ON COMMUTATIVE DIAGRAMS CONSISTING OF LOW TERM EXACT SEQUENCES

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Abstract. We establish several useful commutative diagrams consisting of low term exact sequences attached to Grothendieck spectral sequences, which extends and integrates the previous ones appeared in literature such as Alexei N. Skorobogatov [Beyond the Manin obstruction, Invent. Math. (1999)], and [On the elementary obstruction to the existence of rational points, Mathematical Notes (2007)]. Parts of the diagrams was frequently used in local-global principle to rational points.

1. THE COMMUTATIVE DIAGRAM

Suppose that $\Phi : A \to B$ and $\Psi_t : B \to C$ are left exact additive functors between abelian categories, $t = 1, 2, 3$. Assume that $A$ and $B$ have enough injectives and $\Psi_t$ takes injectives to $\Phi$-acyclics. Then for any $A \in \text{Ob} (A)$, we have the Grothendieck spectral sequence

$$(S)_{\Phi, \Psi_t, A} \quad \Rightarrow \quad E^{p,q}_2 = (R^p \Psi)(R^q \Phi)A \Rightarrow \quad E^{p+q}_2 = R^{p+q}(\Psi \Phi)A$$

and the low term exact sequence

$$(E)_{\Phi, \Psi_t, A} \quad \Rightarrow \quad E^{1,0}_2 \to E^{1}_1 \to E^{2,0}_1 \to E^{2}_1 \to E^{1,1}_1 \to E^{3,0}_1$$

attached to them, where $E^i_1 = \ker (tE^i \to tE^{i,0}_2)$. Let $D^+(A), D^+(B), D^+(C)$ be the corresponding derived category of complexes bounded below, $R\Psi, R\Psi_t$ the corresponding derived functor and $R^s$ (or $H^s$) the hypercohomology functor.

Proposition 1.1. With the previous notation, suppose that there are morphism of functors $u : \Psi_1 \to \Psi_2$ and $v : \Psi_2 \to \Psi_3$ such that for any $F \in D^+(B)$,

$$(1.2) \quad R\Psi_1(F) \xrightarrow{Ru(F)} R\Psi_2(F) \xrightarrow{Rv(F)} R\Psi_3(F) \xrightarrow{Rv_1(F)[1]} R\Psi_1(F)$$

is a distinguished triangle functorial in $F$.

(i) We have the long exact sequence

$$\ldots \rightarrow tE^{i,0}_2 \rightarrow tE^{i}_1 \rightarrow tE^{i-1,1}_2 \rightarrow tE^{i+1,0}_2 \rightarrow \ldots$$

where $tE^{i}_1 = \mathbb{R}^i\Psi_t(\tau_{\leq i}R\Phi(A))$.

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(ii) We have the following commutative diagram with exact rows and columns

\[
\begin{array}{ccccccc}
1E_1^{0,0} & \rightarrow & 1E_1^1 & \rightarrow & 1E_2^{0,1} & \rightarrow & 1E_2^{2,0} & \rightarrow & 1E_3^2 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
2E_2^{1,0} & \rightarrow & 2E_2^1 & \rightarrow & 2E_2^{0,1} & \rightarrow & 2E_2^{2,0} & \rightarrow & 2E_3^2 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
3E_2^{1,0} & \rightarrow & 3E_2^1 & \rightarrow & 3E_2^{0,1} & \rightarrow & 3E_2^{2,0} & \rightarrow & 3E_3^2 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1E_1^{2,0} & \rightarrow & 1E_2^{1,1} & \rightarrow & 1E_2^{3,0} & \rightarrow & 1E_{\leq 1}^3 \\
\end{array}
\]

where \(1E_{\leq 1}^3\) fits into the exact sequence

\[
0 \to 1E_2^1 \to 1E_2^2 \to 1E_2^{0,2} \to 1E_{\leq 1}^3 \to 1E_3^2,
\]

the rows are parts of the low term exact sequences \((E_{\Phi, \Psi, A})_t\) attached to the spectral sequences \((S)_{\Phi, \Psi, A}\) with \(t\) numbered on the left upper corner of each object, and the columns are induced by taking cohomology at 1 of \((1.2)\) in which \(F\) is substituted with \(\tau_0 R\Phi(A), \tau_{\leq 1} R\Phi(A), \tau_{[1]} R\Phi(A), \tau_{[0]} R\Phi(A)[1], \tau_{\leq 1} R\Phi(A)[1]\), respectively.

(iii) Suppose we are given \(\beta \in 3E_2^1\) and \(\gamma \in 2E_2^{0,1}\) such that they map to the same element in \(3E_2^{0,1}\). Then there exists \(\alpha \in 1E_2^{2,0}\) such that \(\alpha, \beta, \gamma\) map to the same element in \(1E_2^1\) and \(-\alpha, \gamma\) map to the same element in \(2E_2^{2,0}\). In other words, we have the zig-zag diagram

\[
\begin{array}{ccccccc}
\alpha & \rightarrow & b & \rightarrow & \beta & \rightarrow & \gamma \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
1E_2^{2,0} & \rightarrow & 1E_1^2 & \rightarrow & 2E_2^{0,1} & \rightarrow & 2E_2^{2,0} \\
\end{array}
\]

(iv) The statement of (iii) is also correct if we move our focus one step right. That is, we are given \(\beta \in 3E_2^{0,1}\) and \(\gamma \in 2E_2^{2,0}\), and so on.

Proof. The proof widely extends [3 Lem. 3] and [7 Prop. 1.1]. For any \(F \in D^+(A)\), the truncation functors determine the distinguished triangle

\[
\tau_{\leq 0} F \to F \to \tau_{\geq 1} F \to (\tau_{\leq 0} F)[1].
\]

Note that \(R\Phi_t, t = 1, 2, 3\) are triangulated. Along with the functorial distinguished triangles \((1.2)\) in which \(F\) is substituted with \(\tau_{\leq 0} F, F\) and \(\tau_{\geq 1} F\) respectively, we obtain the following commutative
Let \( i, j, k \) and note that \( \Psi \)

\[
(\tau \leq 0) \rightarrow \tau \leq 1 \rightarrow (\tau \leq 1)[1] \rightarrow \tau \leq 1[2] \rightarrow \tau \leq 1[2]
\]

Let \( F = \tau \leq 1 \Phi \Phi(A) = \tau_{0, 1} \Phi \Phi(A) \) and taking cohomology at 1, the diagram becomes

\[
\begin{align*}
R^1 \Psi_1(\tau_{0, 1}) &\rightarrow R^1 \Psi_1(\tau \leq 1) \rightarrow R^1 \Psi_1(\tau \leq 1)[1] \rightarrow R^1 \Psi_1(\tau \leq 1)[2] \rightarrow R^1 \Psi_1(\tau \leq 1)[2]
\end{align*}
\]

with exact rows and columns.

We now identify the objects appearing in (1.7) with the ones in (1.3). Clearly for any \( t = 1, 2, 3 \) and \( i, j, k \in \mathbb{Z} \) with \( j, i - j + k \geq 0 \), we have

\[
R^i \Psi_t(\tau_{0, 1})[k] = (R^{i-j+k} \Psi_t)(R^j \Phi)A = t E_2^{i-j+k+j}.
\]

It remains to identify \( R^1 \Psi_t(\tau_{0, 1})[1] \) with \( E_1^1 \) and \( R^1 \Psi_t(\tau_{0, 1})[2] \) with \( E_1^2 \).

For any \( F \) consider the distinguished triangle

\[
\tau \leq 1 F \rightarrow F \rightarrow \tau \leq 2 F \rightarrow (\tau \leq 1)[1]
\]

and note that \( \Psi_t(\tau \geq 2) \) is acyclic in 0 and 1. Then we have the long exact sequences

\[
\begin{align*}
R^0 \Psi_t(\tau \geq 2) &\rightarrow R^1 \Psi_t(\tau \leq 1) \rightarrow R^1 \Psi_t(F) \rightarrow R^1 \Psi_t(\tau \geq 2) \rightarrow R^2 \Psi_t(\tau \leq 1) \rightarrow R^2 \Psi_t(F) \rightarrow R^2 \Psi_t(\tau \geq 2) \\
&\rightarrow R^3 \Psi_t(\tau \leq 1) \rightarrow R^3 \Psi_t(F)
\end{align*}
\]

where \( R^i \Psi_t(\tau \geq 2) = 0 \) with \( t = 1, 2, 3 \) and \( i = 0, 1 \). Now we take \( F = \Phi \Phi(A) \). It follows that

\[
R^1 \Psi_t(\tau_{0, 1})[1] = R^1 \Psi_t(\Phi \Phi(A)) = t E_1^1
\]

where the last equality follows from the isomorphism of functors

\[
(\Phi \Phi)(\Phi) \cong \Phi(\Phi)\Phi.
\]
In a same manner, (1.8) and (1.9) yield
\[
R^1\Psi_t(\tau_{\leq 1} R\Phi(A))[1] = \ker(\mathbb{R}^2\Psi_t(\Phi(A)) \rightarrow \mathbb{R}^2\Psi_t(\tau_{\geq 2} R\Phi(A)))
\]
and
\[
R^2\Psi_t(\tau_{\geq 2} R\Phi(A)) = \ker(\mathbb{R}^2(\Psi_t\Phi)A \rightarrow \mathbb{R}^2\Psi_t(\tau_{\leq 0} F))[1].
\]
Then the low term exact sequence attached to the hypercohomology spectral sequence [4, Appendix C (g)]
\[
E_2^{p,q} = R^p g(H^q(F)) \Rightarrow R^{p+q} g(F)
\]
gives the isomorphism when \( F = \tau_{\geq 2} R\Phi(A) \) and \( g = \Psi_t \)
\[
R^2\Psi_t(\tau_{\geq 2} R\Phi(A)) \cong \Psi_t(R^2\Phi(A)),
\]
and the exact sequence (1.4) is also deduced from (1.8) and (1.9). This completes the identification of objects.

Finally, in diagram (1.7), the identification of the vertical arrows is clear. For that of the horizontal ones, it follows from a general fact for such spectral sequences. See, for example, [5, Appendix B], which shows that \( \mathbb{R}^1\Psi_t(\tau_{\leq 1} R\Phi(A)) \rightarrow \mathbb{R}^1\Psi_t(\tau_{\leq 0} F)[1] \) is exactly the edge map \( \mathbb{E}_1 \rightarrow \mathbb{E}_2 \). This completes the proof of (ii) as well as (i).

Consider the subdiagram of (1.6)
\[
\begin{align*}
\mathbb{R}\Psi_2(F) & \longrightarrow \mathbb{R}\Psi_2(\tau_{\geq 1} F) \longrightarrow \mathbb{R}\Psi_2(\tau_{\leq 0} F)[1] \longrightarrow \\
\downarrow & \downarrow \downarrow \downarrow \\
\mathbb{R}\Psi_3(F) & \longrightarrow \mathbb{R}\Psi_3(\tau_{\geq 1} F) \longrightarrow \mathbb{R}\Psi_3(\tau_{\leq 0} F)[1] \longrightarrow \\
\downarrow & \downarrow \downarrow \downarrow \\
\mathbb{R}\Psi_1(F)[1] & \longrightarrow \mathbb{R}\Psi_1(\tau_{\geq 1} F)[1] \longrightarrow \mathbb{R}\Psi_1(\tau_{\leq 0} F)[2] \longrightarrow \\
\downarrow & \downarrow \downarrow \downarrow \\
\end{align*}
\]
whose rows and columns are all distinguished triangles. Since up to an isomorphism, every distinguished triangle in a derived category arises from some short exact sequence of complexes [2, Chap. IV.2 8. Prop.], we may view (1.10) as a commutative diagram consisting of three rows and three columns of short exact sequences of complexes. Then the result follows from [6, Lem. 4.3.2] by taking \( i = 1 \). This completes the proof of (iii).

(iv) is similar as (iii). The proof is complete.

Next we describe a variant of Proposition 1.1.

**Proposition 1.11.** Keeping assumptions in Proposition 1.1, suppose that there are \( A \in D^+(A) \) and \( B \in D^+(B) \) with a morphism
\[
f : B \rightarrow \tau_{\leq 1} R\Phi(A).
\]
Let $\Delta = \Delta(\Phi, A, B, f)$ be the cone of $-f[1]$. Denote $^tF^p = \mathbb{R}^p\Psi_1B$ and $^tG^p = \mathbb{R}^p\Psi_1\Delta$. Then we have the long exact sequence
\[ \ldots \rightarrow F^i \rightarrow F_{\leq 1}^i \rightarrow G^{i-1} \rightarrow F^{i+1} \rightarrow \ldots \]
where $F_{\leq 1}^i = \mathbb{R}^i\Psi_1(\tau_{\leq 1}\mathbb{R}\Phi(A))$ and the following commutative diagram with exact rows and columns

(1.12)\[
\begin{array}{cccccccc}
1F^1 & \rightarrow & 1E^1 & \rightarrow & 1G^0 & \rightarrow & 1F^2 & \rightarrow & 1E_1^2 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
2F^1 & \rightarrow & 2E^1 & \rightarrow & 2G^0 & \rightarrow & 2F^2 & \rightarrow & 2E_1^2 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
3F^1 & \rightarrow & 3E^1 & \rightarrow & 3G^0 & \rightarrow & 3F^2 & \rightarrow & 3E_1^2 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1F^2 & \rightarrow & 1E^2 & \rightarrow & 1G^1 & \rightarrow & 1F^3 & \rightarrow & 1E_1^3 \\
\end{array}
\]

where $1E_{\leq 1}^3$ fits into the exact sequence
\[ 0 \rightarrow 1E_1^2 \rightarrow 1E^2 \rightarrow 1E_{0,2}^2 \rightarrow 1E_{\leq 1}^3 \rightarrow 1E^3. \]

Moreover, similar statements as (iii), (iv) in Proposition 1.1 hold. That is, if we are given $\beta \in 3E^1$ (resp. $3G^0$) and $\gamma \in 2G^0$ (resp. $2F^2$), then the corresponding zig-zag diagram as in Proposition 1.1 is correct.

**Proof.** The same as Proposition 1.1 except that in the diagram (1.7) we replace the distinguished triangle

(1.13)\[ \tau_{[0]}\mathbb{R}\Phi(A) \rightarrow \tau_{\leq 1}\mathbb{R}\Phi(A) \rightarrow \tau_{[1]}\mathbb{R}\Phi(A) \rightarrow \tau_{[0]}\mathbb{R}\Phi(A)[1] \]
by
\[ B \rightarrow \tau_{\leq 1}\mathbb{R}\Phi(A) \rightarrow \Delta[-1] \rightarrow B[1]. \]

\[ \square \]

**Remark 1.14.** (a) Obviously, Proposition 1.1 is the special case of Proposition 1.1 where $f$ is the canonical map $\tau_{[0]}\mathbb{R}\Phi(A) \rightarrow \tau_{\leq 1}\mathbb{R}\Phi(A)$.

(b) If we replace the distinguished triangle (1.13) by
\[ \tau_{[0]}\mathbb{R}\Phi(A) \rightarrow \mathbb{R}\Phi(A) \rightarrow \tau_{\geq 1}\mathbb{R}\Phi(A) \rightarrow \tau_{[0]}\mathbb{R}\Phi(A)[1] \]
we also have the long exact sequence
\[ \ldots \rightarrow tE_{2,0}^i \rightarrow tE^i \rightarrow tE_2^i \rightarrow tE_{2+1,0}^i \rightarrow \ldots \]
where $tE_{\geq 1}^i = \mathbb{R}^i\Psi_1(\tau_{\geq 1}\mathbb{R}\Phi(A))$.\[ \square \]
2. Applications

Suppose that \( f_* : \mathcal{A} \to \mathcal{B} \) is a left exact additive functor between two abelian categories which has a left adjoint \( f^* \). Assume that \( \mathcal{A} \) and \( \mathcal{B} \) has enough injectives and \( f^* \) is exact. For example, \( f : X \to Y \) is a morphism of topoi, and \( \mathcal{A} = \text{Mod}(X, \Lambda) \), \( \mathcal{B} = \text{Mod}(Y, \Lambda) \). Let \( M \in \text{Ob}(\mathcal{B}) \) and \( N \in \text{Ob}(\mathcal{A}) \). Then we have the Grothendieck spectral sequences

\[
M E_2^{p,q} = \text{Ext}_A^p(M, R^q f_* N) \Rightarrow \text{Ext}_B^{p+q}(f^* M, N).
\]

For simplicity, we omit the category letter in \( \text{Ext}'s \) if it does not cause a confusion.

**Corollary 2.2.** Let \( C, A \in \text{Ob}(\mathcal{B}) \) and \( u \in \text{Ext}^1(C, A) \) be the element representing the extension

\[
0 \to A \overset{j}{\to} B \overset{\delta}{\to} C \to 0.
\]

Define

\[
\text{Ext}_\leq 3(f^* M, N) = \ker \left( \text{Ext}_2^1(f^* M, N) \to \text{Hom}(M, R^2 f_* N) \right),
\]

\[
\text{Ext}_{\leq 2}^3(f^* M, N) = \text{Ext}_3^3(M, \tau_{\leq 1} R f_* N).
\]

(i) We have the following commutative diagram with exact rows and columns

\[
\begin{array}{cccc}
\text{Ext}^1(C, f_* N) & \longrightarrow & \text{Ext}^1(f^* C, N) & \longrightarrow & \text{Hom}(C, R^1 f_* N) & \longrightarrow & \text{Ext}^2(C, f_* N) & \longrightarrow & \text{Ext}^3_1(f^* C, N) \\
\downarrow j' & & \downarrow f^*(j') & & \downarrow j^* & & \downarrow j^* & & \downarrow f^*(j^*) \\
\text{Ext}^1(B, f_* N) & \longrightarrow & \text{Ext}^1(f^* B, N) & \longrightarrow & \text{Hom}(B, R^1 f_* N) & \longrightarrow & \text{Ext}^2(B, f_* N) & \longrightarrow & \text{Ext}^3_1(f^* B, N) \\
\downarrow i' & & \downarrow f^*(i') & & \downarrow i^* & & \downarrow i^* & & \downarrow f^*(i^*) \\
\text{Ext}^1(A, f_* N) & \longrightarrow & \text{Ext}^1(f^* A, N) & \longrightarrow & \text{Hom}(A, R^1 f_* N) & \longrightarrow & \text{Ext}^2(A, f_* N) & \longrightarrow & \text{Ext}^3_1(f^* A, N) \\
\downarrow w & & \downarrow f^*(w) & & \downarrow u & & \downarrow u & & \downarrow f^*(w) \\
\text{Ext}^2(C, f_* N) & \longrightarrow & \text{Ext}^2(f^* C, N) & \longrightarrow & \text{Ext}^1(C, R^1 f_* N) & \longrightarrow & \text{Ext}^3(C, f_* N) & \longrightarrow & \text{Ext}^3_1(f^* C, N) \\
\end{array}
\]

where the rows are parts of the low term exact sequences attached to \( A E_2^{p,q} \) and \( C E_2^{p,q} \) defined in (2.1).

(ii) The statement of Proposition 1.1 (iii) (resp. (iv)) is also correct if we put \( \beta, \gamma \) in the corresponding positions. That is, we are given \( \beta \in \text{Ext}^1(f^* A, N) \) (resp. \( \text{Hom}(A, R^1 f_* N) \)) and \( \gamma \in \text{Hom}(A, R^1 f_* N) \) (resp. \( \text{Ext}^2(B, f_* N) \)), and so on.

**Proof.** We shall use Proposition 1.1. Let

\[
A \to B \to C \to A[1]
\]

be the distinguished triangle in \( D^+(\mathcal{B}) \) determined by (2.3). Take \( C = Ab, \Psi_1 = \text{Hom}(C, -), \Psi_2 = \text{Hom}(B, -), \Psi_3 = \text{Hom}(A, -) \) and \( \Phi = f_* \), which clearly satisfy the assumptions in Proposition 1.1 (c.f. [8] Thm. 10.7.4). Then the result follows. \( \square \)

Let \( k \) be a field with characteristic 0 and \( \Gamma = \text{Gal}(\overline{k}/k) \) where \( \overline{k} \) is a fixed algebraic closure of \( k \). Let \( p : X \to \text{Spec} k \) be a \( k \)-variety and \( X = X \times_{\overline{k}} \overline{k} \). In Corollary 2.2 take \( \mathcal{B} \) be the category of discrete \( \Gamma \)-modules, \( \mathcal{A} \) the category of étale sheaves on \( X \). We write \( \text{Ext}_h \) for \( \text{Ext}_B \), \( \text{Ext}_X \) for \( \text{Ext}_A \) and

\[
\text{Ext}_h^1(p^* T, \mathbb{G}_m) = \ker \left( \text{Ext}_X^2(p^* T, \mathbb{G}_m) \to \text{Hom}_h(T, \text{Br} X) \right) \quad \text{for any } \Gamma \text{-module } T.
\]
Note that $\text{Ext}^1_k(\mathbb{Z}, -) = H^1(k, -)$.

**Corollary 2.4.** With previous notation, let $u \in H^1(k, M)$ be the element representing the extension

\begin{equation}
0 \to M \xrightarrow{i} S \xrightarrow{j} \mathbb{Z} \to 0.
\end{equation}

Define $H^3_{\leq 1}(X, G_m) = H^3(X, \tau_{\leq 1} R_{\pi_*} G_m)$.

(i) We have the following commutative diagram with exact rows and columns

\begin{equation}
\begin{array}{cccccccc}
H^1(k, \bar{k}[X]^\times) & \xrightarrow{\gamma^*} & \text{Pic} X & \xrightarrow{\partial} & H^2(k, \bar{k}[X]^\times) & \xrightarrow{\delta^*} & H^3(k, \bar{k}[X]^\times) & \xrightarrow{\partial^*} \text{Br}_1 X \\
\downarrow j^* & & \downarrow j & & \downarrow j^* & & \downarrow j^* & \\
\text{Ext}^1_k(S, \bar{k}[X]^\times) & \xrightarrow{p^*(j^*)} & \text{Ext}^1_X(p^* S, G_m) & \xrightarrow{\partial} & \text{Hom}_k(S, \text{Pic} X) & \xrightarrow{\partial} & \text{Ext}^2_k(S, \bar{k}[X]^\times) & \xrightarrow{p^*} \text{Ext}^2_1(p^* S, G_m) \\
\downarrow i^* & & \downarrow i^* & & \downarrow i^* & & \downarrow i^* & \\
\text{Ext}^1_k(M, \bar{k}[X]^\times) & \xrightarrow{p^*(i^*)} & \text{Ext}^1_X(p^* M, G_m) & \xrightarrow{\partial} & \text{Hom}_k(M, \text{Pic} X) & \xrightarrow{\partial} & \text{Ext}^2_k(M, \bar{k}[X]^\times) & \xrightarrow{p^*} \text{Ext}^2_1(p^* M, G_m) \\
\downarrow u_{+} j^- & & \downarrow u_{+} j^- & & \downarrow u_{+} j^- & & \downarrow u_{+} j^- & \\
H^2(k, \bar{k}[X]^\times) & \xrightarrow{r} \text{Br}_1 X & \xrightarrow{d} H^3(k, \bar{k}[X]^\times) & \xrightarrow{\partial^*} H^3_{\leq 1}(X, G_m)
\end{array}
\end{equation}

where the top, middle and bottom row is a part of the low term exact sequences attached to the $E_2$ spectral sequences

\begin{equation}
\text{Ext}^p_k(M, R^n p_* G_m) \Rightarrow \text{Ext}^p_{X}(p^* M, G_m),
\end{equation}

and

\begin{equation}
\text{Ext}^p_k(S, R^n p_* G_m) \Rightarrow \text{Ext}^p_{X}(p^* S, G_m)
\end{equation}

respectively.

(ii) Let $\beta \in \text{Hom}_k(M, \text{Pic} X)$ be such that $u \cup \beta \in \text{im} r$. Then there exists $\alpha \in \text{Br}_1 X$ and $\gamma \in \text{Ext}^2_k(S, \bar{k}[X]^\times)$ such that $r(\alpha) = u \cup \beta$, $i^*(\gamma) = \partial(\beta)$ and $p^*(j^*)(-\alpha) = p^*(\gamma)$. In other words, we have the diagram

\begin{equation}
\begin{array}{ccc}
\gamma & \xrightarrow{-\alpha} & \text{Br}_1 X \\
\downarrow & & \downarrow \\
\text{Ext}^2_k(S, \bar{k}[X]^\times) & \xrightarrow{p^*} & \text{Ext}^2_1(p^* S, G_m) \\
\downarrow & & \downarrow \\
\partial(\beta) & \xrightarrow{} & \partial(\beta) \\
\downarrow & & \downarrow \\
\text{Hom}_k(M, \text{Pic} X) & \xrightarrow{} & \text{Ext}^2_k(M, \bar{k}[X]^\times) \\
\downarrow & & \downarrow \\
\text{Br}_1 X & \xrightarrow{} & H^1(k, \text{Pic} X)
\end{array}
\end{equation}
Proof. Use Corollary 2.2. Take $f_*=p_*$, $N=\mathbb{G}_m$ and (2.3) to be (2.6). Then (i) follows from the facts that for $p,q \geq 0$, $R^p\mathbb{G}_m=H^q(\overline{X},\mathbb{G}_m)$ and $H^p(X,\mathbb{G}_m)=\text{Ext}^p_c(p^*\mathbb{Z},\mathbb{G}_m)$, (see [6] p. 23 and [1] Prop. 1.4.1, respectively).

The existence of $\gamma$ follows from an easy diagram chase. Since $u \cup \beta \in \text{im}\ Br_1 X$,

$$u \cup \partial(\beta) = d(u \cup \beta) = 0.$$ 

Then there exists $\gamma \in \text{Ext}^2_\mathbb{K}(S,\overline{k}[X])$ such that $i^*(\gamma) = \partial(\beta)$. Then (ii) follows. \hfill \Box

Remark 2.7. We have some remarks on Corollary 2.4

(1) One may use the variant Proposition 1.11 to replace Pic $\overline{X}$ (resp. $\overline{k}[X]^\times$) appearing in the diagrams by $K\mathbb{D}'(X)$ (resp. $\overline{k}[X]^\times$). To be precise, take $f$ in Proposition 1.11 to be $\mathbb{G}_{m,k} \to \tau_{\leq 1}\mathbb{R}^p_\pi\mathbb{G}_{m,X}$. Then $\Delta = K\mathbb{D}'(X)$ (see [3]).

(2) If moreover $M$ is finitely generated (hence so is $S$), then its Catier dual $\hat{M}$ is a group of multiplicative type. It can be shown that

$$H^p(X,\hat{M}) = \text{Ext}^m_c(p^*\hat{T},\mathbb{G}_m),$$

$$H^p(k,\hat{M}) = \text{Ext}^m_c(p^*\hat{T},\mathbb{G}_m)$$

for $p \geq 0$ and $T = M$ or $S$ (c.f. [1] Prop. 1.4.1). Write $H^2_\mathbb{D}(X,T) = \text{ker}(H^2(X,T) \to \text{Hom}_k(T,\text{Br} X))$. Now (2.6) becomes

\[
\begin{array}{ccccccccc}
H^1(k,\overline{k}^\times) & \to & \text{Pic} X & \to & \mathbb{H}^0(k,\mathbb{D}'(X)) & \to & \text{Br} k & \to & \text{Br}_1 X \\
\downarrow j^* & & \downarrow p^*(j^*) & & \downarrow j^* & & \downarrow p^*(j^*) & & \downarrow j^* \\
H^1(k,\hat{S}) & \to & H^1(X,\hat{S}) & \to & \text{Hom}_{\mathbb{D}(k)}(S,\mathbb{D}'(X)) & \to & H^2(k,\hat{S}) & \to & H^2_\mathbb{D}(X,\hat{S}) \\
i^* & \downarrow p^*(i^*) & \downarrow i^* & \downarrow i^* & \downarrow p^*(i^*) & & & & \\
H^1(k,\hat{M}) & \to & H^1(X,\hat{M}) & \to & \text{Hom}_{\mathbb{D}(k)}(M,\mathbb{D}'(X)) & \to & H^2(k,\hat{M}) & \to & H^2_\mathbb{D}(X,\hat{M}) \\
u_\mathbb{D} & \downarrow p^*(u) & \downarrow u\mathbb{D} & \downarrow u\mathbb{D} & \downarrow p^*(u) & & & & \\
\text{Br} k & \to & \text{Br}_1 X & \to & \mathbb{H}^1(k,\mathbb{D}'(X)) & \to & H^3(k,\overline{k}^\times) & \to & H^3_\mathbb{D}(X,\mathbb{G}_m) \\
\end{array}
\]

where the second and third rows are fundamental exact sequences for open varieties and the maps $\lambda$ are so-called extended type. See [3].

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