LETTER

New exact solutions of the Dirac equation of a charged particle interacting with an electromagnetic plane wave in a medium

Sándor Varró

Wigner Research Centre for Physics, Hungarian Academy of Sciences, Institute for Solid State Physics and Optics, Budapest, Hungary

E-mail: varro.sandor@wigner.mta.hu

Received 19 May 2013, in final form 16 June 2013
Accepted for publication 17 June 2013
Published 30 July 2013
Online at stacks.iop.org/LPL/10/095301

Abstract

Exact solutions are presented of the Dirac equation of a charged particle moving in a classical monochromatic electromagnetic plane wave in a medium of index of refraction $n_m < 1$. The found solutions are expressed in terms of new complex trigonometric polynomials, which form a doubly infinite set labelled by two integer quantum numbers. These quantum numbers represent quantized spectra of the energy–momentum components of the charged particle along the polarization vector and along the propagation direction of the applied electromagnetic plane wave field (which is considered as a laser field of arbitrarily high intensity propagating in an underdense plasma). The found solutions may serve as a basis for the description of possible quantum features of mechanisms for the acceleration of electrons by high-intensity laser fields.

(Some figures may appear in colour only in the online journal)

1. Introduction

The Volkov states have long been an important tool in the theoretical description of fundamental processes taking place in strong laser fields (Fedorov 1997). They, and their scalar or nonrelativistic variants, are exact solutions of the Dirac equation (Wolkow 1935) and Klein–Gordon equation (Gordon 1927) or the Schrödinger equation (Keldish 1964, 1965, Bunkin and Fedorov 1965, 1966, Popov 1974) of an electron or other charged particle moving in a true plane electromagnetic wave in vacuum, or the dipole approximation, respectively. On the basis of these wavefunctions, the interaction with the laser field in the initial, final or in the intermediate states of an electron can be taken into account exactly, i.e. up to ‘infinite order’ (in the terminology of perturbation theory). Recently there has been a renewed interest in the physical applications and mathematical properties of these states, concerning the theory of laser–matter interactions. In the relativistic regime the closed analytic form of these states relies on the vacuum dispersion relation of the classical electromagnetic (EM) plane wave, which represents the laser field. Many mathematical details (e.g. orthogonality and completeness) of these solutions have been investigated over decades (see Brown and Kibble 1964, Eberly 1969, Neville and Rohrlich 1971, Ritus and Nikishov 1979, Bergou and Varró 1980), and even now their study is undergoing a certain ‘renaissance’ (see e.g. Zakowicz 2005, Salamin et al 2006, Musakhayan 2008, Ehlotzky et al 2009, Boca and Florescu 2009, 2010, Harvey et al 2012). We note that the solutions of the Dirac equation can also be expressed in a closed form for a quantized electromagnetic plane wave, as was first shown by Bersons (1969a, 1969b). Besides the generalizations of these quantized states (Fedorov and Kazakov 1973, Bersons and
Valdmans 1973), they have also been used to treat nonlinear Compton scattering beyond the semiclassical approximation (Bergou and Varró 1981b). The generalization of the Volkov states with quantized radiation has also been used to describe multiphoton stimulated bremsstrahlung in the nonrelativistic regime (Bergou and Varró 1981a), a process that has been investigated (Burenkov and Tikhonova 2010) from the point of view of possible nonclassical effects. The appearance of various versions of entanglement in strong-field laser–matter interactions has also been investigated by Fedorov et al (2006) for breakup processes, and by Varró (2008, 2010a, 2010b) for the photon–electron system itself. In the present letter we shall stay in the frame of the semiclassical description of the radiation field, and will not discuss many-particle correlations and entanglement in the presence of strong laser fields.

If the EM field propagates in a medium (e.g. with a real index of refraction \( n_m < 1 \) or \( n_m > 1 \)), or the strong laser field is modelled by a time-periodic electric field, then the previously studied solutions of the relativistic wave equations can be expressed by solutions of the corresponding Mathieu and Hill equations (see Narozhny and Nikishov 1974a, 1974b, Becker 1977, Cronström and Noga 1977). Such an analysis has, for example, been applied in the mathematical study of pair creation in strong fields (Nikishov and Ritus 1967a, 1967b, Nikishov 1970). This phenomenon has received a renewed interest recently (see e.g. Narozhny et al 2004, Popov 2004, Dunne 2004, 2009), owing to the considerable technological developments in extremely high power and ultrashort pulse laser systems (see e.g. Krausz and Ivanov 2009, Mourou et al 2006, 2013). Concerning the mathematical description, except for very special cases (see e.g. Feldberger and Marburger 1975), the solutions cannot be expressed in a closed finite form; in the simplest case they are related to the (transcendental) Mathieu functions.

In the present letter we show that there are exact closed form solutions of the Dirac equation of a charged particle moving in a monochromatic classical plane EM field in a medium of index of refraction \( n_m < 1 \). It is well known that such a radiation field can be transformed to a homogeneous oscillating electric field (Narozhny and Nikishov 1974a, 1974b, Becker 1977) by going over to a suitably chosen Lorentz frame, thus our considerations are relevant for discussing this type of interaction as well. The solutions to be derived are proportional to finite complex trigonometric polynomials whose arguments are integer multiples of the EM wave’s phase, and they form a doubly infinite discrete set of solutions labelled by two integer quantum numbers. This latter property is a completely new feature in comparison with both the Volkov states (in vacuum) parameterized by continuous four momenta, and the Mathieu-type solutions characterized by ‘stability charts’ resulting in a band structure of the allowed parameter.

In section 2 we construct the differential equations for the scalar coefficient functions of the constant bispinor basis, from which the complete Dirac bispinor is built up. In the course of the derivation we also make a brief comparison with the method leading to the usual Volkov states or to the Mathieu-type wavefunctions. Section 3 is devoted to the determination of the complex polynomial solutions, which are associated with the eigenvalue problem of special finite tri-diagonal matrices. The solutions are labelled by two integer quantum numbers, which represent a quantized spectrum of the electron’s momentum components along the (linear) polarization direction and along the propagation of the laser field. In section 4 we show the main mathematical properties of the found solutions and discuss a number of physical implications associated with these solutions. Some numerical examples will also be presented, for illustrative purposes, of the momentum spectrum and the temporal behaviour and harmonic structure of the wavefunctions. In section 5 a brief summary with conclusions closes our letter.

2. Construction of the wave equations for the scalar coefficient functions of the Dirac bispinors

In an external electromagnetic field characterized by the four-vector potential \( A(x) \) the Dirac equation of a spinor particle of charge \( e \) and of mass \( m \) has the form

\[
[y \cdot \Pi - \kappa] \psi = 0, \quad \Pi \equiv i \partial - eA, \quad e = e/\hbar c, \quad \kappa = mc/\hbar, \tag{1}
\]

where \( c \) is the velocity of light in vacuum, and \( \hbar \) is Planck’s constant divided by \( 2\pi \). In (1) we have introduced the operator of the kinetic four-momentum \( \Pi \) of the charged particle. In a medium of index of refraction \( n_m \), a general transverse electromagnetic plane wave of wavevector \( k \) can be represented by a vector potential

\[
A(x) = eA_1(\xi) + e_2A_2(\xi), \quad \xi = k \cdot x,
\]

\[
k \cdot A = 0, \quad k = (k^\mu) = k^0(1, n_m e_3), \tag{2}
\]

where \( \{e_1, e_2, e_3\} \) form a right system of mutually orthogonal unit vectors. In principle, the scalar functions \( A_{1,2}(\xi) \) in (2) may have arbitrary form (satisfying, of course, certain regularity conditions). For a purely monochromatic field \( A_{1,2}(\xi) \) are simple sine and cosine functions and \( k^0 = 0 = \omega_0/c \), where \( \omega_0 = 2\pi v_0 \) is the circular frequency. As has long been

\[1\] The Minkowski metric tensor \( g_{\mu\nu} = g^{\mu\nu} \) has the components \( g_{00} = -g_{i\iota} = 1 \) (\( i = 1–3 \)) and \( g_{\mu\nu} = 0 \) if \( \mu \neq \nu \) (\( \mu, \nu = 0–3 \)). The scalar product of two four-vectors \( a \) and \( b \) is \( a \cdot b = g_{\mu\nu} a^\mu b^\nu \), i.e. \( a \cdot b = a_0 b_0 = \hat{a}^\dagger \hat{b} = a \cdot b \). Space–time coordinates are denoted by \( x^\mu \), where \( x^0 = (ct, r) \). The four-gradient is \( \partial = (\partial/\partial x^0, -\partial/\partial r) \), and \( \partial_0 = \partial/\partial x^0 \). In the standard representation the Dirac matrices \( \alpha = (\alpha_0, \alpha_x, \alpha_y) \) and \( \beta \) have the form

\[
\alpha = \begin{pmatrix} 0 & a \\ \bar{a} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & a \\ -\bar{a} & 0 \end{pmatrix},
\]

\[
\sigma_\pi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

In the first three equations the ‘0’ and ‘1’ denote \( 2 \times 2 \) zero and unit matrices, respectively. In the last three equations \( \sigma_x, \sigma_y, \sigma_z \) are the usual \( 2 \times 2 \) Pauli matrices. The \( \gamma \) matrices are defined as \( \gamma^\dagger \gamma = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \gamma_0 \gamma_2 = -\gamma_1 \gamma_3 \), and \( \gamma_0 = \beta \), their commutation relations are \( \gamma_\mu^\dagger \gamma_\nu = \gamma_\nu^\dagger \gamma_\mu = 2\delta_\mu^\dagger \gamma_\nu \), and \( \gamma_\mu^\dagger \gamma_\nu \gamma_0 = \gamma_\nu \gamma_0^\dagger \gamma_\mu \), where \( \gamma_\mu^\dagger \gamma_\nu \) denotes the adjoint (transposed conjugate) of \( \gamma_\mu \) (cf Bjorken and Drell 1964).
shown (Wolkow 1935), in the case when \( n_m = 1 \), i.e. \( k^2 = 0 \), the solutions of the above Dirac equation can be expressed in a simple closed form; these are the Volkov states. Various physical applications and mathematical details of these states (concerning the boundary conditions, positive and negative energy solutions, orthogonality and completeness) can be found for instance in Brown and Kibble (1964), Eberly (1969), Bergou and Varró (1980), Zakowicz (2005) and Boca and Florescu (2010).

In proceeding towards the solution of the Dirac equation for the bispinor \( \psi \), it is customary to go over to the second-order Dirac equation for a new bispinor \( \Psi \), by using the Ansatz

\[
\Psi = (\gamma \cdot \Pi + \kappa) \Psi,
\]

where we have introduced the spin tensor \( \sigma_{\mu\nu} \) and the electromagnetic field strength tensor \( F^{\mu\nu} \), one can easily show that the matrix part \( \sigma \cdot F \) on the left-hand side of the second-order Dirac equation has the explicit form \( \sigma \cdot F = 2i(\gamma \cdot k)[\gamma \cdot A(\hat{\xi})] \), where the prime denotes derivative with respect to \( \hat{\xi} \). By considering a modified plane wave solution, labelled by a four-momentum parameter \( p_\mu \), we have

\[
\Psi = \Psi_p(\xi) \exp(-ip \cdot x),
\]

\[
-k^2 \frac{d^2 \Psi_p}{d\xi^2} + 2ik \cdot p \frac{d \Psi_p}{d\xi} + \left[ p^2 - k^2 - 2e_p \cdot A + e^2 A^2 - \frac{1}{2} \xi^2 \sigma \cdot F \right] \Psi_p = 0,
\]

In vacuum \((n = 1, k = k_0(1, e_3), k^2 = 0)\) the factor of the second derivative in (4) is zero, and the matrix part is nilpotent, \((\sigma \cdot F)^2 = 0\), because \((\gamma \cdot k)^2 = k^2 = 0\) in this case. The remaining first-order equation can be directly integrated, yielding the Volkov states. The mathematical complexity of the problem is largely increased if one considers the interaction with a plane wave propagating in a medium of index of refraction \( n_m \neq 1 \). This is due to the presence of the non-vanishing second derivative in (4), since now \( k^2 = k_0^2(1 - n_m^2) \neq 0 \). Anyway, in this case we can eliminate the first derivative in (4), or an a priori use the following modified Ansatz at the outset,

\[
\Psi = \Psi_p^{(\pm)}(\xi) \exp \left( \mp i \left[ p - \frac{(k \cdot p)}{k^2} \hat{k} \right] \cdot x \right)
\]

\[
k^2 > 0,
\]

\[
\tilde{p} \equiv \tilde{k} \cdot p/(k^2)^{1/2}, \quad \tilde{x} \equiv \tilde{k} \cdot x/(k^2)^{1/2},
\]

where the ambient sign \( \mp \) refers to the ‘positive and negative energy solutions’. The plane wave factor in the first equation in (5a) has also been written as \( \exp[\mp i(-\tilde{p} \cdot \tilde{x} - p_1x_1 - p_2x_2)] \), where \( p_1 = \epsilon_1 \cdot p \) and \( p_2 = \epsilon_2 \cdot p \) are the transverse momentum components. We have introduced the ‘complementary wavevector’ \( \tilde{k} = -\theta(\tilde{n}_m, e_3) (k_0^2 = -k^2 \) and \( \tilde{k} \cdot k = 0 \), with the help of which one can derive

\[
 p = \frac{(k \cdot p)}{k^2} - \frac{(\tilde{k} \cdot p)}{k^2} \tilde{k} - (p \cdot e_1)e_1 - (p \cdot e_2)e_2,
\]

\[
\tilde{p} = \frac{(k \cdot \tilde{p})}{\tilde{k}^2} \cdot \tilde{x} = -\tilde{p} \tilde{x} - p_1x_1 - p_2x_2.
\]

This kind of expansion can be performed for any other four-vectors, of course. If \( n_m^2 < 1 \), then \( \tilde{k} \) is space-like, and \( \tilde{p} \equiv \tilde{k} \cdot p/(k^2)^{1/2} \) plays the role of a momentum component conjugated to the ‘position variable’ \( \tilde{x} \equiv \tilde{k} \cdot x/(k^2)^{1/2} \) (see e.g. Becker 1977). Thus, the Ansatz in (5a) means a separation of variables, where \( \tilde{k} \cdot p, p_1 \) and \( p_2 \) are ‘three-momentum type parameters’. Accordingly, \( p_\tilde{k} = (k \cdot \tilde{p}) \) may be said to be an ‘energy type parameter’. One should keep in mind that \( p_\tilde{k} \) does not explicitly show up in the solution, because \( (p_\tilde{k}^2/k^2)\tilde{n}_m \) is subtracted from \( p_\mu \), as is shown by the second equation of (5d).

At this point we would like to note that for a circularly polarized monochromatic plane wave one has \( A_{cIR}(x) = A_0(\epsilon, \cos \xi + e \sin \xi) \) and \( A_{cIR}^2 = -A_0^2 = \) constant. By neglecting the spin interaction term \( \sigma \cdot F \) in (5c) (or by a priori considering a Klein–Gordon particle of wavefunction \( \Phi_p^{(\pm)}(\mp x \cdot p) \)), we immediately obtain a Mathieu equation for the scalar modulation function \( \Phi_p^{(\pm)}(\xi) \) (see e.g. Becker 1977, Cronström and Noga 1977). In the case of a linearly polarized monochromatic wave \( A_{\text{lin}}(x) = A_0(\epsilon, \cos \xi) \) with \( A_{\text{lin}}^2 = -A_0^2 = (1 + \cos 2\xi)/2 \neq \) constant, the scalar modulation function \( \Phi_p^{(\pm)}(\xi) \) would satisfy the so-called Whittaker–Hill equation (or Hill’s three-term equation, see Arscott 1964). If one were to attempt to solve this equation in terms of a trigonometric series then that procedure would lead to five-term recurrence relations between the coefficients, in contrast to the three-term expressions encountered with the Mathieu equation. Thus, the standard procedure (the technique of continued fractions) cannot be taken over from the theory of Mathieu equations. In their study of pair production by a periodic electric field, Narozhny and Nikishov (1974a, 1974b) also considered the sub-case \( p \cdot A = 0 = p_\xi \), when the resulting differential equation can also be brought to a Mathieu equation.

In the rest of the present letter we shall study the general equation (5c); however, we emphasize that we shall not carry out a complete analysis of this equation, which has a quite unexplored infinite set of transcendental solutions. We shall rather restrict our analysis to the special class of solutions, proportional to polynomial expressions (finite complex Fourier sums), which form a subset of all solutions. The exceptional feature of these solutions is that they form a doubly infinite countable set, corresponding to discrete values of the transverse and longitudinal momentum parameters.
Henceforth, in equation (5c) we specialize $A = A(\xi)$ to represent a monochromatic linearly polarized plane wave,

$$A(x) = e_x A_0 \cos \xi, \quad \xi = k \cdot x,$$

$$\{ \xi \} = \frac{\alpha_0}{c} (1, 0, n_m, 0), \quad \{ \xi \}^* = (0, 1, 0, 0), \quad \lambda_0 = F_0/k_0,$$

where $\alpha_0 = 2\pi v_0$ is the circular frequency and $F_0$ denotes the amplitude of the electric field strength. This is an $x$-polarized waveform which propagates in the positive $y$-direction in the medium, i.e. the explicit form of the argument of the cosine is $\xi = k \cdot x = \omega_0 (t - n_m y/c)$. In this case the matrix part $\sigma \cdot F = 2i(\gamma \cdot k)(\gamma \cdot A'(\xi))$ in (5c) is proportional to $(\gamma \cdot k)(\gamma \cdot e_j) = -k_0 (1 + n_m \sigma_x) \alpha_x$, which is a complex expression in the standard representation (see footnote 1). In the Majorana representation\(^2\) however, we have a real matrix for this combination, which considerably simplifies the formulae,

$$\begin{align*}
(\gamma \cdot k)(\gamma \cdot e_j)/k_0 &= +(1 + n_m \beta) \alpha_s u = \lambda u \\
&= \begin{pmatrix}
0 & (1 + n_m) \alpha_s x \\
(1 - n_m) \alpha_s x & 0
\end{pmatrix}
\begin{pmatrix}
0 & 1 + n_m \\
1 - n_m & 0
\end{pmatrix}
\end{align*}
\tag{7a}$$

The eigenvalue equation $(1 + n_m \beta) \alpha_s u = \lambda u$ of the above matrix (7a), has the four (normalized) solutions

$$\begin{align*}
u_1 &= \frac{1}{\sqrt{2}} \begin{pmatrix}
+\sqrt{1 + n_m} & 0 \\
0 & -\sqrt{1 - n_m}
\end{pmatrix} \\
u_2 &= \frac{1}{\sqrt{2}} \begin{pmatrix}
+\sqrt{1 + n_m} & 0 \\
0 & -\sqrt{1 - n_m}
\end{pmatrix} \\
u_3 &= \frac{1}{\sqrt{2}} \begin{pmatrix}
-\sqrt{1 + n_m} & 0 \\
0 & +\sqrt{1 - n_m}
\end{pmatrix} \\
u_4 &= \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & -\sqrt{1 + n_m} \\
-\sqrt{1 - n_m} & 0
\end{pmatrix}
\end{align*}
\tag{7b}$$

\(^2\) The Dirac matrices in the Majorana representation are obtained from the standard representation by using the unitary (and also self-adjoint) transformation matrix $U_M = (\beta + \alpha_i)/2^{1/2}$, for which $U_M^{-1} = U_M^\dagger = U_M$. In this representation $\sigma_e^\dagger = U_M \sigma_e U_M^{-1} = -\alpha_e \gamma_z, \quad \sigma_e = U_M \sigma_e U_M^{-1} = \beta, \quad \beta = U_M \beta U_M^{-1} = \alpha_x.$

which belong to two real, twofold degenerate eigenvalues

$$\lambda = \pm \sqrt{1 - n_m^2}, \quad \lambda_3 = \lambda_4 = -\sqrt{1 - n_m^2} \quad (n_m < 1). \quad \tag{7c}$$

The eigenvectors $u_1$ and $u_2$ (similarly $u_3$ and $u_4$) associated with the common eigenvalue are orthogonal to each other, moreover, $u_1$ is also orthogonal to $u_4$, and $u_2$ is orthogonal to $u_3$. However, $u_1$ and $u_3$ are not orthogonal, and $u_2$ and $u_4$ are not, either. Anyway, this system is complete, and can be orthogonalized by the standard procedure. In the appendix it is shown that the non-orthogonal expansion (7d), $\Psi_p^{(\pm)} = \sum_{s=1}^{4} \Psi_{ps}^{(\pm)} u_s$, leads to an equivalent set of scalar equations (8) below, to that of (A.7), obtained by using the orthogonalized basis of bispinors $\hat{u}_1, \hat{u}_2, \hat{u}_3, \hat{u}_4$, in expansion (A.4b). We also note that the pure electric field case ($n_m \to 0$) the system (7b) itself already forms an orthonormalized basis.

It is clear that if $\Psi_p^{(\pm)}$ in (5c) is proportional to one of the eigenvectors $u_{1,2}$ or $u_{3,4}$, then the matrix containing the spin interaction term in (6) can be replaced by a scalar factor, corresponding to the eigenvalues $\lambda_{1,2} = +\sqrt{1 - n_m^2}$ or $\lambda_{3,4} = -\sqrt{1 - n_m^2}$, respectively. More generally, a solution of (5c) can always be written as an expansion in terms of the eigenvectors $u_{1,2}$ and $u_{3,4}$,

$$\Psi_p^{(\pm)} = \sum_{s=1}^{4} \Psi_{ps}^{(\pm)} u_s, \quad \tag{7d}$$

and then the scalar coefficient functions $\Psi_{ps}^{(\pm)}$ will satisfy the following four second-order scalar differential equations,

$$\frac{d^2 \Psi_{ps}^{(\pm)}}{dz^2} + (\theta_0 + 2\theta_1 \cos 2z + 2\theta_2 \cos 4z)
\begin{pmatrix}
0 & 1 + n_m \\
1 - n_m & 0
\end{pmatrix}
\begin{pmatrix}
\Psi_{ps}^{(\pm)} \\
\Psi_{ps}^{(\pm)}
\end{pmatrix} = 0, \quad (s = 1–4) \quad \tag{8}$$

$$\begin{align*}
\theta_0 &= \frac{4}{-k^2} [-(k \cdot p)^2 / k^2 + p^2 - k^2 - \varepsilon_2 A_0^2 / 2], \\
\theta_1 &= \frac{4}{-k^2} [\pm 2p_z e A_0], \\
\theta_2 &= \frac{4}{-k^2} [-\varepsilon_2 A_0^2 / 2], \\
\theta_1^{1/2} &= \frac{e A_0}{k_p}, \\
k_p &= \frac{\varepsilon_2}{\varepsilon_0} = \left(\frac{\varepsilon_0}{\varepsilon_2}\right)^{1/2} \\
2\theta_2 &= \frac{4}{-k^2} (e A_0 k_0 \sqrt{1 - n_m^2} - \frac{\varepsilon_2}{\varepsilon_0}) 4\theta_1^{1/2}. \quad \tag{8c}
\end{align*}$$

In (8a) we have defined the independent variable $z = \xi / 2$ (as is usual in the theory of differential equations with periodic coefficients), and followed the standard notations in introducing the parameters $\theta_{0,1,2}$. We should keep in mind that in (8) the ambient sign $\pm$ in front of the coefficient $\theta_1$ refers to the different eigenvalues given by (7c). We also note that without this term (which stems from the EM-field–spin interaction) each equation for $\Psi_p^{(\pm)}$ in (8) would be a real Whittaker–Hill type equation, corresponding
to the wave equation of a Klein–Gordon particle (Varró 2013). This mathematical background has been used by Narozhny and Nikishov (1974a, 1974b) see equation (24) in their paper. We also note that the differential equations in (8) for spinor particles have long been used by Nikishov (1970) and by Narozhny and Nikishov (1974a, 1974b), see their equation (39), where they considered the eigenvalue problem, using Whittaker’s method, and presented analytic perturbative results.

As has already been mentioned above, the standard procedure to solve equations such as (8) in terms of a trigonometric series would lead to five-term recurrence relations between the coefficients, in contrast to the three-term expressions encountered in the case of the Mathieu equation.

In looking for finite-term solutions, we overcome this difficulty by using a transformation originally due to Ince (see references in Arscott 1964). This is the basic new element in our present approach. In the spirit of this method, in order to solve the complex equation in (8), we proceed by introducing the following Ansatz for \( \psi_{ps}^{(±)} \):

\[
\psi_{ps}^{(±)} = f \exp(-\theta_2^{1/2} \cos 2z),
\]

\[
d^2f \over dz^2 + 4\theta_2^{1/2} \sin 2z df \over dz + \left[ \theta_0 + 2\theta_2 + (2\theta_1 + 4\theta_2^{1/2}) \right] \cos 2z \pm 2i\theta_1 \sin 2z f = 0,
\]

where the parameters \( \theta_{0,1,2} \) have already been defined in (8a)–(8c). Henceforth we take a negative test charge (electron), \( \varepsilon < 0 \) in (8a)–(8c), and consider ‘positive solutions’. We show the details of the derivation only for the coefficient functions \( \psi_{p_1}^{(±)} \) and \( \psi_{p_2}^{(±)} \) of the spin eigenvectors \( u_1 \) and \( u_2 \) in (8), which belong to the eigenvalue \( +\sqrt{1-n_m^2} \).

In this case, from (9b) we obtain

\[
d^2f \over dz^2 + a \sin 2z \left( df \over dz + if \right) + (\eta - qa \cos 2z)f = 0, \quad \eta \equiv 2p_e/k_p - 1,
\]

\[
a = 4\theta_2^{1/2} = 4|\varepsilon|A_0/k_p, \quad q \equiv 2p_e/k_p - 1,
\]

\[
2p_x = (q + 1)k_p, \quad k_p \equiv k_0 \sqrt{1 - n_m^2} \equiv \omega_p/c, \quad \eta \equiv 4(\mu_0^2/p_c^2 + \varepsilon^2A_0^2/\omega_p^2 - p^2/2k_p^2),
\]

\[
p \equiv (k \cdot p), \quad \eta = 4|\varepsilon| \sqrt{2} \sin k_0 z - \varepsilon^2A_0^2/k_p^2.
\]

We would like to emphasize that (10a) is a mathematically new complex equation, which, to our best knowledge, has not been considered so far. In (10b) we have introduced the new parameters \( q \) and \( k_p \equiv k_0 \sqrt{1 - n_m^2} \equiv \omega_p/c \), where the subscript ‘p’ refers to the word ‘plasma’, though at the present stage we do not need to specify the nature of the medium (in section 4 we shall deal with this interpretation). In deriving the third equation of (10c) we have used the relations (valid for an arbitrary four-vector \( p_{\gamma} \))

\[
p_{0+}^2 - p_{0-}^2 = p_{1+}^2/k_p^2 - p^2,
\]

\[
p^2 = p_{0+}^2 - p_{0-}^2 - p_{1+}^2/k_p^2 - p^2 - p_{1-}^2.
\]

where \( \hat{p} \) and \( p_{\pm} \) have been defined in (5b) and (10e), respectively. We note that the sum of the two terms \( \kappa^2 \) and \( \varepsilon^2A_0^2 \) may be combined to \( \kappa^2 + \varepsilon^2A_0^2 \), which is equivalent to introducing the intensity-dependent mass shift \( \Delta m = m_e - m \equiv m_0(1 + \mu_0^2)/m \), where \( \mu_0 = eF_0/mc\omega_0 \) is the well-known dimensionless intensity parameter (see e.g. Brown and Kibble 1964, Harvey et al 2012). We also note that equation (10a) is unchanged under the simultaneous substitutions \( z \to z + \pi/2 \) and \( a \to -a \), thus we need not consider the case \( \varepsilon > 0 \) and \( \varepsilon < 0 \) separately.

3. Basically periodic, finite solutions of the scalar coefficient functions

For the scalar coefficient functions of the bispinor solutions of the second-order Dirac equation, introduced in (8), by using the Ansatz (9a), we have derived (10a), which is in fact a complex generalization of Ince’s original equation (according to the terminology suggested by Arscott 1964). Our procedure to follow is analogous to that of Ince’s method, which we have extended to treat the complex equation (10a) and the associated complex trigonometric polynomials. We note that, though the procedure itself is analogous to that used by Arscott (1964), the complex wavefunctions obtained by us have quite different properties to those of the real Ince polynomials (Varró 2013).

Let us first expand the solution of equation (10a) as a complex Fourier series,

\[
f = \sum_{r=-\infty}^{\infty} D_r \exp(-2irz).
\]

After substituting into (10a) we receive the recurrence relations for the unknown coefficients,

\[
\begin{align*}
[q - (2r)^2]D_r + [(2r - 1)q - (a/2)D_{r-1} - [(2(r + 1) - 1)q + (a/2)D_{r+1} = 0, \quad (11b)
\end{align*}
\]

By shifting the index \( r \to r + 1 \), and multiplying the resulting equation by \( (−1)^r \), we have

\[
\begin{align*}
\left[ \frac{1}{2} (q + 1) \right] D_r aD_r + [4(r + 1)^2 - \eta]D_{r+1} + \frac{1}{2} (q - 1) - 2aD_{r+2} = 0. \quad (11c)
\end{align*}
\]

Now, if we assume that \( 1/2(q + 1) = n \geq 1 \) is a positive integer, then \( 1/2(q - 1) = n - 1 \) is also an integer in the third term in (11c). Moreover, if we assume that \( \eta \) has been chosen so that \( D_{n+1} = 0 \), then the factor in the first term on the left-hand side of (11c), with \( r = n \), is also zero, regardless of the value of \( D_{n=0} \). Accordingly, we have \( D_{n+2} = 0 \), too. Thus, by using the recurrence relation for \( r = n + 1, n + 2, \ldots \) successively we derive \( D_{n=3} = D_{n=4} = \cdots = 0 \), which means that the series terminates with the term \( D_n \exp(-2inz) \). By ‘going backward’, down to the negative values of \( r \), and using the same reasoning with \( D_{−n} = 0 \), one can show that the \( D_r \) for all \( r < −n + 1 \) are also zero. Thus, if \( (1/2(q + 1) = n \geq 1 \) then the (potentially infinite) series contains only a finite number \((2n)\) of terms with coefficients \( D_{−n+1}, D_{−n+2}, \ldots, D_0, D_1, \ldots, D_n \). According
to the definition of $q$ in (10b), we have found that if the transverse momentum component $p_z$ is an integer multiple of the ‘plasma momentum’ $k_p$, i.e. $p_z = 1/2(q + 1)k_p = nk_p$, then there are finite-term, polynomial solutions of the second-order Dirac equation. In this case the finite set of recurrence relations are

$$[4((-n) + 1)^2 - \eta D_{-n+1}]
+ [(n - 1) + (-n) + 2]aD_{-n+2} = 0,$$

$$[(n - r)aD_r + [4(r + 1)^2 - \eta]D_{r+1}]
+ [(n - 1) + r + 2]aD_{r+2} = 0$$

$$(-n + 2 \leq r \leq n - 2),$$

$$[(n) - (n - 1)]aD_{n-1} + [4(\pm n)^2 - \eta]D_n = 0.$$  

(12a) (12b) (12c)

The system of equation (12a)–(12c) represents the algebraic eigenvalue problem of a tri-diagonal $(2n) \times (2n)$ matrix,

$$
\begin{bmatrix}
4(-n + 1)^2 & (1+a) & 0 & 0 & 0 \\
(2n-1)a & 4(-n+2)^2 & \ldots & \ldots & 0 \\
0 & (2n-2)a & \ldots & \ldots & 0 \\
0 & 0 & \ldots & 4(n-1)^2 & (2n-1)a \\
0 & 0 & 0 & 0 & (1+a) 4n^2
\end{bmatrix}
\begin{bmatrix}
D_{-n+1} \\
D_{-n+2} \\
\vdots \\
D_{n-1} \\
D_n
\end{bmatrix}
= \eta
\begin{bmatrix}
D_{-n+1} \\
D_{-n+2} \\
\vdots \\
D_{n-1} \\
D_n
\end{bmatrix}.
$$

(13a)

The eigenvalues $\eta$ determine through equations (6), (7) and (10c) the possible values of the momentum-like constant of motion $\hat{p}$. By using the notations introduced by Arscott (1964), the tri-diagonal matrix in (13a) is written as

$$
\begin{bmatrix}
(1+a) & \ldots & (2n-2)a & (2n-1)a \\
4(-n + 1)^2 & 4(-n+2)^2 & \ldots & \ldots & 4n^2 \\
(2n-1)a & (2n-2)a & \ldots & \ldots & (1+a)
\end{bmatrix}
= M_{2n}(a).
$$

(13b)

The eigenvalue equation (13a) and the associated characteristic polynomial then takes the form

$$M_{2n}(a) \cdot D_{2n} = \eta \cdot D_{2n},
$$

$$X_{2n}(\eta, a) = \det[M_{2n}(a) - \eta \cdot I_{2n}],$$

(13c)

where the superscript $T$ denotes transpose, and $I_{2n}$ is the $(2n) \times (2n)$ unit matrix. The $(2n)$ eigenvalues are given by the zeros of the characteristic polynomial $X_{2n}(\eta, a) = \prod_{k=1}^{2n} (\eta - \eta_k^{(k)})$. For such tri-diagonal matrices as our $M_{2n}(a)$, the eigenvalues are all real and different, they are all simple roots of the equation $X_{2n}(\eta, a) = 0$ (this follows from lemma 1.a stated in subsection 1.8 of Arscott 1964). There are $(2n)$ real, linearly independent vectors $\{D_j^{(k)}\}$, associated with the eigenvalues $\eta_k^{(k)}$ ($1 \leq k \leq 2n$). The general solution (11a) of (10a) reduces to a complex trigonometric polynomial, whose coefficients are the components of the eigenvectors which satisfy the eigenvalue equation in (13c),

$$f = \phi_{n}(\xi|a, +) = \sum_{r=-n+1}^{n} D_r^{(k)}(a|2n) \exp(-2ir\xi),
$$

$$\zeta = \xi/2, \quad (n = 1, 2, \ldots), \quad (1 \leq k \leq 2n).$$

(14)

These solutions may be called an ‘even solution’ of (11a), where, according to (11b), $a = 4\sqrt{d}/a_0/k_p$ and $p_x = 1/2(q + 1)k_p = nk_p$, with $k_p = \sqrt{k^2} = k_0 \sqrt{1 - n^2}$. In $D_r^{(k)}(a|2n)$ the second argument refers to there being $2n$ sets of the coefficients (eigenvectors), and the superscript labels the $k$th set.

By considering still the case $\varepsilon < 0$ and ‘positive solutions’, and the coefficient functions $\psi_{p1}^{(\pm)}$ and $\psi_{p2}^{(\pm)}$ of the spinor eigenvectors $u_1$ and $u_2$ in (8), belonging to the eigenvalue $+\sqrt{1 - n^2}$, we can construct also ‘odd solutions’ to (10a), by expanding into the following complex trigonometric series,

$$\psi = \sum_{r=-\infty}^{\infty} D_r \exp[-(2r + 1)i\zeta].$$

(15a)

We can essentially repeat the steps which led to the even solutions above. After substitution into (10a) we obtain the recurrence relations for the unknown coefficients,

$$[\eta - (2r + 1)^2]D_r + [(2(r + 1) + 1) - 1 - q(a)/2]D_{r+1}
- [(2(r + 1) + 1) - 1 + q(a)/2]D_{r+1} = 0,$$

and by shifting the index $r \rightarrow r + 1$, and multiplying the resulting equation by $(-1)$, we have

$$\frac{1}{2}q - r)D_r + [(2(r + 1) + 1)^2 - \eta]D_{r+1}
+ \frac{1}{2}(q + (r + 2))aD_{r+2} = 0.$$  

(15c)

Now, if we assume that $(1/2)^q = n \geq 0$ is a non-negative integer, then $p_x = (1/2)(q + 1)k_p = k_p(n + 1/2)$; thus in this case $p_x$ contains a ‘zero-point momentum’ $1/2k_p$ too. By assuming that $\eta$ has been choosen so that $D_{-n+1} = 0$, the factor in the first term on the left-hand side of (15c), with $r = n$, is also zero, regardless of the value of $D_{n+1}$. Accordingly, we have $D_{n+1} = 0$ too, and then $D_{n+3} = D_{n+4} = \ldots = 0$, thus the series terminates with the term $D_n \exp[(2n+1)i\zeta]$. The series terminates with the $(-n)$th term, too. The system of equations now represents the algebraic eigenvalue problem of a tri-diagonal $(2n + 1) \times (2n + 1)$ matrix,

$$
\begin{bmatrix}
[2(-n + 1)^2] & (1+a) & 0 & 0 & 0 \\
(2n) & [2(-n + 1) + 1]^2 & \ldots & \ldots & 0 \\
0 & (2n-1)a & \ldots & \ldots & 0 \\
0 & 0 & \ldots & 2(n-1)^2 & (2n)a \\
0 & 0 & 0 & (1+a) & [2(n) + 1]^2
\end{bmatrix}
\begin{bmatrix}
D_{-n} \\
D_{-n+1} \\
\vdots \\
D_{n-1} \\
D_n
\end{bmatrix}
= \eta
\begin{bmatrix}
D_{-n} \\
D_{-n+1} \\
\vdots \\
D_{n-1} \\
D_n
\end{bmatrix}.
$$

(16a)
By using the notations for the tri-diagonal matrix in (16a)

\[
\begin{pmatrix}
(2n+1)a & \cdots & (2n-1)a & (2na) \\
(2n-1)a & \cdots & (2n+1)a & (2na) \\
(2n)a & \cdots & (2n-1)a & (2na) \\
(2na) & \cdots & (2n+1)a & (2n+1)a
\end{pmatrix}
\]

the eigenvalue equation (16a) and the associated characteristic polynomial then takes the form

\[
N_{2n+1}(a) \cdot D_{2n+1} = \eta \cdot D_{2n+1},
\]

\[D_{2n+1} = [D_{-n}, D_{-n+1}, \ldots, D_{n-1}, D_n],
\]

\[Y_{2n+1}(\eta, a) = \det[N_{2n+1}(a) - \eta \cdot I_{2n+1}],
\]

where \(I_{2n+1}\) denotes the \((2n + 1) \times (2n + 1)\) unit matrix. The \((2n + 1)\) eigenvalues are given by the zeros of the characteristic polynomial \(Y_{2n+1}(\eta, a) = \prod_{k=1}^{2n+1} (\eta - \eta_n(k))\). For the tri-diagonal matrix \(N_{2n+1}(a)\), the eigenvalues are all real, and they are all different right simple roots of the equation \(Y_{2n+1}(\eta, a) = 0\). There are \((2n + 1)\) real linearly independent vectors associated with these eigenvalues \(\eta_n(k)\) \((1 \leq k \leq 2n + 1)\). The general solution (15a) of (10a) reduces to a complex trigonometric polynomial, whose coefficients are the components of the eigenvectors which satisfy the eigenvalue equation in (16a)–(16c).

\[
f = h_n^k(\xi |a, +|) = \sum_{r=-n}^{n} D_r^{(k)}(a)[2n + 1] \exp[-(2r + 1)iz],
\]

\[z = \xi/2, \quad (n = 1, 2, \ldots), \quad (1 \leq k \leq 2n + 1).
\]

We note that it is a very remarkable property of these solutions that they contain half-integer harmonics of the incoming radiation, i.e. they are not 2\(\pi\)-periodic, but only 4\(\pi\)-periodic. There are \((2n + 1)\) such linearly independent vectors \([D_r^{(k)}]\), associated with the eigenvalues \(\eta_n(k)\) \((1 \leq k \leq 2n + 1)\).

According to (9b), the coefficient functions \(\Phi_p^{(e)}\) and \(\Phi_p^{(o)}\) of the spinor eigenvectors \(u_{3,4}\) and \(u_{4,5}\) in (8), belonging to the eigenvalue \(-1 + \eta_n(1)\), satisfy an analogous equation to (10a), with a negative sign in front of the complex imaginary unit. It can be shown that the corresponding even and odd solutions are just the complex conjugates of the functions given in (14) and (17), respectively. This is because the \(D^*\) coefficients in their Fourier expansion satisfy exactly the same eigenvalue equations as (13a) or (17), i.e. \(D^* = D\) in each case. In this way, on the basis of (14) and (17), we have

\[
f = g_n^k(\xi |a, -|) = [g_n^k(\xi |a, +|)]^* = \sum_{r=-n}^{n} D_r^{(k)}(a)[2n + 1] \exp[(2r + 1)iz],
\]

\[n = 1, 2, \ldots, \quad (1 \leq k \leq 2n + 1),
\]

From equation (5a), (8a), (9a), (14), (17)–(19) we summarize the finite polynomial solutions of the ‘even’ and ‘odd’ scalar functions associated with the second-order Dirac equation (3),

\[
\Psi_{p1,2}^{(e)} = \exp[+i(\hat{p}x + p_x + p_z)],
\]

\[
\Psi_{p1,2}^{(o)} = \exp[+i(\hat{p}x + p_x + p_z)],
\]

\[
\Psi_{p3,4}^{(e)} = \exp[+i(\hat{p}x + p_x + p_z)]
\]

\[
\Psi_{p3,4}^{(o)} = \exp[+i(\hat{p}x + p_x + p_z)]
\]

\[
\exp[-(a/4) \cos \xi |g_n^k(\xi |a, +|)]^*,
\]

\[
\exp[-(a/4) \cos \xi |g_n^k(\xi |a, +|)]^*.
\]

For each \(n\)-value (corresponding to a discrete set of transverse momenta \(2p_{m} = (q + 1)k_{p}\)), the coefficients of the polynomials \(g_n^k\) and \(h_n^k\) satisfy the eigenvalue equations (13c) and (16c), respectively. In this way, the eigenfunctions and the eigenvalues \(\eta_n(k)\) form a doubly infinite set labelled by the integer numbers \(n\) and \(k\). By taking any linear combinations of \(\Psi_{p}\) of (20a)–(20d), as is shown in (7d), and applying on them the matrix differential operator \((\gamma \cdot \Pi + k)\), we receive an exact solution of the original Dirac equation (1). At present there is no need to show the details of this straightforward calculation.

We note that a new set of exact solutions of the corresponding Klein–Gordon equation found recently by us (Varró 2013) has a similar structure to \(\Psi_{p}\), namely

\[
\Phi_p = \exp[+i(\hat{p}x + p_x + p_z)]
\]

\[
\exp[-(a/4) \cos \xi |\varphi_p^k(\xi |a)|].
\]

where the \(\varphi_p^k(\xi |a)\) satisfy the Whittaker–Hill equation (which one obtains from (10a) by leaving out the term with the imaginary factor in front of the sine). The functions \(\varphi_p^k(\xi |a)\) are real trigonometric polynomials, which coincide with the so-called Ince polynomials (this terminology has been suggested by Arscott 1964). The interrelations between the Ince polynomials and our new complex polynomials (14), (17)–(19) have not yet been explored. Another important open question is how should one perform the ‘vacuum limit’ (the limit when \(n_m \rightarrow 1\), i.e. \(k_{p} \rightarrow 0\)) so that the new solutions (20a)–(20d) go over to the usual Volkov states, which are solutions to the Dirac equation in vacuum \((n_m = 1)\). For instance, in the case of the even solutions the transverse \(p_z = nk_{p}\) can be kept finite, by simultaneously taking the double limit of very large excitations \(n \rightarrow \infty\) with \(k_{p} \rightarrow 0\). However, the fundamental parameter \(a = 4|\xi|k_0/k_{p}\), defined in (10b), diverges in this limit if we keep the prescribed input amplitude \(A_{0}\) fixed. We conjecture that the proper procedure for carrying out the ‘vacuum limit’ should be based on some ‘coherent superposition’ of the eigenstates (20a)–(20d), whose nature is not known for us. The mathematical analysis of the asymptotic behaviour of the new complex polynomials is beyond the scope of the present letter.
4. Basic properties and some numerical illustrations of the new exact solutions

Since the emphasis in the present letter has been on proving the existence of and constructing the general exact polynomial solutions (20a)–(20d) of the Dirac equation, below we will briefly discuss only one possible physical interpretation of these solutions. At the end of this section we will then give just a few numerical illustrations. Before doing so, we summarize the basic mathematical properties of these solutions.

On the basis of the original differential equation (8) it can be shown that the functions in (14), (17)–(19) satisfy the following orthogonality relations

\[ \int_{-\pi}^{\pi} e^{-(a/2) \cos \xi} (g_n^k)^* g_i^k \, d\xi = 0, \]
\[ \int_{-\pi}^{\pi} e^{-(a/2) \cos \xi} (h_n^k)^* h_i^k \, d\xi = 0 \quad (l \neq k). \]  

(22a)

We normalize these solutions as follows

\[ \sum_{r=-\infty}^{\infty} |D_r^{(k)}(a|2n)|^2 = 1, \]
\[ \sum_{r=-\infty}^{\infty} |D_r^{(k)}(a|2n+1)|^2 = 1. \]  

(22b)

Each of the solutions (20a)–(20d) contains the exponential factor \( e^{-(a/4) \cos \xi} \), which can be expanded into an infinite Fourier series (Gradshteyn and Ryzhik 1980 or Abramowitz and Stegun 1965),

\[ \exp[-(a/4) \cos \xi] \]
\[ = \sum_{l=-\infty}^{\infty} I_l(a/2) \exp[i l(\xi - \pi)] \]
\[ = I_0(a/2) + 2 \sum_{l=1}^{\infty} I_l(a/2) \cos[l(\xi - \pi)], \]  

(23a)

where \( I_l(z) \) denote modified Bessel functions of the first kind of order \( l \). This means that, though the polynomials \( g_n^k \) and \( h_n^k \) are finite-term expressions, according to (23a), the wavefunctions in (20a)–(20d) contain all the higher harmonics of the fundamental frequency. If \( a \gg 1 \), then this function is peaked at the points \( \xi = \pi + 2k\pi \) with an exponentially large contrast, and we can approximately represent it as a sequence of ‘delta-kicks’,

\[ I_l(a/2) = \frac{\omega_l^{1/2}}{\sqrt{\pi a}} [1 + O(1/a)], \]
\[ \exp[-(a/4) \cos \xi] \approx \frac{\omega_l^{1/2}}{\sqrt{\pi a}} 2\pi \sum_{l=-\infty}^{\infty} \delta[\xi - (2l+1)\pi] \quad (a \gg 1). \]  

(23b)

The differential equation (10a) has led to a two-parameter eigenvalue problem, where we have considered \( a = 4|\varepsilon|A_0/k_0 \) as a fundamental parameter, and \( 2p_\perp = (q + 1)k_0 \) and \( \eta \) are disposable parameters. We have seen that the condition \( q + 1 = 2n \) has led to even solutions, and \( q + 1 = 2n + 1 \) to odd solutions. In (10c) we have presented two equivalent forms of the eigenvalues \( \eta \), expressed in terms of momentum parameters which are determined by the characteristic equations \( \lambda_2(\eta, a) = 0 \) and \( \lambda_{2n+1}(\eta, a) = 0 \) in (13c) and (16c), respectively,

\[ \dot{p} = (k_0/2\sqrt{\eta - (q + 1)^2 - (2p_\perp/k_0)^2} - (2\xi/k_0)^2 - (a/2)^2), \]
\[ 2p_\perp/k_0^2 = \pm \sqrt{\eta - (a/2)^2}. \]  

(24)

The second equation of a simpler form in (24) has been obtained by imposing (in addition) the free mass-shell condition \( p^2 = m^2 \) for the momentum parameter \( p_{\perp} \), where the definition of \( p_\perp \) in (10c) has also been used. Not all the eigenvalues give physically acceptable solutions of the Dirac equation. More precisely, if \( \dot{p} \) becomes purely imaginary, then the wavefunctions necessarily contain the exponentially growing factor \( \exp[\pm i (k_0^2/2\eta)(\gamma - n_{\text{max}})] \) in the space-like direction \( \hat{k} \). This would not be an acceptable solution, except for the case when the interaction is limited in a finite space–time region.

The fundamental parameter \( a \) can be expressed in terms of various combinations of parameters which characterize the applied monochromatic field (shortly; laser field) and the medium. If we are allowed to take the Drude free electron model of a plasma medium, with a dielectric permittivity \( \varepsilon_m(\omega) = 1 - \omega_p^2/\omega^2 = n_e^2(\omega) \), then \( \omega_p \) in (10b) means the plasma frequency of an underdense electron gas, i.e. \( 4\pi n_e e^2/m = \omega_p^2 < \omega^2 \), where \( n_e \) is the electron density. In fact, in this case the electromagnetic plane wave under discussion has the dispersion relation \( \omega(k_\perp) = \sqrt{\omega_p^2 + (ck_\parallel)^2} \).

Then the fundamental parameter \( a \) can be written as the work done by the electric force along the plasma wavelength divided by the photon energy. The ratio of the photon density and the electron density also naturally appears,

\[ a = 4\varepsilon A_0/k_0 = 4\frac{eF_0\lambda_p}{h\omega_0}, \]
\[ = 4\sqrt{(2mc^2/h\omega_0)(n_{\text{ph}}/n_e)} = 2\mu_0(2mc^2/h\omega_p), \]  

(25a)

\[ n_{\text{ph}} = \frac{I_0}{c\hbar\omega_0} = 2.08 \times 10^8 \times (S/E_{\text{ph}}) \quad (\text{cm}^{-3}), \]
\[ \mu_0 = \frac{eF_0}{mc\omega_0} = 1.06 \times 10^{-9} \times S^{1/2}/E_{\text{ph}}. \]  

(25b)

\[ \omega_p = \sqrt{4\pi n_e e^2/m}, \quad k_\perp = \omega_p/c = 1/k_0 = 2\pi/\lambda_p, \]  

(25c)

We have used the definition \( k_\perp = k_0\sqrt{1 - n_e^2} \) of the ‘plasma wavenumber’ in (10b), and introduced the plasma frequency \( \omega_p \). For example, a plasmon energy \( h\omega_p = 1 \text{ eV} \) corresponds to an electron density \( n_e = 7.242 \times 10^{20} \text{ cm}^{-3} \) (in which case the plasma wavelength is \( \lambda_p = 1240 \text{ nm} \)).

In the numerical expressions for the photon number density \( n_{\text{ph}} \) and for the well-known ‘dimensionless intensity parameter’
eigenvalues, this condition still holds for the underdense plasma. In this case a field case can only be real if energy

\[ \mu I \]

used a Ti:Sa laser of intensity \( 100 \text{ MW cm}^{-2} \) has been considered as an example. We have taken \( h\omega_0 = 1 \text{ eV} \) and \( n = 20 \), i.e. the transverse momentum of the electron (in original units) is \( p_y = 20 \times \hbar \omega_0 \), where \( k_0 = \omega_0/c \). In the interval \( -2\pi \leq \omega_0(t - n\omega_0/c) \leq 2\pi \), the real part (b), imaginary part (c) and absolute value (d) of the wavefunction \( g^{\text{(1)}}(\xi, a, +) \) in (14), associated to the fifth eigenvalue \( n_0(5) = 718.092 \times 8584.742 \ldots \) are shown. In (a) one sees that the lower index eigenvalues appear in pairs, those with \( k = (2-3), (4-5), (6-7) \) and (8-9) cannot be distinguished by eye in the figure. In fact they differ, but only from their 12th decimal digits. That is why we have displayed it with 12 decimal digits, though this accuracy is of no importance here. We note that such a ‘hyperfine structure’ is absent in the Klein–Gordon case (Varró 2013).

Figure 1. Shows the eigenvalues (a), which are the roots of the characteristic polynomial in (13c), for \( a = 12 \). The interaction with a Ti:Sa laser field of photon energy \( h\omega_0 = 1.563 \text{ eV} \) and pink intensity \( I_0 = 100 \text{ MW cm}^{-2} \) has been considered as an example. We have taken \( h\omega_0 = 1 \text{ eV} \) and \( n = 20 \), i.e. the transverse momentum of the electron (in original units) is \( p_y = 20 \times \hbar \omega_0 \), where \( k_0 = \omega_0/c \). In the interval \( -2\pi \leq \omega_0(t - n\omega_0/c) \leq 2\pi \), the real part (b), imaginary part (c) and absolute value (d) of the wavefunction \( g^{\text{(1)}}(\xi, a, +) \) in (14), associated to the fifth eigenvalue \( n_0(5) = 718.092 \times 8584.742 \ldots \) are shown. In (a) one sees that the lower index eigenvalues appear in pairs, those with \( k = (2-3), (4-5), (6-7) \) and (8-9) cannot be distinguished by eye in the figure. In fact they differ, but only from their 12th decimal digits. That is why we have displayed it with 12 decimal digits, though this accuracy is of no importance here. We note that such a ‘hyperfine structure’ is absent in the Klein–Gordon case (Varró 2013).

In figure 2 we show the distribution of the harmonic strengths \( |D^{(k)}_I(a)|^2 \) in the expansion of the polynomial factors, in the case of interaction with a Ti:Sa laser field of photon energy \( h\omega_0 = 1.563 \text{ eV} \) and pink intensity \( I_0 = 100 \text{ MW cm}^{-2} \) in a plasma medium of electron density \( n_e = 7.242 \times 10^{20} \text{ cm}^{-3} \) \( (h\omega_0 = 1 \text{ eV}) \). Of course, the complete physical spectrum may in principle be influenced by the exponential prefactor leading to the modified Bessel function series in (23a), but in the present letter we would like to emphasize the properties of the still unknown polynomial factors. According to figure 2, these spectra are in general qualitatively different from each other.

Finally, we give an overview of the harmonic spectra in figure 3. The further numerical study of the exact solutions (20a)–(20d) is out of the scope of the present letter. The discussion of their possible physical applications is also left for future work.

5. Summary
We have presented closed form exact solutions of the Dirac equation of a charged particle moving in a monochromatic classical plane electromagnetic field in a medium of index of refraction \( n_m < 1 \). The solutions found are finite (complex) trigonometric polynomials which form a doubly infinite discrete set labelled by two integer quantum numbers that...
Figure 2. Shows the strength $|D^{(k)}(a|2n)|^2$ of the harmonic coefficients of the polynomial $g_k^a(\xi|a, +)$ for four eigenvalues labelled by the upper index $k = 5, 14, 24, 27$. These eigenvalues are $\eta_{15}^{(5)} = 718.1 \ldots, \eta_{15}^{(14)} = 355.5 \ldots, \eta_{15}^{(24)} = -163.1 \ldots, \eta_{15}^{(27)} = 81.6 \ldots$. We are considering the case when $a = 12$ and $n = 15$, and the input parameters are the same as in figure 1, where the temporal shape of $g_{15}^{a}(\xi|a, +)$ has been shown.

Figure 3. Shows an overview of the strength $|D^{(k)}(a|2n)|^2$ of the harmonic coefficients on a three-dimensional list plot when $a = 12$ and $n = 15$. This figure summarizes the behaviour of these quantities for different eigenvalues (of indices $k = 1, 2, \ldots, 30$, which are drawn on the right axis). The discrete points are connected by a smoothened surface in order to guide the eye. For the lowest index $k = 1$ the distribution is concentrated towards positive $r$-values (left axis) in a single peak, the negative index Fourier coefficients are practically zero. For medium $k$-values a double peak structure develops, and for larger values the two peaks merge to an oscillatory distribution. The character of these oscillations has already been illustrated in figures 2(c) and (d).

These interpretations of these solutions, we have considered the medium as an underdense plasma. In this case one of the quantum numbers $n$ characterizes the transverse momentum $p_x = nk_p$ or $p_x = (n + 1/2)k_p$ of the test electron, where $k_p = \omega_p/c$, with $\omega_p$ being the plasma frequency. The other quantum number determines the possible values of the energy parameter, and in certain ranges it is associated with a sort of ‘gap state’ in the interval $(-mc^2, mc^2)$. The existence of the ‘bound states’ we have found may have relevance concerning possible quantum features of mechanisms of laser acceleration of electrons by high-intensity laser fields and wake-fields in an underdense plasma (see e.g. Xia et al 2012, Mourou et al 2013). The fundamental parameter $a$ (see (25a)), which determines the strength of the interaction, is the work done on the electron by the electric force of the laser field along a plasma wavelength divided by the photon energy. This $a$ is a quantum parameter, and it is typically by many orders of magnitude larger than the usual intensity parameter $\mu$ (see (25b)).

In section 2 we have derived the wave equations for the scalar coefficient functions of the Dirac bispinors, and made a brief comparison with the derivation of the usual Volkov states and the Mathieu-type wavefunctions in a medium. In section 3 we have derived the basically periodic, finite solutions of the scalar coefficient functions, which are complex trigonometric polynomials. These are associated with the eigenvalue problem of finite special tri-diagonal matrices (13a), (13b), (16a) and (16b). We have restricted our analysis only to these special class of solutions, proportional to polynomial expressions (finite complex Fourier sums), which form a subset of all solutions. In section 4 we have presented the
orthogonal system, as is seen from the following list of all.

\[ u \]

satisfy the following algebraic eigenvalue equations, yielding

\[ p_2 \]

\[ u \]

\[ \lambda \]

\[ p_1 \]

\[ u \]

\[ \xi \]

\[ c \]

\[ d \]

\[ n \]

\[ \beta \]

\[ p \]

\[ A \]

\[ n \]

\[ m \]

\[ \sigma \]

\[ F \]

\[ \xi \]

\[ a \]

\[ \lambda \]

\[ \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \sqrt{\lambda} \]

\[ n_m < 1. \]

\[ \alpha \]

\[ \beta \]

\[ \lambda_1 = \lambda_2 = \sqrt{1 - n_m^2} \]

\[ \lambda_3 = \lambda_4 = -\sqrt{1 - n_m^2} \]

Here we have taken into account the explicit form of the spin–EM-field interaction in the Majorana representation

\[ \sigma \cdot F \rightarrow \sigma^\prime \cdot F = -2i\hat{F}_0(\sin \xi)(1 + n_m^2)\alpha_s. \]  

\[ \hat{u}_3 = 1 \]

\[ \hat{u}_4 = 1 \]

\[ \hat{u}_3 = \frac{1}{\sqrt{1 - n_m^2}}(n_m u_1 + u_3), \]

\[ \hat{u}_4 = \frac{1}{\sqrt{1 - n_m^2}}(n_m u_2 + u_4), \]

\[ u_1 \otimes u_1^\dagger + u_2 \otimes u_2^\dagger + u_3 \otimes u_3^\dagger + u_4 \otimes u_4^\dagger = 14. \]  

In equation (A.3c) we have displayed the explicit completeness relation of the new set of bispinors \( u_1, u_2, \hat{u}_3 \) and \( \hat{u}_4 \). Of course, the effect of the original matrix \( (1 + n_m^2)\alpha_s \) on \( \hat{u}_3 \) and \( \hat{u}_4 \) is not a simple multiplication by an eigenvalue, but additional terms appear,

\[ (1 + n_m^2)\alpha_s \hat{u}_3 = n_m u_1 - u_3 = -\lambda \hat{u}_3 + 2n_m u_1, \]

\[ (1 + n_m^2)\alpha_s \hat{u}_4 = n_m u_2 - u_4 = -\lambda \hat{u}_4 + 2n_m u_2. \]
Since the new set has been obtained from the original (non-orthogonal) system by the usual Gram–Schmidt orthogonalization procedure, the equation (A.3c) also proves that the original system formed by $u_1, u_2, u_3$ and $u_4$ may also serve as a basis set, thus the expansion $\psi_p^{(\pm)} = \sum_{m=1}^{4} \psi_{p,m}^{(\pm)} u_m$, (7d), is truly a general expression. This statement can also be explicitly proved, by applying the completeness relation for expanding a general biphoton. By taking (A.3c) into account, we have

$$\psi_p^{(\pm)} = \psi_{p,1}^{(\pm)} u_1 + \psi_{p,2}^{(\pm)} u_2 + \psi_{p,3}^{(\pm)} u_3 + \psi_{p,4}^{(\pm)} u_4,$$

$$\psi_{p,1,2}^{(\pm)} = u_1^\pm \cdot \psi_p^{(\pm)}, \quad \psi_{p,3,4}^{(\pm)} = u_3^\pm \cdot \psi_p^{(\pm)}. \quad (A.4d)$$

It is clear that this same expansion should also apply for the more general function $\Psi$, which satisfies the second-order Dirac equation (3) with the vector potential (6) and spin interaction term (A.1b),

$$\Psi = \psi_1 u_1 + \psi_2 u_2 + \psi_3 u_3 + \psi_4 u_4,$$

$$\psi_{1,2} = u_1 \pm \cdot \Psi, \quad \psi_{3,4} = u_3 \pm \cdot \Psi. \quad (A.4b)$$

We continue our considerations in this more general case, because the algebraic part of the problem does not depend on the de Broglie plane wave Ansatz we have used in the main text. According to (A.3c), the explicit form of the second-order Dirac equation (3) can be brought to the form

$$\Pi^2 - \kappa^2 + \frac{i}{2} \epsilon \sigma \cdot F \Psi = \Pi^2 - \kappa^2 \Psi + i \epsilon F_0 (\sin \xi) \left[ \lambda \hat{\Psi}_1 + m_n \hat{\Psi}_3 \right] u_1$$

$$+ \left( \lambda \hat{\Psi}_2 + m_n \hat{\Psi}_4 \right) u_2 - \lambda \hat{\Psi}_1 \dot{u}_3 - \lambda \hat{\Psi}_4 \dot{u}_4 \right], \quad (A.5a)$$

where we have used (A.1a), and introduced the linear combinations

$$\hat{\Psi}_1 \equiv \Psi_1 + (n_m/\lambda) \Psi_3,$$

$$\hat{\Psi}_2 \equiv \Psi_2 + (n_m/\lambda) \Psi_4. \quad (A.5b)$$

By projecting to the mutually orthogonal directions (i.e., by taking the scalar products with the bispinors $u_1, u_2, u_3$ and $u_4$ of (A.5a)), from equation (A.5a) we can derive the four scalar equations

$$\left[ \Pi^2 - \kappa^2 + \frac{i}{2} \epsilon 2i F_0 (\sin \xi) \lambda \right] \hat{\Psi}_1 = \left( \Pi^2 - \kappa^2 \right) (n_m/\lambda) \hat{\Psi}_4 - i \epsilon F_0 (\sin \xi) m_n \hat{\Psi}_3, \quad (A.6a)$$

$$\left[ \Pi^2 - \kappa^2 + \frac{i}{2} \epsilon 2i F_0 (\sin \xi) \lambda \right] \hat{\Psi}_2 = \left( \Pi^2 - \kappa^2 \right) (n_m/\lambda) \hat{\Psi}_4 - i \epsilon F_0 (\sin \xi) m_n \hat{\Psi}_3, \quad (A.6b)$$

$$\left[ \Pi^2 - \kappa^2 - \frac{i}{2} \epsilon 2i F_0 (\sin \xi) \lambda \right] \hat{\Psi}_3 = 0, \quad (A.6c)$$

$$\left[ \Pi^2 - \kappa^2 - \frac{i}{2} \epsilon 2i F_0 (\sin \xi) \lambda \right] \hat{\Psi}_4 = 0. \quad (A.6d)$$

According to (A.6c) and (A.6d), the right-hand sides of (A.6a) and (A.6b) are zero, thus we obtain

$$\left[ \Pi^2 - \kappa^2 + \frac{i}{2} \epsilon 2i F_0 (\sin \xi) \lambda \right] \hat{\Psi}_1 = 0. \quad (A.6a')$$

By using the same de Broglie plane wave Ansatz which has led to (5c) or (A.1) above, we can similarly derive from (A.6a'), (A.6b'), (A.6c) and (A.6d) the scalar equations

$$\frac{d^2 \hat{\Psi}_p^{(\pm)}}{dz^2} + (\dot{t}_0 + 2\dot{t}_1 \cos 2z + 2\dot{t}_2 \cos 4z$$

$$\pm 2i\dot{t}_1 \sin 2z) \hat{\Psi}_p^{(\pm)} = 0, \quad (s = 1–4). \quad (A.7)$$

Equations (A.7) have the same form as (8) in section 2, with the same parameters defined in (8a)–(8c). This means at the same time that our original non-orthogonal expansion (7d), $\psi_p^{(\pm)} = \sum_{m=1}^{4} \psi_{p,m}^{(\pm)} u_m$, leads to an equivalent set of scalar equations (8) to that in (A.7), obtained by the orthogonal expansion (A.4b).

References

Abramowitz M and Stegun I A 1965 *Handbook of Mathematical Functions* (New York: Dover) pp 375–8 1965, 9th printing

Arscott F M 1964 *Periodic Differential Equations. An introduction to Mathieu, Lamé, and Allied Functions* (Oxford: Pergamon)

Becker W 1977 Relativistic charged particles in the field of an electromagnetic plane wave in a medium *Physica A* 87 601–13

Bergou J and Varro S 1980 Wavefunctions of a free electron in an external field and their application in intense field interactions: II. Relativistic treatment *J. Phys. A: Math. Gen.* 13 2823–37

Bergou J and Varro S 1991a Nonlinear scattering processes in the presence of a quantized radiation field: I. Nonrelativistic treatment *J. Phys. A: Math. Gen.* 14 1469–82

Bergou J and Varro S 1991b Nonlinear scattering processes in the presence of a quantized radiation field: II. Relativistic treatment *J. Phys. A: Math. Gen.* 14 2281–303

Berson S Y 1969a *Zh. Eksp. Teor. Fiz.* 56 1627–33

Berson S Y 1969b *Sov. Phys.—JETP* 29 871 (Engl. transl.)

Berson S Y and Valdimanis J 1973 Electron in the two field of two monochromatic electromagnetic waves *J. Math. Phys.* 14 1481–4

Bjorken J D and Drell S D 1964 *Relativistic Quantum Mechanics* (New York: McGraw-Hill)

Boca M and Florescu V 2009 Nonlinear Compton scattering with a laser pulse *Phys. Rev. A* 80 053402

Boca M and Florescu V 2010 The completeness of Volkov spinors *Rom. J. Phys.* 55 111–25

Brown L S and Kibble T W B 1964 Interaction of intense laser beams with electrons *Phys. Rev.* 133 A705–19

Bunkin F V and Fedorov M V 1965 Bremsstrahlung in a strong radiation field *Zh. Eksp. Teor. Fiz. (USSR)* 49 1215–21

Bunkin F V and Fedorov M V 1966 *Sov. Phys.—JETP* 22 844–7 (Engl. transl.)

Burenkov I A and Tikhonova O V 2010 Features of multiphoton-stimulated bremsstrahlung in a quantized field *J. Phys. B: At. Mol. Opt. Phys.* 43 235401

Cronström C and Naga M 1977 Photon induced relativistic band structure in dielectrics *Phys. Lett. A* 60 137–9

Dunne G V 2004 Heisenberg–Euler effective Lagrangians: basics and extensions arXiv:hep-th/0406216v123

Dunne G V 2009 New strong-field QED effects at extreme light infrastructure. Nonperturbative vacuum pair production *Eur. Phys. J. D* 55 327–40

Eberly J H 1969 Interaction of very intense light with free electrons *Progress in Optics VII* ed E Wolf (Amsterdam: North-Holland) pp 359–415
Ehlotzky F, Krajewska K and Kaminski J Z 2009 Fundamental processes of quantum electrodynamics in laser fields of relativistic power Rep. Prog. Phys. 72 046401

Fedorov M V 1997 Atomic and Free Electrons in a Strong Laser Field (Singapore: World Scientific)

Fedorov M V, Efremov M A, Volkov P A and Eberly J H 2006 Short-pulse and strongfield breakup processes: a route to entangled wave-packets J. Phys. B: At. Mol. Opt. Phys. 39 S467–83

Fedorov M V and Kazakov A E 1973 An electron in a quantized plane wave and in a constant magnetic field Z. Phys. 261 191–202

Feldberger F S and Marburger J H 1975 New class of exact solutions of the Dirac equation J. Math. Phys. 16 2089–92

Gordon W 1927 Der Comptoneffekt nach der Schrödinger’schen Theorie Z. Phys. 40 117–33

Gradshteyn I S and Ryzhik I M 1980 Table of Integrals, Series, and Products (New York: Academic)

Harvey Ch, Heinzl Th, Ilderton A and Marklund M 2012 Intensity-dependent electron mass shift in a laser field: existence, universality, and detection Phys. Rev. Lett. 109 100402

Keldish L V 1964 Ionization in the field of a strong electromagnetic wave Zh. Eksp. Teor. Fiz. (USSR) 47 1945–57

Keldish L V 1965 Sov. Phys.—JETP 20 1307–14 (Engl. transl.)

Kiefer D et al 2013 Relativistic electron mirrors from nanoscale foils for coherent frequency upshift to the extreme ultraviolet Nature Commun. 4 1763

Krausz F and Ivanov M 2009 Attosecond physics Rev. Mod. Phys. 81 163–234

Mourou G, Brocklesby B, Tajima T and Limpert T 2013 The future is fiber accelerators Nature Photon. 7 258–61

Mourou G A, Tajima T and Bulanov S V 2006 Relativistic optics Rev. Mod. Phys. 78 309–71

Musakhayev V 2008 An exact solution of Dirac’s equation in the field of plane EM wave: physical implications. On the solutions of Lorentz, Dirac and Lorentz–Dirac equations Eur. Phys. J. Spec. Top. 160 311–8

Narozhny N B, Bulanov S S, Mur V D and Popov V S 2004 e+ e−-pair creation by a focused laser pulse in vacuum Phys. Lett. A 330 1–6

Narozhny N B and Nikishov A I 1974a Pair production by a periodic electric field Zh. Eksp. Teor. Fiz. 65 862–74

Narozhny N B and Nikishov A I 1974b Sov. Phys.—JETP 38 427 (Engl. transl.)

Neville R A and Rohrlich F 1971 Quantum field theory on null planes Il Nuovo Cimento A 1 625–44

Nikishov A I 1970 Barrier scattering in field theory removal of Klein paradox Nucl. Phys. B 21 346–58

Nikishov A I and Ritus V I 1967a Zh. Eksp. Teor. Fiz. (USSR) 52 1707–19

Nikishov A I and Ritus V I 1967b Pair production by a photon and photon emission by an electron in the field of an intense electromagnetic wave and in a constant field Sov. Phys.—JETP 25 1135–43 (Engl. transl.)

Popov V S 1974 Method of indirect time for periodical fields 1974 Yad. Fiz. 19 1140–56 (in Russian)

Popov V S 2004 Tunnel and multiphoton ionization of atoms and ions in a strong laser field (Keldish theory) Phys.—Usp. 47 855–85

Ritus V I and Nikishov A I 1979 Quantum electrodynamics of phenomena in intense fields Works of the Lebedev Physical Institute 111 5–278 (in Russian)

Salamin Y I, Hu S X, Hatsagortsian K Z and Keitel Ch H 2006 Relativistic high-power laser–matter interactions Phys. Rep. 427 41–155

Varró S 2008 Entangled photon–electron states and the number-phase minimum uncertainty states of the photon field New J. Phys. 10 053028

Varró S 2010a Entangled states and entropy remnants of a photon–electron system Phys. Scr. T140 014038

Varró S 2010b Intensity effects and absolute phase effects in nonlinear laser–matter interactions Laser Pulse Phenomena and Applications ed F J Duarte (Rijeka: InTech) chapter 12, pp 243–66

Varro S 2013 A new class of exact solutions of the Klein–Gordon equation of a charged particle interacting with an electromagnetic plane wave in a medium Laser Phys. Lett. submitted arXiv:1306.0097 [quant-ph]

Wolkow D M 1935 Über eine Klasse von Lösungen der Diracschen Gleichung Z. Phys. 94 250–60

Xia G, Assmann R, Fonseca R A, Huang C, Mori W, Silva L O, Viera J, Zimmermann J F and Muggli P 2012 A proposed demonstration of an experiment of proton-driven plasma wakefield acceleration based on CERN SPS J. Plasma Phys. 78 347–53

Zakowicz S 2005 Square-integrable wave packets from Volkov solutions J. Math. Phys. 46 032304