Solitons in topological field theories

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Abstract

We present a topological Lagrangian field theory that is geometrically similar to Yang-Mills(-Higgs) theory, and study the Bogomol’nyi solitons contained within this theory. The topological field theory may provide an example of a dual field theory. The existence of a dual field theory to Yang-Mills(-Higgs) theory was conjectured by Montonen and Olive.

1 Introduction

Recently a class of Lagrangian topological field theories possessing a ‘minimizing’ Bogomol’nyi structure has been introduced on oriented, compact, connected four-manifolds [8]. The associated Bogomol’nyi equations are reminiscent of the self-duality equations in Yang-Mills theory, and the solutions to the topological Bogomol’nyi equations share much in common with solutions to the self-duality equations (instantons). Like the Yang-Mills instanton, for example, solutions to the topological Bogomol’nyi equations can be translated into geometrical structure on an appropriate holomorphic vector bundle, and, the moduli space of solutions forms a Hausdorff differentiable manifold. We shall call solutions to the topological Bogomol’nyi equations ‘topological instantons’. The topological field theories studied in [8] achieve these results with relatively little hard analysis and algebraic geometry when
compared with the Yang-Mills instanton theory \[2\]. The reason for this is that topological instantons are essentially equivalent to the differential geometric formulation of ‘stable vector bundles’ due to Kobayashi \[6\]. The differences between Yang-Mills instantons and topological instantons are also significant. We mention three differences. First, non-trivial topological instantons can exist on pseudo-Riemannian space-times, while Yang-Mills instantons are trivial on space-times. Second, topological instantons have a larger gauge group, \(U(n)\). Third, topological instantons by virtue of their non-triviality on space-times have a space-of-motions equivalent to the moduli space; Yang-Mills instantons are pseudo-particles and do not possess a space-of-motions. It is well-known that self-dual instantons in Yang-Mills theory and BPS magnetic monopoles in Yang-Mills-Higgs theory are closely related \[1\]. BPS magnetic monopoles are non-singular, finite-energy solutions to the self-duality equations reduced to three spatial dimensions with a gauge symmetry in the (imaginary) time direction. A similar process can be applied to the topological instanton, leading to the theory of topological monopoles. The topological instanton and the topological monopole obtained by dimensional reduction are the subject of this paper.

In the next section we discuss the differential geometry of the class of topological field theories on four-manifolds introduced in \[8\], and expose the Bogomol’nyi structure. Solutions to the Bogomol’nyi equations (topological instantons) are shown to be projectively flat. The physical stability of the topological instanton field configuration is argued from the topology of the underlying four-manifold. In section three, we dimensionally reduce the four-dimensional topological field theory to three spatial dimensions. The Bogomol’nyi structure survives the dimensional reduction. Topological monopoles are the solutions to the Bogomol’nyi equations in three dimensions. Although the theory of topological monopoles is very similar to the theory of BPS magnetic monopoles, there is an interesting difference between the Bogomol’nyi structures of the two theories. In the theory of BPS magnetic monopoles the Bogomol’nyi equations appear as a completed square in the Lagrangian, while in the theory of topological monopoles they do not. The Bogomol’nyi equations in our class of TFTs consist of two equations, either of which will saturate the Bogomol’nyi energy. This added flexibility in saturating the Bogomol’nyi energy allows greater freedom in constructing solitonic particles with either an electric or magnetic charge.
2 Instantons in topological field theories

The Lagrangian theories in [8] are defined by the Lagrangian Action functional:

\[ L(A, B) = \int_M < (H^A \otimes I_E) \wedge (I_E \otimes K^B) > - \frac{1}{2} < (I_E \otimes K^B)^2 > \]  

defined on the product space \( \mathcal{A}(P) \times \mathcal{A}(P) \). Interpreting \( H^A \) and \( K^B \) as curvatures in the Lagrangian Action requires that the real dimension of \( M \) be four. \( I_E \) is the identity transformation on the adjoint bundle, \( E \). The brackets \( < > \) remind us that a choice of adjoint-invariant, real-valued inner product on the adjoint bundle is needed. The Action functional introduces an artificial asymmetry in \( H^A \) and \( K^B \) which is not supported by a physical argument; we will return to this later. The variational field equations for (1) are

\[ D^A K^B = 0, \quad D^B H^A = 0, \]  

where we have made use of the Bianchi identity \( D^B K^B = 0 \). The set of solutions is clearly neither empty nor entirely trivial. The physical stability of a class of nontrivial, nonsingular, finite-Action solutions to the variational equations (3) can be demonstrated by a topological argument. The Lagrangian (1) can be rewritten as

\[ 2L(A, B) = - \int_M < (H^A \otimes I_E - I_E \otimes K^B)^2 > + \int_M < (H^A \otimes I_E)^2 > \]  

The inner product structure defines a Weyl polynomial of degree two. Let \( E_A \) and \( E_B \) be the vector bundle \( E \) equipped with either the connection \( A \) or \( B \), respectively. The first term in the Lagrangian \( L \) in equation (3) is a topological invariant for the tensor product bundle \( E_A \otimes E_B^* \). Recall that the curvature of \( E_A \otimes E_B^* \) is given by \( \Omega_{E_A \otimes E_B^*} = H^A \otimes I_E - I_E \otimes K^B \). The Bogomol’nyi equations,

\[ H^A \otimes I_E = I_E \otimes K^B, \]  

are therefore a vanishing curvature condition on the tensor product bundle \( E_A \otimes E_B^* \). Solutions to (4) automatically satisfy the variational field equations (2). An indice computation for (4), \( H_{ab}^{cd} \delta_{cd} = \delta_{cd} K_{cd}^B \), shows that the curvature forms \( H^A \) and \( K^B \) are projectively flat. That is,

\[ H^A = K^B = iFI_r, \]  

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where $F$ is a real-valued two form on $M$, and $I_r$ is the identity endomorphism for the vector bundle, $E$, of rank $r$. The Bianchi identity imposes a simple condition on $F$, that $dF = 0$, so that $F \in H^2(M, \mathbb{R})$. Since $M$ is compact, $H^2(M, \mathbb{R})$ is of finite dimension. If $F$ is a curvature on $M$, then the second term in (3) is a topological invariant of the underlying four-manifold, $M$. Topologically non-trivial solutions to the Bogomol’nyi equations will be said to be 'physically stable’ if $F$ is a curvature of $M$ and if the solutions have a fixed non-zero Action given by

$$2\mathcal{L} = -\int_M F \wedge F = -24\pi \text{sgn}(M) \neq 0,$$

where the topological signature of the manifold, $M$, is denoted by $\text{sgn}(M)$. Physically stable, non-trivial solutions to the Bogomol’nyi equations (5) on the vector bundle $(E, <>)$ are called topological instantons [8].

3 Monopoles in topological field theories

We now examine static, non-singular solutions to the Bogomol’nyi equations (5). By assuming a gauge symmetry in the direction of time, $X_t$, we can dimensionally reduce the four-dimensional theory on $\mathbb{R}^4$ defined by (1), to a theory on $\mathbb{R}^3$. The reductions are performed using the gauge symmetry equations,

$$H^A(X_t, \cdot) = -D^A\Phi_A,$$
$$K^B(X_t, \cdot) = -D^B\Phi_B.$$  \hspace{1cm} (6)

Dimensional reduction introduces the equivariant Lie algebra valued fields, $\Phi_A, \Phi_B \in \Lambda^0(\mathbb{R}^3, \text{End}(E))$, defined by $\Phi_A = A(X_t)$ and $\Phi_B = B(X_t)$ [4]. We can either reduce the Bogomol’nyi field equations (5) directly, or, reduce the full variational field equations. In the first case, the Bogomol’nyi equations locally reduce to

$$D^A\Phi_A = D^B\Phi_B = EI_E,$$
$$H^A|_{\mathbb{R}^3} = K^B|_{\mathbb{R}^3} = F I_E.$$  \hspace{1cm} (7)

$E$ is the one-form obtained by contracting $F$ in (5) on the infinitesimal time displacement. In the second equation in (7) $F$ denotes the restriction of $F$ in four-dimensions restricted to the leaves in the foliation defined by $X_t$. 

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Alternatively, the Lagrangian Action (1) after reduction becomes
\[ E(A, B) = \int_{M^3} < (I_E \otimes K^B) \land (I_E \otimes D^B \Phi_B) > - < (H^A \otimes I_E) \land (I_E \otimes D^B \Phi_B) >, \] (8)
and the dimensionally reduced field equations become
\[ D^B H^A = 0, \quad D^B D^A \Phi_A = [H^A, \Phi_B], \]
\[ D^A K^B = 0, \quad D^A D^B \Phi_B = [K^B, \Phi_A]. \] (9)

The energy functional (8) can be rewritten as
\[ E(A, B) = \int_{M^3} < (H^A \otimes I_E - I_E \otimes K^B) \land (D^A \Phi_A \otimes I_E - I_E \otimes D^B \Phi_B) > - \int_{M^3} < (D^A \Phi_A \otimes I_E) \land (H^A \otimes I_E) > \] (10)

It is clear from (10) that the reduced topological instanton equations (7) continue to saturate the Bogomol'nyi bound, given by the second integral in (10). In Yang-Mills theory, solutions to the time-reduced instanton equations are called BPS magnetic monopoles. We call solutions to the time-reduced topological instanton equations: topological monopoles. Unlike Yang-Mills theory, however, the energy functional (10) is saturated at the Bogomol'nyi bound with either equation in (7). We need not insist that both equations in (7) be satisfied in order to saturate the bound, although of course the field configurations must still satisfy the second-order variational field equations.

To be observable to conventional detectors, \( U(n) \) field configurations must be broken. The symmetry breaking mechanism for BPS magnetic monopoles is very attractive [3], so we will use it here. Imagine a solitonic core region at the origin. Let \( G \) and \( H \) be compact and connected gauge groups, where the group \( H \) is assumed to be embedded in \( G \). The gauge group of the core region \( G \) is spontaneously broken to \( H \) outside of the core region when the Higgs field is covariantly constant, \( D \Phi = 0 \). In regions far from the core \((r \to \infty)\) where we assume that \( D^A \Phi_A = 0 \), it can be shown that
\[ H^A = \Phi_A F_A, \] (11)
where \( F_A \in \Lambda^2(M_3, E_H) \), is any closed two-form on \( M_3 \) taking values in the \( H \)-Lie algebra bundle, denoted by \( E_H \) here. A similar expression to (11), \( K^B = \Phi_B F_B \), can be written when \( D^B \Phi_B = 0 \). We assume that
$<\Phi_A\Phi_A>=1$ when $r>>1$ and where spontaneous symmetry breaking has occurred. When $G=U(n)$ and $H=U(1)$, $F_A$ becomes a pure imaginary two-form on $M_3$. Consider the Bogomol’nyi solitons defined by (7). Solutions to (7) have an energy topologically fixed by

$$E = \int_{M_3} <(D^A\Phi_A \otimes I_E) \wedge (H^A \otimes I_E) > = \int_{M_3} d <\Phi_A H^A > = -\int_{S^2} <\Phi_A H^A >, (12)$$

where $S^2$ is a large sphere surrounding the monopole core and lying completely in a region where $D^A\Phi_A = 0$. Substituting (11) into (12) and using the normalisation condition $<\Phi_A\Phi_A>=1$, the energy is fixed by $\int F_A$. As in the case of the BPS magnetic monopole, $\int F_A$ would be interpreted as the magnetic charge.

4 Conclusion

In this short contribution we have introduced a class of topological field theories in three and four dimensions, exposed their Bogomol’nyi structures, and argued the physical stability of solutions. But we believe that the theories presented here are incomplete because there is a physical asymmetry in the gauge fields present in (1). Symmetry in the Action is easily regained, however, by exchanging $H^A$ and $K^B$, and adding it to the Lagrangian (1). The variational field equations and the Bogomol’nyi equations are unchanged by the symmetrization. In four dimensions, the stability of the topological instanton is only slightly different—the symmetrized Action is twice that of the asymmetric Action. In three dimensions, the saturated energy functional becomes

$$E = -\int_{M_3} (D^A\Phi_A \otimes I_E) \wedge (H^A \otimes I_E) - \int_{M_3} (D^B\Phi_B \otimes I_E) \wedge (K^B \otimes I_E) >. (13)$$

We argued stability from the topological interpretation that can be given to (13). The symmetrization of the topological field theory implies that the solitonic particle is topologically stable if either of the integrals in (13) is non-vanishing. The integrals should correspond to the magnetic and electric charge of the soliton given by $\int_{S^2} F_A$ and $\int_{S^2} F_B$, respectively.

A particularly glaring omission in the solitonic particle spectrum in YMH theory is the electric monopole. The Montonen-Olive conjecture addresses
this by proposing with some compelling evidence that there exists a dual field theory to YMH theory which would replace the BPS magnetic monopole with solitonic intermediate vector bosons: $W^\pm$, $Z^0$ [5]. Although there is still much study needed, we believe that theory of topological monopoles may be an example of a dual field theory [9]. If so, then in order to ensure the stability of the $Z_0$ particle, the $Z_0$ must be a magnetic monopole.

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