Stabilization of Itô Stochastic T-S Models via Line Integral and Novel Estimate for Hessian Matrices

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Abstract—This paper proposes a line integral Lyapunov function approach to stability analysis and stabilization for Itô stochastic T-S models. Unlike the deterministic case, stability analysis of this model needs the information of Hessian matrix of the line integral Lyapunov function which is related to partial derivatives of the basis functions. By introducing a new method to handle these partial derivatives and using the property of state-dependent matrix with rank one, the stability conditions of the underlying system can be established via a line integral Lyapunov function. These conditions obtained are more general than the ones which are based on quadratic Lyapunov functions. Based on the stability conditions, a controller is developed by cone complementarity linearization algorithm. A non-quadratic Lyapunov function approach is thus proposed for the stabilization problem of the Itô stochastic T-S models. It has been shown that the problem under concern are solved by optimizing sum of traces for a group of products of matrix variables with linear constraints. Numerical examples are given to illustrate the effectiveness of the proposed control scheme.

Index Terms—Hessian matrix, line integral Lyapunov function, quadratic optimization, stochastic nonlinear system, T-S model.

I. INTRODUCTION

The past few decades have witnessed the significant advancements of Takagi-Sugeno (T-S) fuzzy systems, see, for example [1]–[4] and references therein. The main motivation should be attributed to their capabilities that allow them to represent and handle some sort of qualitative information such as operators’ experience, experts’ knowledge and so forth involved in the plants. The practical experience of these skilled operators and experts’ knowledge are usually summarized and expressed in language, and fuzzy set theory can convert such sort of qualitative information into quantitative data in light of membership functions. For stability analysis and stabilization of fuzzy systems, many researchers have resorted to a quadratic Lyapunov function approach [5]–[7]. The main drawback concerning this approach lies in the fact that it is difficult for a common positive definite matrix to satisfy the stability conditions of all local linear models. In this situation, the non-quadratic Lyapunov function is a helpful alternative [8]–[12]. It has been shown that the stability and stabilization results based on non-quadratic Lyapunov function are less conservative than those based on common quadratic ones. It is worth noting that unlike their discrete-time counterpart, continuous-time T-S models induce the problem of handling derivatives of the fuzzy basis functions that emerge when the basis dependent Lyapunov function is employed to derive stability and stabilization results [9], [13], [14]. In such case, the upper bound of the time derivatives of the fuzzy basis functions are used to analyze stability performance and to develop a controller. Since the time derivatives of fuzzy basis functions are related to the vector field that governs the T-S models and are dependent on the state and control of the systems, their upper bounds are not always readily available, and sometimes they do not even exist. To overcome these difficulties, the authors in [15] proposed a novel fuzzy Lyapunov function that was formulated as a line integral of a vector along a path from the origin to the current state. By using this line integral function, the stability results did not involve the upper bound of the time derivatives of fuzzy basis functions. The main issues in [15] are that the T-S fuzzy model have a specific premise structure and that the stabilization results are provided by bilinear matrix inequalities.

When a fuzzy system is disturbed by random perturbations, the system turns to be a stochastic fuzzy one. Recently, T-S fuzzy descriptions have been introduced to study the analysis and synthesis problem of stochastic nonlinear systems [14], [16]–[18]. The local uniqueness results for the solutions of fuzzy stochastic differential equations were provided in [19]. For uncertain nonlinear stochastic time-delay systems, a fuzzy controller was developed to guarantee the robust asymptotic stability and attenuation performance [20]. In [21], by using Lyapunov method and stochastic analysis approaches, the delay-dependent stability criterion was presented for T-S fuzzy Hopfield neural networks with parametric uncertainties and stochastic perturbations. For a class of T-S model based stochastic systems, the stabilization problem was investigated in [7]. It should be pointed out that most analysis or synthesis results of the T-S model based stochastic systems mentioned above are obtained by using or partly using a common quadratic Lyapunov function.

Motivated by [15] and [7], this paper proposes a line integral Lyapunov function approach to stability analysis and stabilization for a class of Itô stochastic T-S model based nonlinear systems. By using the line integral Lyapunov function proposed in [15], more general sufficient conditions for the stochastic asymptotic stability of the class of systems can be established. The control design is facilitated by the
The introduction of some additional matrix variables and a cone complementarity linearization algorithm. These variables decouple the coupling terms in the obtained stability conditions and make the control design feasible. It will show that the solution of the controller design problem can be obtained by solving a minimization problem with linear constraints.

The organization of this paper is as follows. Section 2 formulates the problem and presents some preliminary results. In Section 3, stochastic stability analysis is given. Controller design is presented in Section 4 and a numerical example is given in Section 5 to illustrate the effectiveness of the proposed approach. Finally, the paper is concluded in Section 6.

Throughout this paper, a real symmetric matrix $P > 0$ (≥ 0) denotes $P$ being a positive definite (or positive semidefinite) matrix, and $A > (≥) B$ means $A - B > (≥) 0$. $I$ is used to denote an identity matrix with proper dimensions. The trace of a square matrix $A$ is denoted by $\text{tr} A$ and the notation $A^\tau$ represents the transpose of the matrix $A$. The expression $A^S$ stands for the sum of $A$ and $A^\tau$. For symmetric matrices, we use $(*)$ as an ellipsis for a block matrix that is induced by symmetry. The notation $(\Omega, \mathcal{F}, \mathcal{P})$ represents the probability space with $\Omega$ the sample space, $\mathcal{F}$ the $\sigma$-algebra of subsets of the sample space and $\mathcal{P}$ the probability measure on $\mathcal{F}$. Matrices, if not explicitly stated, are assumed to be compatible for algebraic operations.

II. Problem Formulation and Preliminaries

Consider an Itô stochastic T-S model described by the following rules:

**Plant Rule** $i$: If $x_1$ is $F^{(a1)}_1$, · · · , and $x_n$ is $F^{(an)}_n$, then

$$dx = (A_i x + B_i u) dt + C_i x dW(t)$$

where $i = 1, \cdots, s$, $s$ is the total number of fuzzy rules; $x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n$, $u \in \mathbb{R}^p$ are the state vector and the control input, respectively; for each pair $(i, j) \in \{1, \cdots, s\} \times \{1, \cdots, n\}$, $F^{(ai)}_i$ is the fuzzy set in the $i$th rule based on the premise variable $x_j$ with $a_{ij}$ specifying which $x_j$-based fuzzy set is used in the $i$th If-Then rule. $W(t)$ is a Wiener process on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ adapted to a filtration $\{\mathcal{F}_t\}_{t \geq 0}$: $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times p}$ and $C_i \in \mathbb{R}^{n \times n}$ are known real constant matrices.

In what follows, for notational convenience, we denote the two sets $\{1, \cdots, s\}$ and $\{1, \cdots, n\}$ by $S$ and $N$ respectively. Assume that for any $j \in N$, the number of all the $x_j$-based fuzzy sets used in the $s$ fuzzy rules is $s_j$ with $s_j$ a certain positive integer. The superscript $a_{ij}$ in the fuzzy set $F^{(ai)}_i$ is an ordered index of all the $s_j$ $x_j$-based fuzzy sets $\{F^{(aj)}_j\}_{a_{ij}=1}^{s_j}$ which will play a role in the following development. If $a_{ij} = \rho_j$, $i \in S$, which means that in the $i$th fuzzy rule the premise variable $x_j(t)$ belongs to the $\rho_j$th fuzzy set $F^{(\rho_j)}_j$, one can see that

$$1 \leq \rho_j \leq s_j, \quad j \in N.$$  

For $i \in S$, $j \in N$ and $x_j \in \mathbb{R}$, let $w^{(ai)}_j(x_j)$ be the membership function of fuzzy set $F^{(ai)}_i$. The normalized membership functions are defined by

$$\mu^{(ai)}_j(x_j) = \frac{w^{(ai)}_j(x_j)}{\sum_{a_{ij}=1}^{s_j} w^{(ai)}_j(x_j)}, \quad i \in S, \quad j \in N. \quad (1)$$

which satisfy

$$0 \leq \mu^{(ai)}_j(x_j) \leq 1, \quad \sum_{a_{ij}=1}^{s_j} \mu^{(ai)}_j(x_j) = 1. \quad (2)$$

Then, the fuzzy basis functions can be defined by

$$h_i(x) = \prod_{j=1}^{n} \mu^{(ai)}_j(x_j), \quad x \in \mathbb{R}^n, \quad i \in S. \quad (3)$$

One also has $0 \leq h_i(x) \leq 1$, $\sum_i h_i(x) = 1$. The Itô stochastic T-S model can be expressed by

$$dx = \sum_{i=1}^{s} h_i(x) (A_i x + B_i u) dt + \sum_{i=1}^{s} h_i(x) C_i x dW(t). \quad (4)$$

The premise variables in the plant rules described above are chosen as the state variables $\{x_j\}_{j \in N}$ which is necessary in stochastic stability analysis and control design of the Itô stochastic T-S model. For an admissible control $u$ and an initial state $x_0$, the trajectory $x(t)$ of system (4) is vector-valued stochastic processes which can be viewed as a mapping $x = (x_1, \cdots, x_n)^T : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}^n$ with $\mathbb{R}_+ := (0, \infty)$. Since the range of the mapping (or state system) $x$ is $\mathbb{R}^n$, the antecedent parts that $x_j$ is $\mathbb{F}^{(ai)}_j$ in the $i$th plant rule make sense. Also $x(t)$ can be regarded as the observation of vector-valued stochastic processes as pointed out in [16].

This paper intends to develop a line integral Lyapunov function approach to stability analysis and stabilization for the Itô stochastic T-S model [4]. We first adopt the stochastic stability concept and the main tool of stability analysis for very general Itô stochastic dynamic systems. Consider an Itô stochastic differential equation on $\mathbb{R}^n$ of the form

$$dx = f(x,t) dt + g(x,t) dW(t) \quad (5)$$

where $f : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}^m$ satisfy the usual linear growth and local Lipschitz conditions for existence and uniqueness of solutions to (5). $W(t)$ is a Wiener process adapted to a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ that contains all $\mathcal{P}$-null sets and is right continuous. Let $x = 0$ be the equilibrium point of (5), that is, $f(0, t) = 0$, $g(0, t) = 0$ and $x(t; t_0, x_0)$ be a trajectory of system (5) with initial value $x(t_0) = x_0 \in \mathbb{R}^n$.

**Definition 1.** [22] The equilibrium point $x = 0$ of equation (5) is said to be

i) stochastically stable or stable in probability if for every pair of $\epsilon \in (0, 1)$ and $r > 0$, there exists a $\delta = \delta(\epsilon, r, t_0) > 0$ such that $P\{|x(t; t_0, x_0)| < r, \text{ for all } t \geq t_0\} \geq 1 - \epsilon$ whenever $|x_0| < \delta$. Otherwise, it is said to be stochastically unstable.

ii) stochastically asymptotically stable if it is stochastically stable and, moreover, for every $\epsilon \in (0, 1)$, there exists a $\delta_0 = \delta_0(\epsilon, t_0) > 0$ such that $P\{|x(t; t_0, x_0)| < \delta, \text{ for all } t \geq t_0\} = 1$ whenever $|x_0| < \delta_0$.

iii) stochastically asymptotically stable in the large if it is stable and, moreover, for all $x_0 \in \mathbb{R}^n$, $P\{|x(t; t_0, x_0)| = 0\} = 1$.

Now, we are in a position to introduce the line integral Lyapunov function that was first proposed in [15]. A line
integral function $V : R^n \rightarrow [0, \infty)$ is defined by
\[ V(x) = 2 \int_{\Gamma(0,x)} \hat{f}^T(\Psi)d\Psi := 2 \int_{\Gamma(0,x)} \sum_{i=1}^{s} h_i(\Psi)\Psi^T P_i d\Psi \]  
(6)

\[ P_i = \hat{P} + D_i > 0 \]  
(7)

\[ \hat{P} = \begin{bmatrix} 0 & p_{12} & \cdots & p_{1n} \\ p_{12} & 0 & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{1n} & p_{2n} & \cdots & 0 \end{bmatrix} \]  
(8)

\[ D_i = \begin{bmatrix} d_{11}^{\alpha_{ij}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_{in}^{\alpha_{ij}} \end{bmatrix} \]  
(9)

where $\Gamma(0,x)$ denotes a path from the origin to the current state $x$; $\Psi \in R^n$ is a dummy vector for the integral and $d\Psi$ is an infinitesimal displacement vector. To illustrate that $V(x)$ defined in (6) is a Lyapunov function, the following fact is adopted from [15].

**Fact 1:** The line integral $V(x)$ defined by (6) is path-independent, continuously differentiable, positive definite and also satisfies
\[ V(x) \rightarrow \infty, \quad ||x|| \rightarrow \infty. \]

It can be also seen from above fact and (6) that the line integral Lyapunov function $V(x)$ in (6) is a non-quadartic one. The following simple fact reveals the relationship between line integral Lyapunov function and quadratic Lyapunov function.

**Fact 2:** For the line integral Lyapunov function $V(x)$ in (6), if there exists a positive definite matrix $P$, such that the matrices $\{P_i\}_{i \in S}$ satisfy $P_1 = \cdots = P_s = P$, then $V(x)$ becomes a quadratic function $x^T P x$, that is, for any $x \in R^n$
\[ V(x) = 2 \int_{\Gamma(0,x)} \Psi^T P d\Psi = x^T P x. \]

For stability analysis of system (4) based on line integral Lyapunov function $V(x)$ in (6), the row gradient vector $\partial V(x)/\partial x^T$ is needed. Due to the path independence of the line integral, $\partial V(x)/\partial x^T$ can be calculated by the following fact 3.

**Fact 3:** Consider the line integral Lyapunov function $V(x)$ in (6). The row gradient vector $\partial V(x)/\partial x^T$ is calculated by
\[ \frac{\partial V(x)}{\partial x^T} = 2x^T P(x) := 2 \sum_{i=1}^{s} h_i(x)x^T P_i. \]  
(10)

The line integral Lyapunov function based stability analysis of the model in (4) is involved in the Hessian matrix $\partial^2 V(x)/\partial x \partial x^T$ of $V(x)$ in (6) which is related to partial derivatives of the fuzzy basis functions $\{h_i(x)\}_{i=1}^{s}$. This also leads to the problem of handling derivatives of membership functions (MFs) [9], [13], [14]. In what follows, we give a new method to deal with the derivatives of MFs. Consider the normalized membership functions $\{\mu_j^{\alpha_{ij}}(x_j), (i,j) \in S \times N\}$ defined by (1). Notice that for a fixed pair $(i,j) \in S \times N$, the function $\mu_j^{\alpha_{ij}}(x_j)$ is a scalar-valued one with only one argument $x_j$. So we can give an assumption for the derivatives of these functions. The assumption will form our starting point for stochastic stability analysis of the model (4).

Assumption 1: Assume for any $(i, j) \in S \times N$ and $x_j \in \mathbb{R}$, that the normalized membership function $\mu_j^{\alpha_{ij}}(x_j) : \mathbb{R} \rightarrow [0, 1]$ is differentiable and that there exists a known constant $\beta_{ij} > 0$ satisfying
\[ \left| x_j \frac{d}{dx_j} (\mu_j^{\alpha_{ij}}(x_j)) \right| \leq \beta_{ij}. \]  
(11)

**Remark 1:** It is important to point out that normalized membership functions $\mu_j^{\alpha_{ij}}(x_j), i \in S, j \in N$ defined by (1) are state $x_j$ dependent and not the trajectory $x_j(t)$ dependent. Therefore, the derivative $\frac{d}{dx_j}(\mu_j^{\alpha_{ij}}(x_j))$ is the derivative of a scalar function with respect to its one argument $x_j$. If the membership functions are regarded as trajectory dependent and their derivatives are regarded as time derivatives, the time derivatives of fuzzy basis functions will be related to the vector field that governs the T-S model. Such a case makes the time derivatives of fuzzy basis functions difficult to handle [9]. So Assumption 1 is different from the ones in [13], [14], [23] where upper bounds of time derivatives of fuzzy basis functions are required.

**Remark 2:** Assumption 1 implies that not only all the normalized membership functions $\mu_j^{\alpha_{ij}}(x_j)$ are differentiable, but also they should satisfy
\[ \frac{-\beta_{ij}}{x_j} \leq \frac{d}{dx_j}(\mu_j^{\alpha_{ij}}(x_j)) \leq \frac{\beta_{ij}}{x_j}, \quad x_j > 0 \]
\[ \frac{-\beta_{ij}}{x_j} \leq \frac{d}{dx_j}(\mu_j^{\alpha_{ij}}(x_j)) \leq \frac{\beta_{ij}}{x_j}, \quad x_j < 0. \]

Assumption 1 for the normalized membership function $\mu_j^{\alpha_{ij}}(x_j)$ seems to be reasonable and the upper bound of inequalities (11) can be fulfilled for the exponential type of membership functions (24) (See Example 2 in Section 5). Fig. 1 shows the region of $\frac{d}{dx_j}(\mu_j^{\alpha_{ij}}(x_j))$ satisfying Assumption 1 with $\beta_{ij} = 0.0125$.

![Fig. 1: Region of $\frac{d}{dx_j}(\mu_j^{\alpha_{ij}}(x_j))$ satisfying Assumption 1](image)

It can be shown that the fuzzy basis functions $\{h_i(x)\}_{i=1}^{s}$ are differentiable. An estimate for all the partial derivatives of these fuzzy basis functions $\{h_i(x)\}_{i=1}^{s}$ can be derived by (11), and will be used later on.

**Lemma 1:** Under Assumption 1, the fuzzy basis functions $\{h_i(x)\}_{i=1}^{s}$ given by (5) are differentiable and satisfy
\[ \left| x_j \frac{\partial h_i(x)}{\partial x_j} \right| \leq \beta_{ij}, \quad i \in S, \quad j \in N. \]  
(12)
Proof: By virtue of (2) and (3), one has
\[
x_j \frac{\partial h_i(x)}{\partial x_j} = x_j \frac{\partial}{\partial x_j} \left( \prod_{k=1}^{n} \mu_{ik}^{\alpha_{ik}}(x_k) \right) = \left| x_j \frac{\partial}{\partial x_j} (\mu_{ij}^{\alpha_{ij}}(x_j)) \right| \leq x_j \frac{d}{dx_j} (\mu_{ij}^{\alpha_{ij}}(x_j))
\]
which together with (11) leads to the inequalities (12).

Now, we give the new results of Hessian matrix for the line integral Lyapunov function \( V(x) \) in (6) which will be adopted in the following section.

**Lemma 2:** Consider the fuzzy basis functions \( \{ h_i(x) \}_{i=1}^{s} \) in (3) with the normalized membership functions \( \mu_{ij}^{\alpha_{ij}}(x_j) \) satisfying (11) in Assumption 1. For matrices \( D_i > 0 \) defined in (9), a positive definite matrix \( D \) with \( D - D_i \geq 0 \) and \( P(x) \) defined in (10), the following inequalities with respect to Hessian matrix hold for \( x, y \in \mathbb{R}^n \)
\[
y^T \frac{\partial^2 V(x)}{\partial x \partial x^T} y = y^T \left( P(x) + \sum_{i=1}^{s} \frac{\partial h_i(x)}{\partial x} x^T D_i \right) y \leq y^T \left( P(x) + \beta D \right) y \quad (13)
\]
with the known constants \( \beta_{ij} > 0 \) given in (11).

**Proof:** For the \( V(x) \) defined in (6), in terms of Fact 3 we have
\[
\frac{\partial V(x)}{\partial t} = 0, \quad \frac{\partial V(x)}{\partial x^T} = 2x^T P(x). \quad (15)
\]
On the other hand, recalling (7)-(9) gives
\[
2x^T P(x) = 2(x_1, \ldots, x_j, \ldots, x_n) \times \left[ \begin{array}{ccc} \sum_{i=1}^{s} h_{i1}^{\alpha_{i1}} & \cdots & p_{1j} & \cdots & p_{1n} \\
\vdots & & \vdots & & \vdots \\
p_{j1} & \cdots & \sum_{i=1}^{s} h_{ij}^{\alpha_{ij}} & \cdots & p_{jn} \\
\vdots & & \vdots & & \vdots \\
p_{1n} & \cdots & p_{nj} & \cdots & \sum_{i=1}^{s} h_{in}^{\alpha_{in}} \end{array} \right] \\
= 2 \left( x_1 \sum_{i=1}^{s} h_{i1}^{\alpha_{i1}} + \cdots + x_k p_{kn} + x_n \sum_{i=1}^{s} h_{in}^{\alpha_{in}} \right).
\]
By (16), the Hessian matrix of the line integral Lyapunov function \( V(x) \) can be calculated by
\[
\frac{\partial^2 V(x)}{\partial x \partial x^T} = \frac{\partial}{\partial x} \left( \frac{\partial V(x)}{\partial x^T} \right) = 2 \frac{\partial}{\partial x} \left( x^T P(x) \right) = 2 \left( x_1 \sum_{i=1}^{s} h_{i1}^{\alpha_{i1}} + \cdots + x_k p_{kn} + x_n \sum_{i=1}^{s} h_{in}^{\alpha_{in}} \right) / \partial x
\]
\[
= 2 \left[ \begin{array}{cccc} \sum_{i=1}^{s} h_{i1}^{\alpha_{i1}} + x_1 \sum_{i=1}^{s} \frac{\partial h_i}{\partial x_1} p_{1i}^{\alpha_{i1}} & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots \\
\sum_{i=1}^{s} h_{in}^{\alpha_{in}} + x_n \sum_{i=1}^{s} \frac{\partial h_i}{\partial x_n} p_{ni}^{\alpha_{in}} \end{array} \right]
\]
\[
= 2 \left( \begin{array}{cccc} \sum_{i=1}^{s} h_{i1}^{\alpha_{i1}} & \cdots & p_{1n} \\
\vdots & \vdots & \vdots & \vdots \\
p_{1n} & \cdots & \sum_{i=1}^{s} h_{in}^{\alpha_{in}} \end{array} \right)
\]
\[
= 2 \left( \begin{array}{cccc} \sum_{i=1}^{s} \frac{\partial h_i}{\partial x} x^T D_i y = y^T D_i^\frac{1}{2} \frac{\partial h_i}{\partial x} x^T D_i^\frac{1}{2} \frac{\partial h_i}{\partial x} x^T D_i^\frac{1}{2} D_i^\frac{1}{2} y. \quad (19)
\end{array} \right)
\]
In order to achieve an upper bound of the Hessian matrix, the upper bound of the second part in (18) is needed to be estimated. For any \( y \in \mathbb{R}^n \), it is clear by \( D_i > 0 \) that
\[
y^T \frac{\partial h_i}{\partial x} x^T D_i y = y^T D_i^\frac{1}{2} \frac{\partial h_i}{\partial x} x^T D_i^\frac{1}{2} \frac{\partial h_i}{\partial x} x^T D_i^\frac{1}{2} y. \quad (19)
\]
In the derivation above, we drop the argument of \( h_i(x) \) for simplicity. In what follows, we will do so in some cases for notational convenience. Since
\[
\text{rank} \left( D_i^\frac{1}{2} \frac{\partial h_i}{\partial x} x^T D_i^\frac{1}{2} \right) \leq \text{rank}(x^T) \leq 1 \quad (20)
\]
there exist an invertible matrix \( P \) such that
\[
P \left( D_i^\frac{1}{2} \frac{\partial h_i}{\partial x} x^T D_i^\frac{1}{2} \right) P^{-1} = \text{diag} \left\{ \text{tr} \left( P D_i^\frac{1}{2} \frac{\partial h_i}{\partial x} x^T D_i^\frac{1}{2} P^{-1} \right), 0, \cdots, 0 \right\} \quad (21)
\]
It follows from (19) and (21) that
\[
y^T \left( \frac{\partial h_i}{\partial x} x^T D_i \right) y = \left( P^T D_i^\frac{1}{2} y \right)^T \left( P D_i^\frac{1}{2} \frac{\partial h_i}{\partial x} x^T D_i^\frac{1}{2} y \right) \quad (22)
\]
\[
\left( P^T D_i^\frac{1}{2} y \right)^T \left( P D_i^\frac{1}{2} y \right).
\]
In light of the operational property of matrix trace, we have
\[
tr(PD_x^2 \frac{\partial h_i}{\partial x} x^T P^{-1}) = tr \left( \frac{\partial h_i}{\partial x} x^T \right) = x^T \frac{\partial h_i}{\partial x}. 
\] (23)

Combining (22) and (23) yield
\[
y^T \left( \frac{\partial h_i}{\partial x} x^T D_i \right) y \leq \left| \left| \frac{\partial h_i}{\partial x} x^T \right| \right| \left( P - \frac{\partial h_i}{\partial x} x^T \right)^{\tau} y 
\]
\[
\left( PD_x^n y \right) = \sum_{j=1}^{n} x_j \frac{\partial h_i}{\partial x_j} y^T D_i y \leq \sum_{j=1}^{n} x_j \frac{\partial h_i}{\partial x_j} y^T D_i y.
\]

In terms of (12) in Lemma 1 and \( D - D_1 \geq 0 \), we obtain
\[
y^T \frac{\partial h_i}{\partial x} x^T D_i y \leq \sum_{j=1}^{n} \beta_{ij} y^T D_i y \leq \sum_{j=1}^{n} \beta_{ij}^2 y^T D y
\]
which together with (14) gives that
\[
y^T \left( \sum_{i=1}^{n} \frac{\partial h_i}{\partial x} x^T D_i \right) y \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_{ij} y^T D y \leq \beta y^T D y.
\]

Therefore, the inequality (13) can be obtained easily.

### III. STOCHASTIC STABILITY ANALYSIS

In this section, relaxed stability conditions for the unforced Itô stochastic T-S model will be given by using the line integral Lyapunov function candidate. It follows from (15), (16) in Lemma 2 and (24), that
\[
\mathcal{L}(x) = \frac{\partial V(x)}{\partial t} + \frac{\partial V(x)}{\partial x} \left( \sum_{i=1}^{s} h_i A_i x \right) + \frac{1}{2} tr \left[ \left( \sum_{i=1}^{s} h_i C_i x \right)^{\tau} \right]
\]
\[
\times \left( 2P(x) + 2\beta D \right) \sum_{i=1}^{s} h_i C_i x \right) = \sum_{i=1}^{s} \sum_{j=1}^{s} h_i h_j h_k \times x^T \left( P_j A_i + \frac{1}{2} C_i^T (P_j + \beta D) C_k \right) x. \quad (30)
\]

Choosing \( y = \left( \sum_{i=1}^{n} h_i C_i x \right) \) in Lemma 2, in terms of (30), one has
\[
\mathcal{L}(x) \leq 2x^T P(x) \left( \sum_{i=1}^{n} h_i A_i x \right) + \frac{1}{2} tr \left[ \left( \sum_{i=1}^{s} h_i C_i x \right)^{\tau} \right]
\]
\[
\times \left( 2P(x) + 2\beta D \right) \sum_{i=1}^{s} h_i C_i x \right) = \sum_{i=1}^{s} \sum_{j=1}^{s} h_i h_j h_k
\]
where \( \beta \) is given in (14).

**Proof:** Suppose that there exist matrices \( \{ D_j \}_{j \in S} \), \( \bar{P} \) and \( D \) as well as matrices \( \{ Q_{ij} : Q_{ij} = Q_{ji} \}_{i,j \in S} \) such that for all \( i,j \in S \), the inequalities hold:
\[
P_j = \bar{P} + D_j \geq 0, \quad D - D_j \geq 0 \quad (25)
\]
\[
(P_j A_i)^{S} + C_i^T (P_j + \beta D) C_i + Q_{ij} < 0 \quad (26)
\]
\[
\Theta_1 = \begin{bmatrix}
Q_{11} & \ldots & Q_{1s} \\
\vdots & \ddots & \vdots \\
Q_{s1} & \ldots & Q_{ss}
\end{bmatrix} > 0 \quad (27)
\]
where \( \beta \) is given in (14).

Using these inequalities and (30), one has that
\[
\mathcal{L}(x) \leq 2x^T P(x) \left( \sum_{i=1}^{s} h_i A_i x \right) + \frac{1}{2} tr \left[ \left( \sum_{i=1}^{s} h_i C_i x \right)^{\tau} \right]
\]
\[
\times \left( 2P(x) + 2\beta D \right) \sum_{i=1}^{s} h_i C_i x \right) = \sum_{i=1}^{s} \sum_{j=1}^{s} h_i h_j h_k
\]
which together with (26) gives
\[
\mathcal{L}(x) < \sum_{i=1}^{s} \sum_{j=1}^{s} h_i h_j h_k \left( [P_j A_i]^S + C_i^T (P_j + \beta D) C_i \right) x. \quad (28)
\]

Then it follows from (27) that for \( x \neq 0, \mathcal{L}(x) < 0 \). Recalling that \( V(x) \) is radially unbounded, we can conclude that the equilibrium point of system (24) is stochastically asymptotically stable in the large by Lemma 3 in [7].

**Remark 3:** For the stability analysis of the unforced Itô stochastic T-S fuzzy system (24), the involved Hessian matrix \( \frac{\partial^2 V(x)}{\partial x^2} \) of the line integral \( V(x) \) is related to partial derivatives of the basis functions. It is clear from (17) that the entry \( \mathcal{H}_{ij} \) in slot \((i,j)\) for Hessian matrix is given by
\[
\mathcal{H}_{ij} = \begin{cases}
\sum_{i=1}^{s} h_i d_{ij} + x_j \sum_{i=1}^{s} \frac{\partial h_i}{\partial x_j} d_{ij}, & i = j \\
P_{ij} + x_j \sum_{i=1}^{s} \frac{\partial h_i}{\partial x_j} d_{ij}, & i \neq j.
\end{cases}
\]

So the Hessian matrix is highly dependent on the state \( x \) and the partial derivatives of the basis functions, which gives rise...
to difficulty and challenging for estimating the upper bound of $\mathcal{L}V(x)$. Fortunately, this Hessian matrix can be decomposed into two terms in $\mathcal{L}V(x)$. The key point is to handle the second term $\sum_{i=1} x_{i}^{2} D_{i} y$ which is involved in $\mathcal{L}V(x)$.

**Remark 4:** To our best knowledge, there is no results on dealing with the Hessian matrix of line integral function, therefore we give a novel Lemma 2 to do so. Facilitating by Assumption 1, an upper bound $y^{T} \beta D y$ of $y^{T} \sum_{i=1} x_{i}^{2} D_{i} y$ can be obtained. The idea behind this lemma is as follows. It is observed that the rank of matrix $D_{i} y$ is less than or equal to 1. So this matrix is equivalent to a diagonal matrix with only one nonzero element $tr(D_{i} y^{T} D_{i} y) = x_{i}^{T}$. This elegant property of matrix with rank one together with Lemma 1 leads to the upper bound $y^{T} D y$. Based on Lemma 2, an estimate of $\mathcal{L}V$ can be achieved which plays an important role on the stability analysis of the unforced stochastic T-S fuzzy system.

Similarly, the following corollary based on the common quadratic Lyapunov function can be obtained.

**Corollary 1:** If there exist positive matrices $\{Q_{i}\}_{i \in S} > 0$ and $P > 0$ satisfying inequalities

\[(PA)^{T} + C_{i}^{T} PC_{i} + Q_{i} < 0, \quad i \in S\]  

then the equilibrium point of the unforced Itô stochastic T-S model is stochastically asymptotically stable in the large.

**Remark 5:** It is noted that for any $i, j \in S$ if $P_{i} = P$, $Q_{i i} = Q_{j}$ and $Q_{i j} = 0$, for $i \neq j$ in inequalities (25)-(27), the conditions in Theorem 1 become conditions in Corollary 1. This means that if inequalities (33) have a set of solutions $\{Q_{i}\}_{i \in S}$ and $P$, then $\{P_{j} = P\}_{j \in S}$, $\{Q_{i i} = Q_{j}\}_{i \in S}$ and $\{Q_{i j} = 0, i \neq j, i, j \in S\}$ are solutions to inequalities (25)-(27). However, if the inequalities (25)-(27) have a set of solutions $\{P_{j}\}_{j \in S}$, $\{Q_{ij}\}_{i,j \in S}$, the inequalities (33) do not necessarily have a solution (See Example 1). Thus, the line integral based stochastic stability result is more general than that derived from quadratic Lyapunov function.

The following example is given to illustrate that the line integral based analysis results of stochastic asymptotic stability in Theorem 1 are less conservative than the ones based on quadratic Lyapunov function.

**Example 1:** For the unforced stochastic T-S model, the fuzzy rules are given as follows:

$R_{1}:$ If $x_{1}$ is $F_{1}^{1}$ and $x_{2}$ is $F_{2}^{1}$, then $dx = A_{1} x dt + C_{1} x dW(t)$

$R_{2}:$ If $x_{1}$ is $F_{1}^{2}$ and $x_{2}$ is $F_{2}^{2}$, then $dx = A_{2} x dt + C_{2} x dW(t)$

$R_{3}:$ If $x_{1}$ is $F_{1}^{3}$ and $x_{2}$ is $F_{2}^{3}$, then $dx = A_{3} x dt + C_{3} x dW(t)$

$R_{4}:$ If $x_{1}$ is $F_{1}^{4}$ and $x_{2}$ is $F_{2}^{4}$, then $dx = A_{4} x dt + C_{4} x dW(t)$

where $a$ and $b$ are real parameters. Then all ordinal numbers $\alpha_{ij}, i = 1, \ldots, 4; j = 1, 2$ read as

$\alpha_{11} = 1, \alpha_{21} = 1, \alpha_{31} = 2, \alpha_{41} = 2$

$\alpha_{12} = 1, \alpha_{22} = 2, \alpha_{32} = 1, \alpha_{42} = 2$

with $s_{1} = s_{2} = 2$. Thus, it is clear that the results in Theorem 1 yield larger regions of stochastic stability than the ones in Corollary 1. As expected, stability results derived from line integral function are less conservative than those derived from quadratic Lyapunov function.

**IV. FUZZY CONTROLLER DESIGN**

In this section, stability analysis and controller design for Itô closed-loop stochastic system will be addressed. More precisely, we are interested in finding a fuzzy controller such that the resulting Itô closed-loop stochastic system is stochastically asymptotically stable. The fuzzy control law can be described as follows:

**Controller Rule i:**

If $x_{1} \in F_{1}^{\alpha_{i1}}$, and $\ldots$, $x_{n} \in F_{n}^{\alpha_{in}}$, then $u = K_{i} x$, $i \in S$. The overall state feedback fuzzy controller is represented by

$$u = \sum_{i=1}^{s} h_{i} K_{i} x.$$  

Substituting (34) into (4) yields

$$dx = \sum_{i=1}^{s} \sum_{j=1}^{n} h_{i} h_{j} A_{ij} x dt + \sum_{i=1}^{s} h_{i} C_{i} x dW(t)$$  

where

$$A_{ij} = A_{i} + B_{i} K_{j}.$$  

For the closed-loop Itô stochastic system, we have the following result.

**Theorem 2:** Under Assumption 1, the equilibrium point of the closed-loop Itô stochastic system is stochastically asymptotically stable in the large, if there exist matrices $\{D_{k}\}_{k \in S}$ and $P$ being of the forms given in (9), a diagonal matrix $D$, as well as matrices $\{Q_{ijk} : Q_{ij} = \tilde{Q}_{ijk}\}_{i,j,k \in S}$ such that for all $i, j, k \in S$, the following inequalities hold:

$$P_{k} = \tilde{P} + D_{k} > 0, \quad D_{k} D_{k}^{T} \geq 0$$
\[(P_k A_{ij})^S + C_i^T (P_k + \beta D) C_i + \tilde{Q}_{ijk} < 0 \] 

(38)  

where \( \beta \) is given in (44). In this case, the fuzzy local feedback gains are given by  

\[ K_j = M_j \Omega_j^{-1}. \]  

(46)  

**Proof:** Assume that there exist matrices \( D_k \) and \( \tilde{P} \) in the form of (40) and matrices \( M_j, \Omega_j, \tilde{P}_k, R_{ijk}, \{Q_{ijk} : Q_{ijk} = Q_{ij k} \} \) being a set of solutions to the minimization problem (41) to (45). It follows from the cone complementarity linearization algorithm in (26) that the matrices \( D_k, P, M_j, \Omega_j, R_{ijk} \) and \( Q_{ijk} \) satisfy  

\[
\begin{bmatrix}
\Lambda_{ijk} & Y_{ij} & 0 & X_k C_i^T & X_k C_i^T \\
* & -\Omega_j & X_k - \Omega_j & 0 & 0 \\
* & * & -R_{ijk} & 0 & 0 \\
* & * & * & -X_k & 0 \\
* & * & * & * & -D^{-1} \beta^{-1}
\end{bmatrix} > 0
\]  

(47)  

where \( X_k = P_k^{-1} \) with \( P_k = \tilde{P} + D_k \). Recalling (36) and (46), the first expression of (47) becomes  

\[
\begin{bmatrix}
\tilde{\Lambda}_{ijk} & A_{ij} \Omega_j & 0 & X_k C_i^T & X_k C_i^T \\
* & -\Omega_j & X_k - \Omega_j & 0 & 0 \\
* & * & -R_{ijk} & 0 & 0 \\
* & * & * & -X_k & 0 \\
* & * & * & * & -D^{-1} \beta^{-1}
\end{bmatrix} > 0
\]  

(48)  

where \( \tilde{\Lambda}_{ijk} = (A_{ij} \Omega_j)^S + R_{ij k} + Q_{ij k} \), which together with the Schur complement equivalence yields  

\[
\begin{bmatrix}
\Phi_{ijk} & \tilde{A}_{ij} \Omega_j & 0 & X_k C_i^T & X_k C_i^T \\
* & -\Omega_j & X_k - \Omega_j & 0 & 0 \\
* & * & -R_{ijk} & 0 & 0 \\
* & * & * & -X_k & 0 \\
* & * & * & * & -D^{-1} \beta^{-1}
\end{bmatrix} < 0
\]  

(49)  

where  

\[ \Phi_{ijk} = (A_{ij} \Omega_j)^S + R_{ij k} + Q_{ij k} + X_k C_i^T (X_k^{-1} + \beta D) C_i. \]  

Applying Lemma 3 to (49), one has  

\[(A_{ij} X_k)^S + Q_{ij k} + X_k C_i^T (X_k^{-1} + \beta D) C_i X_k < 0. \]  

(50)  

\[ (P_k A_{ij})^S + P_k Q_{ij k} P_k + C_i^T (P_k + \beta D) C_i < 0. \]  

(51)  

Letting  

\[ P_k Q_{ij k} P_k = \tilde{Q}_{ijk} \]  

(52)  

and using \( Q_{ijk} = Q_{ij k} \), one can easily check that  

\[ \tilde{Q}_{ijk} = P_k Q_{ij k} P_k = P_k Q_{ij k} P_k = P_k Q_{ij k} P_k = \tilde{Q}_{ijk} = \tilde{Q}_{ij k}. \]  

Inequalities (51) together with (52) gives  

\[(P_k A_{ij})^S + Q_{ij k} + C_i^T (P_k + \beta D) C_i < 0. \]  

(53)  

It follows from (52) and the first expression of (43) that  

\[
\begin{bmatrix}
\tilde{Q}_{11k} & \cdots & \tilde{Q}_{1sk} \\
\vdots & \ddots & \vdots \\
\tilde{Q}_{s1k} & \cdots & \tilde{Q}_{ssk}
\end{bmatrix} =
\begin{bmatrix}
P_k Q_{11k} P_k & \cdots & P_k Q_{1sk} P_k \\
\vdots & \ddots & \vdots \\
P_k Q_{s1k} P_k & \cdots & P_k Q_{ssk} P_k
\end{bmatrix}
\]  

(44)  

\[
\begin{bmatrix}
P_k \Theta_2 \end{bmatrix} > 0.
\]  

(54)  

The conditions \( P_k = \tilde{P} + D_k > 0, D - D_j \geq 0 \) in (42) together with (53) and (54) prove the conclusion by Theorem 2. □
Remark 7: One idea behind Theorem 4 is to decouple $A_{ij}X_k$ in (51) by introducing the additional matrix variables $\Omega_j$ based on Lemma 3. Thus, by tolerating some conservativeness, the nonlinear inequalities (50) can be expressed as inequalities (48), where $B_iK_j\Omega_j$ in $A_{ij}\Omega_j$ makes the determination of the control gains $K_j$ feasible by change of variables. Another idea behind this theorem is that the nonlinear inequalities (47) in which both $X_k$ ($X_k=P_k^{-1}$) and $P_k$ appear, can be converted into the minimization problem (42)-(45), which can be solved by using the cone complementarity linearization algorithm. However, from Lemma 3 one can see that inequalities (48) are a sufficient condition for the inequalities (50) holding, which means that the existence of solutions for (50) does not guarantee the existence of solutions for (48), so the transformation from (50) to (48) leads to some conservativeness. In addition, it should be pointed out that for fixed $j$ in (44)-(45), the same $\Omega_j$ is required to satisfy $s^2$ inequalities when $i, k$ range from 1 to $s$, which inevitably results in some conservativeness.

Solving the quadratic optimization problem in Theorem 3 is an iterative procedure which is summarized in the following Iteration algorithm 1. It is clear when LMIIs (42)-(45) have no solution, so does the minimization problem (41)-(45). For this reason, it is assumed that LMIIs (42)-(45) is feasible in this iteration algorithm.

Iteration algorithm 1:

1. Given $\epsilon > 0$ and a natural number $n_{max}$, determine a feasible solution $I_0 = \left( \tilde{P}_{10}, \cdots, \tilde{P}_{s0}, D_{10}, \cdots, D_{s0}, \tilde{P}(0), \tilde{D}_0, D(0), \tilde{I}_0 \right)$ to LMIIs (42)-(45) where $\tilde{I}$ stands for other matrix variables in Theorem 3 and the index 0's in these symbols denote data appear in the initial iterative step. The iteration error is defined by $E_0 = \sum_{k=1}^{s} \text{tr} \left[ \tilde{P}_{k0} (D_{k0} + \tilde{P}(0)) \right] + \text{tr} (\tilde{D}_0 D(0)) - (s+1)n$.

If $\| E_0 \| < \epsilon$, then exit; or else set $j = 0$ and go to step (2).

2. For $j \geq 1$, assume that the data $I_j = \left( \tilde{P}_{1j}, \cdots, \tilde{P}_{sj}, D_{1j}, \cdots, D_{sj}, \tilde{P}(j), \tilde{D}_j, D(j), \tilde{I}_j \right)$ are obtained in the $j$th iteration step. For data $I_j$ obtained in the $j$th step, solve the LMI problem:

$$\min \left\{ \sum_{k=1}^{s} \text{tr} \left[ \tilde{P}_{kj} (D_k + \tilde{P}) + \tilde{P}_k (D_k + \tilde{P}(j)) \right] + \text{tr} (\tilde{D}_j D + \tilde{D}(j)) \right\}$$

subject to (42)-(45) for determining its matrix variables $\{\tilde{P}_i, D_i, P, I\}_{j=1}^{s}$. Letting $D_{i,j+1} = D_i$, $\tilde{P}_{i,j+1} = \tilde{P}_i$, $P(j+1) = \tilde{P}$, $\tilde{I}_{j+1} = \tilde{I}$ then we can obtain the data $I_{j+1} = (\tilde{P}_{1,j+1} \cdots \tilde{P}_{s,j+1}, D_{1,j+1} \cdots D_{s,j+1}, \tilde{P}(j+1), \tilde{D}_{j+1}, D(j+1), \tilde{I}_{j+1})$.

(3) The $j$th iteration error is defined by

$$E_j = \sum_{k=1}^{s} \text{tr} \left[ \tilde{P}_{k,j+1} (D_{k,j+1} + \tilde{P}(j+1)) \right] + \text{tr} (\tilde{D}_{j+1} D(j+1)) - (s+1)n.$$ 

If $| E_j | < \epsilon$ or $j = n_{max}$, then exit; or else set $j = j + 1$ and go to step (2).

V. NUMERICAL EXAMPLES

Example 2: The Itô stochastic T-S model based control system has the following fuzzy rules:

$R_1$: If $x_1$ is $F_1^1$ and $x_2$ is $F_2^1$, then $dx = (A_1 x + B_1 u)dt + C_1 x dW(t)$

$R_2$: If $x_1$ is $F_1^2$ and $x_2$ is $F_2^2$, then $dx = (A_2 x + B_2 u)dt + C_2 x dW(t)$

$R_3$: If $x_1$ is $F_1^2$ and $x_2$ is $F_2^2$, then $dx = (A_3 x + B_3 u)dt + C_3 x dW(t)$

$R_4$: If $x_1$ is $F_1^2$ and $x_2$ is $F_2^3$, then $dx = (A_4 x + B_4 u)dt + C_4 x dW(t)$

where all the known matrices are listed as follows:

$$A_1 = \begin{bmatrix} -0.5 & 0 \\ -1 & 1.59 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.8 & 1 \\ 0 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -2 \\ 3 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -0.9 & 1.09 \\ 0 & 0.8 \end{bmatrix}, \quad A_4 = \begin{bmatrix} -1 & 1 \\ 0 & 0.4 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 2.0 & 0 \\ 0 & -0.1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0.3 & 0 \\ 0 & -0.2 \end{bmatrix}, \quad B_3 = \begin{bmatrix} -1 \\ 3 \end{bmatrix},$$

$$C_3 = \begin{bmatrix} 0.2 & 0 \\ 0 & -0.3 \end{bmatrix}, \quad C_4 = \begin{bmatrix} 0.35 & 0 \\ 0 & -0.1 \end{bmatrix}, \quad B_4 = \begin{bmatrix} -1.5 \\ 2.6 \end{bmatrix}.$$ 

Then all ordinal numbers $\alpha_{ij}, i = 1, \cdots, 4; j = 1, 2$ read $\alpha_{11} = 1$, $\alpha_{21} = 1$, $\alpha_{31} = 2$, $\alpha_{41} = 2$

$\alpha_{12} = 1$, $\alpha_{22} = 2$, $\alpha_{32} = 1$, $\alpha_{42} = 2$

with $s_1 = s_2 = 2$. It is assumed that the membership functions $\{ w_{ij}^2(x_j), i = 1, 2; j = 1, 2 \}$ of the fuzzy sets $\{ F_i : i, j = 1, 2 \}$ in the fuzzy rules are given by

$$w_{11}^2(x_1) = 0.0169 e^{-x_1^2}, \quad w_{12}^2(x_2) = 0.0024 e^{-0.05(x_2-3)^2}$$

$$w_{21}^2(x_1) = 1 - w_{11}^2(x_1), \quad w_{22}^2(x_2) = 1 - w_{12}^2(x_2).$$

One has by (1) that

$$\mu_{11}^1 (x_1) = \mu_{21}^2 (x_1) = w_{11}^2(x_1) = 0.0169 e^{-x_1^2}$$

$$\mu_{12}^1 (x_1) = \mu_{22}^2 (x_1) = w_{12}^2(x_1) = 0.0169 e^{-x_1^2}$$

$$\mu_{12}^2 (x_2) = w_{22}^2(x_2) = 0.0024 e^{-0.05(x_2-3)^2}$$

$$\mu_{22}^2 (x_2) = \mu_{22}^2 (x_2) = w_{22}^2(x_2) = 1 - 0.0024 e^{-0.05(x_2-3)^2}.$$

It can be checked that Assumption 1 is satisfied for all the $\mu_{ij}^1(x_j), i = 1, \cdots, 4; j = 1, 2$ with all upper bounds $\beta_{ij}, i = 1, \cdots, 4; j = 1, 2$ listed in Table 1.

| $\beta_{11}$ | $\beta_{21}$ | $\beta_{12}$ | $\beta_{22}$ | $\beta_{13}$ | $\beta_{23}$ | $\beta_{14}$ | $\beta_{24}$ |
|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| 0.0125      | 0.0125      | 0.0125      | ×           | ×           | ×           | ×           | ×           |

Table 1: Upper bounds $\beta_{ij}$
The objective is to design a controller such that the resulting Itô closed-loop stochastic system is stochastically asymptotically stable. Solving the minimization problem (41)-(45), we obtain a set of solutions as follows:

\[
\begin{align*}
 P_1 &= 
\begin{bmatrix}
 0.8004 & 0.1785 \\
 0.1785 & 2.4479 
\end{bmatrix}, \quad P_2 = 
\begin{bmatrix}
 0.8217 & 0.1785 \\
 0.1785 & 2.4479 
\end{bmatrix}, \\
 P_3 &= 
\begin{bmatrix}
 0.8217 & 0.1785 \\
 0.1785 & 2.4479 
\end{bmatrix}, \quad P_4 = 
\begin{bmatrix}
 0.8004 & 0.1785 \\
 0.1785 & 2.4479 
\end{bmatrix}, \\
 \bar{P}_1 &= 
\begin{bmatrix}
 1.2701 & -0.0926 \\
 -0.0926 & 0.4153 
\end{bmatrix}, \quad \bar{P}_2 = 
\begin{bmatrix}
 1.2365 & -0.0902 \\
 -0.0902 & 0.4151 
\end{bmatrix}, \\
 \bar{P}_3 &= 
\begin{bmatrix}
 1.2353 & -0.0843 \\
 -0.0843 & 0.3883 
\end{bmatrix}, \quad \bar{P}_4 = 
\begin{bmatrix}
 1.2687 & -0.0866 \\
 -0.0866 & 0.3884 
\end{bmatrix}, \\
 D &= 
\begin{bmatrix}
 1.5380 & 0 \\
 0 & 2.6231 
\end{bmatrix}, \quad \bar{D} = 
\begin{bmatrix}
 0.6502 & 0 \\
 0 & 0.3812 
\end{bmatrix}, \\
 \Omega_1 &= 
\begin{bmatrix}
 1.7249 & 0.0649 \\
 -0.2823 & 0.3543 
\end{bmatrix}, \quad \Omega_2 = 
\begin{bmatrix}
 1.5810 & 0.1353 \\
 -0.3666 & 0.2958 
\end{bmatrix}, \\
 \Omega_3 &= 
\begin{bmatrix}
 1.7120 & 0.0859 \\
 -0.3078 & 0.3397 
\end{bmatrix}, \quad \Omega_4 = 
\begin{bmatrix}
 1.9383 & 0.0697 \\
 -0.3188 & 0.3510 
\end{bmatrix}, \\
 M_1 &= 
\begin{bmatrix}
 0.4650 & -0.3815 \\
 -0.3815 & 0.6144 
\end{bmatrix}, \quad M_2 = 
\begin{bmatrix}
 0.3715 & -0.3311 \\
 -0.3311 & 0.4414 
\end{bmatrix}, \\
 M_3 &= 
\begin{bmatrix}
 0.3924 & -0.3659 \\
 -0.3659 & 0.6144 
\end{bmatrix}, \quad M_4 = 
\begin{bmatrix}
 0.4414 & -0.3802 \\
 -0.3802 & 0.6144 
\end{bmatrix}.
\end{align*}
\]

By Theorem 3, the local feedback gains are given by

\[
K_1 = 
\begin{bmatrix}
 0.0906 & -0.10936 \\
 0.0340 & -0.10855 
\end{bmatrix}, \quad K_2 = 
\begin{bmatrix}
 -0.0222 & -1.1092 \\
 0.0480 & -1.0927 
\end{bmatrix}.
\]

To illustrate the results in Theorem 3, Monte Carlo simulations have been carried out by using a discretization approach [7], [27], [28]. Some initial parameters are given as follows: simulation interval \( t \in [0, T] \) with \( T = 15 \), initial values \( x_a(0) = (-7, 3)^T, x_b(0) = (-5, 5)^T, x_c(0) = (-5, 10)^T, x_d(0) = (12, 10)^T \), normally distributed variance \( \delta t = T/N \) with \( N = 2^8 \), step size \( \Delta t = R \delta t \) with \( R = 2 \). Figures (a)-(d) depict the state responses along 2, 10, 30 and 50 individual Wiener process paths respectively, as well as their mean values over these paths. These figures also show the stochastic stability of the Itô closed-loop stochastic system in (35).

**VI. CONCLUSION**

In this section, the main results and the major features of this paper are summarized as follows. Line integral approach to stability analysis and stabilization is proposed for Itô stochastic T-S models. The stochastic stability analysis of these models needs to handle Hessian matrix of the line integral Lyapunov function. One can see from (18) that the Hessian matrix can be decomposed into two parts: one part is the weighting sum of a set of positive definite matrices \( P_i \); Another is a sum of \( x \) dependent matrices \( \frac{\partial h}{\partial x} x^T D_i \) with rank one. Invoking the property in (21) of matrix with rank one and applying Lemma 1 lead to the upper bound \( \beta y^T D y \sum_{i=1}^n \frac{\partial h_i(x)}{\partial x} x^T D y \). Then an estimate of \( LV \) can also be achieved. It has been shown that the line integral based analysis results of stability are more general than those ones derived from quadratic Lyapunov function due to the fact that a quadratic Lyapunov function is a special line integral. By facilitating the cone complementarity linearization algorithm and introducing some additional matrix variables, a line integral approach is proposed for stabilization of the Itô stochastic T-S model [4].

There is also some conservativeness about the proposed approach to stabilization, one source of which may be that to make controller design feasible, we have to impose certain restrictions on the additional matrix variables. Furthermore, using Lemma 3 to derive the stabilization result in Theorem 3 also results in some conservativeness.

By using the techniques in this paper, the results on the stability and stabilization of the Itô stochastic T-S model can be readily extended to the fuzzy systems with uncertainties of the norm-bounded or linear fractional types. In addition, the problem of output feedback control for this class of systems with (or without) time-delay deserves further attention.

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