A vanishing theorem for Fano varieties in positive characteristic

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1 Introduction

Let \(k\) be a perfect field of characteristic \(p > 0\). \(W = W(k)\) is the ring of Witt vectors of \(k\) and \(K\) is its fraction field.

By a Fano variety or an anti-canonical variety over \(k\), we mean a smooth projective variety \(X\) over \(k\) such that the anti-canonical sheaf \((\Omega^d_X)^*\) is ample, where we denote by \(d\) the dimension of \(X\).

In this note, we wish to prove the following

**Theorem 1** Let \(X\) be a Fano variety over \(k\) of dimension \(d\). Then

\[ H^i(X, W\mathcal{O}_X) \otimes K = 0 \]

for \(i > 0\).

By Poincaré duality, this implies that

\[ H^i(X, W\Omega^d_X) \otimes K = 0 \]

for \(i < d\), where \(W\Omega^d_X\) is the sheaf of De Rham-Witt (DRW) differential forms of degree \(d\) constructed by Bloch and Illusie. This is because \(H^i(W\mathcal{O}_X) \otimes K\) is the part of crystalline cohomology with Frobenius slopes in the interval \([0, 1]\), while \(H^i(W, W\Omega^d_X)\) is the part with slope \(d\) (II.3.5). The author gave an erroneous proof of this corollary (in fact, an integral version) in an earlier preprint. H. Esnault subsequently gave a correct proof using the Bloch-Srinivas decomposition theorem in rational Chow groups and rigid cohomology.

Here, we will use the additional structure provided by the De Rham-Witt complex ([2], [3], [7], [8]) in order to prove the theorem which is a slight strengthening of Esnault’s result.

In fact, the theorem has the following corollary:

**Corollary 1** Let \(X\) be a Fano variety over a finite field with \(q\) elements. Then the number of rational points on \(X\) is congruent to 1 mod \(q\).

The corollary is an easy consequence of the Lefschetz trace formula for crystalline cohomology and slope arguments. Esnault’s theorem gives that the number is congruent to 1 mod \(p\), if \(q = p^n\).
2 Proof

Throughout, if we write $H^i$ without further embellishments, we mean rational crystalline cohomology.

We start with a quick summary of Esnault’s proof (which was an adaptation of Bloch’s proof in characteristic zero [3]): Because $X$ is rationally connected and therefore $CH_0(X) \otimes Q = 0$ ([10]), one gets from the Bloch-Srinivas theorem ([4]) that the diagonal correspondence $\Delta \subset X \times X$ is equivalent in $CH_0(X) \otimes Q$ to a sum
\[ z \times X + Z \]
where $z$ is the class of a closed point and $Z$ is a cycle supported on $X \times U$ for $U \subset X$ the complement of a divisor $D \subset X$. Therefore, if we apply the diagonal correspondence to a class of $H^i(X)$, $i > 0$, then the only thing that acts is the $Z$ part. That is, if $i > 0$, and $\alpha \in H^i(X)$ then $[\Delta]_*(\alpha) = [Z]_*(\alpha)$. We have
\[ H^i(X) \simeq H^i_{\text{rig}}(X) \]
where $H^i_{\text{rig}}$ is Berthelot’s rigid cohomology ([1]), and rigid cohomology has nice properties for open varieties. So if regard $\alpha$ as a class in $H^i_{\text{rig}}(X)$ and pull back to the subset $U$, then $\alpha_U = [Z_{X \times U}]_*(\alpha) = 0$ since $Z$ is supported on $X \times D$. On the other hand, one argues that the map $H^i_{\text{rig}}(X) \to H^i_{\text{rig}}(U)$ is injective on the Frobenius slope zero part. This concludes the argument.

Now we modify this proof. We wish to show that $H^i_{\text{rig}}(X) \to H^i_{\text{rig}}(U)$ is in fact injective on the part with slope in $[0,1]$. First, let $f : Y \to X$ be an alteration ([5]) with the property that $E = f^*(D)$ is of normal crossing. We have a commutative diagram
\[
\begin{array}{ccc}
V & \hookrightarrow & Y \\
\downarrow & & \downarrow \\
U & \hookrightarrow & X
\end{array}
\]
where $V = Y - E$. Since the pullback from $X$ to $Y$ is injective, we just need to show that the map $H^i_{\text{rig}}(Y) \to H^i_{\text{rig}}(V)$ is injective on the part with slope in $[0,1]$. But according to Shiho’s comparison theorem ([11]), we have a commutative diagram
\[
\begin{array}{ccc}
H^i_{\text{rig}}(Y) & \to & H^i_{\text{rig}}(V) \\
\downarrow & & \downarrow \\
H^i(Y) & \to & H^i(Y, E)
\end{array}
\]
where the space $H^i(Y, E)$ refers to the rational log crystalline cohomology of the log scheme $(Y, E)$. The map $H^i(Y) \to H^i(Y, E)$ is induced by a map
\[ W\Omega_Y \to W\Omega_Y(\log E) \]
of De Rham-Witt complexes. This is because we can realize the map at the level of crystalline complexes ([8]) for the two log schemes $Y$ (with trivial log structure) and $(Y, E)$ and the De Rham-Witt complexes are just given level by level as the cohomology sheaves of the crystalline complexes. Meanwhile, the
degree zero part is the same and equal to $W$ for both complexes. So we are done if we can show that the slope spectral sequence for $H^i(Y, E)$ degenerates at $E_1$, as in the case without log structures, and induces an isomorphism between $H^i(W\Omega^r_Y(\log E)) \otimes K$ and the part of $H^i(Y, E)$ with slope in $[j, j+1]$. To see this, one needs only repeat verbatim Bloch’s argument from [2], III.3. This is because the log de Rham-Witt complex $W\Omega_Y(\log E)$ is also equipped with operators $V$ and $F$ satisfying

$$FV = VF = p$$
$$pFd = dF, \quad Vd = pdV$$

which is all that is necessary for Bloch’s argument to apply: In brief, the map given by $p^iF$ on $W\Omega^j_Y(\log E)$ is a map of complexes, and induces on log crystalline cohomology the action of the absolute Frobenius $\phi$. So on each $E_r$ of the spectral sequence, the map induced by $p^iF$ on the subquotient $E_{ji}$ of $H^i(W\Omega^j_Y(\log E)) \otimes K$ is the map that commutes with the differentials. So the slope of $\phi$ on $E_{ji}$ is $\geq j$. However, from $FV = p$ and the fact that $V$ acts topologically nilpotently, we deduce that $\phi|E_{ji}$ has slope $< j + 1$. Therefore, the difference of slopes forces all the differentials to be zero from $E_1$ on, and the Dieudonné-Manin classification of crystals allows us to split the filtration of the spectral sequence.

Let us dispense of the easy corollary 1: As already mentioned, $H^i(W\mathcal{O}_X) \otimes K$ is identified with the part of crystalline cohomology $H^i(X)$ on which the operator $\phi$ induced by the absolute Frobenius of $X$ has slope in $[0, 1]$. Thus, the vanishing shows that all the Frobenius slopes on $H^i(X)$ are $\geq 1$ for $i > 0$. Now when $q = p^n$ and $k$ is the finite field $\mathbb{F}_q$, the Lefschetz trace formula gives us

$$|X(\mathbb{F}_q)| = \sum (-1)^i \text{Tr}(\phi^n|H^i_{cr}(X) \otimes K)$$

which obviously yields our congruence.

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