Analytic moduli for parabolic Dulac germs

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Abstract. This paper gives moduli of analytic classification for parabolic Dulac germs (that is, almost regular germs). Dulac germs appear as first return maps of hyperbolic polycycles. Their moduli are given by a sequence of Écalle–Voronin-type germs of analytic diffeomorphisms. The result is stated in a broader class of parabolic generalized Dulac germs having power-logarithmic asymptotic expansions.

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1. Introduction

First return maps of planar polynomial vector fields are classically used for studying limit cycles, that is, isolated periodic orbits. They are considered when dealing with the possibility of an accumulation of limit cycles (the Dulac problem) or of their creation by deformation (the cyclicity problem related to Hilbert’s 16th problem). Most interesting is the study of first return maps in a neighbourhood of polycycles, which can be hyperbolic or non-hyperbolic. The non-accumulation problem posed by Dulac [4] was solved independently by Écalle [5] and Ilyashenko [8], [9].

We call Dulac germs the type of germs that appear as first return maps of hyperbolic polycycles. In this case, Ilyashenko’s approach [7] consists in extending a Dulac germ (he calls them almost regular germs) from the positive real line to a sufficiently large domain in the complex plane, a so-called standard quadratic domain, which is a biholomorphic image of $\mathbb{C}_+$. This extension ensures the injectivity of the map that assigns to a Dulac germ its power-logarithmic asymptotic expansion, called the Dulac expansion.

However, for non-hyperbolic polycycles, the proof is much more involved ([5] and [8], [9]). Then the first return maps expand asymptotically as transseries that also include iterated logarithms and exponentials. And moreover, determining the domain of their complex extension is not as straightforward.

In this paper we consider parabolic Dulac germs (that is, those tangent to the identity map) and give complete invariants of their analytic classification.

Our construction resembles the construction of Voronin [22] for analytic invariants of parabolic germs analytic at 0, giving the so-called Écalle–Voronin moduli, except that here we get a doubly infinite sequence of pairs of germs giving the moduli. In [15], we study the inverse realization problem for moduli. The problem is solved in a broader class of generalized Dulac germs. This is the reason that here we formulate our results on moduli of analytic classification in this broader class.

The solutions of Dulac’s problem by Ilyashenko and Écalle are still not very well understood. They show the necessity of a very precise analysis of germs of first return maps and, in particular, of the relationship between their asymptotic transserial expansion and the germ itself. In this paper we develop techniques such as integral summability, its iterations, and log-Gevrey asymptotic expansions. They are related to difference equations or differential equations that the objects in question satisfy. We hope that these techniques will be of use also in more general problems of associating germs with given transserial expansions, in particular, in studying the first return map of non-hyperbolic polycycles.

On the other hand, we note that, among Dulac germs, the parabolic germs studied in this paper are the most interesting from the point of view of cyclicity (Hilbert’s 16th problem). Even in the hyperbolic case very little is known about the cyclicity, other than the result of Mourtada [20] in the generic (non-parabolic) case, and some results for polycycles with a few vertices. In the generic case the study
of the cyclicity of a hyperbolic polycycle is based on Mourtada’s normal form [19],
which is a composition of an alternating sequence of powers and translations. The
argument breaks down in the parabolic case, and a more precise knowledge of the
Dulac map is required. We hope that some combinatorial arguments based on our
moduli (and their unfolding) can give this extra information.

2. Main definitions

Let $\mathcal{R}_C$ be a subset of the Riemann surface of the logarithm, given in the loga-
rithmic chart with coordinate $\zeta = -\log z$ by

$$\varphi(\mathbb{C}^+ \setminus \overline{K(0,R)}), \quad \varphi(\zeta) = \zeta + C(\zeta + 1)^{1/2}, \quad C > 0, \quad R > 0,$$

where $\mathbb{C}^+ = \{\zeta \in \mathbb{C}: \text{Re}(\zeta) > 0\}$ and $\overline{K(0,R)} = \{\xi \in \mathbb{C}: |\xi| \leq R\}$. Such a domain
$\mathcal{R}_C$ (or its image in the logarithmic chart of the variable $\zeta = -\log z$) is called
a standard quadratic domain [8], [9] (see Fig. 1).

![Figure 1. The standard quadratic domain $\mathcal{R}_C$, $C > 0$, in the logarithmic chart.](image)

For $C = 0$ we get the Riemann surface of the logarithm with bounded radii, while
for $C > 0$ we get a subdomain of it such that the radii become smaller on each level
of the surface. Moreover, they tend to zero at an exponential rate. Indeed, if by
$\varphi \in [(k - 1)\pi, (k + 1)\pi)$ we denote the $k$th level of the surface $\mathcal{R}_C$, $k \in \mathbb{Z}$, and we
set $\varphi_k = k\pi$, $k \in \mathbb{Z}$, then the radii $r(\varphi_k)$ by levels $k \in \mathbb{Z}$ decrease at most at the
rate

$$C \exp\left(-D\sqrt{\frac{|k|\pi}{2}}\right), \quad |k| \to \infty, \quad \text{for some } C > 0, \quad D > 0. \quad (2.1)$$

By a germ on a standard quadratic domain [8], [9], we mean an equivalence class
of functions that coincide on some standard quadratic domain (for arbitrarily large
$R > 0$ and $C > 0$).

**Definition 2.1** (adapted from [8], [9], [21]). We say that a germ $f$ is a Dulac germ if:

1) it is holomorphic and bounded on some standard quadratic domain $\mathcal{R}_C$ and
real on the set $\{z \in \mathcal{R}_C: \arg(z) = 0\}$;
2) it admits in $R_C$ a Dulac asymptotic expansion\footnote{Uniformly on $R_C$ (see \cite{10}, §24E): for every $\lambda > 0$ there exists an $n \in \mathbb{N}$ such that}

$$\tilde{f}(z) = \sum_{i=1}^{\infty} z^{\lambda_i} P_i(-\log z), \quad c > 0, \quad z \to 0, \tag{2.2}$$

where $\lambda_i \geq 1$, $i \in \mathbb{N}$, are strictly positive, finitely generated,\footnote{In the sense that there exist an $n \in \mathbb{N}$ and $\alpha_1 > 0, \ldots, \alpha_n > 0$ such that each $\lambda_i$, $i \in \mathbb{N}$, is a finite linear combination of the numbers $\alpha_j$, $j = 1, \ldots, n$, with coefficients in $\mathbb{Z}_{\geq 0}$. For Dulac maps that are the first return maps of saddle polycycles the numbers $\alpha_j$, $j = 1, \ldots, n$, are related to the ratios of hyperbolicity of the saddles.} and strictly increasing to $+\infty$, and $P_i$ is a sequence of polynomials with real coefficients, with $P_1 \equiv A$, $A > 0$.

A Dulac germ tangent to the identity map, that is, with $\lambda_1 = 1$ and $P_1 \equiv 1$ in (2.2), is called a parabolic Dulac germ.

A series of the form (2.2) is called a formal Dulac series. Note that all coefficients in the expansion are real.

The choice of the class of Dulac germs is motivated by the fact that the first return maps near hyperbolic saddle polycycles are analytic germs on the open positive real line having Dulac power-logarithmic asymptotic expansions as $x \to 0$ which extend holomorphically to a standard quadratic domain $R_C$. We note that in \cite{8}, \cite{9} Ilyashenko calls Dulac germs almost regular germs. The existence of an extension to a sufficiently large complex domain was important in proving the fact that almost regular germs are uniquely determined by their Dulac asymptotic expansion, which was the key step in proving the non-accumulation theorem for limit cycles (see \cite{4}, \cite{8}, \cite{9}).

By \cite{8}, \cite{9}, Dulac germs form a group under the operation of composition. Here we work with parabolic Dulac germs, that is, with germs tangent to the identity map: $f(z) = z + o(z)$. They form a subgroup under the operation of composition. We expect the hyperbolic cases $f(z) = \lambda z + o(z)$, $|\lambda| \neq 1$, or $f(z) = Az^{\lambda_1} + o(z^{\lambda_1})$, $A \in \mathbb{C}$, $\lambda_1 \neq 1$, to be analytically linearizable, as was the case for germs analytic at $0$ (see, for example, \cite{2}). Parabolic Dulac germs are thus the most interesting from the point of view of cyclicity (see a comment in the Introduction of \cite{17} for more details).

In \cite{15}, we deal with the realization problem for moduli of analytic classification of parabolic Dulac germs. We solve there the realization problem in a broader class of parabolic generalized Dulac germs, which we introduce here. This class contains parabolic Dulac germs. Its definition is well adapted to the solution of the realization problem considered in \cite{15}.

For convenience let

$$\ell := -\frac{1}{\log z}.$$ 

Generalized Dulac expansions and germs are natural generalizations of Dulac expansions and germs: instead of polynomials in $\ell^{-1}$ multiplying powers of $z$ in Dulac
expansions, in generalized Dulac expansions we take series in $\ell$ that are canonically summable on petals (in a certain generalized Gevrey sense) as analytic germs. Here, in order to be able to uniquely define the asymptotic expansion of such a germ, the problem arises of canonical summation at steps with numbers equal to limit ordinals, since we do not assume that the series in $\ell$ multiplying the powers of $z$ reduce to polynomials. This problem is solved by introducing the notion of log-Gevrey summable series (see §4). Then in §9 we construct the moduli of analytic classification for parabolic generalized Dulac germs.

We recall from [16] that by $\hat{L}_1$ (or simply $\hat{L}$) we denote the class of power-logarithmic transseries of the form

$$\hat{f}(z) = \sum_{i=1}^{\infty} \sum_{m=N_i}^{\infty} a_{i,m} z^{\alpha_i} \ell^m, \quad a_{i,m} \in \mathbb{C},$$

where the $\alpha_i \geq 0$ are finitely generated and strictly increasing to $+\infty$, and $N_i \in \mathbb{Z}$ for $i \in \mathbb{N}$. By $\hat{L}_0$, we denote the subclass of $\hat{L}$ consisting of power series with strictly increasing, finitely generated powers. Furthermore, by $\hat{L}_k$, $k \in \mathbb{N}$, we denote the class of power-iterated logarithmic transseries in the variables

$$z, \: \ell, \: \ell_2 = -\frac{1}{\log \ell}, \: \ldots, \: \ell_k = -\frac{1}{\log \ell_{k-1}},$$

where the powers of $z$ are strictly increasing and finitely generated, and the powers of $\ell_i$, $i = 1, \ldots, k$, belong to $\mathbb{Z}$. We put $\hat{L} := \bigcup_{i=0}^{\infty} \hat{L}_i$. If we allow the powers $\alpha_i$ of $z$ to start with negative powers, but to stay finitely generated and strictly increasing to $+\infty$, then we denote the corresponding classes by $\hat{L}_k^\infty$, $k \in \mathbb{N}_0$, and $\hat{L}^\infty$. The subset of all formal transseries of class $\hat{L}_k$ (or $\hat{L}_k^\infty$), $k \in \mathbb{N}_0$, whose leading term does not contain a logarithm is a group under composition. Note also that if $\hat{f} \in \hat{L}$ does not contain a logarithm in the leading term, then $\ell \mapsto -\frac{1}{\log \hat{f}(e^{-1/\ell})}$ is an exponential-power transseries in the variable $\ell$. By $\hat{L}_k^{\text{inv}} \subseteq \hat{L}_k$ we denote the subset of power-logarithmic formal diffeomorphisms (of the form $f(z) = cz + o(z)$, $c \neq 0$), and by $\hat{L}_k^{\text{id}} \subseteq \hat{L}_k$ we denote the subset of transseries tangent to the identity map (that is, of the form $\hat{f}(z) = z + \cdots$). Both are groups under composition. For more details, see [16].

To express that the coefficients of transseries are real, we will write simply $\hat{L}_k(\mathbb{R})$ for $k \in \mathbb{N}_0$, and $\hat{L}(\mathbb{R})$. Parabolic Dulac and parabolic generalized Dulac series belong to $\hat{L}(\mathbb{R})$. They both form groups under compositions (see Proposition 8.2).

In Theorem A of [16] we provided a formal classification of transseries in $\hat{L}(\mathbb{R})$. The results can be directly applied to a formal classification of formal parabolic Dulac series and of formal parabolic generalized Dulac series. The formal classification is provided in the subgroup $\hat{L}_k^{\text{inv}}(\mathbb{R})$ of the group $\hat{L}(\mathbb{R})$ of power-logarithmic formal diffeomorphisms (that is, of the form $f(z) = cz + o(z)$, $c \neq 0$) with real coefficients and with finitely generated support. Let

$$f(z) = z - az^\alpha \ell^m + o(z^\alpha \ell^m), \quad \alpha > 1, \: m \in \mathbb{Z},$$
be a parabolic generalized Dulac series. Then it is formally equivalent in \( \hat{\mathcal{L}}^{\text{inv}}(\mathbb{R}) \) to either of the normal forms
\[
\begin{align*}
\hat{f}_P(z) &= z - z^\alpha \ell^m + \rho z^{2\alpha - 1} \ell^{2m+1}, \\
\hat{f}_1(z) &= \text{Exp}(X_1).\text{id}, \quad X_1(z) = \frac{-z^\alpha \ell^m}{1 + (-\alpha/2)z^\alpha - 1 \ell^m + (m/2 + \rho)z^{\alpha - 1} \ell^{m+1}} \frac{d}{dz}.
\end{align*}
\tag{2.3}
\]

Let \( f_c (c \in \mathbb{R}) \) be the time-\( c \) map of the vector field \( X_1 \):
\[
f_c(z) := \text{Exp}(cX_1).\text{id}.
\]

It is analytic on \( \mathcal{R}_C \). Its (unique) power-logarithmic expansion will be denoted by \( f_c(z) \). The formal invariants of such series are \((\alpha, m, \rho)\), \( \alpha > 1 \), \( m \in \mathbb{Z} \), \( \rho \in \mathbb{R} \).

If \( f \) is normalized, that is, if \( a = 1 \), then the reduction to the normal forms can be done in the smaller subgroup \( \hat{\mathcal{L}}^{\text{id}}(\mathbb{R}) \), consisting of transseries tangent to the identity map.

We note that the formal changes of variables needed for reducing parabolic Dulac series to formal normal forms leave the class of formal Dulac series and are transserial (belong to \( \hat{\mathcal{L}}^{\text{inv}}(\mathbb{R}) \)). Moreover, the blocks in \( \ell \) in them are in general not convergent and do not belong to the class of parabolic generalized Dulac series that we introduce below in Definition 2.3. The class of parabolic generalized Dulac germs is here introduced as the class for which we obtained in [15] a complete result on realization of analytic moduli (namely, realization of every symmetric sequence of diffeomorphisms). For the subclass of parabolic Dulac germs we expect a weaker realization result. The characterization of moduli that can be realized for true parabolic Dulac germs will be the subject of future investigations. On the other hand, the formal changes of variables for parabolic Dulac and parabolic generalized Dulac series belong to a class of so-called block iterated integrally summable series to be introduced in §6.2. It is an even larger subclass of \( \hat{\mathcal{L}}(\mathbb{R}) \), containing the parabolic generalized Dulac series.

To be able to uniquely define generalized Dulac expansions in the definition below of parabolic generalized Dulac germs, we introduce in §4 the definition of log-Gevrey asymptotic expansions on certain cusps that are \( \ell \)-images of sectors, and we state some of their properties. The definition is motivated by the notion of Gevrey asymptotic expansions of order \( k \in \mathbb{N} \) on sectors. For the precise definition of an asymptotic expansion on a sector and of a Gevrey asymptotic expansion of some order on a sector, see, for example, [11], §1, or [1]. The origin of the name log-Gevrey is related to the fact that a log-Gevrey asymptotic expansion on some domain is Gevrey of every order \( k \), but not necessarily convergent at 0, therefore being somewhere between Gevrey-type and convergent.

We will prove in Theorem A, (i), in §8 a more general fact that a flower-like dynamics of a germ \( f \) defined on a standard quadratic domain \( \mathcal{R}_C \) follows merely from the assumption that \( f \) is an analytic germ on \( \mathcal{R}_C \) whose leading terms satisfy the uniform estimate
\[
|f(z) - (z - az^\alpha \ell^m)| \leq c|z^\alpha \ell^{m+1}|, \quad z \in \mathcal{R}_C, \quad a > 0, \quad c > 0.
\]

More precisely, we will prove in §8 the following proposition.
Proposition 2.2 (flower-like dynamics of a parabolic germ). The invariant sets for the dynamics of a parabolic germ $f$ on $\mathcal{R}_C$ satisfying the condition
\[
|f(z) - z + az^\alpha \ell^m| \leq c|z^\alpha \ell^{m+1}|,
\]
$a > 0$, $\alpha > 1$, $m \in \mathbb{Z}$, $c > 0$, $z \in \mathcal{R}_C$, \hspace{1cm} (2.4)
are open attracting petals $V^+_j$ ($j \in \mathbb{Z}$) on the domain $\mathcal{R}_C$ (or possibly $\mathcal{R}_{C'} \subset \mathcal{R}_C$) that have opening $2\pi/(\alpha - 1)$ and are symmetric with respect to the tangential complex directions
\[
1^{-1/(\alpha - 1)}.
\]
Analogously, the invariant sets for the dynamics of the inverse germ $f^{-1}(z) = z + az^\alpha \ell^m + o(z^\alpha \ell^m)$ are open repelling petals $V^-_j$ ($j \in \mathbb{Z}$) on $\mathcal{R}_C$ (or possibly $\mathcal{R}_{C'} \subset \mathcal{R}_C$) that have opening $2\pi/(\alpha - 1)$ and are symmetric with respect to the tangential complex directions
\[
(-1)^{-1/(\alpha - 1)}.
\]
At the intersections of the attracting and repelling petals the orbits for the discrete dynamics are closed.

This in particular applies to parabolic generalized Dulac germs defined below. Therefore, in Definition 2.3 below we may assume in advance the existence of attracting and repelling petals $V^\pm_j$ ($j \in \mathbb{Z}$) on every invariant petal $\mathcal{R}_C$ with opening $2\pi/(\alpha - 1)$ for the dynamics of $f$ ($j \in \mathbb{Z}$). These petals are described in detail in Theorem A, (i), in the next section.

Definition 2.3 (parabolic generalized Dulac germs). A parabolic germ $f$, analytic on a standard quadratic domain $\mathcal{R}_C$, that maps the ray $\{\arg(z) = 0\} \cap \mathcal{R}_C$ to itself and satisfies the estimate
\[
|f(z) - z + az^\alpha \ell^m| \leq c|z^\alpha \ell^{m+1}|,
\]
$a \neq 0$, $\alpha > 1$, $m \in \mathbb{Z}$, $c > 0$, $z \in \mathcal{R}_C$, \hspace{1cm} (2.5)
is called a parabolic generalized Dulac germ if on every invariant petal $V^\pm_j$ ($j \in \mathbb{Z}$) with opening $2\pi/(\alpha - 1)$ it admits an asymptotic expansion of the form
\[
f(z) = z + \sum_{i=1}^n z^{\alpha_i} \hat{R}_i^\pm(\ell) + o(z^{\alpha_n + \delta_n}), \hspace{1cm} \delta_n > 0,
\]
for every $n \in \mathbb{N}$ as $z \to 0$ on $V^\pm_j$. Here $\alpha_1 = \alpha$, the $\alpha_i > 1$ are strictly increasing to $+\infty$ and finitely generated, and the $\hat{R}_i^\pm(\ell)$ are analytic functions on open cusps $\ell(V^\pm_j)$ which admit common log-Gevrey asymptotic expansions\footnote{An adaptation of Gevrey asymptotic expansions of some order (see §4 for the precise definitions).} $\hat{R}_i(\ell)$ of order strictly greater than $(\alpha - 1)/2$ as $\ell \to 0$: \[
\hat{R}_i(\ell) = \sum_{k=N_i}^{\infty} a^i_k \ell^k, \hspace{1cm} a^i_k \in \mathbb{R}, \hspace{1cm} N_i \in \mathbb{Z}.
\]
We then say that the series \( \hat{f} \in \hat{L} \) given by

\[
\hat{f}(z) := z + \sum_{i=1}^{\infty} z^{\alpha_i} \hat{R}_i(\ell)
\]

is the generalized Dulac asymptotic expansion of \( f \). Such an \( \hat{f} \) is called a parabolic generalized Dulac series.

We prove in Proposition 8.2 that the parabolic generalized Dulac germs form a group under composition.

The assumption (2.5) is automatically satisfied for parabolic Dulac germs (see Proposition 8.1), since Dulac asymptotic expansions are uniform on the whole of \( \mathcal{R}_C \). For parabolic generalized Dulac germs, on the other hand, we assume the existence of the asymptotic expansions (2.6) only on petals, and the estimates do not necessarily hold uniformly from petal to petal. The additional uniform condition (2.5) in their definition is therefore important to ensure that the rate of decrease of the radii of petals for the dynamics of a parabolic generalized Dulac germ on levels of the Riemann surface of the logarithm is not greater than the rate dictated by the geometry of a standard quadratic domain (see §8).

We note also that a generalized Dulac asymptotic expansion is an asymptotic expansion in the class of transseries \( \hat{L}(\mathbb{R}) \). As discussed in [17], an asymptotic expansion of a germ in \( \hat{L}(\mathbb{R}) \) is in general not unique and not well defined. A generalized Dulac expansion is a sectional asymptotic expansion that becomes unique after a canonical choice of section functions (summation rules) at limit ordinal steps. See [17] for the precise definition of sectional asymptotic expansions on \( \mathbb{R} \), and for a generalization to complex sectors see Definition 10.1 in the Appendix. The existence of a canonical choice of section functions is here guaranteed by the assumption that the functions \( R^{i,\pm}_i(\ell) \) admit log-Gevrey power asymptotic expansions of some order on cusps which are the \( \ell \)-images of sectors with sufficiently large opening, called \( \ell \)-cusps. Then by Corollary 4.4 (a variation of Watson’s lemma), their log-Gevrey sum on each petal is unique. This log-Gevrey condition is a stronger condition than Gevrey summability of order \( m \) [11] for any \( m > 0 \) (see §4). It ensures unique summability on \( \ell \)-cusps, which do not contain sectors of any positive opening. The assumption of log-Gevrey summability for parabolic generalized Dulac germs generalizes the polynomiality assumption for Dulac germs.

Parabolic Dulac germs are trivially parabolic generalized Dulac germs. In that case we have a canonical choice of polynomial sections. Polynomial functions of \( \ell^{-1} \) are convergent Laurent series in the variable \( \ell \).

In [15] for the converse problem of realizing a sequence of diffeomorphisms as analytic moduli of power-logarithmic germs, we obtain parabolic generalized Dulac germs as realizing germs.

Below we suppose in addition that \( f(z) = z - az^\alpha \ell^m + o(z^\alpha \ell^m) \), where \( a > 0 \), \( \alpha > 1 \), \( m \in \mathbb{Z} \), so that \( \{\arg(z) = 0\} \cap \mathcal{R}_C \) is an invariant attracting direction. (In the case \( a < 0 \) we switch to the inverse germ \( f^{-1} \), which is by Proposition 8.2 also a parabolic generalized Dulac germ.) By a real homothety \( \varphi(z) = a^{-1/(\alpha-1)} z \),
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We transform \( f(z) = z - az^\alpha \ell^m + o(z^\alpha \ell^m) \) into a normalized germ

\[
f(z) = z - z^\alpha \ell^m + o(z^\alpha \ell^m).
\]

(2.7)

Note that we cannot transform the first coefficient \(-1\) to \(+1\) by a homothety conjugation, since we would then lose the invariance of the positive real line. Therefore, below we always suppose that we are working with a (parabolic or parabolic generalized) Dulac germ whose first coefficient \(a\) is equal to \(1\). This first normalization step, performed for simplicity, is not important from the viewpoint of analytic conjugacy of generalized Dulac germs (discussed later), since a homothety is an analytic germ on a standard quadratic domain.

We will work both in the \(z\)-chart (which we will call the original chart) and in the \((\zeta = -\log z)\)-chart (which we will call the logarithmic chart), each time choosing the chart which is more convenient.

Here we are interested in complete invariants of analytic classification, analogous to the Écalle–Voronin moduli for analytic diffeomorphisms. We first define in §5 what we mean by saying that two parabolic (generalized) Dulac germs \( f \) and \( g \) are analytically conjugate. In the case of analytic germs \( \mathbb{C}\{z\} \) it was clear that \( f, g \in \mathbb{C}\{z\} \) are analytically equivalent if there exists a germ of an analytic diffeomorphism \( \varphi \in \mathbb{C}\{z\} \) such that

\[
g = \varphi^{-1} \circ f \circ \varphi.
\]

Obviously, this analytic equivalence implies formal equivalence in the sense of formal Taylor series in the class \( \tilde{\mathcal{L}}_0(\mathbb{R}) \). In this paper we will stay within the limits of the \( \tilde{\mathcal{L}}(\mathbb{R}) \)-formal classes, and we adopt the following definition of analytic equivalence of parabolic generalized Dulac germs.

**Definition 2.4** (analytic conjugacy). We say that two normalized parabolic generalized Dulac germs \( f \) and \( g \) of the form (2.7) defined on a standard quadratic domain \( \mathcal{R}_C \) are **analytically conjugate** if

1) \( f \) and \( g \) are \( \tilde{\mathcal{L}}(\mathbb{R}) \)-formally conjugate,\(^4\) and

2) there exists a germ of a diffeomorphism \( h(z) = z + o(z) \) of a standard quadratic domain \( \mathcal{R}_C \) such that

\[
g = h^{-1} \circ f \circ h \quad \text{on} \quad \mathcal{R}_C.
\]

If we allow non-normalized germs \( f \) and \( g \), then \( h \) must possibly be composed with an additional homothety, so that \( h \) is assumed to be of the more general form \( h(z) = cz + o(z) \), \( c > 0 \). For simplicity we assume that \( f \) and \( g \) are normalized and that \( h \) is tangent to the identity map. Two non-normalized germs are analytically conjugate if and only if their normalizations are analytically conjugate. Definition 2.4 is analyzed in more detail in §5.

We prove Proposition 2.5 below in §7.4. It explains why it is natural to define an analytic conjugacy of two parabolic generalized Dulac germs in Definition 2.4 simply as a bounded analytic germ tangent to the identity map on a standard quadratic domain, without any other asymptotic conditions. Proposition 2.5 states that it is automatically *coherent* with the formal conjugacy in \( \tilde{\mathcal{L}}^{\text{inv}}(\mathbb{R}) \).

\(^4\)That is, have the same formal invariants \((\alpha, m, \rho)\), \( \alpha > 1, m \in \mathbb{Z}, \rho \in \mathbb{R} \).
Proposition 2.5. Let \( f \) and \( g \) be two normalized parabolic generalized Dulac germs that are analytically conjugate on a standard quadratic domain \( \mathcal{R}_C \) (in the sense of Definition 2.4). Let \( h(z) = z + o(z), \ z \to 0, \ z \in \mathcal{R}_C, \) be their analytic conjugacy. Let \( \hat{h} \in \hat{\mathcal{L}}^{\text{id}}(\mathbb{R}) \) be their formal conjugacy:

\[
\hat{g} = \hat{h}^{-1} \circ \hat{f} \circ \hat{h}.
\]

Then \( h \) admits \( \hat{h} \) as its generalized block iterated integral asymptotic expansion, up to the non-uniqueness of the formal conjugacy \( \hat{h} \) described after Proposition 6.14.

In §7.4 we define the sense in which an analytic conjugacy of two parabolic (generalized) Dulac germs has an asymptotic expansion in \( \hat{\mathcal{L}}(\mathbb{R}) \). In fact, because we are working with transserial asymptotic expansions, which are not unique, we need to determine the section function (the summation rule) that we use at each limit ordinal step. The conjugacies are no longer parabolic generalized Dulac germs, but more complicated. We introduce integral sums of length \( k \in \mathbb{N}_0 \) in Definition 6.1, and the notion of generalized block iterated integral summability in Definitions 6.12 and 7.8. We prove in Proposition 7.9 that any analytic conjugacy of two parabolic (generalized) Dulac germs admits a formal conjugacy in \( \hat{\mathcal{L}}(\mathbb{R}) \) as its generalized block iterated asymptotic expansion, up to non-uniqueness of the formal conjugacy described after Proposition 6.14.

3. Main theorems

Theorem A describes the local dynamics of a parabolic (generalized) Dulac germ on a standard quadratic domain \( \mathcal{R}_C \) (considered on the Riemann surface of the logarithm or in the logarithmic chart of the variable \( \zeta = -\log z \) as a ramified neighbourhood of the point \( z = 0 \) or the set \( \text{Re}(\zeta) = +\infty \)). The local dynamics resembles the well-known flower dynamics for parabolic analytic diffeomorphisms given by the Leau–Fatou flower theorem (see, for instance, [13]), but with countably many tangential attracting and repelling directions situated equidistantly on the Riemann surface of the logarithm.

**Theorem A** (Leau–Fatou theorem for parabolic generalized Dulac germs). Let

\[
f(z) = z - az^{\alpha} \ell^m + o(z^{\alpha} \ell^m), \quad \alpha > 1, \ a > 0, \ m \in \mathbb{Z},
\]

be a parabolic generalized Dulac germ on a standard quadratic domain \( \mathcal{R}_C, C > 0 \).

(i) There exist countably many overlapping attracting and repelling open petals\(^6\) \( V_i^+ \) and \( V_i^- \), \( i \in \mathbb{Z} \), respectively, for \( f \) on \( \mathcal{R}_C \), which are maximal invariant domains for the dynamics of \( f \) and \( f^{-1} \), respectively. They are symmetric with respect to the complex directions \( 1^{1/(\alpha-1)} \) and \( (-1)^{1/(\alpha-1)} \), respectively, on \( \mathcal{R}_C \) which are tangential directions for the dynamics of \( f \) and \( f^{-1} \), respectively. The opening angle of each petal is equal to \( 2\pi/(\alpha - 1) \).

(ii) On each open petal \( V_i^\pm \) there exists a (unique up to an additive constant) holomorphic Fatou coordinate \( \Psi_i^\pm \) which conjugates \( f \) to a translation:

\[
\Psi_i^\pm(f(z)) - \Psi_i^\pm(z) = 1, \quad z \in V_i^\pm,
\]

---

\(^5\)For the notion of a generalized block iterated integral asymptotic expansion, see §7.4.

\(^6\)Classically, as in [13], a petal of opening \( \theta \in (0, 2\pi] \) is a union of open sectors of openings continuously increasing to the value \( \theta \), with continuously decreasing radii.
and admits a formal Fatou coordinate $\hat{\Psi} \in \hat{L}_\infty^\infty(\mathbb{R})$ as its integral asymptotic expansion\(^7\) as $z \to 0$ (up to a constant).

(iii) On each open petal $V_{\pm}^i$ the germ $f$ is analytically conjugate by a germ of the form $\varphi_{\pm}^i(z) = a^{-1/(\alpha-1)}z + o(z)$ to its formal normal form $f_1 = \text{Exp}(X_1).\text{id}$ in (2.3). The conjugation on each petal is unique, up to a precomposition $\varphi_i \circ f_1$ with the time-$c$ map ($c \in \mathbb{R}$) along trajectories of the vector field $X_1$. Its block iterated integral asymptotic expansion\(^8\) is, up to precomposition with $\tilde{f}_c$, equal to the formal conjugacy $\hat{\varphi} \in \hat{L}_\text{inv}(\mathbb{R})$.

The formal Fatou coordinate $\hat{\Psi}$ is a formal solution of the Abel equation for the parabolic generalized Dulac expansion $\tilde{f}$ of $f$:  
$$\hat{\Psi}(\tilde{f}) - \hat{\Psi} = 1.$$  

The proof of Theorem A is given in §8. As was done in [17] for the Fatou coordinate of a parabolic Dulac germ, we prove in §8 that the formal Fatou coordinate of a parabolic generalized Dulac germ is unique (up to a constant) in the class $\hat{L}(\mathbb{R})$ and moreover belongs to the class $\hat{L}_\infty^\infty(\mathbb{R})$.

In Proposition 2.2 we prove the statement of Theorem A, (i), under weaker conditions on the germ $f$, without assuming the full generalized Dulac asymptotic expansion, but just the first part of it. However, the proof of the existence of the petalwise analytic Fatou coordinate (and then also of the sectorial conjugacy) in §8 is based on the existence of the transasymptotic expansion of a generalized Dulac germ $f$ at least up to the residual term of the form $z^{2\alpha-1}\mathcal{E}^{2m+1}$. This enables us to deduce the infinite part of the Fatou coordinate in the proof of Theorem A, (ii), and then prove the convergence and analyticity of the infinitesimal remainder using a modified Abel equation.

The local dynamics of a parabolic generalized Dulac germ $f$ is illustrated on a standard quadratic domain in the logarithmic chart in Fig. 2.

In Theorem B we describe the complete system of analytic invariants for a normalized parabolic generalized Dulac germ $f$, with respect to analytic conjugations $\varphi(z) = z + o(z)$, $z \to 0$, on a standard quadratic domain. This result is similar to the result of Écalle and Voronin [22] for parabolic analytic germs of diffeomorphisms, where the analytic class was given by the formal class and a finite number of analytic diffeomorphisms (analytic moduli), after appropriate identifications (see [22] or [13]).

Let $f$ be a normalized parabolic generalized Dulac germ defined on a standard quadratic domain. We suppose for simplicity that $\alpha = 2$. This can always be done without loss of generality, since we can apply an analytic change of variables $h(z) = (\alpha - 1)^{-m/(\alpha-1)}z^{1/(\alpha-1)}$ carrying one standard quadratic domain to another (see Proposition 9.1 for more details). It is just more convenient, since every level of the surface of the logarithm contains only one petal of opening $2\pi$. In this situation it is easier to express the decrease of the size of the domain of definition

\(^7\)For details, see §7.

\(^8\)For details, see §7. Here this expansion is considered after postcomposition of $\hat{\varphi}$ and $\varphi_{\pm}^i$ with the inverse of the first change of variables that is a homothety, that is, by $\varphi_0(z) = a^{1/(\alpha-1)}z$, since we are working with non-normalized germs.
of the analytic moduli (see (3.1) below). The case $\alpha > 1$, $\alpha \neq 2$, can be analyzed in the same way, but then a finite number of moduli that correspond to one level of the Riemann surface of the logarithm ($\lfloor 2\pi/\alpha \rfloor$ of them) have the same radius.

Let

$$f(z) = z - z^2 \ell^m + o(z^2 \ell^m), \quad m \in \mathbb{Z}, \quad z \in \mathbb{R},$$

be a parabolic generalized Dulac germ. Let $(\Psi^i_\pm)_i \in \mathbb{Z}$ be its analytic Fatou coordinates on attracting or repelling petals $V^i_\pm$ (of opening $2\pi$) along the domain. We prove in §9 that there exists a symmetric (with respect to the $\{\arg(z) = 0\}$-axis) sequence $(h^i_0, h^i_\infty)_i \in \mathbb{Z}$ of analytic germs of diffeomorphisms called horn maps for $f$ such that

$$h^i_0(t) := \exp\left\{-2\pi i \Psi^i_{i-1} \circ (\Psi^-_i)^{-1}\left(-\frac{\log t}{2\pi i}\right)\right\}, \quad t \in (\mathbb{C}, 0), \quad i \in \mathbb{Z},$$

and that this sequence and the formal class $(2, m, \rho)$ are a complete system of analytic invariants of the parabolic generalized Dulac germ $f$.

Due to the standard quadratic domain of definition of $f$, the radii $R_i$ of the domains of the sequence of its horn maps are bounded from below:

$$R_i \geq K_1 \exp\left\{-K e^{C \sqrt{|i|}}\right\}, \quad K, K_1, C > 0, \quad i \in \mathbb{Z}. \quad (3.1)$$

Also, the converse holds: if all the horn maps are defined at least on discs of radii given by (3.1), then all the sectorial Fatou coordinates glue together to form an analytic map at least on a standard quadratic domain.

**Theorem B** (analytic invariants of a parabolic generalized Dulac germ). Let $f$ and $g$ be two normalized parabolic generalized Dulac germs that belong to the same $\tilde{L}(\mathbb{R})$-formal class $(2, m, \rho)$, $m \in \mathbb{Z}$, $\rho \in \mathbb{R}$. Let

$$(h^i_0, h^i_\infty; R^f_i)_i \in \mathbb{Z} \quad \text{and} \quad (k^i_0, k^i_\infty; R^g_i)_i \in \mathbb{Z}$$
be their sequences of horn maps with radii of convergence \((R^f_i)\) and \((R^g_i)\), respectively, where \(R^f_i\) and \(R^g_i\) satisfy estimates of the type (3.1). Then \(f\) and \(g\) are analytically conjugate (by an analytic change of variables that is tangent to the identity) on some standard quadratic domain if and only if there exist sequences \((a_i)_i\), \((b_i)_i \in \mathbb{C}^*\) such that
\[
h_0^i(t) = a_{i-1} \cdot k_0^i(b_i t), \quad h_\infty^i(t) = b_i \cdot k_\infty^i(a_i t), \quad t \in (\mathbb{C}, 0), \quad i \in \mathbb{Z}.
\]

More details and the proof of Theorem B are given in §9. We also show in Proposition 9.2 that the horn maps of a parabolic generalized Dulac germ in an \(\mathcal{L}\)-formal class \((2, m, \rho)\) are symmetric with respect to the real axis, because the coefficients are real. That is,
\[
(h_{0}^{-i+1})^{-1}(t) \equiv h_{\infty}^i(\bar{t}), \quad i \in \mathbb{Z},
\]
on their domains of definition.

Note that the assumption that the germs are normalized as in (2.7) is just convenient for simplicity, but is not very important. In fact, two parabolic generalized Dulac germs are analytically conjugate by a change of variables \(\varphi(z) = bz + o(z), \quad z \to 0, \quad b > 0\), on a standard quadratic domain if and only if their normalizations are analytically conjugate by a change of variables tangent to the identity on a standard quadratic domain. Indeed, we only need one additional homothety that is analytic on a standard quadratic domain. Therefore, it suffices to consider the moduli and the question of analytic conjugacy only for normalized generalized Dulac germs. That is, to consider conjugacies that are tangent to the identity maps.

4. log-Gevrey classes \(\text{LG}_m(S)\) on \(\ell\)-cusps

Here we define the log-Gevrey classes of germs and formal series used in Definition 2.3 of parabolic generalized Dulac germs. We also state and prove some of their properties: closedness with respect to addition, almost closedness with respect to multiplication and differentiation. That is needed for describing Fatou coordinates and sectorial analytic reductions to normal form for parabolic generalized Dulac germs in §7, and also in [15]. The proofs are somewhat similar to the proofs in [1], [11] for classical Gevrey function classes which are differential algebras.

Below we define an \(\ell\)-cusp to be an open cusp that is the image of an open sector \(V\) of positive opening at 0 under the change of variables \(\ell = -1/\log z\), and we denote it by \(S = \ell(V)\) (see Fig. 3). Any open \(\ell\)-cusp \(\ell(V') \subset S\), where \(V' \subset V\) is a proper subsector, will be called a proper \(\ell\)-subcusp of \(S\).

It will be shown below in Corollary 4.8 that asymptotic series on such \(\ell\)-cusps inherit the property of termwise differentiation from power asymptotic series on the corresponding sectors of positive opening (a property that cannot be claimed for \(\mathbb{R}_+\) or for general cusps).

**Definition 4.1** (log-Gevrey asymptotic expansions on \(\ell\)-cusps). Let \(F\) be a germ analytic on an \(\ell\)-cusp \(S = \ell(V)\). We say that \(F\) has the series \(\hat{F}(\ell) = \sum_{k=0}^\infty a_k \ell^k\), \(a_k \in \mathbb{C}\), as its log-Gevrey asymptotic expansion of order \(m > 0\) if, for every proper
$\ell$-subcusp $S' = \ell(V') \subset S$, $V' \subset V$, there exists a constant $C_{S'} > 0$ such that for every $n \in \mathbb{N}$ with $n \geq 2$ we have

$$\left| F(\ell) - \sum_{k=0}^{n-1} a_k \ell^k \right| \leq C_{S'} m^{-n} (\log n)^n \exp \left\{ - \frac{n}{\log n} \right\} |\ell|^n, \quad \ell \in S'.$$  \hfill (4.1)

By $\mathbb{L}G_m(S) \subseteq \mathcal{O}(S)$ with $S$ an $\ell$-cusp and $m > 0$ we denote the set of all germs analytic on $S$ that have a log-Gevrey asymptotic expansion of order $m > 0$ on $S$ as $\ell \to 0$. By $\mathbb{LG}_m(S) \subseteq \mathbb{C}[[\ell]]$ we denote the set of their log-Gevrey asymptotic expansions of order $m > 0$ on $S$.

For the definition of generalized Dulac germs, we will need the following generalization of Definition 4.1.

**Definition 4.2** (generalization of Definition 4.1 to formal Laurent expansions). We say that a function $F$ analytic on an $\ell$-cusp $S$ has a formal Laurent series $\widehat{F}(\ell) \in \mathbb{C}((\ell))$, $\widehat{F}(\ell) = \sum_{k=N}^{\infty} a_k \ell^k$, $N \in \mathbb{Z}$, as its log-Gevrey expansion of order $m > 0$ if $F(\ell)\ell^{-k}$ has $\widehat{F}(\ell)\ell^{-k}$ as its log-Gevrey expansion of order $m > 0$ on $S$ in the sense of the above definition.\(^9\) We use the same notation $\mathbb{L}G_m(S)$ for the set of all such germs, and $\mathbb{LG}_m(S) \subseteq \mathbb{C}((\ell))$ for the set of their log-Gevrey asymptotic expansions of order $m > 0$.

Recall that we say that $F(z)$ admits $\widehat{F}(z)$ as its Gevrey asymptotic expansion of order $k > 0$ on a sector $V$ if, for every proper subsector $V' \subset V$, there exist constants $C, D > 0$ such that for all $n \in \mathbb{N}$

$$\left| F(z) - \sum_{k=0}^{n-1} a_k z^k \right| \leq CD^n (n!)^{1/k} |z|^n, \quad z \in V'.$$

We can now check using (4.1) and Stirling’s formula $\sqrt{n} (n/e)^n \sim n!$, $n \to \infty$, that a log-Gevrey asymptotic expansion of some order $m > 0$ is also a Gevrey asymptotic expansion of every order\(^{10}\) $k > 0$, as introduced in [11], [1], and [12].

\(^9\)By $\mathbb{C}((\ell))$ we denote the fraction field of $\mathbb{C}[[\ell]]$, that is, the set of all formal Laurent series in $\ell$ with complex coefficients. And similarly for $\overline{\mathbb{R}}((\ell))$.

\(^{10}\)This can be seen by writing $\left( \log n \right)^n \exp \left\{ - \frac{n}{\log n} \right\} = \exp \{ n \log \log n - \frac{n}{\log n} \}$ and (by Stirling’s formula) $(n!)^{1/k} = \exp \{ \frac{1}{2k} \log n + \frac{n}{k} \log n - \frac{n}{k} \}$, $k > 0$, and by comparing the order of growth of the exponents as $n \to \infty$. 

---

**Figure 3.** $\ell$-cusp.
In this paper we have to consider functions on \( \ell \)-cusps that do not contain any sectors of positive opening. This is why we need a stronger, log-Gevrey, growth estimate in order to apply a Watson-type lemma. Recall that, classically, if \( F \) admits a zero Gevrey asymptotic expansion of order \( k > 0 \) on \( V \), then, for every proper subsector \( V' \subset V \), there exist constants \( C > 0 \) and \( M > 0 \) such that

\[
|F(z)| \leq C \exp \left\{ -\frac{M}{|z|^{m}} \right\}, \quad z \in V'.
\]

Also, the classical Watson’s lemma guarantees that if \( F \) is an analytic function with a zero Gevrey asymptotic expansion of order \( k > 0 \) on a sector of opening strictly greater than \( \pi/k \), then \( F \) is necessarily equal to zero on this sector.

**Proposition 4.3.** Suppose that \( F \) admits a zero log-Gevrey asymptotic expansion of order \( m > 0 \) on an \( \ell \)-cusp \( S = \ell(V) \). Then for every \( \ell \)-subcusp \( S' \subset S \) and every \( \delta > 0 \), there exist constants \( C, M > 0 \) such that

\[
|F(\ell)| \leq C \exp \left\{ -Me^{(\delta-\ell)/|\delta|} \right\}, \quad \ell \in S'.
\]

**Proof.** Let \( S' \subset S \). Putting \( |\ell| = a \) and considering a zero expansion of the form (4.1), we get that

\[
|F(\ell)| \leq Cm^{-n}(\log n)^n \exp \left\{ -\frac{n}{\log n} \right\} a^n, \quad |\ell| = a, \quad n \in \mathbb{N}, \quad n \geq 2.
\]

We maximize the right-hand side by \( n = n(a) \) (take the logarithm, differentiate with respect to \( n \) and find the critical point \( n_a \)). Then \( n_a \sim e^{m/a} \) and \( \log n_a \sim m/a \) as \( a \to 0 \). From (4.3) we get that there exists a \( C > 0 \) such that

\[
|F(\ell)| \leq C \exp \left\{ -\frac{1}{m} \frac{ae^{m/a}}{} \right\}, \quad \ell \in S'.
\]

In other words, returning to \( a = |\ell| \), we get that

\[
|F(\ell)| \leq C \exp \left\{ -\frac{1}{m} |\ell| \frac{e^{m/|\ell|}}{} \right\}, \quad \ell \in S'. \quad \square
\]

**Corollary 4.4** (variant of Watson’s lemma for log-Gevrey expansions). Let \( m > 0 \), and let \( F(\ell) \) be analytic on the \( \ell \)-cusp \( S_m = \ell(V_m) \), where \( V_m \) is a sector of opening strictly greater than \( \pi/m \). Let \( F \) have a zero log-Gevrey asymptotic expansion of order \( m \) on \( S_m \). Then \( F \) is equal to zero on \( S_m \).

**Proof.** Take any \( \ell \)-subcusp \( S' = \ell(V') \subset S_m \), where \( V' \) is any subsector of \( V_m \) of opening strictly less than the opening of \( V_m \). By (4.2), since \( \tilde{F}(\ell) = 0 \), we get that for any \( \delta > 0 \) there exist constants \( C, M > 0 \) such that

\[
|F(\ell)| \leq C \exp \left\{ -Me^{(\delta-\ell)/|\delta|} \right\}, \quad \ell \in S'.
\]

Passing to the variable \( z = e^{-1/\ell} \) and letting \( \tilde{F}(z) := F(\ell) \), we get that (since \( |\log z| \geq -\log |z|, z \in \mathbb{C} \))

\[
|\tilde{F}(z)| \leq C \exp \left\{ -\frac{M}{|z|^{m-\delta}} \right\}, \quad z \in V' \subset V_m.
\]
Now take $V'$ of opening strictly greater than $\pi/m$, and take $\delta > 0$ small enough that $V'$ has opening strictly greater than $\pi/(m - \delta)$. By the classical Watson’s lemma (see [1]), $\tilde{F} \equiv 0$ on $V'$. Therefore, $\tilde{F} \equiv 0$ on the whole of $V_m$ by the uniqueness of the analytic extension. □

Unlike $m$-Gevrey classes (see, for example, [11] or [18] for the 1-Gevrey class), the log-Gevrey classes $\LG_m(S)$ with $m > 0$ and $S$ an $\ell$-cusp in Definition 4.1 are not differential algebras. However, we prove a weaker statement about closedness with respect to the operations $+$, $\cdot$, and $d/d\ell$ in Propositions 4.5–4.7.

**Proposition 4.5** (addition in log-Gevrey classes). Let $m > 0$ and let $S = \ell(V)$ be an $\ell$-cusp such that the opening of the sector $V$ is strictly greater than $\pi/m$. Then the sets $\LG_m(S)$ and $\LG_m(S)$, are groups under the operation $+$. The map $f \mapsto \hat{f}$ associating the series $\hat{f} \in \LG_m(S)$ with a function $f \in \LG_m(S)$ is an isomorphism of these groups with respect to the operation $+$.

**Proof.** The uniqueness of the log-Gevrey asymptotic expansion is obvious. The injectivity follows by Corollary 4.4. The homomorphism property is obvious by the definition of log-Gevrey asymptotic expansions and the formula (4.1). □

**Proposition 4.6** (multiplication in log-Gevrey classes). Let $m, n > 0$, let $S = \ell(V)$ be an $\ell$-cusp, and let $f \in \LG_m(S)$ and $g \in \LG_n(S)$, with asymptotic expansions $\hat{f} \in \LG_m(S)$ and $\hat{g} \in \LG_n(S)$. Let $s := \min\{m, n\}$. Then $fg \in \LG_s(S)$ for every $0 < r < s$. Moreover, $\hat{fg}$ has the formal product $\hat{f} \cdot \hat{g}$ as its log-Gevrey asymptotic expansion of order $r$ on $S$.

The proof is in the Appendix (see §10).

**Proposition 4.7** (differentiation in log-Gevrey classes). Let $f \in \LG_m(S)$ have the series $\hat{f}(\ell)$ as a log-Gevrey asymptotic expansion of order $m$ on an $\ell$-cusp $S = \ell(V)$. Then for every $0 < r < m$ all the derivatives $f^{(k)}$, $k \in \mathbb{N}$, belong to $\LG_r(S)$ and have formal derivatives $\hat{f}^{(k)}(\ell)$ as their log-Gevrey asymptotic expansions of order $r$ on $S$.

The proof is in the Appendix (§10).

In the course of proving Proposition 4.7 we also prove the following corollary. The Cauchy formula in the variable $z$ is used here as in the proof of Proposition 4.7.

**Corollary 4.8** (termwise differentiation of power asymptotic expansions on $\ell$-cusps). The power asymptotic expansions of the form $f(\ell) \sim \sum_{n=0}^{\infty} a_n \ell^n$, $\ell \in S$, $\ell \to 0$, on $\ell$-cusps $S := \ell(V)$, where $V$ is a sector of positive opening at 0, can be differentiated term-by-term. That is,

$$
\frac{d}{d\ell} f(\ell) \sim \sum_{n=1}^{\infty} na_n \ell^{n-1}, \quad \ell \in S, \quad \ell \to 0.
$$

More precisely, we say here that $f$ has an asymptotic expansion $\hat{f}(\ell) = \sum_{n=0}^{\infty} a_n \ell^n$ on an $\ell$-cusp $S = \ell(V)$ if, for every proper $\ell$-subcusp $\ell(V') \subset S$, where $V' \subset V$ is a proper subsector, there exists a constant $C_n > 0$ such that

$$
|f(\ell) - \sum_{k=0}^{n-1} a_k \ell^k| \leq C_n |\ell|^n, \quad \ell \in S', \quad n \in \mathbb{N}.
$$
We remark that in the classical situation, by the Cauchy formula we need a sector of positive opening to be able to differentiate asymptotic expansions term-by-term (see, for example, [23]). Here this property is inherited due to the special form of ℓ-cusps, which are ℓ-images of sectors.

Proposition 4.9 (compositions with analytic germs for log-Gevrey classes). Let \( S = \ell(V) \) be an \( \ell \)-cusp.\(^{11}\) Let \( f, h \in \text{LG}_m(S) \) and \( g \in \text{LG}_p(S) \), \( m, p > 0 \), have log-Gevrey expansions \( \hat{f}, \hat{h} \in \hat{\text{LG}}_m(S) \) and \( \hat{g} \in \hat{\text{LG}}_p(S) \). Assume that \( f(\ell) = o(1) \) as \( \ell \to 0 \) on \( V \). Let \( F \) be an analytic or meromorphic germ at 0, with Taylor expansion or Laurent expansion \( \hat{F} \) at 0. Then:

(i) \( F \circ f \in \text{LG}_r(S) \) for every \( 0 < r < m \), with log-Gevrey expansion \( \hat{F} \circ \hat{f} \in \hat{\text{LG}}_r(S) \) of order \( r \);

(ii) if \( \hat{g} \neq 0 \), then \(^{12}\) \( h/g \in \text{LG}_q(S) \) for every \( 0 < q < \min\{m, p\} \), and \( \hat{h}/\hat{g} \in \hat{\text{LG}}_q(S) \) is its log-Gevrey asymptotic expansion of order \( q \).

The proof is in the Appendix (§10).

5. Analytic equivalence of parabolic generalized Dulac germs

In the case of regular parabolic diffeomorphisms, analytic equivalence implies formal equivalence. For parabolic generalized Dulac germs \( f \) defined on a standard quadratic domain \( \mathcal{R}_C \) the situation is more complicated. We will see that for two parabolic (generalized) Dulac germs on \( \mathcal{R}_C \) an analytic conjugacy tangent to the identity map does not necessarily imply that the asymptotic expansion of the conjugacy germ is in \( \text{LG}(\mathcal{R}) \). However, this will be true in \( \hat{\mathcal{L}}_2(\mathbb{R}) \) (see Lemma 7.5). The reason is the possibility of formal reduction of \( \hat{f}(z) \) to a simpler normal form by a formal conjugacy belonging to a broader class \( \hat{\mathcal{L}}_2(\mathbb{R}) \supset \hat{\mathcal{L}}(\mathbb{R}) \), which allows us also to eliminate the residual term, as in Remark 5.5 below.

However, in this paper when we define the analytic equivalence of two parabolic generalized Dulac germs, we stay within the limits of the \( \mathcal{L}(\mathbb{R}) \)-formal classes. The analytic equivalence of two parabolic generalized Dulac germs is defined in Definition 2.4 of §2.

In the following example we show that in Definition 2.4 the condition 1) does not follow from 2). Indeed, if only 2) is satisfied, then \( h \) has a generalized block iterated integral expansion in the larger class \( \hat{\mathcal{L}}_2^{\text{id}}(\mathbb{R}) \), a subgroup of the class \( \hat{\mathcal{L}}_2(\mathbb{R}) \) of transseries tangent to the identity map.

Example 5.1. There exist parabolic generalized Dulac germs that satisfy 2) in Definition 2.4 but are not formally equivalent in the class \( \mathcal{L}(\mathbb{R}) \). They are in fact formally equivalent in \( \hat{\mathcal{L}}_2(\mathbb{R}) \). This broader class can be used to eliminate the residual invariant term (see Proposition 5.2 below). For example, take two parabolic generalized Dulac germs that are models for two different formal classes with respect

\(^{11}\)Of arbitrarily small radius.

\(^{12}\)Note that \( h/g \) is well defined and analytic on an \( \ell \)-cusp \( S \) (of sufficiently small radius), since \( g \) cannot have an accumulation of zero points in \( S \) at 0. (Otherwise, it is easy to see that its asymptotic expansion \( \hat{g} \) in the power-logarithmic scale on \( S \) would be 0.)
to $\hat{\mathcal{L}}(\mathbb{R})$ and give the time-one maps along trajectories of the two vector fields:

$$f(z) = \exp\left(\frac{z^2}{1 + z - z\ell} \frac{d}{dz}\right) \cdot \text{id} \quad \text{and} \quad g(z) = \exp\left(\frac{z^2}{1 + z} \frac{d}{dz}\right) \cdot \text{id}.$$ 

Both are obviously parabolic Dulac germs, defined on a whole neighbourhood of zero of the Riemann surface $\mathcal{R}$ of the logarithm, and polynomial in $\ell$. They both admit global Fatou coordinates $\Psi_f, \Psi_g : \mathcal{R} \to \hat{\mathcal{R}}$, where $\hat{\mathcal{R}}$ denotes a neighbourhood of the infinity on the Riemann surface of the logarithm. The Fatou coordinates are, up to a complex additive constant, given by

$$\Psi_f(z) = \int \frac{dz}{z^2} + \int \frac{dz}{z} - \int \frac{\ell}{z} dz = -\frac{1}{z} + \log z + \log(-\log z)$$

and

$$\Psi_g(z) = -\frac{1}{z} + \log z, \quad z \in \mathcal{R}.$$ 

The function $\Psi_g$ is injective on $\mathcal{R}$. By the Abel equation for the Fatou coordinate, one analytic conjugacy conjugating $g$ to $f$, defined and analytic on $\mathcal{R}$ and tangent to the identity, is

$$\varphi := \Psi_g^{-1} \circ \Psi_f.$$ 

On the other hand, $f$ and $g$ have different formal invariants $\rho$ in the class $\hat{\mathcal{L}}(\mathbb{R})$. Indeed, the asymptotic expansion $\hat{\varphi}$ of $\varphi$ belongs to the class $\hat{\mathcal{L}}_2(\mathbb{R})$, as is easily seen, for example, from the Taylor expansion of the left-hand side of the equality $\Psi_g(x + h(x)) = \Psi_f$, $\varphi = \text{id} + h$, or by writing $\tilde{\Psi}_f = -e^{1/\ell} - 1/\ell - \log \ell$ and $\tilde{\Psi}_g = e^{1/\ell} - 1/\ell$ in terms of the variable $\ell$. Here it is obvious that we cannot generate the logarithmic term in $\tilde{\Psi}_f$ by precomposing $\tilde{\Psi}_g$ with a power-exponential transseries.

Let $\hat{h} \in \hat{\mathcal{L}}(\mathbb{R})$ denote the formal conjugacy between the generalized Dulac expansions $\hat{f}$ and $\hat{g}$ of two analytically conjugate (in the sense of Definition 2.4) parabolic generalized Dulac germs $f$ and $g$:

$$\hat{g} = \hat{h}^{-1} \circ \hat{f} \circ \hat{h}, \quad \hat{h} \in \hat{\mathcal{L}}(\mathbb{R}).$$

The formal conjugacy $\hat{h} \in \hat{\mathcal{L}}(\mathbb{R})$ is derived in [16]. We will show in §7.4 that, up to some completely determined changes, the $h$ in Definition 2.4 then admits the formal conjugacy $\hat{h} \in \hat{\mathcal{L}}(\mathbb{R})$ as its generalized block iterated integral sectional asymptotic expansion (see Definition 7.8 and Proposition 7.9). In fact, the transseries asymptotic expansions of germs in $\hat{\mathcal{L}}(\mathbb{R})$ are not unique (see [17]). To make them unique we must choose a canonical summation rule at limit ordinal steps. That is, we must fix an appropriate section function (see [17], and Definition 10.1 for a complex version).

We repeat here the $\hat{\mathcal{L}}(\mathbb{R})$-normal form result derived in [16] for parabolic generalized Dulac germs. We give here an alternative proof working block-by-block, not term-by-term as in [16]. As opposed to termwise eliminations done previously in [16], the value of blockwise eliminations is that they determine the form of integrals in each block. This will be important in finding the appropriate integral
asymptotic expansions for conjugacies in §7.4. The proof here is inductive, by block-by-block eliminations. By a block, we mean all monomials in a transseries that have the same powers of $z$. In the proof we need Lemma 5.3 below.

**Proposition 5.2** (formal normal forms of parabolic generalized Dulac transseries; see [16]). Let $f(z) = z - z^\alpha \ell^m + \text{h.o.t.}$, $\alpha > 1$, $m \in \mathbb{Z}$, be a normalized parabolic generalized Dulac transseries. By a formal series $\hat{\varphi} \in \hat{L}^\text{id}(\mathbb{R})$ tangent to the identity map it can be reduced to either of the normal forms

$$
\hat{f}_1(z) = z - z^\alpha \ell^m + \rho z^{2\alpha-1} \ell^{2m+1},
$$

$$
\hat{f}_1(z) = \text{Exp}(X_1).\text{id}, \quad X_1(z) = \frac{-z^\alpha \ell^m}{1 + (\alpha/2)z^{\alpha-1}\ell^m + (m/2 + \rho)z^{\alpha-1}\ell^{m+1}} \frac{d}{dz}.
$$

(5.1)

The $\hat{L}(\mathbb{R})$-formal invariants have the form $(\alpha, m, \rho)$, $\alpha > 1$, $m \in \mathbb{Z}$, $\rho \in \mathbb{R}$.

Clearly, since $\hat{L}^\text{inv}(\mathbb{R})$ is a group under composition, any two parabolic generalized Dulac series $\hat{f}$ and $\hat{g}$ with the same formal invariants $(\alpha, m, \rho)$ are formally conjugate in $\hat{L}^\text{inv}(\mathbb{R})$.

**Lemma 5.3** (blockwise eliminations). Let $\hat{f} \in \hat{L}(\mathbb{R})$, $\hat{f}(z) = z - z^\alpha \ell^m + z^{\beta_i} T_i(\ell) + \text{h.o.t.}$, where $\alpha > 1$, $m \in \mathbb{Z}$, $T_i(\ell) \in \mathbb{R}((\ell))$. The block $z^{\beta_i} T_i(\ell)$, with the possible exception of the monomial $z^{2\alpha-1} \ell^{2m+1}$ in the residual block $\beta_i = 2\alpha + 1$, can be eliminated by the elementary change of variables $\hat{\varphi}_i(z) = z + z^{\gamma_i} \hat{R}_i(\ell)$, where $\gamma_i = \beta_i - \alpha + 1$, and the series $\hat{R}_i(\ell) \in \mathbb{R}((\ell))$ are given by

$$(5.2)\begin{cases}
\hat{R}_i(\ell) = -\frac{1}{z^{\gamma_i-\alpha} \ell^{-m}} \int z^{\gamma_i-\alpha} \ell^{-2m-2} \hat{T}_i(\ell) \, d\ell, & \beta_i > \alpha, \\
\hat{R}_i(\ell) = \frac{1}{\gamma_i} \left( \int \frac{z^{1-\alpha}}{\ell^{m+2}} \, d\ell \right)^{-1} \circ \left( \int \frac{z^{1-\alpha}}{\hat{T}_0(\ell) \ell^2} \, d\ell \right) - 1, & \beta_i = \alpha,
\end{cases}$$

with 0 as the constant of formal integration. Here $\hat{T}_0(\ell) \in \mathbb{R}((\ell))$, $T_0(\ell) = -\ell^m + \text{h.o.t.}$, is the whole first block of $\hat{f}(z)$, the coefficient of $z^\alpha$. In the case $\beta_i = 2\alpha + 1$, $T_i(\ell)$ is the residual block without the term $\rho z^{2\alpha-1} \ell^{2m+1}$, $\rho \in \mathbb{Z}$.

The above integrals are formally integrated by parts, with $dv = e^{(\alpha-\gamma_i)/\ell} \ell^{-2}$ and $u$ equal to the rest of the integrand. We always choose 0 as the constant of formal integration in (5.2), in order that $\hat{\varphi}_i(z)$ be an elementary change of variables with only one block. Note that by taking $\beta_1 = \alpha$ we eliminate the first block, except for the first term in it. The proof of Lemma 5.3 is in the Appendix (§10).

We also state the following generalization of Lemma 5.3, which is proved in the same way as Lemma 5.3, so we omit the proof. We will need Lemma 5.4 in §6.2 to prove the block iterated integral summability of formal conjugacies of two parabolic generalized Dulac germs.

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13Higher-order terms.
14Higher-order blocks.
Lemma 5.4 (formal conjugation of two transseries). Assume that \( \hat{f}, \hat{g} \in \hat{\mathcal{L}}(\mathbb{R}) \) have the same formal invariants \((\alpha, m, \rho)\). For \( n \in \mathbb{N}_0 \) let

\[
\hat{f}(z) = z - z^\alpha \mathcal{L}^m + \sum_{i=1}^{n} z^{\alpha_i} \hat{T}_i(\mathcal{L}) + z^{\alpha_n+1} \hat{F}_{n+1}(\mathcal{L}) + \text{h.o.b.},
\]

\[
\hat{g}(z) = z - z^\alpha \mathcal{L}^m + \sum_{i=1}^{n} z^{\alpha_i} \hat{T}_i(\mathcal{L}) + z^{\alpha_n+1} \hat{G}_{n+1}(\mathcal{L}) + \text{h.o.b.},
\]

where \( \alpha > 1, m \in \mathbb{Z} \), the numbers \( \alpha_i \geq \alpha \) are strictly increasing as \( i \to \infty \), \( \hat{T}_i \in \mathbb{R}(\mathbb{L}), i = 1, \ldots, n, \hat{F}_{n+1}, \hat{G}_{n+1} \in \mathbb{R}(\mathbb{L}) \), and \( \hat{F}_{n+1} \neq 0 \) (\( \hat{G}_{n+1} \) can be 0).

Then

\[
\hat{\varphi}_{n+1} \circ \hat{g} \circ \hat{\varphi}^{-1}_{n+1} = z - z^\alpha \mathcal{L}^m + \sum_{i=1}^{n} z^{\alpha_i} \hat{T}_i(\mathcal{L}) + z^{\alpha_n+1} \hat{F}_{n+1}(\mathcal{L}) + \text{h.o.b.},
\]

where \( \hat{\varphi}_{n+1}(z) = z + z^{\gamma_{n+1}} \hat{R}_{n+1}(\mathcal{L}) \), with \( \gamma_{n+1} = \alpha_{n+1} - \alpha + 1 \) and the series \( \hat{R}_{n+1}(\mathcal{L}) \in \mathbb{R}(\mathbb{L}) \) given by

\[
\begin{align*}
\hat{R}_{n+1}(\mathcal{L}) &= -\frac{1}{z^{\gamma_{n+1}-\alpha} \mathcal{L}^m} \int z^{\gamma_{n+1}-\alpha} \mathcal{L}^{-2m-2}(\hat{G}_{n+1}(\mathcal{L}) - \hat{F}_{n+1}(\mathcal{L})) \, d\mathcal{L} \quad \text{if} \quad \alpha_{n+1} > \alpha, \\
\hat{R}_{n+1}(\mathcal{L}) &= \frac{1}{z} \left[ \left( \int \frac{z^{1-\alpha}}{\hat{F}_1(\mathcal{L}) \mathcal{L}^2} \, d\mathcal{L} \right)^{-1} \circ \left( \int \frac{z^{1-\alpha}}{\hat{G}_1(\mathcal{L}) \mathcal{L}^2} \, d\mathcal{L} \right) \right]^{-1} \quad \text{if} \quad \alpha_{n+1} = \alpha,
\end{align*}
\]

and with 0 as the constant of formal integration. Here \( \hat{F}_1(\mathcal{L}) = -\mathcal{L}^m + \text{h.o.t.} \) and \( \hat{G}_1(\mathcal{L}) = -\mathcal{L}^m + \text{h.o.t.} \), \( \hat{F}_1, \hat{G}_1 \in \mathbb{R}(\mathbb{L}) \), are the whole first blocks of \( \hat{f} \) and \( \hat{g} \), respectively (that is, blocks that are the coefficients of \( z^\alpha \)).

Proof of Proposition 5.2. Let

\[
\hat{f}(z) = z + z^\alpha \hat{T}_0(\mathcal{L}) + z^{\alpha_1} \hat{T}_1(\mathcal{L}) + \cdots,
\]

where \( \hat{T}_i(\mathcal{L}) \in \mathbb{R}(\mathbb{L}) \) for all \( i \in \mathbb{N}_0 \), with the leading term of \( \hat{T}_0 \) given by \( \text{Lt}(\hat{T}_0(\mathcal{L})) = -\mathcal{L}^m, m \in \mathbb{Z} \).

We proceed as in [16], Theorem A, but eliminate block-by-block instead of term-by-term. We first eliminate the first block \( z^\alpha \hat{T}_0(\mathcal{L}) \), which can be eliminated except for the first term \( -z^\alpha \mathcal{L}^m \). Indeed, in each step of the elimination procedure we look for a formal change of variables of the form

\[
\hat{\varphi}_i(z) = z + z^{\gamma_i} \hat{R}_i(\mathcal{L}), \quad \text{ord}(z^{\gamma_i} \hat{R}_i) \succ (1, 0), \quad i \in \mathbb{N}_0,
\]

where \( \hat{R}_i \in \mathbb{R}(\mathbb{L}) \). Here \( \text{ord}(\cdot) \) denotes the lexicographic ordinal number of the first term of a transseries.

By Lemma 5.3, in order to remove the first block \( z^\alpha \hat{T}_0(\mathcal{L}) \) (that is, the part of it that can be eliminated) we need to take \( \gamma_0 = 1 \). But then \( \text{ord}_\mathcal{L}(\hat{R}_0) \geq 1 \). Therefore, the first monomial \( -z^\alpha \mathcal{L}^m \) cannot be eliminated. The remaining terms of the first
block are eliminated by the change \( \hat{\varphi}_0(z) = z + z\hat{R}_0(\ell) \), where \( \hat{R}_0(\ell) \in \ell \mathbb{R}[[\ell]] \) is given by (5.2) (it is deduced by integration by parts and formal composition).

Furthermore, it is obvious from integration by parts in the explicit formula (5.2) in Lemma 5.3 that if \( \gamma_i \neq 1 \) and \( \gamma_i \neq \alpha \), then \( \hat{T}_i(\ell) \in \mathbb{R}(\ell) \) implies that \( \hat{R}_i(\ell) \in \mathbb{R}(\ell) \) and no iterated logarithms are generated. Therefore, we do not need iterated logarithms to remove blocks before the residual block, and consequently all the blocks \( \hat{T}_i(\ell) \) up to the residual block (inclusive) belong to \( \mathbb{R}(\ell) \). Thus, \( \hat{f}_i, \hat{\varphi}_i \in \hat{L}(\mathbb{R}) \) for all steps of the elimination procedure before the residual block, where \( \hat{f}_i \) denotes the initial transseries after the first \( i \) changes of variables, that is,

\[
\hat{f}_i = \left( j_0, \ldots, j_i \right)^{-1} \circ \hat{f} \circ \left( j_0, \ldots, j_i \right), \quad i \in \mathbb{N}_0.
\]

If \( \beta_i = 2\alpha - 1 \) (elimination of the residual block \( z^{2\alpha - 1}\hat{T}_i(\ell), \hat{T}_i(\ell) \in \mathbb{R}(\ell) \)), then by Lemma 5.3 we choose \( \gamma_i = \alpha \) and

\[
\hat{R}_i(\ell) = -\ell^m \int \ell^{-2m-2}\hat{T}_i(\ell) \, d\ell.
\]

It is easy to see that, by an appropriate series \( \hat{R}_i(\ell) \in \mathbb{R}(\ell) \), we can eliminate all terms in \( \hat{T}_i(\ell) \) except for the residual term \( z^{2\alpha - 1}\ell^{2m+1} \). The set of parabolic transseries in \( \hat{L}(\mathbb{R}) \) is a group under composition, so that after eliminating the residual block we have \( \hat{f}_i := \hat{\varphi}_i^{-1} \circ \hat{f}_{i-1} \circ \hat{\varphi}_i \in \hat{L}(\mathbb{R}) \). After this step the other blocks in \( \hat{f}_i \) do not contain iterated logarithms. By Lemma 5.3, for \( \beta_i > 2\alpha - 1 \) (\( \gamma_i > \alpha \)) we can remove all the further blocks \( z^{\beta_i}\hat{T}_i(\ell), \hat{T}_i(\ell) \in \mathbb{R}(\ell) \), using changes of variables with \( \hat{R}_i(\ell) \in \mathbb{R}(\ell) \), as we explained before.

In the inductive procedure, to remove all possible terms we solve a sequence of Lie bracket equations (see [16], Theorem A, for details). But in contrast to [16], where we eliminate term-by-term, here we must solve only countably many equations and not a transfinite number of equations. Indeed, since the powers \( \alpha_i \) in \( \hat{f} \) are finitely generated, the set of all powers \( \gamma_i \) needed in blockwise elimination is also finitely generated (see [16] for details). The proposition is proved.

**Remark 5.5.** If we allow formal conjugation in Proposition 5.2 in the larger class \( \hat{L}_2(\mathbb{R}) \), then we can eliminate also the residual term \( \rho z^{2\alpha - 1}\ell^{2m+1} \). By a formal diffeomorphism \( \hat{\varphi} \in \hat{L}_2(\mathbb{R}) \), a parabolic generalized Dulac series \( \hat{f} \) in Proposition 5.2 can be reduced to either of the two normal forms

\[
f_F(z) = z - z^\alpha \ell^m,
\]

\[
\hat{f}_2(z) = \Exp\left( -z^\alpha \ell^m \frac{d}{dz} \right) \text{id}.
\]

The \( \hat{L}_2(\mathbb{R}) \)-formal invariants are \((\alpha, m)\). The proof is similar to the proof of Proposition 5.2. Now Lemma 5.3 can simply be rewritten with \( \hat{R}_i(\ell), \hat{T}_i(\ell) \in \hat{L}_2^\infty(\mathbb{R}) \) instead of \( \hat{R}_i(\ell), \hat{T}_i(\ell) \in \mathbb{R}(\ell) \). In order to remove the residual term \( z^{2\alpha - 1}\ell^{2m+1} \)

\(^{15}\hat{R}_i(\ell) \) and \( \hat{T}_i(\ell) \) are transseries in integer powers of \( \ell \) and \( \ell_2 \) (see the notation (6.1)).
in some block $\hat{T}_1(\ell)$, we need a double logarithm monomial $\ell^m \ell_2^{-1}$ in the corresponding series $\hat{R}_i(\ell)$ in the formal change of variables $\hat{\varphi}_i$. Therefore, the residual elementary change $\hat{\varphi}_i$ belongs to $\hat{\mathcal{L}}_2(\mathbb{R})$. In general, the blocks of $\hat{f}_i$ after removal of the residual block also contain double logarithms, that is, $\hat{T}_i(\ell) \in \hat{\mathcal{L}}_2(\mathbb{R})$. By a generalization of Lemma 5.3 to $\hat{\mathcal{L}}_2(\mathbb{R})$, we can remove them with the help of transseries $\hat{R}_i(\ell) \in \hat{\mathcal{L}}_2(\mathbb{R})$ in subsequent changes of variables. The parabolic transseries in $\hat{\mathcal{L}}_2(\mathbb{R})$ also form a group under composition.

6. Integral summability

In this section we describe the property of integral summability of blocks in formal conjugacies $\hat{\varphi} \in \hat{\mathcal{L}}(\mathbb{R})$ that reduce a parabolic generalized Dulac germ to its formal normal form, and in its formal Fatou coordinate $\hat{\Psi} \in \hat{\mathcal{L}}_2(\mathbb{R})$. After introducing some necessary definitions, the statement is given in Propositions 6.7 and 6.11 below.

6.1. The notion of integral summability. In this subsection we apply Definition 6.1 of integral summability of length 1 to blocks in formal Fatou coordinates of a generalized Dulac germ (see Proposition 6.7). Definition 6.1 is a generalization of the notion of integral summability introduced earlier in Definition 3.9 of [17] for the formal Fatou coordinate of a parabolic Dulac germ. For simplicity we will again use the same name. However, in the case of formal conjugacies we will need the integral summability of greater lengths than their immediate predecessors $\hat{R}_i(\ell)$, as in the following inductive definition.

The notion of integral summability recalls the notion of block iterated integrals introduced by Chen in [3] and studied extensively in the context of first return maps (see [6] and [14], among others).

Below we will use the notation $\hat{\mathcal{L}}_2(\mathbb{R})$ for the set of transseries in $\ell$ and $\ell_2$ with integer exponents and real coefficients:

$$\hat{F}(\ell) = \sum_{(m,n) \in A \subseteq \mathbb{Z} \times \mathbb{Z}} a_{m,n} \ell^m \ell_2^n, \quad a_{m,n} \in \mathbb{R},$$

where $A \subseteq \mathbb{Z} \times \mathbb{Z}$ is well ordered.

**Definition 6.1** (integral summability of series of length $k \in \mathbb{N}_0$). Let $0 < \theta \leq 2\pi$ and let $S_0 = \ell(V_0)$ be an $\ell$-cusp, where $V_0$ is a sector or a petal of opening $\theta$. Assume that the series $\hat{F}(\ell) \in \hat{\mathcal{L}}_2(\mathbb{R})$ contains at most one term with double logarithm.
1) The series $\hat{F}(\ell) \in \hat{L}_k^\infty(\mathbb{R})$ is said to be \textit{integrally summable of length 0} on $S_\theta$ if there exists an $m > \pi/\theta$ such that $\hat{F}(\ell) \in LG_m(S_\theta)$, and 

$$\ell \mapsto F(\ell) \in LG_m(S_\theta)$$

is called its $0$-integral sum.

2) The series $\hat{F}(\ell) \in \hat{L}_k^\infty(\mathbb{R})$ is said to be \textit{integrally summable of length 1} on $S_\theta$ if $\hat{F}(\ell)$ is not integrally summable of length 0 on $S_\theta$, and there exist exponents $\alpha_1 \in \mathbb{R}$ and $p_1 \in \mathbb{Z}$ and a series $\hat{R}(\ell) \in \hat{L}_k^\infty(\mathbb{R})$ integrally summable of length 0 on $S_\theta$ (with 0-integral sum $R$) such that

$$\frac{d}{d\ell}(e^{-\alpha_1/\ell}p_1\hat{F}(\ell)) = e^{-\alpha_1/\ell}p_1^{2p_1-2}\hat{R}(\ell).$$

The analytic germ

$$F(\ell) := \frac{1}{e^{-\alpha_1/\ell}p_1} \int_{*}^{\ell} e^{-\alpha_1/\eta p_1} e^{-2p_1-2} R(\eta) \, d\eta$$

on $S_\theta$ is called the $1$-integral sum of $\hat{F}$ on $S_\theta$. Here $*$ is equal to 0 if $(\alpha_1, \text{ord}(\hat{R}) + 2p_1 - 1) \succ (0, 0)$, that is, if the integrand is bounded at 0, and is equal to some $\ell_0 \in S_\theta$ otherwise. This germ is unique up to an additive term $Ce^{\alpha_1/\ell}p_1 = C \in \mathbb{R}$. The pair of exponents $(\alpha_1, p_1) \in (\mathbb{R}, \mathbb{Z})$ is called the exponent of integration of $\hat{F}(\ell)$.

3) In general, a series $\hat{F}(\ell) \in \hat{L}_k^\infty(\mathbb{R})$ is said to be \textit{integrally summable of length $k \geq 2$ on $S_\theta$} with respect to a set of series

$$\hat{U} := \{ \hat{R}_{i_1}^{j_1}(\ell) : i = 1, \ldots, k - 1, \; j_i = 1, \ldots, r_i \} \subseteq \hat{L}_k^\infty(\mathbb{R}),$$

where the series $\hat{R}_{i_1}^{j_1}(\ell) \in \hat{U}$, $j_i = 1, \ldots, r_i$, are integrally summable of length $i$ on $S_\theta$, $i = 1, \ldots, k - 1$, with respect to the previous set of series $\{ \hat{R}_{n}^{j_n}(\ell) : n = 1, \ldots, i - 1, \; j_n = 1, \ldots, r_n \} \subseteq \hat{U}$ and where all the elements of $\hat{U}$ have different exponents of integration, if the following two conditions hold.

- The series $\hat{F}(\ell)$ is not of the form

$$\hat{F}(\ell) = \hat{S}_{\leq l}^{R_1^{1}, \ldots, R_1^{1}, \ldots, R_i^{j_i}, \ldots, R_i^{j_i}}(\ell) \in \hat{L}_k^\infty(\mathbb{R}), \quad l \leq k - 1,$$

where $\hat{S}_{\leq l}^{R_1^{1}, \ldots, R_1^{1}, \ldots, R_i^{j_i}, \ldots, R_i^{j_i}}(\ell) \in \hat{L}_k^\infty(\mathbb{R})$ is as follows:

(i) for $l = 1$, $\hat{S}_{\leq l}^{R_1^{1}, \ldots, R_1^{1}}(\ell)$ is a finite algebraic combination (with operations $\cdot, /, d/d\ell$) of series that are integrally summable of length 0 and series $\hat{R}_{j_1}^{1}(\ell)$, $j_1 = 1, \ldots, r_1$, that are not integrally summable of length 0;

(ii) for $l \geq 2$, $\hat{S}_{\leq l}^{R_1^{1}, \ldots, R_1^{1}, \ldots, R_i^{j_i}, \ldots, R_i^{j_i}}(\ell)$ is a finite algebraic combination (with operations $\cdot, /, d/d\ell$) of series that are integrally summable of length 0, series $\hat{R}_{j_1}^{1}(\ell) \in \hat{U}$, $j_1 = 1, \ldots, r_1$, and series $\hat{R}_{i_1}^{j_1}(\ell) \in \hat{U}$, $j_i = 1, \ldots, r_i$, that are integrally summable of length $i$ with respect to the previous series $\{ \hat{R}_{n}^{j_n}(\ell) : n = 1, \ldots, i - 1, \; j_n = 1, \ldots, r_n \} \subseteq \hat{U}$, $1 \leq i \leq l$, and does not have the form $\hat{S}_{\leq l-1}^{R_1^{1}, \ldots, R_1^{1}, \ldots, R_{i-1}^{j_{i-1}}, \ldots, R_{i-1}^{j_{i-1}}}(\ell)$. 


There exists a pair of exponents \((\alpha_k, p_k) \in (\mathbb{R}, \mathbb{Z})\) such that
\[
\frac{d}{d\ell} \left( e^{-\alpha_k/\ell} \ell^p_k \hat{F}(\ell) \right) = e^{-\alpha_k/\ell} \ell^{2p_k-2} \hat{S}_{\leq k-1}^{R_1^1 \ldots \hat{R}_1^1 \ldots \hat{R}_k^{r_k-1}}(\ell),
\]
where \(\hat{S}_{\leq k-1}^{R_1^1 \ldots \hat{R}_1^1 \ldots \hat{R}_k^{r_k-1}}(\ell) \in \mathcal{L}_\ell^\infty(\mathbb{R})\) are of the above described form. Denote by
\[
\hat{S}_{\leq k-1}^{R_1^1 \ldots \hat{R}_1^1 \ldots \hat{R}_k^{r_k-1}}(\ell)
\]
the sum of the series \(\hat{S}_{\leq k-1}^{R_1^1 \ldots \hat{R}_1^1 \ldots \hat{R}_k^{r_k-1}}(\ell)\) (computed in the natural way with the corresponding algebraic operations and with some fixed choice of the constants of integration in the integrals for the germs \(R_i^j(\ell), i = 1, \ldots, l, j_i = 1, \ldots, r_i,\) computed from the previous steps). Then the analytic germ
\[
F(\ell) := \frac{1}{e^{-\alpha_k/\ell} \ell^{p_k}} \int_\ast^\ell e^{-\alpha_k/\eta} \eta^{2p_k-2} \hat{S}_{\leq k-1}^{R_1^1 \ldots \hat{R}_1^1 \ldots \hat{R}_k^{r_k-1}}(\eta) \, d\eta
\]
on \(S_\theta\) is called the \(k\)-integral sum of the series. The choice of a base point \(\ast\) is as in 2). This \(k\)-sum is unique (after a particular sum \(\hat{S}_{\leq k-1}^{R_1^1 \ldots \hat{R}_1^1 \ldots \hat{R}_k^{r_k-1}}(\ell)\) of the series \(\hat{S}_{\leq k-1}^{R_1^1 \ldots \hat{R}_1^1 \ldots \hat{R}_k^{r_k-1}}(\ell)\) is fixed) up to an additive term \(Ce^{\alpha_k/\ell} \ell^{-p_k} = C\ell^{-\alpha_k} \ell^{-p_k}, C \in \mathbb{R}\) (due to the possibility of a different choice of the point \(\ell_0 \in \ell(V_j^\pm)\) if the integrand is not bounded at 0).

Note that solving equation (6.2) in Definition 6.1 is equivalent to solving the non-homogenous linear ordinary differential equation
\[
\ell^2 \frac{d}{d\ell} \hat{F} + (\alpha_1 + p_1) \hat{F} = \ell^{p_1} \hat{R},
\]
and similarly for equation (6.3).

The integration paths in the integrals from \(\ast\) to \(\ell\) do not matter. Indeed, \(\ell(V_\theta)\) is a simply connected cusp with sufficiently small radius (a germ domain) and the integrand is analytic on \(\ell(V_\theta)\).

By the block iterated integral sum
\[
U := \{ R_n^{j_n}(\ell) : n = 1, \ldots, k - 1, j_n = 1, \ldots, r_n \}
\]
of a set
\[
\hat{U} = \{ \hat{R}_n^{j_n}(\ell) : n = 1, \ldots, k - 1, j_n = 1, \ldots, r_n \}
\]
of integrally summable series \(\hat{R}_i^j(\ell)\) with respect to previous series \(\hat{R}_j^j(\ell), 1 \leq j < i,\) of strictly smaller lengths of summation, as described in Definition 6.1 above, we mean that at each step the constant of integration \(\ell_0\) is fixed, and the integrands are determined by the constants of integration from the previous steps.

**Remark 6.2.** If the series \(\hat{F}\) in (6.2) is already integrally summable of length 0 on \(S_\theta,\) then (6.2) is satisfied for every exponent of integration \((\alpha, p) \in (\mathbb{R}, \mathbb{Z})\). Similarly, let \(k \geq 2\). If \(\hat{F}(\ell)\) is already a finite algebraic combination of integrally summable
series of length 0, integrally summable series $\hat{R}^{j_1}_1(\ell)$, $j_1 = 1, \ldots, r_1$, of length 1, and integrally summable series $\hat{R}^{j_l}_l(\ell)$, $j_l = 1, \ldots, r_l$, of arbitrary lengths $2 \leq l \leq k - 1$ with respect to the previous series $\{\hat{R}^{j_n}_n(\ell) : n = 1, \ldots, l - 1, \; j_n = 1, \ldots, r_n\}$, then (6.3) holds with any exponents of integration $(\alpha, p) \in (\mathbb{R}, \mathbb{Z})$. Then we may take their sums in the natural way, performing the corresponding operations (with ambiguity in the choice of additive terms in each step). That is why, in each step of the inductive Definition 6.1, we exclude such trivial cases, in order to have uniqueness for the exponents of integration.

**Proposition 6.3** (uniqueness of integral sums of length $k \geq 1$). Let the series $\hat{F}(\ell) \in \hat{\mathcal{L}}^\infty(\mathbb{R})$ be integrally summable of length 1 on an $\ell$-cusp $S_0$, or integrally summable of length $k \geq 2$ on $S_0$ with respect to the series $\{\hat{R}^{j_n}_n(\ell) : n = 1, \ldots, k - 1, \; j_n = 1, \ldots, r_n\} \subset \hat{\mathcal{L}}^\infty(\mathbb{R})$, which are themselves integrally summable of lengths $1, \ldots, k - 1$, respectively, on $S_0$, corresponding to Definition 6.1. Then the exponent of integration $(\alpha, p) \in (\mathbb{R}, \mathbb{Z})$ of $\hat{F}$ is uniquely determined.

The proof, similar to that in [17], is in the Appendix (§10). Proposition 6.3 states that the 1-integral sums and the $k$-integral sums with respect to given integrally summable series of smaller length are unique, up to the choice of the constants of integration in the inductive Definition 6.1.

In Definition 6.1 we also implicitly assume that if $\hat{S}^{\hat{R}^{i_1}_1, \ldots, \hat{R}^{i_k}_k}(\ell) \in \hat{\mathcal{L}}^\infty(\mathbb{R})$ can be represented as a finite algebraic combination (with the operations $+, \cdot, /, d/d\ell$) of integrally summable series $\hat{R}^{i}_i(\ell)$, $i = 1, \ldots, r_i$, of lengths $i \in \{0, 1, \ldots, k\}$, respectively, with respect to the previous series $\{\hat{R}^{j_n}_n(\ell) : n = 1, \ldots, l - 1, \; j_n = 1, \ldots, r_n\}$, then it can be given a unique sum (up to addition of the constants of integration). We prove this in Proposition 6.5 below. In the proof we use the following auxiliary lemma, Lemma 6.4, whose proof is in the Appendix (§10).

**Lemma 6.4.** Any finite algebraic combination

$$\hat{S}^{\hat{R}^{i_1}_1, \ldots, \hat{R}^{i_k}_k}(\ell) \in \hat{\mathcal{L}}^\infty(\mathbb{R}), \quad k \in \mathbb{N},$$

with respect to the operations $+, \cdot, /, d/d\ell$, where the series $\hat{R}^{i}_i(\ell)$, $i = 1, \ldots, r_i$, are integrally summable of length $i$, $i = 1, \ldots, k$, as in Definition 6.1, is equal\(^{16}\) to a rational function of the series $\hat{R}^{j_n}_n(\ell)$, $n = 1, \ldots, k$, $j_n = 1, \ldots, r_n$, and of integrally summable series of length 0.

**Proposition 6.5.** Let

$$\hat{U} = \{\hat{R}^{j_n}_n(\ell) : n = 1, \ldots, k, \; j_n = 1, \ldots, r_n\} \subseteq \hat{\mathcal{L}}^\infty(\mathbb{R})$$

be a set of consecutively integrally summable series with respect to previous series of strictly smaller lengths, according to Definition 6.1, with a corresponding sequence of strictly increasing exponents of integration $\{\beta_1, \beta_2, \ldots, \beta_{r_1 + \ldots + r_k}\}$. Here the subscripts of the elements of the set $\hat{U}$ denote the length of integral summability of the corresponding transseries.

\(^{16}\)After all differentiations.
Let \( \{R_{n}^j(\ell) : n = 1, \ldots, k, j_n = 1, \ldots, r_n\} \) be a fixed choice of their integral sums, as described after Definition 6.1. Let \( \hat{S} \in \hat{\mathcal{L}}_{\infty}^{\mathbb{R}} \) be a transseries that can be represented as a rational function of elements of the set \( \hat{U} \) and of integrally summable series of length 0. Then this representation is unique,\(^{17}\) Therefore, the sum of the series \( S \), after the corresponding algebraic operations, is unique with respect to fixed sums \( \{R_{n}^j(\ell) : n = 1, \ldots, k, j_n = 1, \ldots, r_n\} \) of elements of the set \( \hat{U} \).

Note that, by Lemma 6.4, Proposition 6.5 states that, if the series \( \hat{S}(\ell) \in \hat{\mathcal{L}}_{\infty}^{\mathbb{R}} \) is representable as a finite algebraic combination of elements of \( \hat{U} \) and of integrally summable series of length 0, then its sum, given the corresponding algebraic operations, is uniquely determined with respect to fixed sums of the series in \( \hat{U} \).

**Proof.** Suppose that there exists a rational function of elements in \( \hat{U} \) and of 0-integrally summable series that equals 0. Then its numerator is a polynomial in integrally summable series of length 0 and in series in \( \hat{U} \). We prove that, up to some operations on integrally summable series of length 0, the algebraic expression of this polynomial, as described in footnote 17, is equal to 0. In this case the sum, taken in the natural way after the algebraic operations, is zero, and the assertion is proved.

Suppose that the length 0 \( \leq l_0 \leq k \) is the maximal length of integral summability of series in this polynomial. We now express the monomial of greatest length in the variables \( \hat{R}_{l_0}^{j_0}(\ell), j_0 = 1, \ldots, r_{l_0} \), as a rational function of monomials of different length \( (\leq) \) in the variables \( \hat{R}_{l_0}^{j_0}(\ell), j_0 = 1, \ldots, r_{l_0} \), and of monomials formed from series integrally summable of length 0 and series \( \hat{R}_{n}^{j_n}(\ell), n = 1, \ldots, l_0 - 1, j_n = 1, \ldots, r_{l_0-1} \), summable of strictly smaller lengths. In the denominator there are only monomials in integrally summable series of length 0 and in series \( \hat{R}_{n}^{j_n}(\ell), n = 1, \ldots, l_0 - 1, j_n = 1, \ldots, r_{l_0-1} \), summable of strictly smaller lengths. We decrease the length by multiplying this maximal-length monomial by suitable factors of the form \( e^{-\alpha/\ell^n} \) (as many factors as there are series in this monomial) and differentiating. Then the length of the monomials on the left-hand side has been reduced by 1. We multiply by the denominator of the right-hand side and repeat the process for the monomial of greatest length in the variables \( \hat{R}_{l_0}^{j_0}(\ell), j_0 = 1, \ldots, r_{l_0} \), whose length is now equal or strictly smaller. In each step, we simplify the algebraic expression as described in footnote 17, up to operations on integrally summable series of length 0. If in some step we get a trivial algebraic combination, then the initial rational function was trivial, and the procedure stops (meaning that the initial polynomial numerator was equal to zero, which is what we wanted to prove); if not, then we proceed. After repeating the procedure finitely many times, if the procedure did not stop before, then we end up with a simple monomial \( \hat{R}_{l_0}^{r}(\ell), r \in \{1, \ldots, r_{l_0}\} \), of length of summation \( l_0 \) expressed as a polynomial in

\(^{17}\)We say that two rational functions of elements in \( \hat{U} \) and of integrally summable series of length 0 are equal if, up to the operations \(+, \cdot, \text{ and } / \) on integrally summable series of length 0, they are equal in the sense of algebraic expressions. This means that, after replacing each series (in \( \hat{U} \) or an integrally summable series of length 0) participating in these rational expressions with a formal variable, the rational algebraic expressions thus obtained are the same up to standard algebraic simplifications.
integrally summable series of strictly smaller lengths and in one other monomial 
$\tilde{R}_{t_0}^{r_1}(\ell)$, $r_1 \neq r$, of length of summation $t_0$, divided by a polynomial in integrally summable series of lengths strictly smaller than $t_0$. Since $\tilde{R}_{t_0}^{r_1}(\ell)$ and $\tilde{R}_{t_0}^{r}(\ell)$ have different exponents of integration by assumption, by one more multiplication by a suitable factor of the form $e^{-\alpha/\ell^p}$ and differentiation we end up with a series $\tilde{R}_{t_0}^{r_1}(\ell)$ summable of length $t_0$ that is equal to a rational function of series of strictly smaller lengths, which is a contradiction. □

We remark that item 2) of Definition 6.1, integral summability of length 1, corresponds to Definition 3.9 in [17], since $\frac{d}{dz} = \frac{\ell^2}{z \, d\ell}$.

Remark 6.6. It can be checked directly using (6.2) that the sum of two integrally summable series $\tilde{F}(\ell)$ and $\tilde{G}(\ell)$ of length 1 with the same exponents of integration $(\alpha, m)$ can either be integrally summable of length 0 or integrally summable of length 1 with the same exponent of integration $(\alpha, m)$.

The sum of two integrally summable series $\tilde{F}(\ell)$ and $\tilde{G}(\ell)$ of length $k \geq 2$ with respect to the same integrally summable series of smaller lengths, where the exponents of integration of $\tilde{F}$ and $\tilde{G}$ are not necessarily equal, is equal to some algebraic combination of the series $\hat{S}_{\leq l}(\ell)$, for some $l \leq k$, with respect to the same integrally summable series of smaller lengths. Indeed, one can show by differentiation and using the formulae (6.2) and (6.3) that the sum satisfies the formula (6.3) with an algebraic combination of the series $\hat{S}_{\leq l}(\ell)$, $l \leq k$, on the right-hand side and with any exponent of integration. Let $(\alpha, m)$ and $(\beta, n)$ be the exponents of integration of $\tilde{F}(\ell)$ and $\tilde{G}(\ell)$, respectively (in general not equal). For every $(\gamma, q) \in \mathbb{R}$

$$
\frac{d}{d\ell} \left( e^{-\gamma/\ell^q} (\tilde{F}(\ell) + \tilde{G}(\ell)) \right) = \frac{d}{d\ell} \left( e^{-\alpha/\ell^m \tilde{F}(\ell)} \cdot e^{-\gamma/\ell^q} + e^{-\beta/\ell^n \tilde{G}(\ell)} \cdot e^{-\gamma/\ell^q} \right) = e^{-\alpha/\ell^2m-2 \hat{R}_1(\ell)} \cdot e^{-\gamma/\ell^q} + e^{-\beta/\ell^n \hat{R}_2(\ell)} \cdot e^{-\gamma/\ell^q} \cdot \left( ((q-n)\ell + (\gamma-\beta)) \cdot \tilde{F}(\ell) \right)
$$

Here, since $\tilde{F}(\ell)$ and $\tilde{G}(\ell)$ are integrally summable of length $k$, the series $\hat{R}_1(\ell)$ and $\hat{R}_2(\ell)$ are both of type $\hat{S}_{\leq (k-1)}(\ell)$. The term in square brackets on the right-hand side is then obviously of type $\hat{S}_{\leq l}(\ell)$ for some $l \leq k$ (due to possible cancellations). In particular, if $(\alpha, m) = (\beta, n)$, then $\tilde{F}(\ell) + \tilde{G}(\ell)$ is by (6.4) integrally summable of length $k$ with the same exponent of integration $(\gamma, q) := (\alpha, m) = (\beta, n)$.

By (6.4), for any exponents of integration $(\gamma, q) \neq (\delta, p)$ the series

$$
\frac{d}{d\ell} \left( e^{-\gamma/\ell^q} (\tilde{F}(\ell) + \tilde{G}(\ell)) \right) \quad \text{and} \quad \frac{d}{d\ell} \left( e^{-\delta/\ell^p} (\tilde{F}(\ell) + \tilde{G}(\ell)) \right)
$$
are of type \( \hat{S}_{\leq l_1}(\ell) \) and \( \hat{S}_{\leq l_2}(\ell) \), respectively, for some \( l_1, l_2 \leq k \). Then

\[
e^{-\gamma/\ell} \ell^{2q-2} \hat{R}_3(\ell) = \frac{d}{d\ell} \left( e^{-\gamma/\ell} \ell^q \left( \hat{F}(\ell) + \hat{G}(\ell) \right) \right)
\]

\[
= \frac{d}{d\ell} \left( e^{-\delta/\ell} \ell^p \left( \hat{F}(\ell) + \hat{G}(\ell) \right) \cdot e^{-({\gamma-\delta})/\ell} \ell^{q-p} \right)
\]

\[
= e^{-\gamma/\ell} \ell^{2p-2} \hat{R}_4(\ell) \cdot e^{-({\gamma-\delta})/\ell} \ell^{q-p}
\]

\[
+ e^{-\gamma/\ell} \ell^{p} \left( \hat{F}(\ell) + \hat{G}(\ell) \right) \cdot e^{-({\gamma-\delta})/\ell} \ell^{q-p-2} ((q-p)\ell + (\gamma - \delta))
\]

\[
e^{-\gamma/\ell} \left[ \ell^{p+q-2} \hat{R}_4(\ell) + \ell^{q-2} \left( \hat{F}(\ell) + \hat{G}(\ell) \right) \right] (q-p)\ell + (\gamma - \delta)). \quad (6.5)
\]

Here \( \hat{R}_3(\ell) \) is of type \( \hat{S}_{\leq l_1}(\ell) \), and \( \hat{R}_4(\ell) \) is of type \( \hat{S}_{\leq l_2}(\ell) \). Comparing the sides in (6.5), we conclude that the sum \( \hat{F}(\ell) + \hat{G}(\ell) \) is of type \( \hat{S}_{\leq l}(\ell) \) for some \( l \leq \max\{l_1, l_2\} \). Note that \( \max\{l_1, l_2\} \leq k \), so that \( \hat{F}(\ell) + \hat{G}(\ell) \) is of type \( S_{\leq l}(\ell) \) for some \( l \leq k \).

A similar result holds for products \( \hat{F}(\ell) \cdot \hat{G}(\ell) \). In the same way as for (6.4) and (6.5) above, it can be proved that if both \( \hat{F}(\ell) \) and \( \hat{G}(\ell) \) are integrally summable of length \( k \geq 2 \) with respect to the same integrally summable series of smaller lengths, then their product is of the same type \( \hat{S}_{\leq l}(\ell), l \leq k \), with respect to the same integrally summable series of smaller lengths. This holds independently of the exponents of integration of \( \hat{F}(\ell) \) and \( \hat{G}(\ell) \).

**Proposition 6.7** (formal Fatou coordinate for a parabolic generalized Dulac germ).

Let \( f \) be a parabolic generalized Dulac germ on a standard quadratic domain, and let \( \hat{f}(z) = z - z^\alpha \hat{R}_1(\ell) + \text{h.o.b.}, \alpha > 1, m \in \mathbb{Z}, \) be its generalized Dulac expansion. Then there exists a unique, up to an additive constant, formal Fatou coordinate \( \hat{\Psi} \in \hat{L}_\infty^1(\mathbb{R}) \). Moreover, \( \hat{\Psi} \in \hat{L}_\infty^2(\mathbb{R}) \) and it has the form

\[
\hat{\Psi}(z) = \sum_{i \in \mathbb{N}} z^{\beta_i} \hat{T}_i(\ell),
\]

where the \( \hat{T}_i(\ell) \in \hat{L}_\ell^\infty(\mathbb{R}) \) are integrally summable of length 1 and have exponent of integration \( (\beta, 0) \) in the sense of Definition 6.1 on \( \ell \)-cusps \( \ell(V^j_{\pm}) \) of petals\(^{18} \) \( V^j_{\pm} \) \( (j \in \mathbb{Z}) \) with opening \( 2\pi/(\alpha - 1) \), and the exponents \( (\beta_i)_i \in \mathbb{R} \) are finitely generated and strictly increasing to \( +\infty \), finitely many of them are negative, and \( \beta_1 = -\alpha + 1 \).

**Proof.** The proof is similar to the proof of the theorem in [17] about the formal Fatou coordinate for a parabolic Dulac germ, with use of the Abel difference equation and blockwise construction of the formal Fatou coordinate. The only difference is that, instead of polynomials, in each block of \( \hat{f} \) we have log-Gevrey series in \( \ell \), which nevertheless are canonically summable in view of § 4. Accordingly, our Definition 6.1 of series integrally summable of length 1 is a generalization of the notion of integrally summable series in Definition 3.8 of [17], which was used in constructing the formal Fatou coordinate of parabolic Dulac germs.

The proof follows the same lines as the proof of Theorem A, (ii). Let

\[
\hat{g}(z) = \hat{f}(z) - \text{id} = z^\alpha \hat{R}_1(\ell) + z^{\alpha_1} \hat{R}_2(\ell) + \text{h.o.b.}, \quad \alpha_1 > \alpha > 1,
\]

\(^{18}\)For the definition of the petals, see Theorem A in § 3 and its proof in § 8.
where $\hat{R}_1(\ell)$ and $\hat{R}_2(\ell)$ are log-Gevrey series of order strictly greater than $(\alpha - 1)/2$ on $\ell(V_\pm^1)$. We construct the formal Fatou coordinate block-by-block, using the formal Taylor expansion of the Abel equation $\hat{\Psi}(\hat{f}) - \hat{\Psi} = 1$:

$$
\hat{\Psi}'(z)\hat{g}(z) + \frac{1}{2!}\hat{\Psi}''(z)\hat{g}^2(z) + \cdots = 1.
$$

(6.6)

Let $\hat{\Psi}_1$ be the lowest-order block of $\hat{\Psi}$. Since the orders (in $z$) of the terms in the Taylor expansion are strictly increasing,

$$
\hat{\Psi}_1'(z) \cdot z^\alpha \hat{R}_1(\ell) = 1.
$$

Because $\hat{R}_1(\ell)$ is a log-Gevrey series of order strictly greater than $(\alpha - 1)/2$ on $\ell(V_\pm^1)$ by the definition of generalized Dulac expansions, and because by Proposition 4.9 the series $1/\hat{R}_1(\ell)$ is also log-Gevrey of order strictly greater than $(\alpha - 1)/2$ on $\ell(V_\pm^1)$, it follows that the series $z^{\alpha - 1}\hat{\Psi}_1(z) = z^{\alpha - 1}\int z^{-\alpha}dz$ is, by Definition 6.1, integrally summable of length 1 in the variable $\ell$ with exponent of integration $(-\alpha + 1, 0)$ on $\ell(V_\pm^1)$ ($j \in \mathbb{Z}$). We have $\beta_1 := -\alpha + 1$.

To find the second block $\hat{\Psi}_2(\ell)$, we put $\hat{\Psi}(z) = \hat{\Psi}_1(z) + \hat{\Psi}_2(z) + \text{h.o.b.}$ in the equation (6.6). After cancellations, and after comparing the lowest-order blocks on the right-hand and left-hand side as above, we get that

$$
\hat{\Psi}'_2(z) \cdot z^\alpha \hat{R}_1(\ell) = \begin{cases} 
  \frac{z^{\alpha - \alpha}\hat{R}_2(\ell)}{\hat{R}_1(\ell)}, & \alpha_1 < 2\alpha - 1, \\
  z^{\alpha - 1}\left(-\frac{1}{2}\right)(\alpha\hat{R}_1 + \hat{R}_1' \cdot \ell^{2}), & \alpha_1 > 2\alpha - 1, \\
  \text{the sum of both}, & \alpha_1 = 2\alpha - 1.
\end{cases}
$$

(6.7)

In the denominator we get only powers of the series $\hat{R}_1(\ell) \neq 0$. Since the log-Gevrey expansions are closed with respect to the algebraic operations and differentiation (Propositions 4.5–4.7), we conclude that the series $z^{-\min\{\alpha - \alpha_1 + 1, -\alpha + 2\}}\hat{\Psi}_2(z)$ is again integrally summable of length 1 on $\ell(V_\pm^1)$, with exponent of integration $(\min\{\alpha - \alpha_1 + 1, -\alpha + 2\}, 0)$. Let $\beta_2 := \min\{\alpha - \alpha_1 + 1, -\alpha + 2\}$. We continue the construction by induction. More precisely, in the $i$th step of the induction ($i \in \mathbb{N}$) we suppose that the series $z^{-\beta_i}\hat{\Psi}_1, \ldots, z^{-\beta_i}\hat{\Psi}_i$, are integrally summable of length 1 on $\ell(V_\pm^1)$ with exponents of integration $(\beta_1, 0), \ldots, (\beta_i, 0)$, respectively.

Therefore, the series $\hat{\Psi}_1'(z), \ldots, \hat{\Psi}_i'(z)$ are products of some power of $z$ by some log-Gevrey series of order strictly greater than $(\alpha - 1)/2$ on $\ell(V_\pm^1)$. Due to the closedness of log-Gevrey series with respect to the algebraic operations, all their derivatives are of the same form. Moreover, all the blocks in $\hat{g}^k(z)$ ($k \in \mathbb{N}$), are products of powers of $z$ by log-Gevrey series of order strictly greater than $(\alpha - 1)/2$, since $\hat{f}$ is a generalized Dulac expansion. Now it follows from (6.6) that $\hat{\Psi}_{i+1}(z)$ is again a product of some power of $z$ by a log-Gevrey series of order strictly greater than $(\alpha - 1)/2$ on $\ell(V_\pm^1)$, which is well defined since only powers of $\hat{R}_1(\ell) \neq 0$ appear in the denominator. Therefore, there exists a unique exponent $\beta_{i+1}$ such that $z^{-\beta_{i+1}}\hat{\Psi}_{i+1}(z)$ is integrally summable of length 1 in $\ell$ on the domain $\ell(V_\pm^1)$. The exponent of integration is $(\beta_{i+1}, 0)$, and $\beta_{i+1} > \beta_i$ by our algorithm. □
6.2. The notion of block iterated integral summability. Definition 6.8 below of block iterated integral summability is adapted to formal normalizations of parabolic generalized Dulac germs. The technical complication in the definition of the final composition with the transformation $\hat{h}_0^{-1}$ comes from the elimination of the first block in reducing a parabolic generalized Dulac transseries to its normal form. This first change of variables is more complicated in form than the changes considered in Lemma 5.3 by which we eliminate higher-order blocks. By removing the first block in the first step, we would essentially change the type of blocks in the initial parabolic generalized Dulac expansion (they would no longer be log-Gevrey, that is, integrally summable of length 0). This would involve technical difficulties in describing subsequent changes using the notion of integral summability of length $k \geq 1$ introduced in Definition 6.1. We avoid these difficulties by first reducing the initial parabolic generalized Dulac transseries to a parabolic generalized Dulac transseries with the same initial block, which we call an auxiliary normal form. Finally, applying one additional change of variables, $\hat{h}_0^{-1}$, we reduce this transseries to a normal form in (5.1). This will be clarified in the proof of Proposition 6.10 below.

Definition 6.8 (block iterated integrally summable transseries). Let $\hat{\varphi} \in \hat{L}^{id}(\mathbb{R})$, $\hat{\varphi}(z) = z + z\hat{R}_0(\ell) + \text{h.o.b.}$, where $\hat{R}_0(\ell) \in \ell \mathbb{R}[[\ell]]$, be a parabolic transseries. The case $\hat{R}_0 = 0$ is also allowed.

We say that the series $\hat{\varphi}$ is block iterated integrally summable with parameters $(\alpha, m, \rho) \in \mathbb{R}_{>1} \times \mathbb{Z} \times \mathbb{R}$ on a petal $V$ of opening $2\pi/(\alpha - 1)$ if the following conditions 1) and 2) hold.

1) There exists a series $\hat{T}_0(\ell) \in \mathbb{R}((\ell))$, $\hat{T}_0(\ell) = -\ell^m + \text{h.o.t.}$, that is log-Gevrey summable of order strictly greater than $(\alpha - 1)/2$ on the $\ell$-cusp $\ell(V)$ and such that

$$
\left(- \int \frac{z^{1-\alpha}}{\ell^{m+2}} \, d\ell\right) \circ (z + z\hat{R}_0(\ell)) = \int \frac{z^{1-\alpha}}{\hat{T}_0(\ell)\ell^2} \, d\ell
$$

(6.8)

(with 0 as the constant of formal integration).

2) If $\hat{h}_0 \in \hat{L}^{id}(\mathbb{R})$ is a transseries given by the formula (with the constant of formal integration equal to 0)

$$
\hat{h}_0(z) := \left(- \int \frac{z^{1-\alpha}}{\ell^{m+2}} \, d\ell - \frac{\alpha}{2} \ell^{-1} + \left(\frac{m}{2} + \rho\right)\ell^{-1}\right)^{-1} \circ \left(\int \frac{z^{-\alpha+1}}{\hat{T}_0(\ell)\ell^2} \, d\ell - \frac{\alpha}{2} \ell^{-1} + b\ell^{-1}\right),
$$

(6.9)

where $b \in \mathbb{R}$ is uniquely determined by the condition that $\hat{h}_0(z)$ does not contain iterated logarithms, then the composition $\hat{\varphi} \circ \hat{h}_0$ can be decomposed into a sequence of compositions $\hat{\varphi} \circ \hat{h}_0 = \circ_{i \in \mathbb{N}} \hat{\varphi}_i$ of elementary changes of variables of the form

$$
\hat{\varphi} \circ \hat{h}_0 = \circ_{i \in \mathbb{N}} \hat{\varphi}_i, \quad \hat{\varphi}_i(z) = z + z^{\beta_i} \hat{R}_i(\ell),
$$

where $(\beta_i)_{i \in \mathbb{N}}, \beta_i > 1$, is a strictly increasing sequence of real numbers, either finite or tending to $+\infty$, and the transseries $\hat{R}_i(\ell) \in \mathbb{R}((\ell))$, $\hat{R}_i \neq 0$, $i \in \mathbb{N}$, form a sequence

$$
\hat{\mathcal{H}} = \{\hat{R}_i(\ell) : i \in \mathbb{N}\} \subseteq \mathbb{R}((\ell))
$$

(6.10)

19Here and below, $\circ_{i \in \mathbb{N}} \hat{\varphi}_i := \hat{\varphi}_1 \circ \hat{\varphi}_2 \circ \cdots$. 

with the following properties:
(a) each series $\hat{R}_i(\ell) \in \hat{\mathcal{H}}$ is either integrally summable of some length $n_i \in \mathbb{N}_0$, $n_i \leq i$, with respect to all the previous elements of $\hat{\mathcal{H}}$ that are integrally summable of strictly smaller lengths, or an algebraic combination of all the previous series $\hat{R}_1(\ell), \ldots, \hat{R}_{i-1}(\ell)$;
(b) the lengths $i \mapsto n_i$ are not necessarily increasing.

Note that we can also present the block iterated integrally summable transseries $\hat{\varphi}$ in Definition 6.8 as the composition

$$\hat{\varphi} = \left( \circ_{i \in \mathbb{N}} \hat{\varphi}_i \right) \circ \hat{h}_0^{-1}, \quad (6.11)$$

where $\hat{h}_0$ and $\hat{\varphi}_i (i \in \mathbb{N})$ are as described in that definition.

We remark also that we can associate with $\hat{\mathcal{H}}$ a collection (non-unique, depending on a choice of consecutive constants of integration) of analytic integral sums of its terms on $\ell(V)$:

$$\mathcal{H}^{h_0} = \{ R_i(\ell) : i \in \mathbb{N} \}. \quad (6.12)$$

The sums in $\mathcal{H}^{h_0}$ also depend on the choice of the constant in the definition of $h_0$. Here the $R_i(\ell)$ are the integral sums of given length of the series $\hat{R}_i(\ell) (i \in \mathbb{N})$ on $\ell(V)$ with respect to previous integral sums, as in Definition 6.1. They differ by the choice of the constants of integration in inductive definitions of the integral sums. The series $\hat{R}_i(\ell)$ are their common asymptotic expansions.

**Proposition 6.9** (uniqueness of the decomposition). Let

$$\hat{\varphi} \in \hat{L}^{\text{id}}(\mathbb{R}), \quad \hat{\varphi}(z) = z + z\hat{R}_0(\ell) + \text{h.o.b.}, \quad \hat{R}_0(\ell) \in \ell\mathbb{R}[[\ell]],$$

be a block iterated integrally summable transseries with parameters $(\alpha, m, \rho)$ on a petal $V$ of opening $2\pi/(\alpha - 1)$. Then a decomposition of it

$$\hat{\varphi} = \left( \circ_{i \in \mathbb{N}} \hat{\varphi}_i \right) \circ \hat{h}_0^{-1}$$

with $\hat{h}_0(z)$ and elementary changes $\hat{\varphi}_i(z)$ as described in Definition 6.8 is unique.

**Proof.** Since $\alpha$, $m$, and $\hat{R}_0(\ell)$ are given, the transseries $\hat{T}_0(\ell) \in \mathbb{R}((\ell))$ is uniquely determined from equation (6.8), and it has the form $\hat{T}_0(\ell) = -\ell^m + \text{h.o.t.}$ Then $\hat{h}_0(z)$ is uniquely given by (6.9), where 0 is taken as the constant of formal integration. We note that $\hat{h}_0 \in \hat{L}^{\text{id}}(\mathbb{R})$ is parabolic, so $\hat{\varphi} \circ \hat{h}_0(z)$ is also parabolic. Let us single out its first block: $\hat{\varphi} \circ \hat{h}_0(z) = z + z^\beta \hat{R}_1(\ell) + \text{h.o.b.}$, where $\hat{R}_1(\ell) \in \mathbb{R}((\ell))$. Further, since $\hat{\varphi} \circ \hat{h}_0(z)$ admits a decomposition $\circ_{i \in \mathbb{N}} \hat{\varphi}_i$, where the exponents $\beta > 1$ are strictly increasing, the first change of variables $\hat{\varphi}_1$ should have the form $\hat{\varphi}_1(z) = z + z^{\beta} \hat{R}_1(\ell)$, and moreover, the series $\hat{R}_1(\ell) \in \mathbb{R}((\ell))$ is integrally summable of length 0 or 1. We now compose on the left with the inverse change $\hat{\varphi}_1^{-1}$ and, continuing by induction, uniquely determine the changes $\hat{\varphi}_i (i \in \mathbb{N})$. $\square$
Proposition 6.10 (block iterated integral summability of formal reduction to the formal normal form). Let \( \hat{f}(z) = z - z^\alpha \ell^m + \text{h.o.t.} \), where \( \alpha > 1 \) and \( m \in \mathbb{Z} \), be a normalized parabolic generalized Dulac transseries belonging to the formal class \((\alpha, m, \rho), \alpha > 1, m \in \mathbb{Z}, \rho \in \mathbb{R}\). Let \( V \) be a petal of opening \( 2\pi/(\alpha - 1) \). Let \( \hat{f}_1 \) be the formal normal form of this transseries defined in (5.1). Then:

(i) there exists a formal normalization \( \hat{\varphi} \in \hat{\mathcal{L}}^{id}(\mathbb{R}) \) which reduces \( \hat{f} \) to the form \( \hat{\tilde{f}}_1 \) and which is block iterated integrally summable with parameters \((\alpha, m, \rho)\) on \( V \);

(ii) every formal normalization \( \hat{\varphi} \in \hat{\mathcal{L}}^{id}(\mathbb{R}) \) reducing \( \hat{f} \) to \( \hat{f}_1 \) is, up to a precomposition with \( \hat{f}_c \) \((c \in \mathbb{R})\), block iterated integrally summable with parameters \((\alpha, m, \rho)\) on \( V \), where \( f_c = \text{Exp}\left( cX_1 \frac{d}{dz}\right) \).id \((c \in \mathbb{R})\), and the vector field \( X_1 \) is given in (5.1).

Proof. (i) We construct \( \hat{\varphi} \) algorithmically, eliminating block after block from \( \hat{f} \), and choosing 0 as the constant of formal integration for each block. Let \( \hat{f} = z + z^\alpha \hat{T}_0(\ell) + \text{h.o.b.}, \) where \( \hat{T}_0(\ell) = -\ell^m + \text{h.o.t.} \) is a log-Gevrey series of order strictly greater than \((\alpha - 1)/2\) on \( V \). We first reduce \( \hat{f} \) to an auxiliary normal form \( \hat{f}_2 \), given as the time-one map along a trajectory of the formal vector field:

\[
\hat{f}_2(z) = \text{Exp}\left( \frac{z^\alpha \hat{T}_0(\ell)}{1 + (\alpha/2)z^{\alpha - 1}\hat{T}_0(\ell) + b z^{\alpha - 1}\hat{T}_0(\ell) \ell d\ell} \right) \text{id}.
\]  

(6.13)

It can easily be checked that \( \hat{f}_2 \) is again a parabolic generalized Dulac transseries, since all blocks in it are log-Gevrey of order strictly greater than \((\alpha - 1)/2\) on the \( \ell \)-cusp \( \ell(V) \) in view of the closedness of the log-Gevrey classes with respect to addition, multiplication, and differentiation (Propositions 4.5–4.7). Moreover, \( \hat{f}_2 \) has the unchanged initial block \( z^\alpha \hat{T}_0(\ell) \), and \( b \in \mathbb{R} \) is chosen (uniquely) so that \( \hat{f}_2 \) has the same formal invariants \((\alpha, m, \rho)\) as \( \hat{f}_1 \). Also, its formal Fatou coordinate is given by the simple formula

\[
\hat{\Psi}_2 = \int \frac{z^{\alpha + 1} \ell \ell}{\hat{T}_0(\ell) \hat{T}_0(\ell) \ell^2} d\ell - \frac{\alpha}{2} \ell^{-1} + b \ell^{-1}.
\]

Therefore, the auxiliary normal form \( \hat{f}_2 \) can be reduced to the normal form \( \hat{f}_1 \) of \( \hat{f} \) given in (5.1) simply by one additional parabolic change of variables \( \hat{h}_0^{-1} \in \hat{\mathcal{L}}^{id}(\mathbb{R}) \) that can be explicitly expressed as

\[
\hat{h}_0^{-1} := \hat{\Psi}_2^{-1} \circ \hat{\Psi}_0 = \left( \int \frac{z^{\alpha + 1} \ell \ell}{\hat{T}_0(\ell) \ell^2} d\ell - \frac{\alpha}{2} \ell^{-1} + b \ell^{-1} \right)^{-1} \circ \left( -\int \frac{z^{1-\alpha} \ell \ell}{\ell^{m+2}} d\ell - \frac{\alpha}{2} \ell^{-1} + \left( \frac{m}{2} + \rho \right) \ell^{-1} \right).
\]

(6.14)

Here we fix the constant of formal integration to be 0. Note that \( \hat{h}_0^{-1} \) belongs to \( \hat{\mathcal{L}}^{id}(\mathbb{R}) \), since \( b \) is chosen so that \( \hat{f}_2 \) and \( \hat{f}_1 \) belong to the same \( \hat{\mathcal{L}} \)-formal class. We now reduce \( \hat{f} \) to its auxiliary normal form \( \hat{f}_2 \) by a parabolic reduction \( \hat{h} \in \hat{\mathcal{L}}^{id}(\mathbb{R}) \).
Since all blocks in \( \hat{f} \) and \( \hat{f}_2 \) are integrally summable of length 0 (are log-Gevrey) and we do not eliminate the first block, it is obvious from the construction of the blockwise changes of variables in the algorithm for eliminating blocks in Lemma 5.4 and from Definition 6.8 that \( \hat{h} = \circ \mathcal{G}_i \), where the \( \mathcal{G}_i(z) = z + z^\beta \hat{R}_i(\ell) \) are elementary changes of variables, the exponents \( \beta_i \) are strictly increasing, and the transseries \( \hat{R}_i(\ell) \) in the expressions for \( \hat{\phi}_i \) \((i \in \mathbb{N})\) belong to a sequence \( \mathcal{H} \) of integrally summable series and algebraic combinations of them of the form (6.10). We can reach this conclusion by carefully keeping account of the blocks in the reduced transseries \( \hat{f} \) after each elimination of a block, and also by using the formula (5.3) in Lemma 5.4 for elimination of blocks. It is important to note that in using (5.3) to get elementary changes of variables (containing only one block after \( z \)) we choose 0 as the constant of formal integration. Note also by (5.3) that the exponents \( p \) in Definition 6.1 of integral summability of the series \( \hat{R}_i(\ell) \) are equal to \( m \) for all \( \hat{R}_i(\ell) \in \mathcal{H} \) that are integrally summable of some length.

Finally, we reduce \( \hat{f}_2 \) to \( \hat{f}_1 \) by the transformation \( \hat{h}_0^{-1} \) given above. The transformation \( \hat{\phi} := \hat{h} \circ \hat{h}_0^{-1} \) reducing \( \hat{f} \) to \( \hat{f}_1 \) is then block iterated integrally summable in the sense of Definition 6.8. Indeed, let \( \hat{R}_0(\ell) \in \ell R[[\ell]] \) be the first block of \( \hat{\phi} \), \( \hat{\phi} = z + z\hat{R}_0(\ell) + \text{h.o.t} \). It eliminates the first block of \( \hat{f} \) and is therefore, by Lemma 5.3, connected with the first block \( \hat{T}_0(\ell) \) of \( \hat{f} \) by the formula (6.8).

(ii) In Proposition 6.11 below we prove that if \( \hat{\phi} \) is a formal normalization of \( \hat{f} \), then \( \text{any} \) other formal normalization of \( \hat{f} \) has the form \( \hat{\phi} \circ \hat{f}_c \) \((c \in \mathbb{R})\). Therefore, any formal normalization \( \hat{\phi} \) of \( \hat{f} \) admits the decomposition (6.11) as its unique decomposition, up to precomposition with \( \hat{f}_c \). \( \Box \)

The algorithm in Lemma 5.3 for reducing generalized Dulac series to \( \hat{L}(\mathbb{R}) \)-normal form \( \hat{f}_1 \) by blockwise eliminations is not unique. The following proposition shows, however, that the formal changes of variables differ from one another in a controlled way.

**Proposition 6.11.** Let \( \hat{f}(z) = z - z^\alpha \ell^m + \ldots \alpha > 1, m \in \mathbb{Z} \), be a normalized parabolic generalized Dulac series, and let \( \hat{\Psi} \in \hat{L}_{\Sigma}^\infty(\mathbb{R}) \) be its formal Fatou coordinate \(^{20}\) (with constant term equal to 0). Let \( \hat{f}_1(z) \) be the normal form in (5.1), and let

\[
\hat{\Psi}_0(z) = -\int \frac{dz}{z^\alpha \ell^m} + \frac{\alpha}{2} \log z + \left( \frac{m}{2} + \rho \right) \ell_2^{-1}
\]

be its formal Fatou coordinate (with constant term equal to 0). Then:

(i) the formal change of variables \( \hat{\phi} \in \hat{L}_{\Sigma}^{1d}(\mathbb{R}) \) such that \( \hat{f} = \hat{\phi} \circ \hat{f}_1 \circ \hat{\phi}^{-1} \) is uniquely determined up to precomposition with \( \hat{f}_c \) \((c \in \mathbb{R})\), where \( \hat{f}_c = \text{Exp} \left( c X_1 \frac{d}{dz} \right). \text{id} \) \((c \in \mathbb{R})\), and \( X_1 \) is defined in (5.1);

(ii) for any formal change of variables \( \hat{\phi} \) reducing \( \hat{f} \) to \( \hat{f}_1 \) there exists a constant \( C \in \mathbb{R} \) such that \( \hat{\phi} + C = \hat{\Psi}_0 \circ \hat{\phi}^{-1} \).

---

\(^{20}\)A formal Fatou coordinate is a formal solution of the Abel equation \( \hat{\Psi} \circ \hat{f} - \hat{\Psi} = 1 \), which is unique in \( \hat{L}(\mathbb{R}) \) up to an additive constant. It is related to embedding in a vector field, since it represents flow time. For more information, as well as the construction of a formal Fatou coordinate for parabolic Dulac germs, see [17].
The proof is well known and can be found in the Appendix (§10).

6.3. The notion of generalized block iterated integral summability. The following definition is introduced for the purpose of describing formal conjugacies between two parabolic generalized Dulac transseries in the same formal class, and not just formal reductions to normal forms (see Proposition 6.14 below).

Definition 6.12. Let \( \widehat{\varphi} \in \widehat{\mathcal{L}}^{id}(\mathbb{R}) \), let \((\alpha, m, \rho) \in \mathbb{R}_{>1} \times \mathbb{Z} \times \mathbb{R} \), and let \( V \) be a petal of opening \( 2\pi/(\alpha - 1) \). We say that the transseries \( \widehat{\varphi} \) is generalized block iterated integrally summable on \( V \) with parameters \((\alpha, m, \rho)\) if there exists a decomposition

\[
\widehat{\varphi} = \varphi_1 \circ \varphi_2^{-1},
\]

(6.15)

where \( \varphi_1 \) and \( \varphi_2 \) are, up to precompositions with \( f_c \ (c \in \mathbb{R}) \), block iterated integrally summable on \( V \) with parameters \((\alpha, m, \rho)\). Here \( f_c \ (c \in \mathbb{R}) \) is the time-\( c \) map along trajectories of the \((\alpha, m, \rho)\)-model field \( X_1 \) as in (5.1).

Note that we do not ask for uniqueness of this decomposition.

Proposition 6.13. Two (normalized) parabolic generalized Dulac germs \( f \) and \( g \) are \( \widehat{\mathcal{L}}(\mathbb{R}) \)-formally conjugate\(^{21}\) if and only if they belong to the same \( \widehat{\mathcal{L}}(\mathbb{R}) \)-formal class \((\alpha, m, \rho)\).

Proof. Let \( f_1 \) be the \((\alpha, m, \rho)\)-normal form as in (5.1). Let \( \varphi_1 \) be a formal normalization of \( f \) that reduces it to \( f_1 \): \( \varphi_1^{-1} \circ f \circ \varphi_1 = f_1 \). Putting this into the equality \( \widehat{g} = \varphi_1^{-1} \circ f \circ \varphi_1 \), we get that \( \varphi_2 := \varphi_1^{-1} \circ \varphi_1 \in \widehat{\mathcal{L}}^{id}(\mathbb{R}) \) is a normalization of \( \widehat{g} \) to the same normal form \( f_1 \). On the other hand, if \( \varphi_1 \in \widehat{\mathcal{L}}^{id}(\mathbb{R}) \) is a formal normalization of \( f \) and \( \varphi_2 \in \widehat{\mathcal{L}}^{id}(\mathbb{R}) \) is a formal normalization of \( g \), then \( \widehat{\varphi} := \varphi_1 \circ \varphi_2^{-1} \in \widehat{\mathcal{L}}^{id}(\mathbb{R}) \) is a formal conjugacy of \( f \) to \( g \).

\( \square \)

Proposition 6.14. Let \( \widehat{f} \) and \( \widehat{g} \) be two normalized parabolic generalized Dulac transseries belonging to the formal class \((\alpha, m, \rho), \alpha > 1, m \in \mathbb{Z}, \rho \in \mathbb{R} \). Let \( V \) be a petal of opening \( 2\pi/(\alpha - 1) \). Then:

(i) there exists a parabolic formal change of variables \( \varphi \in \widehat{\mathcal{L}}^{id}(\mathbb{R}) \) conjugating\(^{22}\) \( \widehat{f} \) to \( \widehat{g} \) which is generalized block iterated integrally summable on \( V \) with parameters \((\alpha, m, \rho)\);

(ii) every conjugacy \( \varphi \in \widehat{\mathcal{L}}^{id}(\mathbb{R}) \) conjugating \( \widehat{f} \) to \( \widehat{g} \) is generalized block iterated integrally summable.

Proof. (i) Put \( \widehat{\varphi} := \varphi_1 \circ \varphi_2^{-1} \), where \( \varphi_1 \in \widehat{\mathcal{L}}^{id}(\mathbb{R}) \) is a reduction of \( \widehat{f} \) to its \((\alpha, m, \rho)\)-normal form \( \widehat{f}_1 \) and \( \varphi_2 \in \widehat{\mathcal{L}}^{id}(\mathbb{R}) \) is a reduction of \( \widehat{g} \) to its \((\alpha, m, \rho)\)-normal form \( \widehat{f}_1 \), and both are block iterated integrally summable with parameters \((\alpha, m, \rho)\) on \( V \) by Proposition 6.10.

(ii) Every such element \( \varphi \) can be decomposed as

\[
\widehat{\varphi} = \varphi_1 \circ \varphi_2^{-1},
\]

\(^{21}\)There exists a formal change of variables \( \varphi \in \widehat{\mathcal{L}}^{id}(\mathbb{R}) \) such that for their generalized Dulac expansions \( f \) and \( g \) we have \( \widehat{g} = \varphi^{-1} \circ f \circ \varphi \).

\(^{22}\)\( \varphi^{-1} \circ f \circ \varphi = \widehat{g} \).
where \( \varphi_2 \) is a reduction of \( \widehat{g} \) to \( \widehat{f}_1 \) and \( \varphi_1 \) is a reduction of \( \widehat{f} \) to \( \widehat{f}_1 \). Indeed, take \( \varphi_1 \) to be any normalization of \( \widehat{f} \). By Proposition 6.10, up to a precomposition with \( f_c \) it is block iterated integrally summable with parameters \((\alpha, m, \rho)\). Then \( \varphi_1^{-1} \circ \varphi_1 \) is a normalization of \( g \). By Proposition 6.10, up to a precomposition with \( f_c \), it is also block iterated integrally summable with parameters \((\alpha, m, \rho)\). □

The proof of Proposition 6.14 gives us an exact description of possible non-uniqueness of the conjugacy \( \varphi \) of \( \widehat{g} \) and \( \widehat{f} \), namely,

\[
\varphi_1 \circ \widehat{f}_c \circ \varphi_2^{-1}, \quad c \in \mathbb{R},
\]

where \( \varphi_1 \) and \( \varphi_2 \) are \textit{any} normalizations of \( \widehat{f} \) and \( \widehat{g} \), respectively. Indeed, by Proposition 6.11, such normalizations are unique up to precomposition with \( \widehat{f}_c \) \((c \in \mathbb{R})\).

Note also that the decomposition (6.15) of a generalized block iterated integrally summable change of variables \( \varphi \in \mathcal{L}^{id}(\mathbb{R}) \) may not be unique. Indeed, suppose that \( \varphi \) is a conjugacy between two parabolic generalized Dulac transseries that is generalized block iterated integrally summable. In general, there may exist different pairs of parabolic generalized Dulac transseries \( \widehat{f}, \widehat{g} \) and \( \widehat{f}_1, \widehat{g}_1 \) that are conjugate by \( \varphi \).

We denote the class of generalized block iterated integrally summable parabolic transseries with parameters \((\alpha, m, \rho)\) on some petal \(^{23} V \) of opening \( 2\pi/(\alpha - 1) \) by

\[
\mathcal{S}^{\alpha, m, \rho}(V) \subset \mathcal{L}^{id}(\mathbb{R}).
\]

For simplicity of notation, below we will not specify the sector \( V \), although the notion of (generalized, iterated) integral summability is always relative to some sector.

As a trivial case of Proposition 6.10, \( \varphi := id \) is trivially block iterated integrally summable for any triple of parameters and on any petal \( V \). Therefore, the block iterated integrally summable parabolic transseries with parameters \((\alpha, m, \rho)\) form a subclass of the class of generalized block iterated integrally summable parabolic transseries with parameters \((\alpha, m, \rho)\).

Moreover, parabolic generalized Dulac series \( \widehat{f}(z) = z - z^\alpha \ell^m + \cdots \) belong to the class \( \mathcal{S}^{\alpha, m, \rho}(V) \), where \( V \) is any petal of opening strictly less than \( 2\pi/(\alpha - 1) \), for any triple of parameters \((\alpha, *, *)\). This can be proved by decomposition into a sequence of elementary changes whose blocks are integrally summable of length 0, using Proposition 8.2.

\textbf{Remark 6.15.} We note that \( \mathcal{L}^{id}(\mathbb{R}) \) is a group under composition. However, we are not able to prove the group property for the class \( \mathcal{S}^{\alpha, m, \rho}(V) \) of all parabolic generalized block iterated integrally summable series on \( V \) with parameters \((\alpha, m, \rho)\). We are not even able to prove the group property for the subclass of all parabolic transseries that are \textit{realizable} as formal conjugacies of two parabolic generalized Dulac transseries in the formal class \((\alpha, m, \rho)\).

Indeed, even if \( \varphi \in \mathcal{S}^{\alpha, m, \rho}(V) \) can be realized as a conjugacy between two parabolic generalized Dulac series \( \widehat{f} \) and \( \widehat{g} \) in the class \((\alpha, m, \rho)\), and \( \widehat{\eta} \in \mathcal{S}^{\alpha, m, \rho}(V) \) can

\(^{23}\)Here the petal is considered as a germ of the domain (its radius is not important).
be realized as a conjugacy between another two parabolic generalized Dulac series $\hat{h}$ and $\hat{k}$ in the same formal class $(\alpha, m, \rho)$, we see no reason in general to believe that the composition $\hat{g} \circ \hat{\varphi}$ will be a conjugacy between two parabolic generalized Dulac series if the pairs are not chainable. But if they do form a chain, that is, if $\hat{g} = \hat{h}$, then $\hat{\varphi} \circ \hat{g}$ is again a conjugacy between $\hat{f}$ and $\hat{k}$, and by Proposition 6.14, this composition is indeed generalized block iterated integrally summable with parameters $(\alpha, m, \rho)$.

Moreover, we do not assert that the formal inverse of a parabolic generalized block iterated integrally summable series is again a parabolic generalized block iterated integrally summable series. However, if $\hat{\varphi}$ is a parabolic generalized block iterated integrally summable series with parameters $(\alpha, m, \rho)$ that can be realized as a formal conjugacy between two parabolic generalized Dulac series $\hat{f}$ and $\hat{g}$, then its formal inverse $\hat{\varphi}^{-1}$ is a formal conjugacy between $\hat{g}$ and $\hat{f}$. Furthermore, these Dulac series belong to the formal class $(\alpha, m, \rho)$. Therefore, by Proposition 6.14, $\hat{\varphi}^{-1}$ is again generalized block iterated integrally summable with parameters $(\alpha, m, \rho)$.

7. Asymptotic expansions of sectorially analytic reductions for parabolic generalized Dulac germs

Let $f$ be a parabolic generalized Dulac germ and let

$$\hat{f}(z) = z - z^\alpha \ell^m + \text{h.o.t.}, \quad \alpha > 1, \quad m \in \mathbb{Z},$$

be its generalized Dulac expansion. Let $\hat{\Psi} \in \hat{\mathcal{L}}_\infty^\circ(\mathbb{R})$ be the formal Fatou coordinate for $f$ and let $\hat{\varphi} \in \hat{\mathcal{L}}^{\text{id}}(\mathbb{R})$ be a formal change of variables that reduces it to its $\hat{\mathcal{L}}(\mathbb{R})$-formal normal form $\hat{f}_1(z) = \text{Exp}(X_1).\text{id}$, where $X_1$ is the vector field in (5.1). In this section we suppose that we have already proved the existence of sectorially analytic Fatou coordinates $\Psi_j^\pm$ on the petals $V_j^\pm (j \in \mathbb{Z})$ of a standard quadratic domain $\mathcal{R}_C$. In fact, this will be proved by a construction in the proof of Theorem A, (i), in §8. For this we use only the assumptions that $f$ is analytic on $\mathcal{R}_C$ and that it admits a uniform estimate (2.5) in the first two terms. It is well known that the relation between the reduction to normal form and the Fatou coordinate is given through the Fatou coordinate $\Psi_0^\pm$ of the formal normal form $\hat{f}_1$, which is defined and analytic on the whole of $\mathcal{R}_C$ (up to an additive constant in $\Psi$, that is, up to a precomposition of $\varphi$ with the map $f_c = \text{Exp}(cX_1).\text{id}$, $c \in \mathbb{R}$):

$$\hat{\Psi} = \Psi_0 \circ \hat{\varphi}, \quad \Psi_j^\pm = \Psi_0 \circ \varphi_j^\pm \text{ on } V_j^\pm, \quad j \in \mathbb{Z}.$$

Sectorial analyticity of the Fatou coordinate therefore immediately implies sectorial analyticity of the changes of variables $\varphi_j^\pm$ on the sectors $V_j^\pm (j \in \mathbb{Z})$.

In Propositions 7.1 and 7.4 in §§7.1 and 7.2 we state that by the choice of integral sections the formal Fatou coordinate and the formal conjugacy can be made the unique sectional asymptotic expansions of the sectorial Fatou coordinates and conjugacies, respectively. For more on section functions and sectional transserial expansions, see [17] and Definition 10.1 in the Appendix (in the complex version).
7.1. Sectorially analytic Fatou coordinates. In Proposition 7.1 below, by integral asymptotic expansion we mean a sectional asymptotic expansion where section functions at limit ordinal steps (that is, sums of series in $\ell$ in each block) are chosen as integral sums of length 1, as defined in Definition 6.1, 2).

**Proposition 7.1** (integral asymptotic expansion of the Fatou coordinate of a parabolic generalized Dulac germ). Let $f$ be a parabolic generalized Dulac germ, and let $\hat{f}(z) = z - z^\alpha \ell^m + \text{h.o.t.}$, $\alpha > 1$, $m \in \mathbb{Z}$, be its generalized Dulac expansion. Let\(^\text{24}\) $\Psi_j^\pm(z) \sim_{z \to 0} -\frac{1}{\alpha - 1}z^{-\alpha + 1}\ell^{-m}$ be its analytic Fatou coordinates on the petals $V_j^\pm$ ($j \in \mathbb{Z}$), which are constructed in Theorem A.

(i) Up to a constant, the Fatou coordinates $\Psi_j^\pm$ admit a common integral asymptotic expansion $\hat{\Psi}$ as $z \to 0$ on $V_j^\pm$.

(ii) Let $\Phi_j^\pm$ be any other Fatou coordinates for $f$ on petals $V_j^\pm$ such that

$$\Phi_j^\pm(z) \sim_{z \to 0} -\frac{1}{\alpha - 1}z^{-\alpha + 1}\ell^{-m}.$$ 

Then on every petal $V_j^\pm$ the Fatou coordinate $\Phi_j^\pm$ differs from $\Psi_j^\pm$ only by an additive constant. Up to a constant, the coordinates $\Phi_j^\pm$ also admit $\hat{\Psi}$ as their integral asymptotic expansion as $z \to 0$ on $V_j^\pm$.

**Proof.** (i) The proof is similar to the proof of the theorem for parabolic Dulac germs in [17]. The only difference is that the coefficients now are not polynomials in $\ell^{-1}$, but log-Gevrey sums on $\ell$-cusps ($\ell$-images of petals). Suppose that $f(z) = z - z^\alpha R_j^{\pm}(\ell) + O(z^{\alpha + \delta})$, $z \in V_j^\pm$, $\alpha > 1$, $\delta > 0$. Here the $R_j^{\pm}(\ell)$ are analytic log-Gevrey sums of orders strictly greater than $(\alpha - 1)/2$ of their formal analogue, the series $\hat{R}_1(\ell)$, on the $\ell$-cusps $\ell(V_j^\pm)$, where the $V_j^\pm$ are petals of opening $2\pi/\alpha - 1$). Thus, for the Fatou coordinate of a generalized Dulac germ the integral sums of length 1 on $\ell$-cusps $\ell(V_j^\pm)$ in Definition 6.1 (and not the integral sums considered earlier) turn out to be the right choice of sections in each block. By Propositions 4.5–4.7 in § 4 the class of all log-Gevrey summable series of order strictly greater than some fixed $r > 0$ on $\ell(V_j^\pm)$ remains closed with respect to addition, multiplication, and differentiation. Note also that the germ $\ell \mapsto R_j^{\pm}(\ell)$ in the first block of $f$ does not have an accumulation of singularities at 0 in $\ell(V_j^\pm)$ ($j \in \mathbb{Z}$).

Indeed, if it had, then its asymptotic expansion $\hat{R}_1(\ell)$ in the power-logarithmic scale on $\ell(V_j^\pm)$ would be 0. Therefore, the algorithm for constructing the Fatou coordinate block-by-block by Taylor expansion of the Abel equation works as in the proof of the theorem in [17].

(ii) Fix one petal, say, $V_j^+$ ($j \in \mathbb{Z}$). Then

$$\Phi_j^+ \circ (\Psi_j^+)^{-1}(w) = w + o(w), \quad w \in \Psi_j^+(V_j^+), \quad |w| \to \infty. \quad (7.1)$$

\(^{24}\) $\Psi_j^+(z) \sim_{z \to 0} -\frac{1}{\alpha - 1}z^{-\alpha + 1}\ell^{-m}$ means that

$$\lim_{z \to 0, z \in V_j^+} \frac{\Psi_j^+(z)}{-(\alpha - 1)^{-1}z^{-\alpha + 1}\ell^{-m}} = 1 \quad \text{on} \quad V_j^+.$$
Indeed, $\Psi^+_j(z)$ is injective on $V^+_j$ sufficiently close to 0. This can easily be checked in the logarithmic chart $w = \log z$, where the petal $V^+_j$ is a convex set. Suppose that $\widetilde{\Psi}^+_j(w) := \Psi^+_j(e^w)$ is not injective in the image of $V^+_j$ in the logarithmic chart in a neighbourhood of the set $\{\Re w = -\infty\}$. By the complex Rolle’s theorem, there exist sequences $(w_n)$ and $(v_n)$ such that $\Re w_n, \Re v_n \to -\infty$ and such that $\Re((\widetilde{\Psi}^+_j)'(w_n)) = \Im((\widetilde{\Psi}^+_j)'(v_n)) = 0$. By the continuity of the real and imaginary parts of the Fatou coordinate, we have $\Psi^+_j(0) = 0$. This contradicts the condition $(\Psi^+_j)'(z) \sim z^{-\alpha \ell - m}$ as $z \to 0$ on $V^+_j$, which holds by the construction of the Fatou coordinate $\Psi^+_j$ in Theorem A.

Now from the asymptotic behaviour of $\Psi^+_j(z)$ we easily get the exact asymptotic behaviour of the inverse map $(\Psi^+_j)^{-1}(w)$ as $|w| \to \infty$, $w \in \Psi^+_j(V^+_j)$. It can be computed directly by letting

$$w = \Psi^+_j(z) = -(\alpha - 1)^{-1} z^{-\alpha + 1} \ell^{-m} \left(1 + o(1)\right),$$

taking the logarithm, and expressing $z$ in terms of $w$. Then the assertion (7.1) follows simply by composing with

$$\Phi^+_j(z) = -(\alpha - 1)^{-1} z^{-\alpha + 1} \ell^{-m} + o(z^{-\alpha + 1} \ell^{-m}), \quad z \to 0.$$

Since $\Psi^+_j(V^+_j)$ contains $\{w: \Re w > R\}$ for some $R > 0$, and since, as in the proof of Lemma 7.5 below, the map $\Phi^+_j \circ (\Psi^+_j)^{-1} - \id$ is periodic on this domain, we can extend it by periodicity to the whole of $\mathbb{C}$ and conclude, using Liouville’s theorem, that $\Phi^+_j - \Psi^+_j = C_j$, $C_j \in \mathbb{R}$.

### 7.2. Sectorially analytic normalizations.

Let

$$\tilde{\varphi} \in \tilde{S}^{\alpha,m,\rho}(V), \quad \tilde{\varphi}(z) = z + z\tilde{R}_0(\ell) + \text{h.o.b.},$$

where $\tilde{R}_0(\ell) \in \ell \mathbb{R}[[\ell]]$, be a block iterated integrally summable series with parameters $(\alpha, m, \rho) \in \mathbb{R}_{>1} \times \mathbb{Z} \times \mathbb{R}$ on a petal $V$ of opening $2\pi/(\alpha - 1)$ in the sense of Definition 6.8. Let $\tilde{h}_0(z)$ be as defined from $\tilde{R}_0(\ell)$ in (6.9) in Definition 6.8. Let $h_0(z)$ be the analytic germ on $V$ given by the analytic variant of the formula (6.9) for $\tilde{h}_0(z)$:

$$h_0(z) := \left(-\int_{\ell_0}^{\ell} \frac{e^{(\alpha-1)/\eta} d\eta}{\eta^{m+2}} - \frac{\alpha}{2} (\ell^{-1} - \ell_0^{-1}) + \left(\frac{m}{2} + \rho\right) (\ell_0^{-1} + \log \ell_0)\right)^{-1} \circ \left(\int_{\ell_0}^{\ell} \frac{e^{(\alpha-1)/\eta} d\eta}{T_0(\eta) \eta^2} - \frac{\alpha}{2} (\ell^{-1} - \ell_0^{-1}) + b (\ell_0^{-1} + \log \ell_0)\right), \quad z \in V.$$

(7.2)

Here $T_0$ is the log-Gevrey sum of the expansion $\tilde{T}_0$ in (6.9) on the cusp $\ell(V)$, and $\ell_0 \in \ell(V)$. Note that $h_0$ is unique up to the choice of a constant of integration $\ell_0 \in \ell(V)$.

Let $\tilde{\varphi} \circ \tilde{h}_0$ decompose as in Definition 6.8, as

$$\tilde{\varphi} \circ \tilde{h}_0 = \circ_{i \in \mathbb{N}} \tilde{\varphi}_i, \quad \tilde{\varphi}_i(z) = z + z^{\beta_i} \tilde{R}_i(\ell), \quad \text{(7.3)}$$
where the \( \hat{\varphi}_i(z) \) are elementary changes of coordinates, the exponents \( \beta_i > 1 \) are strictly increasing, either a finite sequence or tending to \( +\infty \), and the series \( \hat{R}_i(\ell) \) (\( i \in \mathbb{N} \)) belong to a sequence \( \hat{\mathcal{H}} \) of the form (6.10) of integrally summable series and algebraic combinations of them.

**Definition 7.2** (block iterated integral asymptotic expansion). Let \( \varphi(z) = z + o(z) \) be analytic on some open petal \( V \) of opening \( 2\pi/(\alpha - 1) \) for some \( \alpha > 1 \), and let the series \( \hat{\varphi} \in \hat{S}^{\alpha,m,\rho}(V) \) be block iterated integrally summable on \( V \) with respect to parameters \( (\alpha, m, \rho) \). Let \( \hat{h}_0 \in \hat{L}_{\text{id}}(\mathbb{R}) \), and let \( h_0 \) be the analytic transformation on \( V \) defined in (7.2). Let \( \hat{\mathcal{H}} \) be the sequence described above in the decomposition \( \hat{\varphi} \circ \hat{h}_0 \).

We say that the germ \( \varphi \), up to a precomposition with \( f_c \) (\( c \in \mathbb{R} \)), admits \( \hat{\varphi} \) as its **block iterated integral asymptotic expansion** on \( V \) with parameters \( (\alpha, m, \rho) \) if, for every constant of integration \( \ell_0 \in \ell(V) \) in (7.2) for \( h_0 \), and for every integral sum \( \mathcal{H}^{h_0} = \{ R_i(\ell): i \in \mathbb{N} \} \) of the sequence \( \hat{\mathcal{H}} \) on \( \ell(V) \), there exists a \( c \in \mathbb{R} \) such that \( (\varphi \circ f_c) \circ h_0 \) can be written as the following infinite asymptotic composition:

\[
(\varphi \circ f_c) \circ h_0 \sim \circ_{i \in \mathbb{N}} \varphi_i, \quad z \in V, \quad z \to 0,
\]

where \( \varphi_i(z) = z + z^{\beta_i} R_i(\ell) \). Here the phrase **asymptotic composition** \( \sim \) in (7.4) means that there exists a sequence \( (\gamma_n)_n \) of strictly increasing positive numbers tending to \( +\infty \) such that, for every \( n \in \mathbb{N} \),

\[
(\varphi \circ f_c) \circ h_0 = \varphi_1 \circ \cdots \circ \varphi_n + o(z^{\gamma_n}),
\]

that is,

\[
\varphi_n^{-1} \circ \cdots \circ \varphi_1^{-1} \circ ((\varphi \circ f_c) \circ h_0) = z + o(z^{\gamma_n}), \quad z \in V, \quad z \to 0.
\]

For the change \( \varphi_i(z) \) the series \( \hat{\varphi}_i(z) = z + z^{\beta_i} \hat{R}_i(\ell) \) (\( i \in \mathbb{N} \)) are the Poincaré asymptotic expansions, since \( R_i(\ell) \) is an integral sum of \( \hat{R}_i(\ell) \) of some length. This can be checked using Definition 6.1.

**Remark 7.3.** We do not claim uniqueness of a block iterated integral asymptotic expansion of a general analytic germ \( \varphi \) on \( V \). However, if \( \varphi \) is a normalization on \( V \) of a parabolic Dulac germ \( \bar{f} \) in a formal class \( (\alpha, m, \rho) \) that admits a block iterated integral asymptotic expansion, then its block iterated integral asymptotic expansion on \( V \) with parameters \( (\alpha, m, \rho) \) (if it exists) is unique up to a precomposition with \( \bar{f}_c \), where \( \bar{f}_c \) is the time-\( c \) map along trajectories of the \( (\alpha, m, \rho) \)-model field \( X_1 \) in (5.1). Indeed, every block iterated integral asymptotic expansion of an analytic normalization on \( V \) is a formal normalization of the series \( \bar{f} \), by (7.5) and since \( \varphi_i \) admits \( \hat{\varphi}_i \) as its Poincaré asymptotic expansion. On the other hand, a formal normalization of \( \bar{f} \) is by Proposition 6.11 unique up to a precomposition with \( \bar{f}_c \) (\( c \in \mathbb{R} \)).

**Proposition 7.4** (block iterated integral asymptotic expansions of analytic reductions to the formal normal form). Let \( f(z) = z - z^{\alpha} \ell^m + o(z^{\alpha} \ell^m) \), \( \alpha > 1 \), \( m \in \mathbb{Z} \),

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\( ^{25} \)This assumption is weaker than the pointwise convergence of the composition \( \circ_{i \in \mathbb{N}} \varphi_i \) to \( \varphi \circ h_0 \).

\( ^{26} \)The sums \( \varphi_i(\ell) \) depend on the choice of the constant of integration in \( h_0(z) \) and on the choice of the constants of integration in all the previous germs \( R_k(\ell) \), \( k \leq i \).
be a parabolic generalized Dulac germ on $\mathcal{R}_C$ in the formal class $(\alpha, m, \rho)$. Let $\hat{f}$ be its generalized Dulac expansion, and let $V_j^\pm (j \in \mathbb{Z})$ be the petals of $f$ on $\mathcal{R}_C$. Let $\varphi \in \hat{\mathcal{L}}^\text{id}(\mathbb{R})$ be the formal change of variables that reduces $\hat{f}$ to its $\hat{\mathcal{L}}(\mathbb{R})$-normal form $\hat{f}_1$, and is block iterated integrally summable (see Proposition 6.10).

(i) There exist on the open petals $V_j^\pm$ analytic changes of variables $\varphi_j^\pm(z) = z + o(z) (j \in \mathbb{Z})$ conjugating $\hat{f}$ to $f_1$, which, up to a precomposition with $f_c (c \in \mathbb{R})$, admit the formal change of variables $\varphi$ as their block iterated integral asymptotic expansion with parameters $(\alpha, m, \rho)$ as $z \to 0$ on $V_j^\pm$. Here different choices of the constant $c$ are connected with different choices of the constants in integral sums.

(ii) Let $\eta_j^\pm$ be any other analytic changes of variables on $V_j^\pm$, conjugating $f$ to $f_1$, with $\eta_j^\pm(z) = z + o(z)$, $z \in V_j^\pm$. Then there exist constants $c_j (j \in \mathbb{Z})$ such that $\eta_j^\pm = \varphi_j^\pm \circ f_c$. Moreover, up to a precomposition with $f_c (c \in \mathbb{R})$ the $\eta_j^\pm$ admit $\varphi$ as a block iterated integral asymptotic expansion with parameters $(\alpha, m, \rho)$ as $z \to 0$ on $V_j^\pm$.

Due to the importance (in view of Definition 2.4) of analytic normalization of parabolic Dulac germs on a standard quadratic domain, we reformulate the assertion (ii) of Proposition 7.4 as the separate Lemma 7.5 (whose proof is in the Appendix, §10), which shows that in Definition 2.4 we do not need any assumption about the asymptotic expansion of the normalization in the power-logarithmic scale, since it necessarily follows from the construction.

**Lemma 7.5** (analytic normalizations tangent to the identity map for parabolic Dulac germs necessarily admit power-logarithmic asymptotic expansions). Let $f$ be a normalized parabolic generalized Dulac germ in the formal class $(\alpha, m, \rho)$ and let $f_1$ be its $\hat{\mathcal{L}}^\text{id}(\mathbb{R})$-normal form, with formal conjugacy $\varphi \in \hat{\mathcal{L}}^\text{id}(\mathbb{R})$. Let $V_j^\pm$ be petals of $f$, as in Theorem A. Let $h_j^\pm$ be analytic conjugacies of $f$ to $f_1$ on $V_j^\pm (j \in \mathbb{Z})$, with $h_j^\pm(z) = z + o(z)$. Then the germs $h_j^\pm$ admit the formal conjugacy $\varphi \in \hat{\mathcal{L}}^\text{id}(\mathbb{R})$ as their common block iterated integral asymptotic expansion with parameters $(\alpha, m, \rho)$ as $z \to 0$ on $V_j^\pm$, up to a precomposition with $f_c$, $c_j \in \mathbb{R}$ (j \in \mathbb{Z})$.

The same statement holds for a $\hat{\mathcal{L}}_2^\text{id}(\mathbb{R})$-formal normal form, with formal conjugacy $\varphi \in \hat{\mathcal{L}}_2^\text{id}(\mathbb{R})$.

Proof of Proposition 7.4. We prove (i).

Step 1: existence of a petalwise analytic normalization $\varphi_j^\pm$ on $V_j^\pm$. The existence of analytic conjugating changes $\varphi_j^\pm$ on petals $V_j^\pm (j \in \mathbb{Z})$ of opening $2\pi/(\alpha - 1)$, where the petals are as in Theorem A, is proved by the existence (by construction) of an analytic Fatou coordinate $\Psi_j^\pm$ on $V_j^\pm$ in the proof of Theorem A in §8. Let $\Psi_0$ be the Fatou coordinate of the normal form $f_1$ on $V_j^\pm$, and for $f$ let $\Psi_j^\pm$ be the analytic Fatou coordinate on $V_j^\pm (j \in \mathbb{Z})$ constructed in the proof of Theorem A.

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27That is, such that $f \circ \varphi_j^\pm = \varphi_j^\pm \circ f_1$, $z \in V_j^\pm$.

28Unique only as a class, up to a precomposition with $f_c$.

29Here $o(z)$ holds uniformly on the petal, but may depend on $j$. 

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Then
\[ \Psi_j^\pm(z), \Psi_0(z) \sim_{z \to 0} -\frac{1}{\alpha - 1} z^{-\alpha + 1} \ell^{-m}. \]

Consequently, the map \( \varphi_j^\pm \) defined by
\[ \varphi_j^\pm(z) := (\Psi_j^\pm)^{-1} \circ \Psi_0(z) = z + o(z), \quad z \in V_j^\pm, \quad (7.6) \]
is a petalwise analytic normalization of \( f \).

In the definition (7.6), we might have chosen another Fatou coordinate of \( f \) differing from \( \Psi_j^\pm \) by some additive constants on petals. As in the final step of the proof of Lemma 7.5, this gives a conjugacy \( \hat{\varphi}_j^\pm = \varphi_j^\pm \circ f_{c_j} \) \( (j \in \mathbb{Z}) \) that differs from \( \varphi_j^\pm \) by a precomposition with \( f_{c_j}, c_j \in \mathbb{R} \).

**Step 2**: proof that there are block iterated integral asymptotic expansions of analytic normalizations. We now prove that any conjugacy \( \varphi_j^\pm \) tangent to the identity admits the block iterated integrally summable formal conjugacy \( \hat{\varphi} \in \hat{S}^{\alpha, m, \rho}(V_j^\pm) \) as its block iterated integral asymptotic expansion on \( V_j^\pm \) with parameters \( (\alpha, m, \rho) \), in the sense of Definition 7.2.

Recall that \( \hat{\varphi} \) decomposes uniquely (see (7.3)) as:
\[ \hat{\varphi} \circ \hat{h}_0 = \circ_{i \in \mathbb{N}} \hat{\varphi}_i, \]
where \( \hat{h}_0^{-1} \) is a formal normalization of \( \hat{f}_2 \) (defined in (6.13)) given by (6.14), with 0 taken as the constant of formal integration for uniqueness and with \( b \in \mathbb{R} \) such that the expression for \( \hat{h}_0^{-1} \) does not contain iterated logarithms. Moreover, the
\[ \hat{\varphi}_i(z) = z + z^{\beta_i} \hat{R}_i(\ell), \quad i \in \mathbb{N}, \quad (7.7) \]
with strictly increasing exponents \( \beta_i > 1 \), are blockwise elementary changes reducing \( \hat{f} \) to \( \hat{f}_2 \) and obtained as in Lemma 5.4, with 0 taken as the constant of formal integration in (5.3) for \( \hat{R}_i(\ell) \), in order to really get elementary changes of variables. Let \( \hat{\mathcal{H}} = \{ \hat{R}_i(\ell) : i \in \mathbb{N} \} \) be as described in (7.3).

On the other hand, let \( (\hat{h}_0^{j, \pm})^{-1}(z) \), given by the formula (7.2) for a fixed choice of the constant of integration \( \ell_0 \in \ell(V_j^\pm) \), be an analytic normalization of the germ \( f_2^{j, \pm} \) on \( V_j^\pm \). Compared to the formal formula (6.14), it is an analytic analogue of \( \hat{h}_0^{-1}(z) \). Here \( f_2^{j, \pm}(z), z \in V_j^\pm, \) are defined by (6.13):
\[ f_2^{j, \pm}(z) = \text{Exp} \left( \frac{z^{\alpha} T_0^{j, \pm}(\ell)}{1 + (\alpha/2)z^{\alpha - 1} T_0^{j, \pm}(\ell) + bz z^{\alpha - 1} T_0^{j, \pm}(\ell) \ell \, dz} \right) \cdot \text{id}, \]
where \( \hat{T}_0(\ell) \) is the first block of the generalized Dulac expansion \( \hat{f}_j \), and the \( T_0^{j, \pm}(\ell) \) are its log-Gevrey sums on \( \ell(V_j^\pm) \) \( (j \in \mathbb{Z}) \). Note that \( f_2^{j, \pm} \) is not necessarily a parabolic generalized Dulac germ, but only a collection of germs on petals which do not necessarily glue together on intersections of the petals.

Let \( \hat{\mathcal{H}}^{h_0^{j, \pm}} := \{ \hat{R}_i^{j, \pm}(\ell) : i \in \mathbb{N} \} \) be a fixed sequence of integral sums on the cusps \( \ell(V_j^\pm) \) for a fixed choice of the constants of integration \( \ell_0 \in \ell(V_j^\pm) \) in the expressions for \( h_0^{j, \pm}(z) \), and for fixed choices of the constants of integration \( \ell_0 \in \ell(V_j^\pm) \) in the
successive integral sums (in each analytic integral in the formula (5.3) defining the germs $R_i^{j,\pm}(\ell)$; see Definition 6.1). We now let

\[
\varphi_i^{j,\pm}(z) := z + z^{b_i} \widehat{R}_i(\ell), \quad i \in \mathbb{N}, \quad z \in V_j^{\pm}. \tag{7.8}
\]

The procedure is explained in more detail in the following algorithm for block-by-block changes reducing $f$ to $f_2$, carried out in each step formally and analytically in parallel.

**Description of the block-by-block algorithm.** For simplicity we do not indicate here the indices of petals, but just let $V := V_j^{\pm}$, for any petal. Similarly, $f_2 := f_2^{j,\pm}$, $\varphi_i := \varphi_i^{j,\pm}$, and $h_0 := h_0^{j,\pm}$ are analytic on $V$.

We construct the sequence $(\widehat{\varphi}_i)_{i \in \mathbb{N}}$ in (7.7) of formal elementary changes of variables and simultaneously the sequence $(\varphi_i)_{i \in \mathbb{N}}$ in (7.8) of analytic changes of variables on $V$ by blockwise eliminations that transform $f$ into $f_2$ on $V$.

0. Put $\widehat{1f} := \widehat{f}$ and $1f := f$.
1. Let $i \in \mathbb{N}$. By Lemma 5.4 we find the $i$th elementary change $\widehat{\varphi}_i(z) = z + z^{b_i} \widehat{R}_i(\ell)$, where

\[
\widehat{R}_i(\ell) = -e^{-\beta_i}\ell^m \int e^{\beta_i\ell} \ell e^{-2m-2 \widehat{T}_i(\ell)} d\ell. \tag{7.9}
\]

In the first step ($i = 1$), $\widehat{T}_1(\ell)$ is the difference of the series in $\ell$ in the second blocks (which are log-Gevrey on $V$) of $\widehat{f}$ and of $\widehat{f}_2$. Further, for $i \in \mathbb{N}$, $i > 1$, we compute

\[
\widehat{f}(z) := \widehat{\varphi}_{i-1} \circ i^{-1} f \circ \widehat{\varphi}_{i-1}^{-1}(z) = \widehat{f}_2(z) + (z^{\gamma_i} \widehat{T}_i(\ell) + \text{h.o.b.}).
\]

Here the $\gamma_i > \alpha$ strictly increase to $+\infty$, and $\widehat{T}_i(\ell)$ (like every other coefficient of the new series $\widehat{f}$) is an algebraic combination (with the operations $+, \cdot, d/d\ell$) of series integrally summable of length strictly less than $i$ (that is, an algebraic combination of series $\widehat{R}_j(\ell)$ appearing in previous elementary changes, $1 \leq j < i$, of coefficients of the series $i^{-1} f$, and of coefficients of the series $\widehat{f}_2$). The constant of formal integration is always taken to be 0, in order to get elementary, one-block changes of variables $\widehat{\varphi}_i$.

We put the sequence of $\widehat{R}_i(\ell)$ obtained in this formal step-by-step construction in the set $\widehat{R}$. The corresponding integral exponents are by (7.9) equal to $\beta_i$, and they are strictly increasing to $+\infty$ by the algorithm.

2. On the other hand, (7.9) can be simultaneously solved analytically on the $\ell$-cusp $\ell(V)$,

\[
R_i(\ell) = -e^{-\beta_i}\ell^m \int_*^\ell e^{\beta_i\eta} \eta^{-2m-2} T_i(\eta) d\eta, \quad \ell \in \ell(V). \tag{7.10}
\]

Here $*$ is 0 if $(\beta_i - \alpha, \text{ord}(\widehat{T}_i) - 2m - 1) > (0, 0)$ (lexicographically) in this step, and $\ell_0 \in \ell(V)$ otherwise (if the integrand is not bounded at 0). We remark that in the first finitely many steps we may choose $\ell_0$ arbitrarily, which changes the constant of integration. The integration path is not important as long as it stays in $\ell(V)$, since the integrand is analytic on $\ell(V)$ and the cusp is simply connected.
Since $\tilde{T}_i(\ell)$ is an algebraic expression (with respect to the operations $+, \cdot, d/d\ell$) of integrally summable series in $\tilde{\mathcal{H}}$ of lengths strictly smaller than $i$ (of coefficients of $i^{-1}f_1$, of $f_2$, and of the series $\tilde{R}_i(\ell)$ in all the previous elementary changes, $1 \leq j < i$), after we have chosen the constants of integration in the previous steps, it is uniquely summable by Proposition 6.5. Under the assumption that we have chosen some constants of integration for the germs $R_i(\ell)$ in (7.10) in all the previous steps $j < i$, this sum is uniquely determined and analytic on $V$, and we denote it by $T_i(\ell)$. Now we again freely chose the $i$th constant of integration in (7.10) in the $i$th step. This free choice is possible in the first finitely many steps, as long as $(\beta_i - \alpha, \text{ord}(\tilde{T}_i) - 2m - 1) \prec (0, 0)$; in later steps we always choose 0. So the sum $R_i(\ell)$ of the series $\tilde{R}_i(\ell)$ is unique up to a choice of the constants of integration in (7.10) in all the previous steps up to and including the $i$th. As in (6.12), we denote by $\mathcal{H}^{ho} = \{R_i(\ell) : i \in \mathbb{N}\}$ the sequence corresponding to one such choice of the constants of integration in all steps. It thus becomes one integral sum of $\tilde{\mathcal{H}}$.

We now let

$$\varphi_i(z) := z + z^{\beta_i}R_i(\ell).$$

The function $\varphi_i$ is analytic on $V$. In this way we get analytic changes $\varphi^{\mathcal{C}}_{j,\pm}(z)$ on each petal $V_j^\pm (j \in \mathbb{Z})$.

Note that the integration path in (7.10) does not matter. For any petal $V = V_j^\pm$ and any step of iteration, we may take any path lying inside the petal. Indeed, since $f$ is a generalized Dulac germ, all function coefficients in blocks of $f$ and $f_2$ are analytic functions on the corresponding $\ell$-cusps $\partial(V)$. Moreover, according to our algorithm the operations in the algebraic expression $\tilde{T}_i(\ell)$ do not include division. Therefore, the integrand in the formula (7.10) for $R_i(\ell)$, $i \in \mathbb{N}$, does not have any singularities on the $\ell$-cusps $\partial(V)$, which are simply connected.

The $\varphi^\ell_{j,\pm}$, $i \in \mathbb{N}$, in (7.8) obtained by the above algorithm are analytic on the petals $V_j^\pm (j \in \mathbb{Z})$, but in general they do not glue together to form globally defined analytic germs $\varphi^\ell(z)$ on $\mathcal{R}_C$. We also do not assert that the $^i f(z)$, for any $i > 1$, are analytic on $\mathcal{R}_C$ (they are at best analytic, petalwise).

Every elementary change $\varphi_i(z)$ is an analytic function on $V$, by (7.10). For $i$ such that $(\beta_i - \alpha, \text{ord}(\tilde{T}_i) - 2m - 1) \succ (0, 0)$ we choose a canonical method of integration

$$\int_0^\epsilon * dw,$$

so that the integral is a unique analytic function on $V$. Otherwise, we arbitrarily choose the initial point $\ell_0 \in \ell(V)$ in the integral $\int_{\ell_0}^\epsilon * dw$, so the integral is unique only up to a term of the form $C_i e^{-(\alpha - \beta_i) / \ell} \ell^m$, $C_i \in \mathbb{R}$. This ambiguity corresponds to adding an exponentially small term of the form $C_i e^{-(\alpha - \beta_i) / \ell} \ell^m$ to $\varphi_i(z)$, $C_i \in \mathbb{R}$, in all changes of variables until we come to the residual block $(\beta_i \leq \alpha)$.

Again let $V := V_j^\pm$. Fix an arbitrary choice of the constant of integration $\ell_0 \in \ell(V)$ in $h_0(z)$ (that is, take any analytic normalization $h_0^{-1}$ of $f_2$ on $V$) and any choice of integral sums $\mathcal{H}^{ho} = \{R_i(\ell) : i \in \mathbb{N}\}$ of $\tilde{\mathcal{H}}$ (choices of the constants of integration $\ell_0 \in \ell(V))$, in the above algorithm. In the above algorithm the germs $R_i(\ell) \in \mathcal{H}^{ho}$, $i \in \mathbb{N}$, are found by solving the corresponding Lie bracket equation...
for block-by-block eliminations. Now let

$$\varphi_i(z) := z + z^{\beta_i} R_i(\ell), \quad i \in \mathbb{N}. $$

Since in the algorithm we eliminate block after block, there exists a strictly increasing sequence of exponents $(\gamma_n)_n$, $\gamma_n > 1$, tending to $+\infty$ such that for every $n \in \mathbb{N}$

$$ (\varphi_1^{-1} \circ \cdots \circ \varphi_n^{-1}) \circ f \circ (\varphi_1 \circ \cdots \circ \varphi_n) = f_2 + o(z^{\gamma_n}) $$

$$ = h_0^{-1} \circ f_1 \circ h_0 + o(z^{\gamma_n}), \quad z \to 0, \quad (7.11) $$
on $V$. On the other hand, for any analytic normalization $\varphi$ of $f$ on $V$

$$ \varphi^{-1} \circ f \circ \varphi = f_1. \quad (7.12) $$

Putting (7.12) in (7.11) and defining $T_n := \varphi^{-1} \circ (\varphi_1 \circ \cdots \circ \varphi_n)$, we now get that

$$ T_n^{-1} \circ f_1 \circ T_n = h_0^{-1} \circ f_1 \circ h_0 + o(z^{\gamma_n}), \quad z \to 0, $$

so that

$$ h_0 \circ T_n^{-1} \circ f_1 \circ T_n \circ h_0^{-1} = f_1 + o(z^{\gamma_n}), \quad z \to 0. \quad (7.13) $$

Using the equality $f_c \circ f_1 = f_1 \circ f_c$ ($c \in \mathbb{R}$), and defining $r_n := T_n \circ h_0^{-1} - f_c$, $r_n(z) = O(z^{>1})$, $n \in \mathbb{N}$, we get from (7.13) that

$$ f_1 \circ (f_c + r_n) = (r_n + f_c) \circ f_1 + o(z^{\gamma_n}), $$

so that

$$ f_1 \circ (f_c + r_n) - f_1 \circ f_c = r_n \circ f_1 + o(z^{\gamma_n}). $$

Comparing the leading terms on both sides in the last equality, we conclude that

$$ r_n(z) = o(z^{\gamma_n - \alpha}), \quad z \to 0, \quad n \in \mathbb{N}. $$

Thus,

$$ \varphi \circ f_c = (\varphi_1 \circ \cdots \circ \varphi_n) \circ h_0^{-1} + o(z^{\gamma_n - \alpha}). $$

By Definition 7.2, this proves that $\varphi$ is the block iterated integral asymptotic expansion of any analytic normalization $\varphi$ of $f$ on $V$.

Proof of (ii): see Lemma 7.5.

Proposition 7.4 is proved. \(\square\)

We note that the existence of a power-logarithmic asymptotic expansion is not obvious, neither for a sectorial Fatou coordinate nor for a sectorial conjugacy of a parabolic generalized Dulac germ, as the following Example 7.6 shows. However, it can be verified if we assume the same asymptotic behaviour as in Propositions 7.1 and 7.4, (ii). See also Lemma 7.5.

Example 7.6 (a Fatou coordinate for a parabolic generalized Dulac germ which does not admit a sectional asymptotic expansion in the class $bL_\infty(\mathbb{R})$). In Theorem A and its proof in §8 we construct a holomorphic Fatou coordinate $\Psi$ of a parabolic generalized Dulac germ $f$ on an invariant petal $V$ of opening $2\pi/(\alpha - 1)$, with $\Psi$ admitting an integral sectional asymptotic expansion equal to the formal Fatou
coordinate $\hat{\Psi} \in \hat{L}_2^\infty(\mathbb{R})$ up to a constant. Here we define another holomorphic Fatou coordinate $\Psi_1$ on $V$ by

$$\Psi_1(z) := \Psi(z) + g_0(e^{2\pi i \Psi(z)}), \quad z \in V,$$

where $g_0$ is any analytic germ on the doubly punctured sphere (without the poles $0$ and $\infty$). For example, we can take

$$\Psi_1(z) := \Psi(z) + \sin(2\pi \Psi(z)), \quad z \in V,$$

or

$$\Psi_2(z) := \Psi(z) + ce^{2\pi i \Psi(z)}, \quad c \in \mathbb{R}, \quad z \in V.$$

Due to the unbounded exponential term $\sin(2\pi \Psi(z))$ or $ce^{2\pi i \Psi(z)}$, the functions $\Psi_{1,2}$ are Fatou coordinates of $f$ that do not admit sectional asymptotic expansions in the class $bL_\infty^2(\mathbb{R})$ on $V$ as $z \to 0$.

### 7.3. Examples on $\mathbb{R}_+$. By Propositions 7.1 and 7.4, there exists a Fatou coordinate $\Psi$, that is, a conjugacy $\varphi$, which is unique up to a simple transformation and analytic on a petal of opening $2\pi/(\alpha - 1)$, and which admits a sectional asymptotic expansion in $\hat{L}_\infty^\infty(\mathbb{R})$. Moreover, the expansion is then necessarily an (iterated) integral sectional expansion. However, for parabolic Dulac germs defined only on $\mathbb{R}_+$, as in [17], this is not the case.

Indeed, let $f$ be a parabolic Dulac germ on $\mathbb{R}_+$. We construct different Fatou coordinates analytic on $(0, d)$, $d > 0$, that admit sectional power-logarithmic asymptotic expansions in $\hat{L}_\infty^\infty(\mathbb{R})$. In [17] it is proved that for a parabolic Dulac germ $f$ there exists a unique, up to an additive constant, Fatou coordinate on $(0, d)$ that admits an integral sectional expansion as $x \to 0$. Moreover, up to a constant this expansion is equal to the formal Fatou coordinate $\hat{\Psi} \in \hat{L}_2^\infty(\mathbb{R})$.

In the next example we give two different Fatou coordinates of a parabolic Dulac germ on $\mathbb{R}_+$ with the same sectional asymptotic expansion in $\hat{L}_\infty^\infty(\mathbb{R})$ as $x \to 0$, but with respect to different section functions.

**Example 7.7.** Take a parabolic Dulac germ $f$ on $(0, d)$. Let $\Psi$ be its analytic Fatou coordinate on $(0, d)$ constructed algorithmically as in the theorem in [17]. It admits an integral sectional asymptotic expansion, unique up to an additive constant and equal to the formal Fatou coordinate $\hat{\Psi} = \sum_{i=1}^{\infty} \hat{T}_i(\ell)x^{\alpha_i} \in \hat{L}_2^\infty(\mathbb{R})$, $\hat{T}_i \in \hat{L}_\ell^\infty(\mathbb{R})$, with the sequence $(\alpha_i)_i$ strictly increasing to $+\infty$ or finite. Let us now define another Fatou coordinate $\hat{\Psi}_1$ on $(0, d)$ by

$$\hat{\Psi}_1(x) := \Psi(x) + \sin(2\pi \Psi(x)), \quad x \in (0, d).$$

The germ $\hat{\Psi}_1$ is obviously also a Fatou coordinate for $f$ and is analytic on $(0, d)$. Let

$$\Psi(x) = T_1(\ell)x^{\alpha_1} + T_2(\ell)x^{\alpha_2} + T_3(\ell)x^{\alpha_3} + \text{h.o.b.}$$

be the integral sectional asymptotic expansion for $\Psi$ constructed algorithmically in [17]. Here the $T_i(\ell)$ are analytic on $(0, d)$ and admit the Poincaré asymptotic
expansion \( \widehat{T}_i(\ell) \in \widehat{\mathcal{L}}_\ell^\infty(\mathbb{R}) \), \( i \in \mathbb{N} \). Obviously, \( \alpha_1 < 0 \). The Fatou coordinate \( \Psi_1 \) admits the same sectional asymptotic expansion \( \widetilde{\Psi} \) as \( x \to 0 \):
\[
\Psi_1(x) = \widehat{T}_1(\ell)x^{\alpha_1} + \widehat{T}_2(\ell)x^{\alpha_2} + \widehat{T}_3(\ell)x^{\alpha_3} + \text{h.o.b.,}
\]
but with the choice of sections
\[
\widehat{T}_1(\ell) = T_1(\ell) + \sin(2\pi \Psi(e^{-1/\ell}))e^{\alpha_1/\ell}, \quad \ell \in (0, d).
\]

Obviously, for the Poincaré power expansions of \( \widehat{T}_1 \) and \( T_1 \) as \( \ell \to 0 \), we have \( \widehat{T}_1(\ell) \neq T_1(\ell) \), since \( \sin(2\pi \Psi(e^{-1/\ell}))e^{\alpha_1/\ell} \) for \( \alpha_1 < 0 \) is exponentially small in comparison to \( \ell \) as \( \ell \to 0 \). This is due to the boundedness of the sine function on \( \mathbb{R} \). Note that on a sector of arbitrarily small opening in \( \mathbb{C} \) around the \( x \)-axis this is no longer true.

### 7.4. Sectorially analytic conjugacies.

In Proposition 7.4 of §7.2 we proved that there exists a unique, up to a precomposition with \( f_c \) (\( c \in \mathbb{R} \)), sectorially analytic reduction of a generalized Dulac germ \( f \) to its formal normal form \( f_1 \), and the formal reduction to normal form \( \widehat{\varphi} \in \widehat{\mathcal{L}}^{id}(\mathbb{R}) \) is the block iterated integral asymptotic expansion for it. In this subsection we derive similar results, just slightly more complicated, for the conjugacies conjugating two parabolic generalized Dulac germs.

We now introduce the following definition.

**Definition 7.8.** Let \( f \) and \( g \) be two normalized parabolic generalized Dulac germs which are \( \widehat{\mathcal{L}}(\mathbb{R}) \)-formally conjugate and let \( V_j^\pm \) be their common\(^{30} \) petals \( (j \in \mathbb{Z}) \). Let \( \varphi_j^\pm \) be analytic conjugacies of \( g \) and \( f \) on the petals \( V_j^\pm \) \( (j \in \mathbb{Z}) \). We say that the \( \varphi_j^\pm \) admit \( \widehat{\varphi} \) as their **generalized block iterated integral asymptotic expansion** if they can be decomposed as \( \varphi_j^\pm = \varphi_j^\pm \circ (\varphi_{j+1}^\pm)^{-1} \) \( (j \in \mathbb{Z}) \), where:

(a) \( \varphi_{j,1}^\pm \) and \( \varphi_{j,2}^\pm \) are analytic reductions of \( g \) and \( f \), respectively, to the formal normal form \( f_1 \) on \( V_j^\pm \);

(b) \( \varphi_{j,1}^\pm \) and \( \varphi_{j,2}^\pm \) admit formal conjugacies \( \widehat{\varphi}_1 \) and \( \widehat{\varphi}_2 \), respectively, as their block iterated integral asymptotic expansions.

We see that if for analytic reductions of \( f \) and \( g \) to normal form such a decomposition exists, then it is unique up to precomposition of \( \varphi_1 \) and \( \varphi_2 \) with \( f_c \) (\( c \in \mathbb{R} \)), because of the uniqueness of analytic reductions to normal form in Proposition 7.4. Furthermore, we showed in §7.2 that block iterated integral asymptotic expansions are always meant only up to such precompositions (Definition 7.2).

Now we can finally state the more general variant of Proposition 7.4 for sectorial analytic conjugacies between two parabolic generalized Dulac germs. The proof is similar so is omitted.

**Proposition 7.9** (generalized block iterated integral asymptotic expansions of sectorially analytic conjugacies). Let \( f \) and \( g \) be \( \widehat{\mathcal{L}}(\mathbb{R}) \)-formally conjugate normalized

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\(^{30}\)Since \( f \) and \( g \) belong to the same formal class, their invariants \( \alpha \) and \( m \) are the same, so by Theorem A, (i), they share common petals \( V_j^\pm \) of opening \( 2\pi/(\alpha - 1) \) \( (j \in \mathbb{Z}) \).
parabolic generalized Dulac germs on $\mathcal{R}_C$. Let $\hat{\varphi} \in \hat{\mathcal{S}}(V_j^\pm)$ be a formal change of variables\footnote{Unique up to an intermediate composition with the flow $\hat{f}_c$ ($c \in \mathbb{R}$), as described after the proof of Proposition 6.14.} conjugating $\hat{f}$ to $\hat{g}$. Then on the open petals $V_j^\pm$ there exist analytic changes of variables $\varphi_j^\pm(z) = z + o(z)$ ($j \in \mathbb{Z}$) conjugating $f$ to $g$ and admitting the formal change of variables $\hat{\varphi}$ as their generalized block iterated integral asymptotic expansion as $z \to 0$.

Moreover, any other sectorially analytic changes of variables on $V_j^\pm$ conjugating $f$ to $g$ and having the form $\eta_j^\pm(z) = z + o(z)$, $z \in V_j^\pm$, admit $\hat{\varphi}$ as their generalized block iterated integral asymptotic expansion as $z \to 0$.

Note that the second part of Proposition 7.9 is similar to Lemma 7.5, but for analytic conjugacies between two parabolic generalized Dulac germs. It proves Proposition 2.5 in §2. It states that, if two parabolic generalized Dulac germs in the same $\hat{L}(\mathbb{R})$-formal class are analytically conjugate in the weak sense of Definition 2.4, then their conjugacy has an asymptotic expansion in the class of generalized block iterated integrally summable expansions. This will be important in the proof of Theorem B, which states that $f$ and $g$ are analytically conjugate in the class of generalized block iterated integrally summable expansions if and only if they have the same horn maps.

**Proof of Proposition 2.5.** This is a direct corollary of Proposition 7.9 in the case where $f$ and $g$ are analytically conjugate. In this case all sectorial changes of variables $\varphi_j^\pm$ ($j \in \mathbb{Z}$) glue together to form an analytic conjugacy $\varphi$ on a standard quadratic domain. □

8. **Proof of Theorem A: the flower dynamics and the Fatou coordinate of a parabolic generalized Dulac germ**

Let

$$f(z) = z - az^\alpha \ell^m + o(z^\alpha \ell^m), \quad z \in \mathcal{R}_C, \quad a > 0, \quad \alpha > 1, \quad m \in \mathbb{Z},$$

be a parabolic generalized Dulac germ defined on a standard quadratic domain $\mathcal{R}_C$, $C > 0$ (see Definition 2.3).

We prove here Theorem A in §2. We first prove (i), which is stated separately in the more general Proposition 2.2 in §2. Proposition 2.2 states that the dynamics of a germ $f$ satisfying the uniform estimates (2.4) on $\mathcal{R}_C$ is a generalization of the Leau–Fatou ‘flower-like dynamics’ for parabolic analytic germs (see, for example, [13] for a description), to the case of a standard quadratic domain. More precisely, we show, using only the uniform estimate (2.5) of the asymptotic behaviour of $f$ and the fact that $f$ is analytic on $\mathcal{R}_C$, that there exist infinitely many overlapping attracting and repelling dynamically invariant petals $V_j^+$ and $V_j^-$ ($j \in \mathbb{Z}$), respectively, along the domain $\mathcal{R}_C$ (or along a smaller standard quadratic domain, in the sense of ‘germs’ of domains). These petals are symmetric with respect to a countable family of complex directions $1/(\alpha - 1)$ and $(-1)^{1/(\alpha - 1)}$ in $\mathcal{R}_C$. Each petal is of opening angle $2\pi/(\alpha - 1)$. Note here that $\alpha - 1 > 0$ is in general a real
number, and not necessarily an integer. The estimate (2.5) is important for proving that the size of invariant petals for the dynamics decreases to 0 no faster than prescribed by the shape of the standard quadratic domain.

The following proposition shows that the uniform estimate (2.5) in the definition of a parabolic generalized Dulac germ is automatically satisfied for parabolic Dulac germs, due to the uniform Dulac asymptotics.

**Proposition 8.1.** Let

\[ f(z) = z - az^\alpha \ell^m + o(z^\alpha \ell^m), \quad a > 0, \quad \alpha > 1, \quad m \in \mathbb{N}_0, \]

be a parabolic Dulac germ defined on a standard quadratic domain \( \mathcal{R}_C \). Then there exists a constant \( c > 0 \) uniform for the whole domain such that

\[ |f(z) - z + az^\alpha \ell^m| \leq c|z^\alpha \ell^{m+1}|, \quad z \in \mathcal{R}_C. \]

The proof is in the Appendix (§10).

Since on repelling petals we work with the germ \( f^{-1} \) instead of \( f \), we need the following proposition.

**Proposition 8.2.** (i) The formal parabolic generalized Dulac series form a group under composition.

(ii) The parabolic generalized Dulac germs on a standard quadratic domain\(^{32}\) form a group under composition.

The proof is in the Appendix (§10).

**Proof of Theorem A.** We adapt the methods developed for parabolic analytic germs in, for example, Theorem 2.4.1 in [13] or [12]. Estimates similar to those we use in the proof were obtained earlier in [17] for parabolic Dulac germs. We repeat here only the crucial steps.

**Proof of statement (i): petals and local discrete dynamics.**

**Proof of Proposition 2.2.** Consider the coordinate

\[ w = -\frac{1}{a(\alpha - 1)} z^{-\alpha+1} \ell^{-m}. \quad (8.1) \]

This is a multivalued function which is single-valued on \( \mathcal{R}_C \). It maps \( \mathcal{R}_C \) bijectively onto a neighbourhood of the infinity on the Riemann surface \( \mathcal{R} \) of the logarithm, which we will denote as \( w \in \tilde{\mathcal{R}} \). That is, \( \tilde{\mathcal{R}} \) is the image of \( \mathcal{R}_C \) in the \( w \)-chart. In this new variable, \( z \mapsto f(z) \) transforms into \( w \mapsto F(w) \), which is almost a translation by 1. Note that \( -[a(\alpha - 1)]^{-1} z^{-\alpha+1} \ell^{-m} \) is in essence the first term of the asymptotics of the Fatou coordinate for \( f \). Indeed, by the above change of variables, (2.4) transforms directly into:

\[ |F(w) - (w + 1)| \leq c(\log R)^{-1}, \quad c > 0, \quad (8.2) \]

\(^{32}\)‘Germified’ in the sense that the domain of the composition can be a subset of the intersection of standard quadratic domains of both elements, and itself containing a standard quadratic domain.
where \( w \in \overline{R} \). For the change of variables and for a detailed computational proof of (8.2), see the proofs of Proposition 6.2 and the formulae (6.15), (6.16) in [17]. Here \( R > 0 \) can be taken arbitrarily large, if \( |z| \) is made sufficiently small (that is, if \( |w| \) is taken sufficiently large, \( |w| > R \)). The constant \( c \) does not depend on the level of the Riemann surface \( \overline{R} \) of the logarithm, that is, there exists a global constant for all values of the argument as \( |z| \to 0 \), due to the uniform estimate (2.4).

We now find invariant domains for the dynamics of \( F(w) \) in the \( w \)-chart on \( \overline{R} \) and show that they correspond to open attracting petals at the origin in the dynamics of the germ \( f(z) \) on \( R_C \). Then, repeating the same arguments for the inverse germ \( f^{-1}(z) = z + az^\alpha \ell^m + o(z^\alpha \ell^m) \), which, by Proposition 8.2, satisfies the estimate \( |f^{-1}(z) - z - az^\alpha \ell^m| \leq d|z^\alpha \ell^{m+1}|, \ d > 0, \ z \in R_C, \) we get the repelling petals at the origin on \( R_C \).

As in [17], we consider an angle \( \alpha_R > 0 \) such that

\[
\sin(\alpha_R) := c(\log R)^{-1}.
\]

Obviously, \( \alpha_R \to 0 \) as \( R \to \infty \). By (8.2) there exists an \( R_0 > 0 \) such that, for every sufficiently large \( R > R_0 \), the sectors \( W_{R,\alpha_R} \cap \overline{R} \) at infinity (of radius \( |w| > R \) and opening \( 2\alpha_R \)), which are symmetric with respect to the directions \( 2k\pi, \ k \in \mathbb{Z} \), are invariant with respect to the dynamics. We note that \( \alpha_R \) does not change with the level of \( R_C \). Increasing the radius \( R \to \infty \), we get petals at infinity of opening \( 2\pi \) that are symmetric with respect to the directions \( 2k\pi, \ k \in \mathbb{Z} \), and have union covering the domain \( R_C \). Returning to the original variable \( z \), the result follows. Indeed, after returning to \( z \), the pre-images of the directions \( 2k\pi \) are tangential to \( 1^{-1/(\alpha-1)} \) at the point \( z = 0 \). We repeat the procedure for the inverse germ \( f^{-1} \), using Proposition 8.2. We denote the invariant petals on \( R_C \) constructed in this way by \( V_j^\pm \ (j \in \mathbb{Z}) \). Proposition 2.2 is proved. □

The statement (i) in Theorem A now follows by using Propositions 8.1 and 8.2, which prove that a parabolic generalized Dulac germ and its inverse satisfy the uniform estimates (2.4), and then by applying Proposition 2.2 directly.

Proof of statement (ii): the sectorial Fatou coordinates and their asymptotic expansion. We now repeat the procedure in §6 of [17] for constructing the Fatou coordinate of a parabolic Dulac germ, but for a parabolic generalized Dulac germ on a complex domain (on the invariant attracting petals constructed above).

The construction of the Fatou coordinate is carried out block-by-block using the Abel equation, formally and simultaneously in the sense of germs on the petals \( V_j^\pm \).

By the formal construction of the Fatou coordinate \( \Psi \) of \( \hat{f} \) in Proposition 6.7 below, the infinite part of the Fatou coordinate of a parabolic generalized Dulac germ is a sum of finitely many first blocks \( \Psi_1, \ldots, \Psi_n, \ n \in \mathbb{N}, \) given in view of (6.7) by integrals of the form

\[
\hat{\Psi}_i(z) = \int e^{-\beta_i/\ell} \cdot 2\hat{H}_i(\ell) \ d\ell,
\]

where the exponents \( \beta_i, \ i \in \{1, \ldots, n\}, \) are strictly increasing and such that \( (\beta_i, \text{ord}_\ell(\hat{H}_i)) \preceq (0, -1) \), and the series \( \hat{H}_i(\ell) \) is integrally summable of length 0
on $\ell(V_j^\pm)$. We note that only finitely many of the first blocks found consecutively from the Abel equation contain negative powers of $z$, since the exponents $\alpha_n$ in $\hat{f}(z)$ are strictly increasing and finitely generated.

Let $\Psi_j^{1,\pm}, \ldots, \Psi_j^{n,\pm}$ be holomorphic analogues of the series $\hat{\Psi}_1, \ldots, \hat{\Psi}_n$ on $V_j^\pm$ given by integrals of the form

$$\Psi_j^{i,\pm}(z) = \int_{\ell_0}^{\ell} e^{-\beta_i/w} H_i^{j,\pm}(w) \, dw,$$

where $\ell_0 \in \ell(V_j^\pm)$, and $H_i^{j,\pm}(\ell)$ is the 0-integral sum of the series $\hat{H}_i(\ell)$ on $\ell(V_j^\pm)$.

Now let $\Psi_j^{\pm} := \Psi_j^{1,\pm} + \cdots + \Psi_j^{n,\pm}$. This sum is well defined and holomorphic on $V_j^\pm$ ($j \in \mathbb{Z}$), and we call it the infinite part of the analytic Fatou coordinate $\Psi_j^\pm$ on $V_j^\pm$. Different choices of the initial point $\ell_0 \in V_j^\pm$ just correspond to different additive real constants in $\Psi_j^{\pm}$. Therefore, $\Psi_j^{\pm}$ is a holomorphic function on $V_j^\pm$ that is unique up to an additive constant.

The Abel equation

$$\Psi_j^+(f(z)) - \Psi_j^+(z) = 1, \quad z \in V_j^+, \quad \Psi_j^-(f(z)) - \Psi_j^-(z) = 1, \quad z \in V_j^-,$$

now becomes the equation

$$R_j^+(f(z)) - R_j^+(z) = \delta_j^+(z), \quad z \in V_j^+, \quad \delta_j^-(z) = O(z^m), \quad z \in V_j^-,$$

where $\Psi_j^\pm = \Psi_j^{\pm} + R_j^\pm$ ($j \in \mathbb{Z}$). To prove the existence (and uniqueness) of a petalwise analytic Fatou coordinate $\Psi_j^\pm$, we still need to prove the existence (and uniqueness) of holomorphic functions $R_j^\pm(z) = o(1)$ on $V_j^\pm$ satisfying (8.3).

We show as in [17] that

$$\delta_j^+(z) = O(z^\alpha L^{n+2}), \quad z \in V_j^+, \quad \delta_j^-(z) = O(z^\alpha L^{n+2}), \quad z \in V_j^-,$$

where the function $\delta_j^+(z) := 1 - \Psi_j^{\pm}(f(z)) + \Psi_j^{\pm}(z)$ is analytic on $V_j^\pm$. Indeed, by continuation of the formal algorithm in Proposition 6.7, $\bar{R}(z)$ is a power-logarithmic transseries, and is infinitesimally small, so that the order of its lowest-order monomial is at least $\ell$. Since $\bar{R}(z)$ is the asymptotic expansion of the $R_j^\pm(z)$ as $z \to 0$, we conclude from the Cauchy integral formula that $(R_j^\pm)'(z) = O(\ell^2/z)$. Therefore,

$$R_j^+(f(z)) - R_j^+(z) \sim (R_j^+)'(z) \cdot (-z^\alpha L^n) = O(z^\alpha \ell^{n+2}) \quad \text{in} \quad V_j^+ \quad \text{as} \quad z \to 0.$$

We now prove the existence of a (unique) analytic map $R_j^\pm = o(1)$ on $V_j^\pm$ satisfying (8.3). We prove it for the attracting petals $V_j^+ (j \in \mathbb{Z})$. On the repelling petals we work with the inverse germ $f^{-1}$, since by the Abel equation $\Psi_j := -\hat{\Psi}_{j-1}$. Consider again the $w$-chart, as defined in (8.1) in the proof of part (i). Using (8.2), we get the estimate

$$|F^w(w) - w - n| \leq cn(\log R)^{-1},$$

33We may assume that $R_j^\pm(z) = o(1)$ as $z \to 0$ on $V_j^\pm$, since we have eliminated the infinite part of the Fatou coordinate, and we require the analytic Fatou coordinate to have in the power-logarithmic scale an asymptotic expansion equal to the formal Fatou coordinate.
so that
\[|F^{on}(w)| \geq C_R n, \quad C_R > 0, \quad w \in W_{R,\alpha_R} \cap \tilde{\mathcal{R}}, \quad (8.5)\]
where $\tilde{\mathcal{R}}$, $\alpha_R$, and the domain $W_{R,\alpha_R}$, $R > R_0$, are as in the proof of (i). Returning to the variable $z$, we get that
\[|f^{on}(z)| \leq D_R n^{-1/(\alpha-1)}|\ell(F^{on}(w))|^{-m/(\alpha-1)}, \quad w = -[a(\alpha - 1)]^{-1}z^{-\alpha+1}\ell^{-m}, \quad D_R > 0, \quad (8.6)\]
on the sectors in $\mathcal{R}_C$ which are the pre-images of $W_{R,\alpha_R} \cap \tilde{\mathcal{R}}$ under the map $w = -[a(\alpha - 1)]^{-1}z^{-\alpha+1}\ell^{-m}$. The union of these sectors forms the petal $V_j^+$. Note that $\alpha_R$, $C_R$, and $D_R$ are the same for all petals $V_j^\pm$, so that (8.6) holds uniformly on the levels $j \in \mathbb{Z}$ of the Riemann surface, on subsectors of the same opening of the petals $V_j^+$.

Under the assumption that the $R_j^+(z)$ have an asymptotic behaviour as $z \to 0$, we have $R_j^+(z) = o(1)$ as $z \to 0$ on $V_j^+$, up to an additive constant. Iterating (8.3) and using the indicated relation $R_j^+(z) = o(1)$, we get that $R_j^+$ is necessarily of the form
\[R_j^+(z) = -\sum_{i=1}^{\infty} \delta(f^{oi}(z)), \quad z \in V_j^+, \quad (8.7)\]
where $\delta(z) = O(z^{\alpha-1}\ell^{m+2})$, uniformly on all petals. By (8.6), the series converges uniformly on each subsector which is a pre-image of $W_{R,\alpha_R} \cap \tilde{\mathcal{R}}$. Indeed, using (8.4) and (8.6), we conclude that
\[|\delta(f^{oi}(z))| \leq C_R |f^{oi}(z)|^{\alpha-1}|\ell(f^{oi}(z))|^m + 2 \leq D_R i^{-1}|\ell(F^{oi}(w))|^{-m}|\ell(f^{oi}(z))|^m + 2. \]
Since $f^{oi}(z) = h^{-1} \circ F^{oi}(w)$, where $h(z) := -[a(\alpha - 1)]^{-1}z^{-\alpha+1}\ell^{-m}$, there exists a constant $c > 0$ such that $|\ell(f^{oi}(z))|^m + 2 \leq c|\ell(F^{oi}(w))|^m + 2$. By (8.5), there exist $d_R > 0$ and $d'_R > 0$ such that
\[|\delta(f^{oi}(z))| \leq d_R i^{-1}|\ell(F^{oi}(w))|^2 \leq d'_R i^{-1}(-\ell(i))^2, \quad i \geq 2. \]

By the Weierstrass theorem, the series (8.7) converges to an analytic function $R_j^+$ on $V_j^+$ ($j \in \mathbb{Z}$). This defines the analytic Fatou coordinate $\Psi_j^+ := \Psi^{i,\pm} + R_j^+$ on $V_j^+$.

It is left to prove that the analytic Fatou coordinate constructed on each petal as the limit of the uniformly convergent series (8.7) admits an integral sectional asymptotic expansion which is, up to an additive constant, equal to the formal Fatou coordinate $\hat{\Psi} \in \mathcal{L}_2^\infty(\mathbb{R})$. We argue as in [12] and as in §6.1.4 in [17], where the Fatou coordinate of a parabolic Dulac germ on the real line was considered. We continue to subtract blocks (analytic analogues of formal blocks) from the Fatou coordinate just as we did with $\hat{\Psi}$, at the beginning of the proof. We find the asymptotics after the subtraction of each block, using the modified Abel equation
\[h(f(z)) - h(z) = \delta(z), \]
where $\delta(z) = O(z^\gamma)$ and $\gamma$ tends to $+\infty$ as we subtract blocks. Therefore, we use the following auxiliary proposition, whose proof is in the Appendix (§10).
Proposition 8.3 (generalization of Proposition 6.2 in [17] to the complex domain; see also [12]). Let \( f \) be a parabolic generalized Dulac germ on \( R_C \) having the form

\[
f(z) = z - az^\alpha \ell^m + o(z^\alpha \ell^m), \quad a > 0, \quad \alpha > 1, \quad m \in \mathbb{Z}.
\]

Let \( \delta \) be a holomorphic germ \(^{34}\) on an attracting petal \( V_j^+ \) (\( j \in \mathbb{Z} \)), with the form \( \delta(z) = O(z^\gamma \ell^r) \), \( \gamma > 0, \ r \in \mathbb{Z}, \ (\gamma, r) \succeq (\alpha - 1, m + 2) \). Then the series

\[
h_j(z) := -\sum_{k=0}^{\infty} \delta(f^\circ k(z)) \quad (8.8)
\]

converges uniformly on each compact subsector \( W \) of the attracting petal \( V_j^+ \), and thus defines a holomorphic function \( h_j \) on \( V_j^+ \). Moreover, for every small \( \delta > 0 \) such that \( \gamma - (\alpha - 1) - \delta > 0 \), there exists a \( C_W > 0 \) such that

\[
|h_j(z)| \leq \begin{cases} 
C_W \ell \ell(|z|)^{r-m-1}, & \gamma = \alpha - 1, \ r \geq m + 2, \\
C_W |z|^{\gamma-(\alpha-1)-\delta}, & \gamma > \alpha - 1,
\end{cases} \quad (8.9)
\]

on every subsector \( W \) of the attracting petal \( V_j^+ \).

We now apply Proposition 8.3 on the right-hand side \( \delta(z) \) of the Abel equation and use the estimate (8.9) to prove the asymptotic expansion. Moreover, each block in the Fatou coordinate clearly has a Poincaré asymptotic expansion corresponding to the associated formal block.

Finally, the proof of uniqueness up to a constant of the Fatou coordinate which is analytic on each petal and admits the integral sectional asymptotic expansion

\[ \bar{\Psi} \in \hat{L}^\infty_2(\mathbb{R}) \]

is given in Proposition 7.1.

Proof of (iii): the sectorial conjugacies and their asymptotic expansion. This is proved in Proposition 7.4, up to the first change of variables given by the homothety \( \varphi_0(z) = a^{-1/(\alpha-1)} z \).

Theorem A is proved. \( \square \)

9. Proof of Theorem B: moduli of analytic classification for parabolic generalized Dulac germs

For simplicity we confine ourselves to parabolic generalized Dulac germs with \( a = 1 \) and \( \alpha = 2 \):

\[ f(z) = z - z^2 \ell^m + o(z^2 \ell^m), \quad z \in R_C, \ m \in \mathbb{Z}. \]

This simplifies the computations a bit, since there is then only one petal (attracting or repelling) on each level of the Riemann surface \( R_C \). It is additionally justified by the following proposition.

Proposition 9.1. Two parabolic generalized Dulac germs

\[ f(z) = z - z^\alpha \ell^m + o(z^\alpha \ell^m) \]

and

\[ g(z) = z - z^\alpha \ell^m + o(z^\alpha \ell^m) \]

\(^{34}\)Obtained by blockwise subtractions in the Abel equation as described above.
are analytically conjugate on a standard quadratic domain if and only if the parabolic generalized Dulac germs

\[ \tilde{f} = h_{\alpha}^{-1} \circ f \circ h_{\alpha} = z - z^2 \ell^m + o(z^2 \ell^m) \]

and

\[ \tilde{g} = h_{\alpha}^{-1} \circ g \circ h_{\alpha} = z - z^2 \ell^m + o(z^2 \ell^m), \]

are analytically conjugate on the standard quadratic domain, where \( h_{\alpha}(z) := (\alpha - 1)^{-m/(\alpha - 1)} z^{1/(\alpha - 1)}. \)

**Proof.** Let \( h_{\alpha}(z) := (\alpha - 1)^{-m/(\alpha - 1)} z^{1/(\alpha - 1)}. \) This is an analytic bijection from a standard quadratic domain to a standard quadratic domain. It is easy to check that \( \tilde{f} \) and \( \tilde{g} \) keep the generalized Dulac expansion property, that is, they are again parabolic generalized Dulac germs. To a conjugation \( \phi \) between \( f \) and \( g \) there corresponds the conjugation \( \tilde{\phi} = h_{\alpha}^{-1} \circ \phi \circ h_{\alpha} \) between \( \tilde{f} \) and \( \tilde{g} \), and it is obviously parabolic. Moreover, \( f \) and \( g \) are \( \mathcal{L}(\mathbb{R}) \)-formally conjugate if and only if \( \tilde{f} \) and \( \tilde{g} \) are. \( \square \)

**Construction of moduli (proof of Theorem B).** Let \( f(z) = z - z^2 \ell^m + o(z^2 \ell^m) \) be a parabolic generalized Dulac germ on a standard quadratic domain \( \mathcal{R}_C \).

Let \( V_j^- \) denote the maximal repelling petals for the dynamics of \( f \) in \( \mathcal{R}_C \). They are constructed in Theorem A, (i), and they include points with arguments in the intervals \((2(j - 1)\pi, 2j\pi) \) \( (j \in \mathbb{Z}). \) By the proof of Theorem A, (i), these petals, along the length of the standard quadratic domain, have the same shape (due to the uniform estimates for \( \alpha_R \)), only getting smaller as dictated by the decrease of the radius of points of the standard quadratic domain. Let \( r_j^{0,-} \) and \( r_j^{\infty,-} \) denote the radii of the petal \( V_j^- \) measured along the directions \( 2j\pi - \pi/2 \) and \( 2j\pi - \pi/2, \) respectively \( (j \in \mathbb{Z}). \) Similarly, let \( V_j^+ \) denote the maximal attracting petals of points with arguments in the intervals \( ((2j - 1)\pi, (2j + 1)\pi), \) and let \( r_j^{\infty,+} \) and \( r_j^{0,+} \) denote the radii of the petal \( V_j^+ \) measured along the directions \( 2j\pi - \pi/2 \) and \( 2j\pi + \pi/2, \) respectively \( (j \in \mathbb{Z}). \) We now define

\[ r_{0,j+1}^{+1} := \min\{r_{j+1}^{0,-}, r_{j+1}^{0,+}\} \quad \text{and} \quad r_{\infty}^j := \{r_{j}^{\infty,-}, r_{j}^{\infty,+}\}, \quad j \in \mathbb{Z}. \]

Note that the radii \( r_{0,j+1}^{+1} \) and \( r_{\infty}^j \) \( (j \in \mathbb{Z}) \) correspond to the radii of points in the intersections \( V_0^{j+1} \) and \( V_{\infty}^j \) of consecutive petals (defined in (9.2) below) and lying on the axes of symmetry of these intersections with arguments \( 2j\pi + \pi/2 \) and \( 2j\pi - \pi/2. \)

Since a parabolic generalized Dulac germ is defined in any case on a standard quadratic domain, there exist \( C > 0 \) and \( K > 0 \) such that

\[ r_j^0, r_{\infty}^j \geq Ke^{-C\sqrt{|j|}}, \quad j \in \mathbb{Z}. \]  \hspace{1cm} (9.1)

That is, the rate of decrease of the radii \( r_j^0 \) and \( r_{\infty}^j \) is bounded below by the rate of decrease of the radii of points with arguments \( 2j\pi + \pi/2 \) and \( 2j\pi - \pi/2 \) in a standard quadratic domain as \( |j| \rightarrow \infty \) (see (2.1)). This means that the radii of the petals do not tend to zero too quickly along levels of \( \mathcal{R}_C. \)
Let \( \hat{\Psi} \in \hat{L}_2(\mathbb{R}) \) be the formal Fatou coordinate of the generalized Dulac asymptotic expansion \( \hat{f} \) of \( f \). Let \( \Psi_j^\pm (j \in \mathbb{Z}) \) be sectorial analytic Fatou coordinates admitting the integral sectional asymptotic expansion \( \hat{\Psi} \) as \( z \to 0 \) on the corresponding petals \( V_j^\pm (j \in \mathbb{Z}) \), as in Theorem A, (ii). They are unique up to additive constants.

Denote by

\[
V_j^{i+1} := V_{j+1}^i \cap V_j^+ \quad \text{and} \quad V_j^i := V_j^- \cap V_j^+,
\]

the intersections of attracting and repelling petals, of opening \( \pi \) (see Fig. 4). The petals \( V_0^j \) and \( V_\infty^j \) satisfy the lower bound \( (9.1) \) on their radii \( r_0^j \) and \( r_\infty^j \).

The construction of horn maps now mimics the construction of Voronin [5], [22] for parabolic analytic germs.

At each intersection of petals, the difference \( \Psi_+^j - \Psi_-^j \) of suitable realizations of the Fatou coordinate is constant along the closed trajectories of \( f \). We represent the space of trajectories of each petal by a doubly punctured Riemann sphere, using the composition of the Fatou coordinate on the petal with the exponential map. Each trajectory corresponds to one point of the sphere. The closed trajectories correspond to points in punctured neighbourhoods of the poles 0 and \( \infty \). The germs \( (h_0^j, \tilde{h}_\infty^j)_{j \in \mathbb{Z}} \) below are defined on punctured neighbourhoods of 0 and \( \infty \), respectively, and represent the horn maps of \( f \). They map the corresponding closed orbits on neighbouring spheres, as dictated by the dynamics of \( f \) on the intersection of petals, where the trajectories are closed:

\[
h_0^j(t) := \exp \left\{ -2\pi i \Psi_{j-1}^+ \circ (\Psi_j^-)^{-1} \left( -\frac{\log t}{2\pi i} \right) \right\}, \quad t \approx 0,
\]

\[
\tilde{h}_\infty^j(t) := \exp \left\{ -2\pi i \Psi_j^- \circ (\Psi_j^+)^{-1} \left( -\frac{\log t}{2\pi i} \right) \right\}, \quad t \approx \infty,
\]

\[
h_\infty^j(t) := \exp \left\{ 2\pi i \Psi_j^- \circ (\Psi_j^+)^{-1} \left( \frac{\log t}{2\pi i} \right) \right\}, \quad t \approx 0,
\]

Figure 4. The position of petals along a standard quadratic domain on the Riemann surface of the logarithm.
for \( j \in \mathbb{Z} \). By construction, \( h^j_0 \) and \( h^j_{\infty} \) \((j \in \mathbb{Z})\) are analytic germs of diffeomorphisms defined on punctured neighbourhoods of 0 and tending to 0 as \( t \to 0 \). Therefore, they can be analytically extended to 0 by Riemann’s theorem on removable singularities.

In this way we get an infinite necklace of Riemann spheres, indexed by \( \mathbb{Z} \) and connected by analytic diffeomorphisms in neighbourhoods of their poles. Each sphere is connected at one pole with the preceding sphere, and at the other pole with the following sphere (see Fig. 5).

![Figure 5. The necklace of spheres and the horn maps of a parabolic generalized Dulac germ.](image)

The maximal radii of convergence of \( h^j_0 \) and \( h^j_{\infty} \) \((j \in \mathbb{Z})\) are \( R_j \) such that

\[
|t| < R_j \iff \left| (\Psi^j_\pm)^{-1} \left( \pm \frac{\log t}{2\pi i} \right) \right| < r_j, \tag{9.4}
\]

where the \( r_j \) are as in (9.1) (their rate of decrease is prescribed by the shape of the standard quadratic domain). Since \( \Psi^j_\pm(z) \sim -z^{-1} \ell^{-m} \) uniformly\(^{35} \) with respect to \( j \in \mathbb{Z} \) as \( z \to 0 \), we get that

\[
\left| (\Psi^j_\pm)^{-1} \left( \pm \frac{\log t}{2\pi i} \right) \right| \sim \frac{1}{-\log(|t|) \log^m(-\log |t|)}, \quad t \to 0,
\]

uniformly with respect to \( j \in \mathbb{Z} \). From (9.4) we get that the \( R_j \) are such that the set

\[
\left\{ R_j \exp \left\{ \frac{C}{r_j (-\log r_j)^m} \right\} : j \in \mathbb{Z} \right\}
\]

\(^{35}\)By the uniform estimate (2.5) for \( f \) (see [15], Lemma 5.2).
is bounded from above and below by positive constants. The estimate (9.1) of the rate of decrease of \( r_j \) now gives the estimate below for the rate of decrease of \( R_j \) as \( j \to \infty \). There exist constants \( C, K, K_1 > 0 \) such that

\[
R_j \geq K_1 \exp\{-Ke^{C\sqrt{|j|}}(\sqrt{|j|})^m\}, \quad j \in \mathbb{Z}.
\]

Equivalently, there exist (some other) constants \( C, K, K_1 > 0 \) such that

\[
R_j \geq K_1 \exp\{-Ke^{C\sqrt{|j|}}\}, \quad j \in \mathbb{Z}.
\]

Let us now justify the identifications (3.2) in Theorem B. With this equivalence relation, the horn maps become equivalence classes. Indeed, it is easy to see that the addition of constant terms to the Fatou coordinates of a parabolic generalized Dulac germ \( f \) on petals results in change of its horn maps as described in (3.2). Alternatively, this can be considered as reparametrizations of doubly punctured spheres fixing the poles. (Möbius transforms fixing the poles are just homotheties.) Moreover, we can always restrict a parabolic generalized Dulac germ \( f \) to a smaller standard quadratic domain, resulting in a change in the asymptotics of the radii of convergence described in (3.1). However, the character of the asymptotics remains the same.

**Proof of Theorem B.** Suppose that \( f \) and \( g \) are two analytically conjugate normalized parabolic generalized Dulac germs on some standard quadratic domain \( \mathcal{R}_C \). Then there exists a global conjugacy \( \varphi(z) = z + o(z) \) tangent to the identity and admitting the formal conjugacy \( \hat{\varphi} \in \mathcal{L}(\mathbb{R}) \) as its generalized block iterated integral asymptotic expansion (see Proposition 7.9). Since \( f \) and \( g \) belong to the same \( \mathcal{L}(\mathbb{R}) \)-formal class, we can identify their petals. Let \( \hat{\Psi}_f, \hat{\Psi}_g \in \mathcal{L}_2^\infty(\mathbb{R}) \) be the formal Fatou coordinates for \( f \) and \( g \), respectively, which exist and are unique (up to an additive constant) by Proposition 6.7. Using Theorem A, (ii), denote by \( f^j \hat{\Psi}_\pm \) the analytic Fatou coordinates of \( f \) on the petals \( V_j^\pm \), with the formal Fatou coordinate \( \hat{\Psi}_f \) as their integral asymptotic expansion as \( z \to 0 \) on \( V_j^\pm \). Then by the Abel equation, the compositions \( f^j \hat{\Psi}_\pm \circ \varphi \) are analytic Fatou coordinates for \( g \) on the petals \( V_j^\pm \), with formal Fatou coordinate \( \hat{\Psi}_g \) as their integral asymptotic expansion. By Theorem A, (ii), such Fatou coordinates for \( g \) are unique on each petal, up to a constant. Therefore, there exists a choice of Fatou coordinates for \( f \) and \( g \) on petals (that is, of additive constants) such that

\[
(9^j \hat{\Psi}_j^+) \circ (9^j \hat{\Psi}_j^-)^{-1} = (f^j \hat{\Psi}_j^+) \circ \varphi \circ \varphi^{-1} \circ (f^j \hat{\Psi}_j^-)^{-1} = (f^j \hat{\Psi}_j^+) \circ (f^j \hat{\Psi}_j^-)^{-1}, \quad j \in \mathbb{Z},
\]

and

\[
(9^j \hat{\Psi}_j^-) \circ (9^j \hat{\Psi}_j^+)^{-1} = (f^j \hat{\Psi}_j^-) \circ \varphi \circ \varphi^{-1} \circ (f^j \hat{\Psi}_j^+)^{-1} = (f^j \hat{\Psi}_j^-) \circ (f^j \hat{\Psi}_j^+)^{-1}, \quad j \in \mathbb{Z},
\]

on the corresponding images of the intersections \( V_j^\pm \) of petals \( (j \in \mathbb{Z}) \). Thus, the horn maps given by (9.3) are equal. Suppose now that \( f \) and \( g \) have the same (up to equivalence) horn maps. We can take the petals to be common for them, since \( f \) and \( g \) belong to the same formal class. By Theorem A, on the corresponding petals \( V_j^\pm \) there exist analytic Fatou coordinates \( f^j \hat{\Psi}_\pm \) for \( f \) and \( g \).
admitting the formal Fatou coordinates as their integral asymptotic expansions. On each petal we define an analytic conjugacy function by

$$\varphi_j^\pm(z) := (f^\Psi_j^\pm)^{-1} \circ (g^\Psi_j^\pm)(z), \quad z \in V_j^\pm.$$ 

We show that in the Fatou coordinates for \( f \) and \( g \) on the petals the constants can be chosen so that the \( \varphi_j^\pm \) glue together analytically on the intersections \( V_0^j \) and \( V_0^- \) of consecutive repelling and attracting petals to form a global analytic conjugacy germ \( \varphi \) on the standard quadratic domain. But this is exactly ensured by the equality of the horn maps. Theorem B is proved. □

**Proposition 9.2** (symmetry of horn maps). Let \( f(z) = z - z^2 \ell^m + o(z^2 \ell^m) \) be a parabolic generalized Dulac germ on a standard quadratic domain, and let \((h_0^\pm, h_\infty^\pm)_{j\in\mathbb{Z}}\) be its sequence of horn maps. Then up to the identifications (3.2)

\[(h_0^{j+1})^{-1}(t) \equiv h^j_\infty(\overline{t}), \quad j \in \mathbb{Z}.\] (9.5)

That is, the necklace of spheres is symmetric with respect to the real axis. The horn maps with negative indices are symmetric in the sense of (9.5) to the horn maps with positive indices.

**Proof.** Let \( f \) be a normalized parabolic generalized Dulac germ on a standard quadratic domain \( \mathcal{R}_C \). Then \( f(\{\arg(z) = 0\} \cap \mathcal{R}_C) \subseteq \{\arg(z) = 0\} \cap \mathcal{R}_C \) and \( f \) is holomorphic on \( \mathcal{R}_C \). By the Schwarz reflection principle, \( f(z) = \overline{f(\overline{z})} \) on the whole of \( \mathcal{R}_C \). It is then easy to see from the analyticity of the Fatou coordinate and from the Abel equation that

\[
\Psi_{-j+1}^- = K \circ \Psi_j^- \circ K, \quad z \in V_{-j+1}^-, \quad j \in \mathbb{N},
\]

\[
\Psi_{-j}^+ = K \circ \Psi_j^+ \circ K, \quad z \in V_{-j}^+, \quad j \in \mathbb{N},
\]

\[
\Psi_0^+|_{V_0^\infty} = K \circ \Psi_0^-|_{V_0^\infty} \circ K,
\] (9.6)

where \( K(z) = \overline{z} \) is the complex conjugation on the Riemann surface of the logarithm. By (9.3) and (9.6), (9.5) follows. More precisely, by (9.3) and (9.6),

\[
(h_0^{-j+1})^{-1}(t) = \exp \left\{ -2\pi i \Psi_{-j+1}^- \circ (\Psi_j^+)^{-1} \left( -\frac{\log t}{2\pi i} \right) \right\},
\]

\[
= \exp \left\{ -2\pi i K \circ \Psi_j^- \circ (\Psi_j^+)^{-1} \circ K \left( -\frac{\log t}{2\pi i} \right) \right\}, \quad t \in (\mathbb{C}, 0).
\]

Since \( K \left( -\frac{\log t}{2\pi i} \right) = \frac{\log \overline{t}}{2\pi i}, \) \( t \in (\mathbb{C}, 0) \), and \( K(e^{-2\pi i z}) = e^{2\pi iz}, \) \( z \in \mathbb{C} \), we get that

\[
(h_0^{-j+1})^{-1}(t) = h_\infty^j(\overline{t}), \quad t \in (\mathbb{C}, 0).
\] □

**10. Appendix**

In this section we give a generalization to complex sectors of the definition of sectional asymptotic expansions on \( \mathbb{R}_+ \) (introduced earlier in [17]).
Definition 10.1 (sectional asymptotic expansions in $\mathbb{C}$). We say that an analytic germ $f$ on a petal $P$ admits on $P$ a sectional asymptotic expansion

$$\hat{f}(z) = \sum_{i=1}^{\infty} \hat{f}_i(\ell)z^{\alpha_i} \in \hat{L}_n^\infty,$$

where $(\alpha_i)_i$, $\alpha_i \in \mathbb{R}$, is a strictly increasing sequence, either finite or tending to $+\infty$, if there exist germs $f_i$ that are analytic on the $\ell$-cusps $\ell(P)$, admit $\hat{f}_i \in \hat{L}_n^{-1}$ as their sectional asymptotic expansions on $\ell(P)$, and are such that on every proper subsector $V \subset P$

$$f(z) = \sum_{i=1}^{n} f_i(\ell)z^{\alpha_i} = o(z^{\alpha_n+1-\delta}), \quad \delta > 0, \quad n \in \mathbb{N}, \quad z \to 0, \quad z \in V. \quad (10.1)$$

The bounding constants in (10.1) may vary with the sector $V \subset P$, but the germs $f_i$ and the sequence $(\alpha_i)_i$ are the same for all sectors.

We showed in [17] that, for a given choice of section, the sectional asymptotic expansion of a germ is unique.

Proof of Proposition 4.6. Let

$$\hat{f}(\ell) = \sum_{k=0}^{\infty} a_k \ell^k \quad \text{and} \quad \hat{g}(\ell) = \sum_{k=0}^{\infty} b_k \ell^k$$

be the log-Gevrey asymptotic expansions of $f$ and $g$. First, by the classical theory of asymptotic expansions, it follows that the formal product $\hat{f}(\ell) \cdot \hat{g}(\ell)$ is the power asymptotic expansion of $f \cdot g$ on $S$. Let $(c_k)_k$ be the coefficients of the power series $\hat{f}(\ell)\hat{g}(\ell)$. The sequence $(c_k)_k$ is then the convolution of the sequences of coefficients $(a_k)_k$ and $(b_k)_k$. The power asymptotic expansion of $f \cdot g$ on the $\ell$-cusp $S$ is obviously unique. It is left to prove the log-Gevrey bounds of order $r$ for $f \cdot g$ on $S$, for all $0 < r < s$.

Let $S' \subset S$ be an $\ell$-subcusp of $S$. By direct multiplication of series as in Theorem 1 of [1], we get that

$$\left| f(\ell)g(\ell) - \sum_{k=0}^{N-1} c_k \ell^k \right| \leq \left| f(\ell) \right| \left| g(\ell) - \sum_{k=0}^{N-1} c_k \ell^k \right|$$

$$+ \sum_{k=0}^{N-1} \left( |b_k| |\ell|^k \left| f(\ell) - \sum_{i=0}^{N-1-k} a_i \ell^i \right| \right), \quad \ell \in S', \quad (10.2)$$

36 In the following sense: for every proper $\ell$-subcusp $\ell(V) \subset \ell(P)$, the germ $f_i(\ell)$ admits $\hat{f}_i(\ell)$ as its sectional asymptotic expansion as $\ell \to 0$.

37 Following [12], we say that a complex germ admits a power asymptotic expansion on a domain $D$ with zero on its boundary if this asymptotic expansion holds on every proper subsector $V \subset D$ with vertex at the origin (possibly with different rates of convergence of the remainders, depending on the sector).
for every $N \in \mathbb{N}$. Since $f$ and $g$ are log-Gevrey of order $m$ and $n$, respectively, there exists a constant $C > 0$ such that

$$ |b_k| |\ell|^k \leq \left| g(\ell) - \sum_{j=0}^{k} b_j \ell^j \right| + \left| g(\ell) - \sum_{j=0}^{k-1} b_j \ell^j \right| $$

$$ \leq C n^{-k-1} (\log(k+1))^{k+1} \exp\left\{ - \frac{k+1}{\log(k+1)} \right\} |\ell|^{k+1} $$

$$ + C n^{-k} (\log k)^k \exp\left\{ - \frac{k}{\log k} \right\} |\ell|^k $$

$$ \leq C n^{-k} (\log k)^k \exp\left\{ - \frac{k}{\log k} \right\} (|\ell|^k + |\ell|^{k+1}), \quad \ell \in S', \quad k \in \mathbb{N}, \quad k \geq 2. \quad (10.3) $$

To derive the last inequality, we use the fact that the function

$$ x \mapsto n^{-x} (\log x)^x \exp\left\{ - \frac{x}{\log x} \right\} $$

is strictly increasing on $[2, +\infty)$. Moreover,

$$ \lim_{k \to \infty} \frac{n^{-k-1} (\log(k+1))^{k+1} \exp\{-(k+1)/\log(k+1)\}}{n^{-k} (\log k)^k \exp\{-k/\log k\}} \log \sqrt{k} = 1. $$

For $0 < p < n$ we have $(n/p)^{-k} \log \sqrt{k} \to 0$ as $k \to \infty$. Dividing (10.3) by $|\ell|^k$ and passing to the limit as $|\ell| \to 0$, we now get that there exists a constant $C > 0$ such that for every $0 < p < n$

$$ |b_k| \leq C p^{-k} (\log k)^k \exp\left\{ - \frac{k}{\log k} \right\}, \quad k \in \mathbb{N}, \quad k \geq 2. \quad (10.4) $$

Using the fact that $f \in \text{LG}_m(S)$ and $g \in \text{LG}_n(S)$ and the inequality (10.4), we get from (10.2) that there exist constants $A > 0$, $B > 0$, $D > 0$, and $E > 0$ such that
\[ f(\ell)g(\ell) - \sum_{k=0}^{N-1} c_k \ell^k \leq Bn^{-N} (\log N)^N \exp \left\{ -\frac{N}{\log N} \right\} |\ell|^N \]

\[ + AC \sum_{k=2}^{N-2} p^{-k}(\log k)^k \exp \left\{ -\frac{k}{\log k} \right\} |\ell|^k m^{-(N-k)}(\log(N-k))^{N-k} \]

\[ \times \exp \left\{ -\frac{N-k}{\log(N-k)} \right\} |\ell|^{N-k} \]

\[ \leq Ds^{-N} (\log N)^N \exp \left\{ -\frac{N}{\log N} \right\} N|\ell|^N \]

\[ \leq Er^{-N} (\log N)^N \exp \left\{ -\frac{N}{\log N} \right\} |\ell|^N, \quad \ell \in S', \quad N \in \mathbb{N}, \quad N \geq 2. \quad (10.5) \]

Here we let \( s := \min\{m, p\} \) and use the inequalities

\[ \log k, \log(N-k) \leq \log N \quad \text{and} \quad e^{-1/\log k}, e^{-1/\log(N-k)} \leq e^{-1/\log N} \]

for \( 2 \leq k \leq N-2 \). In the last line in (10.5) we take any \( 0 < r < s \). The proposition is proved. □

**Proof of Proposition 4.7.** Let \( \hat{f}(\ell) = \sum_{k=0}^{\infty} a_k \ell^k \). We carry out the proof by induction, first showing that the first derivative is log-Gevrey of every order \( 0 < r < m \) on \( S \), then passing to the second derivative and arguing similarly (using the estimates for the first derivative), and so on. Since \( f \) admits \( \hat{f} \) as a log-Gevrey asymptotic expansion of order \( m > 0 \), for each \( \ell \)-subcusp \( S' = \ell(V') \subset S \), where \( V' \) is a proper subsector of \( V \), there exists a constant \( C > 0 \) such that

\[ \left| f(\ell) - \sum_{k=0}^{n-1} a_k \ell^k \right| \leq Cm^{-n}(\log n)^n \exp \left\{ -\frac{n}{\log n} \right\} |\ell|^n, \quad \ell \in S', \quad n \in \mathbb{N}, \quad n \geq 2. \quad (10.6) \]

Take a sufficiently small \( \delta > 0 \) and consider a proper \( \ell \)-subcusp \( S_\delta \subset S' \) such that \( S_\delta = \ell(V_\delta) \), where \( V_\delta \subset V' \) is obtained from \( V' \) by shrinking by the angle \( \arcsin(\delta) \) from both sides. We show that there exist a constant \( D > 0 \) and a function \( \delta \mapsto e(\delta) > 0 \) such that

\[ \left| \frac{d}{d\ell} f(\ell) - \sum_{k=1}^{n-1} ka_k \ell^{k-1} \right| \leq D(m - e(\delta))^{-n}(\log n)^n \exp \left\{ -\frac{n}{\log n} \right\} |\ell|^{n-1}, \quad (10.7) \]

\[ \ell \in S_\delta, \quad n \in \mathbb{N}, \quad n \geq 2. \]

We also show that \( e(\delta) \searrow 0 \) as \( \delta \to 0 \). Since \( S_\delta \not\supset S' \) as \( \delta \to 0 \), this proves that \( \frac{d}{d\ell} f(\ell) \) admits the formal derivative \( \frac{d}{d\ell} \hat{f}(\ell) \) as a log-Gevrey asymptotic expansion of every order \( 0 < r < m \) on the whole of \( S \).
We now prove (10.7). Let

\[ \tilde{H}_n(z) := f \left( \frac{1}{-\log z} \right) - \sum_{k=0}^{n-1} a_k \left( -\frac{1}{\log z} \right)^k, \quad z \in V', \quad n \in \mathbb{N}, \]

where \( z = e^{-1/\ell} \). By the Cauchy integral formula with respect to the variable \( z \) and the estimate (10.6), we get that

\[
\left| \frac{d}{dz} \tilde{H}_n(z) \right| \leq \int_{K(z,\delta|z|)} \frac{|\tilde{H}_n(\xi)|}{|z-\xi|^2} d|\xi|
\]

\[
\leq C m^{-n} (\log n)^n \exp \left\{ -\frac{n}{\log n} \delta^2 |z|^2 \sup_{|w|=\delta|z|} \left| \frac{1}{-\log(z+w)} \right|^{n} \cdot 2\pi \delta |z| \right\}
\]

\[
\leq D m^{-n} (\log n)^n \exp \left\{ -\frac{n}{\log n} \frac{1}{-\log z} \right\}
\]

\[
\times \sup_{|w|=\delta|z|} \left| 1 + \frac{\log(1+w/z)}{\log z} \right|^{-n} \frac{1}{|z|}
\]

\[
\leq D \left( m(1-e(\delta)) \right)^{-n} (\log n)^n \exp \left\{ -\frac{n}{\log n} \right\}
\]

\[
\times |\ell|^n \frac{1}{|z|}, \quad z \in V_\delta, \quad n \in \mathbb{N}, \quad n \geq 2,
\]

(10.8)

where \( e(\delta) := c\delta \) for \( \delta > 0 \), and \( c > 0 \) depends only on the radius of \( V \). Obviously, \( e(\delta) \to 0 \) as \( \delta \to 0 \). The constants in the estimates do not depend on \( n \in \mathbb{N} \). Indeed, by the triangle inequality,

\[
\left| 1 + \frac{\log(1+w/z)}{\log z} \right|^{-n} \leq \left( 1 - \frac{\log(1+w/z)}{|\log z|} \right)^{-n}, \quad z \in V_\delta, \quad |w|=\delta|z|.
\]

For \( \delta = |w/z| < 1 \) there exists a constant \( c_1 > 0 \) (independent of \( \delta > 0 \)) such that

\[
\frac{|\log(1+w/z)|}{|\log z|} \leq \frac{c_1 |w/z|}{-\log |z|} = \frac{c_1 \delta}{-\log |z|} \leq \frac{c_1 \delta}{-\log R}, \quad z \in V_\delta, \quad |w|=\delta|z|,
\]

where \( R \) is the radius of \( V \). Now put \( c := c_1/(-\log R) \) and \( e(\delta) := c\delta \). The last inequality in (10.8) follows.

Furthermore,

\[
\frac{d}{dz} \tilde{H}_n(z) = \frac{d}{dz} \tilde{H}_n(z) = \frac{d}{d\ell} \frac{d}{d\ell} \tilde{H}_n(z) = \frac{\ell^2}{z} \left( \frac{d}{d\ell} f(\ell) - \sum_{k=1}^{n-1} k a_k \ell^{k-1} \right). \quad (10.9)
\]
From (10.8) and (10.9),
\[
\left| \frac{d}{d\ell} f(\ell) - \sum_{k=1}^{n-1} k a_k \ell^{k-1} \right| \leq \frac{|z|}{|\ell|^2} D(m(1 - e(\delta)))^{-n} (\log n)^n \exp \left\{ -\frac{n}{\log n} \right\} |\ell|^{n-1} |z|
\]
\[
\leq D(m(1 - e(\delta)))^{-n} (\log n)^n \exp \left\{ -\frac{n}{\log n} \right\} |\ell|^{n-2}, \quad \ell \in S_\delta, \ n \in \mathbb{N}, \ n \geq 2.
\]
(10.10)

To get (10.7) from (10.10) we estimate \(|a_{n-1}|\) for \(n \in \mathbb{N}\) with \(n \geq 3\) by using (10.4). There exists a constant \(C_1 > 0\) such that
\[
|a_{n-1}| \leq C_1 m^{-(n-1)} (\log(n - 1))^{n-1} \exp \left\{ -\frac{n - 1}{\log(n - 1)} \right\}
\]
\[
\leq C_1 m^{-n} (\log n)^n \exp \left\{ -\frac{n}{\log n} \right\}, \quad n \in \mathbb{N}, \ n \geq 3.
\]
(10.11)

From (10.10),
\[
\left| \frac{d}{d\ell} f(\ell) - \sum_{k=0}^{n-2} k a_k \ell^{k-1} \right| \leq \left| \frac{d}{d\ell} f(\ell) - \sum_{k=1}^{n-1} k a_k \ell^{k-1} \right| + (n - 1)|a_{n-1}| |\ell|^{n-2}, \quad n \in \mathbb{N}.
\]

Therefore, by (10.11) there exists a constant \(D_1 > 0\) such that
\[
\left| \frac{d}{d\ell} f(\ell) - \sum_{k=1}^{n-2} k a_k \ell^{k-1} \right| \leq D_1 (m(1 - e(\delta)))^{-n} (\log n)^n \exp \left\{ -\frac{n}{\log n} \right\} |\ell|^{n-2},
\]
\[
\ell \in S_\delta, \ n \in \mathbb{N}, \ n \geq 3.
\]

Thus, (10.7) is proved. Finally, using the estimate (10.7) for the first derivative, we repeat the same arguments for the second derivative, and so on. The proposition is proved. \(\square\)

**Proof of Proposition 4.9.** (i) Let \(0 < r < m\). Define \(r_F := \text{ord}(F) \in \mathbb{Z}\) and \(r_f := \text{ord}(f) \geq 1\). Then \(r_{F \circ f} := \text{ord}(F \circ f) = r_F \cdot r_f\). Let
\[
\ell^{-r_{F \circ f}} \cdot (\hat{F} \circ \hat{f})(\ell) := \sum_{i=0}^{\infty} a_i \ell^i, \quad a_i \in \mathbb{R}, \ i \in \mathbb{N}.
\]
(10.12)

Using Faà di Bruno’s formula for derivatives, we prove that on every subcusp \(V \subseteq S\) there exists a constant \(C_V > 0\) such that
\[
\sup_{\ell \in V} \left| (\ell^{-r_{F \circ f}} \cdot (F \circ f))^{(n)}(\ell) \right| \leq C_V n! r^{-n} (\log n)^n \exp \left\{ -\frac{n}{\log n} \right\}, \quad n \in \mathbb{N}.
\]

This will be sufficient, since
\[
\frac{d^n}{d\ell^n} \left( \ell^{-r_{F \circ f}} \cdot (F \circ f)(\ell) - \sum_{i=0}^{n-1} a_i \ell^i \right) = \frac{d^n}{d\ell^n} \left( \ell^{-r_{F \circ f}} (F \circ f)(\ell) \right), \quad n \in \mathbb{N},
\]
where the $a_i$, $i \in \mathbb{N}$, are as in (10.12), and since

$$
\frac{d^k}{d\ell^k} \left( \ell^{-r_{F,o}} \cdot (F \circ f)(\ell) - \sum_{i=0}^{n-1} a_i \ell^i \right)(0) = 0, \quad k = 0, 1, \ldots, n - 1.
$$

The last line follows from the standard result that the map $\ell \mapsto \ell^{-r_{F,o}} \cdot (F \circ f)(\ell)$ admits $\ell^{-r_{F,o}} \cdot (\widehat{F} \circ \widehat{f})(\ell)$ as its power asymptotic expansion as $\ell \to 0$.

Since $F$ is analytic, $f$ admits a log-Gevrey expansion of order $m$, and $\text{ord}(f) \geq 1$, for every subcusp $V \subseteq S$ there exist constants $C_V > 0$, $D_V > 0$, and $E_V > 0$ such that

$$
\sup_{\ell \in V} \left| (\ell^{-\min\{r_F,0\}} \cdot F)^{(k)}(\ell) \right| \leq C_V k! D_V^k, \quad k \in \mathbb{N},
$$

$$
\sup_{\ell \in V} \left| f^{(k)}(\ell) \right| \leq E_V k! r^{-k} (\log k)^k \exp\left\{ -\frac{k}{\log k} \right\}, \quad k \in \mathbb{N}. \quad (10.13)
$$

Because $\ell^{-r_F} F$ is analytic at 0, $D_V > 0$ can be made arbitrarily small by decreasing the radius of the domain $S$.

By Faà di Bruno’s formula [11] and the estimates (10.13), for every $n \in \mathbb{N}$

$$
\left| \left( (\ell^{-\min\{r_F,0\}} \cdot F) \circ f \right)^{(n)} \right|
= \left| \sum_{0 \leq k_1 \leq n \atop \sum_{j=1}^n j k_j = n} \frac{n!}{k_1! \cdots k_n!} (\ell^{-\min\{r_F,0\}} \cdot F)^{(k)}(f(\ell)) \prod_{j=1}^{n} \left( \frac{f^{(j)}(\ell)}{j!} \right)^{k_j} \right|
\leq C_V n! (\log n)^n \exp\left\{ -\frac{n}{\log n} \right\} m^{-n} \sum_{0 \leq k_1 \leq n \atop \sum_{j=1}^n j k_j = n} \frac{k!}{k_1! \cdots k_n!} D_V^k E_V^k
= C_V n! (\log n)^n \exp\left\{ -\frac{n}{\log n} \right\} m^{-n} E_V D_V (1 + E_V D_V)^{n-1}
\leq C_V' n! (\log n)^n \exp\left\{ -\frac{n}{\log n} \right\} (m(1 - \delta))^{-n},
$$

where $\delta > 0$ can be made arbitrarily small by decreasing the radius of $S$ (and thereby making $D_V > 0$ arbitrarily small). Here $k := k_1 + \cdots + k_n$. We use the inequality $\log j \leq \log n$, $j = 1, \ldots, n$, and estimate the last sum by the multinomial theorem. Now let $r := m(1 - \delta)$, so that $r$ can be any number with $0 < r < m$.

We have proved that

$$
(\ell^{-\min\{r_F,0\}} \cdot F) \circ f \in \text{LG}_r(V) \quad (10.14)
$$

for each $0 < r < m$, and this germ has $(\ell^{-\min\{r_F,0\}} \cdot \widehat{F}(\ell)) \circ \widehat{f}(\ell)$ as its log-Gevrey expansion of any order $0 < r < m$. We now show that by (10.14)

$$
\ell^{-\min\{r_{F,o},0\}} \cdot (F \circ f)(\ell) \in \text{LG}_r(V) \quad (10.15)
$$

for every $0 < r < m$, and this germ admits $\ell^{-\min\{r_{F,o},0\}}(\widehat{F} \circ \widehat{f})(\ell)$ as its log-Gevrey asymptotic expansion of order $r$. This finally proves the statement (i).
Indeed, if \( r_F \geq 0 \), then in view of the relations \( r_{F \circ f} = r_F \cdot r_f \) and \( r_f > 0 \), (10.15) follows directly from (10.14). If \( r_F < 0 \), then \( r_{F \circ f} = r_F \cdot r_f < 0 \) and
\[
(\ell^{-\min\{r_F, 0\}} \cdot F) \cdot (f(\ell)) = (\ell^{-r_F} \cdot F)(f(\ell)) = f(\ell)^{-r_F} \cdot (F \circ f)(\ell)
\]
where \( a \neq 0 \) is such that \( \ell^{-r_f} \cdot f = a + o(1) \) as \( \ell \to 0 \). Note that \( x \mapsto (1 + x)^{r_f} \) is an analytic function at 0, and \( \ell^{-r_f} \cdot (f - a^{r_f}) \in \text{LG}_r(V), \) \( 0 < r < m \), with \( \text{ord}(\ell^{-r_f} \cdot (f - a^{r_f})) \geq 1 \). By (10.14), \( (\ell^{-r_f} \cdot F)(f(\ell)) \in \text{LG}_r(V), \) \( 0 < r < m \), hence it follows from (10.16) and Proposition 4.6 that \( \ell^{-r_f \cdot r_f} \cdot (F \circ f)(\ell) \in \text{LG}_r(V), \) \( 0 < r < m \).

(ii) Let \( r_g := \text{ord}(g) \). We write
\[
\frac{h(\ell)}{g(\ell)} = h(\ell) \ell^{-\text{ord}(g)+1} \frac{1}{\ell^{-\text{ord}(g)+1} g(\ell)}.
\]
The function \( y \mapsto 1/y \) is a meromorphic germ at 0, so the statement to be proved follows directly from the statement (i) and Proposition 4.6.

Proposition 4.9 is proved. \( \square \)

Proof of Lemma 5.3. 1. A special case is the removal of the first block \( z^\alpha \hat{T}_0(\ell), \hat{T}_0(\ell) = -\ell^m + \text{h.o.t.} \) We look for an elementary change of variables \( \hat{\varphi}_0(z) = z + z\hat{R}_0(\ell), \hat{R}_0(\ell) \in \ell \mathbb{R}[[\ell]], \) which eliminates the first block except for the first term \( -\ell^m \), which by [16] cannot be eliminated in the class \( \hat{\mathfrak{L}}(\mathbb{R}) \). This is obvious, because \( \hat{R}_0(\ell) \) contains only non-negative powers of \( \ell \). Thus, we look for a \( \hat{\varphi}_0 \) such that
\[
\hat{\varphi}_0^{-1} \circ \hat{f} \circ \hat{\varphi}_0(z) = z - z^\alpha \ell^m + z^\beta_i \hat{T}_i(\ell) + \text{h.o.b.}, \quad \beta_i > \alpha.
\]
Since \( \text{ord}_z(\hat{\varphi}_0 - \text{id}) = 1 \) for the first block, the \( \hat{\varphi}_0 \) in (10.17) does not satisfy a simple Lie bracket equation of the type (10.19), as is the case for the higher-order blocks later. As a consequence, this expansion cannot be expressed as a solution (10.20) of a linear ordinary differential equation. Instead, to get a formula for \( \hat{\varphi}_0 \), we proceed as follows. Let \( \hat{\Psi} \) be the formal Fatou coordinate of \( \hat{f} \) and let \( \hat{\Psi}_0 \) be the formal Fatou coordinate of its normal form \( \hat{f}_1 \) in (5.1). Then by the Abel equation \( \hat{\Psi} \circ \hat{f} - \hat{\Psi} = 1 \), such an expansion \( \hat{\varphi}_0 \) (with the property that \( \hat{\varphi}_0^{-1} \circ \hat{f} \circ \hat{\varphi}_0 - \hat{f}_1 \) contains blocks of order in \( z \) strictly greater than \( \alpha \)) exists if and only if
\[
\text{Lb}(\hat{\Psi}_0) = \text{Lb}(\hat{\Psi} \circ \hat{\varphi}_0) = \text{Lb}(\hat{\Psi}) \circ \hat{\varphi}_0, \quad \hat{\varphi}_0 = z + z\hat{R}_0(\ell), \quad \hat{R}_0 \in \ell \mathbb{R}[[\ell]],
\]
where \( \text{Lb}(\cdot) \) denotes the leading block of a transseries. By the formal Taylor expansion for the Abel equation we see that
\[
\text{Lb}(\hat{\Psi}) = \int \frac{z^{-\alpha}}{\hat{T}_0(\ell)} \ dz = \int \frac{z^{1-\alpha}}{\hat{T}_0(\ell)\ell^2} \ d\ell
\]
and, similarly, 
\[ \text{Lb}(\hat{\Psi}_0) = \int \frac{z^{1-\alpha}}{\ell^{m+2}} \, d\ell. \]

Since \( \hat{\varphi}_0(z) = z(1 + \hat{R}_0(\ell)) \), the statement (5.2) for \( \beta_1 = \alpha \) follows from (10.18). It can easily be checked by formal integration and composition that the series \( \hat{R}_0(\ell) \) given by (5.2) belongs to the class \( \ell \mathbb{R}[[\ell]] \).

2. After the first block is removed, \( \hat{f} \) will be transformed into (10.17). To remove the first block \( z^{\beta_i} \hat{T}_i(\ell), \hat{T}_i(\ell) \in \mathbb{R}(\ell) \), \( \beta_i > \alpha \), we look for a \( \gamma_i > 1 \) and an \( \hat{R}_i(\ell) \in \mathbb{R}(\ell) \) that solve the Lie bracket\(^{38} \) equation (for details, see the proof of Theorem A in [16])
\[
[z^{\alpha}, z^{\gamma_i} \hat{R}_i(\ell)] = -z^{\beta_i} \hat{T}_i(\ell). \tag{10.19}
\]

Evaluating the Lie bracket equation, we have
\[
-z^{\gamma_i+\alpha-1} \left[ ((\alpha - \gamma_i)\ell^m + m\ell^{m+1}) \hat{R}_i(\ell) - \ell^{m+2} \hat{R}_i'(\ell) \right] = -z^{\beta_i} \hat{T}_i(\ell).
\]

We choose \( \gamma_i = \beta_i - \alpha + 1 \) and take \( \hat{R}_i \) as a formal solution of the linear ordinary differential equation
\[
((\alpha - \gamma_i)\ell^{-2} + m\ell^{-1}) \hat{R}_i(\ell) - \hat{R}_i'(\ell) = \ell^{-m-2} \hat{T}_i(\ell).
\]

This solution with constant of integration 0 is given by
\[
\hat{R}_i(\ell) = -z^{\alpha-\gamma_i} \ell^m \int z^{\gamma_i-\alpha} \ell^{-2m-2} \hat{T}_i(\ell) \, d\ell. \tag{10.20}
\]

The lemma is proved. \( \square \)

**Proof of Proposition 6.3.** We carry out the proof for length \( n = 1 \). Suppose that there exist two exponents of integration for \( \hat{F} \): \( (\alpha, p) \neq (\beta, q) \in (\mathbb{R}, \mathbb{Z}) \). Then there exist \( \hat{P}_1, \hat{Q}_1 \in \hat{L}G_r(S_\theta) \) with \( r > \pi/\theta \) such that
\[
\frac{d}{d\ell} (z^{\alpha} \ell^p \hat{F}(\ell)) = z^{\alpha} \ell^{2p-2} \hat{P}_1(\ell) \quad \text{and} \quad \frac{d}{d\ell} (z^{\beta} \ell^q \hat{F}(\ell)) = z^{\beta} \ell^{2q-2} \hat{Q}_1(\ell).
\]

Computations show that
\[
z^{\alpha} \ell^{2p-2} \hat{P}_1(\ell) = \frac{d}{d\ell} (z^{\alpha} \ell^p \hat{F}(\ell)) = \frac{d}{d\ell} (z^{\alpha-\beta} \ell^{p-q} z^{\beta} \ell^q \hat{F}(\ell)) = \frac{d}{d\ell} (z^{\alpha-\beta} \ell^{p-q} \cdot z^{\beta} \ell^q \hat{F}(\ell) + z^{\alpha-\beta} \ell^{p-q} z^{\beta} \ell^{2q-2} \hat{Q}_1(\ell)) = z^{\alpha} \ell^{p-2} ((\alpha - \beta) + (p - q)\ell) \cdot \hat{F}(\ell) + z^{\alpha} \ell^{p+q-2} \hat{Q}_1(\ell).
\]

Thus,
\[
((\alpha - \beta) + (p - q)\ell) \hat{F}(\ell) = \ell^p \hat{P}_1(\ell) - \ell^q \hat{Q}_1(\ell).
\]

\(^{38}\)Given two germs \( f \) and \( g \) of functions of \( z \), we define (by abuse of terms) their Lie bracket \([f, g]\) as the function \( f'g - g'f \). In fact, considering the vector fields \( X_f = f \frac{\partial}{\partial z} \) and \( X_g = g \frac{\partial}{\partial z} \) and their Lie bracket \([X_f, X_g]\), we have \([X_f, X_g] = [f, g] \frac{\partial}{\partial z}\). We commit the same abuse of language on the formal level. This notation was introduced in [16], §3.
By Propositions 4.5–4.7 the expression \( \ell^p \hat{P}_1(\ell) - \ell^q \hat{Q}_1(\ell) \) belongs to the class \( \hat{L}_G(S_0) \) for some \( s \) with \( r > s > \pi/\theta \). If \((\alpha,p) \neq (\beta,q)\), this is a contradiction, since the right-hand side is integrally summable of length 0, while the left-hand side is not.

Finally, by a similar proof, if the series \( \hat{F} \) is integrally summable of length \( n \in \mathbb{N} \) with \( n \geq 2 \), then its exponent of integration \((\alpha,p) \in (\mathbb{R},\mathbb{Z})\) is unique. \( \square \)

**Proof of Lemma 6.4.** Consider a series \( \hat{R}_k^r(\ell) \), \( 1 \leq r \leq r_k \), which is integrally summable of length \( k \leq n \) with respect to \( \{\hat{R}_i^j(\ell), i = 1, \ldots, k-1, j_i = 1, \ldots, r_i\} \), with exponent of integration \((\alpha_k,p_k)\). That is,

\[
\hat{R}_k^r(\ell) = \frac{1}{e^{-\alpha_k/\ell} \ell^{-2p_k-2}} \int e^{-\alpha_k/\ell} \ell^{-2p_k-2} \hat{S}_{\leq k-1}^{\hat{R}_1^1;\ldots;\hat{R}_{k-1}^1;\ldots;\hat{R}_{k-1}^{r_k-1}}(\ell) \, d\ell,
\]

where \( \hat{S}_{\leq k-1}^{\hat{R}_1^1;\ldots;\hat{R}_{k-1}^1;\ldots;\hat{R}_{k-1}^{r_k-1}}(\ell) \) is an algebraic combination (with operations +, ·, /, \( d/d\ell \)) of series \( \hat{R}_i^j(\ell) \), \( i = 1, \ldots, k-1, j_i = 1, \ldots, r_i \), that are integrally summable with respect to previous series of the same type and of strictly smaller lengths and of integrally summable series of length 0. For the derivatives of this series we have

\[
\frac{d}{d\ell} \hat{R}_k^r(\ell) = \frac{1}{e^{-2\alpha_k/\ell} \ell^{2p_k}} \left( e^{-2\alpha_k/\ell} \ell^{-p_k-2} \hat{S}_{\leq k-1}^{\hat{R}_1^1;\ldots;\hat{R}_{k-1}^1;\ldots;\hat{R}_{k-1}^{r_k-1}}(\ell) - \hat{R}_k(\ell) e^{-2\alpha_k/\ell} \ell^{2p_k-2} (\alpha_k - p_k) \right)
\]

The rest of the proof is by induction. \( \square \)

**Proof of Proposition 6.11.** (i) Let \( \hat{\varphi}_1, \hat{\varphi}_2 \in \widehat{\mathcal{L}}(\mathbb{R}) \) be two formal changes that reduce \( \hat{f} \) to \( \hat{f}_1 \):

\[
\hat{f} = \hat{\varphi}_1 \circ \hat{f}_1 \circ \hat{\varphi}_1^{-1} = \hat{\varphi}_2 \circ \hat{f}_1 \circ \hat{\varphi}_2^{-1}.
\]

As in the statement, let \( \hat{\Psi}_0 \) be a formal Fatou coordinate for \( \hat{f}_1 \) with constant term 0. Directly by the Abel equation, \( \hat{\Psi}_0 \circ \hat{\varphi}_1^{-1}, \hat{\Psi}_0 \circ \hat{\varphi}_2^{-1} \in \widehat{\mathcal{L}}_2^\infty(\mathbb{R}) \) are also Fatou coordinates for \( \hat{f} \). Since by [17] the Fatou coordinate of \( \hat{f} \) is unique in the class \( \widehat{\mathcal{L}}^\infty(\mathbb{R}) \) up to a constant term, there exists a constant \( c \in \mathbb{R} \) such that

\[
\hat{\Psi}_0 \circ \hat{\varphi}_1^{-1} = \hat{\Psi}_0 \circ \hat{\varphi}_2^{-1} + c.
\]

Composing with \( \hat{\varphi}_2 \in \widehat{\mathcal{L}}(\mathbb{R}) \) from the right, we get that

\[
\hat{\Psi}_0 \circ \hat{\varphi}_1^{-1} \circ \hat{\varphi}_2 = \hat{\Psi}_0 + c \quad \text{and} \quad \hat{\varphi}_1^{-1} \circ \hat{\varphi}_2 = \hat{\Psi}_0^{-1} \circ (\hat{\Psi}_0 + c) = \hat{f}_c.
\]

(ii) By the Abel equation, \( \hat{\Psi}_0 \circ \hat{\varphi}^{-1} \) is a Fatou coordinate for \( \hat{f} \) in the class \( \widehat{\mathcal{L}}^\infty(\mathbb{R}) \), so it differs from its other Fatou coordinate \( \hat{\Psi} \) by a constant \( C \in \mathbb{R} \). \( \square \)

**Proof of Lemma 7.5.** (The idea of the proof is due to Loïc Teyssier.) For simplicity, denote by \( V \) a fixed petal \( V_j^\pm \) and put \( h := h_j^\pm \). Then \( h(z) = z + o(z) \) is analytic
on $V$, and so is $f = h \circ f_1 \circ h^{-1}$. Let $\Psi_0$ be the Fatou coordinate of $f_1$ on $V$. By Proposition 7.4, (i), there exists an analytic conjugacy $\varphi$ of the form $\varphi(z) = z + o(z)$ on $V$ conjugating $f$ to $f_1$ and admitting, up to a precomposition with $f_c (c \in \mathbb{R})$, the block iterated integral asymptotic expansion $\hat{\varphi}$. By the Abel equation, $\Psi_1 := \Psi_0 \circ h^{-1}$ and $\Psi_2 := \Psi_0 \circ \varphi^{-1}$ are two analytic Fatou coordinates of $f$ on $V$. Since $h$ and $\varphi$ are both tangent to the identity map, $\Psi_1(z), \Psi_2(z) \sim_{z \to 0} \frac{1}{\alpha - 1} z^{-\alpha + 1} \ell^m$. Indeed, they have the same asymptotic behaviour as $\Psi_0$ as $z \to 0$. For sufficiently large $R$ the image $\Psi_2(V)$ contains the half-plane $\{w \in \mathbb{C}: \text{Re}(w) > R\} \subset \mathbb{C}$. Now consider the composition

$$
\Psi_1 \circ \Psi_2^{-1}(w + 1) = \Psi_1 \circ \Psi_2^{-1}(w) + 1, \quad \text{Re}(w) > R.
$$

Here $\Psi_1 \circ \Psi_2^{-1} - \text{id}$ is obviously a periodic analytic function on the half-plane $\text{Re}(w) > R$ for some $R > 0$. By periodicity, it can be holomorphically extended to the whole plane $\mathbb{C}$. On the other hand,

$$
\Psi_1 \circ \Psi_2^{-1}(w) - w = \Psi_0 \circ h^{-1} \circ \varphi \circ \Psi_0^{-1}(w) - w, \quad \text{Re}(w) > R.
$$

Note that $h^{-1} \circ \varphi(z) = z + o(z)$ as $z \to 0$ on $V$, and $\Psi_0(z)$ has the asymptotic behaviour $\Psi_0(z) \sim z^{-\alpha + 1} \ell^m + o(z^{-\alpha + 1} \ell^m)$, therefore,

$$
\Psi_1 \circ \Psi_2^{-1}(w) - w = o(w), \quad |w| \to \infty, \quad \text{Re}(w) > R,
$$

and

$$
\frac{\Psi_1 \circ \Psi_2^{-1}(w) - w}{w} = o(1), \quad |w| \to \infty, \quad \text{Re}(w) > R.
$$

By the periodicity of the function in the numerator, we conclude that for the entire periodic extension $L(w)$ of the function $\Psi_1 \circ \Psi_2^{-1}(w) - w$ in the numerator defined on the image $\Psi_2(V)$ we have

$$
\frac{L(w)}{w} = o(1), \quad |w| \to \infty.
$$

The function in the numerator is entire, so the function on the left-hand side has a pole at 0. Subtracting $C/w$ with $C \in \mathbb{C}$ the corresponding residue, we get an entire function on the left-hand side and on the right-hand side an expression of the form

$$
\frac{L(w)}{w} - \frac{C}{w} = o(1) - \frac{C}{w}, \quad |w| \to \infty.
$$

But the right-hand side is then also entire, and moreover it is obviously bounded as $|w| \to \infty$ and tends to zero. Hence, by Liouville’s theorem it is identically equal to zero. It follows that $L(w) = C$ on $\mathbb{C}$ and $\Psi_1 \circ \Psi_2^{-1}(w) - w = C$ on the image $\Psi_2(V)$. Therefore, $\Psi_1$ and $\Psi_2$ differ on each petal $V$ only by a constant, and on $V$ we have

$$
\Psi_2 - \Psi_1 = C,
$$

$$
\Psi_0 \circ h^{-1} \circ \varphi - \Psi_0 = C,
$$

$$
h^{-1} \circ \varphi = \Psi_0^{-1}(\Psi_0 + C) = f_C,
$$

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Ψ 1 ◦ Ψ 2 −1(w + 1) = Ψ 1 ◦ Ψ 2 −1(w) + 1, Re(w) > R. Here Ψ 1 ◦ Ψ 2 −1 − id is obviously a periodic analytic function on the half-plane Re(w) > R for some R > 0. By periodicity, it can be holomorphically extended to the whole plane C. On the other hand,

Ψ 1 ◦ Ψ 2 −1(w) − w = Ψ 0 ◦ h −1 ◦ Ψ 0 −1(w) − w, Re(w) > R.

Note that h −1 ◦ Ψ 0(z) = z + o(z) as z → 0 on V, and Ψ 0(z) has the asymptotic behaviour Ψ 0(z) ∼ z −α+1 ℓ m + o(z −α+1 ℓ m), therefore,

Ψ 1 ◦ Ψ 2 −1(w) − w = o(w), |w| → ∞, Re(w) > R,

and

Ψ 1 ◦ Ψ 2 −1(w) − w = o(1), |w| → ∞, Re(w) > R.

By the periodicity of the function in the numerator, we conclude that for the entire periodic extension L(w) of the function Ψ 1 ◦ Ψ 2 −1(w) − w in the numerator defined on the image Ψ 2(V) we have

L(w) w = o(1), |w| → ∞.

The function in the numerator is entire, so the function on the left-hand side has a pole at 0. Subtracting C/w with C ∈ C the corresponding residue, we get an entire function on the left-hand side and on the right-hand side an expression of the form

L(w) w − C w = o(1) − C w, |w| → ∞.

But the right-hand side is then also entire, and moreover it is obviously bounded as |w| → ∞ and tends to zero. Hence, by Liouville’s theorem it is identically equal to zero. It follows that L(w) = C on C and Ψ 1 ◦ Ψ 2 −1(w) − w = C on the image Ψ 2(V). Therefore, Ψ 1 and Ψ 2 differ on each petal V only by a constant, and on V we have

Ψ 2 − Ψ 1 = C,

Ψ 0 ◦ h −1 ◦ Ψ 0 −1(w) − w = C,

h −1 ◦ Ψ 0 ◦ Ψ −1 0 + C = f C,
where $C \in \mathbb{R}$. Then $h = \varphi \circ f^{-1}_C$. Since $\varphi$ admits $\hat{\varphi}$ as its block iterated integral asymptotic expansion, which is (by Definition 7.2) well defined up to precomposition with $f_c$ ($c \in \mathbb{R}$), it follows that $h$ also admits $\hat{\varphi}$ as its block iterated integral asymptotic expansion. □

**Proof of Proposition 8.1.** By § 24E in [10], for small $\delta > 0$ there exists a constant $C > 0$ such that

$$|f(z) - z - z^\alpha P(-\log z)| \leq C|z|^{|\alpha + \delta|}, \quad z \in \mathcal{R}_C,$$

where $P(-\log z)$ denotes the first polynomial block of $f$. Moreover, $z^\alpha \ell^m = o(z^\beta \ell^k)$ uniformly as $|z| \to 0$ on $\mathcal{R}_C$ if and only if $(\beta, k) < (\alpha, m)$, because on a standard quadratic domain the arguments of points are controlled by their radii ($\varphi < \log^2 r$ for $z = re^{i\varphi}$ in a standard quadratic domain). We conclude that there exists a uniform constant $D > 0$ such that

$$|f(z) - z + az^\alpha \ell^m| \leq |f(z) - z - z^\alpha P(-\log z)| + |z|^\alpha |P(-\log z) + a\ell^m|$$

$$\leq D|z|^\alpha |\ell|^{m+1}, \quad z \in \mathcal{R}_C. \quad \Box$$

**Proof of Proposition 8.2.** We prove (i) and (ii) simultaneously, considering inverse germs and compositions of germs separately.

**Step 1: germ of the inverse map.** Since $f(z) = z + o(z)$ is a germ tangent to the identity, with $o(z)$ as $z \to 0$ uniformly on $\mathcal{R}_C$, it is injective and by the inverse function theorem is analytically invertible near $z = 0$ on the standard quadratic domain $\mathcal{R}_C$. This can be checked in the logarithmic chart, as in the proof of Proposition 7.1, (ii). Note that by the Cauchy formula $f'(z) = 1 + o(1)$ as $z \to 0$, where $o(1)$ is uniform on $\mathcal{R}_C$. By bijectivity, $f^{-1}$ maps the positive real line to itself. Moreover, the fact that $f^{-1}(w) = w + o(w)$, with $o(w)$ uniform on $\mathcal{R}_C$ as $w \to 0$, follows directly from the similar estimate for $f$. We also have the following uniform estimate for $f^{-1}(w)$:

$$|f^{-1}(w) - w - aw^\alpha \ell^m| \leq D|w^\alpha \ell^{m+1}|, \quad w \in \mathcal{R}_C, \quad D > 0,$$

as is directly seen by putting $z = f^{-1}(w)$ in the uniform estimate (2.5). By Proposition 2.2 the attracting petals for $f$ thus become repelling for the inverse $f^{-1}$, and conversely.

We now prove that $f^{-1}(w)$ has a generalized Dulac asymptotic expansion on every petal $V_j^\pm$, using the fact that $f$ admits such a generalized Dulac expansion on $V_j^\pm$. First we prove by the operator formula (see, for example, [16]) that the formal inverse $\hat{f}^{-1}$ of its generalized Dulac expansion $\hat{f}$ is again a parabolic generalized Dulac series:

$$\hat{f}^{-1} = \text{id} + \hat{g} + (\hat{g} \circ \hat{f} - \hat{g}) + ((\hat{g} \circ \hat{f} - \hat{g}) \circ \hat{f} - (\hat{g} \circ \hat{f} - \hat{g})) + \cdots$$

$$= \text{id} + \hat{g} + \left(\hat{g}' \cdot \hat{g} + \frac{1}{2!} \hat{g}'' \cdot \hat{g}^2 + \cdots\right)$$

$$+ \left(\left(\hat{g}' \cdot \hat{g} + \frac{1}{2!} \hat{g}'' \cdot \hat{g} + \cdots\right) \cdot \hat{g} + \frac{1}{2!} \left(\hat{g}' \cdot \hat{g} + \frac{1}{2!} \hat{g}'' \cdot \hat{g} + \cdots\right) \cdots \cdot \hat{g}^2\right) + \cdots,$$

(10.21)
where \( \hat{g} = \text{id} - \hat{f} \). Indeed, from this formula we can see that the coefficient of each block of the expansion \( \hat{f}^{-1} \) is a finite combination (with operations +, \( \cdot \), \( d/d\ell \)) of the coefficients \( \hat{R}_i(\ell) \) of the blocks in \( \hat{f} \). Since \( f \) is a generalized Dulac germ, \( \hat{R}_i(\ell) \in \hat{L}\Gamma_m(\ell(V_j^\pm)) \) for some order \( m > (\alpha - 1)/2, i \in \mathbb{N} \). By Propositions 4.5–4.7, the finite algebraic combinations of such coefficients again lie in \( \hat{L}\Gamma_r(\ell(V_j^\pm)) \) for any \( r < m \) which is sufficiently close to \( m \). We take \( r > (\alpha - 1)/2 \). Therefore, \( \hat{f}^{-1} \) is a parabolic generalized Dulac series. We now prove that this series is the generalized Dulac expansion of \( f^{-1} \) on the petals.

Indeed, computing the expansion for \( f^{-1}(w) \) block-by-block from the expansion for \( f \) (where the blocks correspond to strictly increasing powers of \( w \)), we can verify that the coefficients of powers of \( \ell \) in each block are log-Gevrey sums of order \( r \) of their formal analogues in the formal inverse series (10.21).

**Step 2: the composition.** The proof for the composition is similar and is based on the fact that the germs and their expansions are parabolic. First we prove that the formal composition of two parabolic generalized Dulac series is again a parabolic generalized Dulac series. Then we prove that for parabolic generalized Dulac germs \( f \) and \( g \), \( \hat{f} \circ \hat{g} \) is the generalized Dulac expansion for the composition \( f \circ g \) on suitable petals.

The proposition is proved. \( \square \)

**Proof of Proposition 8.3.** Exactly as in [12], we compute that, for the sector \( W_\theta \) at infinity of opening \( 2\pi - 2\theta \), there exists for any small \( \theta > 0 \) a sufficiently large radius \( R_\theta \) such that for \( w \in W_\theta \cap \bar{R} \) with \( |w| > R_\theta \) there is a constant \( C_\theta > 0 \) for which

\[
\left| F^{\circ n}(w) \right| \geq C_\theta (|w| + n), \quad n \in \mathbb{N}, \quad w \in W_\theta, \quad |w| > R_\theta. \tag{10.22}
\]

Note that \( R_\theta \) does not depend on the level of the Riemann surface \( \bar{R} \). Since

\[
f^{\circ n}(z) = h^{-1} \circ F^{\circ n}(w), \quad \text{where} \quad w = h(z) = -\frac{1}{a(\alpha - 1)} z^{-\alpha + 1} \ell^{-m},
\]

there exists a \( C > 0 \) such that

\[
|f^{\circ n}(z)| \leq C |F^{\circ n}(w)|^{-1/(\alpha - 1)} |\ell(F^{\circ n}(w))|^{-m/(\alpha - 1)}.
\]

Thus, for every subsector \( W \) of an attracting petal \( V_+^j \), there exists a constant \( C_W > 0 \) such that

\[
|f^{\circ n}(z)| \leq C_W (|w| + n)^{-1/(\alpha - 1)} |\ell(F^{\circ n}(w))|^{-m/(\alpha - 1)}, \quad n \in \mathbb{N}, \quad z \in W. \tag{10.23}
\]

Furthermore,

\[
|f^{\circ n}(z)| \leq C_W (|w| + n)^{-\gamma/(\alpha - 1)} |\ell(F^{\circ n}(w))|^{-m\gamma/(\alpha - 1)}
\]

\[
= C_W (|z|^{-\alpha + 1} |\ell|^{-m + n})^{-\gamma/(\alpha - 1)} |\ell(F^{\circ n}(w))|^{-m\gamma/(\alpha - 1)}.
\]

We now distinguish two cases.
Case 1. Let $\gamma = \alpha - 1$. Then $r \geq m + 2$, and by (10.22)

$$
|\delta(f^\circ n(z))| \leq C_W (|z|^{-\alpha + 1} |\ell|^{-m} + n)^{-1} \left( \frac{1}{\log(|z|^{-\alpha + 1} |\ell|^{-m} + n)} \right)^{r - m}
$$

$$
\leq C_W |z|^{\alpha - 1} |\ell|^m (|z|)^{r - m} \frac{1}{1 + n|z|^{\alpha - 1} |\ell|^m}
$$

$$
\times \left( \frac{1}{1 - m(\alpha - 1)^{-1} \ell(|z|) \log |\ell| + (\alpha - 1)^{-1} \ell(|z|) \log (1 + n|z|^{\alpha - 1} |\ell|^m)} \right)^{r - m}.
$$

Here we used the fact that

$$
\frac{1}{\log F^\circ n(w)} \leq \frac{1}{\log |F^\circ n(w)|} \leq \frac{1}{\log (|w| + n)}
$$

and $r - m > 0$.

Case 2. Suppose that $\gamma > \alpha - 1$. Then $\delta(z) = O(z^{\gamma - \delta_1})$ for $\delta_1 > 0$ and $\epsilon > 0$. From (10.23) it follows that

$$
|f^\circ n(z)| \leq C_W (|w| + n)^{-1/(\alpha - 1) + \epsilon}, \quad n \in \mathbb{N}, \quad z \in W.
$$

By (10.22),

$$
|\delta(f^\circ n(z))| \leq C_W (|z|^{-\alpha + 1} |\ell|^{-m} + n)^{-\gamma/(\alpha - 1) + \delta_2}
$$

$$
\leq C_W |z|^{-\delta_2(\alpha - 1)} |\ell|^m (\gamma/(\alpha - 1) - \delta_2)(1 + n|z|^{\alpha - 1} |\ell|^m)^{-\gamma/(\alpha - 1) + \delta_2}.
$$

By a suitable choice of $\epsilon > 0$ and $\delta_1 > 0$ we can make $\delta_2 := \frac{\delta_1}{\alpha - 1} + \epsilon(\gamma - \delta_1)$ small enough that still $\frac{\gamma}{\alpha - 1} - \delta_2 > 1$. Now $\delta := \delta_2(\alpha - 1)$ is small enough that $\gamma - (\alpha - 1) - \delta > 0$.

Putting the estimates in Cases 1 or 2 into (8.8), we get uniform convergence of the series in each subsector $W \subset V^j_+$, so that the germ $h_j$ is analytic on $V^j_+$ by the Weierstrass theorem. Using an integral approximation of the series as in [17], we get the estimate (8.9), and the proposition is proved. \qed

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