Is conformal symmetry really anomalous?

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Abstract

The conformal anomaly (also known as the stress-energy trace anomaly) of an interacting quantum theory, associated with violation of Weyl (conformal) symmetry by quantum effects, can be amended if one endows the theory with a dilatation current coupled to a vector field that is the gauge connection of local Weyl symmetry transformations. The natural candidate for this Weyl connection is the trace of the geometric torsion tensor, especially if one recalls that pure (Cartan-Einstein) gravity with torsion is conformal. We first point out that both canonical and path integral quantisation respect Weyl symmetry. The only way quantum effects can violate conformal symmetry is by the process of regularization. However, if one calculates an effective action from a conformally invariant classical theory by using a regularisation procedure that is conform with Weyl symmetry, then the conformal Ward identities will be satisfied. In this sense Weyl symmetry is not broken by quantum effects. This work suggests that Weyl symmetry can be treated on equal footing with gauge symmetries and gravity, for which an infinite set of Ward identities guarantees that they remain unbroken by quantum effects.

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I. INTRODUCTION

The discovery of conformal anomaly dates back to 1974 to the seminal work of Capper and Duff [1], in which the authors showed that the Ward identities of conformal symmetry are broken by the 1-loop quantum fluctuations. Next Capper and Duff found that the one-loop photon contributes anomalously to the graviton self-energy through the time-ordered energy-momentum tensor (TT) correlator on Minkowski space, \( \langle T_\mu \cdot T_\lambda (x) T_\sigma (0) \rangle \neq 0 \). Their results show that, while the counter-terms proportional to \( 1/(D-4) \) respect Weyl symmetry, where \( D \) denotes the dimension of spacetime, the finite contribution does not, yielding an anomalous contribution to the TT correlator. In Ref. [2] a second type of anomaly, related to the Euler characteristic of the (Euclidean) manifold, appeared in the trace anomaly, given in four dimensions by the Gauss-Bonnet density, namely \( \langle T\mu\mu \rangle \propto R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} \), where \( R_{\mu\nu\lambda\sigma} \) is the curvature tensor and \( R_{\mu\nu} = g^{\lambda\sigma}R_{\lambda\mu\sigma\nu} \), \( R = g^{\mu\nu}R_{\mu\nu} \).

Inspired by these early results, much work has been done on conformal anomalies in the past 40 years. For example, Ref. [3] has argued that the anomaly is responsible for Hawking radiation. Furthermore, the general form of the anomaly in four dimensions was deduced from general covariance and conformal invariance [4], based on which Riegert showed to follow from a non-local action [5]. More precisely, Riegert showed that all the anomalous terms – including \( \Box R \), Weyl\(^2 \) and the Gauss-Bonnet term – follow from variation of the Riegert (nonlocal) action, where Weyl stands for the Weyl tensor and \( \Box \) is the d’Alembertian. The exception is the anomalous term \( \propto R^2 \), for which Riegert had shown that it cannot be obtained by variation of a non-local action.

The conformal anomaly has found many intriguing physical applications. For example, it has been argued to be the explanation for the dark energy of the Universe [6]. Furthermore, conformal anomaly may be responsible for formation of cosmological perturbations [8], may induce non-Gaussianities in the cosmic microwave background radiation [7] and could play an important role in the formation of compact stellar objects such as gravastars [9], which have been proposed as an alternative to black holes.

The common explanation for these results found in literature is that Weyl symmetry may be a symmetry of classical theory, but it is generally violated in quantum theory. This can be seen from the non-invariance of the path integral measure \( \mathcal{D}\phi \) under the Weyl transformations, \( g_{\mu\nu} \to \Omega^2(x)g_{\mu\nu} \), \( \phi \to \Omega^{-\frac{D-2}{2}} \) (see for example [10]). In this work we show
that this argument is not true and that both canonical and path integral quantization (in its phase space form) are manifestly Weyl invariant thanks to the non-trivial contributions to the path integral measure coming from the gravitational field.  

A violation of Weyl symmetry is usually introduced by the procedure of regularization of a (perturbative) effective action. That is in fact necessary, since usual regularization scheme necessarily introduce a scale in the theory, thus breaking the symmetry. However, as we will show in section IV there might be a way of working around this, if a compensating field for conformal transformations is present in the theory.

II. GRAVITY WITH TORSION AND ITS SYMMETRIES

Recently we have shown [11] (for reviews see [12, 13]) that geometric torsion can be used to define an exact conformal structure on the manifold induced by the transformation laws of the metric tensor $g_{\mu\nu}$ and connection $\Gamma^\lambda_{\mu\nu}$,

$$\begin{align*}
g_{\mu\nu} &\rightarrow \Omega^2 g_{\mu\nu}, \\
\Gamma^\lambda_{\mu\nu} &\rightarrow \Gamma^\lambda_{\mu\nu} + \delta^\lambda_{\mu} \partial_\nu \log \Omega,
\end{align*}$$

(1)

where $\Omega = \Omega(x)$ is an arbitrary scalar field on the spacetime manifold $\mathcal{M}$. The set of local transformations (1) requires the introduction of an additional geometric structure on the spacetime manifold, as the transformed connection is not symmetric anymore, and thus generates torsion. In particular, the transformation (1) generates a longitudinal component of the torsion trace one-form $\propto \partial_\mu \log \Omega$. For this reason we have proposed in [11] to consider the torsion trace as the gauge connection of localized conformal (Weyl) transformations, an analogously to how the electromagnetic field is the gauge connection of the localized $U(1)$ group.

In Ref. [11] we have studied a realization of such a theory, where by using the metric formalism of spacetime with torsion we defined the covariant derivative acting on an arbitrary

\footnote{In this work we take gravity to be classical (non-dynamical). The effects of dynamical gravity will be discussed in a separate publication.}

\footnote{As in a large fraction of literature, we often refer to Weyl transformations [11] as conformal transformations, even though strictly speaking Weyl transformations constitute just one element of the much larger conformal group, which consists of the Poincaré group augmented by special conformal and Weyl transformations.}
representation of the Lorentz group with conformal weight \( \omega \) as,

\[
\bar{\nabla}_\mu \Psi = \nabla_\mu \Psi + (\omega_g - \omega) T_\mu \Psi ,
\]  

(2)

where \( \omega_g \) is the geometric weight (scaling dimension) of \( \Psi \), namely its scaling under the diffeomorphisms, \( x^\mu \to x^\mu + \lambda x^\mu \), and \( \omega \) is the scaling dimension of the field. \( \nabla_\mu \) is the spacetime covariant derivative with torsion, whose connection one can solve for using the metric compatibility condition, to get, when \( \nabla_\mu \) acts on a vector field,

\[
\nabla_\mu V^\lambda = \partial_\mu V^\lambda + \Gamma^\lambda_{\sigma \mu} V^\sigma = \partial_\mu V^\lambda + \{^\lambda_{\sigma \mu}\} V^\sigma + K^{\lambda}_{\sigma \mu} V^\sigma ,
\]

\[
\{^\lambda_{\sigma \mu}\} = \frac{g^{\lambda \rho}}{2} (\partial_\sigma g_{\mu \rho} + \partial_\mu g_{\sigma \rho} - \partial_\rho g_{\mu \sigma}) ,
\]

\[
K^{\lambda}_{\sigma \mu} = T^{\lambda}_{\sigma \mu} + T^{\lambda}_{\sigma \mu \lambda} + T^{\lambda}_{\mu \sigma \lambda} \& T^{\lambda}_{\sigma \mu \lambda} \equiv \Gamma^\lambda_{\sigma \mu} .
\]

The derivative operator \( \bar{\nabla} \) in (2) denotes conformal covariant derivative which commutes with the transformations (1), \( \bar{\nabla}_\mu \Omega = \Omega \bar{\nabla}_\mu \). Sometimes we also use the notation \( \bar{\nabla}_\mu \) to indicate the general relativistic covariant derivative, that is the derivative computed by using Christoffel symbols.

The conformal covariant derivative in Eq. (2) satisfies metric compatibility and conservation of the volume form, since in \( D = 4 \) spacetime dimensions, \( \bar{\nabla}_\mu \epsilon^{\alpha \beta \gamma \delta} = 0 \). Finally (2) satisfies Stokes theorem if the integral is dimensionless (i.e. if its scaling dimension is zero), namely,

\[
\int d\omega = \int \nabla_\mu \omega_{\mu_2 \cdots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} = \int \partial_\Sigma \omega_{\mu_2 \cdots \mu_p} dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_p} ,
\]

(4)

if the scaling dimension of \( \omega_{\mu_2 \cdots \mu_p} \) is equal to 0. This is easily verified using that the factor appearing in the definition of the conformal covariant derivative (2) \((\omega_g - \omega)\) would be \(+p\).

III. WEYL SYMMETRY IN THE QUANTUM THEORY

We begin this section by showing why neither canonical quantisation nor path integral quantisation can break local Weyl symmetry and how this becomes apparent when coupling to (non-dynamical) gravity is accounted for.

\[3\] A generalization to general \( D \) spacetime dimensions is straightforward, \( \bar{\nabla}_\mu \epsilon^{\alpha_1 \alpha_2 \cdots \alpha_D} = 0 \).
Canonical quantisation in curved spaces requires the existence of a time-like vector, $n^\mu$, along which the canonical momentum is defined. This in turn induces a foliation on space-time with spatial slices, $\Sigma$, on which one can rigorously define field quantisation which is, as we shall now show, Weyl invariant on curved spacetimes.

Let us begin by considering a conformally coupled scalar field $\phi(x)$ whose classical action is:

$$S_\phi = \int d^Dx \sqrt{-g} \left\{ -\frac{1}{2} g^{\mu\nu} \left( \partial_\mu \phi + \frac{D-2}{2} T_\mu \phi \right) \left( \partial_\nu \phi + \frac{D-2}{2} T_\nu \phi \right) \right\} = \int d^Dx \sqrt{-g} \left\{ -\frac{1}{2} \left[ \left( g^{\mu\nu} - \frac{n^\mu n^\nu}{\|n\|^2} \right) \nabla_\mu \phi \nabla_\nu \phi + \frac{n^\mu n^\nu}{\|n\|^2} \nabla_\mu \phi \nabla_\nu \phi \right] \right\},$$

(5)

(6)

where $T_\mu$ is the torsion trace 1-form (see Ref. [11]), our metric convention is sign $g_{\mu\nu} = (-1, 1, 1, \ldots)$ and $\phi \rightarrow \tilde{\phi} = \Omega^{-\frac{D-2}{2}} \phi$. Then the usual definition of canonical momentum implies,

$$\pi_\phi = \frac{\delta S_\phi}{\delta (n^\nu \partial_\nu \phi)} = \sqrt{-g} \frac{n_\nu}{\|n\|^2} g^{\mu\nu} \nabla_\mu \phi, \quad \Rightarrow \quad \pi_\phi \rightarrow \tilde{\pi}_\phi = \Omega^{(D-2)/2} \pi_\phi,$$

(7)

(8)

where $\|n\|^2 = g(n, n) = g_{\mu\nu} n^\mu n^\nu$ is the norm-squared of $n^\mu$, whose canonical dimension is 2 (if $t = \text{constant on } \Sigma$ then $n^\mu = \delta_0^\mu$) and $n^\mu$ does not scale (its scaling dimension is zero). Eqs. (7–8) in turn imply, 5

$$[\phi, \pi_\phi] = [\tilde{\phi}, \tilde{\pi}_\phi] = i \mathbb{1},$$

(9)

such that canonical quantisation respects conformal symmetry. This invariance is a simple consequence of the fact the canonically conjugate variables have opposite scaling dimension, see (7–8).

One can easily show that the same is true for a fermionic field $\psi$. To see that consider a fermion whose action is,

$$S_\psi = \int d^Dx \sqrt{-g} \left[ \frac{i}{2} \epsilon^{a}_a \left( \overleftarrow{\nabla}_\mu \psi \gamma^a \overrightarrow{\nabla}_\mu \psi - \overleftarrow{\psi} \gamma^a \overrightarrow{\nabla}_\mu \psi \right) + \mathcal{L}_{\psi,\text{int}} \right],$$

(10)

4 The action (5) is conformal in arbitrary $D$ spacetime dimensions. The quantization procedure would go through for a self-interacting scalar field in $D = 4$, in which the interaction term $S_{\text{int}} = \int d^4x \sqrt{-g} \mathcal{L}_{\text{int}} = -\int d^4x \sqrt{-g} (\lambda/4) \phi^4$.

5 Throughout this work we work in natural units in which, $\hbar = 1 = c$ and 1 in (10) is a shorthand for $\delta^{D-1}(\vec{x} - \vec{x}')$ times a Kronecker delta over internal indices, if the field contains more than one component.
where $L_{\psi,\text{int}}$ is some (conformal) interaction term (which may include coupling to other matter fields but contains no derivatives), $e_\mu^a(x)$ is a tetrad field, $\gamma^a$ are the Dirac matrices on tangent space, $\{\gamma^a,\gamma^b\} = 2\eta^{ab}$, on which the metric is flat Minkowski, $\eta^{ab} = \text{diag}(-1,1,..)$ and $\bar{\psi} = \psi^\dagger \gamma^0$. The action (10) then implies a canonical momentum,

$$\pi_\psi = \sqrt{-g} \frac{n^\mu}{\|n\|^2} e_\mu^a \bar{\psi} \gamma^a,$$

$$\pi_\psi \rightarrow \tilde{\pi}_\psi = \Omega^{(D-1)/2} \pi_\psi.$$  \hspace{1cm} (11)

Taking into account Fermi statistic, the quantisation condition for fermions becomes,

$$\{\psi, \pi_\psi\} = \{\tilde{\psi}, \tilde{\pi}_\psi\} = 1,$$  \hspace{1cm} (12)

proving that canonical quantisation of fermions respects conformal symmetry.

Next we consider conformal symmetry in the context of path integral quantisation. It turns out that the symmetry of quantisation is made manifest in the phase space version of path integral, which is usually taken to define the path integral quantisation.

For an interacting scalar field theory, for example, the vacuum-to-vacuum scattering amplitude reads,  \hspace{1cm} (6)

$$\langle \text{in} | \text{out} \rangle = \int D\phi D\pi_\phi \exp \left\{ i \int d^{D-1} \bar{\chi} d t (\pi_\phi n^\mu \partial_\mu \phi - H_\phi) \right\},$$  \hspace{1cm} (13)

where $n^\mu \partial_\mu \phi = \dot{\phi}$ if the spatial hypersurface $\Sigma$ is chosen to be a constant time hypersurface, $H_\phi(\phi, \pi_\phi) = \pi_\phi n^\mu \partial_\mu \phi - \sqrt{-g} L_\phi(\phi, \partial_\mu \phi)$ is the Hamiltonian density and $L_\phi$ is the Lagrangian density (which for now needs not be specified). With Eqs. (7) and (8) in mind we immediately see that the measure in (13) is Weyl invariant, such that the path integral in (13) and thus also the scattering amplitude must be Weyl invariant if $L_\phi$ is conformal.

The only thorny issue that might spoil conformal symmetry is related to the question of whether the path integral (13) is well defined. That indeed may pose a problem in the sense that the amplitude (13) is generally divergent and since any regularisation of (13) violates Weyl symmetry, it can make it ‘anomalous.’ However, as we argue below, a suitable regularisation scheme can make Weyl symmetry non-anomalous.

\hspace{3cm} 6 For fermions the measure is,

$$D\psi D\pi_\psi = D\psi D\bar{\psi} \det \left( \sqrt{-g} \|n\|^{-2} n^\nu g_{\nu\mu} \gamma^\mu \right),$$

where the determinant is taken both on spinor indices and on spacetime continuous indices. The measure is both diffeomorphism and Weyl invariant.
It is worth remarking that in literature one often finds a path integral formulation in which the integration over the momentum is performed and in which Weyl symmetry of the path integral does not seem manifest. To show that this is not the case, let us perform the Gaussian integral over the (suitably shifted) momentum,

$$
\int D\tilde{\pi} \exp \left\{ i \int d^Dx \frac{n\|n\|^2\tilde{\pi}^2}{\sqrt{-g}} \right\} = \sqrt{\det \left( \sqrt{-g}\|n\|^{-2}\delta_D(x-y) \right)} = \prod_x \left( \frac{\sqrt{-g(x)}}{\|n(x)\|^2} \right)^{\frac{1}{2}}. \quad (14)
$$

With this result in mind, Eq. (13) can be written as,

$$
\langle \text{in} | \text{out} \rangle = \int D\phi e^{iS_\phi}, \quad (15)
$$

where $S_\phi = \int d^Dx \sqrt{-g}L_\phi$ and the barred measure is

$$
\bar{D}\phi = \prod_x d\phi(x) \left( \frac{\sqrt{-g(x)}}{\|n(x)\|^2} \right)^{\frac{1}{2}}, \quad (16)
$$

which is obviously Weyl invariant. Note the dependence on the metric tensor in (16), which is usually omitted from the measure, but is essential for Weyl symmetry.

For systems with constraints – such as gauge theories or gravity – one can also show that the phase space path integral measure is conformal. Since a proper analysis of that question is rather subtle, we relegate the details on how that works in an Abelian gauge theory to Appendix A. The discussion of gravity and non-Abelian gauge theories we postpone to future work.

In this section we have presented cogent arguments in favour of preservation of Weyl symmetry. Namely, since both canonical (or Dirac) and path integral quantisation preserve conformal symmetry, if Weyl (conformal) symmetry is respected by classical theory, it will remain symmetry of the quantum theory provided one uses regularisation scheme that does not violate the symmetry. This is to be contrasted with the literature on conformal anomalies, which states that conformal symmetry is anomalous in the sense that quantum effects generically break it.

In the next section we proceed with further building evidence and show that, if a classical theory is conformal, the conformal Ward identities – when suitably modified – are obeyed in the quantum theory.
IV. WARD IDENTITIES FOR WEYL SYMMETRY

This is the principal section of this paper in which we address the central question: “What is the origin of conformal anomalies?” If we accept the argument of the previous section – according to which quantisation does not break conformal symmetry – then conformal symmetry should be a symmetry of the quantum theory, at least if the classical action is itself Weyl invariant. Instead, the literature claims that conformal symmetry is broken by quantisation, as it is corroborated/evidenced by breaking of the conformal Ward identity, \( \langle T^\mu_\mu \rangle = 0 \). In what follows we show that the problem of conformal anomalies can be solved in theories in which a compensating field for Weyl symmetry is added. The claimed breaking of the quantum identity, \( \langle T^\mu_\mu \rangle = 0 \), should be re-interpreted as breaking of the global (rescaling) symmetry, while the local symmetry is left unbroken.

A. Fundamental Ward identity

The idea we want to pursue here is to consider Weyl symmetry as a gauge transformation, which is compensated by a one-form (Weyl) field which transforms as, \( \mathcal{T}_\mu \to \mathcal{T}_\mu + \partial_\mu \log \Omega(x) \), under Weyl transformations defined in (1). As we show below, this then instigates the following modification of the fundamental Ward identity for Weyl symmetry,

\[
\langle T^\mu_\mu \rangle + \langle \bar{\nabla}_\mu \Pi^\mu \rangle = 0 \quad (17)
\]

where \( \Pi^\mu \) is the dilatation current that sources the Weyl field \( \mathcal{T}_\mu \) and the brackets denote the time ordered product of operators. Equation (17) is the fundamental Ward identity for local Weyl symmetry. The identity simply states that there exists a current \( \Pi^\mu \) whose divergence equals to the trace of the energy-momentum tensor.

If a theory is globally scale invariant, we would be led to the stronger requirement that \( \langle T^\mu_\mu \rangle = 0 \) (since for global scale transformations, \( \partial_\mu \log \Omega = 0 \)). In such a case, at least for flat spacetimes, there exists a conserved current, the dilatation current, which is conserved, namely, \( D^\mu = -T^\mu_\nu x^\nu \). From these observations it then follows that requiring \( \langle T^\mu_\mu \rangle = 0 \) is equivalent to demanding that the global scale transformation is a symmetry of the theory.

\[\text{7 Conformal anomalies is the commonly used term signifying any anomaly associated with the breaking of Weyl symmetry.}\]
which is not the case if the symmetry is e.g. broken by quantum effects. In other words, one can try to construct a classical action by using only the metric tensor and matter fields that is Weyl invariant. In constructing such a theory, however, one usually makes no distinction between global and local coformal symmetry and a breaking of global scale symmetry implies a breaking of local conformal symmetry.

The crucial observation is that in flat space there always exists a dilatation current $D^\mu$ such that,

$$\partial_\mu D^\mu = -T^\mu_\mu,$$

which is divergence-free only if global scale symmetry is realised. Our proposal is to elevate the current $D^\mu$ to the source for the Weyl gauge field $T_\mu$ on general curved spacetimes. If such a Weyl field exists it could be used to generate the source current via $\Pi^\mu = (-g)^{-1/2} \delta S / \delta T_\mu$. Hence the physical meaning of $\Pi^\mu$ is the curved spacetime generalisation of the dilatation current $D^\mu$. Such a current is in general independent of the energy-momentum tensor and moreover – as we shall see – can be written as a local function of the fields. That fact of Nature seems to hint at the existence of a new symmetry and it would be foolish not to make use of it.

As we will see next, there are several operators for which $\Pi^\mu$ is non trivial, for example all dimension four curvature operators with torsion and the scalar field kinetic terms, as in (5). All these contributions can get sourced by a non vanishing energy-momentum tensor trace, such to respect the identity (17).

In order to see that the dilatation current naturally arises and that it can be written as a local function of the fields, let us consider an interacting, scale-invariant field theory (in $D = 4$) of $N$ scalar fields,

$$S_{\{\phi^a\},N} = \int d^4 x \sqrt{-g} \left( -\frac{1}{2} \zeta_{ab} \partial_\mu \phi^a \partial^\mu \phi^b + \frac{\lambda_{abcd}}{4} \phi^a \phi^b \phi^c \phi^d + \frac{\xi_{ab}}{2} \phi^a \phi^b R \right),$$

where $\zeta_{ab}$, $\xi_{ab}$ and $\lambda_{abcd}$ are constants. It is easy to show that the trace of the energy-

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8 The nonlocal expressions, $\Pi^\mu(x) = -\int_{\mathcal{E}(0)} d^4 \bar{x} T^\mu_\nu(\bar{x})$ and $\Pi^\mu(x) = - (\partial^\mu / \Box) T^\mu_\nu(x)$, would obviously do. However, such forms for $\Pi^\mu$ would be obtained by variation of the corresponding nonlocal effective actions. One could make these actions local by introducing an auxiliary field, whose physical meaning is that of a Weyl field $T_\mu$. We may as well bypass the nonlocal step and from the very beginning work with a local formulation in which $T_\mu$ exists as an independent field. That is the approach advocated here.
momentum tensor, \( T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} \), satisfies,
\[
T_{\mu}^\mu = \nabla_{\mu} \left[ (\zeta_{ab} + 12\xi_{ab}) \phi^a \partial^\mu \phi^b \right] \implies D^\mu = -(\zeta_{ab} + 12\xi_{ab}) \phi^a \partial^\mu \phi^b .
\] (20)

Hence our prescription for the dilatation source, namely \( D^\mu = (-g)^{-1/2} \frac{\delta S}{\delta T_{\mu}} \), yields naturally to the modification, \( \partial_{\mu} \rightarrow \bar{\nabla}_{\mu} \), \( R \rightarrow \bar{R} \), in the action (19), where \( \bar{\nabla}_{\mu} \) is the conformal covariant derivative and \( \bar{R} \) is the curvature scalar with torsion. We are then led to the action (5) generalised to \( N \) interacting scalars with non-minimal coupling to the curvature scalar and quartic interactions. Then for all values of \( \zeta_{ab} \) and \( \xi_{ab} \) \((a,b,=1,\ldots,N)\) we would have a Weyl invariant action whose energy momentum tensor is the divergence of a vector current.

In Ref. [11] we showed that the natural candidate for the gauge field of Weyl transformations is torsion trace. Indeed, torsion trace generates scale transformations on vectors that are parallel transported on the manifold and, if the expected transformations \( g_{\mu\nu} \rightarrow \Omega^2 g_{\mu\nu} \), \( T_{\mu} \rightarrow T_{\mu} + \partial_{\mu} \log \Omega \) are performed, one finds that the curvature tensor and the geodesic equation are left invariant. Moreover, since torsion trace is a geometric field, it couples universally to all matter fields. For all these reasons we conclude that the torsion trace can be considered as the Weyl gauge field, and that is what we propose in this paper.

In what follows we consider an interacting, conformal scalar theory and prove that the fundamental Ward identity (17) is implied by the Ehrenfest theorem for the equations of motion. The vacuum-to-vacuum scattering amplitude is
\[
\langle \text{in} | \text{out} \rangle = \int \mathcal{D}\phi e^{iS_\phi},
\] (21)
where \( \mathcal{D}\phi \) is the Weyl invariant measure given in Eq. (13) and \( S_\phi \) is a conformal scalar action, whose kinetic part is given by (5).

Requiring that infinitesimal Weyl transformations, \( \Omega(x) \rightarrow 1 + \omega(x) \), under which the fields transform as,
\[
\phi \rightarrow \phi' = \phi - \frac{D}{2} - 2 \frac{\omega \phi}{\phi}, \quad g_{\mu\nu} \rightarrow g'_{\mu\nu} = g_{\mu\nu} + 2\omega g_{\mu\nu}, \quad \Gamma^\alpha_{\mu\nu} \rightarrow \Gamma'^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} + \delta^\alpha_{\mu} \partial_\nu \omega ,
\] (22)

\footnote{Having in mind the invariant path integral measure for fermions given in footnote 6, the generalisation for fermions of the derivation leading to the identity (27) is straightforward and we do not consider it here separately; see, however, section IV C below.}
do not change the in-out amplitude (21) yields,

\[ \langle \text{in}|\text{out} \rangle = \int \mathcal{D}\phi e^{iS_\phi} \]

\[ = \int \mathcal{D}\phi e^{iS_\phi} \left[ 1 + i \int d^Dx \sqrt{-g} \left( \frac{D - 2}{2\sqrt{-g}} \delta S_{\phi \phi} \omega(x) \phi(x) + \frac{2}{\sqrt{-g}} \frac{\delta S_{\phi \phi}}{\delta g^{\mu\nu}(x)} \omega(x) g^{\mu\nu} \right. \right. \]
\[ \left. \left. + \nabla_\mu \left( \frac{1}{\sqrt{-g}} \frac{\delta S_{\phi \phi}}{\delta T_\mu(x)} \right) \omega(x) \right) \right] . \]

Since this must be true for any arbitrary infinitesimal \( \omega(x) \), Eq. (23) then implies,

\[ \int \mathcal{D}\phi e^{iS_\phi} (T_\mu^\mu + \nabla_\mu \Pi^\mu) = 0 , \]

where,

\[ T_\mu^\mu = \frac{2}{\sqrt{-g}} \frac{\delta S_{\phi \phi}}{\delta g^{\mu\nu}} , \]
\[ \Pi^\mu = \frac{1}{\sqrt{-g}} \frac{\delta S_{\phi \phi}}{\delta T_\mu(x)} , \]

and we have used the Ehrenfest theorem (81) from Appendix B. Upon dividing (24) by \( \langle \text{in}|\text{out} \rangle \) we finally get,

\[ \langle T_\mu^\mu \rangle + \langle \nabla_\mu \Pi^\mu \rangle = 0 , \]

proving thus (17). The angular brackets in (27) denote an expectation value of the time-ordered product and all the derivatives must be evaluated inside the time-ordered product. The identity (27) is the main result of this work. In order to elucidate its meaning, in the remainder of this section we discuss some useful examples.

### B. Conformal anomaly in an interacting scalar theory

Consider now the following self-interacting scalar theory,

\[ S_\phi = \int d^Dx \sqrt{-g} \left( \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - \lambda \phi^4 \right) , \]

which is conformal in \( D = 4 \). The action (28) then implies,

\[ T_\mu^\mu = -\nabla_\mu \phi \nabla^\mu \phi + D \lambda \phi^4 , \]
\[ \nabla_\mu \Pi^\mu = \nabla_\mu \phi \nabla^\mu \phi + \phi \nabla_\mu \nabla^\mu \phi , \]

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and the identity (17) leads to,
\[ \langle \phi (\bar{\nabla}_\mu \bar{\nabla}^\mu \phi + 4\lambda \phi^3) \rangle + (D - 4)\lambda \langle \phi^4 \rangle = (D - 4)\lambda \langle \phi^4 \rangle, \tag{31} \]
where we applied again Eq. (81). Naïvely one might think that the term on the right hand side may generate a finite contribution in dimensional regularisation, as \( \lambda \langle \phi^4 \rangle \propto 1/(D - 4) \).

However, this cannot be so since \( \lambda \langle \phi^4 \rangle \) contributes to the energy-momentum tensor as,
\[ \langle T_{\mu\nu} \rangle \supset g_{\mu\nu} \lambda \langle \phi^4 \rangle, \]
such that any primitive divergence must be regulated, implying that \( \lambda \langle \phi^4 \rangle \) must be finite. This immediately implies that, after regularisation and when the limit \( D \rightarrow 4 \) is taken, the term \( (D - 4)\lambda \langle \phi^4 \rangle \) in (31) will vanish and thus the fundamental Ward identity (17) will be respected.

C. Conformal anomaly in Yukawa theory

In order to further motivate the identity (17), let us consider the following Yukawa theory, whose action is conformal in \( D = 4 \),
\[ S_{Yu} = \int d^Dx \sqrt{-g} \left( \frac{1}{2} \bar{\nabla}_\mu \phi \bar{\nabla}^\mu \phi + i \frac{1}{2} (\bar{\psi} \gamma_\mu \bar{\nabla}^\mu \psi - y \phi \bar{\psi} \psi) \right), \tag{32} \]
where \( \psi \) and \( \phi \) represent fermionic and scalar fields and \( y \) is a (constant) Yukawa coupling.

For this theory the energy-momentum tensor and divergence of the torsion source are given by,
\[ T_{\mu\nu} = i \left( \bar{\psi} \gamma_\mu \bar{\nabla}_\nu \psi \right) + \bar{\nabla}_\mu \phi \bar{\nabla}_\nu \phi - g_{\mu\nu} \left( \frac{1}{2} \bar{\nabla}_\alpha \phi \bar{\nabla}^\alpha \phi + i \frac{1}{2} (\bar{\psi} \gamma_\alpha \bar{\nabla}^\alpha \psi - y \phi \bar{\psi} \psi) \right), \tag{33} \]
\[ \bar{\nabla}_\mu \Pi^\mu = \bar{\nabla}_\mu \phi \bar{\nabla}^\mu \phi + \phi \bar{\nabla}_\mu \bar{\nabla}^\mu \phi. \tag{34} \]

Summing (33) and (34) then leads to,
\[ -\frac{3}{2} \langle \bar{\psi} (i\bar{\nabla} \psi - y\phi \bar{\psi}) \rangle + \frac{3}{2} \langle (i \bar{\nabla} \bar{\psi} + y \phi \bar{\psi}) \psi \rangle + \left( \phi \left( \bar{\nabla}_\mu \bar{\nabla}^\mu \phi + y \bar{\psi} \psi \right) \right) + (D - 4)y \langle \phi \bar{\psi} \psi \rangle. \tag{35} \]

The first three angular brackets in (35) constitute expectation values of composite operators, each containing a product of a field and an equation of motion for either \( \psi \), \( \bar{\psi} \) or \( \phi \). Therefore, all of them must vanish by (the Yukawa-theory version of) the Ehrenfest theorem, c.f. Eq. (81). Similarly as above one can argue that the term \( (D - 4)y \langle \phi \bar{\psi} \psi \rangle \) in (35) must vanish in \( D = 4 \). Indeed, if that was not the case it would have lead to a nonvanishing divergence in the energy momentum tensor which contains a term of the form, \( \langle T_{\mu\nu} \rangle \supset g_{\mu\nu} y \langle \phi \bar{\psi} \psi \rangle \).
D. Conformal anomaly in an interacting Yang-Mills

Another type of interacting theory that occurs in the standard model is a gauge theory that couples to a charged scalar current. The action is,

\[ S_{YM} = \int d^D x \sqrt{-g} \text{Tr} \left( -\frac{1}{4} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} + g^{\mu\nu} (\bar{D}_\mu \phi)^\dagger (\bar{D}_\nu \phi) \right) \]  

(36)

where the gauge field is taken to be in adjoint representation and the scalar in fundamental representation (just like as they are in the standard model), and the Tr is taken on the group indices. The field strength is given by,

\[ F_{\mu\nu} = \bar{\nabla}_\mu A_\nu - \bar{\nabla}_\nu A_\mu + e [A_\mu, A_\nu] = F_{\mu\nu}^a \lambda^a_{\text{adj}}, \quad F_{\mu\nu}^a = \bar{\nabla}_\mu A_\nu^a - \bar{\nabla}_\nu A_\mu^a + e f^{abc} A_b^\mu A_c^\nu, \]  

(37)

where \( f^{abc} \) are the adjoint representation structure constants, \( \lambda^a_{\text{adj}} \) the group generators also in the adjoint representation, \( e \) is the gauge coupling constant and

\[ \bar{\nabla}_\mu A_\nu^a = \partial_\mu A_\nu^a + \frac{D-4}{2} T_{\mu A_\nu^a} \]  

(38)

is the conformal exterior derivative as it acts on the gauge field. Note that the conformal derivative become the usual exterior derivative in \( D = 4 \), but it breaks gauge symmetry away from \( D = 4 \). Since we are ultimately interested in \( D = 4 \), the conformal derivative for gauge fields was introduced for regularisation purposes only. Varying the action \( S_{YM} \) with respect to the matter fields \( A_\mu, \phi \) (here \( \phi = \phi^a A^a_{\text{fund}} \), \( \lambda^a_{\text{fund}} \) are the basis matrices of the fundamental representation of the gauge group) and \( \phi^\dagger \) gives the following equations of motion,

\[ \bar{D}_\mu (\bar{D}^\mu \phi) = 0, \quad \bar{D}_\mu (\bar{D}^\mu \phi)^\dagger = 0, \]  

(39)

\[ \bar{D}_\mu = \bar{\nabla}_\mu - \frac{D-2}{2} T_\mu + i e A_\mu, \quad \bar{D}^\mu = g^{\mu\nu} (\bar{\nabla}_\nu + \frac{D-2}{2} T_\nu + i e A_\nu) \]  

\[ \bar{D}_\mu F^{\mu\nu} = i e g^{\mu\nu} \left[ \phi^\dagger \bar{D}_\nu \phi - (\bar{D}_\nu \phi)^\dagger \phi \right], \]  

(40)

where \( \bar{\nabla}_\mu \) is the general relativity covariant derivative, that is the space-time covariant derivative computed using the Christoffel symbols. When acting on a scalar, it equals

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10 The coupling to fermions can be also included and we leave it to the reader as an exercise. One can in fact consider the action \( S_{YM} \) with minimal coupling to gauge fields, \( \nabla_\mu \rightarrow D_\mu \). Proceeding in the analogous way as we do here, one would then find an extra contribution to Eq. \( (40) \) proportional to \( (D-4) \langle A_\mu \bar{\psi} \gamma^\mu \psi \rangle \), which again drops in \( D = 4 \) since the operator \( \langle A_\mu \bar{\psi} \gamma^\mu \psi \rangle \) must be finite.
the partial derivative $\hat{\nabla}_\mu \phi = \partial_\mu \phi$, while when acting on a vector it has the expression, $\hat{\nabla}_\mu V^\mu = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} V^\mu)$. Note that in (39–40) the conformal covariant derivative always comes with conformal weight of the associated tensor density. Thus conformal weight of $\phi$ is $-(D-2)/2$, of $\sqrt{-g} D^\mu \phi$ is $+(D-2)/2$ while conformal weight of $\sqrt{-g} F^{\mu\nu}$ is $+(D-4)/2$.

On the other hand, varying (36) with respect to the geometric fields $g^{\mu\nu}$ and $T_\mu$ results in,

$$T_{\mu\nu} = \text{Tr} \left\{ F_{\mu\nu} F_{\rho\sigma} g^{\alpha\beta} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F_{\gamma\delta} g^{\alpha\gamma} g^{\beta\delta} + 2 \langle \bar{D}_\mu \phi \rangle \bar{D}_\nu \phi - g_{\mu\nu} \left[ g^{\alpha\beta} (\bar{D}_\alpha \phi) \right] \right\}.$$  \hspace{1cm} (41)

$$\Pi^\mu = \text{Tr} \left\{ - \frac{D-4}{2} A_\mu F_{\gamma\delta} g^{\mu\delta} g^{\mu\gamma} + \frac{D-2}{2} g^{\mu\nu} \left[ \phi^\dagger (D_\nu \phi) + (D_\nu \phi)^\dagger \phi \right] \right\}. \hspace{1cm} (42)$$

Taking a trace and expectation value \(^{11}\) of $T_{\mu\nu}$ we get,

$$\langle T^\mu_\mu \rangle = \text{Tr} \left\{ \frac{D-4}{4} \langle F_{\mu\nu} F_{\rho\sigma} \rangle g^{\mu\rho} g^{\nu\sigma} - (D-2) \langle \langle \bar{D}_\mu \phi \rangle \rangle \bar{D}_\nu \phi \rangle \right\} \hspace{1cm} (43)$$

while taking a (conformal) \(^{12}\) covariant divergence and expectation value of $\Pi^\mu$ we obtain,

$$\langle \hat{\nabla}_\mu \Pi^\mu \rangle = \text{Tr} \left\{ - \frac{D-4}{4} \langle F_{\mu\nu} F_{\rho\sigma} \rangle g^{\mu\rho} g^{\nu\sigma} + (D-2) \langle \langle \bar{D}_\mu \phi \rangle \rangle \bar{D}_\nu \phi \rangle \right\} \hspace{1cm} (44)$$

$$- \frac{D-4}{2} i e \langle A_\mu \left[ \phi^\dagger (\bar{D}_\mu \phi) - (\bar{D}_\mu \phi)^\dagger \phi \right] \rangle. \hspace{1cm} (45)$$

By combining (41) and (42) we see that the fundamental identity (17) is not quite satisfied,

$$\langle \hat{\nabla}_\mu \Pi^\mu \rangle + \langle T^\mu_\mu \rangle = - \frac{D-4}{2} i e \text{Tr} \left\langle A_\mu \left[ \phi^\dagger (\bar{D}_\mu \phi) - (\bar{D}_\mu \phi)^\dagger \phi \right] \right\rangle + \frac{D-4}{2} e \langle A_\mu \bar{\psi} \gamma^\mu \psi \rangle. \hspace{1cm} (45)$$

To see that the terms on the right hand side must vanish, note that the last term in the (expectation value of the) energy-momentum tensor (41) can be decomposed as,

$$\text{Tr} \left\langle (\bar{D}_\mu \phi) (\bar{D}_\mu \phi) \right\rangle = \text{Tr} \left\langle (\hat{\nabla}_\mu \phi)^\dagger (\hat{\nabla}_\mu \phi) \right\rangle - i e \text{Tr} \left\langle A_\mu \left[ \phi^\dagger (\bar{D}_\mu \phi) - (\bar{D}_\mu \phi)^\dagger \phi \right] \right\rangle. \hspace{1cm} (46)$$

Upon renormalisation is exacted both terms must be finite. To see that observe that the first term in (46) is present also when the gauge field vanishes and therefore must be finite

\(^{11}\) In taking an expectation value one ought to integrate over fields fluctuations weight by the action. By making use of the phase space version of the path integral quantisation, in Appendix A we show that such a quantisation respects conformal symmetry in Abelian gauge theories in the sense that the corresponding path integral measure is conformal. Here we assume that that is also the case with the measure of non-Abelian gauge fields, but leave the proof to future publication.

\(^{12}\) The conformal weight of $\sqrt{-g} \Pi^\mu$ is zero, such that taking covariant and conformal covariant derivatives of $\Pi^\mu$ coincide.
by itself and thus the second term must be also finite (because the whole energy-momentum tensor must be finite), implying that the right hand side of (45) must vanish and therefore the fundamental conformal Ward identity (17) is satisfied. Analogously, since \( A_\mu \bar{\psi} \gamma^\mu \psi \) also appears in the energy momentum tensor, with a contribution that vanishes upon setting \( A_\mu \) to zero, and so it must be finite by itself to guarantee that the energy momentum tensor is finite.

In conclusion, we have considered all types of interacting field theories that occur in the standard model and we have showed that the fundamental identity (17) is satisfied for all of them, implying that it is also satisfied in the standard model.

E. Boundary terms and local anomaly

There are terms that contribute to \( \langle T^\mu_\mu \rangle \) as total derivatives. To expound on the meaning of such terms, let us consider the scalar 1-loop effective action around a general gravitational background which before regularisation is of the form [12],

\[
\Gamma_\phi = \frac{1}{4\pi^2} \int d^Dx \sqrt{-g} \left[ \Gamma \left( 1 - \frac{D}{2} \right) \alpha \bar{R}^\frac{D}{2} + \Gamma \left( 2 - \frac{D}{2} \right) \beta C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta} + \gamma \mathcal{E}_4 \left( \alpha \bar{R}^\frac{D-4}{2} \right) \right],
\]

where \( \bar{R} \) is the Ricci scalar formed from the curvature tensor with torsion, \( C_{\alpha\beta\gamma\delta} \) is the Weyl tensor which is independent of the torsion trace, \( \mathcal{E}_4 \) is the Euler density which is in four dimensions a total divergence [17] and \( \alpha, \beta \) and \( \gamma \) are constants. The action (47) is divergent (in the sense that it yields divergent contributions to the Einstein’s equation) and thus it ought to be renormalised. The first step in the renormalisation procedure is to identify the finite parts of the action.

To do that let us firstly analyse the contribution to the stress-energy tensor from the Euler density (48). Its variation gives a finite contribution to the stress-energy tensor and as such does not need any counter term. To see that let us vary the contribution of \( \mathcal{E}_4 \) to the effective action (47). We have,

\[
\frac{\delta}{\delta g^{\rho\sigma}(z)} \int d^Dx \sqrt{-g} \epsilon^{\mu\nu\lambda\sigma} \epsilon_{\alpha\beta\gamma\delta} \bar{R}^\alpha_{\gamma\mu\nu} \bar{R}^\beta_{\delta\lambda\sigma} \Rightarrow \frac{1}{D!} \epsilon^{\mu\nu\lambda\sigma\rho_1\cdots\rho_{D-4}\epsilon_{\alpha\beta\gamma\delta}} \bar{R}_\epsilon \gamma_{\mu\nu\rho_1\cdots\rho_{D-4}} \bar{R}_\gamma \delta_{\lambda\sigma},
\]

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\[
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\]
were we dropped the following two terms,
\[
\frac{\delta \left( \sqrt{-g} \epsilon^{\mu \nu \lambda \sigma \rho_1 \cdots \rho_{D-4}} \right)}{\delta g^{\rho \tau}(z)} \quad \text{&} \quad \sqrt{-g} \epsilon^{\mu \nu \lambda \sigma \rho_1 \cdots \rho_{D-4}} \frac{\delta \left( \bar{R}^\alpha_{\gamma \mu \nu} \bar{\bar{R}}^\beta_{\delta \lambda \sigma} \right)}{\delta g^{\rho \tau}(z)},
\]
the first one because the factors of $\sqrt{-g}$ cancel between $\sqrt{-g}$ and the Levi-Civita tensor and the second one because it vanishes due to the Bianchi identities. The term that is left in (50) is identically zero in $D = 4$ since in four dimensions $\epsilon_{\alpha \beta \gamma \delta}$ yields contributions that are independent of the metric tensor. Taking account of this we finally arrive at the expression,
\[
\frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{\rho \tau}(z)} \int d^D x \sqrt{-g} \epsilon^{\mu \nu \lambda \sigma \rho_1 \cdots \rho_{D-4}} \epsilon_{\rho_1 \cdots \rho_{D-4} \alpha \beta \gamma \delta} \bar{R}^\alpha_{\gamma \mu \nu} \bar{\bar{R}}^\beta_{\delta \lambda \sigma} = \frac{D - 4}{D} g_{\rho \tau} E_4,
\]
which can be verified directly from (50) by evaluating $\delta \left( \epsilon_{\rho_1 \cdots \rho_{D-4} \alpha \beta \gamma \delta} \right)/\delta g^{\rho \tau}(z)$. This shows that we get a finite contribution to the stress-energy tensor from the divergent contribution proportional to the Euler density term in the effective action (47) and thus we do not have to add a counter term to renormalise it.

To be consistent, we should also check that the same term gives a finite contribution to the Weyl field source, $\Pi^\mu$. Indeed, upon noticing that $E_4 = \bar{\nabla}_\mu V^\mu$, where $V^\mu$ has scaling dimension $-4$ under Weyl transformations, we can see that this is the case. Using the conformal Stokes theorem (4) one can show that in general $D$,
\[
\bar{\nabla}_\mu V^\mu = \frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} V^\mu \right) - (D - 4) T_\mu V^\mu,
\]
(50)
since the length dimension of $V^\mu$ is $-3$ (it contains 3 derivatives acting on the metric), and thus $\int d^D x \sqrt{-g} \bar{\nabla}_\mu V^\mu$ is only dimensionless in $D = 4$. We can then conclude that, since the first term in Eq. (50) is a boundary term in any dimension, the Euler density contribution to the torsion source is,
\[
\frac{1}{\sqrt{-g}} \frac{\delta}{\delta T_\mu} \int d^D x \sqrt{-g} E_4 = -(D - 4) V^\mu,
\]
(51)
which shows that the fundamental Ward identity (17) is in fact respected by this contribution. Note that this is not possible to achieve in a theory containing the metric only, since necessarily the Gauss-Bonnet contribution is finite and spoils the identity $\langle T^\mu_\mu \rangle = 0$.

However, this is not yet the end of the story. In order to renormalise (47) one has to make all the terms (except possibly $E_4$) finite in $D = 4$ and the only way of doing that within
dimensional regularisation is to add scale dependent counterterms that in this way introduce a scale dependence in the renormalized action, thus breaking conformal symmetry. In the next section we discuss how to cure such an apparent violation of conformal Ward identities.

F. The role of scale $\mu$

In subsection IV E we have seen that – in presence of geometric or scalar field condensates, perturbative renormalised effective actions typically contain scale dependent terms that may violate the conformal Ward identity (17). Indeed, consider the first term in the effective action (47). Due to the presence of $\Gamma(1-D/2) \propto 1/(D-4)$, that term is divergent in $D = 4$. Using standard dimensional regularisation, by which one adds a scale dependent counterterm $\propto \frac{1}{(D-4)}$ to remove the divergence, one ends up with the following renormalised, scale dependent, effective action,

$$\Gamma_{\text{eff}} = \int d^4x \sqrt{-g} L_{\text{eff}} = \int d^4x \sqrt{-g} \tilde{\alpha} \tilde{R}^2 \ln \left( \frac{\bar{R}}{\mu^2} \right),$$

(52)

where $\tilde{\alpha}$ is a (finite) coupling constant that does not depend on $\mu$. Upon varying this action with respect to $g_{\mu\nu}$ and taking a trace, then upon varying with respect to $T_{\mu}$ and taking a divergence, one ends up with,

$$g_{\mu\nu} \frac{2}{\sqrt{-g}} \frac{\delta \Gamma_{\text{eff}}}{\delta g^{\mu\nu}} + \nabla_\alpha \frac{1}{\sqrt{-g}} \frac{\delta \Gamma_{\text{eff}}}{\delta T_\alpha} = 2\tilde{\alpha} \tilde{R}^2 = -\mu \frac{dL_{\text{eff}}}{d\mu}.$$

(53)

This shows that a standard renormalisation procedure (such as dimensional regularisation) leads to scale-dependent effective action that breaks the conformal Ward identity (53), see Eq. (17). In order to understand how to deal with such a situation without breaking the local Weyl symmetry, we should carefully ponder on the meaning of the renormalisation scale $\mu$.

From the perspective of renormalisation, $\mu$ is not a physical quantity, and it can be chosen to correspond to any scale we pick to probe the physics. On the other hand, renormalisation group flow is constructed by demanding that changing this scale does not change the physics, which is expressed by the following requirement on an effective action $\Gamma$,

$$\mu \frac{d\Gamma[\mu]}{d\mu} = 0.$$  

(54)
This means that the effective action $\Gamma[\mu]$, and consequently all observables derived from it, cannot depend on $\mu$.\textsuperscript{13}

As a consequence, we typically require $\mu$ to be a spacetime constant, $\partial_\alpha \mu = 0$. That is just a probing mass scale which does not change the dynamics. In this context of this work however, $\partial_\alpha \mu = 0$ is not a gauge invariant statement, in the sense that it transforms non-trivially under Weyl transformations. Therefore, it is much more natural to demand instead,

$$\bar{\nabla}_\alpha \mu = \partial_\alpha \mu + \frac{D-2}{2} T_\alpha \mu = 0,$$

which is the conformally invariant expression for spacetime constancy of $\mu$. In writing (55) we have assumed that the conformal dimension of $\mu$ is equal to that of a classical scalar field, \textit{i.e.} $w_\mu = -(D-2)/2$, which is the natural choice. When Eq. (55) is enforced on $\mu$ we have (on-shell),

$$T_\alpha = -\frac{2}{D-2} \partial_\alpha \log(\mu).$$

Note that $T_\alpha$ is purely longitudinal, \textit{i.e.} its transverse part is zero. Eq. (56) does not really mean that $\mu$ is spacetime dependent. Indeed all that dependence is pure gauge as there always exists a (gauge) frame defined by $T_\alpha = 0$ in which $\mu$ is truly constant. The true spacetime constancy of $\mu$ is therefore retained by the gauge invariant statement, $\frac{1}{\mu} \bar{\nabla}_\alpha \mu = 0$ and any other choice of $\mu$ (including the standard one) will be gauge dependent and thus will break conformal symmetry. The problem with the standard choice $\mu = \text{constant}$ in the effective action is that off-shell $\mu = \mu[\mathcal{T}_\mu]$ and neglecting that dependence would be simply wrong (in the sense that it breaks conformal symmetry). In other words, the gauge $\mathcal{T}_\mu = 0$ is allowed only after variation is exacted, \textit{i.e.} on-shell.

That (55) is the natural choice can be argued as follows. Since at low energies the Weyl symmetry is (spontaneously) broken, there exists a Goldstone boson $\theta$ which transforms under the Weyl scaling as, $\theta \rightarrow \theta + \log \Omega$. Then the quantity $e^{\frac{D-2}{2} \theta} \mu$ is a (gauge-invariant)

\textsuperscript{13} This statement should not be mixed with the statement that \textit{e.g.} coupling constants (which determine physical scattering rates) run with some physical scale. When such statements are made, it is always assumed that the scale has a physical meaning, an example being an invariant energy in scattering processes. In fact, one should understand (55) as follows. There exists a map that generates a characteristic flow of the effective action which maps $\Gamma[\mu, \lambda_i, \psi_j]$ onto $\Gamma[\mu', \lambda'_i, \psi'_j]$ such that $\Gamma[\mu, \lambda_i, \psi_j] = \Gamma[\mu', \lambda'_i, \psi'_j]$. Since in general the couplings $\lambda_i \neq \lambda'_i$ and the fields $\psi_j \neq \psi'_j$, the form (\textit{i.e.} the dependence on the couplings and fields) of $\Gamma[\mu, \lambda_i, \psi_j]$ and $\Gamma[\mu', \lambda'_i, \psi'_j]$ are not in general the same. Nevertheless, both actions are equivalent in the sense that they describe exactly the same physics.
scalar field combination and therefore can be used as an invariant probe of some physical scale.

As a consequence of the requirement (55), the torsion source $\Pi^\mu$ acquires an extra contribution $\Delta_\mu \Pi^\alpha$ of the form, \(^14\)

$$\Delta_\mu \Pi^\alpha = -\frac{2}{D-2} \frac{\partial}{\partial \mu} \frac{1}{\Box} \frac{\partial L_{\text{eff}}}{\partial \mu}.$$  \hspace{1cm} (57)

Taking a (conformal) divergence of $\Delta_\mu \Pi^\alpha$ yields,

$$\bar{\nabla}_\alpha (\Delta_\mu \Pi^\alpha) = \bar{\nabla}_\alpha (\Delta_\mu \Pi^\alpha) = -\frac{2}{D-2} \mu \frac{\partial L_{\text{eff}}}{\partial \mu},$$  \hspace{1cm} (58)

where we took account of the fact that the conformal weight of torsion source $\sqrt{-g} \Pi^\alpha$ is zero.

Adding (58) to (53) and taking $D = 4$ we obtain that the fundamental Ward identity (17) is satisfied. This is not a coincidence and in fact it works in general. Namely, demanding that regularisation respects conformal symmetry by requiring that any scale dependence introduced by regularisation procedure is conformally invariant will result in an effective action which satisfies the fundamental conformal Ward identity (17).

In the above procedure we have introduced non-locality in the source of torsion such that we do no longer meet the requirement that $T_\mu^\mu = -\nabla_\mu D^\mu$, with $D^\mu$ being a local dilatation current. This signals a breaking of the global Weyl symmetry, meaning that the physical state of the theory will not be invariant under dilatations or special conformal transformations, but the local Weyl symmetry will in fact not be violated.

Finally, one might worry that this unphysical dependence on $\mu$ changes the equations of motion for $\theta$. This is not the case however, since $\theta$ is a pure gauge field and, since the equations of motion are gauge independent, once on-shell one can always go to the gauge $\theta = \text{const}$. In this gauge, the Goldstone degree of freedom is incorporated in some other field (which will typically be the metric tensor) and since the equations of motion are gauge independent the equation of motion for $\theta$ is just a constraint implied by the equations of motion of the field that has “eaten-up” the Golstone field.

\(^14\) Note that in Eq. (57) the derivative operator and inverse Laplacian are torsion independent and contain only contributions on the metric. However, since the scaling dimension of $\mu \frac{\partial L_{\text{eff}}}{\partial \mu}$ is equal to $-D$, the conformal divergence of $\Delta_\mu \Pi^\alpha$ satisfies $\bar{\nabla}_\alpha \Delta_\mu \Pi^\alpha = \bar{\nabla}_\alpha \Delta_\mu \Pi^\alpha = \mu \frac{\partial L_{\text{eff}}}{\partial \mu}$, as in Eq. (58). This in turn implies that $\Delta_\mu \Pi^\alpha$ scales as $\mu \frac{\partial L_{\text{eff}}}{\partial \mu}$, that is with conformal weight $-D$, even though this is not manifest in Eq. (57).
G. Higher order Ward identities

To conclude, we shall list the higher order Ward identities that follow from invariance of the quantum theory under conformal transformations. Some identities – in particular the identities involving a three-point function – can receive anomalous contributions from interacting fermions. Although we do not analyse such contributions in detail here, we are confident that the arguments provided in this paper apply also to conformal Ward identities for higher point functions. Once again, this belief is motivated by the fact that quantisation on curved spacetimes is Weyl invariant.

Consider the following $n$-point function of the theory,

$$iG^{(n)}(x_1, \ldots, x_n) \equiv \langle T \{ \phi(x_1) \cdots \phi(x_n) \} \rangle = \int D\phi e^{iS(\phi)} \phi(x_1) \cdots \phi(x_n), \quad (59)$$

and performing a conformal transformation, $\phi \to (1 + \gamma \omega)\phi$ we would find,

$$\delta iG^{(n)}(x_1, \ldots, x_n) = \gamma \int \omega(x) \sum_{i=1}^{n} \frac{\delta(x - x_i)}{\sqrt{-g(x_i)}} iG^{(n)}(x_1, \ldots, x_n) \quad (60)$$

$$= i \int D\phi e^{iS(\phi)} \int_x \left( -\frac{D - 2}{2} \sqrt{-g} \frac{\delta S}{\delta \phi(x)} \omega(x) \phi(x) - \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^\mu\nu(x)} \omega(x) g^\mu\nu(x) \right) \phi(x_1) \cdots \phi(x_n), \quad (61)$$

where $\int_x \equiv \int d^D x \sqrt{-g}$, and $\gamma$ is the scaling dimension of the quantum field $\hat{\phi}$. The first term in (61) vanishes due to the Ehrenfest theorem (78) and hence – up to a boundary term – we are left with,

$$\langle T \left\{ \left[ \nabla_\mu \Pi^\mu(x) + T^\mu_\mu(x) \right] \phi(x_1) \cdots \phi(x_n) \right\} \rangle = -\gamma \sum_{i=1}^{n} \frac{\delta(x - x_i)}{\sqrt{-g(x_i)}} G^{(n)}(x_1, \ldots, x_n). \quad (62)$$

Upon expanding the time-ordering operator in (62) and using the fact that $\delta(y^0 - x^0) [\Pi^0(y), \phi(x)] = -[(D - 2)/2] i \delta^D(x - y)$, it is easy to see that Eq. (62) can be rewritten as,

$$\nabla_\mu \langle T \left\{ \Pi^\mu(x) \phi(x_1) \cdots \phi(x_n) \right\} \rangle + \langle T \left\{ T^\mu_\mu(x) \phi(x_1) \cdots \phi(x_n) \right\} \rangle \quad (63)$$

$$= - \left( \frac{\gamma - (D - 2)}{2} \right) \sum_{i=1}^{n} \frac{\delta(x - x_i)}{\sqrt{-g(x_i)}} G^{(n)}(x_1, \ldots, x_n).$$

The factor $\gamma - (D - 2)/2$ on the right hand side of Eq. (63) is just the anomalous dimension of $\hat{\phi}$.

A proper understanding of the higher conformal Ward identities (63) is important and we intend to consider their full ramifications for conformal anomalies in a separate publication.
V. CONCLUSION

A vast amount of literature agrees that conformal (Weyl) symmetry is a very special local symmetry in the sense that, if it is symmetry of a classical theory, it generically gets broken by quantum effects. It is interesting to note that all other local symmetries that are realised in Nature – including gauge symmetries and diffeomorphisms – are widely believed to be respected at the quantum level provided they are realised classically, with the notable exception of chiral symmetry.

In this work we argue that there is a way of preserving Weyl symmetry at the quantum level. The trick is to add to the action a dilatation current (that is independent of the stress-energy tensor) and a corresponding compensating Weyl field. In light of our former work [11] (see section II for a brief introduction into gravity theory with torsion) the torsion trace appears as the natural candidate for the Weyl field since torsion makes gravity conformal and in addition it is geometric and thus couples universally to all matter fields, respecting thus the equivalence principle.

In fact the usual conformal anomalies found in literature can be reinterpreted as the anomalies associated with breaking of global scaling symmetry. Indeed, we argue in section IV that a non-vanishing of the trace of the energy-momentum tensor, \( \langle T_{\mu}^{\mu} \rangle \neq 0 \), is a telltale sign for a breakdown of global rescaling symmetry by quantum effects.

On the other hand, the story of local Weyl symmetry is completely different. Indeed, in section IV we show that local Weyl symmetry needs not get broken by quantum effects. In particular there we show that, in presence of a compensating Weyl field \( T_{\mu} \), the conformal Ward identities get modified, but remain satisfied. For example, the fundamental conformal Ward identity gets modified to (17), i.e. any anomalous contribution to the trace of the energy-momentum, \( \langle T_{\mu}^{\mu} \rangle \), is not interpreted as a violation of local Weyl symmetry, but instead it just sources divergence of the current \( \Pi^{\mu} \), which acts as a source for the Weyl field and can be thus interpreted as the curved space generalisation of the dilatation current. Unlike in flat spacetimes however, where the dilatation current depends on the energy-momentum tensor, in curved spaces it generally does not because in curved spaces there exists an independent Weyl one-form which can be naturally identified with the torsion trace.

We proceed and in IV B, IV C and IV D we consider several simple interacting theories
of matter fields that are classically conformal in four spacetime dimensions and show that they all satisfy the fundamental Ward identity (17). This then lends strong support to the statement that, if the standard model is made classically conformal (which can be done e.g. by replacing the Higgs mass term, $\mu^2 H^\dagger H$, by an interaction term with a scalar singlet $\phi$, $-\lambda_{\phi}H\phi^2 H^\dagger H$), it will remain conformal at the quantum level. The importance of this statement cannot be underestimated for building conformal extensions of the standard model, in which the symmetry is respected in the ultraviolet, but ‘broken’ by scalar condensate(s) mediated by quantum effects such as in the Coleman-Weinberg mechanism.

Furthermore, we point out in [IVF] that - even though quantisation procedures in general respect conformal symmetry (as shown in [III]) – it is generically broken by all standard regularisation procedures. We first point out that, in light of Weyl transformations, scale dependence introduced by conventional regularisation schemes can be thought of as a constant scale in a particular conformal gauge (given by $T^\mu_\alpha \equiv \partial_\alpha \theta = 0 \implies \theta = \text{constant}$). Next we show that, elevating the usual notion of spacetime constancy ($\partial_\alpha \mu = 0 \implies \mu = \text{constant}$) to the suitable gauge independent notion ($\nabla_\alpha \mu = 0 \implies T_\alpha = -[2/(D-2)]\partial_\alpha \ln(\mu)$) restores the conformal Ward identity (17) implied by any effective action. This is of course not a coincidence, as our procedure restores Weyl symmetry in regularisation procedures. It is interesting to point out that one can pick the gauge $\mu = \text{constant}$ at the level of equations of motion (on-shell), such that regarding any physical quantity $O_i[\mu]$ obtained by variation of an effective action, there is no difference whatsoever between our procedure and conventional regularisation procedures. Indeed, since $O_i[\mu]$ is an on-shell quantity, one can choose the scale $\mu$ in $O_i[\mu]$ to be constant (which is what is usually done) and then set $\mu$ to a convenient physical quantity (such as an invariant energy $E$ in scattering processes) to finally obtain $O_i[E]$. Such a procedure fully respects the conformal Ward identity (17). It is worth pointing out that our proposal to modify standard regularisation schemes is supported by our consideration of a representative sample of interacting quantum field theories in sections [IVB] [IVD] in the sense that the operator methods and effective action approach yield equivalent conformal Ward identities.

The Gauss-Bonnet term $E_4$ (48) in section [IVF] is a geometric scalar that contributes to the conformal anomaly and deserves additional reflection. Namely, $E_4$ is topological in $D = 4$, which means that it can be written as a derivative of a vector, $E_4 = \nabla_\mu \gamma_4^\mu$. This then implies that the Gauss-Bonnet integral, which in Euclidean spaces gives the Euler
characteristic of the manifold that can be represented as an alternating sum of the Betti numbers, is related to the topology of the spacetime manifold and therefore, ultimately, to the quantum state of the gravitational field. Then quantum states with different Betti numbers fall into distinct topological classes which might be of fundamental importance in specifying the vacuum state of the gravitational field and, via its connection to the dilatation current, it can lead to creation of particles.

In order to get a better idea on what it all means, let us recall the well known the chiral anomaly in particle physics, which states that the chiral current in the standard model is anomalous in the sense that its divergence is sourced by the Chern-Simons’ density (which is a pseudo-scalar proportional to the product of electric and magnetic fields). Then a change in the Chern-Simons’ number (which is the spatial volume integral of the Chern-Simons’ density) signals creation of chiral fermions out of the Dirac sea. Analogously, a change in the Euler characteristic signifies creation of particles associated with the dilatation current $\Pi^\mu$, which are scalar particles. These particle deserve a name and can be called conformalons or weylons.

While we show in section III that quantisation of simple constrained systems is conformal – see Appendix A for a consideration of quantisation of an Abelian gauge theory – we postpone the analysis of non-Abelian gauge theories and gravity to future work.

Finally, for completeness in IV G we show how to derive higher order conformal Ward identities, but leave to future work to rigorously prove that they are indeed satisfied.

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Appendix A: Abelian Gauge Theory

Gauge theories and gravity constitute constrained systems and therefore their quantization requires a special attention. For simplicity here we consider the symplest case: path
integral quantization of an Abelian gauge theory, but we expect analogous results to hold
for non-Abelian gauge theories and gravity.

For a theory with constraints Dirac quantisation \cite{14} is the method of choice. In this
procedure, instead of replacing Poisson brackets of quantities \( A, B \) with commutators as it
is prescribed in canonical quantisation, \( \{ A, B \} \rightarrow [\hat{A}, \hat{B}]/(i\hbar) \), one constructs Dirac brackets
\( \{·,·\}_D \) and replaces them with commutators according to,
\[
\{ A, B, \}_D \rightarrow \left[ \hat{A}, \hat{B} \right] \frac{1}{i\hbar}.
\]

Dirac brackets are constructed from an extended Hamiltonian, in which all (independent)
constraints are enforced by the introduction of Lagrange multipliers. The choice of Lagrange
multipliers is in principle arbitrary, and the freedom of their choice is known as gauge free-
dom. A particular choice of how Lagrange multipliers depend on fields and their canonical
momenta corresponds to a choice of gauge (this notion of gauge generalises the usual notion
of gauge fixing in gauge theories). One can then show that different gauge choices yield
identical answers for expectation (on-shell) values of Hermitean operators (physical observ-
ables). This independence of gauge of on-shell quantities is then the precise sense in which
gauge freedom exists. For brevity in this Appendix we focus primarily on the path integral
quantization and refer to Ref. \cite{15}, in which it was proved that Dirac quantization and path
integral quantization presented here are equivalent in the sense that they yield identical
in-out scattering amplitudes.

We begin our consideration by noting the classical action for an Abelian gauge field \( A_\mu \),
\[
S_{EM} = \int d^Dx \sqrt{-g} \left[ -\frac{1}{4} g^{\mu\lambda} g^{\nu\sigma} F_{\mu\nu} F_{\lambda\sigma} \right] =
\int d^4x \sqrt{-g} \left[ -\frac{1}{4} \left( g^{\mu\lambda}_\perp + g^{\mu\lambda}_\parallel \right) \left( g^{\nu\sigma}_\perp + g^{\nu\sigma}_\parallel \right) F_{\mu\nu} F_{\lambda\sigma} \right],
\]
where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the gauge field strength, \( g^{\mu\lambda}_\perp = g^{\mu\lambda} - \frac{n^\mu n^\lambda}{\|n\|^2} \) is the induced
(inverse) metric on a space-like hyper-surface \( \Sigma \) (\( n^\mu \perp \Sigma \)) and \( g^{\mu\lambda}_\parallel = \frac{n^\mu n^\lambda}{\|n\|^2} \). The canonical
momentum of \( A_\mu \) is given by,
\[
\pi^\mu = \frac{\delta S}{\delta(n^\nu \partial_\nu A_\mu(x))} = -\sqrt{-g} \frac{n^\sigma}{\|n\|^2} g^{\mu\lambda}_\perp F_{\sigma\lambda} \implies n^\mu F_{\mu\nu} = -\frac{\pi^\mu g_{\mu\nu}}{\sqrt{-g}\|n\|^2} \& n_\mu \pi^\mu = 0.
\]
The last equation in Eq. (67) is a constraint. When one chooses time such \( \Sigma \) is a hyper-
surface of constant time, then \( n_\mu = \delta^0_\mu \) and the condition \( n_\mu \pi^\mu = \pi^0 = 0 \) sets the temporal
momentum of the gauge field to zero. Other (secondary) constraints are obtained by tak-
ing the commutator of (67) with the Hamiltonian, which is obtained by taking a Legendre
transform of the Lagrangian in (65),
\[ H = \int d^{D-1}x \left( -\frac{\hat{g}_{\mu\nu}}{2\sqrt{-g||n||}} \pi^\mu \pi^\nu + \pi^\mu n^\nu \partial_\mu A_\nu + \frac{\sqrt{-g}}{4} g_{\alpha\beta} g^{\beta\delta} F_{\alpha\beta} F^{\gamma\delta} + \lambda_0 n_\mu \pi^\mu \right), \] (68)
where, in order to enforce the constraint \( n_\mu \pi^\mu = 0 \), we have introduced a Lagrange multi-
plier, \( \lambda_0 \). Secondary constraints can be obtained by computing the Poisson bracket of the
constraint \( n_\mu \pi^\mu \) with the Hamiltonian. Using \( \{ \pi^\mu(\vec{x}), A_\nu(\vec{y}) \} = \delta_\nu^\sigma \delta(\vec{x} - \vec{y}) \), we find,
\[ \Phi = n_\sigma \{ \pi^\sigma, H \} = -n_\sigma \partial_\mu (n^\sigma \pi^\mu) - \sqrt{-g} n_\beta \nabla_\alpha \left( g^{\alpha\beta} g^{\gamma\delta} F_{\gamma\delta} \right). \] (69)
One can then verify that no further independent constraints are generated by taking other
Poisson brackets. Note also that \( \Phi \) is a covariant constraint: if the definition of the momen-
tum \( \pi^\mu \) is plugged in we get,
\[ \Phi = -n_\rho \partial_\mu \left( \sqrt{-g} n^\rho n^\sigma \hat{g}^{\mu\lambda} F_{\sigma\lambda} \right) = -\sqrt{-g} ||n|| \nabla_\mu (F^{\mu\rho}). \] (70)
The physical meaning of \( \Phi \) can be divulged/disclosed by making use of the Stokes’ theorem.
Since the conformal dimension of the dual 2-form of the gauge field strength, \( \tilde{F}_{\alpha\beta} = \epsilon_{\alpha\beta}^{\mu\nu} F_{\mu\nu} \),
is \( w = \frac{D-4}{2} \), we can apply the conformal Stokes theorem [4] in \( D = 4 \) to \( d\tilde{F} \) on some subset
of the spatial slice, \( I \subset \Sigma \),
\[ \int_I \nabla_\lambda \tilde{F}_{\mu\nu} dx^\lambda \wedge dx^\mu \wedge dx^\nu = \int_I \partial_\lambda \tilde{F}_{\mu\nu} dx^\lambda \wedge dx^\mu \wedge dx^\nu = \]
\[ = 4 \int_I \frac{n_\rho}{||n||} \nabla_\mu (F^{\mu\rho}) \sqrt{g^{\perp}} d^3x = 4 \int_I \Phi d^3x = \]
\[ = 4 \int_I \left( \nabla_\mu \left( g^{\mu\nu} \frac{n^\alpha}{||n||} F_{\nu\alpha} \right) - F^{\lambda\nu} \nabla_\lambda \left( \frac{n_\rho}{||n||} \right) \right) \sqrt{g^{\perp}} d^3x \]
\[ = 4 \int_I \left( g^{\perp \mu \nu} \nabla_\mu \left( g^{\perp \nu \rho} \frac{n^\alpha}{||n||} F_{\nu\alpha} \right) \right) \sqrt{g^{\perp}} d^3x = \int_{\partial I} \tilde{F}_{\mu\nu} dx^\mu \wedge dx^\nu, \] (71)
where we used the notation,
\[ \frac{n_\rho}{||n||} dV = \frac{n_\rho}{||n||} \sqrt{-g} dxdydz = \frac{1}{3!} \epsilon_{\rho\sigma\beta\gamma} dx^\alpha \wedge dx^\beta \wedge dx^\delta \] (72)
to denote the induced volume form on the surface \( I \subset \Sigma \), which transforms as a vector of
scaling dimension 4 under Weyl transformations. Furthermore, \( \sqrt{g^{\perp}} d^3x = \sqrt{-g} ||n||^{-1} d^3x \)
is the induced volume form on \( \Sigma \). From Eq. (71) we see that the last equality represents the
electric flux associated with the field strength \( F_{\mu\nu} \) through the two-dimensional surface \( \partial I \).
In presence of a source the constraint (69) gets modified from $\Phi = 0$ to $\Phi = n_\mu J^\mu_\gamma$, where $J^\mu_\gamma$ is the electromagnetic current. This then modifies (71) to,

$$\int_{\partial I} \tilde{F}_{\mu \nu} d\xi \wedge dx^\nu = \int_{\partial I} \frac{n_\mu}{||n||} J^\mu_\gamma \sqrt{g^\perp} d^3x. \tag{73}$$

Hence Eq. (73) is the curved space generalization of the Gauss’ law, $\int_{\partial I} \tilde{E} \cdot d\tilde{S} = \int_I \rho dV$.

In Eq. (73) the role of the electric field is played by the spatial vector, $g^\perp_{\mu \nu} F^\alpha_{\nu \alpha}$, which on flat spaces with flat foliation results in $F^0_{\mu \nu}$. The gauge condition that we can associate with the constraint $n_\mu \pi^\mu = 0$ is naturally $n_\mu A^\mu$, while the one to associate with $n_\sigma \partial_\mu (n^\sigma \pi^\mu) = \sqrt{g^\perp} (g^\perp)^{\mu}_\nu \tilde{\nabla}_\mu \frac{x^\nu}{\sqrt{g^\perp}}$ should be, in light of the previous observations, $(g^\perp)^{\beta}_\alpha \partial^\beta (A_\nu g^\perp_{\alpha \nu})$. Thus we can construct the path integral representation of the scattering amplitude in the gauge, $n_\mu A^\mu = 0, \tilde{\nabla}^A = 0$. The Poisson brackets of the constraints with their associated canonical variables are,

$$\left\{ n_\mu \pi^\mu, n^\nu A_\nu \right\} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{g^\perp} (g^\perp)^{\mu}_\nu \tilde{\nabla}_\mu \frac{x^\nu}{\sqrt{g^\perp}}, \tilde{\nabla}^\nu A^\mu_\perp \end{pmatrix},$$

where

$$\Box^A \delta^{D-1}(\vec{x} - \vec{y}) = \frac{1}{\sqrt{g^\perp}} \left( \partial_\mu - \left(\frac{D-2}{2}\right) T_\mu \right) \sqrt{g^\perp} g^\perp_{\mu \nu} \partial_\nu \delta^{D-1}(\vec{x} - \vec{y}), \tag{75}$$

transforms covariantly under Weyl rescalings in $D = 4$, acting on a function with scaling dimension 0.

Following [15] we can now write the path integral representation of the in-out scattering matrix as,\(^{15}\)

$$\langle in|out \rangle = \int \prod_\sigma D\pi^\sigma D\pi^\sigma \delta (n_\mu \pi^\mu) \delta (n^\nu A_\nu) \delta (\partial_\alpha \pi^\alpha_\perp) \delta \left( \tilde{\nabla}^\beta A^\beta_\perp \right) \left| \det \left( ||n||^2 \delta^{D-1}(\vec{x} - \vec{y}) \right) \right| \times \left| \det \left( \Box^A \delta^{D-1}(\vec{x} - \vec{y}) \right) \right| \exp \left( i \int d^Dx \left[ \pi^\mu n^\nu \partial_\nu A_\mu - H \right] \right). \tag{76}$$

\(^{15}\) Even though the path integral (76) is derived for the free theory, its expression is valid also in interacting theories. The constraints in interacting theories get modified in the sense that they involve sources. This modification does not change the invariance of the path integral measure under gauge or conformal transformations, but might introduce non-trivial field dependence in the determinant factor, $\det \left( \tilde{\nabla}_\nu \partial^\beta (\vec{x} - \vec{y}) \right)$, since some constraints and gauge conditions may not commute with each other.
In Ref. [15] it was shown that the scattering amplitude (76) is independent on the gauge condition chosen. In addition Eq. (76) is manifestly covariant (diffeomorphism invariant) and conformal (Weyl invariant), as can be easily checked.

Appendix B: Ehrenfest Theorem

In this appendix we derive the Ehrenfest theorem, which can be reinterpreted as the Ward identity for infinitesimal fields translations. This theorem is useful for proving the fundamental conformal Ward identity in section IV A. Consider now the time ordered product of $n$-fields, represented by the path integral,

$$\langle in \mid T\{\phi(x_1)\cdots\phi(x_n)\} \mid out \rangle = \frac{1}{\langle in \mid out \rangle} \int \mathcal{D}\phi e^{\int d^D x \mathcal{L}_\phi(x_1)\cdots\phi(x_n)} .$$

(77)

The path integral (77) is invariant under the field independent local shifts, $\phi(x) \rightarrow \phi'(x) = \phi(x) + \xi(x)$, for which one can then write, $\langle in \mid T\{\phi(x_1)\cdots\phi(x_n)\} \mid out \rangle = \langle in \mid T\{\phi'(x_1)\cdots\phi'(x_n)\} \mid out \rangle$, from which it follows that [18],

$$\left\langle \langle in \mid T \left\{ \frac{\delta}{\delta \phi(x)} \int d^D x \mathcal{L}_\phi \right\} \phi(x_1)\cdots\phi(x_n) \mid out \right\rangle = 0 ,$$

(78)

where any derivative operator acts inside the time-ordered product [18]. When, on the other hand, the derivatives are pulled outside of the time-ordered product, one gets,

$$\square_x \left\langle \langle in \mid T \{\phi(x)\phi(x_1)\cdots\phi(x_n)\} \mid out \right\rangle + \langle in \mid T \left\{ \frac{\partial \mathcal{L}_{\text{int}}}{\partial \phi(x)} \right\} \phi(x_1)\cdots\phi(x_n) \mid out \right\rangle = \sum_{i=1}^{n} \langle T \{\phi(x_1)\cdots\delta^D (x - x_i)\cdots\phi(x_n)\} \rangle ,$$

(79)

where we have assumed that the Lagrangian density can be split into the free part ($\mathcal{L}_0$) and the part that contains (polynomial) interactions ($\mathcal{L}_{\text{int}}$) as,

$$\mathcal{L}_\phi = \mathcal{L}_0 + \mathcal{L}_{\text{int}} \, , \quad \mathcal{L}_0 = \frac{1}{2} g^{\mu\nu} (\nabla_\mu \phi) (\nabla_\nu \phi) .$$

(80)

The non-vanishing right hand side in (79) is due to the non-commuting of the time ordering operation and the derivative operators appearing in the Lagrangian (80).  

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[18] Equation (79) can be proven by noting that one can rewrite time ordering operation as,

$$\left\langle \langle in \mid T \{\phi(x_1)\cdots\phi(x_n)\} \mid out \right\rangle = \sum_{\sigma \in S_n} \theta \left( x^0_{\sigma(1)} - x^0_{\sigma(2)} \right) \theta \left( x^0_{\sigma(2)} - x^0_{\sigma(3)} \right) \cdots \theta \left( x^0_{\sigma(n-1)} - x^0_{\sigma(n)} \right) \times \left\langle \langle in \mid \phi(x_{\sigma(1)})\cdots\phi(x_{\sigma(n)}) \mid out \right\rangle ,$$

(27)
In particular, Eq. (77) implies the identity,

\[ \langle \text{in} \left| T \left\{ \left( \frac{\delta S}{\delta \phi(x)} \right) \phi(y) \right\} \right| \text{out} \rangle = 0 , \tag{81} \]

where all the derivatives act inside of the time-ordered product. This equation – also known as the Ehrenfest theorem – is just the statement that the field operator satisfies its equations of motion multiplied by one (or more) field(s), upon time ordering and expectation values are exacted.

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