Global boundedness and asymptotic behavior of
time-space fractional nonlocal reaction-diffusion
equation

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Abstract
The global boundedness and asymptotic behavior are investigate for the solution of time-space fractional non-local reaction-diffusion equation (TSFN-RDE)

$$\frac{\partial^\alpha u}{\partial t^\alpha} = -(-\Delta)^s u + \mu u^2(1 - kJ * u) - \gamma u, \quad (x, t) \in \mathbb{R}^N \times (0, +\infty),$$

where $s \in (0, 1), \alpha \in (0, 1), N \leq 2$. The operator $\partial_t^\alpha$ is the Caputo fractional derivative, which $-(-\Delta)^s$ is the fractional Laplacian operator. For appropriate assumptions on $J$, it is proved that for homogeneous Dirichlet boundary condition, this problem admits a global bounded weak solution for $N = 1$, while for $N = 2$, global bounded weak solution exists for large $k$ values by Gagliardo-Nirenberg inequality and fractional differential inequality. With further assumptions on the initial datum, for small $\mu$ values, the solution is shown to converge to 0 exponentially or locally uniformly as $t \to \infty$. Furthermore, under the condition of $J \equiv 1$, it is proved that the nonlinear TSFNRDE has a unique weak solution which is global bounded in fractional Sobolev space with the nonlinear fractional diffusion terms $-(-\Delta)^s u^m (2 - \frac{2}{N} < m < 1)$.

Keywords: Time-space fractional reaction-diffusion equation, Global boundedness, Asymptotic Behavior, Non-local, Nonlinear

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1. Introduction

In this work we study the time-space fractional non-local reaction-diffusion equation

\[
\frac{\partial^\alpha u}{\partial t^\alpha} = -(-\Delta)^s u + \mu u^2 (1 - k J * u) - \gamma u, \quad (1.1)
\]

\[
u(x, 0) = u_0(x), \quad x \in \mathbb{R}^N, \quad (1.2)
\]

with \((x, t) \in \mathbb{R}^N \times (0, +\infty), 0 < \alpha < 1, N \leq 2, \mu, k > 0, \gamma > 1\). According to [1], the nonlocal operator \((-\Delta)^s\), known as the Laplacian of order \(s\), is defined for any function \(g\) in the Schwartz class through the Fourier transform:

\[
\hat{h}(\xi) = |\xi|^{2s} \hat{g}(\xi). \quad (1.3)
\]

If \(0 < s < 1\), by [2], we can also use the representation by means of a hypersingular kernel,

\[
(-\Delta)^s u = C_{N,s} P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy, \quad (1.4)
\]

where \(C_{N,s} = \frac{4^s s! \Gamma(\frac{N}{2} + s)}{\pi^N \Gamma(1-s) \Gamma(\frac{N}{2} + s)}\), and \(P.V.\) is the principal value of Cauchy. Considering the Sobolev space

\[
H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|}{(|x - y|)^{N+2s}} dxdy < \infty \right\}.
\]

And \(\partial^\alpha_t\) denote the left Caputo fractional derivative that is usually defined by the formula

\[
\partial^\alpha_t u(x, t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u}{\partial s}(x, s)(t - s)^{-\alpha} ds, \quad 0 < \alpha < 1 \quad (1.5)
\]

in the formula \(1.5\), the left Caputo fractional derivative \(\partial^\alpha_t u\) is a derivative of the order in \(3\). By \(4\), Here \(J(x)\) is a competition kernel with

\[
0 \leq J \in L^1(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} J(x) dx = 1, \quad \inf_{\mathbb{R}^N} J > \eta \quad (1.6)
\]

for some \(\eta > 0\), and

\[
J * u(x, t) = \int_{\mathbb{R}^N} J(x - y)u(y, t) dy,
\]
and $F(x,t) = \mu u^2(1 - k J * u) - \gamma u$ is represented as sexual reproduction in the population dynamics system, where $-\gamma u$ is population mortality. By [5], assume $\{\Phi_j\}_{j \geq 1}$ is an orthonormal eigenbasis in $L^2(\Omega), \Omega \subset \mathbb{R}^N$, associated with the eigenvalues $\{\lambda_j\}_{j \geq 1}$ such that $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots, \lim_{j \to \infty} \lambda_j = \infty$. For all $0 < s < 1$, we denote by $D((-\Delta)^s)$ the space defined by

$$D((-\Delta)^s) := \left\{ u \in L^2(\Omega) : \sum_{j=1}^{\infty} \lambda_j^{2s} |(u, \Phi_j)|^2 < \infty \right\},$$

then if $u \in D((-\Delta)^s)$, we define the operator $(-\Delta)^s$ by

$$(-\Delta)^s u := \sum_{j=1}^{\infty} \lambda_j^s (u, \Phi_j) \Phi_j,$$

which $(-\Delta)^s : D((-\Delta)^s) \to L^2(\Omega)$, with the following equivalence

$$\|u\|_{D((-\Delta)^s)} = \|(-\Delta)^s u\|_{L^2(\Omega)}.$$

Suppose $\{\lambda_j^s, \Phi_j\}$ be the eigenvalues and corresponding eigenvectors of the Laplacian operator $-\Delta$ in $\Omega \in \mathbb{R}^N$ with Dirichlet boundary condition on $\partial \Omega$:

$$\left\{ \begin{array}{ll}
-\Delta \Phi_j = \lambda_j^s \Phi_j, & \text{in } \Omega, \\
\Phi_j = 0, & \text{on } \partial \Omega.
\end{array} \right.$$

Therefore, we will study the solution for equation (1.1)-(1.2) is global boundedness and asymptotic behavior under the above Dirichlet boundary condition in one dimensional space and two dimensional space.

Fractional calculus has gained considerable importance due to its application in various disciplines such as physics, mechanics, chemistry, engineering, etc[6, 7]. Therefore, fractional-order ordinary and partial differential equations have been widely studied by many authors, see [8, 9]. In the process of practical application, researchers found that the solution of fractional differential equation has a lot of properties, and when the solution of the fractional differential equation is proof and analyzed, Laplace transform [10], Fourier transform [11] and Green function method [12]. At present, the research of fractional diffusion equations has become a new field of active research. More than a dozen universities and research institutes at home and abroad have
engaged in the research of time fractional, space fractional, time-space fractional diffusion equations and special functions related to it such to Wright, Mittag-Leffler function and so on.

Let us first recall some previous results on fractional diffusion equation. Since there is a large amount of papers for these equations, we mention the ones related to our results.

When \( \alpha = 1 \) and \( s = 1 \), then problem (1.1)-(1.2) reduces to the following non-local reaction-diffusion equation in [4]

\[
\frac{\partial u}{\partial t} = \Delta u + \mu u^2(1 - k * u) - \gamma u,
\]

it denotes by \( u(x, t) \) the density of individuals having phenotype \( x \) at time \( t \) and formulate the dynamics of the population density. However, with the further deepening of scientific research, compared with the traditional reaction-diffusion equation, the non-local reaction-diffusion equation has new mathematical characteristics and richer nonlinear dynamic properties [13]. From [13], Kolmogorov and Fish research the following reaction-diffusion equations

\[
\frac{\partial u}{\partial t} = \Delta u + u(1 - u), \quad x \in \mathbb{R},
\]

in order to describe the row phenomenon of foreign invasion species and animals the transmission process of excellent genes in an infinite habitat. Through the widespread research of the diffusion equations, they found that there are many important applications in many fields such as spatial ecology, evolution of species, and disease dissemination of the non-local reaction-diffusion equation [15, 16, 17].

When \( s = 1 \) and \( 0 < \alpha < 1 \), the fractional operator \(-(-\Delta)^s\) become the standard Laplacian \( \Delta \), which is a time fractional non-local reaction-diffusion equation. Many researchers have studied the corresponding property for solution of reaction-diffusion model with Caputo fractional derivative. In order to study the global boundary of the solution of fractional linear reaction-diffusion equation, [18] uses the maximum regularity to verify the existence of such equations. In general space, we often use the convolutional definition of the Caputo fractional derivative. However, in [19], from the perspective of micro-division operator theory, studied the following time fractional reaction-diffusion equation in fractional Sobolev spaces

\[
\frac{\partial^\alpha}{\partial t^\alpha} u(x, t) = -Lu(x, t) + F(x, t), \quad x \in \Omega \subset \mathbb{R}^n, 0 < t \leq T
\]
where $L$ is a differential operator of elliptic type and $\partial^\alpha_t$ denotes the left Caputo fractional derivative that is usually defined by (1.5). In [3], the time fractional reaction-diffusion model with non-local boundary conditions is used to predict the invasion of tumors and its growth. In addition, the Faedo-Galerkin method has also been used to verify that the model has the only weak solution. From [20], Ahmad considers the following time fractional reaction-diffusion equation with boundary conditions

$$^cD^\alpha u = \Delta u - u(1-u), \quad x \in \Omega, \ t > 0$$

with the initial condition $u(x,0) = u_0(x), \ x \in \Omega$. the initial data $u_0(x)$ is a given positive and bounded function. Moreover, [20, 21, 22] research the existence, global boundary and blow-up of time fractional reaction-diffusion equation with boundary conditions. The study of time fractional diffusion equations has recently attracted a lot of attention. It is worth mentioning that we often use time fractional Duhamel principle [23] and Laplace transform to obtain the solution of time fractional PDE.

When $\alpha = 1$ and $0 < s < 1$, then problem (1.1)-(1.2) reduces to the non-local reaction-diffusion equation with fractional Laplacian operator, which is called fractional reaction-diffusion equation. Over the past few decades, due to the existence and non-existent research significance of fractional Laplacian equation, which has attracted many scholars to invest in it. For example, [24] study Liouville theorem of fractional Laplacian equation to verify existence/non-existent of the solution. In particular, scholars conducted a lot of research on fractional reaction-diffusion equation and obtained many important results. [25] establish the global existence of weak solutions of non-local energy-weighted fractional reaction–diffusion equations for any bounded smooth domain. In [26], considering the asymptoticity of the following nonlinear non-autonomous fractional reaction–diffusion equations

$$u_t(x,t) = -(-\Delta)^{\alpha/2} - h(t) F(u), \quad x \in \mathbb{R}^N, \ t > 0,
\quad u(x,0) = u_0(x) \geq 0, \quad x \in \mathbb{R}^N.$$ 

Among them, the initial function $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), \ h : [0,\infty) \to [0,\infty)$ is a continuous function. [27] study the existence and blow-up of solution of semilinear reaction-diffusion system with the fractional Laplacian. In addition, [28] studied the blow-up and asymptoticity of solution of the following initial value problem for the reaction–diffusion equation with the
anomalous diffusion

\[ \partial_t u = -(-\Delta)^\alpha + \lambda u^p, \quad x \in \mathbb{R}^N, \quad t > 0, \]

\[ u(x, 0) = u_0(x), \]

where \( 0 < \alpha \leq 2, \lambda \in \{-1, 1\} \) and \( p > 1 \). The fractional powers of the classical Laplace operator, namely \(-\Delta)^\alpha\) are particular cases of the infinitesimal generators of Lévy stable diffusion processes and appear in anomalous diffusions in plasmas, flames propagation and chemical reactions in liquids, population dynamics, geophysical fluid dynamics [29, 30].

The fractional diffusion equation \( \partial_t u = \Delta u \) describes a cloud of spreading particles at the macroscopic level. The space-time fractional diffusion equation \( \partial_t^n u = -(-\Delta u)^{\alpha/2} \) with \( 0 < \beta < 1 \) and \( 0 < \alpha < 2 \) is used to model anomalous diffusion [31]. The fractional derivative in time is used to describe particle sticking and trapping phenomena and the fractional space derivative is used to model long particle jumps [31]. These two effects combined together produces a concentration profile with a sharper peak, and heavier tails [32]. In [33], basing on a new unique continuation principle for the eigenvalues problem associated with the fractional Laplace operator subject to the zero exterior boundary condition, it study the controllability of the space-time fractional diffusion equation. [34] concerned with boundary stabilization and boundary feed-back stabilization for time-space fractional diffusion equation. In these applications, it is often important to consider boundary value problems. Hence it is useful to develop solutions for space–time fractional diffusion equations on bounded domains with Dirichlet boundary conditions. This paper will consider the following Dirichlet boundary condition [35, 36].

In the classic diffusion equation, the time derivation of the Integer into a time derivation is turned into a time fractional diffusion equation, which is usually used to describe the ultra-diffusion and secondary diffusion phenomenon. However, in some practical situations, part of boundary data, or initial data, or diffusion coefficient, or source term may not be given and we want to find them by additional measurement data which will yield some fractional diffusion inverse problems. [35] analyzed the inverse source problem for the following space-time fractional diffusion equation by setting up
an operator equation

\[ \frac{\partial^\beta}{\partial t^\beta} u(t, x) = -r^\beta(-\Delta)^{\alpha/2} u(t, x) + f(x)h(t, x), \quad (t, x) \in \Omega_T, \]

\[ u(t, -1) = u(t, 1) = 0, \quad 0 < t < T, \]

\[ u(0, x) = 0, \quad x \in \Omega, \]

where \((t, x) \in \Omega_T := (0, T) \times \Omega\) and \(\Omega = (-1, 1)\). For the time fractional diffusion equations cases, the uniqueness of inverse source problems have been widely studied. Basing on the eigenfunction expansion, \[37\] established the unique existence of the weak solution and the asymptotic behavior as the time \(t\) goes to \(\infty\) for fractional diffusion-wave equation. Li and Wei, in \[38\] proved the existence and uniqueness of a weak solution for the following time-space fractional diffusion equation

\[ \partial_{0+}^\alpha u(x, t) = -(-\Delta)^{\alpha/2} u(x, t) + f(x)p(t), \quad (x, t) \in \Omega_T, \]

\[ u(x, 0) = \phi(x), \quad x \in \bar{\Omega}, \]

\[ u(x, t) = 0 \quad x \in \partial\Omega, t \in (0, T], \]

where \(\Omega_T := (0, T) \times \Omega, \Omega \subset \mathbb{R}^d\) and \(\alpha \in (0, 1), \beta \in (1, 2)\). This paper uses the method of proof Theorem 3.2 in \[38\] to proof global existence and uniqueness of a weak solution. In \[39\], Wei and Zhang solved an inverse space-dependent source problem by a modified quasi-boundary value method. Wei et al. in \[40\] identified a time-dependent source term in a multidimensional time-fractional diffusion equation from the boundary Cauchy data. Compared with the problem of classical inverse initial value, the recovery of the fractional inverse initial value is easier and more stable. Through the above two paragraphs, this paper will use the method of verifying the initial value of the inverse initial value under the Dirichlet boundary conditions to verify the existence and uniqueness of time and space fractional non-local reaction diffusion equations.

To the best of our knowledge, until recently there has been still very little works on deal with the existence, decay estimates and blow-up of solutions for time-space fractional diffusion equations. \[41\] studied the global and local existence, blow-up of solutions of the following time-space fractional diffusion

\[ \frac{\partial^\beta}{\partial t^\beta} u(t, x) = -r^\beta(-\Delta)^{\alpha/2} u(t, x) + f(x)h(t, x), \quad (t, x) \in \Omega_T, \]

\[ u(t, -1) = u(t, 1) = 0, \quad 0 < t < T, \]

\[ u(0, x) = 0, \quad x \in \Omega, \]
problem by applying the Galerkin method

\[
\partial_t^\beta u + (-\Delta)^\alpha u + (-\Delta)^\beta \partial_t^\beta u = \lambda f(x,u) + g(x,t), \quad \text{in } \Omega \times \mathbb{R}^+,
\]

\[
u(x,t) = 0, \quad \text{in } (\mathbb{R}^N \setminus \Omega) \times \mathbb{R}^+,
\]

\[
u(x,0) = u_0(x), \quad \text{in } \Omega,
\]

where \( \Omega \subset \mathbb{R}^N, 0 < \alpha < 1, 0 < \beta < 1 \) is a bounded domain with Lipschitz boundary. \([42]\) studies the blow-up, and global existence of solutions to the time-space fractional diffusion problem and give an upper bound estimate of the life span of blowing-up solutions. In \([3]\), By transforming the time-space fractional diffusion equations into an operator equation it investigates the existence, the uniqueness and the instability for the problem. In addition, \([40]\) study the existence and uniqueness of a weak solution of following time-space fractional diffusion equation with homogeneous Dirichlet boundary

\[
\partial_t^\alpha u(x,t) = -(-\Delta)^{\frac{s}{2}} u(x,t) + f(x)p(x), \quad (x,t) \in \Omega_T,
\]

where \( \Omega_T := \Omega \times (0,T), \Omega \subset \mathbb{R}^d \) and \( \alpha \in (0,1), \beta \in (1,2) \) are fractional orders of the time and space derivatives, respectively, \( T > 0 \) is a fixed final time. Moreover, \([43]\) consider the following time-space non-local fractional reaction-diffusion equation

\[
\partial_t^\alpha u(x,t) + (-\Delta)^s u(x,t) = -u(1-u), \quad x \in \Omega, t > 0,
\]

\[
u = 0, \quad x \in \mathbb{R}^n \setminus \Omega, t > 0
\]

\[
u(x,0) = u_0(x), \quad x \in \Omega.
\]

And for realistic initial conditions, studying global existence, blow-up in a finite time, asymptotic behavior of bounded solutions of equation \((1.7)\). In the process of blow-up of TSFNRDE, this paper uses the method of the proof of \([\text{blow-up}, 43]\). However, in addition to the existence and blow-up of time-space fractional diffusion equation mentioned above, we also further study the global boundedness and asymptotic behavior of solution for TSFNRDE.

The outline of this paper is as follows. The global boundedness for the solution of TSFNRDE is analyzed in section 2: First, we introduce the existence and uniqueness of solutions of TSFNRDE with homogeneous Dirichlet boundary condition by Appendix A.2. Secondly, The blow-up for the solution of TSFNRDE is analyzed in Lemma 2.16 for the initial data, blow-up in a finite time \( T_{\text{max}} \) satisfies bi-lateral estimate and when \( t \to T_{\text{max}} \), then
Finally, we use the analytical formula for the solution of TSFNRDE and use the fractional Sobolev inequality, Sobolev embedding inequality conversion to the solution of equation to obtain the global boundedness of equation (1.1)-(1.2). In section 3, we introduce the asymptotic behavior of solutions of TSFNRDE in Hilbert space. In section 4, under the condition of \( J = k = \mu = \gamma = 1 \), we first verify existence unique weak solution for problem (1.8)-(1.9). Next, we mainly use Gagliardo-Nirenberg inequality to prove that nonlinear TSFNRDE in \( L^r \) estimates and \( L^\infty \) estimates, which is global boundedness of solution in fractional Sobolev space with the nonlinear fractional diffusion terms \(-(-\Delta)^s u\).

**Theorem 1.1.** Suppose (1.6) holds, \( 0 < u_0 \in D((-\Delta)^s) \cap L^\infty(\mathbb{R}^N) \). Denote \( k^* = 0 \) for \( N = 1 \) and \( k^* = (\mu C_{GN}^2 + 1)\eta^{-1} \) for \( N = 2 \), \( C_{GN} \) is the constant appears in Gagliardo-Nirenberg inequality in Lemma 2.12 then for any \( k > k^* \), \( T > 0 \), the nonnegative weak solution of (1.1)-(1.2) exists and is globally bounded in time, that is, there exist

\[
K = \begin{cases} 
K(\|u_0\|_{L^\infty(\mathbb{R}^N)}), & N = 1, \\
K(\|u_0\|_{L^\infty(\mathbb{R}^N)}), & N = 2,
\end{cases}
\]

such that

\[
0 \leq u(x,t) \leq K, \quad \forall (x,t) \in \mathbb{R}^N \times (0, \infty).
\]

**Theorem 1.2.** Denote \( u(x,t) \) the globally bounded solution of (1.1)-(1.2).

(i). For any \( \gamma > 1 \), there exist \( \mu^* > 0 \) and \( m^* > 0 \) such that for \( \mu \in (0, \mu^*) \) and \( \|u_0\|_{L^\infty(\mathbb{R}^N)} < m^* \), we have

\[
\|u(x,t)\|_{L^\infty(\mathbb{R}^N \times [0,+,\infty))} \leq \frac{\tau}{\mu},
\]

and thus

\[
\|u(x,t)\|_{L^\infty(\mathbb{R}^N)} \leq \|u_0\|_{L^\infty(\mathbb{R}^N)} e^{-\alpha \|u(x,t)\|_{L^\infty(\mathbb{R}^N \times [0,+,\infty))}} > 0.
\]

(ii). If \( 1 < \gamma < \frac{4}{4k} \), there exist \( \mu^{**} > 0 \) and \( m^{**} > 0 \) such that for \( \mu \in (0, \mu^{**}) \) and \( \|u_0\|_{L^\infty(\mathbb{R}^N)} < m^{**} \), we have

\[
\|u(x,t)\|_{L^\infty(\mathbb{R}^N \times [0,+,\infty))} < a,
\]

and thus

\[
\lim_{t \to \infty} u(x, t) = 0
\]

locally uniformly in \( \mathbb{R}^N \).
Theorem 1.3.
\[
\frac{\partial^\alpha u}{\partial t^\alpha} = -(-\Delta)^s u^m + u^2(1 - \int_{\mathbb{R}^N} u dx) - u(x, t) \quad (1.8)
\]
\[
u(x, 0) = u_0(x) \quad x \in \mathbb{R}^N \quad (1.9)
\]

with \((x, t) \in \mathbb{R}^N \times (0, T] \). Then the problem (1.8)-(1.9) has a unique weak solution
\[
0 \leq u(x, t) \leq M, \quad (x, t) \in \mathbb{R}^N \times (0, T].
\]

2. Global boundedness of solutions for TSFNRDE

Definition 2.1. \([44]\) The Mittag-Leffler function in two parameters is defined as
\[
E_{\alpha, \beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C}
\]

where \(\alpha > 0, \beta > 0, \mathbb{C} \) denote the complex plane.

Lemma 2.1. \([45]\) If \(0 < \alpha < 1, \eta > 0\), then there is \(0 \leq E_{\alpha, \alpha}(-\eta) \leq \frac{1}{\Gamma(\alpha)}\). In addition, for \(\eta > 0\), \(E_{\alpha, \alpha}(-\eta)\) is a monotonically decreasing function.

Lemma 2.2. \([46]\) If \(0 < \alpha < 1, t > 0, w > 0\), for Mittag-Leffler function \(E_{\alpha, 1}(wt^\alpha)\), then there is a constant \(C\) such that
\[
E_{\alpha, 1}(wt^\alpha) \leq Ce^{w^\frac{1}{\alpha}t}. \quad (2.1)
\]

Lemma 2.3. \([44]\) Assume \(0 < \alpha < 2\), for any \(\beta \in \mathbb{R}\), there is a constant \(\mu\) such that \(\frac{\alpha}{2} < \mu < \min \{\pi, \pi \alpha\}\), then there is a constant \(c = c(\alpha, \beta, \mu) > 0\), such that
\[
|E_{\alpha, \beta}(z)| \leq \frac{c}{1 + |z|}, \quad \mu \leq |\text{arg}(z)| \leq \pi.
\]

Lemma 2.4. \([45]\) If \(0 < \alpha < 1, t > 0\), then there is \(0 < E_{\alpha, 1}(-t) < 1\). In addition, \(E_{\alpha, 1}(-t)\) is completely monotonous that is
\[
(-1)^n \frac{d^n}{dt^n} E_{\alpha, 1}(-t) \geq 0, \quad \forall n \in \mathbb{N}
\]
Lemma 2.5. \([47]\) Let \(0 < \alpha < 1\) and \(u \in C([0, T], \mathbb{R}^N), u' \in L^1(0, T; \mathbb{R}^N)\) and \(u\) be monotone. Then

\[
v(t)\partial_t^\alpha v(t) \geq \frac{1}{2} \partial_t^\alpha v^2(t), \quad t \in (0, T].
\]  

(2.2)

Lemma 2.6. \([47]\) Let \(0 \leq s \leq 1, x \in \mathbb{R}^N\) and \(u \in C^2_0(\mathbb{R}^N)\). Then the following inequality holds

\[
2u(-\Delta)^su(x) \geq (-\Delta)^su^2(x).
\]  

(2.3)

Lemma 2.7. \([48]\) Suppose \(u : [0, \infty) \times \mathbb{R}^N \to \mathbb{R},\) the left Caputo fractional derivative with respect to time \(t\) of \(u\) is defined by (1.5). Then there is

\[
\int_{\mathbb{R}^N} \partial_t^\alpha u dy = \partial_t^\alpha \int_{\mathbb{R}^N} u dy.
\]  

(2.4)

Definition 2.2. \([49]\) Assume that \(X\) is a Banach space and let \(u : [0, T] \to X\). The Caputo fractional derivative operators of \(u\) is defined by

\[
C_0^\alpha D_t^\alpha u(t) = \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - s)^{-\alpha} \frac{d}{ds} u(s) ds.
\]  

(2.5)

where \(\Gamma(1 - \alpha)\) is the Gamma function. The above integrals are called the left-sided and the right-sided the Caputo fractional derivatives.

Lemma 2.8. \([47]\) Let \(0 < \alpha < 1\) and \(u \in C([0, T], \mathbb{R}^N), u' \in L^1(0, T; \mathbb{R}^N)\) and \(u\) be monotone. And when \(n \geq 2,\) the Caputo fractional derivative with respect to time \(t\) of \(u\) is defined by (2.5). Then there is

\[
u_n^{-1}(C_0^\alpha D_t^\alpha u) \geq \frac{1}{n} (C_0^\alpha D_t^\alpha u^n).
\]

Lemma 2.9. \([48]\) Suppose \(u : [0, \infty) \times \mathbb{R}^N \to \mathbb{R},\) the Caputo fractional derivative with respect to time \(t\) of \(u\) is defined by (2.5). Then there is

\[
\int_{\mathbb{R}^N} (C_0^\alpha D_t^\alpha u) dy = C_0^\alpha D_t^\alpha \int_{\mathbb{R}^N} u dy.
\]

Lemma 2.10. \([48]\) Suppose that a nonnegative function \(y(t) \geq 0\) satisfies

\[
C_0^\alpha D_t^\alpha y(t) + c_1 y(t) \leq b
\]

for almost all \(t \in [0, T],\) where \(b, c_1 > 0\) are all constants. then

\[
y(t) \leq y(0) + \frac{bT^\alpha}{\alpha \Gamma(\alpha)}.
\]
Definition 2.3. \[50\] Let \( H^s(\mathbb{R}^N) \) denote the completion of \( C^\infty_0(\mathbb{R}^N) \) with respect to the Gagliardo norm
\[
[u]_{H^s} = \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{1/2}.
\]

Remark 2.1. The embedding \( H^s(\mathbb{R}^N) \hookrightarrow L^r \) is continuous, that is
\[
\|u\|_{L^r(\mathbb{R}^N)} \leq C_s \|u\|_{H^s(\mathbb{R}^N)}
\]
for all \( u \in H^s(\mathbb{R}^N) \), and \( C_s = c(N) \frac{a(1-a)}{(N-2s)} \) by Theorem 1 of \[51\].

Lemma 2.11. \[52\] Assume \( 1 \leq p < N \), then \( \forall f \in C^\infty_c(\mathbb{R}^N) \), there is
\[
\|f\|_{L^q(\mathbb{R}^N)} \leq C_1 \|\nabla f\|_{L^p(\mathbb{R}^N)}.
\]
Which is \( q = \frac{Np}{N-p} \), and \( C_1 \) only dependent on \( p, N \).

Lemma 2.12. \[53\] Let \( \Omega \) be an open subset of \( \mathbb{R}^N \), assume that \( 1 \leq p,q \leq \infty \) with \( (N-q)p < Nq \) and \( r \in (0,p) \). Then there exists constant \( C_{GN} > 0 \) only depending on \( q,r \) and \( \Omega \) such that for any \( u \in W^{1,q}(\Omega) \cap L^p(\Omega) \)
\[
\int_{\Omega} u^p dx \leq C_{GN} (\|\nabla u\|_{L^q(\Omega)}^{(\lambda^*)p} \|u\|_{L^p(\Omega)}^{(1-\lambda^*)p} + \|u\|_{L^p(\Omega)}^p)
\]
holds with
\[
\lambda^* = \frac{\frac{N}{r} - \frac{N}{p}}{1 - \frac{N}{q} + \frac{N}{r}} \in (0,1).
\]

Definition 2.4. \[44\] The Mittag-Leffler function in two parameters is defined as
\[
E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C}
\]
where \( \alpha > 0, \beta > 0, \mathbb{C} \) denote the complex plane.

Lemma 2.13. \[54\] \( \forall a, b \geq 0 \) and \( \varepsilon > 0 \), for \( 1 < p, q < \infty, \frac{1}{p} + \frac{1}{q} = 1 \), then there is
\[
a \cdot b \leq \varepsilon \frac{ap}{p} + \varepsilon \frac{bq}{q}.
\]
Definition 2.5. A function \( u \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; D((-\Delta)^s)) \) is said to be a weak solution to equation (1.1)-(1.2) if the following conditions hold

- \( \partial_t^su(t) + (-\Delta)^su(t) = F(t) \) holds in \( L^2(\Omega) \) for \( t \in (0, T] \);
- \( \partial_t^su(t) \in C((0, T]; L^2(\Omega)) \cap L^2(0, T; L^2(\Omega)) \);
- \( \lim_{t \to 0^+} \|u(t) - u_0\| = 0 \).

Lemma 2.14. Let \( 0 < \alpha < 1, u_0 \in D((-\Delta)^s), F(t) = \mu u^2(1-kJ^s u) - \gamma u \in L^\infty(0, T; D((-\Delta)^s)) \). Then there exists a unique weak solution to the problem (1.1)-(1.2) and the solution is given by

\[
    u(t) = \sum_{j=1}^{\infty} E_{\alpha,1}(-\lambda_j t^\alpha) u_{0,j} \Phi_j + \sum_{j=1}^{\infty} \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_j(t-\tau)^\alpha) F_j d\tau \Phi_j. \tag{2.8}
\]

Here \( u_{0,j} = (u_0, \Phi_j) \) and \( F_j = (F(\tau), \Phi_j) \).

Remark 2.2. The conclusion of the Lemma 2.14 can be concluded in reference [5]. But in [5], “For brevity, we leave the detail to the reader.” [p255, proof of Theorem 1, [5]]. We have done the above related proof (See Appendix A.2). Therefore, we will use the properties of the eigensystem for the operator \((-\Delta)^s\) and use the method as in the proof of [37, Theorems 2.1-2.2] or [38, Theorem 3.2] to obtain the existence and uniqueness of the solution (2.9) to the problem.

Remark 2.3. The solution of the above equation (1.1)-(1.2) is also expressed in [55]. And we set

\[
    U(t) u_0 = \sum_{j=1}^{\infty} E_{\alpha,1}(-\lambda_j t^\alpha) u_{0,j} \Phi_j,
\]

and

\[
    V(t) F = \sum_{j=1}^{\infty} t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_j t^\alpha) F_j \Phi_j,
\]

then equation (2.8) can be written in the following form

\[
    u(t) = U(t) u_0 + \int_0^t V(t - \tau) F(\tau) d\tau. \tag{2.9}
\]
Definition 2.6. Suppose eigenfunction $e_1 > 0$ associated to the first eigenvalue $\lambda_1 > 0$ that satisfies the fractional eigenvalue problem
\begin{equation}
(-\Delta)^se_1(x) = \lambda_1 e_1(x), \quad x \in \Omega, 
\end{equation}
\begin{equation}
e_1(x) = 0, \quad x \in \mathbb{R}^N \setminus \Omega,
\end{equation}
normamlized such that $\int_{\Omega} e_1(x) dx = 1$.

Lemma 2.15. The Caputo fractional derivative of an absolutely continuous function $w(t)$ of order $0 < \alpha < 1$ is defined by (2.5). The function $w$ satisfies
\begin{equation}
\begin{cases}
_{0}^{\mathcal{C}}D_{t}^{\alpha}w(t) = w(t)(1 + w(t)), \\
w(0) = w_0.
\end{cases}
\end{equation}
Then, we get the following formula for the solution of problem (2.11)
\begin{equation}
w(t) = E_{\alpha}(t^{\alpha})w_0 + \int_{0}^{t}(t - s)^{\alpha - 1}E_{\alpha,\alpha}((t - s)^{\alpha})w^2(s)ds.
\end{equation}

Lemma 2.16. Assume the initial data $0 < u_0 \in D((-\Delta)^s) < 1$. Then there are a maximal existence time $T_{max} \in (0, \infty)$, $u \in C([0, T_{max}]; H^s(\mathbb{R}^N)) \cap L^2(0, T_{max}; D((-\Delta)^s))$ such that $u$ is the unique nonnegative weak solution of (1.1)-(1.2). Furthermore, if $1 + \lambda_1 \leq \int_{\Omega} u_0(x)e_1(x)dx = H_0$, and $\lambda_1, e_1(x)$ is the value in Definition 2.6 then the solution of problem (1.1)-(1.2) blow-up in a finite time $T_{max}$ that satisfies the bi-lateral estimate
\begin{equation}
\left(\frac{\Gamma(\alpha + 1)}{4(H_0 + 1/2)}\right)^{\frac{1}{\alpha}} \leq T_{max} \leq \left(\frac{\Gamma(\alpha + 1)}{H_0}\right)^{\frac{1}{\alpha}}.
\end{equation}
Furthermore, if $T_{max} < \infty$, then
\begin{equation}
\lim_{t \to T_{max}} \| u(\cdot, t) \|_{L^\infty(\mathbb{R}^N)} = \infty.
\end{equation}

Proof. Multiplying equations (1.1) by $e_1(x)$ and integrating over $\Omega$, we obtain
\begin{equation}
\begin{aligned}
\partial_t^{\alpha} \int_{\Omega} u(x, t)e_1(x)dx + \int_{\Omega} (-\Delta)^s u(x, t)e_1(x)dx \\
= \int_{\Omega} [\mu(1 - kJ * u)u(x, t) - \gamma]u(x, t)e_1(x)dx.
\end{aligned}
\end{equation}
By (2.10), we have
\[
\int_{\Omega} (-\Delta)^s u(x,t) e_1(x) dx = \int_{\Omega} u(x,t) (-\Delta)^s e_1(x) dx = \lambda_1 \int_{\Omega} u(x,t) e_1(x) dx,
\]
as \( u = 0, e_1(x) = 0, x \in \mathbb{R}^N \setminus \Omega \) and
\[
\left( \int_{\Omega} u(x,t) e_1(x) dx \right)^2 \leq \int_{\Omega} u^2(x,t) e_1(x) dx,
\]
let function \( H(t) = \int_{\Omega} u(x,t) e_1(x) dx \), then satisfies
\[
\partial_t^\alpha \tilde{H}(t) + (\gamma + \lambda_1) \tilde{H}(t) \geq \mu \tilde{H}(t)^2. \tag{2.14}
\]
Let \( \bar{H}(t) = H(t) - (\gamma + \lambda_1) \) and \( \mu(1 - kJ * u) > 1, \lambda_1 > 0, \gamma > 1 \), then from (2.14), we get
\[
\partial_t^\alpha \tilde{H}(t) \geq \tilde{H}(t)(\tilde{H}(t) + \gamma + \lambda_1) \geq \tilde{H}(t)(\tilde{H}(t) + 1). \tag{2.15}
\]
It is seen that if \( u(t) \to \infty \) as \( t \to T_{\text{max}} \), then \( H(t) \to \infty \) as \( t \to T_{\text{max}} \) and vice versa. That is \( H \) and \( u \) will have the same blow-up time. As \( 0 \leq \tilde{H}_0 = \tilde{H}(0) \), then from the results in [57], the solution of inequality (2.15) blows-up in a finite time. The proof of the Lemma is complete.

**Remark 2.4.** We will use the method as in the proof of [43, Theorem 2.1] and [57, Theorem 3.2]. In addition, in the process of proof, we also need to use comparative principle to proof of blow-up in a finite time.

**Proof of Theorem 1.1.** We first know that the equation (1.1)-(1.2) has the only weak solution by Lemma 2.14. Next, we will prove the global boundary of this weak solution in one dimensional and two dimensional space. For any \( x \in \mathbb{R}^N \), multiply (1.1) by \( 2u\varphi_\varepsilon \), where \( \varphi_\varepsilon(\cdot) \in C_0^\infty(\mathbb{R}^N) \), and \( \varphi_\varepsilon(\cdot) \to 1 \) locally uniformly in \( \mathbb{R}^N \) as \( \varepsilon \to 0 \). Integrating by parts over \( \mathbb{R}^N \), we obtain
\[
\int_{\mathbb{R}^N} 2u\varphi_\varepsilon \frac{\partial^\alpha u}{\partial t^\alpha} dy = -\int_{\mathbb{R}^N} 2u\varphi_\varepsilon (-\Delta)^s u dy + \int_{\mathbb{R}^N} 2u^3 \mu \varphi_\varepsilon (1 - kJ * u) dy
\]
\[= -2\gamma \int_{\mathbb{R}^N} u^2 \varphi_\varepsilon dy,
\]
by Lemma 2.5 and Lemma 2.7, then we can get
\[ \int_{\mathbb{R}^N} 2u\varphi_\varepsilon \frac{\partial^\alpha}{\partial t^\alpha} u \, dy \geq \int_{\mathbb{R}^N} u^2 \varphi_\varepsilon \, dy, \]
and by fractional Laplacian (1.4) and (2.3), then we can get
\[ \int_{\mathbb{R}^N} 2u\varphi_\varepsilon (-\Delta)^s u \, dy \geq \int_{\mathbb{R}^N} \varphi_\varepsilon (-\Delta)^s u^2 \, dy, \quad (2.16) \]
and a symmetrical estimate of the nuclear function can be obtained (2.16)
\[ \int_{\mathbb{R}^N} \varphi_\varepsilon (-\Delta)^s u^2 \, dy \]
\[ = C_{N,s} P.V. \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \varphi_\varepsilon \frac{u^2(x,t) - u^2(y,t)}{|x-y|^{N+2s}} \, dxdy \]
\[ \geq C_{N,s} P.V. \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \varphi_\varepsilon \frac{(u(x,t) - u(y,t))^2}{|x-y|^{N+2s}} \, dxdy = C_{N,s} P.V. [u]^2_{H^s}. \]
By definition 2.3 and Taking \( \varepsilon \to 0 \), we obtain
\[ \frac{\partial^\alpha}{\partial t^\alpha} \int_{\mathbb{R}^N} u^2 \, dy + C_{N,s} P.V. [u]^2_{H^s} \leq 2\mu \int_{\mathbb{R}^N} u^3 (1 - \eta J * u) \, dy - 2\gamma \int_{\mathbb{R}^N} u^2 \, dy \]
Knowing from (1.6), we have used the fact that \( \forall y, z \in \mathbb{R}^N \), then \( J(z-y) \geq \eta \), and then
\[ J * u(y,t) = \int_{\mathbb{R}^N} J(y-z)u(z,t) \, dz \geq \eta \int_{\mathbb{R}^N} u(y,t) \, dy \]
Therefore
\[ \frac{\partial^\alpha}{\partial t^\alpha} \int_{\mathbb{R}^N} u^2 \, dy + C_{N,s} P.V. [u]^2_{H^s} \]
\[ \leq 2\mu \int_{\mathbb{R}^N} u^3 \, dy - 2\mu \eta k \int_{\mathbb{R}^N} u^3 \, dy \int_{\mathbb{R}^N} u \, dy - 2\gamma \int_{\mathbb{R}^N} u^2 \, dy. \quad (2.17) \]
Now we proceed to estimate the term \( \int_{\mathbb{R}^N} u^3 \, dy \).
Firstly using Gagliardo-Nirenberg inequality in Lemma 2.12 let \( p = 3, q = r = 2, \lambda^s = \frac{N}{6} \), there exists constant \( C_{GN} > 0 \), such that
\[ \int_{\mathbb{R}^N} u^3 \, dy \leq C_{GN}(N)(\| \nabla u \|_{L^2(\mathbb{R}^N)}^\frac{N}{6} + \| u \|_{L^2(\mathbb{R}^N)}^{3 - \frac{N}{6}}), \quad (2.18) \]
On the one hand, by Lemma 2.13 we can make $a = \|\nabla u\|_{L^2(\mathbb{R}^N)}^{\frac{N}{2}-\frac{N}{2}}$, $\varepsilon = \frac{1}{2N}$, $p = \frac{4}{3N}$, $q = \frac{4}{4-N}$, then obtain

$$C_{GN}(N)\|\nabla u\|_{L^2(\mathbb{R}^N)}^{\frac{N}{2}-\frac{N}{2}} \leq \frac{1}{\mu} \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 + \mu \frac{4}{N} C_{4GN}^4 (N) \|u\|_{L^2(\mathbb{R}^N)}^{\frac{2(6-N)}{4-N}}.$$  \hspace{1cm} (2.19)

Here by Young’s inequality, let $a = C_{GN}(N), b = \|u\|_{L^2(\mathbb{R}^N)}^3, p = \frac{2(6-N)}{3(4-N)}, q = \frac{2(6-N)}{N}$, then

$$C_{GN}(N) \|u\|_{L^2(\mathbb{R}^N)}^3 \leq \|u\|_{L^2(\mathbb{R}^N)}^{\frac{2(6-N)}{4-N}} + \frac{\mu}{N} C_{GN}^4 (1).$$  \hspace{1cm} (2.20)

By interpolation inequality, we obtain

$$\|u\|_{L^2(\mathbb{R}^N)}^3 \leq \left( \|u\|_{L^1(\mathbb{R}^N)}^{\frac{1}{2}} \|u\|_{L^3(\mathbb{R}^N)}^{\frac{3}{2}} \right)^{\frac{6-N}{4(4-N)}}.$$  \hspace{1cm} (2.21)

Combing (2.18)-(2.21), we obtain

$$2\mu \int_{\mathbb{R}^N} u^3 dy \leq 2 \int_{\mathbb{R}^N} |\nabla u|^2 dy + 2\mu (1 + \mu \frac{N}{4-N} C_{4GN}^4 (N)) \left( \int_{\mathbb{R}^N} u^3 dy \int_{\mathbb{R}^N} u dy \right)^{\frac{6-N}{2(4-N)}} + 2\mu C_{GN}^4 (1).$$  \hspace{1cm} (2.22)

Next we consider the cases $N = 1$ and $N = 2$ respectively.

**Case 1.** $N = 1$. From (2.22), by Lemma 2.13 let $\varepsilon = \eta k, p = \frac{6}{5}, q = 6$, we get

$$2\mu \int_{\mathbb{R}^N} u^3 dy \leq 2 \int_{\mathbb{R}^N} |\nabla u|^2 dy + 2\mu \eta k \int_{\mathbb{R}^N} u^3 dy \int_{\mathbb{R}^N} u dy + 2\mu (\mu \frac{4}{5} C_{4GN}^4 (1))^6 (\eta k)^{-5} + 2\eta C_{GN}^{10} (1),$$  \hspace{1cm} (2.23)

inserting (2.23) into (2.17), we have

$$\frac{\partial^\alpha}{\partial t^\alpha} \int_{\mathbb{R}^N} u^2 dy + C_{N,s} P.V.[u]_{H^s}^2 + 2\gamma \int_{\mathbb{R}^N} u^2 dy \leq 2 \int_{\mathbb{R}^N} |\nabla u|^2 dy + 2\mu \eta k (\mu \frac{4}{5} C_{4GN}^4 (1))^6 (\eta k)^{-5} + C_{GN}^{10} (1).$$  \hspace{1cm} (2.24)
From Sobolev embedding inequality (Lemma 2.11) and proof of Theorem 1 in [58], we know there exists an embedding constant \( C_2 > 0 \) such that
\[
\| \nabla u \|_{L^p(\mathbb{R}^N)} \geq C_1 \| u \|_{L^2(\mathbb{R}^N)}^2
\]
where \( s_2 \geq p \) will be determined later. Here we set \( s_2 = p = 2 \), then there is
\[
\| \nabla u \|_{L^2(\mathbb{R}^N)} \geq C_2 \| u \|_{L^2(\mathbb{R}^N)}^2.
\]
By fractional Sobolev inequality (2.6), we can obtain
\[
C_{N,s} P.V. [u]_{H^s}^2 \geq C_{N,s} P.V. C^* \| u \|_{L^2(\mathbb{R}^N)}^2
\]
for all \( u \in H^s(\mathbb{R}^N) \). Let
\[
C^{-2} = \frac{C_{N,s} P.V.}{C_*}, \quad Q_1 = 2 \mu \left( (\mu + C^4_{GN}(1))^6 (\eta k)^{-5} + C^{10}_{GN}(1) \right)
\]
then, the equation (2.24) is configured above, you can get
\[
\frac{\partial^\alpha}{\partial t^\alpha} \int_{\mathbb{R}^N} u^2 dy + C^{-2} \int_{\mathbb{R}^N} u^2 dy + 2\gamma \int_{\mathbb{R}^N} u^2 dy \leq C_2 \int_{\mathbb{R}^N} u^2 dy + Q_1.
\]
Denote \( w(t) = \int_{\mathbb{R}^N} u^2 dy \), the solution of the following fractional differential equation
\[
\left\{
\begin{array}{l}
\partial^\alpha t^\alpha \int_{\mathbb{R}^N} u^2 dy + C^{-2} \int_{\mathbb{R}^N} u^2 dy + 2\gamma \int_{\mathbb{R}^N} u^2 dy \leq C_2 \int_{\mathbb{R}^N} u^2 dy + Q_1 \\
w(0) = 2\delta \| u_0 \|_{L^\infty(\mathbb{R}^N)}^2
\end{array}\right.
\]
for \( \forall (x,t) \in \mathbb{R} \times (0,T_{\text{max}}) \) and from Lemma 2.10 let
\[
C_4 = 2\gamma + C^{-2} - C_2
\]
and \( w(0) = \eta = 2\delta \| u_0 \|_{L^\infty(\mathbb{R}^N)}^2 \), we obtain
\[
\int_{\mathbb{R}^N} u^2 dy = \| u \|_{L^2(\mathbb{R}^N)}^2 \leq \| w(t) \|
\]
\[
\leq w(0) + \frac{(K + |Q_1|) T^\alpha}{\alpha \Gamma(\alpha)} C e^{(\mu + C^4_{GN}(1))^6 (\eta k)^{-5} + C^{10}_{GN}(1)}}T^\alpha
\]
\[
\leq 2\delta \| u_0 \|_{L^\infty(\mathbb{R}^N)}^2 + \frac{(K + |2\mu (\mu + C^4_{GN}(1))^6 (\eta k)^{-5} + C^{10}_{GN}(1))) T^\alpha}{\alpha \Gamma(\alpha)} C
\]
\[:= M_1.\]
Case 2. $N = 2$. From (2.22), we have

$$2\mu \int_{\mathbb{R}^N} u^3 dy \leq 2 \int_{\mathbb{R}^N} |\nabla u|^2 dy + 2\mu(1 + \mu C_{GN}^2(2))$$

$$+ \left( \int_{\mathbb{R}^N} u^3 dy \int_{\mathbb{R}^N} u dy \right) + 2\mu C_{GN}^4(2).$$

(2.25)

Inserting (2.25) into (2.17), for $k \geq k^* = \frac{\mu C_{GN}^2(2) + 1}{\eta}$, we obtain

$$\partial_\alpha \frac{\partial}{\partial t} \int_{\mathbb{R}^N} u^2 dy + C_{*}^{-2} \int_{\mathbb{R}^N} u^2 dy + 2\gamma \int_{\mathbb{R}^N} u^2 dy \leq C_2 \int_{\mathbb{R}^N} u^2 dy + 2\mu C_{GN}^4(2).$$

Denote $w(t) = \int_{\mathbb{R}^N} u^2 dy$ the solution of the following ordinary differential equation

$$\begin{cases}
C_0 D_\alpha^t w(t) + (2\gamma + C_{*}^{-2} - C_2) w(t) = 2\mu C_{GN}^4(2), \\
w(0) = (2\delta)^2 \|u_0\|^2_{L^\infty(\mathbb{R}^N)},
\end{cases}$$

(2.26)

for $\forall (x, t) \in \mathbb{R}^2 \times (0, T_{\text{max}})$ and let $C_5 = 2\gamma + C_{*}^{-2} - C_2$, we obtain

$$\int_{\mathbb{R}^N} u^2 dy = \|u\|^2_{L^2(\mathbb{R}^N)} \leq \|w(t)\|$$

$$\leq w(0) + \frac{(K + |2\mu C_{GN}^4(2)|T^\alpha)}{\alpha \Gamma(\alpha)} C e^{(-C_5) \frac{1}{\alpha} + \sigma|t - \sigma T}}$$

$$\leq (2\delta)^2 \|u_0\|^2_{L^\infty(\mathbb{R}^N)} + \frac{(K + |2\mu C_{GN}^4(2)|T^\alpha)}{\alpha \Gamma(\alpha)} C$$

$$:= M_2.$$  

In conclusion, for any $(x, t) \in \mathbb{R}^N \times (0, T_{\text{max}})$, we have

$$\|u\|^2_{L^2(\mathbb{R}^N)} \leq M = \begin{cases}
\sqrt{M_1} & N = 1 \\
\sqrt{M_2} & N = 2
\end{cases}$$

(2.27)

and then

$$\|u\|^2_{L^1(\mathbb{R}^N)} \leq M.$$  

(2.28)

Now we proceed to improve the $L^2$ boundedness of $u$ to $L^\infty$, which is based on the fact that for all $(x, t) \in \mathbb{R}^N \times (0, T_{\text{max}})$ and equation (2.9), let $F(t) = \mu u^2(1 - kJ * u) - \gamma u$, then we have the solution of equation (1.1)-(1.2)

$$u(x, t) = U(t) u_0 + \int_0^t V(t - r) F(r) dr.$$
By [55] and Lemma 2.3, we can obtain
\[
\|U(t)u_0\| = \left\| \sum_{j=1}^{\infty} E_{\alpha,1}(-\lambda_j^* t^\alpha)u_{0,j}\Phi_j \right\| \leq \sum_{j=1}^{\infty} C_6(u_0, \Phi_j) \leq C_6 \|u_0\|.
\]

From using Lemma 2.3 and Parseval identity, we have for some function \(F(t) \in L^2(\mathbb{R}^N)\)
\[
\|V(t-r)F(r)\| = \left\| \sum_{j=1}^{\infty} (t-r)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_j^* (t-r)^\alpha)F_j \Phi_j \right\|
\leq C_7 \left( \sum_{j=1}^{\infty} (t-r)^{2\alpha-2} \frac{1}{1 + \lambda_j^* (t-r)^\alpha} \right)^{\frac{1}{2}}
\leq C_7 \left( (t-r)^{2\alpha-2} \right)^{\frac{1}{2}} \left( \sum_{j=0}^{\infty} |F_j| \right)^{\frac{1}{2}} \left( \sum_{j=0}^{\infty} (t-r)^{\frac{2}{\alpha}} \right)^{\frac{1}{2}}
\leq C_7(t-r)^{\frac{2}{\alpha}-1} \|F(r)\|.
\]
So there is
\[
0 \leq u(x,t) \leq \|U(t)u_0\| + \int_0^t \|V(t-s)F(s)\| \, ds
\leq C_6 \|u_0\|_{L^\infty(\mathbb{R}^N)} + C_7 \int_0^t (t-s)^{\frac{2}{\alpha}-1} \|F(s)\|_{L^\infty(\mathbb{R}^N)} \, ds
\leq C_6 \|u_0\|_{L^\infty(\mathbb{R}^N)} + \mu C_7 \int_0^t (t-s)^{\frac{2}{\alpha}-1} \|u^2(s)\|_{L^\infty(\mathbb{R}^N)} \, ds
\leq C_6 \|u_0\|_{L^\infty(\mathbb{R}^N)} + \mu C_7 M^2 \int_0^t (t-s)^{\frac{2}{\alpha}-1} \, ds
\leq C_6 \|u_0\|_{L^\infty(\mathbb{R}^N)} + \mu C_8 M^2 T^{\frac{2}{\alpha}} \frac{1}{\frac{2}{\alpha}}.
\]

\[
0 \leq u(x,t) \leq \|u_0\|_{L^\infty(\mathbb{R}^N)} + \mu M^2 C_2(N,T,\gamma), \quad (2.29)
\]

with
\[
M = \begin{cases} 
\sqrt{2\delta \|u_0\|^2_{L^\infty(\mathbb{R}^N)} + \frac{(K+2\mu(\mu C^2_{\alpha N}(1)+1/\gamma)^{\alpha})}{\alpha \Gamma(\alpha)} T^\alpha} C, & N = 1 \\
(2\delta)^2 \|u_0\|^2_{L^\infty(\mathbb{R}^N)} + \frac{(K+2\mu C^2_{\alpha N}(2))}{\alpha \Gamma(\alpha)} T^\alpha C, & N = 2
\end{cases}
\]
defined in (2.27). As a summary, we have proved that the solution $u$ is globally bounded in time, the blow-up criterion in Lemma 2.16 shows that $u$ is the unique weak solution of (1.1)-(1.2) on $(x, t) \in \mathbb{R}^N \times (0, +\infty)$. Theorem 1.1 is thus proved.

3. Long time behavior of solutions

Now, we consider the long time behavior of the weak solution of (1.1)-(1.2). To study the long time behavior of solutions for (1.1)-(1.2), by [4], we denote 

$$F(u) := \mu u^2 (1 - kJ \ast u) - \gamma u.$$ 

For $1 < \gamma < \frac{\mu}{4k}$, there are three constant solutions for $F(u) = 0$: $0, a, A$, where

$$a = \frac{1 - \sqrt{1 - \frac{4k^2}{\mu^2} \gamma}}{2k}, \quad A = \frac{1 + \sqrt{1 - \frac{4k^2}{\mu^2} \gamma}}{2k},$$

and satisfy $1 < \frac{\gamma}{\mu} < a < A$.

**Lemma 3.1.** Let $u \in C^2_0(\mathbb{R}^N)$ and $\Phi$ be a convex function of one variable. Then

$$\Phi'(u) (-\Delta)^s u(x) \geq (-\Delta)^s \Phi(u(x)).$$

**Proposition 3.1.** Under the assumptions of Theorem 1.1 there is $\|u(x, t)\|_{L^\infty(\mathbb{R}^N \times (0, +\infty))} < a$, the function

$$H(x, t) = \int_{B(x, \delta)} h(u(y, t)) dy$$

with

$$h(u) = A \ln \left(1 - \frac{u}{A}\right) - a \ln \left(1 - \frac{u}{a}\right)$$

is nonnegative and satisfies

$$\frac{\partial^\alpha H(x, t)}{\partial t^\alpha} \leq -(-\Delta)^s H(x, t) + \int_{B(x, \delta)} |\nabla u(y, t)|^2 \, dy - D(x, t) \tag{3.2}$$

with

$$D(x, t) = \frac{1}{2} (A - a) \mu k \int_{B(x, \delta)} u^2(y, t) \, dy.$$
Proof. Fix \( x_0 \triangleq (x_0^0, \cdots, x_0^N) \in \mathbb{R}^N \), choose \( 0 < \delta < \frac{1}{2} \delta \), and denote
\[
B(x, \delta) := \left\{ x \triangleq (x_1, \cdots, x_N) \in \mathbb{R}^N \mid |x_i - x_0^i| \leq \delta, 1 \leq i \leq N \right\}.
\]

Let \( k = \|u(x, t)\|_{L^\infty(\mathbb{R}^N \times [0, \infty))} \), then \( 0 < k < a \). From the definition of \( h(\cdot) \), it is easy to verify that
\[
h'(u) = \frac{a}{a-u} - \frac{A}{A-u} = \frac{(A-a)u}{(A-u)(a-u)}. \tag{3.3}
\]
and
\[
h''(u) = \frac{a}{(a-u)^2} - \frac{A}{(A-a)^2} = \frac{(A-a^2)(A-a)}{(A-u)^2(a-u)^2}. \tag{3.4}
\]

Test (1.1) by \( h'(u)\varphi_\varepsilon \) with \( \varphi_\varepsilon(\cdot) \in C_0^\infty(B(x, \delta)) \), \( \varphi_\varepsilon(\cdot) \to 1 \) in \( B(x, \delta) \) as \( \varepsilon \to 0 \). Integrating by parts over \( B(x, \delta) \). From Definition 2.3 and Lemma 3.1 we obtain
\[
h'(u)(-\Delta)^s u(x) \geq (-\Delta)^s h(u(x)).
\]

By using Lagrange mean value theorem
\[
(-\Delta)^s \int_{B(x, \delta)} h(u)dy = C_{N,s}P.V. \int_{B(x, \delta)} \left| \int_{B(x, \delta)} h(u(x))dx - \int_{B(x, \delta)} h(u(y))dy \right| dy
\]
\[
= C_{N,s}P.V. \int_{B(x, \delta)} \int_{B(x, \delta)} \frac{|h(u(x)) - h(u(y))|}{|u(x) - u(y)|^{N+2s}} dxdy
\]
\[
\leq \int_{B(x, \delta)} (-\Delta)^s h(u)dy.
\]

From Lemma 2.7 and Equation (1.4), we get
\[
\int_{B(x, \delta)} h'(u)\partial^a_i udy \geq \int_{B(x, \delta)} \partial^a_i h(u)dy = \partial^a_i \int_{B(x, \delta)} h(u)dy.
\]

Then
\[
\frac{\partial^a}{\partial t^a} \int_{B(x, \delta)} h(u)\varphi_\varepsilon dy
\]
\[
\leq -\int_{B(x, \delta)} h'(u)(-\Delta)^s u\varphi_\varepsilon dy + \int_{B(x, \delta)} [\mu u^2(1 - kJ * u) - \gamma u]h'(u)\varphi_\varepsilon dy
\]
\[
\leq -(-\Delta)^s \int_{B(x, \delta)} \varphi_\varepsilon h(u)dy + \int_{B(x, \delta)} [\mu u^2(1 - kJ * u) - \gamma u]h'(u)\varphi_\varepsilon dy.
\]
Taking $\varepsilon \to 0$, we obtain

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}} \int_{B(x,\delta)} h(u) dy \leq -\int_{B(x,\delta)} (-\Delta)^{\alpha} h(u) dy + \int_{B(x,\delta)} [\mu u^2(1 - kJ * u) - \gamma u] h'(u) dy \leq -(-\Delta)^{\alpha} \int_{B(x,\delta)} h(u) dy + \int_{B(x,\delta)} [\mu u^2(1 - kJ * u) - \gamma u] h'(u) dy,$$

which is

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}} H(x,t) \leq -(-\Delta)^{\alpha} H(x,t) + \int_{B(x,\delta)} [\mu u^2(1 - kJ * u) - \gamma u] h'(u) dy. \quad (3.5)$$

By (3.1), we can get

$$\mu u^2(1 - ku) - \gamma u = k\mu u(A - u)(u - a),$$

and

$$\int_{B(x,\delta)} h'(u)[\mu u^2(1 - kJ * u) - \gamma u] dy = \int_{B(x,\delta)} h'(u)[\mu u^2(1 - ku) - \gamma u] dy + \mu k \int_{B(x,\delta)} h'(u)u^2(u - J * u) dy \quad (3.6)$$

$$= -(A - a)\mu k \int_{B(x,\delta)} u^2(y, t) dy + \mu k \int_{B(x,\delta)} h'(u)u^2(u - J * u) dy.$$

Noticing that when $0 \leq u \leq k$, there is

$$0 \leq h'(u)u \leq \frac{(A - a)k^2}{(A - k)(a - k)}.$$
From Young’s inequality and the median value theorem, we can get

\[\mu k \int_{\theta} h'(u)u^2(u - J * u)dy\]

\[\leq \mu k \int_{\theta} h'(u)u^2(y, t)(u(y, t) - u(z, t))J(y - z)dzdy\]

\[\leq \frac{(A - a)K^2}{(A - K)(a - K)} \mu k \int_{\theta} (u(y, t) - u(z, t))J(y, z)dzdy\]

\[\leq \frac{(A - a)K^2}{2(A - K)^2(a - K)^2} \mu k \int_{\theta} (u(z, t) - u(y, t))^2 J(z - y)dzdy\]

\[+ \frac{1}{2}(A - a)\mu k \int_{\theta} u^2(y, t)dy\]

\[\leq \frac{(A - a)K^2}{2(A - K)^2(a - K)^2} \mu k \int_{\theta} \int_{\theta} |\nabla u(y + \theta(z - y), t)|^2 |z - y|^2 J(z - y)dz\]

changing the variables \(y' = y + \theta(z - y), z' = z - y\), then

\[\left| \frac{\partial y'}{\partial y}, \frac{\partial z'}{\partial z} \right| = \left| \begin{matrix} 1 - \theta & \theta \\ -1 & 1 \end{matrix} \right| = 1 - \theta + \theta = 1.\]

For any \(\theta \in [0, 1], y, z \in B(x, \delta)\), we have \(y' \in B((1 - \theta)x + \theta z, (1 - \theta)\delta), z' \in B(x - y, \delta)\). Noticing \(B((1 - \theta)x + \theta z, (1 - \theta)\delta) \subseteq B(x, \delta)\) and \(B(x - y, \delta) \subseteq B(0, 2\delta)\), we obtain

\[\frac{(A - a)K^2}{2(A - K)^2(a - K)^2} \mu k \int_{\theta} \int_{\theta} |\nabla u(y + \theta(z - y), t)|^2 |z - y|^2 J(z - y)dz\]

\[\leq \frac{(A - a)K^2}{2(A - K)^2(a - K)^2} \mu k \int_{\theta} \int_{\theta} |\nabla u(y', t)|^2 |z'|^2 J(z')dz'dy'\]

\[\leq \frac{(A - a)K^2}{2(A - K)^2(a - K)^2} \mu k (2\delta)^2 \int_{\theta} |\nabla u(y, t)|^2 dy.\]

(3.8)
Combining (3.6)-(3.8), we obtain

\begin{align*}
\int_{B(x,\delta)} h'(u)|\mu u^2(1-kJ\ast u) - \gamma u| dy & \leq -\frac{1}{2}(A-a)\mu k \int_{B(x,\delta)} u^2(y,t) dy \\
& \quad + \frac{(A-a)k^4\mu k(2\delta)^2}{2(A-a)^2(a-k)^2} \int_{B(x,\delta)} |\nabla u(y,t)|^2 dy.
\end{align*}

(3.9)

From (3.4), noticing $0 \leq u < a$, we obtain

$$h''(u) > \frac{(A-a)^2}{A^2a},$$

inserting (3.9) into (3.5), we obtain

\begin{align*}
\frac{\partial^\alpha}{\partial t^\alpha} H(x,t) & \leq -(-\Delta)^s H(x,t) + \frac{(A-a)k^4\mu k(2\delta)^2}{2(A-k)^2(a-k)^2} \int_{B(x,\delta)} |\nabla u(y,t)|^2 dy \\
& \quad - \frac{1}{2}(A-a)\mu k \int_{B(x,\delta)} u^2(y,t) dy.
\end{align*}

(3.10)

By choosing $\delta$ sufficiently small such that

$$\frac{(A-a)k^4\mu k(2\delta)^2}{2(A-k)^2(a-k)^2} \leq 1,$$

then

$$\frac{\partial^\alpha}{\partial t^\alpha} H(x,t) \leq -(-\Delta)^s H(x,t) + \int_{B(x,\delta)} |\nabla u(y,t)|^2 dy - D(x,t),$$

making

$$D(x,t) = \frac{1}{2}(A-a)\mu k \int_{B(x,\delta)} u^2(y,t) dy.$$

**Lemma 3.2.** [55][59] Assume $T > 0$ is a final time, $0 < \alpha, s < 1$ and $f$ is a given function. Let $u(x,t)$ is the solution of the following one-dimensional space-time fractional diffusion problem

\begin{align*}
\left\{ \begin{array}{ll}
\frac{\partial^\alpha}{\partial t^\alpha} u(x,t) = -(-\Delta)^s u(x,t), & -1 < x < 1, 0 < t < T, \\
u(-1,t) = u(1,t) = 0, & 0 < t < T, \\
u(x,0) = f(x), & -1 < x < 1.
\end{array} \right.
\end{align*}

(3.11)
Then, we get the following useful formula for the weak solution of the direct problem (3.11)

\[ u(x, t) = \sum_{n=1}^{\infty} \langle f, \psi_n \rangle E_{\alpha}(-\lambda_n t^\alpha)\psi_n(x). \] (3.12)

the series is convergent in \( C((0, T]; H^{2s}(-1, 1)) \) where \( \lambda_n = (\bar{\lambda}_n)^s \), \( \bar{\lambda}_n \) and \( \{\psi_n\}_{n \geq 1} \) are eigenvalues and eigenvectors of the classical Laplace operator \( \Delta \). \( \langle \cdot, \cdot \rangle \) denotes the standard inner product on \( L^2(-1, 1) \).

**Lemma 3.3.** \[60\] Let \( \Omega \subset \mathbb{R}^N \) be a bounded domain with the \( C^2 \) boundary \( \partial \Omega \), and define an unbounded linear operator \( A_{\Omega} \) on \( L^2(\Omega) \) as follows:

\[
\begin{aligned}
D(A_{\Omega}) &= H^2(\Omega) \cap H^1_0(\Omega), \\
A_{\Omega} \varphi &= -\Delta \varphi, \forall \varphi \in D(A_{\Omega}).
\end{aligned}
\]

We have that

\[
\begin{aligned}
&\begin{cases}
\partial_t^\alpha y - A_{\Omega}^s y = -\lambda y & \text{in } (0, +\infty) \\
y(0) = y_0.
\end{cases} \\
\end{aligned}
\] (3.13)

The solution to (3.13) is

\[ y(x, t) = \sum_{j=0}^{\infty} E_{\alpha,1}(-(\lambda_j^s + \lambda)t^\alpha)y_{0,j}e_j(x). \] (3.14)

where \( 0 < \alpha, s < 1, y_{0,j} = \langle y_0, e_j \rangle_{L^2(\Omega)} \) and denote by \( \{\lambda_j\}_{j=1}^{\infty} \) with \( 0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \) the eigenvalues of \( A_{\Omega} \) and \( \{e_j\}_{j=1}^{\infty} \) with \( |e_j|_{L^2(\Omega)} = 1 \) the corresponding eigenvectors.

**Definition 3.1.** \[61\] Let \( X_1 \) be a Banach space, \( z_0 \) belong to \( X_1 \), and \( f \in L^1(0, T; X_1) \). The function \( z(x, t) \in C([0, T]; X_1) \) given by

\[ z(t) = e^{-t}e^{t\Delta}z_0 + \int_0^t e^{-(t-s)} \cdot e^{(t-s)\Delta} f(s)ds, \quad 0 \leq t \leq T, \] (3.15)

is the mild solution of (3.15) on \([0, T] \), where \( (e^{t\Delta} f)(x, t) = \int_{\mathbb{R}^N} G(x - y, t)f(y)dy \) and \( G(x, t) \) is the heat kernel by \( G(x, t) = \frac{1}{(4\pi t)^{N/2}}exp(-\frac{|x|^2}{4t}). \)
Lemma 3.4. \[\text{Let } 0 \leq q \leq p \leq \infty, \frac{1}{q} - \frac{1}{p} < \frac{1}{N} \text{ and suppose that } z \text{ is the function given by } (3.15) \text{ and } z_0 \in W^{1,p}(\mathbb{R}^N). \text{ If } f \in L^\infty(0, \infty; L^q(\mathbb{R}^N)), \text{ then}
\begin{align*}
\| z(t) \|_{L^p(\mathbb{R}^N)} & \leq \| z_0 \|_{L^p(\mathbb{R}^N)} + C \cdot \Gamma(\gamma) \sup_{0 < s < t} \| f(s) \|_{L^q(\mathbb{R}^N)}, \\
\| \nabla z(t) \|_{L^p(\mathbb{R}^N)} & \leq \| \nabla z_0 \|_{L^p(\mathbb{R}^N)} + C \cdot \Gamma(\tilde{\gamma}) \sup_{0 < s < t} \| f(s) \|_{L^q(\mathbb{R}^N)},
\end{align*}
for } t \in [0, \infty), \text{ where } C \text{ is a positive constant independent of } p, \Gamma(\cdot) \text{ is the gamma function, and } \gamma = 1 - \left( \frac{1}{q} - \frac{1}{p} \right) \cdot \frac{N}{2}, \tilde{\gamma} = \frac{1}{2} - \left( \frac{1}{q} - \frac{1}{p} \right) \cdot \frac{N}{2}.
\]

Proof of Theorem 1.2. From the proof of Theorem 1.1, for any } K' > 0, \text{ from (2.29) and the definition of } M, \text{ there exist } \mu^*(K') > 0 \text{ and } m_0(K') > 0 \text{ such that } \forall \mu \in (0, \mu^*(K')) \text{ and } \| u_0 \|_{L^\infty(\mathbb{R}^N)} < m_0(K'), \text{ then for any } (x, t) \in \mathbb{R}^N \times [0, +\infty), \text{ } M \text{ is sufficiently small such that}
\| u(x, t) \|_{L^\infty(\mathbb{R}^N \times [0, +\infty))} < K'.
\]

(i). \textbf{The case: } \| u(x, t) \|_{L^\infty(\mathbb{R}^N \times [0, +\infty))} < K' = \frac{4}{\mu}.

Noticing } \sigma = \gamma - \mu \| u(x, t) \|_{L^\infty(\mathbb{R}^N \times [0, +\infty))} > 0, \text{ we have
\begin{align*}
\frac{\partial^\alpha u}{\partial t^\alpha} & = -(-\Delta)^{\alpha} u + \mu u^2 (1 - k J * u) - \gamma u \\
& = -(-\Delta)^{\alpha} u + u[\mu u(1 - k J * u) - \gamma] \\
& = -(-\Delta)^{\alpha} u - u[\gamma - \mu u(1 - k J * u)] \\
& \leq -(-\Delta)^{\alpha} u - u[s].
\end{align*}
\]
(3.16)

From Lemma 3.2 and Lemma 3.3, Consider the following equation
\[\begin{cases}
\mathcal{D}_t^\alpha w + (-\Delta)^{\alpha} w = -\sigma w, \\
w(0) = \| u_0 \|_{L^\infty(\mathbb{R}^N)}.
\end{cases}\]
(3.17)
then, we obtain
\[w(x, t) = \sum_{j=0}^{\infty} E_{\alpha,1}(-\lambda_j + \sigma) \langle \xi_j, e_j \rangle_{L^2(\Omega)} e_j(x). \]
(3.18)
Therefore, we have
\[
\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq \|w(x, t)\|_{L^\infty(\mathbb{R}^N)}
\]
\[
\leq \left\| \sum_{j=0}^{\infty} E_{\alpha,1} \left( -(\lambda_j^s + \sigma)t^\alpha \right) \langle y_0, e_j \rangle_{L^2(\Omega)} e_j(x) \right\|_{L^\infty(\mathbb{R}^N)}
\]
\[
\leq \|u_0\|_{L^\infty(\mathbb{R}^N)} e^{-\left(\lambda t^s + \sigma\right)\frac{1}{t^\frac{1}{\alpha}}}. \tag{28}
\]

(ii). The case: \(\|u(x, t)\|_{L^\infty(\mathbb{R}^N \times [0, +\infty))} < K' = a\).

Denoting \(H_0(y) = H(y, 0)\), from (3.2) in Proposition 3.1 for any \((x, t) \in \mathbb{R}^N \times [0, +\infty)\), we have
\[
H(x, t) \leq \|H_0\|_{L^\infty(\mathbb{R}^N)} + \int_0^t V(t - r) \|\nabla u\|_2^2 \, dr - \int_0^t V(t - r) D(r) \, dr,
\]
from which we obtain
\[
\int_0^t V(t - r) D(r) \, dr \leq \|H_0\|_{L^\infty(\mathbb{R}^N)} + \int_0^t V(t - r) \|\nabla u\|_2^2 \, dr.
\]
Due to the fact that \(u\) is a classical solution, we have that
\[
\int_0^t V(t - r) D(r) \, dr \in C^{2,1}(\mathbb{R}^N \times [0, \infty)),
\]
which implies that for all \(x \in \mathbb{R}^N\), the following limit holds:
\[
\lim_{t \to \infty} \lim_{r \to t} V(t - r) D(r) = 0,
\]
or equivalently
\[
\lim_{t \to \infty} \lim_{r \to t} V(t - r) \int_{\mathbb{R}^N} u^2(z, r) \, dzdy = 0,
\]
which together with the fact that the heat kernel converges to delta function as \(r \to t\), we have that for any \(x \in \mathbb{R}^N\),
\[
\lim_{t \to \infty} \int_{\mathbb{R}^N} u^2(y, t) \, dy = 0.
\]
Furthermore, with the uniform boundedness of $u$ on $\mathbb{R}^N \times [0, \infty)$, following Lemma 3.3, we can obtain the global boundedness of $\|\nabla u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)}$, from which and Lemma 2.12 with $\Omega = \mathbb{R}^N, p = q = \infty, r = 2$, the convergence of $\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)}$ follows from the convergences of $\|u(\cdot, t)\|_{L^2(\mathbb{R}^N)}$ immediately. This is, we obtain

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \to 0$$

as $t \to 0$. Therefore, for any compact set in $\mathbb{R}^N$, by finite covering, we obtain that $u$ converges to 0 uniformly in that compact set, which means that $u$ converges locally uniformly to 0 in $\mathbb{R}^N$ as $\to \infty$. The proof is complete.

4. Global boundness of solutions for a nonlinear TSFNRDE

In the section, we will make $J = k = \mu = \gamma = 1$ and replace fractional diffusion $(-\Delta)^s u$ as nonlinear fractional diffusion $(-\Delta)^s u^m$ in (1.1)-(1.2) to get equation (1.8)-(1.9). And we make $f(x, t) = u^2(1 - \int_{\mathbb{R}^N} u dx) - u$, then equation (1.8)-(1.9) can write the following form

$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha} + (-\Delta)^s (u^m) = f(x, t), & x \in \mathbb{R}^N \times (0, T) \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^N. \end{cases} \quad (4.1)$$

Definition 4.1. A function $u$ is a weak solution to the problem (4.1) if:

- $u \in L^2((0, T); H^1_0(\mathbb{R}^N)) \cap C((0, T]; L^1(\mathbb{R}^N))$ and
  $$|u|^{m-1} u \in L^2_{\text{loc}}((0, T]; H^s(\mathbb{R}^N));$$

- identity
  $$\int_0^T \int_{\mathbb{R}^N} u \frac{\partial^\alpha \varphi}{\partial t^\alpha} dx dt + \int_0^T \int_{\mathbb{R}^N} (-\Delta)^{s/2} (u^m) (-\Delta)^{s/2} \varphi dx dt = \int_0^T \int_{\mathbb{R}^N} f \varphi dx dt$$
  holds for every $\varphi \in C^1_0(\mathbb{R}^N \times (0, T));$

- $u(\cdot, 0) = u_0 \in L^1(\mathbb{R}^N)$ almost everywhere.

Definition 4.2. We say that a weak solution $u$ to the problem (4.1) is a strong solution if moreover $\frac{\partial^\alpha u}{\partial t^\alpha} \in L^\infty((\tau, \infty); L^1(\mathbb{R}^N)), \tau > 0$. 

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Remark 4.1. The above two definitions are mainly combined with references [1, 62]. On the one hand, [62] gives the definition of weak existence for time-fractional nonlinear diffusion equations. On the other hand, [1] gives the definition of existence of weak solution/strong solution of fractional nonlinear diffusion equations. For specific content, you can view Appendix A.3.

Lemma 4.1. Let $\varphi \in C^2(\mathbb{R}^N)$ and positive real function that is radially symmetric and decreasing in $|x| \geq 1$. Assume also that $\varphi(x) \leq |x|^{-\beta}$ and that $|D^2 \varphi(x)| \leq c_0 |x|^{-\beta-2}$, for some positive constant $\beta$ and for $|x|$ large enough. Then, for all $|x| \geq |x_0| \gg 1$ we have

$$|(-\Delta)^s \varphi(x)| \leq \begin{cases} \frac{c_0}{|x|^{\beta+2s}}, & \text{if } \beta < N, \\ \frac{c_2 (\log |x|)}{|x|^{\beta+2s}}, & \text{if } \beta = N, \\ \frac{c_3}{|x|^{\beta+2s}}, & \text{if } \beta > N, \end{cases}$$

with positive constant $c_1, c_2, c_3 > 0$ that depend only on $\beta, s, N$ and $\|\varphi\|_{C^2(\mathbb{R}^N)}$. For $\beta > N$ the reverse estimate holds from below if $\varphi \geq 0: |(-\Delta)^s \varphi(x)| \geq c_4 |x|^{-(N+2s)}$ for all $|x| \geq |x_0| \gg 1$.

Lemma 4.2. (Weighted $L^1$ estimates). Let $u \geq v$ be two ordered solutions to Eq. (4.1), with $0 < m < 1$. Let $\varphi_R = \varphi(x/R)$ where $R > 0$ and $\varphi$ is as in the previous lemma with $0 \leq \varphi \leq |x|^{-\beta}$ for $|x| \gg 1$ and

$$N - \frac{2s}{1-m} < \beta < N + \frac{2s}{m}.$$ 

Then, for all $0 \leq \tau, t < T$ we have

$$\left( \int_{\mathbb{R}^N} (u(x, t) - v(x, t)) \varphi_R(x) dx \right)^{1-m} \leq \left( \int_{\mathbb{R}^N} (u(x, \tau) - v(x, \tau)) \varphi_R(x) dx \right)^{1-m} + \frac{C_1 \varepsilon T^{\alpha(1-m)}}{(\alpha \Gamma(\alpha))^{1-m} R^{2s-N(1-m)}}$$

with $C_1, \varepsilon > 0$ that depends only on $\beta, m, N$.

Proof. Step 1. A fractional differential inequality for the weighted $L^1$-norm.
If $\psi$ is a smooth and sufficiently decaying function and Lemma 2.9, we have

$$\left| \int_{\mathbb{R}^N} (\frac{C}{0} D_t^\alpha u(x,t) - \frac{C}{0} D_t^\alpha v(x,t))\psi(x)dx \right|$$

$$= \left| \frac{C}{0} D_t^\alpha \int_{\mathbb{R}^N} (u(x,t) - v(x,t))\psi(x)dx \right| = \left| \int_{\mathbb{R}^N} ((-\Delta)^s u^m - (-\Delta)^s v^m)\psi dx \right|$$

$$= (a) \left| \int_{\mathbb{R}^N} (u^m - v^m) (-\Delta)^s \psi dx \right| \leq (b) 2^{1-m}\int_{\mathbb{R}^N} (u - v)^m |(-\Delta)^s \psi| dx$$

$$\leq (c) 2 \left( \int_{\mathbb{R}^N} (u - v) \psi dx \right)^m \left( \int_{\mathbb{R}^N} \frac{|(-\Delta)^s \psi|^{\frac{1}{1-m}}}{\psi^{\frac{m}{1-m}}} dx \right)^{1-m}. $$

Notice that in (a) we used that the fact that $(-\Delta)^s$ is a symmetric operator, while in (b) we have used that $(u^m - v^m) \leq 2^{1-m}(u - v)^m$, where $u^m = |u|^{m-1} u$ as mentioned. In (c) we have used Hölder inequality with conjugate exponents $1/m > 1$ and $1/(1 - m)$. If the last integral factor is bounded, then we get

$$\left| \frac{C}{0} D_t^\alpha \int_{\mathbb{R}^N} (u(x,t) - v(x,t))\psi(x)dx \right| \leq C_\psi^{1-m} \left( \int_{\mathbb{R}^N} (u(x,t) - v(x,t))\psi(x)dx \right)^m$$

and let $y(t) = \int_{\mathbb{R}^N} (u(x,t) - v(x,t))\psi(x)dx$, we can get fractional differential inequality on $(\tau, t)$

$$\frac{C}{0} D_t^\alpha y(t) \leq C_\psi^{1-m} y^m(t)$$

By Young inequality 2.13, let $a = C_\psi^{1-m}, b = y^m(t), q = \frac{1}{m} > 1, p = \frac{1}{1-m}$, then

$$C_\psi^{1-m} y^m(t) \leq \varepsilon^{\frac{1}{1-m}} C_\psi + \frac{m}{\varepsilon^m} y(t),$$

So we have

$$\frac{C}{0} D_t^\alpha y(t) \leq \varepsilon^{\frac{1}{1-m}} C_\psi + \frac{m}{\varepsilon^m} y(t),$$

and by Lemma 2.10, then we have

$$\left( \int_{\mathbb{R}^N} (u(x,t) - v(x,t))\varphi_R(x)dx \right)^{1-m} - \left( \int_{\mathbb{R}^N} (u(x,\tau) - v(x,\tau))\varphi_R(x)dx \right)^{1-m} \leq C_\psi^{1-m} \varepsilon \left( \frac{T^\alpha}{\alpha \Gamma(\alpha)} \right)^{1-m}.$$ 

Now, we estimate the constant $C_\psi$, for a convenient choice of test.
Step 2: Estimating the constant $C_\psi$.

Choose $\psi(x) = \varphi_R(x) := \varphi(x/R) = \varphi(y)$, with $\varphi$ as in Lemma 4.1 and $y = x/R$, so that $(-\Delta)^s \varphi(y) = (-\Delta)^s \varphi(x) = R^{-2s} (-\Delta)^s \varphi(y)$, then

$$C_\psi = \int_{\mathbb{R}^N} \frac{|(-\Delta)^s \varphi_R(x)|^{1-\frac{2s}{m}}}{\varphi_R(x)^{\frac{2s}{m}}} \, dx = R^{N-\frac{2s}{m}} \int_{\mathbb{R}^N} \frac{|(-\Delta)^s \varphi(y)|^{1-\frac{2s}{m}}}{\varphi(y)^{\frac{2s}{m}}} \, dy$$

$$= R^{N-\frac{2s}{m}} \left[ \int_{B_2} \frac{|(-\Delta)^s \varphi(y)|^{1-\frac{2s}{m}}}{\varphi(y)^{\frac{2s}{m}}} \, dy + \int_{B_2^c} \frac{|(-\Delta)^s \varphi(y)|^{1-\frac{2s}{m}}}{\varphi(y)^{\frac{2s}{m}}} \, dy \right]$$

where it is easy to check that first integral is bounded, since $\varphi \geq k_2 > 0$ on $B_2$, and when $|y| > |x_0|$ with $|x_0| \gg 1$ we know by estimate (4.2) that

$$\frac{|(-\Delta)^s \varphi(y)|^{1-\frac{2s}{m}}}{\varphi(y)^{\frac{2s}{m}}} \leq \begin{cases} \frac{k_1}{|y|^\beta}, & \text{if } \beta < N, \\ \frac{k_1|\log|y||}{|y|^{N-\beta+2s/m}}, & \text{if } \beta = N, \\ \frac{k_1|\log|y||}{|y|^{N-\beta+2s/m}}, & \text{if } \beta > N, \end{cases} \quad (4.4)$$

therefore $k_1$ is finite whenever $N - \frac{2s}{1-m} < \beta < N + \frac{2s}{m}$. Note that all the constants $k_i, i = 1, 2, 3, 4, 5$ depend only on $\beta, m, N$.

Remark 4.2. Using the method of proof in [Theorem 2.2, [63]], and when $0 < m < m_c = (N - 2s)/N$ solution corresponding to $u_0 \in L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ with $p \geq N(1-m)/2s$. On the other hand, when $m_c < m < 1$, the estimate implies the conservation of mass by letting $R \to \infty$. By [1], the above estimates provide a lower bound for the extinction time in such a case, just by letting $\tau = T, T > 0$ and $t = 0$ in the above estimates:

$$\frac{1}{C_1 R^{N(1-m)-2s}} \left( \int_{\mathbb{R}^N} u_0 \varphi_R \, dx \right)^{1-m} \leq T.$$

Lemma 4.3. Suppose that a nonnegative function $u(t) \geq 0$ satisfies

$$C_0 \mathcal{D}^\alpha u(t) + c_1 u(t) \leq f(t) \quad (4.5)$$

for almost all $t \in [0, T]$, where $c_1 > 0$, and the function $f(t)$ is nonnegative and integrable for $t \in [0, T]$. Then

$$u(t) \leq u(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \, ds. \quad (4.6)$$
Lemma 4.4. Assume the function $y_k(t)$ is nonnegative and exists the Caputo fractional derivative for $t \in [0, T]$ satisfying

$$C_0^D_0^\alpha y_k(t) \leq -C_9(y_k(t))^\frac{k+m-1}{k} - y_k + a_k(y_k^{\gamma_1}_{k-1}(t) + y_k^{\gamma_2}_{k-1}(t)),$$  \hspace{1cm} (4.7)

where $a_k = \bar{a}3^{r_k} > 1$ with $\bar{a}, r$ are positive bounded constants and $0 < \gamma_2 < \gamma_1 \leq 3$. Assume also that there exists a bounded constant $K \geq 1$ such that $y_k(0) \leq K^{3^k}$, then

$$y_k(t) \leq \frac{y_k(0)}{2} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds$$

Proof. From Lemma 4.3 and Equation (4.6), we make

$$f(t) = a_k(y^{\gamma_1}_{k-1}(t) + y^{\gamma_2}_{k-1}(t)) \leq 2a_k \max \left\{ 1, \sup_{t \in [0,T]} y_k^{3}_{k-1}(t) \right\}, C_9(y_k(t))^\frac{k+m-1}{k} \geq 0$$

then

$$y_k(t) \leq y_k(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds$$

$$\leq K^{3^k} + \frac{2a_k}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \max \left\{ 1, \sup_{t \in [0,T]} y_k^{3}_{k-1}(s) \right\} ds$$

$$\leq K^{3^k} + \frac{2a_k}{\Gamma(\alpha)} \max \left\{ 1, \sup_{t \in [0,T]} y_k^{3}_{k-1}(t) \right\} \int_0^t (t-s)^{\alpha-1} ds$$

$$\leq K^{3^k} + \frac{T^\alpha}{\alpha \Gamma(\alpha)} 2a_k \max \left\{ 1, \sup_{t \in [0,T]} y_k^{3}_{k-1}(t) \right\}$$

$$\leq \frac{T^\alpha}{\alpha \Gamma(\alpha)} 2a_k \max \left\{ K^{3^k}, \sup_{t \in [0,T]} y_k^{3}_{k-1}(t) \right\}.$$
Then from (4.9) after some iterative steps we have
\[ y_k(t) \leq 2a_k(2a_{k-1})^3(2a_{k-2})^3^2(2a_{k-3})^3^3 \cdots (2a_1)^{3^{k-1}} \max \left\{ K^{3^k}, \sup_{t \in [0, T]} y_0^3(t) \right\} \]
\[ \frac{T^\alpha}{\alpha \Gamma(\alpha)} = (2\bar{a})^{1+3^2+3^3+\cdots+3^{k-1}} \max \left\{ K^{3^k}, \sup_{t \in [0, T]} y_0^3(t) \right\} \]
\[ = (2\bar{a})^{\frac{3^k-1}{2}} \max \left\{ K^{3^k}, \sup_{t \in [0, T]} y_0^3(t) \right\} \frac{T^\alpha}{\alpha \Gamma(\alpha)}. \]

**Remark 4.3.** The proof of this Lemma is based on the process of Lemma 4 in [64], where the nature of \( y_k(t) \geq 0, k = 0, 1, 2, \ldots \) is scratched and the desired result is obtained.

**Lemma 4.5.** Suppose \( 0 < k < m < 1 \) and \( b(t) \) is continuous and boundary. And let \( y(t) \geq 0 \) be a solution of the fractional differential inequality
\[ C_0^\alpha D_t^\alpha y(t) + \alpha y^k(t) + \beta y(t) \leq b(t)y^m + c_4. \] (4.10)
For almost all \( t \in [0, T] \), then
\[ y(t) \leq y(0) + \left[ \frac{[\lambda_k y^{1-k}(0) + (c_4 - \alpha)(1-k)T^\alpha]}{\alpha \Gamma(\alpha)} \right]^{\frac{1}{1-k}} \]
\[ + (1-m)^{\frac{1}{1-k}} \frac{T^\alpha}{\alpha \Gamma(\alpha)} \frac{1}{1-k} \beta b^{\frac{1}{1-m}}(t). \]
where \( \lambda_k = -\frac{m-k}{\varepsilon} - \beta(1-k) \) and \( \alpha, \beta, c_4, \varepsilon > 0 \) are all contants.

**Proof.** Multiplying \( (1-k)y^{-k} \) to both side of (4.10) yields
\[ C_0^\alpha D_t^\alpha y^{1-k}(t) + \alpha(1-k) + \beta(1-k)y^{1-k}(t) \leq (1-k)b(t)y^{m-k} + c_4(1-k)y^{-k}(t). \]
Let \( z(t) = y^{1-k}(t), 0 < y^{-k}(t) < 1, \) then we have
\[ C_0^\alpha D_t^\alpha z(t) \leq b(t)(1-k)z^{\frac{m-k}{1-k}}(t) - \beta(1-k)z(t) + (c_4 - \alpha)(1-k). \]
By Young's inequality, let $1 - \bar{\beta} = \frac{a}{1-m}$, $a = b(t)$, $b = (z(t))^{1-\beta}$, $\frac{1}{1-\beta} = q > 1$, $p = \frac{1}{\beta}$, then we have

$$b(t)(1-k)z^{\frac{m-k}{1-m}}(t) \leq (1-m)\varepsilon^p b^p(t) + \frac{m-k}{\varepsilon^{1-\beta}} z(t),$$

then we can get

$$C_0 D_0^\alpha z(t) \leq \left[ \frac{m-k}{\varepsilon^{1-\beta}} - \beta(1-k) \right] z(t) + (c_4 - \alpha)(1-k) + (1-m)\varepsilon^p b^p(t).$$

Let $\lambda_k = -\frac{m-k}{\varepsilon^{1-\beta}} - \beta(1-k)$ and By Lemma 4.3, then we obtain

$$z(t) \leq z(0) + \frac{[\lambda_k z(0) + (c_4 - \alpha)(1-k)]T^\alpha}{\alpha \Gamma(\alpha)}$$

$$+ (1-m)\varepsilon^p \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} b^p(s) ds$$

then the solution of (4.10) can be estimated as

$$y(t) \leq y^{1-k}(0) + \frac{[\lambda_k y^{1-k}(0) + (c_4 - \alpha)(1-k)]T^\alpha}{\alpha \Gamma(\alpha)}$$

$$+ (1-m)\varepsilon^p \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} b^p(s) ds^{\frac{1}{1-k}}.$$  \hspace{1cm} (4.11)

Using the inequality $\sqrt{A+B} \leq \sqrt{A} + \sqrt{B}, A, B > 0$, we can transform (4.11) into

$$y(t) \leq y(0) + \left[ \frac{[\lambda_k y^{1-k}(0) + (c_4 - \alpha)(1-k)]T^\alpha}{\alpha \Gamma(\alpha)} \right]^{\frac{1}{1-k}}$$

$$+ (1-m)\varepsilon^p \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} b^p(s) ds^{\frac{1}{1-k}}.$$

**Remark 4.4.** By using Young inequality, we first will inequality (4.10) Write the form of fractional differential inequality (4.3). Then using the method of proof [Theorem 1[65]], we can the solution of above mentioned inequality (4.10).
Lemma 4.6. Assume that \((a, b) \in (\mathbb{R}^+)^2, 0 < \alpha < 1\), then there exist \(c_1, c_2, c_3 > 0\), such that
\[
(a + b)^\alpha \leq c_1 a^\alpha + c_2 b^\alpha
\]
and
\[
(a - b)(a^\alpha - b^\alpha) \geq c_3 \left| a^{\frac{\alpha}{2}} - b^{\frac{\alpha}{2}} \right|^2
\]

(4.12)

Lemma 4.7. Assume that \(0 < s < 1\), then there exists a positive constant \(S \equiv S(N, s)\) such that for all \(v \in C_0^\infty(\mathbb{R}^N)\),
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |v(x) - v(y)|^2 |x - y|^{N+2s} \, dx \, dy \geq S \left( \int_{\mathbb{R}^N} |v(x)|^{p_+^s} \, dx \right)^{\frac{2}{p_+^s}},
\]
where \(p_+^s = \frac{2N}{N-2s}\).

Lemma 4.8. Let \(N \geq 3, q > 1, m > 1 - 2/N\), assume \(u \in L^1_1(\mathbb{R}^N)\) and \(u^{\frac{m+q-1}{q}} \in H^1(\mathbb{R}^N)\), then
\[
\left( \|u\|_q^q \right)^{\frac{1}{q} + \frac{m-1+2/N}{q-1}} \leq S_N^{-1} \|\nabla u^{(q+m-1)/2}\|_2^2 \|u\|_1^{\frac{1}{2} (2q/N + m-1)}.
\]

Lemma 4.9. When the parameters \(p, q, r\) meet any of the following conditions:

(i) \(q > N \geq 1, r \geq 1\) and \(p = \infty\);

(ii) \(q > \max\{1, \frac{2N}{N+2}\}, 1 \leq r < \sigma\) and \(r < p < \sigma + 1\) in

\[
\sigma := \begin{cases} 
\frac{(q-1)N+q}{N-q}, & q < N, \\
\infty, & q \geq N.
\end{cases}
\]

Then the following inequality is established
\[
\|u\|_{L^p(\mathbb{R}^N)} \leq C_{GN} \|u^{1-\lambda^*}\|_{L^r(\mathbb{R}^N)} \|\nabla u\|_{L^r(\mathbb{R}^N)}^{\lambda^*}
\]
among
\[
\lambda^* = \frac{qN(p-r)}{p[N(q-r) + qr]}.
\]
Lemma 4.10. \[ \text{Let } N \geq 1. \ p \text{ is the exponent from the Sobolev embedding theorem, i.e.} \\
\begin{cases}
p = \frac{2N}{N-2}, & N \geq 3 \\
2 < p < \infty, & N = 2 \\
p = \infty, & N = 1
\end{cases} \quad (4.13)
\]

1 \leq r < q < p \text{ and } \frac{q}{r} < \frac{2}{r} + 1 - \frac{2}{p}, \text{ then for } v \in H'(\mathbb{R}^N) \text{ and } v \in L^r(\mathbb{R}^N), \text{ it holds}

\[ ||v||_{L^q(\mathbb{R}^N)}^q \leq C(N)c_0^{\frac{\gamma q}{r}} ||v||_{L^p(\mathbb{R}^N)}^\gamma + c_0 \left\| \nabla u \right\|_{L^2(\mathbb{R}^N)}^2, \quad N > 2, \]

\[ \text{and } \]

\[ ||v||_{L^q(\mathbb{R}^N)}^q \leq C(N)(c_0^{\frac{\gamma q}{r}} + c_1^{\frac{\gamma q}{r-2-q}}) ||v||_{L^r(\mathbb{R}^N)}^\gamma + c_0 \left\| \nabla u \right\|_{L^2(\mathbb{R}^N)}^2 \quad (4.14) \]

Here \( C(N) \) are constants depending on \( N, c_0, c_1 \) are arbitrary positive constants and

\[ \lambda = \frac{1}{p} - \frac{1}{q} \in (0, 1), \quad \gamma = \frac{2(1 - \lambda)q}{2 - \lambda q} = \frac{2(1 - \frac{4}{p})}{\frac{2 - q}{r} - \frac{2}{p} + 1}. \]

Proof of Theorem 1.3.

Step 1. Existence and unique weak solution for problem (4.1).

(1) We first verify \( u \in L^2((0, T]; H^1_0(\mathbb{R}^N)) \) and \( |u|^{m-1} u \in L^2_{loc}((0, T]; H^s(\mathbb{R}^N)). \)

By [62], we denote the space \( L^2(0, T; H^1_0(\mathbb{R}^N)) \) by \( V \) and define \( Lu := \langle \hat{L}u, \varphi \rangle, \hat{L}u := \frac{\partial^\alpha u}{\partial t^\alpha} \) with domain \( D(\hat{L}) \subset V \rightarrow V^* \) with the domain

\[ D(\hat{L}) = \left\{ u \in V : \frac{\partial^\alpha u}{\partial t^\alpha} \in V^* \right\}. \]

We note that the operator \( L \) is linear, densely defined and \( m \)-accretive, see [70]. By Proposition 31.5 in [71], the operator \( L \) is a maximal monotone operator. Next, We verify \( |u|^{m-1} u \in L^2_{loc}((0, T]; H^s(\mathbb{R}^N)). \) By [1], If \( \psi \) and \( \varphi \) belong to the Schwartz class, definition (1.3) of the fractional Laplacian together with Plancherel’s theorem yields

\[ \int_{\mathbb{R}^N} (-\Delta)^s \psi \varphi dx = \int_{\mathbb{R}^N} |\xi|^{2s} \hat{\psi}(\xi) \hat{\varphi}(\xi) dx \]

\[ = \int_{\mathbb{R}^N} |\xi|^{s} \hat{\psi}(\xi) \xi^s \hat{\varphi}(\xi) dx = \int_{\mathbb{R}^N} (-\Delta)^{s/2} \psi(-\Delta)^{s/2} \varphi dx. \]
Therefore, if we multiply the equation in (4.1) by a test function $\varphi$ and integrate by parts as usual on $t \in [0, T]$, we obtain
\[
\int_0^T \int_{\mathbb{R}^N} u \frac{\partial^\alpha \varphi}{\partial t^\alpha} \, dx \, dt + \int_0^T \int_{\mathbb{R}^N} (-\Delta)^{s/2}(u^m)(-\Delta)^{s/2} \varphi \, dx \, dt
= \int_0^T \int_{\mathbb{R}^N} f \varphi \, dx \, dt
\]  
(4.15)

This identity will be the basis of our definition of a weak solution. The integrals in (4.15) make sense if $u$ and $|u|^{m-1}u$ belong to suitable spaces. The correct space for $|u|^{m-1}u$ is the fractional Sobolev space $H^s(\mathbb{R}^N)$.

(2) We identity (4.15) hold for every $\psi \in C_0^\infty([0, T] \times \mathbb{R}^N)$.

Let $0 \leq u_{n,n-1} \leq u_{0,n}$, converging monotonically to $u \in L^1(\mathbb{R}^N, \varphi \, dx)$, i.e., such that \( \int_{\mathbb{R}^N} (u_0 - u_{n,0}) \phi \, dx \to 0 \) as \( n \to \infty \), where $\phi$ is as in Lemma 4.2 with decay at infinity $|x|^{-\beta}, N - \frac{2s}{1-m} \leq \beta < N + \frac{2s}{m}$. Consider the unique solution $u_n(x, t)$ of Eq. (4.1) with initial data $u_{0,n}$. The weighted estimates (4.3) show that the sequence is bounded in $L^1(\mathbb{R}^N, \varphi \, dx)$ uniformly in $t \in [0, T]$. By the monotone convergence theorem in $L^1(\mathbb{R}^N, \varphi \, dx)$, we know that the solution $u_n(x, t)$ converge monotonically as $n \to \infty$ to a function $u(x, t) \in L^\infty((0, T); L^1(\mathbb{R}^N, \varphi \, dx))$. Indeed, the weighted estimates (4.3) show that when $u_0 \in L^1(\mathbb{R}^N, \varphi \, dx)$ then
\[
\left( \int_{\mathbb{R}^N} u(x, t) \varphi(x) \, dx \right)^{1-m} \leq \lim_{n \to \infty} \left( \int_{\mathbb{R}^N} u_n(x, t) \varphi(x) \, dx \right)^{1-m} + C \int_0^T \left( \int_{\mathbb{R}^N} u_n(x, t) \varphi(x) \, dx \right)^{1-m} \frac{e^{-C_1T^\alpha(1-m)}}{(\alpha \Gamma(\alpha))^{1-m} R^{2s-N(1-m)}}
\]  
(4.16)

At this point we need to show that the function $u(x, t)$ constructed as above is a very weak solution to Eq. (4.1) on $[0, T] \times \mathbb{R}^N$. By definition 4.2, we make that each $u_n$ is a bounded strong solutions, since the initial data $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, therefore for all $\psi \in C_0^\infty([0, T] \times \mathbb{R}^N)$ we have
\[
\int_0^T \int_{\mathbb{R}^N} u_n \frac{\partial^\alpha \psi}{\partial t^\alpha} \, dx \, dt + \int_0^T \int_{\mathbb{R}^N} (-\Delta)^{s/2}(u_n^m)(-\Delta)^{s/2} \psi \, dx \, dt
= \int_0^T \int_{\mathbb{R}^N} f \psi \, dx \, dt
\]  
(4.17)

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Now, for any $\psi \in C^\infty_0([0,T] \times \mathbb{R}^N)$ we easily have that
\[
\lim_{n \to \infty} \int^T_0 \int_{\mathbb{R}^N} u_n \frac{\partial^\alpha \psi}{\partial t^\alpha} \, dx \, dt = \int^T_0 \int_{\mathbb{R}^N} u \frac{\partial^\alpha \psi}{\partial t^\alpha} \, dx \, dt
\]
since $\psi$ is compactly supported and we already know that $u_n(x,t) \to u(x,t)$ in $L^1_{loc}$. On the other hand, for any $\psi \in C^\infty_0([0,T] \times \mathbb{R}^N)$ we have that
\[
\lim_{n \to \infty} \int^T_0 \int_{\mathbb{R}^N} (-\Delta)^{s/2}(u_n^m) (-\Delta)^{s/2}\psi \, dx \, dt = \int^T_0 \int_{\mathbb{R}^N} (-\Delta)^{s/2}(u^m) (-\Delta)^{s/2}\psi \, dx \, dt
\]
since $u_n \leq u$ and
\[
0 \leq \int^T_0 \int_{\mathbb{R}^N} (-\Delta)^{s/2}u^m(x,t) - (-\Delta)^{s/2}u_n^m(x,t) (-\Delta)^{s/2}\psi \, dx \, dt
\]
\[
\leq \int^T_0 \int_{\mathbb{R}^N} (-\Delta)^{s/2}(u^m(x,t) - u_n^m(x,t)) (-\Delta)^{s/2}\psi \, dx \, dt
\]
\[
\leq \int^T_0 \int_{\mathbb{R}^N} (u^m(x,t) - u_n^m(x,t)) (-\Delta)^{s}\psi \, dx \, dt
\]
\[
\leq \int^T_0 \left( \int_{\mathbb{R}^N} |u(x,t) - u_n(x,t)|^m \varphi^m(x) \frac{|(-\Delta)^s\psi|}{\varphi^m(x)} \, dx \right) \left( \int_{\mathbb{R}^N} \frac{|(-\Delta)^s\psi|}{\varphi^m(x)} \, dx \right)^{1-m} \, dt
\]
\[
\leq C \int^T_0 \int_{\mathbb{R}^N} (u(x,t) - u_n(x,t)) \varphi \, dx \, dt \to 0
\]
where we have used Hölder inequality with conjugate exponents $1/m$ and $1/(1 - m)$, and we notice that
\[
\left( \int_{\mathbb{R}^N} \frac{|(-\Delta)^s\psi|}{\varphi^m(x)} \, dx \right)^{1-m} \leq C
\]
since $\psi$ is compactly supported, therefore by Lemma 2.12 we know that $|(-\Delta)^s\psi(x,t)| \leq c_3 |x|^{-(N+2s)}$, and quotient
\[
\frac{|(-\Delta)^s\psi|}{\varphi^m(x)} \leq \frac{c_3}{|x|^\frac{N+2s-m}{4-m}}
\]
is integrable when \( N + 2s - m \beta > N \) that is when \( \beta < N + (2s/m) \). In the last step we already know that \( \int_{\mathbb{R}^N} (u(x,t) - u_n(x,t)) \varphi dx \to 0 \) when \( \varphi \) is as above, i.e. as in Lemma 4.2. Therefore we can let \( n \to \infty \) in (4.17) and obtain (4.15).

(3) We verify \( u(\cdot,0) = u_0 \in L^1(\mathbb{R}^N) \) almost everywhere.

For the solution constructed above, the weighted estimates (4.3) show that when \( 0 \leq u_0 \in L^1(\mathbb{R}^N, \varphi dx) \) imply

\[
\left| \int_{\mathbb{R}^N} u(x,t) \varphi_R dx - \int_{\mathbb{R}^N} u(x,\tau) \varphi_R dx \right| \leq 2^{\frac{1}{1-m}} C_1 T^\alpha \frac{C_1 T^\alpha}{\alpha \Gamma(\alpha)} R^{N - \frac{2s}{1-m}}
\]

which gives the continuity in \( L^1(\mathbb{R}^N, \varphi dx) \). Therefore, the initial trace of this solution is given by \( u_0 \in L^1(\mathbb{R}^N) \).

In summary, by definition 4.1 and Theorem 3.1 in [63], we have proved existence of solutions corresponding to initial data \( u_0 \) that can grow at infinity as \( |x|^{(2s/m) - \varepsilon} \) for any \( \varepsilon > 0 \) for problem (4.1). For the uniqueness of the solution, it can be proved through Theorem 3.2 in [63]. Therefore, equation (1.8)-(1.9) existence the unique weak solution.

Next, we will use Gagliardo-Nirenberg inequality, Young inequality and interpolation inequality, and so on. On the other hand, we use \( \int_{\mathbb{R}^N} |u|^2 dx \) and \( \int_{\mathbb{R}^N} u^{k+1} dx \int_{\mathbb{R}^N} u dx \) to control nonlinear item \( \int_{\mathbb{R}^N} u^{k+1} dx \), and get (1.8)-(1.9) \( L^r \) estimate. In the proof process of the section, \( C(N,k,m) \) and \( C_i(N,k,m)(i = 1,2) \) represent the constant that depends on \( N,k,m \).

Step 2. The \( L^r \) estimates.

For any \( x \in \mathbb{R}^N \), multiply (1.8) by \( u^{k-1}, k > 1 \) and integrating by parts over \( \mathbb{R}^N \), by the proof of Theorem 5.2 in [66], we obtain

\[
\int_{\mathbb{R}^N} u^{k-1} (-\Delta)^s u^m dx = C_{N,s} P.V. \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} u^{k-1}(x) \frac{u^m(x) - u^m(y)}{|x - y|^{N+2s}} dxdy
= \frac{1}{2} C_{N,s} P.V. \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (u^{k-1}(x) - u^{k-1}(y)) \frac{u^m(x) - u^m(y)}{|x - y|^{N+2s}} dxdy,
\]

from Lemma 4.6 and equation (4.12), let \( a = u^m(x), b = u^m(y), \alpha = \frac{k-1}{m} \), then we have

\[
(u^m(x) - u^m(y))(u^{k-1}(x) - u^{k-1}(y)) \geq c_3 \left| u^{\frac{k+m-1}{2}}(x) - u^{\frac{k+m-1}{2}}(y) \right|^2.
\]
By Sobolev inequality (Lemma 4.7), we reach that
\[
\int_{\mathbb{R}^N} u^{k-1}(-\Delta)^s u^m dx \geq \frac{c_3}{2} \left\| u^{\frac{k+m-1}{2}} - u^{\frac{k+m-1}{2}} \right\|^2 \geq \frac{c_3 S}{2} \left( \int_{\mathbb{R}^N} \right|u^{\frac{k+m-1}{2}}(x,t)\right|^{p^*_s} dx \right)^{\frac{2}{p^*_s}},
\]
where \(p^*_s = \frac{2N}{N-2s}\). Then, we obtain
\[
\frac{1}{k} (C_0 D_t^\alpha \int_{\mathbb{R}^N} u^k dx) + \frac{c_3 S}{2} \left( \int_{\mathbb{R}^N} \right|u^{\frac{k+m-1}{2}}(x,t)\right|^{p^*_s} dx \right)^{\frac{2}{p^*_s}} \leq \int_{\mathbb{R}^N} u^{k+1} dx (1 - \int_{\mathbb{R}^N} u^k dx) - \int_{\mathbb{R}^N} u^k dx
\]
(4.18)
The following estimate \(k \int_{\mathbb{R}^N} u^{k+1} dx\). when \(m \leq 3\) and
\[
k > \max \left\{ \frac{1}{4} (3-m)(N-2) - (m-1), (1 - \frac{m}{2})N - 1, N(2-m) - 2 \right\},
\]
from Lemma 4.9, we obtain
\[
k \int_{\mathbb{R}^N} u^{k+1} dx = k \left\| u^{\frac{k+m-1}{2}} \right\|^{\frac{2(k+1)}{k+m+1}}_{L^{\frac{k+m+1}{k+m+1}}(\mathbb{R}^N)} \leq k \left\| u^{\frac{k+m-1}{2}} \right\|^{\frac{2(k+1)}{k+m+1} \lambda^*}_{L^{\frac{k+m+1}{k+m+1}}(\mathbb{R}^N)} \left\| \nabla u^{\frac{k+m-1}{2}} \right\|^{\frac{2(k+1)(1-\lambda^*)}{k+m+1}}_{L^2(\mathbb{R}^N)}.
\]
Knowing from (4.19)
\[
\lambda^* = \frac{1+\frac{(k+m-1)N}{2(k+1)} - \frac{N}{2}}{1+\frac{(k+m-1)N}{k+2} - \frac{N}{2}} \in \left\{ \max \left\{ 0, \frac{2-m}{k+1} \right\}, 1 \right\},
\]
using Young’s inequality, there are
\[
k \int_{\mathbb{R}^N} u^{k+1} dx \leq k \left\| u^{\frac{k+m-1}{2}} \right\|^{\frac{2(k+1)}{k+m+1} \lambda^*}_{L^{\frac{k+m+1}{k+m+1}}(\mathbb{R}^N)} \left\| \nabla u^{\frac{k+m-1}{2}} \right\|^{\frac{2(k+1)(1-\lambda^*)}{k+m+1}}_{L^2(\mathbb{R}^N)} \leq \frac{2k(k-1)}{(m+k-1)^2} \left\| \nabla u^{\frac{k+m-1}{2}} \right\|_{L^2(\mathbb{R}^N)}^2 + C_1(N, k, m) \left\| u^{\frac{k+m-1}{2}} \right\|_{L^{\frac{k+m+1}{k+m+1}}(\mathbb{R}^N)}^{Q_2}
\]
(4.20)
where is $Q_2 = \frac{2(k+1)\lambda^*}{m-2+(k+1)\lambda^*}$. Next estimate $\|u^{\frac{k+m-1}{2}}\|_{L^{\frac{k+2}{k+m-1}}(R^N)}^{Q_2}$. We will use the interpolation inequality to get

$$
\|u^{\frac{k+m-1}{2}}\|_{L^{\frac{k+2}{k+m-1}}(R^N)}^{Q_2} \leq \left( \|u\|_{L^{\frac{k+m-1}{2}}(R^N)}^{Q_2} \|u\|_{L^{\frac{2(k+1)}{k+m-1}}(R^N)}^{Q_2(1-\lambda)} \right)^\frac{2}{Q_2(k+m-1)} \left( \|u\|_{L^{\frac{k+m-1}{2}}(R^N)} \right)^\frac{Q_2(k+m-1)}{2(k+1)}.
$$

in $\lambda = \frac{k+1}{k+2}$, and

$$Q_2(1-\lambda - \frac{\lambda}{k+1}) = 0.
$$

Then

$$
C_1(N, k, m) \|u^{\frac{k+m-1}{2}}\|_{L^{\frac{k+2}{k+m-1}}(R^N)}^{Q_2} \leq C_1(N, k, m) \left( \|u\|_{L^{\frac{k+m-1}{2}}(R^N)}^{Q_2} \|u\|_{L^{\frac{2(k+1)}{k+m-1}}(R^N)}^{Q_2(1-\lambda)} \right)^\frac{2}{Q_2(k+m-1)} \left( \|u\|_{L^{\frac{k+m-1}{2}}(R^N)} \right)^\frac{Q_2(k+m-1)}{2(k+1)}.
$$

Noticing that when $m > 2 - \frac{2}{N}$, it is easy to verify

$$
\frac{Q_2\lambda(k + m - 1)}{2(k + 1)} = \frac{(k + 1)(m - 2) + \lambda^*(k + 1)(3 - m)}{(k + 1)(k + m - 1)\lambda^*} < 1,
$$

using Young’s inequality, then

$$
C_1(N, k, m) \left( \|u\|_{L^{\frac{k+m-1}{2}}(R^N)}^{Q_2} \|u\|_{L^{\frac{2(k+1)}{k+m-1}}(R^N)}^{Q_2(1-\lambda)} \right)^\frac{2}{Q_2(k+m-1)} \left( \|u\|_{L^{\frac{k+m-1}{2}}(R^N)} \right)^\frac{Q_2(k+m-1)}{2(k+1)}
\leq k \left( \|u\|_{L^{\frac{k+m-1}{2}}(R^N)}^{Q_2} \|u\|_{L^{\frac{2(k+1)}{k+m-1}}(R^N)}^{Q_2(1-\lambda)} \right)^\frac{2}{Q_2(k+m-1)} \left( \|u\|_{L^{\frac{k+m-1}{2}}(R^N)} \right)^\frac{Q_2(k+m-1)}{2(k+1)} + C_2(N, k, m).
$$

Substitute (4.20)-(4.23) into (4.18) to get

$$
\epsilon_t^D \int_{R^N} u^k dx + C_3 Sk \left( \int_{R^N} |u^{\frac{k+m-1}{2}}(x, t)|^{p_0^*} dx \right)^\frac{2}{p_0^*} + k \int_{R^N} u^k dx
\leq \frac{2k(k-1)}{(m+k-1)^2} \left( \|\nabla u^{\frac{k+m-1}{2}}\|_{L^2(R^N)}^2 \right) + C_2(N, k, m).
$$

(4.24)
Let $k \to \infty$, then $\frac{2k(k-1)}{(m+k-1)^2} \to 2$, so $\frac{2k(k-1)}{(m+k-1)^2} \leq 2$. And from Lemma 4.8 and $N \geq 3$, we obtain

$$\frac{1}{S_N^{-1} \| u \|_1^{2k \gamma}} (\| u \|_k)^{1+\frac{m-1+2N}{k-1}} \leq \| \nabla u \|^{(k+m-1)/2}_2.$$

Let $k = \frac{N(1-m)}{2s}$, then $\frac{k+m-1}{2}p_s = k$, $\frac{2}{p_s} = \frac{k+m-1}{k}$. Therefore, we can get

$$C_{0} D_{t}^{\alpha} \int_{\mathbb{R}^{N}} u_{k}^{k} dx + \frac{c_{3} S_{k}}{2} \left( \int_{\mathbb{R}^{N}} u_{k}^{k} dx \right)^{k+1} + \int_{\mathbb{R}^{N}} u_{k}^{k} dx \leq \frac{2}{S_N^{-1} \| u \|_1^{2k \gamma}} (\| u \|_k)^{1+\frac{m-1+2N}{k-1}} + C_{2}(N, k, m).$$

Let $a = 1 + \frac{m-1+2/d}{k-1}, f(t) = \frac{2}{S_N^{-1} \| u \|_1^{2k \gamma}} (\| u \|_k)^{1+\frac{m-1+2N}{k-1}}, \beta = \frac{k+m-1}{k}, y(t) = \int_{\mathbb{R}^{N}} u_{k}^{k} dx$.

Then, when $0 < m < 1$, the above-mentioned inequality can be written as

$$C_{0} D_{t}^{\alpha} y(t) + \frac{c_{3} S_{k}}{2} y^{\beta}(t) + k y(t) \leq f(t)y^{a}(t) + C_{2}(N, k, m) \quad (4.25)$$

By Lemma 1.3 and $0 < \beta < a, t \in [0, T]$, fractional differential inequality (4.25) has following solution

$$y(t) \leq y(0) + \left[ \frac{[\lambda_{k} y^{1-\beta}(0) + (C_{2}(N, k, m) - \frac{c_{3} S_{k}}{2})(1-\beta)]T^{\alpha}}{\alpha \Gamma(\alpha)} \right]^{\frac{1}{\alpha-\beta}}$$

$$+ (1-a)^{\frac{1}{\alpha-\beta}} \varepsilon^{\frac{1}{\alpha-a}} \frac{T^{\alpha}}{\alpha^{\frac{1}{\alpha-\beta}} \Gamma(\alpha)} f^{\frac{1}{\alpha-a}}(t).$$

Therefore, we have

$$y(t) = \int_{\mathbb{R}^{N}} u_{k}^{k} dx \leq y(0) + \left[ \frac{[\lambda_{k} y^{1-\beta}(0) + (C_{2}(N, k, m) - \frac{c_{3} S_{k}}{2})(1-\beta)]T^{\alpha}}{\alpha \Gamma(\alpha)} \right]^{\frac{1}{\alpha-\beta}}$$

$$+ (1-a)^{\frac{1}{\alpha-\beta}} \varepsilon^{\frac{1}{\alpha-a}} \frac{T^{\alpha}}{\alpha^{\frac{1}{\alpha-\beta}} \Gamma(\alpha)} f^{\frac{1}{\alpha-a}}(0). \quad (4.26)$$

where $\lambda_{k} = \frac{\alpha-a}{\varepsilon^{\alpha-a}} - k(1-\beta), y(0) = \parallel u_{0} \parallel_{L^{k}(\mathbb{R}^{N})}^{k}$ and $f(0) = \frac{2}{S_N^{-1} \| u_{0} \|_1^{2k \gamma}} (\| u_{0} \|_k)^{1+\frac{m-1+2N}{k-1}}$. 

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Step 3. The $L^\infty$ estimates.

On account of the above arguments, our last task is to give the uniform boundedness of solution for any $t > 0$. Denote $q_k = 2^k + 2$, by taking $k = q_k$ in (4.18), we have

\[
\frac{1}{q_k} C_0 D_t^\alpha \int_{\mathbb{R}^N} u^{q_k} dx + \frac{c_3 S q_k}{2} \left( \int_{\mathbb{R}^N} \left| u^{\frac{q_k + m - 1}{2}} (x, t) \right|^{p^*_s} dx \right)^{\frac{2}{p^*_s}} \leq \int_{\mathbb{R}^N} u^{q_k + 1} dx (1 - \int_{\mathbb{R}^N} u dx) - \int_{\mathbb{R}^N} u^{q_k} dx
\]

(4.27)

armed with Lemma 4.10, letting

\[
v = \frac{m + q_k - 1}{2}, \quad q = \frac{2(q_k + 1)}{m + q_k - 1}, \quad r = \frac{2q_k - 1}{m + q_k - 1}, \quad c_0 = c_1 = \frac{1}{2q_k},
\]

one has that for $N \geq 3,$

\[
\|u\|_{L^{q_k+1}(\mathbb{R}^N)}^{q_k+1} \leq C(N)c_0^{\gamma_1 - 1} \left( \int_{\mathbb{R}^N} u^{q_k-1} dx \right)^{\gamma_1} + \frac{1}{2q_k} \left\| \nabla u^{\frac{m+q_k-1}{2}} \right\|_{L^2(\mathbb{R}^N)}^2 + \frac{1}{2q_k} \|u\|_{L^{m+q_k-1}(\mathbb{R}^N)}^{m+q_k-1},
\]

(4.28)

where

\[
\gamma_1 = 1 + \frac{q_k + q_k - 1 + 1}{q_k - 1 + \frac{p(m-2)}{p-2}} \leq 2,
\]

\[
\delta_1 = \frac{(m + q_k - 1) - 2q_k - 1}{q_k - q_k - 1 + 1} = O(1).
\]

Substituting (4.28) into (4.27) and with notice that $\frac{4q_k(q_k-1)}{(m+q_k-1)} \geq 2$. It follows

\[
C_0 D_t^\alpha \int_{\mathbb{R}^N} u^{q_k} dx + \frac{c_3 S q_k}{2} \left( \int_{\mathbb{R}^N} \left| u^{\frac{q_k + m - 1}{2}} (x, t) \right|^{p^*_s} dx \right)^{\frac{2}{p^*_s}} + q_k \int_{\mathbb{R}^N} u^{q_k} dx
\]

\[
\leq C(N)q_k^{\gamma_1 - 1} \left( \int_{\mathbb{R}^N} u^{q_k-1} dx \right)^{\gamma_1} + \frac{1}{2} \left\| \nabla u^{\frac{m+q_k-1}{2}} \right\|_{L^2(\mathbb{R}^N)}^2
\]

\[
+ \frac{1}{2} \|u\|_{L^{m+q_k-1}(\mathbb{R}^N)}^{m+q_k-1} - \int_{\mathbb{R}^N} u dx \int_{\mathbb{R}^N} u^{q_k+1} dx
\]

(4.29)

Applying Lemma 4.10 with

\[
v = u^{\frac{m+q_k-1}{2}}, \quad q = 2, \quad r = \frac{2q_k - 1}{m + q_k - 1}, \quad c_0 = c_1 = \frac{1}{2}
\]

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noticing \( q_{k-1} = \frac{(q_k+1)+1}{2} \), and using Young’s inequality, we obtain
\[
\frac{1}{2} \| u \|^m_{L^m(q_k-1)(\mathbb{R}^N)} = \frac{1}{2} \int_{\mathbb{R}^N} u^{m+q_k-1} dx \\
\leq c_2(N) (\int_{\mathbb{R}^N} u^{q_k-1} dx)^{\gamma_2} + \frac{1}{2} \left\| \nabla u^{\frac{m+q_k-1}{2}} \right\|^2_{L^2(\mathbb{R}^N)} \tag{4.30}
\]
where
\[\gamma_2 = 1 + \frac{m + q_k - q_{k-1} - 1}{q_k} < 2.\]
By summing up (4.29) and (4.30), with the fact that \( \gamma_1 \leq 2 \) and \( \gamma_2 < 2 \), we have
\[
C_0 D_t^\alpha \int_{\mathbb{R}^N} u^{q_k} dx + \frac{C_3 S q_k}{2} \left( \int_{\mathbb{R}^N} \left| u^{\frac{q_k-1}{2}}(x,t) \right|^{p_{\alpha}^*} dx \right)^{\frac{1}{p_{\alpha}^*}} + q_k \int_{\mathbb{R}^N} u^{q_k} dx \\
\leq C(N) q_k^{\delta_1} \left( \int_{\mathbb{R}^N} u^{q_k-1} dx \right)^{\gamma_1} + \left\| \nabla u^{\frac{m+q_k-1}{2}} \right\|^2_{L^2(\mathbb{R}^N)} + C_3(N) \\
\leq \max \{ C(N), C_3(N) \} q_k^{\delta_1} \left( \int_{\mathbb{R}^N} u^{q_k-1} dx \right)^{\gamma_1} + \left\| \nabla u^{\frac{m+q_k-1}{2}} \right\|^2_{L^2(\mathbb{R}^N)} \\
\leq 2 \max \{ C(N), C_3(N) \} q_k^{\delta_1} \max \left\{ \left( \int_{\mathbb{R}^N} u^{q_k-1} dx \right)^{2}, 1, \left\| \nabla u^{\frac{m+q_k-1}{2}} \right\|^2_{L^2(\mathbb{R}^N)} \right\}.
\]
Let \( q_k = \frac{N(1-m)}{2\alpha} \), then \( \frac{q_k+1}{2} p_{\alpha}^* = q_k, \frac{p_{\alpha}^*}{2} = \frac{q_k+1}{2} q_k \). Therefore, we can get
\[
C_0 D_t^\alpha \int_{\mathbb{R}^N} u^{q_k} dx + \frac{C_3 S q_k}{2} \left( \int_{\mathbb{R}^N} u^{q_k} dx \right)^{\frac{q_k+1}{q_k}} + q_k \int_{\mathbb{R}^N} u^{q_k} dx \\
\leq 2 \max \{ C(N), C_3(N) \} q_k^{\delta_1} \max \left\{ \left( \int_{\mathbb{R}^N} u^{q_k-1} dx \right)^{2}, 1, \left\| \nabla u^{\frac{m+q_k-1}{2}} \right\|^2_{L^2(\mathbb{R}^N)} \right\}.
\]
Let
\[K_0 = \max \left\{ 1, \| u_0 \|_{L^1(\mathbb{R}^N)}, \| u_0 \|_{L^\infty(\mathbb{R}^N)}, \left\| \nabla u_0 \right\|^2_{L^2(\mathbb{R}^N)} \right\}, \]
we have the following inequality for initial data
\[
\int_{\mathbb{R}^N} u_0^{q_k} dx \leq \left( \max \left\{ \| u_0 \|_{L^1(\mathbb{R}^N)}, \| u_0 \|_{L^\infty(\mathbb{R}^N)}, \left\| \nabla u_0 \right\|^2_{L^2(\mathbb{R}^N)} \right\} \right)^{q_k} \leq K_0^{q_k}.
\]
Let \( d_0 = \frac{\delta_1}{\delta_1 - 1} \), it is easy to that \( q_k^{d_0} = (2^k + 2)^{d_0} \leq (2^k + 2^{k+1})^{d_0} \). By taking \( \bar{a} = \max \{ C(N), C_3(N) \} 3^{d_0} \) in the Lemma 4.4 we obtain
\[
\int u^{q_k} dx \leq (2\bar{a})^{2^{k-1} - 1} 2^{d_0(2^{k+1} - k - 2)} \max \left\{ \sup_{t \geq 0} \left( \int_{\mathbb{R}^N} u^q dx \right)^{2^k}, k^{q_k} \right\} \frac{T^\alpha}{\alpha \Gamma(\alpha)}. \tag{4.31}
\]
Since \( q_k = 2^k + 2 \) and taking the power \( \frac{1}{q_k} \) to both sides of \( (4.31) \), then the boundedness of the solution \( u(x, t) \) is obtained by passing to the limit \( k \to \infty \)
\[
\|u(x, t)\|_{L^\infty(\mathbb{R}^N)} \leq 2\bar{a} 2^{d_0} \max \left\{ \sup_{t \geq 0} \int_{\mathbb{R}^N} u^{q_0} dx, K_0 \right\} \frac{T^\alpha}{\alpha \Gamma(\alpha)}. \tag{4.32}
\]
On the other hand, by \( (4.26) \) with \( q_0 > 2, t \in [0, T] \), we know
\[
\int_{\mathbb{R}^N} u^{q_0} dx \leq \int_{\mathbb{R}^N} u^{3} dx \leq \left[ \frac{[\lambda_k y^{1-\beta}(0) + (C_2(N, k, m) - \frac{3c_3 S}{\beta})(1 - \beta)]T^\alpha}{\alpha \Gamma(\alpha)} \right]^{\frac{1}{1-\beta}} + y(0) + (1 - a) \frac{1}{1-\beta} \frac{T^\alpha}{\alpha \Gamma(\alpha)} \int_{\mathbb{R}^N} f^{\frac{1}{1-\beta}}(0).
\]
where \( \lambda_k = \frac{a - \beta}{\epsilon^{1-\beta}} - 3(1 - \beta), y(0) = \|u_0\|_{L^1(\mathbb{R}^N)}^3 \) and \( f(0) = \frac{2}{S_N^{-1} ||u_0||_{L^1(\mathbb{R}^N)}}. \)

Therefore we finally have
\[
\|u(x, t)\|_{L^\infty(\mathbb{R}^N)} \leq C(N, \|u_0\|_{L^1(\mathbb{R}^N)}, \|u_0\|_{L^\infty(\mathbb{R}^N)}, \|\nabla u_0\|_{L^2(\mathbb{R}^N)}^2, T^\alpha) = M.
\]

**Remark 4.5.** The solution for problem \( (1.8)-(1.9) \) constructed above only need to be integrable with respect to the weight \( \varphi \), which has a tail of order less than \( N + 2s/m \). And the method to proof main reference \[63\]. On the other hand, The method to prove the \( L^r \) estimate to the \( L^\infty \) estimate of the equation is also mentioned in \[64\]. This section mainly uses fractional differential inequalities, Gagliardao-Nirenberg inequalities and so on. Therefore, the \( L^r \) estimate is obtained, and if \( k = q_k \) is estimated on \( L^\infty \), the global boundedness of the solution for the nonlinear TSFNRDE is proved in \( \mathbb{R}^N, N \geq 3 \).

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6. Appendix A. Definitions, Related Lemma, and complements

A.1 Related Lemma to proof existence weak solution for (1.1)-(1.2)

Lemma 6.1. [40] Let $0 < \alpha < 1$ and $\lambda > 0$, then we have

$$\frac{d}{dt} E_{\alpha,1}(-\lambda t^\alpha) = -\lambda t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha), \quad t > 0.$$

Lemma 6.2. [40] Let $0 < \alpha < 1$ and $\lambda > 0$, then we have

$$\partial_t^\alpha E_{\alpha,1}(-\lambda t^\alpha) = -\lambda E_{\alpha,\alpha}(-\lambda t^\alpha), \quad t > 0.$$

Lemma 6.3. [72] For $0 < \alpha < 1$, $\lambda > 0$ and Let $AC[0,T]$ be the space of functions $f$ which are absolutely continuous on $[0,T]$, if $q(t) \in AC[0,T]$ then we have

$$\partial_t^\alpha \int_0^t (t - \tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t - \tau)^\alpha) d\tau = q(t) - \lambda \int_0^t q(\tau)(t - \tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t - \tau)^\alpha) d\tau, \quad t \in (0,T].$$

Lemma 6.4. [40] Suppose $p(t) \in L^\infty(0,T), 0 < \alpha < 1, \lambda \geq 0$, denote

$$g(t) = \int_0^t p(\tau)(t - \tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t - \tau)^\alpha) d\tau, \quad t \in (0,T],$$

and defines $g(0) = 0$, then $g(t) \in C[0,T]$

A.2 The proof of existence weak solution for (1.1)-(1.2)

Remark 6.1. By consulting the relevant reference, we can get the Lemma 2.14 by [5], but “For brevity, we leave the detail to the reader.” [p255, Theorem 1, 2]. Therefore, we will give the proof process of Lemma 2.14 as shown below.
Proof of Lemma 2.14. We will show that (2.8) certainly gives a weak solution to (1.1)-(1.2). In the following proof, we denote $C$ as a generic positive constant and make $q(t) = p(t) = 1$ for Lemma 6.3 and Lemma 6.4. Denote $g_j(t) = \int_0^t (t - \tau)^{\alpha - 1} E_{\alpha,\alpha}(-\lambda_j^\alpha(t - \tau)^\alpha) d\tau$, then we know
\begin{align*}
|g_j(t)| &\leq \frac{1}{\Gamma(\alpha + 1)T^\alpha}, & t \in [0, T], \\
|g_j(t)| &\leq \frac{(1 - E_{\alpha,1}(-\lambda_n T^{\alpha}))}{\lambda_j^\alpha}, & t \in [0, T].
\end{align*}

The proof is divided into several steps.

(1) We first verify $u \in C([0, T]; L^2(\Omega))$ and $\lim_{t \to 0^+} \|u(t) - u_0\| = 0$. Define
\begin{align*}
u_1 := \sum_{j=1}^\infty E_{\alpha,1}(-\lambda^\alpha_j t^\alpha)(u_0, \Phi_j) \Phi_j, \\
u_2 := \sum_{j=1}^\infty g_j(t)(F(\tau), \Phi_j) \Phi_j.
\end{align*}

Then we have $u(\cdot, t) = u_1(\cdot, t) + u_2(\cdot, t)$. We estimate each term separately. For fixed $t \in [0, T]$, by Lemma 2.4 and (6.2), we have
\begin{align*}
\|u_1(\cdot, t)\|^2_{L^2(\Omega)} = \sum_{j=1}^\infty E_{\alpha,1}^2(-\lambda_j^\alpha t^\alpha)(u_0, \Phi_j)^2 \leq \|u_0\|^2_{L^2(\Omega)},
\end{align*}
and
\begin{align*}
\|u_2(\cdot, t)\|^2_{L^2(\Omega)} = \sum_{j=1}^\infty g_j^2(t)(F(\tau), \Phi_j)^2 \leq \sum_{j=1}^\infty \frac{(F(\tau), \Phi_j)^2}{\Gamma^2(\alpha + 1)} t^{2\alpha}.
\end{align*}

By (6.5), we know
\begin{align*}
\lim_{t \to 0^+} \|u_2(\cdot, t)\|_{L^2(\Omega)} = 0.
\end{align*}

Thus define $u_2(x, 0) = 0$. From (6.4)-(6.5), we obtain
\begin{align*}
\|u_2(\cdot, t)\|_{L^2(\Omega)} \leq C_1(\|u_0\|_{L^2(\Omega)} + \|F\|_{L^2(\Omega)}), & t \in [0, T],
\end{align*}
where $C_1 = \max \left\{ 1, \frac{T^n}{T^{n+1}} \right\}$. For $t, t + h \in [0, T]$, we have

$$u(x, t + h) - u(x, t) = \sum_{j=1}^{\infty} \left( E_{\alpha, 1}(-\lambda_j^s(t + h)^\alpha) - E_{\alpha, 1}(-\lambda_j^s t^\alpha) \right)(u_0, \Phi_j)\Phi_j$$

$$+ \sum_{j=1}^{\infty} (g_j(t + h) - g_j(t))(F(\tau), \Phi_j)\Phi_j.$$  

$$=: I_1(x; t; h) + I_2(x; t; h).$$

We estimate each term separately. In fact, by Lemma 2.4, we have

$$\|I_1(\cdot; t; h)\|_{L^2(\Omega)}^{2} = \sum_{j=1}^{\infty} \left| E_{\alpha, 1}(-\lambda_j^s(t + h)^\alpha) - E_{\alpha, 1}(-\lambda_j^s t^\alpha) \right|^2 (u_0, \Phi_j)^2$$

$$\leq 4 \|u_0\|_{L^2(\Omega)}^{2},$$

since $\lim_{h \to 0} \left| E_{\alpha, 1}(-\lambda_j^s(t + h)^\alpha) - E_{\alpha, 1}(-\lambda_j^s t^\alpha) \right| = 0$, by using the Lebesgue theorem, we have

$$\lim_{h \to 0} \|I_1(\cdot; t; h)\|_{L^2(\Omega)}^{2} = 0.$$  

By (6.2), we have

$$\|I_2(\cdot; t; h)\|_{L^2(\Omega)}^{2} = \sum_{j=1}^{\infty} (g_j(t + h) - g_j(t))^2(F(\tau), \Phi_j)^2 \leq C \|F\|_{L^2(\Omega)}^{2}.$$  

Similarly, by using the Lebesgue theorem and Lemma 6.4 we can prove

$$\lim_{h \to 0} \|I_2(\cdot; t; h)\|_{L^2(\Omega)}^{2} = 0.$$  

Therefore, $u \in C([0, T]; L^2(\Omega))$. By Lemma 2.4, we know

$$\|u(\cdot; t) - u_0(\cdot)\|_{L^2(\Omega)} \leq \left( \sum_{j=1}^{\infty} (u_0, \Phi_j)^2 (E_{\alpha, 1}(-\lambda_j^s t^\alpha) - 1)^2 \right)^{\frac{1}{2}} + \|u_2(\cdot; t)\|_{L^2(\Omega)}$$

$$\leq \|u_0\|_{L^2(\Omega)} + \|u_2(\cdot; t)\|_{L^2(\Omega)}.$$  

Since $\lim_{t \to 0}(E_{\alpha, 1}(-\lambda_j^s t^\alpha) - 1) = 0$ and (6.6), we have

$$\lim_{t \to 0^+} \|u(t) - u_0\| = 0.$$  

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We verify \( u \in L^2(0, T; D((-\Delta)^s)) \). By (2.8), we know

\[
(-\Delta)^s u(x, t) = \sum_{j=1}^{\infty} \lambda_j^s E_{\alpha,1}(-\lambda_j^s t^\alpha)(u_0, \Phi_j) \Phi_j + \sum_{j=1}^{\infty} \lambda_j^s g_j(t)(F(\tau), \Phi_j) \Phi_j
\]

\[
:= v_1(x, t) + v_2(x, t),
\]

where \( g_j(t) \) is defined in (6.1). For \( 0 < t \leq T \), by Definition 2.4, we obtain

\[
\|v_1(\cdot, t)\|^2_{L^2(\Omega)} = \sum_{j=1}^{\infty} (\lambda_j^s E_{\alpha,1}(-\lambda_j^s t^\alpha)(u_0, \Phi_j))^2 \leq \sum_{j=1}^{\infty} \left( \lambda_j^s(u_0, \Phi_j)^2 \left( \frac{c \sqrt{\lambda_j^{s\alpha}}}{1 + \lambda_j^{s\alpha}} \right)^2 \right) \leq C_2 \|u_0\|_{D((-\Delta)^s)}^2 \frac{\|u_0\|_{D((-\Delta)^s)}}{t^\alpha}.
\]

(6.7)

where \( C \) is a constant depending on \( \alpha \) only. For the second term \( v_2 \), by (6.3) we can deduce that

\[
\|v_2(\cdot, t)\|^2_{L^2(\Omega)} = \sum_{j=1}^{\infty} \lambda_j^{2s} g_j^2(t)(F(\tau), \Phi_j)^2 \leq \sum_{j=1}^{\infty} (F(\tau), \Phi_j)^2 \leq \|F\|^2_{L^2(\Omega)}.
\]

(6.8)

By estomates (6.7)-(6.8), we know \( v_1, v_2 \in L^2(0, T; L^2(\Omega)) \), hence \((-\Delta)^s u \in L^2(0, T; L^2(\Omega))\). Moreover, we can obtain the following estimate from (6.7)-(6.8)

\[
\|u\|_{L^2(0,T;D((-\Delta)^s))} = \|(-\Delta)^s u\|_{L^2(0,T;L^2(\Omega))} \leq C_2(\alpha, T, \Omega)(\|u_0\|_{D((-\Delta)^s)} + \|F\|^2_{L^2(\Omega)}).
\]

where \( C_2 \) is a positive constant. Therefore, \( u \in L^2(0, T; D((-\Delta)^s)) \).

(3) We prove that \( \partial_t \partial^\alpha u(x, t) \in C((0, T]; L^2(\Omega)) \cap L^2(0, T; L^2(\Omega)) \) and (1.1)-(1.2) holds in \( L^2(\Omega) \) for \( t \in (0, T] \). By Lemma 6.3 and Lemma 6.2, we have

\[
\partial_t \partial^\alpha u(x, t) = -\sum_{j=1}^{\infty} \lambda_j^s E_{\alpha,1}(-\lambda_j^s t^\alpha)(u_0, \Phi_j) \Phi_j + \sum_{j=1}^{\infty} (F(\tau), \Phi_j)[1 - \lambda_j^s \int_0^t (t - \tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_j^s (t - \tau)^\alpha) d\tau] \Phi_j \]

\[
= F(t) - (-\Delta)^s u(x, t).
\]
Hence $\partial_t^\alpha u(x, t) \in C((0, T]; L^2(\Omega)) \cap L^2(0, T; L^2(\Omega))$ and (1.1)-(1.2) holds in $L^2(\Omega)$ for $t \in (0, T]$.

(4) We prove the uniqueness of the weak solution to (1.1)-(1.2). Under the condition $F(t) = \mu u^2(1 - kJ^+ u) - \gamma u = 0$, $u_0 = 0$, we need to prove that systems (1.1) has only a trivial solution. We take the inner product of (1.1) with $\Phi_j(x)$. Using the Green formula and $\Phi_j(x)|_{\partial \Omega} = 0$ and setting $u_j(t) := (u(\cdot, t), \Phi_j(x))$, we obtain

$$\begin{cases}
\partial_t^\alpha u_j(t) = -\lambda_j^\alpha u_j(t), \quad t \in (0, T], \\
u_j(0) = 0.
\end{cases}$$

Due to the existence and uniqueness of the ordinary fractional differential equation in [44], we obtain that $u_j(t) = 0, j = 1, 2, \cdots$. Since $\{\Phi_j\}_{j \geq 1}$ is an orthonormal basis in $L^2(\Omega)$, we have $u = 0$ in $\omega \times (0, T]$. Thus the proof is complete.

**Remark 6.2.** The method of this Lemma is used by [37]. However, in reference [37], there is non-local term $p(t)f(x)$. In the paper, There is non-local $F(x, t) = \mu u^2(1 - kJ^+ u) - \gamma u$ which is different from [37]. From [5], we obtain the conclusion, but “For brevity, we leave the detail to the reader.” [p255, Theorem 1, [5]]. Therefore, We have done the above related proof.

A.3 Definition of weak and strong solution for nonlinear fractional diffusion equation

We call here the definition of weak and strong solution taken from [1]. Considering the following Cauchy problem

$$\begin{cases}
\partial_t u + (-\Delta)^{\sigma/2}(|u|^{m-1} u) = 0, \quad x \in \mathbb{R}^N, t > 0 \\
u(x, 0) = f(x), \quad x \in \mathbb{R}^N
\end{cases}$$

(6.9)

**Definition 6.1.** [1] A function $u$ is a weak solution to the problem (6.9) if:

- $u \in C((0, \infty); L^1(\mathbb{R}^d))$ and $|u|^{m-1} u \in L^2_{lo}(0, T; \dot{H}^{\sigma/2}(\mathbb{R}^d))$;

- The identity

$$\int_0^\infty \int_{\mathbb{R}^d} u \frac{\partial \varphi}{\partial t} dx dt + \int_0^\infty \int_{\mathbb{R}^d} (-\Delta)^{\sigma/4}(|u|^{m-1} u)(-\Delta)^{\sigma/4} \varphi dx dt = 0$$

holds for every $\varphi \in C^1_0(\mathbb{R}^d \times (0, \infty))$;
• \( u(\cdot, 0) = u_0 \in L^1(\mathbb{R}^N) \) almost everywhere.

Note that the fractional Sobolev space \( \dot{H}^{\sigma/2}(\mathbb{R}^d) \) is defined as the completion of \( C_0^\infty(\mathbb{R}^d) \) with the norm

\[
\|\psi\|_{\dot{H}^{\sigma/2}} = \left( \int_{\mathbb{R}^d} |\xi|^{\sigma} |\hat{\psi}|^2 \, d\xi \right)^{1/2} = \|(-\Delta)^{\sigma/4}\psi\|_2
\]

**Definition 6.2.** \([1]\) We say that a weak solution \( u \) to the problem (6.9) is a strong solution if \( \partial_t u \in L^\infty((\tau, \infty); L^1(\mathbb{R}^N)), \tau > 0. \)

On the other hand, we recall the definition of weak solution taken from [62].

Considering the following direct problem:

\[
\begin{aligned}
\frac{\partial^\beta u}{\partial t^\beta} &= \nabla \cdot (a(u) \nabla u) + f(x, t), \quad (x, t) \in \Omega_T, \\
u(x, 0) &= 0, \quad x \in \Omega, \\
u(x, t) &= 0, \quad (x, t) \in \Gamma_1 \times [0, T], \Gamma_1 \subset \partial \Omega, \\
a(u) \frac{\partial u}{\partial n} &= \varphi(x, t), \quad (x, t) \in \Gamma_2 \times [0, T], \Gamma_2 \subset \partial \Omega,
\end{aligned}
\tag{6.10}
\]

where \( \Omega_T := \Omega \times (0, T), \) the domain \( \Omega \subset \mathbb{R}^n(n \geq 1) \) is assumed to be bounded simple connected with a piecewise smooth boundary \( \Gamma \) and \( \Gamma_1 \cap \Gamma_2 = \emptyset, \Gamma_1 \cup \Gamma_2 = \Gamma, \) mean \( (\Gamma_i) \neq 0, i = 1, 2. \)

**Definition 6.3.** \([62]\) A weak solution of problem (6.11) is a function

\[ u \in L^2(0, T; H^1_0(\Omega)) \cap W^\beta_2(0, T; L^2(\Omega)) \]

such that the following integral identity holds for a.e \( t \in [0, T] : \)

\[
\int_{\Omega} \frac{\partial^\beta u}{\partial t^\beta} vdx + \int_{\Omega} a(u) \nabla u \cdot \nabla vdx = \int_{\Omega} f vdx + \int_{\Gamma_2} \varphi vdx,
\]

for each \( v \in L^2(0, T; H^1_0(\Omega)) \cap W^\beta_2(0, T; L^2(\Omega)), \) where

\[ W^\beta_2(0, T) := \left\{ u \in L^2[0, T] : \frac{\partial^\beta u}{\partial t^\beta} \in L^2[0, T] \text{ and } u(0) = 0 \right\} \]

is the fractional Sobolev space of order \( \beta. \)
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