A Lower Bound of the First Dirichlet Eigenvalue of a Compact Manifold with Positive Ricci Curvature

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Abstract

We give a new estimate on the lower bound for the first Dirichlet eigenvalue for a compact manifold with positive Ricci curvature in terms of the in-diameter and the lower bound of the Ricci curvature. The result improves the previous estimates.

1 Introduction

If \((M, g)\) is an \(n\)-dimensional compact Riemannian manifold whose Ricci curvature has a positive lower bound \((n - 1)K\) for some constant \(K > 0\) and whose non-empty boundary \(\partial M\) has nonnegative mean curvature with respect to the outward normal, Reilly \([10]\) gave the following lower bound of the first Dirichlet eigenvalue \(\lambda\) of the Laplacian on \(M\)

\[
\lambda \geq nK.
\]

This estimate gives no information when the above constant \(K\) vanishes. In such case, Li-Yau \([5]\) and Zhong-Yang \([14]\) provided another lower bound for the first non-zero eigenvalue of a closed manifold

\[
\lambda \geq \frac{\pi^2}{d^2}.
\]

It is an interesting problem to find a unified lower bound of the first Dirichlet eigenvalue \(\lambda\) in terms of the lower bound \((n - 1)K\) of the Ricci curvature and the diameter \(d\), in-diameter \(\tilde{d}\) and other geometric quantities, which do not vanish as \(K\) vanishes, of the manifold with positive Ricci curvature. D. Yang \([12]\) proved that

\[
\lambda \geq \frac{1}{4}(n - 1)K + \frac{\pi^2}{(d)^2}.
\]

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where $\tilde{d}$ is the diameter of the largest interior ball in $M$.

In this paper we give a new estimate on the lower bound of the first Dirichlet eigenvalue $\lambda$. We have the following result.

**Theorem 1.** If $(M, g)$ is an $n$-dimensional compact Riemannian manifold with boundary. Suppose that Ricci curvature $\text{Ric}(M)$ of $M$ is bounded below by $(n - 1)K$ for some constant $K > 0$

\begin{equation}
\text{Ric}(M) \geq (n - 1)K
\end{equation}

and that the mean curvature of the boundary $\partial M$ with respect to the outward normal is nonnegative, then the first Dirichlet eigenvalue $\lambda$ of the Laplacian $\Delta$ of $M$ has the following lower bound

\begin{equation}
\lambda \geq \frac{1}{2}(n - 1)K + \frac{\pi^2}{(\tilde{d})^2},
\end{equation}

where $\tilde{d}$ is the diameter of the largest interior ball in $M$, that is, $\tilde{d} = 2\sup_{x \in M}\{\text{dist}(x, \partial M)\}$.

Our result improves Yang’s bound \(^2\) by doubling the coefficient before $(n - 1)K$. In the proof, we use a function $\xi$ that the author constructed in \(^8\) for the construction of the suitable test function instead of using the Zhong-Yang’s canonical function. That provides a new way to sharpen the bound. In the next section, we derive some preliminary estimates and conditions for test functions first and we construct the needed test function and prove the main result in the last section.

## 2 Preliminary Estimates

The first basic estimate is of Lichnerowicz-type. Recall that the classic Lichnerowicz Theorem \(^6\) states that if $M$ is an $n$-dimensional closed manifold whose Ricci curvature satisfies \(^3\) then the first non-zero eigenvalue has a lower bound \(^1\). Reilly \(^10\) proved that this Lichnerowicz-type estimate remains true for the first Dirichlet eigenvalue $\lambda$ as well if the manifold has the same lower bound for the Ricci curvature and has non-empty boundary whose mean curvature with respect to the outward normal is nonnegative. For the completeness and consistency, we use gradient estimate in \(2\)-\(5\) and \(11\) to derive the Lichnerowicz-type estimate.

**Lemma 1.** Under the conditions in Theorem 1 the estimate \(^7\) holds.
Proof. Let \( v \) be a normalized eigenfunction of the first Dirichlet eigenvalue such that

\[
\begin{align*}
\sup_M v &= 1, \quad \inf_M v = 0.
\end{align*}
\]

The function \( v \) satisfies the following

\[
\begin{align*}
\Delta v &= -\lambda v \quad \text{in } M \\
v &= 0 \quad \text{on } \partial M.
\end{align*}
\]

Take an orthonormal frame \( \{e_1, \ldots, e_n\} \) of \( M \) about \( x_0 \in M \). At \( x_0 \) we have

\[
\nabla_{e_j} (|\nabla v|^2)(x_0) = \sum_{i=1}^n 2v_i v_{ij}
\]

and

\[
\Delta(|\nabla v|^2)(x_0) = 2 \sum_{i,j=1}^n v_{ij} v_{ij} + 2 \sum_{i,j=1}^n v_i v_{jj} + 2 \sum_{i,j=1}^n R_{ij} v_i v_j
\]

\[
\geq 2 \sum_{i=1}^n v_i^2 + 2|\nabla v| \nabla v(\Delta v) + 2(n-1)K|\nabla v|^2
\]

\[
\geq \frac{2}{n} \frac{(\Delta v)^2}{(\Delta v)^2} - 2|\nabla v|^2 + 2(n-1)K|\nabla v|^2.
\]

Thus at all point \( x \in M \),

\[
\frac{1}{2} \Delta(|\nabla v|^2) \geq \frac{1}{n} \lambda^2 v^2 + [(n-1)K - \lambda]|\nabla v|^2.
\]

On the other hand, after multiplying (6) by \( v \) and integrating both sides over \( M \) and using (7), we have

\[
\int_M \lambda v^2 \, dx = - \int_M v \Delta v \, dx
\]

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\[ \int_{\partial M} v \frac{\partial}{\partial \nu} v \, ds + \int_M |\nabla v|^2 \, dx = \int_M |\nabla v|^2 \, dx, \]

where and below \( \nu \) is the outward normal of \( \partial M \). That the integral on the boundary vanishes is due to (7). Integrating (8) over \( M \) and using the above equality, we get

(9) \[ \frac{1}{2} \int_{\partial M} \frac{\partial}{\partial \nu} (|\nabla v|^2) \, dx \geq \int_M (nK - \lambda) \frac{n-1}{n} \lambda v^2 \, dx. \]

We need show that \( \frac{\partial}{\partial \nu} (|\nabla v|^2) \leq 0 \) on \( \partial M \). Take any \( x_0 \in \partial M \). If \( \nabla v(x_0) = 0 \), then it is done. Assume now that \( \nabla v(x_0) \neq 0 \). Choose a local orthonormal frame \( \{ e_1, e_2, \cdots, e_n \} \) of \( M \) about \( x_0 \) so that \( e_n \) is the unit outward normal vector field near \( x_0 \in \partial M \) and \( \{ e_1, e_2, \cdots, e_{n-1} \} \mid_{\partial M} \) is a local frame of \( \partial M \) about \( x_0 \). The existence of such local frame can be justified as the following. Let \( e_n \) be the local unit outward normal vector field of \( \partial M \) about \( x_0 \in \partial M \) and \( \{ e_1, \cdots, e_{n-1} \} \) the local orthonormal frame of \( \partial M \) about \( x_0 \). By parallel translation along the geodesic \( \gamma(t) = \exp_{x_0} t e_n \), we may extend \( e_1, \cdots, e_{n-1} \) to local vector fields of \( M \). Then the extended frame \( \{ e_1, e_2, \cdots, e_n \} \) is what we need. Note that \( \nabla e_i e_i = 0 \) for \( i \leq n - 1 \).

Since \( v |_{\partial M} = 0 \), we have \( v_i(x_0) = 0 \) for \( i \leq n - 1 \). Using (5)-(7) in the following arguments, then we have that at \( x_0 \),

\[ \frac{\partial}{\partial \nu} (|\nabla v|^2)(x_0) = \sum_{i=1}^{n} 2 v_i v_{in} = 2 v_n v_{nn} \]

\[ = 2 v_n (\Delta^M v - \sum_{i=1}^{n-1} v_{ii}) = 2 v_n (-\lambda v - \sum_{i=1}^{n-1} v_{ii}) \]

\[ = -2 v_n \sum_{i=1}^{n-1} v_{ii} = -2 v_n \sum_{i=1}^{n-1} (e_i e_i v - \nabla^M_{e_i} e_i v) \]

\[ = 2 v_n \sum_{i=1}^{n-1} \nabla^M_{e_i} e_i v = 2 v_n \sum_{i=1}^{n-1} \sum_{j=1}^{n} g(\nabla^M_{e_i} e_i, e_j) v_j \]

\[ = 2 v_n^2 \sum_{i=1}^{n-1} g(\nabla^M_{e_i} e_i, e_n) = -2 v_n^2 \sum_{i=1}^{n-1} g(\nabla^M_{e_i} e_n, e_i) \]

\[ = -2 v_n^2 \sum_{i=1}^{n-1} h_{ii} = -2 v_n^2 (x_0) m(x_0) \]

(10) \[ \leq 0 \quad \text{by the non-negativity of} \ m, \]
where $g(\cdot, \cdot)$ is the Riemann metric of $M$, $(h_{ij})$ is the second fundamental form of $\partial M$ with respect to the outward normal $\nu$ and $m$ is the mean curvature of $\partial M$ with respect to $\nu$. Therefore (1) holds. 

**Lemma 2.** Let $v$ be, as the above, the normalized eigenfunction for the first Dirichlet eigenvalue $\lambda$. Then $v$ satisfies the following

\begin{equation}
|\nabla v|^2 \leq \frac{\lambda}{b^2 - v^2},
\end{equation}

where $b > 1$ is an arbitrary constant.

**Proof.** Consider the function

\begin{equation}
P(x) = |\nabla v|^2 + Av^2,
\end{equation}

where $A = \lambda(1 + \epsilon)$ for small $\epsilon > 0$. Function $P$ must achieve its maximum at some point $x_0 \in M$. We claim that

\begin{equation}
\nabla P(x_0) = 0.
\end{equation}

If $x_0 \in M \setminus \partial M$, (13) is obviously true. Suppose that $x_0 \in \partial M$. Take the same local orthonormal frame $\{e_1, e_2, \ldots, e_n\}$ of $M$ about $x_0$ as in the proof of Lemma 1, where $e_n$ is the unit outward normal vector field near $x_0 \in \partial M$, $\{e_1, e_2, \ldots, e_{n-1}\}|_{\partial M}$ is a local frame of $\partial M$ about $x_0$ and $\nabla_{e_n} e_i = 0$ for $i \leq n - 1$. Since $v|_{\partial M} = 0$, we have $v_i(x_0) = 0$ for $i \leq n - 1$. $P(x_0)$ is the maximum implies that

\begin{equation}
P_i(x_0) = 0 \quad \text{for } i \leq n - 1
\end{equation}

and

\begin{equation}
P_n(x_0) \geq 0.
\end{equation}

Using argument in proving (10) and the non-negativity of the mean curvature $m$ of $\partial M$ with respect to the outward normal, we get

\[
\nabla_{e_n}(|\nabla v|^2)(x_0) \leq 0.
\]

Noticing that $v|_{\partial M} = 0$, we have

\begin{equation}
P_n(x_0) = \nabla_{e_n}(|\nabla v|^2)(x_0) + 2A v(x_0) v_n(x_0)) \leq 0.
\end{equation}

Now (14), (15) and (16) imply that $P_n(x_0) = 0$ and $\nabla P(x_0) = 0$. 

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Thus (13) holds, no matter $x_0 \not\in \partial M$ or $x_0 \in \partial M$. By (13) and the Maximum Principle, we have

$$
\nabla P(x_0) = 0 \quad \text{and} \quad \Delta P(x_0) \leq 0.
$$

We are going to show further that $\nabla v(x_0) = 0$. If on the contrary, $\nabla v(x_0) \neq 0$, then we rotate the local orthonormal frame about $x_0$ such that

$$|v_1(x_0)| = |\nabla v(x_0)| \neq 0 \quad \text{and} \quad v_i(x_0) = 0, \ i \geq 2.$$

From (17) we have at $x_0$,

$$0 = \frac{1}{2} \nabla_i P = \sum_{j=1}^n v_j v_{ji} + Av_i,$$

(18) \quad $v_{11} = -Av$ \quad and \quad $v_{1i} = 0 \ i \geq 2,$

and

$$0 \geq \frac{1}{2} \Delta P(x_0) = \sum_{i,j=1}^n (v_{ji} v_{ji} + v_j v_{jji} + Av_i v_i + Avv_{ii})$$

$$= \sum_{i,j=1}^n (v_{ji}^2 + v_j (v_{ii})_j + R_{ji} v_j v_i + Av_i^2 + Avv_{ii})$$

$$\geq v_{11}^2 + \nabla v \nabla (\Delta v) + \text{Ric}(\nabla v, \nabla v) + A|\nabla v|^2 + Av \Delta v$$

$$\geq v_{11}^2 + \nabla v \nabla (\Delta v) + (n-1)K|\nabla v|^2 + A|\nabla v|^2 + Av \Delta v$$

$$= (-Av)^2 - \lambda |\nabla v|^2 + (n-1)K|\nabla v|^2 + A|\nabla v|^2 - \lambda Av^2$$

$$= (A - \lambda + (n-1)K)|\nabla v|^2 + Av^2(A - \lambda),$$

where we have used (18) and (3). Therefore at $x_0$,

$$0 \geq (A - \lambda)|\nabla v|^2 + A(A - \lambda)v^2,$$

(19) \quad $0 \geq (A - \lambda)|\nabla v|^2 + A(A - \lambda)v^2,$

that is,

$$|\nabla v(x_0)|^2 + \lambda(1 + \epsilon)v(x_0)^2 \leq 0.$$

Thus $\nabla v(x_0) = 0$. This contradicts $\nabla v(x_0) \neq 0$.  

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Therefore in any case, if $P$ achieves its maximum at a point $x_0$, then
\[
\nabla v(x_0) = 0.
\]
Thus at $x_0$
\[
P(x_0) = |\nabla v(x_0)|^2 + Av(x_0)^2 = Av(x_0)^2 \leq A.
\]
and at all $x \in M$
\[
|\nabla v(x)|^2 + Av(x)^2 = P(x) \leq P(x_0) \leq A.
\]
Letting $\epsilon \to 0$ in the above inequality, the estimate (11) follows.

We want to improve the upper bound in (11) further and proceed in the following way.
Define a function $F$ by
\[
Z(t) = \max_{x \in M, t = \sin^{-1}(v(x)/b)} |\nabla v|^2 b^2 - v^2 / \lambda.
\]
The estimate in (11) becomes
\[
(20) \quad Z(t) \leq 1 \quad \text{on } [0, \sin^{-1}(1/b)]
\]
For convenience, in this paper we let
\[
(21) \quad \alpha = \frac{1}{2}(n - 1)K \quad \text{and} \quad \delta = \alpha / \lambda.
\]
By (1) we have
\[
(22) \quad \delta \leq \frac{n - 1}{2n}.
\]
We have the following conditions for the test function $Z$.

**Theorem 2.** If the function $z : [0, \sin^{-1}(1/b)] \mapsto \mathbb{R}^1$ satisfies the following

1. $z(t) \geq Z(t)$, $t \in [0, \sin^{-1}(1/b)]$,
2. there exists some $x_0 \in M$ such that at point $t_0 = \sin^{-1}(v(x_0)/b)$
   $z(t_0) = Z(t_0)$,
3. $z(t_0) > 0$,
4. $z$ extends to a smooth even function, and
5. $z'(t_0) \sin t_0 \geq 0$,
then we have the following
\begin{equation}
0 \leq \frac{1}{2} z''(t_0) \cos^2 t_0 - z'(t_0) \cos t_0 \sin t_0 - z(t_0) + 1 - 2\delta \cos^2 t_0.
\end{equation}

Proof. Define
\[
J(x) = \left\{ \frac{|\nabla v|^2}{b^2 - v^2} - \lambda z \right\} \cos^2 t,
\]
where \( t = \sin^{-1}(v(x)/b) \). Then
\[
J(x) \leq 0 \quad \text{for} \quad x \in M \quad \text{and} \quad J(x_0) = 0.
\]
So \( J(x_0) \) is the maximum of \( J \) on \( M \). If \( \nabla v(x_0) = 0 \), then
\[
0 = J(x_0) = -\lambda z \cos^2 t.
\]
This contradicts the Condition 3 in the theorem. Therefore
\[
\nabla v(x_0) \neq 0.
\]
We claim that
\begin{equation}
\nabla J(x_0) = 0.
\end{equation}
If \( x_0 \in M \setminus \partial M \), (24) is obviously true. Suppose that \( x_0 \in \partial M \). Take the same local orthonormal frame \( \{e_1, e_2, \cdots, e_n\} \) of \( M \) about \( x_0 \) as in the proof of Lemma [11] where \( e_n \) is the unit outward normal vector field near \( x_0 \in \partial M \), \( \{e_1, e_2, \cdots, e_{n-1}\}_{|\partial M} \) is a local frame of \( \partial M \) about \( x_0 \) and \( \nabla e_i e_i = 0 \) for \( i \leq n - 1 \). Since \( v|\partial M = 0 \), we have \( v_i(x_0) = 0 \) for \( i \leq n - 1 \). \( J(x_0) \) is the maximum implies that
\begin{equation}
J_i(x_0) = 0 \quad \text{for} \quad i \leq n - 1
\end{equation}
and
\begin{equation}
J_n(x_0) \geq 0.
\end{equation}
Using argument in proving (10) and the non-negativity of the mean curvature \( m \) of \( \partial M \) with respect to the outward normal, we get
\[
(|\nabla v|^2)_n \bigg|_{x_0} \leq 0.
\]
The Dirichlet condition \( v(x_0) = 0 \) implies that \( t(x_0) = 0 \) and \( z'(t(x_0)) = z'(0) = 0 \), since by the Condition 4 in the theorem \( z \) extends to a smooth even function. Therefore

\[
J_n(x_0) = \frac{1}{b^2} (|\nabla v|^2)_n - \lambda \cos t[z' \cos t - 2z \sin t],
\]

so \( J_n(x_0) \leq 0 \).

Now \( 25 \), \( 26 \) and \( 27 \) imply \( 24 \).

Thus \( 24 \) holds, no matter \( x_0 \notin \partial M \) or \( x_0 \in \partial M \). By \( 24 \) and the Maximum Principle, we have

\[
\nabla J(x_0) = 0 \quad \text{and} \quad \Delta J(x_0) \leq 0.
\]

\( J(x) \) can be rewritten as

\[
J(x) = \frac{1}{b^2} |\nabla v|^2 - \lambda z^2 t.
\]

Thus \( 28 \) is equivalent to

\[
\frac{2}{b^2} \sum_i v_i v_{ij} \bigg|_{x_0} = \lambda \cos t[z' \cos t - 2z \sin t] t_j \bigg|_{x_0}
\]

and

\[
0 \geq \frac{2}{b^2} \sum_{i,j} v_{ij}^2 + \frac{2}{b^2} \sum_{i,j} v_i v_{i,j,j} - \lambda (z'' |\nabla t|^2 + z' \Delta t) \cos^2 t
\]

\[
+ 4 \lambda z' \cos t \sin t |\nabla t|^2 - \lambda z \cos^2 t \bigg|_{x_0}
\]

Rotate the frame so that \( |v_1(x_0)| = |\nabla v(x_0)| \neq 0 \) and \( v_i(x_0) = 0, \quad i \geq 2 \).

Then \( 29 \) implies

\[
v_{11} \bigg|_{x_0} = \lambda b \frac{1}{2} (z' \cos t - 2z \sin t) \bigg|_{x_0} \quad \text{and} \quad v_{1i} \bigg|_{x_0} = 0 \quad \text{for} \quad i \geq 2.
\]
Now we have

\[
|\nabla v|^2 \bigg|_{x_0} = \lambda b^2 z \cos^2 t \bigg|_{x_0},
\]

\[
|\nabla t|^2 \bigg|_{x_0} = \frac{|\nabla v|^2}{b^2 - \nu^2} = \lambda z \bigg|_{x_0},
\]

\[
\frac{\Delta v}{b} \bigg|_{x_0} = \Delta \sin t = \cos t \Delta t - \sin t |\nabla t|^2 \bigg|_{x_0},
\]

\[
\Delta t \bigg|_{x_0} = \frac{1}{\cos t} (\sin t |\nabla t|^2 + \frac{\Delta v}{b})
\]

\[
= \frac{1}{\cos t} [\lambda z \sin t - \frac{\lambda}{b} \nu] \bigg|_{x_0}, \text{ and}
\]

\[
\Delta \cos^2 t \bigg|_{x_0} = \Delta \left(1 - \frac{\nu^2}{b^2}\right) = -\frac{2}{b^2} |\nabla v|^2 - \frac{2}{b^2} \nu \Delta v
\]

\[
= -2\lambda z \cos^2 t + \frac{2}{b^2} \nu^2 \bigg|_{x_0}.
\]

Therefore,

\[
\frac{2}{b^2} \sum_{i,j} v^2_{ij} \bigg|_{x_0} \geq \frac{2}{b^2} v^2_{11}
\]

\[
= \frac{\lambda^2}{2} (z')^2 \cos^2 t - 2\lambda z \cos t \sin t + 2\lambda^2 \nu^2 \cos^2 t \bigg|_{x_0},
\]

\[
\frac{2}{b^2} \sum_{i,j} v_iv_{ijj} \bigg|_{x_0} = \frac{2}{b^2} (\nabla v \nabla (\Delta v) + \text{Ric}(\nabla v, \nabla v))
\]

\[
\geq \frac{2}{b^2} (\nabla v \nabla (\Delta v) + (n - 1)K |\nabla v|^2)
\]

\[
= -2\lambda^2 z \cos^2 t + 4\alpha \lambda z \cos^2 t \bigg|_{x_0},
\]

\[
-\lambda (z'' |\nabla t|^2 + z' \Delta t) \cos^2 t \bigg|_{x_0}
\]

\[
= -\lambda z \nu^2 \cos^2 t - \lambda^2 \nu z \cos t \sin t
\]

\[
+ \frac{1}{b} \lambda^2 \nu \cos t \bigg|_{x_0},
\]
Putting these results into (30) we get

\[
0 \geq -\lambda^2 z z'' \cos^2 t + \frac{\lambda^2}{2} (z')^2 \cos^2 t - \lambda^2 z' \cos t (z \sin t + \sin t)
\]
(32)

\[
+ 2\lambda^2 z^2 - 2\lambda^2 z + 4\alpha \lambda \cos^2 t \Big|_{x_0},
\]

where we used (31). Now

\[
z(t_0) > 0,
\]
(33)

by the Condition 3 in the theorem. Dividing two sides of (32) by \(2\lambda^2 z\big|_{x_0}\), we have

\[
0 \geq -\frac{1}{2} z''(t_0) \cos^2 t_0 + \frac{1}{2} z'(t_0) \cos t_0 \left( \sin t_0 + \frac{\sin t_0}{z(t_0)} \right) + z(t_0)
\]

\[
- 1 + 2\delta \cos^2 t_0 + \frac{1}{4z(t_0)} (z'(t_0))^2 \cos^2 t_0.
\]

Therefore,

\[
0 \geq -\frac{1}{2} z''(t_0) \cos^2 t_0 + z'(t_0) \cos t_0 \sin t_0 + z(t_0) - 1 + 2\delta \cos^2 t_0
\]
(34)

\[
+ \frac{1}{4z(t_0)} (z'(t_0))^2 \cos^2 t_0 + \frac{1}{2} z'(t_0) \sin t_0 \cos t_0 \left( \frac{1}{z(t_0)} - 1 \right).
\]

Conditions 1, 2 and 5 in the theorem imply that \(0 < z(t_0) = Z(t_0) \leq 1\) and \(z'(t_0) \sin t_0 \geq 0\). Therefore the last two terms in (34) are nonnegative and (23) follows.

### 3 Proof of Main Result

**Proof of Theorem 1.** Let

\[
z(t) = 1 + \delta \xi(t),
\]

(35)
where $\xi$ is the functions defined by (45) in Lemma 3. We claim that
\begin{equation}
Z(t) \leq z(t) \quad \text{for } t \in [0, \sin^{-1}(1/b)].
\end{equation}

Lemma 3 implies that for $t \in [0, \sin^{-1}(1/b)]$, we have the following
\begin{equation}
\frac{1}{2}z'' \cos^2 t - z' \cos t \sin t - z = -1 + 2\delta \cos^2 t,
\end{equation}
\begin{equation}
z'(t) \sin t \geq 0,
\end{equation}
\begin{equation}
z \text{ is a smooth even function},
\end{equation}
\begin{equation}
0 < 1 - \left(\frac{\pi^2}{4} - 1\right)\frac{n-1}{2n} \leq 1 - \left(\frac{\pi^2}{4} - 1\right)\delta = z(0) \leq z(t), \quad \text{and}
\end{equation}
\begin{equation}
z(t) \leq z\left(\frac{\pi}{2}\right) = 1.
\end{equation}

Let $P \in \mathbb{R}^1$ and $t_0 \in [0, \sin^{-1}(1/b)]$ such that
\begin{equation}
P = \max_{t \in [0, \sin^{-1}(1/b)]} (Z(t) - z(t)) = Z(t_0) - z(t_0).
\end{equation}
Thus
\begin{equation}
Z(t) \leq z(t) + P \quad \text{for } t \in [0, \sin^{-1}(1/b)] \quad \text{and} \quad Z(t_0) = z(t_0) + P.
\end{equation}
Suppose that $P > 0$. Then $z + P$ satisfies the conditions in Theorem 2 and therefore satisfies (23). So we have
\begin{align*}
z(t_0) + P &= Z(t_0) \\
&\leq \frac{1}{2} (z + P)''(t_0) \cos^2 t_0 - (z + P)'(t_0) \cos t_0 \sin t_0 + 1 - 2\delta \cos^2 t_0 \\
&= \frac{1}{2} z''(t_0) \cos^2 t_0 - z'(t_0) \cos t_0 \sin t_0 + 1 - 2\delta \cos^2 t_0 \\
&= z(t_0).
\end{align*}
This contradicts the assumption $P > 0$. Thus $P \leq 0$ and (36) must hold. That means
\begin{equation}
\sqrt{\lambda} \geq \frac{\|\nabla f\|}{\sqrt{z(t)}}.
\end{equation}

Take $q_1$ on $M$ such that $v(q_1) = 1 = \sup_M v$ and and $q_2 \in \partial M$ such that distance $d(q_1, q_2) = \text{distance } d(q_1, \partial M)$. Let $L$ be the minimum geodesic
segment between $q_1$ and $q_2$. We integrate both sides of (43) along $L$ and change variable and let $b \to 1$. Let $\hat{d}$ be the diameter of the largest interior ball in $M$. Then

$$\sqrt{\lambda} \frac{\hat{d}}{2} \geq \int_L \sqrt{z(t)} |\nabla t| \, dl = \int_0^{\pi/2} \frac{1}{\sqrt{z(t)}} \frac{\pi}{4} \frac{dz(t)}{z(t)} \geq \left( \frac{\pi^3}{2} \int_0^{\pi/2} z(t) \, dt \right)^{1/2}.$$  

Square the two sides. Then

$$\lambda \geq \frac{\pi^3}{2 (\hat{d})^2 \int_0^{\pi/2} z(t) \, dt}.$$  

Now

$$\int_0^{\pi/2} z(t) \, dt = \int_0^{\pi/2} [1 + \delta \xi(t)] \, dt = \frac{\pi}{2} (1 - \delta),$$  

by (45) in Lemma 3. That is,

$$\lambda \geq \frac{\pi^2}{(1 - \delta) (\hat{d})^2} \quad \text{and} \quad \lambda \geq \frac{1}{2} (n - 1) K + \frac{\pi^2}{(\hat{d})^2}.$$  

We now present a lemma that is used in the proof of Theorem 1.

**Lemma 3.** Let

$$\xi(t) = \frac{\cos^2 t + 2t \sin t \cos t + t^2 - \frac{\pi^2}{4}}{\cos^2 t} \quad \text{on} \quad \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right].$$
Then the function $\xi$ satisfies the following

1. $\frac{1}{2} \xi'' \cos^2 t - \xi' \cos t \sin t - \xi = 2 \cos^2 t \quad \text{in } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, \hfill (46)

2. $\xi' \cos t - 2 \xi \sin t = 4t \cos t$ \hfill (47)

3. $\int_0^\frac{\pi}{2} \xi(t) \, dt = -\frac{\pi}{2}$ \hfill (48)

1. $1 - \frac{\pi^2}{4} = \xi(0) \leq \xi(t) \leq \xi(\pm \frac{\pi}{2}) = 0 \quad \text{on } [-\frac{\pi}{2}, \frac{\pi}{2}]$, \hfill (49)

2. $\xi'$ is increasing on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\xi'(\pm \frac{\pi}{2}) = \pm \frac{2\pi}{3}$, \hfill (50)

3. $\xi''(\pm \frac{\pi}{2}) = 2$, $\xi''(0) = 2(3 - \frac{\pi^2}{4})$ and $\xi''(t) > 0$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$, \hfill (51)

4. $(\frac{\xi'(t)}{t})' > 0$ on $(0, \pi/2)$ and $2(3 - \frac{\pi^2}{4}) \leq \frac{\xi'(t)}{t} \leq \frac{4}{3}$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$, \hfill (52)

5. $\xi'''(\frac{\pi}{2}) = \frac{8\pi}{15}$, $\xi'''(t) < 0$ on $(-\frac{\pi}{2}, 0)$ and $\xi'''(t) > 0$ on $(0, \frac{\pi}{2})$. \hfill (53)

**Proof.** For convenience, let $q(t) = \xi'(t)$, i.e.,

$q(t) = \xi'(t) = \frac{2(2t \cos t + t^2 \sin t + \cos^2 t \sin t - \frac{\pi^2}{4} \sin t)}{\cos^3 t}.$ \hfill (54)

Equation (46) and the values $\xi(\pm \frac{\pi}{2}) = 0$, $\xi(0) = 1 - \frac{\pi^2}{4}$ and $\xi'(\pm \frac{\pi}{2}) = \pm \frac{2\pi}{3}$ can be verified directly from (45) and (49). The values of $\xi''$ at 0 and $\pm \frac{\pi}{2}$ can be computed via (46). By (47), $(\xi(t) \cos^2 t)' = 4t \cos^2 t$. Therefore $\xi(t) \cos^2 t = \int_0^t \cos^2 s \, ds$, and

$$\int_0^\frac{\pi}{2} \xi(t) \, dt = 2 \int_0^\frac{\pi}{2} \xi(t) \, dt = -8 \int_0^\frac{\pi}{2} \left( \frac{1}{\cos^2(t)} \int_t^\frac{\pi}{2} s \cos^2 s \, ds \right) \, dt$$

$$= -8 \int_0^\frac{\pi}{2} \left( \int_0^s \frac{1}{\cos^2(t)} \, dt \right) s \cos^2 s \, ds = -8 \int_0^\frac{\pi}{2} s \cos s \sin s \, ds = -\pi.$$ \hfill (55)

It is easy to see that $q$ and $q'$ satisfy the following equations

1. $\frac{1}{2} q'' \cos t - 2q' \sin t - 2q \cos t = -4 \sin t$, \hfill (56)
and

\[
\frac{\cos^2 t}{2(1 + \cos^2 t)} (q')'' - \frac{2 \cos t \sin t}{1 + \cos^2 t} (q')' - 2(q') = -\frac{4}{1 + \cos^2 t}.
\]

The last equation implies \( q' = \xi'' \) cannot achieve its non-positive local minimum at a point in \( (-\frac{\pi}{2}, \frac{\pi}{2}) \). On the other hand, \( \xi''(\pm \frac{\pi}{2}) = 2 \), by equation (46), \( \xi(\pm \frac{\pi}{2}) = 0 \) and \( \xi'(\pm \frac{\pi}{2}) = \pm \frac{2\pi}{3} \). Therefore \( \xi''(t) > 0 \) on \( [-\frac{\pi}{2}, \frac{\pi}{2}] \) and \( \xi' \) is increasing. Since \( \xi'(t) = 0 \), we have \( \xi'(t) < 0 \) on \( (-\frac{\pi}{2}, 0) \) and \( \xi'(t) > 0 \) on \( (0, \frac{\pi}{2}) \). Similarly, from the equation

\[
\frac{\cos^2 t}{2(1 + \cos^2 t)} (q'')'' - \frac{\cos t \sin t (3 + 2 \cos^2 t)}{(1 + \cos^2 t)^2} (q'')' - \frac{2(5 \cos^2 t + \cos^4 t)}{(1 + \cos^2 t)^2} (q'') = -\frac{8 \cos t \sin t}{(1 + \cos^2 t)^2}.
\]

we get the results in the last line of the lemma.

Set \( h(t) = \xi''(t) t - \xi'(t) \). Then \( h(0) = 0 \) and \( h'(t) = \xi'''(t) t > 0 \) in \( (0, \frac{\pi}{2}) \). Therefore \( \frac{\xi'(t)}{t} = \frac{h(t)}{t^2} > 0 \) in \( (0, \frac{\pi}{2}) \). Note that \( \frac{\xi'(\frac{\pi}{2})}{\frac{\pi}{2}} = \frac{\xi'(t)}{t} \), \( \frac{\xi'(t)}{t} \big|_{t=0} = \xi'(0) = 2(3 - \frac{\pi^2}{4}) \) and \( \frac{\xi'(t)}{t} \big|_{t=\pi/2} = \frac{4}{3} \). This completes the proof of the lemma. \( \square \)

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