A mapping property of the heat volume potential

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Abstract

We consider the volume potential associated with the heat operator and we prove a mapping property in the space of distributions which are the time derivative of Hölder continuous functions. As an application we solve the Dirichlet and Neumann problems for the heat equation with a non-homogeneous term in such space of distributions.

Keywords: Heat equation; volume potential; regularity theory for integral operators; initial-boundary value problems.

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1 Introduction

One of the most useful tools to deal with non-homogeneous equations is of course the volume potential. For this reason many authors have investigated the mapping properties of operators of volume potential type in different functional settings and for several partial differential operators. While the elliptic framework is better understood, parabolic volume potentials are less investigated. For example, it is well know that if $F$ is a $C^{0,\alpha}$-vector field defined on a sufficiently regular bounded open subset of $\mathbb{R}^d$ then the Newtonian volume potential $\tilde{P}[\operatorname{div} F]$, i.e. the volume potential associated with the Laplace operator and applied on $\operatorname{div} F$, is of class $C^{1,\alpha}$ (see, e.g., Dalla Riva, Lanza de Cristoforis and Musolino [5]). This property allows to use the Newtonian volume potential $\tilde{P}[\cdot]$ to deal with the Poisson equation when the non-homogeneous term is the distributional divergence of a $C^{0,\alpha}$-vector field $F$, that is

$$\Delta u = \operatorname{div} F.$$ 

The parabolic analog of the above equation is

$$\partial_t u - \Delta u = \operatorname{div} G.$$ 

(1)

Boundary value problems for equation (1) under several Hölder regularity assumptions on the vector field $G$ were considered in Lieberman [13] and Lunardi and Vespri [15] with different techniques. Instead, to the best of the author knowledge, the classical parabolic theory does not cover the case

$$\partial_t u - \Delta u = \partial_t f,$$

(2)

where $f$ is a $\frac{1+\alpha}{2}$-Hölder continuous in time and $\beta$-Hölder continuous in space. Motivated by the above example, in the present paper we develop a theory for the volume potential associated with the heat operator acting on the space of distributions of the form $\partial_t f$. As
a consequence, we show how to solve the Dirichlet and Neumann problems for equation (2). We note that in principle one could also try to deal with (2) with a semigroup approach following, e.g., Lunardi [14]. However, our aim is to consider (2) from the point of view of potential theory and develop some tools that we plan to exploit to analyze perturbation problems for the heat equation via potential theory.

For the classical results on elliptic volume potentials we mention here Gilbarg and Trudinger [8] and Miranda [18]. We also note that a potential theoretic approach has recently revealed to be very effective to deal with elliptic problems in singularly perturbed domains. For this reason, mapping properties of elliptic volume potentials have been also considered in view of applications to perturbation problems (see Dalla Riva, Lanza de Cristoforis and Musolino [3, 4]). More details on the potential theoretic approach to perturbation problems for elliptic equations and results on volume potentials can be found in the monograph by Dalla Riva, Lanza de Cristoforis and Musolino [5].

For what concerns the parabolic case, regularity properties of the heat volume potential have been considered in Friedmann [7]. Ladyženskaja, Solonnikov, and Ural’ceva [10] proved a series of mapping properties of the heat volume potential in parabolic Schauder and Sobolev spaces. In Cherepova [2] the author considered the heat volume potential acting on parabolic Hölder continuous functions that are allowed to blow up at the parabolic boundary. Finally, Karazym and Suragan [9] have considered the volume potential associated with a degenerate parabolic equation. However we note that, up to the author knowledge, no results for the heat volume potential on spaces of distributions are available in the literature.

The results of the present paper continue the line of the works [11, 12, 17] on the properties of integral operators of potential type appearing in the framework of parabolic theory. Incidentally, we mention that our interest in proving these kind of mapping properties for the heat volume potential is also of technical nature since an equation of type (2) arises when one tries to pull-back the heat equation to another domain requiring only optimal regularity assumptions on the domains. In a subsequent paper by Dalla Riva and the author [6] we will indeed use the results of the present paper to prove perturbation results for layer heat potentials.

The paper is organized as follows. In Section 2 we introduce the functional spaces that we need, i.e. parabolic Schauder spaces. Then, in Section 3 we recall the definitions of the heat volume potential and of the Newtonian volume potential. Moreover, we show that the heat and Newtonian volume potentials coincide, up to the sign, whenever the density is time-independent. Section 4 contains the main result of the paper. More in detail, here we introduce the heat volume potential on a space of distributions that are the time derivative of Hölder continuous functions and we prove a mapping property of this operator. Finally, in Section 5 we apply the results to the Dirichlet and Neumann problems. For the clarity of exposition, we have postponed a known result of functional analysis regarding quotient spaces to Appendix A.

## 2 Schauder spaces

Let Ω be a bounded open subset of \( \mathbb{R}^n \). Let \( k \in \mathbb{N} \) and \( \alpha \in [0, 1[ \). For the definition of sets and functions of the Schauder class \( C^{k,\alpha} \) we refer, e.g., to Gilbarg and Trudinger [8]. Next we pass to recall the definitions of the parabolic analog of Schauder spaces. Let \( T \in ]-\infty, +\infty[ \). Let \( \mathbb{D} \subseteq \mathbb{R}^n \). For the sake of brevity, we set

\[
\mathbb{D}_T \equiv ]-\infty, T[ \times \mathbb{D}, \quad \partial_T \mathbb{D} \equiv ]-\infty, T[ \times \partial \mathbb{D}.
\]

We now introduce the definition of an anisotropic Hölder space where Hölder regularity with respect to time and space directions can differ. Let \( \alpha, \beta \in ]0, 1[ \). Then \( C^{\alpha;\beta}(\mathbb{D}_T) \)
denotes the space of bounded continuous functions \( u \) from \( \mathbb{D}_T \) to \( \mathbb{R} \) such that
\[
\| u \|_{C^{\alpha, \beta}(\mathbb{D}_T)} \equiv \sup_{\mathbb{D}_T} |u| + \sup_{t_1, t_2 \in \mathbb{R}} \sup_{x \in \mathbb{D}} \frac{|u(t_1, x) - u(t_2, x)|}{|t_1 - t_2|^\alpha}
\]
\[
+ \sup_{t \in \mathbb{R}} \sup_{x_1, x_2 \in \mathbb{D}} \frac{|u(t, x_1) - u(t, x_2)|}{|x_1 - x_2|^\beta} < +\infty.
\]

We also denote by \( C^{\alpha, 0}(\mathbb{D}_T) \) the space of bounded continuous functions \( u \) from \( \mathbb{D}_T \) to \( \mathbb{R} \) such that
\[
\| u \|_{C^{\alpha, 0}(\mathbb{D}_T)} \equiv \sup_{\mathbb{D}_T} |u| + \sup_{t_1, t_2 \in \mathbb{R}} \sup_{x \in \mathbb{D}} \frac{|u(t_1, x) - u(t_2, x)|}{|t_1 - t_2|^\alpha} < +\infty.
\]

With the aim of considering boundary value problems, we will also need higher order parabolic Hölder space, i.e., parabolic Schauder spaces. Let \( \alpha \in [0, 1] \). Let \( \Omega \) be an open subset of \( \mathbb{R}^n \). We denote by \( C^{\frac{1}{2} + \alpha, 1 + \alpha}(\overline{\Omega}_T) \) the space of bounded continuous functions \( u \) from \( \overline{\Omega}_T \) to \( \mathbb{R} \) that are continuously differentiable with respect to the space variables and such that
\[
\| u \|_{C^{\frac{1}{2} + \alpha, 1 + \alpha}(\overline{\Omega}_T)} \equiv \sup_{\overline{\Omega}_T} |u| + \sup_{t_1, t_2 \in \mathbb{R}} \sup_{x \in \overline{\Omega}} \frac{|u(t_1, x) - u(t_2, x)|}{|t_1 - t_2|^\alpha}
\]
\[
+ \sum_{i=1}^n \| \partial_x \} u \|_{C^{\frac{1}{2} + \alpha, \alpha}((\overline{\Omega}_T)} < +\infty.
\]

If \( \Omega \) is of class \( C^{1, \alpha} \), by local parametrizations it is possible to naturally define the space \( C^{\frac{1}{2} + \alpha, 1 + \alpha}(\partial_T \Omega) \). We refer to Ladyženskaja, Solonnikov, and Ural’ceva [10] and Lanza de Cristoformis and Luzzini [11, 12] for more detailed definitions of parabolic Schauder spaces.

Since we will consider the heat volume potential on a specific space of distributions, we need the following definition. We denote by \( C^{-1 + \alpha, \beta}(\overline{\Omega}_T) \) the space of distributions in \( \Omega_T \) that are the (distributional) time derivative of a function in \( C^{\alpha, \beta}(\overline{\Omega}_T) \), endowed with the quotient norm. That is
\[
C^{-1 + \alpha, \beta}(\overline{\Omega}_T) \equiv \{ \partial_t u : u \in C^{\alpha, \beta}(\overline{\Omega}_T) \}
\]
and
\[
\| f \|_{C^{-1 + \alpha, \beta}(\overline{\Omega}_T)} \equiv \inf \{ \| u \|_{C^{\alpha, \beta}(\overline{\Omega}_T)} : f = \partial_t u \}.
\]

It can be easily seen that all the above spaces endowed with their respective norms are Banach spaces (also see Theorem 1.1 for the case of \( C^{-1 + \alpha, \beta}(\overline{\Omega}_T) \)). Finally, when \( T > 0 \), with a subscript 0 in the above spaces we mean the Banach subspace made of functions that are zero before zero. For example,
\[
C^{\alpha, \beta}_0(\mathbb{D}_T) \equiv \{ u \in C^{\alpha, \beta}(\mathbb{D}_T) : u(t, \cdot) = 0 \quad \forall t \leq 0 \}.
\]
The spaces \( C^{\alpha, 0}_0(\mathbb{D}_T) \), \( C^{\frac{1}{2} + \alpha, 1 + \alpha}_0(\overline{\Omega}_T) \), and \( C^{\frac{1}{2} + \alpha, 1 + \alpha}_0(\partial_T \Omega) \) can be defined in the same way. Similarly
\[
C^{-1 + \alpha, \beta}_0(\overline{\Omega}_T) \equiv \{ \partial_t u : u \in C^{\alpha, \beta}(\overline{\Omega}_T), \supp (\partial_t u) \subseteq [0, +\infty) \}.
\]
3 The heat volume potential

Let \( S_n : \mathbb{R}^{1+n} \setminus \{0,0\} \to \mathbb{R} \) denote the fundamental solution of the heat operator, that is

\[
S_n(t, x) = \begin{cases} 
\frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} & \text{if } (t, x) \in ]0, +\infty[ \times \mathbb{R}^n, \\
0 & \text{if } t \in ]-\infty, 0[ \times \mathbb{R}^n \setminus \{(0,0)\}.
\end{cases}
\]

As it is well known, \( S_n \in C^\infty(\mathbb{R}^{1+n} \setminus \{0,0\}) \) and solves the heat equation in \( \mathbb{R}^{1+n} \setminus \{(0,0)\} \). We recall a known bound for \( S_n \) which can be found e.g. in Ladyzhenskaya, Solonnikov and Urall’ceva [10, p. 274]: for all \( \eta \in \mathbb{N}^n \) and for all \( h \in \mathbb{N} \) there exists a constant \( C_{\eta,h} > 0 \) such that

\[
\left| \partial_t^\eta \partial_x^h S_n(t, x) \right| \leq C_{\eta,h} t^{-\frac{n}{2}-h} e^{-\frac{|x|^2}{4t}} \quad \forall (t, x) \in ]0, +\infty[ \times \mathbb{R}^n. \tag{3}
\]

Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^n \) and \( T \in ]-\infty, +\infty[ \). Let \( x_0 \in \Omega \). If \( f \in L^\infty(\Omega_T) \), we define the heat volume potential \( P[f] \) to be the function from \( \overline{\Omega_T} \) to \( \mathbb{R} \) defined by

\[
P[f](t, x) = \int_{-\infty}^{\infty} \int_{\Omega} \left( S_n(t-\tau, x-y) - \delta_{2,n} S_n(-\tau, x_0-y) \right) f(\tau, y) \, dyd\tau \quad \forall (t, x) \in \overline{\Omega_T}, \tag{4}
\]

where \( \delta_{i,j} \) denotes the Kronecker delta. We note that the above definition, in the case \( n = 2 \), depends on the choice of \( x_0 \in \Omega \). Indeed, a different choice of \( x_0 \) would provide a volume potential that differs by a constant. However, if \( T \in ]0, +\infty[ \) and \( \text{supp } f \subseteq [0,T] \times \Omega \) (this is the case needed when one considers an initial-boundary value problem with initial condition at \( t = 0 \)), then the volume potential \( P[f] \) no longer depends on \( x_0 \) neither in the case \( n = 2 \) and

\[
P[f](t, x) = \int_0^T \int_{\Omega} S_n(t-\tau, x-y) f(\tau, y) \, dyd\tau \quad \forall (t, x) \in [0,T] \times \Omega,
\]

which is the classical definition of heat volume potential. The above definition with the term \( \delta_{2,n} S_n(-\tau, x_0-y) \) is needed to avoid summability issues of the kernel as \( \tau \to -\infty \) in the case \( n = 2 \). Indeed \( S_2(t-\tau, x-y) \) behaves as \( (t-\tau)^{-1} \) as \( \tau \to -\infty \), while \( S_2(t-\tau, x-y) - S_2(-\tau, x_0-y) \) does not have the same problem. To see this fact, we fix \( \tau < \min \{0,t\} \) and \( x, y \in \Omega \). One has

\[
\left| S_2(t-\tau, x-y) - S_2(-\tau, x_0-y) \right| \\
\leq \left| S_2(t-\tau, x-y) - S_2(t-\tau, x_0-y) \right| \\
+ \left| S_2(t-\tau, x_0-y) - S_2(-\tau, x_0-y) \right|.
\]

Then, if we denote by \( \{e_j\}_{j=1,..,n} \) the standard basis of \( \mathbb{R}^n \), the fundamental theorem of calculus and the estimates \((3)\) for the fundamental solution \( S_n \) imply that

\[
\left| S_2(t-\tau, x-y) - S_2(t-\tau, x_0-y) \right| \\
\leq \sum_{j=1}^n |x_j - x_0| \int_0^1 \left| \partial_x e_j S_2(t-\tau, \lambda x + (1-\lambda)x_0 - y) \right| \, d\lambda \\
\leq \sum_{j=1}^n C_{e_j,0} |x_j - x_0| \int_0^1 e^{-\frac{|(\lambda x + (1-\lambda)x_0 - y)^2}{4\tau(t-\tau)}} \, d\lambda \\
\leq \sum_{j=1}^n C_{e_j,0} |x_j - x_0| \int_0^1 \frac{1}{(t-\tau)^\frac{n}{2}} \, d\lambda.
\]
By the changes of variable

the mental solution of the Laplace equation is

We first consider the case

which show that the kernel of (4) is summable for \( \tau \to -\infty \).

We will also need the Newtonian volume potential and then we recall that the fundamental solution of the Laplace equation is

\[
\tilde{S}_n(x) = \begin{cases} 
\frac{1}{s_n} \log |x| & \forall x \in \mathbb{R}^n \setminus \{0\}, \quad \text{if } n = 2, \\
\frac{1}{(2-n)s_n} |x|^{2-n} & \forall x \in \mathbb{R}^n \setminus \{0\}, \quad \text{if } n \geq 3,
\end{cases}
\]

where

\[
s_n \equiv \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}
\]
denotes the \((n-1)\)-dimensional measure of the unit sphere \( \partial B_n(0,1) \) and \( \Gamma \) denotes the Euler Gamma function. If \( h \in L^\infty(\Omega) \), the harmonic volume potential \( \tilde{P}[f] \) is the function from \( \overline{\Omega} \) to \( \mathbb{R} \) defined by

\[
\tilde{P}[h](x) \equiv \int_\Omega \tilde{S}_n(x - y) h(y) \, dy \quad \forall x \in \overline{\Omega}.
\]

Likewise other potential-type operators, the heat volume potential of an autonomous (i.e. time-independent) density is autonomous and coincides up to the sign with the corresponding Newtonian volume potential. That is, we have the following.

**Lemma 3.1.** Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^n \) and \( T \in ]-\infty, +\infty[ \). Let \( x_0 \in \Omega \). Let \( h \in L^\infty(\Omega) \). Then

\[
P[h](t,x) = -\left( \tilde{P}[h](x) - \delta_{2,n} \tilde{P}[h](x_0) \right) \quad \forall (t, x) \in \overline{\Omega_T}.
\]

**Proof.** We follow the lines of the proof of [10, Lemma A.3, Lemma A.4] where the analog relation between heat and harmonic layer potentials has been proved. Let \((t, x) \in \overline{\Omega_T}\).

We first consider the case \( n = 2 \). Then

\[
P[h](t, x) = \int_{-\infty}^{+\infty} \int_\Omega \left( S_2(t - \tau, x - y) - S_2(-\tau, x_0 - y) \right) h(y) \, dy \, d\tau
\]

By the changes of variable \( t - \tau = \frac{|x - y|^2}{4\xi} \) in the first term inside the integral and \( -\tau = \frac{|x_0 - y|^2}{4\xi} \) in the second term, we get

\[
P[h](t, x) = \lim_{\sigma \to +\infty} \left\{ \int_{\Omega} \int_{\frac{|x - y|^2}{4(\xi + \sigma)}}^{+\infty} \frac{1}{4\pi\xi} e^{-\xi} h(y) \, d\xi \, dy - \int_{\Omega} \int_{\frac{|x_0 - y|^2}{4\sigma}}^{+\infty} \frac{1}{4\pi\xi} e^{-\xi} h(y) \, d\xi \, dy \right\}
\]

5
Let $g$ be the function from $\mathbb{R}$ to $\mathbb{R}$ defined by
\[ g(\xi) = \begin{cases} \frac{e^{-\xi} - 1}{-\xi} & \text{if } \xi \neq 0, \\ 1 & \text{if } \xi = 0. \end{cases} \]

It is easy to see that $g$ is continuous in $\mathbb{R}$ and that
\[ \xi^{-1}e^{-\xi} = \xi^{-1} - g(\xi) \quad \forall \xi \in \mathbb{R} \setminus \{0\} \]

Accordingly, the dominated convergence theorem implies that
\[
P[h](t, x) = \lim_{\sigma \to +\infty} \left\{ \int_{\Omega} \int_{|x-y|^2/(4(t+\sigma))} 1\ 4\pi d\xi h(y)dy \right\} - \lim_{\sigma \to +\infty} \left\{ \int_{\Omega} \int_{|x-y|^2/(4(t+\sigma))} g(\xi) 4\pi d\xi h(y)dy \right\}
\]
\[
= \lim_{\sigma \to +\infty} \left\{ \int_{\Omega} \frac{1}{4\pi} \log \left( \frac{|x_0 - y|^2}{4\sigma} \right) h(y)dy - \int_{\Omega} \frac{1}{2\pi} \log(|x - y|) h(y)dy \right\}
= - (P[h](x) - P[h](x_0)).
\]

Next we pass to the case $n \geq 3$. By the change of variable $|x-y|^2 s = 4(t-\tau)$ we have that
\[
P[h](t, x) = \int_{-\infty}^{t} \int_{\Omega} \frac{1}{(4\pi(t-\tau))^{n/2}} e^{-|x-y|^2/(4(t-\tau))} h(y)dyd\tau
\]
\[
= \frac{1}{4\pi^{n/2}} \int_{0}^{\infty} s^{n/2 - 1} e^{-1} ds \int_{\Omega} \frac{1}{|x-y|^{n-2}} h(y)dy
\]
\[
= \frac{1}{4\pi^{n/2}} \Gamma \left( \frac{n}{2} - 1 \right) \int_{\Omega} \frac{1}{|x-y|^{n-2}} h(y)dy
\]
\[
= \frac{1}{(n-2)s_n} \int_{\Omega} \frac{1}{|x-y|^{n-2}} h(y)dy
\]
which proves the statement. \[\square\]

4 The heat volume potential on $C^{-1+\alpha,\beta}(\Omega_T)$

In the present section we consider the action of $P[\cdot]$ in the space $C^{-1+\alpha,\beta}(\Omega_T)$. Since $f \in C^{-1+\alpha,\beta}(\Omega_T)$ is a distributions, and in particular is not a function in $L^\infty(\Omega_T)$, we must specify what we mean by $P[f]$. To this aim, we need some preliminary results.

**Proposition 4.1.** Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ and $T \in ]-\infty, +\infty]$. Let $\alpha \in [0,1]$. Then the operator $B$ defined by
\[
B[f](t,x) = \int_{-\infty}^{t} \int_{\Omega} \partial_t S_n(t-\tau, x-y)(f(\tau,y) - f(t,y)) dyd\tau \quad \forall (t,x) \in \overline{\Omega_T}
\]
is linear and continuous from $C^{1+n,0}(\Omega_T)$ to $C^{1+n,1}(\Omega_T)$.
Proof. For convenience, for $\gamma > 0$ we set

$$K_\gamma \equiv \sup_{x \in \Omega} \int_{\Omega} \frac{1}{|x-y|^{n-\gamma}}.$$ 

It is easily seen that for $r \gamma > 0$ one has

$$K_\gamma < +\infty.$$  \hfill (5)

Let $(t, x) \in \overline{\Omega_T}$. By (3) there exists a constant $C_{0,1} > 0$ such that

$$\int_{-\infty}^{t} \int_{-\infty}^{t} \left| \partial_t S_n(t - \tau, x - y) (f(\tau, y) - f(t, y)) \right| dy d\tau$$

$$\leq C_{0,1} \| f \|_{C^{1+\frac{\nu}{2}, \alpha}(\Omega_T)} \int_{-\infty}^{t} \left( t - \tau \right)^{-\frac{n}{2} - 1 + \frac{\nu}{2} e^{-\frac{|x-y|^2}{8(v-\gamma)}}} dy d\tau$$

$$= 8^{n-1}\alpha C_{0,1} \| f \|_{C^{1+\frac{\nu}{2}, \alpha}(\Omega_T)} \int_{0}^{+\infty} s^{-\frac{n}{2} - 1 + \frac{\nu}{2}} e^{-\frac{1}{8} ds \int_{\Omega} \frac{1}{|x-y|^{n-1-\alpha}} dy}$$

$$\leq 8^{n-1}\alpha C_{0,1} K_{1+\alpha} \Gamma \left( \frac{n-1}{2} \right) \| f \|_{C^{1+\frac{\nu}{2}, \alpha}(\Omega_T)}.$$ 

Then, by the above inequality and by the Vitali convergence theorem, $B[\cdot]$ is linear and continuous from $C^{1+\frac{\nu}{2}, \beta}(\Omega_T)$ to $C^0(\Omega_T)$. Next we take $t', t'' \in ]-\infty, T[, t' < t'', x \in \Omega$. Then

$$|B[f](t', x) - B[f](t'', x)|$$

$$\leq \left| \int_{t' - 2|t'' - t'|}^{t' + 2|t'' - t'|} \int_{\Omega} \partial_t S_n(t' - \tau, x - y) (f(\tau, y) - f(t', y)) dy d\tau \right|$$

$$+ \left| \int_{t' - 2|t'' - t'|}^{t' + 3|t'' - t'|} \int_{\Omega} \partial_t S_n(t'' - \tau, x - y) (f(\tau, y) - f(t', y)) dy d\tau \right|$$

$$+ \left| \int_{t' - 2|t'' - t'|}^{t' - 3|t'' - t'|} \int_{\Omega} \partial_t S_n(t' - \tau, x - y) - \partial_t S_n(t'' - \tau, x - y) (f(\tau, y) - f(t', y)) dy d\tau \right|$$

$$+ \left| \int_{t' - 2|t'' - t'|}^{t' - 3|t'' - t'|} \int_{\Omega} \partial_t S_n(t'' - \tau, x - y) (f(t'', y) - f(t', y)) dy d\tau \right|.$$ 

We begin considering the first term in the right hand side of (5). The bounds (3) on $S_n$ imply

$$\left| \int_{t' - 2|t'' - t'|}^{t'} \int_{\Omega} \partial_t S_n(t' - \tau, x - y) (f(\tau, y) - f(t', y)) dy d\tau \right|$$

$$\leq C_{0,1} \| f \|_{C^{1+\frac{\nu}{2}, \alpha}(\Omega_T)} \int_{t' - 2|t'' - t'|}^{t'} \left( t' - \tau \right)^{-\frac{n}{2} - 1 + \frac{\nu}{2} e^{-\frac{|x-y|^2}{8(v-\gamma)}}} dy d\tau$$

$$\leq C_{0,1} \| f \|_{C^{1+\frac{\nu}{2}, \alpha}(\Omega_T)} \int_{t' - 2|t'' - t'|}^{t'} (t' - \tau)^{-\frac{n}{2} - 1 + \frac{\nu}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{8(v-\gamma)}} dy d\tau$$

$$= (8\pi)^{\frac{n}{2}} C_{0,1} \| f \|_{C^{1+\frac{\nu}{2}, \alpha}(\Omega_T)} \int_{t' - 2|t'' - t'|}^{t'} (t' - \tau)^{-\frac{n}{2} - 1 + \frac{\nu}{2}} d\tau$$

$$= (8\pi)^{\frac{n}{2}} \frac{2^{1+\nu}}{1+\alpha} C_{0,1} \| f \|_{C^{1+\frac{\nu}{2}, \alpha}(\Omega_T)} |t'' - t'|^{\frac{1+\alpha}{2}}.$$
The second term in the right hand side of (6) can be estimated in the same way. We then consider the third term. By (3) together with the mean value theorem one can see that there exists a constant $C_{0,1} > 0$ such that

$$|\partial_{h}S_{n}(t' - \tau, x - y) - \partial_{h}S_{n}(t'' - \tau, x - y)| \leq C_{0,1} |t' - t''|^{\frac{1+\alpha}{2}} e^{-\frac{|x-y|^{2}}{8(t'' - \tau)}} \tag{7}$$

for all $x, y \in \Omega, t' < t''$, $\tau < t' - 2|t'' - t'|$. For an explicit derivation of (7) we refer to Lanza de Cristoforis and Luzzini [11] Lem. 4.3 (iii). Then

$$\left| \int_{-\infty}^{t'' - 2|t'' - t'|} \int_{\Omega} (\partial_{h}S_{n}(t'' - \tau, x - y) - \partial_{h}S_{n}(t'' - \tau, x - y)) (f(t'' - t'')) dy d\tau \right|$$

$$\leq C_{0,1} \|f\|_{C_{1+\alpha/2,\alpha}^{0}(\Omega)} |t'' - t'| \int_{-\infty}^{t'' - 2|t'' - t'|} (t'' - \tau)^{-\frac{n}{2} - 2 + \frac{1+\alpha}{2}} \int_{\Omega} e^{-\frac{|x-y|^{2}}{8(t'' - \tau)}} dy d\tau$$

$$\leq C_{0,1} \|f\|_{C_{1+\alpha/2,\alpha}^{0}(\Omega)} |t'' - t'| \int_{-\infty}^{t'' - 2|t'' - t'|} (t'' - \tau)^{-\frac{n-3\alpha+\alpha}{2}} \int_{\mathbb{R}^{n}} e^{-\frac{|x-y|^{2}}{8(t'' - \tau)}} dy d\tau$$

$$= (8\pi)^{\frac{1}{2}} C_{0,1} \|f\|_{C_{1+\alpha/2,\alpha}^{0}(\Omega)} |t'' - t'| \int_{-\infty}^{t'' - 2|t'' - t'|} (t'' - \tau)^{-\frac{3\alpha}{2}} d\tau$$

$$= \frac{2^{1+\alpha}}{1 - \alpha} (8\pi)^{\frac{1}{2}} C_{0,1} \|f\|_{C_{1+\alpha/2,\alpha}^{0}(\Omega)} |t'' - t'|^{\frac{1+\alpha}{2}}.$$  

Next, we consider the last term in the right hand side of (6).

$$\left| \int_{-\infty}^{t'' - 2|t'' - t'|} \int_{\Omega} \partial_{h}S_{n}(t'' - \tau, x - y) (f(t'' - t')) dy d\tau \right|$$

$$\leq \|f\|_{C_{1+\alpha/2,\alpha}^{0}(\Omega)} |t'' - t'|^{\frac{1+\alpha}{2}} \int_{-\infty}^{t'' - 2|t'' - t'|} \int_{\Omega} \left| \partial_{h}S_{n}(t'' - \tau, x - y) \right| d\tau dy$$

$$= \|f\|_{C_{1+\alpha/2,\alpha}^{0}(\Omega)} |t'' - t'|^{\frac{1+\alpha}{2}} \int_{-\infty}^{t'' - 2|t'' - t'|} \left| S_{n}(3(t'' - t'), x - y) \right| dy$$

$$= \|f\|_{C_{1+\alpha/2,\alpha}^{0}(\Omega)} |t'' - t'|^{\frac{1+\alpha}{2}} \int \frac{1}{(12\pi(t'' - t'))^\frac{n}{2}} e^{-\frac{|x-y|^{2}}{12(t'' - t')}} dy$$

$$\leq \|f\|_{C_{1+\alpha/2,\alpha}^{0}(\Omega)} |t'' - t'|^{\frac{1+\alpha}{2}}.$$  

It remains to prove that for all $i \in \{1, \ldots, n\}$, the map $\partial_{x_{i}}B[\cdot]$ is linear and continuous from $C_{1+\alpha,\beta}^{0}(\Omega)$ to $C_{\alpha+\epsilon}^{0}(\Omega)$. Let $(t, x) \in \Omega T$. Let $i \in \{1, \ldots, n\}$. By classical differentiation theorems for integrals depending on a parameter and by the bound (3) there exists a constant $C_{e,1,1} > 0$ such that

$$|\partial_{x_{i}}B[f](t, x)| \leq C_{e,1,1} \|f\|_{C_{1+\alpha/2,\alpha}^{0}(\Omega)} \int_{-\infty}^{t} \int_{\Omega} (t - \tau)^{-\frac{n-1+\alpha}{2}} \frac{1}{2} e^{-\frac{|x-y|^{2}}{8(t'' - \tau)}} dy d\tau$$

$$= 8^{\frac{n-1+\alpha}{2}} C_{e,1,1} \|f\|_{C_{1+\alpha/2,\alpha}^{0}(\Omega)} \int_{0}^{\infty} u^{-\frac{n}{2} - 1 + \frac{\alpha}{2}} e^{-\frac{1}{u}} du \int_{\Omega} \frac{1}{|x - y|^{n-\alpha}} dy$$

$$\leq 8^{\frac{n-1+\alpha}{2}} C_{e,1,1} K_{\alpha} \Gamma \left( \frac{n - \alpha}{2} \right) \|f\|_{C_{1+\alpha/2,\alpha}^{0}(\Omega)}.$$  

For what concerns the time $\frac{\alpha}{2}$-Hölder norm, it can be estimated exactly in the same way of the first part of the proof just by noting that the kernel is more singular by a term $(t - \tau)^{-\frac{\alpha}{2}}$ but we need to obtain an $\frac{\alpha}{2}$-Hölder regularity instead of an $\frac{1+\alpha}{2}$-regularity. Now
we consider the spatial $\alpha$-Hölder regularity. Let $x', x'' \in \Omega$, $t \in ]-\infty, T[$. By classical differentiation theorems for integrals depending on a parameter we have

$$
|\partial_{x_i} B[f](t, x') - \partial_{x_i} B[f](t, x'')| 
= \left| \int_0^{+\infty} \int_{\Omega} \partial_{x_i} \partial_t S_n(\tau, x' - y) \left( f(t - \tau, y) - f(t, y) \right) dx d\tau 
- \int_0^{+\infty} \int_{\Omega} \partial_{x_i} \partial_t S_n(\tau, x'' - y) \left( f(t - \tau, y) - f(t, y) \right) dx d\tau \right|
\leq \int_0^{+\infty} \int_{\Omega} \partial_{x_i} \partial_t S_n(\tau, x' - y) \left( f(t - \tau, y) - f(t, y) \right) dx d\tau
+ \int_0^{+\infty} \int_{\Omega} \partial_{x_i} \partial_t S_n(\tau, x'' - y) \left( f(t - \tau, y) - f(t, y) \right) dx d\tau
$$

We consider the first term in the right hand side of (8). By the bound (3) we have

$$
\left| \int_0^{+\infty} \int_{\Omega} \partial_{x_i} \partial_t S_n(\tau, x' - y) \left( f(t - \tau, y) - f(t, y) \right) dx d\tau \right|
\leq C_{e_1, \alpha} \|f\|_{C^{\frac{1+\alpha}{2} \tilde{a}(\tilde{T})}} \int_0^{+\infty} \int_{\Omega} \partial_{x_i} \partial_t S_n(\tau, x' - y) \left( f(t - \tau, y) - f(t, y) \right) dx d\tau
= (8\pi)^{\frac{3}{2}} C_{e_1, \alpha} \|f\|_{C^{\frac{1+\alpha}{2} \tilde{a}(\tilde{T})}} \int_0^{+\infty} \frac{\tau^{n_2 - \frac{3}{2} + \frac{1+\alpha}{2} \alpha}}{\tau^{\frac{3}{2}}} d\tau
= \frac{2}{\alpha} (8\pi)^{\frac{3}{2}} C_{e_1, \alpha} \|f\|_{C^{\frac{1+\alpha}{2} \tilde{a}(\tilde{T})}} |x' - x''|^\alpha.
$$

The second term in the right hand side of (8) can be estimated in the same way. Finally we consider the last term. The fundamental theorem of calculus and the bound (3) imply

$$
\left| \int_0^{+\infty} \int_{\Omega} \partial_{x_i} \partial_t S_n(\tau, x' - y) \left( f(t - \tau, y) - f(t, y) \right) dx d\tau \right|
\leq \|f\|_{C^{\frac{1+\alpha}{2} \tilde{a}(\tilde{T})}} \int_0^{+\infty} \int_{\Omega} \partial_{x_i} \partial_t S_n(\tau, x' - y) \left( f(t - \tau, y) - f(t, y) \right) dx d\tau
\leq \|f\|_{C^{\frac{1+\alpha}{2} \tilde{a}(\tilde{T})}} \sum_{j=1}^n |x'_j - x''_j|
\times \int_0^{+\infty} \int_{\Omega} \partial_{x_i} \partial_t S_n(\tau, \lambda x' + (1 - \lambda) x'' - y) \left( f(t - \tau, y) - f(t, y) \right) dx d\tau
\leq (8\pi)^{\frac{3}{2}} \|f\|_{C^{\frac{1+\alpha}{2} \tilde{a}(\tilde{T})}} \sum_{j=1}^n C_{e_1, +e_j, \lambda} |x'_j - x''_j|
\times \int_0^{+\infty} \int_{\Omega} \partial_{x_i} \partial_t S_n(\tau, \lambda x' + (1 - \lambda) x'' - y) \left( f(t - \tau, y) - f(t, y) \right) dx d\tau
\leq (8\pi)^{\frac{3}{2}} \|f\|_{C^{\frac{1+\alpha}{2} \tilde{a}(\tilde{T})}} \sum_{j=1}^n C_{e_1, +e_j, \lambda} |x'_j - x''_j| \int_0^{+\infty} \int_{\Omega} \partial_{x_i} \partial_t S_n(\tau, \lambda x' + (1 - \lambda) x'' - y) \left( f(t - \tau, y) - f(t, y) \right) dx d\tau
\leq \sum_{j=1}^n C_{e_1, +e_j, \lambda} \|f\|_{C^{\frac{1+\alpha}{2} \tilde{a}(\tilde{T})}} |x' - x''| \int_0^{+\infty} \int_{\Omega} \partial_{x_i} \partial_t S_n(\tau, \lambda x' + (1 - \lambda) x'' - y) \left( f(t - \tau, y) - f(t, y) \right) dx d\tau
\leq \sum_{j=1}^n C_{e_1, +e_j, \lambda} \|f\|_{C^{\frac{1+\alpha}{2} \tilde{a}(\tilde{T})}} |x' - x''| \int_0^{+\infty} \int_{\Omega} \partial_{x_i} \partial_t S_n(\tau, \lambda x' + (1 - \lambda) x'' - y) \left( f(t - \tau, y) - f(t, y) \right) dx d\tau
"
\[ \frac{2}{1 - \alpha} \sum_{j=1}^{n} C_{\varepsilon_j, \varepsilon_j, 0} \| f \|_{C^{1+\alpha, 0}(\Omega_T)} |x' - x''|^\alpha. \]

Next we show that the operator \( B[f] \) of Proposition 4.1 coincides with \( \partial_t P[f] \) whenever \( f \in C^{1+\alpha, \beta}(\Omega_T) \).

**Lemma 4.2.** Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^n \) and \( T \in [-\infty, +\infty] \). Let \( \alpha, \beta \in ]0, 1[. \) Let \( f \in C^{1+\alpha, \beta}(\Omega_T) \). Then \( P[f] \) is continuously differentiable with respect to \( t \) and

\[ \partial_t P[f](t, x) = \int_{-\infty}^{t} \int_{\Omega} \partial_t S_n(t - \tau, x - y)(f(\tau, y) - f(t, y)) \, dyd\tau \quad \forall (t, x) \in \Omega_T. \]

**Proof.** Let \( (t, x) \in \Omega_T \). Since \( f \) is \( \beta \)-Hölder continuous in space, by Friedman [17, Thm. 9 p. 21] the volume potential \( P[f] \) is continuously differentiable with respect to the time variable and two time continuously differentiable with respect to the space variables. Moreover

\[ \partial_t P[f](t, x) = f(t, x) + \Delta P[f](t, x). \]

By the properties of the Newtonian volume potential (see e.g. Gilbarg and Trudinger [8, §4.2]), we have

\[ \Delta \tilde{P}[f(t, \cdot)](x) = f(t, x). \]

Since by Lemma 3.1 the heat and Newtonian volume potential coincide up to the sign, we have that

\[ \partial_t P[f](t, x) = \Delta \left( P[f](t, x) - P[f(t, \cdot)](t, x) \right) = \Delta P[f - f(t, \cdot)](t, x). \]

The bound \( \| \cdot \|_{C^{1+\alpha}} \) for the fundamental solution and \( \| \cdot \|_{C^{0}} \) imply that

\[
\int_{-\infty}^{t} \int_{\Omega} |\Delta S_n(t - \tau, x - y)(f(\tau, y) - f(t, y))| \, dyd\tau \\
\leq \sum_{j=1}^{n} C_{2\varepsilon_j, 0} \| f \|_{C^{1+\alpha, 0}(\Omega_T)} \int_{-\infty}^{t} \int_{\Omega} (t - \tau)^{-\frac{n}{2} - 1 + \frac{1+\alpha}{2} \theta} e^{-\frac{|x - y|^2}{8(\tau - \theta)}} \, dyd\tau \\
\leq 8 \sum_{j=1}^{n} C_{2\varepsilon_j, 0} \| f \|_{C^{1+\alpha, 0}(\Omega_T)} \int_{0}^{+\infty} u^{-\frac{n}{2} - 1 + \frac{1+\alpha}{2} \theta} e^{-\frac{u}{8}} \, du \int_{\Omega} \frac{1}{|x - y|^{n-1-\alpha}} \, dy \\
\leq 8 \sum_{j=1}^{n} C_{2\varepsilon_j, 0} K_{1+\alpha} \Gamma \left( \frac{n - 1 - \alpha}{2} \right) \| f \|_{C^{1+\alpha, 0}(\Omega_T)}. \]

Accordingly the statement follows by standard differentiation theorems for integral depending on a parameter and by recalling that \( S_n \) solves the heat equation in \( \mathbb{R}^{1+n} \setminus \{(0, 0)\} \).

In order to define the heat volume potential in \( C^{1+\alpha, \beta}(\Omega_T) \), which is a quotient space, we need to show that our definition is independent on the choice of the representative in the equivalence class. To this aim, we need the following.

**Lemma 4.3.** Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^n \) and \( T \in [-\infty, +\infty] \). Let \( \alpha, \beta \in ]0, 1[. \) Let \( f \in C^{1+\alpha, \beta}(\Omega_T) \) be such that \( \partial_t f = 0 \) in the sense of distributions. Then

\[ B[f](t, x) = \int_{-\infty}^{t} \int_{\Omega} \partial_t S_n(t - \tau, x - y)(f(\tau, y) - f(t, y)) \, dyd\tau = 0 \quad \forall (t, x) \in \Omega_T. \]
Proof. Since $B[f]$ is continuous in $\overline{\Omega_T}$ by Proposition 4.1 it suffices to show equality (9) in $\Omega_T$. Let $(t, x) \in \Omega_T$ be fixed. Since $S_n(t - \tau, x - y)(f(\tau, y) - f(t, y))$ is continuous in $(\tau, y) \in \overline{\Omega_t \setminus \{(t, x)\}}$, it has a distributional $\tau$-derivative which, since $\partial_t f = 0$ in the sense of distributions, equals
\[
g(\tau, y) \equiv -\partial_t S_n(t - \tau, x - y)(f(\tau, y) - f(t, y)) \quad (\tau, y) \in \Omega_t \setminus \{(t, x)\}.
\]
Let $\varepsilon > 0$. Since the function $g(\tau, y)$ is continuous in $\overline{\Omega_{t-\varepsilon}}$, then by Lemma 4.2 we have
\[
-\int_{-\infty}^{t-\varepsilon} \int_{\Omega} \partial_t S_n(t - \tau, x - y)(f(\tau, y) - f(t, y)) \, dy \, d\tau = \int_{\Omega} S_n(\varepsilon, x - y)(f(t - \varepsilon, y) - f(t, y)) \, dy
d\tau
- \lim_{\tau \to -\infty} \int_{\Omega} S_n(t - \tau, x - y)(f(\tau, y) - f(t, y)) \, dy
= \int_{\Omega} S_n(\varepsilon, x - y)(f(t - \varepsilon, y) - f(t, y)) \, dy.
\]
For all $y \in \Omega$ we have
\[
\left| S_n(\varepsilon, x - y)(f(t - \varepsilon, y) - f(t, y)) \right| \leq \left\| f \right\|_{C^{\alpha, \beta}(\overline{\Omega_T})} \frac{1}{(4\pi)^{n/2}} e^{-\frac{|x-y|^2}{4\varepsilon}} \leq \left\| f \right\|_{C^{\alpha, \beta}(\overline{\Omega_T})} \sup_{\xi > 0} \left( \frac{\xi^{n+1+\alpha}}{e^{1+\frac{\alpha}{4\xi}}} \right) \frac{1}{|x-y|^{n-1}}.
\]
By the dominated convergence theorem we obtain (9) by letting $\varepsilon \to 0$. \qed

We are now ready to define the volume potential in $C^{-\frac{1+\alpha}{2}, \beta}(\overline{\Omega_T})$ as
\[
P[g](t, x) \equiv \int_{-\infty}^{\tau} \int_{\Omega} \partial_t S_n(t - \tau, x - y)(f(\tau, y) - f(t, y)) \, dy \, d\tau \quad \forall (t, x) \in \overline{\Omega_T},
\]
for all $g = \partial_t f \in C^{-\frac{1+\alpha}{2}, \beta}(\Omega_T)$. Our main result is the following.

**Theorem 4.4.** Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ and $T \in ]-\infty, +\infty[$. Let $\alpha, \beta \in [0, 1[$. Then

i) If $g \in C^{-\frac{1+\alpha}{2}, \beta}(\overline{\Omega_T})$, then $\partial_t P[g] - \Delta P[g] = g$ in the sense of distributions in $\Omega_T$.

ii) $P[\cdot]$ is a bounded linear operator from $C^{-\frac{1+\alpha}{2}, \beta}(\overline{\Omega_T})$ to $C^{\frac{1+\alpha}{2}, 1+\alpha}(\overline{\Omega_T})$.

**Proof.** We start considering i). If $g \in C^{-\frac{1+\alpha}{2}, \beta}(\overline{\Omega_T})$ and $g = \partial_t f$ with $f \in C^{\frac{1+\alpha}{2}, \beta}(\overline{\Omega_T})$, then by Lemma 4.2 we have
\[
P[g] = \partial_t P[f] \quad \forall (t, x) \in \overline{\Omega_T}.
\]
Hence
\[
\partial_t P[g] - \Delta P[g] = \partial_t (\partial_t P[f] - \Delta P[f]) = \partial_t f = g,
\]
in the sense of distributions. The second equality in (10) follows by classical results for the heat volume potential (see, e.g., Friedman [7] Thm. 9, p.21).

Statement ii) simply follows by the definition of $P$, by Proposition 4.1 and by Theorem A.1 of the Appendix. \qed
5 The Dirichlet and Neumann problems

As an application, we show the solvability of the Dirichlet and Neumann problem for equation (2). Let \( \alpha, \beta \in [0, 1] \). Let \( T > 0 \). Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^n \) of class \( C^{1, \alpha} \). Let \( f \in C^{\frac{1+\alpha}{2}, \beta}(\Omega \times T) \), \( g \in C^{\frac{1+\alpha}{2}, 1+\alpha}(\partial \Omega \times T) \), and \( h \in C^{\frac{1+\alpha}{2}, \alpha}(\partial \Omega \times T) \). The Dirichlet problem for equation (2) is

\[
\begin{aligned}
\partial_t u - \Delta u &= \partial_t f \quad \text{in } [0, T] \times \Omega, \\
u &= g \quad \text{on } [0, T] \times \partial \Omega, \\
u(0, \cdot) &= 0 \quad \text{in } \Omega,
\end{aligned}
\]  

(11)

while the Neumann problem reads

\[
\begin{aligned}
\partial_t v - \Delta v &= \partial_t f \quad \text{in } [0, T] \times \Omega, \\
\partial_n v &= h \quad \text{on } [0, T] \times \partial \Omega, \\
v(0, \cdot) &= 0 \quad \text{in } \Omega.
\end{aligned}
\]  

(12)

We note that following the lines of the present section one can also show the solvability of problems for equation (2) with boundary conditions other than Dirichlet and Neumann, as for example Robin or transmission boundary conditions. For the sake of simplicity, here we treat only problems (11) and (12). We have the following existence and uniqueness result which is an immediate consequence of classical parabolic theory together with Theorem 4.4.

**Theorem 5.1.** Let \( \alpha, \beta \in [0, 1] \). Let \( T > 0 \). Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^n \) of class \( C^{1, \alpha} \). Let \( f \in C^{\frac{1+\alpha}{2}, \beta}(\Omega \times T) \), \( g \in C^{\frac{1+\alpha}{2}, 1+\alpha}(\partial \Omega \times T) \), and \( h \in C^{\frac{1+\alpha}{2}, \alpha}(\partial \Omega \times T) \). Then problem (11) admits a unique solution \( u \in C^{\frac{1+\alpha}{2}, 1+\alpha}(\Omega \times T) \) and problem (12) admits a unique solution \( v \in C^{\frac{1+\alpha}{2}, 1+\alpha}(\Omega \times T) \).

**Proof.** Since (11) and (12) are linear problems, the uniqueness of their solutions in the space \( C^{\frac{1+\alpha}{2}, 1+\alpha}(\Omega \times T) \) is well known (cf, e.g., Friedman [7]).

Next we pass to consider existence. By Theorem 4.4 \( P[\partial_t f] \in C^{\frac{1+\alpha}{2}, 1+\alpha}(\Omega \times T) \) and solves equation (2). Thus, the existence of a solution for problems (11) and (12) can be reduced to the existence of a solution for

\[
\begin{aligned}
\partial_t u - \Delta u &= 0 \quad \text{in } [0, T] \times \Omega, \\
u = g - P[\partial_t f] \quad \text{on } [0, T] \times \partial \Omega, \\
u(0, \cdot) &= 0 \quad \text{in } \Omega,
\end{aligned}
\]  

(13)

and for

\[
\begin{aligned}
\partial_t v - \Delta v &= 0 \quad \text{in } [0, T] \times \Omega, \\
\partial_n v &= h - \partial_n P[\partial_t f] \quad \text{on } [0, T] \times \partial \Omega, \\
v(0, \cdot) &= 0 \quad \text{in } \Omega,
\end{aligned}
\]  

(14)

respectively. It is classical that problems (13) and (14) admit a solution in the space \( C^{\frac{1+\alpha}{2}, 1+\alpha}(\Omega \times T) \). For a proof of this result based on potential theoretic methods we refer to Baderko [1] (see also Lunardi and Vespri [15]).

Appendix A

In this Appendix we collect a well-know result in functional analysis regarding quotient spaces. For a proof we refer to Schaefer [19, p. 42] (see also Dalla Riva, Lanza de Cristoforis and Musolino [5]).

**Theorem A.1.** Let \((X, \| \cdot \|_X)\) be a normed space. Let \(V\) be a closed subspace of \(X\). Then the norm on the quotient \(X/V\) defined by

\[
\| [x] \|_{X/V} \equiv \inf_{v \in V} \| x + v \|_X \quad \forall [x] \in X/V
\]

generates the quotient topology on \(X/V\), i.e the strongest topology on \(X/V\) such that the canonical projection \(\pi\) of \(X\) onto \(X/V\) is continuous.

If \(X\) is complete, then \((X/V, \| \cdot \|_{X/V})\) is complete. If \(Y\) is a normed space and if \(T\) is a linear map from \(X/V\) to \(Y\), then \(T\) is continuous if and only if \(T \circ \pi\) is continuous.

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