Unit Regular Clean Rings

1Israa Th. Younis  2 Prof. Dr. Nazar H. Shuker

1Department of Mathematics, college of computer science and Mathematics, University of Mosul.
2 Department of Mathematics, college of computer science and Mathematics, University of Mosul.

israthanon8080@gmail.com

Abstract. A ring \( R \) is called unit regular clean, if every element is the sum of an idempotent and a unit regular elements. In this paper we introduce the notion of unit regular clean ring. we investigate some of it's basic properties and it's relation with clean ring.

Keyword: Clean ring ,unit regular ring ,unit regular element ,r-clean ring

1- Introduction:

Throughout this paper, \( R \) is an associative ring with identity. \( U(R) \), \( Ur(R) \) and \( Id(R) \) are respectively, the set of units, unit regular and idempotent elements. \( J(R) \) is the Jacobson radical of \( R \).

An element \( x \) of a ring \( R \) is said to be clean if \( x = e + u \) for some \( u \in U(R) \) and \( e \in Id(R) \). A ring \( R \) is called clean if each of its element is clean. Clean ring, was firstly presented by Nicholson [7]. Many researchers worked on this subject and investigated properties of clean rings, see for example [2, 5, 8, 10]. In 1936 Von Neumann defined that; an element \( r \in R \) is called regular if \( r = yr \) for some \( y \in R \). The ring \( R \) is said to be regular if each of its element is regular, some of the properties of regular rings have been studied in [6]. A ring \( R \) is called abelian if every idempotent in \( R \) is central [3].

A ring \( R \) is said to be unit regular if for each \( a \in R \), there exists a unit \( u \in R \) such that \( a = auu \). Camillo and Yu [5, Theorem 5] proved that: “every unit regular ring is clean”.

In [7], Nicholson and Varadrajan proved that the converse is not necessarily be true. In [1] Ashrafi and Nasibi defined that a ring \( R \) is said to be r-clean if every element of it can be written as the sum of idempotent and regular elements.

We say that an element \( x \) of a ring \( R \) is a unit regular clean (briefly, ur- clean) if \( x = e + a \) where \( a \in (R) \) and \( e \in Id(R) \).
A ring $R$ is said to be ur-clean if each of its element is ur-clean.

Clearly unit regular rings and clean rings are ur-clean, we also provide an example of ur-clean ring which is not clean. In this work we give some properties of ur-clean rings and its relation with clean ring.

2- Unit regular clean ring

In this section we introduce the notion of unit regular clean ring, we give some of its properties and provide some examples.

Definition 2.1 An element $x$ of a ring $R$ is unit regular clean, (briefly, ur-clean) if $x = r + e$ where $r \in ur(R)$ and $e \in Id(R)$. A ring $R$ is ur-clean if each of its elements is ur-clean.

Clearly, unit regular rings and clean rings are ur-clean. But the converse is not necessarily be true as the following example shows.

Example 2.2 The ring of integers, Modulo 4, $Z4$ is not unit regular because 2 is not unit regular in $Z4$. However it is easy to check that $Z4$ is ur-clean. If general ur-clean is not necessarily be clean see [11, Theorem 4.1].

Next, we shall give part of basic properties of ur-clean rings.

Proposition 2.3: If $R$ is a ring, then $x \in R$ is ur-clean element if and only if $(1 - x)$ is ur-clean element.

Proof: Let $x$ is ur-clean element then $x = e + a$ where, $e \in Id(R)$ and $a \in Ur(R)$, then

$1 - x = (1 - e) + (-a)$, but $(1 - e)$ is idempotent since

$(1 - e)^2 = 1 - 2e + e^2 = 1 - 2e + e = 1 - e$. Clearly $(-a) \in Ur(R)$ since $[a = e.u, a$ is unit regular then $-a = e(-u)$ is a unit regular]

Hence $1 - x$ is ur-clean element.

Conversely: let $(1 - x)$ is ur-clean element then

$1 - x = e + a$ where $e \in Id(F)$ and $a \in Ur(R)$

$-x = e + a - 1 \Rightarrow x = (1 - e) + a = (1 - e) + (-a)$

$(1 - e)$ is an idempotent and $-a$ is unit regular which implies that $x$ is ur-clean element.

Note that, for any ring $R$, and any ideal $I$ of $R$, if $R/I$ is ur-clean then $R$ is not necessarily to be ur-clean as the following examples shows.

Example 2.4:

1- If $P$ is prime number then $Z/p \cong Zp$ is ur-clean, but the ring $Z$ is not clean.

2- The ring of integers modulo 12, $Z12$. Let $I = \{0, 3, 6, 9\}$ be an ideal of $Z12$. Now $Z12/I$ is ur-clean since $Z12/I$ is a field; but $Z12$ is not ur-clean ring.
Following [9], idempotent can be lifted modulo, as one sided ideal $I$ of a ring $R$, if for $x \in R$ with $x - x^2 \in I$, there exists an idempotent $e \in R$ such that $e - x \in I$.

The following result, gives a sufficient condition for $R$ to be $ur$-clean.

**Theorem 2.5:** Let $I \subseteq J(R)$ be any ideal of a ring $R$ then $R$ is $ur$-clean if and only if the quotient ring $R/I$ is $ur$-clean and idempotent lift modulo $I$.

**Proof:** Let $x + I \in J/I, x \in R$ such that $x = e + a$ where $e$ is an idempotent and $a$ is a unit regular element.

Now, $x + I = e + a + I = (e + I) + (a + I)$.

Clearly, $(e + 1)$ is an idempotent element of $R/I$ and

$$(a + I) = (aua + I) = (a + I)(u + I)(a + I).$$

So $(a + 1)$ is unit regular then $R/I$ is $ur$-clean ring.

Conversely: Suppose that the quotient ring $R/I$ is $ur$-clean and idempotent lift modulo $I$ and let $r$ be any element in $R$, since $R/I$ is $ur$-clean we can write

$$r + 1 = x + e + 1$$

for some unit regular $x + 1$, and idempotent lift modulo $I$, we assume $e$ is an idempotent of the ring $R$, since $r - e + 1 = x + I$ is unit regular element of $R/I$. So $r - e$ is unit regular of $R$, it follows that $r$ may be written as the sum of idempotent and unit regular of $R$ by writing $r = (r - e) + e$, This proves the sufficiency. $lacksquare$

**Theorem 2.6:** If $R$ is abelian $ur$-clean ring and $2 \in U(R)$, then every element of $R$ can be written as a sum of idempotent and two units.

**Proof:** Let $x \in R$, then $x = e + a$, where $e \in \text{Id}(R), a = au = u$ since $u$ is idempotent say $e'$, then $a = u.e'$

Let $v = 2e' - 1$, clearly $v^2 = 1$. So $2e' = v + 1$, Since $2 \in U(R)$ then $e' = 2 - 1v + 2 - 1$. So $a = u.e' = u(2 - 1v + 2 - 1)$

$$= u(2 - 1v + u2 - 1) = u1 + u2$$

Hence $x = e' + u1 + u2$.

In [4] Camillo and Khurana gave the following result.

If $a$ is unit regular element then $a = e + u$ and $a \cap e R = 0.0$.

**Theorem 2.7:** Let $R$ be abelian $ur$-clean ring, for any $x \in R$ there exists an idempotent $e$, such that $ex$ is idempotent.

**Proof:** Since $x$ is $ur$-clean, then $x = e_1 = a$ where $e_1$ is idempotent and $a$ is unit regular then $a = e + u$ and $a R \cap e R = 0$. 


Since \( ae = ea \in aR \cap eR \), then \( ae = 0 \). So \( ex = ee_1 + ea \), then \( ex = e e_1 \), since \( e \) and \( e_1 \) are central idempotents, then \( e e_1 \) is idempotent. ■

In [1] Ashrafi proved that 'if \( R \) be an abelian \( r \)-clean ring, then \( eRe \) is also \( r \)-clean ring'. We do like wise of \( ur \)-clean ring.

**Theorem 2.8:** Let \( R \) be an abelian \( ur \)-clean ring then \( eRe \) is also \( ur \)-clean ring.

**Proof:** Let \( a \in eRe \subseteq R \), then \( a = e_1 + r \) and \( e_1 r = re_1 \) where \( e_1 \) is idempotent and \( r \in U_{r}(R) \) where \( R \) is \( ur \)-clean.

Since \( a \in eRe \), then \( a = ee_1 c + er'e \), it follows that \( a = e_1 e + r'e \) we want to show that \( re \) is unit regular and \( e_1 e \) is idempotent.

for this consider \((e_1 c)^2 = (e_1 e) \cdot (e_1 c) = e_1 (ee_1) e = e_1 (e_1 c) e = e_1 (e_1 e_1) (ee) = (e_1^2 e_2) = e_1 e\)

Therefore \( e_1 e \) is idempotent.

Now consider \( eue \in eRe \)

\[(re)(eue)(re) = (re)(eue)(er) = (re)(u)(er)\]

\[= (er)u(er) = e(rue) e = ere \in eRe \]

Then \( re \) is unit regular, implies that \( eRe \) is \( ur \)-clean ring. ■

**Theorem 2.9:** Let \( R \) be a ring with every \( a \in R \) there is \( b \in R \), such that \( a + b \in J(R) \) and \( a, b = a \), Then \( R \) is \( ur \)-clean.

**Proof:** Let \( a \in R \), then there is \( b \in R \) such that \( a + b \in J(R) \) and \( a, b = a \)

Then \( a + b - 1 \in U(R) \). Let \( a + b - 1 = u \). Now \( au = a(a + b - 1) = a^2 + ab - a = a^2 \)

So \( au = a^2 \), and hence \( a = a^2 u - 1 \)

Therefore \( a \) is unit regular. If we write \( a = 0 + a \), then \( R \) is \( ur \)-clean. ■

**Theorem 2.10:** Let \( R \) be a ring with every \( a \in R \) there is \( a \in R \) such that \( a + b \) is unit and \( a, b = 0 \), Then \( R \) is reduced \( ur \)-clean ring.

**Proof:** Let \( a \in R \), then there exists \( b \in R \) such that \( a + b \) is unit and \( a, b = 0 \)

Now, if we set \( a + b = v \) then \( av = a(a + b) = a^2 + ab = a^2 \). Clearly \( R \) is reduced ring if \( a^2 = 0 \), then \( av = 0 \) implies that, \( a = 0 \).

So \( a = a^2 v - 1 \) this implies \( a(1 - av - 1) = 0 \). Hence \( 1 - av - 1 \in (a) = \ell (a) \).
Then \((1 - av - 1)a = 0\). Hence \(a = av - 1\), so it unit regular.

If we set \(a = 0 + a\) then \(a\) is ur -clean. □

3- The relation between ur-clean and clean rings

In this section we give the relationship between ur-clean and clean rings. Clearly every clean ring is ur-clean ring since unit is unit regular, but the converse is not necessarily be true.

**Theorem 3.1:** Let \(R\) be an abelian ring, then \(R\) is ur-clean if and only if \(R\) is clean.

**Proof:** One direction is trivial.

Conversely: let \(R\) be ur-clean ring and \(x \in R\), then \(x = e + r\) where \(e \in \text{Id} (R)\) and \(r \in \text{Ur} (R)\). So there is \(u \in R\) such that \(uru = r\)

Clearly \(e^2 = r, u\) and \(u, r\) are idempotents and

\[ (re^2 + (1 - e^2)(ue^2 + (1 - e^2))) = 1, \] also since \(R\) is abelian we have

\[ (ue^2 + (1 - e^2))(re^2 + (1 - e^2)) = 1 \text{ ther} \]

\[ (re^2 + (1 - e^2)) \text{ is unit and hence } e'u + (1 - e') \text{ is unit} \]

\[-(e'u + (1 - e')) \text{ is a unit ,since } 1 - e \text{ is idempotent} \]

So, \(-r = (1 - e') + (-e'u + (1 - e'))\) is clean that is \(x\) is clean. □

**Theorem 3.2:** Let \(R\) be abelian ur-clean ring such that each pair of distinct idempotents in \(R\) are orthogonal then \(R\) is clean.

**Proof:** Since every abelian regular ring is clean ther for each \(e \in R\), \(x\) can be written as \(x = e_1 + e_2 + a\) where \(e_1, e_2 \in \text{Id} (R)\) and \(a \in \text{Ur}(R)\)

Now since \(e_1, e_2\) are orthogonal then \(e = e_1 + e_2 \in \text{Id} (R)\) and hence \(x = e + a\) which shows that \(R\) is clean. □

**Theorem 3.3:** If \(R\) is a directly finite ur-clean ring, and 0 and 1, are the only idempotents in \(R\), then \(R\) is clean.

**Proof:** Since \(R\) is ur-clean ring, each \(x \in R\) can be written as \(x = r + e\), where \(r\) is a unit regular element and \(e\) is an idempotent element of \(R\).

If \(r = 0\), then

\(x = e = (2e - 1) + (1 - e)\). Also, since \((2e - 1)\) is a unit of \(R\) and \((1 - e)\) is an idempotent element of \(R\), so \(x\) is a clean. Hence \(R\) is clean.

If \(r \neq 0\), then there exists \(u \in R\) such that \(uru = r\). Thus \(ru\) an idempotent element of \(R\).
So by hypothesis, \( ru = 0 \) or \( ru = 1 \).

Now if \( ru = 0 \), then \( r = rur = 0 \), which is contradiction. Therefore \( ru = 1 \) and since \( R \) is directly finite so \( ur = ru = 1 \).

Thus, \( r \) is a unit of \( R \). So \( x \) is clean element, and hence \( R \) is clean ring. 

**Theorem 3.4**: Let \( R \) be abelian ring and for every \( a \in R \), there exists \( b \in R \) such that \( a + b \in U_r(R) \) and \( c,b = 0 \), then there is \( e \) in \( R \) such that \( ae \) is clean element.

**Proof**: Let \( a \in R \), and \( a + b \in U_r(R) \) then \( a + b = e_1 \cdot u \) where \( e \) is idempotent and \( u \) is unit.

Now,

\[
a(a + b) = ae_1 \cdot u \quad \text{so, } a^2 = ae_1 \cdot u \quad \text{and hence } \quad ae_1 = a^2 \cdot u - 1, \quad \text{so } ae_1 = (ae_1)^2 \cdot u - 1 \quad \text{clearly } \quad ae_1 \cdot e_2 + (e_2 - 1) \quad \text{is unit since} \\
(ae_1 \cdot e_2 + (e_2 - 1)) (e_2 \cdot u - 1 + (e_2 - 1)) = 1 \quad \text{where } e_2 = ae_1 \cdot u - 1 \\
So \quad ae_1 \cdot e_2 + e_2^{-1} = v \quad \text{and hence } \quad ae_1 \cdot e_2 = 1 - e_2 + v \quad \text{but } e_1 \cdot e_2 \text{ is idempotent say } e \text{ so } \quad ae = (1 - e_2) + v \\
\]

This means that \( ae \) is clean element and hence it is \( ur \)-clean. 

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