A CUP PRODUCT LEMMA FOR CONTINUOUS PLURISUBHARMONIC FUNCTIONS

TERRENCE NAPIER AND MOHAN RAMACHANDRAN

Abstract. A version of Gromov’s cup product lemma in which one factor is the (1, 0)-part of the differential of a continuous plurisubharmonic function is obtained. As an application, it is shown that a connected noncompact complete Kähler manifold that has exactly one end and admits a continuous plurisubharmonic function that is strictly plurisubharmonic along some germ of a 2-dimensional complex analytic set at some point has the Bochner–Hartogs property; that is, the first compactly supported cohomology with values in the structure sheaf vanishes.

Introduction

Versions of Gromov’s cup product lemma have been applied in various settings in order to obtain results concerning the holomorphic structure of a connected noncompact complete Kähler manifold $(X, g)$. For example, in [Gro1], [Li], [Gro2], [GroS], [NR1], [DelG], [NR5], and [NR6], versions of the cup product lemma, along with analogues of the classical Castelnuovo–de Franchis theorem, yield conditions under which $X$ admits a proper holomorphic mapping onto a Riemann surface. We may illustrate the general approach to such results as follows. According to Theorem 0.2 of [NR6], which may be viewed as a version of the cup product lemma, $\theta_1 \wedge \theta_2 \equiv 0$ for any pair of closed holomorphic 1-forms $\theta_1$ and $\theta_2$ on $X$ such that $\theta_1$ is bounded, $\text{Re} \theta_1$ is exact, and $\theta_2$ is in $L^2$. If these 1-forms are linearly independent and $X$ has bounded geometry, then Stein factorization of the mapping $\theta_1/\theta_2$ of $X$ into $\mathbb{P}^1$ gives a proper holomorphic mapping onto a Riemann surface (see Theorem 0.1 and Corollary 0.3 of [NR6]).

Corollary 5.4 of [NR6] is a version of the above cup product lemma in which the factor $\theta_1$ is replaced with $\partial \varphi$, where $\varphi$ is a $C^\infty$ plurisubharmonic function with bounded gradient.

Date: October 16, 2016.

2010 Mathematics Subject Classification. 32E40.

Key words and phrases. Bochner–Hartogs property, Kähler.

Preprint of an article submitted for consideration in Journal of Topology and Analysis, © 2016 World Scientific Publishing Company, www.worldscientific.com/worldscinet/jta.
One can remove the boundedness condition on the gradient of $\varphi$ by instead requiring $\theta_2$ to be in $L^2$ with respect to the (complete) Kähler metric $h \equiv g + \mathcal{L}(e^{2\varphi})$ (we denote the Levi form of a function $\psi$ by $\mathcal{L}(\psi)$), since $e^{\varphi}$ has bounded gradient with respect to $h$. The main goal of the present paper is to obtain analogues in which the plurisubharmonic function $\varphi$ is allowed to be only continuous. Not surprisingly, for the proofs, one applies the above construction of $h$ to terms of a sequence of $C^\infty$ approximations of $\varphi$ (in place of $\varphi$) provided by a theorem of Greene and Wu (see Corollary 2 of Theorem 4.1 of [GreW] as well as [Ri] and [Dem1]). The following is the simplest form of these analogues:

**Theorem 0.1.** Let $\varphi$ be a continuous plurisubharmonic function on a connected noncompact complete Kähler manifold $(X, g)$. Then there exists a (complete) Kähler metric $h$ on $X$ such that $h \geq g$ and such that $\partial\varphi \wedge \theta \equiv 0$ as a current for every (closed) holomorphic 1-form $\theta$ on $X$ that is in $L^2$ with respect to $h$.

Theorem 0.1 is applied in this paper to obtain a result which is related to those of [Ra], [NR2], [NR4], and [NR7], and which we now describe. A connected noncompact complex manifold $X$ for which $H^1_c(X, \mathcal{O}) = 0$ is said to have the Bochner–Hartogs property (see Hartogs [Har], Bochner [Bo], and Harvey and Lawson [HavL]). Equivalently, for every $C^\infty$ compactly supported form $\alpha$ of type $(0, 1)$ with $\overline{\partial}\alpha = 0$ on $X$, there is a $C^\infty$ compactly supported function $\beta$ on $X$ such that $\overline{\partial}\beta = \alpha$. A connected noncompact complex manifold $X$ with the Bochner–Hartogs property must satisfy certain analytic and topological conditions. For example, $X$ must have exactly one end, every holomorphic function on the connected complement of a compact subset of $X$ must extend holomorphically to $X$, and $X$ cannot admit a proper holomorphic mapping onto a Riemann surface. Some further elementary consequences are described in [NR7]. Examples of manifolds of dimension $n > 1$ having the Bochner–Hartogs property include strongly $(n - 1)$-complete complex manifolds (Andreotti and Vesentini [AnV]) and strongly hyper-$(n - 1)$-convex Kähler manifolds (Grauert and Riemenschneider [GraR]). According to Theorem 3.3 of [NR7], a one-ended connected noncompact complete Kähler manifold $X$ that is hyperbolic (in the sense that $X$ admits a positive symmetric Green’s functions) and has no nontrivial $L^2$ holomorphic 1-forms has the Bochner–Hartogs property. This observation and Theorem 0.1 together lead to the following (cf. Proposition 4.4 of [NR3]):
Theorem 0.2. Let \((X,g)\) be a connected noncompact complete Kähler manifold with exactly one end. Assume that \(X\) admits a continuous plurisubharmonic function \(\varphi\) whose restriction to some 2-dimensional germ of an analytic set at some point \(p \in X\) is strictly plurisubharmonic (for example, a \(C^2\) plurisubharmonic function on \(X\) whose Levi form has at least two positive eigenvalues at some point has this property). Then \(H_c^1(X,\mathcal{O}) = 0\).

Sketch of the proof. We may assume that \(\varphi \geq 0\). Given a \(\bar{\partial}\)-closed compactly supported \(C^\infty\) \((0,1)\)-form \(\alpha\) on \(X\), we may fix a nonempty connected compact set \(K\), a positive constant \(a < \sup \varphi\), and a connected component \(\Omega\) of \(\{ x \in X \mid \varphi(x) < a \}\) such that \(\text{supp} \alpha \subset K \subset \Omega\), \(\Omega \setminus K\) is connected, and \(p \in K\). Nakano [Nk], Greene and Wu [GreW], and Demailly [Dem1] provide a complete Kähler metric \(h\) on \(\Omega\) with respect to which \(\Omega\) is hyperbolic, and we may choose \(h\) as in Theorem 0.1. The condition on \(\varphi\) then implies that \((\Omega, h)\) has no nontrivial \(L^2\) holomorphic 1-forms, and Theorem 3.3 of [NR7] then gives the claim. \(\square\)

Further applications of the versions of the cup product lemma appearing in this paper will be considered elsewhere (see [NR8]).

In Section 1 we recall some terminology and basic facts concerning ends. In Section 2 we recall some terminology and facts from potential theory, as well as some notation concerning Hermitian metrics. In Section 3 we recall some facts concerning \(C^\infty\) approximation of plurisubharmonic functions. Section 4 contains the statements and proofs of the desired versions of the cup product lemma, of which Theorem 0.1 is a special case. Section 5 contains the proof of Theorem 0.2, which is obtained as special case of a version that allows for multiple ends.

Acknowledgement. The authors would like to thank Cezar Joita for some useful references.

1. ENDS

In this section we recall some terminology and basic facts concerning ends. By an end of a connected manifold \(M\), we will mean either a component \(E\) of \(M \setminus K\) with noncompact closure, where \(K\) is a given compact subset of \(M\), or an element of

\[
\lim_{\leftarrow} \pi_0(M \setminus K),
\]

where the limit is taken as \(K\) ranges over the compact subsets of \(M\) (or equivalently, the compact subsets of \(M\) for which the complement \(M \setminus K\) has no relatively compact
components, since the union of any compact subset of $M$ with the relatively compact connected components of its complement is compact). The number of ends of $M$ will be denoted by $e(M)$. For a compact set $K$ such that $M \setminus K$ has no relatively compact components, we will call

$$M \setminus K = E_1 \cup \cdots \cup E_m,$$

where $\{E_j\}_{j=1}^m$ are the distinct components of $M \setminus K$, an ends decomposition for $M$. The following elementary lemma will allow us to modify ends decompositions and to pass to ends decompositions in domains:

**Lemma 1.1.** Let $M$ be a connected noncompact $C^\infty$ manifold.

(a) Given an ends decomposition $M \setminus K = E_1 \cup \cdots \cup E_m$, there is a connected compact set $K' \supset K$ such that any domain $\Theta$ in $M$ containing $K'$ has an ends decomposition $\Theta \setminus K = E'_1 \cup \cdots \cup E'_m$, where $E'_j = E_j \cap \Theta$ for $j = 1, \ldots, m$.

(b) If $\Omega$ and $\Theta$ are domains in $M$ with $\Theta \subset \Omega$, and both $M \setminus \Omega$ and $\Omega \setminus \Theta$ have no compact components, then $M \setminus \Theta$ has no compact components.

(c) If $E$ is an end of $M$, then there exists an end $A_0$ such that $\overline{E} \cap E \setminus A_0 \subset M$.

(d) If $E$ is an end of $M$, $F_1, \ldots, F_k \subset E$ are disjoint ends of $M$ for which $k > 1$ and $E \setminus (F_1 \cup \cdots \cup F_k) \Subset M$, and $s \in \{2, \ldots, k\}$, then there exist disjoint ends $A_1, \ldots, A_s \subset E$ of $M$ such that $\overline{A_j} \subset F_j$ and $F_j \setminus A_j \Subset M$ for $j = 1, \ldots, s - 1$, and $A_s \supset F_s \cup \cdots \cup F_k$.

**Proof.** Parts (a) and (b) are provided by Lemma 1.1 of [NR7]. We will prove part (d), and leave to the reader the proof of part (c) (which is easier and mostly contained in the proof of part (d)). The claim is trivial in dimension 1, so we may assume that $n \equiv \dim M > 1$. We may fix a nonempty $C^\infty$ domain $\Omega_0$ such that $\overline{E} \setminus (F_1 \cup \cdots \cup F_k) \subset \Omega_0 \Subset M$, $M \setminus \Omega_0$ has no compact connected components, and $E \cap \Omega_0$, as well as $F_j \cap \Omega_0$ for $j = 1, \ldots, k$, are connected. Suppose $G_1$ and $G_2$ are two distinct components of $F_1 \setminus \overline{\Omega_0}$ and therefore, of $E \setminus \overline{\Omega_0}$ and of $M \setminus \overline{\Omega_0}$. Fixing a boundary component $B_i$ of $G_i$ for $i = 1, 2$, we see that $B_1$ and $B_2$ are distinct boundary components of $\Omega_0$, and there exists a connected compact set $C \subset (F_1 \cap \Omega_0) \cup B_1 \cup B_2$ such that $C \cap B_1$ and $C \cap B_2$ are singletons, and the sets $\Omega_0 \setminus C$, $E \cap \Omega_0 \setminus C$, and $F_j \cap \Omega_0 \setminus C$ for $j = 1, \ldots, k$, are connected, and are therefore ends of $\Omega_0 \setminus C$. For example, we may take $C$ to be the image of an embedded $C^\infty$ path in $M$ such that the initial point lies in $B_1$, the terminal point lies in $B_2$, all other points lie in
$F_1 \cap \Omega_0$, and the path meets $B_1$ and $B_2$ transversely. In dimension $n > 1$, such a path is locally nonseparating, while in dimension $n = 1$, $B_1$ and $B_2$ admit disjoint neighborhoods in $F_1$ such that the intersection of each of these neighborhoods with $\Omega_0$ is a coordinate annulus whose intersection with $C$ is a radial line segment. One connected component of $M \setminus (\Omega_0 \setminus C) = (M \setminus \Omega_0) \cup C$ meets, and therefore contains, the connected noncompact set $\overline{G}_1 \cup \overline{G}_2 \cup C$. The remaining connected components are the (finitely many) connected components of $M \setminus \Omega_0$ which do not meet this set, and are in fact, precisely the connected components of $(M \setminus \Omega_0) \setminus (\overline{G}_1 \cup \overline{G}_2)$. Thus each connected component of $M \setminus (\Omega_0 \setminus C)$ is noncompact and $F_1 \setminus (\Omega_0 \setminus C)$ has strictly fewer components than $F_1 \setminus \Omega_0$ (and $M \setminus (\Omega_0 \setminus C)$ has strictly fewer components than $M \setminus \Omega_0$). We may fix a $C^\infty$ domain $\Omega_1$ such that

$$(\Omega_0 \setminus C) \setminus \Omega_1$$

has no compact connected components, and $(E \cap \Omega_0 \setminus C) \cap \Omega_1 = E \cap \Omega_1$, as well as $(F_j \cap \Omega_0 \setminus C) \cap \Omega_1 = F_j \cap \Omega_1$ for $j = 1, \ldots, k$, are ends of $\Omega_1$. Thus we get a $C^\infty$ domain $\Omega_2 \equiv \Omega_1 \cup (\Omega_0 \setminus F_1)$ such that $\overline{E} \setminus (F_1 \cup \cdots \cup F_k) \subset \Omega_2 \subset \Omega_0 \setminus C$; $(\Omega_0 \setminus C) \setminus \Omega_2$ and $M \setminus (\Omega_0 \setminus C)$, and therefore $M \setminus \Omega_2$, have no compact components; and $E \cap \Omega_2$, as well as $F_j \cap \Omega_2$ for $j = 1, \ldots, k$, are connected. Moreover, $F_j \setminus \Omega_2 = F_j \setminus \Omega_0$ for $j = 2, \ldots, k$, and each component of $F_1 \setminus \Omega_2 = F_1 \setminus \Omega_1$ contains a component of $F_1 \setminus (\Omega_0 \setminus C)$, so $F_1 \setminus \Omega_2$ has strictly fewer components than $F_1 \setminus \Omega_0$. Proceeding inductively, we get a $C^\infty$ domain $\Omega$ such that $\overline{E} \setminus (F_1 \cup \cdots \cup F_k) \subset \Omega \Subset M$; $E \cap \Omega$ and $F_j \cap \Omega$ for $j = 1, \ldots, k$ are connected; and for each $j = 1, \ldots, s - 1$, $A_j \equiv F_j \setminus \overline{\Omega}$ is connected and satisfies $\overline{A}_j \subset F_j$. The ends $A_1, \ldots, A_{s-1}, A_s \equiv (E \cap \Omega) \cup F_s \cup \cdots \cup F_k$ then have the properties listed in part (d).

2. **Green’s functions and Hermitian metrics**

In this section we recall some terminology and facts from potential theory, as well as some notation concerning Hermitian metrics. A connected noncompact oriented Riemannian manifold $(M, g)$ is called *hyperbolic* if there exists a positive symmetric Green’s function $G(x, y)$ on $M$; otherwise, $M$ is called *parabolic*. Equivalently, $M$ is hyperbolic if given a relatively compact $C^\infty$ domain $\Omega$ for which no connected component of $M \setminus \Omega$ is compact, there are a connected component $E$ of $M \setminus \overline{\Omega}$ and a (unique) greatest $C^\infty$ function $u_E : \overline{E} \to [0, 1)$ such that $u_E$ is harmonic on $E$, $u_E = 0$ on $\partial E$, and $\sup_E u_E = 1$. We will also call $E$, and any end containing $E$, a *hyperbolic* end. An end that is not hyperbolic is called *parabolic*, and we set $u_E \equiv 0$ for any parabolic end component $E$ of $M \setminus \overline{\Omega}$. We call
the function \( u : M \setminus \Omega \to [0, 1) \) defined by \( u|_E = u_E \) for each connected component \( E \) of \( M \setminus \overline{\Omega} \), the harmonic measure of the ideal boundary of \( M \) with respect to \( M \setminus \overline{\Omega} \). A sequence \( \{x_\nu\} \) in \( M \) with \( x_\nu \to \infty \) and \( G(\cdot, x_\nu) \to 0 \) (equivalently, \( u(x_\nu) \to 1 \)) is called a regular sequence. Such a sequence always exists (for \( M \) hyperbolic). A sequence \( \{x_\nu\} \) tending to infinity with \( \liminf_{\nu \to \infty} G(\cdot, x_\nu) > 0 \) (i.e., \( \limsup_{\nu \to \infty} u(x_\nu) < 1 \) or equivalently, \( \{x_\nu\} \) has no regular subsequences) is called an irregular sequence. Clearly, every sequence tending to infinity that is not regular admits an irregular subsequence. We say that an end \( E \) of \( M \) is regular (irregular) if every sequence in \( E \) tending to infinity in \( M \) is regular (respectively, there exists an irregular sequence in \( E \)). Another characterization of hyperbolicity is that \( M \) is hyperbolic if and only if \( M \) admits a nonconstant negative continuous subharmonic function \( \varphi \). In fact, if \( \{x_\nu\} \) is a sequence in \( M \) with \( x_\nu \to \infty \) and \( \varphi(x_\nu) \to 0 \), then \( \{x_\nu\} \) is a regular sequence.

The energy (or Dirichlet integral) of a suitable function \( \varphi \) (for example, a function with first-order distributional derivatives) on a Riemannian manifold \( M \) is given by \( \int_M |\nabla \varphi|^2 \, dV \). As is well known, the harmonic measure of the ideal boundary of an oriented connected noncompact Riemannian manifold has finite energy.

Let \( X \) be a complex manifold with almost complex structure \( J : TX \to TX \). By a Hermitian metric on \( X \), we will mean a Riemannian metric \( g \) on \( X \) such that \( g(Ju, Jv) = g(u, v) \) for every choice of real tangent vectors \( u, v \in T_pX \) with \( p \in X \). We call \((X, g)\) a Hermitian manifold. We will also denote by \( g \) the complex bilinear extension of \( g \) to the complexified tangent space \((TX)_C\). The corresponding real \((1, 1)\)-form \( \omega \) is given by \( (u, v) \mapsto \omega(u, v) \equiv g(Ju, v) \). The corresponding Hermitian metric (in the sense of a smoothly varying family of Hermitian inner products) in the holomorphic tangent bundle \( T^{1,0}X \) is given by \( (u, v) \mapsto g(u, \bar{v}) \). Observe that with this convention, under the holomorphic vector bundle isomorphism \((TX, J) \xrightarrow{\cong} T^{1,0}X\) given by \( u \mapsto \frac{1}{2}(u - iJu) \), the pullback of this Hermitian metric to \((TX, J)\) is given by \( (u, v) \mapsto \frac{1}{2}g(u, v) - \frac{i}{2}\omega(u, v) \). In a slight abuse of notation, we will also denote the induced Hermitian metric in \( T^{1,0}X \), as well as the induced Hermitian metric in \( \Lambda^*(TX)_C \otimes \Lambda^*(T^*X)_C \), by \( g \). The corresponding Laplacians
are given by:

\[ \Delta = \Delta_d \equiv -(dd^* + d^*d), \]
\[ \Delta_{\bar{\partial}} = -\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}, \]
\[ \Delta_{\partial} = -(\partial\partial^* + \partial^*\partial). \]

If \((X, g, \omega)\) is Kähler, i.e., \(d\omega = 0\), then \(\Delta = 2\Delta_{\bar{\partial}} = 2\Delta_{\partial}\).

### 3. Smooth approximation of plurisubharmonic functions

For the proof of (the generalizations of) Theorem 0.1, we will form suitable \(C^\infty\) approximations of continuous plurisubharmonic functions by applying the following theorem of Greene and Wu (see Corollary 2 of Theorem 4.1 of \cite{GreW} as well as \cite{Ri}, \cite{Dem1}, and Lemma 2.15 of \cite{Dem3}):

**Theorem 3.1** (Greene–Wu). Let \((X, g)\) be a Hermitian manifold, let \(K \subset X\) be a closed subset, let \(\varphi\) be a continuous plurisubharmonic function on \(X\) that is of class \(C^\infty\) on some neighborhood of \(K\), and let \(\delta\) be a positive continuous function on \(X\). Then there exists a \(C^\infty\) function \(\psi\) such that \(\psi = \varphi\) on a neighborhood of \(K\), and \(\varphi \leq \psi < \varphi + \delta\) and \(\mathcal{L}(\psi) \geq -\delta g\) on \(X\).

That the function \(\psi\) may be chosen to be equal to \(\varphi\) near \(K\) is not included explicitly in the statement in \cite{GreW}. As in the proof of Lemma 2.15 of \cite{Dem3}, one may obtain the above version by applying a \(C^\infty\) version of the maximum appearing in \cite{Dem2} (where Demailly applied it to obtain a quasi-plurisubharmonic function with logarithmic singularities along a given analytic subset). The authors have not found the theorem stated in precisely this form in the literature, so a proof is provided in this section for the convenience of the reader. Natural modifications of the proof give analogous statements for other classes of functions (see for example, \cite{JNR}). For example, a continuous strictly plurisubharmonic function \(\varphi\) on a complex manifold (or complex space) \(X\) that is \(C^\infty\) on a neighborhood of a closed set \(K\) may be approximated on \(X\) by a \(C^\infty\) strictly plurisubharmonic function that is equal to \(\varphi\) on a neighborhood of \(K\).

The \(C^\infty\) maximum is given by the following lemma, the proof of which is left to the reader:
Lemma 3.2 (Demaily [Dem2]). Let $\kappa : \mathbb{R} \to [0, \infty)$ be a $C^\infty$ function such that $\text{supp} \kappa \subset (-1, 1)$, $\int_{\mathbb{R}} \kappa(u) \, du = 1$, and $\int_{\mathbb{R}} u \kappa(u) \, du = 0$. For each $m \in \mathbb{Z}_{>0}$ and each $r = (r_1, \ldots, r_m) \in (\mathbb{R}_{>0})^m$, let $\mathcal{M}_r : \mathbb{R}^m \to \mathbb{R}$ be the function given by

$$
\mathcal{M}_r(t) \equiv \int_{\mathbb{R}^m} \left[ \max_{1 \leq j \leq m} (t_j + r_j u_j) \right] \prod_{1 \leq j \leq m} \kappa(u_j) \, du_j \quad \text{for } t = (t_1, \ldots, t_m) \in \mathbb{R}^m.
$$

Then for each $r = (r_1, \ldots, r_m) \in (\mathbb{R}_{>0})^m$, $\mathcal{M}_r$ has the following properties:

(a) For each $t = (t_1, \ldots, t_m) \in \mathbb{R}^m$,

$$
\mathcal{M}_r(t) = \int_{\mathbb{R}^m} \left[ \max_{1 \leq j \leq m} (t_j + u_j) \right] \prod_{1 \leq j \leq m} r_j^{-1} \kappa(u_j/r_j) \, du_j
$$

$$
= \int_{\mathbb{R}^m} \left[ \max_{1 \leq j \leq m} u_j \right] \prod_{1 \leq j \leq m} r_j^{-1} \kappa((u_j - t_j)/r_j) \, du_j.
$$

(b) $\mathcal{M}_r$ is $C^\infty$ and convex, and for each $t = (t_1, \ldots, t_m) \in \mathbb{R}^m$ and each permutation $\sigma$ of $\{1, \ldots, m\}$, $\mathcal{M}_r(t) = \mathcal{M}_r(r_{\sigma(1)}, \ldots, r_{\sigma(m)})(t_{\sigma(1)}, \ldots, t_{\sigma(m)})$.

(c) For each $j = 1, \ldots, m$, $0 \leq \partial \mathcal{M}_r/\partial t_j \leq 1$.

(d) For each $s \in \mathbb{R}$, we have $\mathcal{M}_r(t_1 + s, \ldots, t_m + s) = \mathcal{M}_r(t_1, \ldots, t_m) + s$.

(e) For every $t = (t_1, \ldots, t_m) \in \mathbb{R}^m$, $\max_{1 \leq j \leq m} t_j \leq \mathcal{M}_r(t) \leq \max_{1 \leq j \leq m}(t_j + r_j)$.

(f) If $t' = (t_0, t_1, \ldots, t_m) = (t_0, t) \in \mathbb{R}^{m+1}$ and $r' = (r_0, r_1, \ldots, r_m) \in (\mathbb{R}_{>0})^{m+1}$ with $t_0 + r_0 \leq t_1 - r_1$, then $\mathcal{M}_{r'}(t') = \mathcal{M}_r(t)$.

(g) For $m = 1$, $\mathcal{M}_r(t) = t$ for each $t \in \mathbb{R}$.

(h) If $\varphi = (\varphi_1, \ldots, \varphi_m)$ is an $m$-tuple of $C^\infty$ real-valued functions on a complex manifold $X$, then

$$
\mathcal{L}(\mathcal{M}_r(\varphi))(v, v) \geq \min_{1 \leq j \leq m} \mathcal{L}(\varphi_j)(v, v) \quad \forall v \in T^{1,0}X.
$$

Proof of Theorem 3.1 Let $n \equiv \dim X$. Fixing a function $\kappa$ as in Lemma 3.2, we may form the corresponding family of $C^\infty$ functions $\{\mathcal{M}_r\}$. We may choose open sets $\{\Omega_k\}_{k=0}^3$, $\{U_\nu\}_{\nu \in N}$, $\{V_\nu\}_{\nu \in N}$, and $\{W_\nu\}_{\nu \in N}$ such that

(i) We have $K \subset \Omega_0 \subset \overline{\Omega}_0 \subset \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2 \subset \overline{\Omega}_2 \subset \Omega_3$;

(ii) We have $\lambda \in C^\infty(X)$, $0 \leq \lambda \leq 1$ on $X$, $\lambda \equiv 1$ on $\Omega_1$, and $\text{supp } \lambda \subset \Omega_2$;

(iii) The function $\varphi$ is of class $C^\infty$ on $\Omega_3$;

(iv) For each $\nu \in N$, we have $U_\nu \subset V_\nu \subset W_\nu \subset X \setminus K$ and, for $k = 1, 2, 3$, $W_\nu \subset \Omega_k$ if $W_\nu \cap \overline{\Omega}_{k-1} \neq \emptyset$ and $W_\nu \subset X \setminus \overline{\Omega}_{k-1}$ if $W_\nu \cap X \setminus \Omega_k \neq \emptyset$;
(v) \( \{W_{\nu}\} \) is locally finite in \( X \) and \( \{U_{\nu}\} \) covers \( X \setminus \Omega_0 \);

(vi) For each \( \nu \in N \) and each \( \epsilon > 0 \), there exists a \( C^\infty \) plurisubharmonic function \( \rho \) on \( W_{\nu} \) such that \( \varphi \leq \rho \leq \varphi + \epsilon \) on \( W_{\nu} \) (if \( W_{\nu} \subset \Omega_3 \), then we may take \( \rho = \varphi|W_{\nu} \)); and

(vii) For each \( \nu \in N, \alpha_{\nu} \in C^\infty(W_{\nu}) \), \( 0 \leq \alpha_{\nu} \leq 1 \) on \( W_{\nu} \), \( \alpha_{\nu} \equiv 0 \) on \( U_{\nu} \), and \( \alpha_{\nu} \equiv 1 \) on \( W_{\nu} \setminus V_{\nu} \).

Given positive constants \( \{\epsilon_{\nu}\}_{\nu \in N} \) with \( \epsilon_{\nu} < \delta \) on \( W_{\nu} \) and

\[
\epsilon_{\nu}|L(\alpha_{\nu})(v,v)| \leq \delta(x)|v|_g^2 \quad \forall x \in W_{\nu}, \ v \in T_x^{1,0}X
\]

for each \( \nu \), we may, for each \( \nu \in N \), set

\[
r_{\nu} \equiv \frac{1}{\mu} \min\{\epsilon_{\mu} \mid \mu \in N, W_{\mu} \cap W_{\nu} \neq \emptyset\},
\]

and choose a \( C^\infty \) plurisubharmonic function \( \rho_{\nu} \) on \( W_{\nu} \) with \( \varphi \leq \rho_{\nu} \leq \varphi + r_{\nu} \) on \( W_{\nu} \). If \( W_{\nu} \subset \Omega_3 \), then we set \( \rho_{\nu} \equiv \varphi|W_{\nu} \). Thus we may define a function \( \beta \) on \( X \setminus \overline{\Omega}_0 \) as follows. Given a point \( p \in X \setminus \overline{\Omega}_0 \), we let \( \nu_1, \ldots, \nu_m \in N \) be the distinct indices with \( p \in W_{\nu} \) if and only if \( \nu \in \{\nu_1, \ldots, \nu_m\} \), we set \( r = (r_{\nu_1}, \ldots, r_{\nu_m}) \) and \( \rho = (\rho_{\nu_1} - \epsilon_{\nu_1}\alpha_{\nu_1}, \ldots, \rho_{\nu_m} - \epsilon_{\nu_m}\alpha_{\nu_m}) \), and we set \( \beta(p) \equiv M_r(\rho(p)) \). We will show that \( \beta \) is of class \( C^\infty \), \( \varphi \leq \beta < \varphi + \delta \), and \( L(\beta) \geq -\delta g \) on \( X \setminus \overline{\Omega}_0 \). We will then show that if the constants \( \{\epsilon_{\nu}\} \) are sufficiently small, then the function \( \psi \equiv \lambda \cdot \varphi + (1 - \lambda) \cdot \beta \in C^\infty(X) \) satisfies \( L(\psi) \geq -\delta g \) at each point in \( \overline{\Omega}_2 \setminus \Omega_1 \), and it will follow that \( \psi \) has the required properties.

For \( p \in X \setminus \overline{\Omega}_0 \) and \( \beta(p) = M_r(\rho(p)) \), where \( \nu_1, \ldots, \nu_m \in N \), \( r = (r_{\nu_1}, \ldots, r_{\nu_m}) \), and \( \rho = (\rho_{\nu_1} - \epsilon_{\nu_1}\alpha_{\nu_1}, \ldots, \rho_{\nu_m} - \epsilon_{\nu_m}\alpha_{\nu_m}) \) are as in the previous paragraph, we may assume that \( p \in U_{\nu_1} \), and we may choose a relatively compact neighborhood \( Q \) of \( p \) such that \( Q \subset U_{\nu_1}(V_{\nu}, W_{\nu}, \Omega_k, X \setminus \overline{V}_{\nu}, X \setminus \overline{W}_{\nu}, X \setminus \overline{\Omega}_k) \) whenever \( p \in U_{\nu} \) (respectively, \( V_{\nu}, W_{\nu}, \Omega_k, X \setminus \overline{V}_{\nu}, X \setminus \overline{W}_{\nu}, X \setminus \overline{\Omega}_k \)). In particular, by part (e) of Lemma 3.2, we have

\[
\varphi(p) \leq \rho_{\nu_1}(p) - \epsilon_{\nu_1}\alpha_{\nu_1}(p) \leq \max_{1 \leq j \leq m} (\rho_{\nu_j}(p) - \epsilon_{\nu_j}\alpha_{\nu_j}(p)) \leq \beta(p) \\
< \max_{1 \leq j \leq m} (\rho_{\nu_j}(p) - \epsilon_{\nu_j}\alpha_{\nu_j}(p) + r_{\nu_j}) < \varphi(p) + \delta(p).
\]

After reordering, we may assume that for some \( k \in \{1, \ldots, m\}, \nu = \nu_1, \ldots, \nu_k \in N \) are precisely those indices for which \( p \in \overline{V}_{\nu} \) (i.e., for which \( \overline{Q} \cap \overline{V}_{\nu} \neq \emptyset \)). Setting \( s \equiv (r_{\nu_1}, \ldots, r_{\nu_k}) \), we then have \( \beta = M_s(\rho_{\nu_1} - \epsilon_{\nu_1}\alpha_{\nu_1}, \ldots, \rho_{\nu_k} - \epsilon_{\nu_k}\alpha_{\nu_k}) \) on

\[
Q \subset U_{\nu_1} \cap W_{\nu_2} \cap \cdots \cap W_{\nu_m}.
\]
For if $\nu \in N \setminus \{\nu_1, \ldots, \nu_k\}$, then on $Q \cap W_\nu \subset W_\nu \setminus W_\nu$, we have

$$\rho_\nu - \epsilon_\nu \alpha_\nu + r_\nu + r_{\nu_1} \leq \varphi + r_\nu - \epsilon_\nu + r_\nu + r_{\nu_1} < \varphi \leq \rho_{\nu_1} = \rho_{\nu_1} - \epsilon_{\nu_1} \alpha_{\nu_1}.$$ 

Part (f) of Lemma 3.2 now gives the expression for $\beta$. Hence $\beta$ is smooth, and part (h) of the lemma implies that $L(\beta) \geq -\delta g$.

If in the above $p \in \overline{\Omega}_2 \setminus \Omega_1$, then $W_{\nu_j} \Subset \Omega_3 \setminus \Omega_0$ and $\rho_{\nu_j} = \varphi|_{W_{\nu_j}}$ for $j = 1, \ldots, m$, and hence by part (d) of Lemma 3.2, $\beta = \varphi + \xi$ and $\psi = \varphi + (1 - \lambda) \xi$ on $Q$, where $\xi = M_s(0, \alpha_{\nu_1}, \ldots, \alpha_{\nu_k})$. Moreover, for each point $x \in Q$ and each tangent vector $v \in T_{x,0}X$, we have

$$L((1 - \lambda) \xi)(v,v) = (1 - \lambda(x))L(\xi)(v,v) - 2\operatorname{Re} \left[ \partial \lambda(v) \frac{\partial \xi(v)}{\partial v} \right] - L(\lambda)(v,v)\xi(x) \geq -(1 - \lambda(x)) \max_{1 \leq j \leq k} \epsilon_{\nu_j} L(\alpha_{\nu_j})(v,v)$$

$$- 2|\partial \lambda(v)| \left( \sum_{j=1}^k \epsilon_{\nu_j} |\partial \alpha_{\nu_j}(v)| \right) - |L(\lambda)(v,v)| \max_{1 \leq j \leq k} \epsilon_{\nu_j}.$$ 

Forming a locally finite covering of $\overline{\Omega}_2 \setminus \Omega_1$ by such neighborhoods $Q$, we see that for sufficiently small $\{\epsilon_\nu\}$, we have

$$L(\psi) \geq L((1 - \lambda) \xi) \geq -\delta g$$

at each point in $\overline{\Omega}_2 \setminus \Omega_1$, and it follows that $\psi$ has the required properties on $X$. □

4. Cup product lemmas

In this section, we consider the promised versions of the cup product lemma, of which Theorem 0.1 is a special case. Theorem 0.1 suffices for the applications considered in this paper, but the more general versions are required for applications to be considered elsewhere (see [NR8]). According to Theorem 5.2 of [NR6] (see also Corollary 5.4 of [NR6]), if $\varphi$ is nonconstant $C^\infty$ plurisubharmonic function with bounded gradient on a connected complete Kähler manifold $(X,g)$, then $\partial \varphi \wedge \theta \equiv 0$ for every $L^2$ holomorphic 1-form $\theta$. One may obtain the bounded gradient condition by replacing $\varphi$ with $e^\varphi$ and $g$ with $g + L(e^{2\varphi})$. We will obtain the following version for $\varphi$ continuous (in fact, for a countable family of continuous plurisubharmonic functions):
A CUP PRODUCT LEMMA

**Theorem 4.1.** Let \((X, g, \omega_0)\) be a connected noncompact complete Kähler manifold, let \(K\) be a closed subset of \(X\), and let \(\{\varphi_j\}_{j \in J}\) be a countable collection of continuous plurisubharmonic functions on \(X\), each of which is locally constant on a neighborhood of \(K\) in \(X\). Then for every constant \(\epsilon \in (0, 1)\), there exists a complete Kähler metric \(h_0\) on \(X\) such that \(h_0 \geq (1 - \epsilon)g\) on \(X\), \(h_0 = g\) at each point in \(K\), and for every connected covering space \(\Upsilon: \hat{X} \to X\) and every (complete) Kähler metric \(h\) on \(\hat{X}\) with \(h \geq \hat{h}_0 \equiv \Upsilon^* h_0\), the set \(\hat{K} \equiv \Upsilon^{-1}(K)\) and the continuous plurisubharmonic functions \(\{\hat{\varphi}_j\}\) given by \(\hat{\varphi}_j \equiv \Upsilon^* \varphi_j\) for each \(j \in J\) have the following properties:

(i) If \(\theta\) is a \(C^\infty, (1, 0)\)-form on \(\hat{X}\) for which \(d\theta|_{\hat{X} \setminus \hat{K}} \equiv 0\) and \(\theta|_{\hat{X} \setminus \hat{K}}\) is in \(L^2\) with respect to \(h\), then \(\partial \hat{\varphi}_j \wedge \theta \equiv 0\) as a current on \(\hat{X}\) for each \(j \in J\). In particular, for each connected component \(U\) of \(\hat{X} \setminus \hat{K}\) on which \(\theta\) is not everywhere zero, \(\hat{\varphi}_j\) is constant on each leaf of the holomorphic foliation determined by \(\theta\) in \(U\) for each \(j \in J\).

(ii) If \(\theta_1\) and \(\theta_2\) are two \(C^\infty, (1, 0)\)-forms on \(\hat{X}\) for which \(d\theta_k|_{\hat{X} \setminus \hat{K}} \equiv 0\) and \(\theta_k|_{\hat{X} \setminus \hat{K}}\) is in \(L^2\) with respect to \(h\) for \(k = 1, 2\), and \(U\) is a connected component of \(\hat{X} \setminus \hat{K}\) on which \(\hat{\varphi}_j\) is nonconstant for some \(j \in J\), then \(\theta_1 \wedge \theta_2 \equiv 0\) on \(U\).

Theorem 4.1 is a direct consequence of the following more general version:

**Theorem 4.2.** Let \((X, g_0, \omega_0)\) be a connected noncompact complete Hermitian manifold of dimension \(n\), let \(K\) be a closed subset of \(X\), let \(\{\varphi_j\}_{j \in J}\) be a countable collection of continuous plurisubharmonic functions on \(X\), each of which is locally constant on a neighborhood of \(K\) in \(X\), and let \(\delta: X \to (0, 1)\) be a continuous function. Then there exists a nonnegative \(C^\infty\) function \(\psi\) on \(X\) such that \(\mathcal{L}(\psi) \geq -\delta g_0\) on \(X\), \(\psi\) is constant on each connected component of \(K\), the derivatives of \(\psi\) of all orders vanish at each point in \(K\), and for every constant \(\epsilon \in (0, 1)\), every connected covering space \(\Upsilon: \hat{X} \to X\), and every (complete) Hermitian metric \(g\) on \(\hat{X}\) with \(g \geq \hat{g}_0 \equiv \Upsilon^* g_0\), the \(C^\infty\) function \(\hat{\psi} \equiv \Upsilon^* \psi\), the (complete) Hermitian metric \(h \equiv g + \epsilon \mathcal{L}(\hat{\psi})\) with associated \((1, 1)\)-form \(\omega_h\), the set \(\hat{K} \equiv \Upsilon^{-1}(K)\), and the continuous plurisubharmonic functions \(\{\hat{\varphi}_j\}\) given by \(\hat{\varphi}_j \equiv \Upsilon^* \varphi_j\) for each \(j \in J\) have the following properties:

(i) If \(\beta\) is a \(C^\infty, (1, 1)\)-form on \(\hat{X}\) for which \(\beta|_{\hat{X} \setminus \hat{K}} \geq 0\), \(d(\beta \wedge \omega_h^n)\) in \(L^1\) with respect to \(h\), then \((\partial \hat{\varphi}_j) \wedge \beta \equiv 0\) and \((\partial \hat{\varphi}_j) \wedge \beta \equiv 0\) as currents on \(X\) for each \(j \in J\). Moreover, if \(U \subset \hat{X}\) is a domain on which \(\beta = i\theta \wedge \bar{\theta}\) for some \(C^\infty, (1, 0)\)-form \(\theta\) on \(U\), then \((\partial \hat{\varphi}_j) \wedge \theta \equiv 0\) as a current on \(U\) for each \(j \in J\).
In particular, if this form $\theta$ is a nontrivial closed holomorphic 1-form, then $\hat{\varphi}_j$ is constant on each leaf of the holomorphic foliation determined by $\theta$ in $U$ for each $j \in J$.

(ii) If $\beta_1$ and $\beta_2$ are two $C^\infty$ $(1, 1)$-forms on $\hat{X}$ for which $\beta_k|_{\hat{X}\setminus \hat{K}} \geq 0$, $d(\beta_k \wedge \omega^{n-2})|_{\hat{X}\setminus \hat{K}} \equiv 0$, and $\beta_k|_{\hat{X}\setminus \hat{K}}$ is in $L^1$ with respect to $h$ for $k = 1, 2$, $\theta_1$ and $\theta_2$ are closed holomorphic 1-forms on a domain $U \subset \hat{X}$, $\beta_k|_U = i\theta_k \wedge \overline{\theta}_k$ for $k = 1, 2$, and $\hat{\varphi}_j|_U$ is nonconstant for some $j \in J$, then $\theta_1 \wedge \theta_2 \equiv 0$ on $U$.

We first consider some elementary observations.

**Lemma 4.3.** Let $(X, g, \omega)$ be a Hermitian manifold of dimension $n$, and let $\theta$ be a $C^\infty$ form of type $(1, 0)$ on $X$.

(a) If $h$ is the Hermitian metric with associated $(1, 1)$-form $\omega_h \equiv \omega + i\theta \wedge \overline{\theta}$, then $|\theta|^2_h = (1 + |\theta|_g^2)^{-1}|\theta|_g^2 \leq 1$, $dV_h = (1 + |\theta|_g^2)dV_g$, and $|\theta|^2_h dV_h = |\theta|^2_g dV_g$. Moreover, $|v|^2_h = |v|^2_g + 2|\theta(v)|^2 = |v|^2_g + 2|\text{Re} \theta(v)|^2 + 2|\text{Im} \theta(v)|^2$ for every real tangent vector $v \in TX$.

(b) We have $\|\theta\|^2_{L^2(X, g)} = \int_X \frac{\sqrt{-1}}{(n-1)!} |\theta \wedge \overline{\theta} \wedge \omega^{n-1}|$. Consequently, if $g'$ is any Hermitian metric with associated $(1, 1)$-form $\omega'$ and $\theta \wedge \omega = \theta \wedge \omega'$, then $\|\theta\|_{L^2(X, g')} = \|\theta\|_{L^2(X, g')}$.

**Proof.** For part (a), observe that $h = g$ at any point $p \in X$ at which $\theta_p = 0$, while at any point $p \in X$ at which $\theta_p \neq 0$, one may verify the first group of equalities by writing $g$ and $h$ in terms of a $g$-orthonormal basis $e_1, \ldots, e_n$ for $T_{p,1}X$ with dual basis $e^*_1 = |\theta_p|_g^{-1} \theta_p, e^*_2, \ldots, e^*_n$. The verification of the last equality in part (a) and the verification of part (b) are also straightforward. \hfill $\square$

**Lemma 4.4** (cf. Lemma 5.1 of [NR6]). Let $(V, J)$ be a complex vector space of dimension $n > 1$, let $g$ be a Hermitian inner product on $V$ with associated real skew-symmetric $(1, 1)$-form $\omega$, let $\alpha$ and $\beta$ be real skew-symmetric forms of type $(1, 1)$ on $V$ with $\alpha \geq 0$ and $\beta \geq 0$, and let $\eta$ and $\theta$ be $(1, 0)$-forms on $V$. Then

(i) $\frac{1}{n!}|\beta|_g \omega^n \leq \frac{1}{(n-1)!}|\beta \wedge \omega^{n-1}| \leq \frac{\sqrt{n}}{n!} |\beta|_g \omega^n$;

(ii) $\frac{1}{n!} |\alpha \wedge \beta|_g \omega^n \leq \frac{1}{(n-2)!} |\alpha \wedge \beta \wedge \omega^{n-2}| \leq \frac{\sqrt{n(n-1)!}}{n!} |\alpha \wedge \beta|_g \omega^n$;

(iii) $\frac{1}{n!} |\eta \wedge \beta|_g \omega^n \leq \frac{1}{(n-2)!} |\beta|_g \eta \wedge \eta \wedge \beta \wedge \omega^{n-2}| \leq \frac{2n}{n!} |\eta|_g^2 |\beta|_g \omega^n$; and

(iv) $\frac{1}{n!} |\eta \wedge \theta|_g \omega^n = \frac{1}{(n-2)!} (\eta \wedge \eta) \wedge (i\theta \wedge \overline{\theta}) \wedge \omega^{n-2} \leq \frac{1}{n!} |\eta|_g^2 |\theta|_g^2 \omega^n$. 

Proof. We may choose a basis \( \zeta_1, \ldots, \zeta_n \) for \( (V^*)^1 \) so that
\[
\omega = \sum_i \iota \zeta_i \wedge \bar{\zeta}_i \quad \text{and} \quad \beta = \sum \lambda_i \iota \zeta_i \wedge \bar{\zeta}_i,
\]
where \( 0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \). We then have
\[
\alpha = \sum A_{jk} \iota \zeta_j \wedge \bar{\zeta}_k \quad \text{and} \quad \eta = \sum r_j \zeta_j,
\]
where \( A_{jk} = \overline{A_{kj}} \) for all \( j, k \). Thus
\[
|\beta|_g = \left( \sum \lambda_j^2 \right)^{1/2}
\]
and
\[
\beta \wedge \omega^{n-1} = i^n (n-1)! \sum_{j=1}^n \sum_{k=1}^n \lambda_j \zeta_j \wedge \bar{\zeta}_j \wedge \zeta_1 \wedge \cdots \wedge \zeta_k \wedge \cdots \wedge \zeta_n \wedge \bar{\zeta}_n
\]
\[
= i^n (n-1)! \sum_{j=1}^n \lambda_j \zeta_1 \wedge \cdots \wedge \zeta_n \wedge \bar{\zeta}_n = \frac{1}{n} \sum_{j=1}^n \lambda_j \omega^n,
\]
and the claim (i) follows.

We also have
\[
\alpha \wedge \beta = i^2 \sum_{j \neq k, l} A_{kl} \lambda_j \zeta_k \wedge \bar{\zeta}_l \wedge \zeta_j \wedge \bar{\zeta}_j
\]
\[
= i^2 \sum_{j,k,l \text{ are distinct}} \lambda_j A_{kl} \zeta_j \wedge \bar{\zeta}_j \wedge \zeta_k \wedge \bar{\zeta}_k + i^2 \sum_{j<k} (\lambda_j A_{kk} + \lambda_k A_{jj}) \zeta_j \wedge \bar{\zeta}_j \wedge \zeta_k \wedge \bar{\zeta}_k.
\]
Hence
\[
\alpha \wedge \beta \wedge \omega^{n-2}
\]
\[
= i^{n-2} (n-2)! \sum_{j<k} \alpha \wedge \beta \wedge \zeta_1 \wedge \cdots \wedge \zeta_j \wedge \cdots \wedge \zeta_k \wedge \cdots \wedge \zeta_n \wedge \bar{\zeta}_n
\]
\[
= i^n (n-2)! \sum_{j<k} (\lambda_j A_{kk} + \lambda_k A_{jj}) \zeta_1 \wedge \cdots \wedge \zeta_n \wedge \bar{\zeta}_n
\]
\[
= \frac{1}{n(n-1)} \left( \sum_{j<k} (\lambda_j A_{kk} + \lambda_k A_{jj}) \right) \omega^n,
\]
and
\[
|\alpha \wedge \beta|_g^2 = \sum_{j,k,l \text{ are distinct}} \lambda_j^2 |A_{kl}|^2 + \sum_{j<k} (\lambda_j A_{kk} + \lambda_k A_{jj})^2 \leq \sum_{j,k,l \text{ are distinct}} \lambda_j^2 A_{kk} A_{ll} + \sum_{j<k} (\lambda_j A_{kk} + \lambda_k A_{jj})^2
\]
\[
\leq \left( \sum_{j<k} (\lambda_j A_{kk} + \lambda_k A_{jj}) \right)^2 \leq \frac{n(n-1)}{2} \sum_{j<k} (\lambda_j A_{kk} + \lambda_k A_{jj})^2 \leq \frac{n(n-1)}{2} |\alpha \wedge \beta|_g^2,
\]
so (ii) follows.

From the above, we have

\[ i\eta \wedge \bar{\eta} \wedge \beta \wedge \omega^{n-2} = \frac{1}{n(n-1)} \left( \sum_{j \neq k} \lambda_j |r_k|^2 \right) \omega^n, \]

while

\[
|\eta \wedge \beta|^2_g = \left| \sum_{j \neq k} \lambda_j g r_k \zeta_k \wedge \zeta_j \right|_g^2 \leq \sum_{j \neq k} \lambda_j^2 |r_k|^2 \leq |\beta|_g^2 \sum_{j \neq k} \lambda_j |r_k|^2 \\
\leq |\beta|_g^2 \sum_{j \neq k} |r_k|^2 = (n-1) |\eta|_g^2 |\beta|_g^2,
\]

so (iii) follows.

Finally, letting \( \beta = i\theta \wedge \bar{\theta} \), we get \( \lambda_j = 0 \) for \( j = 1, \ldots, n-1 \), \( \lambda_n = |\theta|_g^2 \),

\[ (i\eta \wedge \bar{\eta}) \wedge (i\theta \wedge \bar{\theta}) \wedge \omega^{n-2} = \frac{1}{n(n-1)} \left( \sum_{k=1}^{n-1} |\theta|_g^2 |r_k|^2 \right) \omega^n, \]

and

\[
|\eta \wedge \theta|^2_g = \left| \sum_{k=1}^{n-1} r_k |\theta|_g \zeta_k \wedge \zeta_n \right|_g^2 \leq \sum_{k=1}^{n-1} |\theta|_g^2 |r_k|^2 \leq |\eta|_g^2 |\theta|_g^2,
\]

so (iv) follows.

\[ \square \]

Given a Hermitian inner product \( g \) with associated real skew-symmetric \((1,1)\)-forms \( \omega \) on a complex vector space \((\mathcal{V}, J)\) of dimension \( n \), and a real skew-symmetric form \( \alpha \) of type \((1, 1)\) on \( \mathcal{V} \), we have the orthogonal decomposition \( \mathcal{V}^{1,0} = \mathcal{V}^{-0}_- \oplus \mathcal{V}^{-0}_+ \oplus \mathcal{V}^{+}_+ \), where \( \mathcal{V}^{1,0}_- (\mathcal{V}^{1,0}_+, \mathcal{V}^{1,0}_+ ) \) is the sum of the eigenspaces for the positive eigenvalues (respectively, the eigenspace for the zero eigenvalue, the sum of the eigenspaces for the negative eigenvalues). Letting \( \text{pr}_+: \mathcal{V}^{1,0} \to \mathcal{V}^{1,0}_+ \) and \( \text{pr}_-: \mathcal{V}^{1,0} \to \mathcal{V}^{1,0}_- \) be the corresponding orthogonal projections, we get \( \alpha = \alpha^+ - \alpha^- \), where \( \alpha^+ \) and \( \alpha^- \), the positive part of \( \alpha \) and negative part of \( \alpha \), respectively, are the nonnegative \((1,1)\)-forms given by

\[ \alpha^+(u, \bar{v}) = \alpha(\text{pr}_+ u, \text{pr}_+ \bar{v}) \quad \text{and} \quad \alpha^-(u, \bar{v}) = -\alpha(\text{pr}_- u, \text{pr}_- \bar{v}) \quad \forall \ u, v \in \mathcal{V}^{1,0}. \]

If \( \{ \zeta_j \} \) is a basis for \((\mathcal{V}^*)^{1,0}\) in which \( \omega = \sum i\zeta_j \wedge \bar{\zeta}_j \) and \( \alpha = \sum \lambda_j i\zeta_j \wedge \bar{\zeta}_j \), then

\[ \alpha^+ = \sum \lambda_j^+ i\zeta_j \wedge \bar{\zeta}_j \quad \text{and} \quad \alpha^- = \sum \lambda_j^- i\zeta_j \wedge \bar{\zeta}_j. \]
Lemma 4.5. Let $\alpha$ be a real $(1,1)$-form on a Hermitian manifold $(X,g,\omega)$. If $\alpha$ is continuous (measurable), then the associated positive and negative parts, $\alpha^+$ and $\alpha^-$, are continuous (respectively, measurable).

Proof. Let $n = \dim X$. If $\alpha$ is continuous but $\alpha^+$ is not, then there exist a point $p \in X$, tangent vectors $u, v \in (T_pX)^{1,0}$, sequences $\{x_\nu\}$ in $X$ and $\{u_\nu\}$ and $\{v_\nu\}$ in $(TX)^{1,0}$, and a positive constant $\epsilon$ such that $u_\nu, v_\nu \in (T_{x_\nu}X)^{1,0}$ for each $\nu$, $x_\nu \to p$, $u_\nu \to u$, and $v_\nu \to v$, but

$$|\alpha^+(u_\nu, \bar{v}_\nu) - \alpha^+(u, \bar{v})| \geq \epsilon \quad \forall \nu = 1, 2, 3, \ldots.$$ 

For each $\nu$, we may fix an (orthonormal) basis $\{\zeta_j^{(\nu)}\}_{j=1}^n$ for $(T^*_x X)^{1,0}$ such that

$$\omega_{x_\nu} = \sum_{j=1}^n i\zeta_j^{(\nu)} \wedge \bar{\zeta}_j^{(\nu)} \quad \text{and} \quad \alpha_{x_\nu} = \sum_{j=1}^n \lambda_j^{(\nu)} i\zeta_j^{(\nu)} \wedge \bar{\zeta}_j^{(\nu)},$$

where $\lambda_1^{(\nu)} \leq \cdots \leq \lambda_n^{(\nu)}$. By the continuity of $\alpha$, the eigenvalues $\{\lambda_j^{(\nu)}\}$ are uniformly bounded, so by replacing the above with a suitable subsequence, we may assume that

$$\zeta_j^{(\nu)} \to \zeta_j \quad \text{and} \quad \lambda_j^{(\nu)} \to \lambda_j \quad \forall j = 1, \ldots, n,$$

and hence that $\omega_p = \sum i\zeta_j \wedge \bar{\zeta}_j$ and $\alpha_p = \sum \lambda_j i\zeta_j \wedge \bar{\zeta}_j$. However, this then implies that

$$\alpha^+(u_\nu, \bar{v}_\nu) = \sum (\lambda_j^{(\nu)})^+ i\zeta_j^{(\nu)}(u_\nu)\bar{\zeta}_j^{(\nu)}(v_\nu) \to \sum \lambda_j^+ i\zeta_j(v)\bar{\zeta}_j(v) = \alpha^+(u, \bar{v}).$$

Thus we have arrived at a contradiction, and hence continuity of $\alpha$ implies that of $\alpha^+$.

For $\alpha$ a measurable form, there exists a sequence of continuous real $(1,1)$-forms $\{\alpha_\nu\}$ converging to $\alpha$ almost everywhere in $X$, and an argument similar to the above shows that $\alpha^+_\nu \to \alpha^+$ a.e. in $X$. Hence $\alpha^+$ is also measurable. \hfill $\square$

For the proof of Theorem 4.2, after forming $C^\infty$ approximations of the given plurisubharmonic functions and modifying the metric, we will apply the following:

Lemma 4.6. Let $\{\alpha_\nu\}_{\nu=1}^\infty$ and $\beta$ be continuous real $(1,1)$-forms on a Hermitian manifold $(X, g, \omega)$ of dimension $n$ such that $\beta \geq 0$ and $\alpha_\nu \wedge \beta \wedge \omega^{n-2} \to 0$ in $L^1_{\text{loc}}$.

(a) If $\alpha_\nu \wedge \beta \to 0$ in $L^1_{\text{loc}}$, then $\alpha_\nu \wedge \beta \to 0$ in $L^1_{\text{loc}}$.

(b) If for each $\nu$, $\alpha_\nu = i\eta_\nu \wedge \bar{\eta}_\nu$ for some continuous $(1,0)$-form $\eta_\nu$, then $\eta_\nu \wedge \beta \to 0$ in $L^2_{\text{loc}}$.

(c) If $\beta = i\theta \wedge \bar{\theta}$ for some continuous $(1,0)$-form $\theta$ and $\alpha_\nu = i\eta_\nu \wedge \bar{\eta}_\nu$ for some continuous $(1,0)$-form $\eta_\nu$ for each $\nu$, then $\eta_\nu \wedge \theta \to 0$ in $L^2_{\text{loc}}$. 
Proof. Assuming that $\alpha_\nu^+ \wedge \beta \rightarrow 0$ in $L^1_{\text{loc}}$, Lemma 4.4 implies that $\alpha_\nu^+ \wedge \beta \wedge \omega^{n-2} \rightarrow 0$ in $L^1_{\text{loc}}$, and hence that

$$\alpha_\nu^+ \wedge \beta \wedge \omega^{n-2} = \alpha_\nu \wedge \beta \wedge \omega^{n-2} + \alpha_\nu^- \wedge \beta \wedge \omega^{n-2} \rightarrow 0 \quad \text{in} \quad L^1_{\text{loc}}.$$ 

Applying the lemma once more, we get $\alpha_\nu^+ \wedge \beta \rightarrow 0$ in $L^1_{\text{loc}}$, and part (a) follows. Parts (b) and (c) follow immediately from parts (iii) and (iv), respectively, of Lemma 4.4.

After we obtain the main parts of Theorem 4.2, the following elementary observations will give the remaining conclusions:

Lemma 4.7. Let $\varphi$ be a nonconstant real-valued continuous function on a connected complex manifold $X$.

(a) For any nontrivial closed holomorphic 1-form $\theta$ on $X$, the following are equivalent:

(i) As a current, $\partial \varphi \wedge \theta \equiv 0$.

(ii) As a current, $\partial \varphi \wedge \theta \wedge \bar{\theta} \equiv 0$.

(iii) The restriction of $\varphi$ to each leaf of the holomorphic foliation determined by $\theta$ is constant (equivalently, for every holomorphic function $f$ with $\theta = df$ on an open set $U$, $\varphi$ is constant on each level of $f$).

(b) If $\theta_1$ and $\theta_2$ are closed holomorphic 1-forms on $X$ and $\partial \varphi \wedge \theta_j \equiv 0$ for $j = 1, 2$, then $\theta_1 \wedge \theta_2 \equiv 0$.

Proof. Let $n \equiv \dim X$. For the proof of (a), given a point $p \in X$ at which $\theta_p \neq 0$, we may fix a local holomorphic coordinate neighborhood $(U, (z_1, \ldots, z_n))$ of $p$ in which $U$ is a coordinate polydisk and $\theta |_U = d\bar{z}_1$. For each $j = 2, \ldots, n$, and for each function $u \in \mathcal{D}(U)$, we then have

$$\langle \partial \varphi \wedge \theta \wedge \bar{\theta}, u \, dz_2 \wedge d\bar{z}_2 \wedge \cdots \wedge d\bar{z}_j \wedge \cdots \wedge dz_n \wedge d\bar{z}_n \rangle$$

$$= \langle \partial \varphi \wedge \theta, u \, d\bar{z}_1 \wedge d\bar{z}_2 \wedge \cdots \wedge d\bar{z}_j \wedge \cdots \wedge dz_n \wedge d\bar{z}_n \rangle$$

$$= \langle \partial \varphi, u \, dz_1 \wedge d\bar{z}_2 \wedge \cdots \wedge d\bar{z}_j \wedge \cdots \wedge dz_n \wedge d\bar{z}_n \rangle$$

$$= -\int_U \varphi \frac{\partial u}{\partial z_j} \, dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n.$$ 

Thus the conditions $\partial \varphi \wedge \theta \equiv 0$ and $\partial \varphi \wedge \theta \wedge \bar{\theta} \equiv 0$ in $U$ are each equivalent to the condition

$$\left( \frac{\partial \varphi}{\partial z_j} \right)_{\text{distr.}} = \left( \frac{\partial \varphi}{\partial z_j} \right)_{\text{distr.}} = 0 \quad \text{for} \quad j = 2, \ldots, n.$$
Hence the above conditions are equivalent to the condition that \( \varphi \upharpoonright_U \) is a function of \( z_1 \), i.e., that \( \varphi \) is constant on the leaves of the holomorphic foliation determined by \( \theta \upharpoonright_U \).

Given an arbitrary point \( p \in X \), we may choose a connected relatively compact neighborhood \( U \) and a bounded holomorphic function \( f \) on \( U \) such that \( \theta \upharpoonright_U = df \), \( f(p) = 0 \), the fiber \( L \equiv f^{-1}(0) \) through \( p \) is connected, and \( \theta \) is nonvanishing on \( U \setminus L \). By continuity of intersections (see [Ste] or [TW] or Theorem 4.23 in [ABCKT]), after fixing a sequence of points \( \{x_{\nu}\} \) in \( U \setminus L \) converging to \( p \), letting \( L_{\nu} \) be the level of \( f \) through \( x_{\nu} \) for each \( \nu \), and passing to a subsequence, we get a sequence of analytic sets \( \{L_{\nu}\} \) converging to \( L \) in \( U \).

By the above, if \( \partial \varphi \wedge \theta \equiv 0 \) or \( \partial \varphi \wedge \theta \wedge \bar{\theta} \equiv 0 \) in \( U \), then \( \varphi \upharpoonright_{L_{\nu}} \) is constant for each \( \nu \), and hence \( \varphi \upharpoonright_L \) is constant. Conversely, assuming that \( \varphi \) is constant on each level of \( f \), let us fix a constant \( m < \inf_U \varphi \), and let us set

\[
\varphi_{\nu} \equiv \max(\varphi + \nu^{-1} \log |f|, m) \quad \text{for } \nu = 1, 2, 3, \ldots.
\]

Then \( \{\varphi_{\nu}\} \) is a uniformly bounded sequence of continuous functions converging pointwise almost everywhere to \( \varphi \upharpoonright_U \). Moreover, each term is constant on a neighborhood of \( L \) as well as on each level of \( f \). Applying the previous observations and the dominated convergence theorem, we see that for every form \( \alpha \in \mathcal{D}^{n-2,n}(U) \),

\[
0 = \langle \partial \varphi_{\nu} \wedge \theta, \alpha \rangle \to \langle \partial \varphi \wedge \theta, \alpha \rangle.
\]

Thus \( \partial \varphi \wedge \theta \equiv 0 \), and hence \( \partial \varphi \wedge \theta \wedge \bar{\theta} \equiv 0 \), on \( U \). The claim (a) now follows.

For the proof of (b), observe that each point in the complement of the zero set \( Z \) of \( \theta_1 \wedge \theta_2 \) admits a local holomorphic coordinate polydisk neighborhood \((U,(z_1,\ldots,z_n))\) such that \( dz_1 = \theta_1 \upharpoonright_U \) and \( dz_2 = \theta_2 \upharpoonright_U \). The restriction \( \varphi \upharpoonright_U \) is then both a function of \( z_1 \) and a function of \( z_2 \), and is therefore constant. Hence the nonconstant function \( \varphi \) is constant on the connected set \( X \setminus Z \), which is either dense or empty, so we must have \( Z = X \). \( \square \)

The main step in the proof of Theorem 4.2 is the following:

**Lemma 4.8.** Let \((X,g,\omega)\) be a connected noncompact complete Hermitian manifold of dimension \( n \), let \( K \) be a closed subset of \( X \), let \( \rho \) be a real-valued \( \mathcal{C}^\infty \) function on \( X \), let \( \beta \) be a \( \mathcal{C}^\infty \) \((1,1)\)-form on \( X \), let \( \gamma \equiv i\partial \bar{\partial} \rho \wedge \beta \wedge \omega^{n-2} \), and let \( R \) be a bounded positive continuous function on \( X \). Assume that

(i) \( |d\rho|_g \) is bounded on \( X \), \( i\partial \bar{\partial} \rho \geq -R \omega \) on \( X \), and \( \rho \) is locally constant on some neighborhood of \( K \); and
(ii) We have \( \beta|_{X\backslash K} \geq 0 \), \( d(\beta \wedge \omega^{n-2})|_{X\backslash K} \equiv 0 \), and \( \beta|_{X\backslash K} \) is in \( L^1 \).

Then

\[
\int_X \gamma^+ = \int_X \gamma^- \leq \int_{X\backslash K} \frac{R}{\sqrt{n}} |\beta|_g \omega^n < \infty
\]

(where at each point, \( \gamma^\pm \equiv (\gamma/\omega_n^0)^\pm \omega_n^0 \) for any positive \((1,1)\)-form \( \omega_0^0 \)). In particular, \( \gamma \) is integrable, \( \int_X \gamma = 0 \), and

\[
\|\gamma\|_{L^1} = \int_X \gamma^+ + \int_X \gamma^- \leq \int_{X\backslash K} \frac{2R}{\sqrt{n}} |\beta|_g \omega^n.
\]

Proof. As in [Ga], fixing a point \( p \in X \) and setting

\[
\tau(s) \equiv \begin{cases} 
1 & \text{if } s \leq 1 \\
2 - s & \text{if } 1 < s < 2 \\
0 & \text{if } 2 \leq s
\end{cases}
\]

and

\[
\tau_r(x) \equiv \tau \left( \frac{\text{dist}(p,x)}{r} \right) \quad \forall x \in X, r > 0,
\]

we get a collection of nonnegative Lipschitz continuous functions \( \{\tau_r\}_{r>0} \) such that for each \( r > 0 \), we have \( 0 \leq \tau_r \leq 1 \) on \( X \), \( \tau_r \equiv 1 \) on \( B(p;r) \), \( \tau_r \equiv 0 \) on \( X \backslash B(p;2r) \), and \( |d\tau_r|_g \leq 1/r \).

We then have

\[
\int_X 2\tau_r \gamma = \int_X \tau_r d^c \rho \wedge \beta \wedge \omega^{n-2} = -\int_{(B(p;2r))\backslash B(p;r)\backslash K} d\tau_r \wedge d^c \rho \wedge \beta \wedge \omega^{n-2},
\]

where \( d^c = -i(\partial - \bar{\partial}) \). For some positive constant \( C \), we have

\[
|d\tau_r \wedge d^c \rho \wedge \beta \wedge \omega^{n-2}|_g \leq \frac{C}{r} |\beta|_g,
\]

and hence \( \int_X \tau_r \gamma \rightarrow 0 \) as \( r \rightarrow \infty \). Since \( i\partial \bar{\partial} \rho \geq -R\omega \), Lemma 4.4 implies that on \( X \backslash K \),

\[
\gamma \geq -R\beta \wedge \omega^{n-1} \geq -\frac{R}{\sqrt{n}} |\beta|_g \omega^n.
\]

Applying the monotone convergence theorem, we get

\[
\int_X \gamma^+ = \lim_{r \to \infty} \int_X (\tau_r \gamma)^+ = \lim_{r \to \infty} \int_X (\tau_r \gamma)^- = \int_X \gamma^- \leq \int_{X\backslash K} \frac{R}{\sqrt{n}} |\beta|_g \omega^n.
\]

\[\square\]

Proof of Theorem 4.2. Let \( \{\delta_{ij}\}_{\nu \in \mathbb{Z}_{>0}, j \in J} \) be a family of bounded positive continuous functions on \( X \), the elements of which we will later choose to be sufficiently small. For each
$\nu \in \mathbb{Z}_{>0}$ and each $j \in J$, the $C^\infty$ approximation theorem of Greene and Wu (Theorem 3.1) provides a $C^\infty$ function $\varphi_{\nu j}$ such that $\varphi_{\nu j} = \varphi_j$ on a neighborhood of $K$, and

$$\varphi_j \leq \varphi_{\nu j} < \varphi_j + \delta_{\nu j} \quad \text{and} \quad \mathcal{L}(\varphi_{\nu j}) \geq -\delta_{\nu j} g_0 \quad \text{on } X.$$ 

So that we may work with positive functions, let us set $\psi_{\nu j} \equiv e^{\varphi_{\nu j}}$ for each $\nu \in \mathbb{Z}_{>0}$ and $j \in J$. We will obtain the required $C^\infty$ function $\psi$ as a weighted sum of the squares of the functions $\{\psi_{\nu j}\}$. By Lemma 4.3, the liftings of the functions $\{\psi_{\nu j}\}$ to a covering space will then have bounded gradient with respect to the associated modified metric. While these functions are not plurisubharmonic, Lemma 4.8 will imply that the cup product property holds in an approximate sense, and we will then pass to a limit.

For each $\nu \in \mathbb{Z}_{>0}$ and $j \in J$, we have

$$i\partial \bar{\partial} \psi_{\nu j} = e^{\varphi_{\nu j}} \left( i\partial \bar{\partial} \varphi_{\nu j} + i\partial \varphi_{\nu j} \wedge \bar{\partial} \varphi_{\nu j} \right) \geq -e^{\varphi_j + \delta_{\nu j}} \delta_{\nu j} \omega_0$$

and

$$i\partial \bar{\partial} \psi_{\nu j}^2 = 2\psi_{\nu j} i\partial \bar{\partial} \psi_{\nu j} + 2i\partial \psi_{\nu j} \wedge \bar{\partial} \psi_{\nu j} \geq -2e^{2\varphi_j + 2\delta_{\nu j}} \delta_{\nu j} \omega_0 + 2i\partial \psi_{\nu j} \wedge \bar{\partial} \psi_{\nu j} \geq -2e^{2\varphi_j + 2\delta_{\nu j}} \delta_{\nu j} \omega_0.$$ 

Hence, choosing $\delta_{\nu j}$ so small that $0 < 2e^{2\varphi_j + 2\delta_{\nu j}} \delta_{\nu j} < \delta < 1$, we get

$$i\partial \bar{\partial} \psi_{\nu j}^2 \geq -\delta \omega_0 + 2i\partial \psi_{\nu j} \wedge \bar{\partial} \psi_{\nu j} \geq -\delta \omega_0.$$ 

If $\{\epsilon_{\nu j}\}_{\nu \in \mathbb{Z}_{>0}, j \in J}$ is a family of sufficiently small positive constants (depending on the choice of $\{\delta_{\nu j}\}$), then $\sum_{\nu, j} \epsilon_{\nu j} < 1$; the series

$$\sum_{\nu \in \mathbb{Z}_{>0}, j \in J} \epsilon_{\nu j} \psi_{\nu j}^2 = \sum_{\nu \in \mathbb{Z}_{>0}, j \in J} \epsilon_{\nu j} e^{2\varphi_{\nu j}}$$

converges to a nonnegative $C^\infty$ function $\psi$; for each $k = 0, 1, 2, \ldots$, any term-by-term $k$th order derivative series for the above series converges uniformly and absolutely on compact subsets of $X$ to the corresponding $k$th order derivative of $\psi$; and each of the series

$$\sum_{\nu, j} \epsilon_{\nu j} \psi_{\nu j} i\partial \bar{\partial} \psi_{\nu j} \quad \text{and} \quad \sum_{\nu, j} \epsilon_{\nu j} i\partial \psi_{\nu j} \wedge \bar{\partial} \psi_{\nu j}$$

converges uniformly on compact subsets of $X$ to a continuous real $(1, 1)$-form. Hence

$$i\partial \bar{\partial} \psi = \sum_{\nu, j} \epsilon_{\nu j} i\partial \bar{\partial} \psi_{\nu j}^2 \geq -\delta \omega_0 + \sum_{\nu, j} 2\epsilon_{\nu j} i\partial \psi_{\nu j} \wedge \bar{\partial} \psi_{\nu j} \geq -\delta \omega_0.$$

Let us now fix a connected covering space $\hat{\mathcal{X}} \rightarrow X$, a Hermitian metric $g \geq \hat{g}_0 \equiv \Upsilon^*g_0$, and a constant $\epsilon \in (0, 1)$. Let $\dot{\psi} \equiv \Upsilon^*\dot{\psi}$, let $h \equiv g + \epsilon \mathcal{L}(\dot{\psi})$, let $\hat{K} \equiv \Upsilon^{-1}(K)$, let $\dot{\varphi}_j \equiv \Upsilon^*\varphi_j$ for each $j \in J$, let $\dot{\varphi}_{\nu j} \equiv \Upsilon^*\varphi_{\nu j}$, $\dot{\psi}_{\nu j} \equiv \Upsilon^*\psi_{\nu j} = e^{\dot{\varphi}_{\nu j}}$, and $\delta_{\nu j} \equiv \Upsilon^*\delta_{\nu j}$ for all $\nu \in \mathbb{Z}_{>0}$ and $j \in J$, let $\delta \equiv \Upsilon^*\delta$, and let $\omega_{\hat{g}_0}$, $\omega_g$, and $\omega_h = \omega_g + \epsilon i\partial\bar{\partial}\dot{\psi}$ be the $(1,1)$-forms associated to $\hat{g}_0$, $g$, and $h$, respectively. Suppose $\beta$ is a $\mathcal{C}^\infty$ $(1,1)$-form on $\hat{\mathcal{X}}$ for which $\beta\big|_{\hat{\mathcal{X}} \setminus \hat{K}} \geq 0$, $d(\beta \wedge \omega_h^{n-2})\big|_{\hat{\mathcal{X}} \setminus \hat{K}} \equiv 0$, and $\beta\big|_{\hat{\mathcal{X}} \setminus \hat{K}}$ is in $L^1$ with respect to $h$. For each $\nu \in \mathbb{Z}_{>0}$ and $j \in J$, we have

\[\omega_h - \epsilon \nu_j i\partial\dot{\psi}_{\nu j} \wedge \bar{\partial}\dot{\psi}_{\nu j} \geq \omega_{\hat{g}_0} - \epsilon \delta \omega_{\hat{g}_0} > 0,\]

and therefore, by Lemma 4.3,

\[|d\dot{\psi}_{\nu j}|_h \leq \sqrt{\frac{2}{\epsilon \nu_j}}.\]

Since

\[i\partial\bar{\partial}\dot{\psi}_{\nu j} \geq -e^{\dot{\varphi}_j + \delta_{\nu j}}\delta_{\nu j}\omega_{\hat{g}_0} \geq -e^{\dot{\varphi}_j + \delta_{\nu j}}\delta_{\nu j}(1 - \epsilon)^{-1}\omega_h,\]

Lemma 4.8 implies that

\[\int_{\hat{\mathcal{X}}} \left[ i\partial\bar{\partial}\dot{\psi}_{\nu j} \wedge \beta \wedge \omega_h^{n-2} \right]^+ = \int_{\hat{\mathcal{X}}} \left[ i\partial\bar{\partial}\dot{\psi}_{\nu j} \wedge \beta \wedge \omega_h^{n-2} \right]^- \leq \int_{\hat{\mathcal{X}} \setminus \hat{K}} \frac{e^{\dot{\varphi}_j + \delta_{\nu j}}\delta_{\nu j}}{(1 - \epsilon)\sqrt{n}} |\beta|_h \omega_h^n.\]

Thus if we choose the sequence of functions $\{\delta_{\nu j}\}_{\nu=1}^\infty$ for each $j \in J$ so that $e^{\dot{\varphi}_j + \delta_{\nu j}}\delta_{\nu j} \rightarrow 0$ uniformly on $X$ as $\nu \rightarrow \infty$, then the sequence of $\mathcal{C}^\infty$ forms $\{i\partial\bar{\partial}\dot{\psi}_{\nu j} \wedge \beta \wedge \omega_h^{n-2}\}_{\nu=1}^\infty$ must converge to 0 in $L^1$ (with respect to any Hermitian metric, since the forms are of type $(n,n)$). Applying Lemma 4.4, we also get

\[\int_{\hat{\mathcal{X}}} \frac{1}{n!} |(i\partial\bar{\partial}\dot{\psi}_{\nu j})^- \wedge \beta|_h \omega_h^n \leq \int_{\hat{\mathcal{X}}} \frac{1}{(n - 2)!} |(i\partial\bar{\partial}\dot{\psi}_{\nu j})^- \wedge \beta \wedge \omega_h^{n-2} \leq \int_{\hat{\mathcal{X}} \setminus \hat{K}} \frac{e^{\dot{\varphi}_j + \delta_{\nu j}}\delta_{\nu j}}{(n - 2)!(1 - \epsilon)} |\beta| \omega_h^{n-1} \leq \int_{\hat{\mathcal{X}} \setminus \hat{K}} \frac{e^{\dot{\varphi}_j + \delta_{\nu j}}\delta_{\nu j}}{(n - 2)!(1 - \epsilon)\sqrt{n}} |\beta|_h \omega_h^n \rightarrow 0,\]

and hence, by Lemma 4.6 the sequence of $\mathcal{C}^\infty$ forms $\{i\partial\bar{\partial}\dot{\psi}_{\nu j} \wedge \beta\}_{\nu=1}^\infty$ converges to 0 in $L^1_{\text{loc}}$. In order to obtain the same property for $\{\dot{\varphi}_{\nu j}\}$, observe that

\[\dot{\psi}_{\nu j}^{-1} i\partial\bar{\partial}\dot{\psi}_{\nu j} \wedge \beta = \left[ (i\partial\bar{\partial}\dot{\varphi}_{\nu j})^+ + i\partial\dot{\varphi}_{\nu j} \wedge \bar{\partial}\dot{\varphi}_{\nu j} \right] \wedge \beta - (i\partial\bar{\partial}\dot{\varphi}_{\nu j})^- \wedge \beta \quad \forall \nu,
\]

and since the functions $\{\dot{\varphi}_{\nu j}\}_{\nu=1}^\infty$ are uniformly bounded on compact sets, the left hand side converges to 0 in $L^1_{\text{loc}}$. Since $\{\delta_{\nu j}\}$ converges to 0 uniformly on compact sets, the sequence
\( \{ (i\bar{\partial}\hat{\phi}_{\nu j})^- \wedge \beta \}_{\nu = 1}^{\infty} \) must also converge to 0 in \( L^1_{\text{loc}} \), and therefore so must the sequence

\[
\left\{ \left[ (i\bar{\partial}\hat{\phi}_{\nu j})^- + i\partial\hat{\phi}_{\nu j} \wedge \bar{\partial}\hat{\phi}_{\nu j} \right] \wedge \beta \right\}.
\]

Hence by Lemma 4.4, the sequence

\[
\left\{ \left[ (i\bar{\partial}\hat{\phi}_{\nu j})^- + i\partial\hat{\phi}_{\nu j} \wedge \bar{\partial}\hat{\phi}_{\nu j} \right] \wedge \beta \wedge \omega_h^{n-2} \right\},
\]

and therefore, the sequences

\[
\left\{ (i\bar{\partial}\hat{\phi}_{\nu j})^+ \wedge \beta \wedge \omega_h^{n-2} \right\} \quad \text{and} \quad \left\{ (i\partial\hat{\phi}_{\nu j} \wedge \bar{\partial}\hat{\phi}_{\nu j}) \wedge \beta \wedge \omega_h^{n-2} \right\},
\]

must converge to 0 in \( L^1_{\text{loc}} \). Applying Lemma 4.6 we see that the sequences

\[
\left\{ (\partial\hat{\phi}_{\nu j}) \wedge \beta \right\} \quad \text{and} \quad \left\{ (\hat{\phi}_{\nu j}) \wedge \beta \right\},
\]

must also converge to 0 in \( L^1_{\text{loc}} \) and \( L^2_{\text{loc}} \), respectively. Since \( \hat{\phi}_{\nu j} \to \hat{\phi}_j \) uniformly on compact sets, it follows that \((\partial\hat{\phi}_j) \wedge \beta = 0 \) and \((\hat{\phi}_j) \wedge \beta = 0 \) for each \( j \in J \). Moreover, if on some open set \( U \), \( \beta = i\theta \wedge \bar{\theta} \) for some \( C^\infty (1,0) \)-form \( \theta \), then Lemma 4.6 implies that for each \( j \in J \), \{\( \partial\hat{\phi}_{\nu j} \) \wedge \theta\} converges to 0 in \( L^2_{\text{loc}} \) in \( U \), and hence \( (\partial\hat{\phi}_j) \wedge \theta = 0 \). The remaining claims of the theorem follow from Lemma 4.7. \( \Box \)

5. Strict plurisubharmonicity and the Bochner–Hartogs property

The main goal of this section is a proof of Theorem 0.2, which we will obtain as an application of Theorem 0.1. The two main ingredients of the proof are Theorem 0.1 and the following:

**Theorem 5.1** (see Theorem 3.3 of [NR7]). Let \( X \) be a connected noncompact hyperbolic complete Kähler manifold with no nontrivial \( L^2 \) holomorphic 1-forms.

(a) For every compactly supported \( \bar{\partial} \)-closed \( C^\infty \) form \( \alpha \) of type \((0,1)\) on \( X \), there exists a bounded \( C^\infty \) function \( \beta \) with finite energy on \( X \) such that \( \bar{\partial}\beta = \alpha \) on \( X \) and \( \beta \) vanishes on every hyperbolic end \( E \) of \( X \) that is contained in \( X \setminus \text{supp} \alpha \).

(b) In any ends decomposition \( X \setminus K = E_1 \cup \cdots \cup E_m \), exactly one of the ends, say \( E_1 \), is hyperbolic, and moreover, every holomorphic function on \( E_1 \) admits a (unique) extension to a holomorphic function on \( X \).

(c) If \( e(X) = 1 \) (equivalently, every end of \( X \) is hyperbolic), then \( H^1_c(X, \mathcal{O}) = 0 \).

For the proof of Theorem 0.2 we will apply Theorem 5.1 to suitable sublevels of \( \varphi \). In order to complete the Kähler metric on these sublevels, we will apply Lemma 5.2 and
Lemma 5.3 below, which are contained implicitly in the work of Nakano [Nk], Greene and Wu [GreW], and Demailly [Dem1]:

**Lemma 5.2** (cf. [Nk] and [Dem1]). Let \((X, g, \omega_g)\) be a connected Hermitian manifold with distance function \(d_g(\cdot, \cdot)\), let \(\epsilon \in (0, 1)\), let \(\delta: X \to (0, \epsilon]\) be a continuous function, and let \(U\) be an open subset.

(a) If \(Y\) is a domain in \(X\) for which the restricted distance function \(d_g|_{Y \setminus U}\) is complete and \(\psi\) is a \(C^\infty\) function on \(Y\) such that \(\psi < 0, \psi \to 0\) at \(\partial Y\), and \(L(\psi) \geq \delta \psi g\), then

\[
h \equiv g + L(-\log(-\psi)) \geq (1 - \delta)g
\]

is a Hermitian metric on \(Y\) (which is Kähler if \(g|_Y\) is Kähler) for which the restriction \(d_h|_{Y \setminus U}\) of the associated distance function \(d_h(\cdot, \cdot)\) in \(Y\) is complete.

(b) Suppose \(\psi\) is a positive \(C^\infty\) function on \(X\) such that \(2\psi L(\psi) \geq -\delta g\) on \(X\) and such that there exists a nonempty closed connected set \(K \subset X\) for which \(\psi\) is bounded on \(K\) and for every \(a > \sup_K \psi\), the restriction \(d_g|_{Y \setminus U}\) of \(d_g(\cdot, \cdot)\), where \(Y\) is the connected component of \(\{ x \in X \mid \psi(x) < a \}\) containing \(K\), is complete. Then

\[
h \equiv g + L(\psi^2) \geq (1 - \delta)g
\]

is a Hermitian metric on \(X\) (which is Kähler if \(g\) is Kähler) for which the restriction \(d_h|_{X \setminus U}\) of the associated distance function \(d_h(\cdot, \cdot)\) in \(X\) is complete.

**Remark.** While it is convenient to have part (b) stated in the above form, if the condition holds for some choice of \(K\), then it holds for any nonempty closed connected set on which \(\psi\) is bounded.

**Proof of Lemma 5.2.** For the proof of part (a), observe that the \((1, 1)\)-form associated to \(h\) is given by

\[
\omega_h \equiv \omega_g + i \partial \overline{\partial}(-\log(-\psi)) = \omega_g - \psi^{-1} i \partial \overline{\partial} \psi + \psi^{-2} i \partial \psi \wedge \overline{\partial} \psi \geq (1 - \delta) \omega_g + \psi^{-2} i \partial \psi \wedge \overline{\partial} \psi \geq (1 - \epsilon) \omega_g.
\]

Given two points \(p, q \in Y\) and a \(C^\infty\) path \(\gamma\) in \(Y\) from \(p\) to \(q\), we have

\[
\ell_h(\gamma) = \int_0^1 |\dot{\gamma}(t)|_h dt \geq \int_0^1 \frac{1}{\sqrt{2(\psi(\gamma(t)))}} \left| \frac{d\psi(\gamma(t))}{\psi(\gamma(t))} \right| dt.
\]

Hence \(\text{dist}_h(p, q) \geq (1 - \epsilon) \text{dist}_g(p, q)\) and \(\text{dist}_h(p, q) \geq \frac{1}{\sqrt{2}} \left| \log \left( \frac{\psi(q)}{\psi(p)} \right) \right|\), and the claim follows. The proof of part (b) is similar. \(\square\)
Lemma 5.3 (cf. [Nk] and [Dem1]). Suppose \((X, g, \omega_g)\) is a connected noncompact Kähler manifold with distance function \(d_g(\cdot,\cdot)\), \(E\) is an end of \(X\), \(U \subset X\) is an open set, \(K\) is a nonempty closed set that contains \(X \setminus E\), and \(\delta: X \to (0,1)\) is a continuous function.

(a) If \(\varphi\) is a continuous plurisubharmonic on \(X\), \(a\) is a constant with \(\sup_E \varphi > a > \sup_{K \cap \overline{E}} \varphi\), and \(Y\) is a connected component of \((X \setminus E) \cup \{x \in E \mid \varphi(x) < a\}\) for which \(Y \supset K\) and \(d_g|_{\overline{Y} \setminus U}\) is complete, then there exists a Kähler metric \(h\) with distance function \(d_h(\cdot,\cdot)\) on \(Y\) such that \(h \geq (1 - \delta) g\) on \(Y\), \(h = g\) at each point in \(K\), and \(d_h|_{\overline{Y} \setminus U}\) is complete.

(b) If \(K \cap \overline{E}\) is compact, \(d_g|_{X \setminus (E \cup U)}\) is complete, and there exists a continuous plurisubharmonic function \(\varphi\) on \(X\) that exhausts \(\overline{E}\), then there exists a Kähler metric \(h\) with distance function \(d_h(\cdot,\cdot)\) on \(X\) such that \(h \geq (1 - \delta) g\) on \(X\), \(h = g\) at each point in \(K\), and \(d_h|_{\overline{E} \cup (X \setminus U)}\) is complete.

Proof. For the proof of part (a), let us fix a constant \(b\) with \(\sup_E \varphi > a > b \geq \sup_{K \cap \overline{E}} \varphi\). Then the function \(\rho\) defined by

\[
\rho \equiv \begin{cases} 
\max(\varphi - a, b - a) & \text{on } Y \cap \overline{E} \\
 b - a & \text{on } K 
\end{cases}
\]

is a negative continuous plurisubharmonic function on \(Y\) that approaches 0 at \(\partial Y\) and is equal to \(b - a\) on a neighborhood of \(K\). Fixing a continuous function \(\eta\) with \(\rho < \eta < 0\) on \(Y\), the \(C^\infty\) approximation theorem of Greene and Wu (Theorem 3.1) then provides a \(C^\infty\) function \(\psi\) on \(Y\) such that \(\psi \equiv b - a\) on a neighborhood of \(K\), \(\rho \leq \psi < \eta < 0\) on \(Y\), and \(\mathcal{L}(\psi) \geq \frac{1}{2}\delta \eta g \geq \frac{1}{2}\delta \psi g\). By Lemma 5.2 the Hermitian metric \(h \equiv g + \mathcal{L}(-\log(-\psi))\) has the required properties. The proof of part (b) is similar. \(\blacksquare\)

We will actually prove a more general version of Theorem 0.1 for a manifold with multiple ends. For this, it will be convenient to first recall some terminology.

Definition 5.4. For \(S \subset X\) and \(k\) a nonnegative integer, we will say that a Hermitian manifold \((X, g)\) of dimension \(n\) has bounded geometry of order \(k\) along \(S\) if for some constant \(C > 0\) and for every point \(p \in S\), there is a biholomorphism \(\Psi\) of the unit ball \(B \equiv B_{g_{\mathbb{C}^n}}(0; 1) \subset \mathbb{C}^n\) onto a neighborhood of \(p\) in \(X\) such that \(\Psi(0) = p\) and such that on \(B\),

\[
C^{-1} g_{\mathbb{C}^n} \leq \Psi^* g \leq C g_{\mathbb{C}^n} \quad \text{and} \quad |D^m \Psi^* g| \leq C \quad \text{for } m = 0, 1, 2, \ldots, k.
\]
Definition 5.5 (cf. Definition 2.2 of [NR5]). We will call an end $E$ of a connected non-compact complete Hermitian manifold $X$ special if $E$ is of at least one of the following types:

(BG) $X$ has bounded geometry of order 2 along $E$;
(W) There exists a continuous plurisubharmonic function $\varphi$ on $X$ such that

$$\{ x \in E \mid \varphi(x) < a \} \subset X \quad \forall a \in \mathbb{R};$$

(RH) $E$ is a regular hyperbolic end (i.e., $E$ is a hyperbolic end and the Green’s function on $X$ vanishes at infinity along $E$); or

(SP) $E$ is a parabolic end, the Ricci curvature of $g$ is bounded below on $E$, and there exist positive constants $R$ and $\delta$ such that $\text{vol} \left( B(x; R) \right) > \delta$ for all $x \in E$.

We will call an ends decomposition in which each of the ends is special a special ends decomposition.

While bounded geometry and special ends do not play fundamental roles in this paper, these conditions have been shown to strongly determine the holomorphic structure of complete Kähler manifolds. In particular, according to [Gro1], [Li], [Gro2], [GroS], [NR1], [DelG], [NR5], and [NR6], a connected noncompact complete Kähler manifold that admits a special ends decomposition and has at least three (filtered) ends admits a proper holomorphic mapping onto a Riemann surface. Applications of the versions of the cup product lemma from Section 4 in the above context will be considered elsewhere (see [NR8]).

Theorem 0.2 is an immediate consequence of the following (cf. Proposition 4.4 of [NR3]):

Theorem 5.6. Let $(X, g)$ be a connected noncompact complete Kähler manifold that admits a continuous plurisubharmonic function $\varphi$ whose restriction to some 2-dimensional germ of an analytic set at some point is strictly plurisubharmonic.

(a) For every compactly supported $\bar{\partial}$-closed $C^\infty$ form $\alpha$ of type $(0, 1)$ on $X$, there exists a bounded $C^\infty$ function $\beta$ on $X$ such that $\bar{\partial}\beta = \alpha$ on $X$ and $\beta$ vanishes on every end $E$ of $X$ for which $E \cap \text{supp} \alpha = \emptyset$ and $\sup_E \varphi > \sup_{\partial E} \varphi$ (here, we take $\sup \emptyset = -\infty$).

(b) In any ends decomposition $X \setminus K = E_1 \cup \cdots \cup E_m$ with $K \neq \emptyset$, $\sup_{E_j} \varphi > \sup_{\partial E_j} \varphi$ for exactly one choice of $j$, say $j = 1$, the remaining ends $E_2, \ldots, E_m$ are parabolic and not special, and every holomorphic function on $E_1$ admits a (unique) extension to a holomorphic function on $X$. Moreover, for every nonconstant continuous
plurisubharmonic function $\psi$ on $X$, $\sup_{E_j} \psi > \sup_{\partial \Omega E_j} \psi$ while $\sup_{E_j} \psi = \sup_{\partial \Omega E_j} \psi$ for $j = 2, \ldots, m$.

(c) If $e(X) = 1$ (i.e., $\sup_E \varphi > \sup_{\partial \Omega} \varphi$ for every end $E$), then $H^1_c(X, \mathcal{O}) = 0$.

Proof. Let $n \equiv \dim X$. It follows from the condition on $\varphi$ that there exists a 2-dimensional connected complex submanifold $Z$ of a nonempty relatively compact domain $U$ in $X$ such that $\varphi|_Z$ is strictly plurisubharmonic, and by replacing $\varphi$ with $e^\varphi$ if necessary, we may assume that $\varphi > 0$ on $X$. Given an ends decomposition $X \setminus K = E_1 \cup \cdots \cup E_m$ for $X$ with $K \neq \emptyset$, which we may order so that $\sup_{E_1} \varphi = \sup \varphi$, part (a) of Lemma 1.1 provides a connected compact set $K'$ such that $K \cup U \subset K'$ and any domain $Y \supset K'$ has an ends decomposition $Y \setminus K = E_1' \cup \cdots \cup E_m'$, where $E_j' = E_j \cap Y$ for $j = 1, \ldots, m$. Fixing a constant $a$ with $\max_{K'} \varphi < a < \sup \varphi$, we may let the above domain $Y$ be the connected component of $\{ x \in X \mid \varphi(x) < a \}$ containing $K'$.

Lemma 5.3 and Theorem 0.1 (or Theorem 4.1) together provide a complete Kähler metric $h$ on $Y$ such that $\partial \varphi \wedge \theta \equiv 0$ for every $L^2$ (closed) holomorphic 1-form $\theta$ on $(Y, h)$. However, if $f$ is a nonconstant holomorphic function on a nonempty domain $W \subset U$, $p \in W \cap Z$, and $L$ is the level of $f$ through $p$, then $L \cap Z$ is an analytic set of positive dimension at $p$ and $\varphi|_{(L \cap Z)}$ is strictly plurisubharmonic. It follows from Lemma 4.7 that $\partial \varphi \wedge df$ is not everywhere 0 in $W$, and hence $(Y, h)$ has no nontrivial $L^2$ holomorphic 1-forms.

Applying part (b) of Theorem 5.1 we see that for $j = 2, \ldots, m$, $E_j'$ is a parabolic end of $(Y, h)$, and hence $E_j = E_j'$ and $\sup_{E_j} \varphi \leq \max_{\partial \Omega E_j} \varphi$. If $f \in \mathcal{O}(E_1)$, then there exists a holomorphic function $u_0$ on $Y \supset (X \setminus E_1) \cup K'$ such that $u_0 = f$ on $E_1' = E_1 \cap Y$, and hence the function $u$ on $X$ given by $u \equiv f$ on $E_1$ and $u \equiv u_0$ on $Y$ is a holomorphic extension of $f$ to $X$. Furthermore, if there exists a continuous plurisubharmonic function $\psi$ on $X$ with $\sup_{E_j} \psi > \max_{\partial \Omega E_j} \psi$ for some $j > 1$, then after replacing $\psi$ with the composition of a suitable nondecreasing convex function and $\psi$, we may assume that $\sup_{E_j} \psi = \infty$. But then for $\epsilon > 0$ sufficiently small, the plurisubharmonic function $\varphi_0 \equiv \varphi + \epsilon \psi$, which satisfies the condition placed on $\varphi$ in the statement of the theorem, must also satisfy $\sup_{E_j} \varphi_0 > \max_{\partial \Omega E_j} \varphi_0$ and $\sup_{E_j} \varphi_0 = \infty = \sup \varphi_0$, which as we have seen, is impossible. Thus such a function $\psi$ cannot exist.

Part (b) will follow if we prove that for each $j = 2, \ldots, m$, with respect to $g$ in $X$, $E_j$ is neither a hyperbolic end nor a special end. For this, let us first fix an ends decomposition
$X \setminus C = A_0 \cup A_1$ such that $A_0 \subset E_j$, $E_j \setminus A_0 \subset X$, and $A_1 \supset (E_1 \cup \cdots \cup \hat{E}_j \cup \cdots \cup E_m)$, as provided by part (d) of Lemma 1. We may also choose the above domain $Y$ so large that $C \subset Y$ and we have the ends decomposition $Y \setminus C = A_0 \cup A_1'$, where $A_1' = A_1 \cap Y$ (and $A_0 = A_0 \cap Y$). Applying Lemma 5.3 we get a complete Kähler metric $k$ on $Y$ such that $k = g$ at each point in the closed set $(X \setminus E_1) \cup K' \supset A_0$. In particular, $A_1'$ is a hyperbolic end in $(Y, k)$, and if $E_j$ is a hyperbolic end or a special end of type (SP) with respect to $g$ (the latter holding if, for example, $E_j$ is parabolic and of type (BG)), then $A_0$ must be hyperbolic or of type (SP) with respect to $k$, and by applying Theorem 3.6 of [NR6] in $(Y, k)$, we are able to produce a nonnegative nonconstant $C^\infty$ plurisubharmonic function on $Y$ that vanishes on $Y \setminus E_j$ and in particular, extends by 0 to a $C^\infty$ plurisubharmonic function $\psi$ on $X$ with $\sup_{E_j} \psi > 0 = \max_{\partial E_j} \psi$. But as we have seen, such a function $\psi$ cannot exist. Similarly, $E_j$ cannot be of type (W). Thus part (b) is proved.

For the proof of part (a), letting $\alpha$ be a nontrivial compactly supported $\bar{\partial}$-closed $C^\infty$ form of type $(0, 1)$ on $X$, we may choose the above compact set $K$ to contain $\text{supp} \alpha$. Part (a) of Theorem 5.3 then provides a bounded $C^\infty$ function $\beta_1$ with finite $h$-energy such that $\bar{\partial} \beta_1 = \alpha$ on $Y \supset K$ and $\beta_1 \equiv 0$ on any hyperbolic end of $Y$ that does not meet $\text{supp} \alpha$. In particular, $\beta_1 \equiv 0$ on $E_1'$, so $\beta_1$ extends to a bounded $C^\infty$ function $\beta$ on $X = Y \cup E_1$ that vanishes on $E_1$. Furthermore, if $E \subset X \setminus \text{supp} \alpha$ is an end of $X$ and $\sup_{E} \varphi > \sup_{\partial E} \varphi$, then there is end $E_0$ of $X$ such that $E_0 \subset E_j \cap E$ for some $j$ and $\sup_{E_0} \varphi > \sup_{\partial E_0} \varphi$. On the one hand, since $E_l$ is a parabolic end and $\varphi$ is bounded on $E_l$ for each $l = 2, \ldots, m$, we must have $j = 1$; i.e., $E_0 \subset E_1 \cap E$. On the other hand, by the above, there exists a bounded $C^\infty$ function $\beta'$ on $X$ such that $\bar{\partial} \beta' = \alpha$ and $\beta' \equiv 0$ on $E$. The holomorphic function $\beta - \beta'$ vanishes on $E_0$, and therefore on $X$, so we must have $\beta = \beta'$, and hence $\beta \equiv 0$ on $E$.

Part (c) follows from parts (a) and (b).

\begin{thebibliography}{99}

[ABCKT] J. Amorós, M. Burger, K. Corlette, D. Kotschick, D. Toledo, \textit{Fundamental groups of compact Kähler manifolds}, Math. Surveys and Monographs, 44, American Mathematical Society, Providence, RI, 1996.

[AnV] A. Andreotti, E. Vesentini, \textit{Carleman estimates for the Laplace-Beltrami equation on complex manifolds}, Inst. Hautes Études Sci. Publ. Math. 25 (1965), 81–130.

[Bo] S. Bochner, \textit{Analytic and meromorphic continuation by means of Green’s formula}, Ann. of Math. 44 (1943), 652–673.
\end{thebibliography}
[DelG] T. Delzant, M. Gromov, *Cuts in Kähler groups*, Infinite groups: geometric, combinatorial and dynamical aspects, Proceedings of the conference in honor of R. Grigorchuk, 31–55, Progr. Math., 248, Birkhäuser, Basel, 2005.

[Dem1] J.-P. Demailly, *Estimations $L^2$ pour l’opérateur $\bar{\partial}$ d’un fibré vectoriel holomorphe semi-positif au-dessus d’une variété Kählerienne complète*, Ann. Sci. École Norm. Sup. 15 (1982), 457–511.

[Dem2] J.-P. Demailly, *Cohomology of $q$-convex spaces in top degrees*, Math. Z. 204 (1990), 283–295.

[Dem3] J.-P. Demailly, *Regularization of closed positive currents and intersection theory*, J. Algebraic Geom. 1 (1992), no. 3, 361–409.

[Ga] M. Gaffney, *A special Stokes theorem for Riemannian manifolds*, Ann. of Math. 60 (1954), 140–145.

[GraR] H. Grauert, O. Riemenschneider, *Kählersche Mannigfaltigkeiten mit hyper-$q$-konvexen Rand*, Problems in analysis (A Symposium in Honor of S. Bochner, Princeton 1969), Princeton University Press, Princeton, 1970, pp. 61–79.

[GreW] R. E. Greene, H. Wu, $C^\infty$ approximation of convex, subharmonic, and plurisubharmonic functions, Ann. Sci. École Norm. Sup. 12 (1979), 47–84.

[Har] F. Hartogs, *Zur Theorie der analytischen Functionen mehrerer unabhangiger Veränderlichen insbesondere über die Darstellung derselben durch Reihen, welche nach Potenzen einer Veränderlichen fortschreiten*, Math. Ann. 62 (1906), 1–88.

[HavL] F. R. Harvey, H. B. Lawson, *Boundaries of complex analytic varieties I*, Math. Ann. 102 (1975), 223–290.

[JNR] C. Joita, T. Napier, M. Ramachandran, *Generically $q$-convex complex spaces*, in preparation.

[Li] P. Li, *On the structure of complete Kähler manifolds with nonnegative curvature near infinity*, Invent. Math. 99 (1990), 579–600.

[Nak] S. Nakano, *Vanishing theorems for weakly 1-complete manifolds II*, Publ. R.I.M.S. Kyoto 10 (1974), 101–110.

[NR1] T. Napier, M. Ramachandran, *Structure theorems for complete Kähler manifolds and applications to Lefschetz type theorems*, Geom. Funct. Anal. 5 (1995), 809–851.

[NR2] T. Napier, M. Ramachandran, *The Bochner–Hartogs dichotomy for weakly 1-complete Kähler manifolds*, Ann. Inst. Fourier (Grenoble) 47 (1997), 1345–1365.

[NR3] T. Napier, M. Ramachandran, *The $L^2$ $\bar{\partial}$-method, weak Lefschetz theorems, and the topology of Kähler manifolds*, J. Amer. Math. Soc. 11, no. 2 (1998), 375–396.

[NR4] T. Napier, M. Ramachandran, *Hyperbolic Kähler manifolds and proper holomorphic mappings to Riemann surfaces*, Geom. Funct. Anal. 11 (2001), 382–406.

[NR5] T. Napier, M. Ramachandran, *Filtered ends, proper holomorphic mappings of Kähler manifolds to Riemann surfaces, and Kähler groups*, Geom. Funct. Anal. 17 (2007), 1621–1654.

[NR6] T. Napier, M. Ramachandran, *$L^2$ Castelnuovo–de Franchis, the cup product lemma, and filtered ends of Kähler manifolds*, J. Topol. Anal. 1 (2009), no. 1, 29–64.

[NR7] T. Napier, M. Ramachandran, *The Bochner–Hartogs dichotomy for bounded geometry hyperbolic complete Kähler manifolds*, to appear in Ann. Inst. Fourier.

[NR8] T. Napier, M. Ramachandran, *Weakly special filtered ends of complete Kähler manifolds, proper holomorphic mappings to Riemann surfaces, and the Bochner–Hartogs dichotomy*, in preparation.

[Ra] M. Ramachandran, *A Bochner–Hartogs type theorem for coverings of compact Kähler manifolds*, Comm. Anal. Geom. 4 (1996), 333–337.

[Ri] R. Richberg, *Stetige streng pseudokonvexe Funktionen*, Math. Ann. 175, (1968), 257–286.
[Ste] K. Stein, *Maximale holomorphe und meromorphe Abbildungen, I*, Amer. J. Math. 85 (1963), 298–315.

[TW] P. Tworzewski, T. Winiarski, *Continuity of intersection of analytic sets*, Ann. Polon. Math. 42 (1983), 387–393.

Department of Mathematics, Lehigh University, Bethlehem, PA 18015, USA

E-mail address: tjn2@lehigh.edu

Department of Mathematics, University at Buffalo, Buffalo, NY 14260, USA

E-mail address: ramac-m@buffalo.edu