THE CHARACTERIZATION OF PLANAR, 4-CONNECTED, $K_{2,5}$-MINOR-FREE GRAPHS

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Abstract. We show that every planar, 4-connected, $K_{2,5}$-minor-free graph is the square of a cycle of even length at least six.

1. Introduction

All graphs in this paper are finite and simple. A graph is a minor of another graph if the first can be obtained from a subgraph of the second by contracting edges and deleting resulting loops and parallel edges. We say that a graph $G$ is $H$-minor-free if $H$ is not a minor of $G$.

In [5] Wagner showed that $K_5$ and $K_{3,3}$-minor-free graphs are precisely the planar graphs; this is probably the most well-known result concerning characterizations of minor-free graphs. A related result that follows from a different formulation of Wagner’s theorem is that a 2-connected graph is $K_{2,3}$-minor-free if and only if it is $K_4$ or outerplanar. More results concerning $H$-minor-free graphs include Dirac’s [2] characterization of all $K_4$-minor-free graphs and more recently, Ding and Liu’s [1] description of $H$-minor-free graphs for all 3-connected graphs $H$ on at most eleven edges. In [3], Ellingham et. al. provide a complete characterization of all $K_{2,4}$-minor-free graphs. This type of questions have acquired more attention recently since the conclusion of Robertson and Seymour’s Graph Minors Project which proved that minor-closed families of graphs can be characterized by a finite set of forbidden minors.

In this paper we focus on $K_{2,5}$-minor-free graphs. We suspect that this family is large and rather complex so we restrict our attention here to 4-connected planar $K_{2,5}$-minor-free graphs. Note that the similar question with 5-connected graphs has a simple answer since every 5-connected graph either has 5 disjoint paths between a non-adjacent pair of vertices (by Menger’s theorem) and hence a $K_{2,5}$ minor, or is $K_6$. 

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To state our main result we need one more definition. The *square* of a graph $G$, denoted by $G^2$, is a graph on the same vertex set as $G$, with pairs of vertices adjacent in $G^2$ if they are at distance at most two in $G$ (see Fig. 3).

**Theorem 1.** A graph is planar, 4-connected and $K_{2,5}$-minor-free if and only if it is the square of a cycle of even length at least 6.

1.1. **Notation.** Let us first introduce an equivalent definition of minor, which we will use in this paper. $H$ is a *minor* of $G$ if for every vertex $v \in H$, there exists a connected subset of vertices $B_v \subseteq G$ called the *branch set* of $v$ such that the branch sets of distinct vertices are disjoint and for each edge $vw$ of $H$, there is an edge in $G$ connecting $B_v$ and $B_w$. If $G$ has $K_{t,s}$ as a minor, often we will denote this minor by $\{B_1, B_2, \ldots, B_t; S\}$, where $B_i$’s are the branch sets of the vertices of $K_{t,s}$ in the same bipartition class with $t$ vertices and $S$ is the union of the branch sets of the vertices in the other side.

For a given graph $G$ and any vertex $v \in G$, the *open neighborhood* $N(v)$ denotes the set of vertices of $G$ adjacent to $v$. Similarly, for vertices $v_1, \ldots, v_n \in G$, $N(v_1, \ldots, v_n) = \left(\bigcup_{i=1}^n N(v_i)\right) \setminus \{v_1, \ldots, v_n\}$. The *closed neighborhood* is defined to be $N[v] := N(v) \cup \{v\}$ and $N[v_1, \ldots, v_n] := N(v_1, \ldots, v_n) \cup \{v_1, \ldots, v_n\}$.

Given a graph $G$, its *line graph* $L(G)$ is a graph with vertex set $V(L(G)) = E(G)$ and two vertices of $L(G)$ are adjacent if and only if their corresponding edges share a common endpoint.

2. **Proof of Theorem 1**

Our proof of Theorem 1 uses the following result of Martinov from [4]. To state his result, we need the following definition. A *cyclically 4-edge-connected* graph is a 3-edge-connected graph with no 3-edge-cuts that leave a cycle in each component.

**Theorem 2** (Martinov, [4]). A 4-connected graph that is 4-regular and has every edge in a triangle is either a squared cycle of length at least five or the line graph of a cubic, cyclically 4-edge-connected graph.

Our proof of the main result follows by combining Martinov’s result along with the following auxiliary lemmas, each of them exploiting the structure of planar, 4-connected, $K_{2,5}$-minor-free graphs. Together they imply that all planar, 4-connected, $K_{2,5}$-minor-free graphs must be the squares of cycles of length at least five. Then with a bit of more work, we rule out the case of odd cycles, thus proving our main result.

**Lemma 1.** If $G$ is a 4-connected planar graph, then for every vertex $v \in V(G)$ the graph $G \setminus N[v]$ is connected.
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**Proof.** Suppose, to the contrary, that for some $v \in V(G)$, $N[v]$ is a cut set. Let $S \subseteq N(v)$ be a minimal cut set of the 3-connected graph $G \setminus v$. Note that $|S| \geq 3$. Let $C_1$ and $C_2$ to be two distinct components of $G \setminus (\{v\} \cup S)$. It is easy to see that $(\{v\}, C_1, C_2; S)$ gives a $K_{3,|S|}$ minor in $G$, in particular a $K_{3,3}$ minor, contradicting planarity. □

**Lemma 2.** If $G$ is a 4-connected planar, $K_{2,5}$-minor-free graph, then it is 4-regular.

**Proof.** Suppose, to the contrary, there exists a vertex $v$ of $G$ with degree $d := d(v) \geq 5$. Take the planar embedding of the graph $G$ and label $v$'s neighbors by $w_1, \ldots, w_d$, ordered clockwise around $v$.

Note that $w_i$ cannot be adjacent to $w_j$ for $j \neq i \pm 1$, where we take the indices of the vertices modulo $d$, since the opposite would contradict either the planarity of the embedding or the 4-connectivity of the graph $G$. In the latter case, $\{v, w_i, w_j\}$ becomes a 3-cut.

By Lemma 1, $G \setminus N[v]$ is connected, denote it by $C$. We claim that $C$ must be adjacent to exactly four vertices among $w_1, w_2, \ldots, w_d$. Let $W$ be the set of these vertices. Indeed, the fact that $|W| \geq 4$ follows from the 4-connectivity of $G$. On the other hand, if $|W| \geq 5$, then $(C, \{v\}; W)$ is a $K_{2,5}$ minor. Now since $|W| = 4$ and $d \geq 5$, it follows that there must be some $i$ such that $w_i$ is not adjacent to any vertex in $C$, hence by our first observation, $d(w_i) \leq 3$, a contradiction. This shows that $G$ is 4-regular. □

**Lemma 3.** If $G$ is a 4-connected planar, $K_{2,5}$-minor-free graph, then every edge of $G$ is in a triangle.

**Proof.** Assume, to the contrary, that there is an edge, say $ab$, not in a triangle. By Lemma 2, $G$ is 4-regular, thus $a$ and $b$ each have three neighbors, all distinct vertices. Let $c, d, e$ and $f, g, h$ be the neighbors of $a$ and $b$, respectively and suppose that they appear in the planar embedding of $G$ in the order as in Figure 1. It is not hard to check that 4-connectivity of $G$ implies that $G \setminus N[a, b]$ is not empty.

As in the previous lemma, one can easily check that any component of $G \setminus N[a, b]$ must be adjacent to exactly four vertices in $N(a, b)$ in order for $G$ to be 4-connected and $K_{2,5}$-minor-free. Now let us consider the following two cases.

**Case 1:** Suppose $G \setminus N[a, b]$ has only one component, $C$. Let $x, y \in N(a, b)$ such that $C$ is not adjacent to these two vertices. If $x$ and $y$ have a common neighbor $z \in N(a, b)$, then $G$ has a $K_{2,5}$ minor given by $(C \cup \{z\}, \{a, b\}; N(a, b) \setminus z)$. If $x$ and $y$ do not have a common neighbor, then to have $d(x) = d(y) = 4$, they must be adjacent. And in this
Case 2: $G \setminus N[a,b]$ has more than one component. Take any two of them, $C_1$ and $C_2$. If $C_1$ and $C_2$ together are adjacent to all of $N(a,b)$, then let $x \in N(a,b)$ be one of the two vertices adjacent to both. Then $G$ has a $K_{2,5}$ minor given by $(\{a, b\}, \{x\} \cup C_1 \cup C_2; N(a,b) \setminus x)$. Otherwise, $C_1$ and $C_2$ must have (at least) three common neighbors, let us denote it by $S \subseteq N(a,b)$. But now $G$ has a $K_{3,3}$ minor given by $(C_1, C_2, \{a, b\}; S)$, contradicting planarity. □

Lemma 4. The line graph of any cubic, 3-connected graph $H$ has $K_{2,5}$ as a minor, unless $H \cong K_4$.

Proof. Consider any cubic, 3-connected graph $H$ not isomorphic to $K_4$. Thus $H$ must have an edge, say $uv$, not in a triangle. Let $w, x$ and $y, z$ be the distinct neighbours of $u$ and $v$, respectively. Let $s$ and $t$ be the two neighbors of $w$ distinct $v$. Note that, unlike Figure 2, they are not necessarily distinct from $x, y$, and $z$.

Since $H$ is 3-connected, $H \setminus \{u, w\}$ is connected. Note that $v$ cannot be a cut vertex of $H \setminus \{u, w\}$, so $H \setminus \{u, v, w\}$ is connected. It is not hard to see that $H \setminus \{u, v, w\}$ must contain an edge, so it will induce a connected subgraph of $L(H)$ which avoids the edges $uv, ux, uy, vz, vw, ws$, and $wt$. Then $L(H)$ has a $K_{2,5}$ minor given by $(\{uv, vw\}, L(H) \setminus N[uv, vw]; N(uv, vw))$. See Figure 2 □

Now we are ready to prove our main result.

Proof of Theorem 1. First note that a cubic, cyclically 4-edge-connected graph is, in particular, 3-connected. Thus Theorem 2, along with Lemmas 2, 3 and 4 show that a planar, 4-connected, $K_{2,5}$-minor-free graph must be a squared cycle of length at least five. However, it is not hard
to see that every $C^2_n$ contains $C^2_{n-2}$ as a minor, therefore, for all odd $n \geq 5$, $C^2_n$ will have $C^2_5 \cong K_5$ as a minor, which shows that for all such $n$, $C^2_n$ is non-planar. This finishes the necessary direction of the theorem.

For the other direction, fix an even $n \geq 6$ and consider $C^2_n$. Note that $C^2_n$ can be embedded on the plane such that there are two vertex-disjoint, degree $n/2$ faces $F_1$ and $F_2$ which are connected by $n$ triangular faces (see Fig. 3). This shows the planarity of $C^2_n$. It is also easy to see that $C^2_n$ is 4-connected, since any cut set must contain at least two vertices from both $F_1$ and $F_2$.

Now suppose $C^2_n$ has a $K_{2,5}$ minor given by $(R_1, R_2; S)$. We can suppose that $S$ is a set of only five vertices. Then, without loss of generality, $F_1$ contains three vertices of $S$, denote the set of these vertices
Consider a new graph $G'$ by adding a new vertex to $C_n^2$ and connecting it to the vertices of $S'$. Clearly, this graph is planar. However, $G'$ contains a $K_{3,3}$ minor given by $(R_1, R_2, \{v\}; S')$, a contradiction. This finishes the proof of the main theorem. □

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