Abstract. We extend an equivariant Mountain Pass Theorem, due to Bartsch, Clapp and Puppe for compact Lie groups to the setting of infinite discrete groups satisfying a maximality condition on their finite subgroups. As an application, we examine the critical set of generalizations of periodic functionals first studied by Rabinowitz. As well as the solutions to symmetric systems of ODE associated with them.

Since the early days of Variational Analysis, symmetries have played a fundamental role in the analysis of critical points and sets of functionals. The development of Equivariant Algebraic Topology, particularly Equivariant Homotopy Theory, has given a number of tools to conclude the existence of critical points in problems which are invariant under the action of a compact Lie group.

The extension of methods of Equivariant Topology to the setting of actions of infinite groups and their applications to problems in nonlinear analysis is the object of this paper. The main result of this note is the modification of a result by Bartsch, Clapp and Puppe originally proven for compact Lie groups, to infinite discrete groups with appropriate families of finite subgroups inside them. Our methods differ from that of [9] in that we use classifying spaces for families, the analysis of the behavior of elements under the differentials in the Atiyah-Hirzebruch spectral sequence, a notion of “universal proper length”, as well as the use of an equivariant quantitative deformation Lemma, which replaces the typical apriori compactness assumptions related to either the Palais-Smale condition or the $G$-strong deformation property of [9]. See example [5] for more on symmetric mountain pass situations without apriori compactness hypothesis, and [38] for further related results.

**Theorem 1.1** (Mountain Pass Theorem). Let $G$ be an infinite discrete group acting linearly on a real Banach space $E$ of infinite dimension. Suppose that $G$ satisfies the maximality condition and that the linear action is proper outside 0. Let $\phi : E \to \mathbb{R}$ be a $G$-invariant functional. For any value $a \in \mathbb{R}$, define the set below $a$, $\phi^a = \{ x \in E \mid \phi(x) \leq a \}$ and the critical set $K = \cup_{c \in \mathbb{R}} K_c$, where $K_c$ is the critical set at level $c$, $K_c = \{ u \mid \|\phi'(u)\| = 0 \phi(u) = c \}$. Suppose that

- $\phi(0) \leq a$ and there exists a linear subspace $\tilde{E} \subset E$ of finite codimension such that $\tilde{E} \cap \phi^a$ is the disjoint union of two closed subspaces one of which is bounded and contains 0.
- The functional $\phi$ satisfies a quantitative deformation property with respect to the subset $S \subset E$, which is assumed to be a fundamental deformation region.
- The group $G$ satisfies the maximal finite subgroups condition.

Then, the equivariant Lusternik-Schnirelmann category of $E$ relative to $\phi^a$, $G$–cat($\phi^a$, $E$) is infinite. If moreover, the critical sets $K_c$ are cocompact under the

Key words and phrases. Mountain Pass Theorem, Critical point theory, Equivariant cohomotopy.
Conditions \[\text{1.2}\] restrict maximal finite subgroups and their conjugacy relations.

**Condition 1.2.** Let \( G \) be a discrete group and \( M \) be a subset of finite subgroups. \( G \) satisfies the maximality condition if

- There exists a prime number \( p \) such that every nontrivial finite subgroup is contained in a unique maximal \( p \)-group \( M \in \text{MAX} \).
- \( M \in \text{MAX} \implies N_G(M) = M \), where \( N_G(M) \) denotes the normalizer of \( M \) in \( G \).

Notice that in particular, the finite subgroups of \( G \) are all finite \( p \)-groups.

These conditions are satisfied in several important cases. Among them:

1. **Extensions 1 \( \to \mathbb{Z}^n \to G \to K \to 1 \)** by a finite \( p \)-group given by a representation \( K \to \mathbb{Z}^n \) acting free outside from the origin \[27\], Lemma 6.3.

2. **Fuchsian groups**, more generally NEC (non-euclidean crystallographic groups) for which the isotropy consists only of \( p \)-groups. \[27\].

3. **One relator groups** \( G = \langle q_i \mid r \rangle \) for which the finite subgroups consists of \( p \)-groups. See \[28\], Propositions 5.17, 5.18, 5.19. in pages 107 and 108.

The quantitative deformation property with respect to a set \( S \) \[2,7\] is an equivariant Version of Willem’s deformation lemma, 2.3 in \[37\], page 38. We use a consequence of it, namely, that if the set \( S \) is a fundamental deformation region, then there exists deformations of sets below critical values out of neighborhoods of the critical set, see Corollary 2.9.

This paper is organized as follows: in the first section, the usual facts concerning the relation between critical points, cohomology length, Lusternik-Schnirelmann category and equivariant deformation theorems are stated, being modified slightly from \[37\] lemma 2.3 page 38, respectively \[6, 12\]. We introduce the notion of Universal Proper Length.

In the second section, we use some algebraic properties of the classifying space for proper actions of groups with an appropriate family of maximal finite subgroups in order to conclude the unboundedness of critical values.

This is done adapting a construction of elements in the Burnside Ring of a finite group, originally due to Bartsch, Clapp and Puppe \[9\] to the infinite group setting, using mainly the Atiyah-Hirzebruch spectral sequence, as well as a version of the Segal Conjecture for families of finite groups inside infinite groups \[20, 7\].

Finally, we present an example of a Banach space with an action of an infinite discrete group to illustrate the applications of the methods handled in this work, extending results of Rabinowitz \[33\].

This work was financially supported by the Hausdorff Center for Mathematics at the University of Bonn.

The author thanks an anonymous referee for valuable remarks improving the content of the paper.

2. **Proper Lusternik-Schnirelmann Category, Universal Proper Length and Critical Points**

The notion of a proper \( G \)-space provides an adequate setting for the study of non compact transformation groups.
Definition 2.1. Let $G$ be a second countable, Hausdorff locally compact group. Let $X$ be a second countable, locally compact Hausdorff space. Recall that a $G$-space is proper if the map

$$G \times X \to X \times X$$

$$_{(g,x) \mapsto (x,gx)}$$

is proper.

Definition 2.2. Recall that a $G$-CW complex structure on the pair $(X,A)$ consists of a filtration of the $G$-space $X = \bigcup_{-1 \leq n} X_n$, $X_{-1} = \emptyset$, $X_0 = A$ and for which every space $X_n$ is inductively obtained from the previous one by attaching cells in pushout diagrams of the form

$$\coprod_i S^{n-1} \times G/H_i \to X_{n-1}$$

$$\coprod_i D^n \times G/H_i \to X_n$$

We say that a proper $G$-CW complex is finite if it consists of a finite number of cells $G/H \times D^n$.

Definition 2.3. Let $G$ be a discrete group. A metrizable proper $G$-space $X$ is an Absolute Neighborhood retract if given any closed subset $A \subset X$ and a $G$-equivariant map $f : A \to Y$, there exist a $G$-neighborhood $U$ of $A$ and an extension $\tilde{f} : U \to Y$.

It is proven in [4], Theorem 1.1 that proper $G$-ANR are $G$-homotopy equivalent to $G$-CW Pairs when $G$ is a locally compact Hausdorff group.

The following result enumerates some facts which will be needed in the following, which are proven in chapter one of [21]:

Proposition 2.4. Let $(X, A)$ be a proper $G$-CW pair

(i) The inclusion $A \to X$ is a closed cofibration.

(ii) $A$ is a neighborhood $G$-deformation retract, in the sense that there exists a neighborhood $A \subset U$, of which $A$ is a $G$-equivariant deformation retract. The neighborhood can be chosen to be closed or open. In particular, all $G$-CW complexes are $G$-ANR.

Definition 2.5. Let $F$ be family of closed subgroups closed under conjugation and intersection inside the locally compact second countable Hausdorff group $G$. A $F$-numerable $G$-space is a $G$-space $X$ if there exists an open covering $\{U_i, \mid i \in I\}$ by $G$-subspaces such that there is for each $i \in I$ a $G$-map $U_i \to G/G_i$ for some $G_i \in F$ and there is a locally finite partition of unity $\{e_{i|i \in I}\}$ subordinate to $\{U_i\}$ by $G$-invariant functions. Notice that we do not require that the isotropy groups of $X$ lie in $F$.

The Slice Theorem 2.3.3, in page 313 of [31] implies that proper actions of Lie groups on completely regular spaces are precisely numerable spaces with respect to the family of compact subgroups for which the isotropy groups of points are all compact subgroups.

Specializing to Lie groups acting properly on $G$-CW complexes, the conditions boil down to the fact that all stabilizers are compact, see [21], Theorem 1.23. In particular for a cellular action of a discrete group $G$ on a $G$-CW complex, a proper action reduces to the finiteness of all stabilizer groups. Notice that any (continuous) action of a compact Lie group or a finite group on a locally compact, Hausdorff space is proper.
The notion of $G$-category of a space with an action of a compact Lie group was introduced by Marzantowicz in [29]. Ayala-Lasheras-Quintero [3] introduced the corresponding notion for proper actions of Lie Groups.

**Definition 2.6.** Let $X' \subset X$ be paracompact proper $G$-spaces. The $G$-category of $(X, X')$, $G$-cat$(X, X')$ is the smallest number $k$ such that $X$ can be covered by $k + 1$ open $G$-subsets $X_0, X_1, \ldots, X_k$ with the following properties:

- $X' \subset X_0$ and there is a homotopy $H : (X_0, X') \times I \to (X_0, X')$ starting with the inclusion and $H(x, 1) = x$.
- For every $i \in \{1, \ldots, k\}$ there exist $G$-maps $\alpha_i : X_i \to A_i$ and $\beta_i : A_i \to Y$ with $A_i$ a $G$-orbit $G/H_i$ such that the restriction of $f$ to $X_i$ is the is $G$-homotopic to the composition $\beta_i \circ \alpha_i$.

If no such a number exists, then we write $G - \text{cat}(X, X') = \infty$.

The next ingredient for a Mountain Pass Theorem Type result is a deformation lemma. This is achieved using the notion of pseudogradient vector field, and the quantitative deformation lemma of Willem, lemma 2.3 in [37], page 37, interpreted in an equivariant setting. In the following we will deal with a $G$-Banach Space $X$ with a proper action outside 0 of the discrete group $G$. Recall [31], the existence of a $G$-invariant metric $d$. If $S \subset X$ is a set, the $\delta$-inflated set $S_\delta$ is defined to be the set $\{y \in X \mid d(y, S) \leq \delta\}$.

**Lemma 2.7** (Equivariant Quantitative Deformation). Let $G$ be a discrete group acting properly outside of the origin on a Banach space $X$. Let $\Phi : X \to \mathbb{R}$ a $G$-invariant $C^1$-functional, $S \subset X$, $c \in \mathbb{R}$, $\epsilon > 0$ such that for any $u \in \Phi^{-1}[c - 2\epsilon, c + 2\epsilon]$ the inequality

$$|\Phi'(u)| \geq \frac{8\epsilon}{\delta}$$

holds. Then, there exists a $G$-equivariant deformation $\eta : X \times I \to X$ satisfying:

- For all $t$, the map $\eta_t := \eta(\cdot, t) : X \to X$ is an homeomorphism.
- $\eta(u, 0) = u$
- $\eta(u, t) = u$ whenever $u \notin \Phi^{-1}[c - 2\epsilon, c + 2\epsilon] \cap S_{2\delta}$.
- $\eta(\Phi^\epsilon \cap S, 1) \subset \Phi^{\epsilon - \epsilon}$.
- $|\eta(u, t) - u| < \delta$.
- $\phi(\eta(u, t)) < c$ for $u \in \phi^\epsilon \cap S_\delta$.

**Proof.** Put $W_G(v) = \sum_{g \in G} g^{-1}(W(gv))$. Notice that due to the slice theorem, this is a locally finite sum, indexed by the elements of a finite stabilizer of a point outside from 0. Since the metric might be assumed to be invariant under the $G$-action, $W_G$ is a $G$-equivariant locally Lipschitz pseudogradient vector field on $\bar{X} = \mathcal{W} - \{v \mid \Phi'(v) = 0\}$.

The proper action allows to construct a $G$-invariant, locally Lipschitz function $\varphi : X \to [0, 1] \subset \mathbb{R}$ such that $\varphi|_{\Phi^{-1}([c - \epsilon, c + \epsilon]) \cap S_\delta} = 1$ and $\varphi|_{X - S_{2\delta} \cup X - \Phi^{-1}[c - 2\epsilon, c + 2\epsilon]} = 0$. Define the $G$-equivariant maximal descent vector field $\psi : X \to X$

$$\psi(v) = \begin{cases} \frac{\dot{\varphi}(v)}{W_G(v)} W_G(v) & v \in \Phi^{-1}([c - 2\epsilon, c + 2\epsilon]) \cap S_{2\delta} \\ 0 & \text{otherwise} \end{cases}$$

The $G$-equivariant problem $\dot{w}(t) = \psi((w(\epsilon)))$ with the initial condition $w(0) = u$ admits a solution defined on $[0, \infty)$. It is then routine that the deformation defined by $\eta(t, u) = w(\delta - \eta, u)$ satisfies i to iv. 

□
In this situation, we will say that the functional has the quantitative deformation property relative to the subspace $S$ at the level $c$.

**Definition 2.8.** Let $\phi$ be a functional with a quantitative deformation property with respect to the Set $S$. A Set $S$ is called a fundamental deformation region if:

- For any $c$, the set $K_c := K_c \cap S$ is compact.
- For any $c$, $K_c$ is a fundamental region for the critical set at level $c$, $K_c$, i.e., the action of $G$ on $K_c$ gives all of $K_c$.

If the set $S$ happens to be such that $K_c \cap S$ is a fundamental deformation region, then the following refinement of the quantitative deformation is possible:

**Corollary 2.9.** Let $\phi : X \to \mathbb{R}$ be a functional having an quantitative deformation property relative to the fundamental deformation region $S$. Then, for every invariant neighborhood $U$ of the critical set $K_c$, every $c > a$, every $0 < \delta < c - a$ there exist $\epsilon > 0$ and a continuous equivariant map $\eta : X \times I \to X$ such that

- $\eta(u, 0) = u$.
- $\eta(\phi^{c-\epsilon} - U, 1) \subset \phi^{c-\epsilon}$ and $\eta(x, t) = x$ for all $x \in \phi^{c-\delta}$.
- If $K_c = \emptyset$, then $\eta(1, \phi^{c+\epsilon}) \subset \phi^{c-\epsilon}$.

**Proof.** Since $K_c \cap S$ is compact, there exists $\alpha > 0$ such that $(K_c \cap S)_\alpha \subset U$. The $G$-invariance of $U$ and the fact that $S$ is a fundamental deforming region imply that $K_{c_\alpha} = G(K_c \cap S)_\alpha \subset U$. The compactness of $K_c \cap S$ implies that there exist a number $b > 0$ and $\epsilon$ such that $|\phi(q)| > b$ for $q \in \phi^{-1}[c - \epsilon, c + \epsilon] \cap S_\alpha$. Choose $0 < \epsilon < \min\{1, \frac{b}{2}, \frac{b}{4}\}$ such that $0 < \frac{b}{2} < b$, then the deformation obtained from the deformation lemma satisfies the required conditions.

**Proposition 2.10.** Let $M$ be a proper, paracompact $G$-Banach, $C^1$- manifold. Let $\phi : M \to \mathbb{R}$ be a $G$-invariant $C^1$-function satisfying the deformation property with respect to neighborhoods of critical sets, as in corollary 2.9. Suppose that for every critical value $c$ the critical set $K_c$ is cocompact.

- If the function is bounded below, then the number of critical points of $\phi$ with values $> a$ in $M$ is at least $G - \text{cat}(M, \phi^a)$.
- If $G - \text{cat}(M, \phi^a)$ is greater than the number of critical values of $\phi$ above $a$, then there is at least one $c > a$ such that the critical set $K_c$ has positive covering dimension, in particular $\phi$ has infinitely many critical orbits with values above $a$.
- If $G - \text{cat}(M, K) = \infty$, then $\phi$ has an unbounded sequence of critical values.

**Proof.** The proofs given in [12], theorem 2.3 and Corollary 2.4, pages 606 and 607, and [13], Theorem 1.1 extend to the proper setting. The point is that the equivariant Lusternik-Schnirelmann Category for proper spaces satisfies subadditivity, deformation monotonicity, and continuity (Proposition 2.3 in [8] in the absolute case, and the obvious modification extends to the relative category).

We recall the notion of the classifying space for a family of subgroups.

**Definition 2.11.** Let $\mathcal{F}$ be a collection of subgroups in a discrete group $G$ closed under conjugation and intersection. A model for the Classifying Space for the family $\mathcal{F}$ is a $G$-CW complex $X$ satisfying

- All isotropy groups of $X$ lie in $\mathcal{F}$.
- For any $G$-CW complex $Y$ with isotropy in $\mathcal{F}$, there exists up to $G$-homotopy a unique $G$-equivariant map $f : Y \to X$. 

Particularly relevant is the classifying space for proper actions, the classifying space for the family $\mathcal{F}$ of finite subgroups, denoted by $\mathcal{E}G$.

The classifying space for proper actions always exists, is unique up to $G$-homotopy and admits several models. The following list includes some examples. We remit to [24] for further discussion.

- If $G$ is a compact group, then the singleton space is a model for $\mathcal{E}G$.
- Let $G$ be a group acting properly and cocompactly on a CAT(0) space $X$, in the sense of [11]. Then $X$ is a model for $\mathcal{E}G$.
- Let $G$ be a Coxeter Group. The Davis Complex is a model for $\mathcal{E}G$.
- Let $G$ be a Mapping Class Group of an orientable surface. The Teichmüller Space is a model for $\mathcal{E}G$.
- Let $G$ be a Gromov Hyperbolic Group. The Rips Complex is a model for the classifying space for $\mathcal{E}G$.

The following version of the classifying space for a family extends the notion to $\mathcal{F}$-numerable spaces.

**Definition 2.12 (Numerable Version for the Classifying space of a family).** Let $\mathcal{F}$ be a family of subgroups. A model $J_{\mathcal{F}}(G)$ for the classifying numerable $G$-space for the family $\mathcal{F}$ is a $G$-space which has the following properties:

- $J_{\mathcal{F}}(G)$ is $\mathcal{F}$-numerable
- For any $\mathcal{F}$-numerable space $X$ there is up to $G$-homotopy precisely one map $X \rightarrow J_{\mathcal{F}}(G)$.

**Remark 2.13.** There exists up to $G$-homotopy a unique $G$-equivariant map $\mathcal{E}G \rightarrow J_{\mathcal{F}}(G)$. This map is proven to be a $G$-homotopy equivalence for a discrete group in Theorem 3.7, part ii of [24].

Recall the notion of an Equivariant Cohomology Theory, [23].

**Definition 2.14.** Let $G$ be a group and fix an associative ring with unit $R$. A $G$-Cohomology Theory with values in $R$-modules is a collection of contravariant functors $\mathcal{H}_G^n$ indexed by the integer numbers $\mathbb{Z}$ from the category of $G$-CW pairs together with natural transformations $\partial_G^n : \mathcal{H}_G^n(A) := \mathcal{H}_G^n(A,\emptyset) \rightarrow \mathcal{H}_G^{n+1}(X,A)$, such that the following axioms are satisfied:

(i) If $f_0$ and $f_1$ are $G$-homotopic maps $(X,A) \rightarrow (Y,B)$ of $G$-CW pairs, then $\mathcal{H}_G^n(f_0) = \mathcal{H}_G^n(f_1)$ for all $n$.

(ii) Given a pair $(X,A)$ of $G$-CW complexes, there is a long exact sequence

$$\cdots \mathcal{H}_G^{n-1}(i) \xrightarrow{\partial_G^{n-1}} \mathcal{H}_G^n(X,A) \xrightarrow{\partial_G^n} \mathcal{H}_G^n(A) \xrightarrow{\partial_G^n} \mathcal{H}_G^{n+1}(X,A) \xrightarrow{\partial_G^n} \cdots$$

where $i : A \rightarrow X$ and $j : X \rightarrow (X,A)$ are the inclusions.

(iii) Let $(X,A)$ be a $G$-CW pair and $f : A \rightarrow B$ be a cellular map. The canonical map $(F,f) : (X,A) \rightarrow (X \cup_f B,B)$ induces an isomorphism

$$\mathcal{H}_G^n(X \cup_f B,B) \cong \mathcal{H}_G^n(X,A)$$

(iv) Let $\{X_i \mid i \in \mathcal{I}\}$ be a family of $G$-CW-complexes and denote by $j_i : X_i \rightarrow \coprod_{i \in \mathcal{I}} X_i$ the inclusion map. Then the map

$$\Pi_{i \in \mathcal{I}} \mathcal{H}_G^n(j_i) : \mathcal{H}_G^n(\coprod_i X_i) \cong \Pi_{i \in \mathcal{I}} \mathcal{H}_G^n(X_i)$$

is bijective for each $n \in \mathbb{Z}$. 

A G-CoA homology Theory is said to have a multiplicative structure if there exist natural, graded commutative $\cup$- products

$$H^n_G(X, A) \otimes H^m_G(X, A) \to H^{n+m}_G(X, A)$$

Let $\alpha : H \to G$ be a group homomorphism and $X$ be a $H$-CW complex. The induced space $\text{ind}_\alpha X$, is defined to be the $G$-CW complex defined as the quotient space $G \times X$ by the right $H$-action given by $(g, x) \cdot h = (g\alpha(h), h^{-1}x)$.

An Equivariant Cohomology Theory consists of a family of G-CoA homology Theories $H^n_G$ together with an induction structure determined by graded ring homomorphisms

$${\mathcal H}^n_G(\text{ind}_\alpha(X, A)) \to {\mathcal H}^n_H(X, A)$$

which are isomorphisms for group homomorphisms $\alpha : H \to G$ whose kernel acts freely on $X$ satisfying the following conditions:

(i) For any $n$, $\partial^n_G \circ \text{ind}_\alpha = \text{ind}_{\alpha^n} \circ \partial^n_H$.

(ii) For any group homomorphism $\beta : G \to K$ such that $\ker \beta \circ \alpha$ acts freely on $X$, one has

$$\text{ind}_{\alpha \circ \beta} = {\mathcal H}^n_K(f_1 \circ \text{ind}_\beta \circ \text{ind}_\alpha) : {\mathcal H}^n_K(\text{ind}_{\beta \circ \alpha}(X, A)) \to {\mathcal H}^n_H(X, A)$$

where $f_1 : \text{ind}_{\beta \circ \alpha} \to \text{ind}_{\beta \circ \alpha}$ is the canonical $G$-homeomorphism.

(iii) For any $n \in \mathbb{Z}$, any $g \in G$, the homomorphism

$$\text{ind}_{c(g)} : {\mathcal H}^n_G(X, A) \to {\mathcal H}^n_G(X, A)$$

agrees with the map $H^n_G(f_2)$, where $f_2 : (X, A) \to \text{ind}_{c(g)} : G \to G$ sends $x$ to $(1, g^{-1}x)$ and $c(g)$ is the conjugation isomorphism in $G$.

**Remark 2.15 (Extensions of G-Cohomology theories to more general spaces).** Let ${\mathcal H}^*_G$ be a $G$-cohomology theory defined on proper $G$-CW complexes. Using a functorial $G$-CW approximation for proper $G$-ANR of as in [4] for locally compact Hausdorff groups, an equivariant cohomology theory may be extended to the category of proper $G$-ANR.

More generally the Čech- expansion of [30] provides a Čech extension of a $G$-cohomology theory to arbitrary pairs of proper $G$-spaces. That is, a family of $R$-mod valued functors $H^n_G$ defined on pairs of proper $G$-spaces and natural transformations $\delta^n_{X,A} : H^n_G(A, \emptyset) \to H^{n+1}_G(X, A)$ satisfying the axioms:

- $G$-homotopy invariance.
- Long exact sequences for $G$- pairs.
- Excision. Let $X_1, X_2 \subset X$ be proper $G$- invariant spaces such that $X_2 - X_1 \cap X_1 - X_2 = \emptyset = X_2 - X_1 \cap X_1 - X_2$. The inclusion map $(X_2, X_1 \cap X_2) \to (X_1 \cup X_2, X_1)$ induces a natural isomorphism.

For the purposes of this work we need an extension of an especific cohomology theory to a certain proper space which is contractible after forgetting the action and is exhausted by finite $G$-CW complexes. This is done by an adhoc construction, see definition [11].

Recall [13], [23], that for any Equivariant Cohomology Theory $H^*_G$ on finite $G$-CW complexes there exists a spectral sequence with $E^2$-term given in terms of Bredon Cohomology

$$E^{p,q}_2 = H^p_{ZD_r(G)}(X, H^{q-}(G/H))$$

converging to $H^*_G(X)$. 
Proposition 2.16. Let $X$ be an $l$-dimensional $G$-CW complex. Suppose that for $r = 2, 3, \ldots, l$ the differential $d_r$ appearing in the Atiyah-Hirzebruch spectral sequence for $X$ and $H^*_G$ vanishes rationally. Then, for any element 

$$x \in H^0_{G}(X, \mathcal{H}^0(G/?,?))$$

there exists some positive integer $k$ such that $x^k$ is contained in the image of $H^0_{G}(X)$ under the edge homomorphism

$$\text{Edge}_G : H^0_{G}(X) \rightarrow H^0_{G/G}(X, \mathcal{H}^0(G/?,?))$$

Proof. Let $x \in H^0_{G}(X, \mathcal{H}^0(G/?,?))$. The proof reduces to construct inductively positive integers $k_2, \ldots, k_{l-1}$ such that $x^{\prod_{i=r}^{l-1} k_i} \in E^0_{r+1}$ for $r = 1, \ldots, l - 1$ and $k_r d^0_{r}(x^{\prod_{i=r}^{l-1} k_i})$ for $r = 2, \ldots, l - 1$. Since $x^1 = x^{k_2} \in E^0_{2}$, the induction step $r = 1$ is clear. Assume inductively that there are $k_2, \ldots, k_{r-1}$ and $x^{\prod_{i=r}^{l-1} k_i} \in E^0_{r}$. Choose $k_r$ such that $k_r d^0_{r}(x^{\prod_{i=r}^{l-1} k_i}) = 0$. Now $d^0_{r}(x^{\prod_{i=r}^{l-1} k_i}) = k_r d^0_{r}(x^{\prod_{i=r}^{l-1} k_i}) d^{0}_{r}(x^{\prod_{i=r}^{l-1} k_i})$. And $x^{\prod_{i=r}^{l-1} k_i} \in E^0_{r+1}$ for $k = \prod_{i=r}^{l-1} k_i$, the $l$-dimensionality of $X$ implies $x^k \in E^0_{\infty}$ and hence in the image under the edge homomorphism.

\[ \square \]

Definition 2.17. (Universal Cohomology Length relative to a family of subgroups)

Let $A = \{G/H_i\}$ be a collection of orbit spaces representing all homogeneous $G$-spaces with isotropy in some family $\mathcal{F}$ of subgroups of $G$. Let $M$ be a module over the ring $\pi^*_G(E_G(\mathcal{F}))$. The $H_A$-length of the module $M$ is the smallest number $k$ such that there exists spaces $A_1, \ldots, A_k \in A$ such that for any $\gamma \in M$ and $\omega_i$ in the kernel of the map

$$H^0_G(E_G(\mathcal{F})) \rightarrow H^0_G(G/H_i)$$

given by the composition of the edge homomorphism

$$H^0_G(E_G(\mathcal{F})) \rightarrow \lim_{H_i} H^0_G(G/H_i)$$

and the structure map $\lim_{H_i} H^0_G(G/H_i) \rightarrow H^0_G(G/H_i)$, the product $\gamma \omega_1 \ldots \omega_k$ is zero.

3. Computations in Burnside Rings

We specialize now to equivariant stable cohomotopy for proper actions.

We give a quick summary of important facts of Equivariant Stable Cohomotopy for finite groups.

Theorem 3.1. Let $G$ be a finite group. Then

- The $0$-th equivariant cohomotopy group of a point, $\pi^0_G(\{\bullet\})$ is isomorphic to the Burnside ring, denoted by $A(G)$, the Grothendieck ring of isomorphism classes of finite $G$-sets.
- The Burnside ring $A(G)$ is provided with maps $\varphi_H : A(G) \rightarrow \mathbb{Z}$, each one for every conjugacy class of subgroups in $G$. These extend to an injective map $A(G) \rightarrow \prod_{H \in \text{ccs}(G)} \mathbb{Z}$, where $\text{ccs}(G)$ denotes the set of conjugacy classes of subgroups in $G$.
- The prime ideals in $A(G)$ are given by the sets $\mathcal{P}_{K,p} = \{x \mid \varphi_H(x) \equiv 0(p)\}$, $\mathcal{P}_{H,0} = \{x \mid \varphi_H(x) = 0\}$, where $p$ is a prime number. The augmentation ideal $I_G$ is defined as the Ideal $\{x \mid \varphi_e(x) = 0\}$.
- There exists an element, the Bartsch element $0 \neq x \in A(G)$ with the property that $\varphi_H(x) = 0$ for every subgroup $H$.
- If $p$ is a prime number and $G$ is a finite $p$-group, then the completion map $A(G) \rightarrow A(G)_{I_G}$ is injective and the $I_G$-adic topology and the $p$-adic topologies coincide.
Equivariant Cohomotopy for proper actions of infinite discrete groups on finite $G$-CW complexes was defined in [22] via finite dimensional equivariant vector bundles for proper, finite $G$-CW complexes. An alternative approach is given by the nonlinear cocycles, which allow actions of noncompact Lie groups on finite $G$-CW complexes. These approaches are compared in [7]. For convenience, we give the definition from [22].

**Definition 3.2.** A $G$-vector bundle over a $G$-CW-complex $X$ consists of a real vector bundle $\xi : E \to X$ together with a $G$-action on $E$ such that $\xi$ is equivariant and each $g \in G$ acts on $E$ and $X$ via vector bundle isomorphisms. Let $S^k$ denote its fibrewise one-point compactification.

**Definition 3.3.** Let $X$ be a proper $G$-CW-complex. Let $\text{SPHB}^G(X)$ be the category with

- $\text{Ob}(\text{SPHB}^G(X)) = \{G\text{-vector bundles over } X\}$; and
- a morphism from a vector bundle $\xi : E \to X$ to vector bundle $\mu : F \to X$ is given by a bundle map $u : S^k \to S^k$ which covers the identity $id : X \to X$ and fiberwise preserves the basepoint.

(It is not required that $u$ is a fiberwise homotopy equivalence.)

Let $\mathbb{R}^k$ denote the trivial vector bundle $X \times \mathbb{R}^k \to X$.

**Definition 3.4.** Fix $n \in \mathbb{Z}$. Let $\xi_0, \xi_1$ be two $G$-vector bundles over $X$, and let $k_0$ and $k_1$ be two non-negative integers such that $k_i + n \geq 0$ for $i = 0, 1$. Then two morphisms

$$u_i : S^{k_i} \oplus \mathbb{R}^{k_i} \to S^{k_i} \oplus \mathbb{R}^{k_i+n}$$

are called equivalent, if there are objects $\mu_i$ in $\text{SPHB}^G(X)$ for $i = 0, 1$ and isomorphisms of $G$-vector bundles $v : \mu_0 \oplus \xi_0 \cong \mu_1 \oplus \xi_1$ such that the following diagram in $\text{SPHB}^G$ commutes up to homotopy.
\[ S^{\mu_0 \oplus \mathbb{R}^{k_0}} \wedge X \to S^{\mu_0 \oplus \mathbb{R}^{k_0} + 1} \]
\[ S^{\mu_1 \oplus \mathbb{R}^{k_1}} \wedge X \to S^{\mu_1 \oplus \mathbb{R}^{k_1} + 1} \]

**Definition 3.5.** For a proper $G$-CW-complex $X$ define
\[
\pi^n_G(X) = \{ \text{equivalence classes of morphisms } u \text{ as above} \}
\]

By introducing triviality conditions on a $G$-CW pair, (considering morphisms which are fibrewise constant with the value the point at infinity), equivariant cohomotopy groups are extended to an equivariant cohomology theory with multiplicative structure.

We introduce a Burnside ring for infinite groups, making out of Segal’s remark, part 1 in Theorem 3.1, our definition for finite groups:

**Definition 3.6.** Let $G$ be a group with a finite model for the classifying space for proper actions $E(G)$. The Burnside ring for $G$ is the 0-th equivariant cohomotopy ring of the classifying space for proper actions. In symbols
\[
A(G) = \pi^0_G(E(G))
\]

Denote by $A^\lim(G) = \lim_{H \in \mathcal{F} N} A(H)$ the inverse limit of the Burnside rings of the finite subgroups of $G$. Notice that this agrees with the 0,0-entry of the $E^2$-term of the equivariant Atiyah-Hirzebruch spectral sequence. The following relations between the Burnside ring and the inverse-limit Burnside ring are easy consequences of the rational collapse of the Atiyah-Hirzebruch spectral sequence:

**Lemma 3.7.** Let $G$ be a discrete group admitting a finite model for the classifying space for proper actions $E(G)$. The Burnside ring for $G$ is the 0-th equivariant cohomotopy ring of the classifying space for proper actions. In symbols

(i) The edge Homomorphism $e : A(G) \to A^\lim(G)$ has nilpotent kernel and cokernel. Its kernel is the nilradical.

(ii) The edge homomorphism gives an isomorphism between the set of prime ideals in $A(G)$ and $A^\lim(G)$ (in fact an homeomorphism in the Zariski topology), by assigning a prime ideal $I \subset A^\lim(H)$ its inverse image $e^{-1}(I) \in A(G)$.

(iii) The rationalized Burnside ring $\pi^0_G(E(G)) \otimes \mathbb{Q}$ does not contain nilpotent elements.

In the rest of the section we will describe a completion theorem for families of $p$-groups inside finite subgroups of discrete groups, which is the main computational tool for the computation of equivariant cohomology lengths needed for the proof of Theorem 1.1. This amounts to a generalization of the Segal Conjecture for families [1]. The result was proved in [7], Theorem 13 in page 58, though similar results have been proved in [25, 26] and [20], from where the crucial ideas and methods come.

Let $G$ be a discrete group and $\mathcal{F}$ be a family of finite subgroups of $G$, closed under conjugation and under subgroups. Fix a finite proper $G$-CW complex $X$ and a finite dimensional proper $G$-CW complex $Z$ whose isotropy subgroups lie in $\mathcal{F}$. Let $f : X \to Z$ be a $G$-map. Regard $\pi^0_G(X)$ as a module over $\pi^0_G(Z)$.
Definition 3.8. The augmentation ideal with respect to the family $\mathcal{F}$ is defined to be the kernel of the homomorphism

$$I = I_{G,\mathcal{F},\mathcal{Z}} = (\pi^0_G(Z) \xrightarrow{\text{res}^G_H \pi^0_H} \prod_{H \in \mathcal{F}} \pi^0_H(Z^0))$$

Proposition 3.9. Let $\mathcal{F}$ be a family of finite $p$-subgroups. Assume that there is an upper bound for the order of subgroups in $\mathcal{F}$.

Let $\mathcal{P} \subset \pi^0_H(\{\bullet\})$ be a prime ideal.

Then, the ideal

$$I_{H,\mathcal{F} \cap H,\{\bullet\}} := \ker \pi^0_H(\{\bullet\}) \to \prod_{K \in \mathcal{F}} \pi^0_{K \cap H}(\{\bullet\})$$

is contained in $\mathcal{P}$ if $\mathcal{P}$ contains the image of the structure map for $H$

$$\phi_H : \lim_{K \in \mathcal{F}} \pi^0_K(\{\bullet\}) \to \pi^0_H(\{\bullet\})$$

Proof. Let $m$ be a positive integer number divided by all orders of subgroups in $\mathcal{F}$. For a given subgroup $K$ in the family, let $u = \{u_1, \ldots, u_m\}$ be a finite set of cardinality $m$ with a free $K$-action. (For example, $u$ may be chosen to be a disjoint union of $\frac{m}{p}$ copies of $K$). This gives an injective homomorphism into the symmetric group in $m$ letters, $\rho : K \to S_m$. For a prime $p$, let $Syl_p$ be the $p$-Sylow subgroup of $S_m$.

Let $S_m/p$ be the set $S_m$ with the free $K$-action given by $k, s \mapsto \rho(h)(s)$ and $S_m/Syl_p$ be the set with the induced $K$-action. Notice that the fixed point set $S_m/Syl^p$ is nonempty if and only if $L$ is a $p$-subgroup. This construction is compatible with morphisms between subgroups in $\mathcal{F}$ in the sense that an homomorphism $K \to K'$ between groups in the family induces a map taking the free $K'$-set $S_m$ to the free $K$-set $S_m$ and the same for the homogeneous set $S_m/Syl_p$.

Consider the elements

$$\{(S_m - | S_m \mid K/K)\}_{K \in \mathcal{F}}$$

$$\{(S_m/Syl_p - | S_m/Syl_p \mid K/K)\}_{K \in \mathcal{F}}$$

Let $\mathcal{P}$ be a prime ideal containing the image of the structure map under $\phi_H$.

By the structure of the prime ideal spectrum, $\mathcal{P}$ is of the form $\mathcal{P}(M, p)$, where $M$ is a subgroup of $H$ and $p$ is a prime number or zero. By assumption, $\mathcal{P}$ contains the image under the structure map of the elements above. Since $\varphi^M(S_m - | S_m \mid K/K) = |\varphi^M(S_m/Syl_p - S_m/Syl_p)\mid M - | S_m/Syl_p \mid$ and both elements belong to $p\mathbb{Z}$, because $S_m/Syl_p$ has order prime to $p$, we conclude that either $p = 0$ or $M$ is a $p$-group.

If $M$ is a $p$-group, then $\mathcal{P}(M, p) = \mathcal{P}(\{e\}, p) \supset \mathcal{P}(\{e\}, 0) \supset I_{\mathcal{F},H,\{\bullet\}}$. If $p=0$, then $| S^M | = | S_m | = 0$, and hence $M = \{e\}$. For any subgroup $K'$ of every element $K \in \mathcal{F} \cap H$, $\mathcal{P}(K', 0) = \mathcal{P}(\{e\}, 0)$, since $K'$ is a $p$-group, hence $\mathcal{P}$ contains the intersection of all such ideals, which is $I_{\mathcal{F},H,\{\bullet\}}$.

□

Proposition 3.10. Let $L$ be an $n$-dimensional $G$-CW complex with isotropy in the family $\mathcal{F}$ consisting of finite $p$-subgroups inside the discrete group $G$. Let $f : G/H \to L$ be a $G$-map and $\mathcal{P} \subset \pi^0_H(\{\bullet\})$ be a prime ideal. Then $I_{\mathcal{F},H,\{\bullet\}} := \ker \pi^0_H(\{\bullet\}) \to \prod_{K \in \mathcal{F}} \pi^0_{K \cap H}(\{\bullet\})$ is contained in $\mathcal{P}$ if $\mathcal{P}$ contains the image of $I_{\mathcal{F},\mathcal{Z}}$ under $\text{ind}_{H \to G} \circ f^* : \pi^0_H(L) \to \pi^0_H(\{\bullet\})$

Proof. Let $\mathcal{P}$ be a prime ideal containing $I_{\mathcal{F},H,\{\bullet\}}$. By the previous proposition, we can assume that $\mathcal{P}$ contains the image of the structural map $\phi_H$. 
Let \( \psi : H^0_{\text{Zor}(G)}(E\mathcal{F}, \pi_G(G/\emptyset)) \to \lim_K \pi_K^0(\{\bullet\}) \) be the isomorphism given by assigning to an element \( x \in H^0_{\text{Zor}(G)}(E\mathcal{F}; \pi_K^0(\{\bullet\})) \) the element whose component under the structural map \( \phi_K \) is the image image under the map induced by the \((G\text{-homotopically})\) unique map \( u_K : G/K \to E\mathcal{F}, \) followed by the induction isomorphism

\[
H^0_{\text{Zor}(G)}(E\mathcal{F}; \pi_G^0(G/\emptyset)) \to H^0_{\text{Zor}(G)}(G/K, \pi_G^0(G/\emptyset)) \to H^0_{\text{Zor}(K)}(\{\bullet\}, \pi_K^0(K/\emptyset)) \cong \pi_K^0(\{\bullet\})
\]

Given an element \( a \in \lim_K I_{F,G,L}^{K,\emptyset}, \) denote by \( x \) its image under \( \psi^{-1}. \) By proposition 2.10 there exist a positive integer \( k \) and an element \( y \in \pi_G^0(E\mathcal{F}) \) such that \( \text{edge}(y) = x^k, \) which is furthermore an element of \( I_{F,G,L}. \)

Due to compatibility, the structure map \( \phi_H : \lim \pi_K^0(\{\bullet\}) \to \pi_H^0(\{\bullet\}) \) maps \( a^k \) to \( \mathcal{P}. \) Because \( \mathcal{P} \) is a prime ideal, the map \( \text{ind} \circ f^* \) maps \( a \) to \( \mathcal{P}. \)

**Theorem 3.11** (Segal Conjecture for families of finite subgroups). Let \( G \) be a discrete group and \( \mathcal{F} \) be a family of finite \( p\)-subgroups of \( G \) closed under conjugation and subgroups. Fix a finite proper \( G\text{-CW complex} \) \( X \) and a finite dimensional proper \( G\text{-CW complex} \) \( Z \) whose isotropy subgroups lie in \( \mathcal{F} \) and have bounded order. Let \( f : X \to Z \) be a \( G\text{-map}. \) Regard \( \pi_G^0(X) \) as a module over \( \pi_G^0(Z) \) and set

\[
I = I_{F,Z} = \ker(\pi_G^0(Z) \xrightarrow{\text{res}^G_H \circ \psi} \prod_{H \in \mathcal{F}} \pi_H^0(Z^0))
\]

then

\[
\lambda_{X,F,f}^m : \{\pi_G^m(X)/I^n \cdot \pi_G^m(X)\} \to \{\pi_G^m(E\mathcal{F} \times X^{n-1})\}
\]

is an isomorphism of pro-groups. Also, the inverse system

\[
\{\pi_G^m((E\mathcal{F} \times X)^n)\}_{n \geq 1}
\]

satisfies the Mittag-Leffler condition. In particular

\[
\lim^1 \pi_G^m((E\mathcal{F} \times X)^n) = 0
\]

and \( \lambda_{X,F,f}^m \) induces an isomorphism

\[
\pi_G^m(X)_f \xrightarrow{\cong} \pi_G^m(E\mathcal{F} \times X) \cong \lim_n \pi_G^m((E\mathcal{F} \times X)^n)
\]

**Proof.** Since both functors have Mayer-Vietoris sequences, both of the systems satisfy the Mittag-Leffler condition and in view of the 5-lemma for pro-modules, \([5]\), section 2, an inductive argument can be used to reduce the problem to the situation of \( X = G/H, \) and where \( H \) is a finite group.

In this case, there exists a commutative diagram

\[
\begin{array}{ccc}
\pi_G^0(Z) & \xrightarrow{f^*} & \pi_G^m(G/H) \\
& \downarrow \text{ind}^m_{n \to 0} & \\
A(H) & \xrightarrow{\sim} & \pi_H^0(\{\bullet\})
\end{array}
\]

Hence, the map of pro-modules

\[
\lambda_{X,F,f}^m : \{\pi_G^m(X)/I^n \cdot \pi_G^m(X)\} \to \{\pi_G^m(E\mathcal{F} \times X^{n-1})\}
\]

factorizes as follows
\[
\begin{align*}
\frac{\pi^n_G(G/H)/I^n \cdot \pi^n_G(G/H)}{\pi^n_H(\bullet)/J^n} \\
\frac{\pi^n_G(E_F \times G/H^n-1)}{\pi^n_H(\bullet)/I^n_{\mathcal{F}_{\Gamma H,H,\{\bullet\}}}} \\
\cong \\
\end{align*}
\]

Where \( J \) is the ideal generated by the image of \( I \) under \( \text{ind} \circ f^* \) and the lower horizontal map is an isomorphism due to the completion theorem for \( \text{families for} \) finite groups of \( \mathbb{H} \), the right vertical map is induced by \( f \). Due to proposition 3.10, the prime ideals containing \( J \) and \( I_{\mathcal{F}_{\Gamma H,H,\{\bullet\}}} \) agree and the right vertical map is an isomorphism.

\[ \square \]

**Corollary 3.12.** Let \( p \) be a prime number. For any group satisfying conditions \( \mathcal{J\bar{E}} \) for which the maximal finite subgroups are finite \( p \)-groups, the groups \( \pi^n_G(EG) \otimes \mathbb{Z}_p \) and \( \pi^n_G(EG)_{\text{max}} \) are isomorphic.

**Proof.** The morphism of pro-groups \( \{ \pi^n_G(X)/p^n \pi^n_G(X) \} \rightarrow \{ \pi^n_G(X \times EMAX)^{n-1} \} \) is proved to be an isomorphism for \( X = G/H \) with \( H \) a \( p \)-group. The prime ideals in \( \pi^n_H(\bullet) \) containing \( I_{\text{MAX},G,H,\{\bullet\}} \) and the one generated by the image of \( I_{\text{MAX},G,H,\{\bullet\}} \) under \( \text{ind} \circ f^* \) agree by the previous argument. Because \( H \) is a \( p \)-group, these agree with the ones containing \( I_{\mathcal{F}_{\Gamma,EG,H}} \) for the trivial family. Due to part 5 of theorem 3.1, these agree with the ones containing \( \pi^n_{\{\bullet\}} \).

Since both functors have Mayer-Vietoris sequences, the result follows by induction on the dimension of \( X \).

\[ \square \]

**Proposition 3.13.** Let \( G \) be a discrete group satisfying conditions \( \mathcal{J\bar{E}} \)

There exists a “Generalized Bartsch element” \( w \in \pi^0_G(EG) \) for which the map \( \pi^0_G(EG) \rightarrow H^0_{\mathbb{Z}_G}(EG, \pi^0_G(\bullet)) = \lim_{K \in \text{Sub}(G)} \pi^0(G, \bullet) \overset{WM}{\longrightarrow} \pi^0_{\mathbb{H}}(\bullet) \) given by the composition of the edge homomorphism and the structural map for the inverse limit maps \( w \) to a power of the element constructed in \( \mathcal{E} \) for any maximal subgroup \( M \).

**Proof.** Let \( X_M = \pi^0_M(\bullet) \) be the Bartsch element constructed in Theorem 3.1 part 4. Put \( x = \{x_M\} \in \lim_H \pi^0_H(\bullet) \). Choose an element \( w \) and a power \( k \) such that \( w \) is mapped to \( x^k \) under the edge homomorphism.

\[ \square \]

4. END OF PROOF

**Definition 4.1.** Let \( X \) be a numerable \( \mathcal{FLN} \)-space, which is contractible after forgetting the group action. Assume \( X \) is exhausted by a collection \( \{X_n\} \) of finite proper \( G \)-CW complexes. Define

\[ \tilde{\pi}^0_G(X) = \lim_n \pi^0_G(X_n) \otimes \mathbb{Q}_p \]

**Proposition 4.2.** Let \( G \) be a discrete group satisfying \( \mathcal{J\bar{E}} \). Let \( X \) be a numerable \( \mathcal{FLN} \)-space and let \( X_n \subset X_{n+1}, \ldots \) be a collection of proper \( G \)-CW complexes exhausting \( X \), where

- Each \( X_n \) is a finite proper \( G \)-CW complex.
- \( X \) is contractible after forgetting the group action.

Then, the maps \( X_n \rightarrow EG \) together with the \( G \)-homotopy equivalence \( EG \rightarrow J_{\mathcal{FLN}}(G) \) induce isomorphisms.
\[ \tilde{\pi}^0_G(J_{FLN}(G)) \rightarrow \tilde{\pi}^0_G(EG) \xrightarrow{=} \lim_n \pi^0_G(X_n) \]

in particular, the definition does not depend on the exhausting sequence.

Proof. The point is the existence of long exact sequences for the functor \( \tilde{\pi}^*_G(X,A) \), which is guaranteed by the natural equivalence with the Equivariant Cohomology Theory defined by \( (X,A) \mapsto \pi_G^0((E_\mathbb{R},0) \times (X,A)) \) on finite \( G \)-CW pairs. \( \square \)

**Proposition 4.3.** Let \( G \) be a group satisfying conditions [4.7]. Let \( X \) be a \( FLN \)-numerable space as in [4.7]. Then, there exists an element \( w \in \pi^0_G(EG) \otimes \mathbb{Q} \) such that

- \( w \in \ker \pi^0_G((EG) \otimes \mathbb{Q} \rightarrow \pi^0_G(G/H) \otimes \mathbb{Q} \) for all finite \( H \).
- \( w \in \ker \pi^0_G((EG) \otimes \mathbb{Q} \rightarrow \pi^0_G(X_0) \otimes \mathbb{Q} \).
- For every \( k > 0 \) there exists an \( n > 0 \) such that the image if \( w^k \) under \( \tilde{\pi}^0_G(EG) \rightarrow \pi^0_G(X_n) \otimes \mathbb{Q} \) is not zero.

Proof. Let \( v \in \pi^0_G(EG) \otimes \mathbb{Q} \cong \Pi_{H \in MAX} A(H) \otimes \mathbb{Q} \) be the element constructed in proposition [4.1]. Let \( m = G\text{-cat}(X_0) \) and put \( w = v^m \). As in [9], the following diagram commutes:

\[
\begin{array}{ccc}
\pi^0_G(EG) \otimes \mathbb{Q} & \xrightarrow{\sim} & \lim_n \pi^0_G(X_n) \otimes \mathbb{Q} \\
\downarrow & & \downarrow \\
\pi^0_G(EG)_{lo,M} \otimes \mathbb{Q} & \xrightarrow{\sim} & \tilde{\pi}^0_G(EG) \xrightarrow{\sim} \tilde{\pi}^0_G(X)
\end{array}
\]

as the left and right vertical maps are isomorphisms, and there are no nilpotent elements in the rationalized Burnside Ring \( \pi^0_G(EG) \otimes \mathbb{Q} \), there are no nilpotents in \( \pi^0_G(EG) \otimes \mathbb{Q} \), and so there exists a natural number \( n \) such that the third condition holds.

\( \square \)

Let \( \hat{E} \subset E \) be a \( G \)-invariant linear subspace with a finite dimensional, \( G \)-invariant complement \( F_0 \) satisfying the mountain pass condition 1 in [11]. For any finite dimensional subspace \( \hat{F} \), the sum \( F = F_0 \oplus \hat{F} \) satisfies

\[ F = B_{\alpha}(F) \subset \phi^\alpha \]

**Lemma 4.4.** There is a \( G \)-map \( f \) such that the diagram

\[
\begin{array}{ccc}
(F,F - B_{\alpha}(F)) & \xrightarrow{i_F} & (E - \{0\},\phi^\alpha) \\
\downarrow f & & \downarrow \\
(F,F - S(F_0 \oplus F)) & \xrightarrow{j_F} & (E - \{0\},S(\hat{E}))
\end{array}
\]

commutes, where \( i_F \) and \( j_F \) are given by inclusions.

Proof. Compare lemma 5.2 in [13]. Define a map \( f : E \rightarrow \hat{E} \) by sending the bounded closed subspace \( A \) in theorem [11] to 0, mapping \( \hat{E} \cap \phi^\alpha \) into \( \hat{E} - B_{\alpha}(\hat{E}) \) and extending to all of \( E \), since \( \hat{E} \) is proper, \( G \)- absolute retract, Theorem 3.9 in page 1953 of [3]. \( \square \)

The same argument as in Proposition 5.3, [13], page 17 yields:

**Proposition 4.5.** For any equivariant Cohomology Theory, \( H^*_G \),

\[ G - cat(E,\phi^\alpha) \geq H^*_G \text{length } S(F_0 \oplus F) \rightarrow S(F_0 \oplus \hat{E},S(F_0)) \]
We now finish the proof of Theorem 1.1. This follows the proof of proposition 3.2 in [9].

**Proposition 4.6.**

\[ G - \text{cat}(E, \phi^n) = \infty \]

**Proof.** Let \( F_n \) be an increasing sequence of finite dimensional linear \( G \)-subspaces of \( \hat{E} \) such that \( \hat{F} = \cup F_n \) is infinite dimensional. as in [9], the length of the inclusion \( S(F_0 \oplus F_n) \rightarrow S(F_0 \oplus \hat{E}, S(F_0)) \)

becomes arbitrarily large as \( n \) tends to infinity.

The contractible, \( F\text{LIN} \)-numerable space \( S(\hat{E}) \) satisfies the hypothesis of lemma 4.2.

Hence there is an element \( v \in \pi^0_G(E,G) \) satisfying conditions 1 to 3 in [4.2] let \( v \) and \( v_n \) be the images of \( W \) along the homomorphism induced by the universal maps \( S(F_0 \oplus \hat{E}) \rightarrow E,G \), respectively \( S(F_0 \oplus F_n) \rightarrow E,G \). Since the diagram

\[
\begin{align*}
\pi^0_G(S(F_0 \oplus \hat{F}_n)) & \xrightarrow{j^*} \pi^0_G(S(F_0 \oplus \hat{E}), S(F_0)) \\
\pi^0_G(S(F_0) & \xrightarrow{j^*} \pi^0_G(S(F_0 \oplus \hat{E}))
\end{align*}
\]

commutes up to homotopy, \( v_n \in \text{im}(j^*_n) \), and proposition 4.2 yields that for any \( k \) there is an \( n \) with \( \pi^0_G - \text{length} j_n \geq k \).

\[ \square \]

5. Example

In this section, we shall discuss an example of Banach \( G \)-space with an action of an infinite group \( G \) satisfying the hypothesis in the Mountain Pass Theorem. The example illustrates also the lack of compactness assumptions on the critical set.

**Example 5.1** (An improvement of a result by Rabinowitz). Let \( P \) be a \( p \)-group, \( \beta : P \rightarrow \text{GL}_n(\mathbb{Z}) \) be a group homomorphism. Consider the \( P \)-action on \( V := \mathbb{R}^n \) induced by \( \beta \). Assume that

- The \( P \)-action on \( V := \mathbb{R}^n \) is free outside 0.
- There exist real numbers \( \{T_1, \ldots, T_n\} \) (called in the subsequent periods) such that the abelian subgroup generated by the vectors \( (T_1,0,\ldots,0),(0,T_2,0,\ldots,0),\ldots,T_n,0,\ldots,0,0) \) is \( P \)-invariant.

In this situation one has a proper action on \( V \) of the group \( \mathbb{Z}^n \times P \) on the space \( \mathbb{R}^n \) furnished with the action induced by \( \beta \).

Let \( T_0 \) be a real number. Consider the Sobolev space \( E := W^{1,2}_{T_0}(\mathbb{R}, V) \) of \( T_0 \)-periodic functions with the norm

\[
|q| = \sqrt{\int_0^{T_0} |q'(t)|^2 dt + \int_0^{T_0} q(s)^2 ds}
\]

Let \( L \in C^1(\mathbb{R} \times V, \mathbb{R}^{n^2}) \), \( W \in C^1(\mathbb{R} \times V, \mathbb{R}) \), \( f \in C(\mathbb{R}, V) \) be such that

- For every \( t \in \mathbb{R} \) and \( q = (q_1, \ldots, q_n) \) \( \lambda(t,q) \) is a symmetric matrix, and there exists an \( \alpha > 0 \) such that \( \langle \lambda(t,q) \xi, \xi \rangle \geq \alpha \langle \xi, \xi \rangle \).
- \( L \) is \( T_0 \)-periodic in \( t \) and \( T_t \)-periodic in \( q_t \).
- \( W \) is \( T_0 \)-periodic in \( t \) and \( T_t \)-periodic in \( q_t \).
- \( f \) is \( T_0 \)-periodic in \( t \) and \( \frac{1}{T_0} \int_0^{T_0} f(t)dt = 0 \).
Consider the functional
\[
\phi(q) := \int_0^{T_0} \left[ \frac{1}{2} (L(t, q) \dot{q}, \dot{q}) - W(t, q) + \langle f(t), q(t) \rangle \right] dt
\]
Notice that \( \phi \) is invariant under the proper action of the group \( G := \mathbb{Z}^n \rtimes P \).

Let \( c \) be a real number and \( K_c := \{ q \mid \phi(q) = c \} \) be the critical set at level \( c \). Let \( S \) be the subset consisting of those \( q = (q_1(t), \ldots, q_n(t)) \) for which \( 0 < \frac{1}{T_0} \int_0^{T_0} q_i < T_i \). Consider the copy of \( V \) given by \( \{ u \in W^{1,2}_{T_0} \mid \frac{1}{T_0} \int_0^{T_0} u dt = u \} \) and denote by \( \hat{V} \) the orthogonal complement.

By an argument parallel to lemma 2.12 in [33], the set \( S := \{ u = (u_i) \mid \forall i \in \{0, \ldots, n\}, 0 \leq \frac{1}{T_0} \int_0^{T_0} u_i dt \leq T_i \} \) is a fundamental deformation region. The crucial point is the compacity of \( K_c \cap S \). We reproduce the argument of Rabinowitz, [33], page 309.

There exists a description of \( \phi' \) as the functional \( D(P_0 + P) \), where \( D : E \rightarrow E^* \) is the dualization map, and \( P_0 \) is a compact operator and \( P_1 \) is the operator assigning to \( q \) the element \( w \) such that \( \int_0^{T_0} \langle L(t, q) \dot{q}, \phi \rangle = \int_0^{T_0} \langle \dot{w}, \phi \rangle \) for all \( \phi \in W^{1,2}_{T_0}(\mathbb{R}, V) \).

Given a sequence \( Q_n = \xi_n + Y_n \subset K_c \cap S \), with \( \xi_n \in V \) and \( Y_n \in \hat{V} \), the ellipticity assumption and the definition of \( K_c \cap S \) imply that \( Y_n \) is bounded, and the definition of \( S \) imply that \( \xi_n \) is bounded. Hence, a subsequence \( Q_m \) converges weakly in \( E \) and strongly in \( L^\infty \) to \( Q \), \( P(Q_m) \) converges in \( E \). The expression
\[
\dot{Q}_n = L^{-1}(t, Q_m) \left[ \frac{1}{T_0} \int_0^{T_0} L(\tau, Q_m) \dot{Q}_m d\tau - \frac{d}{dt} P(Q_m) \right]
\]
implies that the subsequence \( Q_m \) converges in \( L^2 \), and \( \phi'(Q) = 0 \).

Moreover, the hypothesis on the action give that the critical set \( K_c \) is cocompact. This is due to the fact that \( G \setminus K_c \) is a quotient of the compact space \( P \setminus K_c \cap S \) after eventual identification. Theorem [11] Applied to the functional \( \phi \) and the subspace \( \hat{V} \), the orthogonal complement of \( V \) in \( W^{1,2}_{T_0}(\mathbb{R}, V) \) guarantee the existence of an unbounded sequence of critical values above a value \( a \) for which the mountain pass situation occurs. The critical points of \( \phi \) are classical solutions of the system of ordinary differential equations
\[
\frac{d}{dt} L(t, q, \dot{q}) \dot{q} - \frac{1}{2} \frac{\partial L}{\partial q}(q, \dot{q}) + V_q(t, q) = f(t)
\]

6. CONCLUDING REMARKS

Paraphrasing Willem, [37], page 3 Minimax-Type Theorems usually consist of different parts:

- Deformation lemma using some pseudogradient vector field.
- Construction of Palais-Smale typical sequences, which converge either due to some apriori compactness condition, or which give critical points using additional a posteriori information, typically topological intersection properties, like the intermediate value theorem, the Borsuk-Ulam theorem, degree notions, etc.

Remark 6.1 (Borsuk-Ulam Type Theorems). In this work, the proof given by Bartsch-Clapp Puppe was adapted using an equivariant Borsuk-Ulam-Type Theorem, which may be deduced from [11] and [11]. The problem of classifying the groups satisfying equivariant Borsuk-Ulam-Type theorem has deserved particular attention [11], among others.

Let \( G \) be a discrete, linear group which acts properly and linearly on finite dimensional representation spheres \( S^{V} \). Define the Borsuk-Ulam function \( b_G(n) \) as
the maximal natural number $k$ such that if there exists a $G$-map $S^V \to S^W$ where $\dim V \geq n$, then $\dim W \geq k$

**Problem 6.2.** Classify all linear, discrete groups satisfying

$$\lim_{n \to \infty} b_G(n) = \infty$$

as in [8], [19], and in this work, condition [12] the answer should involve restrictions for the number of primes dividing the cardinality of the isotropy groups.

**Remark 6.3** (Topological Noncompact Groups of Symmetry). In the context of Hamiltonian Systems, some proper actions of noncompact Lie groups appear [24]. Equivariant Cohomotopy Theory has been extended in [7] for these class of symmetries. The use of Equivariant Algebraic Topology, particularly Equivariant Cohomotopy may be useful. However, in this context, the Segal Conjecture (which was the main homotopy theoretical input of theorem [14] crucially in the proof of the Borsuk-Ulam-type result) is not true, as it is not even true for compact Lie groups, see [15], [10].

**Remark 6.4** (Equivariant Degree Notions for Infinite Discrete Groups). In [7], an equivariant degree notion for proper actions of discrete group is defined. This assigns to a quadruple $(E,F,T,c)$ consisting of locally trivial $G$-Hilbert bundles over a proper, cocompact $G$-CW complex, a fibrewise Fredholm operator $T$ and a fibrewise compact nonlinearity satisfying the property that the map $T_x + c_x : E_x \to F_x$ defined on the fibers $E_x, F_x$ over each point $x$ is proper, an element in the equivariant cohomotopy $\pi^*_G(X)$, as defined in definition [5]. We will analyze the applicability of this degree notion to equivariant variational problems elsewhere.

**References**

[1] J. F. Adams, J.-P. Haeberly, S. Jackowski, and J. P. May. A generalization of the Segal conjecture. Topology, 27(1):7–21, 1988.
[2] A. Ambrosetti and P. H. Rabinowitz. Dual variational methods in critical point theory and applications. J. Functional Analysis, 14:349–381, 1973.
[3] N. Antonyan, S. A. Antonyan, and L. Rodríguez-Medina. Linearization of proper group actions. Topology Appl., 156(11):1946–1956, 2009.
[4] S. A. Antonyan and E. Ellingsen. The equivariant homotopy type of $G$-ANR’s for proper actions of locally compact groups. In Algebraic topology—old and new, volume 85 of Banach Center Publ., pages 155–178. Polish Acad. Sci. Inst. Math., Warsaw, 2009.
[5] M. F. Atiyah and I. G. Macdonald. Introduction to commutative algebra. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.
[6] R. Ayala, F. F. Lasheras, and A. Quintero. The equivariant category of proper $G$-spaces. Rocky Mountain J. Math., 31(4):1111–1132, 2001.
[7] N. Bárdenas. Nonlinearity, proper actions and homotopy theory. Ph.D Thesis, 2010.
[8] T. Bartsch. On the existence of Borsuk-Ulam theorems. Topology, 31(3):533–543, 1992.
[9] T. Bartsch, M. Clapp, and D. Puppe. A mountain pass theorem for actions of compact Lie groups. J. Reine Angew. Math., 419:55–66, 1991.
[10] S. Bauer. On the Segal conjecture for compact Lie groups. J. Reine Angew. Math., 400:134–145, 1989.
[11] M. R. Bridson and A. Haefliger. Metric spaces of non-positive curvature, volume 319 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1999.
[12] M. Clapp and D. Puppe. Invariants of the Lusternik-Schnirelmann type and the topology of critical sets. Trans. Amer. Math. Soc., 298(2):603–620, 1986.
[13] M. Clapp and D. Puppe. Critical point theory with symmetries. J. Reine Angew. Math., 418:1–29, 1991.
[14] J. F. Davis and W. Lück. Spaces over a category and assembly maps in isomorphism conjectures in $K$- and $L$-theory. $K$-Theory, 15(3):201–252, 1998.
[15] A. Dress. A characterisation of solvable groups. Math. Z., 110:213–217, 1969.
[16] M. Feshbach. The Segal conjecture for compact Lie groups. Topology, 26(1):1–20, 1987.
17. W. Krawcewicz and W. Marzantowicz. Lusternik-Schnirelman method for functionals invariant with respect to a finite group action. *J. Differential Equations*, 85(1):105–124, 1990.

18. E. Laitinen. On the Burnside ring and stable cohomotopy of a finite group. *Math. Scand.*, 44(1):37–72, 1979.

19. E. Laitinen and M. Morimoto. Finite groups with smooth one fixed point actions on spheres. *Forum Math.*, 10(4):479–520, 1998.

20. W. Lück. The segal conjecture for infinite groups. Unpublished.

21. W. Lück. Transformation groups and algebraic K-theory, volume 1408 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1989. Mathematica Gottingensis.

22. W. Lück. The Burnside ring and equivariant stable cohomotopy for infinite groups. *Pure Appl. Math. Q.*, 1(3, part 2):479–541, 2005.

23. W. Lück. Equivariant cohomological Chern characters. *Internat. J. Algebra Comput.*, 15(5-6):1025–1052, 2005.

24. W. Lück. Survey on classifying spaces for families of subgroups. In *Infinite groups: geometric, combinatorial and dynamical aspects*, volume 248 of *Progr. Math.*, pages 269–322. Birkhäuser, Basel, 2005.

25. W. Lück and B. Oliver. Chern characters for the equivariant K-theory of proper G-CW-complexes. In *Cohomological methods in homotopy theory* (Bellaterra, 1998), volume 196 of *Progr. Math.*, pages 217–247. Birkhäuser, Basel, 2001.

26. W. Lück and B. Oliver. The completion theorem in K-theory for proper actions of a discrete group. *Topology*, 40(3):585–616, 2001.

27. W. Lück and R. Stamm. Computations of K- and L-theory of cocompact planar groups. *K-Theory*, 21(3):249–292, 2000.

28. R. C. Lyndon and P. E. Schupp. *Combinatorial group theory*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1977 edition.

29. W. Marzantowicz. A G-Lusternik-Schnirelman category of space with an action of a compact Lie group. *Topology*, 28(4):403–412, 1989.

30. T. Matumoto. Equivariant CW complexes and shape theory. *Tsukuba J. Math.*, 13(1):157–164, 1989.

31. R. S. Palais. On the existence of slices for actions of non-compact Lie groups. *Ann. of Math. (2)*, 73:295–323, 1961.

32. P. H. Rabinowitz. *Minimax methods in critical point theory with applications to differential equations*, volume 65 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1986.

33. P. H. Rabinowitz. On a class of functionals invariant under a $\mathbb{Z}^n$ action. *Trans. Amer. Math. Soc.*, 310(1):303–311, 1988.

34. M. Roberts, C. Wulff, and J. S. W. Lamb. Hamiltonian systems near relative equilibria. *J. Differential Equations*, 179(2):562–604, 2002.

35. G. B. Segal. Equivariant stable homotopy theory. In *Actes du Congrès International des Mathématiciens (Nice, 1970)*, Tome 2, pages 59–63. Gauthier-Villars, Paris, 1971.

36. T. tom Dieck. *Transformation groups*, volume 8 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1987.

37. M. Willem. *Minimax theorems*. Progress in Nonlinear Differential Equations and their Applications, 24. Birkhäuser Boston Inc., Boston, MA, 1996.

38. H. Zhang and Z. Li. Periodic solutions for a class of nonlinear discrete Hamiltonian systems via critical point theory. *J. Difference Equ. Appl.*, 16(12):1381–1391, 2010.

Hausdorff Center for Mathematics, Endenicher Allee 60, D-53115 Bonn, Germany

E-mail address: barcenas@math.uni-bonn.de

URL: http://wwwmath.uni-muenster.de/u/barcenas