Lagrangian formulation of the Palatini action

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July 3, 2018

Abstract

We work on the Lagrangian formulation of the Palatini action. We find that we must assume the metric compatibility condition for the Palatini action to describe General Relativity, which condition should hold in quantization. We find that we must also assume one of the torsion zero condition or the tetrad compatibility condition. Our results will hold for any action in terms of the tetrad and the internal connection which describes General Relativity.

Loop Quantum Gravity is a quantization program of General Relativity based on Gauge Theory [1]. Although it has been studied more than 30 years, yet it is still obscure what should be assumed beforehand and what are derived afterward from the Euler-Lagrange equations in the beginning Lagrangian formulation of this program. In this paper, we clear this up once and for all. This makes the Hamiltonian formulation richer than previously known.

The Palatini action is

\[ S_p(e, w) = \frac{1}{2} \int_M \sqrt{-g} e^a_I e^b_J F_{ab}^{IJ} \] (1)

where \( e^a_I \) is a tetrad and \( F_{ab}^{IJ} \) is a curvature tensor of an internal connection \( w^a_{IJ} \):

\[ F_{ab}^{IJ} = 2 \partial_{[a}w_{b]}^I J + [w_a, w_b]^I J. \] (2)

We can obtain a torsion tensor from \( e^a_I \) and \( w^a_{IJ} \):

\[ T_{ab}^{I} = 2\partial_{[a}e_{b]}^I + w_a^I J e_b^J - w_b^I J e_a^J. \] (3)

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In the Palatini action, $e^a_I$ and $w^I_aJ$ are the basic independent variables. The Palatini action becomes the Einstein-Hilbert action when the tetrad compatibility condition, $D_a e^I_I = 0$, holds where $D_a$ is a generalized derivative operator defined on a generalized tensor field $H_{aI}$ by

$$D_a H_{bI} \equiv \partial_a H_{bI} + A_{ab}^c H_{cI} + w_{aI}^J H_{bJ}$$  \hspace{1cm} (4)

where $A_{ab}^c$ is a spacetime connection. In Riemannian geometry, $D_a e^I_I$ is always zero and we can have the Riemann curvature tensor and the torsion tensor from either $A_{ab}^c$ or $w_{aI}^J$, which makes two actions equivalent. In this case, $A_{ab}^c$ and $w_{aI}^J$ are determined by the tetrad and the torsion. From $D_a e^I_I = 0$, $D_a g_{bc} = 0$. With this

$$A_{ab}^c = \Gamma_{ab}^c + \frac{1}{2} \{-T_a^c b - T_b^c a + T_{ab}^c\}$$  \hspace{1cm} (5)

where $\Gamma_{ab}^c$ is the Christoffel symbols:

$$\Gamma_{ab}^c = -\frac{1}{2} g^{cd} \{ \partial_b g_{ad} + \partial_a g_{bd} - \partial_d g_{ab} \},$$  \hspace{1cm} (6)

and $T_{ab}^c$ is the spacetime torsion tensor defined by non-commutativity of the derivative operator on a scalar field. With $D_a e^I_I = 0$ and (5), $w_{aI}^J$ are determined by the tetrad and the torsion, which can be also obtained by solving (3). If $D_a e^I_I \neq 0$, the Palatini action does not become the Einstein-Hilbert action. The curvature and the torsion from (2) and (3) are not equal to those obtained from $A_{ab}^c$. In this case, (2) and (3) do not have the geometrical meanings of Riemannian geometry.

The variational calculations of the Palatini action are incorrect in many literatures. There seems to have been confusion about what is the correct statement on the relation between the metric compatibility condition, the torsion zero condition and the tetrad compatibility condition. The correct statement is this: If $D_a g_{bc} = 0$ and $T_{ab}^c = 0$, then $T_{ab}^J = 0$ if and only if $D_a e^b_I = 0$. Note that Stokes’s theorem holds for a torsion-free derivative operator on a orientable manifold and Gauss’s theorem holds when the metric compatibility condition is satisfied once a volume element is chosen by a metric. Because great care must be taken without $D_a g_{bc} = 0$ or the torsion zero condition, let’s work on a simple model first:

$$S \equiv \int_M \sqrt{-g} P^a Q^b D_a R_b.$$  \hspace{1cm} (7)

If $D_a g_{bc} = 0$ and $T_{ab}^c = 0$,

$$\partial_a (\sqrt{-g} P^a) = \sqrt{-g} D_a P^a$$  \hspace{1cm} (8)
where we used the formula:
\[ \partial_a \sqrt{-g} = \frac{1}{2} \sqrt{-g} g^{bc} \partial_a g_{bc} = -\sqrt{-g} (A_{ba}^b - T_{ba}^b). \] (9)

Generally without assuming \( D_a g_{bc} = 0 \),
\[ \partial_a (\sqrt{-g} P^a) = \sqrt{-g} D_a P^a + (\partial_a \sqrt{-g} + \sqrt{-g} A_{ca}^c) P^a. \] (10)

If \( D_a g_{bc} = 0 \) but \( T_{ab}^c \neq 0 \),
\[ \partial_a (\sqrt{-g} P^a) = \sqrt{-g} D_a P^a - \sqrt{-g} T_{ba}^b P^a. \] (11)

Let’s see what we have when we vary \( R_a \). From \( \delta S = 0 \), we have
\[ \sqrt{-g} D_a (P^a Q^b) + (\partial_a \sqrt{-g} + \sqrt{-g} A_{ca}^c) P^a Q^b = 0 \] (12)
where we used \( \delta R_a = 0 \) on the boundary. Note that the second term does not disappear as in (10). If \( D_a g_{bc} = 0 \), we have
\[ D_a (P^a Q^b) + T_{ca}^c P^a Q^b = 0. \] (13)

We can see that a solution \( D_a (P^a Q^b) = 0 \) is obtained only when \( D_a g_{bc} = 0 \) and \( T_{ab}^c = 0 \).

Let’s work on the Palatini action (1) with the variational method. To see what we have when we vary \( e^a_I \), it is easier when we use
\[ \tilde{\eta}^{abcd} \epsilon_{IJKL} e^K_c e^L_d = -4 \sqrt{-g} e_I^a e_J^b \] (14)
where \( \tilde{\eta}^{abcd} \) is the Levi-Civita tensor density of weight 1 and \( \epsilon_{IJKL} \) is the volume element of the Minkowski metric \( \eta_{IJ} \). Both of them are -1 or 0 or 1 depending on their indices, so they are independent of \( e^a_I \). With this, varying \( e^a_I \) gives
\[ \tilde{\eta}^{abcd} \epsilon_{IJKL} F_{cd}^{KL} = 0. \] (15)

For \( w_a^{IJ} \), we need the following formula:
\[ \delta F_{ab}^{IJ} = 2 D_{[a} \delta w_{b]}^{IJ} - T_{ab}^c \delta w_c^{IJ}. \] (16)

We can see immediately that the variational calculations of the Palatini action with respect to \( w_a^{IJ} \) are very similar to those of our simple action (7). If we assume \( D_a g_{bc} = 0 \) and \( T_{ab}^c = 0 \), varying \( w_a^{IJ} \) gives us
\[ D_a (e_I^a e_J^b) = 0. \] (17)
To figure out what (17) gives, we express $D_a$ in terms of the unique, torsion-free generalized derivative operator $\nabla_a$ compatible with $e^a_I$, and $C_{ai}^J$ defined by

$$D_aH_I = \nabla_aH_I + C_{ai}^JH_J. \quad (18)$$

If we express (17) with (18), it is not difficult to show that $C_{ai}^J = 0$. There are 24 homogeneous linear equations of 24 variables $C_{ai}^J$, so $C_{ai}^J = 0$. Since $D_a = \nabla_a$, $F_{ab}^{IJ}$ becomes the Riemann curvature tensor and we have the 3+1 vacuum Einstein’s equations from (15). If we do not assume $D_ag_{bc} = 0$ but only assume $T_{ab}^c = 0$, we have another term as in (10):

$$\sqrt{-g}D_a(e^a_I e^b_J) + (\partial_a \sqrt{-g} + \sqrt{-g}A_{ca}^c)(e^a_I e^b_J) = 0. \quad (19)$$

If we express (19) with (18), there are 24 inhomogeneous linear equations of 24 variables $C_{ai}^J$, so $C_{ai}^J \neq 0$. In this case, $D_a e^b_I$ is not zero. The Palatini action does not become the Einstein-Hilbert action and we do not have the Einstein’s equations. Finally if we assume $D_ag_{bc} = 0$ but do not assume $T_{ab}^c = 0$, varying $w_{IJ}^a$ gives us

$$2D_a(e^a_I e^b_J) + e^c_J e^b_J T_{ac}^b + e^c_J e^b_J T_{ca}^c = 0. \quad (20)$$

We do not have $D_a e^b_I = 0$ as the above and therefore we do not have the Einstein’s equations. If we assume $D_a e^b_I = 0$, we multiply $e^b_I$ to both sides and obtain $T_{ac}^c = 0$. Thus we have $T_{ab}^c = 0$. In this case we have the Einstein’s equations.

Since $D_a e^b_I = 0$ means $D_ag_{bc} = 0$, we can see that we must assume $D_ag_{bc} = 0$ to have the Einstein’s equations from the Palatini action. Because this condition is assumed from the beginning, it must be preserved in quantization. We also need to assume either $T_{ab}^c = 0$ or $D_a e^b_I = 0$ to have the Einstein’s equations, which should be also preserved in quantization. The conditions $D_ag_{bc} = 0$ and $T_{ab}^c = 0$ are what Einstein assumed when he constructed General Relativity [2]. With these two conditions, geodesic is an extremal length between two spacetime points, which is related to the Principle of Equivalence. On the other hand, assuming $D_a e^b_I = 0$ is based on Riemannian geometry.

Considering that $T_{ab}^c$ and $w_{IJ}^a$ have the same number of components, our results will hold for any action in terms of $e^a_I$ and $w_{IJ}^a$ which describes General Relativity. If we include the cosmological constant and the standard model fields to the Palatini action, we expect that the results are the same. Because two approaches are equivalent classically, it is unlikely that only one approach describes General Relativity. Our results will impose restrictions on the standard model action in curved spacetime. Finally our results make a clear distinction between the tetrad description and the metric description of General relativity, which will be shown in the Hamiltonian formulation. Although two descriptions
are equivalent classically, quantum theories are different. Two approaches we found in the Lagrangian formulation will directly go through the Hamiltonian formulation, giving a hint to solve 2nd class constraints. This interplay between two formulations might guide us to complete this program. We will work on these and which approach leads to quantum gravity in future.

We thank Sang Pyo Kim for useful discussions and financial support. This work was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2015R1D1A1A01060626).

References

[1] E. S. Abers and B. W. Lee, Phys. Rept. C9, 1-141 (1973).

[2] H. C. Ohanian and R. Ruffini, Gravitation and Spacetime, (Norton, New York, 1994).