On asymptotic behavior of U-statistics for associated random variables

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Abstract
Let \( \{X_n, n \geq 1\} \) be a sequence of stationary associated random variables. For such a sequence, we discuss the limiting behavior of U-statistics based on kernels which are of bounded Hardy-Krause variation. As an application, we obtain the asymptotic distribution of Gini’s mean difference. We also give simulation results to illustrate the asymptotic normality of the statistic under the dependent setup.

Keywords: Associated random variables; Central limit theorem; Gini’s mean difference; Hardy-Krause variation; U-statistics.

1 Introduction
Dewan and Prakasa Rao (2001) gave a central limit theorem for U-statistics based on stationary associated random variables using an orthogonal expansion of the underlying kernel. They also obtained a central limit theorem for U-statistics with continuous component-wise monotonic kernels of degree 2 using the Hoeffding’s decomposition in (2002). Garg and Dewan (2014) extended the results of Dewan and Prakasa Rao (2002) to continuous component-wise monotonic kernels of higher degree and to kernels which are non-monotonic. Continuing with the investigations, in this paper we discuss the asymptotic behavior of U-statistics which are based on kernels of bounded Hardy-Krause variation. Apropos our discussion, we first give the following definition given by Esary et al. (1967).

Definition 1.1. A finite collection of random variables \( \{X_j, 1 \leq j \leq n\} \) is said to be associated, if for any choice of component-wise nondecreasing functions \( h, g : \mathbb{R}^n \rightarrow \mathbb{R} \), we have,

\[ \text{Cov}(h(X_1, \ldots, X_n), g(X_1, \ldots, X_n)) \geq 0 \]

whenever it exists. An infinite collection of random variables \( \{X_j, j \geq 1\} \) is associated if every finite sub-collection is associated.

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Associated random variables have been widely used in reliability studies, statistical mechanics, and percolation theory. A set consisting of independent random variables is associated (cf. Esary et al. [1967]). Monotonic functions of associated random variables are associated. Other examples of associated random variables are: positively correlated normal random variables, the sequence \( \{X_n = Y + X'_n, n \geq 1\} \), where \( \{X'_n, n \geq 1\} \) are independent random variables and \( Y \) is another random variable independent of \( \{X'_n, n \geq 1\} \) (cf. Barlow and Proschan [1975]), the stationary autoregressive process \( \{X_k\} \) of order \( p \) given by \( X_n = \phi_1 X_{n-1} + \cdots + \phi_p X_{n-p} + e_n \) with \( \phi_i \geq 0, 1 \leq i \leq p \), where \( \{e_n; n \geq 1\} \) is a sequence of independent random variables with zero mean and unit variance (cf. Nagaraj and Reddy [1993]). Detailed presentation of the probabilistic results and examples relating to associated sequences can be found in Bulinski and Shashkin (2007) and (2009), Prakasa Rao (2012), and Oliveira (2012).

Given stationary associated observations \( \{X_i, 1 \leq i \leq n\} \), the U-statistic, \( U_n \), of degree \( k \) \((1 \leq k \leq n)\) based on a symmetric kernel \( \rho : \mathbb{R}^k \to \mathbb{R} \) is given by,

\[
U_n = \binom{n}{k}^{-1} \sum_{(n,k)} \rho(X_{i_1}, X_{i_2}, \ldots, X_{i_k}),
\]

where \((n,k)\) indicates all subsets \(1 \leq i_1 < i_2 < \cdots < i_k \leq n\) of \(\{1, 2, \ldots, n\}\).

Let \( F \) be the distribution function of \( X_1 \) and \( U_n \) be the U-statistic based on the 2 degree kernel \( \rho(x_1, x_2) \). Let

\[
\theta = \int_{\mathbb{R}^2} \rho(x_1, x_2) \, dF(x_1) dF(x_2).
\]

The Hoeffding-decomposition (H-decomposition) for \( U_n \) is given by (see Lee [1990])

\[
U_n = \theta + 2H^{(1)}_n + H^{(2)}_n,
\]

where \( H^{(j)}_n \) is the U-statistics of degree \( j \) based on the kernel \( h^{(j)} \), \( j = 1, 2 \).

\[
h^{(1)}(x_1) = \rho_1(x_1) - \theta,
\]

\[
h^{(2)}(x_1, x_2) = \rho(x_1, x_2) - \rho_1(x_1) - \rho_1(x_2) + \theta.
\]

and \( \rho_1(x_1) = \int_{\mathbb{R}} \rho(x_1, x_2) \, dF(x_2) \). When the observations are i.i.d, \( E(U_n) = \theta \).

Next, we discuss the concept of Hardy-Krause variation. The following is from Beard [2009].

**Definition 1.2.** A function \( f : [a, b] \to \mathbb{R}, [a, b] = \{x \in \mathbb{R}^k : a \leq x \leq b\} \), is of bounded Hardy-Krause variation if and only if it can be written as a difference of two bounded functions \( g \) and \( h \), such that \( g_I \) and \( h_I \) are monotone for all non-empty \( I \subseteq \{1, 2, \ldots, k\} \). Here, given a non-empty set \( I \subseteq \{1, 2, \ldots, k\} \) and a function \( f : [a, b] \to \mathbb{R}, f_I \) denotes the real valued function on \( \prod_{i \in I}[a_i, b_i] \) obtained by setting the \( i^{th} \) argument of \( f \) equal to \( b_i \) whenever \( i \notin I \). A function \( g : [a, b] \to \mathbb{R} \) is said
to be monotone if $\Delta_{Rg} \geq 0$ for all $k$-dimensional rectangles $R \subseteq [a, b]$. Here, if $R = [c, d]$, a $k$-dimensional rectangle contained in $[a, b]$, then,

$$\Delta_{Rg} = \sum_{I \subseteq \{1, 2, \ldots, k\}} (-1)^{|I|} g(x_I), \quad (1.2)$$

where, $x_I$ is the vector in $\mathbb{R}^k$ whose $i$th element is given by $c$ if $i \in I$, or by $d$ if $i \notin I$, $g_\emptyset = g(b)$. For instance, if $k = 2$ and $R = [c_1, d_1] \times [c_2, d_2]$, we have,

$$\Delta_R f = f(d_1, d_2) - f(c_1, d_2) - f(d_1, c_2) + f(c_1, c_2).$$

When $k = 1$, the Hardy-Krause variation is equivalent to the standard definition of total variation.

Examples of functions which are of bounded Hardy-Krause variation include the kernel of Gini mean difference and empirical joint distribution functions. The asymptotic distribution of these statistics is discussed in section 4.

The paper is organized as follows. Section 2 includes results and definitions that will be required to prove our main results in section 3. In section 3 of the paper, we obtain a central limit theorem for U-statistics based on functions of bounded Hardy-Krause variation for stationary associated random variables. In section 4, we apply our results to obtain the asymptotic distribution of Gini’s mean difference. We give simulation results in section 5 to investigate the asymptotic normality of the statistic under the dependent setup.

2 Preliminaries

In this section, we give results and definitions which will be needed to prove our main results given in section 3.

**Lemma 2.1. (Serfling [1968])** Consider a sum of the form

$$\Delta = \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{n} \sum_{l=1}^{m} \Delta(i, j, k, l).$$

Suppose,

$$|\Delta(i, j, k, l)| \leq r(\max(|i - k|, |j - l|)) \forall (i, j, k, l),$$

where $r(k)$ is some non-negative function satisfying

$$\sum_{k=1}^{\infty} r(k) < \infty.$$

Then, $|\Delta| = o(mn^2)$ as $m$ and $n \to \infty$, such that $\frac{m}{n} \to c > 0$.

**Lemma 2.2. (Newman [1980])** Let $\{X_n, n \geq 1\}$ be a strictly stationary sequence of associated random variables. Let $\sigma^2 = \text{Var}(X_1) + 2 \sum_{j=2}^{\infty} \text{Cov}(X_1, X_j)$ with $0 < \sigma^2 < \infty$. Then,

$$\frac{n^{-\frac{1}{2}} \sum_{j=1}^{n} (X_j - E(X_j))}{\sigma} \Rightarrow N(0, 1) \quad \text{as} \quad n \to \infty. \quad (2.1)$$
Definition 2.3. \textbf{(Newman (1984))} Let $f$ and $f_1$ be two complex-valued functions on $\mathbb{R}^m$. We say $f \ll f_1$ if $f_1 - \text{Re}(e^{\alpha}f)$ is coordinate-wise nondecreasing for every $\alpha \in \mathbb{R}$. If $f$ and $f_1$ are two real-valued functions on $\mathbb{R}^n$, then $f \ll f_1$ iff $f_1 + f$ and $f_1 - f$ are both coordinate-wise nondecreasing. If $f \ll f_1$, then $f_1$ will be coordinate-wise nondecreasing.

Lemma 2.4. \textbf{(Newman (1984))} Let $\{X_n, n \geq 1\}$ be a stationary sequence of associated random variables. For each $j$, let $Y_j = f(X_j)$ and $\hat{Y}_j = \hat{f}(X_j)$. Suppose that $f \ll \hat{f}$. Let,

$$\sum_{j=1}^{\infty} \text{Cov}(\hat{Y}_1, \hat{Y}_j) < \infty.$$  

Then

$$\frac{n^{-\frac{1}{2}} \sum_{j=1}^{n} (Y_j - E(Y_j))}{\sigma} \xrightarrow{d} N(0,1) \text{ as } n \to \infty,$$  

where

$$\sigma^2 = \text{Var}(Y_1) + 2 \sum_{j=2}^{\infty} \text{Cov}(Y_1, Y_j).$$

Lemma 2.5. \textbf{(Bagai and Prakasa Rao (1991))} Suppose $X$ and $Y$ are associated random variables with bounded continuous densities. Then there exists a constant $C > 0$ such that,

$$\sup_{x,y} |P(X \leq x, Y \leq y) - P(X \leq x)P(Y \leq y)| \leq C \text{Cov}(X, Y)^\frac{1}{2},$$  

(2.3)

Lemma 2.6. \textbf{(Lebowitz (1972))} If the random variables $\{X_j, 1 \leq j \leq n\}$ are associated then,

$$0 \leq H_{A,B} \leq \sum_{i \in A} \sum_{j \in B} H_{\{i\},\{j\}}.$$

Here, for $A$ and $B$, subsets of $\{1, 2, ..., n\}$ and real $x_j$'s,

$$H_{A,B}(x_j, j \in A \cup B) = P[X_j > x_j; j \in A \cup B] - P[X_k > x_k, k \in A]P[X_l > x_l, l \in B].$$

The proofs of the results of Dewan and Prakasa Rao (2002) and Garg and Dewan (2014) were based on Bulinski’s inequality (Bulinski (1996)). For obtaining the asymptotic distribution of U-statistics based on kernels of bounded Hardy-Krause variation we use the following inequality by Beard (2009).

Lemma 2.7. \textbf{(Beard (2009))} Let $Z$ be a random vector taking values in a bounded $u+v$-dimensional rectangle $R = [a, b] \subset \mathbb{R}^{u+v}$, $u, v \in \mathbb{N}$. $R$ is chosen such that each $Z_i$ is equal to $a_i$ with probability zero. For a non-empty set $K \subseteq \{1, ..., u + v\}$, let $R_K = \prod_{k \in K} [a_k, b_k]$, and let $F_K$ denote the joint distribution of those $Z_k$ for which $k \in K$. Let $F_0 = 1$. $X = (Z_1, ..., Z_u)$ and $Y = (Z_{u+1}, ..., Z_{u+v})$ and the two real functions $f$ and $g$ are defined on $R_{\{1, ..., u\}}$ and $R_{\{u+1, ..., u+v\}}$. Suppose $f$ and $g$ are of bounded Hardy-Krause variation $(||f||_{HK}, ||g||_{HK} < \infty)$ and left-continuous. Then we have,

$$|\text{Cov}(f(X), g(Y))| \leq \gamma ||f||_{HK} ||g||_{HK},$$

(2.4)
if, $\gamma < \infty$ is such that,
$$\|F_{I \cup J} - F_I F_J\|_\infty \leq \gamma,$$
for all non-empty sets $I \subseteq \{1, 2, ..., u\}$ and $J \subseteq \{u + 1, u + 2, ..., u + v\}$. If $f : [a, b] \to \mathbb{R}$ is of bounded Hardy-Krause variation, then for any $x \in (a, b)$ there exists a value, denoted by $f^-(x)$ such that $f(x_m) \to f^-(x)$ for any sequence of points $\{x_m\} \in [a, x]$ that converges to $x$. Set $f^+(x) = f(x)$ for $x \notin (a, b]$. If $f_I = f_I^-$ for all non-empty $I \subseteq \{1, ..., n\}$, then we say $f$ is left-continuous.

**Lemma 2.8.** (Garg and Dewan (2014)) Let $\{X_n, n \geq 1\}$ be a stationary sequence of associated random variables. For each $j$, let $Y_j = f(X_j)$ and $\tilde{Y}_j = \tilde{f}(X_j)$. Suppose that $f \ll \tilde{f}$. Let $\{l_n, n \geq 1\}$ be a sequence of positive integers with $1 \leq l_n \leq n$ and $l_n = o(n)$ as $n \to \infty$. Set $S_j(k) = \sum_{i=j+1}^{j+l} Y_i$, $Y_n = \frac{1}{n} \sum_{j=1}^{n} Y_i$. Assume,
$$\sum_{j=1}^{\infty} \text{Cov}(\tilde{Y}_1, \tilde{Y}_j) < \infty.$$
Then,
$$B_n \to \sigma_f \sqrt{\frac{2}{\pi}} \text{ in } L^2 \text{ as } n \to \infty,$$
where (write $l = l_n$),
$$B_n = \frac{1}{n-l} \left( \sum_{j=0}^{n-l} \frac{|S_j(l) - lY_n|}{\sqrt{l}} \right)$$
and $\sigma_f^2 = \text{Var}(Y_1) + 2 \sum_{j=1}^{\infty} \text{Cov}(Y_1, Y_j)$.

**Lemma 2.9.** (Birkel (1988a)) Let $\{X_n, n \geq 1\}$ be a stationary sequence of associated random variables. with $E(X_j) = 0$ and $|X_j| \leq C < \infty$ for $j \geq 1$. Assume that $u(n) = O(n^{-(r-2)/2})$. Then, there is a constant $B > 0$ not depending on $n$ such that for all $n \geq 1$,
$$\sup_{m \geq 0} E|S_{n+m} - S_m|^r \leq Bn^{r/2},$$
where, $S_n = \sum_{j=1}^{n} X_j$.

### 3 Limiting behavior of U-statistics based on kernels of bounded Hardy-Krause variation.

The main result of this section, Theorem 3.2 in sub-section 3.1, gives the central limit theorem for U-statistics based on a kernel of degree 2 which is of bounded Hardy-Krause variation for a sequence of stationary associated random variables. The extension of this theorem to U-statistics with kernels of a general finite degree $k \geq 3$ are also discussed. We also discuss a strong law of large numbers for U-statistics based on such kernels using the results in Christofides (2004) in the sub-section 3.2.
3.1 Central limit theorem

**Lemma 3.1.** Let \( \{X_n, n \geq 1\} \) be a sequence of stationary associated random variables, such that \( \sum_{j=1}^{\infty} \text{Cov}(X_1, X_j) < \infty \). Assume \( |X_n| < C < \infty, n \geq 1 \). Let \( F \) be the distribution function of \( X_1 \) and \( f \) be its density function which is bounded. Let \( U_n \) be a U-statistic based on a symmetric 2-degree kernel \( \rho(x, y) \) which is of bounded Hardy-Krause variation and left continuous. Let \( \sigma_1^2 = \text{Var}(\rho_1(X_1)) < \infty \). Define,

\[
\sigma_{1j}^2 = \text{Cov}(\rho_1(X_1), \rho_1(X_{1+j})).
\]

Assume

\[
\sum_{j=1}^{\infty} \sigma_{1j}^2 < \infty. \tag{3.1}
\]

Then,

\[
\text{Var}(U_n) = \frac{4\sigma_U^2}{n} + o\left(\frac{1}{n}\right),
\]

where

\[
\sigma_U^2 = \sigma_1^2 + 2 \sum_{j=1}^{\infty} \sigma_{1j}^2. \tag{3.2}
\]

**Proof.** Let \( C_1 \) be a generic positive constant in the sequel. Using H-decomposition,

\[
U_n = \theta + 2H_n^{(1)} + H_n^{(2)}.
\]

Then,

\[
\text{Var}(U_n) = 4\text{Var}(H_n^{(1)}) + \text{Var}(H_n^{(2)}) + 4\text{Cov}(H_n^{(1)}, H_n^{(2)}).
\]

Since \( H_n^{(1)} = \frac{1}{n} \sum_{j=1}^{n} h^{(1)}(X_j) \) and \( \sum_{j=1}^{\infty} \sigma_{1j}^2 < \infty \), from Dewan and Prakasa Rao (2002), we get,

\[
\text{Var}(H_n^{(1)}) = \frac{1}{n} (\sigma_1^2 + 2 \sum_{j=1}^{\infty} \sigma_{1j}^2) + o\left(\frac{1}{n}\right). \tag{3.3}
\]

Now,

\[
E(H_n^{(2)}) = 0, \tag{3.4}
\]

and

\[
E(H_n^{(2)})^2 = \binom{n}{2}^{-2} \sum_{1 \leq i < j \leq n} \sum_{1 \leq k < l \leq n} E\left\{h^{(2)}(X_i, X_j)h^{(2)}(X_k, X_l)\right\}.
\]

When \( i = k \) and \( j = l \),

\[
\sum_{1 \leq i < j \leq n} E\left\{(h^{(2)}(X_i, X_j))^2\right\} = O(n^2). \tag{3.5}
\]

Since \( \rho \) is of bounded Hardy-Krause variation, so is \( h^{(2)} \), hence using Lemma 2.7,

\[
E\left\{h^{(2)}(X_i, X_j)h^{(2)}(X_k, X_l)\right\} \leq \|F_{i\cup j} - F_i F_j\|_\infty \|h^{(2)}\|_{HK}^2.
\]
When $I = \{i, j\}$ and $J = \{k, l\}$, using Lemma 2.3 and Lemma 2.5

\[ \|F_{i,j} - F_{i} F_{j}\|_{\infty} = \sup_{x_1, x_2, x_3, x_4} |P(X_i \leq x_1, X_j \leq x_2, X_k \leq x_3, X_l \leq x_4) - P(X_i \leq x_1, x_j \leq x_2)P(X_k \leq x_3, X_l \leq x_4)| \leq C_1[Cov(X_i, X_k) + Cov(X_i, X_l)] + Cov(X_j, X_k) + Cov(X_j, X_l)] \]

(3.6)

When $I = \{i\}$ and $J = \{k\}$, using Lemmas 2.3

\[ \|F_{i,j} - F_{i} F_{j}\|_{\infty} = \sup_{x, y} |P(X_i \leq x, X_k \leq y) - P(X_i \leq x)P(X_k \leq y)| \leq C_1[Cov(X_i, X_k)] \]

(3.7)

Other cases can be handled similarly. Therefore,

\[ E\left\{h^{(2)}(X_i, X_j)h^{(2)}(X_k, X_l)\right\} \leq C_1[Cov(X_i, X_k) + Cov(X_i, X_l)] + Cov(X_j, X_k) + Cov(X_j, X_l)] \]

\[ = |r([i - k]) + r([i - l]) + r([j - k]) + r([j - l])|, \text{ (say)} \]

Using Lemma 2.1 and \( \sum_{j=1}^{\infty} Cov(X_1, X_j)^{1/3} < \infty \), we have,

\[ \sum_{1 \leq i < j \leq n} \sum_{1 \leq k < l \leq n} E\left\{h^{(2)}(X_i, X_j)h^{(2)}(X_k, X_l)\right\} = o(n^3), \]

and hence,

\[ \text{Var}(H_n^{(2)}) = o\left(\frac{1}{n}\right). \]

(3.8)

From 3.3 and 3.8 and using Cauchy-Schwartz Inequality we have,

\[ |Cov(H_n^{(1)}, H_n^{(2)})| \leq o(n^{-1}). \]

(3.9)

From 3.3, 3.8 and 3.9 we have,

\[ \text{Var}(U_n) = \frac{4\sigma_U^2}{n} + o\left(\frac{1}{n}\right). \]

(3.10)

The following gives the central limit theorem for a U-statistic based on a stationary sequence of associated observations with a kernel of bounded Hardy-Krause variation.

**Theorem 3.2.** Assume the conditions of Lemma 3.1 hold and $0 < \sigma_U^2 < \infty$. Suppose $\exists$ a function $\hat{\rho}_1(\cdot)$ such that $\rho_1 \ll \hat{\rho}_1$ and,

\[ \sum_{j=1}^{\infty} Cov(\hat{\rho}_1(X_i), \hat{\rho}_1(X_j)) < \infty. \]

(3.11)

Then,

\[ \frac{\sqrt{n}(U_n - \theta)}{2\sigma_U} \xrightarrow{L} N(0, 1) \text{ as } n \to \infty, \]

(3.12)

where $\sigma_U^2$ is defined by 3.2.
Proof. The H-decomposition for $U_n$ is

$$U_n = \theta + 2H_n^{(1)} + H_n^{(2)}.$$  

Then,

$$n^{1/2} \frac{(U_n - \theta)}{2\sigma_U} = n^{-1/2} \sum_{j=1}^n \frac{h^{(1)}(X_j)}{\sigma_U} + n^{1/2} \frac{H_n^{(2)}}{\sigma_U}. \quad (3.13)$$

In addition,

$$E(n^{1/2} H_n^{(2)}) = 0,$$

$$n \text{Var}(H_n^{(2)}) \to 0 \text{ as } n \to \infty.$$

from 3.8. Hence,

$$n^{1/2} \frac{H_n^{(2)}}{\sigma_U} \overset{p}{\to} 0 \text{ as } n \to \infty. \quad (3.14)$$

From 3.11 and Lemma 2.4, we get that,

$$n^{-1/2} \sum_{j=1}^n \frac{h^{(1)}(X_j)}{\sigma_U} \overset{p}{\to} N(0, 1) \text{ as } n \to \infty. \quad (3.15)$$

Relations 3.13, 3.14, and 3.15 prove the theorem.

Note. The above results can be easily extended to a U-statistic based on a kernel of any finite degree $k$. Let $U_n$ be the U-statistic based on a finite $k$-degree symmetric kernel $\rho(x_1, x_2, \ldots, x_k)$ which is of bounded Hardy-Krause variation. Suppose $\sigma_U^2 = \text{Var}(\rho_1(X_1)) < \infty$ and $\sum_{j=1}^\infty \sigma_{1j}^2 = \sum_{j=1}^\infty \text{Cov}(\rho_1(X_1), \rho_1(X_j)) < \infty$. Then,

$$\text{Var}(U_n) = \frac{k^2 \sigma_U^2}{n} + o\left(\frac{1}{n}\right). \quad (3.16)$$

If conditions of Theorem 3.2 hold, then

$$\sqrt{n}(U_n - \theta) \xrightarrow{L} N(0, 1) \text{ as } n \to \infty, \quad (3.17)$$

where $\sigma_U^2$ is defined by 3.3.

### 3.2 Strong law of large numbers

Christofides (2004) showed that $\{S_n = \binom{n}{k}U_n, n \geq k\}$ ($U_n$ defined by 1.1), is a demimartingale when $E(\rho) = 0$ and $\rho$ is component-wise nondecreasing. Using the concept of demimartingales, he proved a strong law of large numbers for $U_n$ under restrictions on moments of $\rho$. He also extended the result to U-statistics based on kernels $\rho : [a, b] \to \mathbb{R}$ where $[a, b] = [a_1, b_1] \times \ldots \times [a_k, b_k]$ is a $k$-dimensional rectangle and $\rho = h - g$ where, $h, g : [a, b] \to \mathbb{R}$ are two component-wise nondecreasing functions and $\Delta_R h \geq 0$ and $\Delta_R g \geq 0$ (defined by 1.2), $\forall R = [c_1, d_1] \times \ldots \times [c_k, d_k]$ and $a_i \leq c_i < d_i \leq b_i \forall i = 1, 2, \ldots, k$.

We observe that kernels which are of bounded Hardy-Krause variation fall into the class of kernels discussed by Christofides (2004). Hence, under restrictions on the moments of the kernel, as discussed in Theorem 2.1 and Lemma 2.2 of Christofides (2004) a strong law of numbers is true for U-statistics based on kernels of bounded Hardy-Krause variation.
4 Applications

4.1 Gini’s mean difference

Suppose we want a measure of variability for observations from a distribution \( F \). A possible index of variability is \( \text{Mean difference} \), \( \theta \), given by,

\[
\theta = \int_{\mathbb{R}^2} |x - y|dF(x)dF(y).
\] (4.1)

Given a sample \( \{X_j, 1 \leq j \leq n\} \) from \( F \), an estimator for \( \theta \) is the Gini’s mean difference, \( U_n \), defined by,

\[
U_n = \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} \rho(X_i, X_j),
\] (4.2)

where the kernel \( \rho(x, y) = |x - y| \).

When the observations are independent and identically distributed, the asymptotic distribution of \( U_n \), as discussed in Hoeffding (1948) is,

\[
\frac{\sqrt{n}(U_n - \theta)}{2\sqrt{V}} \xrightarrow{L} N(0, 1) \text{ as } n \to \infty,
\]

where,

\[
V = \mathcal{F} - \theta^2.
\]

Here,

\[
\mathcal{F} = \int_{\mathbb{R}^3} |x - y||x - z|dF(x)dF(y)dF(z).
\]

We now obtain the limiting distribution of \( U_n \) when the observations are stationary and associated.

Theorem 4.1. Let \( \{X_n, n \geq 1\} \) be a sequence of stationary associated random variables having one dimensional marginal distribution function \( F \) and a bounded density function. \( |X_n| < C < \infty, n \geq 1 \). Let,

\[
\sum_{j=1}^{\infty} \text{Cov}(X_1, X_j)^{1/3} < \infty.
\] (4.3)

Then,

\[
\frac{\sqrt{n}(U_n - \theta)}{2\sigma_U} \xrightarrow{L} N(0, 1) \text{ as } n \to \infty,
\]

where, \( U_n \) and \( \sigma_U^2 \) are defined by 3.2 and 3.2 respectively.

Proof.

\[
U_n = \frac{1}{(n\choose 2)} \sum_{1 \leq i < j \leq n} \rho(X_i, X_j).
\]

Then, by H-decomposition,

\[
U_n = \theta + 2H_n^{(1)} + H_n^{(2)},
\]

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where $\theta$ is given by 4.1. We observe $\rho(x, y)$ is a function of bounded Hardy-Krause variation (Definition 1.2). Now,

$$
\rho_1(x) = \int_{-\infty}^{\infty} |x - y| dF(y).
$$

For any $x, z \in \mathbb{R}$, we have,

$$
|\rho_1(x) - \rho_1(z)| = \left| \int_{-\infty}^{\infty} |x - y| dF(y) - \int_{-\infty}^{\infty} |z - y| dF(y) \right|
\leq \int_{-\infty}^{\infty} |x - z| dF(y)
\leq |x - z|.
$$

i.e $\rho_1(\cdot)$ is Lipschitzian. Using 4.3 and Theorem 3.2, we have

$$
\sqrt{n} \left( U_n - \theta \right) \xrightarrow{L} N(0, 1) \quad \text{as} \quad n \to \infty.
$$

$$
\sigma_U^2 = \text{Var}(\rho_1(X_1)) + 2 \sum_{j=2}^{\infty} \text{Cov}(\rho_1(X_1), \rho_1(X_j)). \quad \text{Here,} \quad \text{Var}(\rho_1(X_1)) = \mathbb{F}^2 - \theta^2,
$$

and $\text{Cov}(\rho_1(X_1), \rho_1(X_j)) = \mathbb{E}[|X_1 - Y||X_j - Y|] - \theta^2$,

where, $Y$ is a r.v with the distribution function $F$ and is independent of $(X_1, X_j)$.

Note. The limiting distribution of Gini’s mean difference cannot be obtained using results of Garg and Dewan (2014). Though the kernel $\rho(x, y) = |x - y|$ is continuous, it does not satisfy the conditions of Bulinski’s inequality (Bulinski (1996)). There exist infinitely many points $(x, y) \in \mathbb{R}^2$ at which $\frac{\partial^+ \rho(x, y)}{\partial x} \neq \frac{\partial^+ \rho(x, y)}{\partial y}$ and $\frac{\partial^+ \rho(x, y)}{\partial y} \neq \frac{\partial^+ \rho(x, y)}{\partial y}$.

The next result is needed in simulation analysis. For details, see (5) in section 5.

**Theorem 4.2.** Let $\{X_n, n \geq 1\}$ be a sequence of stationary associated random variables having one dimensional marginal distribution function $F$. $|X_n| \in I \text{ } n \geq 1$, where $I = [-C, C]$, $0 < C < \infty$. Let,

$$
\sum_{j=n+1}^{\infty} \text{Cov}(X_1, X_j) = O(n^{-(l-2)/2}),
$$

(4.4)

for some $l > 2$. Then,

$$
\sup_{x \in I} \left| \frac{\sum_{j=1}^{n}(|X_j - x| - \mathbb{E}|X_j - x|)}{b_n} \right| \to 0 \ a.s \ as \ n \to \infty,
$$

(4.5)

where $b_n = O(n^{1+u/2-p})$, for some $u > 1$ and $p \in (0, 1)$ such that $\frac{l}{2}(1+u) - (l+1)p > 1$. 

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Proof. Divide the interval \( I \) into \( d_n = O(n^p) \), for \( p \in (0, 1) \) small intervals as follows:

Let \( -C = y_{n_0} < y_{n_1} < \ldots < y_{n_d} = C \). The \( d_n \) intervals are denoted as \( I_{n_i} = [y_{n_{i-1}}, y_{n_i}], i = 1, 2, \ldots, d_n \), each of length \( \delta_n = \frac{2C}{d_n} \). Let \( x_{n_i} \in I_{n_i} \).

\[
\sup_{x \in I} \sum_{j=1}^n \frac{|X_j - x - E(|X_j - x|)|}{b_n} = \max_i \sup_{x \in I_{n_i}} \left| \sum_{j=1}^n \frac{|X_j - x - E(|X_j - x|)|}{b_n} \right |
\]

\[
= \max_i \sup_{x \in I_{n_i}} \left| \sum_{j=1}^n \frac{|X_j - x - E(|X_j - x|)|}{b_n} \right | + \max_i \sup_{x \in I_{n_i}} \left| \sum_{j=1}^n \frac{|X_j - x - E(|X_j - x|)|}{b_n} \right |
\]

\[
\leq \max_i \sup_{x \in I_{n_i}} \left| \sum_{j=1}^n \frac{|X_j - x - E(|X_j - x|)|}{b_n} \right | + \max_i \sup_{x \in I_{n_i}} \left| \sum_{j=1}^n \frac{|X_j - x - E(|X_j - x|)|}{b_n} \right |
\]

\[
= I_1 + I_2 + I_3 \text{ (say).} \quad (4.6)
\]

Now, \( \frac{\sum_{j=1}^n (|X_j - x| - |X_j - x_{n_i}|)}{b_n} \leq \frac{\delta_n}{b_n} \). Hence, \( I_1 \leq \frac{\delta_n}{b_n} \). Similarly, \( I_3 \leq \frac{\delta_n}{b_n} \).

\[
I_2 = \max_i \sup_{x \in I_{n_i}} \left| \sum_{j=1}^n \frac{|X_j - x - E(|X_j - x|)|}{b_n} \right | = \max_i \left| \sum_{j=1}^n \frac{|X_j - x - E(|X_j - x|)|}{b_n} \right |
\]

For any \( \epsilon > 0 \),

\[
P\left[ \max_i \left| \sum_{j=1}^n \frac{|X_j - x - E(|X_j - x|)|}{b_n} \right | > \epsilon \right ]
\]

\[
\leq \sum_i P \left[ \left| \sum_{j=1}^n \frac{|X_j - x - E(|X_j - x|)|}{b_n} \right | > \epsilon \right ]
\]

\[
= \sum_i P \left[ \frac{|S_n(x_{n_i})|}{b_n} > \epsilon \right ] \leq d_n \frac{E|S_n(x_{n_i})|^l}{b_n^l \epsilon^l}. \quad (4.7)
\]

where \( S_n(x_{n_i}) = \sum_{j=1}^n (|X_j - x| - E|X_j - x|) \). Let \( B \) be a generic positive constant in the sequel.

\[
\frac{E|S_n(x_{n_i})|^l}{b_n^l \epsilon^l} = \frac{1}{2^l} \frac{E|S_n(x_{n_i}) - \tilde{S}_n(x_{n_i}) + S_n(x_{n_i}) + \tilde{S}_n(x_{n_i})|^l}{b_n^l \epsilon^l}, \quad (4.8)
\]

where, \( \tilde{S}_n(x_{n_i}) = \sum_{j=1}^n B(X_j - x_{n_i}) \). Observe, \( |X_j - x| \ll B(X_j - x) \forall x \in I \).

Using the equality \( (a + b)^l = \sum_{j=0}^l \binom{l}{j} a^j b^{l-j} \) and the Cauchy-Schwarz inequal-

\[
ity, we have, \( E|S_n(x_{n_i}) - \tilde{S}_n(x_{n_i}) + S_n(x_{n_i}) + \tilde{S}_n(x_{n_i})|^l \leq \sum_{j=0}^l \binom{l}{j} E|S_n(x_{n_i}) - \tilde{S}_n(x_{n_i})|^2 E|S_n(x_{n_i}) + \tilde{S}_n(x_{n_i})|^2(l-j) \) \)^{1/2}.

Both \( \{S_n(x_{n_i}) + \tilde{S}_n(x_{n_i}); n \geq 1\} \) and \( \{S_n(x_{n_i}) - S_n(x_{n_i}); n \geq 1\} \) form an associated sequence. Now,

\[
\sum_{j=n+1}^\infty \text{Cov}(|X_1 - x| - E|X_j - x| - B(X_j - x), |X_j - x| - E|X_j - x| - B(X_j - x)) \leq B \sum_{j=n+1}^\infty \text{Cov}(X_1 - x, X_j - x) = B \sum_{j=n+1}^\infty \text{Cov}(X_1, X_j) = O(n^{-(l-2)/2}), \forall x \in I.
\]
Similarly, \( \sum_{j=n+1}^{\infty} \text{Cov}(|X_1-x|-E|X_j-x|+B(X_j-x), |X_j-x|-E|X_j-x|+B(X_j-x)) = O(n^{-(l-2)/2}), \forall x \in I. \) Using Lemma 2.9, we get,

\[
\frac{E|S_n(x_n)|}{b_n^e} \leq B \frac{(n^j n^{-j})^{1/2}}{b_n^e 2^l}.
\] (4.9)

From 4.6 we have,

\[
\sup_{x \in I} \left| \sum_{j=1}^{n}(X_j-x-E|X_j-x|) \right| \leq \frac{2n\delta_n}{b_n} + \max_{i} \left| \sum_{j=1}^{n}(X_j-x_n-E|X_j-x_n|) \right|.
\] (4.10)

Hence,

\[
P\left[ \sup_{x \in I} \left| \sum_{j=1}^{n}(X_j-x-E|X_j-x|) \right| > \epsilon \right] \leq P\left[ \frac{2n\delta_n}{b_n} > \frac{\epsilon}{2} \right] + P\left[ \max_{i} \left| \sum_{j=1}^{n}(X_j-x_n-E|X_j-x_n|) \right| > \frac{\epsilon}{2} \right]
\]

\[
\leq P\left[ \frac{4Cn}{b_n d_n} > \frac{\epsilon}{2} \right] + B \frac{n^{l/2} d_n}{b_n^e 2^l}
\]

\[
\leq \left( \frac{8Cn}{b_n d_n \epsilon} \right)^2 + B \frac{n^{l/2} d_n}{b_n^e 2^l}.
\] (4.11)

Finally,

\[
\sum_{n=1}^{\infty} P\left[ \sup_{x \in I} \left| \sum_{j=1}^{n}(X_j-x-E|X_j-x|) \right| > \epsilon \right] \leq B \sum_{n=1}^{\infty} \left\{ \left( \frac{n}{b_n d_n} \right)^2 + \frac{n^{l/2} d_n}{b_n^e 2^l} \right\}.
\] (4.12)

Result follows if \( b_n = O(n^{1+u/2-p}), \) for some \( u > 1 \) and \( \frac{1}{\gamma} (1 + u) - (l + 1)p > 1. \)

### 4.2 Empirical joint distribution functions

Let \( \{X_n, n \geq 1\} \) be a sequence of stationary associated random variables. The asymptotics for the empirical estimator of survival function for this sequence was discussed in Bagai and Prakasa Rao (1991). Henriques and Oliveira (2003) had discussed the asymptotics for the histogram estimator for the two-dimensional distribution function of \((X_1, X_{k+1}).\) In both the cases, the kernel is of bounded Hardy-Krause variation. Similarly, the kernel of the histogram estimator for any finite k-dimensional distribution function is also of bounded Hardy-Krause variation. Hence, the results in section 3 can be used for studying the asymptotic behavior of U-statistics based on these kernels.
5 Simulation Analysis

The asymptotic normality of Gini’s mean difference based on stationary and associated observations were investigated via simulations.

Let \( \{Y_i, i \geq 1\} \) be i.i.d from \( \text{Exp}(1/m) \) for some \( m \in \mathbb{N} \). Putting \( X_i = \min(Y_i, ..., Y_{i+m-1}) \), \( \forall \ 1 \leq i \leq n \), \( \{X_i, 1 \leq i \leq n\} \) forms a set of stationary associated random variables such that \( X_i's \) are standard exponential variables (\( \text{Exp}(1) \)). Similarly, in order to obtain stationary associated random variables \( \{X_i, 1 \leq i \leq n\} \) such that \( X_i's \) are standard normal variables (\( \mathcal{N}(0,1) \)), we can set \( X_i = Y_i + ... + Y_{i+m-1} \), where \( \{Y_i, i \geq 1\} \) are i.i.d from \( \mathcal{N}(0,1/m) \) for some \( m \in \mathbb{N} \).

1. We used the statistical software R (http://www.r-project.org; R Development Core Team (2011)) for our simulations. The samples \( \{X_k, 1 \leq k \leq n\} \) were generated as follows,

(S1) \( X_k = \min(Y_k, Y_{k+1}) \), and \( \{Y_k, k \geq 1\} \) are pseudo-random numbers from \( \text{Exp}(1/2) \) using \( \text{rexp} \) function in R.

(S2) \( X_k = \min(Y_k, Y_{k+1}, Y_{k+2}) \), and \( \{Y_k, k \geq 1\} \) are pseudo-random numbers from \( \text{Exp}(1/3) \) using \( \text{rexp} \) function in R.

(S3) \( X_k = \min(Y_k, ..., Y_{k+9}) \), and \( \{Y_k, k \geq 1\} \) are pseudo-random numbers from \( \exp(1/10) \) using \( \text{rexp} \) function in R.

(S4) \( X_k = Y_k + Y_{k+1} \), and \( \{Y_k, k \geq 1\} \) are pseudo-random numbers from \( \mathcal{N}(0,1/2) \), using \( \text{rnorm} \) function in R.

(S5) \( X_k = Y_k + Y_{k+1} + Y_{k+2} \), and \( \{Y_k, k \geq 1\} \) are pseudo-random numbers from \( \mathcal{N}(0,1/3) \), using \( \text{rnorm} \) function in R.

(S6) \( X_k = Y_k + ... + Y_{k+9} \), and \( \{Y_k, k \geq 1\} \) are pseudo-random numbers from \( \mathcal{N}(0,1/10) \), using \( \text{rnorm} \) function in R.

2. The results are based on 10,000 replications.

3. \( \alpha = 0.05 \).

4. For our simulations, we used Lemma 2.8 for the estimation of \( \sigma_U \). We chose \( l_n = \lfloor n^{3/5} \rfloor \), smallest integer less than or equal to \( n^{3/5} \).

5. In Lemma 2.8 \( Y_i = \rho_1(X_i) \) and \( \tilde{Y}_i = C_1X_i \), for some constant \( C_1 \), \( C_1 > 0 \), \( \forall i \geq 1 \), as \( \rho_1 \) is lipshitz. For practical applications, the distribution function of the underlying population \( F \) will be unknown. Hence, an estimator for \( \rho_1(x) \) is needed. Let \( \hat{B}_n \) be analogous to \( B_n \) with \( S_j(k) \) replaced by \( \hat{S}_j(k) = \sum_{i=j+1}^{j+k} \hat{\rho}_1(X_i) \), and \( \hat{Y}_n \) by \( \hat{Y}_n = \sum_{i=1}^{n} \hat{\rho}_1(X_i) \), where \( \hat{\rho}_1(x) = \sum_{i=1}^{n} \frac{|X_i-x|}{n} \).

\[
|B_n - \hat{B}_n| = \left| \frac{1}{n-l+1} \sum_{j=1}^{n-l} \frac{|S_j(l) - l\hat{Y}_n|}{\sqrt{l}} - \frac{1}{n-l+1} \sum_{j=1}^{n-l} \frac{\hat{S}_j(l) - l\hat{Y}_n}{\sqrt{l}} \right|
\leq \frac{1}{(n-l+1)\sqrt{l}} \sum_{j=1}^{n-l} |S_j(l) - \hat{S}_j(l) - l(\hat{Y}_n - \hat{Y}_n)|
\leq \frac{1}{(n-l+1)\sqrt{l}} \sum_{j=1}^{n-l} \left( \sum_{i=j+1}^{j+k} |Z_i| + \frac{j}{n} \sum_{i=1}^{n} |Z_i| \right),
\]

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where $|Z_i| = 2 |\hat{\rho}_1(X_i) - \rho_1(X_i)|$. Therefore,

$$|B_n - \hat{B}_n| \leq \frac{2}{(n - l + 1)\sqrt{l}} l \sup_x |\hat{\rho}_1(x) - \rho_1(x)|$$

$$= 2 \sqrt{l} \sup_x \frac{n}{n} \sum_{j=1}^{n} |X_j - x| - \rho_1(x)$$

Putting $s = \frac{3}{10}$, we get $\sqrt{l} = O(1)$. In Theorem 4.2 putting $l = 10$, $p = 17/20$, $u = 11/10$ and assuming $\sum_{j=n+1}^{\infty} \text{Cov}(X_1, X_j) = O(n^{-1})$, we get,

$$\sup_{x} |\hat{\rho}_1(x) - \rho_1(x)| = 2 \sqrt{\ln s}$$

$$= \frac{n}{s} \sum_{j=1}^{n} |X_j - x| - \rho_1(x)$$

$$n^{-1/2} \rightarrow 0 \text{ a.s as } n \rightarrow \infty$$.

In the following tables,

1) $\bar{g}$ denotes the mean of the $r = 10,000$ sample gini mean difference values, $g_i$, $1 \leq i \leq r$;
2) E.M.S.E (g) = $\frac{1}{r} \sum_{i=1}^{r} (g_i - \bar{g})$, where E.M.S.E denotes Estimated M.S.E;
3) C.P (g) = $\frac{N}{r}$, where $N = \#g_i \in \{\bar{g} - \hat{B}_n \times \frac{z_{0.025}}{\sqrt{n}}, \bar{g} + \hat{B}_n \times \frac{z_{0.025}}{\sqrt{n}}\}$, $\hat{B}_n = \frac{1}{r} \hat{B}_n(i)$, $z_{0.025} = 1.959964$;
4) E.M.S.E (\hat{B}_n) = $\frac{1}{r} \sum_{i=1}^{r} (\hat{B}_n(i) - \bar{\hat{B}}_n)$;
5) Median (g), Skewness (g), and Kurtosis (g) are the corresponding characteristics of the $r$ sample statistic values.

### Table 5.1 Simulation Results for Exp(1)

| $(S1)$ (m=2) | n=50   | n=100  | n=200  | n=300  | n=500  | n=1000 |
|--------------|--------|--------|--------|--------|--------|--------|
| $\bar{g}$    | 0.983618 | 0.993008 | 0.998001 | 0.998375 | 0.998474 | 0.998946 |
| E.M.S.E (g)  | 0.036769 | 0.019313 | 0.009706 | 0.006435 | 0.003937 | 0.001931 |
| C.P (g)      | 0.8974  | 0.9133  | 0.9258  | 0.9316  | 0.9342  | 0.9403  |
| Median (g)   | 0.971188 | 0.984801 | 0.994159 | 0.994734 | 0.997815 | 0.998264 |
| Skewness (g) | 0.39651 | 0.34662 | 0.20073 | 0.20064 | 0.12031 | 0.089361 |
| Kurtosis (g) | 3.25643 | 3.21907 | 3.02696 | 3.00296 | 3.00163 | 3.02015 |

| $(S2)$ (m=3) | n=50   | n=100  | n=200  | n=300  | n=500  | n=1000 |
|--------------|--------|--------|--------|--------|--------|--------|
| $\bar{g}$    | 0.973299 | 0.987549 | 0.994166 | 0.997279 | 0.997227 | 0.998071 |
| E.M.S.E (g)  | 0.049721 | 0.026710 | 0.013384 | 0.008551 | 0.005316 | 0.002624 |
| C.P (g)      | 0.8804  | 0.9002  | 0.9201  | 0.9241  | 0.9296  | 0.9396  |
| Median (g)   | 0.971188 | 0.984801 | 0.994159 | 0.994734 | 0.997815 | 0.998264 |
| Skewness (g) | 0.521839 | 0.351775 | 0.275627 | 0.240718 | 0.160427 | 0.128438 |
| Kurtosis (g) | 3.513031 | 3.167723 | 3.106484 | 3.15176  | 3.057265 | 3.060678 |

| $(S3)$ (m=10) | n=50   | n=100  | n=200  | n=300  | n=500  | n=1000 |
|--------------|--------|--------|--------|--------|--------|--------|
| $\bar{g}$    | 0.875857 | 0.93135 | 0.962747 | 0.977913 | 0.986905 | 0.994086 |
| E.M.S.E (g)  | 0.128077 | 0.071580 | 0.038874 | 0.026338 | 0.016484 | 0.008293 |
| C.P (g)      | 0.7579  | 0.8134  | 0.8478  | 0.8723  | 0.8889  | 0.9097  |
| Median (g)   | 0.826053 | 0.903345 | 0.949436 | 0.967137 | 0.979577 | 0.990814 |
| Skewness (g) | 0.915153 | 0.629382 | 0.427241 | 0.399389 | 0.350814 | 0.228297 |
| Kurtosis (g) | 4.462582 | 3.537275 | 3.239824 | 3.336504 | 3.231984 | 3.017788 |
Observations

(i) Estimation of $\sigma_U$: As discussed earlier, we have used an estimator for $\sigma_U$ for simulations. \([5]\) and Lemma \([2,8]\) implies that $\sqrt{\pi/2}B_n$ is also a consistent estimator for $\sigma_U$. For the sample generated from $\text{Exp}(1)$, using \((S1), (S2),\), and \((S3)\), we analyse the performance of the estimator by comparing $2\sqrt{\pi/2}B_n$ with the actual values ($2\sigma_U$). The following table shows that as the sample size increases the value of bias reduces. As expected, $E.M.S.E$ (Estimated M.S.E) also reduces with the increase in the sample size. For $m = 2, 3$, the rate of convergence is faster than for $m = 10$.

(ii) Asymptotic Normality: For a fixed $m$, we observe that as the sample size increases, the approximation to the normal distribution is better. For $m = 2, 3$, the convergence to normality is faster, as expected, as the variables are “almost independent”. For $m = 10$, we see that the approximation is good only for much larger values of $n$. The use of the estimator of $\sigma_U$ could also affect the convergence as the bias and $E.M.S.E$ (Estimated M.S.E) reduces much faster for $m = 2, 3$ than for $m = 10$. 

### Table 5.2 Simulation Results for $N(0, 1)$

| $(S4)$ $(m = 2)$ | $n=50$ | $n=100$ | $n=200$ | $n=300$ | $n=500$ | $n=1000$ |
|-----------------|--------|--------|--------|--------|--------|----------|
| $\bar{g}$      | 1.114259 | 1.120531 | 1.125395 | 1.126033 | 1.127356 | 1.127414  |
| E.M.S.E $(g)$  | 0.01965189 | 0.00962333 | 0.004895764 | 0.003354603 | 0.001963522 | 0.000978631 |
| C.P $(g)$      | 0.9077  | 0.9262  | 0.9317  | 0.9327  | 0.9413  | 0.9441    |
| Median $(g)$   | 1.111408 | 1.119902 | 1.124416 | 1.124914 | 1.127125 | 1.127346  |
| Skewness $(g)$ | 0.1287656 | 0.1264063 | 0.09834354 | 0.06234944 | 0.07745284 | 0.04384436 |
| Kurtosis $(g)$ | 2.993197 | 2.923884 | 2.967621 | 2.907618 | 2.947061 | 2.880582  |

| $(S5)$ $(m = 3)$ | $n=50$ | $n=100$ | $n=200$ | $n=300$ | $n=500$ | $n=1000$ |
|-----------------|--------|--------|--------|--------|--------|----------|
| $\bar{g}$      | 1.102877 | 1.114272 | 1.122001 | 1.122333 | 1.125143 | 1.126909  |
| E.M.S.E $(g)$  | 0.02673821 | 0.01331181 | 0.006773474 | 0.004577988 | 0.002688516 | 0.00137823 |
| C.P $(g)$      | 0.8923  | 0.9173  | 0.9259  | 0.9317  | 0.9425  | 0.9483    |
| Median $(g)$   | 1.096844 | 1.11083 | 1.119453 | 1.121584 | 1.124243 | 1.126417  |
| Skewness $(g)$ | 0.2345972 | 0.1703191 | 0.1451524 | 0.08957866 | 0.05842091 | 0.04335457 |
| Kurtosis $(g)$ | 3.027241 | 3.073061 | 2.992162 | 3.030902 | 3.099202 | 2.987061  |

| $(S6)$ $(m = 10)$ | $n=50$ | $n=100$ | $n=200$ | $n=300$ | $n=500$ | $n=1000$ |
|------------------|--------|--------|--------|--------|--------|----------|
| $\bar{g}$       | 1.004597 | 1.064742 | 1.096209 | 1.10654 | 1.113758 | 1.121415  |
| E.M.S.E $(g)$   | 0.06772895 | 0.0381105 | 0.02010571 | 0.01393451 | 0.008248388 | 0.004292068 |
| C.P $(g)$       | 0.7678  | 0.8503  | 0.8878  | 0.9186  | 0.9425  | 0.9677    |
| Median $(g)$    | 0.9787768 | 1.051586 | 1.088681 | 1.101294 | 1.112035 | 1.119691  |
| Skewness $(g)$  | 0.6010408 | 0.4948426 | 0.2736039 | 0.2290044 | 0.2082488 | 0.1407511  |
| Kurtosis $(g)$  | 3.470702 | 3.252502 | 3.034935 | 3.044965 | 3.075823 | 3.000726  |

### Table 5.3 Performance of $\hat{B}_n$ for $\text{Exp}(1)$

| $(S1)$ $(m=2)$ | $2\sigma_U = 1.393864$ | $n=50$ | $n=100$ | $n=200$ | $n=300$ | $n=500$ | $n=1000$ |
|----------------|------------------------|--------|--------|--------|--------|--------|----------|
| $2\sqrt{\pi/2B_n}$ | 1.113067 | 1.206764 | 1.26211 | 1.285339 | 1.312719 | 1.335088  |
| Bias $= 2\sqrt{\pi/2B_n} - \sigma_U$ | 0.280797 | 0.187100 | 0.131754 | 0.108525 | 0.081145 | 0.058776  |
| E.M.S.E $(2\sqrt{\pi/2B_n})$ | 0.3941304 | 0.1271084 | 0.09828345 | 0.0641774 | 0.0455669 | 0.029956 |

| $(S2)$ $(m=3)$ | $2\sigma_U = 1.639871$ | $n=50$ | $n=100$ | $n=200$ | $n=300$ | $n=500$ | $n=1000$ |
|----------------|------------------------|--------|--------|--------|--------|--------|----------|
| $2\sqrt{\pi/2B_n}$ | 1.217808 | 1.345743 | 1.434806 | 1.470134 | 1.506485 | 1.544193  |
| Bias $= 2\sqrt{\pi/2B_n} - \sigma_U$ | 0.422063 | 0.294128 | 0.205065 | 0.169737 | 0.133386 | 0.095678  |
| E.M.S.E $(2\sqrt{\pi/2B_n})$ | 0.3784683 | 0.19717176 | 0.1303082 | 0.1002306 | 0.150937 | 0.046944  |

| $(S3)$ $(m=10)$ | $2\sigma_U = 2.897561$ | $n=50$ | $n=100$ | $n=200$ | $n=300$ | $n=500$ | $n=1000$ |
|----------------|------------------------|--------|--------|--------|--------|--------|----------|
| $2\sqrt{\pi/2B_n}$ | 1.453125 | 1.764125 | 2.013354 | 2.149753 | 2.297722 | 2.469504  |
| Bias $= 2\sqrt{\pi/2B_n} - \sigma_U$ | 1.462436 | 1.36136 | 0.886207 | 0.747808 | 0.599841 | 0.428507  |
| E.M.S.E $(2\sqrt{\pi/2B_n})$ | 0.7394865 | 0.6412837 | 0.4948429 | 0.431475 | 0.3136523 | 0.205857  |
(iii) **Estimation of the mean difference:** When $X'_i$'s are $Exp(1)$, the value of the mean difference, $\theta$, is 1. From Table 5.1 it can be seen that when $m = 2, 3$, the convergence of the mean of 10,000 sample gini mean difference values to 1 is faster than when $m = 10$. This is expected as greater dependence leads to a slower rate. Similar results are observed from Table 5.2. Here $\theta = 1.128379$.

(iv) **Comparison with i.i.d setup:** A comparison of the simulation results with the results of Greselin and Zenga (2006) who had performed the simulations for the statistic under the i.i.d setup, indicate that larger sample sizes are needed for applying the asymptotic results under the dependent setup than under the i.i.d setup.

6 Conclusions

In this paper, we gave the limiting distribution of U-statistics based on kernels of bounded Hardy-Krause variation when the underlying sample consists of stationary associated observations. As an application, we obtain the asymptotic distribution of Gini’s mean difference under the dependent setup. The limiting distribution of Gini’s mean difference cannot be obtained using results of Garg and Dewan (2014) as the kernel $\rho(x, y) = |x - y|$ though continuous, does not satisfy the conditions of Bulinski’s inequality (Bulinski (1996)). Simulation results performed for the statistic indicate that reasonable sample sizes are needed for using the normality approximation. Greater the dependence, larger the sample sizes needed for a viable use of the asymptotic normality results.

Asymptotic distribution for U-statistics based on continuous kernels for associated random variables are discussed in Dewan and Prakasa Rao (2002) and Garg and Dewan (2014). Results for dis-continuous kernels that are not component-wise monotonic and are not functions of bounded Hardy-Krause variations are being presented elsewhere.

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