Doped Mott Insulators Break $\mathbb{Z}_2$ Symmetry of a Fermi Liquid: Stability of Strongly Coupled Fixed Points

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While the Mott transition from a Fermi liquid is correctly believed to obtain without the breaking of any continuous symmetry, we show that in fact a discrete $\mathbb{Z}_2$ symmetry of the Fermi surface is broken. The extra $\mathbb{Z}_2$ symmetry of the Fermi liquid appears to be little known although it was pointed out by Anderson and Haldane[1] and we use it here to classify all possible Fermi liquids topologically by invoking K-homology. It is this $\mathbb{Z}_2$ symmetry breaking that signals the onset of particle-hole asymmetry, a widely observed[2][10] phenomenon in strongly correlated systems. In addition from this principle, we are able to classify which interactions suffice to generate the $\mathbb{Z}_2$-symmetry-broken phase. As this is a symmetry breaking in momentum space, the local-in-momentum space interaction of the Hatsugai-Kohmoto (HK)[11] model suffices as well as the Hubbard interaction as it contains the HK interaction. We then use the Bott topological invariant to establish the stability of a Luttinger surface. Our proof demonstrates that the strongly coupled fixed point only corresponds to those Luttinger surfaces with co-dimension $p+1$ with $p$ odd. We conclude that the Hubbard and HK models both lie in the same high temperature universality class and are controlled by this fixed point.

Symmetry is a fundamental organizing principle of nature. A case in point is the simplest example of symmetry, namely permutations. This symmetry helps organize identical fundamental particles into two groups: fermions, odd under interchange and bosons, even under permutation. Since the permutation group has a finite number of elements, $\pm 1$, it is an example of a discrete symmetry. What we show here is that a group as simple as the permutation group, namely $\mathbb{Z}_2$, controls the transition from a non-interacting collection of electrons constituting a Fermi surface to a state that strongly violates the traditional theory of metals, namely the Mott paramagnetic state which insulates although the band is half full. The Fermi surface retains $\mathbb{Z}_2$ symmetry but the Mott state does not.

A manifestation of this symmetry breaking is the resultant asymmetry upon particle-hole addition or removal. In a non-interacting electron system, adding or subtracting an electron is a symmetrical process. However, cuprate superconductors as varied as underdoped Bi$_2$Sr$_2$CaCu$_2$O$_{8+d}$ (Bi-2212) and Ca$_{2-x}$Na$_x$CuO$_2$Cl$_2$ (Na-CCOC) all exhibit scanning tunneling spectra[9][10][12][13] with a distinct asymmetry in terms of particle addition and removal. The cuprates are not alone here as there are numerous electronic systems[6][8] which exhibit particle-hole asymmetry at low energies upon the addition or removal of an electron. Although it is now commonplace to attribute particle-hole asymmetry to strong correlations[2][3][5], no universal operative principle has been enunciated except for the general phenomenon of Mottness[4]. In his parting words in 2016, P. W. Anderson[14] chided condensed matter theorists for not facing up to this problem: “I remain baffled by the almost universal refusal of theorists to confront this evident fact of hole-particle asymmetry head-on.” It is this task we take on in this paper. What all cuprates have in common is that the parent material cannot be understood without considering the interactions. The minimal model thought to be relevant in this context is due to Hubbard in which electrons move on a square lattice but pay an energy cost whenever opposite-spin electrons reside on the same site. Since this model is unsolvable in any dimension other than $d = 1$, it is difficult to pin-point a clear organizing principle, other than that the interactions are important, as the root cause of the asymmetry. An added complication is that the Mott insulating state that arises from the local interactions is thought to be featureless above any temperature associated with ordering, just as is the Fermi liquid, the non-interacting limit. Consequently, appealing to some sort of symmetry breaking appears to be a non-starter.

We propose here that such an organizing principle can be unearthed by focusing on the full symmetry group of a Fermi liquid and analyzing which symmetries in the Fermi liquid survive the transition to the paramagnetic Mott insulator which is well-established to exist based on numerics[15]. While it is common to use the Hubbard model to study this transition, our key point here is that the essence of the Mott transition is captured by a simpler model.

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which breaks the fundamental local-in-momentum space \( \mathbb{Z}_2 \) symmetry of the Fermi liquid state. This \( \mathbb{Z}_2 \) symmetry serves as an organizing principle for Mott physics. We find that both local on-site Hubbard and local-in-momentum (as in the exactly solvable Hatsugai-Kohmoto model\([11, 16, 17]\) (HK) interactions fall into the same universality class as they both break \( \mathbb{Z}_2 \) symmetry. We then use K-theory to show that the surface of zeros that characterizes the Mott phase is stable to perturbations, thereby establishing the existence of a fixed point. Our work here is analogous to that of Hofava’s\([18]\) on the stability of a Fermi surface.

### I. RELEVANCE OF THE HK INTERACTION

Part of the motivation for this work is that there seem to be two disparate ways of generating a Mott transition with no apparent relationship between them. These constitute the Hatsugai-Kohmoto\([11]\)(HK) and Hubbard models. While both models contain the standard kinetic term, the HK model contains a non-standard local in momentum space interaction

\[
H_{\text{int}}^{HK} = U \sum_k n_{k\uparrow} n_{k\downarrow}, \quad (1)
\]

and the Hubbard model, the standard real-space

\[
H_{\text{int}}^{\text{Hubb}} = U \sum_i n_{i\uparrow} n_{i\downarrow} \quad (2)
\]

interaction. Even with the kinetic energy, the former model is solvable exactly\([11, 16]\), yielding an insulating state with a hard gap should \( U > W \), where \( W \) is the bandwidth. Only numerics\([15]\) on the Hubbard model support a gap indicative of Mott physics. Hence, it is worth comparing both models. As shown in Fig. 1, the spectral functions for both models are roughly identical. Nonetheless, that the HK model is capable of capturing Mott physics is not widely appreciated. Unearing why these two quite different models yield the same physics is the primary goal of this paper.

The first thing that must be established with the HK model is why does the interaction \( H_{\text{int}}^{HK} \) destroy Fermi liquid behaviour. Two distinct arguments will be adopted here. First, we appeal to the renormalization principle\([19, 20]\) for fermions and show that \( H_{\text{int}}^{HK} \) is a relevant perturbation. The correct starting point for renormalization of fermions is to demand that the kinetic term in the action,

\[
S_0 = \int dt d^3p \psi_\sigma(p) (i\partial_t - (\epsilon_p - \epsilon_F)) \psi_\sigma(p) \quad (3)
\]

has zero scaling dimension under the distortion \( \epsilon(p) = k + s \ell \) where \( k \) is along the Fermi surface, \( \ell \) is perpendicular to it and \( s \) the scaling parameter which will be set to 0 to preserve the Fermi surface. Expanding the dispersion relationship of an electron around the Fermi surface,

\[
\epsilon(p) = \epsilon_F + \ell \frac{\partial \epsilon}{\partial p} + O(\ell^2), \quad (4)
\]

we find that demanding \( [S_0] = 0 \) requires that \( \psi_\sigma(p) \to s^{-1/2} \psi_\sigma(p) \). The irrelevance of a generic interaction term

\[
S_{\text{int}} = \int dt \prod_{i=1}^4 d^d k_i d\ell_i V(k_1, \cdots, k_4) \psi_{\sigma_1}(p_1) \psi_{\sigma_2}(p_2) \psi_{\sigma_3}(p_3) \psi_{\sigma_4}(p_4) \delta^d(p_1 + p_2 - p_3 - p_4). \quad (5)
\]

follows because \( [S_{\text{int}}] = 1 \) (arising from \( s^4 \) from the four factors of \( dt \), \( \ell^{-4} \) from \( d\ell \) and \( s^{-4/2} \) from the four fermion fields) and hence vanishes in the \( s \to 0 \) limit. This conclusion holds even if loop corrections with \( L \) loops are included as they scale as \( s^L \), thereby vanishing for a generic interaction. Note an interaction of the form \( U n_{i\sigma} n_{i\bar{\sigma}} \) has a tree-level scaling dimension of \( -1 \) and hence contributes the same order as does the chemical potential. That is, it leads to the mass renormalization of Fermi liquid theory. Loop corrections of this term once again contribute \( s^L \) and hence generate no self-energy corrections. Within this scheme, the only exception arises when electrons scatter with momenta on opposite sides of the Fermi surface. In this case, the \( \delta \)-function factorizes and the interaction is marginal and leads to an instability should \( V < 0 \). Note the form of the kinetic energy term is irrelevant to this argument. All that is necessary is the expansion in Eq. (4).

Contrarily, the local-in-momentum space interac-
FIG. 1. a.) Spectral function of HK and Hubbard models from exact diagonalization with parameters shown. At half-filling there is little difference between the models showing that the HK interaction accurately models the U-scale physics of Hubbard. b.) Density of states of the HK model at filling $\langle n \rangle = 0.8$, showing a strong particle-hole asymmetry at the Fermi energy.

The key point is that because Fermi liquids possess separately conserved currents for up and down spins, the full symmetry group for each point on the Fermi surface is $O(4)$, the real group of rotations in 4-space. The determinant of an $O(4)$ matrix is either +1 or −1 thus exhibiting the disconnected nature of this Lie group. Namely, the proper group $SO(4)$ where the determinant is +1 cannot be continuously deformed into those whose determinant is −1. To understand what remains, we consider the quotient $O(4)/SO(4)$ which is isomorphic to $\mathbb{Z}_2$. The $\mathbb{Z}_2$ arises simply because there are 2 connected components of $O(4)$. That is, $\pi_0(O(4)) \simeq \mathbb{Z}_2$ (here $\pi_p(G)$ is the group of homotopy classes of maps of the p-dimensional sphere to G) which is equal to the group consisting of the identity, $I$, and reflections, $R$. A reflection $R$ through a hyperplane is represented by $R_{ij} = \delta_{ij} - n_i n_j$ if $(n_0, \cdots, n_3)$ is the orthonormal vector to the hyperplane. There is of course a quite distinct $\mathbb{Z}_2$ which lurks because $\pi_1(SO(4)) \simeq \mathbb{Z}_2$ which tells us that there is a simply connected double cover of $SO(4)$ called $Spin(4)$ (the spin group) which is isomorphic to $SU(2) \times SU(2)$, one $SU(2)$ for the spin and the other for the charge pseudospin. As a result, in terms of the particle-hole spinor, $\psi^\dagger_p = (c^\dagger_{p \uparrow}, c_{p \downarrow})$, we can write the Hamiltonian for a Fermi liquid as

$$H_{FL} = \sum_p \psi^\dagger_p (\epsilon_p - \epsilon_F) \tau_3 \psi_p + \cdots$$

which lays plain the inherent $SU(2)$ invariance of the charge sector as proposed initially by Anderson[23] and Nambu[24] and the existence of an infinite number of conserved currents, $n_{k\sigma}$. Here $\tau_3$ is the $z$-
component of the traditional Pauli matrices. The ellipses stand for any interaction terms that renormalize to zero or terms that contribute at the same level as the chemical potential which lead to the mass renormalization of Fermi liquid\textsuperscript{[19–22].} The extra $\mathbb{Z}_2$ symmetry obtains only for the electrons precisely at the Fermi surface. As the kinetic energy vanishes for such electrons, extra symmetries emerge. The relevant symmetry that emerges within $O(4)$ is that the sign of only one of the spin currents can be changed without any consequence to the underlying theory. That is, at the Fermi surface, a particle-hole transformation on one species $c_{p\uparrow} \rightarrow c_{p\downarrow}^\dagger$ or $n_{p\downarrow} \rightarrow 1-n_{p\uparrow}$ but preserving $n_{p\downarrow} \rightarrow n_{p\downarrow}^\dagger$ can be made with impunity. The remaining electrons do not enjoy this symmetry. In this sense, the $\mathbb{Z}_2$ symmetry is emergent in a Fermi liquid as it is exact only at the Fermi surface. In the presence of generic short-range interactions, the precise manifestation of this symmetry is detailed in the Supplementary Materials. It is this discrete $\mathbb{Z}_2$ symmetry that a Fermi surface possesses which ultimately accounts for the inherent particle-hole symmetry at low energies.

There is a subtlety here that points to more than $O(4)$ defining the group structure of Fermi liquids. To establish this, we note that from the $\mathbb{Z}_2$ symmetry (of order $\frac{1}{2}$ detailed in the Supplementary Materials), we can view the $O(4)$ action as giving an $O(4)$-bundle structure to the fermions on the Fermi surface. The $\mathbb{Z}_2$ symmetry in this context is related to orientability (a consistent orthonormal frame that remains invariant upon parallel transport through a loop as illustrated in Fig. 2) of the bundle. The first step\textsuperscript{[25]} is to realize that a Fermi liquid augmented by a number of trivial bands (in a sense we will explain below) has the same properties as the original system. We consider general free Hamiltonians

\[ H = \sum_{\sigma \sigma'} \psi_{\sigma}^\dagger(p) A_{\sigma\sigma'}(p) \psi_{\sigma'}(p). \]  

(8)

We can think of $A_{\sigma\sigma'}$ as a map from the Fermi surface (since we are interested in the $\mathbb{Z}_2$-symmetry described thus far) to a matrix group and we impose two such $A$‘s, say $A_1$ and $A_2$ to be equivalent when

\[ A_1 \sim A_2 \quad \text{if} \quad A_1 \oplus A_{\text{trivial}} \sim_{\text{hom}} A_2 \oplus A_{\text{trivial}}, \]  

(9)

where $A_{\text{trivial}}$ represents the trivial system

\[ A_{\text{trivial}} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} |p|^2, \]  

(10)

and $\sim_{\text{hom}}$ means homotopically equivalent. The homotopy equivalence is reflected in being able to continuously deform the eigenvalues without changing the determinant. This equivalence class gives rise to the set of maps from the Fermi surface (which we assume to be homotopic to a sphere) to a classifying space $C_q$ or $R_q$, complex or real, respectively. The only classifying spaces for which $\pi_0(G) \simeq \mathbb{Z}_2$ corresponds to either $G = O(n)$ or $G = O(2n)/U(n)$ as is evident from the tables in the Supplementary Materials. This means the additional group $O(4)/U(2)$ is a possible candidate to describe Fermi liquids. However, such a group would not allow a description in terms of $H_{\text{FL}}$. The types of Fermi liquids described by $O(4)/U(2)$ is beyond the scope of this paper.

From the analysis above, it is clear that any interaction of the form $n_{p\uparrow} n_{p\downarrow}$ maximally breaks the momentum-space $\mathbb{Z}_2$ symmetry of a Fermi surface as it changes sign under $n_{p\uparrow} \rightarrow -n_{p\downarrow}$ with $n_{p\downarrow} \rightarrow n_{p\uparrow}$. Since this term is a relevant perturbation of a Fermi liquid, it is not a surprise that it breaks the $\mathbb{Z}_2$ invariance of the would-be Fermi surface. Further, because it is the $\mathbb{Z}_2$ symmetry that is relevant, the only part of the Hubbard interaction that is pertinent to this discussion is the HK contribution which can be seen from Fourier transformation. Hence, both the HK and Hubbard interactions break the $\mathbb{Z}_2$ symmetry of a Fermi surface and as a result the transition from a Fermi liquid to a Mott insulator involves the breaking of a discrete $\mathbb{Z}_2$ symmetry. The presence of a charge gap but gapless spin degrees of freedom in the half-filled state are manifestations of the breaking of the discrete $\mathbb{Z}_2$ symmetry as the spin and charge currents can no longer be rotated freely. In the doped state, it is well known\textsuperscript{[2–5]} that the density of states of a doped Mott insulator (see Fig. 3) lacks particle-hole symmetry as must be the case if $\mathbb{Z}_2$ symmetry in momentum space is absent. Conse-
In this section, we concern ourselves with explaining how the RG flow works for non-local theories and propose a K-theory stability analysis for the underlying fixed point. The notion of renormalizability is in general ill-posed as normally stated, as one generally neglects to mention the space of operators within which a theory is renormalizable. More explicitly, consider a certain theory described by a classically local action $S(\phi_i)$ of some (not necessarily scalar) fields $\phi_1, \ldots, \phi_n$. One fixes a certain energy scale $\Lambda$ and integrates out fields whose energy is higher than $\Lambda$ so that the effective action $S_{\Lambda}$ obtains. This is done by integrating out the fields whose frequency
local
for some
a fixed point, one can write

\[ \int D\phi e^{iS(\phi)} = \int D\phi_L e^{iS_L(\phi_L)} \]

where \( S_L(\phi_L) = -i \log (\int D\phi_H e^{iS(\phi_L, \phi_H)}) \). If \( S_* \) is a fixed point, one can write

\[ S_A = S_* + \int d^d x \sum_i g_i O_i \]

for some local operators \( O_i \) (they are local, despite the integration of high frequency fields, because we focus on fields with \( \omega < \Lambda \)). The core of renormalization is in the observation that there is a dimension (of operators) \( D = D(d, H_0) \) (where \( H_0 \) is the Hamiltonian of the Free energy), above which the operators are irrelevant, and the number of local operators \( O_i \) whose dimension is less than (or equal) to \( D \) is finite (this is because classically local operators are polynomials in the fields \( \phi \) and their derivative \( \partial_x \phi \), having used the multi-index notation. Since there are finitely many of these, one can make sense of such theories. The point we want to make is that this makes sense only because we restrict ourselves to a class of operators allowed (in this case the classically local operators \( O_i \)). But this argument can be generalized to a non-local theory in real space whose Fourier transform is of course local. Hence, non-locality in real space poses no real hurdle to the renormalization program. Here locality in momentum space is the standard notion of locality in which position is replaced by momentum.

For the sake of simplicity, we explain this procedure in the case in which the Hamiltonian is \( H = H_0 + H_1 \) where \( H_0 \) arises from a classically local operator (such as kinetic energy) and \( H_1 \) is non-local in real space but its Fourier transform is local as in the example of \( S_{\text{int}} \) in the HK model. We now simply allow operators \( O_i \) whose Fourier transform is local and can be written as a combination of fundamental operators; i.e., operators which are either classically local in position space or that are polynomials in the operator components of \( H_1 \). Since the degree of these polynomials has to be bounded in order for the dimension of the operators to be bounded, there are only finitely many of this latter type as well.

Stability of the Mott fixed point is tantamount to showing that the defining feature of Mott physics [27], the Luttinger surface (defined in Eq. (11)) though not necessarily related to the particle density [28] [29], is stable to perturbations, for example the non-HK terms in the Hubbard model. To this end, we show that the Luttinger surface under perturbations of the Hamiltonian is determined by Bott periodicity [30] [31] and ultimately K-theory much the way a Fermi surface is as shown by Hofstätter [13]. For our purposes, the importance of RG, besides the existence of the fixed point, is that for small values of the parameters \( g_i \), the Green function changes continuously by applying perturbation theory to \( Z(g_i) = \int D\phi_L e^{iS_L(\phi_L)} \int D\phi_H e^{iS_H(\phi_H)} \). Consider the Green function

\[ G(k, \omega) = \langle \psi(0, 0) \psi^\dagger(k, \omega) \rangle = \frac{1}{\omega - \xi_k + \Sigma(k, \omega)} \]

for some Hamiltonian which vanishes along a surface of zeros, the Luttinger surface. In a \( d+1 \)-dimensional \((k, \omega)\) space, we will regard the Luttinger surface, \( \Omega \) to have dimension \( d - p \) and hence its co-dimension is \( p + 1 \). Here \( \Sigma \) is the exact self energy. The precise equation denoting the zero surface as in Eq. (11) is determined by the locus of \( (k_L, \omega) = 0 \) points at which \( \Sigma \) diverges. We assume the fields, \( \psi(k, \omega) \) represent complex Fermions consisting of \( N \) components. We consider a point \( k_L \) in momentum space which is an element of the Luttinger locus, i.e. \( \Omega := \{ \det G = 0 \} \) and consider a \( p \)-sphere of radius \( \epsilon \) centered at a point \( k_L \) in the normal directions (i.e. \( k_L \) is in the normal bundle \( \nu_\Omega \) to \( \Omega \) and we take a fiber of the \( \epsilon \)-tubular neighborhood of \( \Omega \) identified via the exponential map with the \( \epsilon \)-sphere bundle \( S_\Omega(\epsilon) = \{(k_L, k_L) \in \nu_\Omega : |k_L - k_L| = \epsilon \} \). A perturbation which preserves the Luttinger surface moves the zero of \( G \) along \( k_L \). If not, it moves it elsewhere in which case the Luttinger surface is destroyed. We appeal to topology to show that the latter does not obtain. At points in \( S_\Omega(\epsilon) \) we have that the complex \( N \times N \) matrix \( G \) is non-degenerate, since by definition the locus of points in momentum space on which it is degenerate is \( \Omega \). Therefore, we obtain a continuous analytic map

\[ \Omega_\epsilon : S_\Omega(\epsilon) \to GL(N, \mathbb{C}). \] (16)

Here \( GL(N, \mathbb{C}) \) is the group of invertible complex matrices with \( N \times N \) entries. Fixing a point \( k_L \in \Omega \), we have that the relevant set is then \( S_\Omega(\epsilon)_{k_L} = \{ k_L \ | k_L = k_L \} \) (which, in the language of fiber bundles, is the fiber of \( S_\Omega(\epsilon) \) at \( \Omega_L \) and this set \( S_\Omega(\epsilon)_{k_L} \) is an \( S^p \) sphere and the map \( \Omega \) at fixed \( k_L \) is

\[ \Omega'_\epsilon : S^p \to GL(N, \mathbb{C}). \] (17)

Any deformation \( H_M + g H_2 \) of the Hamiltonian \( H_M \) (here we think of this as the Hamiltonian of the fixed point which exhibits a MI nature) will deform this map \( \Omega \) continuously, thus preserving its homotopy class. Now, the main observation is
that if the homotopy class of $\Omega'_p$ in the $p$-th fundamental group $\pi_p(GL(N, \mathbb{C}))$ is non-zero, the Luttinger surface $\Omega$ must be stable under small deformations. In fact, if the image via $\Omega'_p$ of $S^p$ were the trivial class, then the map would be homotopic (so continuously deformable) to a constant map (i.e. mapping the whole of $S^p$ to a constant invertible matrix). But this would mean that the map $\Omega'_p$ could be extended to a map from the solid ball $B_p(k_L) = \{(k_L, k_{-1}) \in \mu_1 : |k_L - k_{-1}| = \epsilon \}$ centered at $k_L$ and of radius $\epsilon$ to $GL(N, \mathbb{C})$. This is impossible because $G$ is degenerate at $k_L$ by definition of the Luttinger surface. As a result, stability follows. Higher fundamental groups are notoriously complicated to calculate, but fortunately for classification purposes, via the use of Morse theory, R. Bott [30] was able to prove that they are periodic (and the period depends on the group)

$$\pi_k \left( \lim_{\to} GL(N, \mathbb{C}) \right) = \pi_{k+2} \left( \lim_{\to} GL(N, \mathbb{C}) \right)$$

and that in particular in the so-called stable regime or $N$ sufficiently large compared to $p$ ($N > \frac{2}{p}$ suffices),

$$\pi_k \left( \lim_{\to} GL(N, \mathbb{C}) \right) = \begin{cases} 0 & \text{if } k \text{ is even} \\ \mathbb{Z} & \text{if } k \text{ is odd} \end{cases}$$

We have thus established the fact that Luttinger surfaces of codimension $p + 1$ in momentum-energy space are stable for $p$ odd, and unstable for $p$ even, much like the Fermi surface case.

Our work here shows that models exhibiting a Luttinger surface, that is, a surface of zeros ultimately have a rigorous stability condition based in K-theory. It would be mistaken to associate the winding number $\pi_k \left( \lim_{\to} GL(N, \mathbb{C}) \right)$ with the charge density because in the Fermi surface case, the winding number counts the multiplicity of the poles and because each pole has a quasiparticle interpretation, the winding is equivalent to knowing the charge. For the zero surface [28, 29], no quasiparticle interpretation of zeros obtains. Hence, their multiplicity as indicated by the non-trivial winding number $\pi_k \left( \lim_{\to} GL(N, \mathbb{C}) \right)$ has no physical significance. This ultimately sheds light on why deviations from the Luttinger count [32] with the charge density have been so numerous [28, 29]. The existence of our stability condition implies that the details of the underlying Hamiltonian are irrelevant. The only quantity of relevance is the Luttinger surface. Consequently, our analysis puts all models with Luttinger surfaces under the same umbrella as they are controlled by a fixed point whose stability is ultimately controlled by K-theory and lack the $Z_2$ symmetry of a Fermi surface. The superconducting transition found earlier [16] should then be a generic feature of this fixed point. It is from the breaking of the discrete $Z_2$ symmetry that the particle-hole asymmetry (see Fig. 3) arises naturally, thereby leading to a direct response to Anderson’s reappraisal.

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SUPPLEMENTARY MATERIALS

A. 1/N expansion

While the $\mathbb{Z}_2$ symmetry is protected because of the renormalization of short-range interactions to zero at a Fermi surface, this symmetry can also be viewed from the perspective of an underlying 1/N expansion. As pointed out by Shankar[22], the renormalization principle for fermions can be recast as an effective 1/N expansion where $N = \frac{\Lambda}{\Lambda'}$ with $\Lambda$ the cut-off. The effective action is given by

$$S_F = \int \frac{d\theta}{2\pi} \int \frac{dk}{(2\pi)^d-1} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{\psi}(\omega k\theta)(i\omega - k)\psi(\omega k\theta)$$

with $F_{ij}$ containing the functional form of the informations. The $\Lambda \to 0$ (i.e. $N \to \infty$) limit is the Landau Fermi liquid theory. Therefore, in the large $N$ limit, the $\mathbb{Z}_2$ symmetry becomes exact; this of course corresponds to the fact that in the large $N$ limit one reduces to the Fermi surface and therefore making precise the heuristics that the symmetry is exact on the Fermi surface. A consequence of this is that unless the HK interaction is turned on, the RG flow of the Hamiltonian has a fixed point as shown in Fig. 1a. In the presence of the HK term, the flow is to the new strong-coupling fixed point.

B. K-theory

We review here the standard classifying scheme[33] in K-theory. Because one has that $S^d \sim \mathbb{R}^d \cup \{\infty\}$, one can calculate such homotopy classes of maps by considering continuous maps from $S^d$ as continuous maps from $\mathbb{R}^d$ with a boundary condition at $\infty$ (e.g., the maps converge to a fixed matrix, or equivalence class of matrices, as $|x| \to +\infty$). In K-theory this corresponds to the fact that

$$KO^{-i}(\mathbb{R}^d) = KO^{-i}(\{pt\}) \oplus KO^{-i}(\mathbb{R}^d) = KO^{-i}(\{pt\}) \oplus KO^{d-i}(\{pt\}).$$

The classifying spaces are denoted by $C_q$ in the complex case and $R_q$ in the real case. The index $q$ is an integer taken to be modulo 2 in the complex case and modulo 8 in the real one by Bott periodicity. Up to homotopy, the classifying spaces are given by

| $q$ | 0 | 1 |
|-----|---|---|
| $C_q$ | $(U(k+m))/(U(k) \times U(m)) \times \mathbb{Z}$ | $U(n)$ |
| $\pi_0(C_q)$ | $\mathbb{Z}$ | 0 |

and the real ones up to $q = 4$ (modulo 8)

| $q$ | 0 | 1 | 2 | 3 | 4 |
|-----|---|---|---|---|---|
| $R_q$ | $(O(k+m))/(O(k) \times O(m)) \times \mathbb{Z}$ | $O(n)$ | $O(2n)$ | $U(n)$ | $Sp(2n)$ | $Sp(k+m)/(Sp(k) \times Sp(m)) \times \mathbb{Z}$ |
| $\pi_0(R_q)$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | 0 | $\mathbb{Z}$ |

and finally

In order for our Hamiltonian to homotopically have the $\mathbb{Z}_2$ symmetry, we need our Hamiltonian to be equivalent in the sense of Eq. (9) to the free Fermi gas up to 1/N-corrections. In other words, because we want to preserve the $\mathbb{Z}_2$ symmetry (or the full $O(4)$ symmetry), the classifying space must arise from the real K-theory ones, $R_q$ and further there are only two possibilities for the classifying space. Either $R_q \sim O(n)$ for $n$ large (here $q$ is defined mod. 8) or $R_q \sim O(2n)/U(n)$. This gives a complete topological classification of Fermi liquids. As mentioned in the text only $O(4)$ Fermi liquids preserve the structure of $H_{FL}$. 
TABLE III. Real classifying spaces—the rest

| q | 5 | 6 | 7 |
|---|---|---|---|
| $R_q$ | $Sp(n)$ | $Sp(n)/U(n)$ | $U(n)/O(n)$ |
| $\pi_0(C_q)$ | 0 | 0 | 0 |

To unlock the underlying geometric structure, recall that a vector bundle $E \to X$ on a topological space $X$ is said to have real structure if it is endowed with an anti-linear bundle isomorphism $i: E \to E$ such that $i^2 = id_E$. If the involution is the trivial one then the bundle is simply a bundle over $\mathbb{R}$. The K-theory of bundle with real structures is denoted by $KO^i(X)$ and the one of real vector bundle is denoted by $KR^i(X)$.

Bott periodicity then says

$$KR^{i+8}(X) \sim KR^i(X) \quad \text{and} \quad KO^{i+8}(X) \sim KO^i(X) \quad (22)$$

The $KO$ groups of a point $X = \{pt\}$, and therefore of $\mathbb{R}^n$ since it is contractible, is

TABLE IV. $KO$ and $KR$ Bott periodicity

| $i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-----|---|---|---|---|---|---|---|---|
| $KO^{-i}$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | 0 | $\mathbb{Z}$ | 0 | 0 | 0 |

$KR$ groups of a sphere one writes $S^d = \mathbb{R}^d \cup \{\infty\}$ and then calculates

$$KR^{-i}(S^d) = KR^{-i}(\{pt\}) \oplus KR^{-i}(\mathbb{R}^d) = KO^{-i}(\{pt\}) \oplus KO^{-i}(\{pt\}) \quad (23)$$

and then use the table above. The same equation holds for $KO^{-i}(S^d)$. Note that Bott periodicity for real K-group in table IV is the same as the one in tables II and III.

The geometric meaning of the $\mathbb{Z}_2$ corresponding to the two sheets of $O(4)$ is related to orientability of the bundle. The structure group of a bundle being $O(n)$ means that the bundle is endowed with a (real) vector bundle metric. If the bundle is orientable, then the structure group (the group where the transition maps are taken) reduces to $SO(n)$ (which corresponds to fixing a quantization). A typical non-orientable bundle is the Möbius bundle.