Heavy quark potential with dynamical flavors: a first order transition

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We study the static potential between external quark-antiquark pairs in a strongly coupled gauge theory with a large number of colors and massive dynamical flavors, using a dual string description. When the constituent mass of the dynamical quarks is set below a certain critical value, we find a first order phase transition between a linear and a Coulomb-like regime. Above the critical mass the two phases are smoothly connected. We also study the dependence on the theory parameters of the quark-antiquark separation at which the static configuration decays into specific static-dynamical mesons.

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1. Introduction. The study of non-perturbative effects of dynamical quarks is a notoriously difficult problem to address with the present computational techniques in QCD. Phenomenological models, such as the screened version [1] of the Cornell potential for heavy quarks [2], are of fundamental importance but an understanding of the physics of these effects from first principles is still missing. In fact, the best tool for the study of QCD at strong coupling, i.e. its lattice formulation, is still partially limited by the lack of suitable computational power when dealing with dynamical light flavors (for our purposes the relevant reference is [3]). In the latest years, string theory has developed as a possible tool for studying strong coupling effects in quantum field theory. It is clearly not an ideal setting, since it allows to study mainly the ’t Hooft coupling, large $N_c$ (number of colors) limit of theories which typically include a number of other fields besides (possibly) the QCD ones. Nevertheless, apart from its obvious theoretical interest, string theory has proven to be a valuable way for understanding processes which are universal enough, such as specific Quark-Gluon Plasma ones.

In this letter we study the “connected part” (i.e. we do not include the mixing with the meson/anti-meson states) of the static potential between two heavy test quarks in a SQCD-like quantum field theory, which includes light dynamical quarks of mass $m_q$, by using its dual string theory description. Below a critical mass $m_c$ of the dynamical quarks, we observe a first order phase transition in the potential $V(L)$ as we increase the distance $L$ between the two test quarks. The transition takes place in the region where the potential turns from Coulomb-like to linear. Instead, for masses larger than $m_c$, the connection between the two regions is perfectly smooth. We also study the “string breaking length” $L_{sb}$ at which the heavy quark configuration decays into a specific pair of heavy-light mesons. It is shown that $L_{sb}$ is a decreasing function of the number $N_f$ and the mass $m_q$ of the dynamical flavors, the latter being a fully non-perturbative effect.

While the theory we study is not QCD, it is possible that the main features of the static potential are sufficiently universal. Hence, we provide an analysis of non-perturbative effects of light dynamical quarks in a strongly coupled quantum field theory, in a very straightforward and simple way.

Sections 2 and 3 provide the technical details on the string set-up. Section 4 and 5, which can be read independently, contain the results. We define $x \equiv \frac{N_f}{N_c}$ and limit our analysis to the range $0 < x < 1.5$ where we reach sufficient numerical precision.

2. Details on the dual string description. The SQCD-like theory we consider is realized at the four dimensional intersection of $N_c \gg 1$ “color” D5-branes wrapped on a $S^2$ and $N_f \sim N_c$ “flavor” D5-branes. It is a particular version of $\mathcal{N} = 1$ SQCD deformed by a quartic superpotential as considered in [4]. In these models, the flavor branes are homogeneously smeared along the (large) internal manifold [5]. The typical separation between two branes is much larger than the string scale so the flavor symmetry is in fact $U(1)^{N_f}$. We find solutions of IIB supergravity coupled to the flavor brane sources. These sources are represented as a sum of DBI+WZ actions [6]. Notice that even if the DBI action for $N_f$ branes is not valid when $g_sN_f$ is not small, here we can use it because the branes are separated far apart so what we have is a sum of actions for individual branes. The string frame metric reads

$$ds^2 = \alpha' e^\phi N_c \left[ \frac{1}{\alpha' g_s N_c} dx_1^2 + 4Y d\rho^2 + H(d\theta^2 + \sin^2 \theta d\varphi^2) + G(d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\varphi}^2) + \frac{a}{2} \cos \psi (d\theta d\tilde{\theta} - \sin \theta \sin \tilde{\theta} d\varphi d\tilde{\varphi}) + \frac{a}{2} \sin \psi (\sin \theta d\tilde{\theta} d\varphi + \sin \tilde{\theta} d\theta d\tilde{\varphi}) \right]$$
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\[ +Y \left( d\psi + \cos \hat{\theta} d\hat{\varphi} + \cos \theta d\varphi \right)^2 \]. \quad (1)

The dilaton \( \phi \) and the functions \( Y, H, G, a \) depend only on the radial coordinate \( \rho \); we used the notation \( g_s = e^{\phi \Omega} \), where \( \phi_{1R} \equiv \phi(\rho = 0) \). In the massless flavor case, which was studied in [4], the \( N_f \) smeared flavor branes were extended along \( \rho \) from the origin up to infinity. In the following we will consider the case in which all the flavor branes are extended up to a finite distance \( \rho_q > 0 \) from the origin. This will correspond to turning on a mass for all the flavors in the gauge theory dual.

The full supergravity background contains also a RR three-form given by

\[ F = \alpha' g_s N_c \left[ -\frac{1}{4} \sin \hat{\theta} d\hat{\theta} \wedge d\hat{\varphi} \wedge \hat{\omega}_3 + \frac{1}{4} (d\theta \wedge d\hat{\theta} - \sin \theta \sin \hat{\theta} d\varphi \wedge d\hat{\varphi}) \right. \]

\[ + \left. \frac{1}{4} \left( -\sin \hat{\theta} d\theta \wedge d\varphi + \sin \theta d\theta \wedge d\hat{\varphi} \right) \right] (\partial_\rho b \sin \psi d\rho - \frac{x - 1}{4} \sin \theta d\theta \wedge d\varphi \wedge \hat{\omega}_3) \],

where \( \hat{\omega}_3 = d\psi + \cos \hat{\theta} d\hat{\varphi} + \cos \theta d\varphi \). In [4], a set of differential equations and constraints for \( H, G, b, a, Y, \phi \) was found. They read, in the present notation,

\[ H = \frac{C(a + b)}{4} - \frac{2 - x}{8}, \quad G = \frac{C(a - b)}{4} + \frac{2 - x}{8}, \]

\[ \partial_\rho b = -\frac{2bc + x - 2}{S}, \quad \partial_\rho a = -\frac{2ac + x - 8Y}{S}, \]

\[ \partial_\rho Y = 8SY \frac{2a^2 C - b(2bC - 2 + x) - a(x + 8Y)}{4a^2 S^2 - (2bC - 2 + x)^2}, \]

\[ \partial_\rho \phi = \frac{1}{2} \partial_\rho \log \left( \frac{32aS^2Y}{(4a^2 S^2 - (2bC - 2 + x)^2)} \right), \]

where \( C = \cosh(2\rho) \), \( S = \sinh(2\rho) \). This system of equations is valid for arbitrary \( N_f, N_c \).

The expressions (2) make the addition of massive flavors relatively easy. As it was noticed in a similar context [7], this can be effectively achieved by just replacing the constant \( x \) with a function of \( \rho, x(\rho) \). In this way the expression for the three-form flux is unchanged, modulo the above redefinition of \( x \). Indeed the Bianchi identity gets changed, but since in the supersymmetric variations for the background fermions only the field \( F \) (and not its derivative \( dF \)) enters, the whole BPS equations will not be modified in form. On the field theory side, we can neglect the quark masses at high energies. At energies smaller than the quark masses, the quarks can be integrated out and the theory looks like the unflavored one (a confining SYM-like theory in the present case). This suggests, via the holographic radius-energy relation, that we can choose, as a model for \( x(\rho) \), just a Heaviside step function \( x(\rho) = x\Theta(\rho - \rho_q) \). At the level of the supergravity equations of motion, the only constraint on \( x(\rho) \) is that \( x(\rho') \geq 0 \) [7]. The exact form of the function should follow from the smearing of the massive embedding associated to the flavor branes. Nevertheless, the Heaviside approximation is expected to correctly capture the qualitative features of the system under study, as it also happens in other similar setups [8]. Since the first two equations in (2) are constraints and not differential equations, not all the functions can be continuous if \( x(\rho) \) is discontinuous. We impose that the metric is continuous so \( H, G, a, Y, \phi \) are continuous at \( \rho = \rho_q \). This is the equivalent procedure to matching the couplings when one integrates out quarks in the field theory. The only function which does not enter the metric is \( b \). In any case, what should be continuous - see eq. (2) - is \( C + b \). The solution for \( b \) consistent with the differential equations and this condition is:

\[ b = \frac{2\rho}{\sinh(2\rho)} + x\Theta(\rho - \rho_q) \left( \frac{\rho_q - \rho - \frac{\sin(2\rho_q)}{2\cosh(2\rho_q)}}{\sinh(2\rho)} \right) \].

Below \( \rho_q \), we have the unflavored system, so we can take the regular solution, found in Section 8 of the first paper in [4], depending on a parameter \( \mu \). Tuning \( \mu \) for a given pair \( (x, \rho_q) \), it is possible to get a solution with the expected UV which is uniquely determined up to the additive constant \( \phi_{1R} \). The system can be solved numerically. The plots of figure 1 depict a typical case for the functions \( Y, \phi \).

3. The quark-antiquark potential. We focus on the interaction energy between two test quarks \( Q, \bar{Q} \) put as external probes of our theory. The string dual of the \( QQ \) system is a macroscopic open string attached to a probe flavor brane which is very far from the bottom of the potential \( V(M_Q) \). From this we can deduce the UV limit relation, where \( V(L) \) is the potential (renormalized energy), i.e. the total energy minus the total quark mass \( 2M_Q \). The open string embedding is chosen as \( t = \tau, y = \sigma, \rho = \rho(y) \) where \( y \in [-L/2, L/2] \) is one of the spatial Minkowski directions. The string worldsheet action reads

\[ S = -\frac{1}{2\pi \alpha'} \int dt dy \sqrt{g_{tt}(\rho)g_{yy}(\rho) + (\partial_\rho \rho)^2 g_{rr}(\rho)} \].

![FIG. 1: The Y, \phi functions in the case x = 0.8, \rho_q = 0.5.](image-url)
Defining \( f \equiv \sqrt{\beta \xi \xi_0} = e^{\phi - \phi_{IR}} \), \( g \equiv \sqrt{\beta \xi \xi_0} = 2 e^{\phi - \phi_{IR}} \sqrt{\beta g_s N_c \alpha'} \), the (constituent) quark mass is given by the energy of a string extended from the bottom of the space at \( \rho = 0 \) to the bottom of the corresponding flavor brane, \( m_q = \frac{1}{2 \pi \alpha'} f_0 g \int_0^\rho \rho \, d\rho \), \( M_Q = \frac{1}{2 \pi \alpha'} \int_0^{\rho_0} g \rho \, d\rho \).

We write the length \( L \) and potential \( V \) as functions of \( \rho_0 \) and with UV cutoff \( \rho_Q \) as (the 0 subindex means that the quantity is evaluated at \( \rho = \rho_0 \)):

\[
L(\rho_0) = 2 \int_{\rho_0}^{\rho_0} \frac{g f_0}{f \sqrt{f^2 - f_0^2}} \, d\rho ,
\]

\[
V(\rho_0) = 2 \frac{\pi \alpha'}{2} \left[ \int_{\rho_0}^{\rho_0} \frac{g f}{\sqrt{f^2 - f_0^2}} \, d\rho - \int_0^{\rho_0} g \rho \, d\rho \right].
\]

It is not difficult to check the relation \( \frac{dV}{d\rho_0} = \frac{f_0}{2 \pi \alpha'} \frac{dL}{d\rho_0} \).

This for instance means that \( \frac{dV}{d\rho_0} \) and \( \frac{dL}{d\rho_0} \) have the same sign. In some cases they can change sign simultaneously at some value of \( \rho_0 \) so that the \( V(L) \) plot turns around.

We can now plug our numeric solutions into the definitions of the quark-antiquark potential \( V(\rho_0) \) and distance \( L(\rho_0) \), to see how they depend on the dynamical massive flavors. In doing so we consider a finite (large) UV cutoff, and so a finite \( M_Q \gg m_q \), which we keep fixed as \( m_q \) and \( x \) are varied (we will show the results for \( M_Q = 200 \sqrt{g_s N_c / \alpha'} \) but their qualitative features do not depend on this choice).

4. A first order transition in the potential.— In the \( m_q = 0 \) case, the \( \bar{Q}Q \) string breaks at a certain \( L_{\text{max}} \), where the \( V(L) \) plot turns around [4]. At \( m_q \neq 0 \), this turn-around (that one would expect at least in the small mass limit), should be followed by a second turn-around, such that the potential comes back to an increasing linear behavior at large \( L \) (i.e. in the far IR, where the quarks are integrated out and the theory is SYM-like).

Our numerical analysis confirms this expectation, adding a crucial piece of information: the existence of a critical value \( m_c \) for the dynamical quark mass. For \( m_q > m_c \), the \( V(L) \) plot is monotonic, a Coulomb-like behavior at small \( L \) smoothly joining with the linear one for large \( L \). Instead, in the small mass regime, \( m_q < m_c \), we find a first order [13] transition between the Coulomb-like and the linear phase, with a corresponding double turn-around behavior of \( V(L) \) at intermediate distances, see figure 2. In the figure, the energetically favored branch of the graph is the one with largest \( L \) at a given \( V \) (the solid line). So, there is a discontinuity in the derivative of \( V(L) \), with a sudden decrease of the “local string tension”, such that at a certain energy scale and length the increase of the energy of the flux tube with the length has a deceleration. The Coulomb-like regime does not connect smoothly with the linear regime [14].

The existence of a critical mass \( m_c \) above which there is no phase transition, is evident by analyzing the \( L(\rho_0) \) curve at different values of \( m_q \). For example, for \( x=0.8 \), one can check from plots as those in figure 3 that below \( m_c \approx 0.09 \), there is a first order phase transition between a large \( \rho_0 \) string and a small \( \rho_0 \) one. Actually these plots share some qualitative features with the pressure/volume isothermal curves in a van der Waals gas, where the phase transition, occurring below a certain critical temperature, is between a liquid and a gas phase. It might prove useful to model the “connected part” of the phenomenological potential between two heavy quarks with some function, similar to the Gibbs free energy/pressure relation in the van der Waals system, exhibiting a discontinuity in the first derivative for small dynamical quark mass. We checked that the quark distance at which the transition occurs is a decreasing function of \( x \) and an increasing function of \( m_q \). On the other hand, the value of the potential at the transition is a decreasing function of both \( x, m_q \). Finally, \( m_c \to 0 \) for \( x \to 0 \), where one recovers the quenched approximation.

5. String breaking length.— In the presence of dy-
FIG. 4: The string breaking length $L_{sb}$ as a function of: (left) the number of flavors $x$ for $m_q = 0.05, 0.1, 0.2$ (bottom to top) and (right) the mass $m_q$ for $x = 0, 0.3, 0.8, 1.2$ (top to bottom).

namical quarks there is a screening length, defined as the length at which the $QQ$ string will break by production of quark pairs. It is determined by the condition that the energy of the Wilson loop equals the energy of two mesons composed by one test and one dynamical quark. In the case at hand, due to the smearing procedure, the lighter of such mesons are nearly massless. In fact, in the string language they are localized at the intersection of the test brane and the dynamical brane. In fact, in the string language they are localized at the intersection of the test brane and the dynamical brane. Their mass should scale as $M_Q/(g_s N_c)$, with $g_s N_c \gg 1$ [11]. This is parametrically smaller than the typical energy $E_{QQ}(L) = 2M_Q + V(L)$ computed above. Thus, the Wilson loop has always enough energy to decay into these mesons and the screening length is not a relevant quantity. Nevertheless, again due to the smearing, the decay in this channel (and in every channel corresponding to a specific flavor) is $1/N_c$ suppressed. If we ask for a decay rate which is not $1/N_c$ suppressed we have to consider a larger $Q, Q$ separation, such that the $QQ$ state can decay into a sizable fraction of the $N_f$ types of heavy-light mesons. Given such a fraction, one could study how the separation $L_{sb}$, required for the decay on all the included flavors to happen, depends on the parameters $m_q, x$. Clearly, the choice of the fraction would be arbitrary.

In order to get an idea of the general trend, we choose to define $L_{sb}$ as the length at which the Wilson loop has enough energy to decay by production of the particular flavor which has precisely the same charge (internal or R-symmetry quantum numbers) as the test quarks. The string dual to the corresponding specific mesons, whose mass is $E_{QQ} = M_Q - m_q$, is stretched only along the radial direction of the geometry (1), from the bottom of the test brane to that of the “parallel” dynamical one [12]. We call $L_{sb}$ the “string breaking length”.

The results are in figure 4. The string breaking length is a decreasing function of the number of flavors $x$: the more flavors there are in the theory, the easiest is the decay. Finally, the string breaking length is a decreasing function of $m_q$: this is a genuine large coupling effect, and may change if one chooses a different definition of $E_{QQ}$.

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[13] We are grateful to Marco Caldarelli for pointing this to us.
[14] One may think that the behavior in Fig 2 is due to a level crossing with a hybrid potential. We thank Joan Soto for a discussion on this point.