Finite size corrections to the Fermi’s golden rule (2) Scattering probability . Wavefunctions of interacting many-body states at finite time interval: wave packets

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(Dated: March 8, 2022)

Abstract

Field theory is formulated with normalized one-particle states, and scattering probability for the finite-time interval is studied from the probability principle of the quantum mechanics following von Neumann. Using normalized one-particle states, we find the many-body wavefunctions without encountering the divergence difficulty of plane waves. Owing to Using the wave packet representations, the wave functions of the interacting systems and the transition probability, which become finite at the finite time owing to their compact natures and space-time dependence, are obtained. The specific properties at finite time interval remain at $t \to \pm \infty$, and appear in scattering processes at finite $t$ and $t \to \pm \infty$. A scalar theory and Quantum Electrodynamics are studied and the finite-size corrections to the Fermi’s golden rule which are not included in the standard S-matrix are obtained. They have distinctive and intriguing properties, and may become sizable, which lead new phenomena beyond those derived from the golden rule. New perspectives of halos and other wave-like phenomena are presented.
I. EVOLUTION OF MANY-BODY STATES

1. Solving time-dependent Schroedinger equation of continuous energy spectrum

2. Initial wave function and its asymptotic behaviors

3. Green’s function

Wave functions satisfying a many-body Schrödinger equation of a Hamiltonian \( H = H_0 + H_{\text{int}} \), where \( H_0 \) is a free part and \( H_{\text{int}} \) is an interaction part of fields, are obtained with perturbative expansion with respects to \( H_{\text{int}} \) using eigenstates of \( H_0 \) of eigenvalue \( E_i \). These states are plane waves, and their matrix elements are described by the Dirac delta function. The Dirac delta function diverges but its integration converges when that is combined with smooth and well-behaved functions which decrease rapidly. Using them, solutions at macroscopic distances can be computed uniquely normally. Transition amplitude is defined with these wave functions. Particularly those which are defined at the infinite time interval, S-matrix, under the interaction \( e^{-\epsilon |t|} H_{\text{int}} \), where \( \epsilon \) is infinitesimal positive number and the limit \( \epsilon \to 0 \) is taken before the limit of \( t \to \pm \infty \), is used as the standard method. Obviously this interaction switches off adiabatically (ASI) artificially. In this method, the physical quantities are convergent, and the diverging norms of the plane waves cause no difficulty, as are described in most literature and textbooks. These have been applied in wide area and lead successful results. However, these are applicable in idealistic situations at \( t \to \infty \) of ASI, but not in many realistic cases. There are many physical systems where this condition is not met. In fact, the time interval is not infinite in real systems, and the interaction does not switch off rapidly but remains. It is not clear if ASI is applicable for the finite time interval. If the standard one under ASI is applied, inconsistency with experiments or divergence in the calculations were found. One of these difficulties was noticed for the transition in a finite time interval sometime ago by Stuckelberg. At a finite time, the interaction remains. For the plane waves, the converging factor is absent, and the integral becomes not well defined. The probability also becomes divergent at a finite \( t \). Obviously this difficulty is connected with divergence of the norm of the plane waves, and should not appear for the wave packets, as the wave functions are normalized. With these wave functions at the finite \( t \), the probability is finite, and leads new physical quantities, as by product. The physical quantities diverged in ASI, but become converged in the proper treatment may show prominent effects for extended wave-like states. These have intriguing
properties and imply new phenomena.

**Many-body wave function**

The wave functions under the interaction $H_{\text{int}}$ reveal the physical states of separating each others and overlapping ones. The former is expressed by ASI but the latter is not. These states hold wavelike properties caused by remaining interactions. That continue while they overlap and the effects remain in the many-body wave function even after they separate, since the Schrödinger equation is first order in time. These can be studied with neither ASI nor the plane waves. Using a wave packet formalism, which uses a complete set of normalized functions, the many-body wave functions are solved without encountering the difficulty.

The wave functions at $t = T$ of $T \to \infty$ under the interaction $e^{-\epsilon|t|}H_{\text{int}}$ (ASI), where the limit $T \to \infty$ is taken first while $\epsilon$ is kept small, were useful to study fluctuations in short distance region in renormalization program [3–5]. These express the particle-like states in the asymptotic region $|t| \to \infty$.

In the overlapping waves, the interaction energy $\langle \Psi(t)|H_{\text{int}}|\Psi(t) \rangle$ is finite. These are expressed by the states of the final energy $E_f \neq E_i$, where $E_i$ is the initial energy and are included in solutions of the Schrödinger equation $|\Psi(t)\rangle$ of $\langle \Psi(t)|H_{\text{int}}|\Psi(t) \rangle \neq 0$ at $t = T_1$. Due to this expectation value, which we call interaction energy, the states satisfy $E_f \neq E_i$ and give unique contributions. We call this state a quasi-stationary composite state (QCS). These wave functions at $T$ are written explicitly with the wave packet of a spatial size, $\sigma$, and effects of QCS were found in [12–14].

QCS is superposition of states of continuous spectrum of the kinetic energy $E_f$ different from the initial value $E_i$, and appears in various channels of decaying one-particle state or many-body scattering states. The wave packets are normalized states, and are specified by not only the momentum $\vec{P}$ but also the position $\vec{X}$ at an instant of time. Because the momentum and the position are transformed under the space-time transformation, the theory is invariant manifestly but the probability is slightly modified under the transformations. Unusual enhancement in a forward direction at high energy was also found. This is characterized by a new characteristic length $\frac{\hbar E}{m^2c^3}$, where $\hbar = \frac{h}{2\pi}$, $h$, $E$, and $m$ are the Planck constant, the energy, and the mass, which is longer than the de Broglie wave length $\frac{\hbar}{p}$, where $p$ is the momentum, in many situations. This is of macroscopic length for the light particles, and can be longer than the size of the detector. Detailed account of these was given in [13].
Transition probability

The transition probability involving QCS is studied with the wave packets without ASI, and denoted as \( S[T] \) and satisfies the necessary boundary conditions. The standard S-matrix \( S[\infty] \) does not give \( T \)-dependence from its definition and is useless.

A transition of an eigenstate of \( H_0 \) of eigenvalue \( E_i \) due to \( H_{\text{int}} \) in ASI occurs to final states of the energy \( E_f = E_i \) but QCS leads to those of \( E_f \neq E_i \). The former contribution reveals of particle-like states in the asymptotic region \( |t| \to \infty \). The latter computed with the wave packets is finite, but divergent if the plane waves are used due to these waves of energy \( E_f \neq E_i \), which includes \( E_f \to \infty \) as was found by Stueckelberg [6]. Towards its resolutions, diffuse boundaries [6], Schrödinger representation of the field theory [7], nonequilibrium dynamics [8], and analysis from different viewpoints [11] have been studied. An importance of the problem was emphasized in [9] and [10]. However, its connection with the von Neumann’s fundamental principle of the quantum mechanics (FQM) has not been paid attention, and the connections with experimental observations and the natural phenomena have been unclear. Difficulties of computing transition probabilities for overlapping waves in quantum mechanics were pointed by (Peierles), Feynman, (Sakurai, Greiner), Stuckelberg, Goldberger-Watson. The difficulty remains. The present series of works compute the rigorous transition probability that includes QCS, by which natural phenomena are governed.

Using the wave function \( |i,0\rangle \) at \( t = 0 \) and \( |f,T\rangle \) at \( t = T \), the transition probability is expressed from FQM as, \( P(T) = |\langle f,T|i,0 \rangle|^2 \), for normalized states. For plane waves or stationary states, a modification introduced by Dirac [1, 2] is, for \( P(T) \ll 1 \), to use the average rate \( \Gamma = \frac{P(T) - P(T_1)}{T - T_1} \) between a small \( T_1 \) and a large \( T \) is used normally. The average transition probability of stationary states was expressed by \( \Gamma \), known as the Fermi’s golden rule for the final state of continuous spectrum, which is computed easily with the Dirac delta function. Solving \( P(T) \) at a large \( T \),

\[
P(T) = \Gamma T + P^{(d)}, \quad P^{(d)} = P(T_1) - \Gamma T_1.
\]

(1)

Unless \( P^{(d)} \ll \Gamma T \), both of \( \Gamma \) and \( P^{(d)} \) are inevitable for physical phenomena. In the previous works, \( P(T) \) in various weak and electromagnetic decays have been shown to resolve puzzles, [12–14]. We compute Eq.(1) using the wave packets in a scalar field theory and Quantum Electrodynamics (QED). The connections of the divergences found by Stuckelberg and those due to the intermediate states are clarified. The latter one is subtracted by the universal
counter terms, by the renormalized mass and the renormalized charge \[3–5\], and the former one becomes finite due to the wave packets and affects the transitions.

The $\Gamma T$ comes from the dynamics in the de Broglie wave length region, which represents the transitions of the states that separate quickly. The de Broglie wave length is $10^{-10}$ meters for the electron of 1 KeV or $10^{-15}$ for 200 MeV, and is much shorter than the typical length 1 meter of the experiments or natural phenomena, which can be approximated with $T = \infty$. This corresponds to the Fermi’s golden rule \[2, 15, 16\] or its extension, an S-matrix $S[\infty]$, in quantum field theory \[17–23\] under ASI. The states are described by the free Hamiltonian, and S-matrix satisfies $[S[\infty], H_0] = 0$. The states of $E_f = E_i$, where $E_i$ and $E_f$ are eigenvalues of $H_0$, contribute. The cross sections or decay rates defined by the ratios of the fluxes are manifestly Lorentz covariant. The amplitudes are expressed with Green’s functions defined with ASI. $\Gamma$, cross sections, or other corresponding rates computed from the ratios of the fluxes of the stationary states of the Schrödinger equation under ASI reveal in fact the particle properties.

Contrary to $\Gamma$, $P^{(d)}$ depends on the spatial size of the initial and final waves $\sigma$, \[12–14\], and is enhanced for large $\sigma$. They can satisfy $P^{(d)} \geq \Gamma T$ or $P^{(d)} \gg \Gamma T$, and lead rapid transitions at small $T$, and macroscopic phenomena for light particles. Importance of transitions of overlapping waves were pointed out in \[24, 25\]. The probability in the laboratory frame, in which the measurement is made, are computed and compared directly with the experimentally observed values \[26, 27\].

Using perturbative expansions with respect to a relevant coupling $g$ in a scalar theory and QED, $P^{(d)}$ and $\Gamma$ are expressed as,

\[
P^{(d)} = g^2 C_0 + g^3 C_1 + \cdots
\]

\[
\Gamma = g^2 D_0 + g^3 D_1 + \cdots
\]

where $C_i$ and $D_i$ are numerical constants and $g$ is the renormalized constant. Although the constants $D_i$ are known long \[28, 29\], in these theories $C_i$ are not. The lowest order term $g^2 C_0$ is a unique consequence of the theory. These are compared with experiments. Experiments are hard and still unknown in many processes. Nevertheless transition processes in nature follow the probability $\Gamma T + P^{(d)}$. $P^{(d)}$ is similar to a correction to the scattering-into-cones theorem \[30\] in a potential scattering, but is connected with the fundamental physical quantities. Thomson scattering, pair annihilation, and two photon scattering in
finite angles are known long, but new corrections in the extreme forward direction are found in the present work. The large effects of intriguing properties in dilute systems are derived. These would be crucial for environmental problems. In majority places the natural unit $\hbar = c = 1$ is used, and $t$ and $T$ are used for a time and a time-interval respectively.

Plane wave limit $\sigma \to \infty$ is unusual but gives an interesting insight to field theory. $P^{(d)} \to \infty$ while $\Gamma T$ remains the same. Accordingly, the total probability and the physical phenomena are governed by $P^{(d)}$ and the relative weight of $\Gamma T$ vanish. Interestingly, this is very close to the Haag’s theorem, i.e., an absence of the scattering in a manifestly invariant formalism [31]. In the standard method for physicists employing ASI [3–5] and the plane waves, the limit $T \to \infty$ is taken first and $\epsilon \to 0$ is taken next. Finite $T$ is not allowed and the probability at finite $T$ is not computable as is seen in the divergent probability by Stueckelberg. FQM bridges the Stueckelberg divergence for the plane wave of the relativistic field theory with the Haag theorem.

This paper is organized in the following manner. In Section 2, the scattering formalism beyond ASI in a scalar theory and unique properties of QCS in the particle decays are presented. In Section 3 and 4, these formalism is applied to QED, and the probability $P(T)$ of the Thomson scattering is presented. In Section 5, unique features of QCS are summarized. In Section 6 and 7, implications and summary are given. Miscellaneous problems of QCS and $P^{(d)}$ such as the connections with the principles of the statistical mechanics are presented in the Appendix.
II. SCATTERING PROBABILITY BEYOND THE FERMI’S GOLDEN RULE

One dimensional potential scattering for initial state of finite spatial extension that satisfies an initial condition is suitable for clarifying a distinctive property of the transition probability in the quantum mechanics. Namely, for a complete description of the scattering process the cross section is not sufficient and another component, which corresponds to the correction to the Fermi’s golden rule, is necessary.

In classical dynamics, the intensity of the particles or of the waves is a physical quantity, which is measured directly, hence the cross section defined by the intensity is necessary and sufficient for describing scattering processes. In quantum mechanical scattering, the transition probability is a physical quantity measured directly in experiments. That is a square of an inner product of the initial and final states and uniquely defined for the normalizes states. The quantum mechanical description is not identical to the classical one.

A. Square well potential in one dimension

III. EVOLUTION OF MANY-BODY STATES WITHOUT ADIABATIC SWITCHING OF INTERACTION AND QUASI-STATIONARY COMPOSITE STATES

Many-body wave functions are solved in a scalar field theory with the wave packets without encountering a divergence appeared for the plane waves. A new component of intriguing properties is found to appear. In Eq.(8) in a following section and sub-section 2.3, \(\hbar\) is written explicitly, although the natural unit is used in the majority of places.

A free system of scalar fields \(\varphi_1(x)\) and \(\varphi_2(x)\) described by a Lagrangian

\[
L_0 = \frac{1}{2} \sum_l \partial_\mu \varphi_l(x) \partial^\mu \varphi_l(x) - \frac{1}{2} \sum_l m_1^2 \varphi_l(x)^2, \tag{3}
\]

where \(m_2 > 2m_1\) is studied. The theory is translationally invariant and rotationally invariant and the energy-momentum tensor and the rotation tensors

\[
T_{\mu\nu} = \sum_l \partial_\mu \varphi_l(x) \partial_\nu \varphi_l(x) - g_{\mu\nu} L, \tag{4}
\]

\[
K^{\mu\nu\lambda} = (x^\nu T^{\mu\lambda} - x^\lambda T^{\mu\nu})
\]

are conserved. The energy-momentum vector and the rotation and boost operators

\[
P^\nu = \int d\vec{x} T^{0\nu}, K^{\nu\lambda} = \int d\vec{x} K^{0\nu\lambda} \tag{5}
\]
are constant and form Poincare algebra.

A. Representation: plane waves vs wave packets

1. Plane wave

The Hamiltonian \( H = P^0 \) describes the evolution of the state. The field is expanded in a set of momentum eigenstates of an energy \( E_l(\vec{p}) = \sqrt{p^2 + m_l^2} \),

\[
\varphi_l(x, t) = \int d\vec{p} e^{i\vec{p} \cdot \vec{x}} \rho_l(\vec{p}) a_l(\vec{p}, t) + \text{h.c.}, \rho_l(\vec{p}) = \frac{1}{\sqrt{(2\pi)^3 2E_l(\vec{p})}},
\]

\[
[a_{l_1}(\vec{p}_1, t_1), a_{l_2}^\dagger(\vec{p}_2, t_2)] \delta(t_1 - t_2) = \delta_{l_1 l_2} \delta(\vec{p}_1 - \vec{p}_2) \delta(t_1 - t_2), l_1, l_2 = 1, 2,
\]

where h.c. stands for a Hermitian conjugate. The states defined by creation and annihilation operators

\[
|0\rangle, a_{l_1}^\dagger(\vec{p}, t)|0\rangle, a_{l_2}^\dagger(\vec{p}_2, t)a_{l_1}^\dagger(\vec{p}_1, t)|0\rangle, \ldots,
\]

\[
a_{l_i}(\vec{p}, t)|0\rangle = 0
\]

are eigenstates of the Hamiltonian,

\[
H_0|\vec{p}_1, \vec{p}_2, \ldots\rangle = E_0|\vec{p}_1, \vec{p}_2, \ldots\rangle, E_0 = \sum_i \sqrt{p_i^2 + m_i^2},
\]

\[
H_0 = \sum_i \int d\vec{p} E_l(p) a_{l_i}^\dagger(\vec{p}, t)a_{l_i}(\vec{p}, t).
\]

They satisfy a Schrodinger equation,

\[
i\hbar \frac{\partial}{\partial t} |\Psi(t)\rangle = H_0 |\Psi(t)\rangle,
\]

\[
|\Psi(t)\rangle = e^{\frac{E_0 t}{\hbar}} |\Psi_0(0)\rangle.
\]

These form many-body states satisfying the completeness condition

\[
I = |0\rangle\langle 0| + \sum_i \int d\vec{p} a_{l_i}^\dagger(\vec{p}, t)|0\rangle \langle 0|a_{l_i}(\vec{p}, t) + \text{(many-body states)},
\]

\[
|\Psi(T)\rangle = N_0(T)|0\rangle + N_1(\vec{p}, T)|\vec{p}\rangle + \text{many-body states},
\]

\[
N_0(T) = \langle 0|\Psi(T)\rangle, N_1(\vec{p}, T) = \langle \vec{p}|\Psi(T)\rangle, N_2(\vec{p}_1, \vec{p}_2, T) = \langle \vec{p}_1, \vec{p}_2|\Psi(T)\rangle.
\]

One particle states are normalized with the Dirac’s delta function,

\[
\langle \vec{p}_1, l_1|\vec{p}_2, l_2\rangle = \delta_{l_1 l_2} \delta(\vec{p}_1 - \vec{p}_2),
\]

\[
\langle \vec{p}, l_1|\vec{p}_2, l_2\rangle = \delta_{l_1 l_2} \delta(0) = \infty,
\]
and transformed by Poincare transformation $U(\Lambda, d)$, where $\Lambda$ represents a rotation and a boost and $d$ represents a four-dimensional translation, and constructed by the operators of Eq. (5)

$$U(\Lambda, d)|\vec{p}\rangle = e^{ip\cdot d}|\Lambda\vec{p}\rangle.$$  \hspace{1cm} (12)

Under a rotation of the coordinates $\tilde{\vec{p}}_i = \vec{R}_i\vec{p}_i$, matrix elements of a scalar operator satisfy

$$\langle \vec{p}_1, \vec{p}_2, \cdots |O(0)|\vec{p}_1', \vec{p}_2', \cdots \rangle = \langle \vec{p}_1, \vec{p}_2, \cdots |U^{-1}(\theta)O(0)U(\theta)|\vec{p}_1', \vec{p}_2', \cdots \rangle$$

$$= \langle \vec{p}_1, \vec{p}_2, \cdots |O(0)|\vec{p}_1', \vec{p}_2', \cdots \rangle,$$  \hspace{1cm} (13)

and the same equality holds for Lorentz transformations. The matrix elements are invariant under the transformations of the momenta, and are functions of the invariant combinations of the momenta. The norm is divergent but the rigorous treatments \[32\] proved that the normal physical quantities are finite in a space of distributions. Cross sections and decay rates are computed from the ratios of fluxes for the stationary states, despite of the diverging norm. They are not identical to the probability computed from the normalized states.

2. A wave packet representation

A representation of a field operator in terms of normalized base functions \[27\] is applied hereafter. Dirac’s brackets notation $\langle A|B \rangle$ is used for one-particle space and for many-body space. In the former, the states $|A\rangle$ and $|B\rangle$ represent of one particle states, and in the latter they represent of the many-body states. In the following equations from Eq.(14) to Eq.(18), the brackets are those of one-particles.

A normalized solution of the free wave equation of a central momentum $\vec{P}$, position $\vec{X}$, and a time $T_0$ in the momentum space is

$$\langle t, \vec{p}|\vec{P}, \vec{X}, T_0, l \rangle = (\frac{\sigma}{\pi})^{3/4} e^{-iE_l(\vec{p})(t-T_0)-i\vec{p}\cdot \vec{X}-\frac{\sigma}{2}(\vec{p}-\vec{P})^2}.$$  \hspace{1cm} (14)

If the time $t - T_0$ is not very large, the solution in the configuration space is approximated well with

$$w(\vec{P}, \vec{x}, t; \vec{X}) = Ne^{-\frac{1}{2}(\vec{x}-\vec{X}-\vec{v}_l(t-T_0))^2} e^{-iE_l(\vec{P})(t-T_0)+i\vec{P} \cdot (\vec{x}-\vec{X})},$$

$$N = (\pi \sigma)^{-3/4}, (\vec{v}_l)_i = \frac{\partial}{\partial p_i}E_l(\vec{p})|_{\vec{p}=\vec{P}},$$  \hspace{1cm} (15)
where \( \sigma \) shows a size in the configuration space, and a case of \( \sigma \vec{P}^2 \gg 1 \) is studied. For the sake of simplicity, we use the Gaussian form in most parts of this paper. For a Fourier transformation, 

\[
\langle \vec{x} | \vec{p} \rangle = (2\pi)^{-3/2} e^{i \vec{p} \cdot \vec{x}}
\]
is used. In this region, a spreading effect is negligible , and the center moves with a velocity \( \vec{v}_l \). This paper concentrates this region. For brevity, \( \chi \) is used for \( (\vec{P}, \vec{X}) \). From Eq.(14)

\[
\langle \chi_1, T_1, l | \chi_2, T_2, l \rangle = \left( \frac{\sigma}{\pi} \right)^{3/2} e^{-\frac{\sigma}{4} (\vec{P}_1 - \vec{P}_2)^2} \int d\vec{p} e^{-i(E_l(\vec{p}(T_1 - T_2) - \vec{p}(\vec{X}_1 - \vec{X}_2)) - \sigma(\vec{p} - \vec{P}_0)^2)},
\]

(16)

\[
\vec{P}_0 = \frac{\vec{P}_1 + \vec{P}_2}{2},
\]

and the detailed property of the matrix element was given in [27]. The states satisfy one-particle completeness relation,

\[
1 = \int d\chi |\chi, T_0\rangle \langle \chi, T_0|, \quad d\chi = \frac{d\vec{X} d\vec{P}}{(2\pi)^3},
\]

(17)

\[
|\psi\rangle = \int d\chi |\chi, T_0\rangle \langle \chi, T_0|\psi\rangle, \quad \langle \psi_1 | \psi_2 \rangle = \int d\chi \langle \psi_1 | \chi, T_0 \rangle \langle \chi, T_0 | \psi_2 \rangle.
\]

Creation and annihilation operators of the wave-packet states defined as

\[
A^{\dagger}_l(\chi, t, T_0) = \int d\vec{p} a^{\dagger}_l(\vec{p}, t) \langle t, \vec{p} | \chi, T_0 \rangle, \quad A_l(\chi, t, T_0) = \int d\vec{p} \langle \chi, T_0 | \vec{p}, t \rangle a_l(\vec{p}, t),
\]

(18)

form many-body states

\[
|0\rangle, \quad A^{\dagger}_l(\chi, t, T_0)|0\rangle, \quad A^{\dagger}_l(\chi_1, t, T_0)A^{\dagger}_l(\chi_2, t, T_0)|0\rangle, \cdots,
\]

(19)

\[
a_l(\vec{p}, t)|0\rangle = 0,
\]

satisfying the completeness condition

\[
I = |0\rangle \langle 0| + \sum_l \int d\chi A^{\dagger}_l(\chi, t, T_0)|0\rangle \langle 0| A_l(\chi, t, T_0) + (\text{many-body states}). \tag{20}
\]

**Lemma 1**

(Expansion of a state)

From Eq.(20), a many-body state is expanded with unique coefficients

\[
|\Psi(T)\rangle = N_0(T)|0\rangle + N_1(\chi, T)|\chi\rangle + (\text{many-body states}),
\]

(21)

\[
N_0(T) = \langle 0|\Psi(T)\rangle, \quad N_1(T) = \langle 0|A_l(\chi, t, T_0)|\Psi(T)\rangle, \quad \text{many-body-components},
\]

\[
\langle \Psi_1(T)|\Psi_2(T)\rangle = (N_0^1(T))^* N_0^2(T) + \int d\chi (N_1^1(\chi, T))^* N_1^2(\chi, T) + (\text{many-body states}),
\]

and \( \langle \Psi(T)|\Psi(T)\rangle = 0 \) only when \( |\Psi(T)\rangle = 0 \).
The field operator is expanded as

\[ \varphi_l(x) = \int d\chi (C_l(\bar{x}; \chi)A_l(\chi, t, T_0) + C^*_l(\bar{x}; \chi)A^*_l(\chi, t, T_0)), \] (22)

\[ C_l(\bar{x}; \chi) = \int d\vec{p} \sqrt{2E_l(\vec{p})} \langle \bar{x}|\vec{p}\rangle \langle \vec{p}|\chi, T_0, l \rangle, \] (23)

and is substituted to the free Hamiltonian,

\[ H_0 = \sum_l \int d\vec{p} E_l(\vec{p}) a^*_l(\vec{p}, t) a_l(\vec{p}, t) \]

\[ = \sum_l \int d\chi_1 d\chi_2 d\vec{p} A^*_l(\chi_1, t, T_0) \langle \chi_1, T_0, l | \vec{p} \rangle E_l(\vec{p}) \langle \vec{p} | \chi_2, T_0 \rangle A_l(\chi_2, t, T_0). \]

The states Eq.(19) are normalized solutions of the Schrödinger equation, Eq.(9) and satisfies

\[ \langle 0 | A_l(\vec{P}_1, \vec{X}_1, t, T) A^*_l(\vec{P}_2, \vec{X}_2, t, T) | 0 \rangle = \langle \vec{P}_1, \vec{X}_1, T | \vec{P}_2, \vec{X}_2, T \rangle, \] (24)

where \( \vec{P}, \vec{X} \) are used instead of \( \chi \), and the matrix element of the scalar operator satisfies

\[ \langle \vec{P}_1, \vec{X}_1; \vec{P}_2, \vec{X}_2; \cdots | O(0) | \vec{P}_1', \vec{X}_1'; \vec{P}_2', \vec{X}_2'; \cdots \rangle = \langle \vec{P}_1, \vec{X}_1; \vec{P}_2, \vec{X}_2; \cdots | O(0) | \vec{P}_1', \vec{X}_1'; \vec{P}_2', \vec{X}_2'; \cdots \rangle, \] (25)

where \( \vec{X} \) and \( \vec{P} \) are \( R \vec{X} \) and \( R \vec{P} \). Not only momenta but also positions in the configuration space are transformed. For a matrix element that is factorized in the form,

\[ \langle \vec{P}_1, \vec{X}_1; \vec{P}_2, \vec{X}_2; \cdots | O(0) | \vec{P}_1', \vec{X}_1'; \vec{P}_2', \vec{X}_2'; \cdots \rangle = \langle \vec{P}_1, \vec{P}_2; \cdots | O(0) | \vec{P}_1', \vec{P}_2'; \cdots \langle \vec{P}_1, \vec{X}_1; \vec{P}_2, \vec{X}_2; \cdots | \vec{P}_1', \vec{X}_1'; \vec{P}_2', \vec{X}_2'; \cdots \rangle \] (26)

the sum over the positions is made easily. By a Poincare transformation, \( U(\Lambda, d) \), \( \vec{P}, \vec{X}, T_0 \) is transformed to \( \vec{P}', \vec{X}', T_0' \). The invariance of the theory is reduced to complicated relations of the matrix elements Eq.(25) generally. For the short-range correlations, Eq.(26), the momentum dependent term is equivalent to Eq.(13), and the probability integrated over the positions are expressed by \( |\langle \vec{P}_1, \vec{P}_2; \cdots | O(0) | \vec{P}_1', \vec{P}_2'; \cdots \rangle|^2 \), which becomes manifestly invariant. A experimental test of the transformation property of the short-range term is straightforward, but that of the long-range term is difficult, because the transformed position \( \vec{X}' \) depends on \( \vec{P} \) and \( T_0 \). A detection at this coordinate is actually impossible. Various aspects of the manifest Lorentz invariance in the finite-size correction to the golden rule has been presented in [13] and will be studied in a latter section.

Thus a complete set of normalized states are constructed and will be used for the computation of the probability in the laboratory frame, and for the transitions in nature.
B. Many-body Schrödinger equation

The interacting scalar fields is described by the Langrangian in terms of bare fields,

\[ L = L_0 + L_{\text{int}} \]  

\[ L_{\text{int}} = -\frac{g_2}{2!}(\phi_2(x)\phi_1(x)^2 + h.c.), \]  

in the classical level. Short-distance fluctuations of intermediate states in the perturbative expansions induce the divergences. They are eliminated by the counter terms in the renormalized theory \[3–5\]. In the present theory, divergence appears only in the mass terms, and the Lagrangian in terms of the renormalized fields is

\[ L = \frac{1}{2} \sum_l \partial_\mu \phi^*_l(x) \partial^\mu \phi^*_l(x) - \frac{1}{2} \sum_l (m_l^2 - \delta m_l^2) \phi^*_l(x)^2 - \frac{g_2}{2!}(\phi_2(x)\phi^*_1(x)^2 + h.c.). \]  

In the interaction representation \[3, 4\], a wavefunction \(|\Psi(t)\rangle\) satisfies a Schrödinger equation,

\[ i\hbar \frac{\partial}{\partial t} |\Psi_i(t)\rangle = H_{\text{int}}(t)|\Psi_i(t)\rangle, \]  

\[ |\Psi(t)\rangle = e^{\frac{iH_{\text{int}}t}{\hbar}}|\Psi(t)\rangle, H_{\text{int}}(t) = e^{\frac{iH_{\text{int}}t}{\hbar}}e^{-\frac{iH_{\text{int}}t}{\hbar}}, \]  

Solutions are obtained with perturbative expansions with respect to \(H_{\text{int}}\), and are expressed with the operator \(U(t,t_0)\) as

\[ |\Psi(t)\rangle = U(t,t_0)|\Psi(t_0)\rangle, \]  

\[ U(t,t_0) = \int_{t_0}^t dt_1 \mathcal{T} e^{-iH_{\text{int}}t_1/\hbar} = 1 + \int_{t_0}^t \frac{dt_1}{\hbar} (-iH_{\text{int}}(t_1)) \]  

\[ + \int_{t_0}^t \frac{dt_1}{\hbar} \int_{t_0}^{t_1} \frac{dt_2}{\hbar} ((-i)^2 H_{\text{int}}(t_1) H_{\text{int}}(t_2)) + \cdots, \]  

\[ = 1 + \sum_{n=1}^{12} \frac{1}{n!} \int_{t_0}^t dt_1 dt_2 \cdots dt_n \mathcal{T}(H_{\text{int}}(t_1) \cdots H_{\text{int}}(t_n)) \]  

where \(\mathcal{T}\) stands for the time-ordered product.

1. Plane waves: Stueckelberg divergences

The solution Eq. (32) is expressed with plane waves. That in the first order of the coupling constant, for an initial state \(|\Psi_1(0)\rangle = |\vec{p}_1\rangle\), \(H_0|\Psi_1(0)\rangle = E_1(\vec{p}_1)|\Psi_1(0)\rangle\), \(E_1 = \sqrt{p_1^2 + m^2}\), is

\[ |\Psi_1(t)\rangle = N_1(|\Psi_1(0)\rangle + \int dn D(\Delta E, t)|n\rangle\langle n|(-iH_{\text{int}}(0)|\Psi_1(0)\rangle), \]  

\[ D(\Delta E, t) = e^{-\frac{\Delta E t}{2\hbar}} \frac{2 \sin \frac{\Delta E t}{2\hbar}}{\Delta E}, \Delta E = E_0 - E_n, \]
where $N_1$ is a normalization constant, the state $|n\rangle$ denotes a state of the energy $E_n$ of the volume element $dn$. The inner product of the state $|\Psi_1(t)\rangle$ with a state $|\Psi_2(t)\rangle$ of momentum $\vec{p}_2$ is given by

$$
\langle \Psi_1(t)|\Psi_2(t)\rangle = \delta(\vec{p}_1 - \vec{p}_2)|N_1|^2(1 + \int dn|D(\Delta E, t)|^2|\langle n|(-iH_{\text{int}}(0)|\Psi(0)\rangle|^2)
$$

$$
= \delta(\vec{p}_1 - \vec{p}_2)|N_1|^2(1 + \prod_i \frac{d^3k_i}{2E(k_i)(2\pi)^3}(k_i)^9(2\sin\frac{\Delta E t}{2\hbar})^2(2\pi)^3\delta(\vec{p}_1 - \sum_i \vec{k}_i))
$$

where in the integral $\vec{p}_1 = \vec{p}_2$ was substituted and $q = 0$ now. The integral Eq.(35) behaves differently depending upon a spectrum of $\Delta E = E_n - E_0$. We write s a minimum value as $E_g$, i.e., $\Delta E \geq E_g$. For positive definite $E_g$, this shows an energy gap between the initial and final states. There is no state of satisfying $\Delta E = 0$. For negative $E_g$, there is no energy gap, and there are states of $\Delta E = 0$. In the integrand in Eq.(35),

$$
(\frac{2\sin\frac{\Delta E t}{2\hbar}}{\Delta E})^2 \approx t\delta(\Delta E)
$$

due to a sharp peak at $\Delta E \approx 0$, and is proportional to $t$. The proportional constant is given by the integral over the phase space and converges. Outside this region, this is not proportional to $t$. This function does not vanish there and at $\Delta E \to \infty$, $(2\sin\frac{\Delta E t}{2\hbar})^2 \approx 2$.

The integrand is independent of $t$, and converges for $q = 0$, but diverges for $q \geq 1$. The norm is proportional to $\delta(\vec{p}_1 - \vec{p}_2)$, and the proportional constant diverges. This leads the divergence found by Stueckelberg of the transition probability at the finite $t$ diverges for the plane waves. This behaves differently for the wave packets.

### 2. Wave packets: Stationary states

The wavefunctions are obtained with the wave packet representation, in which the interaction Hamiltonian is expressed with the operators $A_{l_1}(\chi_1, t)$ and $A_{l_1}^\dagger(\chi_1, t)$,

$$
H_{\text{int}} = \frac{-g}{2}\int d\vec{x}(C_{l_1}(\vec{x}; \chi_1)A_{l_1}(\chi_1, t) + C_{l_1}(\vec{x}; \chi_1)^*A_{l_1}^\dagger(\chi_1, t))(C_{l_2}(\vec{x}; \chi_2)A_{l_2}(\chi_2, t)
$$

$$
+ C_{l_2}(\vec{x}; \chi_2)^*A_{l_2}^\dagger(\chi_2, t))(C_{l_3}(\vec{x}; \chi_3)A_{l_3}(\chi_3, t) + C_{l_3}(\vec{x}; \chi_3)^*A_{l_3}^\dagger(\chi_3, t)),
$$

using the coefficients $C_{l_i}(\vec{x}; \chi_1)$ given in Eq.(22). Substituting the expressions Eq.(20) to Eq.(32), the coefficients of the solutions are determined. They become simple forms for stationary states.
FIG. 1. Diagram of the vacuum amplitude that three particles are produced.

**Vacuum**

The ground state $|0\rangle$ is expressed in the first order of $g$ as,

$$|\Psi_0(T)\rangle = N_0(T)|0\rangle + \int d^3\chi N_3(\chi_1, \chi_2, \chi_3, T)A_i^\dagger(\chi_1, t)A_j^\dagger(\chi_2, t)A_k^\dagger(\chi_3, t)|0\rangle; \quad i, j, k = 1, 2,$$

$$N_0(T) = 1 + O(g^2), d^3\chi = d\chi_1d\chi_2d\chi_3,$$

(38)

where three-particle-state is disconnected part of the energy gap $E_g = 2m_1 + m_2$ shown in Fig. 1. $N_3(\chi_1, \chi_2, \chi_3, T)$ is the Gaussian integral over space-time coordinate, and found in Appendix A and B as

$$|N_0(T)|^2 = 1, \int d^3\chi |N_3(\chi_1, \chi_2, \chi_3, T)|^2 = 0,$$

(39)

$$\langle \Psi_0(T)|H_{int}|\Psi_0(T)\rangle = 0.$$

The ground state is stationary and unique. From Appendix A and B, the following lemma holds.

**Lemma 2**

The integral over the space-time coordinate of the Gaussian functions for the states of the energy gap $E_g$ in the bulk region is proportional to $e^{-\frac{\sigma_t E_g}{2}}$, where $\sigma_t$ is the square of the size in the temporal direction of the overlapping region of the waves. The space-time positions which contribute to the boundary term are restricted to microscopic region and the probability is negligible.

**Stable one-particle state: lightest one**

One-particle state of lighter scalar $A_i^\dagger(\chi)|0\rangle$ is expressed in the order $g$ by

$$|\Psi_1^L(T)\rangle = N_1^L(T)A_i^\dagger(\chi)|0\rangle + \int d^2\chi N_2^L(\chi_1, \chi_2, T)A_i^\dagger(\chi_1)A_2^\dagger(\chi_2)|0\rangle$$

$$+ \int d^4\chi N_4^L(\chi_1, \chi_2, \chi_3, \chi_4, T)A_i^\dagger(\chi_1)A_2^\dagger(\chi_2)A_3^\dagger(\chi_3)A_4^\dagger(\chi_4)|0\rangle,$$

(40)

$$|N_1^L(T)|^2 = 1, \int d^2\chi |N_2^L(\chi_1, \chi_2, T)|^2 = 0, \int d^4\chi |N_4^L(\chi_1, \chi_2, \chi_3, \chi_4, T)|^2 = 0; \quad i, j, k = 1, 2,$$

14
In the above equation, $N_2^i(\chi_l, T)$ and $N_4^i(\chi_l, T)$ are the Gaussian integrals in the bulk of the finite energy gaps of $\sigma_t \neq 0$. From Lemma, they vanish. The four particle state in disconnected part has also a large energy gap and vanishes from the same reasons. The contribution from the boundary also negligible as in the vacuum. The lightest particle is stationary and unique.

For the vacuum state and the stable one-particle state at $t = 0$, the first-order corrections do not modify the wavefunctions.

3. **Quasi-stationary composite states**

Heavier scalar decays to two lighter scalars, and is not stationary. The many-body wavefunction of the decaying state is a superposition of the parent and daughters in a finite time. In a space-time region where these waves fully overlap, which is called a bulk, that is a symmetric superposition, whereas in a boundary regions that is asymmetrical. That behaves differently in two cases.

**Metastable one-particle state: heavier one**

We assume that the heavier field is almost stable and described by a field operator. A solution Eq.(32) starting from the heavier state of the momentum and position $(\vec{P}, \vec{X})$ at $t = 0$ is

$$|\Psi_1^H(T)\rangle = N_1^H(T)A_2^i(\chi)|0\rangle + |\Psi_2(T)\rangle, \quad \text{(41)}$$

$$|\Psi_2(T)\rangle = \int d^3\chi (N_2^H(\chi, T)A_1^i(\chi_1)A_1^i(\chi_2) + \cdots)|0\rangle,$$

where the disconnected part written by "\cdots" vanishes due to the energy gap and is dropped. $N_1^H(T)$ and $N_2^H(\chi, T)$ are finite and were studied in [12, 13] and are briefly summarized hereafter, using the variables $(\vec{P}, \vec{X})$ instead of $\chi$, and the matrix element $F_{\beta,\varphi_2}(\omega) = \langle \beta|H_{int}(0)|\varphi_2\rangle$, $\omega = E_{\varphi_2} - E_{\beta}$. The initial state is an approximate eigenstate of $H_0$.

The coefficient $N_2^H(\chi, T)$ is the integral over the space-time coordinate

$$N_2^H(\chi, T) = \mathcal{M} = \int_{T_1}^{T_2} dt \int d^3x (\vec{k}_1, \vec{k}_1'; \vec{X}_1, \vec{X}_1')|H_{int}(x)|\vec{p}_2, \vec{X}_2\rangle, \quad \text{(42)}$$

which gets contributions from the bulk and the boundary from Appendix A, B, and C.

**Bulk contribution**

The integral over $t$ from $-\sqrt{2\sigma_t} < t - T_0 < \sqrt{2\sigma_t}$ is given in Eq.49 of Appendix A as

$$N_2^H(\chi, T; \text{bulk}) = N_{\text{bulk}}(\vec{X})e^{-\frac{1}{2\sigma_t}(\delta \vec{p})^2 - \frac{1}{2\sigma_t}(\delta \omega)^2}, \quad \text{(43)}$$

15
where $N_{\text{bulk}}(\vec{X})$ is independent of the center position in time. From the exponential behavior, the wave function decreases rapidly with $\delta \omega$. This component shows the conservation of kinetic energy, which is characteristic of point particles.

**Boundary contribution**

The integral over $t$ from the boundary is given in Eq. (A10) of Appendix A as

$$N^H_2(\chi, T; \text{boundary}) = N_{\text{boundary}}(\vec{X}) e^{-\frac{1}{2\sigma_t}(\delta \vec{p})^2} \frac{\sqrt{2\sigma_t}}{1 + i \sqrt{2\sigma_1}(\delta \omega)}. \quad (44)$$

where $N_{\text{boundary}}(\vec{X})$ is defined at a boundary in time. This follows a power behavior instead of the exponential one, and the wavefunction decreases slowly with $\omega$. This shows the non-conservation of kinetic energy, which is characteristic of waves. The wavefunction is a sum,

$$|\Psi_2(T_1)\rangle = |\Psi_{(2,p)}(T_1)\rangle + |\Psi_{(2,w)}(T_1)\rangle, \quad (45)$$

where $|\Psi_{(2,p)}(T_1)\rangle$ is derived from $\omega \approx 0$, which is computed in a standard method from $|D(\omega, T_1)|^2 \approx 2\pi T_1 \delta(\omega)$, and $|\Psi_{(2,w)}(T_1)\rangle$ is from outside of $\omega$ of $|D(\omega, T_1)|^2 \approx \text{constant}$. Their norms are expressed by

$$\langle \Psi_{(2,p)}(T_1) | \Psi_{(2,p)}(T_1) \rangle = \Gamma T_1, \quad \Gamma = \int d\beta \delta(\omega)|F_{\varphi_{2,\beta}}(0)|^2 2\pi \tilde{\rho}(0), \quad (46)$$

$$\langle \Psi_{(2,w)}(T_1) | \Psi_{(2,w)}(T_1) \rangle = P^{(d)} = \int_{|\omega| \geq \frac{1}{\sqrt{2\sigma_t}}} d\vec{p}1 d\vec{p}2 |F_{\varphi_{2,\beta}}(\omega)|^2 |D(\omega, T_1)|^2.$$ 

where $\tilde{\rho}(0)$ is the density of the states at $\omega = 0$.

The norm of the parent and the expectation value of the interaction Hamiltonian of the state $|\Psi_2(T)\rangle$, which we call interaction energy for brevity, up to $g^2$ are,

$$|N^H_2(T)|^2 = 1 - TT - P^{(d)}$$

$$E_{\text{int}}(T) = \langle \Psi_2(T) | H_{\text{int}} |\Psi_2(T)\rangle = \int d\vec{p}1 d\vec{p}2 \omega |F_{\varphi_{2,\beta}}(\omega)|^2 |D(\omega, T)|^2. \quad (47)$$

$|\Psi_{(2,p)}(T)\rangle$ denotes the state of $|\omega| \approx 0$, and $|\Psi_{(2,w)}(T)\rangle$ denotes the state of $|\omega| \neq 0\left( > \frac{1}{\sqrt{2\sigma_t}} \right)$, and satisfy

$$\frac{\langle \Psi_{(2,p)}(T) | H_{\text{int}} |\Psi_{(2,p)}(T)\rangle}{\langle \Psi_{(2,p)}(T) | \Psi_{(2,p)}(T)\rangle} \leq \frac{1}{\sqrt{2\sigma_t}}, \quad (48)$$

$$\frac{\langle \Psi_{(2,w)}(T) | H_{\text{int}} |\Psi_{(2,w)}(T)\rangle}{\langle \Psi_{(2,w)}(T) | \Psi_{(2,w)}(T)\rangle} > \frac{1}{\sqrt{2\sigma_t}}.$$
The former is included in ASI \[34–39\]. Now, \( |\Psi_{(2,w)}(T)\rangle \) is a correlated state similar to a stationary state, and gives various physical effects \[12–14, 25, 40\].

\( |\Psi_{(2,w)}(T)\rangle \) is a quasi-stationary composite states (QCS) of features; (1) does not vanish at macroscopic \( T \), (2) the continuous spectrum of kinetic energy, (3) \( T \)-independent norm at large \( T \), Eq. (17), (4) finite interaction energy Eq.(63). QCS is accompanied by the decaying particle states in the normal cases. In an extremely small \( T \), the integral Eq.(47) is not exactly constant, but varies steeply to reach the constant \[12, 13\]. The final states expressed by QCS appear rapidly, and its probability \( P^{(d)} \) remains at later times.

In an interaction Eq. (C1), the coupling at \( \omega = 0 \) vanishes \( F_{\varphi_{\mu,\beta}}(0) = 0 \), and the state is expressed as

\[
|\Psi(T)\rangle = N_1^H(T)|\varphi_{\mu}\rangle + |\Psi_{(2,w)}(T)\rangle, |N_1^H(T)|^2 = 1 - P^{(d)},
\]

\[
P^{(d)} = \frac{\sqrt{\pi} \omega_{12}}{4\pi g_1^2} \int d^3 p_1 \log(y) \theta(m_1^2 - 2 P_V \cdot p_1) \neq 0,
\]

where \( P_V \) is the energy-momentum of the initial state, and \( y \) is given in Eq.(62) \[14\]. The state is composed of QCS and the parent, and varies with time. The normal component of the decaying particles \( |\Psi_{(2,p)}\rangle \) is not included.

C. Quasi Stationary Composite States (QCS): Correlations

QCS has unique properties which do not arise in the normal transition under ASI.

1. Kinetic energy vs interaction energy

From Eqs.(11) and (63), the expectation values of the total kinetic energy at \( T_1 \) is

\[
E_{\text{kin}}(T_1) = \langle \Psi_1^H(T_1)|H_0|\Psi_1^H(T_1)\rangle = E_2(\vec{P}) - E_{\text{int}}(T_1),
\]

\[
E_{\text{int}}(T_1) = 2 \int d\vec{p}_1 \left\{ \frac{E_{\text{total}}}{2} - E(\vec{p}_1) \right\} \int d\vec{p}_2 |D(\omega,T_1)|^2 |F_{\varphi_{\mu,\beta}}(\omega)|^2,
\]

where \( E_{\text{total}} \) is the total energy and agrees with \( E_2(\vec{P}) \). In the right-hand side of Eq.(52), a symmetric nature of the two particle wave function was used. Thus the interaction energy depends on the single particle spectrum. The \( |\Psi_{(2,w)}(T)\rangle \) has interaction energy and a continuous kinetic energy which differs from that of an isolated particle.
2. **Space-time symmetry**

QCS is a superposition of a continuous mass spectrum \( P_{\mu}P^\mu = (E_1 + E_2)^2 - (\vec{p}_1 + \vec{p}_2)^2 \), which is that of the visible energy and momentum as is shown later and is not a constant. This differs from a bound state which belongs to an irreducible representation of a definite mass.

3. **Order parameter : interaction energy**

QCS is the state that has an expectation value of the interaction Hamiltonian

\[
O_{\text{qcs}}(\pm \frac{T}{2}) = \lim_{t \to \pm \frac{T}{2}} \langle \Psi(t)|H_{\text{int}}(t)|\Psi(t)\rangle. \tag{53}
\]

Accordingly, \( O_{\text{qcs}}(\pm \frac{T}{2}) \) is an order parameter for QCS. A state is QCS if \( O_{\text{qcs}}(\pm \frac{T}{2}) \neq 0 \), and is not if \( O_{\text{qcs}}(\pm \frac{T}{2}) = 0 \). |\( \Psi_{(2,w)}(x) \rangle \) in Eq. (15) and |\( \Psi_{(2,w)}(x) \rangle \) in Eq. (55) have \( O_{\text{qcs}}(T) \neq 0 \) and are QCS, but |\( \Psi_{(2,p)}(x) \rangle \) has \( O_{\text{qcs}}(T) = 0 \), and is not. QCS in scattering processes are also identified by this order parameter. If all the states of Eq.(31) at \( t \to \infty \) have the value \( O_{\text{qcs}}(\infty) = 0 \), QCS does not appear, and states are composed of particle-like states.

In ASI,

\[
O_{\text{qcs}}(\pm \infty) = \lim_{t \to \pm \infty} \langle \Psi(t)|e^{-\epsilon|t|}H_{\text{int}}|\Psi(t)\rangle = 0. \tag{54}
\]

Accordingly all the asymptotic states are particle-like states that satisfy the kinetic-energy conservation. If ASI is not required, there may be states of \( O_{\text{qcs}}(\pm \infty) = 0 \) and those of \( O_{\text{qcs}}(\pm \infty) \neq 0 \). The amplitude and probability in the space of \( O_{\text{qcs}}(\pm \infty) = 0 \), and of \( O_{\text{qcs}}(\infty) \neq 0 \) describe the physical processes. Contributions from the latter are obtained without ambiguity in the tree level of the perturbative expansion, where the scattering is studied without assuming ASI. Accordingly, the method of Tomonaga-Schwinger-Feynman developed for the higher order corrections in QED under ASI is not applied, but a naive Schrödinger equation is applied. Using it, the wave functions for the interaction Hamiltonian \( H_{\text{int}} \) instead of \( e^{-\epsilon|t|}H_{\text{int}} \) are obtained and the transition probabilities are computed.

4. **Norm**

The norm of the initial state and those of the final states are modified at later times, and affect the transition probability. Higher order terms in \( g \) relevant to the probability are
found. The operator in Eq. (32) is written as,

\[ U(t, t_0) = 1 + iK(t, t_0), \quad K(t, t_0) = \int_{t_0}^{t} dt_1 \tilde{K}(t_1), \]  

\[ i(K(t, t_0) - K^\dagger(t, t_0)) + K(t, t_0)K^\dagger(t, t_0) = 0, \]

where \( \tilde{K}(t_1) = H_{\text{int}}(t_1) + \int_{t_0}^{t_1} dt_2 H_{\text{int}}(t_1)H_{\text{int}}(t_2) + \ldots \), is the integrand of \( K(t, t_0) \). Substituting a one-particle state \(|\alpha\rangle\) and a complete set \(|n\rangle\) of approximate eigenstates of \( H_0 \) of the energies \( E_\alpha \) and \( E_n \),

\[ \langle n | K(t, t_0) | \alpha \rangle = D(\omega, \delta t) \langle n | \tilde{K}(t_0) | \alpha \rangle, \quad \omega = E_\alpha - E_n, \quad \delta t = t - t_0, \]  

\[ \langle \alpha | K(t, t_0)K^\dagger(t, t_0) | \alpha \rangle = |D(\omega, \delta t)|^2 |\langle \alpha | \tilde{K}(t_0) | n \rangle|^2, \]

where the phase factors of the states are extracted to define \( D(\omega, \delta t) \), and the rest is denoted as \( \tilde{K}(t_0) \),

\[ \langle \alpha | K(t, t_0) - K^\dagger(t, t_0) | \alpha \rangle = -i |D(\omega, \delta t)|^2 |\langle \alpha | \tilde{K}(t_0) | n \rangle|^2. \]

**D. Transition probability**

1. Wave packets of \( H_0 \)

We assume that \( |\Psi_\alpha(0)\rangle \) is a normalized state defined by an approximate eigenstate of \( H_0 \) of the central energy \( E_0 \), and \( |\Psi_\beta(T)\rangle \) at \( t = T \) be normalized and an approximate eigenstate of the central energy \( E_1 \), then a product of their matrix element,

\[ P_{\beta,\alpha}(T) = |\langle \Psi_\beta(T) | \Psi_\alpha(0) \rangle|^2 = ||D(\Delta E, T)||^2 |\langle \beta | H_{\text{int}} | \Psi_\alpha(0) \rangle|^2, \]

where \( \Delta E = E_1 - E_0 \) satisfies

\[ 0 \leq P_{\beta,\alpha}(T) \leq 1, \quad \sum_{\beta} P_{\beta,\alpha}(T) = 1. \]

By substituting the relation for the large \( T \)

\[ \left| \frac{D(\Delta E, T)}{T - T_1} \right|^2 - |D(\Delta E, T_1)|^2 = 2\pi \delta(\Delta E), \]

the summation over \( \beta \) of these around a center \( \tilde{\beta} \), \( P_{\tilde{\beta}\alpha}(T) = \sum_{\beta; \tilde{\beta}} P_{\beta,\alpha} \) is expressed from Eq. (34) for a large \( T \) as

\[ P_{\tilde{\beta}\alpha}(T) = P_{\tilde{\beta}\alpha}(T_1) + (T - T_1) \Gamma_{\tilde{\beta}\alpha} \]

\[ \Gamma_{\tilde{\beta}\alpha} = \int_{\tilde{\beta}} d\beta |\langle \beta | H_{\text{int}} | \Psi_\alpha(0) \rangle|^2 2\pi \delta(\Delta E). \]
The coefficient $\Gamma_{\beta\alpha}$ is derived from the wavefunctions of $\Delta E = 0$. $P_{\beta\alpha}(0) = 0$ but $\frac{d^2 P_{\beta\alpha}(T_1)}{dT_1^2}$ for the plane waves corresponds to the integral Eq. (35) of $q = 2$, and diverges at $T_1 = 0$ due to the Stueckelberg divergence. Now $P_{\beta\alpha}(T_1) - \Gamma_{\beta\alpha} T_1$ with the normalized states is convergent $^{[12]}$ Next we study the many-body wavefunctions explicitly.

$P^{(d)}$ is the contribution form the boundary of the interval $\sqrt{2\sigma_t}$, and its convergence is proved in Appendix B. Hence, $T_1 = \sqrt{2\sigma_t}$. For a case that the wave packet of the final state $\sigma_1$ and that of the initial state $\sigma_2$, and $\sigma_1 < \sigma_2$, $P^{(d)}$ is computed easily at a high energy region with the correlation function obtained by interchanging the order of the integrations over the space-time positions and momenta, in which the light-cone singularity derived from $|\vec{p}_2| \to \infty$ gives the most important contribution $^{[12]}$,

$$P^{(d)} = \frac{8g^2}{E_2(\vec{P})} \sigma_1 \int \frac{d^3 p_1}{(2\pi)^3 E(\vec{p}_1)} T_1 \tilde{g}(y) \theta(m_2^2 - m_1^2 - 2P \cdot p_1), y = \frac{m_1^2 T_1}{2E_1(\vec{p}_1)}, \quad (62)$$

where the function $\tilde{g}(y)$ is derived from the light-cone singularity and proportional to $\frac{1}{y}$ at large $y$ region and is $\pi$ at small $y$. $T_1 \tilde{g}(y)$ is independent of $T_1$ in the former region and is proportional to $T_1$ in the latter region. This component shows a wider energy distribution which shifts to lower energy region than $\Gamma$ $^{[12], [13], [33]}$.

The norm of the parent and the expectation value of the interaction Hamiltonian of the state $|\Psi_2(T)\rangle$, which we call interaction energy for brevity, up to $g^2$ are,

$$|N_1^H(T)|^2 = 1 - TT - P^{(d)}$$
$$E_{int}(T) = \langle \Psi_2(T)|H_{int}|\Psi_2(T)\rangle = \int d\vec{p}_1 d\vec{p}_2 \omega |F_{\varphi_2,\beta}(\omega)|^2 |D(\omega, T)|^2. \quad (63)$$

$|\Psi_{(2,p)}(T)\rangle$ denotes the state of $|\omega| \approx 0$, and $|\Psi_{(2,w)}(T)\rangle$ denotes the state of $|\omega| \neq 0$($> \frac{1}{\sqrt{2\sigma_t}}$), and satisfy

$$\frac{\langle \Psi_{(2,p)}(T)|H_{int}|\Psi_{(2,p)}(T)\rangle}{\langle \Psi_{(2,p)}(T)|\Psi_{(2,p)}(T)\rangle} \leq \frac{1}{\sqrt{2\sigma_t}}, \quad \frac{\langle \Psi_{(2,w)}(T)|H_{int}|\Psi_{(2,w)}(T)\rangle}{\langle \Psi_{(2,w)}(T)|\Psi_{(2,w)}(T)\rangle} > \frac{1}{\sqrt{2\sigma_t}}. \quad (64)$$

The former is included in ASI $^{[34], [39]}$. Now, $|\Psi_{(2,w)}(T)\rangle$ is a correlated state similar to a stationary state, and gives various physical effects $^{[12], [14], [25], [40]}$.

$|\Psi_{(2,w)}(T)\rangle$ is a quasi-stationary composite states (QCS) of features; (1) does not vanish at macroscopic $T$, (2) the continuous spectrum of kinetic energy, (3) $T$-independent norm at large $T$, Eq. (47), (4) finite interaction energy Eq.(63). QCS is accompanied by the
decaying particle states in the normal cases. In an extremely small $T$, the integral Eq. (47) is not exactly constant, but varies steeply to reach the constant \[ 12, 13 \]. The final states expressed by QCS appear rapidly, and its probability $P^{(d)}$ remains at later times.

In an interaction Eq. (C1), the coupling at $\omega = 0$ vanishes $F_{\varphi_{\mu, \beta}}(0) = 0$, and the state is expressed as

\[
|\Psi(T)\rangle = N_1^H(T)|\varphi_{\mu}\rangle + |\Psi_{(2, w)}(T)\rangle, |N_1^H(T)|^2 = 1 - P^{(d)},
\]

\[
P^{(d)} = \frac{\sqrt{\pi}g^2}{4\pi} \frac{1}{E_V} \int d^3p_1 \log(y) \theta(m_2^2 - 2P_V \cdot p_1) \neq 0,
\]

where $P_V$ is the energy-momentum of the initial state, and $y$ is given in Eq. (62) \[ 14 \]. The state is composed of QCS and the parent, and varies with time. The normal component of the decaying particles $|\Psi_{(2, \rho)}\rangle$ is not included.

**Single particle spectrum**

A single-particle distribution of the particle $\varphi_1$ from $P^{(d)}$ for large $y$ is

\[
\frac{dP^{(d)}(\vec{p}_1, T)}{d\vec{p}_1} = \frac{8g^2}{E_2(\vec{P})} \sigma_1 \frac{1}{(2\pi)^3 m_1^2} \theta(m_2^2 - m_1^2 - 2P \cdot p_1),
\]

from Eq. (62). The average kinetic energy is about a half of the kinetic-energy-conserving value. Accordingly the interaction energy is positive, and the total energy $E_{\text{total}}$ is larger than an energy estimated from the single-particle distribution by about a factor two. A kind of Virial theorem holds.

2. **Wave packets of $H$: Uniqueness of the scattering amplitude**

We show that the scattering amplitude for wave packets defined from the eigenstates of $H$,

\[
H|E(\vec{k}), \vec{k}\rangle = E(\vec{k})|E(\vec{k}), \vec{k}\rangle,
\]

\[
H|E(\vec{p}_1 + \vec{p}_2), \vec{p}_1, \vec{p}_2\rangle = E(\vec{p}_1 + \vec{p}_2)|E(\vec{p}_1 + \vec{p}_2), \vec{p}_1, \vec{p}_2\rangle,
\]

is equivalent to that defined from the eigenstates of $H_0$. The eigenstates in the perturbative expansions with respect to $H_{\text{int}}$ are

\[
|E(\vec{k}), \vec{k}\rangle = |\vec{k}\rangle + \mathcal{P} \int d\vec{p}_1 d\vec{p}_2 \frac{1}{E(\vec{k}) - E(\vec{p}_1) - E(\vec{p}_2)} |\vec{p}_1, \vec{p}_2\rangle \langle \vec{p}_1, \vec{p}_2|H_{\text{int}}|\vec{k}\rangle,
\]

\[
|E(\vec{p}_1 + \vec{p}_2), \vec{p}_1, \vec{p}_2\rangle = |\vec{p}_1, \vec{p}_2\rangle + \mathcal{P} \int d\vec{k} \frac{1}{-E(\vec{k}) + E(\vec{p}_1) + E(\vec{p}_2)} |\vec{k}\rangle \langle \vec{k}|H_{\text{int}}|\vec{p}_1, \vec{p}_2\rangle,
\]
up to the first order, where $\mathcal{P}$ stands for the principle value, and $|E(\vec{k}), \vec{k} \rangle$ is the one particle state of the heavier scalar and $|E(\vec{p}_1 + \vec{p}_2), \vec{p}_1, \vec{p}_2 \rangle$ is the two particle state of the lighter scalar. These are orthogonal each other

$$\langle E(\vec{k}), \vec{k} | E(\vec{p}_1 + \vec{p}_2), \vec{p}_1, \vec{p}_2 \rangle = 0,$$

(70)

but are not normalized. Wave packets constructed from the states of Eq.(69) as

$$|\Psi_\alpha \rangle = \int d\vec{k} w(\vec{P}, \vec{X}, \vec{k}) |E(\vec{k}), \vec{k} \rangle,$$

(71)

$$|\Psi_\beta \rangle = \int d\vec{p}_1 d\vec{p}_2 w(\vec{P}_1, \vec{X}_1, \vec{p}_1) w(\vec{P}_2, \vec{X}_2, \vec{p}_2) |E(\vec{p}_1 + \vec{p}_2), \vec{p}_1, \vec{p}_2 \rangle,$$

are normalized and satisfy $\langle \Psi_\alpha | \Psi_\beta \rangle = 0$. As the state is evolved as

$$|\Psi_\alpha (T) \rangle = U(T, 0) |\Psi_\alpha (0) \rangle,$$

(72)

the transition amplitude for these states is given by

$$\langle \Psi_\beta | U(T, 0) |\Psi_\alpha \rangle = \langle \chi_1 \chi_2 \rangle \int_0^T dt \frac{d}{dt} H_{int}(t) |\chi_0 \rangle,$$

(73)

where Eq.(51) was substituted. This is in agreement with that of the wave packets defined from the eigenstates of $H_0$. Accordingly, the transition amplitude at the finite time interval is universal.

E. Large $T$ behavior

1. $\Gamma T$: $T$-dependent norm

From Eq.(65),

$$2Im \langle \alpha | K(t, t_0) |\alpha \rangle = 2\pi \delta(0) \Gamma = \delta t \Gamma,$$

(74)

$$\Gamma = 2\pi \int d\beta \delta(\omega) |\langle \beta | K(t_0) |\alpha \rangle|^2,$$

where $\beta$ represents a final state. From the unitarity relation at an arbitrary $\delta t$,

$$|\langle \alpha | U(t, t_0) |\alpha \rangle|^2 = 1 - \int_{\beta \neq \alpha} d\beta |\langle \beta | U(t, t_0) |\alpha \rangle|^2.$$

(75)

For a small $|K(t, t_0)|$,

$$|\langle \alpha | U(t, t_0) |\alpha \rangle|^2 = 1 - \int_{\beta \neq \alpha} d\beta |\langle \beta | U(t, t_0) |\alpha \rangle|^2 = 1 - \delta t \Gamma.$$

(76)
For $1 - \delta t \Gamma < 0$, Eq.(76) is invalid, and higher order terms are added. For $|\langle \alpha|U(t, t_0)|\alpha \rangle| \gg |\langle \beta|U(t, t_0)|\alpha \rangle|$ for $\beta \neq \alpha$, ignoring the off-diagonal matrix element for a positive small $t' - t$,

$$|\langle \alpha|U(t', t_0)|\alpha \rangle|^2 = |\langle \alpha|U(t', t)U(t, t_0)|\alpha \rangle|^2 = |\langle \alpha|U(t, t_0)|\alpha \rangle|^2(1 - (t' - t)\Gamma). \quad (77)$$

Integrating over the time difference for a large $T$, it follows that

$$|\langle \alpha|U(t + T, t)|\alpha \rangle|^2 = e^{-T\Gamma}, \quad (78)$$

$$\sum_{\beta \neq \alpha} |\langle \beta|U(t + T, t)|\alpha \rangle|^2 = 1 - e^{-T\Gamma}.$$ 

This is equivalent to Weisskopf- Wigner formula [41].

2. $P^{(d)}$: $T$-independent norm

For the interaction Eq. (C1),

$$|\langle \beta|H_{int}|\alpha \rangle|_{\omega=0} = 0. \quad (79)$$

The transition occurs to the states of $\omega \neq 0$, and the probability varies rapidly in small $t$ from Eq.(66). For a small $|K(t, t_0)|$, a variation of the initial state is negligible, and

$$\int_{\omega \neq 0} d\beta |\langle \beta|U(t, t_0)|\alpha \rangle|^2 = P^{(d)}, \quad (80)$$

$$|\langle \alpha|U(t, t_0)|\alpha \rangle|^2 = 1 - P^{(d)}.$$ 

For $1 - P^{(d)} < 0$, which occurs at large $K(t, t_0)$, the higher order effect of the $t$-independent $P^{(d)}$ is added. That becomes to $1 - P^{(d)} + (-P^{(d)})^2 + \cdots$, and

$$\int_{\omega \neq 0} d\beta |\langle \beta|U(t, t_0)|\alpha \rangle|^2 = \frac{P^{(d)}}{1 + P^{(d)}}, \quad (81)$$

$$|\langle \alpha|U(t, t_0)|\alpha \rangle|^2 = \frac{1}{1 + P^{(d)}}.$$ 

The initial state $|\alpha \rangle$ and the final state $|\beta \rangle$ co-exist at later times. This feature that the initial norm decreases with $P^{(d)}$ substantially arose also in the the decay $0^- \rightarrow l + \nu$ of violating the helicity suppression [12, 13] and in the decay $1^+ \rightarrow 2\gamma$ of violating the Landau-Yang’s theorem [14]. $\Gamma$ is suppressed, but $P^{(d)}$ is not suppressed. The rapid change in an extremely small $T$ region is a feature of $P^{(d)}$. 

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3. \( \Gamma \) and \( P^{(d)} \)

For general case of \( P^{(d)} \gg \Gamma T \),

\[
|\langle \alpha | U(T + t, t) | \alpha \rangle |^2 = \frac{1}{1 + P^{(d)}} e^{-\Gamma T},
\]

\[
\sum_{\beta, \omega \approx 0} |\langle \beta^{(n)} | U(T + t, t) | \alpha \rangle |^2 = \frac{1}{1 + P^{(d)}} (1 - e^{-\Gamma T}),
\]

\[
\sum_{\beta, \omega \neq 0} |\langle \beta^{(d)} | U(T + t, t) | \alpha \rangle |^2 = \frac{P^{(d)}}{1 + P^{(d)}},
\]

where \( |\beta^{(n)}\rangle \) and \( |\beta^{(d)}\rangle \) are the kinetic-energy-conserving and non-conserving states. At a large \( \Gamma T \), the initial state disappears, and the final states include the states of conserving kinetic energy and those of non-conserving it. The former has the kinetic energy \( E_\alpha \) and the latter is assumed \( E_\beta \approx \frac{E_\alpha}{r_{ke}}, \) \( \) where \( r_{ke} > 1 \). In that case, the ratio of the visible energy over the total energy is

\[
r_{\text{visible}} = \frac{E_{\text{visible}}}{E_{\text{total}}} = \frac{1}{1 + P^{(d)}} \left( \frac{P^{(d)}}{r_{ke}} + 1 \right),
\]

which becomes the unity at \( P^{(d)} = 0 \), and agrees with \( \frac{1}{r_{ke}} \) at \( P^{(d)} = \infty \).

F. Higher order effects: renormalization

In higher order corrections with respect to the coupling constant, the integration are made over more than two space-time positions. A boundary term of the integral of the second variable, which is proportional to \( \frac{1}{\omega} \), appears at large momenta of the intermediate states. The phase factor \( \Phi \) which depends on the energy- momentum and the spatial position of the intermediate states remains in the integrand, and the integrals over these variables for the loop diagrams diverge. The divergence due to the intermediate states for the wave packets is equivalent to that for the plane waves from the completeness. These divergences caused by the fluctuations of the infinite momentum arise in the extremely short-distance region, and are subtracted by the local counter terms in the Lagrangian Eq.(29). Thus the ultraviolet divergences due to the intermediate states are subtracted and both of \( \Gamma \) and \( P^{(d)} \) are expressed by the renormalized quantities.
IV. QUANTUM ELECTRO-DYNAMIS (QED)

The wave functions of QED at a finite time interval are studied with the wave packets. The vacuum and one particle states are equivalent to those in ASI, but a new component of overlapping waves of finite interaction energy appears in two particle states.

The QED Lagrangian

\[ L = \bar{\psi}(x)(\gamma^\mu - m_e)\psi(x) - e\bar{\psi}(x)\gamma^\mu A_\mu(x) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \]  

(84)
is written with the renormalized fields,

\[ \psi(x) = Z_2^{1/2}\psi_r(x), \quad \bar{\psi}(x) = Z_2^{1/2}\bar{\psi}_r(x), \quad A_\mu(x) = Z_3^{1/2}A_\mu(x)_r \]  

(85)

and decomposed into the free parts and the interaction parts

\[ L = \bar{\psi}^r(x)(\gamma^\mu - m_e^r)\psi^r(x) - \frac{1}{4}F_{\mu\nu}^rF^{\mu\nu} + Z_2\delta m\bar{\psi}^r(x)\psi^r(x) \]

\[ + (Z_2 - 1)\bar{\psi}^r(x)(\gamma^\mu - m_e^r)\psi^r(x) - eZ_2Z_3^{1/2}\bar{\psi}^r(x)\gamma^\mu A^r_\mu(x) - \frac{1}{4}(Z_3 - 1)F_{\mu\nu}^rF^{\mu\nu}. \]

(86)

In the above equations, \( m_e^r \) and \( eZ_2Z_3^{1/2} \) are the physical mass and charge of the electron and \( Z_2 - 1, Z_3 - 1, \) and \( \delta m \) are order \( e^2 \) and higher orders. The wave functions in a tree level, i.e., the lowest order in \( \hbar \), is studied in this section by substituting the free part \( H_0 \) and the interaction part \( H_{int} \) to the solution Eq.(32). Connections with the renormalization procedure will be given in a latter section.

Hereafter \( \psi(x) \) and \( A_\mu(x) \) express the renormalized fields. These fields in the interaction representation are expanded with the wave packets,

\[ \psi(x) = \sum_l \int d\chi(U(\vec{x}; \chi, l)B(\chi, l) + V^\dagger(\vec{x}; \chi, l)D^\dagger(\chi, l)), \]  

(87)

\[ A_\mu(x) = \sum_s \int d\chi(E_\mu(\vec{x}; \chi, s)A(\chi, s) + E^*_\mu(\vec{x}; \chi, s)A^\dagger(\chi, s)), \]

with spinors and polarization vectors

\[ U(\vec{x}; \chi, l) = \int d\vec{k}\rho_e(\vec{k})e^{i\vec{k}(\vec{x} - \vec{x}_e)}e^{-iE_e(\vec{k})(t-t_0) - \frac{\vec{k}^2}{2m_e}}u(\vec{k}, l), \]  

(88)

\[ V^\dagger(\vec{x}; \chi, l) = \int d\vec{k}\rho_e(\vec{k})e^{i\vec{k}(\vec{x} - \vec{x}_e)}e^{-iE_e(\vec{k})(t-t_0) - \frac{\vec{k}^2}{2m_e}}v^\dagger(\vec{k}, l), \]

\[ E_\mu(\vec{x}; \chi, s) = \int d\vec{k}\rho_\gamma(\vec{k})e^{i\vec{k}(\vec{x} - \vec{x}_\gamma)}e^{-iE_\gamma(\vec{k})(t-t_0) - \frac{\vec{k}^2}{2m_\gamma}}e_\mu(\vec{k}, s), \]
where \( \rho_\gamma(k) = \left( \frac{1}{2E_\gamma(k)(2\pi)^3} \right)^{\frac{1}{2}} \) and \( \rho_e(k) = \left( \frac{m_e}{E_e(k)(2\pi)^3} \right)^{\frac{1}{2}} \). \( \sigma_\gamma \) is the range in space covered by the nucleus or atomic wave function that the photon interacts with, and \( \sigma_e \) is that for the electron, which are estimated later.

A. Vacuum and one-body states

The Schrödinger equation Eq.(31) for QED is expressed by

\[
H_{\text{int}} = e \int d\vec{x} \bar{\psi}(x) \gamma_\mu \psi(x) A^\mu(x),
\]

in which Eq.(87) is substituted, and is solved with the unitary operator \( U(t,t_0) \) of Eq.(32).

1. Stationary states: vacuum and one particle states

The solution satisfying the Schroedinger equation and the initial condition \( |0⟩ \), which satisfies \( B(\chi, l)|0⟩ = D(\chi, l)|0⟩ = A(\chi, l)|0⟩ = 0 \) at \( t = 0 \) is,

\[
|\Psi_0(T)⟩ = (N_0(T) + \int d^3\chi N_3(\chi, T) A^\dagger(\chi_1, s_1) B^\dagger(\chi_2, s_2) D^\dagger(\chi_3, s_3))|0⟩,
\]

where the second term shows the state of the energy gap \( E_g = 2m_e \). As is shown in Appendix B, the coefficients are given by

\[
N_3(\chi, T) = 0, \quad N_0(T) = 1.
\]

\( \Gamma = 0, \quad P^{(d)} = 0, \) and the vacuum is stable.

An electron and a positron have the mass \( m_e \), and a photon has no mass. Due to \( U(1) \) symmetry, the electric charge is conserved, and an electron and a positron do not decay. A massless photon does not decay to a pair of massive electron and positron. Thus one particle states are stable and stationary.

B. Electron and photon : two particle state

For an initial state of an electron and a photon of an energy \( E_{e_1} + E_{\gamma_1} \), final states of \( \Delta E = 0 \) exist and \( \Gamma \neq 0 \). If these waves overlap long, they have a finite interaction energy
as in the decay, Eq.(63). The wave component appears and gives $P^{(d)}$ in the scattering probability. For a two-particle state of the electron and photon at $t = 0$

$$|\Psi(0)\rangle = A^\dagger(\zeta_1, s_{\gamma_1})B^\dagger(\chi_2, s_{\epsilon_1})|0\rangle, \zeta_1 = (\vec{P}_{\gamma_1}, \vec{X}_{\gamma_1}), \chi_2 = (\vec{P}_{\epsilon_2}, \vec{X}_{\epsilon_2}),$$

the matrix element of the operator $U(T, 0)$ in Eq.(32) in the second order in $\epsilon$, is expressed by,

$$U(T, 0) = \frac{1}{(i\hbar)^2} \int_0^T dt_1 \int_0^{t_1} dt_2 F,$$

$$F = \langle e(\chi_2), \gamma(\zeta_2)|H_{int}(t_1)|H_{int}(t_2)|e(\chi_1), \gamma(\zeta_1)\rangle$$

$$= \langle e(\chi_2), \gamma(\zeta_2)|H_{int}(t_1)|e(\chi_3)\rangle \langle e(\chi_3)|H_{int}(t_2)|e(\chi_1), \gamma(\zeta_1)\rangle$$

$$+ \langle e(\chi_2), \gamma(\zeta_2)|H_{int}(t_1)|e(\chi_3), \gamma(\zeta_1)\rangle \langle \gamma(\zeta_1), \gamma(\zeta_2)|e(\chi_3)|H_{int}(t_2)|e(\chi_1), \gamma(\zeta_1)\rangle.$$

The electron state $|e(\chi_3)\rangle$ is an intermediate state and summed over. In the case of plane waves, time-dependent exponential factors in the $t_1$ and $t_1 - t_2$ are,

$$\int_0^T dt_1 e^{i\delta E_{\text{total}}t_1} \int_0^T dt_1 (t_1 - t_2) e^{i\delta E_{\text{rel}}(t_1 - t_2)}, \delta E_{\text{total}} = E_{\gamma_1} + E_{\epsilon_1} - E_{\gamma_2} - E_{\epsilon_2},$$

$$\delta E_{\text{rel}} = E_{\gamma_1} + E_{\epsilon_1} - E_{\epsilon_3}, \delta E_{\text{rel}}' = E_{\epsilon_1} - E_{\epsilon_3} - E_{\gamma_2},$$

where $E_{\text{rel}}$ and $\delta E'_{\text{rel}}$ correspond to the first and the second terms in Eq.(93) of Fig.2 and Fig.3. The probability from the boundary in time converges as in the scalar decay of Appendix B from the energy denominators $\frac{1}{\delta E_{\text{rel}}\delta E'_{\text{rel}}}$. Neither $\delta E_{\text{rel}} = 0$ nor $\delta E'_{\text{rel}} = 0$ has solution for $E_{\epsilon_1}$, and the intermediate state lives a short period. Because the deviations of these integrals at a finite $T$ from those of $T = \infty$ are of order $e^{-\frac{T^2}{\sigma t}}, \sigma t \neq 0$, they can be evaluated with $T = \infty$ or with ASI. Accordingly, the effective interaction for an electron

FIG. 2. One of the diagrams of the Thomson scattering. Solid line shows the electron and wavy line shows the photon.
FIG. 3. One of diagrams of the Thomson scattering. Solid line shows the electron and wavy line shows the photon.

moving in the time-like direction is expressed by

$$S_{\text{int}} = -i \frac{1}{2!} e^2 \int d^4 x d^4 y \psi(x) \gamma_\mu S_F(x - y) \gamma_\nu \psi(y) A^\mu_o(x) A^\nu_i(y),$$  \hspace{1cm} (95)

where $S_F(x - y)$ is the Feynman propagator, and $A^\mu_o(x)$ expresses the out-going photon and $A^\nu_i(y)$ expresses the in-coming photon. Adding the term where the two photons are interchanged, and substituting $\int d^4(x - y) S_F(x - y) = i \frac{e}{m_e}$, the effective interaction in the long-distance region is

$$S_{\text{int}} = \frac{e^2}{2m_e} \int d^4 x \psi(x) g_{\mu\nu} \psi(x) A^\mu(x) A^\nu(x).$$  \hspace{1cm} (96)

Thus the integral over time in the amplitude

$$U(T, 0) = -i \frac{e^2}{2m_e} \int_0^T dt d\vec{x}_1 \langle e(\chi_2) | \psi(x) \psi(x) | e(\chi_1) \rangle \langle e(\zeta_2) | A_\mu(x) A^\nu(x) | e(\zeta_1) \rangle,$$  \hspace{1cm} (97)

is almost equivalent to that of the scalar theory, and composed of the bulk and boundary terms as in Appendix A. $\sigma_s$, $\sigma_t$, $\vec{x}_0(t)$, and $T_0$ are expressed with the parameters of the one particle states as in Appendix A.

1. **Integration over time in bulk**

The integration over the time $t_1$ in the bulk $T_i + \sqrt{2\sigma_t} \leq T_0^r \leq T_f - \sqrt{2\sigma_t}$ is made with the Gaussian integral and is proportional to $e^{-\frac{\delta\omega^2}{2\sigma_t}}$. Thus the amplitude decreases rapidly with $\delta\omega$ and is equivalent to the golden rule term.
2. Integration over time at boundary

The integration in the boundary $T_0^r \leq T_i + \sqrt{2\sigma_t}$ and $T_f - \sqrt{2\sigma_t} \leq T_0^r$ is inversely proportional to $\delta \omega$ and behaves differently from the bulk.

V. THOMSON SCATTERING

The new component of the wave functions leads a new component to the transition probability of the Thomson scattering. The low energy theorem, \[42\] holds, nevertheless. Various cases which are classified by a number of small or large wave packets, $n_s$ and $n_l$ denoted as $(n_s, n_l)$, appear in nature. The formula for $(4, 0)$ case is presented first, and others are next. In ground laboratory, wave packets are small and $(4, 0)$ is applied normally, but in nature others are realized often. Scattering for the time interval $T$ of $(1, 3)$ case shows distinctive properties of observable and is studied in details.

A. Four small wave-packet; $(4, 0)$ case

For the wave packets of photons and electrons, the transition amplitude for the time interval $T$ is given by

$$\mathcal{M} = -i \frac{e^2}{2m_e} \int_0^T dt \int d^3x \langle \vec{p}_{e_2}, s_{e_2} | J(x) | \vec{p}_{e_1}, s_{e_1} \rangle \langle \vec{k}_{\gamma_2}, \vec{X}_{\gamma_2}, s_{\gamma_2}, T_{\gamma_2} | A_\mu(x) A^\mu(x) | \vec{k}_{\gamma_1}, s_{\gamma_1} \rangle,$$\hspace{1cm}(98)$$

$$\langle \vec{k}_{\gamma}, \vec{X}_{\gamma}, s_{\gamma}, T_{\gamma} | A_\mu(x) | 0 \rangle = N_{\gamma} \int d\vec{k}_2 \rho_{\gamma}(\vec{k}_2) e^{-\frac{i}{2}((\vec{k}_2 - \vec{k}_{\gamma_2})^2 + i(E(\vec{k}_2)T_{\gamma_2} - \vec{k}_2 \cdot (\vec{x} - \vec{X}_{\gamma_2}))} k_\mu e^\mu(\vec{k}_2, s_{\gamma_2}),$$

$$\langle \vec{k}_{\gamma_1}, s_{\gamma_1} | A_\mu(x) | 0 \rangle = \rho_{\gamma}(\vec{k}_{\gamma_1}) e^\mu(\vec{k}_{\gamma_1}, s_{\gamma_1}) e^{i(E(\vec{k}_{\gamma_1})t - \vec{k}_{\gamma_1} \cdot \vec{x})},$$

$$\langle \vec{p}_{e_2}, s_{e_2} | J(x) | \vec{p}_{e_1}, s_{e_1} \rangle = (2\pi)^\frac{3}{2} \rho_{\gamma}(\vec{p}_{e_1}) \rho_{\gamma}(\vec{p}_{e_2}) \bar{u}_{\gamma}(\vec{p}_{e_1}, s_{e_1}) u(\vec{p}_{e_2}, s_{e_2}) e^{-i(E_{e_1} - E_{e_2})t - (\vec{p}_{e_1} - \vec{p}_{e_2}) \cdot \vec{x}}.$$

The initial state is normalized, and the numerical constant in the last equation has a coefficient $(2\pi)^\frac{3}{2}$, and

$$k_\mu e^\mu(k) = 0, N_{\gamma} = \left(\frac{\sigma_{\gamma}}{\pi}\right)^{\frac{3}{2}}.$$

Let

$$\lambda = (t - T_{\gamma})^2 - (\vec{x} - \vec{X}_{\gamma})^2,$$

$$\xi(x) = \frac{1}{2\sigma_{\gamma}}((\vec{x} - \vec{X}_{\gamma})_L - \vec{v}_{\gamma}(t - T_{\gamma}))^2 + \frac{1}{2\sigma_T}(\vec{x} - \vec{X}_{\gamma})^2,$$
where \( \sigma^\gamma \) for the longitudinal \( L \) and the transverse \( T \) directions are given by

\[
\sigma^L = \sigma_\gamma, \\
\sigma^T = \sigma_\gamma - \frac{i}{E_\gamma}(t - T_\gamma).
\] (101)

Substituting the integral over \( \vec{k}_2 \) of Eq.(110) \[14\], \( \mathcal{M} \) is written as

\[
\mathcal{M} = -i \frac{e^2}{2m_e} N \int_0^T dt \int d^3x e^{-i(p_{e_1} - p_{e_2} + k_{\gamma_1}) \cdot x} e^{i(E_{\gamma_2}(t - T_{\gamma_2}) - \vec{k}_{\gamma_2} \cdot (\vec{x} - \vec{X}_{\gamma_2})) - \xi(x)} \mathcal{T},
\] (102)

\[
\mathcal{T} = \bar{u}(p_{e_2}) u(p_{e_1}) \epsilon^\mu(\vec{k}_{\gamma_1}) \epsilon_\mu(\vec{k}_{\gamma_2} + \delta k(x)),
\]

\[
N = \left( \frac{(2\pi)^3}{\sigma_\gamma \sigma^T} \right)^{1/2} N_{\gamma_1}(2\pi)^{3/2} \rho_\gamma(\vec{k}_{\gamma_1}) \rho_\gamma(\vec{k}_{\gamma_2} + \delta \vec{k}) \frac{1}{(2\pi)^{3/2}} \left( \frac{m_e^2}{E_{e_1} E_{e_2}} \right)^{1/2},
\]

where \( \delta k(x)^\mu \) represents corrections due to the expansion of the wave packet and in the leading order in \( \frac{1}{\sigma_\gamma} \),

\[
\delta k(x)^\mu = 0, \sigma_T = \sigma_\gamma.
\] (103)

In this case,

\[
\mathcal{M} = -i \frac{e^2}{2m_e} N \mathcal{T} I,
\] (104)

\[
I = \int_0^T dt \int d^3x e^{-i(p_{e_1} - p_{e_2} + k_{\gamma_1}) \cdot x} e^{i(E_{\gamma_2}(t - T_{\gamma_2}) - \vec{k}_{\gamma_2} \cdot (\vec{x} - \vec{X}_{\gamma_2})) - \xi(x)}.
\] (105)

where \( I \) agrees with \( I_{\text{bulk}} \) of Eq.(A9) or \( I_{\text{boundary}} \) of Eq.(A10) depending on the central time \( T_0^r \).

The probability is expressed by

\[
P(T) = \frac{1}{V} \int d\vec{X}_{\gamma_2} d\vec{k}_{\gamma_2} d\vec{p}_{e_2} \sum_{\text{spin}} |\mathcal{M}|^2,
\] (106)

where the normalization volume \( V \) for the initial state expressed by the wave packet size, \[13\], satisfies

\[
\frac{1}{V} \int d\vec{X}_{\gamma_2} = 1.
\] (107)

The transition probability is finite and the integrations over the space-time position \( (t, \vec{x}) \) and of the momentum of the final state can be interchanged. Thus the transition probability is reduced to

\[
\int d\vec{p}_{e_2} \sum_{\text{spin}} |\mathcal{M}|^2 = (e^2)^2 N^2 \int d^4x_1 d^4x_2 \int d\vec{p}_{e_2} e^{-i(p_{e_1} - p_{e_2} + k_{\gamma_1}) \cdot (x_1 - x_2)}
\]

\[
\times e^{i(E_{\gamma_2}(t_1 - t_2) - \vec{k}_{\gamma_2} \cdot (\vec{x}_1 - \vec{X}_{\gamma_2})) - \xi(x_1) - \xi(x_2)} \sum_{\text{spin}} |\mathcal{T}|^2.
\] (108)
The sum over the final spin and the average over the initial spin
\[ \frac{1}{4} \sum_{\text{spin}} |T|^2 = 2, \] (109)
is substituted. The integral over times in Eq. (118) consists of the short-range term derived from the region of \(|t_1 - t_2| \approx 0\) and the long-range one derived from large \(|t_1 - t_2|\). The former gives \(\Gamma T\) and the latter gives \(P^{(d)}\).

B. QCS scattering; (1,3) case

The transition amplitude for the time interval \(T\) is given by
\[ \mathcal{M} = -i \frac{e^2}{2m_e} \int_0^T dt \int d^3x \langle \vec{p}_{e_2}, s_{e_2} | J(x) | \vec{p}_{e_1}, s_{e_1} \rangle \langle \vec{k}_{\gamma_2}, \vec{X}_{\gamma_2}, s_{\gamma_2}, T_{\gamma_2} | A_\mu(x) A^{\mu}(x) | \vec{k}_{\gamma_1}, s_{\gamma_1} \rangle, \] (110)
\[ \langle \vec{k}_{\gamma_1}, \vec{X}_{\gamma_1}, s_{\gamma_1}, T_{\gamma_1} | A^{\mu}(x) | 0 \rangle = N_{\gamma} \int d\vec{k}_{2} \rho_{\gamma}(\vec{k}_{2}) e^{-\frac{2i}{\gamma} (\vec{k}_{2} - \vec{k}_{e_2})^2 + i(E(\vec{k}_{2}) - t_{\gamma_2} - \vec{k}_{2} \cdot (\vec{x} - \vec{X}_{\gamma_2}))} \epsilon^{\mu}(\vec{k}_{\gamma_2}, s_{\gamma_2}), \]
\[ \langle \vec{k}_{\gamma_1}, s_{\gamma_1} | A^{\mu}(x) | 0 \rangle = \rho_{\gamma}(\vec{k}_{\gamma_1}) \epsilon^{\mu}(\vec{k}_{\gamma_1}, s_{\gamma_1}) e^{i(E(\vec{k}_{\gamma_1}) - E(\vec{k}_{\gamma_1}))}, \]
\[ \langle \vec{p}_{e_2}, s_{e_2} | J(x) | \vec{p}_{e_1}, s_{e_1} \rangle = (2\pi)^{\frac{3}{2}} \rho_{e}(\vec{p}_{e_1}) \rho_{e}(\vec{p}_{e_2}) \tilde{u}_{e}(\vec{p}_{e_1}, s_{e_1}) u(\vec{p}_{e_2}, s_{e_2}) e^{-i(E_{e_1} - E_{e_2}) - (\vec{p}_{e_1} - \vec{p}_{e_2} - \vec{x})}. \]

The initial state is normalized, and the numerical constant in the last equation has a coefficient \((2\pi)^{\frac{3}{2}}\), and
\[ k_{\mu} e^{\mu}(k) = 0, N_{\gamma} = \left(\frac{\sigma_{\gamma}}{\pi}\right)^{\frac{3}{4}}. \] (111)
Let
\[ \lambda = (t - T_{\gamma})^2 - (\vec{x} - \vec{X}_{\gamma})^2, \] (112)
\[ \xi(x) = \frac{1}{2\sigma_{\gamma}}((\vec{x} - \vec{X}_{\gamma}) - \tilde{v}_{\gamma}(t - T_{\gamma}))^2 + \frac{1}{2\sigma_{T}}(\vec{x} - \vec{X}_{\gamma})^2, \]
where \(\sigma^L_{\gamma}\) for the longitudinal \(L\) and the transverse \(T\) directions are given by
\[ \sigma^L_{\gamma} = \sigma_{\gamma}, \] (113)
\[ \sigma^T_{\gamma} = \sigma_{\gamma} - \frac{i}{E_{\gamma}}(t - T_{\gamma}). \]
Substituting the integral over \(\vec{k}_2\) of Eq. (110) into \(\mathcal{M}\), \(\mathcal{M}\) is written as
\[ \mathcal{M} = -i \frac{e^2}{2m_e} N \int_0^T dt \int d^3x e^{-i(p_{e_1} - p_{e_2} + k_{\gamma_1}) \cdot x} e^{i(E_{e_2} (t - T_{\gamma_2}) - \vec{k}_{\gamma_2} \cdot (\vec{x} - \vec{X}_{\gamma_2}))} \xi(x) \mathcal{T}, \] (114)
\[ \mathcal{T} = \tilde{u}_{e}(p_{e_1}) u_{e}(p_{e_2}) e^{\mu}(\vec{k}_{\gamma_1}) \epsilon_{\mu}(\vec{k}_{\gamma_2} + \delta k(x)), \]
\[ N = \left(\frac{(2\pi)^{\frac{3}{2}}}{\sigma_{\gamma} \sigma_{T}^{\frac{3}{2}}}\right)^{\frac{1}{2}} (2\pi)^{\frac{3}{2}} \rho_{\gamma}(\vec{k}_{\gamma_1}) \rho_{\gamma}(\vec{k}_{\gamma_2} + \delta k) \frac{1}{(2\pi)^{\frac{3}{2}}} \left(\frac{m_e^2}{E_{e_1} E_{e_2}}\right)^{\frac{1}{2}}, \]
where $\delta k(x)^\mu$ represents corrections due to the expansion of the wave packet and in the leading order in $1/\sigma_\gamma$,

$$\delta k(x)^\mu = 0, \sigma_T = \sigma_\gamma. \quad (115)$$

In this case,

$$\mathcal{M} = -ie^2 N T I, \quad (116)$$

$$I = \int_0^T dt \int d^3 x e^{-i(p_{e_1} - p_{e_2} + k_{\gamma_1}) \cdot x} e^{i(E_{\gamma_2} (t - T_{\gamma_2}) - k_{\gamma_2} \cdot (\vec{x} - \vec{x}_{\gamma_2})) - \xi(x)}, \quad (117)$$

where $I$ agrees with $I_{\text{bulk}}$ of Eq.(A9) or $I_{\text{boundary}}$ of Eq.(A10) depending on the central time $T_r^\gamma$. The transition probability is finite and the integrations over the space-time position $(t, \vec{x})$ and of the momentum of the final state can be interchanged. Thus the transition probability is reduced to

$$\int d\vec{p}_{e_2} \sum |\mathcal{M}|^2 = (e^2)^2 N^2 \int d^4 x_1 d^4 x_2 \int d\vec{p}_{e_2} e^{-i(p_{e_1} - p_{e_2} + k_{\gamma_1}) \cdot (x_1 - x_2)} \times e^{i(E_{\gamma_2} (t_1 - t_2) - k_{\gamma_2} \cdot (\vec{x}_1 - \vec{x}_2)) - \xi(x_1) - \xi(x_2)} \sum |T|^2. \quad (118)$$

The sum over the final spin and the average over the initial spin

$$\frac{1}{4} \sum_{\text{spin}} |T|^2 = 2, \quad (119)$$

is substituted. The integral over times in Eq.(118) consists of the short-range term derived from the region of $|t_1 - t_2| \approx 0$ and the long-range one derived from large $|t_1 - t_2|$. The former gives $\Gamma T$ and the latter gives $P^{(d)}$.

The cross section in the low energy limit agrees with the classical Thomson cross section

$$\sigma_{\text{Thomson}} = \frac{8\pi}{3} r_e^2, \quad r_e = \frac{\alpha}{m_e}. \quad (120)$$

C. $P^{(d)}_\gamma$ in the (1,3) case

In (1,3) case, $\sigma_t = \infty$, and $T_1 = T$ is substituted. From Eqs.(106), (107), and (118), $\Gamma$ is derived from the term proportional to $(2\pi)^4 \delta(4)(p_{e_1} + k_{\gamma_1} - p_{e_2} - k_{\gamma_2})$, whereas $P^{(d)}_\gamma$ is derived from the states of $E_{\gamma_2} \neq E_i$. This is computed easily in the configuration space with the light-cone singularity [43] of Eq.(118) as in decays [12, 13]. After tedious calculations,
it is found that the differential probabilities with respect to $\vec{k}_{\gamma_2}$, which are integrated over the electron’s momentum, corresponding to $\Gamma_{Thom} T$ and $P_{Thom}^{(d)}$ are

$$\frac{d\Gamma_{Thom} T}{d\vec{k}_{\gamma_2}} = T \frac{1}{2E_e E_{\gamma_1} E_{\gamma_2} E_{e_2} \pi^2} \delta(E_{e_1} + E_{\gamma_1} - E_{e_2} - E_{\gamma_2})(e^2)^2, \quad (121)$$

$$\frac{dP_{Thom,\gamma}^{(d)}}{d\vec{k}_{\gamma_2}} = \sigma_{\gamma} \frac{1}{E_{e_1} E_{\gamma_1} E_{\gamma_2} (2\pi)^3} e^{4/2} T \tilde{g}(\omega_{\gamma_2} T) \theta(s - m_e^2 - 2k_{\gamma_2}(p_{e_1} + k_{\gamma_1})), \quad (122)$$

where $E_{e_2} = \sqrt{(k_{\gamma_1} + \vec{p}_{e_1} - \vec{k}_{\gamma_2})^2 + m_e^2}$, $s = (p_{e_1} + k_{\gamma_1})^2$, and $\omega_{\gamma_2} = \frac{m_e^2}{2E_{\gamma_2}}$. $m_\gamma$ is zero in the vacuum and is the photon’s effective mass determined by the plasma frequency in medium at high energy. Matter effect is described by a refractive index, $[14]$ in extreme low energy. We study the situation that the photon has an effective mass in this paper. The function $\tilde{g}(\omega_{\gamma_2} T)$ and the modified phase space in $P^{(d)}$ are almost equivalent to those in the decays Ref.$[12, 13]$. The average energy of the photon in $P^{(d)}$ is lower than that of the golden rule.

In a system of a finite electron density $n_e$, the total probability for one photon to make a transition is

$$P(total) = n_e(\Gamma_{Thom} T + P_{Thom,\gamma}^{(d)}). \quad (123)$$

The transition probability has $P^{(d)}$ in addition to $\Gamma T$, which is equivalent to the decay probability. If $P(total)$ is not small, the reduction of the photons in the initial states are included, and the total probability behaves as Eq.$[82]$.

1. Electron at rest: photon distribution

The photon distribution in the rest system of the electron, $p_{e_1} = (m_e, \vec{0})$ and the photon of momentum $k_{\gamma_1} = (E_{\gamma_1}, 0, 0, k_{\gamma_1})$ is given by the phase space

$$m_e(E_{\gamma_1} - E_{\gamma_2}) - E_{\gamma_1} E_{\gamma_2} (1 - \cos \theta) \geq 0, \quad (124)$$

where $\theta$ is the angle between the final photon and the initial photon. The integration is made over the region

$$k_{\gamma_2} \leq k_{\gamma_2}^{\max} = \frac{m_e E_{\gamma_2}}{E_{\gamma_1} (1 - \cos \theta) + m_e}. \quad (125)$$
At a small $\omega_2 T$, $\bar{g}(\omega_2 T) = \pi$, the differential probability and the total probability are proportional to $T$,

$$\frac{dP^{(d)}}{d \cos \theta} = T \left( \frac{e^2}{2m_e} \right)^2 \sigma_\gamma \frac{1}{4\pi} \frac{k_{\gamma_1}}{m_e} \left( \frac{2m_e + E_{\gamma_1} (k_{\gamma_2})_{max}^2}{2} - \frac{(k_{\gamma_2})_{max}^3}{3} \right),$$

$$P^{(d)} = \frac{\Gamma_{eff}}{T} = \left( \frac{e^2}{2m_e} \right)^2 \sigma_\gamma \frac{1}{4\pi} \frac{k_{\gamma_1}^3}{m_e} \left[ \frac{1}{6} \xi + \frac{1}{6} \xi^2 \right], \quad \xi = \frac{m_e}{m_e + 2E_{\gamma_1}}. \tag{126}$$

Depending on the size $\sigma_\gamma$, $\Gamma_{eff}$ varies,

$$\sigma_\gamma E_\gamma^2 \gg 1; \quad \Gamma_{eff} \gg \Gamma_{Thom}$$

$$\sigma_\gamma E_\gamma^2 \approx 1; \quad \Gamma_{eff} \approx \Gamma_{Thom}$$

$$\sigma_\gamma E_\gamma^2 \ll 1; \quad \Gamma_{eff} \ll \Gamma_{Thom}. \tag{128}$$

At a large $\sigma_\gamma$, $\Gamma_{eff}$ is larger than $\Gamma_{Thom}$. In the low energy limit, $E_\gamma \rightarrow 0$, the probability agrees with the classical Thomson cross section.

At a large $\omega_2 T$, $T \bar{g}(\omega_2 T) = \frac{4E_{\gamma_2}}{m_e^2}$, the differential and the total probability are

$$\frac{dP^{(d)}}{d \cos \theta} = \left( \frac{e^2}{2m_e} \right)^2 \sigma_\gamma \frac{1}{4\pi} \frac{k_{\gamma_1}}{m_e} \left( \frac{2m_e + E_{\gamma_1} (k_{\gamma_2})_{max}^3}{3} - \frac{(k_{\gamma_2})_{max}^4}{4} \right),$$

$$P^{(d)} = \left( \frac{e^2}{2m_e} \right)^2 \sigma_\gamma \frac{1}{6\pi^2} \frac{1}{m_e^2} m_e E_{\gamma_1}^3 \left[ E_{\gamma_1} + 4m_e - \frac{2m_e + E_{\gamma_1}^3 \xi^2 + E_{\gamma_1} \xi^3}{2} \right]. \tag{130}$$

Next we study the energy dependence in the region $E_{\gamma_1} \approx m_e$. From Eq.\ref{eq:124} the angle is integrated in the region,

$$1 - \cos \theta \leq \frac{m_e (E_{\gamma_1} - E_{\gamma_2})}{E_{\gamma_1} E_{\gamma_2}}. \tag{131}$$

Using a constant $C = \left( \frac{e^2}{2m_e} \right)^2 \sigma_\gamma \frac{1}{4\pi} \frac{2m_e^2}{k_{\gamma_1}^2}$, the energy spectrum is expressed as

$$\frac{dP^{(d)}}{dE_{\gamma_2}} = \begin{cases} 
CT \bar{g}(0)(E_{\gamma_1} - E_{\gamma_2}) \\
C \frac{2}{m_e^2} (E_{\gamma_1} - E_{\gamma_2}),
\end{cases} \tag{132}$$

and the average fraction of the kinetic energy $r_{ke} = \frac{E_{\gamma_2}}{E_{\gamma_1}}$ is

$$r_{ke} = \frac{E_{\gamma_2}}{E_{\gamma_1}} = \begin{cases} 
\frac{1}{2} & \text{for } P^{(d)} \text{ in the small } \omega_2 T \\
\frac{1}{3} & \text{for } P^{(d)} \text{ in the large } \omega_2 T.
\end{cases} \tag{133}$$
2. High energy electron: energy conversion from the electron to the photon

In a high energy region, the effective interaction between the electron and the photon is that of the Compton scattering. Nevertheless it is useful to know the behavior and the magnitude of those of the Thomson scattering.

For the parallel initial states,

\[ p_{e_1} = (E_{e_1}, 0, 0, p_{e_1}), \quad p_{\gamma_1} = (E_{\gamma_1}, 0, 0, E_{\gamma_1}), \quad E_{e_1} \gg E_{\gamma_1}, \]

the momentum satisfies

\[ k_{\gamma_2} \leq k_{\gamma_2}^{\text{max}} = \frac{k_{\gamma_1}(E_{e_1} - p_{e_1})}{(p_{e_1} + k_{\gamma_1})(1 - \cos \theta) + (E_{e_1} - p_{e_1})}, \]

and the probability is given, for \( \omega_{\gamma_2} T \ll 1 \), by

\[ \frac{dP^{(d)}}{dk_{\gamma_2} d\cos \theta} = A(2\pi)Tg(0)k_{\gamma_2}^2(2m_e^2 + (E_{e_1} - p_{e_1})k_{\gamma_1} - k_{\gamma_2}(E_{e_1} - p_{e_1} \cos \theta))\theta(B), \]

\[ A = \sigma_\gamma \frac{2k_{\gamma_1}}{E_{e_1}(2\pi)^3} \left( \frac{e^2}{2m_e} \right)^2, \]

\[ B = (k_{\gamma_1} - k_{\gamma_2})(E_{e_1} - p_{e_1}) - (p_{e_1} + k_{\gamma_1})k_{\gamma_2}(1 - \cos \theta), \]

and for \( \omega_{\gamma_2} T > 1 \), by

\[ \frac{dP^{(d)}}{dk_{\gamma_2} d\cos \theta} = A(2\pi) \frac{4}{m_\gamma^2} k_{\gamma_2}^2(2m_e^2 + (E_{e_1} - p_{e_1})k_{\gamma_1} - k_{\gamma_2}(E_{e_1} - p_{e_1} \cos \theta))\theta(B). \]

For the anti-parallel case,

\[ p_{e_1} = (E_{e_1}, 0, 0, p_{e_1}), \quad p_{\gamma_1} = (E_{\gamma_1}, 0, 0, -E_{\gamma_1}), \quad E_{e_1} \gg E_{\gamma_1}, \]

the momentum satisfies

\[ k_{\gamma_2} \leq k_{\gamma_2}^{\text{max}} = \frac{k_{\gamma_1}(E_{e_1} + p_{e_1})}{-(p_{e_1} + k_{\gamma_1})(1 - \cos \theta) + (E_{e_1} + p_{e_1})}, \]

and the probability is given, for \( \omega_{\gamma_2} T \ll 1 \), by

\[ \frac{dP^{(d)}}{dk_{\gamma_2} d\cos \theta} = A(2\pi)Tg(0)k_{\gamma_2}^2(2m_e^2 + (E_{e_1} + p_{e_1})k_{\gamma_1} - k_{\gamma_2}(E_{e_1} - p_{e_1} \cos \theta))\theta(B), \]

\[ B = k_{\gamma_1}(E_{e_1} + p_{e_1}) - k_{\gamma_2}(E_{e_1} + k_{\gamma_1} - (p_{e_1} - k_{\gamma_1}) \cos \theta), \]

and for \( \omega_{\gamma_2} T > 1 \), by

\[ \frac{dP^{(d)}}{dk_{\gamma_2} d\cos \theta} = A(2\pi) \frac{4}{m_\gamma^2} k_{\gamma_2}^2(2m_e^2 + (E_{e_1} - p_{e_1})k_{\gamma_1} - k_{\gamma_2}(E_{e_1} - p_{e_1} \cos \theta))\theta(B). \]

A is proportional to \( \sigma_\gamma \), which is determined not only by the microscopic quantity but also by the macroscopic quantities. The probability is enhanced in a large \( \sigma_\gamma \).
D. $P_{e(d)}$ for the electron in the (1,3) case

For the events that the electron is detected or interacts with other microscopic objects, the wave function of the electron in the final state determined by the reaction process is used. That in a solid state is normally atomic size, but is large in a dilute gas. $\Gamma_{Thom}$ for the electron is equivalent to that for the photon, but $P_{e(d)}$ is derived from the light-cone singularity of the photon. $\omega_e = \frac{m_e^2}{2E_e}$ is much larger than $\omega_\gamma$, Ref.[12, 13]. At the small $\omega_eT$, $\Gamma_{eff}$ is the same as that of the photon, if $\sigma_e = \sigma_\gamma$. At a large $\omega_eT$,

$$P_{Thom,e}^{(d)} = \frac{e^2}{2m_e^2} 4\sigma_e E_{\gamma 1} \int \frac{dK_{e2}}{2E_{e2}(2\pi)^3} (p_{e1} \cdot p_{e2} + m_e^2)T \bar{g}(\omega_{e2}T)\theta(s - m_e^2 - 2p_{e2} \cdot (p_{e1} + k_{\gamma 1})).$$

(142)

Eq. (142) is almost identical to Eq. (122), in its form. Nevertheless, the magnitude is much smaller, since the angular frequency $\omega_e$ is much larger than $\omega_\gamma$.

For the system of a photon density $n_\gamma$, the total probability $P_{e(d)}^{(total)}$ that one photon makes a transition is

$$P_{e(d)}^{(total)} = n_\gamma \times P_{e(d)}.$$  

(143)

1. electron acceleration

In the case $p_{e1} = (E_{e1}, 0, 0, p_{e1}), p_{\gamma 1} = (E_{\gamma 1}, 0, 0, E_{\gamma 1})$, the electron’s energy distribution is obtained by integrating the photon momentum in the final state first. Accordingly, the phase space and the angular velocity are changed to

$$s - m_e^2 - 2p_{e2} \cdot (p_{e1} + k_{\gamma 1}) \geq 0,$$

(144)

$$\omega_e = E_e(p, e) - p_e.$$

For $E_e \gg m_e, E_{\gamma 1} > 2p_{e1}$, the frequency is

$$\omega_{e2} = \frac{m_{e2}^2}{2p_{e2}},$$

(145)

and the probability is

$$P^{(d)} = \frac{e^2}{2m_e^2} \sigma_e \frac{8 \cdot 2k_{\gamma 1}^2}{3\pi^2} \frac{2\om_{\gamma 1}^2}{m_{e1}^2} dE_{e2} E_{e2}^2 = \frac{e^2}{2m_e^2} \sigma_e \frac{2 \cdot 2k_{\gamma 1}^2}{3\pi^2} \frac{2\om_{\gamma 1}^2}{m_{e1}^2} (E_{\gamma 1} - p_{e1}^2)^3.$$  

(146)
The probability that the electrons get the energy from the higher energy photon is determined by this $P^{(d)}$. The lower energy photon does not give the energy.

The probability $P^{(d)}$ for the photon and the electron are proportional to the square of the electron radius $\frac{e^2}{m_e}$, and to the range in space covered by the photon wave function $\sigma_\gamma$ or the electron wave function $\sigma_e$. By tuning the latter parameter, the enhancement of the probability is possible. The asymptotic values for the photon is proportional to $\frac{1}{m_\gamma^2}$ and for the electron is proportional to $\frac{1}{m_e^2}$. Eq. (146) is inversely proportional to $m_e^2$. Since the electron mass is much larger than the photon’s effective mass, $P^{(d)}$ for the electron is much smaller than that of the photon. However, that is still sizable for a large $\sigma_e$, which may be realized in dilute gas such as the atmosphere and the space.

2. The case :(2,2)

In the case that one of the initial states and one of the final states are plane waves and others have small sizes, (2,2) case, the transition amplitude is given by

$$\mathcal{M} = -ie^2\int d^4x \langle p_{e_2}|J(x)|p_{e_1}\rangle \langle \vec{k}_\gamma_2,\vec{X}_\gamma_2,T_\gamma_2|A_\mu(x)A^\mu(x)|\vec{k}_\gamma_1,\vec{X}_\gamma_1,T_1\rangle.$$ 

The probability is written with the amplitude. $T_1\bar{g}(\omega_\gamma T_1)$ is substituted instead of $T\bar{g}(\omega_\gamma T)$ in Eq.(121) and the others, where $T_1$ is determined by the initial wave packet.

3. $e + E(B) \rightarrow e + \gamma$

One photon production in the electron scattering with the electric or magnetic field has the probability $\Gamma T$ and $P^{(d)}$. The cross sections for the bremsstrahlung or the synchrotron radiation are known well. $P^{(d)}$ in the processes that the electron interacts with a macroscopic electric or magnetic field gives a scattering probability in the extreme forward direction. The range in space covered by these photon’s wave functions can be much larger than the atomic size, and $P^{(d)}$ can be enhanced over $\Gamma T$.

E. Other QED processes

Other QED processes, $e^+ + e^- \rightarrow 2\gamma$ and $\gamma + \gamma \rightarrow \gamma + \gamma$ and others are also subject of the corrections of $P^{(d)}$. $P^{(d)}$ of these processes are expressed in the same manner as that of
the Thomson scattering.

F. Magnitude of \( P^\gamma(d) \)

\( P(T) \) at a small \( T \) is mainly given by \( P^\gamma(d) \), and at a large \( T \) by \( \Gamma T \) and \( P(d) \).

1. \( \omega_\gamma T \ll 1 \)

From Eq.(128), \( P^\gamma(d) \) is proportional to \( T \) in the region \( \omega_\gamma T \ll 1 \). The effective rate defined by \( \Gamma^\gamma_{\text{eff}} = \frac{P^\gamma(d)}{T} \) is proportional to the range in space covered by the wave function of the object that the photon interacts with, \( \sigma_\gamma \), which is evaluated from its mean free path \( l_{\text{mfp}} \) as

\[
\sigma_\gamma = \pi l_{\text{mfp}}^2,
\]

(147)

and

\[
\sigma_\gamma E^2_\gamma = \pi \left( \frac{l_{\text{mfp}}}{\lambda_\gamma} \right)^2, \quad \lambda_\gamma = \frac{\hbar}{p_\gamma}.
\]

(148)

If the mean free path is longer than the wave length, \( \Gamma^\gamma_{\text{eff}} \) is larger than \( \Gamma \). The mean free path is determined by

\[
l_{\text{mfp}} = \frac{1}{n_{\text{charge}} \times \sigma_{\text{cross section}}},
\]

(149)

where the cross section \( \sigma_{\text{cross section}} \) is the Rutherford cross section for the charged particles in plasma and for the neutral atoms is \( \pi (r_{\text{atom}})^2 \), where \( r_{\text{atom}} \) is the atomic size. The mean free path is as long as \( 10^5 \) meter or longer in dilute gas, and \( \Gamma^\gamma_{\text{eff}} \) is more important than \( \Gamma \). That is not the case in liquid and solid.

2. \( \omega_\gamma T > 1 \)

In the asymptotic region of \( \omega_\gamma T \gg 1 \) and \( \Gamma T < 1 \), the probability behaves as Eq.(82). \( P^\gamma(d) \) is proportional to \( r_d = c \hbar n_e \frac{\sigma_\gamma}{m_\gamma} \), which depends on the photon’s effective mass, \( m_\gamma \). That is determined by the plasma frequency,

\[
m_\gamma = \hbar \sqrt{\frac{4 \pi \alpha N_e}{m_e}}, \quad N_e \text{ the total electron density},
\]

(150)
and is expressed as
\[ m_\gamma c^2 = 30 \sqrt{\frac{N_e}{N_e^0}} eV, N_e^0 = 10^{30} \text{m}^{-3}. \] (151)

Combining this with the \( \sigma_\gamma \), and substituting \( \hbar = 2 \times 10^{-7} \text{eV m} \),
\[ r_d = c \hbar n_e \sigma_\gamma \frac{m_\gamma}{m_e^2} = c \hbar n_e \pi l_{mfp}^2 \frac{N_e^0}{N_e} \frac{1}{900} = \frac{n_e \pi l_{mfp}^2 N_e^0}{N_e} \frac{900}{900} 2 \times 10^{-7} \text{eV m} \] (152)
is large in dilute systems such as the ionosphere, the solar corona, and the inter space. In the Ionosphere, parameters are roughly \( l = 10^7 \text{m}, n_e = 10^{10} / \text{m}^3 \), and \( N_e = 10^{15} / \text{m}^3 \), and \( r_d > 1 \).

At larger \( T \) of \( \Gamma T > 1 \), the average life-time determines the exponential behavior.

In the normal scattering of \( P^{(d)} \approx 0 \) and \( \Gamma \neq 0 \), the sum of the kinetic energies of final states agrees with the initial energy, but in the processes due to \( P^{(d)} \), the substantial portion of the final states have less kinetic energy than the initial energy. The rest of the energy is stored in the interaction energy. Thus the physical effect due to \( P^{(d)} \) remains.

G. Magnitude of \( P^{(d)}_e \)

The product \( n_\gamma \frac{\sigma_\gamma}{m_e^2} \) determines the magnitude of \( P^{(d)}_e \) at \( \omega_e T > 1 \). The electron mass is much heavier than the photon’s effective mass, and \( P^{(d)}_e \) is much smaller than \( P^{(d)}_\gamma \), if \( \sigma_e \approx \sigma_\gamma \).

VI. UNIQUE FEATURES OF QCS

In this section \( c \) and \( \hbar \) are written. Unique features of QCS were obtained using the wave packets constructed from the eigenstates of the free Hamiltonian \( H_0 \). As these wave packets form a complete set, the results are universal. The case that the initial normalized states are defined from the eigenstates of the \( H \) was studied in Section 2 H, and other cases are studied in Appendix D.

In QED, \( \Gamma_{Thom} \) is smooth in the angle and characterized by the de Broglie wave length \( \frac{\hbar}{p} \). Now, \( P^{(d)} \) peaks at \( \theta = 0 \) and the value is determined by \( \sigma_\gamma \) and \( \frac{\hbar E}{c^4 m_\gamma^2} \), which in fact is of a macroscopic magnitude for the light particles. This component is unique in the quantum mechanics and has no classical counterpart. Even in the low energy limit, the additional component \( P^{(d)}_{Thom} \) does not vanish and appears in the extreme forward direction.
A. Lorentz transformation and $P^{(d)}$

$\frac{\hbar E}{c^2 m^2}$ characterizing $P^{(d)}$ is huge and may be longer than the distance between the positions of the source and the detector, $L = cT$. $P^{(d)}$ is transformed differently under the Lorentz transformation. The length $L$ in the rest system is transformed to $\frac{L}{\sqrt{1-(v/c)^2}}$ in the moving frame of the velocity $v$. The space time positions $(t, 0, 0, z)$ is transformed to $(t', 0, 0, z')$ by the boost in z-direction,

$$(t, z) \rightarrow (t', z') = (t \cosh \xi + z \sinh \xi, t \sinh \xi + z \cosh \xi),$$

where $\cosh \eta = 1/\sqrt{1-(v/c)^2}$. Two space-time positions $X_1$ and $X_2$ of the distance $L_0$ are transformed as

$$X_1, (0, 0, 0, 0); X_2, (0, 0, 0, L_0) \rightarrow X'_1, (0, 0, 0, 0); X'_2, (L_0 \sinh \eta, 0, 0, L_0 \cosh \eta),$$

where $L$ is the transformed distance. The energy-momentum of a particle of the rest mass $m$ is transformed as

$$(m, 0, 0, 0) \rightarrow (E, 0, 0, p), E = m \cosh \eta, p = m \sinh \eta.$$  

Thus the ratio $\frac{T}{E}$ satisfies

$$\frac{T}{E} = \frac{T_0 \cosh \eta}{m \cosh \eta} = \frac{T_0}{m}. \quad (156)$$

In the region $\omega T_1 \approx 0$, $P^{(d)}$ of the decay and the Thomson scattering are expressed from Eq.(62) and (121) as

$$P^{(d)}|_{\text{scalar}} = \frac{T_1}{E_2(P)} 8g^2 \sigma_1 \tilde{g}(0) \int \frac{d^3p_1}{(2\pi)^3 E(p_1)} \theta(m_2^2 - m_1^2 - 2P \cdot p_1),$$

$$P^{(d)}|_{\text{Thom,}\gamma \times E_{e_1}} = \frac{T_1}{E_{\gamma_1}} 2e^4 \sigma_\gamma \tilde{g}(0) \int \frac{dk_2}{(2\pi)^3 E_{\gamma_2}} \theta(s - m_e^2 - 2k_{\gamma_2} \cdot (p_{e_1} + k_{\gamma_1})), \quad (157)$$

where the phase space are defined by the step functions. The first factors in the right-hand sides are the same in the moving frame from Eq.(156) and so are the integrals. Eq.(157) and Eq.(158) are manifestly Lorentz invariant.

In the region $\omega T \gg 1$,

$$P^{(d)} = 8g^2 \sigma_1 \frac{2}{m_1^2} \int \frac{d^3p_1}{(2\pi)^3 E(p_1)} \theta(m_2^2 - m_1^2 - 2P \cdot p_1),$$

$$P^{(d)}|_{\text{Thom,}\gamma \times E_{e_1}} = 2e^4 \sigma_\gamma \frac{2}{m_1^2} \int \frac{dk_2}{(2\pi)^3 E_{\gamma_2}} \theta(s - m_e^2 - 2k_{\gamma_2} \cdot (p_{e_1} + k_{\gamma_1})). \quad (159)$$

40
From Eq. (155), the right-hand sides of Eqs. (159) and (160) are invariant, despite of complicated form. This is understandable because $P^{(d)}$ gets contributions from the long distance. In general situations, $P^{(d)}$ from Eqs. (62) and (121) are more complicated functions. The variable $y = \frac{m_1^2 T_1}{2 E}$ is the invariant combination, and $P^{(d)}$ in the decay and $P^{(d)}_e \times E_e$ in the scattering are invariant. $P^{(d)}_e$ is the same.

B. QCS interaction with matter

Interactions of QCS with matter are governed by the local interaction of the fundamental fields at the same or nearly the same positions, and the transition amplitude and the probability are composed of $\Gamma T$ and $P^{(d)}$. The former has the typical scale of the de Broglie length $\frac{\hbar}{p}$, and the latter has the scale of $\frac{\hbar E}{m_c^2}$. Two particles in an event in the former have a short correlation length in time, whereas those in the latter have a long length.

1. Interaction of QCS with microscopic states

A reaction of QCS with microscopic states is analyzed for an initial state $|\Psi\rangle$ that includes QCS,

$$|\Psi\rangle = |B\rangle \otimes [ |C'\rangle + |\Psi_{QCS}(C,\gamma)\rangle].$$

(161)

where $|B\rangle$ and $|C'\rangle$ are the particle state in a detector and in the beam, and QCS composed of a particle $C$ and a photon is expressed by $|\Psi_{QCS}(C,\gamma)\rangle$. For simplicity, the interaction of $C$ with matters is assumed local and negligibly weak, and a photon interacts with $B$ in the detector by the interaction,

$$H_{\text{int}}(t) = \int d\vec{x} J^\mu(x) A_\mu(x), J^\mu = \bar{\psi}_{B'}(x) \Gamma^\mu \psi_B(x).$$

(162)

The probability amplitude for the state $|\Psi\rangle$ to be transformed to a final state $|\text{out}\rangle$ is

$$\mathcal{M} = \langle \text{out} | \int dt H_{\text{int}}(t) |\Psi\rangle, \langle \text{out} | = \langle C | \otimes \langle B' |.$$

(163)

$|C'\rangle$ and $|\Psi_{QCS}(C,\gamma)\rangle$ interact independently. Scattering of the former is described by a normal amplitude,

$$\mathcal{M} = \langle \text{out} | \int dt H_{\text{int}}(t) |B, C'\rangle,$$

(164)
which depends upon the energies of $B$, $B'$, $C$, and $C'$. That for QCS is,

$$\mathcal{M}_{QCS} = \langle \text{out} \mid \int dt H_{\text{int}}(t) \mid \Psi_{QCS}(C,\gamma), B \rangle = \langle B' \mid \int dt H_{\text{int}}(t) \mid B, \gamma \rangle \langle C \mid \Psi_{QCS} \rangle, \quad (165)$$

where the photon in QCS interacts with $B$, but others do not. That is expressed in Fig. 4. Transition of the photon is described by either $\Gamma$ or $P^{(d)}$. The former amplitude is proportional to the following integral over the time that depends on the kinetic energy,

$$\int dt e^{i(E_B+E_{\gamma}-E_{B'})t} F_{B,B'} \approx 2\pi \delta(E_B+E_{\gamma}-E_{B'}) \epsilon_{\mu} F_{B,B'}^\mu, \quad (166)$$

$$F_{B,B'}^\mu = \int d\vec{x} \langle B' \mid J^\mu(0,\vec{x}) \mid B \rangle \langle C \mid \Psi_{QCS} \rangle.$$

The kinetic energy of the photon is expressed from $E_B$ and $E_{B'}$

$$E_{\gamma} = E_{B'} - E_B. \quad (167)$$

The transition amplitude depends upon this energy, and so does the probability. The kinetic energy of $C$ and the interaction energy decouple from the amplitude, and are irrelevant to the transition. Thus the energy $E_{\gamma}$ is detectable in the experiments, and is able to transfer to other states in the natural processes, but the rest is not. For the process governed by $P^{(d)}$, the energies satisfy

$$E_{\gamma} \leq E_{B'} - E_B, \quad (168)$$
instead of Eq. (167).

If the particle $C$ has some charge that interacts with one of gauge fields, the kinetic energy of $C$ is also detectable in experiments and is able to transfer to other states. In general QCS composed of the constituent particles that interact with an atom or a nucleus, their kinetic energy is detectable, but the interaction energy decouples and is not detectable.

The fact that kinetic energy is extractable, but interaction energy is not, is equivalent to a transmission law of energy in classical mechanics. Kinetic energy of a massive body is determined by the velocity, and its transmission to others is easily made by a contact with another of lower velocity. Now potential energy is determined by the position, and is the same between two bodies in contact with each other, and does not transmit directly. To transmit potential energy, that is transferred to kinetic energy first. For instance, a potential energy of the water at a higher position is transferred first to its kinetic energy, and is transmitted next to a turbine or others. That can be used for generating an electricity in the hydro-electric generation. Similarly, interaction energy of QCS does not appear in the transition amplitude, and is neither detectable with a normal detector nor transmittable to other matter.

It is heuristic to make a comparison of QCS with a bound state (BS). QCS is loosely bound, but BS is tightly bound in a microscopic region of a finite energy gap and is a stationary state that is stable under the Poincare transformation. The matrix element of an operator $J(x)$ is written by the four dimensional momentum,

$$\langle BS; \vec{p}_1 | J(x) | BS; \vec{p}_2 \rangle = e^{i(p_2 - p_1) \cdot x} \langle BS; \vec{p}_1 | J(0) | BS; \vec{p}_2 \rangle. \quad (169)$$

This is valid for any operator and bound state that are composed of fundamental fields. Furthermore, for a Poincare covariant state, the matrix element is described by a boost operator $U(\vec{p}_1, \vec{0})$ as

$$\langle BS; \vec{p}_1 | J(0) | BS; \vec{p}_2 \rangle = \langle BS; \vec{0} | U^\dagger(\vec{p}_1, \vec{0}) J(0) U(\vec{p}_2, \vec{0}) | BS; \vec{0} \rangle. \quad (170)$$

Kinetic energy and interaction energy are not separable, and the amplitude is written with total energy. Total energy is measured in BS, whereas interaction energy is separated and decouples from the transition in QCS.

For a weakly correlated state, such as QCS, kinetic energy is separable from total energy, and is detected independently. Interaction energy decouples and is not detectable by microscopic processes. An example is Feynman's parton model for a nucleon in deep inelastic
scattering. That is expressed by three valence quarks and soft sea quarks and anti-quarks and gluons. Their kinetic energies can be probed by the photon or the weak bosons, but the interaction energy is undetectable, and so is invisible. That is like a missing energy. Interaction energy or an effect of boost that includes the interaction is non-detectable by the scatterings through the electroweak currents also.

Weakly correlated states appear in wide area. They include halos in nuclei, atoms, molecules, and other larger physical systems such as the star and galaxy. It is challenging to measure the interaction energy in these systems [44–46].

2. Interaction with a large wave in space

If \( B \) and \( B' \) are not in the normal matter, but are extended in wide area, their interaction with QCS is different from those of the previous case. Large \( P^{(d)} \) may appear in their processes, and determines the transition. Due to the positive semi-definite interaction energy, kinetic energy shows

\[
\omega = E_\gamma - E_{B'} + E_B \leq 0.
\]

(171)

\( \sigma \) is large in dilute systems, or in highly correlated quantum states such as superconductor, super-fluid, and others. A magnetic field or an electric field can be uniform in large area, and \( \sigma \) of quantum states become macroscopic size in magnitude. That may couple with QCS. A large \( P^{(d)} \) there may be detectable in laboratory or observable in natural processes.

3. Coupling with gravity

Energy momentum tensor is composed of the kinetic part and the interaction part. The latter is ignorable for uncorrelated plane waves \(^1\) but is sizable for QCS. Thus the energy momentum tensor is written as,

\[
T_{\mu\nu} = T_{\mu\nu}^{(0)} + T_{\mu\nu}^{(d)},
\]

(172)

where the first term is from the kinetic energy and the second term is from the interaction energy, which is positive from a Virial theorem in Sec.2.3. The former is equivalent to that

\(^1\) If the fields are normalized in a volume \( V \), the integral of n-th power form of the fields is proportional to \( V^{\frac{n}{2}} \), and vanishes for \( n \geq 3 \) in \( V \to \infty \). The interaction energy of QCS is in the forward angle does not follow this, and remains.
of the particle energy and the latter is proportional to the metric $g_{\mu\nu}$.

$$T^{(0)}_{\mu\nu} = T_{\mu\nu}(\text{matter}), T^{(d)}_{\mu\nu} = g_{\mu\nu}\Lambda^{(d)},$$

(173)

where $\Lambda^{(d)}$ is a constant. The tensor Eq.(172) becomes a source of the gravitational field. The positive definite energy-momentum tensor proportional to the metric $g_{\mu\nu}$ from QCS are added constructively, and those of a macroscopic number can form a macroscopic gravitational field. Consequently they affect a motion of a massive body there. Conversely the interaction energy of QCS affects the macroscopic motion of a massive object, but is not detected by the ordinary measurement that uses the standard interaction. The ratio of the two components is

$$r = \frac{T^{(0)}}{T^{(d)}} = \frac{2r_{ke}}{P^{(d)}},$$

(174)

Interaction energy gives the equivalent effects as a dark matter or a dark energy [47].

C. Fraction of the QCS and of invisible energy

The average kinetic energy of QCS in the scalar decays Eq.(62), and in the Tomson scattering Eq.(133) are studied for the case of $P \ll 1$ and of $P^{(d)} \geq 1$.

1. Fraction of QCS

For $P^{(d)}_{\gamma}(\text{total}) \ll 1$, the total probability that one photon is scattered in a gas of the electron density $n_e$, at $T$ is $P = \Gamma(\text{total})T + P^{(d)}(\text{total})$, where $\Gamma(\text{total}) = n_e\Gamma, P^{(d)}(\text{total}) = n_eP^{(d)}_{\gamma}$. Suppose that $P$ becomes the unity at a time $T_{max}$,

$$\Gamma(\text{total})T_{max} = 1 - P^{(d)}_{\gamma}(\text{total}).$$

(175)

$T_{max}$ is the mean free time of the initial state. By the time $T_{max}$, due to the scattering with the electrons all states are scattered, and transformed to the final states. The number of the final states at $T_{max}$ is proportional to those that are transferred in the time interval between $T = 0$ and $T = T_{max}$,

$$P_{\text{conserv.}} = \frac{1}{2}T_{max}(1 - P^{(d)}_{\gamma}(\text{total})),$$

$$P_{\text{non-conserv.}} = T_{max}P^{(d)}_{\gamma}(\text{total}).$$

(176)
For $P^{(d)}_{\gamma}(total) \geq 1$, the total probability reduced to

$$P = \frac{\Gamma(total)T}{1 + P^{(d)}_{\gamma}(total)} + \frac{P^{(d)}_{\gamma}(total)}{1 + P^{(d)}_{\gamma}(total)},$$

(177)

and becomes $P = 1$ at $T_{\text{max}} = \frac{1}{\Gamma(total)}$. The probabilities Eq.(176) become

$$P_{\text{conserv}} = \frac{T_{\text{max}}}{2(1 + P^{(d)}_{\gamma}(total))},$$

(178)

$$P_{\text{non-conserv.}} = \frac{T_{\text{max}}P^{(d)}_{\gamma}(total)}{1 + P^{(d)}_{\gamma}(total)}.$$

We denote a fraction of the average kinetic energy of the QCS as $r_{\text{ke}}$. That in the scalar decays Eq.(62) and the Thomson scattering Eq.(133) is,

$$r_{\text{ke}} = \begin{cases} 
\frac{1}{2}; \text{scalar decay} \\
\frac{1}{2}, \frac{2}{3}; \text{Thomson scattering.}
\end{cases}$$

(179)

The average fraction of the observed energy for $P^{(d)}_{\gamma}(total) < 1$ is

$$\frac{E_{\text{visible}}}{E_{\text{total}}} = \frac{1}{2}(1 - P^{(d)}_{\gamma}(total)) + P^{(d)}_{\gamma}(total)r_{ke},$$

(180)

and for $P^{(d)}_{\gamma}(total) \geq 1$,

$$\frac{E_{\text{visible}}}{E_{\text{total}}} = \frac{1}{2(1 + P^{(d)}_{\gamma}(total))} + \frac{P^{(d)}_{\gamma}(total)}{1 + P^{(d)}_{\gamma}(total)}r_{ke}.$$  

(181)

If $P^{(d)}_{\gamma}(total) \gg 1$, the majority of the energy is stored in the interaction energy, and is invisible with the detectors that use microscopic processes.

The statistical behaviors of the final states due to $\Gamma(total)$ is determined by Gibbs ensemble, but those due to $P^{(d)}$ is not. As is shown in Appendix E, they show behaviors of non-stationary states.

VII. CONFIRMATION OF QCS

A. Comparison with previous experiments

QCS and the probability $P^{(d)}$ have been barely paid attentions by the researchers. This is because experimental signals are wide in energy and narrow in forward direction in the configuration space. Hence identifications are hard, [48]. These signals have been considered
as backgrounds, and serious studies have been barely made. This does not mean that they do not exit. When $P^{(d)}$ becomes larger than $\Gamma T$, that is necessary for correct understandings of the phenomena.

**B. Comparison with the axiomatic field theory**

Scatterings of plane waves derived from the limit $\sigma \to \infty$ have the infinity large $P^{(d)}$. $\Gamma T$ remains finite, hence its relative weight vanishes. All the final states are QCS, and the scattering process described $P^{(d)}$, which arises in extreme forward angle, takes place. The scattering expressed by the golden rule does not occur. This is partly equivalent to the Haag theorem [31]. The theorem proved that the Lorentz invariant S-matrix is the unity, whereas the present paper showed that normal scattering disappears in the limit $\sigma \to \infty$. They are equivalent. The remaining scattering in the extreme forward angle is not invariant in the 3+1 dimension but in the 1+1 dimension.

**C. Experimental confirmation**

There are various ways to test QCS.

1. Direct observation of the missing energy.

The kinetic energy is visible by its interaction with atoms, but the interaction energy of QCS is invisible. Hence the energy measured with an ordinary detector is a partial portion, and deviates from the total energy. Observation of the missing energy and the kinetic energy transferred from the interaction energy may be able to confirm QCS.

2. Observation of the absolute probability.

Another way to test QCS experimentally is to measure the absolute transition probability $P(T)$ and $\Gamma T$. It would be hard to distinguish the events due to $P^{(d)}$ from backgrounds. If that is made, and a clear difference between two is found, the transition due to QCS will be proved.

3. Observation of the rapid transitions.

The rapid change of $P^{(d)}$ in the region $\omega \gamma T \ll 1$, is independent of the thermal effects. Furthermore, in the dilute system $P^{(d)}$ is enhanced and governs the total transition probability. It would be possible to identify these features using advanced laser technology.
(4) Interaction with gravity.

The interaction energy carried by the overlapping waves becomes a source of the gravitational field. Hence, macroscopic number of QCS is observable through their energy and momentum.

VIII. SUMMARY

The scattering theory for the finite time intervals was formulated with the wave packets and the finite probability was derived. Based on FQM, the difficulty raised by Stueckelberg for the plane waves that the probability diverges was resolved by using the normalized one-particle states. It was found that the transition probability at $T$, $P(T)$, is convergent and composed of $\Gamma T$ and $P^{(d)}$. $\Gamma T$ is derived from the bulk states and computed easily at the asymptotic space-time region under ASI, where the interaction Hamiltonian is $e^{-\epsilon|t|}H_{int}$, and vanishes at $|t| \to \infty$. The states there are composed of independent free waves. They are particle-like states, of $O_{qcs} = 0$, in the initial and final states, and the transitions among them are expressed by the average rate $\Gamma T$ from the ratios of the fluxes. $P^{(d)}$ is derived from the boundary states, which includes the wave-like solutions of the many-body Schrödinger equation, QCS. QCS is the state of the finite-interaction energy, and appears in the final states of the scattering or decaying states, and contributes to the transition probability at later times. Thus the transition probability $\Gamma T + P^{(d)}$ governs the physical phenomena.

Due to intriguing properties of $P^{(d)}$, new phenomena are expected to be observed. They reveal the rapid transition in the extreme forward angle. Although these experiments have been difficult so far, they are getting feasible.

In the Thomson scattering, QCS in the extreme forward direction and $P^{(d)}$ were computed. Because $P^{(d)}$ is proportional to the new scale $\frac{\hbar E}{m^2c^2} \sigma$, where $m$ is the effective mass of the photon, that is large. Moreover, $P^{(d)}$ behaves differently from $\Gamma$ in energy and in time. That is widely spread in kinetic energy and changes rapidly at short time and remains constant later. From these behaviors, $P^{(d)}$ has not been paid attentions. However, $P^{(d)}$ plays important roles in natural phenomena, and new experiments dedicated to $P^{(d)}$ may supply valuable information. The transition probability of the Thomson scattering is modified in intermediate energy but the low-energy theorem [42] is valid in the low energy limit. The critical energy depends on the mean free path of the charged matter and becomes extremely
low in the dilute system.

Interaction energy of QCS is not observed like kinetic energy of particles. Normally kinetic energy of the wave proportional to the frequency by the Einstein relation $E = h\nu$ transmits from the initial state to another state in the microscopic transition processes. This energy is detectable with the normal detectors, and transmittable to other systems in natural phenomena, and visible. Contrary to kinetic energy, interaction energy is stored in overlapping waves and neither transmittable nor detectable unless that is converted to kinetic energy. Hence this is invisible with microscopic processes. Nevertheless, this non-detectable energy is one part of the energy momentum tensor and becomes the source of the gravitational field and affects the macroscopic motion of a massive body. For $P^{(d)} \geq 1$, the dominant part of the energy of the final states is invisible.

$P^{(d)}$ depends upon the fundamental constants of the underlining theory, and has intriguing properties. This could have many implications, especially in dilute systems, and changes also basic understandings of the natural phenomena \[49, 50\]. $P^{(d)}$ gives effects similar to a new interaction.

**Acknowledgements.** The present work is partially supported by a Grant-in-Aid for Scientific Research (Grant No.24340043), and JSPS KAKENHI Grant Number 15H05885(J-Physics). The authors thank Dr. Takashi Kobayashi, Dr. Takasumi Maruyama, Dr. Tsuyoshi Nakaya, and Dr.Koichiro Nishikawa for useful discussions on the neutrino experiments, Dr. Asao Arai, Dr. Shoji Asai, Dr. Tomio Kobayashi, Dr. Makoto Minowa, Dr. Toshinori Mori, Dr. Sakue Yamada, Dr.Terry Sloan, Dr.Kin-ya Oda, Dr. Izumi Ojima, Mr. Hiromasa Nakatsuka, Dr. Toshiki Tajima, Dr. Kiyoshi Kuramoto, Dr. Mitsuteru Sato, Dr. Masaki Takesada, Dr. Tomobumi Mishina, Dr. Shigeto Watanabe, and Dr. Junji Watanabe for useful discussions on quantum interferences.

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Appendix A: Integration over space-time coordinates

1. Integration in the bulk and in the boundary

The amplitude that a state is transformed to other state in the lowest order of the coupling strength is the integral over the space-time coordinates of the product of the initial and final wavefunctions in the coordinates space. For a transition that the parent of an average momentum $\vec{p}_2$ at $X_2 = (T_2, \vec{X}_2)$ and the daughters of average momenta $\vec{k}_1_i$ at $X_1_i = (T_1_i, \vec{X}_1_i), i = 1, 2$

$$\mathcal{M} = \int_{T_2}^{T_1} dt \int d^3x \langle \vec{k}_1_i, \vec{k}_1_i'; \vec{X}_1_i, \vec{X}_1_i'| H_{int}(x)|\vec{p}_2, \vec{X}_2 \rangle.$$  \hspace{1cm} (A1)

Substituting

$$\langle 0|\varphi_2(x)|\vec{p}_2, X_2 \rangle = N_2 \rho_2(2\pi)^{-3/2} \frac{1}{\sigma_2} e^{-\frac{1}{2\sigma_2}((\vec{x}-\vec{X}_2)(t-T_2))^2} e^{-i(E(\vec{p}_2)(t-T_2)-\vec{p}_2(\vec{x}-\vec{X}_2))},$$

$$\langle \vec{k}_1_i, X_1_i|\varphi_1(x)|0 \rangle = N_1 \rho_1(2\pi)^{-3/2} \frac{1}{\sigma_1_i} e^{-\frac{1}{2\sigma_1_i}((\vec{x}-\vec{X}_1_i)(t-T_1_i))^2} e^{i(E(\vec{p}_1_i)(t-T_1_i)-\vec{p}_1_i(\vec{x}-\vec{X}_1_i))},$$

$$N_{2(1_i)} = \frac{\sigma_{2(1_i)}}{\pi} \sqrt{\frac{1}{2E_{2(1_i)}(2\pi)^3}}, \rho_{2(1_i)} = \frac{1}{2E_{2(1_i)}(2\pi)^3} \left( (x - X_{2(1)})^2 \right)^{1/2}, x - X_{2(1)} \geq 0, \hspace{1cm} (A2)$$

that is reduced to

$$\mathcal{M} = g N_1 \int_{T_2}^{T_1} dt \int d^3x e^{-\frac{1}{2\sigma_s}(x-x_0(t))^2 - \frac{1}{2\sigma_t}(t-T_0)^2 + i\delta\omega(t-T_0) - i\delta p \cdot \vec{v}_0 + R + i\Phi},$$ \hspace{1cm} (A3)

$$\frac{1}{\sigma_s} = \frac{1}{\sigma_2} + \frac{1}{\sigma_1_i} + \frac{1}{\sigma_{12}}, \frac{1}{\sigma_t} = \sum_j \frac{\vec{v}_j^2}{\sigma_j}, \vec{v}_0 = \sigma_s \sum \vec{v}_j, \hspace{1cm} (A4)$$

$$\delta\vec{p} = \vec{p}_2 - \vec{k}_1_i - \vec{k}_{12}, \delta E = E_2 - E_{1_i} - E_{12}, \delta\omega = \delta E - \delta p \cdot \vec{v}_0, \hspace{1cm} (A5)$$
where $\Phi$ depends on the momenta and positions of the initial and final states $|\phi\rangle$, and

$$T_0 = \sigma(t) \frac{1}{\sigma_s} \vec{v}_0 \cdot \vec{x}_0 - \sum_j \frac{1}{\sigma_j} \vec{v}_j \vec{X}_j, \quad \vec{x}_0(t) = \vec{v}_0 t + \vec{x}_0, \quad \vec{x}_0 = \sigma_s \left( \sum_l \frac{1}{\sigma_l} \vec{X}_l \right),$$

(A6)

$$R = - \sum_j \frac{1}{2\sigma_j} \vec{X}_j^2 + 2\sigma_s \left( \sum_j \frac{1}{2\sigma_j} \vec{X}_j^2 \right) + 2\sigma_i \left( \frac{\vec{v}_0}{\sigma_s} - \frac{\vec{v}_j}{\sigma_j} \right) \cdot \vec{X}_j^2,$$

(A7)

$$N_1 = i \left( (4\pi)^3 \sigma_1 \sigma_2 \sigma_3 \right)^{-\frac{3}{2}} (E_1, E_2, E_3)^{-\frac{3}{2}} \vec{X}_j - \vec{v}_j T_j.$$

(A8)

In Eq. (A3), the variable $\vec{x}$ is unlimited and the integration over the space is constant irrespective of the center position. On the other hand, $t$ has an upper and lower bounds, and the integral varies depending on the central position $T_0$. For $T_0 < 0 - \sqrt{2\sigma_t}$ and $T + \sqrt{2\sigma_t} < T_0$ that is negligibly small, but for $0 + \sqrt{2\sigma_t} \ll T_0 \ll T - \sqrt{2\sigma_t}$, and $0 - \sqrt{2\sigma_t} < T_0 < 0 + \sqrt{2\sigma_t}, T - \sqrt{2\sigma_t} < T_0 < T + \sqrt{2\sigma_t}$ that is sizable. In the former, the central position is inside the bulk region, and the integral over $t$ is equivalent to that of the region $-\infty \leq t \leq \infty$,

$$I_{\text{bulk}} = \int_0^T dt \int d\vec{x} e^{-\frac{1}{2\sigma_t}(t-T_0)^2 - \frac{1}{2\sigma_s}(\vec{x} - \vec{x}_0)^2} e^{i\delta\omega (t-T_0) - i\vec{\delta p} \cdot (\vec{x} - \vec{x}_0)} = (2\pi)^2 (\sigma_t \sigma_s)^{3/2} e^{-\frac{\sigma_t}{2} (\delta\omega)^2} - \frac{\sigma_s}{2} (\delta p)^2,$$

(A9)

$I_{\text{bulk}}$ decreases exponentially with $|\delta\omega|$ and $|\delta p|$. In the latter, the central position is at the boundaries, and the integral is approximately

$$I_{\text{boundary}} = \int_{T_1}^{T_2} dt \int d\vec{x} e^{-\frac{1}{2\sigma_t}(t-T_0)^2 - \frac{1}{2\sigma_s}(\vec{x} - \vec{x}_0)^2} e^{i\delta\omega (t-T_0) - i\vec{\delta p} \cdot (\vec{x} - \vec{x}_0)}$$

(A10)

$$= \int_0^\infty dt \int d\vec{x} e^{-\frac{1}{2\sigma_t} t^2 - \frac{1}{2\sigma_s} (\vec{x} - \vec{x}_0)^2 + i\delta\omega t - i\vec{\delta p} \cdot \vec{x}}$$

(A11)

$$= \left\{ \begin{array}{ll}
(2\pi)^3/2 (\sigma_s^3)^{1/2} e^{-\frac{\sigma_s}{2} (\delta p)^2}; & \delta\omega \to \infty.
\end{array} \right.$$

$I_{\text{boundary}}$ decreases inversely proportional to $|\delta\omega|$, which gives a large contribution from the large $|\delta\omega|$ region. A number of states of different positions $\vec{X}$ in the former is proportional to the length in temporal direction $T$, whereas that of the latter is independent of $T$. The former corresponds to $\Gamma T$ and the latter corresponds to $P^{(d)}$.

$I_{\text{bulk}}(\delta\omega)$ decreases fast with $\delta\omega$, but $I_{\text{boundary}}(\delta\omega)$ decreases slowly, and their square are given by

$$|I_{\text{bulk}}(\delta\omega)|^2 = (2\pi)^4 (\sigma_1 \sigma_2 \sigma_3)^{3/2} e^{-\frac{\sigma_s}{2} (\delta p)^2}$$

(A12)

$$|I_{\text{boundary}}(\delta\omega; +)|^2 + |I_{\text{boundary}}(\delta\omega; -)|^2 = 2(2\pi)^3 (\sigma_3^3)^{1/2} e^{-\frac{\sigma_s}{2} (\delta p)^2} \frac{1}{(\delta\omega)^2 + \frac{1}{c^2}}; \quad \frac{1}{c^2} = \pi \sigma_t,$$
in which the phase factor $\Phi$ drops.

In higher order diagrams, the integration is made over more than one variables. That in one variable has the boundary terms which decrease slowly as $\frac{1}{\delta \omega}$ or $\frac{1}{|\delta \vec{p}|}$ of the corresponding momentum. The integral of the amplitude over the intermediate states is affected by the phase factor $\Phi$, and results to the common divergence in the bulk and the boundary. These divergences are subtracted by the counter terms.

The position dependences are expressed by $R$ in Eq. (A.3). $R \geq 0$ in general and for the scattering of the wave packets in which the trajectories do not intersect, $R > R_{\text{min}}$ with $R_{\text{min}} > 0$, where $R_{\text{min}}$ is determined by the distance of trajectories. Then the probability is suppressed by a factor $e^{-R_{\text{min}}}$. For the scattering of wave packets of intersecting trajectories, $R_{\text{min}} = 0$, and the probability is not suppressed.

Appendix B: Amplitudes and probabilities of wave packets

1. Vacuum amplitude

The amplitude that the vacuum state is transformed to three particle state, Eq. (38), is

$$N_3(\chi_1, \chi_2, \chi_3, T) = g \int_0^T dt \int d\vec{x} \langle 0 | \varphi_1(\vec{x}) \varphi_2(\vec{x}) A_1(\chi_1) A_1(\chi_2) A_2(\chi_3) | 0 \rangle$$

$$= g \int_0^T dt \int d\vec{x} C_1(\vec{x}; \chi_1) C_1(\vec{x}; \chi_2) C_2(\vec{x}; \chi_3)$$

$$= g N_0 \int_0^T dt \int d\vec{x} e^{-\frac{1}{2\sigma_s}(\vec{x}-\vec{x}_0)^2 - \frac{1}{2\sigma_t}(t-t_0)^2} e^{R+i\phi},$$

$$N_0 = i \left((4\pi)^3 \sigma_1 \sigma_2 \sigma_3\right)^{-\frac{3}{2}} (E_1 E_2)^{-\frac{1}{2}}.$$

where $\vec{x}_0$ and $t_0$ are functions of coordinates $\chi_i$ [27], and $\phi$ is a phase. $0 \ll t_0 \ll T$, and for symmetric cases $\sigma_t = \sigma, \sigma_s = \frac{\sigma}{3},$

$$\frac{1}{\sigma_t} = \sum_l \frac{v_l^2}{\sigma_l} - \sigma_s \sum_l \frac{\vec{v}_l}{\sigma_l} = \frac{1}{3\sigma} \sum_{l \neq m} (\vec{v}_l - \vec{v}_m)^2,$$

$$R_{\text{momentum}} = -\frac{\sigma_t}{2} (\delta E - \vec{v}_0 \cdot \delta \vec{p})^2 - \frac{\sigma_s}{2} (\delta \vec{p})^2, \delta E = \sum_l E_l, \delta \vec{p} = \sum_l \vec{p}_l.$$

Now,

$$|R_{\text{momentum}}| > \frac{\sigma_t}{2} (E_g)^2,$$
\[ |N_3(\chi_1, \chi_2, \chi_3, T)| < N_0 g (2\pi\sigma_s)^{3/2} (2\pi\sigma_i)^{1/2} e^{-\frac{E_g^2}{4\sigma_s^2}}. \] (B4)

The integral over the momenta and positions of final states converges. The last factor in Eq. (A5) satisfies \( e^{-\frac{E_g^2}{4\sigma_s^2}} < e^{-10^{10}} \) for parameters of our interests \( \sigma = (10^{-8})^2 \) meter\(^2\) and \( E_g = 1 \) MeV. Thus the probability from the bulk \( P = 0 \).

The contributions from states of finite energy gap vanish for \( E_g \geq 1 \) MeV and \( \sigma \geq (10^{-8})^2 \) meter\(^2\), or for \( E_g \geq 10^{-1} \) eV and \( \sigma \geq (10^{-1})^2 \) meter\(^2\).

A contribution from the boundary in time is inversely proportional to \( \frac{1}{\omega} \) Eq. (A12), which is not exponentially suppressed. The classical trajectories starting from the positions \( \vec{X}_i \) should be one space-time point, and these positions are restricted to a region of a length \( \sqrt{\sigma_s} \). The total probability integrated over these \( \vec{X}_i \) is less than that of QCS by a factor \( \frac{\sigma_s^{3/2}}{V_D} \) and is negligible, where \( V_D \) is the volume of the detector. The probability from the boundary can be considered \( P = 0 \).

The probability for the vacuum vanishes.

2. Particle decay probability

The decay \( 2 \to 1 + 1' \) is described by Eq. (A11) and occurs for \( m_2 > 2m_1 \), and \( E_g = 0 \). A probability for a particle 1 is found for a large \( \sigma_2 \), from the light-cone singularity due to \( p_1' \to \infty \) was presented in [12, 13]. Here the probability is studied without using the light-cone singularity.

Substituting Eq. (A12), we have the squares

\[ |\mathcal{M}|^2 = |\mathcal{M}_{\text{bulk}}|^2 + |\mathcal{M}_{\text{boundary}}|^2 \] (B5)

\[ |\mathcal{M}_j|^2 = g^2 N_1^2 |I_j(\delta \omega)|^2 e^{2R}; \quad j = \text{bulk, boundary.} \] (B6)

a. integral over \( \vec{X}_i \)

The process \( 1 \to 2 + 1' \) has the gap \( E_g = m_2 \), and \( \Gamma = 0 \). \( P^{(d)} = 0 \), as a causality condition,

\[ (p_1 - p_2)^2 - m_1^2 = 0 \] (B7)

is not satisfied.
b. **Integral over the momenta**

It is shown that (1) the total probability for a scalar decay for wave packets is convergent in $\varphi_1^2 \varphi_2$ interaction but that is divergent for $(\partial \varphi_1)^2 \varphi_2$ interaction, and (2) the probability for $\cos \Theta + 1 \leq \epsilon$ is convergent, where $\Theta$ is the angle between two particles in the final states and $\epsilon$ is a constant. Because the bulk term converges and is in agreement with the golden rule, we focus on the boundary term. That is given by

$$\int \prod_i \frac{d^3 \vec{k}_i}{2E(k_i)(2\pi)^3} (k_i)^q e^{-\sigma_s (\delta \bar{p})^2} \frac{1}{(\delta \omega)^2 + a^2}.$$  \hspace{1cm} (B8)

where $a$ is a constant, and $q = 0$ for $\varphi_1^2 \varphi_2$ interaction or $q = 1$ for $(\partial \varphi_1)^2 \varphi_2$ interaction. We study the initial state at rest, $|\vec{p}_2| = 0$, and mass-less final state $m_1 = 0$.

Two particle final states are written as

$$\vec{k}_i = k_i (\sin \theta_i \cos \phi_i, \sin \theta_i \sin \phi_i, \cos \theta_i); i = 1, 2 \hspace{1cm} (B9)$$

$$d\vec{k}_i = k_i^2 dk_i d\Omega_i, d\Omega_i = d\cos \theta_i d\phi_i$$

$$(\vec{k}_1, \vec{k}_2) = k_1 k_2 \cos \Theta, \cos \Theta = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos (\phi_1 - \phi_2).$$

with these variables

$$(\delta \bar{p})^2 = k_1^2 + k_2^2 + 2k_1 k_2 \cos \Theta \hspace{1cm} (B10)$$

$$(\delta \omega)^2 = ((k_1 + k_2)(1 - \frac{\sigma_s}{\sigma_1}(1 + \cos \Theta)) - m_2)^2.$$

Using spherical functions $P_l(\cos \Theta)$ a function of $\Theta$ is expanded,

$$f(\cos \Theta) = \sum_l a_l P_l(\cos \Theta), a_l = \frac{2l + 1}{2} \int_{-1}^1 d \cos \theta f(\cos \Theta) P_l(\cos \Theta) \hspace{1cm} (B11)$$

and its integral over the angles $d\Omega_i = d\cos \theta_i d\phi_i, -1 \leq \cos \theta_i \leq 1, 0 \leq \phi_i \leq 2\pi$, is

$$\int d\Omega_1 d\Omega_2 f(\cos \Theta) = \sum_l a_l \int d\Omega_1 d\Omega_2 P_l(\cos \Theta) \hspace{1cm} (B12)$$

$$= \sum_l a_l \int d\Omega_1 d\Omega_2 [P_l(\cos \theta_1) P_l(\cos \theta_2) + 2 \sum_{m=1}^{\infty} (-1)^m P^m_P(\cos \theta_1) P_{l-m}^m(\cos \theta_2) \cos m(\phi_1 - \phi_2)]$$

$$= \sum_l a_l (2\pi)^2 \int d\cos \theta_1 d\cos \theta_2 [P_l(\cos \theta_1) P_l(\cos \theta_2)]$$

$$= \sum_l a_l (2\pi)^2 2^2 \delta_{l,0} \delta_{l,0} = (4\pi)^2 a_0.$$
Because now $\bar{p}_2 = 0$,

$$g(\cos \Theta) = \frac{1}{((k_1 + k_2)(1 - \frac{\sigma_s}{\sigma_1}(1 + \cos \Theta)) - m_2)^2 + a^2 e^{x \cos \Theta}}, \quad x = 2\sigma s k_1 k_2 \tag{B13}$$

$$I_g = \int d\Omega_1 d\Omega_2 g(\cos \Theta) = (4\pi)^2 b_0 \tag{B14}$$

$$b_0 = \frac{1}{2} \int_{-1}^{1} dt g(t). \tag{B15}$$

For large $k_1 + k_2$,

$$((k_1 + k_2)(1 - \frac{\sigma_s}{\sigma_1} 2) - m_2)^2 \leq ((k_1 + k_2)(1 - \frac{\sigma_s}{\sigma_1}(1 + \cos \Theta)) - m_2)^2 \leq ((k_1 + k_2) - m_2)^2$$

$$\frac{1}{((k_1 + k_2) - m_2)^2 + a^2 e^{x \cos \Theta}} \leq g(t) \leq \frac{1}{((k_1 + k_2)(1 - \frac{\sigma_s}{\sigma_1} 2) - m_2)^2 + a^2 e^{x \cos \Theta}},$$

and

$$\frac{1}{((k_1 + k_2) - m_2)^2 + a^2} \leq b_0 \leq \frac{1}{((k_1 + k_2)(1 - \frac{\sigma_s}{\sigma_1} 2) - m_2)^2 + a^2} \tag{B16}$$

$$\frac{e^x - e^{-x}}{2x(((k_1 + k_2) - m_2)^2 + a^2)} \leq b_0 \leq \frac{e^x - e^{-x}}{2x(((k_1 + k_2)(1 - \frac{\sigma_s}{\sigma_1} 2) - m_2)^2 + a^2)}. \tag{B17}$$

Thus

$$\frac{e^{2\sigma_s k_1 k_2}}{2\sigma s k_1 k_2((k_1 + k_2) - m_2)^2 + a^2} \leq b_0 \leq \frac{e^{2\sigma_s k_1 k_2}}{2\sigma s k_1 k_2((k_1 + k_2)(1 - \frac{\sigma_s}{\sigma_1} 2) - m_2)^2 + a^2} \tag{B18}$$

The total probability for $q = 0$

$$P = \int \prod_i \frac{d^3 k_i}{2E(k_i)(2\pi)^3} (k_i)^q e^{-\sigma_s (\delta \rho)^2} \frac{1}{(\delta \omega)^2 + a^2}, \tag{B19}$$

$$= (4\pi)^2 \int \prod_i \frac{k_i^2 dk_i}{2E(k_i)(2\pi)^3} e^{-\sigma_s (k_i^2 + k^2)} b_0,$$

is bounded as $P_{\text{min}} \leq P \leq P_{\text{max}}$, where

$$P_{\text{min}} = (4\pi)^2 \int \frac{dk_+ dk_-}{2^2(2\pi)^6} e^{-\sigma_s k^2} \frac{1}{2\sigma s(2k_+ - m_2)^2 + a^2} \tag{B20}$$

$$= \frac{(4\pi)^2}{2^2(2\pi)^6 2\sigma_s} \sqrt{\frac{\pi}{\sigma_s}} \int_0^\infty \frac{dk_+}{(2k_+ - m_2)^2 + a^2} \tag{B21}$$

$$P_{\text{max}} = (4\pi)^2 \int \frac{dk_+ dk_-}{2^2(2\pi)^6} e^{-\sigma_s k^2} \frac{1}{2\sigma s(2k_+(1 - \frac{\sigma_s}{\sigma_1}) - m_2)^2 + a^2}$$

$$= \frac{(4\pi)^2}{2^2(2\pi)^6 2\sigma_s} \sqrt{\frac{\pi}{\sigma_s}} \int_0^\infty \frac{dk_+}{(2(1 - \frac{\sigma_s}{\sigma_1})k_+ - m_2)^2 + a^2},$$

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where \( k_+ = \frac{k_1 + k_2}{2}, k_- = k_1 - k_2 \). \( P_{\text{min}}, P_{\text{max}} \) are finite, and \( P \) is finite.

For \( q = 1 \), \( k_+^2 \) is multiplied to the above integrands, and the integrals are divergent. \( P_{\text{min}}, P_{\text{max}} \) are infinite, and \( P \) is infinite.

For \( \cos \Theta + 1 \leq \epsilon \), the exponential suppression factor \( e^{-2k_1k_2\epsilon} \) is multiplied to the above integrands, and the integrals converge for arbitrary \( q \). This is the case for large \( |\vec{p}_2| \), in which it is worthwhile to estimate the integrals using the behaviors at small angles \( \theta_i \leq \epsilon \). For \( \vec{p}_2 \) in the z-direction, the integrand is given by

\[
g(\cos \Theta; \cos \theta_1, \cos \theta_2) = e^{2\sigma_s p_2(k_1 + k_2)} \frac{1}{((k_1 + k_2)(1 - 2\sigma_s \sigma_1) - m_2)^2 + a^2} e^{-2\sigma_s k_1 k_2}, \tag{B22}
\]

The probability is bounded as \( P_{\text{min}} \leq P \leq P_{\text{max}} \), where

\[
P_{\text{min}} = \frac{(4\pi)^2}{2^2(2\pi)^62\sigma_s} \int_0^\infty dk_+ \int_{-k_+}^{k_+} dk_- (k_+^2 - \frac{1}{4}k_-^2)^q \frac{e^{-\sigma_s(p_2 - 2k_1)^2}}{(2k_+ - m_2)^2 + a^2}
\]

\[
P_{\text{max}} = \frac{(4\pi)^2}{2^2(2\pi)^62\sigma_s} \int_0^\infty dk_+ \int_{-k_+}^{k_+} dk_- (k_+^2 - \frac{1}{4}k_-^2)^q \frac{e^{-\sigma_s(p_2 - 2k_1)^2}}{(2k_+(1 - 2\sigma_s \sigma_1) - m_2)^2 + a^2}
\]

\( P_{\text{min}}, P_{\text{max}} \) are finite due to the exponential factor, and \( P \) is finite even for arbitrary \( q \).

3. Divergences due to the intermediate states in higher order corrections

The summation over the intermediate states of the central values of \((\vec{X}_i, \vec{P}_i)\) for the wave packets in higher order corrections is equivalent to that of the plane waves and lead divergences of the universal properties from Section (2-I). The divergences are the same in the bulk and the boundary, and are subtracted by the counter terms in the Lagrangian. Accordingly, the ultraviolet divergences in the intermediate states are subtracted with the counter terms and \( P^{(d)} \) in the renormalized theory are computed in the standard manners of the renormalization procedure.

The contributions form the bulk in intermediate states of the higher order corrections have \( e^{-\sigma(\omega)^2} \), and \( \delta \omega \geq E_g \). These vanish practically as is given in Appendix, and the contribution from the boundary is inversely proportional to \( \delta \omega \) and finite. is and intermediate states ultraviolet divergences in the corrections of the order \( e^2 \) from the intermediate states are due to the boundary terms, and are cancelled by the counter terms. Now, the boundary effects arise in the lowest order of \( e \) and in the higher orders, and are long-distance effects. They are convergent for the normalized initial and final states. Accordingly, that in the
lowest order is studied in detail. The divergences due to the intermediate states in the higher order corrections are common in the bulk terms and the boundary terms for the plane waves and for the wave packets. These divergences caused by the fluctuations of the fields in the extremely short-distance region are subtracted by the local counter terms in the Lagrangian Eq. (86). \( P^{(d)} \) is expressed with the physical masses and the coupling constants.

**Appendix C: Interaction Lagrangian of the total derivative**

An example of showing \( \Gamma = 0, P^{(d)} \neq 0 \) was given in [14], and another is a system of a scalar field \( \varphi(x) \) of the mass \( m_1 \) and a vector field \( \varphi_\mu(x) \) of the mass \( m_2 \) described by a standard kinetic action of fields of masses \( m_2 > 2m_1 \) and \( L_{\text{int}} = -g\partial_\mu(\varphi_\mu(x)\varphi(x)^2) \). The action integral becomes a surface term

\[
S_{\text{int}} = -g \int d^3S_\mu(\varphi_\mu(x)\varphi(x)^2),
\]

and the interaction does neither modify the equation of motion in the bulk nor the canonical structure of the variables. If ASI used in the Tomonaga-Feynman-Schwinger formalism were applied, the vertex part of the interaction Eq. (C1) in the momentum representation vanishes from the property of the Dirac’s delta function. Thus \( \Gamma = 0 \), but the amplitude in configuration space,

\[
\mathcal{M} = -igN(2\pi)^4 (p_V - p_1 - p_1')\delta^4(p_V - p_1 - p_1')\epsilon^\mu(p_V),
\]

vanishes from the property of the Dirac’s delta function. Thus \( \Gamma = 0 \), but the amplitude in configuration space,

\[
\mathcal{M} = -igN \int_{\lambda \geq 0} d^4x \frac{\partial}{\partial x_\mu} \left[ e^{i(-p_V + p_1)\cdot x + ip_1\cdot(x - X_\mu) - \xi(x)} \right] \epsilon^\mu(p_V), \lambda = (x - X)^2
\]

does not vanish. \( P^{(d)} \) for a large \( \sigma_1 \) is given in Eq. (66). The probability computed with the Fermi’s golden rule vanishes, but \( P^{(d)} \neq 0 \). \( P^{(d)} \) is not computed with ASI, but by the FQM with the normalized state.

**Appendix D: Changing the initial states**

The initial states satisfy the free wave equations in many processes. This is partly because fundamental interactions are local, and the range of the binding force is short. The atom, nucleus, and hadrons are formed almost instantaneously. Furthermore, lighter particles produced in the decays separate quickly due to higher speed than the parent. The subsequent
transition of these products are studied by the wave functions of this initial condition and by the interaction that switches on at this instant of time \( t = 0 \). Now, for a case of strong \( H_{\text{int}} \) which acts rapidly, the initial state may transfer instantaneously to the eigenstate of \( H \). The amplitude then is expressed with the wave packets defined from eigenstates of \( H \). From Section 2H, that gives the equivalent transition amplitude with those of \( H_0 \). General cases are studied here.

1. **Wavefunction**

For a normalized state of a linear combinations Eq.(20),

\[
|\Psi_\alpha(0)\rangle = |\Psi_\alpha(0), \vec{P}, \vec{X}, T_0\rangle + \sum_M C^M_\alpha |M, \chi_i, i = 1, M\rangle, C^M_\alpha = \langle \chi_i, M | \Psi_\alpha(0) \rangle,
\]  
(D1)

\[
\sum_\alpha C^M_\alpha (C^M_\alpha)^* = \langle \chi_i, i = 1 - M | \chi_j, j = 1 - M' \rangle,
\]

where \( |M, \chi_i, i = 1, M\rangle \) are states constructed from \( H_0 \), which are denoted as \( \chi \) states. The wave function at \( T \) for the initial state \( |\Psi_\alpha(0)\rangle \) is expanded with the \( \chi \) states

\[
|\Psi(T)\rangle = |\Psi_\alpha(0)\rangle = U(T)|\Psi_\alpha(0)\rangle = \sum_{\chi_1} (|\Psi_{(p)}(\chi_1)\rangle |\chi_1\rangle + |\Psi_{(w)}(\chi_1)\rangle |\chi_1\rangle |\Psi_\alpha(0)\rangle) \tag{D2}
\]

where the decomposition in the \( \chi \) space,

\[
U(T)|\chi\rangle = |\Psi_{(p)}(T), \chi\rangle + |\Psi_{(w)}(T), \chi\rangle \tag{D3}
\]
of two components was substituted.

2. **S-matrix : \( S[T] \)**

The S-matrix \( S[T, \alpha] \) for the initial states \( |\Psi_\alpha\rangle \) is written as

\[
\langle \alpha, \chi_1 | S[T] | \alpha, \chi_2 \rangle = \sum_{\chi'_1, \chi'_2} \langle \alpha, \chi_1 | \chi'_1 \rangle \langle \chi'_1 | S[T] | \chi'_2 \rangle \langle \chi'_2 | \alpha, \chi_2 \rangle
\]

\[
= \sum_{\chi'_1, \chi'_2} \langle \alpha, \chi_1 | \chi'_2 \rangle [\langle \chi'_1 | S[T] | \chi'_2 \rangle (p) + \langle \chi'_1 | S[T] | \chi'_2 \rangle (w)] \langle \chi'_2 | \alpha, \chi_2 \rangle. \tag{D4}
\]

The probability is

\[
P = |\langle \alpha, \chi_1 | S[T] | \alpha, \chi_2 \rangle|^2, \tag{D5}
\]
and the total probability to certain states is

\[ \sum_{\text{final states}} P_f = \sum_{\chi_1} |\langle \alpha, \chi_1 | S[T] | \alpha, \chi_2 \rangle|^2 \]

\[ = \sum_{\chi_1, \chi_1'} \langle \alpha, \chi_1 | \chi_1' \rangle \langle \chi_1' | S[T] | \alpha, \chi_2 \rangle \langle \alpha, \chi_2 | S[T] | \chi_1'' \rangle \langle \chi_1'' | \alpha, \chi_1 \rangle \]

\[ = \sum_{\chi_1} |\langle \chi_1 | S[T] | \alpha, \chi_2 \rangle|^2 \]

(D6)

and the probability averaged over the initial states is

\[ \bar{P}_f = \sum_{\chi_1, \chi_1'} |\langle \chi_1' | S[T] | \chi_1'' \rangle|^2. \]

(D7)

**Appendix E: Gibbs ensemble, Boltzmann equation, and Ergodic hypothesis**

Probability due to \( \Gamma T \) comes from the microscopic region of the order of de Broglie wave length, and satisfies the kinetic-energy conservation. This term leads the Boltzmann equation consistent with the Gibbs ensemble. The states in the space of the wave functions \( |\Psi(p)_(w)(t)\rangle \) reveal these particle properties and the probabilities Eqs. (78), and (79). Transitions occur independently, the distribution follows the principle of equal a priori probability and the Boltzmann distribution,

\[ P_G = e^{-\beta H}, \]

(E1)

using an average energy per each freedom \( \frac{1}{\beta} = k_b T \), here \( k_b \) is the Boltzmann constant and \( T \) is the temperature. The states follow the normal thermodynamics expressed by the Gibbs ensemble [51]. Many phenomena in fact follow this distribution.

Now \( P^{(d)} \) due to QCS, \( |\Psi_(w)(t)\rangle \), revealing the wave’s overlap is rapid, and remains the same long period. That does neither satisfy the conservation law of the kinetic energy, nor the Markov nature. In the decays, they follow the time-independent power law Eq. (81) instead of the exponential behavior. The probability \( P^{(d)} \) determines the initial probability for the Boltzmann equation, and follows the power law in the kinetic energy, and is independent of the temperature. This kind of behavior has been observed in many area in dilute systems, and will be studied in a forthcoming work. The total statistical ensemble of these waves are
the product of this term and that of Gibbs ensemble

\[ P = P_G \times P_D, \]  

(E2)

where \( P_D \) is the distribution corresponding to \( P^{(d)} \).

The probability in the scattering or the decay from \( \Gamma T \) covers the wide angle in each reaction. Accordingly the whole phase space is covered ultimately, and leads us to the Ergodic hypothesis. Those from \( P^{(d)} \), however, is restricted to the extreme forward angle in the time-independent manner, and does not cover the whole phase space. Hence this component does not follow the Ergodic hypothesis.