GLOBAL EXISTENCE FOR THE 2D NAVIER-STOKES FLOW IN
THE EXTERIOR OF A MOVING OR ROTATING OBSTACLE

SHUGUANG SHAO
College of Applied Sciences
Beijing University of Technology
Beijing 100124, China
and
School of Mathematics and Statistics
Nanyang Normal University
Nanyang 473061, China

SHU WANG AND WEN-QING XU
College of Applied Sciences
Beijing University of Technology
Beijing 100124, China

BIN HAN∗
School of Sciences
Hangzhou Dianzi University
Hangzhou 310018, China

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Abstract. We consider the global existence of the two-dimensional Navier-
Stokes flow in the exterior of a moving or rotating obstacle. Bogovski operator
on a subset of \( \mathbb{R}^2 \) is used in this paper. One important thing is to show that
the solution of the equations does not blow up in finite time in the sense of
some \( L^2 \) norm. We also obtain the global existence for the 2D Navier-Stokes
equations with linearly growing initial velocity.

1. Introduction. In this paper we consider the flow of an incompressible, viscous
fluid past a moving or rotating obstacle in two-dimensional space. The equations
describing the flow are those of Navier-Stokes in an exterior domain which varies
with time \( t \). Whereas the description of the Navier-Stokes flow in the exterior
of a fixed obstacle can be regarded as fairly well understood (for instance, see
[1, 7, 17, 19, 24, 26, 27, 28, 29]), this is not the case for rotating or moving obstacles.
The latter is the objective of this paper.

After a suitable change of coordinates, the Navier-Stokes equations in the exterior
of a moving domain can be expressed as the Navier-Stokes equations in a fixed
domain \( \Omega \) but with the usual Stokes operator \( \mathbb{P} \Delta \) replaced by

\[
Au = \mathbb{P} (\Delta u + M x \cdot \nabla u - Mu).
\]

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∗ Corresponding author: Bin Han.
Here $P$ denotes the Helmholtz projection from $L^p(\Omega)$ into the space $L^p_{\sigma}(\Omega)$ and $M$ denotes a real $2 \times 2$ matrix. The main difficulty then arises from the fact that the lower order term of $A$ has unbounded coefficients. This implies that, similar as in the study of Ornstein-Uhlenbeck processes, whenever $A$ generates a semigroup $T$ on $L^p_{\sigma}(\Omega)$, $T$ is not expected to be analytic. Thus, for the existence of a local mild solution the usual fixed point argument does not seem to apply since smoothing properties of $T$ as well as gradient estimates for $T$ no longer follow from standard theory.

Now we describe the problem precisely. Consider a compact set $O \subset \mathbb{R}^2$, the obstacle, with boundary $\Gamma = \partial O$ of class $C^{1,1}$. Set $\Omega = \mathbb{R}^2 \setminus O$. For $t > 0$ and a real $2 \times 2$ matrix $M$ we set
\[ \Omega(t) := \{ y(t) = e^{tM}x : x \in \Omega \} \]
and
\[ \Gamma(t) := \{ y(t) = e^{tM}x : x \in \Gamma \}. \]

We are concerned with the following model of incompressible viscous fluids past a moving or rotating obstacle:

\[ \begin{aligned}
\partial_t w + w \cdot \nabla w - \Delta w + \nabla q &= 0, \quad y \in \Omega(t), \ t > 0, \\
\text{div}w &= 0, \quad y \in \Omega(t), \ t > 0, \\
w &= My, \quad y \in \Gamma(t), \ t > 0, \\
w(y,0) &= w_0(y), \quad y \in \Omega(t).
\end{aligned} \]

Here $w(y,t), q(y,t)$ denote the velocity and pressure of the fluid respectively. The boundary condition on $\Gamma(t)$ is the usual no-slip boundary condition. Suppose that $M = -M^T$, by the change of variables $x = e^{-tM}y$
and by setting
\[ u(x,t) = e^{-tM}w(y,t), \]
we obtain the following set of equations defined on the fixed domain $\Omega$:

\[ \begin{aligned}
\partial_t u - \Delta u + u \cdot \nabla u - Mx \cdot \nabla u + Mu + \nabla p &= 0, \quad x \in \Omega, \ t > 0, \\
\text{div}u &= 0, \quad x \in \Omega, \ t > 0, \\
u &= Mx, \quad x \in \Gamma, \ t > 0, \\
u(x,0) &= u_0(x) = w_0(x), \quad x \in \Omega.
\end{aligned} \]

We note that the coefficient of the convection term $Mx \cdot \nabla u$ is unbounded, which implies that this term cannot be treated as a perturbation of the Laplacian or the Stokes operator. It is well known since Leray’s pioneering work [25] that there exists a global smooth solution if the initial data $u_0$ is in $L^2(\mathbb{R}^2)$, or in other words, the initial kinematic energy is finite. For such initial data the global existence of solution is proved by a priori estimate called an energy equality
\[ \|u\|_{L^2}^2(t) + 2 \int_0^t \|\nabla u\|_{L^2}^2(\tau) d\tau = \|u_0\|_{L^2}^2, \quad t > 0. \]

This is formally obtained by multiplying $u$ with the momentum equation. This motivates us to use such method to our problem.

Let us briefly review the history of the study on system (2). On the one hand, when $\Omega$ is an exterior domain in $\mathbb{R}^3$, in the special case
\[ M = R, \]
where $R$ denotes a rotation matrix
\[
\begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]
in $\mathbb{R}^3$, problem (2) were first considered by Borchers [6] and by Chen and Miyakawa [13] in the framework of suitable weak solutions. Later, Hishida in [21] and [22], proved, again for $M = R$, that the solution of the linearized equations is governed by a strongly continuous semigroup on $L^2_*(\Omega)$. Moreover, he proved the existence of a unique local mild solution in $L^2_*(\Omega)$ to problem (2) provided the initial data $w_0$ belongs to a certain fractional power space of $A$ in $L^2(\Omega)$ (see the definition of $A$ in [21] or later in the paper). M. Geissert, H. Heck and M. Hieber in [18] proved the local existence of the mild solution of system (2). We wonder whether the global solution of system (2) exists in dimension two in exterior domain $\Omega$, and this is our aim in the paper.

As to the Cauchy problem for the Navier-Stokes equations with linearly growing initial data
\[
\begin{align*}
\partial_t v - \Delta v + v \cdot \nabla v + \nabla p &= 0, \quad x \in \mathbb{R}^N, \quad t > 0, \\
\text{div} v &= 0, \quad x \in \mathbb{R}^N, \quad t > 0, \\
v(x, 0) &= v_0(x) - Mx, \quad x \in \mathbb{R}^N,
\end{align*}
\]
we say it has the same structure as system (3). Here $M$ is an $N \times N$ real matrix.

Indeed, if we set $u = v + Mx$, then system (3) can be rewritten as
\[
\begin{align*}
\partial_t u - \Delta u + u \cdot \nabla u - Mx \cdot \nabla u + Mu + \nabla \tilde{p} &= 2Mu, \quad x \in \mathbb{R}^N, \quad t > 0, \\
\text{div} u &= 0, \quad x \in \mathbb{R}^N, \quad t > 0, \\
u(x, 0) &= u_0(x), \quad x \in \mathbb{R}^N,
\end{align*}
\]
where
\[
\nabla p = \nabla \tilde{p} - MMx.
\]
In this sense we may say system (3) is the limit case when $\Omega$ tends to the whole space $\mathbb{R}^N$. We write the equations in general dimension $N$ here. It is known that Babin, Mahalov and Nicolaenko [2], [3] proved the existence of a local mild solution to (4) provided $u_0$ lies in $L^p_\sigma$ with $1 < p < \infty$ or $u_0$ is a periodic function satisfying certain properties. We recall that all these results were obtained for the particular case of $Mx = \omega \times x$, where $\omega = (1, 0, 0)$ and $x \in \mathbb{R}^3$. For general $M$ with $\text{tr} M = 0$, M. Hieber and O. Sawada in [23] proved the local existence of the mild solution of system (4) in $L^p$ spaces with $p \in [N, \infty)$. We call a function $u \in C([0, T]; L^p(\mathbb{R}^N))$ a mild solution of (4) if $u$ satisfies the integral equation
\[
u(t) = e^{-tA}u_0 - \int_0^t e^{-(t-s)A}P(u \cdot \nabla u)ds + 2\int_0^t e^{-(t-s)A}P(Mu)ds,
\]
for $T > 0$ and $t \in [0, T)$ which has the similar form as the mild solution of (2) stated in [18]. Here
\[
Au = -\Delta u - Mx \cdot \nabla u + Mu
\]
and $P$ is the Helmholtz projection. Recently, the last author and others have also done some works on it. In [15], the authors have proved the local existence on the density dependent problem of system (4) in the framework of Besov space. Later in [16], we have proved the global existence of the density dependent problem of (4) based on the local existence in [15] in dimension two. In the third section of this
paper, we consider the global existence problem of Navier-Stokes equations with linearly growing initial velocity in $L^q$ space for $2 \leq q < \infty$ in $\mathbb{R}^2$.

2. The functional tool box.

**Proposition 1.** [5] Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Then there exists an operator

$$B : C^\infty_c(\Omega) \rightarrow (C^\infty_c(\Omega))^N$$

such that

$$\text{div}(Bf) = f$$

for all $\int_\Omega f dx = 0$. Moreover, if $1 < p < \infty$, $B$ can be continuously extended to a bounded operator from

$$W^{1,p}_0(\Omega) \rightarrow (W^{1,p+1}_0(\Omega))^N.$$

**Proposition 2.** [5] Let $1 < p < \infty$ and $p \leq q \leq \infty$.

(i) Let $T > 0$. Then there exists a constant $C > 0$ such that

$$\|e^{-tA}f\|_q \leq Ct^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{q})}\|f\|_p, \quad 0 < t < T;$$

$$\|\nabla e^{-tA}f\|_p \leq Ct^{-\frac{n}{2}}\|f\|_p, \quad 0 < t < T. \quad (6)$$

(ii) Assume in addition that

$$\|e^{-tM}\| \leq C, \quad \forall t > 0$$

holds. Then there exists a constant $C > 0$ such that

$$\|e^{-tA}f\|_q \leq Ct^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{q})}\|f\|_p, \quad t > 0;$$

$$\|\nabla e^{-tA}f\|_p \leq Ct^{-\frac{n}{2}}\|f\|_p, \quad t > 0. \quad (7)$$

(iii) Moreover, for $1 < p < q \leq \infty$ and $f \in L^p(\mathbb{R}^n)^n$,

$$t^{\frac{n}{2}(\frac{1}{p} - \frac{1}{q})}\|e^{-tA}f\|_q \rightarrow 0 \quad \text{as} \quad t \rightarrow 0, \quad (8)$$

$$t^{\frac{n}{2}}\|\nabla e^{-tA}f\|_p \rightarrow 0 \quad \text{as} \quad t \rightarrow 0. \quad (9)$$

3. Main results. In this section we state the main results of this paper. To this end, recall that $M$ denotes an $2 \times 2$ matrix with real coefficients.

**Theorem 3.1.** Let

$$M = -M^T, u_0 \in L^2_\sigma(\Omega),$$

where $L^2_\sigma(\Omega)$ is the solenoidal closed subspace of $L^2(\Omega)$ in which the divergence of all functions are free. Then there exists a unique global solution $u$ of system (2) such that

$$u \in C(\mathbb{R}^+; L^2_\sigma(\Omega)).$$

The main result for system (3) can be stated as follows:

**Theorem 3.2.** Assume that

$$M = -M^T, u_0 \in L^q_\sigma(\mathbb{R}^2)$$

for some $2 \leq q < \infty$, where $L^q_\sigma(\mathbb{R}^2)$ is the solenoidal closed subspace of $L^q(\mathbb{R}^2)$. Then there exists a unique global mild solution $u(t)$ of system (4) which belongs to $C(\mathbb{R}^+; L^q_\sigma(\mathbb{R}^2))$. 
Remark 1. As usual, $L^p_\sigma(\mathbb{R}^N)$ denotes the closure of the set
\[
\{ u \in C^\infty_c(\mathbb{R}^N), \text{div} u = 0 \}
\]
with respect to the $\| \cdot \|_p$ norm.

4. Proof of the Theorem 3.1. Observe that a usual localization procedure does not guarantee the solenoidal condition for the candidate solution. We therefore introduce the Bogovskii operator which concerns the solution of the equation $\text{div} u = f$ in suitable function spaces.

Let $\xi \in C^\infty_c(\mathbb{R}^2)$ be a cut-off function with $0 \leq \xi \leq 1$ and $\xi = 1$ near $\Gamma$. Further let $\text{supp} \nabla \xi := K$. We then define $b : \mathbb{R}^2 \to \mathbb{R}^2$ by
\[
b(x) := \xi Mx - B_K((\nabla \xi) \cdot Mx),
\]
where $B_K$ is the Bogovskii operator associated to $K$. Then $\text{div} b = 0$ and $b(x) = Mx$ on $\Gamma$. Setting $\tilde{u} = u - b$, it follows that $\tilde{u}$ satisfies
\[
\begin{aligned}
\partial_t u - \Delta u + u \cdot \nabla u - Mu + b \cdot \nabla u + \nabla p & = F \\
\text{div} u & = 0 \\
\left. u \right|_{\Gamma} & = 0 \\
u(0) & = u_0 - b
\end{aligned}
\]
in $\Omega \times [0,T)$,
\[
\begin{aligned}
\partial_t u - \Delta u + u \cdot \nabla u - Mu + b \cdot \nabla u + \nabla p & = F \\
\text{div} u & = 0 \\
\left. u \right|_{\Gamma} & = 0 \\
u(0) & = u_0 - b
\end{aligned}
\]
in $\Omega$,
\[
\nabla \cdot (u_0 - b) = 0
\]
and
\[
F = \Delta b + Mx \cdot \nabla b + Mb + b \cdot \nabla b,
\]
provided $u$ satisfies (2). Applying the Helmholtz projection $P_\Omega$ to (12), we may rewrite (12) as an evolution equation in $L^2_\sigma(\Omega)$
\[
\begin{aligned}
\left\{ u' - A_{\Omega,b} u + P_\Omega (u \cdot \nabla u) & = P_\Omega F \\
u(0) & = u_0 - b
\end{aligned}
\]
in $\Omega \times [0,T)$,
\[
\begin{aligned}
\left\{ u' - A_{\Omega,b} u + P_\Omega (u \cdot \nabla u) & = P_\Omega F \\
u(0) & = u_0 - b
\end{aligned}
\]
in $\Omega$,
\[
A_{\Omega,b} u = P(\Delta u + Mx \cdot \nabla u - Mu + b \cdot \nabla u + u \cdot \nabla b).
\]
Note that the compatibility condition on $u_0$ implies $u_0 - b \in L^2_\sigma(\Omega)$. In [18], the authors obtained a unique local mild solution
\[
u \in C([0,T_0], L^2_\sigma(\Omega))
\]
of (13) such that for all $2 \leq q < \infty$,
\[
t \to t^{\frac{1}{2} - \frac{1}{q}} u(t) \in C([0,T_0], L^2_\sigma(\Omega)), \quad t \to t^{1 - \frac{1}{q}} \nabla u(t) \in C([0,T_0], L^q(\Omega)).
\]
According to this result, in order to get global existence, it suffices to show that the $L^2$ norm of $u$ does not blow up in finite time. Therefore, the aim of the following is to prove that the mapping
\[
t \mapsto \| u \|_{L^2(\Omega)}
\]
is bounded on $[0, T)$ for all $T \in (0, T_0)$, where $T_0$ is the maximal time existence of the solution $u$. If we take the inner product in $L^2(\Omega)$ of (13) with $u$ and integrate by parts, we obtain

$$
\frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \leq C \left( \|u\|_{L^2}^2 + \left| \int_{\mathbb{R}^2} u \cdot \nabla b \cdot u \, dx \right| + \int_{\mathbb{R}^2} Fu \, dx \right)
$$

$$
\leq C \left( \|F\|_{L^2}^2 + \|u\|_{L^2}^2 + \left| \int_{\mathbb{R}^2} u \cdot \nabla b \cdot u \, dx \right| \right)
$$

(14)

The main problem is to deal with the third term of the right hand side. Note that

$$
\int_{\mathbb{R}^2} u \cdot \nabla b \cdot u \, dx = \int_{\mathbb{R}^2} \sum_{i,j} u^i \partial_i b^j u^j \, dx
$$

$$
= \sum_{i,j} \int_{\mathbb{R}^2} u^i \partial_i b^j u^j \, dx
$$

$$
= - \sum_{i,j} \int_{\mathbb{R}^2} b^j u^i \partial_i u^j \, dx
$$

$$
= - \int_{\mathbb{R}^2} bu \cdot \nabla u \, dx
$$

$$
= I_1 + I_2,
$$

(15)

where

$$
I_1 = - \int_{\mathbb{R}^2} \xi M x u \cdot \nabla u \, dx,
$$

$$
I_2 = \int_{\mathbb{R}^2} B_K((\nabla \xi) M x) u \cdot \nabla u \, dx.
$$

The term $I_1$ may be bounded by the following

$$
I_1 \leq \int_{\mathbb{R}^2} |\xi M x| \cdot |u \cdot \nabla u| \, dx
$$

$$
\leq C \|\xi M x\|_{L^\infty} \|u\|_{L^2} \|\nabla u\|_{L^2}
$$

$$
\leq C \|u\|_{L^2}^2 + \frac{1}{4} \|\nabla u\|_{L^2}^2.
$$

(16)

The term $I_2$ is more difficult to deal with because $B_K((\nabla \xi) M x)$ no longer belongs to $L^\infty$. To get an appropriate bound, we first introduce some definitions of BMO and Hardy spaces on domains as well as properties associated to such spaces, which one can find in [9], [10], [14], [12].

In [11], two Hardy spaces are defined on a domain $\Omega$ of $\mathbb{R}^N$, one which is reasonably speaking the largest, and the other which in a sense is the smallest. The largest, $H^1_+^\infty(\Omega)$, arises by restricting to $\Omega$ arbitrary elements of $H^1(\mathbb{R}^N)$. The other, $H^1_-(\Omega)$, arises by restricting to $\Omega$ elements of $H^1(\mathbb{R}^N)$ which are zero outside $\Omega$. Norms on these spaces are defined as follows

$$
\|f\|_{H^1_+^\infty(\Omega)} = \inf \|F\|_{H^1(\mathbb{R}^N)},
$$

the infimum being taken over all functions $F \in H^1(\mathbb{R}^N)$ such that

$$
F|_{\Omega} = f, \|f\|_{H^1_+(\Omega)} = \inf \|F\|_{H^1(\mathbb{R}^N)},
$$

where $F$ is the zero extension of $f$ to $\mathbb{R}^N$. 
From \cite{8} the dual of $\mathcal{H}^1_\text{r}(\Omega)$ is $\text{BMO}_r(\Omega)$, a space of locally integrable functions with
\[
\|f\|_{\text{BMO}_r(\Omega)} = \sup_{Q \subset \Omega} \left( \frac{1}{|Q|} \int_Q |f(x) - f_Q|^2 dx \right)^{\frac{1}{2}} < \infty,
\]
where
\[
f_Q = \frac{1}{|Q|} \int_Q f(x) dx,
\]
and the supremum is taken over all cubes $Q$ in the domain $\Omega$. The dual of $\mathcal{H}^1_\text{r}(\Omega)$ is $\text{BMO}_z(\Omega)$, the space of all functions in $\text{BMO}(R^N)$ supported in $\bar{\Omega}$ equipped with the norm
\[
\|f\|_{\text{BMO}_z(\Omega)} = \|f\|_{\text{BMO}(R^N)}.
\]

**Proposition 3.** Let $\Omega$ be a Lipschitz domain in $R^N$.

(i) If $g \in \text{BMO}_r(\Omega)$, then there exists a unique linear functional $L$ in the dual space of $\mathcal{H}^1_\text{z}(\Omega)$ such that
\[
L(f) = \int_{\Omega} f(x)g(x)dx
\]
for all $f \in \mathcal{H}^1_\text{z}(\Omega)$. Conversely, if $L$ is in the dual space of $\mathcal{H}^1_\text{z}(\Omega)$, then there exists a unique $g \in \text{BMO}_r(\Omega)$ such that \eqref{eq:8} holds. The correspondence $L \mapsto g$ given by \eqref{eq:8} is a Banach space isomorphism between $\text{BMO}_r(\Omega)$ and the dual space of $\mathcal{H}^1_\text{z}(\Omega)$.

(ii) If $g \in \text{BMO}_z(\Omega)$, then there exists a unique linear functional $L$ in the dual space of $\mathcal{H}^1_\text{r}(\Omega)$ such that \eqref{eq:8} holds for all $f \in \mathcal{H}^1_\text{r}(\Omega)$. Conversely, if $L$ is in the dual space of $\mathcal{H}^1_\text{r}(\Omega)$, then there exists a unique $g \in \text{BMO}_z(\Omega)$ such that \eqref{eq:8} holds. The correspondence $L \mapsto g$ given by \eqref{eq:8} is a Banach space isomorphism between $\text{BMO}_z(\Omega)$ and the dual space of $\mathcal{H}^1_\text{r}(\Omega)$.

**Proposition 4.** Suppose $v$ and $w$ are vector fields on an open set $\Omega \subset R^N$, satisfying
\[
v \in L^p(\Omega), \quad w \in L^q(\Omega), \quad 1 < p < \infty, \quad \frac{1}{p} + \frac{1}{q} = 1,
\]
and
\[
div v = 0, \quad \text{curl} w = 0.
\]
in the sense of distribution on $\Omega$. Then $v \cdot w$ belongs to the Hardy space $\mathcal{H}^1_\text{r}(\Omega)$ with
\[
\|v \cdot w\|_{\mathcal{H}^1_\text{r}(\Omega)} \leq C\|v\|_{L^p(\Omega)}\|w\|_{L^q(\Omega)}.
\]

Based on the above two facts, we estimate $I_2$ with
\[
I_2 \leq \int_{R^2} |B_K((\nabla \xi)Mx)| \cdot |v \cdot \nabla u|dx
\leq C\|B_K((\nabla \xi)Mx)|_{\text{BMO}_r(\Omega)}\|v \cdot \nabla u\|_{\mathcal{H}^1_\text{r}(\Omega)}
\leq C\|B_K((\nabla \xi)Mx)|_{\text{BMO}_z(\Omega)}\|v\|_{L^2(\Omega)}\|\nabla u\|_{L^2}.
\]

Let $\Psi$ be the zero extension of $B_K((\nabla \xi)Mx)$ to the whole space $R^2$, equipped with the norm
\[
\|\Psi\|_{\text{BMO}(R^2)} = \|B_K((\nabla \xi)Mx)|_{\text{BMO}(\Omega)},
\]
where in the first inequality we have used the fact that $H^1(\mathbb{R}^2) \hookrightarrow BMO(\mathbb{R}^2)$, see [4] or [30] (Chapter 31.)

Combining with the estimates of $I_1$ and $I_2$, we have

$$
\frac{d}{dt} \left( \|u\|^2_{L^2} + \int_0^t \|\nabla u(\tau)\|^2_{L^2} d\tau \right) \leq C \left( \|F\|^2_{L^2} + \|u\|^2_{L^2} \right)
$$

(20)

Gronwall’s lemma implies that for any $t \in [0, T)$

$$
\|u\|^2_{L^2} + \int_0^t \|\nabla u(\tau)\|^2_{L^2} d\tau \leq \|u_0\|^2_{L^2} + C e^{Ct} \int_0^t \|F\|^2_{L^2} d\tau.
$$

This shows the $L^2$ norm of $u$ does not blow up in finite time which implies the solution of system (2) can be extended to a global solution by standard continuity argument.

5. Navier-Stokes equations with linearly growing initial velocity. In this section we consider the two-dimensional case and $\Omega$ is the whole space $\mathbb{R}^2$. As stated in the introduction, this problem is equivalent to the Navier-Stokes equations with linearly growing initial velocity.

$$
\begin{aligned}
\frac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u - M x \cdot \nabla u + Mu + \nabla p = 2Mu, & \quad x \in \mathbb{R}^2, \ t > 0, \\
div u = 0, & \quad x \in \mathbb{R}^2, \ t > 0, \\
u(0) = u_0, & \quad x \in \mathbb{R}^2.
\end{aligned}
$$

(21)

For initial data $u_0 \in L^q_2(\mathbb{R}^2)$ the authors in [23] obtained a unique local-in-time mild solution $u(t)$ of (21), which belongs to $C([0, T_0]; L^q_2(\mathbb{R}^2))$ with $2 \leq q < \infty$. Thus to prove Theorem 3.2, it is enough to show the following a priori estimate. We use the notation $w = rot u$.

**Theorem 5.1.** Assume that $M = -M^T$, $u_0 \in L^q_2(\mathbb{R}^2)$ for $2 \leq q < \infty$, where $L^q_2(\mathbb{R}^2)$ is the solenoidal closed subspace of $L^q(\mathbb{R}^2)$. Let $u(t)$ be a mild solution of (21). Then there exists a positive $C$ which depends only on $q$, such that

$$
\|u(t)\|_{L^q} \leq C \|u(0)\|_{L^q} \exp \left( C \|v_0\|_{L^q} t \right),
$$

where $v_0 = rot u(0)$.

**Proof of Theorem 5.1.** By the expression of the solution which was obtained in [23]

$$
u(t) = e^{-tA}u_0 - \int_0^t e^{-(t-s)A}P(u \cdot \nabla u) ds + 2 \int_0^t e^{-(t-s)A}P(Mu) ds
$$

for $t \in [0, T]$. We have known that the operator $P$ is also a projection operator of $L^q_2(\mathbb{R}^2)$ in $L^q(\mathbb{R}^2)$ for any $1 \leq q < \infty$. Thus applying Proposition 3.4 in [23] yields

$$
\|e^{-(t-s)A}P(u \cdot \nabla u)\|_{L^q} \leq C \|e^{-(t-s)A}(u \cdot \nabla u)\|_{L^q}
$$
Now we employ the inequality
\[ \| \nabla u \|_{L^q} \leq \frac{q^2}{q-1} \| v \|_{L^q} \]
to get
\[ \| u(t) \|_{L^q} \leq \| u(0) \|_{L^q} + \int_0^t \frac{Cq}{(t-s)^{\frac{1}{q}}} \cdot \| u \|_{L^q} \cdot \| v \|_{L^q} \, ds + C \int_0^t \| u \|_{L^q} \, ds. \]

On the other hand, by applying rot to the equation we obtain that the vorticity \( \text{rot} u \) satisfies
\[
\begin{cases}
\partial_t v - \Delta v + u \cdot \nabla v - Mx \cdot \nabla v = 0, & x \in \mathbb{R}^2, \ t > 0, \\
\text{div} v = 0, & x \in \mathbb{R}^2, \ t > 0, \\
v(0) = \text{rot} u_0.
\end{cases}
\] (23)

The standard energy estimate allows us to get
\[ \| v \|_{L^q} \leq 2 \| v_0 \|_{L^q}. \]
Hence we have
\[ \| u(t) \|_{L^q} \leq \| u_0 \|_{L^q} + \| v_0 \|_{L^q} + C \int_0^t \left( \frac{1}{(t-s)^{\frac{1}{q}}} + 1 \right) \cdot \| u \|_{L^q} \, ds. \]

By the Gronwall inequality we have the desired estimate. 

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Received March 2016; revised May 2016.

E-mail address: ssg@emails.bjut.edu.cn, shaoshuguang0927@126.com

E-mail address: wangshu@bjut.edu.cn

E-mail address: xwq@bjut.edu.cn

E-mail address: hanbinxy@163.com