LOW MACH NUMBER LIMIT OF A PRESSURE CORRECTION MAC SCHEME FOR COMPRESSIBLE BAROTROPIC FLOWS

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ABSTRACT. We study the incompressible limit of a pressure correction MAC scheme [3] for the unstationary compressible barotropic Navier-Stokes equations. Provided the initial data are well-prepared, the solution of the numerical scheme converges, as the Mach number tends to zero, towards the solution of the classical pressure correction \( \inf\)-\( \sup \) stable MAC scheme for the incompressible Navier-Stokes equations.

1. Introduction

Let \( \Omega \) be parallelepiped of \( \mathbb{R}^d \), with \( d \in \{2, 3\} \) and \( T > 0 \). The unsteady barotropic compressible Navier-Stokes equations, parametrized by the Mach number \( \varepsilon \), read for \((x, t) \in \Omega \times (0, T)\):

\[
\begin{align*}
\partial_t \rho^\varepsilon + \text{div}(\rho^\varepsilon \mathbf{u}^\varepsilon) &= 0, \\
\partial_t (\rho^\varepsilon \mathbf{u}^\varepsilon) + \text{div}(\rho^\varepsilon \mathbf{u}^\varepsilon \otimes \mathbf{u}^\varepsilon) - \text{div}(\mathbf{\tau}(\mathbf{u}^\varepsilon)) + \frac{1}{\varepsilon^2} \nabla \psi(\rho^\varepsilon) &= 0, \\
\mathbf{u}^\varepsilon \big|_{\partial \Omega} &= 0, \quad \rho^\varepsilon \big|_{t=0} = \rho_0^\varepsilon, \quad \mathbf{u}^\varepsilon \big|_{t=0} = \mathbf{u}_0^\varepsilon,
\end{align*}
\]

where \( \rho^\varepsilon > 0 \) and \( \mathbf{u}^\varepsilon = (u_1^\varepsilon, \ldots, u_d^\varepsilon)^T \) are the density and velocity of the fluid. The pressure satisfies the ideal gas law \( \psi(\rho^\varepsilon) = (\rho^\varepsilon)^\gamma \), with \( \gamma \geq 1 \), and

\[
\text{div}(\mathbf{\tau}(\mathbf{u})) = \mu \Delta \mathbf{u} + (\mu + \lambda) \nabla (\text{div} \mathbf{u}),
\]

where the real numbers \( \mu \) and \( \lambda \) satisfy \( \mu > 0 \) and \( \mu + \lambda > 0 \). The smooth solutions of (1) are known to satisfy a kinetic energy balance and a renormalization identity. In addition, under assumption on the initial data, it may be inferred from these estimates that the density \( \rho^\varepsilon \) tends to a constant \( \bar{\rho} \), and the velocity tends, in a sense to be defined, to a solution \( \bar{\mathbf{u}} \) of the incompressible Navier-Stokes equations [4]:

\[
\begin{align*}
div \bar{\mathbf{u}} &= 0, \\
\bar{\rho} \partial_t \bar{\mathbf{u}} + \bar{\rho} \text{div}(\bar{\mathbf{u}} \otimes \bar{\mathbf{u}}) - \mu \Delta \bar{\mathbf{u}} + \nabla \pi &= 0,
\end{align*}
\]

where \( \pi \) is the formal limit of \((\psi(\rho^\varepsilon) - \psi(\rho))/\varepsilon^2\).

In this paper, we reproduce this theory for a pressure correction scheme, based on the Marker-And-Cell (MAC) space discretization: we first derive discrete analogues of the kinetic energy and renormalization identities, then establish from these relations that approximate solutions of (1) converge, as \( \varepsilon \to 0 \), towards the solution of the classical projection scheme for the incompressible Navier-Stokes equations (2).

For this asymptotic analysis, we assume that the initial data is “well prepared”: \( \rho_0^\varepsilon > 0, \rho_0^\varepsilon \in L^\infty(\Omega), \mathbf{u}_0^\varepsilon \in H^1_0(\Omega)^d \) and, taking without loss of generality \( \bar{\rho} = 1 \), there exists \( C \) independent of \( \varepsilon \) such that:

\[
\|\mathbf{u}_0^\varepsilon\|_{H^1(\Omega)^d} + \frac{1}{\varepsilon} \|\text{div} \mathbf{u}_0^\varepsilon\|_{L^2(\Omega)} + \frac{1}{\varepsilon^2} \|\rho_0^\varepsilon - 1\|_{L^\infty(\Omega)} \leq C.
\]

Consequently, \( \rho_0^\varepsilon \) tends to 1 when \( \varepsilon \to 0 \); moreover, we suppose that \( \mathbf{u}_0^\varepsilon \) converges in \( L^2(\Omega)^d \) towards a function \( \bar{\mathbf{u}} \in L^2(\Omega)^d \) (the uniform boundedness of the sequence in the \( H^1(\Omega)^d \) norm already implies this convergence up to a subsequence).
primal cells: $K$, $L$.

dual cell for the $y$-component of the velocity: $D_\sigma = D_{K,\sigma} \cup D_{L,\sigma}$.

primal and dual cells $d$-dimensional measures: $|K|$, $|D_\sigma|$, $|D_{K,\sigma}|$.

faces $(d-1)$-dimensional measures: $|\sigma|$, $|\varepsilon|$.

vector normal to $\sigma$ outward $K$: $n_{K,\sigma}$.

**Figure 1.** Notations for control volumes and faces.

### 2. The numerical scheme

Let $\mathcal{M}$ be a MAC mesh (see e.g. [1] and Figure 1 for the notations). The discrete density unknowns are associated with the cells of the mesh $\mathcal{M}$, and are denoted by $\{\rho_K, K \in \mathcal{M}\}$. We denote by $\mathcal{E}$ the set of the faces of the mesh, and by $\mathcal{E}^{(i)}$ the subset of the faces orthogonal to the $i$-th vector of the canonical basis of $\mathbb{R}^d$. The discrete $i$-th component of the velocity is located at the centre of the faces $\sigma \in \mathcal{E}^{(i)}$, so the whole set of discrete velocity unknowns reads $\{u_{\sigma,i}, \sigma \in \mathcal{E}^{(i)}, 1 \leq i \leq d\}$. We define $\mathcal{E}_{ext} = \{\sigma \in \mathcal{E}, \sigma \subset \partial \Omega\}$, $\mathcal{E}_{int} = \mathcal{E} \setminus \mathcal{E}_{ext}$, $\mathcal{E}^{(i)}_{int} = \mathcal{E}_{int} \cap \mathcal{E}^{(i)}$ and $\mathcal{E}^{(i)}_{ext} = \mathcal{E}_{ext} \cap \mathcal{E}^{(i)}$. The boundary conditions (1c) are taken into account by setting $u_{\sigma,i} = 0$ for all $\sigma \in \mathcal{E}^{(i)}_{ext}$, $1 \leq i \leq d$. Let $\delta t > 0$ be a constant time step. The approximate solution $(\rho^n, u^n)$ at time $t_n = n\delta t$ for $1 \leq n \leq N = \lfloor T/\delta t \rfloor$ is computed as follows: knowing $(\rho^{n-1}_K, u^{n-1})_{K \in \mathcal{M}} \subset \mathbb{R}$ and $(u_{\sigma,i}^n)_{\sigma \in \mathcal{E}^{(i)}_{int}, 1 \leq i \leq d} \subset \mathbb{R}$, find $(\rho^{n+1}_K)_{K \in \mathcal{M}} \subset \mathbb{R}$ and $(u_{\sigma,i}^{n+1})_{\sigma \in \mathcal{E}^{(i)}_{int}, 1 \leq i \leq d} \subset \mathbb{R}$ by the following algorithm:

**Pressure gradient scaling step:**

(4a) For $1 \leq i \leq d$, $\forall \sigma \in \mathcal{E}^{(i)}_{int}$, $(\nabla p)^n_{\sigma,i} = \left(\frac{\rho^n_{\sigma,i}}{\rho^n_{\sigma,i} \cdot 1/2}\right)(\nabla p^n)_{\sigma,i}$.

**Prediction step** – Solve for $\tilde{u}^{n+1}$:

(4b) For $1 \leq i \leq d$, $\forall \sigma \in \mathcal{E}^{(i)}_{int}$,

\[
\frac{1}{\delta t} \left(\rho^n_{\sigma,i} \tilde{u}_{\sigma,i}^{n+1} - \rho^{n-1}_n u_{\sigma,i}^n\right) + \text{div}(\rho^n \tilde{u}_{\sigma,i}^{n+1} u^n)_{\sigma} - \text{div} F(\tilde{u}^{n+1})_{\sigma,i} + \frac{1}{\varepsilon^2} (\nabla p^n)_{\sigma,i} = 0.
\]

**Correction step** – Solve for $\rho^{n+1}$ and $u^{n+1}$:

(4c) For $1 \leq i \leq d$, $\forall \sigma \in \mathcal{E}^{(i)}_{int}$,

\[
\frac{1}{\delta t} \left(\rho^n_{\sigma,i} u_{\sigma,i}^{n+1} - \rho^{n+1}_n u_{\sigma,i}^{n+1}\right) + \frac{1}{\varepsilon^2} (\nabla p^{n+1})_{\sigma,i} - \frac{1}{\varepsilon^2} (\nabla p^n)_{\sigma,i} = 0,
\]

(4d) $\forall K \in \mathcal{M}$,

\[
\frac{1}{\delta t} (\rho^{n+1}_K - \rho^n_K) + \text{div}(\rho^{n+1} u^{n+1})_{K} = 0,
\]

(4e) $\forall K \in \mathcal{M}$,

\[
\rho^{n+1}_K = \varphi(\rho^{n+1}_K),
\]

where the discrete densities and space operators are defined below (see also [3, 2]).

**Mass convection flux** – Given a discrete density field $\rho = \{\rho_K, K \in \mathcal{M}\}$, and a velocity field $u = \{u_{\sigma,i}, \sigma \in \mathcal{E}^{(i)}, 1 \leq i \leq d\}$, the convection term in (4d) reads:

(5) $\text{div}(\rho u)_K = \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}(K)} F_{K,\sigma}(\rho, u), \quad K \in \mathcal{M},$

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where $F_{K,\sigma}(\rho, u)$ stands for the mass flux across $\sigma$ outward $K$. This flux is set to 0 on external faces to account for the homogeneous Dirichlet boundary conditions; it is given on internal faces by:

\begin{equation}
F_{K,\sigma}(\rho, u) = |\sigma| \rho_{\sigma} u_{K,\sigma}, \quad \sigma \in \mathcal{E}_{\text{int}}, \sigma = K|L,
\end{equation}

where $u_{K,\sigma} = u_{\sigma,i} n_{K,\sigma} \cdot e^{(i)}$, with $e^{(i)}$ the $i$-th vector of the orthonormal basis of $\mathbb{R}^d$. The density at the face $\sigma = K|L$ is approximated by the upwind technique, i.e. $\rho_{\sigma} = \rho_K$ if $u_{K,\sigma} \geq 0$ and $\rho_{\sigma} = \rho_L$ otherwise.

**Pressure gradient term** – In (4a) and (4e), the term $(\nabla p)_{\sigma,i}$ stands for the $i$-th component of the discrete pressure gradient at the face $\sigma$. Given a discrete density field $\rho = \{\rho_K, K \in \mathcal{M}\}$, this term is defined as:

\begin{equation}
(\nabla p)_{\sigma,i} = \frac{|\sigma|}{|D_{\sigma}|} (p(\rho_L) - p(\rho_K)) n_{K,\sigma} \cdot e^{(i)}, \quad 1 \leq i \leq d, \quad \sigma \in \mathcal{E}_{\text{int}}^{(i)}, \sigma = K|L.
\end{equation}

Defining for all $K \in \mathcal{M}$, $(\nabla p)_{K} = \text{div}(1 \times u)_K$ (see (5)), the following discrete duality relation holds for all discrete density and velocity fields $(\rho, u)$:

\begin{equation}
\sum_{K \in \mathcal{M}} |K| p_K (\nabla u)_K + \sum_{i=1}^{d} \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} |D_{\sigma}| u_{\sigma,i} (\nabla p)_{\sigma,i} = 0.
\end{equation}

The MAC scheme is $\text{inf-sup}$ stable: there exists $\beta > 0$, depending only on $\Omega$ and the regularity of the mesh, such that, for all $p = \{p_K, K \in \mathcal{M}\}$, there exists $u = \{u_{\sigma,i}, \sigma \in \mathcal{E}^{(i)}, 1 \leq i \leq d\}$ satisfying homogeneous Dirichlet boundary conditions with:

$$
\|u\|_{1,M} = 1 \quad \text{and} \quad \sum_{K \in \mathcal{M}} |K| p_K (\nabla u)_K \geq \beta \|p - \frac{1}{|\Omega|} \int_{\Omega} p \, d\Omega\|_{L^2(\Omega)},
$$

where $\|u\|_{1,M}$ is the usual discrete $H^1$-norm of $u$ (see [1]).

**Velocity convection operator** – Given a density field $\rho = \{\rho_K, K \in \mathcal{M}\}$, and two velocity fields $u = \{u_{\sigma,i}, \sigma \in \mathcal{E}^{(i)}, 1 \leq i \leq d\}$ and $v = \{v_{\sigma,i}, \sigma \in \mathcal{E}^{(i)}, 1 \leq i \leq d\}$, we build for each $\sigma \in \mathcal{E}_{\text{int}}$ the following quantities:

- an approximation of the density on the dual cell $\rho_{D_{\sigma}}$, defined as:

\begin{equation}
|D_{\sigma}| \rho_{D_{\sigma}} = |D_{K,\sigma}| \rho_K + |D_{L,\sigma}| \rho_L, \quad \sigma \in \mathcal{E}_{\text{int}}, \sigma = K|L,
\end{equation}

- a discrete divergence for the convection on the dual cell $D_{\sigma}$:

\begin{equation}
\text{div}(\rho v_{\sigma,i})_{\sigma} = \sum_{\varepsilon \in E(D_{\sigma})} F_{\sigma,\varepsilon}(\rho, u) v_{\varepsilon,i}, \quad \sigma \in \mathcal{E}_{\text{int}}^{(i)}, 1 \leq i \leq d.
\end{equation}

For $i \in \{1, \ldots, d\}$, and $\sigma \in \mathcal{E}_{\text{int}}^{(i)}, \sigma = K|L$,

- if the vector $e^{(i)}$ is normal to $\varepsilon$, $\varepsilon$ is included in a primal cell $K$, and we denote by $\varepsilon'$ the second face of $K$ which, in addition to $\sigma$, is normal to $e^{(i)}$. We thus have $\varepsilon = D_{\sigma}|D_{\varepsilon'}$. Then the mass flux through $\varepsilon$ is given by:

\begin{equation}
F_{\sigma,\varepsilon}(\rho, u) = \frac{1}{2} (F_{K,\sigma}(\rho, u) n_{D_{\sigma,\varepsilon}} \cdot n_{K,\sigma} + F_{K,\sigma'}(\rho, u) n_{D_{\sigma',\varepsilon}} \cdot n_{K,\sigma'}).
\end{equation}

- if the vector $e^{(i)}$ is tangent to $\varepsilon$, $\varepsilon$ is the union of the halves of two primal faces $\tau$ and $\tau'$ such that $\tau \in \mathcal{E}(K)$ and $\tau' \in \mathcal{E}(L)$. The mass flux through $\varepsilon$ is then given by:

\begin{equation}
F_{\sigma,\varepsilon}(\rho, u) = \frac{1}{2} (F_{K,\tau}(\rho, u) + F_{L,\tau'}(\rho, u)).
\end{equation}

With this definition, the dual fluxes are locally conservative through dual faces $\varepsilon = D_{\sigma}|D_{\varepsilon'}$ (i.e. $F_{\sigma,\varepsilon}(\rho, u) = -F_{\sigma',\varepsilon}(\rho, u)$), and vanish through a dual face included in the boundary of $\Omega$. For this reason, the values $v_{\varepsilon,i}$ are only needed at the internal dual faces, and are chosen centered, i.e., for $\varepsilon = D_{\sigma}|D_{\varepsilon'}$, $v_{\varepsilon,i} = (v_{\varepsilon,i} + v_{\varepsilon',i})/2$.

As a result, a finite volume discretization of the mass balance (1a) holds over the internal dual cells. Indeed, if $\rho^{n+1} = \{\rho_K^{n+1}, K \in \mathcal{M}\}$, $\rho^n = \{\rho_K^n, K \in \mathcal{M}\}$ and $u^{n+1} = \{u_{\sigma,i}^{n+1}, \sigma \in \mathcal{E}^{(i)}, 1 \leq i \leq d\}$ are density and velocity fields satisfying (4d), then, the dual quantities $\{\rho_{D_{\sigma}}^{n+1}, \rho_{D_{\sigma}}^n, \sigma \in \mathcal{E}_{\text{int}}\}$ and the dual fluxes
\{ F_{\sigma,\varepsilon}(\rho^{n+1}, u^{n+1}), \sigma \in E_{\text{int}}, \varepsilon \in \mathcal{E}(D_{\sigma}) \} satisfy a finite volume discretization of the mass balance (1a) over the internal dual cells:

\begin{equation}
\frac{|D_{\sigma}|}{\delta t} (\rho_{D_{\sigma}}^{n} - \rho_{D_{\sigma}}^{n}) + \sum_{\varepsilon \in \mathcal{E}(D_{\sigma})} F_{\sigma,\varepsilon}(\rho^{n+1}, u^{n+1}) = 0, \quad \sigma \in E_{\text{int}}.
\end{equation}

**Diffusion term** – The discrete diffusion term in (4b) is defined in [2] and is coercive in the following sense: for every discrete velocity field \( u \) satisfying the homogeneous Dirichlet boundary conditions, one has:

\begin{equation}
- \sum_{i=1}^{d} \sum_{\varepsilon \in E_{\text{int}}} |D_{\sigma}| u_{\sigma,i} \text{div}\tau(u_{\sigma,i}) \geq \mu \| u \|_{1,\Sigma}^{2}.
\end{equation}

The initialization of the scheme (4) is performed by setting

\[
\forall K \in M, \rho_{K}^{0} = \frac{1}{|K|} \int_{K} \rho_{0}^{0}(x) \, dx \quad \text{and} \quad \forall \sigma \in E_{\text{int}}^{(i)}, 1 \leq i \leq d, u_{\sigma,i}^{0} = \frac{1}{|\sigma|} \int_{\sigma} u_{\sigma}^{0}(x) \cdot e^{(i)} \, dx,
\]

and computing \( \rho^{-1} \) by solving the backward mass balance equation (4d) for \( n = -1 \) where the unknown is \( \rho^{-1} \) and not \( \rho^{0} \). This allows to perform the first prediction step with \( \{ \rho_{D_{\sigma}}, \rho_{E_{\sigma}}^{0}, \sigma \in E_{\text{int}} \} \) and the dual mass fluxes \( \{ F_{\sigma,\varepsilon}(\rho^{0}, u^{0}), \sigma \in E_{\text{int}}, \varepsilon \in \mathcal{E}(D_{\sigma}) \} \) satisfying the mass balance (12). Moreover, since \( \rho_{0}^{0} > 0 \), one clearly has \( \rho_{K}^{0} > 0 \) for all \( K \in M \) and therefore \( \rho_{D_{\sigma}}^{0} > 0 \) for all \( \sigma \in E_{\text{int}} \). The positivity of \( \rho^{-1} \) is a consequence of the following Lemma.

**Lemma 2.1.** If \( (\rho_{0}^{0}, u_{0}^{0}) \) satisfies (3), then there exists \( C \), depending on the mesh but independent of \( \varepsilon \) such that:

\begin{equation}
\frac{1}{\varepsilon^{2}} \max_{K \in M} |\rho_{K}^{0} - 1| + \frac{\varepsilon^{2}}{1} \max_{1 \leq i \leq d, \sigma \in E_{\text{int}}^{(i)}} |(\nabla p)_{\sigma,i}^{0}| + \frac{1}{\varepsilon^{1}} \max_{K \in M} |\rho_{K}^{-1} - 1| \leq C.
\end{equation}

**Proof.** We sketch the proof. The boundedness of the first two terms is a straightforward consequence of (3).

For the third term that we remark, again by (3):

\[
\forall K \in M, \quad \rho_{K}^{-1} - 1 = \frac{\rho_{K}^{0} - 1}{\varepsilon^{2}} + \frac{\varepsilon^{2}}{1} (\text{div}^{u})_{K}^{0} + \frac{\varepsilon^{2}}{1} \sum_{\sigma \in E(K)} |\sigma| \frac{|(\rho^{0} - \rho_{K}^{0})u_{K,\sigma}^{0}|}{|K|}.
\]

\[\square\]

3. **Asymptotic analysis of the zero Mach limit**

By the results of [3], there exists a solution \( (\rho^{n}, u^{n})_{0 \leq n \leq N} \) to the scheme (4) and any solution satisfies the following relations:

- a discrete kinetic energy balance: for all \( \sigma \in E_{\text{int}}^{(i)}, 1 \leq i \leq d, 0 \leq n \leq N - 1:

\begin{equation}
\frac{1}{2\delta t} \left( \rho_{D_{\sigma}}^{n} |u_{\sigma,i}^{n+1}|^{2} - \rho_{D_{\sigma}}^{n-1} |u_{\sigma,i}^{n}|^{2} \right) + \frac{1}{2|D_{\sigma}|} \sum_{\varepsilon \in \mathcal{E}(D_{\sigma})} F_{\sigma,\varepsilon}(\rho^{n}, u^{n}) u_{\sigma,i}^{n+1} u_{\sigma,i}^{n+1} - \text{div}\tau(u_{\sigma,i}^{n+1}) + \frac{1}{\varepsilon^{2}} (|\nabla p|_{\sigma,i}^{n+1} u_{\sigma,i}^{n+1} + \frac{\varepsilon^{2}}{4} \left( \frac{|\nabla p|_{\sigma,i}^{n+1}|^{2}}{2\rho_{D_{\sigma}}^{n}} - \frac{|\nabla p|_{\sigma,i}^{n}|^{2}}{2\rho_{D_{\sigma}}^{n-1}} \right) + R_{\sigma,i}^{n+1} = 0,
\end{equation}

with

\[R_{\sigma,i}^{n+1} = \frac{1}{2\delta t} (\rho_{D_{\sigma}}^{n-1} (\bar{u}_{\sigma,i}^{n+1} - u_{\sigma,i}^{n})^{2},
\]

- a discrete renormalization identity: for all \( K \in M, 0 \leq n \leq N - 1:

\begin{equation}
\frac{1}{\delta t} \left( \Pi_{\gamma}(\rho_{K}^{n+1}) - \Pi_{\gamma}(\rho_{K}^{n}) \right) + \text{div} \left( b_{\gamma} (\rho^{n+1}) u^{n+1} - b_{\gamma}^{0} (1) \rho^{n+1} u^{n+1} \right) + b_{\gamma}^{0} (1) \rho^{n+1} \text{div}(u^{n+1}) + R_{K}^{n+1} = 0,
\end{equation}

with \( R_{K}^{n+1} \geq 0 \), where the function \( b_{\gamma} \) is defined by \( b_{\gamma}(\rho) = \rho \log \rho \) if \( \gamma = 1 \), \( b_{\gamma}(\rho) = \rho^{2}/(\gamma - 1) \) if \( \gamma > 1 \) and satisfies \( \rho b_{\gamma}'(\rho) - b_{\gamma}(\rho) = \rho^{\gamma} = \psi(\rho) \) for all \( \rho > 0 \), and \( \Pi_{\gamma}(\rho) = b_{\gamma}(\rho) - b_{\gamma}(1) - b_{\gamma}'(1)(\rho - 1) \).
Summing (15) and (16) over the primal cells from one side, and over the dual cells and the components on the other side, and invoking the grad-div duality relation (8), we obtain a local-in-time discrete entropy inequality, for $0 \leq n \leq N - 1$:

$$
\frac{1}{2} \sum_{i=1}^{d} \sum_{\sigma \in \mathcal{E}^{(i)}_{in}} |D_{\sigma} (\rho_{\sigma,i}^n - \rho_{\sigma,i}^{n-1})^2| + \frac{1}{\varepsilon} \sum_{K \in M} |K| (\Pi_{\gamma}(\rho_{K}^{n+1}) - \Pi_{\gamma}(\rho_{K}^{n}))
+ \mu \delta t \|	ilde{u}^n\|_{1,M}^2 + \frac{1}{\varepsilon} \sum_{i=1}^{d} \sum_{\sigma \in \mathcal{E}^{(i)}_{in}} |D_{\sigma} \delta t^2 \left( \frac{|(\nabla \rho)^{n+1}_{\sigma,i}|^2}{2 \rho_{\sigma,i}^{n+1}} - \frac{|(\nabla \rho)^{n}_{\sigma,i}|^2}{2 \rho_{\sigma,i}^{n}} \right) + R^{n+1} \leq 0
$$

where $R^{n+1} = \sum_{i=1}^{d} \sum_{\sigma \in \mathcal{E}^{(i)}_{in}} R_{\sigma,i}^{n+1} + \frac{1}{\varepsilon^2} \sum_{K \in M} R_{\sigma,i}^{n+1} \geq 0$.

The function $\Pi_{\gamma}$ has some important properties:

(18a) For all $\gamma \geq 1$ there exists $C_{\gamma}$ such that: $\Pi_{\gamma}(\rho) \leq C_{\gamma} |\rho - 1|^2$, $\forall \rho \in (0, 2)$.

(18b) If $\gamma \geq 2$ then $\Pi_{\gamma}(\rho) \geq |\rho - 1|^2$, $\forall \rho > 0$.

If $\gamma \in [1, 2)$ then for all $R \in (2, +\infty)$, there exists $C_{\gamma,R}$ such that:

(18c) $\Pi_{\gamma}(\rho) \geq C_{\gamma,R} |\rho - 1|^2$, $\forall \rho \in (0, R)$.

$\Pi_{\gamma}(\rho) \geq C_{\gamma,R} |\rho - 1|^2$, $\forall \rho \in [R, \infty)$.

Lemma 3.1 (Global discrete entropy inequality). Under assumption (3), there exists $C_0 > 0$ independent of $\varepsilon$ such that the solution $(\rho^n, u^n)_{0 \leq n \leq N}$ to the scheme (4) satisfies, for $\varepsilon$ small enough, and for $1 \leq n \leq N$:

$$
\frac{1}{2} \sum_{i=1}^{d} \sum_{\sigma \in \mathcal{E}^{(i)}_{in}} |D_{\sigma} |\rho_{\sigma,i}^{n-1}|^2 |u_{\sigma,i}^n|^2 + \mu \sum_{k=1}^{n} \delta t \|	ilde{u}^n\|^2_{1,M} + \frac{1}{\varepsilon} \sum_{K \in M} |K| \Pi_{\gamma}(\rho_{K}^n) + \frac{1}{\varepsilon} \sum_{i=1}^{d} \sum_{\sigma \in \mathcal{E}^{(i)}_{in}} \frac{|D_{\sigma} \delta t^2 |(\nabla \rho)^{n}_{\sigma,i}|^2}{2 \rho_{\sigma,i}^{n+1}} \leq C_0.
$$

Proof. Summing (17) over $n$ yields the inequality (19) with

$$
C_0 = \frac{1}{2} \sum_{i=1}^{d} \sum_{\sigma \in \mathcal{E}^{(i)}_{in}} |D_{\sigma} |\rho_{\sigma,i}^{n-1}|^2 |u_{\sigma,i}^n|^2 + \frac{1}{\varepsilon^2} \sum_{K \in M} |K| \Pi_{\gamma}(\rho_{K}^n) + \frac{1}{\varepsilon} \sum_{i=1}^{d} \sum_{\sigma \in \mathcal{E}^{(i)}_{in}} \frac{|D_{\sigma} \delta t^2 |(\nabla \rho)^{n}_{\sigma,i}|^2}{2 \rho_{\sigma,i}^{n+1}}.
$$

By (14), for $\varepsilon$ small enough, one has $\rho_{K}^{n-1} \leq 2$ for all $K \in M$ and therefore $\rho_{\sigma,i}^{n-1} \leq 2$ for all $\sigma \in \mathcal{E}^{(i)}_{in}$ and $1 \leq i \leq d$. Hence, since $u_0^n$ is uniformly bounded in $H^1(\Omega)^d$ by (3), a classical trace inequality yields the boundedness of the first term. Again by (14), one has $|\rho_{K}^{n} | - 1 \leq C \varepsilon^2$ for all $K \in M$. Hence, by (18a), the second term vanishes as $\varepsilon \to 0$. The third term is also uniformly bounded with respect to $\varepsilon$ thanks to (14).

Lemma 3.2 (Control of the pressure). Assume that $(\rho_0^n, u_0^n)$ satisfies (3) and let $(\rho^n, u^n)_{0 \leq n \leq N}$ satisfy (4). Let $p^n = \varphi(\rho^n)$ and define $\delta p^n = \{\delta p^n_K, K \in M\}$ where $\delta p^n_K = (p^n_K - |\Omega|^{-1} \int_{\Omega} p^n \mathrm{d}x) / \varepsilon^2$. Then, one has, for all $1 \leq n \leq N$:

$$
\|\delta p^n\| \leq C_{M, \delta t},
$$

where $C_{M, \delta t} \geq 0$ depends on the mesh and $\delta t$ but not on $\varepsilon$, and $\| \cdot \|$ stands for any norm on the space of discrete functions.

Proof. By (19), the discrete pressure gradient is controlled in $L^\infty_{M, \delta t} \varepsilon^2$, so that $\nabla (\delta p^n)$ is bounded in any norm independently of $\varepsilon$. Using the discrete $(H^{-1})^d$-norm (see e.g. [1]), invoking the gradient divergence duality (8) and the inf-sup stability of the scheme, $\|\nabla (\delta p^n)\|_{-1,M} \leq C_{M, \delta t}$ implies that $\|\delta p^n\|_{L^2} \leq \beta^{-1} C_{M, \delta t}$. □
Theorem 3.3 (Incompressible limit of the MAC pressure correction scheme).
Let $(\varepsilon^{(m)})_{m \in \mathbb{N}}$ be a sequence of positive real numbers tending to zero, and let $(\rho^{(m)}, \mathbf{u}^{(m)})$ be a corresponding sequence of solutions of the scheme (4). Then the sequence $(\rho^{(m)})_{m \in \mathbb{N}}$ converges to the constant function $\rho = 1$ when $m$ tends to $+\infty$ in $L^\infty((0,T), L^q(\Omega))$, for all $q \in [1, \min(\gamma,2)]$.

In addition, the sequence $(\mathbf{u}^{(m)}, \delta p^{(m)})_{m \in \mathbb{N}}$ tends, in any discrete norm, to the solution $(\mathbf{u}, \delta p)$ of the usual MAC pressure correction scheme for the incompressible Navier-Stokes equations, which reads:

**Prediction step – Solve for $\tilde{\mathbf{u}}^{n+1}$:****

For $1 \leq i \leq d$, $\forall \sigma \in \mathcal{E}_i^{(i)}$, \[
\frac{1}{\delta t} (\tilde{u}_{\sigma,i}^{n+1} - u_{\sigma,i}^n) + \text{div}(\tilde{u}_{\sigma,i}^{n+1} \mathbf{u}^n) - \text{div}\tau(\tilde{u}_{\sigma,i}^{n+1}) + (\nabla(\delta p)^n)_{\sigma,i} = 0.
\]

**Correction step – Solve for $(\delta p)^{n+1}$ and $\mathbf{u}^{n+1}$:**

For $1 \leq i \leq d$, $\forall \sigma \in \mathcal{E}_i^{(i)}$, $\forall K \in \mathcal{M}$, \[
\frac{1}{\delta t} (u_{\sigma,i}^{n+1} - u_{\sigma,i}^n) + (\nabla(\delta p)^{n+1})_{\sigma,i} - (\nabla(\delta p)^n)_{\sigma,i} = 0,
\]
and estimate (19).

Proof. By (18b) and the global entropy estimate (19), one has for $\gamma \geq 2$, \[
\|\rho^{(m)}(t) - 1\|_{L^2(\Omega)} \leq \int_0^t \Pi_\gamma(\rho^{(m)}(t)) \leq C_0 \varepsilon^2, \forall t \in (0,T).
\]

For $1 \leq \gamma \leq 2$, invoking (18c) and estimate (19), we obtain for all $t \in (0,T)$ and for all $R \in (2, +\infty)$:

(i) \[
\|\rho^{(m)}(t) - 1\|_{L^2(\Omega)} \leq \frac{1}{C_{\gamma,R}} \int_0^t \Pi_\gamma(\rho^{(m)}(t)) \leq C \varepsilon^2, \forall t \in (0,T),
\]

(ii) \[
\gamma \|\rho^{(m)}(t) - 1\|_{L^2(\Omega)} \leq \frac{1}{C_{\gamma,R}} \int_0^t \Pi_\gamma(\rho^{(m)}(t)) \leq C \varepsilon^2, \forall t \in (0,T),
\]
which proves the convergence of $(\rho^{(m)})_{m \in \mathbb{N}}$ to the constant function $\rho = 1$ as $m \to +\infty$ in $L^\infty((0,T), L^q(\Omega))$ for all $q \in [1, \min(\gamma,2)]$. Using again (19), the sequence $(\mathbf{u}^{(m)})_{m \in \mathbb{N}}$ is bounded in any discrete norm and the same holds for the sequence $(\delta p^{(m)})_{m \in \mathbb{N}}$ by Lemma 3.2. By the Bolzano-Weierstrass theorem and a norm equivalence argument, there exists a subsequence of $(\mathbf{u}^{(m)}, \delta p^{(m)})_{m \in \mathbb{N}}$ which tends, in any discrete norm, to a limit $(\mathbf{u}, \delta p)$. Passing to the limit cell-by-cell in (4), one obtains that $(\mathbf{u}, \delta p)$ is a solution to (21). Since this solution is unique, the whole sequence converges, which concludes the proof. \qed

References

[1] T. Gallouët, R. Herbin, J.-C. Latché, and K Mallem. Convergence of the Marker-And-Cell scheme for the incompressible Navier-Stokes equations on non-uniform grids. *Found Comput Math*, 2016.

[2] D. Grapsas, R. Herbin, W. Kheriji, and J.-C. Latché. An unconditionally stable staggered pressure correction scheme for the compressible Navier-Stokes equations. *SMAI-JCM*, 2:51–97, 2016.

[3] R. Herbin, W. Kheriji, and J.-C. Latché. On some implicit and semi-implicit staggered schemes for the shallow water and Euler equations. *M2AN*, 48:1807–1857, 2014.

[4] P.-L. Lions and N. Masmoudi. Incompressible limit for a viscous compressible fluid. *Journal de Mathématiques Pures et Appliquées*, 77:595–627, 1998.