The problem of the least prime number in an arithmetic progression and its applications to Goldbach’s conjecture

SHAOHUA ZHANG

School of Mathematics, Shandong University, Jinan, Shandong, 250100, PRC
E-mail address: shaohuazhang@mail.sdu.edu.cn

Abstract: The problem of the least prime number in an arithmetic progression is one of the most important topics in Number Theory. In [11], we are the first to study the relations between this problem and Goldbach’s conjecture. In this paper, we further consider its applications to Goldbach’s conjecture and refine the result in [11]. Moreover, we also try to generalize the problem of the least prime number in an arithmetic progression and give an analogy of Goldbach’s conjecture.

Keywords: least prime number, arithmetic progression, Kanold’s hypothesis, Chowla’s hypothesis, Goldbach’s conjecture, Dirichlet’s theorem, Dickson’s conjecture, Chinese Remainder Theorem, prime map, prime number, prime point

2000 MR Subject Classification: 11A41, 11A99, 11B25, 11P32

1 Introduction

Let \( k, l \) denote positive integers with \((k, l) = 1\) and \(1 \leq l \leq k - 1\). Denote by \( p(k, l) \) the least prime \( p \equiv l \pmod{k} \). Let \( p(k) \) be the maximum value of \( p(k, l) \) for all \( l \) with \((k, l) = 1\) and \(1 \leq l \leq k - 1\). In 1944, Linnik \([1]\) proved that \( p(k) \ll k^{5.5} \). In 1957, Pan \([2]\) claimed \( L \leq 10000 \). In 1958, he \([3]\) was the first to prove that \( L \leq 5448 \). In 1992, Heath-Brown \([4]\) proved \( p(k) \ll k^{5.5} \). Recently, Xylouris \([5]\) improved this result to \( p(k) \ll k^{5.2} \). In 1989, Bombieri, Friedlander and Iwaniec \([6]\) proved \( L \leq 2 \) for almost all integers. Kanold \([7,\]
8] (also independently made by Schinzel and Sierpiński [9]) conjectured that $p(k) < k^2$ for every positive integer $k > 1$. In [4], Heath-Brown proved $p(k) < (\varphi(k) \log k)^2$ assuming the Generalized Riemann Hypothesis. Chowla [10] has observed that $p(k) \ll k^{2+\varepsilon}$ for every $\varepsilon > 0$ assuming the Generalized Riemann Hypothesis. He further conjectured $p(k) \ll k^{1+\varepsilon}$ for every $\varepsilon > 0$. Thus, we have the following weakened form of Chowla’s hypothesis:

**Conjecture 1:** For any positive real number $0 < \varepsilon < \frac{1}{2}$, there is a positive constant $C_1$ depending on $\varepsilon$ such that for every sufficiently large positive integer $k > C_1$, $p(k) < k^{2-\varepsilon}$.

From the aforementioned rich achievements and advancements, one see that the problem of the least prime number in an arithmetic progression is very interesting. It is one of the most important topics in Number Theory. In 2008, we [11] found that this problem closely ties up Goldbach’s conjecture. In this paper, we try to refine the result in [11]. Moreover, we also try to generalize the problem of the least prime number in an arithmetic progression and give an analogy of Goldbach’s conjecture.

## 2 Main results

**Lemma 1:** For any integer $n > 6$, there must be two distinct odd primes $p, q$ such that $\gcd(pq, n) = 1$ and $p < n, q < n$.

**Proof of Lemma 1:** By the refined Bertrand-Chebyshev theorem which states that there exists at least two distinct primes in the interval $(m, 2m)$ when $m = 4$ or $m > 5$, it is easy to prove Lemma 1 holds since any prime in the interval $(\frac{n}{2}, n)$ is coprime to $n$.

Goldbach’s famous conjecture states that every even integer $2n \geq 4$ is the sum of two primes. Due to it is trivial that it is true for infinitely many even integers: $2p = p + p$ (for every prime $p$), we give Goldbach’s conjecture a slightly different expression that every even integer $2n \geq 8$ is the sum of two distinct primes. Thus, by Lemma 1, we get a necessary condition of Goldbach’s conjecture as follows.

**Conjecture 2:** For integer $n > 6$, there exists a natural number $r$ such that $2n - p_r$ is coprime to each of $2n - p_1, ... , 2n - p_{r-1}, 2n - p_{r+1}, ... , 2n - p_k$, where $p_1, ... , p_{r-1}, p_r, p_{r+1}, ... , p_k$ are all old primes smaller than $n$, $p_r$ satisfies $\gcd(p_r, n) = 1$ and $1 \leq r \leq k = \pi(n-1) - 1$, where $\pi(x)$ is the prime counting
function giving the number of primes less than or equal to a given number $x$.

**Theorem 1:** Conjecture 1 and Conjecture 2 imply that every sufficiently large even integer may be written as the sum of two distinct primes.

**Lemma 2:** Denote the least prime coprime to $m$ by $q(m)$ for any positive integer $m$. For every integer $k \geq 1$, there is a positive integer $C_2$ depending on $k$ such that for every integer $m \geq C_2$, we have $(q(m))^k < m$.

**Proof of Lemma 2:** If $k = 1$, the proof is straightforward. Let’s consider the case $k > 1$. By Pósa’s result [12] or Prime Numbers Theorem, there is a positive integer $n_k$ such that $p_{n+1} < p_{1}p_{2}\cdots p_{n}$ for all $n \geq n_k$, where $p_i$ is the $i$th prime. Let $C_2 = p_{1}p_{2}\cdots p_{n_k}$.

We claim that for every integer $m \geq C_2$, we have $(q(m))^k < m$. Write $p_{1}p_{2}\cdots p_{r} \leq m < p_{1}p_{2}\cdots p_{r+1}$. Since $m \geq C_2 = p_{1}p_{2}\cdots p_{n_k}$, hence $r \geq n_k$. So, $p_{r+1} < p_{1}p_{2}\cdots p_{r}$. If $q(m) \leq p_{r+1}$, then $(q(m))^k \leq p_{r+1}^k < p_{1}p_{2}\cdots p_{r} \leq m$ and Lemma 2 holds. If $q(m) > p_{r+1}$, then $m$ is divisible by $p_{1}p_{2}\cdots p_{r+1}$ because $q(m)$ is the least prime coprime to $m$. Therefore, $m \geq p_{1}p_{2}\cdots p_{r+1}$. It is a contradiction by our assumption on $p_{1}p_{2}\cdots p_{r} \leq m < p_{1}p_{2}\cdots p_{r+1}$. So Lemma 2 holds.

**Lemma 3:** For any integer $k \geq 1$ and real number $\alpha > 0$, there is a positive integer $C_3$ such that for every integer $n \geq C_3$ and any positive integer $m < n^\alpha$, we have $(q(m))^k < n$.

**Proof of Lemma 3:** Let $r = \lceil \alpha \rceil + 1$ be the least integer more than $\alpha$. By Lemma 2, there is a least positive integer $C_4$ such that for every integer $x \geq C_4$, we have $(q(x))^k r < x$. Let $C_5 = (C_4 + 1)^k$. We will prove that for every integer $n \geq C_5$ and any positive integer $m < n^\alpha$, we have $(q(m))^k < n$.

If $C_4 \leq m$, then $(q(m))^k r < m < n^\alpha < n^r$ and $(q(m))^k < n$. If $C_4 > m$, then $(q(m))^k \leq (m + 1)^k < (C_4 + 1)^k = C_5 \leq n$. This shows that Lemma 3 holds.

**Corollary 1:** For any given $\varepsilon$ with $0 < \varepsilon < 0.5$, there is a positive integer $C_6$ such that for every integer $n \geq C_6$ and any positive integer $m < n^{2-\varepsilon}$, we have $2^\frac{1}{r}(q(m))^{\frac{2-\varepsilon}{r}} < n$.

**Proof of Theorem 1:** For any given $\varepsilon$ with $0 < \varepsilon < 0.5$, there is a positive integer $C_6$ such that for every prime $p \geq C_6$ and any positive integer $m < p^{2-\varepsilon}$, we have $2^\frac{1}{r}(q(m))^{\frac{2-\varepsilon}{r}} < p$ by Corollary 1.

By the prime number theorem in an arithmetic progression, it is easy to prove that for any prime $p$ with $p \leq \max\{C_1, C_6\}$, $(C_1$ is the positive
constant in Conjecture 1), there exists a positive constant $C_7$ such that for every positive integer $n > C_7$, when $(p, n) = 1$, there exist two distinct odd primes $p_1$ and $p_2$ satisfying $2n \equiv p_1 \equiv p_2 \pmod{p}$ and $p_1, p_2 \in \mathbb{Z}_n^* = \{x | 1 \leq x \leq n, (x, n) = 1\}$.

Let $n$ be an integer $> C_7$. Since we assume Conjecture 2, there exists $r > 1$ such that $p_r < n$, $(p_r, n) = 1$ and $2n - p_r$ is coprime to every $2n - p$ when $p$ ranges the odd primes $\leq n$ and different from $p_r$. We will show that $2n - p_r$ is prime. If this is the case, our proof is over, so let us suppose we can write $2n - p_r = pm$, where $p$ is the least prime factor of $2n - p_r$. Thus, $2n > p^2$.

We have $p > \max\{C_1, C_6\}$. Indeed, if $p$ is smaller, we can find two odd primes say $q_1$ and $q_2$, not more than $n$ and prime to $2n$, such that $2n \equiv q_1 \equiv q_2 \pmod{p}$. At most one of them, say $q_1$, can be equal to $p_r$. This means that $2n - p_r$ is not coprime to $2n - q_2$, contrarily to our hypothesis on $p_r$.

Note that $p_r \neq p$ since $(p_r, n) = 1$. If $p_r < p$, then $p + p_r < p^{2-\varepsilon}$ and there is a prime $q$ coprime to $p + p_r$ and such that $2^{1/7}p^{2-\varepsilon} < p$ by Corollary 1. Since we suppose that Conjecture 1 holds, hence there is a prime $x$ such that $x \equiv p + p_r \pmod{pq}$ and $x < (pq)^{2-\varepsilon} < \frac{p^2}{2} < n$. Clearly, $p_r \neq x$. But $p|(2n - p_r, 2n - x)$. It is a contradiction by our assumption on $p_r$.

Hence $p_r > p$. We write $p_r = pl + v$ with $1 \leq v < p$. If $l \geq p^{1-\varepsilon}$, there is a prime $y$ such that $y \equiv v \pmod{p}$ and $y < p^{2-\varepsilon} < p_r$ (since we suppose Conjecture 1). But we have also $p|(2n - p_r, 2n - y)$, it is contrary to our assumption on $p_r$ again. So we have $l < p^{1-\varepsilon}$, $lv < p^{2-\varepsilon}$ and there is a prime $q$ coprime to $lv$ and such that $2^{1/7}p^{2-\varepsilon} < p$ by Corollary 1 again. Note that there is a prime $z$ such that $z \equiv v \pmod{pq}$ and $z < (pq)^{2-\varepsilon} < \frac{p^2}{2} < n$ (since we suppose that Conjecture 1 holds). Obviously, we have $z \neq p_r$, since $(q, l) = 1$. But $p|(2n - p_r, 2n - z)$. The contradiction implies that $2n - p_r$ is a prime number. This completes the proof of Theorem 1.

**Remark 1:** It is interesting that in [11], we proved that if $p(k) < k^2$ and the necessary condition of Goldbach’s conjecture hold, then every sufficiently large even integer may be written as the sum of a prime and the product of at most two primes. Namely, our assumptions imply Chen’s theorem [13]. In this paper, we have proved that if $p(k) \ll k^{2-\varepsilon}$ and the necessary condition of Goldbach’s conjecture hold, then every sufficiently large even integer may be written as the sum of two distinct primes. However, it can be further improved, we think. We hope it can be improved to $p(k) \ll k^{2+\varepsilon}$. Thus, based on work of Chowla [10], one will see that the Generalized Riemann
Hypothesis implies Goldbach’s conjecture. How far \( p(k) \ll k^{2-\varepsilon} \) is from \( p(k) \ll k^{2+\varepsilon} \)?

Very naturally, one might ask whether Chowla’s hypothesis is true or not. Of course, due to a limited knowledge of the author, he cannot answer well. However, papers [4] and [6] give some witnesses. Also based on the structural beauty of Mathematics itself, the author believes that there is a prime in each row (resp. column) of the following matrix. This further supports Chowla’s hypothesis.

\[
M = (m_{i,j}) = \begin{pmatrix}
    a_1 + 1 \times n, & \cdots, & a_1 + \varphi(n) \times n \\
    \cdots, & \cdots, & \cdots \\
    a_\varphi(n) + 1 \times n, & \cdots, & a_\varphi(n) + \varphi(n) \times n
\end{pmatrix},
\]

where \( a_i \) is the \( i \)-th positive integer which is coprime to \( n \) for \( 1 \leq i \leq \varphi(n) \).

Moreover, for any given integer \( n > 1 \), let \( b_i \) be the \( i \)-th positive integer which is coprime to \( n \) for \( 1 \leq i \leq \varphi(n) \), where \( \varphi(n) \) is Euler totient function, one might prove that there is a permutation \( a_1, \ldots, a_\varphi(n) \) of \( 1 \) to \( \varphi(n) \) such that

\[
F_1 = \begin{cases}
    f_1(x) = a_1 x + b_1 \\
    \cdots \\
    f_\varphi(n)(x) = a_\varphi(n)x + b_\varphi(n)
\end{cases}
\]

is admissible, moreover,

\[
F_2 = \begin{cases}
    f_1(x) = a_1 nx + b_1 \\
    \cdots \\
    f_\varphi(n)(x) = a_\varphi(n)nx + b_\varphi(n)
\end{cases}
\]

is admissible, too. Thus, by Dickson’s conjecture [15], \( f_1(x), \ldots, f_\varphi(n)(x) \) represent simultaneously prime numbers for infinitely many integers \( x \). Therefore, it is very possible that there is a prime in each row (resp. column) of the aforementioned matrix. For the definition of ‘admissible’, see [20].

By Chowla’s hypothesis, there is a prime in each row of \( M \). By Grimm’s conjecture which implies there are two primes between two square numbers [21], there is a prime in each column of \( M \). From this, we see that many problems in Mathematics are not isolated again.

On the other hand, we must prove the necessary condition of Goldbach’s conjecture holds without a proviso. This question looks easy in analytic number theorists’ eyes. But, the author has not been able to work out a
complete proof. We left this question to the readers who are interested in it. Next, we will try to consider another interesting problems.

3 A generalization of the problem of the least prime number in an arithmetic progression

Clearly, the problem of the least prime number in an arithmetic progression closely relates to the famous Dirichlet’s theorem [14]. In fact, Dirichlet’s theorem guarantees us the existence of the least prime number in an arithmetic progression. In 1904, Dickson [15] generalized Dirichlet’s theorem by concerning the simultaneous values of several linear polynomials, which implies Green-Tao theorem [16] (The primes contain arbitrarily long arithmetic progressions). Unfortunately, Dickson’s generalization still is a conjecture by now. The author would like to call it Dickson’s conjecture on \( \mathbb{N} \), where \( \mathbb{N} \) is the set of all positive integers.

In 2006, Green and Tao [18] considered Dickson’s conjecture in the multi-variable case by generalizing Hardy-Littlewood estimation [22]. The brilliant work Green and Tao [16] [18] shows that it is possible to generalize Dickson’s conjecture on \( \mathbb{N} \) to the general case. In [19], the author gave an equivalent form of Dickson’s Conjecture on \( \mathbb{N} \) and further considered Dickson’s conjecture on \( \mathbb{N}^n \). Moreover, in [20], we gave Dickson’s conjecture on \( \mathbb{Z}^n \) which actually is an equivalent form of Green-Tao’s conjecture [18], where \( \mathbb{Z} \) is the set of all integers.

Well, now, let’s assume that Dickson’s conjecture on \( \mathbb{N}^n \) (or \( \mathbb{Z}^n \)) holds. How to generalize the problem of the least prime number in an arithmetic progression? What do the general forms of this problem look like?

First, let’s go back to Linnik’s theorem [1] again, which states that \( p(k) < k^L \), where \( L \) is an absolute constant. This well-known result can be restated as follows: For given positive integer \( k \), there is a positive integer \( C_k \) depending on \( k \) such that \( p(k) < (C_k)^L \), where \( L \) is an absolute constant. Why do we consider \( C_k \)? Because for given \( k \) and \( C_k \), there are only finite many integers \( l \) satisfying \( |l| < C_k \) such that \( f(x) = kx + l \) is admissible. Here \( f(x) = kx + l \) represents infinitely many prime numbers if and only if \( f(x) = kx + l \) is admissible. In this case, we might set \( C_k = k \). Based on this simple observation, one could give a generalization of the problem of the least prime number in an arithmetic progression as follows:
A generalization of the problem of the least prime number in an arithmetic progression (A naive approach):

Let $A = (a_{i,j}) = \begin{pmatrix} a_{11}, \cdots, a_{1n} \\ \cdots, \cdots, \cdots \\ a_{m1}, \cdots, a_{mn} \end{pmatrix}$ be an integral matrix in which any two row vectors are not the same such that for any positive constant $C$, there is an integral point $X = (x_1, \ldots, x_n) \in \mathbb{Z}^n$ such that $F = \begin{cases} f_1(X) = a_{11}x_1 + \ldots + a_{1n}x_n > C \\ \vdots \\ f_m(X) = a_{m1}x_1 + \ldots + a_{mn}x_n > C \end{cases}$.

Then there is a positive integer $C_A$ depending on $A$ such that $p(A) < (C_A)^L$, where $L$ is an absolute constant, and $p(A)$ be the longest prime vector of $p(A, B)$ for all $B = (b_1, \ldots, b_m) \in \mathbb{Z}^n$ with $(\sum_{i=1}^{m}(b_i)^2)^{\frac{1}{2}} < C_A$ such that

$$G = \begin{cases} g_1(X) = a_{11}x_1 + \ldots + a_{1n}x_n + b_1 \\ \vdots \\ g_m(X) = a_{m1}x_1 + \ldots + a_{mn}x_n + b_m \end{cases}$$

is admissible, and where $p(A, B)$ is the shortest prime vector (point) represented by $G$. In [17], we have pointed out it is significative to estimate the upper bound of $p(A, B)$ if $p(A, B)$ exists. What does $C_A$ look like?

**Remark 2:** The condition $F = \begin{cases} f_1(X) = a_{11}x_1 + \ldots + a_{1n}x_n > C \\ \vdots \\ f_m(X) = a_{m1}x_1 + \ldots + a_{mn}x_n > C \end{cases}$ is necessary because if $G = \begin{cases} g_1(X) = a_{11}x_1 + \ldots + a_{1n}x_n + b_1 \\ \vdots \\ g_m(X) = a_{m1}x_1 + \ldots + a_{mn}x_n + b_m \end{cases}$ represents infinitely many prime points, then for any positive constant $C$, there is an integral point $X = (x_1, \ldots, x_n) \in \mathbb{Z}^n$ such that $F = \begin{cases} f_1(X) > C \\ \vdots \\ f_m(X) > C \end{cases}$.

## 4 An analogy of Goldbach’s conjecture

Goldbach’s conjecture states that every even integer $2n \geq 8$ is the sum of two distinct primes. Namely, we have $2n = p + q$, where $p, q$ are prime with $p < q$. If we look upon $x$ as the value of number-theoretic function $f(x)$, then when $f(x) = x$, Goldbach’s conjecture can be re-stated as $2f(n) = f(u) + f(v)$ when $n > 3$, where $f(u) = u, f(v) = v$ are prime. Notice that
\( f(x) = x \) represents infinitely many prime numbers. More generally, one could expect that for \( \gcd(k, l) = 1 \), \( f(x) = kx + l \) has this property. Namely, for every sufficiently large integer \( n = kw + l \) of the form \( f(x) = kx + l \), \( 2n = 2f(w) = f(u) + f(v) \), where \( f(u) = ku + l, f(v) = kv + l \) are prime. This gives an analogy of Goldbach’s conjecture. For example, let \( f(x) = 5x + 2 \).

Then for every \( n > 9 \), \( 5n + 2 \) may be written as the sum of two distinct primes of the form \( 5x + 2: 2 \times 52 = 104 = 7 + 97, 2 \times 57 = 114 = 17 + 97, 2 \times 62 = 124 = 17 + 107, 2 \times 67 = 134 = 7 + 127, 144 = 17 + 127, 154 = 17 + 137, 164 = 7 + 157, 174 = 17 + 157, 184 = 17 + 167, 194 = 37 + 157, 204 = 7 + 197, 214 = 17 + 197, 224 = 97 + 127, 234 = 7 + 227... \)

In this section, we give the weakened form of analogy of Goldbach’s conjecture (see Conjecture 3). We further prove that this weakened form and the weakened form of Chowla’s hypothesis (Conjecture 4) imply the analogy of Goldbach’s conjecture.

**Conjecture 3:** Let \( k, l \) be given positive integers satisfying \( (k, l) = 1 \) and \( 1 \leq l < k \). Let \( Q_i \) be the \( i \)th prime of the form \( kx + l \). There is a positive constant \( C_8 \) such that every integer \( n > C_8 \), there exists \( r > 1 \) such that \( kn + l > Q_r, (kn + l, Q_r) = 1 \) and \( 2(kn + l) - Q_r \) is coprime to every \( 2(kn + l) - Q \) when \( Q \leq kn + l \) ranges the primes of the \( kx + l \) and different from \( Q_r \).

By Chinese Remainder Theorem, Chowla’s hypothesis implies the following conjecture 4.

**Conjecture 4:** Let \( \varepsilon \) with \( 0 < \varepsilon < 0.5 \) be a real number and \( k, l \) be given positive integers satisfying \( (k, l) = 1 \) and \( 1 \leq l < k \). There is a positive constant \( C_9 \) such that for every integer \( d \) satisfying \( d > C_9 \), \( (d, k) = 1 \) and any positive integer \( a \) with \( 1 \leq a < d \) and \( (a, d) = 1 \), there is a prime \( q \) such that \( q < (dk)^{2-\varepsilon}, q \equiv a \pmod{d} \) and \( q \equiv l \pmod{k} \).

**Theorem 2:** Let \( k, l \) be given positive integers satisfying \( (k, l) = 1 \) and \( 1 \leq l < k \). If Conjecture 3 and Conjecture 4 hold, then for every sufficiently large integer \( n, 2(kn + l) \) may be written as the sum of two distinct primes \( p, q \) satisfying \( p, q \equiv l \pmod{k} \).

By the prime number theorem in an arithmetic progression, more precisely, by the result of Ch. de la Vallée-Poussin [23] which states that

\[
\sum_{p \equiv l \pmod{k}, p \leq x} \log p
\]
equals $\frac{x}{\varphi(k)}$ asymptotically, we have that, for every sufficiently large integer $n$, $\sum_{i=1}^{\lfloor n \rfloor+1} \log Q_i$ equals $Q_{\lfloor n \rfloor+1}^\varphi(k)$ asymptotically. It shows immediately that the following Lemma 4 holds.

**Lemma 4:** For every integer $h > 1$, there is a positive integer $n_h$ such that $Q_{n_h}^h < Q_1Q_2 \cdots Q_n$ for all $n \geq n_h$.

Note that $f(x) = l + kx$ takes infinitely many primes when $(k, l) = 1$. Therefore, for any positive integer $m$, there is a least prime of the form $l + kx$ which is coprime to $m$. Denote this least prime by $Q(m)$. By Lemma 4, one can prove the following lemma 5 holds.

**Lemma 5:** For every integer $r \geq 1$, there is a positive integer $C_{10}$ depending on $r$, such that for every integer $m \geq C_{10}$, we have $(Q(m))^r < m$.

**Lemma 6:** For any integer $r \geq 1$ and real number $\delta > 0$, there is a positive integer $C_{11}$ depending on $r, \delta$ such that for every integer $n \geq C_{11}$ and any positive integer $m < n^\delta$, we have $(Q(m))^r < n$.

**Proof of Lemma 6:** Let $e = [\delta] + 1$ be the least integer more than $\delta$. By Lemma 5, there is a least positive integer $C_{12}$ such that for every integer $m \geq C_{12}$, we have $(Q(m))^re < m$. Let $Q_g$ be the least prime of the form $l + kx$ which is larger than $C_{12}$. Moreover, by Lemma 4, there is a least positive integer $f$ such that for every integer $h \geq f$, $Q_{h+1}^e < Q_1Q_2 \cdots Q_h$. Let $t = \max\{g, f\}$ and $C_{11} = Q_1Q_2 \cdots Q_t$. We will prove that for every integer $n \geq C_{11}$ and any positive integer $m < n^\delta$, we have $(Q(m))^r < n$.

We write $Q_1Q_2 \cdots Q_s \leq n < Q_1Q_2 \cdots Q_{s+1}$. Since $n \geq C_{11}$, hence $s \geq t$. For any positive integer $m < n^\delta$, if $Q(m) \leq Q_{s+1}$, then $(Q(m))^re \leq Q_{s+1}^e < Q_1Q_2 \cdots Q_s$ since $s \geq t \geq f$. So, $(Q(m))^re < n$ and $(Q(m))^r < n$. If $Q(m) > Q_{s+1}$, then $m$ is divisible by $Q_1Q_2 \cdots Q_{s+1}$ and $m \geq Q_1Q_2 \cdots Q_{s+1}$. So, $m > Q_s \geq Q_t \geq Q_g$. Moreover, $m > C_{12}$ since $Q_g > C_{12}$. Therefore, $(Q(m))^re < m < n^\delta < n^e$ and $(Q(m))^r < n$. This completes the proof of Lemma 6.

**Corollary 2:** For given $\varepsilon$ in Conjecture 4 and $k$ in Theorem 2, there is a positive integer $C_{13}$ depending on $\varepsilon, k$ such that for every prime $p \geq C_{13}$ and any positive integer $m < k^{3-\varepsilon}p^{2-\varepsilon}$, we have $2^{\frac{1}{2}}(Q(m))^\frac{k}{2} < p$.

**Lemma 7:** For given $k$ and $l$ in Theorem 2 and for any odd prime $p$ satisfying $(p, k) = 1$ and $p \leq \max\{k+1, C_8, C_{10}, C_{13}\}$ ( $C_8, C_{10}, C_{13}$ are the aforementioned constants), there exists a positive constant $C_{14}$ such that for
every positive integer \( n > C_{14} \), when \( (p, kn + l) = 1 \), there exist two distinct odd primes \( p_1 \) and \( p_2 \) satisfying \( p_1 \equiv p_2 \equiv l( \mod k) \), \( 2(kn + l) \equiv p_1 \equiv p_2( \mod p) \) and \( p_1, p_2 \in Z_{kn+l}^* = \{x|1 \leq x \leq kn + l, (x, kn + l) = 1\} \).

**Proof of Lemma 7:** Let \( D = \max\{k + 1, C_8, C_{10}, C_{13}\} \) where \( C_8, C_{10}, C_{13} \) are the aforementioned constants. For given \( k \) and \( l \) in Theorem 2, any odd prime \( p \) satisfying \( (p, k) = 1 \) and \( p \leq D \), and for any integer \( n \) with \( (p, kn + l) = 1 \), there exists infinitely many primes \( q \) such that \( q \equiv l( \mod k) \) and \( q \equiv 2(kn + l)( \mod p) \) by Chinese Remainder Theorem and Dirichlet’s prime theorem in an arithmetic progression. Note that when \( n \) ranges positive integers \( \leq p \), \( 2(kn + l)( \mod p) = \{0, 1, 2, \ldots, p - 1\} \). By the result of Ch. de la Vallée-Poussin again, for any \( r \in \{1, 2, \ldots, p - 1\} \), there exists a positive constant \( C_{p,r} \) such that for every positive integer \( m \) satisfying \( m > C_{p,r} \) and \( 2(kn + l) \equiv r( \mod p) \), there exist two distinct odd primes \( p_1 \) and \( p_2 \) satisfying \( p_1 \equiv p_2 \equiv l( \mod k) \), \( 2(kn + l) \equiv p_1 \equiv p_2( \mod p) \) and \( p_1, p_2 \in Z_{kn+l}^* = \{x|1 \leq x \leq kn + l, (x, kn + l) = 1\} \). Let \( C_{14} = \max\{p,k\}=1,p \leq D \max_{r \in \{1,2,\ldots,p-1\}} C_{p,r}. \) It shows that Lemma 7 holds.

**Proof of Theorem 2:** For given \( \varepsilon \) in Conjecture 4 and \( k,l \) in Theorem 2, Corollary 2 shows that there is a positive integer \( C_{13} \) depending on \( \varepsilon, k \) such that for every prime \( p \geq C_{13} \) and any positive integer \( m < k^{3-\varepsilon}p^{2-\varepsilon} \), we have \( 2^*(Q(m))^\frac{1}{1+\varepsilon}\frac{1}{2} < p \).

Lemma 7 shows that for any odd prime \( p \) satisfying \( (p, k) = 1 \) and \( p \leq \max\{k + 1, C_8, C_{10}, C_{13}\} \) ( \( C_8, C_{10}, C_{13} \) are the aforementioned constants), there exists a positive constant \( C_{14} \) such that for every positive integer \( n > C_{14} \), when \( (p, kn + l) = 1 \), there exist two distinct odd prime \( p_1 \) and \( p_2 \) satisfying \( p_1 \equiv p_2 \equiv l( \mod k) \), \( 2(kn + l) \equiv p_1 \equiv p_2( \mod p) \) and \( p_1, p_2 \in Z_{kn+l}^* = \{x|1 \leq x \leq kn + l, (x, kn + l) = 1\} \).

Let \( n \) be an integer \( > C_{15} = \max\{C_8, C_{14}\} \). Since we assume Conjecture 3, there exists \( r > 1 \) such that \( kn+l > Q_r, (kn+l,Q_r) = 1 \) and \( 2(kn+l) - Q_r \) is coprime to every \( 2(kn+l) - Q \) when \( Q \leq kn + l \) ranges the primes of the \( kx + l \) and different from \( Q_r \). We will show that \( 2(kn+l) - Q_r \) is prime. If this is the case, \( 2(kn+l) - Q_r \) is also a prime of the form \( kx + l \) and our proof is over, so let us suppose we can write \( 2(kn+l) - Q_r = pm \), where \( p \) is the least prime factor of \( 2(kn+l) - Q_r \). Thus, \( 2(kn+l) > p^2, (p, kn+l) = 1 \).

We have \( p > \max\{k + 1, C_8, C_{10}, C_{13}\} \). Indeed, if \( p \) is smaller, we can find two odd primes of the form \( kx + l \), say \( q_1 \) and \( q_2 \), not more than \( kn + l \) and prime to \( 2(kn+l) \), such that \( 2(kn+l) \equiv q_1 \equiv q_2( \mod p) \). At most
one of them, say \(q_1\), can be equal to \(Q_r\). This means that \(2(kn + l) - Q_r\) is not coprime to \(2(kn + l) - q_2\), contrarily to our hypothesis on \(Q_r\).

Note that \(Q_r \neq p\) since \((kn + l, Q_r) = 1\). If \(Q_r < p\), then \((p + Q_r)k < k^{3 - \varepsilon} p^{2 - \varepsilon}\) and there is a prime \(q\) coprime to \((p + Q_r)k\) and such that \(2^\frac{1}{2} q^{\frac{2 - \varepsilon}{2 - \varepsilon}} k^{\frac{2 - \varepsilon}{2 - \varepsilon}} < p\) and \((pqk)^{2-\varepsilon} < \frac{p^2}{2}\) by Corollary 2. Since we suppose that Conjecture 4 holds, hence there is a prime \(A\) of the form \(kx + l\) such that \(A \equiv p + Q_r(\mod pq)\) and \(A < (pqk)^{2-\varepsilon} < \frac{p^2}{2} < kn + l\). Clearly, \(Q_r \neq A\). But \(p(2(kn + l) - Q_r, 2(kn + l) - A)\). It is a contradiction by our assumption on \(Q_r\).

Hence \(Q_r > p\). We write \(Q_r = pq + z\) with \(1 \leq z < p\). If \(y \geq p^{1 - \varepsilon} k^{2 - \varepsilon}\), there is a prime \(B\) of the form \(kx + l\) such that \(B \equiv z(\mod p)\) and \(B < (pk)^{2 - \varepsilon} < Q_r\) (since we suppose Conjecture 4). But we have also \(p(2(kn + l) - Q_r, 2(kn + l) - B)\), it is contrary to our assumption on \(Q_r\) again. So we have \(y < p^{1 - \varepsilon} k^{2 - \varepsilon}\), \(yzk < p^{2 - \varepsilon} k^{3 - \varepsilon}\) and there is a prime \(q\) coprime to \(yzk\) and such that \(2^\frac{1}{2} q^{\frac{2 - \varepsilon}{2 - \varepsilon}} k^{\frac{2 - \varepsilon}{2 - \varepsilon}} < p\). Note that there is a prime \(E\) of the form \(kx + l\) such that \(E \equiv z(\mod pq)\) and \(E < (pqk)^{2-\varepsilon} < \frac{p^2}{2} < kn + l\) (since we suppose that Conjecture 4 holds). Obviously, we have \(E \neq Q_r\) since \((q, y) = 1\). But \(p(2(kn + l) - Q_r, 2(kn + l) - E)\). The contradiction implies that \(2(kn + l) - Q_r\) is a prime number. Therefore, when \(n > C_{15}\), \(2(kn + l)\) may be written as the sum of two distinct primes \(p, q\) satisfying \(p \equiv q \equiv l(\mod k)\) assuming Conjectures 3 and 4. This completes the proof of Theorem 2.

**Remark 3:** Based on Euclid’s algorithm, in [27], we find a special sequence which is called W sequence. By studying W sequences in the case of non-consecutive positive integers, we give Conjectures 2 and 3. Conjecture 4 can be generalized: Let \(\varepsilon\) with \(0 < \varepsilon < 0.5\) be a real number and \(k_i, l_i\) be given positive integers satisfying \((k_i, l_i) = 1\) and \(1 \leq l_i < k_i\) for \(i = 1, ..., n\), where \((k_i, k_j) = 1\) for \(1 \leq i \neq j \leq n\). There is a positive constant \(C_{16}\), such that for every integer \(d\) satisfying \(d > C_{16}\), \((d, k_1...k_n) = 1\) and any positive integer \(a\) with \(1 \leq a < d\) and \((a, d) = 1\), there is a prime \(q\) such that \(q < (dk_1...k_n)^{2-\varepsilon}, q \equiv a(\mod d)\) and \(q \equiv l_i(\mod k_i)\). This can be deduced by Chinese Remainder Theorem and Chowla’s hypothesis.
5 A generalization of analogy of Goldbach’s conjecture (A naive approach)

It is known that \( f(x) = x \) on \( Z \) is the simplest polynomial which represents infinitely many primes. By Dirichlet’s famous theorem, for any positive integer \( l, k \) with \( (l, k) = 1 \), \( f(x) = l + kx \) is a simpler polynomial which also represents infinitely many primes. If we view \( f(x) = l + kx \) as an analogy of \( f(x) = x \), Theorem 2 shows that it is possible to give an analogy of Goldbach’s conjecture. Lev Landau said tastily: ‘Why add prime numbers? Prime numbers are made to be multiplied, not added.’ This time, we are afraid of ‘Prime numbers might be made to be added.’ If in the higher-dimension case, we have a similar the problem of the least prime number in an arithmetic progression, maybe, there is also a similar Goldbach’s conjecture. We will try to consider this problem in this section. Very naturally, We would like to consider a general problem: might prime points be made to be added? In order to clearly explain this problem, firstly, let’s do an interesting thing as follows:

Based on the point of view that a number is a map, we view an integer \( x \) as the simplest polynomial map on \( Z: f(x) = x \) from \( Z \) to \( Z \). Notice that such a map takes infinitely many prime numbers. More generally, let’s consider the map \( F: Z^n \to Z^m \) for all integral points \( x = (x_1, ..., x_n) \in Z^n \), \( F(x) = (f_1(x), ..., f_m(x)) \) for distinct polynomials \( f_1, ..., f_m \in Z[x_1, ..., x_n] \), where \( m, n \in N \). In this case, we call \( F \) a polynomial map on \( Z^n \). We say that these multivariable integral polynomials \( f_1(x), ..., f_m(x) \) on \( Z^n \) represent simultaneously prime numbers for infinitely many integral points \( x \), if for any \( 1 \leq i \leq m \), \( f_i(x) \) itself can represent prime numbers for infinitely many integral points \( x \), moreover, there is an infinite sequence of integral points \( (x_{11}, ..., x_{n1}), ..., (x_{1i}, ..., x_{ni}), ... \) such that for any positive integer \( r \), \( f_1(x_{1r}, ..., x_{nr}), ..., f_m(x_{1r}, ..., x_{nr}) \) represent simultaneously prime numbers, and for any \( i \neq j \), \( f_1(x_{1i}, ..., x_{ni}) \neq f_1(x_{1j}, ..., x_{nj}), ..., f_m(x_{1i}, ..., x_{ni}) \neq f_m(x_{1j}, ..., x_{nj}) \) hold simultaneously. In this case, we also say that the polynomial map \( F \) on \( Z^n \) represents infinitely many prime points. Such a polynomial map \( F \) is called a prime map. In short, a prime map is a polynomial map on \( Z^n \) which represents infinitely many prime points. For instance: \( F = f(x) = x \), \( F = f(x) = ax + b \) with \( \gcd(a, b) = 1 \), \( F = f(x, y) = x^2 + y^2 \), \( F = f(x, y) = x^3 + y^2 + 1 \), \( F = f(x, y) = x^3 + 2y^3 \), \( F = f(x, y) = x^2 + y^4 \), \( F = (f_1(x, y) = x, f_2(x, y) = x^2 + y^2) \), \( F = f(x, y, z, w) = x^2 + y^2 + z^2 + w^2 \) and so on are all prime maps. This gives a generalization of \( f(x) = x \) on \( Z \).
Due to the fact that \( g(x) = ax + b \) with \( \gcd(a, b) = 1 \) is the unique known prime map on \( \mathbb{Z} \), we want to know more properties about the arithmetic progressions. By the analogy of Goldbach’s conjecture, we further hope to find more interesting analogies between Integers and Arithmetic progressions. For example, for every sufficiently large integer \( n \), if \( g(x) > n \), then there is a prime of the form \( g(x) \) in the interval \( (g(x), 2g(x)) \), which can be viewed as the analogy of Bertrand-Chebyshev theorem, especially, there is a prime of the form \( 3k+1 \) in the interval \( (3x+1, 2(3x+1)) \) for each positive integer \( x \). These problems we will study in other papers.

In [20], we find an interesting property of prime maps and generalize the analogy of Chinese Remainder Theorem as follows: Let \( F = (f_1, \ldots, f_m) \) be a prime map. If \( \gcd(a_i, a_j) = 1 \) for \( 1 \leq i \neq j \leq k \), and there exist integral point \( x^{(j)} \in \mathbb{Z}^n \) such that \( F(x^{(j)}) \) is in \( (\mathbb{Z}^*_a \setminus \{1\})^m \) for \( 1 \leq j \leq k \), then there exists an integral point \( z \) such that \( F(z) \) is in \( (\mathbb{Z}^*_{a_1 \ldots a_k} \setminus \{1\})^m \).

Note that the prime map \( F = (f_1(x, y) = x, f_2(x, y) = x^2 + y^2) \) implies that for any \( 1 \leq m \leq n \), there is a prime map \( F(x) = (f_1(x), \ldots, f_m(x)) \) for distinct polynomials \( f_1, \ldots, f_m \in \mathbb{Z}[x_1, \ldots, x_n] \). However, when \( m > n \), we do not know whether there are always such prime maps. Especially, when \( n = 1, m > 1 \) and \( f_i \) is linear, it is a famous open problem (Dickson’s conjecture). Anyway, one might expect that prime maps have many fascinating properties like integers. We expect that prime points have some interesting properties like prime numbers. The author wishes that in the higher-dimension case, we have a similar Prime Number Theorem.

We call a prime map \( F(x) = (f_1(x), \ldots, f_m(x)) \) on \( \mathbb{Z}^n \) is standard if \( F(1, \ldots, 1) \) is a prime point (vector). For example, \( f(x) = x + 1 \) is a standard prime map. \( F = f(x) = ax + b \) a standard prime map if and only if \( a + b \) is a prime number. Bertrand-Chebyshev theorem implies that for any positive integer \( a > 1 \), there is a positive integer \( b < a \) such that \( ax + b \) is a standard prime map. Clearly, a prime maps can be reduced to a standard prime map. Let \( F = (f_1, \ldots, f_m) \) on \( \mathbb{Z}^n \) be a standard prime map. Then for every sufficiently large integer \( r \), if there is an integral point \( x \) such that each coordinate of \( F(x) = \alpha \) is greater than \( r \), then \( 2\alpha = \beta + \gamma \), where \( \beta, \gamma \) are distinct prime points represented by \( F \). This explains the aforementioned problem. From this, one will see that this problem and the prime map are equivalent. Particularly, Goldbach’s conjecture and the infinitude of primes are equivalent although without any proof. This perhaps is another property of primes maps. But, this problem is the author’s naive viewpoint. The author also finds several propositions which are equivalent to the in-
finitude of prime numbers by considering prime maps, see [Appendix]. For this reason, we revisit Euclid's Number Theory and focus on the essence of integers. Gödel's incompleteness theorem [25] states that all consistent axiomatic formulations of number theory include undecidable propositions. Along this research line, we do not know whether one will meet those undecidable propositions in Number Theory. We hope that people further consider them.

6 Acknowledgements

I am very thankful to Professor Heath-Brown for his comments improving the presentation of the paper, and also to my supervisor Professor Xiaoyun Wang for her help. Thank Professor Xianmeng Meng for her suggestions. Thank the key lab of cryptography technology and information security in Shandong University and the Institute for Advanced Study in Tsinghua University, for providing me with excellent conditions. This work was partially supported by the National Basic Research Program (973) of China (No. 2007CB807902) and the Natural Science Foundation of Shandong Province (No. Y2008G23).

References

[1] Y.V. Linnik, On the least prime in an arithmetic progression I. The basic theorem, Rec. Math. (Mat. Sbornik) N.S. 15(57) (1944), 139-178.

[2] C.D. Pan, On the least prime in an arithmetical progression, Sci. Record (N.S.) 1 (1957), 311-313.

[3] C.D. Pan, On the least prime in an arithmetical progression, Acta Sci. Natur. Univ. Pekinensis 4 (1958), 1-34.

[4] D. R. Heath-Brown, Zero-free regions for Dirichlet L-functions, and the least prime in an arithmetic progression, Proc. London Math. Soc. (3) 64, no. 2, (1992), 265-338.

[5] T. Xylouris, On Linnik’s constant.
Available at: http://arxiv.org/abs/0906.2749

[6] E. Bombieri, J. B. Friedlander and H. Iwaniec, Primes in Arithmetic Progressions to Large Moduli. III, Journal of the American Mathematical Society 2(2) (1989), 215C224.
[7] H. Kanold, Über Primzahlen in arithmetischen Folgen. (German) Math. Ann. 156 (1964), 393-395.

[8] H. Kanold, Über Primzahlen in arithmetischen Folgen. II. (German) Math. Ann. 157 (1965), 358-362.

[9] A. Schinzel and W. Sierpiński, Sur certaines hypothses concernant les nombres premiers. (French) Acta Arith. 4 (1958), 185-208; erratum 5 (1958), 259.

[10] S. Chowla, On the least prime in an arithmetic progression, J. Indian Math. Soc., 1(2), (1934), 1-3.

[11] S.H. Zhang, Goldbach conjecture and the least prime number in an arithmetic progression. Available at http://arxiv.org/abs/0812.4610

[12] L. Pósa, Über eine Eigenschaft der Primzahlen. (Hungarian) Mat. Lapok 11 (1960), 124-129.

[13] J.R. Chen, On the representation of a large even integer as the sum of a prime and the product of at most two primes. II. Sci. Sinica 21 (1978), no. 4, 421-430.

[14] G.L. Dirichlet, Beweis des Satzes daß jede unbegrenzte arithmetische Progression deren erstes Glied und Differenz ganze Zahlen ohne gemeinschaftlichen Faktor sind unendlich viele Primzahlen enthält, Werke, Leipzig: G. Reimer, 1889, I, pp313-342, (Original 1837).

[15] L. E. Dickson, A new extension of dirichlet’s theorem on prime numbers, Messenger of Mathematics, 33, (1904), 155-161.

[16] B. Green and T. Tao, The primes contain arbitrarily long arithmetic progressions, Ann. of Math. (2) 167, no. 2, 481-547, (2008).

[17] S.H. Zhang, On the infinitude of some special kinds of primes, available at: http://arxiv.org/abs/0905.1655

[18] B. Green and T. Tao, Linear equations in primes, preprint, available at: http://arxiv.org/abs/math/0606088 Ann. of Math. (2), in press

[19] S.H. Zhang, Notes on Dickson’s Conjecture, available at: http://arxiv.org/abs/0906.3850

[20] S.H. Zhang, Dickson’s conjecture on $Z^n$—An equivalent form of Green-Tao’s conjecture, available at: http://arxiv.org/abs/0911.3679
Appendix: Euclid’s Number Theory Revisited

Remark: This appendix is self-contained.

From Euclid’s famous Elements [24], (Proposition 20, Book ), we see that Euclid (300 B.C.) proved that \( f(x) = x \) represents infinitely many prime numbers.

It is difficult to image what would happen if there was only finite many prime numbers: many theorems and conjectures do not hold any more. For example, Bertrand-Chebyshev theorem, Dirichlet’s theorem, Prime number theorem, The fundamental theorem of arithmetic, Chinese Remainder Theorem, Goldbach’s conjecture, Landau’s problems and so on are not true if there is only finite many prime numbers. Therefore, Hardy said: ‘Euclid’s theorem which states that the number of primes is infinite is vital for the whole structure of arithmetic. The primes are the raw material out of which
we have to build arithmetic, and Euclid’s theorem assures us that we have plenty of material for the task. For this reason, in this section, we would like to revisit Euclid’s Number Theory and give some equivalent propositions of Euclid’s second theorem.

From Book 7, 8 and 9 of his *Elements*, we see that Euclid had established elementarily Theory of Divisibility and the greatest common divisor. Euclid began his number-theoretical work by giving some definitions and his algorithm (the Euclidean algorithm) (See [24]: Book 7, Propositions 1 and 2) as follows:

11. A prime number is that which is measured by a unit alone.
12. Numbers prime to one another are those which are measured by a unit alone as a common measure.
13. A composite number is that which is measured by some number.

Proposition 1 (Book 7): Two unequal numbers being set out, and the less being continually subtracted in turn from the greater, if the number which is left never measures the one before it until a unit is left, the original numbers will be prime to one another.

Proposition 2 (Book 7): Given two numbers not prime to one another, to find their greatest common measure.

Proposition 31 (Book 7): Any composite number is measured by some prime number.

Proposition 20 (Book 9): Prime numbers are more than any assigned multitude of prime numbers. Namely, there are infinitely many primes.

Now, let’s go back to Euclid’s proof for the infinitude of prime numbers: Supposed that there are only finitely many primes, say $k$ of them, which denoted by $p_1, ..., p_k$. Consider the number $E = 1 + \prod_{i=1}^{k} p_i$. If $E$ is prime, it leads to the contradiction since $E \neq p_i$ for any $1 \leq i \leq k$. If $E$ is not prime, $E$ has a prime divisor $p$ by Proposition 31 (Book 7). But $p \neq p_i$ for any $1 \leq i \leq k$. Otherwise, $p$ divides $\prod_{i=1}^{k} p_i$. Since it also divides $1 + \prod_{i=1}^{k} p_i$, it will divide the difference or unity, which is impossible.

In his proof, we see that Euclid used Proposition 31 (Book 7). Of course, he also used a unexpressed axiom which states that if $A$ divides $B$, and also divides $C$, $A$ will divide the difference between $B$ and $C$. 
Well, let’s look at the proof of Proposition 31 (Book 7): Let $A$ be a composite number. By the definition, there must be a number $B$ ($1 < B < A$) which divides $A$. If $B$ is prime, then Proposition 31 holds. If $B$ is not prime, there must be a number $C$ ($1 < C < B$) which divides $B$. If $C$ is prime, then Proposition 31 holds since $C$ also divides $A$. If $C$ is not prime, by repeating this process, in finite many steps, there must be a prime which divides $A$ and Proposition 31 holds. From this proof, we see that Euclid used a unexpressed axiom which states that if $A$ divides $B$, and $B$ divides $C$, then $A$ divides $C$. In his book [24], Thomas Little Heath had noted that Euclid used the aforementioned axioms. We would be quite surprised if he did use these axioms because on one hand, Proposition 31 (Book 7) and Proposition 20 (Book 9) can be deduced early by definitions, on the other hand, we expect him to make use of his algorithm which is his first number-theoretical proposition in his *Elements*. Then, let’s try to supplement this work.

Now, let’s use these axioms again, Euclid’s definitions on a prime number and a composite number, and his algorithm (Propositions 1, 2) to prove Euclid’s second theorem and some equivalent propositions of the infinitude of prime numbers.

**Theorem 1:** Any composite number is divided by some prime number.

**Proof:** By the definition of a composite number and the axiom which that if $A$ divides $B$, and $B$ divides $C$, then $A$ divides $C$, it is easy to prove that Theorem 1 is true.

**Theorem 2:** For any positive integer $a$, $a$ is co-prime to $a + 1$.

**Proof:** By Euclid’s algorithm (Proposition 1, Book 7), it shows immediately that Theorem 2 holds.

**Corollary 1:** For any positive integer $a$, there is a positive integer $b$ such that $b > 1$ and $b$ is co-prime to $a$.

**Proof:** Let $b = a + 1$. By Theorem 2, it is easy to prove Corollary 1 holds.

**Corollary 2:** For any positive integer $a$, there is a least integer $b$ such that $b > 1$ and $b$ is co-prime to $a$.

**Proof:** By Corollary 1 and the axiom which states there is a least element in any non-empty subset of natural numbers, Corollary 2 holds.
**Theorem 3:** For any positive integer \(a\), let \(b\) be the least integer \(b\) such that \(b > 1\) and \(b\) is co-prime to \(a\). Then \(b\) is prime.

**Proof:** By Corollary 2, we get that for any positive integer \(a\), there is a least integer \(b\) such that \(b > 1\) and \(b\) is co-prime to \(a\). If \(b\) is not prime, by Theorem 1, \(b\) is divided by some prime number \(p\). Of course, \(p\) is co-prime to \(a\) and \(p < b\). But \(b\) is the least. The contradiction shows that \(b\) is prime and Theorem 3 holds.

**Corollary 3:** 2 and 3 are all prime numbers.

**Proof:** We do not want to factor 2 or 3 but prove directly Corollary 3 holds. By Theorem 2, we know that 1 is co-prime to 2. Note that 2 is the least integer such that \(2 > 1\) and 2 is co-prime to 1. Let \(a = 1\). By Theorem 3, we deduce that 2 is prime. Similarly, one can prove that 3 also is prime.

Corollary 3 gives us a method for generating whole prime numbers: Let \(p_i\) be the \(i\)-th prime. By Corollary 3, \(p_1 = 2\), \(p_2 = 3\). \(p_{n+1}\) is the least prime which is co-prime to \(\prod_{i=1}^{n} p_i\).

**Corollary 4:** There are infinitely many prime numbers.

**Proof:** The existence of prime numbers is very clear. for example, 2 is a prime number by Corollary 3. Supposed that there are only finitely many prime numbers, say \(k\) of them, which denoted by \(p_1, ..., p_k\). Let \(a = \prod_{i=1}^{k} p_i\). By Theorem 3, let \(b\) be the least integer \(b\) such that \(b > 1\) and \(b\) is co-prime to \(a\). Then \(b\) is prime. Of course, \(b \neq p_i\) for any \(1 \leq i \leq k\). The contradiction shows that Corollary 4 is true.

From Corollary 3, we see that Propositions 1 and 31 (Book 7) in Euclid’s Number Theory implies the infinitude of prime numbers. Next, we will give some equivalent propositions that there are infinitely many prime numbers. The author wonders why this occurs.

**Theorem 4:** There are infinitely many prime numbers if and only if for any positive integer \(a\), there is a positive integer \(b\) such that \(b > 1\) and \(b\) is co-prime to \(a\).

**Proof:** If there are infinitely many prime numbers, then for any positive integer \(a\), there must be a prime \(p\) which is greater than \(a\). Let \(b = p\) and the necessity holds obviously. On the other hand, if for any positive integer \(a\), there is a positive integer \(b\) such that \(b > 1\) and \(b\) is co-prime to \(a\), then there must be a least integer \(c\) such that \(c > 1\) and \(c\) is co-prime to \(a\).
By Theorem 3, $c$ is prime. Thus the existence of prime numbers has been proved. Supposed that there are only finitely many prime numbers, say $k$ of them, which denoted by $p_1, \ldots, p_k$. Let $d = \prod_{i=1}^{k} p_i$. By Theorem 3 again, let $e$ be the least integer such that $e > 1$ and $e$ is co-prime to $d$. Then $e$ is prime. Of course, $e \neq p_i$ for any $1 \leq i \leq k$. The contradiction shows that the sufficiency is true. Therefore, Theorem 4 holds.

From Theorem 4, we see that the polynomial $f(x) = x$ takes infinitely many prime numbers if and only if it is admissible.

**Lemma 1:** Euclid’s algorithm, Division algorithm and Bezout’s equation are all equivalent.

**Proof:** See [26].

Since we aforehand assume Euclid’s algorithm, hence, by Lemma 1, we can logically deduce many number theoretical results in any number theoretical textbooks. Especially, we get the following theorems 5, 6 and 7.

**Theorem 5:** For any positive integer $a$, there is a positive integer $b$ such that $b > 1$ and $b$ is co-prime to $a$ if and only if there is a positive integer $c$ such that for any positive integer $m > c$, there is a positive integer $k$ such that $1 < k < m$ and $k$ is co-prime to $m$.

**Proof:** First, we prove that the latter implies the former. When $a > c$, since for any positive integer $m > c$, there is a positive integer $k$ such that $1 < k < m$ and $k$ is co-prime to $m$, hence there is a positive integer $b$ such that $b > 1$ and $b$ is co-prime to $a$. When $1 < a \leq c$, clearly, there is a positive integer $r$ such that $a^r > c$. Thus, there is a positive integer $b$ such that $b > 1$ and $b$ is co-prime to $a^r$. Of course, $b$ is co-prime to $a$, too. When $a = 1$, we can choose $b = 2$.

Next, we will prove that the former implies the latter. One might believe that for any positive integer $m > 2$, there is a positive integer $k = m - 1$ such that $1 < k < m$ and $k$ is co-prime to $m$ by Theorem 2. Thus, it seems that the former is not related to the latter. However, we do not do so. We will strictly prove that for any positive integer $m \geq 15$, there is a positive integer $k$ such that $1 < k < m$ and $k$ is co-prime to $m$. Clearly, if $3$ (resp. $5$) is co-prime to $m$, we choose $k = 3$ (resp. $k = 5$). So, when $m \geq 15$, we only consider the case that $m$ is divisible by 15. We write $m = 15t$ with $t \geq 1$. If $t$ is not divisible by 2, we can choose $k = 2$. Well, now we assume that $t$ is divisible by 2. We write $t = 3^e d$ with $\gcd(3, d) = 1$. Since $2|t$, hence $d > 1$. 20
Note that there is a positive integer \( r > 1 \) which is co-prime to \( 3t \) because we assume that for any positive integer \( a \), there is a positive integer \( b \) such that \( b > 1 \) and \( b \) is co-prime to \( a \). By the linear congruence theorem, there is a positive integer \( h \) with \( 0 \leq h < d \leq t \) such that \( 2 + 3h \equiv r \pmod{d} \). Notice that either \( 2 + 3h \) or \( 2 + 3h + 3t \) is co-prime to \( m = 15t \), moreover, \( 1 < 2 + 3h < 2 + 3h + 3t < 15t = m \). Let \( c = 15 \). This shows that Theorem 5 holds.

**Theorem 6:** There are infinitely many prime numbers if and only if there is a positive constant \( c \) such that for any positive integer \( a > c \), there is a positive integer \( b \) such that \( 1 < b < a \) and \( b \) is co-prime to \( a \).

**Proof:** By Theorems 4 and 5, it immediately shows that Theorem 6 is true. From Theorem 6, we see that the polynomial \( f(x) = x \) takes infinitely many primes if and only if it is strongly admissible.

**Corollary 5:** There are infinitely many prime numbers if and only if for any positive integer \( a > 2 \), there is a positive integer \( b \) such that \( 1 < b < a \) and \( b \) is co-prime to \( a \).

**Proof:** By Theorem 5, the infinitude of prime numbers implies for any positive integer \( a > 14 \), there is a positive integer \( b \) such that \( 1 < b < a \) and \( b \) is co-prime to \( a \). Further, one can directly test that it is also true when \( 2 < a < 15 \). So Corollary 5 holds.

**Theorem 7:** Euclid’s second theorem and the analogy of Chinese Remainder Theorem (which states that if there exist a positive integer \( a \) such that \( 1 < a \) is in \( \mathbb{Z}^*_n \) and a positive integer \( b \) such that \( 1 < b \) is in \( \mathbb{Z}^*_m \), then there exists a positive integer \( c \) such that \( 1 < c \) is in \( \mathbb{Z}^*_{mn} \) when \( \gcd(m, n) = 1 \)) are equivalent.

**Proof:** First, we prove that the analogy of Chinese Remainder Theorem implies Euclid’s second theorem. By Corollary 3, we know that 2 and 3 are all prime numbers. Supposed that there are only finitely many prime numbers, say \( k \) of them, which denoted by \( p_1 = 2, p_2 = 3, ..., p_k \). Let \( d = \prod_{i=2}^{k} p_i \). Clearly, 2 is in \( \mathbb{Z}^*_d \). 3 is in \( \mathbb{Z}^*_4 \). Notice that \( \gcd(d, 4) = 1 \). By our assumption, there exists a least positive integer \( c \) such that \( 1 < c \) is in \( \mathbb{Z}^*_{4d} \). By Theorem 3, \( c \) is prime. Of course, \( c \neq p_i \) for any \( 1 \leq i \leq k \). The contradiction shows that the analogy of Chinese Remainder Theorem implies Euclid’s second theorem.
Secondly, we prove that Euclid’s second theorem implies the analogy of Chinese Remainder Theorem. By Corollary 5, we only need to prove that for any positive integer \( d > 2 \) there is a positive integer \( k \) such that \( 1 < k < d \) and \( \gcd(k, d) = 1 \) implies the analogy of Chinese Remainder Theorem. In fact, if there exist a positive integer \( a \) such that \( 1 < a \) is in \( \mathbb{Z}_n^* \) and a positive integer \( b \) such that \( 1 < b \) is in \( \mathbb{Z}_m^* \), then \( m \geq 3, n \geq 3 \). Consequently \( mn \geq 9 > 2 \). By our assumption that Euclid’s second theorem holds, equivalently, when \( mn > 2 \), there is a positive integer \( c \) such that \( 1 < c < mn \) and \( \gcd(c, mn) = 1 \), we deduce that the analogy of Chinese Remainder Theorem holds. This completes the proof of Theorem 7.

One might find more equivalent propositions. By the aforementioned discussion, we believe that one of substantive characteristics of the set of all integers is that it contains infinitely many prime numbers. Therefore, it should be reasonable that we generalize Integers to Prime maps. Based on such a belief, we revisited Euclid’s Number Theory and added this appendix.