Are numerical theories irreplaceable?
A computational complexity analysis.

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Abstract
It is widely known that numerically integrated orbits are more precise than analytical theories for celestial bodies. However, calculation of the positions of celestial bodies via numerical integration at time t requires the amount of computer time proportional to t, while calculation by analytical series is usually asymptotically faster.

The following question then arises: can the precision of numerical theories be combined with the computational speed of analytical ones? We give a negative answer to that question for a particular three-body problem known as Sitnikov problem.

A formal problem statement is given for the initial value problem (IVP) for a system of ordinary dynamical equations. The computational complexity of this problem is analyzed. The analysis is based on the result of Alexeyev (1968–1969) about the oscillatory solutions of the Sitnikov problem that have chaotic behavior. We prove that any algorithm calculating the state of the dynamical system in the Sitnikov problem needs to read the initial conditions with precision proportional to the required point in time (i.e. exponential in the length of the point’s representation). That contradicts the existence of an algorithm that solves the IVP in polynomial time of the length of the input.

Introduction
Analytical theories of planets (Brumberg 1991, Simon et al. 2013) and the Moon (Chapront and Francou 2003, Ivanova 2014) allow to determine the positions of bodies using series over the time parameter t. Technically, those series work
on arbitrarily large $t$, requiring no more than $P(\text{LENGTH}(t))$ of computation time, where $P$ is some polynomial and $\text{LENGTH}(t)$ is the length of $t$’s machine representation. For fixed-precision $t$, $\text{LENGTH}(t)$ is $\lceil \log t \rceil$ plus constant.

Conversely, planetary and lunar theories used in modern applied astronomy (Pitjeva 2015, Folkner et al. 2014, Fienga et al. 2014) are based on numerical integration, which requires at least $O(t)$ time to calculate the positions of bodies at time $t$, regardless of precision.

Numerical theories are used because their accuracy matches the precision of modern astronomical observations (radio observations of spacecraft, lunar laser ranging). Analytical theories fail to provide that level of accuracy. More to say, they are often fit to the existing numerical theories rather than to observations.

Facing the problem of analyzing the behavior of the Solar system for billions of years into the future or into the past, one could wonder whether it is possible to use analytical theories in order to get the $O(P(\log t))$ computation time factor instead of $O(t)$. At this point, one can not ignore the phenomenon of dynamical chaos: high sensitivity of a dynamical system to initial conditions in presence of quasi-periodic orbits.

Clearly, existing analytical theories are not purposed to calculate chaotic orbits. In this work, we take a more general view on the question of the computational complexity of the initial value problem, not bound to any particular algorithm of computation.

In the next sections, a very short review is given about the chaoticity of the Solar system; then the computational complexity of the (theoretical) three-body problem is considered; then for a special artificial case of this problem (Sitnikov problem) we show that no analytical theory or any other kind of $O(P(\log t))$ computation for for that particular problem can exist.

### Chaos in the Solar system

Analyzing the presence of chaos in the planetary motion has always been a hard piece of work. At the end of 18-th century, Laplace and Lagrange proved that (their mathematical model of) the Solar system is stable. Later, Le Verrier showed that high-order terms of the expansion of Newtonian laws, discarded in their proof, can completely change the picture on a large interval of time. However, the question of whether the Solar system is actually stable or chaotic remains unsolved from the theoretical point of view.

In the computer era, numerical experiments have been performed by various researches to estimate the Lyapunov exponent of planetary orbits, and hence their chaoticity. Laskar (1994) found that the inner planets of the Solar system have chaotic behavior and the outer ones do not. Sussman and Wisdom (1992) ran their own simulations that showed the chaotic behavior of the outer planets. Hayes (2007) has demonstrated that the outer planets may show chaotic or stable behavior depending on initial conditions, in both cases within the observational accuracy.

From now on, we shift from the Solar system with its uncertain parameters,
incomplete models and often error-prone numerical integration methods, to the three body problem in its pure mathematical formulation.

The computational complexity of a dynamical system

We study the computational complexity of the initial value problem (IVP) for the dynamical system

\[
\begin{align*}
\dot{x} &= f(x) \\
x(0) &= x_0
\end{align*}
\]

where \( x \in D, x_0 \in D \) is a real vector, and \( f : D \to \mathbb{R}^n \) is a computable real vector-valued function (open set \( D \subseteq \mathbb{R}^n \) is the phase space of the system). We deal with the case when the solution \( x^*(t) : \mathbb{R} \to D \): (i) exists on the whole \( \mathbb{R} \); (ii) is unique; (iii) is a computable real vector-valued function.

The input data for a problem will be: the initial conditions \( x_0 \), the point \( t \) in time, and the precision \( \varepsilon \). A Turing machine TM implementing a (numerically integrated or other) solution is supposed to consume the input and produce the output—an approximate state of the system at time \( t \)—that matches the actual state of the system up to \( \varepsilon \). \( t \) and \( \varepsilon \) are treated as rationals, while the initial conditions are provided by an oracle \( \varphi \) that gives increasingly precise numbers on demand (Kawamura, 2014).

**Definition 1.** The solution function of an initial value problem (1) is the function \( S(x_0, t) : D \times \mathbb{Q} \to D \), where \( S|_{x=x_0} : \mathbb{Q} \to D \) is a computable real vector-valued function, whose closure on the real axis is the solution of (1).

**Definition 2.** Turing machine that computes the solution function of an IVP is a Turing machine that accepts rational \( t \) and \( \varepsilon \) as input; has an oracle \( \varphi \) that instruments \( x_0 \) as a computable real vector; and produces the value of the solution \( x(t) \) corresponding to given \( x_0 \) and \( t \), with the precision \( \varepsilon \).

We say that an IVP has **polynomial complexity** if there exists a Turing machine that computes its solution function in time bounded by \( P(\text{LENGTH}(t), \text{LENGTH}(\varepsilon)) \).

Computational complexity of the IVP for the N-body problem

Gravitational N-body problem is concerned with the Newtonian motion of \( N \) point-masses in three dimensions. The system of ODEs for this problem is the
following:

\[ \begin{align*}
\dot{p}_i &= v_i, \quad i = 1..N \\
\dot{v}_i &= \sum_{j=1, j \neq i}^{N} \mu_j \frac{p_j - p_i}{|p_j - p_i|^3}, \quad i = 1..N
\end{align*} \tag{2} \]

where \( \mu_i \in \mathbb{R}, \mu_i \geq 0, p_i \in \mathbb{R}^3, v_i \in \mathbb{R}^3. \)

The initial state of the system is given by a \((7N)\)-vector

\[ x_0 = (\mu_1, \mu_2, \mu_3, p_{1,1}, \ldots, p_{N,3}, v_{1,1}, \ldots, v_{N,3}), \]

while the system \((2)\) defines a computable real vector-valued function \( \dot{x} = f(x) \).

With \( N = 2 \), there is an algebraic solution involving \( t \) as a parameter to a periodic function, and it is not hard to show that the IVP for the two body problem has polynomial complexity (if the solution exists).

With \( N = 3 \), as shown by Poincaré, no algebraic solution exists. Sundman (1912), however, derived a solution in the form of converging series. Unfortunately, the estimate of the number of terms required to calculate the series at point \( t \) with a sensible precision is exponential in \( t \) (Belorizky, 1930).

There has been developed no series (or, generally speaking, no algorithm) able to calculate the state of a three-body system in a time polynomial in \( \log t \). Intuitively, that is expectable: in a system with a high sensitivity to initial conditions (i.e. nonzero Lyapunov exponent), any algorithm must consume at least \( O(t) \) input data, otherwise it does not know the initial state with enough precision to calculate the answer. Since \( O(t) \) dominates any \( P(\log t) \), a polynomial algorithm must not be possible.

What do we really know about the sensitivity to initial conditions in three body systems? Poincaré’s discovery was qualitative and did not include a measure of sensitivity. A number of papers exist with numerical estimation of Lyapunov exponents (Froeschlé 1970, Quarles et al. 2011), but no theoretical lower bound of the Lyapunov exponent value is known (however, there is a work by Shevchenko (2004) with analytical upper bounds).
Two body problem | General Case of Three body problem
---|---
Two-dimensional phase space | Phase space of three or more dimensions
Algebraic solution known | Algebraic solution proven not to exist
Regular orbits | Chaotic orbits
Zero Lyapunov exponent | Supposedly nonzero Lyapunov exponent
Orbits in practice are calculated in $O(\text{LENGTH}(t)^2)$ time using trivial arithmetical operations, trigonometric functions and Kepler equation solving | Orbits in practice are calculated in $O(t)$ time using numerical integrators

Table 1: Fundamental differences between two body and three body problems—all related to each other

The objectives of this work is to strengthen the link between the chaoticity of dynamical systems and computational complexity, and strictly prove the computational complexity of a particular case of the three body problem.

Sitnikov problem

From now on, we will focus on a special case of the three-body problem probably first proposed by Andrei Kolmogorov to Kirill Sitnikov, who was his student at the time.

In this problem, two of the bodies are of equal positive mass, while the third body is massless and lies on a line perpendicular to the plane of the motion of...
the first two bodies and passing through their center of mass (Fig. 2). Hence, the two bodies follow the unperturbed (Keplerian) orbit; in this problem, the elliptic orbit is the case.

Let us place the center of mass at the origin, and the $Z$ axis along the line where the third body is. Let us denote $r(t)$ the distance from the first body (and the second, as their trajectories are symmetric) to the origin.

Following Newtonian laws (2), the coordinate of the third body, denoted as $z$, obeys the following differential equation:

$$
\ddot{z} = -\frac{2\mu z}{\sqrt{z^2 + r(t)^2}}.
$$

(3)

where $\mu$ is the gravitational constant of the first and second bodies. Periodic function $r(t)$ comes from the elliptical solution of the two-body problem:

$$
\begin{align*}
r(t) &= a(1 - e \cos E(t)) \\
E(t) - e \sin E(t) &= \sqrt{\frac{2\mu}{a^3}}(t - t_0)
\end{align*}
$$

(4)

Semimajor axis $a$, eccentricity $e$, and epoch $t_0$ are constants that can be calculated from the initial state of the two bodies. $E(t)$ is the eccentric anomaly angle. The period of $r(t)$ is $P = 2\pi \sqrt{\frac{a^3}{2\mu}}$.

The initial values in the Sitnikov problem are:

$\triangleright$ $a > 0, e \in (0..1), \mu > 0$ — parameters of the orbit of the two bodies;

$\triangleright$ $z_0 = z(0)$ — initial position of the third body in the $Z$ axis.

$\triangleright$ $v_0 = \dot{z}(0)$ — initial velocity of the third body in the $Z$ axis.

$\triangleright$ $\phi = E(0), 0 \leq \phi < 2\pi$ — initial value of the eccentric anomaly of the orbit of the two bodies.

The state vector of the system is accordingly $x = (a, e, \mu, z, v, E)$. $a$, $e$ and $\mu$ do not depend on time; $\dot{z} = v$; $\dot{v} = \ddot{z}$ from (3); $\dot{E}$ follows from (4):

$$
\begin{align*}
\dot{x} &= f(x) = (0, 0, 0, v, \ddot{z}, E) \\
\dot{z} &= -\frac{2\mu z}{\sqrt{z^2 + a^2(1 - e \cos E)^2}} \\
\dot{E} &= \frac{\sqrt{2\mu}}{\sqrt{a^3(1 - e \cos E)}}
\end{align*}
$$

(5)

This very simple system became a rare example of mathematically proven chaos in gravitational dynamics. Sitnikov (1961) showed the existence of oscillatory trajectories. Alexeyev (1968–1969) significantly extended his result, not only discovering the existence of all the classes of final motions in this problem, but developing the whole theory of symbolic dynamics for it, and finally proving the following:
Theorem 1. For any sufficiently small eccentricity $e > 0$ there exists an $m(e)$ such that for any double-infinite sequence $\{s_n\}_{n \in \mathbb{Z}}, s_n \geq m$ there exists a solution $z(t)$ of the equation (3) whose roots satisfy the equation

$$\left| \frac{\tau_{k+1} - \tau_k}{P} \right| = s_k, \forall k \in \mathbb{Z}. \quad (6)$$

Figure 3: Different orbits obtained in the Sitnikov problem by small variations of initial parameters. Plots of $z(\frac{t}{\tau_2})$ are drawn, showing high sensitivity to parameters $e$ and $v_0$. $z(0) = 0, a = 1, \mu = 0.5$.

A shortened version of the original theorem is given, excluding the finite and semi-infinite sequences. Alexeyev also proved a generalization of his theorem to the case when the third body has a nonzero mass. A simpler proof was later obtained by Moser (1973).

Computational complexity of the IVP for the Sitnikov problem

In (Vasiliev and Pavlov, 2016), it is shown that the solution of the IVP for the Sitnikov problem always exist (i.e. does not have singularities), is unique and computable. Then, the following theorem is proven:

Theorem 2. The time complexity of an initial value problem for the Sitnikov problem with any fixed value of eccentricity does not have a polynomial upper bound.

In other words, no Turing machine can compute the solution function in time polynomial in $\text{LENGTH}(t)$. The proof is based on the fact such a Turing machine also could restore a sequence (6) from Alexeyev’s theorem—any sequence from an exponential (in $t$) number of sequences—reading no more than $P(\text{LENGTH}(t))$ input data, which is impossible. Lyapunov exponent is not derived nor used in the proof.

Since the Sitnikov problem can be (in polynomial time) reduced to the general three-body problem, the above result means that there can not be a polynomial-time Turing machine for the latter, too.
Upper bound of computational complexity

After proving the lower \((O(t))\) bound for the computational complexity, it is natural to wonder whether the IVP for three body problem is actually solvable in \(O(t)\) (or, even, solvable at all). Numerical integrators widely used in practice, though run in \(O(t)\) time, rarely compute the solution function with arbitrary precision as it is required by the definition. The vast majority of these integrators suffer from saturation: the step size being small enough, the error grows upon further decrease of the step size. Therefore, these integrators can not in principle obtain a solution up to an arbitrary precision.

One exception is a modification of Picard–Lindelöf method (Matculevich et al. 2013) that computes the solution function (regardless of how much time it needs) if \(f\) is Lipschitz-continuous on \(D\).

More recently (Pouly and Graça, 2016), an algorithm was given for computing the solution function of a polynomial ODE in a time polynomial in \(\text{LENGTH}(\varepsilon)\) and the length of the solution curve from 0 to \(t\)—which, in many cases of the problem, is linearly bounded by \(t\).

Implications from computational complexity bounds

The proven non-polynomiality is a theoretical asymptotic statement. Even for the Sitnikov problem, there are not yet known obstacles to develop an analytical theory with a fixed but practically sufficient precision available for a sufficiently large period of time. For other cases of the three body problem, or for the Solar system, there may exist yet another special techniques.

Numerical theories are irreplaceable—for a specially constructed dynamical system and for unlimited requirements for the precision and the time span. There can not exist quickly converging series for that case, and hence for the general case. For other particular cases, including the Solar system, we do not know yet.

Connection between computational complexity and dynamical chaos

Looking at Table[4] and adding into it a new partition “polynomial – not polynomial” one could suggest a link between integrability, computational complexity, Lyapunov exponents, and chaotic orbits. The choice of the three-body problem and oscillatory trajectories is not principal. We believe that similar results can be obtained in other systems, where, with the help of methods of symbolic dynamics, complex dynamical behavior can be shown and analyzed.

For the integrable dynamical systems—those who have computable integrals of motion with good complexity bounds in \(t\) and \(\varepsilon\)—it is possible to derive complexity bounds for the initial value problem in our formal statement. Those
bounds will be polynomial by \( \log(t) \) and \( \log(1/\varepsilon) \). That can point to a link between computational complexity of the IVP and integrability.

A strong link between computational complexity and dynamical systems is known: Ercsey-Ravasz and Toroczkai (2011) showed that an NP-complete problem (SAT) can be reduced to a problem of finding an attractor of a dynamical system. A number or other NP-complete problems has been since then reduced to finding various properties of dynamical systems (Ahmadi et al. 2013). However, those strong links are one-way: to our knowledge, there is no bridge from continuous dynamical systems to computational complexity.

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