A FOCK SPACE APPROACH TO REPRESENTATION THEORY OF $\text{osp}(2|2n)$

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Abstract. A Fock space is introduced that admits an action of a quantum group of type $A$ supplemented with some extra operators. The canonical and dual canonical basis of the Fock space are computed and then used to derive the finite-dimensional tilting and irreducible characters for the Lie superalgebra $\text{osp}(2|2n)$. We also determine all the composition factors of the symmetric tensors of the natural $\text{osp}(2|2n)$-module.

Introduction

Supersymmetry usually manifests itself as concrete representations of the relevant Lie superalgebras, and thus the representation theory of Lie superalgebras plays an essential role in the study of supersymmetry. A crucial difference between simple Lie algebras and simple Lie superalgebras is that the categories of finite dimensional representations of the latter are in general not semi-simple [K1, K2]. Partially inspired by earlier works of Lascoux-Leclerc-Thibon [LLT] and Serganova [Se], Brundan in [B1] developed a new approach to the representation theory of the general linear superalgebra (and in [B3] for $\mathfrak{q}(n)$), where he formulated the Kazhdan-Lusztig theory in terms of canonical bases of a Fock space. The Fock space approach enabled Brundan to establish, among other results, a conjecture of [VZ] in the affirmative. The Fock space approach is not only technically powerful, it also leads to a new conceptual framework relating the representations of general linear algebras to those of general linear superalgebras via a fundamental duality [CWZ].

The main purpose of this paper is to extend Brundan’s Fock space approach of Kazhdan-Lusztig theory to the Lie superalgebras $\text{osp}(2|2n)$. After reviewing some background materials in Section 1 we introduce in Section 2 a Fock space with the action of a quantum group of type $A$ supplemented with some extra generators. We remark that the action of the quantum group arises from the two opposite comultiplications. In contrast to [B1], a weight space of the Fock space here may correspond to a unique block or sometimes to two blocks. A bar involution is defined and the canonical and dual canonical bases of the Fock space are worked out explicitly. Theorem 3.2 shows that the canonical basis elements when specialized to $q = 1$ correspond to the finite dimensional tilting modules of $\text{osp}(2|2n)$ while

Partially supported by NSC of R.O.C..
Partially supported by NSA and NSF.
Partially supported by Australian Research Council.
the standard monomials correspond to the Kac modules. This together with the
general theory of tilting modules [So, B2] gives an explicit determination of the
composition factors of Kac modules (Corollary 3.3).

We point out that the finite dimensional irreducible representations of osp(2|2n)
were studied in the work of van der Jeugt [V], who in particular established the
composition factors of Kac modules in a completely different way. This readily
implies a Bernstein-Leites type character formula for such representations as noted in [V]. Also the projective covers in the category of finite-dimensional osp(2|2n)-modules were understood in [Zou]. The new Fock space approach here appears to be conceptually interesting and simple, and it is our hope that it may provide new
insights into the representation theory of other Lie superalgebras.

In spite of the power of the Kazhdan-Lusztig-Brundan theory, the structure of
various naturally constructed modules of Lie superalgebras often remains unclear.
The skew-symmetric tensor of the natural osp(2|2n)-module is easily seen to be
irreducible. In contrast to the classical Lie algebra setup, the symmetric tensors
of the natural module of osp(2|2n) is not completely reducible in general and it is
a rather nontrivial problem to determine the composition factors. In Section 4 we
offer a complete solution to this problem. There exists a surjective homomorphism
via a Laplacian operator ∆ from the $k$-th symmetric tensor to the $(k - 2)$-th
symmetric tensor for each $k$. We show that the kernel of ∆ has 1, 3, or 2 composition
factors depending on whether $k \leq n$, $n < k \leq 2n$, or $k > 2n$. The simplest case
when $k \leq n$ can be also found in [Lee].

**Acknowledgments.** We thank all three host institutions of the authors and
NCTS-Taipei office for the hospitality and support.

### 1. Preliminaries

In this section we present some background material for the use in later sections.

#### 1.1. Lie superalgebra $osp(2|2n)$

Throughout this paper, we shall denote by $\mathfrak{g}$ the Lie superalgebra $osp(2|2n)$ whose standard Dynkin diagram together with the simple roots is given by:

\[
\begin{array}{c}
\bigotimes & \circ & \cdots & \circ & \equiv & \circ \\
\epsilon - \delta_1 & \delta_1 - \delta_2 & \delta_{n-1} - \delta_n & 2\delta_n
\end{array}
\]

Here $\epsilon - \delta_1$ is odd. The set of positive roots is a union of the even and the odd ones: $\Delta^+ = \Delta^+_0 \cup \Delta^+_1$. Denote by $\{e_i, f_i, h_i\}$ for $i = 0, 1, \ldots, n$ the corresponding Chevalley generators of $\mathfrak{g}$. Let $\rho = -n\epsilon + \sum_{i=1}^{n} (n - i + 1)\delta_i$, which is half the graded sum of the positive roots of $\mathfrak{g}$. Let $\mathfrak{b}$ be the Borel subalgebra, and $\mathfrak{h} \subset \mathfrak{b}$ be the Cartan subalgebra of $\mathfrak{g}$ compatible with the above choice of the simple roots.

The space $\mathfrak{h}^*$ is endowed with a non-degenerate symmetric bilinear form

\[(\epsilon, \epsilon) = 1, \quad (\delta_i, \delta_j) = -\delta_{ij}, \quad (\epsilon, \delta_i) = 0, \quad \forall i, j.
\]

A weight $\mu$ is called *atypical* if there is an odd positive root $\gamma = \epsilon - \delta_i$ or $\epsilon + \delta_i$ for some $i$ such that $(\mu + \rho, \gamma) = 0$, and is called *typical* otherwise (cf. [K2]).
Denote by $\mathcal{O}^+$ the category of finite dimensional $\mathbb{Z}_2$-graded $g$-modules of integral weights, that is, the weights of every module belong to the $\mathbb{Z}$-span of $\epsilon, \delta_i, i \geq 1$. We denote by $X_+^{1|n}$ the set of the dominant integral weights of $g$, namely,

\[ X_+^{1|n} = \left\{ \lambda = \lambda_1 \epsilon + \sum_{i=1}^{n} \lambda_i \delta_i \mid \lambda_i \in \mathbb{Z}, \forall i; \; \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \right\}. \]

The Lie superalgebra $g$ has a $\mathbb{Z}$-grading $g = g_- + g_0 + g_+$, where $g_0$ is its even subalgebra and $g_+$ (respectively $g_-$) is the subalgebra spanned by the odd positive (respectively negative) root vectors. The Kac module is defined as $K(\lambda) := U(g) \otimes_{U(g_0 + g_+)} L^0(\lambda)$, where $L^0(\lambda)$ denotes the irreducible $g_0$-module of highest weight $\lambda$ (extended trivially to $g_0 + g_+$). Then the irreducible module $L(\lambda)$ and Kac module $K(\lambda)$ with highest weight $\lambda$ belong to $\mathcal{O}^+$ if and only if $\lambda \in X_+^{1|n}$. It is known that $K(\lambda) = L(\lambda)$ if $\lambda$ is typical.

1.2. The Bruhat order. The Weyl group $W$ of $g$ is defined to be the Weyl group of the $\text{sp}(2n)$ subalgebra, which is generated by the reflections corresponding to the even simple roots of $g$. If an element $\lambda \in X_+^{1|n}$ is atypical with respect to an odd positive root $\gamma$, we define

\[ \lambda^L := w(\lambda + \rho - k\gamma) - \rho. \] (1.1)

Here $k$ is the smallest positive integer such that $\lambda + \rho - k\gamma$ is $\text{sp}(2n)$-regular in the sense that $(\lambda + \rho - k\gamma, \alpha_i) \neq 0, \forall i \geq 1$, and $w$ is the unique element in the Weyl group of $g$ rendering $\lambda^L$ dominant. For example, if $\lambda = \epsilon$, then $\lambda^L = -\epsilon + \delta_1 + \delta_2$. A more explicit description of this “$L$-operator” will be given below. A representation-theoretical interpretation is given in Corollary 33. Given $\lambda \in X_+^{1|n}$, we shall write

\[ \lambda^{(0)} = \lambda, \quad \lambda^{(l+1)} = (\lambda^{(l)})^L, \quad l \geq 0. \] (1.2)

The Bruhat order on $X_+^{1|n}$ is the partial order such that for $\lambda, \mu \in X_+^{1|n}$, $\mu \prec \lambda$ if and only if $\mu = \lambda^{(l+1)}$ for some $l \geq 0$.

Denote by $Y_+^{1|n}$ the set of the $(n+1)$-tuples $f = (f_{-1} \mid f_1, f_2, \ldots, f_n)$ of integers such that $f_1 < f_2 < \cdots < f_n < 0$. There is a bijection

\[ X_+^{1|n} \to Y_+^{1|n}, \quad \lambda \mapsto f_\lambda, \] (1.3)

where $f_\lambda$ is specified by

\[ (f_\lambda)_{-1} = (\lambda + \rho, \epsilon), \quad (f_\lambda)_i = (\lambda + \rho, \delta_i), \quad i \geq 1. \]

An element $f_\lambda \in Y_+^{1|n}$ will also be called atypical (respectively typical) if $\lambda \in X_+^{1|n}$ is atypical (respectively typical). Note that $f \in Y_+^{1|n}$ is atypical if $|f_{-1}| = -f_i$ for some $i \geq 1$. If $f = f_\lambda$ for some atypical $\lambda \in X_+^{1|n}$, we set $f^L = f_{\lambda^L}$. The Bruhat order on $X_+^{1|n}$ induces a partial order on $Y_+^{1|n}$ via the bijection (1.3).

The description of $f^L$ for a given atypical $f$ is divided into three cases as follows.

(1) $f_{-1} = f_i < 0$ for some $1 \leq i \leq n$. Let $d$ be the largest integer such that $d < f_i$ and $d \notin \{f_1, \ldots, f_n\}$. Then, $f^L = (d|f_1, \ldots, \hat{f_i}, \ldots, f_n, d)^+$, where $\hat{f_i}$ denotes
the removal of $f_i$, and $+$ denotes the rearrangement of $f_1, \ldots, \hat{f}_i, \ldots, f_n, d$ in a decreasing order.

(II) $f_{-1} = -f_i > 0$ for some $1 \leq i \leq n$, and $\{f_1, f_2, \ldots, f_n\}$ does not contain $\{-1, -2, \ldots, f_i\}$ as a subset. Let $c$ be the largest integer such that $-1 \geq -c > f_i$ and $-c \notin \{f_1, \ldots, f_n\}$. Then, $f^c_i = (c|f_1, \ldots, \hat{f}_i, \ldots, f_n, -c)^+$.

(III) $f_{-1} = -f_i > 0$ for some $1 \leq i \leq n$, and the set $\{f_1, f_2, \ldots, f_n\}$ contains $\{-1, -2, \ldots, f_i\}$ as a subset. Then $f^c_i = (-f_{-1}|f_1, \ldots, f_n)$.

**Lemma 1.1.** Let $\beta = 2n\epsilon$, the sum of all odd positive roots. Let $w_0$ be the longest element in $W$. Then, $f_{\beta - w_0 \lambda}$ is obtained from $f_\lambda$ by changing the sign on the $(-1)$st component. That is, $f_{\beta - w_0 \lambda} = -w_0 f_\lambda$.

**Proof.** Note that $w_0 \lambda = \lambda_{-1} \epsilon - \sum_{i=1}^{n} \lambda_i \delta_i$ for $\lambda = \lambda_{-1} \epsilon + \sum_{i=1}^{n} \lambda_i \delta_i$. The lemma now follows by unravelling the definitions. \hfill \Box

It follows from Lemma 1.1 and (I), (II), (III) above that

$$\beta - w_0 \lambda = (\beta - w_0 \lambda)^L. \quad (1.4)$$

### 1.3. Quantum group

The quantum group $U_q(\mathfrak{gl}(\infty))$ is the $\mathbb{Q}(q)$-algebra generated by $E_a, F_a, K_a^\pm, a \in \mathbb{Z}$, subject to the following relations

$$K_a K_a^{-1} = K_a^{-1} K_a = 1,$$

$$K_a E_b K_a^{-1} = q^{\delta_{a,b}-\delta_{a,b+1}} E_b,$$

$$K_a F_b K_a^{-1} = q^{\delta_{a,b+1}-\delta_{a,b}} F_b,$$

$$E_a F_b - F_b E_a = \delta_{a,b} (K_a a+1 - K_a a), (q - q^{-1}),$$

$$E_a E_b = E_b E_a, \quad F_a F_b = F_b F_a, \quad \text{if } |a - b| > 1,$$

$$E_a^2 E_b + E_b E_a^2 = (q + q^{-1}) E_a E_b E_a, \quad \text{if } |a - b| = 1,$$

$$F_a^2 F_b + F_b F_a^2 = (q + q^{-1}) F_a F_b F_a, \quad \text{if } |a - b| = 1.$$

Here and further $K_{a+1} := K_a K_a^{-1}$, $a \in \mathbb{Z}$. Define the bar involution on $U_q(\mathfrak{gl}(\infty))$ to be the anti-linear automorphism — such that $\overline{E_a} = E_a, \overline{F_a} = F_a, \overline{K_a} = K_a^{-1}$. As usual anti-linear means $q \mapsto q^{-1}$. We will sometimes also write $\overline{E_a} = E_{a+1}$ and $\overline{F_a} = F_{a+1}$.

Let $\mathbb{V}$ be the natural $U_q(\mathfrak{gl}(\infty))$-module with basis $\{v_a\}_{a \in \mathbb{Z}}$ and $\mathbb{V}^* := \mathbb{V}^*$ the dual module with basis $\{w_a\}_{a \in \mathbb{Z}}$ such that

$$w_a(v_b) = (-q)^{-a} \delta_{a,b}. \quad (1.5)$$

The action of the Chevalley generators on these basis elements are given explicitly by:

$$K_a v_b = q^{\delta_{a,b}} v_b, \quad E_a v_b = \delta_{a+1,b} v_a, \quad F_a v_b = \delta_{a,b} v_{a+1},$$

$$K_a w_b = q^{-\delta_{a,b}} w_b, \quad E_a w_b = \delta_{a,b} w_{a+1}, \quad F_a w_b = \delta_{a+1,b} w_a.$$

We shall use the same comultiplication $\Delta$ on $U_q(\mathfrak{gl}(\infty))$ as in \cite{B1, OWZ}:

$$\Delta(E_a) = 1 \otimes E_a + E_a \otimes K_{a+1},$$

$$\Delta(F_a) = F_a \otimes 1 + K_{a+1} \otimes F_a,$$

$$\Delta(K_a) = K_a \otimes K_a. \quad (1.6)$$
We denote the Iwahori-Hecke algebra of type $A$ by $\mathcal{H}_n$, which is the $\mathbb{Q}(q)$-algebra generated by $H_i$, where $1 \leq i \leq n - 1$, subject to the relations

$$(H_i - q^{-1})(H_i + q) = 0, \quad H_i H_{i+1} H_i = H_{i+1} H_i H_{i+1}, \quad H_i H_j = H_j H_i \quad (|i - j| > 1).$$

For $x \in S_n$ with a reduced expression $x = s_{i_1} \cdots s_{i_r}$, we set $H_x := H_{i_1} \cdots H_{i_r}$. The bar involution $-\cdot$ on $\mathcal{H}_n$ is the unique anti-linear automorphism defined by

$$\overline{H_x} = H_{x^{-1}}^{-1}$$

for all $x \in S_n$. We let $H_0 := \sum_{x \in S_n} (-q)^{\ell(x) - \ell(\sigma_0)} H_x$ where $\sigma_0$ is the longest element in $S_n$.

2. Canonical and dual canonical bases on a Fock space

2.1. The Fock space. Denote by $V_+, V_0$, and $V_-$ the subspaces of $V$ spanned by $v_i$ with $i > 0$, $i = 0$, and $i < 0$, respectively. Denote by $U$ (respectively $U_+$) the subalgebras of $U_q(\mathfrak{gl}(\infty))$ generated by $E_i, F_i, K_i, K_i^{-1}$, $i \leq -2$ (respectively by $E_i, F_i, K_i, K_i^{-1}$, $i \geq 1$). Then $V_-$ (respectively $V_+$) is the natural $U$ (respectively $U_+$) module.

Lemma 2.1. (1) There is an algebra isomorphism $\phi : U_+ \cong U$ given by

$$E_i \mapsto F_{-i-1}, \quad F_i \mapsto E_{-i-1}, \quad K_i^{\pm 1} \mapsto K_i^{\pm 1} \quad (i > 0).$$

(2) The composition of $\phi^{-1}$ with the natural action of $U_+$ on $V_+$ defines a $U$-module structure on $V_+$. Furthermore the linear map $V_+ \to V_-$, $v_i \mapsto v_{-i}$, $i > 0$, is an isomorphism of $U$-modules.

Proof. Part (1) follows by checking the defining relations, while (2) follows from the explicit formulae of the actions of $U$ on $V_-$ and $U_+$ on $V_+$. \hfill \Box

Denote by $\mathbb{W}_-$ the $U$-module which is dual to the natural $U$-module $V_-$, with generators $w_i$, $i < 0$, normalized as in (1.5). Define the $U$-module $\bigotimes^n \mathbb{W}_-$ via the usual comultiplication $\Delta(n-1) = (\text{id} \otimes (n-2) \otimes \Delta) \cdots (\text{id} \otimes \Delta)$. By the Schur-Jimbo duality, $\mathcal{H}_n$ acts on $\bigotimes^n \mathbb{W}_-$ and this action commutes with the action of $U$. Define $\wedge^n \mathbb{W}_-$ to be the quotient of $\bigotimes^n \mathbb{W}_-$ by the kernel of $H_0$. Denote the image of $w_{f_1} \otimes \cdots \otimes w_{f_n}$ in $\wedge^n \mathbb{W}_-$ by $w_{f_1} \wedge \cdots \wedge w_{f_n}$, for $w_{f_1}, \ldots, w_{f_n} \in \mathbb{W}_-$.

Consider the following Fock space

$$\mathcal{F} := V \otimes \bigwedge^n \mathbb{W}_-.$$

For $f = (f_1|f_1, \cdots, f_n) \in V_+^{\otimes n}$, let

$$K_f := v_{f_{-1}} \otimes w_{f_1} \wedge w_{f_2} \wedge \cdots \wedge w_{f_n}.$$ 

(2.1)

Then the $K_f$ form a basis of $\mathcal{F}$.

Define an action of the quantum group $U$ on the space $\mathcal{F}$ in the following way. Let $\mathcal{F}_+ := V_+ \otimes \bigwedge^n \mathbb{W}_-$ for $\bullet = +, -, 0$. The action of $U$ on $\mathcal{F}_-$ is defined exactly as in (1.6) via $\Delta$. The action of $U$ on $\mathcal{F}_+$ is via $\Delta' = (1 \otimes \phi) \circ \Delta \circ \phi^{-1}$, which is a
mixture of the comultiplication $\Delta$ and the isomorphism $\phi$: for $i > 0$,

\[
\Delta'(E_{-i,-i}) = 1 \otimes E_{-i,-i} + E_{i,i+1} \otimes K_{i-1,-i},
\]

\[
\Delta'(E_{-i-1,-i}) = E_{i+1,i} \otimes 1 + K_{i+1,i} \otimes E_{-i-1,-i},
\]

\[
\Delta'(K_{-i}) = K_{i} \otimes K_{-i}.
\]

The action of $\mathcal{U}$ on $\mathcal{F}_0$ is defined by $x \mapsto 1 \otimes x$ for every $x \in \mathcal{U}$ which is compatible with either $\Delta$ or $\Delta'$. Putting together, we have defined an action of $\mathcal{U}$ on $\mathcal{F} = \mathcal{F}_+ \oplus \mathcal{F}_0 \oplus \mathcal{F}_-$.

We also define the following operators on $\mathcal{F}$:

\[
E_{-1} := E_{-1,0} \otimes (K_{-1})^{-1} + E_{1,0} \otimes 1, \quad F_{-1} := E_{0,-1} \otimes 1 + E_{0,1} \otimes K_{-1}.
\]

The following lemmas can be proved by straightforward calculations.

**Lemma 2.2.** For $g = (g_{-1}|g_1,\ldots,g_n) \in Y_{+}^{1|n}$, let $g^\pm = (g_{-1} \pm 1|g_1,\ldots,g_n)$. The actions of $E_{-1}$ and $F_{-1}$ on $K_g$ vanish unless $g_{-1} = 0, \pm 1$, or $g_n = -1$. In these cases,

1. if $g_n = -1$, then

\[
E_{-1}(K_g) = K_g^+ + qK_{g^-}, \quad \text{if } g_{-1} = 0,
\]

\[
F_{-1}(K_g) = q^{-1}K_{g^-}, \quad \text{if } g_{-1} = 1,
\]

\[
F_{-1}(K_g) = K_g^+, \quad \text{if } g_{-1} = -1;
\]

2. if $g_n \neq -1$, then

\[
E_{-1}(K_g) = K_{g^+} + K_{g^-}, \quad \text{if } g_{-1} = 0,
\]

\[
F_{-1}(K_g) = K_{g^-}, \quad \text{if } g_{-1} = 1,
\]

\[
F_{-1}(K_g) = K_{g^+}, \quad \text{if } g_{-1} = -1.
\]

From now on, by abuse of notation, and we will refer to $E_a, F_a (a \geq -1)$ as the Chevalley generators.

**2.2. The canonical and dual canonical bases.**

**Proposition 2.3.** For every atypical $f \in Y_{+}^{1|n}$, there exist a typical $g \in Y_{+}^{1|n}$ and a sequence of Chevalley generators $X_1,\ldots,X_r$ such that

\[
X_1 \cdots X_r(K_g) = K_f + qK_{\mu}.
\]

**Proof.** We explicitly construct the sequence of Chevalley generators $X_1,\ldots,X_r$ and a typical weight $g$ for every atypical $f$. There are three cases to consider according to Subsection [1.2]

1. if $f_{-1} = f_i$ for a fixed $i \geq 1$. Set $k = -f_{-1}$. There exists an $l \geq 0$ such that

\[
f = (-k|f_1,\ldots,f_{i-1},-k-l,-k-l+1,\ldots,-k;f_{i+1},\ldots,f_n),
\]

\[
f^{[l]} = (-k-l-1|f_1,\ldots,f_{i-1},-k-l-1,-k-l,\ldots,-k-1,f_{i+1},\ldots,f_n),
\]

with $f_{i-1} < -k - l - 1$. Then,

\[
E_{-k-l-1}E_{-k-l-1} \cdots E_{-k-1}(K_g) = K_f + qK_{\mu}.
\]
with the typical element
\[ g := (-k|f_1, \ldots, f_{i-1}, -k - l - 1, -k - l, \ldots, -k - 1, f_{i+1}, \ldots, f_n). \]

(II) \( f_{-1} = -f_i \) for some \( i \geq 1 \), and \( f_{-1}^2 > 0 \). There is an \( l \geq 0 \) such that
\[ f = (k + l + 1|f_1, \ldots, f_{i-1}, -k - l - 1, -k - l, \ldots, -k - 1, f_{i+l+1}, \ldots, f_n), \]
\[ f^- = (-k|f_1, \ldots, f_{i-1}, -k - l - 1, -k - l + 1, \ldots, -k, f_{i+l+1}, \ldots, f_n), \]
with \( f_{i+l+1} > -k \), where \( k = f_{-1}^2 \). Then,
\[ F_{-k-1}F_{-k-2} \cdots F_{-k-1}(K_g) = K_f + qK_f. \]

with the typical element
\[ g := (k + l + 1|f_1, \ldots, f_{i-1}, -k - l, -k - l + 1, \ldots, -k, f_{i+l+1}, \ldots, f_n). \]

(III) \( f_{-1} = -f_i \) for some \( i \geq 1 \), and \( f_{-1}^2 < 0 \). Then
\[ f = (k|f_1, \ldots, f_{n-k}, -k, -k + 1, \ldots, -1), \]
\[ f^- = (-k|f_1, \ldots, f_{n-k}, -k, -k + 1, \ldots, -1), \]
where \( k = f_{-1} \). Then
\[ E_{-k} \cdots E_{-2}E_{-1}(K_g) = K_f + qK_f. \]

with the typical element \( g := (0|f_1, \ldots, f_{n-k}, -k, -k + 1, \ldots, -1). \) \( \square \)

We define a bar-involution \( \bar{\cdot} \) on \( \mathcal{U} \) by declaring that it fixes all the Chevalley generators and sends \( K_{-1} \) to \( K_{-1}^{-1} \).

**Theorem 2.4.**

1. There exists a unique anti-linear bar involution \( \bar{\cdot} \) on a suitable completion \( \hat{\mathcal{F}} \) of \( \mathcal{F} \) such that
   
   (i) \( \bar{K}_f = K_f \) for all typical \( f \in Y_+^{[n]} \);
   
   (ii) \( \bar{X}u = \bar{X}u, \bar{E}_{-1}u = \bar{E}_{-1}u \) and \( \bar{F}_{-1}u = \bar{F}_{-1}u \), for all \( X \in \mathcal{U} \) and \( u \in \hat{\mathcal{F}} \).

2. There exists unique canonical basis \( \{U_f\} \) and dual canonical basis \( \{L_f\} \), where \( f \in Y_+^{[n]} \), for \( \hat{\mathcal{F}} \) such that
   
   (i) \( \bar{U}_f = U_f \) and \( \bar{L}_f = L_f \);
   
   (ii) \( U_f \in K_f + \sum_{q \neq f} qZ[q]K_g \) and \( L_f \in K_f + \sum_{q \neq f} q^{-1}Z[q^{-1}]K_g \).

3. \( U_f = L_f = K_f \) for typical \( f \in Y_+^{[n]} \). For every atypical \( f \in Y_+^{[n]} \), we have
   
   \[ U_f = K_f + qK_{f^1}, \quad L_f = K_f + \sum_{l=1}^{\infty} (-q^{-1})^l K_{f^{(l)}} \]  \tag{2.3}

   where \( f^{(1)} = f^2 \) and \( f^{(l+1)} = (f^{(l)})^2 \).

**Proof.** Proposition 2.3 and the requirement (ii) of the bar map imply that \( K_f + qK_{f^1} \) for every atypical \( f \) is bar-invariant. Thus, \( K_f + qK_{f^1} \) for all atypical \( f \in Y_+^{[n]} \) together with \( K_f \) for all typical \( f \in Y_+^{[n]} \) form a bar-invariant basis of \( \hat{\mathcal{F}} \). This proves the uniqueness of the bar map.

Since \( \bar{K}_f = K_f + qK_{f^1} \), for \( f \) atypical, we obtain
\[ \bar{K}_f = K_f + qK_{f^1} - q^{-1}K_{f^1}. \]  \tag{2.4}
By iterating the relation (2.4) we obtain that
\[ \overline{K_f} = K_f + (q - q^{-1}) \sum_{i=1}^{\infty} (-q)^{1-i} K_{f(i)}. \]

It follows that the bar map is indeed an involution with the property that \( \overline{K_f} \) equals \( K_f \) plus lower terms in Bruhat order for every \( f \in Y_+^{1/n} \). The existence and uniqueness of the canonical and dual canonical bases now follows routinely from the bar involution with such a property [KL].

Clearly, for every typical \( f \in Y_+^{1/n} \), we have \( U_f = L_f = K_f \). By the uniqueness of the canonical basis, \( U_f = K_f + qK_f^2 \) for \( f \) atypical. Denote the RHS of (2.3) by \( \mathcal{L}_f \). It follows from (2.4) that \( \overline{\mathcal{L}_f} = K_f - q^{-1} \mathcal{L}_f \). Iterating this relation we obtain \( \overline{\mathcal{L}_f} = \mathcal{L}_f \). Thus \( L_f = \mathcal{L}_f \) by the uniqueness of the dual canonical basis.

It remains to check that the bar map on \( \mathcal{F} \) indeed satisfies the compatibility condition (ii) of (1). For the generators \( E_{-1} \) and \( F_{-1} \), this follows from Lemma 2.2.

Now consider the Chevalley generators of \( \mathcal{U} \). This requires a tedious (albeit elementary) case by case verification that \( X_{-a}(U_f) = X_{-a}(U_f) \), for all \( f \in Y_+^{1/n} \) and all Chevalley generators \( X_{-a} \) of \( \mathcal{U} \). If \( f \in Y_+^{1/n} \) is typical, then for any Chevalley generator \( X_{-a} \) of \( \mathcal{U} \), \( X_{-a}(U_f) \) is either zero or equal to some \( U_g \). This can be established by separately analyzing the two cases with \( \pm f \), for all \( i > 0 \) and with \( \pm f_1 \) for some \( i > 0 \) respectively.

We divide the atypical elements of \( Y_+^{1/n} \) into three cases as in the proof of Proposition 2.3. (I) \( f \) and \( f^l \) are given by equation (2.4). If \( l \geq 1 \), we can show that for all the Chevalley generators \( X_{-a} \) of \( \mathcal{U} \), \( X_{-a}(U_f) \) is either zero or equal to some \( U_g \). This is also true for all the \( X_{-a} \) but \( E_{-k-1} \) when \( l = 0 \). In the latter case, \( E_{-k-1}(U_f) = (q + q^{-1}) U_g \) with \( g = (-k - 1 | f_1, \ldots, f_{i-1}, -k, f_{i+1}, \ldots, f_n) \). The case (II) is analogous, thus we omit the details. In the case (III), \( X_{-a}(U_f) \) is either zero or equal to some \( U_g \) if \( X_a \neq F_{-k} \). When \( X_a = F_{-k} \), we have
\[ F_{-k}(U_f) = \begin{cases} 0, & \text{if } f_{n-k} = -k - 1, \\ U_{g_+} + U_{g_-}, & \text{if } f_{n-k} \neq -k - 1, \end{cases} \]
where \( g_{\pm} = (\pm k | f_1, \ldots, f_{n-k}, -k - 1, -k + 1, \ldots, -1) \). This completes the proof of the theorem.\qed

3. Representation theory of \( \mathfrak{g} \)

3.1. Characters of the tilting and irreducible \( \mathfrak{g} \)-modules. Let \( \sum_{i=0}^{\infty} \mathbb{Z} \epsilon_i \) denote the free abelian group with basis \( \epsilon_i \), \( i = 0, -1, -2, \ldots \). We define a map \( \text{wt} : Y_+^{1/n} \mapsto \sum_{i=0}^{\infty} \mathbb{Z} \epsilon_i \) by
\[ f = (f_{-1} | f_1, f_2, \ldots, f_n) \mapsto \text{wt}(f) = \epsilon_{|f_{-1}|} - \epsilon_{f_1} - \epsilon_{f_2} - \cdots - \epsilon_{f_n}. \]

It was stated in [K2] and proved in [Pe, Theorem 1.2] that \( \lambda, \mu \in X_+^{1/n} \) correspond to the same central character only if \( \text{wt}(f_{\lambda}) = \text{wt}(f_{\mu}) \). On the other hand, exactly when \( \lambda \) and \( \mu \) are atypical, \( \text{wt}(f_{\lambda}) = \text{wt}(f_{\mu}) \) implies that \( \lambda \) and \( \mu \) are in the same
block. The block corresponding to $\lambda$ will be denoted by $O^+_{\lambda}$. Evidently $O^+$ is a
direct sum of blocks corresponding to different central characters.

Given a $g$-module $M$ in the category $O^+$, we endow the dual $M^*$ with the usual
g-module structure. Further twisting the $g$-action on $M^*$ with the automorphism of $g$
given by $e_i \mapsto -f_i, f_i \mapsto -e_i, h_i \mapsto h_i$ for $1 \leq i \leq n$ and $e_0 \mapsto f_0, f_0 \mapsto
-e_0, h_0 \mapsto h_0$, we obtain another $g$-module denoted by $M^\tau$. We have $(M^\tau)^\tau \cong M$.

We shall consider translation functors on the category $O^+$. For any $M \in O^+$
belonging to the block of a weight $\lambda \in X^+_0$, we define for $a = 1, 2, \ldots$
\[E_{-a}(M) = \text{pr}_{\text{wt}(f_{\lambda})+\epsilon_{-a}-\epsilon_{-a+1}}(C^{2|2n} \otimes M),\]
\[F_{-a}(M) = \text{pr}_{\text{wt}(f_{\lambda})+\epsilon_{-a+1}-\epsilon_{-a}}(C^{2|2n} \otimes M).\]
Here for $\gamma \in \sum_{i=0}^{\infty} \mathbb{Z}\epsilon_i$, $\text{pr}_\gamma : O^+ \to \bigoplus_{\text{wt}(\mu)=\gamma} O^+$ stands for the canonical projection. Such functors are exact and their left and right adjoints are of the same
form.

Let $K(O^+)$ denote the Grothendieck group of $O^+$. For $M \in O^+$ the expression
$[M]$ denotes the corresponding element in $K(O^+).$ We shall use the same notation
to denote the operators on $K(O^+)$ corresponding to $E_{-a}$ and $F_{-a}$ respectively. By checking the tensor product of $K(\lambda)$ with the natural module $C^{2|2n}$, we can easily
prove the following result.

**Proposition 3.1.** The linear map $j : K(O^+) \to F|_{q=1}$, $K(\lambda) \mapsto K_{f_{\lambda}}$, is an
isomorphism of vector spaces. Furthermore, for $a \leq -1$ we have
\[E_{-a}j(-) = j(E_{-a}(-)), \quad F_{-a}j(-) = j(F_{-a}(-)).\]

We say that an object $M \in O^+$ has a Kac flag, if $M$ has a filtration of submodules from $O^+$ such that each successive quotient is isomorphic to some Kac module. The
general theory of finite dimensional tilting modules as explained in [Sc] [B2] applies to the Lie superalgebra $g$ as well. We denote by $U(\lambda) \in O^+$ the tilting module
associated with $\lambda \in X^+_0$. It is the unique indecomposable object in $O^+$ satisfying:
(1) $U(\lambda)$ has a Kac flag with $K(\lambda)$ at the bottom; (2) $\text{Ext}^1_{O^+}(K(\mu), U(\lambda)) = 0$ for all $\mu \in X^+_0$. Denote by $(U(\lambda) : K(\mu))$ the multiplicity of the Kac module $K(\mu)$ in a
Kac flag of $U(\lambda)$, and by $[K(\mu) : L(\nu)]$ the multiplicity of $L(\nu)$ in a composition
series of $K(\mu)$. Recall $\beta = 2n\epsilon$. Following [Bl1] [Bl2], we have
\[K(\lambda)^* \cong K(\beta - w_0\lambda); \quad U(\lambda)^* \cong U(\beta - w_0\lambda),\]
\[(U(\lambda) : K(\mu)) = [K(\beta - w_0\mu) : L(\beta - w_0\lambda)]. \quad (3.1)\]

Below we have the following analogue of Theorem 4.37 in [Bl1].

**Theorem 3.2.** Let $\lambda \in X^+_0$. Then,
(1) If $\lambda$ is typical, then $U(\lambda) = K(\lambda) = L(\lambda)$. If $\lambda$ is atypical, there exist a
sequence of translation functors $X_1, \ldots, X_r$ and a typical $\mu$ in $X^+_0$ such
that $U(\lambda) = X_1 \cdots X_r U(\mu)$. Furthermore, $U(\lambda)$ has the following 2-step
Kac flag: $0 \to K(\lambda) \to U(\lambda) \to K(\lambda^2) \to 0$.
(2) $j([-U(\lambda)]) = U_{f_{\lambda}}|_{q=1}$. 

(3) \( U(\lambda) \) is the projective cover of \( L(\lambda^1) \).
(4) \( U(\lambda) \cong U(\lambda)^\tau \).

Proof. Let us first assume the validity of (1). Part (2) immediately follows from Proposition 3.1 and Proposition 2.3. Part (1) implies that there is an epimorphism \( U(\lambda) \rightarrow L(\lambda^1) \) and \( U(\lambda) \) is indecomposable. We have \( \text{Hom}_{\mathcal{O}^+}(U(\lambda), M) \cong \text{Hom}_{\mathcal{O}^+}(\lambda, Y_r \cdots Y_1 M) \), where \( Y_a \) is the translation functor corresponding to the adjoint functor of \( X_a \). Thus \( U(\lambda) \) is projective and (3) holds. Part (4) also follows readily from (1) by an induction argument, since \( \tau \) commutes with the translation functors.

So it remains to prove (1). The typical case is clear. Now let us fix an atypical \( \lambda \). Then by Proposition 2.3 there exists a typical \( \nu \) and a sequence of Chevalley generators \( X_1, \ldots, X_r \) such that \( U_{f_{\lambda}} = X_1 \cdots X_r(U_{f_{\nu}}) \). By abuse of notation, we shall also denote by \( X_1, \ldots, X_r \) the corresponding translation functors.

Clearly (1) holds for a typical \( \nu \). By Proposition 3.1 we have
\[
j[X_1 \cdots X_r U(\nu)] = X_1 \cdots X_r U_{f_{\nu}}(1) = U_{f_{\lambda}}(1).
\]
This and the formula for the canonical basis element \( U_{f_{\lambda}} = K_{f_{\lambda}} + qK_{f_{\lambda}} \) imply the following identity in \( K(\mathcal{O}^+) \):
\[
[X_1 \cdots X_r U(\nu)] = [K(\lambda)] + [K(\lambda^1)].
\] (3.2)

Now it is easy to see that if \( M \in \mathcal{O}^+ \) has a Kac flag, then the translation functor applied to \( M \) produces a module with a Kac flag, which in turn implies that any direct summand of it also has a Kac flag. Since \( \text{Ext}_{\mathcal{O}^+}^1(K(\mu), X_1 \cdots X_r U(\nu)) = \text{Ext}_{\mathcal{O}^+}^1(Y_r \cdots Y_1 K(\mu), U(\nu)) = 0 \), we have \( \text{Ext}_{\mathcal{O}^+}^1(K(\mu), U) = 0 \) for any direct summand \( U \) of \( X_1 \cdots X_r U(\nu) \). Hence by (3.2) \( X_1 \cdots X_r U(\nu) \) is a direct sum of tilting modules and contains \( U(\lambda) \) as a direct summand.

If we can show that \( K(\lambda^1) \) (besides the obvious one \( K(\lambda) \)) appears in a Kac flag for \( U(\lambda) \), then by (3.2) again there will be no more tilting module as a direct summand of \( X_1 \cdots X_r U(\nu) \) and we will be done. This latter claim is indeed true, since by (1.4) and (3.1),
\[
(U(\lambda) : K(\lambda^1)) = [K(\beta - w_0 \lambda^1) : L(\beta - w_0 \lambda)]
\]
\[
= [K(\beta - w_0 \lambda^1) : L((\beta - w_0 \lambda^1)^1)] \geq 1.
\]
The last inequality \( [K(\mu) : L(\mu^1)] \geq 1 \) for every atypical \( \mu \) will be established in Lemma 3.6 below, independently of van der Jeugt’s theorem \( V \). \( \square \)

Using (1.4), (3.1) and Theorem 3.2 we have obtained a new proof of van der Jeugt’s main theorem.

**Corollary 3.3.** \( V \) For \( \lambda \in X_+^{1,n} \) atypical, there is a short exact sequence of \( \mathfrak{g} \)-modules
\[
0 \rightarrow L(\lambda^1) \rightarrow K(\lambda) \rightarrow L(\lambda) \rightarrow 0.
\]
Let \( \rho_0 = \frac{1}{2} \sum_{\alpha \in \Delta_0^+} \alpha \). Write
\[
D_0 := \sum_{w \in W} (-1)^{l(w)} w(e^{\rho_0}).
\]
**Corollary 3.4.** Let \( \lambda \in X^{1|n}_+ \) be atypical weight with \((\lambda + \rho, \gamma) = 0 \) for some \( \gamma \in \Delta_1^+ \). Then

\[
\text{ch} L(\lambda) = \frac{1}{D_0} \sum_{w \in W} (-1)^{l(w)} w \left( e^{\lambda + \rho_0} \prod_{\alpha \in \Delta_1^+ \setminus \{\gamma\}} (1 + e^{-\alpha}) \right).
\]

**Proof.** This follows from Corollary 3.3 and the fact that

\[
\text{RHS of (3.3)} = \frac{\prod_{\alpha \in \Delta_1^+} (1 + e^{-\alpha})}{\prod_{\alpha \in \Delta_0^+} (1 - e^{-\alpha})} \sum_{w \in W} (-1)^{l(w)} w \left( e^{\lambda + \rho_0} / (1 + e^{-\gamma}) \right)
\]

\[
= \frac{\prod_{\alpha \in \Delta_1^+} (1 + e^{-\alpha})}{\prod_{\alpha \in \Delta_0^+} (1 - e^{-\alpha})} \sum_{k \geq 0} (-1)^k \sum_{w \in W} (-1)^{l(w)} w \left( e^{\lambda + \rho_0 - k\gamma} e^{-\rho_0} \right)
\]

\[
= \sum_{k \geq 0} (-1)^k \prod_{\alpha \in \Delta_1^+} (1 + e^{-\alpha}) \sum_{w \in W} (-1)^{l(w)} e^{w(\lambda + \rho_0 - k\gamma) - \rho_0}
\]

\[
= \sum_{i \geq 0} (-1)^i \text{ch} K(\lambda^i).
\]

In the last identity we have used the Weyl character formula and the fact that \( \rho_0 - \rho \) is \( W \)-invariant. \( \square \)

**Remark 3.5.** In [Zou], \( \text{Ext}^{i+}_O(K(\mu), L(\lambda)) \) was computed explicitly. Denote the dual canonical basis element \( L_f = \sum_{q \in W} l_{gf}(q) K_g \). These Kazhdan-Lusztig polynomials \( l_{gf}(q) \) have been computed in (2.3) for atypical \( f \), and \( l_{gf}(q) = \delta_{gf} \) for typical \( f \). Comparing with [Zou], we see the Serganova-Zou’s Kazhdan-Lusztig polynomials coincide with ours: \( l_{f, f_\lambda}(-q^{-1}) = \sum_{i \geq 0} q^i \text{Ext}^{i+}_O(K(\mu), L(\lambda)) \). From the theory of highest weight categories of Cline, Parshall and Scott (cf. [B1] 4-f) for adaptation to superalgebras, we have

\[
\sum_{i \geq 0} \dim \text{Ext}^{i+}_O(L(\mu), L(\lambda)) q^i = \sum_{\nu \in X^{1|n}} a_{\nu \mu} (-q^{-1})^i \nu^\lambda (-q^{-1}).
\]

### 3.2. A technical lemma.

The following was used in the proof of Theorem 3.2.

**Lemma 3.6.** \([K(\lambda) : L(\lambda^i)] \geq 1 \) for every atypical \( \lambda \).

**Proof.** Assume that \( \lambda = (\lambda_1 |\lambda_1, \cdots, \lambda_n) \) is atypical. There are two possibilities:

\((\lambda_1 - n) + (\lambda_i + n - i + 1) = 0\), or \( \lambda_1 - n = \lambda_i + n - i + 1 \) for some \( i \). We will treat in detail below the first case when

\[
\lambda_1 + \lambda_i - i + 1 = 0
\]

and leave the other similar case to the reader.

Let \( T_- \) be the product of all odd negative root vectors and let \( v_\lambda \) be a highest weight vector of the Kac module \( K(\lambda) \). Then the vector \( T_- v_\lambda \) has weight \((\lambda_1 - 2n|\lambda_1, \cdots, \lambda_n) \). Note that \( T_- v_\lambda \) is highest weight with respect to the Borel
subalgebra containing the same even part but the opposite odd part of the standard Borel $\mathfrak{b}$. We apply now odd reflections in the following order to get back to the standard Borel:

$$\epsilon + \delta_1, \epsilon + \delta_2, \ldots, \epsilon + \delta_n, \epsilon - \delta_n, \epsilon - \delta_{n-1}, \ldots, \epsilon - \delta_1.$$ 

Here the usual rule of odd reflection is that if $(\mu, \alpha) = 0$ then the highest weight vector is unchanged, and if $(\mu, \alpha) \neq 0$ then the highest weight vector is obtained by applying the positive root vector corresponding to $\alpha$ to the previous highest weight vector (cf. for example, [25]).

Note that (3.4) implies that $\lambda_{-1} - n < 0$ and thus $\lambda_{-1} - 2n \neq \lambda_1$. So after the first step the weight is $(\lambda_{-1} - 2n + 1|\lambda_1 + 1, \ldots, \lambda_n)$. Repeating the process with the first $n$ odd roots, we end up with the weight $(\lambda_{-1} - n|\lambda_1 + 1, \ldots, \lambda_n + 1)$. We continue by using now the odd root $\epsilon - \delta_n$. If $\lambda_{-1} - n + \lambda_n + 1 = 0$, then $i = n$. So if $n \neq i$, then we need to add $\epsilon - \delta_n$ and get $(\lambda_{-1} - n + 1|\lambda_1 + 1, \ldots, \lambda_{n-1} + 1, \lambda_n)$. Finally we end up with the weight

$$\lambda^L = (\lambda_{-1} - i + j - 1|\lambda_1, \ldots, \lambda_{j-1}, \lambda_j + 1, \ldots, \lambda_i + 1, \lambda_{i+1}, \ldots, \lambda_n).$$

Here $j$ is determined by that $\lambda_j = \lambda_{j+1} = \cdots = \lambda_i$ and $\lambda_{j-1} > \lambda_j$. Note that in the process we did not add $\epsilon - \delta_1, \epsilon - \delta_{j-1}, \ldots, \epsilon - \delta_{i}$ since $\lambda_{-1} - i + \lambda_i + 1 = 0$. In this way, we have obtained a highest weight vector (relative to $\mathfrak{b}$) of highest weight $\lambda^L$. 

\[ \Box \]

4. The composition factors of symmetric tensors

Let $x, \bar{x}$ be 2 even variables and $\xi_1, \cdots, \xi_n, \bar{\xi}_1, \cdots, \bar{\xi}_n$ be $2n$ odd variables. If we let $C^{2|2n}$ stand for the standard representation of $osp(2|2n)$, then we may identify the symmetric algebra $S(C^{2|2n})$ with $C[x, \bar{x}, \xi_i, \bar{\xi}_i]$, the polynomial algebra in the variables $x, \bar{x}$ and $\xi_1, \cdots, \xi_n, \bar{\xi}_1, \cdots, \bar{\xi}_n$. In this identification the action of $\mathfrak{g}$ gets identified with the action of certain linear differential operators whose explicit formulas are easily written down. The positive simple root vectors $e_0, e_1, \cdots, e_n$ and the negative simple root vectors $f_0, f_1, \cdots, f_n$ are:

$$e_0 = x \frac{\partial}{\partial \xi_1} + \bar{x} \frac{\partial}{\partial \xi_1}, \quad f_0 = \xi_1 \frac{\partial}{\partial x} - \bar{\xi}_1 \frac{\partial}{\partial \bar{x}},$$

$$e_i = \xi_i \frac{\partial}{\partial \xi_{i+1}} - \bar{\xi}_{i+1} \frac{\partial}{\partial \xi_i}, \quad f_i = \xi_{i+1} \frac{\partial}{\partial \xi_i} - \bar{\xi}_i \frac{\partial}{\partial \bar{\xi}_{i+1}}, \quad i = 1, \cdots, n - 1,$$

$$e_n = \xi_n \frac{\partial}{\partial \xi_n}, \quad f_n = \bar{\xi}_n \frac{\partial}{\partial \bar{\xi}_n}.$$ 

By declaring all the variables to have degree 1 the algebra $C[x, \bar{x}, \xi_i, \bar{\xi}_i]$ acquires a $\mathbb{Z}$-grading

$$C[x, \bar{x}, \xi_i, \bar{\xi}_i] = \bigoplus_{j=0}^{\infty} C[x, \bar{x}, \xi_i, \bar{\xi}_i] = \bigoplus_{j=0}^{\infty} S^j(C^{2|2n}).$$
Now the Laplace operator
\[ \Delta = \frac{\partial}{\partial x} \frac{\partial}{\partial \bar{x}} - \sum_{i=1}^{n} \frac{\partial}{\partial \xi_i} \frac{\partial}{\partial \bar{\xi}_i} : S^k(\mathbb{C}^{2|2n}) \to S^{k-2}(\mathbb{C}^{2|2n}) \]
is surjective of degree \(-2\) for each \(k \geq 0\). One checks that \(\Delta\) commutes with the action of \(\mathfrak{g}\). This establishes the following.

**Lemma 4.1.** The map \(\Delta : S^k(\mathbb{C}^{2|2n}) \to S^{k-2}(\mathbb{C}^{2|2n})\) is surjective homomorphism of \(\mathfrak{g}\)-modules, and \(S^k(\mathbb{C}^{2|2n})/\ker \Delta \cong S^{k-2}(\mathbb{C}^{2|2n})\) as \(\mathfrak{g}\)-modules.

Consider the case \(0 \leq k \leq n\). In this case using the combinatorial character formula of [Lee, Theorem 3.7] or applying directly (3.3) we see that the character of \(\ker \Delta \subset S^k(\mathbb{C}^{2|2n})\) is equal to the character of the irreducible module of highest weight \((k|0, \ldots , 0)\). This immediately implies the following proposition.

**Proposition 4.2.** For \(0 \leq k \leq n\) the \(\mathfrak{g}\)-module \(\ker \Delta \subset S^k(\mathbb{C}^{2|2n})\) is isomorphic to the irreducible highest weight module of highest weight \((k|0, \ldots , 0)\).

Next we consider the case \(k \geq 2n + 1\). The following lemma is easy to verify.

**Lemma 4.3.** Let \(k \geq 2n + 1\). Then \(\dim S^k(\mathbb{C}^{2|2n}) - \dim S^{k-2}(\mathbb{C}^{2|2n}) = 2^{2n+1}\).

**Lemma 4.4.** Let \(\Phi_1 := \sum_{i=1}^{n} \xi_i \bar{\xi}_i\) and \(k \geq 2n\). Set
\[ \Gamma := \sum_{i=0}^{n} (-1)^i \binom{k-n}{i} \bar{x}^{k-n-i} x^{n-i} \Phi_1 \in S^k(\mathbb{C}^{2|2n}). \]
Then \(\Gamma \neq 0\) and we have \(\Delta(\Gamma) = 0\) and \(e_i \Gamma = 0\), for \(i = 0, \ldots , n\).

**Proof.** Follows by a direct computation. \(\square\)

**Proposition 4.5.** Let \(k \geq 2n + 1\). There is an isomorphism of \(\mathfrak{g}\)-modules:
\[ \ker \Delta \cong L(k|0, \ldots , 0) \oplus L(2n-k|0, \ldots , 0). \]

**Proof.** It is clear that \(x^k \in \text{Ker}\Delta\) is a highest weight vector of weight \((k|0, \ldots , 0)\). By Lemma 4.3 the irreducible \(\mathfrak{g}\)-module of highest weight \((2n-k|0, \ldots , 0)\) is also a composition factor of \(\ker \Delta\). However, both weights are typical, and hence the irreducible modules are equal to the corresponding Kac modules which are of dimension \(2^{2n}\). Now Lemma 4.3 implies that \(\ker \Delta\) has only these two composition factors. Finally, the weights \((k|0, \ldots , 0)\) and \((2n-k|0, \ldots , 0)\) belong to different blocks and so indeed we have a direct sum. \(\square\)

**Remark 4.6.** Let \(k \geq 2n + 1\). Consider a new set of simple roots of \(\mathfrak{g}\) associated with the following Dynkin diagram:

\[
\begin{array}{ccccccc}
\delta_1 & \delta_2 & \delta_2 & \delta_3 & \cdots & \delta_{n-1} & \delta_n \\
\delta_1 - \delta_2 & \delta_2 - \delta_3 & \delta_{n-1} - \delta_n & \delta_n + \epsilon & \delta_n - \epsilon
\end{array}
\]
Here $\delta_n \pm \epsilon$ are odd roots. We can show via the method of odd reflections that the highest weights of the two summands of ker $\Delta$ in Proposition 1.3 with respect to this new Borel have Dynkin labels indicated as follows (with the convention here and below that the unmarked ones are 0):

Note that they are related via a Dynkin diagram automorphism.

It remains to consider the case $n + 1 \leq k \leq 2n$. Denote by $\chi_l$ the irreducible character of $\text{sp}(2n)$ of highest weight $\sum_{i=1}^{l} \delta_i$. The proofs of the following two lemmas are straightforward and omitted.

**Lemma 4.7.** Let $\lambda = \sum_{i=1}^{n} k_i \delta_i$, where $k_i = 0, 1$, for all $i$. Suppose there exists $k_i = 0$ and $k_j = 1$ with $i < j$. Let $x$ be an indeterminate. Then we have

$$\sum_{w \in W} (-1)^{l(w)} w \left( e^{\lambda + \rho_0} \prod_{i=1}^{n} (1 + xe^{-\delta_i}) \right) = 0.$$ 

**Lemma 4.8.** Let $\lambda = \sum_{i=1}^{l} \delta_i$, and $l \leq n$. Let $x$ be an indeterminate. Then we have

$$\frac{1}{D_0} \sum_{w \in W} (-1)^{l(w)} w \left( e^{\lambda + \rho_0} \prod_{i=1}^{n} (1 + xe^{-\delta_i}) \right) = \sum_{j=0}^{l} x^{l-j} \chi_j.$$ 

**Corollary 4.9.** For $n + 1 \leq k \leq 2n$, the character $\text{ch} L(-\epsilon + \sum_{i=1}^{2n-k+1} \delta_i)$ equals

$$\chi_0 \left( e^{-k\epsilon} + e^{(-k+3)\epsilon} + \cdots + e^{(-k+2(k-n-1))\epsilon} \right) + \chi_1 \left( e^{(-k+1)\epsilon} + e^{(-k+3)\epsilon} + \cdots + e^{(-k+2(k-n-1))\epsilon} \right) + \cdots + \chi_n e^{(n-k)\epsilon}. \tag{4.2}$$

**Proof.** By (3.3), the character $\text{ch} L(-\epsilon + \sum_{i=1}^{2n-k+1} \delta_i)$ is equal to

$$\frac{1}{D_0} \sum_{w \in W} (-1)^{l(w)} w \left( e^{-\epsilon + \sum_{i=1}^{2n-k+1} \delta_i + \rho_0} \times \left( 1 + e^{-\epsilon + \delta_n} \right) \cdots \left( 1 + e^{-\epsilon + \delta_{2n-k+2}} \right) \prod_{i=1}^{n} (1 + e^{-\epsilon - \delta_i}) \right),$$
which can then be written by Lemmas 4.7 and 4.8 (with $x = e^{-\epsilon}$) as

$$e^{-\epsilon}\left(e^{(-2n+k-1)\epsilon}\chi_0 + e^{(-2n+k)\epsilon}\chi_1 + \cdots + \chi_{2n-k+1}\right)$$

$$+ e^{-2\epsilon}\left(e^{(-2n+k-2)\epsilon}\chi_0 + e^{(-2n+k-1)\epsilon}\chi_1 + \cdots + \chi_{2n-k+2}\right) + \cdots$$

$$+ e^{(n-k)\epsilon}\left(e^{-n\epsilon}\chi_0 + e^{(-n+1)\epsilon}\chi_1 + \cdots + \chi_n\right).$$

The corollary now follows by collecting the coefficients of the $\chi_i$. □

**Proposition 4.10.** Let $n + 1 \leq k \leq 2n$. We have

$$\text{ch} \ker \Delta = \text{ch}L(ke) + \text{ch}L((2n - k)e) + \text{ch}L(-\epsilon + \sum_{i=1}^{2n-k+1} \delta_i).$$

**Proof.** Note that

$$\text{ch}K(ke) = e^{ke} \prod_{i=1}^{n}(1 + e^{-\epsilon+\delta_i})(1 + e^{-\epsilon-\delta_i}),$$

and it can be rewritten as

$$e^{ke}\left(\chi_0 + e^{-\epsilon}\chi_1 + e^{-2\epsilon}(\chi_2 + \chi_0) + e^{-3\epsilon}(\chi_3 + \chi_1) + \cdots + e^{-n\epsilon}(\chi_n + \chi_{n-2} + \cdots) + e^{-2n\epsilon}\chi_0 + e^{(-2n+1)\epsilon}\chi_1 + e^{(-2n+2)\epsilon}(\chi_2 + \chi_0) + e^{(-2n+3)\epsilon}(\chi_3 + \chi_1) + \cdots + e^{(-n+1)\epsilon}(\chi_{n-1} + \chi_{n-3} + \cdots)\right). \quad (4.3)$$

Now a straightforward calculation shows that

$$\text{ch} \ker \Delta = (e^{(k-n)\epsilon} + e^{(n-k)\epsilon})\chi_n$$

$$+ (e^{(k+1-n)\epsilon} + e^{(k-1-n)\epsilon} + e^{(n-k+1)\epsilon} + e^{(n-k-1)\epsilon})\chi_{n-1} + \cdots$$

$$+ (e^{2(k-n-1)\epsilon} + \cdots + \hat{1} + \cdots + e^{-2(k-n-1)\epsilon})\chi_{2n+2-k} + \cdots$$

$$+ (e^{(k-1)\epsilon} + e^{(k-3)\epsilon} + \cdots + e^{-(k-1)\epsilon})\chi_1$$

$$+ (e^{ke} + e^{(k-2)\epsilon} + \cdots + e^{-ke})\chi_0$$

where $\hat{1}$ means as usual omission. One checks that (4.4) = (4.2) + (4.3). The proposition now follows from the fact that $\text{ch}K(ke) = \text{ch}L(ke) + \text{ch}L((2n-k)e)$. □

Clearly, $S^0(\mathbb{C}^{2|2n}) \cong \mathbb{C}$ and $S^1(\mathbb{C}^{2|2n}) \cong \mathbb{C}^{2|2n}$. The composition factors of $S^k(\mathbb{C}^{2|2n})$ for every $k$ are now described explicitly by combining Lemma 4.1, Propositions 4.2, 4.5 and 4.10.

**Remark 4.11.** Let $n + 1 \leq k \leq 2n$. We can show via the method of odd reflections that the highest weights of the three summands of $\ker \Delta$ in Proposition 4.10 with respect to the set of simple roots 4.11 have Dynkin labels as indicated below:
Note that all three weights are in the same block and two of them are related by a diagram automorphism.

For the sake of completeness, we remark that \( \ker \Delta \) with \( k \leq n \) (see Proposition 4.2) with respect to the new Dynkin diagram have the following Dynkin labels:

\[
\begin{align*}
\text{(} k < n \text{)} & \quad \begin{array}{c}
\circ \cdots \circ \\
\delta_k - \delta_{k+1}
\end{array} \\
\text{(} k = n \text{)} & \quad \begin{array}{c}
\circ \cdots \circ \\
1
\end{array}
\end{align*}
\]

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