A NOTE ON APPROXIMATE SUBGROUPS OF $\text{GL}_n(\mathbb{C})$
AND UNIFORMLY NONAMENABLE GROUPS

EMMANUEL BREUILLARD, BEN GREEN, AND TERENCE TAO

Abstract. The aim of this brief note is to offer another proof of a theorem of Hrushovski that approximate subgroups of $\text{GL}_n(\mathbb{C})$ are almost nilpotent. This approach generalizes to uniformly non amenable groups.

Contents

1. Introduction
2. Six lemmas
3. Proof of the main theorem
References

1. Introduction

Throughout this paper $K \geq 2$ is a real number. We begin by recalling, very briefly, the notions of $K$-approximate group and of control. For a more leisurely introduction we refer the reader to our paper [7].

Definition 1.1. Suppose that $A$ is a finite set in some group which is symmetric in the sense that the identity lies in $A$ and $a^{-1} \in A$ whenever $a \in A$. Then we say that $A$ is a $K$-approximate group if there is some symmetric set $X$, $|X| \leq K$, such that $A^2 \subseteq AX$. If $B$ is some other set then we say that $A$ is $K$-controlled by $B$ if $|B| \leq K|A|$ and if there is some set $X$, $|X| \leq K$, such that $A \subseteq BX \cap XB$.

If $P$ is some property that a group may have, such as being linear, nilpotent or solvable, then by a $P$ $K$-approximate group we mean an approximate group $A$ for which the group $\langle A \rangle$ generated by $A$ has property $P$.

Modulo a little fairly standard multiplicative combinatorics, the following result was proved by Hrushovski [10, Corollary 5.10].

Theorem 1.2. Suppose that $A \subseteq \text{GL}_n(\mathbb{C})$ is a $K$-approximate group. Then $A$ is $O_{K,n}(1)$-controlled by $B$, a solvable $K^{O(1)}$-approximate group.

By combining this with the main result of [5] one immediately obtains the following extension.

Theorem 1.3. Suppose that $A \subseteq \text{GL}_n(\mathbb{C})$ is a $K$-approximate group. Then $A$ is $O_{K,n}(1)$-controlled by $B$, a nilpotent $K^{O(1)}$-approximate group.
Hrushovski’s result was proven using model theory and so did not lead to an explicit form for the $O_{K,n}(1)$ term in Theorem 1.3 even in principle. A different proof of this result was established by the authors in [7], and this gave a stronger result in the sense that the $O_{K,n}(1)$ term was shown to vary polynomially in $K$. We obtained a bound of the form $C_n K^{C'n}$ where $C_n$ could have been computed if desired (and would take the form $\exp(n^{O(1)})$), but $C_n$ could not as a consequence of our dependence on ultrafilters to prove quantitative algebraic geometry estimates in [7]. A more explicit approach to these estimates is taken in the paper of Pyber and Szabo [11] who, subsequent to our work in [7], obtain a result (Theorem 10 of their paper) which is in principle explicit.

Our aim here is to show how Theorem 1.3 follows quickly from various results in the literature, the most substantial of these being the so-called uniform Tits alternative of the first author. The argument is largely distinct from both of the previous two proofs of Theorem 1.3. It gives a term $O_{K,n}(1)$ of the form $\exp(K^{O(m(n) \log K)})$, where $m(n)$ is the constant appearing in the uniform Tits alternative (see Proposition 2.1 below). This is effective in principle. However, computing an explicit bound for $m(n)$ would not only involve chasing constants in [3], but would also require the effective version of §2 of [4] proven in [2]. The exponential nature of $O_{K,n}(1)$ with respect to $K$ stems from the bound in Proposition 2.2, which is a consequence of a recent result of Sanders [12] and Croot-Sisask [8].

Finally, we note that the proof of Theorem 1.3 above generalizes easily to uniformly nonamenable groups. Say that a group $G$ is $\kappa$-uniformly nonamenable if whenever $X$ is a finite subset of $G$ which generates a non-amenable subgroup, then $|AX| \geq (1 + \kappa)|A|$ for every finite subset $A$ in $G$ and some $\kappa > 0$. The uniform Tits alternative implies that $\text{GL}_d(\mathbb{C})$ is $\kappa_d$-uniformly non-amenable, for some $\kappa_d > 0$. Similarly it can be shown (see Koubi [9]) that $\delta$-hyperbolic groups are uniformly nonamenable, and it is known that their amenable subgroups are infinite cyclic-by-bounded. In this setting we obtain the following result.

**Theorem 1.4.** Let $G$ be a $\kappa$-uniformly nonamenable group. Suppose that $A \subseteq G$ is a $K$-approximate group. Then $A$ is $\exp(K^{O(\log K/\kappa)})$-controlled by $B$, an amenable $K^{O(1)}$-approximate group.

Although Theorem 1.4 applies to $\delta$-hyperbolic groups, and thus says that any approximate subgroup is controlled by an approximate subgroup of a cyclic subgroup, it is likely that our bound can be improved a lot for these groups in the spirit of Safin’s bound [13] in the free group case (he obtained a bound of the form $O_{\varepsilon}(K^{2+\varepsilon})$).

---

1. What we defined here is a slightly stronger notion of uniform non amenability than the one defined in [1], because we do not require $X$ to generate the group.

2. Strictly speaking, Koubi’s theorem requires $X$ to be a generating set of $G$, however his proof extends easily to the case when $X$ generates a non-elementary subgroup.
2. Six lemmas

In this section we assemble the tools from the literature that we require to prove Theorem 1.3. The title of the section is hardly accurate, since some of these results are quite substantial.

**Proposition 2.1.** Let \( n \geq 1 \) be an integer. Then there is an integer \( m = m(n) \) with the following property: if \( A \subseteq \text{GL}_n(\mathbb{C}) \) is a finite symmetric set then either the group \( \langle A \rangle \) generated by \( A \) is virtually solvable, or else \( A^m \) contains two elements generating a free subgroup of \( \text{GL}_n(\mathbb{C}) \).

**Proof.** This is the “Uniform Tits Alternative” of the first author \[3\]. □

The following result is a straightforward consequence of a lemma of Sanders \[12\]. For very closely related results, see \[8\] and \[14\].

**Proposition 2.2.** Suppose that \( A \) is a \( K \)-approximate group. Then for any \( \varepsilon > 0 \) and \( m \in \mathbb{N} \) there are sets \( A' \subseteq A^5 \), \( B \subseteq A^4 \) with \( |A'| \geq |A| \), \( |B| \gg_{K,m,\varepsilon} |A| \) and \( |A'B^m| \leq (1 + \varepsilon)|A'| \).

**Proof.** Let \( k = k(\varepsilon, K) \) be an integer to be specified later. It is shown in Sanders \[12\] that there is a symmetric set \( B \) containing the identity such that \( |B| \geq \exp(-K^{O(km)}) |A| \) and \( B^{km} \subseteq A^4 \). We have the nesting

\[
A \subseteq AB^m \subseteq AB^{2m} \subseteq \cdots \subseteq AB^{km} \subseteq A^5.
\]

Note that \( |A^5| \leq K^4 |A| \). Supposing that

\[
(1 + \varepsilon)^k \geq K^4,
\]

it follows from the pigeonhole principle that there is some \( j, 0 \leq j < k \) such that, setting \( A' := AB^{jm} \), we have

\[
|A'B^m| \leq (1 + \varepsilon)|A'|.
\]

These sets \( A' \) and \( B \) then satisfy the requirements of the proposition. □

**Proposition 2.3.** Suppose that \( A \) is a subset of some group \( G \) and that \( X \) is a further subset of \( G \) containing the identity and two elements generating a nonabelian free group. Then \( |AX| \geq \frac{5}{4} |A| \).

**Proof.** This is basically the observation, originally due to von Neumann, made whilst proving that every group containing a non-abelian free subgroup \( F_2 \) is not amenable using Følner’s criterion; see for example the notes of the third author \[15\] or the remarks on page 4 of \[3\]. □

**Proposition 2.4.** Suppose that \( G \leq \text{GL}_n(\mathbb{C}) \) is virtually solvable. Then there is a solvable subgroup \( H \leq G \) such that \( [G : H] = O_n(1) \).

**Proof.** This result of Mal’cev-Platonov is proved in \[7\], Appendix B. □
Proposition 2.5. Let $K \geq 2$, and suppose that $A$ is a $K$-approximate solvable subgroup of $\text{GL}_n(\mathbb{C})$. Then $A$ is $K^{O_n(1)}$-controlled by a $K^{O(1)}$-approximate nilpotent subgroup.

Proof. This is essentially the main result of [5]. There is one difference, in that here we obtain a $K^{O(1)}$-approximate group rather than a $K^{O_n(1)}$-approximate group. However, this stronger statement follows easily from the weaker one and the additive-combinatorial lemma below.

Finally, we require a standard additive combinatorics lemma.

Lemma 2.6. Let $0 < \delta < 1$, $k \geq 1$ and $K \geq 2$ be parameters. Suppose that $A$ is a $K$-approximate group of some ambient group $G$, and that $H \leq G$ is a subgroup with the property that $A^k$ intersects some coset $Hx$ in a set of size $\delta|A|$. Then $A^2 \cap H$ is a $2K^3$-approximate group which $2K^{2k+4}/\delta$-controls $A$.

Proof. According to [6, Lemma 3.3], $B := A^2 \cap H$ is a $2K^3$-approximate group and $|A^2 \cap H| \leq K^{2k-1}|B|$. Moreover $|AB| \leq |A^3| \leq K^2|A| \leq K^{2k+1}|B|$. An appeal to Ruzsa’s covering lemma ends the proof.

3. Proof of the main theorem

This is a very short task given the ingredients we assembled in the previous section. Let $A \subseteq \text{GL}_n(\mathbb{C})$ be a $K$-approximate group. By Proposition 2.2 applied with $\varepsilon = 0.1$ (say) and with $m = m(n)$ the quantity appearing in Proposition 2.1 there are sets $A' \subseteq A^5$, $B \subseteq A^4$ with $|A'| \geq |A|$ and $|B| \gg_{n,K} |A|$ such that $|A'B^m| < \frac{5}{4}|A'|$. By Proposition 2.3, $B^m$ cannot contain two elements generating a nonabelian free group. By the uniform Tits alternative, $B$ must generate a virtually solvable group $G$. By Proposition 2.4 this $G$ contains a solvable subgroup $H$ with $[G:H] = O_n(1)$, and so by the pigeonhole principle there is some $x$ such that $|B \cap Hx| \gg_n |B|$. This implies that

$$|A^4 \cap Hx| \geq |B \cap Hx| \gg_n |B| \gg_{n,K} |A|.$$ 

At this point we may apply Lemma 2.6 to conclude that $A^2 \cap H$ is a $2K^3$-approximate solvable group which $O_{K,n}(1)$-controls $A$. Finally, an application of Proposition 2.5 completes the proof of Theorem 1.3.

We leave the proof of Theorem 1.4, which proceeds along almost identical lines but does not require Proposition 2.4 or Proposition 2.5, to the reader.

Acknowledgement. We thank Thomas Delzant for useful discussions regarding hyperbolic groups.
References

[1] G.N. Arzhantseva, J. Burillo, M. Lustig, L. Reeves, H. Short,, E. Ventura, *Uniform non-amenability*, Advances in Mathematics *197* (2005) 499–522.
[2] E. Breuillard, *Effective estimates for the spectral radius of a bounded set of matrices*, preprint.
[3] E. Breuillard, *A strong Tits alternative*, preprint, [arXiv:0804.1395](http://arxiv.org/abs/0804.1395)
[4] E. Breuillard, *A height gap theorem for finite subsets of SL_n(Q) and non-amenable subgroups*, preprint, [arXiv:0804.1391](http://arxiv.org/abs/0804.1391)
[5] E. Breuillard and B. J. Green, *Approximate groups II: the solvable linear case*, Quart. J. of Math (Oxford), published online April 20th 2010.
[6] E. Breuillard and B. J. Green, *Approximate unitary groups*, preprint, [arXiv:1006.5160](http://arxiv.org/abs/1006.5160)
[7] E. Breuillard, B. J. Green and T. C. Tao, *Approximate subgroups of linear groups*, preprint, [arXiv:1005.1881](http://arxiv.org/abs/1005.1881)
[8] E. Croot and O. Sisask, *A probabilistic technique for finding almost-periods of convolutions*, preprint, [arXiv:1003.2978](http://arxiv.org/abs/1003.2978)
[9] M. Koubi, *Croissance uniforme pour les groupes hyperboliques*, Annales Inst. Fourier, tome 48 5, (1998), 1441–1453
[10] E. Hrushovski, *Stable group theory and approximate subgroups*, preprint, [arXiv:0909.2190](http://arxiv.org/abs/0909.2190)
[11] L. Pyber and E. Szabo, *Growth in finite simple groups of Lie type*, preprint.
[12] T. Sanders, *On a nonabelian Balog-Szemerédi-type lemma*, J. Aust. Math. Soc. *89* (2010), no. 1, 127–132.
[13] S. Safin, *Powers of sets in free groups*, preprint, [arXiv:1005.1820](http://arxiv.org/abs/1005.1820)
[14] T. Schoen, *Near optimal bounds in Freiman’s Theorem*, preprint.
[15] T. C. Tao, *Amenability*, in An epsilon of room, I: Real Analysis, pages from year three of a mathematical blog, AMS 2011.

Laboratoire de Mathématiques, Bâtiment 425, Université Paris Sud 11, 91405 Orsay, FRANCE

E-mail address: emmanuel.breuillard@math.u-psud.fr

Centre for Mathematical Sciences, Wilberforce Road, Cambridge CB3 0WA, ENGLAND

E-mail address: b.j.green@dpmms.cam.ac.uk

Department of Mathematics, UCLA, 405 Hilgard Ave, Los Angeles CA 90095, USA

E-mail address: tao@math.ucla.edu