Coupling independent walkers and the inclusion process

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Abstract

We show propagation of local equilibrium for the symmetric inclusion process (SIP) after diffusive rescaling of space and time, as well as the local equilibrium property of the non-equilibrium steady state in the boundary driven SIP. The main tool is self-duality and a coupling between $n$ SIP particles and $n$ independent random walkers.

1 Introduction

The symmetric inclusion process (SIP) was introduced in [GKRV] and studied in more detail in [GRV]. It is a process where particles move on the lattice $\mathbb{Z}^d$ according to symmetric nearest neighbor random walk jumps at rate $m/2$, and on top there is an attractive interaction governed by “inclusion jumps” where particles at nearest neighbor positions invite each other (invitations which are always followed up) at rate 1. The attractive interaction as opposed to exclusion in the well-known exclusion process is responsible for interesting phenomena. In the limit of small diffusion, condensation phenomena occur [GRVc], and there is an analogue of Liggett’s comparison inequality which leads to “propagation of positive correlations” for appropriately chosen initial product measures [GRV].

The fundamental property which makes this model tractable, despite the interaction is self-duality. Self-duality, combined with a good coupling of exclusion walkers and independent random walkers has lead in the context of the exclusion process to the proof of the hydrodynamic limit, in the sense of propagation of local equilibrium; see e.g. [MP], [MIPP] for two good reference on this approach based on $\nu$-functions. The construction of a good coupling between exclusion walkers and independent random walkers relies quite strongly on particular properties of the exclusion process,
in particular monotonicity. The SIP is not monotone because of the attractive interaction, and therefore presents new challenges both for coupling and for a strong hydrodynamic limit behavior. By self-duality it is easy to see that the (linear) diffusion equation is the correct macroscopic equation. However, because the stationary measures are products of discrete Gamma distributions, their tails are exponential and therefore the standard and powerful general method for hydrodynamic limits of gradient systems, namely the GPV (Guo-Papanicolaou-Varadhan) approach (see [KL]) cannot be applied, neither can we use later developed methods relying on monotonicity. This will of course become an even more serious problem when we pass to the hydrodynamics of the asymmetric inclusion process, where an equation of Burger’s type is expected after hyperbolic rescaling of space and time.

In this paper we show that SIP particles and independent random walkers can be coupled such that their distance at time $t$ grows less than $\sqrt{t}$, which is enough to prove both propagation of local equilibrium and the local equilibrium property of the non-equilibrium steady state. The idea of the coupling is very similar to what happens for simple symmetric exclusion: random walk jumps are performed together and inclusion jumps by inclusion particles only. The discrepancies created on the time scale of potential creation of discrepancies, i.e., when at least two inclusion particles are at nearest neighbor positions, are behaving as a simple random walk. Since inclusion particles in a state of potential creation of discrepancies will get out of it after an exponential time, with parameter bounded from below uniformly over all such configurations, we have that the time at which discrepancies can be built up is at most $t^{1/2+\epsilon}$ with probability close to one as $t \to \infty$. In this time window the differences between the SIP walkers and corresponding independent random walkers can be decomposed into a sum of quantities behaving as continuous-time simple random walk. As a consequence, since the time window is at most $t^{1/2+\epsilon}$, the total discrepancy built up will not exceed $t^{1/4+\epsilon/2}$ with probability close to one.

The rest of our paper is organized as follows. In sections 2.1–2.2 we introduce the SIP and basic properties such as self-duality and invariant product measures. In sections 2.3–2.5 we introduce macroscopic profiles, propagation of local equilibrium, the boundary driven SIP, and the local equilibrium property of its unique non-equilibrium stationary measure. In these sections we use the coupling between independent walkers and SIP walkers. In section 3 we prove the essential estimate controlling the distance between the independent walkers and corresponding SIP walkers in this coupling.
2 Definition and basic properties of the $SIP(m)$

The simple symmetric inclusion process on $\mathbb{Z}$ with parameter $m$ is defined to be the Markov process with state space $\Omega = \mathbb{N}^{\mathbb{Z}}$ (where $\mathbb{N}$ denotes natural numbers including zero) with generator

$$L^{SIP(m)} f(\eta) = \sum_{i \in \mathbb{Z}} \sum_{j \in \{i-1, i+1\}} \eta_i \left( \frac{m}{2} + \eta_j \right) \left( f(\eta^ij) - f(\eta) \right)$$

where $\eta^ij$ denotes the configuration obtained from $\eta \in \Omega$ by removing one particle at $i$ and putting it at $j$: $\eta^ij = \eta - \delta_i + \delta_j$ where $\delta_x$ denotes the configuration with a single particle at location $x \in \mathbb{Z}$ and no particles elsewhere.

The interpretation of the process with generator (1) is that every particle makes the jumps of a continuous-time symmetric nearest neighbor random walk at rate $m/2$ and on top of that there is interaction by inclusion. This interaction is described as follows: every pair of particles at nearest neighbor positions have a Poisson clock which rings at rate one: if this “invitation clock” rings for a pair of particles at positions $i, i+1$, then with probability $1/2$ the particle at $i$ joins the particle at $i+1$ or vice versa. The semigroup associated to the generator (1) is denoted by $S_t$. It is well-defined on multivariate polynomials in the variables $\eta_i, i \in \mathbb{Z}$, depending on a finite number of variables. This is a consequence of self-duality, see section 2.2 below. In fact, $S_t f$ is well-defined for much more functions $f$ but we will need only those (polynomials) here. We denote by $S_t f$ the semigroup working on a function $f$, and $\mu S_t$ for the semigroup working on a probability measure $\mu$ on $\Omega$, i.e., the distribution of $SIP(m)$ at time $t > 0$ when started initially from $\mu$.

2.1 Equilibrium and local equilibrium measures

For $p : \mathbb{Z} \to [0, 1)$ we define the product measure

$$\nu_p(d\eta) = \bigotimes_{i \in \mathbb{Z}} \nu_{p(i)}(d\eta_i)$$

where, with small abuse of notation, we denote for $\lambda \in [0, \infty)$, by $\nu_\lambda$ the discrete Gamma distribution on $\mathbb{N}$, with scale parameter $\lambda$ and shape parameter $m/2$, defined via

$$\nu_\lambda(n) = \frac{1}{Z_\lambda} \frac{\lambda^n \Gamma \left( \frac{m}{2} + n \right)}{n! \Gamma \left( \frac{m}{2} \right)}$$

with $Z_\lambda = (1 - \lambda)^{-m/2}$ the normalizing constant. To the “parameter profile” $p : \mathbb{Z} \to [0, 1)$ corresponds the “density profile”

$$\rho(x) = \int \eta_x \nu_p(d\eta) = \frac{m}{2} \frac{p(x)}{1 - p(x)}.$$
For $p$ a constant, i.e., $p(x) = \lambda$ for all $x \in \mathbb{Z}$, the corresponding homogeneous product measure $\nu_\lambda(d\eta)$ is reversible and ergodic for the SIP($m$) [GRV]. It is natural to expect, but at present not yet proved that these measures are the only ergodic measures of the SIP.

2.2 Self-duality

In order to formulate self-duality of the SIP($m$), we need some more notation. Define for $k, n \in \mathbb{N}$, $k \leq n$

$$d(k, n) = \frac{n!}{(n-k)!} \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2} + k\right)}$$

and for $k > n$, $d(k, n) = 0$. The relation between this polynomial (in $n$) and the discrete Gamma distribution (3) is given by

$$\sum_{n=0}^{\infty} d(k, n) \nu_\lambda(n) = \left(\frac{\lambda}{1-\lambda}\right)^n$$

We denote by $\Omega_f$ the set of configurations in $\Omega$ with a finite number of particles, i.e. $\Omega_f = \{\xi \in \Omega : \sum_{x \in \mathbb{Z}} \xi_x < \infty\}$.

For $\xi \in \Omega_f$ we define

$$\mathcal{D}(\xi, \eta) = \prod_{i \in \mathbb{Z}} d(\xi_i, \eta_i).$$

Self-duality of the SIP($m$) is then the following result, proved in [GKRV], and see also [GRV] for more details on the self-duality of SIP($m$).

**Theorem 2.1.** For $\xi \in \Omega_f$, $\eta \in \Omega$ we have

$$\mathbb{E}_{\eta}^{\text{SIP}(m)} \mathcal{D}(\xi, \eta(t)) = \mathbb{E}_{\xi}^{\text{SIP}(m)} \mathcal{D}(\xi(t), \eta).$$

The relation between the product measures $\nu_p$ from (2) and the duality functions is given by the following formula [GRV] which easily follows from (6):

$$\int \mathcal{D}\left(\sum_{i=1}^{n} \delta_{x_i}, \eta\right) \nu_p(d\eta) = \prod_{i=1}^{n} \frac{p(x_i)}{1-p(x_i)}.$$  

In particular, for the homogeneous product measures we have

$$\int \mathcal{D}\left(\sum_{i=1}^{n} \delta_{x_i}, \eta\right) \nu_\lambda(d\eta) = \left(\frac{\lambda}{1-\lambda}\right)^n.$$  

For a finite configuration $\xi \in \Omega_f$, we can write $\xi = \sum_{i=1}^{n} \delta_{x_i}$ (notice that for different $i$, the corresponding $x_i$ are allowed to be equal), and we will then
denote expectation in the $SIP(m)$ also by $E_{SIP(m)}$, and the locations of the corresponding $n$ SIP($m$) particles by $X_1(t), \ldots, X_n(t)$.

An important role will be played by the “correlation functions”

\[
\mathcal{V}(\nu, x_1, \ldots, x_n; t) = \int E_{SIP(m)} \mathcal{D} \left( \sum_{i=1}^{n} \delta_{x_i}(t) \right) \nu_p(d\eta) - \prod_{i=1}^{n} \int E_{SIP(m)} \mathcal{D} \left( \delta_{x_i}(t) \right) \nu_p(d\eta). \tag{11}
\]

By self-duality, (9) and the fact that a single inclusion walker is a random walk jumping at rate $m^2$, this can be rewritten as

\[
\mathcal{V}(\nu, x_1, \ldots, x_n; t) = E_{SIP(m)} x_1, \ldots, x_n \left( \prod_{i=1}^{n} \frac{p(X_i(t))}{1 - p(X_i(t))} \right) - E_{IRW(m)} x_1, \ldots, x_n \left( \prod_{i=1}^{n} \frac{p(X_i(t))}{1 - p(X_i(t))} \right). \tag{12}
\]

Here $IRW(m)$ denote $n$ independent random walkers moving at rate $\frac{m}{2}$, and $E_{IRW(m)}$ is the expectation in this process, initially started at $(x_1, \ldots, x_n)$. So we see that the function $\mathcal{V}(\nu, x_1, \ldots, x_n; t)$ measures the difference between expectation over $n$ SIP particles and $n$ random walkers.

Moreover, for a function $\phi: \mathbb{Z} \to \mathbb{R}$, the expectation $\psi(t, x) := E_{IRW(m)} \phi(X_t)$ solves the discrete diffusion equation with diffusion constant $\frac{m}{2}$

\[
\frac{\partial \psi(t, x)}{\partial t} = \frac{m}{2} (\psi(t, x + 1) + \psi(t, x - 1) - 2\psi(t, x)) \tag{13}
\]

with initial condition $\psi(0, x) = \phi(x)$, which after suitable scaling converges to the continuous diffusion equation.

From (12), as we will see later, it follows that if we can show that in some sense $n$ independent random walkers and $n$ inclusion walkers can be coupled so that their distance is not too large (as a function of time) then we have, in the sense of propagation of local equilibrium, that the inclusion process has the diffusion equation with diffusion constant $m$ (13) as its hydrodynamic limit.

### 2.3 Macro profiles

We define a macro profile to be a smooth function $\pi: \mathbb{R} \to [0, 1)$. We say that a sequence $p_N: \mathbb{Z} \to [0, 1), N \in \mathbb{N}$ of profiles corresponds to the macro profile $\pi: \mathbb{R} \to [0, 1)$ if for all $y_1, \ldots, y_n \in \mathbb{R}$

\[
\lim_{N \to \infty} \int \mathcal{D} \left( \sum_{i=1}^{n} \delta_{\lfloor Ny_i \rfloor}, \eta \right) \nu_{p_N}(d\eta) = \prod_{i=1}^{n} \frac{\pi(y_i)}{1 - \pi(y_i)}. \tag{14}
\]

We denote this property by “$p_N \approx \pi$”. Intuitively this means that around the macro point $y \in \mathbb{R}$, and corresponding micro point $\lfloor Ny \rfloor$, the profile
$p_N([Ny])$ is close to $\pi(y)$, and as a consequence the product measure $\nu_{p_N(y)}$ is close to the product measure $\nu_{\pi(y)}$.

A simple example of a possible choice for the profiles $p_N$ is given by $p_N(i) = \pi(i/N)$.

More generally, we say that a sequence of probability measures $\mu_N, N \in \mathbb{N}$ on the configuration space $\Omega$ satisfies the local equilibrium property (LEP) with profile $\pi$ if for all $y_1, \ldots, y_n \in \mathbb{R}$

$$\lim_{N \to \infty} \int \mathcal{D} \left( \sum_{i=1}^n \delta_{[Ny_i]}, \eta \right) \mu_N(d\eta) = \prod_{i=1}^n \frac{\pi(y_i)}{1 - \pi(y_i)}.$$  (15)

We denote this property by $\mu_N \approx LEP(\pi)$.

Because $\nu_{p_N}$ is a product measure, we have in particular that if a profile $p_N$ corresponds to the macro profile $\pi$, then $\mu_N = \nu_{p_N}$ satisfies LEP with macro profile $\pi$, i.e., $\nu_{p_N} \approx \pi$ implies $\nu_{p_N} \approx LEP(\pi)$.

If $p_N \approx \pi$ then we consider the evolution of the local equilibrium state $\nu_{p_N}$ after a macroscopic time $N^2 t$. By duality, and the scaling of simple random walk to Brownian motion, we have, for $y_1, \ldots, y_n \in \mathbb{R}$

$$\int \mathbb{E}_\eta \mathcal{D} \left( \sum_{i=1}^n \delta_{[Ny_i]}, \eta(N^2 t) \right) \nu_{p_N}(d\eta)$$

$$= \mathbb{E}^{\text{SIP}(m)}_{[Ny_1], \ldots, [Ny_n]} \left( \prod_{i=1}^n \frac{p_N(X_i(N^2 t))}{1 - p_N(X_i(N^2 t))} \right)$$

$$= \mathbb{V}(\nu_{p_N}, [Ny_1], \ldots, [Ny_n], N^2 t) + \prod_{i=1}^n \mathbb{E}^{\text{IRW}(m)}_{[Ny_i]} \left( \frac{p_N(X_i(N^2 t))}{1 - p_N(X_i(N^2 t))} \right)$$

$$= \mathbb{V}(\nu_{p_N}, [Ny_1], \ldots, [Ny_n], N^2 t) + \prod_{i=1}^n \psi(t, y_i) + o(1)$$  (16)

where $o(1)$ goes to zero when $N \to \infty$, and where $\psi(t, y_i)$ is the solution of the diffusion equation

$$\frac{\partial \psi(t, y)}{\partial t} = \frac{m}{2} \frac{\partial^2 \psi(t, y)}{\partial y^2}$$  (17)

with initial condition

$$\psi(0, y) = \frac{\pi(y)}{1 - \pi(y)}.$$  (18)

Therefore, in order to obtain that $\nu_{p_N}$ after time $N^2 t$ satisfies LEP with macro profile $\psi(t, y)$, it is crucial to prove that the $\mathbb{V}$ functions vanish in that limit. More precisely, if we can prove that for all $y_1, \ldots, y_n \in \mathbb{R}$

$$\lim_{N \to \infty} \mathbb{V}(\nu_{p_N}, [Ny_1], \ldots, [Ny_n], N^2 t) = 0$$  (19)
then we have that \( p_N \approx \pi \) implies \( \nu_{p_N} S_{N^2t} \approx \text{LEP}(\psi(t,y)) \) where \( \psi(t,y) \) satisfies \([17], [18]\). We say then that on the macro scale "propagation of local equilibrium" holds, with corresponding macro equation \([17]\).

In turn, because we speed up time by a factor \( N^2 \), using \([12], [19]\) holds if we can find a coupling \((X_1(t), \ldots, X_n(t); \tilde{X}_1(t), \ldots, \tilde{X}_n(t))\) between \( n \) inclusion walkers and \( n \) independent random walkers such that, for all \( i \), \(|X_i(t) - \tilde{X}_i(t)|/\sqrt{t}\) converges to zero in probability as \( t \to \infty \).

## 2.4 Boundary driven SIP(m)

The boundary driven \( \text{SIP}(m) \) on \( N \) lattice sites is a process on the configuration space \( \Omega_N = \mathbb{N}^{\{1, \ldots, N\}} \) defined via its generator

\[
\mathcal{L}^{\text{SIP}} f(\eta) = \mathcal{L}_a^{\text{SIP}} f(\eta) + \mathcal{L}_0^{\text{SIP}} f(\eta) + \mathcal{L}_b^{\text{SIP}} f(\eta)
\]

\[
= \alpha \left( \frac{m}{2} + \eta_i \right) (f(\eta^{i,0}) - f(\eta)) + \gamma \eta_i (f(\eta^{1,0}) - f(\eta))
\]

\[
+ \sum_{i=1}^{N-1} \eta_i \left( \frac{m}{2} + \eta_{i+1} \right) (f(\eta^{i,i+1}) - f(\eta)) + \eta_{i+1} \left( \frac{m}{2} + \eta_i \right) (f(\eta^{i+1,i}) - f(\eta))
\]

\[
+ \delta \left( \frac{m}{2} + \eta_N \right) (f(\eta^{N+1,N}) - f(\eta)) + \beta \eta_N (f(\eta^{N,N+1}) - f(\eta)) .
\]

Here, we denote, for \( \eta \in \Omega_N \), with slight abuse of notation

\[
\eta^{0,1} = \eta + \delta_1, \quad \eta^{1,0} = \eta - \delta_1
\]

and similarly

\[
\eta^{N+1,N} = \eta + \delta_N, \quad \eta^{N,N+1} = \eta - \delta_N
\]

this notation turns out to be useful in the dual process, where we have two more sites, namely 0 and \( N + 1 \) associated to the reservoirs. In the presence of boundary generators, we have the following duality

**Theorem 2.2.** The process \((\eta(t))_{t \geq 0}\) defined by \([20]\) is dual to the absorbing boundaries process \((\xi(t))_{t \geq 0}\) with configuration space \( \Omega_{\text{Dual}} = \mathbb{N}_0^{0,1,...,N,N+1} \) with generator

\[
\mathcal{L}^{\text{Dual}} f(\xi) = (\gamma - \alpha) \xi_1 (f(\xi^{1,0}) - f(\xi))
\]

\[
+ \sum_{i=1}^{L-1} \xi_i \left( \frac{m}{2} + \xi_{i+1} \right) (f(\xi^{i,i+1}) - f(\xi)) + \xi_{i+1} \left( \frac{m}{2} + \xi_i \right) (f(\xi^{i+1,i}) - f(\xi))
\]

\[
+ (\beta - \delta) \xi_N (f(\xi^{N,N+1}) - f(\xi)) ,
\]

with duality function

\[
\mathcal{D}^{\text{SIP}} (\xi, \eta) = \left( \frac{\alpha}{\gamma - \alpha} \right) \xi_0 \left( \prod_{i=1}^{N} \frac{\eta_i!}{(\eta_i - \xi_i)!} \frac{\Gamma \left( \frac{m}{2} \right)}{\Gamma \left( \frac{m}{2} + \xi_i \right)} \right) \left( \frac{\delta}{\beta - \delta} \right)^{\xi_{N+1}} .
\]

7
In the dual process the reservoirs have been replaced by absorbing boundaries. This is a considerable simplification, because in the dual process eventually all particles will end up at 0 or \( N + 1 \).

Denoting \( \rho_L = \frac{\alpha}{1 - \alpha} \), \( \rho_R = \frac{\delta}{\delta - \beta} \) we obtain from the duality \([22]\) the following expression for the unique stationary distribution \( \nu_{\rho_L, \rho_R} \) (called the non-equilibrium steady state) of the boundary driven process with generator \([20]\).

\[
\int \mathcal{D}^{\text{SIP}}(\xi, \eta) \nu_{\rho_L, \rho_R}(d\eta) = \sum_{k+i=||\xi||} \rho_L^k \rho_R^i \mathbb{P}^{\text{SIP}, \text{abs},N}_{\xi}(\xi(\infty) = k\delta_0 + l\delta_{N+1}) \tag{23}
\]

where \( ||\xi|| = \sum_{x=0}^{N+1} \xi_x \) denotes the total number of dual particles, and where \( \mathbb{P}^{\text{SIP}, \text{abs},N}_{\xi} \) denotes the path space measure of the dual absorbing SIP process with generator \([21]\) starting from \( \xi \). I.e., the correlation functions in the non-equilibrium steady state are completely determined by the absorption probabilities of the dual SIP (m).

From now on we will choose the parameters \( \gamma = (\rho_L + 1)m/2, \alpha = \rho_L m/2, \delta = \rho_R m/2, \beta = (\rho_R + 1)m/2 \), which implies that a single dual particle is a continuous-time random walk moving at rate \( m/2 \) and absorbed at 0 or \( N + 1 \). Since the absorption probability of such a random walker is linear as a function of the starting point, we obtain as a particular case of \([23]\) the linear density profile in the non-equilibrium steady state:

\[
\int D(\delta_i, \eta) \nu_{\rho_L, \rho_R}(d\eta) = \rho_L + (\rho_R - \rho_L)i =: \rho_{R,L}^i. \tag{24}
\]

When we consider this profile on the macro scale we obtain, for \( x \in [0, 1] \) the linear macro profile

\[
\lim_{N \to \infty} \int D(\delta_{\lfloor xN \rfloor}, \eta) \nu_{\rho_L, \rho_R}(d\eta) = \rho_L + (\rho_R - \rho_L)x =: \rho_{R,L,\text{macro}}(x). \tag{25}
\]

2.5 Local equilibrium property of non-equilibrium steady states

Denote by \( \mathbb{E}^{\text{SIP}, \text{abs},N}_{x_1, \ldots, x_n} \) the expectation in the process with \( n \) SIP particles, moving on \( \{0, \ldots, N+1\} \) according to the generator \([21]\), and \( \mathbb{E}^{\text{IRW}, \text{abs},N}_{x_1, \ldots, x_n} \) the corresponding expectation for independent random walkers. The particles under \( \mathbb{E}^{\text{SIP}, \text{abs},N}_{x_1, \ldots, x_n} \) (resp. \( \mathbb{E}^{\text{IRW}, \text{abs},N}_{x_1, \ldots, x_n} \)) move as SIP particles (resp. independent random walkers) on \( \{1, \ldots, N\} \) and are absorbed at the left end (resp. right end) 0 (resp. \( N + 1 \)) at rate \( \gamma - \alpha \) (resp. \( \beta - \delta \)). Absorbed and non-absorbed
Intuitively, this property means that around the macro point \( x \) where we defined \( \rho \) property with profile \( \nu \) equilibrium product measure measures on \( N \)

\[
\text{If we can couple } n \text{ SIP particles and } n \text{ independent random walkers on } Z \text{ (in a Markovian way) such that they are at distance } o(\sqrt{t}) \text{ apart when}
\]

\[
\text{where we defined } \rho(0) = \rho_L, \rho(N + 1) = \rho_R.
\]

We can now define the local equilibrium property of \( \nu_{PL,PR}^N \) as follows. Intuitively, this property means that around the macro point \( x \in [0, 1] \) (corresponding micro point \( [xN] \)), the measure \( \nu_{PL,PR}^N \) looks like the SIP(\( m \)) equilibrium product measure \( \nu_{\rho(x)} \) where \( \rho(x) \) is the macroscopic linear profile \( \rho^{L,R,macro} \) defined in (25).

\[
\text{Definition 2.1. Let } \rho : [0, 1] \rightarrow [0, \infty), \text{ and } \mu_N \text{ a collection of probability measures on } \mathbb{N}^{[1..N]}. \text{ The sequence } \mu_N, N \in \mathbb{N} \text{ satisfies the local equilibrium property with profile } \rho \text{ (notation } \mu_N \approx \text{LEP}(\rho) \text{ if for all } n \in \mathbb{N} \text{ and for all } x_1, \ldots, x_n \in [0, 1]:
\]

\[
\lim_{N \to \infty} \int \mathcal{D} \left( \sum_{i=1}^{n} \delta_{[x_i,N]}, \eta \right) \mu_N(d\eta) = \prod_{i=1}^{n} \rho(x_i) \tag{27}
\]

In view of (26) we obtain then the following statement. If for all \( x_1, \ldots, x_n \in [0, 1] \) and \( a_1, \ldots, a_n \in \{0, N + 1\} \)

\[
\lim_{N \to \infty} \prod_{i=1}^{n} \mathbb{P}_{[x_1,N], \ldots, [x_n,N]}(X_1(\infty) = a_1, \ldots, X_n(\infty) = a_n)
\]

\[
= \lim_{N \to \infty} \prod_{i=1}^{n} \mathbb{P}_{[x_i,N]}(X_i(\infty) = a_i)
\]

\[
= \lim_{N \to \infty} \prod_{i=1}^{n} \mathbb{P}_{[x_1,N], \ldots, [x_n,N]}(X_1(\infty) = a_1, \ldots, X_n(\infty) = a_n) \tag{28}
\]

then \( \nu_{PL,PR}^N \approx \text{LEP}(\rho) \), where \( \rho \) is the macroscopic linear profile \( \rho^{L,R,macro} \) defined in (25). In words, asymptotic (as \( N \to \infty \)) factorization of the absorption probabilities implies the local equilibrium property of the non-equilibrium steady state.

This factorization (28) in turn is implied by the existence of a coupling \((X_1(t), \ldots, X_n(t)), (\tilde{X}_1(t), \ldots, \tilde{X}_n(t))\) of \( n \) SIP(\( m \)) particles (with absorption at \( 0, N + 1 \)) with \( n \) independent random walkers (with absorption at \( 0, N + 1 \)) both starting from initial state \(([x_1N], \ldots, [x_nN])\) such that

\[
\lim_{N \to \infty} \mathbb{P} \left( X_i(\infty) \neq \tilde{X}_i(\infty) \text{ for some } i \in \{1, \ldots, n\} \right) = 0 \tag{29}
\]

If we can couple \( n \) SIP particles and \( n \) independent random walkers on \( Z \) (in a Markovian way) such that they are at distance \( o(\sqrt{t}) \) apart when
$t \to \infty$, then we can also use that coupling to achieve (29). Indeed, if the SIP particle $i$ is absorbed at 0, then this happens before time $bN^2$ for $b$ large enough, with probability close to one. As long as the random walker and the corresponding SIP are not absorbed, they both move as if they were on the infinite lattice. I.e., if the random walk particle is not yet absorbed at the moment that the SIP particle is absorbed, it is at distance $\delta N$ from 0, with probability close to one, for any $\delta > 0$. Therefore, the probability that it will be absorbed at the other end $N + 1$ is at most $\delta$. Hence the probability that they will be absorbed at different ends tends to zero as $N \to \infty$.

3 Coupling $n$-SIP particles and $n$-independent random walkers

The aim of this section is to construct a Markovian coupling for the $n$-SIP($m$) and $n$-independent random walkers. Here we will consider the configuration space $\Omega_f^{(n)} = \{ \eta \in \Omega : \sum_{x \in \mathbb{Z}} \eta_x = n \}$ and its product. As we saw above each configuration $\eta \in \Omega_f^{(n)}$ takes the form $\eta = \sum_{i=1}^n \delta_{x_i}$, for some $(x_1, \ldots, x_n) \in \mathbb{Z}^n$, where $\delta_x$ is a configuration with only one particle at site $x \in \mathbb{Z}$. In the sequel we will identify a configuration with its vector of particle positions. This naturally assigns labels to the particles and the way labels are assigned is not important. Further, for any $x \in \mathbb{Z}^n$ and $\epsilon \in \{1, -1\}$, define $x^{i,i+\epsilon}$ to be such that for any $1 \leq j \leq n$

$$x^{i,i+\epsilon}_j = \begin{cases} x_j & \text{if } j \neq i \\ x_i + \epsilon & \text{otherwise.} \end{cases}$$

(30)

The coupling of interest is the markov process with generator $L_{\text{SIP(m)}, \text{IRW(m)}}$, that acts on functions $f : \Omega_f^{(n)} \times \Omega_f^{(n)} \to \mathbb{R}$ as

$$L_{\text{SIP(m)}, \text{IRW(m)}} f(x, \tilde{x}) = \sum_{i=1}^n \sum_{\epsilon=\pm1} \frac{m}{2} \left[ f(x^{i,i+\epsilon}, \tilde{x}^{i,i+\epsilon}) - f(x, \tilde{x}) \right]$$

$$+ \sum_{i=1}^n \sum_{k=1}^n \sum_{\epsilon=\pm1} I(x_i + \epsilon = x_k) \left[ f(x^{i,i+\epsilon}, \tilde{x}) - f(x, \tilde{x}) \right].$$

(31)

Under this coupling the SIP($m$) particles and the random walkers perform the same random walk jumps but the invitation jumps are only made by the SIP($m$) particles.

Next we state a results that quantifies the distance between the SIP($m$) particles and the random walkers. To this end we denote by $X(t) = (X_1(t), \ldots, X_n(t))$ and $\tilde{X}(t) = (\tilde{X}_1(t), \ldots, \tilde{X}_n(t))$ the processes of the positions of the $n$-SIP($m$) particles and the $n$-independent random walkers respectively.
**Theorem 3.1.** In the coupling \((X(t), \tilde{X}(t))\) between the \(n\)-SIP(m) particles and \(n\)-independent random walkers moving at rate \(\frac{m}{2}\), starting at the same initial positions (i.e. \(X(0) = \tilde{X}(0)\)), we have

\[
\lim_{t \to \infty} \frac{|X_i(t) - \tilde{X}_i(t)|^2}{t} = 0, \quad \text{for all} \quad 1 \leq i \leq n, \quad (32)
\]

where the above limit is in probability.

**Proof.** We will in fact show a stronger result, namely, for all \(\alpha > \frac{1}{2}\) there are \(A, B, \text{ and } \delta > 0\) such that

\[
P \left( |X_i(t) - \tilde{X}_i(t)| < t^\alpha \right) \geq 1 - B e^{-At^\delta}, \quad \forall i = 1, \ldots, n. \quad (33)
\]

This result is obtained for the symmetric exclusion process in [MIPP]. The proof here follows the line of thought of that given in [MIPP]. The proof involves identifying the sources of discrepancies in \(X_i(t) - \tilde{X}_i(t)\) and estimating the probability that a given source contributes too much discrepancy.

More precisely, we will identify the events that switch on the invitation clocks, since under the coupling the discrepancies in \(X_i(t) - \tilde{X}_i(t)\) are caused by the invitation jumps made by the \(i\)th SIP(m) particle. Each invitation jump made by this particle is a collective action of all the particles at the destination site. Therefore to properly account for the discrepancies in \(X_i(t) - \tilde{X}_i(t)\) we need to identify the particles that caused each such invitation jump. To this end we denote by \(\mathcal{P}_{i,n}\), the set of all nonempty subsets of \(\{1, 2, \ldots, n\} \setminus \{i\}\), for \(1 \leq i \leq n\). We say that \(A \in \mathcal{P}_{i,n}\), is a source of discrepancy in \(X_i(\cdot) - \tilde{X}_i(\cdot)\) at time \(s\) if at this time the \(i\)th SIP(m) particle made a jump at the invitation of the SIP(m) particles labelled by the set \(A\) (see Fig. 1 below).

(a) Discrepancy source is \(A=\{u,v\}\)

(b) Discrepancy source is \(A=\{a,b,c\}\)

![Figure 1: Discrepancy source](image-url)

Further, define \(D_{i,A}(t)\) as the sum of all the discrepancies in \(X_i(\cdot) - \tilde{X}_i(\cdot)\)
up to time $t$ caused by the source $A$. Then we have that

$$X_i(t) - \tilde{X}_i(t) = \sum_{A \in \mathcal{P}_{i,n}} D_{i,A}(t). \tag{34}$$

Note that $D_{i,A}$ changes by $\pm 1$ each time $A$ is the discrepancy source and this happens with equal probability, since the particles labelled by $A$ can be in any of the two nearest neighboring sites to the site occupied by the $i$th particle. Further, the changes in $D_{i,A}$ occur independently over time. This means that conditionally on the time scale where we observe displacements in $D_{i,A}$, $D_{i,A}$ behaves like a simple random walk. Note that

$$P\left(|X_i(t) - \tilde{X}_i(t)| < t^\alpha \right) = 1 - P\left(|X_i(t) - \tilde{X}_i(t)| \geq t^\alpha \right)$$

$$= 1 - 2P\left(\sum_{A \in \mathcal{P}_{i,n}} D_{i,A}(t) \geq t^\alpha \right) \geq 1 - 2P\left(\max_{A \in \mathcal{P}_{i,n}} D_{i,A}(t) \geq \frac{t^\alpha}{|\mathcal{P}_{i,n}|} \right)$$

$$= 1 - 2P\left(\exists A \in \mathcal{P}_{i,n}; D_{i,A}(t) \geq c t^\alpha \right) \geq 1 - 2 \sum_{A \in \mathcal{P}_{i,n}} P(D_{i,A}(t) \geq c t^\alpha). \tag{35}$$

Here in the second equality uses the symmetry of the $D_{i,A}$’s. The proof now boils down to estimating the probabilities $P(D_{i,A}(t) \geq c t^\alpha)$. To keep the presentation simple we will write $t^\alpha$ instead of $c t^\alpha$. To proceed we will need the following notation. Let $I_{i,A}(t)$ be the total time, up to time $t$, spent by the $i$th SIP(m) particle sitting nearest neighbor to a site occupied precisely by the particles labeled by $A$. Let $\epsilon > 0$ be such that $\alpha = (\frac{1}{2} + \epsilon)^2$. Then

$$P(D_{i,A}(t) \geq t^\alpha) \leq P\left(D_{i,A}(t) \geq t^\alpha \left| I_{i,A}(t) \geq t^{1/2+\epsilon} \right. \right) P\left(I_{i,A}(t) \geq t^{1/2+\epsilon} \right)$$

$$+ P\left(D_{i,A}(t) \geq t^\alpha \left| I_{i,A}(t) \leq t^{1/2+\epsilon} \right. \right) P\left(I_{i,A}(t) \leq t^{1/2+\epsilon} \right)$$

$$\leq P\left(D_{i,A}(t) \geq t^\alpha \left| I_{i,A}(t) \geq t^{1/2+\epsilon} \right. \right) P\left(I_{i,A}(t) \geq t^{1/2+\epsilon} \right)$$

$$+ P\left(I_{i,A}(t) \geq t^{1/2+\epsilon} \right) \tag{36}$$

Recall from the remark below (34) that $D_{i,A}$ is a continuous-time simple random walk conditionally on $I_{i,A}(t)$. Therefore by the local limit theorem we have the estimate

$$P\left(D_{i,A}(t) \geq I_{i,A}(t)^{1/2+\epsilon} \left| I_{i,A}(t) \leq t^{1/2+\epsilon} \right. \right) \leq b e^{-a I_{i,A}(t)^{2\epsilon}}, \tag{37}$$
where \( a \) and \( b \) are some positive constants. Then we can estimate
\[
\mathbb{P} \left( D_{i,A}(t) \geq t^{\alpha} \mid I_{i,A}(t) \leq t^{1/2+\epsilon} \right) \mathbb{P} \left( I_{i,A}(t) \leq t^{1/2+\epsilon} \right)
\]
\[
\leq \mathbb{P} \left( D_{i,A}(t) \geq I_{i,A}(t)^{1/2+\epsilon}, I_{i,A}(t) \geq t^{\alpha} \right)
\]
\[
\leq \mathbb{E} \left[ b e^{-a I_{i,A}(t)^{2\epsilon}} I \left( I_{i,A}(t) \geq t^{\alpha} \right) \right]
\]
\[
\leq b e^{-a t^{2\epsilon \alpha}},
\]
where \( I(\cdot) \) denotes the indication function.

Going back to (36) we now have to estimate the probability \( \mathbb{P} \left( I_{i,A}(t) \geq t^{1/2+\epsilon} \right) \). To better understand how to proceed for any general \( n \in \mathbb{N} \), we first consider the case \( n = 2 \). Here we put \( i = 1 \) and \( A = \{2\} \).

Let \( N_{i,A}(t) \) be the number of encounters, i.e. instances, \( s \in [0,t) \), where at \( s- \) particles 1 and 2 are not at neighboring positions and at time \( s+ \) they are, plus one if at time zero they are at neighboring position. Let \( X_k \) be the duration of the time \( k \)th encounter, for \( 1 \leq k \leq N_{i,A}(t) \). Then we have the inequality
\[
I_{i,A}(t) \leq \sum_{k=1}^{N_{i,A}(t)} X_k.
\]

Note that in the present case the \( X_k \)’s are independent exponentially distributed random variables which are stochastically dominated by an exponential with rate \( m/2 \), since the two particles wait an exponential time in a n.n. position and exit that state by either increasing the distance between them by one at rate \( m/2 \) or decreasing it by one at rate \( m/2 + 1 \). Note that
\[
\mathbb{P} \left( I_{i,A}(t) \geq t^{1/2+\epsilon} \right) \leq \mathbb{P} \left( \sum_{k=1}^{N_{i,A}(t)} X_k \geq t^{1/2+\epsilon} \right)
\]
\[
\leq \mathbb{P} \left( \sum_{k=1}^{N_{i,A}(t)} X_k \geq t^{1/2+\epsilon}, N_{i,A}(t) \leq t^{1/2+\epsilon} \right) + \mathbb{P} \left( N_{i,A}(t) > t^{1/2+\epsilon} \right)
\]
\[
\leq \mathbb{P} \left( \frac{1}{\left[ t^{1/2+\epsilon} \right]} \sum_{k=1}^{\left[ t^{1/2+\epsilon} \right]} X_k \geq t^{\frac{\epsilon}{2}} \right) + \mathbb{P} \left( N_{i,A}(t) > t^{1/2+\epsilon} \right)
\]
\[
\leq C_1 e^{-C_2 t^{\frac{\epsilon}{2} + \frac{1}{2}}} + C_3 e^{-C_4 t^{\epsilon}},
\]
where \( C_j \)'s are positive constants for \( 1 \leq j \leq 4 \). In the last inequality we used standard large deviations for iid exponentials for the estimation of the first probability. For the estimation of the second term \( \mathbb{P} \left( N_{i,A}(t) > t^{1/2+\epsilon} \right) \) we use that \( N_{i,A}(t) \) is dominated by the number of instances that a discrete-time simple random walk on \( \mathbb{Z} \) visits the set \( \{-1,1\} \) (see e.g. [dHNR], Lemma 13).
1). This in turn follows because the two SIP particles move as independent continuous-time random walkers as long as they are not neighbors, and the difference of the two walkers behaves like a single continuous-time random walk twice the rate.

For the general case, let us call $\Delta$ the set of those $n$-particle configurations where at least one pair of particles is at neighboring positions, i.e., the set of configurations where potentially new discrepancies are created. Outside the set $\Delta$, the $n$ SIP particles behave as $n$ independent random walkers. Let us call $I_\Delta(t)$ the total time spent by the $n$-particle SIP configuration $(X_1(t), \ldots, X_n(t))$ inside the set $\Delta$. Observe that, once the set $\Delta$ is entered at a particle configuration $(a_1, \ldots, a_n)$ say, then, the set $\Delta$ is exited at a rate $\lambda = \lambda(n, m)$, bounded from below uniformly in $(a_1, \ldots, a_n)$. This can easily be concluded from the fact that from every configuration $(a_1, \ldots, a_n) \in \Delta$, there is a path consisting of a finite number of jumps that leaves $\Delta$, and every jump has strictly positive rate (see Fig.2 below). Moreover, the maximal number of jumps needed is bounded by a constant only depending on $n$ and not on the configuration $(a_1, \ldots, a_n) \in \Delta$. Notice however that the maximal length of this exiting path increases with $n$, and correspondingly the estimate based on it will become bad when $n \to \infty$.

Therefore, if we denote $N_\Delta(t)$ the number of times the SIP $n$-particle configuration $(X_1(t), \ldots, X_n(t))$ enters the set $\Delta$ (plus one if we start from a point in $\Delta$), we have the estimate

$$I_{i,A}(t) \leq I(t) \leq \sum_{k=1}^{N_\Delta(t)} X_k$$

where $X_k$ are i.i.d. random variables which satisfy the tail estimate

$$\mathbb{P}(X_k \geq t) \leq e^{-\lambda t}$$

for all $t$ large enough.

We can then proceed as before in the two particle case. Indeed, to estimate $\mathbb{P}(N_\Delta(t) > t^{1+\varepsilon})$, we use that for $n$ independent random walker,
the total time $T_i$ in $[0,t]$ that at least two spent at neighboring positions reduces to a sum of two particle cases, i.e., denoting $T_{ij}(t)$ the total time that the $i$-th and $j$-th walker spend at neighboring positions, we have

$$\mathbb{P} \left( T_i > t^{1/2+\frac{\epsilon}{2}} \right) \leq \sum_{i,j=1}^{n} \mathbb{P} \left( T_{ij} > \frac{t^{1/2+\frac{\epsilon}{2}}}{n^2} \right) \leq C_1 e^{-c_2 t^\epsilon} \quad (42)$$

By passing from discrete to continuous-time and vice versa, we obtain an identical estimate for $\mathbb{P}(N_{\Delta}(t) > t^{1/2+\frac{\epsilon}{2}})$.

Finally, the large deviation estimate of the first term in the third line of (40) is identical because we have (41).

\[\square\]

**REMARK 3.1.** In this final remarks we indicate some generalizations and further perspectives.

1. To generalize to more general symmetric and translation invariant random walks in $d \geq 1$: the essential ingredients of the proof of Theorem 3.1 are the decomposition of the discrepancies according to their “sources” [34], and the fact that in the time interval $[0,t]$, independent random walkers visit the set $\Delta$ where possibly discrepancies are created only during a time window of size $t^{1/2+\epsilon}$ with probability close to one as $t \to \infty$. Both statements remain true for general symmetric and translation invariant random walk with bounded jumps size in $d \geq 1$. The set $\Delta$ is then the set of all configurations $(a_1, \ldots, a_n)$ where at least two positions $a_i$ are at distance less than or equal to the maximal jump size. The fact that in the time interval $[0,t]$, independent random walkers visit this set $\Delta$ where possibly discrepancies are created only during a time window of size $t^{1/2+\epsilon}$ with probability close to one as $t \to \infty$ remains identically true. In fact, in $d = 2$ the set $\Delta$ is visited during a time window of order $\log(t)$ and in $d \geq 3$ of order $1$, so in fact the estimates can even be made stronger, but certainly the result of Theorem 3.1 remains valid. The consequences of the coupling result for the SIP are based on self-duality, and the invariance principle for a single random walk. For self-duality to hold it is essential that the underlying random walk is symmetric.

2. The SIP is dual to an interacting diffusion process with state space $[0,\infty)^Z$ (energies associated to lattices sites), called the Brownian energy process, and for integer values of $m$ also to an interacting diffusion process called the Brownian momentum process with state space $\mathbb{R}^Z$ (momenta associated to lattice sites), see [GKRV], [GRV], and [CGGR] for an overview of all these models. As a consequence, the results on propagation of local equilibrium and of the local equilibrium
property of the non-equilibrium steady state immediately apply to these models.

3. Finally, the essential property of the set $\Delta$ where possibly discrepancies are created is the fact that it is visited only in a time window of order $t^{1+\frac{\epsilon}{2}}$, during the interval $[0,t]$, with probability close to one, and the discrepancy created while in this set is equally probably $\pm 1$. This means that for other modifications of independent random walkers of the same type, i.e., the behavior is only different in a set visited during a time window of order $t^{1+\frac{\epsilon}{2}}$, and the deviation from random walk behavior is “symmetric around 0”, the same result of theorem 3.1 still holds. This is useful e.g. when considering “thermalized SIP” [CGGR], of which the dual KMP model [KMP] is one example: in these processes multiple particles can jump when they are at neighboring sites.

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