HYPERCRITICAL DEFORMED HERMITIAN-YANG-MILLS EQUATION

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Abstract. In this work, we study the deformed Hermitian-Yang-Mills equation on compact Kähler manifold. We introduce the notions of coerciveness and properness of the $J$-functional on the space of almost calibrated $(1,1)$-forms and show that they are both equivalent to the existence of solutions to the hypercritical deformed Hermitian-Yang-Mills equation.

1. Introduction

Informally, mirror symmetry describes the relation between complex geometry and symplectic geometry on Calabi-Yau manifolds. It predicts a duality between the underlying complex and symplectic structure on a manifold. On the holomorphic side, the deformed Hermitian-Yang-Mills (dHYM) equation is corresponding to the special Lagrangian equation in the setting of the Strominger-Yau-Zaslow mirror symmetry [40]. This was first appeared in [32] from the mathematical side drawing from the physics literature [33]. Analytically, on an $n$-dimensional compact Kähler manifold $(X^n, \omega)$ associated with a closed real $(1,1)$-form $\alpha$, the dHYM equation is given by

\[
\begin{align*}
\text{Im} \left( e^{-\sqrt{-1}\theta_0} (\alpha_\varphi + \sqrt{-1} \alpha \omega)^n \right) &= 0, \\
\text{Re} \left( e^{-\sqrt{-1}\theta_0} (\alpha_\varphi + \sqrt{-1} \omega)^n \right) &> 0,
\end{align*}
\]

where $\alpha_\varphi = \alpha + \sqrt{-1} \partial \bar{\partial} \varphi$ and $\theta_0$ is a constant. By integrating (1.1), it is easy to see that $\theta_0$ is uniquely determined (modulo $2\pi$) by a cohomological condition:

\[
\int_X (\alpha + \sqrt{-1} \omega)^n \in e^{\sqrt{-1} \theta_0} \cdot \mathbb{R}_{>0}
\]

provided that the integral on the left hand side does not vanish. It is believed that the dHYM equation plays a fundamental role in mirror symmetry and its solvability is expected to be related to deep notions of stability in algebraic geometry. We refer the reader to [11, 13] and the references therein for an introduction to the physical and mathematical aspects of the dHYM equation.

On the analytic side, the program of solving the dHYM equation was initiated by Jacob-Yau [30] where they used a parabolic flow to prove the existence of solutions when $(X, \omega)$ has positive bisectional curvature, and the initial data is sufficiently positive. Since then, the solvability of the dHYM equation has

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been studied extensively. In [10], Collins-Jacob-Yau characterized the existence of solutions to dHYM in the supercritical phase case in the analytic viewpoint, see also the work of Székelyhidi [41] for the study of a very general class of Hessian type equations on compact Hermitian manifolds under similar analytic assumption. It is also conjectured by Collins-Jacob-Yau [10] that the solvability to dHYM equation is equivalent to a Nakai-Moishezon type criterion in the supercritical phase case, see [4, 18, 29, 9] for the recent progress to this conjecture. For more related works, we refer readers to [27, 43, 35, 36, 26, 42] and the references therein.

On the other hand, motivated by work of Solomon [38], Thomas [44] and Thomas-Yau [45] in symplectic geometry, Collins-Yau [15] proposed to study the dHYM equation using an infinite dimensional GIT (Geometric Invariant Theory) approach where they introduced the space $H$ of almost calibrated $(1, 1)$-forms in the class $[\alpha]$: 

$$
H = \left\{ \phi \in C^\infty(X) \mid \text{Re} \left( e^{-\sqrt{-1}\theta_0} (\alpha \phi + \sqrt{-1}\omega)^n \right) > 0 \right\}.
$$

The space $H$ is a (possibly empty) open subset of the space of smooth, real valued functions on $X$, and hence inherits the structure of an infinite dimensional manifold. On $H$, Collins-Yau [15] also introduced the notion of $\varepsilon$-geodesic and the $J$-functional, and proved (1) $J$-functional is convex along the $\varepsilon$-geodesic for $\varepsilon \geq 0$; (2) $\phi$ is the critical point of $J$ if and only if $\phi$ is the solution of the dHYM equation. Furthermore, it was later shown by the authors and Collins [8] that $H$ admits a Riemannian structure and a Levi-Civita connection with non-positive sectional curvature. In fact, this is closely related, by mirror symmetry, to the work [38] of Solomon on the space of positive (or almost calibrated) Lagrangians. We refer readers to the arxiv version [14] of [15] for more discussion on the relation between the GIT approach and the algebraic obstruction on the existence.

In this work, we will continue the study of the existence problem to the dHYM equation from the viewpoint of $H$. We assume $H$ to be non-empty so that $\theta_0$ is well-defined modulo $2\pi$, see [30, 13]. Without loss of generality, we will assume $0 \in H$. And we will work on the case when $[\alpha]$ has hypercritical phase, i.e., $\theta_0 \in (0, \frac{\pi}{2})$. The following is an equivalent description of space $H$ (see (2.4)): 

$$
H = \left\{ \phi \in C^\infty(X) \mid Q_\omega(\alpha \phi) \in \left( \theta_0 - \frac{\pi}{2}, \theta_0 + \frac{\pi}{2}\right) \right\},
$$

where $Q_\omega(\alpha \phi)$ is the phase operator, see (2.1). Here we used the notion of phase operator to emphasis the branch we are using.

The $J$-functional should be understood as the Kempf-Ness functional in the GIT framework, and thus plays an important role in the study of the solvability for the dHYM equation. In fact, in Kähler geometry, the properness or coercivity of certain functional is usually related to the existence of

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1The formulation taken here is equivalent that in [15], see Lemma 2.2.
canonical metric. For example, Tian’s properness conjecture \cite{16, 48} predicts that the existence of constant scalar curvature Kähler (cscK) metric on compact Kähler manifold \((X, \omega)\) is equivalent to the properness of the Mabuchi \(K\)-energy defined on the space of Kähler potentials:

\[ H_{\text{psh}} = \{ \varphi \in C^\infty(X) \mid \omega + \sqrt{-1} \partial \bar{\partial} \varphi > 0 \}. \]

There are many progress toward this conjecture in the case of Kähler-Einstein metrics. In \cite{20}, Ding-Tian proved one direction (properness implies existence). When \(X\) has trivial automorphism group, Tian \cite{47} confirmed the converse direction (existence implies properness) under a technical assumption. This assumption was later removed by Tian-Zhu \cite{49}. When \(X\) has non-trivial automorphism group, Darvas-Rubinstein \cite{17} gave a counterexample to the direction (existence implies properness). By rephrasing the properness of the Mabuchi \(K\)-energy in terms of \(d_1\)-distance of \(H_{\text{psh}}\), they proposed a modified version of Tian’s conjecture, and confirmed it in the Kähler-Einstein metric case.

For the general cscK metric case of the above modified conjecture, one direction (existence implies properness) was established by Berman-Darvas-Lu \cite{2}, while the converse direction (properness implies existence) was proved by Chen-Cheng \cite{3, 6, 7}. In particular, if the automorphism group is discrete, the existence of cscK metric will be implied by the properness the Mabuchi \(K\)-energy in terms of \(d_1\)-distance of \(H_{\text{psh}}\). They also showed that cscK metric exists if and only if the Mabuchi \(K\)-energy is coercive. For the closely related \(J\)-equation, the similar result was obtained by Collins-Székelyhidi \cite{12}.

Inspired by the above mentioned works, in order to understand the existence problem of the dHYM equation deeply, we introduce the notions of coercivity and properness of \(J\)-functional.

**Definition 1.1.** We say the \(J\)-functional is coercive if there exists constants \(\delta, C > 0\) such that for any \(\varphi \in \mathcal{H}\),

\[ J(\varphi) \geq \delta \int_X (-\varphi) \left( \text{Im}(\alpha \varphi + \sqrt{-1} \omega)^n - \text{Im}(\alpha + \sqrt{-1} \omega)^n \right) - C. \]

This is analogous to \cite{12}, Definition 20] in the study of \(J\)-equation since \(\text{Im}(\alpha \varphi + \sqrt{-1} \omega)^n\) on \(X\) is a natural volume form for \(\varphi \in \mathcal{H}\). To define the properness, we will need to introduce the \(d_p\) pseudo-distance on \(\mathcal{H}\) for \(p \geq 1\) which is a generalization of the \(d_2\)-distance considered in \cite{8}.

For \(p \geq 1\) and a smooth path \(\varphi(t), t \in [0, 1]\) in \(\mathcal{H}\), define

\[ E_p(\varphi, t) = \int_X |\varphi_t|^p \text{Re} \left( e^{-\sqrt{-1} \theta \phi} (\alpha \varphi + \sqrt{-1} \omega)^n \right), \quad \text{length}_p(\varphi) = \int_0^1 E_p^\frac{1}{p}(\varphi, t) dt. \]

For any \(\varphi_0, \varphi_1 \in \mathcal{H}\), the \(d_p\)-distance between \(\varphi_0, \varphi_1\) is defined to be

\[ d_p(\varphi_0, \varphi_1) = \inf \{ \text{length}_p(\varphi) \mid \varphi \text{ is a smooth path in } \mathcal{H} \text{ connecting } \varphi_0 \text{ and } \varphi_1 \}. \]
We will show that $d_p$ is indeed a distance on $\mathcal{H}$ and can be realized by the limit of $\varepsilon$-geodesic as $\varepsilon \to 0$. This might be of independent interest on its own.

**Theorem 1.2.** For each $p \geq 1$, the space $(\mathcal{H}, d_p)$ is a metric space. For any $\varphi_0, \varphi_1 \in \mathcal{H}$, let $\varphi^\varepsilon$ be the $\varepsilon$-geodesic connecting $\varphi_0$ and $\varphi_1$. Then

$$d_p(\varphi_0, \varphi_1) = \lim_{\varepsilon \to 0} \text{length}_p(\varphi^\varepsilon).$$

Since we are working on the hypercritical phase, we can rewrite $\mathcal{H}$ as

$$\mathcal{H} = \bigcup_{c \in (0, \theta_0)} \mathcal{H}_c,$$

where

$$\mathcal{H}_c = \left\{ \varphi \in \mathcal{H} \mid \text{Im} \left( e^{-\sqrt{-1}c} (\alpha \varphi + \sqrt{-1} \omega)^n \right) > 0 \right\}.$$ 

Clearly, we have $\mathcal{H}_{c_2} \subset \mathcal{H}_{c_1}$ for $c_1 \leq c_2$ and $\mathcal{H} = \mathcal{H}_0$. In other words, the volume form $\text{Im}(\alpha \varphi + \sqrt{-1} \omega)^n$ is uniformly positive on the subspace $\mathcal{H}_c \subset \mathcal{H}$ and its positivity increases with $c$, see (2.5).

We define the properness of $J$ as follows.

**Definition 1.3.**

(i) We say the $J$-functional is weakly proper if for each $c \in (0, \theta_0)$ there exist constants $\delta_c, A_c > 0$ such that for any $\varphi \in \mathcal{H}_c$ with $\sup_X \varphi = 0$,

$$J(\varphi) \geq \delta_c d_1(\varphi, 0) - A_c.$$

(ii) We say the $J$-functional is proper if there exist constants $\delta, A > 0$ such that the following holds. For any $c \in (0, \theta_0)$ and $\varphi \in \mathcal{H}_c$ with $\sup_X \varphi = 0$,

$$J(\varphi) \geq \delta \sin(c) d_1(\varphi, 0) - A.$$

Let us explain the motivation of introducing the subspace $\mathcal{H}_c$ and Definition 1.3. Unlike the definition of properness in the Kähler setting, the constant $\delta_c$ and $\delta \sin(c)$ in Definition 1.3 depend on $c$, which is not uniform for all $\varphi \in \mathcal{H}$. The reason of this difference is that there is only one canonical volume form for each $\varphi \in \mathcal{H}_{psh}$, while there are two canonical volume forms for each $\varphi \in \mathcal{H}$.

More precisely, in the Kähler setting, for any $\varphi \in \mathcal{H}_{psh}$, $\omega_\varphi = \omega + \sqrt{-1} \partial \overline{\partial} \varphi$ is a Kähler metric and so the most natural volume form is $\omega^n_\varphi$. However, the situation becomes more subtle in the dHYM setting. For any $\varphi \in \mathcal{H}$, there are two available volume forms which are both natural. The first one is $\text{Re}(e^{-\sqrt{-1} \theta_0} (\alpha \varphi + \sqrt{-1} \omega)^n)$, which appears in the definition of $\mathcal{H}$. This volume form is used to define the Riemannian structure and $d_p$-distance of $\mathcal{H}$, and so is the canonical volume form in the viewpoint of geometry. The second one is $\text{Im}(\omega + \sqrt{-1} \alpha \varphi)^n$, which appears in the definitions of functionals $J$ and $J_\varepsilon$. Recall the $J$-functional serves as the Kempf-Ness functional in the GIT framework. This volume form should be understood as the canonical one in
the viewpoint of analysis. The quotient of these two volume forms can be computed as follows:

\[
\frac{\text{Im}(\alpha_\varphi + \sqrt{-1}\omega)^n}{\text{Re}(e^{-\sqrt{-1}\theta_0(\alpha_\varphi + \sqrt{-1}\omega)})} = \frac{\sin(Q_\omega(\alpha_\varphi))}{\cos(Q_\omega(\alpha_\varphi) - \theta_0)}.
\]

On the other hand, roughly speaking, one equivalent characterization of the properness is that $d_1$-distance can be controlled by $J$-functional. This means the positive lower bound of the above quotient should be as the constant $\delta$ in the definition of properness. Recall that $Q_\omega(\alpha_\varphi) \in (0, \theta_0 + \frac{\pi}{2})$ and $\theta_0 \in (0, \frac{\pi}{2})$. It is clear that there is no positive uniform lower bound for all $\varphi \in \mathcal{H}$, especially when $Q_\omega(\alpha_\varphi)$ is close to zero. This non-uniformity is the main difference between the Kähler and dHYM settings, which motivates us to consider the subspace $\mathcal{H}_c = \{\varphi \in C^\infty(X) \mid Q_\omega(\alpha_\varphi) \in (c, \Theta_0)\}$ (this is an equivalent definition of $\mathcal{H}_c$, see (2.5)) and introduce Definition 1.3.

With the above notions of properness and coercivity in hand, we now state our criterion of the existence of solutions to the dHYM equation in the hypercritical phase case.

**Theorem 1.4.** Let $(X, \omega)$ be an $n$-dimensional compact Kähler manifold with $n \geq 4$. Suppose that $\alpha$ is a real closed $(1, 1)$-form on $X$ so that $[\alpha]$ has hypercritical phase, then the followings are equivalent:

1. The $J$-functional is coercive.
2. The $J$-functional is proper.
3. The $J$-functional is weakly proper.
4. There exists $\varphi \in C^\infty(X)$ solving the dHYM equation (1.1).

**Remark 1.1.** It is interesting to see whether the above result holds when $n \leq 3$. The main obstruction is that the proper twisted dHYM operator $F_\varepsilon$ may be not concave, see Lemma 2.4.

The paper is organized as follows: In Section 2, we will collect some preliminaries of the operators that will be used throughout this work. In Section 3, we will study the $d_p$-distance using the $\varepsilon$-geodesic. In particular, we will prove Theorem 1.2. In Section 4, we will study the twisted dHYM flow which is the negative gradient flow of the twisted $J_\varepsilon$-functional. We will study its long-time existence and convergence. In Section 5, we will establish some technical lemmas which are important ingredients for the proof of main theorem. In Section 6 and 7, we will prove our main result, Theorem 1.4.

2. Preliminaries

In this section, we will introduce some basic operators and properties which we will use in this work.
2.1. Basic operators and properties. For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ and $0 < \theta < \Theta < \pi$, we define

$$P(\lambda) = \max_{1 \leq k \leq n} \sum_{i \neq k} \arccot(\lambda_i), \quad Q(\lambda) = \sum_{i=1}^{n} \arccot(\lambda_i)$$

and

$$\Gamma_{\theta, \Theta} = \{ \lambda \in \mathbb{R}^n \mid P(\lambda) < \theta, Q(\lambda) < \Theta \}.$$ 

It is clear that the operators $P$ and $Q$ are independent of permutation. For a positive definite $n \times n$ Hermitian matrix $A$ and $n \times n$ Hermitian matrix $B$, we will define

$$(2.1) \quad P_A(B) = P(\lambda), \quad Q_A(B) = Q(\lambda)$$

where $\lambda$ is the eigenvalue of $B$ with respect to $A$ and

$$\Gamma_{A,\theta,\Theta} = \{ B \in \text{Herm}(n) \mid P_A(B) < \theta, Q_A(B) < \Theta \}.$$ 

We have the following useful properties of the above operators.

**Lemma 2.1.** For any positive definite matrix $A$ and $0 < \theta < \Theta < \pi$,

(i) The space $\overline{\Gamma}_{A,\theta,\Theta}$ is convex;

(ii) The function $\cot(Q_A)$ is concave on $\Gamma_{A,\theta,\Theta}$.

**Proof.** (i) is proved in [4, Lemma 5.6 (10)]. (ii) is a special case of [4, Lemma 5.6 (9)] by choosing $f = 0$. The difference between $\cot(Q_A)$ and $F$ in [4, Lemma 5.6] is a constant term $\cot(\theta_0)$, which does not affect the concavity. \(\square\)

It is sometimes useful to consider

$$(2.2) \quad \hat{Q}_A(B) = \frac{n\pi}{2} - Q_A(B) = \sum_{i=1}^{n} \arctan(\lambda_i)$$

and $\hat{\theta}_0 = \frac{n\pi}{2} - \theta_0$ where $\hat{\theta}_0 \in \left(\frac{(n-1)\pi}{2}, \frac{n\pi}{2}\right)$ since $\theta_0 \in (0, \frac{\pi}{2})$.

We have the following equivalent forms of some expressions.

**Lemma 2.2.** For any $\varphi \in \mathcal{H}$,

$$\Re\left(e^{-\sqrt{-1}\theta_0}(\omega + \sqrt{-1}\alpha_{\varphi})^n\right) = \Re\left(e^{-\sqrt{-1}\theta_0}(\alpha_{\varphi} + \sqrt{-1}\omega)^n\right),$$

$$\Im\left(e^{-\sqrt{-1}\theta_0}(\omega + \sqrt{-1}\alpha_{\varphi})^n\right) = -\Im\left(e^{-\sqrt{-1}\theta_0}(\alpha_{\varphi} + \sqrt{-1}\omega)^n\right).$$
Proof. Let $\lambda_i$ be the eigenvalues of $\alpha\varphi$ with respect to $\omega$. Then

$$\frac{\text{Re} \left( e^{-\sqrt{-1}\theta_0}(\omega + \sqrt{-1}\alpha\varphi)^n \right)}{\omega^n} = \cos(\hat{Q}_\omega(\alpha\varphi) - \hat{\theta}_0) \prod_{i=1}^n \sqrt{1 + \lambda_i^2} = \cos(\hat{Q}_\omega(\alpha\varphi) - \theta_0) \prod_{i=1}^n \sqrt{1 + \lambda_i^2} = \frac{\text{Re} \left( e^{-\sqrt{-1}\theta_0}(\alpha\varphi + \sqrt{-1}\omega)^n \right)}{\omega^n}. \tag{2.3}$$

The second equality can be proved by the similar argument. \hfill \Box

If we denote $\Theta_0 = \theta_0 + \pi$, then using Lemma 2.2 and (2.3), we have several equivalent descriptions of $\mathcal{H}$:

$$\mathcal{H} = \left\{ \varphi \in C^\infty(X) \mid \text{Re} \left( e^{-\sqrt{-1}\theta_0}(\alpha\varphi + \sqrt{-1}\omega)^n \right) > 0 \right\} = \left\{ \varphi \in C^\infty(X) \mid \text{Re} \left( e^{-\sqrt{-1}\theta_0}(\omega + \sqrt{-1}\alpha\varphi)^n \right) > 0 \right\} \tag{2.4}$$

$$\mathcal{H}_c = \left\{ \varphi \in \mathcal{H} \mid \text{Im} \left( e^{-\sqrt{-1}\theta_0}(\alpha\varphi + \sqrt{-1}\omega)^n \right) > 0 \right\} = \left\{ \varphi \in C^\infty(X) \mid \text{Im} \left( e^{-\sqrt{-1}\theta_0}(\omega + \sqrt{-1}\alpha\varphi)^n \right) > 0 \right\} \tag{2.5}$$

For any $\varphi \in \mathcal{H}_c$, let $\lambda_i$ be the eigenvalues of $\alpha\varphi$ with respect to $\omega$. Then

$$\frac{\text{Im}(\omega + \sqrt{-1}\alpha\varphi)^n}{\omega^n} = \sin(Q_\omega(\alpha\varphi)) \prod_{i=1}^n \sqrt{1 + \lambda_i^2} \geq \sin(c) \prod_{i=1}^n \sqrt{1 + \lambda_i^2}. \tag{2.6}$$

Hence, the constant $c$ is measuring the positivity of the volume form $\text{Im}(\omega + \sqrt{-1}\alpha\varphi)^n$.

Finally, we note that any element of $\mathcal{H}$ is quasi-plurisubharmonic.

**Lemma 2.3.** Define $\chi = \tan(\theta_0)\omega + \alpha$. Then for any $\varphi \in \mathcal{H}$,

$$\chi_\varphi = \chi + \sqrt{-1}\partial\bar{\partial}\varphi > 0,$$

or equivalently, $\varphi \in \text{PSH}(X, \chi)$. 

Proof. Let $\lambda_1 \geq \ldots \geq \lambda_n$ be the eigenvalues of $\alpha_\varphi$ with respect to $\omega$. Then
\[
\arccot(\lambda_n) \leq Q_\omega(\alpha_\varphi) < \Theta_0 = \theta_0 + \frac{\pi}{2},
\]
which implies $\lambda_n > -\tan(\theta_0)$ and so $\alpha_\varphi > -\tan(\theta_0)\omega$. Hence,
\[
\chi_\varphi = \chi + \sqrt{-1}\partial\bar{\partial}\varphi = \tan(\theta_0)\omega + \alpha_\varphi > 0.
\]
\[
\square
\]

2.2. **Twisted operator.** In this subsection, we will introduce the twisted dHYM operator. For $0 < \theta < \Theta < \pi$ and $\lambda \in \Gamma_{\theta,\Theta}$, consider $F_\varepsilon : \Gamma_{\theta,\Theta} \to \mathbb{R}$ which is given by
\[
F_\varepsilon(\lambda) = \Re \prod_{k=1}^n \left( \lambda_k + \sqrt{-1} \right) + \varepsilon \Im \prod_{k=1}^n \left( \lambda_k + \sqrt{-1} \right).
\]

The following basic properties of $F_\varepsilon$ are very important in the upcoming discussion.

**Lemma 2.4.** For $0 < \theta < \Theta < \pi$, there exists $\varepsilon_0(\theta, \Theta, n)$ such that if $n \geq 4$ and $\varepsilon \in (0, \varepsilon_0)$, then the map $F_\varepsilon$ satisfies

(i) For each $i$, $\frac{\partial F_\varepsilon}{\partial \lambda_i} > 0$;

(ii) If $\lambda_i \geq \lambda_j$, then $\frac{\partial F_\varepsilon}{\partial \lambda_i} \leq \frac{\partial F_\varepsilon}{\partial \lambda_j}$;

(iii) $F_\varepsilon$ is concave.

**Proof.** This lemma is a special case of [4, Lemma 5.6] by choosing $f = -\varepsilon$. The difference between $F_\varepsilon$ and $F$ of [4, Lemma 5.6] is a constant term $\cot(\theta_0)$, which does not affect the derivative and concavity. $\square$

For any $\varepsilon \in [0, 1]$ and $\varphi \in \mathcal{H}$, let $\lambda_i$ be the eigenvalues of $\alpha_\varphi$ with respect to $\omega$. We define the operator $F_{\omega,\varepsilon}$ by $F_{\omega,\varepsilon}(\alpha_\varphi) = F_\varepsilon(\lambda)$, or equivalently,
\[
F_{\omega,\varepsilon}(\alpha_\varphi) = \frac{\Re(\alpha_\varphi + \sqrt{-1}\omega)^n + \varepsilon\omega^n}{\Im(\alpha_\varphi + \sqrt{-1}\omega)^n}.
\]

It is clear from the definition that $F_{\omega,\varepsilon}$ is independent of permutation of $\lambda$. Denote the first and second derivatives of $F_{\omega,\varepsilon}$ by $F_{\omega,\varepsilon}^{ij}$ and $F_{\omega,\varepsilon}^{ij,pq}$. For any $x_0 \in X$, if we choose a coordinate such that at $x_0$,
\[
\omega^{ij} = \delta_{ij}, \quad (\alpha_\varphi)^{ij} = \lambda_i\delta_{ij},
\]
then we have (see e.g. [1, 23, 37])
\[
(2.7) \quad F_{\omega,\varepsilon}^{ij} = \frac{\partial F_\varepsilon}{\partial \lambda_i} \delta_{ij}, \quad F_{\omega,\varepsilon}^{ij,pq} = \frac{\partial^2 F_\varepsilon}{\partial \lambda_i \partial \lambda_p} \delta_{ij} \delta_{pq} + \frac{\partial F_\varepsilon}{\partial \lambda_i} - \frac{\partial F_\varepsilon}{\partial \lambda_j}(1 - \delta_{ij})\delta_{iq}\delta_{jp}.
\]

If $\lambda_i = \lambda_j$, we will then regard the quotient appeared on the last term as the limit. For notational convenience, we will denote
\[
\frac{\partial F_\varepsilon}{\partial \lambda_i} = (F_\varepsilon)_i \quad \text{and} \quad \frac{\partial^2 F_\varepsilon}{\partial \lambda_i \partial \lambda_j} = (F_\varepsilon)_{ij}.
2.3. Twisted $\mathcal{F}$-functional and gradient flow. Motivated by the work in [12], we will study the twisted $\mathcal{F}$-functional. For $\varepsilon \in [0, 1]$ and $\varphi \in \mathcal{H}$, we define the $\mathcal{J}_\varepsilon$-functional on $\mathcal{H}$ by setting $\mathcal{J}_\varepsilon(0) = 0$ and

$$
d\mathcal{J}_\varepsilon(\varphi)(\psi) = -\int_X \psi \left( F_{\omega, \varepsilon}(\alpha_\varphi) - \cot(\theta_0) - a_0 \varepsilon \right) \text{Im}(\alpha_\varphi + \sqrt{-1}\omega)^n
$$

$$
= -\int_X \psi \left( \text{Re}(\alpha_\varphi + \sqrt{-1}\omega)^n + \varepsilon \omega^n - (\cot(\theta_0) + a_0 \varepsilon) \text{Im}(\alpha_\varphi + \sqrt{-1}\omega)^n \right),
$$

where $\varphi \in \mathcal{H}$, $\psi \in T_\varphi \mathcal{H} = C^\infty(X)$ and the constant $a_0$ is given by

$$
a_0 = \frac{\int_X \omega^n}{\int_X \text{Im}(\alpha + \sqrt{-1}\omega)^n} > 0.
$$

Clearly, for any $C \in \mathbb{R}$, we have $\mathcal{J}_\varepsilon(\varphi + C) = \mathcal{J}_\varepsilon(\varphi)$.

We now compare the $\mathcal{J}_\varepsilon$-functional and the $\mathcal{J}$-functional introduced by Collins-Yau [15]. Recall that $\mathcal{J}$ is defined by

$$
\mathcal{J}(0) = 0, \quad d\mathcal{J}(\varphi)(\psi) = -\int_X \psi \text{Im} \left( e^{-\sqrt{-1}\theta_0} (\omega + \sqrt{-1}\alpha_\varphi)^n \right).
$$

**Lemma 2.5.** For any $\varphi \in \mathcal{H}$, we have

$$
\mathcal{J}(\varphi) = \sin(\theta_0) \mathcal{J}_0(\varphi).
$$

**Proof.** By Lemma 2.2 we compute

$$
d\mathcal{J}(\varphi)(\psi) = \int_X \psi \text{Im} \left( e^{-\sqrt{-1}\theta_0} (\alpha_\varphi + \sqrt{-1}\omega)^n \right)
$$

$$
= \int_X \psi \left[ \cos(\theta_0) \text{Im}(\alpha_\varphi + \sqrt{-1}\omega)^n - \sin(\theta_0) \text{Re}(\alpha_\varphi + \sqrt{-1}\omega)^n \right]
$$

$$
= -\sin(\theta_0) \int_X \psi \left[ \text{Re}(\alpha_\varphi + \sqrt{-1}\omega)^n - \cot(\theta_0) \text{Im}(\alpha_\varphi + \sqrt{-1}\omega)^n \right]
$$

$$
= \sin(\theta_0) d\mathcal{J}_0(\varphi)(\psi).
$$

Since $\mathcal{J}(0) = \mathcal{J}_0(0) = 0$, we have $\mathcal{J}(\varphi) = \sin(\theta_0) \mathcal{J}_0(\varphi)$. \qed

From the variational viewpoint, it will be natural to consider a parabolic flow which is the negative gradient flow of $\mathcal{J}_\varepsilon$-functional. We therefore consider the following twisted dHYM flow. For any $\varepsilon \in [0, 1]$ and $\varphi_0 \in \mathcal{H}$, the twisted dHYM flow is an one parameter family of $\varphi(t)$ such that

$$
\begin{cases}
\partial_t \varphi = F_{\omega, \varepsilon}(\alpha_\varphi) - \cot(\theta_0) - a_0 \varepsilon, \\
\varphi(0) = \varphi_0.
\end{cases}
$$

Clearly, the twisted dHYM flow (2.8) is the negative gradient flow of the functional $\mathcal{J}_\varepsilon$ whenever $\varphi(t) \in \mathcal{H}$:

$$
\frac{d\mathcal{J}_\varepsilon(\varphi)}{dt} = -\int_X (F_{\omega, \varepsilon}(\alpha_\varphi) - \cot(\theta_0) - a_0 \varepsilon)^2 \text{Im}(\alpha_\varphi + \sqrt{-1}\omega)^n < 0.
$$
On the other hand, by Lemma 2.4 and (2.7), there is \( \varepsilon_0(\varphi_0, \alpha, \omega, X) > 0 \) such that if \( \varepsilon < \varepsilon_0 \), then the twisted dHYM flow (2.8) is strictly parabolic at \( t = 0 \) and hence it admits a short-time solution on \( X \). In section 4 we will study the twisted dHYM flow in more details.

3. Metric space \((\mathcal{H}, d_p)\)

In this section, we will introduce the \( d_p \)-distance on \( \mathcal{H} \) and discuss its properties which ultimately prove Theorem 1.2. This is motivated by the work of Darvas [16] which considered the space of Kähler potentials. We will also derive some \( d_p \)-distance inequalities for later purpose.

3.1. \( \varepsilon \)-geodesic. In this subsection, we will recall the notion and properties of the \( \varepsilon \)-geodesic introduced by Collins-Yau in [15]. Consider

\[
\mathcal{X} = X \times A, \quad A = \{ z \in \mathbb{C} \mid e^{-1} \leq |z| \leq 1 \}.
\]

Denote the projection map from \( \mathcal{X} \) to \( X \) by \( \pi \). Let \( \partial, \bar{\partial} \) and \( D, \bar{D} \) be the complex differential operators on \( X \) and \( X \) respectively. For a smooth path \( \varphi(t), t \in [0, 1] \) on \( X \), define the function on \( \mathcal{X} \) by

\[
\Phi(x, z) = \varphi(x, -\log |z|).
\]

Following [15], the path \( \varphi \) is said to be a geodesic from \( \varphi_0 \in \mathcal{H} \) to \( \varphi_1 \in \mathcal{H} \) if and only if the function \( \Phi \) satisfies

\[
\begin{align*}
\text{Im} \left( e^{-\sqrt{-1} \theta_0} \left( \pi^* \omega + \sqrt{-1} (\pi^* \alpha + \sqrt{-1} D \bar{D} \Phi^\varepsilon) \right)^{n+1} \right) &= 0, \\
\text{Re} \left( e^{-\sqrt{-1} \theta_0} \left( \omega + \sqrt{-1} \alpha \Phi^\varepsilon \right)^n \right) &> 0, \\
\Phi^\varepsilon|_{\{ |z| = 1 \}} &= \varphi_0, \quad \Phi^\varepsilon|_{\{ |z| = e^{-1} \}} = \varphi_1.
\end{align*}
\]

Since \( \pi^* \omega \) is a degenerate metric on \( \mathcal{X} \), the above equation is degenerated. To overcome the degeneracy, Collins-Yau [15] considered the Kähler metric

\[
\hat{\omega}_\varepsilon = \pi^* \omega + \varepsilon^2 \sqrt{-1} dz \wedge d\bar{z}
\]

on \( \mathcal{X} \) for \( \varepsilon > 0 \), and introduced the \( \varepsilon \)-geodesic equation:

\[
\begin{align*}
\text{Im} \left( e^{-\sqrt{-1} \theta_0} \left( \hat{\omega}_\varepsilon + \sqrt{-1} (\pi^* \alpha + \sqrt{-1} D \bar{D} \Phi^\varepsilon) \right)^{n+1} \right) &= 0, \\
\text{Re} \left( e^{-\sqrt{-1} \theta_0} \left( \omega + \sqrt{-1} \alpha \Phi^\varepsilon \right)^n \right) &> 0, \\
\Phi^\varepsilon|_{\{ |z| = 1 \}} &= \varphi_0, \quad \Phi^\varepsilon|_{\{ |z| = e^{-1} \}} = \varphi_1.
\end{align*}
\]

Using the operator \( \hat{Q}_{\hat{\omega}_\varepsilon} \) in (2.2), we may rewrite the above equation as

\[
\begin{align*}
\hat{Q}_{\hat{\omega}_\varepsilon} (\pi^* \alpha + \sqrt{-1} D \bar{D} \Phi^\varepsilon) &= \theta_0, \\
\text{Re} \left( e^{-\sqrt{-1} \theta_0} \left( \omega + \sqrt{-1} \alpha \Phi^\varepsilon \right)^n \right) &= 0, \\
\Phi^\varepsilon|_{\{ |z| = 1 \}} &= \varphi_0, \quad \Phi^\varepsilon|_{\{ |z| = e^{-1} \}} = \varphi_1.
\end{align*}
\]
This is a Dirichlet problem for the specified Lagrangian phase equation on $X$. If (3.2) admits a smooth solution, then the maximum principle shows

$$\Phi^\varepsilon(\cdot, z) = \Phi^\varepsilon(\cdot, |z|).$$

Therefore, it is natural to consider

$$\varphi^\varepsilon(\cdot, t) = \Phi^\varepsilon(\cdot, e^{-t}), \quad t \in [0, 1].$$

And we say that the path $\varphi^\varepsilon$ is the $\varepsilon$-geodesic joining from $\varphi_0$ to $\varphi_1$. By [8, Lemma 2.3], the $\varepsilon$-geodesic $\varphi^\varepsilon$ satisfies

$$\varphi^\varepsilon_{tt} \text{Re} \left( e^{-\sqrt{-1} b_0} \left( \omega + \sqrt{-1} \alpha \varphi^\varepsilon \right)^n \right) + n \sqrt{-1} \partial \varphi^\varepsilon \wedge \bar{\partial} \varphi^\varepsilon \wedge \text{Im} \left( e^{-\sqrt{-1} b_0} \left( \omega + \sqrt{-1} \alpha \varphi^\varepsilon \right)^{n-1} \right) = -4e^{-2t} \varepsilon^2 \text{Im} \left( e^{-\sqrt{-1} b_0} \left( \omega + \sqrt{-1} \alpha \varphi^\varepsilon \right)^n \right).$$

(3.3)

In [15], Collins-Yau solved the Dirichlet problem (3.1) (or equivalently, (3.2)) and established the weak $C^{1,1}$ estimate of the solution. Based on this work, the authors and Collins [8] extended it to the full $C^{1,1}$ estimate.

**Theorem 3.1** (Theorem 1.2 of [15], Theorem 6.1 of [8]). For any $\varphi_0, \varphi_1 \in \mathcal{H}$, there exists a unique solution of (3.1) (or equivalently, (3.2)). Furthermore, there exists a constant $C(\varphi_0, \varphi_1, \alpha, \omega, X)$ such that

$$\sup_M \left( |\Phi^\varepsilon| + |D\Phi^\varepsilon| + |D^2\Phi^\varepsilon| \right) \leq C,$$

or equivalently,

$$\sup_X \left( |\varphi^\varepsilon| + |\varphi_t^\varepsilon| + |\varphi_{tt}^\varepsilon| + |\nabla \varphi^\varepsilon| + |\nabla^2 \varphi^\varepsilon| \right) \leq C.$$

(3.4)

For later use, we collect more estimates of $\varepsilon$-geodesic.

**Lemma 3.2** (Lemma 3.1 and 3.4 of [8]). For any $\varphi_0, \varphi_1 \in \mathcal{H}$, let $\varphi^\varepsilon$ be the $\varepsilon$-geodesic connecting $\varphi_0$ and $\varphi_1$. Then there exists $C(\varphi_0, \varphi_1, \alpha, \omega, X)$ such that

(i) $\varphi^\varepsilon_{tt} \geq -C \varepsilon^2$;

(ii) $|\varphi_t^\varepsilon| \leq \| \varphi_0 - \varphi_1 \|_{L^\infty} + C \varepsilon^2$.

**Proof.** (i) is proved in [8] Lemma 3.1. (ii) follows from the proof of [8] Lemma 3.4. ∎

For notational convenience, for any $\varphi \in \mathcal{H}$, we will define

$$\Omega_\varphi = \omega + \sqrt{-1} \alpha \varphi = \omega + \sqrt{-1}(\alpha + \sqrt{-1} \partial \bar{\partial} \varphi).$$

(3.5)
3.2. $d_p$-distance. Let $\varphi(t), t \in [0, 1]$ be a smooth path in $\mathcal{H}$. For $p \in [1, \infty)$, we define the $L^p$-energy and $L^p$-length of $\varphi$ by

$$E_p(\varphi, t) = \int_X |\varphi_t|^p \text{Re}(e^{-\sqrt{-1}\alpha_0 \Omega^\varphi^n}), \quad \text{length}_{p}(\varphi) = \int_0^1 E_p^\frac{1}{p}(\varphi, t) dt,$$

where $\Omega^\varphi = \omega + \sqrt{-1} \alpha_\varphi$. For any $\varphi_0, \varphi_1 \in \mathcal{H}$, the $d_p$-distance between $\varphi_0$ and $\varphi_1$ is then given by

$$d_p(\varphi_0, \varphi_1) = \inf \{ \text{length}_{p}(\varphi) \mid \varphi \text{ is a smooth path in } \mathcal{H} \text{ connecting } \varphi_0 \text{ and } \varphi_1 \}.$$

In the rest of this subsection, we will prove Theorem 1.2. Since $| \cdot |^p$ is not differentiable at zero for general $p \geq 1$, for any $\delta > 0$, we introduce the following approximated $L^p$-energy and $L^p$-length:

$$E_{p,\delta}(\varphi, t) = \int_X \sqrt{\varphi_t^{2p} + \delta^2} \text{Re}(e^{-\sqrt{-1}\alpha_0 \Omega^\varphi^n}), \quad \text{length}_{p,\delta}(\varphi) = \int_0^1 E_{p,\delta}^\frac{1}{p}(\varphi, t) dt,$$

then for any path $\varphi$ in $\mathcal{H}$, there exists $C(p, \alpha, \omega, X)$ such that

$$(3.6) \quad |E_p(\varphi, t) - E_{p,\delta}(\varphi, t)| \leq C\delta, \quad |\text{length}_{p}(\varphi) - \text{length}_{p,\delta}(\varphi)| \leq C\delta^{\frac{1}{p}}.$$

Note that the constant $C$ is uniform for any path in $\mathcal{H}$. In application, we will choose the path to be the $\varepsilon$-geodesic so that the path depends on the endpoints.

**Lemma 3.3.** For any $\varphi_0, \varphi_1 \in \mathcal{H}$, let $\varphi^\varepsilon$ be the $\varepsilon$-geodesic connecting $\varphi_0$ and $\varphi_1$. Denote the corresponding energies by $E_{p,\delta}^\varepsilon$, then there exists $C(\varphi_0, \varphi_1, p, \alpha, \omega, X)$ such that for any $\delta > 0$,

(i) $|\partial_t E_{p,\delta}^\varepsilon| \leq C\varepsilon^2$;

(ii) The following inequality holds

$$E_{p,\delta}^\varepsilon(t) \geq \max \left\{ \int_{\{\varphi_0 > \varphi_1\}} |\varphi_0 - \varphi_1|^p \text{Re}(e^{-\sqrt{-1}\alpha_0 \Omega^\varphi^n_{\varphi_0}}), \right. $$

$$\left. \int_{\{\varphi_1 > \varphi_0\}} |\varphi_0 - \varphi_1|^p \text{Re}(e^{-\sqrt{-1}\alpha_0 \Omega^\varphi^n_{\varphi_1}}) \right\} - C\varepsilon^2.$$
Proof. Using the $\varepsilon$-geodesic equation (3.3), we compute
\[
\frac{dE_{p,\delta}^\varepsilon}{dt} = p \int_X \frac{(\varphi_t^\varepsilon)^{2p-1} \varphi_t^\varepsilon}{\sqrt{(\varphi_t^\varepsilon)^{2p} + \delta^2}} \text{Re}(e^{-\sqrt{-1}t_0} \Omega_n^\varphi) \\
- n \int_X \sqrt{(\varphi_t^\varepsilon)^{2p} + \delta^2} \sqrt{-1} \overline{\partial} \varphi_t^\varepsilon \wedge \text{Im}(e^{-\sqrt{-1}t_0} \Omega_n^\varphi^{-1}) \\
= p \int_X \frac{(\varphi_t^\varepsilon)^{2p-1} \varphi_t^\varepsilon}{\sqrt{(\varphi_t^\varepsilon)^{2p} + \delta^2}} \text{Re}(e^{-\sqrt{-1}t_0} \Omega_n^\varphi) \\
+ np \int_X \frac{(\varphi_t^\varepsilon)^{2p-1}}{\sqrt{(\varphi_t^\varepsilon)^{2p} + \delta^2}} \sqrt{-1} \partial \varphi_t^\varepsilon \wedge \overline{\partial} \varphi_t^\varepsilon \wedge \text{Im}(e^{-\sqrt{-1}t_0} \Omega_n^\varphi^{-1}) \\
= - 4pe^{-2t}e^{2} \int_X \frac{(\varphi_t^\varepsilon)^{2p-1}}{\sqrt{(\varphi_t^\varepsilon)^{2p} + \delta^2}} \text{Im}(e^{-\sqrt{-1}t_0} \Omega_n^\varphi).\
\]

Then (i) follows from $p \geq 1$ and the estimate from Theorem 3.1.

To prove (ii). By (i) in Lemma 3.2, we have $\varphi_t^\varepsilon(0) \leq \varphi_t^\varepsilon(1) - \varphi_t^\varepsilon(0) + C\varepsilon^2 = \varphi_1 - \varphi_0 + C\varepsilon^2$.

Then on the set $\{\varphi_0 > \varphi_1\}$,
\[
|\varphi_t^\varepsilon(0)| \geq |\varphi_0 - \varphi_1| - C\varepsilon^2
\]
and so
\[
E_{p,\delta}^\varepsilon(0) \geq \int_{\{\varphi_0 > \varphi_1\}} |\varphi_0 - \varphi_1|^p \text{Re}(e^{-\sqrt{-1}t_0} \Omega_n^\varphi) - C\varepsilon^{2p}.
\]
Similarly, we have
\[
E_{p,\delta}^\varepsilon(1) \geq \int_{\{\varphi_1 > \varphi_0\}} |\varphi_0 - \varphi_1|^p \text{Re}(e^{-\sqrt{-1}t_0} \Omega_n^\varphi) - C\varepsilon^{2p}.
\]
Combining the above with (i), we obtain (ii). \qed

Lemma 3.4. Let $\psi(s)$, $s \in [0, s_0]$ be a smooth path in $\mathcal{H}$, and $\hat{\psi}$ be a fixed point in $\mathcal{H}$ such that $\hat{\psi} \not\equiv \psi([0, s_0])$. For any $s \in [0, s_0]$, let $\varphi^\varepsilon(s, t)$, $t \in [0, 1]$ be the $\varepsilon$-geodesic joining from $\psi(s)$ to $\hat{\psi}$. Then there exists a constant $C(\psi, \hat{\psi}, p, \alpha, \omega, X)$ such that
\[
\text{length}_{p,\delta}(\varphi^\varepsilon(0, \cdot)) \leq \text{length}_{p,\delta}(\psi(\cdot)) + \text{length}_{p,\delta}(\varphi^\varepsilon(s_0, \cdot)) + C\varepsilon^2 \delta^{-3}.
\]
Proof. Define
\[
\ell_\delta(s) = \text{length}_{p,\delta}(\psi|_{[0, s]}), \quad \hat{\ell}_\delta(s) = \text{length}_{p,\delta}(\varphi^\varepsilon(s, \cdot)).
\]
It suffices to show that
\[
\ell'_\delta(s) + \hat{\ell}'_\delta(s) \geq -C\varepsilon^2 \delta^{-3}.
\]
It is clear that

\[
\ell'_\delta(s) = E^{\frac{1}{p}}_{p,\delta}(\psi, s) = \left( \int_x \sqrt{\psi_{s}^{2p} + \delta^2} \text{Re}(e^{-\sqrt{-1} \theta_0 \Omega^n_{\varphi}}) \right)^\frac{1}{p}
\]

and

\[
\hat{\ell}'_\delta(s) = \frac{1}{p} \int_0^1 (E^\varepsilon_{\delta}(s, t))^{\frac{1}{p}-1} \partial_s E^\varepsilon_{\delta}(s, t) dt,
\]

where

\[
E^\varepsilon_{\delta}(s, t) = \int_x \sqrt{(\varphi_t^2)^{2p} + \delta^2} \text{Re}(e^{-\sqrt{-1} \theta_0 \Omega^n_{\varphi}}).
\]

For notational convenience, we omit superscript \(\varepsilon\) and subscript \(\delta\), i.e.,

\[
\ell = \ell_\delta, \quad \hat{\ell} = \hat{\ell}_\delta, \quad \varphi = \varphi^\varepsilon, \quad E = E^\varepsilon_{\delta}.
\]

We compute

\[
\frac{\partial E(s, t)}{\partial s} = p \int_x \frac{\varphi_t^{2p-1} \varphi_s}{\sqrt{\varphi_t^{2p} + \delta^2}} \text{Re}(e^{-\sqrt{-1} \theta_0 \Omega^n_{\varphi}}) - n \int_x \sqrt{\varphi_t^{2p} + \delta^2} \sqrt{-1} \partial \bar{\partial} \varphi_s \wedge \text{Im}(e^{-\sqrt{-1} \theta_0 \Omega^n_{\varphi}})
\]

\[= p \frac{\partial}{\partial t} \left( \int_x \frac{\varphi_t^{2p-1} \varphi_s}{\sqrt{\varphi_t^{2p} + \delta^2}} \text{Re}(e^{-\sqrt{-1} \theta_0 \Omega^n_{\varphi}}) \right) - p \int_x \varphi_s \frac{\partial}{\partial t} \left( \frac{\varphi_t^{2p-1}}{\sqrt{\varphi_t^{2p} + \delta^2}} \text{Re}(e^{-\sqrt{-1} \theta_0 \Omega^n_{\varphi}}) \right) + np \int_x \frac{\varphi_t^{2p-1} \varphi_s}{\sqrt{\varphi_t^{2p} + \delta^2}} \sqrt{-1} \partial \bar{\partial} \varphi_t \wedge \bar{\partial} \varphi_s \wedge \text{Im}(e^{-\sqrt{-1} \theta_0 \Omega^n_{\varphi}}) \right).
\]

For the second term in (3.9), direct calculation shows

\[
- p \int_x \varphi_s \frac{\partial}{\partial t} \left( \frac{\varphi_t^{2p-1}}{\sqrt{\varphi_t^{2p} + \delta^2}} \text{Re}(e^{-\sqrt{-1} \theta_0 \Omega^n_{\varphi}}) \right)
\]

\[= - p \int_x \left( (p - 1) \varphi_t^{4p-2} + (2p - 1) \delta^2 \varphi_t^{2p-2} \right) \varphi_t \varphi_s \text{Re}(e^{-\sqrt{-1} \theta_0 \Omega^n_{\varphi}})
\]

\[+ np \int_x \frac{\varphi_t^{2p-1} \varphi_s}{\sqrt{\varphi_t^{2p} + \delta^2}} \sqrt{-1} \partial \bar{\partial} \varphi_t \wedge \text{Im}(e^{-\sqrt{-1} \theta_0 \Omega^n_{\varphi}}) \right).\]
For the third term of \( (3.9) \), integrating by parts yields
\[
np \int_X \frac{\varphi_t^{2p-1}}{\sqrt{\varphi_t^{2p} + \delta^2}} \sqrt{-1} \partial \varphi_t \wedge \bar{\partial} \varphi_s \wedge \text{Im}(e^{-\sqrt{-1} \bar{\theta}_0} \Omega^n) = - np \int_X (p - 1) \varphi_t^{4p-2} + (2p - 1) \delta^2 \varphi_t^{2p-2} \varphi_s \sqrt{-1} \partial \varphi_t \wedge \bar{\partial} \varphi_t \wedge \text{Im}(e^{-\sqrt{-1} \bar{\theta}_0} \Omega^n)
\]
\[
- \int_X \varphi_t^{2p-1} \varphi_s \sqrt{-1} \partial \bar{\partial} \varphi_t \wedge \text{Im}(e^{-\sqrt{-1} \bar{\theta}_0} \Omega^n).
\]

Substituting the above into \( (3.9) \) and using the \( \varepsilon \)-geodesic equation \( (3.3) \),
\[
\frac{\partial E(s, t)}{\partial s} = p \frac{\partial}{\partial t} \left( \int_X \frac{\varphi_t^{2p-1} \varphi_s}{\sqrt{\varphi_t^{2p} + \delta^2}} \text{Re}(e^{-\sqrt{-1} \bar{\theta}_0} \Omega^n) \right)
\]
\[
+ 4p e^{-2t \varepsilon^2} \int_X (p - 1) \varphi_t^{4p-2} + (2p - 1) \delta^2 \varphi_t^{2p-2} \varphi_s \text{Im}(e^{-\sqrt{-1} \bar{\theta}_0} \Omega^n).
\]
Applying the operator \( \partial_s \) to \( (3.3) \) and using the maximum principle, we obtain \( |\varphi_s| \leq C \). Thus, by the estimate of \( \varepsilon \)-geodesic from Theorem \( 3.11 \)
\[
\frac{\partial E(s, t)}{\partial s} \geq p \frac{\partial}{\partial t} \left( \int_X \frac{\varphi_t^{2p-1} \varphi_s}{\sqrt{\varphi_t^{2p} + \delta^2}} \text{Re}(e^{-\sqrt{-1} \bar{\theta}_0} \Omega^n) \right) - C \varepsilon^2 \delta^{-3}.
\]
Substituting this into \( (3.8) \),
\[
\hat{\mathcal{E}}'(s) = \frac{1}{p} \int_0^1 E_{\mathcal{E}}^{1-p}(s, E) dt
\]
\[
\geq \left. \left( E_{\mathcal{E}}^{1-p} \int_X \frac{\varphi_t^{2p-1} \varphi_s}{\sqrt{\varphi_t^{2p} + \delta^2}} \text{Re}(e^{-\sqrt{-1} \bar{\theta}_0} \Omega^n) \right) \right|_{t=1} - \left. \left( E_{\mathcal{E}}^{1-p} \int_X \frac{\varphi_t^{2p-1} \varphi_s}{\sqrt{\varphi_t^{2p} + \delta^2}} \text{Re}(e^{-\sqrt{-1} \bar{\theta}_0} \Omega^n) dt - C \varepsilon^2 \delta^{-3} \int_0^1 \hat{E}_{\mathcal{E}}^{1-p} dt.\right.
\]
By the definition of \( \varphi \), we see that \( \varphi(s, 0) = \psi(s) \) and \( \varphi(s, 1) = \tilde{\psi} \), which implies \( \varphi_s(s, 0) = \psi_s(s) \) and \( \varphi_s(s, 1) = 0 \). On the other hand, Lemma \( 3.3 \) shows \( E \geq C^{-1} \) and \( |\partial_t E_{\mathcal{E}}^{1-p}| \leq C \varepsilon^2 \) provided that \( \varepsilon \) is sufficiently small. Using Theorem \( 3.11 \) and \( |\varphi_s| \leq C \) again,
\[
\hat{\mathcal{E}}'(s) \geq - E(s, 0) \hat{E}_{\mathcal{E}}^{1-p} \int_X \frac{\varphi_t^{2p-1} \psi_s}{\sqrt{\varphi_t^{2p} + \delta^2}} \text{Re}(e^{-\sqrt{-1} \bar{\theta}_0} \Omega^n) - C \varepsilon^2 \delta^{-3}.
\]
By Young’s inequality,
\[ \int_X \frac{\frac{2p-1}{2} \psi_s}{\sqrt{\psi_t^p + \delta^2}} \text{Re}(e^{-\sqrt{-\Delta_0} \Omega_\psi^n}) \leq E(s, o) \left( \int_X |\psi_s|^p \text{Re}(e^{-\sqrt{-\Delta_0} \Omega_\psi^n}) \right)^{\frac{1}{p}}. \]

Recalling \( \ell'(s) = \left( \int_X \sqrt{\psi_t^p + \delta^2} \text{Re}(e^{-\sqrt{-\Delta_0} \Omega_\psi^n}) \right)^{\frac{1}{p}} \) from (3.7), we obtain
\[ \ell'(s) + \ell'(s) \geq -C\varepsilon^2 \delta^{-3}, \]
as required. \( \square \)

**Lemma 3.5.** For any \( \varphi_0, \varphi_1 \in \mathcal{H} \), let \( \varphi^\varepsilon \) be the \( \varepsilon \)-geodesic connecting \( \varphi_0 \) and \( \varphi_1 \). Then we have

(i) \( d_p(\varphi_0, \varphi_1) = \lim_{\varepsilon \to 0} \text{length}_p(\varphi^\varepsilon) \);

(ii) \( d'_p(\varphi_0, \varphi_1) = \lim_{\varepsilon \to 0} E_p(\varepsilon(t)) \) for any \( t \in [0, 1] \);

(iii) The following inequality holds
\[ d_p^p(\varphi_0, \varphi_1) \geq \max \left\{ \int_{\{s > 0\}} |\varphi_0 - \varphi_1|^p \text{Re}(e^{-\sqrt{-\Delta_0}(\alpha \varphi_0 + \sqrt{-1}\omega)}) \right\}. \]

**Proof.** (ii) and (iii) are immediate corollaries of (i), Lemma 3.3 (3.6) and Lemma 2.7. It remains to prove (i). By the definition of \( d_p \),
\[ d_p(\varphi_0, \varphi_1) \leq \liminf_{\varepsilon \to 0} \text{length}_p(\varphi^\varepsilon). \]
It suffices to prove
\[ \limsup_{\varepsilon \to 0} \text{length}_p(\varphi^\varepsilon) \leq d_p(\varphi_0, \varphi_1). \]

Let \( \psi(s), s \in [0, 1] \) be any smooth path connecting \( \varphi_0 \) and \( \varphi_1 \). We assume without loss of generality that \( \varphi_1 \notin \psi([0, 1]) \). For each \( s \in [0, 1] \), let \( \varphi^\varepsilon(s, t) \) be the \( \varepsilon \)-geodesic connecting \( \psi(s) \) to \( \varphi_1 \). For any \( s_0 \in (0, 1) \), using Lemma 3.4
\[ \text{length}_{p, \delta}(\varphi^\varepsilon(0, \cdot)) \leq \text{length}_{p, \delta}(\psi[0, s_0]) + \text{length}_{p, \delta}(\varphi^\varepsilon(s_0, \cdot)) + C\varepsilon^2 \delta^{-3}. \]
Combining this with (3.6) and \( \varphi^\varepsilon(0, \cdot) = \varphi^\varepsilon \), we obtain
\[ \text{length}_p(\varphi^\varepsilon) \leq \text{length}_p(\psi[0, s_0]) + \text{length}_p(\varphi^\varepsilon(s_0, \cdot)) + C\delta^\frac{1}{p} + C\varepsilon^2 \delta^{-3}. \]
By Lemma 3.2 (ii),
\[ \text{length}_p(\varphi^\varepsilon(s_0, \cdot)) \leq C\|\psi(s_0) - \psi(1)\|_{L^\infty} + C\varepsilon^2. \]
Hence,
\[ \limsup_{\varepsilon \to 0} \text{length}_p(\varphi^\varepsilon) \leq \text{length}_p(\psi[0, s_0]) + C\|\psi(s_0) - \psi(1)\|_{L^\infty} + C\delta^\frac{1}{p}. \]
By letting $\delta \to 0$ and $s_0 \to 1$,
\[
\lim_{\varepsilon \to 0} \sup_p \text{length}_p(\varphi^\varepsilon) \leq \text{length}_p(\psi).
\]
Since $\psi$ is arbitrary path connecting $\varphi_0$ and $\varphi_1$,
\[
\lim_{\varepsilon \to 0} \sup_p \text{length}_p(\varphi^\varepsilon) \leq d_p(\varphi_0, \varphi_1),
\]
as required. $\square$

Now we are in a position to prove Theorem 1.2.

**Proof of Theorem 1.2.** The non-negativity and symmetry of $d_p$ are trivial. The triangle inequality follows from Lemma 3.4 (3.6) and Lemma 3.5 (i). The positivity of $d_p$ follows from Lemma 3.5 (iii). $\square$

For later purpose, we will derive some inequalities on the $d_p$-distance.

**Lemma 3.6.** For any $\varphi_0, \varphi_1, \varphi_2 \in \mathcal{H}$ such that $\varphi_0 \leq \varphi_1 \leq \varphi_2$, we have
\[
d^p_p(\varphi_0, \varphi_1) \leq d^p_p(\varphi_0, \varphi_2) \leq \int_X |\varphi_0 - \varphi_2|^p \text{Re}(e^{-\sqrt{-1}i\theta_0} (\alpha \varphi_0 + \sqrt{-1} \omega)^n).
\]

**Proof.** For $i = 0, 1, 2$, let $\varphi^{\varepsilon,i}$ and $\Phi^{\varepsilon,i}$ be the $\varepsilon$-geodesic joining from $\varphi_0$ to $\varphi_i$. Denote the corresponding $L^p$-energy by $E^\varepsilon_p$. By comparison principle,
\[
\Phi^{\varepsilon,0} \leq \Phi^{\varepsilon,1} \leq \Phi^{\varepsilon,2}
\]
and so
\[
\varphi^{\varepsilon,0} \leq \varphi^{\varepsilon,1} \leq \varphi^{\varepsilon,2}.
\]
This implies
\[
\varphi^{\varepsilon,0}_t(0) \leq \varphi^{\varepsilon,1}_t(0) \leq \varphi^{\varepsilon,2}_t(0).
\]

Thanks to Lemma 3.2
\[
\varphi^{\varepsilon,2}_t(0) \leq \varphi_2 - \varphi_0 + C \varepsilon^2, \quad |\varphi^{\varepsilon,0}_t(0)| \leq C \varepsilon^2.
\]
It then follows that
\[
-C \varepsilon^2 \leq \varphi^{\varepsilon,0}_t(0) \leq \varphi^{\varepsilon,1}_t(0) \leq \varphi^{\varepsilon,2}_t(0) \leq \varphi_2 - \varphi_0 + C \varepsilon^2
\]
and
\[
E^{\varepsilon,1}_p(0) \leq E^{\varepsilon,2}_p(0) + C \varepsilon^{2p} \leq \int_X |\varphi_0 - \varphi_2|^p \text{Re}(e^{-\sqrt{-1}i\theta_0} \Omega^n_{\varphi_0}) + C \varepsilon^{2p}.
\]
Letting $\varepsilon \to 0$ and using Lemma 3.5 (ii), we obtain
\[
d^p_p(\varphi_0, \varphi_1) \leq d^p_p(\varphi_0, \varphi_2) \leq \int_X |\varphi_0 - \varphi_2|^p \text{Re}(e^{-\sqrt{-1}i\theta_0} \Omega^n_{\varphi_0}).
\]
Recalling the definition of $\Omega_{\varphi_0}$ (see (3.5)) and using Lemma 2.2
\[
\text{Re}(e^{-\sqrt{-1}i\theta_0} \Omega^n_{\varphi_0}) = \text{Re}(e^{-\sqrt{-1}i\theta_0} (\omega + \sqrt{-1} \alpha \varphi_0)^n) = \text{Re}(e^{-\sqrt{-1}i\theta_0} (\alpha \varphi_0 + \sqrt{-1} \omega)^n).
\]
Then we obtain the required inequality. $\square$
Lemma 3.7. For \( c_0 \in (0, \Theta_0) \), there exists \( C(c_0, \alpha, \omega, X) \) such that for any \( \varphi \in \mathcal{H} \) with \( \sup_X \varphi = 0 \) and \( Q_\omega(\alpha_\varphi) \leq \Theta_0 - c_0 \),

\[
d_p^p(\varphi, 0) \geq \frac{1}{C} \int_X |\varphi|^p \chi^p_n,\]

where \( \Theta_0 = \theta_0 + \frac{\pi}{2} \) and \( \chi, \chi_\varphi \) are defined in Lemma 2.3.

Proof. By Lemma 3.6 and 3.5 (iii),

\[
d_p^p(\varphi, 0) \geq d_p^p \left( \varphi, \frac{\varphi}{2} \right) \geq \frac{1}{2^p} \int_X |\varphi|^p \text{Re}(e^{-\sqrt{-1} \theta_0} \Omega^\varphi_n)\]

\[
= \frac{1}{2^p} \int_X |\varphi|^p \text{Re} \left( e^{-\sqrt{-1} \theta_0} (\alpha_\varphi^2 + \sqrt{-1} \omega)^n \right),
\]

where we used Lemma 2.2 in the last equality.

Let \( \lambda_i \) and \( \mu_i \) be the eigenvalues of \( \alpha_\varphi \) and \( \alpha_\varphi^2 \) with respect to \( \omega \). It is clear that

\[
\alpha_\varphi^2 = \frac{1}{2} \alpha + \frac{1}{2} \alpha_\varphi.
\]

Thanks to the concavity of \( \cot(Q_\omega) \), \( Q\omega(\alpha) \leq \Theta_0 - c'_0 \) and \( Q\omega(\alpha_\varphi) \leq \Theta_0 - c_0 \),

\[
Q\omega(\alpha_\varphi^2) \leq \Theta_0 - c''_0.
\]

On the other hand, by Weyl’s inequality, for each \( i \),

\[
\left| \mu_i - \frac{\lambda_i}{2} \right| \leq C.
\]

Then we have

\[
\frac{\text{Re}(e^{-\sqrt{-1} \theta_0} (\alpha_\varphi^2 + \sqrt{-1} \omega)^n)}{\omega^n} \geq \cos(Q\omega(\alpha_\varphi^2) - \theta_0) \prod_{i=1}^n \sqrt{1 + \mu_i^2} \geq \frac{1}{C} \prod_{i=1}^n \sqrt{1 + \lambda_i^2}.
\]

Recalling \( \chi_\varphi = \tan(\theta_0) \omega + \alpha_\varphi \),

\[
\frac{\chi_n^\varphi}{\omega^n} = \prod_{i=1}^n (\tan(\theta_0) + \lambda_i) \leq C \prod_{i=1}^n \sqrt{1 + \lambda_i^2} \leq C \cdot \frac{\text{Re}(e^{-\sqrt{-1} \theta_0} (\alpha_\varphi^2 + \sqrt{-1} \omega)^n)}{\omega^n}.
\]

Substituting this into (3.10), we obtain

\[
d_p^p(\varphi, 0) \geq \frac{1}{C} \int_X |\varphi|^p \chi^p_n,
\]

as required. \( \square \)
4. Twisted dHYM flow

In this section, we will study the twisted dHYM flow starting from \( \varphi_0 \in \mathcal{H} \):

\[
\begin{aligned}
\partial_t \varphi &= F_{\omega,\varepsilon}(\alpha_\varphi) - \cot \theta_0 - a_0 \varepsilon; \\
\varphi(0) &= \varphi_0.
\end{aligned}
\]

(4.1)

Recall that the twisted dHYM flow admits an unique short-time solution on \( X \) thanks to Lemma \[2.4\] and \[2.7\]. In fact, the linearised operator is given by

\[
\Box_L = \partial_t - F_{\omega,\varepsilon}^j \nabla_i \nabla_j
\]

which is parabolic as long as the eigenvalues of \( \alpha_\varphi \) with respect to \( \omega \) lies inside \( \Gamma_{\theta,\Theta} \). We let \( T_{\text{max}} > 0 \) be the maximal existence time of (4.1). In this section, we will study its long-time existence and behaviour. We first show that the \( \Box_L \) will remain parabolic as long as the flow exists.

**Lemma 4.1.** There are \( \varepsilon_0(\varphi_0, \alpha, \omega, X), c_0(\varphi_0, \alpha, \omega, X) > 0 \) such that if \( \varepsilon < \varepsilon_0 \), then for all \( t \in [0, T_{\text{max}}] \),

\[
c_0 \leq Q\omega(\alpha_\varphi) \leq \Theta_0 - c_0.
\]

In particular, \( \alpha_\varphi \in \Gamma_{\omega,\theta_0-c_0,\theta_0} \), and \( \alpha_\varphi > -(\tan \theta_0) \omega \) for all \( t \in [0, T_{\text{max}}) \).

**Proof.** Let \( c_0 > 0 \) be a constant to be chosen. Define

\[
S = \sup\{ s > 0 \mid Q\omega(\alpha_\varphi) \in [c_0, \Theta_0 - c_0] \text{ for } t < s \}.
\]

Since \( \varphi_0 \in \mathcal{H} \), when \( c_0 \) is sufficiently small, the set on the right hand side is not empty and so the number \( S \) is well-defined. If \( S = T_{\text{max}} \), then we are done. It suffices to rule out the case \( S < T_{\text{max}} \). By differentiating (4.1) with respect to \( t \),

\[
\Box_L F_{\omega,\varepsilon}(\alpha_\varphi) = 0.
\]

By maximum principle and passing \( t \to S \), we conclude that for all \( t \in [0, S] \),

\[
\inf_X F_{\omega,\varepsilon}(\alpha_{\varphi_0}) \leq F_{\omega,\varepsilon}(\alpha_\varphi) \leq \sup_X F_{\omega,\varepsilon}(\alpha_{\varphi_0}).
\]

Since \( \varphi_0 \in \mathcal{H} \), \( c_1 \leq Q\omega(\alpha_{\varphi_0}) \leq \Theta_0 - c_1 \) for some \( c_1(\varphi_0, \alpha, \omega, X) > 0 \) and hence

\[
\cot(\Theta_0 - c_1) \leq F_{\omega,\varepsilon}(\alpha_{\varphi_0}) \leq \cot(c_1) + \varepsilon \csc(c_1).
\]

Therefore,

\[
\cot(Q\omega(\alpha_\varphi)) \leq F_{\omega,\varepsilon}(\alpha_\varphi) \leq \cot(c_1) + \varepsilon \csc(c_1) = \cot(c_2),
\]

which shows \( Q\omega(\alpha_\varphi) \geq c_2 \) for some \( c_2(\varphi_0, \alpha, \omega, X) > 0 \) on \( [0, S] \).

Next, we will show \( Q\omega(\alpha_\varphi) < \Theta_0 - \frac{a_1}{2} \) on \( [0, S] \). If \( Q\omega(\alpha_\varphi) \geq \Theta_0 - \frac{a_1}{2} \) at some point, then by

\[
\cot(Q\omega(\alpha_\varphi)) + \frac{\varepsilon \omega^n}{\Im(\alpha_\varphi + \sqrt{-1} \omega)^n} = F_{\omega,\varepsilon}(\alpha_\varphi) \geq \inf_X F_{\omega,\varepsilon}(\alpha_{\varphi_0}) \geq \cot(\Theta_0 - c_1),
\]

we obtain

\[
\frac{\varepsilon}{\sin(Q\omega(\alpha_\varphi))} \geq \frac{\varepsilon \omega^n}{\Im(\alpha_\varphi + \sqrt{-1} \omega)^n} \geq \cot(\Theta_0 - c_1) - \cot \left( \Theta_0 - \frac{c_1}{2} \right).
\]
\[ Q_\omega(\alpha_\varphi) \leq c_3 \varepsilon \]

for some \( c_3(\varphi_0, \alpha, \omega, X) > 0 \), which contradicts with \( Q_\omega(\alpha_\varphi) \geq \Theta_0 - \hat{\alpha}_0 \) after decreasing \( \varepsilon_0 \) if necessary.

We now have \( c_2 \leq Q_\omega(\alpha_\varphi) < \Theta_0 - \hat{\alpha}_0 \) on \([0, S]\). Choose \( c_0 = \min(\frac{c_2}{2}, c_2) \), then \( Q_\omega(\alpha_\varphi) \in (c_0, \Theta_0 - c_0) \) for all \( t \in [0, S] \), which contradicts with the maximality of \( S \). This shows \( S < T_{\text{max}} \) is impossible, or equivalently, \( S = T_{\text{max}} \). Since \( P_\omega(\alpha_\varphi) \leq Q_\omega(\alpha_\varphi) \), then \( \alpha_\varphi \in \Gamma_{\omega, \Theta_0 - c_0, \Theta_0} \). The lower bound of \( \alpha_\varphi \) follows from Lemma 2.3.

It is known that the twisted dHYM flow is the negative gradient flow of \( J_\varepsilon \)-functional which is basically from construction. For the later purpose, we will consider the \( Z \)-functional defined in Collins-Yau [15] which is defined to be \( Z = e^{-\sqrt{-1} \alpha / 2} \) CYC in terms of the Calabi-Yau functional. Or equivalently, it is given by \( Z(0) = 0 \) and

\[ \delta Z(u) = \int_X (\delta u) e^{-\sqrt{-1} \alpha / 2} (\omega + \sqrt{-1} \alpha_u)^n. \]

By the similar calculation of Lemma 2.2, we obtain

\[ \text{Im} \left( e^{-\sqrt{-1} \alpha / 2} (\omega + \sqrt{-1} \alpha_u)^n \right) = -\text{Im} \left( \alpha_u + \sqrt{-1} \omega \right)^n \]

which gives us

\[ \delta(\text{Im} Z)(u) = -\int_X (\delta u) \text{Im} \left( \alpha_u + \sqrt{-1} \omega \right)^n. \]

Lemma 4.2. Along the twisted dHYM flow (4.1), we have for all \( t \in [0, T_{\text{max}}) \),

\[ \text{Im}(Z(\varphi(t))) = \text{Im}(Z(\varphi_0)). \]

Proof. Direct calculation shows

\[
\frac{d}{dt} \text{Im} \left( Z(\varphi) \right) = -\int_X (F_{\omega, \varepsilon}(\alpha_\varphi) - \cot \theta_0 - a_0 \varepsilon) \text{Im} \left( \alpha_\varphi + \sqrt{-1} \omega \right)^n \\
= -\int_X \text{Re} \left( \alpha_\varphi + \sqrt{-1} \omega \right)^n - \cot \theta_0 \cdot \text{Im} \left( \alpha_\varphi + \sqrt{-1} \omega \right)^n \\
- \int_X \varepsilon \omega^n - a_0 \varepsilon \text{Im} \left( \alpha_\varphi + \sqrt{-1} \omega \right)^n \\
= 0.
\]

We also need the following evolution equation along the twisted dHYM flow.

Lemma 4.3. The twisted dHYM flow (4.1) satisfies

\[ \square_L(\alpha_\varphi)_{ij} = F_{\omega, \varepsilon}^{pq, kl} \nabla_i \alpha_p \nabla_j \alpha_k \nabla_l + F_{\omega, \varepsilon}^{pq} \left( R_{pji}^k \alpha_k - R_{pjq}^k \alpha_i \right). \]
Proof. In the following, all connections are computed with respect to $\omega$. We will omit the subscript for notational convenience. We compute using $d$-closed of $\alpha$ and Ricci identity to obtain:

$$
\partial_t \alpha_{ij} = \partial_i \partial_j \partial_t \varphi \\
= \partial_i \partial_j F_{\varphi} \\
= \nabla_j (F_{\varphi} \nabla_i \alpha_{pq}) \\
= F_{\varphi} \nabla_i \alpha_{pq} \nabla_j \alpha_{kl} + F_{\varphi} \nabla_j \nabla_i \alpha_{pq} \\
= F_{\varphi} \nabla_i \alpha_{pq} \nabla_j \alpha_{kl} + F_{\varphi} \nabla_j \nabla_i \alpha_{iq} \\
= F_{\varphi} \nabla_i \alpha_{pq} \nabla_j \alpha_{kl} + F_{\varphi} \left( \nabla_p \nabla_j \alpha_{iq} + R_{pji} \alpha_{kq} - R_{pjq} \alpha_{ik} \right) \\
= F_{\varphi} \nabla_i \alpha_{pq} \nabla_j \alpha_{kl} + F_{\varphi} \left( \nabla_p \nabla_q \alpha_{ij} + R_{pji} \alpha_{kq} - R_{pjq} \alpha_{ik} \right). 
$$

4.1. Long-time existence. In this subsection, we will show that if $\varepsilon$ is sufficiently small, the corresponding twisted dHYM flow (4.1) will exist for all $t > 0$.

Theorem 4.4. There is $\varepsilon_0(\varphi_0, \alpha, \omega, X) > 0$ such that for all $\varepsilon < \varepsilon_0$, the twisted dHYM flow (4.1) admits a unique solution on $X \times [0, +\infty)$.

Proof. Suppose on the contrary, $T_{\text{max}} < +\infty$. Since $\varphi$ is uniformly bounded along (4.1) by Lemma 4.1, $\varphi$ is uniformly bounded in finite time. Moreover, Lemma 4.1 implies $\varphi \in H$ for $t \in [0, T_{\text{max}})$. Since $F_{\varphi, \omega}$ is concave, by standard parabolic theory, if $|\sqrt{-1} \partial \bar{\partial} \varphi|$ is uniformly bounded on $[0, T_{\text{max}})$, then $\varphi$ is bounded in $C^k$ for all $k$ on $[0, T_{\text{max}})$. This will contradict the maximality of $T_{\text{max}} < +\infty$. Thanks to the lower bound of $\alpha_\varphi$ from Lemma 4.1, it suffices to estimate $\text{tr}_\omega \alpha_\varphi$. In the following, we will use $C_\varphi$ to denote constants depending only on $\varphi_0, \omega, X$ and omit subscript $\varphi, \varepsilon$ for notational convenience.

Denote the associated Riemannian metric of $\omega$ by $g$. Taking trace on the equation from Lemma 4.3, we have

$$
\Box_L \log \text{tr}_\omega \alpha \leq \frac{1}{\text{tr}_\omega \alpha} g^{ij} F_{pq,kl} \nabla_j (\alpha_\varphi)_{pq} \nabla_i (\alpha_\varphi)_{kl} \\
+ \frac{1}{(\text{tr}_\omega \alpha)^2} g^{pq} g^{kl} F_{ij} \nabla_i \alpha_{pq} \nabla_j \alpha_{kl} + \frac{C_\alpha}{\text{tr}_\omega \alpha_\varphi} |F_{ij}| |\text{Rm}| |\alpha|.
$$

At each $(x, t)$, we will work on the coordinate so that

$$
g_{ij} = \delta_{ij}, \quad (\alpha_\varphi)_{ij} = \lambda_i \delta_{ij}.
$$

Hence using (2.7) and the concavity from Lemma 2.4, we conclude that

$$
\Box_L \log \text{tr}_\omega \alpha_\varphi \leq \frac{F_{ii}}{(\text{tr}_\omega \alpha)^2} |\nabla_i \log \text{tr}_\omega \alpha_\varphi|^2 + C_0 \sum_{i=1}^n F_i.
$$
Recall that

\[ F_\varepsilon(\lambda) = \cot Q(\lambda) + \frac{\varepsilon \csc Q(\lambda)}{\prod_{j=1}^n \sqrt{\lambda_j^2 + 1}}. \]

Together with the bound of \( Q(\lambda) \) from Lemma 4.1,

\[ (4.3) \quad F_i = \frac{\partial F_\varepsilon}{\partial \lambda_i} = \frac{\varepsilon \csc Q(\lambda)}{\lambda_i} + \frac{\varepsilon \csc Q(\cot Q - \lambda_i)}{(\lambda_i^2 + 1) \cdot \prod_{j=1}^n \sqrt{\lambda_j^2 + 1}} \leq C_1. \]

Therefore, the function \( G = \log \text{tr}_\omega \alpha_\varphi - 2C_2t \) satisfies

\[ \Box_L \log \text{tr}_\omega \alpha_\varphi \leq \frac{F_i}{(\text{tr}_\omega \alpha)^2} |\nabla_i \log \text{tr}_\omega \alpha_\varphi|^2 - C_2. \]

For any \( S < T_{\max} \), if \( G \) attains its maximum at \( (x_0, t_0) \in X \times [0, S] \) where \( t_0 > 0 \), then at this point,

\[ 0 \leq \Box_L G \leq -C_2 \]

which is impossible. Here we have used the fact that \( \nabla G|_{(x_0, t_0)} = 0 \). By passing \( S \to T_{\max} \), we have shown that for all \( t \in [0, T_{\max}) \),

\[ \log \text{tr}_\omega \alpha_\varphi \leq C_3(t + 1) \leq C_3(T_{\max} + 1). \]

This completes the proof. \( \square \)

### 4.2. Long-time behaviour

In this section, we will study the convergence of twisted dHYM flow assuming the existence of a \( C \)-subsolution. The following is the main result of this subsection which is a twisted version of result in [22].

**Theorem 4.5.** Suppose that the dHYM equation \((1.1)\) admits a solution \( \hat{\varphi}_0 \). There exists \( \varepsilon_0(\hat{\varphi}_0, \alpha, \omega, X) \) such that for any \( \varepsilon \in (0, \varepsilon_0) \), the twisted dHYM flow \((4.1)\) will converge smoothly to \( \hat{\varphi}_\varepsilon \) as \( t \to +\infty \), where \( \varphi_\varepsilon \) is the solution of twisted dHYM equation, i.e.,

\[ F_{\omega, \varepsilon}(\alpha_{\varphi_\varepsilon}) = \cot(\theta_0) + a_0\varepsilon. \]

By Theorem 4.4, it is known that the twisted dHYM flow \((4.1)\) exists for all \( t \geq 0 \). It remains to study its convergence. Before we prove the uniform \( C^k \) estimate along the flow, we first recall the notion of \( C \)-subsolution from [34] in the content of parabolic twisted dHYM equation.

**Definition 4.6.** A smooth function \( u \) is said to be a (parabolic) \( C \)-subsolution of \((1.1)\), if there exist constants \( \delta, K > 0 \), so that for any \( (z, t) \in X \times [0, T) \), the condition

\[ F(\lambda(\alpha_\omega) + \mu) - \partial_t u + \tau = \cot(\theta_0) + a_0\varepsilon, \quad \mu + \delta I_n \in \Gamma_n, \quad \tau > -\delta \]

implies that \( |\mu| + |\tau| \leq K \), where \( \lambda(\alpha_\omega) \) denotes the eigenvalue of \( \alpha_\omega \) with respect to \( \omega \), and \( I_n \) denotes the vector \((1, \ldots, 1)\).
By the work of [10], it is now known that existence of subsolution is equivalent to the existence of dHYM equation. The following Lemma shows that a dHYM equation is indeed a parabolic $C$-subsolution in the sense of [34] as expected, see also [42, Section 5].

**Lemma 4.7.** Suppose the dHYM equation (1.1) admits a solution $\hat{\varphi}_0$, then there is $\varepsilon_0(\hat{\varphi}_0, \alpha, \omega, X) > 0$ such that for all $\varepsilon < \varepsilon_0$, $\hat{\varphi}_0$ is a (parabolic) $C$-subsolution of (4.1).

**Proof.** Note that $u = \hat{\varphi}_0$ is time independent and so $\partial_t u \equiv 0$. Combining this with $(\star)$ and the monotonicity of $F$ from Lemma 2.4 (i), we see that

$$\tau = \cot \theta_0 + a_0\varepsilon - F(\lambda(\alpha_u) + \mu) \leq \cot \theta_0 + a_0\varepsilon - F(\lambda(\alpha_u) - \delta I_n).$$

It remains to consider $\mu$. The equation $(\star)$ can be rewritten as

$$\cot Q + \frac{\varepsilon}{\sin Q \cdot \prod_{i=1}^n \sqrt{(\lambda_i + \mu_i)^2 + 1}} = \cot \theta_0 + a_0\varepsilon - \tau,$$

where $Q = \sum_{i=1}^n \arccot(\lambda_i + \mu_i)$. Suppose $\mu_j \geq \Lambda$ for some $j$, where $\Lambda$ is a large constant to be specified. Then

$$\sum_{i \neq j} \arccot(\lambda_i - \delta) \geq \sum_{i \neq j} \arccot(\lambda_i + \mu_i)$$

$$\geq Q - \arccot(\lambda_j + \Lambda)$$

$$\geq \arccot(\cot \theta_0 + a_0\varepsilon + \delta) - \arccot(\lambda_j + \Lambda)$$

$$= \arccot \left[ \cot \left( \sum_{i=1}^n \arccot(\lambda_i) \right) + a_0\varepsilon + \delta \right] - \arccot(\lambda_j + \Lambda).$$

If we choose $\Lambda$ sufficiently large and $\varepsilon_0, \delta$ sufficiently small, it will be impossible and hence $\mu_j < \Lambda$ for all $j$. This gives the upper bound of $\mu$. □

**Lemma 4.8.** There exist $C_0, \varepsilon_0 > 0$ depending only on $\varphi_0, \hat{\varphi}_0, \alpha, \omega, X$ such that for all $\varepsilon < \varepsilon_0$, the twisted dHYM flow (4.1) satisfies

$$\sup_{X \times [0, +\infty)} |\varphi| \leq C_0.$$

**Proof.** By Lemma 4.2 we have

$$\int_X (\varphi - \varphi_0) \int_0^1 \text{Im} \left( \alpha_{s\varphi + (1-s)\varphi_0} + \sqrt{-1}\omega \right)^n ds = 0.$$

and $\varphi$ is uniformly quasi-plurisubharmonic for all $t \geq 0$ thanks to Lemma 4.1.

Since $\hat{\varphi}_0$ serves as a $C$-subsolution by Lemma 4.7 and the operator $F_{\omega,\varepsilon}$ is elliptic by Lemma 2.4 (i), the proof of [22, Lemma 3.4, 3.5] which is based on the work [34] can now be carried over, see also [42, Lemma 5.3]. □
It is known that the solution to $dHYM$ equation naturally provides a $C$-subsolution to the twisted $dHYM$ equation. This will provide us a barrier to establish higher order estimates along (4.1). We first need the following key properties of $C$-subsolutions.

**Lemma 4.9.** There exist $\rho, K, \varepsilon_0 > 0$ depending only on $\varphi_0, \hat{\varphi}_0, \alpha, \omega, X$ such that for $\varepsilon < \varepsilon_0$, if $|\alpha\varphi - \alpha\hat{\varphi}_0| > K$, then either

$$\Box_L(\varphi - \hat{\varphi}_0) > \rho \sum_{i=1}^{n} F_{\omega,\varepsilon}^{\alpha\varphi}$$

or for all $1 \leq i \leq n$,

$$F_{\omega,\varepsilon}^{\alpha\varphi} > \rho \sum_{j=1}^{n} F_{\omega,\varepsilon}^{\alpha\hat{\varphi}}.$$

**Proof.** Since $\hat{\varphi}_0$ is a $C$-subsolution by Lemma 4.7 and $F_{\omega,\varepsilon}$ is concave from Lemma 2.4 (iii). The argument in [34, Lemma 3] can now be carried over, see also [42, Lemma 5.4].

Before we prove the second order estimate, we first show that the first order is controlling the second order in an improved rate.

**Lemma 4.10.** There exist $C, \varepsilon_0 > 0$ depending only on $\varphi_0, \hat{\varphi}_0, \alpha, \omega, X$ such that for all $T \in (0, +\infty)$, $\varepsilon < \varepsilon_0$,

$$\sup_{X \times [0, T]} |\sqrt{-1} \partial \bar{\partial} \varphi| \leq C(1 + \sup_{X \times [0, T]} |\nabla \varphi|).$$

**Proof.** We will always assume $\varepsilon_0$ is sufficiently small so that the previous lemmas hold. Lemma 4.1 shows $\alpha\varphi > -(\tan \theta_0) \omega$. Then it suffices to estimate the upper bound of $\alpha\varphi$. We will follow the argument in [22] where the test function only involves the zeroth and second order quantities. In the following, we will use $C_i$ to denote any constants depending only on $n, \varphi_0, \hat{\varphi}_0, \alpha, \omega, X$. We will also omit the subscript on $\alpha\varphi, F_{\omega,\varepsilon}$ for notational convenience.

On any compact time interval $[0, T]$, consider the function

$$G = \log \lambda_{\max} + \phi(u),$$

where $\lambda_{\max}$ is the largest eigenvalue of $\alpha\varphi$ with respect to $\omega$, $u = \varphi - \hat{\varphi}_0$ and $\phi$ is non-increasing function on $\mathbb{R}$ which will be chosen later. Note that $u$ is uniformly bounded by Lemma 4.8 and hence $\phi$ can be chosen so that its derivatives are uniformly bounded for all time.

On $[0, T]$, suppose $G$ attains its maximum at $(x_0, t_0) \in X \times (0, T]$. We note that although $G$ is only continuous in general, we will use the perturbation technique as in [11, Section 4]. We therefore will assume $G$ to be differentiable when we apply maximum principle. We will choose a coordinate at $(x_0, t_0)$ so that

$$g_{ij} = \delta_{ij} \quad \text{and} \quad (\alpha\varphi)_{ij} = \lambda_i \delta_{ij}.$$
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with \( \lambda_1 \geq \ldots \geq \lambda_n \) so that \( \lambda_1 = \lambda_{\text{max}} \). As in \([41]\), we may assume \( \lambda_1 > \lambda_2 \). Furthermore, we may assume \( \lambda_1 \geq 1 \), otherwise the result holds trivially. Recall the formulas of the variation of the largest eigenvalue:

\[
\begin{align*}
\nabla_i \lambda_1 &= \nabla_i \alpha_{11} \\
\nabla_i \nabla_i \lambda_1 &= \nabla_i \nabla_i \alpha_{11} + \sum_{k=2}^{n} \frac{|\nabla_i \alpha_{1k}|^2 + |\nabla_i \alpha_{k1}|^2}{\lambda_1 - \lambda_k}.
\end{align*}
\]

Combining this with Lemma 4.3, we compute

\[
\Box_L \log \lambda_1 = \lambda_1^{-1} \Box_L \lambda_1 + \lambda_1^{-2} F^{ij} |\nabla_i \lambda_1|^2
\]

\[
= \lambda_1^{-1} \Box_L \alpha_{11} + \lambda_1^{-2} F^{ij} |\nabla_i \lambda_1|^2 - \lambda_1^{-1} F^{ij} \sum_{k=2}^{n} \frac{|\nabla_i \alpha_{1k}|^2 + |\nabla_i \alpha_{k1}|^2}{\lambda_1 - \lambda_k}
\]

\[
= \lambda_1^{-1} \left[ F^{pq,k \bar{l}} \nabla_i \alpha_{pq} \nabla_i \alpha_{kl} + F^{pq} \left( R^{p_{\bar{j}}} R^{q}_{\bar{j} k \bar{l}} - R^{p}_{\bar{j} q k \bar{l}} \alpha_{i \bar{l}} \right) \right]
\]

\[
+ \lambda_1^{-2} F^{ij} |\nabla_i \lambda_1|^2 - \lambda_1^{-1} F^{ij} \sum_{k=2}^{n} \frac{|\nabla_i \alpha_{1k}|^2 + |\nabla_i \alpha_{k1}|^2}{\lambda_1 - \lambda_k}.
\]

Using (2.7) and (4.3), we see that

\[
\Box_L \log \lambda_1 \leq \lambda_1^{-1} F^{i}_{pk} |\nabla_i \alpha_{pi}|^2 + \lambda_1^{-1} \sum_{p \neq q} \frac{F_{p} - F_{q}}{\lambda_p - \lambda_q} |\nabla_i \alpha_{pq}|^2
\]

\[
- \lambda_1^{-1} F_i \sum_{k=2}^{n} \frac{|\nabla_i \alpha_{1k}|^2 + |\nabla_i \alpha_{k1}|^2}{\lambda_1 - \lambda_k} + \lambda_1^{-2} F_i |\nabla_i \lambda_1|^2 + C_0
\]

\[
(4.5)
\]

\[
\leq \lambda_1^{-1} \sum_{i=2}^{n} \frac{F_i - F_1}{\lambda_i - \lambda_1} |\nabla_i \lambda_1|^2 + \lambda_1^{-2} \sum_{i=1}^{n} F_i |\nabla_i \lambda_1|^2 + C_0
\]

\[
= \lambda_1^{-2} \sum_{i=2}^{n} \left( \frac{F_i - F_1}{\lambda_i - \lambda_1} + F_i \right) |\nabla_i \lambda_1|^2 + \lambda_1^{-2} F_1 |\nabla_1 \lambda_1|^2 + C_0,
\]
where we have used concavity of $F$ and $d\alpha = 0$ in the second-to-last line. To deal with the first term on the right hand side, we use (4.3) again and compute

\[
\frac{\lambda_1 (F_i - F_1)}{\lambda_i - \lambda_1} + F_i
\]

\[
= -\csc^2 Q \frac{\lambda_i^2 + \lambda_1 \lambda_i}{1 + \lambda_i^2} + \frac{\epsilon \lambda_1 \csc Q}{\prod_{j=1}^{n} \sqrt{\lambda_j^2 + 1}} \left( -\frac{(\lambda_1 + \lambda_i) \cot Q + \lambda_1 \lambda_i - 1}{(1 + \lambda_i^2)(1 + \lambda_1^2)} \right)
\]

\[
+ \frac{\csc^2 Q}{1 + \lambda_i^2} \cdot \frac{\epsilon \csc Q (\cot Q - \lambda_1)}{(\lambda_i^2 + 1) \cdot \prod_{j=1}^{n} \sqrt{\lambda_j^2 + 1}}
\]

\[
= \frac{\csc^2 Q}{1 + \lambda_i^2} \cdot \frac{1 - \lambda_1 \lambda_i}{1 + \lambda_i^2} + \frac{\epsilon \csc Q}{(\lambda_i^2 + 1) \cdot \prod_{j=1}^{n} \sqrt{\lambda_j^2 + 1}} \left[ \cot Q (1 - \lambda_1 \lambda_i) - (\lambda_1 + \lambda_i) \right]
\]

\[
= \frac{1 - \lambda_1 \lambda_i}{1 + \lambda_i^2} F_i - \frac{\epsilon \csc Q}{\prod_{j=1}^{n} \sqrt{\lambda_j^2 + 1}} \cdot \frac{\lambda_1}{1 + \lambda_i^2}.
\]

We now squeeze more negativity from the first term. We will use an observation which is originated in [50, 51]. Lemma 4.1 shows $Q \leq \Theta_0 - c_0$. Then we see that $
abla \log \lambda_n \geq -\tan \theta_0$ and $Q > c_0$. Hence, we may assume $\lambda_1 > 2C_1$ so that $Q - \lambda_1 \lambda_i < 0$ for $1 < i < n$ since otherwise the final result holds trivially. Combining this (4.5) and (4.6), we arrive at

\[
\square L \log \lambda_1 \leq \frac{1 - \lambda_1 \lambda_n}{1 + \lambda_1^2} F_n |\nabla_1 \log \lambda_1|^2 + F_1 |\nabla_1 \log \lambda_1|^2 + C_0.
\]

Since $G$ attains its maximum at $(x_0, t_0)$,

\[
\nabla \log \lambda_1 = -\phi' \nabla u,
\]

so that

\[
0 \leq \square L G \leq \frac{1 - \lambda_1 \lambda_n}{1 + \lambda_1^2} F_n (\phi')^2 |\nabla_1 u|^2 + F_1 (\phi')^2 |\nabla_1 u|^2 + C_0
\]

\[
+ \phi' \square L u - \phi'' F_i |\nabla_1 u|^2.
\]

Let $\rho, K$ be the constant obtained from Lemma 4.9. Using Lemma 4.11 we have $\lambda_n \geq -\tan \theta_0$ and $Q > c_0$. Then

\[
c_0 \leq \sum_{i=1}^{n} \arccot(\lambda_i) \leq n\arccot(\lambda_n)
\]

and hence

\[
|\lambda_n| \leq C_2.
\]
Denote $\arccot \lambda_i$ by $\theta_i$. Then $F_i$ can be written as

$$F_i = \csc^2 Q \frac{1}{1 + \lambda_i^2} - \varepsilon \csc^2 Q \frac{\sin(Q - \theta_i)}{1 + \lambda_i^2} \prod_{j \neq i} \frac{1}{\sqrt{1 + \lambda_j^2}}.$$ 

We see that if $\varepsilon_0$ is sufficiently small, then for all $i$,

$$\frac{\csc^2 Q}{1 + \lambda_i^2} \geq F_i \geq \frac{1}{2} \frac{\csc^2 Q}{1 + \lambda_i^2} .$$

Therefore if we further assume

$$\lambda_1 \geq \max \left\{ K + \max_X |\alpha_0|, \frac{10 \rho}{1 + C_2} \right\}$$

then $|\alpha - \alpha_0| > K$ and $F_1 \leq \rho F_n \leq \rho \sum_{i=1}^n F_i$. Combining this with Lemma 4.9 and (4.9),

$$\Box_L u > \rho \sum_{i=1}^n F_i \geq \rho F_n \geq \frac{\rho}{2} \cdot \frac{\csc^2 Q}{1 + \lambda_i^2} > C_3^{-1},$$

where we used Lemma 4.1 and (4.8) in the last inequality.

We also choose $C_4 > 0$ so that $C_4 > \sup_{X \times [0, +\infty)} |u|$ which is possible thanks to Lemma 4.8. Pick $\phi(s) = \frac{s^2}{2} - (C_5 + C_4)s$ with $C_5 = \max\{1, C_4, C_3(1 + C_0)\}$ so that $2C_5 > -\phi' > C_5$ and $\phi'' = 1$ at $s = u$. Substituting it and (4.10) into (4.7), we have

$$0 \leq \frac{1 - \lambda_1}{1 + \lambda_i^2} F_n(\phi')^2 |\nabla_n u|^2 + F_1(\phi')^2 |\nabla_1 u|^2 + C_0 + C_5^{-1} \phi' - \phi'' F_1 |\nabla_i u|^2$$

$$\leq F_n |\nabla_n u|^2 \left( \frac{1 - \lambda_1}{1 + \lambda_i^2} - 1 \right) + 4C_5^2 F_1 |\nabla_1 u|^2 + (C_0 - C_5^{-1} C_5)$$

$$\leq F_n |\nabla_n u|^2 (C_2 \lambda_i^{-1} - 1) + C_6 \lambda_i^{-2} |\nabla_1 u|^2 - 1,$$

where we have used (4.8), (4.9) and Lemma 4.1 in the last inequality. Hence, if $\lambda_1 > 2C_2$, the first bracket in evolution equation is negative and thus

$$\lambda_1(x_0, t_0) \leq C_7 \sup_{X \times [0, T]} |\nabla u| .$$

In conclusion, we have shown that

$$\lambda_1(x_0, t_0) \leq \max \left\{ C_7 \sup_{X \times [0, T]} |\nabla u|, 2C_2, K + \max_X |\alpha_0|, \frac{10 \rho}{1 + C_2^2}, 2C_1 \right\} .$$

Since $\nabla u$ is uniformly comparable to $\nabla \varphi$, the assertion now follows by applying maximum principle on $G$ since $u$ is uniformly bounded for all $t \geq 0$.

With Lemma 4.10, we can control the complex Hessian of $\varphi$ using exactly the same blow-up argument as in [42, Lemma 5.7], see also [10, Proposition 5.1].
Lemma 4.11. There exist $C, \varepsilon_0 > 0$ depending only on $\varphi_0, \dot{\varphi}_0, \alpha, \omega, X$ such that for all $\varepsilon < \varepsilon_0$,
$$
\sup_{X \times [0,T]} |\sqrt{-1} \partial \bar{\partial} \varphi| \leq C.
$$

Proof. Since the argument in [12, Lemma 5.7] is only based on zero order estimate, quasi-subharmonicity and the fact that the complex Hessian is controlled by its gradient quadratically. The proof can be carried over thanks to Lemma 4.1, 4.8 and 4.10.

Proof of Theorem 4.5. This is by now standard using the concavity. We include a sketch of the proof for the reader’s convenience. Thanks to Lemma 4.1, 4.8 and 4.10, $\varphi, \dot{\varphi}$ are uniformly bounded for all $t \geq 0$. By [21, 31], the concavity of $F_{\omega, \varepsilon}$ implies a uniform $C^{2,\alpha}$ estimates for all $t \geq 0$, e.g., see the proof of [24, Theorem 4.1] which only relies on the concavity. The higher order estimates follows from standard parabolic theory.

It remains to establish the convergence. Since $\dot{\varphi}$ satisfies
$$
\Box_L \dot{\varphi} = 0
$$
and $F_{\omega, \varepsilon}^{\bar{i} j} \partial_i \partial_j$ is uniformly elliptic, we can apply the standard argument (see e.g. [24, Section 6]) to show
$$
\text{osc}_X \dot{\varphi} \leq C e^{-c^{-1} t}
$$
for some $C(\varphi_0, \dot{\varphi}_0, \alpha, \omega, X) > 0$. Moreover, by (1.1) and (1.2),
$$
\int_X \dot{\varphi} \text{Im} \left( \alpha \varphi + \sqrt{-1} \omega \right)^n = 0.
$$
This shows for all $t \geq 0$, there is $x_t \in X$ so that $\dot{\varphi}(x_t, t) = 0$. It then follows that $\dot{\varphi}$ decays to zero exponentially fast as $t \to +\infty$. Together with the boundedness of $\varphi$, it is clear that $\varphi(t) \to \varphi_{\infty, \varepsilon}$ as $t \to +\infty$ for some function $\varphi_{\infty, \varepsilon}$. By the higher order estimates, this convergence is $C^\infty$ and so $\varphi_{\infty, \varepsilon}$ is a smooth function which solves the twisted dHYM equation
$$
F_{\omega, \varepsilon}(\alpha \varphi_{\infty, \varepsilon}) = \cot \theta_0 + a_0 \varepsilon.
$$
This completes the proof.

5. Technical lemmas

In this section, we will prove some technical lemmas, which will be used in the proof of Theorem 1.4.

5.1. Continuity of operators. In this subsection, we want to discuss the continuity of the operators $P_\omega, Q_\omega$ and $F_{\omega, \varepsilon}$ with respect to $\omega$. We begin with the continuity of $P_\omega$ and $Q_\omega$. 
Lemma 5.1. For any $c_0 \in (0, \pi)$, there exists a constant $\sigma_0(c_0, n)$ such that the following holds. Let $\omega_1$ and $\omega_2$ be two metrics and $\alpha$ be a real $(1,1)$-form satisfying
\[(1 - \sigma^5)\omega_1 \leq \omega_2 \leq (1 + \sigma^5)\omega_1\]
for some $\sigma \in (0, \sigma_0)$. Then
(i) $Q_{\omega_2}(\alpha) \leq Q_{\omega_1}(\alpha) + \sigma$ if $Q_{\omega_1}(\alpha) \in (0, \pi - c_0)$;
(ii) $P_{\omega_2}(\alpha) \leq P_{\omega_1}(\alpha) + \sigma$ if $P_{\omega_1}(\alpha) \in (0, \pi - c_0)$.

Proof. The proofs of (i) and (ii) are similar, which are almost identical to that of [9, Proposition 2.5]. For the reader’s convenience, we give a sketch of the proof of (i). Since
\[Q_{\omega_2}(\alpha) \leq Q_{\omega_2}(\alpha + \sigma \omega_2) + n \sigma,\]
after relabelling, it suffices to prove
\[Q_{\omega_2}(\alpha + \sigma \omega_2) \leq Q_{\omega_1}(\alpha) + \sigma.\]
Write $\theta = Q_{\omega_1}(\alpha)$ and $\theta' = \theta + \sigma$. The above inequality is equivalent to
\[\text{Im} \left( e^{-\sqrt{-1} \theta'} (\alpha + \sigma \omega_2 + \sqrt{-1} \omega_2)^n \right) \leq 0.\]

Set $\omega_d = \omega_2 - \omega_1$ so that
\[\alpha + \sigma \omega_2 + \sqrt{-1} \omega_2 = (\alpha + \sqrt{-1} \omega_1) + (\sigma \omega_2 + \sqrt{-1} \omega_d).\]

We compute
\[\text{Im} \left( e^{-\sqrt{-1} \theta'} (\alpha + \sigma \omega_2 + \sqrt{-1} \omega_2)^n \right) = \sum_{k=0}^{n} \binom{n}{k} \text{Im} \left( e^{-\sqrt{-1} \theta'} (\alpha + \sqrt{-1} \omega_1)^{n-k} \wedge (\sigma \omega_2 + \sqrt{-1} \omega_d)^k \right),\]

For each $0 \leq k \leq n$, we write
\[T_k = \text{Im} \left( e^{-\sqrt{-1} \theta'} (\alpha + \sqrt{-1} \omega_1)^{n-k} \wedge (\sigma \omega_2 + \sqrt{-1} \omega_d)^k \right).\]

It is clear that $T_0 \leq 0$ since $Q_{\omega_1}(\alpha) \leq \theta'$. It suffices to show that $T_k \leq 0$ for any $1 \leq k \leq n$. We compute
\[T_k = \sum_{l=0}^{k-1} \binom{k}{l} \text{Im} \left( e^{-\sqrt{-1} \theta'} (\alpha + \sqrt{-1} \omega_1)^{n-k} \wedge (\sigma \omega_2)^l \wedge (\sqrt{-1} \omega_d)^{k-l} \right) + \text{Im} \left( e^{-\sqrt{-1} \theta'} (\alpha + \sqrt{-1} \omega_1)^{n-k} \wedge (\sigma \omega_2)^k \right).\]

By the almost identical argument of [9, Claim 2.7] (replacing $\omega, \chi_2, \chi_3, \chi_d$ by $\alpha, \omega_1, \omega_2, \omega_d$),
\[\text{Im} \left( e^{-\sqrt{-1} \theta'} (\alpha + \sqrt{-1} \omega_1)^{n-k} \wedge (\sigma \omega_2)^l \wedge (\sqrt{-1} \omega_d)^{k-l} \right) \leq -2\sigma^{k+2} \text{Im} \left( e^{-\sqrt{-1} \theta'} (\alpha + \sqrt{-1} \omega_1)^{n-k} \wedge \omega_2^k.\]
Then we obtain
\[ T_k \leq (\sigma^k - C_n \sigma^{k+2}) \text{Im} \left( e^{-\sqrt{-1} \theta'} (\alpha + \sqrt{-1} \omega_1)^{n-k} \right) \land \omega_2^k \leq 0, \]
for some constant $C_n$ depending only on $n$. Choosing $\sigma_0$ sufficiently small, we are done. \(\square\)

**Lemma 5.2.** Let $\omega_1$ and $\omega_2$ be two metrics, and $\alpha$ be a real $(1,1)$-form. Suppose that $Q_{\omega_1}(\alpha) \in (c_0, \pi - c_0)$. There exists a constant $\varepsilon_0(c_0, n)$ such that the following holds. For any $\varepsilon \in (0, \varepsilon_0)$, there exists $\sigma(\varepsilon, c_0, n)$ such that if
\[ (1 - \sigma^5) \omega_1 \leq \omega_2 \leq (1 + \sigma^5) \omega_1 \]
for some $\sigma \in (0, \sigma_\varepsilon)$, then
\[ F_{\omega_2, \varepsilon}(\alpha) \geq F_{\omega_1, \varepsilon}(\alpha) - \varepsilon^2. \]

**Proof.** For $k = 1, 2$, recall
\[ F_{\omega_k, \varepsilon}(\alpha) = \cot(Q_{\omega_k}(\alpha)) + \frac{\varepsilon \omega_k^n}{\text{Im}(\alpha + \sqrt{-1} \omega_k)^n}. \]

For the first term in (5.1), using Lemma 5.1 twice,
\[ |Q_{\omega_1}(\alpha) - Q_{\omega_2}(\alpha)| \leq \sigma, \quad \frac{c_0}{2} < Q_{\omega_1}(\alpha), \quad Q_{\omega_2}(\alpha) < \pi - \frac{c_0}{2}, \]
and so
\[ |\cot(Q_{\omega_1}(\alpha)) - \cot(Q_{\omega_2}(\alpha))| \leq C\sigma. \]

Next we deal with the second term of (5.1). Let $\lambda_1 \geq \ldots \geq \lambda_n$ and $\mu_1 \geq \ldots \geq \mu_n$ be eigenvalues of $\alpha$ with respect to $\omega_1$ and $\omega_2$. Then
\[ \frac{\varepsilon \omega_1^n}{\text{Im}(\alpha + \sqrt{-1} \omega_1)^n} = \frac{\varepsilon}{\sin(Q_{\omega_1}(\alpha)) \prod_{i=1}^n \sqrt{1 + \lambda_i^2}} \]
and
\[ \frac{\varepsilon \omega_2^n}{\text{Im}(\alpha + \sqrt{-1} \omega_2)^n} = \frac{\varepsilon}{\sin(Q_{\omega_2}(\alpha)) \prod_{i=1}^n \sqrt{1 + \mu_i^2}}. \]

We split the proof into two cases.

**Case 1:** $\lambda_1 \geq \varepsilon^{-3}$.

By (5.2) and (5.4), $\lambda_1 \geq \varepsilon^{-3}$ implies
\[ \frac{\varepsilon \omega_1^n}{\text{Im}(\alpha + \sqrt{-1} \omega_1)^n} \leq \varepsilon^3. \]

Combining this with (5.3),
\[ F_{\omega_2, \varepsilon}(\alpha) - F_{\omega_1, \varepsilon}(\alpha) \geq -C\sigma - \frac{\varepsilon \omega_1^n}{\text{Im}(\alpha + \sqrt{-1} \omega_1)^n} \geq -C\sigma - \varepsilon^3. \]
After choosing $\varepsilon_0$ and $\sigma_\varepsilon$ sufficiently small, we are done.
Case 2: $\lambda_1 < \varepsilon^{-3}$.

Since $Q_{\omega_1}(\alpha) \in (c_0, \pi - c_0)$, we have $|\lambda_i| \leq C\lambda_1$ for each $i$. By Weyl’s inequality, for each $i$

$$|\lambda_i - \mu_i| \leq C\sigma \lambda_1 \leq \varepsilon^{-3}\sigma.$$ 

By choosing $\sigma_\varepsilon$ sufficiently small, for any $\sigma \in (0, \sigma_\varepsilon)$, we have $|\lambda_i - \mu_i| \leq \varepsilon^5$.

Combining this with (5.2), we have

$$|\lambda_i - \mu_i| \leq \varepsilon^5.$$ 

Recalling (5.3), the above shows

$$F_{\omega_2,\varepsilon}(\alpha) - F_{\omega_1,\varepsilon}(\alpha) \geq -C\sigma - C\varepsilon^4.$$ 

After shrinking $\varepsilon_0$ and $\sigma_\varepsilon$ if necessary, we are done. \hfill \Box

Lemma 5.3. For $0 < \theta < \Theta < \pi$, there exist constants $c_0(\theta, \Theta, n)$ and $\varepsilon_0(\theta, \Theta, n)$ such that if $\varepsilon \in (0, \varepsilon_0)$, $Q_\omega(\alpha) < \Theta$ and $F_{\omega,\varepsilon}(\alpha) \geq \cot(\theta) + \varepsilon$, then

$$P_\omega(\alpha) \leq \theta - c_0\varepsilon.$$ 

Proof. It suffices to show that

$$\cot(P_\omega(\alpha)) \geq \cot(\theta) + \frac{\varepsilon}{2}.$$ 

Indeed, this implies

$$P_\omega(\alpha) \leq \arccot \left(\cot(\theta) + \frac{\varepsilon}{2}\right) \leq \theta - c_0\varepsilon.$$ 

Let $\lambda_1 \geq \ldots \geq \lambda_n$ be the eigenvalues of $\alpha$ with respect to $\omega$. Then we have

$$P_\omega(\alpha) = \sum_{i=2}^n \arccot(\lambda_i), \quad Q_\omega(\alpha) = \arccot(\lambda_1) + P_\omega(\alpha)$$ 

and

$$F_{\omega,\varepsilon}(\alpha) = \cot(Q_\omega(\alpha)) + \frac{\varepsilon}{\sin(Q_\omega(\alpha)) \prod_{i=1}^n \sqrt{1 + \lambda_i^2}}.$$ 

Without loss of generality, we may assume

$$0 < \frac{\theta}{2} \leq P_\omega(\alpha) < Q_\omega(\alpha) < \Theta < \pi.$$ 

This shows

$$\sin(Q_\omega(\alpha)) \geq \frac{1}{C_1} > 0.$$ 

We split the proof into two cases.

Case 1: $\lambda_1 \geq 2C_1$.
In this case, we have
\[
\sin(Q_\omega(\alpha)) \prod_{i=1}^{n} \sqrt{\lambda_i^2 + 1} \geq \frac{1}{C_1} \lambda_1 \geq 2.
\]
Substituting this into (5.5) and using assumption
\[
F_{\omega,\varepsilon}(\alpha) \geq \cot(\theta) + \varepsilon,
\]
which implies
\[
\cot(Q_\omega(\alpha)) \geq \cot(\theta) + \frac{\varepsilon}{2}.
\]
as required.

**Case 2:** $\lambda_1 < 2C_1$.

In this case, (5.6) shows
\[
\frac{\theta}{2} \leq P_\omega(\alpha) < \Theta.
\]
By the mean value theorem, there is a constant $C_2$ such that
\[
(5.7) \quad \cot(Q_\omega(\alpha)) = \cot(P_\omega(\alpha) + \arccot(\lambda_1))
\leq \cot(P_\omega(\alpha)) - C_2^{-1} \arccot(\lambda_1)
\leq \cot(P_\omega(\alpha)) - C_2^{-1} \arccot(2C_1).
\]
On the other hand, using (5.6) again, we obtain
\[
(5.8) \quad \frac{\varepsilon}{2} \leq \sin(Q_\omega(\alpha)) \prod_{i=1}^{n} \sqrt{1 + \lambda_i^2} \leq C_3 \varepsilon.
\]
Using (5.7), (5.8) and assumption $F_{\omega,\varepsilon}(\alpha) \geq \cot(\theta) + \varepsilon,$
\[
(5.9) \quad \cot(\theta) + \varepsilon \leq F_{\omega,\varepsilon}(\alpha) \leq \cot(P_\omega(\alpha)) - C_2^{-1} \arccot(2C_1) + C_3 \varepsilon.
\]
It then follows that
\[
\cot(P_\omega(\alpha)) \geq \cot(\theta) + C_2^{-1} \arccot(2C_1) - C_3 \varepsilon.
\]
Choosing $\varepsilon_0$ sufficiently small, so that
\[
\cot(P_\omega(\alpha)) \geq \cot(\theta) + \frac{\varepsilon}{2},
\]
for any $\varepsilon \in (0, \varepsilon_0)$, as required. \hfill \Box

**5.2. Some inequalities.**

**Lemma 5.4.** There exists $C(\alpha, \omega, X)$ such that for any $\varphi \in \mathcal{H}$ and $s \in \left[\frac{1}{2}, 1\right]$,
\[
\text{Im}(\alpha_{s\varphi} + \sqrt{-1}\omega)^n \geq \frac{1}{C} \text{Im}(\alpha_\varphi + \sqrt{-1}\omega)^n.
\]
Proof. Let \( \lambda_1 \geq \ldots \geq \lambda_n \) and \( \mu_1 \geq \ldots \geq \mu_n \) be the eigenvalues of \( \alpha_\varphi \) and \( \alpha_{s\varphi} \) with respect to \( \omega \). Let \( A_0 \) be the constant such that \(-A_0 \omega \leq \alpha \leq A_0 \omega \). Using \( \alpha_{s\varphi} = (1-s)\alpha + s\alpha_\varphi \) and Weyl’s inequality, for each \( i \),

\[
|\mu_i - s\lambda_i| \leq A_0. 
\]

Since \( s \in \left[ \frac{1}{2}, 1 \right] \), the above shows

\[
\frac{\text{Im}(\alpha_{s\varphi} + \sqrt{-1}\omega)^n}{\text{Im}(\alpha_\varphi + \sqrt{-1}\omega)^n} = \frac{\sin(\varphi_\omega(\alpha_{s\varphi}))}{\sin(\varphi_\omega(\alpha_\varphi))} \cdot \prod_{i=1}^{n} \sqrt{1 + \mu_i^2} \geq \frac{1}{C} \cdot \frac{\sin(\varphi_\omega(\alpha_{s\varphi}))}{\sin(\varphi_\omega(\alpha_\varphi))}. 
\]

We split the proof into two cases.

**Case 1:** \( \lambda_n < 100A_0 \).

In this case, \( (5.9) \) shows \( \mu_n < 100A_0 \) and

\[
\Theta_0 > Q_\omega(\alpha_{s\varphi}) > \arccot(\mu_n) \geq \arccot(100A_0) > 0.
\]

which implies

\[
\frac{\text{Im}(\alpha_{s\varphi} + \sqrt{-1}\omega)^n}{\text{Im}(\alpha_\varphi + \sqrt{-1}\omega)^n} \geq \frac{1}{C} \cdot \frac{\sin(\varphi_\omega(\alpha_{s\varphi}))}{\sin(\varphi_\omega(\alpha_\varphi))} \geq \frac{1}{C}.
\]

**Case 2:** \( \lambda_n \geq 100A_0 \).

If \( \lambda_n \geq 100A_0 \), then \( (5.9) \) shows that for each \( i \),

\[
\frac{1}{4} \lambda_i \leq \mu_i \leq 2\lambda_i.
\]

Increasing \( A_0 \) if necessary, we have

\[
Q_\omega(\omega_{s\varphi}) \leq \sum_{i=1}^{n} \arccot \left( \frac{\lambda_i}{4} \right) \leq n \arccot (25A_0) \leq \frac{\pi}{2}
\]

and

\[
Q_\omega(\omega_{s\varphi}) \geq \sum_{i=1}^{n} \arccot (2\lambda_i) \geq \frac{1}{C} \sum_{i=1}^{n} \arccot (\lambda_i) \geq \frac{1}{C} \cdot Q_\omega(\alpha_\varphi) > 0.
\]

Then

\[
\sin(\varphi_\omega(\alpha_{s\varphi})) \geq \sin \left( \frac{1}{C} \cdot Q_\omega(\alpha_\varphi) \right) \geq \frac{1}{C} \sin(\varphi_\omega(\alpha_\varphi)),
\]

which implies

\[
\frac{\text{Im}(\alpha_{s\varphi} + \sqrt{-1}\omega)^n}{\text{Im}(\alpha_\varphi + \sqrt{-1}\omega)^n} \geq \frac{1}{C} \cdot \frac{\sin(\varphi_\omega(\alpha_{s\varphi}))}{\sin(\varphi_\omega(\alpha_\varphi))} \geq \frac{1}{C}.
\]

\( \square \)
Lemma 5.5. For $c_0 \in (0, \Theta_0)$, there exists $C(c_0, \alpha, \omega, X)$ such that for any \( \phi \in \mathcal{H} \) with $\sup_x \phi = 0$ and $Q_\omega(\alpha_\phi) \leq \Theta_0 - c_0$,
\[
\int_0^1 \int_X |\phi|^p \text{Im}(\alpha_{t_\phi} + \sqrt{-1}\omega)^n dt \leq C d_p^0(\phi, 0).
\]

Proof. By Lemma 5.4 (choosing $s = \frac{1}{2}$), for any $t \in [0, 1]$, we have
\[
\text{Im}(\alpha_{t_\phi} + \sqrt{-1}\omega)^n \leq C \text{Im}(\alpha_{t_\phi} + \sqrt{-1}\omega)^n.
\]
It then follows that
\[
(5.10) \quad \int_0^1 \int_X |\phi|^p \text{Im}(\alpha_{t_\phi} + \sqrt{-1}\omega)^n dt
\]
\[
= \int_0^{\frac{1}{2}} \int_X |\phi|^p \text{Im}(\alpha_{t_\phi} + \sqrt{-1}\omega)^n dt + \int_{\frac{1}{2}}^1 \int_X |\phi|^p \text{Im}(\alpha_{t_\phi} + \sqrt{-1}\omega)^n dt
\]
\[
\leq \int_0^{\frac{1}{2}} \int_X |\phi|^p \text{Im}(\alpha_{t_\phi} + \sqrt{-1}\omega)^n dt + C \int_{\frac{1}{2}}^1 \int_X |\phi|^p \text{Im}(\alpha_{t_\phi} + \sqrt{-1}\omega)^n dt
\]
\[
\leq C \int_0^1 \int_X |\phi|^p \text{Im}(\alpha_{t_\phi} + \sqrt{-1}\omega)^n dt.
\]

On the other hand, by Lemma 3.6 and 3.5 (iii),
\[
d_p^0(\phi, 0) = 2 \int_0^{\frac{1}{2}} d_p^0(\phi, 0) dt \geq 2 \int_0^{\frac{1}{2}} d_p^0(\phi, t_\phi) dt
\]
\[
\geq 2 \int_0^{\frac{1}{2}} \int_X |\phi - t_\phi|^p \text{Re} \left( e^{-\sqrt{-1}\theta_0}(\alpha_{t_\phi} + \sqrt{-1}\omega)^n \right) dt
\]
\[
\geq \frac{1}{2p-1} \int_0^{\frac{1}{2}} \int_X |\phi|^p \text{Re} \left( e^{-\sqrt{-1}\theta_0}(\alpha_{t_\phi} + \sqrt{-1}\omega)^n \right) dt.
\]

For any $t \in [0, \frac{1}{2}]$, we have $\alpha_{t_\phi} = (1-t)\alpha + t\alpha_{t_\phi}$. Suppose that $Q_\omega(\alpha) \leq \Theta_0 - c_0$. Thanks to the concavity of $\cot(Q_\omega)$ and $Q_\omega(\alpha_\phi) \leq \Theta_0 - c_0$, we obtain
\[
Q_\omega(\alpha_{t_\phi}) \leq \Theta_0 - c^n_0 = \theta_0 + \frac{\pi}{2} - c^n_0,
\]
which implies
\[
\cos(Q_\omega(\alpha_{t_\phi}) - \theta_0) \geq \frac{1}{C} \geq \frac{1}{C} \sin(Q_\omega(\alpha_{t_\phi})).
\]
It then follows that
\[
\text{Re} \left( e^{-\sqrt{-1}\theta_0}(\alpha_{t_\phi} + \sqrt{-1}\omega)^n \right) \geq \frac{1}{C} \text{Im}(\alpha_{t_\phi} + \sqrt{-1}\omega)^n.
\]
Combining this with (5.10) and (5.11),
\[
d_p^0(\phi, 0) \geq \frac{1}{C} \int_0^{\frac{1}{2}} \int_X |\phi|^p \text{Im}(\alpha_{t_\phi} + \sqrt{-1}\omega)^n dt \geq \frac{1}{C} \int_0^1 \int_X |\phi|^p \text{Im}(\alpha_{t_\phi} + \sqrt{-1}\omega)^n dt,
\]
as required.

6. PROOF OF (1) ⇒ (2) ⇒ (3) ⇒ (4) IN THEOREM 1.4

In this section, we give the proof of (1) ⇒ (2) ⇒ (3) ⇒ (4) in Theorem 1.4.

Proof of (1) ⇒ (2) ⇒ (3) in Theorem 1.4. The part (2) ⇒ (3) is trivial. For (1) ⇒ (2), it suffices to prove the case when \( c \) is small. Without loss of generality, we assume that \( c < \pi - \Theta_0 \). For any \( \varphi \in H_c \) with \( \sup_X \varphi = 0 \), we have \( Q_\omega(\alpha_\varphi) \in (c, \Theta_0) \) and so

\[
\sin(Q_\omega(\alpha_\varphi)) > \sin(c).
\]

Let \( \lambda_i \) be the eigenvalues of \( \alpha_\varphi \) with respect to \( \omega \). Then

\[
\Re \left( e^{-\sqrt{-1} \theta_0 (\alpha_\varphi + \sqrt{-1} \omega)^n} \right) = \cos(Q_\omega(\alpha_\varphi) - \theta_0) \prod_{i=1}^n \sqrt{1 + \lambda_i^2}.
\]

\[
< \frac{\sin(Q_\omega(\alpha_\varphi))}{\sin(c)} \cdot \prod_{i=1}^n \sqrt{1 + \lambda_i^2} = \frac{1}{\sin(c)} \cdot \frac{\Im(\alpha_\varphi + \sqrt{-1} \omega)^n}{\omega^n}.
\]

Combining this with Lemma 3.6,

\[
d_1(\varphi, 0) \leq \frac{1}{\sin(c)} \int_X |\varphi| \Im(\alpha_\varphi + \sqrt{-1} \omega)^n.
\]

Using \( \varphi \in \text{PSH}(X, \chi) \) from Lemma 2.3 and \( \sup_X \varphi = 0 \),

\[
\int_X |\varphi| \Im(\alpha + \sqrt{-1} \omega)^n \leq C.
\]

Then the coerciveness of \( J \) shows

\[
J(\varphi) \geq \delta \int_X |\varphi| \Im(\alpha_\varphi + \sqrt{-1} \omega)^n - C \geq \delta \sin(c)d_1(\varphi, 0) - C,
\]

which implies \( J \) is proper.

We will prove (3) ⇒ (4) in the rest of this section. For the dHYM equation (1.1), by [10, Theorem 1.2], the existence of subsolution implies the existence of solution. Therefore, it suffices to construct a subsolution.

**Theorem 6.1.** If the \( J \)-functional is weakly proper, then there exists \( \tilde{\varphi} \in C^\infty(X) \) such that \( \alpha_{\tilde{\varphi}} \in \Gamma_{\omega, \theta_0, \Theta_0} \) on \( X \) for some \( \Theta_0 \in (\theta_0, \pi) \).

We consider the twisted dHYM flow starting from zero function:

\[
(6.1) \quad \begin{cases}
\partial_t \varphi = F_{\omega, \varepsilon}(\alpha_\varphi) - \cot(\theta_0) - a_0 \varepsilon, \\
\varphi(0) = 0,
\end{cases}
\]

where the constant \( \varepsilon \) will be determined later.
6.1. Lower bound of $J_{\varepsilon}$.

**Lemma 6.2.** There exist $\varepsilon_0(\alpha, \omega, X)$ and $C(\alpha, \omega, X)$ such that the following holds. For any $\varepsilon \in (0, \varepsilon_0)$, along the twisted dHYM flow \[(6.1),\] we have

$$J_{\varepsilon}(\varphi) \geq \frac{1}{C} \int_{X} (-\varphi)(\chi_{\varphi}^{n} - \chi^{n}) - C.$$

In particular, $J_{\varepsilon}(\varphi) \geq -C$.

**Proof.** Write $u = \varphi - \sup_{X} \varphi$, then the required inequality is equivalent to

$$J_{\varepsilon}(u) \geq \frac{1}{C} \int_{X} (-u)(\chi_{u}^{n} - \chi^{n}) - C.$$

Thanks to Lemma 4.1 we have $u \in \text{PSH}(X, \chi)$ and so $\int_{X} |u|\chi^{n} \leq C$. It then suffices to prove

$$J_{\varepsilon}(u) \geq \frac{1}{C} \int_{X} |u|\chi_{u}^{n} - C.$$

By the definition of $J_{\varepsilon}$ and $J = \sin(\theta_0)J_0$ (see Lemma 2.5), we have

$$J_{\varepsilon}(u) = J_0(u) + \varepsilon \int_{0}^{1} \int_{X} u \left( a_0 \text{Im}(\alpha_{tu} + \sqrt{-1}\omega)^n - \omega^n \right) dt$$

$$= \frac{J(u)}{\sin(\theta_0)} - a_0 \varepsilon \int_{0}^{1} \int_{X} |u|\text{Im}(\alpha_{tu} + \sqrt{-1}\omega)^n dt + \varepsilon \int_{X} |u|\omega^n.$$

Lemma 4.1 implies $Q_\omega(\alpha_u) > c_0$. Then by the weakly properness of $J$, $J(u) \geq \delta_c d_1(u, 0) - A_c$.

Thanks to Lemma 3.7 and 5.5

$$J(u) \geq \frac{\delta_c}{2} d_1(u, 0) + \frac{\delta_c}{2} d_1(u, 0) - A_c$$

$$\geq \frac{\delta_c}{2C} \int_{X} |u|\chi_{u}^{n} + \frac{\delta_c}{2C} \int_{0}^{1} \int_{X} |u|\text{Im}(\alpha_{tu} + \sqrt{-1}\omega)^n dt - A_c.$$ 

Substituting this into \[(6.2),\]

$$J_{\varepsilon}(u) \geq \left( \frac{\delta_c}{2C \sin(\theta_0)} - a_0 \varepsilon \right) \int_{0}^{1} \int_{X} |u|\text{Im}(\alpha_{tu} + \sqrt{-1}\omega)^n dt$$

$$+ \frac{\delta_c}{2C \sin(\theta_0)} \int_{X} |u|\chi_{u}^{n} - \frac{A_c}{\sin(\theta_0)}.$$ 

This completes the proof by choosing $\varepsilon_0 = \frac{\delta_c}{4a_0 C \sin(\theta_0)}$. □
6.2. Limiting function $u_\infty$. Thanks to Lemma 6.2, $J_\varepsilon(\varphi)$ is bounded from below. Combining this with $\partial_t J_\varepsilon(\varphi) < 0$, there exists a sequence $\{t_k\}_{k=1}^\infty$ such that $\lim_{k \to \infty} t_k = \infty$ and

$$\lim_{k \to \infty} \left. \frac{\partial J_\varepsilon(\varphi)}{\partial t} \right|_{t=t_k} = 0.$$ 

Write $\varphi_{t_k} = \varphi(\cdot, t_k)$, then the above implies

$$\lim_{k \to \infty} \int_X (F_\omega,\varepsilon(\alpha_{\varphi_{t_k}}) - \cot(\theta_0) - a_0\varepsilon)^2 \text{Im}(\alpha_{\varphi_{t_k}} + \sqrt{-1}\omega)^n = 0.$$ 

By (2.6) and Lemma 4.1, we obtain

$$\text{Im}(\alpha_{\varphi_{t_k}} + \sqrt{-1}\omega)^n \geq \sin(c_0)\omega^n.$$ 

It then follows that

$$\lim_{k \to \infty} \int_X (F_\omega,\varepsilon(\alpha_{u_k}) - \cot(\theta_0) - a_0\varepsilon)^2 \omega^n = 0.$$ 

We normalize

$$u_k = \varphi_{t_k} - \sup_X \varphi_{t_k},$$ 

so that

$$\lim_{k \to \infty} \int_X (F_\omega,\varepsilon(\alpha_{u_k}) - \cot(\theta_0) - a_0\varepsilon)^2 \omega^n = 0.$$ 

By Lemma 4.1 we have $\varphi_{t_k} \in \text{PSH}(X, \chi)$ and $u_k \in \text{PSH}(X, \chi)$. Passing to a subsequence, we may assume that $u_k$ converges to $u_\infty$ in the $L^1$ sense, where $u_\infty \in \text{PSH}(X, \chi)$. By Lemma 6.2

$$\int_X (-u_k)(\chi_{u_k} - \chi^n) = \int_X (-\varphi_{t_k})(\chi_{\varphi_{t_k}} - \chi^n) \leq C J_\varepsilon(\varphi_{t_k}) + C \leq C J_\varepsilon(0) + C,$$ 

where we have used that monotonicity of $J_\varepsilon$ along the twisted dHYM flow (6.1). Combining this with [25, Proposition 2.2, Corollary 2.7], we obtain

$$u_\infty \in \mathcal{E}^1(X, \chi) \subset \mathcal{E}(X, \chi).$$ 

By [25, Corollary 1.8], the function $u_\infty$ has zero Lelong number everywhere.

6.3. Local regularization. For $\sigma > 0$, we choose a finite open cover $\{B_{4R} = B_{4R}(p_i)\}$ of $X$, where $p_i \in X$ and $B_{4R}(p_i)$ denotes the Euclidean ball with center $p_i$ and radius $4R$. We choose $R$ sufficiently small so that $\{B_R(p_i)\}$ is also a covering of $X$. On each $B_{4R}$, there exists a constant Kähler form $\omega_i$ such that

$$(1 - \sigma^5)\omega_i \leq \omega \leq (1 + \sigma^5)\omega_i.$$ 

We further assume that

$$\alpha = \sqrt{-1}\partial\bar{\partial}\varphi_{\alpha,i}, \quad \omega = \sqrt{-1}\partial\bar{\partial}\varphi_{\omega,i}, \quad \chi = \sqrt{-1}\partial\bar{\partial}\varphi_{\chi,i},$$ 

where three potential functions $\varphi_{\alpha,i}$, $\varphi_{\omega,i}$ and $\varphi_{\chi,i}$ satisfy

$$(1 - \sigma)|z|^2 \leq \varphi_{\omega,i} \leq (1 + \sigma)|z|^2.$$
and

\[ |\varphi_{\alpha,i}| + |\varphi_{\omega,i}| + |\varphi_{\chi,i}| + |\nabla \varphi_{\alpha,i}| + |\nabla \varphi_{\omega,i}| + |\nabla \varphi_{\chi,i}| \leq K \]

for some constant \( K > 0 \).

For any \( k \), define

\[ u_{k,i} = u_k + \varphi_{\alpha,i}, \quad u_{\infty,i} = u_\infty + \varphi_{\alpha,i}. \]

It is clear that

\[ \sqrt{-1} \partial \bar{\partial} u_{k,i} = \sqrt{-1} \partial \bar{\partial} u_k + \alpha = \alpha_{u_k}. \]

**Definition 6.3.** For any real \((1,1)\)-form \( \beta \) on \( B_{i,R} \), the local regularization of \( \beta \) is defined by

\[ \beta^{(r)}(x) = \int_{B_1(0)} r^{-2n} \rho \left( \frac{|y|}{r} \right) \beta(x-y) \, d\text{vol}_{g_E}(y) \]

for \( r \in (0,1) \), where \( B_1(0) \) denotes the unit ball in \( \mathbb{C}^n \), \( g_E \) denotes the Euclidean metric, and \( \rho \) is a non-negative mollifier supporting on \([0,1]\) such that \( \int_{B_1(0)} \rho \, d\text{vol}_{g_E} = 1 \).

**Lemma 6.4.** There exist constants \( \varepsilon_0(\alpha, \omega, X) \) and \( c_0(\alpha, \omega, X) \) such that the following holds. For \( \varepsilon \in (0, \varepsilon_0) \), there exist constant \( \sigma_0(\varepsilon, \alpha, \omega, X) \), \( r_0(\varepsilon, \alpha, \omega, X) \) such that for any \( i, \varepsilon \in (0, \varepsilon_0) \), \( \sigma \in (0, \sigma_0) \) and \( r \in (0, r_0) \),

1. \( Q_\omega(\sqrt{-1} \partial \bar{\partial} u_{k,i}^{(r)}) \leq \Theta_0 - \frac{c_0}{2} \),
2. \( P_\omega(\sqrt{-1} \partial \bar{\partial} u_{k,i}^{(r)}) \leq \theta_0 - \frac{a_0 c_0}{2} \).

**Proof.** For (i), by definitions of \( u_k \) and \( u_{k,i} \), and Lemma 4.1

\[ Q_\omega(\sqrt{-1} \partial \bar{\partial} u_{\infty,i}^{(r)}) = Q_\omega(\alpha_{u_k}) = Q_\omega(\alpha_{\varphi_{\alpha,i}}) \in (c_0, \Theta_0 - c_0). \]

Using Lemma 5.1 (i),

\[ Q_\omega(\sqrt{-1} \partial \bar{\partial} u_{k,i}^{(r)}) \leq \Theta_0 - c_0 + \sigma \leq \Theta_0 - \frac{2c_0}{3}. \]

Thanks to the concavity of \( \cot(Q_\omega) \),

\[ Q_\omega(\sqrt{-1} \partial \bar{\partial} u_{k,i}^{(r)}) = Q_\omega((\sqrt{-1} \partial \bar{\partial} u_{k,i}^{(r)})^\alpha) \leq \Theta_0 - \frac{2c_0}{3}. \]

Using Lemma 5.1 (i) again,

\[ Q_\omega(\sqrt{-1} \partial \bar{\partial} u_{\infty,i}^{(r)}) \leq \Theta_0 - \frac{2c_0}{3} + \sigma \leq \Theta_0 - \frac{c_0}{2}. \]

Letting \( k \to \infty \), we obtain

\[ Q_\omega(\sqrt{-1} \partial \bar{\partial} u_{\infty,i}^{(r)}) < \Theta_0 - \frac{c_0}{2}. \]

For (ii), we first claim

\[ F_{\omega,\varepsilon}(\sqrt{-1} \partial \bar{\partial} u_{\infty,i}^{(r)}) \geq \cot(\theta_0) + \frac{2a_0 \varepsilon}{3}. \]
To prove (6.6), combining (6.4) and Lemma 5.2,
\[ F_{\omega,\epsilon}(\sqrt{-1} \partial \bar{\partial} u_{k,i}) \geq F_{\omega,\epsilon}(\sqrt{-1} \partial \bar{\partial} u_{k,i}) - \epsilon^2 = F_{\omega,\epsilon}(\alpha_{u_k}) - \epsilon^2. \]
The concavity of \( F_{\omega,\epsilon} \) shows
\[
F_{\omega,\epsilon}(\sqrt{-1} \partial \bar{\partial} u_{k,i}^{(r)}) = F_{\omega,\epsilon}((\sqrt{-1} \partial \bar{\partial} u_{k,i})^{(r)}) \\
\geq \int_{B_1} r^{-2n} \rho \left( \frac{|y|}{r} \right) F_{\omega,\epsilon}(\sqrt{-1} \partial \bar{\partial} u_{k,i})(x - y) d\text{vol}_{\omega}(y) \\
\geq \int_{B_1} r^{-2n} \rho \left( \frac{|y|}{r} \right) F_{\omega,\epsilon}(\alpha_{u_k})(x - y) d\text{vol}_{\omega}(y) - \epsilon^2.
\]
Recall that \( F_{\omega,\epsilon}(\alpha_{u_k}) \) converges to \( \cot(\theta_0) + a_0 \epsilon \) in \( L^2 \) (see (6.3)). Letting \( k \to \infty \),
\[ F_{\omega,\epsilon}(\sqrt{-1} \partial \bar{\partial} u_{\infty,i}^{(r)}) \geq \cot(\theta_0) + a_0 \epsilon - \epsilon^2. \]
Decreasing \( \epsilon_0 \) if necessary, we obtain (6.6).

By (6.3) (letting \( k \to \infty \)), we obtain
\[
Q_{\omega}(\sqrt{-1} \partial \bar{\partial} u_{\infty,i}^{(r)}) < \Theta_0.
\]
Combining this with (6.6) and Lemma 5.3
\[
P_{\omega}(\sqrt{-1} \partial \bar{\partial} u_{\infty,i}^{(r)}) \leq \theta_0 - \frac{2a_0 c_0 \epsilon}{3}.
\]
Using Lemma 5.1 (ii), we obtain (ii). \( \square \)

6.4. Gluing argument. In each \( B_{i,3R} \), define the function \( \tilde{u}_{\infty,i} \) by
\[ \tilde{u}_{\infty,i} = u_{\infty} + \varphi_{\chi,i}. \]
Since \( u_{\infty} \in \text{PSH}(X, \chi) \), then \( \tilde{u}_{\infty,i} \) is a plurisubharmonic function on \( B_{i,3R} \). For \( p \in B_{i,3R} \) and \( r \in (0, R) \), we use \( B_{i,r}(p) \subset B_{i,3R} \) to denote the Euclidean ball with center \( p \) and radius \( r \). Write
\[
M_{\tilde{u}_{\infty,i}}(p, r) = \sup_{B_{i,r}(p)} \tilde{u}_{\infty,i}, \quad \nu_{\tilde{u}_{\infty,i}}(p, r) = \frac{M_{\tilde{u}_{\infty,i}}(p, R) - M_{\tilde{u}_{\infty,i}}(p, r)}{\log R - \log r}.
\]
It is well-known that \( \nu_{\tilde{u}_{\infty,i}}(p, r) \) converges decreasingly to the Lelong number \( \nu_{\tilde{u}_{\infty,i}}(p) \) as \( r \to 0 \), i.e.,
\[ \lim_{r \to 0} \nu_{\tilde{u}_{\infty,i}}(p, r) = \nu_{\tilde{u}_{\infty,i}}(p). \]
In fact, \( \nu_{\tilde{u}_{\infty,i}}(p) \) is independent of \( i \). Precisely, for any \( i \) such that \( p \in B_{i,3R} \),
\[ \nu_{\tilde{u}_{\infty,i}}(p) = \nu_{u_{\infty}}(p), \]
where \( \nu_{u_{\infty}}(p) \) denotes the Lelong number of \( \chi \)-plurisubharmonic function \( u_{\infty} \) at \( p \).

By adapting the idea from Blocki-Kołodziej [3], Chen [4] proved the following lemma, which characterizes the behaviours of \( M_{\tilde{u}_{\infty,i}}(p, r) \) and \( \tilde{u}_{\infty,i}^{(r)}(p) \) with respect to \( \nu_{\tilde{u}_{\infty,i}}(p, r) \).
Lemma 6.5 (Lemma 4.2 of [4]). For any $p \in B_{i,3R}$ and $r \in (0, R)$, we have

(i) $0 \leq M_{\tilde{u}_{\infty,i}}(p, r) - M_{\tilde{u}_{\infty,i}}(p, r/2) \leq (\log 2) \nu_{\tilde{u}_{\infty,i}}(p, r)$,

(ii) $0 \leq M_{\tilde{u}_{\infty,i}}(p, r) - \tilde{u}_{\infty,i}(r) \leq \tau_0 \nu_{\tilde{u}_{\infty,i}}(p, r)$, where $\tau_0$ is a universal constant depending only on $n$.

By Lemma 6.4, we have

$$Q_\omega(\sqrt{-1}\partial\bar{\partial}u_{\infty,i}^{(r)} - \varepsilon^2\omega) \leq Q_\omega(\sqrt{-1}\partial\bar{\partial}u_{\infty,i}^{(r)}) + n\varepsilon^2 \leq \Theta_0 - \frac{a_0 \varepsilon}{2} + n\varepsilon^2$$

and

$$P_\omega(\sqrt{-1}\partial\bar{\partial}u_{\infty,i}^{(r)} - \varepsilon^2\omega) \leq P_\omega(\sqrt{-1}\partial\bar{\partial}u_{\infty,i}^{(r)}) + (n - 1)\varepsilon^2 \leq \theta_0 - \frac{a_0 c_0 \varepsilon}{2} + (n - 1)\varepsilon^2.$$

Choosing $\varepsilon$ sufficiently small (this fixes the value of $\varepsilon$), we obtain

$$\sqrt{-1}\partial\bar{\partial}u_{\infty,i}^{(r)} - \varepsilon^2\omega \in \Gamma_{\omega,\theta_0,\Theta_0}.$$

Define

$$\tilde{\varphi}_{i,r} = u_{\infty,i}^{(r)} - \varphi_{\alpha,i} - \varepsilon^2 \varphi_{\omega,i},$$

so that

$$(6.7) \quad \alpha_{\tilde{\varphi}_{i,r}} \in \Gamma_{\omega,\theta_0,\Theta_0}.$$

Recalling the definitions of $u_{\infty,i}$ and $\tilde{u}_{\infty,i}$, we have

$$u_{\infty,i}^{(r)} = (u_{\infty} + \varphi_{\alpha,i})^{(r)} = (\tilde{u}_{\infty,i} - \varphi_{\chi,i} + \varphi_{\alpha,i})^{(r)},$$

which implies

$$\tilde{\varphi}_{i,r} = \tilde{u}_{\infty,i}^{(r)} - \varphi_{\chi,i}^{(r)} + \varphi_{\alpha,i}^{(r)} - \varphi_{\alpha,i} - \varepsilon^2 \varphi_{\omega,i}.$$  

Lemma 6.6. There exists $r_0(\varepsilon, \alpha, \omega, X)$ such that for any $r \in (0, r_0)$ and any $p \in X$, we have

$$\max_{\{ij\in B_{i,3R}\setminus B_{i,2R}\}} \tilde{\varphi}_{j,r}(p) < \max_{\{ij\in B_{i,R}\}} \tilde{\varphi}_{i,r}(p) - \frac{\varepsilon^2 R^2}{2}.$$

Proof. Suppose that $p \in (B_{i,3R} \setminus B_{i,2R}) \cap B_{i,R}$. For plurisubharmonic function $\tilde{u}_{\infty,i}$ on $B_{i,3R}$, the maximum function $M_{\tilde{u}_{\infty,i}}(\cdot, r)$ is continuous, and so $\nu_{\tilde{u}_{\infty,i}}(\cdot, r)$ is also continuous. Since $\nu_{\tilde{u}_{\infty,i}}(\cdot, r)$ is decreasing with respect to $r$ and the limit function is the Lelong number $\nu_{u_{\infty}}(\cdot)$. By Dini-Cartan lemma (see e.g. [28, Lemma 2.2.9]), choosing $r_0$ sufficiently small, for any $r \in (0, r_0)$,

$$\nu_{\tilde{u}_{\infty,i}}(p, r) \leq \nu_{u_{\infty}}(p) + A^{-1}\varepsilon^2 = A^{-1}\varepsilon^2,$$

where we used that $u_{\infty}$ has zero Lelong number everywhere and $A$ is a large constant to be determined later. Similarly,

$$\nu_{\tilde{u}_{\infty,i}}(p, r) \leq A^{-1}\varepsilon^2.$$
First, we establish the upper bound of $\tilde{\varphi}_{j,r}(p)$. Combining (6.9) and Lemma 6.5,

$$
\tilde{u}_{\infty,j}^{(r)}(p) \leq M_{u_{\infty,j}}(p, r) \leq M_{\tilde{u}_{\infty,j}} \left( \frac{p}{2} \right) + (\log 2) A^{-1} \varepsilon^2
$$

$$
= \sup_{B_{j,\xi}(p)} \tilde{u}_{\infty,j} + (\log 2) A^{-1} \varepsilon^2
$$

$$
= \sup_{B_{j,\xi}(p)} (u_{\infty} + \varphi_{\chi,j}) + (\log 2) A^{-1} \varepsilon^2
$$

$$
\leq \sup_{B_{j,\xi}(p)} u_{\infty} + \sup_{B_{j,\xi}(p)} \varphi_{\chi,j} + A^{-1} \varepsilon^2.
$$

Then we compute

$$
\tilde{\varphi}_{j,r}(p) = \tilde{u}_{\infty,j}^{(r)}(p) - \varphi_{\chi,j}^{(r)}(p) + \varphi_{\alpha,j}^{(r)}(p) - \varepsilon^2 \varphi_{\omega,j}(p)
$$

$$
\leq \sup_{B_{j,\xi}(p)} u_{\infty} + \left( \sup_{B_{j,\xi}(p)} \varphi_{\chi,j} - \varphi_{\chi,j}(p) \right) + (\varphi_{\chi,j}(p) - \varphi_{\chi,j}^{(r)}(p))
$$

$$
+ \left( \varphi_{\alpha,j}^{(r)}(p) - \varphi_{\alpha,j}(p) \right) + A^{-1} \varepsilon^2 - \varepsilon^2 \varphi_{\omega,j}(p)
$$

$$
\leq \sup_{B_{j,\xi}(p)} u_{\infty} + \sup_{B_{j,\xi}(p)} \varphi_{\chi,j} - A^{-1} \tau_0 \varepsilon^2 + 3 Kr - A^{-1} \tau_0 \varepsilon^2 - 3 \varepsilon^2 R^2.
$$

Next, we apply the similar argument to establish the lower bound of $\varphi_{i,r}(p)$. Combining (6.8) and Lemma 6.5,

$$
\tilde{u}_{\infty,i}^{(r)}(p) \geq M_{u_{\infty,i}}(p, r) - A^{-1} \tau_0 \varepsilon^2
$$

$$
= \sup_{B_{i,r}(p)} \tilde{u}_{\infty,i} - A^{-1} \tau_0 \varepsilon^2
$$

$$
= \sup_{B_{i,r}(p)} (u_{\infty} + \varphi_{\chi,i}) - A^{-1} \tau_0 \varepsilon^2
$$

$$
\geq \sup_{B_{i,r}(p)} u_{\infty} + \inf_{B_{i,r}(p)} \varphi_{\chi,i} - A^{-1} \tau_0 \varepsilon^2.
$$

Then we compute

$$
\tilde{\varphi}_{i,r}(p) = \tilde{u}_{\infty,i}^{(r)}(p) - \varphi_{\chi,i}^{(r)}(p) + \varphi_{\alpha,i}^{(r)}(p) - \varphi_{\omega,i}(p)
$$

$$
\geq \sup_{B_{i,r}(p)} u_{\infty} + \left( \inf_{B_{i,r}(p)} \varphi_{\chi,i} - \varphi_{\chi,i}(p) \right) + (\varphi_{\chi,i}(p) - \varphi_{\chi,i}^{(r)}(p))
$$

$$
+ \left( \varphi_{\alpha,i}^{(r)}(p) - \varphi_{\alpha,i}(p) \right) - A^{-1} \tau_0 \varepsilon^2 - \varepsilon^2 \varphi_{\omega,i}(p)
$$

$$
\geq \sup_{B_{i,r}(p)} u_{\infty} - 3 Kr - A^{-1} \tau_0 \varepsilon^2 - 2 \varepsilon^2 R^2.
$$
Now, combining the upper bound of \( \tilde{\varphi}_{j,r}(p) \) and lower bound of \( \tilde{\varphi}_{i,r}(p) \), we see that

\[
\tilde{\varphi}_{i,r}(p) - \tilde{\varphi}_{j,r}(p) \geq \left( \sup_{B_{i,r}(p)} u_{\infty} - \sup_{B_{j,r}(p)} u_{\infty} \right) - 6Kr - (\tau_0 + 1)A^{-1} \varepsilon^2 + \varepsilon^2 R^2.
\]

Using \( B_{j,r}(p) \subset B_{i,r}(p) \) and choosing \( A = \frac{3\varepsilon R^3}{K} \),

\[
\tilde{\varphi}_{i,r}(p) - \tilde{\varphi}_{j,r}(p) \geq -6Kr + \frac{2\varepsilon^2 R^2}{3}.
\]

Choosing \( r_0 = \frac{\varepsilon^2 R^2}{36 K} \), we are done. \( \Box \)

6.5. **Proof of Theorem 6.1.** Now we are in a position to prove Theorem 6.1.

Proof of Theorem 6.1. Let \( \tilde{\varphi} \) be the regularized maximum of \( (B_{i,3R}, \tilde{\varphi}_{i,r}) \). Lemma 6.6 shows that \( \tilde{\varphi} \in C^\infty(X) \) by \([19, \text{Lemma I.5.18}]\). Thanks to (6.7),

\[
\alpha_{\tilde{\varphi}} \in \Gamma_{\omega,\theta_0,\Theta_0} \text{ on } X,
\]

which implies \( \tilde{\varphi} \) is the required subsolution. \( \Box \)

7. **Proof of (4) \( \Rightarrow \) (1) in Theorem 1.4**

In this section, we give the proof of (4) \( \Rightarrow \) (1) in Theorem 1.4.

**Lemma 7.1.** There exists \( C(\alpha, \omega, X) \) such that for any \( \varphi \in \mathcal{H} \),

\[
\int_0^1 \int_X (-\varphi) \left( a_0 \text{Im}(\alpha t \varphi + \sqrt{-1} \omega)^n - \omega^n \right) dt \\
\geq \frac{a_0}{C} \int_X (\varphi) \left( \text{Im}(\alpha \varphi + \sqrt{-1} \omega)^n - \text{Im}(\alpha + \sqrt{-1} \omega)^n \right) - C.
\]

**Proof.** We assume without loss of generality that \( \sup_X \varphi = 0 \). Since \( \varphi \in \text{PSH}(X, \chi) \) from Lemma 2.3 then \( \int_X (-\varphi) \omega^n \leq C \). It suffices to show

\[
\int_0^1 \int_X (-\varphi) \text{Im}(\alpha t \varphi + \sqrt{-1} \omega)^n dt \geq \frac{1}{C} \int_X (-\varphi) \text{Im}(\alpha \varphi + \sqrt{-1} \omega)^n.
\]

Thanks to Lemma 5.4,

\[
\int_0^1 \int_X (-\varphi) \text{Im}(\alpha t \varphi + \sqrt{-1} \omega)^n dt \\
\geq \frac{1}{2} \int_X (-\varphi) \text{Im}(\alpha \varphi + \sqrt{-1} \omega)^n dt \\
\geq \frac{1}{C} \int_X (-\varphi) \text{Im}(\alpha \varphi + \sqrt{-1} \omega)^n,
\]

as required. \( \Box \)

Now we are in a position to prove (4) \( \Rightarrow \) (1) in Theorem 1.4.
Proof of (4) ⇒ (1) in Theorem 1.4. For any \( \varphi \in \mathcal{H} \), we consider the twisted dHYM flow starting from \( \varphi \). By Theorem 4.5, if the dHYM equation admits a solution, then for \( \varepsilon \) sufficiently small, the twisted dHYM flow admits a long-time solution and will converge to a stationary solution \( \hat{\varphi}_\varepsilon \). Fixing such an \( \varepsilon \) and using \( J_\varepsilon \) is decreasing along twisted dHYM flow, we obtain

\[
J_\varepsilon (\varphi) \geq J_\varepsilon (\hat{\varphi}_\varepsilon) \geq -C.
\]

By definitions of \( J_0 \) and \( J_\varepsilon \),

\[
J_0 (\varphi) = J_\varepsilon (\varphi) + \varepsilon \int_0^1 \int_X (-\varphi) (a_0 \text{Im}(\alpha_{\varphi} + \sqrt{-1}\omega)^n - \omega^n) \, dt.
\]

Thanks to Lemma 7.1

\[
J_0 (\varphi) \geq \frac{a_0 \varepsilon}{C} \int_X (-\varphi) (\text{Im}(\alpha_{\varphi} + \sqrt{-1}\omega)^n - \text{Im}(\alpha + \sqrt{-1}\omega)^n) - C.
\]

Combining this with \( J = \sin(\theta_0)J_0 \) (see Lemma 2.5), we obtain \( J \) is coercive.

\[\square\]

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