D-dimensional Randall–Sundrum models from Brans–Dicke theory and Kaluza–Klein modes

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Abstract. We investigate the spectroscopy of scalar and vector Kaluza–Klein modes that arise in a deformed Randall–Sundrum model that is constructed from Brans–Dicke theory. The non-minimal coupling in the Brans–Dicke theory translates into a deformation of the Randall–Sundrum geometry that depends on the Brans–Dicke parameter $\omega$. We find that the $\omega$ parameter has a nontrivial effect in the spectroscopy of scalar and vector Kaluza–Klein modes. Our results suggest the interpretation of $\omega$ as a fine-tuning parameter.

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1. Introduction

The hierarchy between gravitational and electromagnetic forces motivated, in the early years, the Dirac cosmological model [1], which considers a time-dependent gravitational constant. This model inspired some field theory approaches such as the Jordan model [2], in which the gravitational constant is taken as a function of some scalar field. A complete scalar-tensor theory of gravitation was proposed in 1961 by Brans and Dicke, where the gravitational constant is inversely related to the scalar field [3].

Kaluza–Klein theories and string theory motivated several models involving extra dimensions and branes, the most interesting one being that proposed by Randall and Sundrum [4]. This model considers a configuration of two four-dimensional (4D) branes in 5D space-time with a negative cosmological constant. The hierarchy problem between the Planck and electroweak scale is solved by the warp factor present in the 5d metric. An important problem in the Randall–Sundrum scenario is the fixing of the extra dimension size \( L \). The first attempt to fix \( L \) was to consider a 5D scalar field with brane potentials [5]. Including the backreaction of this field on the metric led to a 5D scalar-tensor model [6] that differs from the original Randall–Sundrum solution. Recently, a 5D Brans–Dicke model with branes was proposed in [7]. Working in the Jordan–Fierz frame, the 5D Brans–Dicke action can lead to metric solutions very similar to the original Randall–Sundrum metric. The model of [7] includes the backreaction, and the solution is stable because the size of the extra dimension is fixed by the scalar field.

In this paper we construct \( D \)-dimensional Randall–Sundrum models from Brans–Dicke theory. We consider a BPS-like mechanism that translates the second-order differential equations coming from the Brans–Dicke action into first-order ones. In this way we find a special class of scalar potentials that simplifies the background solutions. A particular choice of the scalar potential leads to a Randall–Sundrum solution for the metric, which can be stabilized following a procedure similar to [7]. We analyze the possible implications of the \( D \)-dimensional Brans–Dicke parameter by performing a Kaluza–Klein decomposition of a massless scalar fluctuation living in the bulk. We find an interesting dependence of the \( D−1 \) dimensional scalar masses on the \( D \)-dimensional Brans–Dicke parameter. We also discuss the effect of the Brans–Dicke parameter on the Kaluza–Klein modes arising on a recent Higgless model for electroweak symmetry breaking [8]. Our results suggest the possibility of considering the Brans–Dicke parameter as a fine tuning for the \( W \) and \( Z \) resonances.

We begin in section 2 with a review of the Randall–Sundrum metric. In section 3 we show how this metric arises from the Brans–Dicke theory via a BPS-like mechanism. In section 4 we analyze the Kaluza–Klein modes arising from the decomposition of a massless scalar fluctuation, while in section 5 we discuss the gauge field Kaluza–Klein modes of a Higgless electroweak model. We end with conclusions in section 6.

2. The Randall–Sundrum metric in D-dimensions

The anti-de-Sitter (Ads) space-time is a maximally symmetric solution of Einstein equations with a negative cosmological constant \( \Lambda \). This space-time can be interpreted as a hyperboloid of radius \( \ell \) related to the cosmological constant by \(-\Lambda \ell^2 = (D−1)(D−2)\). The Poincaré chart cuts the hyperboloid into two regions (see [9] for details). The metric of each region can be
written as
\[ \text{d} \tilde{s}^2 = \frac{1}{k^2 z^2} [-\text{d}t^2 + \text{d} \tilde{x}_i^2 + \text{d}z^2], \]
where \( k = 1/\ell, \) \( \text{d} \tilde{x}_i^2 = \sum_{i=1}^{D-2} \text{d} x_i^2 \) and \( z > 0 \) (or \( z < 0 \)). The Randall–Sundrum metric can be constructed by considering two slices of the \( z > 0 \) region. For this purpose, it is convenient to define a new coordinate \( \Omega \) by \( z = \frac{1}{k} e^{\lambda \Omega} \). The two AdS slices are given by \( 0 < \omega \leq L \) and \( -L \leq \Omega < 0 \) and can be joined at \( \Omega = 0 \). The relation between \( z \) and \( \Omega \) is plotted in figure 1. The metric in terms of \( \Omega \) reads
\[ \text{d} \tilde{s}^2 = e^{2\sigma(\Omega)} [-\text{d}t^2 + \text{d} \tilde{x}_i^2] + \text{d}\Omega^2, \]
where \( \sigma(\Omega) = -k|\Omega| \) and \( -L \leq \Omega \leq L \). Identifying \( \Omega \) with \( -\Omega \), we obtain the orbifold space \( S^1/Z_2 \). The metric (2) naturally satisfies this condition.

The Randall–Sundrum metric was obtained from Einstein equations originating from a \( D \)-dimensional gravitational action with negative cosmological constant in the presence of two \( (D-1) \)-branes located at \( \Omega = 0 \) and \( \Omega = L \) with opposite tensions. We will see in the next section how this metric also arises from a \( D \)-dimensional Brans–Dicke theory.

3. Brans–Dicke theory and the deformation of Randall–Sundrum geometry

In this section we will use a BPS–like mechanism to solve the field equations of motion arising from a \( D \)-dimensional Brans–Dicke theory with two \( (D-1) \)-brane potentials. In this theory, there is a scalar field non-minimally coupled to gravity. The total action is given by
\[ S = \int d^{D-1}x \sqrt{-\tilde{g}} \left[ \tilde{\Phi} \tilde{R} - \frac{\omega}{\tilde{\Phi}} \tilde{g}^{MN} \partial_M \tilde{\Phi} \partial_N \tilde{\Phi} - \tilde{V}(\tilde{\Phi}) \right] \]
\[ - \int_{\Omega=0} \text{d}^{D-1}x \sqrt{-\tilde{h} \lambda_1(\tilde{\Phi})} - \int_{\Omega=L} \text{d}^{D-1}x \sqrt{-\tilde{h} \lambda_2(\tilde{\Phi})}, \]

Figure 1. The \( z \) dependence on \( \Omega \).
where the coordinates \( x^M = (x^\mu, \Omega) \) consist of \( D - 1 \) non-compact coordinates \( x^\mu \) and a compact coordinate \( \Omega \) defined in the interval \(-L \leq \Omega \leq L\) with the identification \( \Omega \to -\Omega \). The Ricci scalar of the metric \( \tilde{g}_{MN} \) is denoted by \( \tilde{R} \) and we work with the signature \((-+, \cdots, +)\). We denote by \( \tilde{h}_{\mu \nu} \) the induced metric on the branes. The term \( \tilde{V}(\tilde{\Phi}) \) is a bulk potential while \( \tilde{\lambda}_1 \) and \( \tilde{\lambda}_2 \) are brane potentials. The constant \( \omega \) is the \( D \)-dimensional Brans–Dicke parameter. The orbifold condition in \( \Omega \) implies

\[
\tilde{g}_{\mu \nu}(x, -\Omega) = \tilde{g}_{\mu \nu}(x, \Omega), \quad \tilde{g}_{\Omega \Omega}(x, -\Omega) = \tilde{g}_{\Omega \Omega}(x, \Omega),
\]

\[
(4)
\]

The action (3) leads to the following background equations:

\[
\tilde{R}^{\Omega \Omega} - \frac{1}{2} \tilde{g}^{\Omega \Omega}(\tilde{R} - \frac{\tilde{V}}{\tilde{\Phi}}) + \frac{\omega}{\tilde{\Phi}^2} \partial_M \tilde{\Phi} \partial_N \tilde{\Phi} \left[ \frac{1}{2} g^{\Omega \Omega} \tilde{g}^{MN} - \tilde{g}^{\Omega \Omega} \tilde{g}^{MN} \right] + \frac{\tilde{\Phi}_{M,N}}{\tilde{\Phi}} \left[ \tilde{g}^{\Omega \Omega} \tilde{g}^{MN} - \tilde{g}^{\Omega \Omega} \tilde{g}^{MN} \right] = 0,
\]

\[
(5)
\]

\[
\tilde{R}^{\mu \nu} - \frac{1}{2} \tilde{g}^{\mu \nu}(\tilde{R} - \frac{\tilde{V}}{\tilde{\Phi}}) + \frac{\omega}{\tilde{\Phi}^2} \partial_M \tilde{\Phi} \partial_N \tilde{\Phi} \left[ \frac{1}{2} g^{\mu \nu} \tilde{g}^{MN} - \tilde{g}^{\mu \nu} \tilde{g}^{MN} \right] + \frac{\tilde{\Phi}_{M,N}}{\tilde{\Phi}} \left[ \tilde{g}^{\mu \nu} \tilde{g}^{MN} - \tilde{g}^{\mu \nu} \tilde{g}^{MN} \right] + \frac{1}{2} \tilde{g}^{\mu \nu} \tilde{\lambda}_2 \delta(\Omega - L) + \frac{1}{2} \sqrt{g_{\Omega \Omega}} \tilde{\Phi} \delta(\Omega - L) = 0,
\]

\[
(6)
\]

\[
\frac{\omega}{\tilde{\Phi}^2} \partial_M \tilde{\Phi} \partial_N \tilde{\Phi} \tilde{g}^{MN} + \frac{2}{\sqrt{g}} \partial_M \left[ \sqrt{-\tilde{g}} \frac{\omega}{\tilde{\Phi}} \tilde{g}^{MN} \partial_N \tilde{\Phi} \right] + \tilde{R} - \frac{\partial \tilde{V}}{\partial \tilde{\Phi}} - \frac{1}{\sqrt{g_{\Omega \Omega}}} \frac{\partial \tilde{\lambda}_2}{\partial \tilde{\Phi}} \delta(\Omega - L) - \frac{1}{\sqrt{g_{\Omega \Omega}}} \frac{\partial \tilde{\lambda}_1}{\partial \tilde{\Phi}} \delta(\Omega - L) = 0.
\]

\[
(7)
\]

We consider the following ansatz for the metric and scalar field:

\[
ds^2 = e^{2\sigma(\Omega)} \eta_{\mu \nu} dx^\mu dx^\nu + d\Omega^2, \quad \tilde{\Phi} = \tilde{\Phi}(\Omega),
\]

\[
(8)
\]

where \( \tilde{\Phi}(\Omega) \) and \( \sigma(\Omega) \) are even functions in \( \Omega \). The background equations above then translate into a system of second-order differential equations:

\[
\frac{1}{2} (D - 2) (D - 1) \sigma^2 \tilde{\Phi} + \frac{\tilde{V}}{2} - \frac{w}{2 \tilde{\Phi}} \tilde{\Phi}^2 + (D - 1) \sigma^2 \tilde{\Phi}' = 0,
\]

\[
(9)
\]

\[
\tilde{\Phi}'' + \frac{w}{\tilde{\Phi}} \tilde{\Phi}^2 - \sigma' \tilde{\Phi}' + (D - 2) \sigma'' \tilde{\Phi} + \frac{1}{2} \tilde{\lambda}_1 \delta(\Omega - L) + \frac{1}{2} \tilde{\lambda}_2 \delta(\Omega - L) = 0,
\]

\[
(10)
\]

\[
\frac{w}{\tilde{\Phi}} \tilde{\Phi}'' + (D - 1) w \frac{\sigma'}{\tilde{\Phi}} \tilde{\Phi}' - (D - 1) \sigma'' - \frac{1}{2} D (D - 1) \sigma^2 - \frac{w}{2 \tilde{\Phi}^2} \tilde{\Phi}'' - \frac{1}{2} \frac{\partial \tilde{V}}{\partial \tilde{\Phi}} - \frac{1}{2} \frac{\partial \tilde{\lambda}_1}{\partial \tilde{\Phi}} \delta(\Omega - L) - \frac{1}{2} \frac{\partial \tilde{\lambda}_2}{\partial \tilde{\Phi}} \delta(\Omega - L) = 0.
\]

\[
(11)
\]
Finding a solution of these differential equations is in general complicated for an arbitrary potential $\tilde{V}(\tilde{\Phi})$. We could also invert the problem and solve the equations for the scalar field solution and potential once we know the metric. In this work, we use a BPS-like mechanism that simplifies the background equations and leads to a special class of potentials. The Randall–Sundrum solution for the metric arises from a particular potential belonging to this class.

If we substitute ansatz (8) in the Lagrangian density of equation (3), we find

$$L = -e^{(D-1)\sigma} \left\{ (D - 1) \tilde{\Phi} (2\sigma'' + D\sigma'^2) + \omega \frac{\tilde{\Phi}^2}{\Phi} + \tilde{V} + \tilde{\lambda}_1 \delta(\Omega) + \tilde{\lambda}_2 \delta(\Omega - L) \right\}. \quad (12)$$

In order to have periodicity in the coordinate $\Omega$ and justify the presence of the $\delta(\Omega - L)$ function, the Lagrangian density has to be integrated from $-L + \epsilon$ to $L + \epsilon$ and has to make $\epsilon \to 0$ at the end. The Lagrangian density can be rewritten in the following form:

$$L = -e^{(D-1)\sigma} \left\{ \left( \omega + \frac{D - 1}{D - 2} \right) \tilde{\Phi} \left( \frac{\tilde{\Phi}'}{\Phi} - \tilde{W} + (D - 2) \tilde{\Phi} \frac{\partial \tilde{W}}{\partial \tilde{\Phi}} \right)^2 - (D - 1)(D - 2) \tilde{\Phi} \left( \sigma' + \frac{1}{D - 2} \frac{\tilde{\Phi}'}{\Phi} - \left( \omega + \frac{D - 1}{D - 2} \right) \tilde{W} \right)^2 \right. \right.$$  

$$+ \left. \left[ \tilde{V} + [(D - 2)\omega + (D - 1)] \left[ (D - 1)\omega + D \right] \tilde{\Phi} \tilde{W}^2 + \tilde{\Phi}^2 \tilde{W} \frac{\partial \tilde{W}}{\partial \tilde{\Phi}} - (D - 2) \tilde{\Phi}^3 \left( \frac{\partial \tilde{W}}{\partial \tilde{\Phi}} \right)^2 \right] \right] \right.$$  

$$+ \left. \left[ 2[(D - 2)\omega + (D - 1)] \frac{\partial \tilde{W}}{\partial \Omega} \tilde{\Phi} + \tilde{\lambda}_1 \delta(\Omega) + \tilde{\lambda}_2 \delta(\Omega - L) \right] \right\} - 2(D - 1) \left[ \sigma' \tilde{\Phi} e^{(D-1)\sigma} \right]' + 2[(D - 2)\omega + (D - 1)] \left[ \tilde{\Phi} \tilde{W} e^{(D-1)\sigma} \right]', \quad (13)$$

where we have introduced an arbitrary odd function

$$\tilde{W}(\tilde{\Phi}) = \begin{cases} W(\tilde{\Phi}), & \text{if } 0 < \Omega < L, \\ -W(\tilde{\Phi}), & \text{if } -L < \Omega < 0. \end{cases} \quad (14)$$

The last two terms in (13) are total derivatives, so they vanish using the periodicity of $\Omega$. The first two terms are square terms, which are zero when

$$\tilde{\Phi}' = \left[ \tilde{\Phi} \tilde{W} - (D - 2) \tilde{\Phi}^2 \frac{\partial \tilde{W}}{\partial \tilde{\Phi}} \right], \quad (15)$$

$$\sigma' = \left[ (\omega + 1) \tilde{W} + \tilde{\Phi} \frac{\partial \tilde{W}}{\partial \tilde{\Phi}} \right]. \quad (16)$$
Assuming that the equations above are satisfied by the scalar field and the metric, we find that the following class of bulk potentials,

\[ \tilde{V} = -[(D - 2)\omega + (D - 1)] \left[ ((D - 1)\omega + D)\tilde{\Phi} \tilde{W}^2 + 2\tilde{\Phi}^2 \frac{\partial \tilde{W}}{\partial \tilde{\Phi}} - (D - 2)\tilde{\Phi}^3 \left( \frac{\partial \tilde{W}}{\partial \tilde{\Phi}} \right)^2 \right], \tag{17} \]

with the brane conditions

\[ 2[(D - 2)\omega + (D - 1)] \frac{\partial \tilde{W}}{\partial \Omega} = -\frac{\tilde{\lambda}_1}{\Phi} \delta(\Omega) - \frac{\tilde{\lambda}_2}{\Phi} \delta(\Omega - L), \tag{18} \]

leads to a vanishing action. Using (14) the brane conditions read

\[ [(D - 2)\omega + (D - 1)]W(\tilde{\Phi})|_{\Omega=0^+} = -\frac{\tilde{\lambda}_1}{4\tilde{\Phi}}, \tag{19} \]

\[ [(D - 2)\omega + (D - 1)]W(\tilde{\Phi})|_{\Omega=L^-} = \frac{\tilde{\lambda}_2}{4\tilde{\Phi}}, \tag{20} \]

and similarly for the derivatives in \( \tilde{\Phi} \). It is straightforward to show that the system of equations (15)–(20) gives background solutions that also satisfy the background equations (9)–(11). In this way we find a BPS-like mechanism that gives background solutions for second-order differential equations by solving first-order equations that appear inside the square terms in the Lagrangian density. Because the square terms appear with opposite signs, there is no Bogomolnyi bound. This mechanism is similar to that found in [6]. Note that for the case \( D = 5 \), our equations (15)–(20) reduce to those obtained in [7].

A Randall–Sundrum solution for the metric is obtained for the case of constant \( W \), where the potential and background solutions reduce to

\[ \tilde{V}(\tilde{\Phi}) = \Lambda \tilde{\Phi}, \quad \sigma = -k|\Omega|, \tag{21} \]

\[ \tilde{\Phi} = C \exp \left( \frac{\sigma}{\omega + 1} \right), \tag{22} \]

with

\[ C = \frac{1}{16\pi G_D}, \quad W = -\frac{k}{(w + 1)}, \tag{23} \]

\[ \Lambda = -[(D - 2)\omega + (D - 1)][(D - 1)\omega + D)]W^2. \]

The value of \( C \) was chosen for convenience. The brane potentials in this case are

\[ \tilde{\lambda}_1 = \lambda \tilde{\Phi}, \quad \tilde{\lambda}_2 = -\lambda \tilde{\Phi}, \tag{24} \]

with

\[ \lambda = 4 \sqrt{\frac{(D - 2)w + (D - 1)}{(D - 1)w + D}} \sqrt{-\Lambda}. \]

Note that although we have obtained the Randall–Sundrum metric (2), the scenario given by equations (21)–(23) is different from the traditional Randall–Sundrum scenario because the metric couples non-minimally with a nontrivial background scalar field. The traditional Randall–Sundrum scenario can be obtained in the limit \( \omega \to \infty \) in which the scalar field becomes trivial, as discussed in [7].

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The Einstein frame

If we perform the following background transformations,

\[ \tilde{g}_{MN} = e^{2\alpha} g_{MN}, \quad \tilde{\Phi} = \frac{1}{16\pi G_D} e^{-(D-2)\alpha} \Phi, \]

\[ \tilde{V}(\tilde{\Phi}) = e^{-D\alpha} V(\Phi), \quad \tilde{\lambda}_i(\tilde{\Phi}) = e^{-(D-1)\alpha} \lambda_i(\Phi), \]

we go from the Jordan–Fierz frame (in which the Brans–Dicke theory is originally formulated) to the Einstein frame. These background transformations are known in the literature as conformal transformations \[10\]. Note that this transformation imposes a reality condition for the Brans–Dicke parameter: \( w > -\frac{D-1}{D-2} \). The total action (3) becomes

\[ S = \int d^D x \sqrt{-\tilde{g}} \left( \frac{1}{16\pi G_D} R - \frac{1}{2} \tilde{g}^{MN} \partial_M \tilde{\Phi} \partial_N \tilde{\Phi} - V(\tilde{\Phi}) \right) \]

\[ - \int_{\Omega=0} d^{D-1} x \sqrt{-\tilde{h}} \lambda_1(\tilde{\Phi}) - \int_{\Omega=L} d^{D-1} x \sqrt{-\tilde{h}} \lambda_2(\tilde{\Phi}). \]

(27)

In the Einstein frame, the background solutions of (22) become

\[ ds^2 = e^{2\sigma(2w + 1)} \left[ \eta_{\mu\nu} \, dx^\mu \, dx^\nu + d\Omega^2 \right], \]

\[ \Phi = -\frac{1}{(D-2)\alpha} \left[ \frac{\sigma}{w + 1} \right]. \]

(28)

In terms of the coordinate \( z = 1/k e^{-\sigma(\Omega)} \), the metric reads

\[ ds^2 = f_\omega(z) \left( \frac{1}{(kz)^{2w + 1}} \left[ \eta_{\mu\nu} \, dx^\mu \, dx^\nu + dz^2 \right] \right), \quad f_\omega(z) \equiv (kz)^{-\frac{2}{(2w + 1)}}. \]

(29)

This metric can be interpreted as a deformed Randall–Sundrum metric where the deformation is given by \( f_\omega(z) \). Note that the Planck brane is localized at \( z = 1/k \) while the TeV brane is localized at \( z = (1/k)e^{kL} \). In the limit \( \omega \to \infty \), the deformation factor \( f_\omega(z) \) goes to 1 and we recover the original Randall–Sundrum metric.

4. Spectroscopy of scalar Kaluza–Klein modes

Now we consider the compactification of a massless scalar field fluctuation in the Einstein frame. This frame is well motivated for many reasons, the most important one being the positive sign of the energy density \[10\]. A scalar field fluctuation can be described by the following action:

\[ S = -\frac{1}{2} \int d^{D-1} x \int d\Omega \sqrt{-\tilde{g}} \tilde{g}^{MN} \partial_M \varphi \partial_N \varphi. \]

(30)

This action can be decomposed as

\[ S = -\frac{1}{2} \int d^{D-1} x \int d\Omega \left[ \sqrt{-\tilde{g} h(\Omega)\eta^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - \varphi \partial_\Omega \left( \sqrt{-\tilde{g}} \tilde{g}^{\Omega\Omega} \partial_\Omega \varphi \right) \right]. \]

(31)
where \( h(\Omega) \) is defined by \( g^{\mu\nu} = \eta^{\mu\nu} h(\Omega) \). The Kaluza–Klein decomposition of \( \varphi(x, \Omega) \) is

\[
\varphi(x, \Omega) = \frac{1}{\sqrt{L}} \sum_n \phi_n(x) \chi_n(\Omega).
\]

(32)

If the modes \( \chi_n(\Omega) \) satisfy the relations

\[
\frac{1}{L} \int_{-L}^{L} d\Omega \sqrt{\eta} h(\Omega) \chi_n(\Omega) \chi_m(\Omega) = \delta_{nm},
\]

(33)

\[
\frac{2}{\sqrt{\Omega}} \frac{d}{d\Omega} \left( \sqrt{\eta} \Omega \frac{d\chi_n}{d\Omega} \right) = -m_n^2 \sqrt{\eta} h(\Omega) \chi_n,
\]

(34)

then we get the \((D - 1)\)-dimensional action for \( \phi_n(x) \):

\[
S_{\text{eff}} = -\frac{1}{2} \sum_n \int d^{D-1} x \left[ \eta^{\mu\nu} \partial_\mu \varphi_n \partial_\nu \varphi_n + m_n^2 \varphi_n^2 \right].
\]

(35)

As in the usual Kaluza–Klein compactifications, the bulk field \( \phi(x, \Omega) \) manifests to a \((D - 1)\)-dimensional observer as an infinite tower of scalars \( \phi_n(x) \) with masses \( m_n \).

The tower of masses \( m_n \) can be obtained by solving equation (34), which can be rewritten as

\[
e^{2\sigma} \frac{1}{v(\Omega)} \frac{d}{d\Omega} \left( v(\Omega) \frac{d\chi_n}{d\Omega} \right) = -m_n^2 \chi_n,
\]

(36)

where

\[
v(\Omega) = e^{[\sigma/(w + 1)]((D - 1)w + D)}, \quad h(\Omega) = e^{(-2\sigma/(w + 1))[(w + (D - 1)/(D - 2))]}.
\]

(37)

It is convenient to solve this equation in terms of the coordinate \( z = \frac{1}{\Omega} e^{-\sigma} \),

\[
z^u \frac{d}{dz} \left[ z^{-u} \frac{d}{dz} \chi_n \right] = -m_n^2 \chi_n,
\]

(38)

where \( u = [(D - 2)\omega + D - 1]/[(\omega + 1)] \). This equation has a zero-mode solution corresponding to \( m_n = 0 \) of the form

\[
\chi_n(z) = c_1 + c_2 z^{(u+1)}.
\]

(39)

For \( m_n > 0 \) the solution is a combination of Bessel \( J \) and Bessel \( Y \) functions of argument \( m_n z \). In terms of \( \Omega \), the solution reads

\[
\chi_n = \frac{e^{-\nu \sigma}}{N_n} \left[ J_\nu \left( \frac{m_n}{k} e^{-\sigma} \right) + b_n Y_\nu \left( \frac{m_n}{k} e^{-\sigma} \right) \right].
\]

(40)

where \( \sigma = -k|\Omega| \) and \( \nu = [(D - 1)w + D]/[2(w + 1)] = (u + 1)/2 \) and \( N_n \) is a normalization constant. Besides the condition \( w > -1 \), it is interesting to note that in order to find finite \( \nu \) we need \( \omega \neq -1 \). The limit \( \omega \to \infty \) leads to the result found in [11] for the massless case. Our modes solutions are even functions in \( \Omega \). To guarantee continuity at the orbifold points \( \Omega = 0 \) and \( \Omega = L \), we impose Neumann boundary conditions. The boundary condition at \( \Omega = 0 \) leads to

\[
b_n^{\nu} = \frac{-J_{\nu-1}(\chi_n e^{-kL})}{Y_{\nu-1}(\chi_n e^{-kL})},
\]

(41)
Figure 2. Behavior of the first Kaluza–Klein modes $x_{1\nu}$, $x_{2\nu}$ and $x_{3\nu}$ as functions of the Brans–Dicke parameter $\omega$ for the case $D = 5$.

where we have defined $x_{n\nu} = (m_n/k)e^{kL}$. The boundary condition at $\Omega = \pi$ gives the important equation

$$x_{n\nu}^2 e^{-kL} \left[ J_{\nu-1}(x_{n\nu}) Y_{\nu-1}(x_{n\nu} e^{-kL}) - Y_{\nu-1}(x_{n\nu}) J_{\nu-1}(x_{n\nu} e^{-kL}) \right] = 0.$$  \hspace{1cm} (42)

The Kaluza–Klein modes $x_{n\nu}$ are obtained by solving this equation. We choose $kL = 12$ as considered in the original Randall–Sundrum model. We present in figure 2 our results for the first modes as functions of the Brans–Dicke parameter $\omega$ in the particular case $D = 5$. We see from that figure that the modes grow rapidly when $\omega \to -1$ and approach constant functions for large $\omega$ (for instance, $x_{1\nu} \to 3.83$ for large $\omega$). In this way the distance between these modes is preserved at large $\omega$. Figure 3 shows how the first mode $x_{1\nu}$ increases with dimension.

The normalization constant $N_n$ appearing in the modes solutions can be calculated by performing the integral of equation (33). This integral is not simple in general because it involves products of Bessel $J$ and Bessel $Y$ functions. However, for the lower modes the dominant contribution to the integral comes from the square of Bessel $J$. For these cases the normalization constant can be approximated by

$$N_n \approx \frac{1}{\sqrt{kL}} e^{kL} J_\nu(x_{n\nu}),$$ \hspace{1cm} (43)

where we have supposed in this approximation that $kL$ is large, as expected for the resolution of the hierarchy problem [7].

5. The effect of the Brans–Dicke parameter in electroweak phenomenology

We analyze in this section an interesting application of the Brans–Dicke Randall–Sundrum scenario considered in this paper. This application concerns the Higgless model of [8] (a review can be found in [14]). The model consists of an SU$(2)_L \times SU(2)_R \times U(1)_{B–L}$ gauge group living in a 5D AdS metric limited by flat 3-branes (the Randall–Sundrum scenario revised in New Journal of Physics 12 (2010) 053038 (http://www.njp.org/))
Figure 3. Influence of dimension $D$ on the first Kaluza–Klein mode. Each line corresponds to $x_{1\nu}$ as a function of the Brans–Dicke parameter $\omega$ for a particular dimension. This mode has asymptotic values 3.14, 3.83, 4.49 and 5.16 for the cases $D = 4, 5, 6$ and 7, respectively.

section 2). The gauge symmetry is broken by imposing gauge field boundary conditions on the 3-branes while the Kaluza–Klein towers arising from gauge field fluctuations are interpreted as $W^\pm$ and $Z$ resonances, being the lowest modes associated with the experimentally observed $W$ and $Z$ particles.

In our case the metric contains an extra degree of freedom, which is the Brans–Dicke parameter. As seen in the last section, this parameter acts as a fine tuning for the Kaluza–Klein masses arising from scalar fluctuations. We will see in this section how the $W^\pm$ and $Z$ resonances of the Higgless model will depend on the Brans–Dicke parameter as well.

We begin with the $SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ action

$$S = -\frac{1}{4} \sum_{a=1}^{3} \int d^4x \int dz \sqrt{-g} \left[ F^{aM}_a F^{aM} + F^{aN}_a F^{aN} + B^{MN} B_{MN} \right],$$

where

$$B_{MN} = \partial_M B_N - \partial_N B_M, \quad F^{a(L,R)}_{MN} = \partial_M A^a_N - \partial_N A^a_M + g_5 f^{abc} A^b M A^c_N,$$

with $M = \{z, \mu\}$. We denote as $g_5$ the coupling constant of $SU(2)_L, SU(2)_R$ and as $\tilde{g}_5$ the $U(1)$ coupling constant. In order to cancel the interaction terms between the $z$ and $\mu$ components, we must add gauge fixing terms of the form

$$S_{gf} = \frac{1}{2\tilde{g}_5} \int d^4x dz \sqrt{-g} h^2(z) \left[ \eta^{\mu\nu} \partial_\mu A_\nu - \frac{\xi}{\sqrt{-g} h^2(z)} \partial_z (\sqrt{-g} h^2(z) A_z) \right]^2$$

with $A_M = \{A^a_M(L), A^a_M(R), B_M\}$ and

$$\sqrt{-g} h^2(z) = (kz)^{-1} \sqrt{f_\omega(z)}.$$
The bulk fields can be decomposed in the following way:

\[ B_\mu = g_5 a_0 \gamma_\mu(x) + \sum_{n=1}^{\infty} Z^{(n)}_\mu(x) \psi^n_B(z), \]  

\[ A^{3(L,R)}_\mu = \frac{g}{\sqrt{g}} a_0 \gamma_\mu(x) + \sum_{n=1}^{\infty} Z^{(n)}_\mu(x) \psi^{3(L,R)}_n(z), \]  

\[ A^{\pm(L,R)}_\mu = \sum_{n=1}^{\infty} W^{\pm(n)}_\mu(x) \psi^{\pm(L,R)}_n(z). \]

The boundary conditions on the Planck brane \( z = \frac{1}{k} \) are

\[ \bar{g} B_\mu - g A^{3(R)}_\mu = 0, \]  

\[ \partial_z [g B_\mu + \bar{g} A^{3(R)}_\mu] = 0, \quad \partial_z A^{3L}_\mu = 0, \]  

\[ \partial_z A^{\pm(L)}_\mu = 0, \quad A^{\pm(R)}_\mu = 0. \]

These conditions lead to the symmetry breaking \( \text{SU}(2)_R \times \text{U}(1)_{B-L} \to \text{U}(1)_Y \). The boundary conditions at the TeV brane \( z = \frac{1}{k} e^{kL} \) are

\[ A^{3(L)}_\mu - A^{3(R)}_\mu = 0, \]  

\[ \partial_z [A^{3(L)}_\mu + A^{3(R)}_\mu] = 0, \quad \partial_z B_\mu = 0, \]  

\[ \partial_z [A^{\pm(L)}_\mu + A^{\pm(R)}_\mu] = 0, \quad A^{\pm(L)}_\mu - A^{\pm(R)}_\mu = 0, \]

which lead to the symmetry breaking \( \text{SU}(2)_L \times \text{SU}(2)_R \to \text{SU}(2)_D \). According to the Kaluza–Klein decomposition (48)–(50), the kinetic terms read

\[
S_{\text{kin}} = -\frac{1}{4} \int d^4 x \, dz \, \sqrt{-g} h^2(z) \left\{ \eta^{\mu\alpha} \eta^{\nu\beta} \sum_{n,m=1}^{\infty} Z^{(n)}_\mu Z^{(m)}_\nu (\Psi_n^Z)^T \Psi_m^Z + 2 \sum_{n,m=1}^{\infty} \eta^{\mu\nu} Z^{(n)}_\mu Z^{(m)}_\nu (\Psi_n^Z)^T \frac{\partial_z [\sqrt{-g} h^2(z) \partial_z \Psi_n^Z]}{\sqrt{-g} h^2(z)} + \sum_{a=\pm} \sum_{n,m=1}^{\infty} (\Psi_n^W)^T \eta^{\mu\nu} \eta^{\rho\sigma} W^{(n)}_{\mu\nu} W^{(m)}_{\rho\sigma} (\Psi_m^W)^T + 2 \eta^{\mu\nu} W^a_{(n)} W^{a(m)} \frac{\partial_z [\sqrt{-g} h^2(z) \partial_z \Psi_m^W]}{\sqrt{-g} h^2(z)} \right\},
\]

where we defined the vectors \( \Psi^Z_\mu \equiv \{ \psi^{(B)}_n, \psi^{3(L)}_n, \psi^{3(R)}_n \} \) and \( \Psi^W_n \equiv \{ \psi^{L}_n, \psi^{R}_n \} \). This decomposition suggests the normalization conditions

\[
a_0 (g_5^2 + 2g_5^2) \int_{-L}^L dz \sqrt{-g} h^2(z) = 1, \quad \int_{-L}^L dz \sqrt{-g} h^2(z) \Psi_n^Z \Psi_m^Z = \delta_{mn},
\]

\[
\int_{-L}^L dz \sqrt{-g} h^2(z) (\Psi_n^W)^T \Psi_m^W = \delta_{mn},
\]

\[a_0 (g_5^2 + 2g_5^2) \int_{-L}^L dz \sqrt{-g} h^2(z) = 1, \quad \int_{-L}^L dz \sqrt{-g} h^2(z) \Psi_n^Z \Psi_m^Z = \delta_{mn},\]

\[\int_{-L}^L dz \sqrt{-g} h^2(z) (\Psi_n^W)^T \Psi_m^W = \delta_{mn},\]
and the following equation of motion:

\[ z^2 \frac{d}{dz} \left[ z^{2 \tilde{u}} \frac{d}{dz} \Psi^Z_n \right] = - \left( m^Z_n \right)^2 \Psi^Z_n, \]

\[ z^2 \frac{d}{dz} \left[ z^{2 \tilde{u}} \frac{d}{dz} \Psi^a_n \right] = - \left( m^a_n \right)^2 \Psi^a_n, \]

where \( \tilde{u} = (\omega + 4/3) / (\omega + 1) \). The solution to equation (59) is

\[ \psi_n(z) = \left( \frac{kz}{N_n} \right) \left[ J_i(m_n z) + b_{vi} Y_i(m_n z) \right], \]

where \( \tilde{v} = (\omega + 7/6) / (\omega + 1) \) and \( i = [Z, W] \). By substituting decompositions (48) and (49) into the boundary conditions (52), (51), (55) and (54), we obtain the mass equation for the boson Z:

\[ (R_{\tilde{v} - 1} - \tilde{R}_{\tilde{v} - 1})(R_{\tilde{v}} - \tilde{R}_{\tilde{v}}) + (R_{\tilde{v} - 1} - \tilde{R}_{\tilde{v}})(R_{\tilde{v}} - \tilde{R}_{\tilde{v} - 1}) + 2 \frac{g^2_s}{g^2_5}(R_{\tilde{v} - 1} - \tilde{R}_{\tilde{v}})(R_{\tilde{v}} - \tilde{R}_{\tilde{v} - 1}) = 0, \]

where

\[ R_{\tilde{v}} = -J_{\tilde{v}}(x_{n\tilde{v}} e^{-kL}) / \tilde{Y}_{\tilde{v}}(x_{n\tilde{v}} e^{-kL}), \quad \tilde{R}_{\tilde{v}} = -J_{\tilde{v}}(x_{n\tilde{v}}) / \tilde{Y}_{\tilde{v}}(x_{n\tilde{v}}), \]

with \( x_{n\tilde{v}} = (m_n / k) e^{kL}, \tilde{a} = \{ \tilde{v}, \tilde{v} - 1 \} \) and we assumed that \( g^2_s > 0 \). Similarly, substituting (50) into (53) and (56), we find the \( W^\pm \) mass equation

\[ (R_{\tilde{v} - 1} - \tilde{R}_{\tilde{v} - 1})(R_{\tilde{v}} - \tilde{R}_{\tilde{v}}) + (R_{\tilde{v} - 1} - \tilde{R}_{\tilde{v}})(R_{\tilde{v}} - \tilde{R}_{\tilde{v} - 1}) = 0. \]

The mass equations (61) and (63) reduce to the usual Higgless model [8] for \( \tilde{v} = 1 \). The main difference here is that the index \( \tilde{v} \) varies with the Brans–Dicke parameter \( \omega \) so that the Z and W boson masses depend on \( \omega \) as well. By numerical analysis of equation (63), we conclude that the effect of the Brans–Dicke parameter \( \omega \) is the following: when decreasing \( \omega \), the masses of the first \( W \) and \( Z \) modes decrease while the masses of the higher modes increase. This behavior is shown in figure 4 for \( kL = 12 \) and \( g^2_s/g^2_5 = 0.426 \). Note that when \( \omega \to -1 \), the first mode vanishes while the higher modes diverge.

Another interesting result is the evolution of the quotient \( m^2_W / m^2_Z \) with the Brans–Dicke parameter, where \( m_W \) and \( m_Z \) are the masses of the \( W \) and \( Z \) resonances. This quotient is lower than 1 for the first and third modes and increases when decreasing \( \omega \), whereas for the second mode it is greater than 1 and decreases when decreasing \( \omega \). This behavior is shown in figure 5 for \( kL = 12 \) and \( g^2_s/g^2_5 = 0.426 \).

These values were chosen to obtain a realistic value for the quotient \( m^2_W / m^2_Z \) in the limit \( \omega \to \infty \). Indeed, in this limit we obtain the result \( m^2_W / m^2_Z \sim 0.764 \), which can be compared with the asymptotic expression

\[ \frac{m^2_W}{m^2_Z} \approx \frac{1 + g^2_s/g^2_5}{1 + 2g^2_s/g^2_5} \sim 0.770 \]

obtained in [8] for the limits \( kL \gg 1 \) and \( k \gg 1 \). According to [8] (and also [14]), we can also relate the couplings \( g_s \) and \( g_5 \) to the effective standard model couplings \( g \) and \( g' \) by adding matter fields. This leads to the asymptotic relations

\[ g^2 \approx \frac{g^2_5}{L}, \quad g'^2 \approx \frac{g^2_5 g^2}{L(g^2_5 + g^2_5)}. \]
Figure 4. Kaluza–Klein modes for the $Z$ and $W_{\pm}$ bosons as a function of the Brans–Dicke parameter $\omega$ for $kL = 12$ and $\bar{g}_5^2/g_5^2 = 0.426$. A similar behavior is obtained for other values of $\bar{g}_5^2/g_5^2$.

Figure 5. Quotient $m_W^2/m_Z^2$ as a function of the Brans–Dicke parameter $\omega$ for $kL = 12$ and $\bar{g}_5^2/g_5^2 = 0.426$. A similar behavior is obtained for other values of $\bar{g}_5^2/g_5^2$.

Then for $\bar{g}_5^2/g_5^2 = 0.426$, we obtain

$$\tan^2 \theta_W = \frac{g_5^2}{g^2} = \frac{\bar{g}_5^2/g_5^2}{1 + \bar{g}_5^2/g_5^2} \approx 0.299,$$

$$\cos^2 \theta_W \approx \frac{1 + \bar{g}_5^2/g_5^2}{2 g_5^2/g_5^2} \approx \frac{m_W^2}{m_Z^2},$$

where $\theta_W$ is the Weinberg angle. Relation (67) is characteristic of Higgless models that preserve the SU(2) custodial symmetry.

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6. Conclusions

In this paper, we have constructed $D$-dimensional Randall–Sundrum models from Brans-Dicke theory by using a BPS-like mechanism for solving the background equations. We have also studied the Kaluza–Klein decomposition of massless scalar and gauge fields and showed how the Kaluza–Klein modes depend on the Brans–Dicke parameter $\omega$. In particular, we saw how the Brans–Dicke parameter acts as a fine-tuning parameter for the $W$ and $Z$ resonances of a Higgless electroweak model.

We have considered, in our analysis of scalar and vector Kaluza–Klein modes, a wide range of values for the Brans–Dicke parameter $\omega$. We also assumed that $kL$ is large, as is expected for solving the Planck-weak hierarchy problem. However, it is important to remark that the stability of this model requires the addition of scalar field potentials on the Planck and TeV branes. As mentioned in [7], after introducing stabilizing potentials, a large value of $\omega$ is needed in order to avoid a new hierarchy for the scalar field.

In Brans–Dicke theory, the presence of a background scalar field was crucial. A possible future investigation would be studying the effect of other background fields, such as a Kalb–Ramond field, which is motivated by string theory (see for instance [12, 13]).

Another interesting feature to be explored is the effect of the Brans–Dicke parameter on scalar and gauge field interactions and in the presence of fermionic fields.

Acknowledgment

The authors are financially supported by CNPq.

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New Journal of Physics 12 (2010) 053038 (http://www.njp.org/)

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