I. INTRODUCTION

Stripe formation in the high-$T_c$ oxides has been widely considered by many theorists in the recent years. The present experimental evidences point to a curious electronic order, where carriers introduced by doping (holes) segregate into ‘rivers of charge’. Those structures, generically called stripes, are considered as ‘nearly one dimensional’ objects immersed in an insulating antiferromagnetic sea of spins. This picture was inferred more accurately after recent neutron diffraction experiments\(^1\),\(^2\), which observed incommensurable antiferromagnetic domains separated by charge stripes localized in the Néel wall regions. It is found that the magnetic scattering peaks define a superzone which is half the size of the one associated with the charge order, indicating that stripes introduce a $\pi$-phase shift between neighboring magnetic domains. The above structure results from competing effects, which in conjunction, reduce the energy cost for breaking antiferromagnetic bonds and minimize the kinetic energy of holes.

The incommensurability of the peaks with the reciprocal lattice is a strong argument in favor of the idea that holes condense in stripes of finite width oriented along a given direction in the insulating planes. The temperature dependence of the peak splitting, showing that in some systems the charge-ordered peaks appear at higher temperatures than the magnetic ones\(^3\), also discards the hypothesis of nested Fermi surfaces. Currently, the stripe picture is considered as a good candidate for explaining many of the properties (normal and superconducting) of the high-$T_c$ oxides,\(^4\)\(^5\)\(^6\)(holes in most cases) can hop to neighboring sites. An-
tiferromagnetism also produce an attractive force among stripes, frustrating the tendency to macroscopic phase separation. From the view point of continuous models, the hole repulsion competes with the “helicity” of the spiral spin order , meaning that the antiphase of domains is crucial to achieve the thermodynamic stability of the stripe regime\(^7\). The other possibility, magnetic domains in phase, should be unstable in relation to the homogeneous state. This fact, also rules out the Ising limit, with very narrow walls between domains.

The physics involved in the present problem, embraces as much the long range scale of charge interactions as the microscopic scale of spin fluctuations, where theoretical considerations in the basis of a macroscopic mean field may seem meaningless at first sight. However, for the stripe regime, spin-spin and charge-charge correlation lengths should be comparable in a certain region of doping and temperature, with non-vanishing two-dimensional antiferromagnetic correlations. In spite that long range antiferromagnetic order is not attained in two dimensions, typical correlation lengths are of the order of
100 – 200 lattice parameters. We then define a two-dimensional \(T_N\) as the temperature where the magnetic neutron diffraction peaks are not resolved any more. In a similar way, a critical point \(T_S\) for the stripe order is assumed. The above facts justify a theoretical treatment using the Ginzburg-Landau (GL) approach to elucidate the internal structure of stripes.

The central idea of this paper is to assess the properties of the phase diagram, shown in a qualitative fashion in Fig. 1 and to theorize about the nature of the corresponding phase transitions. We will concentrate our study in two regions, where controversial statements are found in the literature, namely region LI, for the incompressible stripe phase below \(T_N\), and region HI, for the paramagnetic phase immediately below the stripe critical point \(T_S\). We will show that the charge distribution suffers a transition of commensurability, which evolves from the low incommensurable regime in phase LI, characterized by a typical soliton-like distribution, to a highly incommensurable one, which extends until phase HI and collapses at \(T_S\), following possibly a second order transition.

![Fig. 1](image_url)

**FIG. 1:** Cuprate phase diagram temperature vs doping. AF is the true long range antiferromagnetic order, below \(x \sim 0.02\). Inside the quarter of dome is the 2D (quasi-long range) antiferromagnetic phase, coexisting with the superconducting one (SC). LI is the spin driven Low Incommensurable phase of stripes. Regions I and II are High Incommensurable phases with and without spin order, respectively. The harmonic High Incommensurable phase is denoted by HI, right below the High Symmetry (HS) phase, where stripes disappear. \(T_0\) indicates a second order transition separating high and low incommensurable phases.

In the first step of the calculation, we study the marginal stripe phase transition below \(T_N\), building a GL free energy with the staggered magnetization as the main order parameter (OP) and the charge density playing the role of a secondary OP. This construction is based on a microscopic picture, which is supplemented by symmetry considerations. In contrast, in the second part of the paper, we study the highly incommensurable phase in the absence of antiferromagnetism, the transition being driven by the charge density. We develop theoretical arguments entirely based on symmetry to build the GL free energy for the HI phase, where the CDW is harmonic in the vicinity of \(T_S\). We calculate the amplitude and wavelength of the CDW and estimate the limits of the harmonic approximation. This one and other conjectured phases are displayed in the diagram of Fig. 1. We end the paper with a discussion of our results.

### II. LOW INCOMMENSURABLE STRIPE PHASE

Attention has been paid by some theorists to the tendency of attraction between domain walls at large distance. Priadko and coworkers believe that this fact leads to an ubiquitous tendency to phase segregation that renders the stripes unstable in the small doping limit. They demonstrated the existence of an attractive force between asymptotically separated charged domain walls when the charge effect is considered in first order by means of a chemical potential. Here, we argue that the spin exchange "traps" the holes inside domain walls, leading to a direct competition between the spin order attraction and the long range repulsion of holes, which stabilizes the inter-stripe separation. In spite that charges play the role of a secondary OP, their interactions must be included at least in second order in the free energy. In the cuprate case, it is experimentally known that for the doping range \(0.05 < x < \frac{1}{8}\), the effect of doping increasing is counterbalanced by the reduction of the domain width \(d\), and vice-versa, following the empirical relation \(x \sim \frac{1}{d}\). In this phase, the stripes are internally charge incompressible and the low doping regime may be simply thought as a re-scaling of the non-interacting stripe system.

First of all, we propose a constraint in the form of a global charge conservation

\[
\int \text{Dr} \rho(r) = Q ,
\]

where the integration is performed over a two dimensional system with linear dimensions \((l, l_z)\), with the \(z\)-axis along the charge modulation direction. From now on, the charge of holes, the electron spin, and the lattice constant are normalized to unity and will be omitted. If the stripes are non-interacting, the system can be rescaled along the direction \(z\) in terms of the domain width \(d\), considering that two neighboring stripes are infinitely distant (i.e. \(d \to \infty\)). This way, the charge conservation principle can be simply restated as

\[
\Pi \equiv \lim_{d \to \infty} \int_{-d/2}^{d/2} \text{d}z \rho(z) = h(T) ,
\]

where \(h(T)\) is a scalar function of temperature, \(h(T)\) being proportional to a linear charge density \(Q/l\) divided by the number of domains \(l_z/d\) along the modulation di-
or $x \propto \frac{1}{\xi}$. Therefore, the constraint (1) is quite appropriate for the low doping regime.

The total energy of each plane must take into account the long range potential for holes along with the exchange interaction of spins. The simplest static Hamiltonian for a square lattice of $N$ sites is

\[ H = H_e + H_s = \sum_{i,\delta} \frac{1}{\epsilon_{\delta}} (1 - n_i)(1 - n_{i+\delta}) + \]

\[ - \sum_i J_i n_i (\alpha_1 S_{i+1}^x + \alpha_2 S_{i+1}^2) + \sum_{i,\delta} J_{i,\delta} n_i n_{i+\delta} (\beta_1 S_{i+1} S_{i+\delta} + \beta_2 S_{i+1}^z S_{i+\delta}^z) \tag{2} \]

with $n_i = 0, 1$, the spin occupation number. The interactions are extended to distant neighbors, which are labeled by $\delta$. The first term in (2) yields the repulsive interactions of (spinless) holes with coupling $(1/\epsilon_{\delta})$. The second term includes the crystal field contribution, with anisotropy parameters $\alpha_1$ and $\alpha_2$. The last term considers the exchange interactions among spins ($J_{i,\delta} > 0$), which in general are assumed to be anisotropic, with parameters $\beta_1$ and $\beta_2$. The spins are assumed to be classical variables and are constrained to lay on the plane. Measurements taken in a considerable variety of layered compounds reveal that the inplane spin exchange interactions are isotropic to a high accuracy, meaning that the magnetic crystalline anisotropy dominates. Therefore, we consider $\beta_1 \sim \beta_2$ and adopt $\alpha_1$ and $\alpha_2$ as the source of anisotropy in our model.

To take the continuous limit of Eq. (2), we partition the antiferromagnetic lattice in two ferromagnetic sublattices, and define a mean field for each sublattice as $\mathbf{m}_{\perp,\parallel}(r) = \langle n_i S_i \rangle_{B(r)}$, calculating the spin average in a ball centered at the point $r$, in real space. In terms of the staggered fields ($\mathbf{m}_1, \mathbf{m}_2$), we define a continuous spin OP as

\[ \mathbf{m}(r) \equiv (\mathbf{m}_1 - \mathbf{m}_2)(r), \]

in the limit when the lattice constant goes to zero. Its norm $m$ is a positive number between 0 and 1. In the absence of dynamics, $\mathbf{m}_1$ and $\mathbf{m}_2$ are locally antiparallel and thus $m$ is maximum in the saturation regime, for $T \ll T_N$. We also define a renormalized spin occupation density $\eta$, varying in the interval $[0, 1]$, which plays the role of the secondary OP (it does not vanish at $T_N$). To remove the criticality at $T_N$, we impose

\[ \eta(r) = m(r) t^{-\beta}, \tag{3} \]

where $t = 1 - T/T_N$ is the reduced temperature, and $\beta > 0$ corresponds to the OP critical exponent. If we consider now the OP components along the directions parallel and transverse to the stripes, meaning $m_\perp = m \cos \theta$ and $m_\parallel = m \sin \theta$, and use relation (4), we find that the macroscopic regime is well defined in terms of the two OP degrees of freedom, $\eta$ and $\theta$. The symmetry group of the OP is the subgroup $C_{2v}$ of $O(2)$, which includes inversion of the spins and two reflection planes along the two principal axes of the magnetization. The continuous limit of the microscopic Hamiltonian (2) leads to the GL free energy density (see the Appendix)

\[ f(r) = (t^2 b_0 - a_0)(\dot{\eta})^2 + \eta^2 (\dot{\theta})^2 - 2a_2 \eta + \left[ a_2 + c_2 (t_2^2 - t_{12}^2) \beta_2 \alpha_\parallel \cos^2 \theta + \beta_2 \sin^2 \theta \right] \eta^2 - 2c_2 t_{23}^2 \eta^3 + \left[ c_2 t_{23}^2 + 4d_2^4 \right] \eta^4. \tag{4} \]

The dot means a spatial derivative with respect to $z$. We assume $a_2$ and $c_2$ to be regular functions of temperature, say $a_2(T)$ and $c_2(T)$, while $b_2 = bt$ is a standard Landau second order term parameter, with $b$ a positive constant.

To satisfy the continuity condition of the magnetic field in the stripe interface with the antiferromagnetic environment, we must have $\mathbf{m} \cdot \hat{e}_z = 0$ inside the bulk of domains. It leads to fixing the direction of easiest magnetization along $\hat{e}_z$, what is just equivalent to impose $\alpha_\perp < \alpha_\parallel$ to the anisotropy parameters. The $C_{2v}$ point group of the spin ordered phase admits two invariant terms, $m_\perp^2$ and $m_\parallel^2 \cos(2\theta)$, which generate the polynomial expansion of the free energy potential (4). The second order anisotropy term $b_2 m_\parallel^2 \alpha_\parallel \sin^2 \theta + \alpha_\parallel \cos^2 \theta$ can be decomposed in such basis as $\frac{b_2}{2} \left[ (\alpha_\parallel + \alpha_\perp) m_\parallel^2 + (\alpha_\parallel - \alpha_\perp) m_\perp^2 \cos(2\theta) \right]$, making evident the competition between the full symmetry term of the paramagnetic phase ($O(2)$ point group) and the $C_{2v}$ symmetry one. From this, the proper expression for the inplane crystalline anisotropy is

\[ 0 < \varepsilon \equiv \frac{\alpha_\perp - \alpha_\parallel}{\alpha_\parallel + \alpha_\perp} \leq 1. \tag{5} \]

The appropriate set of boundary conditions (BC) that applies to the free energy (4) is

\[ \theta = \begin{cases} 0 & z \to -\infty \\ \pi & z \to \infty \end{cases} \tag{6} \]

\[ \dot{\theta} = 0, \quad z \to \pm \infty \tag{7} \]

and

\[ \eta = 1, \quad z \to \pm \infty \tag{8} \]

\[ \dot{\eta} = 0, \quad z \to \pm \infty \tag{9} \]

The SDW modulates transversely to the stripes, in order to satisfy $\rho = 1 - \eta$. If we denote the energy of a single domain by $E \equiv \int_{-\infty}^{\infty} f(dz)$, variation under the constrain of the global charge conservation (6), yields

\[ \delta(E + \Lambda \Pi) = \delta \int dz \left\{ f(\theta, \eta, \dot{\theta}, \dot{\eta}, z) - \Lambda \eta \right\} = 0, \]
where $\Lambda$ is a Lagrange multiplier. The variational equation above results in
\begin{equation}
\cos \theta = - \tanh \left( \lambda^{-1} z \right),
\end{equation}
and
\begin{equation}
A \dot{\eta} - (S - U \sech^2(\lambda^{-1} z)) \eta - 3C(1 - \eta^2) + 2D(1 - \eta^3) + S = 0,
\end{equation}
where $\lambda^{-1} = [a_1 - a_\perp b_2/b_0, b_2]$ is the domain wall width in the spiral order. Similarly to Bloch walls in ferromagnets, a finite $\lambda$ results from the competition between the exchange and the anisotropy results in the magnetic energy. In expression (11), we have the quantities
\begin{align*}
A &= b_0 t^{2\beta} - a_0, \\
S &= a_2 - t^{2\beta} b_2 a_\parallel + c_2 t^{2\beta}, \\
U &= -2(a_\parallel - a_\perp) b_2 t^{2\beta}, \\
C &= c_2 t^{2\beta}, \\
D &= c_2 t^{2\beta} + t^{4\beta} b_4,
\end{align*}
which are defined in terms of the original GL parameters given in (8). As usual, $a_0$ and $b_0$ have no dependence on temperature and are assumed to be independent of $(a_2, b_2, c_2 t^{2\beta}, b_4 t^{4\beta})$ in the parameter space. The limit $a_0, b_0 \to 0$, with $\lambda^{-1} \to \infty$, reproduces the Ising case where (14) becomes a $\eta$-independent equation
\begin{equation}
S(1 - \eta) - 3C(1 - \eta^2) + 2D(1 - \eta^3) = 0,
\end{equation}
which yields a constant $\eta = 1 - x$. Substituting $\eta$ in terms of the doping $x$ in (12) one finds
\begin{equation}
S = 3C(2 - x) - 2D \left[ 3(1 - x) + x^2 \right],
\end{equation}
for $x \neq 0$. In order (13) to be valid in a continuous interval of temperature, we must have $c_2(T) \propto t^2, c_2(T) \propto t$ and the OP critical exponent $\beta = \frac{1}{2}$.

Now we solve equation (11) in a perturbative way, in relation to the parameters $(c_2, b_4)$ which are associated to the cubic and quartic terms in the free energy density (8). We write
\begin{equation}
\eta(z) = \eta_L(z) + c \psi(z),
\end{equation}
with $\eta_L$ satisfying eq. (11) in lowest order and $\psi$ corresponding to higher order corrections due to $c_2$ and $b_4$, which are small parameters in the GL sense. Imposing the limit $c_2, b_4 \to 0$, condition (13) becomes $S = 0$ for $x \neq 0$, and we obtain from (11)
\begin{equation}
A \dot{\eta}_L + U \sech^2(\lambda^{-1} z) \eta_L = 0,
\end{equation}
whose solution is the hypergeometric function
\begin{equation}
\eta_L(z) = F \left( \Gamma_-, \Gamma_+; 1; \sech^2(\lambda^{-1} z) \right),
\end{equation}
with $\Gamma_\pm = \frac{1}{8}(1 \pm \sqrt{1 + 4U/\Lambda^2/A})$. Solving the linearized $\psi$ equation resulting from (11) in the asymptotic limit $\eta_L \sim 1$, where $\eta_L$ satisfies the equation (13), we encounter that
\begin{equation}
\psi(z) \sim g(z) \left[ \sech(\lambda^{-1} z) \right]^{\sqrt{V}},
\end{equation}
where $g(z)$ is a smooth and convergent hypergeometric function
\begin{equation}
g(z) = F \left( \Lambda_-, \Lambda_+; 1 + \lambda \sqrt{V}; \sech^2(\lambda^{-1} z) \right),
\end{equation}
with $\Lambda_\pm = \Gamma_\pm + \frac{1}{2} \sqrt{V}$ and
\begin{equation}
V = \frac{1}{A} x [2D(3 - x) - 3C]
\end{equation}
as an explicitly doping dependent quantity.

Going one more step further, we apply the constraint (11) to the perturbed solution $\eta_L + c \psi$, which yields
\begin{equation}
c = \frac{\int_{-\infty}^{\infty} du \left[ 1 - \eta_L(u) \right] - x \frac{2}{3}}{\int_{-\infty}^{\infty} du \psi(u)},
\end{equation}
where $u = z/\lambda$ is the coordinate scaled by the soliton width.

### A. Estimation of parameters

A numerical calculation shows that $\int_{-\infty}^{\infty} du \left( 1 - \eta_L \right) \cong 0.81 U \lambda^2 / A^2$, with $u = z/\lambda$. In lowest order we write the constraint (11) as $\int du \left( 1 - \eta_L \right) = \frac{d}{\lambda} x$, and find that
\begin{equation}
\frac{d}{\lambda} x \sim \frac{4 U}{5} \lambda^2 = \frac{8}{5} \frac{b_0 t}{a_0 - b_0 t}.
\end{equation}
Up to this order, the amplitude of the CDW peaks depends on the adimensional parameter $U \lambda^2 / A$ only. This latter quantity is numerically limited to the interval $[0, 2]$ in order to have $0 \leq \rho(z) \leq 1$. As a consequence, relation (13) leads to the approximate inequality
\begin{equation}
0 \leq \frac{d}{\lambda} x \lesssim \frac{8}{5}.
\end{equation}
The above condition also allows us to estimate the limitation of the temperature to saturate the peak. Within our simplified model we get $t_{\text{low}} \sim (a_0/2 b_0) \lesssim 1$, when the peaks have maximum amplitude (for $U \lambda^2 / A \sim 2$). Two examples are shown in Fig. 2.

We observe that the limit $d/\lambda \to 1_+$ for the ratio between the domain width and the domain wall width, is indicative of a CDW commensurability transition from the present low incommensurable (LI) phase to some incommensurable one. Because the validity of this LI phase model requires that $d > \lambda$, we get the condition
\begin{equation}
x \leq \frac{8}{5} \frac{b_0 t}{a_0 - b_0 t},
\end{equation}
which delimits region LI from above in the phase diagram displayed in Fig. 1.
III. STRIPE PHASE TRANSITION

In the spirit of GL commensurability transitions, we now pursue the phenomenological description of the stripe critical regime, where magnetic order is absent. For high incommensurability, the appropriate OP is \( \delta \rho e^{i \phi} \), which describes the charge modulation around the uniform high-symmetry charge distribution \( \rho_0 \). In the vicinity of the critical point \( T_s \), the modulation is soft and therefore we assume that \( (\delta \rho) = 0 \), and we describe the CDW as a periodic modulation with constant amplitude \( \rho(z) - \rho_0 = \delta \rho e^{i \phi(z)} \).

The generalized GL free energy density is written as a contribution of an explicitly dependent term on the OP derivatives \( K \) plus a potential \( U(\phi) \). The most general “kinetic” term has the form

\[
K \equiv \sum_{ij} [b_{ij}(\dot{\phi}_i \dot{\phi}_j - \phi_i \phi_j) + a_{ij} \phi_i \phi_j] .
\]

The invariant part of the antisymmetric term in \( K \) is called the Lifshitz invariant, and is the essential ingredient to stabilize a modulated incommensurate OP phase. The Lifshitz invariant comes from the most general expansion which includes first order derivatives, after one removes contributions which are not extensive in character (terms which contribute to the total free energy \( F \) as surface integrals).

The point group of the OP is chosen with a low symmetry, just noting that a sinusoidal charge distribution locally displays mirror symmetry in the phase \( \phi \) at maxima and minima of the distribution. Thus, with an appropriate phase choice, we include in the symmetry group a reflection \( \sigma \) to describe the operation \( \phi \to -\phi \) and the point group is \( C_{1v} = \{e, \sigma \} \), where \( e \) is the identity. The real components of the OP, \( (\delta \rho \cos \phi, \delta \rho \sin \phi) \), transform differently: \( \delta \rho \cos \phi \) is invariant and \( \delta \rho \sin \phi \) transform after the antisymmetric representation. Therefore, the polynomial basis to generate invariants of the group is \( \{ (\delta \rho)^2, \delta \rho \cos \phi \} \). We observed that the latter basis displays particle-hole symmetry, since

\[
-\delta \rho \cos \phi = \delta \rho \cos (\phi - \pi) = \delta \rho \cos \phi',
\]

i.e. the change in sign of the charge may be included in a redefinition of the phase variable (interchanges maxima and minima of the charge distribution, leaving the free energy invariant).

If we write the ‘kinetic’ energy \( K \) using the real components of the OP, and using the fact that \( (\delta \rho) = 0 \), we get \( K \) in its most symmetric form, with functions in front of the derivatives which are invariants of the symmetry group. Note that \( (\delta \rho)^2 \cos^2 \phi \) is not included, if we additionally require the ‘kinetic’ term to be invariant under an arbitrary phase shift:

\[
K \equiv -a_{01}(\delta \rho)^2 \phi + a_0(\delta \rho)^2 (\dot{\phi})^2 .
\] (21)

The above ‘kinetic’ Hamiltonian \( K \) represents the elastic energy due to the distortion of the OP in relation to the homogeneous case. The phenomenological parameter \( a_{01} \) is related to the Lifshitz invariant, which is linear in \( \phi \). It is then clear the role of this term: while ‘kinetic’ and ‘potential’ energies compete, the Lifshitz invariant will try to stabilize a phase with a non vanishing \( \phi \).

The ‘potential energy’ is expanded as a power series of the polynomial basis of invariants \( \{(\delta \rho)^2, \delta \rho \cos \phi \} \) up to third order. This way, the most general form of the potential up to third order is

\[
U \equiv a_1 (\delta \rho \cos \phi + a_2 (\delta \rho)^2 + a_{21} (\delta \rho)^2 \cos^2 \phi + a_3 (\delta \rho)^3 \cos \phi + a_{31} (\delta \rho)^3 \cos^3 \phi + a_4 (\delta \rho)^4 \}
\] (22)

In first place, note that particle-hole symmetry (which has been used in the choice of the base), excludes a term of the type \( (\delta \rho)^3 \cos^3 \phi \). This is a key point in relation to the order of the transition at \( T_s \). More on this later. As usual in GL theories, the second order coefficients \( a_2 \) and \( a_{21} \) are assumed to be strongly temperature dependent. At last, concerning the fourth order terms in \( U \), a comment is pertinent. Our aim in this section is to describe the transition between the harmonic incommensurate phase (HI in Fig. 1) and the high symmetry phase (HS in Fig. 1), where the OP amplitude \( \delta \rho \) is small (note that the Lifshitz invariant term is proportional to \( (\delta \rho)^2 \)). A single fourth order term is then included to provide for a global stability of our solution, assuming \( a_4 > 0 \). Note that a different physics may be incorporated with other fourth order terms. Take for instance the case of \( (\delta \rho)^4 \cos^4 \phi \), with a negative coefficient. Clearly, this is a ‘lock-in’ term that should favor a homogeneous commensurate solution for large \( \delta \rho \). For simplicity, this case will not be considered here.

The free energy \( (K + U) \) does not explicitly depend on the variable \( z \). In this case, the variational Euler-Lagrange minimization leads to a first integral of the form...
\[
\Lambda = (K + U) - \phi \frac{\partial (K + U)}{\partial \phi}, \quad (23)
\]
where \(\Lambda\) is a constant. The explicit calculation of \(\text{[20]}\) results in the equation
\[
\left(\frac{\dot{\phi}}{\phi}\right)^2 = \Omega \left[1 - \frac{v}{\Omega} \sin^2 \left(\frac{\phi}{2}\right)\right] + \frac{a_{21}}{a_0 \Omega} \cos^2 \phi + \\
+ \frac{a_{31}}{a_0 \Omega} \cos^3 \phi, \quad (24)
\]
which couples \(\phi\) with \(\delta \rho\), with the definitions
\[
\frac{v}{2} = \frac{a_1 + a_3 (\delta \rho)^2}{a_0 \delta \rho}, \quad (25)
\]
and
\[
\Omega \equiv \frac{a_1 \delta \rho + a_2 (\delta \rho)^2 + a_3 (\delta \rho)^3 + a_4 (\delta \rho)^4 - \Lambda}{a_0 (\delta \rho)^2}, \quad (26)
\]
which are constants that parametrize our solutions.

If we take the limit \(a_{21}, a_{31} \to 0\) in \(\text{[21]}\), the remaining expression is a time-independent sine-Gordon equation, which describes highly incommensurable solutions in the expression is a time-independent sine-Gordon equation, \(\Omega\) which are constants that parametrize our solutions.

\[
\int_0^{\phi/2} d\varphi \frac{1}{\sqrt{1 - \frac{v}{\Omega} \sin^2 \varphi}} = \frac{1}{2} \Omega z + \\
\frac{a_{21}}{8a_0 \Omega} \left(1 + \frac{3}{4} \frac{v}{\Omega}\right) \left(\phi + \frac{1}{2} \sin(2\phi)\right) + \\
+ \frac{1}{4a_0 \Omega} \left(a_{31} \delta \rho - \frac{3}{16} a_{21} \frac{v}{\Omega}\right) \sin \phi \left(1 - \frac{1}{3} \sin^2 \phi\right). \quad (27)
\]

The left hand side of \(\text{[27]}\) is an elliptical integral of the first kind \(\text{[24]}\). For \(a_{21}, a_{31} \to 0\), \(\phi(z) = 2am(u) \sim \Omega^{1/2} z = k_0 z\), for \(\frac{v}{\Omega} \ll 1\), where \(am\) is the amplitude of the JacobianElliptic function. Carrying on this procedure perturbatively up to terms of the order of \((\delta \rho)^3\), we get
\[
\phi(z) \sim k_0 + \frac{a_{21}}{8a_0 \Omega} \left(1 + \frac{3}{4} \frac{v}{\Omega}\right) \sin(2k_0 z) + \\
\frac{1}{2a_0 \Omega} \left(a_{31} \delta \rho - \frac{3}{16} a_{21} \frac{v}{\Omega}\right) \sin(k_0 z) \times \\
\times \left(1 - \frac{1}{3} \sin^2(k_0 z)\right) \equiv k_0 + \epsilon(z), \quad (28)
\]
with
\[
k \approx k_0 \left[1 + \frac{a_{21}}{4a_0 \Omega} \left(1 + \frac{3}{4} \frac{v}{\Omega}\right)\right].
\]

From this we can compute the CDW energy \(E = \int_0^{l_z} dz (K + U)\) and obtain \(k\) and \(\delta \rho\) which stabilize the stripe phase at the global minimum condition \(\partial E/\partial (\delta \rho) = \partial E/\partial k = 0\). Since \(\Omega \sim 1/\delta \rho^2\), the terms \(a_{21}/\Omega\) and \(a_{31} \delta \rho/\Omega\) are of orders 2 and 3 in \(\delta \rho\), respectively. This implies that the only term that carries corrections from \(\epsilon\) of the order of \((\delta \rho)^3\) in the free energy, is \(a_1 \delta \rho \cos(k z + \epsilon)\), which expanded within this order becomes
\[
a_1 \delta \rho \cos(k z + \epsilon) = a_1 \delta \rho \cos(k_0 z) + \\
- a_1 \left(\frac{a_{21} \delta \rho}{8a_0 \Omega}\right) \sin(k_0 z) \sin(2k_0 z) + o(\delta \rho^4) \quad (29)
\]

For \(l_z \gg 2\pi/k_0\), the integrals \(\int_0^{l_z} dz \{\sin(k_0 z) \sin(2k_0 z)\}\) and \(\int_0^{l_z} dz \cos^n(k z)\), with \(n\) odd, are non-extensive quantities in the energy and may be ignored. The energy will be given by the simple expression
\[
E = \sum_{l_z} \left[ (\delta \rho)^2 \left(- a_0 \bar{k} + a_0 k_0^2 + a_2 + \frac{a_{21}}{2}\right) + \bar{a}_4 (\delta \rho)^4\right], \quad (30)
\]
where \(\bar{a}_4\) is a renormalized fourth order coefficient, which includes small contributions from \(a_{31}\) and \(a_{21}\). Therefore, we assume that the condition \(\bar{a}_4 > 0\) is fulfilled. Note that third order terms in \(\delta \rho\) do not contribute to \(E\) in the thermodynamic limit. The expression \(\text{[30]}\) is a minimum for
\[
\frac{k}{a_{21}} = \frac{a_{21}}{2a_0}, \quad (31)
\]
and
\[
\delta \rho = \left(\frac{a_{21}^2 - 2a_0 (a_{21} + a_2)}{8a_0 \bar{a}_4}\right)^{\frac{1}{2}}. \quad (32)
\]

In the limit of highly incommensurable solutions, the potential \(U\) is irrelevant for the stabilization of the ground state. The stability is reached through the competition of the \(a_0\) squared gradient term (which usually fixes the scale of energy in such classes of phenomenological models), with the Lifshitz invariant coefficient \(a_0\), what we interpret as a minimization of the CDW elastic energy.

When \(T \to T_S\) from below, \(\delta \rho \to 0\), and the Lifshitz invariant coefficient \(a_0\) competes with the second order coefficients \((a_2, a_{21})\) of the potential \(U\), meaning they ‘cooperate’ with \(a_0\). But far below the stripe critical point \(T_S\), the importance of the Lifshitz invariant term is small in comparison to the ‘potential’ energy contribution of the several cosine terms, which are known to stabilize LI solutions, as we have shown in Section II. Therefore, the second order coefficients \((a_2, a_{21})\) should change sign in between of these two limits. Thus, the precedent analysis requires the coefficients \(a_2\) and \(a_{21}\) to be defined in terms of a new critical parameter \(\tau \equiv T/T_0 - 1\), \(T_0 < T_S\), which drives a CDW commensurability transition.

Writing \(a_2 = a' \tau\) and \(a_{21} = a'' \tau\) \((a, a' > 0)\), the stripe phase transition at \(T_S\) will be of second order if \(T_0\) and
$T_S$ are related by the constraint
\begin{equation}
T_S = T_0 \left( 1 + \frac{a_{01}^2}{2 a_0 (a' + 2a)} \right) > T_0.
\end{equation}
Replacing \( \rho \) in (22) we find
\begin{equation}
\delta \rho = \sqrt{\frac{2a_0 (a' + 2a) + a_{01}^2}{8a_0 a_4} \left( \frac{T_S - T}{T_S} \right)^{1/2}},
\end{equation}
which goes to zero at $T_S$ under the usual mean field square-root law.

The periodic $\epsilon(z)$ function in (28) introduces non-periodical corrections in the CDW phase $\phi = k z + \epsilon(z)$. The only way to preserve the CDW periodic nature is to set $\epsilon(z) \equiv 0$, i.e. $a_{21}, a_{31} = 0$. In this picture, the CDW solution is just $\rho(z) = x + \delta \rho$ ($2a_0 (a' + 2a)$, using appropriate units, with $x$ being the doping. Applying the global constraint $\int_0^{l_z} \rho(z) \, dz = Q$ in the limit $l_z \gg \frac{2a_0}{k}$, we encounter that $\rho_0 = x$. Since $\delta \rho \leq \rho_0$, the $(\delta \rho) \approx 0$ region is delimited by the inequality $x \geq \delta \rho$ in the phase diagram depicted qualitatively in Fig. 1.

### IV. DISCUSSION

We present here a brief summary of the paper and we discuss the results of the last two sections. Phase I describes the LI spin driven regime of stripes, which takes place in our phase diagram for low doping concentrations. This phase is also experimentally associated with incompressible stripes (for $x < \frac{1}{8}$, in the cuprate case). This is an evidence that they interact weakly, despite the existence of ‘long-range’ spin-spin and charge-charge interactions (in both cases, the correlation length is of the order of few hundred angstroms). Separately, each of them would drive the system to macroscopically homogeneous phases. However, their competition, even in the absence of dynamical correlations, reduces considerably the effective range of the interactions and stabilizes non-homogeneous phases in a smaller scale.

We showed in Section I that a simple static Hamiltonian, like the one given by (23), captures in a qualitative way the physics of competing effects which leads to low temperature CDW formation in the presence of two-dimensional magnetism. The second conclusion is that this soliton-like CDW scenario changes drastically with temperature. Moving in region LI at constant doping and increasing temperatures (see Fig. 1), our solution shows that the ratio between the inter-stripe distance and the stripe width decreases (as well as the amplitude of the peaks) and goes to the limit $d/\lambda \to 1$, which corresponds to the commensurability transition that separates the LI phase from the high incommensurable (but still spin driven) phase I.

In contrast, in the absence of 2D antiferromagnetism, we may ask which is the mechanism behind the CDWs, since there are no magnetic correlations to compete with the repulsion of holes. For this phase, we construct the free energy from general theoretical arguments coming entirely from symmetry. We conjecture that this free energy includes, in a stationary fashion, dynamical phenomena that occur at high temperature, such as elastic and electronic coupled effects that induce a polarization of the lattice. In an actual phase diagram of cuprates, as the one shown in Fig. 4 of Ref. 17, this phenomenon is located in the high temperature sector, considerably far from the tetragonal-orthorhombic structural phase transition. This latter is still in the interior of the 2D antiferromagnetic correlated region, and has no relation with the high temperature physics that we are dealing with here.

The $C_{1v}$ point group of the OP for the HI phase implies the occurrence of odd terms in the free energy density. However, particle-hole symmetry makes those terms non-extensive in character, and they can be neglected in the thermodynamic limit. This means, in principle, that one may not rule out the possibility of a second order phase transition for stripes, in contrast to other phenomenological approaches.

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### Appendix

In this appendix we will demonstrate that the GL free energy (1) is equivalent to Hamiltonian (2) in mean field approximation. Starting from the definition of our staggered mean field $\mathbf{m}$, we calculate macroscopic average of the spin term in (2), which yields
\begin{align}
H_s &= - \int d^2 r \sum_{j=1}^2 \alpha_j m_j^2(r) \\
&\quad - \int d^2 r d^2 \epsilon \sum_{j=1}^2 \beta_j m_j^2(r) m_j^2(r + \epsilon),
\end{align}
in the limit where the lattice parameter goes to zero. The charge term calculation is similar, and we may write
\begin{equation}
H_c = \int d^2 r d^2 \epsilon \left[ \frac{1}{\epsilon} \right] \left[ 1 - \eta(r) \right] \left[ 1 - \eta(r + \epsilon) \right]
\end{equation}
Observing that the OP is by definition a continuous function that varies slowly in space, we assume that $\eta(r + \epsilon)$ and $\mathbf{m}(r + \epsilon)$ can be expanded up to terms of lowest order. Performing the integration in $\epsilon$, the resulting free
energy is a functional of the form
\[ f(r) = -a_0(\dot{\eta})^2 + a_2(1 - \eta)^2 + \sum_{j=1}^{2} \left[ b_0(\dot{m}_j)^2 - b_2\alpha_j m_j^2 \right], \]
with \( \alpha_j \) taken as an effective anisotropy parameter.

Note that the charge \((1 - \eta)\) acts as a secondary order parameter, with no critical behavior. In order to achieve a minimum for the thermodynamic potential, we must include higher order powers of the primary OP \( m \) in the above expression. Since the odd ones break the free energy invariance under reflections, the lowest correction is of fourth order. So, we heuristically introduce the simplest term \( b_4 m^4 \), which represents an isotropic magnetic quadrupole interaction. In the same way, we may consider a coupling term for the spin and charge degrees of freedom. We rule out the OP coupling with odd powers of \((1 - \eta)\), due to particle-hole symmetry. If we assume that the coupling does not depend on the crystal directions, the lowest order term is \( c_2 (1 - \eta)^2 m^2 \). This closes expression (36) in our model.

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