Supercharacter theories for Sylow $p$-subgroups of the Ree groups

Yujiao Sun
School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, P. R. China
E-mail: yujiaosun@bit.edu.cn

Abstract

We determine a supercharacter theory for Sylow $p$-subgroups $2G_{2syl}(3^{2m+1})$ of the Ree groups $2G_2(3^{2m+1})$, calculate the conjugacy classes of $2G_{2syl}(3^{2m+1})$, and establish the character table of $2G_{2syl}(3)$.

Keywords: supercharacter theory; character table; Sylow $p$-subgroup

2010 Mathematics Subject Classification: Primary 20C15, 20D15. Secondary 20C33, 20D20

1 Introduction

Let $p$ be a fixed odd prime, $q$ a fixed power of $p$, $\mathbb{F}_q$ the finite field with $q$ elements, $\mathbb{N}^*$ the set of positive integers, and $A_n(q)$ ($n \in \mathbb{N}^*$) the group of upper unitriangular $n \times n$-matrices over $\mathbb{F}_q$. Thus $A_n(q)$ is a Sylow $p$-subgroup of the Chevalley group of Lie type $A_{n-1}$ ($n \geq 2$) over $\mathbb{F}_q$. It is known that determining the conjugacy classes of $A_n(q)$ for all $n$ and $q$ is a wild problem. Higman’s conjecture states that the number of conjugacy classes of $A_n(q)$ is an integer polynomial in $q$ depending on $n$. Lehrer [22] and later Isaacs [18] refined Higman’s conjecture. However, Higman’s conjecture is still open, see e.g. [12, 23, 28].

A generalization of the character theory, supercharacter theory, was introduced in [10]. The supercharacter theory replaces irreducible characters, conjugacy classes and character table by supercharacters, superclasses and supercharacter table, respectively. André [1] using the Kirillov orbit method [20], and Yan [29] using an algebraic and combinatorial method determined the André-Yan supercharacter theory for $A_n(q)$. André and Neto [3, 4, 5] studied the supercharacter theories for the Sylow $p$-subgroups of untwisted types $B_n$, $C_n$, and $D_n$. These supercharacters arise as restrictions of supercharacters of overlying full upper unitriangular groups $A_N(q)$ to the Sylow $p$-subgroups, and the superclasses arise as intersections of supersclasses of $A_N(q)$ with these groups. The construction of [3, 5] has been extended to Sylow $p$-subgroups of finite classical groups of untwisted Lie type in a uniform way in [2]. Andrews [6, 7] reproved the construction and extended it once more to the Sylow $p$-subgroups of twisted type $2A_n$.

Jedličský introduced the monomial linearisation method for a finite group in his doctoral thesis [19]. As a result, he decomposed the André-Neto supercharacters for Sylow $p$-subgroups of Lie type $D_n$ into much smaller characters. The smaller characters are pairwise orthogonal, and each irreducible character is a constituent of exactly one of the smaller characters. Thus these characters look like finer supercharacters for the Sylow $p$-subgroups of type $D$. So far
there are no corresponding finer superclasses for the Sylow $p$-subgroups of type $D$. Recently, 
and some modules affording André-Neto supercharacters are decomposed into a direct sum of
submodules in \[14\]. One may ask, if there is a general supercharacter theory for Sylow $p$-subgroups of all finite groups of Lie type based on the monomial linearisation method.

2 The construction of monomial modules

In this paper, we introduce Jedlitschky’s construction of monomial modules in Section 2. For 
the matrix Sylow $p$-subgroup $U := 2G_2^{syl}(3^{2m+1})$ of the Ree group \(2G_2(3^{2m+1})\) (see Section 3), 
the explicit construction of a monomial $A_8(q)$-module $\mathbb{C}U$ is determined in Section 4. In Section 5, we classify the $2G_2^{syl}(3^{2m+1})$-orbit modules which lead to the supercharacters in Section 7. 
After that, we calculate all of the conjugacy classes of $2G_2^{syl}(3^{2m+1})$ in Section 6 which satisfy 
Higman’s conjecture. In Section 5, at the same time, we get a partition which is proved to be a set of superclasses in Section 7. In the last section, we further determine the character table for the special case of $2G_2^{syl}(3)$ (i.e. $m = 0$).

Supercharacter theories raise other questions in particular concerning algebraic combinatorics. For example, the connection between supercharacter theories and Schur rings is obtained in [16].

Here we fix some notations: Let $K$ be a field, $K^*$ the multiplicative group $K \setminus \{0\}$ of $K$, $K^+$ the additive group of $K$, $\mathbb{C}$ the complex field, and $\mathbb{N}$ the set of all non-negative integers. Let $\text{Mat}_{8 \times 8}(K)$ be the set of all $8 \times 8$ matrices over $K$. If $m \in \text{Mat}_{8 \times 8}(K)$, then set $m := (m_{i,j})$, where $m_{i,j} := m_{i,j} \in K$ denotes the $(i,j)$-entry of $m$. We set $e_{ij} := e_{i,j} \in \text{Mat}_{8 \times 8}(K)$ the matrix unit with $1$ in the $(i,j)$-position and $0$ elsewhere. Let $A^T$ denote the transpose of $A \in \text{Mat}_{8 \times 8}(K)$. Let $I_8$ be the $8 \times 8$ identity matrix $I_{8 \times 8}$, and $1$ be the identity element of a finite group.

2 The construction of monomial modules

In this section, we recall the construction of the monomial modules, and mainly refer to [19].

Let $G$ be a finite multiplicative group, $\text{Irr}(G)$ the set of all complex irreducible characters of $G$, $V$ a finite abelian additive group and $K$ a field. If $V$ is a $K$-vector space, then let it be finite dimensional. If $X$ is a set, then $KX$ denotes the $K$-vector space with the $K$-basis $X$. Let $M$ be a right $KG$-module and $\ast$ be the module operation: $\ast : M \times G \to M : (m, g) \mapsto m \ast g$. Then the right $KG$-module $M$ is also denoted by $(M, \ast)_{KG}$, or by $M_{KG}$ for short.

Let $K^G := \{\tau : G \to K \mid \tau \text{ is a map}\}$. Define addition and scalar multiplication on $K^G$ as follows: For all $\tau, \sigma \in K^G$ and $\lambda \in K$, we set $(\tau + \sigma)(g) = \tau(g) + \sigma(g)$ and $(\lambda \tau)(g) = \lambda \tau(g)$ for all $g \in G$, then $K^G$ is a $K$-vector space. For $g \in G$, set $\tau_g : G \to K : h \mapsto \begin{cases} 1, & g = h \\ 0, & g \neq h \end{cases} = \delta_{g,h}$, where $\delta_{g,h}$ is the Kronecker delta. We have that $\{\tau_g \mid g \in G\}$ is a $K$-basis of $K^G$. In particular, $\tau = \sum_{g \in G} \tau(g)\tau_g$ for all $\tau \in K^G$. The map $\Phi : K^G \to KG$ induced by $\tau_g \mapsto g$ is a $K$-isomorphism. In particular, $\Phi(\tau) = \sum_{g \in G} \tau(g)g$ for all $\tau \in K^G$. Let $KG$ be the group algebra
with the multiplication
\[
(\sum_{g \in G} \alpha_g g)(\sum_{h \in G} \beta_h h) = \sum_{x \in G} \sum_{g \in G} \alpha_g \beta_g x.
\]

For \(\tau, \sigma \in K^G\), the multiplication \(\tau \sigma\) is defined by
\[
\tau \sigma : G \to K : y \mapsto \sum_{g \in G} \tau(g) \sigma(g^{-1} y),
\]
then \(K^G\) is an associative \(K\)-algebra, and \(\Phi : K^G \to KG : \tau \mapsto \sum_{g \in G} \tau(g) g\) is an algebra isomorphism. In particular, \(\tau_g \tau_h = \tau_{gh}\) for all \(g, h \in G\).

For a finite abelian group \(V\), let \(\hat{V} := \text{Hom}(V, \mathbb{C}^*)\). Then \(\text{Irr}(V) = \hat{V} \subset \mathbb{C}^V\) is a linearly independent subset of the \(\mathbb{C}\)-vector space \(\mathbb{C}^V\). We have \(\dim_\mathbb{C} \mathbb{C}^V = |\hat{V}| = |V| = \dim_\mathbb{C} \mathbb{C}V = \dim_\mathbb{C} \mathbb{C}^V\), so \(\mathbb{C}V = \mathbb{C}^V\) (as \(\mathbb{C}\)-vector spaces).

2.1 Lemma (see \[19\] §2.1 and \[13\] 2.6). Let \(f : G \to V\) be a map. Then \(f^* : \mathbb{C}^V \to \mathbb{C}^G : \phi \mapsto f^*(\phi) = \phi f\) defines a \(\mathbb{C}\)-linear map, and \(f\) is surjective (bijective, injective) if and only if \(f^*\) is injective (bijective, surjective). If \(f\) is surjective, then \(\{\hat{x} f \mid \hat{x} \in \hat{V}\}\) is a \(\mathbb{C}\)-basis of \(\text{im} f^* = f^*(\mathbb{C}^V)\).

2.2 Corollary. Let \(f : G \to V\) be a surjective map, and \(U \leq G\) such that \(f|_U\) is bijective. Then \(f|_U^* : \mathbb{C}^V \to \mathbb{C}^U : \phi \mapsto f|_U^*(\phi) = \phi f|_U = f^*(\phi)|_U\) defines a \(\mathbb{C}\)-isomorphism. In particular, \(\{\hat{x} f|_U \mid \hat{x} \in \hat{V}\}\) is a \(\mathbb{C}\)-basis of \(\mathbb{C}^U\).

2.3 Definition (1-cocycle). Let \(V\) be an abelian group. Suppose \(G\) acts on \(V\), \(A, g \mapsto A \circ g\) \((A \in V, g \in G)\), as automorphisms. Then a map \(f : G \to V\) is called a (right) 1-cocycle of \(G\) in \(V\) if it satisfies
\[
f(xg) = f(x) \circ g + f(g) \quad \text{for all } x, g \in G. \quad (2.4)
\]

In the rest of this section, suppose that \(f : G \to V\) is a surjective 1-cocycle and \(U\) is a subgroup of \(G\) such that \(f|_U\) is bijective (i.e. \(f|_U\) is a bijective 1-cocycle of \(U\) in \(V\)). Then \(\mathbb{C}V\), \(\mathbb{C}^V\), \(\mathbb{C}U\), \(\mathbb{C}^U\) and \(\text{im } f^*\) are pairwise \(\mathbb{C}\)-isomorphic:

\[
\begin{align*}
\mathbb{C}\{\hat{x} f|_U \mid \hat{x} \in \hat{V}\} &= \text{im } f|_U, \\
\mathbb{C}\{\hat{x} \mid \hat{x} \in \hat{V}\} &= \mathbb{C}V, \\
\mathbb{C}\{\hat{x} f \mid \hat{x} \in \hat{V}\} &= \text{im } f^*
\end{align*}
\]

where \(\Psi : \hat{x} \mapsto \sum_{B \in V} \hat{x}(B) B, \Phi : \hat{x} f|_U \mapsto \sum_{u \in U} \hat{x}f(u)u, f^* : \hat{x} \mapsto \hat{x} f, f|_U^* : \hat{x} \mapsto \hat{x} f|_U, \) and \(f|_U^* : u \mapsto f(u)\) is the extension of \(f|_U\) to \(\mathbb{C}U\) by linearity. Let \(\chi := \hat{x} f\),
\[
[\hat{x}] := \sum_{B \in V} \hat{x}(B) B \quad \text{and} \quad [\hat{x} f|_U] := [\chi|_U] := \sum_{u \in U} \chi(u)u.
\]

Then \(\mathbb{C}V = \mathbb{C}\{[\hat{x}] \mid \hat{x} \in \hat{V}\}\) and \(\mathbb{C}U = \mathbb{C}\{[\hat{x} f|_U] \mid \hat{x} \in \hat{V}\}\).

Let \(V, W\) be \(K\)-vector spaces (or abelian groups) and \(\varphi : V \to W\) a \(K\)-isomorphism (or group isomorphism). Suppose that \(V\) is a \(KG\)-module \((V, .)_{KG}\) and that the elements of \(G\) act on \(V\) as
$K$-automorphisms (or as group automorphisms). Define a new operation by $− * − : W × G → W : (w, g) → w * g := ϕ(ϕ^{−1}(w).g)$. We extend the operation by linearity

$$w * \left( \sum_{g \in G} α_g g \right) = \sum_{g \in G} α_g (w * g) \quad \text{for all } w \in W, \sum_{g \in G} α_g g \in K^G,$$

then $W$ is also a $K^G$-module $(W, *)_{K^G}$, the elements of $G$ act on $W$ as $K$-automorphisms (or as group automorphisms) and $ϕ$ is a $K^G$-module isomorphism.

2.5 Definition/Lemma. Let $K$ be an arbitrary field and $G$ be a finite group. Define an operation by $− * − : K^G × G → K^G : (ϕ, g) → ϕ * g$, where $ϕ * g(x) = ϕ(ϕg(x))$ for all $x \in G$. Then $K^G$ becomes a right $K^G$-module $(K^G, *)_{K^G} \cong K^G$, where $K^G$ is the regular right module.

2.6 Definition/Lemma. Let $V$ be an abelian group on which $G$ acts from the right as automorphisms. Then the group action of $G$ on $V$ induces a group action $− * −$ of $G$ on $V$ given by

$$− * − : \hat{V} × G → \hat{V} : (χ, g) → \chi * g,$$

where $(\chi * g)(A) = \chi(A o g)$ for all $A \in V$.

2.7 Lemma. Let $f : G → V$ be a surjective 1-cocycle. Then for $\chi \in \hat{V}$ and $g \in G$,

$$(\chi f) * g = (\chi f)(g^{-1}) \cdot (\chi g) = \chi(g^{-1}) \cdot (\chi g),$$

where $− * −$ is the scalar multiplication. By extending the operation $− * −$ linearly, im $f^*$ becomes a monomial module $\text{im } f^*, *_{CG}$.

Proof. Let $\hat{x} \in \hat{V}$ and $g \in G$. Then for $x \in G$, 

$$((\hat{x} f) * g)(x) = (\hat{x} f)(x g^{-1}) \cdot (\chi g) = \hat{x} f(x) \cdot g^{-1} + f(g^{-1}) \cdot (\chi g),$$

2.8 Theorem (Monomial $CG$-modules, [Jedlitschky, [12, 2.1.11]].) Let $f : G → V$ be a surjective 1-cocycle, and $U \leq G$ such that $f|_U$ is bijective. Then the $C$-vector spaces $CV$, $CU$, $C^V$ and $CU$ can be made into monomial $CG$-modules by extending the following operations linearly: for all $\chi \in \hat{V}$ and $g \in G$, we have that

\[ \begin{align*} 
\hat{x} * g &:= \hat{x} \cdot g, \\
|x| * g &:= x \cdot g, \\
(\chi f|_{V}) * g &:= \chi(f(g^{-1})) \cdot (\chi g)|_{V}, \\
[\chi] * g &:= \chi(f(g^{-1})) \cdot [\chi g].
\end{align*} \]

and that $(CV, *)_{CG}$, $(CU, *)_{CG}$, $(CU, *)_{CG}$ and $(CU, *)_{CG}$ are isomorphic to $(\text{im } f^*, *)_{CG}$. We say these $CG$-modules arise from the 1-cocycle $f$.

2.9 Corollary (Monomial $CU$-modules). The vector spaces $CU$, $CV$, $CU$, $CV$, and $\text{im } f^*$ can be made into monomial isomorphic $CU$-modules by extending the restriction of the operations $− * −$ linearly. In particular, the operation $− * −$ of $U$ on $CU$ is the usual right operation of $U$ on $CU$, i.e. for all $\chi \in \hat{V}$ and $x \in U$, we obtain

\[ \sum_{u \in U} \hat{x} f(u) u * x = \hat{x} \cdot f(x^{-1}) \cdot \sum_{u \in U} (\chi x) f(u) u = \sum_{u \in U} \hat{x} f(u) u x, \]

so $(CU, *)_{CU} = CU_{\text{CU}}$.

2.10 Lemma. Let $H := \{ g \in G \mid f(g) = 0 \}$. Then $H \cap U = \{ 1 \}$ and $G = HU$. 

YUJIAO SUN
Proof. Since $H \leq G$ and $U \leq G$, we have $G \supseteq HU$.

Let $X$ be a complete set of right coset representatives of $H$ in $G$. If $g \in G$, then there exist $h \in H$ and $x \in X$ such that $g = hx$. We have $f(g) = f(x) \in V$. Then there exists $u \in U$ such that $f(u) = f(x)$ since $f|_U$ is bijective. We know $f(xu^{-1}) = f(x) \circ u^{-1} + f(u^{-1}) = f(u) \circ u^{-1} + f(u^{-1}) = f(1) = 0$, so $x = h_xu$ for some $h_x \in H$. Thus $g = hh_xu \in HU$, i.e. $G \subseteq HU$. Therefore $G = HU$.

If $g \in H \cap U$, then $f(g) = 0 = f|_U(g)$. So $g = 1_G$ since $f|_U$ is bijective.

2.11 Proposition (see [11, 2.8]). Let
\[ e := \sum_{h \in H} h, \quad \tau_e := \sum_{h \in H} \tau_h \quad \text{and} \quad [\chi] := [\chi f] := \sum_{g \in G} \chi(g) g \quad \text{for all } \chi \in \hat{V}, \]
and $C_H = C\{e\}$ be a trivial $H$-module. Then
\[ \text{Ind}_{H}^{G} C_H = eCg = C\{eu \mid u \in U\} = eCU = C\{e[\chi|_U] \mid \chi \in \hat{V}\} = C\{[\chi] \mid \chi \in \hat{V}\} \quad \text{(as } C\text{-vector space)}, \]
\[ \text{im } f^* = C\{\chi \mid \chi \in \hat{V}\} = C\{\tau_e \ast [\chi|_U] \mid \chi \in \hat{V}\} = \tau_e \ast C U \]
\[ = C\{\tau_e \ast u \mid u \in U\} = \tau_e \ast C G = \tau_e C^G \quad \text{(as } C\text{-vector space)} . \]

In particular, $\text{im } f^*, * \}CG \cong \text{Ind}_{H}^{G} C_H$ and $\chi = \tau_e \ast [\chi|_U]$.

Proof. It is enough to prove that $\chi = \sum_{g \in G} \chi(g) \tau_g = \sum_{h \in H} \sum_{u \in U} \chi(hu) \tau_{hu} = \sum_{h \in H} \sum_{u \in U} \chi(u) \tau_h \ast u$
\[ = (\sum_{h \in H} \tau_h) \ast (\sum_{u \in U} \chi(u)u) = \tau_e \ast [\chi|_U]. \]

Now we give a summary of the isomorphic monomial $CG$-modules and the $C$-bases:

\[
\begin{align*}
C\{[\chi|_U] \mid \chi \in \hat{V}\} &= (\text{C}U, *)_{CG} \xrightarrow{\Phi} (\text{C}U, *)_{CG} \quad = C\{[\chi|_U] \mid \chi \in \hat{V}\} \\
C\hat{V} &= (\text{C}V, *)_{CG} \xrightarrow{\Psi} (\text{C}V, *)_{CG} \quad = C\{[\chi] \mid \chi \in \hat{V}\} \\
C\{\tau_e \ast [\chi|_U] \mid \chi \in \hat{V}\} &= (\text{im } f^*, *)_{CG} \xrightarrow{\gamma_{\text{im } f^*}} \text{Ind}_{H}^{G} C_H \quad = C\{e[\chi|_U] \mid \chi \in \hat{V}\} \\
C\{\chi \mid \chi \in \hat{V}\} &= \gamma \quad \text{where } \gamma : C^G \to CG : \tau \mapsto \sum_{g \in G} \tau(g) g.
\end{align*}
\]

3 Sylow $p$-subgroup $2G_2^{syl}(3^{2m+1})$

In this section, we construct a Sylow 3-subgroup $2G_2^{syl}(3^{2m+1})$ of the Ree group $2G_2(3^{2m+1})$ (see 3.1).

Define the elements of $\text{Mat}_{8 \times 8}(\mathbb{C})$ as follows: $h_1 := e_{11} - e_{88}$, $h_2 := e_{22} - e_{77}$, $h_3 := e_{33} - e_{66}$, $h_4 := e_{44} - e_{55}$. A subspace of $\text{Mat}_{8 \times 8}(\mathbb{C})$ is $\mathcal{H} := \mathbb{C}\text{-span}\{h_1 - h_2 + 2h_3, h_2 - h_3\} = \ldots$
Let $\tilde{H}^*$ be the dual space of $\tilde{H}$, $\tilde{h} := \sum_{i=1}^{3} \lambda_i h_i \in \tilde{H}$, linear maps $\alpha : \tilde{H} \to \mathbb{C} : h \mapsto \sum_{i=1}^{3} \lambda_i h_i$ and $\beta : \tilde{H} \to \mathbb{C} : \tilde{h} \mapsto \lambda_3 - \lambda_2$. We set $\Phi_{G_2} = \pm \{ \alpha, \beta, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta \}$, and $\tilde{\Phi}_{G_2} = \{ \alpha, \beta, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta \}$. Let $V_{G_2}$ be a $\mathbb{R}$-vector subspace of $\tilde{H}^*$ spanned by $\Phi_{G_2}$, and become a Euclidean space (see [9] §5.1). Then $\Delta_{G_2} = \{ \alpha, \beta \}$ is a basis of $V_{G_2}$. Define the elements of Mat$_{8 \times 8}(\mathbb{C})$ as follows:

$$
e_\alpha := (e_{12} - e_{78}) + (e_{34} - e_{56}) + (e_{35} - e_{46}), \quad e_\beta := e_{23} - e_{67},$$

$$e_{\alpha + \beta} := -(e_{13} - e_{68}) + (e_{24} - e_{57}) + (e_{25} - e_{47}), \quad e_{3\alpha + \beta} := -(e_{16} - e_{38}),$$

$$e_{2\alpha + \beta} := -(e_{14} - e_{58}) - (e_{26} - e_{37}) - (e_{15} - e_{48}), \quad e_{3\alpha + 2\beta} := -(e_{17} - e_{28}),$$

and $e_{-r} := e_r^T$ and $h_r := [e_r, e_{-r}]$ for all $r \in \Phi^+$. Then a Lie algebra of type $G_2$ is determined, denoted by $L_{G_2}$, which has a Chevalley basis $\{ h_\alpha, h_\beta \} \cup \{ e_r | \ r \in \Phi \}$ (see [26] 2.1). Let $r := x_1 \alpha + x_2 \beta \in V_{G_2}$, $s := y_1 \alpha + y_2 \beta \in V_{G_2}$. Then we write $r < s$ if $\sum_{i=1}^{2} x_i < \sum_{i=1}^{2} y_i$, or if $\sum_{i=1}^{2} x_i = \sum_{i=1}^{2} y_i$ and the first non-zero coefficient $x_i - y_i$ is positive. The total order on $\Phi^+_{G_2}$ is determined: $0 < \alpha < \beta < \alpha + \beta < 2\alpha + \beta < 3\alpha + \beta < 3\alpha + 2\beta$. The Lie algebra $L_{G_2}$ has the following structure constants: $N_{\alpha, \beta} = -1$, $N_{\alpha, \alpha + \beta} = -2$, $N_{\alpha, 2\alpha + \beta} = 3$ and $N_{3\alpha + 3\alpha + \beta} = 1$. In particular, $N_{\alpha, \alpha + \beta} = -2N_{3\alpha + 3\alpha + \beta}$ and $N_{\alpha, 2\alpha + \beta} = 3N_{3\alpha + 3\alpha + \beta}$.

Set a matrix group $G_2(q) := \{ \exp(te_r) | r \in \Phi_{G_2}, \ t \in \mathbb{F}_q \}$, and the Chevalley group of type $L_{G_2}$ over the field $\mathbb{F}_q$ is $G_2(q) := \{ \exp(t \text{ad} e_r) | r \in \Phi_{G_2}, \ t \in \mathbb{F}_q \}$. For all $r \in \Phi_{G_2}$ and $t \in \mathbb{F}_q$, set $y_r(t) := \exp(te_r) = I_3 + te_r + \frac{1}{2}t^2 e_r$. Let $y_1(t) := y_\alpha(t)$, $y_2(t) := y_\beta(t)$, $y_3(t) := y_{\alpha + \beta}(t)$, $y_4(t) := y_{2\alpha + \beta}(t)$, $y_5(t) := y_{3\alpha + \beta}(t)$, $y_6(t) := y_{3\alpha + 2\beta}(t)$. The positive root subgroups of $G_2(q)$ are $Y_i := \{ y_i(t) | t \in \mathbb{F}_q \}$ for all $i = 1, 2, \ldots, 6$.

Let $t_i \in \mathbb{F}_q$ for all $i = 1, 2, \ldots, 6$ and $[y_i(t_j), y_j(t_j)] := y_i(t_j)^{-1}y_j(t_j)^{-1}y_i(t_i)y_j(t_j)$. Then the non-trivial commutators are determined.

$$
[y_1(t_1), y_2(t_2)] = y_3(-t_2 t_1) \cdot y_4(-t_2 t_1^2) \cdot y_5(t_2 t_1^3) \cdot y_6(-2t_2 t_1^2), \quad [y_1(t_1), y_3(t_3)] = y_4(-2t_1 t_3) \cdot y_5(3t_1^2 t_3) \cdot y_6(3t_1 t_3^2), \quad [y_1(t_1), y_4(t_4)] = y_5(3t_1 t_4), \quad [y_3(t_3), y_4(t_4)] = y_6(3t_3 t_4), \quad [y_2(t_2), y_5(t_5)] = y_6(t_2 t_5).
$$

In particular, if $\text{Char} \mathbb{F}_q = 3$, then

$$
[y_1(t_1), y_2(t_2)] = y_3(-t_2 t_1) \cdot y_4(-t_2 t_1^2) \cdot y_5(t_2 t_1^3) \cdot y_6(t_2 t_1^2), \quad [y_1(t_1), y_3(t_3)] = y_4(-2t_1 t_3) = y_4(t_1 t_3), \quad [y_2(t_2), y_5(t_5)] = y_6(t_2 t_5).
$$

Let $y(t_1, t_2, t_3, t_4, t_5, t_6) := y_2(t_2)y_1(t_1)y_3(t_3)y_4(t_4)y_5(t_5)y_6(t_6)$ for all $t_i \in \mathbb{F}_q$ ($i = 1, 2, \ldots, 6$). Then a matrix Sylow $p$-subgroup $G_2^{sp}(q)$ of $G_2(q)$ (see [26] 2.6) is

$$
G_2^{sp}(q) := \{ y(t_1, t_2, t_3, t_4, t_5, t_6) | t_1, t_2, t_3, t_4, t_5, t_6 \in \mathbb{F}_q \}.
$$

We note that the signs of the structure constants and the Chevalley basis of the Lie algebra $L_{G_2}$ are different from those in [26]. However, the Sylow $p$-subgroup $G_2^{sp}(q)$ and the root subgroups $Y_i (i = 1, 2, \ldots, 6)$ of the Chevalley group $G_2(q)$ are as same as those in [26].

Let $p := 3$, $q := 3^{2m+1}$ ($m \in \mathbb{N}$) and $\theta := 3^m$. There is a field automorphism $F_\theta$ of $G_2(q)$ sending $y_r(t)$ to $y_r(t^\theta) = y_r(t^{3^m})$ for all $r \in \Phi_{G_2}$. Let $\rho : r \mapsto \tilde{r}$ be a non-trivial symmetry of the Dynkin diagram of type $G_2$ (interchanging $\alpha$ and $\beta$). For every $r \in \Phi_{G_2}$, $\tilde{r}$ is obtained by reflecting $r$ in the line bisecting $\alpha$ and $\beta$. 

Let $\epsilon_i = \pm 1$ $(i = 1, 2, 3, 4)$ satisfy that $N_{\alpha, \beta} = \epsilon_1$, $N_{\alpha, 2\alpha + \beta} = 2\epsilon_2$, $N_{\alpha, 2\alpha + \beta} = 3\epsilon_3$ and $N_{\beta, 3\alpha + \beta} = \epsilon_4$. Since the structure constants of $L_{G_2}$ satisfy $-\epsilon_2 = \epsilon_3 = \epsilon_4 = 1$, by [8] 12.4.1 the map

$$y_r(t) \mapsto \begin{cases} y_r(t^\theta), & \bar{r} \text{ is short} \\ y_r(t^{3\theta}), & \bar{r} \text{ is long} \end{cases}$$

for all $r \in \Phi_{G_2}$, $t \in \mathbb{F}_q$ can be extended to a graph automorphism $\tilde{\rho}$ of $G_2(q)$. If $F := \tilde{\rho}F_\theta = F_\theta\tilde{\rho}$, then

$$F: G_2(q) \rightarrow G_2(q) : y_r(t) \mapsto \begin{cases} y_r(t^\theta), & \bar{r} \text{ is short} \\ y_r(t^{3\theta}), & \bar{r} \text{ is long} \end{cases}$$

for all $r \in \Phi_{G_2}$, $t \in \mathbb{F}_q$.

For a subgroup $X$ of $G_2(q)$, we write $X^F := \{ x \in X \mid F(x) = x \}$. By [8] §13.4, $G_2(q)^F = aG_2(q)$ and $G_2(q)^F$ is a subgroup of $2G_2(q)$. By [8] §14, we have that $G_2^{syl}(q)$ is also a Sylow $p$-subgroup of $2G_2(q)$ and that $|G_2(q)| = q^3(q - 1)(q^3 + 1) = 3^{6m+3}(3^{2m+1} - 1)(3^{6m+3} + 1)$.

### 3.1 Proposition (Sylow $p$-subgroup $2G_2^{syl}(3^{2m+1}))$

A matrix Sylow 3-subgroup $2G_2^{syl}(q)$ of the Ree group $2G_2(q)$ is

$$2G_2^{syl}(q) := \{ y_2(t_{t_1}^{3\theta}) \cdot y_1(t_1)y_3(t_3)y_4(t_4) \cdot y_5(t_{t_1}^{3\theta} + t_{t_1}^{2\theta+3})y_6(t_{t_1}^{3\theta} + t_{t_1}^{2\theta+3}) \mid t_1, t_3, t_4, t_2 \in \mathbb{F}_q \},$$

where

$$y_2(t_{t_1}^{3\theta}) \cdot y_1(t_1)y_3(t_3)y_4(t_4) \cdot y_5(t_{t_1}^{3\theta} + t_{t_1}^{2\theta+3})y_6(t_{t_1}^{3\theta} + t_{t_1}^{2\theta+3}) = \begin{pmatrix}
1 & t_1 & -t_3 & t_1t_3 & -t_1t_4 & -t_1t_3^2 & 2t_1t_3t_4 + t_{t_1}^{3\theta+4} + t_{t_1}^{3\theta+3} - t_{t_1}^{3\theta+1} - t_3^2 \\
1 & t_{t_1}^{3\theta+1} & t_{t_1}^{\theta+1} & -t_1^{3\theta+2} & -2t_1^{3\theta+1}t_3 & -t_1^{3\theta+1}t_3 + t_{t_1}^{3\theta+3} + t_{t_1}^{3\theta+1}t_4 & +2t_1^{3\theta+1}t_4 + t_{t_1}^{3\theta+3} + t_{t_1}^{3\theta+1}t_3 + t_{t_1}^{3\theta+3} + t_{t_1}^{3\theta+1}t_4 \\
1 & t_1 & -t_3 & -t_1t_3 & 2t_1t_3 + t_{t_1}^{3\theta+3} + t_{t_1}^{3\theta+1}t_3 + t_{t_1}^{3\theta+3} + t_{t_1}^{3\theta+1}t_4 \\
1 & 0 & -t_1 & -t_3 & -t_1t_3 + t_{t_1}^{3\theta+3} + t_{t_1}^{3\theta+1}t_3 + t_{t_1}^{3\theta+3} + t_{t_1}^{3\theta+1}t_4 \\
1 & 1 & -t_1 & -t_3 & -t_1t_3 + t_{t_1}^{3\theta+3} + t_{t_1}^{3\theta+1}t_3 + t_{t_1}^{3\theta+3} + t_{t_1}^{3\theta+1}t_4 \\
1 & t_{t_1}^{3\theta} & t_{t_1}^{\theta+1} & t_{t_1}^{\theta+1} & -t_1^{3\theta+3} & -t_1^{3\theta+3} & -t_1^{3\theta+2}t_3 + t_{t_1}^{3\theta+3} + t_{t_1}^{3\theta+1}t_4 + t_{t_1}^{3\theta+3} + t_{t_1}^{3\theta+1}t_3 + t_{t_1}^{3\theta+3} + t_{t_1}^{3\theta+1}t_4 \\
1 & t_1 & -t_1 & -t_1t_3 & 2t_1t_3 + t_{t_1}^{3\theta+3} + t_{t_1}^{3\theta+1}t_3 + t_{t_1}^{3\theta+3} + t_{t_1}^{3\theta+1}t_4 & +2t_1^{3\theta+1}t_4 + t_{t_1}^{3\theta+3} + t_{t_1}^{3\theta+1}t_3 + t_{t_1}^{3\theta+3} + t_{t_1}^{3\theta+1}t_4 & +2t_1^{3\theta+1}t_4 + t_{t_1}^{3\theta+3} + t_{t_1}^{3\theta+1}t_3 + t_{t_1}^{3\theta+3} + t_{t_1}^{3\theta+1}t_4 \\
1 & t_1 & -t_1 & -t_1t_3 & 2t_1t_3 + t_{t_1}^{3\theta+3} + t_{t_1}^{3\theta+1}t_3 + t_{t_1}^{3\theta+3} + t_{t_1}^{3\theta+1}t_4 & +2t_1^{3\theta+1}t_4 + t_{t_1}^{3\theta+3} + t_{t_1}^{3\theta+1}t_3 + t_{t_1}^{3\theta+3} + t_{t_1}^{3\theta+1}t_4 \\
1 & t_1 & -t_1 & -t_1t_3 & 2t_1t_3 + t_{t_1}^{3\theta+3} + t_{t_1}^{3\theta+1}t_3 + t_{t_1}^{3\theta+3} + t_{t_1}^{3\theta+1}t_4 & +2t_1^{3\theta+1}t_4 + t_{t_1}^{3\theta+3} + t_{t_1}^{3\theta+1}t_3 + t_{t_1}^{3\theta+3} + t_{t_1}^{3\theta+1}t_4 \\
1 & t_1 & -t_1 & -t_1t_3 & 2t_1t_3 + t_{t_1}^{3\theta+3} + t_{t_1}^{3\theta+1}t_3 + t_{t_1}^{3\theta+3} + t_{t_1}^{3\theta+1}t_4 & +2t_1^{3\theta+1}t_4 + t_{t_1}^{3\theta+3} + t_{t_1}^{3\theta+1}t_3 + t_{t_1}^{3\theta+3} + t_{t_1}^{3\theta+1}t_4
\end{pmatrix}$$
Proof. We know \( \text{Char} \mathbb{F}_q = 3 \) and \( t^{3q^2} = t \) for all \( t \in \mathbb{F}_q \). Let \( t_i \in \mathbb{F}_q \) and \( y(t_1, t_2, t_3, t_4, t_5, t_6) \in G_2^{spl}(q)^F \). Then

\[
y(1, t_2, t_3, t_4, t_5, t_6) = F(y(t_1, t_2, t_3, t_4, t_5, t_6)) = F(\sum (t_1, t_2, t_3, t_4, t_5, t_6) y(t_1, t_2, t_3, t_4, t_5, t_6)) = y(1, t_2, t_3, t_4, t_5, t_6)
\]

Thus,

\[
t_1 = t_2, \quad t_3 = t_2 - t_2^{3q}, \quad t_5 = t_2^{3q} + t_2^{3q+1}, \quad t_5 = t_2^{3q} + t_2^{3q}, \quad t_6 = t_2^{3q} + t_2^{3q+1}.
\]

Hence \( y(t_1, t_2, t_3, t_4, t_5, t_6) = y(t_1, t_2, t_3, t_4, t_3 + t_2^{3q+3}, t_4 + t_1^{3q+3}) \), and \( |G_2^{spl}(q)^F| = q^3 \).

Therefore, \( G_2^{spl}(q) := G_2^{spl}(q)^F \) is a Sylow \( p \)-subgroup of \( G_2(q) \). We get the matrix form by calculation. \( \square \)

3.2 Corollary. \( G_2^{spl}(3^{2m+1}) \leq G_2^{spl}(3^{2m+1}) \).

3.3 Notation/Lemma. For \( i \in \{1, 3, 4\} \) and \( t_i, t_j \in \mathbb{F}_q \) we set

\[
a(t_i) := y(t_i) y(t_i) y(t_i) y(t_i) y(t_i) y(t_i), \quad b(t_3) := y(t_3) y(t_3), \quad c(t_4) := y(t_4) y(t_4).
\]

If \( Y(t_1, t_3, t_4) := a(t_1) b(t_3) c(t_4) \), then

\[
y_2(t_i^{3q}) \cdot y_1(t_1) y_3(t_3) y_4(t_4) \cdot y_3(t_3^{3q} + t_3^{3q+3}) y_6(t_4^{3q} + t_3^{3q+3}) = a(t_1) b(t_3) c(t_4) = Y(t_1, t_3, t_4).
\]

By calculation, we get the following properties.

3.4 Lemma. Let \( i \in \{1, 3, 4\} \) and \( t_i, s_i \in \mathbb{F}_q \). Then

\[
Y(t_1, t_3, t_4) \cdot Y(s_1, s_3, s_4) = Y(t_1 + s_1, t_3 + s_3 - t_1 s_3^{3q}, t_4 + s_4 + t_1 s_4^{3q+1} - t_2 s_4^{3q} - t_3 s_1),
\]

\[
Y(t_1, t_3, t_4)^{-1} = Y(-t_1, -t_3 - t_1^{3q+1}, -t_4 + t_1^{3q+2} - t_1 t_3).
\]

In particular,

\[
a(t_1) \cdot a(s_1) = Y(t_1 + s_1, s_3^{3q+1} + t_1 s_3^{3q}, s_4^{3q+1} - t_2 s_4^{3q}), \quad a(t_1)^{-1} = Y(-t_1, -t_1^{3q+1}, t_1^{3q+2}),
\]

\[
b(t_3) \cdot b(s_3) = b(t_3 + s_3), \quad c(t_4) \cdot c(s_4) = c(t_4 + s_4).
\]

3.5 Lemma. If \( i \in \{1, 3, 4\} \) and \( t_i, s_i \in \mathbb{F}_q \), then the commutators of \( G_2^{spl}(q) \) are

\[
[Y(t_1, t_3, t_4), Y(s_1, s_3, s_4)]
\]

\[
= Y(0, t_1^{3q} s_1 - t_1 s_1^{3q}, t_3 s_3^{3q+1} - t_3^{3q+1} s_1) + (t_2 s_2 - t_2 s_2^{3q}) + (t_3 s_3 - t_3^{3q} s_1),
\]

\[
[Y(t_1, t_3, t_4)^{-1}, Y(s_1, s_3, s_4)^{-1} = Y(0, t_1^{3q} s_1 - t_1 s_1^{3q}, t_3 s_3^{3q+1} - t_3^{3q} s_1) + (t_3 s_3 - t_3^{3q} s_1).
\]

In particular,

\[
[a(t_1), a(s_1)] = b(t_1^{3q} s_1 - t_1 s_1^{3q}) \cdot c(t_1^{3q} s_1^2 - t_1^{3q} s_1^2 + t_1^{3q+1} - t_1^{3q+1} s_1),
\]

\[
[a(t_1)^{-1}, a(s_1)] = b(t_1^{3q} s_1 - t_1 s_1^{3q}) \cdot c(t_1^{3q} s_1^2 - t_1^{3q} s_1^2),
\]

\[
[a(t_1), b(s_3)] = c(t_1 s_3), \quad [a(t_1)^{-1}, b(s_3)^{-1}] = c(t_1 s_3).
3.6 Proposition. Let \( t_1, s_i \in \mathbb{F}_q \) with \( i \in \{1, 3, 4\} \). Then the conjugate of \( Y(t_1, t_3, t_4) \) is

\[
Y(s_1, s_3, s_4) \cdot Y(t_1, t_3, t_4) \cdot Y(s_1, s_3, s_4)^{-1}
= Y(t_1, t_3 + t_1 s_1^3 - t_1^3 s_1, t_4 + (t_1^2 s_1^3 + t_1^3 s_1^2) + t_1^3 s_1 + (t_3 s_1 - t_1 s_3)).
\]

In particular,

\[
Y(s_1, s_3, s_4) \cdot a(t_1) \cdot Y(s_1, s_3, s_4)^{-1} = Y(t_1, t_1 s_1^3 - t_3^3 s_1, (t_1^2 s_1^3 + t_3^3 s_1^2) + t_1^3 s_1 + t_3 s_1 - t_1 s_3),
\]

\[
Y(s_1, s_3, s_4) \cdot b(t_3) \cdot Y(s_1, s_3, s_4)^{-1} = Y(0, t_3, t_3 s_1),
\]

\[
Y(s_1, s_3, s_4) \cdot c(t_4) \cdot Y(s_1, s_3, s_4)^{-1} = c(t_4).
\]

Define the following sets of matrix entry coordinates: \( \Box := \{(i, j) \mid 1 \leq i, j \leq 8\} \), \( \nabla := \{(i, j) \mid 1 \leq i < j \leq 8\} \) and \( \nabla := \{(i, j) \mid i < j < 9 - i\} \). Let \( J := \{(1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (1, 7), (2, 3)\} \subseteq \nabla \), and \( J := \{(1, 2), (1, 3), (1, 4)\} \subseteq J \).

3.7 Comparison (Sylow \( p \)-subgroups). For every element of \( 2G_2^{s\text{pl}}(q) \) in 3.7 we have matrix entries \( t_1 \) and up to sign also \( t_3 \) with positions in \( J \), but \( t_4 \) appears in \( J \) in polynomials involving \( t_1 \) and \( t_3 \). This is similar to that of \( 3D_4^{s\text{pl}}(q^3) \) (see [27 §2]) and that of \( G_2^{s\text{pl}}(q) \) (see [26 §2]).

4 Monomial \( 2G_2^{s\text{pl}}(3^{2m+1}) \)-module

Let \( q = 3^{2m+1}, G := A_8(q) \) and \( U := 2G_2^{s\text{pl}}(3^{2m+1}) \). In this section, we explain the construction of a monomial \( A_8(q) \)-module \( \mathbb{C}U \) that is analogous to that of \( 3D_4^{s\text{pl}}(q^3) \) (see [27]) and that of \( G_2^{s\text{pl}}(q) \) (see [26]).

Let \( V_0 := \text{Mat}_{8\times 8}(q) \). For any subset \( J \subseteq \square \), let \( V_J := \bigoplus_{(i, j) \in I} \mathbb{F}_q e_{i,j} \subseteq V_0 \). In particular, \( V_0 = V_0 \). Then \( V_J \) is an \( \mathbb{F}_q \)-vector subspace. We have \( \dim_{\mathbb{F}_q} V_J = 3 \), since \( J = \{(1, 2), (1, 3), (1, 4)\} \). The map \( \kappa : V_0 \times V_0 \rightarrow \mathbb{F}_q \) : \( (A, B) \rightarrow \text{tr}(AB) \) is a non-degenerate symmetric \( \mathbb{F}_q \)-bilinear form on \( V_0 \) which is called the trace form. Let \( V := V_J \), and \( V^\perp \) denote the orthogonal complement of \( V \) in \( V_0 \) with respect to the trace form \( \kappa \), i.e. \( V^\perp := \{B \in V_0 \mid \kappa(A, B) = 0, \forall A \in V\} \). Then \( V^\perp = V_{\Box \setminus J} \) and \( V_0 = V \perp \). Note that \( \kappa|_{V \times V} : V \times V \rightarrow \mathbb{F}_q \) is a non-degenerate bilinear form. Set \( \pi := \pi_J \), i.e.

\[
\pi : V_0 = V \perp \rightarrow V : A \mapsto \sum_{(i, j) \in J} A_{i,j} e_{i,j} = A_{12} e_{12} + A_{13} e_{13} + A_{14} e_{14}.
\]

Then \( \pi \) is a projection to the first component \( V \) and is an \( \mathbb{F}_q \)-linear map. Suppose \( A, B \in V_0 \) such that \( \text{supp}(A) \cap \text{supp}(B) \subseteq J \). Then \( \kappa(A, B) = \kappa(\pi(A), \pi(B)) = \kappa(\pi(A), \pi(B)) = \kappa|_{V \times V}(\pi(A), \pi(B)) \). If \( A, B \in V \) and \( g, h \in G \), then \( \pi_\alpha(A g^\top) \in V \) and \( \text{supp}(B h^\top) \cap \text{supp}(Ag) \subseteq J \).

4.1 Proposition (Group action of \( G \) on \( V \)). The map

\[
- \circ - : V \times G \rightarrow V : (A, g) \mapsto A \circ g := \pi(AG)
\]

is a group action, and the elements of the group \( G \) act as \( \mathbb{F}_q \)-automorphisms.

Let \( A.g \ (\in V, g \in G) \) denote \( \pi(AG^\top) \). Then this is a group action of \( G \) by [19 §2.1], we get the following new action.
4.2 Corollary. There exists a unique linear action \( \pi \) of \( G \) on \( V \):

\[-\cdot : V \times G \to V : (A, g) \to A.g := \pi(Ag^{-1})\]

such that \( \kappa|_{V \times V}(A.g, B) = \kappa|_{V \times V}(A, B \circ g^{-1}) \) for all \( B \in V \).

4.3 Notation. Set \( f := \pi|_G : G \to V \).

4.4 Proposition. Let \( x, g \in G \). Then \( f(x)g \equiv (x - 1)g \mod V \) and \( f(xg) = f(x) \circ g + f(g) \).

4.5 Proposition (Bijective 1-cocycle of \( 2G^{spl}_2(q) \)). If \( U = 2G^{spl}_2(3^{2m+1}) \), then \( f|_U := \pi|_U : U \to V \) is a bijection. In particular, \( f|_U \) is a bijective 1-cocycle of \( U \).

4.6 Corollary (Monomial linearisation for \( A_8(q) \)). The map \( f := \pi|_G : G \to V \) is a surjective 1-cocycle of \( G \) in \( V \), and \( f|_{V \times V} \) is a monomial linearisation for \( G = A_8(q) \).

4.7 Corollary. \( (f|_U, \kappa|_{V \times V}) \) is a monomial linearisation for \( 2G^{spl}_2(q) \).

Now we establish the monomial \( G \)-module \( C\left(2G^{spl}_2(3^{2m+1})\right) \).

4.8 Theorem (Fundamental theorem for \( 2G^{spl}_2(3^{2m+1}) \)). Let \( G = A_8(q) \), \( U = 2G^{spl}_2(3^{2m+1}) \) and

\[ [A] := \frac{1}{|U|} \sum_{u \in U} \chi_A(u)u \quad \text{for all } A \in V, \]

where \( \chi_A(u) = \partial \kappa(A, f(u)) \). Then the set \( \{[A] \mid A \in V\} \) forms a \( C \)-basis for the complex group algebra \( CU \). For all \( g \in G, A \in V \), let \( [A] * g := \chi_A(g)[A.g] = \partial \kappa(A.g, f(g))[A.g] \). Then \( CU \) is a monomial \( CG \)-module. The restriction of the \(*\)-operation to \( U \) is given by the usual right multiplication of \( U \) on \( CU \), i.e.

\[ [A] * u = [A]u = \frac{1}{|U|} \sum_{y \in U} \chi_A(y)yu, \quad \text{for all } u \in U, A \in V. \]

Proof. By 4.6, \( (f|_U, \kappa|_{V \times V}) \) is a monomial linearisation for \( G \) satisfying that \( f|_U \) is bijective (see 4.5). By 4.2, \( A.u := \pi(Au^{-1}) \). Thus the theorem is obtained by [19, 2.1.35].

4.9 Comparison (Monomial linearisations). Let \( U \) be \( A_n(q), D^{spl}_n(q), 3D^{spl}_4(q^3), G^{spl}_2(q) \) (a fixed power of some odd prime) or \( 2G^{spl}_2(q) \) (\( q = 3^{2m+1} \)), \( G \) an intermediate group of \( U \), \( V_0 := V|_U \), \( V \) a subspace of \( V_0 \), \( f : G \to V \) a surjective 1-cocycle of \( G \) such that \( f|_U \) is injective, \( \kappa : V \times V \to \mathbb{F}_q \) (or \( \mathbb{F}_{q^2} \)) a trace form such that \( (f, \kappa|_{V \times V}) \) is a monomial linearisation for \( G \). Then the monomial linearisations \( (f|_U, \kappa|_{V \times V}) \) for \( A_n(q) \) (see [19 §2.2]), \( D^{spl}_n(q) \) (see [19 §3.1]), \( 3D^{spl}_4(q^3) \) (see [27 §4]), \( G^{spl}_2(q) \) (see [26 §3]), and \( 2G^{spl}_2(q) \) (see [34]) are listed in Table 1.

From now on, we mainly consider the regular right module \( (CU, \ast_{CU}) = CU \).

5 \( 2G^{spl}_2(q) \)-orbit modules

Let \( U := 2G^{spl}_2(q), A \in V \), and \( y_i(t_i) \in U, t_i \in \mathbb{F}_q \) (\( i = t_1, t_3, t_4 \)). In this section, we determine the stabilizers \( \text{Stab}_U(A) \) for all \( A \in V \) and obtain a classification of \( U \)-orbit modules.

For \( A \in V \), the \( U \)-orbit module associated to \( A \) is \( \mathbb{C}O_U([A]) := \mathbb{C}\{[A]u \mid u \in U\} = \mathbb{C}\{[A]u \mid u \in U\} \) or \( \mathbb{C}\{[A]u \mid u \in U\} \). Then \( \mathbb{C}O_U([A]) \) has a \( \mathbb{C} \)-basis \( \{[A]u \mid u \in U\} = \{[C] \mid C \in O_U(A)\} \), where \( O_U(A) := \{A.g \mid g \in U\} \) is the orbit of \( A \) under the operation \( \pi \). The stabilizer \( \text{Stab}_U(A) \) of \( A \) in \( U \) is \( \text{Stab}_U(A) = \{u \in U \mid A.u = A\} \). Two \( CU \)-modules having no nontrivial \( \mathbb{C}U \)-homomorphism between them are called orthogonal. Set \( \hat{x}_{ij}(t) = I_{ij} + te_{ij} \in A_8(q) \) (\( 1 \leq i, j \leq 8 \)).
**5.1 Lemma.** Let $A \in V$, $Y(t_1, t_3, t_4) \in U$ and $t_i \in \mathbb{F}_q$ with $i \in \{1, 3, 4\}$. Then $A.Y(t_1, t_3, t_4)$ and the corresponding figure of moves are determined. The figure describes the way of classifying the orbits.

$$A.Y(t_1, t_3, t_4) = A.(y_2(t_1^y) y_1(t_1) y_3(t_3)) = A.(x_{23}(t_1^y) x_{34}(t_1) x_{24}(t_3)).$$

![Diagram of A.Y(t_1, t_3, t_4)]

The elements of $V$ are called **patterns**. Let $A \in V$. Then $(i, j) \in J$ is a **main condition** of $A$ if and only if $A_{ij}$ is the rightmost non-zero entry in the $i$-th row. We set $\text{main}(A) := \{(i, j) \in J \mid (i, j) \text{ is a main condition of } A\}$. The **verge** of $A \in V$ is $\text{verge}(A) := \sum_{(i, j) \in \text{main}(A)} A_{ij} e_{i,j}$. The pattern $A \in V$ is called the **verge pattern** if $A = \text{verge}(A)$.

**5.2 Notation.** Define the families of $U$-orbit modules as follows: $\hat{\mathcal{S}}_4 := \{\text{CO}_U([A]) \mid A \in V, A_{14} \neq 0\}$, $\hat{\mathcal{S}}_3 := \{\text{CO}_U([A]) \mid A \in V, A_{13} \neq 0, A_{14} = 0\}$, and $\hat{\mathcal{S}}_1 := \{\text{CO}_U([A]) \mid A \in V, A_{12} \neq 0, A_{13} = A_{14} = 0\}$. For $A \in V$, we also say $A \in \hat{\mathcal{S}}_i$ if $\text{CO}_U([A]) \in \hat{\mathcal{S}}_i$.

**5.3 Proposition** ($^2G_2^{spl}(q)$-orbit modules). For $A = (A_{ij}) \in V$, the $U$-orbit module $\text{CO}_U([A])$ is determined.

$$\text{CO}_U([A_{12}e_{12} + A_{13}e_{13} + A_{14}e_{14}])$$

$$= \mathbb{C}\{[(A_{12} - A_{13}t_1^y - A_{14}t_3)e_{12} + (A_{13} - A_{14}t_1)e_{13} + A_{14}e_{14}] \mid t_1, t_3 \in \mathbb{F}_q\}.$$

In particular, every $U$-orbit module contains precisely one verge pattern.

**Proof.** By 5.1 we calculate the orbit modules directly. \qed
5.4 Proposition. Let $A = (A_{ij}) \in V$.

(1) If $A \in \mathfrak{g}_1$, then $\text{Stab}_U(A) = U = 2G_2^{spl}(q)$.

(2) If $A \in \mathfrak{g}_3$ and $A_{13} = A_{13}^* \in \mathbb{F}_q^*$, then $\text{Stab}_U(A) = \{Y(0, t_3, t_4) \mid t_3, t_4 \in \mathbb{F}_q\}$.

(3) If $A \in \mathfrak{g}_4$ and $A_{14} = A_{14}^* \in \mathbb{F}_q^*$, then $\text{Stab}_U(A) = \{Y(0, 0, t_4) \mid t_4 \in \mathbb{F}_q\}$.

Proof. By [5,3], the stabilizers are obtained by straightforward calculation.

Let $A, B \in V$, $\text{Stab}_U(A, B) := \text{Stab}_U(A) \cap \text{Stab}_U(B)$, $\psi_A$ be the character of $CO_U([A])$ and $\psi_B$ denote the character of $CO_U([B])$. Then $\text{Hom}_{COU}(CO_U([A]), CO_U([B])) = \{0\}$ if and only if $\text{Hom}_{\text{Stab}_U(C,D)}(C[C], C[D]) = \{0\}$ for all $C \in \mathcal{O}_U(A)$ and $D \in \mathcal{O}_U(B)$. If $A, B \in V$, then

$$\langle \psi_A, \psi_B \rangle_U = \sum_{D \in \mathcal{O}_U(B)} |\text{Stab}_U(A, D)| \langle \psi_A, \psi_D \rangle_{\text{Stab}_U(A, D)}$$

where $\chi_A$ and $\chi_D$ are the characters of the $\text{CStab}_U(A, D)$-modules $C[A]$ and $C[D]$ respectively.

5.5 Proposition (Classification of $2G_2^{spl}(q)$-orbit modules). Every $U$-orbit module is a verge pattern module in Table 2 and the orbit modules satisfy the following properties.

(1) Let $A, B \in V$, $\text{verge}(A) \neq \text{verge}(B)$. Then $\text{Hom}_{COU}(CO_U([A]), CO_U([B])) = \{0\}$. In particular, if $CO_U([A]) \in \mathfrak{g}_i, CO_U([B]) \in \mathfrak{g}_j$ and $i \neq j$, then $\text{Hom}_{COU}(CO_U([A]), CO_U([B])) = \{0\}$.

(2) In the family $\mathfrak{g}_1$, the $U$-orbit modules are irreducible and pairwise orthogonal.

(3) In the families $\mathfrak{g}_3$ and $\mathfrak{g}_4$, the $U$-orbit modules are reducible.

| Family | $CO_U([A])$ | $A \in V$ | $\dim_C CO_U([A])$ | Irreducible |
|--------|-------------|----------|--------------------|-------------|
| $\mathfrak{g}_4$ | $CO_U([A_{14}^*])$, $A_{14} \in \mathbb{F}_q^*$ | $q^2$ | NO |
| $\mathfrak{g}_3$ | $CO_U([A_{13}])$, $A_{13} \in \mathbb{F}_q^*$ | $q$ | NO |
| $\mathfrak{g}_1$ | $CO_U([A_{12}])$, $A_{12} \in \mathbb{F}_q$ | $1$ | YES |

Proof. By [5,3], every $U$-orbit module is a verge pattern module in Table 2. By calculating the inner products, (1) and the orthogonal properties are proved. The orbit modules in the family $\mathfrak{g}_1$ are 1-dimensional and irreducible, so (2) is obtained.

Let $A = A_{12}e_1 + A_{13}e_3 \in \mathfrak{g}_3$ and $C \in \mathcal{O}_U(A)$. By [5,4], $\text{Stab}_U(A, C) = \text{Stab}_U(A) = \{Y(0, t_3, t_4) \mid t_3, t_4 \in \mathbb{F}_q\}$. The inner product is $\langle \chi_A, \chi_C \rangle_{\text{Stab}_U(A, C)} = 1$. Let $\psi_A$ denote the character of $CO_U([A])$. Then

$$\dim_C \text{Hom}_{COU}(CO_U([A]), CO_U([A])) = \langle \psi_A, \psi_A \rangle_U = \sum_{C \in \mathcal{O}_U(A)} |\text{Stab}_U(A, C)| \dim_C \text{Hom}_{\text{Stab}_U(A, C)}(C[A], C[C]) = q > 1.$$

Thus, $CO_U([A])$ is not irreducible.

If $A \in V$ is a pattern of the family $\mathfrak{g}_4$, then the orbit module $CO_U([A])$ is reducible. Suppose that it is irreducible. Then $(\dim_C CO_U([A]))^2 = q^2 < |U| = q^3$. This is a contradiction. Thus the orbit modules of the family $\mathfrak{g}_4$ are reducible.

5.6 Corollary. If $A, B \in V$, then $CO_U([A]) = \text{Res}_{U}^G CO_G([A])$, and the two orbit modules $CO_U([A])$ and $CO_U([B])$ are either isomorphic or orthogonal.
5.7 Comparison ((Classification of orbit modules). Every \(2G_2^{spl}(q)\)-orbit module has precisely one verge pattern (see 5.5 that is similar to that of \(A_n(q)\) (see [29 Theorem 3.2]). However, this is neither true for \(3D_4^{spl}(q^3)\) (see [27 §6]) nor true for \(G_2^{spl}(q)\) (see [26 §5]).

6 Conjugacy classes of \(2G_2^{spl}(3^{2m+1})\)

Let \(q = 3^{2m+1}, G := A_8(q), U := 2G_2^{spl}(3^{2m+1})\), and \(t^*, t_1^*, t_2^*, t_3^* \in F_q^*\). In this section, we determine the conjugacy classes of \(2G_2^{spl}(3^{2m+1})\) (6.5), which is a set of the superclasses proved in Section 7.

If \(x \in U\), then the conjugate of \(x\) by \(u\) is \(uxu^{-1}\), and the conjugacy class of \(u\) is \(Ux := \{uxu^{-1} | v \in U\}\).

6.1 Lemma. Let \(t \in F_q^*, \theta := 3^m, z_q^+\) be the additive group of \(F_q\) and

\[\gamma_t : z_q^+ \rightarrow z_q^+ : s \mapsto ts^{3\theta} - t^{3\theta}s.\]

Then \(\gamma_t\) is a homomorphism and \(|\text{im } \gamma_t| = 2^\theta = 3^{2m}\).

Proof. We know that \(\gamma_t\) is a homomorphism. We claim that \(\ker \gamma_t = \{0, t, -t\}\). Note that \(\ker \gamma_t \supseteq \{0, t, -t\}\). If \(s \in \ker \gamma_t\), then \(ts^{3\theta} - t^{3\theta}s = 0\). We know \(0 \in \ker \gamma_t \cap \{0, t, -t\}\). If \(s \neq 0\), then \((t^{-1}s)^{3\theta-1} = 1 \implies |t^{-1}s| = 1\) \(\implies (\theta - 1)\). Since \((t^{-1}s)^{3\theta-1} = (t^{-1}s)^{3\theta-1} = 1\), the order \(|t^{-1}s|\) divides the greatest common divisor \((3\theta - 1, 3\theta^2 - 2) = (3^{3m+1} - 1, 3^{3m+1} - 1)\). If \(m = 0\), then \((3^{3m+1} - 1, 3^{3m+1} - 1) = (2, 2) = 2\). If \(m > 0\), then \((3^{m+1} - 1, 3^{3m+1} - 1) = 2\) by the Euclidean algorithm. So, \(|t^{-1}s| = 1 \implies t^{-1}s = \pm 1 \implies s = \pm t\).

Therefore, \(|\text{im } \gamma_t| = \frac{|z_q^+|}{|\ker \gamma_t|} = \frac{3^{2m+1} - 1}{3} = 3^{2m}\.

6.2 Notation. For \(t^* \in F_q^*,\) denote by \(t^* T\) a transversal of \(\text{im } \gamma_{t^*}\) in \(z_q^+\). Thus \(|t^* T| = 3\).

6.3 Proposition (Conjugacy classes of \(2G_2^{spl}(3^{2m+1})\). If \(U = 2G_2^{spl}(3^{2m+1})\), then the conjugacy classes of \(U\) are listed in Table 3.

| Representative \(y \in U\) | Conjugacy Classes \(U^y\) | \(|U^y|\) |
|---------------------------|---------------------------|----------------|
| \(I_8\)                  | \(Y(0, 0, 0)\)            | 1             |
| \(Y(0, t_3^*, 0)\), \(t_3^* \in F_q^*\) | \(Y(0, 0, t_3^*)\) | 1             |
| \(Y(t_1^*, t_3^*, 0)\), \(t_1^*, t_3^* \in F_q^*\) | \(Y(0, t_3^*, s_4)\), \(s_4 \in F_q\) | \(q\) |
| \(Y(t_1^*, t_1^* t_3, 0)\), \(t_1^* \in F_q^*\), \(t_1^* t_3 \in T\) | \(Y(t_1^*, s_3, s_4)\), \(s_3 \in \text{im } \gamma_{t_1^*}, s_4 \in F_q\) | \(q \cdot 3^{4m}\) |

Proof. By 3.6 and 6.1, we get the conjugacy classes of \(U\).

6.4 Remark. We consider the analogue of Higman’s conjecture for \(2G_2^{spl}(3^{2m+1})\). By 6.3, we get \(#\text{Conjugacy Classes of } 2G_2^{spl}(3^{2m+1})\) = \(5q - 4 = 5(q - 1) + 1\). Thus the conjecture is true for \(2G_2^{spl}(3^{2m+1})\).
6.5 Notation/Lemma. Set
\[ C_4(t_4^*) := U Y(0, 0, t_4^*), \quad C_3(t_3^*) := U Y(0, t_3^*, 0), \quad C_1(t_2^*):= \bigcup_{t_i t_j} U Y(t_1^*, t_2^*, 0), \quad C_0 := \{ I_8 \}. \]

Note that the sets form a partition of \( U \), denoted by \( K \), i.e.
\[ K := \{ C_0, C_1(t_2^*), C_2(t_3^*), C_4(t_4^*) \mid t_1^*, t_2^*, t_3^*, t_4^* \in \mathbb{F}_q^* \}. \]

6.6 Comparison. (1) (Superclasses). The superclasses of \( 2G_{2}^{\text{spl}}(q) \) can also be obtained by calculating \( \{ I_8 + (u - I_8)y \mid y \in G \} \cap U = \{ I_8 + (u - I_8)y \mid x, y \in A_8(q) \} \cap 2G_{2}^{\text{spl}}(q) \) for all \( u \in 2G_{2}^{\text{spl}}(q) \) (c.f. [27 §7] and [26 §6]).

(2) (Conjugacy classes). The conjugacy classes of \( 2G_{2}^{\text{spl}}(q) \) are determined by commutator relations that is similar to that of \( 3D_{4}^{\text{spl}}(q^3) \) (see [25 §3]) and that of \( G_{2}^{\text{spl}}(q) \) (see [26 §8]).

7 A supercharacter theory for \( 2G_{2}^{\text{spl}}(3^{2m+1}) \)

In this section, let \( q = 3^{2m+1} \), \( U := 2G_{2}^{\text{spl}}(3^{2m+1}) \), and \( A_{12}, A_{13}, A_{14} \in \mathbb{F}_q^* \). We determine the supercharacter theory for \( 2G_{2}^{\text{spl}}(3^{2m+1}) \) (7.4), and establish the supercharacter table of \( 2G_{2}^{\text{spl}}(3^{2m+1}) \) in Table 4.

7.1 Definition ([10 §2] / [19 3.6.2]). Let \( G \) be a finite group. Suppose that \( \mathcal{K} \) is a partition of \( G \) and that \( \mathcal{X} \) is a set of (nonzero) complex characters of \( G \), such that

(a) \( |\mathcal{X}| = |\mathcal{K}| \),
(b) every character \( \chi \in \mathcal{X} \) is constant on each member of \( \mathcal{K} \),
(c) the elements of \( \mathcal{X} \) are pairwise orthogonal and
(d) the set \( \{ 1 \} \) is a member of \( \mathcal{K} \).

Then \( (\mathcal{X}, \mathcal{K}) \) is called a supercharacter theory for \( G \). We refer to the elements of \( \mathcal{X} \) as supercharacters, and to the elements of \( \mathcal{K} \) as superclasses of \( G \).

7.2 Notation. For \( A = (A_{ij}) \in V \), set \( M(0) := \{ 0 \}, M(A_{12}e_{12}) := \mathcal{C} \mathcal{O} \mathcal{U}([A_{12}e_{12}]), M(A_{13}e_{13}) := \mathcal{C} \mathcal{O} \mathcal{U}([A_{13}e_{13}]), \) and \( M(A_{14}e_{14}) := \mathcal{C} \mathcal{O} \mathcal{U}([A_{14}e_{14}]). \) Denote by \( \mathcal{M} \) the set of all of the above \( \mathcal{C} \mathcal{U} \)-modules, i.e.
\[ \mathcal{M} := \{ M(0), M(A_{12}e_{12}), M(A_{13}e_{13}), M(A_{14}e_{14}) \mid A_{12}, A_{13}, A_{14} \in \mathbb{F}_q^* \} \]
\[ = \{ 0, \mathcal{C} \mathcal{O} \mathcal{U}([A_{12}e_{12}]), \mathcal{C} \mathcal{O} \mathcal{U}([A_{13}e_{13}]), \mathcal{C} \mathcal{O} \mathcal{U}([A_{14}e_{14}]) \mid A_{12}, A_{13}, A_{14} \in \mathbb{F}_q^* \}. \]

7.3 Notation. For \( M \in \mathcal{M} \), the complex character of the \( \mathcal{C} \mathcal{U} \)-module \( M \) is denoted by \( \Psi_M \). We set \( \mathcal{X} := \{ \Psi_M \mid M \in \mathcal{M} \} \). Let \( A \in \mathcal{V} \), and \( \psi_A \) be the character of \( \mathcal{C} \mathcal{O} \mathcal{U}([A]). \) Then \( \Psi_M(0) = \psi_0, \Psi_M(A_{12}e_{12}) = \psi_{A_{12}e_{12}}, \Psi_M(A_{13}e_{13}) = \psi_{A_{13}e_{13}} \), and \( \Psi_M(A_{14}e_{14}) = \psi_{A_{14}e_{14}} \).

7.4 Proposition (Supercharacter theory for \( 2G_{2}^{\text{spl}}(3^{2m+1}) \)). \( (\mathcal{X}, \mathcal{K}) \) is a supercharacter theory for \( 2G_{2}^{\text{spl}}(3^{2m+1}) \), where \( \mathcal{K} \) is defined in [6.5] and \( \mathcal{X} \) is defined in [7.3]. The supercharacter table is shown in Table 4.

Proof. By [6.5] \( \mathcal{K} \) is a partition of \( U \). We know \( \mathcal{X} \) is a set of nonzero complex characters of \( U \).
In this section, we exhibit the conjugacy classes of $\text{2G}^{syl}_{2}(3^{2m+1})$.

Table 4: Supercharacter table of $\text{2G}^{syl}_{2}(3^{2m+1})$

| $\Psi_{M(0)}$ | $C_0$ | $C_1(t_1^*)$ | $C_2(t_2^*)$ | $C_3(t_3^*)$ | $C_4(t_4^*)$ |
|---------------|-------|---------------|---------------|---------------|---------------|
| $\Psi_{M(A_{12}e_{12})}$ | 1     | 1             | 1             | 1             | 1             |
| $\Psi_{M(A_{13}t_{2})}$ | $\vartheta(A_{12}t_1^*)$ | 1             | 1             | 1             | 1             |
| $\Psi_{M(A_{2e_{13}})}$ | $q$   | 0             | $q \cdot \vartheta(-A_{13}^*t_3^*)$ | $q$           |               |
| $\Psi_{M(A_{4e_{14}})}$ | $q^2$ | 0             | 0             | $q^2 \cdot \vartheta(-A_{14}^*t_4^*)$ |               |

(a) **Claim that** $|\mathcal{X}| = |\mathcal{K}|$. By [5.5] [7.2] and [7.3], we have $|\{\Psi_{M(A_{13}^*e_{13})} | A_{13}^* \in \mathbb{F}_q^*\}| = |\{M(A_{13}^*e_{13}) | A_{13}^* \in \mathbb{F}_q^*\}| = |\{C_{3}(t_3^*) | t_3^* \in \mathbb{F}_q^*\}|$. Similarly, we get $|\mathcal{X}| = |\mathcal{K}|$.

(b) **Claim that the characters** $\chi \in \mathcal{X}$ are constant on the members of $\mathcal{K}$.

Let $A \in \mathfrak{S}_3$ (i.e. $A_{14} = 0, A_{13} = A_{13}^* \in \mathbb{F}_q^*$) and $y \in U$. Then

$$
\Psi_{M(A_{13}^*e_{13})}(y) = \sum_{C \in \mathcal{O}_{U}(A_{13}^*)} \chi_{C}(y) = \sum_{C \in \mathcal{O}_{U}(A_{13}^*)} \chi_{C}(y).
$$

If $y = Y(0, t_3, t_4) \in C_0 \cup C_3(t_3^*) \cup C_4(t_4^*) \subseteq K$, then we have $y \in \text{Stab}_U(C)$ for all $C \in \mathcal{O}_{U}(A_{13}^*)$ by [5.4]. Thus

$$
\Psi_{M(A_{13}^*e_{13})}(y) = \sum_{C \in \mathcal{O}_{U}(A_{13}^*)} \chi_{C}(y) = q \cdot \vartheta(-A_{13}^*t_3^*).
$$

If $y \in C_1(t_1^*) \subseteq K$, then $y \notin \text{Stab}_U(C)$ for all $C \in \mathcal{O}_{U}(A_{13}^*)$ by [5.4]. So $\Psi_{M(A_{13}^*e_{13})}(y) = 0$. Similarly, we calculate the other values of the Table 4. Thus, the claim is proved.

(c) The elements of $\mathcal{X}$ are pairwise orthogonal by [5.5].

(d) The set $\{I_6\}$ is a member of $\mathcal{K}$. By [7.1] ($\mathcal{X}, \mathcal{K}$) is a supercharacter theory for $2G_{2}^{syl}(3^{2m+1})$.

**7.5 Comparison** (Supercharacters). Every supercharacter of $G_{2}^{syl}(q)$ is afforded by one orbit module (see [2.2] [7.3] and [4.7]) that is analogous to that of $A_n(q)$ (see [29]). However, this holds neither for $D_{4}^{syl}(q^2)$ (see [27] §8) nor for $G_{2}^{syl}(q)$ (see [26] §7).

**8 Character table of $2G_{2}^{syl}(3)$**

In this section, we exhibit the conjugacy classes of $2G_{2}^{syl}(3)$ (see Table 5), the irreducible characters of $2G_{2}^{syl}(3)$ (see Proposition 8.7), and the character table of $2G_{2}^{syl}(3)$ (see Table 6). Let $q = 3$ (i.e. $m = 0$) and $U := 2G_{2}^{syl}(3)$. Then $\theta = 3^m = 1$.

**8.1 Notation.** Set $Y_a := \{a(t_1) | t_1 \in \mathbb{F}_q\}$, $Y_b := \{b(t_3) | t_3 \in \mathbb{F}_q\}$, and $Y_c := \{c(t_4) | t_4 \in \mathbb{F}_q\}$.

We recall the properties for $2G_{2}^{syl}(3)$.

**8.2 Lemma** (Multiplication). If $i \in \{1, 3, 4\}$ and $t_1, s_i \in \mathbb{F}_3$, then

$$
Y(t_1, t_3, t_4) \cdot Y(s_1, s_3, s_4) = Y(t_1 + s_1, t_3 + s_3 - t_1 s_1, t_4 + s_4 + t_1 s_1^2 - t_2 s_1 - t_3 s_1),
$$

$$
Y(t_1, t_3, t_4)^{-1} = Y(-t_1, -t_3 - t_1, -t_4 + t_1 + t_1 t_3).
$$
In particular,
\[ a(t_1) \cdot a(s_1) = Y(t_1 + s_1, -t_1s_1, t_1s_1^2 - t_1^2s_1), \quad a(t_1)^{-1} = Y(-t_1, -t_1^2, t_1), \]
\[ b(t_3) \cdot b(s_3) = b(t_3 + s_3), \quad c(t_4) \cdot c(s_4) = c(t_4 + s_4). \]

8.3 Corollary. \( Y_0 \) and \( Y_c \) are subgroups of \( 2G_s^{spl}(3) \), but \( Y_a \) is not a subgroup of \( 2G_s^{spl}(3) \).

8.4 Lemma (Commutator relations of \( 2G_s^{spl}(3) \)). Let \( i \in \{1, 3, 4\} \) and \( t_i, s_i \in \mathbb{F}_3 \). Then the commutators of \( 2G_s^{spl}(3) \) are
\[ [Y(t_1, t_3, t_4), Y(s_1, s_3, s_4)] = Y(0, 0, (t_1^2 s_1 - t_1 s_1^2) + (t_1 s_1 - t_3 s_1)), \]
\[ [Y(t_1, t_3, t_4)^{-1}, Y(s_1, s_3, s_4)^{-1}] = Y(0, 0, (t_1^2 s_1 - t_1 s_1^2) + (t_1 s_1 - t_3 s_1)). \]

In particular,
\[ [a(t_1), a(s_1)] = c(t_1^2 s_1 - t_1 s_1^2), \quad [a(t_1)^{-1}, a(s_1)^{-1}] = c(t_1^2 s_1 - t_1 s_1^2), \]
\[ [a(t_1), b(s_3)] = c(t_1 s_3), \quad [a(t_1)^{-1}, b(s_3)^{-1}] = c(t_1 s_3). \]

8.5 Corollary. \( Y_c \) and \( Y_0 Y_c \) are normal subgroups of \( 2G_s^{spl}(3) \). \( Z(U) = Y_c \) and \( Y_c \setminus U \) is abelian.

8.6 Proposition (Conjugacy classes of \( 2G_s^{spl}(3) \)). If \( t_i, s_i \in \mathbb{F}_q \) with \( i \in \{1, 3, 4\} \), then the conjugate of \( Y(t_1, t_3, t_4) \) is
\[ Y(s_1, s_3, s_4) \cdot Y(t_1, t_3, t_4) \cdot Y(s_1, s_3, s_4)^{-1} = Y(t_1, t_3, t_4 + (t_1 s_1^2 - t_1^2 s_1) + (t_3 s_1 - t_1 s_3)). \]

In particular,
\[ Y(s_1, s_3, s_4) \cdot a(t_1) \cdot Y(s_1, s_3, s_4)^{-1} = Y(t_1, 0, t_1 s_1^2 - t_1^2 s_1 - t_3 s_1), \]
\[ Y(s_1, s_3, s_4) \cdot b(t_3) \cdot Y(s_1, s_3, s_4)^{-1} = Y(0, t_3, t_3 s_1), \]
\[ Y(s_1, s_3, s_4) \cdot c(t_4) \cdot Y(s_1, s_3, s_4)^{-1} = c(t_4). \]

Then the conjugacy classes of \( 2G_s^{spl}(3) \) are listed in Table 5

| Representative \( y \in U \) | Conjugacy Classes \( ^U y \) | \( |^U y| \) |
|---------------------------|--------------------------|----------|
| \( t_8 \)                  | \( Y(0, 0, 0) \)          | 1        |
| \( Y(0, 0, t_4), t_4 \in \mathbb{F}_3^* \) | \( Y(0, 0, t_4^4) \) | 1        |
| \( Y(0, t_3, 0), t_3 \in \mathbb{F}_3^* \) | \( Y(0, t_3, s_4), s_4 \in \mathbb{F}_3 \) | 3        |
| \( Y(t_4, t_3, 0), t_4 \in \mathbb{F}_3^*, t_3 \in \mathbb{F}_3 \) | \( Y(t_4, t_3, s_4), s_4 \in \mathbb{F}_3 \) | 3        |

Now we explain the constructions of the irreducible characters of \( 2G_s^{spl}(3) \).

8.7 Proposition. Let \( U := 2G_s^{spl}(3) \), \( \chi \in \text{Irr}(U) \), and \( A_{ij} \in \mathbb{F}_q \), \( A_{ij}^* \in \mathbb{F}_q^* \) \( (1 \leq i, j \leq 8) \).

(1) Let \( \tilde{U} := Y_c \setminus U = Y_\vartheta (\chi_{lin} A_{12}, A_{13})^{\chi_{lin} A_{12}, A_{13}} \in \text{Irr}(\tilde{U}) \), \( \tilde{\chi}_{lin} A_{12}, A_{13}(a(t_1) b(t_3)) := \vartheta(A_{12} t_1) \cdot \vartheta(-A_{13} t_3) \), and \( \chi_{lin} A_{12}, A_{13} \) be the lift of \( \chi_{lin} A_{12}, A_{13} \) to \( U \). Then
\[ \tilde{\chi}_{lin} := \{ \chi \in \text{Irr}(U) \mid Y_c \subseteq \ker \chi \} = \{ \chi_{lin} A_{12}, A_{13} \mid A_{12}, A_{13} \in \mathbb{F}_q \}. \]
(2) Let $H := Y_bY_c$, $\lambda^{A_{14}, A_{13}} \in \text{Irr}(H)$, $\lambda^{A_{14}, A_{13}}(b(t_3)c(t_4)) := \vartheta(-A_{14}t_4 - A_{13}t_3)$, $\chi_{2,q}^{A_{14}} := \text{Ind}_H^U \lambda^{A_{14}, 0}$. Then $\mathcal{F}_2 := \{ \chi \in \text{Irr}(U) \mid Y_c \not\subseteq \ker \chi \} = \{ \chi_{2,q}^{A_{14}} | A_{14}^* \in F_q^* \}$.

Hence $\text{Irr}(U) = \mathcal{F}_{\text{lin}} \dot{\cup} \mathcal{F}_2$.

Proof. Let $\chi \in \text{Irr}(U)$.

(1) Family $\mathcal{F}_{\text{lin}}$, where $Y_c \subseteq \ker \chi$.

Since the commutator subgroup is $Y_c$, all linear characters of $U$ are precisely the lifts of the irreducible characters of the abelian quotient group $Y_c \setminus U$ to $U$.

(2) Family $\mathcal{F}_2$, where $Y_c \not\subseteq \ker \chi$.

Let $H := Y_bY_c$, $\lambda^{A_{14}, A_{13}} \in \text{Irr}(H)$ with $\lambda^{A_{14}, A_{13}}(b(t_3)c(t_4)) := \vartheta(-A_{14}t_4 - A_{13}t_3)$. We note that $Y_c$ is a transversal of $H$ in $U$, and that $Z(U) = Y_c$. For all $s_1 \in F_q$,

$$
(a^{s_1})(b(t_3)c(t_4)) = \lambda^{A_{14}, A_{13}}(a(s_1) \cdot b(t_3)c(t_4) \cdot a(s_1)^{-1})
$$

$$=
\lambda^{A_{14}, A_{13}}(b(t_3)c(t_4 + s_1t_3)) = \vartheta(-A_{14}^*t_4 + s_1t_3) - A_{13}t_3
$$

$$=
\vartheta(-A_{14}t_4 - (A_{13} + s_1A_{14}^*)t_3).
$$

Thus $I_U(\lambda^{A_{14}, A_{13}}) = H$. By Clifford’s Theorem, $\text{Ind}_H^U \lambda^{A_{14}, A_{13}} \in \text{Irr}(U)$ and

$$
\text{Res}_H^UI_U \lambda^{A_{14}, A_{13}} = \sum_{s_1 \in F_q} (\lambda^{A_{14}, A_{13}})^{a(s_1)} = \sum_{s_1 \in F_q} \lambda^{A_{14}, A_{13} + s_1A_{14}} = \sum_{B_{13} \in F_q} \lambda^{A_{14}, B_{13}}.
$$

Let $\chi_{2,q}^{A_{14}} := \text{Ind}_H^U \lambda^{A_{14}, 0}$. By Clifford theory, there are $q - 1$ almost faithful irreducible characters of $U$, i.e. $\mathcal{F}_2 = \{ \chi \in \text{Irr}(U) \mid Y_c \not\subseteq \ker \chi \} = \{ \chi_{2,q}^{A_{14}} | A_{14}^* \in F_q^* \}$.

8.8 Proposition. The character table of $^2G_2^{spl}(3)$ is shown in Table 6.

Table 6: Character table of $^2G_2^{spl}(3)$

|       | $I_8$ | $Y(t_1^*, t_3, 0)$ | $Y(0, t_3^*, 0)$ | $Y(0, 0, t_4^*)$ |
|-------|-------|--------------------|-----------------|-----------------|
| $\chi_{\text{lin}}^{0,0}$ | 1     | 1                  | 1               | 1               |
| $\chi_{\text{lin}}^{A_{12},0}$ | 1     | $\vartheta(A_{12}^*t_1^*)$ | 1               | 1               |
| $\chi_{\text{lin}}^{A_{12}, -A_{13}}$ | 1     | $\vartheta(A_{12}^*t_1^*) \cdot \vartheta(-A_{13}^*t_3)$ | 1               | 1               |
| $\chi_{\text{lin}}^{A_{14}}$ | 3     | 0                  | 0               | 3 $\cdot \vartheta(-A_{14}^*t_4^*)$ |

Proof. We use the notation of Proposition 8.7. Let $u = Y(t_1, t_3, t_4) \in U$, $H = Y_bY_c$, and $\lambda^{A_{14}, A_{13}} \in \text{Irr}(H)$ with $\lambda^{A_{14}, A_{13}}(b(t_3)c(t_4)) := \vartheta(-A_{14}t_4 - A_{13}t_3)$. We have

$$
\chi_{2,q}^{A_{14}}(u) = \text{Ind}_H^U \lambda^{A_{14}, 0}(u) = \frac{1}{|H|} \sum_{g \in U} \lambda^{A_{14}, 0}(g \cdot Y(t_1, t_3, t_4) \cdot g^{-1}).
$$
Then
\[
\chi_{2,q}^{A_{14}}(c(t_4)) = q \cdot \chi_{2,q}^{A_{14},0}(c(t_4)) = q \cdot \vartheta(-A_{14}t_4) = 3 \cdot \vartheta(-A_{14}^*t_4),
\]
\[
\chi_{2,q}^{A_{14}}(b(t_3^s)) = \frac{1}{|H|} \sum_{g = Y(s_1,s_3,s_4) \in U, g \cdot b(t_3^s) \cdot g^{-1} \in H} \lambda^{A_{14},0}(g \cdot b(t_3^s) \cdot g^{-1}) = \sum_{s_1 \in F_q} \lambda^{A_{14},0}(Y(0, t_3^s, t_3^s, s_1))
\]
\[
= \sum_{s_1 \in F_q} \vartheta(-A_{14}^*t_3^s s_1) = 0,
\]
\[
\chi_{2,q}^{A_{14}}(Y(t_1^s, t_3^s)) = 0.
\]

All the other values are determined similarly. \(\square\)

8.9 Proposition (Supercharacters and irreducible characters). If \(q = 3\), then the relations between supercharacters and irreducible characters of \(2G_2^{syl}(3)\) are established.

\[
\Psi_{M(A_{14}^c)} = 3 \cdot \chi_{2,q}^{A_{14}}, \quad \Psi_{M(A_{13}^c)} = \chi_{lin}^{A_{12}, A_{13}}, \quad \Psi_{M(A_{12}^c)} = \chi_{lin}^{A_{12}, 0}, \quad \Psi_{M(0)} = \chi_{lin}^{0, 0}.
\]

Proof. Compare Table 4 and Table 6, the formulae are obtained. \(\square\)

8.10 Remark. Let \(q = 3\), and \#Irr_\(c\) be the number of irreducible characters of \(2G_2^{syl}(3)\) of dimension \(q^c\) with \(c \in \mathbb{N}\). Then \#Irr_1 = \(q - 1 = 2\), \#Irr_0 = \(q^2 = (q - 1)^2 + 2(q - 1) + 1 = 9\), and

\[
\# \text{Irreducible Characters of } 2G_2^{syl}(3) = \# \text{Conjugacy Classes of } 2G_2^{syl}(3) = q^2 + q - 1 = (q - 1)^2 + 3(q - 1) + 1 = 9(2) + 1 = 11.
\]

We consider the analogue of Higman’s conjecture, Lehrer’s conjecture and Isaacs’ conjecture of \(A_n(q)\) for \(2G_2^{syl}(3)\). The conjectures hold for \(2G_2^{syl}(3)\).

8.11 Comparison (Irreducible characters). The irreducible characters of \(2G_2^{syl}(3)\) are determined by Clifford theory that is similar to that of \(3D_4^{syl}(q^4)\) (see [21] and [25, §4]) and that of \(G_2^{syl}(q)\) (see [26, §9]).

Acknowledgements

This paper is a part of my PhD thesis [24] at the University of Stuttgart, Germany, so I am deeply grateful to my supervisor Richard Dipper. I also would like to thank Jun Hu and Mathias Werth for the helpful discussions and valuable suggestions.

References

[1] C. A. M. André. Basic characters of the unitriangular group. J. Algebra, 175(1):287–319, 1995.

[2] C. A. M. André, P. J. Freitas, and A. M. Neto. A supercharacter theory for involutive algebra groups. J. Algebra, 430:159–190, 2015.

[3] C. A. M. André and A. M. Neto. Super-characters of finite unipotent groups of types \(B_n\), \(C_n\), and \(D_n\). J. Algebra, 305(1):394–429, 2006.
[4] C. A. M. André and A. M. Neto. A supercharacter theory for the Sylow $p$-subgroups of the finite symplectic and orthogonal groups. *J. Algebra*, 322(4):1273–1294, 2009.

[5] C. A. M. André and A. M. Neto. Supercharacters of the Sylow $p$-subgroups of the finite symplectic and orthogonal groups. *Pacific J. Math.*, 239(2):201–230, 2009.

[6] S. Andrews. Supercharacters of unipotent groups defined by involutions. *J. Algebra*, 425:1–30, 2015.

[7] S. Andrews. Supercharacter theories constructed by the method of little groups. *Comm. Algebra*, 44(5):2152–2179, 2016.

[8] R. W. Carter. *Simple groups of Lie type*. John Wiley & Sons, London-New York-Sydney, 1972.

[9] R. W. Carter. *Lie algebras of finite and affine type*, volume 96 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2005.

[10] P. Diaconis and I. M. Isaacs. Supercharacters and superclasses for algebra groups. *Trans. Amer. Math. Soc.*, 360(5):2359–2392, 2008.

[11] R. Dipper and Q. Guo. $U_n(q)$ acting on flags and supercharacters at $q$. *Journal of Algebra*, 475(21):80–112, 2016.

[12] A. Evseev. Reduction for characters of finite algebra groups. *J. Algebra*, 325:321–351, 2011.

[13] Q. Guo and R. Dipper. On monomial linearisation and supercharacters of pattern subgroups. *Sci. China Math.*, 61(2):227–252, 2018.

[14] Q. Guo, M. Jedlitschky, and R. Dipper. On coadjoint orbits for $p$-sylow subgroups of finite classical groups. arxiv:1802.08344v1, 2018.

[15] Q. Guo, M. Jedlitschky, and R. Dipper. Orbit method for $p$-Sylow subgroups of finite classical groups. arxiv:1709.03238v3, 2018.

[16] A. F. Hendrickson. Supercharacter theory constructions corresponding to schur ring products. *Communications in Algebra*, 40(12):4420–4438, 2012.

[17] G. Higman. Enumerating $p$-groups. I. Inequalities. *Proc. London Math. Soc. (3)*, 10:24–30, 1960.

[18] I. M. Isaacs. Counting characters of upper triangular groups. *J. Algebra*, 315(2):698–719, 2007.

[19] M. Jedlitschky. *Decomposing André-Neto supercharacters of Sylow $p$-subgroups of Lie type $D$*. PhD thesis, Universität Stuttgart, 2013.

[20] A. A. Kirillov. *Lectures on the orbit method*, volume 64 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2004.

[21] T. Le. Irreducible characters of Sylow $p$-subgroups of the Steinberg triality groups $^3D_4(p^{3m})$. *Bull. Iranian Math. Soc.*, 42(5):1279–1291, 2016.

[22] G. I. Lehrer. Discrete series and the unipotent subgroup. *Compositio Math.*, 28:9–19, 1974.
[23] I. Pak and A. Soffer. On Higman’s $k(U_n(q))$ conjecture. arXiv:1507.00411v1, 2015.

[24] Y. Sun. Supercharacter theories for Sylow $p$-subgroups $^3D_4^{syl}(q^3)$, $G_2^{syl}(q)$ and $^2G_2^{syl}(3^{2m+1})$. PhD thesis, Universität Stuttgart, 2017.

[25] Y. Sun. Character tables of Sylow $p$-subgroups of the Steinberg triality groups $^3D_4(q^3)$. Communications in Algebra, 46(11):5034–5055, 2018.

[26] Y. Sun. Supercharacter theories for Sylow $p$-subgroups of Lie type $G_2$. arXiv:1808.03115v1, 2018.

[27] Y. Sun. A supercharacter theory for Sylow $p$-subgroups of the Steinberg triality groups. Journal of Algebra and its Applications, 18(5):33 pages, 2019.

[28] A. Vera-López and J. M. Arregi. Conjugacy classes in unitriangular matrices. Linear Algebra Appl., 370:85–124, 2003.

[29] N. Yan. Representations of finite unipotent linear groups by the method of clusters. arxiv:1004.2674v1, 2010.