Hinged Kite Mirror Dissection

David Eppstein*

Abstract

Any two polygons of equal area can be partitioned into congruent sets of polygonal pieces, and in many cases one can connect the pieces by flexible hinges while still allowing the connected set to form both polygons. However it is open whether such a hinged dissection always exists. We solve a special case of this problem, by showing that any asymmetric polygon always has a hinged dissection to its mirror image. Our dissection forms a chain of kite-shaped pieces, found by a circle-packing algorithm for quadrilateral mesh generation. A hinged mirror dissection of a polygon with $n$ sides can be formed with $O(n)$ kites in $O(n \log n)$ time.

1 Introduction

A dissection of one polygon to another is a partition of the first polygon into smaller polygonal pieces that can be rearranged to form the second polygon. Dissections are possible when (and only when) the two polygons have the same area, indeed dissection was used by Hilbert as the basis for an axiomatization of area [9]. Dissection puzzles are also popular in recreational geometry, where the main aim is to find dissections of interesting shapes such as regular polygons that use as few pieces as possible [7].

A classic example is the four-piece dissection from an equilateral triangle to a square, often ascribed to Dudeney (but see [7, 8] for speculation on its origin). The four pieces in the dissection can be connected by hinges, points of attachment at which the two attached pieces are free to rotate (Figure 1), while still allowing all four pieces to fold up into both the square and the triangle. This example has sparked much interest in similar hinged dissections [1, 6] but few general results are known, and it remains open whether each equal-area pair of polygons has a hinged dissection. In one of the few theoretical papers in this area, Demaine et al. [6] showed that chains of isosceles right triangles form hinged dissections between any pair of $n$-ominos (Figure 2), and more generally that hinged dissections are possibly between many pairs of polyforms, shapes formed by face-to-face gluing of many copies of the same basic form.

In this paper we demonstrate the existence of another class of hinged dissections: we prove that any asymmetric polygon (including polygons with holes) has a hinged dissection to its mirror image. Our method is to find a dissection that can be unfolded on its hinges into a symmetric form: a chain of kites (quadrilaterals with reflection symmetry across a diagonal) connected end-to-end, so all their axes of symmetry lie on a common line. Clearly, one can then perform the mirror image of the unfolding process to fold the chain back up into the mirror image of the original polygon; Figure 3 shows this unfolding and refolding process applied to a hinged kite dissection of an asymmetric concave hexagon (however this dissection is not one that would be found by the algorithms we describe). We prove that any polygon with $n$ sides has a hinged kite dissection with $O(n)$ pieces, which can be computed by an algorithm running in $O(n \log n)$ time.

Although the set of dissections we find is very limited, we discuss possible implications for more general dissections between any two equal-area polygons. Our technique allows us to reduce the general dissection

---

*Dept. Inf. & Comp. Sci., Univ. of California, Irvine, CA 92697-3425. Email: eppstein@ics.uci.edu.
Figure 1: Hinged dissection of equilateral triangle and square.

Figure 2: Chains of isosceles right triangles form hinged dissections of any polyomino [5].

Figure 3: A chain of kites, hinged along their axes of symmetry, can be unfolded so that all axes are colinear, and refolded to form the mirror image of the original polygon.
problem into the question of whether any two equal-area triangles have a hinged dissection in which some copies of a specified pair of vertices of the first triangle map to a specified pair of vertices of the second.

As in the paper of Demaine et al. [5], we do not consider the question of whether our dissections can be continuously unfolded without any intermediate self-intersections, so in the terminology of Frederickson [8] all our dissections are \textit{wobbly-hinged}.

2 Dissection Process

We now describe the steps by which we find a hinged dissection of an arbitrary polygon into a chain of kites. These steps are also illustrated in Figure 4. Our technique is based on a circle-packing algorithm of Bern and the author [3] for partitioning polygons into well-shaped quadrilaterals in the context of finite element mesh generation; this method is based on previous circle-packing nonobtuse triangulation algorithms by Bern et al. [4, 6] and has also been applied to problems of paper folding and cutting [2].

We dissect the given polygon by the following sequence of steps. Steps 1-3 are taken from the kite meshing algorithm of Bern and the author, while the remaining steps transform the kite mesh into a hinged dissection. In steps 1-3 we pack the polygon by tangent circles, so that the polygon is partitioned by the circles into regions of two types: interiors of circles, and nonconvex \textit{gaps} exterior to the circles. Each gap is bounded by three or more \textit{sides} consisting of segments of polygon boundary and arcs of tangent circles.

1. We begin by placing pairs of congruent circles near each reflex vertex of the polygon, tangent to each other and to the polygon.

2. We place additional circles tangent to each boundary component of the polygon, so that the circles are connected in a cycle by tangencies, with a circle doubly tangent to the polygon near each convex vertex. After this step, each gap involving a segment of polygon boundary (other than the four-sided gaps created in step 1) has exactly three sides. However, the gaps in the interior of the polygon may still have many sides.

3. As long as there is a gap with five or more sides, we place a circle to split it into two simpler gaps.

4. The remaining gaps have three or four sides. We draw line segments between each circle center and the circle’s points of tangencies, partitioning the polygon into triangles and quadrilaterals surrounding each gap, with distinguished points (the tangencies) towards the center of each triangle or quadrilateral edge.

5. We now partition each of these triangles or quadrilaterals into kites, according to a case analysis shown in Figure 5:

(a) In a three-sided gap interior to the polygon, we place a point at the circumcenter of the triangle formed by the three points of tangency, and connect this center point to each tangency.

(b) In a four-sided gap, the four points of tangency are always cocircular [3]. In most such cases, as in the case of three-sided gaps, we place a point at the circumcenter of these four points, and connect this center point to each tangency.

(c) There may be some four-sided gaps in which the center point is not interior to the convex hull of the four tangencies, so that the previous case would lead to the creation of a concave dart shape instead of a kite. Bern et al. [4] call this case a \textit{bad gap} and show that it can always be split into two good gaps by the addition of a single circle tangent to two of the four arcs of the bad gap. These two good gaps can be covered by seven kites (Figure 8, top right).
Figure 4: Steps in our hinged dissection process.
(d) When two circles form a gap with a straight piece of polygon boundary, we can partition this gap into two kites by a line segment through the circle tangency and perpendicular to the line between the two circle centers. The same type of partition also applies to a gap containing a reflex vertex, because we chose the two circles forming this gap to be congruent.

(e) The final case consists of a gap formed by a convex vertex and a single circle, however this type of gap is already in the form of a kite.

6. We now have a partition of the polygon into kites, however we are not finished because it may not be possible to hinge the kites appropriately. We call the kites of this partition large kites to distinguish them from the ones formed in step 7 below. We next find a tree, with one vertex interior to each large kite, where each tree edge connects points from two adjacent large kites. (I.e., this is a spanning tree of the dual graph of the large kite mesh.)

7. We partition each large kite into four smaller pieces by placing a point at the intersection of its two diagonals, and connecting that point to the midpoints of the large kite edges. This partitions the large kite into four pieces, two of which are similar to the original large kite (shown shaded in Figure 4) and the other two of which are rhombi.

8. We arrange the spanning tree of step 6 so that its vertices lie at the interior points added within each large kite, and its edges lie along the connections from these interior points to the large kite edge midpoints. We add a single segment connecting this spanning tree to the midpoint of an edge on the outer boundary of the polygon.

9. Finally, we trace around the boundary of the tree, and form a linear sequence of small kites and rhombs in the order in which they are visited by this trace. We hinge these kites and rhombs at the vertices on the edge midpoints of the large kites. Each small kite is hinged at the two vertices of its axis of symmetry, and each rhomb is hinged at two opposite vertices.

Bern and the author [3] use a somewhat more complicated case analysis in step 5, allowing four-sided gaps involving the edges of the polygon, in order to show that the dissection into kites used here can be performed in time $O(n \log n)$ and that it need only create $O(n)$ pieces.
Another example of a dissection created by this process, of a scalene triangle, is shown in Figure 6 (center). In this case the circle packing consists of a single circle inscribed in the triangle, eventually resulting in a twelve-piece dissection. However, the number of pieces can be improved: as shown in the right of the figure, any scalene triangle has a three-piece hinged mirror dissection into a kite and two isosceles triangles, formed by cutting from the midpoints of the two short sides of the triangle to a third point on the hypotenuse. The third point is the reflection of the hypotenuse midpoint across the perpendicular bisector of the two other midpoints; the line through it and the opposite vertex is perpendicular to the hypotenuse. The three pieces formed by these two cuts are then hinged at the side midpoints.

3 Possible Implications

We still seem to be a long way from solving the question of whether hinged dissections exist between any pair of equal area polygons, or even more generally between any set of equal area polygons. However, our kite dissection can be used to reduce this problem to a seemingly more simple form.

Suppose we have two equal area polygons, both dissected into chains of kites hinged end-to-end. The sequences of areas of the kites can be viewed as partitions of the one-dimensional interval \([0, A]\), and we can find a common refinement of these two partitions by overlaying them. Geometrically, as shown in Figure 7, this corresponds to introducing a sequence of cuts to the two chains of kites, partitioning them into smaller kites and darts, still hinged end-to-end, so that both chains are composed of polygons that form the same sequence of areas. In other words, the first kite or dart from the first chain has the same area as the first kite or dart from the second chain, and so on. The cuts in one chain correspond to the hinges in the other chain and vice versa. If we could then further hinge-dissect each equal-area pair, we could combine these parts into a hinge dissection of the original two polygons. By further splitting the kites and darts along their axes of symmetry, we reduce the problem to one in which we must dissect a sequence of equal area triangles.

Summarizing, we would be able to hinge-dissect any two equal-area polygons, if only we could hinge-dissect the very simple special case of two equal-area triangles, with the restriction that copies of two vertices from the first triangle are mapped to two vertices of the second triangle so that the dissected triangles can be connected to their neighbors in the chain. Our kite dissection method transforms any single dissection problem of two polygons with a total of \(n\) sides into a sequence of \(O(n)\) triangle dissection problems. More generally, we can use the same construction to reduce any \(k\)-way dissection problem to one involving only triangles.
A similar result could be obtained by using Saalfeld’s decomposition of equal-area polygons into combinatorially equivalent equal-area triangulations [10], however his method lacks complexity bounds and seems to use a large number of pieces.

4 Discussion

We have shown that any polygon has a hinged dissection in the form of a chain of kites, that can be unfolded and refolded to form the mirror image of the original polygon. The result also has some possible consequences for the open problem of the existence of hinged dissections between any pair of equal area polygons.

Some questions about our method remain unanswered, for instance whether our dissections or modifications of them can be unfolded in a continuous motion that avoids self-intersections. Also, the number of pieces used by our dissections, although asymptotically optimal, seems large, and Figure 6 shows that it can be reduced by a factor of four in the case of scalene triangles. Is a similar reduction possible more generally?

Acknowledgements

My thanks go to Greg Frederickson for encouraging me to publish these results, to Erik Demaine for extensive comments on a draft of this paper, and to Cinderella for help with the figures. This work was supported in part by NSF grant CCR-9912338.

References

[1] J. Akiyama and G. Nakamura. Dudeney dissection of polygons. Proc. Japan Conf. Discrete & Computational Geometry, December 1998.

[2] M. W. Bern, E. D. Demaine, D. Eppstein, and B. Hayes. A disk-packing algorithm for an origami magic trick. Proc. Int. Conf. Fun with Algorithms (Elba, 1998), pp. 32–42. Carleton Scientific, Proceedings in Informatics 4, 1999. http://www.ics.uci.edu/~eppstein/pubs/BerDemEpp-Fun-98.ps.gz.
[3] M. W. Bern and D. Eppstein. Quadrilateral meshing by circle packing. *Int. J. Computational Geometry & Applications* 10(4):347–360, August 2000, arXiv:cs.CG/9908016.

[4] M. W. Bern, S. A. Mitchell, and J. Ruppert. Linear-size nonobtuse triangulation of polygons. *Discrete & Computational Geometry* 14:411–428, 1995.

[5] E. D. Demaine, M. L. Demaine, D. Eppstein, G. N. Frederickson, and E. Friedman. Hinged dissections of polyominoes and polyforms. ACM Computing Research Repository, October 1999, arXiv:cs.CG/9907018. A preliminary version appeared in *Proc. 11th Canadian Conf. Computational Geometry*, Vancouver, Canada, August 1999.

[6] D. Eppstein. Faster circle packing with application to nonobtuse triangulation. *Int. J. Computational Geometry & Applications* 7(5):485–491, 1997.

[7] G. N. Frederickson. *Dissections: Plane and Fancy*. Cambridge Univ. Press, 1997.

[8] G. N. Frederickson. *Hinged Dissections: Swingin’ n’ Twistin’*. Cambridge Univ. Press, 2002. To appear.

[9] D. Hilbert. *Foundations of Geometry*. Open Court, 1987.

[10] A. J. Saalfeld. Area-preserving piecewise-affine transformations. *Proc. 17th Symp. Computational Geometry*. ACM, June 2001.