AXIOMATIZING MATHEMATICAL CONCEPTUALISM IN THIRD ORDER ARITHMETIC

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Abstract. We review the philosophical framework of mathematical conceptualism as an alternative to set-theoretic foundations and show how mainstream mathematics can be developed on this basis. The paper includes an explicit axiomatization of the basic principles of conceptualism in a formal system CM set in the language of third order arithmetic.

This paper is part of a project whose goal is to make a case that mathematics should be disassociated from set theory. The reasons for wanting to do this, which I discuss in greater detail elsewhere ([22]; see also [19] and [23]), involve both the philosophical unsoundness of set theory and its practical irrelevance to mainstream mathematics.

Set theory is based on the reification of a collection as a separate object, an elementary philosophical error. Not only is this error obvious, it also has the spectacular consequence of immediately giving rise to the classical set theoretic paradoxes. Of course, these paradoxes are not derivable in the standard axiomatizations of set theory, but that is only because these systems were specifically designed to avoid them. In these systems the paradoxes are blocked by means of ad hoc restrictions on the set concept that have no obvious intuitive justification, which has led to the development of a large literature of attempted rationalizations (e.g., [2, 3, 6, 9, 10, 11, 12, 13, 15]). The heterogeneity of these efforts attests to the difficulty of this task. For example, from a platonistic perspective it seems impossible to give a cogent, principled explanation of why it should be legal to form power sets of infinite sets, given that unrestricted comprehension (forming the set of all x such that P(x)) is not supposed to be valid in general. Antiplatonistic attempts to justify set theory, on the other hand, appear doomed from the start because of the massive gap in consistency strength between straightforwardly antiplatonistically justifiable systems like Peano arithmetic and, say, Zermelo-Frankel set theory. The fact that modern mathematics apparently rests on this kind of basis must be considered a major embarrassment for the subject.

Probably the real appeal of set theory comes not from any murky philosophical defense, but rather from the role it plays as the standard foundation for mathematics. However, its concordance with normal mathematical practice is actually quite poor. Cantorian set theory postulates a vast universe of sets containing remote cardinals which bear no relation to the relatively concrete world of ordinary mathematics, where most objects of central interest are essentially countable (i.e., separable for some natural topology). Similarly, set theory as a mathematical discipline is quite isolated from the rest of mathematics, and it could hardly be otherwise.

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given the gap between its subject matter and the subject matter of normal mathematics.

One might still claim that even if set theory does not fit mainstream mathematics very well, it nonetheless does so better than any foundational alternative. The main purpose of the present paper is to show that this is false, by explaining how ordinary mathematics can be developed in a concrete way that avoids the metaphysical extravagance and nonseparable pathology of Cantorian set theory. The general point is not new: many authors have observed that large amounts of mainstream mathematics can be developed in surprisingly weak systems. I have already done something like this myself in [20]. (Also see the introduction to [20] for other references, and particularly see [16], which contains a very thorough development that is relatively close to what we do here.) Actually, for a classically trained reader [20] may be easier to read than the present paper, because the approach taken there was modeled on the usual set-theoretic development of mathematics (in particular, the formal language used there was the usual language of set theory); here the goal is not to mimic set-theoretic mathematical foundations, but rather to find an approach derived more directly from an alternative philosophical basis.

The philosophical approach we adopt, mathematical conceptualism, is a refinement of the predicativist philosophy of Poincaré and Russell. The basic idea is that we accept as legitimate only those structures that can be constructed, but we allow constructions of transfinite length. What makes this “conceptual” [19, 23] is that we are concerned not only with those constructions that we can actually physically carry out, but more broadly with all those that are conceivable (perhaps supposing our universe had different properties than it does). The admission of transfinite processes takes us well beyond intuitionism, which only allows finite constructions, but at the same time our insistence on having some degree of constructivity is far more restrictive than full-blooded platonism. The result is a foundational stance that matches actual mathematical practice much better than either of these two alternatives.

At the level of countable structures there is little practical difference between conceptualism and platonism. However, uncountable structures in general can be only partially realized in the conceptualist framework. For example, although we can (transfinitely) construct individual real numbers (regarded, say, as Dedekind cuts), we have no clear picture of a transfinite construction that would succeed in producing the entire real line. Thus we might say that the real line exists conceptualistically only in an unfinished state.

(A platonist might counter that the well-ordering theorem does give him a picture of how the real line could be sequentially constructed one element at a time. This is not a good argument because it is the set theoretic axioms — specifically, the power set axiom — on the basis of which the well-ordering theorem is proven that are in question here. If we take conceivability as a first principle then uncountable constructions in general become highly dubious; see Section 1.2.)

We regard the idea of a completed surveyable real line in roughly the same way that we regard naïve infinitesimals, as an evocative idealization that does not really have a definite meaning. Admittedly, this is at odds with normal mathematics, which does treat the real line as a completed and in some sense surveyable structure. However, it does not seem that this assumption is actually used in any serious way in mainstream mathematics. The standard developments of all mainstream subjects
can be executed perfectly well in a conceptualistic setting which treats the real line and other structures at a similar level of complexity as only incompletely realizable. The unfinished nature of the real line introduces logical subtleties which we handle by adopting intuitionistic logic when quantifying over all real numbers. This is one of the principal differences between conceptualism and earlier versions of predicativism, where classical logic was used almost exclusively and there was persistent confusion about the legitimacy of second order quantification (e.g., [5]). Alternatively, one could avoid the use of intuitionistic logic by arresting the construction of the mathematical universe at some natural point and reasoning classically about the resulting fixed partial universe; this was the approach adopted in [20].

In the present paper we also go further and allow some reasoning about arbitrary sets of real numbers, although this requires even greater care. This represents a change from the point of view expressed in Section 2.5 of [21], where I would have rejected any reference to arbitrary sets of real numbers. (However, I stand on the main point of that discussion, that self-applicative schematic predicates are prima facie predicatively invalid.) I now believe that reasoning at this level of abstraction may be legitimate provided an even weaker logical apparatus, the minimal logic of Johansson, is used. The justification for this conclusion is explained in [23]. It is perhaps not crucial here because we are going to work with a restricted notion of sets of real numbers to which ordinary intuitionistic logic does apply.

We will present a formal system CM for conceptualist mathematics and outline how core mathematics can be developed within this system. The claim is that virtually all mainstream mathematics can be straightforwardly realized in CM.

Our system CM is similar to systems in [16], and there is a strong resemblance between our development and that in [16] (though the similarity to [20] seems greater). The main differences lie in our use of third order variables, which simplifies the presentation in some ways, and our use of non-classical logic, which is an important theoretical distinction but has surprisingly little practical effect. Most assertions of interest in mainstream subjects can be reduced to questions involving quantification only over countable sets, at which point classical logic can be used.

1. Philosophical motivation

1.1. The concept of a set. Sets are typically defined as “collections of objects”, and, crucially, these collections are themselves supposed to be objects capable of belonging to other sets. This seems to be a simple grammatical confusion. If we can talk sensibly about the set of all books in the Library of Congress, then “the set of all books in the Library of Congress” must be a particular thing, the reasoning apparently goes; it is not a physical object, so it must be a non-physical object. But we can also talk sensibly about the average taxpayer; should we infer that this is an actual (albeit non-physical) person? Is it really coherent to maintain that the set of all taxpayers is a genuine “abstract object” while conceding that the average taxpayer is just a figure of speech?

Despite its nonsensical official rationale, in sufficiently concrete settings the apparatus of set theory can be straightforwardly justified. For example, if we want to talk about sets of natural numbers as if they were actual objects, we can refer instead to infinite sequences of 0’s and 1’s. Philosophical questions can still be raised about the general concept of an infinite sequence of 0’s and 1’s, but these
are questions about the notion of infinity which could be brought against any interpretation of mathematics. The point is that we are no longer postulating the existence of fictional entities based on a grammatical confusion.

In other words, set theory can be legitimized to the extent that we are able to set up a system of token entities which can play the role of sets. And it is clear that most of our intuition about sets comes from such token structures. When I think of a set, it is not the set itself that I picture — how could I, when this is supposed to be an abstract object that has no visual aspect — but rather some structural representation. Perhaps this explains Gödel’s famous comment that we have “something like a perception” of sets ([7], p. 484). We do not have any perception of sets, but we do have a conception, perhaps something like a perception, of structures which can play the role of sets.

The philosophical literature on set theory tends to blur this distinction. In particular, it is highly ambivalent as to whether sets can actually be formed or manipulated in any sense, or whether they are really supposed to be absolutely inert abstractions. The latter is the official platonist position, but language suggesting the former is ubiquitous. This is particularly seen in the most popular explanation of the paradoxes of naive set theory, which invokes an “iterative conception” of sets according to which sets are to be thought of as being iteratively constructed in stages. Of course, this makes no sense if they are simultaneously thought of as being inert abstractions.

(Most authors who write about the iterative conception are clearly aware of this difficulty. This sometimes results in strange comments to the effect that the informal explanation of the iterative conception in quasi-constructive terms is not supposed to be taken literally, which obviously begs the questions of how, then, one is supposed to take it, and what meaning the iterative conception can have if it cannot be explained in a way that makes literal sense.)

In practice, mathematical reasoning about sets consists largely of imagined quasi-physical manipulations that could hardly apply to causally inert metaphysical entities. We order sets, we remove elements from them, we form products, we cut and paste. Practically anything we would call a mathematical construction makes sense only as applied to quasi-physical structures, not abstract sets. This suggests that a rational approach to set theory would drop the nonsensical “abstract objects” interpretation of sets and focus entirely on possible structures [8]. But we then have to ask to what extent traditional set theory can be justified in this approach. In particular, it would not seem to justify set-theoretic “constructions” like the power set operation which do not correspond to any obvious quasi-physical construction.

1.2. Uncountable structures. Our choice of tokens for arbitrary sets of natural numbers is special to that case, but it is natural to assume that a reasonable system of tokens for arbitrary sets of X’s could be found for any choice of X. However, that intuition may be misleading. There is no obvious system of tokens which could be used to represent arbitrary sets of infinite sequences of 0’s and 1’s, for example.

We have to specify more clearly what would count as a token. We need not insist that these actually be physically realizable in our universe. Above I used the term “quasi-physical” with the intention of suggesting something like “physically realizable in some conceivable universe”. This seems like the right criterion to use given that we want to imagine manipulating these tokens in the ways mentioned above.
So: could a meaningful system of tokens for arbitrary sets of infinite sequences of 0’s and 1’s appear in any conceivable universe?

The difficulty here is that in this example the tokens themselves would presumably have to be uncountably large. So the question becomes whether it is possible to clearly conceive of a universe that contains uncountable structures.

It is not so hard to imagine living in a universe that is infinite in extent and contains a countable infinity of physical objects. Indeed, it is quite conceivable that our own universe has this property. One can even imagine a universe that is finite in extent but still contains a countable infinity of non-overlapping physical objects whose sizes decrease rapidly enough that they collectively fill only a finite volume. But what would it be like to live in a universe containing uncountably many non-overlapping physical objects?

Such a universe is difficult to imagine, and there is a good reason for this: the Löwenheim-Skolem theorem. According to this theorem, any formal description one could give of an uncountable universe would be equally true of some countable substructure. Thus, there is nothing we can say, at least formally, that would serve to distinguish an uncountable universe from a countable one. This suggests that we have no a priori conception of any uncountable universe, which presents a serious obstruction to any attempt at setting up a system of tokens to model a classically uncountable structure.

Taking conceivability as a first principle really calls the whole notion of uncountability into question. All classical methods of constructing uncountable sets ultimately rely on the power set operation. But from the conceptualist perspective the latter cannot be justified unless one has a prior concept of uncountable structures. This point may puzzle mathematicians who are accustomed to thinking of the power set operation as just one among many straightforward tools for constructing sets. The crucial distinction that is missed here is between collections that are surveyable (we have a clear picture of what it would mean to exhaustively search such a collection) and those that are merely determinate (we can decide whether any given object belongs to the collection, but have no clear idea even in principle how one would go about searching through all objects in the collection).

The natural numbers are surveyable, their power “set” is merely determinate.

We have a clear idea of how we would go about searching through the natural numbers, but not how we could search through the real numbers. So if “set” means “surveyable collection” then we cannot a priori assume that the real line is a set. But if “set” means “determinate collection” then basic set-theoretic constructions become problematic: for instance, it is not clear that we can take the union of a family of sets indexed by a determinate set since the result might not be determinate. We cannot decide whether any given object belongs to the union because this might require searching through the entire index set.

Somehow the vague sense that the natural numbers and the real line are both “sets” has led us to transfer intuition about one to the other, resulting in the idea that the power set operation is just a straightforward set construction. It is not.

Since uncountability is embedded so firmly in current thought, this point must be emphasized: we have no obvious justification for the existence of uncountable structures that does not rely on the conflation of surveyable and determinate collections. Moreover, the Löwenheim-Skolem theorem gives us a powerful reason to expect that no conceptualistic justification of uncountable structures is possible. If we
cannot describe what it would be like to exist in a universe containing uncountable structures (in a way that would distinguish it from some countable subuniverse), then we presumably cannot imagine it either.

Thus, we reject the notion of actually existing uncountable structures, not just in our universe but in any conceivable universe.

1.3. Countable constructions. As I mentioned in the last section, I think it is quite possible that our universe is infinite in extent and contains infinitely many disjoint physical objects. Regardless of whether this is actually the case, it would be difficult to argue that this possibility is literally inconceivable. Therefore, we can accept that countably infinite structures are legitimately part of the conceptualist landscape.

A more subtle question involves the conceivability of transfinite computations or constructions. By this I mean processes that not only potentially involve any finite number of steps, but literally involve infinitely many steps, and may even continue on after infinitely many steps have been completed. The philosophical term for such a process is “supertask”. For example, we can imagine resolving the Goldbach conjecture by mechanically searching through the natural numbers for a counterexample; if no counterexample is found, the conjecture must be true. This example could require the literal execution of infinitely many steps, since we might not get an answer until after all numbers have been checked.

But can we really imagine carrying out and completing such a process? Can we imagine what it would be like to live in a universe in which such supertasks were possible? If not, then they cannot be admitted into the conceptualist picture.

It may not be obvious that literally infinite constructions are really conceivable. However, several recent proposals for carrying out supertasks make it quite plausible that one can indeed form a perfectly coherent picture of what doing this would be like. My favorite is a suggestion due to Davies [4] that he calls “building infinite machines”.

Briefly, Davies’ idea is something like this. Suppose we want to write down all the natural numbers. To achieve this, we build a machine that accepts as input a natural number \(n\), and on this input it first writes down the number \(n\) (say, in decimal notation) in front of itself; then, to its right, it builds a copy of itself half its size that runs twice as fast; and finally it feeds this smaller copy the input \(n + 1\) and sets it running. Having built such a machine, we feed it the number 1 as input and set it running. It writes down the number 1, builds a smaller and faster copy of itself, and feeds it the number 2 as input; that smaller copy writes down the number 2, builds a still smaller and faster machine, and feeds it the number 3 as input; and so on. Because the machines are speeding up exponentially the entire task is completed in a finite amount of time. Because they are shrinking exponentially it takes place in a finite spatial region.

Davies observes that experiments like this would be possible in a continuous Newtonian universe. I am not so sure that something like this is not actually possible in our universe (with the machines being being “built” as perturbations of the electromagnetic field, say), but for our purposes here merely being able to imagine a universe in which it could take place is sufficient. I see no logical obstruction to a universe in which an experiment of Davies’ type could actually be performed.
The task I have just described is a particularly simple kind of supertask; although it does require the literal completion of infinitely many steps, its goal is merely to write something down. More problematic would be a supertask like the one which checks the truth of the Goldbach conjecture and returns an answer. It is not hard to imagine programming a Davies machine to search through the natural numbers looking for a counterexample, but since the size of the machines carrying out this task goes to zero and we want a final answer to be returned on the scale of the first machine, this seems to require a discontinuity. In a universe whose time evolution is continuous there may well be a logical obstruction to carrying out such a task.

On the other hand, are universes with discontinuous time evolution literally inconceivable? This would come as a surprise to proponents of the Copenhagen interpretation of quantum mechanics. In this interpretation the time evolution in our universe is thought to involve a discontinuous “collapse” of the state vector whenever an observer makes a measurement. This view can be criticized in many ways, but one criticism I have never seen is that it is inconceivable because it involves a discontinuous time evolution. Indeed, it is easy to imagine universes whose time evolution is discontinuous.

If discontinuities are possible, then we can easily imagine building a sequence of Davies machines next to a wire, say, and giving each machine the ability to send a signal along the wire if it finds the counterexample it is looking for. A discontinuity appears in the assumption that we have the ability to detect a signal coming from any of the machines, no matter how small. On the basis of thought experiments like these, we can justify the conceivability not only of countably infinite structures, but also of some transfinite computations or constructions.

1.4. Quantifying over the reals. We have seen that the power set of the natural numbers does not exist conceptualistically as a well-defined structure. But we have also seen that the general notion of a set of natural numbers, realized as an infinite sequence of 0’s and 1’s, makes perfect sense conceptualistically. The point is that even though we have no notion of a construction that would generate all sequences of 0’s and 1’s, we can nonetheless recognize such a sequence when we see one (or at any rate we could build a Davies machine to perform such a check).

Similarly, regarding real numbers as Dedekind cuts, our notion of a general real number is in the preceding sense completely definite, but the entire real line does not exist conceptualistically as a well-defined structure.

Because we do understand the general concept of a real number, we can regard assertions which quantify over all real numbers as intelligible, provided we are slightly careful about interpreting the meaning of the quantifiers. For instance, the classical interpretation of “there exists” only makes sense when quantifying over the objects appearing in some well-defined structure, which we do not have in this case. However, we can still interpret “there exists” constructively, i.e., we can understand an assertion of the existence of a real number with some property to be an assertion that we have some way of constructing it. Similarly, we take an assertion that all real numbers have some property not as meaning that if one checked all real numbers one would find they all have this property — this cannot be done, even in principle — but rather as meaning that we can know in advance that any real number that appears will have the property in question.

This is just the usual intuitionistic interpretation of quantifiers. Indeed, intuitionistic logic is exactly suited to the present situation. I will review this logic in
Section 2.1. Its most salient property is that the law $A \lor \neg A$ is not generally valid. Classically this law can be justified by saying that the truth value of any statement could in principle be determined by a mechanical search (much like the Goldbach conjecture in our discussion above); thus any statement is definitely either true or false. However, when we quantify over real numbers this justification is not available. Since the notion of “all real numbers” is indefinite in the sense that there is no structural amalgamation of all real numbers which could be mechanically surveyed, even in principle, we cannot generally affirm that any statement about arbitrary real numbers has a definite truth value. Or, at least, since there could be statements whose truth value cannot be determined even in principle, such an affirmation would not have any substantive content.

The main point to keep in mind is that we cannot assume that all statements have definite truth values; to a large extent, intuitionistic logic merely codifies the forms of reasoning that one would naturally adopt in such a case.

1.5. Quantifying over sets of reals. Above I have distinguished between fixed, surveyable structures which can be conceived in toto and determinate concepts which can be partially, but never wholly, realized by actual concrete structures. For us classically countable structures generally fall in the first category and classical structures at the same level of complexity as the real line fall in the second. I have also claimed that classical logic is appropriate only when quantifying over surveyable structures and that intuitionistic logic should be used when quantifying over indefinitely extendable but determinate concepts.

It must be emphasized that the individual elements that make up an indefinitely extendable concept such as the real line are themselves concretely realizable structures. The logical subtlety arises when we discuss arbitrary structures of some type, e.g., arbitrary sets of natural numbers.

We now want to consider the classical concept of an arbitrary set of real numbers. Here even an individual — a single set of real numbers — in general can never be concretely realized. So arbitrary sets of real numbers can have concrete reality only as something like rules or prescriptions telling us how to decide, as new real numbers become available, whether they should be accepted into or rejected from the set.

I argue in [23] that the correct logic to use when reasoning at this level of abstraction is Johansson’s minimal logic. This is a weakening of intuitionistic logic which omits the ex falso law that states that any assertion follows from a contradiction. The idea is that, unlike concrete structures, rules or prescriptions have a genuine potential to be contradictory. Thus, we can set up standards for reasoning about arbitrary rules, which is what minimal logic does, but we are not allowed to decree that such rules will be free from contradiction. The subtle point is that the meaning of the rules under discussion depends on the logical apparatus used to reason about them, so incorporating an assertion of consistency into that apparatus would be circular.

This point is pursued further in [23]. It is somewhat peripheral here since we will work in a restricted setting where the ex falso law is valid. (We consider only “determinate” subcollections of $\mathcal{P}(\omega)$ for which the membership relation is decidable, rather than the “definite” subcollections discussed in [23]. Thus the law of excluded middle holds for all atomic formulas, which justifies ex falso; see Section 2.3 below.)
1.6. **Summary.** We reject the proposition that sets literally exist as some kind of ghostly non-physical objects. This assertion has no meaningful content, it is responsible for the classical paradoxes of naive set theory, and it is not even compatible with the grammar of set language in ordinary speech [17].

The way mathematicians use set language is also incompatible with the concept of a set as an inert unitary object existing in some timeless metaphysical realm. Mathematicians handle sets as if they were articulated structures capable of being physically manipulated. For the purposes of understanding the foundations of mathematics, it therefore makes sense to focus on the concept of a possible structure in a conceivable universe.

This radically changes the nature of mathematical foundations, most importantly by delegitimizing the power set operation. There is no obvious “construction” of a structure whose elements correspond to all substructures of a given structure. Moreover, a general argument can be made that this would actually be incompatible with the notion of a conceivable universe. Any conceivable universe should be finitely describable, and hence subject to the Löwenheim-Skolem theorem, which would entail the incompleteness of any putative infinite “power set” structure.

Indeed, the applicability of the Löwenheim-Skolem theorem to conceivable universes casts doubt on the general concept of an actually uncountable structure. However, a good case can still be made for the literal conceivability of transfinite constructions of length $\omega$, $\omega^2$, $\omega^\omega$, etc.

In the foundational picture that emerges countable structures are seen as legitimate but we do not accept the idea of an existing completed uncountable structure. Rather, we treat concepts such as “real number” or “set of natural numbers” as definite concepts which can be only incompletely realized as actual structures. This calls for the use of intuitionistic logic when we quantify over all real numbers or all sets of natural numbers.

We can also think of, for example, sets of real numbers as prescriptions telling us how to decide whether each new real number, as it becomes available, is to be accepted or rejected. It does not seem possible to meaningfully iterate this process any further. Since “legitimate prescription” is not even a sharp concept, it is hard to imagine how one could sensibly model the concept of an arbitrary set of such things.

Our task is now to set out a formal system which expresses the philosophical point of view set out above, and to show that it naturally accommodates the vast bulk of normal mainstream mathematics.

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2. **The system CM**

2.1. **Systems of logic.** Intuitionistic logic is most easily understood in terms of systems of “natural deduction” [14]. For more details the reader may also consult the excellent survey article [18].

Informally, the rules for natural deduction in minimal logic are:

- Given $\phi$ and $\psi$ deduce $\phi \land \psi$; given $\phi \land \psi$ deduce $\phi$ and $\psi$.
- Given either $\phi$ or $\psi$ deduce $\phi \lor \psi$; given $\phi \lor \psi$, a proof of $\sigma$ from $\phi$, and a proof of $\sigma$ from $\psi$, deduce $\sigma$.
- Given a proof of $\psi$ from $\phi$ deduce $\phi \rightarrow \psi$; given $\phi$ and $\phi \rightarrow \psi$ deduce $\psi$.
- Given $\phi(x)$ deduce $(\forall x)\phi(x)$; if the term $t$ is free for $x$, given $(\forall x)\phi(x)$ deduce $\phi(t)$. 

• If the term $t$ is free for $x$, given $\phi(t)$ deduce $(\exists x)\phi(x)$; if $y$ does not occur freely in $\psi$, given $(\exists x)\phi(x)$ and a proof of $\psi$ from $\phi(y)$ deduce $\psi$.

Natural deduction involves the concept of “a proof of $\psi$ from $\phi$”; thus, for example, we may temporarily assume $\phi$, use this to prove $\psi$, and then infer the statement $\phi \rightarrow \psi$. Temporary assumptions can be nested, just as in normal informal reasoning, and one has to be slightly careful about allowing the variables involved in the quantifier rules to appear in active temporary assumptions. See [14] or [18] for a precise exposition.

Despite this minor complication, it should be evident that the rules of natural deduction are indeed very natural and correspond exactly to ordinary informal reasoning. It could be said that these rules are nothing more than a direct expression of the meaning of the various logical symbols.

We let $\bot$ stand for the statement “0 = 1” and define negation by letting $\neg \phi$ stand for $\phi \rightarrow \bot$. Minimal logic contains no special rules for negation; in intuitionistic logic we adopt the ex falso rule “given $\bot$ deduce $\phi$” for any formula $\phi$, and in classical logic we also adopt the law of excluded middle “$\phi \lor \neg \phi$” for any formula $\phi$.

As I explained above in Sections 1.4 and 1.5, this is justified when we are reasoning about objects belonging to a fixed surveyable structure, but not in broader settings.

It is easy to learn to reason intuitionistically using natural deduction. For example, we prove $(\phi \rightarrow \psi) \rightarrow (\neg \psi \rightarrow \neg \phi)$ as follows: (1) assume $\phi \rightarrow \psi$; (2) assume $\neg \psi$; (3) assume $\phi$; (4) deduce $\psi$ from (1) and (3); (5) deduce $\bot$ from (2) and (4); (6) cancel assumption (3) and infer $\neg \phi$ from (5); (7) cancel assumption (2) and infer $\neg \psi \rightarrow \neg \phi$ from (6); (8) cancel assumption (1) and infer $(\phi \rightarrow \psi) \rightarrow (\neg \psi \rightarrow \neg \phi)$ from (7). Exercise: verify $\neg(\phi \lor \psi) \leftrightarrow (\neg \phi \land \neg \psi)$ and $\neg(\phi \lor \neg \psi) \leftrightarrow (\neg \phi \land \psi)$ (both of these can be proven in minimal logic), and convince oneself that the last implication cannot be reversed without using excluded middle. For practice with quantifiers the reader can check that $(\forall x)\neg \phi(x) \leftrightarrow (\exists x)\phi(x)$ and $(\exists x)\neg \phi(x) \rightarrow (\forall x)\phi(x)$ (again, the last implication cannot be reversed without using excluded middle). A simple example of a basic law whose proof requires the use of ex falso is $[(\phi \lor \psi) \land \neg \phi] \rightarrow \psi$.

Minimal logic can also be formulated in the following more standard way. We now eliminate the falsehood symbol $\bot$ and take negation as primitive. There are three logical rules of inference:

1. given $\phi$ and $\phi \rightarrow \psi$ deduce $\psi$
2. given $\phi \rightarrow \psi$ deduce $\phi \rightarrow (\forall x)\psi$
3. if $x$ does not occur freely in $\psi$, given $\phi \rightarrow \psi$ deduce $(\exists x)\phi \rightarrow \psi$.

There are eleven logical axiom schemes:

1. $\phi \rightarrow (\psi \rightarrow \phi)$
2. $(\phi \rightarrow \psi) \rightarrow ((\phi \rightarrow (\psi \rightarrow \sigma)) \rightarrow (\phi \rightarrow \sigma))$
3. $\phi \rightarrow [\psi \rightarrow (\phi \land \psi)]$
4. $\phi \land \psi \rightarrow \phi$
5. $\phi \land \psi \rightarrow \psi$
6. $\phi \rightarrow (\phi \lor \psi)$
7. $\psi \rightarrow (\phi \lor \psi)$
8. $(\phi \rightarrow \sigma) \rightarrow [(\psi \rightarrow \sigma) \rightarrow (\phi \lor \psi \rightarrow \sigma)]$
9. $(\phi \rightarrow \psi) \rightarrow [((\phi \rightarrow \neg \psi) \rightarrow \neg \phi]$
10. $\phi(t) \rightarrow (\exists x)\phi(x)$
11. $(\forall x)\phi(x) \rightarrow \phi(t)$
In axioms 10 and 11, \( t \) is a term which is free for \( x \) in \( \phi \).

Intuitionistic logic includes the axioms
\[
(12) \quad \phi \rightarrow (\neg \phi \rightarrow \psi)
\]
and classical logic additionally includes the axioms
\[
(13) \quad \phi \lor \neg \phi.
\]

2.2. The formal system CM. We introduce the formal language of third order arithmetic. There are three kinds of variables: first order variables \( a, b, c, \ldots \) (thought of as ranging over \( \omega \)), second order variables \( A, B, C, \ldots \) (thought of as ranging over subsets of \( \omega \)), and third order variables \( A, B, C, \ldots \) (thought of as ranging over subsets of \( \mathcal{P}(\omega) \)). More concretely, we think of first order variables as representing the objects appearing in some infinite sequence, second order variables as representing infinite sequences of 0’s and 1’s, and third order variables as representing predicates which take any second order object as input and return either 0 or 1. A helpful heuristic is: first order objects are urelements, second order objects are sets of first order objects, and third order objects are classes of second order objects.

Numerical terms are built up from the number variables and the constant symbol 0 using the successor operation ‘\( = \)’ and the binary operations + and ·. The atomic formulas of the language consist of all formulas of the form \( t_1 = t_2, t_1 \in X, \) and \( X \in X \) where \( t_1 \) and \( t_2 \) are numerical terms, \( X \) is a second order variable, and \( X \) is a third order variable. General formulas are built up from the atomic formulas using the logical connectives \( \land, \lor, \neg, \rightarrow \) and the quantifiers \( \forall n, \exists n, \forall X, \exists X, \forall X \), and \( \exists X \) for any first, second, and third order variables \( n, X, \) and \( X \). Other symbols such as \( \leftrightarrow, \subset, \) etc., are defined in terms of the above symbols in the usual way, except for \( \subseteq \) at the third order level, which we will define in Definition 3.3.

The logical apparatus of CM is intuitionistic logic, as described in Section 2.1. Additionally, we adopt the law of excluded middle \( \phi \lor \neg \phi \) for all formulas \( \phi \) that contain no second or third order quantifiers. We also include the usual axioms for equality between terms.

We now state the non-logical axioms of CM. Throughout the following \( m \) and \( n \) are any first order variables, \( X, Y, \) and \( Z \) are any second order variables, and \( X \) is any third order variable.

I. Number axioms:
\[
\begin{align*}
(1) & \quad \neg(n' = 0) \\
(2) & \quad m' = n' \rightarrow m = n \\
(3) & \quad m + 0 = m \\
(4) & \quad m + n' = (m + n)' \\
(5) & \quad m \cdot 0 = 0 \\
(6) & \quad m \cdot n' = (m \cdot n) + m
\end{align*}
\]

II. Induction and recursion axioms:
\[
\begin{align*}
(7) & \quad [\phi(0) \land (\forall n)(\phi(n) \rightarrow \phi(n + 1))] \rightarrow (\forall n)\phi(n) \\
(8) & \quad (\forall n)(\forall X)(\exists Y)\phi(n, X, Y) \rightarrow (\forall X)(\exists Z)[Z_{(0)} = X \land (\forall n)\phi(n, Z_{(n)}, Z_{(n')})]
\end{align*}
\]
for all formulas \( \phi \).

III. Comprehension axioms:
\[
\begin{align*}
(9) & \quad (\forall n)(\phi(n) \lor \neg \phi(n)) \rightarrow (\exists X)(\forall n)(n \in X \leftrightarrow \phi(n)) \\
(10) & \quad (\forall X)(\phi(X) \lor \neg \phi(X)) \rightarrow (\exists X)(\forall X)(X \leftrightarrow \phi(X))
\end{align*}
\]
for all formulas $\phi$ containing no free occurrences of $X$ in (9), and no free occurrences of $X$ in (10).

The notation $Z(n)$ is defined below in Definition 3.2. Essentially, axiom (8) states that if for all $n$ and $X$ there exists $Y$ such that $\phi(n, X, Y)$, then for any $X$ we can find a sequence $(Z_n)$ such that $Z_0 = X$ and $\phi(n, Z_n, Z_{n+1})$ holds for all $n$.

2.3. Discussion. We adopt the usual intuitionistic interpretation of the logical symbols, e.g., $(\exists n)\phi(n)$ means that we have (in principle) a way to find a value of $n$ satisfying $\phi$; $\phi \rightarrow \psi$ means that we can convert any proof of $\phi$ into a proof of $\psi$; and so on.

The rules of minimal logic presented in Section 2.1 directly express the meanings of the logical symbols, and all atomic formulas satisfy the law of excluded middle. (We have excluded middle for the statement $n \in X$ for any $n$ and $X$ since any sequence of 0’s and 1’s could in principle be surveyed to determine its truth value, and we have excluded middle for the statement $X \in X$ since third order objects are assumed to assign a definite truth value to this statement for any $X$.) This is enough to justify the ex falso law; to see this, for each atomic formula $A = A(x_1, \ldots, x_n)$ introduce a function symbol $f_A$ together with the axiom

$$(A(x_1, \ldots, x_n) \iff f_A(x_1, \ldots, x_n) = 1) \land (\neg A(x_1, \ldots, x_n) \iff f_A(x_1, \ldots, x_n) = 0).$$

This extension of the original system should be unproblematic. But now assuming $\bot$ we can deduce $A \land \bot$, i.e., $\neg A$, for any atomic formula $A = A(x_1, \ldots, x_n)$, then infer $f_A(x_1, \ldots, x_n) = 0$, then (since $0 = 1$) infer $f_A(x_1, \ldots, x_n) = 1$, and finally infer $A$. So in the extended system we can actually deduce any atomic formula $A$ from $\bot$, which is enough to verify ex falso for the original system.

(In [23] we considered a broader notion of third order objects, the “definite” subsets of $P(\omega)$. These do not have a decidable membership relation, so the above justification of the ex falso law would not be valid. On the other hand, this choice of third order objects would support a stronger comprehension scheme according to which $(\exists X)(\forall Y)(X \in X \iff \phi(Y))$ for any formula $\phi$ which contains no third order quantifiers and no free occurrence of $X$.)

The truth value of any formula with no second or third order quantifiers could be determined by a countable computation, so we adopt the law of excluded middle for such formulas.

The number axioms (1) – (6) assert basic facts which are evidently true for any $\omega$-sequence. The induction axioms (7) reflect our acceptance of countable procedures, as they could be verified by deductions of length $\omega$.

The recursion axioms (8) are usually called “dependent choice” [16], but in the context of intuitionistic logic they should not be understood as choice axioms in the traditional sense. Intuitionistically we interpret the assertion $(\forall x)\phi(x)$ as expressing that there is a uniform proof of $\phi(x)$ for all $x$. Thus the premise $(\forall n)(\forall X)(\exists Y)\phi(n, X, Y)$ entails possession of a uniform procedure for constructing the desired sequence $(Z_n)$. In our setting the content of the axiom has to do not with choice but with our ability to perform countable constructions.

The second order comprehension axioms (9) hold because we take subsets of $\omega$ to be modelled by (in principle) actually existing infinite sequences of 0’s and 1’s: if we knew that $\phi(n) \lor \neg \phi(n)$ held for all $n$ then in principle we could determine the truth value of $\phi(n)$ for every $n$ and use this information to construct a corresponding $X$.

The third order comprehension axioms (10) are immediately justified by the fact
that we take third order variables to stand for predicates which assign a definite truth value to the assertion \( X \in X \) for any \( X \). That is, we can take \( X \) to be the formula \( \phi \) itself.

Our conception of second order objects as appearing in well-ordered stages, such that only countably many of them are available at each stage, would also support a genuine choice axiom. The most straightforward way to formalize it would be to augment CM with a second order relation symbol \( \prec \) and add axioms asserting that \( \prec \) is a total ordering, together with the axioms

\[
(\forall X)((\forall Y)(Y \prec X \rightarrow \phi(Y)) \rightarrow \phi(X)) \rightarrow (\forall X)\phi(X)
\]

asserting progressivity of \( \prec \). This should only be asserted for formulas that do not contain \( \prec \), for reasons having to do with the circularity involved in making sense of a relation that is well-ordered with respect to properties that are defined in terms of that relation; see Section 2.5 of [21]. We could also add the axiom

\[
(\forall X)(\exists Z)(\forall Y)[Y \prec X \rightarrow (\exists n)(Y = Z(n))]
\]

expressing the fact that only countably many second order objects are available at any moment. We call the resulting system CM\(^+\). None of the mathematics we develop in Section 3 requires the extra axioms of CM\(^+\).

We record two general facts about CM that will be of use in the next section. First, the comprehension axioms immediately imply *arithmetical comprehension*:

**Theorem 2.1.** (a) Let \( \phi \) be a formula that contains no free occurrences of \( X \) and no second or third order quantifiers. Then CM proves

\[
(\exists X)(\forall n)(n \in X \leftrightarrow \phi(n)).
\]

(b) Let \( \phi \) be a formula that contains no free occurrences of \( X \) and no second or third order quantifiers. Then CM proves

\[
(\exists X)(\forall X)(X \in X \leftrightarrow \phi(X)).
\]

Second, we have a principle of *numerical omniscience*:

**Theorem 2.2.** For any formulas \( \phi \) and \( \psi \), CM proves

\[
(\forall n)(\phi(n) \lor \psi(n)) \rightarrow [(\forall n)\phi(n) \lor (\exists n)\psi(n)].
\]

Intuitively, our ability to perform countable constructions and the fact that we have a uniform proof of \( \phi(n) \lor \psi(n) \) allows us to verify either \((\forall n)\phi(n)\) or \((\exists n)\psi(n)\). The theorem is formally proven as follows. Suppose \((\forall n)(\phi(n) \lor \psi(n))\). Then for any \( n \) there exists \( X \) such that

\[
(\phi(n) \land 0 \in X) \lor (\psi(n) \land 0 \notin X).
\]

Dependent choice (axiom (8)) followed by arithmetical comprehension then yields a set \( Y \) such that

\[
(\phi(n) \land n \in Y) \lor (\psi(n) \land n \notin Y)
\]

holds for all \( n \). Finally, since classical logic holds for formulas without second or third order quantifiers, we have \((\forall n)(n \in Y) \lor (\exists n)(n \notin Y)\), and we can then infer \((\forall n)\phi(n) \lor (\exists n)\psi(n)\).

As a special case of numerical omniscience we have that \((\forall n)(\phi(n) \lor \lnot \phi(n))\) implies both \((\forall n)\phi(n) \lor \lnot(\forall n)\phi(n)\) and \((\exists n)\phi(n) \lor \lnot(\exists n)\phi(n)\). In effect, this means that any formula whose truth can be evaluated in countably many steps satisfies
the law of excluded middle. This is important because it implies that we can use classical logic, in particular proofs by contradiction, in such cases.

As an alternative formulation of CM we could adopt numerical omniscience as an axiom and only assume the law of excluded middle for atomic formulas. It comes to the same thing because numerical omniscience plus excluded middle for atomic formulas implies excluded middle for all arithmetical formulas (by induction on formula complexity).

3. Development of core mathematics

3.1. Preliminaries. The remainder of the paper will sketch how core mathematics can be developed within the framework presented in Section 2. We concentrate on analysis because this is the mainstream area that is, broadly speaking, most resistant to formalization in weak systems. Our development is similar to the one in [20]. Probably the main hurdles to overcome are getting accustomed to the basic set-up established in the present section and familiarizing oneself with the technique of using comprehension axioms to prove existence results. In the following we will reason informally in CM.

Definition 3.1. Let $\tilde{N}$ be a second-order constant denoting the natural numbers (including zero):

$$\forall n (n \in \tilde{N}).$$

$\tilde{N}$ exists by second order comprehension. CM contains the axioms of Peano arithmetic, so elementary number theory in $\tilde{N}$ can be developed as usual.

Definition 3.2. An ordered $k$-tuple of natural numbers ($k \geq 2$) is a nonzero natural number having no prime divisors besides $p_1, \ldots, p_k$, where $p_i$ is the $i$th prime. We write $\langle a_1, \ldots, a_k \rangle = p_1^{a_1} \cdots p_k^{a_k}$. For any $X_1, \ldots, X_k$ ($k \geq 2$) we define

$$X_1 \times \cdots \times X_k = \{\langle a_1, \ldots, a_k \rangle : a_1 \in X_1, \ldots, a_k \in X_k\}.$$  

A sequence of second order objects is a second order object contained in $\tilde{N}^2 = \tilde{N} \times \tilde{N}$. For such an object $X$ and each $a \in \tilde{N}$ we write $X_{(a)} = \{b : \langle a, b \rangle \in X\}$.

At the third order level, for any $X_1, \ldots, X_k$ ($k \geq 2$) we define $X_0 \times \cdots \times X_k$ to be the third order object consisting of all sequences $X$ of second order objects such that $X_{(0)} \in X_0, \ldots, X_{(k)} \in X_k$, and $X_{(l)} = \emptyset$ for all $l > k$; we write $X = \langle X_{(0)}, \ldots, X_{(k)} \rangle$.

In the preceding definition $X_1 \times \cdots \times X_k$, $X_{(a)}$, and $X_0 \times \cdots \times X_k$ all exist by arithmetical comprehension. Also observe that if $X = \langle X_{(n)} \rangle$ is a sequence of second order objects then $\bigcup X_{(n)}$ and $\bigcap X_{(n)}$ exist by arithmetical comprehension.

We now introduce set language. Notice that in classical set theory quotient constructions involve passing to a higher type, an avenue that is not available here. One way to define quotients in the present setting would be to model the quotient structure by selecting one element from each equivalence class. This could generally be done using the third order choice axiom in $CM^+$ (see Section 2.3), but this method introduces extraneous information. In other words, quotients become noncanonical.

We opt instead to model a “set” as a third order object equipped with an equivalence relation. This is reflected in the basic definitions that follow but shows up almost nowhere else: once we agree that we are, in effect, always working up to equivalence, there is no further need to explicitly refer to that equivalence.
Definition 3.3. A set is a third order object \( X \) together with a third order object \( \equiv \) contained in \( X^2 = X \times X \) which is reflexive, symmetric, and transitive. We call \( \equiv \) the identity on \( X \). Notationally, we will typically suppress \( \equiv \) and refer to \( X \) as the set. A subset of a set \( X \) is a third order object \( Y \) that is contained in \( X \) and satisfies \( X \in Y, X \equiv Y \Rightarrow Y \in Y \), together with the identity \( (\equiv \cap Y^2) \). We write \( Y \subseteq X \). A quotient of a set \( X \) with identity \( \equiv \) is the same third order object \( X \) together with an identity \( \equiv' \) that contains \( \equiv \). The product of two sets \( X \) and \( Y \) is the third order object \( X \times Y \) together with the identity defined by \( \langle X, Y \rangle \equiv \langle X', Y' \rangle \) if and only if \( X \equiv X' \) and \( Y \equiv Y' \).

A relation on a set \( X \) is a subset of \( X^2 \). A function from a set \( X \) to a set \( Y \) is a subset \( f \) of \( X \times Y \) such that for every \( X \in X \) there exists exactly one (up to identity) \( Y \in Y \) with \( \langle X, Y \rangle \in f \).

The next proposition should help orient the reader to the kind of thinking involved in reasoning in CM.

Proposition 3.4. Let \( X \) and \( Y \) be sets, let \( f : X \to Y \) be a function, and let \( Y_0 \) be a subset of \( Y \). Then \( f^{-1}(Y_0) \) exists and is a subset of \( X \).

Proof. We show that the condition \( \{f(X) \in Y_0\} \) satisfies excluded middle. By third order comprehension this implies that \( f^{-1}(Y_0) = \{X \in X : f(X) \in Y_0\} \) exists; the fact that it is compatible with the identity on \( X \) is an easy consequence of the fact that \( f \) is a subset of \( X \times Y \) together with the definition of the identity on \( X \times Y \).

To verify the claim, let \( X \in X \). Then by the definition of a function there exists \( Y \in Y \), unique up to identity, such that \( \langle X, Y \rangle \in f \). Since we assume excluded middle for arithmetical formulas, we have \( (Y \in Y_0) \lor (Y \not\in Y_0) \), and since \( Y_0 \) is a subset of \( Y \) this truth value does not depend on the choice of \( Y \) up to identity. Therefore \( f(X) \in Y_0 \) satisfies excluded middle. \( \square \)

In contrast, the push-forward \( f(X_0) \) need not exist in general because we have no way to effectively test whether a given \( Y \in Y \) belongs to the image. The image of \( X_0 \) is continually expanding as new second order objects appear, and given \( Y \in Y \) we may not be able to predict whether some future \( X \in X \) will map to \( Y \). (Inverse images also expand, but given \( X \in X \) the fact that \( f \) is a function means that we can construct \( f(X) \) and we can then immediately check whether \( f(X) \) belongs to \( Y_0 \). Further expansion of \( Y_0 \) cannot affect this result.)

However, if \( X_0 \) is countable (see Definition 3.8 below) then we can use numerical omniscience to check whether \( Y \) belongs to \( f(X_0) \), so images of countable sets do always exist (Proposition 3.9).

3.2. \( \hat{Z} \) and \( \hat{Q} \). We encode the integers using a sign bit (0 for + and 1 for −):

Definition 3.5. Let \( \hat{Z} \) be a second order constant denoting the set of all ordered pairs of natural numbers \( (a, b) \) such that either \( a = b = 0 \) or \( a > 0 \) and \( b = 0 \) or 1.

We define addition in \( \hat{Z} \) by letting \( (a, b) + (a', b') \) be

\[
\begin{align*}
(a + a', b) \text{ if } b = b' \\
(a - a', 0) \text{ if } b = 0, b' = 1, \text{ and } a \geq a' \\
(a' - a, 1) \text{ if } b = 0, b' = 1, \text{ and } a < a' \\
(a - a', 1) \text{ if } b = 1, b' = 0, \text{ and } a > a' \\
(a' - a, 0) \text{ if } b = 1, b' = 0, \text{ and } a \leq a'.
\end{align*}
\]

The product and the order relation are defined by cases in the same way.
The existence of \( \hat{Z} \) is provable in CM by arithmetical comprehension. Basic properties of \( \hat{Z} \) as an ordered ring are straightforwardly provable as facts of first order arithmetic.

We define the rationals as fractions in lowest terms.

**Definition 3.6.** Let \( \tilde{Q} \) be a second order constant denoting the set of all ordered pairs \( (a, b) \) with \( a, b \in \hat{Z} \) relatively prime and \( b \) positive.

Here “relatively prime” and “positive” mean with respect to the product and order on \( \hat{Z} \). \( \tilde{Q} \) exists by arithmetical comprehension.

We define order, addition, multiplication, and division (with nonzero denominator) in \( \tilde{Q} \) in the usual way. All these definitions are arithmetical and basic properties of \( \tilde{Q} \) as an ordered field are straightforwardly provable as facts of first order arithmetic. (For more details on the material of this section, with inessentially different definitions, see Section II.4 of [10].)

**3.3. The real line.** We now define the real line in terms of Dedekind cuts of \( \tilde{Q} \).

**Definition 3.7.** Let \( \mathbf{R} \) be a third order constant satisfying \( X \in \mathbf{R} \) if and only if \( X \subseteq \tilde{Q} \), \( \emptyset \neq X \neq \tilde{Q} \), \( X \) has no greatest element, and

\[
(p \in X, q \in \tilde{Q}, q < p) \Rightarrow q \in X
\]

(where \( < \) is the order relation defined on \( \tilde{Q} \)). We equip \( \mathbf{R} \) with the trivial identity \( x \equiv y \Leftrightarrow x = y \).

\( \mathbf{R} \) contains canonical copies of \( \hat{N}, \hat{Z}, \) and \( \tilde{Q} \), which we denote \( N, Z, \) and \( Q \). For instance, \( X \in Q \) holds if and only if there exists \( p \in \tilde{Q} \) such that \( q \in X \Leftrightarrow q < p \).

Note that by arithmetical comprehension, for any \( X \subseteq \tilde{Q} \) there exists \( X \subseteq Q \) containing the corresponding elements, and vice versa. Similar statements hold for \( N \) and \( Z \). Thus, we can effectively identify \( N \) with \( \hat{N} \), \( Z \) with \( \hat{Z} \), and \( Q \) with \( \tilde{Q} \), and we will generally do so without comment.

Now that we have a third order version of \( N \), we can make the following definitions:

**Definition 3.8.** A set \( X \) is **countable** if there exists a surjective function \( f \) from \( N \) to \( X \). A **sequence of subsets of \( X \)** is a subset \( Y \) of \( N \times X \); we write

\[
Y_{(n)} = \{ X : (n, X) \in Y \}
\]

(and set \( X \equiv Y \) in \( Y_{(n)} \) if \( (n, X) \equiv (n, Y) \) in \( Y \)). The **product** \( \prod Y_{(n)} \) of a sequence \( Y \) of subsets of a set is the set of all sequences \( Y \) of second order objects such that \( Y_{(n)} \in Y_{(n)}^{(n)} \) for all \( n \), with \( Y \equiv Y' \) if \( Y_{(n)} \equiv Y'_{(n)} \) for all \( n \).

For any \( n \in \hat{N} \) and \( X \subseteq \hat{N} \) the ordered pair \( (n, X) \) exists by arithmetical comprehension. It follows that if \( Y \) is a sequence of subsets of \( X \) then \( X \in Y_{(n)}^{(n)} \), for any \( n \) and \( X \). From this one easily sees (using third order comprehension) that \( \bigcup Y_{(n)} \) and \( \bigcap Y_{(n)} \) exist.

**Proposition 3.9.** Let \( f : X \to Y \) be a function and let \( X_0 \subseteq X \) be countable. Then \( f(X_0) \) exists and is a subset of \( Y \).

**Proof.** We can verify excluded middle for the condition \( (\exists X)(X \in X \land Y = f(X)) \) by numerical omniscience. □
The order relation and the algebraic operations on \( \mathbb{R} \) are easily (arithmetically) defined. Also, if \((X_n)\) is a sequence of reals that is bounded above then \( \bigcup X_n \in \mathbb{R} \) is its least upper bound. So \( \mathbb{R} \) is, in this sense, a sequentially complete ordered field. The converse is also true:

**Theorem 3.10.** \( \mathbb{R} \) is a sequentially complete ordered field. Every sequentially complete ordered field is isomorphic to \( \mathbb{R} \).

The second statement of the theorem is proven in the usual way: given any sequentially complete ordered field \( F \), first isolate the countable subfield \( F \) generated by \( 1 \); then establish an isomorphism between \( F \) and \( \mathbb{Q} \); and finally show that \( F \) is dense in \( F \) and use this to define the desired isomorphism between \( F \) and \( \mathbb{R} \). The novelty here is that the existence of \( F \) requires proof. The result we need is stated in the following lemma.

**Lemma 3.11.** Any countable subset of a field \( F \) generates a countable subfield of \( F \).

**Proof.** Let \( S \subseteq F \) be countable. The lemma does not follow from arithmetical comprehension; we must prove directly that the condition “\( X \) is in the subfield generated by \( S \)” is expressible in a way that satisfies the law of excluded middle, so that third order comprehension can be used. Informally, we expand the above to “there is a word in elements of \( S \) which when evaluated in \( F \) produces \( X \), up to identity.” Next we use induction on word length to check that for any word \( w \) the statement “evaluating \( w \) in \( F \) produces \( X \), up to identity” satisfies excluded middle. Enumerating the words as \((w_n)\), second order comprehension then verifies the existence of a sequence \((X_n)\) such that \( X_n \) is the result of evaluating \( w_n \) in \( F \) (or \( X_n = 0 \) if \( w_n \) does not evaluate). Finally, numerical omniscience implies the law of excluded middle for the assertion \((\exists n)(X \equiv X_n)\). Together with third order comprehension, this shows that the subfield of \( F \) generated by \( S \) exists. \( \square \)

The same argument will apply in more general algebraic settings to show the existence of countably generated subobjects.

The preceding proof is a good illustration of the technique of combining numerical omniscience with comprehension for existence results. This will be used repeatedly in the sequel. The general principle is: if we can test whether \( \phi(X) \) holds in countably many steps then \( \{X : \phi(X)\} \) exists.

**Theorem 3.12.** \( \mathbb{R} \) is Cauchy complete.

**Proof.** This follows from the formula

\[
\liminf X_n = \bigcup_{n \in \mathbb{N}} \bigcap_{k \geq n} (X_k - 1/n),
\]

where \( X_k - 1/n \in \mathbb{R} \) is taken literally as a Dedekind cut. \( \square \)

**Proposition 3.13.** Let \( X \subseteq \mathbb{R} \). Then any sequence of functions \( f_n : X \to \mathbb{R} \) which converges pointwise has a pointwise limit \( f : X \to \mathbb{R} \).

**Proof.** By arithmetical comprehension. \( \square \)

Polynomial functions from \( \mathbb{R} \) to itself are arithmetically definable. It easily follows that all of the standard continuous functions from real analysis (\( \sin x, \cos x, \))
Proof. \( e^x, \ln x, \text{etc.} \) may be defined. Standard discontinuous functions such as the Heaviside step function, the comb function, and the characteristic function of the Cantor set are also straightforwardly definable.

3.4. **Topology in \( \mathbb{R} \).** Let \( Q^+ = \{ p \in Q : p > 0 \} \) and \( \mathbb{R}^+ = \{ x \in \mathbb{R} : x > 0 \} \). (We will now start using lowercase letters to denote elements of \( \mathbb{R} \).)

**Definition 3.14.** A subset \( U \subseteq \mathbb{R} \) is open if there is a set \( P \subseteq Q \times Q^+ \) such that \( x \in U \) if and only if \( |p - x| < r \) for some \( \langle p, r \rangle \in P \). We call \( P \) a witness for \( U \).

An open ball in \( \mathbb{R} \) is a set of the form
\[
\text{ball}_r(x) = \{ y \in \mathbb{R} : |x - y| < r \}
\]
for some \( \langle x, r \rangle \in \mathbb{R} \times \mathbb{R}^+ \).

**Proposition 3.15.** (a) Open balls are open.

(b) The union of any sequence of open subsets of \( \mathbb{R} \) is open.

(c) The intersection of any finitely many open subsets of \( \mathbb{R} \) is open.

**Proof.** Parts (a) and (c) are routine. In part (b), given a sequence \( (U_n) \) of open sets we need to use dependent choice to select a sequence of witnesses \( (P_n) \); the union of this sequence is then a witness for \( \bigcup U_n \). \( \square \)

**Definition 3.16.** The closure of a countable set \( C \subseteq \mathbb{R} \) is the set
\[
\{ x \in \mathbb{R} : \text{for every } n \text{ there exists } y \in C \text{ such that } |x - y| < 1/n \}.
\]

A subset \( C \) of \( \mathbb{R} \) is closed if it is the closure of a countable set \( C \subseteq \mathbb{R} \); we call \( C \) a witness for \( C \).

In the definition of closure, observe that for any \( x \) the existence, for every \( n \in \mathbb{N} \), of a point \( y \in C \) such that \( |x - y| < 1/n \) satisfies the law of excluded middle by numerical omniscience, because \( \mathbb{N} \) and \( C \) are countable. Thus the closure of \( C \) exists by third order comprehension. This technique was already introduced in the proof of Lemma 3.11 and from now on we will use it without comment.

**Proposition 3.17.** Closed subsets of \( \mathbb{R} \) are sequentially closed (i.e., contain all limits of Cauchy sequences).

**Proof.** Let \( C \) be the closure of a countable set \( C \) and enumerate the elements of \( C \) as \( (x_k) \). Now let \( (y_n) \) be a Cauchy sequence in \( C \). For each \( n \) let \( k_n \) be the smallest index such that \( |y_n - x_{k_n}| < 1/n \); then the sequence \( (x_{k_n}) \) is Cauchy and converges to the same limit as \( (y_n) \), hence that limit belongs to \( C \). \( \square \)

**Theorem 3.18.** A subset of \( \mathbb{R} \) is closed if and only if its complement is open.

**Proof.** The forward direction is easy: given a witness \( C \) for a closed set, the set of pairs \( \langle p, r \rangle \in Q \times Q^+ \) such that \( |p - x| \geq r \) for all \( x \in C \) is a witness for the complementary open set. For the reverse direction, given a witness \( P \) for an open set \( U \) we construct a witness \( C \) for the complementary closed set \( C \) as follows. First let \( C_0 \) be the set of rationals in \( \mathbb{R} - U \), i.e., the set of \( p' \in Q \) such that \( |p' - p| \geq r \) for all \( \langle p, r \rangle \in P \). Then for each rational \( p \in U \) let \( S_p \) be the set of \( p' \in Q \) such that there is a finite sequence of elements \( \langle p_i, r_i \rangle \in P, 1 \leq i \leq n, \) with \( p' \in \text{ball}_{r_i}(p_i) \), \( p_i \in \text{ball}_r(p_n) \), and \( \text{ball}_{r_i}(p_i) \cap \text{ball}_{r_{i+1}}(p_{i+1}) \neq \emptyset \) for \( 1 \leq i < n \). If \( S_p \) is bounded above then let \( x_p \) be its least upper bound; thus \( x_p \) is the smallest element of \( C \) that is greater than \( p \). Finally, let \( C_1 \) be the set of all such elements \( x_p \) and let \( C = C_0 \cup C_1 \). One easily checks that \( C \) is the closure of \( C \). \( \square \)
Corollary 3.19. (a) The intersection of any sequence of closed subsets of $\mathbb{R}$ is closed.
(b) The union of any finitely many closed subsets of $\mathbb{R}$ is closed.

Proposition 3.20. Let $X \subseteq \mathbb{R}$ and let $X \subseteq X$ be countable. Then every open ball about any point in $X$ intersects $X$ if and only if $X$ is contained in the closure of $X$.

Definition 3.21. We say that $X \subseteq \mathbb{R}$ is separable if it has a countable subset $X$ such that either of the two equivalent conditions in Proposition 3.20 is satisfied. We say that $X$ is dense in $X$.

Lemma 3.22. Every open subset of $\mathbb{R}$ is separable, as is every closed subset of $\mathbb{R}$.

Proposition 3.23. Let $X \subseteq \mathbb{R}$ be a separable subset and let $Y = \mathbb{R} - X$.
(a) $X$ is closed if and only if it is closed under limits of Cauchy sequences.
(b) $Y$ is open if and only if for every $x \in Y$ there exists $r > 0$ such that ball$(r)(x) \subseteq Y$.

Proof. (a) The forward direction was Proposition 3.17. For the reverse direction suppose $X$ is sequentially closed, let $C$ be a countable dense subset of $X$, and let $C$ be the closure of $C$. Then $X$ is contained in $C$ by density and $X$ contains $C$ since it is closed under Cauchy convergence. So $X = C$ and hence $X$ is closed.
(b) The forward direction is trivial; for the reverse direction let $C$ be a countable dense subset of $X = \mathbb{R} - Y$ and verify that $Y$ is disjoint from the closure of $C$. It follows that $X$ is the closure of $C$, hence $X$ is closed, hence $Y$ is open.

Definition 3.24. $K \subseteq \mathbb{R}$ is compact if every sequence of open sets that covers $K$ has a finite subcover.

Theorem 3.25. Let $K$ be a separable subset of $\mathbb{R}$. Then the following are equivalent:
(i) $K$ is closed and bounded;
(ii) $K$ is compact;
(iii) $K$ is bounded and contains the limits of all of its Cauchy sequences;
(iv) every sequence in $K$ has a subsequence which converges to a limit in $K$.

Proof. The proofs of (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are standard and do not use separability. For (iv) $\Rightarrow$ (i), suppose every sequence has a convergent subsequence and let $C$ be a countable dense subset of $K$. Then by numerical omniscience the assertion “$C$ is bounded” satisfies excluded middle, so a proof by contradiction shows that $C$, and hence $K$, must be bounded. The fact that $K$ is closed follows from Proposition 3.23 (a).

Definition 3.26. Let $X \subseteq \mathbb{R}$. We say that a function $f : X \to \mathbb{R}$ is continuous if the inverse image of any open set in $\mathbb{R}$ is the intersection of an open subset of $\mathbb{R}$ with $X$.

Theorem 3.27. Suppose $X \subseteq \mathbb{R}$ is separable and let $f : X \to \mathbb{R}$ be a function. Then the following are equivalent:
(i) $f$ is continuous;
(ii) the inverse image of every closed set in $\mathbb{R}$ is the intersection of a closed set in $\mathbb{R}$ with $X$;
(iii) for any countable set \( C \subseteq X \) with closure \( \overline{C} \) we have \( x \in \overline{C} \cap X \Rightarrow f(x) \in f(C) \);
(iv) \( f \) preserves convergence of sequences;
(v) for every \( x \in X \) and every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that \( d(x, y) < \delta \) implies \( d(f(x), f(y)) < \epsilon \).

**Proof.** The proofs of (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii) \( \Rightarrow \) (iv) are standard and do not use separability. For (iv) \( \Rightarrow \) (v), let \( X \) be a countable dense subset of \( X \). We first show that the \( \epsilon \)-\( \delta \) condition holds for \( x \) and \( y \) in \( X \) and with \( \epsilon \) and \( \delta \) restricted to rational values. If not then we can find \( x \in X \) and a sequence \( (y_n) \subseteq X \) such that \( y_n \to x \) but \( f(y_n) \neq f(x) \), contradicting (iv). Since \( X \) is countable and \( \epsilon \) and \( \delta \) are restricted to the rationals we have excluded middle, so we conclude that the \( \epsilon \)-\( \delta \) condition does hold for \( x \) in \( X \). Since \( X \) is dense in \( X \) and \( f \) preserves convergence of sequences, (v) follows easily.

For (v) \( \Rightarrow \) (i), again let \( X \) be a countable dense subset of \( X \). Also let \( U \subseteq \mathbb{R} \) be an open set with witness \( P \). Let \( P' \) be the set of pairs \( (p', r') \in \mathbb{Q} \times \mathbb{Q}^+ \) such that \( f(X \cap \text{ball}_{r'}(p')) \subseteq \text{ball}_{r-r}(p) \) for some \( (p, r) \in P \) and some \( \epsilon > 0 \). It is then straightforward to verify that \( P' \) is a witness for an open set \( U' \) which satisfies \( U' \cap X = f^{-1}(U) \).

**Theorem 3.28.** The sum and product of two continuous functions from \( X \subseteq \mathbb{R} \) to \( \mathbb{R} \) are continuous. The composition of two continuous functions, if defined, is continuous.

**Theorem 3.29.** Let \( X \subseteq \mathbb{R} \) and let \( f : X \to \mathbb{R} \) be continuous. If \( X \) is separable and compact then \( f \) is bounded and achieves its maximum and minimum. If \( X \) contains the interval \([a, b]\) then \( f \) attains every value between \( f(a) \) and \( f(b) \).

**Proof.** For the first statement, let \( X = (x_n) \) be a countable dense subset of \( X \), pass to a subsequence \( (x_{n_k}) \) such that \( f(x_{n_k}) \) converges to \( \sup f(x_n) \) or \( \inf f(x_n) \) (possibly \( \pm \infty \)), then pass to a subsequence which converges in \( X \), and finally apply Theorem 3.27 (iv) (showing \( \pm \infty \) are not possible maximum and minimum values).

For the second statement, suppose \( f(a) \neq f(b) \) and let \( z \) be any value strictly between \( f(a) \) and \( f(b) \). Consider the disjoint open sets \( U = f^{-1}([z, \infty)) \) and \( V = f^{-1}((\infty, z]) \). Without loss of generality suppose \( a \in U \) and \( b \in V \). Let \( Y \) be the set of rationals \( p > a \) such that every rational in \( [a, p] \) lies in \( U \); then \( x = \sup Y \) cannot lie in \( U \) (since \( U \) is open) or in \( V \) (since \( V \) is open and disjoint from \( U \)). Also \( a < x < b \), so that \( x \in X \). Since \( f(x) \not\in \mathbb{R} - \{z\} \) we must have \( f(x) = z \).

### 3.5. Metric spaces.

**Definition 3.30.** A metric space is a set \( X \) together with a function \( d : X \times X \to [0, \infty) \) which satisfies the usual metric axioms. It is complete if every Cauchy sequence converges. A subset is dense if it intersects every open ball \( \text{ball}_r(x) = \{y \in X : d(x, y) < r\} \) and a space is separable if it contains a countable dense subset.

**Proposition 3.31.** Every metric space densely embeds in a complete metric space. This embedding is unique up to an isometric isomorphism fixing the original space.

**Proof.** The proof is essentially the standard one. The set of Cauchy sequences in a metric space exists by third order comprehension, as does the standard equivalence
relation on Cauchy sequences. We can then use this equivalence relation as the identity in the set of Cauchy sequences. The remainder of the proof is standard. □

This is our first serious use of the convention that identity in sets is determined by equivalence relations.

A subtle point: although there is a function $f$ that embeds $X$ into its completion, the image $f(X)$ need not exist as a set. We noted earlier (at the end of Section 3.1) that images of sets do not exist in general, and this is true even in the present rather special situation.

**Definition 3.32.** Let $X$ be a metric space. A subset $U \subseteq X$ is open if there is a countable set $P \subseteq X \times \mathbb{R}^+$ such that $U = \bigcup_{(x,r) \in P} \text{ball}_r(x)$. A subset $C \subseteq X$ is closed if its complement is open. We call $P$ a witness both for the open set $U$ and the complementary closed set $X - U$.

**Theorem 3.33.** (a) Open balls are open.
(b) The union of any sequence of open subsets of a metric space is open.
(c) The intersection of any finitely many open subsets of a separable metric space is open.

Proof. The only subtlety occurs in part (c), which comes down to showing that the intersection of any two open balls is open. We obtain a witness for $\text{ball}_r(x) \cap \text{ball}_{r'}(x')$ by letting $X$ be a countable dense subset and taking the set of pairs $\langle y, s \rangle$ such that $y \in X$, $s \in \mathbb{Q}^+$, $d(x,y) + s \leq r$, and $d(x',y) + s \leq r'$.

**Corollary 3.34.** (a) The intersection of any sequence of closed subsets of a metric space is closed.
(b) The union of any finitely many closed subsets of a separable metric space is closed.

**Definition 3.35.** The closure of a countable set $C \subseteq X$ is the set of all limits of convergent sequences in $C$. $X$ is totally bounded if for every $n \in \mathbb{N}$ there is a finite set $S \subseteq X$ such that every $x \in X$ satisfies $d(x, s) < 1/n$ for some $s \in S$. $X$ is compact if every sequence of closed sets, any finitely many of which have nonempty intersection, has nonempty intersection. $X$ is boundedly compact if every closed ball $\text{ball}_1(x) = \{y \in X : d(x,y) \leq 1\}$ is compact.

(Closures exist by third order comprehension and a double application of numerical omniscience: first we use it to check that for any $x \in X$ and any $n \in \mathbb{N}$ the condition $\text{ball}_{1/n}(x) \cap C \neq \emptyset$ satisfies excluded middle, then we use it again to check that the condition $(\forall n)(\text{ball}_{1/n}(x) \cap C \neq \emptyset)$ satisfies excluded middle.)

**Theorem 3.36.** Let $X$ be a separable metric space. Then the following are equivalent:
(i) $X$ is compact;
(ii) $X$ is complete and totally bounded;
(iii) every sequence in $X$ has a convergent subsequence.

Proof. (i) $\Rightarrow$ (ii): Let $X$ be a countable dense subset of $X$. Suppose $X$ is compact and let $(x_n)$ be a Cauchy sequence in $X$. For each $k$ let $n_k$ be the smallest natural number such that $d(x_n, x_{n_k}) \leq 1/k$ for all $n > n_k$. Then the set of pairs $(x, q)$ with $x \in X, q \in \mathbb{Q}^+$, and $d(x, x_{n_k}) > q + 1/k$ witnesses an open set $U_k$. Applying the compactness hypothesis to the sequence of complementary closed sets then produces
a limit for \((x_n)\). This shows that \(X\) is complete. To verify total boundedness, enumerate the elements of \(X\) (up to identity) as \((z_n)\) and observe that the assertion “for every \(k\) there exists \(n\) such that every \(z_i\) is within \(1/k\) of some \(z_j\) with \(1 \leq j \leq n\)” satisfies excluded middle. So suppose this statement fails. Then there exists \(k\) such that for every \(n\), some \(z_i\) satisfies \(d(z_i, z_j) \geq 1/k\) for \(1 \leq j \leq n\). Using dependent choice, we can then construct a sequence \((n_k)\) such that \(d(z_{n_k}, z_{n_j}) \geq 1/k\) for all \(i \neq j\). Finally, for each \(i\) the pairs \((x, q)\) with \(x \in X, q \in Q^+\), and \(q < d(x, z_{n_j}) - 1/2k\) for all \(j \geq i\) witness an open set \(U_i\), and the complementary closed sets falsify the compactness condition. Thus \(X\) must be totally bounded.

(ii) \(\Rightarrow\) (iii): Assume (ii) and let \((x_n)\) be any sequence in \(X\). By completeness it will suffice to show that \((x_n)\) has a Cauchy subsequence. Let \(S_1 \subseteq X\) be a finite set such that for all \(x \in X\) there exists \(s \in S_1\) with \(d(x, s) < 1/2\). Find \(s_1 \in S_1\) such that \(d(x_{n_1}, s_1) < 1/2\) for infinitely many \(n_1\), and let \(n_1\) be the smallest number such that \(d(x_{n_1}, s_1) < 1/2\). Then let \(S_2 \subseteq X\) be a finite set such that for all \(x \in X\) there exists \(s \in S_2\) with \(d(x, s) < 1/4\), find \(s_2 \in S_2\) such that \(d(s_1, s_2) < 1/2 + 1/4\) and \(d(x_{n_1}, s_2) < 1/4\) for infinitely many \(n_1\), and let \(n_2\) be the smallest number after \(n_1\) such that \(d(x_{n_2}, s_2) < 1/4\). Continue in this way, with \(d(s_k, s_{k+1}) < 2^{-k} + 2^{-k-1}\) and \(d(x_{n_k}, s_k) < 2^{-k}\). Then \((x_{n_k})\) is the desired Cauchy subsequence.

(iii) \(\Rightarrow\) (i): Assume (iii). Let \((C_n)\) be a sequence of closed subsets with the finite intersection property, and for each \(n\) choose a point \(x_n \in \bigcap_{k=1}^n C_k\). Then \((x_n)\) has a convergent subsequence, and the limit of this sequence belongs to every \(C_n\). \(\square\)

**Proposition 3.37.** If \(X\) is separable then the closure of any countable set is closed. If \(X\) is separable and boundedly compact then every closed set is separable.

*Proof.* The first statement is easy: let \(X\) be a countable dense subset of \(X\); then a witness for the closure of any countable set \(C\) is given by the pairs \((x, r) \in X \times Q^+\) such that \(d(x, y) \geq r\) for all \(y \in C\). For the second statement suppose \(X\) is also boundedly compact and let \(C \subseteq X\) be closed. We construct, for each \(x \in X\) and \(r \in Q^+\) such that \(\text{ball}_r(x)\) intersects \(C\), an element of \(\text{ball}_{r+1/n}(x)\) \(\cap C\). This produces a countable subset of \(C\) that is evidently dense in \(C\). To do this, fix a witness \(P\) for \(C\) and enumerate \(P\) as \(((x_n, r_n))\). Let \(R\) be the set of pairs \((x, r) \in X \times Q^+\) such that for any \(n\) there exists \(y \in X \cap \text{ball}_{r+1/n}(x)\) with \(d(x, y) \geq r_i - 1/n\) for \(1 \leq i \leq n\). This set exists by numerical omniscience. Observe that if \((x, r) \notin R\) then \(\text{ball}_r(x) \cap C = \emptyset\). But if \((x, r) \in R\) then we can find a sequence \((y_n)\) such that \(d(x, y_n) < r + 1/n\) and \(d(x_i, y_n) \geq r_i - 1/n\) for \(1 \leq i \leq n\). Letting \(y\) be a limit point of this sequence (using bounded compactness and Theorem 3.36 (iii)), we must have \(d(x, y) \geq r_i\) for all \(i\), i.e., \(y \in C\). By dependent choice we can select one such \(y\) for each pair \((x, r) \in R\); this is the desired countable dense subset of \(C\). \(\square\)

**Definition 3.38.** A function \(f : X \to Y\) between metric spaces is *continuous* if the inverse image of any open set in \(Y\) is open in \(X\). It is a *homeomorphism* if it is a bijection and its inverse is also continuous.

**Theorem 3.39.** Let \(X\) and \(Y\) be metric spaces, suppose \(X\) is separable, and let \(f : X \to Y\) be a function. Then the following are equivalent:

(i) \(f\) is continuous;

(ii) the inverse image of every closed set in \(Y\) is closed in \(X\);

(iii) for any countable set \(C \subseteq X\) with closure \(\overline{C}\) we have \(x \in \overline{C} \Rightarrow f(x) \in \overline{f(C)}\);
(iv) $f$ preserves convergence of sequences;
(v) for every $x \in X$ and every $\epsilon > 0$ there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $d(f(x), f(y)) < \epsilon$.

(The proof is a straightforward generalization of the proof of Theorem 3.27.)

**Proposition 3.40.** Let $X$ and $Y$ be metric spaces and suppose $X$ is compact.

(a) Every closed subset of $X$ is compact.

(b) If $X$ is separable and $f : X \to Y$ is continuous then $f(X)$ exists and is a separable compact subset of $Y$.

(c) If $X$ is separable and $f : X \to Y$ is a continuous bijection then it is a homeomorphism.

Proof. Part (a) is trivial since every closed subset of a closed subset of $X$ is closed in $X$. For part (b) let $X$ be a countable dense subset of $X$ and let $C$ be the closure of $f(X)$. Theorem 3.39 (iii) implies that $f$ maps every element of $X$ into $C$, and Theorem 3.36 (iii) plus Theorem 3.39 (iv) implies that every element of $C$ is in the image of $f$. So $f(X)$ exists and equals $C$. $C$ is clearly separable, and compactness follows easily from Theorem 3.39 (ii) (considering $f$ as a function from $X$ to $C$).

Part (c) is proven by combining parts (a) and (b) with both parts of Proposition 3.37, using the characterization of continuity of $f^{-1}$ in Theorem 3.39 (ii). (Y is separable because it is the closure of $f(X)$, as in part (b).) □

**Theorem 3.41.** The intersection of any sequence of open dense subsets of a separable complete metric space is dense.

(The proof is identical to the classical proof.)

3.6. **Topological spaces.** We introduce the notion of a family of subsets of a set.

**Definition 3.42.** A family of subsets of a set $X$ is a subset $T$ of $X \times T$ for some set $T$. For each $Y \in T$ we write $T_Y = \{ x \in X : (x, Y) \in T \}$. We say that $Y$ belongs to the family $T$ if $Y = T_Y$ for some $Y \in T$.

A topological space is a set $X$ together with a family of subsets $T$ of $X$ such that (i) $\emptyset$ and $X$ belong to $T$; (ii) the union of any sequence of sets that belong to $T$ belongs to $T$; and (iii) the intersection of any finitely many sets that belong to $T$ belongs to $T$. $T$ is a topology on $X$.

A subset of a topological space is open if it belongs to $T$ and closed if its complement belongs to $T$.

**Definition 3.43.** Let $X$ be a topological space with topology $T \subseteq X \times T$ and let $Y \subseteq X$. The relative topology $T'$ on $Y$ is the family $T' = T \cap (Y \times T)$.

It is easy to see that $T'$ is a topology on $Y$.

Next we indicate how topologies can be generated from bases.

**Proposition 3.44.** Let $X$ be a set and let $B \subseteq X \times B$ be a family of subsets of $X$ such that $\emptyset$ and $X$ belong to $B$ and the intersection of any two sets that belong to $B$ is the union of a sequence of sets that belong to $B$. Let $T$ be the set of $Y$ such that $Y_{(n)} \in B$ for all $n$ and let $T$ be the set of pairs $(x, Y)$ such that $x \in \bigcup_n B(Y_{(n)})$. Then $T$ is a topology on $X$.

**Definition 3.45.** The family $B$ in Proposition 3.44 is a base for the topology $T$. 

Proposition 3.46. Let $X$ be a separable metric space with countable dense subset $X$. Let $B = X \times Q^+$ and let $B \subseteq X \times B$ be the set of pairs $(y, (x, r))$ such that $d(x, y) < r$. Then $B$ is a base for a topology and the open sets are precisely those identified in Definition 3.32.

Definition 3.47. Let $(X_n)$ be a sequence of sets and let $(T^n)$ be a corresponding sequence of topologies. Let $B$ consist of all pairs $\langle m, Y \rangle$ such that $m \in N$ and $Y \in T^1 \times \cdots \times T^m$. The product topology on the product $\prod X_n$ is the topology generated by the base $B \subseteq (\prod X_n) \times B$ consisting of all pairs $\langle x, \langle m, Y \rangle \rangle$ such that $X_n(x) \in T^n(\langle m, Y \rangle)$ for all $n \leq m$.

Definition 3.48. A function between topological spaces is continuous if the inverse image of any open set is open.

Proposition 3.49. The composition of two continuous functions is continuous.

Proposition 3.50. Let $X$ be a topological space and let $(X_n)$ be a sequence of topological spaces. Then a function $f : X \to \prod X_n$ is continuous if and only if $\pi_n \circ f : X \to X_n$ is continuous for all $n$, where $\pi_n$ is the projection onto the $n$th coordinate.

Definition 3.51. A topological space is second countable if it has a countable base (i.e., $B$ is countable). A subset $C$ of a topological space is sequentially closed if the limit of any convergent sequence in $C$ belongs to $C$. The sequential closure of a countable set $C$ in a second countable space is the set of limit points of convergent sequences in $C$.

In the last part of this definition we need $C$ to be countable and the ambient space to be second countable so that the statement “$x$ is the limit of a convergent sequence in $C$” satisfies excluded middle. This statement will be true if and only if every basic open set that contains $x$ intersects $C$.

Proposition 3.52. Any second countable space is separable.

Proposition 3.53. In any topological space, any closed set is sequentially closed. In a second countable space, any separable sequentially closed set is closed.

Proof. The first statement is trivial. For the second, let $C$ be a separable sequentially closed set with countable dense subset $C$; we construct the complementary open set as the union of all basic open sets that do not intersect $C$. Since $C$ is countable the condition $U \cap C = \emptyset$ satisfies excluded middle, so this union exists. Checking that it is the complement of $C$ is straightforward. \qed

Corollary 3.54. In a second countable space the sequential closure of any countable set is closed.

Definition 3.55. A topological space is compact if the intersection of any sequence of closed sets, any finitely many of which have nonempty intersection, is nonempty. It is sequentially compact if every sequence has a convergent subsequence.

Proposition 3.56. Any sequentially compact space is compact. Any compact second countable space is sequentially compact.

Proof. The first assertion is easy: given a sequence of closed sets $C_n$ with the finite intersection property, and assuming sequential compactness, for each $n$ choose
$x_n \in C_1 \cap \cdots \cap C_n$ and then let $x$ be the limit of some convergent subsequence of $(x_n)$. It is easy to see that $x$ must belong to the intersection $\bigcap C_n$.

For the second assertion suppose $X$ is compact and second countable and let $(x_n)$ be a sequence in $X$. Then for each $n$ the sequential closure $C_n$ of the set $(x_k : k \geq n)$ is closed by Corollary 3.54. By compactness the intersection of these sets is nonempty, and any point in this intersection is easily seen (using second countability) to be the limit of some subsequence of $(x_n)$. □

**Theorem 3.57.** Let $(X_n)$ be a sequence of compact second countable spaces. Then $\prod X_n$ is compact and second countable.

**Proof.** The fact that the product of a sequence of second countable spaces is second countable follows easily from the definition of the product topology. For compactness, use the equivalence of compactness and sequential compactness and show that any sequence has a convergent subsequence by successively extracting subsequences that converge on the first $n$ coordinates and diagonalizing. □

### 3.7. Measure theory

Measure theory presents a greater challenge to formalization in CM because its usual development involves uncountable pathology in the form of, for example, the Borel hierarchy on the real line. However, the fact that every measurable subset of $\mathbb{R}$ is a $G_\delta$ set minus a null set strongly suggests that this kind of pathology is not essential to the theory. Every measurable set is nested $\sigma$-set and a $G_\delta$ set with null difference, which motivates the following definition.

**Definition 3.58.** A function on a family of subsets $M \subseteq X \times M$ of a set $X$ is a function $f$ with domain $M$ such that $M_{(Y)} = M_{(Z)}$ implies $f(Y) = f(Z)$. This allows us to define the value of $f$ on $M_Y$ to be $f(Y)$. We may write $f(M_{(Y)})$ for $f(Y)$.

A family of pairs of subsets of a set $X$ is a family of subsets $M$ of $X \times \{0, 1\}$. For each $Y \in M$ and $i = 0, 1$ we write $M_{(Y)} = \{x \in X : (x, i) \in M_{(Y)}\}$, and we also write $M_{(Y)} = (M_{(Y)}^0, M_{(Y)}^1)$. $M$ is a family of nested pairs of subsets if every pair $(Y_0, Y_1)$ in $M$ satisfies $Y_0 \subseteq Y_1$.

A compatible function on a family of nested pairs of subsets of $X$ is a function $\mu$ on a family $M$ of nested pairs of subsets of $X$ with the following property:

if $(Y_0, Y_1)$ and $(Z_0, Z_1)$ belong to $M$ and $Y_0 \cup Z_0 \subseteq Y_1 \cap Z_1$ then

$\mu((Y_0, Y_1)) = \mu((Z_0, Z_1))$.

Since $Y_0 \subseteq Y \subseteq Y_1$ and $Z_0 \subseteq Y \subseteq Z_1$ imply $Y_0 \cup Z_0 \subseteq Y_1 \cap Z_1$, the compatibility condition allows us to define the value of $\mu$ on $Y$ to be $\mu((Y_0, Y_1))$ for any subset $Y \subseteq X$ such that $Y_0 \subseteq Y \subseteq Y_1$. We may write $\mu(Y)$ for $\mu((Y_0, Y_1))$. We say that such a set $Y$ is measurable or $\mu$-measurable.

A measure on a set $X$ is a compatible function $\mu : M \to [0, \infty]$ on a family of nested pairs of subsets of $X$ such that

(i) $\emptyset$ is measurable and $\mu(\emptyset) = 0$;

(ii) if $Y$ is measurable then so is $X - Y$;

(iii) if each set in a sequence $(Y_n)$ is measurable then so is their union, and if the sets are disjoint then $\mu(\bigcup Y_n) = \sum \mu(Y_n)$.

The problem of constructing measures also requires a new technique. We cannot use Carathéodory’s method because it defines measurability using what would be
in our context a third order quantification. However, it is not hard to come up
with a more direct construction that also works. We consider only the case of finite
measures, but passing to $\sigma$-finite measures would be a simple matter of partitioning
into finite measure subspaces.

**Theorem 3.59.** Let $X$ be a set, let $\mathcal{M} \subseteq X \times \bar{M}$ be a nonempty family of subsets
of $X$ which is stable under finite unions and complements, and let $\hat{\mu} : \bar{M} \to [0, a]$ be a function on the family $\mathcal{M}$. Suppose that $\hat{\mu} (\emptyset) = 0$ and $\hat{\mu} (\bigcup X_n) = \sum \hat{\mu} (X_n)$ whenever $(X_n)$ is a disjoint sequence of sets that belong to the family whose union also belongs to the family. Then there is a measure $\mu$ on $X$ such that every set that belongs to the family $\mathcal{M}$ is measurable and $\mu$ agrees with $\hat{\mu}$ on every such set.

**Proof.** We merely indicate the construction of $\mu$. The verification that $\mu$ has the
desired properties is an exercise in measure theory and we omit it.

For any $Y$ and $Z$ in $\mathcal{M}$ define $d(Y, Z) = \hat{\mu} (\mathcal{M}(Y) \Delta \mathcal{M}(Z))$, where $\Delta$ denotes
symmetric difference. This is a pseudometric on $\mathcal{M}$. Then let $\mathcal{M}$ be the set of all
$Y$ such that $Y_{(n)}$ belongs to $\mathcal{M}$ for all $n$ and $d(Y_{(m)}, Y_{(n)}) \to 0$ as $m, n \to \infty$. We
define $\mathcal{M}$ by the prescription $\mathcal{M}(Y) = \lim\inf \mathcal{M}(Y_{(n)})$ and $\mathcal{M}(Y) = \lim\sup \mathcal{M}(Y_{(n)})$,
and we set $\mu(\mathcal{M}(Y), \mathcal{M}(Y)) = \lim \hat{\mu}(\mathcal{M}(Y_{(n)}))$. This completes the construction of $\mu$. \qed

**Definition 3.60.** The function $\hat{\mu}$ in Theorem 3.59 is a premeasure, and $\mu$ is the
measure generated by $\hat{\mu}$. A measure is separable if it is generated by a premeasure
defined on a countable family of subsets.

Theorem 3.59 allows us to construct Lebesgue measure in $[0, 1]^n$ in the usual
way, or in $\mathbb{R}^n$ by partitioning into cubes.

Integration can be defined using similar methods. The definition is framed in
terms of a generating premeasure but it is not hard to see that the integral does
not actually depend on the choice of premeasure.

**Definition 3.61.** Let $\mu$ be a measure generated by a premeasure $\hat{\mu}$. We say that
a function $f : X \to \mathbb{R}$ is simple if it is a finite linear combination of characteristic
functions of sets that belong to the family $\mathcal{M}$. We define the integral of a simple
function $f = \sum a_i \chi_{A_i}$ to be

$$\int f \, d\hat{\mu} = \sum a_i \hat{\mu} (A_i)$$

and we define the $L^1$ distance between two simple functions $f$ and $g$ to be

$$d(f, g) = \int |f - g| \, d\hat{\mu}.$$ 

A function $f : X \to \mathbb{R}$ is integrable if there is a sequence $(f_n)$ of simple functions,
Cauchy for $L^1$ distance, such that

$$\lim\inf f_n \leq f \leq \lim\sup f_n.$$ 

We then define its integral $\int f \, d\mu$ to be

$$\int f \, d\mu = \lim \int f_n \, d\hat{\mu}.$$

**Theorem 3.62.** The integral $\int f \, d\mu$ is well-defined.
Finally, we indicate how to get a version of the Radon-Nikodym theorem. The technique of sequential approximation is again crucial.

**Definition 3.63.** A signed measure on $X$ is a compatible function $\nu : \mathcal{M} \to \mathbb{R}$ on a family of nested pairs of subsets of $X$ that satisfies the same axioms as a measure. A signed measure $\nu$ is absolutely continuous with respect to a measure $\mu$ if every $\mu$-measurable set is $\nu$-measurable and $\mu(Y) = 0$ implies $\nu(Y) = 0$.

**Theorem 3.64.** Let $\mu$ be a separable finite measure on $X$ and let $\nu$ be a (finite) signed measure on $X$ that is absolutely continuous with respect to $\mu$. Then there is a $\mu$-integrable function $f : X \to \mathbb{R}$ such that

$$\nu(A) = \int f \cdot \chi_A \, d\mu$$

for every $\mu$-measurable set $A$.

**Proof.** Again we merely indicate the construction. First, by separability there is a generating premeasure defined on a countable algebra of sets. We can then find a sequence of finite partitions of $X$ by sets in the algebra, such that every finite partition of $X$ by sets in the algebra is refined by some member of the sequence. If the $n$th partition is $X = \bigcup_{j=1}^{k} A_j$ then we define

$$f_n = \sum_{j=1}^{k} \nu(A_j) \chi_{A_j}.$$ 

We then check that the sequence $(f_n)$ is Cauchy and that this implies that it converges absolutely on a set of full measure to a $\mu$-integrable function $f$. This completes the construction of $f$. \hfill $\Box$

### 3.8. Banach spaces

For simplicity we take the scalar field to be real; complex scalars do not carry any additional logical demands.

Our definition of Banach spaces is identical to the classical one. What is noteworthy here is that most of the classical examples require some sort of coding. But little $L^p$ spaces do not:

**Definition 3.65.** For $1 \leq p < \infty$ let $l^p$ be the set of all sequences $(a_n)$ of real numbers such that $\sum |a_n|^p < \infty$, with norm $\|(a_n)\|_p = \left( \sum |a_n|^p \right)^{1/p}$. Let $l^\infty$ be the set of all bounded sequences of real numbers, with norm $\|(a_n)\|_\infty = \sup |a_n|$.

Here the condition $\sum |a_n|^p < \infty$ satisfies excluded middle because it is equivalent to the condition “there exists $K > 0$ such that $\sum_{n=1}^{m} |a_n|^p \leq K$ for all $m$”. The norm itself exists because the sequence of partial sums can be constructed using dependent choice, and the supremum of that sequence can then be taken by Cauchy completeness of $\mathbb{R}$.

Spaces of the form $C(X)$ with $X$ a compact metric space cannot be directly represented in CM because each element is supposed to be a third order object (a function from $X$ into $\mathbb{R}$). However, this is not a serious problem because any continuous function is determined by its values on a dense subset.

**Definition 3.66.** Let $X$ be a separable compact metric space with countable dense subset $X$. We define $C(X)$ to be the set of all uniformly continuous functions from $X$ to $\mathbb{R}$. 
Literally, $C(X)$ is the set of bounded sequences $(a_n) \in l^\infty$ such that the map $x_n \mapsto a_n$ from $X$ to $\mathbb{R}$ is uniformly continuous, for some enumeration $(x_n)$ of $X$. $C(X)$ inherits its Banach space structure from $l^\infty$.

At the Banach space level there is no particular advantage to working with the functions themselves rather than their restrictions to a dense subset. However, we certainly want to be able to work with individual elements of $C(X)$ as continuous functions on $X$. This is easily seen to be possible:

**Proposition 3.67.** Let $X$ be a separable compact metric space with countable dense subset $X$. Then the restriction of any continuous function $f : X \to \mathbb{R}$ to $X$ defines a sequence $(a_n)$ in $C(X)$, and every sequence $(a_n)$ in $C(X)$ is the restriction of precisely one continuous function.

We use the Radon-Nikodym theorem (Theorem 3.64) to encode $L^p$ functions:

**Definition 3.68.** Let $X$ be a separable finite measure space with generating pre-measure $\tilde{\mu}$. We define $L^1(X)$ to be the set of all signed premeasures $\tilde{\nu}$ on the family $\tilde{\mathcal{M}}$ which are absolutely continuous with respect to $\tilde{\mu}$, i.e., for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$\sum |\tilde{\nu}(A_i)| \leq \epsilon \Rightarrow \sum |\tilde{\mu}(A_i)| \leq \delta$$

for any disjoint $A_1, \ldots, A_n$ in the algebra. $L^\infty(X)$ consists of the premeasures which satisfy the stronger condition that there exists $K \geq 0$ such that

$$\sum |\tilde{\nu}(A_i)| \leq K \cdot \sum \tilde{\mu}(A_i)$$

for any disjoint $A_1, \ldots, A_n$ in the algebra. We define $L^p(X)$ for $1 < p < \infty$ to be those premeasures in $L^1(X)$ the $p$th power of whose Radon-Nikodym derivative is bounded.

As for $C(X)$, elements of $L^p(X)$ are literally sequences of real numbers which become premeasures when composed with a bijection from $\tilde{\mathcal{M}}$ to $\mathcal{N}$. Again, the above definition could be extended to the $\sigma$-finite case by partitioning into finite measure subsets.

The following analog of Proposition 3.67 is an immediate consequence of Theorem 3.64.

**Proposition 3.69.** For $1 \leq p < \infty$ the elements of $L^p(X)$ correspond to functions on $X$, modulo alteration on a set of measure zero, the $p$th power of whose absolute value is integrable. The elements of $L^\infty(X)$ correspond to bounded integrable functions on $X$, modulo alteration on a set of measure zero.

In the $\sigma$-finite case we no longer have $L^p(X) \subseteq L^1(X)$, but we still have $L^p(X) \subseteq L^1_{loc}(X)$, so can adapt the above result to this case.

Next we discuss duality.

**Definition 3.70.** Let $E$ be a separable Banach space with countable dense subset $E$. We may assume that $E$ is a vector space over $Q$ (cf. Lemma 3.11). We define the dual Banach space $E'$ to be the set of bounded $Q$-linear maps from $E$ to $\mathbb{R}$. The norm on $E'$ is defined by $\|f\| = \sup\{|f(x)| : x \in E, \|x\| \leq 1\}$.

As before, the elements of $E'$ are modelled as sequences of real numbers.

**Proposition 3.71.** The restriction of any bounded linear functional on $E$ to $E$ defines an element of $E'$, and every element of $E'$ is the restriction of precisely one bounded linear functional on $E$. 
We can prove a version of the Hahn-Banach theorem:

**Theorem 3.72.** Let $E$ be a separable Banach space, let $E_0$ be a separable closed subspace, and let $f_0 : E_0 \to \mathbb{R}$ be a bounded linear functional on $E_0$. Then $f_0$ extends to a bounded linear functional $f$ on $E$ with $\|f\| = \|f_0\|$.

**Proof.** Since $E$ is separable, we can enumerate a dense subset $(x_n)$, and it will suffice to show that $f_0$ extends to $E_0 + \mathbb{R} \cdot x_1$; we can then recursively extend to the span of $E_0$ and $x_1, \ldots, x_n$, use dependent choice to extract a nested sequence of extensions, and amalgamate them.

The extension to $E_0 + \mathbb{R} x_1$ is effected just as in the classical proof. We need $E_0$ to be separable so that the classical inequality

$$\sup_{x \in E_0} (-\|f_0\| \|x_1 + x\| - f_0(x)) \leq f(x_1) \leq \inf_{x \in E_0} (\|f_0\| \|x_1 + x\| - f_0(x))$$

can be restricted to $x$ ranging over a countable dense subset of $E_0$, in order to ensure that the supremum and infimum exist. □

The same result holds, with the same proof, for extensions from separable subspaces of nonseparable spaces, but this requires the well-ordering $\prec$ of $CM^+$. (See Section 2.3.)

The weak* topology on the dual of a separable Banach space $E$ is defined in the usual way. Note that its restriction to the unit ball of $E'$ is second countable, and even metrizable.

**Theorem 3.73.** The closed unit ball of the dual of any separable Banach space is weak* compact.

**Proof.** We verify sequential compactness. This is enough by Proposition 3.56. To do this let $E$ be a countable dense $Q$-linear subspace of a separable Banach space $E$ and let $(f_n)$ be a sequence of bounded linear functionals on $E$, each of norm at most 1. Enumerating $E$ as $(x_n)$, we then successively extract subsequences of $(f_n)$ which converge on $x_1, \ldots, x_k$. Diagonalizing yields a subsequence $(f_{n_k})$ such that the sequence $(f_{n_k}(x_i))$ converges for every $i$. Thus every sequence has a weak* convergent subsequence. (It suffices to verify convergence on a dense set in $E$ since the sequence $(f_n)$ is bounded.) □

We close with a version of Goldstine's theorem. This is interesting because it is a basic theorem about the second dual, yet in general second duals, even of separable Banach spaces, cannot be constructed in CM. Separability of $E$ does not imply separability of $E'$, but we need $E'$ to be separable in order to construct $E''$.

Our inability to form second duals might appear to reveal a serious limitation in our ability to formalize standard functional analysis within CM. But the limitation is not severe because typical applications of $E''$ do not involve its Banach space structure. Rather, they have to do with the behavior of individual elements of $E''$, which are not excluded from CM. (Though as we mentioned just above, if $E'$ is nonseparable we would need to work in $CM^+$ to prove the existence of elements of $E'' - E$.)

Goldstine’s theorem is a good illustration of this phenomenon. Its classical statement is that the unit ball of $E$ is weak* dense in the unit ball of $E''$. But this really comes down to an assertion about weak* approximability of individual elements of $E''$ by elements of $E$. That version of the result can be stated and proven in CM.
Theorem 3.74. Let $E$ be a separable Banach space and let $\phi : E' \to \mathbb{R}$ be a bounded linear functional of norm at most 1. Then for any $f_1, \ldots, f_n$ in $E'$ and any $\epsilon > 0$ there exists $x$ in the unit ball of $E$ such that
\[ |\phi(f_i) - f_i(x)| < \epsilon \]
for $1 \leq i \leq n$.

Proof. In the statement of the theorem we have identified the linear functionals $f_i$ on $E$ with their representatives $\tilde{f}_i$ in $E'$. That is, $f_i$ is the restriction of $\tilde{f}_i$ to a countable dense $Q$-linear subspace $E$ of $E$. Now fix $f_1, \ldots, f_n$ and $\epsilon$; we claim that there exists $x \in E$ with the desired properties. Since $E$ is countable this assertion satisfies excluded middle, so we can prove it by contradiction.

Thus, suppose no $x \in E$ satisfies $\|x\| \leq 1$ and $|\phi(f_i) - f_i(x)| < \epsilon$ for all $i$. Consider the map $T : E \to \mathbb{R}^n$ defined by $T(x) = (f_1(x), \ldots, f_n(x))$. Then the closure $K = T([E]_1)$ is a convex subset of $\mathbb{R}^n$ (here $[E]_1$ denotes the unit ball of $E$) and it is separated from the point $\alpha = (\phi(f_1), \ldots, \phi(f_n))$ by a distance of at least $\epsilon$. So by a separation theorem for separable convex subsets of $\mathbb{R}^n$, which has easy elementary proofs, we can find a linear map $g : \mathbb{R}^n \to \mathbb{R}$ such that $g(\beta) \leq 1 < g(\alpha)$ for all $\beta \in K$. Finally, the map $g \circ T$ belongs to the unit ball of $E'$ but we have $\phi(g \circ T) = g(\alpha) > 1$ by linearity, which contradicts the assumption that $\phi$ has norm at most 1. This shows that the desired $x$ does exist. \qed

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[1] See [http://www.math.wustl.edu/~nweaver/conceptualism.html](http://www.math.wustl.edu/~nweaver/conceptualism.html)