DISCRETE CURVATURE AND TORSION FROM CROSS-RATIOS

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ABSTRACT. Motivated by a Möbius invariant subdivision scheme for polygons, we study a curvature notion for discrete curves where the cross-ratio plays an important role in all our key definitions. Using a particular Möbius invariant point-insertion-rule, comparable to the classical four-point-scheme, we construct circles along discrete curves. Asymptotic analysis shows that these circles defined on a sampled curve converge to the smooth curvature circles as the sampling density increases. We express our discrete torsion for space curves, which is not a Möbius invariant notion, using the cross-ratio and show its asymptotic behavior in analogy to the curvature.

1. Introduction

Many topics in applied geometry like computer graphics, computer vision, and geometry processing in general, cover tasks like the acquisition and analysis of geometric data, its reconstruction, and further its manipulation and simulation. Numerically stable approximations of 3D-geometric notions play here a crucial part in creating algorithms that can handle such tasks. In particular the estimation of curvatures of curves and surfaces is needed in these geometric algorithms [3, 9, 11]. A good understanding of estimating curvatures of curves is often an important step in the direction of estimating curvatures of surfaces.

A different approach to discretized or discrete curvatures than by numerical approximation comes from discrete differential geometry [2]. There the motivation behind any discretization is to apply the ideas and methods from classical differential geometry to discrete objects like polygons and meshes without “simply” discretizing equations or using classical differential calculus. Discrete curvature notions of curves are thus connected to a sensible idea of a curvature circle [8] or a consistent definition of a Frenet-frame [5] or geometric ideas that appear in geometric knot theory [13]. Sometimes discrete definitions of “differential” notions of curves are justified to be sensible by asymptotic analysis and convergence behavior [10, 12].

In the present paper in a way we combine both strategies. For example, we show the invariance of our discrete curvature circle with respect to Möbius transformations or characterize classes of discrete curves that are Möbius equivalent to an arc length parametrization. On the other hand, and therein lies our focus, we use asymptotic analysis to justify the definitions of our discrete notions. For example, in analogy to Sauer [12], we discretize/sample a smooth curve \( s(t) \) by constructing the inscribed polygon \( s(k\varepsilon) \) with \( k \in \mathbb{Z} \) as depicted in Figure 2 (right). And then we use this discrete curve to prove, for example, that our discrete curvature \( \kappa_k \), which is defined at the polygon edge \( k, k+1 \), is a second order approximation of the curvature \( \kappa \) of \( s \), i.e., \( \kappa_k = \kappa + O(\varepsilon^2) \) as \( \varepsilon \to 0 \), as we will see in Theorem 7. Our definition of \( \kappa_k \) will use four consecutive points as input. From our definition of the curvature circle we immediately obtain a discrete Frenet-frame in Theorem 17 and Section 5.2.

In our definition of the discrete curvature circle appears the so called cross-ratio of four points as main ingredient of its definition. The Möbius invariance of the cross-ratio thus implies the same for the curvature circle, which also holds for smooth curves. Even our definition for the
torsion includes the cross-ratio in its definition, however not only it as the torsion is not Möbius invariant.

Our exposition starts with setting the scene in the preliminaries (Sec. 2). Then we investigate a discrete curvature notion for planar curves (Sec. 3) which we generalize to space curves in Section 4. In Section 5 we study a discrete torsion for three-dimensional curves. In Section 6 we consider some special cases and geometric properties of a particular ‘point-insertion-rule’ (Eqn. (3)) that plays an important role in our definition of the discrete curvature. Finally, in Section 7 we perform numerical experiments to verify our discrete notions of curvature and torsion.

2. Preliminaries

2.1. Quaternions. The Hamiltonian quaternions \( \mathbb{H} \) are very well suited for expressing geometry in three dimensional space and in particular for three dimensional Möbius geometry (Sec. 2.2). The quaternions constitute a skew field whose elements can be identified with \( \mathbb{R} \times \mathbb{R}^3 \). In this paper we write quaternions in the following way:

\[
\mathbb{H} = \{ [r, v] \mid r \in \mathbb{R}, v \in \mathbb{R}^3 \}.
\]

The first component \( r = \text{Re} q \) of a quaternion \( q = [r, v] \) is called real part and the second component \( v = \text{Im} q \) imaginary part. Consequently, we write \( \text{Im} \mathbb{H} = \{ q \in \mathbb{H} \mid q = [0, v], \text{with} v \in \mathbb{R}^3 \} \). The addition in this notation of \( \mathbb{H} \) reads \( [r, v] + [s, w] = [r+s, v+w] \), and the multiplication reads \( [r, v] \cdot [s, w] = [rs - \langle v, w \rangle, rw + sv + v \times w] \), where \( \langle \cdot, \cdot \rangle \) is the Euclidean scalar product in \( \mathbb{R}^3 \) and where \( \times \) is the cross product. The conjugation of \( q = [r, v] \) is defined by \( \overline{q} = [r, -v] \) and the square root of the real number \( q \overline{q} \) is called norm of \( q \) and is denoted by \( |q| = \sqrt{q \overline{q}} \). For every \( q \in \mathbb{H} \setminus \{0\} \) its inverse is given by \( q^{-1} = \overline{q}/|q|^2 \).

Any quaternion \( q \in \mathbb{H} \) can be represented in its polar representation \( q = |q| [\cos \phi, v \sin \phi] \) with \( ||v|| = 1 \) and \( \phi \in [0, \pi] \). In that case we can define the square root of \( q \) by \( \sqrt{q} = \sqrt{|q|} [\cos \frac{\phi}{2}, v \sin \frac{\phi}{2}] \). Also when computing the square root of a complex number we will always take the principal square root.

Finally, to express points and vectors of \( \mathbb{R}^3 \) with quaternions we identify \( \mathbb{R}^3 \) with \( \text{Im} \mathbb{H} \) via \( v \leftrightarrow [0, v] \).

2.2. Möbius geometry. A Möbius transformation is a concatenation of a finite number of reflections \( \sigma \) in spheres (center \( c \), radius \( r \)), hence \( \sigma : \mathbb{R}^n \setminus \{\infty\} \to \mathbb{R}^n \setminus \{\infty\} \) with \( \sigma(x) = (x-c)/(|x-c|^2 + c, \sigma(\infty) = c, \sigma(c) = \infty \). Invariants in Möbius geometry are consequently notions and objects that stay invariant under Möbius transformations. An important example of an invariant of planar Möbius geometry is the complex cross-ratio.

2.3. Cross-ratio. The cross-ratio is a fundamental notion in geometry, in particular Möbius geometry. For four quaternionic numbers \( a, b, c, d \in \mathbb{H} \) the cross-ratio is defined as

\[
\text{cr}(a, b, c, d) := (a - b)(b - c)^{-1}(c - d)(d - a)^{-1},
\]

and it is therefore a quaternion itself. The complex numbers \( \mathbb{C} \) constitute a subfield in \( \mathbb{H} \). In our notation \( \mathbb{C} \) can be embedded in \( \mathbb{H} \) as \( \mathbb{C} \cong \{ q \in \mathbb{H} \mid q = [r, (x, 0, 0)] \} \), with \( r, x \in \mathbb{R} \). Consequently, the cross-ratio for complex numbers can be written in the form

\[
\text{cr}(a, b, c, d) = \frac{(a-b)(c-d)}{(b-c)(d-a)},
\]

as \( \mathbb{C} \) is commutative.

It is well known that the cross-ratio of four points in \( \mathbb{R}^3 \) or in \( \mathbb{C} \) is real if and only if the four points are concyclic (see e.g. [1]).
2.4. **Smooth curves.** Our goal is to define a notion of curvature and torsion for discrete curves (Sec. 2.5). We will compare our discrete notions to those of the classical (smooth) differential geometry and as such to parametrized curves \( s : \mathbb{R} \to \mathbb{R}^3 \). We will always assume \( s \) to be sufficiently differentiable. The curvature \( \kappa \) and torsion \( \tau \) of \( s \) are given by (see e.g. [6])

\[
\kappa = \frac{\| s' \times s'' \|}{\| s' \|^3}, \quad \text{and} \quad \tau = -\frac{(s' \times s'', s''')}{\| s' \times s'' \|^2}.
\]

The torsion vanishes if and only if the curve is planar. For a planar curve \( s : \mathbb{R} \to \mathbb{R}^2 \) the curvature is the oriented quantity

\[
\kappa = \det(s', s'') \frac{1}{\| s' \|^3}.
\]

2.5. **Discrete curves.** By a discrete curve we understand a polygonal curve in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \) which is given by its vertices hence by the map \( \gamma : \mathbb{Z} \to \mathbb{R}^3 \). To get visually closer to the notion of a smooth curve we connect for all \( i \in \mathbb{Z} \) consecutive vertices \( \gamma(i) \) by a straight line segment, the edges. However, connecting by line segments is not crucial in this paper except for better visualizations in our illustrations. To shorten the notation we will write \( \gamma_i \) for \( \gamma(i) \). We call the discrete curve planar if it is contained in a plane, i.e., in a two dimensional affine subspace.

3. **Curvature of Planar Discrete Curves**

We first begin our investigation with a discrete curvature notion for planar curves and extend it in Section 4 to curves in \( \mathbb{R}^3 \). We identify the two dimensional plane in which our curves live with the plane of complex numbers \( \mathbb{C} \). Before we proceed to the definition of the curvature (Sec. 3.2) we will consider a ‘point-insertion-rule’ in Section 3.1. We have also considered this point-insertion-rule in the context of a Möbius invariant subdivision method in [15].

3.1. **Point-insertion-rule in \( \mathbb{C} \).** Let \( a, b, c, d \in \mathbb{C} \) be four pairwise distinct points. We construct a new point \( f(a, b, c, d) \in \mathbb{C} \) in an, at a first glance, very unintuitive way:

\[
f(a, b, c, d) := \frac{c(b-a)\sqrt{\text{cr}(c, a, b, d)} + b(c-a)}{(b-a)\sqrt{\text{cr}(c, a, b, d)} + (c-a)} \in \mathbb{C} \cup \infty.
\]

We will explain more about special cases and the geometric relation of \( f \) with respect to \( a, b, c, d \) in Section 6.

**Lemma 1.** The newly inserted point \( f(a, b, c, d) \) fulfills

\[
\text{cr}(c, a, b, f(a, b, c, d)) = -\sqrt{\text{cr}(c, a, b, d)}.
\]

In particular the construction of \( f \) is Möbius invariant.

**Proof.** We expand the cross-ratio on the left hand side and obtain

\[
\frac{(c-a)(b-f)}{(a-b)(f-c)} = -\sqrt{\text{cr}(c, a, b, d)}.
\]

Now simple manipulations of this equation yield (3). The Möbius invariance follows immediately, as \( f \) can be expressed just in terms of cross-ratios. \( \square \)

**Theorem 2.** Let \( a, b, c, d \in \mathbb{C} \) be four pairwise distinct points and consider the four new points obtained from \( f \) by cyclic permutation:

\[
p_{ab} = f(d, a, b, c), \quad p_{bc} = f(a, b, c, d), \quad p_{cd} = f(b, c, d, a), \quad p_{da} = f(c, d, a, b).
\]

Then \( p_{ab}, p_{bc}, p_{cd}, p_{da} \) are concyclic with \( \text{cr}(p_{ab}, p_{bc}, p_{cd}, p_{da}) = -1 \) (see Figure 2 left).
Lemma 1 immediately implies the following two important consequences:
3.2. Curvature for planar curves. Let us consider the planar discrete curve \( \gamma : \mathbb{Z} \to \mathbb{C} \) as illustrated in Figure 2 (left). We assume that any four consecutive vertices of the curve are pairwise distinct. Then, Theorem 2 guarantees the existence of a circle \( k \) passing through \( f(\gamma_{i-1}, \gamma_i, \gamma_{i+1}, \gamma_{i+2}) \), \( f(\gamma_i, \gamma_{i+1}, \gamma_{i+2}, \gamma_{i-1}) \), \( f(\gamma_{i+1}, \gamma_{i+2}, \gamma_1, \gamma_i) \), \( f(\gamma_{i+2}, \gamma_1, \gamma_i, \gamma_{i+1}) \). We use this circle \( k_i \) in the following definition of our discrete curvature.

**Definition 6.** Let \( \gamma : \mathbb{Z} \to \mathbb{C} \) be a planar discrete curve. We call the circle \( k_i \) (discrete) curvature circle at the edge \( \gamma_i \gamma_{i+1} \), the inverse of its radius (discrete) curvature \( \kappa_i \) at the edge \( \gamma_i \gamma_{i+1} \), and its center \( m_i \) (discrete) curvature center. For an illustration see Figure 2 (left).

A ‘good’ discrete definition ‘mimics’ its smooth counterparts. Along these lines we note that our discrete curvature circle is Möbius invariant (Corollary 3 (ii)) as in the smooth case. Furthermore, the curvature circle of a discrete curve with vertices on a circle – we could call
it a discrete circle – is the circumcircle itself, as expected. And Corollary 3 implies that the curvature circle separates the first and the last point of the four points that are involved in its definition (see Figure 2 left). This resembles the local behavior of smooth curves in non-vertex points where the curvature circle separates locally the curve into an ‘inner’ and an ‘outer’ curve.

In the following we continue our argumentation for the reasonableness of this definition of the discrete curvature circle with asymptotic analysis. We will show that the discrete curvature circle (its radius and center) of a sampled curve $s$ converges to the smooth curvature circle as the sampling gets denser and denser. For an illustration of the setting of the following theorem see Figures 2 and 3.

**Theorem 7.** Let $s : \mathbb{R} \rightarrow \mathbb{C}$ be a sufficiently smooth planar curve and let $u, \varepsilon \in \mathbb{R}$. Let further $\gamma : \mathbb{Z} \rightarrow \mathbb{C}$ be the planar discrete curve that samples the smooth curve $s$ in the following way:

$$\gamma_k = \gamma(k) := s(u + (2k - 1)\varepsilon) \quad k \in \mathbb{Z}.$$

Then the discrete curvature $\kappa_0$ of $\gamma$ at the edge $\gamma_0\gamma_1$ is a second order approximation of the smooth curvature $\kappa$ of $s$ at $u$:

$$\kappa_0 = |\kappa(u)| + O(\varepsilon^2). \quad (7)$$

The center $m_0$ of the discrete curvature circle $k_0$ of $\gamma$ converges to the center of the smooth curvature circle of $s$ at the same rate,

$$m_0 = s(u) + \frac{1}{\kappa(u)} N(u) + O(\varepsilon^2), \quad (8)$$

where $N$ denotes the unit normal vector along $s$. Furthermore, $p_{\gamma_0 \gamma_1} = f(\gamma_{-1}, \gamma_0, \gamma_1, \gamma_2)$ is even a third order approximation of $s(u)$, i.e.,

$$p_{\gamma_0 \gamma_1} = s(u) + O(\varepsilon^3). \quad (9)$$

Before we give a proof of this theorem we need a couple of preparatory lemmas. We consider w.l.o.g. the approximation point at $u = 0$. To study the asymptotic behavior of the curvature notions based on our sampled curve we need its Taylor expansion at 0:

$$s(u) = s(0) + us'(0) + \frac{u^2}{2} s''(0) + \frac{u^3}{6} s'''(0) + O(u^4).$$

For the sake of brevity we will just write $s$ instead of $s(0)$, $s'$ instead of $s'(0)$, etc. And until the end of this section we will use the following abbreviations for those four points on which the curvature circle depends:

$$a := \gamma_{-1} = s(-3\varepsilon), \quad b := \gamma_0 = s(-\varepsilon), \quad c := \gamma_1 = s(\varepsilon), \quad d := \gamma_2 = s(3\varepsilon). \quad (10)$$
We will very frequently encounter rational functions depending on \( \varepsilon \) for which we need its Taylor expansion. So at first, a general technical lemma that can easily be verified.

**Lemma 8.** Let \( x_i, y_i \in \mathbb{C} \) with \( y_0 \neq 0 \), then

\[
\sum_{k=0}^{2} x_k y_i = \frac{x_0}{y_0} + \frac{x_1 y_0 - x_0 y_1}{y_0^2} \varepsilon + \frac{x_2 y_0^2 - x_1 y_0 y_1 + x_0 y_1^2 - x_0 y_0 y_2}{y_0^3} \varepsilon^2 + O(\varepsilon^3).
\]

Our first task is to compute the Taylor expansion of the cross-ratio which appears in the ‘inserting’ construction [3]. The following formula illustrates also the close connection between the cross-ratio and the Schwarzian derivative of \( s \) which reads \( \frac{2s''s''' - 3f''^2}{2f^2} \), cf. [4].

**Lemma 9.** Let \( a, b, c, d \) be the four consecutive points of the sampled curve as defined in [10]. Then

\[
\text{cr}(c, a, b, d) = 4 + \frac{12s'^2 - 8s's''}{s'^2} \varepsilon^2 + O(\varepsilon^4), \quad \text{cr}(b, d, a, c) = 4 + \frac{-12s''^2 + 8s'''}{9s'^2} \varepsilon^2 + O(\varepsilon^4).
\]

**Proof.** We start by computing the factors of

\[
\text{cr}(c, a, b, d) = \frac{(c - a)(b - d)}{(a - b)(d - c)}
\]

in terms of the Taylor expansion:

\[
-c - a = s(\varepsilon) - s(-3\varepsilon) = \varepsilon s' + \frac{\varepsilon^2}{2} s'' + \frac{\varepsilon^3}{6} s''' + O(\varepsilon^4) - (-3\varepsilon s' + \frac{9\varepsilon^2}{2} s'' - \frac{9\varepsilon^3}{2} s''' + O(\varepsilon^4))
\]

\[
= 4\varepsilon s' - 4\varepsilon^2 s'' + \frac{14}{3} \varepsilon^3 s''' + O(\varepsilon^4).
\]

And analogously we obtain

\[
(a - b) = -2\varepsilon s' + 4\varepsilon^2 s'' - \frac{13}{3} \varepsilon^3 s''' + O(\varepsilon^4),
\]

\[
b - d = -4\varepsilon s' - 4\varepsilon^2 s'' - \frac{14}{3} \varepsilon^3 s''' + O(\varepsilon^4),
\]

\[
d - c = 2\varepsilon s' + 4\varepsilon^2 s'' + \frac{13}{3} \varepsilon^3 s''' + O(\varepsilon^4).
\]

Now the numerator of the cross-ratio expands to

\[
(c - a)(b - d) = -16s'^2 \varepsilon^2 + \left(16s'^2 - \frac{112s''s'''}{3}\right) \varepsilon^4 + O(\varepsilon^6),
\]

and the denominator to

\[
(a - b)(d - c) = -4s'^2 \varepsilon^2 + \left(16s'^2 - \frac{52s''s'''}{3}\right) \varepsilon^4 + O(\varepsilon^6).
\]

Consequently, after canceling \(-4\varepsilon^2\) the cross-ratio reads

\[
\text{cr}(c, a, b, d) = \frac{4s'^2 - (4s'^2 - \frac{28}{3} s'' s''') \varepsilon^2 + O(\varepsilon^4)}{s'^2 - (4s'^2 - \frac{16}{3} s'' s''') \varepsilon^2 + O(\varepsilon^4)},
\]

which, using Lemma 8 simplifies to

\[
\text{cr}(c, a, b, d) = 4 + \frac{12s'^2 - 8s'' s'''}{s'^2} \varepsilon^2 + O(\varepsilon^4),
\]

the Taylor expansion of the first cross-ratio. The computations for the second one work analogously. \qed
Lemma 10. Let $a, b, c, d$ be as in Lemma 9. Then
\[
\sqrt{cr(c, a, b, d)} = 2 + \frac{3s''^2 - 2s''s'''}{s'} \varepsilon^2 + O(\varepsilon^3).
\]

Proof. This equation follows immediately from the general Taylor expansion for
\[
\sqrt{x_0 + x_2 \varepsilon^2 + O(\varepsilon^3)} = \sqrt{x_0} + \frac{x_2}{2\sqrt{x_0}} \varepsilon^2 + O(\varepsilon^3),
\]
and from Lemma 9.
\[\square\]

Now we are in the position to show the important lemma that guarantees that $p_{\gamma_0 \gamma_1} = p_{bc}$ is a third order approximation of $s$.

Lemma 11. Let $a, b, c, d$ be as in Lemma 9. Then
\[
p_{bc} = s + O(\varepsilon^3).
\]

Proof. We have to compute
\[
p_{bc} = f(a, b, c, d) = \frac{c(b - a)\sqrt{cr(c, a, b, d)} + b(c - a)}{(b - a)\sqrt{cr(c, a, b, d)} + (c - a)},
\]
and start with its components:
\[
c(b - a) = \left(s + \varepsilon s' + \frac{\varepsilon^2}{2} s'' + \frac{\varepsilon^3}{6} + O(\varepsilon^4)\right) \left(2\varepsilon s' - 4\varepsilon^2 s'' + \frac{13}{3} \varepsilon^3 s''' + O(\varepsilon^4)\right)
\]
\[= 2\varepsilon s' + (2s'^2 - 4ss'')\varepsilon^2 + \left(\frac{13}{3} s' s''' - 3s'' s''\right) \varepsilon^3 + O(\varepsilon^4),
\]
and analogously
\[b(c - a) = 4ss' - 4(s'^2 + ss'')\varepsilon^2 + (6s' s'' + \frac{14s'''}{3}) \varepsilon^3 + O(\varepsilon^4).
\]
Putting numerator and denominator together also using Lemma 10 we obtain
\[
p_{bc} = \frac{8s' \varepsilon - 12ss'' \varepsilon^2 + \left(\frac{6s''^2}{3} + \frac{28s'''}{3}\right) s \varepsilon^3 + O(\varepsilon^4)}{8s' \varepsilon - 12ss'' \varepsilon^2 + \left(\frac{6s''^2}{3} + \frac{28s'''}{3}\right) s \varepsilon^3 + O(\varepsilon^4)},
\]
and after canceling $\varepsilon$, applying Lemma 8 concludes the proof.
\[\square\]

Note that Lemma 11 proves Equation 9 in Theorem 7 which says that $p_{bc}$ converges to $s$ at third order. The following lemma can be verified analogously to Lemma 11.

Lemma 12. Let $a, b, c, d$ be as in (10). Then
\[
p_{ab} = s - \sqrt{3s' \varepsilon + \frac{3s''}{2} \varepsilon^2 + O(\varepsilon^3)},
\]
\[
p_{cd} = s + \sqrt{3s' \varepsilon + \frac{3s''}{2} \varepsilon^2 + O(\varepsilon^3)},
\]
\[
p_{da} = s - \frac{2s'^2}{s'''} + \left(5s'' - \frac{20s'''}{3s'''}\right) \varepsilon^2 + O(\varepsilon^3).
\]

Lemma 12 implies that $p_{da} = p_{\gamma_i \gamma_{i-1}}$ is a second order approximation of $s - \frac{2s'^2}{s'''}$. This point $s - \frac{2s'^2}{s'''}$ has an interesting geometric interpretation which we detail in Proposition 13.

After putting these preparatory lemmas in place we can finally turn to the proof of our result on the limit of the curvature circle.
Proof of Theorem 7. Let us first compute the center \( m_0 \) of the discrete curvature circle \( k_0 \). Generally, the circumcenter of a triangle \( a, b, c \in \mathbb{C}^2 \) is given by

\[
\frac{a(\lVert b \rVert^2 - \lVert c \rVert^2) + b(\lVert c \rVert^2 - \lVert a \rVert^2) + c(\lVert a \rVert^2 - \lVert b \rVert^2)}{(a-c)(b-c) - (a-c)(b-c)}.
\]

In our case we want to compute the circumcenter of the four concyclic points \( p_{ab}, p_{bc}, p_{cd}, p_{da} \) from which we choose the three points \( p_{bc}, p_{cd}, p_{da} \) to insert them in the formula above. Let us start with the denominator \( D \):

\[
D = (p_{da} - p_{cd})(p_{bc} - p_{cd}) - (p_{da} - p_{cd})(p_{bc} - p_{cd})
\]

\[
= (-2s'' + \sqrt{3}s' + O(\varepsilon^2))(-2s'' - \sqrt{3}s' + O(\varepsilon^2)) - (\ldots)(\ldots)
\]

\[
= 2\sqrt{3}|s'|^2(s''s' - s's'') + 3(s's'' - s's')(s''s'' + s's''')\varepsilon^2 + O(\varepsilon^3).
\]

We compute the numerator \( N \) in the same way. After a lengthy computation we get

\[
N = 4\sqrt{3}|s'|^2(|s''|^2s' - \frac{1}{2}s(s''s' - s's''))\varepsilon + 6(s''s'' + s's'')(|s'|^2s' - \frac{1}{2}s(s''s' - s's''))\varepsilon^2 + O(\varepsilon^3).
\]

Now, Lemma 8 yields for the center \( m_0 \) of the discrete curvature circle \( k_0 \)

(12) \[
m_0 = \frac{N}{D} = s + \frac{2s's''}{s''s' - s's''} + O(\varepsilon^2).
\]

We need to relate the discrete curvature circle to its smooth counterpart. In order to do that, we rewrite the curvature \( \kappa \) in terms of complex functions. The determinant of a matrix consisting of two column vectors \( a, b \in \mathbb{R}^2 \) is the same as \( \frac{1}{2}(ab - \bar{a}b) \) when \( a \) and \( b \) are expressed as complex numbers. Consequently, the curvature for a curve \( s : \mathbb{R} \to \mathbb{C} \) and its unit normal vector \( N \) can be written in the form

(13) \[
\kappa = \frac{i(s's'' - s's''')}{2|s'|^3}, \quad \text{and} \quad N = \frac{s'}{|s'|}.
\]

as multiplication with \( i \) corresponds to a rotation about the angle \( \pi/2 \). So, we use these notions to rewrite (12):

\[
m_0 = s + \frac{1}{2}(s's'' + s''s'') + O(\varepsilon^2) = s + \frac{|s'|^3}{\det(s', s''')}\frac{s'}{|s'|} + O(\varepsilon^2) = s + \frac{1}{\kappa}N + O(\varepsilon^2).
\]

Consequently, the distance between the center \( m_0 \) of the discrete curvature circle \( k_0 \) and the center \( s + \frac{1}{\kappa}N \) of the smooth curvature circle is of magnitude \( O(\varepsilon^2) \).

Let us now compute the radius of the discrete curvature circle \( k_0 \) by computing the distance of its center \( m_0 \) to a point on the circle, e.g., \( p_{bc} \):

\[
\frac{1}{\kappa_0} = |p_{bc} - m_0| = \left| p_{bc} - \left(s + \frac{1}{\kappa}N\right) + \left(s + \frac{1}{\kappa}N\right) - m_0 \right| = \left| \frac{1}{\kappa}N + O(\varepsilon^2) \right| = \frac{1}{|\kappa|} + O(\varepsilon^2),
\]

which implies Equation (7). Equation (9) follows from Lemma 11. \( \square \)

In Lemma 12 we saw that the point \( p_{da} = p_{\gamma_{i+1}\gamma_{i-1}} \) is a second order approximation of \( s(u) - \frac{2s'^2(u)}{s''(u)} \). In the following proposition we study the geometric meaning of that special point.

Proposition 13. Let \( s : \mathbb{R} \to \mathbb{C} \) be a smooth curve. Then for all \( u \in \mathbb{R} \) the point

\[
\hat{s}(u) := s(u) - \frac{2s'^2(u)}{s''(u)}
\]
is a point on the curvature circle at \( s(u) \) (see Figure 4 top-left). The curve \( \tilde{s} \) is Möbius-invariantly connected to the parametrization of \( s \). Furthermore, the normal vector \( N \) of \( s \) is the angle bisector of \( \tilde{s} - s \) and the second derivative vector \( s'' \) (see Figure 4 top-left).

Proof. To show that \( \tilde{s} \) lies on the curvature circle we show
\[
|\tilde{s} - (s + \frac{1}{\kappa} N)| = \frac{1}{|\kappa|};
\]
\[
|\tilde{s} - (s + \frac{1}{\kappa} N)| = \left| \frac{2s'^2}{s''} + \frac{1}{\kappa} N \right| = \left| \frac{2s'^2}{s''} + \frac{|s'|^3}{2(s' s'' - s''') i s'} \right|
\]
\[
= \left| \frac{2s'^3 s'' - 2s'^2 s' s'' + 2s'^2 s' s''}{s''(s' s'' - s''')} \right| = \left| \frac{|s''|^3}{|s''|^3} \frac{|s'|^3}{|s''|^3} \right| = \frac{1}{|\kappa|}.
\]

Next we show the Möbius invariant property of \( \tilde{s} \). For that, let \( M \) be a Möbius transformation. We have to show
\[
M \circ \tilde{s} = M \circ s = \frac{2(M \circ s)^2}{(M \circ s)^2}.
\]

This equation holds trivially for translations, rotations and scalings. So the only thing left to show is that it is also true for inversions \( M(z) = 1/z \). We start with the right hand side:
\[
\frac{1}{s} - \frac{2(\frac{1}{s})^2}{(\frac{1}{s})^2} = \frac{1}{s} + \frac{2s'^2}{s''s^2 - 2s'^2} = \frac{s''}{s''s - 2s'^2} = \frac{1}{s} - \frac{2s'^2}{s''} = \frac{1}{s} = M \circ \tilde{s}.
\]

Now we show the symmetry property. We have to show that \( \frac{\tilde{s}}{\kappa} \) gets reflected to \( \frac{s''}{\kappa} \) at the symmetry axis \( N \). The reflection of a complex number \( z \) on an axis with direction \( a \) (see Figure 4 bottom-left) is expressible in complex numbers by \( \frac{a}{\bar{a}} \). So we have to show
\[
\frac{N \frac{s - s}{|s - s|}}{N |s - s|} = \frac{s''}{|s''|}.
\]
This equation is equivalent to
\[ \frac{is' - 2s''} {-is' \frac{2s''}{s'}} = \frac{s'}{|s'|} \iff \frac{s'}{|s'|s'} = \frac{s'}{|s'|} \iff \frac{s'}{|s'|s'} = \frac{s''}{s's''}, \]
which is true and therefore implies the symmetry property. \( \square \)

**Remark 14.** If \( s \) is parametrized proportionally to arc length, then \( \bar{s}s \) is a diameter of the curvature circle.

**Corollary 15.** A parametrized curve is Möbius equivalent to a arc length parametrized curve if and only if for all \( u \in \mathbb{R} \) the circles orthogonal to the curvature circle and passing through \( s \) and \( \bar{s}s \) intersect in one common point (see Figure 4 right). The commonly used characterization of arclength parametrizations of discrete curves is by a polygon with constant edgelengths. However, Corollary 15 implies an immediate alternative:

**Definition 16.** We call a discrete curve parametrized proportionally to arclength if \( p_{\gamma_i \gamma_{i+1}} \) and \( p_{\gamma_{i+1} \gamma_{i+2}} \) are opposite points on the discrete curvature circle.

**Theorem 17.** The unit tangent vector \( T_i \) and unit normal vector \( N_i \) of the discrete curvature circle \( k_i \) at \( p_{\gamma_i \gamma_{i+1}} \) are second order approximations of the unit tangent vector \( T \) and unit normal vector \( N \) of the smooth curve (after appropriate orientation), i.e.,
\[ T_i = T + O(\varepsilon^2) \quad \text{and} \quad N_i = N + O(\varepsilon^2). \]

**Proof.** The approximation quality of \( T \) and \( N \) is the same so we just have to prove it for one of them:
\[ N_i = \frac{p_{\gamma_i \gamma_{i+1}} - m_i}{|p_{\gamma_i \gamma_{i+1}} - m_i|} = \frac{s + O(\varepsilon^2)}{s + O(\varepsilon^2)} = \frac{|\kappa|}{\kappa} N + O(\varepsilon^2). \]

After appropriate orientation \( N \) and \( N_i \) differ only about \( O(\varepsilon^2) \). \( \square \)

4. CURVATURE FOR THREE DIMENSIONAL CURVES

Before we generalize discrete curvature from discrete planar curves to space curves we need some more results on the quaternionic cross-ratio for points in three dimensional space. We will use the imaginary quaternions \( \text{Im} \mathbb{H} \) to describe points in three dimensional space \( \mathbb{R}^3 \) (see Sec. 2.1).

4.1. Cross-ratio and geometry. The authors used the quaternionic algebra and the cross-ratio extensively in [14, 15] for applications in regular mesh design and for Möbius invariant subdivision algorithms. The results of this paragraph can also be found there. To prove some technical Lemmas we first consider the following geometric property, which can easily be verified.

**Lemma 18.** Let \( a, b, c \in \mathbb{R}^n \) be three points. Then \( (a - b)(a - c) - (a - c)(a - b) \) is the direction of the tangent of the circumcircle to the triangle \( abc \) at \( a \).

**Definition 19.** Let \( a, b, c \in \text{Im} \mathbb{H} \) be pairwise distinct points. Then we call the imaginary quaternion
\[ t[a, b, c] := (a - b)^{-1} + (b - c)^{-1}, \]
corner tangent.

Note that the identity \( a^{-1} + b^{-1} = a^{-1}(a + b)b^{-1} \) immediately implies
\[ t[a, b, c] = (a - b)^{-1}(a - c)(b - c)^{-1}. \]
Lemma 20. Consider the circumsphere of $a, b, c \in \mathbb{H}$, oriented according to this defining triangle. Then the vector $t[c, a, b]$, placed at $a$, is in oriented tangential contact with the circle (see Figure 5 left).

Proof. Note that $q \in \text{Im} \mathbb{H}$ implies $q^{-1} = -q/|q|^2$. Using the definition of the corner tangent yields 
\[ t[c, a, b] = (c - a)^{-1} + (a - b)^{-1} = -|c - a|^2/|a - b|^2. \]

Consequently, Lemma 18 concludes the proof. \hfill $\Box$

Lemma 21. Let $a, b, c, d \in \text{Im} \mathbb{H}$ be four points not lying on a common circle. Then, the imaginary part of the cross-ratio is the normal of the circumsphere (or plane) at $a$, i.e., for a proper circumsphere with center $m$ we have $\text{Im} \text{cr}(a, b, c, d) \parallel (m - a)$ (see Figure 5 center).

Proof. We compute the cross-ratio in terms of corner tangents abbreviated by $t_1 = t[c, a, b]$ and $t_2 = t[d, a, c]$: 
\[
\text{cr}(a, b, c, d) = (a - b) \cdot (b - c)^{-1} \cdot (c - d) \cdot (d - a)^{-1} \\
= (a - b) \cdot (b - c)^{-1} \cdot (a - c) \cdot (a - c)^{-1} \cdot (c - d) \cdot (d - a)^{-1} \\
= [(a - c)^{-1} \cdot (b - c) \cdot (a - b)^{-1}]^{-1} \cdot [(a - c)^{-1} \cdot (c - d) \cdot (d - a)^{-1}] \\
= [(a - c)^{-1} \cdot (b - c) \cdot (a - b)^{-1}]^{-1} \cdot [(a - c)^{-1} \cdot (d - c) \cdot (d - a)^{-1}] \\

\[\text{Im} \text{cr}(a, b, c, d) = t_1^{-1} \cdot t_2.\]

Since $t_1$ and $t_2$ are both imaginary we can write the cross-ratio as 
\[
\text{cr}(a, b, c, d) = [(t_1^{-1} \cdot t_2), -t_1^{-1} \times t_2].
\]

Lemma 20 implies that $t_1^{-1}$ and $t_2$ are tangent vectors to the circumcircles of the triangles $(abc)$ and $(cda)$, respectively, both at $a$. Consequently, the imaginary part of the above cross-ratio is the cross product of tangent vectors to circles on the circumsphere of $a, b, c, d$ at $a$, hence orthogonal to the tangent plane of the circumsphere at $a$. \hfill $\Box$

Proposition 22. Let $a, b, c, d \in \text{Im} \mathbb{H}$ be four non-concyclic points with $\text{cr}(a, b, c, d) = [r, v]$. Further, let $f \in \mathbb{H}$ be the quaternion that solves 
\[
\text{cr}(a, b, c, f) = [\lambda r, \mu v],
\]
The two occurring cross-ratios can be expressed as (see the proof of Lemma 21)

\[ \text{cr}(a, b, c, d) = t_1 \cdot t_2, \quad \text{and} \quad \text{cr}(a, b, c, f) = t_1 \cdot t_3, \]

where \( t_1 := t[c, a, b]^{-1}, t_2 := t[d, a, c], \) and \( t_3 := t[f, a, c]. \) Consequently, as all \( t_i \in \text{Im} \, \mathbb{H}, \) we have

\[ [r, v] = \langle (t_1, t_2), -t_1 \times t_2 \rangle, \quad \text{and} \quad [\lambda r, \mu v] = \langle (t_1, t_3), -t_1 \times t_3 \rangle. \]

Since all \( t_1, t_2, t_3 \) are orthogonal to \( v \) and are therefore linearly dependent we can express \( t_3 \) in the form \( t_3 = \alpha t_1 + \beta t_2. \) The two vectors \( t_1 \) and \( t_2 \) are linearly independent since otherwise \( t_1 \times t_2 \) would be zero and therefore \( \text{cr}(a, b, c, d) = [r, v] = [-t_1, t_2, 0] \in \mathbb{R} \) which is a contradiction to the four points \( a, b, c, d \) not being concyclic.

After inserting \( t_3 = \alpha t_1 + \beta t_2 \) into the above equations we obtain

\[ \lambda(t_1, t_2) = \lambda r = \langle t_1, t_3 \rangle = \alpha(t_1, t_1) + \beta(t_1, t_2), \]
\[ \mu t_1 \times t_2 = -\mu v = t_1 \times t_3 = \alpha t_1 \times t_1 + \beta t_1 \times t_2. \]

Consequently, \( \beta = \mu \) and \( \alpha = (\lambda - \mu)\langle t_1, t_2 \rangle/|t_1|^2, \) which determines \( t_3 \) uniquely. From the definition of \( t_3 = t[f, a, c] = (f - a)^{-1} + (a - c)^{-1}, \) we then immediately get

\[ f = (t_3 - (a - c)^{-1})^{-1} + a \in \text{Im} \, \mathbb{H}. \]

Furthermore, the circumsphere of \( a, b, c, f \) is the same as the circumsphere of \( a, b, c, d \) since both pass through \( a, b, c \) and both have parallel normal vectors \( (\mu v \text{ and } v, \text{ resp.}) \) at \( a, \) and there is only one such sphere.

4.2. Point-insertion-rule in \( \mathbb{H}. \) Let us now consider the analogous construction of \( \mathbb{H} \) by inserting a new point to given four points \( a, b, c, d \in \text{Im} \, \mathbb{H} \) in three dimensional space. However, the quaternionic square root is not uniquely defined in our formulation (see Sec. 2.1) for negative real numbers. So we must exclude that case in the following which is not a significant restriction as this case (i.e., \( \text{cr}(a, b, a, b) \in \mathbb{R}_{<0} \)) corresponds to a concyclic quadrilateral \( a, b, c, d \) with \( a \) separated from \( d \) by \( b \) and \( c \) on the circumcircle. We exclude such “zigzag” quadrilaterals in the following and consider them as discrete singularities of our polygons.

The quaternionic formula analogous to (3) reads:

\[ f(a, b, c, d) := ((b - a)(c - a)^{-1} \sqrt{\text{cr}(c, a, b, d)} + 1)^{-1} \cdot ((b - a)(c - a)^{-1} \sqrt{\text{cr}(c, a, b, d)c + b}). \]

The notation of this formula is less flexible than in the complex case due to the noncommutativity of \( \mathbb{H}. \) As it will turn out \( f(a, b, c, d) \) is purely imaginary and thus in three space, but note that a priori \( f \) is a quaternion and at first not apparently imaginary. In analogy to Lemma 22 \( f \) is also the solution to a cross-ratio equation:

**Lemma 23.** The newly inserted point \( f(a, b, c, d) \) fulfills

\[ \text{cr}(c, a, b, f(a, b, c, d)) = -\sqrt{\text{cr}(c, a, b, d)}. \]

**Corollary 24.** \( f \) is a point in three dimensional space, i.e., \( f \in \text{Im} \, \mathbb{H}. \) Even more, \( f \) lies on the circumsphere of \( a, b, c, d. \)

**Proof.** The square root of a quaternion \( q = [r, v] \) (see Sec. 2.1) is a quaternion with imaginary part parallel to \( v, \) i.e., parallel to the imaginary part of \( q. \) Consequently, Proposition 22 implies that \( f \) is in \( \text{Im} \, \mathbb{H} \) and in particular on the circumsphere of \( a, b, c, d. \)
4.3. Curvature for discrete space curves. In this section we will relate the curvature and curvature circle of discrete curves in three dimensional space to the planar case (Sec. 3.2). But first let us recall some properties of smooth curves $s : \mathbb{R} \to \mathbb{R}^3$.

Consider a sequence of four points on the curve $s$ which converge to one point $s(0)$. At any time the four points are assumed to uniquely determine a sphere. Consequently, as the four points converge to one point the sequence of spheres defined that way converges to the so-called osculating sphere (see e.g., [6]). The osculating sphere passes through $s(0)$ and has its center at

$$s(0) + \frac{1}{\kappa} N + \frac{\kappa'}{\kappa^2 \tau} B,$$

where $N$ is the unit normal vector, $B$ the binormal unit vector, $\kappa$ the curvature, and $\tau$ the torsion of the curve. The curvature circle at $s(0)$ is the intersection of the osculating plane with the osculating sphere and thus lies on the osculating sphere.

Lemma 25. The osculating sphere has contact of order $\geq 3$ with the curve $s$ which implies that there is a curve $\hat{s}$ on the osculating sphere, s.t.,

$$s(0) = \hat{s}(0), \quad s'(0) = \hat{s}'(0), \quad s''(0) = \hat{s}''(0), \quad s'''(0) = \hat{s}'''(0),$$

This immediately implies the following lemma.

Lemma 26. The curvature and the curvature circle of a space curve $s(u)$ at $u = 0$ is the same as the curvature and the curvature circle of $\hat{s}$ on the osculating sphere at $u = 0$.

Any Möbius transformation that maps the osculating sphere to a plane also transforms the curvature circle to that plane.

Let us now define a curvature circle for discrete space curves. So let us start with a discrete curve $\gamma : \mathbb{Z} \to \mathbb{R}^3$ and set $a = \gamma_{i-1}, b = \gamma_i, c = \gamma_{i+1}, d = \gamma_{i+2}$. In analogy to Theorem 2 we define $p_{ab} = f(d, a, b, c), p_{bc} = f(a, b, c, d), p_{cd} = f(b, c, d, a), p_{da} = f(c, d, a, b)$, but now for the ‘quaternionic’ $f$. Lemma 23 implies that $p_{ab}, p_{bc}, p_{cd}, p_{da}$ lie on the circumsphere of $a, b, c, d$ which we consider as the discrete osculating sphere.

Let us now consider a Möbius transformation that maps the osculating sphere to the $[yz]$-plane of a Cartesian $xyz$-coordinate system. This Möbius transformation (as any Möbius transformation does) keeps the real part as well as the length of the imaginary part of the cross-ratio of four points invariant. The transformed cross-ratios have imaginary parts that are orthogonal to the circumsphere of the new points (Lemma 21). Therefore the transformed cross-ratios have imaginary parts that are parallel to the $x$-axis of the coordinate system. Consequently, the cross-ratios are complex numbers $[x, (x, 0, 0)]$ and we arrive at the case of planar curves (Sec. 3).

So after the Möbius transformation we can apply Theorem 2 which implies that $p_{ab}, p_{bc}, p_{cd}, p_{da}$ lie on a common circle $k_i$ and have a cross-ratio of $-1$. Furthermore, the inverse Möbius transformation maps the circle $k_i$ to a circle $\kappa_i$ on the osculating sphere. And since Möbius transformations map the curvature circle of a curve to the curvature circle of the transformed curve the following definition is sensible.

Definition 27. For discrete space curves $\gamma : \mathbb{Z} \to \mathbb{R}^3$ we call the circle $k_i$ (discrete) curvature circle and the inverse of its radius curvature $\kappa_i$ at the edge $\gamma_i \gamma_{i+1}$. For an illustration see Figure 5 (right).

Theorem 28. Let $s : \mathbb{R} \to \mathbb{R}^3$ be a sufficiently smooth planar curve and let $u, \varepsilon \in \mathbb{R}$. Let further $\gamma : \mathbb{Z} \to \mathbb{R}^3$ be the discrete curve $\gamma_k = \gamma(k) = s(u + (2k - 1)\varepsilon)$. All the approximation results from Theorem 7 apply to space curves in $\mathbb{R}^3$. 

\textbf{Proof.} At first we convince ourselves that it is sufficient to replace the curve \( s \) by the curve \( \hat{s} \) on the osculating sphere (Lemma \[25\]). So instead of \( \gamma_k \) we use \( \hat{\gamma}_k = \hat{s}(u + (2k - 1)\varepsilon) \) for the computation of the discrete curvature circle. We have \( \hat{\gamma}_k = \gamma_k + O(\varepsilon^4) \) and therefore
\[
f(\hat{\gamma}_{i-1}, \hat{\gamma}_i, \hat{\gamma}_{i+1}, \hat{\gamma}_{i+2}) = f(\gamma_{i-1}, \gamma_i, \gamma_{i+1}, \gamma_{i+2}) + O(\varepsilon^4),
\]
i.e., the four points for which we construct the discrete curvature circle are \( O(\varepsilon^4) \)-close to the points on the actual discrete curvature circle. And the center of the replacing curvature circle is therefore also \( O(\varepsilon^4) \)-close to the actual circle since the center of the circumcircle of a triangle \( a, b, c \in \mathbb{R}^3 \) is
\[
\frac{\|a - c\|^2(b - c) - \|b - c\|^2(a - c)) \times ((a - c) \times (b - c))}{2\|(a - c) \times (b - c)\|^2} + c.
\]
So now we know that it is sufficient to show the 3-space version of Theorem \[7\] for \( \hat{s} \) instead of \( s \). After a stereographic projection from the osculating sphere to the complex plane we arrive at the case of planar curves (Sec. \[3\]) for which Theorem \[7\] holds. The only thing left to prove is that the stereographic projection does not change the approximation order of the center of the curvature circle.

Let \( m, r \) denote the center and radius of the smooth curvature circle of the planar curve, and let \( m_0(\varepsilon), r_0(\varepsilon) \) denote the curvature circle of \( \gamma \). From Theorem \[7\] we know that
\[
m = m_0(\varepsilon) + O(\varepsilon^2) \quad \text{and} \quad r = r_0(\varepsilon^2) + O(\varepsilon^2).
\]
After mapping a circle in \( \mathbb{C} \) with center \( m = m_1 + im_2 \) and radius \( r \) stereographically to the sphere we obtain
\[
\phi(m, r) := \frac{1}{r^2 - 2r^2(|m|^2 - 1) + (|m|^2 + 1)^2} \left( \frac{2m_1(1 - r^2 + |m|^2)}{2m_2(1 - r^2 + |m|^2)} \left( r^2 - 1 - |m|^2 \right) \right)
\]
for the new center. Therefore
\[
\phi(m_0, r_0) = \phi(m, r) + O(\varepsilon^2),
\]
i.e., the centers are \( O(\varepsilon^2) \)-close. \( \square \)

5. Torsion

We define the torsion for discrete curves again partially in terms of the cross-ratio. But before we do that we have to consider the right formulation of the torsion of smooth curves.

5.1. Torsion for smooth curves. Let us reformulate the common notation of the torsion \( \tau \) (see Equation \[1\]):
\[
\tau = -\frac{(s' \times s'', s''')}{\|s' \times s''\|^2} = -\frac{\det(s', s'', s''')}{\|s' \times s''\|^2} = -\frac{\det(s'', s', s''')}{\|s' \times s''\|^2} = \frac{\langle s' \times s'', s''' \rangle}{\kappa^2\|s''\|^3}.
\]
The normal unit vector \( N \) is the cross product of the binormal unit vector \( B \) and the tangent unit vector \( T \) and therefore reads
\[
N = B \times T = \frac{s' \times s'''}{\|s' \times s''\|} \times \frac{s'}{\|s'\|} = \frac{s''(s', s') - s'(s', s'')}{\|s'\|\|s' \times s''\|} = \frac{1}{\|s'\|^2\kappa} s'' - \frac{\langle s', s'' \rangle}{\kappa^2\|s''\|^3}s'.
\]
Consequently,
\[
\tau = \frac{\langle s' \times s'', s''' \rangle}{\kappa^2\|s''\|^3} = \frac{\langle s' \times s''', \|s''\|^2\kappa N \rangle}{\kappa^2\|s''\|^6} = \frac{\langle s' \times s''', N \rangle}{\kappa\|s''\|^4}.
\]
We will come back to that formulation of \( \tau \) in the proof of Theorem \[34\].
5.2. **Discrete Frenet frame.** There is a natural way in our setting to define a discrete Frenet frame. Theorem 7 implies that \( p_{bc} = p_{\gamma, i-1} \) is a good discrete candidate for a point where the curvature circle should be in tangential contact with the curve as \( p_{bc} \) is a third order approximation of \( s(u) \). So it is sensible to choose the discrete unit tangent vector \( T_i \) to be in tangential contact with the curvature circle at \( p_{bc} \). It is therefore equally natural to define the normal unit vector \( N_i \) to be the normal of the curvature circle at \( p_{bc} \). Consequently, the binormal vector \( B_i \) should be orthogonal to \( N_i \) and \( T_i \) (see Figure 6).

**Lemma 29.** Let \( s : \mathbb{R} \to \mathbb{R}^3 \) be a sufficiently smooth curve and let \( u, \varepsilon \in \mathbb{R} \). Let further \( \gamma : \mathbb{Z} \to \mathbb{C} \) be the discrete curve \( \gamma_k = s(u + (2k - 1)\varepsilon) \). Then the discrete unit normal \( N_i \) is a second order approximation of the smooth normal \( N \), i.e.,

\[
N_i = N + O(\varepsilon^2).
\]

**Proof.**

\[
N_i = \frac{p_{bc} - m_0}{\|p_{bc} - m_0\|} = \frac{s + O(\varepsilon^3) - m + O(\varepsilon^2)}{\|s + O(\varepsilon^3) - m + O(\varepsilon^2)\|} = \frac{s - m}{\|s - m\|} + O(\varepsilon^2) = N + O(\varepsilon^2),
\]

where we used Theorem 28 at (\( \ast \)). \( \square \)

**Lemma 30.** With the same assumptions as in Lemma 29 we obtain

\[
T_i = T + O(\varepsilon^2).
\]

**Proof.** Theorem 17 implies \( T_i = T + O(\varepsilon^2) \) for the planar case. What remains to verify is that a Möbius transformation does not change this order.

Any vector \( v \) attached at a point \( p \) can be represented as the derivative of a straight line:

\[
[p + tv]_{t=0}.
\]

Consequently, an inversion maps that vector to

\[
\left[ \frac{d}{dt} \right]_{t=0} \left[ \frac{p + tv}{\| p + tv \|^2} \right] = \frac{\| p \|^2 v - 2 \langle p, v \rangle p}{\| p \|^4}.
\]

In our case the vector \( T_i \) is attached at point \( p_{bc} \). Since \( T_i = T + O(\varepsilon^2) \) and \( p_{bc} = s + O(\varepsilon^3) \) for planar curves, we obtain for the tangent vector after inversion

\[
\frac{\|p_{bc}\|^2 T_i - 2 \langle p_{bc}, T_i \rangle p_{bc}}{\|p_{bc}\|^4} = \frac{\| s + O(\varepsilon^3) \|^2 (T + O(\varepsilon^2)) - 2 (s + O(\varepsilon^3), T + O(\varepsilon^2)) (s + O(\varepsilon^3))}{\| s + O(\varepsilon^3) \|^4} = \frac{\| s \|^2 T - 2 \langle s, T \rangle s}{\| s \|^4} + O(\varepsilon^2).
\]

Therefore, Möbius transformations map \( \varepsilon^2 \)-close vectors attached at \( \varepsilon^3 \)-close points to \( \varepsilon^2 \)-close vectors. \( \square \)
Corollary 31. With the same assumptions as in Lemma 29 the discrete Frenet frame \((T_i, N_i, B_i)\) is a second order approximation of the smooth Frenet frame \((T, N, B)\).

5.3. Torsion for discrete curves. In this section we relate the torsion of a discrete curve to the cross-ratio of four successive vertices of the curve. As the real part and the length of the imaginary part of the quaternionic cross-ratio is Möbius invariant but the torsion is not, the definition must also include other quantities that are not Möbius invariant, in our case curvature and length. In Theorem 34 we again use asymptotic analysis to justify our definition of the discrete torsion.

Definition 32. Let \(\gamma : \mathbb{Z} \to \mathbb{R}^3 \cong \mathbb{H}\) be a discrete curve, let \(\kappa_i\) be the discrete curvature at the edge \(\gamma_i \gamma_{i+1}\), and let \(N_i\) denote the discrete normal unit vector. Then, we call
\[
\tau_i := -\frac{9(\text{Im} \text{cr}(\gamma_{i-1}, \gamma_i, \gamma_{i+1}, \gamma_{i+2}), N_i)}{2\kappa_i \|\gamma_i - \gamma_{i+1}\|^2}
\]
(discrete) torsion of \(\gamma\) at the edge \(\gamma_i \gamma_{i+1}\).

Proposition 33. The discrete torsion vanishes for planar discrete curves.

Proof. Planarity of the discrete curve and Lemma 21 imply that the imaginary part of the cross-ratio in the definition of the torsion is perpendicular to that plane. The normal vector \(N_i\) on the other hand is contained in the plane. Therefore the two vectors are orthogonal and the discrete torsion vanishes. \(\Box\)

Theorem 34. Let \(s : \mathbb{R} \to \mathbb{R}^3\) denote a sufficiently smooth curve, let \(u, \varepsilon \in \mathbb{R}\) and let the discrete curve \(\gamma : \mathbb{Z} \to \mathbb{C}\) with \(\gamma_k = \gamma(k) = s(u + (2k - 1)\varepsilon)\) be a sampling of \(s\). Then
\[
\tau_0 = \tau + O(\varepsilon^2).
\]

However, before we prove this theorem we need a preparatory lemma.

Lemma 35. Let \(s : \mathbb{R} \to \mathbb{C}\) denote a sufficiently smooth curve, let \(u, \varepsilon \in \mathbb{R}\) and let the discrete curve \(\gamma : \mathbb{Z} \to \mathbb{C}\) with \(\gamma_k = \gamma(k) = s(u + (2k - 1)\varepsilon)\) be a sampling of \(s\). Let further \(q_0\) denote the cross-ratio of four consecutive vertices \(q_0 := \text{cr}(\gamma_1, \gamma_0, \gamma_1, \gamma_2)\). Then
\[
\text{Re } q_0 = -\frac{1}{3} - \frac{24(s', s'')^2 + 8\|s'\|^2(s', s'') + 12\|s''\|^2}{9\|s''\|^2} \varepsilon^2 + O(\varepsilon^4),
\]
\[
\text{Im } q_0 = -\frac{8\|s''\|^2 s' \times s'' - 24(s', s'')s' \times s''}{9\|s''\|^4} \varepsilon^2 + O(\varepsilon^3).
\]

Proof. We compute
\[
\text{cr}(\gamma_1, \gamma_0, \gamma_1, \gamma_2) = (\gamma_1 - \gamma_0)(\gamma_0 - \gamma_1)^{-1}(\gamma_1 - \gamma_2)(\gamma_2 - \gamma_1)^{-1}
\]
by first expressing each factor in terms of its Taylor expansion:
\[
\gamma_1 - \gamma_0 = -2s' \varepsilon + 4s'' \varepsilon^2 - \frac{13s''}{3} \varepsilon^3 + O(\varepsilon^4),
\]
\[
\gamma_1 - \gamma_2 = -2s' \varepsilon - 4s'' \varepsilon^2 - \frac{13s''}{3} \varepsilon^3 + O(\varepsilon^4),
\]
and now the inverted factors
\[
(\gamma_0 - \gamma_1)^{-1} = \left(-2s' \varepsilon - \frac{s''}{3} \varepsilon^3 + O(\varepsilon^4)\right)^{-1} = \frac{2s' \varepsilon + \frac{s''}{3} \varepsilon^3 + O(\varepsilon^4)}{4\|s''\|^2 \varepsilon^2 + \frac{4(s', s'')}{3} \|s''\|^2 \varepsilon^4 + O(\varepsilon^5)}
\]
\[
= \frac{s'}{2\|s''\|^2 \varepsilon} + \frac{\|s''\|^2 - 2(s', s'')s'}{12\|s''\|^4} \varepsilon + O(\varepsilon^2),
\]
Its cross-ratio is therefore defined as follows immediately from the formula for the center of the osculating sphere, Eq. (15). We can obtain

$$x = \frac{y_n}{y_1} \gamma + \frac{x_0 y_2 - x_0 y_1}{y_1^2} \varepsilon + O(\varepsilon^2).$$

Analogously, we obtain

$$(\gamma_2 - \gamma_1)^{-1} = -\frac{s'}{6\|s''\|^2 \varepsilon} - \frac{-\|s''\|^2 s'' + 2\langle s', s'' \rangle s'}{4\|s''\|^4} \varepsilon + O(\varepsilon^2).$$

The above four factors are all purely imaginary quaternions. Multiplying these factors together in the right order yields the proposed real and imaginary part of the cross-ratio. \(\square\)

So, let us now turn to our approximation result for the torsion:

**Proof of Theorem 34.** We show the formula at \(u = 0\) and therefore \(i = 0\):

$$\tau_0 = -\frac{9}{2} \left( \text{Im} q_0 \right) \gamma = -\frac{9}{2} \left( -8\|s''\|^2 s' \times s'' + 24\langle s', s'' \rangle s' \times s'' \right) \varepsilon^2 + O(\varepsilon^3), N + O(\varepsilon^2)$$

\(\equiv\) \(\|s''\|^2 \frac{s' \times s'' \varepsilon^2}{\|s''\|^2 \varepsilon^2 + O(\varepsilon^2)} + O(\varepsilon^2)\)

where we used

$$\|\gamma_0 - \gamma_1\| = \|s(\varepsilon) - s(\varepsilon)\| = \|s - \varepsilon s' + \varepsilon^2/2 + O(\varepsilon^3) - (s + \varepsilon s' + \varepsilon^2/2 + O(\varepsilon^3))\|$$

$$= \|2\varepsilon s' + O(\varepsilon^3)\|$$

at \((\ast)\) and \(\langle s' \times s'', N \rangle = 0\) at \((\tilde{\gamma})\). \(\square\)

**Remark 36.** We have now a curvature and torsion for a discrete space curve as well as an osculating sphere and osculating circle. In the setting of smooth curves the oriented distance between the center of the osculating circle and the osculating sphere is

$$\frac{\kappa'}{\kappa^2 \tau}$$

as follows immediately from the formula for the center of the osculating sphere, Eq. (15). We can therefore define a discrete version of \(\kappa'\) as that value that fulfills the equation above by replacing smooth notions by their discrete counterparts.

6. Geometric properties

Any quadrilateral is Möbius equivalent to a parallelogram, and even more special, it is Möbius equivalent to a parallelogram \(a, b, c, d\) with \(a = 0\) and \(b = 1\) and \(b - a = d - c\). See Figure 7 (left). Its cross-ratio is

$$\text{cr}(c, a, b, d) = c^2.$$

If \(f\) denotes the intersection point of the diagonals then we obtain

$$\text{cr}(c, a, b, f) = -c,$$
and therefore
$$\mathrm{cr}(c, a, b, f) = -\sqrt{\mathrm{cr}(c, a, b, d)}.$$ 

This together with Lemma 1 immediately implies the following Lemma.

**Lemma 37.** Let $a, b, c, d$ be a parallelogram with $b - a = d - c$. Then the insertion point $f(a, b, c, d)$ corresponds to the intersection point of the diagonals.

Furthermore, in the case of a parallelogram the circumcircles of $a, b, c$ and $b, d, c$ are congruent. Therefore, one of their two bisector circles is the straight line containing the diagonal $bc$ (see Figure 7 right). The same holds for the other pair of circumcircles $a, b, d$ and $a, d, c$. Since Möbius transformations do not change the intersection angles of curves we obtain the following lemma.

**Lemma 38.** The insertion point $f(a, b, c, d)$ is one of the intersection points of the bisector circles of the pairs of circumcircles mentioned above.

Consequently, the discrete curvature circle can be constructed with compass and ruler. In the following lemma we mention three special cases:

**Lemma 39.**

(i) Let $a, b, c, d$ be a parallelogram with $b - a = c - d$. Then the four points $p_{ab}, p_{bc}, p_{cd}, p_{da}$ form a square (see Figure 8 left).

(ii) Let $a, b, c, d$ be a parallelogram with $b - a = d - c$. Then $p_{da} = \infty$ and the curvature circle degenerates to a straight line (see Figure 8 center).

(iii) Let $a, b, c, d$ be symmetric as in Figure 8 (right). Then $p_{ab} = \infty$ and the curvature circle degenerates to a straight line. Thus, this arrangement of points can be seen as a discrete analogue of a cusp on a curve.

**Proof.** ad (i) The rotational symmetry by an angle of $\pi$ of the parallelogram implies that the points $p_{ab}$ and $p_{cd}$ are opposite of the center of rotation as well as $p_{bc}$ and $p_{da}$. A quadrilateral with this property and with a cross-ratio of $-1$ (Theorem 2) must be a square.

ad (ii) and (iii). It follows from simple computations that $p_{da}$ and $p_{ab}$, respectively, vanish to $\infty$. Circles containing this point are straight lines. \qed

7. **Experimental results**

We conducted convergence tests which empirically verify our claims. For this, we used the following seven curves (see Figure 9 for their depiction):

(i) The epitrochoid $c_1(t) = (6\cos(t) - 3\cos(6t), 6\sin(t) - 3\sin(6t))$ (the curve is planar).

(ii) A planar logarithmic spiral $c_2(t) = e^{at}(\cos(t), \sin(t))$, where we use $a = 0.5$.

(iii) A helix $c_3(t) = (\cos(at), \sin(at), bt)$ where we use $a = 4$ and $b = 0.5$.

(iv) A helical spiral $c_4(t) = (e^{at}\cos(4t), e^{at}(\sin(4t), bt))$ where we use $a = 0.4$ and $b = 4$. 

(v) A toroidal “coil” \( c_5(t) = ((a + \sin(bt)) \cos(t), (a + \sin(bt)) \sin(t), \cos(bt)) \) where we use \( a = 2.5 \) and \( b = 20. \)

(vi) The trefoil knot \( c_6(t) = (\sin(t) + 2 \sin(2t), \cos(t) - 2 \cos(2t), -\sin(3t)) \).

(vii) Viviani’s curve \( c_7(t) = (a(1 + \cos(2t)), a \sin(2t), 2a \sin(t)) \) with \( a = 5. \)

For all examples, we used \( t \in [0, 2\pi] \). For simplicity, we assumed all curves are open, and disregarded boundaries; that is we do not compute edge midpoint and consequent quantities for edges adjacent to boundary vertices. The curves are not assumed to be arc-length parametrized.

For any given resolution step \( \varepsilon \), we created a discrete curve by sampling every curve \( c_i(t) \), as explained in Theorem 28. Then, we compute the discrete curvature \( \kappa \), the discrete torsion \( \tau \), and the discrete Frenet frame \( \{T, N, B\} \) for every midedge point. We measure the approximation error to the corresponding quantities of the smooth curve at the sampled points by the \( l^\infty \) norm. This produces the maximum absolute deviation of every discrete quantity from the ground truth. In case of vector quantities (like the Frenet frame), we do so per component. We use \( \varepsilon = 0.1 \times 1.1^l \), where \( l \in \mathbb{N} \) runs between 0 and \( -15 \) in steps of \( -1 \), which creates gradual refinement. To measure convergence rate, we perform linear regression on the logarithmic scale of \( \varepsilon \) vs. \( l^\infty \) error per curve. The graphs of errors can be seen in Figure 10, and the convergence rates are in Table 1. It is evident that we are able to reproduce the quadratic convergence rates that we prove in this paper. Note that we do not measure torsion for \( c_1(t) \) and \( c_2(t) \) as they are planar. Another outlier is the normal error for \( c_3(t) \) which is already initially very low (due to the high regularity of the helix), and thus we only see the effect of numerical noise.

Acknowledgements

The first author gratefully acknowledges the support of the Austrian Science Fund (FWF) through projects P 29981 and I 4868.

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Figure 10. $l^\infty$ errors vs. sampling step $\varepsilon$.

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