Quantum Mechanics, Chaos and the Bohm Theory

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I. INTRODUCTION

Formally Bohm’s Causal Interpretation [1], or de Broglie’s Pilot wave [2] interpretation, of quantum mechanics arises when the substitution \( \psi = Re^{iS/\hbar} \) is made in the Schrödinger equation:

\[
\left( -\frac{\hbar^2}{2m} \nabla^2 + V \right) \psi = i\hbar \frac{\partial \psi}{\partial t} \tag{1}
\]

and the real and imaginary parts are separated yielding the equations,

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \nu) = 0 \tag{2}
\]

and

\[
-\frac{\partial S}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 R + \frac{(\nabla S)^2}{2m} + V. \tag{3}
\]

Where \( \rho = |\psi|^2 \) and

\[
m\nu = \nabla S. \tag{4}
\]

The particle is assumed to have a definite, but unknown, position with a momentum given by (3). Equation (3) can be interpreted as a Hamilton-Jacobi-type of equation with an extra ‘quantum potential’ term \( Q \), \( Q = -\frac{\hbar^2}{2m} \nabla^2 R \). Newton’s second law is modified by the quantum force term and becomes

\[
F = -\nabla (Q + V) = -\nabla V_{\text{eff}}. \tag{5}
\]

II. THE KICKED ROTOR

The kicked rotor has been extensively studied \cite{2,13} in a variety of forms, but here we take as the Hamiltonian

\[
H = p_\theta^2/2I - I\omega_0^2 \cos \theta \sum_{n=-\infty}^{\infty} \delta(t/T - n), \tag{6}
\]

where \( I = ml^2 \) is the inertia (\( m, l \) being the mass and length of the rotor), \( \theta \) is the orientation, \( p_\theta \) is the angular
momentum, \( \omega_0 = \sqrt{g/l} \) is the natural frequency for small displacements, and \( T \) is the period of the delta-function kicking.

Classically this hamiltonian describes a rotor (e.g., a pendulum) subjected to a periodically pulsed gravitational field, and it leads to equations of motion which can be reduced to the Standard Map \([3]\), (which exhibits all the features of a chaotic system). The dynamics of the rotor depend on just one parameter \( K = (\omega_0 T)^2 \). For \( K << 1 \) the behaviour is regular and for \( K >> 1 \) the motion is chaotic.

Quantum mechanically, an arbitrary rotor wavefunction can be expanded in terms of the free rotor simultaneous eigenfunctions of energy and angular momentum according to

\[
\psi(\theta, t) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} a_n(t)e^{in\theta}.
\]

The time evolution of the wavefunction is then reduced to an iteration of the expansion coefficients according to

\[
a_n((N + 1)T^+) = \sum_{r=-\infty}^{\infty} a_r(NT^+)e^{ir\tau/2},
\]

where \( \tau = \hbar T/I, k = (\omega_0^2 T/\hbar) \) and \( J_s(k) \) are ordinary bessel functions of the first kind. Equation \( (8) \) gives the value of the expansion coefficients just after the \((N + 1)^{th}\) kick in terms of the coefficients just after the \(N^{th}\) kick. In practice the summation can be truncated, the bessel functions \( J_s(k) \) become negligible outside the range \( s \approx 2k \) and we can check the accuracy of the truncation by monitoring the conservation of the probability.

Classically, in the chaotic regime (for, say \( K = 5 \)), the average energy grows linearly with time showing a random diffusive-like behaviour of the system, a possible indicator of chaos. Quantum mechanically, however, this growth of \( \langle E \rangle = \frac{\hbar^2}{2I} \sum_n n^2 |a_n|^2 \) is suppressed after a certain time \([3]\). This suppression occurs in the semiclassical regime where \( k = 10, \tau = 1/2 \) and \( K = 5 \).

### III. THE FREE ROTOR AND THE BOHM THEORY

Between kicks the rotor is free and so we begin with an investigation of the Bohm dynamics of the free rotor in various states and then continue with the kicked case. For a simultaneous eigenstate of energy and angular momentum of the free rotor,

\[
\psi(\theta, t) = \frac{1}{\sqrt{2\pi}} e^{-\hbar n_0^2 t/2I} e^{i n_0 \theta},
\]

and the Bohm motion is a subset of the classical motion, the angular momentum of the rotor using \([3]\), is \( p_\theta(t) = \hbar n_0 \).

Next consider a simple superposition of the ground plus equal amounts of the angular momentum +1 and −1 first excited states given by,

\[
\psi(\theta, t) = 1 + 2a \cos \theta e^{-it/2},
\]

where \( a_1 = a_{-1} = a \) is real and set \( \hbar = I = 1 \). The Bohm momentum for this state is readily shown to be

\[
p_\theta(t) = \frac{d\theta}{dt} = \frac{2a \sin \theta \sin t/2}{1 + 4a^2 \cos^2 \theta + 4a \cos \theta \cos t/2}
\]

This can be integrated by making the substitution \( C = \cos t/2 \), giving the following implicit trajectory equation

\[
4a \cos t/2 (\sin \theta_t - \sin \theta_0) = -(1 + 2a^2)(\theta_t - \theta_0) - a^2 (\sin 2\theta_t - \sin 2\theta_0)
\]

Equation \( (12) \) can be seen to be periodic in time, with a period \( T = 4\pi \). Hence here, obviously there can be no divergence in the trajectories.

An alternative approach to the calculation of the system’s motion is to attempt the direct numerical solution of

\[
F = I \frac{\partial^2 \theta}{\partial t^2} = -\frac{\partial Q}{\partial \theta}.
\]

with \( Q \) derived from \( (10) \), for this simple superposition state. If the initial condition \( p_\theta = \nabla S \) is imposed one recovers the trajectories obtained above \([12]\). Without this initial condition the trajectories appear to diverge rapidly. Figure \([1]\) shows the divergent behaviour of two initially close trajectories in this case. Classically, for a given potential and given initial conditions a trajectory may wander over the whole of phase space whereas quantum mechanically it is restricted to \( p = \nabla S \) \([11]\). Trajectories with initial conditions satisfying the latter condition, because of the single-valuedness of the phase, never cross in configuration space-time, but if this constraint is relaxed this feature is lost allowing for the possibility of divergence. Bohm trajectories for superposition states can be exceedingly complicated, but for an arbitrary superposition we would expect the Bohm trajectories to be quasiperiodic. Calculation shows no discernible divergence of trajectories for such a superposition state with arbitrarily chosen \( a_n \)’s.

### IV. THE KICKED ROTOR AND THE BOHM THEORY

We also calculated the trajectories for the kicked rotor, with particular interest in the semiclassical regime,
where \( k = 10 \) and \( \tau = 1/2 \). The initial state of the rotor was taken to be the ground state (giving an initial Bohm momentum of zero) and the trajectories for two initially close positions were calculated. (Figures 2 and 3 are Poincaré sections with a period equal to that of the kick.) Figure 2 shows that there is practically no divergence between two trajectories started at \( \theta = 30^\circ \) and \( \theta = 30.5^\circ \) after eighty or so kicks. Figure 3 shows two trajectories with the same initial position but with slightly different wavefunctions (and therefore momenta): the two trajectories are highly correlated. It turns out that even trajectories which are far apart in orientation of the two trajectories are highly correlated. It turns out that even trajectories which are far apart in orientation do not diverge. Figure 4 shows the evolution of two trajectories initially at \( \theta = 1^\circ \) and \( \theta = 30^\circ \), with the initial state (for both) being a Gaussian wavepacket.

Moreover the results above are typical for Bohm trajectories of the kicked rotor system. Divergence in the trajectories cannot be obtained by varying any of the parameters involved, including the choice of initially highly excited states. The trajectories themselves can be exceedingly complicated, but there is no divergence.

Similar results follow for the Bohm trajectories with the quasiperiodically kicked rotor [17, 18].

V. MEASUREMENT

We conclude by discussing the process of measurement in the Bohm theory and its possible significance in a discussion of quantum chaos. Chaotic systems are those with only a few degrees of freedom in which practically random behaviour arises, but not as a result of an external random influence. In the usual interpretation of quantum mechanics measurement is considered to be an inherently random external process and because of this there is no detailed discussion of measured systems in the quantum chaos literature. If one adopts Bohm’s point of view the situation is rather different. The Bohm theory has a perfectly deterministic description of measurement as a dynamical process and unique measurement results follow from unique initial conditions.

The details of the Bohmian dynamics for an angular momentum measurement using a Stern-Gerlach device have already been given, for hydrogen-like atoms, in [19]. Detailed work is now in progress to investigate repeated angular momentum measurements on the kicked rotor and on kicked spin systems, but some general observations are relevant here. Recently Dürr et al [20] have discussed similar ideas, in the context of a somewhat different system.

We take the initial wave function of the rotor to be

\[
\Psi(z, \theta, 0) = \phi_0(z)\psi_0(\theta)
\]

(14)

where \( \phi(z) \) describes the position of the rotor and has the form of a packet centred at \( z = 0 \) and \( \psi_0(\theta) \) is the ground state of the oscillator. The Stern-Gerlach magnetic field gradients are aligned with the \( z \)-direction and so only motion of the rotor in this direction is significant.

Consider now a sequence in which the system is repeatedly kicked and then measured. Each time the system is kicked the wavefunction becomes a superposition, \( \Psi(z, \theta, t) \), and each time a measurement is carried out, providing we allow enough time for the wave packets associated with different angular momenta to separate along the \( z \)-axis, the rotor’s wave function will become effectively the simple product

\[
\Psi(z, \theta, t) = \phi_n(z, t)\psi_n(\theta, t)
\]

(15)
as the rotor’s actual \( z \)-coordinate evolves to uniquely associate it with just one of the \( \phi_n \). When this has taken place all of the other terms in the superposition can be ignored in the further stages of the calculation. (This is the equivalent of the reduction of the wavepacket, the “empty” packets do not affect the motion of the system.)

At each measurement there will be a series of bifurcation points in the initial packet at the entrance to the Stern-Gerlach apparatus, and the actual value of \( z \) with respect to these points will determine which of the angular momentum eigenstates the system enters, furthermore, the evolution of the wave-function under the kick (the set of \( \phi_n \) produced) depends on the wavefunction just before the kick. Hence the sequence of eigenfunctions of angular momentum produced in a particular sequence of kicks and measurements depends on the precise value of the centre of mass coordinate \( z \) at the outset. The deterministic evolution of the rotor wave function throughout the sequence of kicks and measurements can be calculated and, since the particular evolution depends on the initial \( z \)-value, the possibility exists for the evolution of the rotor wave functions, for two slightly different initial \( z \)-values, to become divergent. If the rotor wavefunctions for different initial conditions can become divergent then the trajectories can cross (they obey \( p = \nabla S \), but \( S \) develops differently in each of the two cases) and we can look for the signatures of chaos in the motion of the centre of mass or in the rotation of the rotor.

In conclusion, it is clear that the Schrödinger evolution of the wave function with its probabilistic interpretation, which removes the possibility to discuss individual systems, will not allow quantum chaos. Even in our repeatedly measured and kicked system the evolution of the complete wavefunction (with all components of the superposition maintained at each stage) will not allow for divergence in the sense that two initially close wavefunctions will remain close. Supplementing the Schrödinger description with the Bohm trajectory description allows an access to the motion of individual systems and we have seen that two isolated systems with slightly different initial conditions, either initial position or initial wavefunction, will not diverge. The possibility to produce

3
divergent effective wave functions for individual systems with slightly different initial conditions does arise if we consider interactions with the system. An alternative to measurement is to consider interaction with an environment [21], but from Bohm’s point of view.

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FIG. 1. Momentum/time plot for the superposition state $\psi(\theta,t) = 1 + 2a \cos \theta e^{-it/2}$, of the free rotor when the initial condition $p_{\theta} = \nabla \theta$ is relaxed. Initially $\theta = 1^\circ$ or $\theta = 1.01^\circ$ and $\dot{\theta} = 1$.

FIG. 2. Orientation/time plot for the kicked rotor with $\omega_0 = (20)^{1/2}$ and $T = 1/2$. The two trajectories are initially close together with initial angles $\theta = 30^\circ$ and $\theta = 30.5^\circ$. The rotor is initially in the ground state $a_0 = 1$.

FIG. 3. Orientation/time plot for the kicked rotor with $\omega_0 = (20)^{1/2}$ and $T = 1/2$. Here we have two trajectories given by the initial states, $a_0 = 1, a_1 = (4637/13313)^{1/2}$ and $a_0 = 1, a_1 = (0.3483)^{1/2}$ and both with the same initial angle, $\theta = 30^\circ$.

FIG. 4. Orientation/time plot for the kicked rotor with $\omega_0 = (20)^{1/2}$ and $T = 1/2$. Two trajectories are taken initially far apart with initial angles $\theta = 1^\circ$ and $\theta = 30^\circ$. The initial state is a gaussian wavepacket with a momentum half-width 0.5 and an initial momentum of two.
