A note on the error estimate of the virtual element methods

Shuhao Caoa,*, Long Chenb, Frank Linb

aDepartment of Mathematics, University of California Irvine, Irvine, CA 92697

Abstract

This short note reports a new derivation of the optimal order of the a priori error estimates for conforming virtual element methods (VEM) on 3D polyhedral meshes based on an error equation. The geometric assumptions, which are necessary for the optimal order of the conforming VEM error estimate in the $H^1$-seminorm, are relaxed for that in a bilinear form-induced energy norm.

Keywords: Virtual elements, polytopal finite elements, polyhedral meshes, Poisson problem

2010 MSC: 65N30, 65N12, 65N15

1. Introduction

Virtual element method (VEM) (e.g., [3, 4, 5]) can be viewed as a universalization of the finite element method (FEM) on simplicial and cubical elements to any polytopal elements, enabling a much greater flexibility in mesh generations. All the relevant integral quantities (e.g., stiffness matrix) can be computed or approximated from the degrees of freedom in the VEM space without explicitly constructing the non-polynomial basis functions.

The aim of this paper is to present the optimal order of error estimates of VEM with relaxed geometric assumptions on a three dimensional mesh. Consider the following weak formulation for a toy model Poisson equation with zero Dirichlet boundary condition in a three dimensional polyhedral domain $\Omega$: given an $f \in L^2(\Omega)$, find $u \in H^1(\Omega)$ such that

$$a(u, v) := (\nabla u, \nabla v) = (f, v) \quad \forall v \in H^1(\Omega), \tag{1.1}$$

where $(\cdot, \cdot)$ is the inner product on $L^2(\Omega)$.

Then $a(\cdot, \cdot)$ is approximated by the following bilinear form $a_h(\cdot, \cdot)$ on the VEM space $V_h$ (2.9) built upon a polyhedral partition $\mathcal{T}_h$ of $\Omega$. $a_h(\cdot, \cdot)$ is the summation of local bilinear forms on each element that contains an orthogonal

*Corresponding author

Email addresses: scao@math.uci.edu (Shuhao Cao), chenlong@math.uci.edu (Long Chen), fmlin@uci.edu (Frank Lin)
$H^1$-projection term $(\nabla \Pi_K(\cdot), \nabla \Pi_K(\cdot))$ and a stabilization term $S_K(\cdot, \cdot)$ to ensure the coercivity. The projection term can be computed exactly through the local degrees of freedom for functions of $u, v \in V_h$:

$$a_h(u, v) = \sum_{K \in \mathcal{T}_h} \left[ (\nabla \Pi_K u, \nabla \Pi_K v)_K + S_K(u, v) \right], \quad (1.2)$$

where $\Pi_K$ is the $H^1$-projection operator (see Definition 2.3) to the space of degree $k$ polynomials on $K$.

In the traditional VEM approaches, the stabilization term is $S_K(\cdot, \cdot)$ constructed to satisfy $k$-consistency, as well as the following norm equivalence that holds between the exact $H^1$-inner product $a(\cdot, \cdot)$ and the approximated form $a_h(\cdot, \cdot)$ on the VEM space,

$$a(u, u) \lesssim a_h(u, u) \lesssim a(u, u), \quad \text{for } u \in V_h, \quad (1.3)$$

in which both constants in the inequalities are independent of $u$, but are dependent of the geometry of the meshes. With this property the finite dimensional approximation problem to the weak formulation ((2.12) in Section 2) using the VEM discretization is well-posed, and it can be proved that the error estimate under the $H^1$-seminorm is optimal (e.g., see [3]). One possible choice of the local stabilization on $K$ is given in [3]:

$$S_{orig}^K(u, v) = \sum_{r=1}^{N_K} \chi_r(u - \Pi_K u)\chi_r(v - \Pi_K v), \quad \text{for } u, v \in V_h, \quad (1.4)$$

where $N_K$ is the number of degrees of freedom on an element $K$ (see Definition 2.2), $\chi_r$ is each individual degree of freedom.

Under certain geometric assumptions of the polytopal mesh, the aforementioned norm equivalence (1.3) is established with a proper choice of the stabilization (e.g., (1.4)). Typical geometric assumptions include that (1) the mesh is star-shaped with chunkiness parameter uniformly bounded below (uniform star shape; see Definition 2.10), and (2) the distance between neighboring vertices are comparable to the diameter of the element (no short edges or small faces). However, it has been observed in numerical experiments that the optimal rates of convergence for the virtual element methods can be achieved extraordinarily with relatively little to none geometric constraints on meshes (see e.g., [8, 7, 5, 13]).

In [6], different choices of stabilization terms are analyzed in detail, and the geometric assumptions are further relaxed for two dimensional mesh by including the case where elements contain short edges. Recently, in [10], it is shown that one can allow small faces on a three dimensional mesh and still achieve the optimal order with only the assumption that the elements are uniform star shape. However, several error estimates still require faces to be uniform star shape and some error estimates depend on the logarithm of the longest to the shortest edge ratio of the faces. This rules out anisotropic faces with unbounded aspect ratio.
To partially explain the robustness of VEM with respect to shapes, we shall use a different approach, which was first proposed in [11] to handle the 2D cut mesh and in [12] for nonconforming VEM, to relax the geometric assumptions further on three dimensional meshes, and still achieve the optimal order error estimates. Instead of working on the stronger $H^1$-seminorm, the error analysis is performed toward a weaker “energy norm” $\| \cdot \| := a_h^{1/2}(\cdot, \cdot)$. Similar to that of the Discontinuous Galerkin (DG)-type methods, an error equation for $\| \cdot \|$, is derived. This error equation breaks down the $\|uh - u_I\|$ into several standard projection and interpolation error estimates. Our method does not rely on the norm equivalence property of the stabilization term. Instead, different from the above so-called identity matrix stabilization (1.4), the stabilization term is concocted from the “boundary term” emerged from the integration by parts (see Section 3 for detail), while equipped with correct weights to remain the optimal order for the error estimates.

The following new stabilization term is proposed in this paper, which is partly inspired by the conjecture in [10] and is “singly conforming” in the sense that the term which keeps the conformity of the method may have a small constant in front it.

$$S_K(u, v) = h_K^{-1} \sum_{F \subset \partial K} \left( (Q_K u - Q_F u, Q_K v - Q_F v)_F + \epsilon_F h_F \sum_{\epsilon \subset \partial F} (u - Q_F u, v - Q_F v)_\epsilon \right), \tag{1.5}$$

where $Q_K, Q_F$ are $L^2$-projection operators (see Definition 2.4) to the local spaces of degree $k$ polynomials on $K$ and $F$, respectively. The $\epsilon_F$ is related to the chunkiness parameter of $\rho_F$ (see Definition 2.10) of each polygonal face $F$ on the boundary of a polyhedral element $K$.

This new approach, comparing to the existing results in [6, 10], allows us to deal with the mesh that has less constraints on the shape regularity. For example, the chunkiness parameter $\rho_F$ of each face $F$ on an element $K$ may no longer be uniformly bounded below, i.e., the constants in the new estimates do not depend on the logarithm of the longest and the shortest edge of each face. As a result, we obtain the optimal order error estimate on a weaker energy norm (4.5) with a set of relaxed geometric assumptions 4.1 that are introduced in Section 4.

The rest of the paper is organized as follows. We begin with a brief review of VEM definitions. Then we go through some error estimate lemmas from the past references (e.g., [10]). For each inequality with constant we will put extra emphasis on whether the hidden constant depends on the chunkiness parameter of the domain or not. After that, our main result, the error equation will be derived, and then based on this error equation, the optimal order of the a priori error estimate under appropriate geometric assumptions can be achieved.
2. Preliminaries

In this section, we introduce the definition of the VEM space, its modified variant, and the corresponding degrees of freedom. Throughout the paper, without explicitly define them, we will use standard notations for differential operators, function spaces and norms that can be found, for examples in [1].

The domain $\Omega$ is partitioned into a three dimensional mesh $T_h$, and for simplicity $\Omega$ is assumed to have a polyhedral boundary so that there is no geometric error of $T_h$ on $\partial \Omega$. Let $K$ be a simple polyhedral element in $T_h$. $F$ denotes a face of the element, and $e$ denotes an edge of a face. $D$ denotes a general domain in two or three dimensions, and $h_D$ is the diameter of $D$.

2.1. VEM spaces

To define the three dimensional VEM space, first we need to define the two dimensional local VEM space $V_k(F)$ and the modified space $W_k(F)$. Notice when defining the local VEM space on a face, the surface Laplacian operator $\Delta_F$ on a face $F$ shall be used. Let $k \in \mathbb{N}$ and let $P^k(D)$ be the space of polynomial functions with degree up to $k$ (where $P_{-1}$ contains only zero polynomial.) on $D$.

Definition 2.1 (Local two dimensional VEM space on a face $F$).

$$V_k(F) := \{ v \in H^1(F) : \Delta_F v \in P_{k-2}(F), v|_{\partial F} \in B_k(\partial F) \}, \quad (2.1)$$

where

$$B_k(\partial F) := \{ v \in C^0(\partial F) : v|_e \in P_k(e) \text{ for all } e \subset \partial F \}. \quad (2.2)$$

The degrees of freedom of the space in Definition 2.1 can be defined using the scaled monomials. Let $D$ be a two dimensional simple polygon or three dimensional simple polygonal domain, and $(x_c, y_c, z_c)$ be the center of mass of $D$. Then the scaled monomials are polynomials of the form $m_\alpha = (x - x_c)^{\alpha_1}(y - y_c)^{\alpha_2}(z - z_c)^{\alpha_3} h_D^{-\alpha}$ where $\alpha_1, \alpha_2, \alpha_3$ are non-negative integers. We define the degree to be $\alpha = \alpha_1 + \alpha_2 + \alpha_3$.

Definition 2.2 (Degrees of freedom on a face). The degrees of freedom of $v_h$ in $V_k(F)$ are defined as follows:

1. The value of $v_h$ at the vertices of $F$.

2. The moments up to order $k - 2$ of $v_h$ in each edge $e$. That is, $\frac{1}{|e|} \int_v v_h m_\alpha$ where $m_\alpha$ is a scaled monomial for $\alpha \leq k - 2$.

3. the moments up to order $k - 2$ of $v_h$ in $F$. That is, $\frac{1}{|F|} \int_F v_h m_\alpha$ where $m_\alpha$ is a scaled monomial for $\alpha \leq k - 2$.

The following projection operator $\Pi_K$ in the gradient inner product can be defined for $H^1(D)$ functions in 2D or 3D, but only can be computed with the degrees of freedom above in 2D.
Definition 2.3 (Gradient orthogonal projection operator). \( \Pi^k_D : H^1(D) \to \mathbb{P}_k(D), v \mapsto \Pi^k_Dv \) satisfies
\[
(\nabla(\Pi^k_Dv - v), \nabla p)_D = 0, \quad \forall p \in \mathbb{P}_k(D).
\]
where the constant kernel is determined by the following constraint:
\[
\int_D (\Pi^k_Dv - v) = 0, \quad k \geq 2,
\]
and
\[
\int_{\partial D} (\Pi^k_Dv - v) = 0, \quad k = 1.
\]

On a polygonal domain \( D \), to compute the gradient projection of \( v_h \in V_k(D) \) to \( \mathbb{P}_k(D) \), it is sufficient to compute \( (\nabla v_h, \nabla q)_D \) for all \( q \in \mathbb{P}_k(D) \). Integration by parts yields
\[
(\nabla v_h, \nabla q)_D = -(v_h, \Delta q)_D + (v_h, \nabla q \cdot n)_{\partial D},
\]
then the first term of the right hand side can be computed via internal moments of \( v_h \) in \( D \) (See definition 2.2), and the second term can be computed because it is a polynomial integral (See definition 2.1).

However, for a three dimensional polyhedron \( D \), a naive generalization of the degrees of freedom for the local space \( V_k(D) \) mimicking what of the polygonal version in 2.2 is not sensible. In the three dimensional case, part of the second term \( (v_h, \nabla q \cdot n)_F \) in (2.5) is a surface moment integral on \( F \) that is not computable if \( F \) is not triangular. The reason is that only the moments of \( v_h \) on a face \( F \subset \partial D \) up to degree \( k-2 \) are given as degrees of freedom (See definition 2.2), yet for \( q \in \mathbb{P}_k(D) \), \( \nabla q \cdot n|_F \in \mathbb{P}_{k-1}(F) \). To compute this, we need to be able to compute the \( L^2 \)-projection onto \( \mathbb{P}_{k-1}(F) \) for a VEM function \( v_h \). To this end, modified face spaces such as \( W_k(F) \) or \( \tilde{W}_k(F) \) are to be introduced.

Definition 2.4 (\( L^2 \)-orthogonal projection operator). \( Q^k_D : L^2(D) \to \mathbb{P}_k(D), v \mapsto Q^k_Dv \) satisfies
\[
(Q^k_Dv - v, q)_D = 0, \quad \forall q \in \mathbb{P}_k(D).
\]

When \( D \) is a polygonal face \( F \) on the boundary of a polyhedron \( K \), the above \( L^2 \) projection is not computable through the internal moment degrees of freedom for \( V_k(F) \) in 2.2, in that the moments \( (v_h, q)_D \) for polynomial \( q \) being degree \( k \) or \( k-1 \) are unknown. However the space \( V_k(F) \) defined above can be enriched in a certain way ([2, 10], see definition 2.5 and 2.8) such that the \( L^2 \)-projection is computable from the same set of degrees of freedom with 2.2. These are the motivations behind defining the modified space such as \( W_k(F) \) and \( \tilde{W}_k(F) \), instead of using a direct generalization from \( V_k(F) \) to \( V_k(D) \) for a polyhedron \( D \).

When the order of the projection operators are omitted, we assume it is \( k \), the same as the order of the VEM space.
Definition 2.5 (Local modified VEM space \([2]\)). Let \(\tilde{P}_k(e)\) be the space of degree exactly \(k\) monomials, then the local modified VEM space can be defined as:

\[
W_k(F) := \left\{ v \in H^1(F) : \Delta_F v \in \tilde{P}_k(F), v|_{\partial F} \in B_k(\partial F), (v, q)_F = (\Pi^k_F v, q)_F, \forall q \in \tilde{P}_k(F) \cup \tilde{P}_{k-1}(F) \right\},
\]

(2.6)

Note that \(W_k\) and \(V_k\) share the same degrees of freedom \((2.2)\), but the \(L^2\)-projection of a function in \(W_k\) is now computable. In \(W_k\) we can replace \((v_h, q)_K\) by \((\Pi^k_F v_h, q)_K\) for \(q\) being degree \(k\) or \(k-1\) and the latter integral is computable.

The three dimensional local VEM space can be defined as follows:

Definition 2.6 (Local three dimensional VEM space on an element \(K\) \([2]\)).

\[
V_k(K) := \left\{ v \in H^1(K) : \Delta v \in \tilde{P}_{k-2}(K), v|_{\partial K} \in B_k(\partial K) \right\},
\]

(2.7)

where \(B_k(\partial K) := \left\{ v \in C^0(\partial K) : v|_F \in W_k(F), v|_e \in \tilde{P}_k(e) \right\}\).

Any function in \(V_k(K)\) can be uniquely determined by its degrees of freedom \((2.2)\) defined in the following paragraph.

Definition 2.7 (Degrees of freedom of the three dimensional VEM space). We can take the following degrees of freedom of \(v_h\) in \(V_k(K)\), where \(K\) is a three dimensional element.

1. The value of \(v_h\) at the vertices of \(K\)

2. The moments on each edge \(e\) up to degree \(k-2\). That is, \(\frac{1}{|e|} \int_e v_h m_\alpha\) where \(m_\alpha\) is the scaled monomials with \(\alpha \leq k - 2\).

3. The moments on each face \(F\) up to degree \(k-2\). That is, \(\frac{1}{|F|} \int_F v_h m_\alpha\) where \(m_\alpha\) is the scaled monomials with \(\alpha \leq k - 2\).

4. The moments on the element \(K\) up to degree \(k-2\). That is, \(\frac{1}{|K|} \int_K v_h m_\alpha\) where \(m_\alpha\) is the scaled monomials with \(\alpha \leq k - 2\).

An alternative definition of the modified VEM space \([10]\), that allows us to compute both \(H^1\) and \(L^2\) projection from degrees of freedom is the following. We denote such a space \(\tilde{W}_k(D)\), where \(D\) can be a polyhedron domain in any dimension. For convenience we shall define \(\tilde{W}_k(e) = \tilde{P}_k(e)\) for \(e\) being 1 dimensional edge and higher dimension spaces are defined recursively.

Definition 2.8 (The modified local \(\tilde{W}_k\) space \([10]\)). Let \(D\) be a two or three dimensional polygon or polygonal domain, define the space \(\tilde{W}_k(D)\) by

\[
\tilde{W}_k(D) := \left\{ v \in H^1(D) : \Delta_D v \in \tilde{P}_k(D), v|_{\partial D} \in C^0(\partial D), v|_F \in \tilde{W}_k(F) \forall F \subset \partial D, \Pi^k_D v_h - Q^k_D v_h \in \tilde{P}_{k-2}(D) \right\},
\]

(2.8)
When computing $L^2$ projection in $\tilde{W}_k$, we first write $Q_K^kv_h = \Pi_K^kv_h + w, \ w \in P_{k-2}$, and the corresponding integrals can be computed using internal degrees up to $k-2$.

We shall henceforth use the only $\tilde{W}_k$ to define the global VEM space for the rest of the paper. Let

$$V_h = \tilde{W}_h := \{ v_h \in H^1_0(\Omega) \cap C^0(\overline{\Omega}) : v_h|_K \in \tilde{W}_k(K), \ \forall K \in T_h \} \quad (2.9)$$

be the global VEM space for the rest of the paper, so that $L^2$ projection is computable for any three dimensional element $K$.

We then have the following natural definition of the nodal interpolation.

**Definition 2.9** (The canonical interpolation). For any $u \in H^1(\Omega)$, on $K \in T_h$, $u_I$ is the local interpolation on $K$ which is defined as a function in $\tilde{W}_k(K)$ that has the same degrees of freedom (2.7) with $u$. Globally, $u_I$ is defined as a function in $V_h = \tilde{W}_h$ that has the same degrees of freedom as $u$.

We use the same notation $u_I$ for both local and global interpolation, but under the proper context it should not be confused.

The following choice of stabilization term is motivated by the error equation that will be derived in section 3. On an element $K$, the stabilization term is defined as follows:

$$S_K(u, v) := h_K^{-1} \sum_{F \subset \partial K} \left[ (Q_Ku - Q_Fu, Q_Kv - Q_Fv)_F \right. \\
+ \left. \epsilon_F h_F \sum_{e \subset \partial F} ((u - Q_Fu), (v - Q_Fv))_e \right], \quad (2.10)$$

where $\epsilon_F \propto \rho_F$ is a mesh dependent constant and the discrete bilinear form is given by

$$a_h(u, v) = \sum_{K \in T_h} \left[ \langle \nabla \Pi_Ku, \nabla \Pi_Kv \rangle_K + S_K(u, v) \right]. \quad (2.11)$$

Then the VEM approximation problem is: to seek $u_h \in V_h \ (2.9)$

$$a_h(u_h, v_h) = \sum_{K \in T_h} \langle f, Q_Kv_h \rangle \ \forall v_h \in V_h. \quad (2.12)$$

### 2.2. Approximation Theory on Star-Shaped Domains

In this section, we shall review some existing results on VEM projection (2.13) and interpolation error estimates (2.14).

**Definition 2.10** (Star-shaped polytope). Let $D$ be a simple polygon or polyhedron. We said $D$ is star-shaped with respect to a disc/ball $B$ if for every point $y \in D$, the convex hull of $\{y\} \cup B$ is contained in $D$. If $D$ is star-shaped with respect to a disc/ball with radius $\rho_D$. We define the supremum of $\rho$ to be chunkiness parameter $\rho_D$. 

7
Lemma 2.11 (Bramble-Hilbert estimates on star-shaped domain [9]). Let $D$ be a star-shaped domain, then we have the following estimates,

$$\inf_{q \in \mathbb{R}} |u - q|_{H^m(D)} \leq C(\rho_D)h_D^{l+1-m} |u|_{H^{l+1}(D)}, \forall u \in H^{l+1}(D), l = 0, 1, \ldots, k, m \leq l$$

(2.13)

where $C(\rho_D)$ is inverse proportional to $\rho_D$.

The following scaled trace inequalities are often used when we need to bound norm on boundary faces by norm on elements.

Lemma 2.12 (Trace inequalities on a star-shaped domain [9]). Let $D$ be a star-shaped domain, then

$$\|u\|^2_{L^2(\partial D)} \lesssim h^{-1}_D \|u\|^2_{L^2(D)} + h_D \|u\|^2_{H^1(D)}, \forall u \in H^1(D),$$

(2.14)

where the constant in $\lesssim$ is inverse proportional to $\rho_D$.

By combining the Bramble-Hilbert estimates, and the stability of projection operators ($Q_D$ and $\Pi_D$) (see [6, 10]) in $L^2$, $H^1$, and $H^2$ norms, we can obtain the following projection error estimates.

Theorem 2.13 (Projection error estimate). Let $D$ be a star-shaped domain. Let $\Pi$ be $\Pi_D$ or $Q_D$ then for $m, l, k \in \mathbb{N}$, $0 \leq m \leq 2$, $\min(1, m) \leq l \leq k$, $u \in H^{l+1}(D)$ we have

$$\|u - \Pi u\|_{m,D} \leq [C(\rho_D)h_D]^{l+1-m} |u|_{l+1,D}$$

(2.15)

where $C(\rho_D)$ is inverse proportional to $\rho_D$.

The optimal order of interpolation operators are much harder to prove. In [10], an auxiliary semi-norm is constructed to prove the following interpolation estimates. We will list the result here and refer the reference for the detail (although the estimate of $|u - Q_Du|_{1,D}$ is not explicitly given in [10], the derivation follows from $H^1$ stability of $Q_D$, and the procedure of deriving the estimate of $|u - \Pi_Du|_{1,D}$ is almost identical). The interpolation estimates for three dimensional element require the uniform star shape condition to be of optimal order.

Theorem 2.14 (Interpolation error estimate [10]). Let $D$ be a star-shaped domain. Let $u_I$ be the nodal interpolation of the function on the local VEM space defined in 2.9. We have, for $1 \leq l \leq k$, $\forall u \in H^{l+1}(D)$

$$|u - u_I|_{1,D} + |u - \Pi_Du|_{1,D} + |u - Q_Du|_{1,D} \lesssim [C(\rho_D)h_D]^l |u|_{l+1,D}$$

(2.16)

$$|u - \Pi_Du|_{2,D} \lesssim [C(\rho_D)h_D]^{l-1} |u|_{l+1,D}$$

(2.17)

$$|u - u_I|_{0,D} + |u - Q_Du|_{0,D} + |u - \Pi_Du|_{0,D} \lesssim [C(\rho_D)h_D]^{l+1} |u|_{l+1,D}$$

(2.18)

The constants $C(\rho_D)$ is inverse proportional to $\rho_D$. 

8
3. A priori error estimate

In this section, we first verify that the discrete bilinear form induces a norm on $V_h$, then an error equation based on the discrete bilinear form is derived. The a priori error estimate can be then derived from this error equation.

Recall that on an element $K$, the bilinear form and the stabilization term are defined in (2.11) and (2.10), and the VEM approximation problem is (2.12). Then the seminorm $\|v\| = a_h^{1/2}(v,v)$ induced by $a_h(\cdot,\cdot)$ is verified to be a norm on the VEM space with the boundary condition imposed in the following lemma.

**Lemma 3.1.** $\| \cdot \|$ is a norm on $V_h \cap H^1_0(\Omega)$.

**Proof.** It suffices to verify that if $\|v_h\| = 0$, then $v_h \equiv 0$. By definition, when $a_h(v_h,v_h) = 0$, we have $\Pi_K v_h = 0$ on each $K$, $Q_K v_h = Q_F v_h$ on each $F \subset \partial K$, and $v_h = Q_F v_h$ on each edge $e$.

By the boundary condition of the space, $Q_F v_h = 0$ for $F \subset \partial\Omega$. Because $Q_K v_h = Q_F v_h$ on each $F \subset \partial K$, that makes $Q_K v_h = 0$ for $K$ that contains at least a boundary face. For the same reasons, $Q_K v_h = 0$ for $K'$ that shares a face with $K$, which is an element that contains at least a boundary face. Repeat this argument we have $Q_K v_h = Q_F v_h = 0$ for each $K$.

In addition, on each edge $v_h = Q_F v_h = 0$. That makes the degrees of freedom of $v_h$ on each $K$ equal 0, which in turn implies $v_h = 0$ by unisolvence of the VEM space [3], which completes the proof.

**Lemma 3.2** (An identity of the approximated bilinear form). For $u$ that is the solution to (1.1), $u_h$ that is the solution to (2.12), and any $v \in V_h$, the following identity holds,

$$a_h(u_h,v) = \sum_{K \in T_h} (\nabla \Pi_K u, \nabla \Pi_K v)_K + \sum_{K \in T_h} (\nabla (u - \Pi_K u) \cdot n, Q_K v - Q_F v)_{\partial K}. \tag{3.1}$$

**Proof.** First we apply the integration by parts to $u, v, \Pi_K u, Q_K v$, and use the definitions of $H^1$-projection $\Pi_K$ and $L^2$-projection $Q_K$ to get

$$-(\Delta u, Q_K v)_K = (\nabla \Pi_K u, \nabla Q_K v)_K + (\nabla u \cdot n, Q_K v)_{\partial K},$$

$$(\Delta \Pi_K u, Q_K v)_K = - (\nabla \Pi_K u, \nabla Q_K v)_K - (\nabla \Pi_K u \cdot n, Q_K v)_{\partial K},$$

and $$-(\Delta \Pi_K u, v)_K = (\nabla \Pi_K u, \nabla v)_K + (\nabla \Pi_K u \cdot n, Q_F v)_{\partial K}.$$

Adding above equations together and notice that $(\Delta \Pi_K u, Q_K v)_K = (\Delta \Pi_K u, v)_K$. By the definition of $Q_K$, we get

$$-(\Delta u, Q_K v)_K = (\nabla \Pi_K u, \nabla v)_K + ((\nabla u - \nabla \Pi_K u) \cdot n, Q_K v)_{\partial K} + (\nabla \Pi_K u \cdot n, Q_F v)_{\partial K}.$$

By definition, the first term can be rewritten as $(\nabla \Pi_K u, \nabla \Pi_K v)_K$. By the continuity of $\nabla u \cdot n$ across the interelement faces, and the fact that $Q_F v$ is
single value on the face $F$, $\sum_{K \in T_h} \langle \nabla u \cdot n, Q_F v \rangle_{\partial K} = 0$. As a result, recalling the VEM approximation problem in equation (2.12), we arrive at the following identity

$$a_h(u_h, v) = - \sum_{K \in T_h} \langle \Delta u, Q_K v \rangle_K$$

$$= \sum_{K \in T_h} \langle \nabla \Pi_K u, \nabla \Pi_K v \rangle_K + \sum_{K \in T_h} \langle \nabla (u - \Pi_K u) \cdot n, Q_K v - Q_F v \rangle_{\partial K}.$$ 

Theorem 3.3 (An error equation). Under the same setting with Lemma 3.2, let $u_I$ be the VEM interpolation in (2.9), and using the stabilization in (2.10), the following identity holds,

$$a_h(u_h - u_I, v_h) = \sum_{K \in T_h} \left[ \langle \nabla \Pi_K (u - u_I), \nabla \Pi_K v_h \rangle_K 
+ \sum_{F \subset \partial K} \langle \nabla (\Pi_K u - u) \cdot n, Q_K v_h - Q_F v_h \rangle_F 
- h_K^{-1} \langle Q_K u_I - Q_F u_I, Q_K v_h - Q_F v_h \rangle_F 
- \epsilon_F h_F \sum_{e \subset \partial F} \langle u_I - Q_F u_I, v_h - Q_F v_h \rangle_e \right].$$

(3.2)

Proof. It follows directly from Lemma 3.2 and Definition 2.10.

Corollary 3.4 (An a priori error bound). For a constant $\epsilon_F \propto \rho_F$, the following a priori error estimate holds for $u_h$ and $u_I$ (defined in 2.9) with a constant independent of the chunkiness parameter for $\rho_F$ of each face in the underlying mesh,

$$\| u_h - u_I \|^2 \lesssim \sum_{K \in T_h} \left[ \| \nabla \Pi_K (u - u_I) \|^2_{0, K} 
+ \sum_{F \subset \partial K} \left( h_K \| \nabla (\Pi_K u - u) \cdot n \|^2_{0, F} 
+ h_K^{-1} \| Q_K u_I - Q_F u_I \|^2_{0, F} 
+ \epsilon_F \sum_{e \subset \partial F} \| u_I - Q_F u_I \|^2_{0, e} \right) \right].$$

(3.3)

Proof. From the error equation, plug in $v_h = u_h - u_I$ and apply the Cauchy-Schwarz inequality, we have
\[ \|u_h - u_I\|^2 \lesssim \sum_{K \in T_h} \left( \|\nabla \Pi_K (u - u_I)\|_{0,K} \|\nabla \Pi_K v_h\|_{0,K} + \sum_{F \subset \partial K} h_K^{1/2} \|\nabla (\Pi_K u - u) \cdot n\|_{0,F} h_K^{-1/2} \|Q_K v_h - Q_F v_h\|_{0,F} + h_K^{-1/2} \|Q_K u_I - Q_F u_I\|_{0,F} h_K^{-1/2} \|Q_K v_h - Q_F v_h\|_{0,F} \right) + \sum_{e \subset \partial F} \epsilon_F \|u_I - \Pi_F u_I\|_{0,e} \epsilon_F \|v_h - \Pi_F v_h\|_{0,e} \right). \]

(3.4)

The second part of each term is clearly parts of \(\|u_h - u_I\|\) and therefore can be bounded by \(\|u_h - u_I\|\). After cancelling \(\|u_h - u_I\|\) we get the estimate. \(\square\)

4. Geometric conditions and error estimations

In this section, based on the a priori error estimate in Corollary 3.4, the energy norm estimate follows from estimating each term in (3.3). The necessary geometry conditions, which are motivated by (3.3) to have optimal order of convergence, are proposed as follows.

**Assumption 4.1** (Geometric conditions). For each element \(K \in T_h\), the following three geometric conditions are met:

1. Number of faces in \(K\) is uniformly bounded.
2. \(K\) is star-shaped with the chunkiness parameter \(\rho_K\) defined in 2.10 bounded below.
3. For each \(F \subset \partial K\), \(F\) is star-shaped, but the chunkiness parameter \(\rho_F\) may not be uniformly bounded below.

An example of the polyhedral element satisfying Assumption 4.1 is shown in Figure 1, on which the \(H^1\)-seminorm error estimate will have an undesirable \(|\ln \epsilon|\) factor related to \(\rho_F\) (see [6, 10]). Subsequently, we will show that on a weaker energy norm \(||\cdot||\), the a priori error estimate is independent of \(\rho_F\) and only dependent on \(\rho_K\).

**Lemma 4.2** (Optimal order error estimate of the stabilization term on a face). Let \(u \in H^{k+1}(K)\), for \(k \geq 1\), and \(u_I\) be the VEM space interpolation defined in 2.9. Suppose the geometric assumptions 4.1 hold, then

\[ h_K^{-1/2} \|Q_K u_I - Q_F u_I\|_{0,F} \lesssim h_K^k |u|_{k+1,K}. \]

(4.1)
Figure 1: Set $\epsilon \to 0$ as $h_K \to 0$. $K$ is a cube without a prismatic slit with $\rho_K > 1/2$ when $\epsilon$ is small, which satisfies all three assumptions in 4.1. However, the chunkiness parameter $\rho_F \to 0$ for the marked face $F$, and this is problematic for the error estimate under $| \cdot |_1$.

Proof. By triangle inequality,

$$h_K^{-1/2}||Q_K u_I - Q_F u_I||_{0,F} = h_K^{-1/2}||Q_F (Q_K u_I - u_I)||_{0,F}$$

$$\lesssim h_K^{-1/2}||Q_K u_I - u_I||_{0,F}$$

$$\lesssim h_K^{-1/2} (||Q_K u_I - u||_{0,F} + ||u - u_I||_{0,F})$$

$$\lesssim h_K^{-1} (||Q_K u_I - u||_{0,K} + ||u - u_I||_{0,K})$$

$$+ (||Q_K u_I - u||_{1,K} + ||u - u_I||_{1,K})$$

$$\lesssim h_K^k |u|_{k+1,K}$$

where Theorems 2.13, 2.14, 2.12 are applied. \qed

**Lemma 4.3** (Optimal order error estimate of stabilization term on an edge). Under the same setting with Lemma 4.2, for a mesh dependent constant $\epsilon_F \propto \rho_F$, we have

$$\epsilon_F ||u_I - Q_F u_I||_{0,e} \lesssim \rho_F^{-1} h_F^k |u|_{k+1,K},$$

(4.3)

**Proof.** By the Theorem 2.12 and triangle inequality, under the star-shaped condition 2.10,

$$||u_I - Q_F u_I||_{0,e} \lesssim \epsilon_F^{-1} (h_F^{-1/2} ||u_I - Q_F u_I||_{0,F} + h_F^{1/2} ||u_I - Q_F u_I||_{1,F})$$

$$\lesssim \epsilon_F^{-1} (h_F^{-1/2} ||u_I - u||_{0,F} + h_F^{1/2} ||u - Q_F u_I||_{0,F})$$

$$+ h_F^{1/2} ||u_I - u||_{1,F} + h_F^{1/2} ||u - Q_F u_I||_{1,F}$$

(4.4)

where each except the last term has optimal error order by Theorem 2.14. In order to use Theorem 2.14 on the face, $\rho_F$ need to be included because we do not assume it is uniformly bounded below, therefore the constant $\epsilon_F \propto \rho_F$ is introduced. For the $||\Pi_F u_I - Q_F u_I||_{1,F}$ term, we apply the inverse inequality on the face $F$ and the triangle inequality.
\[ |u - Q_F u_I|_{1,F} \lesssim |u - \Pi_F u_I|_{1,F} + |\Pi_F u_I - Q_F u_I|_{1,F} \]
\[ \lesssim |u - \Pi_F u_I|_{1,F} + \epsilon_F^{-1} h_F^{-1} \|\Pi_F u_I - Q_F u_I\|_{0,F} \]
\[ \leq |u - \Pi_F u_I|_{1,F} + \epsilon_F^{-1} h_F^{-1} \|\Pi_F u_I - u_I\|_{0,F} \]
\[ + \epsilon_F^{-1} h_F^{-1} \|u_I - Q_F u_I\|_{0,F} \]
\[ (4.5) \]

where each term has optimal error order by Theorem 2.14. Similarly the inverse inequality depends on \( \rho_F [10] \), a mesh dependent constant \( \epsilon_F \propto \rho_F \) is introduced to compensate.

Now we turn to derive the estimates of the other terms in the a priori error bound (3.4).

**Lemma 4.4 (The projection type error estimates).** Under the same setting with Lemma 4.2, we have
\[ \| \nabla \Pi_K (u - u_I) \|_{0,K} \lesssim h_K^k |u|_{k+1,K}, \]  
\[ (4.6) \]

and
\[ h_K^{1/2} \| \nabla (u - \Pi_K u) \|_{0,F} \lesssim h_k^k |u|_{k+1,K}. \]  
\[ (4.7) \]

**Proof.** By Theorem 2.13 and \( \Pi_K \) is the projection under \( |\cdot|_{1,K} \),
\[ \| \nabla \Pi_K (u - u_I) \|_{0,K} \lesssim |u - u_I|_{1,K} \lesssim h_K^k |u|_{k+1,K}. \]

In addition, by Theorems 2.12 and 2.13
\[ h_K^{1/2} \| \nabla (\Pi_K u - u) \|_{0,F} \lesssim |\Pi_K u - u|_{1,K} + h_K |\Pi_K u - u|_{2,K} \lesssim h_K^k |u|_{k+1,K}. \]  
\[ \square \]

**Theorem 4.5 (Energy norm error estimate).** If (1.1) has a sufficiently regular solution \( u \in H^{k+1}(K) \) \((k \geq 1)\), let \( u_h \) be the VEM approximation in (2.12), and \( u_I \) be the interpolation in Definition 2.9, then under the geometric assumptions 4.1, we have the following estimate
\[ \| u_h - u_I \| \lesssim h^k |u|_{k+1}. \]  
\[ (4.8) \]

**Proof.** (4.8) follows immediately from the bound of each term in the a priori error estimate (3.3) by Lemmas 4.4, 4.2, and 4.3.  
\[ \square \]

**Acknowledgement**

This work was supported in part by the National Science Foundation under grant DMS-1418934.
References

[1] R. A. Adams, J. J. Fournier, Sobolev spaces, Vol. 140, Elsevier, 2003.

[2] B. Ahmad, A. Alsaeedi, F. Brezzi, L. D. Marini, A. Russo, Equivalent projectors for virtual element methods, Computers & Mathematics with Applications 66 (3) (2013) 376–391.

[3] L. Beirão da Veiga, F. Brezzi, A. Cangiani, G. Manzini, L. Marini, A. Russo, Basic principles of virtual element methods, Mathematical Models and Methods in Applied Sciences 23 (01) (2013) 199–214.

[4] L. Beirão da Veiga, F. Brezzi, L. Marini, A. Russo, The hitchhiker’s guide to the virtual element method, Mathematical models and methods in applied sciences 24 (08) (2014) 1541–1573.

[5] L. Beirão da Veiga, F. Dassi, A. Russo, High-order virtual element method on polyhedral meshes, Computers & Mathematics with Applications.

[6] L. Beirão da Veiga, C. Lovadina, A. Russo, Stability analysis for the virtual element method, Mathematical Models and Methods in Applied Sciences 27 (13) (2017) 2557–2594.

[7] M. F. Benedetto, S. Berrone, S. Pieraccini, S. Scialò, The virtual element method for discrete fracture network simulations, Computer Methods in Applied Mechanics and Engineering 280 (2014) 135–156.

[8] S. Berrone, A. Borio, Orthogonal polynomials in badly shaped polygonal elements for the virtual element method, Finite Elements in Analysis and Design 129 (2017) 14–31.

[9] S. Brenner, R. Scott, The mathematical theory of finite element methods, Vol. 15, Springer Science & Business Media, 2007.

[10] S. C. Brenner, L.-Y. Sung, Virtual element methods on meshes with small edges or faces, Mathematical Models and Methods in Applied Sciences 28 (7) (2018) 1291–1336.

[11] S. Cao, L. Chen, Anisotropic error estimates of the linear virtual element method on polygonal meshes, SIAM Journal on Numerical Analysis 56 (5) (2018) 2913–2939.

[12] S. Cao, L. Chen, Anisotropic error estimates of the linear nonconforming virtual element methods, arXiv preprint arXiv:1806.09054.

[13] L. Chen, H. Wei, M. Wen, An interface-fitted mesh generator and virtual element methods for elliptic interface problems, Journal of Computational Physics 334 (2017) 327–348.