Mechanism Design for Two-Opposite-Facility Location Games with Penalties on Distance*

Xujin Chen\textsuperscript{1,2}, Xiaodong Hu\textsuperscript{1,2}, Xiaohua Jia\textsuperscript{3}, Minming Li\textsuperscript{3}, Zhongzheng Tang\textsuperscript{1,2,3}, and Chenhao Wang\textsuperscript{1,2,3}

\textsuperscript{1}Academy of Mathematics and Systems Science, Chinese Academy of Sciences, China
\textsuperscript{2}School of Mathematical Sciences, University of Chinese Academy of Sciences, China
\textsuperscript{3}Department of Computer Science, City University of Hong Kong, HKSAR, China

\{xchen,xdhu,tangzhongzheng,wangch\}@amss.ac.cn
\{csjia,minming.li\}@cityu.edu.hk

Abstract. This paper is devoted to the two-opposite-facility location games with a penalty whose amount depends on the distance between the two facilities to be opened by an authority. The two facilities are “opposite” in that one is popular and the other is obnoxious. Every selfish agent in the game wishes to stay close to the popular facility and stay away from the obnoxious one; its utility is measured by the difference between its distances to the obnoxious facility and the popular one. The authority determines the locations of the two facilities on a line segment where all agents are located. Each agent has its location information as private, and is required to report its location to the authority. Using the reported agent locations as input, an algorithmic mechanism run by the authority outputs the locations of the two facilities with an aim to maximize certain social welfare. The sum-type social welfare concerns with the penalized total utility of all agents, for which we design both randomized and deterministic group strategy-proof mechanisms with provable approximation ratios, and establish a lower bound on the approximation ratio of any deterministic strategy-proof mechanism. The bottleneck-type social welfare concerns with the penalized minimum utility among all agents, for which we propose a deterministic group strategy-proof mechanism that ensures optimality.

Keywords: Algorithmic Game Theory · Mechanism Design · Facility Location.

1 Introduction

The facility location game originally models the following scenario in practice: the central authority is going to build one or more facilities on a street (modeled as a line segment) where some selfish agents are located. The authority does not know the exact locations of agents, and thus conduct a survey for all agents. Each agent, who is required to report its own location, wishes to maximize its own utilities (e.g., minimizing its own distances to the facilities). The authority needs to design a mechanism, that maps the reported locations

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of agents to locations where the facilities are to be opened. It is assumed that all agents know the mechanisms that the authority adopts to aggregate agents’ information to the final facility locations. Some agents might have incentive to misreport their location. The goals of the authority are twofold: avoiding such misreports and maximizing some social welfare. The strategy-proofness of a mechanism is emphasized; it guarantees that an agent cannot acquire more utility from misreporting. Group strategy-proof mechanisms are even more encouraged; they discourage simultaneous misreporting of any group of agents.

In this paper, we address (group) strategy-proof mechanism design for the two-opposite-facility location game proposed by [20], where two facilities to be opened have opposite characteristics for agents, that is, all agents want to stay as close as possible to one facility and stay as far away as possible from the other. Nevertheless, for some practical reasons, the two facilities should not be too far away to lose some connection. So, the distance between the two facilities cannot exceed a given constant $C$ (referred to as distance constant). For instance, in order to save the cost of transportation and enhance garbage disposal efficiency, the government should set a limitation to the distance between the refuse collection point and the waste treatment plant. In our model, we relax the distance constraint to be a soft one by introducing to the central authority a penalty which equals to a coefficient $\lambda$ times the amount of distance violation. The more violation a location scheme commits, the heavier penalty the authority carries. In addition to (group) strategy-proofness, we evaluate the efficiency of mechanisms in terms of optimizing certain social welfare: the sum-type one of maximizing the penalized total utility of all agents, and the bottleneck-type one of maximizing the penalized minimum utility of an agent.

**Related Work.** The facility location games with one facility to be opened is widely studied in economics and industries. Moulin [13] first characterized strategy-proof and Pareto efficient mechanisms in the line space. [12,10,8] extended this model for locating two facilities. The characterizations of strategy-proof mechanisms on other networks (e.g., circle metric space) were studied in [15,1].

The approximation performance of strategy-proof mechanisms first received attention from Procaccia and Tennenholtz [14]. They studied the facility location game on the line for both total utilities and minimum agent utilities, and derived several approximation bounds under the constraint of strategy-proofness. Fotakis and Tzamos [9] explored the facility location game with $k$ facilities and showed the strategy-proofness. Filos-Ratsikas et al. [6] extended the single-peaked preference to the double-peaked preference where every agent has two ideal places for the facility on its two sides.

On the topic of two facilities, Procaccia and Tennenholtz [14] provided guaranteed bounds in the line space. Lu et al. [11] improved the lower bound to $\frac{n-1}{2}$ for deterministic strategy-proof mechanisms, and gave a randomized strategy-proof mechanism with approximation ratio of 4. Serafino and Ventre [16,17] initiated the study on two heterogeneous facility location games for minimizing the maximum connection cost and total connection cost of the agents. Yuan et al. [19] extended the study by allowing agents to have optional preference. Fong
et al. [7] proposed a model where each agent has an individual fractional preference over facilities. Zou and Li [20] (independently, Feigenbaum [3]) introduced a two-facility model with dual preference, that is, agents may have different preferences over the two facilities. Zou and Li [20] also proposed the model where two facilities are of opposite characteristics for all agents under the distance constraint. We generalize their model with penalty on the distance between the two opposite facilities in this paper.

Our Contributions

We investigate the two-opposite-facility location game model with a penalty whose amount depends on the distance between these two facilities. Different from the convention on approximation ratios in some literature, we assume in this paper that the approximation ratio w.r.t. our maximization objective is always no more than one.

- For sum-type social welfare, we design both randomized and deterministic group strategy-proof mechanisms. The randomized one achieves an approximation ratio $\frac{1}{2}$, by taking two possible optimal schemes respectively with probability $\frac{1}{2}$. The deterministic one achieves an approximation ratio $\frac{1}{(k-1)R+1}$ (when $n = 2k$), or $\frac{1}{2(k-1)R+1}$ (when $n = 2k - 1$), where $R$ is the ratio of the length of the line segment to the distance constant $C$. Furthermore, we prove that no deterministic strategy-proof mechanism can have an approximation ratio better than $\frac{1}{k-1}$, when $n = 2k$ is even and penalty coefficient $\lambda \in (0, 2)$.

- For bottleneck-type social welfare, we propose a deterministic group strategy-proof mechanism, that achieves the optimality.

Organizations. Preliminaries of the two-opposite-facility location game are presented in Section 2. Mechanism design and inapproximate result concerning with the sum-type social welfare are given in Section 3. An optimal deterministic strategy-proof mechanism for the bottleneck-type social welfare is proposed in Section 4. Future research direction is discussed in Section 5. All proofs and details omitted can be found in Appendix.

2 Preliminaries

We mainly use the notations from [20]. In the two-opposite-facility location game, the decision maker wishes to build two facilities with opposite preferences on some network based on the location information reported by each agent. Let $N = \{1, 2, \ldots, n\}$ denote the set of agents. We use $x_i$ to indicate the location of each agent $i \in N$ and $x = (x_1, \ldots, x_n)$ to represent the location profile of all agents. The two facilities are of opposite characteristics for agents, which means all agents want to stay as close as possible to one facility (denoted as $F_1$) and stay as far away as possible from the other (denoted as $F_0$). Moreover, a building scheme $S = (y_0, y_1)$ is the output of a mechanism where $y_0$ and $y_1$ are locations of $F_0$ and $F_1$, respectively. Denote by $d(x, y)$ the distance between the locations $x$ and $y$. Denote the distance between two facilities as $|S| = d(y_0, y_1)$. We further
require that the length of $S$ satisfies a distance constraint: $|S| \leq C$ for a given constant $C \geq 0$, and this constraint can be violated moderately with a penalty, differing from the hard constraint which must be satisfied strictly in [20]. The penalty is measured by function $p(S) := \lambda(|S| - C)_+$ with a penalty coefficient $\lambda \geq 0$, where $(|S| - C)_+ = \max\{|S| - C, 0\}$.

For a certain location profile with building scheme $S$, the utility of agent $i$ is defined as the difference between its distances towards $F_0$ and $F_1$, i.e., $u(x_i, S) = u(x_i, (y_0, y_1)) = d(x_i, y_0) - d(x_i, y_1)$. For a given location profile $x$, a randomized mechanism in this game outputs a mixed building scheme $MS$, which is a probability distribution over candidate building schemes and for a given agent $i$, $u(x_i, MS) = \mathbb{E}_{S \sim MS}u(x_i, S)$.

We study in this paper the game on a line segment with length $L$. Without loss of generality assume the left end-point of the segment is 0 and the right end-point is $L$. The location of each agent or each facility is on this segment: $x_i, y_j \in [0, L]$ for $i \in N, j \in \{0, 1\}$, and the distance $d(x, y)$ between two locations $x$ and $y$ is their one-dimensional Euclidean distance $|x - y|$.

Each agent reports only its own location, and tends to maximize its utility by misreporting, since the decision maker determines the building scheme based on the location information reported by all $n$ agents, resulting in some room for manipulation. The game model consists of two sub-models as specified below, for maximizing the total utility of all agents and the minimum agent utility, respectively.

**The sum-type social welfare.** Given a location profile $x$ and building scheme $S = (y_0, y_1)$, this type of social welfare consists of two terms: the first term is the sum of the utilities of all $n$ agents, i.e., $\sum_{i=1}^{n} u(x_i, S) = \sum_{i=1}^{n} (d(x_i, y_0) - d(x_i, y_1))$; the second term is the penalty function $p(S)$. Formally, the sum-type social welfare is defined as the difference between the above two terms:

$$su(S, x) = \sum_{i=1}^{n} (d(x_i, y_0) - d(x_i, y_1)) - \lambda(|S| - C)_+$$  \hspace{1cm} (1)

When $\lambda$ tends to infinity, this model is actually equivalent to the model of two-opposite-facility location game with limited distance in [20]. Most of results in [20] are generalized in this paper.

**The bottleneck-type social welfare.** Among all $n$ agents, the agent who has the least utility is called a bottleneck agent, and its utility is called the bottleneck value (or simply, the bottleneck). The penalty on the distance requirement is still to be considered. Formally, given a location profile $x$, we wish to maximize the objective function

$$mu(S, x) = \min_{i \in N} (d(x_i, y_0) - d(x_i, y_1)) - \lambda(|S| - C)_+$$  \hspace{1cm} (2)

### 3 Mechanisms for the Sum-type Social Welfare

In this section, we design strategy-proof mechanisms for maximizing the sum-type social welfare. We establish in Section 3.1 a condition for the optimal
schemes. We present in Section 3.2 both randomized and deterministic mechanisms with provable approximation ratios. We give in Section 3.3 a inapproximation result for deterministic mechanisms.

3.1 The Optimality Condition

Given a location profile \( x \), we assume w.l.o.g. that \( x_1 \leq \cdots \leq x_n \). Denote by \( x_m \) the location of a median agent in \( x \) if \( n \) is odd, i.e., the agent with at most \( \frac{n+1}{2} \) ones located on its left side and the same on the right side, and similarly denote by \( x_{m_1} \) and \( x_{m_2} \) the left and right median locations respectively (\( x_{m_1} = x_{m_2} \) if \( n \) is odd). The maximum sum-type social welfare a scheme can achieve is written as \( \text{OPT}(x) \). For a mechanism \( f \), if there exists a number \( \beta \) such that \( \frac{su(f(x),x)}{\text{OPT}(x)} \geq \beta \) for any location profile \( x \) with \( \text{OPT}(x) \neq 0 \), where \( f(x) \) is the output scheme of mechanism \( f \), then \( \beta \) is the approximation ratio of \( f \). The approximation ratio only makes sense when \( \text{OPT}(x) \neq 0 \). We first give an important proposition about applicability of the approximation ratio for the social welfare.

**Fact 1** Suppose that there are an even number \( n = 2k \) of agents. For any location profile \( x \), \( \text{OPT}(x) \geq 0 \), and the equality holds if and only if \( k \) agents located at 0 and the other \( k \) agents located at \( L \).

Given a location profile \( x \), define a function \( g(y) = \sum_{i=1}^n g_i(y) = \sum_{i=1}^n d(x_i - y) \) over the domain \([0,L]\). By the convexity of \( g_i(y) = d(x_i - y) \) for each \( i \in N \) and the additivity of convex functions, \( g(y) \) is a convex function as Figure 1 shows. Denote by \( g'_-(y) \) and \( g'_+(y) \) the left and right derivatives of \( g(y) \), respectively. The following observation of \( g'_-(y) \) and \( g'_+(y) \) is simple but important:

**Observation 2** Let \( l_y = \{i \in N | x_i < y \} \) and \( r_y = \{i \in N | x_i > y \} \). Then, \( g'_-(y) = 2l_y - n \) and \( g'_+(y) = n - 2r_y \).

It follows from this observation that function \( g \) decreases in \([0,x_{m_1}]\), reaches its minimum point in \([x_{m_1},x_{m_2}]\), and increases in \((x_{m_2},L]\).

Given a scheme \( S = (y_0, y_1) \), the social welfare \( su(S,x) = \sum_{i=1}^n (d(x_i, y_0) - d(x_i, y_1)) - \lambda(|S| - C)_+ = g(y_0) - g(y_1) - \lambda(|S| - C)_+ \). Fixing \( y_0 = 0 \), we wish to find the best location for \( y_1 \) to maximize the social welfare, that is,

\[
\max_y g(0) - g(y) - \lambda \cdot \max\{y-C,0\} \tag{3}
\]

Based on maximization problem (3) and Observation 2 we define \( \text{opt}_1(x) \) by:

\[
\text{opt}_1(x) := \begin{cases} 
  x_{m_1} & \text{if } x_{m_1} \leq C \\
  C & \text{if } x_{m_1} > C \text{ and } \lambda \geq |g'_+(C)| \\
  x_i & \text{if } x_{m_1} > C \text{ and } \lambda < |g'_+(C)|
\end{cases}
\]

where \( x_i \) is the unique solution of equations \( |g'_-(x_i)| > \lambda \) and \( |g'_+(x_i)| \leq \lambda \). There is an intuitive interpretation for \( \text{opt}_1(x) \): the location \( y_1 \) should be as close as
possible to $x_{m_1}$ where it is permitted by the cost, that is, if the penalty coefficient \( \lambda \) is less than the decease rate of $g(y)$ at some potential location $x \leq x_{m_1}$ for $y_1$, which means the social welfare has room for growth, then $y_1$ moves right. It can be easily verified that $opt_l(x)$ is an optimal solution of problem (3).

Fixing $y_0^* = L$, we solve the following optimization problem for $y_1$:

$$
\max_y \ g(L) - g(y) - \lambda(L - y - C) +
$$

Define $opt_r(x)$ by:

$$
\text{opt}_r(x) := \begin{cases} 
  x_{m_2} & \text{if } L - x_{m_2} \leq C \\
  L - C & \text{if } L - x_{m_2} > C & \lambda \geq |g'_-(C)| \\
  x_j & \text{if } L - x_{m_2} > C & \lambda < |g'_-(C)| 
\end{cases}
$$

where $x_j$ is the unique solution of equations $|g'_+(x_j)| > \lambda$ and $|g'_-(x_j)| \leq \lambda$. Similarly, $opt_r(x)$ is an optimal solution of problem (4). An illustration of $opt_l(x)$ and $opt_r(x)$ is depicted in Figure 1.

![Fig. 1. Given x with n = 6. g(y) = |y - 1| + |y - 2| + |y - 4| + |y - 5| + |y - 6| + |y - 7| with parameters L = 10, C = 3 and \( \lambda \) = 3.5. We have \( \text{opt}_l = 3 \) and \( \text{opt}_r = 6 \). In the remainder of this paper, we denote, for brevity, \( \text{opt}_l = opt_l(x) \) and \( \text{opt}_r = opt_r(x) \) when no confusion arises. Now we have results about optimal building schemes. It could be proved that at least one building scheme of $(0, opt_l)$ and $(L, opt_r)$ can optimize the sum-type social welfare.

**Theorem 3.** Either $su((0, opt_l), x) = OPT(x)$ or $su((L, opt_r), x) = OPT(x)$.

### 3.2 Strategy-proof Mechanisms

Both randomized and deterministic mechanisms with guaranteed performance are proposed in this subsection. Given a mechanism for the two-opposite-facility division problem, the social welfare can be optimized.
location game outputting a solution $S$ for a location profile $x$, we say the mechanism is strategy-proof, if for any agent $i \in N$ and its misreported location $x_i'$, we have $u(x_i, S) \geq u(x_i, S')$, where $S'$ is the output of this mechanism with respect to input $x' = (x_{-i}, x_i')$ where $x_{-i}$ is the sub-profile which contains bids of all agents except $x_i$. In addition, it is group strategy-proof if for any group of agents $G \subseteq N$ and misreported location profile $x_G' \in I^G$, $u(x_i, S) \geq u(x_i, S')$ holds for some $i \in G$, where $S'$ is the output of the mechanism with respect to input $x' = (x_{-G}, x_G')$. With strategy-proofness no single agent can improve its utility by unilaterally misreporting its location, and with group strategy-proofness no group of agents can collude to misreport their locations in a way that makes every member better off. For a randomized mechanism, it is universally group strategy-proof if it is a probability distribution over deterministic group strategy-proof mechanisms.

In this subsection, for convenience we denote $\text{opt}_t = \text{opt}_t(x')$ and $\text{opt}_r = \text{opt}_r(x')$. We refer to a misreporting of agent $i \in G$ satisfying $x_i' < x_i$ ($x_i' > x_i$) as a left motion (right motion) of $x_i$ (or simply, of $i$), as Figure 2 shows. We define there is a left (right) motion of $\text{opt}_t$ if $\text{opt}_t' < \text{opt}_t$ ($\text{opt}_t' > \text{opt}_t$). We say there is a left (right) motion of $\text{opt}_r$ if $\text{opt}_r' < \text{opt}_r$ ($\text{opt}_r' > \text{opt}_r$). Before designing strategy-proof mechanisms, we note that for some agent $i \in G$, its any left motion cannot yield a right motion of $\text{opt}_t$, and it makes a difference between $\text{opt}_t$ and $\text{opt}_t'$ only if its motion strides over $\text{opt}_t$ from one side to the other side, which, combining with similar nature for $\text{opt}_r$, gives the following observation:

**Observation 4** a) $\text{opt}_t' < \text{opt}_t$ ($\text{opt}_t' > \text{opt}_t$) only if there is a left (right) motion of some agent $i \in G$ with $x_i \geq \text{opt}_t$ ($x_i \leq \text{opt}_t$).  
b) $\text{opt}_r' < \text{opt}_r$ ($\text{opt}_r' > \text{opt}_r$) only if there is a left (right) motion of some agent $i \in G$ with $x_i \geq \text{opt}_r$ ($x_i \leq \text{opt}_r$).

![Fig. 2. An illustration of left motion and right motion](image)

**Randomized Mechanism** Based on the optimality condition (Theorem 3) that the optimal sum-type social welfare is attained in either $(0, \text{opt}_t(x))$ or $(L, \text{opt}_r(x))$ for a given location profile $x$, we have a somewhat natural randomized mechanism by taking one of these two with some fixed probability.

**Mechanism 1** Given a location profile $x$, output $S = (0, \text{opt}_t(x))$ with probability $\alpha$, and $S = (L, \text{opt}_r(x))$ with probability $1 - \alpha$, where $\alpha \in [0, 1]$ is a constant.

Next we show the universally group strategy-proofness of Mechanism 1 by proving that each candidate mechanism, i.e., $S = (0, \text{opt}_t(x))$ or $S = (L, \text{opt}_r(x))$, is group strategy-proof.

**Lemma 5.** Given a location profile $x$, the deterministic mechanism outputting $S = (0, \text{opt}_t(x))$ is group strategy-proof.
Similarly, we can also show that the deterministic mechanism outputting \( S = (L, \text{opt}_r) \) is group strategy-proof, and combining these two naturally gives the universally group strategy-proofness of Mechanism 1.

**Theorem 6.** Mechanism 1 is universally group strategy-proof.

By Theorem 3, we derive the approximation performance of Mechanism 1.

**Theorem 7.** The approximation ratio of Mechanism 1 is \( \min\{\alpha, 1 - \alpha\} \).

Since \( \alpha \) can be arbitrary value in interval \([0, 1]\), the best approximation ratio is 1/2 when \( \alpha = 1/2 \).

**Corollary 8.** Mechanism 1 with \( \alpha = 1/2 \) is a randomized universally group strategy-proof mechanism that achieves approximation ratio 1/2.

**Deterministic Mechanism** A group strategy-proof deterministic mechanism can be obtained by choosing one of the two possible optimal schemes whichever has a longer distance between the two facilities, breaking ties arbitrarily.

**Mechanism 2** Given location profile \( x \), output \( S = (0, \text{opt}_l(x)) \) if \( \text{opt}_l \geq L - \text{opt}_r \); and output \( S = (L, \text{opt}_r(x)) \) otherwise.

By Lemma 5, the building scheme \( S = (0, \text{opt}_l) \) and \( S = (L, \text{opt}_r) \) are both group strategy-proof.

**Theorem 9.** Mechanism 2 is group strategy-proof.

Because there exists a worst-case example (see Appendix, Example 17) that has not a guaranteed performance if \( C \) can be any length, we introduce a new definition as a parameter to the approximation ratio. Denote the ratio of \( L \) to \( C \) as \( R \). Now we show the approximation performance of Mechanism 2 as follows.

**Theorem 10.** Mechanism 2 has an approximation ratio \( \frac{1}{k-1} \) when \( n = 2k \), and an approximation ratio \( \frac{1}{2(k-1)R+1} \) when \( n = 2k - 1 \).

### 3.3 A Lower Bound on Approximation Ratios of Deterministic Mechanisms

In this subsection, we show a lower bound of \( \frac{1}{k-1} \) for deterministic strategy-proof mechanisms when \( n = 2k \) and \( \lambda \in (0, 2) \).

In order to represent a location profile effectively, we use the notation from [20] in the form of \((d_1 * n_1, d_2 * n_2, ..., d_m * n_m, |d_{m+1} * n_{m+1}, d_{m+2} * n_{m+2}, ..., d_w * n_w)\). The ‘|’ symbol separates the location profile for left agents (including the distance to left endpoint \( d_i \) and occurrence number \( n_i \)) and that of right agents (including the distance to right endpoint \( d_i \) and occurrence number \( n_i \)). Specifically, in the expression, \( d_i \) appears in ascending order in the left part and in a descending order in the right part.
In addition, with respect to a given profile \( x \), we define \( S = (y_0, y_1) \) as **left pattern** if \( y_0 \in [0, x_{m_1}) \) and \( y_1 > y_0 \), or **right pattern** if \( y_0 \in (x_{m_2}, L] \) and \( y_1 < y_0 \). From \( \text{su}((y_0, y_1), x) = g(y_0) - g(y_1) - \lambda(|S| - C) \), and the graph of \( g(y) \), we can easily get \( \text{su}((y_0, y_1), x) \leq 0 \) when \( y_0 \in [0, x_{m_1}) \) and \( y_1 \leq y_0 \), or \( y_0 \in (x_{m_2}, L] \) and \( y_1 > y_0 \), or \( y_0 \in [x_{m_1}, x_{m_2}] \). Because the optimal social welfare for any location profile is non-negative by Fact 11, we have the following lemma.

**Lemma 11.** Given a location profile \( x \) with \( \text{OPT}(x) \neq 0 \), if a building scheme \( S \) satisfies \( \frac{\text{su}(S, x)}{\text{OPT}(x)} > 0 \), then \( S \) must be left pattern or right pattern.

Define function \( c(x, a) \) for \( x \) and \( a \) as \( c(x, a) = \frac{x + a}{x + a} \). The following two lemmas (see, Lemma 12 and 13) gave a series of location profiles, for which the building schemes of a mechanism with approximation ratio better than the lower bound must be right pattern. We complete the proof for the bound in Theorem 14 by showing that right pattern scheme does not exist for one profile of them, which causes a contradiction.

**Lemma 12.** Assume that a deterministic mechanism with a positive approximation ratio \( \beta \) is adopted. For any numbers \( x, y \in [0, L] \), if \( y > c(x, \beta) \) and \( x < L - y \), then the building scheme for the location profile \( x = (0 \ast (k - 1), x \ast 1|y \ast k) \) must be right pattern.

Define a special number \( P \) with respect to \( L \) as \( P = \frac{L}{1 - k} \), and define a function \( t(x, a) = \frac{c(x, a) + x}{2} \). Note when \( a > \frac{1}{k - 1} \) and \( x \leq P \), we have \( c(x, a) < t(x, a) < x \) and \( L - t(x, a) > L - x \geq L - P > P \). These two inequalities are prerequisites for the location profile \( x^0 \) defined in the following lemma. Also, because \( P \neq 0 \), we can guarantee the applicability of the approximation ratio for location profiles discussed in the proof.

**Lemma 13.** Assume that a deterministic strategy-proof mechanism with approximation ratio \( \beta > \frac{1}{k - 1} \) is adopted. Given a location profile \( x^0 = (0 \ast (k - 1), P \ast 1|t(P, \beta) \ast m, 0 \ast (k - m)) \ast 1 \leq m \leq k \), if the building scheme for \( x^0 \) is right pattern, then the building scheme for location profile \( x^1 = (0 \ast (k - 1), P \ast 1|t(P, \beta) \ast (m - 1), 0 \ast (k - m + 1)) \) is right pattern.

**Theorem 14.** When \( n \neq 2k \) and \( \lambda \in (0, 2) \), any deterministic strategy-proof mechanism cannot have an approximation ratio larger than \( \frac{1}{k - 1} \).

**Proof.** Suppose a deterministic strategy-proof mechanism with approximation ratio \( \beta > \frac{1}{k - 1} \) is adopted.

Consider location profile \( x_0 = (0 \ast (k - 1), P \ast 1|t(P, \beta) \ast k) \). Since \( t(P, \beta) > c(P, \beta) \), \( L - t(P, \beta) > P \), by Lemma 12.12. the building scheme \( S_0 \) for \( x_0 \) is right pattern. Then by Lemma 13, we have that the building scheme for \( (0 \ast (k - 1), P \ast 1|t(P, \beta) \ast (k - 1), 0 \ast 1), (0 \ast (k - 1), P \ast 1|t(P, \beta) \ast (k - 2), 0 \ast 2) \) until \( S_1 \) for \( x^1 = (0 \ast (k - 1), P \ast 1|t(P, \beta) \ast (k - k), 0 \ast k) = (0 \ast (k - 1), P \ast 1|0 \ast k) \) is right pattern. However, by the definition of right pattern building scheme, \( S_k \) cannot be a right pattern since \( x_{m_2}^1 = L \), which causes a contradiction. Therefore, any deterministic strategy-proof mechanism cannot have an approximation ratio for the social welfare larger than \( \frac{1}{k - 1} \). \( \square \)
In [20], Zou and Li gave a lower bound of $\frac{1}{k}$ for the performance of any deterministic strategy-proof mechanism in two-opposite-facility location games with rigid constraint of distance. Our result $\frac{1}{k-1}$ is slightly weaker than that due to the involvement of penalty, leading to a lower bound for optimum of $2(k-1)y$ in Lemma 12, rather than $2ky$ in [20].

4 A Mechanism for the Bottleneck-type Social Welfare

In this section, we design a group strategy-proof mechanism for maximizing the bottleneck-type social welfare. Among all $n$ agents in the two-opposite-facility location game, the agent who has the least utility is called a bottleneck agent, and its utility is called the bottleneck value (or simply, the bottleneck). In this section, we consider maximizing the bottleneck value. The soft distance constraint is still to be considered. Formally, given a location profile $x$, we aim to maximize:

$$\mu(S, x) = \min_{i \in N} u(x_i, S) - \lambda(|S| - C)$$  \hspace{1cm} (5)

Denote by $x_{e_1}$ and $x_{e_2}$ the locations of leftmost and rightmost agents respectively. Define

$$v_l(x) = \begin{cases} x_{e_1} & \text{if } \lambda < 1 \\ \min\{C, x_{e_1}\} & \text{if } \lambda \geq 1 \end{cases}$$

and

$$v_r(x) = \begin{cases} x_{e_2} & \text{if } \lambda < 1 \\ \max\{x_{e_2}, L - C\} & \text{if } \lambda \geq 1 \end{cases}$$

It can be easily verified that $v_l(x)$ ($v_r(x)$) is the best locations of facility $F_1$ when $F_0$ is fixed at 0 (at $L$), respectively. Then we propose the following group strategy-proof mechanism which achieves optimal bottleneck value.

**Mechanism 3** If $v_l(x) \geq L - v_r(x)$, output $(0, v_l(x))$; otherwise, output $(L, v_r(x))$.

Notice that, by this mechanism, when the penalty coefficient $\lambda \geq 1$, the building scheme satisfies the distance constraint. In addition, $v_l(x) > L - v_r(x)$ actually implies $\mu((0, v_l(x)), x) > \mu((L, v_r(x)), x)$. The high-level idea is simple: assumed the obnoxious facility $F_0$ is located at the endpoints of the line, if neither $(0, v_l(x))$ nor $(L, v_r(x))$ violates the distance constraint, then $\mu((0, v_l(x)), x) > \mu((L, v_r(x)), x)$ follows from $v_l(x) > L - v_r(x)$; otherwise, $\lambda$ must be less than one, and thus the increase of minimum agent utility outweighs that of penalty value in the objective function whenever $v_l(x)$ ($v_r(x)$) has a right(left) motion to $x_{e_1}$ ($x_{e_2}$). Next we show the group strategy-proofness and optimality of Mechanism 3.

**Theorem 15.** Mechanism 3 is group strategy-proof.

**Theorem 16.** Mechanism 3 outputs an optimal building scheme for the bottleneck-type social welfare.
Proof. Given a location profile \( \mathbf{x} \), denote by \( S \) the output locations. By definition of Mechanism 3 we have \( mu(S, \mathbf{x}) = \max(u_i(\mathbf{x}), L - v_r(\mathbf{x}) - \lambda(|S| - C)_+ \geq 0 \). Consider an arbitrary building scheme \( S' = (y_0', y_1') \) and we prove the optimality of Mechanism 3 by showing \( mu(S', \mathbf{x}) \leq mu(S, \mathbf{x}) \) in the following three cases.

Case 1: \( y_1' < y_0' \). If \( \exists i \in N \) such that \( x_i \geq y_0' \), then \( mu(S', \mathbf{x}) \leq u(x_i, S') = -|S'| < 0 \leq mu(S, \mathbf{x}) \).

If \( \forall i \in N, x_i < y_0' \) and \( \exists i \in N \) with \( x_i > y_1' \), then \( y_1' < x_{e_2} < y_0' \). Obviously \( e_2 \) is the bottleneck agent under \( S' \). The objective function \( mu(S', \mathbf{x}) = u(x_{e_2}, S') - \lambda(|S'| - C)_+ \leq d(x_{e_2}, y_0') - \lambda(d(x_{e_2}, y_0') - C)_+ = mu(S'', \mathbf{x}) = mu((L, L - |S''|), \mathbf{x}) \), where \( S'' = (y_0', x_{e_2}) \). Since \( v_r(\mathbf{x}) \) is the best location of \( F_1 \) when \( F_0 \) is located at \( L \), it derives \( mu(S', \mathbf{x}) \leq mu((L, L - |S''|), \mathbf{x}) \leq mu((L, v_r(\mathbf{x})), \mathbf{x}) \leq mu(S, \mathbf{x}) \).

If \( \forall i \in N, x_i \leq y_1' \), then \( x_{e_2} \leq y_1' \), and \( e_2 \) is still the bottleneck agent under \( S' \). We have \( mu(S', \mathbf{x}) = mu((L, L - |S'|), \mathbf{x}) \leq mu((L, v_r(\mathbf{x})), \mathbf{x}) \leq mu(S, \mathbf{x}) \), where the first inequality comes from the fact that \( v_r(\mathbf{x}) \) is the best location of \( F_1 \) with \( F_0 \) located at \( L \).

Case 2: \( y_1' = y_0' \). Obviously, for any agent \( i \in N, u(x_i, S') = 0 \), hence \( mu(S', \mathbf{x}) = 0 \leq mu(S, \mathbf{x}). \)

Case 3: \( y_1' > y_0' \). The proof for this case is similar to that for Case 1. \( \Box \)

5 Conclusions

In this paper, we investigate the two-opposite-facility location games with penalties on the distance between the two facilities and propose randomized and deterministic group strategy-proof mechanisms with guaranteed performance for maximizing either type of social welfare.

There are some interesting extensions for further study. The game could be extended to the case where \( n \geq 3 \) facilities have heterogeneous characteristics [2], and different agents might have different preferences among these facilities. Some different choices for the social welfare could be taken into account, such as the least-squares objective [13,18] or the \( L_p \) norm of costs [4]. In addition, for other restricted location spaces such as cycles or tree metric spaces [11], there might be some specific strategy-proof mechanisms with guaranteed performance which exploit the internal structures of such spaces.

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Appendix

Proof of Fact 1
Proof. When $n = 2k$ with $k$ agents located at 0 and others located at $L$, consider an arbitrary building scheme $S = (y_0, y_1)$. If $y_0 \leq y_1$, then $u(x_i, S) = -|S|$ for any agent $i$ with $x_i = 0$, and $u(x_j, S) = |S|$ for any agent $j$ with $x_j = L$. Therefore the sum-type social welfare $su(S, x) = \sum_{i \in N} u(x_i, S) = k * (-|S|) + k * |S| - \lambda(|S| - C)_+ \leq 0$ and the equality holds when taking $|S| \leq C$. If $y_0 > y_1$, similarly we have $su(S, x) \leq 0$. In addition, $S = (0, 0)$ provides a sum-type social welfare 0, and thus OPT(x) = 0 follows.

When the above condition is not satisfied, we show that there exists a scheme $S$ such that $su(S, x) > 0$. Consider the following two cases w.r.t. the parity of $n$.

Case 1. $n = 2k - 1$, $k \in \mathbb{N}^+$.

If $x_m = L$, consider the building scheme $S = (L - C, L)$. Let $G = \{i \in N | x_i = L\}$, and obviously $|G| \geq k$. For each $i \in G$, $u(x_i, S) = C$ and for each $i \notin G$, $u(x_i, S) \geq -C$. Thus, we have $su(x, S) \geq |G| \cdot C - (2k - 1 - |G|) \cdot C = (2|G| - 2k + 1) \cdot C \geq C > 0$.

If $x_m < L$, consider the building scheme $S = \{\min\{x_m + C, L\}, x_m\}$ with $|S| > 0$. Let $G = \{i \in N | x_i \leq x_m\}$, and obviously $|G| \geq k$. For each $i \in G$, its utility $u(x_i, S) = |S|$, and for each $i \notin G$, $u(x_i, S) \geq -|S|$. It implies $su(x, S) \geq |G| \cdot |S| - (2k - 1 - |G|) \cdot |S| = (2|G| - 2k + 1) \cdot |S| \geq |S| > 0$.

Case 2. $n = 2k$, $k \in \mathbb{N}^+$. Note that $x_{m_1} = 0$ and $x_{m_2} = L$ cannot hold simultaneously.

If $x_{m_1} \neq 0$, consider $S = \{\max\{x_m - C, 0\}, x_m\}$ with $|S| > 0$. Let $G$ be the set $\{i \in N | x_i \geq x_{m_1}\}$, and obviously $|G| \geq k + 1$. For each $i \in G$, $u(x_i, S) = |S|$, and for each $i \notin G$, $u(x_i, S) \geq -|S|$. Thus we have $su(x, S) \geq |G| \cdot |S| - (2k - |G|) \cdot |S| = (2|G| - 2k) \cdot |S| \geq 2|S| > 0$.

If $x_{m_2} \neq L$, a similar analysis for $S = \{\min\{x_m + C, L\}, x_m\}$ completes the proof. \qed

Proof of Theorem 3
Proof. We show that for any $S = (y_0, y_1)$, $su(S, x)$ is no more than $su((0, opt_l), x)$ or $su(L, opt_r), x)$. Clearly $su((0, opt_l), x) = g(0) - g(opt_l) - \lambda(opt_l - C)_+ \geq 0$.

The range of $y_0$ can be divided into three cases:

Case 1: $y_0 \in [x_{m_1}, x_{m_2}]$. Then we have $su((y_0, y_1), x) = g(y_0) - g(y_1) - \lambda(|S| - C)_+ \leq g(y_0) - g(y_1) \leq 0 \leq su((0, opt_l), x)$, where the second inequality holds since $y_0 = \arg \min g(y)$.

Case 2: $y_0 \in [0, x_{m_1})$. Consider $S' = (y_0, y_1')$ where $y_1' = \min\{y_1, x_{m_1}\}$. Combining $g(y_1') \leq g(y_1)$ and $|S'| \leq |S|$ leads to $su(S', x) \geq su(S, x)$. Additionally, consider $S'' = (0, y_1'')$ where $y_1'' = |S'| = |y_1' - y_0|$. Comparing $su(S'', x)$ and $su(S', x)$, by the equal penalty item and the convexity of $g(y)$ we have $g(0) - g(y_1'') \geq g(y_0) - g(y_1')$. Hence, $su(S'', x) \geq su(S', x) \geq su(S, x)$, which derives $su((0, opt_l), x) \geq su(S'', x) \geq su(S, x)$.

Case 3: $y_0 \in (x_{m_2}, L]$. Similar to the proof in Case 2. \qed

Proof of Lemma 5
Proof. Given the true location profile \( x \), a group of agents \( G \subseteq N \) and misreported location profile \( x'_G \in I^{|G|} \), denote \( x' = (x - G, x'_G) \). Let \( S = (0, \text{opt}_l) \) and \( S' = (0, \text{opt}'_l) \) be the output with respect to input \( x \) and \( x' \), respectively. We consider the following three cases, and claim that at least one agent in group \( G \) cannot increase its payoff by misreporting in each case.

Case 1: \( \text{opt}_l < \text{opt}'_l \). By Observation 4 there exists agent \( i \in G \) such that \( x_i \leq \text{opt}_l < \text{opt}'_l \). For such agent \( i \), we have \( u(x_i, S') = x_i - (\text{opt}'_l - x_i) < x_i - (\text{opt}_l - x_i) = u(x_i, S) \), implying its utility after misreporting is worse by being truthful.

Case 2: \( \text{opt}_l = \text{opt}'_l \). As there is no change in the mechanism output, any agent in \( G \) cannot increase its utility.

Case 3: \( \text{opt}_l > \text{opt}'_l \). In this case, clearly there is some agent \( i \in G \) such that \( x_i \geq \text{opt}_l > \text{opt}'_l \). For such agent \( i \), we have \( u(x_i, S') = \text{opt}'_l < \text{opt}_l = u(x_i, S) \), implying misreporting is not good for it.

\( \square \)

**Proof of Theorem 1**

Proof. Consider a true location profile \( x \) with \( \text{OPT}(x) \neq 0 \), and we denote by \( S_l = (0, \text{opt}_l(x)) \) and \( S_r = (L, \text{opt}_r(x)) \) the two possible outcomes of Mechanism 1. We further discuss about two cases based on \( \text{OPT}(x) \).

Case 1: \( \text{OPT}(x) = su(S_l, x) \). We show that \( su(S_r, x) \) is always positive, which directly gives the approximation ratio \( \alpha \). Let \( N_r \) be the set of agents whose locations are strictly to the right of \( \text{opt}_r \), i.e., \( N_r = \{i \in N|x_i > \text{opt}_r\} \).

Since \( \text{opt}_r \geq x_m \), we have \( |N_r| < n/2 \). It immediately follows that \( su(S_r, x) > (n - |N_r|)(L - \text{opt}_r) + |N_r|\text{opt}_r - L > 0 \).

Case 2: \( \text{OPT}(x) = su(S_r, x) \). Similar to the proof in Case 1, the approximation ratio \( 1 - \alpha \) follows.

\( \square \)

**Proof of Theorem 2**

Proof. For location profile \( x \), we assume \( \text{opt}_l \geq L - \text{opt}_r \), without loss of generality, and Mechanism 2 outputs solution \( S = (0, \text{opt}_l) \). Consider a group of agents \( G \subseteq N \) and misreported location profile \( x'_G \in I^{|G|} \). Denote by \( S' \) the output of Mechanism 2 with respect to input \( x' = (x - G, x'_G) \). We discuss about the following three cases with respect to the range of \( G \).

Case 1: \( \forall i \in G, x_i < \text{opt}_l \). Firstly we consider some single agent \( i \in G \), and show that it has no motivation to change unilaterally. While a left motion of \( x_i \) does not affect \( \text{opt}_l \) or \( \text{opt}_r \), a right motion of \( x_i \) leads to \( \text{opt}_l \leq \text{opt}'_l \) and \( \text{opt}_r \leq \text{opt}'_r \). Hence the output \( S' \) is \( (0, \text{opt}'_l) \) by \( \text{opt}'_l \geq L - \text{opt}'_r \). The utility of agent \( i \) after its unilateral misreporting is \( u(x_i, S') = x_i - (\text{opt}'_l - x_i) \leq x_i - (\text{opt}_l - x_i) = u(x_i, S) \). Then for the whole group \( G \), the simultaneous motion of agents in \( G \) likewise results in an output \( S' = (0, \text{opt}'_l) \) with \( \text{opt}'_l \leq \text{opt}_l \), which implies \( u(x_i, S') \leq u(x_i, S) \) for each \( i \in G \).

Case 2: \( \forall i \in G, \text{opt}_l \leq x_i \). Similarly, we first claim that some single agent \( i \in G \) would not change unilaterally. On one hand, a right motion of \( x_i \) leads to \( \text{opt}_r \leq \text{opt}'_r \), and \( \text{opt}_l = \text{opt}'_l \). The output \( S' \) equals \( S \). On the other hand, a left motion leads to \( \text{opt}'_l \leq \text{opt}_l \) and \( \text{opt}'_r \leq \text{opt}_r \). If \( S' = (0, \text{opt}'_l) \), then the utility of
agent $i$ does not increase. If $S' = (L, \text{opt}_t')$, then \( u(x_i, S') \leq L - \text{opt}_t \leq \text{opt}_t = u(x_i, S) \), where the first inequality holds since $\text{opt}_t' < \text{opt}_t$ only when $x_i \leq \text{opt}_t$. Therefore, any single agent cannot improve its utility by unilateral misreporting. Then consider the whole group $G$. Based on the above discussion, it is necessary to output $S' = (L, \text{opt}_t')$ with $\text{opt}_t' < \text{opt}_t$ in order to improve the utilities of all agents in $G$, otherwise an output of $(0, \text{opt}_t')$ with $\text{opt}_t' \leq \text{opt}_t$ can benefit no agent in $G$. Further, we obtain $\text{opt}_t' < \text{opt}_t$ only when there exists some agent $k \in G$ satisfying $x_k \geq \text{opt}_t$ who has a left motion. For agent $k$, however, we have

\[
u(x_k, S') < L - \text{opt}_t \leq \text{opt}_t = u(x_k, S), \tag{6}
\]

which indicates agent $k$ has a loss of utility.

Case 3: $\exists i, j \in G$ such that $x_i < \text{opt}_t$ and $x_j \geq \text{opt}_t$. If the output after misreporting is $S' = (0, \text{opt}_t')$, it is easy to see that not every agent in $G$ can improve its utility (e.g., if $\text{opt}_t' < \text{opt}_t$ then agent $j$ satisfying $x_j \geq \text{opt}_t$ incurs a loss). Suppose the output is $S' = (L, \text{opt}_t')$. If $\text{opt}_t < \text{opt}_t'$, then we have $u(x_i, S') = L - \text{opt}_t' < L - \text{opt}_t \leq \text{opt}_t = u(x_i, S)$ for agent $i \in G$ satisfying $x_i < \text{opt}_t$. If $\text{opt}_t' < \text{opt}_t$, there must exist some agent $k \in G$ satisfying $x_k \geq \text{opt}_t$ who has a left motion. For agent $k$, \hfill \square

The Worst-case Example for Mechanism 2

Example 17. Given $n = 2k$ is even, $L \geq 6, C = \epsilon \in (0, 2)$ is a real number, we construct a location profile $x = (0*(k-1), (L/2)^*1|(L/3)^*k)$. So, we have $\text{opt}_t = L/2$ and $\text{opt}_r = 2L/3$. The output scheme of Mechanism 2 is $S = (0, \text{opt}_t)$ as $\text{opt}_t \geq L - \text{opt}_r$, where the sum-type social welfare $su(S, x) = 2 \cdot L/2 - \lambda(L/2 - C) = \epsilon \cdot L/2 + (2 - \epsilon) \cdot \epsilon \leq \epsilon(L/2 + 2) \leq \epsilon \cdot L$. However, the optimal sum-type social welfare $OPT(x) = su((L, 2L/3), x) = 2k \cdot L/3 - \lambda(L/3 - C) \geq 2(k - 1) \cdot L/3$. Therefore, the ratio of $OPT(x)$ to $su(S, x)$:

\[
\frac{su(S, x)}{OPT(x)} \leq \frac{\epsilon \cdot L}{2(k - 1) \cdot L/3} = \frac{3\epsilon}{2(k - 1)} \rightarrow 0 \quad \text{as} \quad \epsilon \to 0
\]

Proof of Theorem 10

Proof. Consider an output building scheme $S$ from Mechanism 2 for a location profile $x$ with $OPT(x) \neq 0$. Without loss of generality, we only need to consider $S = (0, \text{opt}_t)$, which indicates $\text{opt}_t \geq L - \text{opt}_r$, but the optimal sum-type social welfare occurs when $(L, \text{opt}_r)$ is used.

- If $x_{m_1} \leq C$, then $\text{opt}_t(x) = x_{m_1}$. The optimal solution is achieved.
- If $x_{m_1} > C$, then
  - Case 1: $\lambda > |g'_C|$, $\text{opt}_t(x) = C$. When $n = 2k$, $su(S, x) \geq (k + 1)\text{opt}_t - (k - 1)\text{opt}_t = 2C$, since the worst case occurs when $k - 1$ agents are located at 0; Also, $OPT(x) \leq 2k(\text{L} - \text{opt}_r) \leq 2k\text{opt}_t = 2kC$ has an upper bound when no agent is located in the right of $\text{opt}_r$. The ratio is no less than $1/k$. When $n = 2k - 1$, $su(S, x) \geq k\text{opt}_t - (k - 1)\text{opt}_t = C$; $OPT(x) \leq (2k - 1)(L - \text{opt}_r) \leq (2k - 1)C$. The ratio is no less than $1/(2k - 1)$.
Case 2: \( \lambda < \|g'_+(C)|, \) \( \text{OPT}_I(x) = x_i \) satisfying that \( |g'_+(x_i)| > \lambda \) and \( |g'_+(x_i)| \leq \lambda \). Obviously, \( \lambda < \|g'_+(C)| \leq n \). Denote the number of agents located at \( \text{OPT}_I \) as \( k_0 \). Let \( k_1 \) be the number of agents located at \([0, \text{OPT}_I] \) and \( k_2 \) be the number of agents located at \((\text{OPT}_I, L] \). Then we have that \( k_0 - k_1 + k_2 > \lambda \geq -k_0 - k_1 + k_2, k_0 + k_1 + k_2 = n \) and \( k_0 \geq 1 \). Then, \( \text{OPT}(x) \leq n(L - \text{OPT}_r) - \lambda(L - \text{OPT}_r - C) \leq (n - \lambda)\text{OPT}_I + \lambda C \) and \( \text{su}(S, x) \geq -k_1 \text{OPT}_I + (k_0 + k_2)\text{OPT}_I - \lambda(\text{OPT}_I - C) = (k_0 - k_1 + k_2 - \lambda)\text{OPT}_I + \lambda C > \lambda C \).

Set \( \lambda = k_0 - k_1 + k_2 - \epsilon \) with \( \epsilon > 0 \). Consider the ratio:

\[
\frac{\text{su}(S, x)}{\text{OPT}(x)} \geq \frac{k_0 - k_1 + k_2 - \lambda + \lambda C/\text{OPT}_I}{n - \lambda + \lambda C/\text{OPT}_I}
\]

\[
= \frac{(k_0 - k_1 + k_2)C/\text{OPT}_I + \epsilon(1 - C/\text{OPT}_I)}{(k_0 - k_1 + k_2)C/\text{OPT}_I + (k_0 + k_1 - k_2 + n) + \epsilon(1 - C/\text{OPT}_I)}
\]

\[
> \frac{(k_0 - k_1 + k_2)C/\text{OPT}_I + (k_0 + k_1 - k_2 + n)}{k_0 - k_1 + k_2 + (k_0 + k_1 - k_2 + n)R} = \frac{n - 2k_1}{(n - 2k_1) + 2k_1 R}
\]

As \( k_1 < n/2, \) \( k_1 \leq k - 1 \).

\[
\frac{\text{su}(S, x)}{\text{OPT}(x)} \geq \frac{n - 2k_1}{(n - 2k_1) + 2k_1 R}
\]

\[
\geq \begin{cases} 
1 & n = 2k. \\
\frac{1}{k - 1} + \frac{1}{R} & n = 2k - 1.
\end{cases}
\]

\( \square \)

**Proof of Lemma 12**

*Proof.* Obviously, \( 0 < \beta \leq 1 \) and \( y > 0 \). By Fact 11 \( \text{OPT}(x) \neq 0 \). Denote the output scheme as \( S = f(x) \). As \( \beta > 0 \), by Lemma 11 \( S \) can only be left pattern or right pattern.

Define two schemes \( S_L = (0, x) \) and \( S_R = (L, L - y) \), and we have \( \text{OPT}(x) = \text{su}(S_L, x) \) or \( \text{OPT}(x) = \text{su}(S_R, x) \) by Theorem 3 since \( \text{OPT}_I = x \) and \( \text{OPT}_r = L - y \). Assume for contradiction that \( S \) is left pattern, we have \( \text{su}(S, x) \leq \text{su}(0, |S|, x) \leq \text{su}(S_L, x) \). It is worth noting that \( \text{su}(S_L, x) \leq 2x \) and \( \text{su}(S_R, x) = 2ky - \lambda(y - C) \geq 2(k - 1)y, \) and we have \( \text{OPT}(x) \cdot \beta \geq su(S_R, x) \cdot \beta \geq 2(k - 1)y \cdot \beta > 2x \geq su(S_L, x) \geq su(S, x) \), which contradicts the approximation ratio of \( \beta \). Therefore \( S \) must be right pattern.

\( \square \)

**Proof of Lemma 13**

*Proof.* Define \( S_0 = f(x^0) \) and \( S_1 = f(x^1) \). As \( \beta > 0 \) is guaranteed and \( \text{OPT}(x^1) \neq 0, S_1 \) is left pattern or right pattern by Lemma 11.

Assume for contradiction that \( S_1 = (y^0_i, y^1_i) \) is left pattern. Given an agent \( i \) with \( x^0_i = L - t(P, \beta) \), since \( S_0 \) is right pattern, we have \( u(x^0_i, S_0) = L - x^0_i = L - t(P, \beta) \).
\[ t(P, \beta) < P. \]

We then consider \( u(x^0_i, S_1) \). Notice that \( x^1 = \langle x^0_i, L \rangle \) if \( y^1_i > x^0_i \), then the utility of \( i \) with \( S_1 \) is \( u(x^0_i, S_1) \geq d(x^0_i, P) - d(x^0_i, L) = L - 2t(P, \beta) - P > P > u(x^0_i, S_0) \), which contradicts with strategy-proofness. So \( y^1_i \leq x^0_i \). Therefore, \(|S_1| = u(x^0_i, S_1) \leq u(x^0_i, S_0) \leq t(P, \beta) < P. \)

**Claim.** Consider location profile \( x = (0 \ast (k-1), P \ast 1 \ast t(P, \beta) \ast (m-1), t(|S_1|, \beta) \ast k', 0 \ast (k - m - k' + 1)) \) for any \( 1 \leq m \leq k, 0 \leq k' \leq k - m \), and corresponding scheme \( S = (y_0, y_1) \). If \( S \) is left pattern with \( |S| = |S_1| \) and \( y_1 < L - t(P, \beta) \), then the building scheme \( S' \) for \( x' = (0 \ast (k-1), P \ast 1 \ast t(P, \beta) \ast (m-1), t(|S_1|, \beta) \ast (k' + 1), 0 \ast (k - m - k')) \) is left pattern with \(|S'| = |S_1| \).

We prove the above claim by considering an agent \( i \) with \( x_i = L \) and \( x'_i = L - t(|S_1|, \beta) \). If \( S' \) is right pattern, then \( u(x'_i, S') \leq t(|S_1|, \beta) < |S| = |S| = u(x'_i, S) \), which contradicts with strategy-proofness. So \( S' \) is left pattern. As \( u(x_i, S) = |S| \) and \( u(x_i, S') = |S'| \), we have \(|S| \geq |S'| \). So \(|S'| = u(x'_i, S') \geq u(x'_i, S) = |S| \). Therefore, \( S' \) is left pattern with \(|S'| = |S_1| \).

Reusing the above claim, we can find that the building scheme \( S_i \) for \( x_i = (0 \ast (k-1), P \ast 1 \ast t(P, \beta) \ast (m-1), t(|S_1|, \beta) \ast (k' + 1), 0 \ast (k - m - k')) \) is left pattern with \(|S| = |S_1| < P \), and the sum-type social welfare is \( su(S_i, x_i) = 2 \cdot |S_1| \). By contrast, consider a right pattern scheme \( S'_i = (L, L - t(|S_1|, \beta)) \), we have \( su(S'_i, x'_i) \geq 2(k - 1) \cdot t(|S_1|, \beta) > 2(k - 1) \cdot c(|S_1|, \beta) = 2(k - 1) \cdot \frac{|S_1|}{k - 1} = 2 |S_1| \). It follows that \( OPT(x') \cdot \beta \geq su(S'_i, x'_i) \cdot \beta > 2 \cdot |S_1| \geq su(S_i, x_i) \), which contradicts the approximation ratio of \( \beta \) for the sum-type social welfare. Hence, \( S_1 \) is right pattern.

**Proof of Theorem 15**

*Proof.* Given a location profile \( x \), a group \( G \subseteq N \) and misreported location profiles \( x'_G \in I^{|G|} \), \( S \) and \( S' \) are the outputs w.r.t. \( x \) and \( x' = (x_G - x'_G, x'_G) \) respectively. By the symmetry of the two building schemes, we assume \( S = (0, v_l(x)) \) without loss of generality, which indicates \( v_l(x) \geq L - v_r(x) \).

Case 1: \( v_l(x') > v_l(x) \). Obviously, there exists \( i \in G \) such that \( x_i = v_l(x) \), who has a right motion. If \( S' = (0, v_l(x')) \), then \( u(x_i, S') = x_i - (v_l(x') - x_i) = x_i - (v_l(x') - v_l(x)) < x_i = u(x_i, S) \). If \( S' = (L, v_r(x')) \), then \( L - v_r(x') > v_l(x') > v_l(x) \geq L - v_r(x) \). Hence, there exists \( j \in G \) such that \( x_j = v_r(x) \), who has a left motion. However, \( u(x_j, S') < L - v_r(x) \leq v_l(x) = u(x_j, S) \), which derives a contradiction.

Case 2: \( v_l(x') \leq v_l(x) \). If \( S' = (0, v_l(x')) \), then any agent \( i \in G \) incurs a loss of utility, since \( x_i \geq v_l(x) \). If \( S' = (L, v_r(x')) \), assume for contradiction that for any agent \( i \in G \), \( u(x_i, S') > u(x_i, S) \). Because for any \( i \in G \), \( u(x_i, S) = v_l(x) \) and \( u(x_i, S') \leq L - v_r(x') \), we have \( L - v_r(x') > v_l(x) \geq L - v_r(x) \). Hence there exists \( j \in G \) with \( x_j = v_r(x) \) who has a left motion. However, \( u(x_j, S') < L - v_r(x) \leq v_l(x) = u(x_j, S) \), which contradicts our previous assumption. \( \square \)