On the Reidemeister torsion of rational homology spheres

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Introduction

In the paper [3] V. Turaev has proved a certain identity involving the Reidemeister torsion of a rational homology sphere. In this very short note we will suitably interpret this identity as a second order finite difference equation satisfied by the torsion which will allow us to prove a general structure result for the \( \mod Z \) reduction of the torsion. More precisely we prove that the mod \( Z \) reduction of the torsion is completely determined by three data.

- a certain canonical spin\(^c\) structure,
- the linking form of the rational homology sphere and
- a constant \( c \in \mathbb{Q}/\mathbb{Z} \).

As a consequence, the constant \( c \) is a \( \mathbb{Q}/\mathbb{Z} \)-valued invariant of the rational homology sphere. Experimentations with lens spaces suggest this invariant is as powerful as the torsion itself.

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1 The Reidemeister torsion

We review briefly a few basic facts about the Reidemeister torsion a rational homology 3-sphere. For more details and examples we refer to [1, 2].

Suppose \( M \) is a rational homology sphere. We set \( H := H_1(M, \mathbb{Z}) \) and use the multiplicative notation to denote the group operation on \( H \). Denote \( \text{Spin}^c(M) \) the \( H \)-torsor of
isomorphism classes of spin c structure on $M$. We denote by $\mathcal{F}$ the space of functions

$$\phi : H \to \mathbb{Q}.$$ 

The group $H$ acts on $\mathcal{F}_M$ by

$$H \times \mathcal{F} \ni (g, \phi) \mapsto g \cdot \phi$$

where

$$(g \cdot \phi)(h) = \phi(hg).$$

We denote by $\int_H$ the augmentation map

$$\mathcal{F}_M \to \mathbb{Q}, \quad \int_H \phi = \sum_{h \in H} \phi(h).$$

According to [3] Reidemeister torsion is a $H$-equivariant map

$$\tau : \text{Spin}^c(M) \to \mathcal{F}_M, \quad \text{Spin}^c(M) \sigma \mapsto \tau_\sigma \in \mathcal{F}_M$$

such that

$$\int_H \tau_\sigma = 0$$

Denote by $\text{lk}_M$ the linking form of $M$,

$$\text{lk}_M : H \times H \to \mathbb{Q}/\mathbb{Z}.$$ 

V. Turaev has proved in [3] that $\tau_\sigma$ satisfies the identity

$$\tau_\sigma(g_1g_2h) - \tau_\sigma(g_1h) - \tau_\sigma(g_2h) + \tau_\sigma(h) = -\text{lk}_M(g_1, g_2) \mod \mathbb{Z} \quad (1.1)$$

$$\forall g_1, g_2, h \in H, \sigma \in \text{Spin}^c(M).$$

2 A second order “differential equation”

The identity (1.1) admits a more suggestive interpretation. To describe it we need a few more notation.

Denote by $\mathcal{S}$ the space of functions $H \to \mathbb{Q}/\mathbb{Z}$. Each $g \in H$ defines a first order differential operator

$$\Delta_g : \mathcal{S} \to \mathcal{S}, \quad (\Delta_g u)(h) := u(gh) - u(h), \quad \forall u \in \mathcal{S}, \quad h \in H.$$ 

If $\Xi = \Xi_\sigma$ denotes the mod $\mathbb{Z}$ reduction of $\tau_\sigma$ then we can rewrite (1.1) as

$$(\Delta_{g_1}\Delta_{g_2}\Xi)(h) = -\text{lk}_M(g_1, g_2) \quad (2.1)$$

We will prove uniqueness and existence results for this equation. We begin with the (almost) uniqueness part.
Lemma 2.1. The second order linear differential equation (2.1) determines $\Xi$ up to an “affine” function.

Proof Suppose $\Xi_1, \Xi_2$ are two solutions of the above equation. Set $\Psi := \Xi_1 - \Xi_2$. $\Psi$ satisfies the equation

$$\Delta_{g_1} \Delta_{g_2} \Psi = 0.$$ 

Now observe that any function $F \in \mathcal{S}$ satisfying the second order equation

$$\Delta_u \Delta_v F = 0, \ \forall u, v \in H$$ 

is affine, i.e. it has the form

$$F = c + \lambda$$

where $c \in \mathbb{Q}/\mathbb{H}$ is a constant and $\lambda : H \to \mathbb{Q}/\mathbb{Z}$ is a character. Indeed, the condition

$$\Delta_u(\Delta_v F) = 0, \ \forall u$$

implies $\Delta_v F$ is a constant depending on $v, c(v)$. Thus

$$F(vh) - F(h) = c(v), \ \forall h.$$ 

The function $G = F - F(1)$ satisfies the same differential equation

$$G(vh) - G(h) = c(v)$$

and the additional condition $G(1) = 0$. If we set $h = 1$ in the above equation we deduce

$$G(v) = c(v).$$

Hence

$$G(vh) = G(h) + G(v), \ \forall v, h$$

so that $G$ is a character and $F = F(1) + G$. Thus, the differential equation (2.1) determines $\Xi$ up to a constant and a character. ■

Lemma 2.2. Suppose $b : H \times H \to \mathbb{Q}/\mathbb{Z}$ is a nonsingular, symmetric bilinear form on $H$. Then there exists a quadratic form $q : H \to \mathbb{Q}/\mathbb{Z}$ such that

$$\Delta q = b$$

where

$$(\Delta q)(uv) := q(uv) - q(u) - q(v).$$
Proof. Let us briefly recall the terminology in this lemma. \( b \) is nonsingular if the induced map \( G \to G^\# \) is an isomorphism. A quadratic map form is a function \( q : H \to \mathbb{Q}/\mathbb{Z} \) such that
\[
q(0) = 0, \quad q(u^k) = k^2 q(u), \quad \forall u \in H, \; k \in \mathbb{Z}
\]
and \( \Delta q \) is a bilinear form.

Suppose \( b \) is a nonsingular, symmetric, bilinear form \( H \times H \to \mathbb{Q}/\mathbb{Z} \). Then, according to [4, §7], \( b \) admits a resolution. This is a nondegenerate, symmetric, bilinear form
\[
B : \Lambda \times \Lambda \to \mathbb{Z}
\]
on a free abelian group \( \Lambda \) such that, the induced monomorphism \( J_B \Lambda \to \mathbb{Z} \)
\[
0 \hookrightarrow \Lambda \xrightarrow{J_B} \Lambda^* \xrightarrow{\pi} H \to 0
\]
and \( b \) coincides with the induced bilinear form on \( \Lambda^*/(J_B\Lambda) \) \((n := \#H)\)
\[
b(\pi(u), \pi(v)) = \frac{1}{n^2} B(J_B^{-1}(nu), J_B^{-1}(nv)) \mod \mathbb{Z}, \; \forall u, v \in \Lambda^*.
\]
Now set
\[
q(\pi(u)) = \frac{1}{2n^2} B(J_B^{-1}(nu), J_B^{-1}(nu)) \mod \mathbb{Z}
\]
It is clear that this quantity is well defined i.e.
\[
\frac{1}{2n^2} B(J_B^{-1}(nu), J_B^{-1}(nu)) = \frac{1}{2n^2} B(J_B^{-1}(nv), J_B^{-1}(nv)) \mod \mathbb{Z}
\]
if \( v = u + J_B\lambda, \; \lambda \in \Lambda \). Clearly
\[
\Delta q = b. \; \blacksquare
\]

We deduce that there exists a constant \( c \), a character \( \lambda : H \to \mathbb{Q}/\mathbb{Z} \) and a quadratic form \( q \) such that
\[
\Xi(h) = \Xi_\sigma(h) = c + \lambda(h) + q(h), \quad \Delta q = \text{lk}_M.
\]
In the above discussion the choice of the \( \text{spin}^c \) structure \( \sigma \) is tantamount to a choice of an origin of \( H \) which allowed us to identify the torsion of \( M \) as a function \( H \to \mathbb{Q} \). Once we make such a non-canonical choice, we have to replace \( \Xi \) with the family of translates
\[
\{ \Xi_g(\bullet) := \Xi(g\bullet); \; g \in H \}
\]
In particular
\[
\Xi_g(h) := \Xi(gh) = c + \lambda(gh) + q(gh) = \left( c + \lambda(g) + q(g) \right) + \left( \lambda(h) + (\Delta q)(g, h) \right) + q(h)
\]
where \( \lambda_g(\bullet) = \lambda(\bullet) + \text{lk}_M(g, \bullet) \). Since the linking form is nondegenerate we can find an unique \( g \) such that \( \lambda_g = 0 \).

We have proved the following result.

\footnote{We are indebted to Andrew Ranicki for suggesting this approach.}
Proposition 2.3. Suppose $M$ is a rational homology sphere. Then there exists an unique spin$^c$-structure $\sigma$ on $M$ so that, with respect to this choice the mod $\mathbb{Z}$ reduction of $\tau_{M,\sigma}$

$$\Xi(h) := \tau_{\sigma}(h) \mod \mathbb{Z}$$

has the form

$$\Xi(h) = c + q(h)$$

where $c \in \mathbb{Q}/\mathbb{Z}$ is a constant while $q(u)$ is the unique quadratic form such that

$$\Delta q = -\text{lk}_M.$$

In particular,

$$\Xi(h) = \Xi(h^{-1}) \mod \mathbb{Z},$$

and the constant $c \in \mathbb{Q}/\mathbb{Z}$ is a topological invariant of $M$.

3 Examples

We want to show on some simple examples that the invariant $c$ is nontrivial.

(a) Suppose $M = L(8,3)$. Then its torsion is (see [2])

$$T_{8,3} \sim -\frac{9}{32}x_7 - \frac{3}{32}x_6 - \frac{9}{32}x_5 + \frac{5}{32}x_4 + \frac{7}{32}x_3 - \frac{3}{32}x_2 + \frac{7}{32}x_1 + \frac{5}{32}$$

where $x_8 = 1$ is a generator of $\mathbb{Z}_8$. Then

$$q(x^n) = -\frac{3k^2n^2}{16}$$

The set of possible values $-\frac{3k^2n^2}{16}$ mod $\mathbb{Z}$ is

$$A := \{0, \frac{-3}{16}, \frac{4}{16}, \frac{5}{16}\}$$

The set possible values of $\Xi(h)$ is

$$B := \{-\frac{9}{32}, -\frac{3}{32}, \frac{5}{32}, \frac{7}{32}\}.$$  

We need to find a constant $c \in \mathbb{Q}/\mathbb{Z}$ such that

$$B - c = A.$$  

Equivalently, we need to figure out orderings $\{a_1, a_2, a_3, a_4\}$ and $\{b_1, b_2, b_3, b_4\}$ of $A$ and $B$ such that $b_i - a_i \mod \mathbb{Z}$ is a constant independent of $i$. A little trial and error shows that

$$\vec{A} = (0, -\frac{3}{16}, \frac{4}{16}, \frac{5}{16}), \quad \vec{B} = (-\frac{3}{32}, -\frac{9}{32}, \frac{5}{32}, \frac{7}{32})$$
and the constant is \( c = -\frac{3}{32} \). This is the coefficient of \( x^2 \). We deduce that (modulo \( \mathbb{Z} \))

\[
F := T_{8,3}(x) + \frac{3}{32} \sim -\frac{3}{16}x^7 - 0 \cdot x^6 - \frac{3}{16}x^5 + \frac{1}{4}x^4 + \frac{1}{4}x^3 - 0 \cdot x^2 + \frac{1}{4}x + \frac{1}{4}
\]

The translation of \( F \) by \( x^{-2} \) is

\[
x^{-2}(T_{8,3} + \frac{3}{32}) = \frac{1}{4}x^7 + \frac{1}{4}x^6 - \frac{3}{16}x^5 - \frac{3}{16}x^3 + \frac{1}{4}x^2 + \frac{1}{4}x.
\]

(b) Suppose \( M = L(7, 2) \). Then, its torsion is (see [2])

\[
T_{7,2} \sim -\frac{2}{7}x^6 + \frac{1}{7}x^5 + \frac{2}{7}x^3 + \frac{1}{7}x - \frac{2}{7}
\]

where \( x^7 = 1 \) is a generator of \( \mathbb{Z}_7 \). We see that in this form \( T_{7,2} \) is symmetric, i.e. the coefficient of \( x^k \) is equal to the coefficient of \( x^{6-k} \). The constant \( c \) in this case must be the coefficient of the middle monomial \( x^3 \), which is \( \frac{2}{7} \).

(c) Suppose \( M = L(7, 1) \). Then

\[
T_{7,1} \sim \frac{2}{7}x^6 + \frac{1}{7}x^5 - \frac{1}{7}x^4 - \frac{4}{7}x^3 - \frac{1}{7}x^2 + \frac{1}{7}x + \frac{2}{7}
\]

This is again a symmetric polynomial and the coefficient of the middle monomial is \(-4/7\). We see that this invariant distinguishes the lens spaces \( L(7, 1) \), \( L(7, 2) \).

(d) For \( M = L(9, 2) \) we have

\[
T_{9,2} \sim -\frac{10}{27}x^8 + \frac{2}{27}x^7 - \frac{1}{27}x^6 + \frac{8}{27}x^5 + \frac{2}{27}x^4 + \frac{8}{27}x^3 - \frac{1}{27}x^2 + \frac{2}{27}x - \frac{10}{27}
\]

Again, this is a symmetric function, i.e the coefficient of \( x^k \) is equal to the coefficient of \( x^{8-k} \), \( x^9 = 1 \). The constant is the coefficient of \( x^5 \), which is \( 2/27 \). We deduce that, mod \( \mathbb{Z} \), we have

\[
T_{9,2} = -\frac{2}{3}x^8 - \frac{2}{9}x^7 - \frac{1}{3}x^6 - \frac{2}{9}x^7
\]

(e) Finally when \( M = L(9, 7) \) we have

\[
T_{9,7} \sim -\frac{8}{27}x^8 - \frac{2}{27}x^7 + \frac{10}{27}x^6 + \frac{1}{27}x^5 - \frac{2}{27}x^4 + \frac{1}{27}x^3 + \frac{10}{27}x^2 - \frac{2}{27}x - \frac{8}{27}
\]

the polynomial is again symmetric so that the constant \( c \) is the coefficient of \( x^4 \) which is \(-2/7\).

It would be very interesting to know whether the invariant \( c \) satisfies any surgery properties. This is not a trivial issue because we cannot relate the potential surgery properties of \( c \) to the surgery properties of the torsion. In the case of torsion the surgery formula involve finite difference operators which kill the constants so \( c \) will not appear in any of them.

6
References

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