Multiplicity and stability of closed characteristics on compact convex P-cyclic symmetric hypersurfaces in $\mathbb{R}^{2n}$

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Abstract

Let $\Sigma$ be a compact convex hypersurface in $\mathbb{R}^{2n}$ which is P-cyclic symmetric, i.e., $x \in \Sigma$ implies $Px \in \Sigma$ with $P$ being a $2n \times 2n$ symplectic orthogonal matrix and satisfying $P^k = I_{2n}$, $\ker(P^l - I_{2n}) = 0$ for $1 \leq l < k$, where $n, k \geq 2$. In this paper, we prove that there exist at least $n$ geometrically distinct closed characteristics on $\Sigma$, which solves a longstanding conjecture about the multiplicity of closed characteristics for a broad class of compact convex hypersurfaces with symmetries (cf., Page 235 of [Eke1]). Based on the proof, we further prove that if the number of geometrically distinct closed characteristics on $\Sigma$ is finite, then at least $2\left[\frac{k}{2}\right]$ of them are non-hyperbolic; and if the number of geometrically distinct closed characteristics on $\Sigma$ is exactly $n$ and $k \geq 3$, then all of them are P-cyclic symmetric, where a closed characteristic $(\tau, y)$ on $\Sigma$ is called P-cyclic symmetric if $y(R) = Py(R)$.

Key words: Compact convex P-cyclic symmetric hypersurfaces, Closed characteristics, Hamiltonian systems, Multiplicity.

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1 Introduction and main results

In this paper, we study the multiplicity and stability of closed characteristics on any compact convex hypersurface with some kind of symmetries in $\mathbb{R}^{2n}$. Let $\Sigma$ be a $C^2$ compact hypersurface in $\mathbb{R}^{2n}$, bounding a strictly convex compact set $U$ with non-empty interior, where $n \geq 2$. Denote the set of all such hypersurfaces by $\mathcal{H}(2n)$. Without loss of generality, we suppose $U$ contains the origin. Let $P$ be a $2n \times 2n$ symplectic orthogonal matrix and $P^k = I_{2n}$, where $k \geq 2$. We denote

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by $\mathcal{H}_P(2n)$ the set of all $P$-cyclic symmetric hypersurfaces in $\mathcal{H}(2n)$, where $\Sigma$ is called $P$-cyclic symmetric if $P\Sigma = \Sigma$, i.e., $x \in \Sigma$ implies $Px \in \Sigma$. We consider closed characteristics $(\tau, y)$ on $\Sigma$, which are solutions of the following problem

$$
\begin{aligned}
\dot{y}(t) &= JN_\Sigma(y(t)), \ y(t) \in \Sigma, \ \forall \ t \in \mathbb{R}, \\
y(\tau) &= y(0),
\end{aligned}
$$

(1.1)

where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$, $I_n$ is the identity matrix in $\mathbb{R}^n$ and $N_\Sigma(y)$ is the outward normal unit vector of $\Sigma$ at $y$ normalized by the condition $N_\Sigma(y) \cdot y = 1$. Here $a \cdot b$ denotes the standard inner product of $a, b \in \mathbb{R}^{2n}$. A closed characteristic $(\tau, y)$ is prime if $\tau$ is the minimal period of $y$. Two closed characteristics $(\tau, x)$ and $(\sigma, y)$ are geometrically distinct, if $x(\mathbb{R}) \neq y(\mathbb{R})$. We denote by $T(\Sigma)$ the set of all geometrically distinct closed characteristics $(\tau, y)$ on $\Sigma$ with $\tau$ being the minimal period of $y$. A closed characteristic $(\tau, y)$ on $\Sigma \in \mathcal{H}_P(2n)$ is called $P$-cyclic symmetric if $y(\mathbb{R}) = Py(\mathbb{R})$, cf. Proposition 1 of [Zha1]. In this paper, we further assume ker$(P^l - I_{2n}) = 0$ holds for any $1 \leq l < k$.

Let $j : \mathbb{R}^{2n} \to \mathbb{R}$ be the gauge function of $\Sigma$, i.e., $j(\lambda x) = \lambda$ for $x \in \Sigma$ and $\lambda \geq 0$, then $j \in C^2(\mathbb{R}^{2n} \setminus \{0\}, \mathbb{R}) \cap C^1(\mathbb{R}^{2n}, \mathbb{R})$ and $\Sigma = j^{-1}(1)$. Fix a constant $\alpha \in (1, 2)$ and define the Hamiltonian $H_\alpha : \mathbb{R}^{2n} \to [0, +\infty)$ by

$$
H_\alpha(x) := j(x)^\alpha
$$

Then $H_\alpha \in C^2(\mathbb{R}^{2n} \setminus \{0\}, \mathbb{R}) \cap C^0(\mathbb{R}^{2n}, \mathbb{R})$ is convex and $\Sigma = H_\alpha^{-1}(1)$. It is well known that the problem (1.1) is equivalent to the following given energy problem of the Hamiltonian system

$$
\begin{aligned}
\dot{y}(t) &= JH_\alpha'(y(t)), H_\alpha(y(t)) = 1, \ \forall \ t \in \mathbb{R}, \\
y(\tau) &= y(0).
\end{aligned}
$$

(1.2)

Denote by $T(\Sigma, \alpha)$ the set of all geometrically distinct solutions $(\tau, y)$ of the problem (1.2), where $\tau$ is the minimal period of $y$. Note that elements in $T(\Sigma)$ and $T(\Sigma, \alpha)$ are in one to one correspondence with each other. Let $(\tau, y) \in T(\Sigma, \alpha)$. We call the fundamental solution $\gamma_y : [0, \tau] \to Sp(2n)$ with $\gamma_y(0) = I_{2n}$ of the linearized Hamiltonian system

$$
\dot{z}(t) = JH_\alpha''(y(t))z(t), \ \forall \ t \in \mathbb{R}.
$$

(1.3)

the associated symplectic path of $(\tau, y)$. The eigenvalue of $\gamma_y(\tau)$ are called Floquet multipliers of $(\tau, y)$. By Proposition 1.6.13 of [Eke1], the Floquet multipliers with their multiplicities and Krein type numbers of $(\tau, y) \in T(\Sigma, \alpha)$ do not depend on the particular choice of the Hamiltonian function in (1.2). As in Chapter 15 of [Lon1], for any symplectic matrix $M$, we define the elliptic height $e(M)$.
of M by the total algebraic multiplicity of all eigenvalues of M on the unit circle \( U \) in the complex plane \( \mathbb{C} \). And for any \((\tau, y) \in \mathcal{T}(\Sigma, \alpha)\) we define \( e(\tau, y) = e(\gamma_y(\tau)) \), and call \((\tau, y)\) hyperbolic if \( e(\tau, y) = 2 \).

There is a long standing conjecture on the number of closed characteristics on compact convex hypersurfaces in \( \mathbb{R}^{2n} \) (cf., Page 235 of [Eke1])

\[
\#\mathcal{T}(\Sigma) \geq n, \quad \forall \Sigma \in \mathcal{H}(2n).
\]  

(1.4)

Since the pioneering works [Rab1] of P. Rabinowitz and [Wei1] of A. Weinstein in 1978 on the existence of at least one closed characteristic on every hypersurface in \( \mathcal{H}(2n) \), the existence of multiple closed characteristics on \( \Sigma \in \mathcal{H}(2n) \) has been deeply studied by many mathematicians. When \( n \geq 2 \), besides many results under pinching conditions, in 1987-1988 I. Ekeland-L. Lassoued, I. Ekeland-H. Hofer, and A. Szulkin (cf. [EkL1], [EkH1], [Szul]) proved

\[
\#\mathcal{T}(\Sigma) \geq 2, \quad \forall \Sigma \in \mathcal{H}(2n).
\]

In [HWZ1] of 1998, H. Hofer-K. Wysocki-E. Zehnder proved that \( \#\mathcal{T}(\Sigma) = 2 \) or \( \infty \) holds for every \( \Sigma \in \mathcal{H}(4) \). In [LoZ1] of 2002, Y. Long and C. Zhu proved

\[
\#\mathcal{T}(\Sigma) \geq \left| \frac{n}{2} \right| + 1, \quad \forall \Sigma \in \mathcal{H}(2n).
\]

In [WHL1], W. Wang, X. Hu and Y. Long proved the conjecture (1.4) for \( n = 3 \). In [Wan2], W. Wang proved the conjecture (1.4) for \( n = 4 \). In [LLZ1], C. Liu, Y. Long and C. Zhu proved the conjecture (1.4) when \( \Sigma \in \mathcal{H}_P(2n) \) for \( P = -I_{2n} \).

In this paper, we prove the conjecture (1.4) for compact convex \( P \)-cyclic symmetric hypersurfaces with \( P \) being very general.

**Theorem 1.1.** For every \( \Sigma \in \mathcal{H}_P(2n) \), we have \( \#\mathcal{T}(\Sigma) \geq n \).

Based on the proof of Theorem 1.1, we can further obtain the following results:

**Theorem 1.2.** For every \( \Sigma \in \mathcal{H}_P(2n) \) satisfying \( \#\mathcal{T}(\Sigma) < +\infty \), there exist at least \( 2\left[ \frac{n}{2} \right] \) geometrically distinct non-hyperbolic closed characteristics on \( \Sigma \).

**Theorem 1.3.** For every \( \Sigma \in \mathcal{H}_P(2n) \) satisfying \( \#\mathcal{T}(\Sigma) = n \) and \( k \geq 3 \), all of the closed characteristics on \( \Sigma \) are \( P \)-cyclic symmetric, where \( k \) satisfies \( P^k = I_{2n} \).

**Remark 1.4.** (i) Let \( P = -I_{2n} \), our Theorem 1.1 is the same as Theorem 1.1 of [LLZ1], thus our theorem extends the main result of [LLZ1] to compact convex hypersurfaces with general symmetries. Our Theorem 1.2 extends and covers Theorem 1.1 of [LLW] which shows Theorem 1.2 holds for \( P = -I_{2n} \). Our Theorem 1.3 is related to the main results of [Wan1] and [LLWZ].
in which the symmetries of closed characteristics were considered for the special cases $2 \leq n \leq 4$. For more studies about closed characteristics on compact convex P-cyclic symmetric hypersurfaces, one can also refer to \cite{DoL1, DoL2, LW, Liu1, LWZ, LiZ1, LiZ2, LiZh, Zha1}.

(ii) Our proofs are more complicated than those of \cite{LLZ1} and \cite{LLW} because we always need to compute Maslov $(P, \omega)$-index for P-cyclic symmetric closed characteristics, which are not considered elsewhere to study the problems of closed characteristics, thus our proofs are different from those of other papers.

(iii) For the special case $n = 2$, one can even obtain the existence of $\mathbb{Z}_k$-symmetric unknotted periodic orbit on $\Sigma$ which is the binding of an open book decomposition and each page of the open book is a disk-like global surfaces of section when $P = e^{\frac{2\pi i}{k}} J$, which has interesting applications to celestial mechanics, for example, in \cite{Kim20} and \cite{Sch20}, the Hénon-Heiles Hamiltonian energy level presents $\mathbb{Z}_3$-symmetry and the Hamiltonian energy level presents $\mathbb{Z}_4$-symmetry in Hill’s lunar problem.

This paper is arranged as follows. In Section 2, we recall briefly the index theory for symplectic paths, especially the Maslov $(P, \omega)$-index theory for symplectic paths and an important formula of Maslov $(P, \omega)$-index for the associated symplectic paths with convex Hamiltonian systems. In Section 3, we briefly review a variational structure for closed characteristics on compact convex hypersurfaces. In Section 4, we use Maslov $(P, \omega)$-index theory to give a key estimation for P-cyclic symmetric closed characteristics in Proposition 4.3 and then prove our main results.

In this paper, let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of natural integers, integers, rational numbers, real numbers, and complex numbers respectively. Denote by $a \cdot b$ and $|a|$ the standard inner product and norm in $\mathbb{R}^{2n}$. Denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the standard $L^2$-inner product and $L^2$-norm. We denote by $\# A$ the number of elements in the set $A$ when it is finite. We define the functions $[a] = \max \{k \in \mathbb{Z} \mid k \leq a\}$.

2 Index theory for symplectic paths

In this section, we recall briefly the index theory for symplectic paths which will be useful in the studies of closed characteristics.

As usual, the symplectic group $\text{Sp}(2n)$ is defined by

$$\text{Sp}(2n) = \{M \in \text{GL}(2n, \mathbb{R}) \mid M^T J M = J\},$$

whose topology is induced from that of $\mathbb{R}^{4n^2}$. For $\tau > 0$ we are interested in paths in $\text{Sp}(2n)$:

$$\mathcal{P}_\tau(2n) = \{\gamma \in C([0, \tau], \text{Sp}(2n)) \mid \gamma(0) = I_{2n}\}.$$
We consider this path-space equipped with the $C^0$-topology. For any $\omega \in U$ the following codimension 1 hypersurface in $\text{Sp}(2n)$ is defined in [Lon2]:

$$\text{Sp}(2n)^0_\omega = \{ M \in \text{Sp}(2n) | \det(M - \omega I_{2n}) = 0 \}.$$ 

For any $M \in \text{Sp}(2n)^0_\omega$, we define a co-orientation of $\text{Sp}(2n)^0_\omega$ at $M$ by the positive direction $\frac{d}{dt} Me^J|_{t=0}$. Let

$$\text{Sp}(2n)^*_\omega = \text{Sp}(2n) \setminus \text{Sp}(2n)^0_\omega,$$

$$\mathcal{P}^*_{\tau, \omega}(2n) = \{ \gamma \in \mathcal{P}_{\tau}(2n) | \gamma(\tau) \in \text{Sp}(2n)^*_\omega \},$$

$$\mathcal{P}^0_{\tau, \omega}(2n) = \mathcal{P}_{\tau}(2n) \setminus \mathcal{P}^*_{\tau, \omega}(2n).$$

For any two continuous arcs $\xi$ and $\eta : [0, \tau] \to \text{Sp}(2n)$ with $\xi(\tau) = \eta(0)$, their concatenation is defined as usual by

$$\eta \ast \xi(t) = \begin{cases} 
\xi(2t), & \text{if } 0 \leq t \leq \tau/2, \\
\eta(2t - \tau), & \text{if } \tau/2 \leq t \leq \tau.
\end{cases}$$

Given any two $2m_k \times 2m_k$ matrices of square block form $M_k = \begin{pmatrix} A_k & B_k \\
C_k & D_k \end{pmatrix}$ with $k = 1, 2$, as in [Lon4], the $\diamond$-product of $M_1$ and $M_2$ is defined by the following $2(m_1 + m_2) \times 2(m_1 + m_2)$ matrix $M_1 \diamond M_2$:

$$M_1 \diamond M_2 = \begin{pmatrix}
A_1 & 0 & B_1 & 0 \\
0 & A_2 & 0 & B_2 \\
C_1 & 0 & D_1 & 0 \\
0 & C_2 & 0 & D_2
\end{pmatrix}.$$ 

Denote by $M^{\odot k}$ the $k$-fold $\diamond$-product $M \diamond \cdots \diamond M$. Note that the $\diamond$-product of any two symplectic matrices is symplectic. For any two paths $\gamma_j \in \mathcal{P}_{\tau}(2n_j)$ with $j = 0$ and 1, let $\gamma_0 \diamond \gamma_1(t) = \gamma_0(t) \diamond \gamma_1(t)$ for all $t \in [0, \tau]$.

A special path $\xi_n$ is defined by

$$\xi_n(t) = \begin{pmatrix} 
2 - \frac{t}{\tau} \\
0 \\
(2 - \frac{t}{\tau})^{-1}
\end{pmatrix}^{\odot n} \text{ for } 0 \leq t \leq \tau.$$ 

**Definition 2.1.** (cf. [Lon2], [Lon4]) For any $\omega \in U$ and $M \in \text{Sp}(2n)$, define

$$\nu_\omega(M) = \dim \ker_C(M - \omega I_{2n}).$$

For any $\tau > 0$ and $\gamma \in \mathcal{P}_{\tau}(2n)$, define

$$\nu_\omega(\gamma) = \nu_\omega(\gamma(\tau)).$$
If $\gamma \in \mathcal{P}_{\tau, \omega}^*(2n)$, define
\[
i_\omega(\gamma) = [\text{Sp}(2n)^0_\omega : \gamma \ast \xi_n],
\]
where the right hand side of (2.1) is the usual homotopy intersection number, and the orientation of $\gamma \ast \xi_n$ is its positive time direction under homotopy with fixed end points.

If $\gamma \in \mathcal{P}_{\tau, \omega}^0(2n)$, we let $\mathcal{F}(\gamma)$ be the set of all open neighborhoods of $\gamma$ in $\mathcal{P}_\tau(2n)$, and define
\[
i_\omega(\gamma) = \sup_{U \in \mathcal{F}(\gamma)} \inf\{i_\omega(\beta) | \beta \in U \cap \mathcal{P}_{\tau, \omega}^*(2n)\}.
\]
Then
\[
(i_\omega(\gamma), \nu_\omega(\gamma)) \in \mathbb{Z} \times \{0, 1, \ldots, 2n\},
\]
is called the index function of $\gamma$ at $\omega$.

Note that when $\omega = 1$, this index theory was introduced by C. Conley-E. Zehnder in [CoZ1] for the non-degenerate case with $n \geq 2$, Y. Long-E. Zehnder in [LZe1] for the non-degenerate case with $n = 1$, and Y. Long in [Lon1] and C. Viterbo in [Vit1] independently for the degenerate case. The case for general $\omega \in U$ was defined by Y. Long in [Lon2] in order to study the index iteration theory (cf. [Lon4] for more details and references).

For any symplectic path $\gamma \in \mathcal{P}_\tau(2n)$ and $m \in \mathbb{N}$, we define its $m$-th iteration $\gamma^m : [0, m\tau] \to \text{Sp}(2n)$ by
\[
\gamma^m(t) = \gamma(t - j\tau)\gamma(\tau)^j, \quad \text{for } j\tau \leq t \leq (j + 1)\tau, \quad j = 0, 1, \ldots, m - 1.
\]
We still denote the extended path on $[0, +\infty)$ by $\gamma$.

**Definition 2.2.** (cf. [Lon2], [Lon4]) For any $\gamma \in \mathcal{P}_\tau(2n)$, we define
\[
(i(\gamma, m), \nu(\gamma, m)) = (i_1(\gamma^m), \nu_1(\gamma^m)), \quad \forall m \in \mathbb{N}.
\]
The mean index $i(\gamma, m)$ per $m\tau$ for $m \in \mathbb{N}$ is defined by
\[
\hat{i}(\gamma, m) = \lim_{k \to +\infty} \frac{i(\gamma, mk)}{k}.
\]

For any $M \in \text{Sp}(2n)$ and $\omega \in U$, the splitting numbers $S_M^\pm(\omega)$ of $M$ at $\omega$ are defined by
\[
S_M^\pm(\omega) = \lim_{\epsilon \to 0^+} i_\omega(\exp(\pm \sqrt{-1}\epsilon))\gamma(\tau) - i_\omega(\gamma),
\]
for any path $\gamma \in \mathcal{P}_\tau(2n)$ satisfying $\gamma(\tau) = M$.

Based on the index theory above, the Maslov $(P, \omega)$-index was defined as follows:
**Definition 2.3.** (cf. [LT1]) For any \( P \in \text{Sp}(2n) \), \( \omega \in \mathbb{U} \) and \( \gamma \in \mathcal{P}_s(2n) \), the Maslov \((P, \omega)\)-index is defined by

\[
i^P_\omega(\gamma) = i_\omega(P^{-1}\gamma * \xi) - i_\omega(\xi),
\]

where \( \xi \in \mathcal{P}_s(2n) \) such that \( \xi(\tau) = P^{-1}\gamma(0) = P^{-1} \), and \((P, \omega)\)-nullity \( \nu^P_\omega(\gamma) \) is defined by

\[
\nu^P_\omega(\gamma) = \dim \ker (\gamma(\tau) - \omega P).
\]

For any \( M \in \text{Sp}(2n) \) and \( \omega \in \mathbb{U} \), the splitting numbers \( pS^\pm_M(\omega) \) of \( M \) at \((P, \omega)\) are defined in Definition 2.4 of [LT2] as follows

\[
pS^\pm_M(\omega) = \lim_{\epsilon \to 0^+} i^P_\omega \exp \left( \pm \sqrt{-1} \epsilon \right) (\gamma) - i^P_\omega(\gamma), \tag{2.4}
\]

for any path \( \gamma \in \mathcal{P}_s(2n) \) satisfying \( \gamma(\tau) = M \).

Note that the Maslov P-index theory for a symplectic path was first studied by Y. Dong and C. Liu in [Dong, LiuC] independently for any symplectic matrix \( P \) with different treatment. The Maslov P-index theory was generalized in [LT1] to the Maslov \((P, \omega)\)-index theory for any \( P \in \text{Sp}(2n) \) and all \( \omega \in \mathbb{U} \). The iteration theory of \((P, \omega)\)-index theory was studied in [LT2]. When \( \omega = 1 \), the Maslov \((P, \omega)\)-index theory coincides with the Maslov P-index theory. In order to study the properties of P-cyclic symmetric closed characteristics, we will use Maslov \((P, \omega)\)-index theory for symplectic paths in Section 4.

Let \( \Omega^0(M) \) be the path connected component containing \( M = \gamma(\tau) \) of the set

\[
\Omega(M) = \{ N \in \text{Sp}(2n) | \sigma(N) \cap \mathbb{U} = \sigma(M) \cap \mathbb{U} \text{ and } \nu_\lambda(N) = \nu_\lambda(M), \forall \lambda \in \sigma(M) \cap \mathbb{U} \}
\]

Here \( \Omega^0(M) \) is called the homotopy component of \( M \) in \( \text{Sp}(2n) \).

In [Lon2]-[Lon4], the following symplectic matrices were introduced as basic normal forms:

\[
D(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \lambda = \pm 2, \tag{2.5}
\]

\[
N_1(\lambda, b) = \begin{pmatrix} \lambda & b \\ 0 & -\lambda \end{pmatrix}, \lambda = \pm 1, b = \pm 1, 0, \tag{2.6}
\]

\[
R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \theta \in (0, \pi) \cup (\pi, 2\pi), \tag{2.7}
\]

\[
N_2(\omega, B) = \begin{pmatrix} R(\theta) & B \\ 0 & R(\theta) \end{pmatrix}, \theta \in (0, \pi) \cup (\pi, 2\pi), \tag{2.8}
\]
where $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$ with $b_i \in \mathbb{R}$ and $b_2 \neq b_3$, $\omega = e^{\theta \sqrt{\tau}}$.

**Lemma 2.4.** (cf. Theorem 1.8.10 of [Lon4]) For any $M \in \text{Sp}(2n)$, there is a path $f : [0, 1] \to \Omega^0(M)$ such that $f(0) = M$ and

$$f(1) = M_1 \circ \cdots \circ M_l,$$

where each $M_i$ is a basic normal form listed in (2.5)-(2.8) for $1 \leq i \leq l$.

Splitting numbers possess the following properties:

**Lemma 2.5.** (cf. Lemma 9.1.5, Lemma 9.1.6, List 9.1.12 and Corollary 9.2.4 of [Lon4]) Splitting numbers $S_M^\pm(\omega)$ are well defined, i.e., they are independent of the choice of the path $\gamma \in \mathcal{P}_f(2n)$ satisfying $\gamma(\tau) = M$ appeared in (2.5). For $M \in \text{Sp}(2n)$, splitting numbers $S_N^\pm(\omega)$ are constant for all $N \in \Omega^0(M)$. Moreover, there hold

$$S_M^+(\omega) = S_M^-(\bar{\omega}), \ \forall \omega \in U.$$

$$S_M^+(\omega) = 0, \ \text{if } \omega \notin \sigma(M).$$

$$S_M^+(z) = \sum_{\omega^m = z} S_M^+(\omega), \ \forall z \in U, m \in \mathbb{N}.$$  

$$S_{N_1(1,a)}^+(1) = \begin{cases} 1, & \text{if } a \geq 0, \\ 0, & \text{if } a < 0. \end{cases}$$

$$S_{N_1(-1,a)}^-(1) = \begin{cases} 1, & \text{if } a \leq 0, \\ 0, & \text{if } a > 0. \end{cases}$$

$$S_{N_2(e^{\sqrt{-1}\theta},B)}^{\pm}(e^{\sqrt{-1}\theta}) = \begin{cases} 1, & \text{if } (b_2 - b_3) \sin \theta > 0, \\ 0, & \text{if } (b_2 - b_3) \sin \theta < 0, \end{cases}$$

where $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$ with $b_i \in \mathbb{R}$ and $b_2 \neq b_3$. For any $M_i \in \text{Sp}(2n_i)$ with $i = 0$ and $1$, there holds

$$S_{M_0 \circ M_1}^\pm(\omega) = S_{M_0}^\pm(\omega) + S_{M_1}^\pm(\omega), \ \forall \omega \in U.$$

**Lemma 2.6.** (cf. Lemma 2.5 and Lemma 2.6 of [LT2]) For any $M \in \text{Sp}(2n)$ and $\omega \in U$, the splitting numbers $p S_M^\pm(\omega)$ are well defined, i.e., they are independent of the choice of the path $\gamma \in \mathcal{P}_f(2n)$ satisfying $\gamma(\tau) = M$ appeared in (2.4). And the following properties hold:

(i) $p S_M^+(\omega) = S_{M^1}^+(\omega) - S_{M^2}^+(\omega).$

(ii) $p S_M^+(\omega) = p S_M^-(\omega).$
(iii) $P^+ S_M^\pm(\omega) = P^- N(\omega)$ if $P^{-1} N \in \Omega^0(P^{-1} M)$.

(iv) $P^+ S_{M_1 \circ M_2}^\pm(\omega) = P^+ S_{M_1}^\pm(\omega) + P^+ S_{M_2}^\pm(\omega)$ for $M_j, P_j \in Sp(2n_j)$ with $n_j \in \{1, \cdots, n\}$ satisfying $P = P_1 \circ P_2$ and $n_1 + n_2 = n$.

(v) $P^+ S_M^\pm(\omega) = 0$ if $\omega \notin \sigma(P^{-1} M) \cup \sigma(P^{-1})$.

For any symplectic path $\gamma \in \mathcal{P}_r(2n)$ and $m \in \mathbb{N}$, we define the $m$-times iteration path $\gamma_P^m : [0, m\tau] \rightarrow Sp(2n)$ of $\gamma$ by

$$
\gamma_P^m(t) = \begin{cases} 
\gamma(t), & t \in [0, \tau], \\
P\gamma(t - \tau)P^{-1}\gamma(\tau), & t \in [\tau, 2\tau], \\
P^2\gamma(t - 2\tau)(P^{-1}\gamma(\tau))^2, & t \in [2\tau, 3\tau], \\
\vdots \\
P^{m-1}\gamma(t - (m - 1)\tau)(P^{-1}\gamma(\tau))^{m-1}, & t \in [(m-1)\tau, m\tau].
\end{cases}
$$

By Lemma 2.7 of [LT2], we have the Bott-type formula for Maslov $(P, \omega)$-index:

**Lemma 2.7.** For any $\gamma \in \mathcal{P}_r(2n)$, $z \in U$ and $m \in \mathbb{N}$, we have

$$
i^m_P(\gamma_P^m) = \sum_{\omega^m = z} i^P_\omega(\gamma),$$

$$
\nu^m_P(\gamma_P^m) = \sum_{\omega^m = z} \nu^P_\omega(\gamma).
$$

The following formula of Maslov $(P, \omega)$-index for the associated symplectic paths with convex Hamiltonian systems will play an important role in the proof of Proposition 4.3 below.

**Lemma 2.8.** (cf. Lemma 3.5 of [LWZ]) Assume $A(t) \in GL(\mathbb{R}^{2n})$ is positive definite for $t \in [0, \tau]$, let $\gamma \equiv \gamma_A \in \mathcal{P}_r(2n)$ be the fundamental solution of the linearized Hamiltonian system

$$
\dot{y}(t) = JA(t)y(t).
$$

Then we have

$$
i^P_\omega(\gamma) = \nu^P_\omega(P^{-1}) + \sum_{0 < s < \tau} \nu^P_\omega(\gamma(s)), \forall \omega \in U,
$$

where $P$ satisfies $\ker(P - I_{2n}) = 0$ as assumed in [LWZ].

### 3 Variational properties for closed characteristics

In this section, we describe the variational properties for closed characteristics.

To solve the given energy problem (1.2), we consider the fixed period problem

$$
\begin{align*}
\dot{x}(t) &= JH'_\omega(x(t)), \\
x(1) &= x(0).
\end{align*}
$$

(3.1)
Define
\[ L^{\frac{1}{n}}_0(S^1, \mathbb{R}^{2n}) = \{ u \in L^{\frac{1}{n}}_0(S^1, \mathbb{R}^{2n}) | \int_0^1 u dt = 0 \}. \]

The corresponding Clarke-Ekeland dual action functional is defined by
\[ \Phi(u) = \int_0^1 \left( \frac{1}{2} J_u \cdot M u + H^*_\alpha(-J u) \right) dt, \quad \forall \ u \in L^{\frac{1}{n}}_0(S^1, \mathbb{R}^{2n}), \]

where \( M u \) is defined by \( \frac{d}{dt} Mu(t) = u(t) \) and \( \int_0^1 M u(t) dt = 0 \), \( H^*_\alpha \) is the Fenchel transform of \( H_\alpha \) defined by \( H^*_\alpha(y) = \sup \{ x \cdot y - H_\alpha(x) | x \in \mathbb{R}^{2n} \} \). By Theorem 5.2.8 of [Eke1], \( \Phi \) is \( C^1 \) on \( L^{\frac{1}{n}}_0(S^1, \mathbb{R}^{2n}) \) and satisfies the Palais-Smale condition. Suppose \( x \) is a solution of (3.1). Then \( u = \dot{x} \) is a critical point of \( \Phi \). Conversely, suppose \( u \) is a critical point of \( \Phi \). Then there exists a unique \( \xi \in \mathbb{R}^{2n} \) such that \( x_u = M u - \xi \) is a solution of (3.1). In particular, solutions of (3.1) are in one to one correspondence with critical points of \( \Phi \). Moreover, \( \Phi(u) < 0 \) for every critical point \( u \neq 0 \) of \( \Phi \). In addition, we have a natural \( S^1 \)-action on \( L^{\frac{1}{n}}_0(S^1, \mathbb{R}^{2n}) \) defined by \( \theta \cdot u(t) = u(\theta + t) \) for all \( \theta \in S^1 \) and \( t \in \mathbb{R} \). Clearly \( \Phi \) is \( S^1 \)-invariant.

Let \( h = H_\alpha(x_u(t)) \) and \( 1/m \) be the minimal period of \( x_u \) for some \( m \in \mathbb{N} \). Define
\[ y_u(t) = h^{\frac{1}{\alpha}} x_u(h^{\frac{2}{\alpha}} t) \quad \text{and} \quad \tau = \frac{1}{m} h^{\frac{2-\alpha}{\alpha}}. \]

Then there hold \( y_u(t) \in \Sigma \) for all \( t \in \mathbb{R} \) and \( (\tau, y_u) \in T(\Sigma, \alpha) \). Note that the period 1 of \( x_u \) corresponds to the period \( m\tau \) of the solution \( (m\tau, y^m_u) \) of (1.2) with minimal period \( \tau \). On the other hand, every solution \( (\tau, y) \in T(\Sigma, \alpha) \) gives rise to a sequence \( \{x^m_y\}_{m \in \mathbb{N}} \) of solutions of the problem (3.1), and a sequence \( \{u^m_y\}_{m \in \mathbb{N}} \) of critical points of \( \Phi \) defined by
\[ x^m_y(t) = (m\tau)^{\frac{1}{\alpha}} y(m\tau t) \]
\[ u^m_y(t) = (m\tau)^{\frac{2}{\alpha}} y(m\tau t) \]  \hspace{1cm} (3.3)

Suppose \( u \) is a nonzero critical point of \( \Phi \). Then the formal Hessian of \( \Phi \) at \( u \) on \( L^{\frac{1}{n}}_0(S^1, \mathbb{R}^{2n}) \) is defined by
\[ Q(v, v) = \int_0^1 (Jv \cdot M v + (H^*_\alpha)'(-J u)Jv \cdot Jv) dt, \]  \hspace{1cm} (3.4)
which defines an orthogonal splitting \( L^{\frac{1}{n}}_0(S^1, \mathbb{R}^{2n}) = E_- \oplus E_0 \oplus E_+ \) into negative, zero and positive subspaces. The Ekeland index of \( u \) is defined by \( i(u) = \dim E_- \) and the nullity of \( u \) is defined by \( \nu(u) = \dim E_0 \). The inequality \( 1 \leq \nu(u) \leq 2n \) always holds, cf. P.219 of [Eke1].

For a closed characteristic \( (\tau, y) \in T(\Sigma, \alpha) \), we denote by \( y^m \equiv (m\tau, y) \) the \( m \)-th iteration of \( y \) for \( m \in \mathbb{N} \). Then we define the index \( i(y^m) \) and nullity \( \nu(y^m) \) of \( (m\tau, y) \) for \( m \in \mathbb{N} \) by
\[ i(y^m) = i(y^m), \quad \nu(y^m) = \nu(y^m). \]  \hspace{1cm} (3.5)
The mean index of \((\tau, y)\) is defined by
\[
\hat{\imath}(y) = \lim_{{m \to \infty}} \frac{i(y^m)}{m}.
\] (3.6)

Note that by Corollary 8.3.2 and Lemma 15.3.2 of [Lon4], there always holds
\[
\hat{\imath}(y) > 2.
\] (3.7)

We define via Definition 2.2 the following
\[
S^+(y) = S^+_{\gamma_y}(\tau),
\] (3.8)
\[
(i(y, m), \nu(y, m)) = (i(\gamma_y, m), \nu(\gamma_y, m)),
\] (3.9)
\[
\hat{\imath}(y, m) = \hat{\imath}(\gamma_y, m),
\] (3.10)
for all \(m \in \mathbb{N}\), where \(\gamma_y\) is the associated symplectic path of \((\tau, y)\), i.e., the fundamental solution of the linearized Hamiltonian system of (1.3) at \((\tau, y)\).

**Lemma 3.1.** (cf. Lemma 1.1 of [LoZ1], Theorem 15.1.1 of [Lon4]) Suppose \((\tau, y) \in \mathcal{T}(\Sigma, \alpha)\). Then we have
\[
i(y^m) \equiv i(m\tau, y) = i(y, m) - n, \quad \nu(y^m) \equiv \nu(m\tau, y) = \nu(y, m), \quad \forall m \in \mathbb{N}.
\]
In particular, (2.2), (3.6) and (3.10) coincide, thus we simply denote them by \(\hat{\imath}(y)\).

Since the Ekeland index is a Morse-type index which is non-negative, then by Lemma 3.1 we have

**Corollary 3.2.** (cf. Corollary 15.1.4 of [Lon4]) It holds that
\[
i(y, 1) \geq n, \quad \forall (\tau, y) \in \mathcal{T}(\Sigma, \alpha).
\]

By Corollary 3.1 of [LoZ1], we have the monotonicity for the index iterations of closed characteristics:

**Lemma 3.3.** For any \((\tau, y) \in \mathcal{T}(\Sigma, \alpha), m \in \mathbb{N}\), we have
\[
i(y, m + 1) - i(y, m) \geq 2, \quad i(y, m + 1) + \nu(y, m + 1) - 1 \geq i(y, m + 1) > i(y, m) + \nu(y, m) - 1. \] (3.11) (3.12)

Following Section V.3 of [Eke1], denote by “\(\text{ind}\)” the \(S^1\)-action cohomology index theory for \(S^1\)-invariant subset of \(L^2_0(S^1, \mathbb{R}^{2n})\) defined in [Eke1]. For \([\Phi]_c \equiv \{u \in L^2_0(S^1, \mathbb{R}^{2n}) \mid \Phi(u) \leq c\}\) define
\[
c_k = \inf \{c < 0 \mid \text{ind}([\Phi]_c) \geq k\}.
\]
Then by Proposition 3 in P.218 of [Eke1], we have

**Lemma 3.4.** Every $c_i$ is a critical value of $\Phi$. If $c_i = c_j$ for some $i < j$, then there are infinitely many geometrically distinct closed characteristics on $\Sigma$.

By Theorem 4 in P.219 of [Eke1], we have the following

**Lemma 3.5.** For any given $k \in \mathbb{N}$, there exists $(\tau, y) \in \mathcal{T}(\Sigma, \alpha)$ and $m \in \mathbb{N}$ such that for $u^m_y$ defined by (3.3)

$$
\Phi'(u^m_y) = 0 \quad \text{and} \quad \Phi(u^m_y) = c_k,
$$

$$
i(u^m_y) \leq 2k - 2 \leq i(u^m_y) + \nu(u^m_y) - 1.
$$

Combining Lemma 3.1, Lemma 3.4 and Lemma 3.5, we obtain

**Lemma 3.6.** (cf. Lemma 3.1 of [LoZ1]) Suppose $\#\mathcal{T}(\Sigma) < +\infty$, there exists an injection map $p = p(\Sigma, \alpha) : \mathbb{N} \to \mathcal{T}(\Sigma, \alpha) \times \mathbb{N}$ such that for any $k \in \mathbb{N}$, $(\tau, y) \in \mathcal{T}(\Sigma, \alpha)$ and $m \in \mathbb{N}$ satisfying $p(k) = ((\tau, y), m)$, there hold

$$
\Phi'(u^m_y) = 0 \quad \text{and} \quad \Phi(u^m_y) = c_k,
$$

$$
i(y, m) \leq 2k - 2 + n \leq i(y, m) + \nu(y, m) - 1,
$$

where $u^m_y$ is defined by (3.3).

4 Proof of the main results

Before we give the proofs of our main results, we first consider the normal forms of the cyclic symplectic orthogonal matrix $P$, which is crucial for our proof of Proposition 4.3 below.

**Proposition 4.1.** Let $P$ be a symplectic orthogonal matrix satisfying $P^k = I_{2n}$ for some integer $k \geq 2$, then there exists some $Q \in Sp(2n)$ such that

$$
QPQ^{-1} = R(\theta_1) \circ R(\theta_2) \circ \cdots \circ R(\theta_n),
$$

where $\theta_i \in [0, 2\pi)$.

**Proof.** By Lemma 2.4, there exists some $Q \in Sp(2n)$ such that

$$
QPQ^{-1} = N_1(1, 1)^{o_p} \circ N_1(1, -1)^{o_p} \circ N_1(-1, 1)^{o_q} \circ N_1(-1, -1)^{o_q} \circ N_2(e^{o_1\sqrt{-1}}, A_1) \circ \cdots \circ N_2(e^{o_r\sqrt{-1}}, A_r) \circ R(\theta_1) \circ \cdots \circ R(\theta_r) \circ D(\pm 2)^{oh},
$$
where \( \alpha_j \in (0, \pi) \cup (\pi, 2\pi) \) for \( 1 \leq j \leq r_* \), \( \theta_j \in [0, 2\pi) \) for \( 1 \leq j' \leq r \) and non-negative integers \( p_-, p_+, q_-, q_+, r, r_*, h \) satisfy

\[
p_- + p_+ + q_- + q_+ + r + 2r_* + h = n.
\]

Note that \( I_{2n} = Q P^k Q^{-1} = (QPQ^{-1})^k \), then we must have

\[
Q P Q^{-1} = N_2(e^{\alpha_1 \sqrt{-1}}, A_1) \circ \cdots \circ N_2(e^{\alpha_* \sqrt{-1}}, A_{r_*}) \circ R(\theta_1) \circ \cdots \circ R(\theta_r), \tag{4.1}
\]

\[
\left( N_2(e^{\alpha_j \sqrt{-1}}, A_j) \right)^k = I_4, \quad 1 \leq j \leq r_* \tag{4.2}
\]

On the other hand, by direct computation we have

\[
\left( N_2(e^{\alpha_j \sqrt{-1}}, A_j) \right)^k = \begin{pmatrix} R(k\alpha_j) & B \\ 0 & R(k\alpha_j) \end{pmatrix}, \tag{4.3}
\]

\[
B = \sum_{i=0}^{k} R(i\alpha_j)A_jR((k-i)\alpha_j). \tag{4.4}
\]

Comparing (4.2) with (4.3), we have \( R(k\alpha_j) = I_2 \) and \( B = 0 \) which together with (4.4) implies

\[
0 = R(\alpha_j)BR(-\alpha_j) = \sum_{i=1}^{k+1} R(i\alpha_j)A_jR((k-i)\alpha_j). \tag{4.5}
\]

Comparing it with (4.4), we get \( A_j = R(\alpha_j)A_jR(-\alpha_j) \). Combining it with (4.5), we obtain \( A_j = 0 \). Thus \( r_* = 0 \) and by (4.1) we complete the proof.

In the following, we fix a \( \Sigma \in \mathcal{H}_P(2n) \) satisfying \( P^k = I_{2n} \) and \( \ker(P^l - I_{2n}) = 0 \) holds for any \( 1 \leq l < k \), and assume that there exist only finitely many geometrically distinct closed characteristics on \( \Sigma \), i.e., \( \mathcal{T}(\Sigma) = \{ (\tau_j, y_j) \}_{1 \leq j \leq S} \). We denote by \( \gamma_j \equiv \gamma_{y_j} \) the associated symplectic path of \( (\tau_j, y_j) \) on \( \Sigma \) for \( 1 \leq j \leq S \). Then by Lemma 1.3 of \cite{LoZ1} or Lemma 15.2.4 of \cite{Lon4}, there exist \( P_j \in \text{Sp}(2n) \) and \( M_j \in \text{Sp}(2n - 2) \) such that

\[
\gamma_j(\tau_j) = P_j^{-1}(N_1(1, 1) \circ M_j)P_j, \quad \forall \ 1 \leq j \leq S. \tag{4.6}
\]

**Proposition 4.2.** Suppose \( (\tau, y) \in \mathcal{T}(\Sigma, \alpha) \), then \( (\tau, P y) \in \mathcal{T}(\Sigma, \alpha) \) and either \( \mathcal{O}(y) = \mathcal{O}(Py) \) or \( \mathcal{O}(y) \cap \mathcal{O}(Py) = \emptyset \), where \( \mathcal{O}(Py) = \{ P y(t) | t \in \mathbb{R} \} \). Moreover, if \( \mathcal{O}(y) \cap \mathcal{O}(Py) \neq \emptyset \), then we have

\[
y \left( t + \frac{\tau}{k} \right) = P^l y(t), \quad \forall t \in \mathbb{R}. \tag{4.7}
\]

for some \( 1 \leq l < k \).
Proof. Since $\Sigma = P\Sigma$, then

$$H_{\alpha}(Py) = H_{\alpha}(y), \quad (4.8)$$

$$H'_{\alpha}(Py) = PH'_{\alpha}(y), \quad (4.9)$$

$$H''_{\alpha}(y) = P^T H''_{\alpha}(Py) P. \quad (4.10)$$

Note that $PJ = JP$ since $P$ is a symplectic orthogonal matrix, which together with (1.2) and (4.9) yields $(\tau, Py) \in T(\Sigma, \alpha)$ for any $(\tau, y) \in T(\Sigma, \alpha)$.

If $O(y) \cap O(Py) \neq \emptyset$, there exists an $s \in [0, \tau)$ such that $y(s) = Py(0)$. Because $y(s + t)$ and $Py(t)$ satisfy the same Hamiltonian system

$$\dot{x} = JH'_{\alpha}(x),$$

and they have the same initial value for $t = 0$, we have

$$y(t + s) = Py(t), \quad \forall t \in \mathbb{R}, \quad (4.11)$$

which implies $O(y) = O(Py)$. Combining (4.11) with the assumption that $P^k = I_{2n}$, we get $y(t + ks) = y(t)$, then there holds

$$s = \frac{j}{k} \tau, \quad (4.12)$$

for some $0 \leq j < k$.

Claim. $j$ and $k$ are co-prime.

If otherwise, there exists some $1 \leq l < k$ such that $jl \equiv 0 \text{ (mod } k)$, then $y(t) = y(t + ls) = P^l y(t)$ by (4.11)-(4.12). Note that $y(t) \neq 0$ for any $t \in \mathbb{R}$ and $\ker(P^l - I_{2n}) = 0$ for $1 \leq l < k$, we get a contradiction and the above claim holds.

By this claim, there exists some $1 \leq l < k$ such that $jl \equiv 1 \text{ (mod } k)$, hence we obtain $y(t + \frac{r}{k}) = y(t + ls) = P^l y(t)$ by (4.11)-(4.12). The proof is complete.

As in [Zha1], we call a closed characteristic $(\tau, y)$ on $\Sigma \in \mathcal{H}_P(2n)$ $P$-cyclic symmetric if $O(y) = O(Py)$. Thus by Proposition 4.2, we obtain that if $(\tau, y)$ is not $P$-cyclic symmetric, then $(\tau, Py)$ and $(\tau, y)$ are geometrically distinct.

For $\Sigma \in \mathcal{H}_P(2n)$, we have $\Phi(Pu) = \Phi(u)$. In fact, by (4.8), we have $H^*_{\alpha}(Py) = H^*_{\alpha}(y)$ which together with (3.2) yields

$$\Phi(Pu) = \int_0^1 \left( \frac{1}{2} JPu \cdot MPu + H^*_{\alpha}(-JPu) \right) dt$$

$$= \int_0^1 \left( \frac{1}{2} PJu \cdot PMu + H^*_{\alpha}(-PJu) \right) dt$$
\[
\int_0^1 \left( \frac{1}{2} J u \cdot M u + H_\alpha^*(-J u) \right) dt = \Phi(u), \quad \forall \, u \in L_0^{\alpha-1}(S^1, \mathbb{R}^{2n}),
\]
where we used the fact that \(PJ = JP\). Note that if \(u \in L_0^{\alpha-1}(S^1, \mathbb{R}^{2n})\) is the critical point of \(\Phi\) corresponding to a closed characteristic \((\sigma, z)\), then \(Pu \in L_0^{\alpha-1}(S^1, \mathbb{R}^{2n})\) is the critical point of \(\Phi\) corresponding to the closed characteristic \((\sigma, Pz)\), thus \(u\) and \(Pu\) have the same critical values. Moreover, by (3.4)-(3.5), Lemma 3.1 and (4.10), we have

\[
i(z, 1) = i(Pz, 1), \quad \nu(z, 1) = \nu(Pz, 1).
\] 

Now we give a key estimation for \(P\)-cyclic symmetric closed characteristics, which is a crucial step for proving our main results. As far as we know, it is not considered by other papers.

**Proposition 4.3.** For any \(P\)-cyclic symmetric closed characteristic \((\tau, y) \in \mathcal{T}(\Sigma, \alpha)\) on \(\Sigma \in \mathcal{H}_P(2n)\), we have

\[
i(y, 1) + 2S^+(y) - \nu(y, 1) \geq n,
\]
where we use the notations in (3.8)-(3.9).

**Proof.** By (4.7), we have

\[
y \left( t + \frac{\tau}{k} \right) = P^l y(t), \quad \forall t \in \mathbb{R},
\]
for some \(1 \leq l < k\). Let \(\bar{P} = P^l\), then \(\Sigma\) is still \(\bar{P}\)-cyclic symmetric, and (4.16) becomes

\[
y \left( t + \frac{\tau}{k} \right) = \bar{P} y(t), \quad \forall t \in \mathbb{R}.
\]
In the following, we use \(\bar{P}\) instead of \(P\) considered in the \((P, \omega)\) index theory of Section 2. Let \(\gamma_y : [0, \tau] \to \text{Sp}(2n)\) with \(\gamma_y(0) = I_{2n}\) be the associated symplectic path of \((\tau, y)\) and \(\gamma = \gamma_y|_{[0, \tau/k]}\). From (4.10) and (4.17), we obtain \(H_\alpha''(y(t)) = \bar{P}^T H_\alpha''(y(t + \frac{\tau}{k})) \bar{P}\), then by (1.5)-(1.6) of [LT1] we have

\[
\gamma_y = \gamma_{\bar{P}}^k,
\]
which is the \(k\)-times iteration path of \(\gamma\) defined as in (2.9).

By Proposition 4.1 and the assumption that \(\text{ker}(P^l - I_{2n}) = 0\) for \(1 \leq l < k\), there exists some \(Q \in \text{Sp}(2n)\) such that

\[
Q \bar{P} Q^{-1} = R(\theta_1) \circ R(\theta_2) \circ \cdots \circ R(\theta_n), \quad 0 < \frac{\theta_i}{\pi} < 2.
\]
Noticing that \( \bar{P}^k = I_{2n} \), then by (4.19) we have
\[
\sum_{\omega = 1, \omega \neq 1} \nu_\omega (\bar{P}^{-1}) = 2n. \tag{4.20}
\]
Let \( \omega_1 = e^{\frac{2\pi \sqrt{-1}}{k}} \) and denote the eigenvalues of \( \bar{P}^{-1}M \) between 1 and \( \omega_1 \) lying on the upper semi-circle in \( U \) by \( \alpha_1, \alpha_2, \cdots, \alpha_r \) anticlockwise, where \( M = \gamma(\tau/k) \). Let \( \alpha_0 = 1 \) and \( \alpha_{r+1} = \omega_1 \). By the definitions of splitting numbers, we have
\[
i^P_{\alpha_i}(\gamma) + pS^+_{M}(\alpha_i) = i^P_{\alpha_{i+1}}(\gamma) + pS^-_{M}(\alpha_{i+1}), i = 0, 1, \cdots, r. \tag{4.21}
\]
Note that by Lemma 2.6(i), it follows that
\[
pS^\pm_M(\omega) = S^\pm_{M-1}(\omega) - S^\pm_{M-1}(\omega). \tag{4.22}
\]
Then by (4.21)-(4.22) we have
\[
i^P_{\alpha_{i+1}}(\gamma) - i^P_{\alpha_i}(\gamma) = S^+_{M-1}(\alpha_i) - S^-_{M-1}(\alpha_{i+1}) + S^-_{M-1}(\alpha_{i+1}) - S^+_{M-1}(\alpha_i), i = 0, 1, \cdots, r,
\]
which implies
\[
i^P_{\omega_1}(\gamma) - i^P_{1}(\gamma) = i^P_{\alpha_{r+1}}(\gamma) - i^P_{\alpha_0}(\gamma)
\]
\[
= \sum_{i=0}^{r} (i^P_{\alpha_{i+1}}(\gamma) - i^P_{\alpha_i}(\gamma))
\]
\[
= \sum_{i=0}^{r} S^+_{M-1}(\alpha_i) - \sum_{i=0}^{r} S^-_{M-1}(\alpha_{i+1}) + \sum_{i=0}^{r} S^-_{M-1}(\alpha_{i+1}) - \sum_{i=0}^{r} S^+_{M-1}(\alpha_i). \tag{4.23}
\]
Note that \( S^\pm_{M-1}(\alpha_i) = 0 \) for \( 0 \leq i \leq r \) since \( \alpha_i \) is not an eigenvalue of \( \bar{P}^{-1} \), then (4.23) becomes
\[
i^P_{1}(\gamma) = i^P_{\omega_1}(\gamma) + \sum_{i=0}^{r} S^-_{M-1}(\alpha_{i+1}) - \sum_{i=0}^{r} S^+_{M-1}(\alpha_i) - S^-_{M-1}(\omega_1). \tag{4.24}
\]
On the other hand, since \( ker(\bar{P} - I_{2n}) = 0 \), then it follows from Lemma 2.8 that
\[
i^P_{\omega_1}(\gamma) \geq \nu_\omega (\bar{P}^{-1}). \tag{4.25}
\]
Combining (4.24) with (4.25), we have
\[
i^P_{1}(\gamma) \geq \sum_{i=0}^{r} S^-_{M-1}(\alpha_{i+1}) - \sum_{i=0}^{r} S^+_{M-1}(\alpha_i)
\]
\[
= \sum_{i=1}^{r} S^-_{M-1}(\alpha_i) - \sum_{i=1}^{r} S^+_{M-1}(\alpha_i) + S^-_{M-1}(\omega_1) - S^+_{M-1}(1), \tag{4.26}
\]
where we used the fact that \( \nu_1(\bar{P}^{-1}) \geq S_{\bar{P}^{-1}}(\omega_1) \) by Lemma 2.5. Then by (3.8)-(3.10), (4.18), Lemma 2.5 and Lemma 2.7, we obtain

\[
i(y, 1) + 2S^+(y) - \nu(y, 1) = i(\gamma_y) + 2S^+_{\gamma_y(\bar{P})(1)} - \nu(\gamma_y) = i(\gamma_{\bar{P}}) + 2S^+_{(\bar{P}^{-1}M)^k(1)} - \nu(\gamma_{\bar{P}}) = \sum_{\omega^k=1, \omega \neq 1} (i^\omega_{\bar{P}}(\gamma) + 2S^+_{\bar{P}^{-1}M}(\nu_\omega(\bar{P}^{-1}M)),
\]

which together with (4.25)-(4.26) and (4.20) implies

\[
i(y, 1) + 2S^+(y) - \nu(y, 1) \geq \sum_{\omega^k=1, \omega \neq 1} (\nu_\omega(\bar{P}^{-1}) + 2S^+_{\bar{P}^{-1}M}(\nu_\omega(\bar{P}^{-1}M))
+ \sum_{i=1}^r S^-_{\bar{P}^{-1}M}(\alpha_i) - \sum_{i=1}^r S^+_{\bar{P}^{-1}M}(\alpha_i) + S^-_{\bar{P}^{-1}M}(\omega_1) - S^+_{\bar{P}^{-1}M}(1)
+ 2S^+_{\bar{P}^{-1}M}(1) - \nu_1(\bar{P}^{-1}M)
= 2n + \sum_{\omega^k=1, \omega \neq 1} (2S^+_{\bar{P}^{-1}M}(\omega) - \nu_\omega(\bar{P}^{-1}M)) + \sum_{i=1}^r S^-_{\bar{P}^{-1}M}(\alpha_i)
- \sum_{i=1}^r S^+_{\bar{P}^{-1}M}(\alpha_i) + S^-_{\bar{P}^{-1}M}(\omega_1) + S^+_{\bar{P}^{-1}M}(1) - \nu_1(\bar{P}^{-1}M)
\geq 2n - \{ \sum_{\omega^k=1, \omega \neq 1} (\nu_\omega(\bar{P}^{-1}M) - 2S^+_{\bar{P}^{-1}M}(\omega)) + \sum_{i=1}^r S^+_{\bar{P}^{-1}M}(\nu_\omega(\bar{P}^{-1}M))
+ (\nu_1(\bar{P}^{-1}M) - S^+_{\bar{P}^{-1}M}(1))\}
\geq 2n - n = n,
\]

where we also used the fact that

\[
\sum_{\omega^k=1, \omega \neq 1} (\nu_\omega(\bar{P}^{-1}M) - 2S^+_{\bar{P}^{-1}M}(\omega)) + \sum_{i=1}^r S^+_{\bar{P}^{-1}M}(\alpha_i) + (\nu_1(\bar{P}^{-1}M) - S^+_{\bar{P}^{-1}M}(1)) \leq n,
\]

in fact, by Lemma 2.5, we can get (4.27) if \( \bar{P}^{-1}M \) is any basic form listed in (2.5)-(2.8) and then (4.27) holds for any symplectic matrix \( \bar{P}^{-1}M \) by Lemma 2.4 and Lemma 2.5. The proof is complete.

\[\blacksquare\]

**Remark 4.4.** In Definition 1.1 of [GM], for the first time, V. Ginzburg and L. Macarini introduced the important notion of strong dynamical convexity for contact forms invariant under a group action, supporting the standard contact structure on the sphere, and in Theorem 1.6 of the same paper, they proved that any compact convex hypersurface which is symmetric with respect to the origin satisfies this strong dynamical convexity condition. Comparing our Proposition 4.3 with Definition 1.1 of [GM], we can easily see that the compact convex P-cyclic symmetric hypersurfaces
correspond to strong dynamical convex contact forms, so our Proposition 4.3 gives a lot of examples satisfying this strong dynamical convexity condition, which also shows the validity of the definition of strong dynamical convexity.

Now we can give the proofs of our main results:

**Proof of Theorem 1.1.**

By Proposition 4.2, we denote the elements in $\mathcal{T}(\Sigma, \alpha)$ by

$$\mathcal{T}(\Sigma, \alpha) = \{(\tau_j, y_j) \mid 1 \leq j \leq s_1\} \cup \{(\tau_l, y_l, (\tau_l, Py_l)) \mid s_1 + 1 \leq l \leq s_1 + s_2\},$$

where $O(y_j) = O(Py_j)$ for $1 \leq j \leq s_1$, and $O(y_l) \cap O(Py_l) = \emptyset$ for $s_1 + 1 \leq l \leq s_1 + s_2$. Since we have assumed $\#\mathcal{T}(\Sigma, \alpha) = S$, then

$$S = s_1 + 2s_2. \tag{4.28}$$

According to Lemma 3.6, we get an injection map $p = p(\Sigma, \alpha) : \mathbb{N} \to \mathcal{T}(\Sigma, \alpha) \times \mathbb{N}$. From (3.3), (4.13)-(4.14) and Lemma 3.6, we can further require that

$$im(p) \subseteq \{(\tau_j, y_j) \mid 1 \leq j \leq s_1 + s_2\} \times \mathbb{N}.$$ 

By (3.7), we have $i(y_j) > 2$ for $1 \leq j \leq s_1 + s_2$, then we can use the common index jump theorem (Theorems 4.3 and 4.4 of [LoZ1], Theorems 11.2.1 and 11.2.2 of [Lon4]) to obtain infinitely many $(T, m_1, \ldots, m_{s_1 + s_2}) \in \mathbb{N}^{s_1 + s_2 + 1}$ such that the following hold for every $j \in \{1, \ldots, s_1 + s_2\}$:

$$\nu(y_j, 2m_j - 1) = \nu(y_j, 1), \tag{4.29}$$

$$i(y_j, 2m_j) \geq 2T - \frac{e(\gamma_j(\tau_j))}{2} \geq 2T - n, \tag{4.30}$$

$$i(y_j, 2m_j) + \nu(y_j, 2m_j) \leq 2T + \frac{e(\gamma_j(\tau_j))}{2} - 1 \leq 2T + n - 1, \tag{4.31}$$

$$i(y_j, 2m_j + 1) = 2T + i(y_j, 1), \tag{4.32}$$

$$i(y_j, 2m_j - 1) + \nu(y_j, 2m_j - 1) = 2T - (i(y_j, 1) + 2S^+(y_j) - \nu(y_j, 1)), \tag{4.33}$$

where $e(M)$ is the elliptic height of $M$ defined in §1. Note that (4.31) holds by Theorem 4.4 of [LoZ1], other parts follows by Theorem 4.3 of [LoZ1].

By Lemma 2.4, we can assume $\gamma_j(\tau_j)$ can be connected within $\Omega^0(\gamma_j(\tau_j))$ to

$$N_1(1, 1)^{p_j,-} \circ I_2^{p_j,0} \circ N_1(1, -1)^{p_j,+} \circ G_j, \quad 1 \leq j \leq s_1 + s_2, \tag{4.34}$$

for some nonnegative integers $p_j,-, p_j,0, p_j,+ \text{ and some symplectic matrix } G_j \text{ satisfying } 1 \notin \sigma(G_j).$

By (4.6), (4.34) and Lemma 2.5 we obtain

$$2S^+(y_j) = 2(p_j,- + p_j,0) \geq 2, \quad 1 \leq j \leq s_1 + s_2. \tag{4.35}$$
By Corollary 3.2, we have

\[ i(y_j, 1) \geq n, \quad 1 \leq j \leq s_1 + s_2. \tag{4.36} \]

By (4.29), (4.33), (4.35) and (4.36) we have

\[ i(y_j, 2m_j - 1) = 2T - (i(y_j, 1) + 2S^+(y_j)) \leq 2T - n - 2. \tag{4.37} \]

Combining (3.11)-(3.12) with (4.37), for \( m \geq 2 \) we have

\[ i(y_j, 2m_j - m) + \nu(y_j, 2m_j - m) - 1 \leq i(y_j, 2m_j - m + 1) - 1 \leq i(y_j, 2m_j - 1) - 1 \leq 2T - n - 3. \tag{4.38} \]

By (3.11), (4.32), and (4.36), for all \( m \geq 2 \) we obtain

\[ i(y_j, 2m_j + m) > i(y_j, 2m_j + 1) = 2T + i(y_j, 1) \geq 2T + n. \tag{4.39} \]

For every \( 1 \leq i \leq n \), Denote by \( p(T - i + 1) = ((\tau_{\rho(i)}, y_{\rho(i)}), \lambda(i)) \), where \( 1 \leq i \leq n \), \( \rho(i) \in \{1, \cdots, s_1 + s_2\} \) and \( \lambda(i) \in \mathbb{N} \). By the definition of \( p \) and (3.13) we have

\[ i(y_{\rho(i)}, \lambda(i)) \leq 2T - 2i + n \leq i(y_{\rho(i)}, \lambda(i)) + \nu(y_{\rho(i)}, \lambda(i)) - 1, \tag{4.40} \]

which together with (4.38)-(4.39) yields

\[ \lambda(i) \in \{2m_{\rho(i)} - 1, 2m_{\rho(i)}\}, \quad \forall 1 \leq i \leq n. \tag{4.41} \]

**Claim 1.** If \( y_{\rho(i)} \) is P-cyclic symmetric, then \( \lambda(i) = 2m_{\rho(i)} \).

In fact, by (4.33) and (4.15) of Proposition 4.3, we have

\[ i(y_{\rho(i)}, 2m_{\rho(i)} - 1) + \nu(y_{\rho(i)}, 2m_{\rho(i)} - 1) - 1 \leq 2T - n - 1. \]

Thus Claim 1 holds by (4.40) and (4.41).

Since the map \( p \) is injective, then by Claim 1 we have

\[ \#\{i \in \{1, \ldots, n\} \mid \rho(i) \leq s_1\} \leq s_1, \tag{4.42} \]

and by (4.41) there holds

\[ \#\{i \in \{1, \ldots, n\} \mid \rho(i) > s_1\} \leq 2s_2. \tag{4.43} \]
Therefore, combining (4.28) with (4.42)-(4.43) we obtain

\[
T(\Sigma) = T(\Sigma, \alpha) = s_1 + 2s_2 \geq n.
\]

The proof is complete.

**Proof of Theorem 1.2.**

Based on the proof of Theorem 1.1, we have the following claims:

**Claim 2.** If \( \lambda(i) = 2m_{\rho(i)} - 1 \), then \( y_{\rho(i)} \) is not \( P \)-cyclic symmetric and non-hyperbolic.

The first statement follows directly from Claim 1. We prove the latter.

In fact, suppose \( y_{\rho(i)} \) for some \( i \in \{1, \ldots, n\} \) is hyperbolic. Then by (4.6), (4.36) and Lemma 2.4 we have

\[
i(y_{\rho(i)}, 1) + 2S^+(y_{\rho(i)}) - \nu(y_{\rho(i)}, 1) = i(y_{\rho(i)}, 1) + 1 \geq n + 1,
\]

which together with (4.33) yields

\[
i(y_{\rho(i)}, 2m_{\rho(i)} - 1) + \nu(y_{\rho(i)}, 2m_{\rho(i)} - 1) - 1 \leq 2T - n - 2 < 2T + n - 2i,
\]

which contradicts to (4.40). Thus Claim 2 holds.

**Claim 3.** When \( n \) is even and \( \lambda(i) = 2m_{\rho(i)} \), \( y_{\rho(i)} \) is non-hyperbolic.

Suppose \( y_{\rho(i)} \) is hyperbolic. Then we have \( \nu(y_{\rho(i)}) = 1 \) and

\[
e(\gamma_{\rho(i)}(\tau_{\rho(i)})) = 2.
\]

By (4.30), (4.31), correspondingly we have

\[
i(y_{\rho(i)}, 2m_{\rho(i)}) + \nu(y_{\rho(i)}, 2m_{\rho(i)}) - 1 \leq 2T - 1 \leq i(y_{\rho(i)}, 2m_{\rho(i)}), \tag{4.44}
\]

On the other hand, by (4.40) we have

\[
i(y_{\rho(i)}, 2m_{\rho(i)}) \leq 2T - 2i + n \leq i(y_{\rho(i)}, 2m_{\rho(i)}) + \nu(y_{\rho(i)}, 2m_{\rho(i)}) - 1, \tag{4.45}
\]

which contradicts to (4.44), because \( n \) is even. Hence Claim 3 holds.

**Claim 4.** When \( n \) is odd, let

\[
\mathcal{I} = \{ i \in \{1, \ldots, n\} \mid \lambda(i) = 2m_{\rho(i)} \text{ holds for } y_{\rho(i)} \}.
\]

Then there exists at most one \( i \in \mathcal{I} \) such that \( y_{\rho(i)} \) is hyperbolic. Here we do not require specially \( y_{\rho(i)} \) is \( P \)-cyclic symmetric or not.

In fact, suppose \( y_{\rho(i)} \) is hyperbolic for some \( i \in \mathcal{I} \), and then by (4.44)-(4.45) we must have \( 2i = n + 1 \). Assume \( y_{\rho(j)} \) is also hyperbolic for some \( j \in \mathcal{I} \setminus \{i\} \). Then we obtain \( 2T - 2j \neq \)
Thus (4.44)-(4.45) with \( i \) replaced by \( j \) imply \( y_{\rho(j)} \) can not be hyperbolic.

This completes the proof of Claim 4.

Now when \( n \) is even, by Claim 2 and Claim 3, the closed characteristics found in Theorem 1.1 are all non-hyperbolic and thus there exist at least \( n \) non-hyperbolic closed characteristics.

When \( n \) is odd, by Claim 2 and Claim 4, there exists at most one of the closed characteristics found in Theorem 1.1 is hyperbolic and thus there exist at least \( n - 1 \) non-hyperbolic closed characteristics. The proof is complete.

**Proof of Theorem 1.3.**

Based on the proof of Theorem 1.1, without of loss of generality, we denote the elements in

\[
\{(\tau_l, y_l) \mid s_1 + 1 \leq l \leq s_1 + s_2\}
\]

by

\[
\{(\tau_l, y_l) \mid s_1 + 1 \leq l \leq s_1 + s_2\} = \{(\tau_j, y_j) \mid s_1 + 1 \leq j \leq s_1 + s_3\} \cup
\{(\tau_l, y_l), (\tau_l, P^2 y_l) \mid s_1 + s_3 + 1 \leq l \leq s_1 + s_3 + s_4\},
\]

(4.46)

where \( \mathcal{O}(y_l) = \mathcal{O}(P^2 y_l) \) for \( s_1 + 1 \leq l \leq s_1 + s_3 \) and \( \mathcal{O}(y_l) \cap \mathcal{O}(P^2 y_l) = \emptyset \) for \( s_1 + s_3 + 1 \leq l \leq s_1 + s_3 + s_4 \). Then by (4.46) we have

\[s_2 = s_3 + 2s_4.\]  
(4.47)

Since \( \Sigma \) is also \( P^2 \)-cyclic symmetric, then \( \Phi(P^2 u) = \Phi(u) \) for any \( u \in L^{\frac{n}{n-1}}(S^1, \mathbb{R}^{2n}) \) from (4.13), and (4.14) with \( P \) replaced by \( P^2 \) also holds, thus by Lemma 3.6, we can further require that

\[\text{im}(p) \subseteq \{(\tau_l, y_l) \mid 1 \leq l \leq s_1 + s_3 + s_4\} \times \mathbb{N}.
\]

Then by the same proof of Theorem 1.1, we obtain

\[\# \mathcal{T}(\Sigma) = \# \mathcal{T}(\Sigma, \alpha) = s_1 + 2s_2 \geq s_1 + 2(s_3 + s_4) \geq n.\]  
(4.48)

On the other hand, by assumption we have

\[S = \# \mathcal{T}(\Sigma) = \# \mathcal{T}(\Sigma, \alpha) = n,
\]

which together with (4.47)-(4.48) implies \( s_4 = 0 \), thus by (4.46)-(4.47) we have

\[\mathcal{O}(y_j) = \mathcal{O}(P^2 y_j), \quad s_1 + 1 \leq j \leq s_1 + s_2.\]  
(4.49)

Now we proceed our proof in two case according to the parity of \( k \) which satisfies \( P^k = I_{2n} \) and \( k \geq 3 \).
Case 1. \( k \) is odd.

In this case, by (4.49) we have
\[
O(y_j) = O(P^2 y_j) = \cdots = O((P^2)^{k+1} y_j) = O(P y_j), \quad s_1 + 1 \leq j \leq s_1 + s_2.
\]

Because \( O(y_j) = O(P y_j) \) for \( 1 \leq j \leq s_1 \), we get all the closed characteristics are \( P \)-cyclic symmetric.

Case 2. \( k \) is even.

In this case, by (4.49), for \( s_1 + 1 \leq j \leq s_1 + s_2 \), the closed characteristics \((\tau_j, y_j)\) are \( P^2 \)-cyclic symmetric. Note that we have \( \ker((P^2)^l - I_{2n}) = 0 \) for \( 1 \leq l < \frac{k}{2} \) since \( \ker(P^l - I_{2n}) = 0 \) for \( 1 \leq l < k \), and \( (P^2)^\frac{k}{2} = I_{2n} \) for \( \frac{k}{2} \geq 2 \) since \( k \geq 3 \) and \( k \) is even, thus we can use Proposition 4.3 with \( P \) replaced by \( P^2 \) to obtain
\[
i(y_j, 1) + 2S^+(y_j) - \nu(y_j, 1) \geq n, \quad s_1 + 1 \leq j \leq s_1 + s_2.
\]

Then by the proof of Claim 1, we have \( \lambda(i) = 2m_\rho(i) \) for all \( 1 \leq i \leq n \), which allow us to use the same proof of Theorem 1.1 and obtain
\[
\# \mathcal{T}(\Sigma) = \# \mathcal{T}(\Sigma, \alpha) \geq s_1 + s_2 \geq n. \quad (4.50)
\]

On the other hand, by assumption we have
\[
S = \# \mathcal{T}(\Sigma) = \# \mathcal{T}(\Sigma, \alpha) = n,
\]
which together with (4.28) and (4.50) implies \( s_2 = 0 \), we get all the closed characteristics are \( P \)-cyclic symmetric. The proof is complete.

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