Some Comments On Lie-Poisson Structure Of Conformal Non-Abelian Thirring Models

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Abstract

The interconnection between self-duality, conformal invariance and Lie-Poisson structure of the two dimensional non-abelian Thirring model is investigated in the framework of the hamiltonian method.

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1 Introduction

In the present paper we will discuss the interconnection between the Siegel symmetry [1], self-duality, conformal invariance and Lie-Poisson structure of the two dimensional non-abelian Thirring model [2]. Our motivations in considering the Thirring model come from string theory. It has been observed that a given conformal Thirring model should correspond to a certain compactification in string theory [3,4]. Therefore, the space of all conformal Thirring models seems to be a good candidate to describe the space of all symmetric string vacua, which could form the space of conformal backgrounds appropriate to the formulation of background independent string field theory [5].

The remarkable universality of Thirring models originates from the invariance of these theories under the symmetry first introduced by W. Siegel [1]. The Siegel symmetry is a crucial property of self-dual fields in two dimensions [1]. We will show that self-duality provides some clues in understanding the geometrical quantization of the non-abelian conformal Thirring models.

2 The equivalence between non-abelian fermionic and bosonic Thirring models

One of the manifestations of universality in the Thirring model is the equivalence between its fermionic and bosonic formulations [2]. The fermionic Thirring model action is given by

$$S_F = \int d^2 x (\bar{\psi}_L \partial \psi_L + \bar{\psi}_R \partial \psi_R + S_{a\bar{a}} J^a_L J^{\bar{a}}_R),$$

(1)

where $\psi_L$ and $\psi_R$ are Weyl spinors (in general carrying a flavor) transforming as the fundamental representations of given groups $G_L$ and $G_R$ respectively. The last term in (1) describes the general interaction between fermionic currents $J^a_L = \bar{\psi}_L t^a \psi_L$, $J^{\bar{a}}_R = \bar{\psi}_R t^{\bar{a}} \psi_R$, where $t^a$, $t^{\bar{a}}$ are the generators in the Lie algebras $G_L$, $G_R$. $S_{a\bar{a}}$ is a coupling constant matrix.

The action of the bosonic Thirring model is formulated as follows

$$S_B = \int [L_L(k_L, g_L) + L_R(k_R, g_R) + L_{int}(g_L, g_R; S)],$$

(2)
where these three terms respectively are given by

\[
4\pi L_L(k_L, g_L) = -k_L[(1/2) \text{tr}_L g_L^{-1} dg_L|^2 + (i/3)d^{-1} \text{tr}_L (g_L^{-1} dg_L)^3],
\]

\[
4\pi L_R(k_R, g_R) = -k_R[(1/2) \text{tr}_R g_R^{-1} dg_R|^2 + (i/3)d^{-1} \text{tr}_R (g_R^{-1} dg_R)^3],
\]

\[
L_{int}(g_L, g_R; S) = -(k_L k_R/4\pi) \text{tr}_L tr_R g_L^{-1} \partial g_L \cdot S \cdot d g_R g_R^{-1},
\]

with the coupling \( S \) belonging to the direct product \( \mathcal{G}_L \otimes \mathcal{G}_R \). Here the fields \( g_L \) and \( g_R \) take their values in the Lie groups \( G_L \) and \( G_R \), respectively, \( k_L, k_R \) are central elements in the affine algebras \( \hat{G}_L, \hat{G}_R \). The symbols \( \text{tr}_L, \text{tr}_R \) indicate tracing over the group indices of \( G_L, G_R \).

Classically the theories (1) and (2) are inequivalent, whatever conditions we may impose upon them. However, at the quantum level the fermionic and bosonic non-abelian Thirring models become indistinguishable under the following conditions: 1) the two Weyl spinors \( \psi_i^R \) and \( \bar{\psi}_i^L \) carry the flavor indices \( i = 1, ..., k_R \) and \( \bar{i} = 1, ..., k_L \); 2) the coupling constant matrix \( S \) in eqs. (1), (2) and (3) is reversible; 3) the fields \( g_L \) and \( g_R \) are left and right moving scalars respectively [2]. When these conditions are fulfilled the statistical sums of the two models are identical [2]

\[
\frac{Z_B(k_L, k_R; S/4\pi)}{Z_B(k_L, k_R; 0)} = \frac{Z_F(k_L, k_R; S)}{Z_F(k_L, k_R; 0)},
\]

where \( Z_B, Z_F \) are defined via usual partition functions of the given two dimensional models.

Apparently, in the limit \( S = 0 \) the identity (4) contains no useful information. It is not surprising because as we demonstrated in [6] in order to fermionize the WZNW models (or \( S = 0 \) Thirring model) with arbitrary levels, we have to use the fermionic Thirring model at the so-called isoscalar Dashen-Frishman conformal points, not at \( S_{ab} = 0 \). Meanwhile, when \( S \neq 0 \), the identity (4) is very fruitful since allows us to establish an equivalence between the conformal points of the fermionic and bosonic versions of the Thirring model as well as to clarify its geometrical meaning [7].
3 Conformal points of the Thirring model

There are considerable merits of Thirring models which make them especially interesting in finding the appropriate unification of both conformal field theories and massive integrable models. Therefore, it would be illuminating if one could explore the Thirring model at all the possible values of the couplings $S_{ab}$. However, this seems to be beyond our present analytical abilities. Most of the difficulty resides in the highly non-linear character of the current-current interaction of the Thirring theory. Given our present knowledge, the theory is tractable only when it possesses either affine symmetry or quantum group symmetry (which might turn out to be a sort of deformation of the former.) In this paper we will not discuss the quantum group symmetry of Thirring models but rather affine symmetries. We will show that affine symmetries are intimately related to the conformal invariance of the Thirring model. The non-abelian Thirring model has been shown to have at least two types of conformal points for different values of the Thirring coupling constants.

The conformal points belonging to the first type are called Higgs conformal points [8]. They may appear in the theory only when $k_L \neq k_R$. For example, in the simplest case $G_L = G_R = G$ and $S = \sigma t^a \otimes t^a$ conformal invariance holds at the following values of $\sigma$ [8]

$$\sigma_{n}^{L,R} = \left( \frac{k_L k_R}{(k_L + c_2(G))(k_R + c_2(G))} \right)^{-n} \sigma_0^{L,R},$$

(5)

where $\sigma_0^{L,R} = 1/k_{L,R}$; $c_2(G)$ is a quadratic Casimir operator eigenvalue referring to the adjoint representation of the group $G$; $n = 0, 1, 2, ..., \infty$. Interestingly, all Higgs conformal points share the same Virasoro central charges [8]

$$c(k_L, k_R, \sigma_n^L) = c(k_L, k_R, \sigma_0^L) = \left( \frac{k_L}{k_L + c_2(G)/2} + \frac{k_R - k_L}{k_R - k_L + c_2(G)/2} \right) \dim G,$$

$$c(k_L, k_R, \sigma_n^R) = c(k_L, k_R, \sigma_0^R) = \left( \frac{k_R}{k_R + c_2(G)/2} + \frac{k_L - k_R}{k_L - k_R + c_2(G)/2} \right) \dim G.$$  

(6)

Meanwhile, the second derivative of the $c$-function of Zamolodchikov [9] at the conformal points (5) varies from one conformal point to another and goes to zero in the limit $n \to \infty$ [8].
From the equivalence between the fermionic and bosonic statistical sums, the Higgs conformal points of the bosonic Thirring model should be the critical points in the fermionic theory. Note that it could be seen also by straightforward calculations of the fermionic partition functions at the given values of the coupling constants. However in the bosonic case the analysis is much simpler [8].

In its turn the fermionic formulation of the Thirring model turns out to be easier in finding the so-called Dashen-Frishman conformal points [6] which are the natural generalization of the isoscalar conformal points discovered by Dashen and Frishman two decades ago [10].

At the Dashen-Frishman conformal points the non-linear Thirring model can be quantized non-perturbatively in the framework of operator quantization [6]. The procedure amounts to quantizing the classical equation of motion

\[ \partial \psi_L = -2S_{ab}J^a_R t^b \psi_L. \]  

We have shown in [6] that the r.h.s of eq. (7) can be well defined at the quantum level if the coupling matrix \( S_{ab} \) satisfies the so-called Virasoro master equation [11]. This result being purely non-perturbative seems somewhat mysterious, because given the conformal points, we cannot employ even feeble perturbative arguments to justify the conformal invariance of the Thirring model at the quantum level. Therefore some more sophisticated arguments are required. We are going to demonstrate that the Dashen-Frishman conformal points are quite natural for the bosonic Thirring model despite the analysis being a little more cumbersome, compared to the fermionic version.

4 Dashen-Frishman conformal points and Lie-Poisson structure of the Thirring model

To clarify the nature of the Dashen-Frishman conformal points we have to consider more carefully the structure of the classical bosonic non-abelian Thirring model. The action of the bosonic Thirring model can be presented in the geometrical form [7]

\[ S_B = \int \alpha_L + \int \alpha_R + S_{\text{int}}, \]  

4
where \( \alpha_L \) and \( \alpha_R \) are canonical one-forms associated to the canonical symplectic structures on the orbits of the affine groups \( \hat{G}_L \) and \( \hat{G}_R \) respectively [12,13].

\[ d\alpha_L = \omega_L, \quad d\alpha_R = \omega_R. \quad (9) \]

The symplectic forms \( \omega_L \) and \( \omega_R \) define canonical variables and their Poisson brackets. We will show that the last term in eq. (8) is a Hamiltonian in the phase space with the symplectic forms given above.

To this end, we rewrite the actions for the free WZNW models in the first order form [15]

\[ A_L = -(1/4\gamma_L) \int tr_L[\partial_0 g_L g_L^{-1} J_{0L} - (1/2)(J_{0L}^2 + J_{1L}^2)]dxdt + WZ_L, \quad \gamma_L = \pi/k_L, \quad \] (10)

\[ A_R = -(1/4\gamma_R) \int tr_R[\partial_0 g_R g_R^{-1} J_{0R} - (1/2)(J_{0R}^2 + J_{1R}^2)]dxdt + WZ_R, \quad \gamma_R = \pi/k_R, \]

where \( WZ_{L,R} \) are WZ-terms, i.e. the second terms in the r.h.s. of \( L_{L,R} \) in (3). The WZ-terms are linear in \( \partial_0 g_{L,R} \) and therefore can be considered as functionals of \( g_L, g_R \) alone [15]. We have also used the notations

\[ J_{1L} = \partial_x g_L \cdot g_L^{-1}, \]

\[ J_{1R} = \partial_x g_R \cdot g_R^{-1}. \quad (11) \]

The variation of the canonical 1-forms in \( A_L, A_R \) leads us to the symplectic forms

\[ \Omega_L = (1/4\pi) \int tr_L(dg_L g_L^{-1} \wedge dJ_{0L} + J_{1L} dg_L g_L^{-1} \wedge dg_L g_L^{-1})dxdt, \]

\[ \Omega_R = (1/4\pi) \int tr_R(dg_R g_R^{-1} \wedge dJ_{0R} + J_{1R} dg_R g_R^{-1} \wedge dg_R g_R^{-1})dxdt. \quad (12) \]

These symplectic forms are a little different from those in (9). The difference is the same as between usual coordinates and light-cone coordinates. We can find the Poisson brackets for variables \( g_L, J_{0L}, g_R, J_{0R} \) by inverting \( \Omega_L, \Omega_R \). We find

\[ \{g_L^1(x), g_L^2(y)\} = 0, \]

\[ \{J_{0L}^1(x), g_L^2(y)\} = -2\gamma_L C_L g_L^2(y)\delta(x-y), \]

\[ \{J_{0L}^1(x), J_{0L}^2(y)\} = -\gamma_L [J_{0L}^1(x) - J_{1L}^1(x) - J_{0L}^2(x) + J_{1L}^2(x), C_L] \delta(x-y), \quad (13) \]
and similar ones for $g_R$, $J_{0R}$. Here $\{A^1(x), B^2(y)\}$ denotes the 2 dim $G_L \times 2$ dim $G_L$ matrix of all Poisson brackets of dim $G_L \times$ dim $G_L$ matrices $A$ and $B$, arranged in the same fashion, as in the product of matrices

$$A^1 = A \otimes I$$

and

$$B^2 = I \otimes B;$$

$C_L$ is a constant 2 dim $G_L \times 2$ dim $G_L$ matrix given by

$$C_L = \sum_a t^a \otimes t^a.$$  

Note that the actions $A_L$, $A_R$ are equivalent to the actions of the WZNW models upon use of the equations of motion for $J_{0L}$, $J_{0R}$. Therefore, the sets $(g_L, J_{0L})$ and $(g_R, J_{0R})$ describe the independent canonical variables in $A_L$, $A_R$. For our aims, however, it is more convenient to introduce new coordinates which are

$$g_L, \quad L = (1/2)g_L^{-1}(J_{0L} + J_{1L})g_L$$

eq (14)$$

for the $A_L$-theory, and

$$g_R, \quad R = (1/2)(J_{0R} - J_{1R})$$

eq (15)$$

for the $A_R$-theory. The Poisson brackets for the new variables follow from eqs. (12) and can be also obtained by inverting the symplectic forms in (9). In particular [15]

\[
\{L^1(x), L^2(y)\} = (\gamma_L/2)[C_L, L^1(x) - L^2(y)]\delta(x - y) + \gamma_L C_L \delta'(x - y),
\]

\[
\{R^1(x), R^2(y)\} = (\gamma_R/2)[C_R, R^1(x) - R^2(y)]\delta(x - y) + \gamma_R C_R \delta'(x - y).
\]

Now the interaction term in eq. (8) can be seen as a hamiltonian in the phase space of variables (14), (15)

$$S_{int} = \int dx^+ dx^- \mathcal{H},$$

with the Hamiltonian density

$$\mathcal{H} = -\frac{4\pi}{\gamma_L \gamma_R} \langle S, L \otimes R \rangle,$$
where \( S = \sum_{aa} S_{aa} t^a \otimes t^a \). The Hamiltonian \( H^+ = \int dx^- \mathcal{H} \) describes the evolution of the system in the \( x^+ \)-direction. The Hamiltonian equation for \( g_R \) is given by

\[
\partial_+ g_R + \left( 4\pi/\gamma_L \right) (\text{tr}_L S \cdot L) g_R = 0. \tag{19}
\]

Thus by solving this equation, we can express \( L \) in terms of \( g_R \). Due to the symmetry between \( g_L \) and \( g_R \), the equation for \( g_L \) has to be as follows

\[
\partial_- g_L + \left( 4\pi/\gamma_R \right) g_L (\text{tr}_R S \cdot R) = 0. \tag{20}
\]

In fact the last equation could be derived as a Hamiltonian equation with the Hamiltonian

\[
H^- = \int dx^+ \mathcal{H}. \tag{21}
\]

Hence, eqs. (19), (20) give rise to the explicit expressions for \( L \) and \( R \) in terms of \( g_R \) and \( g_L \). Thus, the action

\[
A = A_L + A_R + \int dx^+ dx^- \mathcal{H} \tag{22}
\]

becomes a functional of the fields \( g_L \) and \( g_R \) alone. Now it is not very difficult to see that the given functional coincides with the action of the bosonic Thirring model upon use of the Siegel constraints. A noteworthy fact is that the Siegel constraints similar to (19) and (20) appear in the Thirring model as the self-duality conditions [1, 14] via introduction of Lagrangian multipliers [1], whereas in the geometric formulation these same constraints appear as Hamiltonian equations of motion without any auxiliary fields.

Now we may try to quantize the Thirring model by the Hamiltonian method. The method will work as long as the algebraic Poisson structure will be preserved. In other words, to gain the advantage of geometric quantization, we have to promote the affine algebra Poisson brackets (16) to the quantum level. Certainly it will be the case, if the quantum fields \( g_L, g_R \) are elements of the representations of the affine algebras \( R, L \) respectively. As a byproduct the theory should be conformally invariant, since the Virasoro algebra belongs to the enveloping algebra of the affine algebra. It means that the Hamiltonian quantization of the Thirring model will be consistent as long as conformal symmetry will be present.
Therefore, for consistent quantization, we have to impose the following quantum equations of motion

\[
[L_{-1}, g_R] = -(4\pi/\gamma_L)tr_L : S \cdot L \cdot g_R :,
\]

\[
[\bar{L}_{-1}, g_L] = -(4\pi/\gamma_R)tr_R : g_L \cdot S \cdot R :,
\]

Here the double dots :: denote normal ordering between the currents \( L, R \) and their affine primary fields \( g_R, g_L \) respectively. The brackets [], are understood as quantum analogues of the classical Poisson brackets. The operators \( L_{-1} \) and \( \bar{L}_{-1} \) are to be the generators of translations. By definition

\[
L_{-1} = \oint \frac{dz}{2\pi i} T(z), \quad \bar{L}_{-1} = \oint \frac{d\bar{z}}{2\pi i} \bar{T}(\bar{z}),
\]

with \( T(z) \) and \( \bar{T}(\bar{z}) \) the holomorphic and antiholomorphic components of the energy-momentum tensor in the conformal Thirring model.

Let us suppose that \( \hat{G}_L = \hat{G}_R \). Then we can construct the following operators

\[
T(z) = L_{ab} \tilde{L}^a \tilde{L}^b, \quad \tilde{L} = (1/\gamma_L)L,
\]

\[
\bar{T}(\bar{z}) = L_{ab} \tilde{R}^a \tilde{R}^b, \quad \tilde{R} = (1/\gamma_R)R,
\]

which one can use to obtain equations (23). It is not hard to check that eqs. (23) are fulfilled with the given \( T \) and \( \bar{T} \) provided \( L_{ab} = 4\pi S_{ab} \). Moreover, to be components of the conformal energy-momentum tensor, the given operators must form two copies of the Virasoro algebra. In the full analogy with the fermionic Thirring model [6], we can prove that the operators \( T \) and \( \bar{T} \) form the Virasoro algebras if and only if the matrix \( L_{ab} \) satisfies the Virasoro master equation [11]. Thus, we have a one-to-one correspondence between the Dashen-Frishman conformal points of the fermionic Thirring model and the self-consistence conditions of the bosonic Thirring model.
5 Conclusion

In summary, the bosonic Thirring model at the Dashen-Frisman conformal points possesses the explicit affine symmetries realized by the Poisson algebras (16). These symmetries are inherited by the quantum theory from the Lie-Poisson algebra of the classical geometrical formulation. At the same time, due to the constraints (19), (20), we could expect certain algebraic structures for the operators $\partial g_R g_R^{-1}$ and $g_L^{-1}\partial g_L$ despite them not playing any role in the quantization. We might guess that these operators should be generators of quantum algebras with the $r$-matrices depending on $S_{ab}$. It is very tempting to suppose that among all the solutions of the Virasoro master equation there should be solutions to the Yang-Baxter equation. Then at the given conformal points we might find as a byproduct the realization of the quantum algebras in terms of Thirring models.

It is also amusing that at the Dashen-Frisman conformal points, the interaction term in the lagrangian becomes a truly marginal operator. Therefore it would be very interesting to investigate the flows between the underlying free WZNW models and the Thirring models at the Dashen-Frisman conformal points. Then we probably could understand the nature of the finite conformal deformations and $c > 1$ string models. In this respect, the flows between Dashen-Frisman conformal points and Higgs conformal points are also very important.

The hope is that the realization of this program might shed some more light on the space of two dimensional hamiltonian theories [16].

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