Localization of Vector Field on Dynamical Domain Wall

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In the previous works (arXiv:1202.5375 and 1402.1346), the dynamical domain wall, where the four dimensional FRW universe is embedded in the five dimensional space-time, has been realized by using two scalar fields. In this paper, we consider the localization of vector field in three formulations. The first formulation was investigated in the previous paper (arXiv:1510.01099) for the $U(1)$ gauge field. In the second formulation, we investigate the Dvali-Shifman mechanism (hep-th/9612128), where the non-abelian gauge field is confined in the bulk but the gauge symmetry is spontaneously broken on the domain wall. In the third formulation, we investigate the Kaluza-Klein modes coming from the five dimensional graviton. In the Randall-Sundrum model, the graviton was localized on the brane. We show that the $(5, \mu)$ components ($\mu = 0, 1, 2, 3$) of the graviton are also localized on the domain wall and can be regarded as the vector field on the domain wall. There are, however, some corrections coming from the bulk extra dimension if the domain wall universe is expanding.

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I. INTRODUCTION

There is a long history in the scenarios that our universe could be a brane or domain wall embedded in a higher dimensional space-time [11, 22]. After the discovery of the so-called D-brane solution in string theories [3, 4], the brane world scenarios [3, 0] or the domain wall scenario [10, 27] have been well studied. In the studies, the models of the inflationary brane using the trace anomaly have been proposed [21, 22]. The brane can be regarded with a limit that the thickness of the domain wall vanishes. Recently a model where the general FRW universe is embedded in the five dimensional space-time with an arbitrary warp factor by using two scalar fields [24, 25, 1]

In this paper, we investigate the localization of the vector field in the model [24, 25] by using three formulations. The localization of the graviton has been shown in [25] and the localizations of the spinor field and vector field have been also investigated and shown in [27]. The first formulation of the localization of the vector field in this paper is just the review of the work in [27] about the $U(1)$ gauge field by using the action of the five dimensional vector field. The second formulation is an extension of the Dvali-Shifman mechanism in [28], where the non-abelian gauge field is confined in the four dimensional bulk but the gauge symmetry is spontaneously broken on the three dimensional domain wall. An extension of the work in [28] on the static four dimensional domain wall has been investigated in [29] and in this paper, we further extend the mechanism to the dynamical domain wall model. As the third formulation, we investigate the Kaluza-Klein modes coming from the five dimensional graviton. In the third formulation, we consider the vector field coming from the Kaluza-Klein reduction. In the second Randall-Sundrum model [1], the graviton was localized on the brane. The localized graviton can be regarded as a zero mode of the five dimensional graviton. We show that the $(5, \mu)$ components ($\mu = 0, 1, 2, 3$) of the graviton are also localized and can be regarded as the vector field on the four dimensional domain wall. We show that, however, there appear some corrections coming from the bulk extra dimension if the domain wall is dynamical.

In the next section, we briefly review on the formulation of the dynamical domain wall based on [24, 25]. In section III we also review on the localization of the vector field in [27]. In section IV we extend the formulation in [28] and [29] to the four dimensional dynamical domain wall model. In section V we consider the Kaluza-Klein vector field coming from the five dimensional graviton. The last section VI is devoted to the summary of the obtained results.

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1 This formulation is an extension of the formalism of the reconstruction of the domain wall [26]. Before the work, a formulation where only the warp factor of the domain wall is arbitrary has been proposed in [13], it has been proposed.
II. DOMAIN WALL MODEL WITH TWO SCALAR FIELDS

In [24, 25], the formulation of the dynamical domain wall model have been proposed by using two scalar fields. The formulation could be regarded as an extension of the formulation in [31]

The metric of the five dimensional space-time embedded a general spatially flat FRW universe with an arbitrary warp factor is given by

$$ds^2 = dw^2 + L^2 e^{u(w,t)}ds^2_{FRW}.$$  \hspace{1cm} (1)

Here $ds^2_{FRW}$ is the metric of the FRW universe,

$$ds^2_{FRW} = -dt^2 + a(t)^2 \left\{ dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right\}.$$  \hspace{1cm} (2)

In [24, 25], the following action with two scalar fields $\phi$ and $\chi$ were considered,

$$S_{\phi \chi} = \int d^5x \sqrt{-g} \left\{ -\frac{1}{2} G(\phi, \chi) \partial_M \phi \partial^M \phi - B(\phi, \chi) \partial_M \phi \partial^M \chi - \frac{1}{2} C(\phi, \chi) \partial_M \chi \partial^M \chi - V(\phi, \chi) \right\}.$$  \hspace{1cm} (3)

We can construct a model to realize the arbitrary metric (1) by using the model (3). The energy-momentum tensor for the scalar fields $\phi$ and $\chi$ in the model (3) are given by

$$T^\phi_{MN} = g_{MN} \left\{ -\frac{1}{2} A(\phi, \chi) \partial_M \phi \partial^M \phi - B(\phi, \chi) \partial_M \phi \partial^M \chi - \frac{1}{2} C(\phi, \chi) \partial_M \chi \partial^M \chi - V(\phi, \chi) \right\} + A(\phi, \chi) \partial_M \phi \partial_N \phi + B(\phi, \chi) (\partial_M \phi \partial_N \chi + \partial_N \phi \partial_M \chi) + C(\phi, \chi) \partial_M \chi \partial_N \chi.$$  \hspace{1cm} (4)

The variations of $\phi$ and $\chi$ give the following field equations,

$$0 = \frac{1}{2} A_\phi \partial_M \phi \partial^M \phi + A \nabla^M \partial_M \phi + A_\chi \partial_M \phi \partial^M \chi + \left( B_\chi - \frac{1}{2} C_\phi \right) \partial_M \chi \partial^M \chi + B \nabla^M \partial_M \chi - V_\phi ,$$ \hspace{1cm} (5)

$$0 = \left( -\frac{1}{2} A_\chi + B_\phi \right) \partial_M \phi \partial^M \phi + B \nabla^M \partial_M \phi + \frac{1}{2} C_\chi \partial_M \chi \partial^M \chi + C \nabla^M \partial_M \chi + C_\phi \partial_M \phi \partial^M \chi - V_\chi.$$ \hspace{1cm} (6)

Here $A_\phi = \partial A(\phi, \chi)/\partial \phi$, etc. By choosing $\phi = t$ and $\chi = w$, we obtain

$$T^{00}_0 = -\frac{e^{-2u(w,t)}}{2L^2} A - \frac{1}{2} C - V , \quad T^{ij}_i = \delta^{ij} \left( \frac{e^{-2u(w,t)}}{2L^2} A - \frac{1}{2} C - V \right) , \quad T^{55}_5 = \frac{e^{-2u(w,t)}}{2L^2} A + \frac{1}{2} C - V , \quad T^{05}_0 = B.$$ \hspace{1cm} (7)

By using the Einstein equation and the equations in (7), we find $A$, $B$, $C$, and $V$ can be expressed as follows,

$$A = \frac{L^2 e^{u(w,t)}}{\kappa^2} (G^1_1 - G^0_0) = \frac{L^2 e^{u(w,t)}}{\kappa^2} (G^2_2 - G^0_0) = \frac{L^2 e^{u(w,t)}}{\kappa^2} (G^3_3 - G^0_0) = \frac{1}{\kappa^2} \left( u - 2H + \frac{(\dot{u})^2}{2} + \dot{u}H \right) ,$$

$$B = \frac{1}{\kappa^2} G^5_5 = -\frac{3u'}{2\kappa^2 L^2 e^u} (\dot{u} + 2H) ,$$

$$C = \frac{1}{\kappa^2} (G^5_5 - G^1_1) = \frac{1}{\kappa^2} (G^5_5 - G^2_2) = \frac{1}{\kappa^2} (G^5_5 - G^3_3) = \frac{1}{\kappa^2} \left( -\frac{3}{2} u'' - \frac{1}{2 e^u} (\ddot{u} + 2H + (\dot{u})^2 + 5\dot{u}H + 6H^2) \right) ,$$

$$V = \frac{1}{\kappa^2} (G^0_0 + G^5_5) = \frac{1}{\kappa^2} \left( -\frac{3}{4} (u'' + 2(\dot{u})^2) + \frac{1}{4 L^2 e^u} (3\ddot{u} + 6H + 3(\dot{u})^2 + 15\dot{u} + 18H^2) \right) .$$ \hspace{1cm} (8)

Here $G_{\mu \nu}$ is the Einstein tensor. The explicit forms of $A(\phi, \chi)$, $B(\phi, \chi)$, $C(\phi, \chi)$, and $V(\phi, \chi)$ can be obtained by replacing $t$ and $w$ in the r.h.s. of Eqs. (3) by $\phi$ and $\chi$. The obtained expressions in the action (3) gives a model which realize the metric (1). Eqs. (5) and (6) are satisfied automatically, which can be seen by using the Bianchi identity $\nabla^\mu (R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu}) = 0$. 

\hspace{1cm} \hspace{1cm} 2 A similar procedure was also invented for the reconstruction of the FRW universe by single scalar model [31].
III. LOCALIZATION OF VECTOR FIELD

In this section, we review on the localization of the vector field by using the formulation in [27]. We consider the following action of the five dimensional vector field,

\[
S_V = \int d^5x \sqrt{-g} \left\{ -\frac{1}{4} F_{MN} F^{MN} - \frac{1}{2} m(\chi)^2 A_M A^M \right\}, \quad F_{MN} = \partial_M A_N - \partial_N A_M .
\]  

In the background (1) with (2), we assume that \( e^{u(t,w)} \) is given by the product of the \( t \)-dependent part and \( w \)-dependent part, \( e^{u(t,w)} = T(t) W(w) \),

\[
ds^2 = dw^2 + L^2 W(w) T(t) ds^2_{FRW}, \quad ds^2_{FRW} = -dt^2 + a(t)^2 \sum_{i=1}^3 (dx^i)^2 .
\]  

Under the assumption (10), the action (9) has the following form,

\[
S_V = \int d^5x \left\{ -\frac{1}{2} L^2 W(w) T(t) a(t)^3 F_{0a}^2 + \frac{1}{2} L^2 W(w) T(t) a(t) F_{0a}^2 + \frac{1}{2} a(t) F_{0a}^2 - \frac{1}{4} a(t)^{-1} F_{ij}^2 
- \frac{1}{2} m(\chi)^2 \left( L^4 W(w)^2 T(t)^2 a(t)^3 A_0^2 - L^2 W(w) T(t) a(t)^2 A_0^2 + L^2 W(w) T(t) a(t) A_0^2 \right) \right\} .
\]  

The variations of \( A_5, A_0, \) and \( A_i \) give the following equations,

\[
\begin{align*}
0 &= L^2 W(w) \partial_0 \left( T(t) a(t)^3 \left( \partial_5 A_0 - \partial_0 A_5 \right) \right) - L^2 W(w) T(t) a(t) \left( \partial_5 \partial_i A_i - \partial_i^2 A_5 \right) \\
&\quad - m(\chi)^2 L^4 W(w)^2 T(t)^2 a(t)^3 A_5 , \quad (12) \\
0 &= - L^2 T(t) a(t)^3 \partial_0 W(w) \left( \partial_5 A_0 - \partial_0 A_5 \right) + a(t) \left( \partial_0 \partial_i A_i - \partial_i^2 A_0 \right) + m(\chi)^2 L^2 W(w) T(t) a(t)^3 A_0 , \quad (13) \\
0 &= L^2 T(t) a(t) \partial_0 \left( W(w) \left( \partial_5 A_0 - \partial_0 A_5 \right) \right) - \partial_0 \left( a \left( \partial_0 A_i - \partial_i A_0 \right) \right) - a(t)^{-1} \left( \partial_0 \partial_j A_j - \partial_j^2 A_i \right) \\
&\quad - m(\chi)^2 L^2 W(w) T(t) a(t) A_i . \quad (14)
\end{align*}
\]

If we assume

\[
A_5 = 0 , \quad A_\mu = X(w) C_\mu \left( x^\nu \right) , \quad \mu, \nu = 0, 1, 2, 3 ,
\]

and choose

\[
m(\chi = w)^2 = \frac{(W(w) X'(w))^2}{W(w) X(w)} ,
\]

we rewrite Eqs. (12), (13), and (14) as follows,

\[
\begin{align*}
0 &= \partial_5 X(w) \left\{ \partial_0 \left( T(t) a(t)^3 C_0 \right) \right\} - T(t) a(t) \partial_5 C_i , \quad (17) \\
0 &= \partial_0 \partial_i C_i - \partial_i^2 C_0 , \quad (18) \\
0 &= \partial_0 \left( a(t) \left( \partial_0 C_i - \partial_i C_0 \right) \right) + a(t)^{-1} \left( \partial_0 \partial_j C_j - \partial_j^2 C_i \right) . \quad (19)
\end{align*}
\]

Eqs. (13) and (19) are nothing but the field equations of the vector field in four dimensions. On the other hand, Eq. (17) can be regarded as a gauge condition, which is a generalization of the Landau gauge, \( \partial^\mu A_\mu = 0 \).

By choosing \( X(w) \) decreases rapidly enough for large \( |w| \), \( A_\mu \) becomes normalizable. Then if we choose \( m(\chi) \) as in (10), the vector field localizes on the domain wall.

IV. DVALI-SHIFMAN MECHANISM

In [28], the non-abelian vector field on the three dimensional domain wall embedded in the four dimensional space-time was considered. In the bulk space-time, the vector field is confined but on the domain wall, the scalar field which generates the domain wall also change the potential of the Higgs field and there occurs the spontaneous breakdown of the gauge symmetry and massless \( U(1) \) gauge field appears on the domain wall. An extension of the scenario was proposed in [29], where the four dimensional domain wall in the five dimensional space-time was considered and by
the mechanism similar to that in \( \text{(28)} \), the standard model could be realized on the domain wall. In this section, we consider a similar mechanism on the dynamical domain wall.

We consider the following action for the SU(2) gauge field,

\[
S = \int d^5x \sqrt{-g} \left[ -\frac{1}{4} G^a_{MN} G^a_{MN} - \frac{1}{2} (D_M \eta^a)^2 + \frac{1}{2} \lambda (\eta^2 + \kappa^2 - v^2 + v^2 \tanh^2(m \phi))^2 \right],
\]

where \( G^a_{MN} = \partial_M A^a_N - \partial_N A^a_M + g f^{abc} A^b_M A^c_N. \)

Here \( G^a_{MN} \) is the field strength of the SU(2) field and we also include the scalar field \( \eta^a \) which is the adjoint representation of SU(2). The parameters \( \kappa \) and \( v \) have the dimension of mass and \( \lambda \) is a dimensionless positive parameter. We assume \( \kappa^2 - v^2 < 0 \) and also the metric in \( \text{(10)} \).

In the limit of \( |w| = |\chi| \to \infty \), because the potential for the scalar field \( \eta^a \) is given by \( \frac{1}{2} \lambda (\eta^2 + \kappa^2)^2 \), there does not occur the breakdown of the SU(2) gauge symmetry if the gauge coupling is strong enough. On the other hand, on the brane, \( w = \chi \sim 0 \), the potential becomes \( \frac{1}{2} \lambda (\eta^2 + \kappa^2 - v^2)^2 \) and because \( \kappa^2 - v^2 < 0 \), \( \eta^a \) has a vacuum expectation value

\[
\eta^a = \delta_{3a} \eta_0(w) = \delta_{3a} k \cosh^{-1}(mw),
\]

and therefore SU(2) gauge symmetry is spontaneously broken. By substituting the expression of \( \eta^a \) into \( \text{(21)} \) into the equation of the motion

\[
-\eta_0'' + 2\lambda (\eta^2 + \kappa^2 - v^2 + v^2 \tanh^2(m \phi)) \eta_0 = 0,
\]

we obtain

\[
\tanh^2(mw) \left( -2m^2 - 2\kappa^2 + 2\kappa v^2 \right) + m^2 + 2\lambda (k^2 + \kappa^2 - v^2) = 0,
\]

which tells

\[
k^2 = v^2 - 2\kappa^2, \quad m^2 = 2\lambda \kappa^2.
\]

Because the \( \eta^a \) has a vacuum expectation value,

\[
\left( \begin{array}{c} \langle \eta^1 \rangle \\ \langle \eta^2 \rangle \\ \langle \eta^3 \rangle \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \\ \left[(1 - \tanh^2(mw))v^2 - \kappa^2\right]^{1/2} \end{array} \right),
\]

the gauge field obtains a mass,

\[
\Delta \mathcal{L} = -\frac{1}{2} g^2 f^{abc} f^{ab'c'} g^{MN} A^b_M A^{b'}_N \eta^{a} \eta^{c'},
\]

Then by using \( \text{(26)} \), we find

\[
\Delta \mathcal{L} = -\frac{1}{2} g^2 g^{MN} (A^1_M A^1_N + A^2_M A^2_N) \langle \eta^3 \rangle^2
\]

\[
= -\frac{1}{2} g^2 g^{MN} (A^1_M A^1_N + A^2_M A^2_N) \left[(1 - \tanh^2(mw))v^2 - \kappa^2\right]
\]

\[
= -\frac{1}{2} \mu(w)^2 (1 - \delta_{3a}) g^{MN} A^a_M A^a_N.
\]

By substituting \( \text{(10)} \) into the action \( \text{(20)} \), we obtain

\[
S = \int d^5x \left[ \frac{1}{2} a(t)(G^a_{ij})^2 + \frac{1}{2} L^2 W(w) T(t) a(t)^3 (G^a_{05})^2 - \frac{1}{4} a(t)^{-1} (G^a_{ij})^2 - \frac{1}{2} L^2 W(w) T(t) a(t)(G^a_{ij})^2 - \frac{1}{2} \mu(w)^2 L^2 W(w) T(t) a(t) \left[-a(t)^2 \left((A^1_0)^2 + (A^2_0)^2\right) + ((A_1_0)^2 + (A_2_0)^2) + L^2 W(w) T(t) a(t)^2 \left((A^1_0)^2 + (A^2_0)^2\right)\right] \right].
\]

Then the equations for the gauge fields are given by

\[
0 = T(t) a(t)^3 \partial_5 \left[ L^2 W(w)(\partial_0 A^a_5 - \partial_5 A^a_0)\right] + a(t)(\partial_0 \partial_t A^a_0 - \partial_5^2 A^a_0).
\]
Therefore the massless gauge field appears on the domain wall.

In the Randall-Sundrum model, the massless graviton in four dimensions appears as a zero mode, or normalized

0 = T(t)a(t)∂t\left[ L^2 W(w) \partial_t A_5^5 - \partial_t A_5^5 \right] - \partial_t \left[ a(t)(\partial_t A_5^0 - \partial_t A_0^5) \right] - a^{-1}(\partial_t \partial_t A_5^0 - \partial_t^2 A_0^5)

For the massless vector field A_M^3, by choosing

A_9^3 = 0, \quad A_3^3 = X(w)C(\mu)(x^\nu), \quad \mu = 0, 1, 2, 3,

and

(W(w)X'(w))' = 0,

in the order of O(g^0), Eqs. 29, 30, and 31 reduce to the equations for the vector field and the gauge fixing condition,

0 = a(t)X(w)(\partial_t \partial_t C_1 - \partial_t^2 C_0),

0 = \partial_t \left[ a(t)(\partial_t C_1 - \partial_t C_0) \right] + a^{-1}(\partial_t \partial_t C_1 - \partial_t^2 C_0),

0 = X'(w)\left[ -T(t)a(t)\partial_t C_1 + \partial_t (T(t)a(t)^3 C_0) \right].

Therefore the massless gauge field appears on the domain wall.

In 29, the confinement in the bulk space-time was assumed but in the dimensions higher than four, there could be a phase transition and the confinement could occur only in the strong coupling region. Then we may consider the scalar field, which also plays a role of the gauge coupling. The scalar field plays a role on the coordinate w in the extra dimension and the gauge coupling can become strong and the confinement always occurs in the bulk space-time.

V. KALUZA-KLEIN REDUCTION

In the Randall-Sundrum model, the massless graviton in four dimensions appears as a zero mode, or normalized and localized mode, of the five dimensional graviton. On the other hand, in the standard Kaluza-Klein model, the vector field appears as the fluctuation h_\delta w of the (5, \mu) components of the metric (\mu = 0, 1, 2, 3). Therefore if the (5, \mu) components are also localized on the brane or the domain wall, the modes can be regarded as the vector field in four dimensions. In this section, we investigate the possibility that the vector field appears due to the Kaluza-Klein reduction.

We consider the fluctuation around the background space-time, g_{AB} = g^{(0)}_{AB} + h_{AB}. Then we obtain the following expressions,

$$\sqrt{-g} = \sqrt{-g^{(0)}} \left( 1 + \frac{1}{2} h^A_A + \frac{1}{8} (h^A_A)^2 - \frac{1}{4} h_{AB} h^{AB} \right),$$

$$R = R^{(0)} - R^{(0)}_{AB} h^{AB} + \nabla^{(0)} A^{(0)} B h_{AB} - \nabla^{(0)} A^{2} h^{A},$$

$$+ \frac{3}{4} \left( \nabla^{(0)} C h^{A} \right) \nabla^{(0)} B h^{CB} + \frac{3}{4} h^{AB} \nabla^{(0)} A \nabla^{(0)} B h^{D} - \frac{1}{2} \nabla^{(0)} A h^{AE} \nabla^{(0)} B h^{BE}.$$
Then by the variation of the action with respect to $h$, $\mu$,
Because we are interested in the (5, $\mu$) component, we put $h_{\mu\nu} = h_{55} = 0$. Then (5, $\mu$) component of Eq. (40) has the following form,

$$0 = \frac{1}{2\kappa^2} \left( \nabla_{\nu}^2 h_{AB} - \frac{1}{2} R_{\mu\nu}^5 h_{55} + \frac{1}{2} R_{\mu\nu}^{55} h_{55}^5 + \frac{1}{2} R_{\mu\nu}^{5\nu} h_{5\nu}^5 - 2 R_{\mu\nu}^{5\nu} h_{5\nu}^5 \right) - \mathcal{L}_m h_{5\mu}^5. \quad (41)$$

A. Localization on Flat Domain Wall

Before considering the FRW universe, we first consider the case that the four dimensional domain wall is flat as in the Randall-Sundrum model. Then the metric has the following form,

$$ds^2 = e^{u(w)}\eta_{\mu\nu}dx^\mu dx^\nu + dw^2. \quad (42)$$

Then Eq. (41) has the following form,

$$\left[ \frac{1}{2\kappa^2} \left( e^{-u} \partial_{\nu}^5 \partial_{\nu}^5 + \partial_{\nu}^5 \partial_{\nu}^5 + 2u' + u^2 \right) - \mathcal{L}_m \right] h_{5\mu}^5 = 0. \quad (43)$$

The deviation of (43) is given in the Appendix A. By assuming $h_{5\mu}(x^\nu, w) = N(w) A_\mu(x^\nu)$, we consider the following Lagrangian density of the scalar field instead of (3) (see Ref. [24]),

$$\mathcal{L}_m = -\frac{1}{2} C(\chi) \partial_A \chi \partial_A \chi - \mathcal{V}(\chi) = \frac{3}{2} u'' + \frac{3}{2} u'^2. \quad (44)$$

The Lagrangian density is given by putting $A(\phi, \chi) = B(\phi, \chi) = 0, \tilde{C}(\chi) = C(\phi, \chi)|_{\phi = 0}$, and $\mathcal{V}(\chi) = V(\phi, \chi)|_{\phi = 0}$. We also used (8) in the second equality in (44). Then we obtain,

$$\left( N e^{-u} \partial_{\nu}^5 \partial_{\nu}^5 + N'' + N' u'' - N u'' - 2N u'^2 \right) A_\mu = 0. \quad (45)$$

If we choose

$$N \propto e^u, \quad (46)$$

Eq. (45) coincides with the expression of the standard equation for the vector field in four dimensions,

$$\partial_\nu \partial^\nu A_\mu = 0. \quad (47)$$

For example, we consider the case, $u(w) = -2\sqrt{w^2 + w_0^2}$, we find $N \propto e^{-2\sqrt{w^2 + w_0^2}} \rightarrow e^{-2|w|}$ ($w_0 \rightarrow 0$) and therefore there occurs the localization of the vector field. If the number of the extra dimensions is not one but there are several extra dimensions and furthermore if the extra dimensions have a structure of the non-abelian group, there could appear the non-abelian gauge theory localized on the domain wall.
B. Localization on the Dynamical Domain Wall

We now consider the case that the domain wall is dynamical, that is, the FRW universe is embedded in five dimensional bulk space-time as in [1]. Then the equation for the graviton is given by

\[
0 = \frac{1}{2\kappa^2} \left\{ (\partial_x^2 + 2u'' + u'\partial_5 + u''') h_{5\mu} + e^{-u} \left( \hat{\nabla}^2 - \frac{4\dot{u}}{a} - \frac{a^2}{2} \right) h_{5\mu} + e^{-u} \left( \hat{\nabla}^2 - \frac{4\ddot{u}}{a} \right) h_{50} - \hat{u} \partial_\mu h_{50} \right\} - L_m h_{5\mu} .
\]  

(48)

The derivation of (48) is given in Appendix [13]. We assume the Lagrangian density \( L_m \) is given by [3]. By substituting the expression of \( L_m \) into (48) and using the gauge fixing condition \( \nabla^4 h_{A5} = 0 \), again, we find

\[
0 = \left( \partial_x^2 + u'\partial_5 - u'' - 2u'' \right) + e^{-u} \left( \hat{\nabla}^2 - \frac{2\ddot{u}}{a} \right) h_{5\mu} + e^{-u} \left( \dot{u} + \frac{u''}{2} + \frac{4\ddot{u}}{a} \right) h_{50} \delta_{\mu}^0 - e^{-u} \hat{u} \partial_\mu h_{50} .
\]  

(49)

By assuming \( h_{5\mu}(x', w) = N(w)A_{\mu}(x') \), Eq. (49) can be rewritten as

\[
0 = (N'' + N'u' - Nu'' - 2Nu') A_{\mu} + Ne^{-u} \left( \hat{\nabla}^2 - \frac{2\ddot{u}}{a} \right) A_{\mu} + \left( \dot{u} + \frac{u''}{2} + \frac{4\ddot{u}}{a} \right) A_0 \delta_{\mu}^0 - \hat{u} \partial_\mu A_0
\]

\[
= (N'' + N'u' - Nu'' - 2Nu') A_{\mu} + Ne^{-u} \left( \hat{\nabla}^2 - \frac{2\ddot{u}}{a} \right) A_{\mu} + \left( \dot{u} + \frac{u''}{2} + \frac{5\ddot{u}}{a} \right) A_0 \delta_{\mu}^0 - \hat{u} \hat{\nabla}_{\mu} A_0
\]

(50)

In case that \( u \) can be separated into a sum of \( \omega \)-dependent part \( u_\omega(w) \) and \( t \)-dependent part \( u_\mu(t) \), that is, \( u(w, t) = u_\omega(w) + u_\mu(t) \) as in [11] \((W(w) \propto e^{u_\omega(w)}, T(t) \propto e^{u_\mu(t)})\), if we choose \( N(w) \propto e^{u_\omega(w)} \) as in [10], the first term vanishes in (50) and we obtain

\[
0 = \left( \hat{\nabla}^2 - \frac{2\ddot{u}}{a} \right) A_{\mu} + \left( \dot{u} + \frac{u''}{2} + \frac{5\ddot{u}}{a} \right) A_0 \delta_{\mu}^0 - \hat{u} \hat{\nabla}_{\mu} A_0 .
\]  

(51)

On the other hand, the vector field in the four dimensional FRW space-time obeys the following equation,

\[
0 = \hat{\nabla}^2 A_{\mu} - \hat{\nabla}_\nu \hat{\nabla}_\mu A_{\nu} = \hat{\nabla}^2 A_{\mu} - \hat{\nabla}_\mu \hat{\nabla}_\nu A_{\nu} - \hat{R}_{\lambda\mu\nu} A_{\lambda}
\]

\[
= \hat{\nabla}^2 A_{\mu} - \hat{\nabla}_\mu \hat{\nabla}_\nu A_{\nu} - \left( \frac{\ddot{u}}{a} + \frac{u''}{2} \right) A_{\mu} - \left( \frac{2\ddot{u}}{a} - \frac{2\ddot{u}}{a} \right) A_0 \delta_{\mu}^0 .
\]  

(52)

There are some differences between (51) and (52) even if we choose the gauge condition \( \hat{\nabla}_{\mu} A_{\nu} = 0 \). Therefore the vector field can be localized on the domain wall even if dynamical but there appear some corrections from the extra dimensions.

VI. SUMMARY

In summary, the localization of vector field in the model [24, 25] has been investigated by using three formulations.

1. The first formulation was just the review of the work in [27], where we have used the action of the five dimensional vector field.

2. The second formulation was an extension of those in [26] and [28]. In this formulation, the non-abelian gauge field is confined in the bulk space-time but massless \( U(1) \) gauge field appears due to the spontaneous breakdown of the gauge symmetry. In [29], the confinement in the bulk space-time was assumed in the five dimensional bulk space-time. It is known, however that there could be a phase transition in the dimensions higher than four and the confinement could occur only in the strong coupling region. Then we may consider the model where the scalar field plays a role of the gauge coupling. The strong coupling phase can be always realized if the gauge coupling is given by the scalar field depending on the coordinate in the extra dimension and the coupling becomes strong enough and the confinement always occurs in the bulk space-time.
3. The third formulation was given by the Kaluza-Klein modes coming from the five dimensional graviton. In the second Randall-Sundrum model, the graviton was localized on the brane. The localized graviton can be regarded as a zero mode of the five dimensional graviton. We have shown that the $(5, \mu)$ components ($\mu = 0, 1, 2, 3$) of the graviton are also localized on the domain wall and can be regarded as the vector field on the four dimensional domain wall. We found that, however, some corrections appear from the bulk extra dimension if we consider the dynamical domain wall. An interesting point is that if we have several extra dimensions and the extra dimensions have a symmetry under the non-abelian group transformation, there could appear the non-abelian gauge theory localized on the domain wall.

Then it might be interesting to realize the GUT on the domain wall.

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**Appendix A: Derivation of (43)**

Here we consider the derivation of (43). We have the following expressions of the connection

$$\Gamma^5_{\mu\nu} = -\frac{u'}{2} e^{\mu\nu}, \quad \Gamma^\mu_{\nu5} = \frac{u'}{2} \delta^\mu_5,$$  \hspace{1cm} (A1)

the Riemann tensor,

$$R^{(0)}_{\mu\nu5} = -e^\mu \left( \frac{1}{2} u'' + \frac{1}{4} u'^2 \right) \eta_{\nu5},$$ \hspace{1cm} (A2)

Ricci tensor,

$$R^{(0)}_{\mu\nu} = -e^\mu \left( \frac{u''}{2} + u'^2 \right) \eta_{\mu\nu}, \quad R^{(0)}_{55} = -2u'' + u'^2,$$ \hspace{1cm} (A3)

and the scalar curvature

$$R^{(0)} = -4u'' - 5u'^2.$$ \hspace{1cm} (A4)

Then we find

\begin{align*}
g^{AB} \nabla_B \nabla_A (h_{5\mu}) = & g^{AB} \left( \partial_A \nabla_B (h_{5\mu}) - \Gamma^C_{AB} \nabla_C (h_{5\mu}) - \Gamma^C_{A5} \nabla_B (h_{C\mu}) - \Gamma^C_{A\mu} \nabla_B (h_{5C}) \right) \\
= & g^{AB} \partial_A \left( \nabla_B h_{5\mu} - \Gamma^C_{B5} h_{C\mu} - \Gamma^C_{B\mu} h_{5C} \right) - g^{AB} \Gamma^C_{AC} \left( \partial_C h_{5\mu} - \Gamma^D_{C5} h_{D\mu} - \Gamma^D_{C\mu} h_{5D} \right) \\
- & g^{AB} \Gamma^C_{A5} \left( \partial_B h_{C\mu} - \Gamma^D_{B5} h_{D\mu} - \Gamma^D_{B\mu} h_{5D} \right) - g^{AB} \Gamma^C_{A\mu} \left( \partial_B h_{5C} - \Gamma^D_{B5} h_{D\mu} - \Gamma^D_{B\mu} h_{5D} \right) \\
= & \partial^2_5 h_{5\mu} + e^{-u} \tilde{g}^{\alpha\beta} \partial_\alpha \partial_\beta h_{5\mu} - \partial_3 \Gamma^5_{55} h_{5\mu} - e^{-u} \tilde{g}^{\alpha\beta} \partial_\alpha \Gamma^\beta_{55} h_{5\mu} - 2\Gamma^5_{55} \partial_5 h_{5\mu} - e^{-u} \tilde{g}^{\alpha\beta} \Gamma^\beta_{55} \partial_\alpha h_{5\mu} \\
- & \partial_5 \Gamma^5_{5\mu} - e^{-u} \tilde{g}^{\alpha\beta} \partial_\alpha \Gamma^\beta_{55} h_{5\mu} - 2\Gamma^5_{55} \partial_5 h_{5\mu} - e^{-u} \tilde{g}^{\alpha\beta} \Gamma^\beta_{55} \partial_\alpha h_{5\mu} \\
- & \Gamma^C_{55} \partial_C h_{5\mu} - e^{-u} \tilde{g}^{\alpha\beta} \Gamma^C_{\alpha\beta} \partial_C h_{5\mu} + \Gamma^C_{55} \Gamma^5_{5\mu} + e^{-u} \tilde{g}^{\alpha\beta} \Gamma^C_{\alpha\beta} \Gamma^5_{5\mu} + \Gamma^C_{5\mu} \Gamma^5_{5\mu} + e^{-u} \tilde{g}^{\alpha\beta} \Gamma^C_{\alpha\beta} \Gamma^5_{5\mu} + \Gamma^C_{5\mu} \Gamma^5_{5\mu} + e^{-u} \tilde{g}^{\alpha\beta} \Gamma^C_{\alpha\beta} \Gamma^5_{5\mu} + \Gamma^C_{5\mu} \Gamma^5_{5\mu} + e^{-u} \tilde{g}^{\alpha\beta} \Gamma^C_{\alpha\beta} \Gamma^5_{5\mu} + \Gamma^C_{5\mu} \Gamma^5_{5\mu}
\end{align*}
\[ \begin{align*}
+ e^{-u} g^{\alpha \beta} \Gamma_{\alpha\beta}^{C} \nabla_{B} h_{5\mu} &= \partial_{\alpha}^{2} h_{5\mu} - \partial_{\alpha} \Gamma_{\beta_{\mu}}^{C} h_{5\gamma} - 2 \Gamma_{\beta_{\mu}}^{C} \partial_{\beta} h_{5\gamma} + \Gamma_{\alpha_{\mu}}^{C} \Gamma_{\beta_{\gamma}}^{D} h_{55} \\
+ e^{-u} g^{\alpha \beta} \left[ \partial_{\alpha} \partial_{\beta} h_{5\mu} - \partial_{\alpha} \Gamma_{\beta_{\mu}}^{C} h_{5\gamma} - 2 \Gamma_{\beta_{\mu}}^{C} \partial_{\alpha} h_{5\gamma} - \Gamma_{\alpha_{\mu}}^{C} \Gamma_{\beta_{\gamma}}^{D} h_{55} + \Gamma_{\alpha_{\mu}}^{C} \Gamma_{\beta_{\gamma}}^{D} h_{55} \right] \\
= \partial_{\alpha}^{2} h_{5\mu} - \frac{u''}{2} h_{5\mu} - u' \partial_{\nu} h_{5\mu} + \frac{u'^{2}}{4} h_{5\mu} \\
+ e^{-u} g^{\alpha \beta} \left[ \partial_{\alpha} \partial_{\beta} h_{5\mu} + \frac{u'}{2} e^{u} \eta_{\alpha \beta} \partial_{\beta} h_{5\mu} - \frac{u'^{2}}{4} e^{u} \eta_{\alpha \beta} h_{5\mu} - \frac{u'^{2}}{4} e^{u} \eta_{\alpha \beta} h_{5\mu} \right] \\
- \frac{u'^{2}}{4} e^{u} \eta_{\alpha \beta} h_{5\mu} = \left( \frac{\partial_{\alpha}^{2} - \frac{u''}{2} + u' \partial_{\beta} - \frac{5 u'^{2}}{2} }{2} \right) h_{5\mu} + e^{-u} g^{\alpha \beta} \partial_{\alpha} \partial_{\beta} h_{5\mu}, \\
- R^{(0)} h_{5\mu} + R^{(0)55} h_{5\mu} + R^{(0)\nu\mu} h_{5\nu} - 2 R^{(0)}_{\mu}^{5 \nu} h_{5\nu} \\
= - \left( -4 u'' - 5 u'^{2} \right) h_{5\mu} + (2 u'' - u'^{2}) h_{5\mu} - \left( \frac{u''}{2} + u'^{2} \right) \delta_{\mu}^{\nu} h_{5\nu} - 2 \left( \frac{u''}{2} - \frac{u'^{2}}{4} \right) \delta_{\mu}^{\nu} h_{5\nu} \\
= \left( \frac{u''}{2} + \frac{7 u'^{2}}{2} \right) h_{5\mu}.
\end{align*} \]

Then by substituting the above expressions into (11), we obtain Eq. (13).

**Appendix B: Derivation of (18)**

We now consider the derivation of (18). The expressions of the connection and curvatures are given by

\[ \begin{align*}
\Gamma_{\nu0}^{\mu} &= \left( \frac{\dot{a}}{a} + \frac{\dot{\mu}}{2} \right) \delta_{\nu}^{\mu} - \frac{\dot{a}}{a} \delta_{\nu}^{\mu}, \\
\Gamma_{ij}^{\mu} &= \left( \frac{\dot{a}}{a} + \frac{\dot{\mu}}{2} \right) \delta_{ij}^{\mu} + \Gamma_{\nu5}^{\mu} = -e^{u} \frac{u'}{2} \delta_{\nu}^{\mu}, \\
R^{(0)} &= -4 u'' - 5 u'^{2} + 3 e^{-u} \left( \frac{\dot{u}'}{2} + \frac{3 \dot{a} \dot{u}}{2} + \frac{2 \ddot{a}}{a} + \frac{2 \dot{a}^2}{a^2} \right), \\
R^{(0)55} &= -2 u'' - u'^{2}, \\
R^{(0)\nu\mu} &= \left[ -e^{u} \left( \frac{u''}{2} + u'^{2} \right) + \frac{3 \ddot{a} \dot{u}}{2a} \right] \delta_{\nu}^{\mu} + \ddot{R}_{0\nu}, \\
R^{(0)\nu\mu} &= \left[ -e^{u} \left( \frac{u''}{2} + u'^{2} \right) + \frac{2 \ddot{a} \dot{u}}{2a} \right] \delta_{\nu}^{\mu} + \dddot{R}_{0\nu}, \\
R^{(0)\mu5\nu} &= - \left( \frac{u''}{2} + \frac{u'^{2}}{4} \right) g_{\nu\mu}.
\end{align*} \]

We also find

\[ \nabla^{(0)2} h_{5\mu} = g^{AB} \left( \partial_{A} \nabla_{B} h_{5\mu} - \Gamma_{AB}^{C} \nabla_{B} h_{5\mu} - \Gamma_{B5}^{C} \nabla_{B} h_{5\mu} - \Gamma_{B5}^{C} \nabla_{B} h_{5\mu} \right) \]

\[ = g^{AB} \partial_{A} \left( \partial_{B} h_{5\mu} - \Gamma_{B5}^{C} \nabla_{B} h_{5\mu} - \Gamma_{B5}^{C} \nabla_{B} h_{5\mu} \right) - g^{AB} \Gamma_{AB}^{C} \left( \partial_{C} h_{5\mu} - \Gamma_{C5}^{D} h_{5\mu} - \Gamma_{C5}^{D} h_{5\mu} \right) \]

\[ - g^{AB} \Gamma_{AB}^{C} \left( \partial_{B} h_{5\mu} - \Gamma_{B5}^{C} \nabla_{B} h_{5\mu} - \Gamma_{B5}^{C} \nabla_{B} h_{5\mu} \right) - g^{AB} \Gamma_{AB}^{C} \left( \partial_{B} h_{5\mu} - \Gamma_{B5}^{C} \nabla_{B} h_{5\mu} - \Gamma_{B5}^{C} \nabla_{B} h_{5\mu} \right) \]

\[ = g^{AB} \partial_{A} \partial_{B} h_{5\mu} + g^{AB} \Gamma_{AB}^{C} \partial_{C} h_{5\mu} + g^{AB} \Gamma_{AB}^{C} \partial_{C} h_{5\mu} + g^{AB} \Gamma_{AB}^{C} \partial_{C} h_{5\mu} \]

\[ + g^{AB} \Gamma_{AB}^{C} \partial_{C} h_{5\mu} + 2 g^{AB} \Gamma_{AB}^{C} \Gamma_{B5}^{D} h_{55} + g^{AB} \Gamma_{AB}^{C} \Gamma_{B5}^{D} h_{55} \]

\[ = \partial_{\alpha}^{2} h_{5\mu} + e^{-u} \dot{g} \delta^{\alpha} \partial_{\beta} h_{5\mu} - \partial_{5} \Gamma_{55}^{5} h_{5\mu} - e^{-u} \dot{g} \delta^{\alpha} \partial_{5} \Gamma_{55}^{5} h_{5\mu} - 2 \Gamma_{55}^{5} h_{5\mu} - 2 e^{-u} \dot{g} \delta^{\alpha} \Gamma_{55}^{5} \partial_{5} h_{5\mu} \]

\[ - \partial_{5} \Gamma_{55}^{5} h_{5\gamma} - e^{-u} \dot{g} \delta^{\alpha} \partial_{5} \Gamma_{55}^{5} h_{5\gamma} - 2 \Gamma_{55}^{5} \partial_{5} h_{5\gamma} - 2 e^{-u} \dot{g} \delta^{\alpha} \Gamma_{55}^{5} \partial_{5} h_{5\gamma} \]

\[ - \Gamma_{55}^{5} \partial_{5} h_{5\mu} - e^{-u} \dot{g} \delta^{\alpha} \Gamma_{55}^{5} \partial_{5} h_{5\mu} + \Gamma_{55}^{5} \partial_{5} h_{5\mu} + e^{-u} \dot{g} \delta^{\alpha} \Gamma_{55}^{5} \partial_{5} h_{5\mu} + \Gamma_{55}^{5} \partial_{5} h_{5\mu} \]
We obtain
\[
\Gamma^{a}_{\beta \gamma} = \frac{1}{2} g^{aM} (g_{M\beta, \gamma} + g_{M\gamma, \beta} - g_{\beta \gamma, M})
\]
\[
= \frac{1}{2} g^{aM} (g_{\mu \beta, \gamma} + g_{\mu \gamma, \beta} - g_{\beta \gamma, \mu} + g_{\beta \mu, \gamma} \partial_{\gamma} u + g_{\beta \gamma, \mu} \partial_{\mu} u - g_{\beta \gamma, u} \partial_{\beta} u)
\]
\[
= \Gamma^{a}_{\beta \gamma} + \frac{1}{2} (\delta^{a}_{\gamma} \partial_{\gamma} u + \delta^{a}_{\beta} \partial_{\beta} u - g_{\beta \gamma, \mu} \hat{g}^{\alpha \beta} \partial_{\alpha} u)
\];
(B4)

we obtain
\[
\hat{g}^{\alpha \beta} \partial_{\alpha} \Gamma^{\gamma}_{\beta \mu} h_{5\gamma} = \hat{g}^{\alpha \beta} \partial_{\alpha} \left[ \frac{\Gamma^{\gamma}_{\beta \mu} + \frac{1}{2} \left( \delta^{\gamma}_{\beta} \partial_{\beta} u + \delta^{\gamma}_{\mu} \partial_{\mu} u - \hat{g}^{\alpha \mu} \hat{g}^{\gamma \beta} \partial_{\beta} u \right) }{h_{5\gamma}} \right]
\]
\[
= \hat{g}^{\alpha \beta} \partial_{\alpha} \hat{\Gamma}^{\gamma}_{\beta \mu} h_{5\gamma} + \frac{1}{2} (\partial_{\alpha} \partial_{\mu} u) \hat{g}^{\alpha \gamma} h_{5\gamma} + \frac{1}{2} \hat{g}^{\alpha \beta} (\partial_{\alpha} \partial_{\beta} u) h_{5\mu}
\]
\[
- \frac{1}{2} \hat{g}^{\alpha \beta} (\partial_{\alpha} \hat{g}^{\beta \mu})(\partial_{\mu} u) \hat{g}^{\gamma \beta} h_{5\gamma} - \frac{1}{2} (\partial_{\mu} \hat{g}^{\gamma \beta})(\partial_{\beta} u) h_{5\gamma} - \frac{1}{2} (\partial_{\beta} \hat{g}^{\gamma \beta})(\partial_{\mu} u) h_{5\gamma}
\]
\[
= \hat{g}^{\alpha \beta} \partial_{\alpha} \hat{\Gamma}^{\gamma}_{\beta \mu} h_{5\gamma} + \frac{1}{2} \hat{g}^{\alpha \beta} (\partial_{\alpha} \partial_{\beta} u) h_{5\mu}
\]
\[
= \hat{g}^{\alpha \beta} \partial_{\alpha} \Gamma^{\gamma}_{\beta \mu} h_{5\gamma} - \frac{1}{2} \hat{u} h_{5\mu}
\];
(B5)

\[
\hat{g}^{\alpha \beta} \Gamma^{\gamma}_{\beta \mu} \partial_{\alpha} h_{5\gamma} = \hat{g}^{\alpha \beta} \left[ \frac{\Gamma^{\gamma}_{\beta \mu} + \frac{1}{2} \left( \delta^{\gamma}_{\beta} \partial_{\beta} u + \delta^{\gamma}_{\mu} \partial_{\mu} u - \hat{g}^{\alpha \mu} \hat{g}^{\gamma \beta} \partial_{\beta} u \right) }{h_{5\gamma}} \right] \partial_{\alpha} h_{5\gamma}
\]
\[
= \hat{g}^{\alpha \beta} \Gamma^{\gamma}_{\beta \mu} \partial_{\alpha} h_{5\gamma} + \frac{1}{2} (\partial_{\alpha} \partial_{\mu} u) \hat{g}^{\alpha \gamma} \partial_{\alpha} h_{5\gamma} + \frac{1}{2} \hat{g}^{\alpha \beta} (\partial_{\alpha} \partial_{\beta} u) \partial_{\alpha} h_{5\mu} - \frac{1}{2} (\partial_{\beta} u) \hat{g}^{\gamma \beta} \partial_{\beta} h_{5\gamma}
\]
\[
= \hat{g}^{\alpha \beta} \Gamma^{\gamma}_{\beta \mu} \partial_{\alpha} h_{5\gamma} + \frac{1}{2} (\partial_{\alpha} \partial_{\mu} u) \hat{g}^{\alpha \gamma} \partial_{\alpha} h_{5\gamma} - \frac{1}{2} \hat{u} \partial_{\alpha} h_{5\mu} + \frac{1}{2} \hat{u} \partial_{\alpha} h_{5\mu}
\];
(B6)

\[
\hat{g}^{\alpha \beta} \Gamma^{\alpha \beta}_{\gamma} \partial_{\alpha} h_{5\mu} = \hat{g}^{\alpha \beta} \left[ \frac{\Gamma^{\alpha \beta}_{\gamma} + \frac{1}{2} \left( \delta^{\alpha}_{\beta} \partial_{\beta} u + \delta^{\alpha}_{\gamma} \partial_{\gamma} u - \hat{g}^{\alpha \beta} \hat{g}^{\gamma \beta} \partial_{\beta} u \right) }{h_{5\gamma}} \right] \partial_{\gamma} h_{5\mu} + \hat{g}^{\alpha \beta} \Gamma^{\alpha \beta}_{\gamma} \partial_{\gamma} h_{5\mu}
\]
\[
= \hat{g}^{\alpha \beta} \Gamma^{\alpha \beta}_{\gamma} \partial_{\gamma} h_{5\mu} + \frac{1}{2} (\partial_{\beta} u) \hat{g}^{\alpha \beta} \partial_{\gamma} h_{5\mu} + \frac{1}{2} (\partial_{\alpha} \partial_{\beta} u) \hat{g}^{\alpha \beta} \partial_{\gamma} h_{5\mu} - 2 (\partial_{\beta} u) \hat{g}^{\alpha \beta} \partial_{\gamma} h_{5\gamma} + \hat{g}^{\alpha \beta} \Gamma^{\alpha \beta}_{\gamma} \partial_{\gamma} h_{5\mu}
\]
\[
= \hat{g}^{\alpha \beta} \Gamma^{\alpha \beta}_{\gamma} \partial_{\gamma} h_{5\mu} + \frac{1}{2} (\partial_{\beta} u) \hat{g}^{\alpha \beta} \partial_{\gamma} h_{5\mu} - 2 (\partial_{\beta} u) \hat{g}^{\alpha \beta} \partial_{\gamma} h_{5\gamma} + \hat{g}^{\alpha \beta} \Gamma^{\alpha \beta}_{\gamma} \partial_{\gamma} h_{5\mu}
\]
\[
= \hat{g}^{\alpha \beta} \Gamma^{\alpha \beta}_{\gamma} \partial_{\gamma} h_{5\mu} + \frac{1}{2} (\partial_{\beta} u) \hat{g}^{\alpha \beta} \partial_{\gamma} h_{5\mu} - 2 (\partial_{\beta} u) \hat{g}^{\alpha \beta} \partial_{\gamma} h_{5\gamma} + \frac{1}{2} (\partial_{\beta} u) \hat{g}^{\alpha \beta} \partial_{\gamma} h_{5\mu}
\];
(B7)

We now find the explicit forms of Eqs. (B5), (B6), (B7), (B8), (B9), (B10), and (B11). Because
\[
\Gamma^{a}_{\beta \gamma} = \frac{1}{2} g^{aM} (g_{M\beta, \gamma} + g_{M\gamma, \beta} - g_{\beta \gamma, M})
\]
\[
= \frac{1}{2} g^{aM} (g_{\mu \beta, \gamma} + g_{\mu \gamma, \beta} - g_{\beta \gamma, \mu} + g_{\beta \mu, \gamma} \partial_{\gamma} u + g_{\beta \gamma, \mu} \partial_{\mu} u - g_{\beta \gamma, u} \partial_{\beta} u)
\]
\[
= \Gamma^{a}_{\beta \gamma} + \frac{1}{2} (\delta^{a}_{\gamma} \partial_{\gamma} u + \delta^{a}_{\beta} \partial_{\beta} u - g_{\beta \gamma, \mu} \hat{g}^{\alpha \beta} \partial_{\alpha} u)
\];
(B4)
\[+ \frac{1}{2}(\partial_{\alpha} u)\hat{g}^{\alpha \beta} \partial_{\beta} h_{55} + \frac{1}{4}(\partial_{\alpha} u)(\partial_{\mu} u)\hat{g}^{\alpha \beta} h_{55} + \frac{1}{4}(\partial_{\alpha} u)\hat{g}^{\alpha \gamma} (\partial_{\gamma} u)h_{5\mu} - \frac{1}{4}(\partial_{\mu} u)(\partial_{\nu} u)\hat{g}^{\nu \rho} h_{55} - 2(\partial_{\alpha} u)\hat{g}^{\gamma \lambda} \partial_{\gamma} h_{55} - (\partial_{\lambda} u)(\partial_{\gamma} u)\hat{g}^{\gamma \lambda} (\partial_{\gamma} u)h_{5\mu} + (\partial_{\nu} u)(\partial_{\rho} u)\hat{g}^{\nu \rho} h_{55}\]
\[+ \hat{g}^{\alpha \beta} \Gamma_{\alpha \beta \gamma} \partial_{\gamma} h_{55}\]
\[= \hat{g}^{\alpha \beta} \Gamma_{\alpha \beta \gamma} \partial_{\gamma} h_{55} + \frac{1}{2} \hat{g}^{\alpha \beta} \Gamma_{\alpha \beta \gamma} (\partial_{\mu} u)h_{5\gamma} + \frac{1}{2} \hat{g}^{\alpha \beta} \Gamma_{\alpha \beta \gamma} (\partial_{\nu} u)h_{5\mu} - \frac{1}{2} \hat{g}^{\alpha \beta} \Gamma_{\alpha \beta \gamma} (\partial_{\nu} u)(\partial_{\rho} u)\hat{g}^{\rho \sigma} h_{55}\]
\[- (\partial_{\alpha} u)\hat{g}^{\alpha \gamma} \partial_{\gamma} h_{55} - \frac{1}{2} (\partial_{\alpha} u)\hat{g}^{\alpha \beta} (\partial_{\beta} u)h_{5\mu} + \hat{g}^{\alpha \beta} \Gamma_{\alpha \beta \gamma} \partial_{\gamma} h_{55}\]
\[= \hat{g}^{\alpha \beta} \Gamma_{\alpha \beta \gamma} \partial_{\gamma} h_{55} + \frac{3\hat{\alpha} u}{2a} h_{50} + \frac{3\hat{\alpha} u}{2a} h_{55} - \frac{3\hat{\alpha} u}{2a} h_{50} h_{55}\]
\[+ \hat{g}^{\alpha \beta} \Gamma_{\alpha \beta \gamma} \partial_{\gamma} h_{55} - \frac{\hat{\alpha} u}{a} h_{50} h_{55} + \frac{5\hat{\alpha} u}{2a} h_{55} + \frac{\hat{\alpha} u}{2a} h_{55} - \hat{u}^2 h_{55},\]  
\[\hat{g}^{\alpha \beta} \Gamma_{\alpha \beta \gamma} \partial_{\gamma} h_{55} = -e^u u^2 h_{55},\]  
\[\hat{g}^{\alpha \beta} \Gamma_{\alpha \beta \gamma} \partial_{\gamma} h_{55} = -e^u u^2 h_{55},\]  
\[\hat{g}^{\alpha \beta} \Gamma_{\alpha \beta \gamma} \partial_{\gamma} h_{55} = \hat{g}^{\alpha \beta} \Gamma_{\alpha \beta \gamma} \partial_{\gamma} h_{55} + \hat{g}^{\alpha \beta} \Gamma_{\alpha \beta \gamma} \partial_{\gamma} h_{55}\]
\[= \hat{g}^{\alpha \beta} \Gamma_{\alpha \beta \gamma} \partial_{\gamma} h_{55} + \frac{1}{2} \hat{g}^{\alpha \beta} \partial_{\alpha} (\partial_{\beta} u)\hat{g}^{\alpha \beta} h_{55} + \frac{1}{2} \hat{g}^{\alpha \beta} \partial_{\alpha} (\partial_{\beta} u)\hat{g}^{\alpha \beta} h_{55} - \frac{1}{2} \hat{g}^{\alpha \beta} \partial_{\alpha} (\partial_{\beta} u)\hat{g}^{\alpha \beta} h_{55}\]
\[+ \hat{g}^{\alpha \beta} \Gamma_{\alpha \beta \gamma} \partial_{\gamma} h_{55} + \frac{1}{2} \hat{g}^{\alpha \beta} \Gamma_{\alpha \beta \gamma} (\partial_{\mu} u)h_{5\gamma} + \frac{1}{2} \hat{g}^{\alpha \beta} \Gamma_{\alpha \beta \gamma} (\partial_{\nu} u)h_{5\mu} - \frac{1}{2} \hat{g}^{\alpha \beta} \Gamma_{\alpha \beta \gamma} (\partial_{\nu} u)(\partial_{\rho} u)\hat{g}^{\rho \sigma} h_{55}\]
\[+ \hat{g}^{\alpha \beta} \Gamma_{\alpha \beta \gamma} \partial_{\gamma} h_{55} - \frac{\hat{\alpha} u}{a} h_{50} h_{55} + \frac{5\hat{\alpha} u}{2a} h_{55} + \frac{\hat{\alpha} u}{2a} h_{55} - \hat{u}^2 h_{55},\]  
\[\hat{g}^{\alpha \beta} \Gamma_{\alpha \beta \gamma} \partial_{\gamma} h_{55} = \hat{g}^{\alpha \beta} \Gamma_{\alpha \beta \gamma} \partial_{\gamma} h_{55} + \hat{g}^{\alpha \beta} \Gamma_{\alpha \beta \gamma} \partial_{\gamma} h_{55}\]
\[= \hat{g}^{\alpha \beta} \Gamma_{\alpha \beta \gamma} \partial_{\gamma} h_{55} + \frac{1}{2} \hat{g}^{\alpha \beta} \partial_{\alpha} (\partial_{\beta} u)\hat{g}^{\alpha \beta} h_{55} + \frac{1}{2} \hat{g}^{\alpha \beta} \partial_{\alpha} (\partial_{\beta} u)\hat{g}^{\alpha \beta} h_{55} - \frac{1}{2} \hat{g}^{\alpha \beta} \partial_{\alpha} (\partial_{\beta} u)\hat{g}^{\alpha \beta} h_{55}\]
\[+ \frac{3\hat{\alpha} u}{2a} h_{50} + \frac{3\hat{\alpha} u}{2a} h_{55} - \frac{3\hat{\alpha} u}{2a} h_{50} h_{55}\]
\[+ \frac{5\hat{\alpha} u}{2a} h_{55} + \frac{\hat{\alpha} u}{2a} h_{55} - \hat{u}^2 h_{55},\]  
\[\nabla^{(0)} h_{55} = \partial_{\beta} h_{55} - \frac{u^\mu}{2} h_{55} - u^\mu \partial_{5} h_{55} + \frac{u^2}{4} h_{55}\]
\[+ e^{-u} \left[ \hat{g}^{\alpha \beta} \partial_{\alpha} (\partial_{\beta} u)\hat{g}^{\alpha \beta} h_{55} - \hat{g}^{\alpha \beta} \partial_{\alpha} \Gamma_{\beta \gamma} h_{5\gamma} + \frac{\hat{u}}{2} h_{55} \right.\]
\[- 2 \hat{g}^{\alpha \beta} \Gamma_{\alpha \beta \gamma} \partial_{\gamma} h_{55} - \hat{u} \hat{g}^{\alpha \gamma} \partial_{\alpha} h_{55} \hat{\gamma} - \hat{u} \hat{\partial}_{\gamma} h_{55} \]
\[+ \frac{3\hat{\alpha} u}{2a} h_{50} + \frac{3\hat{\alpha} u}{2a} h_{55} - \hat{u}^2 h_{55} - \frac{u^2}{4} h_{55}\]
\[ \frac{\dot{u}^2}{2} h_{5\mu} + \frac{\dot{u}^2}{2} h_{50\delta^0_\mu} + \frac{2\dot{u}}{a} h_{50\delta^0_\mu} - \dot{u} \dot{g}^{\alpha\gamma} \partial_\alpha h_{5\gamma\delta^0_\mu} \].

(12)

Then by combining the above expressions, we find

\[
- R^{(0)} h_{5\mu} + R^{(0)\nu}_{\mu\nu} h_{5\nu} + 2 R^{(0)} g^{\mu\nu} h_{5\nu} \\
= \left[ 2u'' + 4u' - 3e^{-u} \left( \frac{\dot{u}^2}{2} + 3\dot{u} \frac{\dot{u}}{a} + \frac{2\dot{u}^2}{a^2} \right) \right] h_{5\mu} \\
+ \left[ \frac{-u''}{2} - u' + e^{-u} \left( \frac{\dot{u}^2}{2} + \frac{5\dot{u}^2}{2a} \right) \right] h_{5\mu} + e^{-u} \left( \frac{\dot{u}}{2} - \frac{\dot{a}}{a} \right) h_{50\delta^0_\mu} + e^{-u} \dot{R}^{\mu}_{\mu\nu} h_{5\nu} \\
+ 2 \left( \frac{-u''}{2} + \frac{u'^2}{4} \right) h_{5\mu} \\
= \left[ \frac{5u''}{2} + \frac{7u'^2}{2} - e^{-u} \left( \frac{5\dot{u}}{2} + \dot{u} + 13\dot{a} \frac{\dot{u}}{a} + 6\dot{a} \frac{\dot{a}}{a^2} \right) \right] h_{5\mu} + e^{-u} \left( \frac{\dot{u}}{2} - \frac{\dot{a}}{a} \right) h_{50\delta^0_\mu}. 
\]

(13)

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