Shape Invariant Natanzon Potentials from Potential Algebra

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Abstract

Using the underlying algebraic structures of Natanzon potentials, we discuss conditions that generate shape invariant potentials. In fact, these conditions give all the known shape invariant potentials corresponding to a translational change of parameters. We also find that while the algebra for the general Natanzon potential is \(SO(2, 2)\), a subgroup \(SO(2, 1)\) suffices for all the shape invariant problems of Natanzon type.

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I. Introduction

Natanzon potentials\[1\] form a complete set of exactly solvable potentials of nonrelativistic quantum mechanics for which the Schrödinger equation reduces to the hypergeometric equation. In refs. \[2, 3\], Alhassid et al. have studied the group structure of these quantum mechanical systems, related their Hamiltonians to the Casimir operator of an underlying SO(2,2) algebra, and determined all their quantum states by group theoretical methods.

In supersymmetric quantum mechanics (SUSY-QM)\[4\] one applies a different algebraic method. The exactly solvable problems in SUSY-QM are described by superpotentials \(W(x,a)\) that obey a special integrability condition,

\[
W^2(x,a_0) + \frac{d}{dx}W(x,a_0) = W^2(x,a_1) - \frac{d}{dx}W(x,a_1) + R(a_0) ,
\]

known as shape invariance\[5, 6\]. \(R(a_0)\) is a constant and the parameter \(a_1\) is a function of \(a_0\), i.e. \(a_1 = f(a_0)\). For a shape invariant system, the entire spectrum can be determined algebraically by a procedure similar to that of the one dimensional harmonic oscillator, without ever referring to the underlying differential equations. Although most of the known shape invariant potentials (SIP) belong to the Natanzon class, there are a few exceptions \[7, 8\].

In a previous work\[9\], we have shown that problems for which 1) there is a translational change of parameters \(a_1 = f(a_0) = a_0 + \text{constant}\) and 2) \(R(a_0)\) is a linear function of \(a_0\), the shape invariance condition of eq. \(1\) implies the presence of a \(SO(2,1)\) dynamical algebra. Hence these problems are solvable by either method. As shown in ref. \[4\], potentials of this type includes Morse, Scarf I, Scarf II, and generalized Pöschl-Teller. However, the formalism used in ref. \[9\] is not readily extendable to other shape invariant potentials.

In this paper, we generalize our work of ref. \[9\]. However, we have used a different approach here which is closely based on the work of Alhassid et. al. \[4\]. The authors of ref. \[4\] have shown that a Hamiltonian with a general Natanzon potential has a \(SO(2,2)\) symmetry. We study here the algebra of Natanzon potentials that are also shape invariant. We find that general Natanzon potentials when subjected to a further constraint give the entire set of shape invariant potentials. The shape invariant potentials that reduce to the confluent hypergeometric equation can be obtained as a limit\[10\]. We also find that while the algebra for the general Natanzon potential is \(SO(2,2)\), a subgroup \(SO(2,1)\) suffices for
all the shape invariant problems of Natanzon type.

Thus, this paper connects all the shape invariant potentials of translational type \( a_1 = a_0 + \) constant) to an algebraic structure which has many interesting consequences. Some time ago it was discovered that spectra of potentials with translational shape invariance can be exactly determined by the supersymmetric WKB method [11], which usually only gives approximate results. The reason for this exactness was very puzzling. However, in light of this group theoretical connection, this result may not be that difficult to understand as various authors have demonstrated the exactness of WKB results on a group manifold [12].

In sec. II, we will quickly review the formalism of SUSY-QM and shape invariance. In sec. III, we will briefly describe our previous work [9] where we connected a subset of SIPs to \( SO(2,1) \) potential algebra. In sec. IV, we discuss the potential algebra of a general Natanzon potential. This section will closely follow ref. [2]. In sec. V, we identify conditions under which the general Natanzon potential reduces to a shape invariant potential. We then show that this condition has a finite number of solutions for shape invariant potentials; however, they generate all the known shape invariant potentials of translational type.

II. SUSY-QM and Shape Invariance

In this section, we very briefly describe supersymmetric quantum mechanics (SUSY-QM), and also show how SUSY-QM with shape invariance allows one to completely determine the spectrum of a quantum system. For a more detailed description, see ref. [4].

A quantum mechanical system given by a potential \( V_-(x) \) can alternatively be described by its ground state wavefunction \( \psi_0^{(-)} \). From the Schrödinger equation for the ground state wavefunction \( -\psi_0'' + V(x)\psi_0 = 0 \), it follows that the potential can be written as, \( V_-(x) = \left( \frac{\psi_0''}{\psi_0} \right) \), where prime denotes differentiation with respect to \( x \). [It is assumed that the potential is properly adjusted to make the ground state energy \( E_0 = 0 \)]. In SUSY-QM, it is customary to express the system in terms of the superpotential \( W(x) = -\left( \frac{\psi_0'}{\psi_0} \right) \). The ground state wavefunction is then given by \( \psi_0 \sim \exp \left( -\int_{x_0}^x W(x)dx \right) \); \( x_0 \) is an arbitrarily
chosen reference point. The Hamiltonian $H_-$ can now be written as

$$H_- = \left( -\frac{d^2}{dx^2} + V_-(x) \right) = \left( -\frac{d^2}{dx^2} + W^2(x) - \frac{dW(x)}{dx} \right) \quad (2)$$

$$= \left( -\frac{d}{dx} + W(x) \right) \left( \frac{d}{dx} + W(x) \right)$$

(We are using units with $\hbar$ and $2m = 1$.) In analogy with the harmonic oscillator raising and lowering operators, we have introduced two operators $A = \left( \frac{d}{dx} + W(x) \right)$, and and its Hermitian conjugate $A^+ = \left( -\frac{d}{dx} + W(x) \right)$. Thus $H_- = A^+ A$.

However, one can easily construct another Hermitian operator $H_+ = A A^\dagger$ and show that the eigenstates of $H_+$ are iso-spectral with excited states of $H_-$. The Hamiltonian $H_+$, with potential $V_+(x) = (W^2(x) + \frac{dW(x)}{dx})$, is called the superpartner of the Hamiltonian $H_-$. To show the iso-spectrality mentioned above, let us denote the eigenfunctions of $H_\pm$ that correspond to eigenvalues $E_n^\pm$, by $\psi_n^{(\pm)}$. For $n = 1, 2, \cdots$,

$$H_+ \left( A \psi_n^{(-)} \right) = A A^+ \left( A \psi_n^{(-)} \right) = A \left( A^+ A \psi_n^{(-)} \right) = AH_- \left( \psi_n^{(-)} \right) = E_n^- \left( A \psi_n^{(-)} \right) \quad (3)$$

Hence, except for the ground state which obeys $A \psi_0^{(-)} = 0$, all other states $\psi_n^{(-)}$ of $H_-$ there exists a state $\psi_{n-1}^{(+)} \propto A \psi_n^{(-)}$ of $H_+$ with exactly the same energy, i.e. $E_n^+ = E_{n+1}^-$, where $n = 0, 1, 2, \cdots$. Conversely, one also has $A^+ \psi_n^{(+)} \propto \psi_{n+1}^{(-)}$.

Thus, if the eigenvalues and the eigenfunctions of $H_-$ were known, one would automatically obtain the eigenvalues and the eigenfunctions of $H_+$, which is in general a completely different Hamiltonian.

Now, let us consider the special case where $V_-(x, a_0)$ is a shape invariant potential. For such systems, potentials $V_+(x, a_0) = V_-(x, a_1) + R(a_0)$. Their superpotential $W$ obeys the integrability condition of eq. (1). Since potentials $V_+(x, a_0)$ and $V_-(x, a_1)$ differ only in an additive constant, their common ground state wavefunction is given by $\psi_0^{(-)}(x, a_1) \sim \exp \left( -\int_{x_0}^{x} W(x, a_1) dx \right)$. The ground state energy of $H_+(x, a_0)$ is $R(a_0)$, because the ground state energy of $V_-(x, a_1)$ vanishes. Now using SUSY-QM algebra, the first excited state of $H_-(x, a_0)$ is given by $A^+(x, a_0) \psi_0^{(-)}(x, a_1)$ and the corresponding eigenvalue is $R(a_0)$. By iterating this procedure, the $(n+1)$-th excited state is given by

$$\psi_{n+1}^{(-)}(x, a_0) \sim A^+(a_0) A^+(a_1) \cdots A^+(a_n) \psi_0^{(-)}(x, a_n),$$

and corresponding eigenvalues are given by

$$E_0 = 0; \quad \text{and} \quad E_n^{(-)} = \sum_{k=0}^{n-1} R(a_k) \quad \text{for} \ n > 0.$$
To avoid notational complexity, we have suppressed the \( x \)-dependence of operators \( A(x, a_0) \) and \( A^+(x, a_0) \). Thus, for a shape invariant potential, one can obtain the entire spectrum of \( H \) by the algebraic methods of SUSY-QM.

Most of the known exactly solvable problems possess a spectrum generating algebra (SGA)\(^2\)\(^3\)\(^3\). We would like to find out if there is any connection between SGA and shape invariance of these systems. As we shall see later, the type of SGA that is most relevant to us is known as potential algebra, studied extensively by Alhassid et al.\(^2\)\(^3\). In potential algebra, the Hamiltonian of the system is written in terms of the Casimir operator \( (C_2) \) of the algebra, and the energy of states specified by an eigenvalue \( \omega \) of \( C_2 \) is fixed. Different states with fixed \( \omega \) represent eigenstates of a set of Hamiltonians that differ only in values of parameters and share a common energy. For a system with a \( SO(2,1) \) potential algebra, the different values of parameters are eigenvalues of operator \( J_3 \), chosen to form a complete set of commuting observables. This is very similar to the case of shape invariant potentials. In the next section, we will attempt to establish this connection in a more concrete fashion. In fact, for a set of solvable quantum mechanical systems we shall explicitly show that shape invariance leads to a potential algebra.

III. Shape Invariance and Connection to Algebra

Let us consider a generic shape invariant potential \( V_-(x, a_0) \) with a translational change of parameters \( a_{m+1} = a_m + \delta = a_0 + (m+1)\delta \), where \( \delta \) is a constant. For the superpotential \( W(x, a_m) \equiv W(x, m) \), the shape invariance condition is

\[
W^2(x, m) + W'(x, m) = W^2(x, m+1) - W'(x, m+1) + R(m) .
\]  

It is natural to ask whether the change of parameters can be formally accomplished by the action of a ladder type operator. With this in mind, we define an operator \( J_3 = -i\frac{\partial}{\partial \phi} \equiv -i\partial_\phi \), analogous to the \( z \)-component of the angular momentum operator. It acts upon functions in the space described by two coordinates \( x \) and \( \phi \), and its eigenvalues \( m \) will play the role of the parameter of the potential. We also define two more operators, \( J^- \) and its Hermitian conjugate \( J^+ \) by

\[
J^\pm = e^{\pm i\phi} \left[ \pm \frac{\partial}{\partial x} - W \left( x, -i \partial_\phi \pm \frac{1}{2} \right) \right] .
\]
The factors $e^{\pm i\phi}$ in $J^\pm$ ensure that they indeed behave as ladder operators for the quantum number $m$. Operators $J^\pm$ are basically of the same form as the $A^\pm$ operators of SUSY-QM, except that the parameter $m$ of the superpotential $W$ is replaced by operators $(J_3 \pm \frac{1}{2})$. Explicit computation shows that

$$[J_3, J^\pm] = \pm J^\pm,$$  \hspace{1cm} (6)

and hence operators $J^\pm$ change the eigenvalues of the $J_3$ operator by unity, similar to the ladder operators of angular momentum ($SO(3)$). Now let us determine the commutator $[J^+, J^-]$

$$[J^+, J^-] = J^+ J^- - J^- J^+ = \left[ -\frac{\partial^2}{\partial x^2} + W^2 \left( x, J_3 - \frac{1}{2} \right) - W' \left( x, J_3 - \frac{1}{2} \right) \right]$$

$$- \left[ -\frac{\partial^2}{\partial x^2} + W^2 \left( x, J_3 + \frac{1}{2} \right) + W' \left( x, J_3 + \frac{1}{2} \right) \right]$$

$$= -R \left( J_3 + \frac{1}{2} \right),$$ \hspace{1cm} (7)

where we have used the shape invariance condition \(^4\). Thus, we see that shape invariance enables us to close the algebra of $J_3$ and $J^\pm$ to

$$[J_3, J^\pm] = \pm J^\pm, \quad [J^+, J^-] = -R \left( J_3 + \frac{1}{2} \right).$$ \hspace{1cm} (8)

Now, if the function $R(J_3)$ is linear in $J_3$, the algebra of eq. (8) reduces to that of $SO(3)$ or $SO(2,1)$ \(^4\). Several SIP’s are of this type, among them are the Morse, Scarf I, Scarf II, and generalized Pöschl-Teller potentials. For these potentials, $R \left( J_3 + \frac{1}{2} \right) = 2 J_3 \frac{1}{2}$, and eq. (8) reduces to an $SO(2,1)$ algebra and hence establishes a connection between shape invariance and potential algebra. With a slightly different formalism, Balantekin arrived at a similar conclusion for these SIPs \(^{14}\). However, there are many other important systems like Coulomb, Eckart etc. where $R(a_0)$ is not linear in $a_0$, and these cases will be discussed later.

**IV. Differential Realization of SO(2,2)**

Before establishing a connection between a general Natanzon Hamiltonian and a $SO(2,2)$ potential algebra, we will discuss a realization of $SO(2,2)$ algebra in terms of differential operators on a $(2,2)$-hyperboloid. For consistency, we use the formalism and the notations of refs. 2, 3.
A \((2,2)\)-hyperboloid is defined by

\[ x_1 = \rho \cosh \chi \cos \phi, \quad x_2 = \rho \cosh \chi \sin \phi \]
\[ x_3 = \rho \sinh \chi \cos \theta, \quad x_4 = \rho \sinh \chi \sin \theta, \]

(9)

where \(\phi\) and \(\theta\) are rotation angles in the \(x_1, x_2\) and \(x_3, x_4\) planes respectively \([0 \leq \phi, \theta < 2\pi]\).

Six generators of the algebra, \(J_i\) and \(K_i\) \((i = 1, \cdots, 3)\) can be chosen as

\[ J_1 = (x_2 p_3 + x_3 p_2), \quad J_2 = -(x_1 p_3 + x_3 p_1), \quad J_3 = (x_1 p_2 - x_2 p_1) \]
\[ K_1 = (x_1 p_4 + x_4 p_1), \quad K_2 = (x_2 p_4 + x_4 p_2), \quad K_3 = (x_3 p_4 - x_4 p_3). \]

(10)

Operators \(p_i\) represent derivatives \(-i \frac{\partial}{\partial x_i}\). The algebraic relations obeyed by these operators are given by

\[ [J_1, J_2] = -i J_3, \quad [J_2, J_3] = i J_1, \quad [J_3, J_1] = i J_2, \]
\[ [K_1, K_2] = -i K_3, \quad [K_2, K_3] = i J_1, \quad [K_3, K_1] = i J_2, \]
\[ [J_1, K_2] = -i K_3, \quad [J_2, K_3] = i J_1, \quad [J_3, K_1] = i J_2, \]
\[ [K_1, J_2] = -i K_3, \quad [K_2, J_3] = i J_1, \quad [K_3, J_1] = i J_2. \]

(11)

The above algebra can be decomposed in terms of two commuting \(SO(2,1)\) algebras generated by

\[ A_i = \frac{1}{2} (J_i + K_i); \quad B_i = \frac{1}{2} (J_i - K_i). \]

(12)

These operators commute, i.e. \([A_i, B_j] = 0\). Using eqs. (9) and (11), the differential realization can be written explicitly as \([2,3]\)

\[ A^\pm \equiv A_1 \pm A_2 = \frac{1}{2} e^{\pm i(\phi+\theta)} \left[ \mp \frac{\partial}{\partial \chi} + \tanh \chi (-i \partial_\phi) + \coth \chi (-i \partial_\theta) \right]; \]
\[ A_3 = -\frac{i}{2} (\partial_\phi + \partial_\theta); \]
\[ B^\pm \equiv B_1 \pm B_2 = \frac{1}{2} e^{\pm i(\phi-\theta)} \left[ \mp \frac{\partial}{\partial \chi} + \tanh \chi (-i \partial_\phi) + \coth \chi (+i \partial_\theta) \right]; \]
\[ B_3 = -\frac{i}{2} (\partial_\phi - \partial_\theta). \]

The \(SO(2,1)\) algebra obeyed by these operators is

\[ [A_3, A^\pm] = \pm A^\pm, \quad [A^+, A^-] = -2 A_3. \]
and a similar one for the $B$'s. The Casimir operator $C_2$ is given by

$$C_2 = 2 \left( A_3^2 - A_+ A_- - A_3 \right) + 2 \left( B_3^2 - B_+ B_- - B_3 \right)$$

$$= \left[ \frac{\partial^2}{\partial \chi^2} + (\tanh \chi + \coth \chi) \frac{\partial}{\partial \chi} + \text{sech}^2 \chi (-i\partial_\phi)^2 - \text{cosech}^2 \chi (-i\partial_\theta)^2 \right]. \quad (14)$$

Operators $A_3$, $B_3$ and $C_2$ can be simultaneously diagonalized, and their actions on their common eigenstate are given by

$$C_2 |\omega, m_1, m_2 \rangle = \omega(\omega + 2) |\omega, m_1, m_2 \rangle ;$$
$$A_3 |\omega, m_1, m_2 \rangle = m_1 |\omega, m_1, m_2 \rangle ;$$
$$B_3 |\omega, m_1, m_2 \rangle = m_2 |\omega, m_1, m_2 \rangle . \quad (15)$$

It is worth mentioning at this point that the Casimir operator given above is indeed self-adjoint with respect to a measure $\sinh \chi \cosh \chi d\chi d\phi d\theta$.

V. The Natanzon Potentials

The Schrödinger equation for any Natanzon potential can be reduced by a point canonical transformation (a general similarity transformation followed by an appropriate change of independent variable) \[10, 15, 16\] to the hypergeometric equation. A general potential $U(r)$ of the Natanzon type is implicitly defined by \[1\]

$$U[z(r)] = \frac{-fz(1-z) + h_0(1-z) + h_1 z}{R(z)} - \frac{1}{2} \{z, r\} , \quad (16)$$

where $R(z) = az^2 + b_0 z + c_0 = a(1-z)^2 - b_1 (1-z) + c_1$ and $f, h_0, h_1, a, b_0, b_1, c_0, c_1$ are constants. The Schwarzian derivative $\{z, r\}$ is defined by

$$\{z, r\} \equiv \frac{d^2 z/dr^3}{dz/dr} - 3 \left[ \frac{d^2 z/dr^2}{dz/dr} \right]^2 . \quad (17)$$

The relationship between variables $z$ ($0 < z < 1$) and $r$ is implicitly given by

$$\left( \frac{dz}{dr} \right)^2 = \frac{2z(1-z)}{\sqrt{R(z)}} . \quad (18)$$

To avoid a singularity in $U(z(r))$, one assumes that $R(z)$ has no singularity in the domain $(0,1)$. The Schrödinger equation is given by

$$\left[ \frac{d^2}{dr^2} + \left( \frac{dz}{dr} \right)^2 I(z) + \frac{1}{2} \{z, r\} \right] = 0,$$
where
\[ I(z) = \frac{(1 - \lambda_0^2) (1 - z) + (1 - \lambda_1^2) z + (\mu^2 - 1) z (1 - z)}{4 z^2 (1 - z)^2} \]
and
\[
(1 - \mu^2) = a \ E - f, \quad (1 - \lambda_0^2) = c_0 \ E - h_0, \quad (1 - \lambda_1^2) = c_1 \ E - h_1.
\] (19)

To connect the Casimir operator \( C_2 \) of the \( SO(2, 2) \) algebra [eq. (14)] to the general Natanzon potential, we will first perform a similarity transformation on \( C_2 \) by a function \( F \) and then follow that up by an appropriate change of variable \( \chi \rightarrow g(r) \). Under the similarity transformation,
\[
\frac{d}{d\chi} \rightarrow F \frac{d}{d\chi} F^{-1} = \left( \frac{d}{d\chi} - \frac{\dot{F}}{F} \right), \quad \frac{d^2}{d\chi^2} \rightarrow \left( \frac{d^2}{d\chi^2} - \frac{2 \ddot{F}}{F} \frac{d}{d\chi} + \frac{2 \dot{F}^2}{F^2} - \frac{\ddot{F}}{F} \right),
\]
where dots represent derivatives with respect to \( \chi \). The Casimir operator \( C_2 \) of eq. (14) transforms as:
\[
C_2 \rightarrow \tilde{C}_2 = \left[ \frac{d^2}{dr^2} + \left( \tanh \chi + \coth \chi - \frac{2 \dot{F}}{F} \right) \frac{d}{dr} + \frac{2 F''}{F^2} - \frac{\ddot{F}}{F} - \left( \tanh \chi + \coth \chi \right) \frac{\dot{F}}{F} + \text{sech}^2 \chi \left( -i \partial_\phi \right)^2 - \text{cosech}^2 \chi \left( -i \partial_\theta \right)^2 \right].
\] (20)

Now, let us carry out a change of variable from \( \chi \) to \( r \) via \( \chi = g(r) \). We are going to denote differentiation with respect to \( r \) by a prime. Operators \( \frac{d}{d\chi} \) and \( \frac{d^2}{d\chi^2} \) transform as
\[
\frac{d}{d\chi} = \frac{1}{g'} \frac{d}{dr}, \quad \frac{d^2}{d\chi^2} = \frac{1}{g'^2} \left[ \frac{d^2}{dr^2} - \frac{g''}{g'} \frac{d}{dr} \right].
\]
The operator \( \tilde{C}_2 \) now transforms into
\[
\tilde{C}_2 = \frac{1}{g'^2} \left[ \frac{d^2}{dr^2} + \left\{ - \frac{g''}{g'} - \frac{2 F'''}{F} + g' \left( \tanh g + \coth g \right) \right\} \frac{d}{dr} \right. \\
+ \frac{2 F'^2}{F^2} - \frac{F''}{F} + \frac{F' \ g''}{F \ g'} \\
\left. \frac{F''}{F} \left( \tanh g + \coth g \right) + g'^2 \left( \text{sech}^2 g \left( -i \partial_\phi \right)^2 - \text{cosech}^2 g \left( -i \partial_\theta \right)^2 \right) \right].
\] (21)

In order for \( g'^2 \tilde{C}_2 \) to be a Schrödinger Hamiltonian, we require the expression inside the curly brackets in eq. (21) to vanish. This constrains the relationship between the two functions \( F \) and \( g \) to be
\[
- \frac{g''}{g'} - \frac{2 F'}{F} + g' \left( \tanh g + \coth g \right) = 0,
\] (22)
which yields
\[ F \sim \left( \frac{\sinh(2g)}{g'} \right)^{\frac{1}{2}}. \] (23)

Thus, the operator $\tilde{C}_2$, transforms into
\[ \tilde{C}_2 = \frac{1}{g'^2} \left[ \frac{d^2}{dr^2} + g'^2 \left( \frac{(1 - \tanh^2 g)^2 - 4 \tanh^2 g}{4 \tanh^2 g} \right) \right. \]
\[ + \frac{1}{2} \{g, r\} + g'^2 \left[ \text{sech}^2 g (-i\partial_\phi)^2 - \text{cosech}^2 g (-i\partial_\theta)^2 \right] \] . (24)

This Casimir operator now has a form of
\[ \tilde{C}_2 = -\frac{1}{g'^2} H, \]
where $H$ is a one-dimensional Hamiltonian with the potential $U(r)$ given by
\[ E - U(r) = g'^2 \left( \frac{(1 - \tanh^2 g)^2 - 4 \tanh^2 g}{4 \tanh^2 g} \right) + \frac{1}{2} \{g, r\} + g'^2 \left[ \text{sech}^2 g (-i\partial_\phi)^2 - \text{cosech}^2 g (-i\partial_\theta)^2 \right] . \]

Now, for this potential to take the form of a general Natanzon potential, we have to relate variables $g$ and $z$ in such a way that the potential in terms of $z$ is given by eq. (16). Since the potential has to be a ratio of two quadratic functions of $z$, this is accomplished with the identification $z = \tanh^2 g$, which leads to
\[ U(z(r)) = \frac{E}{R} \left( R \left[ -\frac{7}{4} + \frac{5}{2} z - \frac{7}{4} z^2 \right] - z(1-z)(-i\partial_\phi)^2 + (1-z)(-i\partial_\theta)^2 \right) - \frac{1}{2} \{z, r\} \]
\[ + \left\{ (a + b_0 + c_0)E - 1 \right\} / R(z) - \frac{1}{2} \{z, r\} . \] (25)

We have used
\[ g' = \frac{dg}{dz}, \quad z' = \frac{1}{2\sqrt{z(1-z)}} \frac{2z(1-z)}{R} = \sqrt{\frac{z}{R}} . \]

Now, with the following identification
\[ f = aE - \frac{7}{4} + (-i\partial_\phi)^2, \]
\[ h_0 = c_0E - \frac{7}{4} + (-i\partial_\theta)^2, \]
\[ h_1 = (a + b_0 + c_0)E - 1 , \] (26)
the potential of eq. (25) indeed has the form of a general Natanzon potential [eq. (16)].

VI. Shape Invariant Natanzon Potentials from Potential Algebra

At this point we go back to the operators $A^\pm$ [eq. (14)] and see how they transform under the similarity transformation given by $F \sim \left( \frac{\sinh(2\phi)}{\phi} \right)^{1/2} \sim \sqrt{z^*}$. This transformation carries operators $A^\pm$ to

$$A^\pm \rightarrow \tilde{A}^\pm = \frac{e^{\pm i(\phi+\theta)}}{2} \left[ \mp \left( \frac{d}{d\chi} + \frac{1}{2z'} \frac{dz'}{d\chi} - \frac{1}{2z} \frac{dz}{d\chi} \right) + \tanh(\chi) (\mp i\partial_\phi) + \coth(\chi) (\mp i\partial_\theta) \right].$$

(27)

If the expression $\left( \frac{1}{2z^*} \frac{dz'}{d\chi} - \frac{1}{2z} \frac{dz}{d\chi} \right)$ can be written as a linear combination of $\tanh(\chi)$ and $\coth(\chi)$, operators $\tilde{A}^\pm$ can be cast in a form similar to the operators $J^\pm$ of eq. (5), and connection with shape invariance is established.

Hence to get shape invariant potentials, we require,

$$\left( \frac{1}{2z^*} \frac{dz'}{d\chi} - \frac{1}{2z} \frac{dz}{d\chi} \right) = \alpha \tanh(\chi) + \beta \coth(\chi).$$

This leads to $z' = z^{1+\beta}(1-z)^{-\alpha-\beta}$, which is another restriction on the relationship between variables $z$ and $r$. Since these variables are already constrained by eq. (18), only a handful of solutions would be compatible with both restrictions. Thus $z(r)$’s that are compatible with both equations are given by

$$z^{1+\beta}(1-z)^{-\alpha-\beta} = \frac{2z(1-z)}{\sqrt{R(z)}},$$

(28)

where $R(z)$ is a quadratic function of $z$. After some computation, we find that there are only a finite number of values of $\alpha, \beta$ which satisfy eq. (28). These values are listed in Table 1, and they exhaust all known shape invariant potentials that lead to the hypergeometric equation. It is also interesting to note that while the potential algebra of a general Natanzon system is $SO(2,2)$, and requires two sets of raising and lowering operators $A^\pm$ and $B^\pm$, all known shape invariant potentials need only one such set. For all SIPs of Table 4.1 of ref. [1], one finds that all partner potentials are connected by change of just one independent parameter (although other parameters which don’t change are also present.) Thus there is a series of potentials that only differ in one parameter. From the potential algebra perspective, all these potentials differ only by the eigenvalue of an operator that is a linear combination of $A_3$ and $B_3$, and all are characterized by a common eigenvalue of $C_2$. 

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Thus, these shape invariant potentials can be associated with a $SO(2,1)$ potential algebra generated by operators $A^+, A^-$ and the same linear combination of $A_3$ and $B_3$.

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| $\alpha$ | $\beta$ | $z(r)$ | Superpotential | Potential |
|------|------|------|---------------|----------|
| 0    | 0    | $z = e^{-r}$ | $\tilde{m}_1 \coth \frac{r}{2} + \tilde{m}_2$ | Eckart |
| 0    | $-\frac{1}{2}$ | $z = \sin^2 \frac{r}{2}$ | $\tilde{m}_1 \csc r + \tilde{m}_2 \cot r$ | Gen. Pöschl-Teller trigonometric |
| 0    | $-1$  | $z = 1 - e^{-r}$ | $\tilde{m}_1 \coth \frac{r}{2} + \tilde{m}_2$ | Eckart |
| $-\frac{1}{2}$ | 0    | $z = \sech^2 \frac{r}{2}$ | $\tilde{m}_1 \text{cosech} r + \tilde{m}_2 \coth r$ | Pöschl-Teller II |
| $-\frac{1}{2}$ | $-\frac{1}{2}$ | $z = \tanh^2 \frac{r}{2}$ | $\tilde{m}_1 \tanh \frac{r}{2} + \tilde{m}_2 \coth \frac{r}{2}$ | Gen. Pöschl-Teller |
| $-1$  | 0    | $z = 1 + \tanh \frac{r}{2}$ | $\tilde{m}_1 \tanh \frac{r}{2} + \tilde{m}_2$ | Rosen Morse |

Table 1
Table Caption: Table 1 shows all allowed value of $\alpha$, $\beta$ and the superpotentials that they generate. Constants $\hat{m}_i$, ($i = 1, 2$) are linear functions of $m_1$ and $m_2$s of eq. (13).