TRIANGULATED CATEGORIES WITH A SINGLE COMPACT GENERATOR AND A BROWN REPRESENTABILITY THEOREM

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Abstract. We generalize a theorem of Bondal and Van den Bergh. A corollary of our main results says the following: Let \( X \) be a scheme proper over a noetherian ring \( R \). Then the Yoneda map, taking an object \( D \) in the category \( \text{D}^{\text{coh}}(X) \) to the functor \( \text{Hom}(\cdot, D) \big|_{\text{D}^{\text{perf}}(X)} : \text{D}^{\text{perf}}(X)^{\text{op}} \to R\text{-mod} \), is an equivalence of \( \text{D}^{\text{coh}}(X) \) with the category of finite \( R \)-linear cohomological functors \( H : \text{D}^{\text{perf}}(X)^{\text{op}} \to R\text{-mod} \).

A cohomological functor \( H \) is finite if \( \oplus_{i=-\infty}^{\infty} H^i(C) \) is a finite \( R \)-module for every \( C \in \text{D}^{\text{perf}}(X) \).

Bondal and Van den Bergh proved the special case where \( R \) is a field and \( X \) is projective over \( R \).

But our theorems are more general. They work in the abstract generality of triangulated categories with coproducts and a single compact generator, satisfying a certain approximability property. At the moment I only know how to prove this approximability for the categories \( \text{D}^{\text{qeq}}(X) \) with \( X \) a quasicompact, separated scheme, for the homotopy category of spectra, and for the category \( \text{D}(R) \) where \( R \) is a (possibly non-commutative) negatively graded dg algebra.

The work was inspired by Jack Hall’s elegant new proof of a vast generalization of GAGA, a proof based on representability theorems of the type above. The generality of Hall’s result made me wonder how far the known representability theorems could be improved.

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0. Introduction

This paper begins with the observation that triangulated categories with coproducts and a single compact generator have a preferred class of $t$–structures. This allows us to define thick subcategories $\mathcal{T}^-$, $\mathcal{T}^+$ and $\mathcal{T}^b$. A slightly subtler definition is that of $\mathcal{T}^-_c$. The full subcategory $\mathcal{T}^-_c \subset \mathcal{T}$ makes sense unconditionally, and it is thick as long as there exists a compact generator $G$ and an integer $A > 0$ so that $\text{Hom}(G, \Sigma^i G) = 0$ for all $i \geq A$. We also define a subcategory $\mathcal{T}^-_c = \mathcal{T}^-_c \cap \mathcal{T}^b$.

In the special case where $\mathcal{T} = \mathcal{D}_{qc}(X)$, with $X$ a quasicompact, separated scheme, the preferred class of $t$–structures contains the standard $t$–structure, the subcategories $\mathcal{T}^-$, $\mathcal{T}^+$ and $\mathcal{T}^b$ are nothing other than $\mathcal{D}^-_{qc}(X)$, $\mathcal{D}^+_{qc}(X)$ and $\mathcal{D}^b_{qc}(X)$, and if $X$ is noetherian the subcategories $\mathcal{T}^-_c \subset \mathcal{T}^b$ can be proved to be $\mathcal{D}^b_{coh}(X) \subset \mathcal{D}^-_{coh}(X)$. What we have learned so far is that these standard categories have an intrinsic description. There is a method to construct them out of $\mathcal{T}$ in purely triangulated-category terms.

Still in the world of triangulated categories with coproducts and a single compact generator: the category $\mathcal{T}$ may be approximable. We will define this concept later in the introduction, and study its properties in the body of the paper. For now we note that the category $\mathcal{D}_{qc}(X)$ is approximable, as long as $X$ is a quasicompact, separated scheme. The homotopy category of spectra is also approximable.

To show that this abstraction can be useful we will prove representability theorems. To state them we begin with

**Definition 0.1.** Let $R$ be a commutative ring, let $\mathcal{T}$ be an $R$–linear triangulated category and let $\mathcal{B} \subset \mathcal{T}$ be a full, replete subcategory with $\Sigma \mathcal{B} = \mathcal{B}$. A $\mathcal{B}$–cohomological functor is an $R$–linear functor $H : \mathcal{B}^{\text{op}} \rightarrow R$–Mod which takes triangles to long exact sequences. This means that, if we have a triangle $x \rightarrow y \rightarrow z \rightarrow \Sigma x$ with all three of $x, y, z$ belonging to $\mathcal{B}$, then $H$ takes it to a long exact sequence in $R$–Mod.

Suppose the ring $R$ is noetherian, and let $G \in \mathcal{B} \subset \mathcal{T}$ be an object. The $\mathcal{B}$–cohomological functor $H : \mathcal{B}^{\text{op}} \rightarrow R$–Mod is called $G$–locally finite if

- (i) $H(\Sigma^i G)$ is a finite $R$–module for all $i \in \mathbb{Z}$.
- (ii) $H(\Sigma^i G) = 0$ for $i \ll 0$.

The $\mathcal{B}$–cohomological functor $H$ is $G$–finite if, in addition to the above, we have

- (iii) $H(\Sigma^i G) = 0$ for $i \gg 0$.

**Remark 0.2.** Let $\mathcal{T}$ be an $R$–linear triangulated category, $\mathcal{B}$ a full, replete subcategory with $\Sigma \mathcal{B} = \mathcal{B}$, and $H$ a $\mathcal{B}$–cohomological functor. If $H$ is $G$–locally finite (respectively finite) for every $G \in \mathcal{B}$ we leave out the $G$, and just say that $H$ is locally finite (respectively finite).

Note that if $H$ is $G$–locally finite (respectively finite) then it is also $G'$–locally finite for any $G'$ obtainable from $G$ by forming in $\mathcal{B}$ finite direct sums, direct summands, suspensions or triangles. Thus local finiteness (respectively finiteness) can be checked on any classical generator.
Our main theorem says that

**Theorem 0.3.** Let $R$ be a noetherian ring, and $\mathcal{I}$ an $R$–linear triangulated category with coproducts. Assume $\mathcal{I}$ has a compact generator $G$ with $\text{Hom}(\cdot,G)$ a $G$–locally finite cohomological functor. Suppose further that $\mathcal{I}$ is approximable.

Then the restricted Yoneda functor $\mathcal{Y} : \mathcal{I} \rightarrow \text{Hom}[(\mathcal{I}^c)^{\text{op}}, R\text{-Mod}]$, that is the functor taking an object $t \in \mathcal{I}$ to the restriction to $\mathcal{I}^c$ of the representable functor $\text{Hom}(\cdot,t)$, restricts on $\mathcal{I}^c_0 \subset \mathcal{I}$ to a full functor. In fact more is true: any map $\varphi : \mathcal{Y}(s) \rightarrow \mathcal{Y}(t)$, with $s \in \mathcal{I}^c_0$ and $t \in \mathcal{I}$, is equal to $\mathcal{Y}(f)$ for some $f : s \rightarrow t$. Furthermore the essential image of $\mathcal{I}^c_0$ is precisely the category of locally finite $\mathcal{I}^c$–cohomological functors.

If $f : s \rightarrow t$ is a morphism from $s \in \mathcal{I}^c_0$ to $t \in \mathcal{I}^+$, then $\mathcal{Y}(f) = 0$ implies $f = 0$. It follows that on the subcategory $\mathcal{I}^c_b \subset \mathcal{I}^c_0$ the functor $\mathcal{Y}$ is fully faithful. Furthermore the essential image of $\mathcal{I}^c_b$ is the category of finite $\mathcal{I}^c$–cohomological functors.

From this we will deduce

**Corollary 0.4.** Let $\mathcal{I}$ be as in Theorem 0.3, but assume further that $\mathcal{I}^c$ is contained in $\mathcal{I}^c_b$. Let $(\mathcal{I}^{\leq 0}, \mathcal{I}^{\geq 0})$ be one of the preferred $t$–structures.

Assume $\mathcal{L} : \mathcal{I}^c_b \rightarrow S$ is an $R$–linear triangulated functor such that

(i) For any pair of objects $(t,s)$, with $t \in \mathcal{I}^c$ and $s \in S$, the $R$–module $\text{Hom}(\mathcal{L}(t),s)$ is finite.

(ii) For any object $s \in S$ there exists an integer $A > 0$ with $\text{Hom}(\mathcal{L}(\mathcal{I}^c_b \cap \mathcal{I}^{\leq -A}), s) = 0$.

(iii) For any object $t \in \mathcal{I}^c$ and any object $s \in S$ there exists an integer $A$ so that $\text{Hom}(\mathcal{L}(\Sigma^m t), s) = 0$ for all $m \leq -A$.

Then $\mathcal{L}$ has a right adjoint $R : S \rightarrow \mathcal{I}^c_b$.

In the special example of $\mathcal{I} = \text{D}^{\text{qc}}(X)$ Theorem 0.3 specializes to

**Corollary 0.5.** If $X$ is a scheme proper over a noetherian ring $R$, then the restricted Yoneda functor $\mathcal{Y}$ gives an equivalence from the category $\text{D}^{\text{b}}_{\text{coh}}(X)$ to the category of finite cohomological functors $\text{D}^{\text{perf}}(X)^{\text{op}} \rightarrow R\text{-Mod}$.

On the larger category $\text{D}^{\text{coh}}_{\text{perf}}(X)$, the functor $\mathcal{Y}$ is full and the essential image is the category of locally finite cohomological functors $\text{D}^{\text{perf}}(X)^{\text{op}} \rightarrow R\text{-Mod}$.

**Proof.** If $X$ is proper over $R$ then it is separated and quasicompact, hence the category $\mathcal{I} = \text{D}^{\text{qc}}(X)$ is approximable. But properness also guarantees that, for any compact generator $G \in \mathcal{I}$ and any $i \in \mathbb{Z}$, the $R$–module $\text{Hom}(\Sigma^i G, G)$ is finite. The vanishing of $\text{Hom}(\Sigma^i G, G)$ for $i \ll 0$ is true more generally, it doesn’t require properness. Anyway: the functor $\text{Hom}(\cdot,G)$ is $G$–locally finite and Theorem 0.3 applies. □

**Remark 0.6.** If $R$ is a field and $X$ is projective over $R$, then the part of Corollary 0.5 concerning the objects in the image of $\mathcal{Y}$ is known—see Bondal and Van den Bergh [5] Theorem A.1]. Bondal and Van den Bergh’s theorem says nothing about the functor $\mathcal{Y}$ being fully faithful.
The existing proofs of variants of Theorem $0.3$ including the current one, proceed in two steps. Starting with a finite $\mathcal{T}^c$-cohomological functor $H$ one first proves that $H \cong Y(t)$ for some $t \in \mathcal{T}$, and then shows that $t$ must actually belong to $\mathcal{T}^b_c$. Bondal and Van den Bergh \cite[Theorem A.1]{5} and Jack Hall \cite[Proposition 4.1]{9} rely on suitable special features that allow the functor $H : \mathcal{T}^c = D_{\text{perf}}(X) \rightarrow R\text{-Mod}$ to extend to a cohomological functor on all of $\mathcal{T} = D_{\text{qc}}(X)$, and then use the usual Brown representability theorem for $D_{\text{qc}}(X)$. For Bondal and Van den Bergh the key is forming the double dual—this works since $R$ is assumed a field, and a finite-dimensional vector space over $R$ is canonically isomorphic to its double dual. Jack Hall relies on the fact that his functors come from morphisms of ringed spaces $c : X \rightarrow X$, and formal properties then provide adjoints

$$D_{\text{qc}}(X) \xrightarrow{\text{natural}} \text{D}(X) \xrightarrow{Lc^*} \text{D}(X)$$

We should recall one more result in the literature: although Ben-Zvi, Nadler and Preygel \cite[Section 3]{4} is not technically either a special case or a generalization of Theorem $0.3$, the reader is nonetheless encouraged to look at it—there are interesting parallels. Enhancements play a role in \cite{4}, as well as the construction of an explicit generator and estimates similar to those of \cite[Theorem 4.1]{13}.

What’s different here is the generality. Let $H$ be any locally finite $\mathcal{T}^c$-cohomological functor. Under hypotheses weaker than approximability (see Proposition $7.10$ for the precise statement) we prove that $H \cong Y(t)$ where $t \in \mathcal{T}$ is some object—the existence of $t$ is formal, not special to narrow classes of $\mathcal{T}$’s or $H$’s. And by combining a careful analysis of the proof of Proposition $7.10$ with the theory developed in Section $2$, we will deduce—under only the approximability hypothesis—that $t$ must belong to $\mathcal{T}^b_c$.

**Remark 0.7.** The work was inspired by the lovely new proof of a vast generalization of GAGA to be found in Jack Hall \cite{9}. More precisely: it was inspired by the original idea, which is to be found in \cite[Section 2]{9}—as Hall’s paper became more general it developed a different tack. Let me sketch the simple idea and encourage the reader to look at \cite{9} for more detail, as well as for far greater generality. Because this is only a tiny glimpse I will confine myself to the original, analytic version of GAGA in Serre \cite{19}, not to the other incarnations.

Suppose $X$ is a scheme proper over $\mathbb{C}$. Then $X^{\text{an}}$ is a compact, complex analytic space, and analytification defines a functor $\mathcal{L} : D^b_{\text{coh}}(X) \rightarrow D^b_{\text{coh}}(X^{\text{an}})$, taking a complex of $\mathcal{O}_X$-modules with bounded, coherent cohomology to its analytification. With $\mathcal{T} = D_{\text{qc}}(X)$ and $\mathcal{T}^b_c = D^b_{\text{coh}}(X)$ we have exhibited a $\mathbb{C}$-linear triangulated functor $\mathcal{L} : \mathcal{T}^b_c \rightarrow \mathcal{S}$, where $\mathcal{S} = D^b_{\text{coh}}(X^{\text{an}})$. It is a tiny exercise to show that this functor satisfies the criteria of Corollary $0.4$, hence has a right adjoint $\mathcal{R} : \mathcal{S} \rightarrow \mathcal{T}^b_c$.

To prove GAGA it suffices to show that $\mathcal{R}$ is an essential inverse of $\mathcal{L}$. The gist of the argument is that, when you have a pair of adjoint triangulated functors between triangulated categories, there are standard techniques that render it easy to check if...
these functors are inverse equivalences. And the stunning feature of this particular application is that the global data one needs, about coherent analytic sheaves on the compact, complex analytic space $X^{an}$, are just the minimal ones used to prove that the hypotheses of Corollary 0.4 are satisfied. Once you have the right adjoint $\mathcal{R}$, the proof that $\mathcal{L}$ and $\mathcal{R}$ are essential inverses reduces to applying Nakayama’s lemma to some suitable finite modules over the noetherian local rings $\mathcal{O}_{X,x}$ and $\mathcal{O}_{X^{an},x}$, the stalks at a closed point $x \in X$ of the structure sheaves $\mathcal{O}_X$ and $\mathcal{O}_{X^{an}}$. This is starkly different from the existing proofs of GAGA in the literature, which (after a great deal of reduction) hinge on the computation of $H^i(X^{an}, S)$ for a (short) list of coherent analytic sheaves $S$.

We have already mentioned that part of the interest of the paper is that natural objects, like the subcategories $D^b_{coh}(X) \subseteq D_{coh}(X)$ of the category $\mathcal{I} = D_{qc}(X)$, have an intrinsic description. The definitions are not hard to give, we include them in the Introduction. Before all else we recall some standard notation.

**Reminder 0.8.** Let $\mathcal{I}$ be a triangulated category. We define

(i) If $\mathcal{A} \subseteq \mathcal{I}$ is a full subcategory, then $\text{smd}(\mathcal{A})$ is the full subcategory of all direct summands of objects of $\mathcal{A}$.

(ii) If $\mathcal{A} \subseteq \mathcal{I}$ is a full subcategory, then $\text{add}(\mathcal{A})$ is the full subcategory of all finite direct sums of objects of $\mathcal{A}$.

(iii) If $\mathcal{I}$ has small coproducts and $\mathcal{A} \subseteq \mathcal{I}$ is a full subcategory, then $\text{Add}(\mathcal{A})$ is the full subcategory of all coproducts of objects of $\mathcal{A}$.

(iv) If $\mathcal{A}, \mathcal{B}$ are two full subcategories of $\mathcal{I}$, then $\mathcal{A} \star \mathcal{B}$ is the full subcategory of all objects $y \in \mathcal{I}$ such that there exists a triangle $a \to y \to b \to$ with $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

(v) Given an object $G \in \mathcal{I}$ and two integers $A \leq B$, let $\mathcal{C} \subseteq \mathcal{I}$ be the full subcategory with objects $\{\Sigma^{-i}G \mid A \leq i \leq B\}$. For integers $n > 0$ we define the subcategories $\coprod_n(G[A, B])$, inductively on the integer $n$, by the formulas

$$\coprod_1(G[A, B]) = \text{add}(\mathcal{C}),$$

$$\coprod_{n+1}(G[A, B]) = \coprod_1(G[A, B]) \ast \coprod_n(G[A, B]).$$

(vi) Given an object $G \in \mathcal{I}$ and three integers $A \leq B$, $n > 0$ we define the subcategories $\langle G \rangle_n^{[A, B]}$ by the formula $\langle G \rangle_n^{[A, B]} = \text{smd}([\coprod_n(G[A, B])])$.

(vii) We adopt the following conventions:

$$\langle G \rangle_n^{(-\infty, B]} = \bigcup_A \langle G \rangle_n^{[A, B]}, \quad \langle G \rangle_n^{[A, \infty)} = \bigcup_B \langle G \rangle_n^{[A, B]}, \quad \langle G \rangle_n = \bigcup_{A \leq B} \langle G \rangle_n^{[A, B]},$$

$$\langle G \rangle_n = \bigcup_{n > 0} \langle G \rangle_n, \quad \langle G \rangle_n^{[A, B]} = \bigcup_{n > 0} \langle G \rangle_n^{[A, B]}, \quad \langle G \rangle_n^{(-\infty, B]} = \bigcup_A \langle G \rangle_n^{[A, B]},$$

$$\langle G \rangle_n^{[A, \infty)} = \bigcup_B \langle G \rangle_n^{[A, B]}.$$

(viii) Suppose $\mathcal{I}$ has coproducts, let $G$ be an object, and let $A \leq B$ be two integers. We define $\mathcal{C} \subseteq \mathcal{I}$ to be the full subcategory with objects $\{\Sigma^{-i}G \mid A \leq i \leq B\}$. For
integers \( n > 0 \) we define the subcategories \( \text{Coprod}_n(G[A,B]) \), inductively on the integer \( n \), by the formulas

\[
\text{Coprod}_1(G[A,B]) = \text{Add}(\mathcal{C}) , \\
\text{Coprod}_{n+1}(G[A,B]) = \text{Coprod}_1(G[A,B]) \star \text{Coprod}_n(G[A,B]) .
\]

In other words the difference between Coprod and coprod is that in Coprod we allow infinite coproducts in the formation of Coprod_1. The inductive procedure is unaltered.

(iii) We allow \( A \) and \( B \) to be infinite in (viii). For example \( \text{Coprod}_1(G(-\infty,B]) \) is defined to be \( \text{Add}(\mathcal{C}) \) with \( \mathcal{C} = \{\Sigma^{-i}G \mid i \leq B\} \).

(vi) Let \( A \leq B \) be integers, possibly infinite. Then \( \text{Coprod}(G[A,B]) \) is the smallest full subcategory \( S \subseteq \mathcal{T} \), closed under coproducts, with \( S \circ S \subseteq S \), and with \( \Sigma^{-i}G \in S \) for \( A \leq i \leq B \).

(v) For triples of integers \( A \leq B, n > 0 \) we let \( \text{Coprod}_n(G[A,B]) \). In this formula we also allow \( A \) and \( B \) to be infinite.

(vi) For pairs of integers \( A \leq B \) we let \( \text{Coprod}(G[A,B]) \). In this formula we also allow \( A \) and \( B \) to be infinite, but as it happens for infinite \( A \) we obtain nothing new. The categories

\[
\text{Coprod}(G(-\infty,B]), \quad \text{Coprod}(G(-\infty,\infty))
\]

are closed under coproducts and (positive) suspensions, and therefore contain all direct summands of their objects.

The following lemma is an easy consequence of the definitions.

**Lemma 0.9.** Suppose \( G, H \) are objects in a triangulated category \( \mathcal{T} \). We show

(i) If \( H \in \langle G \rangle \) then there exists an integer \( A > 0 \) with \( H \in \langle G \rangle_A^{\leq A,A} \),

(ii) If \( \langle G \rangle = \langle H \rangle \) then there exists an integer \( A > 0 \) with \( H \in \langle G \rangle_A^{\leq A,A} \) and \( G \in \langle H \rangle_A^{\leq A,A} \).

**Proof.** For (i) the assumption is \( H \in \langle G \rangle = \cup_{A>0} \langle G \rangle_A^{\leq A,A} \), hence \( H \) belongs to one of the sets in the union. For (ii) observe that \( \langle G \rangle = \langle H \rangle \) implies \( H \in \langle G \rangle \) and \( G \in \langle H \rangle \) and apply (i). \( \square \)

Now we come to the first new definition.

**Definition 0.10.** Suppose we are given two \( t \)-structures on a triangulated category \( \mathcal{T} \), that is we are given two pairs of subcategories \( (\mathcal{T}^{\leq 0}_1,\mathcal{T}^{\geq 0}_1) \) and \( (\mathcal{T}^{\leq 0}_2,\mathcal{T}^{\geq 0}_2) \) satisfying the conditions in \( [3] \) Définition 1.3.1]. These \( t \)-structures are equivalent if and only if there exists an integer \( A > 0 \) with \( \mathcal{T}^{\leq A}_1 \subset \mathcal{T}^{\leq 0}_2 \subset \mathcal{T}^{\geq A}_1 \).

**Observation 0.11.** For any \( t \)-structure \( (\mathcal{T}^{\leq 0},\mathcal{T}^{\geq 0}) \) we have \( \mathcal{T}^{\leq 0} = \Sigma^{-1}(\Sigma \mathcal{T}^{\geq 0}) \) and \( \mathcal{T}^{\geq 0} = (\Sigma \mathcal{T}^{\leq 0})^\perp \). It immediately follows that two \( t \)-structures \( (\mathcal{T}^{\leq 0}_1,\mathcal{T}^{\geq 0}_1) \) and \( (\mathcal{T}^{\leq 0}_2,\mathcal{T}^{\geq 0}_2) \) are equivalent if and only if there exists an integer \( A > 0 \) with \( \mathcal{T}^{\geq A}_1 \subset \mathcal{T}^{\geq 0}_2 \subset \mathcal{T}^{\geq A}_1 \).
Observation 0.12. Recall that, for any $t$-structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$, the categories $\mathcal{T}^-$, $\mathcal{T}^+$ and $\mathcal{T}^b$ are defined by

$$\mathcal{T}^- = \bigcup_{m > 0} \mathcal{T}_1^{\leq -m}, \quad \mathcal{T}^+ = \bigcup_{m > 0} \mathcal{T}_2^{\geq -m}, \quad \mathcal{T}^b = \mathcal{T}^- \cap \mathcal{T}^+.$$ 

If $(\mathcal{T}_1^{< 0}, \mathcal{T}_1^{\geq 0})$ and $(\mathcal{T}_2^{< 0}, \mathcal{T}_2^{\geq 0})$ are equivalent $t$-structures we note

(i) $\mathcal{T}_1^- = \mathcal{T}_2^-$, $\mathcal{T}_1^+ = \mathcal{T}_2^+$ and $\mathcal{T}_1^b = \mathcal{T}_2^b$.

(ii) If $\mathcal{T}^-$ [respectively $\mathcal{T}^+$, respectively $\mathcal{T}^b$] contains a compact generator $G \in \mathcal{T}^c$, then $\mathcal{T}^-$ [respectively $\mathcal{T}^+$, respectively $\mathcal{T}^b$] contains all of $\mathcal{T}^c$.

Proof. We prove (i) and (ii) for $\mathcal{T}^-$ and leave $\mathcal{T}^+$ and $\mathcal{T}^b$ to the reader. To prove (i) observe that the inclusions $\mathcal{T}_1^{\leq -A} \subset \mathcal{T}_2^{\leq 0} \subset \mathcal{T}_1^{\leq A}$ imply

$$\bigcup_{m > 0} \mathcal{T}_1^{\leq -A+m} \subset \bigcup_{m > 0} \mathcal{T}_2^{\leq m} \subset \bigcup_{m > 0} \mathcal{T}_1^{\leq A+m}$$

that is $\mathcal{T}_1^- \subset \mathcal{T}_2^- \subset \mathcal{T}_1^-$. For the proof of (ii) the assumption is that $G \in \mathcal{T}^-$. This makes $\mathcal{T}^- \subset \mathcal{T}$ a thick subcategory containing $G$, hence $\mathcal{T}^c = \langle G \rangle \subset \mathcal{T}^-$. \qed

Example 0.13. Let $\mathcal{T}$ be a triangulated category with coproducts. Given any compact object $G \in \mathcal{T}$, from Alonso, Jeremías and Souto [2, Theorem A.1] we learn that $\mathcal{T}$ has a unique $t$-structure $(\mathcal{T}_G^{< 0}, \mathcal{T}_G^{> 0})$ generated by $G$. In the notation of Reminder 0.8 the aisle $\mathcal{T}_G^{< 0}$ of this $t$-structure is nothing other than $\mathcal{T}_G^{< 0} = \langle G \rangle^{[-\infty, 0]}$. It follows formally that both $\mathcal{T}_G^{< 0}$ and $\mathcal{T}_G^{> 0}$ are closed under coproducts and direct summands—the closure under direct summands is true for any aisle and co-aisle of a $t$-structure, the closure of $\mathcal{T}_G^{< 0}$ under coproducts is also true for any aisle, while the fact that $\mathcal{T}_G^{> 0}$ is closed under coproducts may be found in [2, Proposition A.2]; it comes from the compactness of the object $G$.

If $G, H$ are two compact objects of $\mathcal{T}$ with $\langle G \rangle = \langle H \rangle$, Lemma 0.9(ii) tells us that there exists an integer $A > 0$ with $H \in \langle G \rangle_A^{[-A, A]}$ and $G \in \langle H \rangle_A^{[-A, A]}$. Hence $\langle H \rangle^{[-\infty, -A]} \subset \langle G \rangle^{[-\infty, 0]} \subset \langle H \rangle^{[-\infty, A]}$, that is $\mathcal{T}_H^{\leq -A} \subset \mathcal{T}_G^{< 0} \subset \mathcal{T}_H^{\leq A}$. Thus the $t$-structures generated by $G$ and $H$ are equivalent. This leads us to

Definition 0.14. If the compactly generated triangulated category $\mathcal{T}$ has a single compact object $G$ that generates it, then the preferred equivalence class of $t$-structures is the one containing the $t$-structure $(\mathcal{T}_G^{< 0}, \mathcal{T}_G^{> 0})$ generated by $G$.

Remark 0.15. For any compact generator $G$ we have that $\langle G \rangle = \mathcal{T}^c$, the full subcategory of all compact objects. Any two compact generators $G, H$ satisfy $\langle G \rangle = \mathcal{T}^c = \langle H \rangle$, and Example 0.13 says that $G$ and $H$ generate equivalent $t$-structures. Thus the preferred equivalence class of $t$-structures does not depend on the choice of compact generator.

Now [2, Proposition A.2] guarantees that, in the preferred equivalence class, there will exist some $t$-structures with $\mathcal{T}^{\leq 0}$ and $\mathcal{T}^{> 0}$ both closed under coproducts—just take $(\mathcal{T}_G^{< 0}, \mathcal{T}_G^{> 0})$ for a compact generator $G$. The reader should note that this property is not
stable under equivalence. In general there will be \( t \)-structures in the preferred equivalence class where \( T \geq 0 \) is not closed in \( T \) under coproducts.

From Observation \( 0.12(\text{i}) \) we learn that, as long as we stick to the preferred equivalence class of \( t \)-structures, the categories \( T^-, T^+ \) and \( T^b \) are intrinsic.

And now for the next formal construction.

**Definition 0.16.** Suppose \( T \) is a triangulated category with coproducts and let \( (T^\leq 0, T^\geq 0) \) be a \( t \)-structure.

An object \( F \) belongs to the subcategory \( T^- \subset T \) if, for any integer \( m > 0 \), there exists a triangle \( E \rightarrow F \rightarrow D \) with \( E \in T^c \) and \( D \in T^\leq -m \).

The subcategory \( T^b \) is defined by \( T^b = T^- \cap T^b \).

**Remark 0.17.** Note that the definition of \( T^- \) depends on the choice of a \( t \)-structure, but not much—equivalent \( t \)-structures lead to the same \( T^- \). For any choice of \( t \)-structure the category \( T^- \) contains \( T^c \). After all if \( F \) is compact then the triangle \( F \rightarrow F \rightarrow 0 \) has \( F \in T^c \) and \( 0 \in T^\leq -m \), for every \( m \) and every \( t \)-structure.

**Remark 0.18.** Assume the \( t \)-structure \( (T^\leq 0, T^\geq 0) \) is such that there is a compact generator \( G \) contained in \( T^- \); any \( t \)-structure in the preferred equivalence class is an example, after all \( G \in (G)^{-\infty,0} = T^\leq G \subset T^- \). Observation \( 0.12(\text{ii}) \) gives that \( T^c \subset T^- \), and Definition \( 0.16 \) tells us that, for any integer \( m > 0 \),

\[
T^- \subset T^c \cap T^\leq -m \subset T^- = T^-.
\]

Still in gorgeous generality we will prove

**Proposition 0.19.** Let \( T \) be a triangulated category with coproducts, and let \( (T^\leq 0, T^\geq 0) \) be a \( t \)-structure. If there exists an integer \( A > 0 \) and a compact generator \( G \in T \) with \( \text{Hom}(\Sigma^{-A}G, T^\leq 0) = 0 \) then \( T^b \subset T^- \) are triangulated subcategories of \( T \). If furthermore \( G \in T^- \), then \( T^b \subset T^- \subset T \) are thick subcategories of \( T^- \).

**Remark 0.20.** We are most interested in the special case where the \( t \)-structure \( (T^\leq 0, T^\geq 0) \) is in the preferred equivalence class and \( T^c \subset T^- \) are independent of choices.

Suppose there exists a compact generator \( G \) and an integer \( A > 0 \), so that \( \text{Hom}(G, \Sigma^iG) = 0 \) for all \( i \geq A \). Define the full subcategory \( S \) by

\[
S = \{ S \in T \mid \text{Hom}(\Sigma^{-A}G, S) = 0 \}.
\]

The compactness of \( G \) says that \( S \) is closed under coproducts, by hypothesis \( S \) contains \( \Sigma^iG \) for all \( i \geq 0 \), while obviously \( S \) is closed under direct summands and \( S \) is \( S \). Therefore \( S \) contains \( (G)^{-\infty,0} = T^\leq_G \). We deduce that \( \text{Hom}(\Sigma^{-A}G, T^\leq_G) = 0 \). Since \( G \) is obviously in \( T^\leq_G \subset T^- \), Proposition \( 0.19 \) informs us that \( T^b \subset T^- \) are thick subcategories of \( T^- \).
For the structure defined so far we needed very little. To go further it turns out to be useful to estimate how much effort it takes to approximate an object in $\mathcal{T}^-$ by a compact generator $G$. This leads us to

**Definition 0.21.** Let $\mathcal{T}$ be a triangulated category with coproducts. The category $\mathcal{T}$ is called weakly approximable if there exists a compact generator $G$, a $t$–structure $(\mathcal{T}^\leq 0, \mathcal{T}^\geq 0)$ and an integer $A > 0$ so that

(i) $\Sigma^A G \in \mathcal{T}^\leq 0$ and $\text{Hom}(\Sigma^{-A} G, \mathcal{T}^\leq 0) = 0$.

(ii) Every object $F \in \mathcal{T}^\leq 0$ admits a triangle $E \rightarrow F \rightarrow D$ with $E \in \overline{\langle G \rangle}^{-A, A}$ and $D \in \mathcal{T}^\leq -1$.

The category $\mathcal{T}$ is called approximable if the integer $A$ can be chosen to further satisfy

(iii) In the triangle $E \rightarrow F \rightarrow D$ of (ii) above we may strengthen the condition on $E$, we may assume $E \in \langle G \rangle_A \subset \langle G \rangle^{-A, A}$.

The following are easy to prove, they will be part of a string of formal consequences of approximability, see Section 2.

**Facts 0.22.** Let $\mathcal{T}$ be a triangulated category with coproducts. If $\mathcal{T}$ is weakly approximable then

(i) The $t$–structure $(\mathcal{T}^\leq 0, \mathcal{T}^\geq 0)$, which is part of Definition 0.21 and is assumed to satisfy some hypotheses, must belong to the preferred equivalence class.

(ii) For any compact generator $G$ and any $t$–structure $(\mathcal{T}^\leq 0, \mathcal{T}^\geq 0)$ in the preferred equivalence class there must exist an integer $A$, depending on $G$ and on the $t$–structure $(\mathcal{T}^\leq 0, \mathcal{T}^\geq 0)$, which satisfies Definition 0.21(i) and (ii). If $\mathcal{T}$ is approximable the integer $A$ may be chosen to satisfy (iii) as well.

Thus in proving that $\mathcal{T}$ is (weakly) approximable we can choose our compact generator and $t$–structure to suit our convenience. Once we know the category is approximable, it follows that the convenient $t$–structure is in the preferred class, and any compact generator and any $t$–structure in the preferred equivalence class fulfill the approximability criteria.

**Facts 0.23.** As stated in the first few paragraphs of the introduction [before we presented the definitions] we will prove that, if $X$ is a quasicompact, separated scheme, then $\mathcal{T} = D_{\text{qc}}(X)$ is approximable and the standard $t$–structure is in the preferred equivalence class. If $X$ is noetherian then $\mathcal{T}^b_c \subset \mathcal{T}^-_c$ are just $D_{\text{coh}}^b(X) \subset D_{\text{coh}}^<(X)$, for non-noetherian $X$ the description of $\mathcal{T}^b_c \subset \mathcal{T}^-_c$ is slightly more complicated, but still classical—see Example 3.4. The fact that the standard $t$–structure is in the preferred equivalence class tells us that $\mathcal{T}^- = D_{\text{qc}}^>(X), \mathcal{T}^+ = D_{\text{qc}}^<(X)$ and $\mathcal{T}^b = D_{\text{qc}}^b(X)$.

Another example is the homotopy category $\mathcal{T}$ of spectra. In this case we can take $\mathcal{T}^\leq 0 \subset \mathcal{T}$ to be the subcategory of connective spectra—the $t$–structure this defines is in the preferred equivalence class. The category $\mathcal{T}$ turns out to be approximable, and the subcategory $\mathcal{T}^-_c$ is the category of spectra $X$ whose stable homotopy groups $\pi_i(X)$ are
finitely generated \( \mathbb{Z} \)-modules and \( \pi_i(X) = 0 \) for \( i \ll 0 \). And \( \mathcal{J}_c^b \subset \mathcal{J}_c^- \) is the subcategory where all but finitely many of the \( \pi_i(X) \) vanish.

The representability we prove in Theorem \( \text{0.3} \) applies to this example but the result is not new. There is a theorem of Adams \( \text{[1]} \) which says that every cohomological functor \( H \) on \( \mathcal{J} \) is the restriction of a representable one on \( \mathcal{J} \), and it is easy to show that finiteness or local finiteness of \( H \) translate to saying that the representing object must lie in \( \mathcal{J}_c^b \) or \( \mathcal{J}_c^- \). But the theorem of Adams does not generalize to \( D_{\text{perf}}(X) \subset D_{\text{qc}}(X) \); see \( \text{[15, 8]} \).

We will not give a proof but the interested reader can check that, if \( X \) is a quasicompact, separated scheme and \( \mathcal{Z} \subset X \) is a closed subset with quasicompact complement, then the category \( \mathcal{J} = D_{\text{qc},Z}(X) \), the subcategory of \( D_{\text{qc}}(X) \) of all complexes supported on \( Z \), is weakly approximable but not approximable. The standard \( t \)-structure is in the preferred equivalence class. If \( X \) is noetherian the categories \( \mathcal{J}_c^+ \) and \( \mathcal{J}_c^b \) are (respectively) the intersections of \( D_{\text{coh}}^+(X) \) and \( D_{\text{coh}}^b(X) \) with the category \( D_{\text{qc},Z}(X) \).

The definitions have all been made and the reader can go back to the statements of Theorem \( \text{0.3} \) and Corollary \( \text{0.4} \) which are now precise. Note that in both results \( \mathcal{J} \) has to be approximable, weakly approximable is not enough.

We have discussed what we know, but should point out that there are many more potential examples. After all: let \( R \) be a commutative ring and let \( T \) be a dg \( R \)-algebra. Then the category \( \mathcal{J} = \mathcal{H}^0(T-\text{Mod}) \) is a triangulated category with coproducts and a single compact generator \( T \). It has a preferred equivalence class of \( t \)-structures, one can define the intrinsic subcategories \( \mathcal{J}^- , \mathcal{J}^+ , \mathcal{J}^b , \mathcal{J}_c^- \) and \( \mathcal{J}_c^b \), and in general I have no idea what they are. If \( H^i(T) = 0 \) for \( i \gg 0 \) then the subcategories \( \mathcal{J}_c^- \) and \( \mathcal{J}_c^b \) are thick, this follows from Remark \( \text{0.20} \). If \( H^i(T) = 0 \) for all \( i > 0 \) we are in the trivial case (see Remark \( \text{3.3} \)), where it’s easy to prove the category \( \mathcal{H}^0(T-\text{Mod}) \) approximable and work out explicitly what are \( \mathcal{J}^- , \mathcal{J}^+ , \mathcal{J}^b , \mathcal{J}_c^- \) and \( \mathcal{J}_c^b \).

So far the only other general result, producing further examples of approximable triangulated categories, is \( \text{[6, Theorem 4.1]} \). It says that, under reasonable hypotheses, the recollement of two approximable triangulated categories is approximable. But for \( T \) a general dga, satisfying \( H^i(T) = 0 \) for \( i \gg 0 \), I have no idea when the categories \( \mathcal{H}^0(T-\text{Mod}) \) are approximable. In view of Theorem \( \text{0.3} \) and Corollary \( \text{0.4} \) it would be interesting to find out, especially since the categories \( \mathcal{H}^0(T-\text{Mod}) \) are of so much current active interest—their study is at the core of noncommutative algebraic geometry. Who knows, there might be a noncommutative generalization of GAGA.

1. Basics

Since \( t \)-structures will play a big part in the article we begin with a quick reminder of some elementary facts.

**Reminder 1.1.** In this section \( \mathcal{J} \) will be a triangulated category and \( (\mathcal{J}^{\leq 0}, \mathcal{J}^{\geq 0}) \) will be a \( t \)-structure on \( \mathcal{J} \). The category \( A = \mathcal{J}^{\leq 0} \cap \mathcal{J}^{\geq 0} \) is abelian, it is called the heart of the
Lemma 1.2. Let $\mathcal{T}$ be a triangulated category and let $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ be a $t$–structure on $\mathcal{T}$. If $F$ is an object of $\mathcal{T}^-$, and $\mathcal{H}^\ell(F) = 0$ for all $\ell > -i$, then $F$ belongs to $\mathcal{T}^{\leq -i}$.

Proof. We are given that $F$ belongs to $\mathcal{T}^- = \cup_n \mathcal{T}^{\leq n}$, hence $F \in \mathcal{T}^{\leq n}$ for some $n$ and the map $F^{\leq n} \to F$ is an isomorphism. But now the triangle $F^{\leq \ell-1} \to F^{\leq \ell} \to \Sigma^{-\ell} \mathcal{H}^\ell(F)$ informs us that, as long as $\ell > -i$, the map $F^{\leq \ell-1} \to F^{\leq \ell}$ is also an isomorphism. Composing the string of isomorphisms $F^{\leq -i} \to F^{\leq -i+1} \to \cdots \to F^{\leq n} \to F$ we have that $F^{\leq -i} \to F$ is an isomorphism—therefore $F \in \mathcal{T}^{\leq -i}$. □

Lemma 1.3. If there is an integer $A$ and a generator $G \in \mathcal{T}$ with $\text{Hom}(G, \mathcal{T}^{-A}) = 0$, then

(i) Any object $F \in \mathcal{T}^-$, with $\mathcal{H}^\ell(F) = 0$ for all $\ell$, must vanish.

(ii) If $f : E \to F$ is a morphism in $\mathcal{T}^-$ such that $\mathcal{H}^\ell(f)$ is an isomorphism for every $\ell \in \mathbb{Z}$, then $f$ is an isomorphism.

Proof. To prove (i) assume $\mathcal{H}^\ell(F) = 0$ for all $\ell$; Lemma 1.2 says that $F$ belongs to $\bigcap_i \mathcal{T}^{\leq i}$. But then $\text{Hom}(\Sigma^i G, F) = 0$ for all $i \in \mathbb{Z}$, and as $G$ is a generator this implies $F = 0$.

(ii) follows by applying (i) to the mapping cone of $f$. □

Lemma 1.4. Suppose the category $\mathcal{T}$ has coproducts, and the $t$–structure is such that both $\mathcal{T}^{\leq 0}$ and $\mathcal{T}^{\geq 0}$ are closed under the coproducts of $\mathcal{T}$. Then:

(i) The functors $(-)^{\leq 0}$ and $(-)^{\geq 0}$ both respect coproducts.

(ii) The heart $\mathcal{A}$ is closed in $\mathcal{T}$ under coproducts, and the functor $\mathcal{H} : \mathcal{T} \to \mathcal{A}$ respects coproducts.

(iii) The abelian category $\mathcal{A}$ satisfies [AB4], that is coproducts are exact.

(iv) If $E_1 \to E_2 \to E_3 \to \cdots$ is a sequence of objects and morphisms in $\mathcal{T}$, then there is a short exact sequence in the heart $\mathcal{A}$ of the $t$–structure

\[
\begin{array}{c}
0 \to \operatorname{colim} \mathcal{H}^\ell(E_i) \to \mathcal{H}^\ell \left( \operatorname{Hocolim} E_i \right) \to \operatorname{colim} \mathcal{H}^{\ell+1}(E_i) \to 0
\end{array}
\]

Proof. Suppose we are given in $\mathcal{T}$ a collection of objects $\{E_\lambda, \lambda \in \Lambda\}$. For each $\lambda$ we have a canonical triangle $E^{\leq 0}_\lambda \to E_\lambda \to E^{\geq 1}_\lambda \to \Sigma E^{\leq 0}_\lambda$. The coproduct of these triangles is a triangle

\[
\bigoplus_{\lambda \in \Lambda} E^{\leq 0}_\lambda \to \bigoplus_{\lambda \in \Lambda} E_\lambda \to \bigoplus_{\lambda \in \Lambda} E^{\geq 1}_\lambda \to \bigoplus_{\lambda \in \Lambda} \Sigma E^{\leq 0}_\lambda
\]

By hypothesis $\bigoplus_{\lambda \in \Lambda} E^{\leq 0}_\lambda$ belongs to $\mathcal{T}^{\leq 0}$ and $\bigoplus_{\lambda \in \Lambda} E^{\geq 1}_\lambda$ belongs to $\mathcal{T}^{\geq 1}$, and the triangle above must be canonically isomorphic to

\[
\bigoplus_{\lambda \in \Lambda} (E^{\leq 0}_\lambda) \to \bigoplus_{\lambda \in \Lambda} E_\lambda \to \bigoplus_{\lambda \in \Lambda} (E^{\geq 1}_\lambda) \to \bigoplus_{\lambda \in \Lambda} (\Sigma E^{\leq 0}_\lambda)
\]
This proves (i).

Since \( \mathcal{T}^{\leq 0} \) and \( \mathcal{T}^{\geq 0} \) are closed in \( \mathcal{T} \) under coproducts so is their intersection \( \mathcal{A} = \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0} \). By (i) we know that the functors \((-)^{\leq 0}\) and \((-)^{\geq 0}\) both respect coproducts, hence so does their composition \( \mathcal{H}(-) = \left[ (-)^{\leq 0} \right]^{\geq 0} \). This proves (ii).

The category \( \mathcal{T} \) has coproducts and its subcategory \( \mathcal{A} \) is closed under these coproducts, hence \( \mathcal{A} \) has coproducts—it satisfies [AB3]. Now suppose we are given a set \( \{ f_\lambda : A_\lambda \rightarrow B_\lambda, \lambda \in \Lambda \} \) of morphisms in \( \mathcal{A} \). Complete these to triangles \( A_\lambda \rightarrow B_\lambda \rightarrow C_\lambda \rightarrow \Sigma A_\lambda \) and form the coproduct

\[
\bigoplus_{\lambda \in \Lambda} A_\lambda \xrightarrow{\bigoplus_{\lambda \in \Lambda} f_\lambda} \bigoplus_{\lambda \in \Lambda} B_\lambda \rightarrow \bigoplus_{\lambda \in \Lambda} C_\lambda \rightarrow \bigoplus_{\lambda \in \Lambda} \Sigma A_\lambda
\]

which is a triangle. The long exact sequence obtained by applying \( \mathcal{H} \) to this triangle tells us that the kernel of the map \( \bigoplus_{\lambda \in \Lambda} f_\lambda \) is \( \mathcal{H}^{-1} \left( \bigoplus_{\lambda \in \Lambda} C_\lambda \right) \), but (ii) informs us that this is \( \bigoplus_{\lambda \in \Lambda} \mathcal{H}^{-1}(C_\lambda) \), which is \( \bigoplus_{\lambda \in \Lambda} \ker(f_\lambda) \). The right exactness of coproducts is formal, completing the proof of (iii).

Finally (iv) follows by applying the functor \( \mathcal{H} \) to the triangle

\[
\bigoplus_{i=1}^\infty E_i \rightarrow \bigoplus_{i=1}^\infty E_i \rightarrow \operatorname{Hocolim} E_i \rightarrow \bigoplus_{i=1}^\infty E_i
\]

and using (ii) to compute the long exact sequence. \( \square \)

Remark 1.5. Remark 0.15 tells us that, if \( \mathcal{T} \) is a triangulated category with coproducts and a single compact generator, then the preferred equivalence class contains \( t \)-structures \( (\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}) \) with \( \mathcal{T}^{\leq 0} \) and \( \mathcal{T}^{\geq 0} \) both closed under coproducts. This is the situation in which we will apply Lemma 1.4. Note also that Remark 0.15 warns us that not every \( t \)-structure in the preferred equivalence class need satisfy the property.

We will mostly use Lemma 1.4(iv) in the special case where the sequences \( \mathcal{H}^\ell(E_1) \rightarrow \mathcal{H}^\ell(E_2) \rightarrow \mathcal{H}^\ell(E_3) \rightarrow \cdots \) eventually stabilize for every \( \ell \). When this happens the \( \operatorname{colim}^1 \) terms all vanish, and the natural map is an isomorphism \( \operatorname{colim} \mathcal{H}^\ell(E_i) \rightarrow \mathcal{H}^\ell(\operatorname{Hocolim} E_i) \).

2. The fundamental properties of approximability

Lemma 2.1. Let \( \mathcal{T} \) be a triangulated category with a \( t \)-structure \( (\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}) \), and let \( \mathcal{S} \subset \mathcal{T} \) be a full subcategory with \( \Sigma \mathcal{S} = \mathcal{S} \). Assume \( \mathcal{A} \) is also a full subcategory of \( \mathcal{T} \), and define \( \mathcal{A}(m) \) inductively by

(i) \( \mathcal{A}(1) = \mathcal{A} \).

(ii) \( \mathcal{A}(m+1) = \mathcal{A}(m) \star \Sigma^m \mathcal{A} \).

Suppose every object in \( F \in \mathcal{S} \cap \mathcal{T}^{\leq 0} \) admits a triangle \( E_1 \rightarrow F \rightarrow D_1, \) with \( E_1 \in \mathcal{A} \) and \( D_1 \in \mathcal{S} \cap \mathcal{T}^{\leq -1} \). Then we can construct a sequence \( E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow \cdots \),
with a map from the sequence to \( F \) and so that, if we complete \( E_m \to F \) to a triangle \( E_m \to F \to D_m \), then \( E_m \in A(m) \) and \( D_m \in S \cap \mathcal{T}^{\leq -m} \).

**Proof.** We are given the case \( m = 1 \); assume we have constructed the sequence as far as an integer \( m > 0 \), and we want to extend it to \( m + 1 \). Take any object \( F \in S \cap \mathcal{T}^{\leq 0} \), and by the inductive hypothesis construct the sequence up to \( m \). In particular choose a triangle \( E_m \to F \to D_m \) with \( E_m \in A(m) \) and \( D_m \in S \cap \mathcal{T}^{\leq -m} \). Now apply the case \( m = 1 \) to \( \Sigma^{-m}D_m \); we produce a triangle \( E' \to D_m \to D_{m+1} \) with \( D_{m+1} \in S \cap \mathcal{T}^{\leq -m-1} \) and \( E' \in \Sigma^m A \). Form an octahedron from the composable morphisms \( F \to D_m \to D_{m+1} \), that is

```
\[
\begin{array}{ccc}
E_m & \to & E_m \\
\downarrow & & \downarrow \\
E_{m+1} & \to & F & \to & D_{m+1} \\
\downarrow & & \downarrow & & \downarrow \\
E' & \to & D_m & \to & D_{m+1}
\end{array}
\]
```

The object \( D_{m+1} \) belongs to \( S \cap \mathcal{T}^{\leq -m-1} \) by construction. The triangle \( E_m \to E_{m+1} \to E' \) tells us that \( E_{m+1} \in A(m) * \Sigma^m A = A(m + 1) \), and we have factored the map \( E_m \to F \) as \( E_m \to E_{m+1} \to F \) so that, in the triangle \( E_{m+1} \to F \to D_{m+1} \), we have \( E_{m+1} \in A(m + 1) \) and \( D_{m+1} \in S \cap \mathcal{T}^{\leq -m-1} \).

**Corollary 2.2.** Let \( \mathcal{T} \) be a triangulated category with coproducts, let \( G \in \mathcal{T} \) be an object, and let \(( \mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}) \) be a \( t \)-structure. The following is true.

**2.2.1.** Suppose every object \( F \in \mathcal{T}^{\leq 0} \) admits a triangle \( E_1 \to F \to D_1 \), with \( E_1 \in \langle G \rangle^{[-A,A]} \) and \( D_1 \in \mathcal{T}^{\leq -1} \). Then we can extend to a sequence \( E_1 \to E_2 \to E_3 \to \cdots \), with a map from the sequence to \( F \) and so that, if we complete \( E_m \to F \) to a triangle \( E_m \to F \to D_m \), then \( E_m \in \langle G \rangle^{[-m-A,A]} \) and \( D_m \in \mathcal{T}^{\leq -m} \).

**2.2.2.** Suppose every object \( F \in \mathcal{T}^{\leq 0} \) admits a triangle \( E_1 \to F \to D_1 \), with \( E_1 \in \langle G \rangle^{[A]} \) and \( D_1 \in \mathcal{T}^{\leq -1} \). Then we can extend to a sequence \( E_1 \to E_2 \to E_3 \to \cdots \), with a map from the sequence to \( F \) and so that, if we complete \( E_m \to F \) to a triangle \( E_m \to F \to D_m \), then \( E_m \in \langle G \rangle^{[-m-A,A]} \) and \( D_m \in \mathcal{T}^{\leq -m} \).

**2.2.3.** For a full subcategory \( S \subset \mathcal{T} \) with \( \Sigma S = S \), suppose every object \( F \in S \cap \mathcal{T}^{\leq 0} \) admits a triangle \( E_1 \to F \to D_1 \), with \( E_1 \in \langle G \rangle^{[-A,A]} \) and \( D_1 \in S \cap \mathcal{T}^{\leq -1} \). Then we can extend to a sequence \( E_1 \to E_2 \to E_3 \to \cdots \), with a map from the sequence to \( F \) and so that, if we complete \( E_m \to F \) to a triangle \( E_m \to F \to D_m \), then \( E_m \in \langle G \rangle^{[-m-A,A]} \) and \( D_m \in S \cap \mathcal{T}^{\leq -m} \).

**2.2.4.** For a full subcategory \( S \subset \mathcal{T} \) with \( \Sigma S = S \), suppose every object \( F \in S \cap \mathcal{T}^{\leq 0} \) admits a triangle \( E_1 \to F \to D_1 \), with \( E_1 \in \langle G \rangle^{[-A,A]} \) and \( D_1 \in S \cap \mathcal{T}^{\leq -1} \). Then
we can extend to a sequence \( E_1 \to E_2 \to E_3 \to \cdots \), with a map from the sequence to \( F \) and so that, if we complete \( E_m \to F \) to a triangle \( E_m \to F \to D_m \), then \( E_m \in (G)^{1-m-A,A}_{mA} \) and \( D_m \in S \cap \mathcal{T}^{-m} \).

**Proof.** In each case we apply Lemma 2.1 with a suitable choice of \( A \) and \( S \).

To prove (2.2.1) let \( S = \mathcal{T} \) and let \( A = (G)^{-A,A}_{A} \). By induction we see that \( A(m) \subset \langle G \rangle^{1-m-A,A}_{mA} \) and the result follows.

To prove (2.2.2) let \( S = \mathcal{T} \) and let \( A = (G)^{-A,A}_{A} \). By induction we see that \( A(m) \subset \langle G \rangle^{1-m-A,A}_{mA} \) and the result follows.

To prove (2.2.3) let \( \mathcal{A} = (G)^{-A,A}_{A} \). By induction we see that \( A(m) \subset \langle G \rangle^{1-m-A,A}_{mA} \) and the result follows.

To prove (2.2.4) let \( \mathcal{A} = (G)^{-A,A}_{A} \). By induction we see that \( A(m) \subset \langle G \rangle^{1-m-A,A}_{mA} \) and the result follows.

**Lemma 2.3.** Suppose \( \mathcal{T} \) is a compactly generated triangulated category, \( G \) is a compact generator and \( (\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}) \) a \( t \)-structure. Suppose there exists an integer \( B \) with \( \text{Hom}(\Sigma^{-B}G, \mathcal{T}^{\leq 0}) = 0 \).

With any sequence \( E_1 \to E_2 \to E_3 \to \cdots \) mapping to \( F \), and such that in the triangles \( E_m \to F \to D_m \) we have \( D_m \in \mathcal{T}^{\leq -m} \), the (non-canonical) map \( \text{Hocolim} \ E_m \to F \) is an isomorphism.

**Proof.** For any \( n \geq 0 \) we have \( \mathcal{T}^{\leq -n} \subset \mathcal{T}^{\leq 0} \), hence \( \text{Hom}(\Sigma^{-B}G, \mathcal{T}^{\leq -n}) = 0 \). By shifting we deduce that \( \text{Hom}(\Sigma^{-\ell}G, \mathcal{T}^{\leq -m}) = 0 \) as long as \( m + \ell \geq B \).

The triangle \( E_m \to F \to D_m \), with \( D_m \in \mathcal{T}^{\leq -m} \), tells us that if \( m > \max(1, B - \ell) \) then the functor \( \text{Hom}(\Sigma^{-\ell}G, -) \) takes the map \( E_m \to F \) to an isomorphism. Now [14, Lemma 2.8], applied to the compact object \( G \in \mathcal{T} \) and the map from the sequence \( \{ E_m \} \) to \( F \), tells us that \( \text{Hom}(\Sigma^{-\ell}G, -) \) takes the map \( \text{Hocolim} \ E_m \to F \) to an isomorphism. But \( G \) is a generator, hence the map \( \text{Hocolim} \ E_m \to F \) must be an isomorphism.

**Proposition 2.4.** Suppose the triangulated category \( \mathcal{T} \), the generator \( G \) and the \( t \)-structure \( (\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}) \) are as in the hypotheses of weakly approximable categories of Definition 0.21. We remind the reader: \( \mathcal{T} \) has coproducts, \( G \) is a compact generator, and there is an integer \( A > 0 \) so that

(i) \( \Sigma^A G \in \mathcal{T}^{\leq 0} \) and \( \text{Hom}(\Sigma^{-A}G, \mathcal{T}^{\leq 0}) = 0 \).

(ii) Every object \( F \in \mathcal{T}^{\leq 0} \) admits a triangle \( E \to F \to D \) with \( E \in \langle G \rangle^{-A,A}_{A} \) and \( D \in \mathcal{T}^{\leq -1} \).

Then the \( t \)-structure \( (\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}) \) is in the preferred equivalence class.

**Proof.** By (i) we have \( \Sigma^A G \in \mathcal{T}^{\leq 0} \), hence \( \Sigma^m G \in \mathcal{T}^{\leq 0} \) for all \( m \geq A \). Therefore \( \mathcal{T}^{\leq 0} \) contains \( \langle G \rangle^{(-\infty,-A)}_{-A} = \mathcal{T}^{\leq -A} \). It remains to show an inclusion in the other direction.

But (2.2.1) constructed, for every object \( F \in \mathcal{T}^{\leq 0} \), a sequence \( E_1 \to E_2 \to E_3 \to \cdots \) with \( E_m \in \langle G \rangle^{1-m-A,A}_{mA} \subset \langle G \rangle^{(-\infty,-A)} \). In Lemma 2.3 we proved that \( F \) is isomorphic
to $\text{Hocolim} E_m$. There exists a triangle
\[
\bigoplus_{m=1}^\infty E_m \longrightarrow F \longrightarrow \sum_{m=1}^\infty \bigoplus_{m=1}^\infty E_m
\]
where the outside terms obviously lie in $\langle G \rangle^{[-\infty,A]} \subseteq T_G^A$. Hence $F \in T_G^A$, and since $F \in T^0$ is arbitrary we conclude that $T^0 \subseteq T_G^A$.

Lemma 2.5. Let $\mathcal{T}$ be a compactly generated triangulated category, let $G$ be a compact generator, and let $(T_1^{\leq 0}, T_1^{\geq 0})$ and $(T_2^{\leq 0}, T_2^{\geq 0})$ be two equivalent $t$–structures. Let $A > 0$ be an integer so that, with $k$ we must replace $A$ by $B$. Then, after increasing the integer $A$ if necessary, (i) and (ii) will also hold for $k = 2$. Furthermore if (iii) below holds for $k = 1$

(i) $\Sigma^A G \in T_1^{\leq 0}$ and $\text{Hom}(\Sigma^{-A} G, T_1^{\leq 0}) = 0$.

(ii) Every object $F \in T_1^{\leq 0}$ admits a triangle $E \longrightarrow F \longrightarrow D$ with $E \in \langle G \rangle^{[-A,A]}$ and $D \in T_1^{\leq -1}$. both hold. Then, after increasing the integer $A$ if necessary, (i) and (ii) will also hold for $k = 2$. Furthermore if (iii) below holds for $k = 1$

(iii) In the triangle $E \longrightarrow F \longrightarrow D$ of (ii) above we may strengthen the condition on $E$, we may assume $E \in \langle G \rangle^{[-A,A]} \subset \langle G \rangle^{[-A,A]}$.

then the integer $A$ may be chosen large enough so that (iii) will hold for $k = 2$.

Proof. Because the $t$–structures are equivalent we may choose an integer $B$ so that $T_2^{\leq -B} \subset T_1^{\leq 0} \subset T_2^{\leq B}$. Hence $\text{Hom}(\Sigma^{-A-B} G, T_2^{\leq 0}) \cong \text{Hom}(\Sigma^{-A} G, T_2^{\leq -B}) = 0$, where the vanishing is because $T_2^{\leq -B} \subset T_1^{\leq 0}$ and $\text{Hom}(\Sigma^{-A} G, T_1^{\leq 0}) = 0$. Also $\Sigma^A G \in T_1^{\leq 0} \subset T_2^{\leq B}$ implies $\Sigma^{A+B} G \in T_1^{\leq 0}$. This proves (i) for $k = 2$, as long as we replace $A$ by $A+B$.

If $F$ is an object in $T_2^{\leq 0} \subset T_2^{\leq B}$ we may, using (ii) in combination with (2.2.1) applied to $\Sigma^B F \in T_2^{\leq 0}$, construct a triangle $E_{2B+1} \longrightarrow F \longrightarrow D_{2B+1}$ with $E_{2B+1} \in \langle G \rangle^{[-B-A,B+1]}$ and $D_{2B+1} \in T_2^{\leq -B-1} \subset T_2^{\leq -1}$. Thus (ii) also holds for $k = 2$, as long as $A$ is replaced by $A+B$.

It remains to prove the assertion (iii) for $k = 2$, assuming it holds for $k = 1$. By (2.2.2) applied to $\Sigma^B F \in T_2^{\leq 0}$, we may construct the triangle $E_{2B+1} \longrightarrow F \longrightarrow D_{2B+1}$ with $E_{2B+1} \in \langle G \rangle^{[-B-A,B+1]}$ and $D_{2B+1} \in T_2^{\leq -B-1} \subset T_2^{\leq -1}$. Thus assertion (iii) holds, but we must replace $A$ by $A = \max \left[ A + B, A(2B + 1) \right]$.

Proposition 2.6. Suppose $\mathcal{T}$ is a weakly approximable triangulated category, $H$ is a compact generator, and $(T_1^{\leq 0}, T_1^{\geq 0})$ is any $t$–structure in the preferred equivalence class. Then there exists an integer $A > 0$ so that

(i) $\Sigma^A H \in T_1^{\leq 0}$ and $\text{Hom}(\Sigma^{-A} H, T_1^{\leq 0}) = 0$.

(ii) Every object $F \in T_1^{\leq 0}$ admits a triangle $E \longrightarrow F \longrightarrow D$ with $E \in \langle H \rangle^{[-A,A]}$ and $D \in T_1^{\leq -1}$.
If the category \( \mathcal{T} \) is approximable then the integer \( A \) may be chosen to further satisfy

(iii) In the triangle \( E \rightarrow F \rightarrow D \) of (ii) above we may strengthen the condition on \( E \), we may assume \( E \in \langle G \rangle_A^{[-A,A]} \subset \langle G \rangle^{[-A,A]} \).

**Proof.** The definition of weakly approximable categories gives us a compact generator \( G \), a \( t \)-structure \((\mathcal{T}^\leq, \mathcal{T}^\geq)\) and an integer \( A \) satisfying (i) and (ii), plus (iii) if \( \mathcal{T} \) is approximable. Proposition 2.4 guarantees that \((\mathcal{T}^\leq, \mathcal{T}^\geq)\) is in the preferred equivalence class of \( t \)-structures. By assumption so is \((\mathcal{T}^\leq_1, \mathcal{T}^\geq_1)\), hence the \( t \)-structures \((\mathcal{T}^\leq, \mathcal{T}^\geq)\) and \((\mathcal{T}^\leq_1, \mathcal{T}^\geq_1)\) are equivalent. By Lemma 2.5 we can, by modifying the integer \( A \), also have the conditions (i), (ii) and [when appropriate] (iii) hold for the the \( t \)-structure \((\mathcal{T}^\leq_1, \mathcal{T}^\geq_1)\) and the compact generator \( G \). Thus we may assume that the \( t \)-structures are the same. We have a single \( t \)-structure \((\mathcal{T}^\leq, \mathcal{T}^\geq) = (\mathcal{T}^\leq_1, \mathcal{T}^\geq_1)\), and two compact generators \( G \) and \( H \). There exists an integer \( A \) that works for \( G \) and the \( t \)-structure \((\mathcal{T}^\leq, \mathcal{T}^\geq)\), and we need to produce an integer that works for \( H \) and the \( t \)-structure \((\mathcal{T}^\leq, \mathcal{T}^\geq)\).

We are given that \( G \) and \( H \) are compact generators of \( \mathcal{T} \), hence \( \langle G \rangle = \mathcal{T}^c = \langle H \rangle \), and Lemma 0.9(ii) allows us to choose an integer \( B > 0 \) with \( G \in \langle H \rangle^{[-B,B]} \) and \( H \in \langle G \rangle^{[-B,B]} \). By (i) for \( G \) we know that \( \Sigma^A G \in \mathcal{T}^\leq \) and \( \text{Hom}(\Sigma^{-A} G, \mathcal{T}^\leq) = 0 \). It immediately follows that \( \langle G \rangle_B^{[-A-2B,-A]} \subset \mathcal{T}^\leq \) and that \( \text{Hom}(\langle G \rangle_B^{[A,A+2B]}, \mathcal{T}^\leq) = 0 \), and as \( \Sigma^A H \in \langle G \rangle_B^{[-A-2B,-A]} \) and \( \Sigma^{-A} H \in \langle G \rangle_B^{[A,A+2B]} \) we deduce that \( \Sigma^A H \in \mathcal{T}^\leq \) and that \( \text{Hom}(\Sigma^{-A} H, \mathcal{T}^\leq) = 0 \). This established (i) for \( H \), if we replace \( A \) by \( A + B \).

Now for (ii) and (iii): for any \( F \in \mathcal{T}^\leq \) we know that there exists a triangle \( E \rightarrow F \rightarrow D \) with \( D \in \mathcal{T}^{\leq-1} \), with \( E \in \langle G \rangle^{[-A,A]} \), and if \( \mathcal{T} \) is approximable we may even choose \( E \) to lie in \( \langle G \rangle_A^{[-A,A]} \). But \( G \) belongs to \( \langle H \rangle_B^{[-B,B]} \), and therefore

\[
\langle G \rangle_A^{[-A,A]} \subset \langle H \rangle^{[-A-B,A+B]} \quad \text{while} \quad \langle G \rangle_A^{[-A,A]} \subset \langle H \rangle_B^{[-A-B,A+B]} .
\]

Thus (ii) and [when appropriate] (iii) hold for \( H \) if \( A \) is replaced by \( \max(A + B, AB) \). \( \square \)

**Remark 2.7.** We have so far proved Facts 0.22. Proposition 2.4 amounts to 0.22(i) and Proposition 2.6 to 0.22(ii). The remainder of the section will be devoted to the basic properties of the subcategory \( \mathcal{T}^c \) of Definition 0.16.

**Lemma 2.8.** Suppose \( \mathcal{T} \) is a triangulated category with coproducts and let \((\mathcal{T}^\leq, \mathcal{T}^\geq)\) be a \( t \)-structure. Assume there exists a compact generator \( G \) and an integer \( A > 0 \) so that \( \text{Hom}(\Sigma^{-A} G, \mathcal{T}^\leq) = 0 \).

Then for any compact object \( H \in \mathcal{T} \) there exists an integer \( B > 0 \), depending on \( H \), with \( \text{Hom}(\Sigma^{-B} H, \mathcal{T}^\leq) = 0 \).

**Proof.** Let \( H \in \mathcal{T} \) be a compact object. The fact that \( G \) is a compact generator gives the equality in \( H \in \mathcal{T}^c = \langle G \rangle \); Lemma 0.9(i) allows us to deduce that \( H \in \langle G \rangle^{[-C,C]} \)
for some $C > 0$. Thus $\Sigma^{-A-C}H \in \langle G \rangle^{[A,A+2C]}$, and as $\text{Hom}(\langle G \rangle^{[A,A+2C]}, \mathcal{T}^{\leq 0}) = 0$ the Lemma follows, with $B = A + C$. \hfill \Box

**Lemma 2.9.** Suppose $\mathcal{T}$ is a compactly generated triangulated category and let $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ be a $t$–structure. Assume there exists a compact generator $G$ and an integer $A > 0$ so that $\text{Hom}(\Sigma^{-A}G, \mathcal{T}^{\leq 0}) = 0$.

Then the subcategory $\mathcal{T}_c^{-} \subset \mathcal{T}$ is triangulated.

**Proof.** It is clear that $\mathcal{T}_c^{-}$ is closed under all suspensions and is additive. We must show that, if $R \rightarrow S \rightarrow T \rightarrow \Sigma R$ is a triangle so that $R$ and $T$ belong to $\mathcal{T}_c^{-}$, then $S$ must also belong to $\mathcal{T}_c^{-}$.

Choose any integer $m > 0$. Because $T$ belongs to $\mathcal{T}_c^{-}$ we may choose a triangle $T' \rightarrow T \rightarrow T''$ with $T' \in \mathcal{T}_c$ and $T'' \in \mathcal{T}^{\leq -m}$. Since $T'$ is compact, Lemma 2.8 says that we may choose an integer $B > 0$ with $\text{Hom}(T', \mathcal{T}^{\leq -B}) = 0$.

Now $R$ belongs to $\mathcal{T}_c^{-}$, allowing us to choose a triangle $R' \rightarrow R \rightarrow R''$ with $R' \in \mathcal{T}_c$ and $R'' \in \mathcal{T}^{\leq -m-B}$. We have a diagram

\[
\begin{array}{ccc}
T' & \rightarrow & T \\
\downarrow & & \downarrow \\
\Sigma R' & \rightarrow & \Sigma R \\
\downarrow & & \downarrow \\
\Sigma R'' & \rightarrow & \Sigma R''
\end{array}
\]

The composite from top left to bottom right is a map $T' \rightarrow \Sigma R''$, with $\Sigma R'' \in \mathcal{T}^{\leq -m-B-1} \subset \mathcal{T}^{\leq -B}$. Since $B > 0$ was chosen so that $\text{Hom}(T', \mathcal{T}^{\leq -B}) = 0$ the map $T' \rightarrow \Sigma R''$ must vanish, hence the composite $T' \rightarrow \Sigma R$ must factor through $\Sigma R' \rightarrow \Sigma R$. We produce a commutative square

\[
\begin{array}{ccc}
T' & \rightarrow & T \\
\downarrow & & \downarrow \\
\Sigma R' & \rightarrow & \Sigma R
\end{array}
\]

which we may complete to a $3 \times 3$ diagram where the rows and columns are triangles

\[
\begin{array}{ccc}
R' & \rightarrow & R \\
\downarrow & & \downarrow \\
R'' & \rightarrow & \Sigma R'
\end{array}
\quad
\begin{array}{ccc}
S' & \rightarrow & S \\
\downarrow & & \downarrow \\
S'' & \rightarrow & \Sigma R'
\end{array}
\quad
\begin{array}{ccc}
T' & \rightarrow & T \\
\downarrow & & \downarrow \\
T'' & \rightarrow & \Sigma T'
\end{array}
\quad
\begin{array}{ccc}
\Sigma R' & \rightarrow & \Sigma R \\
\downarrow & & \downarrow \\
\Sigma R'' & \rightarrow & \Sigma^2 R''
\end{array}
\]
Because \( R' \) and \( T' \) are compact, the triangle \( R' \to S' \to T' \) tells us that \( S' \) must be compact. Also \( T'' \in \mathcal{T}^{\leq -m} \) and \( R'' \in \mathcal{T}^{\leq -m-\beta} \subset \mathcal{T}^{\leq -m} \), and the triangle \( R'' \to S'' \to T'' \) implies that \( S'' \in \mathcal{T}^{\leq -m} \). The triangle \( S' \to S \to S'' \) now does the job for \( S \). □

**Proposition 2.10.** Suppose \( \mathcal{T} \) is a compactly generated triangulated category and let \((\mathcal{T}^{\leq 0},\mathcal{T}^{\geq 0})\) be a \( t \)-structure. Assume there exists a compact generator \( G \) and an integer \( A > 0 \) so that \( \Sigma^A G \in \mathcal{T}^{\leq 0} \) and \( \text{Hom}(\Sigma^{-A}G, \mathcal{T}^{\leq 0}) = 0 \).

Then the subcategory \( \mathcal{T}_{-} \) is thick.

**Proof.** We already know that \( \mathcal{T}_{-} \) is triangulated, we need to prove it closed under direct summands. Suppose therefore that \( S \oplus T \) belongs to \( \mathcal{T}_{-} \), we must prove that so does \( S \).

Consider the map \( 0 \oplus \text{id} : S \oplus T \to S \oplus T \). Complete to a triangle

\[
S \oplus T \quad \xrightarrow{0 \oplus \text{id}} \quad S \oplus T \quad \to \quad S \oplus \Sigma S
\]

By Lemma 2.9 we deduce that \( S \oplus \Sigma S \) belongs to \( \mathcal{T}_{-} \). Induction on \( n \) allows us to prove that, for any \( n \geq 0 \), the object \( S \oplus \Sigma^{2n+1}S \) belongs to \( \mathcal{T}_{-} \). To spell it out: we have proved the case \( n = 0 \) above. For any \( n \) we know that \( \Sigma^{2n+1}(S \oplus \Sigma S) \cong \Sigma^{2n+2}S \oplus \Sigma^{2n+1}S \) lies in \( \mathcal{T}_{-} \), and induction on \( n \) allows us to assume that so does \( S \oplus \Sigma^{2n+1}S \). The triangle

\[
\Sigma^{2n+2}S \oplus \Sigma^{2n+1}S \quad \xrightarrow{0 \oplus \text{id}} \quad S \oplus \Sigma^{2n+1}S \quad \to \quad S \oplus \Sigma^{2n+3}S
\]

then informs us that \( S \oplus \Sigma^{2n+3}S \) belongs to \( \mathcal{T}_{-} \).

By Remark 0.18 the category \( \mathcal{T}_{c} \) is contained in \( \mathcal{T}_{-} \), and the object \( S \oplus \Sigma S \) must belong to \( \mathcal{T}_{\leq \ell} \) for some \( \ell > 0 \). Hence \( S \) belongs to \( \mathcal{T}_{\leq \ell} \) and, for every integer \( m > 0 \), we have that \( \Sigma^{m+\ell}S \in \mathcal{T}^{\leq -m} \). Choose an integer \( n \geq 0 \) with \( 2n + 2 \geq \ell + m \); then \( \Sigma^{2n+2}S \in \mathcal{T}^{\leq -m} \). Since the object \( S \oplus \Sigma^{2n+1}S \) belongs to \( \mathcal{T}_{-} \) we may choose a triangle \( K \to S \oplus \Sigma^{2n+1}S \to P \) with \( K \in \mathcal{T}_{c} \) and \( P \in \mathcal{T}^{\leq -m} \). Now form the octahedron on the composable morphisms \( K \to S \oplus \Sigma^{2n+1}S \to S \). We obtain

\[
\begin{array}{c}
K \\
\downarrow \\
\downarrow \\
K \\
\downarrow \\
\downarrow \\
S \\
\downarrow \\
\downarrow \\
\Sigma^{2n+2}S \\
\downarrow \\
\downarrow \\
\Sigma^{2n+2}S \\
\end{array}
\]

The triangle \( P \to Q \to \Sigma^{2n+2}S \), together with the fact that both \( P \) and \( \Sigma^{2n+2}S \) belong to \( \mathcal{T}^{\leq -m} \), tell us that \( Q \) must belong to \( \mathcal{T}^{\leq -m} \). Now the triangle \( K \to S \to Q \) does the trick for \( S \). □

The next few results work out how \( \mathcal{T}_{c} \) behaves when \( \mathcal{T} \) is approximable or weakly approximable.
Lemma 2.11. Let us fix a weakly approximable [or approximable] triangulated category \( \mathcal{T} \). Choose a compact generator \( G \) and a t-structure \((\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})\) in the preferred equivalence class. Choose an integer \( A \) as in Proposition 2.6 and let \( m > 2A + 1 \) be an integer.

Then for any \( K \in \mathcal{T}^\circ \cap \mathcal{T}^{\leq 0} \) there exists an object \( L \) and a triangle \( E \to K \oplus L \to D \) with \( E \in \langle G \rangle^{1-m-A,A} \) and \( D \in \langle G \rangle^{(-\infty,-m+A]} \).

If \( \mathcal{T} \) is approximable we may further assume \( E \in \langle G \rangle_{mA}^{1-m-A,A} \).

Proof. Because \( K \) belongs to \( \mathcal{T}^{\leq 0} \) the result (2.2.1) permits us to construct, for every integer \( m > 0 \), a triangle \( E_m \to K \to D_m \) with \( E_m \in \langle G \rangle^{1-m-A,A} \) and \( D_m \in \mathcal{T}^{\leq -m} \). If the category \( \mathcal{T} \) is approximable we may assume \( E_m \in \langle G \rangle_{mA}^{1-m-A,A} \).

The object \( K \) is assumed compact and Lemma 2.8 produces for us a positive integer \( B \), which we may assume \( \geq m+2A \), with \( \text{Hom}(K, \mathcal{T}^{\leq -B}) = 0 \). Since we chose \( B \geq m+2A \) we have \( \text{Hom}(\Sigma \langle G \rangle^{1-m-A,A}, \mathcal{T}^{\leq -B}) = 0 \), and in particular \( \text{Hom}(\Sigma E_m, \mathcal{T}^{\leq -B}) \). From the triangle \( K \to D_m \to \Sigma E_m \) and the fact that \( \text{Hom}(K, \mathcal{T}^{\leq -B}) \) and \( \text{Hom}(\Sigma E_m, \mathcal{T}^{\leq -B}) \) both vanish we deduce that \( \text{Hom}(D_m, \mathcal{T}^{\leq -B}) = 0 \).

From \( D_m \in \mathcal{T}^{\leq -m} \) we construct a triangle \( E' \to D_m \to Q \) with \( E' \in \langle G \rangle^{1-B-A,-m+A} \) and \( Q \in \mathcal{T}^{\leq -B} \). Since \( \text{Hom}(D_m, \mathcal{T}^{\leq -B}) = 0 \) the map \( D_m \to Q \) must vanish, hence the map \( E' \to D_m \) must be a split epimorphism. Since \( E' \) belongs to \( \langle G \rangle^{1-B-A,-m+A} \subset \langle G \rangle^{(-\infty,-m+A]} \) so does its direct summand \( D_m \).

We have learned that \( K \) belongs to \( \langle G \rangle^{1-m-A,A} \ast \langle G \rangle^{(-\infty,-m+A]} \), and if \( \mathcal{T} \) is approximable \( K \) even belongs to the smaller \( \langle G \rangle_{mA}^{1-m-A,A} \ast \langle G \rangle^{(-\infty,-m+A]} \).

Now set

\[
\begin{align*}
\mathcal{X}_1 &= \text{Coprod}(G[1 - m - A, A]) , \\
\mathcal{X}_2 &= \text{Coprod}_{mA}(G[1 - m - A, A]) , \\
\mathcal{Z} &= \text{Coprod}(G(-\infty, -m + A)) .
\end{align*}
\]

Then \( \mathcal{Z} = \text{smd}(\mathcal{Z}) \) is closed under direct summands so \( \mathcal{Z} = \langle G \rangle^{(-\infty,-m+A]} \), while

\[
\langle G \rangle^{1-m-A,A} = \text{smd}(\mathcal{X}_1) \quad \text{and} \quad \langle G \rangle_{mA}^{1-m-A,A} = \text{smd}(\mathcal{X}_2) .
\]

We are given that \( K \) belongs to \( \text{smd}(\mathcal{X}_i) \ast \mathcal{Z} \subset \text{smd}(\mathcal{X}_i \ast \mathcal{Z}) \) with \( i = 1 \) or \( 2 \), depending on whether \( \mathcal{T} \) is approximable. Choose an object \( K' \) in one of the categories \( \mathcal{X}_i \ast \mathcal{Z} \) above, so that \( K \) is a direct summand and \( K \to K' \to K \) is a pair of morphisms composing to the identity. Now put

\[
\begin{align*}
\mathcal{A}_1 &= \text{coprod}(G[1 - m - A, A]) , \\
\mathcal{A}_2 &= \text{coprod}_{mA}(G[1 - m - A, A]) , \\
\mathcal{C} &= \text{coprod}(G(-\infty, -m + A)) .
\end{align*}
\]
By [16, Lemma 1.7] any morphism from an object in \( \mathcal{T}^c \), to any of \( X_1, X_2 \) or \( Z \), factors (respectively) through an object in \( A_i, A_2 \) or \( C \). Now the map \( f : K \to K' \) is a morphism from \( K \in \mathcal{T}^c \) to an object \( K' \in X_i \ast Z \), with \( i = 1 \) or \( i = 2 \). By [16, Lemma 1.5] it factors as \( K \to K'' \to K' \) with \( K'' \in A_i \ast C \), with \( i = 1 \) or \( 2 \). Since the composite \( K \to K'' \to K' \) is the identity we deduce that \( K \) is a direct summand of the object in \( K'' \in A_i \ast C \), proving the Lemma.

**Lemma 2.12.** Let us fix a weakly approximable [or approximable] triangulated category \( \mathcal{T} \). Choose a compact generator \( G \) and a \( t \)-structure \((\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})\) in the preferred equivalence class.

There exists an integer \( B > 0 \) so that, for any object \( K \in \mathcal{T}^c \cap \mathcal{T}^{\leq 0} \), there exists a triangle \( E \to K \to D \) with \( E \in \langle G \rangle^{[-B,B]} \) and \( D \in \mathcal{T}^{\geq -1} \).

If \( \mathcal{T} \) is approximable we may further assume \( E \in \langle G \rangle^{[-B,B]}_B \).

**Proof.** Choose an integer \( A \) as in Proposition 2.6. We apply Lemma 2.11 to the object \( K \), with \( m = 4A + 1 \), and obtain a triangle \( E \to K \to L \to D \) with \( E \in \langle G \rangle^{[-5A,A]} \) and \( D \in \langle G \rangle^{(-\infty,-3A-1]} \); if the category is approximable we may even assume \( E \in \langle G \rangle^{[-5A,A]}_{4A+1} A \).

Now \( \langle G \rangle^{(-\infty,-3A-1]} \subseteq \mathcal{T}^c \) hence the object \( D \) is compact, and it belongs to \( \mathcal{T}^{\leq -2A-1} \) since \( G \in \mathcal{T}^{\leq A} \). Applying Lemma 2.11 to the object \( \Sigma^{-2A-1} D \) and with \( m = 6A \) we obtain a triangle \( E' \to D \oplus M \to D' \) with \( E' \in \langle G \rangle^{[-9A,-A-1]} \) and \( D' \in \langle G \rangle^{(-\infty,-7A-1]} \); if the category is approximable we may even assume \( E' \in \langle G \rangle^{[-9A,-A-1]}_{6A^2+4A} A \).

Now complete the composable maps \( K \oplus L \oplus M \to D \oplus M \to D' \) to an octahedron

\[
\begin{array}{c}
E \\
\uparrow \\
E \\
\downarrow \\
E \\
\rightarrow \quad \rightarrow \quad \rightarrow \quad \rightarrow \\
E'' \\
\uparrow \\
E'' \\
\downarrow \\
E'' \\
\rightarrow \quad \rightarrow \quad \rightarrow \quad \rightarrow \\
K \oplus L \oplus M \\
\downarrow \\
D \oplus M \\
\downarrow \\
D' \\
\end{array}
\]

We know that \( E \in \langle G \rangle^{[-5A,A]} \) and \( E' \in \langle G \rangle^{[-9A,-A-1]} \), and the triangle \( E \to E'' \to E' \) tells us that \( E'' \) belongs to

\[
\langle G \rangle^{[-5A,A]} \ast \langle G \rangle^{[-9A,-A-1]} \subseteq \langle G \rangle^{[-9A,A]};
\]

if \( \mathcal{T} \) is approximable \( E'' \) belongs to \( \langle G \rangle^{[-9A,A]}_{10A^2+4A} A \).

Now the object \( D' \) belongs to \( \langle G \rangle^{(-\infty,-7A-1]} \subseteq \mathcal{T}^{\leq -6A-1} \). The object \( E' \) belongs to \( \langle G \rangle^{[-9A,-A-1]} \subseteq \mathcal{T}^{\leq -1} \) and the triangle \( E' \to D \oplus M \to D' \) guarantees that \( D \oplus M \) and therefore its direct summand \( M \) belongs to \( \mathcal{T}^{\leq -1} \). Summarizing we have

(i) The object \( E \) belongs to \( \langle G \rangle^{[-5A,A]} \), the object \( E'' \) belongs \( \langle G \rangle^{[-9A,A]} \), the object \( D \) belongs to \( \langle G \rangle^{(-\infty,-3A-1]} \subseteq \mathcal{T}^{2A-1} \), the object \( M \) belongs to \( \mathcal{T}^{\leq -1} \) and the object \( D' \) belongs to \( \mathcal{T}^{\leq -6A-1} \).
(ii) If the category $\mathcal{T}$ is approximable then the objects $E$ and $E''$ were chosen so that $E \in \langle G \rangle_{[-5A,A]}$ and $E'' \in \langle G \rangle_{[-9A,A]}$.

Now consider the following diagram

\[
\begin{array}{ccc}
E & \longrightarrow & K \oplus L \\
\downarrow & & \downarrow \\
E'' & \longrightarrow & K \oplus L \oplus M
\end{array}
\]

where the vertical map is the direct sum of $\text{id} : L \longrightarrow L$ with the zero map. The composite from top left to bottom right is a morphism $E \longrightarrow D'$, with $E \in \langle G \rangle_{[-5A,A]}$ and $D' \in \mathcal{T}^{\leq -6A-1}$, hence must vanish. Therefore the composite $E \longrightarrow K \oplus L \longrightarrow K \oplus L \oplus M$ must factor through $E'' \longrightarrow K \oplus L \oplus M$. We deduce a commutative square

\[
\begin{array}{ccc}
E & \longrightarrow & K \oplus L \\
\downarrow & & \downarrow \\
E'' & \longrightarrow & K \oplus L \oplus M
\end{array}
\]

which we may complete to a $3 \times 3$ diagram whose rows and columns are triangles

\[
\begin{array}{ccc}
E & \longrightarrow & K \oplus L \\
\downarrow & & \downarrow \\
E'' & \longrightarrow & K \oplus L \oplus M \\
\downarrow & & \downarrow \\
\tilde{E} & \longrightarrow & K \oplus L \oplus M \\
\downarrow & & \downarrow \\
\tilde{E} & \longrightarrow & K \oplus L \oplus M \\
\downarrow & & \downarrow \\
\Sigma^2 K \oplus \Sigma M & \longrightarrow & \Sigma^2 K \oplus \Sigma M
\end{array}
\]

The triangle $D' \longrightarrow D'' \longrightarrow \Sigma D$, together with the fact that $D \in \mathcal{T}^{-2A-1}$ and $D' \in \mathcal{T}^{-6A-1}$, tell us that $D'' \in \mathcal{T}^{\leq -2A-2}$. The triangle $E'' \longrightarrow \tilde{E} \longrightarrow \Sigma E$, combined with the fact that $E \in \langle G \rangle_{[-5A,A]}$ and $E'' \in \langle G \rangle_{[-9A,A]}$, tell us that $\tilde{E} \in \langle G \rangle_{[-9A,A]}$; if $\mathcal{T}$ is approximable we have that $E \in \langle G \rangle_{[-5A,A]}$ and $E'' \in \langle G \rangle_{[-9A,A]}$ and therefore $\tilde{E}$ belongs to $\langle G \rangle_{[14A^2+8A]}$. Now complete the composable maps $\tilde{E} \longrightarrow K \oplus \Sigma K \oplus M \longrightarrow K$ to an octahedron

\[
\begin{array}{ccc}
\tilde{E} & \longrightarrow & K \oplus \Sigma K \oplus M \\
\downarrow & & \downarrow \\
\tilde{E} & \longrightarrow & K \\
\downarrow & & \downarrow \\
\Sigma^2 K \oplus \Sigma M & \longrightarrow & \Sigma^2 K \oplus \Sigma M \\
\end{array}
\]

We have that $\Sigma^2 K$ and $\Sigma M$ both belong to $\mathcal{T}^{\leq -2}$ and $D''$ belongs to $\mathcal{T}^{\leq -2A-2}$. Hence $\tilde{D} \in \mathcal{T}^{\leq -2}$, and the triangle $\tilde{E} \longrightarrow K \longrightarrow \tilde{D}$ satisfies the assertion of the Lemma. \qed
Proposition 2.13. Let us fix a weakly approximable [or approximable] triangulated category \( \mathcal{T} \). Choose a compact generator \( G \) and a t–structure \((\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})\) in the preferred equivalence class. Choose an integer \( B > 0 \) as in Lemma 2.12.

Then for any object \( F \in \mathcal{T}_c^{-} \cap \mathcal{T}^{\leq 0} \) there exists a triangle \( E \rightarrow F \rightarrow D \) with \( E \in \langle G \rangle^{[-B,B]} \) and \( D \in \mathcal{T}^{\leq -1} \).

If \( \mathcal{T} \) is approximable we may further assume \( E \in \langle G \rangle^{[1-m,B]} \).

Proof. Because \( F \) belongs to \( \mathcal{T}_c^{-} \) we may choose a triangle \( K \rightarrow F \rightarrow D_1 \) with \( K \in \mathcal{T}_c^{c} \) and \( D_1 \in \mathcal{T}^{\leq -1} \). The triangle \( \Sigma^{-1}D_1 \rightarrow K \rightarrow F \), coupled with the fact that both \( \Sigma^{-1}D_1 \) and \( F \) belong to \( \mathcal{T}^{\leq 0} \), tell us that \( K \in \mathcal{T}^{\leq 0} \). Thus \( K \in \mathcal{T}_c^{c} \cap \mathcal{T}^{\leq 0} \).

We may therefore apply Lemma 2.12 there exists a triangle \( E \rightarrow K \rightarrow D_2 \) with \( E \in \langle G \rangle^{[-B,B]} \) and \( D_2 \in \mathcal{T}^{\leq -1} \). If \( \mathcal{T} \) is approximable the object \( E \) may be chosen in \( \langle G \rangle^{[1-m,B]} \). Now complete the composable maps \( E \rightarrow K \rightarrow F \) to an octahedron

\[
\begin{array}{ccc}
E & \rightarrow & K \\
\downarrow & & \downarrow \\
F & \rightarrow & D \\
\downarrow & & \downarrow \\
D_1 & \rightarrow & D_2
\end{array}
\]

The triangle \( D_2 \rightarrow D \rightarrow D_1 \), coupled with the fact that \( D_2 \) and \( D_1 \) both lie in \( \mathcal{T}^{\leq -1} \), tell us that \( D \in \mathcal{T}^{\leq -1} \). And the triangle \( E \rightarrow F \rightarrow D \) satisfies the assertion of the Proposition. \( \square \)

Corollary 2.14. Let \( \mathcal{T} \) be a weakly approximable triangulated category. Let \( G \) be a compact generator and let \((\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})\) be a t–structure in the preferred equivalence class. Choose an integer \( B > 0 \) as in Lemma 2.12.

For any object \( F \in \mathcal{T}_c^{-} \cap \mathcal{T}^{\leq 0} \) there exists a sequence of objects \( E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow \cdots \) mapping to \( F \), and so that \( E_m \in \langle G \rangle^{[1-m,B]} \) and in each triangle \( E_m \rightarrow F \rightarrow D_m \) we have \( D_m \in \mathcal{T}^{\leq -m} \). For any such sequence the non-canonical map \( \text{Hocolim} \ E_m \rightarrow F \) is an isomorphism.

If the category is approximable we may construct the \( E_m \) to lie in \( \langle G \rangle^{[1-m,B]} \).

Proof. The fact that any such sequence would deliver a non-canonical isomorphism \( \text{Hocolim} \ E_m \rightarrow F \) is contained in Lemma 2.3. We need to prove the existence of the sequence.

In Proposition 2.13 we constructed a triangle \( E_1 \rightarrow F \rightarrow D_1 \) with \( E_1 \in \langle G \rangle^{[-B,B]} \) and \( D_1 \in \mathcal{T}^{\leq -1} \). But \( \langle G \rangle^{[-B,B]} \subset \mathcal{T}_c^{c} \), and in Remark 0.17 we noted that \( \mathcal{T}_c^{c} \subset \mathcal{T}_c^{-} \). In the triangle \( E_1 \rightarrow F \rightarrow D_1 \) we have that both \( E_1 \) and \( F \) lie in \( \mathcal{T}_c^{-} \), while Lemma 2.9 proved that the category \( \mathcal{T}_c^{-} \) is triangulated. Therefore \( D_1 \in \mathcal{T}_c^{-} \cap \mathcal{T}^{\leq -1} \). If we let \( S = \mathcal{T}_c^{-} \) we are in the situation of Corollary 2.2, more specifically the hypotheses of (2.2.3) hold;
if $\mathcal{T}$ is approximable the hypotheses of (2.2.1) hold. The current Corollary is simply the conclusions of (2.2.3) and (2.2.4).

\[ \square \]

3. Examples

In Section 2 we developed some abstraction, and it’s high time to look at examples. We begin with the trivial ones.

**Example 3.1.** Let $R$ be a ring, and put $\mathcal{T} = \mathbf{D}(R)$ its unbounded derived category. The category $\mathcal{T}$ has coproducts and $R$ is a compact generator. Let $\mathcal{T}$ be the standard $t$–structure. Then $\text{Hom}(\Sigma^{-1}R, \mathcal{T}^{\leq 0}) = 0$ and $\Sigma R \in \mathcal{T}^{\leq 0}$.

Let $F$ be any object in $\mathcal{T}^{\leq 0}$. That is, we take a cochain complex $F$ with $H^\ell(F) = 0$ for all $\ell > 0$. Such a complex has a free resolution; it is isomorphic in $\mathcal{T}$ to the cochain complex

$$\cdots \longrightarrow F^{-3} \longrightarrow F^{-2} \longrightarrow F^{-1} \longrightarrow F^0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

with $F^i$ free $R$–modules. The brutal truncation produces for us a short exact sequence of cochain complexes

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow F^0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

$$\cdots \longrightarrow F^{-3} \longrightarrow F^{-2} \longrightarrow F^{-1} \longrightarrow F^0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

and this is a triangle $E \longrightarrow F \longrightarrow D$ with $E \in \langle R \rangle_{[0,1]}$ and $D \in \mathcal{T}^{\leq -1}$. The category is approximable, the standard $t$–structure is in preferred equivalence class, and $\mathcal{T}_c^b$ is just $\mathbf{D}^b(R$–proj), the category of bounded-above complexes which admit projective resolutions by finitely generated projectives.

The category $\mathcal{T}_c^b$ is the intersection $\mathcal{T}_c^b \cap \mathcal{T}^b$; it consists of the objects in $\mathbf{D}^b(R$–proj) with only finitely many nonzero cohomology groups. We have an inclusion $\mathbf{D}^b(R$–proj) $\subset \mathcal{T}_c^b$, and for $R$ general I don’t know much about the difference $\mathcal{T}_c^b - \mathbf{D}^b(R$–proj). When $R$ is noetherian we have $\mathcal{T}_c^b = \mathbf{D}^b(R$–mod), which is usually much larger than $\mathbf{D}^b(R$–proj).

**Example 3.2.** A very similar analysis works when $\mathcal{T}$ is the homotopy category of spectra. The sphere $S^0$ is a compact generator. Consider the $t$–structure where $\mathcal{T}^{\leq 0}$ is the category of connective spectra—these are the spectra $F$ with $\pi_i(F) = 0$ when $i < 0$. Then $\text{Hom}(\Sigma^{-1}S^0, \mathcal{T}^{\leq 0}) = 0$ and $\Sigma S^0 \in \mathcal{T}^{\leq 0}$. And any object $F \in \mathcal{T}^{\leq 0}$ admits a triangle $E \longrightarrow F \longrightarrow D$ with $E \in \langle S^0 \rangle_{[0,1]}$ and $D \in \mathcal{T}^{\leq -1}$; this just says that we may choose a bouquet of zero-spheres $E$ and a map $E \longrightarrow F$ which is surjective on $\pi_0$. The category is approximable and the $t$–structure above is in the preferred equivalence class.
Remark 3.3. Examples \(3.1\) and \(3.2\) should be viewed as the baby case. If \(\mathcal{T}\) has a compact generator \(G\), such that \(\text{Hom}(G, \Sigma^i G) = 0\) for all \(i > 0\), then \(\mathcal{T}\) is approximable. Just take the \(t\)-structure \((\mathcal{T}_G^{\leq 0}, \mathcal{T}_G^{\geq 0})\); then \(\Sigma G \in \mathcal{T}_G^{\leq 0}\) and \(\text{Hom}(\Sigma^{-1} G, \mathcal{T}_G^{\leq 0}) = 0\), and every object \(F \in \mathcal{T}_G^{\leq 0}\) admits a triangle \(E \to F \to D\) with \(E \in \langle G \rangle_{1,0}^{\leq 0}\) and \(D \in \mathcal{T}_G^{\leq -1}\).

Example 3.4. If \(X\) is a quasicompact, quasiseparated scheme then \(\mathcal{T} = D_{\text{qc}}(X)\) has a single compact generator, see Bondal and Van den Bergh [5 Theorem 3.1.1(ii)]. Let \(G\) be any such compact generator; \([5\) Theorem 3.1.1(ii)] tells us that \(G\) is a perfect complex. Let \((\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})\) be the standard \(t\)-structure. It is easy to show that there exists an integer \(A > 0\) so that \(\Sigma^A G \in \mathcal{T}^{\leq 0}\) and \(\text{Hom}(\Sigma^{-A} G, \mathcal{T}^{\leq 0}) = 0\). It follows formally, from Proposition \(2.10\) that \(\mathcal{T}^{-}\) is thick.

But in this case we happen to know \(\mathcal{T}^{-}\) explicitly. To state the results it might be helpful to imagine two classes of objects in \(\mathcal{T} = D_{\text{qc}}(X)\).

(i) The objects belonging to \(\mathcal{T}^{-} = D_{\text{qc}}(X)^{-}\), for the \(\mathcal{T}^{-}\) defined using the standard \(t\)-structure.

(ii) The objects locally with this property. That is the objects \(P \in D_{\text{qc}}(X)\) such that, for any open immersion \(j : \text{Spec}(R) \to X\), the object \(L_j^* P \in D_{\text{qc}}(\text{Spec}(R)) \cong D(R)\) is in \(D(R)^{-}\). See Example \(3.1\) for a description of \(D(R)^{-}\); note that in the case of \(D(R)\) the standard \(t\)-structure is in the preferred equivalence class.

The objects satisfying (ii) are classically called pseudocoherent, they were first studied in Illusie’s exposés [11, 12] in SGA6. Now [13 Theorem 4.1] is precisely the statement that the objects satisfying (ii) all satisfy (i). It is trivial to check that the objects satisfying (i) must satisfy (ii); this means that, for the standard \(t\)-structure on \(\mathcal{T} = D_{\text{qc}}(X)\), the subcategory \(\mathcal{T}^{-}\) is just \(D_{\text{qc}}(X)^{p} \subset D_{\text{qc}}(X)\), the subcategory of pseudocoherent complexes. If \(X\) happens to be noetherian then pseudocoherence simplifies to something more familiar: for noetherian \(X\) we have \(\mathcal{T}^{-} = D_{\text{qc}}(X) = D_{\text{coh}}(X)\).

Still in the general case, where \(X\) is only assumed quasicompact and quasiseparated: Since \(G\) is compact it is perfect, and there exists an integer \(A > 0\) so that \(\text{Hom}(G, \Sigma^i G) = 0\) for all \(i \geq A\). Remark \(0.20\) applies and teaches us that the \(\mathcal{T}^{-}\) corresponding to the preferred equivalence class of \(t\)-structures is also thick.

This ends what I know in glorious generality. In this kind of generality I have no idea if \(D_{\text{qc}}(X)\) is approximable, or how the \(\mathcal{T}^{-}\) obtained from a \(t\)-structure in the preferred equivalence class compares to \(\mathcal{T}^{-} = D_{\text{qc}}(X)^{p}\), the subcategory \(\mathcal{T}^{-}\) that comes from the standard \(t\)-structure on \(\mathcal{T} = D_{\text{qc}}(X)\). But when \(X\) is separated we can prove

Lemma 3.5. Let \(X\) be a separated, quasicompact scheme, let \(\mathcal{T} = D_{\text{qc}}(X)\) be its derived category, and let \((\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})\) be the standard \(t\)-structure. Then there is a compact generator \(G^0 \in \mathcal{T}\) and an integer \(A > 0\), so that every object \(F \in \mathcal{T}^{\leq 0}\) admits a triangle \(E \to F \to D\) with \(E \in \langle G^0 \rangle_{A,0}^{\leq 0}\) and \(D \in \mathcal{T}^{\leq -1}\).

Proof. Absolute noetherian approximation, that is Thomason and Trobaugh [21 Theorem C.9] or [20 Tags 01YT and 081A], allows us to choose a separated scheme \(Y\), of finite
type over \( \mathbb{Z} \), and an affine map \( f : X \to Y \). From [16] Definition 5.2 and Theorem 5.8, in the special case where \( S = D_{\text{qc}}(Y)^{\leq 0} \) for the standard \( t \)-structure and \( m = 0 \), we learn:

(i) There exists a compact generator \( G \in D_{\text{qc}}(Y) \) and an integer \( A > 0 \), so that every \( F' \in D_{\text{qc}}(Y)^{\leq 0} \) admits a triangle \( E' \to F' \to D' \) with \( E' \in \langle G \rangle_{A}^{[-A,A]} \) and \( D' \in D_{\text{qc}}(Y)^{\leq -1} \).

Hall and Rydh [10] Lemma 8.2] tells us that \( G' = Lf^*G \) is a compact generator for \( D_{\text{qc}}(X) \); this is our choice of \( G' \) for the Lemma. Now take any object \( F \in D_{\text{qc}}(X)^{\leq 0} \).

Since \( f : X \to Y \) is affine we have \( Rf_*F \cong f_*F \in D_{\text{qc}}(Y)^{\leq 0} \), and (i) above permits us to find a triangle \( E' \to f_*F \to D' \) with \( E' \in \langle G \rangle_{A}^{[-A,A]} \) and \( D' \in D_{\text{qc}}(Y)^{\leq -1} \). Applying the functor \( Lf^* \), and remembering that \( Lf^*D_{\text{qc}}(Y)^{\leq 0} \subset D_{\text{qc}}(X)^{\leq 0} \), we deduce

(ii) There is in \( D_{\text{qc}}(X) \) a triangle \( Lf^*E' \to Lf^*f_*F \to Lf^*D' \), with \( Lf^*D' \in D_{\text{qc}}(X)^{\leq -1} \) and \( Lf^*E' \in Lf^*\langle G \rangle_{A}^{[-A,A]} \subset \langle Lf^*G \rangle_{A}^{[-A,A]} \).

But the counit of adjunction gives a map \( \varepsilon : Lf^*f_*F \to F \), and the fact that the maps \( f_*F \xrightarrow{\eta f_*} f_*Lf^*f_*F \xrightarrow{\varepsilon} f_*F \) compose to the identity tells us that the functor \( f_* \) takes \( \varepsilon : Lf^*f_*F \to F \) to a split epimorphism. In particular \( f_*\varepsilon \) induces an epimorphism on cohomology sheaves and, because \( f \) is affine, this means that \( \varepsilon \) induces an epimorphism of cohomology sheaves already over \( X \). We have a morphism \( \varepsilon : Lf^*f_*F \to F \) in \( D_{\text{qc}}(X)^{\leq 0} \) and, if we complete it to a triangle, the long exact sequence of cohomology sheaves gives

(iii) In the triangle \( Lf^*f_*F \xrightarrow{\varepsilon} F \to D'' \) we have \( D'' \in D_{\text{qc}}(X)^{\leq -1} \).

Next we form the octahedron

\[
\begin{array}{ccc}
Lf^*E' & \to & Lf^*E' \\
\downarrow & & \downarrow \\
Lf^*f_*F & \xrightarrow{\varepsilon} & F & \to & D'' \\
\downarrow & & \downarrow & & \downarrow \\
Lf^*D' & \to & D & \to & D''
\end{array}
\]

and (ii) tells us that \( Lf^*E' \in \langle Lf^*G \rangle_{A}^{[-A,A]} \) and \( Lf^*D' \in D_{\text{qc}}(X)^{\leq -1} \), while (iii) gives that \( D'' \in D_{\text{qc}}(X)^{\leq -1} \). The triangle \( Lf^*D' \to D \to D'' \) tells us that \( D \in D_{\text{qc}}(X)^{\leq -1} \), and the triangle \( Lf^*E' \to F \to D \) does the trick. \( \square \)

**Example 3.6.** Assume \( X \) is separated and quasicompact, and let the \( t \)-structure on \( \mathcal{T} = D_{\text{qc}}(X) \) be the standard one. Lemma [3.5] finds a generator \( G' \) and an integer \( A > 0 \) so that, for every object \( F \in \mathcal{T}^{\leq 0} \), there exists a triangle \( E \to F \to D \) with \( E \in \langle G' \rangle_{A}^{[-A,A]} \) and \( D \in \mathcal{T}^{\leq -1} \). From Example [3.4] we know that, for the compact generator \( G' \in \mathcal{T} \), there is an integer \( A' > 0 \) with \( \text{Hom}(\Sigma^{-A'}G', \mathcal{T}^{\leq 0}) = 0 \) and with \( \Sigma A' G' \in \mathcal{T}^{\leq 0} \).
Putting this together we have that $\mathcal{T}$ is approximable; it satisfies Definition 0.21 for the compact generator $G'$, the standard $t$-structure, and the integer $\max(A, A')$. And now Proposition 2.6 informs us that

(i) The standard $t$-structure is in the preferred equivalence class.

By (i) the two $\mathcal{T}^c$ discussed in Example 3.4 agree. Hence

(ii) For the $\mathcal{T}^c$ coming from a $t$-structure in the preferred equivalence class we have $\mathcal{T}^c = D_{\text{qc}}(X)$, the category of pseudocoherent complexes. If $X$ is noetherian this simplifies to $D_{\text{coh}}(X)$.

Up until now we have simply figured out that the standard $t$-structure on $D_{\text{qc}}(X)$ is in the preferred equivalence class, that $D_{\text{qc}}(X)$ is approximable and that $D_{\text{qc}}(X)^c$ is nothing other than $D_{\text{qc}}(X)$. Together this tells us first that $D_{\text{qc}}(X)$ is an example of the general theory, and then works out what $D_{\text{qc}}(X)^c$ is.

When we apply Corollary 2.14 we discover something new.

(iii) Let $X$ be a quasicompact, separated scheme, and let $G$ be a compact generator of $D_{\text{qc}}(X)$. There exists an integer $B > 0$ so that, for any integer $m > 0$ and any object $F \in D^p_{\text{qc}}(X) \cap D_{\text{qc}}(X)_{\leq 0}$, there is a triangle $E_m \rightarrow F \rightarrow D_m$ with $E_m \in \langle G \rangle_{mB, B}$ and $D_m \in D_{\text{qc}}(X)_{\leq -m}$.

4. Approximating systems

In this short section we collect some elementary facts about countable direct limits of representable functors. The generality that will suffice for us is $R$–linear functors between $R$–linear categories, where $R$ is a commutative ring.

**Definition 4.1.** Let $R$ be a commutative ring, let $\mathcal{T}$ be an $R$–linear category, let $A,B$ be full subcategories of $\mathcal{T}$, and let $H : B^{\text{op}} \rightarrow R$–Mod be an $R$–linear functor. An $A$–approximating system for $H$ is a sequence in $A$ of objects and morphisms $E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow \cdots$, so that

(i) There is a cofinal subsequence of $E_*$ whose objects belong to $A \cap B$.

(ii) We are given an isomorphism $\colim \rightarrow \text{Hom}(-, E_i) \rightarrow H(-)$.

In this article we will mostly consider the case where $A$ is contained in $B$, but in a sequel we will need the more flexible notion.

Since we will freely use approximating systems in our constructions, it is comforting to know that they are all the same up to subsequences. More precisely we have

**Lemma 4.2.** Suppose we have an $R$–linear functor $H : B^{\text{op}} \rightarrow R$–Mod, and two $A$–approximating systems $E_*$ and $F_*$ for $H$. Then the systems $E_*$ and $F_*$ are ind-isomorphic. We remind the reader: this means that there exists an $A$–approximating system $L_*$ for the functor $H$, more explicitly $L_1 \rightarrow L_2 \rightarrow L_3 \rightarrow \cdots$, and subsequences $E'_* \subset E_*$, $F'_* \subset F_*$ and $L'_* \subset L_*$ with $E'_* = L'_*$ and $F'_* = L''_*$. 
Proof. Since we may pass to subsequences we will assume that both \( E_\ast \) and \( F_\ast \) belong to \( A \cap B \). To slightly compress the argument we will extend the sequences by zero; we will set \( E_0 = F_0 = L_0 = 0 \), look at the sequences \( E_0 \to E_1 \to E_2 \to \cdots \) and \( F_0 \to F_1 \to F_2 \to \cdots \), and out of them construct the sequence \( L_0 \to L_1 \to L_2 \to \cdots \).

Let \( L_0 = 0 \) and \( L_1 = E_1 \), and inductively proceed as follows. Assume that, for some \( m \geq 1 \), the sequence \( L_0 \to L_1 \to \cdots L_{m-1} \to L_m \) and the map from the sequence to \( H(\_ \_ \_ ) \) have been defined, in such a way that \( L_1 \to L_3 \to L_5 \to \cdots \) is a subsequence of \( E_\ast \) and \( L_0 \to L_2 \to L_4 \to L_6 \to \cdots \) is a subsequence of \( F_\ast \). We wish to extend to \( m + 1 \). There are two cases, \( m \) can be odd or even, but up to interchanging \( E_\ast \) and \( F_\ast \) in the argument below they are the same. We will therefore assume \( m \) odd and leave to the reader the even case.

Then \( L_m \) is equal to \( E_J \) for some \( J > 0 \) while \( L_{m-1} \) is \( F_I \) for some \( I \geq 0 \). The map \( \text{Hom}(\_ \_ \_ , L_m) = \text{Hom}(\_ \_ \_ , E_J) \to H(\_ \_ \_ ) = \text{colim} \text{Hom}(\_ \_ \_ , F_I) \) is a natural transformation from the representable functor \( \text{Hom}(\_ \_ \_ , E_J) \) to the colimit. Yoneda tells us that it corresponds to an element in \( H(E_J) = \text{colim} \text{Hom}(E_J, F_I) \), where the colimit is over \( i \). We may therefore choose an \( I' > J \) and a morphism \( L_m = E_J \to F_{I'} \) which delivers the right element in the colimit. The composite \( F_I = L_{m-1} \to L_m \to F_{I'} \) does not have to agree with the map \( F_I \to F_{I'} \) of the sequence \( F_\ast \), but they have the same image in \( H(F_I) = \text{colim} \text{Hom}(F_I, F_i) \) (where the colimit is over \( i \)). That is: after composing with some \( F_{I'} \to F_{I''} \) in the sequence \( F_\ast \) they become equal. Set \( L_{m+1} = F_{I''} \). \( \square \)

Lemma 4.3. Let \( J \) be and \( R \)-linear category let \( A, B \) be full subcategories, let \( H, H' : B^{\text{op}} \to R\text{-Mod} \) be two \( R \)-linear functors, and assume we are given for \( H \) an \( A \)-approximating system \( E_1 \to E_2 \to E_3 \to \cdots \). Replacing the sequence \( E_\ast \) by a subsequence belonging to \( A \cap B \), there is a natural isomorphism \( H(H, H') \cong \lim H'(E_m) \).

Proof. We have isomorphisms
\[
\text{Hom}(H, H') = \text{Hom}(\text{colim} \text{Hom}(\_ \_ \_ , E_m), H'(-)) \\
= \lim \text{Hom}(\text{Hom}(\_ \_ \_ , E_m), H'(-)) \\
= \lim H'(E_m)
\]
where the last isomorphism is by Yoneda. \( \square \)

Corollary 4.4. Suppose we are given \( R \)-linear categories \( A \subset B \) and two \( R \)-linear functors \( H, H' : B^{\text{op}} \to R\text{-Mod} \). If \( H \) has an \( A \)-approximating system then restriction to the subcategory \( A \subset B \) is a natural bijection \( \text{Hom}(H, H') \to \text{Hom}(H|_A, H'|_A) \).

Proof. Choose an \( A \)-approximating system \( E_1 \to E_2 \to E_3 \to \cdots \) for \( H \). Lemma 4.3 tells us that both sets are in bijection with \( \lim H'(E_m) \), and the bijection commutes with the restriction map. \( \square \)
Lemma 4.5. Let \( \mathcal{B} \) be an \( R \)-linear category, let \( H, H' : \mathcal{B}^{\text{op}} \to R\text{-Mod} \) be two \( R \)-linear functors, and let \( \varphi : H \to H' \) be a natural transformation. If each of \( H, H' \) has a \( \mathcal{B} \)-approximating system, let’s say \( E_1 \to E_2 \to E_3 \to \cdots \) for \( H \) and \( E'_1 \to E'_2 \to E'_3 \to \cdots \) for \( H' \), then, after replacing \( E'_1 \to E'_2 \to E'_3 \to \cdots \) by a subsequence, we can produce a map of sequences \( f_* : E_* \to E'_* \) so that \( \varphi : H \to H' \) is the colimit of the image of \( f_* \) under Yoneda.

Moreover: if we are given \( n > 0 \) pairs of subsequences of \( E_* \) and \( E'_* \), then we construct the subsequence of the map \( f_* : E_* \to E'_* \) to respect the given subsequences.

Proof. By Lemma 4.3 we have an isomorphism
\[
\text{Hom}(H, H') \cong \varprojlim H'(E_i) = \varprojlim \text{colim}_j \text{Hom}(E_i, E'_j).
\]

Our element \( \varphi \in \text{Hom}(H, H') \) must therefore correspond to an inverse system of elements \( \varphi_i \in \text{colim}_j \text{Hom}(E_i, E'_j) \). We proceed inductively.

(i) Choose a some integer \( j_1 \) and a preimage in \( \text{Hom}(E_1, E'_{j_1}) \) of \( \varphi_1 \in \text{colim}_j \text{Hom}(E_1, E_j) \).

Call this map \( f_1 : E_1 \to E'_{j_1} \). If \( E_1 \) belongs to one of the prescribed subsequences of \( E_* \) then choose \( j_1 \) so that \( E'_{j_1} \) belongs to the matching subsequence of \( E'_* \).

(ii) Suppose the sequence has been constructed up to an integer \( m \geq 1 \). In particular we have a map \( E_m \to E'_{j_m} \), whose image under the natural map \( \text{Hom}(E_m, E'_{j_m}) \to \text{colim}_j \text{Hom}(E_m, E'_j) \) is \( \varphi_m \).

We have the element \( \varphi_{m+1} \in \text{colim}_j (E_{m+1}, E'_j) \), we can choose a preimage in \( \text{Hom}(E_{m+1}, E'_j) \) for some integer \( J \), and we may assume \( J > j_m \). This gives us a map \( f' : E_{m+1} \to E'_J \). Now the square
\[
\begin{array}{ccc}
E_m & \xrightarrow{f_m} & E'_{j_m} \\
\downarrow & & \downarrow \\
E_{m+1} & \xrightarrow{f'} & E'_J \\
\end{array}
\]

need not commute, but the two composites both go, via the map \( \text{Hom}(E_m, E'_j) \to \text{colim}_j \text{Hom}(E_m, E'_j) \), to the same element \( \varphi_m \). Hence replacing \( E'_j \) by some \( E'_{j_{m+1}} \) with \( j_{m+1} > J \), we may assume the square commutes. And if \( E_{m+1} \) belongs to one of the prescribed subsequences of \( E_* \), choose \( j_{m+1} > J \) so that \( E'_{j_{m+1}} \) belongs to the matching subsequence of \( E'_* \).

We have replaced \( E'_* \) by a subsequence and produced a map of sequences \( f_* : E_* \to E'_* \). The reader can check that, if we apply Yoneda to the map of sequences \( f_* : E_* \to E'_* \) and then take colimits, we recover \( \varphi : H \to H' \).

Remark 4.6. In the remainder of the paper we will use approximating systems in the following situation. We will work in some ambient \( R \)-linear triangulated category \( \mathcal{T} \), and will assume that \( \mathcal{T} \) has coproducts. What is special in this case is that, given a functor
$H : \mathcal{B}^{\text{op}} \to R\text{-Mod}$ and an $\mathcal{A}$–approximating system $E_1 \to E_2 \to E_3 \to \cdots$ for $H$, we can construct in $\mathcal{T}$ the homotopy colimit $F = \text{Hocolim} E_i$. For $(-)$ in the category $\mathcal{B}$ we have a natural map

$$H(-) \xrightarrow{\text{colim}} \text{Hom}(-, E_i) \xrightarrow{} \text{Hom}(-, F)$$

and we will be interested in approximating sequences for which this map $H(-) \to \text{Hom}(-, F)$ is an isomorphism.

In this situation we will say that $E_1 \to E_2 \to E_3 \to \cdots$ is an $\mathcal{A}$–approximating system for $F$ over $\mathcal{B}$.

**Remark 4.7.** In this article the case of interest is where $\mathcal{B} \subset \mathcal{T}^c$, that is the objects of $\mathcal{B}$ are all compact. From [14, Lemma 2.8] we know that, for any compact object $K \in \mathcal{T}$ and any sequence of objects of $\mathcal{T}$, the natural map is an isomorphism $H(K) \cong \text{colim} \text{Hom}(K, E_i) \to \text{Hom}(K, \text{Hocolim} E_i)$. Thus we’re automatically in the situation of Remark 4.6; any sequence $E_\ast$ in $\mathcal{A}$ is an $\mathcal{A}$–approximating system for $F = \text{Hocolim} E_i$, over any $\mathcal{B} \subset \mathcal{T}^c$. Late in the article (meaning in Section 7) we will therefore allow ourselves to occasionally leave unspecified the category $\mathcal{B} \subset \mathcal{T}^c$, and just say that $E_\ast$ is an $\mathcal{A}$–approximating system for $F$.

For now we are careful to specify $\mathcal{B}$, because we plan to use the lemmas in other contexts in future articles.

In the generality of Remark 4.6 we note the following little observation.

**Lemma 4.8.** Let $\mathcal{T}$ be a triangulated category with coproducts, let $\mathcal{A}, \mathcal{B}$ be subcategories, and assume $E_1 \to E_2 \to E_3 \to \cdots$ is a $\mathcal{A}$–approximating system for $F \in \mathcal{T}$ over $\mathcal{B}$.

If $G$ is another object of $\mathcal{T}$, and if $\varphi : \text{Hom}(-, F)|_{\mathcal{B}} \to \text{Hom}(-, G)|_{\mathcal{B}}$ is a natural transformation of functors on $\mathcal{B}$, then there exists in $\mathcal{T}$ a (non-unique) morphism $f : F \to G$ with $\varphi = \text{Hom}(-, f)|_{\mathcal{B}}$.

**Proof.** By Lemma 4.3 the natural transformation $\varphi : \text{Hom}(-, F) \to \text{Hom}(-, G)$ corresponds to an element in $\lim \text{Hom}(E_i, G)$. Thus for each $i$ we are given in $\mathcal{T}$ a morphism $f_i : E_i \to G$, compatibly with the sequence maps $E_i \to E_{i+1}$. The compatibility means that the composite

$$\bigoplus_{i=1}^{\infty} E_i \xrightarrow{1\text{-shift}} \bigoplus_{i=1}^{\infty} E_i \xrightarrow{(f_1, f_2, f_3, \ldots)} G$$

must vanish. Hence the map $\bigoplus_{i=1}^{\infty} E_i \to G$ factors (non-uniquely) through $F = \text{Hocolim} E_i$, which is the third edge in the triangle

$$\bigoplus_{i=1}^{\infty} E_i \xrightarrow{1\text{-shift}} \bigoplus_{i=1}^{\infty} E_i \xrightarrow{(f_1, f_2, f_3, \ldots)} \text{Hocolim} E_i.$$

$\square$
5. An easy representability theorem

It’s time to start proving representability theorems. The main theorems in the article are a little technical—they depend on taking homotopy colimits carefully. For this reason I thought it best to illustrate the methods in a simple case, which involves no homotopy colimits. In this section we give a simple proof of an old result of Rouquier, generalizing an even older result of Bondal and Van den Bergh.

**Lemma 5.1.** Let $R$ be a commutative, noetherian ring, let $S$ be an $R$–linear triangulated category, let $G \in S$ be an object and, with the notation of Definition 0.1 and Remark 0.2, assume $\text{Hom}(-, G)$ is a $G$–finite cohomological functor. Let $k \geq 0$ be an integer and $H$ a finite $\langle G \rangle_{2k}$–cohomological functor. Then there exists an object $F \in \langle G \rangle_{2k}$ and an epimorphism $\text{Hom}(-, F)|_{\langle G \rangle_{2k}} \rightarrow H(-)$.

**Proof.** The proof is by induction on $H$. Suppose $k = 0$: by hypothesis $H(\Sigma^{-i}G)$ is a finite $R$–module, and vanishes for $i$ outside a bounded interval $[-A, A]$. For each $i$ with $-A \leq i \leq A$ choose a finite set of generators $\{f_{ij}, j \in J_i\}$ for the $R$–module $H(\Sigma^{-i}G)$. Yoneda tells us that each $f_{ij}$ corresponds to a natural transformation $\varphi_{ij} : \text{Hom}(-, \Sigma^{-1}G)|_{\langle G \rangle_{1}} \rightarrow H(-)$. Define $F$ to be $F = \bigoplus_{i = -A}^{A} \bigoplus_{j \in J_i} \Sigma^{-i}G$, and let $\varphi : \text{Hom}(-, F)|_{\langle G \rangle_{1}} \rightarrow H(-)$ be the composite

$$
\text{Hom}(-, F)|_{\langle G \rangle_{1}} \longrightarrow \bigoplus_{i = -A}^{A} \bigoplus_{j \in J_i} \text{Hom}(-, \Sigma^{-i}G)|_{\langle G \rangle_{1}} \longrightarrow H(-)
$$

Obviously $F$ belongs to $\langle G \rangle_{1}$ and $\varphi$ is surjective.

Now suppose we know the Lemma for $k \geq 0$, and let $H$ be a finite $\langle G \rangle_{2k+1}$–cohomological functor. Then the restriction of $H$ to $\langle G \rangle_{2k}$ is a finite $\langle G \rangle_{2k}$–cohomological functor, and induction permits us to find an object $F_1 \in \langle G \rangle_{2k}$ and an epimorphism $\varphi : \text{Hom}(-, F_1)|_{\langle G \rangle_{2k}} \rightarrow H(-)|_{\langle G \rangle_{2k}}$. Complete the map $\varphi$ to a short exact sequence

$$
0 \rightarrow H'(-) \rightarrow \text{Hom}(-, F_1)|_{\langle G \rangle_{2k}} \rightarrow \text{Hom}(-, F_2)|_{\langle G \rangle_{2k}} \rightarrow 0.
$$

Then $H'$ is a finite $\langle G \rangle_{2k}$–cohomological functor, and induction applies again to tell us that there exists an object $F_2 \in \langle G \rangle_{2k}$ and an epimorphism $\rho : \text{Hom}(-, F_2)|_{\langle G \rangle_{2k}} \rightarrow H'(-)$. Combining the results we deduce an exact sequence of functors

$$
\text{Hom}(-, F_2)|_{\langle G \rangle_{2k}} \longrightarrow \text{Hom}(-, F_1)|_{\langle G \rangle_{2k}} \longrightarrow \text{Hom}(-, F_1)|_{\langle G \rangle_{2k}} \longrightarrow 0.
$$

Because $F_1$ and $F_2$ both lie in $\langle G \rangle_{2k}$ the functors $\text{Hom}(-, F_i)|_{\langle G \rangle_{2k}}$ are representable for $i \in \{1, 2\}$—Yoneda’s lemma applies. The natural transformation $\sigma \rho : \text{Hom}(-, F_2)|_{\langle G \rangle_{2k}} \rightarrow \text{Hom}(-, F_1)|_{\langle G \rangle_{2k}}$ is $\text{Hom}(-, \alpha)|_{\langle G \rangle_{2k}}$ for some morphism $\alpha : F_2 \rightarrow F_1$, the natural transformation $\varphi : \text{Hom}(-, F_1)|_{\langle G \rangle_{2k}} \rightarrow H(-)|_{\langle G \rangle_{2k}}$ corresponds to some element $y \in H(F_1)$,
and the vanishing of the composite \( \varphi \sigma \rho \) says that \( H(\alpha) : H(F_1) \to H(F_2) \) must take \( y \) to zero.

Complete \( \alpha : F_2 \to F_1 \) to a triangle \( F_2 \to F_1 \to F \to \Sigma F_2 \). As \( F_1 \) and \( \Sigma F_2 \) belong to \( \langle G \rangle_{2k} \), the triangle tells us that \( F \) must be in \( \langle G \rangle_{2k} * \langle G \rangle_{2k} \subset \langle G \rangle_{2k+1} \). And now we remember that \( H \) is actually a \( \langle G \rangle_{2k+1} \)-cohomological functor. The sequence \( H(F) \to \to H(F_1) \to H(F_2) \) is exact, and the vanishing of \( H(\alpha)(y) \) says that there exists an element \( x \in H(F) \) so that \( H(\beta) : H(F) \to H(F_1) \) takes \( x \in H(F) \) to \( y \in H(F_1) \).

By Yoneda \( x \) corresponds to a natural transformation \( \psi : \text{Hom}(-, F) |_{\langle G \rangle_{2k+1}} \to H(\cdot) \). The fact that \( H(\beta)x = y \) translates, via Yoneda, to the assertion that the composite

\[
\text{Hom}(-, F_1) |_{\langle G \rangle_{2k+1}} \xrightarrow{\text{Hom}(\cdot, \beta) |_{\langle G \rangle_{2k+1}}} \text{Hom}(-, F) |_{\langle G \rangle_{2k+1}} \xrightarrow{\psi} H(\cdot)
\]

restricts to be \( \varphi \) on the category \( \langle G \rangle_{2k} \).

We assert that \( \psi \) is surjective. Take any object \( C' \in \langle G \rangle_{2k+1} \); we need to show the surjectivity of the map \( \psi : \text{Hom}(C', F) \to H(C') \). Now \( \langle G \rangle_{2k+1} = \text{smd}(\langle G \rangle_{2k} * \langle G \rangle_{2k}) \), so there exists an object \( C'' \in S \) with \( C' \subset C'' \subset \langle G \rangle_{2k} * \langle G \rangle_{2k} \). Put \( C = C' \cup C'' \) and it clearly suffices to prove the surjectivity of \( \psi : \text{Hom}(C, F) \to H(C) \). Choose a triangle \( A \xrightarrow{\alpha'} B \xrightarrow{\beta'} C \xrightarrow{\gamma'} \Sigma A \) with \( A, B \in \langle G \rangle_{2k} \).

Take any \( z \in H(C) \). The map \( H(\beta') : H(C) \to H(B) \) takes \( z \) to an element \( H(\beta')z \in H(B) \). But the map \( \varphi : \text{Hom}(B, F_1) \to H(B) \) is surjective, hence there is an element \( g \in \text{Hom}(B, F_1) \) with \( \varphi(g) = H(\beta')(z) \). Now \( 0 = H(\alpha')H(\beta')z = H(\alpha')\varphi(g) = \varphi \text{Hom}(\alpha', F_1)g \), where the last equality is by the naturality of the map \( \varphi : \text{Hom}(\cdot, F_1) |_{\langle G \rangle_{2k}} \to H(\cdot) |_{\langle G \rangle_{2k}} \). Therefore \( \text{Hom}(\alpha', F_1)g = ga' \in \text{Hom}(A, F_1) \) lies in the kernel of \( \varphi : \text{Hom}(A, F_1) \to H(A) \). The exact sequence

\[
\text{Hom}(\cdot, F_2) |_{\langle G \rangle_{2k}} \xrightarrow{\text{Hom}(\cdot, \alpha) |_{\langle G \rangle_{2k}}} \text{Hom}(\cdot, F_1) |_{\langle G \rangle_{2k}} \xrightarrow{\varphi} H(\cdot) |_{\langle G \rangle_{2k}}
\]

tells us that there is an \( f \in \text{Hom}(A, F_2) \) with \( ga' = \text{Hom}(A, \alpha)f = \alpha f \).

In concrete terms we have produced a commutative diagram

```
A \xrightarrow{\alpha'} B \xrightarrow{\beta'} C \xrightarrow{\gamma'} \Sigma A
\downarrow f \quad \downarrow g \quad \downarrow \gamma
F_2 \xrightarrow{\alpha} F_1 \xrightarrow{\beta} F \xrightarrow{\gamma} \Sigma F_2
```

where the rows are triangles, which we may complete to a morphism of triangles

```
A \xrightarrow{\alpha'} B \xrightarrow{\beta'} C \xrightarrow{\gamma'} \Sigma A
\downarrow f \quad \downarrow g \quad \downarrow h \quad \downarrow \gamma
F_2 \xrightarrow{\alpha} F_1 \xrightarrow{\beta} F \xrightarrow{\gamma} \Sigma F_2
```
We now have an element \( h \in \text{Hom}(C, F) \) with \( h \beta' = \beta g \), or in more complicated terms \( \text{Hom}(\beta', F)h = \text{Hom}(B, \beta)g \). This buys us the third equality below
\[
H(\beta')z = \varphi(g) = \psi \text{Hom}(B, \beta)(g) = \psi \text{Hom}(\beta', F)h = H(\beta')\psi(h)
\]
The first equality is the choice of \( g \), the second is because on the category \( \langle G \rangle_{2k} \) we have \( \varphi = \psi \circ \text{Hom}(\cdot, \beta) \), and the fourth equality is the naturality of \( \psi \). We deduce that \( H(\beta') : H(C) \to H(B) \) annihilates \( z - \psi(h) \).

But the exact sequence \( H(\Sigma A) \xrightarrow{H(\gamma')} H(C) \xrightarrow{H(\beta')} H(B) \) tells us there exists a \( z' \in H(\Sigma A) \) with \( H(\gamma')z' = z - \psi(h) \). Since \( \Sigma A \) belongs to \( \langle G \rangle_{2k} \) the map \( \varphi : \text{Hom}(\Sigma A, F_1) \to H(\Sigma A) \) must be surjective; there is a \( \lambda \in \text{Hom}(\Sigma A, F_1) \) with \( \varphi(\lambda) = z' \). Therefore
\[
z - \psi(h) = H(\gamma')\varphi(\lambda) = H(\gamma')\psi \text{Hom}(\Sigma A, \beta)(\lambda) = \psi \text{Hom}(\gamma', F)\text{Hom}(\Sigma A, \beta)(\lambda) = \psi(\beta\lambda\gamma')
\]
where the second equality is the fact that on the category \( \langle G \rangle_{2k} \) we have \( \varphi = \psi \circ \text{Hom}(\cdot, \beta) \), the third is the naturality of \( \psi \), and the fourth is obvious. Hence \( z = \psi(h + \beta\lambda\gamma') \) is in the image of \( \psi \).

When \( R \) is a field the theorem below is due to Bondal and Van den Bergh [5, Theorem 1.3], and in the generality below it may be found in Rouquier [18, Theorem 4.16 and Corollary 4.18]. We include this new proof because it contains the simple ideas, whose more technical adaptation will yield the theorems of Section 7.

**Theorem 5.2.** Let \( R \) be a noetherian, commutative ring, let \( S \) be an \( R \)-linear triangulated category, and assume that \( G \in S \) is a strong generator—we remind the reader, this means that there exists some integer \( n > 0 \) with \( \langle G \rangle_n = S \). Assume \( H \) is a finite cohomological functor, as is \( \text{Hom}(\cdot, G) \). Then there exists a cohomological functor \( H' \) with \( H \oplus H' \) representable. If \( S \) is Karoubian, meaning idempotents split, then \( H \) is representable.

**Proof.** Choose an integer \( k \) with \( \langle G \rangle_{2k} = S \). Applying Lemma 5.1 to the \( \langle G \rangle_{2k} \)-cohomological functor \( H \) we can find an epimorphism \( \text{Hom}(\cdot, F) \to H(\cdot) \). Complete to an exact sequence \( 0 \to H'(\cdot) \to \text{Hom}(\cdot, F) \to H(\cdot) \to 0 \), and it follows that \( H' \) is also a finite cohomological functor. Applying Lemma 5.1 again we have a surjection \( \text{Hom}(\cdot, F') \to H'(\cdot) \), and this assembles to an exact sequence \( \text{Hom}(\cdot, F') \to \text{Hom}(\cdot, F) \to H(\cdot) \to 0 \).
Yoneda’s lemma tells us that the natural transformation $\text{Hom}(-, F') \to \text{Hom}(-, F)$ must be of the form $\text{Hom}(-, \alpha)$ for some $\alpha : F' \to F$, that the natural transformation $\text{Hom}(-, F) \to H(-)$ corresponds to an element $y \in H(F)$, and that the vanishing of the composite $\text{Hom}(-, F') \to \text{Hom}(-, F) \to H(-)$ means that the map $H(\alpha) : H(F) \to H(F')$ must take $y \in H(F)$ to zero. Now complete $\alpha : F' \to F$ to a triangle $F' \xrightarrow{\alpha} F \xrightarrow{\beta} F'' \to$. The exactness of $H(F'') \xrightarrow{H(\beta)} H(F) \xrightarrow{H(\alpha)} H(F')$, coupled with the fact that $H(\alpha)y = 0$, means that there must be an element $x \in H(F'')$ with $y = H(\beta)x$. By Yoneda this means that we obtain a natural transformation $\text{Hom}(-, F'') \to H(-)$ so that the diagram below commutes

\[
\begin{array}{ccc}
\text{Hom}(-, F''') & \to & \text{Hom}(-, F) \to \text{Hom}(-, F') \\
\downarrow & & \downarrow \\
\text{Hom}(-, F') & \to & \text{Hom}(-, F) \to H(-) \to 0
\end{array}
\]

and the rows are exact. It immediately follows that the map $\text{Hom}(-, F'') \to H(-)$ is a split epimorphism.

Choose a splitting $H(-) \to \text{Hom}(-, F'')$, for example the one coming from the diagram above. The composite $\text{Hom}(-, F'') \to H(-) \to \text{Hom}(-, F'')$ is an idempotent natural endomorphism of a representable functor, therefore of the form $\text{Hom}(-, e)$ where $e : F'' \to F''$ is idempotent. If $e$ splits then $H$ is representable. \hfill \Box

6. A couple of technical lemmas

In Section 7 we will prove Theorem 0.3. The proof will rely heavily on a couple of technical lemmas—in this section we state these in great generality, to cover both the application to come in Section 7 and the one to be found in the proof of [17, Theorem 4.6]. Let us therefore set up a little notation.

**Notation 6.1.** Throughout this section $R$ will be a commutative ring, $\mathcal{T}$ will be an $R$–linear triangulated category with coproducts, and $\mathcal{S} \subset \mathcal{T}$ will be a triangulated subcategory. The Yoneda functor $\mathcal{Y} : \mathcal{T} \to \text{Hom}_{R}[\mathcal{S}^{\text{op}}, R–\text{Mod}]$ will be the map taking $t \in \mathcal{T}$ to $\text{Hom}(-, t)$, where $\text{Hom}(-, t)$ is viewed as an $R$–linear functor $\mathcal{S}^{\text{op}} \to R–\text{Mod}$.

We remind the reader of Definition 4.1: suppose $A$ is a full subcategory of $\mathcal{T}$ closed under direct summands, finite coproducts and suspensions, and $H$ is a $\mathcal{S}$–cohomological functor, meaning $H : \mathcal{S}^{\text{op}} \to R–\text{Mod}$ is an $R$–linear cohomological functor. Then a $\mathcal{A}$–approximating system for $H$ is a sequence $E_1 \to E_2 \to E_3 \to \cdots$ in $\mathcal{A}$, with a subsequence in $\mathcal{A} \cap \mathcal{S}$, and an isomorphism $\text{colim} \mathcal{Y}(E_i) \to H(-)$.

**Definition 6.2.** Let $\mathcal{T}$ be an $R$–linear triangulated category with coproducts and let $\mathcal{S}$ be a triangulated subcategory. If $H$ is a $\mathcal{S}$–cohomological functor, we define $\Sigma H$ by the rule $\Sigma H(s) = H(\Sigma^{-1}s)$. 

\[
\begin{array}{ccc}
\text{Hom}(-, F'') & \to & \text{Hom}(-, F) \to \text{Hom}(-, F') \\
\downarrow & & \downarrow \\
\text{Hom}(-, F') & \to & \text{Hom}(-, F) \to H(-) \to 0
\end{array}
\]
A weak triangle in the category $\text{Hom}_R[\mathcal{S}^{\text{op}}, R{-}\text{Mod}]$ is a sequence of cohomological functors $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} \Sigma A$ such that any rotation of the following is true: given any triangle $a \xrightarrow{w'} b \xrightarrow{v'} c \xrightarrow{w'} \Sigma a$ in the category $\mathcal{S}$ and a commutative diagram

$$
\begin{array}{c}
y(a) \xrightarrow{y(u')} y(b) \xrightarrow{y(v')} y(c) \xrightarrow{y(w')} y(\Sigma a) \\
\downarrow f \quad \downarrow g \quad \downarrow h \\
A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} \Sigma A
\end{array}
$$

there is an extension to a commutative diagram

$$
\begin{array}{c}
y(a) \xrightarrow{y(u')} y(b) \xrightarrow{y(v')} y(c) \xrightarrow{y(w')} y(\Sigma a) \\
\downarrow f \quad \downarrow g \quad \downarrow h \\
A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} \Sigma A
\end{array}
$$

A diagram $\widehat{A} \xrightarrow{\widehat{u}} \widehat{B} \xrightarrow{\widehat{v}} \widehat{C} \xrightarrow{\widehat{w}} \Sigma \widehat{A}$ in the category $\mathcal{T}$ is called a weak triangle if the functor $\mathcal{Y}$ takes it to a weak triangle in $\text{Hom}_R[\mathcal{S}^{\text{op}}, R{-}\text{Mod}]$.

**Remark 6.3.** We remind the reader of Remark 4.6 and Lemma 4.8: if $A$, $B$ and $C$ have $\mathcal{T}$--approximating systems $\mathcal{A}_s$, $\mathcal{B}_s$ and $\mathcal{C}_s$, we may form in $\mathcal{T}$ the homotopy colimits $\widehat{A} = \text{Hocolim}\mathcal{A}_s$, $\widehat{B} = \text{Hocolim}\mathcal{B}_s$ and $\widehat{C} = \text{Hocolim}\mathcal{C}_s$. Remark 4.6 tells us that there are canonical maps $\alpha : A \rightarrow \mathcal{Y}(\widehat{A})$, $\beta : B \rightarrow \mathcal{Y}(\widehat{B})$ and $\gamma : C \rightarrow \mathcal{Y}(\widehat{C})$. Since our plan is to apply the lemmas in this section to prove representability theorems, we will mostly be interested in cases where $\alpha$, $\beta$ and $\gamma$ are isomorphisms. In this case Lemma 4.8 says that the maps $u$, $v$ and $w$ may be lifted (non-uniquely) to $\mathcal{T}$; we may form in $\mathcal{T}$ a diagram $\widehat{A} \xrightarrow{\widehat{u}} \widehat{B} \xrightarrow{\widehat{v}} \widehat{C} \xrightarrow{\widehat{w}} \Sigma \widehat{A}$ whose image under $\mathcal{Y}$ is (canonically) isomorphic to $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} \Sigma A$.

**Lemma 6.4.** Suppose $A \rightarrow B \rightarrow C \rightarrow \Sigma A$ is a weak triangle in $\text{Hom}_R[\mathcal{S}^{\text{op}}, R{-}\text{Mod}]$. For any $s \in \mathcal{S}$ the functor $\mathcal{Y}(s, -)$ takes it to an exact sequence.

**Proof.** Given any map $f : \mathcal{Y}(s) \rightarrow A$, we can consider the commutative diagram

$$
\begin{array}{c}
\mathcal{Y}(s) \xrightarrow{\mathcal{Y}(f)} 0 \xrightarrow{0} \mathcal{Y}(\Sigma s) \\
\downarrow f \\
A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} \Sigma A
\end{array}
$$

and the fact that this commutative diagram can be extended gives the vanishing of $vu \mathcal{Y}(f)$.

A morphism $g : \mathcal{Y}(s) \rightarrow B$ so that $vg = 0$ gives a commutative diagram

$$
\begin{array}{c}
\mathcal{Y}(s) \xrightarrow{\mathcal{Y}(f)} 0 \xrightarrow{0} \mathcal{Y}(\Sigma s) \\
\downarrow g \\
A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} \Sigma A
\end{array}
$$
Lemma 6.5. With the conventions of Notation [6.1] and Definition [6.2], suppose we are given:

(i) A morphism $\alpha : A \rightarrow B$ in the category $\text{Hom}_R[S^{\text{op}}, R-\text{Mod}]$.
(ii) A morphism $\alpha_* : A_* \rightarrow B_*$ of sequences in $\mathcal{T}$, and an isomorphism in $\text{Hom}_R[S^{\text{op}}, R-\text{Mod}]$ of $\alpha : A \rightarrow B$ with the colimit of $\gamma(\alpha_* ) : \gamma(A_*) \rightarrow \gamma(B_*)$.
(iii) The sequence $\alpha_*$ is assumed to have a subsequence in $S$.

With just these hypotheses we may complete $\alpha_* : A_* \rightarrow B_*$ to a sequence $A_* \xrightarrow{\alpha_*} B_* \xrightarrow{\beta_*} C_* \xrightarrow{\gamma_*} \Sigma_* A_*$ of triangles in $\mathcal{T}$, and the colimit of $\gamma(\alpha_* ) : \gamma(A_*) \rightarrow \gamma(B_*) \rightarrow \gamma(C_*) \rightarrow \Sigma_* \gamma(A_*)$ is a weak triangle $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} \Sigma_* A$.

Suppose we add the following assumptions:

(iv) We are given two subcategories $A \subset B \subset S$, closed under finite coproducts, direct summands and suspensions.
(v) There is a subsequence of $\alpha_* : A_* \rightarrow B_*$ such that the $A_i$ belong to $A$ and the $B_i$ belong to $B$. Put $C = \text{smd}(B \ast A)$.
(vi) Assume furthermore that we are given a $C$-cohomological functor $H$ and a natural transformation of $B$-cohomological functors $\varphi : B|_B \rightarrow H|_B$. Assume that, on the category $A \subset B$, the composite

$$A|_A \xrightarrow{\alpha|_A} B|_A \xrightarrow{\varphi|_A} H|_A$$

vanishes.

(vii) Assume further that the approximating system $A_*$ for $A$ is such that each morphism $A_* \rightarrow A_{i+1}$ is a split monomorphism.

Then there exists a map $\psi : C|_C \rightarrow H$ so that $\varphi : B|_B \rightarrow H|_B$ is equal to the composite

$$B|_B \xrightarrow{\beta|_B} C|_B \xrightarrow{\psi|_B} H|_B.$$ 

Proof. We are given a morphism of sequences $\alpha_* : A_* \rightarrow B_*$, meaning for each $m > 0$ we have a commutative square

$$\begin{array}{ccc}
A_m & \xrightarrow{\alpha_m} & B_m \\
\downarrow \alpha_m & & \downarrow \beta_m \\
A_{m+1} & \xrightarrow{\alpha_{m+1}} & B_{m+1}
\end{array}$$

We extend this to a morphism of triangles

$$\begin{array}{ccc}
A_m & \xrightarrow{\alpha_m} & B_m & \xrightarrow{\beta_m} & C_m & \xrightarrow{\gamma_m} & \Sigma A_m \\
\downarrow \alpha_m & & \downarrow \beta_m & & \downarrow \gamma_m & & \downarrow \Sigma A_m \\
A_{m+1} & \xrightarrow{\alpha_{m+1}} & B_{m+1} & \xrightarrow{\beta_{m+1}} & C_{m+1} & \xrightarrow{\gamma_{m+1}} & \Sigma A_{m+1}
\end{array}$$
This produces for us in \( \mathcal{T} \) the sequence of triangles \( A_m \xrightarrow{\alpha_m} B_m \xrightarrow{\beta_m} C_m \xrightarrow{\gamma_m} \Sigma A_m \), with a subsequence in \( \mathcal{S} \), and it is easy to see that the colimit of \( y(\mathcal{A}_n) \xrightarrow{y(\alpha_n)} y(\mathcal{B}_n) \xrightarrow{y(\beta_n)} y(\mathcal{C}_n) \) is a weak triangle \( A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} \Sigma A \).

It remains to prove the part with the further assumptions added. Note that, by passing to a subsequence, we may assume \( A \in \mathcal{A} \) and \( \mathcal{B} \in \mathcal{B} \), and hence \( \mathcal{C} \in \mathcal{B} \ast \mathcal{A} \subseteq \mathcal{C} \). As \( H \) is a cohomological functor on \( \mathcal{C} \) and \( \mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C} \), we have that, for each integer \( m \), the sequence \( H(\Sigma A_m) \xrightarrow{H(\gamma_m)} H(\mathcal{C}_m) \xrightarrow{H(\beta_m)} H(\mathcal{B}_m) \xrightarrow{H(\alpha_m)} H(\mathcal{A}_m) \) must be exact. As \( m \) increases this gives an inverse system of exact sequences, which we now propose to analyze. The short exact sequences \( 0 \rightarrow \ker(H(\gamma_m)) \rightarrow H(\Sigma A_m) \rightarrow \text{Im}(H(\gamma_m)) \rightarrow 0 \) give an exact sequence

\[
\lim_1 H(\Sigma A_m) \longrightarrow \lim_1 \text{Im}(H(\gamma_m)) \longrightarrow \lim_2 \ker(H(\gamma_m))
\]

In (vii) we assumed that the maps \( \mathcal{A}_m \rightarrow \mathcal{A}_{m+1} \) are split epimorphisms, hence the maps \( H(\Sigma \mathcal{A}_m) \rightarrow H(\Sigma \mathcal{A}_{m+1}) \) are split epimorphisms, making the sequence Mittag-Leffler. Therefore \( \lim_1 H(\Sigma \mathcal{A}_m) = 0 \). We have \( \lim_2 \ker(H(\gamma_m)) = 0 \) just because we’re dealing with a countable limit. We conclude that \( \lim_1 \text{Im}(H(\gamma_m)) = 0 \).

Now consider the inverse system of short exact sequences \( 0 \rightarrow \text{Im}(H(\gamma_m)) \rightarrow H(\mathcal{C}_m) \rightarrow \text{Im}(H(\beta_m)) \rightarrow 0 \). Passing to the limit we obtain an exact sequence

\[
\lim_1 H(\mathcal{C}_m) \longrightarrow \lim_1 \text{Im}(H(\beta_m)) \longrightarrow \lim_1 \text{Im}(H(\gamma_m))
\]

We have proved the vanishing of \( \lim_1 \text{Im}(H(\gamma_m)) \), allowing us to conclude that the map \( \lim_1 H(\mathcal{C}_m) \longrightarrow \lim_1 \text{Im}(H(\beta_m)) \) is an epimorphism. Finally we observe the exact sequences \( 0 \rightarrow \text{Im}(H(\beta_m)) \rightarrow H(\mathcal{B}_m) \rightarrow H(\mathcal{A}_m) \) and, since inverse limit is left exact, we deduce the exactness of

\[
0 \longrightarrow \lim_1 \text{Im}(H(\beta_m)) \longrightarrow \lim_1 H(\mathcal{B}_m) \longrightarrow \lim_1 H(\mathcal{A}_m)
\]

Combining the results we have the exactness of

\[
\lim_1 H(\mathcal{C}_m) \longrightarrow \lim_1 H(\mathcal{B}_m) \longrightarrow \lim_1 H(\mathcal{A}_m)
\]

Now Lemma \[\text{Lemma 6.3}\] tells us that \( \varphi : B|_B \rightarrow H|_B \) corresponds to an element \( f \in \lim_1 H(\mathcal{B}_m) \), and the vanishing of the composite \( A|_A \rightarrow B|_A \rightarrow H|_A \) translates to saying that the image of \( f \) under the map \( \lim_1 H(\mathcal{B}_m) \rightarrow \lim_1 H(\mathcal{A}_m) \) vanishes. The exactness tells us that \( f \) is in the image of the map \( \lim_1 H(\mathcal{C}_m) \rightarrow \lim_1 H(\mathcal{B}_m) \). This exactly says that there is a natural transformation \( \psi : C|_C \rightarrow H \) with \( \varphi = \psi \circ \beta \). We have proved the “extra assumptions” part. \( \square \)

**Lemma 6.6.** With the conventions of Notation \[\text{Notation 6.1}\] and Definition \[\text{Definition 6.3}\] suppose we are given:
(i) Two full subcategories \( \mathcal{A} \subset \mathcal{B} \) of the category \( \mathcal{S} \), closed under finite coproducts, direct summands and suspensions.

(ii) Put \( \mathcal{C} = \text{smd}(\mathcal{B} \ast \mathcal{A}) \). Assume we are also given a \( \mathcal{C} \)-cohomological functor \( H \).

(iii) We are given a weak triangle \( A \overset{\alpha}{\longrightarrow} B \overset{\beta}{\longrightarrow} C \overset{\gamma}{\longrightarrow} \Sigma A \), and a natural transformation of \( \mathcal{C} \)-cohomological functors \( \psi : C|_\mathcal{C} \rightarrow H \).

(iv) The composite \( (\psi \beta)|_B : B|_B \rightarrow H|_B \) is surjective.

(v) The sequence

\[
\begin{array}{ccc}
A|_\mathcal{A} & \overset{\alpha|_\mathcal{A}}{\longrightarrow} & B|_\mathcal{A} \\
& \downarrow{\psi|_\mathcal{A}} & \downarrow{\psi|_B} \\
H|_\mathcal{A} & \overset{\beta|_B}{\longrightarrow} & 0
\end{array}
\]

is exact.

Then the map \( \psi : C|_\mathcal{C} \rightarrow H \) an epimorphism.

**Proof.** We need to show the surjectivity of the map \( \psi : C(c) \rightarrow H(c) \) for every \( c \in \mathcal{C} = \text{smd}(\mathcal{B} \ast \mathcal{A}) \); without loss of generality we may assume \( c \in \mathcal{B} \ast \mathcal{A} \). Choose a triangle \( a \overset{\alpha'}{\longrightarrow} b \overset{\beta'}{\longrightarrow} c \overset{\gamma'}{\longrightarrow} \Sigma a \) with \( b \in \mathcal{B} \) and \( a \in \mathcal{A} \). Given any element \( y \in H(c) \), the map \( H(\beta') : H(c) \rightarrow H(b) \) takes \( y \) to an element \( H(\beta')(y) \) which must be in the image of the surjective map \( \psi \beta : B(b) \rightarrow H(b) \). After all \( b \) is an object in \( \mathcal{B} \), and the map \( \psi \beta : B(b) \rightarrow H(b) \) is an epimorphism on objects \( b \in \mathcal{B} \). Choose an element \( g \in B(b) \) mapping under \( \psi \beta \) to \( H(\beta')(y) \). The naturality of \( \psi \beta \) means that the square below commutes

\[
\begin{array}{ccc}
B(b) & \overset{B(\alpha')}{\longrightarrow} & B(a) \\
\downarrow{\psi \beta} & & \downarrow{\psi \beta} \\
H(c) & \overset{H(\beta')}{\longrightarrow} & H(a)
\end{array}
\]

If we apply the equal composites in the square to \( g \in B(b) \) we discover that it goes to \( H(\alpha')H(\beta')(y) = 0 \), where the vanishing is because \( \beta'\alpha' = 0 \). Therefore the map \( B(\alpha') \) takes \( g \in B(b) \) to an element in the kernel of \( \psi \beta : B(a) \rightarrow H(a) \). As \( a \) belongs to \( \mathcal{A} \), the map \( \alpha : A(a) \rightarrow B(a) \) surjects onto this kernel; there is an element \( f \in A(a) \) with \( \alpha(f) = B(\alpha')(g) \).

We have produced elements \( f \in A(a) \) and \( g \in B(b) \), and Yoneda allows us to view them as natural transformations \( f : \mathcal{Y}(a) \rightarrow A \) and \( g : \mathcal{Y}(b) \rightarrow B \). The equality \( \alpha(f) = B(\alpha')(g) \) transforms into the assertion that the square below commutes

\[
\begin{array}{ccccc}
\mathcal{Y}(a) & \overset{\mathcal{Y}(\alpha')}{\longrightarrow} & \mathcal{Y}(b) & \overset{\mathcal{Y}(\beta')}{\longrightarrow} & \mathcal{Y}(c) & \overset{\mathcal{Y}(\gamma')}{\longrightarrow} & \mathcal{Y}(\Sigma A) \\
\downarrow{f} & & \downarrow{g} & & \downarrow{\gamma} & & \downarrow{\Sigma A} \\
A & \overset{\alpha}{\longrightarrow} & B & \overset{\beta}{\longrightarrow} & C & \overset{\gamma}{\longrightarrow} & \Sigma A
\end{array}
\]
The top row is the image under Yoneda of a triangle in $S$, while the bottom row is a weak triangle; hence we may complete to a commutative diagram

\[
\begin{array}{cccccc}
Y(a) & \xrightarrow{y(a')} & Y(b) & \xrightarrow{y(b')} & Y(c) & \xrightarrow{y(\gamma')} \Sigma f \\
\downarrow f & & \downarrow g & & \downarrow h & \\
A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \xrightarrow{\gamma} \Sigma A
\end{array}
\]

We have produced a morphism $h : Y(c) \rightarrow C$, which we may view as an element $h \in C(c)$. And the commutativity of the middle square translates, under Yoneda, to the statement that $C(\beta') : C(c) \rightarrow C(b)$ takes $h \in C(c)$ to $\beta(g) \in C(b)$. Applying $\psi : C \rightarrow H$ we obtain the second equality below

\[
H(\beta')(y) = \psi \beta(g) = \psi C(\beta')(h) = H(\beta') \psi(h).
\]

The first equality is by construction of $g \in B(b)$, and the third is the naturality of $\psi$. Therefore the map $H(\beta') : H(c) \rightarrow H(b)$ annihilates $y - \psi(h)$. Because $H$ is cohomological there is an element $x \in H(\Sigma a)$ with $H(\gamma')(x) = y - \psi(h)$. But $a \in \mathcal{A} \subset \mathcal{B}$ and we may choose an $\theta \in B(\Sigma a)$ with $\psi \beta(\theta) = x$. We have

\[
y - \psi(h) = H(\gamma') \psi \beta(\theta) = \psi C(\gamma') \beta(\theta)
\]

where the first equality is the construction of $\theta$, and the second is the naturality of $\psi : C \rightarrow H$. These equalities combine to the formula $y = \psi[h + C(\gamma') \beta(\theta)]$, which exhibits $y \in H(c)$ as lying in the image of $\psi : C(c) \rightarrow H(c)$. □

The next lemma will not be used in the current manuscript, but will be needed in the proof of [17, Theorem 4.6].

**Lemma 6.7.** With the conventions of Notation 6.1 and Definition 6.2 suppose we are given:

(i) Two full subcategories $\mathcal{A} \subset \mathcal{B}$ of the category $S$, closed under finite coproducts, direct summands and suspensions.

(ii) Put $\mathcal{C} = \text{smd}(\mathcal{B} \ast \mathcal{A})$. 
(iii) In the category $\text{Hom}_R[\mathcal{S}^{op}, R\text{-Mod}]$ we are given a diagram of cohomological functors

$$
\begin{array}{ccc}
A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \xrightarrow{\gamma} & \Sigma A \\
& & \downarrow{\delta} & & \downarrow{\psi} & & \downarrow{\psi} \\
& & B & & C & & A
\end{array}
$$

where the middle row is a weak triangle.

(iv) The composite $\psi\beta\delta : \tilde{B} \rightarrow H$ is surjective.

(v) The kernel of the map $(\psi\beta\delta)_|_B : \tilde{B}|_B \rightarrow H|_B$ is annihilated by $\delta : \tilde{B} \rightarrow B$

(vi) The sequence

$$
\begin{array}{ccc}
A|_A & \xrightarrow{A|_A} & B|_A & \xrightarrow{(B\beta)_|_A} & H|_A & \rightarrow & 0
\end{array}
$$

is exact.

Then the map $(\beta\delta) : \tilde{B} \rightarrow C$ annihilates the kernel of $(\psi\beta\delta)|_C : \tilde{B}|_C \rightarrow H|_C$.

**Proof.** We need to show that, if $c \in \mathcal{C} = \text{smd}(\mathcal{B} \ast \mathcal{A})$ and $y \in \tilde{B}(c)$ is annihilated by the map $\psi\beta\delta : \tilde{B}(c) \rightarrow H(c)$, then $y$ is already annihilated by the shorter map $\beta\delta : \tilde{B}(c) \rightarrow C(c)$. Note that without loss of generality we may assume $c \in \mathcal{B} \ast \mathcal{A}$.

Choose therefore a triangle $a \xrightarrow{\alpha} b \xrightarrow{\beta} c \xrightarrow{\gamma} \Sigma a$ with $b \in \mathcal{B}$ and $a \in \mathcal{A}$, and consider the commutative diagram with exact rows

$$
\begin{array}{ccc}
\tilde{B}(c) & \xrightarrow{\tilde{B}(\beta')} & \tilde{B}(b) \\
B(\Sigma b) & \xrightarrow{B(\Sigma \alpha')} & B(\Sigma a) & \xrightarrow{B(\gamma')} & B(c) & \xrightarrow{B(\beta')} & B(b) \\
H(\Sigma b) & \xrightarrow{H(\Sigma \alpha')} & H(\Sigma a) & \xrightarrow{H(\gamma')} & H(c) & \xrightarrow{H(\beta')} & H(b)
\end{array}
$$

We are given an element $y \in \tilde{B}(c)$ such that the vertical composite in the third column annihilates it. Therefore $\tilde{B}(\beta')(y)$ is an element of $\tilde{B}(b)$ annihilated by the vertical composite in the fourth column. By assumption (v) the element $\tilde{B}(\beta')(y)$ is already killed by $\delta : \tilde{B}(b) \rightarrow B(b)$, and we conclude that the equal composite in the top-right square annihilate $y$. Hence the map $\delta : \tilde{B}(c) \rightarrow B(c)$ must take $y \in \tilde{B}(c)$ to an element in the image of $B(\gamma')$, and we may therefore

(vii) Choose an $x \in B(\Sigma a)$ with $B(\gamma')(x) = \delta(y)$.

Now recall that the vertical composite in the third column kills $y$, hence the equal composites in the middle square at the bottom must annihilate $x$. Therefore $\psi\beta(x) \in$
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\[ H(\Sigma a) \text{ lies in the kernel of } H(\gamma'), \text{ which is the image of } H(\Sigma \alpha') : H(\Sigma b) \rightarrow H(\Sigma a). \]

And since the map \( \psi \beta : B(\Sigma b) \rightarrow H(\Sigma b) \) is surjective we may lift further to \( B(\Sigma b) \); we can choose an element \( w \in B(\Sigma b) \) whose image under the equal composites in the bottom-left square are equal to \( \psi \beta(x) \). Therefore we have that, with \( x \in B(\Sigma a) \) as in (vii) and \( w \in B(\Sigma b) \) as above, the element \( x - B(\Sigma \alpha')(w) \) is annihilated by \( \psi \beta : B(\Sigma a) \rightarrow H(\Sigma a) \).

Now (vi) tells us that

(viii) We may choose an element \( v \in A(\Sigma a) \) whose image under \( \alpha : A(\Sigma a) \rightarrow B(\Sigma a) \) is equal to \( x - B(\Sigma \alpha')(w) \).

To complete the proof consider the commutative diagram with vanishing horizontal and vertical composites

\[
\begin{array}{ccc}
A(\Sigma a) & \xrightarrow{A(\gamma')} & A(c) \\
\downarrow{\alpha} & & \downarrow{\alpha} \\
B(\Sigma b) & \xrightarrow{B(\Sigma \alpha')} & B(\Sigma a) \\
\downarrow{\beta} & & \downarrow{\beta} \\
C(\Sigma a) & \xrightarrow{C(\gamma')} & C(c)
\end{array}
\]

The horizontal map in the top row takes the element \( v \in A(\Sigma a) \) constructed in (viii) to \( A(\gamma')(v) \), which must be annihilated by the vertical composite in the third column. By the commutativity of the top square coupled with (viii), this means that \( x - B(\Sigma \alpha')(w) \) is an element of \( B(\Sigma a) \) annihilated by the equal composites in the bottom square. In particular the horizontal map \( B(\gamma') \) takes \( x - B(\Sigma \alpha')(w) \) to an element of the kernel of \( \beta : B(c) \rightarrow C(c). \) But the map \( B(\gamma') \) annihilates \( B(\Sigma \alpha')(w) \), and by (vii) it takes \( x \) to \( \delta(y) \). We conclude that \( \beta \delta(y) = 0. \) □

7. The main theorems

It’s time to prove Theorem 0.3; we should focus the general lemmas of Section 6 on the situation at hand. Thus in this section we make the following global assumptions:

Notation 7.1. We specialize the conventions of Notation 6.1 by setting \( S = T^c \), that is \( S \) is the subcategory of compact objects in \( T \). Thus in this section the functor \( y \) of Notation 6.1 specializes to \( y : T \rightarrow \text{Hom}_R([T^c]^\text{op}, R-\text{Mod}) \), which takes an object \( t \in T \) to \( y(t) = \text{Hom}(-, t)|_{T^c}. \)

In the generality of Notation 6.1 we considered \( S \)-cohomological functors \( H \) and \( B \)-approximating systems \( E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow \cdots \). Because we are now in the special case where \( S = T^c \) Remark 4.7 applies: if \( F = \text{Hocolim} \ E_i \) then the natural map \( H \rightarrow y(F) \) must be an isomorphism.

We will furthermore assume that we have chosen in \( T \) a single compact generator \( G \). We will suppose given a \( t \)-structure \((T^{\leq 0}, T^{\geq 0})\) in the preferred equivalence class, with \( T^{\geq 0} \) closed under coproducts. For example we could let \((T^{\leq 0}, T^{\geq 0})\) be equal to
Let $\mathcal{A}$ be the heart of the $t$-structure, and $\mathcal{K}: \mathcal{T} \rightarrow \mathcal{A}$ the homological functor of Remark \[1.1\]. We will assume given an integer $A > 0$ with $\text{Hom}(\Sigma^{-A}G, \mathcal{T} \leq 0) = 0$. The existence of such an $A$ is equivalent to the hypothesis that $\text{Hom}(G, \Sigma^iG) = 0$ for $i \gg 0$. And what is important for us is that this guarantees that the category $\mathcal{T}_c^-$ is a thick subcategory of $\mathcal{T}$.

Finally and most importantly: as in Notation \[6.1\] the triangulated category $\mathcal{T}$ is assumed to be $R$-linear for some commutative ring $R$. But from now on we add the assumption that the ring $R$ is noetherian and, with $G$ as in the last paragraph, the $R$-module $\text{Hom}(G, \Sigma^iG)$ is finite for every $i \in \mathbb{Z}$. Combining this paragraph with the last: the functor $\mathcal{Y}(G)$ is $G$-locally finite.

Under some additional approximability assumptions, Theorem \[0.3\] describes the essential image of the functor $\mathcal{Y}$ taking $F \in \mathcal{T}_c^-$ to $\mathcal{Y}(F) = \text{Hom}(-, F)|_{\mathcal{T}_c}$, and tells us that the functor $\mathcal{Y}$ is full. To show that $\mathcal{Y}(F)$ lies in the expected image one doesn’t need any hypotheses beyond the ones above, we prove

**Lemma 7.2.** With the assumptions of Notation \[7.1\], for any $F \in \mathcal{T}_c^-$ the functor $\mathcal{Y}(F): [\mathcal{T}_c]^0 \rightarrow R\text{-Mod}$ is a locally finite $\mathcal{T}_c$-cohomological functor.

**Proof.** We are given that the functor $\mathcal{Y}(G)$ is $G$-locally finite. In particular: for $i \ll 0$ we have $\text{Hom}(\Sigma^iG, G) = 0$. With $(\mathcal{T} \leq 0, \mathcal{T} \geq 0)$ as in Notation \[7.1\] that is our fixed $t$-structure in the preferred equivalence class, Remark \[0.20\] coupled with Lemma \[2.8\] tell us that, for any object $K \in \mathcal{T}_c$, there is an integer $B > 0$ so that $\text{Hom}(\Sigma^{-B}K, \mathcal{T} \leq 0) = 0$. Remark \[0.18\] gives the inclusion in $F \in \mathcal{T}_c^- \subset \mathcal{T}^-$, hence we may choose an integer $A > 0$ with $\Sigma^AF \in \mathcal{T} \leq 0$. We deduce that $\text{Hom}(\Sigma^iK, F) = 0$ for all $i \leq -A - B$.

The fact that the functor $\mathcal{Y}(G)$ is $G$-locally finite also means that, for every integer $i \in \mathbb{Z}$, the $R$-module $\text{Hom}(\Sigma^iG, G)$ is finite. The full subcategory $\mathcal{L} \subset \mathcal{T}_c$ defined by

$$\mathcal{L} = \{ L \in \mathcal{T}_c \mid \text{Hom}(\Sigma^iG, L) \text{ is a finite } R\text{-module for all } i \in \mathbb{Z} \}$$

is thick and contains $G$, hence $\mathcal{T}_c = \langle G \rangle \subset \mathcal{L}$. Now take any $L \in \mathcal{T}_c$ and define the full subcategory $\mathcal{K}(L) \subset \mathcal{T}_c$ by

$$\mathcal{K}(L) = \{ K \in \mathcal{T}_c \mid \text{Hom}(\Sigma^iK, L) \text{ is a finite } R\text{-module for all } i \in \mathbb{Z} \}$$

Then $\mathcal{K}(L)$ is thick and contains $G$, hence $\mathcal{T}_c = \langle G \rangle \subset \mathcal{K}(L)$. We conclude that $\text{Hom}(\Sigma^iK, L)$ is a finite $R$-module for all $K, L \in \mathcal{T}_c$ and all $i \in \mathbb{Z}$.

Now fix the integer $i$, the object $K \in \mathcal{T}_c$ and the object $F \in \mathcal{T}_c^-$, and we want to prove that $\text{Hom}(\Sigma^iK, F)$ is a finite $R$-module. The first paragraph of the proof produced an integer $B > 0$ with $\text{Hom}(\Sigma^{-B}K, \mathcal{T} \leq 0) = 0$, and since $F$ is approximable there exists a triangle $L \rightarrow F \rightarrow D$ with $L \in \mathcal{T}_c$ and $D \in \mathcal{T} \leq -i-B-1$. In the exact sequence

$$\text{Hom}(\Sigma^iK, \Sigma^{-1}D) \rightarrow \text{Hom}(\Sigma^iK, L) \rightarrow \text{Hom}(\Sigma^iK, F) \rightarrow \text{Hom}(\Sigma^iK, D)$$

we have that $\text{Hom}(\Sigma^iK, D) = 0 = \text{Hom}(\Sigma^iK, \Sigma^{-1}D)$, and $\text{Hom}(\Sigma^iK, F) \cong \text{Hom}(\Sigma^iK, L)$ must be a finite $R$-module by the second paragraph of the proof. \[\square\]
In Corollary 2.14 we learned that, under some approximability hypotheses, objects in $\mathcal{T}_c$ can be well approximated by sequences with special properties. We don’t need all these properties yet; for the next few lemmas we formulate what we will use.

**Definition 7.3.** Adopting the conventions of Notation 7.1, a strong $(G)_n$-approximating system is a sequence of objects and morphisms $E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow \cdots$

(i) Each $E_m$ belongs to $(G)_n$.

(ii) The map $\mathcal{H}^i(E_m) \rightarrow \mathcal{H}^i(E_{m+1})$ is an isomorphism whenever $i \geq -m$.

In this definition we also allow $n = \infty$, we simply declare $\langle G \rangle_\infty = \langle G \rangle = \mathcal{T}_c$.

Suppose we are also given an object $F \in \mathcal{T}$, together with

(iii) A map of the approximating system $E_0$ to $F$.

(iv) The map in (iii) is such that $\mathcal{H}^i(E_m) \rightarrow \mathcal{H}^i(F)$ is an isomorphism whenever $i \geq -m$.

Then we declare $E_*$ to be a strong $(G)_n$-approximating system for $F$.

**Remark 7.4.** Although Definition 7.3 was phrased in terms of the particular choice of $t$-structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{> 0})$, made in Notation 7.1, it is robust—up to passing to subsequences a strong $(G)_n$-approximating system for $F$ will work for any equivalent $t$-structure.

**Lemma 7.5.** With the conventions of Definition 7.3 we have

(i) Given an object $F \in \mathcal{T}^-$ and a strong $(G)_n$-approximating system $E_*$ for $F$, then the (non-canonical) map $\text{Hocolim } E_i \rightarrow F$ is an isomorphism.

(ii) Any object $F \in \mathcal{T}^-$ has a strong $\mathcal{T}^c$-approximating system.

(iii) Any $(G)_n$-strong approximating system $E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow \cdots$ is a strong $(G)_n$-approximating system of the homotopy colimit $F = \text{Hocolim } E_i$. Moreover $F$ belongs to $\mathcal{T}^{\leq n}$.

**Proof.** We begin by proving (iii). Suppose $E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow \cdots$ is a strong $(G)_n$-approximating system, and let $F = \text{Hocolim } E_i$. The objects $E_i$ all belong to $(G)_n \subset \mathcal{T}_c \subset \mathcal{T}^-$. Choose an integer $n > 0$ with $E_1 \in \mathcal{T}^{\leq n}$. The fact that $\mathcal{H}^i(E_1) \rightarrow \mathcal{H}^i(E_m)$ is an isomorphism for all $i \geq -1$ means that $\mathcal{H}^i(E_m) = 0$ for all $i > n$ and all $m$, and Lemma 1.2 gives that the $E_m$ all lie in $\mathcal{T}^{\leq n}$. Hence the homotopy colimit $F$ also belongs to $\mathcal{T}^{\leq n}$.

Now Remark 1.3 tells us that the map $\text{colim } \mathcal{H}^i(E_m) \rightarrow \mathcal{H}^i(F)$ is an isomorphism for every $i \in \mathbb{Z}$. By the previous paragraph the triangle $E_m \rightarrow F \rightarrow D_m$ lies in $\mathcal{T}^-$, and as $\mathcal{H}^i(E_m) \rightarrow \mathcal{H}^i(F)$ is an isomorphism for $i \geq -m$ we deduce that $\mathcal{H}^i(D_m) = 0$ for all $i \geq -m$. Lemma 1.2 guarantees that $D_m \in \mathcal{T}^{\leq -m-1}$, and as $E_m \in (G)_n \subset \mathcal{T}_c$ and $m > 0$ is arbitrary we have that $F$ satisfies the criterion for belonging to $\mathcal{T}^c$.

Next we prove (i). By (iii) the map $\text{Hocolim } E_i \rightarrow F$ is a morphism from $\text{Hocolim } E_i \in \mathcal{T}^c$ to $F \in \mathcal{T}^-$, hence it is a morphism in $\mathcal{T}^-$. The hypothesis of (i), coupled with Remark 1.3(ii) tell us that $\mathcal{H}^i(\text{Hocolim } E_i) \rightarrow \mathcal{H}^i(F)$ is an isomorphism for every $i \in \mathbb{Z}$. By Lemma 1.3(ii) the map $\text{Hocolim } E_i \rightarrow F$ is an isomorphism.
It remains to prove (ii). Note that, if we assume more approximability hypotheses on $\mathcal{T}$, then (ii) is immediate from Corollary 2.11. But let us see that we don’t yet need any strong assumptions.

Take any $F \in \mathcal{T}_c^-$. There exists a triangle $E_1 \to F \to D_1$ with $E_1 \in \mathcal{T}^c$ and $D_1 \in \mathcal{T}^{\leq -3}$. When $i \geq -1$ exact sequence $\mathcal{H}^{i-1}(D_1) \to \mathcal{H}^i(E_1) \to \mathcal{H}^i(F) \to \mathcal{H}^i(D)$ has $\mathcal{H}^{i-1}(D_1) = 0 = \mathcal{H}^i(D)$, starting the construction of $E_\ast$.

Suppose now that we have constructed the sequence up to an integer $n > 0$, that is we have a map $f_m : E_m \to F$, with $E_m \in \mathcal{T}^c$, and so that $\mathcal{H}^i(f_m)$ is an isomorphism for all $i \geq -m$. Lemma 2.8 allows us to choose an integer $N > 0$ so that $\text{Hom}(E_m, \mathcal{T}^{\leq -N}) = 0$. Because $F$ belongs to $\mathcal{T}_c^-$ we may choose a triangle $E_{m+1} \to F \to D_{m+1}$ with $E_{m+1} \in \mathcal{T}^c$ and $D_{m+1} \in \mathcal{T}^{\leq -N-m-3}$. As in the paragraph above we show that the map $\mathcal{H}^i(E_{m+1}) \to \mathcal{H}^i(F)$ is an isomorphism for all $i \geq -m - 1$. And since the composite $E_m \xrightarrow{f_n} F \to D_{m+1}$ vanishes, the map $f_n$ must factor as $E_m \to E_{m+1} \to F$. \qed

**Remark 7.6.** Lemma 7.5(iii) and Remark 4.7 combine to tell us that a strong $\langle G \rangle_n$–approximating system for $F$, in the sense of Definition 7.3 is in fact an approximating system for $F$ as defined in Remark 4.6. Our terminology isn’t misleading.

**Remark 7.7.** Let us now specialize Lemma 6.5 to the framework of this section. Assume we are given

(i) A morphism $\hat{\alpha} : \hat{A} \to \hat{B}$ in the category $\mathcal{T}_c^-$.

(ii) Two integers $n'$ and $n$, as well as a strong $\langle G \rangle_{n'}$–approximating system $\mathfrak{A}_\ast$ for $\hat{A}$ and a strong $\langle G \rangle_n$–approximating system $\mathfrak{B}_\ast$ for $\hat{B}$.

Lemma 4.5 allows us to choose a subsequence of $\mathfrak{B}_\ast \subset \mathfrak{B}_\ast$ and a map of sequences $\alpha_\ast : \mathfrak{A}_\ast \to \mathfrak{B}_\ast$ compatible with $\hat{\alpha} : \hat{A} \to \hat{B}$. A subsequence of a strong $\langle G \rangle_n$–approximating sequence is clearly a strong $\langle G \rangle_n$–approximating sequence, hence $\mathfrak{B}_\ast$ is a strong $\langle G \rangle_n$–approximating sequence for $\hat{B}$. Now as in Lemma 6.5 we extend $\alpha_\ast : \mathfrak{A}_\ast \to \mathfrak{B}_\ast$ to a sequence of triangles, in particular for each $m > 0$ this gives a morphism of triangles

\[
\begin{array}{cccccc}
\mathfrak{A}_m & \xrightarrow{\alpha_m} & \mathfrak{B}_m & \xrightarrow{\beta_m} & \mathfrak{C}_m & \xrightarrow{\gamma_m} & \Sigma \mathfrak{A}_m & \xrightarrow{\Sigma \alpha_m} & \Sigma \mathfrak{B}_m \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathfrak{A}_{m+1} & \xrightarrow{\alpha_{m+1}} & \mathfrak{B}_{m+1} & \xrightarrow{\beta_{m+1}} & \mathfrak{C}_{m+1} & \xrightarrow{\gamma_{m+1}} & \Sigma \mathfrak{A}_{m+1} & \xrightarrow{\Sigma \alpha_{m+1}} & \Sigma \mathfrak{B}_{m+1}
\end{array}
\]

Applying the functor $\mathcal{H}^i$ with $i \geq -m$ yields a commutative diagram in the heart of $\mathcal{T}$ where the rows are exact, and where the vertical maps away from the middle are isomorphisms. By the 5-lemma the middle vertical map, i.e. the map $\mathcal{H}^i(\mathfrak{C}_m) \to \mathcal{H}^i(\mathfrak{C}_{m+1})$, must also be an isomorphism when $i \geq -m$. We conclude that $\mathfrak{C}_\ast$ is a strong $\langle G \rangle_{n'+n}$–approximating system. Put $\hat{\mathfrak{C}} = \text{Hocolim} \mathfrak{C}_\ast$. By Lemma 7.3(iii) the object $\hat{\mathfrak{C}}$ belongs to $\mathcal{T}_c^-$ and $\mathfrak{C}_\ast$ is a strong $\langle G \rangle_{n'+n}$–approximating system for $\hat{\mathfrak{C}}$, while Remark 6.5 guarantees that the weak triangle $\hat{A} \xrightarrow{u} \hat{B} \xrightarrow{v} \hat{C} \xrightarrow{w} \Sigma \hat{A}$ of Lemma 6.5 is isomorphic to the image under $\mathcal{Y}$ of a weak triangle $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} \Sigma A$ in the category $\mathcal{T}_c^-$.

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Furthermore: the homological functor \( \mathcal{H} \) takes each of the triangles \( \mathcal{A}_m \xrightarrow{\alpha_m} \mathcal{B}_m \xrightarrow{\beta_m} \mathcal{C}_m \xrightarrow{\gamma_m} \Sigma \mathcal{A}_m \) to a long exact sequence, and by Remark 1.3 the (eventually stable) colimit is \( \mathcal{H} \) of the weak triangle \( \hat{\mathcal{A}} \xrightarrow{\hat{u}} \hat{\mathcal{B}} \xrightarrow{\hat{v}} \hat{\mathcal{C}} \xrightarrow{\hat{w}} \Sigma \hat{\mathcal{A}} \). Hence \( \mathcal{H} \) takes the weak triangle \( \hat{\mathcal{A}} \xrightarrow{\hat{u}} \hat{\mathcal{B}} \xrightarrow{\hat{v}} \hat{\mathcal{C}} \xrightarrow{\hat{w}} \Sigma \hat{\mathcal{A}} \) to a long exact sequence.

**Lemma 7.8.** Let the conventions be as in Notation 7.1. Assume \( H \) is a locally finite \( (G) \)-cohomological functor. Then there exists an object \( F \in \mathcal{T}^{-} \) and an epimorphism of \( (G) \)-cohomological functors \( \varphi : \mathcal{Y}(F)|_{(G)} \xrightarrow{} H \). Furthermore the object \( F \) may be chosen to have a strong \( (G) \)-approximating system.

We will in fact prove a refinement of the above. Since \( H \) is assumed to be a locally finite \( (G) \)-cohomological functor its restriction to \( (G) \), for any integer \( m < n \), is a locally finite \( (G) \)-cohomological functor. Hence for any \( m < n \) the first paragraph delivers an object \( F_m \in \mathcal{T}^{-} \), with a strong \( (G) \)-approximating system, and a surjective natural transformation \( \varphi_m : \mathcal{Y}(F_m)|_{(G)} \xrightarrow{} H|_{(G)} \). We will actually construct these \( F_m \)'s compatibly. We will produce in \( \mathcal{T}^{-} \) a sequence \( F_1 \rightarrow F_2 \rightarrow \cdots \rightarrow F_{n-1} \rightarrow F_n \), with compatible maps \( \varphi_m : \mathcal{Y}(F_m)|_{(G)} \xrightarrow{} H \), so that

(i) For each \( m > 0 \) the object \( F_m \) has a strong \( (G) \)-approximating system, and the map \( \varphi_m|_{(G)} : \mathcal{Y}(F_m)|_{(G)} \xrightarrow{} H|_{(G)} \) is an epimorphism.

(ii) The sequence is such that the kernel of the map \( \varphi_m : \mathcal{Y}(F_m)|_{(G)} \rightarrow H|_{(G)} \) is annihilated by the map \( \mathcal{Y}(F_m)|_{(G)} \rightarrow \mathcal{Y}(F_{m+1})|_{(G)} \).

**Proof.** The proof is by induction on \( n \). In the case \( n = 1 \) we prove the refinement that allows the induction to proceed

(iii) Suppose \( H \) is a locally finite \( (G) \)-cohomological functor. Then we may construct an object \( F \in \mathcal{T}^{-} \) and an epimorphism \( \mathcal{Y}(F)|_{(G)} \xrightarrow{} H \). Furthermore the object \( F \) can be chosen to have a strong \( (G) \)-approximating system \( E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow \cdots \) in which every morphism \( E_i \rightarrow E_{i+1} \) is a split monomorphism.

The proof of (iii) is easy: we have that \( H(\Sigma^i G) \) is a finite \( R \)-module for every \( i \in \mathbb{Z} \), and vanishes is \( i \ll 0 \). For each \( i \) with \( H(\Sigma^i G) \neq 0 \) choose a finite number of generators \( \{ f_{ij}, j \in J_i \} \) for the \( R \)-module \( H(\Sigma^i G) \). By Yoneda every \( f_{ij} \in H(\Sigma^i G) \) corresponds to a morphism \( \varphi_{ij} : \mathcal{Y}(\Sigma^i G) \xrightarrow{} H \). Let \( F \) be defined by

\[
F = \coprod_{i \in \mathbb{Z}} \bigoplus_{j \in J_i} \Sigma^i G
\]

and let the morphism \( \varphi : \mathcal{Y}(F) \xrightarrow{} H \) be given by

\[
\mathcal{Y}(F)|_{(G)} \xrightarrow{} \bigoplus_{i \in \mathbb{Z}} \bigoplus_{j \in J_i} \mathcal{Y}(\Sigma^i G) \xrightarrow{\varphi_{ij}} H
\]

where \( (\varphi_{ij}) \) stands for the row matrix with entries \( \varphi_{ij} \); on the \( i,j \) summand the map is \( \varphi_{ij} \). Finally: because the \( t \)-structure is in the preferred equivalence class there is an
integer $B > 0$ with $\Sigma^B G \in \mathcal{T}_{\leq 0}$. For $m > 0$ we define

$$E_m = \bigoplus_{i \leq m + B} \bigoplus_{j \in J_i} \Sigma^j G$$

The sum is finite by hypothesis, making $E_m$ an object of $(G)_{1}$. The obvious map $E_m \rightarrow E_{m+1}$ is a split monomorphism, and in the decomposition $F \cong E_m \oplus \widetilde{F}$ we have that $\widetilde{F}$, being the coproduct of $\Sigma^j G$ for $i = m + B + 1$, belongs to $\mathcal{T}_{\leq -m-1}$. Therefore the map $\mathcal{H}^i(E_m) \rightarrow \mathcal{H}^i(F)$ is an isomorphism when $i \geq -m$, making $E_*$ is a strong $(G)_1$-approximating system for $F$.

Now for the induction step. Suppose $n \geq 1$ is an integer and we know the Lemma for all integers $\leq n$. We wish to show it holds for $n + 1$. Let $H$ be a locally finite $(G)_{n+1}$-cohomological functor. Then the restriction of $H$ to $(G)_{n}$ is a locally finite $(G)_n$-cohomological functor, and we may apply the induction hypothesis to produce in $\mathcal{T}_c$ a sequence $F_1 \rightarrow F_2 \rightarrow \cdots \rightarrow F_{n-1} \rightarrow F_n$, with compatible surjections $\varphi_m : \gamma(F_m)\langle G\rangle_n \rightarrow H\langle G\rangle_n$. In particular the map $\varphi_n : \gamma(F_n)\langle G\rangle_n \rightarrow H\langle G\rangle_n$ is an epimorphism. Complete the natural transformation $\varphi_n$ to a short exact sequence

$$0 \rightarrow H' \rightarrow \gamma(F_n)\langle G\rangle_n \xrightarrow{\varphi} H\langle G\rangle_n \rightarrow 0$$

of functors on $(G)_{n}$. Since $H\langle G\rangle_n$ and $\gamma(F_n)\langle G\rangle_n$ are locally finite $(G)_n$-cohomological functors so is $H'$, and induction applies. For any $m \leq n$ we may choose a surjection $\varphi'_m : \gamma(F')\langle G\rangle_n \rightarrow H'\langle G\rangle_n$, as in (i) and (ii). We wish to consider the special case $m = 1$, where we can assume our $F'$ is as in (iii). That is we choose an object $F' \in \mathcal{T}_c$, which admits a strong $(G)_1$-approximating system and a surjection $\varphi' : \gamma(F')\langle G\rangle_1 \rightarrow H'\langle G\rangle_1$. And we may further assume that our approximating system $E'_1 \rightarrow E'_2 \rightarrow E'_3 \rightarrow \cdots$ for $F'$ is such that every morphism $E'_i \rightarrow E'_{i+1}$ is a split monomorphism.

We have natural transformations $\gamma(F')\langle G\rangle_1 \rightarrow H'\langle G\rangle_1 \rightarrow \gamma(F_n)\langle G\rangle_1$, and as $F'$ admits a $(G)_1$-approximating system Lemma [4.8] tells us that the composite is equal to $\gamma(\alpha_n)\langle G\rangle_1$ for some morphism $\alpha_n : F_n \rightarrow F_n$ in the category $\mathcal{T}$. And now Lemma [5.5] applies; see Remark [7.7] for an elaboration of how it specializes to the current context. We learn that

(iv) There exists a weak triangle $F' \xrightarrow{\alpha_n} F_n \xrightarrow{\beta_n} F_{n+1}$ in the category $\mathcal{T}_c$, with $F_{n+1}$ admitting a strong $(G)_{n+1}$-approximating system.

(v) There is natural transformation $\varphi_{n+1} : \gamma(F_{n+1})\langle G\rangle_{n+1} \rightarrow H$, such that $\varphi_n : \gamma(F_n)\langle G\rangle_n \rightarrow H\langle G\rangle_n$ is equal to the composite

$$\gamma(F_n)\langle G\rangle_n \xrightarrow{\gamma(\beta_n)\langle G\rangle_n} \gamma(F_{n+1})\langle G\rangle_n \xrightarrow{\varphi_{n+1} \langle G\rangle_n} H\langle G\rangle_n.$$
Comparing the two exact sequences
\[
\begin{array}{c}
\varphi(F_1)|_{(G)} \rightarrow \varphi(F_2)|_{(G)} \rightarrow \varphi(F_3)|_{(G)} \rightarrow \cdots
\end{array}
\]
we conclude that \( \varphi_n|_{(G)} \) and \( \varphi(\beta_n)|_{(G)} \) have the same kernel, that is the map \( \beta_n : F_n \rightarrow F_{n+1} \) satisfies (i).

To finish the proof of (i) it remains to show that \( \varphi_{n+1} : \varphi(F_{n+1})|_{(G)} \rightarrow H \) is an epimorphism, but this is now immediate from Lemma 6.6.

Remark 7.9. Let the conventions be as in Notation 7.1 and assume \( H \) is a locally finite \( \mathcal{T} \)-cohomological functor. For every integer \( n > 0 \) the restriction of \( H \) to \( \langle G \rangle_n \) is a locally finite \( \langle G \rangle_n \)-homological functor, and Lemma 7.8 permits us to construct a sequence \( F_1 \overset{\beta_1}{\rightarrow} F_2 \overset{\beta_2}{\rightarrow} F_3 \overset{\beta_3}{\rightarrow} \cdots \) in the category \( \mathcal{T} \), together with compatible epimorphisms \( \varphi_n : \varphi(F_n)|_{(G)} \rightarrow H|_{(G)} \). Since \( F_n \) is constructed to have a \( \langle G \rangle_n \)-approximating system, Corollary 4.4 says that each \( \varphi_n \) lifts uniquely to a natural transformation (which we will also call \( \varphi_n \)) of the form \( \varphi_n : \varphi(F_n) \rightarrow H \). The triangle
\[
\begin{array}{ccc}
\varphi(F_n) & \rightarrow & \varphi(F_n+1) \\
\varphi_n & \downarrow & \varphi_{n+1} \\
& & H
\end{array}
\]
commutes when restricted to the subcategory \( \langle G \rangle_n \subset \langle G \rangle \), and the fact that \( F_n \) has a \( \langle G \rangle_n \)-approximating system coupled with the uniqueness assertion of Corollary 4.4 tells us that the triangle commutes on the nose.

Proposition 7.10. Let the conventions be as in Notation 7.1 and assume \( H \) is a locally finite \( \mathcal{T} \)-cohomological functor. Then there exists an object \( F \in \mathcal{T} \) and an isomorphism \( \varphi : \varphi(F) \rightarrow H \).

Now let \( \mathcal{S} = \bigoplus_{C \in \mathcal{T}} C \); for those worried about set theoretic issues this means that \( \mathcal{S} \) is the coproduct, over the isomorphism classes of objects in \( \mathcal{T} \), of a representative in the isomorphism class. Then \( F \) may be chosen to lie in \( \langle \mathcal{S} \rangle \).

Proof. In Remark 7.9 we noted that Lemma 7.8 constructs for us a sequence \( F_1 \overset{\beta_1}{\rightarrow} F_2 \overset{\beta_2}{\rightarrow} F_3 \overset{\beta_3}{\rightarrow} \cdots \) in the category \( \mathcal{T} \), together with maps \( \varphi_n : \varphi(F_n) \rightarrow H \); there is an induced map \( \colim \varphi(F_n) \rightarrow H \); If we define \( F \) to be \( F = \hocolim F_n \), then we have an object \( F \in \mathcal{T} \), and [14 Lemma 2.8] tells us that the natural map \( \colim \varphi(F_n) \rightarrow \varphi(F) \) is an isomorphism. We have constructed a map \( \varphi : \varphi(F) \rightarrow H \) and will prove that \( \varphi \) is an isomorphism.

Let us consider the restriction of the natural transformation \( \varphi \) to the subcategory \( \langle G \rangle_1 \subset \mathcal{T} \). The natural transformation \( \varphi|_{(G)_1} : \varphi(F)|_{(G)_1} \rightarrow H|_{(G)_1} \) is the map to
\[ H|_{\langle G \rangle_1} \] from the colimit of the sequence

\[ y(F_1)|_{\langle G \rangle_1} \xrightarrow{y(\beta_1)|_{\langle G \rangle_1}} y(F_2)|_{\langle G \rangle_1} \xrightarrow{y(\beta_2)|_{\langle G \rangle_1}} \cdots \]

and Lemma 7.12(ii) says the sequence is such that each map \( y(\beta_n)|_{\langle G \rangle_1} \) factors as \( y(F_n)|_{\langle G \rangle_1} \to H|_{\langle G \rangle_1} \to y(F_{n+1})|_{\langle G \rangle_1} \). Hence the colimit agrees with the colimit of the ind-isomorphic constant sequence \( H|_{\langle G \rangle_1} \to H|_{\langle G \rangle_1} \to H|_{\langle G \rangle_1} \to \cdots \), and this proves that the restriction of \( \varphi : y(F) \to H \) to the category \( \langle G \rangle_1 \) is an isomorphism. Concretely: for every \( i \in \mathbb{Z} \) the map \( \varphi : \text{Hom}(\Sigma^i G, F) \to H(\Sigma^i G) \) is an isomorphism. The full subcategory \( \mathcal{K} \subset \mathcal{T}^c \) defined by

\[ \mathcal{K} = \bigg\{ K \in \mathcal{T}^c \bigg| \forall i \in \mathbb{Z} \text{ the map } \varphi : \text{Hom}(\Sigma^i K, F) \to H(\Sigma^i K) \text{ is an isomorphism} \bigg\} \]

is thick and contains \( G \), hence \( \mathcal{K} \subset \mathcal{T}^c = \langle G \rangle \subset \mathcal{K} \).

It remains to prove that the \( F \) we constructed belongs to \( \langle G \rangle_4 \). We begin with the observation that each \( F_n \) in the sequence \( F_1 \xrightarrow{\beta_1} F_2 \xrightarrow{\beta_2} F_3 \xrightarrow{\beta_3} \cdots \) of Remark 7.9 has a strong \( \langle G \rangle_n \)-approximating system. This means that \( F_n = \text{Hocolim} E^n_i \), with each \( E^n_i \in \langle G \rangle_n \). The triangle

\[
\begin{array}{ccc}
\bigoplus_{i=1}^{\infty} E^n_i & \to & \bigoplus_{i=1}^{\infty} E^n_i \\
\to & \to & \to \\
\bigoplus_{i=1}^{\infty} \Sigma E^n_i & \to & \bigoplus_{i=1}^{\infty} \Sigma E^n_i \\
\end{array}
\]

tells us that \( F_n \) must belong to \( \langle G \rangle_1 \ast \langle G \rangle_1 \subset \langle G \rangle_2 \). But now \( F = \text{Hocolim} F_n \), and the triangle

\[
\begin{array}{ccc}
\bigoplus_{n=1}^{\infty} F_n & \to & \bigoplus_{n=1}^{\infty} F_n \\
\to & \to & \to \\
\bigoplus_{n=1}^{\infty} \Sigma F_n & \to & \bigoplus_{n=1}^{\infty} \Sigma F_n \\
\end{array}
\]

gives that \( F \) belongs to \( \langle G \rangle_2 \ast \langle G \rangle_2 \subset \langle G \rangle_4 \). □

**Notation 7.11.** This is as far as we get with the assumptions of Notation 7.11. From now on we will assume further that \( \mathcal{T} \) is weakly approximable.

**Lemma 7.12.** Let the conventions be as in Notation 7.11. Assume \( H \) is a locally finite \( \mathcal{T}^c \)-cohomological functor. There exists an integer \( \bar{A} > 0 \) and, for any \( n > 0 \), an object \( F_n \in \mathcal{T}_c \cap \mathcal{T}^{\leq \bar{A}} \) as well as a natural transformation \( \varphi_n : y(F_n) \to H \) which is surjective when restricted to \( \langle G \rangle_n \).

**Proof.** Remark 7.9 produced for us an object \( F_n \in \mathcal{T}_c \) and a natural transformation \( \varphi_n : y(F_n) \to H \), so that the restriction to \( \langle G \rangle_n \) of \( \varphi_n \) is surjective. The new assertion is that we may choose \( F_n \) to lie in \( \mathcal{T}^{\leq \bar{A}} \) for some \( \bar{A} > 0 \) independent of \( n \).

Because \( H \) is locally finite there exists an integer \( B' > 0 \) with \( H(\Sigma^{-i} G) = 0 \) for all \( i \geq B' \). Since \( H \) is cohomological it follows that \( H(E) = 0 \) for all \( E \in \langle G \rangle[B',\infty) \).
And Corollary 2.14 gives us an integer \( B > 0 \) so that, for every integer \( m > 0 \), every
object \( F \in \mathcal{T}_c^- \cap \mathcal{T}^{\leq 0} \) admits a triangle \( E_m \rightarrow F \rightarrow D_m \) with \( D_m \in \mathcal{T}^{\leq -m} \) and
\( E_m \in \langle G \rangle_{[1-m-B,B]} \subset \langle G \rangle_{[1-m-B,\infty]} \). I assert that \( \hat{A} = B + B' \) works.

Let us begin with the \( F_n \) provided by Remark 7.9. Because it belongs to \( \mathcal{T}_c^- \subset \mathcal{T}^- \) there exists an integer \( \ell > 0 \) with \( F_n \in \mathcal{T}^{\leq \ell} \). Applying Corollary 2.14 to the object
\( \Sigma^\ell F_n \in \mathcal{T}_c^- \cap \mathcal{T}^{\leq 0} \), with \( m = \ell - B - B' \), we learn that there exists a triangle \( E \xrightarrow{\alpha} F_n \rightarrow D \) with \( D \in \mathcal{T}^{\leq \ell-m} = \mathcal{T}^{\leq B+B'} = \mathcal{T}^{\leq \hat{A}} \) and \( E \in \langle G \rangle_{[\ell+1-m-B,\infty]} = \langle G \rangle_{[1+B',\infty]} \). In particular \( E \in \mathcal{T}^c \), and \( H(E) = 0 \) by the choice of \( B' \). Hence the composite
\[
\begin{array}{ccc}
y(E) & \xrightarrow{y(\alpha)} & y(F_n) \\
\downarrow & & \downarrow \phi \\
H & & H
\end{array}
\]
vanishes.

Now we apply Lemma 6.5 with \( n' = n = \infty \). The object \( F_n \in \mathcal{T}_c^- \), of Lemma 7.8 and
Remark 7.9, comes with a strong \( \langle G \rangle_n \)-approximating system, which is certainly a strong \( \mathcal{T}^c \)-approximating system. The object \( E \in \mathcal{T}^c \) comes with the trivial strong \( \mathcal{T}^c \)-approximating system \( E \xrightarrow{id} E \xrightarrow{id} E \xrightarrow{id} \cdots \). In this system the connecting maps are all identities, which are split monomorphisms. The hypotheses of Lemma 6.5 and
Remark 7.7 hold and the Lemma produces for us, in \( \mathcal{T}_c^- \), a weak triangle \( E \xrightarrow{\alpha} F_n \xrightarrow{\beta} D \rightarrow \Sigma E \) and a factorization of \( \varphi : y(F_n) \rightarrow H \) as a composite
\[
\begin{array}{ccc}
y(F_n) & \xrightarrow{y(\beta)} & y(D) \\
\downarrow & & \downarrow \psi \\
H & & H
\end{array}
\]
The surjectivity of the restriction to \( \langle G \rangle_n \) of \( \varphi \) implies the surjectivity of the restriction to \( \langle G \rangle_n \) of \( \psi \).

It remains to show that \( D \in \mathcal{T}_c^- \) belongs to \( \mathcal{T}_c^- \cap \mathcal{T}^{\leq \hat{A}} \). We know that, in the triangle
\( E \rightarrow F_n \rightarrow D \rightarrow \Sigma E \), the object \( D \) belongs to \( \mathcal{T}^{\leq \hat{A}} \). The long exact sequence
\( \mathcal{H}^{i-1}(D) \rightarrow \mathcal{H}^i(E) \rightarrow \mathcal{H}^i(F_n) \rightarrow \mathcal{H}^i(D) \) tells us that \( \mathcal{H}^i(\alpha) : \mathcal{H}^i(E) \rightarrow \mathcal{H}^i(F) \) is surjective if \( i = 1 + \hat{A} \) and is an isomorphism when \( i > 1 + \hat{A} \). The long exact sequence
\( \mathcal{H}^i(E) \rightarrow \mathcal{H}^i(F_n) \rightarrow \mathcal{H}^i(D) \rightarrow \mathcal{H}^{i+1}(E) \rightarrow \mathcal{H}^{i+1}(F_n) \) says that \( \mathcal{H}^i(D) = 0 \) if \( i \geq 1 + \hat{A} \). By Lemma 1.2 we conclude that \( D \in \mathcal{T}^{\leq \hat{A}} \). \( \square \)

**Notation 7.13.** This is as far as we get with the assumptions of Notation 7.11 from
now on we assume further that \( \mathcal{T} \) is approximable—weak approximability will no longer
be enough.

**Lemma 7.14.** Let the conventions be as in Notation 7.13 and assume \( H \) is a locally
finite \( \mathcal{T}^c \)-cohomological functor. Choose an integer \( B > 0 \) as in Lemma 2.12.

Suppose \( \widetilde{F}, F' \) are objects in \( \mathcal{T}_c^- \cap \mathcal{T}^{\leq 0} \), \( E \) is an object in \( \mathcal{T}^c \cap \mathcal{T}^{\leq 0} \), and we have a
morphism \( \alpha : E \rightarrow \widetilde{F} \) in \( \mathcal{T}_c^- \). Assume we are given an integer \( m > 0 \), as well as
natural transformations \( \bar{\varphi} : y(F) \rightarrow H \) and \( \varphi' : y(F') \rightarrow H \), so that \( \bar{\varphi} \) restricts to an
epimorphism on \( \langle G \rangle_{mB} \) and \( \varphi' \) restricts to an epimorphism on \( \langle G \rangle_{(m+1)B} \).
Then there exists in $\mathcal{T}_c^{-} \cap \mathcal{T}^{\leq 0}$ a commutative diagram

$$
\begin{array}{ccc}
E & \xrightarrow{\varepsilon} & E' \\
\downarrow{\alpha} & & \downarrow{\alpha'} \\
F & \xrightarrow{\gamma} & F'
\end{array}

and there exists a natural transformation $\varphi' : \mathcal{Y}(\bar{F}') \to H$ so that

(i) The object $E'$ belongs to $\mathcal{T}^c \subset \mathcal{T}_c^{-}$.

(ii) The maps $\mathcal{H}(\alpha')$ and $\mathcal{H}(\gamma)$ are isomorphisms for all $i \geq -m + 2$.

(iii) The triangle

$$
\begin{array}{ccc}
\mathcal{Y}(\bar{F}') & \xrightarrow{\mathcal{Y}(\beta')} & \mathcal{Y}(F') \\
\downarrow{\varphi'} & & \downarrow{\varphi'} \\
H & & H
\end{array}
$$

commutes; the surjectivity of the restriction to $\langle G \rangle_{m+1}^{B,B} \subset \langle G \rangle_{mB}^{B,B}$ of $\varphi'$ therefore implies the surjectivity of the restriction to $\langle G \rangle_{m+1}^{B,B}$ of $\varphi'$.

(iv) The square below commutes

$$
\begin{array}{ccc}
\mathcal{Y}(E') & \xrightarrow{\mathcal{Y}(\gamma)} & \mathcal{Y}(\bar{F}') \\
\downarrow{\mathcal{Y}(\alpha')} & & \downarrow{\varphi'} \\
\mathcal{Y}(\bar{F}') & \xrightarrow{\varphi'} & H
\end{array}
$$

Proof. Corollary 2.14 applied to the object $F' \in \mathcal{T}_c^{-} \cap \mathcal{T}^{\leq 0}$ and the integer $m$, permits us to construct in $\mathcal{T}_c^{-}$ a triangle $E'_m \xrightarrow{a} F' \xrightarrow{b} D_m$ with $D_m \in \mathcal{T}^{\leq -m}$ and $E'_m \in \langle G \rangle_{mB}^{[1-m-B,B]} \subset \langle G \rangle_{mB}$. Because $E'_m$ belongs to $\mathcal{T}^c$ the composite

$$
\mathcal{Y}(E'_m) \xrightarrow{\mathcal{Y}(a)} \mathcal{Y}(F') \xrightarrow{\varphi'} H
$$

is a natural transformation from a representable functor on $\mathcal{T}^c$ to $H$, and corresponds to an element $x \in H(E'_m)$. As $E'_m$ belongs to $\langle G \rangle_{mB}$ the morphism $\bar{\varphi} : \text{Hom}(E'_m, \bar{F}) \to H(E'_m)$ is surjective; there is a map $f : E'_m \to \bar{F}$ with $\bar{\varphi}(f) = x$. Yoneda translates this to mean that the square below commutes

$$
\begin{array}{ccc}
\mathcal{Y}(E'_m) & \xrightarrow{\mathcal{Y}(f)} & \mathcal{Y}(\bar{F}) \\
\downarrow{\mathcal{Y}(a)} & & \downarrow{\bar{\varphi}} \\
\mathcal{Y}(F') & \xrightarrow{\varphi'} & H
\end{array}
$$

We now have a morphism $(\alpha, f) : E \oplus E'_m \to \bar{F}$. The object $E \oplus E'_m$ belongs to $\mathcal{T}^c$ while the object $\bar{F} \in \mathcal{T}_c^{-}$ has a strong $\mathcal{T}^c$–approximating system—see Lemma 7.5(ii).
There exists a sequence $\tilde{E}_1 \to \tilde{E}_2 \to \tilde{E}_3 \to \cdots$ in $\mathcal{T}^c$, and a map of $\tilde{E}_* \to \tilde{F}$, and so that $\mathcal{H}^i(\tilde{E}_m) \to \mathcal{H}^i(\tilde{F})$ is an isomorphism whenever $i \geq -m$. The morphism $E \oplus E'_m \to \tilde{F}$ must factor through some $\tilde{E}_i$—see Lemma 7.5(ii) and [14 Lemma 2.8]. Choose such an $\tilde{E}_i$, pick $i \geq m$, declare $E'_i = \tilde{E}_i$ and let $\varepsilon : E \to E'$, $g : E'_m \to E'$ and $\gamma : E' \to \tilde{F}$ be the obvious maps. Then $\gamma \varepsilon = \alpha$ and $\gamma g = f$.

By construction the map $\mathcal{H}^\ell(\gamma) : \mathcal{H}^\ell(E') \to \mathcal{H}^\ell(\tilde{F})$ is an isomorphism for all $\ell \geq -m$. Recalling the triangle $E'_m \to F'_m \to D'_m$ with $D'_m \in \mathcal{T}^{-m}$, the exact sequence

$$
\mathcal{H}^{\ell-1}(D'_m) \to \mathcal{H}^\ell(E'_m) \to \mathcal{H}^\ell(F') \to \mathcal{H}^\ell(D'_m)
$$

teaches us that

(v) The maps $\mathcal{H}^\ell(\gamma)$ and $\mathcal{H}^\ell(a)$ are isomorphisms for all $\ell \geq -m + 2$.

Since $m \geq 1$ by assumption, we learn in particular that if $\ell \geq 1$ then $\mathcal{H}^\ell(E') \cong \mathcal{H}^\ell(\tilde{F}) = 0$ and $\mathcal{H}^\ell(E'_m) \cong \mathcal{H}^\ell(F') = 0$. As both $E'$ and $E'_m$ are objects of $\mathcal{T} \subset \mathcal{T}^c$, Lemma 1.2 informs us that

(vi) $E', E'_m$ both lie in $\mathcal{T}^{-0}$.

Now consider the commutative square

$$
\begin{array}{ccc}
\mathcal{Y}(E'_m) & \xrightarrow{y(g)} & \mathcal{Y}(E') \\
\downarrow{y(a)} & & \downarrow{\rho = \varphi \circ y(\gamma)} \\
\mathcal{Y}(F') & \xrightarrow{\varphi'} & \mathcal{Y}
\end{array}
$$

In other words: we have in $\mathcal{T}^c \cup \mathcal{T}^{-0}$ a morphism $\sigma = \begin{pmatrix} \varepsilon_g & a \end{pmatrix} : E'_m \to E' \oplus F'$, as well as a natural transformation $\rho : \mathcal{Y}(E' \oplus F') \to H$, and the composite

$$
\begin{array}{ccc}
\mathcal{Y}(E'_m) & \xrightarrow{y(\sigma)} & \mathcal{Y}(E' \oplus F') \\
\downarrow{y(\sigma)} & & \downarrow{(\rho, \varphi')} \\
\mathcal{Y}(E' \oplus F') & \xrightarrow{(\rho, \varphi')} & \mathcal{Y} \to H
\end{array}
$$

vanishes. The object $E' \oplus F' \in \mathcal{T}^c$ has a strong $\mathcal{T}^c$–approximating system by Lemma 7.5(ii), and the object $E'_m \in \mathcal{T}^c$ has the trivial strong $\mathcal{T}^c$–approximating system $E'_m \xrightarrow{id} E'_m \xrightarrow{id} E'_m \xrightarrow{id} \cdots$. We may apply Lemma 6.5 as specialized in Remark 7.4 with $n = n' = \infty$, to deduce that $E'_m \xrightarrow{\sigma} E' \oplus F'$ may be completed in $\mathcal{T}^c$ to a weak triangle

$$
E'_m \xrightarrow{\sigma} E' \oplus F' \xrightarrow{(\alpha', \beta)} \tilde{F}' \xrightarrow{\tau} \Sigma E'_m
$$

in such a way that the morphism $(\rho, \varphi') : \text{Hom}(\cdot, E' \oplus F')|_{\mathcal{T}^c} \to H(-)$ factors as

$$
\varphi' \circ \text{Hom}(\cdot, (\alpha', \beta))
$$

for some natural transformation $\varphi' : \text{Hom}(\cdot, \tilde{F}')|_{\mathcal{T}^c} \to H(-)$. We
have constructed in $\mathcal{T}_c^-$ a commutative square

\[
\begin{array}{ccc}
E'_m & \xrightarrow{g} & E' \\
\downarrow^a & & \downarrow^\alpha' \\
F' & \xrightarrow{\beta} & F'
\end{array}
\]

and a natural transformation $\tilde{\varphi}' : \text{Hom}(-, \tilde{F}')|_{\mathcal{T}_c^-} \rightarrow H(-)$ so that the diagram below commutes

\[
\begin{array}{ccc}
y(E'_m) & \xrightarrow{y(g)} & y(E') \\
y(a) & & y(\alpha') \\
y(F') & \xrightarrow{y(\beta)} & y(\tilde{F}')
\end{array}
\]

This finishes our construction of the diagram

\[
\begin{array}{ccc}
E & \xrightarrow{\epsilon} & E' \\
\downarrow^\alpha & & \downarrow^\alpha' \\
F & \xrightarrow{\gamma} & F'
\end{array}
\]

and the natural transformation $\tilde{\varphi}' : y(\tilde{F}') \rightarrow H$. The assertions (i), (iii) and (iv) of the Lemma have already been proved, as well as half of (ii). It remains to show that $\tilde{F}' \in \mathcal{T}_c^-$ also belongs to $\mathcal{T}^{\leq 0}$, that the morphisms $\mathcal{H}^\ell(\alpha')$ are isomorphisms for $\ell \geq -m + 2$.

To prove this we recall that the commutative square

\[
\begin{array}{ccc}
\mathcal{H}^\ell(E'_m) & \xrightarrow{\mathcal{H}^\ell(g)} & \mathcal{H}^\ell(E') \\
\downarrow^{\mathcal{H}^\ell(a)} & & \downarrow^{\mathcal{H}^\ell(\alpha')} \\
\mathcal{H}^\ell(F') & \xrightarrow{\mathcal{H}^\ell(\beta)} & \mathcal{H}^\ell(\tilde{F}')
\end{array}
\]

comes from applying $\mathcal{H}^\ell$ to the weak triangle

\[
\begin{array}{ccc}
E'_m & \xrightarrow{\sigma} & E' \oplus F' \\
\downarrow^{(\alpha',\beta)} & & \downarrow^{\tau} \\
\Sigma E'_m
\end{array}
\]

The morphism $\mathcal{H}^\ell(\tau) : \mathcal{H}^\ell(\tilde{F}') \rightarrow \mathcal{H}^{\ell+1}(E'_m)$ fits in a long exact sequence. In (v) we proved that $\mathcal{H}^\ell(a) : \mathcal{H}^\ell(E'_m) \rightarrow \mathcal{H}^\ell(F')$ is an isomorphism for $\ell \geq -m + 2$, which makes the map $\mathcal{H}^\ell(\sigma) : \mathcal{H}^\ell(E'_m) \rightarrow \mathcal{H}^\ell(E' \oplus F')$ a split monomorphism for all $\ell \geq -m + 2$. 
The exactness tells us first that $\mathcal{H}^\ell(\tau) = 0$ for all $\ell \geq -m + 1$, and then that the square

$$
\begin{array}{ccc}
\mathcal{H}^\ell(E'_m) & \xrightarrow{\mathcal{H}^\ell(g)} & \mathcal{H}^\ell(E') \\
\Downarrow \mathcal{H}(a) & & \Downarrow \mathcal{H}(a') \\
\mathcal{H}^\ell(F') & \xrightarrow{\mathcal{H}^\ell(\beta)} & \mathcal{H}^\ell(\bar{F}')
\end{array}
$$

is bicartesian for all $\ell \geq -m + 2$. As long as $\ell \geq -m + 2$, the fact that $\mathcal{H}^\ell(a)$ is an isomorphism forces $\mathcal{H}^\ell(a')$ to also be.

In particular: since $m \geq 1$ we have that $-m + 2 \leq 1$, and deduce that $\mathcal{H}^\ell(E') = 0$ for all $\ell \geq 1$. Lemma 7.12 now gives that $\bar{F}'$ belongs to $\mathcal{T}^{\leq 0}$.

\[\square\]

**Lemma 7.15.** Let the conventions be as in Notation 7.13 and assume $H$ is a locally finite $\mathcal{T}^-\text{-cohomological functor}$. There exists an object $F \in \mathcal{T}^-_c$ and an epimorphism $\text{Hom}(\cdot, F)|_{\mathcal{T}^-} \to H(\cdot)$.

**Proof.** By Lemma 7.12 we may assume given: an integer $\bar{A} > 0$, objects $F_n \in \mathcal{T}^-_c \cap \mathcal{T}^{\leq \bar{A}}$ and natural transformations $\varphi_n : \mathcal{Y}(F_n) \to H$ which restrict to epimorphisms on $\langle G \rangle_n$.

Replacing the functor $H$ by $H(\Sigma^A \cdot)$ we may assume $\bar{A} = 0$. Let $B > 0$ be the integer whose existence is given Lemma 7.12. Next we need to make our construction.

We will proceed inductively, using Lemma 7.14, to construct in $\mathcal{T}^-_c \cap \mathcal{T}^{\leq 0}$ a sequence

$$
E_1 \xrightarrow{\varepsilon_2} E_2 \xrightarrow{\varepsilon_3} E_3 \xrightarrow{\cdot} \ldots
$$

as well as natural transformations $\tilde{\varphi}_i : \mathcal{Y}(\tilde{F}_i) \to H$, satisfying the following

(i) The restriction of $\tilde{\varphi}_i$ to the subcategory $\langle G \rangle_{(i+3)B}$ is surjective.

(ii) For each $i > 0$ the square

$$
\begin{array}{ccc}
\mathcal{Y}(E_{i+1}) & \xrightarrow{\mathcal{Y}(\alpha_{i+1})} & \mathcal{Y}(\tilde{F}_{i+1}) \\
\Downarrow \mathcal{Y}(\gamma_{i+1}) & & \Downarrow \tilde{\varphi}_{i+1} \\
\mathcal{Y}(\tilde{F}_i) & & \mathcal{Y}(H)
\end{array}
$$

commutes.

(iii) The morphisms $\mathcal{H}^\ell(\gamma_i)$ and $\mathcal{H}^\ell(\alpha_i)$ are isomorphisms whenever $\ell \geq -i$.

To start the induction we declare $\tilde{F}_1 = F_{4B}$ and $\tilde{\varphi}_1 = \varphi_{4B}$. Choose a triangle $E_1 \xrightarrow{\alpha_1} \tilde{F}_1 \to D_1$ with $D_1 \in \mathcal{T}^{\leq -3}$; we immediately have that $\mathcal{H}^\ell(\alpha_1)$ is an isomorphism for $\ell \geq -1$. In particular for $\ell \geq 1$ we have $\mathcal{H}^\ell(E_1) \cong \mathcal{H}^\ell(F_{4B}) = 0$; hence $E_1 \in \mathcal{T}^-_c \cap \mathcal{T}^{\leq 0}$.

Suppose our induction has proceeded as far as $n$. In particular: we have produced in $\mathcal{T}^-_c \cap \mathcal{T}^{\leq 0}$ a morphism $\alpha_n : E_n \to \tilde{F}_n$, with $E_n \in \mathcal{T}^-_c$, as well as a natural transformation
\[ \varphi_n : Y(F_n) \rightarrow H \] which is surjective when restricted to \( \langle G \rangle_{(n+3)B} \). And we have done it in such a way that \( \mathcal{H}^\ell(\alpha_n) \) is an isomorphism for \( \ell \geq -n \). We wish to go on to \( n+1 \).

Now the first paragraph of the proof gives us an object \( F_{(n+4)B} \in T_c^c \cap T^{\leq 0} \), as well as a natural transformation \( \varphi_{(n+4)B} : Y(F_{(n+4)B}) \rightarrow H \) whose restriction to \( \langle G \rangle_{(n+4)B} \) is surjective. Lemma 7.14 with \( m = n + 3 \), allows us to construct in \( T_c^c \cap T^{\leq 0} \) the diagram

\[
\begin{array}{ccc}
E_n & \xrightarrow{\epsilon_{n+1}} & E_{n+1} \\
\downarrow{\alpha_n} & & \downarrow{\alpha_{n+1}} \\
F_n & \xrightarrow{\gamma_{n+1}} & F_{n+1}
\end{array}
\]

as well as the natural transformation \( \tilde{\varphi}_{n+1} : Y(F_{n+1}) \rightarrow H \) satisfying lots of properties: Lemma 7.14(i) tells us that \( E_{n+1} \) belongs to \( T_c^c \). Lemma 7.14(ii) gives that \( \mathcal{H}^\ell(\alpha_{n+1}) \) and \( \mathcal{H}^\ell(\gamma_{n+1}) \) are isomorphisms if \( \ell \geq -n - 3 + 2 = -n - 1 \). Lemma 7.14(iii) says that \( \tilde{\varphi}_{n+1} \) is an epimorphism when restricted to \( \langle G \rangle_{(n+4)B} \), and Lemma 7.14(iv) says that the square in (ii) above commutes. This finishes the induction.

It remains to see how to deduce the Lemma. We have produced a sequence \( E_1 \xrightarrow{\varepsilon_2} E_3 \xrightarrow{\varepsilon_4} E_3 \xrightarrow{\varepsilon_4} \cdots \) and, for each \( i > 0 \)

(iv) We define \( \psi_i : Y(E_i) \rightarrow H \) to be the composite

\[
\begin{array}{ccc}
Y(E_i) & \xrightarrow{Y(\alpha_i)} & Y(F_i) \\
\downarrow{\psi_i} & & \downarrow{\tilde{\varphi}_i} \\
& & H
\end{array}
\]

With these definitions we will prove

(v) The following triangles commute

\[
\begin{array}{ccc}
Y(E_i) & \xrightarrow{Y(\epsilon_{i+1})} & Y(E_{i+1}) \\
\downarrow{\psi_i} & & \downarrow{\psi_{i+1}} \\
& & H
\end{array}
\]

This permits us to make the next definitions

(vi) We put \( F = \text{Hocolim} E_i \), and let \( \psi : Y(F) \rightarrow H \) be

\[
\begin{array}{ccc}
Y(F) & \xrightarrow{\text{colim}} & Y(E_i) \\
& & \downarrow{\psi_i} \\
& & H
\end{array}
\]

that is the colimit of the maps \( \psi_i \).

And with the definitions made, we will prove

(vii) The object \( F \) belongs to \( T_c^c \).

(viii) The map \( \psi : Y(F)|_{T_c} \rightarrow H \) is an epimorphism.

Together, (vii) and (viii) contain the assertion of the Lemma. All that remains is to prove (v), (vii) and (viii).
To prove (v) the reader should consider the diagram

\[
\begin{array}{ccc}
\gamma(E_i) & \xrightarrow{y(E_{i+1})} & \gamma(E_{i+1}) \\
\alpha_i & & \gamma(\alpha_{i+1}) \xrightarrow{y(\alpha_{i+1})} \gamma(F_{i+1}) \\
\epsilon_{i+1} & & \epsilon_i \\
F_i & \xrightarrow{\gamma_{i+1}} & F_{i+1}
\end{array}
\]

We wish to prove the commutativity of the perimeter. The square commutes by (ii), and the triangle by applying the functor \(\gamma\) to the commutative triangle

\[
\begin{array}{ccc}
E_i & \xrightarrow{\epsilon_{i+1}} & E_{i+1} \\
\alpha_i & & \gamma_{i+1} \\
\tilde{F}_i & \xrightarrow{\gamma_{i+1}} & \tilde{F}_{i+1}
\end{array}
\]

Also: we may apply \(\mathcal{H}^\ell\) to the commutative triangle. From (iii) we know that \(\mathcal{H}^\ell(\alpha_i)\) is an isomorphism for \(\ell \geq -i\) and also that \(\mathcal{H}^\ell(\gamma_{i+1})\) is an isomorphism for \(\ell \geq -i - 1\). Since \(\mathcal{H}^\ell(\alpha_i) = \mathcal{H}^\ell(\gamma_{i+1})\mathcal{H}^\ell(\epsilon_{i+1})\) we learn that \(\mathcal{H}^\ell(\epsilon_{i+1})\) is an isomorphism when \(\ell \geq -i\). Therefore \(E_s\) is a strong \(\mathcal{T}^e\)-approximating system as in Definition 7.3, and Lemma 7.5 informs us that \(F = \text{Hocolim} E_i\) belongs to \(\mathcal{T}_c\)—that is we have proved (vii). It remains only to prove (viii).

Suppose therefore that \(C\) is an object \(\mathcal{T}^e\). There exists an \(n > 0\) with \(C \in \langle G \rangle_{(n+3)B}\), and Lemma 2.8 says that we may also choose \(n\) so that \(\text{Hom}(C, \mathcal{T}^{-n-1}) = 0\). Because \(\tilde{\varphi}_n : y(\tilde{F}_n) \rightarrow H\) is surjective on \(\langle G \rangle_{(n+3)B}\) we have that the map \(\tilde{\varphi}_n : \text{Hom}(C, \tilde{F}_n) \rightarrow H(C)\) is surjective. In the triangle \(E_n \xrightarrow{\alpha} \tilde{F}_n \rightarrow \tilde{D}_n\) we have that \(\mathcal{H}^\ell(\alpha_n)\) is an isomorphism for \(\ell \geq -n\), therefore \(\mathcal{H}^\ell(\tilde{D}_n) = 0\) for \(\ell \geq -n\), therefore \(\tilde{D}_n \in \mathcal{T}^{-n-1}\). In the exact sequence

\[
\text{Hom}(C, E_n) \xrightarrow{\text{Hom}(C, \alpha_n)} \text{Hom}(C, \tilde{F}_n) \xrightarrow{\tilde{\varphi}_n} \text{Hom}(C, \tilde{D}_n)
\]

we have \(\text{Hom}(C, D_n) = 0\) since \(\tilde{D}_n \in \mathcal{T}^{-n-1}\). The map \(\psi_n : \text{Hom}(C, E_n) \rightarrow H(C)\) of (iv) is the composite of the two epimorphisms

\[
\text{Hom}(C, E_n) \xrightarrow{\text{Hom}(C, \alpha_n)} \text{Hom}(C, \tilde{F}_n) \xrightarrow{\tilde{\varphi}_n} H(C)
\]

But it factors through \(\psi : \text{Hom}(C, F) \rightarrow H(C)\), which must therefore be epi. \(\square\)

**Reminder 7.16.** Let \(\mathcal{T}\) be a triangulated category with coproducts. A morphism \(f : D \rightarrow E\) is called **phantom** if, for every compact object \(C \in \mathcal{T}\), the induced map \(\text{Hom}(C, f) : \text{Hom}(C, D) \rightarrow \text{Hom}(C, E)\) vanishes. The phantom maps form an ideal: if \(f, f' : D \rightarrow E\) are phantom then so is \(f + f'\), and if \(D' \xrightarrow{e} D \xrightarrow{f} E \xrightarrow{g} E'\) are composable morphisms with \(f\) phantom, then \(gfe : D' \rightarrow E'\) is also phantom.
Corollary 7.17. Let the conventions be as in Notation 7.13. Let $F' \in \mathcal{T}$ be an object such that the functor $H = \mathcal{Y}(F')$ is a locally finite $\mathcal{T}^c$–cohomological functor. There exists an object $F \in \mathcal{T}_c$ and a triangle $F \xrightarrow{f} F' \xrightarrow{g} D$ with $g$ phantom.

Proof. Lemma 7.15 gives us an object $F \in \mathcal{T}_c$ and an epimorphism $\varphi : \mathcal{Y}(F) \twoheadrightarrow H = \mathcal{Y}(F')$. Since $F$ belongs to $\mathcal{T}_c$ Lemma 7.5(ii) produces for $F$ a (strong) $\mathcal{T}^c$–approximating system. Lemma 4.8 allows us to realize the natural transformation $\varphi$ as $\mathcal{Y}(f) : \mathcal{Y}(F) \to \mathcal{Y}(F')$ for some (non-unique) $f : F \to F'$. Complete $f$ to a triangle $F \xrightarrow{f} F' \xrightarrow{g} D$.

For every object $C \in \mathcal{T}$ we have an exact sequence

$$\text{Hom}(C, F) \xrightarrow{\text{Hom}(C,f)} \text{Hom}(C, F') \xrightarrow{\text{Hom}(C,g)} \text{Hom}(C, D)$$

and if $C$ is compact the map $\text{Hom}(C, f) = \mathcal{Y}(f)(C)$ is surjective. It follows that $\text{Hom}(C, g)$ is the zero map. □

Now that it’s time to state the main theorem we include all the hypotheses explicitly.

Theorem 7.18. Let $R$ be a noetherian, commutative ring. Let $\mathcal{T}$ be an $R$–linear triangulated category with coproducts, and suppose it has a compact generator $G$ such that $\text{Hom}(–, G)$ is a $G$–locally finite cohomological functor. Assume further that $\mathcal{T}$ is approximable.

Let $\mathcal{T}_c \subset \mathcal{T}$ be the category of Definition 0.16, where the $t$–structure with respect to which we define it is in the preferred equivalence class. Then the functor $\mathcal{Y} : \mathcal{T}_c \to \text{Hom}((\mathcal{T}^c)^{\text{op}}, R\text{-Mod})$, taking $F \in \mathcal{T}_c$ to $\mathcal{Y}(F) = \text{Hom}(–, F)|_{\mathcal{T}^c}$, satisfies

(i) The objects in the essential image of $\mathcal{Y}$ are the locally finite $\mathcal{T}^c$–cohomological functors.

(ii) The functor $\mathcal{Y}$ is full.

Proof. The fact that, for any object $F \in \mathcal{T}_c$, the functor $\mathcal{Y}(F)$ is a locally finite $\mathcal{T}^c$–cohomological functor was proved in Lemma 7.2. In Lemma 7.3(ii) we saw that any $F \in \mathcal{T}_c$ admits a $\mathcal{T}^c$–approximating system, and Lemma 4.8 guarantees that any natural transformation $\varphi : \mathcal{Y}(F) \to \mathcal{Y}(F')$ can be expressed as $\varphi = \mathcal{Y}(f)$ for some $f : F \to F'$; that is the functor is full. It remains to show that any locally finite $\mathcal{T}^c$–cohomological functor $H : (\mathcal{T}^c)^{\text{op}} \to R\text{-Mod}$ is in the essential image; we must show it is isomorphic to $\mathcal{Y}(F)$ for some $F \in \mathcal{T}_c$.

Proposition 7.10 produced a candidate $F$; we have an $F \in (\mathcal{G})_4$ and an isomorphism $H \cong \mathcal{Y}(F)$. We wish to show that $F$ belongs to $\mathcal{T}_c$. We proceed by induction to prove

(iii) Let $\mathcal{G}$ be the ideal of phantom maps. For each integer $n > 0$ there exists a triangle $F_n \to F \xrightarrow{\beta_n} D_n$ with $F_n \in \mathcal{T}_c$ and $\beta_n \in \mathcal{T}^n$.

We prove (iii) by induction on $n$. The case $n = 1$ is given by Corollary 7.17. Now for the inductive step: assume that, for some $n \geq 1$, we are given a triangle $F_n \to F \xrightarrow{\beta} D_n$
with \( F_n \in \mathcal{T}_c^- \) and \( \beta_n \in \mathcal{T}^n \). We know that both \( \mathcal{Y}(F_n) \) and \( \mathcal{Y}(F) \) are locally finite \( \mathcal{T}^-\)-cohomological functors, and the exact sequence

\[
0 \to \mathcal{Y}(\Sigma^{-1}D_n) \to \mathcal{Y}(F_n) \to \mathcal{Y}(F) \to 0
\]

says that so is \( \mathcal{Y}(\Sigma^{-1}D_n) \). Corollary \[7.17\] permits us to construct a triangle \( F' \to D_n \to D_{n+1} \) with \( F' \in \mathcal{T}_c^- \) and \( \gamma \in \mathcal{I} \). Let \( \beta_{n+1} : F \to D_{n+1} \) be the composite \( F \to T \), \( \gamma \to D_{n+1} \). Since \( \beta_n \in \mathcal{T}^n \) and \( \gamma \in \mathcal{I} \) we deduce that \( \beta_{n+1} \in \mathcal{I}^{n+1} \). If we complete \( \beta_{n+1} \) to a triangle \( F_{n+1} \to F \), the octahedral axiom allows us to find a triangle \( F_n \to F_{n+1} \to F' \). Since \( F_n \) and \( F' \) lie in \( \mathcal{T}_c^- \), so does \( F_{n+1} \). This completes the proof of (iii).

Now consider the triangle \( F_4 \to F \to D_4 \). The morphism \( F \to D_4 \) is in \( \mathcal{T}^4 \), but \( F \) belongs to \( \langle \mathcal{I} \rangle_4 \). One easily shows that \( \langle \mathcal{I} \rangle_4 \) is a projective class as in Christensen \[7\] Definition 2.2, and \[7\] Theorem 1.1] tells us that so is \( \langle \mathcal{I} \rangle_4 \). The map \( F \to D_4 \) is a morphism in \( \mathcal{T}^4 \) out of an object in \( \langle \mathcal{I} \rangle_4 \) and must vanish, making \( F \) a direct summand of \( F_4 \in \mathcal{T}_c^* \). Proposition \[2.10\] tells us that \( \mathcal{T}_c^- \) is thick, and therefore \( F \in \mathcal{T}_c^* \).

\[\boxdot\]

**Lemma 7.19.** Let the assumptions be as in Theorem \[7.18\]. Suppose \( f : F \to F' \) is a morphism in \( \mathcal{T}_c^- \) and assume \( F' \) belongs to \( \mathcal{T}_c^b \). Then \( \mathcal{Y}(f) = 0 \) implies \( f = 0 \).

**Proof.** Because \( F' \) belongs to \( \mathcal{T}_c^b \) there must be an integer \( \ell \) with \( F' \in \mathcal{T}^{\geq \ell} \); without loss of generality we may assume \( \ell = 0 \). Now \( F \) belongs to \( \mathcal{T}_c^- \), hence there must exist a triangle \( E \to g F \to h D \) with \( E \in \mathcal{T}_c \) and \( D \in \mathcal{T}_c^{-\leq 1} \). Choose such a triangle.

The vanishing of \( \mathcal{Y}(f) \) means that \( \text{Hom}(E,F) : \text{Hom}(E,F) \to \text{Hom}(E,F') \) must take \( g \in \text{Hom}(E,F) \) to zero; that means \( fg = 0 \). But the triangle tells us that \( f : F \to F' \) must factor as \( F \to h D \to F' \). As \( D \in \mathcal{T}_c^{-\leq 1} \) and \( F' \in \mathcal{T}_c^{\geq 0} \) we have \( \text{Hom}(D,F') = 0 \), hence \( f = 0 \).

\[\boxdot\]

**Theorem 7.20.** Let the assumptions be as in Theorem \[7.18\]. The restriction of the functor \( \mathcal{Y} \) to the subcategory \( \mathcal{T}_c^b \) is fully faithful, and the essential image is the class of finite \( \mathcal{T}_c^-\)-cohomological functors.

**Proof.** The functor is full on all of \( \mathcal{T}_c^- \), and Lemma \[7.19\] guarantees that on the subcategory \( \mathcal{T}_c^b \) it is faithful. It remains to identify the essential image. Let \( F \) be an object in \( \mathcal{T}_c^- \), we need to show that \( \mathcal{Y}(F) \) is finite if and only if \( F \in \mathcal{T}_c^b \).

Suppose \( F \in \mathcal{T}_c^b \subset \mathcal{T}_c^+ \) and \( C \in \mathcal{T}_c \subset \mathcal{T}_c^- \). We can choose an integer \( \ell > 0 \) so that \( \text{Hom}(\Sigma^{i}C,F) = 0 \) for all \( i \geq \ell \), which implies that \( \text{Hom}(-,F|_{\mathcal{T}_c}) \) is \( C \)-finite. Since this is true for every \( C \in \mathcal{T}_c \) we have that \( \mathcal{Y}(F) \) is finite.

Conversely: suppose \( \mathcal{Y}(F) \) is finite and choose a compact generator \( G \). Because \( \mathcal{Y}(F) = \text{Hom}(-,F|_{\mathcal{T}_c}) \) is \( G \)-finite there is an integer \( \ell \) so that \( \text{Hom}(\Sigma^{i}G,F) = 0 \) for all \( i \geq \ell \). But then \( \text{Hom}(T,F) = 0 \) for all \( T \in \langle \mathcal{G} \rangle_{(-\infty,-\ell]}^{\leq -\ell} \) and \( F \) must belong to \( \mathcal{T}_c^- \cap \mathcal{T}_c^{\leq -\ell + 1} \subseteq \mathcal{T}_c^b \).

\[\boxdot\]
8. Applications: the construction of adjoints

We prove Corollary [0.4], a restricted version of which was the key tool in Jack Hall’s original, simple proof of GAGA—see Remark [0.7]. Hall’s later proof of a more general result, see [9], sidesteps the representability theorems presented here.

**Theorem 8.1.** Let $R$ be a noetherian, commutative ring. Let $\mathcal{T}$ be an $R$–linear triangulated category with coproducts, and assume that it is approximable. Let $\mathcal{T}^b_c \subset \mathcal{T}^-_c$ be the subcategories of Definition [0.16], constructed using a $t$–structure in the preferred equivalence class. Assume the category $\mathcal{T}^c$ is contained in $\mathcal{T}^b_c$. Assume further that $\mathcal{T}$ has a compact generator $G$ so that $\text{Hom}_\mathcal{T}(\cdot, G)$ is a $G$–locally finite cohomological functor.

Let $\mathcal{L} : \mathcal{T}^b_c \longrightarrow \mathcal{S}$ be an $R$–linear triangulated functor, and let $(\mathcal{T}^{\leq 0},\mathcal{T}^{\geq 0})$ be any $t$–structure in the preferred equivalence class. Assume further:

(i) For any pair of objects $(t,s)$, with $t \in \mathcal{T}^c$ and $s \in \mathcal{S}$, the $R$–module $\text{Hom}(\mathcal{L}(t), s)$ is finite.

(ii) For any object $s \in \mathcal{S}$ there exists an integer $A > 0$ with $\text{Hom}(\mathcal{L}(\mathcal{T}^b_c \cap \mathcal{T}^{\leq -A}), s) = 0$.

(iii) For any object $t \in \mathcal{T}^c$ and any object $s \in \mathcal{S}$ there exists an integer $A$ so that $\text{Hom}(\mathcal{L}(\Sigma^m t), s) = 0$ for all $m \leq -A$.

Then $\mathcal{L}$ has a right adjoint $\mathcal{R} : \mathcal{S} \longrightarrow \mathcal{T}^b_c$.

**Proof.** For any pair of objects $t \in \mathcal{T}^c$, $s \in \mathcal{S}$ and any integer $m \in \mathbb{Z}$, from (i) we learn that $\text{Hom}(\mathcal{L}(\Sigma^m t), s)$ is a finite $R$–module. Now (ii) and (iii) guarantee that it vanishes whenever $m \gg 0$ or $m \ll 0$. Thus $\text{Hom}(\mathcal{L}(\cdot), s)$ is a finite $\mathcal{T}^c$–cohomological functor.

The assignment taking $s \in \mathcal{S}$ to the functor $\text{Hom}(\mathcal{L}(\cdot), s)$ is a functor from $\mathcal{S}$ to the category of finite $\mathcal{T}^c$–cohomological functors; by Theorem [7.18] we can lift it through the equivalence of categories $\mathcal{Y}$. There is a functor $\mathcal{R} : \mathcal{S} \longrightarrow \mathcal{T}^b_c$ so that, for all objects $t \in \mathcal{T}^c$ and all objects $s \in \mathcal{S}$, we have a natural isomorphism

$$\text{Hom}(\mathcal{L}(t), s) \xrightarrow{\varphi} \text{Hom}(t, \mathcal{R}(s)).$$

Fix $t' \in \mathcal{T}^b_c$ and consider the following composite, which is natural in $t \in \mathcal{T}^c$, $t' \in \mathcal{T}^b_c$

$$\text{Hom}(t, t') \xrightarrow{\mathcal{L}} \text{Hom}(\mathcal{L}(t), \mathcal{L}(t')) \xrightarrow{\varphi} \text{Hom}(t, \mathcal{R}\mathcal{L}(t')).$$

We have objects $t', \mathcal{R}\mathcal{L}(t') \in \mathcal{T}^b_c$ and a natural transformation $\mathcal{Y}(t') \longrightarrow \mathcal{Y}(\mathcal{R}\mathcal{L}(t'))$, and Theorem [7.20] allows us to express it uniquely as $\mathcal{Y}(\alpha_{t'})$ for some morphism $\alpha_{t'} : t' \longrightarrow \mathcal{R}\mathcal{L}(t')$ in $\mathcal{T}^b_c$. We leave it to the reader to check that $\alpha_{t'}$ is natural in $t'$; it gives a natural transformation $\alpha : \text{id} \longrightarrow \mathcal{R}\mathcal{L}$.

Now we define a natural transformation $\psi : \text{Hom}(\mathcal{L}(\cdot), \cdot) \longrightarrow \text{Hom}(\cdot, \mathcal{R}(\cdot))$. For objects $t \in \mathcal{T}^b_c$, $s \in \mathcal{S}$ the map is

$$\text{Hom}(\mathcal{L}(t), s) \xrightarrow{\mathcal{R}} \text{Hom}(\mathcal{R}\mathcal{L}(t), \mathcal{R}(s)) \xrightarrow{\text{Hom}(\alpha_{t}, \mathcal{R}(s))} \text{Hom}(t, \mathcal{R}(s)).$$
When restricted to \( t \in \mathcal{T}^c \subset \mathcal{T}^b_c \) the map \( \psi \) agrees with \( \varphi \) and is an isomorphism. It suffices to prove that \( \psi \) is an isomorphism for all \( t \in \mathcal{T}^b \) and all \( s \in \mathcal{S} \).

Fix \( t \in \mathcal{T}^b_c \) and \( s \in \mathcal{S} \). By (ii) we can choose an integer \( A > 0 \) with \( \text{Hom}(\mathcal{L}(\mathcal{T}^b_c \cap \mathcal{T}^{\leq -A}), s) = 0 \). Because \( \mathcal{R}(s) \) belongs to \( \mathcal{T}^b_c \subset \mathcal{T}^+ \) we may choose an integer \( A' > 0 \) so that \( \text{Hom}(\mathcal{T}^{\leq -A'}, \mathcal{R}(s)) = 0 \). Now take \( m \geq 1 + \max(A, A') \), and choose a triangle \( \Sigma^{-1}d \to e \to t \to d \) with \( e \in \mathcal{T}^c \) and \( d \in \mathcal{T}^{\leq -m} \). Because \( t \in \mathcal{T}^b_c \) and \( e \in \mathcal{T}^c \subset \mathcal{T}_c^b \) we have that \( d \in \mathcal{T}^b_c \cap \mathcal{T}^{\leq -m} \). Consider the commutative diagram with exact rows

\[
\begin{array}{ccc}
\text{Hom}(\mathcal{L}(d), s) & \longrightarrow & \text{Hom}(\mathcal{L}(t), s) \\
& \downarrow b & \downarrow c \\
\text{Hom}(d, \mathcal{R}(s)) & \longrightarrow & \text{Hom}(t, \mathcal{R}(s))
\end{array}
\]

\[
\begin{array}{ccc}
\longrightarrow & \longrightarrow & \longrightarrow \\
\downarrow & \downarrow & \downarrow \\
\text{Hom}(\mathcal{L}(\Sigma^{-1}d), s) & \longrightarrow & \text{Hom}(\Sigma^{-1}d, \mathcal{R}(s))
\end{array}
\]

By our choice of \( m \) we know that

\[
\text{Hom}(\mathcal{L}(d), s) = 0 = (\mathcal{L}(\Sigma^{-1}d), s), \quad \text{Hom}(d, \mathcal{R}(s)) = 0 = \text{Hom}(\Sigma^{-1}d, \mathcal{R}(s)).
\]

Hence \( a, a' \) are isomorphisms. But \( c \) is an isomorphism by the compactness of \( e \), and therefore \( b \) is an isomorphism. \( \square \)

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