Uniqueness, Comparison and Stability for Scalar BSDEs with \( L \exp(\mu \sqrt{2 \log(1 + L)}) \)-integrable terminal values and monotonic generators

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ABSTRACT
In a new viewpoint, we continue to study a backward stochastic differential equation (BSDE for short) with \( L \exp(\mu \sqrt{2 \log(1 + L)}) \)-integrable terminal value which was recently introduced together with the proof of the existence part in [Hu and Tang, 2018]. Through Girsanov change, we associate a solution to this equation with an \( L^1 \)-solution to a certain BSDE with integrable parameters. Using this technique, we show that the One-Sided Osgood condition (extended monotonicity) on generator rather than Lipschitz continuity is sufficient to guarantee the uniqueness of the solution. Next, we show the comparison principle of solutions under both strict monotonicity and Lipschitz conditions. We also study the stability of the dynamics under One-Sided Osgood condition.

KEYWORDS
backward stochastic differential equation; \( L \exp(\mu \sqrt{2 \log(1 + L)}) \)-integrability; uniqueness; comparison; stability monotonic generator; One-Sided Osgood condition

1. Introduction

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space, \(T > 0\) a finite time and \(W\) be a standard \(d\)-dimensional Brownian motion. Let \(\mathbb{F} := \{\mathcal{F}_t\}_{0 \leq t \leq T}\) be a completion of the filtration generated by the Brownian motion.

We consider the following backward stochastic differential equation (BSDE for short).

\[
y_t = \xi + \int_t^T f(s, y_s, z_s) \, ds - \int_t^T z_s \, dW_s, \quad t \in [0, T].
\]

where the generator \(f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^{1 \times d} \to \mathbb{R}\) is a predictable function and terminal value \(\xi\) is an \(\mathbb{F}_T\)-measurable random variable.

The theory of BSDEs is a powerful tool to treat important issues arising in many applied fields such as finance and optimal control. A general nonlinear pricing problem of the European contingent claim in standard complete market is equivalent to solve the BSDE (1.1). In this case, \(\xi\) is the contingent claim to hedge and \(T\) is the maturity date. Let us assume that (1.1) has a solution \((y_t, z_t)\) in an appropriate space. If the
generator is uniformly Lipschtz in $z$ (with Lipschtz constant $b$), we can apply the Girsanov measure change to the equation which leads to

$$y_t = \xi + \int_t^T f(s, y_s, 0) \, ds - \int_t^T z_s \, dW^Q_s, \quad t \in [0, T]. \tag{1.2}$$

where

$$Q := \exp\left( \int_0^T g(s, y_s, z_s) \, dW_s - \frac{1}{2} \int_0^T g^2(s, y_s, z_s) \, ds \right) \cdot \mathbb{P},$$

$$g(s, y_s, z_s) = \frac{f(s, y_s, z_s) - f(s, y_s, 0)}{|z_s|^2} \cdot z_s |_{|z_s| \neq 0},$$

and $W^Q := W - \int_0^T g(s, y_s, z_s) \, ds$ is a $Q$-Brownian motion.

In the financial view, $Q$ is called risk-neutral measure or martingale measure ([5]). For the convenience, we shall assume that $f$ only depends on $z$ (hence $f = f(s, z)$ and $f(s, 0) = 0$. Then we have

$$y_t = \xi - \int_t^T z_s \, dW^Q_s.$$

When $\xi$ is square-integrable, it is well-known that the fair price of $\xi$ is evaluated as the expectation of the claim under $Q$ (see e.g. [5]), that is,

$$y_t = \mathbb{E}_Q[\xi | \mathcal{F}_t]. \tag{1.3}$$

At this point, one can naturally try to look for an “optimal” integrability condition on terminal value under which the price is allowed to be represented by the risk-neutral measure. The paper of Ankirchner, Imkeller and Popiér [1] gives the partial result on this subject. Motivated by the expression (1.3), they introduced the notation of measure solution which is benefit to give an efficient formula of pricing contingent claim by martingale measure. In Lipschtz setting, they showed the existence of the measure solution when the terminal value is $L^p$-integrable for $p > 1$. In this case, we can use the Hölder’s inequality and the boundness of moments of the exponential martingale to show $\mathbb{E}_Q[\xi] < \infty$. If the terminal value is assumed to be only integrable (hence $L^1$-integrable), we can not guarantee that $\mathbb{E}_Q[\xi] < \infty$, so the measure solution does not exist in general. That is, we need a stronger integrability on terminal value. Consequently, we want to find a sufficient integrability condition which is weaker than $L^p$-integrability for any $p > 1$ and is stronger than $L^1$-integrability.

Obviously, the expression (1.3) has a meaning if and only if the following condition holds.

$$\mathbb{E}_Q[|\xi|] = \mathbb{E}[|\xi| \exp \left( \int_0^T g(s, z_s) \, dW_s - \frac{1}{2} \int_0^T g^2(s, z_s) \, ds \right)] < \infty. \tag{1.4}$$
As \( |g(s, z_s)| \leq b \), above condition is equivalent to
\[
\mathbb{E}[|\xi| \exp(\int_0^T g(s, z_s) \, dW_s)] < \infty.
\]

In Hu and Tang [8], they showed the following useful inequalities.

- \( e^x y \leq e^{\frac{x^2}{2}} + e^{2\mu^2} y \exp(\mu \sqrt{2 \log (1 + y)}) \).
- \( \mathbb{E}[\exp(\frac{1}{2\mu^2} \int_0^T q_s \, dW_s^2)] \leq [1 - \frac{b^2}{\mu^2} T]^{-1/2} \) if \( |q_s| \leq b, \mu > b\sqrt{T} \).

From these two inequalities, we can deduce
\[
\mathbb{E}[|\xi| \exp(\int_0^T g(s, z_s) \, dW_s)] \leq [1 - \frac{b^2}{\mu^2} T]^{-1/2} + e^{2\mu^2} \mathbb{E}[|\xi| \exp(\mu \sqrt{2 \log (1 + |\xi|))}].
\]

So, we can get one sufficient condition to guarantee (1.4) such that
\[
\mathbb{E}[|\xi| \exp(\mu \sqrt{2 \log (1 + |\xi|))} < \infty.
\]

That is, \( \xi \) is required to be \( L \exp(\mu \sqrt{2 \log (1 + L)}) \)-integrable. Furthermore, if the condition (1.4) is true, then \( \xi \) will be integrable under the measure \( \mathbb{Q} \), so the BSDE (1.1) is transferred into the BSDE (1.2) with integrable parameters whose solution is called an \( L^1 \)-solution. Also, the generating function of the equation (1.2) does not depend on \( z \) and this allows us to make a guess that the additional assumption appeared in the study of \( L^1 \)-solution may be omitted (see (1.5)).

Pardoux and Peng [10] first introduced the notion of nonlinear BSDE and studied the \( L^2 \)-solution under Lipschitz condition on generator.

Briand et al [2] discussed a solution of the BSDE in \( L^p \)-setting \((p \geq 1)\) whose generator satisfies the monotonicity condition. For the comparisons for \( L^p \)-solutions \((p > 1)\) under monotonicity condition, we can refer to [11]. Later, Fan [6] studied the wellposedness and comparison theorem of \( L^p \)-solutions \((p > 1)\) under various kind of extended monotonicity conditions. Also, Fan [7] showed the existence and uniqueness of \( L^1 \)-solution to BSDE under One-Sided Osgood condition which is one of the extended monotonicity conditions. However, we could not find any result on comparison principle for \( L^1 \)-solution. On the other hand, it was shown in [2] that the following sub-linear growth assumption on generator is needed to ensure the wellposedness of \( L^1 \)-solution.

\[
|f(t, y, z) - f(t, 0, 0)| \leq a|y| + b|z|^q, \quad (t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^{1 \times d} \quad (1.5)
\]

for some \( q \in [0, 1) \).

Recently, Hu and Tang [8] studied the solution to scalar BSDE in \( L \exp(\mu \sqrt{2 \log (1 + L)}) \)-setting such that \( \mu > \mu_0 \) for some critical value \( \mu_0 \), that is, the terminal value is \( L \exp(\mu \sqrt{2 \log (1 + L)}) \)-integrable. This integrability is stronger than \( L \log L \)-integrability and weaker than \( L^p \)-integrability for any \( p > 1 \). They showed the existence of the solution of that BSDE under the linear growth condition on generator. Furthermore they gave counterpart examples which show that \( L \log L \)-integrability is not sufficient to guarantee the existence of the solution. Perhaps, to the best of our knowledge, the \( L \exp(\mu \sqrt{2 \log (1 + L)}) \)-integrability seems to be “optimal” to guarantee the existence of a strong solution as well as a measure solution under “standard” conditions.
As a good continuation, Buckdahn, Hu and Tang [3] improved the existence result in [8] and gave the uniqueness result under Lipschitz condition. In their proof of uniqueness, the Lipschitz assumption plays a crucial role because the representation of a solution to linear BSDE is used.

In this paper, we state the uniqueness result under One-Sided Osgood condition which is an extended form of monotonicity. The next subject of this paper is to state the comparison principles. As it is well known, the comparisons for BSDEs are fundamental in the theory of nonlinear expectations, particularly in constructing the dynamic risk measures. In Cohen et al [4], they showed a general comparison theorem by means of the super-martingale measure which is corresponded to the "no-arbitrage" condition in financial sense. In that paper, the terminal value is only abstractly assumed to guarantee the existence of a solution and the existence of certain super-martingale measure is also assumed, independently. For the BSDEs with $L^1$-solution, the Lipschtz assumption plays a crucial role because the representation of a solution to linear BSDE is used.

Then it has the following properties ([3, 8]).

**2. Notations and Assumptions**

- For $A \in \mathcal{F}$ and $\mathcal{F}$-measurable random variable $\eta$, we define $\mathbb{E}^Q[\eta; A] := \int_A \eta dQ$. And $\mathbb{E}^Q[\eta] := \mathbb{E}^Q[\eta; \mathbb{Q}]$.
- $\mathcal{T}(0, T)$ is a set of stopping times $\tau$ such that $0 \leq \tau \leq T$.
- For any predictable process $\phi$, $\mathcal{E}(\phi \circ W) := \exp\left(\int_0^\tau \phi_r \, dW_r - \frac{1}{2} \int_0^\tau \phi_r^2 \, dr\right)$.
- We say that the process $Y = \{Y_t\}_{0 \leq t \leq T}$ belongs to class $(D)$ if the family $\{Y_\tau, \tau \in \mathcal{T}(0, T)\}$ is uniformly integrable.
- $|\cdot|$ means the standard euclidean norm.
- $M^p([0, T], \mathbb{R}^{1 \times d}; \mathbb{Q})$ is the space of predictable processes $Z$ with values in $\mathbb{R}^{1 \times d}$ such that $|Z|_{M^p} := \mathbb{E}^Q[(\int_0^T |Z_s|^2 \, ds)^{p/2}]^{1/p} < \infty$. If $\mathbb{Q} = \mathbb{P}$, then we denote by $M^p([0, T]; \mathbb{R}^{1 \times d})$.
- $H^1_1(\mathbb{Q})$ is the space of real càdlàg, adapted processes $Y$ such that $\mathbb{E}^Q[\sup_{t \in [0, T]} |Y_t|] < \infty$. If $\mathbb{Q} = \mathbb{P}$, then we use $H^1_{1,1}$.
- The solution of (1.1) is denoted by a pair $\{(Y_t, Z_t), t \in [0, T]\}$ of predictable processes with values in $\mathbb{R} \times \mathbb{R}^{1 \times d}$ such that $\mathbb{P}$-a.s., $Y$ is continuous, $Z \in M^2([0, T]; \mathbb{R}^{1 \times d})$ and $\mathbb{P}$-a.s., $(Y, Z)$ satisfies the equation (1.1).
- For any real valued function $g$, we define $g^+ := \max(g, 0)$.

Define the real function $\psi$:

$$\psi(x, \mu) := x \exp(\mu \sqrt{2 \log(1 + x)}), \quad (x, \mu) \in [0, \infty) \times (0, +\infty).$$

Then it has the following properties ([3, 8]).
For any $x \in \mathbb{R}$ and $y \geq 0$, we have
\[ e^{xy} \leq e^{x^2} + e^{2\mu^2} \psi(y, \mu). \] (2.1)

Let $\mu > b \sqrt{T}$. Then for any $d$-dimensional adapted process $q$ with $|q_t| \leq b$ a.s., for any $t \in [0, T]$,
\[ \mathbb{E}[\exp \left( \frac{1}{2\mu^2} \int_t^T |q_s| dW_s \right)^2 |\mathcal{F}_t] \leq \left[ 1 - \frac{b^2}{\mu^2} (T-t) \right]^{-1/2}. \] (2.2)

For any $\mu > 0$, $\psi(\cdot, \mu)$ is convex, that is, for any $0 \leq \lambda \leq 1$ and $x, y \in [0, +\infty)$,
\[ \psi(\lambda x + (1-\lambda)y, \mu) \leq \lambda \psi(x, \mu) + (1-\lambda)\psi(y, \mu). \] (2.3)

For any $l > 1, x \leq 0$, we have
\[ \psi(lx, \mu) \leq \psi(l, \mu) \psi(x, \mu). \] (2.4)

We present some useful assumptions on generator below.

(A1) $f$ satisfies the One-Sided Osgood condition with respect to $y$, that is there exists a non-decreasing and concave function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ with $\rho(0) = 0, \rho(t) > 0$ for $t > 0$ and $\int_{\mathbb{R}^+} \frac{dt}{\rho(t)} = +\infty$ such that for any $y, y' \in \mathbb{R}$ and $z \in \mathbb{R}^{1 \times d}$,
\[ \frac{y - y'}{|y - y'|} 1_{|y-y'| \neq 0} (f(t, y, z) - f(t, y', z)) \leq \rho(|y - y'|).
\]

(A2) $f$ is strictly monotone in $y$, that is, for any $y, y' \in \mathbb{R}$ and $z \in \mathbb{R}^{1 \times d}$,
\[ (y - y') (f(t, y, z) - f(t, y', z)) \leq 0.
\]

(A3) $f$ is uniformly Lipschtz in $z$, that is, there exists a constant $b$ such that for any $y \in \mathbb{R}$ and $z, z' \in \mathbb{R}^{1 \times d}$,
\[ |f(t, y, z) - f(t, y, z')| \leq b |z - z'|.
\]

(A4) The map $y \mapsto f(t, y, z)$ is continuous.

(A5) $f$ is linear growth with respect to $y$, that is, there exists a constant $a \geq 0$ such that for any $y, y' \in \mathbb{R}$ and $z \in \mathbb{R}^{1 \times d}$,
\[ |f(t, y, z) - f(t, 0, z)| \leq a |y|.
\]

(A6) $f$ is uniformly Lipschtz in $y$.

3. Uniqueness

**Theorem 3.1.** Let assumptions (A1), (A3) hold. Then, BSDE (1.1) has at most one solution $(Y, Z)$ such that $\psi(Y, c)$ belongs to class $(D)$ for some $c > 0.$
Proof. For $i = 1, 2$, let $(Y^i, Z^i)$ be a solution to (1.1) such that $\psi(Y^i, c^i)$ belongs to the class $(D)$ for some $c^i > 0$. Since $(x, \mu)$ is non-decreasing in $\mu$, both $\psi(Y^1, c)$ and $\psi(Y^2, c)$ belong to class $(D)$ for $c = c^1 \land c^2$.

Define $(\bar{Y}, \bar{Z}) := (Y^1 - Y^2, Z^1 - Z^2)$. For any $\tau \in \mathbb{T}(0, T)$, by (2.3), (2.4),

$$\psi(|\bar{Y}_\tau|, c) \leq \psi(|Y^1_\tau| + |Y^2_\tau|, c) = \frac{1}{2} \psi(2|Y^1_\tau|, c) + \frac{1}{2} \psi(2|Y^2_\tau|, c) \leq \frac{1}{2} \psi(2, c)[\psi(|Y^1_\tau|, c) + \psi(|Y^2_\tau|, c)].$$

So, $\psi(|\bar{Y}|, c)$ is also belongs to class $(D)$. We first restrict our discussion to the case where $T < \frac{c^2}{b^2}$, so $c > b\sqrt{T}$. Obviously, $(\bar{Y}, \bar{Z})$ satisfies the following equation.

$$\bar{Y}_t = \int_T^T \bar{f}(s, \bar{Y}_s, \bar{Z}_s) \, ds - \int_T^T \bar{Z}_s \, dW_s, \quad t \in [0, T]. \quad (3.1)$$

where $\bar{f}(s, y, z) := f(s, y + Y^2_s, z + Z^2_s) - f(s, Y^1_s, Z^1_s)$.

Define $\bar{g}(s, y, z) := 1_{|z| \neq 0} \frac{f(s, y, z) - f(s, y, 0)}{|z|^2}$, then

$$\bar{g}_s := \bar{g}(s, \bar{Y}_s, \bar{Z}_s) = 1_{|\bar{Z}_s| \neq 0} \frac{f(s, Y^1_s, Z^1_s) - f(s, Y^1_s, Z^2_s)}{|Z^1_s|^2} \bar{Z}_s.$$

From the assumption (A3), we get $|\bar{g}| \leq b$, a.s. and so $\mathbb{E}(\bar{g} \mathcal{E} \cdot W)_t = \exp \left( \int_0^t \bar{g}_s \, dW_s - \frac{1}{2} \int_0^t \bar{g}^2_s \, ds \right)$ is an uniformly integrable martingale.

By the virtue of Girsanov change, we have

$$\bar{Y}_t = \int_T^T \bar{f}(s, \bar{Y}_s, 0) \, ds - \int_T^T \bar{Z}_s \, dW^\mathbb{Q}_s, \quad t \in [0, T].$$

where $\mathbb{Q} := \mathcal{E}(\bar{g}_s \mathcal{W} |_T \cdot \mathbb{P})$, and $W_\mathbb{Q} := W - \int_0^T \bar{g}_s \, ds$.

Note that $\mathbb{Q}$ is a probability measure equivalent to $\mathbb{P}$ and $W^\mathbb{Q}$ is a Brownian motion under $\mathbb{Q}$. Then, for any $\tau \in \mathbb{T}(0, T)$ and $A \in \mathcal{F}$, by (2.1), (2.2),

$$\mathbb{E}^{\mathbb{Q}}(|\bar{Y}_\tau|; A) = \mathbb{E}(|\bar{Y}_\tau| \cdot \mathcal{E}(\bar{g} \mathcal{W})_\tau; A)$$

$$\leq \mathbb{E}[|\bar{Y}_\tau| \exp \left( \int_0^T \bar{g}_s \, dW_s \right); A] \leq \mathbb{E}[\exp \left( \frac{1}{2} \int_0^T \bar{g}_s \, dW_s \right)^2 + e^{4e^2} \psi(|\bar{Y}_\tau|, c); A]$$

$$\leq (1 - \frac{b^2}{c^2} T)^{-1/2} + \mathbb{E}[e^{4e^2} \psi(|\bar{Y}_\tau|, c); A].$$

So, $\bar{Y}$ belongs to class $(D)$ under $\mathbb{Q}$. Now we give an estimate on $\bar{Z}$ under $\mathbb{Q}$.

Let $1 < p < \infty$, $p^\prime = p - 1$, $q^\prime = 1$ and $k = \frac{\sqrt{p}}{2(\sqrt{p} - 1)}$. Then for any $\tau \in \mathbb{T}(0, T)$,

$$\mathcal{E}((k\bar{g}) \cdot W)_\tau = \exp \left( \int_0^T k\bar{g}_s \, dW_s \right) \exp \left( -\frac{1}{2} \int_0^T k^2 \bar{g}^2_s \, ds \right) \geq \exp \left( \int_0^T k\bar{g}_s \, dW_s \right) \exp \left( -\frac{1}{2} b^2 k^2 T \right).$$
Due to $\mathbb{E}[[\mathcal{E}((k\tilde{g}) \bullet W)_T]] = 1$, sup$_{s \in (0,T]} \mathbb{E}[[\exp (\int_0^T k\tilde{g}_s dW_s)]] \leq \exp \left( \frac{1}{2} b^2 k^2 T \right)$. Therefore, according to [9, Theorem 1.5], $\mathcal{E}(\tilde{g} \bullet W)$ is an $L^q$-bounded martingale.

So, by the Holder’s inequality, we obtain

$$
\mathbb{E}[[\int_0^T \bar{Z}_s^2 ds]^{1/p}] = \mathbb{E}[[\int_0^T \bar{Z}_s^2 ds]^{1/p} \mathcal{E}(\bar{g} \bullet W)_T] \\
\leq \mathbb{E}[[\int_0^T \bar{Z}_s^2 ds]^{1/p} \mathcal{E}(\bar{g} \bullet W)_T] \leq \mathbb{E}[[\int_0^T \bar{Z}_s^2 ds]^{1/p} \mathbb{E}[\mathcal{E}(\bar{g} \bullet W)_T^{1/q}] < \infty.
$$

Taking $\bar{p} := \frac{2}{p}$, then $\bar{Z} \in M^{\bar{p}}(0,T, \mathbb{R}^{1 \times d}; \mathbb{Q})$. (Hence $\bar{Z} \in M^{\bar{p}}(0,T, \mathbb{R}^{1 \times d}; \mathbb{Q})$, for any $0 < \bar{p} < 2$.)

Therefore, $(\bar{Y}, \bar{Z})$ is an $(L^1 -)$ solution of the following BSDE such that $\bar{Y}$ belongs to class $(D)$ and $\bar{Z} \in M^{\bar{p}}(\mathbb{R}^{1 \times d}; \mathbb{Q})$.

$$
y_t = \int_t^T \bar{f}(s,y,0) ds - \int_t^T z_s dW_s^Q, \quad t \in [0,T].
$$

(3.2)

Also, due to $\bar{f}(s,0,0) = 0$, a pair $(0,0)$ is also a solution of (3.2).

On the other hand, for any $y, y' \in \mathbb{R}$,

$$
\frac{y - y'}{|y - y'|} 1_{|y - y'| > 0}(\bar{f}(s,y,0) - \bar{f}(s,y',0))
= \frac{y - y'}{|y - y'|} 1_{|y - y'| > 0}(f(s,y + Y_s^2, Z_s^2) - f(s,y' + Y_s^2, Z_s^2)) \leq \rho(|y - y'|).
$$

Therefore, according to the uniqueness of $L^1$-solution of BSDEs with generators of One-Sided Osgood type ([7], Theorem 1), we have $(\bar{Y}, \bar{Z}) = (0,0)$. For larger value of $T$, we discuss on interval $[T - \delta, T]$ for small $\delta > 0$ from which we get $(\bar{Y}_t, \bar{Z}_t) = 0$ for $T - \delta \leq t \leq T$ and then with the terminal value $\bar{Y}_{T-\delta} = 0$, we discuss on interval $[T - 2\delta, T - \delta]$ from which $(\bar{Y}_t, \bar{Z}_t) = 0$ for $T - 2\delta \leq t < T - \delta$ and so on by an inductive argument. This provides $(\bar{Y}, \bar{Z}) = 0$ on the whole interval $[0,T]$. That is, we have $Y^1 = Y^2$ and $Z^1 = Z^2$.

Combined with the existence result ([3, 8]), we get the following result.

**Corollary 3.2.** Suppose that $(A1)$, $(A3)$, $(A4)$ and $(A5)$ hold. We further assume that there exist $\mu > b\sqrt{T}$ such that

$$
\psi(|\xi| + \int_0^T |f(t,0,0)| dt, \mu) \in L^1(\Omega, \mathbb{P}).
$$

Then, BSDE (1.1) has a unique solution $(Y, Z)$ such that $\psi(Y, c)$ belongs to the class $(D)$ for some $c > 0$. Moreover we have the following estimate on $Y$.

$$
|Y_t| \leq \frac{1}{\sqrt{1 - \frac{b^2}{\mu^2}(T-t)}} \exp \{a(T-t)\} + e^2 \mu^2 + a(T-t) \mathbb{E}[\psi(|\xi|) + \int_t^T |f(s,0,0)| ds, \mu] [\mathcal{F}_t].
$$

(3.4)
Remark 3.1. We also have an estimate on $\psi(Y, \mu)$ (See the last inequality of the proof of [3, Theorem 2.4]). For some constants $A, B \leq 0$, it holds that

$$\psi(|Y_t|, \mu) \leq A + B \cdot \mathbb{E}[\psi(\mu, |\xi|) + \int_0^T |f(s, 0, 0)| ds] \mathcal{F}_t].$$

(3.5)

Moreover, by referring to [2], Lemma 6.1, we can see that for any $\beta \in (0, 1)$,

$$\mathbb{E}[\sup_{t \in [0, T]} \psi(|Y_t|, \mu)] \leq \frac{1}{1 - \beta} (A + B \cdot \mathbb{E}[\psi(\mu, |\xi|) + \int_0^T |f(s, 0, 0)| ds] \mathcal{F}_t])^\beta < \infty. \quad (3.6)$$

4. Comparisons

We first show the comparison principle for BSDE (1.1) with Lipschitz generator.

Theorem 4.1. Let $(\xi, f)$ and $(\xi', f')$ be any two pairs of terminal value and generator of (1.1), respectively. Let $(Y, Z)$ and $(Y', Z')$ be associated solutions such that $\psi(Y, c)$ and $\psi(Y', c')$ belong to class (D) for some $c, c' > 0$. Suppose that $f$ satisfies (A3) and (A6). If $\xi \geq \xi'$ and $f(t, Y, Z) \geq f'(t, Y', Z')$ then $Y_t \geq Y'_t$ for all $t \in [0, T]$, $\mathbb{P}$-a.s. Moreover this comparison is strict, that is, if $Y^1_t = Y^2_t$, $\mathbb{P}$ - a.s. on $A \in \mathcal{F}_t$, then $Y^1_s = Y^2_s$ on $[t, T] A$ up to evanescence.

Proof. As we showed in the beginning of the proof of theorem 3.1, $\psi(Y - Y', c_0)$ belongs to class (D) for some $c_0 = c \wedge c'$. We first discuss for small $T$ such that $c_0 > b\sqrt{T}$. Define $\Gamma_s := \frac{f(s, Y_s, Z_s) - f(s, Y'_s, Z'_s)}{|Z_s - Z'_s|^2} 1_{|Z_s - Z'_s| = 0}(Z_s - Z'_s)$ which is uniformly bounded. The measure $\mathbb{Q}$ is defined as follows.

$$\frac{d\mathbb{Q}}{d\mathbb{P}} := \mathcal{E}(\Gamma \cdot W)_T = \int_0^T \Gamma_s dW_s - \frac{1}{2} \int_0^T \Gamma^2_s ds.$$

Then, $W^2 := W - \int_0^T \Gamma_s ds$ is $\mathbb{Q}$-Brownian motion. As we showed in preceding discussion, $Z - Z' \in M^q(R^{1 \times d}; \mathbb{Q})$ for any $q \in (1, 2)$. Therefore,

$$- \int_0^t (f(s, Y'_s, Z_s) - f(s, Y'_s, Z'_s)) ds + \int_0^t (Z_s - Z'_s) dW_s = \int_0^t (Z_s - Z'_s) dW^2_s.$$

is a $\mathbb{Q}$-martingale. On the other hand,

$$\mathbb{E}[\int_0^T (f(s, Y_s, Z_s) - f(s, Y'_s, Z'_s)) ds] \leq a \mathbb{E}[\int_0^T |Y_s - Y'_s| ds]

= a \mathbb{E}[\int_0^T |Y_s - Y'_s| \mathcal{E}(\Gamma \cdot W)_T ds]

\leq a \mathbb{E}[\int_0^T (1 - \frac{b^2}{c_0^2} T)^{-1/2} + e^{2c_0^2} \psi(\xi, c_0)] d\tau

\leq a (1 - \frac{b^2}{c_0^2} T)^{-1/2} + a T e^{2c_0^2} \sup_{\tau \in T} \mathbb{E}[\psi(\xi, c_0)] < \infty.$$
Therefore by [4], Theorems 1, 2 and 3, both the comparison and strict comparison theorems clearly hold.

**Remark 4.1.** As an immediate consequence of Theorem 4.1, we can see that the solution of the BSDE (1.1) is unique. So, that provides another method for the proof of the uniqueness part in Lipschtz setting.

Now we shall discuss under strict monotonicity condition.

**Theorem 4.2.** Suppose that $f$ satisfies (A2) and (A3). Then the comparisons principle such as in theorem 4.1 still holds.

**Proof.** Set $\bar{\xi} := \xi - \xi', \bar{f} := f - f', \delta f(s, y, z) := f(s, y + Y'_s, z + Z'_s) - f(s, Y'_s, Z'_s)$. The pair $(\bar{Y}, \bar{Z}) := (Y - Y', Z - Z')$ satisfies

$$\bar{Y}_t = \bar{\xi} + \int_t^T [\delta f(s, \bar{Y}_s, \bar{Z}_s) + \bar{f}(s, Y'_s, Z'_s)] ds - \int_t^T \bar{Z}_s dW_s.$$  

After an application of the Girsanov change, we have

$$\bar{Y}_t = \bar{\xi} + \int_t^T [\delta f(s, \bar{Y}_s, 0) + \bar{f}(s, Y'_s, Z'_s)] ds - \int_t^T \bar{Z}_s dW_s^Q.$$  

where the probability measure $Q$ is similarly defined as before.

Note that $E^Q[\bar{\xi}] \leq \infty, \delta f(s, 0, 0) = 0$ and $\bar{Y}_t$ belongs to class $(D) \text{ under } Q$. We introduce for $n \geq 1$,

$$\tau_n := \inf\{t \in [0, T]; \int_0^t (|Z_s|^2 + |Z'_s|^2) ds \geq n\} \land T.$$  

Then we have

$$\bar{Y}_{t \land \tau_n} = \bar{Y}_{\tau_n} + \int_{t \land \tau_n}^{\tau_n} [\delta f(s, \bar{Y}_s, 0) + \bar{f}(s, Y'_s, Z'_s)] ds - \int_{t \land \tau_n}^{\tau_n} \bar{Z}_s dW_s^Q. \tag{4.1}$$  

Applying the minor version of Tanakas formula (see [11], pp.107) to (4.1),

$$\bar{Y}^+_{t \land \tau_n} = \bar{Y}^+_{\tau_n} + \int_{t \land \tau_n}^{\tau_n} \theta(\bar{Y}_s)[\delta f(s, \bar{Y}_s, 0) + f(s, Y'_s, Z'_s)] ds - \int_{t \land \tau_n}^{\tau_n} \theta(\bar{Y}_s)Z_s dW_s^Q - \int_{t \land \tau_n}^{\tau_n} dP_s,$$

where $P$ is an increasing, continuous process and the function $\theta$ is defined as follows.

$$\theta(x) := 1_{x > 0} + \frac{1}{2}1_{x = 0}.$$

Using the facts that $f$ is strictly monotone in $y$, $\delta f(s, 0, 0) = 0$ and $\bar{f}(s, Y'_s, Z'_s) \leq 0$, we can deduce $\theta(\bar{Y}_s)[\delta f(s, \bar{Y}_s, 0) + f(s, Y'_s, Z'_s)] \leq 0$. So we have

$$\bar{Y}^+_{t \land \tau_n} \leq E^Q[\bar{Y}^+_{\tau_n} | \mathcal{F}_t].$$
Noting that $\bar{Y}^+$ is continuous and belongs to class $(D)$ under $\mathbb{Q}$, we have $\mathbb{Q}$-a.s.,

$$\bar{Y}_{\tau_n}^+ \to \bar{Y}_T^+ = \bar{\xi}^+ = 0, \quad n \to \infty.$$ 

By $L^1$-continuity of conditional expectation, we get $\mathbb{E}^\mathbb{Q}[\bar{Y}(\tau_n)^+|\mathcal{F}_t] \xrightarrow{ucp} 0$.

Taking a subsequence, we get $\mathbb{Q}$-a.s., $\bar{Y}_t^+ = 0$. As $\mathbb{Q}$ is equivalent to $\mathbb{P}$, we have $\mathbb{P}$-a.s., $\bar{Y}_t^+ = 0$ which implies $Y_t^1 \leq Y_t^2$.

5. Stability

In this section, we state the stability result of the BSDE (1.1). We shall restrict to the case where the generator is linear with respect to $z$. The more general case is left to the future work. Before the main discussion, we study a solution in norm space.

Lemma 5.1. Suppose that the generator satisfies $\textbf{(A1)}, \textbf{(A3)}, \textbf{(A4)}$ and $\textbf{(A5)}$. Instead of (3.3), we assume that

$$\sup_{t \in [0,T]} (\mathbb{E}[\psi(|\xi| + \int_0^T |f(t,0,0)|dt,\mu)|\mathcal{F}_t]) \in L^1(\Omega,\mathbb{P}), \mu > b\sqrt{T}.$$ 

Then the BSDE (1.1) has a unique solution such that $\psi(|Y|,\mu) \in H^1_T$.

Proof. By Corollary 3.2, (1.1) has a unique solution $(Y,Z)$ such that $\psi(Y,\mu)$ belongs to class $(D)$. Due to (3.5), we can see that $\psi(|Y|,\mu) \in H^1_T$.

Theorem 5.2. For each $n \in \mathbb{N}_0$, let us consider the following BSDEs depending on parameter $n$:

$$Y^n_t = \xi^n + \int_t^T f^n(s,Y^n_s,Z^n_s) ds - \int_t^T Z^n_s dW_s, \quad t \in [0,T].$$

We introduce the following assumptions.

(1) For all $n$, $\xi^n$ and $f^n$ satisfy $\textbf{(A1)}, \textbf{(A3)}, \textbf{(A4)}$ and $\textbf{(A5)}$ with the same parameters $\rho(\cdot), a, b$.

(2) $f^0$ is linear with respect to $z$, that is, $f^0(s,y,z) = f^0(s,y,0) + bz$.

(3) There exists a constant $\mu > b\sqrt{T}$ such that

$$\psi(\xi^0 + \int_0^T f^0(t,0,0) dt,\mu) \in L^1(\Omega,\mathbb{P}).$$

(4) There exists a non-negative real sequence $(l_n)_{n=1,2,...}$ which converges to 0 such that for each $n$, for any $(y,z) \in \mathbb{R} \times \mathbb{R}^d$,

$$|f^n(s,y,z) - f^0(s,y,z)| \leq l_n, \quad d\mathbb{P} \times dt - a.s.$$ 

(5) There exists a random variable $\eta$ satisfying $\psi(\eta,\mu) \in L^1(\Omega,\mathbb{P})$ such that $|\xi^n - \xi^0| \leq \eta$ for any $n \geq 1$ and

$$\mathbb{E}[|\xi^n - \xi^0|] \to 0, \quad n \to \infty.$$
(6) There exists a constant $\mu > b\sqrt{T}$ such that

$$\sup_{t \in [0,T]} (\mathbb{E}[\psi(|\xi^0| + \int_0^T |f^0(t,0,0)|dt, \mu)|\mathcal{F}_t]) \in L^1(\Omega, \mathbb{P}), \mu > b\sqrt{T}.$$ 

(7) There exists a random variable $\eta$ satisfying $\psi(\sup_{t \in [0,T]} \mathbb{E}[|\eta|, \mathcal{F}_t], \mu) \in L^1(\Omega, \mathbb{P})$ such that $|\xi^n - \xi^0| \leq \eta$ for any $n \geq 1$ and

$$\mathbb{E}\left[ \sup_{t \in [0,T]} \mathbb{E}[|\xi^n - \xi^0||\mathcal{F}_t]\right] \to 0, \quad n \to \infty.$$

Then the following assertions hold.

(1) Under assumptions 1-5, we have

$$\sup_{t \in [0,T]} \mathbb{E}[|Y^n_t - Y^0_t|] + \mathbb{E}\left[\left(\int_0^T |Z^n_s - Z^0_s|^2 ds\right)^{1/2}\right] \to 0, \quad n \to \infty.$$ 

and for any $\beta \in (0,1)$,

$$\mathbb{E}\left[ \sup_{t \in [0,T]} |Y^n_t - Y^0_t|^{\beta}\right] \to 0, \quad n \to \infty.$$

(2) Under assumptions 1, 2, 4, 6, 7, we have

$$\mathbb{E}\left[ \sup_{t \in [0,T]} |Y^n_t - Y^0_t| + \left(\int_0^T |Z^n_s - Z^0_s|^2 ds\right)^{1/2}\right] \to 0, \quad n \to \infty.$$

**Proof.** (1) By the virtue of Girsanov change, we have for each $n \in \mathbb{N}_0$,

$$Y^n_t = \xi^n + \int_t^T f^n(s,Y^n_s,0) \, ds - \int_t^T Z^n_s \, dW^n_s.$$

where

$$\frac{d\mathbb{Q}^n}{d\mathbb{P}} := \mathcal{E}(g^n \bullet W)_{T}, \quad W^\mathbb{Q}^n := W - \int_0^T g^n(s) \, ds.$$

$$g^n(s) := \frac{f^n(s,Y^n_s,Z^n_s) - f^n(s,Y^n_s,0)}{|Z^n_s|^2} 1_{|Z^n_s| \neq 0} Z^n_s.$$

We put $\mathbb{Q} := \mathbb{Q}^0$. Clearly, $W^\mathbb{Q}^n = W^\mathbb{Q} - \int_0^T (g^n(s) - g^0(s)) \, ds$ for each $n$. So, we get

$$Y^n_t = \xi^n + \int_t^T \tilde{f}^n(s,Y^n_s,Z^n_s) ds - \int_t^T Z^n_s \, dW^\mathbb{Q}_s.$$

where $\tilde{f}^n(s,y,z) := f^n(s,y,z) - g^0(s)z = f^n(s,y,z) - bz$. In the same way as in the
preceding discussions, we can see that

\[ \mathbb{E}^Q[\eta] < \infty, \quad \mathbb{E}^Q[\xi^n + \int_0^T \tilde{f}^n(s, 0, 0)] = \mathbb{E}^Q[\xi^n + \int_0^T f^n(s, 0, 0)] < \infty. \]

\(Y^n\) belongs to class (D) under \(Q\) and \(Z^n \in M^b([0, T], \mathbb{R}^{1 \times d}; Q)\), for any \(0 < \tilde{p} < 2\).

Remark that \(\bar{f}(s, y, z) = f^0(s, y, 0)\). Moreover \(|\tilde{f}^n(s, y, z) - \tilde{f}^n(s, y, 0)| = |f^n(s, y, z) - f^n(s, y, 0) - bz| \leq 2l_n + |f^0(s, y, z) - f^0(s, y, 0) - bz| = 2l_n\), so \(\tilde{f}^n\) clearly has sublinear growth in \(z\). Therefore, for each \(n\), \((Y^n, Z^n)\) is a unique \(L^1\)-solution of the following BSDE under \(Q\).

\[ y_t = \xi^n + \int_t^T \tilde{f}^n(s, y_s, z_s)ds - \int_t^T z_s dW_s^Q. \quad (5.1) \]

From the assumption, \(\xi^n\) converges to \(\xi^0\) in probability and so does under \(Q\). As \(|\xi^n - \xi^0| \leq \eta, n \in \mathbb{N}_0\) and \(\mathbb{E}^Q[\eta] < \infty\), by Lebesgue’s dominated convergence theorem, we get

\[ \mathbb{E}^Q[|\xi^n - \xi^0|] = 0. \]

Also \(|\tilde{f}^n(s, y, z) - \tilde{f}^0(s, y, z)| = |f^n(s, y, z) - f^0(s, y, z)| \leq l_n\), for any \(n \in \mathbb{N}_0\). Now we can use the stability results of \(L^1\)-solutions to BSDE (5.1). According to [7], Theorem 4, it holds that

\[ \sup_{t \in [0, T]} \mathbb{E}^Q[|Y^n_t - Y^0_t|] \to 0, \quad n \to \infty. \quad (5.2) \]

and for any \(\beta \in (0, 1)\),

\[ \mathbb{E}^Q[\sup_{t \in [0, T]} |Y^{n,0}_t|^\beta + \left( \int_0^T |Z^n_s - Z^0_s|^2 ds \right)^{\beta/2}] \to 0, \quad n \to \infty. \quad (5.3) \]

For the simplicity, we define

\[ Y^{n,0} := Y^n - Y^0, \quad Z^{n,0} := Z^n - Z^0, \quad \xi^{n,0} := \xi^n - \xi^0, \quad \tilde{f}^{n,0} := \tilde{f}^n - \tilde{f}^0 = f^n - f^0 =: f^{n,0}. \]

From the definition, \(\sup_{t \in [0, T]} \mathbb{E}^Q[|Y^{n,0}_t|] = \sup_{t \in [0, T]} \mathbb{E}[|Y^{n,0}_t|] = \mathbb{E}[|Y^{n,0}_T|] = \mathbb{E}[|\mathcal{L}(b \cdot W)_T|]. \)

As \(Y^{n,0}_t \xrightarrow{ucp} 0\) under measure \(Q\) (or equivalently \(Y^{n,0}_t \cdot \mathcal{E}(b \cdot W)_T \xrightarrow{ucp} 0\)), it holds that

\[ Y^{n,0}_t = Y^{n,0}_t \cdot \mathcal{E}(b \cdot W)_T \cdot \mathcal{E}^{-1}(b \cdot W)_T \xrightarrow{ucp} 0. \]
On the other hand,

\[ |Y_{t}^{n,0}| \mathcal{E}(b \cdot W)_{T} \leq \exp \left( \frac{\left( \int_{0}^{T} b dW_{s} \right)^{2}}{2\mu^{2}} \right) + e^{2\mu^{2}\psi(|Y_{t}^{n,0}|, \mu)} \]

\[ \leq \exp \left( \frac{\left| bW_{T} \right|^{2}}{2\mu^{2}} \right) + e^{2\mu^{2}(A + B \cdot \psi(|\xi^{n,0}| + \int_{0}^{T} f^{n,0}(s, 0, 0) ds, \mu))} \]

\[ \leq \exp \left( \frac{\left| bW_{T} \right|^{2}}{2\mu^{2}} \right) + e^{2\mu^{2}(A + B \cdot \psi(\eta + T \sup_{n}l_{n}, \mu))} =: (*) . \]

And \( \mathbb{E}[(*)] \leq (1 - \frac{b^{2}T}{\mu^{2}})^{-1/2} + e^{2\mu^{2}(A + \frac{1}{2}B\psi(2, \mu) \cdot (\psi(\eta, \mu) + \psi(T \sup_{n}l_{n}, \mu)))} < \infty. \)

So, by Lebesgue’s dominated convergence theorem, we obtain

\[ \sup_{t \in [0, T]} \mathbb{E}[|Y_{t}^{n,0}|] \to 0, \ n \to \infty. \]

From the Hölder’s inequality, for any \( \epsilon < (1 - \beta)/\beta, \)

\[ \mathbb{E}[\sup_{t \in [0, T]} |Y_{t}^{n,0}|^{\beta} + (\int_{0}^{T} |Z_{s}^{n,0}|^{2} ds)^{\beta/2}] \]

\[ = \mathbb{E}[\sup_{t \in [0, T]} |Y_{t}^{n,0}|^{\beta} + (\int_{0}^{T} |Z_{s}^{n,0}|^{2} ds)^{\beta/2}] \times [\mathcal{E}(b \cdot W)_{T}]^{1/(1+\epsilon)} \times [\mathcal{E}(b \cdot W)_{T}]^{-1/(1+\epsilon)} \]

\[ \leq C \mathbb{E}[\sup_{t \in [0, T]} |Y_{t}^{n,0}|^{\beta(1+\epsilon)} + (\int_{0}^{T} |Z_{s}^{n,0}|^{2} ds)^{\beta(1+\epsilon)/2}] \mathcal{E}(b \cdot W)_{T}^{1/(1+\epsilon)} \]

\[ \times \mathbb{E}([\mathcal{E}(b \cdot W)_{T}]^{-1/\epsilon})^{\epsilon/(1+\epsilon)} \]

\[ = \mathbb{E}^{Q}[\sup_{t \in [0, T]} Y_{t}^{n,0}]^{\beta(1+\epsilon)} + (\int_{0}^{T} |Z_{s}^{n,0}|^{2} ds)^{\beta(1+\epsilon)/2}]^{1/(1+\epsilon)} \]

\[ \times \mathbb{E}([\mathcal{E}(b \cdot W)_{T}]^{-1/\epsilon})^{\epsilon/(1+\epsilon)}. \]

From (5.2) and (5.3), the last term tends to 0 as \( n \to \infty. \) Consequently, we have

\[ \mathbb{E}[\sup_{t \in [0, T]} |Y_{t}^{n,0}|^{\beta} + (\int_{0}^{T} |Z_{s}^{n,0}|^{2} ds)^{\beta/2}] \to 0, \ n \to \infty. \]

Due to the arbitrariness of \( \beta \in [0, 1) \) and \( Z^{n} \in M^{2}([0, T]; \mathbb{R}^{1 \times d}), \) we see that

\[ \mathbb{E}[\left( \int_{0}^{T} |Z_{s}^{n,0}|^{2} ds \right)^{1/2}] \to 0, \ n \to \infty. \]

(2) We can have very similar procedure as in the proof of the first assertion together with lemma 5.1 and [7], Theorem 5, so we omit the proof.

**Remark 5.1.** One may have attempts to prove directly the stability theorem without using the properties of \( L^{1} \)-solution. But this is not the objective within our framework.
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