BOUNDEDNESS OF WEAK SOLUTIONS OF DEGENERATE QUASILINEAR EQUATIONS WITH ROUGH COEFFICIENTS

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Abstract. We derive local boundedness estimates for weak solutions of a large class of second order quasilinear equations. The structural assumptions imposed on an equation in the class allow vanishing of the quadratic form associated with its principal part and require no smoothness of its coefficients. The class includes second order linear elliptic equations as studied in [GT] and second order subelliptic linear equations as in [SW1, 2]. Our results also extend ones obtained by J. Serrin [S] concerning local boundedness of weak solutions of quasilinear elliptic equations.

1. Introduction

The main purpose of this paper is to prove local boundedness of weak solutions u of rough subelliptic quasilinear equations of the form

\begin{equation}
\text{div}(A(x, u, \nabla u)) = B(x, u, \nabla u)
\end{equation}

in an open set \( \Omega \subset \mathbb{R}^n \). Further regularity results will be studied in a sequel to this paper. We will assume that the vector-valued function A and the scalar function B satisfy specific structural restrictions on their size, but not on their smoothness, relative to a symmetric nonnegative semi-definite matrix \( Q(x) \). Thus the quadratic form \( Q(x, \xi) = \langle Q(x)\xi, \xi \rangle, \quad \xi \in \mathbb{R}^n \), may vanish when \( \xi \neq 0 \).

More precisely, given \( p \) and an \( n \times n \) matrix \( Q \) with \( 1 < p < \infty \) and \( |Q| \in L_{\text{loc}}^{p/2}(\Omega) \), our weak solutions are pairs \((u, \nabla u)\) belonging to an appropriate Banach space \( W_{Q}^{1,p}(\Omega) \). As described in [SW2], \( W_{Q}^{1,p}(\Omega) \) is obtained via isomorphism from the degenerate Sobolev space \( W_{Q}^{1,p}(\Omega) \) defined to be the completion with respect to the norm

\begin{equation}
||u||_{W_{Q}^{1,p}(\Omega)} = \left( \int_{\Omega} |u|^p dx + \int_{\Omega} Q(x, \nabla u)^{\frac{p}{2}} dx \right)^{\frac{1}{2}}
\end{equation}

of the class of functions in \( \text{Lip}_{\text{loc}}(\Omega) \) with finite \( W_{Q}^{1,p}(\Omega) \) norm. We will give some further discussion about these Banach spaces below. The structural conditions which we assume are that there exists a vector \( \tilde{A}(x, z, \xi) \), \( (x, z, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^{n} \), with values in \( \mathbb{R}^{n} \), such that for a.e. \( x \in \Omega \) and all \( (z, \xi) \in \mathbb{R} \times \mathbb{R}^{n} \),

\begin{equation}
\begin{aligned}
(i) \quad & A(x, z, \xi) = \sqrt{Q(x)} \tilde{A}(x, z, \xi), \\
(ii) \quad & \xi \cdot A(x, z, \xi) \geq a^{-1} \sqrt{Q(x)} \xi^p - h|z|^q - g, \\
(iii) \quad & |\tilde{A}(x, z, \xi)| \leq a \sqrt{Q(x)} \xi^{p-1} + b|z|^{q-1} + c, \\
(iv) \quad & |B(x, z, \xi)| \leq c \sqrt{Q(x)} \xi^{\nu-1} + d|z|^{\delta-1} + f,
\end{aligned}
\end{equation}

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where $a, \gamma, \psi, \delta > 1$ are constants, and $b, c, d, e, f, g, h$ are nonnegative functions of $x$. In fact, when dealing with a particular weak solution $(u, \nabla u)$, it will be enough to assume that parts (ii), (iii) and (iv) of (1.3) hold with $z$ and $\xi$ replaced by $u(x)$ and $\nabla u(x)$ respectively.

The sizes of $\gamma, \psi$ and $\delta$ will be further restricted in terms of $p$ and a natural “Sobolev gain factor” $\sigma > 1$ to be described below in (1.13), while the functions $b, c, d, e, f, g, h$ will be assumed to lie in appropriate Lebesgue or Morrey classes related to $p, \sigma, \gamma, \psi$ and $\delta$. For the classical Euclidean metric, non-degenerate $Q$ and $1 < p < n$, the Sobolev gain is $\sigma = n/(n - p)$. In general, we will always restrict $\gamma, \psi, \delta$ to the ranges

$$(1.4) \quad \gamma \in (1, \sigma(p - 1) + 1), \quad \psi \in (1, p + 1 - \sigma^{-1}), \quad \delta \in (1, p\sigma).$$

We will often refer to $a, b, c, d, e, f, g, h$ as structural coefficients, or simply as coefficients. Except for $a$, which is constant, the coefficients must always satisfy certain minimal local integrability requirements: see (2.12).

We remark that the set of structural properties (1.3) is invariant under replacing the symmetric nonnegative semidefinite matrix $Q(x)$ by another symmetric nonnegative semidefinite matrix $M(x)$ which is equivalent to it, i.e., which satisfies

$$(1.5) \quad \frac{1}{C} \langle Q(x)\xi, \xi \rangle \leq \langle M(x)\xi, \xi \rangle \leq C \langle Q(x)\xi, \xi \rangle \quad \text{for all } \xi \in \mathbb{R}^n, \text{ a.e. } x \in \Omega;$$

see Theorem 5.1 in Appendix 2.

We also remark that the structural assumptions (1.3) are equivalent to the following set of assumptions: there exists a nonnegative function $\tilde{a}(x, z, \xi)$, $(x, z, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$, such that for a.e. $x \in \Omega$, all $\eta \in \mathbb{R}^n$ and all $(z, \xi) \in \mathbb{R} \times \mathbb{R}^n$,

$$(1.6) \begin{cases} 
(i) \quad \eta \cdot A(x, z, \xi) \leq \sqrt{Q(x)\eta} \tilde{a}(x, z, \xi), \\
(ii) \quad \xi \cdot A(x, z, \xi) \geq a^{-1} \sqrt{Q(x)\xi}^p - b|z|\gamma - g, \\
(iii) \quad \tilde{a}(x, z, \xi) \leq a \sqrt{Q(x)\xi}^p + b|z|^\gamma + c, \\
(iv) \quad |B(x, z, \xi)| \leq c \sqrt{Q(x)\xi}^p + d|z|^\gamma + f;
\end{cases}$$

see Theorem 5.3 in Appendix 2.

Historically speaking, in the classical elliptic case ($Q(x) = \text{Identity}$), structural conditions more restrictive than (1.3) were considered by J. Serrin [S], who derived a broad class of regularity results for weak solutions of (1.1). Our ranges of the parameters $\gamma, \psi, \delta$ are however wider than those studied in [S], where these parameters are all equal to $p$. In case $p = 2$, our ranges correspond more closely to those in [G p. 176] and [GM], although we miss some endpoint values. These latter papers impose continuity conditions on coefficients which we do not assume, but which lead to stronger regularity and also to results for systems. N. Trudinger [T] also derived regularity results in the elliptic case, relaxing some structural conditions under the assumption of local boundedness of weak solutions, but generally for the same choices as in [S]. We note in passing that the equation for the $p$-Laplacian, namely $\text{div}(\nabla u |\nabla u|^{p-2}) = 0$, as well as Yamabe type equations $\Delta u - Ru + Ru^{n-1} = 0$ for $q < 2n/(n - 2)$ are included in the case $Q(x) = \text{Id}$ (with $p = 2$ for the Yamabe type equations).

In the subelliptic case, by which we mean the case when $Q(x)$ may be singular, regularity results including local boundedness of weak solutions are derived in [SW1, 2] for linear equations with rough coefficients and nonhomogeneous terms. The form of the linear equations studied there is

$$(1.7) \quad \text{div} \left( M(x)\nabla u + H(x)R(x)u + S(x)G(x)u + F(x)u = F_1(x) + T(x)'G_1(x), \right)$$

where $M(x)$ is a symmetric matrix whose quadratic form $M(x, \xi)$ satisfies

$$c_1 Q(x, \xi) \leq M(x, \xi) \leq c_2 Q(x, \xi)$$

for some positive constants $c_1, c_2$, and where $H, G, F, F_1, G_1$ are functions on $\Omega$ and $R, S, T$ are vector fields on $\Omega$ that are subunit with respect to $Q(x)$. Here we say that a vector field $V(x)$ is subunit with respect to $Q$ if $V(x)u(x) = v(x) \cdot \nabla u(x)$ and $|v(x) \cdot \xi|^2 \leq Q(x, \xi)$ for almost all $x \in \Omega,$
all $\xi \in \mathbb{R}^n$ and all Lipschitz continuous functions $u$ on $\Omega$. By direct computation (see Theorems 5.2 and 5.1 and relations (5.10) in Appendix 2), such a linear subelliptic equation satisfies the structural conditions (1.3) with $p = \gamma = \psi = \delta = 2$. Our principal result includes the local boundedness estimates in [SW1, 2] for solutions of (1.7) and can be viewed as an extension of them to solutions of quasilinear equations.

The regularity results in [SW1, 2] for equations of type (1.7) were derived in an axiomatic setting which assumes the existence of appropriate Sobolev-Poincaré estimates in a space of homogeneous type, as well as the existence of sequences of Lipschitz cutoff functions. We will derive our estimates for weak solutions of (1.1) in a quasimetric setting, but our axioms are generally less restrictive than those in [SW1, 2]. For example, we do not need the assumption in [SW1, 2] that Lebesgue measure satisfies the doubling property relative to quasimetric balls. In fact, our main result Theorem 1.2 requires no hypothesis at all about doubling, and Corollaries 1.8–1.11 use only the condition $D_\mu$ listed in Definition 1.7. Also, our main Sobolev-Poincaré assumption will be one of Sobolev type for compactly supported functions. Unlike [SW1, 2], where not only local boundedness but also Hölder continuity of weak solutions is obtained, we will not require any Poincaré estimate for non-compactly supported functions. However, depending on the order of integrability of $|Q|$, we sometimes assume that functions $w$ in $W^{1,p}_Q(\Omega)$ satisfy higher local integrability than order $p$.

In order to state our main theorem, we now briefly describe the axiomatic framework. A fuller discussion can be found in [SW1, 2]. Facts about degenerate Sobolev spaces $W^{1,p}_Q(\Omega)$ for $p = 2$, are given in [R]. Here we recall some of them. Let $\text{Lip}_Q(\Omega)$ denote the class of locally Lipschitz functions with finite $W^{1,p}_Q(\Omega)$ norm. By definition, $W^{1,p}_Q(\Omega)$ is the Banach space of equivalence classes of Cauchy sequences in $\text{Lip}_Q(\Omega)$, i.e., the Banach space with norm (1.8) for $1 \leq p < \infty$.

Identifying measurable $\mathbb{R}^n$-valued functions $f, g$ which satisfy $\|f - g\|_{L^p(\Omega,Q)} = 0$ in $L^p(\Omega,Q)$, we define a norm on the resulting vector space of equivalence classes; we define $L^p(\Omega,Q)$ as the space consisting of these equivalence classes. When $p = 2$, $L^2(\Omega,Q)$ is shown to be a Hilbert space with inner product $(f, g) = \int_{\Omega} f(x)^*Q(x)g(x)\,dx$ in Theorem 4 of [SW2]. and the arguments in the proof there show that $L^p(\Omega,Q)$ is a Banach space with norm (1.8) for $1 \leq p < \infty$. If $\{w_k\}_{k=1}^\infty \subset W^{1,p}_Q(\Omega)$, i.e., if $\{w_k\}_{k=1}^\infty$ is a Cauchy sequence in $W^{1,p}_Q(\Omega)$ of sequences of Lipschitz functions, there is a unique pair $(w, v) \in L^p(\Omega) \times L^p(\Omega,Q)$ such that $w_k \to w$ in $L^p(\Omega)$ and $\nabla w_k \to v$ in $L^p(\Omega,Q)$. The pair $(w, v)$ represents the equivalence class in $W^{1,p}_Q(\Omega)$ which contains the Cauchy sequence $\{w_k\}$. Any pair $(w, v)$ representing an equivalence class in $W^{1,p}_Q(\Omega)$ is said to belong to the space $W^{1,p}_Q(\Omega)$. Thus, $W^{1,p}_Q(\Omega)$ is the image of the isomorphism $\mathcal{J} : W^{1,p}_Q(\Omega) \to L^p(\Omega) \times L^p(\Omega,Q)$ defined by

$$\mathcal{J}(\{w_k\}) = (w, v),$$

where $\{w_k\}$ denotes the equivalence class in $W^{1,p}_Q(\Omega)$ containing the Cauchy sequence $\{w_k\}$. Therefore, $W^{1,p}_Q(\Omega)$ is a closed subspace of $L^p(\Omega) \times L^p(\Omega,Q)$ and hence a Banach space as well. As the spaces $W^{1,p}_Q(\Omega)$ and $W^{1,p}_Q(\Omega)$ are isomorphic, we will often refer to elements $(w, v)$ of $W^{1,p}_Q(\Omega)$ as elements of $W^{1,p}_Q(\Omega)$, where the isomorphism is taken in context. We caution the reader that $v$ is not generally uniquely determined by $w$ for pairs $(w, v)$ in $W^{1,p}_Q(\Omega)$, i.e., the projection $P : W^{1,p}_Q(\Omega) \to L^p(\Omega)$ obtained by mapping a pair onto its first component is not always an injection, as shown by an example in [F]. Nevertheless, we will generally abuse notation and denote representative pairs in $W^{1,p}_Q(\Omega)$ by $(w, \nabla w)$ instead of $(w, v)$. Moreover, we will often abuse notation even further by

$$P : W^{1,p}_Q(\Omega) \to L^p(\Omega).$$
simply writing \( w \) instead of the pair \((w, \nabla w)\). Some additional facts about degenerate Sobolev spaces are listed in Section 2 and Appendix 1 below.

In [SW2], the notion of the regular gradient \( \nabla_{reg} w \) of an element \( w \) in \( W^{1,p}_Q(\Omega) \) is introduced and used to derive results related to regularity of linear subelliptic equations. However, in the present paper, we have been able to avoid this technical device; see the comment which follows Corollary 2.10.

By a quasimetric \( \rho \) on \( \Omega \), we mean a finite nonnegative function on \( \Omega \times \Omega \) such that for some constant \( \kappa \geq 1 \),

\[
\rho(x,y) = 0 \text{ iff } x = y
\]

\[
\rho(x,y) \leq \kappa[\rho(x,z) + \rho(y,z)] \text{ if } x,y,z \in \Omega.
\]

For simplicity, we will also assume that \( \rho \) is symmetric, i.e., that \( \rho(x,y) = \rho(y,x) \) if \( x,y \in \Omega \). For \( x \in \Omega \) and \( r > 0 \), define the sets

\[
B(x,r) = \{ y \in \Omega : \rho(x,y) < r \},
\]

\[
D(x,r) = \{ y \in \Omega : |x - y| < r \},
\]

and assume that \( B(x,r) \) is Lebesgue measurable for every \( r > 0 \), \( x \in \Omega \). We call \( B(x,r) \) the quasimetric ball (or \( \rho \)-ball) with center \( x \) and radius \( r \), and we sometimes write \( B_r(x) \) or simply \( B_r \) instead of \( B(x,r) \). Throughout the paper we will assume that

\[
(1.9) \quad \text{for all } x \in \Omega, \ |x - y| \to 0 \text{ if } \rho(x,y) \to 0,
\]

and in some of our results we will also assume that

\[
(1.10) \quad \text{for all } x \in \Omega, \ \rho(x,y) \to 0 \text{ if } |x - y| \to 0.
\]

We remark that condition (1.9) is equivalent to requiring that

\[
(1.11) \quad \text{for every } x \in \Omega \text{ and every } \epsilon > 0 \text{ there exists } \delta > 0, \text{ depending on } x \text{ and } \epsilon,
\]

such that \( B(x,\delta) \subset D(x,\epsilon) \),

while condition (1.10) is equivalent to

\[
(1.12) \quad \text{for every } x \in \Omega \text{ and every } r > 0 \text{ there exists } s > 0, \text{ depending on } x \text{ and } r,
\]

such that \( D(x,s) \subset B(x,r) \).

Then condition (1.9), or equivalently condition (1.12), implies that \( |B(x,r)| > 0 \) for every \( \rho \)-ball with \( r > 0 \). By Lemma 2.1 in Section 2, condition (1.9) implies that for every \( x \in \Omega \), one has \( B(x,r) \subset \Omega \) if \( r \) is smaller than a suitable \( r_0 = r_0(x) > 0 \); here \( \hat{E} \) denotes the Euclidean closure of a set \( E \subset \Omega \).

Given \( p \), \( 1 < p < \infty \), and a nonnegative semidefinite quadratic form \( Q(x,\xi) = \langle Q(x)\xi,\xi \rangle \), where \( Q(x) \) is a symmetric matrix for each \( x \in \Omega \) and \( |Q| \in L_{loc}^{p/2}(\Omega) \), we need the following Sobolev estimate: there exist \( \sigma > 1 \) and \( C > 0 \) such that for every \( \rho \)-ball \( B_r = B(y,r) \) with \( y \in \Omega \) and \( 0 < r < r_1(y) \) for a suitable \( r_1(y) > 0 \),

\[
(1.13) \quad \left( \frac{1}{|B_r|} \int_{B_r} |w|^{p\sigma} \, dx \right)^{\frac{1}{p\sigma}} \leq C \left[ r \left( \frac{1}{|B_r|} \int_{B_r} Q(x,\nabla w)^\frac{p}{2} \, dx \right)^{\frac{1}{p}} + \left( \frac{1}{|B_r|} \int_{B_r} |w|^p \, dx \right)^{\frac{1}{p}} \right],
\]

for all pairs \( (w, \nabla w) \in (W^{1,p}_Q)_0(B_r) \). Here \( (W^{1,p}_Q)_0(B_r) \) denotes the analogue of the space \( W^{1,p}_Q(B_r) \) defined earlier but now the completion with respect to (1.2), with \( \Omega \) now replaced by \( B_r \), is formed by using Lipschitz functions with compact support in \( B_r \). Even though \( \nabla w \) may not be determined uniquely by \( w \), it follows that (1.13) holds for all \( (w, \nabla w) \in (W^{1,p}_Q)_0(B_r) \) provided it holds for all Lipschitz functions with compact support in \( B_r \). We also note that since \( Q(x,\xi) = |\sqrt{Q(x)}\xi|^2 \), (1.13) can be rewritten as

\[
(1.13) \quad \left( \frac{1}{|B_r|} \int_{B_r} |w|^{p\sigma} \, dx \right)^{\frac{1}{p\sigma}} \leq C \left[ r \left( \frac{1}{|B_r|} \int_{B_r} |\sqrt{Q}\nabla w|^p \, dx \right)^{\frac{1}{p}} + \left( \frac{1}{|B_r|} \int_{B_r} |w|^p \, dx \right)^{\frac{1}{p}} \right].
\]
The number $\sigma$ is a factor which measures the “Sobolev gain” in integrability of $w$, from $L^p(B_r)$ to $L^{p\sigma}(B_r)$ independently of $B_r$; $\sigma$ plays a crucial role in our results.

We will always assume that $r_1(y) \leq r_0(y)$ for every $y \in \Omega$, where $r_1(y)$ is as in (1.13) and $r_0(y)$ is as in Lemma 2.1. In particular, it then follows that the closure of any ball $B(y, r)$ with $r < r_1(y)$ lies in $\Omega$.

We also require the existence of appropriate sequences of Lipschitz cutoff functions (called “accumulating sequences of Lipschitz cutoff functions” in [SW1]), namely, we require that for some collection which satisfies (1.14) except for the second part, simply define a new collection $\{\eta_j\}_{j=1}^\infty$ with the properties

$$
\begin{align*}
\supp \eta_1 & \subset B(y, r) \\
0 & \leq \eta_j \leq 1 \quad \text{for all } j \geq 1 \\
B(y, \tau r) & \subset \{x \in B(y, r) : \eta_j(x) = 1\} \quad \text{for all } j \geq 1 \\
\supp \eta_{j+1} & \subset \{x \in B(y, r) : \eta_j(x) = 1\} \quad \text{for all } j \geq 1 \\
\left(\frac{1}{|B(y, r)|} \int_{B(y, r)} |\sqrt{Q}\nabla \eta_j|^{s^*} dx\right)^{\frac{1}{s^*}} & \leq C_s \frac{N^j}{r} \quad \text{for all } j \geq 1.
\end{align*}
$$

We remark that the above condition is slightly different from that appearing in [SW1], and it is actually weaker. Indeed, the key final property in (1.14) is weaker than its analogue in [SW1] where the exponential growth constant $N^j$ is replaced by $j^N$. Further, it is assumed in [SW1] that $r_1(y) = \delta_0 \text{dist}(y, \partial \Omega)$ for some $\delta_0 > 0$, where “dist” denotes the standard Euclidean distance in $\mathbb{R}^n$. The second property, $0 \leq \eta_j \leq 1$ for every $j$, is not required in [SW1]. However, if $\{\eta_j\}_{j=1}^\infty$ is some collection which satisfies (1.14) except for the second part, simply define a new collection $\{\tilde{\eta}_j\}_{j=1}^\infty$ by

$$
\tilde{\eta}_j(x) = \begin{cases} 
\eta_j(x) & \text{if } 0 < \eta_j(x) < 1, \\
0 & \text{if } \eta_j(x) \leq 0, \\
1 & \text{if } \eta_j(x) \geq 1.
\end{cases}
$$

for each $j$. This new collection then satisfies (1.14) as written. We also remark that since $s^* > p\sigma'$, we may choose a number $s' > \sigma'$ so that $s^* = s'p$. The exponent $s$ which is dual to $s'$, i.e., so that $1/s + 1/s' = 1$, satisfies $1 \leq s < \sigma$ and plays an important role in our results.

Another assumption, generally simpler than (1.14), which we will impose in our main theorem is that there exists $t$, $1 \leq t \leq \infty$, such that for every $\rho$-ball $B(y, r)$ with $0 < r < r_1(y)$ and every $\eta = \eta_j$ in the corresponding sequence $\{\eta_j\}$ provided by (1.14),

$$
\left(\int_{B(y, r)} |\sqrt{Q}\nabla \eta|^{tp} dx\right)^{\frac{1}{p}} < \infty.
$$

In fact, by (1.14), condition (1.15) is automatically satisfied for every $t$ with $1 \leq t \leq s^*/p$. On the other hand, (1.15) might hold for larger values of $t$ independently of (1.14); for example, if $Q(x)$ is bounded, then (1.15) holds with $t = \infty$ for all $B(y, r)$ with closure in $\Omega$ and for every $\eta \in Lip_0(\Omega)$, even if (1.14) is not valid for any $s^*$. To derive some of the preliminary results in Section 2, we will assume (1.15) for more restricted classes of balls $B(y, r)$ and functions $\eta$. In any case, (1.15) as well as (1.14) below are only qualitative conditions, in the sense that the constants involved in both of them will not enter our final estimates.

In our main theorem, (1.15) will be paired with the following assumption, where $t'$ is the usual dual index of $t$ given by $1/t + 1/t' = 1$: for every $\rho$-ball $B(y, r)$ with $0 < r < r_1(y)$, there is a constant $c_2 = c_2(B(y, r))$ so that for all $f \in Lip_0(\Omega)$,

$$
\left(\int_{B(y, r)} |f|^{t'} dx\right)^{\frac{1}{t'}} \leq c_2 \|f\|_{W^{1,p}_{\text{loc}}(\Omega)} = c_2 \left(\int_{\Omega} |\sqrt{Q}\nabla f|^{p} dx + \int_{\Omega} |f|^{p} dx\right)^{\frac{1}{p}}.
$$
It is easy to see that condition (1.16) holds for all elements of \( W^1_p(Q) \) and not just for functions in \( \text{Lip}_{loc}(\Omega) \).

In Section 2, (1.15) and (1.16) will be used to derive a useful version of the product rule. They will also be used to prove that functions in \( W^1_p(Q) \), which are generally without compact support, have sufficiently high local integrability in the presence of the Sobolev estimate (1.13) for compactly supported ones. See Proposition 2.3 for an estimate of \( ||w||_{L^p(B(y,\tau r))}, 0 < \tau < 1 \), in case \( w \in W^1_p(Q) \) and \( B(y,r) \) is any \( p \)-ball with \( 0 < r < r_1(y) \). As is true for (1.15), we sometimes assume in Section 2 that (1.16) holds for a smaller class of balls.

**Remark 1.1.** Note that condition (1.16) becomes weaker as \( t' \) becomes smaller. In particular, if (1.15) holds with \( t = \infty \) (e.g., if \( Q \in L_{loc}^{\infty}(\Omega) \) or if (1.14) is valid with \( s^* = \infty \)), then \( t' = 1 \) and (1.16) is trivially true.

When (1.14) holds for some \( s^* > p\sigma' \), then (1.15) is automatically true with \( t = s^*/p \), and the corresponding \( t' \) in (1.16) satisfies \( 1 \leq t' < \sigma \). In case \( t' < \sigma \), (1.16) is considerably weaker than the Sobolev inequality (1.13) when restricted to Lipschitz functions \( f \) with compact support in \( B(y,r) \). On the other hand, (1.16) is assumed to hold for any locally Lipschitz function whether it is compactly supported in \( B(y,r) \) or not.

In the case the Poincaré inequality

\[
\left( \frac{1}{|B_r|} \int_{B_r} |f - f_{B_r}|^{p'} \, dx \right)^{\frac{1}{p'}} \leq C_{r} \left( \frac{1}{|B_r|} \int_{B_r} Q(x, \nabla f)^{\frac{p}{p-1}} \, dx \right)^{\frac{1}{p}}, \quad f_{B_r} = \frac{1}{|B_r|} \int_{B_r} f \, dx,
\]

holds with \( B_r = B(y,r) \) for all \( f \in \text{Lip}_{loc}(\Omega) \), then (1.10) clearly holds as well.

In many cases of interest, conditions (1.13), (1.14), (1.15) and (1.16) automatically hold. An enormous related literature exists, and we refer to [SW1] for an introduction to it. In particular, (1.14) is known to hold with \( s^* = \infty \) for the subunit balls \( K(x,\xi) \) associated with a quadratic form \( Q(x,\xi) \) that is continuous in \( x \), provided the Fefferman-Phong condition [FP] holds, i.e., provided there are positive constants \( c_0, \epsilon \) such that for every \( K(x,\xi) \) with closure in \( \Omega \), there is a Euclidean ball \( D(x,r) \) satisfying

\[
D(x,r) \subset K(x,\epsilon r^{s^*}).
\]

Notice that this condition in particular implies (1.12), i.e., condition (1.10), for the subunit balls \( K(x,\xi) \). In order to elaborate, we extend (as in [SW1]) the notion of subunit metric to a nonnegative continuous quadratic form \( Q(\xi) \) on \( \Omega \) by defining

\[
\delta(x,y) = \inf \{ r > 0 : \gamma(0) = x, \gamma(r) = y, \gamma \text{ is a Lipschitz subunit curve in } \Omega \},
\]

where a Lipschitz curve \( \gamma : [0,\tau] \to \Omega \) is said to be subunit (with respect to \( Q(x,\xi) \)) if

\[
(\gamma'(t) \cdot \xi)^2 \leq Q(\gamma(t),\xi)
\]

for a.e. \( t \in [0,\tau] \) and all \( \xi \in \mathbb{R}^n \). Then \( \delta(x,y) \) is a symmetric metric on \( \Omega \), although possibly infinite if \( Q \) is degenerate. If \( \delta(x,y) \) is finite for all \( x,y \in \Omega \), the subunit balls \( K(x,\xi) \) are defined by

\[
K(x,r) = \{ y \in \Omega : \delta(x,y) < r \}, \quad x \in \Omega, \quad 0 < r < \infty.
\]

Assuming that \( Q \) is continuous, that \( \delta(x,y) \) is finite, and that the Fefferman-Phong containment condition holds, it is shown in [SW1] (and, under more restrictive assumptions, in the related references listed there) that (1.14) holds with \( s^* = \infty \) for the balls \( K(x,\xi) \).

We say that a pair \((u,\nabla u)\) in \( W^{1,p}_Q(\Omega) \) is a weak solution of (1.1) if

\[
\int_{\Omega} [\nabla \varphi \cdot A(x,u,\nabla u) + \varphi B(x,u,\nabla u)] \, dx = 0 \quad \text{for all } \varphi \in \text{Lip}_0(\Omega),
\]

where \( \text{Lip}_0(\Omega) \) denotes the class of Lipschitz functions with compact support in \( \Omega \).
The main results of this paper are the following theorem and corollaries, in which we will use the notation
\[ \|f\|_{\alpha, E; dx} = \left( \int_{E} |f(x)|^{\alpha} \, dx \right)^{1/\alpha}, \quad \|f\|_{\alpha, E, dx} := \left( \frac{1}{|E|} \int_{E} |f(x)|^{\alpha} \, dx \right)^{1/\alpha} \]
whenever \( E \subset \Omega \) is Lebesgue measurable, \( f \) is a Lebesgue measurable function on \( E \), and \( \alpha > 0 \).

**Theorem 1.2.** Let \( \Omega \) be an open set in \( \mathbb{R}^n \), \( 1 < p < \infty \), and \( Q(x, \xi) = \langle Q(x)\xi, \xi \rangle \) be a symmetric nonnegative semidefinite quadratic form on \( \Omega \) with \( |Q| \in L^{p/2}_{loc}(\Omega) \). Suppose that \( (\Omega, \rho) \) is a quasimetric space, that condition (1.13) holds, and that there exists \( \alpha > 1 \) such that the Sobolev estimate (1.11) holds for all \( (w, \nabla w) \in (W_{Q}^{1,p})_{0}(B) \) for all \( \rho \)-balls \( B = B(y, r) \) with \( 0 < r < r_1(y) \). Let \( A(x, z, \xi) \) and \( B(x, z, \xi) \) satisfy the structural assumptions (1.12) with
\[ \gamma = \delta = p, \quad \psi \in [p, p + 1 - \sigma^{-1}) \]
Suppose that condition (1.14) about Lipschitz cutoff functions holds for some \( \tau \in (0, 1) \) and \( s^* > p\tau \), and that conditions (1.15) and (1.16) hold with \( 1/t + 1/t' = 1 \) for some \( t \geq 1 \) and all \( \rho \)-balls \( B \) as above. Let \( (u, \nabla u) \in W_{Q}^{1,p}(\Omega) \) be a weak solution of (1.1) in \( \Omega \) and let \( B(y, r) \) be a \( \rho \)-ball with \( 0 < r < \tau r_1(y) \).

Furthermore, given \( k > 0 \) and \( \epsilon_1, \epsilon_2, \epsilon_3 \in (0, 1] \), let
\[ \tilde{u} = |u| + k, \quad \tilde{b} = b + k^{1-p}c, \quad \tilde{h} = h + k^{-p}f, \quad \tilde{d} = d + k^{-p}f, \]
and define

\begin{align*}
Z &= 1 + r^{p-1} \|\tilde{h}\|_{p, \alpha', B(y, r), dx} + \left( r^{p} \|\tilde{h}\|_{p, 1 - \tau, B(y, r), dx} \right)^{\frac{1}{\tau_1}} \\
&+ \left( r^{p} \|\tilde{h}\|_{p, \alpha', B(y, r), dx} \right)^{\frac{1}{\tau_2}} + \left( r^{p} \|\tilde{d}\|_{p, \alpha', B(y, r), dx} \right)^{\frac{1}{\tau_3}}.
\end{align*}

Then
\[ \|\tilde{u}\|_{L^{s}(B(y, \tau r))} \leq C\tilde{Z}^{\Psi_0} \|\tilde{u}\|_{sp, B(y, r), dx}, \]
where \( s \) is the dual exponent of the number \( s' \) which satisfies \( s^* = s'p \), \( C \) is a constant independent of \( u, k, B(y, r), b, c, d, e, f, g, h \), and \( \Psi_0 = \frac{s}{s'} \).

**Remark 1.3.** Under the hypothesis of Theorem 1.2, Proposition 2.3 guarantees that the factor \( \|\tilde{u}\|_{sp, B(y, r), dx} \) in (1.22) is finite. If \( Z \) is finite for all \( B(y, r) \) as above and condition (1.10) is satisfied, then Theorem 1.2 gives local boundedness of weak solutions of equation (1.1) in \( \Omega \), i.e., weak solutions in \( W_{Q}^{1,p}(\Omega) \) are bounded in every compact subset of \( \Omega \); see Section 2 for the simple proof.

**Remark 1.4.** The proof of Theorem 1.2 also provides an \( L^p \) estimate for the size of \( \sqrt{Q} \nabla u \) when \( (u, \nabla u) \) is a weak solution. In fact, under the same assumptions as in Theorem 1.2
\[ \|\sqrt{Q} \nabla u\|_{L^p(B(y, \tau r))} \leq C\tilde{Z} \left( \frac{1}{r} \|\tilde{u}\|_{L^p(B(y, r))} + \|\tilde{u}\|_{L^{p\tau}(B(y, r))} \right), \]
where the norms are now unnormalized. This estimate is an analogue of one obtained in [8] in the nondegenerate case. It follows from (3.19) below by choosing \( q = 1 \) and \( \eta = \eta_1 \) there, and by applying (1.10) to the first term on the right in (3.19).

**Remark 1.5.** As mentioned earlier, if we are dealing with a particular weak solution \( (u, \nabla u) \), then parts (ii), (iii) and (iv) of the structural assumptions (1.3) required in Theorem 1.2 (where \( \gamma = \delta = p \)) can be weakened without affecting the conclusion. In particular, it is enough to assume
that they hold when the general variable \( z \in \mathbb{R} \) is replaced by \( u(x) \), \( x \in \Omega \), \( i.e., \) to assume that for a.e. \( x \in \Omega \) and all \( (z, \xi) \in \mathbb{R} \times \mathbb{R}^n \),

\[
\begin{align*}
(i) & \quad A(x, z, \xi) = \sqrt{Q(x)} \bar{A}(x, z, \xi), \\
(ii) & \quad \xi \cdot A(x, u(x), \xi) \geq a^{-1} \left| \sqrt{Q(x)} \xi \right|^{p} - h(x) |u(x)|^{p} - g(x), \\
(iii) & \quad \bar{A}(x, u(x), \xi) \leq a \left| \sqrt{Q(x)} \xi \right|^{p-1} + b(x) |u(x)|^{p-1} + e(x), \\
(iv) & \quad B(x, u(x), \xi) \leq c \left| \sqrt{Q(x)} \xi \right|^{\psi-1} + d(x) |u(x)|^{p-1} + f(x).
\end{align*}
\]

(1.24)

When \( \gamma \) or \( \delta \) exceeds \( p \) and we assume the structural conditions (1.3), this fact will be used in some of the corollaries below to deduce boundedness results from the case \( \gamma = \delta = p \) considered in Theorem 1.2, it will allow us to bundle some powers of \( |u(x)| \) together with the coefficients.

We now turn to the question of estimating the expression \( \bar{Z} \) defined in (1.21), and in particular of determining when it is finite.

In case \( e = f = g = 0 \), we can let \( k \) tend to 0 in (1.22) to obtain (1.22) for \( u \) instead of \( \bar{u} \). In our applications of Theorem 1.2 provided \( e, f, g \) are not all identically 0 in \( B_r = B(y, r) \), we will choose the constant \( k \) to be

\[
k(r) = k(y, r) = \left( r^{p-1} \| e \|_{p, p', B_r, \mathbb{R}^n} \right)^{\frac{1}{p-1}} + \left( r^{p} \| g \|_{p, p', p+1, B_r, \mathbb{R}^n} \right)^{\frac{1}{p}} + \left( r^{p} \| f \|_{p, p', p+1, B_r, \mathbb{R}^n} \right)^{\frac{1}{p}}.
\]

As above, in case \( k = 0 \) in (1.23), then in order to be able to apply Theorem 1.2, we will instead choose any positive number for \( k \) and then let this number tend to 0. In any case, it follows from (1.25) that the three terms of (1.21) corresponding to \( \bar{b} \), \( \bar{h} \) and \( \bar{d} \) satisfy

\[
r^{p-1} \| \bar{b} \|_{p, p', B_r, \mathbb{R}^n} + \left( r^{p} \| \bar{h} \|_{p, p', p+1, B_r, \mathbb{R}^n} \right)^{\frac{1}{p}} + \left( r^{p} \| \bar{d} \|_{p, p', p+1, B_r, \mathbb{R}^n} \right)^{\frac{1}{p}} \leq
\]

\[
1 + 2 \left( r^{p-1} \| e \|_{p, p', B_r, \mathbb{R}^n} \right)^{\frac{1}{p-1}} + \left( r^{p} \| g \|_{p, p', p+1, B_r, \mathbb{R}^n} \right)^{\frac{1}{p}} + \left( r^{p} \| f \|_{p, p', p+1, B_r, \mathbb{R}^n} \right)^{\frac{1}{p}}.
\]

Consequently, with \( k \) defined by (1.25), we may replace \( \bar{Z} \) in (1.21) and (1.22) by the following analogous expression in which \( \bar{b}, \bar{h}, \bar{d} \) are replaced respectively by \( b, h, d \):

\[
Z = 1 + r^{p-1} \| b \|_{p, p', B_r, \mathbb{R}^n} + \left( r^{p} \| c \|_{p, p', p+1, \mathbb{R}^n} \right) \left( r^{p} \| \bar{b} \|_{p, p', p+1, B_r, \mathbb{R}^n} \right) \left( r^{p} \| \bar{d} \|_{p, p', p+1, B_r, \mathbb{R}^n} \right).
\]

Strictly speaking, the additive constant 1 in (1.26) should be replaced by \( 1 + 2^{1/\varepsilon_2} + 2^{1/\varepsilon_3} \), but we can incorporate extra constant factors depending on \( \varepsilon_2, \varepsilon_3 \) in the constant \( C \) in (1.22).

In order to better understand the expression \( Z \) in (1.26), we first note that its form leads naturally to the following definition of spaces of Morrey type for quasimetric balls.
Definition 1.6. Let $\alpha, \beta$ satisfy $0 < \alpha < \infty$ and $0 < \beta \leq \infty$. We say that a measurable function $m(x)$ on $\Omega$ belongs to the Morrey class $M^\beta_\alpha(\Omega)$ if

$$\tag{1.27} \|m\|_{M^\beta_\alpha(\Omega)} = \sup \left\{ r^\alpha \left( \frac{1}{|B(y, r)|} \int_{B(y, r)} |m(x)|^\beta dx \right)^{1/\beta} \right\} = \sup \left\{ r^\alpha \|m\|_{\beta,B(y,r);dx} \right\} < \infty,$$

where the sup is taken over all $\rho$-balls $B_r = B(y, r)$ with $r < \min\{1, r_1(y)\}$. We recall that the closure of any such ball is contained in $\Omega$. In case $\beta = \infty$, (1.27) means that

$$\|m\|_{M^\infty_\alpha(\Omega)} = \sup \left\{ r^\alpha \text{ ess sup}_{B(y,r)} |m| \right\} < \infty.$$

Using this notation, the expression $Z$ in (1.26) satisfies

$$\tag{1.28} Z \leq 1 + \|b\|_{M^{q^*'}_{p-1}(\Omega)} + \|c\|_{M^{\frac{q^*(p-q)}{p}}_{p-1}(\Omega)} + \|h\|_{M^{\frac{q}{p^*}}_{p^*}(\Omega)} + \|d\|_{M^{\frac{q}{p^*}-1}_{p^*}(\Omega)}.$$

However, since $Z$ involves only a single ball, it is more local than the right-hand side of (1.28), and we will often take further advantage of its local nature before using Morrey classes.

In general, there is no simple way to characterize Morrey classes in terms of Lebesgue classes. However, it is possible to combine a Lebesgue condition with an (algebraic) growth condition on $|B_r|$ in order to estimate the size of $r^\alpha \|m\|_{\beta,B(y,r);dx}$ and determine upper bounds for $Z$. To do this, we will use the following simple observations.

For balls as in Definition 1.6, note that

$$r^\alpha \left( \frac{1}{|B(y, r)|} \int_{B(y, r)} |m|^\beta dx \right)^{1/\beta} \leq \left( \sup \frac{r^\alpha}{|B(y, r)|^{1/\beta}} \right) \|m\|_{\beta,B(y,r);dx},$$

where the supremum is taken over all such balls.

Definition 1.7. If $q^*$ satisfies $0 < q^* < \infty$ and there is a positive constant $c_0$ such that

$$\tag{1.29} |B(y, r)| \geq c_0 r^{q^*}$$

for all $\rho$-balls $B(y, r)$ with $r < \min\{1, r_1(y)\}$, we will say that condition $D_{q^*}$ holds.

Condition (1.29) is related to, but weaker than, the local doubling condition

$$\tag{1.30} |B(x, 2r)| \leq C |B(x, r)|, \quad x \in \Omega, 0 < r < \tilde{r}(x),$$

where $	ilde{r}(x) < r_0(x)/2$. It is well-known that (1.30) implies there are positive constants $C, D^*$ such that

$$\tag{1.31} |B(x, R)| \leq C \left( \frac{R}{r} \right)^{D^*} |B(y, r)| \quad \text{if} \quad B(y, r) \subset B(x, R)$$

and $r$ is sufficiently small. Note that (1.29) follows from (1.31) with $D^* = q^*$ if $\rho$ is bounded in $\Omega$ since then by choosing $R = \text{sup}\{\rho(x, y) : x, y \in \Omega\}$, we have for all $x \in \Omega$ that $\Omega \subset B(x, R)$, and consequently $B(x, R) = \Omega$. Moreover, even if $\rho$ is unbounded in $\Omega$, (1.29) follows from (1.31) if

$$\text{inf}\{|B(x, 1)| : x \in \Omega\} > 0.$$

Then for $\alpha, \beta$ as above,

$$\tag{1.32} r^\alpha \left( \frac{1}{|B(y, r)|} \int_{B(y, r)} |m|^\beta dx \right)^{1/\beta} \leq \begin{cases} C \|m\|_{\beta,B(y,r);dx} & \text{if} \ \beta < \infty \ \text{and} \ D_\alpha \beta \ \text{holds} \\ \|m\|_{L^\infty(B(y,r))} & \text{if} \ \beta = \infty \ \text{or} \ \beta < \infty \ \text{and} \ D_\alpha \beta \ \text{holds}. \end{cases}$$

with $C = \tilde{c}_0^{-1/\beta}$, where to obtain the second option we have used $\alpha > 0$ and $r < 1$. Thus $L^\beta(\Omega) \subset M^\beta_\alpha(\Omega)$ if $\beta = \infty$, or if $\beta < \infty$ and $D_\alpha \beta$ holds.
If $m$ is a product, $m(x) = m_1(x)m_2(x)$, and if $\beta_1, \beta_2$ satisfy $0 < \beta_1, \beta_2 \leq \infty$ and $\frac{1}{\beta} = \frac{1}{\beta_1} + \frac{1}{\beta_2}$, then Hölder’s inequality implies that for any $B_r = B(y,r)$,
\[
  r^\alpha \left( \frac{1}{|B_r|} \int_{B_r} |m_1 m_2|^{\beta dx} \right)^{\frac{1}{\beta}} \leq r^\alpha \left( \frac{1}{|B_r|} \int_{B_r} |m_1|^{\beta_1 dx} \right)^{\frac{1}{\beta_1}} \left( \frac{1}{|B_r|} \int_{B_r} |m_2|^{\beta_2 dx} \right)^{\frac{1}{\beta_2}}.
\]
Combining this with (1.32) gives (again we denote $B_r = B(y,r)$)
\[
  (1.33) \quad r^\alpha \left( \frac{1}{|B_r|} \int_{B_r} |m_1 m_2|^{\beta dx} \right)^{\frac{1}{\beta}} \leq C \|m_1\|_{\beta_1, B(y,r)} \left( \frac{1}{|B_r|} \int_{B_r} |m_2|^{\beta_2 dx} \right)^{\frac{1}{\beta_2}} \ 	ext{provided}
\]
\[
  \frac{1}{\beta} = \frac{1}{\beta_1} + \frac{1}{\beta_2}, \ \text{and either} \ \beta_1 = \infty, \ \text{or} \ \beta_1 < \infty \ \text{and} \ D_{\alpha\beta_1} \ \text{holds}.
\]

The fewest technicalities arise when $\gamma = \delta = \psi = p$, and we begin with that case. The result we will state aims at making the weakest possible integrability assumptions on the coefficients, and consequently it makes a strong assumption about the order of the $D$ condition. As is apparent from (1.29) and (1.32), since $r < 1$ and $\alpha \beta$ increases with $\beta$, a general principle is that the better the coefficients are (i.e., the higher their integrability becomes), then the weaker the required $D$ condition becomes.

**Corollary 1.8.** Suppose the same hypotheses and notation as in Theorem 1.2 hold, but now also that $\psi = p$ (i.e., $\gamma = \delta = \psi = p$), that condition $D_q^*$ holds for some $q^* \leq p\sigma'$, and that $b, e \in L^p(\sigma')(B(y,r))$, $c \in L^{p\sigma(1+\epsilon)}(B(y,r))$ and $d, f, h, g \in L^s(1+\epsilon)(B(y,r))$ for some $\epsilon > 0$. Then
\[
  \|u\|_{L^\infty(B(y,r))} \leq C \left\{ \left( \frac{1}{|B(y,r)|} \int_{B(y,r)} |u|^{p\sigma} dx \right)^{\frac{1}{\sigma}} + K(y,r) \right\}, \ 	ext{where}
\]
\[
  K(y,r) = r^{1 - \frac{p\sigma}{p\sigma'}} \left( \left\|e\right\|_{p\sigma', B(y,r)}^{\frac{1}{\sigma'}} + r^{(1 - \frac{p\sigma}{p\sigma'} + \frac{q^*}{p\sigma' + q^*}) \frac{1}{\sigma' - 1}} \left\|f\right\|_{p\sigma(1+\epsilon), B(y,r)}^{\frac{1}{\sigma'}} \right) + r^{1 - \frac{p\sigma}{p\sigma'}} \left\|g\right\|_{p\sigma(1+\epsilon), B(y,r)}^{\frac{1}{\sigma'}}
\]
and $C$ depends on all relevant parameters including $\epsilon$, the constant in the $D_{q^*}$ condition and the sum of the corresponding norms of $b, c, d, h$ over $B(y,r)$, but does not depend on $u, B(y,r), e, f$ or $g$. In particular, if $s = 1$, i.e., if the cutoff condition (1.14) holds in the $L^\infty$ sense, then
\[
  \|u\|_{L^\infty(B(y,r))} \leq C \left\{ \left( \frac{1}{|B(y,r)|} \int_{B(y,r)} |u|^{p} dx \right)^{\frac{1}{p}} + K(y,r) \right\}
\]
with $K(y,r)$ and $C$ as above.

Note that for the case of the standard Euclidean structure, $p\sigma' = n$ and condition $D_n$ automatically holds. Hence, since (1.14) is true with $s^* = \infty$ in this situation, estimate (1.35) then applies and includes the local boundedness result of [5] Theorem 1, p. 555. If $e, f, g$ vanish identically in $B(y,r)$, then $K(y,r) = 0$ in Corollary 1.8. Also, if $q^* < p\sigma'$ and the corresponding norms of $e, f, g$ over all of $\Omega$ are finite, note that $K(y,r) \leq c_{n, \epsilon}$ for some $\eta > 0$ which depends on $q^*$.

The proof of Corollary 1.8 is an application of Theorem 1.2 and follows from (1.26) and (1.32), without needing to use (1.33). We choose $k$ as in (1.25) and drop $k$ on the left side of (1.22), thereby replacing $\tilde{u}$ by $u$ on the left side. However, on the right side, we use the hypotheses
to estimate $Z$ and show that $k \leq CK(y, r)$. As examples of the required computations, let us briefly indicate how to estimate the term of $Z$ which corresponds to $b$ and the term of $k$ which corresponds to $g$. Denoting $B(y, r) = B$ and using the estimate $|B| \geq c_0 p^\sigma$, we have

$$r^{p-1} \|b\|_{L^p(B; dx)} = \left( \frac{r}{|B|^{1/(p\sigma)}} \right)^{p-1} \|b\|_{L^p(B; dx)} \leq \left( c_0 \frac{r}{p\sigma} r^{1 - \frac{tp}{p\sigma}} \right)^{p-1} \|b\|_{L^p(B; dx)} \leq c_0 \frac{1}{p\sigma} \|b\|_{L^p(B; dx)}$$

since $q^* \leq p\sigma$ and $r \leq 1$. Similarly, choosing $\epsilon_2 = c\epsilon/(1 + \epsilon)$, we obtain $p - \epsilon_2 = p/(1 + \epsilon)$ and

$$\left( r^p \|g\|_{L^{p\sigma}(B; dx)} \right)^{\frac{1}{p}} \leq \left( c_0 \frac{1}{p\sigma(1+\epsilon)} r^{1 - \frac{tp}{p\sigma} + \frac{tp}{p\sigma(1+\epsilon)}} \right) \|g\|_{L^{p\sigma(1+\epsilon)}(B; dx)}.$$  

Further details are left to the reader.

Our next corollary gives an estimate when all of $\gamma, \psi, \delta$ are less than $p$. In this case, we can easily replace each of the structural assumptions (1.3) (ii), (iii) and (iv) by a similar one involving only $p$ and modified coefficients. For example, if $\gamma < p$, we can use the simple estimate

$$b|z|^\gamma + e \leq b|z|^{p-1} + (b + e)$$

to see that (1.3) (iii) implies

$$|\tilde{A}(x, z, \xi)| \leq a \sqrt{Q(x)} |\xi|^{p-1} + b|z|^{p-1} + (b + e).$$

Similarly, an analogue of (1.3) (ii) holds with $-h|z|^\gamma - g$ replaced by $-h|z|^p - (h + g)$, and if $\psi, \delta < p$ then (1.3) (iv) gives

$$|B(x, z, \xi)| \leq c \sqrt{Q(x)} |\xi|^{p-1} + d|z|^{p-1} + (f + c + d).$$

It follows that when $\gamma, \psi, \delta$ are less than $p$, (1.3) implies its analogue with $\gamma, \psi, \delta$ all replaced by $p$ and with $e, g, f$ replaced by $b + g, f + c + d$ respectively. Hence we immediately obtain the next corollary from the previous one.

**Corollary 1.9.** Suppose the same hypotheses and notation as in Theorem 1.2 hold with these exceptions: the structural assumptions (1.3) hold for some $\gamma, \psi, \delta < p$; condition $D_{\sigma'}$ holds for some $q^* \leq p\sigma'$; and $b, e \in L^{p\sigma'}(B(y, r))$, $c \in L^{p\sigma'(1+\epsilon)}(B(y, r))$ and $d, f, h, g \in L^{\sigma'(1+\epsilon)}(B(y, r))$ for some $\epsilon > 0$. Then (1.3) is true with

$$K(y, r) = r \left( 1 - \frac{\gamma^p}{p\sigma'} \right) \left( \|e + b\|_{L^{p\sigma'}(B(y, r); dx)} + r^\frac{p\sigma'}{p\sigma' - \gamma} \|g + h\|_{L^{p\sigma'(1+\epsilon)}(B(y, r); dx)} \right)$$

and with $C$ depending on all relevant parameters including $\epsilon$, the constant in the $D_{\sigma'}$ condition and the norms of $b, c, d, h$ over $B(y, r)$, but not on $u, B(y, r), e, f$ or $g$.

Next we list corollaries in case all of $\gamma, \psi, \delta$ in (1.3) exceed $p$. In this situation, we will use the observation in Remark 1.5 in fact, the last three assumptions in either (1.3) or its weaker analogue when $z$ is replaced by $u(x)$ (for a fixed weak solution $u$) yield

$$A(x, z, \xi) = \sqrt{Q(x)} \tilde{A}(x, z, \xi),$$

$$\xi \cdot A(x, u(x), \xi) \geq a \sqrt{Q(x)} |\xi|^{p-1} - \left( h(x)|u(x)|^{\gamma p} - g(x) \right)|u(x)|^p - g(x),$$

$$|\tilde{A}(x, u(x), \xi)| \leq a \sqrt{Q(x)} |\xi|^{p-1} + \left( b(x)|u(x)|^{\gamma p} \right)|u(x)|^{p-1} + c(x),$$

$$|B(x, u(x), \xi)| \leq c \sqrt{Q(x)} |\xi|^{\psi - 1} + \left( d(x)|u(x)|^{\delta p} \right)|u(x)|^{p-1} + f(x)$$

for the same $\gamma, \psi, \delta$ and $a, b, c, d, e, f, g, h$ as in (1.3). Consequently, denoting

$$b^* = b|u|^{-\gamma p}, \quad d^* = d|u|^{-\delta p}, \quad h^* = h|u|^{-\gamma p},$$

we have

$$r^{p-1} \|b\|_{L^p(B; dx)} = \left( \frac{r}{|B|^{1/(p\sigma')}} \right)^{p-1} \|b\|_{L^p(B; dx)} \leq \left( c_0 \frac{r}{p\sigma'} r^{1 - \frac{tp}{p\sigma'}} \right)^{p-1} \|b\|_{L^p(B; dx)} \leq c_0 \frac{1}{p\sigma'} \|b\|_{L^p(B; dx)}$$

since $q^* \leq p\sigma'$ and $r \leq 1$. Similarly, choosing $\epsilon_2 = c\epsilon/(1 + \epsilon)$, we obtain $p - \epsilon_2 = p/(1 + \epsilon)$ and

$$\left( r^p \|g\|_{L^{p\sigma'}(B; dx)} \right)^{\frac{1}{p}} \leq \left( c_0 \frac{1}{p\sigma'(1+\epsilon)} r^{1 - \frac{tp}{p\sigma'} + \frac{tp}{p\sigma'(1+\epsilon)}} \right) \|g\|_{L^{p\sigma'(1+\epsilon)}(B; dx)}.$$
we obtain the conditions (1.24) with $b, d, h$ there replaced respectively by $b^*, d^*, h^*$. Using Remark 1.39 as well as (1.26), where the form of the constant $k$ in the formula $\bar{u} = |u| + k$ is still the same as in (1.25), we may apply Theorem 1.2 with $\bar{C}$ (1.40) and $\bar{\theta}$ (1.42) replaced by

$$Z^* = 1 + r^{p-1} ||u||_{p', B_r, dx} + \left( r^{p} ||c||_{L^{\infty}(B_r)} ||u||_{p, B_r, dx} \right)^{\frac{1}{p}}$$

$$+ \left( r^{p} ||h||_{L^{\infty}(B_r)} ||u||_{p, B_r, dx} \right)^{\frac{1}{p}} + \left( r^{p} ||d||_{L^{\infty}(B_r)} ||u||_{p, B_r, dx} \right)^{\frac{1}{p}}$$

(1.37)

The terms in (1.37) can be treated by applying (1.33), and we obtain the following corollaries. In the first one, we make the strongest possible assumption on the coefficients, namely that they are all bounded. In this case, we require no $D$ condition at all. By using (1.37) together with (1.33) for $\beta_1 = \infty$, we obtain

$$Z^* \leq 1 + ||b||_{L^{\infty}(B_r)} ||u||_{p', B_r, dx} + \left( ||c||_{L^{\infty}(B_r)} ||u||_{p, B_r, dx} \right)^{\frac{1}{p}}$$

$$+ \left( ||h||_{L^{\infty}(B_r)} ||u||_{p, B_r, dx} \right)^{\frac{1}{p}} + \left( ||d||_{L^{\infty}(B_r)} ||u||_{p, B_r, dx} \right)^{\frac{1}{p}}$$

(1.38)

In the third term on the right side of the estimate for $Z^*$, we use (1.26) and the fact that $r \leq 1$ to obtain

$$\bar{u} = |u| + k \leq |u| + \left( ||c||_{p', B_r, dx} \right)^{\frac{1}{p}} + \left( ||g||_{p, B_r, dx} \right)^{1/2} + \left( ||f||_{p, B_r, dx} \right)^{1/2}.$$

Choosing $\epsilon_1, \epsilon_2, \epsilon_3$ small, we then easily obtain from (1.38) that for any sufficiently small $\epsilon > 0$, depending on $\gamma, \delta, \psi, \rho$ and $\sigma$, there are constants $\theta, C_1, C_2, L$ satisfying

$$\theta = \max \left\{ (\gamma - p)p', \frac{1}{p} - 1, (\delta - p)p' \right\},$$

(1.39)

$$C_1 = (\epsilon, \rho, p, \|c\|_{L^\infty(B(y, r), ||c||_{L^\infty(B(y, r))}, ||f||_{L^\infty(B(y, r))}, ||g||_{L^\infty(B(y, r))}}$$

with $C_1 = 0$ when $c \equiv 0$ in $B(y, r)$ or when $e, f, g \equiv 0$ in $B(y, r),$}

(1.40)

$$C_2 = C_2(\epsilon, p, \psi, ||b||_{L^\infty(B(y, r))}, ||c||_{L^\infty(B(y, r))}, ||d||_{L^\infty(B(y, r))}, ||h||_{L^\infty(B(y, r))}$$

with $C_2 = 0$ when $b, c, d, h \equiv 0$ in $B(y, r),$}

(1.41)

such that

$$Z^* \leq 1 + C_1 + C_2 \left[ 1 + \left( \frac{1}{|B(y, r)|} \int_{B(y, r)} |u|^\theta \, dx \right)^{\frac{1}{\theta}} \right].$$

Moreover, for small $\epsilon$, the restrictions (1.4) imply that $\theta < \rho \sigma$, and consequently that $u \in L^\rho(B(y, r))$. Thus we obtain the following estimate.

**Corollary 1.10.** Suppose the same hypotheses and notation as in Theorem 1.2 hold with these exceptions: the structural assumptions (1.3) hold for some $\gamma, \psi, \delta > \rho$ which satisfy (1.3), and the
coefficients $b, c, d, e, f, g, h$ are bounded in $\Omega$. For small $\epsilon > 0$, define $\theta, C_1, C_2$ and $L$ as in (1.39), (1.40) and (1.42), respectively. Then for any $\rho$-ball $B(y, r)$ with $0 < r < \tau r_1(y)$,

$$
\|u\|_{L^\infty(B(y, r))} \leq C \left\{ 1 + C_1 + C_2 \left[ 1 + \left( \frac{1}{|B(y, r)|} \int_{B(y, r)} |u|^{\theta} \, dx \right)^{\frac{1}{\theta}} \right] \right\} \times \left\{ \left( \frac{1}{|B(y, r)|} \int_{B(y, r)} |u|^p \, dx \right)^{\frac{1}{p}} + K(y, r) \right\},
$$

(1.43)

where $K(y, r) = \left( r |e|_{L^\infty(B(y, r))} + r^\sigma \|f\|_{L^\infty(B(y, r))} + r \|g\|_{L^\infty(B(y, r))} \right)$ and $C$ is as in (1.22). In particular, if (1.14) holds in the $L^\infty$ sense, then (1.43) holds with $s = 1$. In this corollary, no $D$ condition is needed.

As noted above, when $\epsilon$ is small, the value of $\theta$ in (1.43) satisfies $\theta < p\sigma$. The largest power of $|u|$ which is a priori locally integrable is $p\sigma$, and our next corollary gives an estimate when $\theta$ is replaced by $p\sigma$, still assuming that all of $\gamma, \psi, \delta, p, \sigma$ exceed $p$. In this situation, the conditions required of the coefficients are weaker than boundedness, but an appropriate restriction in terms of a $D$ condition is required.

**Corollary 1.11.** Suppose the same hypotheses and notation as in Theorem 1.2 hold with these exceptions: the structural assumptions (1.3) hold for some $\gamma, \psi, \delta > p$ which satisfy (1.4); the $D_p$-condition holds for some $q^* > 0$ as described below; for a given $\rho$-ball $B(y, r)$ with $r < \tau r_1(y)$, the coefficients satisfy

$$
b \in L^B(B(y, r)), \quad B \geq \frac{p\sigma}{\sigma(p - 1) + 1 - \gamma}; \quad c \in L^C(B(y, r)), \quad C > \frac{p\sigma}{\sigma(p - 1 - \psi) - 1};
$$

$$
d \in L^D(B(y, r)), \quad D > \frac{p\sigma}{p\sigma - \delta}; \quad e \in L^E(B(y, r)), \quad E \geq p' \sigma';
$$

$$
f \in L^F(B(y, r)), \quad F > \sigma'; \quad g \in L^G(B(y, r)), \quad G > \sigma'; \quad h \in L^H(B(y, r)), \quad H > \frac{p\sigma}{p\sigma - \gamma},
$$

and

$$
q^* \leq \min \{ B(p - 1), E(p - 1), C(p + 1 - \psi), Dp, Fp, Gp, Hp \}.
$$

Then

$$
\|u\|_{L^\infty(B(y, r))} \leq C \left\{ 1 + C'_1 + C'_2 \left[ 1 + \left( \frac{1}{|B(y, r)|} \int_{B(y, r)} |u|^{p\sigma} \, dx \right)^{\frac{1}{p\sigma}} \right] \right\} \times \left\{ \left( \frac{1}{|B(y, r)|} \int_{B(y, r)} |u|^p \, dx \right)^{\frac{1}{p}} + K(y, r) \right\},
$$

(1.44)

where

$$
K(y, r) = \left( r^{1-\frac{q^*}{p\sigma}} \|e\|_{E, B(y, r); dx} + r^{p'(1-\frac{q^*}{p\sigma})} \|f\|_{F, B(y, r); dx} + r^{1-\frac{q^*}{p\sigma}} \|g\|_{G, B(y, r); dx} \right),
$$

$C$ is as in (1.22), $L > 0$ is a constant depending on $\gamma, \psi, \delta, p, \sigma, B, C, D, H$, and the constants $C'_1, C'_2$ are analogous to the constants $C_1, C_2$ in (1.40), (1.41) but with the $L^\infty(\Omega)$ norms of $b, c, d, e, f, g, h$ replaced by their corresponding norms listed above. Also, $C'_1, C'_2$ have the same vanishing properties as $C_1, C_2$ but now depend on the constants in the $D_p$-condition as well as $\gamma, \psi, \delta, p, \sigma, C, D, H$ and the norm of $c$; $C'_1$ depends furthermore on the norms of $e, f, g$, and $C'_2$ on $B$ and the norms of $b, d, h$. Again, in case (1.14) holds in the $L^\infty$ sense, then (1.44) holds with $s = 1$. 
The proof of Corollary 1.11 is computational but straightforward; it is based on (1.33) and (1.32). Details are left to the reader. Further computations show that if \( q^* = p\sigma' \), then all conditions involving \( q^* \) in Corollary 1.11 are satisfied. This is the case in the classical Euclidean setting if \( p < n \) since then \( \sigma' = n/p \), giving \( p\sigma' = n \).

2. Preliminary Definitions and Lemmas

Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set and \( |Q| \in L^2_{\text{loc}}(\Omega) \). For \( w \in \text{Lip}_{\text{loc}}(\Omega) \), recall that

\[
(2.1) \quad \|w\|_{W^{1,p}_Q(\Omega)} = \left( \int_{\Omega} |w|^p \, dx + \int_{\Omega} |\nabla w|^p \, dx \right)^{\frac{1}{p}}.
\]

By definition, \( W^{1,p}_Q(\Omega) \) is the completion with respect to \( \| \cdot \|_{W^{1,p}_Q(\Omega)} \) of those functions in \( \text{Lip}_{\text{loc}}(\Omega) \) with finite \( W^{1,p}_Q(\Omega) \)-norm. Also, \( (W^{1,p}_Q)_0(\Omega) \) denotes the completion with respect to \( \| \cdot \|_{W^{1,p}_Q(\Omega)} \) of \( \text{Lip}_0(\Omega) \).

A sequence \( \{w_i\}_{i \in \mathbb{N}} \subset \text{Lip}_{\text{loc}}(\Omega) \) such that \( \|w_i\|_{W^{1,p}_Q(\Omega)} < \infty \) for every \( i \) and which is Cauchy with respect to \( \| \cdot \|_{W^{1,p}_Q(\Omega)} \) identifies an element of \( W^{1,p}_Q(\Omega) \). Then \( \{w_i\} \) is a Cauchy sequence in \( L^p(\Omega) \) and \( \{\sqrt{Q} \nabla w_i\} \) is a Cauchy sequence in \( [L^p(\Omega)]^n \). Hence, up to subsequences, as \( i \to \infty \),

\[
(2.2) \quad w_i \to w \quad \text{in } L^p(\Omega) \text{ and a.e. in } \Omega,
\]

\[
(2.3) \quad \nabla w_i \to v := \nabla w \quad \text{in } L^p(\Omega) \text{ and}
\]

\[
(2.4) \quad \sqrt{Q} \nabla w_i \to \sqrt{Q} \nabla w \quad \text{in } [L^p(\Omega)]^n \text{ and a.e. in } \Omega.
\]

We adopt the abuses of notation mentioned in the Introduction. Thus, we will not generally distinguish between \( W^{1,p}_Q(\Omega) \) and its isomorphic copy \( W^{1,p}_Q(\Omega) \) defined to be the collection of pairs \((w, \nabla w)\) which arise as in (2.2), (2.3) and (2.4). We will often write simply \( w \) instead of \((w, \nabla w)\) even though \( \nabla w \) may not be uniquely determined by \( w \).

It follows from (2.2), (2.3) and (2.4) that (2.1) also holds for a generic element \( w \in W^{1,p}_Q(\Omega) \). Similarly, it follows by passing to the limit that conditions like (1.16) hold for all functions in \( W^{1,p}_Q(\Omega) \) instead of just for \( \text{Lip}_{\text{loc}}(\Omega) \). In order to deal with the left side of such inequalities when passing to the limit, we generally use Fatou’s lemma.

The role of condition (1.9) is illustrated in the next simple lemma.

**Lemma 2.1.** If (1.9) holds, then for every \( y \in \Omega \) there exists \( r_0 = r_0(y) > 0 \) such that \( B(y, r) \subset \Omega \) for all \( r \in (0, r_0] \).

**Proof:** Let \( y \in \Omega \). Since \( \Omega \) is open, there exists \( \epsilon > 0 \) such that \( D(y, \epsilon) \subset \Omega \). By (1.11), there is \( r_0 > 0 \) for which \( B(y, r_0) \subset D(y, \epsilon/2) \), and it follows that the closure of \( B(y, r_0) \) lies in \( \Omega \). \( \square \)

We now derive a useful version of the product rule. See the comments after Lemma 2.4 for a more global version.

**Proposition 2.2.** Let (1.9) be true. Suppose that (1.15) holds for a particular \( p \)-ball \( B \) with closure in \( \Omega \) and a particular function \( \eta \in \text{Lip}_0(B) \). Suppose also that, for \( t \geq 1 \) as in (1.15), condition (1.16) holds for \( B \) with \( t' \) given by \( 1/t + 1/t' = 1 \). If \( \theta \geq 1 \) and \( w \in W^{1,p}_Q(\Omega) \), then \( \eta^p w \in (W^{1,p}_Q)_0(B) \) and

\[
\sqrt{Q} \nabla (\eta^p w) = \theta \eta^{p-1} w \sqrt{Q} \nabla \eta + \eta^{\theta} \sqrt{Q} \nabla w \quad \text{a.e. in } \Omega.
\]
Proof: Since \( w \in W^{1,p}_Q(\Omega) \), there is a sequence \( \{w_i\} \subset Lip_{loc}(\Omega) \) representing \( w \) which is Cauchy in \( W^{1,p}_Q(\Omega) \). Taking a subsequence, we may assume that (2.2), (2.3) and (2.4) hold. Now fix \( B \) and \( \eta \) as in the hypotheses, and consider the sequence \( \varphi_i = \eta^\theta w_i \). Clearly, \( \varphi_i \in Lip_0(B) \).

Claim 1: \( \varphi_i \to \varphi := \eta^\theta w \) a.e. in \( \Omega \) and in \( L^p(\Omega) \).
In fact, \( \varphi_i \to \eta^\theta w \) a.e. in \( \Omega \) by our assumptions on the sequence \( \{w_i\} \). Also,
\[
\|\varphi_i - \eta^\theta w\|_{\|\cdot\|_{p,\Omega;dx}} \leq C(\theta,\eta)\|w_i - w\|_{\|\cdot\|_{p,\Omega;dx}} \to 0,
\]
which proves the claim.

Claim 2: For a.e. \( x \in \Omega \) and in \( [L^p(\Omega)]^n \) norm,
\[
\sqrt{Q\nabla\varphi_i} \to \theta\eta^{\theta-1}w\sqrt{Q\nabla\eta} + \eta^\theta \sqrt{Q\nabla w}.
\]
By the product rule for Lipschitz functions, \( \sqrt{Q\nabla\varphi_i} = \theta\eta^{\theta-1}w\sqrt{Q\nabla\eta} + \eta^\theta \sqrt{Q\nabla w} \) a.e. in \( \Omega \). Then (2.5) holds for a.e. \( x \in \Omega \) by the convergence properties of \( w_i \) and \( \sqrt{Q\nabla w} \). Moreover
\[
\begin{align*}
\int_{\Omega} \left| \sqrt{Q\nabla\varphi_i} - \theta\eta^{\theta-1}w\sqrt{Q\nabla\eta} - \eta^\theta \sqrt{Q\nabla w} \right|^p dx &
\leq 2^p \int_{\Omega} \left[ |\theta| |\eta|^{(\theta-1)p} |w_i - w|^p |\sqrt{Q\nabla\eta}|^p + |\eta|^{\theta p} |\sqrt{Q\nabla w_i} - \sqrt{Q\nabla w}|^p \right] dx \\
&
\leq C(\theta,\eta) \eta^p \int_{\Omega} \left[ |w_i - w|^p |\sqrt{Q\nabla\eta}|^p + |\sqrt{Q\nabla w_i} - \sqrt{Q\nabla w}|^p \right] dx.
\end{align*}
\]
Now
\[
\int_{\Omega} |\sqrt{Q\nabla w_i} - \sqrt{Q\nabla w}|^p dx = \|\sqrt{Q\nabla w_i} - \sqrt{Q\nabla w}\|_{p,\Omega;dx} \leq \|w_i - w\|_{W^{1,p}_Q(\Omega)} \to 0.
\]
By Hölder’s inequality and (1.16) (recall that (1.16) holds for general elements of \( W^{1,p}_Q(\Omega) \),
\[
\int_{\Omega} |w_i - w|^p |\sqrt{Q\nabla\eta}|^p dx = \int_{\Omega} |w_i - w|^p |\sqrt{Q\nabla\eta}|^p dx
\]
(2.6)\[
\leq \left( \int_{B} |w_i - w|^{pt} dx \right)^{\frac{1}{t}} \left( \int_{B} |\sqrt{Q\nabla\eta}|^{pt} dx \right)^{\frac{1}{t}}
\leq C_2 \|w_i - w\|_{W^{1,p}_Q(\Omega)} \|\sqrt{Q\nabla\eta}\|_{p t;B;dx},
\]
and the last right-hand side tends to 0 by (1.15). Hence,
\[
\|\sqrt{Q\nabla\varphi_i} - (\theta\eta^{\theta-1}w\sqrt{Q\nabla\eta} + \eta^\theta \sqrt{Q\nabla w})\|_{p,\Omega;dx} \to 0.
\]
and in particular \( \{\sqrt{Q\nabla\varphi_i}\} \) is a Cauchy sequence in \( [L^p(\Omega)]^n \). Claim 2 is thus proved.

It follows that the sequence \( \{\varphi_i\} \subset Lip_0(B) \) identifies an element of \( (W^{1,p}_Q)_0(B) \), that \( \varphi_i \) converges to \( \eta^\theta w \) a.e. in \( \Omega \) and in \( L^p(\Omega) \), and that
\[
\sqrt{Q\nabla}(\eta^\theta w) = \theta\eta^{\theta-1}w\sqrt{Q\nabla w} + \eta^\theta \sqrt{Q\nabla w} \quad \text{a.e. in } \Omega.
\]
The proof is now complete. \( \square \)

Next we will derive a result about higher local integrability of functions in \( W^{1,p}_Q(\Omega) \), whether or not they have compact support.

Proposition 2.3. Assume that (1.9) and the Sobolev inequality (1.13) hold. Let \( 0 < \tau < 1 \) and \( B = B(y, r) \) be a \( \rho \)-ball with \( r < r_1(y) \). Suppose that (1.15) holds for \( B \) and a function \( \eta \in Lip_0(B) \) which equals 1 on \( B(y, \tau r) \). With \( t \) as in (1.15), assume that (1.10) holds for \( B \) with \( t' \) given by \( 1/t + 1/t' = 1 \). Then \( w \in L^{p\tau}(B(y, \tau r)) \) for every \( w \in W^{1,p}_Q(\Omega) \), and
\[
\|w\|_{p\tau,B(y,\tau r);dx} \leq C\|w\|_{W^{1,p}_Q(\Omega)},
\]
with \( C > 0 \) depending on \( p, \sigma, \max|\eta|, B \) and the constants which arise in (1.13), (1.15) and (1.16), but independent of \( w \).
We note that under the hypotheses of Theorem \([1.2]\) by using the functions \(\eta\) in \((1.14)\), the hypotheses of Proposition \((2.3)\) are met for all \(p\)-balls \(B = B(y, r)\) with \(r < r_1(y)\), and so the conclusion of Proposition \((2.3)\) holds for all such \(B\).

**Proof:** Let \(w \in W^{1,p}_Q(\Omega)\) and let \(B, \eta\) satisfy the hypotheses. Denote \(\tau B = B(y, \tau r)\). Since \(\eta = 1\) on \(\tau B\),

\[
\int_{\tau B} |w|^{p\sigma} \, dx \leq \int_{\tau B} |\eta w|^{p\sigma} \, dx.
\]

By Proposition \((2.2)\) \(\eta w \in (W^{1,p}_Q)_0(B)\) and satisfies the product rule. Applying \((1.13)\) with constant \(c_1\), we have

\[
\frac{1}{|B|} \int_B |\eta w|^{p\sigma} \, dx \leq (2c_1)^{p\sigma} \left[ \frac{r^p}{|B|} \int_B |\sqrt{Q} \nabla (\eta w)|^p \, dx + \frac{1}{|B|} \int_B |\eta w|^p \, dx \right]^\sigma \leq C_1^{p\sigma} \left[ \frac{r^p}{|B|} \int_B |\eta \sqrt{Q} \nabla w|^p \, dx + \frac{r^p}{|B|} \int_B |\eta \sqrt{Q} \nabla \eta|^p \, dx \right] + \frac{(\max |\eta|)^p}{|B|} \int_B |w|^{p\sigma} \, dx
\]

\[(2.7)\]

where \(C > 0\) is a constant depending on \(p, \sigma, B, \max |\eta|\) and \(c_1\). We will use \((1.15)\) and \((1.16)\) to estimate the last integral on the right; recall again that \((1.10)\) holds for any \(w \in W^{1,p}_Q(\Omega)\). By Hölder’s inequality (cf. \((2.6)\)),

\[
\int_B |w| \sqrt{Q} \nabla \eta |^p \, dx \leq \left( \int_B |w|^{p_1} \, dx \right)^{\frac{p}{p_1}} \left( \int_B |\sqrt{Q} \nabla \eta |^{p_1} \, dx \right)^{\frac{1}{p_1}} \leq c_3^p c_2 \|w\|_{W^{1,p}_Q(\Omega)},
\]

where \(c_3^p = \int_B |\sqrt{Q} \nabla \eta |^{p_1} \, dx\).

Combining estimates gives

\[
\|\eta w\|_{p\sigma, \tau B; dx} \leq C \|w\|_{W^{1,p}_Q(\Omega)}
\]

with \(C > 0\) now also depending on \(c_2\) and \(c_3\).

Condition \((1.10)\) provides a simple way to extend some of our results proved for individual balls to general compact subsets of \(\Omega\). As an example, let us verify Remark \((1.3)\) of the Introduction. Let \(\Omega'\) be a compact set in \(\Omega\) and \(u(x)\) be a function on \(\Omega\) with the property that for all \(B = B(y, r)\) with \(r < r_1(y)\), \(u\) is bounded on \(\tau B\) for some \(\tau \in (0, 1)\). For such \(B\), by using \((1.10)\), there is an open concentric Euclidean ball \(D \subset \tau B\). It follows from the Heine-Borel Theorem that \(\Omega'\) can be covered by a finite number of such \(D\), and so also by a finite number of balls \(\tau B\) in which \(u\) is bounded. Consequently \(u\) is bounded on \(\Omega'\), which verifies Remark \((1.3)\).

Similarly, \((1.10)\) leads to the following extension of Proposition \((2.3)\) whose proof we omit.

**Lemma 2.4.** Assume that \((1.9)\) and \((1.10)\) hold as well as the Sobolev inequality \((1.13)\). Suppose that for each \(y \in \Omega\), there is a ball \(B\) with center \(y\) and radius \(r < r_1(y)\) such that \((1.15)\) holds for some \(\eta \in \text{Lip}_0(B)\) which equals 1 on \(\tau B\) for some \(\tau \in (0, 1)\) and some \(t \geq 1\). Suppose also that \((1.16)\) holds for \(B\) and \(t'\) with \(1/t + 1/t' = 1\). The values of \(\tau, t, t'\) may vary with \(y\). Then for every compact subset \(\Omega'\) of \(\Omega\), there is a constant \(C\) depending on \(\Omega'\) so that

\[
\|w\|_{p\sigma, \Omega'; dx} \leq C \|w\|_{W^{1,p}_Q(\Omega)} \quad \text{for all} \quad w \in W^{1,p}_Q(\Omega).
\]

\[(2.9)\]
In passing, we note that under the same hypotheses as in Lemma 2.4, the product rule in Proposition 2.2 extends to Lipshitz functions $\eta$ supported in $\Omega$ (not just those supported in a ball), provided $\eta$ satisfies the global condition

$$\int_{\Omega} |\sqrt{Q} \nabla \eta|^{p}\, dx < +\infty.$$  

The proof is similar to the one of Proposition 2.2, using the conclusion of Lemma 2.4 to modify the argument for (2.6). We will not use this fact and so we omit the details of its proof.

2.1. Weak Solutions. As in the Introduction, we say that a pair $(u, \nabla u) \in W^{1,p}_Q(\Omega)$ is a weak solution of equation (1.1) if

$$\int_{\Omega} \left[ \nabla \varphi \cdot A(x,u,\nabla u) + \varphi B(x,u,\nabla u) \right] = 0 \quad \text{for all } \varphi \in \text{Lip}_0(\Omega).$$  

If $(u, \nabla u)$ is a weak solution, we will sometimes simply say that $u$ is a weak solution without explicitly mentioning $\nabla u$. If $u$ is a weak solution, the class of functions $\varphi$ for which (2.10) holds can be enlarged from $\text{Lip}_0(\Omega)$; see Proposition 2.14. We shall refer to such functions as test functions.

We start by showing that the notion of a weak solution is well-defined and that the class of test functions can be enlarged from $\text{Lip}_0(\Omega)$.

**Proposition 2.5.** Assume that (1.3) holds with

$$\gamma \in (1, \sigma(p-1)+1), \quad \psi \in (1, p+1-\sigma^{-1}), \quad \delta \in (1, p\sigma),$$

and that

$$c \in L^{\sigma_p-1,\sigma_p(1-\gamma-1)}_{\text{loc}}(\Omega), \quad e \in L^{p'}_{\text{loc}}(\Omega), \quad f \in L^{(\sigma_p)'}_{\text{loc}}(\Omega),$$

$$b \in L^{\sigma_p-1,\gamma+1}_{\text{loc}}(\Omega), \quad d \in L^{p}_{\text{loc}}(\Omega),$$

where $p' = \frac{p}{p-1}$, $(\sigma_p)' = \frac{\sigma_p}{\sigma_p-1}$ are the conjugate exponents of $p$ and $\sigma_p$ respectively. Let $0 < \tau < 1$ and $B = B(y,r)$ be a $\rho$-ball with $r < \tau r_1(y)$, and the hypotheses of Proposition 2.3 are satisfied, but with $r$ there replaced by $r/\tau$. Assume also that (1.3) holds, and let $\tilde{A}(x,z,\xi)$ be defined as in (1.3). Then for every $u \in W^{1,p}_Q(\Omega),$

$$|\tilde{A}(\cdot,u,\nabla u)| \in L^{p'}(B) \quad \text{and} \quad B(\cdot,u,\nabla u) \in L^{(\sigma_p)'}(B),$$

with

$$\|\tilde{A}(\cdot,u,\nabla u)\|_{p',B;dx} \leq C_1 \left( p, \sigma, \gamma, \|\sqrt{Q} \nabla u\|_{p,B;dx}, \|e\|_{p,B;dx}, \|b\|_{\sigma(p-1)-1,\gamma+1,B;dx}, \|u\|_{p\sigma,B;dx} \right),$$

$$\|B(\cdot,u,\nabla u)\|_{(\sigma_p)',B;dx} \leq C_2 \left( p, \sigma, \delta, \psi, \|c\|_{\sigma_p-1,\sigma_p(1-\gamma-1),B;dx}, \|\sqrt{Q} \nabla u\|_{p,B;dx}, \|d\|_{p-1,p,B;dx}, \|f\|_{(\sigma_p)'+B;dx}, \|u\|_{p\sigma,B;dx} \right).$$

**Proof:** Let $B$ be a $\rho$-ball which satisfies the hypotheses. By (1.3),

$$\int_{B} |\tilde{A}(x,u(x),\nabla u(x))|^\frac{p}{p-1} \, dx \leq \int_{B} \left( a|\sqrt{Q} \nabla u|^{p-1} + b|u|^{\gamma-1} + e \right)^\frac{p}{p-1} \, dx$$

$$\leq \int_{B} 3^{\frac{p}{p-1}} \left( a^{\frac{p}{p-1}} |\sqrt{Q} \nabla u|^p + b^{\frac{p}{p-1}} |u|^{p(\gamma-1)} + e^{\frac{p}{p-1}} \right) \, dx.$$
Corollary 2.6. Let the hypotheses of Proposition 2.2 hold and let \( B = B(y, r) \) be a \( p \)-ball as described there. Then for every \( \varphi \in L^{p,q}_0(\Omega) \) and every \( u \in W^{1,p}_Q(\Omega) \),

\[
\int_{\Omega} \left[ \nabla \varphi \cdot A(x,u,\nabla u) + \varphi B(x,u,\nabla u) \right] \, dx < \infty.
\]

**Proof:** Since no confusion should arise, we will use \( B \) to denote both the ball \( B(y,r) \) and the function \( B(x,u,\nabla u) \). Then

\[
\left| \int_{\Omega} \left[ \nabla \varphi \cdot A(x,u,\nabla u) + \varphi B(x,u,\nabla u) \right] \, dx \right| \leq \int_{\Omega} \left[ |\nabla \varphi A(x,u,\nabla u) + |\varphi B(x,u,\nabla u)| \right] \, dx
\]

\[
\leq \int_{\Omega} |\nabla \varphi| |A(x,u,\nabla u)| + |\varphi B(x,u,\nabla u)| \, dx + \int_{\Omega} |\varphi| |B(x,u,\nabla u)| \, dx
\]

\[
\leq \| |\nabla \varphi|_{p,B;dx} \| A \|_{p',B;dx} + |\varphi|_{\sigma p,B;dx} \| B \|_{(\sigma p)'B;dx}.
\]

By the Sobolev inequality (1.13),

\[
\int_{\Omega} \left[ \nabla \varphi \cdot A(x,u,\nabla u) + \varphi B(x,u,\nabla u) \right] \, dx \leq \| |\nabla \varphi|_{W^{1,p}_Q(B)} \| A \|_{p',B;dx} + C(B,p) \| |\varphi|_{W^{1,p}_Q(B)} \| B \|_{(\sigma p)'B;dx},
\]

This is finite by (2.4), (2.12) and Proposition 2.3. In the same way,

\[
\int_{\Omega} |B(x,u,\nabla u)| \, dx \leq \int_{\Omega} \left( c|\sqrt{Q} \nabla u|^{\sigma - 1} + d|u|^{\delta - 1} + f \right) \frac{\sigma_p}{\sigma_p - 1} \, dx
\]

which is finite by (2.4), (2.12) and Proposition 2.3. Indeed,

\[
\int_{\Omega} c^{\frac{\sigma_p}{\sigma_p - 1}} |\sqrt{Q} \nabla u|^{\sigma - 1} \, dx \leq \left( \int_{\Omega} c^{\frac{\sigma_p}{\sigma_p - 1}} |\sqrt{Q} \nabla u|^{\sigma - 1} \, dx \right)^{\frac{\sigma_p - 1}{\sigma_p - 1}} \left( \int_{\Omega} |\sqrt{Q} \nabla u|^{\sigma - 1} \, dx \right)^{\frac{\sigma_p}{\sigma_p - 1}} < \infty,
\]

\[
\int_{\Omega} d^{(\sigma p)} |u|^{\sigma p} \, dx \leq \left( \int_{\Omega} d^{\frac{\sigma p}{\sigma_p - 1}} \, dx \right)^{\frac{\sigma_p - 1}{\sigma_p - 1}} \left( \int_{\Omega} |u|^{\sigma p} \, dx \right)^{\frac{\sigma_p}{\sigma_p - 1}} < \infty.
\]

Thus \( \tilde{A}(\cdot, u, \nabla u) \in L^{p'}(B) \) and \( B(\cdot, u, \nabla u) \in L^{(\sigma p)'}(B) \), and the proposition is established. \( \Box \)
and hence
\begin{equation}
(2.13) \quad \left| \int_{\Omega} \left[ \nabla \varphi \cdot A(x, u, \nabla u) + \varphi B(x, u, \nabla u) \right] \, dx \right| \leq \left( \| \tilde{A} \|_{p', B; dx} + C(B, p) \| B \|_{(\sigma p)', B; dx} \right) \| \varphi \|_{W^{1, p}_{Q}(B)}.
\end{equation}

The last quantity is finite because of Proposition 2.5, the fact that \( \varphi \in \text{Lip}_0(B) \), and our hypothesis that \( |Q| \in L_{\text{loc}}^{\frac{p}{p}}(\Omega) \). \( \square \)

**Proposition 2.7.** Under the hypotheses of Proposition 2.3, the map \( \Lambda : \text{Lip}_0(B(y, r)) \times W^{1, p}_{Q}(\Omega) \rightarrow \mathbb{R}^n \) defined by
\begin{equation}
(2.14) \quad \Lambda(\varphi, u) = \int_{\Omega} \left[ \nabla \varphi \cdot A(x, u, \nabla u) + \varphi B(x, u, \nabla u) \right] \, dx
\end{equation}
can be extended by continuity so as to be defined on \( (W^{1, p}_{Q})_0(B(y, r)) \times W^{1, p}_{Q}(\Omega) \). Also, if \( \varphi \in (W^{1, p}_{Q})_0(B(y, r)) \) and \( u \in W^{1, p}_{Q}(\Omega) \), then
\begin{equation}
(2.15) \quad \varphi_i \rightarrow \varphi \quad \text{in} \quad W^{1, p}_{Q}(B).
\end{equation}

**Proof:** We will again use \( B \) to denote both \( B(y, r) \) and \( B(x, u, \nabla u) \). The map \( \Lambda \) is well-defined on \( \text{Lip}_0(B) \times W^{1, p}_{Q}(\Omega) \) by Corollary 2.6. For fixed \( u \in W^{1, p}_{Q}(\Omega) \), the map \( \varphi \mapsto \Lambda(\varphi, u) \) is linear in \( \varphi \in \text{Lip}_0(B) \), and by (2.13),
\[ |\Lambda(\varphi, u)| \leq C\|\varphi\|_{W^{1, p}_{Q}(B)}, \]
with \( C \) depending on \( B, p, u \) and \( \tilde{A} \) but not on \( \varphi \). Then the linear map is continuous and can be extended by continuity to \( (W^{1, p}_{Q})_0(B) \), since this is the completion of \( \text{Lip}_0(B) \).

In order to prove (2.14), let \( u \in W^{1, p}_{Q}(\Omega), \varphi \in (W^{1, p}_{Q})_0(B), \) and \( \{ \varphi_i \}_{i \in \mathbb{N}} \subset \text{Lip}_0(B) \) be a Cauchy sequence representing \( \varphi \). Then
\begin{equation}
(2.15) \quad \varphi_i \rightarrow \varphi \quad \text{in} \quad W^{1, p}_{Q}(B).
\end{equation}

Moreover, by the previous estimates,
\[ \left| \int_{\Omega} \left[ \sqrt{Q} \nabla \varphi \cdot \tilde{A}(x, u, \nabla u) + \varphi B(x, u, \nabla u) \right] \, dx \right| < \infty. \]

Then
\begin{align*}
\left| \int_{\Omega} \left[ \sqrt{Q} \nabla \varphi \cdot \tilde{A}(x, u, \nabla u) + \varphi B(x, u, \nabla u) \right] \, dx - \Lambda(\varphi_i, u) \right| & = \int_{\Omega} \left| \sqrt{Q} \nabla \varphi \cdot \tilde{A} + \varphi B - \sqrt{Q} \nabla \varphi_i \cdot \tilde{A} - \varphi_i B \right| \, dx \\
& \leq \int_{B} \left[ \| \sqrt{Q} \nabla \varphi - \sqrt{Q} \nabla \varphi_i |\tilde{A} \right] + \| \varphi - \varphi_i \|_{L^p(B)} \, dx \\
& \leq \| \tilde{A} \|_{p', B; dx} \| \sqrt{Q} \nabla \varphi - \sqrt{Q} \nabla \varphi_i \|_{p, B; dx} + \| B \|_{(\sigma p)', B; dx} \| \varphi - \varphi_i \|_{\sigma p, B; dx} \leq \left( \| \tilde{A} \|_{p', B; dx} + C \| B \|_{(\sigma p)', B; dx} \right) \| \varphi - \varphi_i \|_{W^{1, p}_{Q}(B)},
\end{align*}

where we used the Sobolev inequality (1.13) to obtain the last inequality. Since \( \| \varphi - \varphi_i \|_{W^{1, p}_{Q}(B)} \rightarrow 0 \) by (2.15), we get
\[ \Lambda(\varphi, u) := \lim_{i \rightarrow \infty} \Lambda(\varphi_i, u) = \int_{\Omega} \left[ \sqrt{Q} \nabla \varphi \cdot \tilde{A}(x, u, \nabla u) + \varphi B(x, u, \nabla u) \right] \, dx, \]
and (2.14) is established. \( \square \)
2.2. A useful test function. Let $k,l,q,\mu,\beta \in \mathbb{R}$ with $q \geq 1$, $l > k \geq 0$, $\mu = p\sigma - 1$ and $\beta = (\mu + 1)q - \mu$. For any $t \in \mathbb{R}$, set $\bar{t} = |t| + k$. Define

\begin{align*}
F(\bar{t}) &= \begin{cases} 
\bar{t}^q & k \leq \bar{t} \leq l, \\
q^{\beta - 1} - (q - 1)\bar{t}^q & \bar{t} \geq l,
\end{cases} \\
G(t) &= \text{sign}(t) \left\{ F(\bar{t})F'(\bar{t})^\mu - q^\beta k^\beta \right\} \quad \text{for } t \in \mathbb{R}.
\end{align*}

As in the nondegenerate case studied in [S], we would like to use the function $\phi(x) = \eta(x)^p G(u(x))$, $x \in \Omega$, as a test function in (2.10), where $\eta \in \text{Lip}_b(\Omega)$ is any of the cutoff functions provided by (1.14), and $u \in W_{Q}^{1,p}(\Omega)$ is a weak solution of the differential equation (1.11). In order to show that $\phi$ a feasible test function, we begin by showing that there is a sequence $\{l_i\} \subset \mathbb{R}^+$, $l_i \nearrow \infty$, such that if we choose these $l_i$’s in definitions (2.16) and (2.17), then $G(u) \in W_{Q}^{1,p}(\Omega)$.

**Lemma 2.8.** Let $u \in L^\alpha(\Omega)$, $\alpha \in [1, \infty)$, $u \geq 0$ a.e. in $\Omega$. For any $l \in \mathbb{R}^+$, let $E_l = \{x \in \Omega : u(x) = l\}$. Then the set

$$
\Sigma = \{l \in \mathbb{R}^+ : |E_l| > 0\}
$$

is countable.

**Proof:** We claim that for every $\varepsilon > 0$, the set $\Sigma_\varepsilon = \{l \geq \varepsilon : |E_l| > 0\}$ is countable. For $j \in \mathbb{N}$, let $\Sigma_{\varepsilon,j} = \{l \geq \varepsilon : |E_l| > \frac{1}{j}\}$. Then $\Sigma_\varepsilon = \bigcup_{j \in \mathbb{N}} \Sigma_{\varepsilon,j}$, and it is enough to show that each $\Sigma_{\varepsilon,j}$ is countable. Fix $\varepsilon, j$ and let $\{l_i : i \in I\}$ be a sequence of distinct points in $\Sigma_{\varepsilon,j}$. Then

$$
\int_\Omega u^\alpha \, dx \geq \int_{\cup_{i \in I} E_{l_i}} u^\alpha \, dx = \sum_{i \in I} l_i^\alpha |E_{l_i}| \geq \frac{\varepsilon^\alpha}{j} \sum_{i \in I} 1.
$$

Since $u \in L^\alpha(\Omega)$, it follows that $\Sigma_{\varepsilon,j}$ is actually finite, and the claim follows. Since $\Sigma_{\frac{1}{m}}$ is countable for every $m \in \mathbb{N}$, the set

$$
\{l > 0 : |E_l| > 0\} = \bigcup_{m \in \mathbb{N}} \Sigma_{\frac{1}{m}}
$$

is countable too. Thus the set $\Sigma = \{l \geq 0 : |E_l| > 0\}$ is also countable, which proves the lemma. □

**Corollary 2.9.** Given a sequence $\{u_i\}_{i \in \mathbb{N}} \subset L^\alpha(\Omega)$, there is a sequence of positive numbers $l_j \nearrow \infty$ such that

$$
|E_{i,l_j}| = |\{x \in \Omega : |u_i(x)| = l_j\}| = 0 \quad \text{for all } i, j \in \mathbb{N}.
$$

**Proof:** The sets $\Sigma_i := \{l \in \mathbb{R}^+ : |E_{i,l}| > 0\}$ are countable for every $i$ by Lemma [2.8] and hence the set $\Sigma := \bigcup \Sigma_i = \{l \in \mathbb{R}^+ : |E_{i,l}| > 0\}$ for some $i$ is countable. Then $\mathbb{R}^+ \setminus \Sigma$ is uncountable, and in particular there is a sequence $\{l_j\}_{j \in \mathbb{N}} \subset \mathbb{R}^+ \setminus \Sigma$ such that $l_j \nearrow \infty$. Since $E_{i,l_j} \subset \mathbb{R}^+ \setminus \Sigma$ for every $j$, we have

$$
|E_{i,l_j}| = 0 \quad \text{for all } i, j \in \mathbb{N}.
$$

The corollary is proved. □

The next fact can be proved in a similar way.

**Corollary 2.10.** Given a sequence $\{u_i\}_{i \in \mathbb{N}} \subset L^\alpha(\Omega)$, there is a sequence of positive numbers $\lambda_j \searrow 0^+$ such that

$$
|E_{i,\lambda_j}| = |\{x \in \Omega : |u_i(x)| = \lambda_j\}| = 0 \quad \text{for all } i, j \in \mathbb{N}.
$$

Lemma 2.8 and Corollaries 2.9 and 2.10 provide a means to avoid using the notion of regular gradient introduced in SW2. This simplifies some technical aspects in SW2 and leads to relatively short proofs of results like Theorem 2.11 and Lemma 1.2.
Remark 2.12. In the proof of Theorem 2.11, we will use the following facts for every $j \in \mathbb{N}$:

i) $F_j \in C^1([k, \infty))$, with

\[
F'_j(\tilde{t}) = \begin{cases} 
q\tilde{t}^{q-1} & k \leq \tilde{t} \leq l_j, \\
q\tilde{t}^{q-1} - (q - 1)l_j^q & \tilde{t} > l_j,
\end{cases}
\]

\[
F''_j(\tilde{t}) = \begin{cases} 
q(q - 1)\tilde{t}^{q-2} & k < \tilde{t} < l_j, \\
0 & \tilde{t} > l_j.
\end{cases}
\]

ii) $0 \leq F_j(\tilde{t}) \leq q\tilde{t}^{q-1} + l_j^q,
0 \leq F'_j(\tilde{t}) \leq q\tilde{t}^{q-1},
F''_j(\tilde{t}) \geq 0$ and $F''_j(\tilde{t})$ is a bounded function away from $\tilde{t} = 0$.

iii) $G_j \in C^0(\mathbb{R})$ and $G_j$ is differentiable everywhere except at $\pm(l_j - k)$ where it has “corners”. Indeed

\[
G'_j(t) = F'_j(\tilde{t})^{\mu+1} + \mu F'_j(\tilde{t})F''_j(\tilde{t})F'_j(\tilde{t})^{\mu-1}
\]

\[
= \begin{cases} 
q^{\mu+1}(\mu+1) + q^{\mu+1}(\mu+1)(q-1) + \mu(q - 1)q^{\mu+1}(\mu+1)(q-1) & 0 < |t| < l_j - k, \\
q^{\mu+1}l_j^{\mu+1}(q-1) & |t| > l_j - k,
\end{cases}
\]

\[
= \begin{cases} 
\beta q^{\mu+1}l_j^{\mu+1}(q-1) & 0 < |t| < l_j - k, \\
q^{\mu+1}l_j^{\mu+1}(q-1) & |t| > l_j - k,
\end{cases}
\]

\[
= \begin{cases} 
q^{-1}\beta F'_j(\tilde{t})^{\mu+1} & 0 < |t| < l_j - k, \\
F'_j(\tilde{t})^{\mu+1} & |t| > l_j - k.
\end{cases}
\]

The discontinuity of $G'_j$ at 0 is removable.

iv) $|G_j(t)| \leq F_j(\tilde{t})F'_j(\tilde{t})^{\mu} \leq \left(q_l^{p-1}\right)^{\mu} F_j(\tilde{t}),
0 \leq G'_j(t) \leq q^{-1}\beta F'_j(\tilde{t})^{\mu+1} \leq q^{-1}\beta q^{\mu+1}l_j^{\mu+1}(q-1)^{\mu+1} = \beta q^{\mu+1}l_j^{\mu+1}(q-1)^{\mu+1} < \infty$.

Proof of Theorem 2.11. Let $u \in W^{1,p}_Q(\Omega)$ and $\{u_i\} \subset \text{Lip}_{\text{loc}}(\Omega)$ be a sequence representing $u$. Then up to subsequences,

\[
u_i \rightarrow u \quad \text{in } W^{1,p}_Q(\Omega),
\]

\[
u_i \rightarrow u \quad \text{in } L^p(\Omega) \text{ and a.e. in } \Omega \text{ and}
\]

\[
\sqrt{Q\nabla u_i} \rightarrow \sqrt{Q\nabla u} \quad \text{in } [L^p(\Omega)]^n \text{ and a.e. in } \Omega.
\]

Use Corollary 2.9 to choose a sequence $r_j \nearrow \infty$ such that

\[
|E_{i,r_j}| = |\{x \in \Omega : |u_i(x)| = r_j\}| = 0 \quad \text{for } i, j \in \mathbb{N},
\]

\[
|E_{r_j}| = |\{x \in \Omega : |u(x)| = r_j\}| = 0 \quad \text{for } j \in \mathbb{N}.
\]

Define $l_j = r_j + k$ for $j \in \mathbb{N}$. 

Claim 1: $G_j(u_i) \in \text{Lip}_{\text{loc}}(\Omega)$ for every $i,j \in \mathbb{N}$.

By the fundamental theorem of calculus,

$$G_j(t) = \int_0^t G'_j(s) \, ds + G(0) \quad \text{for all } t \in \mathbb{R}, j \in \mathbb{N}.$$  

Then for $x,y \in \Omega$,

$$|G_j(u_i(x)) - G_j(u_i(y))| = \left| \int_{u_i(y)}^{u_i(x)} G'_j(s) \, ds \right| \leq \|G'_j\|_{\infty} |u_i(x) - u_i(y)|.$$  

Since $\|G'_j\|_{\infty}$ is finite by Remark 2.12 and since $u_i$ is locally Lipschitz continuous, it follows that $G_j(u_i) \in \text{Lip}_{\text{loc}}(\Omega)$ for every $i,j \in \mathbb{N}$. In particular, $\nabla(G_j(u_i))$ is well-defined a.e. in $\Omega$.

Claim 2: For almost every $x \in \Omega$,

$$\nabla(G_j(u_i))(x) = G'_j(u_i(x))\nabla u_i(x) \quad \text{for all } i,j \in \mathbb{N},$$

(2.19)  

$$\int_{\Omega} |\sqrt{Q}\nabla(G_j(u_i))| \, dx < \infty.$$  

Indeed, consider $x \in \Omega$ such that

i) $\nabla u_i(x), \nabla(G_j(u_i))(x)$ are defined,

ii) $|u_i(x)| \neq l_j - k = r_j$ for every $i,j$.

Then $G_j(t)$ is $C^1$ in a neighborhood of $u_i(x)$, since $G_j(t)$ has corners only at $t = \pm (l_j - k)$ by Remark 2.12. By the chain rule, formula (2.19) holds at the point $x$.

Since the set of points for which either (i) or (ii) does not hold has Lebesgue measure 0, formula (2.19) holds for a.e. $x \in \Omega$. Thus for every $i$ and $j$,

$$\int_{\Omega} |\sqrt{Q}\nabla(G_j(u_i))| \, dx = \int_{\Omega} |G'_j(u_i(x))| |\sqrt{Q}\nabla u_i(x)| \, dx \leq \|G'_j\|_{\infty} \int_{\Omega} |\sqrt{Q}\nabla u_i(x)| \, dx < \infty.$$  

Claim 2 follows.

Claim 3: $G_j(u_i) \rightarrow G_j(u)$ a.e. in $\Omega$ and in $L^p(\Omega)$ for all $j$.

Since $u_i \rightarrow u$ a.e. in $\Omega$ and $G_j$ is continuous for every $j$, we obviously have that $G_j(u_i) \rightarrow G_j(u)$ a.e. in $\Omega$ for every $j$. Then

$$|G_j(u_i) - G_j(u)|^p \rightarrow 0 \quad \text{a.e. in } \Omega, \text{ for all } j \in \mathbb{N}.$$
We also have for every \( x \in \Omega \) that
\[
|G_j(u_i) - G_j(u)|^p \leq 2^p (|G_j(u_i)|^p + |G_j(u)|^p) \\
\leq 2^p (|F_j(\tilde{u}_i)|^p |F'_j(\tilde{u}_i)|^{p\mu} + |F_j(\tilde{u})|^p |F'_j(\tilde{u})|^{p\mu}) \\
\leq 2^p (q_{t_j}^{-1})^{p\mu} (|F_j(\tilde{u}_i)|^p + |F_j(\tilde{u})|^p) \\
\leq 2^p (q_{t_j}^{-1})^{p\mu} \left( \left( q_{t_j}^{-1} \tilde{u}_i + l_j^1 \right)^p + \left( q_{t_j}^{-1} \tilde{u} + l_j^2 \right)^p \right)
\]
(2.20)
\[
\leq 2^{2p} (q_{t_j}^{-1})^{p\mu} \left( \left( q_{t_j}^{-1} \tilde{u}_i \right)^p \bar{\n}_i^p + \left( q_{t_j}^{-1} \bar{\n}^p + 2l_j^p \right)^p \right)
\leq 2^{2p} \left( q_{t_j}^{-1} \right)^{p\mu} \max \left( \left( q_{t_j}^{-1} \right)^p \bar{\n}_i^p, 2l_j^p \right) \left( \left| \bar{\n}_i^p + \bar{\n}^p + 1 \right| \right)
\]
\[
\leq 2^{3p} \left( q_{t_j}^{-1} \right)^{p\mu} \max \left( \left( q_{t_j}^{-1} \right)^p, 2l_j^p \right) \left( \left| \bar{\n}_i^p + |\bar{\n}|^p + (2k^p + 1) \right| \right)
\leq 2^{3p} \left( q_{t_j}^{-1} \right)^{p\mu} \max \left( \left( q_{t_j}^{-1} \right)^p, 2l_j^p \right) \left( 2k^p + 1 \right) \left( |\bar{\n}_i|^p + |\bar{\n}|^p + 1 \right)
\]
\[
= C(p, \sigma, q, l, k) \left( |u_i|^p + |\bar{\n}|^p + 1 \right),
\]
and \((|u_i|^p + |\bar{\n}|^p + 1) \in L^1(\Omega)\) for every \( i \in \mathbb{N} \).

Since \((|u_i|^p + |\bar{\n}|^p + 1) \rightarrow (2|\bar{\n}|^p + 1)\) for a.e. \( x \in \Omega \), and since
\[
\int_{\Omega} (|u_i|^p + |\bar{\n}|^p + 1) \, dx = \|u_i\|^p_{p, \Omega; dx} + |\bar{\n}|^p_{p, \Omega; dx} + |\Omega|
\rightarrow 2|\bar{\n}|^p_{p, \Omega; dx} + |\Omega| = \int_{\Omega} (2|\bar{\n}|^p + 1) \, dx,
\]
the Lebesgue Sequentially Dominated Convergence Theorem gives
\[ G_j(u_i) \rightarrow G_j(u) \quad \text{in } L^p(\Omega) \quad \text{for all } j. \]

Claim 3 is thus proved.

\[ \text{Claim 4: } \sqrt{Q} \nabla (G_j(u_i)) \rightarrow G'_j(u) \sqrt{Q} \nabla u \text{ a.e. in } \Omega \text{ and in } [L^p(\Omega)]^n, \text{ for all } j \in \mathbb{N}. \]

Consider a point \( x \in \Omega \) such that
\[
\begin{aligned}
i) & \quad \nabla (G_j(u_i))(x) \text{ and } \sqrt{Q} \nabla u(x) \text{ are defined,} \\
ii) & \quad \nabla (G_j(u_i))(x) = G'_j(u_i(x)) \nabla u_i(x), \\
iii) & \quad |u(x)| \neq r_j \neq l_j - k \quad \text{and} \quad |u_i(x)| \neq r_j \text{ for every } i, j, \\
iv) & \quad u_i(x) \rightarrow u(x), \\
v) & \quad \sqrt{Q} \nabla u_i(x) \rightarrow \sqrt{Q} \nabla u(x). \\
\end{aligned}
\]

The set of points which do not satisfy one or more of these conditions has Lebesgue measure 0 for the following reasons, respectively:
\[
\begin{aligned}
i) & \quad \text{because } G_j(u_i) \text{ is locally Lipschitz for every } i, j \text{ by claim 1, and } u \in W^{1,p}_Q(\Omega), \\
ii) & \quad \text{by claim 2,} \\
iii) & \quad \text{by our choice of the sequence } r_j, \\
iv) & \quad \text{because } u_i \rightarrow u \text{ a.e. in } \Omega, \\
v) & \quad \text{because } \sqrt{Q} \nabla u_i \rightarrow \sqrt{Q} \nabla u \text{ a.e. in } \Omega.
\end{aligned}
\]

For any \( x \in \Omega \) satisfying all these conditions,
\[
\sqrt{Q} \nabla (G_j(u_i))(x) = G'_j(u_i(x)) \sqrt{Q} \nabla u_i(x) \rightarrow G'_j(u(x)) \sqrt{Q} \nabla u(x)
\]
since $G'_j(t)$ is continuous everywhere except at $t = \pm(l_j - k)$ while $|u(x)| \neq l_j - k$. Thus,

$$\sqrt{Q}\nabla(G_j(u_i)) \rightarrow G'_j(u)\sqrt{Q}\nabla u \quad \text{a.e in } \Omega.$$ 

On the other hand, a.e. in $\Omega$ and for every $i, j \in \mathbb{N},$

$$\left|\sqrt{Q}\nabla(G_j(u_i)) - G'_j(u)\sqrt{Q}\nabla u\right|^p = \left|G'_j(u_i)\sqrt{Q}\nabla u_i - G'_j(u)\sqrt{Q}\nabla u\right|^p \leq \left((|G'_j(u_i)| + |G'_j(u)|)\right)^p \leq 2^p \left(|G'_j|_\infty^p \sqrt{Q}\nabla u_i|^p + |G'_j|_\infty^p \sqrt{Q}\nabla u|^p\right) \leq 2^p \left(\beta_d \mu^{(\mu+1)(q-1)}\right)^p \left(\sqrt{Q}\nabla u_i|^p + \sqrt{Q}\nabla u|^p\right),$$

and the functions on the right in the last inequality belong to $L^1(\Omega)$ for every $i, j.$

Also, $(\sqrt{Q}\nabla u_i^p + \sqrt{Q}\nabla u|^p) \rightarrow 2\sqrt{Q}\nabla u|^p$ a.e. in $\Omega,$ and

$$\int_\Omega \left(\sqrt{Q}\nabla u_i|^p + \sqrt{Q}\nabla u|^p\right) dx = \|\sqrt{Q}\nabla u_i|_{p,\Omega;dx} + \|\sqrt{Q}\nabla u|^p_{p,\Omega;dx} \rightarrow 2\|\sqrt{Q}\nabla u|^p_{p,\Omega;dx} = \int_\Omega 2\sqrt{Q}\nabla u|^p dx.$$

Then by the Lebesgue Sequentially Dominated Convergence Theorem we have

$$\sqrt{Q}\nabla(G_j(u_i)) \rightarrow G'_j(u)\sqrt{Q}\nabla u \quad \text{in } [L^p(\Omega)]^n \text{ for all } j \in \mathbb{N}.$$ 

Claim 4 is thus proved.

Claims 3 and 4 together prove that $G_j(u) \in W^{1,p}_Q(\Omega),$ that $\{G_j(u_i)\}_{i \in \mathbb{N}}$ is a sequence of locally Lipschitz functions in $\Omega$ which represents $G_j(u)$ and that

$$\sqrt{Q}\nabla(G_j(u)) = G'_j(u)\sqrt{Q}\nabla u \quad \text{a.e in } \Omega,$$

which is formula (2.18). □

3. Proof of Theorem 1.2

Step 1. We will use the notation

$$\|w\|_{\alpha,B_r} := \|w\|_{\alpha,B_r;dx} = \left(\frac{1}{|B_r|} \int_{B_r} |w|^\alpha dx\right)^{\frac{1}{\alpha}} = \left(\int_{B_r} |w|^\alpha dx\right)^{\frac{1}{\alpha}}$$

for any $\alpha \geq 1,$ any function $w$ and any $\rho$-ball $B_r = B(y, r)$ with $0 < r < r_1(y).$ For $k > 0,$ define

$$\bar{z} = |z| + k, \quad z \in \mathbb{R},$$

$$\bar{b}(x) = b(x) + k^{-1}p e(x), \quad x \in \Omega,$$

$$\bar{h}(x) = h(x) + k^{-p}g(x), \quad x \in \Omega,$$

$$\bar{d}(x) = d(x) + k^{-1}p f(x), \quad x \in \Omega.$$ 

Then the following new structural inequalities for the coefficients are easily obtained from (1.3) with $\gamma = \delta = \rho:$

$$\xi \cdot A(x,z,\xi) \geq a^{-1}|\sqrt{Q}(x) : \xi|^p - \bar{h}(x)\bar{z}^p,$$

$$|\bar{A}(x,z,\xi)| \leq a|\sqrt{Q}(x) : \xi|^{p-1} + \bar{b}(x)\bar{z}^{p-1},$$

$$|B(x,z,\xi)| \leq c|\sqrt{Q}(x) : \xi|^{\nu-1} + \bar{d}(x)\bar{z}^{\nu-1}.$$

In fact, when we deal with a specific solution pair $(u, \nabla u),$ as is the case in Theorem 1.2 we will only need to assume the analogue of (3.1) in which $z$ and $\xi$ are replaced respectively by $u(x)$ and $\nabla u(x)$ for all $x \in \Omega.$

Now consider the functions $F_j(t)$ and $G_j(t)$ defined in Theorem 2.11. Let $\eta \in \text{Lip}_0(B_r)$ be any of the Lipschitz cutoff functions provided by (1.14) for a $\rho$-ball $B_r.$ Then by Theorem 2.11...
Corollary 2.2 and Proposition 2.7 each function \( \varphi_j(x) := \eta(x)^p G_j(u(x)) \) is a feasible test function in (2.10).

In order to simplify notation, we will not explicitly show the dependence of \( A, \tilde{A} \) or \( B \) on the variables \( x, u(x) \) and \( \nabla u(x) \). Also, we will often not show the dependence of any function of \( x \) on \( x \).

Step 2. We start by deriving some pointwise estimates which give lower bounds for \( \nabla \varphi_j A + \varphi_j B \). By the structural conditions (3.1),

\[
\nabla \varphi_j \cdot A + \varphi_j B = \sqrt{Q} \nabla \varphi_j \cdot \tilde{A} + \varphi_j B
\]

\[
= \eta^p G_j(u) \sqrt{Q} \nabla \eta \cdot \tilde{A} + \eta^p G_j'(u) \sqrt{Q} \nabla u \cdot \tilde{A} + \eta^p G_j(u) B
\]

\[
\geq \eta^p G_j(u) \left[ a^{-1} \left| \sqrt{Q} \nabla u \right|^p - \tilde{h} \alpha \right] - \eta^p G_j(u) \left[ c \left| \sqrt{Q} \nabla u \right|^{\psi - 1} + \tilde{d} \alpha \right]
\]

\[
- \eta^p G_j(u) \left[ \left| \sqrt{Q} \nabla \eta \right| \left| \sqrt{Q} \nabla u \right|^{p - 1} \right]
\]

Then it follows from Remark 2.12 that

\[
\nabla \varphi_j \cdot A + \varphi_j B \geq \eta^p F_j'(\bar{u})^{\alpha+1} \left| \sqrt{Q} \nabla u \right|^p
\]

\[
- \eta^p F_j'(\bar{u})^{\alpha+1} \left| \sqrt{Q} \nabla D \right|^{\psi - 1} - \tilde{h} \alpha \eta^p D \geq \eta^p F_j'(\bar{u})^{\alpha+1} \left| \sqrt{Q} \nabla u \right|^{p - 1}
\]

\[
- \eta^p \tilde{h} \alpha \eta^p F_j'(\bar{u})^{\alpha+1} \left| \sqrt{Q} \nabla u \right|^{p - 1}
\]

\[
- \tilde{d} \alpha \eta^p F_j'(\bar{u})^{\alpha+1} \left| \sqrt{Q} \nabla u \right|^{p - 1}
\]

\[
\geq \eta^p F_j'(\bar{u})^{\alpha+1} \left| \sqrt{Q} \nabla u \right|^p
\]

Although the last two terms are identical apart from the multiplicative factor \( \beta \), we will treat them separately in order to simplify calculations later in the corollaries following our main theorem. By Lemma 4.2 in Appendix 1, we can replace \( \sqrt{Q} \nabla u \) by \( \sqrt{Q} \nabla \bar{u} \) in the previous inequalities. Setting \( \eta := F_j'(\bar{u}) \), we have \( \sqrt{Q} \nabla v_j = F_j'(\bar{u}) \sqrt{Q} \nabla \bar{u} \) by Lemma 4.1 in Appendix 1. Thus

\[
\nabla \varphi_j \cdot A + \varphi_j B \geq \eta^{-1} F_j'(\bar{u})^{\alpha+1} \left| \sqrt{Q} \nabla v_j \right|^p
\]

\[
- \eta^{-1} F_j'(\bar{u})^{\alpha+1} \left| \sqrt{Q} \nabla v_j \right|^{\psi - 1}
\]

\[
- \eta^{-1} F_j'(\bar{u})^{\alpha+1} \left| \sqrt{Q} \nabla \eta \right| \left| \sqrt{Q} \nabla v_j \right|^{p - 1}
\]

\[
- \eta^{-1} F_j'(\bar{u})^{\alpha+1} \left| \sqrt{Q} \nabla \eta \right| \left| \sqrt{Q} \nabla v_j \right|^{p - 1}
\]

\[
- \eta^{-1} F_j'(\bar{u})^{\alpha+1} \left| \sqrt{Q} \nabla \eta \right| \left| \sqrt{Q} \nabla v_j \right|^{p - 1}
\]

(3.2)

\[
- \eta^{-1} F_j'(\bar{u})^{\alpha+1} \left| \sqrt{Q} \nabla \eta \right| \left| \sqrt{Q} \nabla v_j \right|^{p - 1}
\]

\[
- \eta^{-1} F_j'(\bar{u})^{\alpha+1} \left| \sqrt{Q} \nabla \eta \right| \left| \sqrt{Q} \nabla v_j \right|^{p - 1}
\]

\[
- \eta^{-1} F_j'(\bar{u})^{\alpha+1} \left| \sqrt{Q} \nabla \eta \right| \left| \sqrt{Q} \nabla v_j \right|^{p - 1}
\]
Now, recalling that \( u \) is a weak solution of (1.1) and that \( \eta \) and thus \( \varphi_j \) have support in \( B_r \), we have

\[
(3.3) \quad \int_{B_r} [\nabla \varphi_j \cdot A + \varphi_j B] \, dx = \int_{\Omega} [\nabla \varphi_j \cdot A + \varphi_j B] \, dx = 0.
\]

Integrating (3.2) over \( \Omega \), dividing by \( \beta_\eta \) and rearranging terms we then get

\[
(3.4) \quad a^{-1} \int_{B_r} F'_j(\bar{u})^{\mu+1-p} |\eta \sqrt{Q} \nabla v_j|^p \, dx \leq \frac{1}{\theta B_r} \int_{B_r} F'_j(\bar{u})^{\mu+1-p} |v_j \sqrt{Q} \nabla \eta|^{p-1} \, dx
\]

\[
+ \int_{B_r} c \eta^{p-1} \nabla v_j F'_j(\bar{u})^{\mu+1-p} |\eta \sqrt{Q} \nabla v_j|^{p-1} \, dx
\]

\[
+ \int_{B_r} \eta \sqrt{Q} \nabla \eta j \, dx
\]

\[
+ \int_{B_r} \theta \eta \sqrt{Q} \nabla \eta j \, dx.
\]

**Step 3.** Our next aim is to apply Young’s inequality to the first and third terms on the right side in order to absorb all terms containing \( |\eta \sqrt{Q} \nabla v_j| \) into the left side. We begin by noticing that for any \( \theta > 0 \),

\[
|v_j \sqrt{Q} \nabla \eta| |\eta \sqrt{Q} \nabla v_j|^{p-1} \leq \theta |\eta \sqrt{Q} \nabla v_j|^p + \frac{1}{\theta^{p-1}} |v_j \sqrt{Q} \nabla \eta|^p.
\]

Hence

\[
\int_{B_r} F'_j(\bar{u})^{\mu+1-p} |v_j \sqrt{Q} \nabla \eta| |\eta \sqrt{Q} \nabla v_j|^{p-1} \, dx \leq \theta \int_{B_r} F'_j(\bar{u})^{\mu+1-p} |v_j \sqrt{Q} \nabla \eta|^p \, dx
\]

\[
+ \frac{1}{\theta^{p-1}} \int_{B_r} F'_j(\bar{u})^{\mu+1-p} |v_j \sqrt{Q} \nabla \eta|^{p-1} \, dx.
\]

In order to deal with the third term on the right side of (3.4), we use Young’s inequality with exponents \( \frac{\nu}{\psi-1} \) and \( \frac{\psi}{p+\psi-1} \). Setting \( \nu = \frac{(\mu+1-p)(\psi-1)}{p} \), we get for any \( \theta > 0 \) that

\[
\int_{B_r} c \eta^{p+1-\psi} v_j F'_j(\bar{u})^{\mu+1-\psi} |\eta \sqrt{Q} \nabla v_j|^{\psi-1} \, dx
\]

\[
= \theta \int_{B_r} F'_j(\bar{u})^{\mu+1-\psi} |\eta \sqrt{Q} \nabla v_j|^{\psi-1} \, dx
\]

\[
\leq \theta \int_{B_r} F'_j(\bar{u})^{\mu+1-p} |\eta \sqrt{Q} \nabla v_j|^p \, dx + \frac{1}{\theta^{\psi-1}} \int_{B_r} \eta^p v_j^{p+\psi-1} F'_j(\bar{u})^{\mu+1-\psi-1} \, dx.
\]

We explicitly note that \( \frac{\nu}{\psi-1} > 1 \) and \( \nu > 0 \) by (1.20) and since \( \mu = p \sigma - 1 \); see also Theorem 2.11. Moreover \( \mu + 1 - \psi - \nu > 0 \).
Combining \((3.5)\) and \((3.6)\) with \((3.4)\), and choosing \(\theta\) suitably small, we obtain
\[
\int_{B_r} F'_j(\bar{u})^{\mu+1-p} |\eta \sqrt{Q \nabla v_j}|^p dx \leq C \left\{ \int_{B_r} F'_j(\bar{u})^{\mu+1-p} |v_j \sqrt{Q \nabla \eta}|^p dx + q^{\gamma-1} \int_{B_r} \eta^{p-1} \bar{b} F'_j(\bar{u})^{\mu+1-p} |\sqrt{Q \nabla \eta}| v_j^p dx \right. \\
+ \int_{B_r} c^{p+1-v} \eta^{p} v_j^{\frac{\psi}{p+1-v}} F'_j(\bar{u})^{\mu+1-p} v_j dx \\
+ \beta q^{\gamma-1} \int_{B_r} \bar{h} \eta^{p-1} v_j F'_j(\bar{u})^{\mu+1-p} dx \\
+ q^{p-1} \int_{B_r} \bar{d} \eta^{p} v_j^p F'_j(\bar{u})^{\mu+1-p} dx \left\}
\]
(3.7)
for a positive constant \(C = C(p, a, \sigma)\).

Step 4. Now we would like to pass to the limit as \(j \to \infty\) in (3.7). By Theorem 2.11 both \(\{F_j(\bar{t})\}\) and \(\{F_j(\bar{t})\}\) are nondecreasing for every \(t\). Then the three sequences \(v_j = F_j(\bar{u}), F_j(\bar{u})\) and \(|\sqrt{Q \nabla v_j}| = F_j(\bar{u})|\sqrt{Q \nabla \bar{u}}|\) are nondecreasing. Indeed
\[
v_j \not\to \bar{u}, \quad F_j(\bar{u}) \not\to \bar{q} \bar{u}^{q-1}, \quad |\sqrt{Q \nabla v_j}| \not\to \bar{q} \bar{u}^{q-1}|\sqrt{Q \nabla \bar{u}}|
\]
a.e. in \(\Omega\) as \(j\) tends to \(\infty\). Passing to the limit in (3.7) and using the monotone convergence theorem then yields
\[
q^{\mu+1} \int_{B_r} \bar{u}^{(q-1)(\mu+1)} |\eta \sqrt{Q \nabla \bar{u}}|^p dx \leq C \left\{ \int_{B_r} \bar{u}^{(q-1)(\mu+1)+p} |\sqrt{Q \nabla \eta}|^p dx + q^{\mu} \int_{B_r} \eta^{p-1} \bar{b} |\sqrt{Q \nabla \eta}| \bar{u}^{(q-1)(\mu+1)+p} dx \right. \\
+ q^{\mu} \int_{B_r} c^{p+1-v} \eta^{p} \bar{u}^{(q-1)(\mu+1)+\frac{\psi}{p+1-v}} dx \\
+ \beta q^{\mu} \int_{B_r} \bar{h} \eta^{p-1} \bar{u}^{(q-1)(\mu+1)+p} dx \\
+ q^{p-1} \int_{B_r} \bar{d} \eta^{p} \bar{u}^{(q-1)(\mu+1)+p} dx \left\}
\]
(3.8)
where the integrals may not be finite.

Step 5. We will estimate \(\Pi, \Pi I, \Pi II, \Pi IV\) and \(\Pi V\) separately. Define
\[
(i) \quad Y = (\mu + 1)(q - 1) + p, \\
(ii) \quad t = \frac{p}{p + 1 - \psi} \quad \text{and} \quad \text{and} \\
(iii) \quad T = \mu - \frac{\psi - 1}{p + 1 - \psi}.
\]
We begin with term \(\Pi I:\)
\[
\Pi I = q^{\mu} \int_{B_r} \eta^{p-1} \bar{b} \sqrt{Q \nabla \eta} |\bar{u}^Y| dx \\
= q^{\mu} \int_{B_r} \bar{b} \sqrt{Q \nabla \eta} |\bar{u}^Y| (\eta \bar{u}^Y)^{p-1} dx \\
\leq q^{\mu} \|\bar{b}\| \sqrt{Q \nabla \eta} |\bar{u}^Y|_{\frac{p\sigma}{p\sigma + 1 - \psi}} \|\eta \bar{u}^Y\|_{\frac{p\sigma - 1}{p\sigma - \psi}} \|\bar{u}^Y\|_{\frac{p\sigma - 1}{p\sigma - \psi}} 
\]
(3.10)
by Hölder’s inequality with \(\frac{p - 1}{p\sigma} + \frac{p(\sigma - p + 1)}{p\sigma} = 1\). Use Hölder’s inequality again on the first factor
with \(\frac{p}{p(\sigma - 1) + 1} + \frac{(\sigma - 1)(p - 1)}{p(\sigma - 1) + 1} = 1\) and apply Lemma 4.5 to the second factor to obtain

\[
II \leq C q^\mu B y^p + C q^\mu B y^p \left( \frac{x}{r} \right)^{p - 1} yz^{p - 1} + C q^\mu B y^p \left( \frac{x}{r} \right)^{p - 1} yz^{p - 1}.
\]

We now use Young’s inequality on the second and third terms of (3.12). Fix \(s_1 \in (0, 1)\) to be chosen precisely later in the proof. Then, since \(p' = \frac{p}{p - 1}\) and \(q \sim Y, \)

\[
II \leq C q^\mu B y^p + C q^\mu B y^p \left( \frac{x}{r} \right)^{p - 1} yz^{p - 1} + C q^\mu B y^p \left( \frac{x}{r} \right)^{p - 1} yz^{p - 1}.
\]

We next estimate III. Fix \(\epsilon_1 \in (0, 1]\) as provided in the hypothesis. Then

\[
III = q^T \int_{B_r} c^\epsilon yu^\mu Y^{1 - t - p} dx
\]

\[
= q^T \int_{B_r} c^\epsilon yu^\mu Y^{1 - t - p} (\eta y u^\mu Y) dx
\]

\[
\leq q^T \left\| c^\epsilon yu^\mu Y^{1 - t - p} \right\|_{p(\sigma - p + 1)/p\sigma, dx} \left\| \eta y u^\mu Y \right\|_{p(\sigma - p + 1)/p\sigma, dx}
\]

\[
\leq q^T \left\| c^\epsilon yu^{1 - p}(\eta y u^\mu Y)^{1 - \epsilon_1} \right\|_{p(\sigma - p + 1)/p\sigma, dx} \left\| \eta y u^\mu Y \right\|_{p(\sigma - p + 1)/p\sigma, dx},
\]

where Hölder’s inequality with \(\frac{p - 1}{p\sigma} + \frac{p(\sigma - p + 1)}{p\sigma} = 1\) was used to obtain (3.13). Due to the restrictions on \(p, \sigma, \epsilon_1\) we have

\[
(i) \quad \frac{p\sigma - p + 1}{\sigma\epsilon_1} \geq 1 \quad \text{and} \quad (ii) \quad \frac{\sigma(p - \epsilon_1) + 1 - p}{1 - \epsilon_1} \geq 1.
\]

Therefore, by Hölder’s inequality applied to the triple product in (3.14),
III \leq q^T \|c^T \tilde{u}^{t-p}\|_{\frac{p \sigma}{(\rho - 1)(\rho - 1)}} \|\eta \tilde{u}^{\frac{\gamma}{\rho}}\|_{p;dx} \|\eta \tilde{u}^{\frac{\gamma}{\rho - \sigma}}\|_{p;dx}.

Now use Lemma 4.3 on the last factor to obtain

\[
III \leq Cq^T r^{p-\epsilon_1} \|c^T \tilde{u}^{t-p}\|_{\frac{p \sigma}{(p - 1)(\sigma - 1)}} \|\eta \tilde{u}^{\frac{\gamma}{\rho}}\|_{p;dx}
\]

\[
\times \left[\|\tilde{u}^{\frac{\gamma}{\rho}} \sqrt{Q\nabla \eta}\|_{p;dx}^{p-\epsilon_1} + \left(\frac{Y}{p}\right)^{p-\epsilon_1} \|\tilde{u}^{\frac{\gamma}{\rho}} \sqrt{Q\nabla \tilde{u}}\|_{p;dx}^{p-\epsilon_1} + \frac{1}{r^{p-\epsilon_1}} \|\eta \tilde{u}^{\frac{\gamma}{\rho - \sigma}}\|_{p;dx}^{p-\epsilon_1}\right].
\]

Setting

(3.15)

\[
\tilde{C} = r^p \|c^T \tilde{u}^{t-p}\|_{\frac{p \sigma}{(p - 1)(\sigma - 1)}} \|\eta \tilde{u}^{\frac{\gamma}{\rho}}\|_{p;dx}
\]

and making the substitutions given by (3.11) yields

\[
III \leq Cq^T \tilde{C} \left(\frac{\tilde{s}}{r}\right)^{\frac{\rho}{\epsilon_1}} y^{p-\epsilon_1} + Cq^T \tilde{C} \left(\frac{Y}{p}\right)^{p-\epsilon_1} \left(\frac{\tilde{s}}{r}\right)^{\frac{\rho}{\epsilon_1}} y z^{p-\epsilon_1} + Cq^T \tilde{C} \left(\frac{\tilde{s}}{r}\right)^{p}.
\]

Fix $s_2 \in (0, 1)$ to be chosen later, and note that \(\left(\frac{p}{\epsilon_1}\right)^t = \frac{p}{p - \epsilon_1}\). Then, using Young’s inequality in (3.10) in a similar manner as in (3.12), we have

\[
III \leq Cq^T \tilde{C} \left(\frac{\tilde{s}}{s_2}\right) \left(\frac{\tilde{s}}{s_2}\right)^{\frac{\rho}{\epsilon_1}} \left(y^{p-\epsilon_1} + \frac{1}{r^{p-\epsilon_1}} y z^{p-\epsilon_1} + \left(\frac{\tilde{s}}{s_2}\right)^{p}\right) + s_2^{\frac{p}{p-\epsilon_1}} z^p.
\]

Terms IV and V are estimated in the same way as III. Fix $\epsilon_2, \epsilon_3 \in (0, 1)$. For $s_3, s_4 \in (0, 1)$ to be chosen later, and with

(3.16)

\[
\begin{align*}
(i) \quad \tilde{H} &= r^p \|\tilde{h}\|_{\frac{p \sigma}{(p - 2)(\sigma - 1)}} \|\eta \tilde{u}^{\frac{\gamma}{\rho}}\|_{p;dx} \text{ and } \\
(ii) \quad \tilde{D} &= r^p \|\tilde{d}\|_{\frac{p \sigma}{(p - 3)(\sigma - 1)}} \|\eta \tilde{u}^{\frac{\gamma}{\rho}}\|_{p;dx},
\end{align*}
\]

we have

(3.17)

\[
IV \leq Cq^T \tilde{C} \left(\frac{\tilde{s}}{s_3}\right)^{\frac{p}{\epsilon_2}} \left(\frac{\tilde{s}}{s_3}\right)^{\frac{p}{\epsilon_2}} \left(y^{p-\epsilon_2} + \frac{1}{r^{p-\epsilon_2}} y z^{p-\epsilon_2} + \frac{\tilde{s}}{s_3}^{\frac{p}{p-\epsilon_2}} z^p\right)
\]

and

(3.18)

\[
V \leq Cq^T \tilde{C} \left(\frac{\tilde{s}}{s_4}\right)^{\frac{p}{\epsilon_3}} \left(\frac{\tilde{s}}{s_4}\right)^{\frac{p}{\epsilon_3}} \left(y^{p-\epsilon_3} + \frac{1}{r^{p-\epsilon_3}} y z^{p-\epsilon_3} + \frac{\tilde{s}}{s_4}^{\frac{p}{p-\epsilon_3}} z^p\right).
\]

**Step 6.** Set $\epsilon = \min\{\epsilon_1, \epsilon_2, \epsilon_3\}$ and choose $s_1, s_2, s_3, s_4$ depending only on $p, \epsilon_1, \epsilon_2, \epsilon_3$ so that

\[
\frac{\tilde{s}}{s_1} + \frac{\tilde{s}}{s_2} + \frac{\tilde{s}}{s_3} + \frac{\tilde{s}}{s_4} \leq \frac{1}{2C}
\]

where $C$ is as in (3.8). Then there is a $c_\epsilon = c_\epsilon(\epsilon, p, \sigma) > 0$ so that (3.8) becomes

\[
z^p \leq Cq^c \epsilon \left(1 + \left(\frac{B}{s_1}\right)^p + \left(\frac{C}{s_2}\right)^{\frac{\rho}{\epsilon_2}} + \left(\frac{\tilde{H}}{s_3}\right)^{\frac{\rho}{\epsilon_2}} + \left(\frac{\tilde{D}}{s_4}\right)^{\frac{\rho}{\epsilon_2}}\right) \left(y^{p-\epsilon_2} + \frac{x^p}{r^{p-\epsilon_2}}\right)
\]

\[
\leq Cq^c \epsilon \left(1 + B + \tilde{C} \tilde{C}^{\frac{\rho}{\epsilon_2}} + \tilde{H} \tilde{D}^{\frac{\rho}{\epsilon_2}}\right) \left(y^{p-\epsilon_2} + \frac{x^p}{r^{p-\epsilon_2}}\right),
\]

where $C$ now also depends on $s_1, s_2, s_3$ and $s_4$. Taking $p$th roots and setting $Z = \left(1 + B + \tilde{C} \tilde{C}^{\frac{\rho}{\epsilon_2}} + \tilde{H} \tilde{D}^{\frac{\rho}{\epsilon_2}}\right)$ as in (1.21), it follows that

(3.19)

\[
\|\tilde{u}^{\frac{\gamma}{\rho - \sigma}} \sqrt{Q\nabla \tilde{u}}\|_{p;dx} \leq Cq^c \epsilon Z \left(\|\tilde{u}^{\frac{\gamma}{\rho}} \sqrt{Q\nabla \eta}\|_{p;dx} + \frac{1}{r^{\frac{\rho}{p}}} \|\eta \tilde{u}^{\frac{\gamma}{\rho - \sigma}}\|_{p;dx}\right).
\]
Using Lemma 4.3 and estimate (3.19) we have, with $b_s = c_s + 1$,
\[
\|\eta\bar{u}^{\frac{p}{p-1}}\|_{\rho, dx} \leq C r \|\sqrt{Q} \nabla \left(\eta\bar{u}^{\frac{p}{p-1}}\right)\|_{p, dx} + C \|\eta\bar{u}^{\frac{p}{p-1}}\|_{p, dx}
\]
(3.20)
\[
\leq C r \left(\|u^{\frac{p}{p-1}} \sqrt{Q} \nabla \eta\|_{p, dx} + \left(\frac{p}{q}\right)\|u^{\frac{p}{p-1}} \eta \sqrt{Q} \nabla \bar{u}\|_{p, dx}\right) + C \|\eta\bar{u}^{\frac{p}{p-1}}\|_{p, dx}
\]
\[
\leq C q^b \tilde{Z} \left(\|\eta\bar{u}^{\frac{p}{p-1}}\|_{p, dx} + \|\sqrt{Q} \nabla \eta\|_{p, dx}\right).
\]

Step 7. We now use the accumulating sequence of Lipschitz cutoff functions in (1.14). For any $\Omega' \subset \Omega$, let $\chi_{\Omega'} : \Omega \to \mathbb{R}$ be defined by
\[
\chi_{\Omega'}(x) = \begin{cases} 
1 & \text{if } x \in \Omega' \\
0 & \text{if } x \notin \Omega'. 
\end{cases}
\]
For each $j$, let $S_j = \text{supp} \eta_j$ and recall that $\eta_j = 1$ on $S_{j+1}$. Since $s^* > p\sigma^*$, there exists $s$ so that $1 \leq s < \sigma$ and $s^p = s^*$. Then for each $j$,
\[
\|\bar{u}^{\frac{p}{p-1}} \chi_{S_{j+1}}\|_{p, dx} \leq C q^b \tilde{Z} \left(\|u^{\frac{p}{p-1}} \chi_{S_j}\|_{p, dx} + \|\sqrt{Q} \nabla \eta_j\|_{s^p, B_{r'}; dx}\right) \|u^{\frac{p}{p-1}} \chi_{S_j}\|_{s^p, dx}
\]
(3.21)
\[
\leq C q^b \tilde{Z} N^j \|u^{\frac{p}{p-1}} \chi_{S_j}\|_{s^p, dx}.
\]
Rewriting (3.21) so that $\bar{u}$ appears to power 1 inside each norm, we see that
\[
\|\bar{u} \chi_{S_{j+1}}\|_{\rho, dx} \leq \left(C \tilde{Z} N^j q^b\right)^{\frac{p}{q}} \|\bar{u} \chi_{S_j}\|_{\rho, dx}.
\]
(3.22)

Note that $sY > sY$ since $1 \leq s < \sigma$. Thus, $\bar{u} \in L^{sY}(S_j)$ implies the stronger inclusion $\bar{u} \in L^{sY}(S_{j+1})$. We will use this fact and a M"{o}ser iteration to obtain the conclusion of the Theorem 1.2. Set $\lambda = \frac{s}{q} > 1$ and let $q_0 = 1$. For each $j \in \mathbb{N}$, choose $q_j > 1$ so that $Y_j = (\mu + 1)(q_j - 1) + p$ and $Y_j = p\lambda^j$. With $Y_0 = p$, we have
\[
Y_j = p\lambda^j \quad \text{for } j \geq 0.
\]
(3.23)
Choosing $Y = Y_j$ in (3.22) gives
\[
\|\bar{u} \chi_{S_{j+1}}\|_{s\lambda^{j+1}, dx} \leq (C \tilde{Z})^s \lambda^j N^j \lambda^j q_j^b \|\bar{u} \chi_{S_j}\|_{s\lambda^{j+1}, dx}.
\]
(3.24)

Let $\Psi_1 = \sum_{j=1}^{\infty} \lambda^{j-1}$ and $\Psi_2 = \sum_{j=1}^{\infty} j \lambda^{j-1}$, recalling that $\lambda > 1$. Then, since $q_j \sim \lambda^j$ and $B_{r'} \subset S_j$ for each $j \in \mathbb{N}$, we obtain
\[
\|\bar{u} \chi_{B_{r'}}\|_{s\lambda^{j+1}, dx} \leq (C \tilde{Z})^{\Psi_1} (N \lambda^{b_s})^{\Psi_2} \|\bar{u} \chi_{S_1}\|_{s\lambda^{j+1}, dx}.
\]
(3.25)

Let $\eta_0 \in C_0^\infty(B_r)$ be a nonnegative cutoff function so that $S_1 \subset \{x : \eta_0(x) = 1\}$ and $\eta_0 \leq 1$ in $B_r$. Then since $Y_0 = p$ and $q_0 = 1$, (3.20) and (3.25) imply that
\[
\|\bar{u} \chi_{B_{r'}}\|_{s\lambda^{j+1}, dx} \leq C \tilde{Z}^{\Psi_1+1} (N \lambda^{b_s})^{\Psi_2} \|\bar{u}\|_{s\lambda^{j+1}, dx}.
\]
(3.26)

Since this holds for every $j \in \mathbb{N}$, it follows that
\[
\|\bar{u}\|_{L^\infty(B_{r'})} \leq C \tilde{Z}^{\Psi_0} \|\bar{u}\|_{s\lambda^{j+1}, dx}
\]
(3.27)

where $\Psi_0 = \sum_{j=0}^{\infty} \lambda^{j-1}$ and $C$ are independent of $u, r, b, c, d, e, f, g, h$. This completes the proof of Theorem 1.2. □
4. Appendix 1

In Appendix 1, we will prove some facts used in the proof of Theorem 1.2 that are related to the chain rule and the iteration process. See also [SW2] for results related to the chain rule.

**Lemma 4.1.** Let \((u, \nabla u) \in W^{1,p}_Q(\Omega)\) and \(\phi \in C^1(\mathbb{R})\) with \(\phi' \in L^\infty(\mathbb{R})\). Then \(\phi(u) \in W^{1,p}_Q(\Omega)\) and
\[
\sqrt{Q} \nabla (\phi(u)) = \phi'(u) \sqrt{Q} \nabla u \quad \text{a.e. in } \Omega.
\]

**Proof:** Let \(\{u_j\}_{j \in \mathbb{N}} \subset W^{1,p}_Q(\Omega) \cap \text{Lip}_\text{loc}(\Omega)\) be a representative sequence for \((u, \nabla u) \in W^{1,p}_Q(\Omega)\). Then as usual, up to subsequences,
\[
(4.1) \quad u_j \to u \quad \text{a.e. in } \Omega \text{ and in } L^p(\Omega), \quad \text{and } \sqrt{Q} \nabla u_j \to \sqrt{Q} \nabla u \quad \text{a.e. in } \Omega \text{ and in } (L^p(\Omega))^n.
\]

**Claim 1:** \(\phi(u_j) \in \text{Lip}_\text{loc}(\Omega)\) for every \(j\). Indeed, by the fundamental theorem of calculus, for every \(x, y \in \Omega\),
\[
|\phi(u_j(x)) - \phi(u_j(y))| = \left| \int_{u_j(y)}^{u_j(x)} \phi'(t) \, dt \right| \leq \|\phi'\|\infty|u_j(x) - u_j(y)|,
\]
and the claim follows from \(u_j \in \text{Lip}_\text{loc}(\Omega)\).

**Claim 2:** \(\phi(u_j) \to \phi(u)\) in \(L^p(\Omega)\) and a.e. in \(\Omega\). In fact, since \(u_j \to u\) a.e. in \(\Omega\) and \(\phi\) is continuous, then \(\phi(u_j) \to \phi(u)\) a.e. in \(\Omega\). Also,
\[
|\phi(t)| = |\phi(0) + \int_0^t \phi'(s) \, ds| \leq |\phi(0)| + \|\phi'\|\infty|t| := B_0 + A_0|t|.
\]

Then
\[
|\phi(u_j) - \phi(u)|^p \leq 2^p (|\phi(u_j)|^p + |\phi(u)|^p) \\
\leq 4^p (2B_0^p + A_0^p|u_j|^p + A_0^p|u|^p) \\
\leq c(1 + |u_j|^p + |u|^p).
\]

Since \(1 + |u_j|^p + |u|^p \to 1 + 2|u|^p\) a.e. in \(\Omega\) and in \(L^1(\Omega)\) by relation (1.1), by Lebesgue’s Sequentially Dominated Convergence Theorem implies that
\[
\int_\Omega |\phi(u_j) - \phi(u)|^p \, dx \to 0.
\]

**Claim 3:** \(\sqrt{Q} \nabla (\phi(u_j)) \to \phi'(u) \sqrt{Q} \nabla u\) a.e. in \(\Omega\) and in \((L^p(\Omega))^n\). Indeed, since \(u_j\) and \(\phi(u_j)\) are locally Lipschitz, their gradients exist a.e. in \(\Omega\), and
\[
\sqrt{Q} \nabla (\phi(u_j)) = \phi'(u_j) \sqrt{Q} \nabla u_j \quad \text{a.e. in } \Omega.
\]

Since \(\phi'\) is continuous, (4.1) gives \(\sqrt{Q} \nabla (\phi(u_j)) \to \phi'(u) \sqrt{Q} \nabla u\) a.e. in \(\Omega\). Moreover,
\[
|\sqrt{Q} \nabla (\phi(u_j)) - \phi'(u) \sqrt{Q} \nabla u|^p \leq 2^p \left( |\sqrt{Q} \nabla (\phi(u_j))|^p + \|\phi''\|\infty^p \sqrt{Q} \nabla u|^p \right) \\
\leq 2^p \|\phi''\|\infty^p \left( |\sqrt{Q} \nabla u_j|^p + |\sqrt{Q} \nabla u|^p \right),
\]
and \(|\sqrt{Q} \nabla u_j|^p + |\sqrt{Q} \nabla u|^p \to 2|\sqrt{Q} \nabla u|^p\) a.e. in \(\Omega\) and in \(L^1(\Omega)\) by (1.1). Again by Lebesgue’s Sequentially Dominated Convergence Theorem,
\[
\int_\Omega |\sqrt{Q} \nabla (\phi(u_j)) - \phi'(u) \sqrt{Q} \nabla u|^p \, dx \to 0.
\]

Therefore, the sequence of locally Lipschitz functions \(\{\phi(u_j)\}_{j \in \mathbb{N}} \subset W^{1,p}_Q(\Omega)\) is Cauchy in \(W^{1,p}_Q(\Omega)\) and defines an element of \(W^{1,p}_Q(\Omega)\) having \(\phi(u)\) as its \(L^p\)-part and \(\phi'(u) \sqrt{Q} \nabla u\) as its gradient part. This completes the proof. \(\square\)
Lemma 4.2. Let $u \in W^{1,p}_Q(\Omega)$ and $k \in \mathbb{R}$. Then there exists $\bar{u} \in W^{1,p}_Q(\Omega)$ whose $L^p$-part is $|u| + k$ and whose gradient–part $\nabla \bar{u}$ satisfies

$$\sqrt{Q(x)} \nabla \bar{u}(x) = \begin{cases} \sqrt{Q(x)} \nabla u(x) & \text{if } u(x) \geq 0, \\ -\sqrt{Q(x)} \nabla u(x) & \text{if } u(x) < 0 \end{cases} \quad \text{a.e. in } \Omega.$$ 

(4.2)

Remark 4.3. Choosing $k = 0$, it follows that if $u \in W^{1,p}_Q(\Omega)$ then $|u| \in W^{1,p}_Q(\Omega)$, and the gradient–part of $|u|$ satisfies (4.2).

Remark 4.4. For $u, k$ and $\bar{u}$ as in Lemma 4.2, $|\sqrt{Q} \nabla u| = |\sqrt{Q} \nabla \bar{u}|$ a.e. in $\Omega$.

Proof of Lemma 4.2. For any $\theta > 0$, define $\phi_\theta : \mathbb{R} \rightarrow \mathbb{R}$ by $\phi_\theta(t) = (t^2 + \theta^2)^{\frac{1}{2}}$. Then

- $\phi_\theta \in C^1(\mathbb{R})$ with $\phi_\theta'(t) = t/(t^2 + \theta^2)^{\frac{1}{2}}$ and $\phi_\theta' \in L^\infty(\mathbb{R})$,
- $0 \leq \phi_\theta(t) \leq |t| + \theta$ and $|\phi_\theta'(t)| \leq 1$ for $t \in \mathbb{R}$,
- as $\theta \rightarrow 0$, $\phi_\theta(t) \rightarrow |t|$ and $\phi_\theta'(t) \rightarrow \text{sign}(t) := \begin{cases} 1 & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ -1 & \text{if } t < 0. \end{cases}$

Now let $u \in W^{1,p}_Q(\Omega)$ and $\lambda \geq 0$ be such that $\left|\{x \in \Omega : |u(x)| = \lambda\}\right| = 0$. Let $\{u_j\}_{j \in \mathbb{N}} \subset W^{1,p}_Q(\Omega) \cap \text{Lip}_\text{loc}(\Omega)$ be a representative sequence for $(u, \nabla u)$. We may assume as usual that

$$u_j \rightarrow u \quad \text{a.e. in } \Omega \text{ and in } L^p(\Omega),$$

$$\sqrt{Q} \nabla u_j \rightarrow \sqrt{Q} \nabla u \quad \text{a.e. in } \Omega \text{ and in } (L^p(\Omega))^n.$$ 

Keeping $\lambda$ fixed, let $\varphi_{j,\lambda}(x) = \phi_{\lambda}(u_j(x) + \lambda)$ for $x \in \Omega$. As shown in the proof of Lemma 4.1, $\varphi_{j,\lambda} \in \text{Lip}_\text{loc}(\Omega)$ for every $j$ and $\varphi_{j,\lambda} \rightarrow |u + \lambda|$ a.e. in $\Omega$ and in $L^p(\Omega)$ as $j \rightarrow \infty$. Moreover, since $u \neq -\lambda$ a.e. in $\Omega$,

$$\sqrt{Q} \nabla \varphi_{j,\lambda}(x) \rightarrow \sqrt{Q} \nabla \varphi_{\lambda}(x) := \begin{cases} \sqrt{Q} \nabla u(x) & \text{if } u(x) \geq -\lambda, \\ -\sqrt{Q} \nabla u(x) & \text{if } u(x) < -\lambda \end{cases}$$
a.e. in $\Omega$ and in $(L^p(\Omega))^n$ as $j \rightarrow \infty$. Then $\{\varphi_{j,\lambda}\}_{j \in \mathbb{N}} \subset W^{1,p}_Q(\Omega)$ is a Cauchy sequence and thus defines an element $\varphi_\lambda \in W^{1,p}_Q(\Omega)$ having $|u + \lambda|$ as its $L^p$-part and $\nabla_\lambda$ as its gradient–part.

In case $\left|\{x \in \Omega : |u(x)| = 0\}\right| = 0$, we choose $\lambda = 0$ in the preceding argument and conclude the proof of the lemma. In case $\left|\{x \in \Omega : |u(x)| = 0\}\right| > 0$, choose a sequence $\lambda_m \searrow 0$ such that (see Corollary 2.10)

$$\left|\{x \in \Omega : |u(x)| = \lambda_m\}\right| = 0 \quad \text{for all } m.$$ 

Then $\varphi_{\lambda_m} = |u + \lambda_m| \rightarrow |u|$ a.e. in $\Omega$, and since

$$\int_{\Omega} |u + \lambda_m| - |u|^p \, dx \leq \lambda_m^p |\Omega|,$$

we also have that $\varphi_{\lambda_m} \rightarrow |u|$ in $L^p(\Omega)$.

Let us show that for a.e. $x \in \Omega$,

$$V_{\lambda_m}(x) \rightarrow V(x) := \begin{cases} \sqrt{Q} \nabla u(x) & \text{if } u(x) \geq 0, \\ -\sqrt{Q} \nabla u(x) & \text{if } u(x) < 0. \end{cases}$$

Indeed, if $u(x) \geq 0$ then $u(x) \geq -\lambda_m$, and hence $V_{\lambda_m}(x) = \sqrt{Q} \nabla u(x)$ for all $m \in \mathbb{N}$. On the other hand, if $u(x) < 0$ then $u(x) < -\lambda_m$ for all large $m$, and then $V_{\lambda_m}(x) = -\sqrt{Q} \nabla u(x)$, again for all large $m$.

Since for all $m$,

$$|V_{\lambda_m} - V|^p \leq 2^p(|V_{\lambda_m}|^p + |V|^p) = 2^{p+1} |\sqrt{Q} \nabla u|^p,$$
Lebesgue’s Dominated Convergence Theorem yields that $V_{\lambda_m} \to V$ in $(L^p(\Omega))^n$. Then $\{\varphi_{\lambda_m}\}_{m \in \mathbb{N}}$ is a Cauchy sequence in $W_{Q}^{1, p}(\Omega)$, and it converges to an element of $W_{Q}^{1, p}(\Omega)$ having $|u|$ as its $L^p$-part and $V$ as its gradient–part. We denote this element by $|u|$. Finally, since $|u| \in W_{Q}^{1, p}(\Omega)$ and $k \in W_{Q}^{1, p}(\Omega)$ with $\sqrt{Q} \nabla k = 0$ for every $k \in \mathbb{R}$, we obtain that $\bar{u} = |u| + k \in W_{Q}^{1, p}(\Omega)$ and (1.2) holds. The proof of the lemma is now complete. \hfill $\blacksquare$

**Lemma 4.5.** Let $k \geq 0$, $u \in W_{Q}^{1, p}(\Omega)$ be a weak solution of (1.1), and $\bar{u} = |u| + k$. Let $\eta \in \text{Lip}_0(\Omega)$ and supp$(\eta) \subset B$ for a $p$-ball $B = B(y, r)$ with $r < r_1(y)$. Let (1.9) and Sobolev’s inequality (1.13) be true. Suppose that (1.15) holds for $B$ and $\eta$, and also that, for $t \geq 1$, condition (1.10) holds for $B$ with $t'$ given by $1/t + 1/t' = 1$. If $q \geq 1$, $p > 1$, $\sigma > 1$ and $\theta > 0$, then

$$
\left( \int_B |\eta \bar{u}|^{(q-1)\theta + p} \right)^{\frac{1}{p}} \leq C \left[ \left( \int_B \left| \frac{(q-1)\theta}{p} \eta \bar{u} \frac{q}{\sqrt{Q} \nabla \bar{u}} \right|^p \right)^{\frac{1}{p}} \right] + C \left( \int_B |\eta \bar{u}|^{p} \right)^{\frac{1}{p}},
$$

where the integrals may not be finite.

**Proof:** For any $l > 0$, let

$$
H_l(t) = \begin{cases} 
\frac{(q-1)\theta + p}{p} \left( \frac{(q-1)\theta}{p} t + \frac{(q-1)\theta}{p} \eta \frac{(q-1)\theta + p}{p} \right) & \text{if } t < -l, \\
\frac{(q-1)\theta + p}{p} \text{ sign}(t) & \text{if } |t| \leq l, \\
\frac{(q-1)\theta + p}{p} \left( \frac{(q-1)\theta}{p} t - \frac{(q-1)\theta}{p} \eta \frac{(q-1)\theta + p}{p} \right) & \text{if } t > l.
\end{cases}
$$

Then $H_l \in C^1(\mathbb{R})$ with

$$
H'_l(t) = \begin{cases} 
\frac{(q-1)\theta + p}{p} \frac{(q-1)\theta}{p} & \text{if } t < -l, \\
\frac{(q-1)\theta + p}{p} \text{ sign}(t) & \text{if } |t| \leq l, \\
\frac{(q-1)\theta + p}{p} \left( \frac{(q-1)\theta}{p} t - \frac{(q-1)\theta}{p} \eta \frac{(q-1)\theta + p}{p} \right) & \text{if } t > l,
\end{cases}
$$

and $H'_l \in L^\infty(\mathbb{R})$ with $\|H'_l\|_{\infty} \leq \frac{(q-1)\theta + p}{p} \frac{(q-1)\theta}{p}$. Notice that $H'_l(t)$ is nondecreasing in $l$ for every $t \in \mathbb{R}$, while $H_l(t)$ is nondecreasing in $l$ only for $t \geq 0$.

By Lemmas 4.2 and 4.11 $H_l(\bar{u}) \in W_{Q}^{1, p}(\Omega)$ with $\sqrt{Q} \nabla (H_l(\bar{u})) = H'_l(\bar{u}) \sqrt{Q} \nabla \bar{u}$ a.e. in $\Omega$. Then, by Proposition 2.2 and the assumptions on $\eta$, $\eta H_l(\bar{u}) \in (W_{Q}^{1, p})_0(\Omega)$ with support in $B$ and

$$
\sqrt{Q} \nabla (\eta H_l(\bar{u})) = \eta H'_l(\bar{u}) \sqrt{Q} \nabla \bar{u} + H_l(\bar{u}) \sqrt{Q} \nabla \eta.
$$

By Sobolev’s inequality (1.13),

$$
\left( \int_B |\eta H_l(\bar{u})|^{p} \right)^{\frac{1}{p}} \leq C R \left[ \left( \int_B |\eta H'_l(\bar{u}) \sqrt{Q} \nabla \bar{u}|^{p} \right)^{\frac{1}{p}} + \left( \int_B |H_l(\bar{u}) \sqrt{Q} \nabla \eta|^{p} \right)^{\frac{1}{p}} \right] + C \left( \int_B |\eta H_l(\bar{u})|^{p} \right)^{\frac{1}{p}}.
$$

Since $\bar{u} \geq 0$ in $\Omega$, both $H_l(\bar{u})$ and $H'_l(\bar{u})$ are nondecreasing in $l$ and

$$
H_l(\bar{u}) \nearrow \frac{(q-1)\theta + p}{p} \bar{u}, \quad H'_l(\bar{u}) \nearrow \frac{(q-1)\theta}{p} \bar{u}.
$$
a.e. in $\Omega$ as $l \to \infty$. Passing to the limit in the previous inequality and using the monotone convergence theorem, we get

$$
\left(\int_B |\bar{u}^{(q-1)\theta+p}|^{\rho}\sigma dx\right)^{\frac{1}{\rho}} \leq C r \left[ \frac{(q-1)\theta+p}{p} \left(\int_B |\bar{u}^{(q-1)\theta+p}|^{\sqrt{Q}\nabla u|^p} dx\right)^{\frac{1}{p}} + \left(\int_B |\bar{u}^{(q-1)\theta+p}|^{\sqrt{Q}\nabla u|^p} dx\right)^{\frac{1}{p}} \right] + C \left(\int_B |\eta \bar{u}^{(q-1)\theta+p}|^{p}\sigma dx\right)^{\frac{1}{p}},
$$

where the integrals may not be finite. This completes the proof. \hfill \Box

5. Appendix 2

In Appendix 2, we will prove the following three theorems related to the structural assumptions about equation (1.1).

**Theorem 5.1.** Consider the differential equation (1.1):

$$
\text{div}(A(x,u,\nabla u)) = B(x,u,\nabla u).
$$

Suppose that the structural assumptions (1.3) hold relative to a symmetric nonnegative definite matrix $Q(x)$. If $H(x)$ is another symmetric nonnegative definite matrix and

$$
(5.1) \quad \frac{1}{C} \langle Q(x)\xi,\xi \rangle \leq \langle H(x)\xi,\xi \rangle \leq C \langle Q(x)\xi,\xi \rangle \quad \text{for all } \xi \in \mathbb{R}^n, \text{ a.e. } x \in \Omega,
$$

then there is a vector $\hat{A}(x,z,\xi)$ such that

$$
\begin{align*}
A(x,z,\xi) &= \sqrt{H(x)}\hat{A}(x,z,\xi), \\
\xi \cdot A(x,z,\xi) &\geq (C^{\frac{1}{2}}a)^{-1} |\sqrt{H(x)} \cdot \xi|^p - h|\gamma - g|,
\end{align*}
$$

$$
(5.2) \quad \left| \hat{A}(x,z,\xi) \right| \leq (C^{\frac{1}{2}}a) |\sqrt{H(x)} \cdot \xi|^{p-1} + (C^{\frac{1}{2}}b)|\gamma|^{-1} + (C^{\frac{1}{2}}\gamma),
\left| B(x,z,\xi) \right| \leq (C^{\frac{1}{2}}c) |\sqrt{H(x)} \cdot \xi|^{p-1} + d|\delta|^{-1} + f
$$

for $\xi \in \mathbb{R}^n, z \in \mathbb{R}$ and a.e. $x \in \Omega$. Here, $C$ is the same constant as in (1.1), and $a, b, c, d, e, f, g, h$ are as in (1.3).

Next we will show that many linear equations satisfy (1.3). Consider the linear equation

$$
(5.3) \quad \text{div}(Q(x)\nabla u) + HRu + S'Gu + Fu = f + T'g \quad \text{in } \Omega,
$$

where $Q(x)$ is symmetric and nonnegative definite, $R = \{R_i\}_{i=1}^n$, $S = \{S_i\}_{i=1}^n$, $T = \{T_i\}_{i=1}^n$ are collections of vector fields subunit with respect to $Q(x)$, and where the operator coefficients $H = \{H_i\}_{i=1}^n$, $G = \{G_i\}_{i=1}^n$ and $F$ as well as the inhomogeneous data $g = \{g_i\}_{i=1}^n$ and $f$ are measurable. See also [SW1]. We will prove the following fact about such equations.

**Theorem 5.2.** The linear equation (5.3) satisfies the structural conditions (1.3) with $p = \gamma = \psi = \delta = 2$ relative to the matrix $Q(x)$.

Finally, we will prove the next result concerning conditions (1.3) and (1.6).

**Theorem 5.3.** For the differential equation (1.1), the structural assumptions (1.3) are satisfied if and only if (1.6) is satisfied.

For the proofs, we will need some technical results which we collect in the following lemmas. We state the first two without proofs.

**Notation:** For any $k \in \mathbb{N}$, we will denote the identity $k \times k$ matrix by $I_k$ and the zero $k \times k$ matrix by $0_k$. Also, $\langle \cdot, \cdot \rangle_{\mathbb{R}^k}$ and $|\cdot|_{\mathbb{R}^k}$ will denote respectively the inner product and the norm in $\mathbb{R}^k$. When we work in $\mathbb{R}^n$, i.e., when $k = n$, we will usually omit the subscript $\mathbb{R}^k$. 
Also, let Mat(n, R) be the set of n × n real matrices, O(n) be the set of n × n real orthogonal matrices, and S^n_{sym} = Symm(n, R) be the set of symmetric n × n real matrices. For any Q ∈ S^n_{sym}, we will write Q ≥ 0 if Q is nonnegative definite. Since Q is symmetric and hence diagonalizable, the condition Q ≥ 0 is the same as assuming that Q has nonnegative eigenvalues. Finally, if Q, H ∈ S^n_{sym}, we will write Q ≥ H if Q − H ≥ 0, i.e., if ⟨Qξ, ξ⟩ ≥ ⟨Hξ, ξ⟩ for all ξ ∈ R^n.

**Lemma 5.4.** (i) If Q, H ∈ S^n_{sym} with Q, H ≥ 0, then Q + H ≥ 0.

(ii) If Q ∈ S^n_{sym}, Q ≥ 0 and M ∈ Mat(n, R), then M^TQM ≥ 0, where M^T denotes the transpose of M.

(iii) If Q ∈ S^n_{sym}, Q ≥ 0 and detQ = 0, then Q^{-1} ≥ 0.

**Lemma 5.5.** If Q ∈ S^n_{sym} with Q ≥ 0, there is a unique matrix √Q ∈ S^n_{sym} such that √Q ≥ 0 and √Q^T√Q = Q. Moreover

i) λ ≥ 0 is an eigenvalue for Q with eigenvector v if and only if √λ is an eigenvalue for √Q with eigenvector v.

ii) √Q is invertible if and only if Q is invertible.

**Proposition 5.6.** Let H, Q ∈ S^n_{sym} with Q, H ≥ 0 and suppose that there is a constant C > 0 such that

\[-\frac{1}{C}⟨Qξ, ξ⟩ ≤ ⟨Hξ, ξ⟩ ≤ C⟨Qξ, ξ⟩ \quad \text{for all } ξ ∈ R^n.

Then there is an invertible matrix M ∈ Mat(n, R) such that

1. Q = M^THM,
2. √Q = √HMM^T√H,
3. \frac{1}{√C} |ξ| ≤ |M^Tξ| ≤ √C |ξ| \quad \text{for all } ξ ∈ R^n.

**Proof of Proposition 5.6**

Step 1. We claim that

\[\text{Ker} Q = \text{Ker} H,\]

i.e., Qξ = 0 if and only if Hξ = 0 for any ξ ∈ R^n. Indeed, suppose Qξ = 0. Then ⟨Hξ, ξ⟩ = 0 by (5.4) and

\[|√Hξ|^2 = ⟨√Hξ, √Hξ⟩ = ⟨√H√Hξ, ξ⟩ = ⟨Hξ, ξ⟩ = 0,\]

i.e., √Hξ = 0 and ξ is an eigenvector of √H for the eigenvalue 0. This in turn implies by Lemma 5.5 that ξ is an eigenvector of H for the eigenvalue 0. Thus ξ ∈ KerH. The same argument shows that if Hξ = 0 then ξ ∈ KerQ, and thus the claim is proved.

Step 2. Assume that one of the two matrices is invertible, i.e., has empty kernel. Then by Step 1 the other matrix is also invertible, and so by Lemma 5.5 both √H and √Q are invertible. In this case, we may define

\[M = (√H)^{-1}√Q.\]

Then M^T = √Q^T \left((√H)^{-1}\right)^T = √Q(√H)^{-1} and hence

\[\begin{align*}
\text{i) } M^THM &= (√Q(√H)^{-1}√H(√H)^{-1}√Q = Q, \\
\text{ii) } √Q^TQ &= √Q, \\
\text{iii) } M^T√H &= (√Q(√H)^{-1}√H = √Q.
\end{align*}\]

Thus in this case parts (1) and (2) of the proposition are satisfied. Next note that (5.4) implies that both H − \frac{1}{C}Q, CQ − H ≥ 0. Then Lemma 5.4, part 2, implies that both of the following are also nonnegative definite:

\[\begin{align*}
I_n − \frac{1}{C}(√H)^{-1}Q(√H)^{-1} &= (H − \frac{1}{C}Q)(√H)^{-1}, \\
C(√H)^{-1}Q(√H)^{-1} − I_n &= (CQ − H)(√H)^{-1}.
\end{align*}\]
Equivalently, for every \( \xi \in \mathbb{R}^n \),
\[
|\xi|^2 \geq \frac{1}{C}((\sqrt{H})^{-1}Q(\sqrt{H})^{-1}\xi,\xi).
\]

But then
\[
|M^T\xi|^2 = (M^T\xi,M^T\xi) = (MM^T\xi,\xi) = (\sqrt{H})^{-1}Q(\sqrt{H})^{-1}\xi,\xi \geq \frac{1}{C}|\xi|^2
\]
and \( |M^T\xi|^2 \leq C|\xi|^2 \). Thus we finally obtain that
\[
\frac{1}{\sqrt{C}}|\xi| \leq |M^T\xi| \leq \sqrt{C}|\xi|,
\]
which is part (3) of the statement. This proves the desired result in case both \( Q,H \) are invertible.

Step 3. It remains only to consider the case when neither \( H \) nor \( Q \) is invertible since \( \text{Ker}H = \text{Ker}Q \) by Step 1. Since both matrices are symmetric, each is diagonalizable. Moreover, eigenvectors related to different eigenvalues are orthogonal in \( \mathbb{R}^n \).

Consider the subspaces \( V := \text{Ker}H \subset \mathbb{R}^n \) and \( V^\perp \). Then \( \mathbb{R}^n = V \oplus V^\perp \), and letting \( k = \dim V \), we have \( k \geq 1 \). If \( k = n \) then \( Q = H = 0 \), and the conclusion of Proposition 5.6 is obvious with \( M = I_n \). Thus we may assume \( k \leq n - 1 \).

Now choose an orthonormal basis \( \{v_1,\ldots,v_k\} \) in \( V \) and another one \( \{v_{k+1},\ldots,v_n\} \) in \( V^\perp \). Then \( B := \{v_1,\ldots,v_k,v_{k+1},\ldots,v_n\} \) is an orthonormal basis in \( \mathbb{R}^n \). Let \( B' := \{e_1,\ldots,e_n\} \) be the canonical basis in \( \mathbb{R}^n \), and let \( O \) be the matrix which expresses the change of basis between \( B \) and \( B' \). Then \( O \in \mathcal{O}(n) \) and
\[
Q = O^T \begin{pmatrix} 0_k & 0 \\ 0 & Q_1 \end{pmatrix} O, \quad H = O^T \begin{pmatrix} 0_k & 0 \\ 0 & H_1 \end{pmatrix} O.
\]

Here \( Q_1 \in \text{Mat}(n-k,\mathbb{R}) \) is the invertible matrix associated to the bijective linear map \( T_Q : V^\perp \to V^\perp \) defined by \( T_Q(x) = Qx \), expressed with respect to the basis \( \{v_{k+1},\ldots,v_n\} \) of \( V^\perp \). Similarly \( H_1 \in \text{Mat}(n-k,\mathbb{R}) \) is the invertible matrix associated to the bijective linear map \( T_H : V^\perp \to V^\perp \) defined by \( T_H(x) = Hx \), also expressed with respect to the basis \( \{v_{k+1},\ldots,v_n\} \) of \( V^\perp \).

Since \( Q,H \) are symmetric and nonnegative definite, so are \( Q_1,H_1 \). Then we can apply the result from Step 2 to find an invertible \( M_1 \in \text{Mat}(n-k,\mathbb{R}) \) such that

i) \( M_1^T H_1 M_1 = Q_1 \),
ii) \( \sqrt{H_1} M_1 = \sqrt{Q_1} = M_1^T \sqrt{H_1} \),
iii) \( \frac{1}{\sqrt{C}}|w|_{\mathbb{R}^n-k} \leq |M_1w|_{\mathbb{R}^n-k} \leq \sqrt{C}|w|_{\mathbb{R}^n-k} \) for all \( w \in \mathbb{R}^n-k \).

Now define
\[
M = O^T \begin{pmatrix} I_k & 0 \\ 0 & M_1 \end{pmatrix} O.
\]

Then
\[
M^T H M = O^T \begin{pmatrix} I_k & 0 \\ 0 & M_1^T \end{pmatrix} O O^T \begin{pmatrix} 0_k & 0 \\ 0 & H_1 \end{pmatrix} O O^T \begin{pmatrix} I_k & 0 \\ 0 & M_1 \end{pmatrix} O
= O^T \begin{pmatrix} 0_k & 0 \\ 0 & M_1^T H_1 M_1 \end{pmatrix} O
= O^T \begin{pmatrix} 0_k & 0 \\ 0 & Q_1 \end{pmatrix} O = Q.
\]

Moreover, \( \sqrt{Q} = O^T \begin{pmatrix} 0_k & 0 \\ 0 & \sqrt{Q_1} \end{pmatrix} O \) and \( \sqrt{H} = O^T \begin{pmatrix} 0_k & 0 \\ 0 & \sqrt{H_1} \end{pmatrix} O \), so that
\[
\sqrt{HM} = O^T \begin{pmatrix} 0_k & 0 \\ 0 & \sqrt{H_1} M_1 \end{pmatrix} O = O^T \begin{pmatrix} 0_k & 0 \\ 0 & \sqrt{Q_1} \end{pmatrix} O = \sqrt{Q}.
\]
Similarly, a, b, c, d, e, f, g, h

\[ M^T \sqrt{H} = O^T \left( \begin{array}{cc} 0_k & 0 \\ 0 & M_1^T \sqrt{H_1} \end{array} \right) O = O^T \left( \begin{array}{cc} 0_k & 0 \\ 0 & \sqrt{Q_1} \end{array} \right) O = \sqrt{Q}. \]

Finally let \( \xi \in \mathbb{R}^n \) and \( \eta = O \xi \). Write \( \eta = (v, w) \) with \( w \in \mathbb{R}^{n-k} \) and \( v \in \mathbb{R}^k \). Then \( M^T \xi = O^T \left( \begin{array}{c} 1_k \\ 0 \\ M_1^T \end{array} \right) \eta = O^T \left( \begin{array}{c} v \\ M_1^T w \end{array} \right) \). Since \( O \in \mathcal{O}(n) \), we have

\[
|M^T \xi|^2_{\mathbb{R}^n} = |v|^2_{\mathbb{R}^k} + |M_1^T w|^2_{\mathbb{R}^{n-k}} \\
\leq |v|^2_{\mathbb{R}^k} + C|w|^2_{\mathbb{R}^{n-k}} \\
\leq C|v|^2_{\mathbb{R}^k} + C|w|^2_{\mathbb{R}^{n-k}} \\
= C|\eta|^2_{\mathbb{R}^n} = C|O \xi|^2_{\mathbb{R}^n} = C|\xi|^2_{\mathbb{R}^n}.
\]

Similarly, \( |M^T \xi|^2_{\mathbb{R}^n} \geq \frac{1}{C}|\xi|^2_{\mathbb{R}^n} \). Thus

\[
\frac{1}{\sqrt{C}} |\xi|_{\mathbb{R}^n} \leq |M^T \xi|_{\mathbb{R}^n} \leq \sqrt{C} |\xi|_{\mathbb{R}^n},
\]

and the proof of Proposition 5.6 is complete. \( \square \)

**Corollary 5.7.** Let \( Q(x) \) and \( H(x) \) be symmetric nonnegative definite matrices depending on \( x \in \Omega \), and suppose there is a constant \( C > 0 \) so that

\[
(5.5) \quad \frac{1}{C} \langle Q(x) \xi, \xi \rangle \leq \langle H(x) \xi, \xi \rangle \leq C \langle Q(x) \xi, \xi \rangle
\]

for all \( \xi \in \mathbb{R}^n \) and a.e. \( x \in \Omega \). Then for a.e. \( x \in \Omega \), there is an invertible matrix \( M(x) \) such that

1. \( Q(x) = M^T(x)H(x)M(x) \),
2. \( \sqrt{Q(x)} = \sqrt{H(x)}M(x) = M^T(x)\sqrt{H(x)} \),
3. \( \frac{1}{\sqrt{C}} |\xi| \leq |M^T(x)\xi| \leq \sqrt{C} |\xi| \) for all \( \xi \in \mathbb{R}^n \).

**Proof:** This follows immediately by applying Proposition 5.6 at each point \( x \in \Omega \) where (5.5) holds. \( \square \)

**Proof of Theorem 5.1** Let \( Q \) and \( H \) satisfy the hypothesis of Theorem 5.1. By Corollary 5.7, for a.e. \( x \in \Omega \), there is an invertible matrix \( M(x) \) satisfying properties (1), (2) and (3) relative to \( Q \) and \( H \). For any such \( x \), define

\[
\tilde{A}(x, z, \xi) = M(x)\tilde{A}(x, z, \xi).
\]

Then by property (2) in Corollary 5.7

\[
A(x, z, \xi) = \sqrt{Q(x)}\tilde{A}(x, z, \xi) = \sqrt{H(x)}M(x)\tilde{A}(x, z, \xi) = \sqrt{H(x)}\tilde{A}(x, z, \xi).
\]

Therefore, by properties (2) and (3),

\[
\xi \cdot A(x, z, \xi) \geq a^{-1} |\sqrt{Q(x)} \cdot \xi|^p - h|z|^\gamma - g \\
\geq a^{-1} C^{-\frac{\gamma}{p}} |\sqrt{H(x)} \cdot \xi|^p - h|z|^\gamma - g,
\]

\[
|B(x, z, \xi)| \leq c \sqrt{Q(x)} \cdot \xi |\psi^{-1}| + d|z|^\delta - f \\
\leq C^{\frac{\gamma-1}{p}} c \sqrt{H(x)} \cdot \xi |\psi^{-1}| + d|z|^\delta - f,
\]

where \( a, b, c, d, e, f, g, h \) are as in (1.3). In order to prove the third part of (5.2), we first note that

\[
|\eta| = \sup_{\zeta \in \mathbb{R}^n, ||\zeta||=1} |\langle \zeta, \eta \rangle| \quad \text{for any } \eta \in \mathbb{R}^n.
\]
Then by property (3) in Corollary [5.7]

\[
|M(x)\eta| = \sup_{\zeta \in \mathbb{R}^n, |\zeta| = 1} |\langle \zeta, M(x)\eta \rangle|
\]

\[
= \sup_{\zeta \in \mathbb{R}^n, |\zeta| = 1} |\langle M^T(x)\zeta, \eta \rangle|
\]

\[
\leq \sup_{\zeta \in \mathbb{R}^n, |\zeta| = 1} |M^T(x)\zeta||\eta| \leq \sqrt{C}|\eta|.
\]

Hence

\[
|\tilde{A}(x, z, \xi)| = |M(x)\tilde{A}(x, z, \xi)|
\]

\[
\leq \sqrt{C} |\tilde{A}(x, z, \xi)|
\]

\[
\leq \sqrt{C} \left| a \sqrt{|Q(x)|} \cdot \xi \right|^p + b|z|^{\gamma - 1} + e,
\]

\[
\leq C\hat{a} \sqrt{H(x)} \cdot \xi^p + C\hat{b}|z|^{\gamma - 1} + C\hat{c} e,
\]

which completes the proof. □

**Proposition 5.8.** For \( x \in \Omega \), consider a symmetric nonnegative definite matrix \( Q(x) \) and a vector field \( T(x) = \sum_{j=1}^{n} t_j(x) \frac{\partial}{\partial x_j} = (t_1(x), \ldots, t_n(x)) \) which is subunit with respect to \( Q(x) \), i.e.,

\[
(5.6) \quad \left( \sum_{i=1}^{n} t_i(x)\xi_i \right)^2 \leq \langle Q(x)\xi, \xi \rangle \quad \text{for a.e } x \in \Omega \text{ and all } \xi \in \mathbb{R}^n.
\]

Then there exists a vector \( V(x) \) such that

1. \( T(x) = \sqrt{Q(x)}V(x) \) for a.e. \( x \in \Omega \),
2. \( |V(x)| \leq 1 \) for a.e. \( x \in \Omega \).

**Proof:** Consider any point \( x_0 \in \Omega \) at which (5.6) holds with \( x = x_0 \) for every \( \xi \in \mathbb{R}^n \). Denote \( T = T(x_0) = (t_1(x_0), \ldots, t_n(x_0)) \), \( Q = Q(x_0) \) and \( K = \ker Q = \{ \xi \in \mathbb{R}^n : Q\xi = 0 \} \). Write \( \mathbb{R}^n = K \perp K^\perp \), and accordingly write \( T = T_1 + T_2 \) with \( T_1 \in K \) and \( T_2 \in K^\perp \). Then by (5.6) at \( x_0 \),

\[
|\langle T, \xi \rangle|^2 \leq \langle Q\xi, \xi \rangle \quad \text{for all } \xi \in \mathbb{R}^n.
\]

Choosing \( \xi = T_1 \) gives

\[
|T_1|^2 = |\langle T, T_1 \rangle|^2 \leq \langle QT_1, T_1 \rangle = \langle 0, T_1 \rangle = 0,
\]

hence \( T = T_2 \in K^\perp \).

We may assume that \( K \nsubseteq \mathbb{R}^n \), since otherwise \( Q = 0 \), \( T = 0 \) and then the conclusion of the proposition holds at \( x_0 \) by choosing \( V(x_0) = 0 \). Now note that there is an orthogonal matrix \( O \in \mathcal{O}(n) \) such that

\[
Q = O^T \begin{pmatrix} Q_1 & 0 \\ 0 & 0_k \end{pmatrix} O,
\]

where \( k = \dim K \geq 0 \) and all the eigenvalues of \( Q_1 \) are strictly positive. Then

\[
\sqrt{Q} = O^T \begin{pmatrix} \sqrt{Q_1} & 0 \\ 0 & 0_k \end{pmatrix} O.
\]

Also, \( Q_1 \) is an invertible symmetric matrix which corresponds to the invertible linear operator \( L_{Q,K^\perp} \) defined on \( K^\perp \) by \( L_{Q,K^\perp}(\xi) = Q\xi \). Hence we may define

\[
N = O^T \begin{pmatrix} \left( \sqrt{Q_1} \right)^{-1} & 0 \\ 0 & 0_k \end{pmatrix} O.
\]

The matrix \( N \) is symmetric and

\[
\sqrt{Q}N = O^T \begin{pmatrix} I_{n-k} & 0 \\ 0 & 0_k \end{pmatrix} O.
\]
corresponds to the canonical projection of $R^n$ onto $K^\perp$. Since $T \in K^\perp$, we have $T = \sqrt{Q}NT$. Now set $NT = V$. Then $T = \sqrt{Q}V$ and 

$$|V|^2 = \sup_{|\xi|=1} |(V, \xi)|^2 = \sup_{|\xi|=1} |(NT, \xi)|^2 = \sup_{|\xi|=1} |(T, N\xi)|^2$$

$$\leq \sup_{|\xi|=1} (QN\xi, N\xi) = \sup_{|\xi|=1} (\sqrt{Q}N\xi, \sqrt{Q}N\xi) = 1,$$

where the inequality follows from the fact that $T$ is subunit. Thus the desired result holds at $x_0$ and the proof of Proposition 5.8 is complete. □

Proof of Theorem 5.2. Rewrite (5.3) in the form

$$\text{div}(Q(x)\nabla u) + \sum_{i=1}^{n} S'_i G_i u - \sum_{i=1}^{n} T'_i g_i = f - \sum_{i=1}^{n} H_i R_i u - F u.$$  

Since the vector fields $R_i$, $S_i$, $T_i$ are all subunit with respect to $Q(x)$, Proposition 5.8 shows that they can be expressed as

$$(5.7) \quad R_i(x) = \sqrt{Q(x)} \hat{R}_i(x), \quad S_i(x) = \sqrt{Q(x)} \hat{S}_i(x), \quad T_i(x) = \sqrt{Q(x)} \hat{T}_i(x),$$

where

$$(5.8) \quad |\hat{R}_i(x)| \leq 1, \quad |\hat{S}_i(x)| \leq 1, \quad |\hat{T}_i(x)| \leq 1$$

for every $i$ and a.e. $x \in \Omega$. Now write $R_i$, $S_i$, $T_i$, $\hat{R}_i$, $\hat{S}_i$, $\hat{T}_i$ as

$$R_i(x) = \sum_{j=1}^{n} R_{ij}(x) \frac{\partial}{\partial x_j} = (R_{i1}(x), \ldots, R_{in}(x)),$$

$$\hat{R}_i(x) = \left(\hat{R}_{i1}(x), \ldots, \hat{R}_{in}(x)\right),$$

and similarly,

$$S_i(x) = (S_{i1}(x), \ldots, S_{in}(x)), \quad S_i(x) = (\hat{S}_{i1}(x), \ldots, \hat{S}_{in}(x)), \quad S_i(x) = (\hat{S}_{i1}(x), \ldots, \hat{S}_{in}(x)), \quad T_i(x) = (T_{i1}(x), \ldots, T_{in}(x)), \quad T_i(x) = (T_{i1}(x), \ldots, T_{in}(x)),$$

For every $i, j = 1, \ldots, n$ and a.e $x \in \Omega$, (5.7) gives

$$S_{ij} = (S_{ij})_j = (\sqrt{Q} \hat{S}_i)_j = \sum_{k=1}^{n} (\sqrt{Q})_{jk} \hat{S}_k,$$

and in a similar way,

$$R_{ij} = \sum_{k=1}^{n} (\sqrt{Q})_{jk} \hat{R}_k, \quad T_{ij} = \sum_{k=1}^{n} (\sqrt{Q})_{jk} \hat{T}_k,$$

where in the notation we have suppressed dependence on $x$. Letting $\hat{R}$, $\hat{S}$, $\hat{T}$ denote respectively the matrices $[\hat{R}_{ij}]$, $[\hat{S}_{ij}]$, $[\hat{T}_{ij}]$, we obtain for a.e. $x \in \Omega$ that

$$S'G_u = \sum_{i=1}^{n} S'_i G_i u = -\text{div} \left( \sum_{i=1}^{n} S_{i1} G_i u, \ldots, \sum_{i=1}^{n} S_{in} G_i u \right)$$

$$= -\text{div} \left( \sum_{i,k=1}^{n} (\sqrt{Q})_{ik} \hat{S}_k G_i u, \ldots, \sum_{i,k=1}^{n} (\sqrt{Q})_{nk} \hat{S}_k G_i u \right)$$

$$= -\text{div} \left( \sqrt{Q} S^T G_u \right).$$

In the same way,

$$T'g = -\text{div} \left( \sqrt{Q} T^T g \right)$$

a.e. in $\Omega$. On the other hand,

$$HR_u = \sum_{i,j=1}^{n} H_i R_{ij} \frac{\partial u}{\partial x_j} = \sum_{i,j,k=1}^{n} H_i (\sqrt{Q})_{jk} \hat{R}_k \frac{\partial u}{\partial x_j} = (H, \hat{R} \sqrt{Q} \nabla u).$$
Then we can rewrite (5.3) as follows for a.e. \( x \in \Omega \):

\[
\text{div}\left(Q(x)\nabla u - \sqrt{Q}S^T Gu + \sqrt{Q}T^T g\right) = f - \langle H, \tilde{R}\sqrt{Q}\nabla u \rangle - Fu.
\]

To compare this form with (1.1) and with the structural conditions (1.3) in case all of \( p, \gamma, \psi, \delta \) are equal to 2, let

\[
A(x, z, \xi) = Q(x)\xi - \sqrt{Q(x)}S^T(x)G(x)z + \sqrt{Q(x)}T^T(x)g(x),
\]

\[
\tilde{A}(x, z, \xi) = \sqrt{Q(x)}\xi - S^T(x)G(x)z + T^T(x)g(x),
\]

\[
B(x, z, \xi) = f(x) - \langle H(x), \tilde{R}(x)\sqrt{Q(x)}\xi \rangle - F(x)z.
\]

Then \( A(x, z, \xi) = \sqrt{Q(x)}\tilde{A}(x, z, \xi) \) and (5.9) takes the form (1.1). By (5.8), for a.e \( x \in \Omega \) and every \( \eta \in \mathbb{R}^n \),

\[
|\tilde{R}\eta|^2 = \sum_{i=1}^n \left( \sum_{j=1}^n \tilde{R}_{ij}\eta_j \right)^2 \leq \sum_{i=1}^n \left( \left( \sum_{j=1}^n \tilde{R}_{ij}^2 \right) \left( \sum_{j=1}^n \eta_j^2 \right) \right) = n|\eta|^2,
\]

and in the same way,

\[
|S^T\eta|^2 \leq n|\eta|^2, \quad |T^T\eta|^2 \leq n|\eta|^2.
\]

Then for a.e. \( x \in \Omega \) and every \( z \in \mathbb{R} \) and \( \xi \in \mathbb{R}^n \),

\[
\xi \cdot A(x, z, \xi) = |\sqrt{Q(x)}\xi|^2 - (\xi, \sqrt{Q(x)}S^T(x)G(x)z + (\xi, \sqrt{Q(x)}T^T(x)g(x))
\]

\[
\geq |\sqrt{Q(x)}\xi|^2 - |\sqrt{Q(x)}\xi||S^T(x)G(x)z - |\sqrt{Q(x)}\xi||T^T(x)g(x)|
\]

\[
\geq |\sqrt{Q(x)}\xi|^2 - \frac{1}{4}|\sqrt{Q(x)}\xi|^2 - 4||S^T(x)G(x)z|^2
\]

\[
\geq \frac{1}{2}|\sqrt{Q(x)}\xi|^2 - 4n||G(x)||z|^2 - 4n||g(x)||^2.
\]

Moreover,

\[
|\tilde{A}(x, z, \xi)| \leq |\sqrt{Q(x)}\xi| + |S^T(x)G(x)z| + |T^T(x)g(x)|
\]

\[
\leq |\sqrt{Q(x)}\xi| + \sqrt{n||G(x)||z|} + \sqrt{n||g(x)||},
\]

\[
|B(x, z, \xi)| \leq |f(x)| + |\langle H(x), \tilde{R}(x)\sqrt{Q(x)}\xi \rangle| + |F(x)z|
\]

\[
\leq |\langle H(x)||\tilde{R}(x)\sqrt{Q(x)}\xi \rangle| + |F(x)||z| + |f(x)|
\]

\[
\leq \sqrt{n||H(x)||\sqrt{Q(x)}\xi|} + |F(x)||z| + |f(x)|.
\]

Thus the structural conditions (1.3) hold with \( p = \gamma = \psi = \delta = 2 \) and with

\[
a = 2, \quad d = |F(x)|, \quad g = 4n||g(x)||^2, \quad h(x) = 4n||G(x)||^2
\]

\[
b = \sqrt{n||G(x)||}, \quad e = \sqrt{n||g(x)||}, \quad c(x) = \sqrt{n||H(x)||}, \quad f = |f(x)|.
\]

This completes the proof of Theorem 5.2. \( \Box \)

**Remark 5.9.** By Theorems 5.1 and 5.2, the linear equation (5.3) satisfies the structural assumptions (1.3) with \( p = \gamma = \psi = \delta = 2 \) not only with respect to \( Q(x) \) but also with respect to any other symmetric matrix \( H(x) \geq 0 \) such that for a.e. \( x \in \Omega \) and every \( \xi \in \mathbb{R}^n \),

\[
\frac{1}{C} \langle H(x)\xi, \xi \rangle \leq \langle Q(x)\xi, \xi \rangle \leq C\langle H(x)\xi, \xi \rangle.
\]

**Proof of Theorem 5.3.** Step 1. It is easy to see that if (1.3) is satisfied, then (1.6) is also satisfied with \( \tilde{a}(x, z, \xi) := |A(x, z, \xi)| \) for \( \xi \in \mathbb{R}^n \), \( z \in \mathbb{R} \) and a.e. \( x \in \Omega \). Indeed, if (1.3) holds, then for every \( \eta, \xi \in \mathbb{R}^n \), \( z \in \mathbb{R} \) and a.e. \( x \in \Omega \),

\[
|\eta \cdot A(x, z, \xi)| = |\eta \cdot \sqrt{Q(x)}\tilde{A}(x, z, \xi)| \leq |\sqrt{Q(x)}|\tilde{A}(x, z, \xi)| = |\sqrt{Q(x)}|\tilde{a}(x, z, \xi).
\]
Moreover,
\[ \tilde{a}(x, z, \xi) = |\tilde{A}(x, z, \xi)| \leq a|\sqrt{Q(x)}\xi|^{p-1} + b|z|^{\gamma-1} + e, \]
and thus (1.6) holds.

Step 2. We will now prove that (1.6) implies (1.3). Fix any \( x \in \Omega \) such that (1.6) is satisfied for all \( \xi, \eta \in \mathbb{R}^n \) and all \( z \in \mathbb{R} \). Claim: \( A(x, z, \xi) \in (\ker Q(x))^\perp \) for \( \xi \in \mathbb{R}^n, z \in \mathbb{R} \). Indeed, define \( K = \ker Q(x) \) and recall from Lemma 5.5 that since \( Q(x) \) is symmetric and nonnegative, then also \( K = \ker \sqrt{Q(x)} \). Consider the decomposition \( \mathbb{R}^n = K \oplus K^\perp \) and write \( A(x, z, \xi) = A_1 + A_2 \) with \( A_1 \in K \) and \( A_2 \in K^\perp \). From the first inequality in (1.6) with \( \eta = A_1 \), we get
\[ |A_1|^2 = A_1 \cdot A_1 = A_1 \cdot (A_1 + A_2) = A_1 \cdot A(x, z, \xi) \leq |\sqrt{Q(x)}A_1|\tilde{a}(x, z, \xi) = 0. \]
Hence \( A(x, z, \xi) = A_2 \in K^\perp \), which proves the claim.

Now suppose that \( k := \dim K < n \) and choose an orthogonal matrix \( O \in O(n) \) such that
\[ Q(x) = O^T \begin{pmatrix} Q_1 & 0 \\ 0 & 0_k \end{pmatrix} O, \]
with \( Q_1 \) symmetric, nonnegative and invertible. Then
\[ \sqrt{Q(x)} = O^T \begin{pmatrix} \sqrt{Q_1} & 0 \\ 0 & 0_k \end{pmatrix} O. \]
Next define
\[ N(x) = O^T \begin{pmatrix} (\sqrt{Q_1})^{-1} & 0 \\ 0 & 1_k \end{pmatrix} O, \]
so that
\[ \sqrt{Q(x)}N(x) = O^T \begin{pmatrix} 1_{n-k} & 0 \\ 0 & 0_k \end{pmatrix} O, \]
i.e., the linear mapping \( L : \mathbb{R}^n \to \mathbb{R}^n \) defined by \( L\eta = \sqrt{Q(x)}N(x)\eta \) is the canonical projection of \( \mathbb{R}^n \) onto \( K^\perp \). Since \( A(x, z, \xi) \in K^\perp \), then
\[ A(x, z, \xi) = \sqrt{Q(x)}N(x)A(x, z, \xi) = \sqrt{Q(x)}\tilde{A}(x, z, \xi) \]
where \( \tilde{A}(x, z, \xi) := N(x)A(x, z, \xi) \). Hence
\[ |\tilde{A}(x, z, \xi)| = \sup_{|\eta|=1} |\langle \eta, \tilde{A}(x, z, \xi) \rangle| = \sup_{|\eta|=1} |\langle \eta, N(x)A(x, z, \xi) \rangle| = \sup_{|\eta|=1} |\langle N(x)\eta, A(x, z, \xi) \rangle|. \]
The last term can be estimated by using (1.6) to obtain
\[ |\langle N(x)\eta, A(x, z, \xi) \rangle| \leq |\sqrt{Q(x)}N(x)\eta|\tilde{a}(x, z, \xi) \leq |\sqrt{Q(x)}N(x)| |\eta|\tilde{a}(x, z, \xi) = |\eta|\tilde{a}(x, z, \xi). \]
Therefore,
\[ |\tilde{A}(x, z, \xi)| \leq \tilde{a}(x, z, \xi) \leq a|\sqrt{Q(x)}\xi|^{p-1} + b|z|^{\gamma-1} + e. \]
The proof Theorem 5.3 is now complete. \( \square \)

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