Differential and falsified sampling expansions

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Abstract
Differential and falsified sampling expansions $\sum_{k \in \mathbb{Z}^d} c_k \varphi(M^j x + k)$, where $M$ is a matrix dilation, are studied. In the case of differential expansions, $c_k = Lf(M^{-j} \cdot)(-k)$, where $L$ is an appropriate differential operator. For a large class of functions $\varphi$, the approximation order of differential expansions was recently studied. Some smoothness of the Fourier transform of $\varphi$ from this class is required. In the present paper, we obtain similar results for a class of band-limited functions $\varphi$ with the discontinuous Fourier transform. In the case of falsified expansions, $c_k$ is the mathematical expectation of random integral average of a signal $f$ near the point $M^{-j} k$. To estimate the approximation order of the falsified sampling expansions we compare them with the differential expansions. Error estimations in $L_p$-norm are given in terms of the Fourier transform of $f$.

Keywords differential expansion, falsified sampling expansion, approximation order, matrix dilation, Strang-Fix condition.

AMS Subject Classification: 41A58, 41A25, 41A63

1 Introduction

The well-known sampling theorem (Kotel’nikov’s or Shannon’s formula) states that
\begin{equation}
\widehat{f}(x) = \sum_{k \in \mathbb{Z}} f(-2^{-j} k) \text{sinc}(2^j x + k), \quad \text{sinc}(x) := \frac{\sin \pi x}{\pi x},
\end{equation}
for band-limited to $[-2^{j-1}, 2^{j-1}]$ signals (functions) $f$. This formula is very useful for engineers. It was just Kotel’nikov \textsuperscript{18} and Shannon \textsuperscript{24} who started to apply this formula for signal processing, respectively in 1933 and 1949. Up to now, an overwhelming diversity of digital signal processing applications and devices are based on it and more than successfully use it. However, mathematicians knew this formula much earlier, actually, it can be found in the papers by Ogura \textsuperscript{23} (1920), Whittaker \textsuperscript{32} (1915), Borel \textsuperscript{2} (1897), and even Cauchy \textsuperscript{9} (1841).

Equality (1) holds only for functions $f \in L_2(\mathbb{R})$ whose Fourier transform is supported on $[-2^{j-1}, 2^{j-1}]$. However the right hand side of (1) (the sampling expansion of $f$) has meaning for every continuous $f$ with a good enough decay. The problem of approximation of $f$ by its sampling

\footnotesize{*}This research was supported by Volkswagen Foundation; the first author is also supported by H2020-MSCA-RISE-2014 Project number 645672; the second and the third authors are also supported by grants from RFBR # 15-01-05796-a, St. Petersburg State University # 9.38.198.2015.
expansions as \( j \to +\infty \) was studied by many mathematicians. We mention only some of such results. Brown [5] proved that for every \( x \in \mathbb{R} \)

\[
\left| f(x) - \sum_{k \in \mathbb{Z}} f(-2^{-j}k) \text{sinc}(2^j x + k) \right| \leq C \int_{|\xi| > 2^{-j-1}} |\hat{f}(\xi)| d\xi, \tag{2}
\]

whenever the Fourier transform of \( f \) is summable on \( \mathbb{R} \). It is known that the pointwise approximation by sampling expansions does not hold for arbitrary continuous functions \( f \), even compactly supported. Moreover, Trynin [30] proved that there exists a continuous function vanishing outside of \( (0, \pi) \) such that its deviation from the sampling expansion diverges at every point \( x \in (0, \pi) \). Approximation by sampling expansions in \( L_p \)-norm was also actively studied. Bardaro, Butzer, Higgins, Stens, and Vinti [3], [2], proved that

\[
\Delta_p := \left\| f - \sum_{k \in \mathbb{Z}} f(-2^{-j}k) \text{sinc}(2^j \cdot +k) \right\|_{L_p(\mathbb{R})} \xrightarrow{j \to \infty} 0, \quad 1 \leq p < \infty,
\]

for \( f \in C \cap \Lambda_p \), where \( \Lambda_p \) consists of \( f \) such that

\[
\sum_{k \in \mathbb{Z}} |f(x_k)|^p (x_k - x_{k-1}) < \infty
\]

for some class of admissible partition \( \{x_k\}_{k \in \mathbb{Z}} \) of \( \mathbb{R} \). Also they proved that the Sobolev spaces \( W^r_p(\mathbb{R}) \), \( r \in \mathbb{N} \), are subspaces of \( \Lambda_p \), and that for every \( f \in W^r_p(\mathbb{R}) \)

\[
\Delta_p \leq C \omega(f^{(r)}, 2^{-j}p) 2^{-jr}, \tag{3}
\]

where \( \omega(\cdot)_p \) is the modulus of continuity in \( L_p(\mathbb{R}) \), which is a typical estimate for shift invariant spaces. In [4], the same authors studied a generalized sampling approximation, replacing the sinc-function by certain linear combinations of \( B \)-splines. For the case \( p = \infty \), Butzer, Ries, and Stens [8] proved that if a bounded function \( \varphi \) is such that

\[
\sup_{x \in \mathbb{R}} \sum_{k \in \mathbb{Z}} |x - k|^{r+1} |\varphi(x - k)| < \infty
\]

for some \( r \in \mathbb{N} \), then the estimate

\[
\sup_{x \in \mathbb{R}} \left| \sum_{k \in \mathbb{Z}} f \left( \frac{k}{W} \right) \varphi(W(t - k) - f(t)) \right| \leq CW^{-r} \omega \left( f^{(r)} \frac{1}{W} \right) \tag{4}
\]

holds for all \( f \in C^r(\mathbb{R}) \) and \( W > 0 \) if and only if the Strang-Fix condition of order \( r + 1 \) is satisfied for \( \varphi \). The minimal requirement for the convergence (in terms of Sobolev spaces) is \( f \in W^1_p(\mathbb{R}) \), and, by [3], the minimal approximation order is \( o(2^{-j}) \). Sickel [25], [26] studied error estimations for \( \Delta_p \) in terms of Besov-Triebel-Lizorkin spaces. In particular, his results provide the approximation order \( 2^{-j\gamma} \), \( 1/p \leq \gamma < 1 \), for some functions. The author of [27], [28] investigated approximation by sampling expansions

\[
\sum_{k \in \mathbb{Z}} f(-2^{-j}k) \varphi(2^j x + k)
\]

for a wide class of band-limited functions \( \varphi \). For \( p \geq 2 \), the error analysis was given in terms of the Fourier transform of \( f \). In particular, the approximation order \( 2^{-j(1/p + \alpha)} \) was found for functions \( f \) in Sobolev spaces \( W^1_p(\mathbb{R}) \) with \( f' \in \text{Lip}_{L_1}(\alpha, \alpha > 0 \). In the case \( 1/p + \alpha > 1 \) these estimates give
the same approximation order as (3). Similar results were obtained for the generalized sampling expansions (differential expansions)

\[ \sum_{k \in \mathbb{Z}} Lf(2^{-j}k)(-k)\varphi(2^j x + k), \]

where \( Lf := \sum_{l=0}^{m} \alpha_l f^{(l)}, \) and a function \( \varphi \) satisfies a special condition of compatibility with \( L. \) The approximation order depends on \( m \) in this case. An analog of Brown’s estimate (2) was proved for such expansions in [28].

Multivariate differential expansions

\[ \sum_{k \in \mathbb{Z}^d} Lf(M^{-j}k)(-k)\varphi(M^j x + k), \]

where \( M \) is a matric dilation, were studied in [20]. Error estimations in terms of the Fourier transform of the approximated function were given for a large class of functions \( \varphi. \) In particular, this class contains compactly supported functions, but it does not contain functions with discontinuous Fourier transform.

Differential expansions (5) may be useful for some problems and engineering applications. The analog of Kotelnikov’s formula (1) for differential expansions can be used to solve differential equations. The solution can be represented in analytic form which depends only on sampled values of known function for some equations, see Sec. 4 for details. On the other hand, the integral average of compactly supported functions, but it does not contain functions with discontinuous Fourier transform.

But the latter sum is nothing as \( Lf(2^{-j}k)(k), \) where \( \alpha_l = \frac{1}{(t+1)!} h^l. \) This idea is used in Sec. 4 for error analysis of falsified sampling approximation.

In the present paper we study approximation properties of differential expansions (5) for a class of band-limited functions \( \varphi \) with discontinuous Fourier transform. Error estimations in \( L^p \)-norm are given in terms of the Fourier transform of the approximated function \( f. \) Analogs of the classical sampling theorem and Brown’s inequality (2) are proved. We also study the falsified sampling expansions

\[ \sum_{k \in \mathbb{Z}^d} E(f, M^{-j}k) \varphi(M^j x + k), \]

where \( M \) is a matrix dilation,

\[ E(f, M^{-j}k) = E(f, M^{-j}k, h, w) = \int_0^{\infty} du w(u) \frac{1}{\text{meas } M^{-j} B_h(u)} \int_{M^{-j} B_h(u)} f(M^{-j}k + t) dt, \]

\( h(u) \) is a positive function defined on \((0, \infty)\) and \( u \) is a random value with probability density \( w. \) To estimate the approximation order of falsified sampling expansions we compare them with differential expansions. The error estimations of approximation by differential expansions obtained in [20] as well as new estimations for band-limited functions \( \varphi. \) are used.

The paper is organized as follows: in Section 2 we introduce notation and give some basic facts. In Section 3 we study scaling operators \( \sum_{k \in \mathbb{Z}} (f, \varphi_{jk})\varphi_{jk} \) and their approximation properties for a class of band-limited functions \( \varphi \) with discontinuous Fourier transform. In Section 4 we study approximation properties of generalized sampling expansions defined by differential operators. In Section 5 we obtain estimates of the approximation order of falsified sampling expansions. In Section 6 we give some examples.
2 Notation and basic facts

\( \mathbb{N} \) is the set of positive integers, \( \mathbb{R} \) is the set of real numbers, \( \mathbb{C} \) is the set of complex numbers. \( \mathbb{R}^d \) is the \( d \)-dimensional Euclidean space, \( x = (x_1, \ldots, x_d) \) and \( y = (y_1, \ldots, y_d) \) are its elements (vectors), \((x)_j = x_j \) for \( j = 1, \ldots, d \), \((x,y) = x_1y_1 + \ldots + x_dy_d \), \(|x| = \sqrt{(x,x)} \). \( O = (0, \ldots, 0) \) \( \in \mathbb{R}^d \).

\( B_r = \{ x \in \mathbb{R}^d : |x| \leq r \} \), \( \mathbb{Z}^d \) is the integer lattice in \( \mathbb{R}^d \), \( \mathbb{Z}^d_+ := \{ x \in \mathbb{Z}^d : x \geq 0 \} \).

If \( \alpha, \beta \in \mathbb{Z}_+^d \), \( a, b \in \mathbb{R}^d \), we set \( [\alpha] = \sum_{j=1}^d \alpha_j \), \( [\alpha]! = \prod_{j=1}^d (\alpha_j!) \),

\[ \left( \begin{array}{c} \beta \\ \alpha \end{array} \right) = \frac{\alpha!}{\beta!(\alpha - \beta)!}, \quad a^b = \prod_{j=1}^d a_j^{b_j}, \quad D^\alpha f = \frac{\partial^{[\alpha]} f}{\partial x^{\alpha}} = \frac{\partial^{[\alpha]} f}{\partial x_1 \cdots \partial x_d}, \]

\( \delta_{ab} \) is the Kronecker delta.

A real \( d \times d \) matrix \( M \) whose eigenvalues are bigger than 1 is placed on the diagonal matrix. Throughout the paper we consider that such a matrix \( M \) is fixed and \( m = |\det M| \), \( M^* \) denotes the conjugate matrix to \( M \). Since the spectrum of the operator \( M^{-1} \) is located in \( B_r \), where \( r = r(M^{-1}) := \lim_{j \to +\infty} \|M^{-j}\|^{1/j} \) is the spectral radius of \( M^{-1} \), and there exists at least one point of the spectrum on the boundary of \( B_r \), we have

\[ \|M^{-j}\| \leq C_{M, \vartheta} \vartheta^{-j}, \quad j \geq 0, \quad \text{(6)} \]

for every positive number \( \vartheta \) which is smaller in module than any eigenvalue of \( M \). In particular, we can take \( \vartheta > 1 \), then

\[ \lim_{j \to +\infty} \|M^{-j}\| = 0. \quad \text{(7)} \]

A matrix \( M \) is called isotropic if it is similar to a diagonal matrix with elements \( \lambda_1, \ldots, \lambda_d \) that are placed on the main diagonal and \( |\lambda_1| = \cdots = |\lambda_d| \). Thus, \( \lambda_1, \ldots, \lambda_d \) are eigenvalues of \( M \) and the spectral radius of \( M \) is equal to \( |\lambda| \), where \( \lambda \) is one of the eigenvalues of \( M \). Note that if the matrix \( M \) is isotropic then \( M^* \) is isotropic and \( M^j \) is isotropic for all \( j \in \mathbb{Z} \).

It is well known that for an isotropic matrix \( M \) and for any \( j \in \mathbb{Z} \) we have

\[ C_1|\lambda|^j \leq \|M^j\| \leq C_2|\lambda|^j, \quad \text{(8)} \]

where \( \lambda \) is one of the eigenvalues of \( M \) and the positive constants \( C_1 \) and \( C_2 \) do not depend on \( j \).

If \( \varphi \) is a function defined on \( \mathbb{R}^d \), we set

\[ \varphi_{jk}(x) := m^{j/2}\varphi(M^j x + k), \quad j \in \mathbb{Z}, \quad k \in \mathbb{R}^d. \]

\( L_p \) denotes \( L_p(\mathbb{R}^d) \), \( 1 \leq p \leq \infty \). We use \( W^n_p \) (or \( W^n_p(\mathbb{R}^d) \)), \( 1 \leq p \leq \infty, \ n \in \mathbb{N} \), to denote the Sobolev space on \( \mathbb{R}^d \), i.e. the set of functions whose derivatives up to order \( n \) are in \( L_p \), with the usual Sobolev norm.

If \( f, g \) are functions defined on \( \mathbb{R}^d \) and \( f \varphi \in L_1 \), then \( \langle f, g \rangle := \int_{\mathbb{R}^d} f \varphi \).

For any function \( f \), we set \( f^{-}(x) := f(-x) \).

If \( F \) is a 1-periodic (with respect to each variable) function and \( F \in L_1(\mathbb{T}^d) \), then \( \hat{F}(k) = \int_{\mathbb{R}^d} F(x)e^{-2\pi ik \cdot x} \ dx \) is its \( k \)-th Fourier coefficient. If \( f \in L_1 \), then its Fourier transform is \( \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i \xi \cdot x} \ dx \).

Denote by \( \mathcal{S} \) the Schwartz class of functions defined on \( \mathbb{R}^d \). The dual space of \( \mathcal{S} \) is \( \mathcal{S}' \), i.e. \( \mathcal{S}' \) is the space of tempered distributions. The basic facts from distribution theory can be found, e.g., in [31]. Suppose \( f \in \mathcal{S}, \varphi \in \mathcal{S}' \), then \( \langle \varphi, f \rangle := \langle f, \varphi \rangle := \langle f, \varphi \rangle \). If \( \varphi \in \mathcal{S}' \), then \( \hat{\varphi} \) denotes its Fourier transform defined by \( \langle \hat{f}, \hat{\varphi} \rangle = \langle \hat{f}, \varphi \rangle, \ f \in \mathcal{S} \). If \( \varphi \in \mathcal{S}', \ j \in \mathbb{Z}, \ k \in \mathbb{Z}^d \), then we define \( \varphi_{jk} \) by \( \langle f, \varphi_{jk} \rangle = \langle f_{j-Mjk}, \varphi \rangle \) for all \( f \in \mathcal{S} \).
Given a dilation matrix \( M \) and \( \delta > 0 \), we introduce a special notation for the following integrals if they make sense

\[
T_{j,\gamma,q}^{\text{in}}(g) := \int_{|\xi^{\gamma}| \leq \delta} |\xi|^{\gamma}|g(\xi)||d\xi, \quad T_{j,\gamma,q}^{\text{out}}(g) := \int_{|\xi^{\gamma}| \geq \delta} |\xi|^{\gamma}|g(\xi)||d\xi.
\]

A function \( \varphi \in L_1 \) is said to satisfy the \textit{Strang-Fix condition of order} \( n \) if \( D^{\beta} \hat{\varphi}(k) = 0 \), whenever \( k \in \mathbb{Z}^d \setminus \{0\} \) and \( |\beta| < n \).

Let \( 1 \leq p \leq \infty \). Denote by \( \mathcal{L}_p \) the set

\[
\mathcal{L}_p := \left\{ \varphi \in \mathcal{L}_p : \|\varphi\|_{\mathcal{L}_p} := \left\| \sum_{k \in \mathbb{Z}^d} |\varphi(\cdot + k)| \right\|_{L_p(\mathbb{T}^d)} < \infty \right\}.
\]

With the norm \( \| \cdot \|_{\mathcal{L}_p} \), \( \mathcal{L}_p \) is a Banach space. The simple properties are: \( \mathcal{L}_1 = L_1 \), \( \|\varphi\|_p \leq \|\varphi\|_{\mathcal{L}_p} \), \( \|\varphi\|_{\mathcal{L}_p} \leq \|\varphi\|_{\mathcal{L}_q} \) for \( 1 \leq q \leq p \leq \infty \). Therefore, \( \mathcal{L}_p \subset \mathcal{L}_q \) and \( \mathcal{L}_p \subset \mathcal{L}_\infty \) for \( 1 \leq q \leq p \leq \infty \). If \( \varphi \in \mathcal{L}_p \) and compactly supported then \( \varphi \in \mathcal{L}_p \) for \( p \geq 1 \). If \( \varphi \) decays fast enough, i.e. there exist constants \( C > 0 \) and \( \epsilon > 0 \) such that \( |\varphi(x)| \leq C(1 + |x|)^{-d-\epsilon} \) for all \( x \in \mathbb{R}^d \), then \( \varphi \in \mathcal{L}_\infty \).

The following auxiliary statements will be useful for us.

**Proposition 1 ([14])** Let \( 1 \leq p \leq \infty \). If \( \varphi \in \mathcal{L}_p \) and \( a = \{a_k\}_{k \in \mathbb{Z}^d} \in \ell_p \), then

\[
\left\| \sum_{k \in \mathbb{Z}^d} a_k \varphi_{\text{ok}} \right\|_p \leq \|\varphi\|_{\mathcal{L}_p} \|a\|_{\ell_p}.
\]

**Lemma 2 ([20])** Let \( 1 \leq q < \infty \), \( 1/p + 1/q = 1 \), \( j \in \mathbb{Z}_+ \), \( \varphi \) be a tempered distribution whose Fourier transform \( \hat{\varphi} \) is a function on \( \mathbb{R}^d \) such that \( |\hat{\varphi}(\xi)| \leq C_\varphi |\xi|^N \) for almost all \( \xi \notin \mathbb{T}^d \), \( N = N(\varphi) \geq 0 \), and \( |\hat{\varphi}(\xi)| \leq C_\varphi' \) for almost all \( \xi \in \mathbb{T}^d \). Suppose \( g \in L_q \), \( g(\xi) = O(|\xi|^{-N-d-\epsilon}) \) as \( |\xi| \to \infty \), where \( \epsilon > 0 \); \( \gamma \in \left(N + \frac{d}{p}, N + \frac{d}{p} + \epsilon\right) \) for \( q \neq 1 \), \( \gamma = N \) for \( q = 1 \), and set

\[
G_j(\xi) = G_j(\varphi, g, \xi) := \sum_{l \in \mathbb{Z}^d} g(M^{*j}(\xi + l)) \hat{\varphi}(\xi + l).
\]

Then \( G_j \) is a 1-periodic function in \( L_q(\mathbb{T}^d) \), \( (g, \varphi_{jk}) = m^{j/2} G_j(k) \), and for every \( \delta \in (0, \frac{1}{2}) \)

\[
\left\| G_j - g(M^{*j}) \hat{\varphi} \right\|^q_{L_q(\mathbb{T}^d)} \leq m^{-j}(C, \varphi)^q \|M^{*j} \|_{\mathcal{L}_q} T_{j,\gamma,q}^{\text{out}}(g).
\]

**Lemma 3 ([20])** Let \( g \) and \( \varphi \) be as in Lemma 2. Suppose \( 2 \leq p \leq \infty \), \( \varphi \in \mathcal{L}_p \). Then the series \( \sum_{k \in \mathbb{Z}^d} (g, \varphi_{jk} \varphi_{jk} \) converges unconditionally in \( L_p \).

## 3 Scaling Approximation

Scaling operator \( \sum_{k \in \mathbb{Z}^d} f, \varphi_{jk} \) is a good tool of approximation for many appropriate pairs of functions \( \varphi, \varphi \). We are interested in such operators, where \( \varphi \) is a tempered distribution, e.g., the delta-function or a linear combination of its derivatives. In this case the inner product \( (f, \varphi_{jk}) \) has meaning only for functions \( f \) in \( \mathcal{S} \). To extend the class of functions \( f \) one can replace \( (f, \varphi_{jk}) \) by \( (\hat{f}, \varphi_{jk}) \). In this case we set

\[
Q_j(\varphi, \varphi, f) = \sum_{k \in \mathbb{Z}^d} (\hat{f}, \varphi_{jk}) \varphi_{jk}, \quad j \in \mathbb{Z}_+.
\]
Approximation properties of such operators for certain classes of distributions \( \tilde{\varphi} \) and functions \( \varphi \) were studied in \([20]\). In particular, the following statement is proved.

**Theorem 4 \([20]\)** Let \( 2 \leq p \leq \infty, 1/p + 1/q = 1, N \in \mathbb{Z}_+, \gamma \in (N + \frac{d}{p}, N + \frac{d}{p} + \epsilon) \) for \( p \neq \infty \), and \( \gamma = N \) for \( p = \infty \). Suppose

(a) \( \tilde{\varphi} \) is a tempered distribution whose Fourier transform \( \hat{\tilde{\varphi}} \) is a function on \( \mathbb{R}^d \) such that \( |\hat{\tilde{\varphi}}(\xi)| \leq C\varphi|\xi|^N \) for almost all \( \xi \notin \mathbb{N}^d \), \( N > 0 \), and \( |\hat{\tilde{\varphi}}(\xi)| \leq C\varphi \) for almost all \( \xi \in \mathbb{N}^d \);

(b) \( \varphi \in L_p \) and there exists \( B \varphi > 0 \) such that \( \sum_{k \in \mathbb{Z}^d} |\hat{\varphi}(\xi + k)|^q < B \varphi \) for all \( \xi \in \mathbb{R}^d \);

(c) there exist \( n \in \mathbb{N} \) and \( \delta \in (0, 1/2) \) such that \( \hat{\tilde{\varphi}} \) is boundedly differentiable up to order \( n \) on \( \{ ||\xi| < \delta \} \), \( \hat{\tilde{\varphi}} \) is boundedly differentiable up to order \( n \) on \( \{ ||\xi + l| < \delta \} \) for all \( l \in \mathbb{N}^d \setminus \{0\} \); the function \( \sum_{l \in \mathbb{N}^d, l \neq 0} |\hat{\tilde{\varphi}}(\xi + l)| \) is bounded on \( \{ ||\xi| < \delta \} \) for \( \beta = n \); \( D^\delta(1 - \tilde{\varphi}\tilde{\varphi})(0) = 0 \) for \( \beta < n \); the Strang-Fix condition of order \( n \) holds for \( \varphi \);

\[(d) f \in L_p, \tilde{f} \in L_q, \tilde{f}(\xi) = O(|\xi|^{-N-\epsilon_d}) \text{ as } |\xi| \to \infty, \epsilon > 0.\]

Then
\[
\|f - Q_\delta(\varphi, \tilde{\varphi}, f)\|_p^q \leq C_1 \|M^{* - j}\|_q^q T_{j,q}^\text{Out}(\tilde{f}) + C_2 \|M^{* - j}\|_q^q T_{j,n,q}^\text{In}(\tilde{f}),
\]
where \( C_1 \) and \( C_2 \) do not depend on \( j \) and \( f \).

A compatibility between distribution \( \tilde{\varphi} \) and function \( \varphi \) given in item (c) is required for the error of approximation in the latter theorem. This compatibility is given in terms of the derivatives of \( \hat{\tilde{\varphi}}, \hat{\varphi} \) at the origin up to the order of the Strang-Fix condition of \( \varphi \). In what follows we will also consider a compatibility up to an arbitrary order.

**Definition 5** A tempered distribution \( \tilde{\varphi} \) and a function \( \varphi \) is said to be strictly compatible if there exists \( \delta \in (0, 1/2) \) such that \( \tilde{\varphi}(\xi)\tilde{\varphi}(\xi) = 1 \) a.e. on \( \{ ||\xi| < \delta \} \) and \( \hat{\varphi}(\xi) = 0 \) a.e. on \( \{ ||\xi| - \delta < \delta \} \) for all \( l \in \mathbb{Z}^d \setminus \{0\} \).

The class of functions \( \varphi \) considered in Theorem 3 is large, it contains both compactly supported and band-limited functions. However, it does not include functions whose Fourier transform is discontinuous, for example, the function \( \prod_{k=1}^{d} \text{sinc}(x_k) \) is out of consideration. Here we will make up for this omission.

Let \( \mathcal{B} = \mathcal{B}(\mathbb{R}^d) \) denote the class of functions \( \varphi \) given by
\[
\varphi(x) = \int_{\mathbb{R}^d} \theta(\xi) e^{2\pi i (x, \xi)} d\xi,
\]
where \( \theta \) is supported on a parallelepiped \( S := [a_1, b_1] \times \cdots \times [a_d, b_d] \) and such that \( \theta|_S \in C^d(S) \).

**Proposition 6** Let \( 1 < p < \infty, \varphi \in \mathcal{B}, f \in L_p \). Then
\[
\left( \sum_{k \in \mathbb{Z}^d} |\langle f, \varphi_k \rangle|^p \right)^{\frac{1}{p}} \leq C_{\varphi,p} \|f\|_p.
\]

The proof of Proposition 6 is based on two lemmas. To formulate these lemmas, as well as further results, we need additional notation. Set
\[
U_0^0 = \{ t \in \mathbb{R} : |t| < 1 \} \quad \text{and} \quad U_1^k = \mathbb{R} \setminus U_0^0, \quad k \in \mathbb{Z};
\]
if \( k \in \mathbb{Z}^d, \chi = (\chi_1, \ldots, \chi_d) \in \{0, 1\}^d \), then \( U_k^\chi \) is defined by
\[
U_k^\chi = U_{k_1}^{\chi_1} \times \cdots \times U_{k_d}^{\chi_d}.
\]

The proof of the lemma below follows easily from the proof of Lemma 4 in \([27]\).
Lemma 7 Let $f \in L_p(\mathbb{R})$, $1 < p < \infty$, $u \in \mathbb{R}$. Then

$$\left( \sum_{k \in \mathbb{Z}} \left| \int_{U_k^0} f(t) e^{2\pi i u(t-k)} \frac{dt}{t-k} \right|^p \right)^{\frac{1}{p}} \leq C_p \|f\|_{L_p(\mathbb{R})}.$$ 

The following lemma gives a proof of Proposition \[\] in the case $d = 1$. Since $S = [a_1, b_1]$ if $d = 1$, for convenience, we reddenote $[a_1, b_1]$ by $[a, b]$ for this case.

Lemma 8 Let $f \in L_p(\mathbb{R})$, $1 < p < \infty$, and $\varphi \in \mathcal{B}(\mathbb{R})$. Then, for any $\chi \in \{0, 1\}$, we have

$$\left( \sum_{k \in \mathbb{Z}} \left| \int_{U_k^0} f(t) \varphi(t-k) dt \right|^p \right)^{\frac{1}{p}} \leq C_{S,p} \|\theta\|_{W_p^1(S)} \|f\|_{L_p(\mathbb{R})}. \quad (13)$$

Proof. 1) If $\chi = 0$, then by Hölder’s inequality, we have

$$\sum_{k \in \mathbb{Z}} \left| \int_{U_k^0} f(t) \varphi(t-k) dt \right|^p \leq \left\| \varphi \right\|_{L_p(S)}^p \sum_{k \in \mathbb{Z}} \left( \int_{U_k^0} |f(t)| dt \right)^p \leq C_{S,p} \|\theta\|_{L_p(S)} \sum_{k \in \mathbb{Z}} \int_{U_k^0} |f(t)|^p dt \leq C_{S,p} \|\theta\|_{L_p(S)} \|f\|_{L_p(\mathbb{R})}^p,$$

which implies $13$.

2) Let $\chi = 1$. Using the formula

$$\varphi(x) = \int_a^b \theta(\xi) e^{2\pi i x \xi} d\xi = \frac{\theta(b) e^{2\pi ibx} - \theta(a) e^{2\pi iax}}{2\pi i x} - \frac{1}{2\pi i} \int_a^b \theta'(\xi) e^{2\pi i \xi} d\xi, \quad (14)$$

we obtain

$$\sum_{k \in \mathbb{Z}} \left| \int_{U_k^0} f(t) \varphi(t-k) dt \right|^p \leq C_p \sum_{k \in \mathbb{Z}} \left| \int_{U_k^0} f(t) \frac{\theta(b) e^{2\pi ibx} - \theta(a) e^{2\pi iax}}{t-k} dt \right|^p + \left| \int_{U_k^0} \frac{f(t)}{t-k} \left( \int_a^b \theta'(\xi) e^{2\pi i(t-k) \xi} d\xi \right) dt \right|^p = I_1 + I_2. \quad (15)$$

By Lemma 7 we get

$$I_1 \leq C_p \sum_{a \in (a, b]} |\theta(a)| \sum_{k \in \mathbb{Z}} \left| \int_{U_k^0} f(t) e^{2\pi i u(t-k)} \frac{dt}{t-k} \right|^p \leq C_p \|\theta\|_{L_p(S)} \|f\|_{L_p(\mathbb{R})} \cdot (16)$$

Now, let us consider the sum $I_2$. Using Hölder’s inequality with $1/p + 1/q = 1$ and Lemma 7 we
derive
\[ I_2 = C_p \sum_{k \in \mathbb{Z}} \left| \int_a^b \theta'(\xi) \int_{U_k^1} f(t) \frac{e^{2\pi i (t-k)\xi}}{t-k} \, dt \, d\xi \right|^p \leq \]
\[ C_p \sum_{k \in \mathbb{Z}} \left( \int_a^b |\theta'(\xi)| \left| \int_{U_k^1} f(t) \frac{e^{2\pi i (t-k)\xi}}{t-k} \, dt \right| d\xi \right)^p \leq \]
\[ C_p \sum_{k \in \mathbb{Z}} \left( \int_a^b |\theta'(\xi)|^q \, d\xi \right)^{\frac{p}{q}} \leq \int_{U_k^1} \left| \int_a^b f(t) \frac{e^{2\pi i (t-k)\xi}}{t-k} \, dt \right|^p \, d\xi \leq C_{S,p} \| \theta' \|^p_{L_\infty(S)} \| f \|^p_{L_p(R)}. \]  

Finally, combining (15)–(17), we get (13) for \( \chi = 1 \).

**Proof of Proposition 4.** For convenience, we introduce the following notation. If \( t \in \mathbb{R}^d \), then \( \bar{t} := (t_1, \ldots, t_{d-1}) \in \mathbb{R}^{d-1} \). For a function \( g \) of \( d \) variables \( t_1, \ldots, t_{d-1} \) and \( s \), we set \( g_s(t) = g(t_1, \ldots, t_{d-1}, s) \). Let also \( \tilde{w}_\Sigma(\eta) = F^{-1} \theta_\eta(\bar{x}) \) and \( S = [a_1, b_1] \times \cdots \times [a_{d-1}, b_{d-1}] \).

We have
\[ \sum_{k \in \mathbb{Z}^d} |(f, \varphi_k)|^p = \sum_{k \in \mathbb{Z}^d} \left| \int_{\mathbb{R}^d} f(t) \varphi(t-k) \, dt \right|^p \leq C_p \sum_{\chi \in \{0,1\}^d} I^\chi, \]  

where
\[ I^\chi = \sum_{k \in \mathbb{Z}^d} \left| \int_{U_k^1} f(t) \varphi(t-k) \, dt \right|^p. \]

Thus, to prove (12) it is enough to show that
\[ I^\chi \leq C_{S,p} \| \theta \|^p_{W^{d-1}_\infty(S)} \| f \|^p_{L_p} \]  

for any \( \chi \in \{0,1\}^d \).

We prove (19) by induction on \( d \). For \( d = 1 \), the inequality (19) was proved in Lemma 8. To prove the inductive step \( d-1 \to d \), we assume that for any \( g \in L_p(\mathbb{R}^{d-1}) \) and \( \tilde{\varphi} \in B(\mathbb{R}^{d-1}) \) (more precisely \( \tilde{\varphi} = F^{-1} \tilde{\theta} \), where \( \tilde{\theta} \) is the same as in (11) with \( S \) in place of \( S \)), we have
\[ \sum_{k \in \mathbb{Z}^{d-1}} \left| \int_{U_k^{d-1}} g(t) \tilde{\varphi}(\bar{t} - \bar{k}) \, dt \right|^p \leq C_{S,p} \| \tilde{\theta} \|^p_{W^{d-1}_\infty(S)} \| g \|^p_{L_p(\mathbb{R}^{d-1})}. \]  

For any \( \chi \in \{0,1\}^d \), we can write \( \chi = (\bar{\chi}, \chi_d) \) and
\[ I^\chi = \sum_{\chi_d \in \{0,1\}} I^{(\bar{\chi}, \chi_d)}. \]

Let us estimate \( I^{(\bar{\chi}, \chi_d)} \) for \( \chi_d = 0 \) and \( \chi_d = 1 \).

1) For \( \chi_d = 0 \), we obtain
\[ I^{(\bar{\chi},0)} = \sum_{\bar{k} \in \mathbb{Z}^{d-1}} \sum_{l \in \mathbb{Z}} \left| \int_{U_{\bar{k}}^{d-1}} \int_{U_k^1} f_s(t) \varphi(\bar{t} - \bar{k}, s-l) \, dt \, ds \right|^p. \]  

(21)
Using the above notation and formula (14), we get
\[ \varphi(x) = \mathcal{F}^{-1}\theta(x) = \int_{a_d}^{b_d} \psi \varphi \eta e^{2\pi i s \eta} d\eta = \frac{\psi \varphi \eta e^{2\pi i b_d x_d} - \psi \varphi \eta e^{2\pi i a_d x_d}}{2\pi i x_d} \frac{1}{2\pi i x_d} \mathcal{F}^{-1}\psi \varphi'(x). \] (22)

By (22) and Hölder’s inequality, we derive
\[ \left| \int_{U_\mathcal{K}} \int_{U_\mathcal{K}} f_s(\tilde{t}) \int_{a_d}^{b_d} \psi \varphi \eta e^{2\pi i (s-l) \eta} d\eta ds d\eta \right|^p \leq \left( \int_{U_\mathcal{K}} \int_{U_\mathcal{K}} f_s(\tilde{t}) \psi \varphi(\eta) d\eta ds \right)^p \leq \left( \int_{U_\mathcal{K}} \int_{U_\mathcal{K}} f_s(\tilde{t}) \psi \varphi(\eta) d\eta ds \right)^p \leq (2(b_d - a_d))^{p-1} \int_{a_d}^{b_d} \left| F_{k,\eta}(s) \right|^p ds d\eta, \] (23)

where
\[ F_{k,\eta}(s) = \int_{U_\mathcal{K}} f_s(\tilde{t}) \psi \varphi (\eta) d\eta. \] (24)

Next, combining (21) and (23) and using the induction hypothesis (20), we get
\[ I(\chi,0) \leq C_{S,p} \sum_{k \in \mathbb{Z}^{d-1}} \sum_{l \in \mathbb{Z}} \int_{U_\mathcal{K}} \left| F_{k,\eta}(s) \right|^p ds d\eta \leq C_{S,p} \int_{a_d}^{b_d} \left| F_{k,\eta}(s) \right|^p ds d\eta \leq C_{S,p} \int_{a_d}^{b_d} \left| F_{k,\eta}(s) \right|^p ds d\eta \leq C_{S,p} \int_{a_d}^{b_d} \left| F_{k,\eta}(s) \right|^p ds d\eta \leq C_{S,p} \left\| F_{k,\eta}(s) \right\|^p_{L_p(\mathbb{R})} ds d\eta \] (25)

2) Now consider the case \( \chi_d = 1 \). We have
\[ I(\chi,1) = \sum_{k \in \mathbb{Z}^{d-1}} \sum_{l \in \mathbb{Z}} \int_{U_\mathcal{K}} \int_{U_\mathcal{K}} f_s(\tilde{t}) \varphi (\eta - \tilde{k}, s-l) d\tilde{t} d\eta ds \] (26)

By (22), we get
\[ I(\chi,1) \leq C_p (I_1 + I_2), \] (27)

where
\[ I_1 = \sum_{k \in \mathbb{Z}^{d-1}} \sum_{l \in \mathbb{Z}} \left| \int_{U_\mathcal{K}} \int_{U_\mathcal{K}} f_s(\tilde{t}) \psi \varphi (\eta - \tilde{k}, s-l) \psi \varphi (\eta - \tilde{k}) e^{2\pi i b_d (s-l)} - \psi \varphi (\eta - \tilde{k}) e^{2\pi i a_d (s-l)} \right| ds d\eta \right|^p \] (28)
Using the function (24) and Lemma 7, we obtain
\[ I_1 \leq C_p \sum_{u \in \{a, b_d\}} \sum_{k \in \mathbb{Z}_{d-1}} \left( \sum_{l \in \mathbb{Z}} \left| \int_{U^k_l} \frac{F_{k,u}(s) e^{2\pi i u(s-l)}}{s-l} ds \right|^p \right) \leq C_p \sum_{u \in \{a, b_d\}} \sum_{k \in \mathbb{Z}_{d-1}} \|F_{k,u}\|_{L_p^p}. \] (29)

Now, using the induction hypothesis (20), for any \( u \in \mathbb{R} \), we derive
\[ \sum_{k \in \mathbb{Z}_{d-1}} \|F_{k,u}\|_{L_p^p} = \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}_{d-1}} \left( \int f_s(\bar{t}) F^{-1}_{u} \theta_u(\bar{t} - \bar{k}) d\bar{t} \right)^p ds \leq \]
\[ C_S p \|\theta_u\|_{W^d_{\infty -1}(\mathbb{S})} \int_{\mathbb{R}} \|f_s\|_{L_p^p(R^{d-1})} ds. \] (30)

Hence, combining (29) and (30), we get
\[ I_1 \leq C_{S,p} \sum_{u \in \{a, b_d\}} \|\theta_u\|_{W^d_{\infty -1}(\mathbb{S})} \|f\|_{L_p^p} \leq C_{S,p} \|\theta\|_{W^d_{\infty}(\mathbb{S})} \|f\|_{L_p^p}. \] (31)

Let us consider \( I_2 \). Denoting
\[ F_{k,\eta}^*(s) = \int_{U^k_l} f_s(\bar{t}) \psi_{t-k}(\eta) d\bar{t} \]
and using H"older’s inequality, we obtain
\[ \left| \int_{U^k_l} \int f_s(\bar{t}) F^{-1}_{t-k}(s-l) d\bar{t} ds \right|^p \leq \left( \int_{a_d}^{b_d} \int_{U^k_l} \int f_s(\bar{t}) \psi_{t-k}(\eta) e^{2\pi i (s-l)\eta} \frac{d\bar{t} ds}{s-l} d\eta \right)^p \leq \]
\[ (b_d - a_d)^{p-1} \int_{a_d}^{b_d} \left( \int_{U^k_l} \int f_s(\bar{t}) e^{2\pi i (s-l)\eta} \frac{d\bar{t} ds}{s-l} \right)^p d\eta. \] (32)

Thus, combining (28) and (32), using Lemma 7 and the induction hypothesis (20), we derive
\[ I_2 \leq C_{S,p} \sum_{k \in \mathbb{Z}_{d-1}} \sum_{l \in \mathbb{Z}_{a_d}} \left( \int_{U^k_l} \int F_{k,\eta}^*(s) e^{2\pi i (s-l)\eta} \frac{ds}{s-l} d\eta \right)^p = \]
\[ C_{S,p} \int_{a_d}^{b_d} \sum_{k \in \mathbb{Z}_{d-1}} \sum_{l \in \mathbb{Z}_{a_d}} \left( \int F_{k,\eta}^*(s) e^{2\pi i (s-l)\eta} ds \right)^p d\eta \leq \]
\[ C_{S,p} \int_{a_d}^{b_d} \sum_{k \in \mathbb{Z}_{d-1}} \|F_{k,\eta}\|_{L_p^p} d\eta = \] (33)
\[ C_{S,p} \int_{a_d}^{b_d} \left( \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}_{d-1}} \left( \int f_s(\bar{t}) F^{-1}_{u} \theta_u(\bar{t} - \bar{k}) d\bar{t} \right)^p ds d\eta \right) d\eta \leq \]
\[ C_{S,p} \int_{a_d}^{b_d} \left| \frac{\partial}{\partial \eta} \theta_u \right|_{W^d_{\infty -1}(\mathbb{S})} d\eta \int_{\mathbb{R}} \|f_s\|_{L_p^p(R^{d-1})} ds \leq C_{S,p} \|\theta\|_{W^d_{\infty}(\mathbb{S})} \|f\|_{L_p^p}. \]
Finally, combining (27), (31), and (33), we get (14) for any \( \chi = (\chi_1, \ldots, \chi_{d-1}, 1) \). This and (25) prove the proposition. ∗

**Proposition 9** Let \( 1 < p < \infty, \varphi \in \mathcal{B}, a = \{a_k\}_{k \in \mathbb{Z}^d} \in \ell_p \). Then

\[
\left\| \sum_{k \in \mathbb{Z}^d} a_k \varphi_{0k} \right\|_p \leq C_{\varphi,p} \|a\|_{\ell_p}.
\]

**Proof.** Since

\[
\left\| \sum_{k \in \mathbb{Z}^d} a_k \varphi_{0k} \right\|_p = \left\langle \sum_{k \in \mathbb{Z}^d} a_k \varphi_{0k}, f \right\rangle = \left| \sum_{k \in \mathbb{Z}^d} a_k \langle \varphi_{0k}, f \rangle \right|,
\]

where \( f \in L_q, \|f\|_q \leq 1, \) \( 1/p + 1/q = 1 \), the statement follows immediately from Proposition 6 and Hölder’s inequality. ∗

**Corollary 10** Let \( 2 \leq p < \infty, 1/p + 1/q = 1, g \in L_q, \) and \( \varphi \) be as in Lemma 12. \( \varphi \in \mathcal{B} \). Then the series \( \sum_{k \in \mathbb{Z}^d} \langle g, \varphi_j(k) \rangle \varphi_{jk} \) converges unconditionally in \( L_p \) and

\[
\left\| \sum_{k \in \mathbb{Z}^d} \langle g, \varphi_j(k) \rangle \varphi_{jk} \right\|_p \leq C_{\varphi,q} m^{\frac{1}{p} - \frac{1}{q}} \left( \sum_{k \in \mathbb{Z}^d} |\langle g, \varphi_j(k) \rangle|^p \right)^{\frac{1}{p}}.
\]

(34)

**Proof.** Because of Lemma 12 and the Hausdorff-Young inequality, we have

\[
\left( \sum_{k \in \mathbb{Z}^d} |\langle g, \varphi_j(k) \rangle|^p \right)^{\frac{1}{p}} = m^{\frac{1}{p}} \left( \sum_{k \in \mathbb{Z}^d} |\hat{G}_j(k)|^p \right)^{\frac{1}{p}} \leq m^{\frac{1}{p}} \|G_j\|_{L_q(T^d)} < \infty,
\]

where \( G_j \) is a function from Lemma 12. By Proposition 9 we can state that for every finite subset \( \Omega \) of \( \mathbb{Z}^d \)

\[
\left\| \sum_{k \in \Omega} \langle g, \varphi_j(k) \rangle \varphi_{jk} \right\|_p = m^{\frac{1}{p} - \frac{1}{q}} \left\| \sum_{k \in \Omega} \langle g, \varphi_j(k) \rangle \varphi_{0k} \right\|_p \leq m^{\frac{1}{p} - \frac{1}{q}} C_{\varphi,q} \left( \sum_{k \in \Omega} |\langle g, \varphi_j(k) \rangle|^p \right)^{\frac{1}{p}}.
\]

The series \( \sum_{k \in \mathbb{Z}^d} |\langle g, \varphi_j(k) \rangle|^p \) is convergent, which yields that \( \sum_{k \in \mathbb{Z}^d} \langle g, \varphi_j(k) \rangle \varphi_{jk} \) converges unconditionally. Similarly we obtain (34) ∗

Now we are ready to study approximation properties of the operators

\[
Q_j(\varphi, \varphi, f) = \sum_{j \in \mathbb{Z}^d} \langle \hat{f}, \varphi_j \rangle \varphi_{jk},
\]

where \( \varphi \in \mathcal{B} \). The following theorem is a counterpart of Theorem 4 for such functions \( \varphi \).

**Theorem 11** Let \( 2 \leq p < \infty, 1/p + 1/q = 1, N \in \mathbb{Z}_+, \gamma \in (N + \frac{d}{p}, N + \frac{d}{p} + \epsilon) \). Suppose

(a) \( \varphi \) is a tempered distribution whose Fourier transform \( \hat{\varphi} \) is a function on \( \mathbb{R}^d \) such that \( |\hat{\varphi}(\xi)| \leq C_{\varphi} |\xi|^N \) for almost all \( \xi \notin \mathbb{T}^d \) and \( |\hat{\varphi}(\xi)| \leq C_{\varphi}' \) for almost all \( \xi \in \mathbb{T}^d \);
(b) $\varphi \in B$;
(c) $\tilde{\varphi}$ and $\varphi$ are strictly compatible;
(d) $f \in L_p$, $\tilde{f} \in L_q$, $\tilde{f}(\xi) = O(|\xi|^{-N-d-\varepsilon})$ as $|\xi| \to \infty$, $\varepsilon > 0$.

Then

$$
\|f - Q_j(\varphi, \tilde{\varphi}, f)\|_p \leq C\|M^{*-j}\|_{\mathcal{I}^{\text{Out}}_{j, \gamma, q}(\tilde{f})},
$$

(36)

where $C$ does not depend on $j$ and $f$.

**Proof.** Throughout the proof we denote by $C_1, C_2, \ldots$ different constants which do not depend on $j$ and $f$. It follows from (34), (35), and Lemma 2 with $g = \tilde{f}$ that

$$
\|Q_j(\varphi, \tilde{\varphi}, f)\|_p \leq m^2 C_{\varphi, q} \|G_j\|_{L_q(\mathbb{T}^d)} \leq C_{\varphi, q} m^q \left( m^{-j}(C_{\gamma, \tilde{\varphi}})^q\|M^{*-j}\|_{\mathcal{I}^{\text{Out}}_{j, \gamma, q}(\tilde{f})} \right)^{1/q} + C_{\varphi, q} m^q \left( \int_{\mathbb{T}^d} |\hat{f}(\xi)| \hat{\varphi}(\xi)|^q d\xi \right)^{1/q} \leq C_1 \left( \|M^{*-j}\|_{\mathcal{I}^{\text{Out}}_{j, \gamma, q}(\tilde{f})} \right)^{1/q} + \|\tilde{f}\|_q.
$$

Since $f^{-}$ coincides with $\hat{f}$ a.e. due to the du Bois-Reymond lemma, applying the Hausdorff-Young inequality, we have $\|f\|_p = \|f^{-}\|_p \leq \|\tilde{f}\|_q$, which yields

$$
\|f - Q_j(\varphi, \tilde{\varphi}, f)\|_p \leq C_2 \left( \|M^{*-j}\|_{\mathcal{I}^{\text{Out}}_{j, \gamma, q}(\tilde{f})} \right)^{1/q} + \|\tilde{f}\|_q \leq C_3 \left( \|M^{*-j}\|_{\mathcal{I}^{\text{Out}}_{j, \gamma, q}(\tilde{f})} \right)^{1/q} + \int_{|M^{*-j}\xi| < \delta} |\hat{f}(\xi)| \hat{\varphi}(\xi)|^q d\xi \leq \|M^{*-j}\|_{\mathcal{I}^{\text{Out}}_{j, \gamma, q}(\tilde{f})} \right)^{1/q},
$$

(37)

where $\delta$ is from Definition 5.

For any compact set $K \subset \mathbb{R}^d$, the function $\tilde{f}$ can be approximated in $L_q(K)$ by infinitely smooth functions supported on $K$. So, given $j$, one can find a function $F_j$ such that supp $\tilde{F}_j \subset \{ |M^{*-j}\xi| < \delta \}$, $\tilde{F}_j \in C(\mathbb{R}^d)$, and

$$
\int_{|M^{*-j}\xi| < \delta} |\hat{f}(\xi) - \hat{F}_j(\xi)|^q d\xi \leq \|M^{*-j}\|_{\mathcal{I}^{\text{Out}}_{j, \gamma, q}(\tilde{f})}.
$$

Combining this with (37), where $f$ is replaced by $f - F_j$, and taking into account that $\mathcal{I}^{\text{Out}}_{j, \gamma, q}(\tilde{f}) = \mathcal{I}^{\text{Out}}_{j, \gamma, q}(\tilde{f} - \tilde{F}_j)$, we have

$$
\|(f - F_j) - Q_j(\varphi, \tilde{\varphi}, f - F_j)\|_p \leq \|M^{*-j}\|_{\mathcal{I}^{\text{Out}}_{j, \gamma, q}(\tilde{f})}.
$$

(38)

Due to Carleson’s theorem and Lemma 2 we have

$$
\sum_{k \in \mathbb{Z}^d} \langle \tilde{F}_j, \tilde{\varphi}_{jk} \rangle \varphi_{jk}(\xi) = \sum_{l \in \mathbb{Z}^d} \tilde{F}_j(\xi + M^* l) \overline{\varphi(\xi + M^* l)} \tilde{\varphi}(M^* \xi).
$$

If $l \neq 0$ and $F_j(\xi + M^* l) \neq 0$, then $|M^{*-j}\xi + l| < \delta$ and hence $\tilde{\varphi}(M^{*-j}\xi) = 0$. So,

$$
\sum_{k \in \mathbb{Z}^d} \langle \tilde{F}_j, \tilde{\varphi}_{jk} \rangle \varphi_{jk}(\xi) = \tilde{F}_j(\xi),
$$

12
which yields that $Q_j(\varphi, \tilde{\varphi}, F_j) = F_j$. It follows that

$$\|f - Q_j(\varphi, \tilde{\varphi}, f)\|_p = \|f - Q_j(\varphi, \tilde{\varphi}, f - F_j)\|_p.$$ 

Together with this yields $\clubsuit$.

**Corollary 12** If, under assumptions (a) – (c) of Theorem 11, a function $f$ is such that its Fourier transform is supported in $\{\xi : |M^{-j}\xi| < \delta\}$, where $\delta$ is from Definition 3 then

$$f = Q_j(\varphi, \tilde{\varphi}, f) \text{ a.e.}$$

The latter equality is a generalization of $[1]$.

## 4 Differential expansions

Consider a differential operator $L$ defined by

$$Lf := \sum_{|\beta| \leq N} a_\beta D^\beta f, \quad a_\beta \in \mathbb{C}, \quad a_0 
eq 0,$$  

(39)

where $N \in \mathbb{Z}_+$. The action of the operator $D^\beta$ is associated with the action of the corresponding derivative of the $\delta$-function. In more detail, let $f$ be such that $\int_{\mathbb{R}^d} (1 + |\xi|)^{N+\alpha} |\hat{f}(\xi)| d\xi < \infty$, where $\alpha > 0$, which implies that $f$ is continuously differentiable on $\mathbb{R}^d$ up to the order $N$. Then for $|\beta| \leq N$

$$D^\beta f(M^{-j} \cdot)(-k) = (-1)^{|\beta|} m^j \int_{\mathbb{R}^d} \hat{f}(M^{-j} \xi) (-2\pi i \xi)^\beta e^{-2\pi i (k, \xi)} d\xi =$$

$$(-1)^{|\beta|} \int_{\mathbb{R}^d} \hat{f}(\xi) (2\pi i M^{-j} \xi)^\beta e^{2\pi i (M^{-j} \xi, k)} d\xi = (-1)^{|\beta|} m^{j/2} \langle \hat{f}, D^\beta \delta_{jk} \rangle.$$

If now $\tilde{\varphi} = \sum_{|\beta| \leq N} \overline{a_\beta} (-1)^{|\beta|} D^\beta \delta$ (we say that $\tilde{\varphi}$ is associated with $L$), then

$$m^{-j/2} Lf(M^{-j} \cdot)(-k) = \langle \hat{f}, \tilde{\varphi}_{jk} \rangle, \quad k \in \mathbb{Z}^d.$$ 

Hence

$$Q_j(\varphi, \tilde{\varphi}, F_j) = m^{-j/2} \sum_{k \in \mathbb{Z}^d} Lf(M^{-j} \cdot)(-k) \varphi_{jk}.$$  

(40)

We are interested in the error of approximation of a function $f$ by these operators.

**Theorem 13** ([20]) Let $2 \leq p < \infty$, $1/p + 1/q = 1$, a differential operator $L$ be defined by (39). Suppose

(a) $\tilde{\varphi}$ is the distribution associated with $L$;

(b) $\varphi \in L_p$ and there exists $B_\varphi > 0$ such that $\sum_{k \in \mathbb{Z}^d} |\tilde{\varphi}(\xi + k)|^q < B_\varphi$ for all $\xi \in \mathbb{R}^d$;

(c) there exist $n \in \mathbb{N}$ and $\delta \in (0, 1/2)$ such that $\hat{\varphi} \hat{\tilde{\varphi}}$ is boundedly differentiable up to order $n$ on $\{|\xi| < \delta\}$, $\tilde{\varphi}$ is boundedly differentiable up to order $n$ on $\{|\xi + l| < \delta\}$ for all $l \in \mathbb{Z}^d \setminus \{0\}$; the function $\sum_{l \in \mathbb{Z}^d, l \neq 0} D^\beta \hat{\varphi}(\xi + l)$ is bounded on $\{|\xi| < \delta\}$ for $|\beta| = n$; $D^\beta (1 - \hat{\varphi} \hat{\tilde{\varphi}})(0) = 0$ for $|\beta| < n$; the Strang-Fix condition of order $n$ holds for $\varphi$;

(d) $f \in L_p$, $\tilde{f} \in L_q$, $\hat{f}(\xi) = O(|\xi|^{-N-d-\varepsilon})$ as $|\xi| \to \infty$, $\varepsilon > 0$.

Then the following statements hold:
Theorem 14

Let \( |\xi| \to \infty \), and the next lemma.

Lemma 15

Let \( 1 \leq q < \infty, 1/p + 1/q = 1, N \in \mathbb{Z}_+, \varepsilon > 0, g \in L_q, g(\xi) = O(|\xi|^{-N-d-\varepsilon}) \) as \( |\xi| \to \infty, \gamma < N + \frac{d}{p} + \varepsilon \). Then

\[
\|M^{s-j}\|_{T^g_{\gamma,q}} \leq C\theta^{-j(N+\frac{d}{p}+\varepsilon)},
\]

where \( \theta \) is any positive number which is smaller in module than any eigenvalue of \( M \), \( C \) does not depend on \( j \).
Proof. Throughout the proof we denote by $C_1, C_2, \ldots$ different constants which do not depend on $j$.
Since there exists $A > 0$ such that $|g(\xi)| \leq C_1|\xi|^{-N-d-\varepsilon}$ for any $|\xi| > A$ and the set \{\(|M^{s-j}\xi| \geq \delta\)\} is a subset of \{\(|\xi| \geq \delta/\|M^{s-j}\|\)\}, we have
\[
\|M^{s-j}\|^{\gamma q}T_{j,\gamma,q}^{\text{Out}}(g) = \|M^{s-j}\|^{\gamma q} \int_{|M^{s-j}\xi| \geq \delta} |\xi|^\gamma |g(\xi)|^q d\xi \leq C_1\|M^{s-j}\|^{\gamma q} \int_{|\xi| \geq \delta/\|M^{s-j}\|} \frac{d\xi}{|\xi|^{(N+d+\varepsilon-\gamma)q}}
\]
for all $j > j_0$, where $j_0 \in \mathbb{Z}$ is such that $\frac{\delta}{\|M^{s-n}\|} > A$. Using general polar coordinates with $\rho := |\xi|$ and taking into account that $(N + d + \varepsilon - \gamma)q > d$, we obtain
\[
\int_{|\xi| \geq \delta/\|M^{s-j}\|} \frac{d\xi}{|\xi|^{(N+d+\varepsilon-\gamma)q}} \leq C_2 \int_{\delta/\|M^{s-j}\|}^{+\infty} \frac{1}{\rho^{(N+d+\varepsilon-\gamma)q-d+1}} d\rho \leq C_3\|M^{s-j}\|^{q(N+d/\gamma+\varepsilon)},
\]
and, by (33),
\[
\|M^{s-j}\|^{\gamma q}T_{j,\gamma,q}^{\text{Out}}(g) \leq C_4\|M^{s-j}\|^{q(N+d/\gamma+\varepsilon)} \leq C_4^2 q(N+d/\gamma+\varepsilon). \quad \diamond
\]

Theorem 14 says nothing about $p = \infty$. Now we consider this case and prove a generalization of Brown’s inequality (2).

**Theorem 16** Let a differential operator $L$ be defined by (37). Suppose
\begin{enumerate}
  \item[(a)] $\hat{\varphi}$ is the distribution associated with $L$;
  \item[(b)] $\varphi \in \mathcal{B}$;
  \item[(c)] $\hat{\varphi}$ and $\varphi$ are strictly compatible;
  \item[(d)] $f \in C(\mathbb{R}^d)$, $\hat{f} \in L$, $\hat{f}(\xi) = O(|\xi|^{-N-d-\varepsilon})$ as $|\xi| \to \infty$, $\varepsilon > 0$.
\end{enumerate}
Then for every $x \in \mathbb{R}^d$ and $j \in \mathbb{Z}$, the series
\[
\sum_{k \in \mathbb{Z}^d} Lf(M^{-j}\cdot)(-k)\varphi_{jk}(x),
\]
considered as the limit of cubic partial sums, converges, and
\[
|f(x) - m^{-j/2} \sum_{k \in \mathbb{Z}^d} Lf(M^{-j}\cdot)(-k)\varphi_{jk}(x)| \leq C\|M^{-j}\|^N \int_{|M^{-j}\xi| \geq \delta} |\xi|^N|\hat{f}(\xi)| d\xi \leq C'\theta^{-j(N+\varepsilon)},
\]
where $C$ does not depend on $f$, $j$, and $x$; $C'$ does not depend on $j$ and $x$; $\theta$ is a positive number which is smaller in module than any eigenvalue of $M$.

**Proof.** Let $\varphi$ be given by (11), $x \in \mathbb{R}^d$. Set $\Theta(\xi) := \sum_{s \in \mathbb{Z}^d} \theta(\xi + s)e^{2\pi i(x\cdot\xi + s)}$. By the Poisson summation formula, $\Theta$ is a summable 1-periodic (with respect to each variable) function and its $n$-th Fourier coefficient is
\[
\hat{\Theta}(n) = \int_{\mathbb{R}^d} \theta(\xi)e^{-2\pi i(n\cdot\xi)} d\xi = \hat{\theta}(n - x) = \varphi(x - n).
\]
Since $\Theta$ is a bounded function, the cubic partial Fourier sums are uniformly bounded in $L_\infty$-norm, and the corresponding Fourier series converges to the function almost everywhere. Using this and...
Lebesgue's dominated convergence theorem, for every $\beta \in \mathbb{Z}^d$, $[\beta] \leq N$, we derive

$$
\lim_{M \to \infty} \sum_{n \in \mathbb{Z}^d} D^\beta f(-n) \varphi(x + n) = \lim_{M \to \infty} \int_{\mathbb{R}^d} \sum_{n \in \mathbb{Z}^d} \varphi(x + n) e^{-2\pi i (n, \xi)} (2\pi i \xi)^\beta \hat{f}(\xi) d\xi =
$$

$$
\int_{\mathbb{R}^d} \lim_{M \to \infty} \sum_{n \in \mathbb{Z}^d} \varphi(x - n) e^{2\pi i (n, \xi)} (2\pi i \xi)^\beta \hat{f}(\xi) d\xi = \int_{\mathbb{R}^d} e^{2\pi i (x, \xi)} \sum_{s \in \mathbb{Z}^d} \theta(\xi + s) e^{2\pi i (s, \xi)} (2\pi i \xi)^\beta \hat{f}(\xi) d\xi.
$$

Replacing $x$ by $M^j x$ and $f$ by $f(M^{-j} \cdot)$, after a change of variable, we obtain

$$
\sum_{n \in \mathbb{Z}^d} D^\beta f(M^{-j} \cdot)(-n) \varphi_j n(x) = \lim_{M \to \infty} \sum_{n \in \mathbb{Z}^d} D^\beta f(M^{-j} \cdot)(-n) \varphi_j n(x) = m^{j/2} \int_{\mathbb{R}^d} e^{2\pi i (x, \xi)} \sum_{s \in \mathbb{Z}^d} \theta(M^{-j} \xi + s) e^{2\pi i (M^j x, s)} (2\pi i M^{-j} \xi)^\beta \hat{f}(\xi) d\xi.
$$

This yields

$$
m^{-j/2} \sum_{n \in \mathbb{Z}^d} L f(M^{-j} \cdot)(-n) \varphi_j n(x) = \int_{\mathbb{R}^d} e^{2\pi i (x, \xi)} \sum_{s \in \mathbb{Z}^d} \theta(M^{-j} \xi + s) e^{2\pi i (M^j x, s)} \sum_{|\beta| \leq N} \alpha_\beta (2\pi i M^{-j} \xi)^\beta \hat{f}(\xi) d\xi.
$$

Set

$$
f_1(x) = \int_{|M^{-j} \xi| \leq \delta} \hat{f}(\xi) e^{2\pi i (x, \xi)} d\xi, \quad f_2(x) = f(x) - f_1(x).
$$

If $|M^{-j} \xi| \leq \delta$, then

$$
\sum_{s \in \mathbb{Z}^d} \theta(M^{-j} \xi + s) e^{2\pi i (M^j x, s)} = \hat{\varphi}(M^{-j} \xi)
$$

Taking into account that $\hat{\varphi}(M^{-j} \xi) \hat{\varphi}(M^{-j} \xi) = 1$, we have

$$
\sum_{s \in \mathbb{Z}^d} \theta(M^{-j} \xi + s) e^{2\pi i (M^j x, s)} \sum_{|\beta| \leq N} \alpha_\beta (2\pi i M^{-j} \xi)^\beta = 1.
$$

Hence, it follows from (44) that

$$
m^{-j/2} \sum_{n \in \mathbb{Z}^d} L f_1(M^{-j} \cdot)(-n) \varphi_j n(x) = \int_{|M^{-j} \xi| \leq \delta} e^{2\pi i x \xi} \hat{f}(\xi) d\xi = f_1(x).
$$

Since $|e^{2\pi i (x, \xi)} \sum_{s \in \mathbb{Z}^d} \theta(M^{-j} \xi + s) e^{2\pi i (M^j x, s)}| \leq C_1$, where $C_1$ depends only on $\varphi$, using (44) for $f_2$ and (45), we have

$$
\left| f(x) - m^{-j/2} \sum_{k \in \mathbb{Z}^d} L f(k)(-x) \varphi_j k(x) \right| \leq f_2(x) - m^{-j/2} \sum_{k \in \mathbb{Z}^d} L f_2(k)(-x) \varphi_j k(x) \leq
$$

$$
\int_{|M^{-j} \xi| \geq \delta} \left( 1 + C_1 \sum_{|\beta| \leq N} |\alpha_\beta (2\pi M^{-j} \xi)^\beta| \right) |\hat{f}(\xi)| d\xi \leq C ||M^{-j}||^N \int_{|M^{-j} \xi| \geq \delta} |\xi|^N |\hat{f}(\xi)| d\xi,
$$

which yields the first inequality in (43). For the second inequality it remains to apply Lemma 15 with $\gamma = N$, $q = 1$. \(\diamondsuit\)

16
First we prove the following lemma. Note that if \( h \)"L with \( a \) where \( V \) where \( L \) holds for every \( f \) whose Fourier transform is supported in \( M \centerdot [-1/2, 1/2]^d \).

**Proof.** The first statement is a trivial consequence of Theorem 16. Analysing the proof of this theorem, it is easy to see that the second statement is also true. \( \diamond \)

Corollary 17 is an analog of the classical sampling theorem. Now consider the following differential equation

\[
Lf := \sum_{|\beta| \leq N} a_\beta D^\beta f = g,
\]

where \( a_0 \neq 0 \) and \( g \) is a function band-limited to \([-1/2, 1/2]^d\). Let \( \hat{\varphi} \) is a distribution associated with \( L \). Assume that \( \hat{\varphi} \) does not vanish on \([-1/2, 1/2]^d\) and define \( \varphi \) by

\[
\hat{\varphi}(\xi) = \begin{cases} \frac{1}{\varphi(x)}, & \text{if } \xi \in [-1/2, 1/2]^d, \\ 0, & \text{if } \xi \notin [-1/2, 1/2]^d. \end{cases}
\]

By Corollary 17 we have

\[
f(x) = \sum_{k \in \mathbb{Z}^d} g(k) \varphi(x + k).
\]

For example, if \( d = 1, Lf = f - f'' \), then \( \varphi(x) = \int_0^\pi \frac{\cos tx}{1+t^2} \, dt \). The latter function can be expressed via four special functions: hyperbolic sine, hyperbolic cosine, integral sine and integral cosine, see Wolfram Mathematica.

**5 Falsified sampling expansions**

If exact sampled values of a signal \( f \) are known, then sampling expansions are very useful for applications. Theorems 13, 14, and 16 provide error estimates for this case. Now we discuss what happens if exact sampled values \( f(-M^{-j}k) \) are replaced by average values. We assume that at each point \( M^{-j}k \) one knows the following average value of \( f \)

\[
\frac{1}{V_{h(u)} B_{h(u)}} \int_{B_{h(u)}} f(M^{-j}k + M^{-j}t) \, dt = \frac{m^j}{V_{h(u)} B_{h(u)}} \int_{M^{-j}B_{h(u)}} f(M^{-j}k + t) \, dt =: \text{Av}_{h(u)}(f, M^{-j}k),
\]

where \( h(u) \) is a positive function defined on \((0, \infty)\) and \( u \) is a random value with probability density \( w; V_h \) is the volume of the ball \( B_h \). Set

\[
E(f, M^{-j}k) = E(f, M^{-j}k, h, w) = \int_0^\infty du \, w(u) \text{Av}_{h(u)}(f, M^{-j}k).
\]

Note that if \( h(u) \equiv h > 0 \), then \( E(f, M^{-j}k) = \text{Av}_h(f, M^{-j}k) \).

We are interested in error analysis for the **falsified sampling expansions**

\[
m^{-j/2} \sum_{k \in \mathbb{Z}^d} E(f, M^{-j}k) \varphi_{jk}.
\]

First we prove the following lemma.
Lemma 18 Let \( N \in \mathbb{N} \), let a function \( f \) be continuously differentiable up to order \( N \), and let \( A \) be a real \( d \times d \) matrix. Then for all \( t, x \in \mathbb{R}^d \)

\[
\sum_{|\beta| < N+1} \frac{D^\beta f(Ax)}{\beta!} (At)^\beta = \sum_{|\beta| < N+1} \frac{D^\beta f(A \cdot (x))}{\beta!} t^\beta.
\]

Proof. First we introduce some additional notations. Let \( r \in \mathbb{Z}_+ \), \( O_r = \{ \beta \in \mathbb{Z}_+^d : |\beta| = r \} \). Assume that the set \( O_r \) is ordered by lexicographic order. Namely, \((\beta_1, \ldots, \beta_d)\) is less than \((\alpha_1, \ldots, \alpha_d)\) in lexicographic order if \( \beta_j = \alpha_j \) for \( j = 1, \ldots, i-1 \) and \( \beta_i < \alpha_i \) for some \( i \). Let \( S(A, r) \) be a \((\#O_r) \times (\#O_r)\) matrix which is uniquely determined by

\[
\frac{(At)^\alpha}{\alpha!} = \sum_{\beta \in O_r} [S(A, r)]_{\alpha, \beta} t^\beta.
\]

where \( \alpha \in O_r \), \( t \in \mathbb{R}^d \). It can be verified that

\[
\alpha ![S(A, r)]_{\alpha, \beta} = \beta ![S(A^*, r)]_{\beta, \alpha}.
\]

The above notation and the latter fact is borrowed from [11].

Fix \( \beta \in \mathbb{Z}_+^d \). Let \( \mathcal{E} \) be the set of ordered samples with replacement of size \(|\beta|\) from the set \( \{e_1, \ldots, e_d\} \), where \( e_k \) is the \( k \)-th ort in \( \mathbb{R}^d \). An element \( e \in \mathcal{E} \) is a set \( \{e_{i_1}, \ldots, e_{i_\beta}\} \), where \( i_1 \in \{1, \ldots, d\} \), \( l = 1, \ldots, |\beta| \). \#\( \mathcal{E} = d^{|\beta|} \). For \( e \in \mathcal{E} \) denote by \( (e)_l := e_{i_l} \), \( l = 1, \ldots, |\beta| \). Let \( T \) be a function defined on \( \mathcal{E} \) by \( T(e) := \sum_{i=1}^{|\beta|} (e)_i \). Note that \( T(e) \in \mathbb{Z}_+^d \) and \( T(e) = |\beta| \). Denote by \( b \) an element of \( \mathcal{E} \) so that \( T(b) = \beta \). Such \( b \) is unique up to a permutation. Using the higher chain rule, we have

\[
D^\beta f(A \cdot (x)) = \frac{\partial |\beta| f(A \cdot (x))}{\partial x^\beta}(x) = \sum_{e \in \mathcal{E}} \frac{\partial |\beta| f(y)}{\partial y^{T(e)}} |_{y = Ax} \prod_{i=1}^{|\beta|} \frac{\partial (Ax)^{(e)_i}}{\partial x^{(b)_i},}, \ x \in \mathbb{R}^d,
\]

where \( \prod_{i=1}^{|\beta|} \frac{\partial (Ax)^{(e)_i}}{\partial x^{(b)_i}} \) does not depend on \( x \) and \( D^\beta f(A \cdot (x)) \) does not depend on the choice of \( b \). For different elements \( e, h \in \mathcal{E} \), we may have \( T(e) = T(h) \). Thus, we can group terms in the sum with equal values of \( T(\cdot) \). Namely,

\[
D^\beta f(A \cdot (x)) = \sum_{\alpha \in \mathbb{Z}_+^d, |\alpha| = |\beta|} \frac{\partial |\beta| f(y)}{\partial y^{\alpha}} |_{y = Ax} \sum_{e \in \mathcal{E}, T(e) = \alpha} \prod_{i=1}^{|\beta|} \frac{\partial (Ax)^{(e)_i}}{\partial x^{(b)_i},}.
\]

(50)

If \( f(x) = e^{2\pi i(t \cdot x)}, \ t \in \mathbb{R}^d \), then \( D^\beta f(A \cdot (x)) = D^\beta e^{2\pi i(A^* t \cdot x)} = (A^* t)^\beta e^{2\pi i(t \cdot Ax)} \). On the other hand, by (50),

\[
D^\beta f(A \cdot (x)) = e^{2\pi i(t \cdot Ax)} \sum_{\alpha \in \mathbb{Z}_+^d, |\alpha| = |\beta|} t^\alpha \sum_{e \in \mathcal{E}, T(e) = \alpha} \prod_{i=1}^{|\beta|} \frac{\partial (Ax)^{(e)_i}}{\partial x^{(b)_i},}.
\]

Thus, due to (48) with the matrix \( A \) replaced by \( A^* \), we obtain

\[
\sum_{e \in \mathcal{E}, T(e) = \alpha} \prod_{i=1}^{|\beta|} \frac{\partial (Ax)^{(e)_i}}{\partial x^{(b)_i},} = [S(A^*, r)]_{\beta, \alpha} \frac{\beta!}{\alpha!}.
\]

(51)

Now let \( f \) be an arbitrary function continuously differentiable up to order \( N, 0 \leq r \leq N, \ r \in \mathbb{Z}_+ \). It follows from (49) and (51) that
Proposition 19  Let for every positive number \( \vartheta \)
\[
\sum_{\alpha \in \mathcal{O}_r} \frac{D^\alpha f(Ax)}{\alpha!} (At)^\alpha = \sum_{\alpha \in \mathcal{O}_r} D^\alpha f(Ax) \sum_{\beta \in \mathcal{O}_r} \left[ S(A, r) \right]_{\alpha, \beta} \frac{t^\beta}{\beta!} = \sum_{\beta \in \mathcal{O}_r} t^\beta \sum_{\alpha \in \mathcal{O}_r} D^\alpha f(Ax) \frac{[S(A^*, r)]_{\beta, \alpha}}{\alpha!} =
\]
\[
\sum_{\beta \in \mathcal{O}_r} \frac{t^\beta}{\beta!} \sum_{\alpha \in \mathcal{O}_r} D^\alpha f(Ax) \sum_{e \in \mathcal{E}} \frac{\partial (Ax)^{(e)}_i}{\partial x^{(0)}_i} = \sum_{\beta \in \mathcal{O}_r} \frac{t^\beta}{\beta!} D^\beta f(A)(x).
\]

It remains to sum the latter expression over \( r \) from 0 to \( N \).

Let
\[
L \left[ M^{-j} \right](k) = \sum_{|\beta| < N+1} a_\beta D^\beta \left[ M^{-j} \right](k), \quad a_\beta = \int_0^\infty du \frac{w(u)}{\beta! V_{h(u)}} \left[ M^{-j} \right](k) t^\beta dt.
\]

By Lemma \[18\] we have
\[
L \left[ M^{-j} \right](k) = \int_0^\infty du \frac{w(u)}{V_{h(u)}} \left[ M^{-j} \right](k) t^\beta dt =
\]
\[
\int_0^\infty du \frac{w(u)}{V_{h(u)}} \left[ M^{-j} \right](k) t^\beta dt
\]

and, due to the Tailor formula,
\[
m^j \int_0^\infty du \frac{w(u)}{V_{h(u)}} \left[ M^{-j} \right](k) t^\beta dt = L \left[ M^{-j} \right](k).
\]

To investigate the convergence and approximation order of falsified sampling expansions we can use Theorems \[13 \] \[14 \] \[16\] and estimate the sum \( m^{-j/2} \sum_{k \in \mathbb{Z}^d} \varepsilon_j(-k) \varphi_{jk} \), where
\[
\varepsilon_j(k) := \int_0^\infty w(u) Av_{h(u)}(f, M^{-j} k) du - L \left[ M^{-j} \right](k).
\]

Proposition 19  Let \( d < p \leq \infty \), \( \varphi \in \mathcal{L}_p \) or \( \varphi \in \mathcal{B} \), \( p \neq \infty \). Suppose \( N \in \mathbb{N} \), \( f \in W_p^{N+1} \), the operator \( L \) is defined by \[24\], \( \varepsilon_j(k) \) is defined by \[24\], \( w \) and \( h \) are as in \[4\]. Then
\[
\left\| m^{-j/2} \sum_{k \in \mathbb{Z}^d} \varepsilon_j(-k) \varphi_{jk} \right\|_p \leq C \int_0^\infty w(u)(1 + h^{N+1}(u)) du \| f \|_{W_p^{N+1}} \vartheta^{-j(N+1)}
\]

for every positive number \( \vartheta \) which is smaller in module than any eigenvalue of \( M \) and some \( C \) which does not depend on \( f \), \( h \), \( w \), and \( j \).

Proof. Let us fix \( j \in \mathbb{N} \), \( \varepsilon_j = \{ \varepsilon_j(k) \}_{k \in \mathbb{Z}^d} \). If \( \varphi \in \mathcal{L}_p \), then, due to Proposition \[1\]
\[
\left\| m^{-j/2} \sum_{k \in \mathbb{Z}^d} \varepsilon_j(-k) \varphi_{jk} \right\|_p = m^{-j/p} \left\| \sum_{k \in \mathbb{Z}^d} \varepsilon_j(-k) \varphi_{0k} \right\|_p \leq m^{-j/p} \| \varphi \|_{\mathcal{L}_p} \| \varepsilon_j \|_{l_p}.
\]

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If \( \varphi \in \mathcal{B} \) and \( p \neq \infty \), then, due to Proposition 9,

\[
\left\| m^{-j/2} \sum_{k \in \mathbb{Z}^d} \varepsilon_j(-k) \varphi_j \right\|_p = m^{-j/p} \left\| \sum_{k \in \mathbb{Z}^d} \varepsilon_j(-k) \varphi_{0k} \right\|_p \leq m^{-j/p} C_{p,q} \| \varepsilon_j \|_{L^p}. \tag{57}
\]

By the Taylor formula with integral remainder, we have

\[
f(M^{-j}k + t) = \sum_{[\beta] < N+1} D^\beta f(M^{-j}k) \frac{t^\beta}{\beta!} + \sum_{\beta \in \mathbb{Z}_+^d, [\beta] = N+1} \frac{N+1}{\beta!} \int_0^1 (1 - \tau)^N D^\beta f(M^{-j}k + \tau t) d\tau.
\]

It follows from (53) that

\[
\varepsilon_j(k) = m^j \int_0^\infty du \frac{w(u)}{V_h(u)} \int_{M^{-j}B_{h(u)}} dt \sum_{\beta \in \mathbb{Z}_+^d, [\beta] = N+1} \frac{N+1}{\beta!} \int_0^1 (1 - \tau)^N D^\beta f(M^{-j}k + \tau t) dt.
\]

Hence, taking into account that \( |t^\beta| \leq |t|^{[\beta]} \), we get

\[
|\varepsilon_j(k)| \leq \sum_{\beta \in \mathbb{Z}_+^d, [\beta] = N+1} \frac{N+1}{\beta!} m^j \int_0^\infty du \frac{w(u)}{V_h(u)} \int_{M^{-j}B_{h(u)}} dt |t|^{[\beta]} \int_0^1 |D^\beta f(M^{-j}k + \tau t)| d\tau \leq \]

\[
\sum_{\beta \in \mathbb{Z}_+^d, [\beta] = N+1} \frac{N+1}{\beta!} m^j \int_0^\infty du \frac{w(u)}{V_h(u)} \int_{M^{-j}B_{h(u)}} dt |t|^{[\beta]} \int_0^1 \left| D^{\beta} f(M^{-j}k + M^{-j}t) \right| \frac{dt}{t^{[\beta] + d}} \leq \]

The latter integration is taken over the set \( \{ \tau \in [0,1], t \in \tau B_{h(u)} \} = \{ \tau \in [0,1], |t| \leq \tau h(u) \} \), or equivalently \( \{|t| \in [0,h(u)], \frac{|t|}{h(u)} \leq \tau \leq 1 \} \). Changing the order of integration, we obtain

\[
|\varepsilon_j(k)| \leq \sum_{\beta \in \mathbb{Z}_+^d, [\beta] = N+1} \frac{N+1}{\beta!} \int_0^\infty du \frac{w(u)}{V_h(u)} \int_{B_{h(u)}} dt |M^{-j}t|^{[\beta]} |D^{\beta} f(M^{-j}k + M^{-j}t)| \int_0^1 \frac{dt}{t^{N+d+1}} \leq \]

\[
\sum_{\beta \in \mathbb{Z}_+^d, [\beta] = N+1} \frac{(N+1)}{(N+d)\beta!} \int_0^\infty du \frac{w(u)}{V_h(u)} \int_{B_{h(u)}} dt |M^{-j}t|^{N+1} |D^{\beta} f(M^{-j}k + M^{-j}t)| \left( \frac{h(u)}{|t|} \right)^{N+d} \leq \]

\[
\sum_{\beta \in \mathbb{Z}_+^d, [\beta] = N+1} \frac{\|M^{-j}\|^{N+1}}{\beta!} \int_0^\infty du \frac{w(u)h^{N+d}(u)}{V_h(u)} \int_{B_{h(u)}} dt |D^{\beta} f(M^{-j}k + M^{-j}t)| \left| \frac{h(u)}{|t|^{d-1}} \right|^d \leq \]

\[
\sum_{\beta \in \mathbb{Z}_+^d, [\beta] = N+1} \frac{\|M^{-j}\|^{N+1}}{\beta!} \int_0^\infty du \frac{w(u)h^{N+d}(u)}{V_h(u)} \int_{M^{-j}B_{h(u)}} dt |D^{\beta} f(M^{-j}k + M^{-j}t)| \left| \frac{h(u)}{|M^{-j}t|^{d-1}} \right|^d dt.
\]
If $p = \infty$, then
\[
\int_{M^{-j}B_{h(u)}} \frac{|D^\beta f(M^{-j}k + t)|}{|Mt|^{d-1}} \, dt \leq \int_{M^{-j}B_{h(u)}} \frac{dt}{|Mt|^{d-1}} \|f\|_{W^N_{\infty}}.
\]

If $p \neq \infty$, then using Hölder’s inequality, we have
\[
\int_{M^{-j}B_{h(u)}} \frac{|D^\beta f(M^{-j}k + t)|}{|Mt|^{d-1}} \, dt \leq \left( \int_{M^{-j}B_{h(u)}} \frac{dt}{|Mt|^{q(d-1)}} \right)^{1/p} \left( \int_{M^{-j}B_{h(u)}} \frac{dt}{|Mt|^{q(d-1)}} \right)^{1/q},
\]
where $q = \frac{p}{p-1}$. Since
\[
\int_{M^{-j}B_{h(u)}} \frac{dt}{|Mt|^{q(d-1)}} = m^{-j} \int_{B_{h(u)}} \frac{dt}{|t|^{q(d-1)}} = m^{-j} \int_{B_1} \frac{dt}{|t|^{q(d-1)}} = m^{-j} \int_{B_1} \frac{dt}{|t|^{q(d-1)}}
\]
and $q(d-1) < d$, the latter integral is finite. Summarizing the above estimates, we obtain
\[
|\varepsilon_j(k)|^p \leq C_1 m^j \|M^{-j}\|^{p(N+1)} \int_0^\infty du w(u) \int_{M^{-j}k + M^{-j}B_{h(u)}} \sum_{\beta \in \mathbb{Z}^d, |\beta| = N+1} \|D^\beta f(t)|^p \, dt,
\]
and
\[
\sum_{k \in \mathbb{Z}^d} |\varepsilon_j(k)|^p \leq C_2 m^j \|M^{-j}\|^{p(N+1)} \int_0^\infty \int_{M^{-j}k + M^{-j}B_{h(u)}} w(u) \|h^{p(N-d+1+d/q)}(u)(1 + h^d(u))\|f\|_{W^N_{p^+}}^p \, dt.
\]

where $C_2$ does not depend on $f$, $h$, and $j$. It follows that
\[
\sum_{k \in \mathbb{Z}^d} |\varepsilon_j(k)|^p \leq C_2 m^j \|M^{-j}\|^{p(N+1)} \int_0^\infty \int_{M^{-j}k + M^{-j}B_{h(u)}} w(u) \|h^{p(N-d+1+d/q)}(u)(1 + h^d(u))\|f\|_{W^N_{p^+}}^p \, dt.
\]

Combining this with (55), (57) and (59), we get (55). ◇

**Proposition 20** Let $d < p' < \infty$, $\varphi \in B$. Suppose $N \in \mathbb{N}$, $f \in W^{N+1}$, operator $L$ is defined by (52), $\varepsilon_j(k)$ is defined by (54), $w$ and $h$ are as in (47). Then
\[
\left\| m^{-j/2} \sum_{k \in \mathbb{Z}^d} \varepsilon_j(k) \varphi_{jk} \right\|_{\infty} \leq C \int_0^\infty w(u)(1 + h^{N+1}(u)) \, du \|f\|_{W^{N+1}_{p^+}} \vartheta^{-j(N+1)}
\]
for every positive number $\vartheta$ which is smaller in module than any eigenvalue of $M$ and some $C$ which does not depend on $f$, $h$, $w$ and $j$.

**Proof.** Let us fix $j \in \mathbb{N}$, $\varepsilon_j = \{\varepsilon_j(k)\}_{k \in \mathbb{Z}^d}$. Because of (22), it is easy to see that
\[
\sum_{k \in \mathbb{Z}^d} |\varphi(x + k)|^q < C'_{\varphi,q}
\]
for every $q > 1$ and every $x \in \mathbb{R}^d$. It follows that
\[
\left\| m^{-j/2} \sum_{k \in \mathbb{Z}^d} \varepsilon_j(-k) \varphi_{jk} \right\|_{\infty} = \sup_{x \in \mathbb{R}^d} \left\| \sum_{k \in \mathbb{Z}^d} \varepsilon_j(-k) \varphi(M^j x + k) \right\|_{\infty} \leq (C'_{\varphi,q})^{1/q'} \|\varepsilon_j\|_{\ell_{p'}}.
\]
where $1/q' + 1/p' = 1$. To get (58) it remains to repeat all arguments of the proof of Theorem 19 after relation (57), replacing $p$ and $q$ by $p'$ and $q'$ respectively. ◇
Remark 21 If $M$ is an isotropic matrix for which $\lambda$ is an eigenvalue, then in the proof of Propositions 18, 20 we can use inequality (5) instead of (6). Hence $\vartheta$ in (53) and (58) can be replaced by $\lambda$.

Using the above results we can state the convergence and approximation order of falsified sampling expansions.

**Theorem 22** Let $d < p \leq \infty$, $p \geq 2$, $1/p + 1/q = 1$, $N \in \mathbb{Z_+}$, $M$ be an isotropic matrix dilation and $\lambda$ be its eigenvalue. Suppose $\varphi$ and $n$ are as of Theorem 19, where $\widetilde{\varphi}$ is the distribution associated with the differential operator $L$ given by (52), $w$ and $h$ are as in (47) and

$$f \in L_p, \hat{f} \in L_q, \hat{f}(\xi) = O(|\xi|^{-N-\frac{d}{p} - \varepsilon}) \text{ as } |\xi| \to \infty, \varepsilon > 0. \text{ Then,}$$

$$\left\| f - m^{-j/2} \sum_{k \in \mathbb{Z}^d} E(f, M^{-j}k) \varphi_{jk} \right\|_p \leq \begin{cases} C|\lambda|^{-j(N+1)} & \text{if } n > N + 1, \\ C|\lambda|^{-jn} & \text{if } n \leq N + 1, \end{cases}$$

(59)

where $C$ does not depend on $j$.\[\Box\]

**Proof.** The proof follows from Theorem 13, item (B), Proposition 19 and Remark 21.

**Theorem 23** Let $d < p \leq \infty$, $p \geq 2$, $1/p + 1/q = 1$, $N \in \mathbb{Z_+}$. Suppose $\varphi$ is as in item (b) of Theorem 12 or $\varphi \in \mathcal{B}$, $p \neq \infty$; $\widetilde{\varphi}$ is the distribution associated with the differential operator $L$ given by (52), $\widetilde{\varphi}$ and $\varphi$ are strictly compatible; $w$ and $h$ are as in (47). $f \in L_p, \hat{f} \in L_q, \hat{f}(\xi) = O(|\xi|^{-N-\frac{d}{p} - \varepsilon}) \text{ as } |\xi| \to \infty, \varepsilon > 0. \text{ Then,}$$

$$\left\| f - m^{-j/2} \sum_{k \in \mathbb{Z}^d} E(f, M^{-j}k) \varphi_{jk} \right\|_p \leq C\vartheta^{-j(N+1)}$$

(60)

for every positive number $\vartheta$ which is smaller in module than any eigenvalue of $M$ and some $C$ which does not depend on $j$.\[\Box\]

**Proof.** The proof follows from Theorem 13, item (C), Theorem 14, and Proposition 19.

**Remark 24** If the assumption on $f$ in Theorems 22 and 23 is replaced by $f \in W_p^{N+1}$, $\hat{f} \in L_q, \hat{f}(\xi) = O(|\xi|^{-N-\frac{d}{p} - \varepsilon}) \text{ as } |\xi| \to \infty, \text{ then Theorem 23 remains to be true, and inequality (58) in Theorem 22 must be replaced by}$$

$$\left\| f - m^{-j/2} \sum_{k \in \mathbb{Z}^d} E(f, M^{-j}k) \varphi_{jk} \right\|_p \leq \begin{cases} C|\lambda|^{-j(N+1)} & \text{if } n > N + 1, \\ C(j + 1)^{1/2}|\lambda|^{-j(N+1)} & \text{if } n = N + 1, \\ C|\lambda|^{-jn} & \text{if } n \leq N + 1. \end{cases}$$

Theorem 25 Let $d < p' < \infty$, $1/p' + 1/q' = 1$, $N \in \mathbb{Z_+}$. Suppose $\varphi \in \mathcal{B}$, $\widetilde{\varphi}$ is the distribution associated with the differential operator $L$ given by (52), $\widetilde{\varphi}$ and $\varphi$ are strictly compatible; $w$ and $h$ are as in (47). $f \in L_{p'}, \hat{f} \in L_{q'}, \hat{f}(\xi) = O(|\xi|^{-N-\frac{d}{p'} - \varepsilon}) \text{ as } |\xi| \to \infty, \varepsilon > 0. \text{ Then,}$

$$\left\| f - m^{-j/2} \sum_{k \in \mathbb{Z}^d} E(f, M^{-j}k) \varphi_{jk} \right\|_\infty \leq C\vartheta^{-j(N+1)}$$

(61)

for every positive number $\vartheta$ which is smaller in module than any eigenvalue of $M$ and some $C$ which does not depend on $j$.\[\Box\]
Proof. The proof follows from Theorem 16 and Proposition 20. ▷

Remark 26 If the assumption on \( f \) in Theorem 25 is replaced by \( f \in W_p^{N+1}, \hat{f} \in L_1, \hat{f}(\xi) = O(|\xi|^{-N-d-c}) \) as \( |\xi| \to \infty, \varepsilon > 0 \), then Theorem 25 remains to be true with

\[
\left\| f - m^{-j/2} \sum_{k \in \mathbb{Z}^d} E(f, M^{-j}k) \varphi_{jk} \right\|_\infty \leq C \delta^{-j(N+\min(1,\varepsilon))}
\]

instead of [64].

Observing the proof of Proposition 19, one can see that in the one-dimensional case an analog of (55) holds true for a wider class of functions \( f \) and any \( p \geq 1 \). Indeed, in this case we have

\[
|\varepsilon_j(k)| \leq \frac{1}{(N+1)!} \int_0^\infty du \frac{w(u)}{V_{h(u)}} \int |M^{-j}t|^{N+1} |f^{(N+1)}(M^{-j}k + M^{-j}t)| \left( \frac{h(u)}{|t|} \right)^{N+1} dt \leq
\]

\[
\frac{M^{-j(N+1)}}{(N+1)!} \int_0^\infty du \frac{w(u)h^{N+1}(u)}{V_{h(u)}} \int |f^{(N+1)}(M^{-j}k + M^{-j}t)| dt =
\]

\[
\frac{M^{-jN}}{(N+1)!} \int_0^\infty du \frac{w(u)h^{N+1}(u)}{V_{h(u)}} \int_{M^{-j}B_{h(u)} + M^{-j}k} |f^{(N+1)}(t)| dt.
\]

It follows that

\[
\left\| m^{-j/2} \sum_{k \in \mathbb{Z}} \varepsilon_j(-k) \varphi_{jk} \right\|_p \leq m^{-j/p} \| \varphi \|_p \sum_{k \in \mathbb{Z}} |\varepsilon_j(k)| \leq C \int_0^\infty w(u)h^N(u) du \| f \|_{W_1^{N+1} L^{-j(N+\frac{1}{p})}}.
\]

This yields the following statements.

Proposition 27 Let \( d = 1, p \geq 1, N \in \mathbb{N}, \varphi \in L_p \). Suppose \( f \in W_1^{N+1}, \varepsilon_j(k) \) is defined by (54), \( w \) and \( h \) are as in (47) and

\[
\int_0^\infty w(u)(1 + h^N(u)) du < \infty.
\]

Then

\[
\left\| M^{-j/2} \sum_{k \in \mathbb{Z}} \varepsilon_j(-k) \varphi_{jk} \right\|_p \leq C \| f \|_{W_1^{N+1} L^{-j(N+\frac{1}{p})}}, \tag{62}
\]

where \( C \) does not depend on \( f \) and \( j \).

Theorem 28 Let \( d = 1, 2 \leq p \leq \infty, N \in \mathbb{N} \). Suppose \( \varphi \) and \( n \) are as in of Theorem 13, where \( \tilde{\varphi} \) is the distribution associated with the differential operator \( L \) given by [32], or \( \varphi \in B; w \) and \( h \) are as in Proposition 27, \( f \in W_1^{N+1}, f^{(N+1)} \in \text{Lip}_{L_1}, \varepsilon > 0 \). Then

\[
\left\| f - M^{-j/2} \sum_{k \in \mathbb{Z}} E(f, M^{-j}k) \varphi_{jk} \right\|_p \leq \begin{cases} CM^{-j(N+\frac{1}{p})} & \text{if } n > N + \frac{1}{p}, \\ CM^{-jn} & \text{if } n \leq N + \frac{1}{p}, \end{cases}
\]

where \( C \) does not depend on \( j \).
Theorem 29 Let \( d = 1, 2 \leq p \leq \infty, N \in \mathbb{Z}_+ \). Suppose \( \varphi \) is as in item (b) of Theorem \ref{tw} or \( \varphi \in B \); \( \tilde{\varphi} \) is the distribution associated with the differential operator \( L \) given by \eqref{e52}, \( \tilde{\varphi} \) and \( \varphi \) are strictly compatible; \( w \) and \( h \) are as in Proposition \ref{p27}; \( f \in W^{N+1}_1 \), \( f^{(N+1)} \in \text{Lip}_1, \varepsilon, \varepsilon > 0 \). Then

\[
\left\| f - M^{-j/2} \sum_{k \in \mathbb{Z}} E(f, M^{-j} k) \varphi_{jk} \right\|_p \leq CM^{-j(N + \frac{p}{2})}
\]  

for some \( C \) which does not depend on \( j \).

6 Examples

In this section some examples will be given to illustrate the above results.

I. First we discuss construction of band-limited functions \( \varphi \). For every differential operator \( L \) one can easily construct \( \varphi \) supported on a small neighborhood of zero and such that on the same neighborhood \( \hat{\varphi} = 1 \), where \( \hat{\varphi} \) is a distribution associated with \( L \). The function \( \varphi \) is in \( B \), \( \tilde{\varphi} \) and \( \varphi \) are strictly compatible, so, due to Theorems \ref{t13} and \ref{t23} the approximation order of the corresponding differential and falsified expansions with arbitrary matrix dilation depends only on how smooth is a function \( f \). In particular, in the case \( Lf = f \), the function \( \varphi(x) = \prod_{k=1}^d \frac{\sin \pi \xi_k}{\pi \xi_k} \) can be taken as \( \varphi \). Note that this function \( \varphi \) is appropriate not only for the expansions with exact sampled values, but also for falsified expansions. Indeed, if \( N = 2 \), then \( Lf = f + \sum_{|\beta| = 1} a_\beta D^\beta f \), where \( a_\beta = 0 \) by \eqref{e52}. Thus \( \varphi \) satisfies all conditions of Theorem \ref{t23} with \( N = 1 \), and, according to \eqref{e60}, the approximation order of the corresponding falsified sampling expansions is 2 for smooth enough functions \( f \).

Hence, theoretically, we have a simple and very good solution to our problem. However such expansions are not good from the computational point of view because in the case \( Lf \neq f \) we will not be able to derive explicit formulas which is needed for implementations.

II. Let

\[
\tilde{\varphi}(\xi) = \prod_{k=1}^d \left( \frac{\sin \pi \xi_k}{\pi \xi_k} \right)^2.
\]

Since \( \varphi(x) = \prod_{k=1}^d (1 - |x_k|) \chi_{[-1,1]^d}(x) \), \( \varphi \) is compactly supported and in \( L_p \). Also, the function \( \sum_{k \in \mathbb{Z}^d} |\tilde{\varphi}(\xi + k)|^q \) is bounded, \( \tilde{\varphi} \) is continuously differentiable up to any order, the function \( \sum_{l \in \mathbb{Z}^d, l \neq 0} |D^\beta \tilde{\varphi}(\xi + l)| \) is bounded near the origin for \( |\beta| = 0 \). Also, the Strang-Fix condition of order 2 holds for \( \varphi \). The values of \( \tilde{\varphi} \) and its derivatives at the origin are

\[
\tilde{\varphi}(0) = 1, \quad D^\beta \tilde{\varphi}(0) = 0, \quad |\beta| = 1.
\]

So, if \( \tilde{\varphi} = \delta \), then all assumptions of Theorem \ref{t13} are satisfied. The corresponding sampling expansion of a signal \( f \) interpolates \( f \) at the points \( M^{-j} k, k \in \mathbb{Z}^d \), the approximation order depends on how smooth is \( f \), but, according to \eqref{e10}, it cannot be better than 2. Again \( \tilde{\varphi} \) is associated with the differential operator \( Lf = f + \sum_{|\beta| = 1} a_\beta D^\beta f \), where \( a_\beta = 0 \). Hence the functions \( \varphi, \tilde{\varphi} \) satisfy all conditions of Theorem \ref{t22} with \( n = 2 \), \( N = 1 \), and, according to \eqref{e60}, the approximation order of the corresponding falsified sampling expansions is 2 for smooth enough functions \( f \).

III. Let \( d = 2 \),

\[
\tilde{\varphi}(\xi_1, \xi_2) = \frac{1}{(\pi^2 \xi_1 \xi_2)^3} \left( \sin^3 \pi \xi_1 \sin^3 \pi \xi_2 + b_1 \sin^3 \pi \xi_1 \sin^4 \pi \xi_2 + b_2 \sin^4 \pi \xi_1 \sin^3 \pi \xi_2 \right).
\]

Again \( \varphi \) is compactly supported and belongs to \( L_p \). Also, the function \( \sum_{k \in \mathbb{Z}^d} |\tilde{\varphi}(\xi + k)|^q \) is bounded, \( \tilde{\varphi} \) is continuously differentiable up to any order. Since the trigonometric polynomial in the numerator
of $\widehat{\varphi}$ is bounded, the function $\sum_{l \in \mathbb{Z}, l \neq 0} |D^\beta \widehat{\varphi}(\xi + l)|$ is bounded near the origin for $|\beta| = 3$. Also, the Strang-Fix condition of order 3 holds for $\varphi$. The values of $\widehat{\varphi}$ and its derivatives at the origin are

$$\widehat{\varphi}(0, 0) = 1, \quad D^{(1, 0)}\widehat{\varphi}(0, 0) = D^{(0, 1)}\widehat{\varphi}(0, 0) = D^{(0, 0)}\widehat{\varphi}(0, 0) = 0,$$

Thus, we set $b_1 = b_2 = \frac{1}{2}(1 - 4a_{(2, 0)})$. According to [59], the approximation order of the corresponding falsified sampling expansions is 3 for smooth enough functions $f$.

IV. Let $d = 1$,

$$\widehat{\varphi}(\xi) = \frac{\sin^4 \pi \xi + \int_{0}^{\infty} h(x)^2 \omega(x) dx}{(\pi \xi)^4}.$$
Thus, the coefficients of the function $\hat{\varphi}$ can be easily found using the coefficients of the differential operator $L$. Finally, all conditions of Theorem 13 are satisfied. The approximation order depends on how smooth is $f$, but, according to (41) with $|\lambda| = |M|$, it cannot be better than $n = 4$.

Now we show that the coefficients $b_1, b_2, b_3$ can be chosen such that all conditions of Theorem 22 are satisfied with $n = 4, N = 3$. In this case the differential operator $Lf = a_0 f + a_1 f' + a_2 f'' + a_3 f'''$ is given by (52) and its coefficients are defined as

$$a_0 = 1, \quad a_1 = 0, \quad a_2 = \frac{1}{6} \int_0^\infty h(x)\omega(x)dx, \quad a_3 = 0.$$ 

Using (65), we set $b_1 = 0, b_2 = \frac{\omega}{4} + 4a_2, b_3 = 0$. According to (53), the approximation order of the corresponding falsified sampling expansions is 4 for smooth enough functions $f$.

Note that all conditions of Theorem 28 are also satisfied, which provides approximation order for a wider class of functions $f$. Namely, according to (63), the approximation order is $3 + \frac{1}{p}$, whenever $f \in W_1^4, f^{(IV)} \in \text{Lip}_{L_1}, \varepsilon > 0$. ◇

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