CHARACTERIZATION OF $\gamma$-FACTORS: THE ASAI CASE

GUY HENNIART AND LUIS LOMELÍ

Abstract. Let $E$ be a separable quadratic algebra over a locally compact field $F$ of positive characteristic. The Langlands-Shahidi method can be used to define the Asai $\gamma$-factors for a smooth irreducible generic representation $\pi$ of $\text{GL}_n(E)$. If $\sigma$ is the Weil-Deligne representation of $W_E$ corresponding to $\pi$ under the local Langlands correspondence, then it is shown that the Asai $\gamma$-factor is the same as the $\gamma$-factor on the Galois side corresponding to the representation of $W_E$ obtained from $\sigma$ under tensor induction. This is achieved by proving that Asai $\gamma$-factors are characterized by their local properties together with their role in global functional equations for $L$-functions. An immediate application concerns the stability of $\gamma$-factors under twists by highly ramified characters.

1. Introduction

Let $F$ be a locally compact field of positive characteristic $p$. Let $\psi$ be a non-trivial character of $F$ and $\pi$ a smooth irreducible generic representation of $\text{GL}_n(F)$, where $n$ is a positive integer. If $\rho_n$ denotes the standard representation of $\text{GL}_n(C)$, let $\tau$ be either $\text{Sym}^2 \rho_n$ or $\wedge^2 \rho_n$. In [7] the authors establish the equality of $\gamma$-factors:

$$\gamma(s, \pi, r, \psi) = \gamma(s, r \circ \sigma, \psi),$$

where the factor on the left is defined via the Langlands-Shahidi method [14, 15], and $\sigma$ on the right is the the Weil-Deligne representation corresponding to $\pi$ under the local Langlands correspondence [13]. The same question in characteristic zero remains open, although much progress has been made [6].

In this paper, we address the case of Asai $\gamma$-factors and related $L$- and $\varepsilon$-factors. These factors can be seen as a generalization of those studied in [1] by T. Asai. Let $E/F$ be a separable quadratic extension of locally compact fields of positive characteristic and let $E$ be a separable algebraic closure containing $F$. Let $\pi$ be a smooth irreducible representation of $\text{GL}_n(E)$. The $L$-group of $\text{Res}_{E/F} \text{GL}_n$ is $GL_n(C) \times GL_n(C) \rtimes W_F$, where the Weil group $W_F$ acts via the Galois group $\text{Gal}(E/F) = \{1, \theta\}$. The Asai representation $r_A = r_{A_n}$ can be defined by

$$r_A : GL_n(C) \times GL_n(C) \times \text{Gal}(E/F) \to GL_{n^2}(C),$$

$$r_A(x, y, 1) = (x \otimes y) \text{ and } r_A(x, y, \theta) = (y \otimes x).$$

The Langlands-Shahidi method is used in [15] to define Asai $\gamma$-factors $\gamma_{E/F}(s, \pi, r_A, \psi)$ in characteristic $p$; we rely on that construction in the current paper. Writing $\sigma$ as
the Weil-Deligne representation of $\mathcal{W}_E$ corresponding to $\pi$ under local Langlands, we prove that

$$\gamma_{E/F}(s, \pi, r_A, \psi) = \gamma_{F}^{\text{Gal}}(s, I(\sigma), \psi),$$

where $I(\sigma)$ denotes the representation of $\mathcal{W}_F$ obtained from $\sigma$ by tensor induction (see Theorem 3.3). In the case of characteristic zero, equation (1.1) for $n = 2$ is known [11, 17] (see [6] for progress in the general case).

Theorem 3.3 is proved via a characterization of Asai $\gamma$-factors involving local properties together with their connection with the global theory by means of a functional equation described below (1.2). More precisely, the local properties of $\gamma_{E/F}(s, \pi, r_A, \psi)$ include: a naturality property with respect to isomorphisms of quadratic extensions $E/F$; an isomorphism property pertaining to $\pi$; a dependence on the additive character $\psi$, which can be made explicit; a crucial multiplicativity property with respect to parabolic induction, which reflects the influence of taking tensor induction on a direct sum of Weil-Deligne representations; and finally, (1.1) is needed whenever the representation $\pi$ is the generic component of an unramified principal series.

Let $K/k$ be a quadratic separable extension of global function fields of characteristic $p$, and for a split place $v$ of $K$ we have $K \otimes k_v \cong k_v \times k_v$. Thus, in the local theory, the case of a separable quadratic algebra $E/F$ is treated simultaneously. The connection with the global theory is now given by the global functional equation:

$$L^S(s, \Pi, r_A) = \prod_{v \in S} \gamma_{K_v/k_v}(s, \Pi_v, r_A, \Psi_v) L^S(1 - s, \Pi, r_A),$$

where

$$L^S(s, \Pi, r_A) = \prod_{v \not\in S} L(s, \Pi_v, r_A).$$

In the course of proving our main results, we directly establish a local-to-global argument for the case of a cuspidal, tamely ramified representation $\pi$ of $\text{GL}_n(E)$ (hence $\pi$ of level zero) via the Grundwald-Wang theorem. Then the general problem is reduced to the case of a tamely ramified representation $\pi$. This is done by using a local-to-global result due to Gabber and Katz for $\ell$-adic representations of the Galois group [10], and translating it via the global Langlands correspondence [12]. We note that care must be taken, since we are considering a quadratic extension $E/F$.

In the Langlands-Shahidi method, $\pi$ is assumed to be generic. But, using the Langlands-Zelevinsky classification together with multiplicativity, the definition of $\gamma$-factors can be extended to the general case (see §4.1). Also, we show that the local $L$- and $\varepsilon$-factors are the same as the corresponding Galois factors. In §4.2 we take the opportunity to write down a stability property of $\gamma$-factors that is not known in characteristic zero. Finally, in §5 we give a short proof of the equality of local factors studied in [19] for Rankin-Selberg products of $\text{GL}_{m}$ and $\text{GL}_{n}$.

The second author kindly thanks F. Shahidi for his help and encouragement; he also thanks M. Krishnamurthy and P. Kutzko for useful discussions.
2. Asai $\gamma$-factors and tensor induction

2.1. Fix a prime number $p$ and consider the class $\mathcal{L}_{\text{quad}}(p)$ of triples $(E/F, \pi, \psi)$ consisting of:

- $F$ a locally compact field of characteristic $p$;
- $E$ a separable quadratic algebra over $F$, i.e., either $E/F$ is a separable quadratic extension of local fields or $E \simeq F \times F$;
- $\pi$ a smooth irreducible representation of $\text{GL}_n(E)$, $n \geq 1$;
- $\psi$ a non-trivial character of $F$.

Given a triple $(E/F, \pi, \psi) \in \mathcal{L}_{\text{quad}}(p)$, we say it is of degree $n$ if $\pi$ is a representation of $\text{GL}_n(E)$. Let $x \mapsto \bar{x}$ denote conjugation in $E/F$, i.e., the non-trivial automorphism of $E/F$. Let $q$ be the cardinality of its residue field and let $p$ be its maximal ideal. Given a representation $\sigma$, let $\sigma^{\text{conj}}$ denote the representation obtained from $\sigma$ by conjugation.

Let $G = U(2n)$ be the quasi-split unitary group with respect to $E/F$. This group can be obtained via the hermitian form $h(x, y) = \sum_{i=1}^{2n} \bar{x}_i y_{2n+1-i}$.

Asai $\gamma$-factors in characteristic $p$ are defined in [15] via the Langlands-Shahidi method. They arise from generic representations $\pi$ of $M = \text{Res}_{E/F} \text{GL}_n$ is the Siegel Levi subgroup of $G$ (for general $\pi$ we refer to §4.1 below). Asai $\gamma$-factors give a rule which, to a triple $(E/F, \pi, \psi) \in \mathcal{L}_{\text{quad}}(p)$, associates a rational function $\gamma_{E/F}(s, \pi, r_A, \psi)$ of $\mathcal{C}(q^{-s})$.

2.2. Given a representation $\sigma$ of $\mathcal{W}_E$, we consider $I(\sigma)$ to be tensor induction from $\mathcal{W}_E$ to $\mathcal{W}_F$. (See §13 of [3]). Given $(E/F, \psi, \pi) \in \mathcal{L}_{\text{quad}}(p)$, let $\sigma = \pi(\sigma)$ be the representation of $\mathcal{W}_E$ corresponding to $\pi$ via the local Langlands correspondence. The Galois $\gamma$-factors arising in connection with the Asai $\gamma$-factors will be written: $\gamma_{E/F}^{\text{Gal}}(s, I(\sigma), \psi)$.

These factors satisfy a number of easily established properties, including a multiplicativity property reflecting the decomposition rule:

$$I(\sigma \oplus \tau) \simeq I(\sigma) \oplus I(\tau) \oplus \text{Ind}^F_E(\sigma \otimes \tau^{\text{conj}}).$$

The Asai $\gamma$-factors satisfy the corresponding properties, which we list in the next section.

3. Characterization of Asai factors

3.1. We first give the local properties of Asai $\gamma$-factors:

(i) (Naturality). Let $(E/F, \psi, \pi) \in \mathcal{L}_{\text{quad}}(p)$ be of degree $n$, and let $\eta$ be an isomorphism $\eta : E'/F' \simeq E/F$. Then $\psi' = \psi \circ \eta|_{E'}$ is a non-trivial additive character of $F'$. Also, via $\eta$, $\pi$ defines a smooth irreducible generic representation $\pi'$ of $\text{GL}_n(E')$. Then

$$\gamma_{E/F}(s, \pi, r_A, \psi) = \gamma_{E'/F'}(s, \pi', r_A, \psi').$$

(ii) (Isomorphism). Let \((E/F, \psi, \pi) \in \mathcal{L}_{\text{quad}}(p)\) be of degree \(n\), and let \(\pi'\) be a smooth irreducible generic representation of \(\text{GL}_n(E)\) isomorphic to \(\pi\). Then
\[
\gamma_{E/F}(s, \pi', r_A, \psi) = \gamma_{E/F}(s, \pi, r_A, \psi).
\]
For the relationship with Artin factors, see § 5 of [15]. In the case \(n = 1\), \(\pi\) can be viewed as a character \(\chi\) of \(E^\times\). Then \(\gamma_{E/F}(s, \chi, r_A, \psi)\) is equal to the abelian \(\gamma\)-factor \(\gamma_F(s, \chi|_{F^\times}, \psi)\).

(iii) (Relation with Artin factors). Let \((E/F, \psi, \pi) \in \mathcal{L}_{\text{quad}}(p)\) be of degree \(n\), and assume that \(n = 1\) or that \(\pi\) is the generic component of a principal series. Let \(\sigma = \sigma(\pi)\) be the Weil-Deligne representation of \(W_F\) associated to \(\pi\) via the local Langlands correspondence. Let \(I(\sigma)\) be the representation of \(W_F\) obtained from \(\sigma\) by tensor induction. Then
\[
\gamma_{E/F}(s, \pi, r_A, \psi) = \gamma_F^\text{Gal}(s, I(\sigma), \psi).
\]
We write this as the next property of Asai \(\gamma\)-factors.

(iv) (Dependence on \(\psi\)). Let \((E/F, \psi, \pi) \in \mathcal{L}_{\text{quad}}(p)\) be of degree \(n\), and let \(a \in F^\times\). Then \(\psi^a : x \mapsto \psi(ax)\) is a non-trivial additive character of \(F\) and we have
\[
\gamma_{E/F}(s, \pi, r_A, \psi^a) = \omega_\pi(a)^n|a|_F^{n^2(s-\frac{1}{2})} \gamma_{E/F}(s, \pi, r_A, \psi).
\]
Let us give a short proof of (iv), it relies on the definition given in § 5.1 of [15], which we refer to for any unexplained notation. Consider \(\pi\) as a representation of \(M \cong \text{GL}_n(E)\) and assume it is \(\chi_0\)-generic, where \(\chi_0\) is obtained from \(\psi\). Let
\[
t = \text{diag}(a^{-(n-1)}, a^{-(n-2)}, \ldots, a^{(n-2)}, a^{(n-1)}, a^{n-2}, a^{n-1}, a^n) \in T(E),
\]
where \(a^n = a^{n-1} = a \in F^\times\). Then \(w_0(t)t^{-1}\) lies in the center of \(M\). Let \(\pi_t\) be given by \(\pi_t(x) = \pi(t^{-1}xt)\). The character \(\chi_{0,t}\) given by \(\chi_{0,t}(u) = \chi_0(t^{-1}ut)\) is then obtained from \(\psi^a\) and \(\pi_t\) is \(\chi_{0,t}\) generic. Using the definition and a direct computation we get
\[
\gamma_{E/F}(s, \pi, r_A, \psi^a) = \lambda(\psi, w_0) C_{\mathbb{T}_0}(s, \pi_t, w_0)
= \omega_\pi(a)^n|a|_F^{n^2(s-\frac{1}{2})} \lambda(\psi, w_0) C_{\mathbb{T}_0}(s, \pi, w_0)
= \omega_\pi(a)^n|a|_F^{n^2(s-\frac{1}{2})} \gamma_{E/F}(s, \pi, r_A, \psi).
\]

The following can be found in § 5 of [15]:

(v) (Multiplicativity of \(\gamma\)-factors). For \(i = 1, \ldots, d\), let \((E/F, \psi, \pi_i) \in \mathcal{L}_{\text{quad}}(p)\) be of degree \(n_i\). Let \(n = n_1 + \cdots + n_d\) and let \(\pi\) be the unique generic component of the representation of \(\text{GL}_n(E)\) parabolically induced from \(\pi_1 \otimes \cdots \otimes \pi_d\). Then
\[
\gamma_{E/F}(s, \pi, r_A, \psi) = \prod_{i=1}^d \gamma_{E/F}(s, \pi_i, r_{A_{\pi_i}}, \psi) \prod_{1 \leq j < 2} \gamma_{E}(s, \pi_i \otimes \pi_j^{\text{conj}}; \psi \circ \text{Tr}_{E/F}).
\]
(Here, each \(\gamma_{E}(s, \pi_i \otimes \pi_j^{\text{conj}}; \psi \circ \text{Tr}_{E/F})\) is a Rankin-Selberg factor; see § 5). Notice that (v) and the case \(n = 1\) give (iii).
(vi) (Split case). Let $(E/F, \psi, \pi) \in \mathcal{L}_{\text{quad.}}(p)$ be of degree $n$ and assume $E \simeq F \times F$. Then $\pi \simeq \pi_1 \otimes \pi_2$, where $\pi_1$ and $\pi_2$ are smooth irreducible representations of $\text{GL}_n(F)$ and

$$\gamma_{E/F}(s, \pi, r, \psi) = \gamma_F(s, \pi_1 \times \pi_2, \psi),$$

(Again, $\gamma_F(s, \pi_1 \times \pi_2, \psi)$ is a Rankin-Selberg factor; see §5).

3.2. The link between the local and global theory is provided by the following property (see §5 of [15]):

(v) (Global functional equation). Let $K/k$ be a quadratic separable extension of global function fields of characteristic $p$. Let $\Psi$ be a non-trivial character of $\mathbf{A}_k/k$ and let $\Pi = \otimes_{v} \Pi_v$ be an automorphic cuspidal representation of $(\text{Res}_{K/k} \text{GL}_n)(\mathbf{A}_k) \simeq \text{GL}_n(\mathbf{A}_K)$. Given a place $v$ of $k$, let $K_v = K \otimes k_v$. Let $S$ be a finite set of places such that $K/k$, $\Pi$ and $\Psi$ are unramified outside of $S$. Then

$$L^S(s, \Pi, r, \Lambda) = \prod_{v \in S} \gamma_{K_v/k_v}(s, \Pi_v, r, \Lambda_v) L^S(1 - s, \Pi, r, \Lambda),$$

where

$$L^S(s, \Pi, r, \Lambda) = \prod_{v \not\in S} L(s, \Pi_v, r, \Lambda).$$

3.3. Theorem. There is only one rule on $\mathcal{L}_{\text{quad.}}(p)$ satisfying properties (i)–(v). In particular, for $(E/F, \psi, \pi) \in \mathcal{L}_{\text{quad.}}(p)$ of degree $n$, we have

$$\gamma_{E/F}(s, \pi, r, \psi) = \gamma^\text{Gal}_F(s, \chi),$$

where $\sigma = \sigma(\pi)$ is associated to $\pi$ via the local Langlands correspondence. Moreover, Asai $\gamma$-factors also satisfy:

(vii) (Twisting by unramified characters). Let $(E/F, \psi, \pi) \in \mathcal{L}_{\text{quad.}}(p)$. Then

$$\gamma_{E/F}(s, \pi \otimes |\det|_{E}^{1/n}, r, \psi) = \gamma_{E/F}(s + s_0, \pi, r, \psi).$$

(ix) (Local functional equation).

$$\gamma_{E/F}(s, \pi, r, \psi) \gamma_{E/F}(1 - s, \pi, r, \psi) = 1.$$  

3.4. Proof of Theorem. Property (vii) can be shown directly, and the local functional equation is a property of $\gamma^\text{Gal}_F$ that can be immediately translated to $\gamma_{E/F}$. To prove the main result, we can assume, by multiplicativity, that $\pi$ is cuspidal; as the case $E \cong F \times F$ is given by (vi), we may assume $E/F$ is a quadratic extension. We proceed by induction on $n \geq 1$, where $n = 1$ is given by property (iii). Thus, we consider $(E/F, \psi, \pi) \in \mathcal{L}_{\text{quad.}}(p)$ of degree $n > 1$.

Case 1: $n > 1$, $E/F$ tame, $\pi$ cuspidal and tame. (Thus $\pi$ is of level zero). Let $\sigma$ be the corresponding Galois representation under local Langlands. Then $\sigma$ is irreducible and is given by

$$\sigma = \text{Ind}_{W_{E'}}^{W_{E'}}(\chi),$$

where $E'/E$ is an unramified extension of degree $n$ and $\chi : W_{E'} \to \mathbb{C}^\times$ is a tame character. By class field theory, $\chi$ is the same as a character $\chi : E'/\mathbb{C}^\times \to \mathbb{C}^\times$. Moreover, $\chi$ restricted to $U_{E'}$ is obtained from a regular character of $k_{E'}^\times$. (Notation: given a local field $F$, its residue field is denoted by $k_F$; given a global function field $k$, its field of constants is denoted by $k_k$).
Let $k = k_F(t)$ and let $K/k$ be a separable quadratic extension with $K_0/k_0 \simeq E/F$ and such that $k_K = k_E$. Let $k_n$ be a degree $n$ extension of $k_K$, then the constant field extension $K' = k_n \cdot K$ is a cyclic extension of degree $n$, unramified everywhere. Then $K'/K_0 \simeq E'/E$. Let $w$ be a place of $K$ that splits completely in $K'/K$. (Notice that a place of $K$ splits completely in $K'/K$ if $n$ divides the degree of $w$). Let $S = \{0, w_1', \ldots, w_n'\}$, where $w_i'|w$. We can now proceed as in § 2.3 of \cite{17} and construct a character
\[ \xi : \prod U_{K'_w} \to \mathbb{C}^\times, \]
where the product ranges over all places $w'$ of $K'$, such that:

- $\xi_0 = \chi|_{U_{K'_0}}$;
- $\xi_{w'} = 1$ if $w' \notin S$, and
- $\xi_{|_{k_{K'}}} = 1$.

Then $\xi$ further extends to a gr"ossencharacter
\[ \tilde{\xi} : K'^\times \backslash A_{K'}^\times \to \mathbb{C}^\times. \]

After globally twisting by an unramified character, we can assume that $\tilde{\xi}_0 = \chi$. Also, $\tilde{\xi}_{w'}$ will be unramified for $w' \notin S$.

A gr"ossencharacter $\xi$ as above is the same as a character of $\mathcal{W}_K$, via global class field theory. Then
\[ R = \text{Ind}_{\mathcal{W}_K}^{\mathcal{W}_{K'}} \tilde{\xi} \]
will have $R_0 = \rho$ and $R_v$ will be reducible for all places $v$ of $K$, with $v \neq 0$. Indeed, $R_v$ is unramified for $v \notin \{0, w\}$, and $R_w$ is a sum of characters because $w$ is split in $K'/K$.

Let $\ell, \ell \neq p$, be a fixed prime number, and let $\iota : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ be a fixed field isomorphism. Then, $R$ gives rise to a continuous degree $n$ $\overline{\mathbb{Q}}_\ell$-representation $\Sigma$ of $\mathcal{W}_K$. The global Langlands correspondence, proved in \cite{12}, gives a cuspidal automorphic representation $\Pi = \Pi(\Sigma)$. By the local Langlands correspondence of \cite{13}, $\Pi_v$ corresponds to $\Sigma_v$.

By construction: $\pi$ corresponds to $\Pi_0$, $\Pi_v$ is unramified for $v \notin \{0, w\}$, and $\Pi_w$ is a principal series representation. (If the place $u$ of $k$ lying below $w$ splits in $K/k$, then we can use property (vi) instead).

By properties (i) and (ii), we can assume $F = K_0$ and $\pi = \Pi_0$. By property (iv), we can also assume $\psi = \Psi_0$. The global functional equation then gives
\begin{equation}
\prod_{v \in S} \gamma_{K_v/k_v}(s, \Pi_v, r_A, \Psi_v) = \prod_{v \in S} \gamma_{k_v}^{\text{Gal}}(s, I(\Pi_v), \Psi_v),
\end{equation}
where $S$ is a finite set of places of $k$ containing $0$ and $u$, $w|u$. But, for $v$ distinct from $0$ and $u$, $\Pi_v$ is an unramified principal series. Also, $\Pi_v$ is a (possibly ramified) principal series. Hence, property (vii) (and property (vi) if $u$ splits) gives
\begin{equation}
\gamma_{K_v/k_v}(s, \Pi_v, r_A, \Psi_v) = \gamma_{k_v}^{\text{Gal}}(s, I(\Pi_v), \Psi_v), \ v \in S - \{0\}. \tag{3.2}
\end{equation}
Then \ref{3.1} and \ref{3.2} give equality at 0.

**CASE 2.** $n > 1$, $E/F$ general and $\pi$ cuspidal, but not necessarily of level zero. Let $\sigma = \sigma(\pi)$ be the corresponding irreducible Weil-Deligne representation. Also, twisting by an unramified character if necessary, we can assume $\sigma$ is a representation of the Galois group. Then $\sigma$ factors through some Galois group $\text{Gal}(E'/E)$. Let $\tilde{E}$ be the Galois closure of $E'/F$. Consider $k = k_F(t)$ and find a Galois extension $\tilde{K}$.
of \( k \), such that \( \tilde{K} \) is \( \tilde{E} \) at 0, tame at \( \infty \) and unramified elsewhere. This is possible by the results of Gabber-Katz [11]; moreover, notice that \( \text{Gal}(\tilde{E}/F) \) is the same as \( \text{Gal}(\tilde{K}/k) \) in Katz’s construction. Then, if we let \( K/k \) be the quadratic extension corresponding to \( E/F, \sigma \) gives a representation \( \Sigma \) of \( \text{Gal}(\tilde{K}/k) \). The representation \( \Sigma \) will be \( \sigma \) at 0, tame at \( \infty \) and unramified elsewhere.

By the global Langlands correspondence of [12], there exists an irreducible cuspidal automorphic representation \( \Pi = \Pi(\Sigma) \). The local components \( \Pi_v \) of \( \Pi \) are obtained from \( \Sigma \) globally by an unramified character to ensure that \( \Pi_0 = \pi \), but this does not affect the properties of \( \Pi \). Then we can apply (3.1) and (3.2) to \( \Pi \), with \( S = \{0, \infty\} \), in order to complete the proof. The equality at \( \infty \) is given by Case 1 treated above and by the fact that if \( \Sigma_\infty \) is tame, then so are all its irreducible components. \( \square \)

4. LOCAL \( L \)-FUNCTIONS, ROOT NUMBERS AND STABILITY

4.1. Equality of local factors. Let us recall the definition of local \( L \)-functions and \( \varepsilon \)-factors via the Langlands-Shahidi method. Since the local factors we are studying arise from \( \text{GL}_n \), they can be defined for \( (E/F, \psi, \pi) \in \mathcal{L}_{\text{quad.}}(p) \) where \( \pi \) is any smooth irreducible representation of \( \text{GL}_n(E) \). We note that the following discussion can be used to extend the results of [7], related to \( \wedge^2 \rho_n \) and \( \text{Sym}^2 \rho_n \), to representations that are not necessarily generic.

Let \( (E/F, \psi, \pi) \in \mathcal{L}_{\text{quad.}}(p) \). Let us first assume that \( \pi \) is tempered, then \( \pi \) is generic [22]. Let \( P_\pi(t) \) be the unique polynomial satisfying \( P_\pi(0) = 1 \) and such that \( P_\pi(q^{-s}) \) is the numerator of \( \gamma_{E/F}(s, \pi, r_A, \psi) \). Then
\[
L(s, \pi, r_A) := P_\pi(q^{-s})^{-1}.
\]
Because \( \pi \) is tempered, \( L(s, \pi, r_A) \) is holomorphic for \( \text{Re}(s) > 0 \). If \( \pi \) is parabolically induced from \( \pi_1 \otimes \cdots \otimes \pi_d \), where each \( \pi_i \) is tempered, then multiplicativity of \( \gamma \)-factors gives multiplicativity for the \( L \)-functions:
\[
L(s, \pi, r_A) = \prod_{i=1}^d L(s, \pi_i, r_{A_i}) \prod_{i<j} L(s, \pi_i \times \pi_j^{\text{conj}}).
\]
The local \( \varepsilon \)-factor is defined to satisfy the relation:
\[
\gamma_{E/F}(s, \pi, r_A, \psi) = \varepsilon_{E/F}(s, \pi, r_A, \psi) \frac{L(1-s, \bar{\pi}, r_A)}{L(s, \pi, r_A)}.
\]
Given \( (E/F, \psi, \pi) \in \mathcal{L}_{\text{quad.}}(p) \) in general, we can use Langlands classification to write \( \pi \) as parabolically induced from \( \pi_{\nu,1} \otimes \cdots \otimes \pi_{\nu,d} \), where each \( \pi_{\nu,i} \) is quasi-tempered with a negative Langlands parameter \( \nu \). Each \( \pi_{0,i} \) is tempered, and the \( L \)-functions \( L(s, \pi_{\nu,i}, r_A) \) and \( L(s, \pi_{\nu,i} \times \pi_{\nu,j}) \) are defined by analytic continuation on \( \nu \). Then
\[
L(s, \pi, r_A) := \prod_{i=1}^d L(s, \pi_{\nu,i}, r_{A_i}) \prod_{i<j} L(s, \pi_{\nu,i} \times \pi_{\nu,j}^{\text{conj}}),
\]
and the root numbers are defined to satisfy (4.1).

This is in accordance with the way local \( L \)-functions and \( \varepsilon \)-factors are defined for Weil-Deligne representations. Equality of local factors follows first for tempered representations from Theorem 3.3. Then in general by the above discussion.
4.2. Stability of $\gamma$-factors. Let us briefly recall the stability property of local factors for Weil-Deligne representations \[14 \hspace{1pt} 5\]. Let $\sigma$ be a Weil-Deligne representation. Let $\eta$ be a character of $F^\times$ of level $k$, for $k$ sufficiently large (depending on $\sigma$). Take an element $c = c(\eta, \psi) \in F^\times$ such that $\psi(cx) = \eta(1 + x)$ for $x \in p^{k/2} + 1$. Then

$$
\varepsilon(s, \sigma \otimes \eta, \psi) = \det(\sigma(c))^{-1} \varepsilon(s, \eta, \psi)^{\dim \sigma},
$$

$$
L(s, \sigma \otimes \eta) = L(s, \sigma \otimes \eta^{-1}) = 1.
$$

Because of this, the next property is now a corollary to Theorem 3.3. We phrase it in terms of $\gamma$-factors.

**Corollary (Stability).** Let $(E/F, \psi, \pi_i) \in \mathcal{L}_{\text{quad.}}(p)$, $i = 1, 2$, both of the same degree. Assume that $\pi_1$ and $\pi_2$ have the same central character. Then, for every sufficiently highly ramified character $\eta$ of $F^\times$, we have

$$
\gamma_{E/F}(s, \eta \cdot \pi_1, \rho_A, \psi) = \gamma_{E/F}(s, \eta \cdot \pi_2, \rho_A, \psi).
$$

**Remark 1.** It is clear that the same result holds for local factors corresponding to exterior and symmetric square $L$-functions. To be more precise, we use the notation of \[7\]: Let $(F, \psi, \pi) \in \mathcal{L}(p)$, $i = 1, 2$, both of degree $n$. Assume that $\pi_1$ and $\pi_2$ have the same central character. Then, for every sufficiently highly ramified character $\eta$ of $F^\times$, we have

$$
\gamma_F(s, \eta \cdot \pi_1, \rho_n, \psi) = \gamma_F(s, \eta \cdot \pi_2, \rho_n, \psi).
$$

Notice that, by the discussion in §\[4.1\], the representations $\pi_i$ need not be generic.

5. Rankin Selberg products for representations of $GL_m$ and $GL_n$

5.1. Let $(F, \psi, \pi_i) \in \mathcal{L}_{\text{quad.}}(p)$, $i = 1, 2$. The Rankin-Selberg $\gamma$-factor

$$
\gamma(s, \pi_1 \times \pi_2, \psi) = \varepsilon(s, \pi_1 \times \pi_2, \psi) \frac{L(1 - s, \pi_1 \times \pi_2)}{L(s, \pi_1 \times \pi_2)}
$$

is defined in \[8\]. Consider $M = GL_m \times GL_n$ as a maximal Levi subgroup of $G = GL_{m+n}$ and let $P = MN$ be the maximal standard parabolic subgroup with Levi $M$ and unipotent radical $N$. The adjoint action of $LM$ on $L^1$ is $r \simeq \rho_m \otimes \rho_n$. For this $r$, the $\gamma$-factors $\gamma(s, \pi_1 \otimes \pi_2, r, \psi)$ are defined in \[14 \hspace{1pt} 15\] via the Langlands-Shahidi method. The aim of \[19\] is to establish the equality

$$
\gamma(s, \pi_1 \times \pi_2, \psi) = \gamma(s, \pi_1 \otimes \pi_2, r, \psi)
$$

using completely local methods. What we now provide is a short proof of this result by means of a characterization of $\gamma$-factors.

5.2. **Proof of equation \[5.1\].** Let $B = TU$, be the Borel subgroup of $GL_{m+n}$ consisting of upper triangular matrices. Let $\chi_0$ be the character of $U(F)$ obtained from $\psi$ and, abusing notation, we also write $\chi_0$ for the restriction of $\chi_0$ to $U_M = M(F) \cap U(F)$; they will be $w_0$-compatible in the notation of \[15\]. Consider $\sigma = \pi_1 \otimes \pi_2$ as a representation of the Levi $M$. We may assume $\sigma$ is $\chi_0$-generic.

Both $\gamma(s, \pi_1 \times \pi_2, \psi)$ and $\gamma(s, \pi_1 \otimes \pi_2, r, \psi)$ satisfy naturality and isomorphism properties. The multiplicativity property of the local coefficient implies multiplicativity for $\gamma(s, \pi_1 \otimes \pi_2, r, \psi)$. For $\gamma(s, \pi_1 \times \pi_2, \psi)$, multiplicativity can be found in Theorem 3.1 of \[8\]. The relation with Artin factors when $\pi_1$ and $\pi_2$ are principal series is reduced via multiplicativity to establishing the relation in the case of $GL_2$, which is well known.
For $a \in F^\times$, let $t = \text{diag}(a^{-(m+n-1)}, a^{-(m+n-2)}, \ldots, a, 1)$. Let $\sigma = \pi_1 \otimes \pi_2$ and let $\sigma_\ell$ be given by $\sigma_\ell(x) = \sigma(t^{-1}xt)$. The character $\chi_{0,t}$ given by $\chi_{0,t}(u) = \chi_0(t^{-1}ut)$ is then obtained from $\psi^a$ and $\sigma_\ell$ is $\chi_{0,t}$ generic. Using the definition and a direct computation to compare both local coefficients we obtain:

$$
\gamma(s, \pi_1 \otimes \pi_2, r, \psi) = C_{\chi_{0,t}}(s, \sigma, w_0) = \frac{\omega_{\chi_{0,t}}(\sigma)}{\omega_{\chi_{0,t}}(\pi_1 \otimes \pi_2)} C_{\chi_{0,t}}(s, \sigma_\ell, w_0) = \omega_{\pi_1}(a)^m \omega_{\pi_2}(a)^n |a_F|^{mn(s-\frac{1}{2})} \gamma(s, \pi_1 \otimes \pi_2, r, \psi).
$$

The same relationship holds for $\gamma(s, \pi_1 \times \pi_2, \psi)$.

Finally, we have a global functional equation: Let $K$ be a global function field of characteristic $p$, let $\Psi = \otimes_v \Psi_v$ be a non-trivial character of $K \setminus \mathbb{A}_K$, and let $\Pi_1$ and $\Pi_2$ be cuspidal automorphic representations of $GL_n(\mathbb{A}_K)$ and $GL_n(\mathbb{A}_K)$, respectively. Let $S$ be a finite set of places of $K$ such that $\Psi$ and $\Pi_i$, for $i = 1, 2$, are unramified outside of $S$. Then, Theorem 5.14 of [14] gives

$$
L^S(s, \Pi_1 \times \Pi_2) = \prod_{v \in S} \gamma(s, \Pi_1 \times \Pi_2, \psi_v) L^S(1 - s, \Pi_1 \times \Pi_2).
$$

The functional equation for $\gamma(s, \Pi_1 \times \Pi_2, \psi)$ can be found in [2] [16].

Given local representations $\pi_1$ and $\pi_2$, we use the local-global argument in the proof of Theorem 5.3 to prove (5.11). A brief outline should suffice: by multiplicativity, assume $\pi_1$ and $\pi_2$ are cuspidal. Use Proposition 2.2 of [7] to deal with the case when $\pi_1$ and $\pi_2$ are both cuspidal of level zero. Then, use Proposition 3.1 of [7] for general cuspidal representations.

**Remark 2.** The above argument should also hold in characteristic zero by using Proposition 5.1 of [20] as the link between the local and the global theory, relying on the global functional equation [2] [18]. The theory for archimedean local fields is studied in [9] [21].

**Remark 3.** The local properties of $\gamma(s, \pi_1 \times \pi_2, \psi)$ used above can also be obtained via the local Langlands correspondence in any characteristic.

**References**

[1] T. Asai, *On certain Dirichlet series associated with Hilbert modular forms and Rankin’s method*, Math. Ann. 226 (1977), 81-94.

[2] J. Cogdell, *Notes on $L$-functions for $GL_n$*, ICTP Lecture Notes, 2000.

[3] C. W. Curtis and I. Reiner, *Methods of representation theory I*, John Wiley & Sons, New York, 1981.

[4] P. Deligne, *Les constantes des équations fonctionnelles des fonctions $L$*, Modular functions of one variable II, LNM 349, Springer Verlag, 1973.

[5] P. Deligne and G. Henniart, *Sur la variation, par torsion, des constantes locales d’équations fonctionnelles des fonctions $L$*, Invent. Math. 64 (1981), 89-118.

[6] G. Henniart, *Correspondance de Langlands et fonctions $L$ des carrés extérieur et symétrique*, preprint.

[7] G. Henniart and L. Lomelí, *Local-to-global extensions for $GL_n$ in non-zero characteristic: a characterization of $\gamma(s, \pi, (\pi^\vee)^\vee, \psi_F)$ and $\gamma(s, \pi, \Lambda^2, \psi_F)$*, preprint.

[8] H. Jacquet, I. I. Piatetski-Shapiro and J. A. Shalika, *Rankin-Selberg convolutions*, Am. J. Math. 105 (1983), 367-464.

[9] H. Jacquet and J. Shalika, *Rankin-Selberg convolutions: archimedean theory*, in *Festschrift in Honor of I.I. Piatetski-Shapiro*, Part I, Weizmann Science Press, 1990, 125-207.

[10] N. M. Katz, *Local-to-global extensions of representations of fundamental groups*, Ann. Inst. Fourier 36 (1986), 69-106.
[11] M. Krishnamurthy, The Asai transfer to $GL_4$ via the Langlands-Shahidi method, IMRN 2003, no. 41 (2003), 2221-2254.
[12] L. Lafforgue, Chretoucs de Drinfeld et correspondance de Langlands, Invent. Math. 147 (2002), 1-241.
[13] G. Laumon, M. Rapoport and U. Stuhler, $\mathcal{D}$-elliptic sheaves and the Langlands correspondence, Invent. Math. 113 (1993), 217-338.
[14] L. Lomelí, Functoriality for the classical groups over function fields, International Mathematics Research Notices 2009; Vol. 2009: article ID rnp089, 65 pages, doi:10.1093/imrn/rnp089
[15] L. Lomelí, On the Langlands-Shahidi Method for the classical groups in non-zero characteristic and applications, preprint.
[16] I. I. Piatetski-Shapiro, Zeta functions for $GL(n)$, preprint of the University of Maryland, 1976.
[17] D. Ramakrishnan, Modularity of solvable Artin representations of $GO(4)$-type, IMRN 2002, no. 1 (2002), 1-54.
[18] F. Shahidi, On certain $L$-Functions, Amer. J. Math. 106 (1981), 297-355.
[19] F. Shahidi, Fourier transforms of intertwining operators and Plancherel measures for $GL(n)$, Amer. J. Math. 106 (1984), 67-111.
[20] ______, A proof of Langlands’ conjecture on Plancherel measures; complementary series of $p$-adic groups, Ann. Math. 132 (1990), 273-330.
[21] ______, Local coefficients as Artin factors for real groups, Duke Math. J. 52 (1985), 973-1007.
[22] A. V. Zelevinsky, Induced representations of reductive $p$-adic groups II. On irreducible representations of $GL_n$, Ann. scient. Éc. Norm. Sup., 4$^e$ série, 13 (1980), 165-210.

Guy Henniart, Univ. Paris-Sud, Laboratoire de Mathématiques d’Orsay, CNRS, Orsay cedex F-91405, France
E-mail address: Guy.Henniart@math.u-psud.fr

Luis Lomelí, Department of Mathematics, The University of Iowa, 15 MacLean Hall, Iowa City, IA 52242, USA
E-mail address: llomeli@math.uiowa.edu