On spin asymmetry in muon and tau decays

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The angular asymmetry in decays of polarized muons and tau leptons is discussed. Both the standard $V-A$ Fermi model and the general parameterization via Michel parameters are considered. Numerical importance of contributions suppressed by charged lepton mass ratio is underlined. Contribution of the second order QED correction is estimated in the leading logarithm approximation.

Keywords: muon decay; tau lepton decay; Michel parameters; single spin asymmetry.

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1. Introduction

Extremely accurate experiments on the muon decay lifetime\textsuperscript{12} give the value of the Fermi coupling constant $G_{\text{Fermi}}$ with the precision of about 0.5 ppm. That provides normalization for evaluation of many observables in electroweak physics starting from differential distributions in muon decays up to the $W$ boson mass measurements\textsuperscript{3} high-precision tests of the Standard Model (SM), and new-physics searches. New experiments with high statistics on tau lepton decays provide a good possibility to test the lepton universality hypothesis with increased accuracy.

Effects suppressed by the electron to muon mass ratio in the muon decay spectrum were computed within the standard $V-A$ Fermi model in the $\mathcal{O}(\alpha)$ order\textsuperscript{4} and in the $\mathcal{O}(\alpha^2)$ order\textsuperscript{5} approximations in perturbative QED. Recently in Ref.\textsuperscript{6} we also considered the mass effects in radiative muon and tau lepton decays within the model-independent approach.

Here we would like to continue discussion of the spin asymmetry in muon decay started in Ref.\textsuperscript{7} This quantity is an inclusive observable and can be potentially measured with a high accuracy. We will include in consideration leptonic decay modes of tau lepton. Besides the pure $V-A$ case, the asymmetry will be treated in the model-independent approach with the help of the Michel parameters. Special attention will be payed to the terms suppressed by the ratio of the produced charged lepton mass to the decaying one.

At the Born level the spin asymmetry can be defined as an explicitly inclusive

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observable. Meanwhile taking into account higher order corrections due to emission of real photons and light lepton pairs makes the situation more complicated or even ambiguous when several charged leptons appear in the final state. In order to compute the higher order contributions one should rely on a concrete experimental set-up. Both theoretical and experimental definitions of the asymmetry should be infrared safe, i.e. be numerically stable with respect to variations of soft and collinear radiation. Two choices of such definitions will be discussed below.

The paper is organized as follows. The next section contains the notation and preliminary remarks. Sect. 3 is devoted to the spin asymmetry treated in the two approaches mentioned above. Numerical estimates of the effects due to radiative corrections and the terms suppressed by mass ratio are also given there. Sect. 4 contains concluding remarks.

2. Preliminaries and notation

Within the SM, muon decays are described by interactions of vector currents formed by left fermions. Meanwhile, many models beyond the SM predict contributions of other kinds. Since the energy scale of new physics is (most likely) higher than the electroweak scale, the corresponding contributions can be parameterized by four-fermions interactions with different currents and coupling constants, see e.g.

Let us consider the general decay \( L^- \rightarrow l^- \bar{\nu}_l \nu_L \) in the rest frame of the heavy lepton (\( L \) is either \( \mu \) or \( \tau \), and the light lepton is either electron or muon). The differential distribution in the energy and angle of the final state charged lepton can be described with the help of the Michel parameters \( \rho_L, \xi_L, \delta_L, \eta_L \):

\[
\frac{d\Gamma(L^- \rightarrow l^- \bar{\nu}_l \nu_L)}{dx \, d\cos \theta_l} = \frac{m_l^5 (1 + r^2)^4}{32 \pi^3} \left[ (\frac{x}{2})^2 - x_0^2 G_{0,L} [F_{IS}(x) - F_{AS}(x)] \right] \cdot P_L \cos \theta_l,
\]

\[
F_{IS}(x) = x(1 - x)N + \frac{2N \rho_L}{\theta}(4x^2 - 3x - x_0^2) + N \eta_L x_0(1 - x),
\]

\[
F_{AS}(x) = \frac{N \xi_L \sqrt{x^2 - x_0^2}}{3} \left\{ 1 - x + \frac{2\delta_L(4x - 4 + \sqrt{1 - x_0^2})}{3} \right\},
\]

where \( P_L \) is the polarization degree of the initial lepton. In order to account possible violation of the lepton universality, we introduced the lepton index \( L \) for the effective Fermi constant \( G_{0,L} \) and for the Michel parameters. Angle \( \theta_L \) is chosen between the \( L^- \) spin and the final \( l^- \) momentum. The energy fraction of the produced charged lepton is denoted as \( x = E_l/E_l^{max} \). Its minimal value is \( x_0 = 2r/(1 + r^2) \), where \( r = m_l/m_L \); \( m_l \) and \( m_L \) are the masses of the decaying and produced charged leptons, respectively. The maximal lepton energy is \( E_l^{max} = m_L(1 + r^2)/2 \). The constant \( N \) and its products with the Michel parameters \( N \rho_L, N \xi_L, N \xi_L \delta_L, \) and \( N \eta_L \) are certain bilinear combinations of the four-fermion coupling constants. Their values are defined from the fits of experimental data on differential and integrated decay distributions (or taken from the SM).

\(^a\)In general, the tau lepton Michel parameters might also depend on the light lepton choice.
3. Muon decay spin asymmetry

Let us define the spin asymmetry in the leptonic decays of muon and tau lepton as

$$A_L = \frac{1}{\Gamma_L} \int_{x_0}^{1} dx \left\{ \int_{0}^{\theta_l} d\cos \theta_l \cdot \frac{d\Gamma(L^- \rightarrow l^- \nu \bar{\nu})}{dx d\cos \theta_l} \right\},$$

where $\Gamma_L$ is the total decay width,

$$\Gamma_L = \int_{x_0}^{1} dx \int_{-1}^{1} d\cos \theta_l \frac{d\Gamma(L^- \rightarrow l^- \nu \bar{\nu})}{dx d\cos \theta_l}.$$  (3)

Our definition of the asymmetry differs by factor 2 from the one adapted in Ref. 7 where instead of integration over the angle just the difference between the values at $\cos \theta_l = +1$ and $\cos \theta_l = -1$ was used. We also suggest to normalize the asymmetry by the total decay width instead of the tree level one used in Ref. 7.

3.1. Spin asymmetry in the model-independent approach

In the model-independent approach at the Born level the asymmetry is equal to the integral of function $F_{AS}(x)$ from Eq. (1):

$$A_0 = - \int_{x_0}^{1} F_{AS}(x) dx = - \int_{x_0}^{1} \frac{1}{3} N \xi L (1-x) \sqrt{x^2-x_0^2}$$

$$- \int_{x_0}^{1} dx \frac{1}{3} N \xi L \delta_L \sqrt{x^2-x_0^2} (A r - 4 + \sqrt{1-x_0^2})$$

$$= - \frac{1}{6} \{ N \xi L g_\xi (r) + N \xi L \delta_L g_\delta (r) \},$$

where 100% polarization ($P_L = 1$) is assumed and

$$g_\xi (r) = 1 - 20r^2 + 64r^3 - 90r^4 + 64r^5 - 20r^6 + r^8,$$

$$g_\delta (r) = \frac{80}{3} r^2 - 128r^3 + 240r^4 - \frac{640}{3} r^5 + 80r^6 - \frac{16}{3} r^8.$$  (5)

Note that the coefficients in front of the terms proportional to $r^2$ are numerically large. So these terms might be relevant for the analysis of the asymmetry in the decay $\tau \rightarrow \mu \nu_\mu \nu_\tau$ for which $r = m_\mu / m_\tau \approx 0.06$.

3.2. Asymmetry in the V − A case

One can see that in the pure $V − A$ case where $N \xi L = 1$ and $N \xi L \delta_L = 3/4$ for the Born-level asymmetry (4) we reproduce the known result, see e.g.,[2]

$$A_0(r) = - \frac{1}{6} g_0 (r) \equiv A_0(0) g_0 (r),$$

$$g_0 (r) = 1 - 32r^3 + 90r^4 - 96r^5 + 40r^6 - 3r^8.$$  (6)
Interestingly, the terms proportional to $r^2$ are canceled out in the sum $g_\xi(r) + \frac{3}{4}g_\delta(r)$.

Within the perturbative QED, the spin asymmetry can be presented in the form

$$A = A_0(0) \left[ g_0(r) + a g_1(r) + a^2 g_2(r) + \mathcal{O}(a^3) \right],$$

where $a \equiv \alpha/\pi$, and $\alpha \approx 1/137.036$ is the fine structure constant. The explicit expression for function $G_1(r) = 4A_0 g_1(r)$ can be found in Ref. [4], while here we will use its expansion in $r$:

$$g_1(r) = \frac{617}{72} - 7\zeta(2) + 4r + r^2 \left( 48 - 24\zeta(2) - 2 \ln r \right) + r^3 \left( \frac{284}{9} - \frac{112}{3} \ln r \right) + \mathcal{O}(r^4).$$

As one can see, the one-loop QED correction to the asymmetry is free from mass singularities, i.e. it is finite in the limit $r \to 0$. That is in accord with the Kinoshita–Lee–Nauenberg theorem. Contributions proportional to the logarithms of mass ratios, like $\ln(m_\mu^2/m_e^2) \approx 10.66$, are canceled out in sufficiently inclusive observables, including the total decay width and the spin asymmetry. Nevertheless, the dependence on the mass ratio logarithms remains in the effect of the fine structure constant running which gives $\alpha(0) \approx 1/137.0 \to \alpha(m_\mu) \approx 1/135.9$ and $\alpha(m_\tau) \approx 1/134.4$.

The $\mathcal{O}(a^2)$ contribution to the asymmetry was computed in Ref. [7] under a quite specific assumption of experimental setup. Namely, it was assumed that components of an electron-positron pair emitted at a small angle with respect to the final primary electron are recombined with the parent particle, i.e. a calorimetric event selection is applied. A parameter to define the QED jet resolution was introduced. The QCD-like construction was motivated by the necessity to have an infrared safe definition of the asymmetry.

The suggested definition is infrared safe only under conditions of the calorimetric event selection which is typical for hadron jet registration at high-energy colliders. But the definition has nothing to do with the typical experimental set-up used in muon decay spectrum measurements. Indeed, most such experiments exploit the possibility to define the electron momentum from its curved spiral track in a magnetic field. In particular it is so in the TWIST experiment [13,14]. Obviously, a sufficiently strong magnetic field destroys the structure of any QED jet (consisting either of one electron plus some photons or of several charged particles, like $e^+e^-e^-$), and the observed trajectories of all charged particles become separated from each other. In this case the treatment of events with several charged particles suggested in Ref. [7] is not infrared safe. Moreover, treatment of the spin asymmetry in events with observation of several charged tracks becomes ambiguous.

On the other hand, in experiments on tau lepton decays the momentum of the final charged lepton is typically much higher than in the case of muon decays,
especially if the tau lepton decays in flight like at the LHC. So the set-up considered in Ref.\textsuperscript{7} might be relevant for the experiments on leptonic tau decays in flight.

In order to estimate the $\mathcal{O}(\alpha^2)$ contribution in the case when the presence of a strong magnetic field destroys the QED jet structure, we suggest to use the following (simplified) set-up. Let us assume that if the detector sees more than one charged particle track, such an event is just dropped. It this case decays accompanied by production of a real $e^+e^-$ pair will be not accounted in the integrated decay asymmetry. But if the components of the pair have small energies they will not be detected at all. To simulate this situation, let us impose the simple cut-off on the total energy of the produced pair: events with real pairs with energy exceeding $\Delta m_L/2$ are dropped. Here $m_L$ is the mass of the decaying lepton and $\Delta \ll 1$ is a small parameter. The energy of the detected charged lepton (which exhibits the angular asymmetry) should be above the cut-off $\Delta m_L/2$. This definition of the asymmetry is obviously infrared safe because the domain of soft radiation is integrated out and the events with collinear pair emission are cut off. Numerical results below will be given for $\Delta = 0$ \textsuperscript{1} which corresponds to the maximal $e^+e^-$ pair energy $E_{\text{max}} \sim 10$ MeV in the muon decay and $E_{\text{max}} \sim 90$ MeV in the tau lepton decays $\tau \to e\bar{\nu}_e\nu_\tau e^+e^-$ and $\tau \to \mu\bar{\nu}_\mu\nu_\tau e^+e^-$. For this event selection procedure we will have contributions of the order $\mathcal{O}(\alpha^2 \ln^2(m_L^2/m_\tau^2))$ which are numerically dominant at the two-loop level.\textsuperscript{15–17} Moreover, the pair contribution will be also enhanced by the logarithm of the parameter $\Delta$. The described event selection removes the ambiguity in the treatment of events with two electrons in the final state. This also means that the quantum interference of these electrons is excluded since the energy domains of the primary electron and the secondary one (from the emitted pair) do not overlap.

There is also the so-called singlet channel kinematical situation when the primary electron and the secondary positron are soft, while the secondary electron has a large energy and it is detected instead of the primary one. The corresponding contribution can be estimated by looking at the integral of the singlet part of the electron structure function $D_{e^+e^-}^S(x)$, see e.g.\textsuperscript{15} over the interval $[1 - \Delta, 1]$. It is also enhanced by the square of the large logarithm but it is suppressed by the second power of the small parameter $\Delta$. That allows to neglect the singlet-channel contribution.

One should note that in Eq. (2) we defined the asymmetry as the integral over the total range of the observed electron energy. While as we discussed above the low-energy region (below $\Delta m_L/2$) might be not accessed experimentally. Here we will use the approximation where this region for the primary electron is not dropped. This would not much affect our estimates of the higher-order and mass effects since the electron spectrum at small energy fraction values $x \equiv 2E_e/m_\mu$ behaves as $\sim x^2$. But in a realistic application one should take into account the threshold of electron registration.

In the leading logarithmic approximation under the discussed experimental con-
ditions, the emission of an extra (virtual or real soft) $e^+e^-$ pair gives

$$g_2^{LLA}(r, \Delta) = \frac{1}{4} \ln^2(r^2) \left( 2 \ln \Delta + \frac{3}{2} \right).$$

(9)

Note that for the decay $\tau \rightarrow \mu e^+e^-\bar{\nu}_\mu\bar{\nu}_\tau$ there are two types of mass singularities, that leads to the substitution

$$\ln^2(r^2) \rightarrow \ln(m_\mu^2/m_\tau^2) \cdot \ln(m_e^2/m_\tau^2).$$

The NNLO result for virtual and soft $e^+e^-$ pair corrections to the spin asymmetry in muon decay was presented in Ref.\(^{15}\).

Numerical results for $g_0$, $g_1$, and $g_2^{LLA}$ are presented in Table 1. For tau lepton decays we took into account only the electron-positron pair contribution since its impact is enhanced by the large logarithm $\ln(m_\tau^2/m_e^2) \sim 16.3$ (even squared for $\tau \rightarrow 2e\bar{\nu}_e\bar{\nu}_\tau$) while the logarithm with the muon mass is considerably smaller, $\ln(m_\tau^2/m_\mu^2) \sim 5.6$.

| $r$ | $g_0(r)$ | $g_1(r)$ | $g_2^{LLA}(r)$ |
|-----|----------|----------|----------------|
| 0   | 1.000000 | -2.94509 | -88.3          |
| $m_\nu/m_\mu$ | 0.999996 | -2.92528 | -206.5         |
| $m_e/m_\tau$ | 1.000000 | -2.94394 | -71.5          |
| $m_\mu/m_\tau$ | 0.994327 | -2.64765 |                |

Table 1. Coefficients $g_0$, $g_1$, and $g_2^{LLA}$ vs the mass ratio $r$.

We would like to underline that our estimates of the second order contributions are performed only to have an idea about the size of the effect. If the experimental uncertainty is of the same order or lower, one should perform a Monte Carlo generation of decays with emission of additional photons or electron-positron pairs and let the events pass through a detector simulator.

The numerical result for asymmetry in Ref.\(^{7}\) was given by Eq. (9) in the form

$$A = A_0 \left[ 1 - 2.9451 \bar{a} + 11.2(1) \bar{a}^2 \right],$$

(10)

where $\bar{a} \equiv \alpha_{\overline{MS}}(m_\mu)/\pi$ and $\alpha_{\overline{MS}}(m_\mu) \approx 1/135.9$ is the QED coupling constant in the $\overline{MS}$ scheme at the muon mass scale. Calculations of the second order pair corrections in Ref.\(^{7}\) were also performed in the $\overline{MS}$ scheme. Comparisons of the numerical results for the radiative corrections to muon decay asymmetry given in Eq. (10) and Table 1 show a considerable difference in the $\mathcal{O}(\alpha^2)$ order. The difference consists of two parts: the one in the $g_2$ coefficient value and the one given by the scheme change, $i.e.$ $\bar{a} \cdot 2.9451 - (\alpha/\pi)g_1(0)$. The main source of the deviation is due to the different treatment of the events with real $e^+e^-$ pair creation.

Paper\(^{7}\) states that “the radiative corrections are more important than the electron mass effects.” This statement is not fully correct. Obviously for the muon decay case considered in Ref.\(^{7}\) the $r^3$ mass corrections at the tree level, see Eq. (6), are small compared even to the $\mathcal{O}(\alpha^2)$ contribution of radiative corrections and one

\(^{b}\)This substitution was used to get the number for $g_2^{LLA}$ in the last column in Table 1.
can safely take the limit $g_0(r) \to 1$. But the one-loop QED correction [5] contains the contribution proportional to the first power of the mass ratio $r$, which makes it exactly of the same order as the $\mathcal{O}(\alpha^2)$ contribution found in Ref. [7]. Namely, $4(m_e/m_\mu)\bar{a} \approx 8.3\bar{a}^2$. This contribution was missed in Eq. (10). It is interesting to note that such linear in mass terms are not typical in the first order radiative corrections to isotropic observables while they sometimes appear in asymmetries, see e.g. Ref. [10].

4. Conclusions

Taking into account the spin orientation of the initial particle in decays of muon and tau lepton allows to get additional information about the structure of weak interactions. High statistics on tau lepton decays at several modern and future experiments admits high-precision tests of the lepton universality hypothesis.

In this paper we discussed the spin asymmetry in the muon decay and the leptonic modes of tau decays. Explicit expressions for the asymmetry in model-independent description via the Michel parameters at the Born level are derived with taking into account the terms suppressed by the charged lepton mass ratio. These terms can be relevant for future high-precision studies of the muon decay spectrum and especially for experiments on the decay $\tau \to \mu \nu_\mu \nu_\tau$. We pointed out that the terms suppressed by the charged lepton mass ratio are also numerically relevant in the contribution of the first order radiative corrections. This fact was discovered in Ref. [4] but it was missed in the earlier paper on muon decay asymmetry. [7] Radiative corrections and mass effects were considered for muon and tau lepton decays in parallel. The second order QED corrections in the $V-A$ case were estimated above for a simplified (but still reasonable) experimental set-up with a cut-off on events with production of extra charged particles.

The most accurate measurements of the differential distributions in polarized muon decay were performed by the TWIST collaboration. [14,20,21] The resulting uncertainty for the extracted Michel parameters reached the $10^{-4}$ order. The TWIST experiment did not cover the full angular phase space and it was not suited to measure the decay asymmetry. There were also inclusive experiments which measured the total muon decay width with the precision up to 0.5 ppm. [11,2] But the asymmetry was not measured there as well. We suggest to foresee the asymmetry measurement in future experiments on muon (and hopefully tau lepton) decays. As discussed in the Introduction, a new more precise number for the muon lifetime can not serve to test the Standard Model, since the precision of its theoretical prediction doesn’t have such a high accuracy. But the asymmetry value is sensitive to the presence of non-standard weak interactions and its measurement in future experiments potentially can be performed with the accuracy better than $10^{-4}$ achieved in the TWIST experiment.

In conclusion we would like to underline that the presented results can be used as an estimate of higher order effects while to treat experimental data on spin
asymmetries one would need to perform Monte Carlo simulations.

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