STRONGLY ASYMPTOTICALLY OPTIMAL SCHEMES FOR THE STRONG APPROXIMATION OF NON-LIPSCHITZIAN STOCHASTIC DIFFERENTIAL EQUATIONS WITH RESPECT TO THE SUPREMUM ERROR

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Abstract. Our subject of study is strong approximation of systems of stochastic differential equations (SDEs) with respect to the supremum error criterion, and we seek approximations that perform strongly asymptotically optimal. In this context, we focus on two principal sequences of classes of approximations, namely, the classes of approximations that are based only on the evaluation of the initial value and on at most finitely many sequential evaluations of the driving Brownian motion on average and the classes of approximations that are based only on the evaluation of the initial value and on at most finitely many evaluations of the driving Brownian motion at equidistant sites. On the one hand, for SDEs with globally Lipschitz continuous coefficients, Müller-Gronbach [Ann. Appl. Probab. 12 (2002), no. 2, 664–690] showed that specific Euler–Maruyama schemes corresponding to adaptive and to equidistant time discretizations perform strongly asymptotically optimal in these classes. On the other hand, for SDEs with super-linearly growing coefficients, the main theorem of Hutzenthaler et al. [Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 467 (2011), no. 2130, 1563–1576] implies that the errors of these particular approximations tend to infinity as the numbers of discretization sites tend to infinity. In the present article, we generalize the results of the first mentioned reference such that SDEs of the latter type are incorporated. More precisely, we show under rather mild assumptions on the underlying SDE, notably Khasminskii-type and monotonicity conditions, that specific tamed Euler schemes corresponding to adaptive and to equidistant time discretizations perform strongly asymptotically optimal in the two aforementioned sequences of classes. To illustrate our findings, we numerically analyze the SDE relating to the Heston–3/2–model originating from mathematical finance.

1. Introduction

Let $T \in (0, \infty)$, let $d, m \in \mathbb{N}$, and consider a $d$-dimensional stochastic differential equation (SDE)
\begin{equation}
\begin{aligned}
dX(t) &= \mu(t, X(t)) \, dt + \sigma(t, X(t)) \, dW(t), \\
X(0) &= \xi,
\end{aligned}
\end{equation}
with drift coefficient $\mu : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, diffusion coefficient $\sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d \times m}$, $m$-dimensional Brownian motion $W$, and random initial value $\xi$ such that (1.1) has a unique (strong) solution $(X(t))_{t \in [0, T]}$. In this article, we study the classes of adaptive approximations $(X_N^{ad})_{N \in \mathbb{N}}$ and the classes of equidistant approximations $(X_N^{eq})_{N \in \mathbb{N}}$. More precisely, for each $N \in \mathbb{N}$, the set $X_N^{ad}$ denotes the class of all approximations that are based only on the evaluation of $\xi$ and on at most
N sequential evaluations of $W$ on average and the set $X^q_N$ denotes the class of all approximations that are based only on the evaluation of $\xi$ and of $W(T/N)$, $W(2T/N), \ldots, W(T)$. Moreover, we consider the error criterion

$$e_q(\hat{X}) := \left( E \left[ \sup_{t \in [0,T]} \max_{i \in \{1, \ldots, d\}} |X_i(t) - \hat{X}_i(t)|^q \right] \right)^{1/q},$$

$q \in [1, \infty)$, which measures the $q$th mean supremum distance between the solution and a given approximation $\hat{X}$. For fixed $* \in \{\text{ad, eq}\}$, the task of interest in the strong approximation problem we address is to find approximations that are strongly asymptotically optimal (in the classes $(X^*_N)_{N \in \mathbb{N}}$), i.e., approximations $(\hat{X}_N)_{N \in \mathbb{N}}$ that satisfy $\hat{X}_N \in X^*_N$ for every $N \in \mathbb{N}$ and

$$\lim_{N \to \infty} \inf \left\{ e_q(\hat{X}) \mid \hat{X} \in X^*_N \right\} = 1$$

for certain $q \in [1, \infty)$.

In the case that the coefficients of the SDE (1.1) are globally Lipschitz continuous and of at most linear growth (each with respect to the state variable), Müller-Gronbach [26] showed that specific Euler–Maruyama type schemes perform strongly asymptotically optimal. In particular, the author showed strong asymptotic optimality for, on the one hand, a sequence $(\hat{E}^q_N)_{N \in \mathbb{N}}$ of piecewise-linearly interpolated Euler–Maruyama schemes on suitably constructed adaptive time discretizations in the classes $(X^q_N)_{N \in \mathbb{N}}$ and for, on the other hand, a sequence $(\hat{E}^q_{eq})_{N \in \mathbb{N}}$ of piecewise-linearly interpolated Euler–Maruyama schemes on equidistant time discretizations in the classes $(X^{eq}_N)_{N \in \mathbb{N}}$.

In the present article, we generalize these results to SDEs with coefficients that may be non-globally Lipschitz continuous or super-linearly growing. More precisely, we show under rather mild assumptions on the SDE (1.1), notably Khasminskii-type and monotonicity conditions, strong asymptotic optimality for a sequence $(\hat{X}^{\text{ad}}_N)_{N \in \mathbb{N}}$ of piecewise-linearly interpolated so-called tamed Euler schemes on suitably constructed adaptive time discretizations in the classes $(X^{\text{ad}}_N)_{N \in \mathbb{N}}$ and for, on the other hand, a sequence $(\hat{X}^{\text{eq}}_N)_{N \in \mathbb{N}}$ of piecewise-linearly interpolated tamed Euler schemes on equidistant time discretizations in the classes $(X^{\text{eq}}_N)_{N \in \mathbb{N}}$.

To illustrate our results, we analyze the SDE relating to the Heston–$3/2$–model originating from mathematical finance. The scalar version of this SDE is given by

$$dX(t) = \alpha \cdot X(t) \cdot (\beta - |X(t)|) \ dt + \gamma \cdot |X(t)|^{3/2} \ dW(t), \quad t \in [0, T], \quad X(0) = \xi,$$

with parameters $d = m = 1$ and $T, \alpha, \beta, \gamma, \xi \in (0, \infty)$. In Theorem 4, we show for certain constellations of the parameters $\alpha$, $\beta$, and $\gamma$ that the asymptotic

$$\lim_{N \to \infty} \left( \frac{N}{\log(N)} \right)^{1/2} \inf \left\{ e_q(\hat{X}) \mid \hat{X} \in X^*_N \right\} = C^*_q$$

holds for various values of $q \in [1, \infty)$ and $* \in \{\text{ad, eq}\}$ where $C^*_q \in (0, \infty)$ is defined as in the beginning of Section 3. Since the coefficients of the autonomous SDE (1.3) are not of at most linear growth, we cannot apply the main theorems in Müller-Gronbach [26] to infer that the particular schemes $(\hat{E}^{\text{ad}}_N)_{N \in \mathbb{N}}$ and $(\hat{E}^{\text{eq}}_N)_{N \in \mathbb{N}}$ are strongly asymptotically optimal in the classes $(X^{\text{ad}}_N)_{N \in \mathbb{N}}$ and $(X^{\text{eq}}_N)_{N \in \mathbb{N}}$, respectively. Even worse, Theorem 1 in Huženthaler, Jentzen, and Kloeden [15] implies that for each $q \in [1, \infty)$ the associated errors $e_q(\hat{E}^{\text{ad}}_N)$ and $e_q(\hat{E}^{\text{eq}}_N)$ tend to infinity as $N$ tends to infinity. In contrast, we show in Corollary 6 that—in the same setting as required for (1.4)—the
tamed Euler schemes $\left(\hat{X}_{N}^{ad}\right)_{N \in \mathbb{N}}$ and $\left(\hat{X}_{N}^{eq}\right)_{N \in \mathbb{N}}$ are strongly asymptotically optimal in the classes $\left(X_{N}^{ad}\right)_{N \in \mathbb{N}}$ and $\left(X_{N}^{eq}\right)_{N \in \mathbb{N}}$, respectively. We demonstrate these results by a numerical experiment which substantiates the asymptotic behavior of the errors $e_{2}(\hat{X}_{N}^{ad})$ and $e_{2}(\hat{X}_{N}^{eq})$ as $N$ tends to infinity.

We now provide a concise overview of already existing results in the literature concerning the considered strong approximation problem. Results on strongly asymptotically optimal schemes for the strong approximation of SDEs with respect to the particular supremum error criterion \(1.2\) were first established by Hofmann, Müller-Gronbach, and Ritter \[11\] and Müller-Gronbach \[26, 27\] in the case of SDEs whose coefficients are globally Lipschitz continuous and of at most linear growth. Under essentially the same assumptions, Hofmann, Müller-Gronbach, and Ritter \[10, 12\] and Müller-Gronbach \[27\] showed respective results for the global $L_{q}$-error criterion. Lower error bounds for the strong approximation of SDEs have been extensively studied for the case of coefficients that are globally Lipschitz continuous, see, e.g., Cambanis and Hu \[3\], Hofmann, Müller-Gronbach, and Ritter \[10, 11, 12\], and Müller-Gronbach \[26, 27\]. Recently, the authors in Hefter, Herzwurm, and Müller-Gronbach \[9\] obtained lower error bounds for SDEs with coefficients that may be non-globally Lipschitz continuous but are required to possess continuous derivatives on some interval at least. Whereas most of the previously mentioned results entail lower bounds with polynomial rates of strong convergence for the minimal approximation errors in certain classes, the authors in Jentzen, Müller-Gronbach, and Yaroslavtseva \[16\] and Yaroslavtseva \[33\] constructed SDEs for which no approximation admits any polynomial convergence rate. During the last decades, upper error bounds of specific approximations for the strong approximation of SDEs have been established in the case of globally Lipschitz continuous coefficients, see, e.g., the seminal works by Maruyama \[24\] and Milstein \[25\]. The book of Kloeden and Platen \[19\] contains upper error bounds for various strong Itô–Taylor approximations. The latest progress in this area is due to explicit schemes that converge strongly to the solution even if the coefficients of the considered SDE are non-globally Lipschitz continuous or super-linearly growing. In particular, we mention tamed schemes (see Hutzenthaler, Jentzen, and Kloeden \[14\], Gan and Wang \[6\], Sabanis \[30\], Kumar and Sabanis \[20\], Sabanis and Zhang \[31\]), truncated schemes (see Mao \[23\], Guo et al. \[7\]), projected schemes (see Beyn, Isaak, and Kruse \[1, 2\]), and balanced schemes (see Tretyakov and Zhang \[32\]). For SDEs with possibly discontinuous coefficients, upper error bounds of Euler–Maruyama schemes are addressed in Leobacher and Szölgyenyi \[21\], Ngo and Taguchi \[29\], and Müller-Gronbach and Yaroslavtseva \[28\]. Additionally, we also refer to Faure \[5\] and Hutzenthaler, Jentzen, and Kloeden \[15\] for upper error bounds on piecewise-linearly interpolated Euler–Maruyama and tamed Euler schemes, respectively. In Hefter and Herzwurm \[8\], the authors studied a specific SDE for which the solution is the square of a one-dimensional Bessel-process, and gave upper error bounds with respect to the supremum error criterion \(1.2\) for certain squared piecewise-constantly interpolated projected Euler schemes. We point out that, in contrast to our findings, the asymptotic constants which can be deducted from the whole previously mentioned references are (up to exceptional cases) not given in an explicit form and therefore not known to be sharp.

The remainder of this article is organized as follows. In Section 2, we provide the setting and the notation for the rest of this work. Moreover, we introduce the conditions that will be imposed on the underlying SDE in the subsequent analysis. In Section 3, we formally explain what we mean by an approximation and define the classes of adaptive and of equidistant approximations. In Section 4, we first present a continuous-time tamed Euler scheme. Building upon this scheme, we construct the equidistant and the adapted tamed Euler schemes in full detail afterwards. In Section 5, we state the main results of this paper, i.e., strong asymptotic optimality of the adaptive
and of the equidistant tamed Euler schemes in the respective classes. In Section 6 we illustrate our findings via a numerical experiment. To this end, we review the introductory example, namely, the SDE relating to the Heston–3/2–model. In Section 7 we carry out the proofs of our main theorems. In Section 8 we indicate future research regarding a different error criterion. Finally, Appendix A comprises properties of the solution process and of the continuous-time tamed Euler scheme that will be employed in our proofs.

2. Setting, Notation, and Assumptions

Throughout this article, let \( T \in (0, \infty) \), let \( d, m \in \mathbb{N} \), let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space with a normal filtration \((\mathcal{F}(t))_{t \in [0, T]} \), let \( W : [0, T] \times \Omega \to \mathbb{R}^m \) be a standard \((\mathcal{F}(t))_{t \in [0, T]}\)-Brownian motion on \((\Omega, \mathcal{F}, \mathbb{P})\), let \( \mu : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \) be \((\mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{R}^d))\)-\(\mathcal{B}(\mathbb{R}^d)\)-measurable, let \( \sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d \times m} \) be \((\mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{R}^d))\)-\(\mathcal{B}(\mathbb{R}^{d \times m})\)-measurable, and let \( \xi : \Omega \to \mathbb{R}^d \) be \(\mathcal{F}(0)\)-\(\mathcal{B}(\mathbb{R}^d)\)-measurable with finite second moment. We study the \(d\)-dimensional stochastic differential equation

\[
\begin{align*}
\text{d}X(t) &= \mu(t, X(t)) \text{ d}t + \sigma(t, X(t)) \text{ d}W(t), \quad t \in [0, T], \\
X(0) &= \xi.
\end{align*}
\]

(2.1)

Furthermore, the following notations are used in the sequel. We denote the integer part of \( z \in \mathbb{R} \) by \( \lfloor z \rfloor := \min \{ y \in \mathbb{Z} \mid y \leq z \} \), and we set \( \lfloor x \rfloor \) to be the integer part of \( x \). For an arbitrary set \( M \), we define \( \# M \) to be the cardinality of \( M \) and, in the case that \( M \subseteq \Omega \), we define \( 1_M : \Omega \to \{0, 1\} \) to be the indicator function of \( M \). We denote the Banach space of all continuous functions \( f = (f_1, \ldots, f_d) : [0, T] \to \mathbb{R}^d \) equipped with the norm \( \|f\|_{\infty} := \sup_{t \in [0, T]} \max_{i \in \{1, \ldots, d\}} |f_i(t)| \) by \((C([0, T]; \mathbb{R}^d), \| \cdot \|_{\infty})\). For every \( p \in (0, \infty) \) and for every random variable \( Z : \Omega \to \mathbb{R} \), we put \( \|Z\|_{L^p} := (\mathbb{E}[|Z|^p])^{1/p} \). For a vector \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \), we define \( x^\top \) to be the transpose of \( x \) and \( |x| \) to be the Euclidean norm of \( x \). For a matrix \( A = (A_{i,j})_{i,j \in \{1, \ldots, d\} \times \{1, \ldots, m\}} \in \mathbb{R}^{d \times m} \), we denote by \( |A| := (\sum_{i=1}^{d} \sum_{j=1}^{m} A_{i,j}^2)^{1/2} \) the Frobenius norm of \( A \) and we set \( \|A\| := \max_{i,j \in \{1, \ldots, d\}} (\sum_{j=1}^{m} A_{i,j}^2)^{1/2} \). By \( \log : (0, \infty) \to \mathbb{R} \) we denote the natural logarithm.

In the course of this article, we will impose additional conditions on the initial value and on the coefficients of the SDE (2.1). For \( p \in [0, \infty) \) and \( \varphi \in \{\mu, \sigma\} \), we introduce the following technical assumptions:

**Assumption (I).** The initial value \( \xi \) satisfies \( \mathbb{E}[|\xi|^p] < \infty \).

**Assumption (locL).** The coefficients \( \mu \) and \( \sigma \) are locally Lipschitz continuous with respect to the state variable, i.e., for all \( M \in \mathbb{N} \) there exists a constant \( C \in (0, \infty) \) such that for all \( t \in [0, T] \) and for all \( x, y \in \mathbb{R}^d \) with \( \max \{|x|, |y|\} \leq M \) it holds that

\[
\max \left\{ |\mu(t, x) - \mu(t, y)|, |\sigma(t, x) - \sigma(t, y)| \right\} \leq C \cdot |x - y|.
\]

**Assumption (H).** The coefficients \( \mu \) and \( \sigma \) are Hölder–1/2–continuous with respect to the time variable with a Hölder bound that is linearly growing in the state variable, i.e., there exists a constant \( C \in (0, \infty) \) such that for all \( s, t \in [0, T] \) and for all \( x \in \mathbb{R}^d \) it holds that

\[
\max \left\{ |\mu(s, x) - \mu(t, x)|, |\sigma(s, x) - \sigma(t, x)| \right\} \leq C \cdot |s - t|^{1/2} \cdot (1 + |x|).
\]

**Assumption (K).** The coefficients \( \mu \) and \( \sigma \) satisfy a so-called “Khasminskii-type condition”, i.e., there exists a constant \( C \in (0, \infty) \) such that for all \( t \in [0, T] \) and for all \( x \in \mathbb{R}^d \) it holds that

\[
2 \cdot x^\top \cdot \mu(t, x) + (p - 1) \cdot |\sigma(t, x)|^2 \leq C \cdot (1 + |x|^2).
\]
**Assumption (M_p).** The coefficients $\mu$ and $\sigma$ satisfy a so-called “monotonicity condition”, i.e., there exists a constant $C \in (0, \infty)$ such that for all $t \in [0, T]$ and for all $x, y \in \mathbb{R}^d$ it holds that

$$2 \cdot (x - y)^T \cdot (\mu(t, x) - \mu(t, y)) + (p - 1) \cdot |\sigma(t, x) - \sigma(t, y)|^2 \leq C \cdot |x - y|^2.$$ 

**Assumption (pG^p).** The coefficient $\varphi$ grows at most polynomially in the state variable, i.e., there exists a constant $C \in (0, \infty)$ such that for all $t \in [0, T]$ and for all $x \in \mathbb{R}^d$ it holds that

$$|\varphi(t, x)| \leq C \cdot (1 + |x|^p).$$

**Assumption (pL^p).** The coefficient $\varphi$ is Lipschitz continuous with respect to the state variable and is polynomially growing in the state variable, i.e., there exists a constant $C \in (0, \infty)$ such that for all $t \in [0, T]$ and for all $x, y \in \mathbb{R}^d$ it holds that

$$|\varphi(t, x) - \varphi(t, y)| \leq C \cdot |x - y| \cdot (1 + |x|^p + |y|^p).$$

It is well-known that the Assumptions $[I_2]$, $[\text{locL}]$, and $[K_2]$ ensure that a unique solution $(X(t))_{t \in [0, T]}$ of the SDE (2.1) exists, see, e.g., Theorem 2.3.5 in Mao [22]; moreover, for each $p \in [2, \infty)$, the Assumptions $[I_p]$, $[\text{locL}]$, and $[K_p]$ ensure that $(X(t))_{t \in [0, T]}$ satisfies

$$\sup_{t \in [0, T]} \mathbb{E} \left[ |X(t)|^p \right] < \infty,$$

see, e.g., Theorem 2.4.1 in Mao [22].

### 3. The Classes of Adaptive and of Equidistant Approximations

In the present section, we briefly introduce the essential concepts needed to define the two principal (sequences of) classes of approximations we are interested in. To a great extent, we follow the approaches of Hefter, Herzwurm, and Müller-Gronbach [9, Section 4] and of Müller-Gronbach [20, Section 5].

Every approximation $\tilde{X} : \Omega \to \mathcal{C}([0, T]; \mathbb{R}^d)$ for the strong approximation of the solution of the SDE (2.1) that is based only on the evaluation of the initial value $\xi$ and on finitely many sequential evaluations of the driving Brownian motion $W$ is determined by three sequences

$$\psi := (\psi_k)_{k \in \mathbb{N}}, \quad \chi := (\chi_k)_{k \in \mathbb{N}}, \quad \varphi := (\varphi_k)_{k \in \mathbb{N}},$$

of measurable mappings

$$\psi_k : \mathbb{R}^d \times (\mathbb{R}^m)^{k-1} \to (0, T],$$

$$\chi_k : \mathbb{R}^d \times (\mathbb{R}^m)^k \to \{\text{STOP}, \text{GO}\},$$

$$\varphi_k : \mathbb{R}^d \times (\mathbb{R}^m)^k \to \mathcal{C}([0, T]; \mathbb{R}^d),$$

for $k \in \mathbb{N}$. Here, the sequence $\psi$ is used to obtain the sequential evaluation sites for $W$ in $(0, T]$, the sequence $\chi$ defines when to stop the evaluation of $W$, and the sequence $\varphi$ is used to get the outcome of $\tilde{X}$ once the evaluation of $W$ has stopped. More precisely, let $\omega \in \Omega$ and let $x := \xi(\omega)$ and $w := W(\omega)$ be the corresponding realizations of $\xi$ and $W$, respectively. We start the evaluation of $W$ at the time point $\psi_1(x)$. After $k$ steps, we are given the data $D_k(\omega) := (x, y_1, \ldots, y_k)$ where $y_1 := w(\psi_1(x)), \ldots, y_k := w(\psi_k(x, y_1, \ldots, y_{k-1}))$, and we decide whether to stop or to go on with the evaluation of $W$ according to the value of $\chi_k(D_k(\omega))$. The total number of evaluations of $W$ is given by

$$\nu(\omega) := \min \{ k \in \mathbb{N} \mid \chi_k(D_k(\omega)) = \text{STOP} \}.$$
To exclude non-terminating evaluations of $W$, we require $\nu < \infty$ almost surely. We obtain the realization of the approximation $\hat{X}$ by

$$
\hat{X}(\omega) := \varphi_{\nu}(D_{\nu}(\omega)).
$$

For technical reasons, we may assume without loss of generality that for all $k, \ell \in \mathbb{N}$ with $k < \ell$, for all $x \in \mathbb{R}^d$, and for all $y \in (\mathbb{R}^m)^{\ell-1}$ it holds that $\psi_k(x, y_1, \ldots, y_{k-1}) \neq \psi_\ell(x, y_1, \ldots, y_{\ell-1})$. Furthermore, we denote the average number of evaluations of $W$ of the approximation $\hat{X}$ by $c(\hat{X}) := \mathbb{E}[\nu]$.

As a next step, we specify the two classes of approximations that are studied in this article, namely, the classes of adaptive approximations $(X^\text{ad}_N)_{N \in \mathbb{N}}$ and the classes of equidistant approximations $(X^\text{eq}_N)_{N \in \mathbb{N}}$. To this end, we fix $N \in \mathbb{N}$ for the moment. First, the class $X^\text{ad}_N$ consists of all approximations that are based on the evaluation of $\xi$ and on at most $N$ sequential evaluations of $W$ on average, i.e., we define

$$
X^\text{ad}_N := \{ \hat{X} : \Omega \to C([0, T]; \mathbb{R}^d) \mid \hat{X} \text{ is of the form } (3.1) \text{ with } c(\hat{X}) \leq N \}.
$$

Second, the class $X^\text{eq}_N$ consists of all approximations that are based on the evaluation of $\xi$ and on the evaluation of $W$ at the equidistant sites $kT/N$ for $k \in \{1, \ldots, N\}$, i.e., we define

$$
X^\text{eq}_N := \{ \hat{X} : \Omega \to C([0, T]; \mathbb{R}^d) \mid \hat{X} \text{ is of the form } (3.1) \text{ with } \chi_1 = \cdots = \chi_{N-1} = \text{GO}, \chi_N = \text{STOP}, \text{ and } \psi_k = kT/N \text{ for all } k \in \{1, \ldots, N\} \}.
$$

It is easy to see that $X^\text{eq}_N \subseteq X^\text{ad}_N$ and

$$
X^\text{eq}_N = \{ u(\xi, W(T/N), W(2T/N), \ldots, W(T)) \mid u : \mathbb{R}^d \times (\mathbb{R}^m)^N \to C([0, T]; \mathbb{R}^d) \text{ is measurable} \}.
$$

Note that the classes of adaptive and of equidistant approximations incorporate a multitude of interesting approximations. In particular, classical approximations like Euler–Maruyama type schemes corresponding to suitably chosen adaptive time discretizations (see, e.g., the schemes presented in Fang and Giles [3], Kelly and Lord [17, 18], Hofmann, Müller-Gronbach, and Ritter [10, 11, 12] and Müller-Gronbach [26, 27]) to or to equidistant time discretizations lie in the respective classes. Besides, observe that these classes also contain even possibly non-implementable approximations like conditional expectations of the form $\mathbb{E}[X \mid (\xi, W(T/N), W(2T/N), \ldots, W(T))]$, $N \in \mathbb{N}$, cf. characterization (3.2).

Recall the definition (1.2) of our error criterion. For $N \in \mathbb{N}$ and $q \in [1, \infty)$, we call $\inf \{ e_q(\hat{X}) \mid \hat{X} \in X^\text{ad}_N \}$ and $\inf \{ e_q(\hat{X}) \mid \hat{X} \in X^\text{eq}_N \}$ the $N$th minimal errors in the classes of adaptive and of equidistant approximations, respectively.

**Remark 1.** The classes of adaptive and of equidistant approximations introduced above are clearly not the only classes which may be studied. For example, consider the classes $(X_N)_{N \in \mathbb{N}}$ defined by

$$
X^*_N := \{ \hat{X} : \Omega \to C([0, T]; \mathbb{R}^d) \mid \hat{X} \text{ is of the form } (3.1), \chi_k \text{ is constant for every } k \in \mathbb{N}, \text{ and } \nu = \min\{k \in \mathbb{N} \mid \chi_k = \text{STOP}\} \leq N \}
$$

for each $N \in \mathbb{N}$. For every $N \in \mathbb{N}$, the class $X^*_N$ comprises all approximations that use the same number of observations of the driving Brownian motion for each of its trajectories and satisfies $X^\text{eq}_N \subseteq X^*_N \subseteq X^\text{ad}_N$. In Müller-Gronbach [26], the author analyzes these classes and shows strong asymptotic optimality for specific Euler–Maruyama approximations. Here, we solely focus on the two first-mentioned sequences of classes since these cover, in our opinion, the most interesting approximations appearing in practice.
4. The Equidistant and the Adaptive Tamed Euler Schemes

We now introduce two types of so-called tamed Euler schemes that are based on equidistant and on adaptive time discretizations, respectively. The crucial ingredient for both approximations is a continuous-time tamed Euler scheme which, on the one hand, is suitably close to the solution of the SDE (2.1) and which, on the other hand, possesses a simple recursive structure that will be exploited in the subsequent analysis. These equidistant and adaptive tamed Euler schemes will turn out to be strongly asymptotically optimal in the classes of equidistant and of adaptive approximations, respectively.

Moreover, we point out that the continuous-time tamed Euler scheme presented here is heavily inspired by the one introduced in Sabanis [30]. The reason we do not use the latter is that our approach is more convenient for our analysis; in particular, our scheme satisfies the desired recursion (4.2) below. Nevertheless, observe that both schemes coincide in the case that the SDE (2.1) is autonomous and \( T = 1 \).

4.1. The Continuous-time Tamed Euler Scheme. Let \( N \in \mathbb{N} \) and consider the equidistant time discretization

\[
\ell t^{(N)} := \ell T/N, \quad \ell \in \{0, \ldots, N\}.
\]

Let \( r \in [0, \infty) \). The continuous-time tamed Euler scheme \( \tilde{X}_{N,r}: \Omega \to C([0,T];\mathbb{R}^d) \) is given by

\[
\tilde{X}_{N,r}(t) := \tilde{X}_{N,r}(t^{(N)}_{\ell}) + \frac{\mu(t^{(N)}_{\ell}, \tilde{X}_{N,r}(t^{(N)}_{\ell}))}{1 + (T/N)^{1/2} \cdot |\tilde{X}_{N,r}(t^{(N)}_{\ell})|^r} \cdot (t - t^{(N)}_{\ell}) \]

\[
+ \frac{\sigma(t^{(N)}_{\ell}, \tilde{X}_{N,r}(t^{(N)}_{\ell}))}{1 + (T/N)^{1/2} \cdot |\tilde{X}_{N,r}(t^{(N)}_{\ell})|^r} \cdot (W(t) - W(t^{(N)}_{\ell})) ,
\]

for all \( \ell \in \{0, \ldots, N-1\} \) and for all \( t \in (t^{(N)}_{\ell}, t^{(N)}_{\ell+1}) \).

Observe that almost surely we have

\[
\tilde{X}_{N,r}(t) = \xi + \int_0^t \mu\left(\frac{sN}{T} \cdot T/N, \tilde{X}_{N,r}\left(\frac{sN}{T} \cdot T/N\right)\right) ds + \int_0^t \frac{\sigma\left(\frac{sN}{T} \cdot T/N, \tilde{X}_{N,r}\left(\frac{sN}{T} \cdot T/N\right)\right)}{1 + (T/N)^{1/2} \cdot |\tilde{X}_{N,r}\left(\frac{sN}{T} \cdot T/N\right)|^r} dW(s)
\]

for all \( t \in [0, T] \).

Since the whole path of the driving Brownian motion is used in the construction of the continuous-time tamed Euler scheme, we have \( \tilde{X}_{N,r} \notin \mathcal{F}^d_M \) for any \( M \in \mathbb{N} \).

4.2. The Equidistant Tamed Euler Scheme. Next, based on the continuous-time tamed Euler scheme, we construct an approximation that uses not whole trajectories of the driving Brownian motion but evaluates \( W \) only at equidistant sites.

Let \( N \in \mathbb{N} \), let \( r \in [0, \infty) \), and consider the equidistant discretization (4.1). The equidistant tamed Euler scheme \( \hat{X}_{N,r}^{eq}: \Omega \to C([0,T];\mathbb{R}^d) \) is given by

\[
\hat{X}_{N,r}^{eq}(t^{(N)}_{\ell}) := \tilde{X}_{N,r}(t^{(N)}_{\ell})
\]

for \( \ell \in \{0, \ldots, N\} \) and linearly interpolated between these time points.
By suitably choosing sequences $\psi$, $\chi$, and $\varphi$ as per Section 3, we obtain $\hat{X}_{N,r}^{eq} \in \mathcal{X}_{eq}^{N}$. Clearly, the total number of evaluations of $W$ employed in the approximation $\hat{X}_{N,r}^{eq}$ is given by $N$. 

4.3. The Adaptive Tamed Euler Scheme. The following construction of the adaptive tamed Euler scheme is heavily inspired by the corresponding construction in Müller-Gronbach [26, Subsection 3.1].

Recall that, under suitable regularity assumptions on the coefficients of the SDE (2.1), we have

$$\mathbb{E} [(X_{i}(t+\delta) - X_{i}(t))^2 | X(t)] = \sum_{j=1}^{m} |\sigma_{i,j}(t, X(t))|^2 \cdot \delta + o(\delta)$$

for all $i \in \{1, \ldots, d\}$ and for all $t \in [0, T]$. Hence, the paths of each component $X_{i}$ of the solution of the considered SDE are, in the root mean square sense and conditioned on $X(t)$, locally Hölder–$1/2$–continuous with Hölder constant $(\sum_{j=1}^{m} |\sigma_{i,j}(t, X(t))|^2)^{1/2}$, and the maximum of all these constants over $i \in \{1, \ldots, d\}$ is given by $\|\sigma(t, X(t))\|$. In comparison to equidistant evaluation sites, it is therefore more beneficial to evaluate $W$ more often in regions where the value of $\|\sigma(t, X(t))\|$ is large and vice versa.

Motivated by this idea, we construct our adaptive tamed Euler scheme in two steps. First, we use equidistant time steps in order to roughly approximate the solution and to obtain estimates for the conditional Hölder constants at these sites. Second, we refine our approximation between those equidistant time points for which the corresponding estimated Hölder constant is large in proportion to the totality of the estimated Hölder constants.

Let $r \in [0, \infty)$ and let $(k_{N})_{N \in \mathbb{N}}$ be a sequence of natural numbers. Fix $N \in \mathbb{N}$ and put

$$\mathcal{A}_{k_{N}} := \left( \frac{T}{k_{N}} \sum_{\ell=0}^{k_{N}-1} \| \frac{\sigma(t_{\ell}^{(k_{N})}, X_{k_{N},r}(t_{\ell}^{(k_{N})}))}{1 + (T/k_{N})^{1/2} \cdot |X_{k_{N},r}(t_{\ell}^{(k_{N})})|^r} \|^{2} \right)^{1/2}.$$ 

Let $q \in [1, \infty)$. For each $\ell \in \{0, \ldots, k_{N} - 1\}$, we consider the random discretization

$$t_{\ell}^{(k_{N})} = t_{\ell,0}^{(k_{N})} < t_{\ell,1}^{(k_{N})} < \ldots < t_{\ell,\eta_{\ell}+1}^{(k_{N})} = t_{\ell+1}^{(k_{N})}$$

of $[t_{\ell}^{(k_{N})}, t_{\ell+1}^{(k_{N})}]$ where

$$\eta_{\ell} := 1 \{ A_{k_{N}} > 0 \} \cdot N \cdot A_{k_{N}}^{2q/(q+2)} \cdot \left[ \sum_{\ell=0}^{k_{N}-1} \left( \frac{\| \sigma(t_{\ell}^{(k_{N})}, X_{k_{N},r}(t_{\ell}^{(k_{N})}))}{1 + (T/k_{N})^{1/2} \cdot |X_{k_{N},r}(t_{\ell}^{(k_{N})})|^r} \|^{2} \right) \right]^{-2}$$

and

$$t_{\ell,\kappa}^{(k_{N})} := t_{\ell}^{(k_{N})} + \frac{T}{k_{N}} \cdot \frac{\kappa}{\eta_{\ell} + 1}$$

for all $\kappa \in \{0, \ldots, \eta_{\ell} + 1\}$. The adaptive tamed Euler scheme $\hat{X}_{N,r}^{ad} : \Omega \to C([0, T]; \mathbb{R}^{d})$ is given by

$$\hat{X}_{N,r}^{ad}(t_{\ell}^{(k_{N})}) := \hat{X}_{k_{N},r}(t_{\ell}^{(k_{N})})$$
for all $\ell \in \{0, \ldots, k_N\}$,
\[
\tilde{X}_{N,r,q}^{ad}(\tau_{\ell+1}^{(k_N)}) := \tilde{X}_{N,r,q}^{ad}(\tau_{\ell}^{(k_N)}) + \frac{\mu(t_{\ell}^{(k_N)}, \tilde{X}_{k_N,r}(t_{\ell}^{(k_N)}))}{1 + (T/k_N)^{1/2} \cdot |\tilde{X}_{k_N,r}(t_{\ell}^{(k_N)})|} \cdot \left(\tau_{\ell+1}^{(k_N)} - \tau_{\ell}^{(k_N)}\right) + \frac{\sigma(t_{\ell}^{(k_N)}, \tilde{X}_{k_N,r}(t_{\ell}^{(k_N)}))}{1 + (T/k_N)^{1/2} \cdot |\tilde{X}_{k_N,r}(t_{\ell}^{(k_N)})|} \cdot \left(W(\tau_{\ell+1}^{(k_N)}) - W(\tau_{\ell}^{(k_N)})\right)
\]

for all $\ell \in \{0, \ldots, k_N - 1\}$ and for all $\kappa \in \{0, \ldots, \eta_k\}$, and linearly interpolated between all these time points.

By suitably choosing sequences $\psi$, $\chi$, and $\varphi$ as per Section 3, we obtain $\tilde{X}_{N,r,q}^{ad} \in \mathbb{X}_{[c(\tilde{X}_{N,r,q}^{ad})]}$ if $0 < c(\tilde{X}_{N,r,q}^{ad}) < \infty$. Define $\nu_{N,r,q}^{ad}$ to be the (random) number of evaluations of $W$ employed in the approximation $\tilde{X}_{N,r,q}^{ad}$. Observe that

\[(4.6) \quad \nu_{N,r,q}^{ad} = k_N + \sum_{\ell=0}^{k_N-1} \eta_{\ell} \leq k_N + N \cdot \mathcal{A}_{k_N}^{2q/(q+2)} \]

and

\[(4.7) \quad \nu_{N,r,q}^{ad} \geq \max \left\{ k_N + 1_{\{\mathcal{A}_{k_N} > 0\}}, (N \cdot \mathcal{A}_{k_N}^{2q/(q+2)} - k_N), k_N \right\}. \]

5. **Main Results**

The following theorems entirely specify the asymptotics of the $N$th minimal errors in the classes of adaptive and of equidistant approximations as well as the asymptotics of the errors of the adaptive and of the equidistant tamed Euler schemes. As a consequence, we will conclude strong asymptotic optimality of the two aforementioned schemes in the respective classes. The proofs of all theorems are postponed to Section 7. Finally, we compare our results to those in Müller-Gronbach [26] at the end of this section.

In the case that the SDE (2.1) has a unique solution $(X(t))_{t \in [0,T]}$, put

\[
C_{q}^{ad} := \sqrt{\frac{1}{2} \left\langle \left\langle \left( \int_0^T \left\| \sigma(t, X(t)) \right\|^2 \, dt \right) \right\rangle \right\|_{L_{2q/(q+2)}}},
\]
\[
C_{q}^{eq} := \sqrt{\frac{T}{2} \sup_{t \in [0,T]} \left\| \sigma(t, X(t)) \right\| \left\| \right\|_{L_q}},
\]

for $q \in (0, \infty)$. Note that $C_{q}^{ad} \leq C_{q}^{eq}$ for all $q \in (0, \infty)$. The succeeding remarks provide sufficient conditions for the finiteness of these two constants as well as sufficient and necessary conditions for them being identical (to zero).

**Remark 2.** Let the Assumptions $\{p\}$, $\{\text{locL}\}$, $\{K_p\}$, and $\{pG^2\}$ be satisfied for some $p \in [2, \infty)$ and $r \in [1, \infty)$ with $p \geq 2r$. Then Proposition 1 in Appendix A implies that $C_{(p-2r+2)/r}^{eq} < \infty$.

**Remark 3.** Let the SDE (2.1) have a unique solution $(X(t))_{t \in [0,T]}$. Then for all $q \in (0, \infty)$, we have $C_{q}^{ad} = C_{q}^{eq}$ if and only if almost surely it holds that the mapping $[0,T] \to \mathbb{R}, t \mapsto \|\sigma(t, X(t))\|$, is constant, and we have $C_{q}^{ad} = C_{q}^{eq} = 0$ if and only if almost surely it holds that $\sigma(t, X(t)) = 0$ for all $t \in [0,T]$, see Remark 1 in Müller-Gronbach [26].
First, we specify the asymptotics of the \(N\)th minimal errors in the classes of adaptive and of equidistant approximations. More precisely, we not only state the orders of convergence but also give the sharp asymptotic constants.

**Theorem 4.** Let the Assumptions \((I_p), (H), (K_p), (M_a), \) and \((pL^{p})\) be satisfied for some \(p, a \in [2, \infty)\) and \(r \in [0, \infty)\) with \(p \geq 4r + 2\). Then for all \(q \in [1, \min\{a, p/(2r + 1)\}\) it holds that

\[
\lim_{N \to \infty} \left( \frac{N}{\log(N)} \right)^{1/2} \cdot \inf \left\{ e_q(\hat{X}) \mid \hat{X} \in X_{N}^{ad} \right\} = C_{q}^{ad}
\]

and

\[
\lim_{N \to \infty} \left( \frac{N}{\log(N)} \right)^{1/2} \cdot \inf \left\{ e_q(\hat{X}) \mid \hat{X} \in X_{N}^{ad} \right\} = C_{q}^{eq}.
\]

**Proof.** This result is an immediate consequence of the Lemmas 11, 12, 13, and 14 given in Section 7. □

Next, we specify the asymptotics of the errors of the adaptive and of the equidistant tamed Euler schemes. Again, we not only state the orders of convergence but also give the sharp asymptotic constants.

**Theorem 5.** Let the Assumptions \((I_p), (H), (K_p), (M_a), \) and \((pL^{p})\) be satisfied for some \(p, a \in [2, \infty)\) and \(r \in [0, \infty)\) with \(p \geq 4r + 2\). Then for all \(q \in [1, \min\{a, p/(2r + 1)\}\) with \(C_{q}^{ad} > 0\) it holds that

\[
0 < c(\hat{X}_{N,r,q}^{ad}) < \infty
\]

for each \(N \in \mathbb{N}\) and

\[
\lim_{N \to \infty} c(\hat{X}_{N,r,q}^{ad}) = \infty
\]

as well as

\[
\lim_{N \to \infty} \left( \frac{c(\hat{X}_{N,r,q}^{ad})}{\log(c(\hat{X}_{N,r,q}^{ad})))} \right)^{1/2} \cdot e_q(\hat{X}_{N,r,q}^{ad}) = C_{q}^{ad}
\]

and

\[
\lim_{N \to \infty} \left( \frac{N}{\log(N)} \right)^{1/2} \cdot e_q(\hat{X}_{N,r,q}^{eq}) = C_{q}^{eq}.
\]

**Proof.** This result is an immediate consequence of the Lemmas 8, 11, 12, 13, and 14 given in Section 7. □

Since both the orders of convergence and the asymptotic constants match in the preceding theorems, we obtain strong asymptotic optimality of the adaptive and of the equidistant tamed Euler schemes in their respective classes.

**Corollary 6.** Let the Assumptions \((I_p), (H), (K_p), (M_a), \) and \((pL^{p})\) be satisfied for some \(p, a \in [2, \infty)\) and \(r \in [0, \infty)\) with \(p \geq 4r + 2\). Then for all \(q \in [1, \min\{a, p/(2r + 1)\}\) with \(C_{q}^{ad} > 0\) it holds that

\[
\lim_{N \to \infty} \inf \left\{ e_q(\hat{X}) \mid \hat{X} \in X_{[c(\hat{X}_{N,r,q}^{ad})]}^{ad} \right\} = 1
\]
and 
\[
\lim_{N \to \infty} \frac{e_q(\hat{X}_{N,r}^{eq})}{\inf \{e_q(\hat{X}) \mid \hat{X} \in \mathcal{X}_N^{eq} \}} = 1;
\]
therefore, the approximations \((\hat{X}_{N,r}^{ad})_{N \in \mathbb{N}}\) and \((\hat{X}_{N,r}^{eq})_{N \in \mathbb{N}}\) are strongly asymptotically optimal in the classes \((\mathcal{X}_N^{ad})_{N \in \mathbb{N}}\) and \((\mathcal{X}_N^{eq})_{N \in \mathbb{N}}\), respectively.

**Proof.** This result is an immediate consequence of the Theorems 4 and 5. \(\square\)

The preceding findings generalize the corresponding results in Müller-Gronbach [26], where the author assumes, among other conditions, global Lipschitz continuity of the drift and diffusion coefficients.

**Remark 7.** Assume \(T = 1\), let the Assumptions \([I_p]\) and \([H]\) be satisfied for some \(p^* \in [2, \infty)\), and let the coefficients \(\mu\) and \(\sigma\) be globally Lipschitz continuous and of at most linear growth, i.e., let the Assumptions \([pL^0]\) and \([pL^\infty]\) be satisfied. Theorem 3 in Müller-Gronbach [26] then shows that the Nth minimal errors in the classes of adaptive and of equidistant approximations satisfy (5.1) and (5.2) for all \(q \in [1, p^*]\).

Moreover, in the setting above, it is easy to see that the Assumption \([K_p]\) is satisfied for all \(p \in [0, \infty)\), the Assumption \([M_\alpha]\) is satisfied for all \(\alpha \in [0, \infty)\), and the Assumption \([pL^r]\) is satisfied for all \(r \in [0, \infty)\). By choosing \(p = a = p^*\) and \(r = 0\), Theorem 4 shows that the Nth minimal errors in the classes of adaptive and of equidistant approximations satisfy (5.1) and (5.2) for all \(q \in [1, p^*]\).

The only difference is, hence, that the choice \(q = p^*\) is included in the results of Müller-Gronbach [26] whereas we do not incorporate this case in our findings.

6. Numerical Simulations

We illustrate the findings of the preceding section by a numerical experiment. To this end, we consider the introductory SDE (1.3) regarding the Heston–3/2-model with parameters \(d = 1\), \(m = 1\), \(T = 1\), \(\alpha = 5\), \(\beta = 1\), \(\gamma = 1\), and \(\xi = 1\). Thus, this SDE reads as
\[
\begin{align*}
\text{d}X(t) &= 5 \cdot X(t) \cdot (1 - |X(t)|) \, \text{d}t + |X(t)|^{3/2} \, \text{d}W(t), \quad t \in [0, 1], \\
X(0) &= 1.
\end{align*}
\]

(6.1)

It is easy to see that the SDE (6.1) satisfies all the assumptions of our main theorems. More precisely, we have that the Assumption \([I_p]\) is satisfied for all \(p \in [0, \infty)\), the Assumption \([H]\) is satisfied, the Assumption \([K_p]\) is satisfied for all \(p \in [2, 11]\), the Assumption \([M_\alpha]\) is satisfied for all \(\alpha \in [2, 6]\), and the Assumption \([pL^r]\) is satisfied for all \(r \in [1, \infty)\). For the rest of this section, we fix \(p = 11\), \(a = 6\), \(r = 1\), and \(q = 2\).

In view of Theorem 5, we aim at visualizing that, for large \(N \in \mathbb{N}\), the approximation errors \(e_2(\hat{X}_{N,1,2}^{ad})\) and \(e_2(\hat{X}_{N,1}^{eq})\) of the adaptive and of the equidistant tamed Euler schemes are close to \(C_2^{ad} \cdot (\log(c(\hat{X}_{N,1,2}^{ad}))/c(\hat{X}_{N,1,2}^{ad}))^{1/2}\) and \(C_2^{eq} \cdot (\log(N)/N)^{1/2}\), respectively.

In doing so, we encounter three different approximation issues, namely, the approximation of the asymptotic constants \(C_2^{ad}\) and \(C_2^{eq}\), of the errors \(e_2(\hat{X}_{N,1,2}^{ad})\) and \(e_2(\hat{X}_{N,1}^{eq})\), and of the average number of evaluations \(c(\hat{X}_{N,1,2}^{ad})\).

Regarding the first approximation issue, we do not know numerically suitable closed-form expressions of the constants \(C_2^{ad}\) and \(C_2^{eq}\), nor of the solution, for the particular SDE (6.1). Therefore,
we estimate these constants via Monte Carlo simulations in which we approximate the solution by an equidistant tamed Euler scheme with a sufficiently large number of time steps. More precisely, we estimate $C_{eq}^2$ and $C_{ad}^2$ by

$$
\hat{C}_{eq}^2_{M,N} := \sqrt{\frac{1}{2} \left( \frac{1}{M} \sum_{m=1}^{M} \max_{\ell \in \{0, \ldots, N\}} |\hat{X}_{N,1,m}^{eq}(t_{\ell}^{(N)})|^{3} \right)^{1/2}}
$$

and

$$
\hat{C}_{ad}^2_{M,N} := \sqrt{\frac{1}{2} \left( \frac{1}{M} \sum_{m=1}^{M} \left( \frac{1}{N} \sum_{\ell=0}^{N-1} |\hat{X}_{N,1,m}^{eq}(t_{\ell}^{(N)})|^{3} \right) \right)^{1/2}},
$$

respectively, where $M, N \in \mathbb{N}$ and where the random vectors

$$(\hat{X}_{N,1,m}^{eq}(t_{0}^{(N)}), \ldots, \hat{X}_{N,1,m}^{eq}(t_{N}^{(N)})), \quad m \in \{1, \ldots, M\},$$

are independent copies of $(\hat{X}_{N,1}^{eq}(t_{0}^{(N)}), \ldots, \hat{X}_{N,1}^{eq}(t_{N}^{(N)}))$. Observe that for $C_{ad}^2$, we approximate the integral occurring in its definition by left Riemann sums. Proposition 17 in Appendix A implies that $\hat{C}_{eq}^2_{M,N}$ and $\hat{C}_{ad}^2_{M,N}$ tend to $C_{eq}^2$ and $C_{ad}^2$, respectively, as $M$ and $N$ tend to infinity. Figure 1 depicts simulations of $\hat{C}_{ad}^2_{M,27}$ and $\hat{C}_{eq}^2_{M,27}$ in dependence of $M$ along with their corresponding 95% CLT-based confidence intervals. Furthermore, we utilize the specific approximations $C_{ad}^2 \approx 0.7080$ and $C_{eq}^2 \approx 1.7749$ obtained from realizations of $\hat{C}_{ad}^2_{10^4,27}$ and of $\hat{C}_{eq}^2_{10^4,27}$, respectively, for the black lines featured in Figure 2.

**Figure 1.** Monte Carlo approximations of the asymptotic constants $C_{ad}^2$ and $C_{eq}^2$ for the SDE 6.1.
The remaining two approximation issues are addressed jointly. Similarly to the approximation of the asymptotic constants, we again estimate the solution by a sufficiently accurate equidistant tamed Euler scheme, and we approximate the errors of the equidistant tamed Euler schemes via Monte Carlo simulations. More precisely, for each $N \in \mathbb{N}$ we estimate $e_2(\hat{X}_{N,1}^{eq})$, $e_2(\hat{X}_{N,1,2}^{ad})$, and $c(\hat{X}_{N,1,2}^{ad})$ by

$$
\hat{e}_{2,M,N^*,N}^{eq} := \left( \frac{1}{M} \sum_{m=1}^{M} \max_{\ell \in \{0, \ldots, N^*\}} \left[ \hat{X}_{N^*,1,m}^{eq}(t^{(N^*)}_\ell) - \hat{X}_{N,1,m}^{eq}(t^{(N^*)}_\ell) \right]^2 \right)^{1/2},
$$

$$
\hat{e}_{2,M,N^*,N}^{ad} := \left( \frac{1}{M} \sum_{m=1}^{M} \max_{\ell \in \{0, \ldots, N^*\}} \left[ \hat{X}_{N^*,1,m}^{ad}(t^{(N^*)}_\ell) - \hat{X}_{N,1,2,m}^{ad}(t^{(N^*)}_\ell) \right]^2 \right)^{1/2},
$$

and

$$
\hat{c}_{M,N} := \frac{1}{M} \sum_{m=1}^{M} \nu_{N,1,2,m}^{ad},
$$

respectively, where $M, N^* \in \mathbb{N}$ and where the random vectors

$$(\hat{X}_{N^*,1,m}^{eq}(t^{(N^*)}_0), \ldots, \hat{X}_{N^*,1,m}^{eq}(t^{(N^*)}_{N^*})), \quad m \in \{1, \ldots, M\},$$

are independent copies of $(\hat{X}_{N,1}^{eq}(t^{(N^*)}_0), \ldots, \hat{X}_{N,1}^{eq}(t^{(N^*)}_{N^*}))$, the random vectors

$$(\hat{X}_{N^*,1,m}^{eq}(t^{(N^*)}_0), \ldots, \hat{X}_{N^*,1,m}^{eq}(t^{(N^*)}_{N^*})), \quad m \in \{1, \ldots, M\},$$

are independent copies of $(\hat{X}_{N,1}^{eq}(t^{(N^*)}_0), \ldots, \hat{X}_{N,1}^{eq}(t^{(N^*)}_{N^*}))$, the random vectors

$$(\hat{X}_{N^*,1,2,m}^{ad}(t^{(N^*)}_0), \ldots, \hat{X}_{N^*,1,2,m}^{ad}(t^{(N^*)}_{N^*})), \quad m \in \{1, \ldots, M\},$$

are independent copies of $(\hat{X}_{N,1,2}^{ad}(t^{(N^*)}_0), \ldots, \hat{X}_{N,1,2}^{ad}(t^{(N^*)}_{N^*}))$, and the random variables

$$\nu_{N,1,2,m}^{ad}, \quad m \in \{1, \ldots, M\},$$

are independent copies of $\nu_{N,1,2}^{ad}$.

For the adaptive tamed Euler schemes, we used $k_N := \lceil N \cdot (\log(N + 1))^{-1/2} \rceil$ for each $N \in \mathbb{N}$ on every computation. Numerical estimates $(N, \xi_{2,10^4,2^{27}}^{eq}, N, \xi_{2,10^4,2^{27}}^{ad}, N, \xi_{2,10^4,2^{27}}^{eq}, N, \xi_{2,10^4,2^{27}}^{ad})$, $N \in \{2^7, 2^8, \ldots, 2^{21}\}$, and $(\xi_{10^3,N,2,10^3,2^{27}}^{eq}, N, \xi_{10^3,N,2,10^3,2^{27}}^{ad}, N, \xi_{10^3,N,2,10^3,2^{27}}^{eq}, N, \xi_{10^3,N,2,10^3,2^{27}}^{ad})$, $N \in \{2^7, 2^9, \ldots, 2^{21}\}$, are visualized in Figure 2.

7. Proofs

In this section, we prove our main theorems by showing asymptotic lower bounds relating to (5.1) and (5.2) as well as asymptotic upper bounds relating to (5.3) and (5.4). The structure and the content of these proofs are to a large extent based on techniques developed in Müller-Gronbach [29].

Throughout this section, let the Assumptions [Lp] [H] [Kp] [M0] and [pL2] be satisfied for some $p, a \in [2, \infty)$ and $r \in (0, \infty]$ with $p \geq 4r + 2$. Observe that, in this setting, the Assumptions [locL] [pG(r+1)] [pL(r+1)/2] and [pG(r+2)/2] are also satisfied. In addition, let $(k_N)_{N \in \mathbb{N}}$ be a sequence of natural numbers such that

$$
\lim_{N \to \infty} k_N \frac{1}{N} = 0 = \lim_{N \to \infty} k_N \cdot \log(N)
$$

(7.1)
holds, and let \( c \) denote unspecified positive constants that may vary at every occurrence and that may only depend on \( T, d, m \), and the parameters and constants from the preceding assumptions.

Furthermore, we adopt some notations introduced in Section 6 in Müller-Gronbach [26]. More specifically, for every sequence \((B_\ell)_{\ell \in \mathbb{N}}\) of independent real-valued Brownian bridges on \([0, 1]\) from \(0\) to \(0\), for all \(\alpha_1, \ldots, \alpha_N \in [0, \infty)\), \(N \in \mathbb{N}\), and for all \(q \in [1, \infty)\) we put

\[
G_q(\cdot; \alpha_1, \ldots, \alpha_N) : [0, \infty) \to [0, 1], \quad u \mapsto \mathbb{P}\left( \left\{ \max_{\ell \in \{1, \ldots, N\}} \left( \frac{\alpha_\ell \cdot \sup_{t \in [0, 1]} |B_\ell(t)|^q}{\alpha_\ell} \right) > u \right\} \right),
\]

and

\[
\mathcal{M}_q(\alpha_1, \ldots, \alpha_N) := \mathbb{E}\left[ \max_{\ell \in \{1, \ldots, N\}} \left( \frac{\alpha_\ell \cdot \sup_{t \in [0, 1]} |B_\ell(t)|^q}{\alpha_\ell} \right)^q \right] \in [0, \infty)
\]

as well as

\[
G_q(\cdot; N) := G_q(\cdot; 1, \ldots, 1), \quad \mathcal{M}_q(N) := \mathcal{M}_q(1, \ldots, 1).
\]

### 7.1. Preliminary lemmas

As a first step, we prove that the average numbers of evaluations of the driving Brownian motion of the adaptive tamed Euler schemes are positive, finite, and tend to infinity as the number of discretization sites tends to infinity. We obtain, in particular, that each such approximation does indeed lie in one of the classes of adaptive schemes.
Lemma 8. For all \( q \in [1, \min\{a,p/(2r+1)\}) \) and for all \( N \in \mathbb{N} \) it holds that

\[
(7.2) \quad k_N \leq c(\hat{X}_N^{ad}) < \infty.
\]

Proof. It is easy to see that the inequalities (7.2) immediately follow from the estimates (4.6) and (4.7) of the total number of evaluations of \( \hat{W} \) of the respective adaptive tamed Euler scheme, provided that for the proof of the upper bound one also employs Assumption (pG\((r+2)/2\)) along with Proposition 16 from Appendix A.\( \square \)

In the following lemma, we show that the maximum distance between the solution inserted in the diffusion coefficient and the continuous-time tamed Euler scheme inserted in the tamed diffusion coefficient converges in probability to zero. In fact, this result will turn out to be a crucial tool for the proofs of both the asymptotic lower and the asymptotic upper bounds. In contrast to the case of globally Lipschitz continuous coefficients, our assumptions do not allow us to conclude \( L_\sigma^q \)-convergence right away. To overcome this issue, we will combine convergence in probability with uniform integrability at appropriate situations.

Lemma 9. It holds that

\[
(7.3) \quad \sup_{t \in [0,T]} \left| \sigma(t, X(t)) - \sigma(t, \hat{X}_N(t)) \right| \xrightarrow{p} N \to \infty 0.
\]

Proof. Due to the triangle inequality, it suffices to show that

\[
(7.4) \quad \sup_{t \in [0,T]} \left| \frac{\sigma(t, \hat{X}_N(t))}{1 + (T/N)^{1/2} \cdot |\hat{X}_N(t)|^{r/2}} - \frac{\sigma(t, \hat{X}_N(t))}{1 + (T/N)^{1/2} \cdot |X_N(t)|^{r/2}} \right| \xrightarrow{p} N \to \infty 0.
\]

To this end, we show \( L_\theta \)-convergence of the respective random variables to zero for appropriate values of \( \theta \in (0, \infty) \). First, combining Assumption (pL\((r+2)/2\)), the Cauchy–Schwarz inequality, the triangle inequality, and the Propositions 17, 15, and 16 from Appendix A yields

\[
\left\| \sup_{t \in [0,T]} \left| \sigma(t, X(t)) - \sigma(t, \hat{X}_N(t)) \right| \right\|_{L_\theta} \leq c \cdot \left\| \sup_{t \in [0,T]} \left| X(t) - \hat{X}_N(t) \right| \cdot \left( 1 + \left| X(t) \right|^{r/2} + \left| \hat{X}_N(t) \right|^{r/2} \right) \right\|_{L_\theta} 
\]

\[
\leq c \cdot \left\| \sup_{t \in [0,T]} \left| X(t) - \hat{X}_N(t) \right| \right\|_{L_{2\theta}} \cdot \left\| \sup_{t \in [0,T]} \left( 1 + \left| X(t) \right|^{r/2} + \left| \hat{X}_N(t) \right|^{r/2} \right) \right\|_{L_{2\theta}} 
\]

\[
\leq c \cdot \left\| \sup_{t \in [0,T]} \left| X(t) - \hat{X}_N(t) \right| \right\|_{L_{2\theta}} \cdot \left( 1 + \sup_{t \in [0,T]} \left| X(t) \right|^{r/2} \right) \cdot \left( 1 + \sup_{t \in [0,T]} \left| \hat{X}_N(t) \right|^{r/2} \right) \leq c \cdot N^{-1/2}
\]

for all \( N \in \mathbb{N} \) where \( \theta := \min\{a,p/(2r+1)\}/3 \in [2/3, \infty) \). By letting \( N \) tend to infinity, we eventually obtain (7.3). Second, combining Assumption (pG\((r+2)/2\)), the triangle inequality, and
Proposition 16 yields
\[
\left\| \sup_{t \in [0,T]} \sigma(t, \tilde{X}_{N,r}(t)) - \frac{\sigma(t, \tilde{X}_{N,r}(t))}{1 + (T/N)^{1/2} \cdot |\tilde{X}_{N,r}(t)|^r} \right\|_{L^p} \\
= (T/N)^{1/2} \cdot \sup_{t \in [0,T]} \left\| \frac{\sigma(t, \tilde{X}_{N,r}(t)) \cdot |\tilde{X}_{N,r}(t)|^r}{1 + (T/N)^{1/2} \cdot |\tilde{X}_{N,r}(t)|^r} \right\|_{L^p} \\
\leq (T/N)^{1/2} \cdot \sup_{t \in [0,T]} \left\| \left(1 + |\tilde{X}_{N,r}(t)|^{(r+2)/2}\right) \cdot |\tilde{X}_{N,r}(t)|^r \right\|_{L^p} \\
\leq c \cdot N^{-1/2} \cdot \left( \left\| \sup_{t \in [0,T]} |\tilde{X}_{N,r}(t)|^r \right\|_{L^p} + \left\| \sup_{t \in [0,T]} |\tilde{X}_{N,r}(t)|^{(3r+2)/2} \right\|_{L^p} \right) \\
\leq c \cdot N^{-1/2}
\]
for all $N \in \mathbb{N}$ where $\theta := 2(p - r)/(3r + 2) \in [2, \infty)$. By letting $N$ tend to infinity, we eventually obtain (7.4).

For the convenience of the reader, we also provide a lemma containing a simple subsequence argument that will be employed in the proofs of the asymptotic lower bounds.

Lemma 10. Let $(a_N)_{N \in \mathbb{N}}$ be a sequence of real numbers that is bounded from below and let $C \in \mathbb{R}$. Then the following statements are equivalent:

(i) It holds that $\liminf_{N \to \infty} a_N \geq C$.

(ii) For every subsequence $(a_{N_k})_{k \in \mathbb{N}}$ of $(a_N)_{N \in \mathbb{N}}$ there exists a subsequence $(a_{N_{k_n}})_{n \in \mathbb{N}}$ of $(a_{N_k})_{k \in \mathbb{N}}$ such that $\liminf_{n \to \infty} a_{N_{k_n}} \geq C$.

7.2. Asymptotic lower bounds. First, we prove the sharp asymptotic lower bound with regard to the $N$th minimal errors in the classes of adaptive approximations.

Lemma 11. For all $q \in [1, \min\{a, p/(2r + 1)\}]$ it holds that

\[
\liminf_{N \to \infty} \left( \frac{N}{\log(N)} \right)^{1/2} \inf \left\{ e_q(\tilde{X}) \mid \tilde{X} \in X^\text{ad} \right\} \geq C_q.
\]

Proof. Due to the inverse triangle inequality and Proposition 17 from Appendix A observe that

\[
e_q(\tilde{X}) = \left\| X - \tilde{X} \right\|_{L^q} \geq \left\| \tilde{X}_{k_N,r} - \tilde{X} \right\|_{L^q} - c \cdot k_N^{1/2}
\]
holds for all $N \in \mathbb{N}$ and for all $\tilde{X} \in X^\text{ad}$.

Now fix $N \in \mathbb{N}$ and $\tilde{X} \in X^\text{ad}$ for the moment. Let $D_N$ denote the entire data used by $\tilde{X}$ and define $\Psi_N$ to be the set of evaluation sites of the driving Brownian motion used by the approximation $\tilde{X}$. As a first step, we show that the distance between $\tilde{X}_{k_N,r}$ and $\tilde{X}$ as above is greater or equal than the distance between $\tilde{X}_{k_N,r}$ and $E[\tilde{X}_{k_N,r} \mid D_N]$. Because of the first limit in (7.1), we may actually assume that $\{t_{1N}^{(kN)}, \ldots, t_{KN}^{(kN)}\} \subseteq \Psi_N$. Hence, we have that

\[
\tilde{X}_{k_N,r}(t) - E[\tilde{X}_{k_N,r}(t) \mid D_N] = \frac{\sigma(t_{1N}^{(kN)}, \tilde{X}_{k_N,r}(t_{1N}^{(kN)}))}{1 + (T/k_N)^{1/2} \cdot |\tilde{X}_{k_N,r}(t_{1N}^{(kN)})|^r} \cdot (W(t) - E[W(t) \mid D_N])
\]
holds for all \( \ell \in \{0, \ldots, k_N - 1\} \) and for all \( t \in (t_{\ell}\), \( t_{\ell+1}\) \]. Similarly to the proofs of the Lemmas 1 and 2 in Yaroslavtseva \[33\], one shows that for \( P^D\)-almost all \((x_0, x) \in \mathbb{R}^d \times \bigcup_{n \in \mathbb{N}} \mathbb{R}^m \) it holds that

\[
p \cdot \mathbb{P}_n(D_n) = P \cdot \mathbb{E}_n(D_n).\]

Thus, we conclude that the vectors \((\tilde{X}_{kN}, r - \mathbb{E}[\tilde{X}_{kN}, r | D_N])\) and \((-\tilde{X}_{kN}, r + \mathbb{E}[\tilde{X}_{kN}, r | D_N], D_N)\) are identically distributed. Consequently, we obtain that the processes \(\tilde{X}_{kN} - \tilde{X}_N\) and \(2\mathbb{E}[\tilde{X}_{kN}, r | D_N] - \tilde{X}_{kN, r} - \tilde{X}_N\) are also identically distributed and thereby

\[
\|\tilde{X}_{kN} - \tilde{X}_N\|_L_q \geq \|\tilde{X}_{kN, r} - \mathbb{E}[\tilde{X}_{kN, r} | D_N]\|_L_q.\]

Almost identically to the proof of inequality (12) in Müller-Gronbach \[26\] and the ensuing inequality therein, one subsequently shows that

\[
\mathbb{E}\|\tilde{X}_{kN, r} - \mathbb{E}[\tilde{X}_{kN, r} | D_N]\|_L_q^q \geq A_{kN} \cdot \delta_N^{-q/2} \cdot \mathbb{M}_q(\delta_N)
\]

holds almost surely where \(A_{kN}\) is defined as in \(1.4\) and where

\[
\delta_N := \max \left\{1, \sum_{\ell \in L_N} (\#(\Psi_N \cap (t_{\ell}, t_{\ell+1}) \} + 1)\right\}
\]

By using arguments in a similar way to the ones in the proof of the last inequality on page 681 in Müller-Gronbach \[26\], we arrive at

\[
\left(\frac{N}{\log(N)}\right)^{1/2} \cdot A_{kN} \cdot \delta_N^{-1/2} \cdot \mathbb{M}_q(\delta_N)^{1/2} \geq \mathbb{E}[\tilde{X}_{kN, r} - \mathbb{E}[\tilde{X}_{kN, r} | D_N]\|_L_q^q \cdot \mathbb{M}_q(\delta_N)^{1/2} \cdot \mathbb{1}_{\{\delta_N < e^2\} \cap \{\|\sigma(.X(.))\|_\infty > 0\}} \|L_{2q/\delta_N}\}.
\]

in the case that \(N > \exp(2)\).

Combining (7.6), (7.7), (7.8) and (7.9) yields

\[
\liminf_{N \to \infty} \left(\frac{N}{\log(N)}\right)^{1/2} \cdot e_q(\tilde{X}_N) \geq \liminf_{N \to \infty} \|\alpha(N)\|_{L_{2q/\delta_N}}\]

where

\[
\alpha(N) := A_{kN} \cdot \left(\log(\delta_N)\right)^{-1/2} \cdot \mathbb{M}_q(\delta_N)^{1/2} \cdot \mathbb{1}_{\{\delta_N < e^2\} \cap \{\|\sigma(.X(.))\|_\infty > 0\}}.
\]

Next, we use a subsequence argument as in Lemma 10 to infer (7.5) from (7.10). As a first step, Lemma 9 implies

\[
A_{kN} \to P \quad N \to \infty \quad \left(\int_0^T \|\sigma(t, X(t))\|_\infty^2 dt\right)^{1/2}.
\]

Now let \((\alpha(N))_{N \in \mathbb{N}}\) be a subsequence of \((\alpha(N))_{N \in \mathbb{N}}\). In view of (7.11), there exists a subsequence \((A_{kN})_{N \in \mathbb{N}}\) of \((A_{kN})_{N \in \mathbb{N}}\) such that

\[
\alpha_{kN} \to \alpha_{kN} \quad N \to \infty \quad \left(\int_0^T \|\sigma(t, X(t))\|_\infty^2 dt\right)^{1/2}.
\]
Almost identical to the proof of equation (18) in Müller-Gronbach [26], one shows that
\[
\lim_{N \to \infty} \mathbb{P}\left\{ \left| \lim_{N \to \infty} \delta_N \right| = \infty \right\} \cap \left\{ \left\| \sigma(t, X(\cdot)) \right\|_\infty > 0 \right\} = \mathbb{P}\left\{ \left\| \sigma(t, X(\cdot)) \right\|_\infty > 0 \right\}.
\]
Combining (7.12), (7.13), and an easy generalization of Corollary 2 in Müller-Gronbach [26] with non-negative instead of strictly positive scalars, we conclude that
\[
\liminf_{n \to \infty} \alpha^{(N_n)}(\eps_n) \geq \frac{1}{\sqrt{2}} \cdot \left( \int_0^T \left\| \sigma(t, X(t)) \right\|_2^2 \, dt \right)^{1/2}
\]
holds almost surely. Consequently, Fatou’s lemma gives
\[
\liminf_{n \to \infty} \left\| \alpha^{(N_n)}(\eps_n) \right\|_{L_{2q/(q+2)}^\infty} \geq C_q^\text{ad}.
\]
Finally, employing Lemma 10 finishes the proof of this lemma.

Next, we prove the sharp asymptotic lower bound with regard to the \(N\)th minimal errors in the classes of equidistant approximations.

**Lemma 12.** For all \(q \in [1, \min\{a, p/(2r + 1)\}) \) it holds that
\[
\liminf_{N \to \infty} \left( \frac{N}{\log(N)} \right)^{1/2} \cdot \inf \left\{ c_q(\widehat{X}) \mid \widehat{X} \in X_N^{eq} \right\} \geq C_q^\text{seq}.
\]

**Proof.** Due to the inverse triangle inequality and Proposition 17 from Appendix A, observe that
\[
ce_q(\widehat{X}_N) = \left\| X - \widehat{X}_N \right\|_L_q \geq \left\| \widehat{X}_{N,r} - \widehat{X}_N \right\|_{L_q} - c \cdot N^{-1/2}
\]
holds for all \(N \in \mathbb{N}\) and for all \(\widehat{X}_N \in X_N^{eq}\).

Now fix \(N \in \mathbb{N}\) and \(\widehat{X}_N \in X_N^{eq}\) for the moment. Similarly to the proof of (7.7), one argues that
\[
\left\| \widehat{X}_{N,r} - \widehat{X}_N \right\|_{L_q} \geq \left\| \widehat{X}_{N,r} - \mathbb{E}[\widehat{X}_{N,r} \mid (\xi, W(t_1^{(N)}), \ldots, W(t_N^{(N)}))] \right\|_{L_q}.
\]
Almost identically to the proofs of inequality (12) in Müller-Gronbach [26] and of the inequality in the third detached formula on page 683 therein, one shows that
\[
\mathbb{E}\left[ \left\| \widehat{X}_{N,r} - \mathbb{E}[\widehat{X}_{N,r} \mid (\xi, W(t_1^{(N)}), \ldots, W(t_N^{(N)}))] \right\|_{L_q}^q \mid (\xi, W(t_1^{(N)}), \ldots, W(t_N^{(N)})) \right]
\geq (T/N)^{q/2} \cdot \mathcal{M}_q(\alpha_0^{(N)}, \ldots, \alpha_N^{(N)})
\]
holds almost surely where
\[
\alpha_\ell^{(N)} := \frac{\sigma(t_\ell^{(N)}, \widehat{X}_{N,r}(t_\ell^{(N)}))}{1 + (T/N)^{1/2} \cdot \left| \widehat{X}_{N,r}(t_\ell^{(N)}) \right|^2}
\]
for \(\ell \in \{0, \ldots, N - 1\}\).

Combining (7.15), (7.16), and (7.17) yields
\[
\liminf_{N \to \infty} \left( \frac{N}{\log(N)} \right)^{1/2} \cdot \inf \left\{ c_q(\widehat{X}) \mid \widehat{X} \in X_N^{eq} \right\} \geq \sqrt{T} \cdot \liminf_{N \to \infty} \left\| \log(N) \right\|_{L_q}^{-1/2} \cdot \mathcal{M}_q^{1/q}(\alpha_0^{(N)}, \ldots, \alpha_N^{(N)})_{L_q}.
\]
Next, we use a subsequence argument provided by Lemma 10 to infer (7.14) from (7.18). First of all, Lemma 9 implies

\[(7.19) \quad \alpha^{(N)} := \max_{t \in \{0, \ldots, N-1\}} \alpha_t^{(N)} \xrightarrow{\mathcal{P}} \sup_{t \in [0, T]} \|\sigma(t, X(t))\|.
\]

Now let \((\alpha^{(N)})_{N \in \mathbb{N}}\) be a subsequence of \((\alpha^{(N)})_{N \in \mathbb{N}}\). In view of (7.19), there exists a subsequence \((\alpha^{(N_n)})_{n \in \mathbb{N}}\) of \((\alpha^{(N_n)})_{n \in \mathbb{N}}\) such that

\[\alpha^{(N_n)} \xrightarrow{\text{a.s.}} \sup_{t \in [0, T]} \|\sigma(t, X(t))\|.
\]

Again, an easy generalization of Corollary 2 in Müller-Gronbach [26] with non-negative instead of strictly positive scalars leads to

\[(7.3. \quad \text{Asymptotic upper bounds.} \quad \text{First, we prove the sharp asymptotic upper bound with regard to the errors of the adapted tamed Euler schemes. Finally, employing Lemma 10 finishes the proof of this lemma.})\]

**Lemma 13.** For all \(q \in [1, \min\{a, p/(2r + 1)\})\) it holds that

\[\limsup_{N \to \infty} \left( \frac{c(\hat{X}_{N,r,q})}{\log(c(\hat{X}_{N,r,q}^{ad}))} \right)^{1/2} \cdot e_q(\hat{X}_{N,r,q}) \leq C_{q}^{ad}.
\]

**Proof.** Due to the triangle inequality and Proposition 17 from Appendix A, observe that

\[\text{(7.20)} \quad e_q(\hat{X}_{N,r,q}) = \left\| X - \hat{X}_{N,r,q}^{ad} \right\|_{L_q} \leq \left\| \hat{X}_{k_N, r} - \hat{X}_{N,r,q}^{ad} \right\|_{L_q} + c \cdot k_N^{-1/2}
\]

holds for all \(N \in \mathbb{N}\).

Now fix \(N \in \mathbb{N}\) for the moment. Note that

\[\hat{X}_{k_N, r}(t) - \hat{X}_{N,r,q}^{ad}(t) = \frac{\sigma(t_k^{(k_N)}, \hat{X}_{k_N, r}(t_k^{(k_N)}))}{1 + (T/k_N)^{1/2} \cdot \left\| \hat{X}_{k_N, r}(t_k^{(k_N)}) \right\|} \cdot (W(t) - \bar{W}_N^{ad}(t))
\]

holds for all \(\ell \in \{0, \ldots, k_N - 1\}\) and for all \(t \in (t_k^{(k_N)}, t_{k_N+1}^{(k_N)})\) where \(\bar{W}_N^{ad} : \Omega \times [0, T] \to \mathbb{R}^m\) denotes the piecewise-linear interpolation of \(W\) at the adaptive sites (4.5). Almost identically to the proof of equation (25) in Müller-Gronbach [26], one shows that

\[\mathbb{E}\left[ \left\| \hat{X}_{k_N, r} - \hat{X}_{N,r,q}^{ad} \right\|_{\infty}^q \right] \leq \left( \frac{\log (\nu_N^{ad})}{N} \right)^{1/2} \cdot \left( \frac{1}{\sqrt{T}} \cdot A_{k_N}^{2/(q+2)} \cdot I_{\nu_N^{ad}, r,q} \right)^q.
\]

holds almost surely where

\[I_{\nu_N^{ad}, r,q} := \left( 1 + d \cdot 2^{q/2} \cdot \int_{2^{-q/2}}^{\infty} G_q(u \cdot \log(\nu_N^{ad}))^{q/2} \, du \right).
\]
Hence, we conclude that
\[
\left( \frac{c(X_{N,r,q}^{ad})}{\log(c(X_{N,r,q}^{ad}))} \right)^{1/2} \cdot \left\| X_{kN,r}^{ad} - X_{N,r,q}^{ad} \right\|_{L_q} \\
(7.21)
\leq \frac{1}{\sqrt{2}} \cdot \left( \frac{c(X_{N,r,q}^{ad})}{\log(c(X_{N,r,q}^{ad}))} \cdot \log(N) \right)^{1/2} \cdot \left\| \left( \log(\nu_{N,r,q}^{ad}) \right)^{1/2} \cdot A_{kN}^{2/(q+2)} \cdot I_{\nu_{N,r,q}^{ad}}^{1/q} \right\|_{L_q}
\]
in the case that \( N > 1 \) and \( k_N > 1 \).

Our main task is now to show that the limit of the right hand side of (7.21) is bounded above by \( C_q \) as \( N \) tends to infinity. To this end, note that
\[
A_{kN} \xrightarrow[p]{N \to \infty} \left( \int_0^T \left\| \sigma(t, X(t)) \right\|^2 \, dt \right)^{1/2}
(7.22)
\]
holds due to Lemma 9, and that the sequence \((A_{kN}^{2q/(q+2)})_{N \in \mathbb{N}}\) is uniformly integrable due to Assumption \((pG^q_{(r+2)/2})\) and Proposition 16. Hence, we obtain
\[
A_{kN} \xrightarrow{L_{2q/(q+2)} N \to \infty} \left( \int_0^T \left\| \sigma(t, X(t)) \right\|^2 \, dt \right)^{1/2}.
(7.23)
\]
Next, we analyze the asymptotics of the two principal terms appearing in the right hand side of (7.21) separately. First, it is straightforward to prove
\[
\lim_{N \to \infty} \frac{c(X_{N,r,q}^{ad})}{\log(c(X_{N,r,q}^{ad}))} \cdot \log(N) = \left( \int_0^T \left\| \sigma(t, X(t)) \right\|^2 \, dt \right)^{1/2} \xrightarrow{L_{2q/(q+2)} N \to \infty} \left(7.24\right)
\]
by using (4.6), (4.7), (7.1), and (7.23). Second, observe that (4.6) yields
\[
\left\| \left( \log(\nu_{N,r,q}^{ad}) \right)^{1/2} \cdot A_{kN}^{2/(q+2)} \cdot I_{\nu_{N,r,q}^{ad}}^{1/q} \right\|_{L_q} \leq \left( 1 + \frac{A_{kN}^{2q/(q+2)}}{\log(N)} \right)^{1/2} \cdot A_{kN}^{2/(q+2)} \cdot I_{\nu_{N,r,q}^{ad}}^{1/q}
\]
for all \( N \in \mathbb{N} \) such that \( N > 1 \) and \( k_N \leq N \). Furthermore, note that \( \nu_{N,r,q}^{ad} \) tends to infinity as \( N \) tends to infinity due to (4.7). Hence, Lemma 2 in Müller-Gronbach [26] along with (4.7) implies
\[
I_{\nu_{N,r,q}^{ad}} \xrightarrow{N \to \infty \ a.s.} 1.
(7.25)
\]
Combining (7.22) and (7.25) gives
\[
\left( 1 + \frac{A_{kN}^{2q/(q+2)}}{\log(N)} \right)^{1/2} \cdot A_{kN}^{2/(q+2)} \cdot I_{\nu_{N,r,q}^{ad}}^{1/q} \xrightarrow[p]{N \to \infty} \left( \int_0^T \left\| \sigma(t, X(t)) \right\|^2 \, dt \right)^{2/(q+2)}.
\]
Moreover, the sequence
\[
\left( 1 + \frac{A_{kN}^{2q/(q+2)}}{\log(N)} \right)^{q/2} \cdot A_{kN}^{2q/(q+2)} \cdot I_{\nu_{N,r,q}^{ad}}^{1/q} \xrightarrow{N \in \mathbb{N}}
\]
is uniformly integrable due to Assumption \( pG_{(r+2)/2} \) and Proposition 16. Hence, we obtain

\[
\left( 1 + \frac{A_{kN}^{2q/(q+2)}}{\log(N)} \right)^{1/2} \cdot A_{kN}^{2/(q+2)} \cdot I_{v_{N,r}^q} \xrightarrow{L_q} \left( \int_0^T \left\| \sigma(t, X(t)) \right\|^2 \, dt \right)^{2/(q+2)}.
\]

Finally, combining (7.20), (7.21), (7.24), and (7.26) finishes the proof of this lemma.

Next, we prove the sharp asymptotic upper bound with regard to the errors of the equidistant tamed Euler schemes.

**Lemma 14.** For all \( q \in [1, \min\{a, p/(2r+1)\}) \) it holds that

\[
\limsup_{N \to \infty} \left( \frac{N}{\log(N)} \right)^{1/2} \cdot e_q(\tilde{X}_{N,r}^{eq}) \leq C_q.
\]

**Proof.** Due to the triangle inequality and Proposition 17 from Appendix A we observe that

\[
e_q(\tilde{X}_{N,r}^{eq}) = \left\| X - \tilde{X}_{N,r}^{eq} \right\|_{L_q} \leq \left\| \tilde{X}_{N,r}^{eq} - \tilde{X}_{N,r}^{eq} \right\|_{L_q} + c \cdot N^{-1/2}
\]

holds for all \( N \in \mathbb{N} \).

Now fix \( N \in \mathbb{N} \) for the moment. Note that

\[
\tilde{X}_{N,r}^{eq}(t) - \tilde{X}_{N,r}^{eq}(t) = \frac{\sigma(t_{\ell}^{(N)}), \tilde{X}_{N,r}^{eq}(t_{\ell}^{(N)})}{1 + (T/N)^{1/2} \cdot |\tilde{X}_{N,r}^{eq}(t_{\ell}^{(N)})|^r} \cdot \left( W(t) - \tilde{W}_{N}^{eq}(t) \right)
\]

holds for all \( \ell \in \{0, \ldots, N-1\} \) and for all \( t \in (t_{\ell}^{(N)}, t_{\ell+1}^{(N)}) \) where \( \tilde{W}_N^{eq} : \Omega \times [0, T] \to \mathbb{R}^m \) denotes the piecewise-linear interpolation of \( W \) at the equidistant sites \( t_{\ell}^{(N)} \). Almost identically to the proof of equation (25) in Müller-Gronbach [26], one shows that

\[
\mathbb{E} \left[ \left\| \tilde{X}_{N,r}^{eq} - \tilde{X}_{N,r}^{eq} \right\|_{L_q}^q \left| (\xi, W(t_{\ell}^{(N)}), \ldots, W(t_{N}^{(N)})) \right| \right]
\]

\[
\leq \left( \frac{\log(N)}{N} \right)^{1/2} \cdot \sqrt{T} \cdot \max_{\ell \in \{0, \ldots, N-1\}} \left\| \frac{\sigma(t_{\ell}^{(N)}, \tilde{X}_{N,r}^{eq}(t_{\ell}^{(N)}))}{1 + (T/N)^{1/2} \cdot |\tilde{X}_{N,r}^{eq}(t_{\ell}^{(N)})|^r} \right\|_N^q \cdot I_N
\]

holds almost surely where

\[
I_N := \left( 1 + d \cdot 2^{q/2} \cdot \int_{2^{-q/2}}^{\infty} G_q(u \cdot \log(N)^{q/2}; N) \, du \right).
\]

Thus, we conclude that

\[
\left( \frac{N}{\log(N)} \right)^{1/2} \cdot \left\| \tilde{X}_{N,r}^{eq} - \tilde{X}_{N,r}^{eq} \right\|_{L_q}
\]

\[
\leq \sqrt{\frac{T}{2}} \cdot \max_{\ell \in \{0, \ldots, N-1\}} \left\| \frac{\sigma(t_{\ell}^{(N)}, \tilde{X}_{N,r}^{eq}(t_{\ell}^{(N)}))}{1 + (T/N)^{1/2} \cdot |\tilde{X}_{N,r}^{eq}(t_{\ell}^{(N)})|^r} \right\|_{L_q} \cdot I_N^{1/q}
\]

in the case that \( N > 1 \).

As a final step, we show that the right hand side of (7.28) tends to \( C_q \) as \( N \) tends to infinity. To this end, note that

\[
\max_{\ell \in \{0, \ldots, N-1\}} \left\| \frac{\sigma(t_{\ell}^{(N)}, \tilde{X}_{N,r}^{eq}(t_{\ell}^{(N)}))}{1 + (T/N)^{1/2} \cdot |\tilde{X}_{N,r}^{eq}(t_{\ell}^{(N)})|^r} \right\| \xrightarrow{p} \sup_{t \in [0, T]} \left\| \sigma(t, X(t)) \right\|
\]
holds due to Lemma 9 and that the sequence
\[
\max_{\ell \in \{0, \ldots, N-1\}} \left\| \frac{\sigma(t^N_\ell, X_N, r(t^N_\ell))}{1 + (T/N)^{1/2} \cdot |X_N, r(t^N_\ell)|} \right\|^q_{N \in \mathbb{N}}
\]
is uniformly integrable due to Assumption \((pG_{(r+2)/2})\) and Proposition 16. Hence, we obtain
\[
(7.29) \quad \max_{\ell \in \{0, \ldots, N-1\}} \left\| \frac{\sigma(t^N_\ell, X_N, r(t^N_\ell))}{1 + (T/N)^{1/2} \cdot |X_N, r(t^N_\ell)|} \right\|_{N \to \infty} \sup_{t \in [0, T]} \|\sigma(t, X(t))\|.
\]
Moreover, Lemma 2 in Müller-Gronbach [26] gives
\[
(7.30) \lim_{N \to \infty} I_N = 1.
\]
Finally, combining (7.27), (7.28), (7.29), and (7.30) finishes the proof of this lemma. \(\blacksquare\)

8. Future Work

In this paper, we studied strongly asymptotically optimal approximations with respect to the particular error criterion (1.2). Besides, the \(q\)th mean \(L_q\) distance of an approximation \(\tilde{X}\) given by
\[
\tilde{e}_q(\tilde{X}) := \left( \mathbb{E} \left[ \int_0^T \sum_{i=1}^d \left| X_i(t) - \tilde{X}_i(t) \right|^q \, dt \right] \right)^{1/q}
\]
for \(q \in [1, \infty)\) is another error criterion commonly analyzed in the literature. In the case of SDEs with globally Lipschitz continuous coefficients, Müller-Gronbach [27] showed that specific Milstein schemes corresponding to adaptive and to equidistant time discretizations perform strongly asymptotically optimal in the classes of adaptive and of equidistant approximations, respectively. In order to generalize these findings to SDEs with possibly superlinearly growing coefficients, it appears very promising to switch from Milstein schemes to tamed Milstein schemes (such as the ones defined in Gan and Wang [6] or Kumar and Sabanis [20]). This may constitute the object of future studies.

Appendix A. Properties of the Solution Process and of the Continuous-time Tamed Euler Scheme

As before, we use \(c\) to denote unspecified positive constants that may vary at every occurrence and that may only depend on \(T, d, m\), and the parameters and constants from the assumptions used in the respective propositions.

We initially show finiteness of certain moments of both the solution of the SDE (2.1) and the continuous-time tamed Euler schemes.

**Proposition 15.** Let the Assumptions \((I_p), (locL), (K_p),\) and \((pG_{\sigma r})\) be satisfied for some \(p \in [2, \infty)\) and \(r \in [1, \infty)\) with \(p \geq 2r\). Then it holds that
\[
\mathbb{E} \left[ \sup_{t \in [0, T]} |X(t)|^{p-2r+2} \right] < \infty.
\]

**Proof.** Let \(\bar{p} := p - 2r + 2 \in [2, p]\). For each \(n \in \mathbb{N}\), observe that the mapping
\[
\tau_n : \Omega \to [0, T], \quad \omega \mapsto T \wedge \inf \{ t \in [0, T] \mid n \leq |X(t, \omega)| \},
\]

is a stopping time which satisfies

\[(A.1) \quad \sup_{t \in [0, T]} |X(t \wedge \tau_n)| \leq \max \{n, |\xi|\}\]

almost surely.

Fix \(n \in \mathbb{N}\) for the moment. Applying Itô’s formula and employing Assumption \((K_p)\) yield that almost surely we have

\[
\left(1 + |X(t \wedge \tau_n)|^2\right)^{p/2}
\leq \left(1 + |\xi|^2\right)^{p/2} + c \cdot \int_0^t \left(1 + |X(s \wedge \tau_n)|^2\right)^{p/2} \, ds
\]
\[
+ \mathbb{P} \cdot \int_0^t I_{\{s \leq \tau_n\}} \cdot \left(1 + |X(s \wedge \tau_n)|^2\right)^{(p-2)/2} \cdot X(s \wedge \tau_n)^T \cdot \sigma(s \wedge \tau_n, X(s \wedge \tau_n)) \, dW(s)
\]

for all \(t \in [0, T]\). Thus, Assumption \((I_p)\), Fubini’s theorem, and estimate \((2.2)\) give

\[
\mathbb{E} \left[ \sup_{t \in [0, T]} \left(1 + |X(t \wedge \tau_n)|^2\right)^{p/2} \right]
\leq c + \mathbb{P} \cdot \mathbb{E} \left[ \sup_{t \in [0, T]} \int_0^t I_{\{s \leq \tau_n\}} \cdot \left(1 + |X(s \wedge \tau_n)|^2\right)^{(p-2)/2}
\cdot X(s \wedge \tau_n)^T \cdot \sigma(s \wedge \tau_n, X(s \wedge \tau_n)) \, dW(s) \right].
\]

Next, observe that the Burkholder–Davis–Gundy inequality and the Cauchy–Schwarz inequality imply

\[
\mathbb{E} \left[ \sup_{t \in [0, T]} \int_0^t I_{\{s \leq \tau_n\}} \cdot \left(1 + |X(s \wedge \tau_n)|^2\right)^{(p-2)/2}
\cdot X(s \wedge \tau_n)^T \cdot \sigma(s \wedge \tau_n, X(s \wedge \tau_n)) \, dW(s) \right]
\leq \sqrt{32} \cdot \mathbb{E} \left[ \left( \int_0^T I_{\{s \leq \tau_n\}} \cdot \left(1 + |X(s \wedge \tau_n)|^2\right)^{p-1} \cdot |\sigma(s \wedge \tau_n, X(s \wedge \tau_n)|^2 \, ds \right)^{1/2} \right].
\]
Moreover, Assumption \((pGp)\) and the inequality \(\sqrt{x-y} \leq x/(2\rho) + yho/2\) for all \(x, y \in [0, \infty)\) and \(\rho \in (0, \infty)\) yield

\[
\mathbb{E}
\left[
\left(
\int_0^T \mathbb{I}_{\{s \leq \tau_n\}} \cdot \left(1 + |X(s \wedge \tau_n)|^2\right)^{\gamma - 1} \cdot |\sigma(s \wedge \tau_n, X(s \wedge \tau_n))|^2 \, ds
\right)^{1/2}
\right]
\leq c \cdot \mathbb{E}
\left[
\left(
\sup_{t \in [0, T]} \left(1 + |X(t \wedge \tau_n)|^2\right)^{\gamma/2}
\cdot \int_0^T \mathbb{I}_{\{s \leq \tau_n\}} \cdot \left(1 + |X(s \wedge \tau_n)|^2\right)^{\gamma/4 - 1} \cdot \left(1 + |X(s \wedge \tau_n)|^{2\rho}\right) \, ds
\right)^{1/2}
\right]
\leq c \cdot \mathbb{E}
\left[
\left(
\sup_{t \in [0, T]} \left(1 + |X(t \wedge \tau_n)|^2\right)^{\gamma/2}
\cdot \int_0^T \mathbb{I}_{\{s \leq \tau_n\}} \cdot \left(1 + |X(s \wedge \tau_n)|^2\right)^{\gamma/2} \, ds
\right]
\right]
\leq \frac{1}{2 \cdot \sqrt{32} \cdot \rho} \cdot \mathbb{E}
\left[
\sup_{t \in [0, T]} \left(1 + |X(t \wedge \tau_n)|^2\right)^{\gamma/2}
\right] + c^2 \cdot \sqrt{32} \cdot \rho/2 \cdot \mathbb{E}
\left[
\int_0^T \mathbb{I}_{\{s \leq \tau_n\}} \cdot \left(1 + |X(s \wedge \tau_n)|^2\right)^{\gamma/2} \, ds
\right].
\]

Note that

\[\mathbb{E}\left[\int_0^T \mathbb{I}_{\{s \leq \tau_n\}} \cdot \left(1 + |X(s \wedge \tau_n)|^2\right)^{\gamma/2} \, ds\right] \leq \mathbb{E}\left[\int_0^T \left(1 + |X(s)|^2\right)^{\gamma/2} \, ds\right] \leq c\]

holds, again, due to Fubini’s theorem and \((2.2)\). Combining the inequalities \((A.2)\), \((A.3)\), \((A.4)\), and \((A.5)\) shows

\[
\mathbb{E}\left[\sup_{t \in [0, T]} \left(1 + |X(t \wedge \tau_n)|^2\right)^{\gamma/2}\right] \leq \frac{1}{2} \cdot \mathbb{E}\left[\sup_{t \in [0, T]} \left(1 + |X(t \wedge \tau_n)|^2\right)^{\gamma/4}\right] + c.
\]

To subtract the first summand of the right hand side from the left hand side, we need to ensure that these quantities are actually not infinite. For this purpose, we employ \((A.1)\) and Assumption \((Lp)\) to conclude that

\[
\mathbb{E}\left[\sup_{t \in [0, T]} \left(1 + |X(t \wedge \tau_n)|^2\right)^{\gamma/2}\right] \leq \mathbb{E}\left[\left(1 + \max\{n, |\xi|\}^2\right)^{\gamma/2}\right] < \infty.
\]

Hence, we obtain

\[\mathbb{E}\left[\sup_{t \in [0, T]} |X(t \wedge \tau_n)|^\gamma\right] \leq \mathbb{E}\left[\sup_{t \in [0, T]} \left(1 + |X(t \wedge \tau_n)|^2\right)^{\gamma/2}\right] \leq c.
\]

Using Fatou’s lemma, we derive from \((A.6)\) that

\[
\mathbb{E}\left[\sup_{t \in [0, T]} |X(t)|^\gamma\right] = \mathbb{E}\left[\lim_{n \to \infty} \sup_{t \in [0, T]} |X(t \wedge \tau_n)|^\gamma\right] \leq \liminf_{n \to \infty} \mathbb{E}\left[\sup_{t \in [0, T]} |X(t \wedge \tau_n)|^\gamma\right] \leq c,
\]

which finishes the proof of this proposition. \(\Box\)

**Proposition 16.** Let the Assumptions \((Lp)\), \((locL)\), \((Kp)\), and \((pGp)\) be satisfied for some \(p \in [2, \infty)\) and \(r \in [1, \infty)\) with \(p \geq r + 1\). Then it holds that

\[\sup_{N \in \mathbb{N}} \mathbb{E}\left[\sup_{t \in [0, T]} |\tilde{X}_{N,r}(t)|^{p-r+1}\right] < \infty.
\]
Proof. Fix $N \in \mathbb{N}$ and put $\overline{p} := p - r + 1 \in [2, p]$ as well as $t_N := \lfloor tN/T \rfloor \cdot T/N$ for $t \in [0, T]$.

First, applying Itô’s formula to the process given by (4.3) and employing Assumption $(K_p)$ yield

$$
\mathbb{E} \left[ \sup_{t \in [0,T]} \left( 1 + |\tilde{X}_{N,r}(t)|^2 \right)^{\overline{p}/2} \right] 
\leq c + c \cdot \mathbb{E} \left[ \int_0^T \left( 1 + |\tilde{X}_{N,r}(s)|^2 \right)^{\overline{p}/2} \cdot \left( 1 + |\tilde{X}_{N,r}(\tilde{s})|^2 \right) 
+ \left( 1 + |\tilde{X}_{N,r}(s)|^2 \right)^{\overline{p}/2} \cdot \left( \tilde{X}_{N,r}(s) - \tilde{X}_{N,r}(\tilde{s}) \right)^\top \cdot \frac{\mu(\tilde{s}, \tilde{X}_{N,r}(\tilde{s}))}{1 + (T/N)^{1/2} \cdot |\tilde{X}_{N,r}(\tilde{s})|^r} \right] ds 
\right] 
+ c \cdot \mathbb{E} \left[ \sup_{t \in [0,T]} \int_0^t \left( 1 + |\tilde{X}_{N,r}(s)|^2 \right)^{\overline{p}/2} \cdot \tilde{X}_{N,r}(s)^\top \cdot \frac{\sigma(\tilde{s}, \tilde{X}_{N,r}(\tilde{s}))}{1 + (T/N)^{1/2} \cdot |\tilde{X}_{N,r}(\tilde{s})|^r} dW(s) \right].
$$

Similarly to the proof of the preceding Proposition 15, we then conclude

$$
\mathbb{E} \left[ \sup_{t \in [0,T]} |\tilde{X}_{N,r}(t)|^{\overline{p}} \right] \leq c
$$

where one particularly uses

$$
\sup_{N \in \mathbb{N}} \mathbb{E} \left[ |\tilde{X}_{N,r}(t)|^{\overline{p}} \right] \leq c
$$

which is proved analogously to Lemma 2 in Sabanis [30]. Hence, the desired inequality (A.7) follows immediately.

The next proposition states that the continuous-time tamed Euler scheme is strongly convergent to the solution of the SDE (2.1) with respect to the error criterion (1.2) and possesses strong convergence order 1/2.

**Proposition 17.** Let the Assumptions $(I_p)$, $(H)$, $(K_p)$, $(M_a)$, and $(pL^r_\sigma)$ be satisfied for some $p, a \in [2, \infty)$ and $r \in [0, \infty)$ with $p \geq 4r + 2$. Then for all $q \in (0, \min\{a, p/(2r + 1)\})$ there exists a constant $C \in (0, \infty)$ such that for all $N \in \mathbb{N}$ it holds that

$$
\left\| \sup_{t \in [0,T]} |X(t) - \tilde{X}_{N,r}(t)| \right\|_{L_q} \leq C \cdot N^{-1/2}.
$$

**Proof.** Essentially, the proof of Theorem 3 in Sabanis [30] carries over here and is therefore omitted.

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