Quantum dust collapse in 2+1 dimension

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In this paper we will examine the consequence of a canonical theory of quantum dust collapse in 2+1 dimensions. The solution of the WDW equation describing the collapse indicates that collapsing shells outside the apparent horizon are accompanied by outgoing shells within the apparent horizon during their collapse phase and stop collapsing once they reach the apparent horizon. Taking this picture of quantum collapse seriously, we determine a static solution with energy density corresponding to a dust ball whose collapse has terminated at the apparent horizon. We show that the boundary radius of the ball is larger than the BTZ radius confirming that no event horizon is formed. The ball is sustained by radial pressure which we determine and which we attribute to the Unruh radiation within it.

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I. INTRODUCTION

In the mid-70’s Bekenstein [1] and Hawking [2] studied the behavior of quantum fields in the neighborhood of blackholes and argued that the latter would evaporate thermally by quantum effects so that quantum information would be lost during the evaporation process. This is the famous 'Information Paradox'. The fundamental postulates of Quantum Mechanics say that all the information about a quantum system is encoded in its wavefunction until the latter collapses. The evolution of the wavefunction is determined by an unitary operator and unitarity implies information is conserved in the quantum sense. However, if the system entering a blackhole is in a pure state, the transformation into the mixed state that describes the Hawking radiation would destroy information about the original quantum state. This leads to a breakdown of unitarity in quantum mechanics whenever an event horizon is present. From the 'No-hair theorem' it is expected that Hawking radiation is independent of the form of matter entering the blackhole. Therefore, this paradox is generic.

According to Quantum Field Theory (QFT) in curved spacetime, a single emission of Hawking radiation involves two mutually entangled states where an outgoing particle escapes as Hawking radiation and the infalling one is swallowed by the blackhole. Therefore the exterior is entangled with the interior. For an observer with access only to the exterior, the outgoing particle is in a mixed state and since the quantum numbers of the particle inside the blackhole can never escape, there will only be an exterior mixed state if the black hole evaporates completely. To resolve this paradox, several proposals have been made.

The most straightforward resolution to this paradox would be to assume that the evaporation process leaves behind a remnant by some, as yet unknown, mechanism. This is difficult to imagine because it requires that a relatively small object would possess a large degeneracy while remaining stable. A second option is to assume that Hawking radiation is in fact pure. As proposed by

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Susskind et al [3], this can be achieved if one requires that information is both emitted at the horizon and passes through it so that an observer outside would see it as the Hawking radiation and an observer who falls into it would see it inside, but no single observer would be able to confirm both pictures (so as to avoid cloning). Thought experiments [4] that support Blackhole Complementarity employ three assumptions: a) Hawking radiation is unitary b) Effective Field Theory (QFT in curved spacetime) is valid outside the event horizon and c) the Equivalence Principle holds. However, in 2012, Almehri, Marolf, Polchinsky and Sully (AMPS) [5] argued that the three assumptions above, taken together, are logically inconsistent: if postulates a) and b) are assumed, then a Bogoliubov transformation from the frame of the distant, static observer for whom the quantum field is in a pure state to that of a freely falling observer indicates that the latter will see thermal radiation as she crosses the horizon. This violates the equivalence principle and leads to the AMPS “firewall” at the horizon.

However Hawking [6] proposed an alternative solution by suggesting the possibility that the collapse doesn’t form an event horizon, rather matter stops collapsing once it reaches the apparent horizon. No singularity and event horizon will form and in the absence of an event horizon, the entire discussion of information loss becomes irrelevant.

We attempt to realize Hawking’s proposal in a quantized model of dust collapse in 2+1-dimensions with a negative cosmological constant. We do so because 2+1 dimensional dust collapse is already quite well understood on the classical and semiclassical levels and the BTZ black hole is well understood on the quantum level via the AdS/CFT correspondence. Thus it may be possible to compare the degrees of freedom of the two approaches at a future date. Our approach here will be to exploit an exact canonical quantization of the Lemaître-Tolman-Bondi (LTB) family of solutions [7] and examine the implications of the functional solutions to the Wheeler-DeWitt (WDW) equation [8]. Two kinds of solutions are obtained. In one, matter coalesces on the apparent horizon from the interior and the exterior. In the second, matter moves away from the apparent horizon on both sides of it. In the first solution, the exterior infalling waves represent collapsing shells of dust, which are necessarily accompanied by interior outgoing waves representing the Unruh radiation. In the second solution continued collapse of the dust shells to a central singularity is accompanied by exterior Unruh radiation. To recover the standard picture of gravitational collapse the two solutions should be superposed. However, Hawking’s proposed resolution of the AMPS paradox is captured by the first of these solutions.

If we take seriously the possibility that continued collapse doesn’t occur, we expect to end up with a spherically symmetric, quasi-stable, static configuration of finite size. While no classical extended dust configuration can exist, we will argue that the interior Unruh radiation that accompanies collapsing dust shells will generate the conditions necessary for such a static configuration. The outgoing Unruh radiation leads to a negative mass singularity, weakening the gravitational field, and may eventually cause the matter to expand again [9].

II. DUST COLLAPSE

A. Classical Solutions

Dust collapse in 2+1-dimensions with a negative cosmological constant is described by the LTB family of solutions [7]. In comoving and synchronous coordinates, $(\tau, \rho, \phi)$, the metric takes the
form
\[ ds^2 = -d\tau^2 + \frac{(\partial_\rho R)^2}{2(E - F)} d\rho^2 + R^2 d\phi^2 \] (1)
where the physical radius of shells is given by \( R(\tau, \rho) \) which obeys
\[ (\partial_\tau R)^2 = 2E - \Lambda R^2. \] (2)
The energy density, \( \varepsilon(\tau, \rho) \), is given by
\[ 2\pi G \varepsilon(\tau, \rho) = \frac{\partial_\rho F}{R(\partial_\rho R)}, \] (3)
where \( \Lambda \) is the cosmological constant and \( G \) is Newton’s constant in 2 + 1-dimensions. Using the freedom to rescale the shell labels, \( \rho \), we can set \( R(0, \rho) = \rho \) at some initial epoch. In this case the functions \( F(\rho) \) and \( E(\rho) \) may be given as
\[ F(\rho) = 2\pi G \int_0^\rho \rho' \varepsilon(0, \rho') d\rho' \]
\[ E(\rho) = [\partial_\tau R(0, \rho)]^2 + \Lambda \rho^2 \] (4)
The physical interpretation of these relations is that \( 2F(\rho) \) is the gravitational mass inside the shell labeled by \( \rho \) and \( E(\rho)/2 \) is total energy per unit mass of that shell. Owing to this they are called the “mass function” and the “energy function” respectively. We assume that \( F(\rho) \) is a positive, monotonic increasing function of \( \rho \) and that the initial data disallow shell crossings, i.e., \( R'(\tau, \rho) > 0 \).

The solution to (2) that represents a collapsing dust ball is given by
\[ R(\tau, \rho) = \sqrt{\frac{2E}{\Lambda}} \sin \left( -\sqrt{\Lambda} \tau + \sin^{-1} \sqrt{\frac{\Lambda \rho^2}{2E}} \right) \] (5)
and shows that the collapse inevitably forms a central singularity as each shell shrinks to zero physical radius at the proper time
\[ \tau_0(\rho) = \frac{1}{\sqrt{\Lambda}} \sin^{-1} \left( \sqrt{\frac{\Lambda \rho^2}{2E}} \right). \] (6)
A detailed analysis \[ \square \] also shows each shell, labeled by \( \rho \), becomes trapped when its physical radius crosses the apparent horizon at \( \Lambda R^2 - 2F = 0 \), i.e., when \( R < \sqrt{2E/\Lambda} \). Thus only shells satisfying the condition \( F > 0 \) (therefore \( \rho > 0 \)) will become trapped, each at proper time
\[ \tau_{ah}(\rho) = \frac{1}{\sqrt{\Lambda}} \sin^{-1} \left( \sqrt{\frac{\Lambda \rho^2}{2E}} \right) - \sin^{-1} \sqrt{\frac{F}{E}}. \] (7)
Moreover, by our assumptions about \( F \), the physical radius of the apparent horizon will be a monotonic increasing function of \( \rho \). Notice that \( \tau_{ah}(\rho) < \tau_0(\rho) \), so each trapping surface forms before the shell becomes singular and collapse to the central singularity is not a necessary condition for the formation of a trapping surface\[ \square \]

\[ F(\rho) = 2\pi G \int_0^\rho \rho' \varepsilon(0, \rho') d\rho' - f_0, \]
B. Collapse Wavefunctionals

The canonical dynamics of collapsing dust shells is described by embedding the spherically symmetric ADM metric

\[ ds^2 = -N^2 dt^2 + L^2 (dr + N^r dt)^2 + R^2 d\phi^2 \]  

in the LTB metric \([1]\). Above, \(N\) is the lapse, \(N^r\) is the shift. After a series of canonical transformations \([\square]\), they are described in a phase space of dust proper time, \(\tau(t, r)\), the physical shell radius, \(R(t, r)\), the mass density, \(\Gamma(r) = F'(r)\), and their conjugate momenta, \(P_\tau(t, r)\), \(P_R(t, r)\) and \(P_\Gamma(t, r)\) respectively. In this phase space the Hamiltonian and diffeomorphism constraints are \([10]\)

\[ P_\tau^2 + F P_R^2 - \frac{I^2}{F} = 0 \]
\[ R' P_R - \Gamma P_\Gamma' + \tau' P_\tau = 0 \]  

where the prime denotes a derivative w.r.t. the ADM radial coordinate, \(r\), and \(F \equiv \Lambda R^2 - 2F\) \((10)\).

The apparent horizon occurs when \(F = 0\), which is determined by the vanishing of the null divergence. On the apparent horizon the physical radius of each shell is

\[ R(\tau_{ah}, \rho) = \sqrt{\frac{2F}{\Lambda}} \]  

To transform from classical to quantum, Dirac’s quantization procedure may be applied on the constraints, which act on wave functionals. The Hamiltonian constraint gives the Wheeler-DeWitt equation and the momentum constraint imposes diffeomorphism invariance. We begin with an ansatz for the wave functional

\[ \Psi[\tau, R, \Gamma] = \exp \left[ -i \int dr \Gamma(r) W(\tau(r), R(r), \Gamma(r)) \right] \]  

which automatically obeys the momentum constraint provided \(\mathcal{W}\) doesn’t have any explicit dependence on \(r\). At this point we will consider a one-dimensional lattice with discrete points \(r_i\), a distance \(\sigma\) apart. With this discretization and the ansatz in \([12]\), the WDW equation gives

\[ \omega_i^2 \left[ \left( \frac{\partial W_i}{\partial \tau_i} \right)^2 + \mathcal{F}_i \left( \frac{\partial W_i}{\partial R_i} \right)^2 + \frac{1}{\mathcal{F}_i} \right] + \omega_i \left[ \frac{\partial^2 W_i}{\partial \tau_i^2} + \mathcal{F}_i \frac{\partial^2 W_i}{\partial R_i^2} + A_i \frac{\partial W_i}{\partial R_i} \right] + B_i = 0 \]  

Defining \(W_i = -iW_i\) and equating each co-efficient to zero as it is true for arbitrary \(\omega_i\), we have three independent equations

\[ \left( \frac{\partial W_i(\tau_i, R_i, \Gamma_i)}{\partial \tau_i} \right)^2 + \mathcal{F}_i \left( \frac{\partial W_i(\tau_i, R_i, \Gamma_i)}{\partial R_i} \right)^2 = \frac{1}{\mathcal{F}_i} \left( \frac{\partial^2 W_i(\tau_i, R_i, \Gamma_i)}{\partial \tau_i^2} + \mathcal{F}_i \frac{\partial^2 W_i(\tau_i, R_i, \Gamma_i)}{\partial R_i^2} + A_i(R_i, \Gamma_i) \frac{\partial W_i(\tau_i, R_i, \Gamma_i)}{\partial R_i} \right) = 0 \]  

where \(f_0\) is a positive integration constant. In this case, there will be a critical shell for which \(F(\rho_c) = 0\). It can then be shown that the singularity is timelike for \(\rho < \rho_c\), null for \(\rho = \rho_c\) and spacelike for \(\rho > \rho_c\). In 2+1 dimensions, this does not lead to singular initial data and \(f_0\) does not have the interpretation of a point mass at the center \([7]\).
whose solution is

$$W_i = a_i \tau_i + \int dR_i \sqrt{1 - a_i^2 F_i}$$

(15)

where $a_i = 1/\sqrt{2(E_i - F_i)}$. Therefore, we have

$$\Psi = \lim_{\sigma \to 0} \prod_i \Psi_i(\tau_i, R_i, \Gamma_i) = \lim_{\sigma \to 0} \prod_i \exp \left\{ -i\omega_i \left[ a_i \tau_i \pm \int dR_i \sqrt{1 - a_i^2 F_i} \right] \right\}$$

(16)

with a well-defined continuum limit, where $\omega_i = \sigma \Gamma_i$.

The solutions are defined everywhere except at the apparent horizon where there is an essential singularity. Thus there are “exterior” wave functionals that must be matched to “interior” wave functionals at the apparent horizon. A standard technique is to analytically continue the solutions into the complex plane. This technique is used to derive Hawking radiation as a tunneling process [11]. Analytically continuing to the complex $R_i$-plane, following a semicircular contour in the upper-half plane of radius approaching zero around the pole, we find two sets of matched solutions,

$$\Psi^{(1)}_i = \begin{cases} e^{i\omega_i b_i} \times \exp \left\{ -i\omega_i \left[ a_i \tau_i + \int R_i dR_i \sqrt{1 - a_i^2 F_i} \right] \right\}, & F_i > 0 \\ e^{-\frac{\pi i}{g_i,h}} \times e^{i\omega_i b_i} \times \exp \left\{ -i\omega_i \left[ a_i \tau_i + \int R_i dR_i \sqrt{1 - a_i^2 F_i} \right] \right\}, & F_i < 0 \end{cases}$$

(17)

and

$$\Psi^{(2)}_i = \begin{cases} e^{-\frac{\pi i}{g_i,h}} \times e^{i\omega_i b_i} \times \exp \left\{ -i\omega_i \left[ a_i \tau_i - \int R_i dR_i \sqrt{1 - a_i^2 F_i} \right] \right\}, & F_i > 0 \\ e^{i\omega_i b_i} \times \exp \left\{ -i\omega_i \left[ a_i \tau_i - \int R_i dR_i \sqrt{1 - a_i^2 F_i} \right] \right\}, & F_i < 0 \end{cases}$$

(18)

where

$$g_{i,h} = \partial_R F_i(R_i,h)/2$$

(19)

is the surface gravity of the $i$-th shell at the apparent horizon. The first set [17] represents a flow toward the apparent horizon on both sides of it, that is, an infalling shell in the exterior is accompanied by an interior outgoing shell with relative probability determined by the Boltzmann factor at the Hawking temperature of the shell. The second set [18] represents a flow away from the apparent horizon on both sides, that is, an infalling interior shell representing continued collapse past the apparent horizon to the central singularity, is accompanied by an exterior outgoing shell with relative probability determined by the same factor. It represents thermal (Unruh) radiation in the exterior. Some useful conclusions can be drawn from the solutions. As mentioned in the introduction, one may superpose both the solutions leading to a picture in which continued collapse to a singularity occurs, with accompanying thermal radiation in the exterior. However if we take only [17] as the basis for quantum collapse then there will be thermal Unruh radiation inside the apparent horizon but no thermal radiation outside. There will be no continued collapse to a singularity. The collapse would terminate at the apparent horizon ($F_i = 0$), which agrees with Hawking’s proposal.
C. A quasi-classical configuration

As we see from (17) collapse stops at the apparent horizon for forming a quasi-stable compact object, we expect that there exist solutions to the Einstein equations with finite boundary radius which show the effect of the internal Unruh radiation. The matter should condense on the apparent horizon and the solution should match smoothly with the external BTZ vacuum. Within the dust ball, the metric will have the form,

\[ ds^2 = -e^{2A(r)}dt^2 + e^{2B(r)}dr^2 + r^2d\phi^2 \]

and the corresponding field equations, with \( T_{\mu \nu} = \text{diag}\{-\varepsilon(r), p_r(r), p_\phi(r)\} \), will be

\[
\begin{align*}
\frac{e^{2(A-B)}B'}{r} + \Lambda e^{2A} &= 4\pi G e^{2A}\varepsilon(r) \\
\frac{A'}{r} - \Lambda e^{2B} &= 4\pi G p_r(r) \\
e^{-2B}r^2(A'' - A'B' + A'^2) - \Lambda r^2 &= 4\pi G p_\phi(r)
\end{align*}
\]

where \( \varepsilon(r) \) is the energy density within the dust ball, \( p_r(r) \) is the radial pressure and \( p_\phi(r) \) is the tangential pressure. We can choose two of the stress-energy tensor components arbitrarily and the third will be determined by the Einstein equations. We choose the energy density and set the tangential pressure to zero. According to equation (10), the mass function that is expected of a dust ball whose collapse has stopped at the apparent horizon is \( F(r) = \frac{\Lambda r^2}{2} \). This shows that the energy density will be

\[ \varepsilon(r) = \frac{\Lambda}{2\pi G}, \]

so the \( tt \)-component of the field equations gives

\[ e^{2B} = \frac{1}{C_1 - \Lambda r^2}. \]

For a physically meaningful solution, \( C_1 \) has to be greater than \( \Lambda r^2 \). We will see later that it actually describes a negative point mass source at the center. With this solution for \( B(r) \) if we solve the tangential component of the field equations we find

\[ e^{2A} = \cosh^2 \left( \arctan \left( \frac{\sqrt{\Lambda}r}{\sqrt{C_1 - \Lambda r^2}} \right) + \tilde{C}_2 \right) \]

A singularity occurs at \( r = 0 \). We can calculate the radial pressure directly from the \( rr \)-component of the field equations

\[ p_r = \frac{\Lambda}{\Lambda r^2 - C_1} + \frac{\sqrt{\Lambda} \tanh \left[ \tilde{C}_2 + \arctan \left( \frac{\sqrt{\Lambda}r}{\sqrt{C_1 - \Lambda r^2}} \right) \right]}{r \sqrt{C_1 - \Lambda r^2}}. \]

If \( r_b \) denotes the outer boundary of the collapsed star, we want to match the interior geometry to the outer vacuum described by the BTZ metric

\[ ds^2 = -f(R)dT^2 + \frac{1}{f(R)}dR^2 + R^2d\phi^2 \]
where \( f(R) = (AR^2 - GM_s) \) and \( M_s \) is the BTZ mass of the dust ball. The junction conditions require that, at \( r_b = R_b \),

\[
dT = \left( \frac{e^{2A(r_b)}}{f(r_b)} \right) dt \quad e^{-B(r_b)} = \sqrt{f(R_b)} \quad 2A'(r_b) = (\ln f)' |_{R_b}
\]

These give,

\[
C_1 = \Lambda (2r_b^2 - r_s^2) \\
\tilde{C}_2 = -\tan^{-1}\left( \frac{r_b}{\sqrt{r_b^2 - r_s^2}} \right) + \tanh^{-1}\left( \frac{r_b}{\sqrt{r_b^2 - r_s^2}} \right)
\]

where we define \( GM_s = \Lambda r_s^2 \).

**III. ENERGY EXTRACTION**

Our solutions depend on two constants, which can be taken to be the BTZ radius, \( r_s \), and the boundary radius, \( r_b \), of the dustball. The two are related by the strength of the negative mass singularity at the center as, according to (27),

\[
M_s = M_b - (C_1/G - M_b)
\]

where the quantity \( C_1/G - M_b = M_0 \) is the mass energy extracted from the center of the dustball by the outgoing Unruh radiation. To estimate the value of \( M_0 \), we assume each shell to be a quantum harmonic oscillator with definite energy. A single quantum harmonic oscillator, located at lattice point \( i \), will have mean energy

\[
\langle E_i \rangle = \frac{\omega_i}{2} \coth(\beta_i \omega_i/2)
\]

where \( \beta_i = \frac{1}{kT_i} = \frac{1}{\Lambda r_i} \). Moreover, for our collapse,

\[
\omega_i = \frac{\sigma \Lambda r_i}{G}
\]

and therefore \( \beta_i \omega_i = \frac{\sigma}{G} \). At this point we should notice that \( \sigma \) cannot be arbitrarily small in the continuum limit as the total energy, \( E = \sum_i \langle E_i \rangle \), would then be unbounded. Instead, it will be such that it is microscopically large but macroscopically so that a given lattice spacing contains many Planck lengths. Therefore, assuming \( \sigma \gg G \), the average energy of the Unruh radiation inside the apparent horizon will be \( \langle r_i = i\sigma \rangle \)

\[
\langle E \rangle \approx \frac{1}{2} \sum_{i=1}^{N} \omega_i = \sum_{i=1}^{N} \frac{\Lambda \sigma^2}{2G} = \frac{\Lambda N(N+1)\sigma^2}{4G} = \frac{\Lambda r_b^2}{4G}
\]

where \( N \) is the number of shells to the boundary, i.e., \( r_b = N\sigma \). Equating the mean energy of the Unruh radiation to the mass \( M_0 \) in (28), we find that \( C_1 = 5\Lambda r_b^2/4 \) and therefore

\[
r_s = \frac{\sqrt{3r_b}}{2},
\]

so the BTZ (horizon) radius lies within the boundary of the star and the collapse does not end up forming an event horizon, which supports Hawking’s conjecture.
IV. CONCLUSION

In this paper, we have speculated about what may be expected from quantum collapse in 2+1 dimensions with a cosmological constant. Our interest in this model stems from the fact that the 2+1 dimensional model has served in the past as a toy model of black hole thermodynamics and as an aid to understanding many of the higher dimensional black holes of string theory. Long ago, Strominger \[13\] argued that because the asymptotic symmetry group of 2+1 dimensional gravity with a cosmological constant is generated by two copies of the Virasoro algebra, its degrees of freedom can be described by a two dimensional conformal field theory (CFT) at infinity, with central charges \(c_R = c_L = 3l/2G\). Ever since then, most approaches to describing the BTZ black hole have employed the AdS/CFT correspondence. A connection between the description of the quantum BTZ black hole via the canonical approach and its description via the AdS/CFT correspondence was later found \[14\].

Here we have shown, however, that the collapse process need not lead to the formation of a black hole. It is possible, within the quantum description, that collapse stops when all shells arrive at the apparent horizon. The quantum wavefunctionals indicate that, during the collapse, infalling shells of matter never cross the apparent horizon and are accompanied by outgoing Unruh radiation emanating from the center, which also terminates at the apparent horizon. There is no exterior Unruh radiation. The effect of this outgoing radiation and the absence of continued collapse across the apparent horizon leaves behind a negative mass singularity at the center, as the system settles into a quasi-stable, static configuration. We have found static solutions smoothly matching the BTZ vacuum at the boundary and possessing an energy density describing dust which has coalesced on the apparent horizon together with radial pressure, which is presumed to originate in quantum gravity. These solutions are fully determined by two parameters, viz., the radius of the dust ball and its BTZ radius, which differ by the magnitude of the mass of the central singularity. The boundary lies outside the BTZ (horizon) radius and no Unruh radiation escapes, therefore the information loss problem gets resolved.

The next step is to determine a dynamical collapse model whose end state is the configuration described in this article. Ideally, we would like to find an effective collapse scenario that incorporates the features predicted by the wavefunctional in \[17\], achieves the quasi-stable configuration described in this article and eventually expands, thus preserving CPT invariance. It would be worthwhile finding such a description within the context of the AdS/CFT correspondence.

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