Analysis of the modulation depth of some femtosecond laser pulses in holographic interferometry

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Received: 26 February 2021 / Accepted: 15 July 2021 / Published online: 7 August 2021
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Abstract
In this paper, we discuss the quality of holograms generated by two-femtosecond laser pulses of two different colors based on the interference and holography processes by the calculation of the modulation depth. Some numerical simulations for the two Higher-order sh- and ch-Gaussian temporal profiles (shnGTP and chnGTP) are investigated. These simulations show that, when two lasers with a very large frequency detuning are used, the considered profiles exhibit more precise results and give new frequency detuning zones that allow a very significant value of the modulation depth to fringe contrast (MDFC) ratio to obtain the best hologram compared to Gaussian profile. The proposed theory will be a good basis for the development of some new experiments on holographic interferometry and it will certainly be very useful for the specialists using these femtosecond laser pulses.

Keywords Femtosecond laser pulses · Sh-Gaussian profile · Ch-Gaussian profile · Modulation depth · Frequency detuning · Holograms

1 Introduction
In the last few decades, the dark hollow beams (DHBs) have attracted the attention of many authors because of their extensive applications in modern optics such as trapping of particles, optical communications, and guiding cold atoms (Ito et al. 1996; Yin et al. 2003; Kuga et al. 1997; Ovchinnikov et al. 1997). Several models have been elaborated in the literature to describe the DHBs (Boufalah et al. 2019; Zhou and Zheng 2009; Nos-sir et al. 2020; Belafhal and Saad 2017; Sun et al. 2012; Gu et al. 2009) by using various techniques such as the adaptive optics (Chou et al. 2003), the fiber-optic shaping method (Dickey 2014), the spiral phase plate (Wei et al. 2011; Mawardi et al. 2011) and the vortex grating methods (Zhang et al. 2010) to mention but a few. In 2017, Zou et al. (2017) treated the propagation characteristics of hollow sinh-Gaussian beams through the quadratic-index medium. Earlier, Saad and Belafhal (2021) studied the propagation of the hollow higher-order cosh-Gaussian (HhCG) beam through a quadratic index medium (QIM) and
a Fractional Fourier transform (FRFT), they found that the intensity distribution of HhCG changes periodically during propagation in the QIM.

As we know, the lasers are the basic building block of the technologies for the generation of short light pulses. Shortly after the invention of these sources of light, the duration of the shortest produced pulse had decreased by six orders of magnitude (Rulliere 1998), ranging from microsecond (μs) and nanosecond (ns) regimes in the 1960s and through the picosecond (ps) regime in the seventies and finally to femtosecond (fs) techniques in the eighties (Diels and Rudolph 1996). Holography is a process that takes place as a result of recording the phase, amplitude, or the both, of the wave diffracted by an object, i.e. the object wave. In this recording process, the reference wave interferes with the object wave. The characteristics of the interference pattern formed are recorded on the holographic plate which is reconstructed to obtain holograms. So as to attain high enough contrast of the fringes formed, a light source with large coherence length, generally a laser source, is recommended for best quality holograms. Therefore, holographic techniques had a great success in various areas, such as three-dimensional (3D) display (Schnars and Jueptner 1994), optical metrology (Kreis 2005), medicine (Boyer et al. 1996) and many more.

On the other hand, lithium Niobate (LiNbO₃) has been proven by Sohn et al. (2018) as a photorefractive material successfully used in different field as a holographic medium (Gunter and Huignard 1988). Generally, information stored in the photorefractive material is erased during readout due to the dynamic nature characterizing this material. Furthermore, two-color holography is a way out for non-volatile holography in which a beam of high photon energy is also present while recording for sensitizing the material. This beam is useful for making the readout non-volatile but only at higher values of the wavelength. A large number of Lithium Niobate systems has been constructed to demonstrate non-volatile two-color holography for single hologram storage (Liu et al. 2002; Zhang and Tomita 2000). Multiplexing of two-color holograms was successfully studied through the use of stoichiometric (Lee et al. 2002).

Recently, recording of permanent holographic gratings using laser beams of small frequency difference has been examined by Odoulov et al. (2015). However, Malik and Escarquel (2018) have reported in their work that best holograms can be produced with large frequency detuning. In 2020, Malik and Escarquel (2020) showed that sh⁶GTP is better for the order is equal to unity than the Gaussian profile for holography, especially for higher frequencies differences. However, to the best of our knowledge, no study has treated the modulation depth of sh⁶GTP and ch⁶GTP in terms of the fringe contrast. Hence the interest of our study consists to show that, by using these types of profile, null regions develop and frequency bands are observed favoring the formation of good quality holograms. The remainder of this paper is organized as follows: The theory details for the ratio of modulation depth and fringe contrast is developed in Sect. 2. While Sect. 3 is devoted to some important numerical simulations and discussion of our main results applied in two general cases: sh⁶GTP and ch⁶GTP. Finally, the results are summarized in the conclusion.

2 Theory

In this section, we consider the interference of two femtosecond laser pulses with different amplitudes and different frequencies. We present the theory concerning the modulation depth evaluation of the considered beams. In the plane waves approximation, the two waves have the following electric fields
\[ E_1(r, t) = A_1 f(t) \cos(\omega_1 t - \vec{k}_1 \cdot \vec{r}) \]

and

\[ E_2(r, t) = A_2 f(t) \cos(\omega_2 t - \vec{k}_2 \cdot \vec{r}) \]

where \( A_1 \) and \( A_2 \) are the amplitudes of the electric fields of the two lasers, \( \omega_1 \) and \( \omega_2 \) are the angular frequencies, \( \vec{k}_1 \) and \( \vec{k}_2 \) are the wave-vectors of the electric fields and \( f(t) \) is their temporal profile. The intensity profiles of these beams are given by

\[ I_{1,2}(r, t) = \left| E_{1,2}(r, t) \right|^2 \]

Consequently, in time and space, the intensity of the total electric field is evaluated by

\[ I(\vec{r}, t) = \left| E_1(\vec{r}, t) + E_2(\vec{r}, t) \right|^2 \]

By introducing the frequency detuning of the waves \( \Omega = \omega_2 - \omega_1 \), the difference of the wave vectors \( \vec{k} = \vec{k}_2 - \vec{k}_1 \) and after some algebraic calculations, the intensity distribution of these two waves can be rewritten as (Malik and Escarquel 2018, 2020)

\[ I(\vec{r}, t) = f^2(t) \left[ A_1^2 + A_2^2 + 2A_1A_2 \cos(\Omega t - \vec{k} \cdot \vec{r}) \right] \]

which gives a fringe pattern moving along the wave vector with a velocity given by \( v = \Omega / k \). The fringe contrast is defined by

\[ m = \frac{2A_1A_2}{A_1^2 + A_2^2} \]

We note that the experimental laser beams, used for the creation of the fringe pattern, have their intensity distribution expressed by Eq. (4). This yields to define the total energy density per unit area, given by

\[ \epsilon(\vec{r}) = \int_{-\infty}^{+\infty} I(\vec{r}, t) dt \]

By substituting Eq. (4) in this last equation, \( \epsilon(\vec{r}) \) becomes

\[ \epsilon(\vec{r}) = (I_1 + I_2) + 2\sqrt{I_1I_2} \]

where

\[ (I_1 + I_2) = (A_1^2 + A_2^2) \int_{-\infty}^{+\infty} f^2(t) dt \]

and

\[ \sqrt{I_1I_2} = A_1A_2j \]
By using the Euler’s formula, Eq. (10) can be written as

\[ J = \frac{e^{-ikr}}{2} \int_{-\infty}^{+\infty} f(t) e^{it\Omega} dt + \frac{e^{ikr}}{2} \int_{-\infty}^{+\infty} f(t) e^{-it\Omega} dt \]

and the total energy density per unit area becomes

\[ \varepsilon(\vec{r}) = (I_1 + I_2) \left\{ 1 + M_d \cos(\vec{k}\vec{r}) \right\} \]

and this intensity can be expressed as

\[ \varepsilon(\vec{r}) = (A_1^2 + A_2^2) \int_{-\infty}^{+\infty} f^2(t) dt \left\{ 1 + \frac{m}{2} \frac{e^{-ikr} \int_{-\infty}^{+\infty} f^2(t) e^{it\Omega} dt + e^{ikr} \int_{-\infty}^{+\infty} f^2(t) e^{-it\Omega} dt}{\int_{-\infty}^{+\infty} f^2(t) dt} \right\} \]

Finally, one obtains for the modulation depth

\[ M_d = \frac{m}{2 \cos(kr)} \frac{e^{-ikr} K^+ + e^{ikr} K^-}{K} \]

where

\[ K^+ = \int_{-\infty}^{+\infty} f^2(t) e^{it\Omega} dt \]

\[ K^- = \int_{-\infty}^{+\infty} f^2(t) e^{-it\Omega} dt \]

and

\[ K = \int_{-\infty}^{+\infty} f^2(t) dt \]

Equation (14) is the main result of the present work. Note that for a Fourier transforms spectrometer, the modulation depth is also defined by Eq. (14) where \( \cos(kr) = \cos \delta \) with \( \delta \) is the phase difference of the two coherent beams. If the two waves have the same frequency but the amplitudes are different, in this case \( \Omega = 0 \) and Eq. (14) reduces to \( M_d = m \). So, in this case, the modulation depth is equal to the fringe contrast and the ratio MDFC/m is equal to unity.

In general, this MDFC ratio plays an important role in holography and holographic interferometry, because this ratio yields to the best quality of the holograms. If we have two beams, with the same amplitudes \( (A_1 = A_2 = A) \) and the fringe contrast is equal to unity (m = 1), \( M_d \) is expressed as

\[ M_d = \left| \frac{1}{2 \cos(kr)} \frac{e^{-ikr} \int_{-\infty}^{+\infty} f^2(t) e^{it\Omega} dt + e^{ikr} \int_{-\infty}^{+\infty} f^2(t) e^{-it\Omega} dt}{\int_{-\infty}^{+\infty} f^2(t) dt} \right| \]
In the following sub-section, we will apply our main result to two important cases: sh^nGTP and ch^nGTP.

### 2.1 Case of sh^nGTP

We apply Eq. (14) to evaluate the modulation depth of sh^nGTP given by

\[ I(t) = I_0 f^2(t) = I_0 e^{-t^2/\tau^2} \text{sh}^{2n} \left( \frac{\delta t}{\tau} \right) \]  \hspace{1cm} (17)

where \( \tau \) is the initial pulse duration, \( n \) is the beam order and \( \delta \) is the controller parameter of the central dark spot size of the considered pulse. \( I_0 \) is the amplitude of the profile and for simplicity, we take \( I_0 = 1 \).

To study the modulation depth \( M_{sh}^{\text{shGTP}} \) of sh^nGTP, we will evaluate the terms \( K \), \( K^+ \) and \( K^- \) given by

\[ K = \int_{-\infty}^{+\infty} e^{-t^2/\tau^2} \text{sh}^{2n} \left( \frac{\delta t}{\tau} \right) dt \]  \hspace{1cm} (18a)

\[ K^+ = \int_{-\infty}^{+\infty} e^{-t^2/\tau^2} \text{sh}^{2n} \left( \frac{\delta t}{\tau} \right) e^{i\Omega t} dt \]  \hspace{1cm} (18b)

and

\[ K^- = \int_{-\infty}^{+\infty} e^{-t^2/\tau^2} \text{sh}^{2n} \left( \frac{\delta t}{\tau} \right) e^{-i\Omega t} dt \]  \hspace{1cm} (18c)

To develop these integrals, we recall the following identities (Gradshteyn et al. 2007)

\[ \text{sh}^{2n}(x) = \frac{(-1)^n}{2^{2n}} \left\{ \sum_{k=0}^{n-1} (-1)^{n-k} 2\left( \begin{array}{c} 2n \\ k \end{array} \right) \text{ch}[2(n-k)x] + \left( \begin{array}{c} 2n \\ n \end{array} \right) \right\} \]  \hspace{1cm} (19)

\[ \int_0^{\infty} e^{-q^2t^2} dt = \frac{\sqrt{\pi}}{2q} \]  \hspace{1cm} (20)

and

\[ \int_0^{\infty} e^{-\beta t^2} \text{ch}(at) dt = \frac{1}{2} \sqrt{\frac{\pi}{\beta}} \]  \hspace{1cm} (21)

and Eq. (18.a) becomes

\[ K = \frac{(-1)^n}{2^{2n}} \sqrt{\pi} \tau \left\{ \sum_{k=0}^{n-1} (-1)^{n-k} 2\left( \begin{array}{c} 2n \\ k \end{array} \right) e^{(n-k)^2\delta^2} + \left( \begin{array}{c} 2n \\ n \end{array} \right) \right\} \]  \hspace{1cm} (22)

To calculate the integrals \( K^+ \) and \( K^- \), we consider the following expression
which can be written as

\[ R_f = \int_{-\infty}^{+\infty} e^{-\beta^2 + \gamma t} sh^{2n}(at) dt \]  

(23)

with

\[ R_f = \frac{\gamma^2 / 4\beta}{\sqrt{\beta}} \int_{-\infty}^{+\infty} e^{-u^2} sh^{2n} \left( \frac{au}{\sqrt{\beta}} + \frac{a\gamma}{2\beta} \right) du \]  

(24)

With the help of Eqs. (19) and (20), this last equation can be restated as

\[ R_f = \frac{\gamma^2 / 4\beta}{\sqrt{\beta}} \left( \sum_{k=0}^{n-1} (-1)^{n-k} 2 \binom{2n}{k} S_k + \binom{2n}{n} \sqrt{\pi} \right) \]  

(25)

with

\[ S_k = \int_{-\infty}^{+\infty} e^{-u^2} ch \left[ 2(n - k) \left( \frac{au}{\sqrt{\beta}} + \frac{a\gamma}{2\beta} \right) \right] du \]  

(26)

By using the identity

\[ ch(x + y) = ch(y)ch(x) + sh(y)sh(x) \]  

(27)

and Eq. (21), \( S_k \) becomes

\[ S_k = \sqrt{\pi} ch \left[ (n - k) \frac{a\gamma}{\beta} \right] e^{\frac{(n-k)^2}{\beta}} \]  

(28)

Consequently, \( R_f \) is expressed as

\[ R_f = \frac{\gamma^2 / 4\beta}{\sqrt{\beta}} \left( \sum_{k=0}^{n-1} (-1)^{n-k} 2 \binom{2n}{k} e^{\frac{(n-k)^2}{\beta}} . ch[n\Omega \delta(n-k)] + \binom{2n}{n} \right) \]  

(29)

The two integrals given by Eqs. (18b) and (18c) can be evaluated from Eq. (29) by taking \( \gamma = i\Omega \) and \( \gamma = -i\Omega \), respectively. After tedious algebraic calculations, one obtains

\[ K^+ = e^{-\Omega^2 \tau^2 / 4\tau \sqrt{\pi}} \left( \sum_{k=0}^{n-1} 2 \binom{2n}{k} e^{\delta^2(n-k)^2} . ch[i\Omega \tau \delta(n-k)] + \binom{2n}{n} \right) \]  

(30)

and

\[ K^- = e^{-\Omega^2 \tau^2 / 4\tau \sqrt{\pi}} \left( \sum_{k=0}^{n-1} 2 \binom{2n}{k} e^{\delta^2(n-k)^2} . ch[-i\Omega \tau \delta(n-k)] + \binom{2n}{n} \right) \]  

(31)

Finally, for this temporal profile, the modulation depth can be rearranged as
\[ M_{d}^{sh} = me^{-\Omega^2 \tau^2/4} \left\{ \sum_{k=0}^{n-1} (-1)^{n-k} 2 \left( \frac{2n}{k} \right) e^{\delta^2 (n-k)^2} \cos \left[ \Omega \tau \delta (n-k) \right] + \left( \frac{2n}{n} \right) \right\} \]

It shows that this MDFC ratio depends on the frequency detuning \( \Omega \) of the waves, the beams order \( n \) and the controller parameter \( \delta \), which is valid for \( n \geq 1 \). In the particular case for \( n = 1 \), Eq. (32) becomes

\[ M_{d}^{sh} = me^{-\Omega^2 \tau^2/4} \left\{ e^{\delta^2} \cos(\Omega \tau \delta) - 1 \right\} \]

From this equation, one deduces the expression of the MDFC ratio of this profile with \( n = 1 \)

\[ MDFC = e^{-\Omega^2 \tau^2/4} \left\{ e^{\delta^2} \cos(\Omega \tau \delta) - 1 \right\} \]

which is consistent with the well-known expression given by Malik and Escarquel (2020).

### 2.2 Case of \( ch^n \)GTP

In this case, the temporal profile is given by

\[ I(t) = I_0 f^2(t) = I_0 \cdot e^{-t^2/\tau^2} \cdot ch^{2n} \left( \frac{\delta t}{\tau} \right) \]

where \( \delta \) is the controller parameter of the central dark spot size of the pulse, \( n \) is the beam order and \( \tau \) is the pulse initial duration. As above, the amplitude \( I_0 \) of the temporal profile is taken equal to unity. In this section, we evaluate the integral transforms \( K, K^+ \) and \( K^- \) given by Eq. (15) corresponding to this temporal profile family. By using the following identity (Gradshteyn et al. 2007)

\[ ch^{2n}(x) = \frac{1}{2^{2n}} \left\{ \sum_{k=0}^{n-1} 2 \left( \frac{2n}{k} \right) ch[2(n-k)x] + \left( \frac{2n}{n} \right) \right\} \]

and with the help of Eq. (20), the first integral becomes

\[ K = \sqrt{\frac{\pi \tau}{2^{2n}}} \left\{ \sum_{k=0}^{n-1} 2 \left( \frac{2n}{k} \right) e^{(n-k)^2 \delta^2} + \left( \frac{2n}{n} \right) \right\} \]

For the evaluation of \( K^+ \) and \( K^- \), we determine the integral transform

\[ R'_{\tau} = \int_{-\infty}^{+\infty} e^{-\beta^2 + \gamma t} ch^{2n}(at) dt \]

which can be written, if one uses Eq. (36) and the following change of variable \( x = \frac{a}{\sqrt{\beta}} u + \frac{a}{2\beta} \gamma \), as
By using the expression of $S_k$ given by Eq. (26), $R'_i$ becomes

$$R'_i = e^{\gamma^2/4\beta} \frac{1}{2^{2n}} \left\{ \sum_{k=0}^{n-1} 2 \left( \begin{array}{c} 2n \\ k \end{array} \right) S_k + \left( \begin{array}{c} 2n \\ n \end{array} \right) \sqrt{\pi} \right\}$$  \hspace{1cm} (39)$$

The integrals transforms $K^+$ and $K^−$ correspond to the value of $R'_i$ for $\gamma = i\Omega$ and $\gamma = -i\Omega$, respectively. Consequently, one obtains

$$K^+ = e^{-\Omega^2\tau^2/4\tau} \frac{1}{2^{2n}} \left\{ \sum_{k=0}^{n-1} 2 \left( \begin{array}{c} 2n \\ k \end{array} \right).e^{i\Omega\tau(n-k)} . ch[(n-k)] + \left( \begin{array}{c} 2n \\ n \end{array} \right) \right\}$$  \hspace{1cm} (41)$$

and

$$K^- = e^{-\Omega^2\tau^2/4\tau} \frac{1}{2^{2n}} \left\{ \sum_{k=0}^{n-1} 2 \left( \begin{array}{c} 2n \\ k \end{array} \right).e^{-i\Omega\tau(n-k)} . ch[-i\Omega\tau(n-k)] + \left( \begin{array}{c} 2n \\ n \end{array} \right) \right\}$$  \hspace{1cm} (42)$$

With the help of the previous results, and in the case of $\text{ch}\n\text{GTP}$, Eq. (14) can be written as

$$M_{ch}^{\text{div}} = me^{-\Omega^2\tau^2/4} \left\{ \sum_{k=0}^{n-1} 2 \left( \begin{array}{c} 2n \\ k \end{array} \right).e^{\delta^2(n-k)^2}. cos(\Omega\tau(n-k)) + \left( \begin{array}{c} 2n \\ n \end{array} \right) \right\}$$  \hspace{1cm} (43)$$

It proves that the MDFFC ratio is dependent for this temporal profile on the frequency detuning $\Omega$ of the waves and on the two parameters: the beams order $n$ and the controller parameter $\delta$. One notes that Eq. (43) is the closed-form of the modulation depth of $\text{ch}\n\text{GTP}$ and for the first order $n=1$ of this beams family, one finds the following expression of the modulation depth rearranged as

$$M_{ch} = me^{-\Omega^2\tau^2/4} \left\{ \frac{\delta^2 . cos(\Omega\tau\delta) + 1}{\delta^2 + 1} \right\}$$  \hspace{1cm} (44)$$

From Eqs. (33) and (44), one deduces the MDFFC ratio of a Gaussian profile as follows

$$MDFFC^G = e^{-\Omega^2\tau^2/4}$$  \hspace{1cm} (45)$$

which is consistent with the-known expression for a Gaussian profile (Odoulov et al. 2015).
3 Numerical analysis and discussions

In this section, we will present some numerical simulations to study the dependence of the MDFC ratio on the frequency detuning for sh\(^n\)GTP, ch\(^n\)GTP and especially for the Gaussian profiles. The present analysis is based on the analytical expressions given by Eqs. (32), (43) and (45).

3.1 Case of Gaussian temporal profile

In this sub-section, we illustrate in Fig. 1 the MDFC ratio for the Gaussian pulses given by Eq. (45) in terms of the pulse duration \(\tau\) and the frequency detuning \(\Omega\) for (a) three-dimensional form and (b) in three-dimensional (x–y) plot. From this figure, one observes that the MDFC ratio increases from 0 to 1 as the color changes from blue to yellow (see the rectangular bar). It can also be seen that the value of the MDFC ratio decreases when the pulse duration increases and the frequency detuning is larger for the Gaussian profile. Consequently, the appropriate value of the MDFC is obtained for a higher pulse duration and a small frequency difference.

3.2 Case of sh\(^n\)GTP

We investigate here the MDFC ratio of the temporal profile given by Eq. (32). We illustrate in Fig. 2 the temporal profile of sh\(^n\)GTP expressed by Eq. (17) for two values of the beam order \(n\) and by varying the controlled parameter \(\delta\).

As it is shown from this figure, the central dark spot of this dark hollow temporal profile widens with the increase of the beam order \(n\), and the decrease of the parameter \(\delta\). We also observe that the intensity of the considered profile takes a large value with the increment of the two parameters \(n\) and \(\delta\).

From our main result represented by Eq. (32), we investigate some numerical examples for the present case. For that, we illustrate in Fig. 3 the variation of MDFC ratio with the frequency detuning for different values of \(\delta\) (0.1, 0.5, and 1.3). One observes from this

![Image](a) and (b)

Fig. 1 The MDFC ratio as a function of the pulse duration \(\tau\) and the frequency detuning \(\Omega\) of the Gaussian pulses
figure that the width of the lobes becomes larger as the value of $\delta$ is small, and new secondary lobes appear when the beam order $n$ increases.

We present in Fig. 4 the variation of the MDFC ratio with the frequency detuning for different values of the pulse duration and for a fixed value of the skew parameter ($\delta = 0.1$). We can see clearly the effect of the beam order $n$ on the MDFC ratio, which results in the appearance of new secondary lobes when $n$ increases. The parameter $\tau$ has also an influence on the width of the principal lobe as well as that of the secondary lobes.

To show the effects of the beam order $n$ and the skew parameters $\delta$, we illustrate in Fig. 5 the MDFC ratio as a function of the frequency detuning for the shGTP (dashed curve) and the Gaussian profile (solid curve) for four values of $n$ and two values of $\delta$. It is shown from these plots that for a particular frequency detuning, the MDFC ratio of shGTP is higher than the MDFC ratio of the Gaussian profile (red hatched area) (Odoulov et al. 2015). For example, the MDFC ratio has a value of 0.45 in the case of shGTP when $n = 1$ (Fig. 5a), which is much greater than that the obtained one with Gaussian profile (i.e. MDFC ratio = 0.29); this is more pronounced when the controlled parameter is smaller (i.e.
δ = 0.1). We can also observe side lobes that take place when the beam order increases. It is clear that the total number of the secondary lobes is proportional to the considered profile order. For example, for n = 1, we find only one secondary lobe but when n = 3, we note three lobes and when n = 4, another lobe starts to rise (four secondary lobes). We note that when the controlled parameter and the beam order are equal to unity, the MDFC ratio is identical to that one investigated by Malik and Escarquel (2018).

In Fig. 6, the MDFC ratio is depicted as a function of the pulse duration and the frequency detuning. It’s seen from these plots that the MDFC ratio increases from 0 to 1 as the color changes from blue to yellow (see the rectangular bar). We note that for a particular pulse duration, the MDFC ratio progressively decreases as the frequency detuning increases. After this duration, a thin strip is formed when the MDFC ratio stays zero. As Ω increases further, the MDFC ratio starts to rise and progressively decreases to 0, which gives rise to two null regions. We observe that by increasing the beam order n the new thin strips appear. Hence, the MDFC ratio of sh^nGTP presents discontinuities compared to that one of the Gaussian profile, due to the formation of null region (MDCD ratio = 0). The discontinuities increase with the increase of the beam order and a better quality of the hologram can be obtained for this beam profile. We deduce that the sh^nGTP is more suitable for obtaining the holograms with good quality.

### 3.3 Case of ch^nGTP

In this section, we treat the case of ch^nGTP. We illustrate firstly, the graphical representations of the temporal profile for this beam family given in Fig. 7 for three values of the controlled parameter δ and for two values of the beam order n. We find the Gaussian profile when δ = 0 and as the controlled parameter δ increases and reaches a value of 1.2, we observe the appearance of two small lobes, which grow with the increase of δ.

We represent in Fig. 8 the variation of the modulation depth with the frequency detuning Ω for three values of the skew parameter δ of ch^nGTP having 50 femtosecond duration. We observe the appearance of new side lobes with the increase of the beam order n and the width of the lobes widens with the decrease of the skew parameter δ.

Figure 9 displays the MDFC ratio as a function of the frequency detuning Ω with three values of the pulses duration τ. It’s seen that the total number of secondary lobes starts to
Fig. 5  The MDFC ratio as a function of the frequency detuning with different skew parameters δ and for a fixed value of the pulse duration set as $\tau = 50 \text{fs}$ and a $n = 1$, b $n = 2$, c $n = 3$ and d $n = 4$. 
Fig. 6 The MDFC ratio as a function of the pulse duration and the frequency detuning of the sh\textsuperscript{2}GTP for: a $\delta = 0.1$ and b $\delta = 0.5$ and for different beam orders.
Fig. 7 Temporal profile of ch\(^3\)GTP for different skew parameters \(\delta\) and for: (a) \(n = 1\) and (b) \(n = 2\)

Fig. 8 The MDFC ratio of ch\(^3\)GTP as a function of the frequency detuning \(\Omega\) for: (a) \(n = 1\) and (b) \(n = 2\)

Fig. 9 The MDFC ratio of ch\(^3\)GTP as a function of the frequency detuning \(\Omega\) for: (a) \(n = 1\) and (b) \(n = 2\)
Fig. 10 The MDFC ratio of ch³GTP and Gaussian profile as a function of the frequency detuning for $\tau = 50 \, fs$ and a $n = 1$, b $n = 2$, c $n = 3$ and d $n = 4$.
Fig. 11 The MDFC ratio of \( \text{ch}^2 \text{GTP} \) as a function of the pulse duration and the frequency detuning for:

(a) \( \delta = 1 \) and (b) \( \delta = 1.3 \)
increase with the increase of the beam order \( n \) and the width of the lobes grow with the decrease of the pulse duration \( \tau \).

Basing on Eq. (43), we illustrate in Fig. 10 the MDFC ratio as a function of the frequency detuning \( \Omega \) with different skew parameters \( \delta \), for a fixed value of the pulse duration set as \( \tau = 50 \) fs and for different values of \( n \) (1, 2, 3, and 4). One can easily see from these plots that the ch\(^a\)GTP yields the Gaussian profile when \( \delta = 0 \), and by varying the parameter \( \delta \) we see the appearance of side lobes. For example, when \( n = 2 \) (Fig. 10b) for \( \delta = 1 \), we obtain two lobes and when \( \delta = 1.3 \) another lobe is added.

The plots in Fig. 11 depict the MDFC ratio as a function of the pulse duration and the frequency detuning for ch\(^a\)GTP with two values of \( \delta \). It’s observed from this figure that the decrease from 1 to 0 of the MDFC ratio value as the color becomes darker (see the rectangular bar). Contrary to the Gaussian profile, the ch\(^a\)GTP is characterized by a MDFC ratio which progressively decreases from 1 (at \( \Omega = 0 \)) to 0 as the frequency detuning is increased (see Fig. 11) and then a thin strip is formed while the MDFC ratio keeps null. In the end and as \( \Omega \) increases further, the MDFC ratio begins to enhance and gradually decreases to zero, which results the appearance of two null regions favoring the formation of the holograms.

### 4 Conclusion

In this paper, we have investigated the process of the interference and the holography with two laser beams having different amplitudes and different frequencies. We have evaluated in detail the analytical expression of the modulation depth in terms of the fringe contrast, the frequency detuning and the controller parameter of the Higher-order sh- and ch-Gaussian temporal profiles. Based on our main results of the evaluated modulation depth, we showed that null regions develop favoring the formation of holograms for both profiles which is more suited to obtaining holograms of high quality. We conclude that the hollow laser beams with dark spot, contrary to Gaussian profile, give a new frequency detuning zones where good quality holograms can be produced for a higher frequency detuning. Furthermore, our work allows us to know the exact combination of the pulse duration and the frequency detuning for a greater modulation depth to fringe contrast ratio to obtain the best quality holograms possible. The present work seeks to present in a plain way that shGTP as better than the Gaussian laser for holography, particularly for higher frequency differences.

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