$Z_{2k}$-CODE VERTEX OPERATOR ALGEBRAS

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ABSTRACT. We study a simple, self-dual, rational, and $C_2$-cofinite vertex operator algebra of CFT-type whose simple current modules are graded by $Z_{2k}$. Based on those simple current modules, a vertex operator algebra associated with a $Z_{2k}$-code is constructed. The classification of irreducible modules for such a vertex operator algebra is established. Furthermore, all the irreducible modules are realized in a module for a certain lattice vertex operator algebra.

1. INTRODUCTION

Let $V$ be a simple, self-dual, rational, and $C_2$-cofinite vertex operator algebra of CFT-type. Then the set $\text{Irr}(V)_{sc}$ of equivalence classes of simple current $V$-modules is closed under the fusion product $\boxtimes_V$, and it is graded by a finite abelian group, say $C$. That is, $\text{Irr}(V)_{sc} = \{A^\alpha \mid \alpha \in C\}$ with $A^\alpha$, $\alpha \in C$, being inequivalent to each other, $A^0 = V$, and $A^\alpha \boxtimes_V A^\beta = A^{\alpha+\beta}$ for $\alpha, \beta \in C$. If $D$ is a subgroup of $C$ such that the conformal weight $h(A^\alpha)$ of $A^\alpha$ is an integer for $\alpha \in D$, then the direct sum $\bigoplus_{\alpha \in D} A^\alpha$ has either a simple vertex operator algebra structure or a simple vertex operator superalgebra structure, which extends the $V$-module structure [13, Theorem 3.12]. In such a case $\bigoplus_{\alpha \in D} A^\alpha$ is called a simple current extension of $V$.

The theory of simple current extensions of vertex operator algebras has been developed extensively, see for example [2, 4, 6, 12, 15, 17, 18, 20]. Nowadays it is not hard to construct new vertex operator (super)algebras as simple current extensions of known ones. One of the examples is the $Z_k$-code vertex operator algebra [3], which is a $D$-graded simple current extension of the tensor product $K(\mathfrak{sl}_2, k)^{\otimes \ell}$ of $\ell$ copies of the parafermion vertex operator algebra $K(\mathfrak{sl}_2, k)$ associated with $\mathfrak{sl}_2$ and an integer $k \geq 2$, and $D$ is an additive subgroup of $(Z_k)^{\ell}$ satisfying a certain condition.

In this paper, we study a $D$-graded simple current extension $U_D$ of the tensor product $(U^0)^{\otimes \ell}$ of $\ell$ copies of a vertex operator algebra $U^0$ such that $\text{Irr}(U^0)_{sc}$ is graded by $Z_{2k}$ for an integer $k \geq 2$. Here $D$ is an additive subgroup of $(Z_{2k})^{\ell}$.

Let $N = \sqrt{2}A_{k-1}$ be $\sqrt{2}$ times an $A_{k-1}$ root lattice. Then a vertex operator algebra $V_N$ associated with the lattice $N$ contains a subalgebra isomorphic to

$$L(c_1, 0) \otimes \cdots \otimes L(c_{k-1}, 0) \otimes K(\mathfrak{sl}_2, k),$$

where $L(c_m, 0)$, $1 \leq m \leq k-1$, are the Virasoro vertex operator algebras of discrete series. The vertex operator algebra $U^0$ is defined to be the commutant of $S = L(c_1, 0) \otimes \cdots \otimes L(c_{k-2}, 0)$ in $V_N$. The vertex operator algebra $U^0$ was previously known [2, 5]. There are two descriptions of $U^0$, one is a $Z_{k-1}$-graded simple current extension of $K(\mathfrak{sl}_2, k-1)^{\otimes \ell}V_{Z_{2d}}$ with $\langle d, d \rangle = 2(k-1)k$, and the other is a non-simple current extension of $L(c_{k-1}, 0) \otimes$
K(\mathfrak{s}l_2, k) (Theorem 3.2). We review the irreducible $U^0$-modules (Theorem 3.6) and fusion rules (Theorem 4.1) in our notation. Moreover, we show how the irreducible $U^0$-modules appear in $V_N$ (Lemma 5.1).

The $\mathbb{Z}_{2k}$-grading of the set Irr($U^0$)$_{sc} = \{U^l \mid 0 \leq l < 2k\}$ of equivalence classes of simple current $U^0$-modules corresponds to the $\mathbb{Z}_{2k}$-part of the discriminant group $N^\circ/N \cong (\mathbb{Z}_2)^{k-2} \times \mathbb{Z}_{2k}$ of the lattice $N$ (Eq. (5.2)). We also recall that the $\mathbb{Z}_k$-grading of Irr($K(\mathfrak{s}l_2, k)$)$_{sc}$ used in [4] for $\mathbb{Z}_k$-code vertex operator algebras corresponds to the $\mathbb{Z}_k$-part of $(\mathbb{Z}_2)^{k-2} \times \mathbb{Z}_{2k}$.

Once the necessary properties of the vertex operator algebra $U^0$ are obtained, the construction of the vertex operator algebra $U_D$ is straightforward. In fact, $U_D$ is defined to be the commutant of $S^\otimes \ell$ in a vertex operator algebra $V_{\Gamma_D}$ associated with a certain positive definite integral lattice $\Gamma_D$ (Theorem 5.7). It turns out that $U_D$ is a direct sum of a $D$-graded set of simple current $(U^0)^{\otimes \ell}$-modules $U_{\xi} = U^{\xi_1} \otimes \cdots \otimes U^{\xi_\ell}$, $\xi = (\xi_1, \ldots, \xi_\ell) \in D$. We construct all the irreducible $\chi$-twisted $U_D$-modules for $\chi \in \text{Hom}(D, \mathbb{C}^\times)$ in $V_{(N^\circ)^{\ell}}$, and classify them (Theorems 7.4 and 7.6). The arguments concerning $U_D$ and its irreducible $\chi$-twisted modules are similar to those in Sections 8 and 9 of [4].

If $k = 2$, then $U^0$ is isomorphic to a rank one lattice vertex operator algebra $V_{\mathbb{Z}_2}$. In the case $k = 4$, the $\mathbb{Z}_8$-code vertex operator algebra $U_D$ was studied in [21]. Our result is a generalization of [21] to an arbitrary $k \geq 2$.

This paper is organized as follows. In Section 2, we recall basic properties of the Virasoro vertex operator algebra of discrete series and the parafermion vertex operator algebra associated with $\mathfrak{s}l_2$ and a positive integer $k$. In Section 3 we define the vertex operator algebra $U^0$ and describe it in two ways. We classify irreducible $U^0$-modules as well. Fusion rules for irreducible $U^0$-modules are discussed in Section 4. In Section 4 we introduce the positive definite integral lattice $\Gamma_D$ and the vertex operator algebra or a vertex operator superalgebra $U_D$ for a $\mathbb{Z}_{2k}$-code $D$. Finally, in Section 5, we classify the irreducible $U_D$-modules. The weight and the dimension of the top level of the simple current $U^0$-module $U^l$, $0 \leq l < 2k$, are calculated in Appendix A.

We use the symbol $\boxtimes_{V'}$ to denote the fusion product over a vertex operator algebra $V$. We also use the symbol $\otimes$ to denote the tensor product of vertex operator algebras and their modules as in [10].

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## 2. Preliminaries

In this section, we recall the vertex operator algebra associated with the discrete series of Virasoro algebra and the parafermion vertex operator algebra associated with $\mathfrak{s}l_2$ and a positive integer $k$.

### 2.1. Virasoro vertex operator algebra $L(c_m, 0)$

Let $L(c, 0)$ be the simple Virasoro vertex operator algebra of central charge $c$, and let $L(c, h)$ be its irreducible highest weight module with highest weight $h$. Let

$$c_m = 1 - \frac{6}{(m + 2)(m + 3)}$$
for $m = 1, 2, \ldots$, and
\[
h_{r,s}^{(m)} = \frac{(r(m + 3) - s(m + 2))^2 - 1}{4(m + 2)(m + 3)}
\]
for $1 \leq r \leq m + 1$, $1 \leq s \leq m + 2$. The vertex operator algebra $L(c_m, 0)$ is self-dual, rational, $C_2$-cofinite, and of CFT-type with $L(c_m, h_{r,s}^{(m)})$, $1 \leq s \leq r \leq m + 1$, a complete set of representatives of equivalence classes of irreducible $L(c_m, 0)$-modules [23, Theorem 4.2]. The fusion product of irreducible $L(c_m, 0)$-modules is as follows [23, Theorem 4.3].

\[
L(c_m, h_{r_1,s_1}^{(m)}) \boxtimes L(c_m, h_{r_2,s_2}^{(m)}) = \sum_{i,j} L(c_m, h_{|r_1-r_2|+2i-1,s_1-s_2|+2j-1}^{(m)}),
\]
where the summation is taken over the integers $i$ and $j$ satisfying
\[
1 \leq i \leq \min\{r_1, r_2, m + 2 - r_1, m + 2 - r_2\}, \quad 1 \leq j \leq \min\{s_1, s_2, m + 3 - s_1, m + 3 - s_2\}.
\]

2.2. Parafermion vertex operator algebra $K(sl_2, k)$. We fix the notation for the parafermion vertex operator algebra $K(sl_2, k)$ associated with $sl_2$ and a positive integer $k$. Details about $K(sl_2, k)$ can be found in [3, 4, 10, 14]. If $k = 1$, then $K(sl_2, k)$ reduces to the trivial vertex operator algebra $\mathbb{C}1$. So we assume that $k \geq 2$. Let
\[
L^{(k)} = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_k
\]
with $\langle \alpha_i, \alpha_j \rangle = 2\delta_{i,j}$, and set $\gamma_k = \alpha_1 + \cdots + \alpha_k$.

The vertex operator algebra $V^{(k)}_L$ contains a subalgebra isomorphic to the simple affine vertex operator algebra $L\tilde{sl}_2(k, 0)$ associated with the affine Kac-Moody algebra $\tilde{sl}_2$ at level $k$, and $K(sl_2, k)$ is realized as the commutant of the vertex operator algebra $V_{\mathbb{Z}\gamma_k}$ in $L\tilde{sl}_2(k, 0)$. Let
\[
M^j_{(k)} = \{v \in L\tilde{sl}_2(k, 0) \mid \gamma_k(n)v = -2j\delta_{n,0}v \text{ for } n \geq 0\}
\]
for $0 \leq j < k$. Then $M^0_{(k)} = K(sl_2, k)$, and
\[
L\tilde{sl}_2(k, 0) = \bigoplus_{j=0}^{k-1} M^j_{(k)} \otimes V_{\mathbb{Z}\gamma_k-j\gamma_k/k}.
\]

An irreducible $L\tilde{sl}_2(k, 0)$-module $L\tilde{sl}_2(k, i)$ with $i + 1$ dimensional top level can be constructed in the $V_{L^{(k)}}$-module $V_{(L^{(k)})^0}$ for $0 \leq i \leq k$, where $(L^{(k)})^0 = \frac{1}{2}L^{(k)}$ is the dual lattice of $L^{(k)}$. Let
\[
M^{ij}_{(k)} = \{v \in L\tilde{sl}_2(k, i) \mid \gamma_k(n)v = (i - 2j)\delta_{n,0}v \text{ for } n \geq 0\}
\]
for $0 \leq j < k$. Then $M^0_{(k)} = M^j_{(k)}$, and
\[
L\tilde{sl}_2(k, i) = \bigoplus_{j=0}^{k-1} M^{ij}_{(k)} \otimes V_{\mathbb{Z}\gamma_k+(i-2j)\gamma_k/2k}.
\]

The index $j$ of $M^j_{(k)}$ and $M^{ij}_{(k)}$ can be considered to be modulo $k$. We will use the following properties of $M^0_{(k)}$ and $M^{ij}_{(k)}$. 

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(1) $M_{(k)}^0 = K(\mathfrak{sl}_2, k)$ is a simple, self-dual, rational, and $C_2$-cofinite vertex operator algebra of CFT-type with central charge $2(k - 1)/(k + 2)$.

(2) $M_{(k)}^{i,j}$, $0 \leq i \leq k$, $0 \leq j < k$, are irreducible $M_{(k)}^0$-modules with
\[ M_{(k)}^{i,j} \cong M_{(k)}^{k-i,j-i}, \quad (2.5) \]

and $M_{(k)}^{i,j}$, $0 \leq j < i \leq k$, form a complete set of representatives of equivalence classes of irreducible $M_{(k)}^0$-modules.

(3) The top level of $M_{(k)}^{i,j}$ is one dimensional and its conformal weight is
\[ h(M_{(k)}^{i,j}) = \frac{1}{2k(k + 2)}(k(i - 2j) - (i - 2j)^2 + 2k(i - j + 1)j), \quad (2.6) \]

for $0 \leq j \leq i \leq k$. Eq. (2.6) is valid even in the case $j = i$.

(4) The fusion product of irreducible $M_{(k)}^0$-modules is
\[ M_{(k)}^{i_1,j_1} \boxtimes M_{(k)}^{i_2,j_2} = \sum_{r \in R(i_1,i_2)} M_{(k)}^{(2j_1-i_1+2j_2-i_2+r)/2}, \quad (2.7) \]

where $R(i_1, i_2)$ is the set of integers $r$ satisfying
\[ |i_1 - i_2| \leq r \leq \min\{i_1 + i_2, 2k - i_1 - i_2\}, \quad i_1 + i_2 + r \in 2\mathbb{Z}. \]

In particular, $M_{(k)}^j$, $0 \leq j < k$, are the simple currents, and
\[ M_{(k)}^p \boxtimes M_{(k)}^{i,j} = M_{(k)}^{i,j+p}. \]

(5) If $k \geq 3$, then the automorphism group $\text{Aut} M_{(k)}^0$ of $M_{(k)}^0$ is generated by an involution $\theta$, and $M_{(k)}^{i,j} \circ \theta \cong M_{(k)}^{i,j}$. The automorphism $\theta$ is induced from the $-1$-isometry $\alpha \mapsto -\alpha$ of the lattice $L^{(k)}$.

3. Vertex operator algebra $U^0$

In this section, we discuss a vertex operator algebra $U^0$. The vertex operator algebra $U^0$ and its irreducible modules have already been studied [2], [3], Section 4.4.2, see also [20, Section 3.1]. We describe $U^0$ and all the irreducible $U^0$-modules in a lattice vertex operator algebra and in its module for later use.

We fix an integer $k \geq 2$. Let $\alpha_i$, $1 \leq i \leq k$, and $L^{(k)}$ be as in Section 2.2. Thus $\langle \alpha_i, \alpha_j \rangle = 2\delta_{i,j}$ and $L^{(k)} = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_k$. Let
\[ \gamma_{k-1} = \alpha_1 + \cdots + \alpha_{k-1}, \quad \gamma_k = \alpha_1 + \cdots + \alpha_k, \quad d = \gamma_{k-1} - (k - 1)\alpha_k. \quad (3.1) \]

Then $d = \gamma_k - k\alpha_k$, $\langle \gamma_k, \gamma_k \rangle = 2k$, $\langle d, d \rangle = 2(k - 1)k$, and $\langle \gamma_k, d \rangle = 0$.

For $a = a(k) = (a_1, \ldots, a_k) \in \{0, 1\}^k$, let $\delta_{a(k)} = \frac{1}{2} \sum_{p=1}^{k} a_p \alpha_p$. The vertex operator algebra $V_{L^{(k)}}$ associated with the lattice $L^{(k)}$ contains a subalgebra isomorphic to
\[ L(c_1, 0) \otimes \cdots \otimes L(c_k-1, 0) \otimes \hat{\mathfrak{sl}_2}(k, 0), \]

and
\[ V_{L^{(k)}}, \delta_{a(k)} = \bigoplus_{0 \leq i_s \leq s} L(c_1, h_{i_1+1,i_2+1}) \otimes \cdots \otimes L(c_k-1, h_{i_{k-1}+1,i_k+1}) \otimes \hat{\mathfrak{sl}_2}(k, i_k) \quad (3.2) \]
as an $L(c_{1},0) \otimes \cdots \otimes L(c_{k-1},0) \otimes L\hat{\gamma}_{k}(k,0)$-module, where $b_{s} = \sum_{p=1}^{s} a_{p}$ [19, 22, 24], see also [3, Section 5]. Since $L^{(k-1)} \oplus \mathbb{Z}\alpha_{k} = L^{(k)}$, it follows that

$$V_{L^{(k-1)}+\delta_{\alpha_{(k-1)}}} \otimes V_{\mathbb{Z}\alpha_{k}+\alpha_{k}/2} = V_{L^{(k)}+\delta_{\alpha_{k}}}.$$  \hspace{1cm} (3.3)

Let $\omega^{s}$ be the conformal vector of the Virasoro vertex operator algebra $L(c_{s},0), 1 \leq s \leq k-1$. We apply (3.2) to $V_{L^{(k-1)}+\delta_{\alpha_{(k-1)}}}$ for $k-1$ in place of $k$. Then

$$\{v \in V_{L^{(k-1)}+\delta_{\alpha_{(k-1)}}} \otimes V_{\mathbb{Z}\alpha_{k}+\alpha_{k}/2} | \omega^{s}_{(1)}v = h_{i_{s+1},i_{s+1}+1}^{(s)}v, 1 \leq s \leq k-2\}$$

$$= L\hat{\gamma}_{k}(k-1,i_{k-1}) \otimes V_{\mathbb{Z}\alpha_{k}+\alpha_{k}/2}.$$  \hspace{1cm} (3.4)

We also have

$$\{v \in V_{L^{(k)}+\delta_{\alpha_{(k)}}} | \omega^{s}_{(1)}v = h_{i_{1},i_{1}+1}^{(1)}v, 1 \leq s \leq k-2\}$$

$$= \bigoplus_{0 \leq i_{k} \leq k} L(c_{k-1},h_{i_{k-1}+1,i_{k}+1}) \otimes L\hat{\gamma}_{k}(k,i_{k})$$  \hspace{1cm} (3.5)

and

$$L\hat{\gamma}_{k}(k-1,i_{k-1}) \otimes V_{\mathbb{Z}\alpha_{k}+\alpha_{k}/2} = \bigoplus_{0 \leq i_{k} \leq k} L(c_{k-1},h_{i_{k-1}+1,i_{k}+1}) \otimes L\hat{\gamma}_{k}(k,i_{k})$$  \hspace{1cm} (3.6)

for $0 \leq i_{k-1} \leq k-1$ with $i_{k-1} \equiv b_{k-1} \pmod{2}$.

We set $i = i_{k-1}$ for simplicity of notation. Since $\langle \gamma_{k-1}, \alpha_{k} \rangle = 0$, the left hand side of (3.3) is

$$\bigoplus_{j=0}^{k-2} M_{k-1}^{i_{2},j} \otimes V_{\mathbb{Z}\gamma_{k-1}+(i-2j)\gamma_{k-1}/2(k-1)+\mathbb{Z}\alpha_{k}+\alpha_{k}/2}$$  \hspace{1cm} (3.7)

by (2.4) for $k-1$ in place of $k$. Since $\gamma_{k-1} = \gamma_{k} - \alpha_{k}$, and since $\alpha_{k} = (\gamma_{k} - d)/k$, we have a coset decomposition

$$\mathbb{Z}\gamma_{k-1} + \mathbb{Z}\alpha_{k} = \bigcup_{p=0}^{k-1} \left( \mathbb{Z}d + \mathbb{Z}\gamma_{k} + \frac{p}{k}(\gamma_{k} - d) \right)$$  \hspace{1cm} (3.8)

with $[\mathbb{Z}\gamma_{k-1} + \mathbb{Z}\alpha_{k} : \mathbb{Z}d + \mathbb{Z}\gamma_{k}] = k$, and

$$\mathbb{Z}\gamma_{k-1} + \frac{i - 2j}{2(k-1)}\gamma_{k-1} + \mathbb{Z}\alpha_{k} + \frac{1}{2}\alpha_{k} \alpha_{k}$$

$$= \bigcup_{p=0}^{k-1} \left( \mathbb{Z}d + \frac{1}{2(k-1)}((-k-1)(a_{k} + 2p) + i - 2j)d \right)$$  \hspace{1cm} (3.9)

for $0 \leq j < k-1$. The left hand side of (3.9) is a coset of $\mathbb{Z}\gamma_{k-1} + \mathbb{Z}\alpha_{k}$, and it is determined by $j$ modulo $k-1$, whereas the right hand side is a union of cosets of $\mathbb{Z}d + \mathbb{Z}\gamma_{k}$, and $j$ can not be considered to be modulo $k-1$ for these cosets.

For $0 \leq p < k$, define $0 \leq l < 2k$ by

$$l \equiv i + a_{k} + 2(p-j) \pmod{2k},$$  \hspace{1cm} (3.9)
Then (3.8) can be written as

\[ Z\gamma_{k-1} + \frac{i - 2j}{2(k-1)} \gamma_k + Z d + \frac{i - 2j}{2(k-1)} d + (Z \gamma_k + \frac{l}{2} \gamma_k). \]  

(3.10)

Since \( \langle b \rangle \equiv 0 \pmod{2} \), it follows from (3.6) and (3.10) that

\[ L_{\tilde{\alpha}}(k-1, i) \otimes V_{Z \alpha_k + a_k \alpha_k/2} \]

\[ = \bigoplus_{j=0}^{k-2} \bigoplus_{0 \leq l < 2k \atop l \equiv i + a_k \pmod{2}} M_{(k-1)}^{i,j} \otimes V_{Zd - l-2k + (i-2j)d/2(k-1)} \otimes V_{Z \gamma_k + t \gamma_k/2k}. \]  

(3.11)

Let

\[ U^{i,l} = \{ v \in L_{\tilde{\alpha}}(k-1, i) \otimes V_{Z \alpha_k + a_k \alpha_k/2} \mid \gamma_k(n)v = l \delta_{n,0}v \text{ for } n \geq 0 \} \]

for \( 0 \leq i \leq k-1, 0 \leq l < 2k \), which is the multiplicity of \( V_{Z \gamma_k + t \gamma_k/2k} \) in (3.11). Then

\[ U^{i,l} = \bigoplus_{j=0}^{k-2} M_{(k-1)}^{i,j} \otimes V_{Zd - l-2k + (i-2j)d/2(k-1)}. \]

The index \( l \) of \( U^{i,l} \) can be considered to be modulo \( 2k \).

The right hand side of (3.3) is

\[ \bigoplus_{0 \leq i_k \leq k \atop i_k \equiv b_k \pmod{2}} L(c_{k-1}, h_{i_k+1, i_k+1}^{(k-1)}) \otimes M_{(k)}^{i_k,p} \otimes V_{Z \gamma_k + (i_k-2p) \gamma_k/2k} \]  

(3.12)

by (2.4). Recall that \( l \) is defined in (3.9) and that \( i = i_{k-1} \) satisfies the condition \( i \equiv b_{k-1} \pmod{2} \). Since \( b_k = b_{k-1} + a_k \), we have \( i_k - l \in 2\mathbb{Z} \) for \( i_k \) such that \( i_k \equiv b_k \pmod{2} \). Then \( V_{Z \gamma_k + (i_k - 2p) \gamma_k/2k} \) agrees with \( V_{Z \gamma_k + t \gamma_k/2k} \) if and only if \( p \equiv (i_k - l)/2 \pmod{k} \). Hence the multiplicity of \( V_{Z \gamma_k + t \gamma_k/2k} \) in (3.12) is

\[ \{ v \in \bigoplus_{0 \leq i_k \leq k \atop i_k \equiv b_k \pmod{2}} L(c_{k-1}, h_{i_k+1, i_k+1}^{(k-1)}) \otimes L_{\tilde{\alpha}}(k, i_k) \mid \gamma_k(n)v = l \delta_{n,0}v \text{ for } n \geq 0 \} \]

\[ = \bigoplus_{0 \leq i_k \leq k \atop i_k \equiv b_k \pmod{2}} L(c_{k-1}, h_{i_k+1, i_k+1}^{(k-1)}) \otimes M_{(k)}^{i_k, (i_k-l)/2}. \]  

(3.13)

Therefore, the following theorem is proved.
Theorem 3.1. The multiplicity $U^{i,l}$ of $V_{Z\gamma_k+t\gamma_k/2k}$ on both sides of (3.3) is described in two ways, namely,

$$U^{i,l} = \bigoplus_{j=0}^{k-2} M^{i,j}_{(k-1)} \otimes V_{Zd-jd/2k + (i-2)j/d/(k-1)}$$

$$= \bigoplus_{0 \leq i_k \leq k, i_k \equiv b_k \pmod{2}} L(c_{k-1}, h_{i+1,i_k+1}^{(k-1)}) \otimes M^{i,0}_{(k)}$$

for $0 \leq i \leq k - 1$ with $i \equiv b_k - 1 \pmod{2}$, and $0 \leq l < 2k$ with $l \equiv b_k \pmod{2}$. The first one is a direct sum of irreducible $M^{0}_{(k-1)} \otimes V_{Zd}$-modules, and the second one is a direct sum of irreducible $L(c_{k-1}, 0) \otimes M^{0}_{(k)}$-modules.

In the case $a(k) = (0, \ldots, 0)$, take the commutant of $S = L(c_1, 0) \otimes \cdots \otimes L(c_{k-2}, 0)$ in $V_{L(k-1)} \otimes V_{Z\alpha_k} = V_{L(k)}$. Then we have

$$L_{\hat{sl}_2}(k-1, 0) \otimes V_{Z\alpha_k} = \bigoplus_{j=0}^{\lfloor k/2 \rfloor} L(c_{k-1}, h_{1,2j+1}^{(k-1)}) \otimes L_{\hat{sl}_2}(k, 2j), \quad (3.14)$$

where $\lfloor k/2 \rfloor$ is the largest integer which does not exceed $k/2$.

Let $U^0$ be the commutant of $V_{Z\gamma_k}$ in (3.14). Then $U^0 = U^{0,0}$ in the notation of Theorem 3.1. In particular,

$$U^0 = \bigoplus_{j=0}^{k-2} M^{j}_{(k-1)} \otimes V_{Zd-jd/(k-1)} \quad (3.15)$$

which is a $Z_{k-1}$-graded simple current extension of $M^{0}_{(k-1)} \otimes V_{Zd}$. Hence the following theorem holds by [27, Theorem 2.14], see also [3, 5, Section 4.4.2].

Theorem 3.2. (1) $U^0$ is a simple, self-dual, rational, and $C_2$-cofinite vertex operator algebra of CFT-type with central charge $3(k-1)/(k+1)$.

(2) $U^0$ is described in two ways, namely,

$$U^0 = \bigoplus_{j=0}^{k-2} M^{j}_{(k-1)} \otimes V_{Zd-jd/(k-1)}$$

$$= \bigoplus_{j=0}^{\lfloor k/2 \rfloor} L(c_{k-1}, h_{1,2j+1}^{(k-1)}) \otimes M^{2j,j}_{(k)}.$$

The first one is a $Z_{k-1}$-graded simple current extension of $M^{0}_{(k-1)} \otimes V_{Zd}$, and the second one is a non-simple current extension of $L(c_{k-1}, 0) \otimes M^{0}_{(k)}$.

The fusion product of irreducible $M^{0}_{(k-1)}$-modules $M^{j}_{(k-1)}$ agrees with the fusion product of irreducible $V_{Zd}$-modules $V_{Zd-jd/(k-1)}$ under the correspondence

$$M^{j}_{(k-1)} \leftrightarrow V_{Zd-jd/(k-1)}, \quad 0 \leq j < k - 1.$$
Similarly, the fusion product of irreducible $L(c_{k-1}, 0)$-modules $L(c_{k-1}, h^{(k-1)}_{1,2j+1})$ agrees with the fusion product of irreducible $M^0_{(k)}$-modules $M^{2j, j}_{(k)}$ under the correspondence

$$L(c_{k-1}, h^{(k-1)}_{1,2j+1}) \leftrightarrow M^{2j, j}_{(k)}, \quad 0 \leq j \leq \lfloor k/2 \rfloor.$$ 

In fact, for $0 \leq p \leq q \leq \lfloor k/2 \rfloor$, we have

$$L(c_{k-1}, h^{(k-1)}_{1,2p+1}) \boxtimes L(c_{k-1}, h^{(k-1)}_{1,2q+1}) = \sum_{j=0}^{\min\{2p, k-2q\}} L(c_{k-1}, h^{(k-1)}_{1,2(q-j)+1})$$

by (2.1), and

$$M^{2p, p}_{(k)} \boxtimes M^0_{(k)} \boxtimes M^{2q, q}_{(k)} = \sum_{j=0}^{\min\{2p, k-2q\}} M^{2(p-j), q-p-j}_{(k)}$$

by (2.7).

Since the left hand side of (1.11) is an $L_{\mathfrak{sl}_2} (k-1, 0) \otimes V_{\mathfrak{sl}_2}$-module, and since $U^0$ is the commutant of $V_{\mathfrak{sl}_2}$ in $L_{\mathfrak{sl}_2} (k-1, 0) \otimes V_{\mathfrak{sl}_2}$, it follows that $U^{i,l}$ is a $U^0$-module.

For simplicity of notation, set

$$A^i = M^0_{(k-1)} \otimes V_{\mathfrak{sl}_2} \otimes \mathbb{C}$$

$$X(i, j, l) = M^{i,j}_{(k-1)} \otimes V_{\mathfrak{sl}_2} \otimes \mathbb{C}$$

for $0 \leq i \leq k-1$, $0 \leq j < k-1$, and $0 \leq l \leq 2k$. Then $U^0 = \bigoplus_{k-1}^{k} A^i$ and $U^{i,l} = \bigoplus_{j=0}^{k-1} X(i, j, l)$ as $A^0$-modules. For fixed $i$ and $l$, $X(i, j, l)$, $0 \leq j < k-1$, are inequivalent irreducible $A^0$-modules. Thus the $U^0$-module $U^{i,l}$ is irreducible. In fact, the $U^0$-module structure on the direct sum $\bigoplus_{j=0}^{k-1} X(i, j, l)$ which extends the $A^0$-module structure is unique [23, Proposition 3.8]. Since

$$A^i \boxtimes A^0 X(i, 0, l) = X(i, j, l), \quad (3.16)$$

we have

$$U^{i,l} = U^0 \boxtimes A^0 X(i, 0, l). \quad (3.17)$$

The following lemma is a consequence of the isomorphism $M^{i,j}_{(k-1)} \cong M^{k-1-i,j-i}_{(k-1)}$ of $M^0_{(k-1)}$-modules in (2.5) for $k-1$ in place of $k$.

**Lemma 3.3.** (1) Let $0 \leq i, i' \leq k-1$, $0 \leq j, j' < k-1$, and $0 \leq l, l' < 2k$. Then $X(i, j, l) \cong X(i', j', l')$ as $A^0$-modules if and only if one of the following conditions holds.

(i) $i = i'$, $j = j'$, and $l = l'$.

(ii) $i' = k-1-i$, $j' \equiv j - i \pmod{k-1}$, and $l' \equiv k+l \pmod{2k}$.

(2) $X(i, j, l)$, $0 \leq j < i < k-1$, $0 \leq l < 2k$, are inequivalent to each other.

The above lemma implies the next lemma. **Lemma 3.4.** Let $0 \leq i, i' \leq k-1$ and $0 \leq l, l' < 2k$. Then $U^{i,l} \cong U^{i',l'}$ as $U^0$-modules if and only if one of the following conditions holds.

(i) $i = i'$ and $l = l'$.

(ii) $i' = k-1-i$ and $l' \equiv k + l \pmod{2k}$.

Next, we calculate the difference of the conformal weight of two irreducible $A^0$-modules.
Theorem 3.6. Eq. (3.18) holds.

\[ 0 \leq \frac{j(i-s)}{k-1} \pmod{\mathbb{Z}}. \]  

Indeed, we have \( h(M^{i,j}_{(k-1)}) \) for \( 0 \leq j < i \leq k-1 \) by (2.4) for \( k-1 \) in place of \( k \), and \( h(M^{i,j}_{(k-1)}) \) for \( 0 \leq i \leq j \leq k-1 \) is obtained by using (2.5) for \( k-1 \) in place of \( k \). In fact,

\[ h(M^{i,j}_{(k-1)}) - h(M^{i,0}_{(k-1)}) \equiv \frac{j(i-j)}{k-1} \pmod{\mathbb{Z}}. \]

Since

\[ h(V_{Zd+sd/2(k-1)k-jd/(k-1)}) - h(V_{Zd+sd/2(k-1)k}) \equiv \frac{j(i-s)}{k-1} \pmod{\mathbb{Z}}, \]

Eq. (3.18) holds.

The following theorem holds, see [2, 8, Section 4.4.2], [26, Section 3.1].

**Theorem 3.6.** (1) Any irreducible \( U^0 \)-module is isomorphic to \( U^{i,l} \) for some \( 0 \leq i \leq k-1 \), \( 0 \leq l < 2k \).

(2) The irreducible \( U^0 \)-modules \( U^{i,l} \), \( 0 \leq i \leq k-1 \), \( 0 \leq l < 2k \), are inequivalent to each other except for the isomorphism

\[ U^{i,l} \cong U^{k-1-i,k+l}. \]

(3) There are exactly \( k^2 \) inequivalent irreducible \( U^0 \)-modules.

(4) The conformal weight of any irreducible \( U^0 \)-module except for \( U^0 \) is positive.

**Proof.** Let \( W \) be an irreducible \( U^0 \)-module. Then \( W \) is a direct sum of irreducible \( A^0 \)-modules. Let \( X \) be an irreducible \( A^0 \)-submodule of \( W \). We may assume that \( X = M^{i,0}_{(k-1)} \otimes V_{Zd+sd/2(k-1)k} \) for some \( 0 \leq i \leq k-1 \) and \( 0 \leq s < 2(k-1)k \) by (3.10). Then

\[ A^1 \otimes A^0 X = M^{i,1}_{(k-1)} \otimes V_{Zd+sd/2(k-1)k-jd/(k-1)} \]

is contained in \( W \), so \( h(A^1 \otimes A^0 X) - h(X) \) is an integer. Thus \( s \equiv i \pmod{k-1} \) by Lemma 3.3. Then \( s \equiv i - m(k-1) \pmod{2(k-1)k} \) for some \( 0 \leq m < 2k \). Take \( 0 \leq l < 2k \) such that \( l \equiv i + m \pmod{2k} \). Then \( s \equiv ik - l(k-1) \pmod{2(k-1)k} \), and \( W = U^{i,l} \) by (3.17). Thus the assertion (1) holds.

Lemma 3.4 implies the assertion (2). The assertion (3) is clear from (1) and (2). The conformal weight of any irreducible \( M^0_{(k-1)} \)-module except for \( M^0_{(k-1)} \) is positive. Thus the assertion (4) holds. \( \square \)

There are \( (k-1)^2k^2 \) inequivalent irreducible \( A^0 \)-modules, namely,

\[ \text{Irr}(A^0) = \{ M^{i,j}_{(k-1)} \otimes V_{Zd+sd/2(k-1)k} \mid 0 \leq j < i \leq k-1, 0 \leq s < 2(k-1)k \}. \]

Only \( (k-1)k^2 \) of them can be direct summands of an irreducible \( U^0 \)-module. In fact, let

\[ \text{Irr}^0(A^0) = \{ X(i,j,l) \mid 0 \leq j < i \leq k-1, 0 \leq l < 2k \}. \]

Then each irreducible \( U^0 \)-module is a direct sum of \( k-1 \) inequivalent irreducible \( A^0 \)-modules in \( \text{Irr}^0(A^0) \).
Recall the automorphism $\theta$ of $M_{(k)}^{0}$, which is induced from the $-1$-isometry $\alpha \mapsto -\alpha$ of the lattice $L^{(k)}$. The isometry induces an automorphism of $U^{0}$ of order two as well. We denote the automorphism by the same symbol $\theta$. Then
\[ U^{i, l} \circ \theta \cong U^{i, -l}. \] (3.20)

4. Fusion rule of irreducible $U^{0}$-modules

The fusion rule of the irreducible $U^{0}$-modules was previously known [3, Section 4.4.2], see also [26, Section 3.2]. The fusion rule in our notation for the irreducible $U^{0}$-modules is as follows.

**Theorem 4.1.** Let $0 \leq i_{1}, i_{2} \leq k - 1$ and $0 \leq l_{1}, l_{2} < 2k$. Then
\[ U^{i_{1}, l_{1}} \boxtimes_{U^{0}} U^{i_{2}, l_{2}} = \sum_{r \in R(i_{1}, i_{2})} U^{r, l_{1} + l_{2}}, \] (4.1)
where $R(i_{1}, i_{2})$ is the set of integers $r$ satisfying
\[ |i_{1} - i_{2}| \leq r \leq \min\{i_{1} + i_{2}, 2(k - 1) - i_{1} - i_{2}\}, \quad i_{1} + i_{2} + r \in 2\mathbb{Z}, \]
and $l_{1} + l_{2}$ is considered to be modulo $2k$. The irreducible $U^{0}$-modules $U^{r, l_{1} + l_{2}}$, $r \in R(i_{1}, i_{2})$, on the right hand side of (4.1) are inequivalent to each other.

**Proof.** Let $C_{A^{0}}$ be the category of $A^{0}$-modules, and let $C_{A^{0}}^{0}$ be the full subcategory of $C_{A^{0}}$ consisting of the objects $X$ of $C_{A^{0}}$ such that $U^{0} \boxtimes_{A^{0}} X$ is a $U^{0}$-module [3, Definition 2.66]. Then $\text{Irr}^{0}(A^{0})$ constitutes the simple objects of $C_{A^{0}}^{0}$ [3, Proposition 2.65]. The category $C_{A^{0}}^{0}$ is a $\mathbb{C}$-linear additive braided monoidal category with structures induced from $C_{A^{0}}$, and the functor $F : C_{A^{0}}^{0} \rightarrow C_{U^{0}}$; $X \mapsto U^{0} \boxtimes_{A^{0}} X$ is a braided tensor functor [3, Theorem 2.67], where $C_{U^{0}}$ is the category of $U^{0}$-modules.

We fix $0 \leq i_{1}, i_{2} \leq k - 1$ and $0 \leq l_{1}, l_{2} < 2k$. Since the category $C_{A^{0}}^{0}$ is closed under the fusion product, we have
\[ X(i_{1}, 0, l_{1}) \boxtimes_{A^{0}} X(i_{2}, 0, l_{2}) = \sum_{X(i_{3}, j_{3}, l_{3}) \in \text{Irr}^{0}(A^{0})} n(i_{3}, j_{3}, l_{3}) X(i_{3}, j_{3}, l_{3}), \] (4.2)
where
\[ n(i_{3}, j_{3}, l_{3}) = \dim I^{0}_{A^{0}} \left( X(i_{3}, j_{3}, l_{3}) \right) \]
is the fusion rule, that is, the dimension of the space of intertwining operators of type \(X(i_{1}, 0, l_{1}) \leftarrow X(i_{2}, 0, l_{2})\). Let
\[ n(i_{3}, l_{3}) = \dim I^{0}_{U^{0}} \left( \begin{array}{c} U^{i_{3}, l_{3}} \\ U^{i_{1}, l_{1}} \\ U^{i_{2}, l_{2}} \end{array} \right) \]
be the fusion rule of the irreducible $U^{0}$-modules $U^{i_{p}, l_{p}}$, $p = 1, 2, 3$. Then
\[ U^{i_{1}, l_{1}} \boxtimes_{U^{0}} U^{i_{2}, l_{2}} = \sum_{U^{i_{3}, l_{3}} \in \text{Irr}(U^{0})} n(i_{3}, l_{3}) U^{i_{3}, l_{3}}. \] (4.3)
Since $U^{i,l} = U^0 \boxtimes_{A^0} X(i, j, l)$ for any $0 \leq j < k - 1$, and since the functor $F : \mathcal{C}_{A^0}^0 \to \mathcal{C}_{U^0}$; $X \mapsto U^0 \boxtimes_{A^0} X$ is a braided tensor functor, it follows from (1.2) that
\[
U^{i_1,l_1} \boxtimes_{U^0} U^{i_2,l_2} = \sum_{X(i_3,j_3,l_3) \in \text{Irr}^0(A^0)} n(i_3,j_3,l_3)U^{i_3,l_3}.
\]
Thus $n(i_3,l_3) = \sum_{j_3=0}^{k-2} n(i_3,j_3,l_3)$ by (4.3).

Now,
\[
n(i_3,j_3,l_3) = \dim I_{M^0_{(k-1)}} \left( \begin{array}{cc} M^i_{j_1,l_1}^{j_3,j_3} \\ M^i_{j_1,l_1}^{i_3,j_3} \\ M^i_{j_1,l_1}^{j_1,j_1} \end{array} \right) \cdot \dim I_{V_{Zd}^{k}} \left( \begin{array}{cc} V_{Zd-l_3d/2k+j_1d/2(k-1)} \\ V_{Zd-l_3d/2k+j_2d/2(k-1)} \end{array} \right)
\]
by [4, Theorem 2.10]. The first term of the right hand side of the above equation is 0 or 1, and it is 1 if and only if $i_3 \in R(i_1, i_2)$ and $i_3 - 2j_3 \equiv i_1 + i_2 \pmod{2(k-1)}$ by (2.7) for $k-1$ in place of $k$. The second term of the right hand side is 0 or 1, and it is 1 if and only if
\[
(l_1 + l_2)(k-1) + (i_1 + i_2) \equiv -l_3(k-1) + (i_3 - 2j_3)k \pmod{2(k-1)}k
\]
by the fusion rule for $V_{Zd}^k$ [11, Chapter 12]. Hence $n(i_3,j_3,l_3)$ is 0 or 1, and it is 1 if and only if $i_3 \in R(i_1, i_2)$, $i_3 - 2j_3 \equiv i_1 + i_2 \pmod{2(k-1)}$, and the condition (4.4) is satisfied.

If $i_3 \in R(i_1, i_2)$, then there is a unique $0 \leq j_3 < k - 1$ such that $i_3 - 2j_3 \equiv i_1 + i_2 \pmod{2(k-1)}$. Moreover, if $i_3 - 2j_3 \equiv i_1 + i_2 \pmod{2(k-1)}$, then the condition (4.4) is equivalent to the condition that $l_1 + l_2 \equiv l_3 \pmod{2k}$. Therefore, (4.4) holds.

We have the following two corollaries.

**Corollary 4.2.** Let $\zeta = \exp(2\pi \sqrt{-1}/k)$. Then the map $\varphi : U^{i,l} \mapsto \zeta^i U^{i,l}$ for $0 \leq i \leq k - 1$ and $0 \leq l < 2k$ defines an automorphism of the fusion algebra of $U^0$ of order $k$ which is compatible with the isomorphism (3.13).

**Corollary 4.3.** There are exactly $2k$ inequivalent simple current $U^0$-modules, which are represented by $U^{0,l}$, $0 \leq l < 2k$.

Let $U^l = U^{0,l}$. Then
\[
U^l = \bigoplus_{j=0}^{k-2} M^j_{(k-1)} \otimes V_{Zd-l_3d/2k+j_1j/2(k-1)}
\]
\[
= U^0 \boxtimes_{A^0} X(0,0,l)
\]
(4.5)
and the set of equivalence classes of simple current $U^0$-modules is
\[
\text{Irr}(U^0)_{sc} = \{U^l \mid 0 \leq l < 2k\} \quad \text{with} \quad U^l \boxtimes_{U^0} U^{l'} = U^{l+l'}.
\]
(4.6)

The conformal weight of $U^l$ satisfies $h(U^l) \equiv h(X(0,0,l)) \pmod{Z}$ by (4.5). Hence we have
\[
h(U^l) \equiv \frac{(k-1)^2}{4k} \pmod{Z}.
\]
(4.7)
Since
\[
U^p \boxtimes_{U^0} U^{i,l} = U^{i,l+p}
\]
(4.8)
by (4.1), the isomorphism (3.19) implies the next lemma.

**Lemma 4.4.** Let \(0 \leq i \leq k - 1\) and \(0 \leq p, l < 2k\). Then \(U^p \boxtimes_{L^0} U^{i,l} = U^{i,l}\) if and only if one of the following conditions holds.

(i) \(p = 0\).

(ii) \(k\) is odd, \(i = (k - 1)/2\), and \(p = k\).

The fusion rules of \(M^0_{(k-1)}\)-modules are illustrated as follows. We set \(M^0 = M^0_{(k-1)}\) and \(M^{i,j} = M^{i,j}_{(k-1)}\) for simplicity of notation. The irreducible \(M^0\)-modules are denoted as \(M^{i,j}\) by using \(0 \leq i \leq k - 1\) and \(0 \leq j < k - 1\). There is another description of the irreducible \(M^0\)-modules. Take \(0 \leq q < 2(k - 1)\) such that \(q \equiv i - 2j \pmod{2(k - 1)}\). Let \(\tilde{M}^{i,q} = M^{i,j}\), which is the multiplicity of \(V_{Z_{\gamma(k-1)+q\gamma k-1/2(k-1)}}\) in the decomposition (2.4) of \(L_{\gamma}(k - 1, i)\). Then the fusion product (2.7) for \(k - 1\) in place of \(k\) can be written as

\[
\tilde{M}^{i_1,q_1} \boxtimes_{M^0} \tilde{M}^{i_2,q_2} = \sum_{r \in R(i_1,i_2)} \tilde{M}^{r,q_1+q_2}.
\]

(4.9)

The relationship between (1.1) and (4.9) is clear. Moreover, the isomorphisms \(M^{i,j} \cong M^{k-1-i,-j-i}\) and \(M^{i,j} \circ \theta \cong M^{i,j}\) can be written as

\[
\tilde{M}^{i,q} \cong \tilde{M}^{k-1-i,-j-i}, \quad \tilde{M}^{i,q} \circ \theta \cong \tilde{M}^{i,-q}.
\]

It is known that a map defined by \(\tilde{M}^{i,q} \mapsto \eta^q \tilde{M}^{i,q}\) with \(\eta = \exp(2\pi \sqrt{-1}/(k - 1))\) is compatible with the isomorphism \(\tilde{M}^{i,q} \cong \tilde{M}^{k-1-i,-j-i}\), and it induces an automorphism of the fusion algebra of \(M^0\) of order \(k - 1\).

5. **Vertex operator (super)algebra \(U^D\)**

In this section, we introduce a positive definite integral lattice \(\Gamma_D\) and a vertex operator algebra or a vertex operator superalgebra \(U^D\) for a \(\mathbb{Z}_{2k}\)-code \(D\).

5.1. **Irreducible \(U^0\)-modules in \(V_{N^0}\)**

Let \(L^{(k)}\) be the lattice as in (2.2). We set

\[N = \{\alpha \in L^{(k)} \mid \langle \alpha, \gamma_k \rangle = 0\},\]

where \(\gamma_k \in L^{(k)}\) is as in (3.4). We denote the dual lattice of \(N\) by \(N^0\). The lattice \(N\) is also considered in Section 4 of [3], where \(L^{(k)}\) and \(\gamma_k\) are denoted by \(L\) and \(\gamma\), respectively. We show how the irreducible \(U^0\)-modules \(U^{i,l}\) appear in the \(V_{N^0}\)-module \(V_{N^0}\).

For \(0 \leq j < k\) and \(a = a(k) = (a_1, \ldots, a_k) \in \{0, 1\}^k\), let

\[N(j, a(k)) = N + \delta a(k) - j \alpha_k + \frac{2j - b_k}{2k} \gamma_k\]

be a coset of \(N\) in \(N^0\), which is identical with \(N(j, a)\) of (4.4) in [3] as \(\delta a(k) = \frac{1}{2} \sum_{p=1}^k a_p \alpha_p\) and \(\text{wt}(a(k)) = b_k\). We have

\[V_{L^{(k)}+\delta a(k)} = \bigoplus_{j=0}^{k-1} V_{N(j,a(k))} \otimes V_{Z_{\gamma k}+(b_k-2j)\gamma_k/2k}\]

by (5.2) of [3]. Let \(0 \leq l < 2k\) be such that \(l \equiv b_k - 2j \pmod{2k}\). Then \(j \equiv (b_k - l)/2 \pmod{k}\), and the multiplicity of \(V_{Z_{\gamma k}+l\gamma_k/2k}\) in \(V_{L^{(k)}+\delta a(k)}\) is \(V_{N((b_k-l)/2,a(k))}\). Therefore,

\[U^{i,l} = \{v \in V_{N((b_k-l)/2,a(k))} \mid \omega^{(s)}(1)v = h^{(s)}_{i+1,i+s+1}v, 1 \leq s \leq k - 2\}\]
with \( i = i_{k-1} \) by (3.4) and (3.13). In the case where \( a_1 = \cdots = a_{k-2} = 0 \), we have \( b_{k-1} = a_{k-1} \) and \( b_k = a_{k-1} + a_k \). Thus the following lemma holds.

**Lemma 5.1.** Let \( 0 \leq i \leq k - 1 \) and \( 0 \leq l < 2k \).

1. Define \( a_{k-1}, a_k \in \{0, 1\} \), and \( 0 \leq j < k \) by the conditions
   \[
   i \equiv a_{k-1}, \quad l \equiv a_{k-1} + a_k \pmod{2}, \quad j \equiv (a_{k-1} + a_k - l)/2 \pmod{k}.
   \]

Then
   \[
   U^{i,l} = \{ v \in V_{\langle d,j,(0,\ldots,0,a_k) \rangle} | \omega^s_{(1)}v = 0, 1 \leq s \leq k - 3, \omega^{k-2}_{(1)}v = h^{(k-2)}_{1, i+1}v \}. \]

2. In the case \( i = 0 \), we have \( a_{k-1} = 0, \ l \equiv a_k \pmod{2}, \) and \( j \equiv (a_k - l)/2 \pmod{k} \).

In particular,
   \[
   U^l = \{ v \in V_{\langle d,j,(0,\ldots,0,a_k) \rangle} | \omega^s_{(1)}v = 0, 1 \leq s \leq k - 2 \}. \]

In the assertion (2) of the above lemma, we have \( N(j, (0, \ldots, 0, a_k)) = N - ld/2k \), as \( d = \gamma_k - k\alpha_k \). Thus
   \[
   U^l = \{ v \in V_{\langle d,j,(0,\ldots,0,a_k) \rangle} | \omega^s_{(1)}v = 0, 1 \leq s \leq k - 2 \}. \]

5.2. \( \Gamma_D \) and \( U_D \). The arguments in this subsection are parallel to those in Section 7 of [3]. For simplicity of notation, set
   \[
   \tilde{N}^{(l)} = N - ld/2k, \quad 0 \leq l < 2k.
   \]

We can regard the index \( l \) of \( \tilde{N}^{(l)} \) as \( l \in \mathbb{Z}_{2k} \). Since \( \langle x, y \rangle \in 2\mathbb{Z} \) and \( \langle x, d/2k \rangle \in \mathbb{Z} \) for \( x, y \in N \), and since \( \langle d, d \rangle = 2(k - 1)k \), the following lemma holds.

**Lemma 5.2.** Let \( 0 \leq p, q < 2k \).

1. \( \langle \alpha, \beta \rangle \in \frac{k-1}{2k}pq + \mathbb{Z} \) for \( \alpha \in \tilde{N}^{(p)} \) and \( \beta \in \tilde{N}^{(q)} \).

2. \( \langle \alpha, \alpha \rangle \in \frac{k-1}{2k}p^2 + 2\mathbb{Z} \) for \( \alpha \in \tilde{N}^{(p)} \).

We fix a positive integer \( \ell \). Define a coset \( \tilde{N}(\xi) \) of \( N^\ell \) in \( (N^\circ)^\ell \) by
   \[
   \tilde{N}(\xi) = \{(x_1, \ldots, x_\ell) | x_r \in \tilde{N}(\xi_r), 1 \leq r \leq \ell \}
   \]

for \( \xi = (\xi_1, \ldots, \xi_\ell) \in (\mathbb{Z}_{2k})^\ell \). Then \( \tilde{N}(\xi) + \tilde{N}(\eta) = \tilde{N}(\xi + \eta) \) for \( \xi, \eta \in (\mathbb{Z}_{2k})^\ell \).

For \( \xi = (\xi_1, \ldots, \xi_\ell), \eta = (\eta_1, \ldots, \eta_\ell) \in \mathbb{Z}^\ell \), define an integer \( \xi \cdot \eta \) by
   \[
   \xi \cdot \eta = \xi_1\eta_1 + \cdots + \xi_\ell\eta_\ell,
   \]

and consider a \( \mathbb{Z} \)-bilinear map
   \[
   \mathbb{Z}^\ell \times \mathbb{Z}^\ell \to \mathbb{Q}/\mathbb{Z}; \quad (\xi, \eta) \mapsto \frac{k-1}{2k}\xi \cdot \eta + \mathbb{Z}.
   \]

Since \( \frac{k-1}{2k}\xi \cdot \eta + \mathbb{Z} \) depends only on \( \xi_r \) and \( \eta_r \) modulo 2k, the above \( \mathbb{Z} \)-bilinear map induces a \( \mathbb{Z} \)-bilinear map
   \[
   (\mathbb{Z}_{2k})^\ell \times (\mathbb{Z}_{2k})^\ell \to \mathbb{Q}/\mathbb{Z}; \quad (\xi, \eta) \mapsto \frac{k-1}{2k}\xi \cdot \eta + \mathbb{Z},
   \]

where \( \xi \cdot \eta = \xi_1\eta_1 + \cdots + \xi_\ell\eta_\ell \) is considered for integers \( \xi_r, \eta_r \) such that \( 0 \leq \xi_r, \eta_r < 2k \); \( 1 \leq r \leq \ell \). The \( \mathbb{Z} \)-bilinear map is non-degenerate if \( k \) is even, whereas it is degenerate with radical \( \{0, k\}^\ell \) if \( k \) is odd. The following lemma holds by Lemma 5.2.
Lemma 5.3. Let $\xi, \eta \in (\mathbb{Z}_{2k})^\ell$.

1. $\langle \alpha, \beta \rangle = \frac{k-1}{2k} \xi \cdot \eta + \mathbb{Z}$ for $\alpha \in \tilde{N}(\xi)$ and $\beta \in \tilde{N}(\eta)$.
2. $\langle \alpha, \alpha \rangle = \frac{k-1}{2k} \xi \cdot \xi + 2\mathbb{Z}$ for $\alpha \in \tilde{N}(\xi)$.

Remark 5.4. The Euclidean weight $\mathrm{wt}_E(\xi)$ of $\xi = (\xi_1, \ldots, \xi_\ell) \in (\mathbb{Z}_{2k})^\ell$ is defined as

$$\mathrm{wt}_E(\xi) = \sum_{r=1}^\ell \min\{\xi_r^2, (2k - \xi_r)^2\} \in \mathbb{Z},$$

where $\xi_r$ are considered to be integers such that $0 \leq \xi_r < 2k$, $1 \leq r \leq \ell$. Note that $\frac{1}{2k} \mathrm{wt}_E(\xi) + 2\mathbb{Z} = \frac{1}{2k} \xi \cdot \xi + 2\mathbb{Z}$. The Euclidean weight was used in [21].

Let $D$ be a $\mathbb{Z}_{2k}$-code of length $\ell$, that is, $D$ is an additive subgroup of $(\mathbb{Z}_{2k})^\ell$. Set

$$\Gamma_D = \bigcup_{\xi \in D} \tilde{N}(\xi),$$

which is a sublattice of $(\mathbb{N}^\ell)^\ell$. We consider two cases, namely,

Case A. $\frac{k-1}{2k} \xi \cdot \xi \in 2\mathbb{Z}$ for all $\xi \in D$.

Case B. $\frac{k-1}{2k} \xi \cdot \eta \in \mathbb{Z}$ for all $\xi, \eta \in D$, and $\frac{k-1}{2k} \xi \cdot \xi \in 2\mathbb{Z} + 1$ for some $\xi \in D$.

Lemma 5.3 implies the following lemma.

Lemma 5.5. (1) $\Gamma_D$ is a positive definite even lattice if and only if $D$ is in Case A.

(2) $\Gamma_D$ is a positive definite odd lattice if and only if $D$ is in Case B.

In Case A, $V_{\Gamma_D}$ is a vertex operator algebra. In Case B, set

$$D^0 = \{\xi \in D \mid \frac{k-1}{2k} \xi \cdot \xi \in 2\mathbb{Z}\}, \quad D^1 = \{\xi \in D \mid \frac{k-1}{2k} \xi \cdot \xi \in 2\mathbb{Z} + 1\}.$$

Then $D^0$ is a subgroup of $D$, and $D = D^0 \cup D^1$ is the coset decomposition of $D$ by $D^0$. Let $\Gamma_{D^0} = \bigcup_{\xi \in D^0} \tilde{N}(\xi)$, $p = 0, 1$. Then $V_{\Gamma_{D^0}} = V_{\Gamma_{D^0}^t} \oplus V_{\Gamma_{D^1}^t}$ is a vertex operator superalgebra.

We have $V_{\tilde{N}(\xi)} = V_{\tilde{N}(\xi_1)} \otimes \cdots \otimes V_{\tilde{N}(\xi_\ell)} \subset (V_{\mathbb{N}^\ell})^{\otimes \ell}$, and $V_{\Gamma_D} = \bigoplus_{\xi \in D} V_{\tilde{N}(\xi)}$. Let

$$U_\xi = \{v \in V_{\tilde{N}(\xi)} \mid (\omega_{S^{\otimes \ell}}(1))v = 0\},$$

where $\omega_{S^{\otimes \ell}}$ is the conformal vector of the vertex operator subalgebra $S^{\otimes \ell}$ of $(V_{\mathbb{N}^\ell})^{\otimes \ell}$ with $S = L(c_1, 0) \otimes \cdots \otimes L(c_{k-2}, 0)$. Then $U_\xi = U_{\xi_1} \otimes \cdots \otimes U_{\xi_\ell}$ by (5.2). In particular, $U_0 = (U_0^{\otimes \ell})^{\otimes \ell}$ for the zero codeword $0 = (0, \ldots, 0)$. We see from (1.6) that the set of equivalence classes of simple current $U_0$-modules is

$$\operatorname{Irr}(U_0)_{\text{sc}} = \{U_\xi \mid \xi \in (\mathbb{Z}_{2k})^\ell\} \quad \text{with} \quad U_\xi \boxtimes_{U_0} U_{\xi'} = U_{\xi + \xi'}. $$

The conformal weight $h(U_\xi)$ of $U_\xi$ is

$$h(U_\xi) \equiv \frac{k-1}{4k} \xi \cdot \xi \quad (\text{mod } \mathbb{Z})$$
by \((\ref{eq:1})\). Hence \(h(U_0) \in \mathbb{Z}\) for \(\xi \in D\) if \(D\) is in Case A.

The next proposition follows from Theorems 3.2 and 3.6.

**Proposition 5.6.** \(U_0 = (U^0)^{\otimes \ell}\) is a simple, self-dual, rational, and \(C_2\)-cofinite vertex operator algebra of CFT-type with central charge \(3\ell(k-1)/(k+1)\). Any irreducible \(U_0\)-module except for \(U_0\) itself has positive conformal weight.

Let \(U_D\) be the commutant of \(S^\otimes \ell\) in \(V_{\Gamma_D}\). Then

\[
U_D = \{v \in V_{\Gamma_D} \mid (\omega S^\otimes \ell)(1)v = 0\} = \bigoplus_{\xi \in D} U_\xi,
\]

so \(U_D\) is a \(D\)-graded simple current extension of \(U_0\). We have the next theorem.

**Theorem 5.7.** (1) If \(D\) is in Case A, then \(U_D\) is a simple, self-dual, rational, and \(C_2\)-cofinite vertex operator algebra of CFT-type with central charge \(3\ell(k-1)/(k+1)\).

(2) If \(D\) is in Case B, then \(U_D = U_{D^0} \oplus U_{D^1}\) is a simple vertex operator superalgebra. The even part \(U_{D^0}\) and the odd part \(U_{D^1}\) are given by \(U_{D^p} = \bigoplus_{\xi \in D^p} U_\xi\), \(p = 0, 1\), and \(h(M_{D^1}) \in \mathbb{Z} + 1/2\).

### 6. Representations of \(U_D\)

In this section, we construct all the irreducible \(\chi\)-twisted \(U_D\)-modules for \(\chi \in D^*\) in \(V_{(N_{\chi})^\ell}\), and classify them, where \(D^* = \text{Hom}(D, \mathbb{C}^*)\). We argue as in Sections 8 and 9 of \cite{key-2}.

#### 6.1. Irreducible \(U_D\)-modules: Case A

Let \(D\) be a \(\mathbb{Z}_{2k}\)-code of length \(\ell\) in Case A of Section 5.2. Let \(b_{U^0} : \text{Irr}(U^0)_{sc} \times \text{Irr}(U^0) \to \mathbb{Q}/\mathbb{Z}\) be a map defined by

\[
b_{U^0}(U^p, U^{i;l}) = h(U^p \boxtimes_{U^0} U^{i;l}) - h(U^p) - h(U^{i;l}) + \mathbb{Z}
\]

for \(0 \leq i \leq k-1\) and \(0 \leq p, l < 2k\). Since \(h(U^{i;l}) \equiv h(X(i, 0, l)) \pmod{\mathbb{Z}}\) by (3.17), we obtain by using (1.7) and (1.8) that

\[
b_{U^0}(U^p, U^{i;l}) = p((k-1)l - ki)/2k + \mathbb{Z}. \quad (6.1)
\]

Any irreducible \(U_0\) module is of the form

\[
U_{\mu,\nu} = U^{\mu_1,\nu_1} \otimes \cdots \otimes U^{\mu_\ell,\nu_\ell}
\]

for some \(\mu = (\mu_1, \ldots, \mu_\ell)\) with \(0 \leq \mu_r \leq k-1\), \(1 \leq r \leq \ell\), and \(\nu = (\nu_1, \ldots, \nu_\ell) \in (\mathbb{Z}_{2k})^\ell\). We have \(U_{0,\xi} = U_\xi\), and

\[
U_\xi \boxtimes_{U_0} U_{\mu,\nu} = U_{\mu,\nu + \xi}.
\]

Define a map \(b_{U_0} : \text{Irr}(U_0)_{sc} \times \text{Irr}(U_0) \to \mathbb{Q}/\mathbb{Z}\) by

\[
b_{U_0}(U_\xi, U_{\mu,\nu}) = h(U_\xi \boxtimes_{U_0} U_{\mu,\nu}) - h(U_\xi) - h(U_{\mu,\nu}) + \mathbb{Z}.
\]

Then it follows from (6.1) that

\[
b_{U_0}(U_\xi, U_{\mu,\nu}) = \frac{1}{2k}(\xi((k-1)\nu - k\mu) + \mathbb{Z}, \quad (6.2)
\]
where $(\cdot | \cdot)$ is the standard inner product on $(\mathbb{Z}_{2k})^\ell$. Although each entry $\mu_r$ of $\mu$ is an integer such that $0 \leq \mu_r \leq k - 1$, we can treat it as an element of $\mathbb{Z}_{2k}$ on the right hand side of $(\mathbb{Z}_{2k})^\ell$. Since $U_\eta = U_{0, \eta}$ for $\eta \in (\mathbb{Z}_{2k})^\ell$, this in particular implies that

$$b_{U_0}(U_\xi, U_\eta) = \frac{k - 1}{2k}(\xi|\eta) + \mathbb{Z}.$$  

Let $D^\perp = \{ \eta \in (\mathbb{Z}_{2k})^\ell \mid (D|\eta) = 0 \}$. Then $|D||D^\perp| = |(\mathbb{Z}_{2k})^\ell|$, as $(\cdot | \cdot)$ is a non-degenerate bilinear form on $(\mathbb{Z}_{2k})^\ell$. Consider a map

$$\chi_{U_{\mu, \nu}} : D \rightarrow \mathbb{C}^\times; \quad \xi \mapsto \exp(2\pi \sqrt{-1} b_{U_0}(U_\xi, U_{\mu, \nu})).$$

We have

$$\chi_{U_{\mu, \nu}}(\xi) = \exp(2\pi \sqrt{-1}(\xi|(-1)\nu - k\mu)/2k)$$

by $(\mathbb{Z}_{2k})^\ell$. Hence $\chi_{U_{\mu, \nu}} \in D^\ast$. 

**Lemma 6.1.** (1) $\chi_{U_{\mu, \nu}} = 1$; the principal character of $D$ if and only if $(k - 1)\nu - k\mu \in D^\perp$.

(2) For any $\chi \in D^\ast$, there exists $U_{\mu, \nu} \in \text{Irr}(U_0)$ such that $\chi = \chi_{U_{\mu, \nu}}$.

**Proof.** The assertion (1) is a consequence of $(\mathbb{Z}_{2k})^\ell$ and the definition of $D^\perp$. For any $0 \leq p < 2k$, we have $p \equiv (k - 1)l - k i \pmod{2k}$ for some $0 \leq i \leq k - 1$ and $0 \leq l < 2k$. Hence for any $\eta \in (\mathbb{Z}_{2k})^\ell$, there are $\mu = (\mu_1, \ldots, \mu_\ell)$ with $0 \leq \mu_r \leq k - 1$, $1 \leq r \leq \ell$, and $\nu \in (\mathbb{Z}_{2k})^\ell$ such that $\eta = (k - 1)\nu - k\mu$. Since $(\cdot | \cdot)$ is non-degenerate on $(\mathbb{Z}_{2k})^\ell$, the assertion (2) holds. 

We consider a coset

$$N(\eta, \delta^{(1)}, \delta^{(2)}) = \{(x_1, \ldots, x_\ell) \mid x_r \in N(\eta_r, (0, \ldots, 0, d_r^{(1)}, d_r^{(2)})), 1 \leq r \leq \ell\}$$

of $N^\ell$ in $(N^\circ)^\ell$ for $\eta = (\eta_1, \ldots, \eta_\ell) \in (\mathbb{Z}_k)^\ell$ and $\delta^{(s)} = (d_r^{(s)}, \ldots, d_\ell^{(s)}) \in \{0, 1\}^\ell$, $s = 1, 2$. The next proposition holds by Lemma 6.1.

**Proposition 6.2.** Let $\mu = (\mu_1, \ldots, \mu_\ell)$ with $0 \leq \mu_r \leq k - 1$, $1 \leq r \leq \ell$, and let $\nu = (\nu_1, \ldots, \nu_\ell) \in (\mathbb{Z}_{2k})^\ell$. Define $d_r^{(1)}, d_r^{(2)} \in \{0, 1\}$, and $0 \leq \eta_r < k$ by the conditions

$$\mu_r \equiv d_r^{(1)} \pmod{2}, \quad \nu_r \equiv d_r^{(2)} \pmod{2}, \quad \eta_r \equiv (d_r^{(1)} + d_r^{(2)} - \nu_r)/2 \pmod{k}$$

for $1 \leq r \leq \ell$. Then $V_{N(\eta, \delta^{(1)}, \delta^{(2)})}$ contains the irreducible $U_0$-module $U_{\mu, \nu}$.

Let $0 \leq i \leq k - 1$ and $0 \leq p, l < 2k$. Define $a_{k-1}, a_k \in \{0, 1\}$, and $0 \leq j < k$ by the conditions $(\mathbb{Z}_{2k})^\ell$. Then

$$(\alpha, \beta) \in p((k - 1)l - ki)/2k + \mathbb{Z}$$

for $\alpha \in \tilde{N}(\nu)$ and $\beta \in N(j, (0, \ldots, 0, a_{k-1}, a_k))$. Thus the the following lemma holds by $(\mathbb{Z}_{2k})^\ell$.

**Lemma 6.3.** Let $\mu, \nu, \eta, \delta^{(1)}$, and $\delta^{(2)}$ be as in Proposition 6.2, and let $\xi \in (\mathbb{Z}_{2k})^\ell$. Then

$$(x, y) \in b_{U_0}(U_\xi, U_{\mu, \nu}) \text{ for } x \in \tilde{N}(\xi) \text{ and } y \in N(\eta, \delta^{(1)}, \delta^{(2)}).$$

Let $X \in \text{Irr}(U_0)$. Then $X = U_{\mu, \nu}$ for some $\mu$ and $\nu$. Let $\eta, \delta^{(1)}$, and $\delta^{(2)}$ be as in Proposition 6.2. Then $X$ is contained in $V_{N(\eta, \delta^{(1)}, \delta^{(2)})}$. Since $U_\xi$ is contained in $V_{N(\xi)}$, and since the cosets $\tilde{N}(\xi) + N(\eta, \delta^{(1)}, \delta^{(2)})$ of $N^\ell$ in $(N^\circ)^\ell$ are distinct for all $\xi \in D$, the $\chi_X$-twisted $U_D$-submodule $U_D \cdot X$ of $V_{N(\eta, \delta^{(1)}, \delta^{(2)})}$ generated by $X$ is isomorphic to $U_D \otimes_{U_0} X = \bigoplus_{\xi \in D} U_\xi \otimes_{U_0} X$. If $\chi_X(\xi) = 1$ for all $\xi \in D$, then $N(\eta, \delta^{(1)}, \delta^{(2)}) \subset (\Gamma_D)^\circ$ by Lemma 6.3 and we have $U_D \cdot X \subset V(\Gamma_D)^\circ$. Thus the following theorem holds.
Theorem 6.4. (1) Any irreducible $\chi$-twisted $U_D$-module, $\chi \in D^*$, is contained in $V_{(N^0)^{\chi}}$.
(2) Any irreducible untwisted $U_D$-module is contained in $V_{(I_D)^\psi}$.

Define an action of $D$ on $\text{Irr}(U_0)$ by $X \mapsto U_\xi \boxtimes_{\mathcal{O}_0} X$ for $\xi \in D$ and $X \in \text{Irr}(U_0)$. Let $\text{Irr}(U_0) = \bigcup_{i \in I} \mathcal{O}_i$ be the $D$-orbit decomposition, and let $D_X = \{ \xi \in D \mid U_\xi \boxtimes_{\mathcal{O}_0} X = X \}$ be the stabilizer of $X$. The next lemma holds by Lemma 4.4.

Lemma 6.5. $U_\xi \boxtimes_{\mathcal{O}_0} U_{\mu,\nu} = U_{\mu,\nu}$ for some $\xi \neq 0$ if and only if $k$ is odd, $\xi = (\xi_1, \ldots, \xi_\ell) \in \{0, k\}^\ell$, and $\mu_r = (k-1)/2$ for $1 \leq r \leq \ell$ such that $\xi_r = k$.

We study the structure of $U_D \boxtimes_{\mathcal{O}_0} X$ for $X \in \text{Irr}(U_0)$. If $D_X = 0$, then $U_D \boxtimes_{\mathcal{O}_0} X$ is an irreducible $\chi_X$-twisted $U_D$-module.

Suppose $D_X \neq 0$. Then $k$ is odd, and $D_X \subset \{0, k\}^\ell$ by Lemma 6.3. Let $C = \{(0), (k)\}$ be a $\mathbb{Z}_{2k}$-code of length one consisting of two codewords $(0)$ and $(k)$. The code $C$ is in Case A or in Case B according as $k \equiv 1$ or $k \equiv 3 \pmod{4}$. Hence the $\mathbb{Z}_2$-graded simple current extension $U_C = U_0 \oplus U_k$ of $U_0$ is a simple vertex operator algebra with $h(U_k) \in \mathbb{Z}$ or a simple vertex operator superalgebra with $h(U_k) \in \mathbb{Z} + 1/2$ according as $k \equiv 1$ or $k \equiv 3 \pmod{4}$. We can regard any additive subgroup of $\{0, k\}^\ell \subset (\mathbb{Z}_{2k})^\ell$ as an additive subgroup of $(\mathbb{Z}_2)^\ell$ under the correspondence $0 \mapsto 0$ and $k \mapsto 1$. Since $k$ is odd, the correspondence is the reduction modulo 2, and it gives an isometry from $\{0, k\}^\ell, (\cdot, \cdot)$ to $(\mathbb{Z}_2)^\ell, (\cdot, \cdot))$, where $(\cdot, \cdot)$ is the standard inner product on either $(\mathbb{Z}_{2k})^\ell$ or $(\mathbb{Z}_2)^\ell$. In particular, $D_X \cap D_X^1 \subset (\mathbb{Z}_{2k})^\ell$ corresponds to $D_X \cap D_X^1$ in $(\mathbb{Z}_2)^\ell$. Thus the following theorem holds by Propositions 2.3, 2.5, and 2.6 of [4].

Theorem 6.6. Let $X \in \text{Irr}(U_0)$.
(1) If $D_X = 0$, then $U_D \boxtimes_{\mathcal{O}_0} X$ is an irreducible $\chi_X$-twisted $U_D$-module.
(2) Suppose $k$ is odd and $D_X \neq 0$.
   If $k \equiv 1 \pmod{4}$, then $U_D \boxtimes_{\mathcal{O}_0} X = \bigoplus_{j=1}^{\lfloor D_X \rfloor} V_j$, where $V_j$, $1 \leq j \leq |D_X|$, are inequivalent irreducible $\chi_X$-twisted $U_D$-modules. Furthermore, $V_j \cong \bigoplus_{W \in \mathcal{O}_i} W$ as $U_0$-modules, where $\mathcal{O}_i$ is the $D$-orbit in $\text{Irr}(U_0)$ containing $X$.
   If $k \equiv 3 \pmod{4}$, then $U_D \boxtimes_{\mathcal{O}_0} X = \bigoplus_{j=1}^{\lfloor D_X \cap D_X^1 \rfloor} (V_j)^\otimes m$, where $m = \lceil D_X : D_X \cap D_X^1 \rceil^{1/2}$, and $V_j$, $1 \leq j \leq |D_X \cap D_X^1|$, are inequivalent irreducible $\chi_X$-twisted $U_D$-modules. Furthermore, $V_j \cong \bigoplus_{W \in \mathcal{O}_i} W^\otimes m$ as $U_0$-modules, where $\mathcal{O}_i$ is the $D$-orbit in $\text{Irr}(U_0)$ containing $X$.

Any irreducible $\chi$-twisted $U_D$-module, $\chi \in D^*$, is isomorphic to a direct summand of $U_D \boxtimes_{\mathcal{O}_0} X$ with $\chi = \chi_X$ for some $X \in \text{Irr}(U_0)$. Thus the classification of irreducible $\chi$-twisted $U_D$-modules for any $\chi \in D^*$ is obtained by Theorem 6.6.

We can write $\chi_i$ for $\chi_X$, and $D_i$ for $D_X$ if $X$ belongs to a $D$-orbit $\mathcal{O}_i$ in $\text{Irr}(U_0)$, as $\chi_X$ and $D_X$ are independent of the choice of $X \in \mathcal{O}_i$. Let $I(\chi) = \{ i \in I \mid \chi_i = \chi \}$, which is non-empty by Lemma 5.1.

By the above arguments, we obtain the next theorem.
Theorem 6.7. The number of inequivalent irreducible \( \chi \)-twisted \( U_D \)-modules for \( \chi \in D^* \) is as follows.

\[
\begin{align*}
|I(\chi)| & \quad \text{if } k \text{ is even}, \\
|I(\chi)| & + \sum_{i \in I(\chi)_1} |D_i| \quad \text{if } k \equiv 1 \pmod{4}, \\
|I(\chi)| & + \sum_{i \in I(\chi)_1} |D_i \cap D_i| \quad \text{if } k \equiv 3 \pmod{4},
\end{align*}
\]

where \( I(\chi)_0 = \{ i \in I(\chi) \mid D_i = 0 \} \) and \( I(\chi)_1 = I(\chi) \setminus I(\chi)_0 \).

6.2. Irreducible \( U_D \)-modules: Case B. Let \( D \) be a \( \mathbb{Z}_2k \)-code of length \( \ell \) in Case B of Section 5.2, and let \( D^0 \) and \( D^1 \) be as in Section 5.2. Since \( D^0 \) is a \( \mathbb{Z}_2k \)-code of length \( \ell \) in Case A, we see from Section 6.1 that any irreducible \( U_{D^0} \)-module \( P \) is isomorphic to a direct summand of \( U_{D^0} \otimes_k X \) for some \( X \in \text{Irr}(U_0) \), and that \( X \) is contained in \( V_{N(\eta, \delta(1), \delta(2))} \) for some coset \( N(\eta, \delta(1), \delta(2)) \) of \( N^l \) in \( (\Gamma_{D^0})^o \). Since \( U_D = U_{D^0} \oplus U_{D^1} \), the \( U_D \)-submodule \( U_D \cdot P \) of \( V_{(\Gamma_{D^0})^o} \) generated by \( P \) is isomorphic to \( U_D \otimes_{U_{D^0}} P \). Moreover, \( U_D \otimes_{U_{D^0}} P \) is either an irreducible \( U_D \)-module or a direct sum of two irreducible \( U_D \)-modules. Since any irreducible \( U_D \)-module is obtained in this way, the following theorem holds.

Theorem 6.8. Any irreducible \( U_D \)-module is contained in \( V_{(\Gamma_{D^0})^o} \).

Appendix A. Top level of \( U^l \), \( 0 \leq l < 2k \)

In this appendix, we prove the following theorem on the top level of \( U^l \), \( 0 \leq l < 2k \), defined in (4.3).

Theorem A.1. The weight and the dimension of the top level of the simple current \( U^0 \)-module \( U^l \), \( 0 \leq l < 2k \), are as follows.

(1) If \( l = 0 \), then the weight is 0 and the dimension is 1.

(2) If \( l \) is odd, then the weight is \( l(2k - l)/4k - 1/4 \) and the dimension is 1.

(3) If \( l \neq 0 \) is even, then the weight is \( l(2k - l)/4k \) and the dimension is 2.

Proof. Since \( U^0 \otimes \theta \cong U^{-l} = U^{2k-l} \) by (3.20), it is enough to consider the case \( 0 \leq l \leq k \). The top level of \( U^0 \) is \( \mathbb{C}1 \), and the assertion (1) holds. Thus we assume that \( 1 \leq l \leq k \). If \( k = 2 \), then \( U^l = V_{2d-ld/4} \) with \( \langle d, d \rangle = 4 \), as \( M^0_{(1)} = \mathbb{C}1 \). Hence the theorem holds for \( k = 2 \). So we assume that \( k \geq 3 \).

Recall the notation \( X(i, j, l) \) in Section 3. For a fixed \( l \), let \( P(j) \) be the conformal weight of \( X(0, j, l) \). Since \( U^l = \bigoplus_{j=0}^{k-2} X(0, j, l) \), we need to calculate the minimum value of \( P(j) \) for integers \( j \) in the range \( 0 \leq j < k - 1 \). We have

\[
P(j) = \frac{j(k-1-j)}{k-1} + \frac{(k-1+l+2kj)^2}{4(k-1)k} \]

\[
= \left( j + \frac{l+1}{2} \right)^2 - \frac{(l+1)^2}{4} + \frac{(k-1)^2}{4k} \tag{A.1}
\]

for \( j \) in the range

\[
0 \leq j \leq (k-1)(k-l)/2k, \tag{A.2}
\]
and
\[ P(j) = \frac{j(k - 1 - j)}{k - 1} + \frac{(k - 1)l + 2kj - 2(k - 1)k^2}{4(k - 1)k} \]
\[ = \left( j - \left( k - \frac{l + 1}{2} \right) \right)^2 + \frac{l(2k - l)}{4k} - \frac{1}{4} \]  \hspace{1cm} (A.3)

for \( j \) in the range
\[ (k - 1)(k - l)/2k \leq j < k - 1. \]  \hspace{1cm} (A.4)

The dimension of the top level of \( V_{2d-ld/2k-jd/(k-1)} \) is 2 if \((k - 1)l + 2kj = (k - 1)k\), otherwise it is 1. Since the dimension of the top level of \( M_j^{(k-1)} \) is 1, the dimension of the top level of \( X(0, j, l) \) is 2 if \( l = k \) and \( j = 0 \), otherwise it is 1, as \( 1 \leq l \leq k \).

The minimum value of the quadratic polynomial \( P(j) \) for integers \( j \) in the range \((A.2)\) is \((k - 1)l^2/4k\) at \( j = 0 \) by \((A.1)\). As for the minimum value of \( P(j) \) for integers \( j \) in the range \((A.4)\), note that
\[ (k - 1)(k - l)/2k \leq k - 1 - (l + 1)/2 < k - 1, \]
as \( k \geq 3 \) and \( 1 \leq l \leq k \). We argue the cases \( l = 1, 2, \) and \( l \geq 3 \) separately.

First, assume that \( l = 1 \). Then \( k - (l + 1)/2 = k - 1 \) is not in the range \((A.4)\). So the minimum value of \( P(j) \) for integers \( j \) in the range \((A.4)\) is \( 5/4 - 1/4k \) at \( j = k - 2 \) by \((A.3)\). Since \( P(0) < P(k - 2) \), the assertion \((2)\) holds for \( l = 1 \).

Next, assume that \( l = 2 \). Then \( k - (l + 1)/2 = k - 3/2 \), so the minimum value of \( P(j) \) for integers \( j \) in the range \((A.4)\) is \((k - 1)/k \) at \( j = k - 2 \) by \((A.3)\). Since \( P(0) = P(k - 2) \), the assertion \((3)\) holds for \( l = 2 \).

Now, assume that \( 3 \leq l \leq k \). Suppose \( l \) is odd. Then the minimum value of \( P(j) \) for integers \( j \) in the range \((A.4)\) is \( l(2k - l)/4k - 1/4 \) at \( j = k - (l + 1)/2 \) by \((A.3)\). The minimum value is smaller than \( P(0) \). Thus the assertion \((2)\) holds.

Finally, suppose \( 4 \leq l \leq k \) and \( l \) is even. Then the minimum value of \( P(j) \) for integers \( j \) in the range \((A.4)\) is \( l(2k - l)/4k \) at \( j = k - 1 - l/2 \) and \( k - l/2 \) by \((A.3)\). The minimum value is smaller than \( P(0) \). Thus the assertion \((3)\) holds. The proof is complete. \( \square \)

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