Skeletonization and Reconstruction based on
Graph Morphological Transformations

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Abstract. Multiscale shape skeletonization on pixel adjacency graphs is
an advanced intriguing research subject in the field of image processing,
computer vision and data mining. The previous works in this area almost
focused on the graph vertices. We proposed novel structured based graph
morphological transformations based on edges opposite to the current
node based transformations and used them for deploying skeletonization
and reconstruction of infrared thermal images represented by graphs.
The advantage of this method is that many widely used path based ap-
proaches become available within this definition of morphological opera-
tions. For instance, we use distance maps and image foresting transform
(IFT) as two main path based methods are utilized for computing the
skeleton of an image. Moreover, In addition, the open question proposed
by Maragos et al. about connectivity of graph skeletonization method
are discussed and shown to be quite difficult to decide in general case.

Keywords: Graphs · Mathematical morphology · Infrared imagery ·
reconstruction · Skeletonization · Galois connections.

1 Introduction

Mathematical morphology is a highly used framework for binary and grayscale
image processing [12] (also for infrared image analyses [34567]). The core of
mathematical morphology depends upon fitting a pattern called the structuring
element on an image for extracting useful information about the geometrical
structure of the image by matching small patterns into it at various locations of
the image. By making use of different shapes and sizes of structuring elements,
information such as connectivity and borders considering the shape of different
parts of the image and interrelations between them can be obtained.

Graph based image representations typically deal with pixel adjacency graphs,
i.e, graphs whose nodes are the set of image pixels and whose edge set is deter-
mined by some adjacency relation on image pixels. In a 2D grid environment the
4,6,8-connected adjacency relation applies according to the connectivity property of the grid. Figure 1 indicates graph representation of the two dimensional grid with respect to 4,8-connectivity.

Extensive applications of graphs for clustering, segmentation and edge detection in different areas of image processing and infrared thermography [8],[9],[10],[11] have been proposed recently.

Exposing a gray-level image by a graph express the pixel values of the image by a function from the set of vertices $V$ to the complete lattice of gray values $f : V \rightarrow L$. In binary case where pixels are only black or white, this complete lattice just contains two elements $\{0, 1\}$.

The notion of an image skeleton has been introduced by Bium [12]. He called his idea the "medial axis" and later the "symmetric axis". The intuitive model suggests considering an image as a grass field. At time $t=0$ the image boundary is set to fire and the fire propagates inwards [13]. The set of points remaining after extinguishing the fire is adopted as the skeleton of the image. An applicative and very useful property of the symmetric axis is inherited in its capability of reconstructing the image boundary by propagating the wavefronts backward. Lantujoul employed binary morphological operations for calculating the skeleton of a binary image [14]. He proved that the skeleton $\text{Skel}(M)$ of the image $M$ can be extracted by an iterative routine as follows,

$$\text{skel}_n(M) = (M \ominus nB) - ((M \ominus nB) \circ clB) \quad (1)$$

where $nB$ denotes dilation of the structuring element $B$ with itself iterated for $n$ times and $\ominus, \circ$ and $clB$ correspond to the binary erosion, opening and closed structuring element $B$ respectively. This leads to an iterative approach for calculating the skeleton. The iteration continues by $n$ steps until $M \ominus nB \neq \emptyset$. Kresh [15] applied morphological operations for extracting skeletons while Maragos et al. have utilized the method for encoding binary images [16].

Given the skeleton of an image, the original image can be completely or partially retrieved by the following formula called reconstruction,

$$M \circ nB = (\bigcup n \ | \ k \leq n \leq N : \text{skel}_n(M) \oplus nB) \quad (2)$$

where $k \geq 0$ is an integer controlling the process and $\ominus, \circ$ are dilation and opening operations respectively. By setting $k = 0$, one gets a complete reconstruction of the original image whereas $k \geq 1$ conducts partial reconstruction of original image. The result of reconstruction in this case corresponds to a copy of the opened original image by the structuring element $nB$.

Exploiting binary morphological operations introduced in [11] to graph morphological transformations were first proposed by Maragos [17][18]. We have combined the idea of structured binary morphological operations proposed initially in [19] and edge based morphological operations explained in [20] to attain new morphological operations for graphs. Then the skeletonization procedure is formalized benefiting from these new binary morphological operations. Comparing with Maragos technique that uses vertex based morphological operations our
method opens the door to import many well-known path related methods such as distance mappings into graph based morphological transformations. Moreover, the proposed method was employed for extracting skeleton of some infrared thermal samples. Before proposing our method, we intend to respond the question posed by Maragos in [17] about connectivity of skeletonizing method using graph. The more general question that can be discussed is under which conditions morphological transformations defined for graphs are connected.

Connected morphological operations initially proposed by Salamier and Serra [21,22], has been investigated extensively by many researchers [23, 24, 25]. These kind of operators act on gray-level images owing to the concept of flat zones. Simply in binary case, a connected operator is not capable of modifying background or foreground boundaries of an image. Roughly speaking, connected morphological operators interact with a binary image just by preserving or removing connected components. Figure 2 illustrates an example of a connected and non-connected operator. A connected morphological operation can’t produce the image in the middle since the boundaries of foreground image is moved. Precisely, a connected morphological operator is just capable of disappearing or creating connected components of an image but not moving or breaking any of their boundaries.

However, eroding a graph by another structuring graph may not be classified as a connected operation. Figure 4 illustrates an example of eroding a graph by a structuring graph formed from a triangle consisting of a root and two buds which is not a connected operation. In fact many practical properties of dilation and erosion do not hold in the graph area. One of them is the anti-extensive property of erosion which is witnessed to be violated in figure 4. The reason for extending the foreground stems from the definition of erosion. The erosion of graph $X$ representing a foreground image by the structuring graph $S$ is calculated via:

$$\epsilon_S(X|G) = \{v \in V | N_A(v|G) \subseteq X\}$$

(3)
$X, G$ stands for the graphs representing foreground and whole image and $N_A(v|G)$ is the neighborhood function defined by the formula:

$$N_S(v|G) = \bigcup \theta | \theta \text{ is an embedding of } S \to G \text{ at node } v : \theta(B_S)$$  \hspace{1cm} (4)

Roughly speaking, for deciding whether a node $v$ of the graph presenting a picture will be black or white after the erosion operation, one looks for all embeddings of the structuring graph in the source graph in which a root of the structuring graph is mapped to $v$. If buds of the structuring element are all mapped to nodes with black color (foreground graph), then $v$ becomes black, otherwise white. However, in the case in which not any embedding of the structuring graph into the source exists at node $v$ exists, $N_A(v|G)$ is empty and $v$ becomes black since $\emptyset \subseteq X$ always holds. This confirms that the foreground pixels (black nodes) may be extended via an erosion operation and the erosion may extend the boundaries of the foreground part which contradicts with the definition of connected morphological operations.

It is obvious that the aforementioned erosion is not connected since it changes the boundaries of foreground hexagon and this pretty assures that the skeletonization of a graph by an structuring graph may also be non-connected.

However, one can deduce that deciding about satisfying the connectivity property for structured graph morphological operations can be quite a difficult problem. Deciding on the problem that the structuring graph can be embedded in the source graph reduces to the subgraph isomorphism problem which is itself the generalization of maximum clique problem and is proved to be NP-complete \cite{26}. Investigating the conditions for connectivity of morphological operations like dilation and erosion for non-structured graph morphology looks more interesting since we are not facing the problem of finding the embeddings of the structured graph into the source graph. This is an interesting future research topic.

2 Method

Given a 4,8-connected grid represented by an undirected graph $G = (V_G, E_G)$, suppose an image is represented by $M = (V_M, E_M)$ satisfying $V_M \subseteq V_G, E_M \subseteq E_G$. The inclusion is so trivial since the image is embedded in the grid. We observe just the binary graphs and the method which will be proposed corresponds to this category.

Inspiring from the vertex based structured graph morphology introduced in \cite{27} and the non-structured edge based morphological operations proposed in \cite{28}, we came to the structured edge based morphological operations. We do not include any explanation on aforementioned methods for the sake of brevity but the reader can refer to aforementioned papers. The idea of homomorphisms from the structuring graph to the source graph is the core of node based structured graph morphological transformations. Nodes of the structuring graph are classified to 3 categories called roots, buds and nodes that are neither roots nor
buds. If an injective homomorphism from the structuring graph to the source graph exists in which a root node is mapped to a node \( v \) with value 1 and the mapping of at least one bud is a node with the value 1, then \( v \) is added to the result of dilation operation. Inspiring from this idea, our approach is constructed on injective homomorphisms from the structuring graphs to the source graph as well. The edges of structuring graph are classified as roots, buds, not root, bud edges.

Imagine an edge \( e \) in the source graph. If an injective homomorphism from the structuring graph to the source graph exists that maps a root edge to \( e \), then the mapping of buds are added to the result of dilation operation. This technique is axiomatized as follows.

Given a structuring graph \( S = (V_S, E_S) \), define two sets \( R_S, B_S \subseteq E_S \), as the set of roots and buds respectively. Suppose the regular graph denoting the grid as \( G \) and the graph that represents the image is denoted by \( M \). If \( \theta : V_S \to V_G \) is a homomorphism and \( h : E_S \to E_G \) is defined for any \((u,v) \in E_S\), as \( h(u,v) = (\theta(u),\theta(v)) \). The neighborhood function of an edge \( e \) can now be defined by,

\[
N_S(e|G) = \bigcup \{ h \mid h \text{ maps an edge in } R_S \text{ to } e : h(B_S) \} \tag{5}
\]

In fact this corresponds to the informal description of dilation. if there exist a homomorphism \( h \) which maps an edge of the structuring element defined as root to an edge \( e \) of the foreground image, the mappings of edges defined as buds are along with the vertices coinciding with them are added to the result. The neighborhood function \( N_S(e|G) \) extracts the mapping of buds regarding any homomorphism \( h \). Now the dilation of graph \( G \) by \( S \) as the structuring element is defined by,

\[
\delta(M|G) = (\bigcup \{ h \mid e \in M : h(B_S) \}) \tag{6}
\]
Figure 3 illustrates an example of dilation transformation. It should be noted that the graph dilation operation is denoted by $\delta$ and the erosion by $\epsilon$ and $G$ corresponds to the graph shown by dashed line. The structuring graph consists of 3 nodes and 2 edges with the only root edge labeled by 'r' and the only bud labeled by 'b'. For instance assume the homomorphism $h : r \rightarrow (a, c)$. This embedding results in adding $(d, c)$, since it is the mapping of the only bud. Mapping the root edge of the structuring graph to $(a, c)$ entails adding $(d, e)$ to the result of dilation according to 6. Another example illustrating dilation of an image composed of two disjoint graphs taking a square structuring element including one root edge and one bud edge is also shown in figure 3(b). Getting the dilation in hand, the erosion can be calculated making use of its Galois connection with the dilation.

**Definition 1.**[29] Given sets $P, Q$ and their powersets $\rho(P), \rho(Q)$, a pair of functions $f : \rho(P) \rightarrow \rho(Q), g : \rho(Q) \rightarrow \rho(P)$ constitute a Galois connection if for each $X \in \rho(P) \wedge Y \in \rho(Q)$ we have:

$$X \subseteq f(y) \iff Y \subseteq g(X)$$

The function $f$ is called the lower adjoint of $g$ and $g$ the upper adjoint of $f$.

A very useful property of a Galois connection is that an upper or lower adjoint function uniquely determines the other saying that $f(a)$ is the least $d$ satisfying $a \leq g(d)$ and $g(b)$ is the largest existent $c$ where $f(c) \leq d$[30]. It is pretty known that dilation and erosion engage in a Galois connection [1]. This allows to extract the axiomatization of erosion from dilation. According to the Galois connection between dilation and erosion we can write:

$$\delta_S(M|G) \subseteq P \iff M \subseteq \epsilon_S(P|G).$$
\[ \delta_S(M|G) \subseteq P \]
\[ \Leftrightarrow \quad \langle \text{Definition of dilation } \rangle \]
\[ (\bigcup e \mid e \in E_M : N_S(e|G) \subseteq P) \]
\[ \Leftrightarrow \quad \langle \text{Properties of } \bigcup \rangle \]
\[ (\forall e \mid e \in E_M : N_S(e|G) \subseteq P) \]
\[ \Leftrightarrow \quad \langle \text{Properties of } \forall \rangle \]
\[ G \subseteq (\bigcap e \mid e \in E_G : N_S(e|G) \subseteq P) \]

This yields that the erosion can be calculated by:
\[ \epsilon_S(P|G) = (\bigcap e \mid e \in E_G : N_S(e|G) \subseteq P). \quad (7) \]

Equation (7) provides a procedure for calculating the erosion of a graph \( G \) by means of the structuring element \( S \). Imagine \( G \) represents the 4 or 8-connected grid that corresponds to the whole 2-dimensional plane and \( P \) is the graph representation of the image. If a root edge of the structuring element is mapped to an edge \( e \) in \( P \) while all edges of \( G \) that are neighbors of mappings of buds are included in \( P \), \( e \) is added to the result of erosion.

Figure 3(c) provides an example of the erosion operation. \( G \) corresponds to the whole plane (dashed lines) and \( P \) is indicated in the left. The result of erosion operation is empty. Imagine the embedding \( h : r \rightarrow (a,e) \). Since \((e,d)\) is not an edge of the graph, nothing is added to the result of erosion. So, If the mapping of buds through all existent embeddings are contained in the source graph, the edges which are the mapping of roots are added to the result of erosion.

With dilation and erosion in hand, one can simply express the opening and closing. However, there is another way of specifying opening and closing called structural opening [27]. One can also define the opening transform for structured morphological graph transformations based on edges as follows: \( \alpha_S(M|G) = (\bigcup h \mid h \text{ is a homomorphism and } h(B_S) \subseteq M : h(B_S)) \). This definition states that the opening consists of buds mappings whenever mappings of all buds is included in the source graph. It is not hard to prove that this formula is indeed an opening. For that target we need to prove that \( \alpha_S(M|G) \) is anti-extensive, idempotent and increasing.

**Definition 2.** Let \( O : X \rightarrow X \) be a function acting on complete lattice \( X \). One can express that:

- \( O \) is idempotent if \( O^2 = O \).
- \( O \) is anti-extensive if for every \( u \in X \) then \( O(u) \leq u \).
- \( O \) is increasing if for every \( u,v \in X,u \leq v \Rightarrow O(u) \leq O(v) \).

Proving the anti-extensive property is straightforward since \( h(B_S) \subseteq M \Rightarrow \alpha_S(M|G) \subseteq M \). The proof for increasing property is that \( M_1 \subseteq M_2 \Rightarrow \alpha_S(M_1|G) \subseteq \alpha_S(M_2|G) \).
Algorithm 1 Skeletonization algorithm

1: procedure Skel($G, M, S$)  \> $S$ structuring graph, $M$ source and $G$ the grid
2:     $S \leftarrow \emptyset$, Temp $\leftarrow \delta_S(M|G)$
3:     while Temp $\neq \emptyset$ do
4:         Temp $\leftarrow \epsilon_S(M|G)$ \> erosion-\(\epsilon_S(M|G)^n\)
5:         Temp$_1 \leftarrow \epsilon_S$(Temp$|G$) \> erosion once more- \(\epsilon_S(M|G)^{n+1}\)
6:         Temp$_2 \leftarrow \delta_S$(Temp$_1|G$) \> dilation-\(\delta_S(\epsilon_S(M|G)^{n+1})\)
7:     $S \leftarrow S \cup$ Temp$_2$
8:     end while
9: return Temp$_2$ \> The skeleton is Temp$_2$
10: end procedure

\[ \alpha_S(M_2|G) \text{ because } M_1 \subseteq M_2 \text{ and } h(B_S) \subseteq M_1 \text{ for a homomorphism } h \text{ implies that } h(B_S) \subseteq M_2 \text{. For proving idempotency we have } \alpha_S(\alpha_S(M|G)) \subseteq \alpha_S(M|G) \text{ by the anti-extensive property. For any edge } e \in \alpha_S(M|G), \text{ we have } e \in \alpha_S(\alpha_S(M|G)). \text{ Thus } \alpha_S(\alpha_S(M|G)) = \alpha_S(M|G). \]

The primary procedure for extracting the skeleton of an image requires dilating the structuring element by itself for $n$ times and then eroding the source image by this structuring graph. However, one can erode the image by the structuring element $n$ times instead. Algorithm 1 narrates the procedure performed for extracting the skeleton of an image formed as a graph $M$ with a predefined structuring graph $S$. There $\epsilon_S(M|G)$ denotes eroding the picture represented by graph $M$ with the structuring graph $S$ and $\delta_S(M|G)^{n+1}$ stands for dilating graph $M$ by $S$. Instead of dilating the structuring graph $n$ times with itself, the algorithm performs the dilation with the structuring graph $n$ times which is the common way of computing the skeleton in literature. In fact The algorithm formalizes the following formula for computing the skeleton of an image shown with the graph $M$ which is embedded in the grid indicated by the regular graph $G$.

\[ \text{skel}_n(M) = \epsilon_S(M|G)^n - \delta_S(\epsilon_S(M|G)^{n+1}) \]  \hspace{1cm} (8)

Figure 5 indicates an example of skeletonization. On contrary with the skeleton algorithm for showing what happens more clearly, instead of expressed method performing $n$ consecutive erosions, first the structuring graph is dilated by itself in $n$ times corresponding to the initiative approach. Intuitively, the role of disk with radius $n$ is simulated by regular graphs.

Conversely, We can reconstruct a graph $M$ from its skeleton by the following formula:

\[ \delta_S^k \epsilon_S^k = (\bigcup n \mid k \leq n \leq N : \delta_S^k(\text{skel}_n(M))). \]  \hspace{1cm} (9)

Specifying $k=0$ yields almost the exact reconstruction of the source image. The benefit of reconstruction procedure is that all subsets of the skeleton derived in each iteration of the skeletonization algorithm resembles the original image and the original image can be reconstructed from them. However, if $k > 0$ then the
Algorithm 2 Reconstruction algorithm

1: procedure Recons \(G, k, M_k, S\)
   \(\triangleright S\) structuring-graph, \(M_k\) the result of skel algorithm until step \(k\) and \(G\) the grid
2: \(\text{Count} = k, \text{Temp} = M_k\)
3: while \(\text{Count} \leq N\) do
4: \(\text{Temp} \leftarrow \delta_S(\text{Temp}|G)\)
   \(\triangleright\) dilation-\(\delta_S(M_k|G)\)
5: \(\text{Count} \leftarrow \text{Count} + 1\)
6: end while
7: return \(\text{Temp}\)
   \(\triangleright\) The skeleton is \(\text{Temp}\)
8: end procedure

original image can be partially reconstructed and an opened copy of the original shape is rebuilt according to \(\text{[9]}\). The algorithm \(\text{[2]}\) states the procedure formally.

An advantage of skeletonization and reconstruction built on edge based morphological operations comparing to the vertex based method is that one can import quite rich and applicative concept of graph paths into it. A path is a sequence of vertices in a graph in which every two successive vertices are connected by an edge. If weights are imposed to edges, the length of a path is considered to be the sum of edge weights constituting the path. Otherwise, for an unweighted graph the length of any path is simply carried out by counting the number of its consisting edges.

An important concept directly deduced from paths which is widely utilized in many applications is the distance map. The distance map between two pixels \(p\) and \(q\) simply is the length of shortest path connecting them. The connecting path between any two pixels is calculated regarding to the connectivity policy which can be 4,6,8-connected. A well-known method of computing the skeleton of a binary image \(\text{[31]}\), \(\text{[32]}\) relies on the distance map of foreground pixels calculated from the closest background pixel.

A brief description of skeletonization of a binary image emerged from distance maps is expressed as follows. Refer to \(\text{[31]}\) for the detailed description of the method. The distance map between two pixels \(p\) and \(m\) is computed either by \(D_{\text{odd}}\) or \(D_{\text{even}}\) depending on choosing horizontal/vertical or diagonal steps for the path between them. Figure 6 illustrates \(D_{\text{odd}}\) or \(D_{\text{even}}\). Intuitively, \(D_{\text{odd}}\) is denoted by \(D_{1,3,5,7}\) while \(D_{\text{even}}\) is written as \(D_{2,4,6,8}\). The distance map of pixels in each foreground sections of the image from the closest pixel on the boundary are calculated using either the \(D_{\text{odd}}\) or \(D_{\text{even}}\) pattern. The local extremes of the distance maps corresponding to the image foreground pixels constitute the skeleton. Figure 6d illustrates a \(D_{\text{odd}}\) distance maps of the foreground pixels from the boundary pixels. The pixels with maximal distance maps saying with distance maps 2,3 constitute the skeleton of the image. This distance maps can be calculated by edge based erosion operations. In other words, continuing the edge based erosion of the source graph with the structuring graph until the two pixels \(p\) and \(m\) does not both include in the result provides the distance map between \(p\) and \(m\) by the number of erosions. Different connectivity policies for calculation of distance maps can be easily covered by selecting different edges
of the structuring graph as roots and buds. For instance, one can use the first triangle shown in figure 6(d) for calculating the distance map $D_{\text{odd}}$ whereas the distance map $D_{\text{even}}$ can be calculated making use of the second triangle as the structuring element. If an embedding maps the root in the left triangle to an edge that contains a pixel $p$, the pixels with the horizontal/vertical distance map
1 from p are contained in the erosion of image with that triangle. Similarly, if an embedding maps the root in the right triangle to an edge that contains a pixel p, the pixels with the diagonal distance map 1 from p are contained in the erosion of image with that triangle.

Let us express IFT as another interesting application of edge based morphological operations that is out of reach with the vertex based method. Image foresting transform (IFT) proposed by Falcao et al. [33] reminds a widely used optimal path algorithm in image processing. IFT intuitively suggests searching for minimal paths to pixels from other pixels. In graph based representation of images this resembles the well-known idea of minimum spanning tree but with the difference that the optimal paths needs not to be connected but forming a forest. The idea for skeletonization of a gray-level image nominated in [33] suggests making use of minimal paths from pixels on the boundary of a foreground image to the pixels of foreground for computing its skeleton. This idea can be implemented by our edge based morphological transformations. General definition of erosion and dilation for gray-level images are required for this purpose which are given respectively,

\[
N_S(e|G) = \{h \mid h \text{ maps an edge in } R_S \text{ to } e : h(B_S)\} \quad (10)
\]

\[
\epsilon_S(P|G) = \{e \mid e \in E_G : N_S(e|G) \subseteq P\}. \quad (11)
\]

In simple words in dilation the weight of each edge changes to maximum weight of buds mapped to it through different homomorphisms(h). Similarly, within the erosion the weight of each edge changes to minimum weight of buds mapped to it through different homomorphisms(h). Now, one may reformulate the skeletonization algorithm 1 for gray-scaled images substituting the new definition of edge based dilation and erosion [10][11] for erosion and dilation operations. IFT calls the pixels that are used as the source of minimal paths to a pixel seeds. For calculating the skeleton of an image, it calculates the minimal weight paths from pixels on the boundary to the image pixels. The minimal path from a pixel p to a pixel q can be extracted by iterative gray-scaled erosions containing p, q. For the same reason triangular structuring graphs with different root and buds can collaborate the erosion with one of 4,8-connected policies.

To summarize, morphological operations of a graph with some structuring graph based on edges opens the door to use lots of ideas relevant to optimal maps that far from our access within the vertex based morphological operations.

### 3 Results

We have applied the expressed approach in order to to obtain the skeleton of some infrared thermal images. Since the implementation of the method for grayscale images remains for future research, the sample infrared images were transformed to binary counterparts and then their skeleton was calculated. Figure 7(a) illustrates the sample image. Its skeleton using the structuring graph isomorphic to 3*3 grid is depicted in 7(d). Figure 7(e) represents the skeleton of 7(b) by the
same structuring graph. Figure 7(f) illustrates the result of reconstruction of the source image using the skeleton 7(e). The value $k = 0$ was employed along with the reconstruction operation aiming at reconstructing the exact original image. The partial reconstruction of the original image by $k = 1$ was not so satisfactory.

4 Conclusion

A new method for extracting skeleton of a binary image using graph morphological operations based on edges have been proposed. This approach enables us to import many widespread path based methods into morphological graph operations which are not accessible with the well-known vertex based methods. Some experiments have been conducted using this approach aiming at extracting the skeleton of infrared thermography images and reconstructing the original image from the skeleton. Investigating on edge based implementation of various path based techniques can be a potential research subject.

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