2-CLASS GROUPS OF CYCLOTOMIC TOWERS OF IMAGINARY BIQUADRATIC FIELDS

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Abstract. Let \( d \) be a square-free integer. In this paper we shall investigate the structure of the 2-class group of the cyclotomic \( \mathbb{Z}_2 \)-extension of the imaginary biquadratic number field \( \mathbb{Q}(\sqrt{d}, \sqrt{-1}) \).

1. Introduction

Let \( p \) be a prime number and \( k \) be a number field. Denote by \( k_\infty \) the cyclotomic \( \mathbb{Z}_p \)-extension of \( k \). The field \( k_\infty \) contains a unique cyclic subfield \( k_n \) of degree \( p^n \) over \( k \). The field \( k_n \) is called the \( n \)-th layer of the \( \mathbb{Z}_p \)-extension of \( k \). In 1959, the study of \( p \)-class numbers of number fields with large degree led to a spectacular result due to Iwasawa, that we shall recall here and use later (for \( p = 2 \)). Denote by \( e_n \) the highest power of \( p \) dividing the class number of \( k_n \). There exist integers \( \lambda, \mu \geq 0, \nu \), all independent of \( n \), and an integer \( n_0 \) such that:

\[
e_n = \lambda n + \mu p^n + \nu,
\]

for all \( n \geq n_0 \). The integers \( \lambda, \mu \geq 0 \) and \( \nu \) are called the Iwasawa invariants of \( k_\infty \)(cf. [9]).

Thereafter, the study of cyclotomic \( \mathbb{Z}_2 \)-extensions of \( CM \)-Fields was the subject of many papers and is still of huge interest in algebraic number theory. In 1980, Kida studied the Iwasawa’s \( \lambda^- \)-invariants and the 2-ranks of the narrow ideal class groups in the 2-extensions of \( CM \)-fields (cf. [11]). In 2018, Atsuta (cf. [4]) studied the maximal finite submodule of the minus part of the Iwasawa module attached to \( k_\infty \), while Müller worked on the capitulation in the minus-part in the steps of the cyclotomic \( \mathbb{Z}_p \)-extension of a \( CM \)-field \( k \) (cf. [16]).

In this paper we will concentrate on \( CM \)-fields of the following form: Let \( n \geq 0 \) be an integer, \( d \) be a square-free and \( L_{n,d} := \mathbb{Q}(\zeta_{2^{n+2}}, \sqrt{d}) \). In 2019, Chems-Eddin, Azizi and Zekhnini, computed the rank of the 2-class group of \( L_{n,d} \), the layers of the \( \mathbb{Z}_2 \)-extension of some special Dirichlet fields of the form \( L_{0,d} = \mathbb{Q}(\sqrt{d}, \sqrt{-1}) \) (cf. [1, 6, 5]). Li, Ouyang, Xu and Zhang computed the 2-class groups of these fields for \( d \) being a prime congruent to 3 (mod 8), 5 (mod 8) and 7 (mod 16) (cf. [15]).

2010 Mathematics Subject Classification. 11R29; 11R23; 11R18; 11R20.
Key words and phrases. Cyclotomic \( \mathbb{Z}_p \)-extension, 2-rank; 2-class group.
In the present work we consider two different classes of biquadratic base fields $L_{0,d}$ and determine the structure of the 2-class group of the $n$-th layer of its cyclotomic $\mathbb{Z}_2$-extension. The main aim of this paper is to prove the following theorem.

**Theorem 1.** Let $n \geq 1$. Then the following holds:

- Let $d$ be either a prime congruent to 9 (mod 16) satisfying an extra condition on the biquadratic residue symbol or the product of two different primes congruent to 3 (mod 8). Then, the 2-class group of $L_{n,d}$ is isomorphic to $\mathbb{Z}/2^{n+r-2}\mathbb{Z} \times \mathbb{Z}/2^r\mathbb{Z}$, for a constant $r$ only depending on $d$.

- Let $d = pq$ for two primes $p$ and $q$ such that $p \equiv -q \equiv 5$ (mod 8). Then the 2-class group of $L_{n,d}$ is isomorphic to $\mathbb{Z}/2^{n+r-2}\mathbb{Z}$, for a constant $r$ only depending on $d$. Further, Greenberg’s Conjecture holds for the field $L_{n,d}^+$, i.e., the 2-class number of $L_{n,d}^+$ is uniformly bounded.

In section 2 we will summarize some results on minus parts of 2-class groups of CM number fields. In section 3 we collect results on the rank of the 2-class groups of the fields $L_{n,d}$ and prove some of the main ingredients for the proof of the main theorem. Section 4 contains the proof of the main theorem (cf. Theorems 8 and 9).

**Notations**

Let $k$ be a number field. The next notations will be used for the rest of this article:

- $d$: An odd square-free integer,
- $n$: An integer $\geq 0$,
- $K_n$: $\mathbb{Q}(\zeta_{2^n+2})$,
- $K_n^+$: the maximal real subfield of $K_n$,
- $L_{n,d}$: $K_n(\sqrt{d})$,
- $L_{n,d}^+$: The real maximal subfield of $L_{n,d}$/$L_{0,d}$,
- $\tau$: a topological generator of $Gal(L_{\infty,d}/L_{0,d})$,
- $\Lambda = \mathbb{Z}_p[[T]]$ for $T = \tau - 1$,
- $N$: The application norm for the extension $L_{n,d}$/$K_n$,
- $E_k$: The unit group of $k$,
- $Cl(k)$: The class group of $k$,
- $h_2(d)$: The class number of the quadratic field $\mathbb{Q}(\sqrt{d})$,
- $\left(\frac{d}{p}\right)$: The biquadratic residue symbol,
- $\left(\frac{\alpha,d}{p}\right)$: The quadratic norm residue symbol for $L_{n,d}$/$K_n$,
- $q(L_{1,d}) := (E_{L_{1,d}} : \prod E_{k_i})$, with $k_i$ are the quadratic subfields of $L_{1,d}$. 
2. SOME PRELIMINARY RESULTS ON THE MINUS PART OF THE 2-CLASS GROUP

Let $p$ be a prime and $K$ be an arbitrary CM number field containing the $p$-th roots of unity (the 4-th roots of unity if $p = 2$). Consider the cyclotomic $\mathbb{Z}_p$-extension of $K$, denoted by $K_\infty$. The complex conjugation acts on the $p$ part of the class group $A_n$ of the intermediate fields $K_n$ as well as on the projective limit $\operatorname{lim}_{\infty \leftarrow n} A_n$. Usually one defines the minus part of the class group as $\widehat{A}_n^- = \{a \in A_n \mid ja = -a\}$ and the plus part as $\widehat{A}_n^+ = \{a \in A_n \mid ja = a\}$. For $p \neq 2$ this yields a direct decomposition of $A_n$. Further, it is well known that there is no capitulation on the minus part for $p \neq 2$. For $p = 2$ this is in general not true. To avoid this problem we define $A_\infty^+$ as the group of strongly ambiguous classes with respect to the group $K_n/K_\infty^+$ and $A_\infty^- = A_n/A_n^+$. Note that $A^+ = \widehat{A}^+$ and $A^- \cong \widehat{A}^-$ for $p \neq 2$ (see [16]).

Note that $A_\infty^- = \operatorname{lim}_{\infty \leftarrow n} A_n^-$ is a finitely generated $\Lambda$-torsion module. In the following we will for every $\lambda$-module $M$ denote it’s $\lambda$-invariant by $\lambda(M)$. For the rest of the paper we will only work with 2-class groups.

Lemma 1. Assume that $\mu(A_\infty^-) = 0$. Then $\lambda(A_\infty^-) \geq 2\text{-}\text{rank}(A_n^-)$ for all $n$.

Proof. By [16] Theorem 2.5 there is no finite submodule in $A_n^-$. So if $\mu = 0$ the 2-rank of $A_\infty$ equals it’s $\lambda$-invariant. Thus, the claim is immediate. \hfill $\Box$

Lemma 2. Assume that $\mu(A_\infty^-) = 0$. Then $\lambda(A_\infty^-) = \lambda(\widehat{A}_\infty^-)$.

Proof. Note that $2A_\infty \subset (1+j)A_\infty + (1-j)A_\infty$. Clearly, all elements in $(1+J)A_\infty$ are strongly ambiguous. Thus, if we consider the projection $\pi : A_\infty \rightarrow A_\infty^-$ we see that $(1+J)A_\infty$ lies in the kernel of $\pi$. On the other hand $(1-J)A_\infty$ intersects the kernel of $\pi$ only in a finite submodule $(1-J)a = -(1-j)a$. So if a class $(1-J)a$ is strongly ambiguous then it is of order 2. Hence, the torsion free parts of $(1-J)A_\infty$ and $\pi((1-J)A_\infty)$ are isomorphic. Furthermore, $2A_\infty = \pi(2A_\infty) \subset \pi((1-J)A_\infty)$. As the torsion free part of $A_\infty$ and $2A_\infty$ are equal we see that $\widehat{A}^-$ and $A_\infty^-$ have the same $\lambda$ invariant. \hfill $\Box$

3. MORE PRELIMINARIES ON THE FIELDS $L_{n,d}$ AND $L_{n,d}^+$

The $\lambda$-invariants of $A_n$ are of particular interest. Kida proved the following formula.

Theorem 2. [11] Theorem 3] Let $F$ and $K$ be CM-fields and $K/F$ a finite 2 extension. Assume that $\mu^-(F) = 0$. Then

$$\lambda^-(K) - \delta(K) = [K_\infty : F_\infty] \left(\lambda^-(F) - \delta(F)\right) + \sum (e_\beta - 1) - \sum (e_{\beta^+} - 1),$$

where $\delta(k)$ takes the values 1 or 0 according to whether $K_\infty$ contains the fourth roots of unity or not. The $e_\beta$ is the ramification index of a prime $\beta$ in $K_\infty$ coprime...
to 2 and in $K_\infty/F_\infty$ and $e_{\beta^+}$ is the ramification index for a prime coprime to 2 in $K_\infty^+/F_\infty^+$.

Note that Kida proves results for $\lambda(A^-)$. But due to lemma this $\lambda$-invariant equals the $\lambda$-invariant of $A^-$.

**Theorem 3.** Let $d > 2$ have $r$ prime divisors congruent to 7 or 9 mod 16 and $s$ prime divisors congruent to 3 or 5 mod 8. Then $\lambda^- = 2r + s - 1$.

**Proof.** Let $K = L_{0,d} = \mathbb{Q}(\sqrt{d}, \sqrt{-1})$ and $F = \mathbb{Q}(\sqrt{-1})$. Then $\delta(F) = \delta(K) = 1$ and $\lambda^-(F) = 0$. Every prime congruent to 7 or 9 modulo 16 splits into 4 primes in $K_n$ for $n$ large enough, while it splits only into 2 primes in $K_n^+$ (see [5]). Primes congruent to 3 or 5 modulo 8 decompose into 2 primes in $K_n$, while $K_n^+$ contains only one prime above $p$ (see [6]). As $[K_\infty : F_\infty] = [K_\infty^+ : F_\infty^+] = 2$ all the non trivial terms satisfy $e_\beta = e_{\beta^+} = 2$. Plugging all of this into Kida’s formula we obtain

$$\lambda^- - 1 = 2(0 - 1) + 4r + 2s - 2r - s = 2r + s - 2$$

and the claim follows. \(\square\)

**Theorem 4.** Let $d > 2$ be an odd square-free integer and $n \geq 1$ a positive integer. Then $\text{Cl}_2(L_{n,d})$ is cyclic non-trivial if and only if $d$ takes one of the following forms:

1. $d$ is a prime congruent to 7 (mod 16),
2. $d = pq$, where $p$ and $q$ are two primes such that $q \equiv 3$ (mod 8) and $p \equiv 5$ (mod 8).

**Proof.** By [6] Theorem 6], it suffices to check the case when $p$ is 7 (mod 8).

- Suppose that $p$ is congruent to 15 (mod 16) and let $\sigma$ denote it’s Frobenius homomorphism in $\text{Gal}(\mathbb{Q}(\zeta_{16})/\mathbb{Q})$. Then $\sigma(\zeta_{16}) = \sigma^p(\zeta_{16})$ by the definition of the Frobenius homomorphism. Let $H$ be the group generated by $\sigma$. Then $p$ is totally split in $\mathbb{Q}(\zeta_{16})^H/\mathbb{Q}$. Since $p \equiv 15 \pmod{16}$, $\sigma$ is the complex conjugation. Hence, $p$ is totally split in $\mathbb{Q}(\zeta_{16})^+ / \mathbb{Q}$ and inert in $\mathbb{Q}(\zeta_{16}) / \mathbb{Q}(\zeta_{16})^+$.

On another hand, by the proof of [6] Proposition 2], there are 4 primes of $K_2$ lying over $p$. Thus, $2\text{-rank}(\text{Cl}(L_{2,d})) = 4 - 1 - e$, where $e$ is defined by $e = [E_{K_2} : E_{K_2} \cap N(L_{2,d}^*)]$. The unit group of $K_2$ is given by $E_{K_2} = \langle \zeta_{16}, \zeta_3, \zeta_5, \zeta_7 \rangle$, where $\xi_k = \zeta_{16}^{(1-k)/2} \frac{1 - \zeta_{16}^{1+k}}{1 - \zeta_{16}}$. Since $p$ is inert in $K_2/K_2^+$ we obtain for $k = 3, 5$ or 7

$$\left(\frac{\xi_k}{p}\right)_{K_2} = \left(\frac{N^r(\xi_k)}{p}\right)_{K_2^+} = \left(\frac{\xi_k^2}{p}\right)_{K_2^+} = 1.$$

Then $e$ is at most equals 1. So $2\text{-rank}(\text{Cl}(L_{2,d})) \geq 4 - 1 - 1 = 2$. Hence, the 2-class group of $L_{n,d}$ is not cyclic.

- If now $p$ is congruent to 7 (mod 16), then by [15] Theorem 1, $\text{Cl}_2(L_{n,d})$ is cyclic. Which completes the proof.
Theorem 5. Assume that $d$ takes one of the forms of Theorem 4. Then $\lambda = 1$ and Greenberg’s conjecture holds for $L_{n,d}^+$. 

Proof. By Theorem 4 the 2-class group of $L_{n,d}$ is cyclic. If $d$ is a prime we get $r = 1$ and $s = 0$, hence $\lambda = 2r + s - 1 = 1$. If $d$ is not a prime then $r = 0$ and $s = 2$, so we obtain $\lambda = 0 + 2 - 1 = 1$ in this case as well. Thus, $\lambda = \lambda = 1$ and the first claim follows. Recall that $\lambda(A^-) = \lambda(A^-)$. Note that the group $\hat{A}_n \cap A_n$ is of exponent 2. So if we know that the 2-class group of $L_{n,d}$ is cyclic and $\lambda(\hat{A}_n) = 1$, then $A_n$ contains at most 2 elements. As the capitulation kernel $A_n(L_{n,d}) \to A_n(L_{n,d})$ contains at most 2 elements due to [18, Theorem 10.3], we see that the 2-class group of $L_{n,d}$ is uniformly bounded. □

Theorem 6 ([6, 5]). The rank of the 2-class group of $L_{n,d}$ is 2 in the following cases.
1. $d = pq$ for $p$ and $q$ primes congruent to 3 (mod 8).
2. $d = p$ is a prime congruent to 9 (mod 16).

Theorem 7. Let $d$ an odd be square free integer. Then, the class number of $L_{n,d}^+$ is odd if and only if $d$ takes one of the following forms
1. $d = q_1q_2$ with $q_i \equiv 3 \pmod{4}$ and $q_1$ or $q_2 \equiv 3 \pmod{8}$.
2. $d$ is a prime $p$ congruent to 3 (mod 4).
3. $d$ is a prime $p$ congruent to 5 (mod 8).
4. $d$ is a prime $p$ congruent to 1 (mod 8) and $(\frac{2}{p})_4(\frac{p}{2})_4 = -1$.

Proof. As $L_{n+1,d}/L_{n,d}$ is a quadratic extension that ramifies at the prime ideals of $L_{n,d}^+$ lying over 2 and is unramified elsewhere, for all $n \geq 1$, the class number of $L_{n,d}^+$ is odd implies that the class number of $L_{n,d}^+$ is odd. The converse follows as the extension $L_{n+1,d}/L_{n,d}$ is totally ramified. Hence, the class number of $L_{n,d}^+$ is odd if and only if the class number of $L_{1,d}^+ = \mathbb{Q}(\sqrt{2}, \sqrt{d})$ is odd. See [7, pp. 155, 157] and [8, p. 78] for the rest. □

4. The Main results

Lemma 3. Let $d$ be a square-free integer. We have:
1. $h_2(L_{1,d}) = 2 \cdot h_2(-d)$, if $d = pq$, for two primes $p \equiv 5 \pmod{8}$ and $q \equiv 3 \pmod{8}$.
2. $h_2(L_{1,d}) = h_2(-2d)$, if $d = pq$, for two primes $p \equiv q \equiv 3 \pmod{8}$ or $d = p$ for a prime $p$ such that $p \equiv 9 \pmod{16}$ and $(\frac{2}{p})_4 = 1$.

Proof. Suppose that $d$ takes the first form of the lemma. Denote by $\varepsilon_{2pq}$ the fundamental unit of the quadratic field $\mathbb{Q}(\sqrt{2pq})$. We have $\varepsilon_{2pq} = x + y\sqrt{2pq}$, for
some integer $x$ and $y$. Since $\varepsilon_{2pq}$ has a positive norm, then $x^2 - 2pqy^2 = 1$. Thus $x^2 - 1 = 2pqy^2$. Put $y = y_1y_2$ for $y_i \in \mathbb{Z}$. Assume that we have

$$\begin{cases} x + 1 &= y_1^2 \\ x - 1 &= 2pqy_2^2. \end{cases}$$

Hence $1 = \left(\frac{y_1^2}{p}\right) = \left(\frac{x + 1}{p}\right) = \left(\frac{x + 1}{p}\right) = \left(\frac{2}{p}\right) = -1$, which is impossible. So $x \pm 1$ is not square in $\mathbb{N}$. So from the third and the fourth item of [2, Proposition 3.3], we deduce that $q(L_{1,d}) = 4$. By class number formula (cf. [17]), we have

$$h_2(L_{1,d}) = \frac{1}{2^5}q(L_{1,d})h_2(pq)h_2(-pq)h_2(2pq)h_2(-2qp)h_2(2)h_2(-2)h_2(-1)$$

$$= \frac{1}{2^5}q(L_{1,d})h_2(pq)h_2(-pq)h_2(2pq)h_2(-2qp)$$

$$= \frac{1}{2^5} \cdot 4 \cdot 2 \cdot h_2(-pq) \cdot 2 \cdot 4 \quad \text{(see [7, 10])}$$

$$= 2 \cdot h_2(-pq).$$

We prove similarly the second item using the references [3, 10, 12, 7, 2].

\[ \square \]

**Theorem 8.** Let $d$ be in one of the following cases:

- $d = p$ be a prime congruent to 9 (mod 16) and assume that $\left(\frac{2}{p}\right) = 1$.
- $d = pq$ for two primes congruent to 3 (mod 8).

Let $2^r = h_2(-2d)$. Then for $n \geq 1$ the 2-class group of $L_{n,d}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^{n+r-2}\mathbb{Z}$. In the projective limit we obtain $\mathbb{Z}_2 \times \mathbb{Z}/2\mathbb{Z}$.

**Proof.** By Theorem 3 we know that the 2-rank of the 2-class group of $L_{n,d}$ equals 2. Further $\lambda^- = 1$ due to Theorem 3 and $h_2(L_{1,d}) = 2^r$ by Lemma 3. By Theorem 7 the class number of $L_{n,d}^+$ is odd for all $n$. As there is no capitulation on $A_{\infty}^-$ (see [16] Lemma 2.2) and $\lambda^- = 1$ we see that $A_{\infty}^-$ has rank one for $n$ large enough. That implies that the second generator of the 2-class group of $L_{n,d}$ is a class of a ramified prime in $L_{n,d}/L_{n,d}^+$. As the class number of $L_{n,d}^+$ is odd these ramified classes have order 2 and we obtain that the 2-class group of $L_{n,d}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^{n+r-2}\mathbb{Z}$. In particular, $\lambda = 1$. Let $E$ be the elementary $\Lambda$-module associated to $A_{\infty}$. Then according to [18] page 282-283 $\nu_{n,0}E = 2\nu_{n-1,0}E$ for all $n \geq 2$. Hence, $|E/\nu_{n,0}E| = |E/2^{n-1}E||E/\nu_{1,0}E| = 2^{n-1+c'}$ for $n \geq 1$ and some constant $c' \geq 1$ independent of $n$. Note that we can rewrite this as $|E/\nu_{n,0}E| = 2^{n+c}$. As $E$ has only one $\mathbb{Z}_2$-generator we can assume that the pseudoisomorphism $\phi : A_{\infty} \to E$ is surjective. The maximal finite submodule of $A_{\infty}$ is generated by the classes $(c_n)_{n \in \mathbb{N}}$ of the ramified primes above 2. Let $\tau$ be a generator of $Gal(L_{d,\infty}/L_{0,\infty})$. Then $\tau(c_n) = c_n$ as the primes above 2 are totally ramified in $L_{\infty,d}/\mathbb{Q}(\sqrt{d})$. It follows that $Tc_n = 0$. Hence, for every $n \geq 1$ the kernel of $\overline{\phi} : A_{\infty}/\nu_{n,0}A_{\infty} \to E/\nu_{n,0}E$ is isomorphic to the maximal finite submodule in $A_{\infty}$ and contains 2 elements. Let $Y$ be defined as in [18] page 281.
Then we obtain
\[ |A_n| = |A_\infty /\nu_{n,0}Y| = |A_\infty /\nu_{n,0}A_\infty| / |\nu_{n,0}A_\infty /\nu_{n,0}Y| = 2^{n+c+1} |\nu_{n,0}A_\infty /\nu_{n,0}Y| \]
for \( n \geq 1 \).

As the maximal finite submodule in \( A_\infty \) is annihilated by \( \nu_{n,0} \) we see that the size of \( \nu_{n,0}A_\infty /\nu_{n,0}Y \) is constant independent of \( n \). This shows that in this case we get that the 2-class group of \( L_{n,d} \) is of size \( 2^{n+c+1} \) for all \( n \geq 1 \). Using that \( h_2(L_{1,d}) = 2^r \) we obtain \( n = r - 1 \). This yields \( 2 \cdot 2^{l_n} = 2^{n+r-1} \) and we obtain \( l_n = n + r - 2 \). Noting that \( L_{n,d} \) is the \( n \)-th step of the field \( L_0,d \) finishes the proof of the first claim. As the direct term \( \mathbb{Z}/2\mathbb{Z} \) is normcoherent the second claim is immediate.

**Corollary 1.** Let \( d \) be in one of the following cases:

- \( d = p \) a prime congruent to 9 (mod 16) and assume that \( (\frac{2}{p})_4 = 1 \).
- \( d = pq \) for two primes congruent to 3 (mod 8).

If \( d \) takes the first form, set, \( p = u^2 - 2v^2 \) where \( u \) and \( v \) are two positive integers such that \( u \equiv 1 \mod 8 \).

If \( d \) takes the second form, set \( (\frac{u}{p}) = 1 \) and let the integers \( X, Y, k, l \) and \( m \) such that \( 2q = k^2X^2 + 2lXY + 2mY^2 \) and \( p = l^2 - 2k^2m \). Let \( 2^r = h_2(-d) \). For all \( n \geq 1 \), we have

1. If \( d \) takes the first from, then the 2-class group of \( L_{n,d} \) is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^{n+1}\mathbb{Z} \) if and only if \( (\frac{u}{p})_4 = -1 \).

   Elsewhere, it is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^{n+r-2}\mathbb{Z} \), for some \( r \geq 4 \).

2. If \( d \) takes the second from, then the 2-class group of \( L_{n,d} \) is isomorphic to

   \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^{n+1}\mathbb{Z} \) if and only if \( (\frac{-2}{4k^2X^2 + 2lXY + 2mY^2}) = -1 \).

   Elsewhere, it is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^{n+r-2}\mathbb{Z} \), with \( r \geq 4 \).

**Proof.** By Lemma 3 we have \( h(L_{1,d}) = h_2(-2d) \). Since the 2-rank of \( Cl(L_{1,d}) \) equals 2, and it is not of order 4 (see [11, Theorems 5.7]), then \( h_2(-2d) \) is divisible by 8. Thus [14, Theorem 2] (resp. [10, pp. 356-357]) gives the first (resp. second) item.

We have the following numerical examples that illustrating the above corollary:

1. Set \( p = 89, u = 17 \) and \( v = 10 \). We have \( p = u^2 - 2v^2 \) and \( (\frac{2}{p})_4 = -\left(\frac{u}{p}\right)_4 = 1 \). So the 2-class group of \( L_{n,p} \) is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^{n+1}\mathbb{Z} \), for all \( n \geq 1 \).

2. Let \( p = 11, q = 19, k = 1, l = 3, m = -1, X = 4 \) and \( Y = 1 \). We have : \( p = l^2 - 2k^2m \) and \( 2q = k^2X^2 + 2lXY + 2mY^2 \). Since \( \left(\frac{-2}{4k^2X^2 + 2lXY + 2mY^2}\right) = (\frac{-2}{p}) = -1 \), So the 2-class group of \( L_{n,p} \) is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^{n+1}\mathbb{Z} \), for all \( n \geq 1 \).
Theorem 9. Assume that \( d = pq \) is the product of two primes \( p \equiv -q \equiv 5 \pmod{8} \) and \( 2^r = 2 \cdot h_2(-pq) \). Then for \( n \geq 1 \) the 2-class group of \( L_{n,d} \) is isomorphic to \( \mathbb{Z}/2^{n+r-1}\mathbb{Z} \).

**Proof.** We know already from Theorem 4 that the 2-class group of \( L_{n,d} \) is cyclic and that \( \lambda = 1 \). In particular the module \( A_\infty \) does not contain a finite submodule and is hence isomorphic to it’s elementary module \( E \). Let \( Y \) as above, then there is no \( \nu_{n,0} \)-torsion and we obtain that the size of \( \nu_{n,0} A_\infty/\nu_{n,0} Y \) equals a constant independent of \( n \). Then we obtain \( |A_n| = |A_\infty/\nu_{n,0} A_\infty||\nu_{n,0} A_\infty/\nu_{n,0} Y| = 2^{n+d} \). In particular, Iwasawa’s formula holds for all \( n \geq 1 \). Hence, \( h_2(L_{1,d}) = 2^r = 2^{1+\nu} \) and \( \nu = r - 1 \) and the claim follows. \( \square \)

Corollary 2. Let \( d = pq \) be the product of two primes \( p \) and \( q \) such that \( p \equiv -q \equiv 5 \pmod{8} \). Then for all \( n \geq 1 \), we have

1. If \( \left( \frac{p}{q} \right) = -1 \), then, the 2-class group of \( L_{n,d} \) is isomorphic to \( \mathbb{Z}/2^{n+1}\mathbb{Z} \).
2. If \( \left( \frac{p}{q} \right) = 1 \) and \( \left( \frac{q}{p} \right)_4 = 1 \), then, the 2-class group of \( L_{n,d} \) is isomorphic to \( \mathbb{Z}/2^{n+2}\mathbb{Z} \).

Elsewhere, the 2-class group of \( L_{n,d} \) is isomorphic to \( \mathbb{Z}/2^{n+r-1}\mathbb{Z} \), for some \( r \geq 4 \).

**Proof.** By the previous theorem, the first item is direct from [17] 19.6 Corollary and also the rest is direct from [19] Theorem 3.9 and its proof. \( \square \)

We have the following examples illustrating the above corollary:

1. Let \( d = 13 \cdot 19 \). We have \( \left( \frac{13}{19} \right) = -1 \). So the 2-class group of \( L_{n,p} \) is isomorphic to \( \mathbb{Z}/2^{n+1}\mathbb{Z} \), for all \( n \geq 1 \).
2. Let \( d = 5 \cdot 11 \). We have \( \left( \frac{5}{11} \right) = 1 \) and \( \left( \frac{11}{5} \right)_4 = 1 \). So the 2-class group of \( L_{n,p} \) is isomorphic to \( \mathbb{Z}/2^{n+2}\mathbb{Z} \), for all \( n \geq 1 \).

Let now \( X', Y' \) and \( Z \) three positive integers verifying the Legendre equation

\[ pX'^2 + qY'^2 - Z^2 = 0 \]  \hspace{1cm} (1)

And satisfying

\[ (X', Y') = (Y', Z) = (Z', X') = (p, Y'Z) = (q, X'Z) = 1, \]  \hspace{1cm} (2)

and

\[ X' \text{ odd}, Y' \text{ even and } Z \equiv 1 \pmod{4}. \]  \hspace{1cm} (3)

see [13], for more details about these equations.

Corollary 3. Let \( d = pq \) be the product of two primes \( p \) and \( q \) satisfying \( p \equiv -q \equiv 5 \pmod{8} \), \( \left( \frac{p}{q} \right) = 1 \) and \( \left( \frac{-q}{p} \right)_4 = 1 \). Let \( X', Y' \) and \( Z \) be three positive integers satisfying the equation (1) and the conditions (2) and (3). If \( \left( \frac{p}{q} \right)_4 \neq \left( \frac{2X'}{Z} \right) \), then
the 2-class group of $L_{n,d}$ is isomorphic to $\mathbb{Z}/2^{n+3}\mathbb{Z}$. Elsewhere, it is isomorphic to $\mathbb{Z}/2^{n+r-1}\mathbb{Z}$, for some $r \geq 5$.

Proof. Direct by Theorem 9 and [13, Theorem 2]. □

Now we close this paper with some numerical examples illustrating the above corollary:

(1) Let $p = 5$, $q = 19$ and $d = -pq$. Then $X' = 1$, $Y' = 2$ and $Z = 9$ are solutions of the equation (1) verifying the condition (2) and (3). Furthermore, $\left(\frac{2}{9}\right)_4 = -\left(\frac{9}{2}\right) = -1$. Thus, the 2-class group of $L_{n,d}$ is isomorphic to $\mathbb{Z}/2^{n+3}\mathbb{Z}$.

(2) Let $p = 37$, $q = 11$ and $d = -pq = -407$. Then $X' = 1$, $Y' = 56518$ and $Z = 187449$ are solutions of the equation (1) verifying the condition (2) and (3). Furthermore, $\left(\frac{187449}{47}\right)_4 = \left(\frac{2}{187449}\right) = 1$. Thus, the 2-class group of $L_{n,d}$ is isomorphic to $\mathbb{Z}/2^{n+r-1}\mathbb{Z}$, for some $r \geq 5$. Indeed with these settings $r = 5$ (see [13, p. 230]).

Remark 1. A continuation of the study of these fields will be in a forthcoming paper.

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