A WEAK SPACE-TIME FORMULATION FOR THE LINEAR STOCHASTIC HEAT EQUATION

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Abstract. We apply the well-known Banach-Nečas-Babuška inf-sup theory in a stochastic setting to introduce a weak space-time formulation of the linear stochastic heat equation with additive noise. We give sufficient conditions on the data and on the covariance operator associated to the driving Wiener process, in order to have existence and uniqueness of the solution. We show the relation of the obtained solution to the mild solution and to the variational solution of the same problem. The spatial regularity of the solution is also discussed. Finally, an extension to the case of linear multiplicative noise is presented.

1. Introduction

We consider a linear parabolic stochastic evolution problem of the form
\begin{equation}
\begin{aligned}
&dU(t) + A(t)U(t) \, dt = f(t) \, dt + \Psi(t) \, dW(t), \\
&U(0) = U_0
\end{aligned}
\end{equation}
(1.1)

We assume that \( A(t) \) is a random elliptic operator defined within a Gelfand triple setting as follows. Given separable Hilbert spaces \( V, H \), we consider a Gelfand triple \( V \subset H \subset V^* \), where \( V \) has a compact and dense embedding into \( H \). We denote by \( \langle \cdot, \cdot \rangle_H \) the inner product in \( H \) and by \( \langle \cdot, \cdot \rangle_{V^*V} \) the dual pairing between \( V \) and \( V^* \) with \( \langle u, v \rangle_{V^*V} = \langle u, v \rangle_H \), \( \forall v \in V \) whenever \( u \in H \). Further, we denote by \( \mathcal{L}(H) = \mathcal{L}(H;H) \) the space of bounded linear operators on \( H \) and by \( \mathcal{L}_2(H) = \mathcal{L}_2(H;H) \) the Hilbert-Schmidt operators.

Let \( T \in (0, \infty) \) be fixed and let \( (\Omega, \Sigma, \mathbb{P}) \) be a complete probability space, with normal filtration \( \Sigma = (\Sigma_t)_{t \in [0,T]} \). We assume that a progressively measurable map \( A: \Omega \times [0,T] \times V \to V^* \), coercive and bounded \( \mathbb{P} \otimes dt \)-a.s., is given, with associated bilinear form \( a \) given by \( a(\omega,t;u,v) = \langle A(\omega,t)u,v \rangle_H \). We consider a predictable process with Bochner integrable trajectories \( f \in L^2(\Omega \times (0,T);V^*) \) and we assume that \( W = (W(t))_{t \in [0,T]} \) is a \( Q \)-Wiener process with covariance operator \( Q \), and a predictable operator-valued process \( \Psi \) such that \( \Psi Q^{1/2} \in L^2(\Omega \times (0,T);\mathcal{L}_2(H)) \).

A typical example would be \( V = H^1_0(D) \subset H = L^2(D) \) with a spatial domain \( D \) and a random elliptic operator of the form \( A(\omega,s)u = -\nabla \cdot (a(\omega,s)\nabla u) + b(\omega,s) \cdot \nabla u + c(\omega,s)u \) with suitable assumptions on the coefficients.

In order to give a meaning to (1.1), we have to define what we mean by a solution. In the special case when \( A \) is independent of \( t \) and \( \omega \) and considered as unbounded operator in \( H \), we have the concepts of weak and mild solution, see [6].

Definition 1 (Weak and mild solution). Let the operator \( A \) be possibly unbounded, independent of \( \omega \) and \( t \), and defined on a certain domain \( D(A) \) dense in \( H \), i.e.,
A: \( D(A) \subset H \to H \). A weak solution to (1.1) is an \( H \)-valued, predictable stochastic process \( U(t) \), which is Bochner integrable \( \mathbb{P} \)-a.s. and satisfies

\[
\langle U(t), v \rangle_H = \langle U_0, v \rangle_H - \int_0^t \langle U(s), A^\ast v \rangle_H \, ds + \int_0^t \langle f(s), v \rangle_H \, ds \\
+ \int_0^t \langle \Psi(s) \, dW(s), v \rangle_H, \quad \mathbb{P}\text{-a.s.}, \forall v \in D(A^\ast), \ t \in [0,T].
\]

In particular \(-A\) is the generator a strongly continuous semigroup \((S(t))_{t \geq 0}\) in \( H \) and \( \int_0^T \|S(s)\Psi(s)Q^\frac{1}{2}\|_{\mathcal{F}(H)}^2 \, ds < \infty \), so that the unique weak solution coincides with the mild solution, given by the formula

\[
(1.3) \quad U(t) = S(t)U_0 + \int_0^t S(t-s)f(s) \, ds + \int_0^t S(t-s)\Psi(s) \, dW(s), \quad t \in [0,T].
\]

Within the semigroup framework it is possible to prove results about about spatial regularity and temporal Hölder-continuity of the solution, by defining spaces of fractional order, \( H^\beta := D(A^\frac{\beta}{2}) \), and exploiting the semigroup theory. For example, in the parabolic case, when the semigroup is analytic, it was shown in [15] that if \( U_0 \in L^2(\Omega; H^\beta) \), \( f = 0 \), \( \Psi(t) = I \), and \( \|A^{\frac{\beta-1}{2}}Q^\frac{1}{2}\|_{\mathcal{F}(H)} < \infty \) for some \( \beta \geq 0 \), then the mild solution satisfies

\[
\|U(t)\|_{L^2(\Omega; H^\beta)} \leq C\left(\|U_0\|_{L^2(\Omega; H^\beta)} + \|A^{\frac{\beta-1}{2}}Q^\frac{1}{2}\|_{\mathcal{F}(H)}\right), \quad t \in [0,T].
\]

The concept of mild solution presents however the disadvantage of not being applicable whenever the operator does not generate a semigroup. This fact provides a good reason to look for more general concepts of solution that do not rely at all on such a theory.

For this purpose we recall that, in order to derive the mild solution formula \(1.3\), [6] proceeds from the weak formulation \(1.2\) with time-independent deterministic test functions, to a weak formulation with time-dependent deterministic test functions, c.f. Lemma [5] below:

\[
\langle U(t), v(t) \rangle_H = \langle U_0, v(0) \rangle_H + \int_0^t \langle U(s), \dot{v}(s) - A^\ast v(s) \rangle_H \, ds \\
+ \int_0^t \langle f(s), v(s) \rangle_H \, ds + \int_0^t \langle \Psi(s) \, dW(s), v(s) \rangle_H.
\]

This suggests the possibility of using a weak space-time formulation, which would be to find a pair \((U_1, U_2)\) such that

\[
\int_0^t \langle U_1(s), -\dot{v}(s) + A^\ast v(s) \rangle_H \, ds + \langle U_2, v(t) \rangle_H = \langle U_0, v(0) \rangle_H + \int_0^t \langle f(s), v(s) \rangle_H \, ds + \int_0^t \langle \Psi(s) \, dW(s), v(s) \rangle_H,
\]

for all \( v \) in a suitable class of test functions.

With a proper choice of function spaces, the well-posedness of this problem in the deterministic setting is obtained within the Banach-Nečas-Babuška inf-sup theory, see Section [2] below. In Section [3] we extend this to the stochastic evolution problem (1.1). The equation is solved \( \omega \)-wise and the inf-sup theory allows us to prove that a solution exists, is unique, and satisfies a bound that is expressed in terms of the data \( U_0, f, \Psi, \) and \( W, \mathbb{P} \)-a.s. By taking the expectation.
of this, we achieve a standard estimate for the norm of the solution in the space $L^2(\Omega \times (0, T); V) \cap L^2(\Omega; \mathcal{C}([0, T]; H))$, which is consistent with standard estimates presented, for example, in [11 Chapt. 5]. In particular, under suitable assumptions, our solution coincides with the mild solution. In Section 4, we briefly discuss the spatial regularity under such assumptions.

A more general solution concept is the variational solution, for which a comprehensive theory can be found, for example, in [10, Chapt. 4]. This theory applies to more general quasilinear equations, but we present it here for our linear equation.

**Definition 2** (Variational solution). Assume that $\Psi$ and $Q$ are as before, that is to say, $\Psi Q^{\frac{1}{2}} \in L^2(\Omega \times (0, T); \mathcal{L}_2(H))$. A continuous $H$-valued $\Sigma$-adapted process $(U(t))_{t \in [0, T]}$ is called a variational solution to (1.1), if for its $\mathbb{P} \otimes dt$ equivalence class $\hat{U}$ we have $\hat{U} \in L^2(\Omega \times (0, T), \mathbb{P} \otimes dt; V)$ and, for any $t \in [0, T]$,

$$U(t) = U_0 - \int_0^t A(s) \hat{U}(s) \, ds + \int_0^t f(s) \, ds + \int_0^t \Psi(s) \, dW(s), \quad \mathbb{P}\text{-a.s.,}$$

where $\hat{U}$ is any $V$-valued progressively measurable $\mathbb{P} \otimes dt$ version of $\hat{U}$.

We show in Lemma [5] that our solution coincides with such a solution, in particular, that our $U_1$ and $U_2$ play the roles of the $\hat{U}$ and $U$, respectively.

Finally, the norm bound that we obtain for the solution operator of the linear problem with additive noise allows us to use a standard fixed point technique and extend our theory to the case of multiplicative noise. In Section 5 we present this in the case of linear multiplicative noise. This approach extends to semilinear equations under appropriate global Lipschitz assumptions.

We want to remark that despite the fact that the concept of solution that we present is essentially no more general than the ones already known, it presents two advantages. It allows in fact the development of a theory for existence and uniqueness that is relatively easier than others and it states the problem in a way that can naturally be used for Petrov-Galerkin approximation of the problem. For this second reason our work can be seen as the potential starting point for future works dealing with numerical solutions for (1.1).

2. Preliminaries

2.1. The inf-sup theory. We recall the Banach-Nečas-Babuška (BNB) theorem, see [11][8], for example. Let $V$ and $W$ be Banach spaces, $W$ reflexive, and consider a bounded bilinear form $\mathcal{B}: W \times V \to \mathbb{R}$, with

$$C_B := \sup_{0 \neq w \in W} \sup_{0 \neq v \in V} \frac{\mathcal{B}(w, v)}{\|v\|_V \|w\|_W} < \infty,$$

and the associated bounded linear operator $B: W \to V^*$, i.e., $B \in \mathcal{L}(W, V^*)$, defined by

$$\langle Bw, v \rangle_V := \mathcal{B}(w, v), \quad \forall w \in W, \forall v \in V.$$

The operator $B$ is boundedly invertible if and only if the following conditions are satisfied:

$$c_B := \inf_{0 \neq w \in W} \sup_{0 \neq v \in V} \frac{\mathcal{B}(w, v)}{\|v\|_V \|w\|_W} > 0,$$

$$\forall 0 \neq v \in V, \sup_{0 \neq w \in W} \mathcal{B}(w, v) > 0.$$
The constant $c_B$ is called the inf-sup constant and, whenever both $V$ and $W$ are reflexive and \([\text{BNF1}]\) holds, we have the identity

\[
\inf_{0 \neq w \in W} \sup_{0 \neq v \in V} \frac{\mathcal{B}(w, v)}{\|v\|_V \|w\|_W} = \inf_{0 \neq v \in V} \sup_{0 \neq w \in W} \frac{\mathcal{B}(w, v)}{\|v\|_V \|w\|_W},
\]

which allows to swap the spaces where the infimum and the supremum are taken.

An immediate consequence of this is that the variational problem:

given $F \in V^*$, find $w \in W$: $\mathcal{B}(w, v) = F(v), \quad \forall v \in V,$

i.e., solve $Bw = F$ in $V^*$, and its adjoint:

given $G \in W^*$, find $v \in V$: $\mathcal{B}(w, v) = G(w), \quad \forall w \in W,$

i.e., solve $B^*v = G$ in $W^*$, are well-posed whenever $\text{[BDD]}, \text{[BNB1]}$ and $\text{[BNB2]}$ hold. In particular, the well-posedness of the former is equivalent to the well-posedness of the latter and the respective solutions satisfy

\[
\|w\|_W \leq \frac{1}{c_B} \|F\|_{V^*}, \quad \|v\|_V \leq \frac{1}{c_B} \|G\|_{W^*}.
\]

2.2. The inf-sup theory applied to an abstract parabolic problem. In recent years there has been a renewed interest for the tools presented above in order to deal with the linear heat equation starting from an abstract parabolic equation given in the Gelfand triple framework (see, for example, \([2, 3, 11, 12, 13, 14]\)). Assume indeed that Hilbert spaces $V, H$ are given, forming a Gelfand triple $V \subset H \subset V^*$ with bilinear forms

\[
a(t; \cdot, \cdot): V \times V \to \mathbb{R}, \quad t \in [0, T],
\]

satisfying the following conditions for some positive numbers $A_{\text{min}}, A_{\text{max}}$:

\[
|a(t; u, v)| \leq A_{\text{max}} \|u\|_V \|v\|_V, \quad t \in [0, T], \; u, v \in V,
\]

\[
a(t; v, v) \geq A_{\text{min}} \|v\|_V^2, \quad t \in [0, T], \; v \in V.
\]

For every $t \in [0, T]$, let $A(t)$ be the bounded linear operator from $V$ to $V^*$ associated with the bilinear form, i.e., $A(t) \in \mathcal{L}(V, V^*)$ and

\[
\langle A(t)u, v \rangle_V = a(t; u, v) = \langle u, A^*(t)v \rangle_{V^*}.
\]

Consider now the problem

\[
\begin{align*}
\dot{u}(t) + A(t)u(t) &= f(t) \quad &\text{in } V^*, \quad t \in (0, T), \\
u(0) &= u_0 \quad &\text{in } H,
\end{align*}
\]

where $\dot{u}(t)$ denotes the derivative of $u$ with respect to $t$, i.e., $\dot{u}(t) := \frac{du}{dt}$. Define the Lebesgue-Bochner spaces

\[
\mathcal{Y} = L^2((0, T); V),
\]

\[
\mathcal{X} = L^2((0, T); V) \cap H^1((0, T); V^*),
\]

normed by

\[
\|y\|_\mathcal{Y}^2 = \|y\|_{L^2((0, T); V)}^2 = \int_0^T \|y(t)\|_V^2 \; dt,
\]

\[
\|x\|_\mathcal{X}^2 = \|x\|_{L^2((0, T); V^*)}^2 + \|\dot{x}\|_{L^2((0, T); V^*)}^2 + \|x(0)\|_H^2 + \|x(T)\|_H^2.
\]

The trace theorem for Bochner-Lebesgue spaces (\([7\) Theorem 1, Chapter XVIII.1]), says that $\mathcal{X}$ is densely embedded in $\mathcal{C}([0, T]; H)$, so that $x(0), x(T) \in H$ are defined. Due to the inclusion of the boundary terms in the norm, the embedding constant
whose first space-time variational formulation is given by

\[ \int_0^t \left( v \cdot \langle x, y \rangle_V + v \cdot \langle x, y \rangle_V \right) ds = \langle x(t), y(t) \rangle_H - \langle x(0), y(0) \rangle_H. \]

This implies

\[ \|x(t)\|_H^2 = \|x(0)\|_H^2 + 2 \int_0^t \langle x(s), x(s) \rangle_V ds. \]

Hence, for arbitrary \( x \in \mathcal{X} \) and \( t \in [0, T] \),

\[ \|x(t)\|_H^2 \leq \|x(0)\|_H^2 + 2 \int_0^T \langle x(s), x(s) \rangle_V ds. \]

which leads to the following estimate for the embedding constant:

\[ \sup_{t \in [0,T]} \|x(t)\|_H^2 \leq \|x(0)\|_H^2 + \|x\|_{L^2((0,T);V)}^2 + \|\dot{x}\|_{L^2((0,T);V')}^2 \leq \|x\|_{\mathcal{X}}^2, \]

which can be referred to \[7, \text{Chapter XVIII}\] for a comprehensive presentation of these spaces.

A possible approach to solving the differential problem (2.3) is presented for example in \[11\] and it consists in integrating in time the dual pairing between the initial condition and another test vector \( y_2 \in H \), thus obtaining the following two equations:

\[ \int_0^T \langle v \cdot \langle \dot{u}(t), y_1(t) \rangle_V + a(t; u(t), y_1(t)) \rangle dt = \int_0^T \langle v \cdot \langle f(t), y_1(t) \rangle_V \rangle dt, \]

\[ \langle u(0), y_2 \rangle_H = \langle u_0, y_2 \rangle_H. \]

Adding the equations and defining \( \mathcal{Y}_H := \mathcal{Y} \times H \), Hilbert space normed by its product norm, gives the variational problem

\[ u \in \mathcal{X} : \mathcal{B}(u, y) = \mathcal{F}(y), \quad \forall y = (y_1, y_2) \in \mathcal{Y}_H, \]

where the following bilinear and linear forms are used

\[ \mathcal{B} : \mathcal{X} \times \mathcal{Y}_H \to \mathbb{R}, \]

\[ \mathcal{B}(x, y) := \int_0^T \left( v \cdot \langle \dot{x}(t), y_1(t) \rangle_V + a(t; x(t), y_1(t)) \right) dt + \langle x(0), y_2 \rangle_H, \]

\[ \mathcal{F} : \mathcal{Y}_H \to \mathbb{R}, \]

\[ \mathcal{F}(y) := \int_0^T v \cdot \langle f(t), y_1(t) \rangle_V dt + \langle u_0, y_2 \rangle_H. \]

We call this the first space-time variational formulation of (2.3).

Consider now the backward adjoint problem to (2.3):

\[ -\dot{v}(t) + A^*v(t) = g(t) \quad \text{in } V^*, t \in (0, T), \]

\[ v(T) = \xi \quad \text{in } H, \]

whose first space-time variational formulation is given by

\[ v \in \mathcal{X} : \mathcal{B}^*(y, v) = \mathcal{F}(y), \quad \forall y \in \mathcal{Y}_H. \]
Here the bilinear form is given by
\[ B^*: \mathcal{Y}_H \times \mathcal{X} \to \mathbb{R}, \]
\[ B^*(y, x) := \int_0^T \left( \nu(y_1(t), -\dot{x}(t))_{V^*} + a(t; y_1(t), x(t)) \right) \, dt + \langle y_2, x(T) \rangle_H, \]
and the load functional by
\[ S: \mathcal{Y}_H \to \mathbb{R}, \]
\[ S(y) := \int_0^T \nu(y_1(t), g(t))_{V^*} \, dt + \langle y_2, \xi \rangle_H. \]

Note that \( \mathcal{X} \subseteq \mathcal{Y}_H \) via the embedding \( y_1(t) = x(t), \ y_2 = x(0) \). By considering the restriction of the load functional \( S \) to \( \mathcal{X} \subseteq \mathcal{Y}_H \),
\[ \mathcal{S}: \mathcal{X} \to \mathbb{R}, \]
\[ \mathcal{S}(x) := \int_0^T \nu(x(t), x(t))_{V^*} \, dt + \langle u_0, x(0) \rangle_H, \]
and by interchanging the roles of trial and test spaces, the second (or weak) space-time formulation of the original problem (2.3) is obtained:
\[ (2.9) \quad u = (u_1, u_2) \in \mathcal{Y}_H: B^*(u, x) = \mathcal{S}(x), \ \forall x \in \mathcal{X}. \]

The first and the second formulations are related and the well-posedness of the former is equivalent to the well-posedness of the latter. More precisely, it holds that (by a suitable modification of the proofs in \([11, 13]\))
\[ C_B := \sup_{0 \neq x \in \mathcal{X}} \sup_{0 \neq y \in \mathcal{Y}_H} \frac{B^*(y, x)}{\|x\|_X \|y\|_{\mathcal{Y}_H}} \leq \sqrt{2} \max\{1, A_{\max}^2\}, \]
\[ c_B := \inf_{0 \neq x \in \mathcal{X}} \sup_{0 \neq y \in \mathcal{Y}_H} \frac{B^*(y, x)}{\|x\|_X \|y\|_{\mathcal{Y}_H}} \geq \min\{A_{\min}, A_{\max}^{-1}\}, \]
and, for any \( y \in \mathcal{Y}_H \),
\[ \sup_{0 \neq x \in \mathcal{X}} B^*(y, x) \geq \min\{1, A_{\min}\} \|y\|_{\mathcal{Y}_H}^2. \]

This shows that the operator \( B^* \in \mathcal{L}(\mathcal{X}, \mathcal{Y}_H^*) \), associated with the bilinear form \( B^*(\cdot, \cdot) \) via \( B^*(y, x) = \langle y, B^*x \rangle_{\mathcal{Y}_H^*} \) is boundedly invertible. This, in turn, implies that the operator \( B \in \mathcal{L}(\mathcal{Y}_H, \mathcal{X}^*) \) associated with \( B^*(\cdot, \cdot) \) via \( B^*(y, x) = \langle x, B^*y \rangle_X \) is also boundedly invertible, with the same inf-sup constant, see (2.1). Moreover, for \( f \in L^2((0, T); V^*) \) and \( u_0 \in H \), we have \( \mathcal{S} \in \mathcal{X}^* \). Hence, (2.4) is well-posed.

If a solution of (2.4) has the additional regularity \( u_1 \in \mathcal{X} \), then an integration by parts (2.4) shows that \( u_1 \) is a solution of the first problem (2.0) and that \( u_2 = u_1(T) \). This is the case when \( f \in L^2((0, T); V^*) \), as is easily seen. In this case the second component of the solution, \( u_2 \), is a continuous \( H \)-valued version of \( u_1 \), evaluated at time \( t = T \). Therefore, \( u_2 \) is redundant and in other works, e.g., \([3]\) and \([12]\), the weak space-time formulation is
\[ u \in \mathcal{Y}: B^*(u, x) = \mathcal{S}(x), \ \forall x \in \mathcal{X}_{0,T} := \{ x \in \mathcal{X}: x(T) = 0 \}. \]

Of course, more general functionals \( \mathcal{S} \) may be considered for which \( u_1 \not\in \mathcal{X} \), e.g.,
\[ \mathcal{S}(x) = \int_0^T \nu(x(t), g(t))_{V^*} \, dt. \]
As another example, in the next section we add a noise term to \( \mathcal{S} \). Then we find it useful to keep \( u_2 \).
3. A WEAK SPACE-TIME FORMULATION FOR THE LINEAR STOCHASTIC HEAT EQUATION

3.1. Existence and uniqueness. In order to introduce the weak space-time formulation for the equation (1.1), we will follow the idea outlined in Subsection 2.2. We consider spaces \( X \) and \( Y \) restricted to a time interval \([0, T]\), for fixed but arbitrary \( t \in [0, T] \), endowed with their respective natural norms. We denote these spaces

\[
X^0_t = L^2((0, t); V), \quad X^0_t = L^2((0, t); V) \cap H^1((0, t); V^*),
\]

normed by

\[
\|y\|^2_{X^0_t} = \|y\|^2_{L^2((0, t); V)}, \\
\|x\|^2_{X^0_t} = \|x\|^2_{L^2((0, t); V)} + \|\dot{x}\|^2_{L^2((0, t); V^*)} + \|x(0)\|^2_H + \|x(t)\|^2_H,
\]

with the convention that \( X = X^0_T \) and \( Y = Y^0_T \). The reason for introducing the parameter \( t \in [0, T] \) is that we want to display the time dependence of \( u_\omega \), so that we can take the supremum with respect to \( t \) and obtain norms and spaces consistent with the ones used in (10).

We assume that the family of operators \( A(\omega, s) \) is as in Section 1, i.e., that its bilinear forms satisfy the following conditions for some positive numbers \( A_{\min}, A_{\max} \):

\[
|a(\omega, s; u, v)| \leq A_{\max} \|u\|_V \|v\|_V, \quad (\omega, s) \in \Omega \times [0, T], \quad u, v \in V,
\]

\[
a(\omega, s; v, \cdot) \geq A_{\min} \|v\|^2_V, \quad (\omega, s) \in \Omega \times [0, T], \quad v \in V.
\]

We introduce a family of problems parametrized by \((\omega, t)\), defined by the bilinear forms

\[
\mathcal{B}^\omega_{t, y} : (X^0_t \times H) \times X^0_t \to \mathbb{R},
\]

\[
\mathcal{B}^\omega_{t, y}(y, x) := \int_0^t \left( \langle \dot{y}_1(s), -\dot{x}(s) \rangle_V + a(\omega, s; y_1(s), x(s)) \right) \, ds + \langle y_2, x(t) \rangle_H,
\]

and the load functionals

\[
\mathcal{F}_{\omega, t} : X^0_t \to \mathbb{R}, \quad \mathcal{W}_{\omega, t} : X^0_t \to \mathbb{R},
\]

where

\[
\mathcal{F}_{\omega, t}(x) = \int_0^t \langle f(\omega, s), x(s) \rangle_V \, ds + \langle U_0(\omega), x(0) \rangle_H,
\]

\[
\mathcal{W}_{\omega, t}(x) = \left( \int_0^t \langle \Psi(s) \, dW(s), x(s) \rangle_H \right)(\omega).
\]

The weak space-time formulation reads, for almost every \((\omega, t) \in \Omega \times [0, T]\):

\[
(3.1) \quad U_{\omega, t} \in Y^0_t \times H : \mathcal{B}^\omega_{t, y}(U_{\omega, t}, x) = \mathcal{F}_{\omega, t}(x) + \mathcal{W}_{\omega, t}(x), \quad \forall x \in X^0_t.
\]

Since our assumption on \( a(\omega, s; \cdot, \cdot) \) is uniform with respect to \( \omega, s \) with constants \( A_{\min}, A_{\max} \), we conclude that the bilinear forms \( \mathcal{B}^\omega_{t, y} \) satisfy the inf-sup conditions uniformly in \( \omega, t \) with the same constants \( C_B, c_B \) as in (2.10). This means that, for almost every \((\omega, t) \in \Omega \times [0, T]\), the operator \( B_{\omega, t} \in \mathcal{L}(Y^0_t \times H, (X^0_t)^*) \) associated to \( \mathcal{B}^\omega_{t, y} \) via \( \mathcal{B}^\omega_{t, y}(y, x) = \langle x, (B_{\omega, t} y, x) \rangle_{X^0_t} \) is boundedly invertible. Moreover, the norm of its inverse \( B_{\omega, t}^{-1} \) is bounded by \( c_B^{-1} \), uniformly in \( \omega, t \).
Proof. The constant hidden in $B$ and that $U \in S. LARSSON AND M. MOLTENI$

for almost every $(\omega, t) \in \Omega \times [0, T]$. Hence, by monotonicity in $t$, it follows that

$$E\left[\sup_{t \in [0, T]} \|\mathcal{F}_{\omega, t}\|_{(X_0^\alpha)^*}\right] \lesssim E\left[\|f\|_{L^2((0, T); V^*)} + \|U_0(\omega)\|_H\right].$$

The next step is provided by the following lemma, which shows that $\mathcal{W}_{\omega, t} \in (X_0^\alpha)^*$ with an estimate similar to the one in (3.3). In order to prove this, we let $A_0 \in \mathcal{L}(V, V^*)$ be the operator associated with the bilinear form $a_0(\cdot, \cdot) = \langle \cdot, \cdot \rangle_V$. Then $A_0$ does not depend on $(\omega, t)$ and satisfies the boundedness and coercivity (2.2) with constants $A_{\text{min}} = A_{\text{min}} = 1$. Then $-A_0$ is self-adjoint and the generator of an analytic semigroup $(S_0(t))_{t \geq 0}$, which is also self-adjoint, $(S_0(t))^* = S_0(t)$. Due to the compact embedding $V \subset H$ and the spectral theorem there is an orthonormal eigenbasis for $A_0$ in $H$. We denote the eigenpairs by $(\lambda_j, \phi_j)$, $j = 1, \ldots, \infty$.

In the generic example, where $V = H^1_0(\Omega) \subset H = L^2(\Omega)$ with $\langle u, v \rangle_V = \langle \nabla u, \nabla v \rangle_H$ and elliptic operator of the form $A(\omega, s)u = -\nabla \cdot (a(\omega, s)\nabla u) + b(\omega, s) \cdot \nabla u + c(\omega, s)u$, we would have $A_0 = -\Delta$, the Dirichlet Laplacian.

**Lemma 3.** If $\Psi Q^{\frac{1}{2}} \in L^2(\Omega \times (0, T); L^2(H))$, then there exists a process $K \in L^2(\Omega; C([0, T]; \mathbb{R}))$ such that, for almost every $(\omega, t) \in \Omega \times [0, T]$,

$$\mathcal{W}_{\omega, t}\|_{(X_0^\alpha)^*} \lesssim K(\omega, t)$$

and

$$E\left[\sup_{t \in [0, T]} K(\cdot, t)\right]^2 \lesssim E\left[\int_0^T \|\Psi(t)Q^{\frac{1}{2}}\|^2_{L^2(H)} \, dt\right].$$

Hence, $\mathcal{W}_{\omega, t} \in (X_0^\alpha)^*$ for almost every $(\omega, t) \in \Omega \times [0, T]$ and

$$E\left[\sup_{t \in [0, T]} \mathcal{W}_{\omega, t}\|_{(X_0^\alpha)^*}^2\right] \lesssim E\left[\int_0^T \|\Psi(t)Q^{\frac{1}{2}}\|^2_{L^2(H)} \, dt\right].$$

The constant hidden in $\lesssim$ depends only on numerical factors.

Proof. We consider the adjoint problem (2.7) on $[0, t]$, with $A^*(\cdot)$ replaced by $A_0^* = A_0$. Problem (2.7) is well-posed, i.e., the operator $B_0^* : X_0^\alpha \to (Y_0^\alpha \times H)^*$ associated with the bilinear form in (2.3) is a bijection. From the theory of operator semigroups we recall that the solution operator $(B_0^*)^{-1}$ can be represented by the mild solution formula,

$$v(s) = ((B_0^*)^{-1}(g, \xi))(s) = \int_s^t \int_0^1 S_0(r-s)g(r) \, dr + S_0(t-s)\xi, \quad s \in [0, t].$$

We insert this expression into the weak stochastic integral to get
$$
\langle \Psi(s) \ dW(s), x(s) \rangle_H
$$

$$
= \int_0^t \langle \Psi(s) \ dW(s), \int_s^t S_0(r-s)(-\dot{x}(r) + A_0x(r)) \ dr \rangle_H
$$

$$
+ \int_0^t \langle \Psi(s) \ dW(s), S_0(t-s)x(t) \rangle_H
$$

$$
= \int_0^t \langle \int_0^s S_0(r-s)\Psi(s) \ dW(s), (-\dot{x}(r) + A_0x(r)) \rangle_H \ dr
$$

$$
+ \langle \int_0^t S_0(t-s)\Psi(s) \ dW(s), x(t) \rangle_H.
$$
Here we used the stochastic Fubini theorem and \((S_0(t))^* = S_0(t)\). It follows that
\[
\left| \int_0^t \langle \Psi(s) \, dW(s), x(s) \rangle_H \right| \\
\leq \left( \int_0^t \left\| \int_0^r S_0(r-s) \Psi(s) \, dW(s) \right\|^2 \, dr \right)^{\frac{1}{2}} \left( \int_0^t \left\| -\dot{x}(r) + A_0 x(r) \right\|^2 \, dr \right)^{\frac{1}{2}} \\
+ \left\| \int_0^t S_0(t-s) \Psi(s) \, dW(s) \right\|_H \| x(t) \|_H \\
\lesssim \left( \int_0^t \left\| \int_0^r S_0(r-s) \Psi(s) \, dW(s) \right\|^2 \, dr \right)^{\frac{1}{2}} \left\| \int_0^t S_0(t-s) \Psi(s) \, dW(s) \right\|^2 \, dr \\
+ \left\| \int_0^t S_0(t-s) \Psi(s) \, dW(s) \right\|_H^2 \| x \|_{X^0},
\]
where the constant hidden in \(\lesssim\) depends only on numerical factors.

This implies (3.3) with
\[
K(\cdot, t) := \left( \int_0^t \left\| \int_0^r S_0(r-s) \Psi(s) \, dW(s) \right\|^2 \, dr + \left\| \int_0^t S_0(t-s) \Psi(s) \, dW(s) \right\|^2 \right)^{\frac{1}{2}}.
\]

By monotonicity in \(t\) and by taking the expectation, we obtain
\[
\mathbb{E} \left[ \sup_{t \in [0, T]} K(\cdot, t)^2 \right] \lesssim \mathbb{E} \left[ \left( \int_0^T \left\| \int_0^r S_0(r-s) \Psi(s) \, dW(s) \right\|^2 \, dr \right)^{\frac{1}{2}} \right] \\
+ \mathbb{E} \left[ \left( \int_0^T \left\| \int_0^r S_0(r-s) \Psi(s) \, dW(s) \right\|^2 \, dr \right)^{\frac{1}{2}} \right]\mathbb{E} \left[ \left( \int_0^T \left\| \int_0^r S_0(r-s) \Psi(s) \, dW(s) \right\|^2 \, dr \right)^{\frac{1}{2}} \right].
\]

The proof of (3.5) is now completed by the inequalities
\[
\mathbb{E} \left[ \left( \int_0^T \left\| \int_0^r S_0(r-s) \Psi(s) \, dW(s) \right\|^2 \, dr \right)^{\frac{1}{2}} \right] \leq \frac{1}{2} \mathbb{E} \left[ \left( \int_0^T \left\| \Psi(s) Q^\frac{1}{2} \right\|_{L^2(H)} \, ds \right)^2 \right]
\]
and
\[
\mathbb{E} \left[ \sup_{t \in [0, T]} \left\| \int_0^t S_0(t-s) \Psi(s) \, dW(s) \right\|^2 \right] \leq 16 \mathbb{E} \left[ \left( \int_0^T \left\| \Psi(s) Q^\frac{1}{2} \right\|_{L^2(H)} \, ds \right)^2 \right].
\]

These are proved in [4, Chapt. 3, Lemma 5.2]. We sketch the proof of the second inequality; the first one is proved by an eigenbasis expansion and Parseval’s identity and can be found in the cited reference. We introduce the notation
\[
v(t) := \int_0^t S_0(t-s) \Psi(s) \, dW(s), \quad z(t) := \int_0^t \Psi(s) \, dW(s)
\]
and integrate by parts, using \(\frac{d}{ds} S_0(t-s) = A_0 S_0(t-s)\) and \(dz = \Psi \, dW\), to get
\[
v(t) = z(t) - A_0 \int_0^t S_0(t-s) z(s) \, ds.
\]

By means of an eigenbasis expansion and Parseval’s identity, we have
\[
\left\| A_0 \int_0^t S_0(t-s) z(s) \, ds \right\|_H^2 \leq \sup_{s \in [0, t]} \| z(s) \|^2_{H^*}.
\]
where \( \phi \) we multiply this by arbitrary 
By using Ito’s formula (similarly to [5, Lemma 5.5]) on the process 
\( \langle U \rangle \) ensures that 
Moreover,
Proof. For any \( (U_1, U_2) \) of the form \( \underline{3.7} \) be the unique solution of \( \underline{3.1} \). If we denote by \( \mathcal{U} \) the variational solution to \( \underline{1.1} \) as in Definition \( \underline{2} \) then the following identities hold:
\[
U_1 = \bar{U} \text{ in } L^2((0,T); V), \quad \mathcal{P}\text{-a.s.} \\
U_2 = \mathcal{U} \text{ in } H, \quad \mathbb{P} \otimes dt\text{-a.s.} \\
U_1 = U_2 \text{ in } L^2((0,T); H), \quad \mathbb{P}\text{-a.s.}
\]
Moreover,
\[
U_2(t) = U_0 + \int_0^t \left( -A(s)U_1(s) + f(s) \right) ds + \int_0^t \Psi(s) dW(s), \quad \mathbb{P}\text{-a.s.}
\]
In particular, it follows that \( U_2 \) is an \( H \)-valued continuous version of \( U_1 \).
Proof. For any \( t \in [0,T] \) the variational solution is such that:
\[
\mathcal{U}(t) = U_0 + \int_0^t \left( -A(s)\bar{U}(s) + f(s) \right) ds + \int_0^t \Psi(s) dW(s), \quad \mathbb{P}\text{-a.s.}
\]
We multiply this by arbitrary \( \xi \in V \):
\[
\langle \mathcal{U}(t), \xi \rangle_H = \langle U_0, \xi \rangle_H + \int_0^t \langle -A(s)\bar{U}(s) + f(s), \xi \rangle_H ds + \int_0^t \langle \Psi(s) dW(s), \xi \rangle_H.
\]
By using Ito’s formula (similarly to [5 Lemma 5.5]) on the process \( \langle \mathcal{U}(t), \xi \rangle_H \dot{\phi}(t) \), where \( \phi \in H^1((0,T]; \mathbb{R}) \), we obtain
\[
\langle \mathcal{U}(t), \xi \rangle_H \dot{\phi}(t) = \langle U_0, \xi \rangle_H \dot{\phi}(0) \\
+ \int_0^t \left( \langle \bar{U}(s), \xi \rangle_H \dot{\phi}(s) + \langle -A(s)\bar{U}(s) + f(s), \xi \rangle_V \phi(s) \right) ds \\
+ \int_0^t \dot{\phi}(s) \langle \Psi(s) dW(s), \xi \rangle_H, \quad \mathbb{P}\text{-a.s.}
\]
This is the same as
\[
\langle \mathcal{U}(t), \phi(t)\rangle_H = \langle U_0, \phi(0)\rangle_H + \int_0^t \langle \mathcal{U}(s), \dot{\phi}(s) \rangle_H + A^*(s) \phi(s) \rangle_{V^*} \, ds
\]
\[
+ \int_0^t \langle f(s), \phi(s) \rangle_{V^*} \, ds + \int_0^t \langle \Psi(s) \, dW(s), \phi(s) \rangle_H.
\]
Since functions of the form \(x = \phi \xi\) are dense in \(X_0^d\), we conclude that, for almost all \((\omega, t) \in \Omega \times [0, T]\),
\[
\int_0^t \langle \mathcal{U}(s), -\dot{x}(s) + A^*(s)x(s) \rangle_{V} \, ds + \langle \mathcal{U}(t), x(t) \rangle_H
\]
\[
= \langle U_0, x(0) \rangle_H + \int_0^t \langle f(s), x(s) \rangle_{V^*} \, ds + \int_0^t \langle \Psi(s) \, dW(s), x(s) \rangle_H, \quad \forall x \in X_0^d.
\]
This means that \((\mathcal{U}, \mathcal{U}(t)) \in Y_0^d \times H\) is a solution to (3.1). The conclusions of the lemma now follow by uniqueness of such a solution.

**Theorem 6 (Existence and uniqueness).** If \(U_0 \in L^2(\Omega; H), f \in L^2((0 \times (0, T); V^*)\) and \(\Psi \mathcal{Q}^\sharp \in L^2(\Omega \times (0, T); \mathcal{Z}_2(H))\), then there exists a unique solution \(U = (U_1, U_2) \in L^2(\Omega \times (0, T); Y) \times L^2(\Omega; \mathcal{C}([0, T]; H))\) of the form (3.7) to (3.1). Its norm satisfies the bound
\[
E\left[ \int_0^T \|U_1(t)\|_{V}^2 \, dt + \sup_{t \in [0, T]} \|U_2(t)\|_{H}^2 \right] \lesssim c_B^{-1} E\left[ \int_0^T \|f(t)\|_{V^*}^2 \, dt + U_0 \|_{H}^2 + \int_0^T \|\Psi(t) Q^\sharp\|_{\mathcal{Z}_2(H)}^2 \, dt \right],
\]
where the constant hidden in \(\lesssim\) only depends only on numerical factors.

**Proof.** In view of the \(\omega\)-wise invertibility of the operator \(B_{\omega,t}\), and the bounds for \(\mathcal{F}_{\omega,t}\) in (3.2) and \(\mathcal{W}_{\omega,t}\) in (3.6), we have that for fixed \(\omega\) and for any \(t \in [0, T]\), there exists a unique solution to (3.1), which satisfies the bound
\[
\int_0^t \|U_1(\omega, s)\|_{V}^2 \, ds + \|U_2(\omega, t)\|_{H}^2 \lesssim c_B^{-1} \left( \|\mathcal{F}_{\omega,t}\|_{X_0^d} + \|\mathcal{W}_{\omega,t}\|_{X_0^d} \right)
\]
\[
\lesssim c_B^{-1} \left( \|f(\omega, \cdot)\|_{L^2((0, T); V^*)}^2 + \|U_0(\omega)\|_{H}^2 + K(\omega, t)^2 \right).
\]
In view of (3.3) and (3.5), this leads to
\[
E\left[ \int_0^T \|U_1(t)\|_{V}^2 \, dt + \sup_{t \in [0, T]} \|U_2(t)\|_{H}^2 \right] \lesssim c_B^{-1} E\left[ \sup_{t \in [0, T]} \|\mathcal{F}_{\omega,t}\|_{X_0^d} + \sup_{t \in [0, T]} \|\mathcal{W}_{\omega,t}\|_{X_0^d} \right]
\]
\[
\lesssim c_B^{-1} E\left[ \|f\|_{L^2((0, T); V^*)}^2 + \|U_0\|_{H}^2 + \int_0^T \|\Psi(t) Q^\sharp\|_{\mathcal{Z}_2(H)}^2 \, dt \right].
\]
Together with Lemma 4 this concludes the proof of the theorem.

In the remainder of the manuscript we will sometimes use the alternative notation \(U \in L^2(\Omega \times (0, T); Y) \cap L^2(\Omega; \mathcal{C}([0, T]; H))\), equivalent to \(U = (U_1, U_2) \in L^2(\Omega \times (0, T); Y) \times L^2(\Omega; \mathcal{C}([0, T]; H))\), where the two components of \(U\) are now understood as versions of the same object.
3.2. Connection with the mild solution. We have already shown that a weak space-time solution is a variational solution. If we assume that $-A$ is independent of $\omega$ and $t$ and hence generates an analytic semigroup $(S(t))_{t \geq 0}$, we can also show that a weak space-time solution is a mild solution. The following theorem holds:

**Theorem 7.** Let $U$ be the mild solution (3.3) to the problem (1.1) and assume that $(U_1, U_2(t)) \in \mathcal{L}_0^2 \times H$ is the weak space-time solution to the same problem, i.e., the solution to (3.1). Then, for any $t \in [0, T]$, $U_1 \equiv U$ and $U_2(t) \equiv U(t)$.

**Proof.** For any $t \in [0, T]$ and for any $x \in \mathcal{X}^0_0$, we have $\mathbb{P}$-a.s. that

$$
\int_0^t \langle U_1(s), -\dot{x}(s) + A^*x(s) \rangle_{V'} \, ds + \langle U_2(t), x(t) \rangle_H
$$

(3.8)

$$
= \int_0^t \langle f(s), x(s) \rangle_V \, ds + \langle U_0, x(0) \rangle_H + \int_0^t \langle \Psi(s) \, dW(s), x(s) \rangle_H.
$$

We now choose test functions $v = x$, where $v \in \mathcal{X}^0_0$ is the solution to the deterministic backward equation (2.7) over the time interval $[0, t]$, with arbitrary final data $\xi \in H$ and load function $g \in L^2((0, t); V^*)$. Its variational formulation is given by (2.8), that is,

$$
\int_0^t \langle y_1(s), -\dot{v}(s) + A^*v(s) \rangle_{V'} \, ds + \langle y_2, v(t) \rangle_H
$$

(3.9)

$$
= \int_0^t \langle y_1(s), g(s) \rangle_V \, ds + \langle y_2, \xi \rangle_H,
$$

(3.10)

for all $y \in \mathcal{Y}^0_0 \times H$. The solution is given by the mild solution formula

$$
v(s) = S^*(t-s)\xi + \int_s^t S^*(r-s)g(r) \, dr, \quad s \in [0, t],
$$

(3.11)

where $S^*$ is the semigroup generated by $-A^*$, namely $S^*(s) = e^{-sA^*}$. By substituting $x = v$ in (3.9) and $y = (U_1, U_2(t))$ in (3.8), we obtain

$$
\int_0^t \langle U_1(s), g(s) \rangle_{V'} \, ds + \langle U_2(t), \xi \rangle_H
$$

$$
= \int_0^t \langle f(s), v(s) \rangle_V \, ds + \langle U_0, v(0) \rangle_H + \int_0^t \langle \Psi(s) \, dW(s), v(s) \rangle_H,
$$

which, by (3.11), in its turn is equal to

$$
= \int_0^t \langle f(s), S^*(t-s)\xi \rangle_V \, ds + \int_0^t \langle f(s), \int_s^t S^*(r-s)g(r) \, dr \rangle_V \, ds
$$

$$
+ \langle U_0, S^*(t)\xi \rangle_H + \bigg\langle U_0, \int_0^t S^*(r-s)g(r) \, dr \bigg\rangle_H
$$

$$
+ \int_0^t \langle \Psi(s) \, dW(s), S^*(t-s)\xi \rangle_H + \int_0^t \bigg\langle \Psi(s) \, dW(s), \int_s^t S^*(r-s)g(r) \, dr \bigg\rangle_H.
$$

By manipulating the dual pairings in a suitable way, changing the order of integration (using the stochastic version of Fubini’s theorem), and using the mild solution
formula \((1.3)\), we get

\[
\int_0^t \langle U_1(s), g(s) \rangle_{V^*} \, ds + \langle U_2(t), \xi \rangle_H = \left\langle S(t)U_0 + \int_0^t S(t-s)f(s) \, ds + \int_0^t S(t-s)\Psi(s) \, dW(s), \xi \right\rangle_H
\]

\[
+ \int_0^t \langle S(s)U_0 + \int_0^s S(s-r)f(r) \, dr + \int_0^s S(s-r)\Psi(r) \, dW(r), g(s) \rangle_{V^*} \, ds
\]

\[
= \langle U(t), \xi \rangle_H + \int_0^t \langle U(s), g(s) \rangle_{V^*} \, ds,
\]

which reads

\[
\mathcal{Y}^0_0(U_1 - U, g) + \langle U_2(t) - U(t), \xi \rangle_H = 0.
\]

Since \((g, \xi)\) is arbitrary in \(L^2((0, t); V^*) \times H\), and \(t \in [0, T]\), it follows that

\[
U_1 = U, \quad U_2(t) = \dot{U}(t), \quad t \in [0, T], \text{ a.s.}
\]

\(\Box\)

**Remark 8.** This is consistent with the fact that \(U_1\) is a \(V\)-valued version of \(U_2\) and that \(U_2\) is a continuous \(H\)-valued function of time.

### 4. Regularity

In this section we briefly investigate the regularity properties of the weak space-time solution. In order to simplify the presentation, we assume now that \(A\) is independent of \(\omega\) and \(t\) and self-adjoint in addition to \(\beta = T\).

Then \(-A\) is the generator of an analytic semigroup \(S(t) = e^{-tA}\) and fractional powers \(A^s, s \in \mathbb{R}\), of \(A\) are well defined. We define norms of fractional order \(\|v\|_{\dot{H}^s} := \|A^s v\|_H\) for \(s \in \mathbb{R}\). For \(s \geq 0\) we define the spaces \(\dot{H}^s = D(A^s)\) and for \(s \leq 0\) we define \(\dot{H}^s\) to be the closure of \(H\) with respect to the \(\dot{H}^s\)-norm. These spaces are Hilbert spaces, in particular, \(\dot{H}^0 = H\), \(\dot{H}^1 \simeq V\), and \(\dot{H}^{-s} = (\dot{H}^s)^*\).

For \(\beta \geq 0\), we then consider the spaces

\[
\mathcal{Y}^{t, \beta}_0 := L^2((0, t); \dot{H}^{1+\beta}),
\]

\[
\mathcal{X}^{t, \beta}_0 := L^2((0, t); \dot{H}^{-1-\beta}) \cap H^1((0, t); \dot{H}^{-1-\beta}),
\]

normed by

\[
\|y\|_{\mathcal{Y}^{t, \beta}_0}^2 := \int_0^t \|y(s)\|^2_{\dot{H}^{1+\beta}} \, ds,
\]

\[
\|x\|_{\mathcal{X}^{t, \beta}_0}^2 := \int_0^t (\|x(s)\|^2_{\dot{H}^{-1-\beta}} + \|\dot{x}(s)\|^2_{\dot{H}^{-1-\beta}}) \, ds + \|x(0)\|^2_{\dot{H}^{-\beta}} + \|x(t)\|^2_{\dot{H}^{-\beta}}.
\]

The spaces in the previous sections correspond to \(\beta = 0\). In particular, as before, we use the notation \(\mathcal{Y}^\beta = \mathcal{Y}^{0, \beta}_0\) and \(\mathcal{X}^\beta = \mathcal{X}^{0, \beta}_0\). The space \(\mathcal{Y}^{t, \beta}_0 \times \dot{H}^\beta\) endowed with its product norm \(|| \cdot \|_{\mathcal{Y}^{t, \beta}_0 \times \dot{H}^\beta}\) and the space \(\mathcal{X}^{t, \beta}_0\) endowed with the norm \(\| \cdot \|_{\mathcal{X}^{t, \beta}_0}\) are Hilbert spaces.

There is a dense embedding \(\mathcal{X}^\beta \hookrightarrow C([0, T]; \dot{H}^{-\beta})\), i.e., for any \(x \in \mathcal{X}^\beta\),

\[
\|x\|_{C([0, T]; \dot{H}^{-\beta})} \leq \|x\|_{\mathcal{X}^\beta},
\]
where the embedding constant is the same as in [25]. A proof of this fact can be found in [7, 9], and relies on the properties of the interpolation space
\[(\dot{H}^{1-\beta}, \dot{H}^{-1-\beta})_{\frac{1}{2}} = \dot{H}^{-\beta}.
\]
We introduce a new bilinear form, \(B_{t,\beta}^*\), given by the original one, \(B_t\) with constant operator \(A\), restricted to the newly introduced spaces, that is,
\[B_{t,\beta}^*(\dot{Y}_0^{t,\beta} \times \dot{H}^\beta) \times \Lambda_0^{t,\beta} \rightarrow \mathbb{R},\]
together with new load functionals,
\[\mathcal{F}_{\omega,t,\beta}: \Lambda_0^{t,\beta} \rightarrow \mathbb{R}, \quad \mathcal{M}_{\omega,t,\beta}: \Lambda_0^{t,\beta} \rightarrow \mathbb{R},\]
given by \(\mathcal{F}_{\omega,t}\) and \(\mathcal{M}_{\omega,t}\) defined on the new spaces introduced above.

The weak space-time formulation reads, for almost every \((\omega, t) \in \Omega \times [0,T]::
\begin{align}
U_\beta(\omega, t) &\in \dot{Y}_0^{t,\beta} \times \dot{H}^\beta; \\
B_{t,\beta}^*(U_\beta(\omega, t), x) &= \mathcal{F}_{\omega,t,\beta}(x) + \mathcal{M}_{\omega,t,\beta}(x), \quad \forall x \in \Lambda_0^{t,\beta}.
\end{align}

It is possible to prove that the conditions (BDD), (BNB1) and (BNB2) still hold, with the same constants \(C_B\) and \(c_B\) as before. The proof of this follows from a straightforward modification of the proof for the deterministic framework in [11] or [13], taking into account the remarks made for its extension to the stochastic framework in Section 3. It will therefore be omitted.

In the following lemma we give sufficient conditions on the load functionals in order to have a unique solution.

**Lemma 9.** With the notation introduced above, the following facts hold true:

- If \(f \in \dot{Y}_0^{t,\beta-2}\) and \(U_0 \in \dot{H}^\beta\), \(\mathbb{P}\)-a.s., then \(\mathcal{F}_{t,\beta} \in (\Lambda_0^{t,\beta})^*, \mathbb{P}\)-a.s. Moreover, if \(f \in L^2(\Omega; \dot{Y}_0^{t,\beta-2})\) and \(U_0 \in L^2(\Omega; \dot{H}^\beta)\), then
  \[
  \mathbb{E}\left[\sup_{t \in [0,T]} \|\mathcal{F}_{t,\beta}\|_{\Lambda_0^{t,\beta}} \right] \lesssim \mathbb{E}\left[\|f\|_{\dot{Y}_0^{t,\beta-2}} + \|U_0\|_{\dot{H}^\beta}\right].
  \]

- If \(\Psi Q_\beta^* \in L^2(\Omega \times (0,T); \mathcal{L}_2(\dot{H}, \dot{H}^\beta))\), then \(\mathcal{M}_{t,\beta} \in (\Lambda_0^{t,\beta})^*\), \(\mathbb{P}\)-a.s. Moreover,
  \[
  \mathbb{E}\left[\sup_{t \in [0,T]} \|\mathcal{M}_{t,\beta}\|_{\Lambda_0^{t,\beta}} \right] \lesssim \mathbb{E}\left[\int_0^T \|\Psi(t)Q_\beta^*\|_{\mathcal{L}_2(\dot{H}, \dot{H}^\beta)} dt\right].
  \]

**Proof.** The first statement is obvious. In order to prove the second one, one can use the same notation and techniques as in Section 5 together with the employment of the following inequalities to derive an analogue of Lemma 3
\[
\mathbb{E}\left[\sup_{t \in (0,T]} \left\|A^\beta \int_t^T S(t-s)\Psi(s) dW(s)\right\|_{H}^2\right] \lesssim \mathbb{E}\left[\int_0^T \left\|A^\beta \Psi(t)Q_\beta^*\right\|_{\mathcal{L}_2(H)}^2 dt\right]
\]
and
\[
\mathbb{E}\left[\int_0^T \left\|A^\beta \int t^R S(r-s)\Psi(s) dW(s)\right\|_{H}^2 dr\right] \lesssim \mathbb{E}\left[\int_0^T \left\|A^\beta \Psi(t)Q_\beta^*\right\|_{\mathcal{L}_2(H)}^2 dt\right].
\]
These two properties are direct generalizations of the ones presented in Lemma 3.

The previous lemma, together with the initial remarks about the fulfillment of the conditions (BDD), (BNB1), and (BNB2), gives the following result.
Theorem 10. Let $\beta \geq 0$ and $f \in L^2(\Omega \times (0, T); H^{\beta-1})$, $U_0 \in L^2(\Omega; H^{\beta})$, and $\Psi Q^\frac{1}{2} \in L^2(\Omega \times (0, T); L_\beta^2(H; H^{\beta}))$. Then the problem (5.1) has a unique solution $U \in L^2(\Omega \times (0, T); H^{\beta+1})$ and $L^2(\Omega; C([0, T]; H^{\beta}))$ and its norm is bounded by

$$\mathbb{E}\left[ \int_0^T \|U(t)\|_{H^{\beta+1}}^2 \, dt + \sup_{t \in [0, T]} \|U(t)\|_{H^\beta}^2 \right] \lesssim c_B^{-1} \mathbb{E}\left[ \int_0^T \|f(t)\|^2_{H^{\beta-1}} \, dt + \int_0^T \|\Psi(t)Q^\frac{1}{2}\|^2_{L_{\beta}^2(H; H^\beta)} \, dt + \|U_0\|_{H^\beta}^2 \right],$$

where the constant hidden in $\lesssim$ depends only on numerical factors.

5. Linear multiplicative noise

In this section we use the theory developed in the previous sections to prove existence and uniqueness to the weak space-time solution to the problem

$$dU(t) + A(t)U(t) \, dt = f(t) \, dt + (G(t)U(t)) \, dW(t), \quad t \in (0, T],$$

$$U(0) = U_0.$$  

Here $G(\omega, t) \in \mathcal{L}(H, \mathcal{L}(H))$, with further assumptions on its $(\omega, t)$-dependence to be specified below. As we have done before, we introduce an $\omega$-wise weak formulation. In order to do so we introduce a new load functional $\mathcal{U}_{\omega, t}^\nu$ defined by

$$\mathcal{U}_{\omega, t}^\nu : X_0^t \to \mathbb{R}, \quad \mathcal{U}_{\omega, t}^\nu(x) = \left( \int_0^t \langle (G(s)v(s)) \, dW(s), x(s) \rangle_H \right)(\omega),$$

for $(\omega, t) \in \Omega \times [0, T]$ and $v \in \mathcal{F}_T = L^2(\Omega; L^2((0, T); V)) \cap L^2(\Omega; C([0, T]; H))$. The weak space-time formulation of problem (5.1) reads hence, for almost every $(\omega, t) \in \Omega \times [0, T],

$$U_{\omega, t} \in \mathcal{F}_0^t \times H : \mathcal{U}_{\omega, t}^\nu(U_{\omega, t}, x) = \mathcal{F}_{\omega, t}(x) + \mathcal{U}_{\omega, t}^\nu(x), \quad \forall x \in X_0^t.$$

We use Banach’s fixed point theorem for the linear operator $\mathcal{F} : v \mapsto U$ that maps $v \in \mathcal{F}_T$ to the solution of of the problem

$$U_{\omega, t} \in \mathcal{F}_0^t \times H : \mathcal{U}_{\omega, t}^\nu(U_{\omega, t}, x) = \mathcal{F}_{\omega, t}(x) + \mathcal{U}_{\omega, t}^\nu(x), \quad \forall x \in X_0^t.$$

We will show that $\mathcal{F} : \mathcal{F}_T \to \mathcal{F}_T$ is a contraction, if $T$ is small. We introduce the notation $L_{\beta}^2(H)$ for the space of operators $\Psi$ such that

$$\|\Psi\|_{L_{\beta}^2(H)} := \|\Psi Q^\frac{1}{2}\|_{L_{\beta}^2(H)} < \infty.$$

We make the further assumption that $G$ is predictable, bounded with respect to $\omega$, and $L^p$ in time for some $p > 2$, i.e., for some constant $\kappa$,

$$\text{ess sup}_{\omega \in \Omega} \left( \int_0^T \|G(\omega, t)\|_{L(H, \mathcal{F}_0^t(H))}^p \, dt \right)^{1/p} \leq \kappa.$$

An example is presented in Remark 15 below.

Lemma 11. For any $v \in \mathcal{F}_T$ and $G$ as in (5.4), it holds that

$$\mathbb{E}\left[ \int_0^T \|G(t)v(t)\|^2_{\mathcal{F}_0^t(H)} \, dt \right] \leq T^{p/2} \kappa^2 \|v\|_{\mathcal{F}_T}^2.$$
A weak space-time formulation for the linear stochastic heat equation

Proof. We use Hölder’s inequality to get

\[ \mathbb{E} \left[ \int_0^T \| G(\cdot, t) v(\cdot, t) \|_2^2 \, dt \right] \leq \mathbb{E} \left[ \int_0^T \| G(\cdot, t) \|_V^2 \, dt \right] \]

\[ \leq \mathbb{E} \left[ \sup_{t \in [0, T]} \| v(\cdot, t) \|_H^2 \left( \int_0^T \| G(\cdot, t) \|_V^2 \, dt \right) \right] \]

\[ \leq \mathbb{E} \left[ \sup_{t \in [0, T]} \| v(\cdot, t) \|_H^2 \right] T^{\frac{p}{2}} \sup_{\omega \in \Omega} \left( \int_0^T \| G(\omega, t) \|_L^p \, dt \right)^{\frac{2}{p}} \]

\[ \leq T^{\frac{p}{2}} 2^2 \| v \|_{L^2}^2, \]

where in the last line we used \( (5.3) \). \( \square \)

By combining Lemmas [11] and [3] with \( \Psi = Gv \), we see that \( \mathbb{W}_{\omega, t} \in (\mathcal{X}_0^t)^* \) and

\[ \mathbb{E} \left[ \sup_{t \in [0, T]} \| \mathbb{W}_{\omega, t} \|_{(\mathcal{X}_0^t)^*}^2 \right] \leq \mathbb{E} \left[ \int_0^T \| (G(t)v(t))Q^\frac{1}{2} \|_L^2 \, dt \right] \leq T^{\frac{p}{2}} \| v \|_{L^2}^2. \]

If \( U_0 \in L^2(\Omega; H), f \in L^2(\Omega \times (0, T); V^*), \) \( Q^\frac{1}{2} \in L^2(H) \), then we may refer to Theorem [4] to conclude that \( (5.3) \) has a unique solution with

\[ \mathbb{E} \left[ \int_0^T \| U_1 \|_V^2 \, dt + \sup_{t \in [0, T]} \| U_2 \|_H^2 \right] \leq \mathbb{E} \left[ \int_0^T \| f \|_V^2 \, dt + \| U_0 \|_H^2 \right] + T^{\frac{p}{2}} \| v \|_{L^2}^2. \]

Hence, the solution operator \( \mathcal{F} \) maps \( \mathcal{D} \) to itself. An application of the previous bound with \( f = 0, U_0 = 0 \) shows that it is a contraction, if \( T \) is small. We thus have a unique solution on some short interval \( [0, T_0] \) and, since the interval of existence does not depend on the size of the data \( f, U_0 \), we may repeat the argument and extend it to \( [T_0, 2T_0], [2T_0, 3T_0], \) and so on until we obtain a solution on \( [0, T] \).

We summarize the result in the following theorem:

**Theorem 12** (Existence and uniqueness). If \( U_0 \in L^2(\Omega; H), f \in L^2(\Omega \times (0, T); V^*), \) and \( G \in L^\infty(\Omega; L^p((0, T); \mathcal{L}(H, \mathcal{L}_2^p(H)))) \) for some \( p > 2 \), see \( (5.4) \), then \( (5.2) \) has a unique solution \( U \in L^2(\Omega \times (0, T); V) \cap L^2(\Omega; \mathcal{F}([0, T]; H)). \)

**Remark 13.** This approach extends easily to a semilinear equation of the form

\[ dU(t) + A(t) U(t) \, dt = F(t, U(t)) \, dt + G(t, U(t)) \, dW(t) \]

under appropriate global Lipschitz assumptions on the nonlinear operators \( F, G \).

**Remark 14.** Under the assumptions of Section [4] and, for some \( p > 2 \),

\[ \mathbb{E} \left[ \sup_{\omega \in \Omega} \left( \int_0^T \| G(\omega, t) \|_L^p \, dt \right)^{\frac{1}{p}} \right] \leq \kappa, \]

we may extend the regularity result of Theorem [10] to \( (5.2) \).

**Remark 15.** We present an example of an operator satisfying \( (5.4) \). Let \( H = L^2(D) \) and define, for all \( v, w \in H \) and for some function \( g \),

\[ \left( (G(\omega, t)v)w \right)(\xi) = g(\omega, t, \xi)v(\xi)w(\xi), \quad \xi \in D. \]

Let \( \{ \epsilon_j \}_{j=1}^\infty \subset H \) be an ON basis such that \( \sup_{j \geq 1} \| \epsilon_j \|_{L^\infty(D)} \leq C \). This can be achieved, for example, when \( D \) is a parallellogram in \( \mathbb{R}^d \). Then construct \( Qv = \)
\[ \sum_{j=1}^{\infty} \gamma_j \langle v, e_j \rangle_{\mathcal{H}} e_j, \] where the eigenvalues \( \{\gamma_j\}_{j=1}^{\infty} \) are chosen so that \( \sum_{j=1}^{\infty} \gamma_j = \|Q^\frac{1}{2}\|_{\mathcal{L}_2(\mathcal{H})}^2 < \infty \). Then
\[
\|G(\omega, t)v\|_{\mathcal{L}^2(H)}^2 = \|(G(\omega, t)v)Q^\frac{1}{2}\|_{\mathcal{L}^2(\mathcal{H})}^2 = \sum_{j=1}^{\infty} \|(G(\omega, t)v)Q^\frac{1}{2}e_j\|_H^2 \]
\[
= \sum_{j=1}^{\infty} \gamma_j \|(G(\omega, t)v)e_j\|_H^2 = \sum_{j=1}^{\infty} \gamma_j \|g(\omega, t, \cdot)v e_j\|_{L^2(D)}^2 
\leq \sum_{j=1}^{\infty} \gamma_j \|g(\omega, t, \cdot)\|_{L^\infty(D)}^2 \|v\|_{L^\infty(D)}^2 \|e_j\|_{L^\infty(D)}^2 \lesssim \|g(\omega, t, \cdot)\|_{L^\infty(D)}^2 \|v\|_{L^2(D)}^2.
\]
Therefore,
\[
\|G(\omega, t)\|_{\mathcal{L}^2(H, L^2_D)} \lesssim \|g(\omega, t, \cdot)\|_{L^\infty(D)}
\]
and \eqref{5.4} follows if we assume that \( g \in L^\infty(\Omega; L^p((0, T); L^\infty(D))) \).

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