GROUPS NOT ACTING ON MANIFOLDS

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Abstract. In this article we collect a series of observations that constrain actions of many groups on compact manifolds. In particular, we show that “generic” finitely generated groups have no smooth volume preserving actions on compact manifolds while also producing many finitely presented, torsion free groups with the same property.

1. Introduction

There are a number of interesting conjectures concerning actions of large groups on manifolds, particularly conjectures of Gromov and Zimmer on actions of higher rank lattices and Lie groups. In this context, Gromov conjectured that a random group should not have any smooth actions on any compact manifold. In this paper we show that, in an appropriate model of randomness, a random group has no smooth volume preserving actions on compact manifolds.

We begin by defining the class of groups for which we can prove this result. That this class is in some sense “generic” is justified and discussed in Section 3. In that section we also discuss some “less generic” groups satisfying our hypotheses. While the notion of genericity we use necessarily produces groups that are not finitely presented, we also provide many examples of finitely presented groups satisfying our hypothesis. In both cases, we produce groups that are torsion free. See §4.2 of this paper for further discussion of both Gromov’s conjecture and the meaning of “generic” or “random” group.

Let $\Gamma$ be a finitely generated group. We say $\Gamma$ has no finite quotients if there are no non-trivial homomorphisms from $\Gamma$ to a finite group. We say $\Gamma$ has property ($FHM$) if any $\Gamma$ action on a complete $\text{CAT}(0)$ Hilbert manifold has a fixed point. By a non-positively curved Hilbert manifold, we mean a complete geodesic $\text{CAT}(0)$ metric space all of whose tangent cones are (isometric to) Hilbert spaces. We remark that property ($FHM$) implies property ($FH$), the fixed point property on Hilbert spaces, which (for locally compact groups) is equivalent to property ($T$).

The main result of this paper is:

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Theorem 1.1. Let $\Gamma$ be a finitely generated group with no finite quotients and property (FHM). Then any volume preserving action of $\Gamma$ on a compact manifold is trivial.

Since every finite group admits many actions on compact manifolds, the assumption of no finite quotients is necessary. For weaker statements on groups with property (FHM) but with finite quotients, see Proposition 2.3 and Theorem 2.5. We construct many torsion free groups satisfying the hypotheses of Theorem 1.1. In Section 3 we also discuss other classes of groups which satisfy the conclusion of Proposition 2.3 without having property (FHM). We include some observations concerning groups with no actions by homeomorphisms on any compact manifold in Section 4. These last results depend heavily on torsion elements.

The proof of Theorem 1.1 involves three steps. First, we observe that if a group $\Gamma$ has property (FHM) then any volume preserving action on a compact manifold preserves a measurable Riemannian metric. Then we apply a theorem of Zimmer [22] to show that the invariant measurable metric and the fact that $\Gamma$ has property (T) imply that the action has discrete spectrum, i.e. that the unitary representation of $\Gamma$ on $L^2(M)$ decomposes as a sum of finite dimensional subspaces. The fact that $\Gamma$ has no finite quotients implies that it has no non-trivial finite-dimensional representations. It follows that the representation of $\Gamma$ on $L^2(M)$ is trivial which immediately implies that the $\Gamma$ action on $M$ is trivial.

This article is motivated by the growing interest in many quarters in the conjecture that random groups don’t act on manifolds. No one interested in the conjecture seemed to know the proof of Theorem 1.1 or its application to “generic” finitely generated groups.

Acknowledgements: The trick of combining Zimmer’s theorem from [22] with no finite quotients is first observed in [FM], though in a slightly more roundabout fashion. Many thanks to Furman and Monod for interesting conversations.

The torsion tricks used in section 4.1 were explained to the first author by Benson Farb in April of 2007. They seem to have been observed by many people simultaneously and independently, see e.g. [BV, W]. The application here to Kac-Moody groups appears to be new.

Thanks to Goulnara Arzhantseva and Ashot Minasyan for pointing out the latter two methods of constructing groups without finite quotients in section 3.3. Also thanks to Alain Valette and Yann Ollivier for useful remarks on an earlier version of this paper and to Martin Bridson for sharing the observation that other groups satisfy Corollary 4.2.

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2. Proof of Theorem 1.1

We briefly recall the construction of the space of “$L^2$ metrics” on a manifold $M$. Given a volume form $\omega$ on $M$, we can consider the space of all (smooth) Riemannian metrics on $M$ whose associated volume form is $\omega$. This is the space of smooth sections of a bundle $P \to M$. The fiber of $P$ is $X = \text{SL}(n, \mathbb{R})/\text{SO}(n)$. The bundle $P$ is an associated bundle to the $\text{SL}(n, \mathbb{R})$ sub-bundle of the frame bundle of $M$ defined by $\omega$. The space $X$ carries a natural $\text{SL}(n, \mathbb{R})$-invariant Riemannian metric of non-positive curvature; we denote its associated distance function by $d_X$. This induces a natural notion of distance on the space of metrics, given by $d(g_1, g_2)^2 = \int_M d_X(g_1(m), g_2(m))^2d\omega$. The completion of the sections with respect to the metric $d$ will be denoted $L^2(M, \omega, X)$; it is commonly referred to as the space of $L^2$ metrics on $M$ and its elements will be called $L^2$ metrics on $M$. That this space is CAT(0) follows easily from the fact that $X$ is CAT(0). For more discussion of $X$ and its structure as a Hilbert manifold, see e.g. [FH]. It is easy to check that a volume preserving $\Gamma$ action on $M$ defines an isometric $\Gamma$ action on $L^2(M, \omega, X)$.

More generally, we can replace $X$ by any symmetric space $Y$ of non-compact type and consider the same construction for any $Y$ bundle over $M$. In fact, the same construction applies if $(M, \omega)$ is just a standard finite measure space and does not depend on the differentiable structure of $M$. The resulting space is called a continuum product. One method for obtaining an isometric $\Gamma$ action on $L^2(M, \omega, Y)$ is to have an $\omega$ preserving $\Gamma$ action on $M$ and a cocycle $\alpha : \Gamma \times M \to \text{Isom}(Y)$ satisfying an integral bound (ensuring that the $\Gamma$ action preserves the space $L^2(M, \omega, Y)$). We call such actions cocycle actions. Not all isometric $\Gamma$ actions arise in this way. This construction contains the case $Y = \mathbb{R}^n$, in which case $L^2(M, \omega, Y)$ is a Hilbert space and there are many isometric actions not arising from cocycles over actions on $Y$. This is essentially the only way in which non-cocycle actions arise in this setting, see [FH] for more discussion.

We say a group $\Gamma$ has property (FCP) if for any non-positively curved symmetric space $Y$, any finite measure space $(M, \omega)$ and any isometric $\Gamma$ action on $L^2(M, \omega, Y)$, the $\Gamma$ action has a fixed point. Clearly property (FHM) implies property (FCP). Since we do not assume our actions are cocycle actions, any group with property (FCP) also has property (FH).

Our argument establishes the following strengthening of Theorem 1.1:

**Theorem 2.1.** Let $\Gamma$ be a finitely generated group with property (FCP) and no finite quotients, then any volume preserving $\Gamma$ action on a compact manifold is trivial.

**Lemma 2.2.** Let $\Gamma$ be a group with property (FCP). Then any volume preserving $\Gamma$ action on a compact manifold $M$ preserves an $L^2$ metric.

This observation is immediate from the definitions and appears to be well-known, but does not appear anywhere in the literature.
Combining Lemma 2.2 with a result of Zimmer [Z2, Theorem 1.7], we have:

**Proposition 2.3.** Let $\Gamma$ be a group with property (FCP). Then any volume preserving $\Gamma$ action on a compact manifold $M$ has discrete spectrum.

As mentioned in the introduction, discrete spectrum means that the unitary representation of $\Gamma$ on $L^2(M, \omega)$ splits as an infinite direct sum of finite dimensional representations. In particular, Proposition 2.3 implies that no group with property (FCP) has a volume preserving weak mixing action on compact manifolds. As in [Z2], one can deduce from Proposition 2.3 that the action is measurably isometric, i.e. measurably conjugate to an action defined by embedding $\Gamma$ in a compact group $K$. Much stronger results would follow if one could prove that $K$ was a Lie group. To show this, one can either show that enough of the $\Gamma$ invariant subspaces of $L^2(M)$ are spanned by smooth functions or by proving directly that the invariant metric is smooth (or even just continuous).

Proving that $K$ is a Lie group is a well-known and difficult problem. Here we do not need to establish this – for our purposes it suffices to note that it embeds in a product of compact Lie groups. The proof of Theorem 2.1 is completed by the following proposition.

**Proposition 2.4.** Let $\Gamma$ be a group with no finite images. Then any discrete spectrum action of $\Gamma$ is trivial.

*Proof.* We have a $\Gamma$ action on $M$ whose action on $L^2(M, \omega)$ splits as a sum of finite dimensional representations $\pi_j$ on finite dimensional spaces $V_j$. Since finitely generated linear groups are residually finite, each $\pi_j$ must have trivial image. Therefore the $\Gamma$ action on functions on $M$ is trivial and so is the $\Gamma$ action on $M$. \qed

We remark briefly on one strengthening of our main results. It is possible to have groups with only finitely many finite quotients. For such a group $\Gamma$, there is always a maximal finite quotient $F_\Gamma$. Our methods also yield:

**Theorem 2.5.** Let $\Gamma$ be a group with property (FCP) and with finitely many finite quotients. Then any volume preserving $\Gamma$ action on a compact manifold factors through $F_\Gamma$.

We remark that our results are stronger than the statements of Theorem 2.1, Proposition 2.3 and Theorem 2.5. Indeed, to obtain the conclusion of those theorems, we only require a fixed point in any action on the space $L^2(M, \omega, X)$ coming from a smooth action on $M$ or the even weaker condition of a measurable invariant metric. In particular, the conclusion of Proposition 2.3 also holds for all lattices in higher rank semisimple algebraic groups over fields of positive characteristic. In this context the Zimmer-Margulis approach to super-rigidity for cocycles produces a measurable invariant metric and the groups are known to have property ($T$), so
3. Groups satisfying the assumptions of Theorem 1.1

In this section, we discuss methods of constructing groups satisfying the hypotheses of Theorems 1.1 and 2.1. In the first two subsections, we discuss groups with property \((FHM)\) and property \((FCP)\) respectively. In the final subsection, we discuss various methods which, starting with a hyperbolic group with property \((FHM)\) or \((FCP)\), produce quotients of the given group with no finite quotients.

3.1. Groups with property \((FHM)\).

A criterion for property \((FHM)\) in terms of actions on simplicial complexes is given explicitly in \([IN]\). This criterion builds on earlier work of Wang \([W1, W2]\). A similar criterion is established in an unpublished preprint of Schoen and Wang \([SW]\). Combined with Zuk’s work on random groups in the triangular model, this implies that a random group in the triangular model at density more than 1/3 has property \((FHM)\) with high probability. As remarked in \([O1, I.3.g]\) this then implies the same property for random group in the density model with density more than 1/3 with high probability. More recently Naor-Silberman \([NS]\) have given proofs that property \((FHM)\) (and more) holds with high probability for random groups in the graph model of \([Gr2]\). Also, Silberman has given a simpler proof that property \((FHM)\) holds in the density model at density greater than 1/3 \([S]\). In the context of \([NS]\), we need much less than is used there. Here we can get by with adding relations corresponding to a single graph to a non-abelian free group, rather than considering an infinite sequence of graphs. The existence of such groups is fully justified by \([O2]\). We remark here that all the groups mentioned in this paragraph are, with high probability, aspherical and hyperbolic. This is important for constructing quotients of these groups with no finite factors. Here, a group is aspherical if it has an aspherical presentation. In particular, this implies the group is torsion free.

Any cocompact group of isometries of a building of type \(\tilde{A}_2\) can be shown to have property \((FHM)\) by the methods of \([IN]\). These groups therefore satisfy \([2.3]\). It seems quite likely that the same is true of cocompact groups of isometries of irreducible higher rank buildings. It seems plausible that the same should be true for non-uniform lattices. Since none of these groups is hyperbolic, we cannot use them to build examples with no finite quotients.

3.2. Groups with property \((FCP)\).

It follows from the main results of \([FH]\) that any quotient of a lattice \(\tilde{\Gamma}\) in \(\text{Sp}(1,n)\) by an infinite normal subgroup has property \((FCP)\). Again by standard constructions, one can construct such a quotient \(\Gamma\) which is torsion free and hyperbolic, see \([O1]\) Chapter II for discussion and references. It is not clear that one can construct the quotient to be aspherical, so it is not clear that our first method for
producing groups with no finite quotients can be used for these groups, see below.

3.3. Groups with no finite quotients.

Method One: Given an aspherical hyperbolic group, there is a standard method of using iterated random quotient to produce from it a group with no finite quotients. The finite stages of this process preserve the property of being aspherical. The process is described on [O1 Section IV.k.] and is originally due to Gromov [Gr1]. This process can be applied to any aspherical hyperbolic group with property (FH) or (FCP), to obtain finitely generated, infinitely presented groups with no finite quotients and with property (FH) or property (FCP). This uses the fact that like property (FH), properties (FH) and (FCP) both obviously pass to quotients. We remark here that this method of producing groups without finite quotients depends only on having infinitely many relators chosen at random. The hard part of the construction is guaranteeing that the resulting group is infinite.

It seems plausible that all assertions in the previous paragraph are true for torsion free hyperbolic groups and not just aspherical ones. This is not currently known and is technically a much more difficult question. Because of this difficulty, it is not clear that one apply this construction to the quotients of lattices in \(SP(1,n)\) in §3.2.

Method Two: In [O2], Ol'shanski gives a method of producing infinite groups with no finite quotients from any hyperbolic group. It is not clear from that article that this method can be used to produce torsion free groups though it may be possible to achieve this using in addition arguments from [O1]. The method does produce groups that are finitely presented. This method applies to both random hyperbolic groups and to cocompact lattices in \(Sp(1,n)\).

Method Three: In this method we use results from [Os] and [AMO] in a manner inspired by [ABJLMS].

Let \(F\) be any group with no finite quotients and at least three generators. Then the free product \(K = F * F\) is hyperbolic relative to its two free factors. Also, \(K\) is finitely presented and torsion free if \(F\) is. Let \(H\) be any relatively hyperbolic group. Then by [AMO, Theorem 1.4] and [Os, Theorem 2.4] there exists an infinite relatively hyperbolic group \(\Gamma\) that is a common quotient of \(K\) and \(H\) such that \(\Gamma\) is finitely presented and torsion free if \(K\) and \(H\). The peripheral subgroups of \(G\) are exactly images of the peripheral subgroups of \(K\) and \(H\).

To apply this result in our context it suffices to find groups \(F\) which are finitely presented, torsion free and have no finite quotients. One can use, e.g. the finitely presented torsion free simple groups constructed by Burger and Mozes [BM] or the four generated Higman group with the presentation

\[
\langle a, b, c, d | a^b = a^2, b^c = b^2, c^d = c^2, d^a = d^2 \rangle.
\]
This technique can be applied to produce groups with no finite quotients from random hyperbolic groups and from both uniform and non-uniform lattices in Sp(1, n).

When applying either of the last two methods to lattices in Sp(1, n), one can apply them directly to the lattice. Since the method produces infinite groups with no finite quotients, it is clearly producing quotients of the lattice by infinite normal subgroups of infinite index.

4. Other groups not acting and some questions

We close with two collections of remarks. In the first subsection, we discuss other reasons why torsion impedes non-trivial group actions, mostly to explain our emphasis on producing torsion free groups. In the second subsection, we ask some questions motivated by this work and Gromov’s conjecture.

4.1. Torsion tricks. This section points out some classes of groups with no actions by homeomorphisms on compact manifolds. The main point is the following fact. We learned it from Farb and Whyte, but it appears to be known independently by many people see e.g. [BV, W].

Lemma 4.1. Let \( \Gamma \) be a simple group that contains a copy of \((\mathbb{Z}/p\mathbb{Z})^\infty\). Then \( \Gamma \) has no non-trivial actions by homeomorphisms on any compact manifold.

The lemma is simply the fact, see e.g. [MS], that for any compact manifold \( M \), there is a number \( k = k(M) \) such that a faithful action of \((\mathbb{Z}/p\mathbb{Z})^n\) by homeomorphisms on \( M \) implies \( n \leq k \). Farb and Whyte observed that by results in [Sch] one can construct a two generated simple group \( \Gamma \) containing the lamplighter group \( \mathbb{Z} \wr \mathbb{Z}/p\mathbb{Z} \). The lemma then implies that this group \( \Gamma \) has no non-trivial actions by homeomorphisms on any compact manifold. One can also give a proof of Shmuel Weinberger’s observation [Z3] that \( \text{SL}(\infty, \mathbb{Z}) \) does not act on a compact manifold using the same observations, Margulis’ normal subgroups theorem and the congruence subgroup property for \( \text{SL}(n, \mathbb{Z}) \).

A more constructive method for finding finitely presented simple groups containing \((\mathbb{Z}/p\mathbb{Z})^\infty\) is the theory of Kac-moody groups. Lemma 4.1, the main theorem of [CR] and [R, Proof of Theorem 4.6] imply that:

Corollary 4.2. Let \( \Gamma \) be a split or almost split Kac-Moody group over the finite field \( \mathbb{Z}/p\mathbb{Z} \) with an infinite, irreducible, non-affine Weyl group. If \( p \) is large enough, then any \( \Gamma \) action by homeomorphisms on a compact manifold is trivial.

We remark here that the \( \Gamma \) in the corollary is finitely presented. Bridson has independently observed that Thompson’s group \( V \) and the variants constructed by Higman satisfy a similar conclusion for the same reasons. Kac-Moody groups are of particular interest in this context because they are
lattices in locally compact groups. Historically the motivation for studying questions concerning groups (not) acting on compact manifolds derives from Zimmer’s conjectures concerning actions of lattices in Lie groups and also algebraic groups over other local fields.

In a recent paper [ABJLMS], similar and more elaborate uses of torsion have led to classes of infinite groups with no non-trivial actions on certain classes of non-compact manifolds and also more general spaces.

4.2. Further questions. The results in this paper raise an obvious sequence of questions. Are there finitely generated or even finitely presented, torsion free groups with no non-trivial actions on compact manifolds? One can ask the question either for actions by homeomorphisms or for actions by diffeomorphisms. It seems plausible that most groups will have no smooth actions. Unless the manifold acted on is the circle, there are no known obstructions to any torsion free group acting by homeomorphisms on any compact manifold. While producing a few sporadic examples would already be quite interesting, there is significantly more interest in showing that having no non-trivial actions is typical (“generic”) in a model for random groups. However, it is important to note that groups chosen according to such a model are not “generic” in the way that “random regular graphs” are – there is currently no good notion of a “typical” group.

More concretely, to reduce Gromov’s conjecture to Theorem 1.1, one wants to prove:

**Conjecture 4.3.** In an appropriate model for random groups, any action of a “generic” group \( \Gamma \) on a compact manifold \( M \) preserves a smooth volume form.

The conjecture is interesting in the context of groups satisfying Theorems 1.1, 2.1, 2.3 or even just Proposition 2.3. It seems reasonable to try to approach Conjecture 4.3 as a fixed point problem. It is worth noting that the fixed point property used must be strictly stronger than property \((T)\) as many linear \((T)\) groups admit actions on manifolds that do not preserve a volume form. A version of this conjecture was asked as a question by Nigel Higson at the July 2007 Banff workshop.

Another reasonable and fairly well known question also motivated by the work in this paper and Gromov’s conjecture.

**Question 4.4.** Is there a number \( \frac{1}{2} > d > 0 \) for which a random group with density at least \( d \) has no finite quotients with positive probability? With probability tending to 1?

In particular, a positive answer to this question would produce many hyperbolic groups which are not residually finite.

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