THE BOULWARE STATE AND THE GENERALISED
SECOND LAW OF THERMODYNAMICS

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Abstract

We show that the appropriate vacuum state for the interior of a box with reflecting walls being lowered adiabatically into a Schwarzschild black hole is the Boulware state. This is concordant with the results of Unruh and Wald, who used a different approach to obtain the stress-energy inside the box. Some comments about an entropy bound for ordinary matter, as first conjectured by Bekenstein, are presented.

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I. INTRODUCTION

One of the most remarkable developments in gravitational theory in the last century has been the discovery that fields quantised on a black hole background exhibit thermodynamical properties [1]. This discovery was presaged by the work of Bardeen, Carter and Hawking [2] in which they pointed out an analogy between laws governing certain properties of black holes and the laws of ordinary thermodynamics. In particular, the analogue of the second law of thermodynamics is Hawking’s theorem that the surface area of a black hole is nondecreasing [3], i.e.

\[
\frac{dA_{BH}}{d\tau} \geq 0.
\] (1.1)

It was based on this analogy between (1.1) and the second law of thermodynamics that Bekenstein [4] conjectured a generalised second law of thermodynamics (GSL): *The sum of the black hole entropy and the ordinary entropy in the black hole exterior never decreases.* More precisely, the GSL states that for any physical process

\[
\delta S_{\text{matter}} + \frac{1}{4} \delta A_{BH} \geq 0,
\] (1.2)

(units \( h=c=G=k=1 \)), where \( S_{\text{matter}} \) is the entropy of the matter outside the black hole. In (1.2), \( \frac{1}{4} A_{BH} \), one quarter of the black hole’s surface area, plays the role of the entropy of the black hole. This correspondence between the surface area and entropy of a black hole has become firmly established in the context of black hole thermodynamics, beginning with the Hawking’s discovery of the thermal radiation emitted by a black hole [1].

Bekenstein [5] further argued that an entropy bound on matter was required in order for the GSL to hold. His argument relied on the following *Gedankenexperimente*. Let us imagine that a box of linear dimension \( R \) with reflecting walls is filled with ordinary matter of energy \( E_\infty \) and entropy \( S \) at a very large proper distance from a black hole. The box is then slowly (adiabatically) lowered toward the black hole of mass \( M \). When the box is opened and the matter released into the black hole, the energy of the matter will have been
reduced by the redshift factor $\chi = (1 - 2M/r)^{1/2}$ so that the black hole’s energy is increased by

$$E = \chi E_\infty.$$  \hspace{1cm} (1.3)

Since we can lower the box to approximately the distance $R$ (the dimension of the box) from the event horizon before releasing the energy into the black hole, we can provide the black hole with as little as $E = (1 - 2M/(R + 2M))^{1/2}E_\infty$ energy. However, as Bekenstein demonstrated, this will lead to a change in the black hole entropy of

$$\delta S_{BH} = \frac{1}{4} \delta A_{BH} = 8\pi ME.$$  \hspace{1cm} (1.4)

After the box is emptied, it can be slowly pulled back out to infinity. But observe that, if $R < S/(2\pi E_\infty)$, we will have $\delta S_{BH} < S$ and the GSL will be violated. Therefore, Bekenstein concluded there was a bound on the entropy of matter with energy $E$ that could be placed in a box of dimension $R$,

$$S/E \leq 2\pi R.$$  \hspace{1cm} (1.5)

Unruh and Wald [6] (UW) have pointed out, however, that Bekenstein failed to consider black hole quantum effects in his analysis. In particular, they point out the effect of acceleration radiation on the box as it is being lowered. Since, in the reference frame of the almost stationary (hence accelerated) box, the black hole is surrounded by a bath of thermal radiation, there will be an upward pressure on the box. In fact, when this is taken into account, Unruh and Wald demonstrate that the box will float when the energy contained in the box, $E$, is exactly the same as the energy of the acceleration radiation displaced by the box. In order to lower the box further, one will have to do work against this buoyancy force. Unruh and Wald go on to show that in order to minimize the entropy increase of the black hole, the box must be opened at the floating point. They further show that the matter released at this point will contribute at least enough energy to the black hole to increase its entropy by an amount
\( \delta S_{BH} \geq S, \) \hspace{1cm} \text{(1.6)}

where \( S \) is the entropy of the matter in the box. Thus, they conclude, the GSL will hold independently of the validity of (1.5).

More recently, Li and Liu [7] have stated that the belief of Unruh and Wald that the Hawking radiation is thermal near the black hole is in error. In support of this statement they use the approximate stress-energy tensor for a massless scalar field surrounding a black hole found by Page [9]. They demonstrate that this stress-energy does not have the form of a perfect gas of photons in thermal equilibrium. They go on to derive a new equation of state for the Hawking radiation near the black hole. They find that the pressure of the Hawking radiation near the black hole is not large enough to produce a substantial buoyancy effect, and derive a bound on the entropy very similar to (1.5) of Bekenstein.

Page’s approximate stress-energy is for the Hartle-Hawking state associated with a conformally coupled massless scalar field in a Schwarzschild background. This is an adequate description of the state outside the box as seen by a freely falling observer. However, as demonstrated in UW, because the box is accelerating, it is subject to a bath of acceleration radiation. Further, UW show that this acceleration can be thought of as affecting the energy density inside the box.

How can the acceleration of the reflecting walls of the box affect the energy inside? Let us first answer this question for a box of fixed proper length accelerating in flat space. It is well established that, when quantum effects are considered, a mirror experiencing nonuniform acceleration will radiate two fluxes of energy proportional to the change in acceleration, \( dE \propto da \) [8]. One of these fluxes will be in the direction of the change in the acceleration of the mirror, and will have negative energy. The other, in the opposite direction, will have positive energy.

We first consider the situation from the point of view of an inertial observer watching the mirrored box accelerate from left to right. If the box increases its acceleration, two fluxes of energy will enter the box. The flux from the rear (left) wall will be negative and
the flux from the front (right) wall will be positive. However, these fluxes will not be equal. As the box accelerates, it will undergo Lorentz contraction. The rear wall will therefore be forced to accelerate, and change its acceleration, at a higher rate than the front wall, and will thus emit a larger flux. As a result, the inertial observer sees a negative energy density developing inside the box.

Now, let us consider the situation from the point of view from an observer inside the box, accelerating with it. This observer does not notice a negative energy density developing inside the box. Indeed, this observer, who started in the empty Minkowski vacuum, still believes that the interior of the box is (apart from himself) empty. Thus, with respect to what he sees as the vacuum state, the exterior of the box is filled with a positive energy fluid. This fluid is none other than the acceleration radiation described by UW. It should be emphasised that the bath of acceleration radiation seen by the accelerating observer is an artifact of this observer measuring energy with respect to the vacuum of his non-inertial (accelerated) frame. The true stress-energy for the quantum fields and accelerated mirrors in flat space is properly described by the inertial observer, who sees a negative energy density inside the box.

Let us now return to the case of the rigid box being lowered toward a black hole. Both the top and bottom reflecting walls will undergo a change of acceleration. The positive energy flux from both mirrors will be toward the horizon, the negative energy fluxes away from the horizon. Thus, as we lower the box, positive energy will flow from the mirror at the top of the box into the boxes interior, while at the same time, negative energy will flow from the bottom mirror into the box. But for a box of fixed proper length, the change in acceleration during lowering is larger at the bottom than at the top. Therefore, the flux from the bottom mirror will be larger, and there is a net negative energy flow into the box. The interior of a box which is initially empty will consequently acquire a negative energy density through the lowering process. This negative energy density is exactly what one would expect from the Boulware state.

Using a 1+1 dimensional model, which we believe captures the essential features of the
problem, we will show that energy of the contents of the box is indeed correctly measured with respect to the (negative) energy of the Boulware state. Furthermore, it is easy to show that the measurement of the box’s internal stress-energy with respect to the Boulware state is in full agreement with UW. We wish to stress that we obtain the Boulware vacuum energy from considering only the acceleration of the reflecting walls of the box in a flat background. This, we feel, is a remarkable result which may be exploited more fully in the future.

II. 1+1 DIMENSIONAL BLACK HOLE VACUUM STATES

Let us begin by considering a 1+1 dimensional black hole with metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)}$$

$$f(r_0) = 0, f'(r_0) = 2\kappa$$

(2.1)

where $r_0$ is the horizon radius, $f'$ denotes $df/dr$, and $\kappa$ is a constant. The redshift factor for metric (2.1) is $\chi = \sqrt{f}$ and $dz := dr/\sqrt{f}$ defines $z$ which measures the proper distance from the horizon.

Let us introduce null coordinates $u$ and $v$ defined by

$$dv := dt + \frac{dr}{f(r)} = -n_a dx^a,$$

$$du := dt - \frac{dr}{f(r)} = -l_a dx^a,$$

(2.2)

where indices $a, b, c, \ldots$ range over 0, 1. The expectation value for the stress-energy tensor of a massless field on the background (2.1) can be written in the form

$$T^{ab} = \frac{1}{2} T_c^c g^{ab} + E(l^a l^b + n^a n^b) + F l^a l^b,$$

(2.3)

where the unspecified functions $T_c^c(r), E(r)$ and $F(r)$ correspond to vacuum polarisation, an isotropic radiation field and a net outward flux respectively.

For a massless scalar field, the function $T_c^c$ is given by the “trace anomaly”,

$$T_c^c = \frac{1}{24\pi} R = -\frac{1}{24\pi} f''.$$  

(2.4)
where $R$ is the curvature scalar for metric (2.1). One can obtain the remaining components from the conservation law, $T^{ab}_{\mid a} = 0$, where $\mid$ denotes covariant differentiation. In terms of $E$ and $F$ the conservation law takes the form

$$F'(r) = 0,$$
$$E'(r) = -\frac{1}{4} f(r) \frac{d}{dr} T^c_c(r).$$

(2.5)

The specific vacuum state with respect to which the expectation value of the stress-energy tensor is taken is given by the boundary conditions which are imposed on (2.5).

We will be interested in two types of vacuum stress-energy here. The Boulware state appears empty, apart from the vacuum polarisation represented by (2.4), to stationary observers. This is expressed by the boundary condition $T^{ab} \to 0$ as $r \to \infty$. This condition and the conservation equations (2.5) imply

$$E \equiv E_B = \frac{1}{48\pi} \left( \frac{1}{2} f f'' - \frac{1}{4} f'^2 \right),$$
$$F \equiv F_B = 0.$$  

(2.6)

When (2.4) and (2.6) are substituted into (2.3) the stress-energy takes the form of a stationary fluid with energy density and pressure

$$\rho_B = \frac{1}{24\pi} \left( f'' - \frac{f'^2}{4f} \right),$$
$$P_B = -\frac{1}{24\pi} \frac{f'^2}{4f},$$

(2.7)

(2.8)

respectively.

The Hartle-Hawking state is the one which is appropriate for an eternal black hole inside a cavity with reflecting walls, in thermal equilibrium with its own radiation. It appears empty (modulo vacuum polarisation) to free-falling observers at the horizon. This corresponds to the boundary condition that the stress-energy be regular on both the past and future event horizons. By imposing this boundary condition on equations (2.5) we find that $E$ and $F$ take the form
\[ E \equiv E_{HH} = \frac{1}{48\pi} \left( \frac{1}{2} f f'' + \kappa^2 - \frac{1}{4} f'^2 \right) \]
\[ F \equiv F_{HH} = 0 \quad (2.9) \]

Thus, the expectation value of the stress-energy in the Hartle-Hawking state also takes the form of a stationary fluid with energy density and pressure
\[ \rho_{HH} = \frac{1}{24\pi} \left( f'' - \frac{4\kappa^2 - f'^2}{4f} \right), \quad (2.10) \]
\[ P_{HH} = -\frac{1}{24\pi} \frac{4\kappa^2 - f'^2}{4f}, \quad (2.11) \]
respectively. Notice that as \( r \to \infty \) we have \( P_{HH} \approx \rho_{HH} \approx \kappa^2/24\pi \). This is the thermodynamical equation of state for black-body radiation at temperature
\[ T \equiv T_{BH} = \kappa/2\pi. \quad (2.12) \]

\( T_{BH} \) is taken to be the temperature of the black hole.

### III. ACCELERATION RADIATION AND THE BOULWARE STATE

In UW, the energy flux from a accelerating mirror in 1+1 dimensions is considered. They find, in accordance with Fulling and Davies([8]), that \( dE = -da/12\pi \) from which we obtain
\[ \frac{dE}{dz} = -\frac{1}{12\pi} \frac{da}{dz}. \quad (3.1) \]

For the spacetime (2.1), the magnitude of the four acceleration, \( a \), at a proper distance \( z \) from the horizon is given by
\[ a = \frac{1}{\chi} \frac{d\chi}{dz} = \frac{f'}{2\sqrt{f}}. \quad (3.2) \]

Thus, the magnitude of the energy flux from a mirror is
\[ dE = -\frac{1}{24\pi} \left( f'' - \frac{f'^2}{2f} \right) dz. \quad (3.3) \]

Now, let us consider a rigid box with reflecting walls being lowered adiabatically toward the black hole. Let us assume the top and bottom walls are rigidly separated by a proper
length $\ell$ which is much less than the radius of the black hole. At the surface labeled by $z_0$ in the interior of the box, the energy due to the acceleration radiation will be the energy from the flux from the top of the box, blueshifted to the appropriate value, plus the energy from the flux from the bottom of the box, redshifted to the appropriate value,

$$dE_{\text{net}}(z_0) = \frac{1}{24\pi} \left\{ \frac{\sqrt{f(z_T)}}{\sqrt{f(z_0)}} \left( f'' - \frac{f'^2}{2f} \right)_T - \frac{\sqrt{f(z_B)}}{\sqrt{f(z_0)}} \left( f'' - \frac{f'^2}{2f} \right)_B \right\} dz,$$

(3.4)

where the subscripts 0, B, and T denote quantities at $z_0$, the bottom, and the top of the box respectively. In obtaining (3.4) we have taken advantage of the fact that since the proper length of the box, $\ell$, is assumed constant, $z_B = z_T + \ell$, and therefore $dz_B = dz_T = dz$.

For small $\ell$,

$$\frac{dF}{dz} \simeq \frac{F(z + \ell) - F(z)}{\ell},$$

(3.5)

for any function $F(z)$. Therefore, we rewrite (3.4):

$$dE_{\text{net}}(z_0) \simeq \frac{\ell}{24\pi\sqrt{f}} \left\{ \frac{d}{dz} \left[ \sqrt{f} \left( f'' - \frac{f'^2}{2f} \right) \right] \right\} dz.$$

(3.6)

The total energy is the sum of the energies entering the box as it is lowered from infinity to $z_0$,

$$E_{\text{net}}(z_0) \simeq \frac{\ell}{24\pi} \int_{z_0}^{\infty} \frac{1}{\sqrt{f}} \left\{ \frac{d}{dz} \left[ \sqrt{f} \left( f'' - \frac{f'^2}{2f} \right) \right] \right\} dz$$

$$\simeq \frac{\ell}{24\pi} \left( f'' - \frac{f'^2}{4f} \right)_{z_0};$$

(3.7)

But this is just the energy of the Boulware state (2.7). Thus, the energy of the matter content of the box is properly measured with respect to the Boulware vacuum energy.

Alternatively, let us examine the forces involved in lowering the box toward the black hole. By taking the Boulware state to be the vacuum state inside the box, and the Hartle-Hawking state to be the state outside the box, one can recover the results of UW. To prove this, let us consider the contribution of radiation pressure on the top and bottom of the box to the force needed to lower it. The pressure on each of the reflecting walls is the difference
between the Hartle-Hawking fluid on the outside of the box and the Boulware fluid on the inside,

\[ P_{\text{net}} = P_{\text{HH}} - P_{B} = \frac{\kappa^2}{24\pi f}. \]  

(3.8)

Thus the net contribution to the force needed to lower the box is

\[ F_{\text{net}} = \frac{\kappa^2}{24\pi} \left( \frac{1}{f_T} - \frac{1}{f_B} \right) = \frac{\pi}{6} (T_T^2 - T_B^2), \]  

(3.9)

where \( T_T = T_{BH}/\chi_T \) and \( T_B = T_{BH}/\chi_B \) are the redshifted values of the black hole temperature at the top and bottom of the box respectively. This is precisely the expression obtained by UW for the contribution of the acceleration radiation to the force needed to lower the box.

IV. CONCLUSION

We have examined once more the long debated question of whether the validity of the GSL implies an entropy bound of the type found by Bekenstein (1.5). We considered the standard *Gedankenexperimente* of lowering a box containing matter fields from infinity to a finite proper distance from a black hole. We have shown that the effect of the acceleration radiation on the energy density inside the box is exactly the same as obtained by considering the vacuum state of the interior of the box to be the Boulware state, and concluded that the acceleration of the box has induced a Boulware state inside it.

That the in-vacuum for the box’s interior is the Boulware state is to be expected on general grounds. The interior vacuum is initially the Boulware state (the vacuum for a stationary observer at infinity) and is invariant under the adiabatic (quasi-static) process of lowering. This is in contradiction with Li and Liu [7], who, by ignoring the effect of acceleration radiation, implicitly assume a Minkowski vacuum inside the box. We have, furthermore, shown that using the Boulware state for the interior of the box leads to complete agreement with the results of UW. This is not surprising since we started with a relation
derived by UW, (3.4), in our derivation of the Boulware energy density for the interior of the box.

What may be surprising is that the acceleration of an empty box in flat spacetime can be used to obtain the energy density of the Boulware state. However, this is not difficult to understand. As we mentioned, it is expected that the state inside an adiabatically lowered box is the Boulware state. However, a small enough box sees the black hole field as being essentially homogeneous. By the equivalence principle, such a box is unable to determine whether it is accelerating in a gravitational field or in Rindler space [10]. We have demonstrated how this allows us to obtain the Boulware energy density simply by considering the energy balance between two accelerating mirrors in 1+1 dimensional Rindler space. We are presently investigating possibility of carrying out this procedure in 3+1 dimensions.

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