ON THE COHOMOLOGY OF THE LOSEV–MANIN MODULI SPACE

JONAS BERGSTRÖM AND SATOSHI MINABE

Abstract. We determine the cohomology of the Losev–Manin moduli space \( \overline{M}_{0,2|n} \) of pointed genus zero curves as a representation of the product of symmetric groups \( \mathbb{S}_2 \times \mathbb{S}_n \).

Introduction

The Losev–Manin moduli space \( \overline{M}_{0,2|n} \) was introduced in [6] and it parametrizes stable chains of projective lines with marked points \( x_0 \neq x_\infty \) and \( y_1, \ldots, y_n \), where the points \( y_1, \ldots, y_n \) are allowed to collide, but not with \( x_0 \) nor \( x_\infty \), see Definition 1.1. In [6] this moduli space was denoted by \( \overline{L}_n \), here we have adapted the notation used in [8]. There is a natural action of \( \mathbb{S}_2 \times \mathbb{S}_n \) on \( \overline{M}_{0,2|n} \) by permuting \( x_0, x_\infty \) and \( y_1, \ldots, y_n \) respectively. This makes the cohomology \( H^*(\overline{M}_{0,2|n}, \mathbb{Q}) \) into a representation of \( \mathbb{S}_2 \times \mathbb{S}_n \). The aim of this note is to determine the character of this representation.

The moduli space \( \overline{M}_{0,2|n} \) can also be described as a moduli space of weighted pointed curves which were studied by Hassett [3, Section 6.4]. In this terminology it is the moduli space of genus 0 curves with 2 points of weight 1 and \( n \) points of weight \( 1/n \), and it would be written \( \overline{M}_{0,A} \) where \( A = (1, 1/n, \ldots, 1/n) \).

Another interesting aspect of the space \( \overline{M}_{0,2|n} \) is that it has a structure of toric variety. It is proved in [6] that \( \overline{M}_{0,2|n} \) is isomorphic to the smooth projective toric variety associated with the convex polytope called the permutahedron. This toric variety is obtained by an iterated blow-up of \( \mathbb{P}^{n-1} \) formed by first blowing up \( n \) general points, then blowing up the strict transforms of the lines joining pairs among the original \( n \) points, and so on up to \((n-3)\)-dimensional hyperplanes, see [4, §4.3]. With this perspective, the action of \( \mathbb{S}_2 \times \mathbb{S}_n \) can be seen in the following way. The \( \mathbb{S}_n \)-action comes from permuting the \( n \)-points of the blow-up, and the action of \( \mathbb{S}_2 \) comes from the Cremona transform of \( \mathbb{P}^{n-1} \) induced by the group inversion of the torus \((\mathbb{C}^*)^{n-1} : (t_1, \ldots, t_{n-1}) \mapsto (t_1^{-1}, \ldots, t_{n-1}^{-1}) \).

Alternatively, we can view our moduli space \( \overline{M}_{0,2|n} \) as the toric variety \( X(A_{n-1}) \) associated to the fan formed by Weyl chambers of the root system of type \( A_{n-1} \) (\( n \geq 2 \)), see [1]. The cohomology of \( X(A_{n-1}) \) is a representation of the Weyl group \( W(A_{n-1}) \cong \mathbb{S}_n \) and this representation was studied in [9, 2, 12, 5]. On the other hand, \( X(A_{n-1}) \) has another automorphism coming from that of the Dynkin diagram. This automorphism together with the action of the Weyl group corresponds precisely to the \( \mathbb{S}_2 \times \mathbb{S}_n \)-action on \( \overline{M}_{0,2|n} \).

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The cohomology of the moduli space \( \overline{M}_{0,2|n} \) has also been studied by mathematical physicists, since it corresponds to the solutions of the so-called commutativity equations. For this perspective we refer to \([6, 10]\) and the references therein.

The outline of the paper is as follows. In Section \(1\) we define \( \overline{M}_{0,2|n} \) and we state some known results on its cohomology. Our main result is Theorem \(2.3\) where we give a formula for the \( S_2 \times S_n \)-equivariant Poincaré-Serre polynomial of \( \overline{M}_{0,2|n} \). The main theorem is formulated in Section \(2\) and it is proved in Section \(3\). In Section \(4\) we present a formula for the generating series of the \( S_2 \times S_n \)-equivariant Poincaré-Serre polynomial of \( \overline{M}_{0,2|n} \). In Appendix \(A\) we then show that the result of Procesi in \([9]\) on the \( S_n \)-equivariant Poincaré-Serre polynomial is in agreement with our result. Finally in Appendix \(B\) we list the \( S_2 \times S_n \)-equivariant Poincaré-Serre polynomial of \( \overline{M}_{0,2|n} \) for \( n \) up to 6.

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1. The moduli space \( \overline{M}_{0,2|n} \)

In this note, a curve means a compact and connected curve over \( \mathbb{C} \) with at most nodal singularities and the genus of a curve is the arithmetic genus.

**Definition 1.1.** For \( n \geq 1 \), let \( \overline{M}_{0,2|n} \) be the moduli space of genus 0 curves \( C \) with \( n+2 \) marked points \((x_0, x_\infty|y_1, \ldots, y_n)\) satisfying the following conditions:

(i) all the marked points are non-singular points of \( C \),

(ii) \( x_0 \) and \( x_\infty \) are distinct,

(iii) \( y_1, \ldots, y_n \) are distinct from \( x_0 \) and \( x_\infty \),

(iv) the components corresponding to the ends of the dual graph contain \( x_0 \) or \( x_\infty \),

(v) each component has at least three special (i.e. marked or singular) points.

**Remark 1.2.** In (iii) above, \( y_i \) and \( y_j \) are allowed to coincide. The conditions imply that the dual graph of \( C \) is linear and that each irreducible component must contain at least one marked point in \((y_1, \ldots, y_n)\). This means that \( C \) is a chain of projective lines of length at most \( n \).

The moduli space \( \overline{M}_{0,2|n} \) is a nonsingular projective variety of dimension \( n-1 \), see \([6\) Theorem 2.2\]. It has an action of \( S_2 \times S_n \) by permuting the marked points \((x_0, x_\infty|y_1, \ldots, y_n)\).

1.1. Cohomology of \( \overline{M}_{0,2|n} \). The cohomology ring \( H^*(\overline{M}_{0,2|n}, \mathbb{Q}) \) was studied in \([6]\). It is algebraic, i.e., all the odd cohomology groups are zero and \( H^*(\overline{M}_{0,2|n}, \mathbb{Q}) \) is isomorphic to the Chow ring \( A^*(\overline{M}_{0,2|n}, \mathbb{Q}) \), see \([6\) Theorem 2.7.1\]. The Poincaré-Serre polynomials

\[
E_{2|n}(q) = \sum_{i=0}^{n-1} \dim_{\mathbb{Q}} H^{2i}(\overline{M}_{0,2|n}, \mathbb{Q}) q^i \in \mathbb{Z}[q],
\]
were also computed, see [6] Theorem 2.3.

The action of \( S_2 \times S_n \) on \( \overline{M}_{0,2|n} \) gives the cohomology \( H^*(\overline{M}_{0,2|n}, \mathbb{Q}) \) a structure of \( S_2 \times S_n \) representation. In [9], Procesi computed the \( S_n \)-equivariant Poincaré-Serre polynomial of the toric variety \( X(A_{n-1}) \) (which is isomorphic to \( \overline{M}_{0,2|n} \)), see Appendix A.

Throughout this note the coefficients of all cohomology groups will be \( \mathbb{Q} \).

2. Statement of the result

2.1. Partitions. A partition \( \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots) \) is a non-increasing sequence of non-negative integers which contains only finitely many non-zero \( \lambda_i \)'s. The number \( l(\lambda) \) of positive entries is called the \textit{length} of \( \lambda \). The number \( |\lambda| := \sum \lambda_i \) is called the \textit{weight} of \( \lambda \). If \( |\lambda| = n \) we say that \( \lambda \) is a partition of \( n \). We denote by \( \mathcal{P}(n) \) the set of partitions of \( n \) and by \( \mathcal{P} \) the set of all partitions. A sequence

\[
w \cdot \lambda = (\lambda_{w(1)}, \lambda_{w(2)}, \ldots), \quad w \in S_{l(\lambda)},
\]

obtained by permuting the non-zero elements of \( \lambda \) is called an ordered partition of \( n \). The number \( c_{\lambda} \) of distinct ordered partitions obtained from \( \lambda \) is given by

\[
c_{\lambda} = \frac{l(\lambda)!}{\text{Aut}(\lambda)},
\]

where \( \text{Aut}(\lambda) \) is the subgroup of \( S_{l(\lambda)} \) consisting of the permutations which preserve \( \lambda \). Let \( m_k(\lambda) := \# \{ i \mid \lambda_i = k \} \), we then have

\[
\text{Aut}(\lambda) = \prod_{k \geq 1} (m_k(\lambda)!) .
\]

With this notation a partition \( \lambda \) can also be written as \( \lambda = [1^{m_1(\lambda)} 2^{m_2(\lambda)} \cdots] \). For \( \lambda \in \mathcal{P}(n) \) and \( \mu \in \mathcal{P}(m) \) we then define \( \lambda + \mu \in \mathcal{P}(m+n) \) by \( m_k(\lambda + \mu) := \# \{ i \mid \lambda_i = k \} + \# \{ i \mid \mu_i = k \} \).

2.2. Symmetric functions. For proofs of the statements in this section see for instance [7].

Let \( \Lambda^y := \lim_{n \to \infty} \mathbb{Z}[y_1, \ldots, y_n]^{S_n} \) be the ring of symmetric functions. Similarly we define \( \Lambda^{x|y} := \Lambda^x \otimes \Lambda^y \). It is known that \( \Lambda^x \otimes \mathbb{Q} = \mathbb{Q}[p_1^y, p_2^y, \ldots] \) where \( p_k^y \) are the power sums in the variable \( y \). For \( \lambda \in \mathcal{P} \), we set \( p_{\lambda}^y := \prod_i p_{\lambda_i}^y \).

For a representation \( V \) of \( S_n \), we define \( \text{ch}^y_{\rho}(V) := \frac{1}{n!} \sum_{w \in S_n} \text{Tr}_V(w)p_{\rho(w)}^y \in \Lambda^y \),

where \( \rho(w) \in \mathcal{P}(n) \) is the partition of \( n \) which represents the cycle type of \( w \in S_n \). Similarly we define, for a \( S_2 \times S_n \) representation \( V \),

\[
\text{ch}^{x|y}_{2|n}(V) := \frac{1}{2(n!)} \sum_{(v, w) \in S_2 \times S_n} \text{Tr}_V((v, w))p_{\rho(v)}^x p_{\rho(w)}^y \in \Lambda^{x|y}.
\]

Recall that irreducible representations of \( S_n \) are indexed by \( \mathcal{P}(n) \). For \( \lambda \in \mathcal{P}(n) \), let \( V_\lambda \) be the irreducible representation corresponding to \( \lambda \) and define the Schur polynomial

\[
s_{\lambda}^y := \text{ch}^y_{\rho}(V_\lambda) \in \Lambda^y.
\]
In the following we will use that, if $V_i$ are representations of $S_n$, for $1 \leq i \leq k$, then
\[
\text{ch}^y_{\sum_{i=1}^n} \left( \text{Ind}_{S_{n_1} \times \ldots \times S_{n_k}}^{S_{n_1+\ldots+n_k}} (V_1 \otimes \ldots \otimes V_k) \right) = \prod_{i=1}^k \text{ch}^y_{n_i} (V_i),
\]
\[
\text{ch}^y_{n_1n_2} \left( \text{Ind}_{S_{n_1} \times S_{n_2}}^{S_{n_1+n_2}} (V_1 \otimes V_2 \otimes \ldots \otimes V_2) \right) = \text{ch}^y_{n_1} (V_1) \circ \text{ch}^y_{n_2} (V_2),
\]
where $\sim$ denotes the wreath product, that is, $S_{n_1} \sim S_{n_2} := S_{n_1} \times (S_{n_2})^{n_1}$ where $S_{n_1}$ acts on $(S_{n_2})^{n_1}$ by permutation, see [Appendix A, p. 158]. Plethysm is an operation $\circ : \Lambda^y \times \Lambda^y \rightarrow \Lambda^y$ which we will extend to an operation $\circ : \Lambda^y \times \Lambda^y[q] \rightarrow \Lambda^y[q]$ by putting $p^n \circ q = q^n$.

2.3. The main theorem.

**Definition 2.1.** The $S_2 \times S_n$-equivariant Poincaré-Serre polynomial of $\overline{M}_{0,2|n}$ is defined by
\[
E_{S_2 \times S_n}(q) := \sum_{i=0}^{n-1} \text{ch}^{y,q}_{2|n} (H^2(\overline{M}_{0,2|n})) q^i \in \Lambda^y[q].
\]

The usual Poincaré-Serre polynomial $E_{2|n}(q)$ is recovered from the equivariant one by
\[
\frac{\partial^2}{\partial (p_1^y)^2} \frac{\partial^n}{\partial (p_1^y)^n} E_{S_2 \times S_n}(q) = E_{2|n}(q).
\]

We will make some ad-hoc definitions in order to formulate an explicit formula for $E_{S_2 \times S_n}(q)$. The proof will then furnish an explanation to these definitions.

**Definition 2.2.** First put $g_0^y := 1$, then for any $n \geq 1$ and any (unordered) partition $\lambda$ put
\[
f_\lambda^y := \sum_{i=0}^{n-1} (-1)^i s_{(n-i,1)}^y q^{n-1-i}, \quad F_\lambda^y := \prod_{j=1}^{\ell(\lambda)} f_{\lambda_j}^y, \quad g_\lambda^y := \sum_{i=0}^{n-1} s_{(n-i,1)}^y q^{n-1-i}.
\]

**Theorem 2.3.** We then have
\[
E_{S_2 \times S_n}(q) = \frac{1}{2} (p_1^y)^2 \sum_{\lambda \in \mathcal{P}(n)} c_\lambda F_\lambda^y + \frac{1}{2} p_2^y \sum_{k=0}^{[n/2]} g_{n-k}^y \sum_{\mu \in \mathcal{P}(k)} c_\mu (p_2^y \circ F_\mu^y).
\]

Results for $1 \leq n \leq 6$ obtained from (2.1) are listed in Appendix B.

3. Proof of Theorem 2.3

3.1. Stratification of $\overline{M}_{0,2|n}$. For $k \geq 0$, we denote by $\Delta_{n,k}$ the closed subset of $\overline{M}_{0,2|n}$ consisting of curves with at least $k$ nodes. Let $\Delta_{n,k}^* := \Delta_{n,k} \setminus \Delta_{n,k+1}$ be the open part of $\Delta_{n,k}$ which corresponds to curves with exactly $k$ nodes. It is easy to see that $\Delta_{n,k} \neq \emptyset$ only for $0 \leq k \leq n-1$ and that $\Delta_{n,n-1}^* = \Delta_{n,n-1} = \{pt\}$. Note that $\Delta_{n,k}^*$ is preserved by the $S_2 \times S_n$-action. Hence its cohomology $H^*(\Delta_{n,k}^*)$ is a representation of $S_2 \times S_n$.

**Definition 3.1.** For an ordered partition $\lambda$ of $n$ with length $k+1$, let $\Delta_\lambda^* \subset \Delta_{n,k}^*$ correspond to all chains of projective lines of length $k+1$ such that precisely $\lambda_i$ of the marked points $(y_1, \ldots, y_n)$ are on the $i$th component (where the component with the marked point $x_0$ is the 1st component and the one with $x_\infty$ is the $(k+1)$th).
Note that $\Delta^*_\lambda$ is preserved by $S_n$ (but not necessarily by $S_2 \times S_n$, see below) and hence $H^*(\Delta^*_\lambda)$ is a representation of $S_n$.

**Lemma 3.2.** (i) $\Delta^*_{n, 0} \cong (\mathbb{C}^*)^{n-1}$. (ii) $\Delta^*_\lambda \cong \prod_{i=1}^{\ell(\lambda)} \Delta^*_{\lambda_i, 0}$.

(iii) We have a stratification

$$
\Delta^*_{n, k} = \bigsqcup_{\lambda = (\lambda_1, \ldots, \lambda_{k+1})} \Delta^*_\lambda,
$$

where $\lambda$ runs over all ordered partitions of $n$ with length $k + 1$.

**Proof.** (i) We have $\Delta^*_{n, 0} \cong (\mathbb{P}^1 \setminus \{0, \infty\})^n / \mathbb{C}^* \cong (\mathbb{C}^*)^n / \mathbb{C}^*$. (ii) Clear from the definition. (iii) This is found by considering the ways to distribute $n$ marked points $(y_1, \ldots, y_n)$ on the chain of projective lines of length $k + 1$ so that each irreducible component contains at least one of the points.

It follows from Lemma 3.2 (ii) that $\Delta^*_\lambda$ and $\Delta^*_{\lambda'}$ are $(S_n$-equivariantly) isomorphic when $\lambda$ and $\lambda'$ are different orderings of the same element in $P(n)$.

### 3.2. Cohomology of $\Delta^*_{n, 0}$

Since $\Delta^*_{n, 0} \cong (\mathbb{C}^*)^{n-1}$, $H^i(\Delta^*_{n, 0}) = 0$ for $i \geq n$, and moreover the mixed Hodge structure on $H^2(\Delta^*_{n, 0})$ is a pure Tate structure of weight $2(n-1-i)$, that is,

$$
H^2(\Delta^*_{n, 0}) = \mathbb{C}(n-1-i)^{\oplus (n-1)}.
$$

**Lemma 3.3.** For $0 \leq i \leq n-1$, we have

$$
\text{ch}_{2|n}^{x|y}(H^i(\Delta^*_{n, 0})) = \begin{cases} 
\sum_{s(i) \in (n-1, 1)} s_i \otimes s_{(n-i, 1)} & \text{if } i \text{ is even} \\
\sum_{s(i) \in (n-1, 1)} \otimes s_{(n-i, 1)} & \text{if } i \text{ is odd}.
\end{cases}
$$

**Proof.** Take an isomorphism $\Delta^*_{n, 0} = (\mathbb{C}^*)^n / \mathbb{C}^* \rightarrow (\mathbb{C}^*)^{n-1}$ given by

$$(z_1 : z_2 : \cdots : z_{n-1} : z_n) \mapsto \left(\frac{z_1}{z_n}, \ldots, \frac{z_{n-1}}{z_n}\right) =: (y_1, \ldots, y_{n-1}).$$

Then it is easy to see that $H^1(\Delta^*_{n, 0}) = \otimes_{i=1}^{n-1} \mathbb{C}(\frac{1}{2\pi i} \frac{dy_j}{y_j})$ is the standard representation $s_{(n-1, 1)}$ under the action of $S_n$. The action of $S_2$ is by interchanging 0 and $\infty$, that is by the isomorphism $t \mapsto 1/t$ of $\mathbb{P}^1$, which induces the action $(z_1 : \cdots : z_n) \mapsto (1/z_1 : \cdots : 1/z_n)$ on $\Delta^*_{n, 0}$. This tells us that $(y_1, \ldots, y_{n-1}) \mapsto (1/y_1, \ldots, 1/y_{n-1})$ and since $\frac{d(1/y)}{dy} = -\frac{dy}{y^2}$ we conclude that $H^1(\Delta^*_{n, 0}) = V_{(1,2)} \otimes V_{(n-1,1)}$. Using once more that $\Delta^*_{n, 0} \cong (\mathbb{C}^*)^{n-1}$ we get

$$
H^k(\Delta^*_{n, 0}) \cong \Lambda^k H^1(\Delta^*_{n, 0}) \cong \Lambda^k(V_{(1,2)} \otimes V_{(n-1,1)}) \cong (\otimes^k V_{(1,2)}) \otimes V_{(n-k, 1^k)}.
$$

**Corollary 3.4.** We have the equality

$$
\sum_{i=0}^{n-1} (-1)^i \text{ch}_{2|n}^{x|y}(H^2(\Delta^*_{n, 0}, i)) q^{n-1-i} = \frac{1}{2} (p_1^x)^2 f_n^y + \frac{1}{2} p_2^x g_n^y.
$$
Proof. By Poincaré duality, \( H^2_c(n-1-i)(\Delta_{n,0}^*) \cong H^i(\Delta_{n,0}^*) \), and since every irreducible representation of \( S_2 \times S_n \) is defined over \( \mathbb{Q} \), the dual representation is isomorphic to itself. The equality now follows from the lemma together with the relations \( 2s_{(2)}^* = (p_1^*)^2 + p_2^* \) and \( 2s_{(1,2)}^* = (p_1^*)^2 - p_2^* \). \( \square \)

3.3. Cohomology of \( \Delta_n^* \).

Corollary 3.5. For any ordered partition \( \lambda \) of \( n \) with length \( k + 1 \), \( H^c_c(2(n-k-1)-i)(\Delta_n^*) \) is a pure Hodge structure of weight \( 2(n-k-1-i) \).

Proof. This follows from Lemma 3.2 (ii) and the purity of the cohomology of \( \Delta_n^* \). \( \square \)

Corollary 3.6. For any ordered partition \( \lambda \) of \( n \) with length \( k + 1 \) we have

\[
\sum_{i=0}^{n-k-1} (-1)^i \text{ch}_n^y \left( H^c_c(2(n-k-1)-i)(\Delta_n^*) \right) q^{n-k-1-i} = F^y_n.
\]

Proof. From Lemma 3.2 (ii) we know that \( \Delta_n^* \cong \prod_{s=1}^{k+1} \Delta_{n,s}^* \), and on each \( \Delta_{n,s}^* \) we have an action of \( S_{n,s} \). The action of \( S_n \) on \( H^c_c(\Delta_n^*) \) will thus be the induced action from \( S_{\lambda,1} \times \ldots \times S_{\lambda_k+1} \) to \( S_n \). The result now follows from Corollary 3.4 by forgetting the action of \( S_2 \). \( \square \)

3.4. Proof of Theorem 2.3. We have the following long exact sequence of cohomology with compact support:

\[
\cdots \longrightarrow H^{n-k}(\Delta_{n,n+1}) \longrightarrow H^{n-k}(\Delta_{n,k}) \longrightarrow H^i(\Delta_{n,n}) \longrightarrow H^i(\Delta_{n,k+1}) \longrightarrow \cdots.
\]

This is an exact sequence of both mixed Hodge structures and \( S_2 \times S_n \)-representations. Therefore, using the exact sequence (3.1) inductively (this is just the additivity of the Poincaré-Serre polynomial) we get

\[
E_{S_2 \times S_n}(q) = \sum_{k=0}^{n-1} \left\{ \sum_{i=0}^{n-1} (-1)^i \text{ch}_n^y \left( H^c_c(2(n-i)-1)(\Delta_{n,k}^*) \right) q^{n-1-i} \right\}.
\]

We will now find a formula for \( \text{ch}_n^y(H^c_c(2(n-1)-i)(\Delta_{n,k}^*)) \). Let us begin with a strata \( \Delta_n^* \) for an ordered partition \( \lambda \) of \( n \) with length \( k + 1 \). The action of \( S_2 \) will then send the strata given by \( \lambda \) to the one given by \( \lambda' = (\lambda_{k+1}, \lambda_k, \ldots, \lambda_1) \). We will therefore divide into two cases.

Let us first assume that \( \lambda \neq \lambda' \). Since the action of \( S_2 \) interchanges the two components it will also interchange the factors of \( H^i_c(\Delta_n^* \cup \Delta_{n,k}^*) = H^i_c(\Delta_n^*) \oplus H^i_c(\Delta_{n,k}^*) \) and hence

\[
\text{ch}_n^y(H^i_c(\Delta_n^* \cup \Delta_{n,k}^*)) = (p_1^*)^2 \text{ch}_n^y(H^i_c(\Delta_n^*)).
\]

Let us now assume that \( \lambda = \lambda' \). We can then decompose our space as \( \Delta_n^* = \Delta_1^* \times \Delta_2^* \times \Delta_3^* \) where, if \( k + 1 = 2m \),

\[
\Delta_1^* := \prod_{i=1}^{m} \Delta_{n,i,0}, \quad \Delta_2^* := \{ pt \}, \quad \Delta_3^* := \prod_{i=m+1}^{2m} \Delta_{n,i,0}.
\]
and, if \( k + 1 = 2m + 1 \),

\[
\Delta_i^* := \prod_{i=1}^{m} \Delta_{i,0}^*, \quad \Delta_i^* := \Delta_{i,m+1,0}, \quad \Delta_i^* := \prod_{i=m+2}^{2m+1} \Delta_{i,0}^* .
\]

Let us put \( \alpha := \lambda_{m+1} \) if \( k+1 \) is odd and \( \alpha := 1 \) if \( k+1 \) is even, and in both cases \( \beta := \sum_{i=1}^{m} \lambda_i \).

The action of \( S_2 \) interchanges the \( (S_\beta \text{-equivariant}) \) isomorphic components \( \Delta_1^* \) and \( \Delta_3^* \) and sends the space \( \Delta_2^* \) to itself. Define the semidirect product \( S_2 \rtimes (S_\beta \times S_\alpha \times S_\beta) \) where \( S_2 \) acts as the identity on \( S_\alpha \) and permutes the factors \( S_\beta \times S_\beta \) (i.e. as the wreath product). The group \( S_2 \rtimes (S_\beta \times S_\alpha \times S_\beta) \) naturally embeds, by the map \( i \) say, in \( S_{2\beta + \alpha} = S_n \). Let us then put \( S_2 \rtimes (S_\beta \times S_\alpha \times S_\beta) \) in \( S_2 \times S_n \) by \( (\tau, \sigma) \mapsto (i(\tau, \sigma)) \), where \( \tau \in S_2 \) and \( \sigma \in S_\beta \times S_\alpha \times S_\beta \).

The action of \( S_2 \rtimes S_n \) on \( \Delta_1^* \) will then be the induced action from \( S_2 \rtimes (S_\beta \times S_\alpha \times S_\beta) \) acting naturally on \( \Delta_1^* \times \Delta_2^* \times \Delta_3^* \). Using Corollary 3.3 we conclude that

\[
\text{ch}_2^{\text{inv}}(H_1^*(\Delta_1^*)) = \frac{1}{2} p_1^2 f_0 \left( p_1^{p_1} \circ \text{ch}_1^\beta(H_1^*(\Delta_1^*)) \right) + \frac{1}{2} p_2^2 g_0 \left( p_2^{p_2} \circ \text{ch}_1^\beta(H_1^*(\Delta_1^*)) \right) .
\]

Applying formula (3.3) and formula (3.4) (and using Lemma 3.2 (iii) and Corollary 3.6) to equation (4.2), gives equation (2.1).

4. Generating series

4.1. Generating series of \( E_{S_2 \times S_n}(q) \). For any sequence of polynomials \( h_n \) we have the formal identity,

\[
1 + \sum_{n=1, n \lambda \in P(n)} c_{\lambda} \prod_{j=1}^{l(\lambda)} h_{\lambda_j} = 1 + \sum_{r=1}^{\infty} \left( \sum_{n=1}^{\infty} h_n \right)^r = \left( 1 - \sum_{n=1}^{\infty} h_n \right)^{-1} .
\]

The following proposition follows directly from (1.1) and Theorem 2.3.

**Proposition 4.1.** The generating series of \( E_{S_2 \times S_n}(q) \) is determined by,

\[
1 + \sum_{n=1}^{\infty} E_{S_2 \times S_n}(q) = \frac{1}{2} \left( p_1^2 \right)^2 \left( 1 - \sum_{n=1}^{\infty} f_n \right)^{-1} + \frac{1}{2} p_2^2 \left( 1 + \sum_{n=1}^{\infty} g_n \right) \left( 1 - \sum_{n=1}^{\infty} \left( p_2^2 \circ f_n \right) \right)^{-1} .
\]

**Remark 4.2.** Consider the moduli space \( M \) defined as in Definition 1.1 but with the additional demand that \( y_1, \ldots, y_n \) are distinct from each other. From Carel Faber we learnt the following formula, which is very similar to (1.2), for the generating series of the \( S_2 \times S_n \)-equivariant Poincaré-Serre polynomial of \( M \). Carel Faber obtained the formula as a direct consequence of an equality he learned from Ezra Getzler. These results have not been published.

Let \( h_{n+2}^y \) be the \( S_{n+2} \)-equivariant Poincaré-Serre polynomial of \( M_{0,n+2} \), the moduli space of genus 0 curves with \( n + 2 \) marked distinct points. The \( S_2 \times S_n \)-equivariant Poincaré-Serre polynomial of the open part of \( M \) (defined using the compactly supported Euler-characteristic) consisting of irreducible curves will then equal

\[
\frac{1}{2} \left( p_1^2 \right)^2 f_n^y + \frac{1}{2} p_2^2 g_n^y = \frac{1}{2} \left( p_1^2 \right)^2 \left( \frac{\partial^2 h_{n+2}^y}{\partial p_1^2} \right) + \frac{1}{2} p_2^2 \left( 2 \frac{\partial h_{n+2}^y}{\partial p_2} \right) .
\]
From the proof of Theorem 2.3 we see that replacing $f_n^y$ by $\tilde{f}_n^y$ (and $g_n^y$ by $\tilde{g}_n^y$) in equation (1.2) gives the $S_2 \times S_n$-equivariant Poincaré-Serre polynomial of $M$.

**Remark 4.3.** The polynomials $f_n^y$ and $g_n^y$ can be formulated in terms of $P_n^y(q) \in \Lambda^y[q]$, the Hall–Littlewood symmetric function associated to $\lambda \in \mathcal{P}$ (cf. [7, III-2]). This function is defined as the limit of the following symmetric polynomial:

$$P_\lambda(y_1, \ldots, y_k; q) = \sum_{w \in S_k/\mathcal{S}_k^\lambda} w \left( y_1^{\lambda_1} \cdots y_k^{\lambda_k} \prod_{\lambda_i > \lambda_j} \frac{y_i - qy_j}{y_i - y_j} \right),$$

where $\mathcal{S}_k^\lambda$ is the stabilizer subgroup of $\lambda$ in $S_k$ and $l(\lambda) \leq k$ is assumed. In the special case $\lambda = (n)$, where $n \geq 1$, the following formula is known (cf. [7, p. 214]):

$$P_{(n)}^y(q) = \sum_{r=0}^{n-1} (-q)^r s_{(n-r, 1^r)},$$

hence $f_n^y = q^{n-1}P_{(n)}^y(q^{-1})$ and $g_n^y = q^{n-1}P_{(n)}^y(-q^{-1})$.

### 4.2. Generating series of $E_{S_n}(q)$

The $S_n$-equivariant Poincaré-Serre polynomial of $\overline{M}_{0,2|n}$ equals

$$E_{S_n}(q) := \sum_{i=0}^{n-1} \text{ch}_n^i \left( H^2(\overline{M}_{0,2|n}) \right) q^i = \left. \frac{\partial^2}{\partial (pt)^2} \right|_{pt=1} E_{S_2 \times S_n}(q) \in \Lambda^y[q],$$

and so

$$1 + \sum_{n=1}^{\infty} E_{S_n}(q) = \left( 1 - \sum_{n=1}^{\infty} f_n^y \right)^{-1}.$$

Corollary 3.4 then tells us that the generating series of $E_{S_n}(q)$ is the multiplicative inverse of the generating series (in compactly supported cohomology) of $\Delta^*_{n,0}$, which is the open part of $\overline{M}_{0,2|n}$ consisting of irreducible curves.

If we set $q = 1$, the Hall–Littlewood function $P_{(n)}^y(q^{-1})$ becomes the $n$th power sum $p_n^y$ and formula (1.4) takes a very simple form. Let $\epsilon_{S_n} := E_{S_n}(1) \in \Lambda^y$, be the $S_n$-equivariant Euler characteristic of $\overline{M}_{0,2|n}$. We then have

$$1 + \sum_{n=1}^{\infty} \epsilon_{S_n} z^n = \left( 1 - \sum_{n=1}^{\infty} p_n^y z^n \right)^{-1}.$$

**Appendix A. Consistency with Procesi’s result**

### A.1. Procesi’s recursive formula

In [9], Procesi obtained the following recursive relation among $E_{S_n}(q)$ with respect to $n$.

**Theorem A.1** (Procesi). The $E_{S_n}(q)$ satisfy

$$E_{S_{n+1}}(q) = s_{(n+1)}^y \sum_{i=0}^{n} q^i + \sum_{i=0}^{n-2} s_{(n-i)}^y E_{S_{i+1}}(q) \left( \sum_{k=1}^{n-i-1} q^k \right).$$

As a corollary, we have the following formula which is obtained in [2, 11, 12].
Corollary A.2. We have

\[ 1 + \sum_{n=1}^{\infty} E_{S_n}(q)t^n = \frac{(1 - q)H(t)}{H(qt) - qH(t)}, \]

where \( H(t) = \sum_{r \geq 1} h_r t^r \) is the generating function of the complete symmetric functions in the variable \( y \).

A.2. Equivalence. The following proposition shows the equivalence between our result and Procesi’s by comparing Equation (4.4) and Equation (4.3) to Corollary A.2.

Proposition A.3. We have

\[ \frac{(1 - q)H(t)}{H(qt) - qH(t)} = \left\{ 1 - \sum_{r=1}^{\infty} q^{-1} P_r^{\mu} (q^{-1})(qt)^r \right\}^{-1}. \]

Proof. As in [7] pp. 209–210, we have

\[ \frac{H(qt)}{H(t)} = \prod_{i \geq 1} \frac{1 - ty_i}{1 - qt y_i} = 1 + (1 - q^{-1}) \sum_{i=1}^{n} \frac{y_i q t}{y_i} \prod_{j \neq i} \frac{y_i - y_j^{-1}}{y_i - y_j} = 1 + (1 - q^{-1}) \sum_{r=1}^{\infty} P_r^{\mu} (q^{-1})(qt)^r. \]

An easy manipulation of this formula gives the wanted equality. \( \square \)

APPENDIX B. \( E_{S_2 \times S_n}(q) \) for \( n \) up to 6

| \( n \) | \( E_{S_2 \times S_n}(q) \) |
|---|---|
| 1 | \( s_{(2)}^{(2)}s_{(1)}^{(2)} \) |
| 2 | \( (q + 1)s_{(2)}^{(2)}s_{(2)}^{(2)} \) |
| 3 | \( s_{(2)}^{(2)} \left( (q^2 + q + 1)s_{(3)}^{(2)} + q s_{(2,1)}^{(2)} \right) + q s_{(1)}^{(2)}s_{(3)}^{(2)} \) |
| 4 | \( s_{(2)}^{(2)} \left( (q^2 + 2q^2 + 2q + 1)s_{(4)}^{(1)} + (q^2 + q)s_{(3,1)}^{(2)} + (q^2 + q^2)s_{(2,2)}^{(1)} \right) + s_{(1,2)}^{(2)} \left( (q^2 + q)s_{(4)}^{(1)} + (q^2 + q)s_{(2,1)}^{(1)} \right) \) |
| 5 | \( s_{(2)}^{(2)} \left( (q^4 + 2q^3 + 4q^2 + 2q + 1)s_{(5)}^{(3)} + (2q^3 + 3q^2 + 2q)s_{(4,1)}^{(2)} + (q^2 + 2q^2 + q^3 + 2q^2 + q)s_{(3,2)}^{(1)} + 2q^2 s_{(3,1,2)}^{(1)} \right) \) |
| 6 | \( s_{(2)}^{(2)} \left( (q^5 + 3q^4 + 6q^3 + 6q^2 + 3q + 1)s_{(6)}^{(4)} + (2q^4 + 6q^3 + 6q^2 + 2q)s_{(5,1)}^{(2)} + (2q^4 + 7q^3 + 7q^2 + 2q)s_{(5,2)}^{(2)} + (q^3 + q^2)s_{(4,1)}^{(4)} + (2q^3 + 2q^2)s_{(3,2)}^{(2)} \right) + s_{(1,2)}^{(2)} \left( (2q^4 + 4q^3 + 4q^2 + 2q)s_{(6)}^{(4)} + (q^4 + 6q^3 + 6q^2 + 2q)s_{(5,1)}^{(2)} + (q^4 + 5q^3 + 5q^2 + q)s_{(4,2)}^{(4)} + (2q^3 + 2q^2)s_{(3,2)}^{(2)} \right) + s_{(1,2)}^{(2)} \left( (2q^4 + 4q^3 + 4q^2 + 2q)s_{(6)}^{(4)} + (q^4 + 3q^3 + 3q^2 + q)s_{(3,2)}^{(2)} \right) + (2q^3 + 2q^2)s_{(3,2,1)}^{(1)} \) |
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Matematiska institutionen, Stockholms Universitet, 106 91 Stockholm, Sweden.
E-mail address: jonasb@math.su.se

Department of Mathematics, Tokyo Denki University, 120-8551 Tokyo, Japan
E-mail address: minabe@mail.dendai.ac.jp