S-wave contributions in $\bar{B}_s^0 \to (D^0, \bar{D}^0)\pi^+\pi^-$ within perturbative QCD approach

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The $\bar{B}_s^0 \to (D^0, \bar{D}^0)\pi^+\pi^-$ is induced by the $b \to c\bar{u}s/b \to u\bar{c}s$ transition, and can interfere if a CP-eigenstate $D_{CP}$ is formed. The interference contribution is sensitive to the CKM angle $\gamma$. In this work, we study S-wave $\pi^+\pi^-$ contributions to the process in the perturbative QCD factorization. Under the factorization framework, we adopt two-meson light-cone distribution amplitudes, whose normalization is parametrized by the S-wave time-like two-pion form factor with the resonance contributions from $f_0(500)$, $f_0(980)$, $f_0(1500)$, $f_0(1790)$. We find the branching ratios of $\bar{B}_s^0 \to (D^0, \bar{D}^0)(\pi^+\pi^-)_{S}$ can reach the order of $10^{-6}$, and significant interferences exist in $\bar{B}_s^0 \to D_{CP}(\pi^+\pi^-)_{S}$. The future measurement can not only provide useful constraints on the CKM angle $\gamma$ but is also helpful to explore the multi-body decay mechanism of heavy mesons.

I. INTRODUCTION

In recent years, three-body hadronic $B/B_s$ meson decays have attracted great attentions on the experimental side [1–3]. These processes are capable to provide new sources to study the phenomenology in the Standard Model and probe the new physics effects. For instance, LHCb Collaboration has measured sizable direct CP asymmetries in the various phase space of three-body $B$ decays $[4, 5]$. In addition, they are also valuable for us to understand the mechanism for multi-body heavy meson decays.

On the theoretical side, the perturbative QCD(PQCD) approach, based on the $k_T$ factorization, has been applied to analyze the $B/B_s$ semi-leptonic and two-body decays processes $[6–29]$. The PQCD approach has also been used to study three-body decays $[30–39]$. Generally, the multi-scale decay amplitude might be written as as a convolution, including the nonperturbative wave functions, hard kernel at the intermediate scale and short-distance Wilson coefficients. The factorization is greatly simplified if two of the final hadrons move collinearly. In this case, the three-body decays are reduced to quasi-two-body processes. Therefore, nonperturbative wave functions include two-meson light-cone distributions, which contain both resonant and nonresonant contributions. For instance, the measurement of LHCb $[2]$ of $B_s \to J/\psi(\pi^+\pi^-)_{S}$ supports that the resonances $f_0(500)$, $f_0(980)$, $f_0(1500)$, $f_0(1790)$ of the S-wave $\pi\pi$-pair are dominant, which is confirmed by the theoretical calculation in the frame of PQCD $[40–44]$. In this work, we will focus on the $\bar{B}_s^0 \to D^0(\bar{D}^0)\pi^+\pi^-$, and include the $B_s \to D(f_0(500)+f_0(980)+f_0(1500)+f_0(1790)) \to D[(\pi^+\pi^-)_{S}]$ contributions. More explicitly, a Breit-Wigner(BW) model will be used for the resonance $f_0(500)$, $f_0(1500)$, $f_0(1790)$ and Flatté Model is adopted for the resonance $f_0(980)$ $[46]$. The $\bar{B}_s^0 \to D^0(\bar{D}^0)\pi^+\pi^-$, with CP eigenstate containing the interference of $b \to c\bar{u}s$ ($b \to u\bar{c}s$) amplitude, is sensitive to the angle $\gamma$ of the CKM Unitarity Triangle whose precise measurement is one of the primary objectives in flavour physics.

This paper is organized as follow: In Sec.II, we introduce the wave functions of $B_s$, $D$ and two pion mesons in turn, while Sec.III contains our perturbative calculation within the PQCD framework. In Sec.IV, we study the numerical results, and a conclusion is presented in the last section.

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II. WAVE FUNCTIONS

In general, wave function \( \Phi_{\alpha \beta} \) with Dirac indices \( \alpha, \beta \) can be decomposed into 16 independent components, \( I_{\alpha \beta}, \gamma^\mu_{\alpha \beta}, (\gamma^\mu \gamma^5)_{\alpha \beta}, \gamma_5^\alpha \beta, \sigma_{\alpha \beta}^\mu \). For the pseudoscalar \( B_s \) meson, the light-cone matrix element is defined as

\[
\int_0^1 \frac{d^4 z}{(2\pi)^4} e^{ik_{1z}z} \langle 0|b_\alpha(0)\delta_\beta(z)|B_s(P_{B_s}) \rangle = \frac{i}{\sqrt{2N_c}} \left\{ (P_{B_s} + m_{B_s}) \gamma_5 \phi_{B_s}(k_1) + \frac{n - p \cdot \phi_{B_s}(k_1)}{\sqrt{2}} \right\} \alpha \beta,
\]

(1)

where the light-cone vectors \( n = (1, 0, 0) \) and \( v = (0, 1, 0) \). The two independent structures in \( B_s \) meson light cone distribution amplitudes obey the following normalization conditions.

\[
\int \frac{d^4 k_1}{(2\pi)^4} \phi_{B_s}(k_1) = \frac{f_{B_s}}{\sqrt{2N_c}}, \quad \int \frac{d^4 k_1}{(2\pi)^4} \phi_{B_s}(k_1) = 0,
\]

(2)

with \( f_{B_s} \) as the decay constant of \( B_s \) meson. Since the contribution of \( \bar{\phi}_{B_s}(k_1) \) is numerically small \( 27 \), we neglect it and keep only \( \phi_{B_s}(k_1) \) in the above equation. In momentum space the light cone matrix of \( B_s \) meson can be expressed as follows:

\[
\Phi_{B_s} = \frac{i}{\sqrt{2N_c}} (P_{B_s} + m_{B_s}) \gamma_5 \phi_{B_s}(k_1).
\]

(3)

Usually the hard part is independent of \( k^+ \) or/and \( k^- \), thus one can integrate one of them out from \( \phi_{B_s}(k^+, k^-, k_\perp) \). With \( b \) as the conjugate space coordinate of \( k_\perp \), we can express \( \phi_{B_s}(x, k_\perp) \) in \( b \)-space by

\[
\phi_{B_s}(x, b) = N_{B_s} x^2 (1 - x)^2 \exp \left[ -\frac{m_{B_s}^2 x^2 + (\omega_b b)^2}{2} \right],
\]

(4)

where \( x \) is the momentum fraction of the light quark in \( B_s \) meson. In this paper, we adopt the following expression for \( \phi_{B_s}(x, b) \)

\[
\phi_{B_s}(x, b) = N_{B_s} x^2 (1 - x)^2 \exp \left[ -\frac{m_{B_s}^2 x^2 + (\omega_b b)^2}{2} \right],
\]

(5)

with \( N_{B_s} \) the normalization factor, which is determined by equation at \( b = 0 \). In our calculation, we adopt \( \omega_b = (0.50 \pm 0.05) \text{GeV} \) and \( f_{B_s} = (0.23 \pm 0.03) \text{GeV} \) \( 26 \), from which we determine the \( N_{B_s} = 63.58 \).

The wave function of the charmed D meson, treated as the heavy-light system, is defined by the light cone matrix element as follows \( 10 \).

\[
\int_0^1 \frac{d^4 z}{(2\pi)^4} e^{ik_{1z}z} \langle 0|\bar{c}_\alpha(0)q_\beta(z)|D(0) \rangle = \frac{i}{\sqrt{2N_c}} \left\{ (P_D + m_D^0) \gamma_5 \phi_D(k_2) \right\} \alpha \beta,
\]

(6)

which satisfies the normalization

\[
\int \frac{d^4 k_2}{(2\pi)^4} \phi_D(k_2) = \frac{f_D}{\sqrt{2N_c}}.
\]

(7)

Here \( f_D \) is the decay constant, and the chiral D meson mass is taken as \( m_D^0 = \frac{m_D^2}{m_c + m_d} = m_D + \mathcal{O}(\Lambda) \). For the numerical calculation, we adopt the parametrization \( 47 \).

\[
\phi_D(x_2, b_2) = \frac{f_D}{2\sqrt{2N_c}} \exp \left[ -\frac{\omega_D^2 b_2^2}{2} \right] \left[ 1 + C_D (1 - 2x_2) \right],
\]

(8)

with the free shape parameter \( C_D \) taken as \( C_D = 0.5 \pm 0.1 \), \( f_D, \omega_D \) read as \( f_D = 0.221 \pm 0.018 \) and \( \omega_D = 0.1 \), respectively \( 13 \).
Then the S-wave two-pion distribution amplitudes is given as [44]

\[ \Phi_{\pi\pi}^{S-\text{wave}} = \frac{1}{\sqrt{2N_c}} \left[ m_\pi^2 \Phi_{\pi\pi}^0(z, \xi, m_\pi^2) + m_\pi \Phi_{\pi\pi}^1(z, \xi, m_\pi^2) + m_\pi^2 (q^2 - 1) \Phi_{\pi\pi}^T(z, \xi, m_\pi^2) \right], \]

where \( z \) is the momentum fraction carried by the spectator positive quark, \( \Phi_{\pi\pi}^0, \Phi_{\pi\pi}^1 \) and \( \Phi_{\pi\pi}^T \) are twist-2 and twist-3 distribution amplitudes. \( m_\pi \) is the invariant mass of the pion pair. We consider the two-pion system move in the \( n \) direction. \( \xi \) as the momentum fraction of \( \pi^+ \) in pion pair. The asymptotic forms are parameterized as [48–50]

\[ \Phi_{\pi\pi} = \frac{F_s(m_{\pi\pi}^2)}{2\sqrt{2N_c}} a_2 g z (1-z) 3(2z-1), \quad \Phi_{\pi\pi}^0 = \frac{F_s(m_{\pi\pi}^2)}{2\sqrt{2N_c}}, \quad \Phi_{\pi\pi}^T = \frac{F_s(m_{\pi\pi}^2)}{2\sqrt{2N_c}} (1-2z). \]

Here, \( F_s(m_{\pi\pi}^2) \) and \( a_2 \) are the timelike scalar form factor and the Gegenbauer coefficient respectively. As a first approximation, the S-wave resonances are used to parametrized \( F_s(m_{\pi\pi}^2) \), to include both resonant and nonresonant contributions into the S-wave two-pion distribution amplitudes. Therefore, we take into account \( f_0(980), f_0(1500) \) and \( f_0(1790) \) for the \( s\bar{s} \) density operator, \( f_0(500) \) for the \( u\bar{u} \) density operator:

\[ F_s^{s\bar{s}}(m_{\pi\pi}^2) = \frac{c_1 m_{f_0(980)}^2 e^{i\theta_1}}{m_{f_0(980)}^2 - m_{\pi\pi}^2 - i m_{f_0(980)} \rho_{\pi\pi} + g_{KK} \rho_{KK}} - \frac{c_2 m_{f_0(1500)}^2 e^{i\theta_2}}{m_{f_0(1500)}^2 - m_{\pi\pi}^2 - i m_{f_0(1500)} \Gamma_{f_0(1500)}(m_{\pi\pi}^2)} + \frac{c_3 m_{f_0(1790)}^2 e^{i\theta_3}}{m_{f_0(1790)}^2 - m_{\pi\pi}^2 - i m_{f_0(1790)} \Gamma_{f_0(1790)}(m_{\pi\pi}^2)} \]

\[ F_s^{u\bar{u}}(m_{\pi\pi}^2) = \frac{c_0 m_{f_0(500)}^2}{m_{f_0(500)}^2 - m_{\pi\pi}^2 - i m_{f_0(500)} \Gamma_{f_0(500)}(m_{\pi\pi}^2)}. \]

For the contribution of \( f_0(980) \), the Flatté model has been used, and the phase space factors \( \rho_{\pi\pi} \) and \( \rho_{KK} \) are given as [45]

\[ \rho_{\pi\pi} = \frac{2}{3} \sqrt{1 - \frac{4m_{\pi\pi}^2}{m_{\pi\pi}^2}} + \frac{2}{3} \sqrt{1 - \frac{4m_{\pi\pi}^2}{m_{\pi\pi}^2}}, \quad \rho_{KK} = \frac{1}{2} \sqrt{1 - \frac{4m_{KK}^2}{m_{\pi\pi}^2}} + \frac{1}{2} \sqrt{1 - \frac{4m_{KK}^2}{m_{\pi\pi}^2}}. \]

III. PERTURBATIVE CALCULATIONS

According to factorization theorems, the amplitude for the process can be calculated as an expansion of \( \alpha_s(Q) \) and \( \Lambda/Q \), \( Q \) denotes a large momentum transfer, and \( \Lambda \) is a small hadronic scale. Usually, the factorization formula for the nonleptonic b-meson decays can be expressed as

\[ A \sim \int_0^1 dx_1 dx_2 dx_3 \int d^2 b_1 d^2 b_2 d^2 b_3 C(t) \phi_B(x_1, b_1, t) H(x_1, x_2, x_3, b_1, b_2, b_3, t) \phi_2(x_2, b_2, t) \phi_3(x_3, b_3, t), \]

where the Wilson coefficients \( C(t) \), organizing the large logarithms from the hard gluon corrections, is described by the renormalization-group summation of QCD dynamics between W boson \( m_W \) and the typical scale \( t \). The hard kernel \( H(x_i, b_i, t) \), representing \( b \)-quark decay sub-amplitude, and the nonperturbative meson wave function \( \phi_i(x_i, b_i, t) \), describes the evolution from scale \( t \) to the lower hadronic scale \( \Lambda_{QCD} \). For a review of this approach, see Ref. [7].

The effective Hamiltonian for \( B_s^0 \to D^0(\bar{D}^0)^{\pi^+\pi^-} \) is given as

\[ \mathcal{H}_{eff} = \frac{G_F}{\sqrt{2}} V_{cb} V_{qs}(C_1 O_1 + C_2 O_2), \quad (Q = c, u, q = u, c). \]
with \( O_1 = (\vec{c}_a \vec{b}_b)_{V-A}(\vec{s}_\beta u_a)_{V-A}, \) \( O_2 = (\vec{c}_a \vec{b}_b)_{V-A}(\vec{s}_\beta \bar{u}_b)_{V-A} \) for the \( \bar{B}_s^0 \rightarrow D^0 \pi^+ \pi^- \) process, and \( O_1 = (\vec{u}_a \vec{b}_b)_{V-A}(\vec{s}_\beta c_a)_{V-A}, \) \( O_2 = (\vec{u}_a \vec{b}_b)_{V-A}(\vec{s}_\beta \bar{c}_b)_{V-A} \) for the process of \( \bar{B}_s^0 \rightarrow D^0 \pi^+ \pi^- \). In particular, the penguin operators do not contribute to the processes. Using the above effective Hamiltonian, we obtain the typical Feynman diagrams for the \( \bar{B}_s^0 \rightarrow D^0 \pi^+ \pi^- \) process shown in Fig. [1] in which the first row represents the color-suppressed emission process, and the second row indicates the W-exchange process. In the factorization framework, the factorizable diagrams in Fig. [1a,b,e,f] are relevant to \( a_2 \), and the non-factorizable diagrams in Fig. [1c,d,g,h] are proportional to \( C_2 \), where

\[
a_1 = C_2 + C_1/N_c, \quad a_2 = C_1 + C_2/N_c.
\]

We will work in the light-cone coordinates. The momentum of the mesons are defined as follows:

\[
P_{B_s} = (p_1^+, p_1^0, 0), \quad P_\pi = (p_2^+, 0, 0), \quad P_D = (p_1^+, p_2^+, m_{B_s}^2/(2p_1^+), 0).
\]

Accordingly, the transfer momentum and light-cone components can be achieved as \( q^2 = (P_{B_s} - P_\pi)^2 = (1 - \rho)m_{B_s}^2, \) \( \rho = 1 - \frac{m_0}{m_{B_s}}, \) \( p_1^- = m_{B_s}^2/(2p_1^+) \) and \( p_2^+ = (m_{B_s}^2 - q^2)p_1^+ / m_{B_s}^2. \) In the heavy quark limit, the mass difference of b-quark(c-quark) and \( B_s(D) \) meson is negligible, \( m_{B_s,D} = m_c + \Lambda \) (\( \Lambda \) is the order of QCD scale). Since \( m_{B_s} \gg m_D \gg \Lambda, \) we expand the amplitudes in terms of \( \frac{m_0}{m_{B_s}}, \frac{\Lambda}{m_{B_s}} \) and high order \( \frac{1}{m_{B_s}}. \) At the leading order of expansion, \( \rho \sim 1, q^2 \sim 0. \) The momenta of the light quark in mesons \( (k_1, k_2) \) represent the momentum of light quark in \( B_s \) and \( D \) meson, \( k_3 \) is the momentum of positive quark in pion-pair system are given as

\[
k_1 = (0, x_1 P^+_{B_s}, k_{1\perp}), \quad k_2 = (x_2 P^+_{\pi\pi}, 0, k_{2\perp}), \quad k_3 = (0, x_3 P^+_{D}, k_{3\perp}).
\]

In the \( k_T \)-factorization, the color-suppressed emission Feynman diagrams can be calculated out, with the formulas labelling as \( e_x(x=1,2,3,4) \) in subscript. Thus factorization formulas for the color-suppressed \( D^0 \)-emission diagrams are given as

\[
M_{e12} = \frac{8\pi C_F m_{B_s}^4 f_D}{\sqrt{2}N_c} \int\! dx_1 dx_2 \int\! dx_3 \int\! dx_4 \left[ b_1 b_2 b_3 b_4 \phi_D(x_1, b_1) \{ E_{e_1}(t_{e_1}) h_{e_1}(x_1, x_2, b_1, b_2, a_2(t_{e_1})) \right.
\]

\[
\left[ r_0 (1 - 2x_2) \phi_{\pi\pi}(s\bar{s}, x_2) + \frac{r_0}{s\bar{s}, \pi\pi}(x_2) \right] \left( 2 - x_2 \right) \phi_{\pi\pi}(s\bar{s}, x_2) - 2r_0 \phi_{\pi\pi}(s\bar{s}, x_2) E_{e_2}(t_{e_2}) h_{e_2}(x_1, x_2, b_1, b_2, a_2(t_{e_2})) \right] \}
\]

\[
M_{e34} = \frac{32\pi C_F m_{B_s}^4}{\sqrt{2}N_c} \int\! dx_1 dx_2 dx_3 \int\! dx_4 \left[ b_1 b_2 b_3 b_4 \phi_D(x_1, b_1) \phi_D(x_3, b_3) C_2(t_{e_3}) \right.
\]

\[
\left[ E_{e_3}(t_{e_3}) h_{e_3}(x_1, x_3, b_3, b_1) \{ r_0 \bar{e}_2 \phi_{\pi\pi}(s\bar{s}, x_2) + \phi_{\pi\pi}(s\bar{s}, x_2) x_3 \phi_{\pi\pi}(s\bar{s}, x_2) \} \right]
\]

\[
- E_{e_4}(t_{e_4}) h_{e_4}(x_1, x_2, b_2, b_3) \{ r_0 \bar{e}_2 \phi_{\pi\pi}(s\bar{s}, x_2) - \phi_{\pi\pi}(s\bar{s}, x_2) x_3 + \bar{e}_2 \phi_{\pi\pi}(s\bar{s}, x_2) \} \}
\]

where \( r_0 = \frac{m_0}{m_{B_s}}, C_F \) is the color factor. \( \phi_{\pi\pi}(s\bar{s}, x_2) \) represents the two-pion distribution amplitude defined by \( s\bar{s} \) operator. The hard kernels \( E_{e_x} \) and \( h_{e_x} \) are given in the following.

The factorization formulas for the W-exchange \( D^0 \) diagrams \( M_{w12} \) and \( M_{w34} \) are given as

\[
M_{w12} = \frac{8\pi C_F m_{B_s}^4 f_D}{\sqrt{2}N_c} \int\! dx_1 dx_2 dx_3 \int\! dx_4 \left[ b_2 b_3 b_4 \phi_D(x_3, b_3) \{ E_{w_1}(t_{w_1}) h_{w_1}(x_2, x_3, b_2, b_3, a_2(t_{w_1})) \right.
\]

\[
\left[ x_3 \phi_{\pi\pi}(u\bar{u}, x_2) + 2r_0 r_D(x_3 + 1) \phi_{\pi\pi}(u\bar{u}, x_2) - x_2 \phi_{\pi\pi}(u\bar{u}, x_2) + r_0 r_D(2x_2 + 2) \phi_{\pi\pi}(u\bar{u}, x_2) \right]
\]

\[
+ r_0 r_D(2x_2 + 2) \phi_{\pi\pi}(u\bar{u}, x_2) E_{w_2}(t_{w_2}) h_{w_2}(x_2, x_3, b_3, b_3, a_2(t_{w_2})) \}
\]

\[
M_{w34} = \frac{32\pi C_F m_{B_s}^4}{\sqrt{2}N_c} \int\! dx_1 dx_2 dx_3 \int\! dx_4 \left[ b_1 b_2 b_3 b_4 \phi_D(x_1, b_1) \phi_D(x_3, b_3) \{ E_{w_3}(t_{w_3}) h_{w_3}(x_1, x_2, x_3, b_1, b_2, C_2(t_{w_3})) \right.
\]

\[
\left[ x_2 \phi_{\pi\pi}(u\bar{u}, x_2) + r_0 r_D(x_2 + x_3) \phi_{\pi\pi}(u\bar{u}, x_2) + r_0 r_D(x_2 - x_3) \phi_{\pi\pi}(u\bar{u}, x_2) + x_3 \phi_{\pi\pi}(u\bar{u}, x_2) \right]
\]

\[
- r_0 r_D(x_2 + x_3 + 2) \phi_{\pi\pi}(u\bar{u}, x_2) + r_0 r_D(x_2 - x_3) \phi_{\pi\pi}(u\bar{u}, x_2) E_{w_4}(t_{w_4}) h_{w_4}(x_1, x_2, x_3, b_1, b_2, C_2(t_{w_4})) \}
\]

where \( r_D = \frac{m_0}{m_{B_s}}, \phi_{\pi\pi}(u\bar{u}, x_2) \) represents the distribution amplitude of the \( u\bar{u} \) operator. Due to the helicity suppression, the contribution of factorizable diagrams \( M_{w12} \) is suppressed significantly. Therefore, the dominant contribution comes from the non-factorizable diagrams \( M_{w34}. \)
In the $D^0$-emission process, the two factorization diagrams have the same factorization $\mathcal{M}_{12} = \mathcal{M}_{12}'$. Accordingly, we give the factorization formulas for the nonfactorizable emission diagrams $\mathcal{M}_{12}'$, the factorizable W-exchange diagrams $\mathcal{M}_{12}'$, and the nonfactorizable W-exchange diagrams $\mathcal{M}_{12}'$ as follows:

\[
\mathcal{M}_{12}' = \frac{32\pi C_F m_B^4}{\sqrt{2N_c}} \int_0^1 dx_1 dx_2 dx_3 \int_0^{1/\Lambda} b_1 db_1 b_3 db_3 \phi_B(x_1, b_1) \phi_D(x_3, b_3) \{E_{\epsilon_1}^3(t_{\epsilon_1}) h_{\epsilon_1}(x_1, x_2, x_3, b_1, b_3) C_2(t_{\epsilon_1})[r_0(x_2) \phi_{\pi\pi}^\ast(s_s, x_2) + \phi_{\pi\pi}^T(s_s, x_2)] + x_3 \phi_{\pi\pi}(s_s, x_2)
\]

\[
- E_{\epsilon_2}^3(t_{\epsilon_2}) h_{\epsilon_2}(x_1, x_2, x_3, b_1, b_3) C_2(t_{\epsilon_2})[r_0(x_2) \phi_{\pi\pi}^\ast(s_s, x_2) - \phi_{\pi\pi}^T(s_s, x_2)] + (\bar{x}_2 + \bar{x}_3) \phi_{\pi\pi}(s_s, x_2),\}
\]

\[
\mathcal{M}_{12}' = 8\pi C_F m_B^4 \int_0^1 dx_1 dx_2 dx_3 \int_0^{1/\Lambda} b_2 db_2 db_3 \phi_D(x_3, b_3) \{E_{\epsilon_1}^3(t_{\epsilon_1}) h_{\epsilon_1}(x_2, x_3, b_2) a_2(t_{\epsilon_1}) \}
\]

\[
[(1 - x_2) \phi_{\pi\pi}(u\bar{u}, x_2) + r_0 r_D(2x_2 - 3) \phi_{\pi\pi}^\ast(u\bar{u}, x_2) + r_0 r_D(1 - 2x_2) \phi_{\pi\pi}^T(u\bar{u}, x_2)]
\]

\[
+ [-x_3 \phi_{\pi\pi}(u\bar{u}, x_2) + 2r_0 r_D(3x_3 - 1) \phi_{\pi\pi}^\ast(u\bar{u}, x_2)] \phi_{\pi\pi}(u\bar{u}, x_2),
\]

\[
\mathcal{M}_{12}' = \frac{32\pi C_F m_B^4}{\sqrt{2N_c}} \int_0^1 dx_1 dx_2 dx_3 \int_0^{1/\Lambda} b_1 db_1 b_2 db_2 \phi_B(x_1, b_1) \phi_D(x_2, b_2) \{E_{\epsilon_1}^3(t_{\epsilon_1}) h_{\epsilon_1}(x_1, x_2, x_3, b_1, b_2) C_2(t_{\epsilon_1})
\]

\[
[x_3 \phi_{\pi\pi}(u\bar{u}, x_2) - r_0 r_D(1 - x_2 + x_3) \phi_{\pi\pi}^\ast(u\bar{u}, x_2) + r_0 r_D(x_2 - 3 - 1) \phi_{\pi\pi}^T(u\bar{u}, x_2)] + [(x_2 - 1) \phi_{\pi\pi}(u\bar{u}, x_2)
\]

\[
+ r_0 r_D(x_2 - 3 - 1) \phi_{\pi\pi}^\ast(u\bar{u}, x_2) + r_0 r_D(x_2 - 3 - 1) \phi_{\pi\pi}^T(u\bar{u}, x_2) \phi_{\pi\pi}(u\bar{u}, x_2) \}
\]

\[
\]

In the following, we give the forms for the offshellness of the intermediate gluon $\beta_{e_1}/\beta_{w_1}$ and quarks $\alpha_{e_1}/\alpha_{w_1}$ ($x = 1, 2, 3, 4$) in the $B_s^0 \rightarrow D^0\pi^+\pi^-$ process.

\[
\alpha_{e_1} = (1 - x_2) m_B^2, \beta_{e_1} = x_1 m_B^2, \alpha_{e_2} = (1 - x_2) m_B^2, \alpha_{e_4} = x_1 - x_3 m_B^2, \beta_{e_4} = (1 - x_2)(1 - x_3)m_B^2, \beta_{e_5} = (1 - x_2)(1 - x_3)m_B^2, \beta_{e_6} = x_1 - x_3 m_B^2.
\]

For the $B_s^0 \rightarrow D^0\pi^+\pi^-$, we have

\[
\alpha_{e_1}' = (1 - x_2) m_B^2, \beta_{e_1}' = x_1 m_B^2, \alpha_{e_2}' = (1 - x_2) m_B^2, \beta_{e_2}' = x_1 - x_3 m_B^2, \beta_{e_4}' = (1 - x_2)(1 - x_3)m_B^2, \beta_{e_5}' = (1 - x_2)(1 - x_3)m_B^2, \beta_{e_6}' = x_1 - x_3 m_B^2.
\]

The hard kernel functions $h_{e_1}(h_{e_1}')$ and $h_{w_1}(h_{w_1}')$ are written as

\[
h_{e_1}(x_1, x_2, b_1, b_2) = \theta(b_1 - b_2)I_0(\sqrt{\alpha_{e_1} b_1})K_0(\sqrt{\beta_{e_1} b_1}) + (b_1 \leftrightarrow b_2)K_0(\sqrt{\beta_{e_1} b_1})S_t(\alpha_{e_1}/(m_B^2, b_1)),
\]

\[
h_{e_2}(x_1, x_2, x_3, b_1, b_3) = \theta(b_1 - b_3)I_0(\sqrt{\alpha_{e_2} b_3})K_0(\sqrt{\beta_{e_2} b_1}) + (b_1 \leftrightarrow b_3) \times \left\{ \begin{array}{ll}
K_0(\sqrt{\beta_{e_2} b_1}), & \beta_{e_2} \geq 0, \\
\frac{i}{2}H_0^{(1)}(\sqrt{\beta_{e_2} b_1}), & \beta_{e_2} < 0,
\end{array} \right.
\]

\[
h_{w_1}(x_1, x_2, b_2, b_3) = \left( \frac{\pi}{2} \right)^2 H_0^{(1)}(\sqrt{\alpha_{w_1} b_2})H_0^{(1)}(\sqrt{\alpha_{w_1} b_3}) + (b_2 \leftrightarrow b_3)S_t(\alpha_{w_1}/(m_B^2, b_1)),
\]
FIG. 1: Typical Feynman diagrams for the three-body decays $B_S^0 \rightarrow D^0 (\bar{D}^0) \pi^+ \pi^-$. For the three-body process, the operators in quark level are $\mathcal{O}_1, \mathcal{O}_2$, which correspond with two kinds of Feynman diagrams: the color-suppressed and the W-exchange. The color-suppressed diagrams are drawn in panels (a-d) and (a'-d'), further more, the W-exchange diagrams are shown in (e-h) and (e'-h').

$$h_{w_1}(x_1, x_2, x_3, b_1, b_2) = \frac{\pi}{2} \left[ \theta(b_1 - b_2) H_0^{(1)}(\sqrt{\alpha_{w_1} b_1}) J_0(\sqrt{\alpha_{w_2} b_2}) + (b_1 \leftrightarrow b_2) \right] \times \begin{cases} K_0(\sqrt{\beta_{w_1} b_1}), & \beta_{w_1} \leq 0, \\ \frac{\pi}{2} H_0^{(1)}(\sqrt{\beta_{w_1} b_1}), & \beta_{w_1} > 0, \end{cases}$$

where $i, k = 1, 2$ and $j, l = 3, 4$, the $I_0$, $K_0$ and $H_0 = J_0 + iY_0$ are Bessel functions. The threshold resummation factor $S_t(x)$ follows the parametrization as

$$S_t(x) = \frac{2^{1+2c} \Gamma(3/2+c)}{\sqrt{\pi} \Gamma(1+c)} [x(1-x)]^c,$$

with the parameter $c = 0.4$ in this paper. The evolution factors $E_x(t)$s in the factorization formulas are given by

$$E_{c_1}(t) = \alpha_s(t) \exp(-S_{B_1}(t) - S_{\pi\pi}(t)),$$

$$E_{c_2}(t) = \alpha_s(t) \exp(-S_{B_2}(t) - S_{\pi\pi}(t) - S_D(t))|_{b_1=b_2},$$

$$E_{w_1}(t) = \alpha_s(t) \exp(-S_{\pi\pi}(t) - S_D(t))|_{b_2=b_3},$$

$$E_{w_2}(t) = \alpha_s(t) \exp(-S_{B_1}(t) - S_{\pi\pi}(t) - S_D(t))|_{b_1=b_2},$$

$$E_{w_3}(t) = \alpha_s(t) \exp(-S_{B_2}(t) - S_{\pi\pi}(t) - S_D(t))|_{b_2=b_3}. $$

(23)

(24)

(25)
where

\[ S_{B_s}(t) = s(x_1 m_{B_s}, b_1) + \frac{5}{3} \int_{1/b_1}^{t} \frac{d\mu}{\mu} \gamma_q(\alpha_s(\mu)), \]

\[ S_D(t) = s(x_3 m_{B_s}, b_3) + 2 \int_{1/b_3}^{t} \frac{d\mu}{\mu} \gamma_q(\alpha_s(\mu)), \]

\[ S_{\pi\pi}(t) = s(x_2 m_{B_s}, b_2) + s((1 - x_2) m_{B_s}, b_2) + 2 \int_{1/b_2}^{t} \frac{d\mu}{\mu} \gamma_q(\alpha_s(\mu)), \] (26)

with the quark anomalous dimension \( \gamma_q = -\alpha_s/\pi \). The explicit expression of \( s(Q, b) \) can be found, for example, in Appendix A of Ref [9]. The hard scales are chosen as

\[ t_{e_i} = \max(\sqrt{\alpha_{e_i}}, \sqrt{\beta_{e_i}}, 1/b_1, 1/b_2), \quad t_{e_j} = \max(\sqrt{\alpha_{e_j}}, \sqrt{\beta_{e_j}}, 1/b_1, 1/b_3), \]

\[ t_{w_k} = \max(\sqrt{\alpha_{w_k}}, \sqrt{\beta_{w_k}}, 1/b_2, 1/b_3), \quad t_{w_l} = \max(\sqrt{\alpha_{w_l}}, \sqrt{\beta_{w_l}}, 1/b_1, 1/b_2). \] (27)

Therefore, we obtain the total decay amplitudes,

\[ \mathcal{A}(\bar{B}_s \to D^0 \pi^+ \pi^-) = \frac{G_F}{\sqrt{2}} V_{cb} V_{us}^* (\mathcal{M}_{c12} + \mathcal{M}_{c34} + \mathcal{M}_{w12} + \mathcal{M}_{w34}), \]

\[ \mathcal{A}(\bar{B}_s \to \bar{D}^0 \pi^+ \pi^-) = \frac{G_F}{\sqrt{2}} V_{cb} V_{us}^* (\mathcal{M}_{c'12} + \mathcal{M}_{c'34} + \mathcal{M}_{w'12} + \mathcal{M}_{w'34}). \] (28)

The differential branching ratio for the \( \bar{B}_s^0 \to D^0(\bar{D}^0) \pi^+ \pi^- \) decay follows the formula given as

\[ \frac{d\mathcal{B}}{dm_{\pi\pi}} = \tau_{B_s} \frac{m_{\pi\pi} |\vec{p}_1| |\vec{p}_2|}{4(2\pi)^3 m_{B_s}^3} |A|^2, \] (29)

with the \( B_s \) meson mean lifetime \( \tau_{B_s} \). The kinematic variables \( |\vec{p}_1| \) and \( |\vec{p}_2| \) denote the magnitudes of the \( \pi^+ \) and \( D \) momenta in the center-of-mass frame of the pion pair,

\[ |\vec{p}_1| = \frac{1}{2} \sqrt{m_{\pi\pi}^2 - 4m_{\pi\pi}^2}, \quad |\vec{p}_2| = \frac{1}{2} \sqrt{m_{\pi\pi}^2 - (m_{\pi\pi} + m_D)^2} |(m_{B_s}^2 - (m_{\pi\pi} - m_D)^2). \] (30)

### IV. NUMERICAL RESULTS

We adopt the following inputs (in units of GeV) \[52, 53\]

\[ \Lambda_{MS}^{f=4} = 0.250, \quad m_{B_s} = 5.367, \quad m_{D^0} = 1.869, \quad m_{\pi^0} = 0.140, \quad m_{\pi^0} = 0.135, \quad m_{K^0} = 0.494, \]

\[ m_{K^0} = 0.498, \quad m_b = 4.66, \quad m_s = 0.095, \quad \tau_{B_s} = 1.512 \times 10^{-12} s, \quad G_F = 1.166 \times 10^{-5}, \]

and the CKM matrix elements are taken as:

\[ |V_{us}| = 0.2252, \quad |V_{ub}| = 3.89 \times 10^{-3}, \quad |V_{cs}| = 0.97345, \quad |V_{cb}| = 40.6 \times 10^{-3}. \]

The parameters for the scalar form factor \( F_s(m_{\pi\pi}^2) \) are extracted from the LHCb data in the process of \( B_s \to J/\psi \pi^+ \pi^- \), given as \[44, 54\] (mass and widths are given in units of GeV):

\[ m(f_0(500)) = 0.5, \quad m(f_0(980)) = 0.97, \quad m(f_0(1500)) = 1.5, \quad m(f_0(1790)) = 1.81, \]

\[ \Gamma(f_0(500)) = 0.4, \quad \Gamma(f_0(1500)) = 0.12, \quad \Gamma(f_0(1790)) = 0.32, \]

\[ g_{\pi\pi} = 0.167, \quad g_{KK} = 3.478g_{\pi\pi}, \]

\[ c_0 = 3.500, \quad c_1 = 0.900, \quad c_2 = 0.106, \quad c_3 = 0.066, \]
TABLE I: Branching ratios from the different intermediate resonances.

| Resonances | Branching ratio ($\times 10^{-6}$) |
|------------|-----------------------------------|
| $\bar{B}^0_s \to D^0 f_0(500)[f_0(500) \to \pi^+\pi^-]$ | $(0.14)_{-0.01}^{+0.05}(\omega_b)+0.22(a_2)+0.05(\Lambda_{QCD})$ |
| $\bar{B}^0_s \to D^0 f_0(980)[f_0(980) \to \pi^+\pi^-]$ | $(0.52)_{-0.13}^{+0.14}(\omega_b)+0.69(a_2)+0.10(\Lambda_{QCD})$ |
| $\bar{B}^0_s \to D^0 f_0(1500)[f_0(1500) \to \pi^+\pi^-]$ | $(0.13)_{-0.02}^{+0.04}(\omega_b)+0.09(a_2)+0.03(\Lambda_{QCD})$ |
| $\bar{B}^0_s \to D^0 f_0(1790)[f_0(1790) \to \pi^+\pi^-]$ | $(0.039)_{-0.01}^{+0.03}(\omega_b)+0.19(a_2)+0.08(\Lambda_{QCD})$ |
| $\bar{B}^0_s \to D^0 f_0(500)[f_0(500) \to \pi^+\pi^-]$ | $(0.13)_{-0.05}^{+0.05}(\omega_b)+0.23(a_2)+0.03(\Lambda_{QCD})$ |
| $\bar{B}^0_s \to D^0 f_0(980)[f_0(980) \to \pi^+\pi^-]$ | $(0.19)_{-0.12}^{+0.20}(\omega_b)+0.13(a_2)+0.01(\Lambda_{QCD})$ |
| $\bar{B}^0_s \to D^0 f_0(1500)[f_0(1500) \to \pi^+\pi^-]$ | $(0.044)_{-0.025}^{+0.019}(\omega_b)+0.035(a_2)+0.007(\Lambda_{QCD})$ |
| $\bar{B}^0_s \to D^0 f_0(1790)[f_0(1790) \to \pi^+\pi^-]$ | $(0.013)_{-0.009}^{+0.005}(\omega_b)+0.009(a_2)+0.009(\Lambda_{QCD})$ |

We calculate the branching ratios with the different resonances in S-wave pion-pair function shown in Tab. I. In this table, the first uncertainties are from $\omega_b = 0.50 \pm 0.05$ in the $B_s$ wave function, the second errors arise from $a_2 = 0.2 \pm 0.2$ in the pion-pair wave function, and the third uncertainties come from QCD scale $\Lambda = 0.25 \pm 0.05$. The errors from the parameter of $D$-meson function $C_D$, the variations of CKM matrix elements and the mean lifetime of $B_s$ are tiny, and have been omitted. However the above results are sensitive to $\omega_b$ and $a_2$, namely the $B_s$ and S-wave two-pion wave functions. The future measurements of decay branching fractions will be valuable to understand the $B_s$ physics and the S-wave two-pion resonances.

Including all the S-wave resonances $f_0(500)$, $f_0(980)$, $f_0(1500)$ and $f_0(1790)$ in the scalar form factor, we obtain the total branching ratio

$$B(\bar{B}^0_s \to D^0(\pi^+\pi^-)_S) = (0.87)_{-0.20}^{+0.20}(\omega_b)+1.30(a_2)+0.13(\Lambda_{QCD}) \times 10^{-6}.$$  

$$B(\bar{B}^0_s \to D^0(\pi^+\pi^-)_S) = (0.53)_{-0.18}^{+0.20}(\omega_b)+0.66(a_2)+0.09(\Lambda_{QCD}) \times 10^{-6}.$$  

(31)

We found the $\bar{B}^0_s \to D^0 f_0(500)[f_0(500) \to \pi^+\pi^-]$, $\bar{B}^0_s \to D^0 f_0(980)[f_0(980) \to \pi^+\pi^-]$, $\bar{B}^0_s \to D^0 f_0(1500)[f_0(1500) \to \pi^+\pi^-]$ and $\bar{B}^0_s \to D^0 f_0(1790)[f_0(1790) \to \pi^+\pi^-]$ contributions to be 16.4%, 59.3%, 14.6% and 4.5% of the total $\bar{B}^0_s \to D^0(\pi^+\pi^-)_S$ decay rate. For the $\bar{B}^0_s \to D^0(\pi^+\pi^-)_S$ process, the corresponding rates are 24.6%, 35.2%, 8.3% and 2.4% respectively. It indicates that the $f_0(500)$ and $f_0(980)$ contributions are dominant, and the contribution from $f_0(980)$ is larger than $f_0(500)$ in $D^0(\bar{D})^0$ final state. LHCb collaboration measures the branching ratio with the upper limit of $B(B_s \to D^0 f_0(980)) < 3.1 \times 10^{-6}$, which roughly agrees with our value.

For the comparison of $\bar{B}^0_s \to D^0(\pi\pi)_S$ and $\bar{B}^0_s \to D^0(\pi\pi)_S$, we determine the rate of their branching ratios

$$R_1 = \frac{B(\bar{B}^0_s \to D^0(\pi^+\pi^-)_S)}{B(\bar{B}^0_s \to D^0(\pi^+\pi^-)_S)} \sim 1.64,$$  

(32)

with the quite different CKM ratio factor

$$R_{CKM} = \left| \frac{V_{cb}V_{us}^*}{V_{ub}V_{cs}^*} \right| \sim 5.83.$$  

(33)

The CKM elements of $\bar{B}^0_s \to D^0(\bar{D}^0(\pi^+\pi^-)_S)$ is $V_{cb}V_{us}^*(V_{ub}V_{cs}^*)$, in which $V_{ub}$ is sensitive to the $\gamma$. Therefore, we can achieve the dependence of our results about $\gamma$, by providing a parameter $D_{CP\pm}$ defined as

$$\sqrt{2}A(\bar{B}^0_s \to D_{CP\pm}(\pi^+\pi^-)_S) = A(\bar{B}^0_s \to D^0(\pi^+\pi^-)_S) \pm A(\bar{B}^0_s \to D^0(\pi^+\pi^-)_S).$$  

(34)

Accordingly, the dependence curve of branching ratio $B(\bar{B}^0_s \to D_{CP\pm}(\pi^+\pi^-)_S)$ on $\gamma$ is obtained in Fig. 2a,b. In experimentally side, the corresponding physical observable measurement is defined as

$$R_{CP\pm} = \frac{4B(\bar{B}^0_s \to D_{CP\pm}(\pi^+\pi^-)_S)}{B(\bar{B}^0_s \to D^0(\pi^+\pi^-)_S) + B(\bar{B}^0_s \to D^0(\pi^+\pi^-)_S)}.$$  

(35)
We give the dependencies of $R_{CP \pm}$ on $\gamma$ shown in Fig. 2(c,d). The current bound on $\gamma$ is constrained as $\gamma = (73.5^{+4.2}_{-5.9})^\circ$.

The predicted dependencies of the differential branching ratios $d\mathcal{B}/dm_{\pi\pi}$ on the pion-pair invariant mass $m_{\pi\pi}$ are presented in Fig. 3(a) and Fig. 3(b) for the resonances $f_0(500)$, $f_0(980)$, $f_0(1500)$ and $f_0(1790)$ in the $\bar{B}_s \to D^0 \pi^+ \pi^-$ and $B_s \to D^0 \pi^+ \pi^-$ decay. The graphs show that the main contribution of the two decays lies in the region around the pole mass $m_{f_0(980)} = 0.97$, while the $f_0(500)$ lead to the primary contribution below the region $m_{\pi\pi} = 1\,GeV$. The other resonances $f_0(1500)$ and $f_0(1790)$ still give the considerable contributions to the processes. Therefore, we expect that more precise data from the LHCb and the future KEKB may test our theoretical calculations.

V. CONCLUSIONS

In the past decades, two-body $B$ decays have provided an ideal platform to extract the standard model parameters, and probe the new physics beyond the SM. In this work, we have studied the three-body $\bar{B}_s \to D^0(\bar{D}^0)\pi^+ \pi^-$ decay within the PQCD framework, and in particular the S-wave contribution is explicitly calculated. The S-wave two-pion light-cone distribution amplitudes can receive both resonant $f_0(500)$, $f_0(980)$, $f_0(1500)$, $f_0(1790)$ and non-resonant contributions. Furthermore, the processes proceed via the tree level operators, and branching ratios are found in the range from $10^{-7}$ to $10^{-6}$. It is found that the branching ratios are sensitive to the parameters $\omega_b$ and $a_2$, in the $B_s$ and two-pion distribution amplitudes. Therefore, we expect that the future measurement can help us...
FIG. 3: The differential branching ratios on the pion-pair invariant mass for the resonance $f_0(980)$, $f_0(1500)$ and $f_0(1790)$ in the (a) $\bar{B}_s^0 \to D^0\pi^+\pi^-$ and (b) $\bar{B}_s^0 \to \bar{D}^0\pi^+\pi^-$ decays.

better understanding the multi-body processes, and S-wave two-pion resonance and $B_s$ distribution amplitudes.

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