. (95) — and time was introduced to keep track of how these small changes accumulate; the net result is an evolution equation, eq. (105), driven

\(^{16}\)In hydrodynamics \( \cos \theta \) and \( \phi \) are known as the Clebsch potentials.
by various “potentials” — the phase $\Phi$, the angles $\theta$ and $\phi$ associated to the spin, and the vector potential $A_a$. The result is a consistent dynamics driven by constraints with some peculiar rotational properties.

However, this ED is not yet quantum mechanics. So far the spin variables have been treated as externally prescribed background fields but in a proper quantum dynamics not only the probability $\rho$ but the phase $\Phi$ and the spin variables themselves must also participate in the dynamics.

4 Symplectic structure and Hamiltonian flows

The ED that turns out to be physically relevant is achieved by requiring that the constraints used to update the probability $\rho_t(x)$ are not kept fixed. Instead they are themselves continuously updated in response to the evolving $\rho$. Such an ED describes the coupled evolution of the four variables $(\rho, \Phi, \theta, \phi)$ and is closely analogous to the ED of scalar particles developed in [9]: the criterion for updating the constraints is that some suitably defined symplectic and Riemannian structures are preserved by the dynamics.

4.1 The epistemic phase space

The symplectic structure we seek is chosen so that its preservation implies the ED continuity equation (105). This leads us to identify the phase function $\Phi(x)$ introduced in (97) as the momentum that is canonically conjugate to the probability $\rho(x)$. In this section and the next we tackle the question of how to handle the other two spin variables, $\theta$ and $\phi$, and assign a value to the multiplier $\gamma$.

In the ED of a scalar particle we deal with two configuration spaces. One is the ontic configuration space $X = \mathbb{R}^3$ of positions and the other is the epistemic configuration space or $e$-configuration space of the normalized probabilities,

$$
P = \left\{ \rho \mid \rho(x) \geq 0; \int dx \rho(x) = 1 \right\}.
$$

In the case of a single particle with spin the ontic configuration space $X = \mathbb{R}^3$ remains unchanged but the epistemic configuration space will be enlarged to include spin.

The choice of the configurational spin variable is suggested by an early work of Kramers [38] that was later adapted by Takabayasi [29] and by Bohm, Schiller, and Tiomno [?] to the context of spin and the Pauli equation. Kramers observed that since the orientation of a magnetic dipole with a fixed magnitude is defined by just two degrees of freedom, the angles $\phi$ and $\theta$ that define its orientation, then the minimal Hamiltonian dynamics — minimal in the sense that no additional variables are introduced — is constructed by identifying one of these variables as the coordinate and the other as its conjugate momentum. We shall adopt the azimuthal angle $\phi$ of the spin vector $\vec{s}$ for the coordinate variable so
that the e-configuration space is \( C = \mathbf{P} \times \{ \phi \}. \) For the conjugate momentum we choose the densitized \( z \)-component of the spin vector,

\[
\rho_s = \frac{\hbar}{2} \rho \cos \theta ,
\]

which we shall call the spin density. (The factor \( \hbar/2 \) is included for later convenience.)

To formulate the dynamics we need a framework to study paths in the larger space \( \{ \rho, \Phi, \phi, \rho_s \} \) which we will call the epistemic phase space or e-phase space.

Given a manifold such as \( C \) its cotangent bundle, \( T^*C \), is a geometric object that comes automatically endowed with a rich geometric structure [39]-[48]. Cotangent bundles are symplectic manifolds and this provides a “natural” criterion for dynamical laws, namely those that happen to preserve some privileged symplectic form. Thus, the natural criterion for updating constraints follows from identifying the cotangent bundle \( T^*C \) with e-phase space \( \{ \rho, \Phi, \phi, \rho_s \} \),

\[
T^*C = \{ \rho, \Phi, \phi, \rho_s \} .
\]

As discussed in [9] this identification is highly non-trivial. It amounts to asserting that the phase \( \Phi \) and the spin density \( \rho_s \) transform as the components of a locally defined Poincare 1-form

\[
\Theta = \int dx \left( \Phi \delta \rho_x + \rho_{sx} \delta \phi_x \right) ,
\]

where \( \delta \) is the exterior derivative on \( T^*C \). The corresponding symplectic 2-form \( \Omega = -\delta \Theta \) is

\[
\Omega = \int dx \left( \delta \rho_x \wedge \delta \Phi_x + \delta \phi_x \wedge \delta \rho_{sx} \right) .
\]

By construction \( \Omega \) is locally exact (\( \Omega = -\delta \Theta \)) and closed (\( \delta \Omega = 0 \)).

4.2 Notation

The notation follows closely that adopted in [9]. A point \( X = (\rho, \Phi, \phi, \rho_s) \) will be labelled by its coordinates

\[
X^{\alpha x} = (X^{1x}, X^{2x}, X^{3x}, X^{4x}) = (\rho^x, \Phi^x, \phi^x, \rho_{sx}) ,
\]

where \( (\rho^x, \phi^x) \) represent coordinates on the base manifold \( C \) and \( (\Phi^x, \rho_{sx}) \) represent coordinates on the space \( T^*C_{\rho \phi} \) that is cotangent to \( C \) at the point \( (\rho, \phi) \). We shall use a composite index \( \alpha x \) with the Greek index \( \alpha = 1, 2, 3, 4 \) taken from the beginning of the Greek alphabet.[13]

Curves in \( T^*C \) allow us to define vectors: the vector \( \bar{V} \) tangent to the curve \( X = X(\lambda) \) parametrized by \( \lambda \) at the point \( X(\lambda) \) is written as

\[
\bar{V} = V^{\alpha x} \frac{\delta}{\delta X^{\alpha x}} , \quad \text{where} \quad V^{\alpha x} = \frac{dX^{\alpha x}}{d\lambda} .
\]

\[\text{17}\]Since \( \alpha \) keeps track of whether \( x \) is an upper index (\( \alpha = 1, 3 \)) or a lower index (\( \alpha = 2, 4 \)) from now on we need not distinguish between them: \( \rho_x = \rho^x = \rho(x). \)
where the repeated indices indicate a summation over \( \alpha \) and an integration over \( x \). The directional derivative of a functional \( F[X] \) along the curve \( X(\lambda) \) is
\[
\frac{dF}{d\lambda} = \nabla F[V] = \frac{\delta F}{\delta X^\alpha} V^\alpha ,
\]
where \( \nabla \) is the functional gradient in \( T^*C \),
\[
\frac{dF}{d\lambda} = \tilde{\nabla} F \equiv \frac{\delta F}{\delta X^\alpha} \tilde{\nabla} X^\alpha .
\] (116)

The tilde serves to distinguish the functional gradient \( \tilde{\nabla} \) on \( T^*C \) from the spatial gradient \( \nabla f = \frac{\partial f}{\partial x} \nabla x \) on \( \mathbb{R}^3 \).

The fact that probabilities in (108) are normalized introduces a technical difficulty in that the coordinates \( \rho_x \) are not independent. We work our way around this problem by embedding the \( \infty \)-dimensional manifold \( C \) in an \((\infty + 1)\)-dimensional manifold \( C^{+1} \) where the coordinates \( \rho_x \) are unconstrained.\(^{18}\) Then, since \( \tilde{\nabla} F \) is strictly speaking a covector on \( T^*C^{+1} \), in order to obtain the desired directional derivatives (115) on \( T^*C \) we must impose that the vectors \( \bar{V} \) be restricted to be tangent to \( C \). Such tangent vectors are constrained to obey
\[
\frac{d}{d\lambda} \int dx \rho_x = \int dx \frac{dX^1_x}{d\lambda} = 0 .
\] (117)

Expressed in terms of the functional gradient \( \tilde{\nabla} \) the symplectic form (112) is
\[
\Omega = \int dx \left[ \partial_t \rho_x \otimes \tilde{\nabla} \Phi_x - \tilde{\nabla} \Phi_x \otimes \partial_t \rho_x + \tilde{\nabla} \phi^x \otimes \tilde{\nabla} \rho_{sx} - \tilde{\nabla} \rho_{sx} \otimes \tilde{\nabla} \phi^x \right] .
\] (118)

The action of \( \Omega \) on two vectors \( \bar{V} = d/d\lambda \) and \( \bar{U} = d/d\mu \) is given by
\[
\Omega[\bar{V},\bar{U}] = \int dx \left[ V^1_x U^2_x - V^2_x U^1_x + V^3_x U^4_x - V^4_x U^3_x \right] = \Omega_{\alpha x,\beta x'} V^\alpha_x U^\beta_{x'} ,
\] and the components of \( \Omega \) are
\[
\Omega_{\alpha x,\beta x'} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 \\
\end{bmatrix} \delta(x, x') .
\] (120)

### 4.3 Hamiltonian flows and Poisson brackets

We seek a dynamics that preserves the symplectic structure. Specifically we are interested in those vector fields \( V^\alpha [X] = dX^\alpha /d\lambda \) that generate flows \( X^\alpha = X^\alpha (\lambda) \) that preserve the symplectic form,
\[
\mathcal{L}_V \Omega = 0 ,
\] (121)

\(^{18}\)It is this embedding that eventually leads to quantum states being represented by rays (and not vectors) in a Hilbert space.
where the Lie derivative \([49]\) is given by
\[
(\mathcal{L}_V \Omega)_{\alpha x, \beta x'} = V^{\gamma x'} \nabla_{\gamma x'} \Omega_{\alpha x, \beta x'} + \Omega_{\gamma x', \beta x'} \nabla_{\alpha x} V^{\gamma x'} + \Omega_{\alpha x, \gamma x'} \nabla_{\beta x'} V^{\gamma x'}.
\] (122)

As discussed in \([49][50]\) these vector fields are such that the covector
\[
\Omega_{\alpha x, \beta x'} V^{\beta x'} = -\nabla_{\alpha x} \tilde{V}.
\] (123)

More explicitly, this condition is written as
\[
\frac{d \rho_x}{d \lambda} = \frac{\delta \tilde{V}}{\delta \Phi_x}, \quad \frac{d \Phi_x}{d \lambda} = -\frac{\delta \tilde{V}}{\delta \rho_x}, \quad \frac{d \rho_{sx}}{d \lambda} = -\frac{\delta \tilde{V}}{\delta \phi_x},
\] (124)

which we recognize as Hamilton’s equations for a Hamiltonian function \(\tilde{V}\). The vector \(\tilde{V}\) is called the Hamiltonian vector field associated to the Hamiltonian function \(\tilde{V}\).

From (119) and (124), the action of \(\Omega\) on two Hamiltonian vector fields \(\tilde{V} = d/d\lambda\) and \(\tilde{U} = d/d\mu\) generated respectively by \(\tilde{V}\) and \(\tilde{U}\) is
\[
\Omega[\tilde{V}, \tilde{U}] = \int dx \left[ \frac{\delta \tilde{V}}{\delta \rho_x} \frac{\delta \tilde{U}}{\delta \Phi_x} - \frac{\delta \tilde{V}}{\delta \Phi_x} \frac{\delta \tilde{U}}{\delta \rho_x} + \frac{\delta \tilde{V}}{\delta \phi_x} \frac{\delta \tilde{U}}{\delta \rho_{sx}} - \frac{\delta \tilde{V}}{\delta \rho_{sx}} \frac{\delta \tilde{U}}{\delta \phi_x} \right] \overset{\text{def}}{=} \{ \tilde{V}, \tilde{U} \},
\] (125)

where on the right we introduced the Poisson bracket notation. Thus, the action of \(\Omega\) on two Hamiltonian vector fields is the Poisson bracket of the associated Hamiltonian functions,
\[
\Omega[\tilde{V}, \tilde{U}] = \Omega_{\alpha x, \beta x'} V^{\alpha x} U^{\beta x'} = \{ \tilde{V}, \tilde{U} \}. \quad (126)
\]

The main idea of this section can now be expressed succinctly: The condition for a flow generated by the vector field \(V^{\alpha x}\) to preserve the symplectic structure, \(\mathcal{L}_V \Omega = 0\), is that \(V^{\alpha x}\) be the Hamiltonian vector field associated to a Hamiltonian function \(\tilde{V}\), eq. (124),
\[
V^{\alpha x} = \frac{d X^{\alpha x}}{d \lambda} = \{ X^{\alpha x}, \tilde{V} \}. \quad (127)
\]

At this point our goal of defining an ED that preserves the symplectic structure \(\Omega\) and reproduces the continuity equation \((105)\) is partially achieved: it is given by the Hamiltonian flow of the scalar functional \(\tilde{H}\) in \((106)\). The remaining task is to determine the as yet unspecified integration constant \(F[\rho, \phi, \theta]\) in \((106)\).

4.4 The normalization constraint

The conservation of probability implied by the continuity equation is expressed as a constraint,
\[
\tilde{N} = 0 \quad \text{where} \quad \tilde{N} = 1 - |\rho| \quad \text{and} \quad |\rho| \overset{\text{def}}{=} \int dx \rho(x), \quad (128)
\]
that must be preserved by the Hamiltonian flow,

$$\partial_t \tilde{N} = \{ \tilde{N}, \hat{H} \} = 0.$$  \hspace{1cm} (129)

The constraint $\tilde{N}$ itself generates a Hamiltonian flow given by the vector field

$$\vec{N} = N^{\alpha x} \frac{\delta}{\delta X^{\alpha x}} \text{ with } N^{\alpha x} = \frac{dX^{\alpha x}}{d\sigma} = \{ X^{\alpha x}, \tilde{N} \}, \hspace{1cm} (130)$$

where $\sigma$ is the parameter along the flow lines. More explicitly,

$$N^{1x} = \frac{d\rho_x}{d\sigma} = 0, \quad N^{2x} = \frac{d\Phi_x}{d\sigma} = 1, \quad N^{3x} = \frac{d\phi_x}{d\sigma} = 0, \quad \text{and} \quad N^{4x} = \frac{d\rho_{sx}}{d\sigma} = 0.$$  \hspace{1cm} (131)

The conservation of $\tilde{N}$, eq.(129), implies that $\tilde{N}$ is the generator of a symmetry, namely,

$$\frac{d\tilde{H}}{d\sigma} = \{ \tilde{H}, \tilde{N} \} = 0.$$  \hspace{1cm} (132)

Integrating (131) one finds the integral curves generated by $\tilde{N}$,

$$\rho_x(\sigma) = \rho_x(0), \quad \Phi_x(\sigma) = \Phi_x(0) + \sigma, \quad \phi_x(\sigma) = \phi_x(0), \quad \text{and} \quad \rho_{sx}(\sigma) = \rho_{sx}(0), \hspace{1cm} (133)$$

which shows that the symmetry generated by $\tilde{N}$ is to shift the phase $\Phi$ by a constant $\sigma$ without otherwise changing the dynamics. The interpretation is that the constraint $\tilde{N} = 0$ reduces by one the (infinite) number of independent $\rho_x$ degrees of freedom and it also reduces by one the number of $\Phi_x$s because for any value of $\sigma$ the phases $\Phi_x + \sigma$ and $\Phi_x$ correspond to the same state. The result is a global gauge symmetry. (This is the ED analogue of the fact that in QM states are represented by rays rather than vectors in a Hilbert space.)

The situation thus far can be summarized as follows: the phase space of interest is $T^*C$ but the constraint $|\rho| = 1$ forces us to use coordinates in a larger embedding space $T^*C^{+1}$. The introduction of one superfluous coordinate has led us to introduce the corresponding superfluous momentum. We eliminate the extra coordinate by imposing $\tilde{N} = 0$ and we eliminate the extra momentum by declaring it an unphysical gauge variable.

### 5 The information geometry of e-phase space

We have just seen that a natural criterion to update the constraints is to impose a dynamics that preserves a symplectic structure. To select the particular Hamiltonian dynamics that reproduces the QM of spin we proceed in close analogy to the ED of scalar particles. The procedure is to generalize the metric structure that is naturally available in the statistical manifold $P = \{ \rho \}$ — its information geometry — to the full phase space $T^*C = \{ \rho, \Phi, \phi, \rho_s \}$. Then the criterion we adopt for updating constraints is a dynamics that preserves both the symplectic and the metric structures.
In this section the objective is to transform e-phase space $T^\ast C$ from a manifold that is merely symplectic to a manifold that is both symplectic and Riemannian. Then we will find the corresponding Hamilton-Killing flows (section 6) and the particular Hamiltonians (section 7) that reproduce the ED of spin, eqs.(105, 124).

5.1 The metric on the embedding space $T^\ast C^{+1}$

As we saw earlier the normalization constraint forces us to embed the phase space of interest $T^\ast C$ in a larger unconstrained space $T^\ast C^{+1}$. The first goal is to extend the information geometry of the statistical manifold $\mathcal{P} = \{\rho\}$ to the full e-phase space $T^\ast C^{+1} = \{\rho, \Phi, \phi, \rho_s\}$. In the next subsection we obtain the metric induced on the space $T^\ast C$.

One possible path (pursued in [51]) is to proceed exactly as in the case of scalar particles [9] where the known spherical symmetry of the statistical manifold $\mathcal{P} = \{\rho\}$ is extended to the unconstrained embedding space $T^\ast \mathcal{P}^{+1}$. It is very significant that the geometry of the embedding space $T^\ast \mathcal{P}^{+1}$ is not unique; any spherically symmetric geometry will serve our purposes. We can therefore choose the simplest possible embedding in which the space $T^\ast \mathcal{P}^{+1}$ happens to be a flat. This choice, while strictly optional, turns out to be very convenient for calculational purposes because it allows one to introduce the notion of Hilbert spaces and leads to a linear Schrödinger equation.

Here we follow an alternative path: rather than extending the metric from $\mathcal{P}$ to $T^\ast \mathcal{P}^{+1}$, we start from the already known metric for $T^\ast \mathcal{P}^{+1}$ derived for the scalar case in [9] and extend it to $T^\ast C^{+1}$. Furthermore, from the beginning we choose both spaces $T^\ast \mathcal{P}^{+1}$ and $T^\ast C^{+1}$ to be flat. This approach is designed to guarantee that the resulting metric will have the desired spherical symmetry and that it will correctly reproduce the scalar limit. The argument is simplest when we adopt complex coordinates. In the scalar case the metric of the flat space $T^\ast \mathcal{P}^{+1}$ is given by

$$\delta \ell_0^2 = \frac{1}{4} \int dx \left[ \frac{1}{\rho_x} (\delta \rho_x)^2 + \rho_x (\delta \phi)^2 \right] = \int dx \delta \psi_x^* \delta \psi_x, \tag{134}$$

where

$$\psi = \rho^{1/2} e^{i\phi}$$

is the scalar wave function, $\phi$ is the drift potential, and $\hbar \phi$ is the momentum conjugate to $\rho$.

**Introducing the spinor wave function** — We propose that the natural generalization of (135) that takes the peculiar rotational properties of spin into account is the spinor wave function

$$\Psi = \rho^{1/2} e^{i\phi} U(\theta, \phi, \chi) u_+ . \tag{136}$$

---

19See eq.(91) in [9]. The choice $A = 0$ and $B = 1/2h$ yields a flat geometry. The first term $(\delta \rho)^2 / \rho$ in (134) is the information metric on the base space $\mathcal{P}^{+1}$; the second term $\rho (\delta \phi)^2$ is the metric on the cotangent fibers of the bundle $T^\ast \mathcal{P}^{+1}$. 

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25
Using (52) \( \Psi \) can be written in the alternative forms

\[
\Psi = \rho^{1/2} U(\theta, \phi, \bar{\chi}) u_+ = \rho^{1/2} e^{-i\bar{\chi}/2} U(\theta, \phi, 0) u_+ \quad \text{where} \quad \bar{\chi} = \chi - 2\varphi ,
\]

which shows that the phase of \( \Psi \) can be geometrically interpreted as a rotation of the spin frame by an angle \( \bar{\chi} \).

We can now address the question about the still unspecified value of the multiplier \( \gamma \) in eq.(97). As discussed in [9] the single- or multi-valued nature of the phase \( \Phi \) is severely restricted by experimental facts. On one hand the validity of the superposition principle forces wave functions to be either single- or double-valued functions and, on the other hand the nature of electromagnetic interactions — that is, the specific gauge group \( U(1) \) that generates them — leads to charge quantization. For scalar particles this required that the drift potential be a multi-valued function with the topological properties of an angle — \( \varphi \) and \( \varphi + 2\pi \) represent the same angle. For spin 1/2 these same scalar restrictions on the phase still apply but \( \Phi \) in (97) now receives an additional contribution from the Euler angle \( \chi \).

Equation (137) shows that both the rotational properties of spin and the validity of the superposition principle — a double-valued wave function — can be properly accounted for by choosing \( \gamma \) to be \( 1/2 \). With this choice the momentum conjugate to \( \rho \) is the phase

\[
\Phi = \hbar \left( \varphi - \frac{1}{2} \chi \right) = -\frac{\hbar}{2} \bar{\chi} ,
\]

and the spinor wave function is

\[
\Psi = \rho^{1/2} e^{i\Phi/\hbar} U(\theta, \phi, 0) u_+ .
\]

Since

\[
\Psi^\dagger \Psi = \rho U u_+ u_+^\dagger U^\dagger = \rho U(1 + \bar{e}_3) U^\dagger = \rho(1 + \bar{s})
\]

we have an elegant extension of the Born rule in which both the probability density and the spin density can be written in terms of \( \Psi \Psi^\dagger \),

\[
\langle \Psi \Psi^\dagger \rangle_0 = \rho \quad \text{and} \quad \langle \Psi \Psi^\dagger \rangle_1 = \rho \bar{s} .
\]

**Complex coordinates** — A useful new set of coordinates is suggested by using eq.(48) which leads us to write (137) as

\[
\Psi = \psi_+ u_+ + \psi_- u_- ,
\]

where

\[
\psi_+ = \rho^{1/2} \cos \frac{\theta}{2} e^{-i(\bar{\chi} + \phi)/2} \quad \text{and} \quad \psi_- = \rho^{1/2} \sin \frac{\theta}{2} e^{-i(\bar{\chi} - \phi)/2}
\]

are called the complex “amplitudes” for spin up and down respectively. A straightforward calculation of the symplectic form \( \Omega \), eq.(112), in terms of \( \psi_\pm \),

\[
\Omega = \int dx \left( \delta \psi_+ \wedge \delta (i\hbar \psi_+^*) + \delta \psi_- \wedge \delta (i\hbar \psi_-^*) \right) ,
\]

26
shows that the coordinate transformation
\[(\rho, \Phi, \phi, \rho_s) \rightarrow (\psi_+, i\hbar \psi_+^*, \psi_-, i\hbar \psi_-^*) \text{,} \tag{145}\]
is indeed canonical. The new coordinates of a point \(\Psi\),
\[
\Psi^x = (\Psi^1, x, \Psi^2, x, \Psi^3, x, \Psi^4, x) = (\psi_+ x, i\hbar \psi_+^* x, \psi_- x, i\hbar \psi_-^* x) \text{,} \tag{146}\]
will be labeled by a composite index \(\mu_x\) with the Greek index \(\mu\) taken from the middle of the Greek alphabet. In these coordinates the action \(\Omega\) on two vectors \(\bar{V} = d/d\lambda\) and \(\bar{U} = d/d\mu\) is given by
\[
\Omega[\bar{V}, \bar{U}] = \Omega_{\mu,\nu,\mu,\nu} V^\mu U^\nu \text{,} \tag{147}\]
where the components of \(\Omega\) are
\[
\Omega_{\mu,\nu,\mu,\nu} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 \\
\end{bmatrix} \delta(x, x') \text{.} \tag{148}\]

**The metric on \(T^*\mathbb{C}^{+1}\) —** We propose that the natural generalization of the scalar metric (134) from the space \(T^*\mathbb{P}^{+1}\) to the space \(T^*\mathbb{C}^{+1}\) is
\[
\delta \ell^2 = \int dx \langle \delta \Psi^*_x \delta \Psi_x \rangle_0 \text{.} \tag{149}\]
To get insight recall eq.(137) and introduce the quaternion variable
\[
\Upsilon = \rho^{1/2} U(\theta, \phi, \bar{\chi}) \text{.} \tag{150}\]
Then
\[
\delta \ell^2 = \int dx \langle u^*_x \delta \Upsilon^* \delta \Upsilon u_x \rangle_0 = \int dx \langle (1 + \vec{e}_3) \delta \Upsilon^* \delta \Upsilon \rangle_0 \text{.} \tag{151}\]
Next use (54) and (56) to write
\[
\delta \Upsilon^* \delta \Upsilon = \delta (\rho^{1/2} U^*) \delta (\rho^{1/2} U) = \delta (\rho^{1/2})^2 + \rho \delta U^* \delta U
= \frac{1}{4\rho_x} (\delta \rho_x)^2 + \frac{1}{4} \rho_x (\delta \zeta)^2 \text{,} \tag{152}\]
which is a scalar. Therefore
\[
\delta \ell^2 = \int dx \delta \Upsilon^* \delta \Upsilon = \frac{1}{4} \int dx \left( \frac{1}{\rho_x} (\delta \rho_x)^2 + \rho_x (\delta \zeta)^2 \right) \text{,} \tag{153}\]
which shows that the angle \(\delta \phi\) in the scalar metric, eq.(134), is replaced by the rotation angle
\[
\delta \zeta = \vec{e}_3 \delta \phi + \vec{e}_\phi \delta \theta + \vec{s} \delta \bar{\chi} \text{.} \tag{154}\]
In terms of the complex coordinates $\psi_{\pm}$ the metric (149) takes a particularly simple form,

$$\delta \ell^2 = \int dx \left( \delta \psi_{+x}^* \delta \psi_{+x} + \delta \psi_{-x}^* \delta \psi_{-x} \right).$$

(155)

The scalar product of two vectors $\bar{V} = d/d\lambda$ and $\bar{U} = d/d\mu$ is given by

$$G[\bar{V}, \bar{U}] = G_{\mu x, \nu x'} V^\mu U^{x'},$$

(156)

where the components of $G$ and its inverse are

$$G_{\mu x, \nu x'} = \frac{1}{2i\hbar} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \delta(x, x'), \quad G^{\mu x, \nu x'} = 2i\hbar \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \delta(x, x').$$

(157)

Yet another important aspect of the metric (149) can be made explicit in the following coordinates,

$$\psi_+ = \xi_1 + i\xi_2 \quad \text{and} \quad \psi_- = \xi_3 + i\xi_4.$$  

(158)

Then the normalization condition,

$$\int dx \left( \psi_{+x}^* \psi_{+x} + \psi_{-x}^* \psi_{-x} \right) = \int dx \rho_x = 1,$$

(159)

becomes

$$\int dx \left( \xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 \right) = 1,$$

(160)

which is the equation of a sphere. This shows that our choice (149) which extends the information geometry of the statistical manifold $P = \{\rho\}$ to the space $T^*\mathbb{C}^{+1} = \{\rho, \Phi, \phi, \rho_s\}$ amounts to extending the well-known infinite-dimensional spherical symmetry of the space $P$ to the larger embedding space $T^*\mathbb{C}^{+1}$. Furthermore, the metric (155) becomes

$$\delta \ell^2 = \int dx \left( \delta \xi_1^2 + \delta \xi_2^2 + \delta \xi_3^2 + \delta \xi_4^2 \right),$$

(161)

which is Euclidean. This means that, beyond being spherically symmetric, the particular embedding space $T^*\mathbb{C}^{+1}$ that we have chosen is flat.

**A complex structure on $T^*\mathbb{C}^{+1}$** — The contraction of the symplectic form $\Omega$, eq.(147), with the inverse metric $G^{-1}$ allows us to construct a mixed tensor $J$ with components

$$J^{\mu x}_{\nu x'} = -\frac{1}{2i\hbar} G^{\mu x, \lambda x''} \Omega_{\lambda x'', \nu x'} = \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix} \delta(x, x').$$

(162)
What makes the tensor $J$ special is that its square is
\[ J^{\mu x'} J^{\lambda x''} = -\delta^{\mu \lambda} \delta_{x' x''} . \] (163)

In words, the action of $J^2$ (or $\Omega^2$) is equivalent to multiplying by $-1$. This means that $J$ plays the role of a complex structure. Note however that the $i$ that appears in all these expressions, including the complex structure $J$, is not the imaginary unit familiar from complex analysis but the pseudoscalar $i$, eq. (10), that is geometrically interpreted as a directed volume. Strictly speaking we are not using complex variables.

### 5.2 The metric induced on $T^*C$

As we saw in section 4.4 the e-phase space $T^*C$ is obtained from the embedding space $T^*C^{+1}$ by the restriction $|\rho| = 1$ and by identifying the point $X = (\rho_x, \Phi_x, \phi_x, \rho_{sx})$ with the shifted points $X_\sigma = (\rho_x, \Phi_x + \sigma, \phi_x, \rho_{sx})$ that lie on the same ray.

Consider two neighboring points $X$ and $X' = X + \delta X$. The metric induced on $T^*C$ is defined as the shortest $T^*C^{+1}$ distance between $X$ and points $X'_\sigma$ on the ray defined by $X'$. From (138) the effect of shifting $\Phi_x$ by $\sigma$ is to shift $\bar{\chi}$ by $\bar{\sigma} = -2\sigma / \hbar$. Then, setting $|\rho| = 1$ and recalling (153) and (154), the $T^*C^{+1}$ distance between $X$ and $X'_\sigma$ is given by
\[ \delta \ell^2 = \frac{1}{4} \int dx \left( \frac{1}{\rho} (\delta \rho)^2 + \rho (\delta \vec{\zeta} + \vec{s} \delta \bar{\sigma})^2 \right) , \] (164)

Let
\[ \delta \ell^2_{FS} = \min_{\delta \sigma} \delta \ell^2_\sigma . \] (165)

The value of $\delta \bar{\sigma}$ that minimizes (164) is
\[ \delta \bar{\sigma} = -\int dx \rho \vec{s} \cdot \delta \vec{\zeta} = -\langle \vec{s} \cdot \delta \vec{\zeta} \rangle \] (166)

so that the metric on $T^*C$ is
\[ \delta \ell^2_{FS} = \frac{1}{4} \int dx \left[ \frac{1}{\rho} (\delta \rho)^2 + \rho \left( \delta \vec{\zeta} - \vec{s} \langle \vec{s} \cdot \delta \vec{\zeta} \rangle \right)^2 \right] , \] (167)

which can be recognized as the Fubini-Study metric for a spin-1/2 particle.

**A convenient choice of gauge** — The scalar product between two vectors $\vec{V} = d/d\lambda$ and $\vec{U} = d/d\mu$ is a bit messy,
\[ G(\vec{V}, \vec{U}) = \frac{1}{4} \int dx \left[ \frac{1}{\rho} \frac{dp}{d\lambda} \frac{dp}{d\mu} + \rho_x \left( \frac{d\vec{\zeta}}{d\lambda} - \vec{s} \langle \vec{s} \cdot \frac{d\vec{\zeta}}{d\lambda} \rangle \right) \left( \frac{d\vec{\zeta}}{d\mu} - \vec{s} \langle \vec{s} \cdot \frac{d\vec{\zeta}}{d\mu} \rangle \right) \right] , \] (168)
but can be simplified considerably. First, we note that by virtue of being tangent to \( T^*C \) a vector \( \bar{V} \) satisfies
\[
\frac{d\rho}{d\lambda} = \int dx \frac{d\rho_x}{d\lambda} = 0 .
\] (169)

Furthermore, since points \( X \) on the same ray are equivalent it follows all vectors \( \bar{V} \) and \( \bar{V}' \) that differ by a vector along the gauge direction \( \bar{N} \) are equivalent. To prove this consider a curve \( X(\lambda) \) and a second curve \( X'(\lambda) \) obtained from the first by a shift along the gauge direction. We assume that their coordinates are identical except for
\[
\Phi'_x(\lambda) = \Phi_x'(\lambda) + k(\lambda - \lambda_0)
\] (170)
so that the two curves are equivalent and they cross at \( \lambda = \lambda_0 \). Using eq.131 we see that the tangent vectors at \( \lambda_0 \) are related by
\[
\frac{dX'^{\alpha x}}{d\lambda} = \frac{dX^{\alpha x}}{d\lambda} + kN^{\alpha x} \quad \text{or} \quad \bar{V}' = \bar{V} + k\bar{N} .
\] (171)

To work with such equivalent vectors it desirable to choose a convenient representative, that is, we fix the gauge. Equation (168) suggests that a convenient “Tangent Gauge-Fixed” representative (which we will call the TGF gauge) is
\[
\int dx \rho \vec{s} \cdot d\vec{\zeta} d\lambda = \langle \vec{s} \cdot d\vec{\zeta} d\lambda \rangle = 0
\] (172)
Vectors satisfying (169) and (172) will be called TGF vectors: the first condition enforces a flow tangent to the \( |\rho| = 1 \) surface; the second condition eliminates a superfluous vector component along the gauge direction \( \bar{N} \). In the TGF gauge \( G(\bar{V}, \bar{U}) \) simplifies to
\[
G(\bar{V}, \bar{U}) = \frac{1}{4} \int dx \left[ \frac{1}{\rho} \frac{d\rho}{d\lambda} \frac{d\rho}{d\mu} + \frac{\bar{c} \cdot d\bar{\zeta}}{d\lambda} \right] .
\] (173)

To be explicit: the tensor \( G \) in (173) can act on arbitrary vectors but it is only when they satisfy the TGF condition that \( G \) can be interpreted as the metric on \( T^*C \). The corresponding expression for the length element is
\[
\delta \ell^2_{FS} = \frac{1}{4} \int dx \left[ \frac{1}{\rho} \delta \rho^2 + \rho \delta \bar{\zeta}^2 \right] ,
\] (174)
where it is understood that \( \delta X \) satisfies the TGF conditions,
\[
|\delta \rho| = 0 \quad \text{and} \quad \langle \bar{s} \cdot \delta \bar{\zeta} \rangle = 0 .
\] (175)

**The \( T^*C \) metric in \( \psi_\pm \) coordinates** — The same analysis can be carried out in \( \psi_\pm \) coordinates. The normalization constraint now reads
\[
\bar{N} = 0 \quad \text{with} \quad \tilde{N} = 1 - \int dx \left( \psi_+ \psi_+^* + \psi_- \psi_-^* \right) ,
\] (176)
From eqs. (133), (138) and (143) we see that the Hamiltonian flow generated by $\tilde{N}$ and parametrized by $\sigma$ yields the integral curves

$$\psi_\pm(\sigma) = \psi_\pm(0)e^{i\sigma/\hbar}.$$  \hspace{1cm} (177)

Using

$$\langle \Psi \frac{d\Psi^\dagger}{d\lambda} \rangle_{0+3} = \frac{1}{2} \frac{d\rho}{d\lambda} + i \rho \hat{s} \cdot \frac{d\vec{z}}{d\lambda}$$  \hspace{1cm} (178)

the two real TGF conditions, (169) and (172), are replaced by a single complex condition,

$$\int dx \langle \Psi \frac{d\Psi^\dagger}{d\lambda} \rangle_{0+3} = \int dx \left( \psi_+ \frac{d\psi_+^*}{d\lambda} + \psi_- \frac{d\psi_-^*}{d\lambda} \right) = 0.$$  \hspace{1cm} (179)

In $\psi_\pm$ coordinates the metric on $T^*C$, eq. (174), reads

$$\delta^2 S_{FS} = \int dx \langle \delta \Psi_x \delta \Psi_x^\dagger \rangle_0.$$  \hspace{1cm} (180)

The complex structure on $T^*C$ — It is straightforward to check that the tensor $J_{\mu x\nu x'}$ defined in (162) takes a TGF vector $\tilde{V}$ to another vector $J\tilde{V}$ that is also TGF. This means that $J_{\mu x\nu x'}$ is indeed a tensor on $T^*C$ — it linearly maps vectors to vectors. Since $J^2 = -1$ we conclude that the same $J$ that defines a complex structure on $T^*C^+$ also serves to define a complex structure on $T^*C$.

6 Hamilton-Killing flows

Our next goal is to find those Hamiltonian flows $Q^{\mu x}$ that also happen to preserve the metric tensor, that is, we want $Q^{\mu x}$ to be a Killing vector. The condition that $Q^{\mu x}$ must obey is

$$(L_Q G)_{\mu x,\nu x'} = Q^{\lambda x''} \nabla_{\lambda x''} G_{\mu x,\nu x'} + G_{\lambda x'',\nu x'} \nabla_{\mu x} Q^{\lambda x''} + G_{\mu x,\lambda x''} \nabla_{\nu x'} Q^{\lambda x''} = 0.$$  \hspace{1cm} (182)

In complex coordinates eqs. (180) and (157) gives $\nabla_{\gamma x''} G_{\alpha x,\beta x'} = 0$, and the Killing equation simplifies to

$$(L_Q G)_{\mu x,\nu x'} = G_{\lambda x'',\nu x'} \nabla_{\mu x} Q^{\lambda x''} + G_{\mu x,\lambda x''} \nabla_{\nu x'} Q^{\lambda x''} = 0,$$  \hspace{1cm} (183)

which can be expressed as an $4 \times 4$ matrix equation

$$(L_Q G)_{\mu x,\nu x'} = \begin{bmatrix} (L_Q G)_{+x,+x'} & 0 \\ 0 & (L_Q G)_{-x,-x'} \end{bmatrix} = 0$$  \hspace{1cm} (184)
where the $2 \times 2$ blocks $(\mathcal{L}_Q G)_{+x,+x'}$ and $(\mathcal{L}_Q G)_{-x,-x'}$ are

\[
(\mathcal{L}_Q G)_{+x,+x'} = \frac{1}{2\hbar} \left[ \delta Q^2_{x'} \frac{\delta}{\delta \psi_{+x}} + \delta Q^2_{x} \frac{\delta}{\delta \psi_{+x'}} + \delta Q^2_{x'} \frac{\delta}{\delta \psi_{x'}} + \delta Q^2_{x} \frac{\delta}{\delta \psi_{x}} \right] = 0 \tag{185}
\]

and

\[
(\mathcal{L}_Q G)_{-x,-x'} = \frac{1}{2\hbar} \left[ \delta Q^2_{x'} \frac{\delta}{\delta \psi_{-x}} + \delta Q^2_{x} \frac{\delta}{\delta \psi_{-x'}} + \delta Q^2_{x'} \frac{\delta}{\delta \psi_{x'}} + \delta Q^2_{x} \frac{\delta}{\delta \psi_{x}} \right] = 0. \tag{186}
\]

But $Q_{ix}$ must also generate a Hamiltonian flow, $\mathcal{L}_Q \Omega = 0$. Substituting

\[
Q^1_x = \frac{\delta \tilde{Q}}{\delta \hbar \psi^*_{+x}}, \quad Q^2_x = -\frac{\delta \tilde{Q}}{\delta \psi^*_{+x}}, \quad Q^3_x = \frac{\delta \tilde{Q}}{\delta \hbar \psi^*_{-x}}, \quad \text{and} \quad Q^4_x = -\frac{\delta \tilde{Q}}{\delta \psi_{-x}} \tag{187}
\]

into (185) and (186) gives us the condition for a Hamilton-Killing (or HK) flow,

\[
\frac{\delta^2 \tilde{Q}}{\delta \psi_{+x} \delta \psi_{+x'}} = 0, \quad \frac{\delta^2 \tilde{Q}}{\delta \psi_{-x} \delta \psi_{-x'}} = 0, \quad \frac{\delta^2 \tilde{Q}}{\delta \psi^*_{+x} \delta \psi^*_{+x'}} = 0, \quad \text{and} \quad \frac{\delta^2 \tilde{Q}}{\delta \psi^*_{-x} \delta \psi^*_{-x'}} = 0. \tag{188}
\]

The conclusion is that a flow that preserves both $G$ and $\Omega$ requires a functional $\tilde{Q}[\psi_{+x}, \psi^*_{+x}, \psi_{-x}, \psi^*_{-x}]$ that is at most linear in each of $\psi$, $\psi^*_+$, $\psi_-$, and $\psi^*_-$ This means that $\tilde{Q}$ can be expressed as the sum of scalar (grade zero) functionals

\[
\tilde{Q}[\psi_{+x}, \psi^*_{+x}, \psi_{-x}, \psi^*_{-x}] = \tilde{Q}_0 + \tilde{Q}_1 + \tilde{Q}_2 + \tilde{Q}_3 + \tilde{Q}_4 \tag{189}
\]

where $\tilde{Q}_0$ is a constant, $\tilde{Q}_1$ is first order in $\psi$s, $\tilde{Q}_2$ is quadratic in the $\psi$s, and so on.

The functional form of $\tilde{Q}$ can be restricted as follows. First, since the constant $\tilde{Q}_0$ does not contribute to the flow (187) it can be set equal to zero, $\tilde{Q}_0 = 0$. Next, in order to reflect the rotational properties of spin we shall require that the dependence of $\tilde{Q}$ on $\psi_+$ and $\psi_-$ occurs only through the particular combination of a spinor wave function, that is, $\tilde{Q} = \tilde{Q}[\Psi, \Psi^\dagger]$ with $\Psi = \psi_+ u_+ + \psi_- u_-$. And finally, we require that the HK flows,

\[
\frac{d\Psi_x}{d\lambda} = \frac{\delta \tilde{Q}}{\delta \hbar \Psi_x} \quad \text{and} \quad \frac{d\hbar \Psi_x}{d\lambda} = -\frac{\delta \tilde{Q}}{\delta \Psi_x}, \tag{190}
\]

be such that initial states that are equivalent — states that lie on the same ray — flow to final states that are also equivalent: if the state $\Psi_x(\lambda_f)$ then $e^{i\sigma/\hbar} \Psi_x(\lambda_f)$ must flow to $e^{i\sigma/\hbar} \Psi_x(\lambda_f)$. We shall therefore require that $\tilde{Q}[\Psi, \Psi^\dagger]$ be invariant under global gauge transformations,

\[
\tilde{Q}[\Psi e^{i\sigma/\hbar}, \Psi^\dagger e^{-i\sigma/\hbar}] = \tilde{Q}[\Psi, \Psi^\dagger]. \tag{191}
\]
The linear and cubic terms: The functionals $\tilde{Q}_1$ and $\tilde{Q}_3$ in (189) are ruled out because they violate the global gauge symmetry (191). This is immediately clear once one writes

$$\tilde{Q}_1[\Psi, \Psi^\dagger] = \int dx \langle \Psi_x A_x^\dagger + A_x \Psi_x^\dagger \rangle_0$$

where $A$ is some generic multivector kernel and, similarly, for $\tilde{Q}_3$ which involves integrands of the form

$$\langle A\Psi B\Psi^\dagger C\Psi^\dagger D \rangle_0 \quad \text{or} \quad \langle A\Psi^\dagger B\Psi C\Psi^\dagger D \rangle_0 .$$

The quadratic term: The term $\tilde{Q}_2$ in (189) is bilinear in $\Psi$ and $\Psi^\dagger$. Gauge invariance restricts us to functionals of the form

$$\tilde{Q}_2[\Psi, \Psi^\dagger] = \int dx_1 dx_2 (A_{x_1 x_2} \Psi_{x_1 x_2}^\dagger \Psi_{x_1 x_2}^\dagger)$$

where $A$ and $B$ are multivector kernels. $\tilde{Q}_2$ can be simplified as follows. Introducing a matrix representation,

$$\Psi_{x_1} B_{x_1 x_2} \Psi_{x_2}^\dagger = \begin{bmatrix} \psi_{+x_1} & 0 \\ \psi_{-x_1} & 0 \end{bmatrix} \begin{bmatrix} B_{11, x_1 x_2} & B_{12, x_1 x_2} \\ B_{21, x_1 x_2} & B_{22, x_1 x_2} \end{bmatrix} \begin{bmatrix} \psi_{+x_2}^\dagger & \psi_{-x_2}^\dagger \\ 0 & 0 \end{bmatrix}$$

shows that the matrix elements $B_{12}$, $B_{21}$, and $B_{22}$ do not contribute and can therefore be set to zero. The remaining matrix element $B_{11, x_1 x_2}$, which is just a complex number, can be absorbed into the kernel $A_{x_1 x_2}$. Let $B_{11} A = Q$, then

$$\tilde{Q}_2[\Psi, \Psi^\dagger] = \int dx_1 dx_2 (\Psi_{x_1}^\dagger \tilde{Q}_{x_1 x_2} \Psi_{x_2})_0 .$$

Furthermore, since $\langle M^\dagger \rangle_0 = \langle M \rangle_0$ for any multivector $M$, we have

$$\tilde{Q}_2[\Psi, \Psi^\dagger] = \int dx_1 dx_2 (\Psi_{x_1}^\dagger \tilde{Q}_{x_1 x_2} \Psi_{x_2})_0 = \int dx_1 dx_2 (\Psi_{x_2}^\dagger \tilde{Q}_{x_1 x_2} \Psi_{x_1})_0$$

which implies that the kernel $A$ can be taken to be Hermitian,

$$Q_{x_1 x_2}^\dagger = Q_{x_1 x_2} .$$

More explicitly, we can introduce a matrix representation

$$\Psi_{x_1}^\dagger \tilde{Q}_{x_1 x_2} \Psi_{x_2} = \begin{bmatrix} \psi_{+x_1}^\dagger & \psi_{-x_1}^\dagger \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{Q}_{11, x_1 x_2} & \tilde{Q}_{12, x_1 x_2} \\ \tilde{Q}_{21, x_1 x_2} & \tilde{Q}_{22, x_1 x_2} \end{bmatrix} \begin{bmatrix} \psi_{+x_2} & \psi_{-x_2}^\dagger \\ 0 & 0 \end{bmatrix} ,$$

and write the Hamiltonian functional as

$$\tilde{Q}_2[\Psi, \Psi^\dagger] = \int dx_1 dx_2 (\psi_{+x_1}^\dagger \tilde{Q}_{11, x_1 x_2} \psi_{+x_2} + \psi_{-x_1}^\dagger \tilde{Q}_{12, x_1 x_2} \psi_{-x_2}$$

$$+ \psi_{-x_1}^\dagger \tilde{Q}_{21, x_1 x_2} \psi_{+x_2} + \psi_{+x_1}^\dagger \tilde{Q}_{22, x_1 x_2} \psi_{-x_2})_0$$

which shows the various possible couplings between the spin up and down amplitudes.
The quartic terms: The $\tilde{Q}_4$ functional takes the generic form

$$\tilde{Q}_4 = \int dx_1 dx_2 dx_3 dx_4 \left< K_{x_1 x_2 x_3 x_4} \psi_{+x_1} \psi_{+x_2}^* \psi_{-x_3} \psi_{-x_4}^* \right>_0 ,$$  \tag{200}$$

where the kernel $K$ is a complex number. Its action is to generate non-linear HK flows that extend beyond the reach of the familiar unitary operators that are provided within the linear framework of quantum mechanics. These quartic functionals are ruled out because they are in conflict with rotational invariance. Indeed, from (48) and (49) we see that rotating the wave function $\Psi$, eq.(142), by Euler angles $\{\theta', \phi', \chi'\}$,

$$\Psi'(x) = \psi_+'(x)u_+ + \psi_-'(x)u_-, \tag{201}$$

has the effect of mixing the amplitudes $\psi_+$ and $\psi_-$,

$$\psi_+ = \psi_+ e^{-i(x' + \phi')/2} \cos \frac{\theta'}{2} - \psi_- e^{i(x'-\phi')/2} \sin \frac{\theta'}{2}, \tag{202}$$

$$\psi_- = \psi_+ e^{-i(x'-\phi')/2} \sin \frac{\theta'}{2} + \psi_- e^{i(x'+\phi')/2} \cos \frac{\theta'}{2}. \tag{203}$$

This means that a rotated $\tilde{Q}_4$ will include terms that violate the HK condition (188). Note also that a rotation mixes $\psi_+$ and $\psi_-$ but does not mix $\psi$ and $\psi^*$. Therefore a rotated $\tilde{Q}_2$, eq.(199), does not violate (188) while a rotated $\tilde{Q}_4$ does.

To summarize: The Hamiltonian functionals that generate Hamilton-Killing flows and are compatible with rotational invariance take the form (196) which is linear in $\Psi$ and in its adjoint $\Psi^\dagger$.

7 The e-Hamiltonian and the Pauli equation

We have shown that the condition for the simplest form of dynamics — one that preserves the metric, symplectic, and complex structures — is a Hamilton-Killing flow generated by a Hamiltonian $\tilde{H}$ of the form (196),

$$\tilde{H} = \int dx_1 dx_2 \left< \Psi_{x_1 x_2}^\dagger \tilde{H}_{x_1 x_2} \Psi_{x_2} \right>_0 ,$$  \tag{204}$$

In ED the clock that defines entropic time is provided by the system itself and it is natural to demand that $\tilde{H}$ — the generator of time translations — be defined in terms of the very same clock. Thus, the last ingredient in the construction of $\tilde{H}$ is to require its agreement with $\hat{H}$ in order to reproduce the entropic evolution of $\rho$ given by the continuity eq.(105). To proceed we introduce

$$\tilde{H}_0 = \int dx \left< \hat{H}_0 \Psi \right>_0 \quad \text{with} \quad \hat{H}_0 = \frac{1}{2m} \left( \frac{\hbar}{i} \hat{\beta} - \frac{q}{c} \hat{A} \right)^2 \Psi , \tag{205}$$

34
and use the identity
\[ \rho \frac{1}{2} m v^2 = \left\langle \Psi^\dagger \hat{H}_0 \Psi \right\rangle_0 + \frac{\hbar^2}{2m} \rho^{1/2} \partial^2 \rho^{1/2} - \frac{\hbar^2}{8m} \rho (\partial_a \vec{s})^2 + \frac{\hbar q}{2mc} \rho \vec{B} \cdot \vec{s}, \]
(206)
derived in Appendix A. This allows us to rewrite \( \tilde{H}[\rho, \Phi, \phi, \theta] \) in (106) as
\[ \tilde{H}[\Psi, \Psi^\dagger] = \int dx \left\langle \Psi^\dagger \hat{H}_0 \Psi \right\rangle_0 + F'[\rho, \phi, \theta], \]
(207)where
\[ F'\left[ \rho, \phi, \theta \right] = F[\rho, \phi, \theta] + \int dx \rho \left( \frac{\hbar^2}{2m} \partial^2 \rho^{1/2} - \frac{\hbar^2}{8m} (\partial_a \vec{s})^2 + \frac{\hbar q}{2mc} \vec{B} \cdot \vec{s} \right). \]
(208)Two conditions are to be imposed on \( F'[\rho, \phi, \theta] \): first, in order for \( \tilde{H}[\Psi, \Psi^\dagger] \) to generate an HK flow we require that \( F' \) itself be of the form (196),
\[ F'\left[ \rho, \phi, \theta \right] = \int dx_1 dx_2 \langle \Psi^\dagger x_1 \hat{V}_{x_1 x_2} \Psi x_2 \rangle_0 \]
(209)for some Hermitian kernel \( \hat{V}_{x_1 x_2} \). And second, to reproduce the ED flow given by (105) \( F'[\rho, \phi, \theta] \) must be independent of \( \Phi \),
\[ \frac{\delta F'\left[ \rho, \phi, \theta \right]}{\delta \Phi_x} = 0. \]
(210)Note that \( \vec{s} \) in eq. (51) depends only on \( \theta \) and \( \phi \) so the integral term in (208) is indeed independent of \( \Phi = -\hbar \vec{\chi}/2 \). Substituting (104) into (209) we get
\[ F' = \int dx_1 dx_2 \rho^{1/2} \rho^{1/2} \langle e^{i(\Phi x_2 - \Phi x_1)/\hbar} \hat{U}_x \hat{V}_{x_1 x_2} U_x (1 + \vec{e}_3) \rangle_0 \]
(211)To satisfy (210) for arbitrary choices of \( \Phi_{x_1}, \Phi_{x_2}, U_{x_1}, \) and \( U_{x_2} \) we must impose that the kernel \( \hat{V}_{x_1 x_2} \) be local in \( x \), that is,
\[ \hat{V}_{x_1 x_2} = \delta_{x_1 x_2} \hat{V}_{x_1} = \delta_{x_1 x_2} (\alpha_x + \vec{a}_x + i \vec{b}_x + i \beta_x), \]
(212)where the multivector field \( \hat{V}_{x_1} \) has been expressed in terms of its pure grade components. Substituting into (211),
\[ F' = \int dx_1 dx_2 \rho_x (\alpha_x + \vec{a}_x + \vec{b}_x + i \beta_x) (1 + \vec{s}_x) \rangle_0, \]
(213)leads to
\[ F' = \int dx \rho_x (\alpha_x + \vec{s}_x \cdot \vec{a}_x). \]
(214)We see that the bivector \( \vec{b} \) and pseudoscalar \( i \beta \) components do not contribute and can therefore be set to zero. Then, using (212) in (209) we get
\[ F' = \int dx \langle \Psi^\dagger \hat{V}_x + \vec{a}_x \Psi_x \rangle_0, \]
(215)
where we adopt the standard notation for the scalar potential, $\alpha_x = V_x$. Substituting eqs. (205) and (215) into (207) gives the final expression for the e-Hamiltonian,

$$\tilde{H}[\Psi, \Psi^\dagger] = \int dx \langle \Psi^\dagger \tilde{H} \Psi \rangle_0$$  \hspace{1cm} (216)

where

$$\tilde{H} = \hat{H}_0 + V + \vec{a}.$$ \hspace{1cm} (217)

Below, in section 8.6, we discuss the effect of the vector potential $\vec{a}$; for now we shall set $\vec{a} = 0$.

We are now ready to derive the Pauli equation. From (127),

$$\partial_t \Psi = \{ \Psi, \tilde{H} \} \text{ or } i\hbar \partial_t \Psi = \frac{\delta \hat{H}}{\delta \Psi^\dagger} = \hat{H} \Psi,$$

where we used (27). Then the Pauli equation is

$$i\hbar \partial_t \Psi = \frac{1}{2m} \left( \frac{\hbar}{i} \vec{\sigma} - \frac{q}{c} \vec{A} \right)^2 \Psi + V \Psi,$$

which can be rewritten in its more standard form

$$i\hbar \partial_t \Psi = \frac{1}{2m} \left( \frac{\hbar}{i} \vec{\sigma} - \frac{q}{c} \vec{A} \right) \cdot \left( \frac{\hbar}{i} \vec{\sigma} - \frac{q}{c} \vec{A} \right) \Psi + V \Psi - \frac{\hbar q}{2mc} \vec{B} \Psi.$$  \hspace{1cm} (220)

The last term represents the “anomalous” Zeeman interaction of the magnetic field with the spin. The standard practice is to represent the basis vectors $\vec{e}_a$ by Pauli matrices $\hat{\sigma}_a$. Then the matrix representation of the Zeeman term for the electron ($q = -e$) is

$$- \frac{\hbar q}{2mc} \vec{B} \Psi \sim g \frac{e}{2mc} \frac{\hbar}{2} \hat{\sigma}_a B_a \Psi \quad \text{with} \quad g = 2.$$  \hspace{1cm} (221)

which exhibits the correct gyromagnetic factor.

Once the canonical framework has been established with a Hamiltonian and Poisson brackets it is possible to summarize the whole formalism in terms of an action principle. The idea is to reverse the usual procedure and construct the action so that it reproduces the desired equations. Define the differential

$$\delta \mathcal{A} = \int_{t_1}^{t_2} dt \int_R dx \left\langle \delta \Psi^\dagger \left( i\hbar \partial_t \Psi - \hat{H} \Psi \right) - \left( i\hbar \partial_t \Psi^\dagger + \Psi^\dagger \hat{H} \right) \delta \Psi \right\rangle_0$$  \hspace{1cm} (222)

with the fields $\delta \Psi^\dagger$ and $\delta \Psi$ vanishing at the boundary and then integrate to get the action

$$\mathcal{A} = \int_{t_1}^{t_2} dt \int_R dx \left\langle i\hbar \Psi^\dagger \partial_t \Psi - \Psi^\dagger \hat{H} \Psi \right\rangle_0.$$  \hspace{1cm} (223)

By construction, imposing $\delta \mathcal{A} = 0$ leads to the Pauli equation (218). In the ED framework this construction is a convenient way to summarize the content of the theory but it is not particularly fundamental.
8 Discussion

8.1 Energy

The e-Hamiltonian $\hat{H}$ is the generator of translations in time. It tells us how wave functions evolve and, therefore, it tells us that energy is an epistemic concept associated to the wave function and not to the particle. To gain further insight substitute (218) into (216),

$$\hat{H} = \int dx \langle \Psi^\dagger i\hbar \partial_t \Psi \rangle_0 .$$  \hspace{1cm} (224)

Using (137) a calculation similar to (71) gives

$$\langle \Psi^\dagger i\hbar \partial_t \Psi \rangle_0 = \rho \tilde{\omega}_t \cdot \vec{S}$$  \hspace{1cm} (225)

where a factor $\hbar/2$ has been absorbed into the spin vector to obtain the more familiar normalization

$$\vec{S}(x) = \frac{\hbar}{2} \vec{s}(x) ,$$  \hspace{1cm} (226)

and

$$\tilde{\omega}_t = \vec{e}_3 \partial_t \Phi + \vec{e}_\phi \partial_t \theta + \vec{s} \partial_t \bar{\Phi}$$  \hspace{1cm} (227)

is the angular velocity. Eq. (225) allows us to define a local energy,

$$\varepsilon(x) \overset{\text{def}}{=} \tilde{\omega}_t(x) \cdot \vec{S}(x) = -\partial_t \Phi + \frac{\hbar}{2} \cos \theta \partial_t \phi .$$  \hspace{1cm} (228)

For those special states $\Psi_E$ that happen to be eigenstates of $\hat{H}$,

$$\hat{H} \Psi_E = E \Psi_E ,$$  \hspace{1cm} (229)

eq. (216) gives

$$\hat{H}[\Psi_E , \Psi_E^\dagger] = \int dx \rho E = E \text{ so that } \varepsilon(x) = E .$$  \hspace{1cm} (230)

Thus, energy eigenstates are those for which the local energy $\varepsilon(x)$ is independent of position.

Our concept of a local energy bears an inevitable formal resemblance to the corresponding “local observable” proposed by Hestenes and Gurtler [52] but the conceptual difference is significant: our local energy is an epistemic object, a property of the wave function; their local energy is ontic, a property of the particle.

8.2 Linear momentum

Momentum is the generator of spatial translations. Under a translation $x^a \rightarrow x^a + \xi^a$ the wave function transforms as

$$\Psi(x) \rightarrow \Psi_\xi(x_\xi) = \Psi(x) \text{ or } \Psi_\xi(x) = \Psi(x - \xi)$$  \hspace{1cm} (231)
Then the momentum functional $\tilde{P}_a$ is defined so that
\[ \delta \xi \Psi = \Psi_\xi(x) - \Psi(x) = -\xi^a \partial_a \Psi = \{ \Psi, \tilde{P}_a \xi^a \} \].

(232)

The required $\tilde{P}_a$ is such that
\[ \frac{\delta \tilde{P}_a}{\delta i \hbar \Psi^\dagger} = -\partial_a \Psi \],

(233)

which, with the help of (27), can be integrated to
\[ \tilde{P}_a[\Psi, \Psi^\dagger] = \int dx \langle \Psi^\dagger \frac{\hbar}{i} \partial_a \Psi \rangle_0 \]

(234)

Further insight comes from a calculation similar to (71) which gives the momentum density
\[ \langle \Psi^\dagger \frac{\hbar}{i} \partial_a \Psi \rangle_0 = -\rho \vec{\omega}_a \cdot \vec{S} \]

(235)

where
\[ \vec{\omega}_a = \vec{e}_3 \partial_a \phi + \vec{e}_\phi \partial_a \theta + \vec{s} \partial_a \bar{\chi} \]

(236)

is the dual of the spin connection, $\Omega_a = i \vec{\omega}_a$. Eq. (235) allows us to define a local momentum,
\[ p_a(x) \overset{\text{def}}{=} -\vec{\omega}_a(x) \cdot \vec{S}(x) \],

(237)

or, using eq. (230),
\[ p_a(x) = mv_a(x) + q c A_a(x) \].

(238)

The probability flux,
\[ \rho v_a = \frac{1}{m} \langle \Psi^\dagger (\frac{\hbar}{i} \partial_a - \frac{q}{c} A_a) \Psi \rangle_0 \],

(239)

turns out to involve the same velocity $v^a$ which, as a result of eqs. (96) and (101), can be attributed to the particle. This fact might lead us to the mistaken belief that momentum is a property of the particle, but momentum is what generates translations of the wave function and, therefore, it is an epistemic property of the wave function. The local momentum can be inferred from our knowledge of the wave function but it is not itself directly observable. Just as for energy, the momentum eigenstates are those for which the local momentum (238) is independent of position. In fact it turns out that the outcomes of what are normally called “measurements of momentum” do not reflect the pre-existing values of local momentum; they reflect eigenvalues that are actually “created” during the measurement process and are distributed according to the standard Born rule involving the Fourier transform of the wave function [13][14].

From (236) and (237) we see that the local momentum,
\[ p_a = \partial_a \Phi - \frac{\hbar}{2} \cos \theta \partial_a \phi \],

(240)
depends only on spatial derivatives of the phase $\Phi$ and the spin and not on the
probability density. This might at first look surprising. It implies, for example,
that in the ground state of hydrogen where both $\Phi$ and $\vec{S}$ are uniform in space
the local momentum $p_a$ and the velocity $v^a$ vanish. The electron is at rest; there
is a zero point energy but no zero point motion. How can this be? Why doesn’t
an electron at rest immediately start falling towards the nucleus? What holds
it in place? The answer is that the electron does not obey Newtonian dynamics
and our intuition must be re-educated. In the ED approach there are no forces
acting on the electron. ED is a dynamics of probabilities; what is being pushed
is not the particle but our expectations about where the electron will be found.

8.3 Orbital and spin angular momentum

Angular momentum is the generator of rotations. Under a rotation by an angle
$\xi$ about the axis $\vec{n}$ we have

$$\vec{x} \rightarrow \vec{x}_\xi = R_\xi \vec{x} R_\xi^\dagger \quad \text{where} \quad R_\xi = e^{-i\vec{n}\xi/2}$$

(241)

and the wave function transforms as

$$\Psi(x) \rightarrow \Psi_\xi(x_\xi) = R_\xi \Psi(x) \quad \text{or} \quad \Psi_\xi(x) = R_\xi \Psi(x-\xi) .$$

(242)

In a rotation by the infinitesimal angle $\xi$,

$$\vec{x}_{-\xi} = \vec{x} - \xi \vec{n} \times \vec{x} ,$$

(243)

and the change of the wave function is

$$\delta_\xi \Psi(x) = R_\xi \Psi(x-\xi) - \Psi(x) = -\frac{i}{\hbar} \xi n^a \hat{J}_a \Psi(x)$$

(244)

where the total angular momentum operator,

$$\hat{J}_a = \hat{L}_a + \hat{S}_a ,$$

(245)

includes orbital and spin contributions,

$$\hat{L}_a = \vec{e}_a \cdot (\vec{x} \times \frac{\hbar}{i} \partial) \quad \text{and} \quad \hat{S}_a = \frac{\hbar}{2} \vec{e}_a.$$  

(246)

The angular momentum functional $\tilde{J}_a$ that generates rotations does so ac-

according to the Poisson bracket

$$\delta_\xi \Psi = \{ \Psi, \xi n^a \tilde{J}_a \} \quad \text{so that} \quad \frac{\delta \tilde{J}_a}{\delta \hbar \Psi} = \tilde{J}_a \Psi .$$

(247)

Using (27) this integrates to

$$\tilde{J}_a[\Psi, \Psi^\dagger] = \int dx \langle \Psi^\dagger \tilde{J}_a \Psi \rangle_0 = \int dx \langle \Psi^\dagger \left( \frac{\hbar}{i} \varepsilon_{abc} x_b \partial_c + \frac{\hbar}{2} \vec{e}_a \right) \Psi \rangle_0 .$$  

(248)
To gain further insight into the spin angular momentum we note that in a Pauli matrix representation of GA, \( \vec{e}_a \sim \hat{\sigma}_a \), one recovers the familiar expression for the spin operator, \( \hat{S}_a \sim (\hbar/2) \hat{\sigma}_a \). Furthermore, a calculation similar to (71) allows us to define the local spin density,

\[
\langle \Psi | \frac{\hbar}{2} \vec{e}_a | \Psi \rangle_0 = \frac{\hbar}{2} \rho \langle \vec{e}_a (1 + \vec{s}) \rangle_0 = \frac{\hbar}{2} \rho \vec{e}_a \cdot \vec{s} = \rho S_a
\]

so that the vector spin functional is

\[
\tilde{S} [\Psi, \Psi^\dagger] = \vec{e}_a \int dx \langle \Psi | \frac{\hbar}{2} \vec{e}_a | \Psi \rangle_0 = \int dx \rho \vec{S}.
\]

**What is spinning?** — Once an angular momentum such as spin is identified it is tempting to ask “what is it that rotates?” But this is a classical prejudice that must be avoided. Angular momentum is a mathematical abstraction; it is the generator of rotations but the rotations need not be dynamical. In particular, there exist no small rigid rotators or vortices in some underlying fluid. Nothing is spinning. Once again, the ground state of hydrogen, with \( \vec{v} = 0 \) and a constant \( \vec{S} \), is instructive. This is a situation in which neither the electron is moving nor are the probabilities flowing and yet the integral in (250) gives a total spin of \( \vec{S} = \hbar/2 \).

**The magnitude of spin?** — The spin vector \( \vec{S} \) has magnitude \( \hbar/2 \). Note however that the GA expression

\[
\tilde{S}^2 + \tilde{S}_2^2 + \tilde{S}_3^2 = (\frac{\hbar}{2})^2 (\vec{e}_1^2 + \vec{e}_2^2 + \vec{e}_3^2) = \frac{3}{4} \hbar^2
\]

corresponds to

\[
\tilde{S}^2 \sim (\frac{\hbar}{2})^2 (\hat{\sigma}_1^2 + \hat{\sigma}_2^2 + \hat{\sigma}_3^2)
\]

and exactly reproduces the well-known expression

\[
\tilde{S}^2 \sim \hbar^2 s(s + 1) \hat{1} \quad \text{with} \quad s = \frac{1}{2}.
\]

But the interpretations are completely different: the latter is meant to represent the magnitude squared of the spin operator while the former is just the square of the diagonal of a cube of side \( \hbar/2 \).

**A connection to relativity?** — It may surprise us that in the GA framework the spin operator \( \tilde{S}_a \) in (246) is a vector, while \( \tilde{L}_a \) is a pseudoscalar, so that the orbital angular momentum,

\[
\tilde{L} = \vec{x} \times \frac{\hbar}{i} \vec{\partial},
\]

40
is a bivector. This, of course, is not a problem: the actual generator of rotations is $\tilde{J}_a$ in eq. (248). Nevertheless, the operators for spin $\hat{S}_a$ and for orbital angular momentum $L_a$ appear to be very different objects. The gap between the two can, however, be narrowed in a relativistic context. It turns out that the Pauli algebra of space $\{1, e_a, i\epsilon_a, \epsilon_1 \epsilon_2 \epsilon_3\}$ is isomorphic to the even subalgebra of Minkowski space-time. If we adopt an orthonormal basis for space-time, $\{\gamma_\mu\}$ with $\gamma_0^2 = -1$ and $\gamma_a^2 = 1$, the even subalgebra is $\{1, \gamma_0 \gamma_a, i\gamma_0 \gamma_a, i\gamma_0 \gamma_1 \gamma_2 \gamma_3\}$ and it is natural to identify $\vec{e}_a = \gamma_0 \gamma_a$ (instead of $\vec{e}_a = \gamma_a$) so that the spatial vector $\vec{e}_a$ is a space-time bivector. Here we shall not pursue this subject further except to point out that, as we shall see in [53], this isomorphism is exploited to great advantage when formulating the ED of several particles with spin.

8.4 The electric current and the magnetic dipole moment

When quantities such as mass or charge are interpreted as ontic properties of the particle it is only natural to expect that the corresponding densities and currents be closely related. For example, if the particle flux is $\rho \vec{v}$ one expects the fluxes of mass and charge to be $m \rho \vec{v}$ and $q \rho \vec{v}$ — each moving particle automatically carries an amount $m$ of mass and $q$ of charge. Conversely, when mass and charge are not ontic but epistemic properties assigned to the wave function the question of whether the mass and charge currents coincide must be revisited.

We define the electric charge and electric current through their coupling to the electromagnetic potentials. This is most easily implemented in terms of the action, eq. (223),

$$A = \int dt dx \left\langle \Psi^\dagger \left( i \hbar \frac{\partial}{\partial t} - \frac{1}{2m} \left( \frac{\hbar}{i} \vec{\partial} - \frac{q}{c} \vec{A} \right)^2 - qA^0 \right) \Psi \right\rangle$$

(255)

where the potential $V$ is written as $qA^0$. The electric charge $\rho_e$ and current density $\vec{J}_e$ are defined as functional derivatives with respect to the potentials $A^0$ and $A^a$,

$$\delta A \overset{\text{def}}{=} \int dt dx \left[ -\rho_e \delta A^0 + \frac{1}{c} \vec{J}_e \cdot \delta \vec{A} \right]$$

(256)

A straightforward calculation gives the charge density

$$\rho_e = q\langle \Psi^\dagger \Psi \rangle = q\rho$$

(257)

and an electric current that splits into convective and spin components,

$$J_e^a = \frac{q}{m} \langle \Psi^\dagger (\frac{\hbar}{i} \partial^a - \frac{q}{c} A^a) \Psi \rangle + \frac{q}{m} \varepsilon^{abc} \partial_b \langle \Psi^\dagger \frac{\hbar}{2} \epsilon_c \Psi \rangle .$$

(258)

Using (239) and (249) this is written as

$$\vec{J}_e = q\rho \vec{v} + c \vec{\partial} \times \vec{M} ,$$

(259)
which explicitly shows that the spin contributes to the electric current as a magnetization current with a magnetization $\vec{M}$ proportional to the spin density $\rho \vec{S}$,

$$\vec{M} = \frac{q}{mc} \rho \vec{S}.$$  \hfill (260)

The convective $q \rho \vec{v}$ and magnetization $c \hat{\vec{a}} \times \vec{M}$ currents are quite independent. In the ground state of hydrogen, with $\vec{v} = 0$ and $\vec{S}$ constant, nothing is moving but it is easy to check that the electric current does not vanish. If we take $\vec{S}$ in the $\hat{e}_3$ direction then the current forms a toroidal ring in the vicinity of the $xy$ plane. It is this “current” in which nothing flows that is responsible for the electron’s magnetic dipole moment. Alternatively, we can adopt a manifestly “static” model and explain the dipole moment as arising from a magnetic pole density $-\hat{\vec{a}} \cdot \vec{M}$. Again, it is easy to check that this gives North and South pole densities concentrated around the lower and upper $\hat{e}_3$ axis respectively.

### 8.5 What happens in a spin measurement?

Since position is the only ontic quantity all measurements in ED must be traced back to detecting the presence or absence of a particle within the volume of a device. Position is the only beable and therefore the only observable. All other quantities, such as momentum or energy, are necessarily epistemic and cannot therefore be observed. These quantities can at best be inferred from position detections and, accordingly, it is appropriate to refer to them as inferables. Spin is no exception.

The process of “measuring” spin by means of a Stern-Gerlach (SG) experiment is well known. The central question of what precisely happens during the experiment depends on the particular interpretation of quantum mechanics one adopts. (For a recent review with references to the literature see [54].)

ED resembles Bohmian mechanics in that particles have definite positions and its formalism includes functions $(\Phi, \theta, \phi)$ that play the role of a pilot wave. However, the differences are considerable. Bohmian mechanics attempts to provide a complete description of reality. The wave function $\Psi$ is a real field that lives in configuration space and acts on the particles without the particles reacting back upon it — peculiarities that have stood in the way of a wider acceptance of the Bohmian interpretation. In contrast, ED’s pragmatic goal is limited to make the best possible inferences on the basis of very incomplete information; the ED variables $(\Phi, \theta, \phi)$ are purely epistemic and there is no implication that the particles are carried by a pilot wave or pushed by any other force. In fact ED is silent on the issue of what if anything makes the particles move as they do. The wave function $\Psi$ exerts no causal influence on the particles themselves; what the wave function $\Psi$ does is to guide our expectations of where and when the particles will be. Having described the differences between the ED and

\[\text{The term 'inferable' is due to K. Vanslette. With the recent development of techniques to perform weak measurements the wave function is itself an inferable. (See e.g., [14] and references therein.)}\]
Bohmian frameworks we now proceed to exploit the similarities and borrow from the analysis of the Pauli equation by Dewdney et al. [55].

The particle is prepared in a state that describes an initial wave packet \( \Psi(t_0, \vec{x}) \) with a spin vector \( \vec{S}(t_0) = \hbar \vec{s}/2 \) in some definite but arbitrary direction \( \vec{s} \). Solving the Pauli equation shows that the particle can follow any of a congruence of smooth trajectories. Which trajectory is actually followed depends only on the particle’s initial position within the wave packet \( \Psi(t_0, \vec{x}) \). As the particle traverses the SG device the spin vector field \( \vec{S}(t, \vec{x}) \) evolves continuously. The analysis of Dewdney et al. shows that the wave packet splits into two separate components along the direction \( \vec{e}_z \) of the inhomogeneous magnetic field. Whether the particle ends in the upper or the lower wave packet depends only which trajectory was followed.

The remarkable result is that the evolution leads to a strong correlation between final values of the spin vectors and the particle trajectories: all those trajectories that end up in the upper wave packet will have final spin \( \vec{S} = +\hbar \vec{e}_z/2 \) while all trajectories that end up in the lower wave packet will have final spin \( \vec{S} = -\hbar \vec{e}_z/2 \). The analysis also shows that the actual probabilities of being in the upper of lower wave packet turn out to be given by the standard Born rule. (Note that these are the probabilities of being and not merely being found in one or the other packets.)

It is only at this final stage that an actual measurement happens: the particle’s position is detected. Depending on how precise the position measurement is this allows us not only to infer the trajectory that was actually followed but also the final value of the spin vector being either \( +\hbar/2 \) or \( -\hbar/2 \) along the \( \vec{e}_z \) direction.

This result is remarkable in several ways. The discrete outcomes \( \pm \hbar/2 \) explain the so-called “space quantization” without invoking the eigenvalues of \( \hat{\sigma}_z \). Standard QM is forced to postulate the Born rule and the rule that the possible outcomes of a measurement are given by the eigenvalues of the Hermitian operator being measured. Within ED such postulates are no longer needed. (See also [13][14].) One can also see that the outcomes of this “measurement” of spin do not reflect the initial pre-existing spin value. The outcome of the experiment was in effect “created” by the process of measurement. Indeed, as emphasized long ago by Bell [3] the term ‘measurement’ provides a very inadequate description of what actually happens in the SG experiment.

### 8.6 Other interactions

Here we briefly comment on additional interactions that might be included in (217) through the scalar potential \( V \) or the vector potential \( \vec{a} \). The requirement that the kernel \( V_{x_1,x_2} \) be local, eq.(212), is very restrictive because it rules out interactions that involve spatial derivatives of \( \Psi \). Therefore momentum- and velocity-dependent interactions — see e.g. eq.(240) — are ruled out. These

21Unlike ED, in Bohmian mechanics the identification of the probability \( \rho \) with \( |\Psi|^2 \) requires for its justification additional dynamical arguments that are highly nontrivial.
include relativistic corrections to the kinetic energy that can be described as a scalar perturbation ($\propto p^4$) and the spin-orbit interaction. Indeed, a magnetic dipole $\vec{\mu} \propto \vec{S}$ in motion exhibits an electric dipole $\vec{d}$ which interacts with the electric field in the atom. However, the interaction depends on the velocity $\vec{v}$,

$$\hat{H}_{so} \propto -\vec{d} \cdot \vec{E} = \left(\frac{\vec{v}}{c} \times \vec{\mu}\right) \cdot \vec{E},$$

and is therefore ruled out because it involves derivatives. In order to reproduce these relativistic effects one would need a fully relativistic treatment that from the start takes the positron degrees of freedom into account.

The vector potential $\vec{a}$ in (217) does, however, allow new interactions. In the presence of an external electromagnetic field the only available vectors are the magnetic and the electric fields, $\vec{B}$ and $\vec{E}$, and this allows a Hamiltonian of the form

$$\hat{H} = \hat{H}_0 + V + \kappa_m \vec{B} + \kappa_e \vec{E},$$

for some constants $\kappa_m$ and $\kappa_e$. In the usual matrix representation $\hat{\sigma}_a \sim \hat{\sigma}_a$, the fields $\vec{B}$ and $\vec{E}$ are written as $B_a \hat{\sigma}_a$ and $E_a \hat{\sigma}_a$ where $\hbar \hat{\sigma}_a/2$ are called the spin operators. Then the term $\kappa_m \vec{B} \propto \vec{S} \cdot \vec{B}$ describes the anomalous magnetic moment of say, the proton or the neutron. The analogous electric dipole term $\kappa_e \vec{E} \propto \vec{S} \cdot \vec{E}$ is in principle allowed but, as we show next, its presence would signal a violation of time reversal invariance.

**Time reversal**

The transformation of the spinor wave function $\Psi$ involves taking $t \to -t$ and the spatial inverse which plays the role of complex conjugation. In addition, since spatial inversion has the effect of changing the ideal,

$$(1 + \hat{e}_3)^* = 1 - \hat{e}_3,$$

it is necessary to right multiply by $\hat{e}_1$ in order to restore the original ideal,

$$u_+ = \frac{1 + \hat{e}_3}{\sqrt{2}} \rightarrow u_+^T = u_+^* \hat{e}_1 = \frac{1 - \hat{e}_3}{\sqrt{2}} \hat{e}_1 = \frac{1 + \hat{e}_3}{\sqrt{2}}.$$

(Right multiplication by $\hat{e}_2$ also works.) Therefore the time reversal of $\Psi$ is implemented by

$$\Psi_t^T(x) = \Psi_{-t}^*(x) \hat{e}_1,$$

or, equivalently

$$\Psi_t(x) = \psi_+(x,t)u_+ + \psi_-(x,t)u_- \rightarrow \Psi_t^T(x) = -\psi_+(x,-t)u_+ + \psi_-(x,t)u_-.$$

To find the time-reversed spin vector we appeal to the “Born” rule, eq. (141) and use

$$\Psi^T \Psi^T = \rho U^* u_+^* \hat{e}_1^* u_+^* (1 - \hat{e}_3) U^T = \rho (1 - \hat{s})$$

44
so that
\[ \rho^T = \langle \Psi^T \Psi^\dagger \rangle_0 = \rho \quad \text{and} \quad \rho s^T = \langle \Psi^T \Psi^\dagger \rangle_1 = -\rho s^* . \] (268)
Therefore the time-reversed spin vector is
\[ \vec{S}^T_t(x) = -\vec{S}_{-t}(x) . \] (269)

The time reversal of the Pauli equation (219) involves the time-reversed electromagnetic fields,
\[ \vec{A}^T_t(x) = -\vec{A}_{-t}(x), \quad \vec{E}^T_t(x) = \vec{E}_{-t}(x), \quad \text{and} \quad \vec{B}^T_t(x) = -\vec{B}_{-t}(x) . \] (270)
It is easy to check that the transformed \( \Psi^T_t(x) \) is a solution of the transformed Pauli equation (219), that is, time reversal is a symmetry. However, under \( T \) an additional electric moment interaction \( \kappa_e \vec{E} \) as in (262) changes sign,
\[ \kappa_e \vec{E}_t \Psi_t^T \rightarrow (\kappa_e \vec{E}_{-t} \Psi_{-t})^* \vec{e}_1 = -\kappa_e \vec{E}^T_t \Psi^T_t , \] (271)
and would spoil the symmetry. Thus, time reversal symmetry allows a magnetic dipole but rules out an electric dipole.

### 8.7 Hilbert space

The formulation of the ED of a spin-1/2 particle is now complete. Just as for scalar particles it is important to emphasize that the notion of Hilbert spaces did not turn out to be necessary. However, while strictly unnecessary in principle, the introduction of Hilbert spaces is nevertheless very convenient for calculational purposes.

**A vector space** — The difficulty with dealing with the infinite-dimensional e-phase space — the cotangent bundle \( T^* \mathbb{C} \) — is that due to the normalization constraint the natural coordinates, the probabilities \( \rho_\mathbb{C} \), are not independent. The problem is handled by embedding \( T^* \mathbb{C} \) in a larger space \( T^* \mathbb{C}^{+1} \) but we are still forced to deal with a constrained system with a global gauge symmetry.

As we saw in [9] the choice of the embedding space is a matter of convenience. What is required in order to reproduce the metric structure of QM is that \( T^* \mathbb{C}^{+1} \) be spherically symmetric but any spherically symmetric space will do.

The usefulness of the linearity of the Pauli equation as an equation in the restricted space \( T^* \mathbb{C} \) is severely limited by the normalization constraint. If \( \Psi_1 \) and \( \Psi_2 \) are flows in \( T^* \mathbb{C} \) then the superposition \( \Psi_3 = \alpha_1 \Psi_1 + \alpha_2 \Psi_2 \) is a flow in \( T^* \mathbb{C} \) too but only when the constants \( \alpha_1 \) and \( \alpha_2 \) are such that \( \Psi_3 \) is properly normalized. We can take full advantage of linearity by choosing the embedding space \( T^* \mathbb{C}^{+1} \) to be not just spherically symmetric but also flat. The reason this choice is so convenient is that in a flat \( T^* \mathbb{C}^{+1} \) space all superpositions with arbitrary constants \( \alpha_1 \) and \( \alpha_2 \) are allowed. This means that each point \( \Psi \) is itself a vector and \( T^* \mathbb{C}^{+1} \) is a vector space. Indeed, since the vector tangent to a curve is just a difference of two neighboring \( \Psi \)s it follows that points on
the manifold and vectors tangent to the manifold are objects of the same kind. In other words, the spaces tangent to $T^*C_+^1$ are identical to the space $T^*C_+^1$ itself.

The symplectic form $\Omega$ and the metric tensor $G$ on the extended space $T^*C_+^1$ are given by eqs. (148) and (157). Being tensors they are meant to act on tangent vectors but now they are also free to act on all points $\Psi$ in the flat $T^*C_+^1$ space. For example, the action of the mixed tensor $J$, eq. (162), on a spinor wave function $\Psi$, eq. (146), is

$$J^{\mu x, \nu x'} = \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix} \begin{bmatrix} \psi_{+x} \\ i\hbar \psi^*_{+x} \\ \psi_{-x} \\ i\hbar (i\psi_{+x})^* \end{bmatrix} = \begin{bmatrix} i\psi_{+x} \\ i\hbar (i\psi_{+x})^* \\ i\psi_{-x} \\ i\hbar (i\psi_{-x})^* \end{bmatrix}$$

which indicates that $J$ plays the role of multiplication by $i$, that is, $\psi_{\pm} \rightarrow i\psi_{\pm}$.

**Dirac notation** — We can at this point introduce the Dirac notation to represent the spinor wave functions $\Psi$, eq. (142), as vectors $|\Psi\rangle$ in a Hilbert space. The scalar product $\langle \Psi_1 | \Psi_2 \rangle$ is defined using the metric $G$ and the symplectic form $\Omega$,

$$\langle \Psi_1 | \Psi_2 \rangle \overset{\text{def}}{=} \left( G_{\mu x, \nu x'} + \frac{i}{2\hbar} \Omega_{\mu x, \nu x'} \right) \Psi^{\mu x} \Psi'^{\nu x'}$$

where

$$G_{\mu x, \nu x'} + \frac{i}{2\hbar} \Omega_{\mu x, \nu x'} = \frac{1}{i\hbar} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

A quick calculation leads to the familiar result

$$\langle \Psi_1 | \Psi_2 \rangle = \int dx \left( \psi^*_+ \psi_{2+x} + \psi^*_- \psi_{2-x} \right) .$$

The map $\Psi \leftrightarrow |\Psi\rangle$ is defined by

$$|\Psi\rangle = \int dx \left( \psi_+ |u_+\rangle |x\rangle + \psi_- |u_-\rangle |x\rangle \right),$$

where the spinor basis \{ $|u_+\rangle, |u_-\rangle$ \} and the position basis \{ $|x\rangle$ \} are both orthogonal and complete.

**Hermitian and unitary operators** — The bilinear Hamilton functionals $\tilde{Q}[\Psi, \Psi^*]$ with kernel $\hat{Q}(x, x')$ in eqs. (196) or (199) can now be written in terms of a Hermitian operator $\hat{Q}$ and its matrix elements,

$$\tilde{Q}[\Psi, \Psi^*] = \langle \Psi | \hat{Q} | \Psi \rangle \quad \text{and} \quad \hat{Q}_{\sigma \sigma'}(x, x') = \langle u_{\sigma} | x | \hat{Q} | u_{\sigma'} \rangle |x'\rangle .$$
The corresponding Hamilton-Killing flows are given by
\[ i \hbar \frac{d}{d\lambda} |\Psi\rangle = \hat{Q} |\Psi\rangle . \] (278)

These flows are described by unitary transformations
\[ |\Psi(\lambda)\rangle = \hat{U}_Q(\lambda) |\Psi(0)\rangle \quad \text{where} \quad \hat{U}_Q(\lambda) = \exp \left( -\frac{i}{\hbar} \hat{Q}\lambda \right) . \] (279)

**Commutators** — The Poisson bracket of two Hamiltonian functionals \( \tilde{U}[\Psi, \Psi^*] \) and \( \tilde{V}[\Psi, \Psi^*] \),
\[ \{ \tilde{U}, \tilde{V} \} = \int dx \left( \frac{\delta \tilde{U}}{\delta \Psi_x} \frac{\delta \tilde{V}}{\delta i\hbar \Psi^*_x} - \frac{\delta \tilde{U}}{\delta i\hbar \Psi^*_x} \frac{\delta \tilde{V}}{\delta \Psi_x} \right) , \]
can be written in terms of the commutator of the associated operators, then
\[ \{ \tilde{U}, \tilde{V} \} = \frac{1}{i\hbar} \langle \Psi | [\tilde{U}, \tilde{V}] |\Psi\rangle . \] (280)

Thus the Poisson bracket is the expectation of the commutator. Further parallels between the geometric and the Hilbert space formulation of QM can be found in [39]-[48].

**9 Some final comments**

We conclude with a summary of the main results. In this paper the ED framework has been extended to describe a spin-1/2 particle. In this model the particle’s position is real while the wave function is epistemic. Position is the only ontic variable; all other dynamical variables including spin are properties of the wave function and therefore they are epistemic too.

The language of geometric algebra has been used to describe the rotational properties of spin and to construct the spinor wave function as an element of a minimal left ideal. The input of physical information is carried out by constraints that must be continuously updated. The geometric criterion for these updates requires identifying suitable symplectic and information geometric structures of phase space. The ED that preserves both the symplectic structure (a Hamiltonian flow) and the metric structure (a Killing flow) is described by a linear Hamiltonian flow. The additional condition that the Hamiltonian be the generator of translations in *entropic* time leads to the Pauli equation.

The introduction of Hilbert spaces as an additional structure is optional but it is very useful in that it allows us to exploit the calculational advantages of the linearity of the Pauli equation.

Spin in ED is not a feature of a rotating body, and it is not a feature of the motion of a point particle; it is an epistemic property of the wave function. The spin does not guide the motion of the particle; instead it guides the assignment
of probabilities of the particle’s position. It is noteworthy that since in the ED approach to QM the only probabilities are probabilities of position it makes no sense to talk about an isolated spin degree of freedom divorced from its associated particle. How would such an isolated spin ever be measured?

We have also seen that in ED the spin $\vec{S}$ is not quantized; what turns out to be quantized are the outcomes of Stern-Gerlach experiments. More generally, the statements that probability, mass, momentum, charge, and angular momentum including spin are not ontic but epistemic quantities are not innocent. They force an extreme revision of our intuitions about physics. Probabilities are not substances; they may change but they neither move nor flow. And similar considerations apply to spin. Accepting that spin is an epistemic concept forces us to abandon the classical intuition that an angular momentum betrays the existence of some ontic substance that is actually spinning.

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10 Appendix

Here we derive the identity (206). Start from eq. (96) with $\gamma = 1/2$ and use (68) to write

$$mv_a = -\frac{\hbar}{2} \vec{\omega}_a \cdot \vec{s} - \frac{q}{c} A_a,$$

so that

$$\frac{1}{2}mv^2 = \frac{1}{2m} \left( \frac{\hbar}{2} \vec{\omega}_a \cdot \vec{s} + \frac{q}{c} A_a \right)^2.$$ (281)

On the other hand, substituting $\Psi = \Upsilon u_+$ with $\Upsilon = \rho^{1/2} U$ into (205) gives

$$(\hat{H}_0 \Psi) \Psi^\dagger = (\hat{H}_0 \Upsilon) \Upsilon^\dagger (1 + \vec{s}).$$ (282)

Next expand $\hat{H}_0 \Upsilon$ to write

$$\hat{H}_0 \Upsilon = \left[ -\frac{\hbar^2}{2m} \partial^2 \Upsilon + \frac{q^2}{2mc^2} A^2 \Upsilon \right]$$

$$+ \left\{ \frac{i}{2mc} (\partial_a A_a) \Upsilon + i \frac{hq}{mc} (A_a \partial_a) \Upsilon - \frac{hq}{2mc} \vec{B} \Upsilon \right\}.$$ (283)
The terms in brackets are even; those in braces are odd. Therefore, substituting back into (282) gives

\[
\langle \Psi^\dagger \hat{H}_0 \Psi \rangle_0 = \left\langle -\frac{\hbar^2}{2m}(\partial^2 \Upsilon) \Upsilon^\dagger \right\rangle_0 + \frac{q^2}{2mc^2}A^2 \rho
\]

\[
+ \left\langle i \frac{\hbar q}{mc} A_a (\partial_a \Upsilon) \Upsilon^\dagger \right\rangle_0 - \frac{\hbar q}{2mc} \vec{B} \cdot \vec{s} \rho .
\]

(284)

Next use (64) to write

\[
\partial_a \Upsilon = \frac{1}{2} \left( \frac{\partial_a \rho}{\rho} - i \vec{\omega}_a \right) \Upsilon ,
\]

\[
\partial^2 \Upsilon = \left[ \frac{\partial^2 \rho^{1/2}}{\rho^{1/2}} - \frac{1}{4} \vec{\omega}_a^2 - \frac{i}{2} \partial_a \vec{\omega}_a - i \vec{\omega}_a \vec{\omega}_a \right] \Upsilon .
\]

and substitute back into (284) to get

\[
\langle \Psi^\dagger \hat{H}_0 \Psi \rangle_0 = -\frac{\hbar^2}{2m} \frac{\partial^2 \rho^{1/2}}{\rho^{1/2}} \rho + \frac{\hbar^2}{8m} \vec{\omega}_a^2 \rho + \frac{q^2}{2mc^2}A^2 \rho + \frac{\hbar q}{2mc} A_a \vec{\omega}_a \cdot \vec{s} \rho - \frac{\hbar q}{2mc} \vec{B} \cdot \vec{s} \rho .
\]

(285)

Finally, use

\[
\vec{\omega}_a^2 = (\vec{\omega}_a \cdot \vec{s})^2 + (\vec{\omega}_a \times \vec{s})^2 = (\vec{\omega}_a \cdot \vec{s})^2 + (\partial_a \vec{s})^2 ,
\]

and eq. (281), to get

\[
\langle \Psi^\dagger \hat{H}_0 \Psi \rangle_0 = \rho \left( \frac{1}{2} m v^2 - \frac{\hbar^2}{2m} \frac{\partial^2 \rho^{1/2}}{\rho^{1/2}} + \frac{\hbar^2}{8m} (\partial_a \vec{s})^2 - \frac{\hbar q}{2mc} \vec{B} \cdot \vec{s} \right) ,
\]

(286)

which concludes the derivation of (206).

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