A New Analytical Method for Self-Force Regularization. II

Testing the Efficiency for Circular Orbits

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In a previous paper, based on the black hole perturbation approach, we formulated a new analytical method for regularizing the self-force acting on a particle of small mass \( \mu \) orbiting a Schwarzschild black hole of mass \( M \), where \( \mu \ll M \). In our method, we divide the self-force into the \( \tilde{S} \)-part and \( \tilde{R} \)-part. All the singular behavior is contained in the \( \tilde{S} \)-part, and hence the \( \tilde{R} \)-part is guaranteed to be regular. In this paper, focusing on the case of a scalar-charged particle for simplicity, we investigate the precision of both the regularized \( \tilde{S} \)-part and the \( \tilde{R} \)-part required for the construction of sufficiently accurate waveforms for almost circular inspiral orbits. We calculate the regularized \( \tilde{S} \)-part for circular orbits to 18th post-Newtonian (PN) order and investigate the convergence of the post-Newtonian expansion. We also study the convergence of the remaining \( \tilde{R} \)-part in the spherical harmonic expansion. We find that a sufficiently accurate Green function can be obtained by keeping the terms up to \( \ell = 13 \).

§1. Introduction

We are at the dawn of gravitational wave astronomy. Several ground-based interferometric gravitational wave detectors are in various stages of development.1)–4) R&D studies of a space-based gravitational wave observatory project, the Laser Interferometer Space Antenna (LISA),5) which will observe gravitational waves in the mHz-band, are in rapid progress. There is also a proposal for a DECi hertz Interferometer Gravitational wave Observatory (DECIGO/BBO).6), 7) This will be a laser interferometer gravitational wave antenna in space sensitive to frequencies

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Among the promising targets for LISA are binary systems consisting of a supermassive black hole ($M \sim 10^{5-8} M_\odot$) and a compact object of near solar mass ($\mu \sim 1-10 M_\odot$). Such gravitational wave sources could provide the first high-precision tests of general relativity in very strong gravitational regimes. It is expected that gravitational waves will be observed from the inspiral stage of these binary systems. Because the typical observation time of the inspiral stage is very long, ranging from several months to several years, high-order relativistic corrections should be observable. The standard post-Newtonian approximation\(^8\) seems inappropriate for this type of binary, i.e. with an extremely large mass ratio, because the extension of the PN approximation to higher post-Newtonian orders is not straightforward in this case. However, in such a system, there is another natural perturbation parameter, $\mu/M \sim 10^{-6}$. Therefore, the black hole perturbation approach is suited for such a binary system.\(^9\), \(^10\)

We consider the self-force acting on a particle of small mass $\mu$ orbiting a black hole of mass $M$, where $\mu \ll M$. The central black hole $M$ fixes the metric of the ‘background’ spacetime. At lowest order in the mass ratio, $(\mu/M)^0$, the particle moves along a geodesic in the background geometry. Already at this lowest order, combined with the assumption of adiabatic orbital evolution, this approach has proven to be very powerful for evaluating general relativistic corrections to the gravitational waveforms.\(^9\), \(^10\) At the next order, however, the spacetime geometry is perturbed, and the motion of the particle is affected by its self-force. When we consider the point particle limit, the full self-force diverges at the position of the particle, and hence it needs to be regularized. The field generated by the particle can be expressed in terms of the retarded Green function. The retarded Green function can be decomposed into the direct part and the tail part, and the properly regularized self-force is given by the tail part (or the $R$-part) of the self-field, which is obtained by subtracting the direct part (or the $S$-part) from the full field. This was first shown by DeWitt and Brehme\(^11\) for the scalar and electro-magnetic cases decades ago, and rather recently by Mino, Sasaki and Tanaka,\(^12\) and Quinn and Wald\(^13\) for the gravitational case. An equivalent but more elegant decomposition of the Green function was proposed by Detweiler and Whiting,\(^14\) in which the direct part is replaced by the $S$-part and the tail part by the $R$-part. The $S$-part is defined so as to vanish when the two arguments $x$ and $x'$ are timelike, and to satisfy the same equation as the retarded Green function. The latter condition implies that the $R$-part satisfies the source-free homogeneous equation.

To avoid various technical difficulties associated with the gravitational case, many previous papers focused on the regularization problem of the scalar radiation reaction force.\(^15\)–\(^25\) Even in this simplified case of a scalar-charged particle, however, only special orbits, such as circular or radial orbits, have been considered, and no systematic method for computing the regularized self-force has been given.

Recently,\(^27\) we formulated a new analytical method for regularizing the self-force acting on a particle in the Schwarzschild spacetime. As is commonly the case in the self-force regularization problem, our method also uses the spherical harmonic decomposition to obtain a regularized expression for the self-force.\(^15\)–\(^22\), \(^24\)–\(^26\), \(^28\)–\(^31\)
The novel point of our method is that it provides a new decomposition of the retarded Green function in the frequency domain, which we call the $\tilde{S}$ and $\tilde{R}$-parts. This decomposition guarantees that all the singular behavior is contained in the $\tilde{S}$-part. In the black hole perturbation approach, the Green function is conventionally calculated in the frequency domain, while the $S$-part to be subtracted is given in the time domain. We presented a systematic method for translating the $\tilde{S}$-part into an expression in the time domain, and gave an explicit expression for the regularized $\tilde{S}$-part of the self-force, that is, the $\tilde{(S - S)}$-part for general orbits.

Our method for handling divergences in the $\tilde{S}$-part relies on the slow motion approximation, i.e., the post-Newtonian (PN) expansion. Although the order of the expansion is not technically limited, thanks to the systematic calculation method, the highest PN order attainable in practice is limited by the availability of computational resources. In this paper, we revisit the problem of the self-force acting on a scalar charge moving in a circular orbit around a Schwarzschild black hole, and demonstrate that this practical limitation of our new method is in fact not at all severe. We investigate the convergence of the PN expansion and estimate the required PN order to obtain sufficiently accurate waveforms. We also clearly elucidate the difference between the roles of the conservative part and the dissipative part of radiation reaction forces in this formalism. The role played by the conservative self-force during the orbital evolution has to this time been demonstrated only in toy model scenarios (see, for example, Ref. 32). We also discuss the convergence of the remaining $\tilde{R}$-part.

The paper is organized as follows. In §2, we summarize our new regularization method in the case of the scalar self-force. In §3, we calculate the scalar self-force on a particle in a circular orbit around a Schwarzschild black hole. First, we explicitly obtain an expression for the $(\tilde{S} - S)$ part of the scalar self-force to 4 PN order. It is possible to extend the calculation to an arbitrarily high order systematically. In fact, though not presented explicitly in this paper, the actual calculation is done to 18PN order. Then, using this result, we study the convergence of the $(\tilde{S} - S)$-force in the PN expansion. Subsequently, combining this result with the calculation of the $\tilde{R}$-part with sufficient accuracy, so that it does not spoil the 18PN order accuracy of the $(\tilde{S} - S)$-part, the regularized scalar self-force is evaluated and is compared with the result obtained by Detweiler, Messaritaki and Whiting. In §4, in order to obtain a rough, qualitative estimate of the PN order to which we need to proceed in the realistic gravitational case, we pretend that the scalar charge can be replaced by the gravitational mass and consider the phase error in the gravitational waveform due to the truncation of a series in the PN expansion in our regularization calculation. The final section, §5, contains conclusion and discussion on the implications of our result.
§2. Analytic regularization scheme

We consider a point particle of a scalar charge \( q \) moving in a Schwarzschild background characterized by mass \( M \),

\[
ds^2 = - \left( 1 - \frac{2M}{r} \right) dt^2 + \left( 1 - \frac{2M}{r} \right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \tag{2.1}
\]

The equation for the scalar field induced by the particle is given by

\[
\nabla^\alpha \nabla_\alpha \psi(x) = -q \int_{-\infty}^{\infty} d\tau \frac{\delta^{(4)}(x - z(\tau))}{\sqrt{-g}}. \tag{2.2}
\]

Here, \( \tau \) and \( z(\tau) \) are, respectively, the proper time and trajectory of the particle, and \( g \) is the determinant of the metric. The solution for the (full) scalar field can be obtained using the retarded Green function as

\[
\psi^{\text{full}}(x) = q \int_{-\infty}^{\infty} d\tau \, G^{\text{full}}(x, z(\tau)). \tag{2.3}
\]

The retarded Green function satisfies the Klein-Gordon equation

\[
\nabla^\alpha \nabla_\alpha G^{\text{full}}(x, x') = -\frac{\delta^{(4)}(x - x')}{\sqrt{-g}}, \tag{2.4}
\]

with retarded boundary conditions. Furthermore, due to the spherical symmetry and the static nature of the background spacetime, the Green function can be decomposed in Fourier-spherical harmonics as

\[
G^{\text{full}}(x, x') = \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \sum_{\ell m} g^{\text{full}}_{\ell m \omega}(r, r') Y_{\ell m}(\theta, \phi) Y_{\ell m}^*(\theta', \phi'). \tag{2.5}
\]

Here the functions \( Y_{\ell m}(\theta, \phi) \) are the ordinary spherical harmonics. Then, Eq. (2.4) reduces to an ordinary differential equation for the radial part of the retarded Green function,

\[
\left[ \left( 1 - \frac{2M}{r} \right) \frac{d^2}{dr^2} + \frac{2(r - M)}{r^2} \frac{d}{dr} + \left( \frac{r \omega^2}{r - 2M} - \frac{\ell(\ell + 1)}{r^2} \right) \right] g^{\text{full}}_{\ell m \omega}(r, r') = -\frac{\delta(r - r')}{r^2}. \tag{2.6}
\]

The radial part of the full Green function is expressed in terms of two independent homogeneous solutions of Eq. (2.6) as

\[
g^{\text{full}}_{\ell m \omega}(r, r') = -\frac{1}{W_{\ell m \omega}(\phi^\nu_{\text{in}}, \phi^\nu_{\text{up}})} \left( \phi^\nu_{\text{in}}(r') \phi^\nu_{\text{up}}(r)(r' - r) + \phi^\nu_{\text{up}}(r) \phi^\nu_{\text{in}}(r')(r - r') \right), \tag{2.7}
\]

with \( W_{\ell m \omega} \) being the Wronskian,

\[
W_{\ell m \omega}(\phi^\nu_{\text{in}}, \phi^\nu_{\text{up}}) = r^2 \left( 1 - \frac{2M}{r} \right) \left[ \left( \frac{d}{dr} \phi^\nu_{\text{up}}(r) \right) \phi^\nu_{\text{in}}(r) - \left( \frac{d}{dr} \phi^\nu_{\text{in}}(r) \right) \phi^\nu_{\text{up}}(r) \right].
\]
Here \( \phi^{\nu}_{\text{in}} \) is a homogeneous solution with vanishing flux from the past horizon (when multiplied by \( e^{-i\omega t} \)), and \( \phi^{\nu}_{\text{up}} \) is a homogeneous solution with vanishing flux from the past null infinity. They can be obtained using a systematic method developed by Mano, Suzuki and Takasugi.\(^{33}\) The parameter \( \nu \) is called the renormalized angular momentum, which reduces to \( \ell \) in the limit \( \omega M \to 0 \).\(^{33,34}\)

We can express the homogeneous solutions \( \phi^{\nu}_{\text{in}} \) and \( \phi^{\nu}_{\text{up}} \) in terms of another set of solutions, \( \phi^{\nu}_{c} \) and \( \phi^{-\nu-1}_{c} \), which are given by a series of Coulomb wave functions as\(^{10,33}\)

\[
\phi^{\nu}_{\text{in}} = \phi^{\nu}_{c} + \tilde{\beta}_{\nu} \phi^{-\nu-1}_{c}, \quad \phi^{\nu}_{\text{up}} = \tilde{\gamma}_{\nu} \phi^{\nu}_{c} + \phi^{-\nu-1}_{c}.
\] (2.8)

The properties of the coefficients \( \tilde{\beta}_{\nu} \) and \( \tilde{\gamma}_{\nu} \) are discussed in Ref. 33). As shown in one of our previous papers,\(^{27}\) using these homogeneous solutions, the Green function can be divided into the two parts as

\[
g^{\text{full}}_{\ell\nu\omega}(r, r') = g^{\text{S}}_{\ell\nu\omega}(r, r') + g^{\text{R}}_{\ell\nu\omega}(r, r'),
\] (2.9)

where

\[
g^{\text{S}}_{\ell\nu\omega}(r, r') = -\frac{1}{W_{\ell\nu\omega}(\phi^{\nu}_{c}, \phi^{-\nu-1}_{c})} \left[ \phi^{\nu}_{c}(r)\phi^{-\nu-1}_{c}(r')\theta(r' - r) + \phi^{-\nu-1}_{c}(r)\phi^{\nu}_{c}(r')\theta(r - r') \right],
\]

\[
g^{\text{R}}_{\ell\nu\omega}(r, r') = -\frac{1}{(1 - \tilde{\beta}_{\nu}\tilde{\gamma}_{\nu})W_{\ell\nu\omega}(\phi^{\nu}_{c}, \phi^{-\nu-1}_{c})} \left[ \hat{\beta}_{\nu}\hat{\gamma}_{\nu} \left( \phi^{\nu}_{c}(r)\phi^{-\nu-1}_{c}(r') + \phi^{-\nu-1}_{c}(r)\phi^{\nu}_{c}(r') \right) \right.
\]

\[
+ \tilde{\gamma}_{\nu}\phi^{\nu}_{c}(r)\phi^{\nu}_{c}(r') + \hat{\beta}_{\nu}\phi^{-\nu-1}_{c}(r)\phi^{-\nu-1}_{c}(r') \right].
\] (2.10)

The \( \tilde{S} \)-part of the radial Green function, \( g^{\text{S}}_{\ell\nu\omega} \), is symmetric, and it satisfies the same inhomogeneous equation as the full radial Green function, hence becomes singular when the sum over the spherical harmonic indices is taken. Contrastingly, the \( \tilde{R} \)-part, \( g^{\text{R}}_{\ell\nu\omega} \), satisfies the source free equation, and hence remains regular. It is the \( \tilde{S} \)-part that needs to be regularized. An important fact is that the \( \tilde{S} \)-part contains only positive integer powers of \( \omega \) when expanded. Therefore the frequency integral can be analytically carried out to give the \( \tilde{S} \)-part in the time domain easily. By contrast, it is difficult to obtain the \( \tilde{R} \)-part in the time domain analytically for general orbits, because it includes terms logarithmic in \( \omega \). The contribution of such a term in the time domain cannot be expressed in terms of local quantities at the location of the particle, but it can be expressed non-locally in terms of the integral along the past trajectory. Nevertheless, because the \( \tilde{R} \)-part is regular and the summation over \( \ell \) converges, it should be possible to evaluate it numerically without much difficulty.

The scalar self-force is given by

\[
F_{\alpha} = qP^{\beta}_{\alpha} \partial_{\beta} \psi,
\] (2.11)

where the projection tensor \( P^{\beta}_{\alpha} = \delta^{\beta}_{\alpha} + u_{\alpha}u^{\beta} \) is applied to it in order to keep the scalar charge constant. The regularized self-force is given by the \( \tilde{R} \)-part of the self-field defined by Detweiler and Whiting.\(^{14}\) The \( \tilde{R} \)-part of the self-force in our new
decomposition scheme now takes the form
\[ F^R_\alpha = F^{\text{full}}_\alpha - F^S_\alpha = (F^S_\alpha - F^S_\alpha) + F^R_\alpha. \] (2.12)
Further motivating the above decomposition of the self-force from a physical point of view, we note that it has been shown that the \((\tilde{S} - S)\)-part of the self-force is shown to be purely conservative for generic orbits.\(^{27}\) In other words, the radiative (or dissipative) part of the force is solely contained in the \(\tilde{R}\)-part. This property is an additional merit of our decomposition.

§3. Convergence test of the PN expansion

Because our regularization method relies on the PN expansion, it is necessary to examine if it converges sufficiently rapidly. First we investigate the convergence of a PN series expansion for the \((\tilde{S} - S)\)-part for circular orbits, as a test of the efficiency of our analytic regularization method. To obtain an analytical expression for the \((\tilde{S} - S)\)-part in the time domain for general orbits, it is necessary to expand it in powers of \(\omega\). This means that we adopt the slow motion approximation,
\[ \omega r = O(v), \quad \omega M = O(v^3); \quad v \ll 1, \] (3.1)
where \(v\) is the velocity of the particle. A calculation is said to be of \(n\)-PN order if it is accurate up through \(O(v^{2n})\).

We next study the convergence of the \(\tilde{R}\)-part. In the circular case, it is easy to evaluate even the \(\tilde{R}\)-part analytically in the time domain. For a circular orbit, the trajectory \(z^\alpha(\tau)\) and the four velocity \(u^\alpha(\tau)\) of the particle can be written as
\[ \{z^\alpha(\tau)\} = \left\{ u^t, r_0, \frac{\pi}{2}, u^\phi \right\}; \quad u^t = \sqrt{\frac{r_0}{r_0 - 3M}}, \quad u^\phi = \frac{1}{r_0} \sqrt{\frac{M}{r_0 - 3M}}. \] (3.2)
Then the frequency integral is readily done by substituting \(m\Omega\) for \(\omega\), where \(\Omega = u^\phi / u^t\). However, unlike the case of the \((\tilde{S} - S)\)-part, a time-domain expression for the \(\tilde{R}\)-part cannot be easily obtained for general orbits. For this reason, we do not bother expanding it in \(\omega\) but test the convergence with respect to the summation over \(\ell\).

In these convergence tests, in order to obtain a rough estimate for the corresponding tests in the gravitational case, we translate our results in the scalar case to the gravitational case by identifying \(q/\sqrt{G}\) with the mass \(\mu\) of the particle.

3.1. The \((\tilde{S} - S)\)-part

The transformation of the \(\tilde{S}\)-part into the time domain makes it possible to subtract the divergent \(S\)-part analytically. If we were to perform this subtraction numerically, the fraction to be subtracted would become closer and closer to unity as \(\ell\) increases. Apparently, this would imply a stringent requirement of numerical accuracy. In this sense, we anticipate a clear advantage of the analytical subtraction.

As noted above Eq. (3.1), it is necessary to expand the \((\tilde{S} - S)\)-part in powers of \(\omega\), which corresponds to a PN expansion, to obtain its expression in the time
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domain. Here, however, we would like to emphasize that the PN expansion should not be taken as a limitation of our general scheme. Let us explain the reason. The decomposition into $\ell$ modes is unavoidable, as long as we adopt the mode-sum (mode-by-mode) regularization, in which the large $\ell$ modes are the cause of the divergence. Now, once we regularize the $\tilde{S}$-part by subtracting off the $S$-part, the result is a convergent series in $\ell$ modes, which is essentially a PN series. Thus, we cannot completely avoid PN expansion even if we employ a fully numerical approach.

When the orbit shrinks and approaches the innermost stable circular orbit (ISCO), it is believed that the PN expansion is a poor approximation. This belief comes partly from the expected difficulty in carrying out the PN expansion to very high orders, and partly from the possible worsened convergence of the PN expansion, particularly for orbits close to the ISCO. We demonstrate that this belief is unfounded on both counts. Specifically, with regard to the first point, we calculate to 18PN order to show that systematic calculations to very high PN order are indeed possible. With regard to the second point, which is the central issue of this paper, we find that the convergence is rather rapid, at least for almost circular orbits, and the expansion up to 4PN or 6PN order turns out to be sufficient for the $(\tilde{S} - S)$-part, even for orbits shrinking up to the ISCO.

The force arising from the $(\tilde{S} - S)$-part is purely conservative for generic geodesic orbits. The conservative part of the self-force influences the orbit by pushing a particle off the geodesic orbit, and it also appears in source terms for second order perturbations. Therefore, it clearly affects the waveforms, but with corrections of higher order in $\mu/M$. Specifically, when we investigate the orbital evolution, corrections due to the conservative self-force are, roughly speaking, given by its ratio to the Newtonian force. In contrast, since the radiative part is that responsible for the inspiral, errors due to the truncation of the PN expansion at a finite order are to be evaluated in comparison with its leading term. Hence, errors in the conservative self-force have an additional suppression factor, $\mu/M$, compared with those in the radiative part. This is partly the reason why a rather low PN order, such as 4PN, is adequate to obtain the $(\tilde{S} - S)$-part with sufficient accuracy.

The components of the self-force from the $(\tilde{S} - S)$-part of the scalar field, $F^{\tilde{S} - S}_\alpha = F^{\tilde{S}}_\alpha - F^{S}_\alpha$, have already been obtained for general orbits in Ref. 27). This part of the self-force is expressed in terms of local quantities of the particle, i.e., its position and velocity. Therefore, all we need is to provide in these general formulae information specific to the circular nature of the orbit, Eq. (3.2). Only the $r$-component is non-vanishing for the $(\tilde{S} - S)$-force, because the $t$- and $\varphi$-components are directly related to the rates of change of the energy and angular momentum, and they are purely dissipative for circular orbits. It is given explicitly as

$$F^{\tilde{S} - S}_r = \frac{q^2}{4\pi r_0^2} \left[ \frac{73}{133} + \frac{16151}{21014} V^2 + \frac{395567}{106808} V^4 + \left( \frac{1107284037660637}{400151300487120} + \frac{7}{64} \pi^2 \right) V^6 + \left( \frac{182118981911377689978271}{854863070735138617520} + \frac{29 \pi^2}{1024} \right) V^8 \right] + O(v^9), \quad (3.3)$$

where $V = \sqrt{M/r_0} = r_0 \Omega$. Here we have expanded $F^{\tilde{S} - S}_r$ up to 4PN order. However,
it is computed up to 18PN order in the convergence test below.

First, in Fig. 1, we display the convergence of the \( r \)-component of the \((\tilde{S} - S)\)-force as a function of the PN order for several representative orbital radii \( r_0 \). Here, an estimator of the convergence of the PN expansion is defined by

\[
\Delta_{n}^{S-S}(n) := \left| \frac{F_{\alpha}^{S-S} |_{n} - F_{\alpha}^{S-S} |_{n-1}}{F_{\alpha}} \right|,
\]

(3.4)

where \( F_{\alpha}^{S-S} |_{n} \) denotes the \((\tilde{S} - S)\)-part of the force truncated at \( n \)-PN order, and the denominator \( F_{\alpha} \) denotes the exact (fully relativistic) self-force including the \( \tilde{R} \)-part. In practice, because it is impossible to know its exact value, we use the most accurate result in our calculation. It is found that the convergence of the PN expansion is steady near the ISCO, although it is less rapid there. The convergence improves slightly when the Padé approximation is used near the ISCO. Here, in the Padé approximation, we have chosen the denominator to be quadratic in \( V^2 \).

3.2. The \( \tilde{R} \)-part and the regularized self-force

In the case of circular orbits around a Schwarzschild black hole, each component of the self-force due to the \( \tilde{R} \)-part is formally given by

\[
\begin{align*}
F_t^{\tilde{R}} &= -i q^2 \frac{\Omega}{u^3} \sum_{\ell m} m g_{\ell m, m \Omega} (r_0, r_0) \left| Y_{\ell m} \left( \frac{\pi}{2}, 0 \right) \right|^2, \\
F_r^{\tilde{R}} &= \frac{q^2}{u^3} \sum_{\ell m} \partial_r g_{\ell m, m \Omega} (r, r_0) \left| Y_{\ell m} \left( \frac{\pi}{2}, 0 \right) \right|^2, \\
F_\theta^{\tilde{R}} &= 0, \\
F_\phi^{\tilde{R}} &= -\frac{1}{\Omega} F_t^{\tilde{R}}.
\end{align*}
\]

(3.5)

The \( \tilde{R} \)-force in this case can be completely obtained analytically, because the integration with respect to \( \omega \) is done by substituting \( m \Omega \) for \( \omega \). Also, note that since only modes up to a finite value of \( \ell \) contribute to the self-force for a given PN order, the computation of the \( \tilde{R} \)-force is relatively easy. If we need precision up to \( n \)-PN order inclusive, it is sufficient for us to calculate the modes up to \( \ell \leq n + 1 \). The detailed analysis is summarized in Ref. 27). The 4PN results, after summation over the \( \ell \)-modes, are

\[
\begin{align*}
F_t^{\tilde{R}} &= \frac{q^2}{4 \pi r_0^2} \left[ \frac{73}{133} - \frac{16151}{21014} V^2 - \frac{395567}{106808} V^4 - \left( \frac{4}{3} + \frac{4}{3} \ln(2V) + \frac{119620654887997}{400151300487120} \right) V^6, \\
&+ \left( \frac{59372120592232147984979}{1709726141470277234304} - \frac{14}{3} \ln(V) - \frac{66}{5} \ln(2) - \frac{14}{3} \gamma \right) V^8 + O(v^9) \right] \] ,  \\
F_r^{\tilde{R}} &= \frac{q^2 V}{4 \pi r_0^2} \left[ \frac{1}{3} V^3 - \frac{1}{6} V^5 + \frac{2 \pi}{3} V^6 - \frac{77}{24} V^7 + \frac{9 \pi}{5} V^8 + O(v^9) \right].
\end{align*}
\]  

(3.6)
Fig. 1. The relative error of post-Newtonian formulas in the $r$-component of the ($\tilde{S} - S$)-force at $r_0 = 6M, 10M, 20M$ and $50M$. The horizontal axis is the order of the post-Newtonian expansion. The top figure displays the convergence in the Taylor expansion and the bottom figure is that in the Pade approximation.

Here $\gamma$ is Euler’s constant, $\gamma = 0.57 \cdots$. The temporal component, which represents the energy loss rate, starts at $1.5PN$ order, as is expected for dipole radiation.

To see the convergence of both the $r$-component and $t$-component of the $\tilde{R}$ force, in Fig. 2 we plot $\Delta^R_n(n)$, defined in a manner analogous to $\Delta^S_n(n)$, for several representative orbital radii $r_0$. The $\tilde{R}$-part contains the radiative part, whose errors are to be compared with its leading term, as mentioned above. Thus we need to evaluate the $\tilde{R}$-part with better accuracy than the $(\tilde{S} - S)$-part. In this case, it would seem that the convergence of the naive post-Newtonian expansion given above
is too slow for a small $r_0$. Since the $\tilde{R}$-part plays no role in the divergences, it is not necessary to transform its expression into the time domain analytically. This means that we do not have to expand the $\tilde{R}$-part of the Green function in powers of $\omega$. Here we propose to use just the spherical harmonic expansion ($\ell$-expansion), and to compute the contribution from each $\ell$-mode with sufficient accuracy. A criterion for truncating various series expansions in $g_{\ell m \omega}^{\tilde{R}}(r, r')$ is discussed in the Appendix in some detail. Plots of the $\ell$-expansion similar to that in Fig. 2 are given in Fig. 3.
Fig. 3. Plots of the errors in the $\ell$ expansion of both the $r$-component (upper panel) and the $t$-component (lower panel) of the $\tilde{R}$-force. Here, $\Delta_{\ell}^R(\ell)$ is defined in the same way as in the case of $\Delta_{n}^R(n)$, but in the $\ell$-expansion instead of the post-Newtonian expansion. The convergence is much faster than that of the naive post-Newtonian expansion.

3.3. Comparison with Detweiler, Messariti and Whiting\textsuperscript{25)}

The total self-force in the case of a circular orbit around a Schwarzschild black hole is obtained by summing the $(\bar{S} - S)$-part and the $\bar{R}$-part as

$$F^R_r = \frac{q^2}{4\pi r_0^2} \left[ \left( -\frac{4}{3} \gamma + \frac{7}{64} \pi^2 - \frac{4}{3} \ln(2V) - \frac{2}{9} \right) V^6 + \left( \frac{604}{45} + \frac{29\pi^2}{1024} - \frac{66}{5} \ln(2) - \frac{14}{3} \ln(V) - \frac{14}{9} \gamma \right) V^8 \right] + O(v^9), \quad (3.7)$$
Table I. The $r$-component of the self-force at several radii ($M = 1$ and $q^2 = 4\pi$).

| $r_0$ | 6$M$ | 10$M$ | 20$M$ | 50$M$ |
|-------|------|------|------|------|
| $F^R_r(r_0)$ | $1.676820878 \times 10^{-4}$ | $1.378448171 \times 10^{-5}$ | $4.937905866 \times 10^{-6}$ | $6.346791373 \times 10^{-7}$ |

$$F^R_t = F^R_r.$$ (3.8)

In the $r$-component of the scalar self-force, there is a significant cancellation between the $(\dot{S} - S)$-part and the $\dot{R}$-part, and the total force begins at 3PN order.

Here we compare our result with that obtained by Detweiler, Messoritaki and Whiting. They calculated the radial component of the self-force for the case $r_0 = 10M$, $M = 1$ and $q^2 = 4\pi$, and evaluated the uncertainty using a Monte Carlo simulation. It should be noted that the source term of the field equation in our definition is different from theirs by a factor of $4\pi$. Their result is

$$F^R_r = 1.37844828(2) \times 10^{-5}.$$ 

On the other hand, with the result obtained using a Pade approximation for the $(\dot{S} - S)$-force accurate to 18PN order and with the most accurate $\dot{R}$-force in our calculation including the terms up to $\ell = 18$, we obtain

$$F^R_r = 1.378448171 \times 10^{-5}.$$ 

In Table I we list the numerical values for the $r$-component of the regularized self-force for $r_0 = 6M$ (ISCO), 10$M$, 20$M$ and $r_0 = 50M$. In our computation the accuracy is limited by the $(\dot{S} - S)$-part. Hence, the accuracy of the full regularized force can be read from Fig. I.

§4. Errors in gravitational wave cycles

In this section, we evaluate the PN order necessary to obtain sufficiently accurate gravitational waveforms. For this purpose, we consider the number of wave cycles $N$ when the particle spirals in from an initial radius $r_i$ to a final radius $r_f$.

For the gravitational wave search with known waveforms, the so-called *matched filtering method* is used. In this method, the correlation between the detector’s output and theoretical templates for the waveforms is taken to search for the maximum signal-to-noise ratio (SNR). Because a phase error of order unity in the theoretical templates strongly reduces the SNR, an error in the number of cycles, denoted by $\Delta N$, caused by truncating the force at a given order of the expansion is a good indicator of the significance of the error. Here, because the frequency of the gravitational wave is given by twice the orbital period, we estimate twice the number of cycles of the orbital rotation by the formula

$$N = \frac{1}{\pi} \int_{r_i}^{r_f} \frac{dE}{dt} \frac{dE}{dr_0} dr_0 
\approx \frac{4M}{\mu} \int_{V_i}^{V_f} dV \frac{(1 - 6V^2)}{(1 - 3V^2)^2 f^R_t},$$ (4.1)
where $V_i = \sqrt{M/r_i}$, $V_f = \sqrt{M/r_f}$, and $f^R_i = (4\pi r_0^2/q^2) \times F^R_i$.

First, we consider the correction due to the conservative part of the self-force. The factor $\Omega(dE/dr_0)$ in Eq. (4.1) is determined by the background geodesic motion at lowest order. Hence, the self-force contribution is suppressed by its ratio to the Newtonian force. This ratio is given in the PN expansion as

$$\frac{F^S - S}{\mu M/r^2} = -\frac{\mu}{M} \sum_i a_i V_i^2.$$ (4.2)

Hence, the correction to $N$ due to the $(S - S)$-force at $n$-PN order is given by

$$\Delta N^{S - S}(n) = 4a_n \int_{V_i}^{V_f} dV \frac{(1 - 6V^2)}{(1 - 3V^2)^2} f^R_i V^{2n}.$$ (4.3)

Because the ratio $F^S - S/\mu M/r^2$ contains a factor $\mu/M$, the $\mu$-dependence in this correction appears only through $V_i$ and $V_f$. We set $V_f$ equal to the velocity at the ISCO, where higher PN order corrections become the largest. Explicitly, $V_f$ is fixed to $1/\sqrt{6}$. The value of $V_i$ depends on the masses $\mu$ and $M$ and the observation period, but here we set $V_i$ equal to 0 in order to estimate the maximum error. This makes $\Delta N^{S - S}(n)$ completely independent of $\mu$. Table II lists the error estimates. From this table, it is evident that the expansion up to 4PN order is in most cases sufficient for the $(S - S)$-part of the self-force. However, in the case we expect to detect an inspiral signal with a very large SNR, the observational error in the number of cycles, $\Delta N$, may be made small, say, $\Delta N \lesssim 0.1$. In such cases, in order to extract as much orbital information as possible, it will be necessary to calculate up to 6PN order or higher.

Next we consider the dissipative part of the self-force, which is completely contained in the $\tilde{R}$-part. In particular, in the case of circular orbits, the $t$- and $\varphi$-components of the $\tilde{R}$-force are purely dissipative. Because the self-force is responsible for the orbital energy loss, the energy loss rate $dE/dt$ which appears in Eq. (4.1) is solely determined by $F^R_t$. Therefore, in contrast to the conservative part, the higher PN corrections of the dissipative self-force are not suppressed by a factor of $\mu/M$ in comparison with its leading order term. Hence, the PN corrections through $dE/dt$ is not suppressed by a factor of $\mu/M$. We therefore define the error indicator $\Delta N^{\tilde{R}}(\ell)$ by

$$\Delta N^{\tilde{R}}(\ell) = N^{\tilde{R}}(\ell) - N^{\tilde{R}}(\ell - 1),$$ (4.4)

with

$$N^{\tilde{R}}(\ell) = 4 \frac{M}{\mu} \int_{V_i}^{V_f} dV \frac{(1 - 6V^2)}{(1 - 3V^2)^2} f^{\tilde{R}}_i V^{2n}.$$ (4.5)

Table II. The relative error $\Delta N^{S - S}(n)$ for the conservative self-force.
where $f_{\ell}^R$ is the $\tilde{R}$-part of the force summed up to the $\ell$-th harmonics.

Interestingly, when both the mass of an inspiraling small compact object and the observation period are fixed, the error $\Delta N^R(\ell)$ has a maximum as a function of the mass of the central black hole. Hence, an upper bound on the required order of the $\ell$-expansion can be obtained by evaluating the phase error formula (4.4) at the maximum. The presence of a maximum can be understood by considering two limiting cases:

- **Small $M$ limit:** The largest truncation error comes at the ISCO. Therefore, we fix $V_f$ at the ISCO. Then $V_i$ is fixed for a given observation period. Here, we adopt 1 year for the observation period. When $M$ is small, $V_i$ is small. Thus we can obtain a good upper bound on the error in $\Delta N$ by setting $V_i = 0$. After substituting 0 for $V_i$, the expression (4.4) for $\Delta N$ is manifestly proportional to $M$. Hence, $\Delta N$ decreases as $M$ decreases.

- **Large $M$ limit:** Again, we fix $V_f$ at the ISCO. Then, because $M$ is large, the orbit remains close to the ISCO. In the limiting case, we can regard the integrand in Eq. (4.4) as constant. Then, we have $\Delta N \approx N \Delta F/\bar{F}_t$, and the $M$ dependence appears only through $N$. Because for a given observation period, $N$ decreases as $M$ increases, $\Delta N$ also decreases as $M$ increases.

Because $\Delta N$ decreases for both small and large $M$ limits, it should have a maximum. Therefore, we can identify the “required” order of the $\ell$-expansion for a given observation period independently of $M$. We plot $\Delta N^R(\ell)$ as a function of $M$ in Fig. 4, where we set $\mu$ equal to $1 M_\odot$ and the observation period to 1 year. The plots exhibit a peak near $M = 10^5 M_\odot$, and the error decreases as $M$ moves away from this value on both sides. From these plots, for an inspiraling compact object of solar mass and for a given observation period of $O(1\text{ year})$, the expansion up to $\ell = 13$ is found to be sufficient irrespective of the mass of the central supermassive black hole.

![Fig. 4](image_url)

**Fig. 4.** Error in the number of cycles caused by truncating the $\ell$-expansion in the $\tilde{R}$-force as a function of the mass of the central black hole.
§5. Conclusion and discussion

The new regularization method proposed in Ref. 27 is based on the post-Newtonian (PN) expansion. To exhibit the effectiveness of our regularization method, we have considered the simple case of a scalar charged particle in circular orbits around a Schwarzschild black hole, and we have shown here that one can actually compute the PN expansion up to sufficiently high orders.

We have analytically obtained the $(\tilde{S} - S)$-part of the scalar self-force up to 18PN order (though the explicit results are displayed only up to 4PN). Using this result, we investigated the convergence of the PN expansion for the $(\tilde{S} - S)$-part. The results are shown in Fig. 1. It appears that our result up to 18PN realizes an accuracy of $\sim 10^{-7}$ for $r_0 = 10M$. At the innermost stable circular orbit (ISCO), the convergence slows down, but is found to be steady, and the accuracy of the obtained regularized self-force is high, with relative error of $O(10^{-4})$.

For the $\tilde{R}$-part, we have computed the contribution from each spherical harmonic mode up to $\ell = 18$. The expansions with respect to $\epsilon = 2M\omega$ and $U = M/r_0$ are truncated at a sufficiently high order. Because the $\tilde{R}$-part is unaffected by the regularization procedure, we do not have to perform its PN expansion. Interestingly, the convergence of the $\ell$-expansion is found to be much faster than the PN expansion, although the results truncated at the $\ell$-th harmonics are correct only up to $(\ell+1)$-PN order.

Next, we have considered the phase error in the gravitational waveform due to truncation of the PN expanded series by interpreting the scalar charge as the mass of an orbiting particle. We have found that the 4PN order calculation of the $(\tilde{S} - S)$-part seems to be sufficiently accurate. For the $\tilde{R}$-part, we need a calculation up to $\ell = 13$ to make templates for a one-year observation up to the ISCO, assuming that the mass of the inspiraling star is $1M_\odot$.

Comments are now in order concerning the convergence behavior considered in the Appendix. First, we note here that the expression for the Green function given in Ref. 33 is convergent for any values of $\epsilon$ and $z$. However, the convergence is not guaranteed once we expand the Green function in powers of these parameters. In fact, the expansion of the $(\tilde{S} - S)$-part of the Green function with respect to $\epsilon$ has a finite convergence radius. This can be understood as follows. In the limit $\epsilon \to 0$, we have $\nu \to \ell$. Now, as the value of $\epsilon$ increases, $\nu$ decreases and eventually becomes $\ell - 1/2$. For this value of $\nu$, $\phi^\nu$ and $\phi^{-\nu-1}$ are no longer independent, and the Wronskian $W(\phi^\nu, \phi^{-\nu-1})$ vanishes there. Therefore, we have a simple pole in $g_{\ell m \nu}(r, r')$, and hence the power series expansion fails to converge at this value of $\epsilon$. This problem is as far as circular orbits are concerned, because high frequency contributions are completely suppressed. However, for generic orbits, we have arbitrarily high frequency contributions. For example if we expand the radial component of an orbit with respect to the eccentricity $e$, terms of higher order in $e$ have higher frequencies. Hence, $\omega$ can be arbitrarily large. These high frequency contributions, which cannot be handled by our analytic regularization method, are present in general. Nevertheless, we believe that our scheme is useful in a relatively
wide region of the orbital parameter space.

In addition to the problem mentioned above, the spins of the central black hole and the inspiraling compact star also shift the orbital frequency at the ISCO. We may encounter situations in which relativistic effects become more important than we have considered in this paper. Therefore, developing an alternative numerical method that can play a complimentary role is also important.

Ultimately, our goal is to derive the gravitational self-force on a point particle orbiting a “Kerr” black hole for a generic orbit, and eventually to construct highly precise theoretical template waveforms, to be used in the upcoming era of gravitational wave astronomy. For this purpose, we need to develop the second-order black hole perturbation theory. Our present results suggest that a relatively low PN order is sufficient for the second-order perturbation. This fact encourages us to develop a formalism along the lines of our scheme whose treatment of the singular part exploits the PN expansion efficaciously.

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Appendix

Criteria for Truncating in $g_{\ell m \omega}$

Because the infinite sum over the spherical harmonic modes is practically impossible, we need to decide at which value of $\ell$ to truncate the sum, which should be determined by an accuracy requirement.

For the $t$-component, if we compute the sum up to $\ell = 18$, the accuracy is $\sim 10^{-8}$ for $r_0 = 6M$, as shown in Fig. 3 or Fig. 5. Approximately the same accuracy is obtained by the truncation at $\ell = 18$ for the $r$-component as well. However, because an accuracy of the $r$-component is determined by that of the $(\hat{S} - S)$-part, if we require an accuracy of $\sim 10^{-4}$ for $F_{\ell r}^{\hat{S} - S}$, we only need to compute up to $\ell = 11$ for $F_{\ell}^{R}$.

The above conclusion, however, assumes that $g_{\ell m \omega}$ for each given $\ell$ is com-
Fig. 5. Plots of the error in the ℓ-expansion of both the r-component (upper panel) and the t-component (lower panel) of $F^{1,(8.25)}$, $F^{2,(8.25)}$ and $F^{3,(8.25)}$ (denoted by 1, 2 and 3, respectively) at $r_0 = 6M$.

computed with sufficiently high accuracy. In this appendix, we explain our method for computing $g_{\ell m \omega}^{\tilde{R}}$ and examine its accuracy. Our method is based essentially on the post-Minkowski expansion, in which the black hole mass $M$ plays the role of the expansion parameter. Specifically, we expand the $\tilde{R}$-part of the Green function with respect to $\epsilon = 2M \omega = O(v^3)$ and $U = M/r = O(r^2)$.

A.1. Analytic expressions for the $\tilde{R}$-part

The $\tilde{R}$-part of the Green function is given by

$$g_{\ell m \omega}^{\tilde{R}}(r, r') = \frac{-1}{(1 - \tilde{\gamma}_\nu \tilde{\gamma}_\nu) W_{\ell m \omega} (\phi^\nu, \phi^{-\nu-1})} \left[ \tilde{\beta}_\nu \tilde{\gamma}_\nu (\phi^\nu_c(r) \phi^{-\nu-1}_c(r') + \phi^{-\nu-1}_c(r) \phi^\nu_c(r')) \
+ \tilde{\gamma}_\nu \phi^\nu_c(r') \phi_c^{-\nu-1}(r') + \tilde{\beta}_\nu \phi_c^{-\nu-1}(r) \phi^{-\nu-1}_c(r') \right] . \tag{A.1}$$
Using the variables \( z = \omega r \) and \( \epsilon = 2M\omega \), the function \( \phi_c^\nu \) and the coefficients \( \tilde{\beta} \) and \( \tilde{\gamma} \) are given by

\[
\phi_c^\nu(z) = e^{-i(z-\epsilon)} (2(z-\epsilon))^{\nu} \sum_{n=-\infty}^{\infty} i^n a_n^\nu \frac{\Gamma(n + \nu + 1 + i\epsilon)}{\Gamma(\nu + 1 + i\epsilon)} \frac{\Gamma(2\nu + 2)}{\Gamma(2n + 2\nu + 2)} (2(z-\epsilon))^n \times_1 F_1(n + \nu + 1 + i\epsilon; 2n + 2\nu + 2; 2i(z-\epsilon)),
\]

\[
\tilde{\beta}_\nu = \frac{\Gamma(-\nu + i\epsilon)}{\Gamma(\nu + 1 + i\epsilon)} \frac{\Gamma(2\nu + 2)}{\Gamma(-2\nu)} K_{-\nu-1}^\nu, \tag{A.2}
\]

\[
\tilde{\gamma}_\nu = i \frac{\Gamma(\nu + 1 + i\epsilon)}{\Gamma(-\nu + i\epsilon)} \frac{\Gamma(-2\nu)}{\Gamma(2\nu + 2)} e^{-i\pi\nu} \frac{\sin(\pi(\nu + i\epsilon))}{\sin(\pi(\nu - i\epsilon))}, \tag{A.4}
\]

where \(_1F_1\) is the confluent hypergeometric function, and

\[
K_{\nu} = \frac{2^{-\nu} e^{-\nu} \Gamma(1 - 2i\epsilon)}{\Gamma(1 + \nu + i\epsilon)^2 \Gamma(1 + \nu - i\epsilon)} \sum_{n=0}^{\infty} \frac{\Gamma(n + \nu + 1 + i\epsilon) \Gamma(n + 2\nu + 1)}{n! \Gamma(n + \nu + 1 - i\epsilon)} a_n^\nu \times \left[ \sum_{n=-\infty}^{0} \frac{1}{(-n)! \Gamma(n + 2\nu + 2)} a_n^\nu \right]^{-1}.
\]

We assume \( \omega > 0 \) in the above derivation of the homogeneous solutions. To obtain the solutions for \( \omega < 0 \), we consider their analytic continuation to complex \( \omega \) through the upper half complex \( \omega \)-plane. This is done by replacing \( \omega \) in \( \phi_c^\nu \) with \( |\omega| e^{i\pi} \).

The Wronskian is given by

\[
W_{\ell m \omega}(\phi_c^\nu, \phi_c^{-\nu-1}) = -\frac{2\nu + 1}{2\omega} \left| \sum_{n=-\infty}^{\infty} a_n^\nu \right|^2. \tag{A.5}
\]

It is important to note that we can easily obtain expressions for \( a_n^\nu \) (for \(-N < n < N \) where \( N \) is a large integer) to very high accuracy in powers of \( \epsilon \), for any value of \( \nu \). In fact, because the convergence condition of the quantities \( a_n^\nu \) for \( n \to \pm\infty \) determines \( \nu \), it is the accuracy of \( \nu \) that is important. Therefore, the accuracies of the coefficients \( \tilde{\beta}, \tilde{\gamma} \) and \( W_{\ell m \omega}(\phi_c^\nu, \phi_c^{-\nu-1}) \) essentially depend only on \( \nu \). In other words, we may regard these as functions of only \( \nu \) when we estimate their accuracies.

### A.2. Error analysis

Apart from the dependence on \( \nu \), which we leave unexpanded in all the quantities, we perform the post-Minkowskian expansion by replacing \( z \) with \( \epsilon/(2U) \). We then assume \( \epsilon = O(\nu^3) \) and \( U = O(\nu^2) \). We express the homogeneous solution \( \phi_c^\nu \) as

\[
\phi_c^\nu(r) = e^{\nu} \left( \frac{1}{U} - 2 \right)^{\nu} \bar{\phi}_c^\nu(r). \tag{A.6}
\]

We then write a truncated expression for \( g_{\ell m \omega}^{R,(n_1,n_2)}(r,r') \) as

\[
g_{\ell m \omega}^{R,(n_1,n_2)}(r,r') = g_{\ell m \omega}^{1,(n_1,n_2)}(r,r') + g_{\ell m \omega}^{2,(n_1,n_2)}(r,r') + g_{\ell m \omega}^{3,(n_1,n_2)}(r,r'), \tag{A.7}
\]
where
\[
\begin{align*}
    g^{1,(n_1,n_2)}_{\ell m \omega}(r, r') &\equiv A_1(\nu) \left[ \hat{\Phi}_c^{\nu}(r)\hat{\Phi}_c^{\nu-1}(r') \right]^{(n_1,n_2)} + (r \leftrightarrow r'), \\
    g^{2,(n_1,n_2)}_{\ell m \omega}(r, r') &\equiv A_2(\nu) \left[ \hat{\Phi}_c^{\nu}(r)\hat{\Phi}_c^{\nu}(r') \right]^{(n_1,n_2)}, \\
    g^{3,(n_1,n_2)}_{\ell m \omega}(r, r') &\equiv A_3(\nu) \left[ \hat{\Phi}_c^{\nu-1}(r)\hat{\Phi}_c^{\nu-1}(r') \right]^{(n_1,n_2)}.
\end{align*}
\] (A.8)

The indices \((n_1, n_2)\) indicate that the corresponding quantities are expanded in terms of \(\epsilon\) and \(v^2\) to \(O(\epsilon^{2n_1})\) and \(O(v^{2n_2})\), respectively. Note that \(n_1\) and \(n_2\) correspond to the post-Minkowski order and post-Newtonian order, respectively.

Fig. 6. The error \(\Delta^\nu(n)\) in \(\nu\).

Fig. 7. Plots of the error in the \(t\)-component of \(F^{2,(n_1,25)}_\alpha\) for various values of \(\ell\) as functions of the post-Minkowski order \(n_1\).
coefficients $A_i$ are composed of $\tilde{\beta}, \tilde{\gamma}, W_{\ell m}\omega(\phi_c^\nu, \phi_c^{-\nu-1}), \epsilon^\nu$ and $U^\nu$. As noted above, because their accuracy is essentially determined by the accuracy of $\nu$, we regard the $A_i$ as functions of only $\nu$, and we estimate the error caused by using an approximate value of $\nu$.

First, we investigate the convergence of $\nu$, which is given by a power series in $\epsilon$. We define $\Delta^\nu(n)$ by

$$\Delta^\nu(n) \equiv \left| \nu^{(n)} - \nu^{(n-1)} \right|.$$  \hfill (A.9)

Here, the index $(n)$ indicates that these quantities are expanded in terms of $\epsilon$ to $O(\epsilon^{2n})$. Considering the most relativistic case, we set $r = 6M$ and $\epsilon = 2M\ell\Omega$, where $\Omega$ is the orbital angular velocity. We plot $\Delta^\nu(n)$ for various values of $\ell$ in Fig. 6.

We see that the convergence is very fast. Because the errors in $A_i$ are proportional to that in $\nu$, we may ignore the errors in $A^{(i)}$ if we take, say, $n = 8$.

We now investigate the convergence of the $\tilde{R}$-force. Let us denote the part of the force that is due to $g^{i, (m_1, m_2)}_{\ell m\omega}(r, r')$ by $F^{i, (m_1, m_2)}_{\alpha}$. As seen from Fig. 5 the $\tilde{R}$-force is dominated by $F^{2, (m_1, m_2)}_{\alpha}$. This is in agreement with the order-counting carried out previously. \hfill (27) We define the errors $\Delta^2_{\alpha}(n_1)$ and $\tilde{\Delta}^2_{\alpha}(n_2)$ by

$$\Delta^2_{\alpha}(n_1) \equiv \left| \frac{F^{2, (n_1, n_2)}_{\alpha} - F^{2, (n_1-1, n_2)}_{\alpha}}{F^{R}_{\alpha}} \right|,$$

$$\tilde{\Delta}^2_{\alpha}(n_2) \equiv \left| \frac{F^{2, (n_1, n_2)}_{\alpha} - F^{2, (n_1, n_2-1)}_{\alpha}}{F^{R}_{\alpha}} \right|.$$  \hfill (A.10)

The $t$-components of $\Delta^2_{\alpha}(n_1)$ at $n_2 = 25$ and $\tilde{\Delta}^2_{\alpha}(n_2)$ at $n_1 = 8$ are plotted in Fig. 7 and Fig. 8 respectively. We see very fast convergence. Although not shown here,
we find that the same is true for the other components. These results confirm that our calculations are sufficiently accurate.

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