General Greenberger-Horne-Zeilinger theorem of cluster states

Li Tang,1 Zeqian Chen,2 and Zeng-Bing Chen1

1Hefei National Laboratory for Physical Sciences at Microscale and Department of Modern Physics, University of Science and Technology of China, Hefei, Anhui 230026, China
2State Key Laboratory of Magnetic Resonance and Atomic and Molecular Physics and United Laboratory of Mathematical Physics, Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences, 30 West District, Xiao-Hong-Shan, P.O.Box 71010, Wuhan 430071, China

(Dated: December 2008)

In this paper, we show that there are eight distinct forms of the Greenberger-Horne-Zeilinger (GHZ) argument for the four-qubit cluster state $|\phi_1\rangle$ and forty eight distinct forms for the five-qubit cluster state $|\phi_2\rangle$ in the case of the one-dimensional lattice. The proof is obtained by regarding the pair qubits as a single object and constructing the new Pauli-like operators. The method can be easily extended to the case of the N-qubit system and the associated Bell inequalities are also discussed. Consequently, we present a complete construction of the GHZ theorem for the cluster states of N-qubit in the case of the one-dimensional lattice.

PACS numbers: 03.65.Ud, 03.67.Mn, 03.67.Pp

I. INTRODUCTION

Bell’s inequality [1] indicates that certain statistical correlations predicted by quantum mechanics for measurements on two-qubit ensembles cannot be understood within a realistic picture based on Einstein, Podolsky, and Rosen’s (EPR’s) notion of local realism [2]. However, there is an unsatisfactory feature in the derivation of Bell’s inequality that such a local realistic and, consequently, classical picture can explain perfect correlations and is only in conflict with statistical prediction of quantum mechanics. Strikingly enough, the Greenberger-Horne-Zeilinger (GHZ’s) theorem exhibits that the contradiction between quantum mechanics and local realistic theories arises even for definite predictions on a four-qubit system [3]. Mermin [4] subsequently refined the original GHZ argument on a three-qubit system. Motivated by this discovery, a growing amount of interest in studying the various types of the GHZ argument has been apparent in recent literatures [4]. In particular, the GHZ argument for cluster states is presented in [2]. In this paper, we will present a complete construction of the GHZ argument for the cluster states of N-qubit in the case of the one-dimensional lattice.

As is well known, the cluster states are suitable multi-qubit states for universal, scalable quantum computation [7], and can be used in the quantum error-correction code [8]. There have been a number of experimental systems proposed as candidates for the generation of cluster states, and several of them have been implemented, e.g., creation of six-photon cluster states with verifiable six-party entanglement by a linear optical elements [4].

Following [2] we give a brief review of the definition and main properties of the cluster states in the case of an one-dimensional lattice with an open segment, which we denote $|\phi_N\rangle$. The cluster state $|\phi_N\rangle$ is determined by the set of eigenvalue equations

$$E_a|\phi_N\rangle = |\phi_N\rangle,$$

(1)

with the correlations operators

$$E_a = X_a \bigotimes_{b \in \text{neigh}(a)} Z_b,$$

(2)

where $\text{neigh}(a)$ is the set of all neighbors of $a$, $X = \sigma_x$, $Y = \sigma_y$, and $Z = \sigma_z$. For such a lattice, $|\phi_2\rangle$ and $|\phi_3\rangle$ are locally equivalent to a maximally entangled Bell state and a GHZ state, respectively. Indeed,

$$|\phi_3\rangle = \frac{1}{\sqrt{2}}(|+\rangle|0\rangle|+\rangle + |\rangle|0\rangle|0\rangle).$$

(3)

Hereafter, the one-qubit states are defined as usual as $Z|0\rangle = |0\rangle$, $Z|1\rangle = |1\rangle$, and $X|\pm\rangle = \pm|\pm\rangle$. Moreover, the four-qubit cluster state is

$$|\phi_4\rangle = \frac{1}{2}(|+\rangle|0\rangle|+\rangle|0\rangle + |\rangle|0\rangle|0\rangle|0\rangle + |\rangle|0\rangle|0\rangle|0\rangle + |\rangle|0\rangle|0\rangle|0\rangle).$$

(4)

Note that the state $|\phi_N\rangle$ is not locally equivalent to the N-qubit GHZ state whenever $N \geq 4$. In particular in the case of $|\phi_4\rangle$, a Bell inequality based on the GHZ argument can be derived [2] and has been tested in laboratory [11]. It acts as a strong entanglement witness, takes the quantum upper bound 4 as its maximum, while the four-qubit GHZ state does not violate it at all. However no complete construction of the GHZ theorem has been reported for the cluster states of N-qubit in the case of the one-dimensional lattice.

This paper is organized as follows. In Sec. III, we give a short recall of the general GHZ theorem for the three-qubit cluster state $|\phi_3\rangle$ (as the three-qubit GHZ state...
unitary equivalences, namely, The associated Bell operator reads as

\[ B_3 = Z_1X_2Z_3 + Y_1Y_2Z_3 - Y_1X_2Y_3 + Z_1Y_2Y_3 = (1 + E_1)E_2(1 + E_3), \] (17)

which reaches 4 when evaluated on |φ₃⟩, and no other state can ever give a larger value.

Contrary to the three-qubit case, there are distinct forms of the GHZ argument for the GHZ state of four-qubit. In [12], the authors present a complete construction of the GHZ argument for the four-qubit GHZ state, of which there are nine distinct forms. As shown in [6], the GHZ argument is also valid for the cluster state |φ₃⟩ except for the GHZ state (GHZ₄). Since each form of the GHZ argument admits a Bell inequality, it is meaningful to give a complete construction of the GHZ argument for |φ₄⟩ which will be done in the next section.

III. GENERAL GHZ THEOREM FOR |φ₄⟩ AND |φ₅⟩

The four-qubit cluster state |φ₄⟩ is defined by

\[ X_1Z_2I_3|φ₄⟩ = 1, \quad (E_1) \] (18)
\[ Z_1X_2Z_3|φ₄⟩ = 1, \quad (E_2) \] (19)
\[ I_1Z_2X_3|φ₄⟩ = 1, \quad (E_3) \] (20)
\[ I_1I_2Z_3X_4|φ₄⟩ = 1. \quad (E_4) \] (21)

Specifically, |φ₄⟩ can be projected into a variation of |φ₃⟩ as

\[ |φ₄⟩ = \frac{1}{\sqrt{2}} \left( |+\rangle_1|0\rangle_2|+\rangle_3,4 + |−\rangle_1|1\rangle_2|−\rangle_3,4 \right), \] (22)

with two Bell basis vectors defined on \( \mathbb{C}^2 \otimes \mathbb{C}^2 \)

\[ |+\rangle_{i,j} = \frac{1}{\sqrt{2}} (|0\rangle_i|+\rangle_j + |1\rangle_i|−\rangle_j), \] (23)
\[ |−\rangle_{i,j} = \frac{1}{\sqrt{2}} (|0\rangle_i|+\rangle_j - |1\rangle_i|−\rangle_j). \]

If we regard the pair qubit 3 and 4 as a single object, it makes sense to discuss the GHZ argument as the one in Sec. II since |+⟩₃,4 and |−⟩₃,4 could be treated as an orthonormal basis in \( \mathbb{C}^2 \). This motivates us to introduce four groups of Pauli-like operators as

\[ Y'_{3,4} ∈ \{-X_3Y_4, Y_3Z_4\}, \] (24)
\[ Z'_{3,4} ∈ \{I_3X_4, Z_3I_4\}, \]

with

\[ X'_{3,4} = -iY'_{3,4}Z'_{3,4} ∈ \{X_3Z_4, Y_3Y_4\}, \] (25)

since \( X'_{3,4}, Y'_{3,4}, Z'_{3,4} \) satisfy the usual algebraic identities of Pauli’s matrices [14] (see more details in Table II):

\[ [X'_{3,4}, Y'_{3,4}] = 2iZ'_{3,4}, \]
\[ [Y'_{3,4}, Z'_{3,4}] = 2iX'_{3,4}, \] (26)
\[ [Z'_{3,4}, X'_{3,4}] = 2Y'_{3,4}. \]

The three-qubit cluster state |φ₃⟩ is defined by

\[ X_1Z_2I_3|φ₃⟩ = 1, \quad (E_1) \] (5)
\[ Z_1X_2Z_3|φ₃⟩ = 1, \quad (E_2) \] (6)
\[ I_1Z_2X_3|φ₃⟩ = 1. \quad (E_3) \] (7)

Let us recall that the scenario for the GHZ argument of the cluster state |φ₃⟩ is the following: Particles 1, 2, 3 move away from each other. At a given time, an observer, Alice, has access to particle 1, a second observer, Bob, has access to particle 2 and a third observer, Charlie, has access to particle 3. The GHZ theorem is obtained by involving the algebra of Pauli matrices

\[ Z_1X_2Z_3|φ₃⟩ = 1, \quad (E_2) \] (8)
\[ Y_1Y_2Z_3|φ₃⟩ = 1, \quad (E_1 × E_2) \] (9)
\[ Y_1X_2Y_3|φ₃⟩ = -1, \quad -(E_1 × E_2 × E_3) \] (10)
\[ Z_1Y_2Y_3|φ₃⟩ = 1. \quad (E_2 × E_3) \] (11)

According to EPR’s criterion of local realism, Eqs. (8)-(11) allow three observers Alice, Bob and Charlie to predict the following relation between the values of the elements of reality

\[ z_1x_2z_3 = 1, \quad (12) \]
\[ y_1y_2z_3 = 1, \quad (13) \]
\[ y_1x_2y_3 = -1, \quad (14) \]
\[ z_1y_2z_3 = 1. \quad (15) \]

However, Eqs. (12)-(15) are inconsistent, because when we take the product of Eqs. (12)-(15), the value of the left-hand side is 1, while the right-hand side is −1. This suggests that Eqs. (8)-(11) exactly exhibit an “all versus nothing” contradiction between quantum mechanics and EPR’s local realism. Here \( z_1x_2z_3 \) etc. are shorthand for the values of the elements of reality \( ν(Z_1)ν(X_2)ν(Z_3) \).

It is proved in [12] that not only is the GHZ argument of three qubits valid merely for |φ₃⟩ but also there is only one form of the GHZ argument for |φ₄⟩ under locally unitary equivalence, namely,

\[ \{Z_1X_2Z_3, Y_1Y_2Z_3, Y_1X_2Y_3, Z_1Y_2Y_3\}. \] (16)
TABLE I: $Y'_{3,4}, Z'_{3,4}, X'_{3,4}$.

| $Y'_{3,4}$ | $Z'_{3,4}$ | $X'_{3,4}$ |
|-----------|-----------|-----------|
| $-X_3 Y_4$ | $I_3 X_4$ | $X_3 Z_4$ |
| $-X_3 Y_4$ | $Z_3 I_4$ | $Y_3 Y_4$ |
| $Y_3 Z_4$ | $I_3 X_4$ | $Y_3 Y_4$ |
| $Y_3 Z_4$ | $Z_3 I_4$ | $X_3 Z_4$ |

Choosing $X'_{3,4}$ representation $\{ +'_{3,4}, -'_{3,4} \}$, one has

$X'_{3,4} | +'_{3,4} = \pm | \pm \rangle_{3,4}$,

$Y'_{3,4} | +'_{3,4} = \mp i \mp \rangle_{3,4}$,

$Z'_{3,4} | +'_{3,4} = | \mp \rangle_{3,4}$,

$I'_{3,4} | +'_{3,4} = | \pm \rangle_{3,4}$,

with $I'_{3,4} \in \{ Z_3 X_4, I_3 I_4 \}$. Apparently, according to the unique form Eq. (16) of the GHZ argument for $|\phi_3\rangle$ in Sec. [II] the GHZ argument for $|\phi_4\rangle$ (written as Eq. (22)) can be expressed in terms of Pauli-like operators as

$$\{ Z_1 X_2 Z'_{3,4}, Y_1 Y_2 Z'_3, Y_1 X_2 Y'_{3,4}, Z_1 Y_2 Y'_{3,4} \}.$$  

(28)

Combining Eq. (24) and Eq. (28), we obtain the four distinct forms of the GHZ argument for $|\phi_4\rangle$ as follows:

$$\{ XIX, Y Y I X, Y X X Y, Z Y Y Z \}, \quad (29)$$

$$\{ XIX, Y Y I X, Y X Y Z, Z Y Z Y \}, \quad (30)$$

$$\{ X Z I, Y Y Z I, Y X X Y, Z Y X Y \}, \quad (31)$$

$$\{ X Z I, Y Y Z I, Y X Y Z, Z Y Y Z \}. \quad (32)$$

For simplicity, we denote $Z X I X$ etc. are shortcuts for $Z_1 X_2 I_3 X_4$. Furthermore, we can rewrite $|\phi_4\rangle$ in the following variation of $|\phi_3\rangle$:

$$|\phi_4\rangle = \frac{1}{\sqrt{2}}(| + \rangle_{1,2} | 0 \rangle_3 | + \rangle_4 + | - \rangle_{1,2} | 1 \rangle_3 | - \rangle_4),$$  

(33)

with

$$| + \rangle_{i,j} = \frac{1}{\sqrt{2}}(| + \rangle_i | 0 \rangle_j + | - \rangle_i | 1 \rangle_j),$$

$$| - \rangle_{i,j} = \frac{1}{\sqrt{2}}(| + \rangle_i | 0 \rangle_j - | - \rangle_i | 1 \rangle_j).$$

(34)

Similarly, by regarding the pair qubit 1 and 2 as a single object and introducing another four groups of operators:

$$Y''_{1,2} \in \{ Z_1 Y_2, -Y_1 X_2 \},$$

$$Z''_{1,2} \in \{ X_1 I_2, I_1 Z_2 \},$$

(35)

with

$$X''_{1,2} = -i Y''_{1,2} Z''_{1,2} \in \{ Z_1 X_2, Y_1 Y_2 \},$$

(36)

which also satisfy the algebraic identities Eq. (26), we have the following four forms of the GHZ argument for

$$|\phi_4\rangle = \{ XIX, Z Y Z Y, Z Y X Y, X I Y Y \}, \quad (37)$$

$$\{ XIX, Y X Y Z, Y X X Y, X I Y Y \}, \quad (38)$$

$$\{ I Z X Z, Z Y Y Z, Z Y X Y, I Z Y Y \}, \quad (39)$$

$$\{ I Z X Z, Y X Y Z, Y X X Y, I Z Y Y \}. \quad (40)$$

We would like to point out that the form Eq. (37) is just the one of the GHZ argument presented in [3]. Moreover, Eqs. (37)-(40) can be obtained from Eqs. (29)-(32) by simultaneously permuting qubit 1 and 4, qubit 2 and 3 based on the basic symmetry of $|\phi_4\rangle$ (as shown in Fig. [I]). Since each form of the GHZ argument for the four-qubit system of the one-dimensional lattice can be reduced to the three-qubit case, our method is universal to find all forms of the GHZ argument for the four-qubit system from the three-qubit case. Hence for the four-qubit cluster state $|\phi_4\rangle$, we have obtained all the eight distinct forms of the GHZ argument.

On the other hand, our method suggests that there are some states other than both $|\phi_4\rangle$ and the GHZ state, which also exhibit the GHZ argument, for instance,

$$|\phi_4\rangle = \alpha (| + 0 + 0 \rangle + | - 1 - 1 \rangle) + \beta (| - 0 - 1 \rangle + | + 1 0 \rangle),$$

where $|\alpha|^2 + |\beta|^2 = 1/2$ are the common eigenstates of the observables set Eq. (29) on which those observables assume values that refute the attempt to assign values only required to have them by EPR’s local realism.

In general, an N-qubit cluster state is given by

$$|\phi_N\rangle = \frac{1}{2^{N/2}} \bigotimes_{a=1}^{N} (|0\rangle_a + |1\rangle_a \sigma_2^{a+1}),$$

(41)

with $\sigma_2^{N+1} = 1$. Expanding the last two terms leads to

$$|\phi_N\rangle = \frac{1}{2^{(N-1)/2}} \bigotimes_{a=1}^{N-1} (|0\rangle_a + |1\rangle_a \sigma_2^{a+1})|+\rangle_N$$

$$= \frac{1}{2^{(N-2)/2}} \bigotimes_{a=1}^{N-2} (|0\rangle_a + |1\rangle_a \sigma_2^{a+1})|+\rangle_{N-1,N}$$

$$\Rightarrow |\phi_{N-1}\rangle. \quad \text{(as a variation of } |\phi_{N-1}\rangle)$$

(42)
On the other hand, we may reformulate $|\phi_N\rangle$ as

$$|\phi_N\rangle = \frac{1}{2^{N/2}} \bigotimes_{a=N}^1 \left( |0\rangle_a + |1\rangle_a \sigma_z^{a-1} \right). \quad (43)$$

Expanding the first two terms in Eq. (43) leads to

$$|\phi_N\rangle = \frac{1}{\sqrt{2}} \left( |0\rangle_N + |1\rangle_N \sigma_z^{-1} |\phi_{N-1}\rangle \right)$$

$$= \frac{1}{\sqrt{2}} \left( |+\rangle_N |0\rangle_{N-1} + |\rangle_N |1\rangle_{N-1} \sigma_z^{-2} |\phi_{N-2}\rangle \right)$$

$$= \frac{1}{\sqrt{2}} \left( |0\rangle_{N,N} - |1\rangle_{N,N} \sigma_z^{-2} |\phi_{N-2}\rangle \right)$$

$$\Rightarrow |\phi_{N-1}\rangle. \quad (a \text{ variation of } |\phi_{N-1}\rangle) \quad (44)$$

From Eqs. (41-44), we conclude that the N-qubit cluster state $|\phi_N\rangle$ can be projected into two variations of $|\phi_{N-1}\rangle$ by making proper combination of the N qubits in the above two ways.

For the purpose of further discussion, it is convenient to define the following class of multiqubit Pauli-like operators in the following recursive way:

$$Y'_{k,N} \in \{-X_k'Y'_{k+1,N}, Y_kZ'_{k+1,N}\},$$

$$Z'_{k,N} \in \{I_kX'_{k+1,N}, Z_kI'_{k+1,N}\},$$

$$I'_{k,N} \in \{Z_kX'_{k+1,N}, I_kI'_{k+1,N}\}, \quad (45)$$

with $X'_{k,N} = -iY'_{k,N}Z'_{k,N} \in \{X_kZ'_{k+1,N}, Y_kY'_{k+1,N}\}$, and $1 \leq k \leq N - 1$, and

$$Y''_{1,j} \in \{Z_{1,j-1}Y_j, -Y_{1,j-1}Y_j\},$$

$$Z''_{1,j} \in \{X_{1,j-1}I_j, I_{1,j-1}Z_j\},$$

$$I''_{1,j} \in \{X_{1,j-1}Z_j, I_{1,j-1}I_j\}, \quad (46)$$

with $X''_{1,j} = -iY''_{1,j}Z''_{1,j} \in \{Z_{1,j-1}X_j, Y_{1,j-1}Y_j\}$, and $2 \leq j \leq N$, and

$$X'_{N,N} = X_N, \quad Y'_{N,N} = Y_N, \quad Z'_{N,N} = Z_N,$$

$$X_{1,1} = X_1, \quad Y''_{1,1} = Y_1, \quad Z''_{1,1} = Z_1, \quad (47)$$

$$I''_{N,N} = I_N, \quad I''_{1,1} = I_1.$$

Now, we are in position to present our construction of the GHZ argument for $|\phi_N\rangle$. Obviously the five-qubit cluster state $|\phi_5\rangle$ can be rewritten either of the form

$$|\phi_5\rangle = \frac{1}{2} \left( |+\rangle_{1,2,3}|0\rangle_4|3\rangle_5 + |+\rangle_{1,2,3}|0\rangle_3|4\rangle_5 \right.$$ 

$$+ |\rangle_{1,2,3}|3\rangle_4|5\rangle_5 + |\rangle_{1,2,3}|4\rangle_3|5\rangle_4 \right)$$

or

$$|\phi_5\rangle = \frac{1}{2} \left( |+\rangle_{1,2}|0\rangle_3|4\rangle_5 + |+\rangle_{1,2}|0\rangle_4|3\rangle_5 \right.$$ 

$$+ |\rangle_{1,2}|3\rangle_4|5\rangle_3 + |\rangle_{1,2}|4\rangle_3|5\rangle_4 \right), \quad (48)$$

by regarding the pair qubits 1, 2 and 4, 5 as two single objects respectively. Fig. [2] shows three distinct ways to derive the GHZ argument for $|\phi_5\rangle$. Note that Eq. (48) and Eq. (49) can be transformed to each other by symmetrical actions on the qubits subscripts: $1 \leftrightarrow 5, \ 2 \leftrightarrow 4$, thus we restrict our consideration to the case of the form Eq. (48).

Therefore, the GHZ argument for $|\phi_5\rangle$ expressed in terms of Pauli-like operators is given by

$$\{Z_{1,3}X_4Z_5, Y_{1,3}Y_4Z_5, Y_{1,3}X_4Y_5, Z_{1,3}X_4Y_5\},$$

$$\Rightarrow \quad 16 \text{ forms, \quad (see Fig. [2]a)}$$

$$\{Z_{1,2}X_3Z_{4,5}, Y_{1,2}Y_3Z_{4,5}, Y_{1,2}X_3Y_{4,5}, Z_{1,2}X_3Y_{4,5}\},$$

$$\Rightarrow \quad 16 \text{ forms, \quad (see Fig. [2]b)}$$

$$1 \leftrightarrow 5, \ 2 \leftrightarrow 4, \quad \Rightarrow \quad 16 \text{ forms, \quad (see Fig. [2]c)} \quad (50)$$

where $Z_{1,3}, Y_{1,3}, Z_{1,2}, Y_{1,2}, Z_{4,5}, Y_{4,5}$ are depicted in Table [11]. Hence, there are forty eight distinct forms of the Greenberger-Horne-Zeilinger (GHZ) argument for the five-qubit cluster state $|\phi_5\rangle$.

IV. GENERAL GHZ THEOREM FOR THE N-QUBIT CLUSTER STATE ($N \geq 4$)

Generally speaking, our method used in the previous section can be easily extended to the case of the N-qubit system, since each form of the GHZ argument for the N-qubit system of the one-dimensional lattice can be eventually reduced to the three-qubit case in Sec. [11].

More precisely, the GHZ argument for $|\phi_N\rangle$ can be expressed in terms of Pauli-like operators as

$$C_j = \{Z_{j+1,j}^2X_{j+1,j+2}Z_{j+2,j+2}, Y_{j+1,j}^2Y_{j+1,j+2}Y_{j+2,j+2}, Y_{j+1,j}^2X_{j+1,j+2}, Z_{j+1,j}^2Y_{j+1,j+2}Y_{j+2,j+2}, \}$$

$$(j = [N/2], [N/2] + 1, \ldots, N - 2), \quad (51)$$

and its permutations $C'_j$ as

$$C'_j \leftarrow \frac{S_k}{k=1,2,\ldots,[N/2]} \rightarrow C_j, \quad (52)$$

where $[N/2]$ denotes the largest integer $l$ with $l \leq N/2$, and $S_k$ denotes the permuting action: changing the labeling of the qubits subscripts with

$$k \leftrightarrow (N + 1 - k). \quad (53)$$
FIG. 2: Diagram of the three ways to derive the GHZ argument for $|\phi_5\rangle$. Note that (a) and (c) can be transformed to each other by simultaneously permuting qubit 1 and 5, qubit 2 and 4 based on the basic symmetry of $|\phi_5\rangle$.

Now we utilize the fact that $A_{j+1} = A_j \cup (E_j A_j)$, by mathematics induction we arrive that

$$
\begin{align*}
Y''_{1,j} &\in Z_{j-1} Y_j A_j, \\
Z''_{1,j} &\in Z_j A_j, \\
I''_{1,j} &\in A_j, \\
X''_{1,j} &= -i Y''_{1,j} Z''_{1,j} \\
&\in Z_{j-1} X_j A_j,
\end{align*}
$$

where the operator set $A_j$ is defined as

$$
A_j = \{a_k, k = 1, 2, \ldots, 2^{j-1}\} \quad \text{with} \quad a_k = \prod_{i \in T_k} E_i,
$$

and $T_k$ denotes a subset of the vertices set $\{1, 2, \ldots, j - 1\}$.

Since the definition of $Y''_{j+2,N}$ etc. can be transformed to that of $Y''_{N-j-1,j-1}$ by symmetrical actions $S_k$ on the qubits subscripts set $\{1, 2, \ldots, N - j - 1\}$, it turns out that

$$
\begin{align*}
Y'_{j+2,N} &\in Y_{j+2} Z_{j+3} A'_{N-j-1}, \\
Z'_{j+2,N} &\in Z_{j+2} A'_{N-j-1}, \\
I'_{j+2,N} &\in A'_{N-j-1}, \\
X'_{j+2,N} &= -i Y'_{j+2,N} Z'_{j+2,N} \\
&\in X_{j+2} Z_{j+3} A'_{N-j-1},
\end{align*}
$$

with

$$
A'_{N-j-1} \xleftarrow{S_k \: k = 1, 2, \ldots, N - j - 1} A_{N-j-1}.
$$

Those equations allow us to construct the GHZ argument for the cluster state of N-qubit in detail.

V. BELL INEQUALITIES

As is demonstrated in Sec. [14] we have presented a complete construction of the GHZ argument for the cluster states of N-qubit. In the following we extend the last considerations concerning the associated Bell inequalities to the case of N-qubit from the same conditions.

At the beginning let us define the Bell operator in a standard form:

$$
B_{\phi_N} = \begin{array}{l}
Z'_{1,j} X_{j+1} Z'_{j+2,N} + Y''_{1,j} Y_{j+1} Y'_{j+2,N} \\
- Y''_{1,j} X_{j+1} Y''_{j+2,N} + Z''_{1,j} Y_{j+1} Y_{j+2,N}
\end{array} = (1 + E'_1) E'_2 (1 + E'_3),
$$

with

$$
\begin{align*}
E'_1 &= X'_{1,j} Z'_{j+1}, \\
E'_2 &= Z''_{1,j} X_{j+1} Z''_{j+2,N}, \\
E'_3 &= Z''_{1,j} X'_{j+1} Z''_{j+2,N}.
\end{align*}
$$

It is easy to check that

$$
[E'_m, E'_n] = 0, \quad (E'_n)^2 = I, \quad \forall n, m = 1, 2, 3.
$$

A simple computation yields that

$$
B^2_{\phi_N} = 4(1 + E'_1)(1 + E'_3).
$$

It is concluded that $\|B^2_{\phi_N}\| = 16$ and $\|B_{\phi_N}\| = 4$. This shows that $B_{\phi_N}$ reaches 4 when evaluated on $\phi_N$ ($\phi_N$ is the common eigenstate of $B_{\phi_N}$).

In a local realistic theory, the correlation function of the measurements performed by the observers is the average of the outcomes over many runs of the experiment. The classical correlation functions corresponding to $B_{\phi_N}$ is

$$
\begin{align*}
|\langle x_{1,j} z_{j+1} z'_{j+2,N} \rangle + \langle y'_{1,j} y_{j+1} z'_{j+2,N} \rangle - \langle y''_{1,j} x_{j+1} y'_{j+2,N} \rangle \\
+ \langle x''_{1,j} y_{j+1} y'_{j+2,N} \rangle| \leq 2,
\end{align*}
$$
by grouping 1 and 2 together in an analogous way to that presented in [4, 15].

In Table III we give out all the Bell operators in the form of $B_{\phi_N} = (1 + E'_1)E'_2(1 + E'_3)$ with $N = 4, 5, 6$ which in turn introduce Bell inequalities. Our result generalizes those investigations in [3], and is an extended version of [10]. Moreover, our method can be easily extended to $N$ qubits.

From Eq. (54) and Eq. (56), one can immediately infer that

$$E'_1 \in E_jA_j,$$

$$E'_2 \in \mathcal{A} = E_{j+1}A_j * A'_{N-j-1},$$

$$E'_3 \in E_{j+2}A'_{N-j-1},$$  \hspace{1cm} (62)

where the notion $*$ is defined as $A * B = \{xy, x \in A, y \in B\}$.

In Sec. III it is pointed out that these Bell operators are all maximally violated by the cluster state, but not only by the cluster state, since they are not a GHZ-Mermin experiment at whose some site there is only one measurement. For $\phi_N$ combining $2^{N-3}$ of those Bell operators, we could obtain a new Bell operator such that only $\phi_N$ violate it maximally. The N-qubit Bell operator is then defined as

$$B_{\phi_N} = (1 + E_j) \sum_{\omega \in \mathcal{A}} \omega(1 + E_{j+2})$$

$$= E_{j+1} \prod_{m=1, m \neq j+1}^{N} (1 + E_m).$$  \hspace{1cm} (63)

To give a simple example,

$$B_{\phi_4} = (1 + E_1)E_2(1 + E_3)(1 + E_4),$$

$$B_{\phi_5} = (1 + E_1)(1 + E_2)E_3(1 + E_4)(1 + E_5),$$

$$B_{\phi_6} = (1 + E_1)(1 + E_2)(1 + E_3)E_4(1 + E_5)(1 + E_6),$$

$$B_{\phi_6} = (1 + E_1)(1 + E_2)(1 + E_3)E_4(1 + E_5)(1 + E_6),$$

$$= E_{3} \leftrightarrow E_{4}.$$

(64)

(65)

(66)

VI. CONCLUSION

In conclusion, we have introduced a general method to derive the GHZ argument for the cluster states of N-qubit in the case of the one-dimensional lattice. We show that there are eight distinct forms of the GHZ argument for $\phi_4$ and forty eight distinct forms for $\phi_6$ respectively. The proof is obtained by regarding the pair qubits as a single object and constructing the new Pauli-like operators. Our method can be easily extended to the case of the N-qubit system. In addition, we discuss the associated Bell inequalities and therefore obtain a new Bell operator such that only $\phi_N$ violate it maximally.

| $\phi_N$ | $E'_1$ | $E'_2$ | $E'_3$ |
|----------|--------|--------|--------|
| N = 4 | (1, 2), 3, 4 | $E_2$ | $E_3$ | $E_4$ |
|         | $E_1E_2$ | $E_1E_3$ | $E_4E_5$ |
| N = 5 | (1, 2), (4, 5) | $E_3$ | $E_4$ | $E_5$ |
|         | $E_1E_3$ | $E_1E_4$ | $E_5E_6$ |
|         | $E_2E_3$ | $E_2E_4$ | $E_5E_6$ |
| N = 6 | (1, 3), (4, 5, 6) | $E_3$ | $E_4$ | $E_5$ |
|         | $E_1E_3$ | $E_1E_4$ | $E_5E_6$ |
|         | $E_2E_3$ | $E_2E_4$ | $E_5E_6$ |
|         | $E_1E_2E_3$ | $E_1E_2E_4$ | $E_1E_2E_6$ |
|         | $E_1E_2E_4$ | $E_1E_2E_6$ |
|         | $E_2E_4$ | $E_2E_6$ |
|         | $E_1E_2E_4$ | $E_1E_2E_6$ |
|         | $E_2E_4$ | $E_2E_6$ |
|         | $E_1E_2E_4$ | $E_1E_2E_6$ |
|         | $E_2E_4$ | $E_2E_6$ |

ACKNOWLEDGMENTS

The authors would like to thank Chuang Ye Liu for useful discussions and suggestions. This work is supported by the NNSF of China, the CAS, the National Fundamental Research Program (Grant No. 2006CB921900), and partially supported by the National Science Foundation of China under Grant 10775175.
Phys. Rev. A 71, 042325 (2005); X.-Q. Zhou, C.-Y. Lu, W.-B. Gao, J. Zhang, Z.-B Chen, T. Yang, and J.-W. Pan, Phys. Rev. A 78, 012112 (2008).

[7] R. Raussendorf and H. J. Briegel, Phys. Rev. Lett. 86, 5188 (2001); H. J. Briegel and R. Raussendorf, Phys. Rev. Lett. 86, 910 (2001); R. Raussendorf, D. E. Browne, and H. J. Briegel, Phys. Rev. A 68, 022312 (2003).

[8] D. Gottesman, Phys. Rev. A 54, 1862 (1996); D. Schlingemann and R. F. Werner, Phys. Rev. A 65, 012308 (2001).

[9] C.-Y. Lu, X.-Q. Zhou, O. Gühne, W.-B. Gao, J. Zhang, Z.-S. Yuan, A. Goebel, T. Yang, and J.-W. Pan, Nature Physics 3, 91 (2007).

[10] A. Cabello, Phys. Rev. Lett. 95, 210401 (2005).

[11] P. Walther, M. Aspelmeyer, K. J. Resch, and A. Zeilinger, Phys. Rev. Lett. 95, 020403 (2005); N. Kiesel, C. Schmid, U. Weber, G. Tóth, O. Gühne, R. Ursin, and H. Weinfurter, Phys. Rev. Lett. 95, 210502 (2005); G. Vallone, E. Pomarico, P. Mataloni, F. De Martini, and V. Berardi, Phys. Rev. Lett. 98, 180502 (2007).

[12] Z. Chen, Phys. Rev. A 70, 032109 (2004).

[13] L. Tang, J. Zhong, Y.-F. Ren, M.-S. Zhan, Z. Chen, Acta Math. Sci. B 27, 753 (2007).

[14] W. Pauli, Z. Phys. 43, 601 (1927).

[15] M. Ardehali, Phys. Rev. A 46, 5375 (1992); A. V. Belinskii and D. N. Klyshko, Phys. Usp. 36, 653 (1993).

[16] O. Gühne and A. Cabello, Phys. Rev. A 77, 032108 (2008); A. Cabello, O. Gühne, and D. Rodriguez, Phys. Rev. A 77, 062106 (2008); G. Tóth, O. Gühne, and H. J. Briegel, Phys. Rev. A 73, 022303 (2006).