GEODESIC RAYS AND EXPONENTS IN ERGODIC PLANAR FIRST
PASSAGE PERCOLATION.

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Abstract. We study first passage percolation on the plane for a family of invariant, ergodic
measures on \( \mathbb{Z}^2 \). We prove that for all of these models the asymptotic shape is the \( \ell_1 \) ball and
that there are exactly four infinite geodesics starting at the origin a.s. In addition we determine
the exponents for the variance and wandering of finite geodesics. We show that the variance and
wandering exponents do not satisfy the relationship of \( \chi = 2\xi - 1 \) which is expected for independent
first passage percolation.

1. Introduction

First passage percolation is a widely studied model in statistical physics. One of the main
reasons for interest in first passage percolation is that it is believed that, for independence passage
times (and under mild assumptions on the common distribution) the model belongs to the KPZ
universality class. The study of first passage percolation has centered on the three main sets of
questions below. (Precise definitions are given in the next two sections.)

1. Asymptotic shape. Cox and Durrett proved that every model of first passage percolation
has an asymptotic shape \( B \subset \mathbb{R}^2 \) which is convex and has the symmetries of \( \mathbb{Z}^2 \) [CD81]. We
would like to determine \( B \) or at least describe some of its properties. In particular is the
asymptotic shape strictly convex and is its boundary differentiable?

2. Infinite geodesics from the origin. Are there infinitely many one-sided infinite geodesics
that start at \((0,0)\)? Do these geodesics all have asymptotic directions?

3. Variance and wandering exponents. For any \( \lambda \geq 0 \) does there exist a variance exponent
\( \chi = \chi(\lambda) \) such that

\[ \text{Var}(T(0, (n, \lambda n))) = n^{2\chi + o(1)} \]

Does there exist a wandering exponent \( \xi = \xi(\lambda) \) such that with high probability every edge
in \( \gamma(0, (n, \lambda n)) \) is within distance \( n^{\xi + o(1)} \) of the line segment connecting \( 0 \) and \( (n, \lambda n) \)? Do \( \chi \) and \( \xi \) satisfy the universal scaling relation

\[ \chi = 2\xi - 1 ? \]

It is widely believed that (under mild assumptions) in independent first passage percolation the
answer to all of these questions is yes. However in our models we show that the answer all of these
questions is at least somewhat different than the answers that are expected for the independent
case. Thus our model shows that universality can not be expected to hold for general models of
ergodic first passage percolation. Our results are as follows.

1. For all of our models the asymptotic shape \( B \) is the unit ball in the \( \ell_1 \)–norm.

2. Our models have exactly four one-sided infinite geodesics starting from the origin a.s., each
of which meander through a quadrant.

3. For each value of \( \lambda \) we calculate exact variance and wandering exponents of the geodesic from
\( 0 \) to \((n, \lambda n)\). For all \( \lambda > 0 \) the variance exponent \( \chi \) is zero while the wandering exponent
is 1. For \( \lambda = 0 \) we get variance and wandering exponents that satisfy \( 0 < \chi = \xi < 1 \). In
neither of these cases do the exponents satisfy the universal scaling relation \( \chi = 2\xi - 1 \).
It is already known that there exist models of ergodic first passage percolation whose behavior is different than what is expected for independent first passage percolation. Häggström and Meester showed that for any set $B \subset \mathbb{R}^2$ which is bounded, convex and has non-empty interior and all the symmetries of $\mathbb{Z}^2$ there is a model of ergodic first passage percolation that has $B$ as its limiting shape. [HM95a] The examples we construct show that the there are models of ergodic first passage percolation that have anomalous geodesic structures. More interestingly our models have anomalous variance and wandering exponents and these exponents depend on the direction. We are not aware of any other non-trivial models of ergodic first passage percolation where the variance and wandering exponents have been explicitly calculated.

2. Background on first passage percolation.

In first passage percolation, a nonnegative variable is associated to each edge of a given graph. These variables give rise to a random metric space. Among the fundamental objects of study of this metric space are the scaling properties of balls and the structure of geodesics. By planar first passage percolation, we refer to the model on the lattice, denoted by $\mathbb{Z}^2$, which has vertex set $V = \{(x, y) : x, y \in \mathbb{Z}\}$ and edge set $E \subset V \times V = \{(v, w) : |v - w| = 1\}$ where $|\cdot|$ denotes the taxi cab distance. A configuration of $\mathbb{Z}^2$ is simply a function from the edge set to the non-negative real numbers:

$$t : E \to \mathbb{R}_+$$

We will use the more common notation $t_e$ for $t(e)$. If $\nu$ is a probability measure in $(\mathbb{R}_+)^E$, we denote $\text{FPP}(\nu)$ the random space obtained by taking $t \sim \nu$. The number $t_e$ can be seen as the passage time or length of the edge $e$. Given a configuration $t$ on $\mathbb{Z}^2$ and a path $\pi = \{e_i\}_{i=1}^k$ the length of $\pi$ is

$$\Gamma(\pi) = \sum_{i=1}^k t_{e_i}.$$ 

The distance between two vertices $u$ and $v$ is denoted by $d(u, v)$ and it is defined as

$$d(u, v) = \inf \Gamma(\pi)$$

where the inf is taken over the set of all paths connecting $u$ and $v$. It is not hard to check that $(\mathbb{Z}^2, d(\cdot, \cdot))$ is a metric space for any configuration. The ball of radius $R$ centered at $u$ is

$$B(u, R) = \{v \in V : d(u, v) \leq R\}.$$ 

Cox and Durrett [CD81] studied the behavior of large balls after scaling. They proved that, if $t_e \sim \nu$ satisfying

$$\mathbb{E}(\min\{t_1^2, t_2^2, t_3^2, t_4^2\}) < \infty$$

for independent copies of $\nu$, and the mass at zero is less than the threshold for bond percolation then there is a non-empty set, $B$ compact, convex and symmetric with respect to the origin such that, for any $\epsilon > 0$

$$\mathbb{P}\left((1 - \epsilon)B \subset \frac{B(0, R)}{R} \subset (1 + \epsilon)B \text{ for all large } R\right) = 1.$$ 

Boivin extended this to a wide class of ergodic models of first passage percolation. [Boi90]

The question of which compact sets can be obtained as limit in FPP is almost entirely open for the i.i.d. case. Interestingly, when we consider the bigger set of stationary and ergodic measures on $(\mathbb{R}_+)^E$ it was proved by Häggström and Meester [HM95b] that any compact, convex, symmetric (with respect to the origin) set is the limiting shape for a stationary and ergodic measure, not necessarily i.i.d. It is worth mentioning that the limiting shape $B$ is the unit ball of a norm $\|\cdot\|_{\nu}$:

$$B = \{x \in \mathbb{R}^2 : \|x\|_{\nu} \leq 1\}$$ (2)
induced by the metric defined in \( (1) \).

A geodesic between \( u \) and \( v \) is a path that realizes the infimum in \( (1) \). We denote geodesics by \( \gamma(u,v) \). Geodesics aren’t always unique. A simple condition to guarantee such property for independent edge weights is to consider continuous distribution for \( t_e \). A geodesic ray is an infinite path \( \{v_0, v_1, v_2, \ldots \} \) such that every finite sub-path is a geodesic between its endpoints. We consider two geodesic rays to be distinct if they intersect in only finitely many edges. We denote by \( T_0 \) the set of different geodesic rays starting at the origin. Ahlberg and Hoffman [AH16] recently showed that for a wide class of measures the cardinality of \( T_0 \) is constant almost surely.

3. Statement of Results

The limiting shape \( B \) is closely related to the number and geometry of geodesic rays for ergodic FPP. Let \( \text{sides}(B) \) denote the number of sizes of \( B \) if it is a polygon, and infinity otherwise, Hoffman ([Hof05], [Hof08]) proved that, for any \( k \leq \text{sides}(B) \) there exist \( k \) geodesic rays almost surely, for \emph{good measures}, see section \( 4 \) for details. In particular, his results imply that there exist at least four geodesics a.s. When \( B \) is a polygon, little is known about existence of geodesics rays in the direction of the corners of \( B \). Recently, Alexander and Berger [AB18] exhibit a model for which the limiting shape is an octagon and all (possibly infinitely many) geodesic rays are directed along the coordinate axis. Our first result shows that our model has exactly four geodesic rays a.s.. To the best of our knowledge, this is the first known FPP model for which \( |T_0| \) is finite.

**Theorem 1.** There exists a family of measures \( \{\nu_{\alpha}\}_{0<\alpha<0.2} \) such that \( |T_0| = 4 \nu_{\alpha}\)-almost surely.

Our next result is about the direction of geodesic rays. We start with a definition. The direction, \( \text{Dir}(\Pi) \), of a sequence of points \( \Pi = \{x_k, \ k \geq 0\} \) is the set of limits of \( \{v_k/|v_k|, \ v_k \in \Pi\} \). Thus, \( \text{Dir}(\Pi) \) is a connected subset of \( S^1 \). Damron and Hanson [DH14] were the first to prove directional results for geodesic rays for \emph{good measures} that also have the finite upward energy property. Their results are also dependent on the geometry of \( B \) in the following way. We said that a linear functional \( \rho : \mathbb{R}^2 \to \mathbb{R} \) is tangent to \( B \) if the line \( \{x \in \mathbb{R}^2 : \rho(x) = 1\} \) is tangent to \( B \) at a point of differentiability of the boundary of \( B \). In view of equation \( (2) \), we can write the intersection of this tangent line and the boundary of \( B \) as a set in \( S^1 \):

\[
(3) \quad D_\rho = \{x \in S^1 : \rho(x) = \|x\|_\nu\}.
\]

[DH14] Theorem 1.1 states that for any functional \( \rho \), tangent to \( B \), there is an element \( \gamma \in T_0 \) satisfying \( \text{Dir}(\gamma) \subset D_\rho \). Because of the differentiability condition, their result gives no information about the behavior around corners of \( B \). For our family of measures \( \{\nu_{\alpha}\} \) we are able to completely characterize the directions of geodesic rays.

**Theorem 2.** Let \( \rho \) be a linear functional tangent to the \( \ell_1 \)-ball. There is exactly one geodesic with generalized direction equal to \( D_\rho \).

Lastly, we turn our attention to the study of the geometry of finite geodesics. We prove non universality of ergodic FPP, by explicitly computing the variance and wandering exponent in every direction. Our results are strong enough that they satisfy any reasonable definition of a wandering exponent.

**Theorem 3.** In every direction not parallel to the coordinate axes we have the variance exponent \( \chi = 0 \) and the wandering exponent \( \xi = 1 \). Parallel to the coordinate axes the two exponents are equal with \( \chi = \xi = \frac{\log 5}{\log 2 - \log \alpha} \). In no direction do the exponents satisfy the universal scaling relation.

3.1. Organization of the paper. The rest of the paper is organized as follows. In Section \( 4 \) we define the measures \( \nu_{\alpha} \) and show its main properties. The proof of Theorem \( 1 \) is given in Section \( 5 \) where the limiting shape is also determined. Section \( 6 \) is devoted to the study of the directional properties of geodesic rays and the proof of Theorem \( 2 \). In Sections \( 7 \) and \( 8 \) we prove our final
theorem which determines the exponents in all directions. This is the content of Lemmas\textsuperscript{18} and \textsuperscript{19} for the coordinate directions and Lemma \textsuperscript{14} and Proposition \textsuperscript{15} for the non-coordinate directions.

4. Construction of the measures \{\nu_\alpha\}.

In this section we construct a family of measures \{\nu_\alpha\} \subset \mathcal{M}((\mathbb{R}_+)^E), indexed by a parameter \(0 < \alpha < 0.2\). We state their main properties and study the behavior of geodesics for \(FPP(\nu_\alpha)\).

Let \(\sigma : \Omega \rightarrow \Omega\) be the 5-adic adding machine: \(\sigma\) adds one \((\text{mod } 5)\) to the first coordinate. If the result is not zero then we leave all the other coordinates unchanged. If the result is zero then we add one to the second coordinate. We repeat until we get the first non-zero coordinate. All subsequent coordinates are left unchanged. Thus
\[
\sigma(0, 1, 2, \ldots) = (1, 1, 2, \ldots) \quad \text{and} \quad \sigma(4, 4, 2, 1, \ldots) = (0, 0, 3, 1, \ldots).
\]

We also adopt the convention \(\sigma(444\ldots) = 000\ldots\)

It follows that \(\sigma : \Omega \rightarrow \Omega\) is uniquely ergodic with respect to the uniform measure in \(\Omega\). We use this map to form a \(\mathbb{Z}_2\) action of \(\Omega \times \Omega\). Let \(v = (x(v), y(v)) \in V\) and \(\omega = (\omega_1, \omega_2) \in \Omega \times \Omega\).

Let \(L : \Omega \rightarrow \mathbb{N}\) given by
\[
L(\omega) = \min\{i : \omega(i) > 0\}.
\]

For \((\omega_1, \omega_2) \in \Omega \times \Omega\) fixed and \(v \in \mathbb{Z}^2\) we define:
\[
k(v, v + (1, 0)) = L(\sigma^{x(v)}(\omega_1))
\]
and
\[
k(v, v + (0, 1)) = L(\sigma^{y(v)}(\omega_2))
\]

The following set will be often refer in the paper, so we highlight its definition now.

**Definition 1.** We denote the set of edges \(e \in E\) such that \(k(e) \geq j\) as the \(j\)-grid.

Note that if \(e_1\) and \(e_2\) are two edges in the same horizontal or vertical line then \(k(e_1) = k(e_2)\). Hence, it is not hard to see that for each \(j\), the \(j\)-grid consists of a translation by \(\omega\) of the grid \(5^j \mathbb{Z}^2\). Furthermore, the \(j\)-grids are nested:
\[
1\text{-grid} \supseteq 2\text{-grid} \supseteq \ldots
\]

Let \(\{X_{j,e}\}_{j \in \mathbb{N}, e \in E}\) be a set of independent random variables where \(X_{j,*}\) has the uniform distribution over the interval \([0, \frac{\alpha^j(1-\alpha)}{1000}]\). We finally define the measure \(\nu_\alpha\) by taking
\[
t_e = 1 + \alpha^{k(e)} + X_{k(e),e}
\]
where \(\omega_1\) and \(\omega_2\) are chosen uniformly i.i.d. and independent of the \(\{X_{j,e}\}\). For technical reasons, we let \(0 < \alpha < 0.2\). Note that for every edge \(e\)
\begin{equation}
1 \leq t_e \leq 1.3
\end{equation}

**Remark 1.** The measures \(\nu_\alpha\) fall into the class of good measures introduced in \([\text{Hof05}]\) and \([\text{Hof08}]\).

We recall the definition of good measures. A measure \(\mathbb{P}\) is good if:

(a) \(\mathbb{P}\) is ergodic with respect to the translations of \(\mathbb{Z}^2\).
(b) \(\mathbb{P}\) has all the symmetries of \(\mathbb{Z}^2\).
(c) \(\mathbb{P}\) has unique passage times.
(d) The distribution of \(\mathbb{P}\) on an edge has finite \(2 + \epsilon\) moment.
(e) The limiting shape is bounded.

The construction of \(\nu_\alpha\) is done so properties (a) – (e) are easy to check.
Informally, we think of a realization of $FPP(\nu_{\alpha})$ as building a series of horizontal and vertical highways on the nearest neighbor graph of $\mathbb{Z}^2$. The value of $\omega$ determines where the origin lies with respect to these highways. By construction, edges in the grid of $\kappa$ are faster (i.e., have smaller passage time) than edges in $j'$-grid for $j > j'$. Hence, geodesics are expected to follow one grid until it encounters a faster one. Then the geodesic continues along edges of the faster grid. Globally, we expect to see rays with longer segments parallel to the axes as they move away from the origin. We also suspect that the length of these horizontal or vertical segments is roughly determined by the value of the $j$-grid they are part of. We formalize this intuition in the next sections.

5. Structure of finite geodesics

In this section we present several properties of geodesics in $FPP(\nu_{\alpha})$. The first lemmas describe the geometric properties of finite geodesics along vertices in the $k-$grid, recall definition [1].

**Lemma 4.** Let $C = C(x, y, k)$ a square of side $5^k$ with lower left vertex $(x, y)$ such that all the edges in its boundary are in the $k-$grid. Consider two vertices $v$ and $w$ in the boundary of $C$. Then

(i) $\gamma(v, w)$ is completely contained in $C$.

(ii) If $v$ and $w$ lie in the same or adjacent sides of $C$, $\gamma(v, w)$ lies in the boundary of $C$.

**Proof.** We argue by contradiction. Assume there are vertices $v$ and $w$ in the boundary of $C$ such that $\gamma(v, w)$ intersects the complement of $C$. Because a subpath of a geodesic is also a geodesic, we can assume that the edges of $\gamma(v, w)$ lie entirely in the complement of $C$, by considering a segment of $\gamma(v, w)$ completely in the complement of $C$ and taking $v$ and $w$ to be its end points. Let $d$ denote the length (the $\ell_1-$distance) of the shortest path along the boundary of $C$ connecting $v$ and $w$. If the maximal distance from a vertex in $\gamma(v, w)$ to $C$ is less than $5^k$ then, by construction, all edges in $\gamma(v, w)$ will lie on the $(k - 1)-$grid at most, hence, have passage time at least $1 + \alpha^{k-1}$. Then the passage time of $\gamma(v, w)$ is at least

$$d(1 + \alpha^{k-1}) > d(1 + \alpha^k) + d\alpha^k$$

The right hand side is an upper bound for the passage time of the path from $v$ to $w$ along the boundary of $C$. We conclude that going along the boundary of $C$ will be a shortest path from $v$ to $w$. Hence, there should be a vertex in $\gamma(v, w)$ at distance at least $5^k$ of $C$. Then the passage time of $\gamma(v, w)$ is at least

$$2(5^k) + d$$

where the factor of two appears since we move away from $C$ at least $5^k$ edges and come back to $C$, crossing another $5^k$ edges. Observe that $d \leq 2(5^k)$. Hence,

$$d(1 + \alpha^k) + d\alpha^k = d + 2d\alpha^k < d + 2(5^k),$$

using that $2\alpha^k < 1$ as long as $\alpha < 1/5$. The left hand side above is an upper bound on the passage time of a path connecting $v$ and $w$ along the boundary of $C$. This concludes the proof of part (i).

To prove (ii), assume that $v$ lies in the left side of $C$ and consider two cases for $w$.

**Case 1:** $w$ lies also on the left hand side or the horizontal sides of $C$, but it is not a corner on the right hand side. By (i) we know $\gamma(v, w)$ is contained in $C$. If $\gamma(v, w)$ uses edges in the interior of $C$, we can assume, changing $v$ and $w$ if necessary, that the entire geodesic lies in the interior. This implies that all edges in $\gamma(v, w)$ have passage times at least $(1 + \alpha^{k-1}) > 1 + \alpha^k + \frac{\alpha^k(1 - \alpha)}{1000} \geq t_\epsilon$ for all $e$ in the boundary of $C$. Hence, a path along the boundary will have smaller passage times, which shows that $\gamma(v, w)$ lies on the boundary.

**Case 2:** $w$ is a corner on the right hand side of $C$. We compare the path along the boundary of $C$ to any path using interior edges to cross $C$. 
Such path will have length at least:

$$5^k(1 + \alpha^{k-1}) + |y(w) - y(v)|.$$  

This bound follows from the $5^k$ edges we need to cross in the interior of $C$, each with passage time at least $(1 + \alpha^{k-1})$, and the $|y(w) - y(v)|$ many edges we need to traverse horizontally. A path on the boundary of $C$ connecting $v$ and $w$ has length at most:

$$5^k(1 + \alpha^k) + |y(w) - y(v)|(1 + \alpha^k) + 5^k\frac{\alpha^k(1 - \alpha)}{1000}.$$  

The last summand is an upper bound on the sum of the random portion of the path’s distance. To conclude we need to show that

$$5^k(1 + \alpha^{k-1}) + |y(w) - y(v)| \geq 5^k(1 + \alpha^k) + |y(w) - y(v)|(1 + \alpha^k) + 5^k\frac{\alpha^k(1 - \alpha)}{1000}.$$  

This inequality is equivalent to

$$5^k \left( 1 - \alpha - \frac{\alpha(1 - \alpha)}{1000} \right) \geq |y(w) - y(v)|\alpha.$$  

which follows directly since $5^k \geq |y(w) - y(v)|$ and $1 - \alpha - \frac{\alpha(1 - \alpha)}{1000} \geq \alpha$ for $0 < \alpha < \frac{1}{5}$.

\[\square\]

**Corollary 5.** In the setting of Lemma 4, let $v$ and $w$ be any vertices in the boundary. Assume that $\gamma(v, w)$ visits a corner of $C$. Then $\gamma(v, w)$ is completely contained in the boundary of $C$.

**Proof.** Let $v'$ be a vertex in $\gamma(v, w)$ which is in the corner of $C$. Then $v'$ is in two sides and both the other two sides are adjacent to one of these two sides. Then both the pairs $v$ and $v'$ and $v'$ and $w$ lie in (the same or) adjacent sides of $C$. Thus the corollary follows from Lemma 4 applied to $\gamma(v, v')$ and $\gamma(v', w)$.

\[\square\]

We extend the result above to a large rectangle in the next lemma.

**Lemma 6.** Let $M = M(\alpha, k)$ be an integer, to be defined later, and let $R = R(x, y, k)$ be the rectangle with vertices $(x, y); (x, y + 5^k); (x + 5^kM, y); (x + 5^kM, y + 5^k)$ such that all the edges in its sides are in the $k$-grid. Let $v$ and $w$ be vertices in the boundary of $R$ such that at least one is on the one of the shorter sides of $R$. If $\gamma(v, w)$ is contained in $R$, then it is contained in the $k$-grid.

The lemma above confirms that, once a geodesics enters a fast grid, it will not visit slower edges anymore. Notice that the only edges parallel to the $x$-axis in $\gamma(v, w)$ are in the boundary of $R$. We will see that we may use edges in the interior of $R$ but parallel to the $y$-axis.

**Proof.** To fix ideas, assume $v$ lies on the left side of $R$. Notice that $R$ can be divided into $M$ squares of side $5^k$, each satisfying the condition of Lemma 4, namely, each has boundary edges in the $k$-grid. We denote these squares by $C_1, C_2, \ldots, C_M$ from left to right. Also, for $1 \leq j \leq M$ denote $v_j$ and $w_j$ the first and last vertex that $\gamma(v, w)$ visits in $C_j$, respectively.

If $w$ lies on the boundary of $C_1$ then this is just Lemma 4. Assume $w$ lies in one of the larger (horizontal) sides of $R$. Then this case follows by induction on $M$, with Lemma 4 being the initial step.

It remains to prove the case when $w$ is in the right side of $R$. If $\gamma(v, w)$ visits any of the common corners of $C_{M-1}$ and $C_M$, say $c$, then, by Lemma 4 $\gamma(v, w)$ restricted to $C_M$ is entirely in the boundary and, by the induction above applied to $v \leftrightarrow c$ the entire geodesics is on the boundary. If $\gamma(v, w)$ does not visit the corners of $C_M$, then $v_M \leftrightarrow w_M$ is entirely in the interior of $C_M$, by Corollary 5. The same analysis applied to $v \leftrightarrow v_M$ gives us now that the restriction of $\gamma(v, w)$ to $C_{M-1}$ is entirely in its interior. We conclude that, $\gamma(v, w)$ must be entirely in the interior of $R$. Then its length will be at least:

$$5^k M(1 + \alpha^{k-1})$$
but the shortest path on the boundary of $R$ connecting $v$ and $w$ has length at most:

$$5^k(M + 1)(1 + \alpha^k) + 5^k(M + 1)\frac{\alpha^k(1 - \alpha)}{2}.$$  

We can choose $M = M(\alpha, k)$ such that $5^k M(1 + \alpha^{k-1}) \geq 5^k(M + 1)(1 + \alpha^k) + 5^k(M + 1)(\alpha^k \frac{1 - \alpha}{2})$, which contradicts our assumption that $\gamma(v, w)$ is in the interior of $R$. We have proved that $\gamma(v, w)$ lies on the union of the boundaries of $C_j$, which proves the lemma.

Remark 2. The lemma above is still true if the largest side of the rectangle is parallel to the $y$-axis.

Proposition 7. Let $v, w$ vertices that are end points of edges in the $k$-grid, satisfying $|v - w| \geq 5^2 k M$. Then the geodesic $\gamma(v, w)$ is contained in the $k$-grid.

Proof. Suppose that the path $\gamma(v, w) = \{e_1, e_2, \ldots, e_t\}$ contains edges outside the grid. Let

$$m = \min\{s \geq 1 : e_s \text{ is not on the } k\text{-grid}\}.$$  

Let $R = R(x, y, k)$, as defined in Lemma 6, be the unique rectangle containing $e_m$ in its interior whose boundary shares a vertex with $e_m$, and its larger sizes are parallel to $e_m$. We consider two cases.

1. $w$ is in the complement of $R$. Then the geodesic has to exit $R$ at some point $w'$. Then Lemma 6 and the assumption that $e_m$ is in the interior of the rectangle imply that we can modify $\gamma$ inside $R$ to get a shortest path from $v$ to $w'$, which contradicts the fact the $\gamma$ is a geodesic.

2. $w$ is in the interior of $R$. Because $w$ is on the $k$-grid, it have to be in the boundary of one of the $M$ squares that form $R$. Since we assume that $|v - w| \geq 5^2 k M$, we conclude that $v$ is not in $R$. By definition of $m$, all edges $e_s, s < m$ are in the $k$-grid. Thus, for $\gamma(v, w)$ to traverse $e_m$, it must visit a corner of $R$. Denote it $v'$. Consider the square of size $5^k$ with one corner $v'$ and in whose interior lies $e_m$. Then, by Lemma 4 (ii), the segment of $\gamma(v, w)$ contained in such square is on its boundary, which contradicts the assumptions on $e_m$. This concludes the proof.

To prepare the ground for our next lemma, we draw a few conclusions from Proposition 7. First, notice that any geodesic ray $\gamma$ will have infinitely many vertices in the $k$-grid, for all $k$. If $v_k \in V(A_k)$ is the first such vertex, it follows that all edges in $\gamma$ after $v_k$ are in the corresponding grid. Applying the same reasoning we conclude that, for large values of $k$, the intersection of $\gamma$ and the $k$-grid is a connected set (it could be empty) and $v_k+1$ is the endpoint of such connected set and the starting point for the intersection of $\gamma$ and the $k+1$-grid. We turn our attention to a set of special vertices and introduce the following definition.

Definition 2. We denote by $\mathcal{V}_k(\gamma)$ the (filled in) box in the $k$-grid with minimal area that contains $v$. If $v$ lies on the boundary of more than one such boxes, let $\mathcal{V}_k(\gamma)$ be the unique one to the right and/or above $v$. Denote by $v^k_i(\gamma)$, $1 \leq i \leq 4$ the corners of $\mathcal{V}_k(\gamma)$, starting at the upper right and going counterclockwise.

When there is no confusion, we will drop the dependence on $v$ in $\mathcal{V}_k$ and $v^k_i$. The importance of these vertices is explained in the next lemma.

Lemma 8. Let $\gamma$ be a geodesic ray starting at $v$. For each $k$, there is at least one value $1 \leq i \leq 4$ such that $v^k_i \in \gamma$.

Proof. Consider $w \in \gamma$ be a vertex in the $k$-grid such that $d(v, w) \geq 5^2 k M$, for $M$ which was defined in the proof of Lemma 6. The existence of $w$ derives from the fact that $\gamma$ is an infinite path and there are infinite boxes of edges in $A_k$ around $v$. Let $\hat{v} \in \mathcal{V}_k(v)$ denotes the first vertex in the $k$-grid that we encounter while going along $\gamma$, starting at $v$. We have $d(v, \hat{v}) \leq 5^k$ and thus, by Corollary 7, the subpath from $\hat{v}$ to $w$ is contained in the $k$-grid. Thus, the last vertex that $\gamma$ visits in $\mathcal{V}_k$ is one of its corners.

We are ready to prove Theorem 1.
5.1. **Proof of Theorem 1** Assume there exists five different $\gamma_i \in T_0$, $i \in \{1, 2, 3, 4, 5\}$. Then there is a (random) ball $B$ centered at $v$ sufficiently large such that any two of these five geodesic rays only intersect in the interior of $B$. Take $k$ large such that $\mathcal{V}_k$ has its four corners in the complement of $B$. By Lemma 5 each $\gamma_i$ will visit at least one corner of $\mathcal{V}_k$. This contradicts the intersection property since $\mathcal{V}_k$ has four corners. This proves

$$|T_0| \leq 4$$

Since $\nu_\alpha$ is good, it follows from [Ho08, Theorem 1.2] that $|T_0| \geq 4$ and the result follows.

This result allows us to determine the shape. A direct proof of the shape is also very short.

**Corollary 9.** The limiting shape of $\text{FPP}(\nu_\alpha)$ is the $\ell_1$–ball.

**Proof.** From Theorem 1 and [Ho08, Theorem 1.2] we have that the limiting shape of $\text{FPP}(\nu_\alpha)$ is either proportional to the $\ell_1$–ball or the $\ell_\infty$–ball. As every edge has passage time at least one the limiting shape must be contained in the $\ell_1$–ball. But the speed in the coordinate directions is one so the limiting shape must be the $\ell_1$–ball.

6. **Direction of the geodesic rays and proof of Theorem 2**

Our goal in this section is to completely characterize $\text{Dir}(\gamma)$ for each geodesic ray $\gamma$ in $\text{FPP}(\nu_\alpha)$. We start by combining Theorem 1 and recent results of Ahlberg and Hoffman [AH16] to get further information about the geodesic rays. Recall the definition of the boxes $\mathcal{V}_k$ and its corners $\{v_i^k\}_{i=1}^N$ (2). Denote by $C_i = \{v_i^k, k \in \mathbb{N}\}$ the set of corners in the same quadrant of the coordinate plane.

For any geodesic $\gamma$ recall that $\text{Dir}(\gamma) \subset S^1$. In the remainder of the section we will slightly abuse notation by considering $\text{Dir}(\gamma) \subset [0, 2\pi)$.

**Lemma 10.** For each $1 \leq i \leq 4$ there is a unique geodesic $\gamma_i$ such that the angle $(i - \frac{1}{2})\frac{\pi}{2} \in \text{Dir}(\gamma_i)$ and $\text{Dir}(\gamma_i)$ is in the $i$th quadrant.

**Proof.** For each quadrant there is a linear function $\rho_i$ whose level set $D_{\rho_i}(z) = 1$ is the intersection of the boundary of the $\ell_1$ ball with the $i$th quadrant. By Theorems 1.11 and 4.6 of [DH14] for each $i$ there is a geodesic whose Busemann function is asymptotically linear with growth rate $D_{\rho_i}$ and whose $\text{Dir}(\gamma_i)$ is contained in the $i$th quadrant. As there are only four geodesics a.s., these geodesics are unique. We denote by $\gamma_i$ the only geodesic ray directed on the $i$th quadrant.

For any $v, w \in \mathbb{Z}^d$ we have for all $k$ sufficiently large that $\mathcal{V}_k(v) = \mathcal{V}_k(w)$ a.s. Thus by Lemma 8 we have that the geodesics are coalescing. Thus $\text{Dir}(\gamma_i)$ is an almost sure invariant subset of $S_i$. Either

$$\mathbb{P}(\text{Dir}(\gamma_1) \cap [0, \pi/4] \neq \emptyset) \geq 0$$

or

$$\mathbb{P}(\text{Dir}(\gamma_1) \cap [\pi/4, \pi/2] \neq \emptyset) \geq 0.$$  

By symmetry they must both be greater than zero. By shift invariance they both must have probability one. As $\text{Dir}(\gamma_1)$ is connected subset of $[0, \pi/2]$ then $\pi/4 \in \text{Dir}(\gamma_1)$. The same argument works for the other three quadrants.

**Lemma 11.** With probability one $v_i^k \in \gamma_i$ for all $i$ for all but finitely many $k$.

**Proof.** For each $k$ there exists an $i$ such that both coordinates of $v_i^k$ are at least $5^k/2$ in absolute value. For such a $k$ we have $\text{Dir}(v_i^k) \in (i - 1)\pi/2 + (0, \pi/2 - 1)$. Let $K$ be large enough such that for each $k > K$ we have that $\gamma_i^k$ for each $i$ is in a distinct geodesic. Also for any $i$ and any vertex $v \in \gamma_i$ such that $|v| \geq \min_j |v_j^k|$ we have $\text{dir}(v) \in (\pi/2)(i - 1) + (-0.01, \pi/2 + 0.01)$. Then for this particular $i$ we have that $v_i^k \in \gamma_i$. From this we can conclude that for all other $j \neq i$ we have that $v_j^k \in \gamma_j$ as well.
Lemma 12. Let $\omega_1, \omega_2 \in \Omega$ be sampled according to $\mu$. The position of the origin in the interior of $\mathcal{V}_k$ is completely determined by the first $k$ entries of $\omega_i$, $i = 1, 2$.

Proof. Let $\{e_j\}$ be the canonical base of $\Omega$. The entries of $e_j$ satisfies: $(e_j)_k = \delta_{k=j}$. We will prove the lemma by induction on $k$.

For $k = 1$ and box $\mathcal{V}_1$, there are 25 vertices in $\mathcal{V}_1$ with $\mathcal{V}_1(v) = \mathcal{V}_1$. Assign to each vertex a pair $(a, b)$ given by the distance from the vertex to the bottom side and left side of $\mathcal{V}_1$, respectively. Observe that this is a surjective map from the set of vertices in $\mathcal{V}_1$ and $\{0, 1, 2, 3, 4\}^2$. We can check now that the origin is at vertex $v$ if and only if: $(\omega_1)_1 = a$ and $(\omega_2)_1 = b$. This proves the initial case. To prove the general case, consider $\mathcal{V}_k$ divided into 25 squares of side $5^{k-1}$. We will prove next that the pair $((\omega_1)_k, (\omega_2)_k)$ is enough to determine in which of these squares the origin is. To see this, we argue similarly to the case $k = 1$. Notice that each of the 25 squares can be encode by a pair $(a, b)$ given by the distance to the bottom and left side of $\mathcal{V}_k$, respectively. We can check that the origin lies in the square labeled $(a, b)$ if and only if $(\omega_1)_k = a$ and $(\omega_2)_k = b$. Using the induction hypothesis the proof will follow. □

Lemma 13. Denote by $\theta^k_1$ the argument of $v^k_1$. Fix $\theta \in (0, \pi/2)$. For any $\epsilon > 0$ there are infinitely many values of $k$ such that

$$|\theta - \theta^k_1| < \epsilon.$$ 

Proof. We will do the case $\theta = 0$. We want to show that infinitely many $v^k_1$ are inside the cone bounded by the lines $\theta = 0$ and $\theta = \epsilon$. Let $t$ be a natural number such that $\frac{5^{-t}}{1-\epsilon} < \tan(\epsilon)$. For large values of $k$, denote by $E_k$ the event:

$$\begin{align*}
(\omega_1)_{k-t+1} = (\omega_1)_k = 4 \\
(\omega_2)_{k-t+1} = (\omega_2)_k = 0.
\end{align*}$$

In words, this corresponds to $t$ coordinates been simultaneously equal to 0 and 4 in $\omega_1$ and $\omega_2$, respectively. If follows by Borel-Cantelli that $\{E_k\}$ happens infinitely often. By Lemma 12 this event corresponds to the origin being in the top left $5^{k-t}$ square in $\mathcal{V}_k$. Then

$$0 < \theta^k_1 \leq \arctan\left(\frac{\frac{5^{k-t}}{5^k - 5^{k-t}}}{\frac{5^{k-t}}{5^k - 5^{k-t}}}\right) < \epsilon.$$ 

which completes the proof. □

6.1. Proof of Theorem 2. Let $\rho$ be a functional tangent to the $\ell_1$-ball. Associate to $\rho$ a set $C_i$ of corners, in the natural way. By Lemma 10 there is a unique geodesic ray, $\gamma_i$, with the property that $\text{Dir}(\gamma_i) \subset (i-1)(\pi/2) + (0, \pi/2)$. By Lemma 13 we can find points $u \in \text{Dir}(\gamma_i)$ as close as we want to the endpoints of $(i-1)(\pi/2) + (0, \pi/2)$. Since Dir($\gamma_i$) is connected we conclude that $(i-1)(\pi/2) + (0, \pi/2) \subset \text{Dir}(\gamma_i)$. It follows now that $(i-1)(\pi/2) + (0, \pi/2) = \text{Dir}(\gamma_i)$.

7. Exponents in non-coordinate directions

The next two sections are devoted to the proof of Theorem 3. We start by showing that $T(x, y)$ is well concentrated. denote the origin by $0$.

Lemma 14. Let $0 < \lambda \leq 1$ be a fixed constant and $x^\lambda_n = (n, \lambda n)$. There exists a constant $C = C(\lambda)$ such that

$$|x^\lambda_n|_1 \leq T(0, x^\lambda_n) \leq |x^\lambda_n|_1 + C.$$ 

Proof. The lower bound follows from the fact that $t_n \geq 1$. For the upper bound, we construct a path from $0$ to $x^\lambda_n$ satisfying the desired inequality. Consider the boxes $\{\mathcal{V}_k(0)\}$ and $\{\mathcal{V}_k(x^\lambda_n)\}$ for $1 \leq k \leq N-1$ where $N$ is the minimum $t$ such that the projections of $\mathcal{V}_k(0)$ and $\mathcal{V}_k(x^\lambda_n)$ onto either the $x$ or $y$ axes have nonempty intersection. Note that this definition implies that $n \leq (2/\lambda)5^N$. 

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Next, we choose a corner in \( \{ \mathcal{V}_k(0) \} \) for each \( k \) that is closest to \( x_n^\lambda \), and similarly we choose a corner in \( \{ \mathcal{V}_k(x_n^\lambda) \} \) closest to \( 0 \). We have a sequence of vertices:

\[
0, v_1, v_2, \ldots, v_{N-1}, w_{N-1}, \ldots, w_2, w_1, x_n^\lambda
\]

where \( v_i, w_i \) are the corners with chose above in \( \mathcal{V}_i(0) \) and \( \mathcal{V}_i(x_n^\lambda) \), respectively. Consider the north-east directed paths on the appropriate \( j \)-grid between consecutive vertices on this sequence. Our path from \( 0 \) to \( x_n^\lambda \) is the concatenation of these geodesics. Note that for each \( i \), all edges in the geodesic from \( v_i \) to \( v_{i+1} \) have passage times bounded by \( 1 + 2\alpha^i \), since the geodesic is contained in \( \mathcal{V}_{i+1}(0) \). Also, we cross at most \( 2 \cdot 5^{j+1} \) many edges between \( v_i \) and \( v_{i+1} \). An analogous analysis extends to the vertices \( w_j \). Between \( v_{N-1} \) and \( w_{N-1} \) we have at most 2\( \alpha \)N edges with weights at most \( 1 + 2\alpha^N \). The total length of our path is at most \(|x_n^\lambda|_1\). We put this together to conclude that

\[
T(0, x_n^\lambda) \leq |x_n^\lambda|_1 + 20 \sum_{j=1}^{\infty} 5^j \alpha^j + n 2\alpha^{N} \leq |x_n^\lambda|_1 + C' + (2/\lambda)5^{N}2\alpha^{N} = |x_n^\lambda|_1 + C
\]

as \( \alpha < 1/5 \).

\[\square\]

7.1. The wandering exponent.

**Proposition 15.** Let \( 0 < \lambda \leq 1 \). Let \( \text{Cyl}(x_n^\lambda, cn) \) be the set of all points within distance \( cn \) of the line segment connecting \( 0 \) and \( x_n^\lambda = (n, \lambda n) \). There exists \( c = c(\lambda) > 0 \) such that for all \( n \) sufficiently large \( \gamma(0, (n, \lambda n)) \) is not contained in \( \text{Cyl}(x_n^\lambda, cn) \).

**Proof.** Let \( P \) be a path from \( 0 \) to \( (n, \lambda n) \) with all of its vertices in \( \text{Cyl}(x_n^\lambda, cn) \). Let \( j = \lfloor \log_5(\lambda n) \rfloor - 2 \). Let \( j' \) be the smallest integer such that no horizontal (or vertical) line segment of length \( 5j' - 3 \) lies entirely in \( \text{Cyl}(x_n^\lambda, cn) \). Note that if \( c \) is small and \( n \) is sufficiently large then \( j > j' + 5 \). Let \( z_1 \) be the first (closest to \( 0 \)) vertex of \( P \) in the \( j \)-grid and let \( z_2 \) be the last (closest to \( (n, \lambda n) \)) vertex of \( P \) in the \( j \)-grid. Note that by the choice of \( j \) we have that both the \( x \) and \( y \) coordinates of \( z_2 \) are at least \( 5j' \) greater than the respective \( x \) and \( y \) coordinates of \( z_1 \). Thus there exists a northeast directed path \( P'' \) from \( z_1 \) to \( z_2 \) that is contained entirely in the \( j \)-grid. We will show that there exists a path \( P' \) which is not contained in \( \text{Cyl}(x_n^\lambda, cn) \) which is a faster path from \( 0 \) to \( (n, \lambda n) \). \( P' \) will agree with \( P \) from \( 0 \) to \( z_1 \) and from \( z_2 \) to \( (n, \lambda n) \). Between \( z_1 \) and \( z_2 \) the path \( P' \) is \( P'' \). The choice of \( j \) insures that this is possible and \(|z_1 - z_2| > \frac{1-\lambda}{2\lambda}n\).

As every edge of \( P' \) between \( z_1 \) and \( z_2 \) is in the \( j \)-grid, the sum of the passage times of all of these edges is at most

\[
|z_1 - z_2|(1 + 2 \cdot \alpha^j).
\]

We now show that this is faster than the path \( P \) so \( P \) is not a geodesic.

Define a sequence \( \{z_i^j\}_{i=0}^k \) with \( z_0 = z_1 \) and \( z_k = z_2 \) with each \( z_i^j \) (with \( 0 < i < k \)) the first time that \( P \) hits a new vertical line on the \( j' \)-grid. Note that \( k \) is at least \( 5^5 > 1000 \). If \( 0 < i < k - 1 \) and the path \( P \) between \( z_i^j \) and \( z_i^{j+1} \) hits another vertical line (besides the start and end lines) in the \( j' \)-grid then it has at least \( 3 \cdot 5^{j-1} \) horizontal edges. Similarly we can see that between \( z_1^j \) and \( z_1^{j+1} \) the path \( P \) hits at two horizontal lines in the \((j-1)\)-grid. By the choice of \( j' \) and \( \lambda \leq 1 \) we have \(|z_1^j - z_1^{j+1}| \leq 3 \cdot 5^{j'} \) and the difference in the \( x \) coordinate is \( 5^{j'} \). As all edges have passage times between \( 1 \) and \( 2 \) then \( P \) is not a geodesic.

Otherwise as \( \lambda \leq 1 \) this segment of \( P \) contains edges on the \( j' \)-grid on at most one vertical and two horizontal lines. By the choice of \( j' \) each of these lines contains at most \( 5^{j'-3} \) edges in \( \text{Cyl}(x_n^\lambda, cn) \). Thus this segment of \( P \) contains at most \( 3 \cdot 5^{j'-3} \) edges in the \( j' \)-grid and at least \( 5^{j'} \) edges in total. Thus at least \( 85\% \) of the edges in this segment of \( P \) are not in the \( j' \)-grid and have passage times at least \( 1 + \alpha^{j'} \). As this applies to all but the first and last segments, at least \( 80\% \) of the edges on \( P \) from \( z_1 \) to \( z_2 \) are not in the \( j' \)-grid. As above the first and last segments have at most \( 3 \cdot 5^{j-1} \) edges and thus at most three times as many edges as the shortest intermediate.
segment. Thus the first and last segments make up less than one percent of the length of \( P \) from \( z_1 \) to \( z_2 \).

Thus the total passage time for \( P \) between \( z_1 \) and \( z_2 \) is at least

\[
|z_1 - v_2|(1 + 0.8 \cdot \alpha^j) > |v_1 - v_2|(1 + 2 \alpha^j).
\]

Thus the passage time along \( P \) is more than the passage time along \( P' \) and \( P \) is not a geodesic. This proves that the geodesic does not lie in \( \text{Cyl}(x_n^\lambda, cn) \). 

**Proposition 16.** Let \( 0 < \lambda \leq 1 \). Remember that \( \text{Cyl}(x_n^\lambda, 10n) \) is the set of all points within distance \( 10n \) of the line segment connecting \( 0 \) and \( (n, \lambda n) \). For all \( n \) sufficiently large \( \gamma(0,(n,\lambda n)) \) is contained in \( \text{Cyl}(x_n^\lambda, cn) \).

**Proof.** If a path \( P \) from 0 to \( (n, \lambda n) \) is not in \( \text{Cyl}(x_n^\lambda, 10n) \) then the length of \( P \) is at least \( 10n \). But as \( \lambda \leq 1 \) there is a path \( P' \) which is in \( \text{Cyl}(x_n^\lambda, cn) \) from \( 0 \) to \( (n, \lambda n) \) of length at most \( 2n \). As every edge has weight at most \( 2 \) the length of \( P' \) is at most \( 4n \) and \( P \) is not the geodesic from \( 0 \) to \( (n, \lambda n) \). □

8. **Exponents in the coordinate direction**

In this section we consider \( \gamma(0,(n,0)) \). Define

\[
(5) \quad \beta = \frac{\log 5}{\log 5 - \log \alpha} < 1.
\]

Note that, for any \( j \) we can write

\[
(6) \quad \alpha^{\beta j} = (5^j)^{\beta-1}.
\]

**Lemma 17.** There exists universal constants \( C \) and \( N \) such that for all \( c > C \) and \( n > N \) we have

1. \( T(0,(n,0)) \geq n \)
2. \( \mathbb{P}(T(0,(n,0)) \leq n + 0.01 n^{\beta}) > 10^{-9} \)
3. \( \mathbb{P}(T(0,(n,0)) \geq n + 0.02 n^{\beta}) > 10^{-9} \) and
4. \( \mathbb{P}(T(0,(n,0)) \geq n + 10 n^{\beta}) = 0. \)

**Proof.** The first inequality is true because all passage times are at least 1.

For the second inequality we define \( \Gamma_l \) to be the following path from \( (0,0) \) to \( (n,0) \). The start of \( \Gamma_l \) goes northeast from \( (0,0) \) to the line \( y = y_l \), where \( y_l \) is the lowest non-negative number such that the line \( y = y_l \) is in the \( l \)-grid. Suppose we have defined the path to the point \( (x', y') \) where both the lines \( x = x' \) and \( y = y' \) are in the \( l' \)-grid. Then we extend the path so that it goes east to the \( (l' + 1) \)-grid and then north to the \( (l' + 1) \)-grid. We continue until we have hit the line \( y = y_l \). The final portion of \( \Gamma_l \) is defined in a symmetric manner. It goes northwest from \( (n,0) \) to the line \( y = y_0 \). Then \( \Gamma_l \) connects these two pieces by moving horizontally along the line \( y = y_0 \).

Given \( n \), choose \( j \) such that

\[
(7) \quad 5^j \leq n < 5^{j+1}.
\]

Let \( Q \) be the event that there exists \( y^* \in [0,0.001 \cdot 5^{\beta j}] \) with the line \( y = y^* \) in the \( (\lfloor \beta j \rfloor + 5) \)-grid. If \( Q \) occurs then \( \Gamma_{\lfloor \beta j \rfloor + 5} \) contains:

1. at most \( n + 0.002 \cdot 5^{\beta j} \) edges,
2. at most \( 16 \cdot 4^k \) edges in the \( k \)-grid but not the \( (k + 1) \)-grid for all \( k < \lfloor \beta j \rfloor + 5 \), and,
3. at most \( n \) edges in the \( (\lfloor \beta j \rfloor + 5) \)-grid.

If \( Q \) occurs, from 1–3 above and the definition of the \( X_{k(e),e} \), we have
\[
\sum_{e \in \Gamma_{|\beta j|+5}} \alpha^{k(e)} + X_{k(e),e} \leq \sum_{e \in \Gamma_{|\beta j|+5}} 1.001\alpha^{k(e)} \\
\leq 1.001 \sum_{k} \left(16 \cdot 4^k \alpha^k\right) + 1.001n\alpha^{|\beta j|+5} \\
\leq C + .001 \cdot 5^{\beta j-1} \\
\leq C + .001n^\beta.
\]

Then, if \(Q\) occurs

\[
T((0,0),(n,0)) \leq T(\Gamma_{|\beta j|+5}) \\
\leq |\Gamma_{|\beta j|+5}| + \sum_{e \in \Gamma_{|\beta j|+5}} \alpha^{k(e)} + X_{k(e),e} \\
\leq n + .002 \cdot 5^{\beta j} + C + .001n^\beta \\
\leq C + n + .004n^\beta \\
\leq n + .01n^\beta.
\]

Then

\[
\mathbb{P}(T(0, (n, 0)) \leq n + .01n^\beta) \geq \mathbb{P}(Q) \geq .001 \cdot 5^{-5} \geq 2 \cdot 10^{-9}
\]

and the result follows.

The fourth inequality follows in much the same way as the second except we do not assume that the event \(Q\) occurs. In this case we have that \(\Gamma_{|\beta j|}\) contains

(1) at most \(n + 2 \cdot 5^{\beta j}\) edges
(2) at most \(16 \cdot 4^k\) edges in the \(k\)-grid but not the \((k+1)\)-grid for all \(k < |\beta j|\) and
(3) at most \(n\) edges in the \((|\beta j|)\)-grid.

Then a similar calculation as above proves the claim.

For the third inequality we note that if there does not exist \(y_0\) such that \(|y_0| \leq .1 \cdot 5^{\beta j}\) such that the line \(y = y_0\) is in the \(|\beta j|\)-grid and if \(\Gamma'\) be any path from \((0,0)\) to \((n,0)\) in the cylinder \(\{(x,y) : |y| \leq .1 \cdot 5^{\beta j}\}\). Then

\[
T(\Gamma') \geq n(1 + \alpha^{\beta j-1}) \\
\geq n + \frac{1}{\alpha}n\alpha^{\beta j} \\
\geq n + \frac{1}{\alpha}n(5^j)^{\beta-1} \\
\geq n + \frac{1}{\alpha}n(5^j)^{\beta}(5^j)^{-1} \\
\geq n + \frac{1}{\alpha}(5^j)^{\beta} \\
\geq n + \frac{1}{\alpha}(5^{j+1})^{\beta}5^{-\beta} \\
\geq n + \frac{1}{\alpha5^\beta}n^\beta \\
\geq n + n^\beta.
\]
Now let $\Gamma''$ be any path from $(0,0)$ to $(n,0)$ not contained in the cylinder \{(x,y) : |y| \leq 1 \cdot 5^{\beta j}\}.
Then by (5) and (7)
\[ T(\Gamma'') \geq n + 2 \cdot 5^{\beta j} \]
\[ \geq n + 2 \cdot (5^{j+1})^{\beta} 5^{-\beta} \]
\[ \geq n + 0.04n^\beta. \]
As any path from $(0,0)$ to $(n,0)$ falls into one of these two categories we have that
\[ T((0,0),(n,0)) \geq \min(n + n^\beta, n + 0.04n^\beta) = n + 0.04n^\beta. \]
This happens with probability at least
\[ 1 - \frac{3 \cdot 5^{\beta j}}{5^{\lceil \beta j \rceil}} \geq 10^{-9}. \]
\[ \square \]

We use Lemma 17 to show that the variance exponent is $\beta$ along the axes.

**Lemma 18.** There exists $K > 0$ such that for all $n$ sufficiently large
\[ \frac{1}{K} n^{2\beta} < Var(T(0, (n,0))) < Kn^{2\beta}. \]

**Proof.** The lower bound follows directly from parts 2 and 3 from Lemma 17. The upper bound follows from parts 1 and 4. \[ \square \]

For any $K$ define $\text{Cyl}((n,0), K)$ be the subgraph with vertices \{(x,y) : |y| \leq K\} and all edges between two vertices in the set. Now we show that the fluctuation exponent is also $\beta$.

**Lemma 19.** With the above notation, for any $\epsilon > 0$ the following holds
\[ \mathbb{P}(\gamma(0, (n,0)) \text{ is contained in } \text{Cyl}((n,0), n^{\beta-\epsilon}) = o(1). \]

Also
\[ \mathbb{P}(\gamma(0, (n,0)) \text{ is not contained in } \text{Cyl}((n,0), 10n^\beta) = 0. \]

**Proof.** Define
\[ j = j(n) = \max\{k(e) : e \text{ is an horizontal edge in } \gamma(0, (n,0))\}. \]
We first notice that all horizontal edges in $\gamma(0, (n,0))$ with $k(e) = j$ are contained in the horizontal line that is furthest away from the $x-$axis. Consider a path $P$ that goes up to the $j+1$ grid and connects 0 and $(n,0)$. We have
\[ T(0, (n,0)) \leq T(P). \]
by definition.
For any $\epsilon > 0$ and for all $n$ sufficiently large we will show that
\[ \mathbb{P}(\exists \text{ a path } P \text{ from } 0 \text{ to } (n,0) \text{ contained in } \text{Cyl}((n,0), n^{\beta-\epsilon}) \text{ with } T(P) \leq n + 10n^\beta) = o(1). \]
There are at least $n$ horizontal edges in any path from $0$ to $(n,0).$ If all the horizontal edges in $\text{Cyl}((n,0), n^{\beta-\epsilon})$ have passage time at least $1 + 10n^{\beta-1}$ then the passage time across any path from $0$ to $(n,0)$ entirely contained in $\text{Cyl}((n,0), n^{\beta-\epsilon})$ has passage time at least $n + 10n^\beta.$
By part 4 of Lemma 17 the event that
\[ \gamma(0, (n,0)) \text{ is contained in } \text{Cyl}((n,0), n^{\beta-\epsilon}) \]
is contained in the event that
\[ \text{there exists a path } P \text{ from } 0 \text{ to } (n,0) \text{ contained in } \text{Cyl}((n,0), n^{\beta-\epsilon}) \text{ with } T(P) \leq n + 10n^\beta. \]
This last event is in turn contained in the event that there exists a horizontal edge in $\text{Cyl}(n, 0, n^{\beta-\epsilon})$ with passage time at most $1 + 10n^{\beta-1}$.

This requires that there is a line of the form $y = l$ which is in the $(\lfloor \beta j \rfloor - 3)$-grid with $l \in [-n^{\beta-\epsilon}, n^{\beta-\epsilon}]$. By the choice of $\beta$ and $j$ the probability of this is at most

$$\frac{2n^{\beta-\epsilon} + 1}{5^{\beta j - 3}} \leq Cn^{-\epsilon}.$$ 

The upper bound follows from part 3 of Lemma 17 and the fact that all passage times are at least 1. □

8.1. **Proof of Theorem 3**. For the non-coordinate directions, $\chi = 0$ follows directly from Lemma 14 and $\xi = 1$ follows combining Proposition 15 and Proposition 16. For the coordinate directions, $\chi = \xi = \beta$ is a consequence of Lemma 18 and Lemma 19.

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