Borel-Weil-Bott Theory for Loop Groups

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Loop groups admit a class of representations (those of positive energy) similar to those of semi-simple groups, for which the subgroups of loops that extend holomorphically to a compact Riemann surface play the role of maximal parabolic subgroups. In this note, I outline the corresponding Borel-Weil-Bott theory, as worked out in [T1], [T2], [T3]. The general ideas, in the analytic setting of Conformal Field Theory, are due to Graeme Segal, while my algebraic methods rely on the works [K], [M], [LS], [TY]. There is now a body of literature ([BL], [F], [KNR]) devoted to the “Borel-Weil” theory (spaces of holomorphic sections); my contribution mainly concerns the “Bott” part (higher cohomology), except for the uniform determination of the “fusion rules”, which is a noteworthy application thereof.

Let $G$ be a complex, simple\(^1\) simply connected Lie group, $\Sigma$ a smooth, affine curve, smoothly compactified to $\Sigma^c$, with points at infinity $w_1, \ldots, w_n$. The group $G^c$ (or $G[\Sigma]$) of algebraic $G$-valued functions on $\Sigma$ is contained in the product $\hat{L}G$ of its formal completions $\hat{L}_iG$ at the $w_i$. These are copies of the group of $G$-valued formal Laurent series. Highest-weight representations (HWR’s) of $\hat{L}G$ are tensor products of HWR’s of the $\hat{L}_iG$. The factors, which shall all be chosen at the same, negative level $(-h)$, are direct sums of the (negative) energy eigenspaces. The projective cocycle of such HWR’s splits uniquely over $G^c$. Finite sums of algebraic duals of HWR’s are the positive energy representations (PER’s); they are formally complete, for a total-energy filtration.

Label $m$ points $z_1, \ldots, z_m$ on $\Sigma$ by irreducible representations (irreps) $V_k$ of $G$, and let $G^c$ act on the space $V := V_1 \otimes \cdots \otimes V_m$ by evaluation at the respective points. Consider:

- $X$, the product of the flag varieties $G((z))/G[[z]]$ of the $\hat{L}_iG$;
- $X_\Sigma$, the generalized flag variety $G^c \setminus \hat{L}G$;
- $\mathcal{M}$, the moduli stack of $G$-bundles over $\Sigma^c$;
- $\mathcal{L}$, the generator of $\text{Pic}(\mathcal{M})$ (cf. [LS]);
- $\mathcal{V} := \bigotimes_k \mathcal{V}_k(z_k)$, where $\mathcal{V}_k(z_k)$ is the restriction to $\mathcal{M} \equiv \mathcal{M} \times (z_k)$ of the $V_k$-bundle associated to the universal $G$-bundle on $\mathcal{M} \times \Sigma^c$.

$X$ is an ind-scheme, whereas $X_\Sigma$ is a scheme of infinite type. The “uniformization theorem” in [LS] says that the moduli stack $\mathcal{M}$ is (étale) equivalent to the quotient stack $G^c \setminus X$. (This combines A. Weil’s adèlic description of moduli spaces of vector bundles with the holomorphic double coset construction of [PS].) Using the subgroup $\hat{L}_iG \subset \hat{L}G$ of formal-holomorphic loops, this is also the quotient stack $X_\Sigma/\hat{L}_iG$. The lifting of $\mathcal{V}$ to $X$ is a trivial bundle, but its fiber $V$ carries the $G^c$-action described above. Next, $\text{Pic}(G((z))/G[[z]]) \equiv \mathbb{Z}$, and the lift to $X$ of $\mathcal{L}$, which I denote by $\tilde{\mathcal{L}}$ as well, is the product of the positive, generating line bundles on the factors of $X$. Also call $\mathcal{L}$ and $\mathcal{V}$ the lifts to $X_\Sigma$ of the same bundles over $\mathcal{M}$; their domicile will be contextually clear.

**Theorem 4.** The cohomology groups $H^i(X_\Sigma; \mathcal{L}^{\otimes h} \otimes \mathcal{V})$ are finite sums of HWR’s of $\hat{L}G$ at level $(-h)$, with multiplicity space $H^i(\mathcal{M}; \mathcal{L}^{\otimes h} \otimes \mathcal{V} \otimes \mathcal{U}_i)$ for the HWR with highest-energy spaces $\mathcal{U}_i$.

Here, $\mathcal{U}$ is the evaluation bundle analogous to $\mathcal{V}$, but with the irreps $\mathcal{U}_i$ attached to the points $w_i$.

The result holds even when $\Sigma$ is singular, but in the smooth case I can be more precise.

\(^1\)The extension to semi-simple groups is immediate; see [T3], Sec. V, for the case of non-trivial $\pi_1$. 

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Theorem 3. The cohomology of \( \mathcal{L}^{\otimes h} \otimes \mathcal{V} \) over \( \mathfrak{M} \) lives in a single degree \( \ell \). It vanishes altogether, if \( (\lambda_k + \rho) \cdot \alpha \in (h + c) \cdot \mathbb{Z} \), for any of the highest weights \( \lambda_k \) of \( \mathcal{V}_k \) and roots \( \alpha \) of \( \mathfrak{g} \); otherwise, its dimension is given by a “Verlinde factorization formula”.

The degree \( \ell \) is the sum of the lengths of the affine Weyl transformations taking the \( (\lambda_k + \rho) \) to the positive alcove, at level \( (h + c) \); \( c \) is the dual Coxeter number of \( \mathfrak{g} \).

Proof of Thm. 4: The key statement is the highest-weight property; see [T3], Sec. VIII for the proof. Assuming that, let \( U := \bigotimes_i U_i \), on which \( \hat{L}_G \) acts by evaluation at the \( w_i \). There is a “descent spectral sequence” from \( X_\Sigma \) to \( \mathfrak{M} \), involving the algebraic group cohomology of \( \hat{L}_G \):

\[
E_2^{p,q} = H^p_{\hat{L}_G} \left( H^q \left( X_\Sigma ; \mathcal{L}^{\otimes h} \otimes \mathcal{V} \right) \right) \Rightarrow H^* \left( \mathfrak{M}; \mathcal{L}^{\otimes h} \otimes \mathcal{V} \otimes \mathcal{U}' \right).
\]

(1)

For an HWR \( \hat{H} \), \( H^p_{\hat{L}_G} \left( \hat{H} \otimes \mathcal{U}' \right) \) vanishes for \( p > 0 \), while \( H^0 \) equals \( \mathbb{C} \), rather than zero, precisely when \( \hat{H} \) has highest-energy spaces \( U_i \). The sequence (1), then, collapses at \( E_2 \), and the abutment is the multiplicity space for this same \( \hat{H} \).

Proof of Thm. 3: By a theorem of Kumar [K] and Mathieu [M], we have \( H^0 \left( X; \mathcal{L}^{\otimes h} \right) = \mathcal{H}_{0,h} \) (the “vacuum representation” at level \( h \)), while the higher cohomology vanishes. The collapse at \( E_2 \) of the descent spectral sequence for the morphism \( X \to \mathfrak{M} \) implies \( (p = *, q = 0) \)

\[
H^* \left( \mathfrak{M}; \mathcal{L}^{\otimes h} \otimes \mathcal{V} \right) \cong H^*_{G_{\Sigma}^L} \left( \hat{H}_{0,h} \otimes \mathcal{V} \right),
\]

(2)

the last term being the group cohomology of \( G_{\Sigma}^L \). The theorem follows from the next result.

Theorem 2. For any PER \( H \) of \( \hat{L}_G \) at level \( h \), the group cohomology \( H^*_{G_{\Sigma}^L} (H \otimes \mathcal{V}) \) is concentrated in degree \( \ell \), and satisfies the properties listed in Thm. 3.

Remark. This is modeled on the following rewriting of the Borel-Weil-Bott theorem. Let \( P \subset G \) be a parabolic subgroup, \( E \) an irrep of \( G \), \( F \) one of \( P \). The group cohomology \( H^*_P (E \otimes F) \) of \( P \), with coefficients in \( E \otimes F \), is determined as follows. If \( \mathcal{T} \) is the sheaf of sections of the algebraic vector bundle \( G \times^P F \) over \( G/P \), the descent spectral sequence from \( G \) to \( G/P \),

\[
E_2^{p,q} = H^p_P \left( H^q \left( G; \mathcal{O}_G \otimes F \right) \right) \Rightarrow H^* (G/P; \mathcal{T}),
\]

(3)

collapses at \( E_2 \), because \( G \) is affine. Using the Peter-Weyl theorem for \( G \), we get

\[
H^* (G/P; \mathcal{T}) \cong \bigoplus_{E'} E' \otimes H^*_P (E \otimes F).
\]

(4)

BWB says that the left-hand side lives in a single degree, where it gives an irrep of \( G \); this amounts to the vanishing of \( H^*_P (E \otimes F) \), except possibly in a single degree. Pure-dimensionality of \( H^*_G (H \otimes \mathcal{V}) \) is the corresponding loop group statement. Note, however, that this line of argument does not work for loop groups, for which the Peter-Weyl statement fails.

Proof of Thm. 2: The van Est spectral sequence relates group cohomology to Lie algebra cohomology and to the cohomology of the topological space underlying \( G_{\Sigma}^L \). It reads

\[
E_2^{p,q} = H^p_G \left( H \otimes \mathcal{V} \right) \otimes H^q (G; \mathcal{A}) \Rightarrow H^* (g; H \otimes \mathcal{V}).
\]

(5)

Its collapse at \( E_2 \), and the single-dimensionality theorem, follow from the next two theorems.

Theorem 1. \( G_{\Sigma}^L \) is homotopy equivalent to the group \( C^* (\Sigma; G) \) of smooth maps to \( G \).

Proof: This follows by equating homotopy types in two descriptions of the stack of holomorphic \( G \)-bundles. Regarding it as the analytic stack underlying the stack of algebraic \( G \)-bundles gives the
homotopy quotient $X/G^\Sigma$, by the uniformization theorem. On the other hand, the Atiyah-Bott construction [AB] realizes it as the quotient stack of smooth $(0,1)$-connections modulo complex gauge transformations. This gives the homotopy type $C^\ast(\Sigma; BG)$, which is the homotopy quotient $\Omega G^\Sigma/\Sigma G$. Theorem 1 follows because $X = \Omega G^\Sigma$, by the natural inclusion (cf. [PS], Ch. 8). ▲

**Remark.** The map $H^\ast(L_g; \mathbb{C}) \to H^\ast(LG)$ is onto ([PS], Ch. 4), and Thm. 1 implies that the same holds for the left-edge-homomorphism in (5). The sequence, therefore, collapses at $E_2$.

**Theorem 0.** [T2, Thm. 2.5] The cohomology of the Lie algebra $\mathfrak{g}[\Sigma]$ of $\mathfrak{g}$-valued algebraic functions on $\Sigma$, with coefficients in the representation $H \otimes V$, is given by

$$H^\ast(\mathfrak{g}[\Sigma]; H \otimes V) \cong H^\ast(\mathfrak{g}[\Sigma]; H \otimes V) \otimes H^{-\ast}(C^\ast(\Sigma; G))$$

(6)

The first factor on the right satisfies the properties listed in Thm. 3.

**Remark.** The second factor is $H^\ast(G) \otimes H^\ast(\Omega G^\Sigma N)$, where $N = 2g + n - 1$.

**Proof:** The proof goes by induction over the genus. For the inductive step, let $\Sigma$ degenerate to a curve $\Sigma_0$ of genus $(g - 1)$, with one node and normalization $\tilde{\Sigma}_0$. Shapiro’s lemma gives

$$H^\ast_{\text{G}[\Sigma]}(H \otimes V) \cong H^\ast_{\text{G}[\Sigma]}(\text{Ind}_{\text{G}[\Sigma_0]}(H \otimes V)) \cong \bigoplus H^\ast_{\text{G}[\Sigma]}(H \otimes V \otimes \cup(x') \otimes \cup''(x''))$$

(7)

using evaluation representations at $x'$ and $x''$, the liftings in $\tilde{\Sigma}_0$ of the node. By inductive assumption in Thm. 2, $H^\ast_{\text{G}[\Sigma]}(H \otimes V)$ is given by Verlinde factorization in genus $(g - 1)$, and one shows as in [T2], Prop. 3.8, that the entire group cohomology is $H^\ast_{\text{G}[\Sigma]}(H \otimes V) \otimes H^\ast(\Omega G)$. (This uses Bott’s basis for $H^\ast(\Omega G)$, indexed by alcoves in the positive Weyl chamber.) Further,

$$H^\ast(\mathfrak{g}[\Sigma_0]; H \otimes V) \cong H^\ast_{\text{G}[\Sigma_0]}(H \otimes V) \otimes H^\ast(\mathfrak{g}[\Sigma_0]) \cong H^\ast_{\text{G}[\Sigma_0]}(H \otimes V) \otimes H^{-\ast}(G \times \Omega G^\Sigma N)$$

(8)

by the van Est spectral sequence, the last isomorphism following from $G[\Sigma_0] \sim G \times \Omega G^\Sigma (N - 1)$. Finally, consider the relative cohomology $H^\ast(\mathfrak{g}[\Sigma_0], g; H \otimes V)$: it loses the factor $H^\ast(G)$, so it lives only in degrees of the same parity; this implies that relative cohomology (and then, the absolute one) is rigid under the degeneration of $\Sigma$ to $\Sigma_0$, and is given as in Thm. 0.

In genus zero, further degeneration reduces us to the case of a single puncture, treated in [T1], Thm. 0. The Lie algebra cohomology $^2 H^\ast(\mathfrak{g}[z^{-1}], g; H \otimes V)$ is resolved by a Koszul complex $C^\ast$, sitting inside the Dolbeault complex for $L^\otimes H \otimes V$ over the flag variety $X_0 := G[z^{-1}] \setminus G((z))$. $C^\ast$ carries a natural Hilbert space topology, involving the standard hermitian norms on $H$ and $V$, and the Kähler metric on $X_0$. (Both require us to fix the unit disk in the affine line). The Weitzenböck formula (also known here as Nakano’s identity) expresses the $\bar{\partial}$-Laplacian as a non-negative operator (the $(1,0)$-Laplacian) plus a zero-order term, related to the hermitian curvature of $L^\otimes h \otimes V$ and to the Ricci curvature of $X_0$. All of these preserve $C^\ast$.

When the shifted highest weights $(\lambda_k + \rho)$ lie in the positive alcove, and when the $z_k$ are just outside the unit disk, but not near each other, it turns out that the curvature term is positive on positive-degree forms. There is, then, no higher Lie algebra $L^2$-cohomology $^3$. A direct computation ([T1], Sec. 3.3) shows that any cohomology class in the algebraic Koszul complex — the direct product of energy eigenspaces — has a Hilbert space representative; but this must lie in the range

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2 This differs from the convention in [T1], where $\mathfrak{g}[z]$ was used instead.

3 Under the stated assumptions, Nakano’s identity also ensures that the Hilbert space range of $\bar{\partial}$ is closed; the argument in Sec. 3.3 of [T1] was excessively cautious.
of $\mathfrak{g}$, so all higher “honest” Lie algebra cohomology vanishes as well. In the algebraic context, the restriction on the placement of the $z_k$ can be readily lifted. This yields the case $\ell = 0$ of Thm. 0; the general case follows from the usual reflection argument of Bott ([T1], Sec. 3.5). •

Finally, recall the connection with the fusion rules, originally conjectured by Segal. Bott defines a linear holomorphic induction map $I_h$, from the representation ring of $G$ to the free Abelian group on HWR’s of the loop group at level $(-h)$: $V$ is sent to the Euler characteristic of $\mathcal{L}^{\otimes h} \otimes V$ over $X_0$ ($m = 1$, $V_1 := V$, $\Sigma$ is the affine line). The choice of a marked point $z_1$ is irrelevant; when $V$ is irreducible, $I_h(V)$ (if non-zero) is $\pm$ a HWR of $LG$, with virtual character given by the Kac formula, applied to the highest weight of $V$.

Given $m$ HWR’s of of $LG$, with highest-energy spaces the $G$-irreps $V_1, \ldots, V_m$, Segal defines their fusion product as the space of global sections of $\mathcal{L}^{\otimes h} \otimes V$ over $X_0$. This depends on the choice of $m$ distinct points on the line, but its HWR-multiplicities do not, and can be equated with other definitions of the fusion coefficients, using the degree-zero case of Thms. 2, 3 and 4 (cf. also [T2], Prop. 2.10). (The multiplicity spaces, of course, carry extra structure, namely a flat projective connection and, conjecturally, a distinguished, compatible hermitian metric.) The following was, I think, first conjectured by Bott.

**Corollary 5.** Holomorphic induction is a ring homomorphism, taking tensor product to fusion.

**Proof:** By Thms. 3 and 4, the fusion product is really the Euler characteristic of $\mathcal{L}^{\otimes h} \otimes V$ over $X_0$. This does not change when $V$ is deformed by moving the marked points $z_1, \ldots, z_m$ to $\infty$, so it equals the desired $I_h(V_1 \otimes \cdots \otimes V_m)$. To see this rigidity of the characteristic, note that the fusion multiplicities are given by Lie algebra cohomology groups (Thm. 0); but ([T1], Cor. 3.2.7) these can be resolved by a finite-dimensional complex, where only the differentials depend on the $z_k$. •

**Remark.** This proof (dating from late ’93, in Lie algebra language) realizes an older, heuristic argument of Segal’s, and was probably the first uniform proof published ([T1], Thm.1); although several other arguments, partial or exhaustive (e.g. [F], [Fi]), are slightly prior to it. There seems to be no simple uniform proof: [Fi] uses results of Gelfand-Kazhdan and Kazhdan-Lusztig.

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