Magic square and half-hypermultiplets in F-theory

Rinto Kuramochi, a Shun’ya Mizoguchi, a,b Taro Tani c

a Graduate University for Advanced Studies (Sokendai)
Tsukuba, Ibaraki, 305-0801, Japan
b Theory Center, Institute of Particle and Nuclear Studies, KEK
Tsukuba, Ibaraki, 305-0801, Japan
c National Institute of Technology, Kurume College,
Kurume, Fukuoka, 830-8555, Japan

E-mail: rinto@post.kek.jp, mizoguchi@post.kek.jp, tani@kurume-nct.ac.jp

Abstract: In six-dimensional F-theory/heterotic string theory, half-hypermultiplets arise only when they correspond to particular quaternionic Kähler symmetric spaces, which are mostly associated with the Freudenthal-Tits magic square. Motivated by the intriguing singularity structure previously found in such F-theory models with a gauge group SU(6), SO(12) or E7, we investigate, as the final magical example, an F-theory on an elliptic fibration over a Hirzebruch surface of the non-split I6 type, in which the unbroken gauge symmetry is supposed to be Sp(3). We find significant qualitative differences between the previous F-theory models associated with the magic square and the present case. We argue that the relevant half-hypermultiplets arise at the E6 points, where half-hypermultiplets 20 of SU(6) would have appeared in the split model. We also consider the problem on the non-local matter generation near the D6 point. After stating what the problem is, we explain why this is so by using the recent result that a split/non-split transition can be regarded as a conifold transition.
1 Introduction

F-theory [1–3] is a framework of nonperturbative compactifications of type IIB string theory containing general \((p,q)\)-7-branes. The nonperturbativeness of F-theory arises due to the nonlocality among the 7-branes and the strings, where the \(SL(2,\mathbb{Z})\) identification before and after a move of a string among 7-branes gives rise to open-string-like light pronged objects, string junctions. In the dual M-theory picture, they correspond to wrapped M2-branes around vanishing cycles. These objects account for the emergence of the exceptional gauge symmetry and matter in the spinor representation in a type II setup, which is one of the virtues of F-theory in the application to the phenomenological model building.

In F-theory, matter typically arises at the intersections of 7-branes, where the singularity of the gauge brane with gauge group \(H\) is “enhanced” to that labeled by some another higher-rank group \(G\) [2–6]. ¹ In generic cases, \(G\) is one rank higher than \(H\), and in six

¹The matter localization at the intersection of the spectral cover \(C\) and the zero section \(\sigma_{B_2}\) (in the 4D case) was originally shown in [7, 8] by using the Leray spectral sequence. It is precisely where the singularity gets enhanced on \(B_2\), though of course the spectral cover \(C\) cannot be regarded as the matter 7-brane itself as it intersects with the elliptic fiber. This coincidence was explained in [9, 10] in terms of the Mordell-Weil lattice of a rational elliptic surface [11].
dimensions the matter arising at the intersection is in most cases a hypermultiplet transforming as $G/(H \times U(1))$, which determines a homogeneous Kähler manifold. However, in some cases, matter emerging at the intersection is not a full hypermultiplet but a half-hypermultiplet. For example [4], when $(G, H)$ are $(E_6, SU(6))$, $(E_7, SO(12))$ or $(E_8, E_7)$, half-hypermultiplets in $20, 32$ or $56$ of the respective $H$ appear. They are all pseudo-real representations and correspond, not to homogeneous Kähler manifolds, but to quaternionic Kähler symmetric spaces known as Wolf spaces [12, 13] (see [14] for a review):

$$
\begin{align*}
\frac{E_6}{SU(6) \times SU(2)}, & \quad \frac{E_7}{SO(12) \times SU(2)}, & \quad \frac{E_8}{E_7 \times SU(2)}.
\end{align*}
$$

In [15], an explicit resolution of the codimension-two singularity was carried out for the first example $(G, H) = (E_6, SU(6))$. It was found that the codimension-two singularity was already resolved by blowing up the nearby codimension-one $A_5 = SU(6)$ singularities without any additional blow-up at that point, although the Kodaira fiber type right above the intersection point was $IV^*$, which would mean an $E_6$ singularity. The number of exceptional curves above the codimension-two point is the same as that of the codimension-one loci supporting a fiber of the type $I_6$. It was also found that the intersection diagram at the codimension-two point was different from that of the nearby codimension-one loci, explaining the generation of the half-hypermultiplet at that point. This type of resolution was called an “incomplete resolution” [15]. In [16], a similar analysis was performed for $(G, H) = (E_7, SO(12))$ and $(E_8, E_7)$ to find similar features.

We should note that all these enhancements are relevant in the applications to F-theory GUT model buildings. For instance, the enhancement $SU(6) \to E_6$ is the one at the (codimension-three) Yukawa Kähler point on the $5$ matter curve in the four-dimensional $SU(5)$ F-GUT model. Similarly, the enhancements $SO(12) \to E_7$ and $E_7 \to E_8$ are the ones at the Yukawa points on the $10$ and $27$ curves in the $SO(10)$ and $E_6$ F-GUT models, respectively. Also, the multiple (=higher-rank) enhancement $SU(5) \to E_7$ (or $E_8$) (which includes these special enhancements as intermediate steps) is relevant to the F-theory family unification scenario [17] aiming to implement the supersymmetric $E_7$ coset sigma model [18] in F-theory.

Incidentally, the three symmetric spaces (1.1) are precisely the ones obtained by taking a quotient of the groups of the entries of the Freudenthal-Tits magic square (Table 1). The relation between quaternionic Kähler manifolds and the magic square was noticed some time ago in [14]. Indeed, the $G$’s and $H$’s comprising the symmetric spaces in (1.1) are the groups of the Lie algebras listed in the bottom and the second bottom rows of the rightmost three columns in the table. Motivated by this observation, in this paper we focus on the final remaining column of the magic square and study the corresponding six-dimensional F-theory compactification on an elliptic CY3 over a Hirzebruch surface [2, 3]. We can indeed find in [4] a model with the gauge group $C_3 = Sp(3)$ yielding half-hypermultiplets in $F_4/(Sp(3) \times SU(2)) = 14'$ as a part of the massless matter: the non-split $I_6$ model.

One of our interests is what kind of singularity gives rise to the supermultiplet of chiral matter in this representation. The equation defining the non-split $I_6$ model is obtained by modifying that of the split $I_6$ model [4]. The latter gives the $SU(6)$ unbroken gauge
symmetry with matter fields in $20$, $15$ and $6$ where the singularity is enhanced from $A_5$ to $E_6$, $D_6$ and $A_6$ respectively. A $20$ is a half-hypermultiplet in the split case studied in [15]. We can obtain the equation for the non-split $I_6$ model by a certain change of the sections that characterize the equation of the split $I_6$ model. With this change, the local structures of the singularities at the $E_6$ and $A_6$ points remain intact, but only those at the $D_6$ points are affected, so we examine the singularity structure at the $D_6$ points in the non-split $I_6$ model.

The non-split models are known to have some puzzles regarding the generation of matter fields [4, 19–23]. The equation defining the non-split $I_6$ model is obtained by replacing the square of a particular section $h_{n+2-r}^2$ (see text for the definition) in the split $I_6$ equation with a non-square section $h_{2n+4-2r}$. This global non-factorization implies monodromy among the exceptional fibers, which is interpreted as a feature that causes the gauge group to reduce from the simply-laced $SU(6)$ to the non-simply-laced $Sp(3)$ [4]. However, there is a puzzle here: At each double zero locus of $h_{n+2-r}$ there appears a hypermultiplet in $15$ of $SU(6)$ in the split model. Therefore, the anomaly cancellation requires that the hypermultiplets in $15$ of $SU(6)$ at the double zeros should split in pairs according to the replacement of the section, but the $14$ (not $14'$ - see below) of $Sp(3)$, supposed to arise from the $15$ of $SU(6)$, is a real (not a pseudo-real) representation, which does not allow half-hypermultiplets. This is the first puzzle.

There is another curious feature about this non-split model: As in [15, 16], we consider a local equation which exhibits the singularity structure near a single zero locus of the section $h_{2n+4-2r}$. The resolution of the singularity turns out to be an “incomplete” resolution, meaning that the codimension-two “$D_6$” singularity is already resolved when the resolution of the codimension-one singularity is completed. However, the difference from the previous three magical examples is that the intersection matrix of the exceptional curves at the codimension-two $D_6$ point remains identical to that at a nearby point on the codimension-one singularity. Therefore, the configuration of the exceptional curves generated there does not indicate that any chiral matter field is localized there.

These puzzles require a new understanding of charged matter generation in the non-split model, other than wrapped branes around vanishing cycles [5] or string junctions ending on the intersections of 7-branes [6]. Very recently, it was shown [24] that the split/non-split transition in F-theory can be regarded as, except some exceptional cases, a conifold transition associated with the relevant conifold singularities. In this paper, we will use this fact to discuss how the necessary matter can emerge from the geometry of the non-split model. More precisely, since the non-split model corresponds to the “deformed side” of the conifold transition, there arise three-cycles instead of two cycles on the “resolved side”, which is the split model. We will argue what branes can give chiral matter field with the three-cycles.

On the other hand, as for the question of where the $14’s$ are generated, we argue that they just arise as the $Sp(3)$ decomposition of $20s$ of $SU(6)$ at the $E_6$ points, and not at

Note that this codimension is counted in the base space of the elliptic fibration, and not in the total space of the Calabi-Yau.
the $D_6$ points.

The organization of this paper is as follows: In section 2, we give a brief review of
the Freudenthal-Tits magic square and point out its relation to half-hypermultiplets in
F-theory. In section 3, we consider the global split and non-split $I_6$ models and examine
their matter spectra. In section 4, we perform a concrete blowing-up process of the “$D_6$”
singularity of the non-split $I_6$ local equation. In section 5, we introduce the recent result
of [24] and show how it is used to resolve the issue of non-local matter. The final section is
devoted to conclusions.

2 Magic square and half-hypermultiplets in F-theory

2.1 The Freudenthal-Tits magic square

A Freudenthal-Tits magic square is a four-by-four table whose entries are Lie algebras.
They are determined by specifying a pair of composition algebras $(A, B)$. When these
composition algebras are the ones over the real number field $\mathbb{R}$, they are either one of the
four division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$, or they are one of the “split” algebras of $\mathbb{C}$, $\mathbb{H}$ and
$\mathbb{O}$, which are non-compact analogues of the corresponding division algebras. In this case,
each entry of the magic square is some real form of a complex Lie algebra.

If $(A, B)$ are a pair of either of the four division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$, the magic
square consists of compact Lie algebras with definite signatures (Table 1), while if $(A, B)$
are chosen from the set of $\mathbb{R}$ and the three split algebras, the entries are all split real forms
of the same complexifications as those of the compact Lie algebras in the corresponding
cells. They typically arise (besides a few exceptions) as (Lie algebras of) duality groups or
hidden symmetries of dimensionally reduced maximally symmetric supergravities, bosonic
string or the NS-NS sector effective theory and pure gravities. Finally, if $A$ is a division
algebra and $B$ is a split algebra, the magic square comprises a special set of real forms of
exceptional Lie algebras arising as scalar manifolds of dimensional reductions of $D = 5$
“magical” supergravities [25–28].

The $(A, B)$ entry of the magic square always has the following structure:

$$\text{der } A \oplus \text{der } J^B \oplus (A_0 \otimes J_0^B), \quad (2.1)$$

where $\text{der } A$ and $\text{der } J^B$ are the Lie algebras of the automorphism groups of $A$ and $J^B$, respectively, and $A_0$ and $J_0^B$ denote their traceless parts.

For example, for the compact case $A, B = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ (Table 1), 3

$$\text{der } A = 0, 0, \text{su}(2), g_2, \quad (2.2)$$

$$\text{der } J^B = \text{so}(3), \text{su}(3), \text{sp}(3), f_4, \quad (2.3)$$

$$A_0 = 0, 0, 3, 7 \quad \text{of } \text{der } A, \quad (2.4)$$

$$J_0^B = 5, 8, 14, 26 \quad \text{of } \text{der } J^B. \quad (2.5)$$

---

3In this paper, we use the notations $\text{sp}(n)$ and $\text{Sp}(n)$ to denote the Lie algebra and the Lie group of the $C_n$ type Dynkin diagram.
Table 1. The Freudenthal-Tits magic square for $A, B$ being either of the four division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. They are all compact Lie algebras with definite signatures. If the division algebras are replaced by split composition algebras, the entries become different real forms with the same complexifications.

| $B \setminus A$ | $R$  | $C$  | $H$  | $O$  |
|-----------------|------|------|------|------|
| $R$             | $\mathfrak{so}(3)$ | $\mathfrak{su}(3)$ | $\mathfrak{sp}(3)$ | $\mathfrak{f}_4$ |
| $C$             | $\mathfrak{so}(3)$ | $\mathfrak{su}(3) \oplus \mathfrak{su}(3)$ | $\mathfrak{su}(6)$ | $\mathfrak{e}_6$ |
| $H$             | $\mathfrak{sp}(3)$ | $\mathfrak{su}(6)$ | $\mathfrak{so}(12)$ | $\mathfrak{e}_7$ |
| $O$             | $\mathfrak{f}_4$   | $\mathfrak{e}_6$   | $\mathfrak{e}_7$   | $\mathfrak{e}_8$   |

Then, for instance, $\mathfrak{e}_7$ allows a decomposition

$$E_7 \supset SU(2) \times F_4$$

$$133 = (3, 1) \oplus (1, 52) \oplus (3, 26)$$

for $A = H, B = O$, and also

$$E_7 \supset G_2 \times Sp(3)$$

$$133 = (14, 1) \oplus (1, 21) \oplus (7, 14)$$

for $A = O, B = H$. The other Lie algebras allow similar decompositions.

Remark. In this paper the word “split” is used in three different meanings:

1. This word is used for a “split” composition algebra, which is a noncompact version of $\mathbb{C}, \mathbb{H}$ or $\mathbb{O}$ with an indefinite bilinear form.

2. “Split” is also used for a “split” real form of a complex Lie algebra, which has, besides the Cartan subalgebra, an equal number of positive and negative generators with respect to the invariant bilinear form.

3. Finally, the word “split” appears in the classification of singularities or the fiber types of exceptional curves [4]. Singularities of the “split” type are the ones in which relevant exceptional curves factor globally so that they yield simply-laced gauge symmetries.

The first two are closely related in that split real forms of the item 2 arise in the magic square when the composition algebras are taken to be split ones in the sense of item 1. The third one is, however, a different notion from the two.

2.2 Half-hypermultiplets in F-theory

In [4], a detailed analysis was carried out on the matter spectra of six-dimensional F-theory compactifications on an elliptically fibered Calabi-Yau threefold over a Hirzebruch surface [2, 3] for various patterns of unbroken gauge groups. In particular, it was revealed that there were (essentially) four cases of unbroken gauge groups \(^4\) in which half-hypermultiplets arise as massless matter: the 32 of $SO(11)$. This is also a non-split model ($I_{2}^{ns}$), and this 32 is easily seen to arise at the $E_7$ point, where the corresponding split model ($I_{2}^{s}$) with the $SO(12)$ gauge symmetry also yields 32.

\(^4\)There is, in fact, one more example in [4] where half-hypermultiplets arise as massless matter: the 32 of $SO(11)$. This is also a non-split model ($I_{2}^{ns}$), and this 32 is easily seen to arise at the $E_7$ point, where the corresponding split model ($I_{2}^{s}$) with the $SO(12)$ gauge symmetry also yields 32.
Table 2. Three cases in which half-hypermultiplets appear as massless matter in six-dimensional F-theory on an elliptic CY3 over $F_n$ / heterotic string theory on K3 (quoted from Table 3 of [4]).

(rather than normal hypermultiplets) appeared as massless matter. They are listed in Table 2 and 3. These spectra can be confirmed either by the heterotic index calculation [29] or by the generalized Green-Schwarz mechanism using the divisor data of the Hirzebruch surface [31, 32]. They satisfy the anomaly free constraint for one of the $E_8$ factors with instanton number $12 + n$ [4]

$$n_H - n_V = 30n + 112. \quad (2.8)$$

Table 3. The massless matter spectrum of six-dimensional heterotic string theory on K3 with an unbroken $Sp(3)$ gauge symmetry. This is anomaly free, and also contains half-hypermultiplets.

As we can see, the representations $56, 32, 20$, together with $14'$ and $6$, to which the half-hypermultiplets belong, are precisely the ones of quaternionic Kähler manifolds (or “Wolf spaces”). All but the last $6$ are obtained by taking the Lie groups of the extreme bottom and the third rows of the magic square as the groups of the numerator and denominator of the homogeneous space. The denominator groups also always come with

$\text{tr}_R F^2 = \text{index}(R) \text{tr}_6 F^2$ and $\text{tr}_R F^4 = \text{tr}_6 \text{tr}_4 F^4 + \text{tr}_6 (\text{tr}_6 F^2)^2$. By using these data and assuming that the charged matter spectrum only contains $6, 14$ and $14'$, one can solve the equations of generalized GS mechanism on $F_n$ and obtain the unique solution given in Table 3.

---

| gauge group $H$ | fiber type $G$ | enhancement | matter rep. | multiplicity | homogeneous space |
|-----------------|----------------|-------------|-------------|--------------|------------------|
| $E_7$           | $III^{ss}$     | $E_8$       | $\frac{1}{2} 56$ | $n + 8$ | $E_8 \times SU(2)$ |
|                 |                | $E_7$       | $\frac{1}{2} 32$ | $n + 4$ | $E_6 \times SU(2)$ |
|                 |                | $D_7$       | $2 12$       | $n + 8$ | $SO(12) \times SU(2)$ |
|                 |                |             | $1$          | $2n + 18$ | $SO(12) \times SU(2) \times U(1)$ |
| $D_6$           | $I_2^{ss}$     | $E_7$       | $\frac{1}{2} 20$ | $r$      | $SU(6) \times SU(2)$ |
|                 |                | $D_7$       | $15$         | $n + 2 - r$ | $SU(6) \times SU(2) \times U(1)$ |
|                 |                | $A_6$       | $6$          | $2n + 16 + r$ | $SU(6) \times U(1)$ |
|                 |                |             | $1$          | $3n + 21 - r$ | $SU(6) \times U(1)$ |

| gauge group $H$ | representation | multiplicity |
|-----------------|----------------|--------------|
| $C_3$           | $\frac{1}{2} (14' + 6)$ | $r$           |
|                 | $14$           | $n + 1 - r$  |
|                 | $6$            | $2n + 16 + r$|
|                 | $1$            | $4n + 23 - 2r$|

---

5For $Sp(3)$, the dual heterotic gauge bundle is $SU(2) \times G_2$ since the maximal embedding is $E_8 \supset SU(2) \times G_2 \times Sp(3)$ (see e.g. [30] for the branching rules). The spectrum in Table 3 is obtained by distributing the $12 + n$ instantons as $(4 + r, 8 + n - r)$ in $(SU(2), G_2)$.

6For $Sp(3)$, the relevant indices of a representation $R$ for examining the generalized Green-Schwarz (GS) mechanism are given by $(\text{index}(R), x_R, y_R) = (8, 14, 3), (1, 1, 0), (4, -2, 3)$ and $(5, -7, 6)$ for $R = \text{Adj}, 6, 14$ and $14'$, respectively, where $\text{tr}_R F^2 = \text{index}(R) \text{tr}_6 F^2$ and $\text{tr}_R F^4 = \text{tr}_6 \text{tr}_4 F^4 + \text{tr}_6 (\text{tr}_6 F^2)^2$. By using these data and assuming that the charged matter spectrum only contains $6, 14$ and $14'$, one can solve the equations of generalized GS mechanism on $F_n$ and obtain the unique solution given in Table 3.
an $SU(2)$ factor in contrast to the case of ordinary hypermultiplets, where the denominator group comprises not an $SU(2)$ but a $U(1)$ factor. In the latter case, the symmetric space is a homogeneous Kähler manifold [17]. In the M-theory Coulomb branch analysis of codimension-two or higher singularities [33], the Weyl-group invariant phases of this $SU(2)$ were shown to correspond to the resolutions yielding half-hypermultiplets.

Let us summarize what is known so far, for the three simply-laced split examples of Table 2, about the resolutions of the codimension-two singularities that yield half-hypermultiplets. The resolutions of the third example were studied in [15], and the those of the first and second ones were worked out in [16]. The main relevant features are:\n
(i) As in [2, 3], let $z (z')$ be the affine coordinate of the $\mathbb{P}^1$ fiber ($\mathbb{P}^1$ base) of the Hirzebruch surface $F_n$, respectively. Suppose that we have a codimension-one singularity along the line $z = 0$ with the fiber type specified in the second column of Table 2. Non-singlet matter arises where the singularity is “enhanced” from $H$ to $G$, in the sense that the Kodaira fibers read off at right above that point have intersections specified by the Dynkin diagram of $G$. However, where the half-hypermultiplets appear, the codimension-two singularity is already resolved by blowing up the nearby codimension-one singularities. No additional blow-up at the codimension-two point is required, even though the singularity is “enhanced” there in the sense explained above. Such type of resolution is called an incomplete resolution [15].\n
(ii) In an incomplete resolution, the relevant section that vanishes at codimension two goes like $O(s)$, where $s$ is a local coordinate holomorphic in $z'$, and $s = 0$ is the codimension-two singularity. In this case, although the number of blow-ups required to resolve it is the same as that to resolve the nearby generic codimension-one singularities, the intersection matrix of the exceptional curves at $s = 0$ is not the same as the generic one determined by the Cartan matrix of $H$ (nor that of $G$), but turns out to be a curious non-Dynkin diagram with some nodes having self-intersections $-\frac{3}{2}$.

(iii) In the first three examples of Table 2 studied in [15] and [16], $\frac{3}{2}$ is the length square of the weight vector of the representations to which the half-hypermultiplets belong. It was confirmed that although the intersection matrix was not the (minus of the) Cartan matrix of $G$, the exceptional curves at $s = 0$ formed an extremal ray that could span all the weights of the relevant pseudo-real representation of the half-hypermultiplets.

(iv) In the first two examples, there arise several codimension-one singularities during the intermediate stages of the blow-up process, and there are several options in which singularity we blow up first, and which we do afterwards. Depending on the ordering of the blow-ups, one obtains different intersection diagrams of the exceptional curves at the codimension-two point $s = 0$ [16]. More specifically, the intersection diagram on every other row found in [33] can be obtained in this way, but not all of them.

---

7The local coordinate $s$ parametrizing the base $\mathbb{P}^1$ of $F_n$ will be denoted by $\omega$ in section 4 when we blow up the singularities.
Instead, when the relevant section vanishes like $O(s^2)$ at the codimension-two point, the singularity becomes stronger than the case above so that there arises an additional conifold singularity. A small resolution generates an extra exceptional fiber at that point so that it completes the proper Dynkin diagram of group $G$. This type of resolution is called a complete resolution [15].

3 Six-dimensional $Sp(3)$ global model

3.1 The non-split $I_6$ equation on $\mathbb{F}_n$

In this section we consider a six-dimensional F-theory compactification on an elliptic fibration over a Hirzebruch surface $\mathbb{F}_n$ in which the unbroken gauge symmetry reduces to $Sp(3)$. We work in the $dP_3$ fibration so that we focus on one of the two $E_8$’s of the heterotic dual.

As was shown in [4], the equation of this curve is the one that supports a $I_6$ Kodaira fiber of the non-split type at $z = 0$. A $I_6$ non-split curve may be obtained by replacing the relevant factorized section of a split $I_6$ curve with a non-factorized one. More specifically, consider Tate’s form of the equation describing the elliptic fibration:

$$-(y^2 + a_1 xy + a_3 y) + x^3 + a_2 x^2 + a_4 x + a_6 = 0.$$  \hspace{1cm} (3.1)

As in [2, 3], we use $z$ and $z'$ as the affine coordinates of the base and fiber $\mathbb{P}^1$’s of the Hirzebruch surface. The equation for the theory with the unbroken group $H = SU(6)$ can be obtained by specializing the sections as

$$a_1 = 2\sqrt{3}t_r h_{n-r+2},$$
$$a_2 = -3z t_r H_{n-r+4},$$
$$a_3 = 2\sqrt{3}z^2 u_{r+4} h_{n-r+2},$$
$$a_4 = z^3 (t_r f_{n-r+8} - 3u_{r+4} H_{n-r+4}) + f_8 z^4,$$
$$a_6 = z^5 u_{r+4} f_{n-r+8} + g_{12} z^6,$$ \hspace{1cm} (3.2)

where $t_r$, $h_{n-r+2}$, $H_{n-r+4}$, $u_{r+4}$ and $f_{n-r+8}$ (together with $f_8$ and $g_{12}$) are the sections of appropriate line bundles over the base $\mathbb{P}^1$ specified by their subscripts, which in this case denote nothing but the degrees of the polynomials in $z'$. It can be verified that the equation (3.1) with (3.2) correctly reproduces the anomaly-free heterotic massless spectrum for an unbroken $SU(6)$ gauge group with $SU(3) \times SU(2)$ instanton numbers $(r, 12 + n - r)$ (see e.g.[34]).

Remark. While (3.1) and (3.2) successfully yields a consistent $SU(6)$ model, the vanishing orders of $(a_1, a_2, a_3, a_4, a_6)$ in $z$ are $(0, 1, 2, 3, 5)$, which are the same as those for the split $I_5$ fiber type $I_5^*$ and differ from the “standard” Tate’s orders $(0, 1, 3, 3, 6)$ for the split $I_6$ fiber type $I_6^*$ classified in [4]. Indeed, it can be easily seen that the sections $(a_1, a_2, a_3, a_4, a_6)$ with orders $(0, 1, 3, 3, 6)$ only result in the Weierstrass model (3.3)(3.4)(3.5) with constant $t_r$, that is, no instantons are distributed to the $SU(3)$ factor, and all the $12 + n$ instantons are in the $SU(2)$ factor. In fact, one can redefine $y$ and $x$ so that the vanishing orders of
(a_1, a_2, a_3, a_4, a_6) may become (0, 1, 3, 3, 6) only when t_r \neq 0, but cannot when t_r = 0 since the redefinitions of y and x contain shifts proportional to \frac{1}{t_r}, which diverge at t_r = 0.

By redefining y and x, we obtain the Weierstrass equation

\[
0 = -y^2 + x^3 + f_{SU(6)}(z, z')x + g_{SU(6)}(z, z'),
\]

with a discriminant

\[
4f_{SU(6)}^3 + 27g_{SU(6)}^2 = z^6t_r^4h_{n-r+2}^4P_{2n+r+16} + z^7t_r^2h_{n-r+2}^2Q_{3n+20} + z^8R_{4n+24} + O(z^9),
\]

where P_{2n+r+16}, Q_{3n+20} and R_{4n+24} are some non-factorizable polynomials in z' of degrees specified by the subscripts. In generic cases, any two of t_r, h_{n-r+2} and P_{2n+r+16} do not share a common zero locus, which we assume in this paper. From (3.4), (3.5) and (3.6) we can see that the Kodaira fiber types above the zero loci of t_r, h_{n-r+2} and P_{2n+r+16} are respectively IV^*, I_6^* and I_7, yielding the singularity enhancements from H = SU(6) to G = E_6, D_6 and A_6 as presented in the third column of Table 2. We can also see that the h_{n-r+2}^2-dependence of f_{SU(6)} (3.4) or g_{SU(6)} (3.5) is only through h_{n-r+2}^2, which allows us to replace every h_{n-r+2}^2 in f_{SU(6)} and g_{SU(6)} with a generic polynomial h_{2n-2r+4}. The resulting equation is the one for I_6^{ns} [4].

3.2 The massless spectrum

As we will see explicitly in the next section, the replacement of the section h_{n-r+2}^2 \rightarrow h_{2n-2r+4}^2 in the split I_6 equation results in the global non-factorization of the exceptional curves, which reduces the gauge group from SU(6) to Sp(3). Let us examine what matter multiplets are expected to arise in this model.

In the transition I_6 \leftrightarrow I_6^{ns}, nothing changes in the local singularity structure near the zero loci of t_r and P_{2n+r+16}, where 1 \over 2 \cdot 20 and 6 of SU(6) appear as massless matter in the split theory; the string junctions or the vanishing cycles there do not “know” whether the total equation is of the split type or of the non-split type. The only change they feel is that of the gauge group, so they simply decompose into irreducible representations of Sp(3), which is the gauge group of the non-split theory. Thus, at a zero locus of t_r, a half-hypermultiplet in 20 of SU(6), of which the quaternionic Kähler manifold E_6/(SU(6) \times SU(2)) is comprised, is decomposed into half-hypermultiplets in 14' and 6 of Sp(3), while at a zero of P_{2n+r+16},
a hypermultiplet in 6 of SU(6) entirely becomes one in 6 of Sp(3). Note that 6 is also a pseudo-real representation of Sp(3), and the latter can be regarded as $2n+r+16$ pairs of half-hypermultiplets. The 14' constitutes the quaternionic Kähler manifold $F_4/(Sp(3) \times SU(2))$, while the 6 does $Sp(4)/(Sp(3) \times SU(2))$. This will answer to the original question of where the matter fields corresponding to the final magical coset arise; they arise at the $E_6$ points of the non-split $I_6$ model as an irreducible multiplet in the $Sp(3)$ decomposition of 20 of $SU(6)$.

3.3 A puzzle on matter fields near the $D_6$ points

On the other hand, there is a puzzle as we mentioned in Introduction: With the replacement $h_{n-r+2}^2 \rightarrow h_{2n-2r+4}$, the $n-r+2$ double roots of the equation $h_{n-r+2}^2 = 0$ split into $n-r+2$ pairs of single roots of $h_{2n-2r+4} = 0$. Thus the number of loci where hypermultiplets in 15 of $SU(6)$ occur are doubled. A 15 of $SU(6)$ decomposes into 14 (and not 14' + 1) of $Sp(3)$. Since the adjoint of $SU(6)$ decomposes as $35 = 21 + 14$, where 21 is the adjoint of $Sp(3)$, one 14 of $n-r+2$ hypermultiplets can be thought of as eaten by the $SU(6)$ vector multiplet. Thus the anomaly-free massless matter spectrum shown in Table 3 can be reproduced if the $n-r+2-1$ hypermultiplets in 14 are “distributed” at the $2n-2r+4$ zero loci of $h_{2n-2r+4}$. This, however, seems impossible, since the 14 of $Sp(3)$ is a real representation and does not allow half-hypermultiplets in this representation.

Of course, the original $SU(6)$ spectrum is already anomaly free, so hypermultiplets in 14 can not be present equally at all the $2n-2r+4$ zeros of $h_{2n-2r+4} = 0$ as they are too many to be anomaly free. If they were 14' instead of 14, they could be split into pairs and equally be distributed (up to the eaten ones) at the $2n-2r+4$ zeros, but both the heterotic anomaly analysis and Sadov’s generalized anomaly cancellation mechanism tell us that they must be 14, and not 14'.

This poses a question of how the $n-r+1$ matter in 14 of $Sp(3)$ are generated and where they reside in the non-split $I_6$ model. In the next section, in order to explore what happens near a zero locus of $h_{2n-2r+4}$, we perform an explicit blow-up of the singularity.

4 Resolutions of the singularities

4.1 The local equation

In this section, we carry out the process of blow-up of the codimension-two singularity at a zero locus of $h_{2n-2r+4} = 0$. To this aim, we consider a local equation in which the enhancement of “$A_5$” to “$D_6$” is achieved at codimension two. \(^8\) To obtain such an equation, We first complete the square with respect to $y$ in (3.1) and substitute (3.2) into it. Writing $y + \frac{1}{2}(a_1 x + a_3) = Y$, we have

\[
\begin{align*}
-Y^2 + x^3 + x^2 (3t_r^2 h_{n-r+2}^2 - 3z t_r H_{n-r+4}) \\
+ x (z^2 t_r f_{n-r+8} + f_8 z^2 + 6 z^2 t_r u_{r+4} h_{n-r+2}^2 - 3 z^3 u_{r+4} H_{n-r+4}) \\
+ 3 z^4 u_{r+4}^2 h_{n-r+2} + z^5 u_{r+4} f_{n-r+8} + g_{12} z^6 = 0,
\end{align*}
\]  

\(^8\) Again, they are quoted because they only imply the Lie algebras whose Dynkin diagrams specify the intersections of the Kodaira fibers right above those points with fixed $\beta$.\]
in which \( h_{n-r+2}'s \) appear only in the form \( h_{n-r+2}^2 \). Thus we can make a replacement
\[ h_{n-r+2}^2 \to h_{2n-2r+4} \] in (4.1). By setting\(^9\)
\[
\begin{align*}
h_{n-r+2}^2 &\to h_{2n-2r+4} = w, \\
t_r = H_{n-r+4} = u_{r+4} &= \frac{1}{\sqrt{3}}, \\
f_{n-r+8} = f_8 = g_{12} &= 0,
\end{align*}
\] we can obtain a desired equation, but it is more convenient to make a shift in the \( x \) coordinate \( x + z^2 = X \). In terms of \( X \), the final equation is
\[
-Y^2 + X^3 + X^2 (w - z(3z + 1)) + X(3z + 1)z^3 - z^6 = 0,
\] which we blow up in the following section.

If we write (4.3) as
\[
-Y^2 + X^3 + \frac{b_2}{4}X^2 + \frac{b_4}{2}X + \frac{b_6}{4} = 0,
\] the vanishing orders of the sections \( b_2, b_4, b_6 \) in \( z \) are 0, 3, 6, respectively, which satisfy the criteria for the \( I_6 \) type Kodaira fiber in Tate’s algorithm. This is due to the shift \( x + z^2 = X \), as without it one would have instead the vanishing orders 0, 2, 4. Note that such a shift of the variable \( x \) to eliminate the order-2 term in \( z \) from \( b_4 \) is not possible globally, since near a zero locus of \( t_r \), where a \( \frac{1}{2}20 \) of \( SU(6) \) (or \( \frac{1}{2}(14' \oplus 6) \) of \( Sp(3) \)) appears, the necessary shift becomes divergent. This is why an equation with \( \text{ord}(b_2, b_4, b_6) = (0, 2, 4) \) was used in [15, 16].

### 4.2 Blowing up the singularity

Let us now consider the resolution of the singularity of the local equation (4.3)
\[
\Phi(x, y, z, w) \equiv -y^2 + x^3 + x^2 (w - z(3z + 1)) + x(3z + 1)z^3 - z^6 = 0,
\] where we have replaced \( X, Y \) with \( x, y \). The equation (4.5) has a codimension-one singularity along \( (x, y, z) = (0, 0, 0) \) for arbitrary \( w \).

#### 1st blow up

As was done in the previous works, we replace the complex line \( (x, y, z) = (0, 0, 0) \) with \( \mathbb{P}^2 \times \mathbb{C} \) in \( \mathbb{C}^4 \) and examine the singularities of the local equations in three different charts corresponding to the affine patches of the \( \mathbb{P}^2 \) for some fixed \( w \). We also give the explicit forms of the exceptional curves \( C's \) at \( w \neq 0 \) and \( \delta's \) at \( w = 0 \). (\( \delta \) is defined by the \( w \to 0 \) limit of \( C \) in the chart where \( C \) arises.)

---

\(^9\)In this section, the local coordinates of the base \( \mathbb{P}^1 \) of \( \mathbb{P}_n \) (whose affine coordinate is \( z' \)) will be denoted by \( w \) and not by \( s \), in accordance with [24].
Chart 1_x

\[ \Phi(x, xy_1, xz_1, w) = x^2 \Phi_x(x, y_1, z_1, w), \]
\[ \Phi_x(x, y_1, z_1, w) = w - x^4 z_1^6 + 3x^3 z_1^4 + x^2 (z_1 - 3) z_1^2 - xz_1 + x - y_1^2. \]
\[ C_{p_1}^\pm \text{ in } 1_x : x = 0, \ y_1 = \pm \sqrt{w}. \]
\[ \delta_{p_1} \text{ in } 1_x : x = 0, \ y_1 = 0. \]

Singularities : None. (4.6)

Chart 1_y

\[ \Phi(x_1 y, y, yz_1, w) = y^2 \Phi_y(x_1, y, z_1, w), \]
\[ \Phi_y(x_1, y, z_1, w) = w x_1^2 + x_1^3 y - x_1^2 y z_1 (3 y z_1 + 1) + x_1 y^2 z_1^3 (3 y z_1 + 1) - y^4 z_1^6 - 1. \]
\[ C_{p_1}^\pm \text{ in } 1_y : y = 0, \ x_1 = \pm 1/\sqrt{w}. \]
\[ \delta_{p_1} \text{ in } 1_y : \text{Invisible.} \]

Singularities : None. (4.7)

Chart 1_z

\[ \Phi(x_1 z, y_1 z, z, w) = z^2 \Phi_z(x_1, y_1, z, w), \]
\[ \Phi_z(x_1, y_1, z, w) = w x_1^2 + z (x_1^2 - x_1^2 (3z + 1) + x_1 z (3z + 1) - z^3) - y_1^2. \]
\[ C_{p_1}^\pm \text{ in } 1_z : z = 0, \ y_1 = \pm \sqrt{w} x_1. \]
\[ \delta_{p_1} \text{ in } 1_z : z = 0, \ y_1 = 0. \]

Singularities : \((x_1, y_1, z) = (0, 0, 0)\). (4.8)

Here, the chart 1_x is the affine patch of \( \mathbb{P}^2 \ni (x : y : z) \) for \( x \neq 0 \) in which \((x : y : z) = (1 : y_1 : z_1)\). The other charts are also similar.\(^{10}\)

2nd blow up

As we can see, the only singularity after the first blow up is \((x_1, y_1, z) = (0, 0, 0)\) on the chart 1_z, which is not visible from the other charts. This is codimension one, and we blow up this singularity by similarly inserting a one-parameter (= w) family of \( \mathbb{P}^2 \) along \((x_1, y_1, z, w) = (0, 0, 0, w)\). The computation is similar. We find a singularity in the chart 2_zz, while the blown-up equations are regular for the charts 2_zx and 2_zy. Here we show the result for the relevant charts 2_zx and 2_zy.

\(^{10}\)Note that we have used the same “z_1” in 1_x and 1_y for different coordinate variables, and similarly for \( x_1 \) and \( y_1 \). There will be no confusion as we do not compare equations in different charts.
Chart $2_{\epsilon z}$

\[
\Phi_{z}(x_1, x_1 y_2, x_1 z_2, w) = x_1^2 \Phi_{xx}(x_1, y_2, z_2, w),
\]
\[
\Phi_{xx}(x_1, y_2, z_2, w) = x_1 (z_2 - 1) z_2 - x_1^2 (z_2 - 1)^3 + w - y_2^2.
\]
\[
C_{p_2}^{\pm} \text{ in } 2_{xx} : x_1 = 0, \quad y_2 = \pm \sqrt{w}.
\]
\[
\delta_{p_2} \text{ in } 2_{xx} : x_1 = 0, \quad y_2 = 0.
\]
Singularities : None. \hspace{1cm} (4.9)

Chart $2_{zz}$

\[
\Phi_z(x_2 z, y_2 z, z, w) = z^2 \Phi_{zz}(x_2, y_2, z, w),
\]
\[
\Phi_{zz}(x_2, y_2, z, w) = wx_2^2 + (x_2 - 1) z (x_2^2 z - 2 x_2 z - x_2 + z) - y_2^2.
\]
\[
C_{p_2}^{\pm} \text{ in } 2_{zz} : z = 0, \quad y_2 = \pm \sqrt{wx_2}.
\]
\[
\delta_{p_2} \text{ in } 2_{zz} : z = 0, \quad y_2 = 0.
\]
Singularities : $(x_2, y_2, z) = (0, 0, 0)$. \hspace{1cm} (4.10)

3rd blow up

We finally blow up the codimension-one singularity $(x_2, y_2, z) = (0, 0, 0)$ in the chart $2_{zz}$. It turns out that this completes the resolution process completely without leaving any singularities.

The equations of the exceptional curve (with a definite $w$) in the relevant charts are:

Chart $3_{zzz}$

\[
\Phi_{zzz}(x_2 y_3, x_2 z_3, w) = x_2^3 \Phi_{zzzz}(x_2 y_3, x_2 z_3, w),
\]
\[
\Phi_{zzzz}(x_2 y_3, x_2 z_3, w) = w + (x_2 - 1) z_3 ((x_2 - 1)^2 z_3 - 1) - y_3^2.
\]
\[
C_{p_3} \text{ in } 3_{zzz} : x_2 = 0, \quad y_3^2 = w - (z_3 - 1) z_3.
\]
\[
\delta_{p_3} \text{ in } 3_{zzz} : x_2 = 0, \quad y_3^2 = -(z_3 - 1) z_3.
\]
Singularities : None. \hspace{1cm} (4.11)

Chart $3_{zzzz}$

\[
\Phi_{zzz}(x_3 z, y_3 z, z, w) = z_3^2 \Phi_{zzzz}(x_3 z, y_3 z, z, w),
\]
\[
\Phi_{zzzz}(x_3 z, y_3 z, z, w) = x_3^3 (w - z(3z + 1)) + x_3^3 z^3 + 3x_3 z + x_3 - y_3^2 - 1 = 0.
\]
\[
C_{p_3} \text{ in } 3_{zzzz} : z_2 = 0, \quad y_3^2 = wx_3^2 + x_3 - 1.
\]
\[
\delta_{p_3} \text{ in } 3_{zzzz} : z_2 = 0, \quad y_3^2 = x_3 - 1.
\]
Singularities : None. \hspace{1cm} (4.12)

This completes the blowing-up process, and the space is now smooth. We have seen that conifold singularities do not appear at any stage of the blow up at the $D_6$ points. This is similar to the case of the incomplete resolution at the $E_6$ point in the split $I_6$ model. However, unlike that case, the intersection of the exceptional curves does not change at all at the $D_6$ points, as we will see in the next section.
4.3 Intersections of the exceptional curves

At fixed $w \neq 0$, we have five exceptional curves $C_{P_1}^\pm$, $C_{P_2}^\pm$ and $C_{P_3}$. From the above explicit forms, one finds that their intersection matrix is given by the $A_5$ Dynkin diagram (the top diagram of Figure 1). Although $C_{P_1}^\pm$ and $C_{P_2}^\pm$ are respectively factorized into two lines on this fixed $w \neq 0$ plane, they do not factor in the polynomial ring of $w$. The two lines at some fixed $w \neq 0$ are interchanged with each other at $w = 0$, meaning that this is a non-split type of the singularity. Thus the two lines for $C_{P_1}^\pm$ or $C_{P_2}^\pm$ at fixed $w \neq 0$ comprising the Kodaira fibers of type $I_6$ are identified. Hence we define

$$C_{P_i} = \frac{1}{2}(C_{P_i}^+ + C_{P_i}^-) \quad (i = 1, 2), \quad (4.13)$$

which are the projections onto the components invariant under the diagram automorphism of the $A_5$ Dynkin diagram. Then one can show that the three exceptional curves $C_{P_1}$, $C_{P_2}$ and $C_{P_3}$ form a non-simply-laced Dynkin diagram of $C_3$ (the middle diagram of Figure 1).

At $w = 0$, we again encounter another difference between the present non-split case and the previous examples of singularities associated with the magic square. In the incomplete resolutions for the previous examples $(G, H) = (E_6, SU(6))$, $(E_7, SO(12))$ and $(E_8, E_7)$, while the number of the exceptional fibers at $w = 0$ is the same as that at $w \neq 0$, some of the exceptional fibers at $w = 0$ turn out to be linear combinations of those at $w \neq 0$. Therefore, the intersection diagram of the exceptional fibers at $w = 0$ becomes different from that at $w \neq 0$ as we summarized in section 2.2. Here, we see something different. As in the previous works, by lifting up the exceptional curves from the defining chart into subsequent charts and seeing their relations, one finds that

$$C_{P_1}^\pm \to \delta_{P_1}, \quad C_{P_2}^\pm \to \delta_{P_2}, \quad C_{P_3} \to \delta_{P_3}, \quad (4.14)$$

Substituting them into (4.13), we obtain

$$C_{P_1} \to \delta_{P_1}, \quad C_{P_2} \to \delta_{P_2}, \quad C_{P_3} \to \delta_{P_3}. \quad (4.15)$$

Thus, the intersection matrix remains identical even at the codimension-two point (see the bottom diagram of Figure 1). This is a sharp contrast to the previous examples, where the intersection matrices at $w = 0$ did not coincide with any of (the minus of) the Lie algebra Cartan matrices.

5 Split/non-split transition as a conifold transition

In the previous section, we have seen that there is no sign of local matter fields near the $D_6$ points. In this section, we will use the recent result of [24] to illustrate how the matter fields are considered to arise near the $D_6$ points in the non-split $I_6$ model. In a nutshell, what has been found in [24] is that a transition from the split to the non-split model in F-theory is in most cases a transition from the deformed side to the resolved side in the conifold transition associated with the conifold singularities which arise at $D_{2k}$ points (or $E_7$ points for the non-split $IV^*$, which are irrelevant here). In the present case, they are $D_6$ points, so they are precisely what we have been considering in the previous sections.
If we consider the resolution of the split $I_6$ model instead of the non-split one, we find various conifold singularities (Figure 2). Indeed, by replacing $w$ with $w^2$ in (4.6), we find

\[
\Phi_x(x, y_1, z_1, w^2) = w^2 - x^4 z_1^6 + 3x^3 z_1^4 + x^2 (z_1 - 3) z_1^2 - x z_1 + x - y_1^2 \\
= -y_1^2 + w^2 - x(z_1 + O(x)),
\]  
(5.1)

Figure 1. Intersection diagrams of the exceptional curves: (Top) $w \neq 0$ before the projection (4.13); (Middle) $w \neq 0$ after the projection (4.13); (Bottom) $w = 0$. 

Figure 2.
which shows that
\[ v_{q_1} : (x, y_1, z_1, w) = (0, 0, 0, 0) \] (5.2)
is a conifold singularity. Also, in (4.9), \( \Phi_{zx}(x_1, y_2, z_2, w^2) \) becomes
\[
\Phi_{zx}(x_1, y_2, z_2, w^2) = x_1(z_2 - 1)z_2 - x_1^2(z_2 - 1)^3 + w^2 - y_2^2 \\
= -y_2^2 + w^2 + x_1((z_2 - 1)z_2 + O(x_1)),
\] (5.3)
showing that
\[ v_{q_2} : (x_1, y_2, z_2, w) = (0, 0, 0, 0) \] and \( v_{r_2} : (x_1, y_2, z_2, w) = (0, 0, 1, 0) \) (5.4)
are conifold singularities. In this case, it can be shown that the exceptional curves arising from their small resolutions precisely yield (together with the ones coming from the codimension-one singularities) the \( D_6 \) Dynkin diagram as their intersection diagram (Figure 2).

In both the split and non-split cases, we can say that the \( D_6 \) point are where \( h_{2n-2r+4} \) vanishes, and the split case is when \( h_{2n-2r+4} \) is in the special form \( h_{n-r+2} \). In other words, in the split model, a \( D_6 \) point is a double root of the equation \( h_{2n-2r+4} = 0 \), whereas in the non-split model, it is a single root. So suppose that \( h_{2n-2r+4} = w^2 \) near \( w = 0 \) in the split case. Then, by a deformation of the complex structure \( w^2 \to w^2 - \epsilon^2 = (w + \epsilon)(w - \epsilon) \) for some small deformation parameter \( \epsilon \), the double zero \( w = 0 \) becomes a pair of single roots \( w = \pm \epsilon \), and the split model becomes a non-split model accordingly. On the other hand, as we can see in eqs. (5.1) and (5.3), changing \( w^2 \) to \( w^2 - \epsilon^2 \) is exactly turning a conifold into a deformed conifold. Therefore, we see that, at the stage where we have finished blowing up all the codimension-one singularities and only conifold singularities remain, what we get by a small resolution is a split model, and what we get by a deformation is a non-split model. In other words, the split/non-split transition is a conifold transition [24].

Once this fact is revealed, it is not surprising that the conifold singularity does not appear in the non-split model. Since the non-split model corresponds to a deformed conifold, the two-cycles in the split model that are responsible for the matter generation are replaced by three-cycles in the non-split model.

How do these three-cycles give rise to massless matter fields? In [24], we have discussed several possibilities. One of them is the wrapped M5-branes around \( S^2 \times S^3 \). Since the massless matter in the split model is accounted for by the wrapped M2-branes around the vanishing two-cycles, this would be a natural guess. The total volume of \( S^2 \times S^3 \) will vanish at the apex of the deformed conifold as the volume of \( S_2 \) vanishes there. Also it must contain at least one dimension of the elliptic fiber, for which a small volume limit is taken in the F-theory limit. We cannot say anything conclusive in this paper, so we leave the clarification of the precise mechanism as an issue for the future.

6 Conclusions

Motivated by the coincidence between the three examples of half-hypermultiplets and the entries of the magic square, we have studied a six-dimensional \( \mathcal{N} = 1 \) F-theory compactification on an elliptic fibration over a Hirzebruch surface with a codimension-one singularity.
of the non-split $I_6$ type found in [4]. This model supports an $Sp(3)$ gauge symmetry. The heterotic index and the generalized Green-Schwarz analysis both show that such a compactification gives massless half-hypermultiplets in the $14'$ representation (as well as the 6 representation) of $Sp(3)$, which is $F_4/(Sp(3) \times SU(2)) (Sp(4)/(Sp(3) \times SU(2)))$. We have shown that they are generated at the $E_6$ points, where half-hypermultiplets 20 of $SU(6)$ would have appeared in the split model. In the non-split model, $SU(6)$ is broken to $Sp(3)$, and 20 is decomposed into $14' \oplus 6$ of $Sp(3)$ accordingly, yielding the desired multiplets.

We have also considered the problem on the non-local matter generation near the $D_6$ point. We have pointed out two puzzles: The first one is how the degrees of massless matter fields in the split model can be plausibly assigned at the zero loci of the relevant section $h_{2n+4-2r}$, the number of which is doubled in the transition from the split to non-split models. Second, by performing a singularity resolution, we found no indication of the existence of localized massless matter fields. We have explained why this is so by using the result of [24] that the split/non-split transition can be regarded as a conifold transition.

Acknowledgement

We thank Y. Kimura and H. Otsuka for useful discussions.

References

[1] C. Vafa, Nucl. Phys. B 469, 403 (1996) [hep-th/9602022].
[2] D. R. Morrison and C. Vafa, Nucl. Phys. B 473, 74 (1996) [hep-th/9602114].
[3] D. R. Morrison and C. Vafa, Nucl. Phys. B 476, 437 (1996) [hep-th/9603161].
[4] M. Bershadsky, K. Intriligator, S. Kachru, D.R. Morrison, V. Sadov and C. Vafa, Nucl.Phys. B481 (1996) 215-252 [hep-th/9605200].
[5] S. H. Katz and C. Vafa, Nucl. Phys. B 497, 146 (1997) [hep-th/9606086].
[6] T. Tani, Nucl. Phys. B 602, 434 (2001).
[7] G. Curio, Phys. Lett. B 435, 39 (1998) [hep-th/9803224].
[8] D. E. Diaconescu and G. Ionesci, JHEP 9812, 001 (1998) [hep-th/9811129].
[9] S. Mizoguchi and T. Tani, PTEP 2016 (2016) no.7, 073B05 [arXiv:1508.07423 [hep-th]].
[10] S. Mizoguchi and T. Tani, JHEP 11 (2016), 053 [arXiv:1607.07280 [hep-th]].
[11] K. Oguiso and T. Shioda, Comment. Math. Univ. St. Pauli. 40 (1991) 83.
[12] J. A. Wolf, J. of Math. Mech., 14 (1965), 1033.
[13] D.V. Alekseevskii, Funct. Anal. Appl. 2 (1968), 97; Funct. Anal. Appl. 2 (1968), 106; Math. USSR Izv. 9 (1975), 297.
[14] K. Dasgupta, V. Hussin and A. Wissanji, Nucl. Phys. B 793 (2008), 34-82 [arXiv:0708.1023 [hep-th]].
[15] D. R. Morrison and W. Taylor, JHEP 1201, 022 (2012) [arXiv:1106.3563 [hep-th]].
[16] N. Kan, S. Mizoguchi and T. Tani, [arXiv:2003.05563 [hep-th]]. To appear in JHEP.
[17] S. Mizoguchi, JHEP 1407, 018 (2014) [arXiv:1403.7066 [hep-th]].
[18] T. Kugo and T. Yanagida, Phys. Lett. 134B, 313 (1984).
[19] A. Grassi, J. Halverson, C. Long, J. L. Shaneson and J. Tian, JHEP 09 (2018), 129 [arXiv:1805.06949 [hep-th]].
[20] P. Arras, A. Grassi and T. Weigand, J. Geom. Phys. 123 (2018), 71-97
[21] M. Esole, P. Jefferson and M. J. Kang, [arXiv:1704.08251 [hep-th]].
[22] M. Esole and M. J. Kang, JHEP 02 (2019), 091 [arXiv:1805.03214 [hep-th]].
[23] M. Esole and P. Jefferson, [arXiv:1910.09536 [hep-th]].
[24] R. Kuramochi, S. Mizoguchi and T. Tani, [arXiv:2108.10136 [hep-th]].
[25] M. Gunaydin, G. Sierra and P. K. Townsend, Phys. Lett. B 133 (1983) 72.
[26] M. Gunaydin, G. Sierra and P. K. Townsend, Nucl. Phys. B 242 (1984) 244.
[27] N. Kan and S. Mizoguchi, Phys. Lett. B 762 (2016), 177-183 [arXiv:1605.01904 [hep-th]].
[28] S. Fukuchi and S. Mizoguchi, Phys. Lett. B 781 (2018), 77-82 [arXiv:1802.06555 [hep-th]].
[29] M. B. Green, J. H. Schwarz and P. C. West, Nucl. Phys. B 254 (1985), 327-348
[30] N. Yamatsu, [arXiv:1511.08771 [hep-ph]].
[31] V. Sadov, Phys. Lett. B 388 (1996), 45-50 [arXiv:hep-th/9606008 [hep-th]].
[32] S. Mizoguchi and T. Tani, PTEP 2016 (2016) no.7, 073B05 [arXiv:1508.07423 [hep-th]].
[33] H. Hayashi, C. Lawrie, D. R. Morrison and S. Schafer-Nameki, JHEP 1405, 048 (2014) [arXiv:1402.2653 [hep-th]].
[34] S. Mizoguchi and T. Tani, JHEP 03, 121 (2019) [arXiv:1808.08001 [hep-th]].