HOFTER'S GEOMETRY AND FLOER THEORY UNDER
THE QUANTUM LIMIT

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Abstract. In this paper, we use Floer theory to study the Hofer length
functional for paths of Hamiltonian diffeomorphisms which are suffi-
ciently short. In particular, the length minimizing properties of a short
Hamiltonian path are related to the properties and number of its peri-
odic orbits.

1. Introduction

Throughout this work, $\langle M, \omega \rangle$ will be a closed symplectic manifold of
dimension $2n$. The space of smooth $\omega$-compatible almost complex structures
on $M$ will be denoted by $J(M, \omega)$. For $J$ in $J(M, \omega)$, let $h(J)$ be the
infimum over the symplectic areas of nonconstant $J$-holomorphic spheres
in $M$, and set

$$h = \sup_{J \in J(M, \omega)} h(J).$$

This strictly positive quantity is the quantum limit referred to in the title. In
this paper, we consider paths of Hamiltonian diffeomorphisms whose Hofer
length is less than $h$, and prove several results which relate the length mini-
mizing properties of these paths to the periodic orbits of the corresponding
Hamiltonian flows. To establish these results we develop some new Floer
theoretic tools. These are motivated, in part, by similar constructions in
Lagrangian Floer theory due to Chekanov, [Ch], as well as recent work by
Albers in [Al]. These tools allow one to detect length minimizing Hamilton-
ian paths using homological algebra, [KL Oh2 Oh3 Schw].

1.1. Definitions. Before stating the main results, we first recall some basic
definitions and fix some conventions. For more detailed accounts of this
material the reader is referred to the books [HZ McDSa1 Po].

Let $S^1 = \mathbb{R}/\mathbb{Z}$ be the circle parameterized by $t \in [0, 1]$. A Hamiltonian
on $M$ is a smooth function $H$ in $C^\infty(S^1 \times M)$, which we also view as a loop
of smooth functions $H_t(\cdot) = H(t, \cdot)$ on $M$. We say that $H$ is normalized if

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\item[We use the convention that the infimum over the empty set is equal to infinity.]
\end{itemize}
\[ \int_M H_t \omega^n = 0 \] for each \( t \in [0,1] \). The space of normalized Hamiltonians is denoted by \( C_0^\infty(S^1 \times M) \).

Each Hamiltonian \( H \) determines a time-dependent vector field \( X_H \) on \( M \) via the equation

\[ \omega(X_H, \cdot) = -dH_t(\cdot). \]

The flow of this vector field (also called the Hamiltonian flow of \( H \)) is denoted by \( \phi_t^H \) and is defined for all \( t \in [0,1] \). The group of Hamiltonian diffeomorphisms of \(( M, \omega)\) consists of all the time one maps of such flows.

For every path of Hamiltonian diffeomorphisms, \( \psi_t \), there is a unique normalized Hamiltonian \( H \in C_0^\infty(S^1 \times M) \) such that \( \phi_t^H \circ \psi_0 = \psi_t \). Following Hofer [Ho1], this generating Hamiltonian is used to define the Hofer length of the path \( \psi_t \) as follows

\[
\text{length}(\psi_t) = \|H\| = \int_0^1 \max_M H_t \, dt - \int_0^1 \min_M H_t \, dt
\]

Both \( \|H\|^+ \) and \( \|H\|^− \) provide different measures of the length of \( \psi_t \), called the positive and negative Hofer lengths, respectively.

Let \([\psi_t] \) be the class of Hamiltonian paths which are homotopic to \( \psi_t \) relative to its endpoints. Denote the set of normalized Hamiltonians which generate the paths in \([\psi_t] \) by

\[
C_0^\infty([\psi_t]) = \{ H \in C_0^\infty(S^1 \times M) \mid [\phi_t^H \circ \psi_0] = [\psi_t] \}.
\]

The Hofer seminorm of \([\psi_t] \) is then defined by

\[
\rho_H([\psi_t]) = \inf \{ \|H\| \mid H \in C_0^\infty([\psi_t]) \}.
\]

The positive and negative Hofer seminorms of \([\psi_t] \) are defined similarly as

\[
\rho^±([\psi_t]) = \inf \{ \|H\|^± \mid H \in C_0^\infty([\psi_t]) \},
\]

and we call

\[
\bar{\rho}([\psi_t]) = \rho^+(\psi_t) + \rho^−(\psi_t)
\]

the two-sided Hofer seminorm of \([\psi_t] \).

Since these seminorms are bi-invariant, we need only consider paths which start at the identity and hence have the form \( \phi_t^H \) for some \( H \) in \( C_0^\infty(S^1 \times M) \). To avoid pathologies, we will also assume that our paths are regular. That is, we will only consider Hamiltonian paths \( \phi_t^H \) for which the functions \( H_t \) do not vanish identically for \( t \in (0,1) \).

We say that \( \phi_t^H \) minimizes the Hofer length in its homotopy class if there is no path in \([\phi_t^H] \) with a smaller Hofer length, i.e., \( \|H\| = \rho_H([\phi_t^H]) \). The notion of a path which minimizes the positive, negative or two-sided Hofer length is defined analogously. Note that \( \rho_H([\psi_t]) \geq \bar{\rho}([\psi_t]) \) and so if \( \phi_t^H \) minimizes \( \bar{\rho} \) then it also minimizes \( \rho_H \). It should also be mentioned that length
minimizing paths for the Hofer seminorm need not exist in a given homotopy class. Explicit examples of such classes are constructed by Lalonde and McDuff in \[LMcD\].

We will use Floer theory to relate the length minimizing properties of a Hamiltonian path to the number and properties of its periodic orbits. Hence, we will focus our attention on \(\mathcal{P}(H)\), the set of contractible periodic orbits of \(X_H\) which are 1-periodic (have period equal to one). We say that \(x\) in \(\mathcal{P}(H)\) is nondegenerate if the map \(d\phi^1_\mathcal{H} \colon T_{x(0)}M \to T_{x(0)}M\) does not have one as an eigenvalue. A Hamiltonian \(H\) will be called a Floer Hamiltonian if each \(x\) in \(\mathcal{P}(H)\) is nondegenerate.

A spanning disc for a 1-periodic orbit \(x\) is a smooth map \(u\) from the unit disc \(D^2 \subset \mathbb{C}\) to \(M\), such that \(u(e^{2\pi it}) = x(t)\). It can be used to define two important quantities for \(x\) which can be studied using Floer theory. The first quantity is \(\mu_{CZ}(x, u)\), the Conley-Zehnder index of \(x\) with respect to \(u\). This index is normalized here so that if \(p\) is a nondegenerate critical point of a \(C^2\)-small time-independent Hamiltonian, and \(u\) is the constant spanning disc, then

\[
\mu_{CZ}(p, u) = \text{ind}(p) - n,
\]

where \(\text{ind}(p)\) is the Morse index of \(p\). The second important quantity is the action of \(x\) with respect to \(u\), which is defined by

\[
\mathcal{A}_H(x, u) = \int_0^1 H(t, x(t)) dt - \int_{D^2} u^* \omega.
\]

Both the Conley-Zehnder index and the action of \(x\) with respect to \(u\) depend only on the homotopy class of \(u\).

1.2. Results. If \(\phi^\mathcal{H}_t\) does not minimize \(\rho_\mathcal{H}\) in its homotopy class, then it also fails to minimize \(\rho^+\) or \(\rho^-\). For this reason, we formulate and prove our results for the one-sided seminorms. In particular, we consider the positive Hofer seminorm. The statements and proofs of the corresponding results for \(\rho^-\) are entirely similar (see Corollary 6.1).

**Theorem 1.1.** Let \(H\) be a Floer Hamiltonian such that \(\|H\| < h\). If \(\phi^\mathcal{H}_t\) does not minimize the positive Hofer seminorm in its homotopy class, then there are at least \(\text{rk}(\mathbb{H}(M; \mathbb{Z}))\) contractible 1-periodic orbits \(x_j\) of \(H\) which admit spanning disks \(u_j\) such that

\[
-n \leq \mu_{CZ}(x_j, u_j) \leq n
\]

and

\[
-\|H\|^- \leq \mathcal{A}_H(x_j, u_j) < \|H\|^+.
\]

Here, \(\text{rk}(\mathbb{H}(M; \mathbb{Z}))\) is the rank of the abelian group \(\mathbb{H}(M; \mathbb{Z})\), and the rank of the torsion subgroup is the minimal number of elements needed to generate it.

For a general, possibly degenerate, Hamiltonian we prove:
Theorem 1.2. Let $H$ be a Hamiltonian such that $\|H\| < \hbar$. If $\phi^t_H$ does not minimize the positive Hofer length in its homotopy class, then there is a contractible 1-periodic orbits $y$ of $H$ which admits a spanning disk $w$ such that

$$-\|H\|^- \leq A_H(y, w) < \|H\|^+.$$ 

In the corresponding results for the negative Hofer seminorm, the action values are confined instead to the interval $(-\|H\|^-, \|H\|^+)$, (see, again, Corollary 6.1).

To refine Theorems 1.1 and 1.2 it is useful to restrict one's attention to Hamiltonians which have properties known to be necessary to generate a length minimizing path. Following [BP], a Hamiltonian $H$ is called quasi-autonomous if it has at least one fixed global maximum $P \in M$ and one fixed global minimum $Q \in M$. That is,

$$H(t, P) \geq H(t, p) \geq H(t, Q)$$

for all $p \in M$ and $t \in [0, 1]$.

Theorem 1.3. [BP, LMcD] If the Hamiltonian path generated by $H$ minimizes the Hofer norm in its homotopy class, then $H$ must be quasi-autonomous.

A symplectic manifold $(M, \omega)$ is said to be spherically rational, if the quantity

$$r(M, \omega) = \inf_{A \in \pi_2(M)} \{|\omega(A)| \mid |\omega(A)| > 0\}.$$ 

is strictly positive. In this case, $r(M, \omega) \leq \hbar$ and we prove:

Theorem 1.4. Let $H$ be a Floer Hamiltonian on a spherically rational symplectic manifold $(M, \omega)$ such that $H$ is quasi-autonomous and $\|H\| < r(M, \omega)$. If $\phi^t_H$ does not minimize $\rho_H$ in its homotopy class, then $H$ has at least $SB(M) + 2$ contractible 1-periodic orbits.

Here, $SB(M)$ is equal to $rk(H(M; \mathbb{Q}))$, the sum of Betti numbers of $M$.

The Arnold conjecture for compact symplectic manifolds, [CZ, FH, FP, HS, FO, LT], implies the existence of at least $SB(M)$ contractible 1-periodic orbits of a Floer Hamiltonian. The two extra orbits detected in Theorem 1.4 were also found in the case of symplectically aspherical manifolds in [KL]. One is lead by these results (as well those mentioned below) to the following question.

Question 1.5. If the path generated by a Floer Hamiltonian does not minimize the (positive/negative) Hofer length (in its homotopy class), must it have at least $SB(M) + 2$ contractible 1-periodic orbits?

For paths generated by autonomous Hamiltonians there is a well known conjecture in this direction which is motivated by the work of Hofer in [Ho2].

It states that if a path generated by an autonomous Hamiltonian is not length minimizing in its homotopy class, then there must be a nonconstant contractible periodic orbit of the Hamiltonian flow with period less than or
This conjecture was essentially settled for all compact symplectic manifolds by McDuff and Slimowitz, in [McDSi]. An extra hypothesis in the main theorem of [McDSi], concerning the under-twistedness of all critical points, was later shown to be unnecessary by Schlenk in [Schl].

1.3. Examples. Theorems 1.1, 1.2 and 1.4 can each be viewed from two perspectives. In terms of Hofer’s geometry, they can be phrased as sufficient conditions for a Hamiltonian path to be length minimizing for some Hofer seminorm. In terms of dynamics, these results can also be used to detect the existence of special 1-periodic orbits of a Hamiltonian path which is known not to minimize a Hofer seminorm. In this section, we describe some elementary examples of Hamiltonian paths on the two sphere which one can show are length minimizing in their homotopy class using the results proved here. We also describe a basic setting wherein Theorems 1.1, 1.2 and 1.4 can be used to detect periodic orbits for Hamiltonian flows of independent interest.

1.3.1. Length minimizing Hamiltonian paths on $S^2$. Consider the unit sphere $S^2$ in $\mathbb{R}^3$. One has coordinates

$$\theta: S^2 \setminus \{p_N, p_S\} \to [0, 2\pi]$$

and

$$z: S^2 \to [-1, 1],$$

where $p_N$ and $p_S$ are the north and south poles, $\theta(x, y, z)$ is the angle of the point $(x, y)$ with respect to the positive $x$-axis, and $z$ is the height. The area form $\omega$ on $S^2$ inherited from the Euclidean volume form on $\mathbb{R}^3$, is a symplectic form for which $\hbar = 4\pi$.

The Hamiltonian flow of the function $H_\upsilon(\theta, z) = \upsilon z$ is given by

$$\phi^t_{H_\upsilon}(\theta, z) = (\theta + \upsilon t, z).$$

For $\upsilon < 2\pi$, this flow has no nonconstant periodic orbits of period less than or equal to one. It then follows from [McDSi], that $\phi^t_{H_\upsilon}$ minimizes the two-sided Hofer length in its homotopy class. This is also implied by Theorem 1.1. In particular, the only 1-periodic orbits of the flow are the constant orbits at the poles. The action of $p_N$ with respect to any spanning disc has the form $\upsilon + 4\pi k$ for some integer $k$. Hence, for $\upsilon < 2\pi$ the north pole does not admit a spanning disc for which the corresponding action lies in the interval $[-\|H_\upsilon\|, \|H_\upsilon\|] = [-\upsilon, +\upsilon]$. Since $rk(\mathbb{H}(S^2; \mathbb{Z})) = 2$ and only the south pole admits a spanning disk yielding an action value in $[-\upsilon, +\upsilon]$, it follows from Theorem 1.1 that $\phi^t_{H_\upsilon}$ minimizes the positive Hofer length in its homotopy class. The analogous result for the negative Hofer seminorm implies that $\phi^t_{H_\upsilon}$ also minimizes the negative Hofer length in its homotopy class.

\footnote{It seems reasonable to refine this conjecture to orbits of period equal to one.}
In this setting, one can also construct simple examples of time-dependent Hamiltonians which generate length minimizing paths. Consider a Hamiltonian \( H = H(t, z) \) which does not depend on \( \theta \) and has a fixed global maximum (resp. minimum) at \( z = 1 \) (resp. \( z = -1 \)). The time-\( t \) flow of \( H \) is given by

\[
\phi^t_H(\theta, z) = (\theta + \int_0^t \partial_z H(\tau, z) \, d\tau, z).
\]

If

\[
0 < \int_0^1 \partial_z H(t, z) \, dt < 2\pi \text{ for all } z \in [-1, 1],
\]

then \( \|H\| < 4\pi \) and the only 1-periodic orbits of \( H \) are again the constant orbits at the poles. Arguing as above, Theorem 1.1 implies that \( \phi^t_H \) minimizes both \( \rho^+ \) and \( \rho^- \), and hence \( \bar{\rho} \), in its homotopy class. Note that these length minimizing paths may have many nonconstant periodic orbits with period less than one. They also come in infinite dimensional families (with fixed endpoints).

**Remark 1.6.** These examples also illustrate the limitations of the methods developed here. For example, for \( \pi < \nu < 2\pi \), the path generated by \( H_\nu : S^2 \to \mathbb{R} \) is not length minimizing among all paths (e.g. the flow of \(-H_{2\pi-\nu} \) has the same time one map). However, there is no discernable change in the Floer complexes of the \( H_\nu \) as the parameter \( \nu \) crosses the value \( \pi \).

**Remark 1.7.** At least for \( S^2 \), the problem of characterizing length minimizing Hamiltonian paths above the quantum limit may not be a meaningful. In particular, it not clear to the author whether there exists a Hamiltonian \( H \) on \((S^2, \omega)\) such that \( \|H\| > h \) and \( \phi^t_H \) minimizes the Hofer length in its homotopy class.

### 1.3.2. Examples of short paths which are not length minimizing

We now provide some examples of Hamiltonian paths which satisfy the hypotheses of the theorems proved in this paper. As described below, similar flows appear in several applications of Hofer’s geometry to Hamiltonian dynamics and symplectic topology.

The examples are constructed using Sikorav’s curve shortening procedure for Hamiltonian paths supported on sets with finite displacement energy. The displacement energy of a subset \( V \subset M \) is defined as

\[
e(V) = \inf_{H \in C_0^\infty(S^1 \times M)} \{\|H\| \mid \phi^1_H(V) \cap V = \emptyset\}.
\]

**Proposition 1.8.** Let \( H \) be an autonomous normalized Hamiltonian which is constant on the complement of an open subset \( U \) of \( M \) which has finite displacement energy. If \( \|H\| > 4e(U) \), then

\[
\|H\| > \rho_h([\phi^1_H]).
\]
If $H$ is equal to its minimum value outside of $U$ and $\|H\|^+ > 2e(U)$, then
\[ \|H\|^+ \geq \rho^+([\phi^t_H]) + \frac{1}{2}\|H\|^-. \]

In the first case, $\phi^t_H$ does not minimize the Hofer length in its homotopy class and in the second case it does not minimize the positive Hofer length in its homotopy class. The first part of Proposition 1.8 is proved in [Schl] as Proposition 2.1. The second part can be obtained by repeating the argument from [Schl] with only minor changes.

Let $N \subset M$ be a closed submanifold of $M$. If $2e(N) < h$, then for sufficiently small tubular neighborhoods $U$ of $N$ one can easily construct an autonomous normalized function $H$ which is equal to its minimum value outside of $U$ and satisfies $\|H\| < h$ and $\|H\|^+ > 2e(U)$. By Proposition 1.8, the corresponding path $\phi^t_H$ does not minimize the positive Hofer length. Perturbing this function if necessary, one can then apply Theorem 1.1, 1.2, or 1.4 to obtain information about the periodic orbits of $H$. If $H$ is chosen to be radially symmetric in the normal directions to $N$, then these periodic orbits are often of considerable interest. For example, when $N$ is a Lagrangian submanifold, the flow of $H$ is a reparameterization of a geodesic flow on $N$, and when $N$ is a symplectic submanifold, the flow of $H$ can model the motion of a charged particle in a nondegenerate magnetic field on $N$. The periodic orbits of such Hamiltonian flows have been studied in several works on Hamiltonian dynamics, Lagrangian embeddings and symplectic intersection phenomena. (see, for example, [BPS, CGK, Gi, G ¨ u, Ke1, Schl, Th, Vi]). In subsequent work, [Ke2], we will consider applications of the techniques developed here to such problems.

1.4. Paths with close endpoints. The methods developed in this work also apply to Hamiltonian paths which may be long but whose endpoints are close. The following result implies Theorem 1.4 but will be proved separately.

**Theorem 1.9.** Let $H$ be a Floer Hamiltonian such that $\bar{\rho}([\phi^t_H]) < h$. For every $\epsilon^+ > 0$ and $\epsilon^- > 0$, there are at least $rk(\Pi(M; \mathbb{Z}))$ contractible 1-periodic orbits $x_j$ of $H$ which admit spanning disks $u_j$ such that
\[ -n \leq \mu_{CZ}(x_j, u_j) \leq n \]
and
\[ -\rho^-([\phi^t_H]) - \epsilon^- \leq \mathcal{A}_H(x_j, u_j) \leq \rho^+([\phi^t_H]) + \epsilon^+. \]

**Remark 1.10.** If $\rho^+([\phi^t_H])$ (resp. $\rho^-([\phi^t_H])$) is realized by a Hamiltonian path in class $[\phi^t_H]$, then we can set $\epsilon^+ = 0$ (resp. $\epsilon^- = 0$). One can also set $\epsilon^\pm = 0$ if the symplectic manifold $(M, \omega)$ is rational.

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2. Floer Theory Under the Quantum Limit

2.1. Starting data. Let \( H \) be a Floer Hamiltonian. Denote the space of smooth \( S^1 \)-families of \( \omega \)-compatible almost complex structures on \( M \) by \( \mathcal{J}_{S^1}(M,\omega) \). Fixing a family \( J \) in \( \mathcal{J}_{S^1}(M,\omega) \), we refer to the pair \((H,J)\) as our Hamiltonian data.

Let \( F_s \) be a smooth \( \mathbb{R} \)-family of functions in \( C^\infty(S^1 \times M) \) or elements of \( \mathcal{J}_{S^1}(M,\omega) \). We call \( F_s \) a compact homotopy from \( F^- \) to \( F^+ \), if there is a \( \tau > 0 \) such that

\[
F_s = \begin{cases} 
  F^-, & \text{for } s \leq -\tau; \\
  F^+, & \text{for } s \geq \tau.
\end{cases}
\]

A homotopy triple for our Hamiltonian data \((H,J)\) is a collection of compact homotopies

\[ \mathcal{H} = (H_s, K_s, J_s). \]

Until we state otherwise, \( H_s \) will be a compact homotopy from the zero function to \( H \), \( K_s \) will be a compact homotopy from the zero function to itself, and \( J_s \) will be a compact homotopy in \( \mathcal{J}_{S^1}(M,\omega) \) from some \( J^- \) to \( J \).

The curvature of a homotopy triple \( \mathcal{H} = (H_s, K_s, J_s) \) is the function \( \kappa(\mathcal{H}) : \mathbb{R} \times S^1 \times M \to \mathbb{R} \) given by

\[ \kappa(\mathcal{H}) = \partial_s H_s - \partial_t K_s + \{H_s, K_s\}. \]

We define the positive and negative norms of the curvature by

\[
|||\kappa(\mathcal{H})|||^+ = \int_{\mathbb{R} \times S^1} \max_{p \in M} \kappa(\mathcal{H}) \, ds \, dt,
\]

and

\[
|||\kappa(\mathcal{H})|||^-= -\int_{\mathbb{R} \times S^1} \min_{p \in M} \kappa(\mathcal{H}) \, ds \, dt.
\]

Example 2.1. Let \( \eta : \mathbb{R} \to [0,1] \) be a smooth nondecreasing function such that \( \eta(s) = 0 \) for \( s \leq -1 \) and \( \eta(s) = 1 \) for \( s \geq 1 \). A linear homotopy triple for \((H,J)\) is a triple of the form

\[ \overline{\mathcal{H}} = (\eta(s)H, 0, J_s). \]

For a linear homotopy triple we have

\[ \kappa(\overline{\mathcal{H}}) = \dot{\eta}(s)H, \]

\[
|||\kappa(\overline{\mathcal{H}})|||^+ = \int_{\mathbb{R} \times S^1} \dot{\eta}(s) \left( \max_{p \in M} H(t,p) \right) \, ds \, dt = ||H||^+,
\]

and

\[
|||\kappa(\overline{\mathcal{H}})|||^-= -\int_{\mathbb{R} \times S^1} \dot{\eta}(s) \left( \min_{p \in M} H(t,p) \right) \, ds \, dt = ||H||^-.
\]

For Hamiltonian data \((H,J)\), we will choose a pair of homotopy triples

\[ H = (\mathcal{H}_L, \mathcal{H}_R). \]
This will be referred to as our cap data. The norm of the curvature of $H$ is defined to be

$$|||\kappa(H)||| = |||\kappa(H_R)|||^- + |||\kappa(H_L)|||^-.$$

2.2. Floer caps. Let $H$ be a Floer Hamiltonian. Given a homotopy triple $\mathcal{H} = (H_s, K_s, J_s)$ for $(H, J)$, we consider smooth maps $u : \mathbb{R} \times S^1 \to M$, which satisfy the equation

$$\partial_s u - X_{K_s}(u) + J_s(u)(\partial_t u - X_{H_s}(u)) = 0.\tag{1}$$

When $s \gg 0$ this is Floer’s equation

$$\partial_s u + J(u)(\partial_t u - X_H(u)) = 0,$$

and for $s \ll 0$ it becomes the nonlinear Cauchy-Riemann equation

$$\partial_s u + J^-(u)(\partial_t u) = 0.$$

The maps satisfying (1) are pseudo-holomorphic sections of the bundle $\mathbb{R} \times S^1 \times M \to \mathbb{R} \times S^1$ with respect to the almost complex structure on the total space given by

$$
\begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
X_{H_s} - J_s(X_{K_s}) & X_{K_s} - J_s(X_{H_s}) & J_s
\end{pmatrix}.
$$

The energy of a solution $u$ of (1) is defined as

$$E(u) = \int_{\mathbb{R} \times S^1} \omega(u) \left( \partial_s u - X_{K_s}(u), J_s(\partial_t u - X_{H_s}(u)) \right) ds dt.$$

If the energy of $u$ is finite, then it follows from the usual arguments that

$$u(+\infty) := \lim_{s \to \infty} u(s, t) = x(t) \in \mathcal{P}(H)$$

and

$$u(-\infty) := \lim_{s \to -\infty} u(s, t) = p \in M.$$

Here, convergence is in $C^\infty(S^1, M)$ and the point $p$ in $M$ is identified with the constant loop $t \mapsto p$.

This asymptotic behavior implies that a solution $u$ of (1) with finite energy determines an asymptotic spanning disk for the 1-periodic orbit $u(+\infty) = x$ (see Figure 1). More precisely, for sufficiently large $s > 0$, one can complete and reparameterize $u|_{[-s, 0]}$ to be a spanning disc for $x$ in a homotopy class which is independent of $s$. Since they play the same role for us, we will not distinguish between spanning discs and asymptotic spanning discs.

The set of left Floer caps of $x \in \mathcal{P}(H)$ with respect to $\mathcal{H}$ is

$$\mathcal{L}(x; H) = \left\{ u \in C^\infty(\mathbb{R} \times S^1, M) \mid u \text{ satisfies (1)} \quad , \quad E(u) < \infty, \quad u(+\infty) = x \right\}.$$

For each $u \in \mathcal{L}(x; H)$ we define the action of $x$ with respect to $u$ by

$$A_H(x, u) = \int_0^1 H(t, x(t)) dt - \int_{\mathbb{R} \times S^1} u^* \omega.$$
Each left Floer cap $u \in \mathcal{L}(x; \mathcal{H})$ also determines a unique homotopy class of trivializations of $T^*M|_{x}$ and hence a Conley-Zehnder index $\mu_{CZ}(x,u)$.

For a generic homotopy triple $\mathcal{H}$, i.e., a generic choice of $J_{s}$, each $\mathcal{L}(x; \mathcal{H})$ is a smooth manifold. The dimension of the component containing $u$ is $n - \mu_{CZ}(x,u)$, [PSS]. Hence, for every orbit $x \in \mathcal{P}(\mathcal{H})$ with a (regular) left Floer cap $u$ we have $\mu_{CZ}(x,u) \leq n$.

For any function of the form $F(s, \cdot)$, we will use the notation

$$\overline{F}(s, \cdot) = F(-s, \cdot).$$

Given a homotopy triple $\mathcal{H} = (H_{s}, K_{s}, J_{s})$, we will also consider maps $v: \mathbb{R} \times S^1 \to M$ which satisfy the equation

$$\partial_{s} v + X_{\overline{K_{s}}}(v) + \overline{J_{s}}(v)(\partial_{t} v - X_{\overline{H_{s}}}(v)) = 0. \quad (2)$$

In this way, we obtain for each $x \in \mathcal{P}(\mathcal{H})$ the space of right Floer caps,

$$\mathcal{R}(x; \mathcal{H}) = \{ v \in C^\infty(\mathbb{R} \times S^1, M) \mid v \text{ satisfies (2)}, E(v) < \infty, v(-\infty) = x \}.$$

Every right Floer cap $v \in \mathcal{R}(x; \mathcal{H})$ also determines an asymptotic spanning disc for $x$,

$$\overline{v}(s, t) = v(-s, t).$$

For a generic homotopy triple the space of right Floer caps $\mathcal{R}(x; \mathcal{H})$, is also a smooth manifold. The dimension of the component containing $v$ is $\mu_{CZ}(x, \overline{v}) - n$, and so for every orbit $x \in \mathcal{P}(\mathcal{H})$ with a (regular) right Floer cap $v$ we have $\mu_{CZ}(x, \overline{v}) \geq -n$.

2.3. Standing assumptions and their implications. We now specify three conditions on the Hamiltonian data $(H, J)$ and the cap data $\mathcal{H}$ which will be assumed to hold throughout Section 2. We also describe some implications of these assumptions which will be used repeatedly.

Our first assumption, which has already been stated, is

**A1:** $H$ is a Floer Hamiltonian.

This implies that the elements of $\mathcal{P}(\mathcal{H})$ are isolated and hence finite in number. It also ensures that the limit as $s \to +\infty$ (resp. $-\infty$) of every left (resp. right) Floer cap is a unique element of $\mathcal{P}(\mathcal{H})$.

Secondly, we assume

**A2:** The cap data $\mathcal{H}$ satisfies $|||\kappa(\mathcal{H})||| < h$. 
Cap data with curvature satisfying this estimate exists if \( \|H\| < \hbar \) (see Example 2.1). More generally, one can find such cap data if the path generated by \( H \) satisfies \( \bar{\rho}(\phi^H) < \hbar^3 \).

As described in the next subsection, the energy of the Floer caps we will consider is bounded above by \( |||\kappa(H)||| \) (see inequality (7)). Together with condition A2, this fact will allow us to avoid bubbling in our compactness statements. To achieve this, we must first restrict our choice of the families of almost complex structures, \( J_s \), which appear in our cap data. We begin with the following simple observation.

**Lemma 2.2.** For every \( \delta > 0 \) there is a nonempty open subset \( J^\delta \subset \mathcal{J}(M,\omega) \) such that for every \( J \in J^\delta \) we have \( h(J) \geq h - \delta \).

**Proof.** Assume that no such subset exists. Recall that \( \mathcal{J}(M,\omega) \) with its usual \( C^\infty \)-topology is a complete metric space with a metric defined in terms of a fixed background metric on \( M \). Choose an almost complex structure \( J_0 \) in \( \mathcal{J}(M,\omega) \) with \( h(J_0) \geq h - \delta/2 \). Let \( B(J_0, n) \) be the ball in \( \mathcal{J}(M,\omega) \) with radius \( 1/n \) and center \( J_0 \). By our assumption, in each open ball \( B(J_0, n) \) there is a almost complex structure \( J_n \in \mathcal{J}(M,\omega) \) and a nonconstant \( J_n \)-holomorphic sphere \( u_n: S^2 \to M \) with \( \omega(u_n) < h - \delta \). By the compactness theorem for holomorphic curves (see, for example [McDSa2] Theorem 4.6.1), there is a subsequence of the \( u_n \) which, after reparameterization, converges modulo bubbling to a nonconstant \( J_0 \)-holomorphic sphere with energy no greater than \( h - \delta \). This contradicts the condition \( h(J_0) \geq h - \delta/2 \). \( \square \)

Let \( J^\delta \) be a set as in Lemma 2.2. Since \( J^\delta \) is open in the \( C^\infty \)-topology, it follows from [FHS] that one can achieve transversality for the moduli spaces of Floer caps using only families \( J_s \) which take values in \( J^\delta \). In other words, these moduli spaces can be assumed to be smooth manifolds such that the dimensions of their components agree with their virtual dimensions.

Our final assumption is

**A3:** The families \( J_s \) which appear in the cap data \( H \) only take values in a fixed set \( J^{\delta_H} \), as described in Lemma 2.2 for

\[
\delta_H = \frac{\hbar - |||\kappa(H)|||}{2}.
\]

Moreover, transversality holds for every space of left and right Floer caps for \( H \).

By condition A3, any bubble which forms from a sequence of Floer caps is a nonconstant \( J \)-holomorphic sphere for some \( J \in J^{\delta_H} \). By the definition of \( J^{\delta_H} \) and condition A2 we have

\[
h(J) \geq \frac{1}{2}(\hbar + |||\kappa(H)|||) > |||\kappa(H)|||.\]

---

3This is established in the proof of Theorem 1.9.
Hence, the energy of this bubble must be greater than $|||\kappa(H)|||$. However, as mentioned above, the energy of the Floer caps we will consider is less than $|||\kappa(H)|||$, so no such bubbling can occur.

Since we are avoiding all holomorphic spheres in our compactifications, we can also fix coherent orientations on our moduli spaces, as in [FH]. In particular, we can count zero dimensional moduli spaces with signs, and use the fact that the signed count of boundary components of a compact one dimensional moduli space is zero.

Remark 2.3. The first two conditions do not depend on the choice of the families of almost complex structures which appear in $H$. Hence, if $A_1$ and $A_2$ hold, one can always choose these families of almost complex structures so that $A_3$ is satisfied.

2.4. Central orbits. We say that an orbit $x \in \mathcal{P}(H)$ is central with respect to $H$, if there is a pair of Floer caps $(u, v)$ in $\mathcal{L}(x; \mathcal{H}_L) \times \mathcal{R}(x; \mathcal{H}_R)$ such that the class $[u \# v] \in \pi_2(M)$ is trivial. Here, $u \# v$ denotes the obvious concatenation of maps. In this case, the pair $(u, v)$ will be referred to as a central pair of Floer caps for $x$. The point of these definitions is that there are useful bounds on the indices and actions of central orbits, as well as the energy of Floer caps which appear in central pairs.

**Figure 2.** A central orbit $x$ with central pair $(u, v)$.

The dimension formulas for left and right Floer caps imply that for any $u \in \mathcal{L}(x; \mathcal{H}_L)$ and $v \in \mathcal{R}(x; \mathcal{H}_R)$ we have $\mu_{CZ}(x, u) \leq n$ and $\mu_{CZ}(x, v) \geq -n$. For a central pair $(u, v)$ for $x$ we then get

\[ -n \leq \mu_{CZ}(x, v) = \mu_{CZ}(x, u) \leq n. \]

A straightforward computation also shows that

\[ 0 \leq E(u) = -A_H(x, u) + \int_{\mathbb{R} \times S^1} \kappa(\mathcal{H}_L)(s, t, u) \, ds \, dt, \]

and

\[ 0 \leq E(v) = A_H(x, v) - \int_{\mathbb{R} \times S^1} \kappa(\mathcal{H}_R)(s, t, v) \, ds \, dt. \]

If $(u, v)$ is a central pair for $x$, then $A_H(x, u) = A_H(x, v)$ and equations (4) and (5) imply that

\[ -|||\kappa(\mathcal{H}_R)||| \leq A_H(x, v) = A_H(x, u) \leq |||\kappa(\mathcal{H}_L)|||'. \]
For any central pair \((u, v)\) one obtains from (4), (5) and (6) the uniform energy bounds
\[
E(u), E(v) \leq ||| \kappa(H) |||.
\]

2.5. A lower bound for the number of central periodic orbits. We now prove that there are at least \(r_k(H(M; \mathbb{Z}))\) central periodic orbits of \(H\) with respect to \(H\). The argument relies heavily on the assumption, \(A2\), that \(||| \kappa(H) ||| < \hbar\).

Fix a Morse function \(f\) on \(M\) and a metric \(g\) so that the pair \((f, g)\) is Morse-Smale. In other words, the Morse complex, \((\text{CM}(f), \partial_g)\), is well-defined. The chain group, \(\text{CM}(f)\), is the \(\mathbb{Z}\)-module generated by the critical points of \(f\), and is graded by the Morse index. The boundary map \(\partial_g\) counts solutions \(\sigma: \mathbb{R} \to M\) of the ordinary differential equation
\[
\dot{\sigma}(s) = -\nabla_g f(\sigma(s)).
\]

To be more precise, for critical points \(p\) and \(q\), let
\[
m(p, q) = \{\sigma: \mathbb{R} \to M \mid \sigma \text{ satisfies } (5), \sigma(-\infty) = p, \sigma(+\infty) = q\}.
\]
The Morse boundary map counts the elements of \(m(p, q) / \mathbb{R}\) for \(\text{ind}(p) = \text{ind}(q) + 1\), where \(\mathbb{R}\) acts (freely) by translation on the domains of the maps in \(m(p, q)\).

Let \(\text{CF}(H)\) be the \(\mathbb{Z}\)-module generated by the elements of \(\mathcal{P}(H)\). To detect central orbits, we will construct two \(\mathbb{Z}\)-module homomorphisms,
\[
\Phi_L: \text{CM}(f) \to \text{CF}(H)
\]
and
\[
\Phi_R: \text{CF}(H) \to \text{CM}(f),
\]
whose composition \(\Phi_H = \Phi_R \circ \Phi_L\) is chain homotopic to the identity. These maps are similar to those constructed in Lagrangian Floer theory by Chekanov in [Ch]. In place of Floer continuation trajectories, we use the hybrid moduli spaces of [PSS].

We begin by defining \(\Phi_L\). A left or right Floer cap is called short if its energy is less than \(\hbar\). The subset of short elements in \(\mathcal{L}(x; \mathcal{H}_L)\) is denoted by \(\mathcal{L}^r(x; \mathcal{H}_L)\). Consider the space of left-half gradient trajectories;
\[
\ell(p) = \{\alpha: (-\infty, 0) \to M \mid \dot{\alpha} = -\nabla_g f(\alpha), \alpha(-\infty) = p\}.
\]
For a critical point \(p\) of \(f\) and an orbit \(x\) in \(\mathcal{P}(H)\), set
\[
\mathcal{L}(p, x; f, \mathcal{H}_L) = \{(\alpha, u) \in \ell(p) \times \mathcal{L}^r(x; \mathcal{H}_L) \mid \alpha(0) = u(-\infty)\}.
\]
For generic data, \(\mathcal{L}(p, x; f, \mathcal{H}_L)\) is a smooth manifold and the local dimension of the component containing \((\alpha, u)\) is \(\text{ind}(p) - n - \mu_{cz}(x, u)\), [PSS]. The homomorphism \(\Phi_L: \text{CM}(f) \to \text{CF}(H)\) is now defined on each critical point \(p\) of \(f\) by
\[
\Phi_L(p) = \sum_{x \in \mathcal{P}(H)} \# \mathcal{L}_0(p, x; f, \mathcal{H}_L) x,
\]
where, \( \#L_0(p, x; f, H_L) \) is the number of zero-dimensional components in \( L(p, x; f, H_L) \) counted with signs determined by a fixed coherent orientation. The shortness assumption implies that \( L_0(p, x; f, H_L) \) is compact. Hence, \( \#L_0(p, x; f, H_L) \) is finite and \( \Phi_L \) is well-defined.

It is now easiest to define the composite homomorphism \( \Phi_H \). Consider the space of right-half gradient trajectories

\[
r(q) = \left\{ \beta: [0, +\infty) \to M \mid \dot{\beta} = -\nabla_g f(\beta), \beta(+\infty) = q \right\},
\]

and let \( \mathcal{R}'(x; H_R) \) be the collection of short right Floer caps of \( x \). The spaces

\[
\mathcal{R}(x, q; H_R, f) = \left\{ (v, \beta) \in \mathcal{R}'(x; H_R) \times r(q) \mid v(+\infty) = \beta(0) \right\},
\]

are also smooth manifolds for generic data, and the dimension of the component containing \((v, \beta)\) is \( \mu_{cz}(x, \frac{v}{\|v\|}) - \text{ind}(q) + n \). Let \( \mathcal{R}_0(x, q; H_L, f) \) be the set of zero-dimensional components in \( \mathcal{R}(x, q; H_L, f) \). The map

\[
\Phi_H: \text{CM}(f) \to \text{CM}(f)
\]

is then defined by setting the coefficient of \( q \) in \( \Phi_H(p) \) to be the integer

\[
\sum_{x \in \mathcal{P}(H)} \# \left\{ ((\alpha, u), (v, \beta)) \in L_0(p, x; H_R, f) \times \mathcal{R}_0(x, q; H_R, f)) \mid [u \# v] = 0 \right\}.
\]

The map \( \Phi_H \) preserves the grading by the Morse index.

**Figure 3. Rigid configurations counted by \( \Phi_H \).**

Finally, the map \( \Phi_R: \text{CF}(H) \to \text{CM}(f) \) is determined by \( \Phi_L \) and \( \Phi_H \) as follows. Let \( V_L \) be the submodule of \( \text{CF}(H) \) generated by the orbits \( x \) in \( \mathcal{P}(H) \) which appear in an element in the image of \( \Phi_L \) with a nonzero integer coefficient. The maps \( \Phi_L \) and \( \Phi_H \), together with the condition \( \Phi_H = \Phi_R \circ \Phi_L \), uniquely determine the restriction of \( \Phi_R \) to \( V_L \). Setting \( \Phi_R = 0 \) on the complement of \( V_L \) we obtain the full map \( \Phi_R \). In particular, the coefficient of \( q \) in \( \Phi_R(x) \) is the (signed) count of elements \((v, \beta) \in \mathcal{R}_0(x, q; H_R, f)\) for which there is an element \((u, \alpha)\) in some \( L_0(p, x; H_R, f) \) such that \([u \# v] = 0\).

**Proposition 2.4.** The map \( \Phi_H: \text{CM}(f) \to \text{CM}(f) \) is chain homotopic to the identity.

**Proof.** We first use the cap data, \( H = (H_L, H_R) \), to construct a homotopy of homotopy triples \( \{H^\lambda\} = \{(H_s^\lambda, K_s^\lambda, J_s^\lambda)\} \) for \( \lambda \in [0, +\infty) \). For convenience, we introduce the notation \( \mathcal{H} = \mathcal{H}(s) = (H_s, K_s, J_s) \) to emphasize the \( s \)-dependence of \( H \). Set

\[
\mathcal{H}^\lambda(s) = \begin{cases} 
H_L(s + c_1) & \text{when } s \leq 0, \\
H_R(s - c_1) & \text{when } s \geq 0,
\end{cases}
\]
for a constant \( c_1 > 0 \) which is large enough to ensure that the domains on which \( \mathcal{H}_L(s + c_1) \) and \( \mathcal{H}_R(s - c_1) \) depend on \( s \), do not intersect. For \( \lambda \in [1, +\infty) \), we then define

\[
\mathcal{H}^\lambda(s) = \begin{cases} 
\mathcal{H}_L(s + c(\lambda)) & \text{when } s \leq 0, \\
\mathcal{H}_R(s - c(\lambda)) & \text{when } s \geq 0,
\end{cases}
\]

where \( c(\lambda) \) is a smooth nondecreasing function which equals \( \lambda \) for \( \lambda \gg 1 \) and is equal to \( c_1 \) for a constant on which \( H = 0 \) and equals one near \( \lambda \) one. Furthermore, the \( J \) appearing in these families lies in \( J^3_H \) are chosen so that they equal \( J^0_s \) for \( \lambda \) near zero and equal \( J^1_s \) for \( \lambda \) near one. Furthermore, the \( J^\lambda_s \) are chosen so that each almost complex structure appearing in these families lies in \( \mathcal{J}^3_H \).

The following properties of \( J^\lambda_s \) are easily verified:

- For each \( \lambda \in [0, +\infty) \), \( H^\lambda_s \) and \( K^\lambda_s \) are compact homotopies from the zero function to itself.
- \( \mathcal{H}^0 = (0, 0, J^0_s) \).
- When \( \lambda \) is sufficiently large

\[
\mathcal{H}^\lambda(s) = \begin{cases} 
\mathcal{H}_L(s + \lambda) & \text{for } s \leq 0, \\
\mathcal{H}_R(s - \lambda) & \text{for } s \geq 0.
\end{cases}
\]

- \( |||\kappa(\mathcal{H}^\lambda)|||^+ = \zeta(\lambda)|||\kappa(\mathcal{H})||\) for all \( \lambda \in [0, +\infty) \).

To construct the desired chain homotopy, we consider maps \( w \in \mathcal{C}^\infty(\mathbb{R} \times S^1, M) \) which satisfy the equation

\[
(9) \quad \partial_s w - X_{K^\lambda_s}(w) + J^\lambda_s(w)(\partial_t w - X_{H^\lambda_s}(w)) = 0.
\]

These are perturbed holomorphic cylinders which are asymptotic, at both ends, to points in \( M \). In particular, since the perturbations are compact, each \( w \) can be uniquely completed to a perturbed holomorphic sphere. We consider these maps for all values of \( \lambda \in [0, +\infty) \) and restrict our attention to those maps which represent the trivial class in \( \pi_2(M) \). Let

\[
\mathcal{M}(\mathcal{H}^\lambda) = \{ (\lambda, w) \in [0, +\infty) \times \mathcal{C}^\infty(\mathbb{R} \times S^1, M) \mid w \text{ satisfies } (9), [w] = 0 \}.
\]

For every \( (\lambda, w) \in \mathcal{M}(\mathcal{H}^\lambda) \), we have

\[
E(\lambda, w) = \int_{\mathbb{R} \times S^1} \omega\left( \partial_s w - X_{K^\lambda_s}(w), J^\lambda_s(\partial_s w - X_{K^\lambda_s}(w)) \right) ds \, dt
\]

\[
= \int_{\mathbb{R} \times S^1} \left( \omega(X_{H^\lambda_s}(w), \partial_s w) - \omega(X_{K^\lambda_s}(w), \partial_t w) + \omega(X_{K^\lambda_s}(w), X_{H^\lambda_s}(w)) \right) ds \, dt
\]

\[
= \int_{\mathbb{R} \times S^1} \kappa(\mathcal{H}^\lambda)(s, t, w) ds dt
\]

\[
\leq |||\kappa(\mathcal{H}^\lambda)|||^+.
\]
The last property of $\mathcal{H}^\lambda$ listed above, $|||\kappa(\mathcal{H}^\lambda)|||^+ = \zeta(\lambda)|||\kappa(\mathcal{H})|||$, implies that for every $(\lambda, w) \in \mathcal{M}(\mathcal{H}^\lambda)$ we have the uniform energy bound

$$E(\lambda, w) \leq h.$$  

For a pair of critical points $p$ and $q$ of $f$, consider the set $\mathcal{M}(p, q; \{\mathcal{H}^\lambda\})$ defined by

$$\{(\alpha, (\lambda, w), \beta) \in \ell(p) \times \mathcal{M}(\mathcal{H}^\lambda) \times r(q) \mid \alpha(0) = w(-\infty), w(+\infty) = \beta(0)\}$$

For generic data, each $\mathcal{M}(p, q; \{\mathcal{H}^\lambda\})$ is a manifold of dimension $\text{ind}(p) - \text{ind}(q) + 1$. We define the homomorphism $h: \text{CM}(f) \to \text{CM}(f)$ by

$$h(p) = \sum_{\mu_{\text{CZ}}(q) = \mu_{\text{CZ}}(p) + 1} \# \mathcal{M}(p, q; \{\mathcal{H}^\lambda\})q.$$  

The uniform bound (10) precludes bubbling and implies that the zero dimensional spaces counted by $h$ are compact.

To prove that $h$ is the desired chain homotopy, it suffices to show that for every critical point $p$ of $f$ we have

$$(\text{id} - \Phi_H + h \circ \partial_g + \partial_g \circ h)(p) = 0.$$  

Let $r$ be a critical point of $f$ with $\text{ind}(r) = \text{ind}(p)$. We prove that the coefficients of $r$ in $\text{id} - \Phi_H + h \circ \partial_g + \partial_g \circ h)(p)$ is zero.

Consider the compactification $\overline{\mathcal{M}}(p, r; \{\mathcal{H}^\lambda\})$ of the one dimensional moduli space $\mathcal{M}(p, r; \{\mathcal{H}^\lambda\})$. The energy bound (10) again prevents bubbling for sequences in $\mathcal{M}(p, r; \{\mathcal{H}^\lambda\})$. The gluing and compactness theorems of Floer theory then imply that the boundary of $\overline{\mathcal{M}}(p, r; \{\mathcal{H}^\lambda\})$ can be identified with the union of the following four compact, zero-dimensional manifolds:

(i) $m(p, r)$,

(ii) $\bigcup_{x \in \mathcal{P}(\mathcal{H})} \{(\alpha, u), (v, \beta)\} \in \mathcal{L}_0(p, x; f, \mathcal{H}_L) \times \mathcal{R}_0(x, r; \mathcal{H}_R, f) \mid [u\#v] = 0\}$,

(iii) $\bigcup_{\mu_{\text{CZ}}(q) = \mu_{\text{CZ}}(p) - 1} m(p, q)/\mathbb{R} \times \mathcal{M}(q, r; \{\mathcal{H}^\lambda\}),$

(iv) $\bigcup_{\mu_{\text{CZ}}(q) = \mu_{\text{CZ}}(p) + 1} \mathcal{M}(p, q; \{\mathcal{H}^\lambda\}) \times m(q, r)/\mathbb{R}.$

The signed count of these boundary terms is equal to zero. It is clear from the definitions, that the signed counts of the elements in sets (iii) and (iv) correspond to the coefficient of $r$ in $\partial_g \circ h(p)$ and $\partial_g \circ h(p)$, respectively. As well, the set (i) is empty if $p \neq r$, and consists only of the constant map when $p = r$. Hence, the signed count of elements in $m(p, r)$ is equal to the coefficient of $r$ in $\text{id}(p)$.

It remains to show that the signed count of elements in set (ii) is the coefficient of $r$ in $\Phi_H(p)$. From the definition of $\Phi_H$, we only need to show that for each tuple $((\alpha, u)/(v, \beta))$ in set (ii), both $u$ and $v$ are short. Since $[u\#v] = 0$, the pair $(u, v)$ is central and inequality (7) yields the shortness of $u$ and $v$. □
Proposition 2.5. There are at least \( rk(H(M;\mathbb{Z})) \) central periodic orbits of \( H \). In particular, there are at least \( rk(H(M;\mathbb{Z})) \) contractible 1-periodic orbits \( x_j \) of \( H \) which admit spanning disks \( u_j \) such that

\[
-n \leq \mu_{cz}(x_j, u_j) \leq n
\]

and

\[
-|||\kappa(H_R)||| \leq \mathcal{A}_H(x_j, u_j) \leq |||\kappa(H_L)||| + .
\]

Proof. Recall that \( V_L \) is the submodule of \( CF(H) \) generated by periodic orbits \( x \) which appear in an element in the image of \( \Phi_L \) with a nonzero coefficient. Let \( K_R \) be the submodule of \( CF(H) \) generated by periodic orbits which lie in the kernel of \( \Phi_R \) and let \( p: V_L \to V_L/K_R \) be the projection map. We then have

\[
\Phi_H = \Phi_R \circ p \circ \Phi_L.
\]

Note that any periodic orbit which appears in the image of \( p \circ \Phi_L \) is central with respect to \( H \).

The homology of the Morse complex, \( H(CM(f, \partial_g), \mathbb{Z}) \), is isomorphic to \( H(M;\mathbb{Z}) \). Fix a basis \( \{a_i\} \) for \( H(CM(f, \partial_g), \mathbb{Z}) \) and a set of representatives \( \{A_i\} \) such that \([A_i] = a_i \). By Proposition 2.5 each \( A'_i = \Phi_H(A_i) \) represents the same class as \( A_i \). Let \( W \) (resp. \( W' \)) be the free submodule of rank \( rk(H(M;\mathbb{Z})) \) generated by the \( A_i \) (resp. \( A'_i \)). The restriction of \( \Phi_H \) to \( W \) is then an isomorphism from \( W \) to \( W' \). Together with (11), this implies that

\[
rk(p \circ \Phi_L(W)) = rk(W) = rk(H(M;\mathbb{Z})).
\]

Hence, there are at least \( rk(H(M;\mathbb{Z})) \) central periodic orbits of \( H \). The bounds on the Conley-Zehnder indices and actions follow immediately from (11) and (11). \( \square \)

2.6. Identifying Floer caps for homotopic Hamiltonian paths. We now describe some results which allow one to choose useful capping data.

Proposition 2.6. Let \( \mathcal{G} \) be a homotopy triple for \((G, J_G)\). For any \( H \) in \( C_0^\infty([\phi_G]) \) there is a \( J_H \in \mathcal{J}_{S^1}(M, \omega) \), a homotopy triple \( \mathcal{H} \) for \((H, J_H)\), and bijections

\[
\Psi_{H_G}: \mathcal{L}(x; \mathcal{G}) \to \mathcal{L}(\phi_H^t \circ (\phi_G^t)^{-1}(x); \mathcal{H})
\]

and

\[
\Psi_{R_G}: \mathcal{R}(x; \mathcal{G}) \to \mathcal{R}(\phi_H^t \circ (\phi_G^t)^{-1}(x); \mathcal{H}).
\]

Moreover, these bijections preserve actions and Conley-Zehnder indices.

Proof. Let \( \varrho_t = \phi_H^t \circ (\phi_G^t)^{-1} \). The map \( \Psi_{H_G}: C^\infty(S^1, M) \to C^\infty(S^1, M) \) defined by

\[
\Psi_{H_G}(x)(t) = \varrho_t(x(t)).
\]

takes contractible loops to contractible loops and hence \( \Psi_{H_G}(\mathcal{P}(G)) = \mathcal{P}(H) \).

Set

\[
J_H = d\varrho_t \circ J_G \circ d(\varrho_t^{-1}).
\]
Given the homotopy triple $G = (G_s, K_s, J_s)$ for $(G, J_G)$, we now define the desired homotopy triple $\tilde{\mathcal{H}}$ for $(H, J_H)$ and the bijection

$$
\Psi_{HG}^L : \mathcal{L}(x; G) \to \mathcal{L}(\Psi_{HG}(x); \mathcal{H}).
$$

The construction of the bijection $\Psi_{HG}^R$ is entirely similar.

Let $F_s$ be a compact homotopy from $G$ to $H$ such that $F_s$ belongs to $C_0^\infty([\phi_G])$ for each $s \in \mathbb{R}$. For a sufficiently large $\tau > 0$, we may assume that the compact homotopies $G_s$ and $F_s$ are both independent of $s$ for $|s| > \tau$. Thus,

$$
\tilde{H}_s = \begin{cases} 
0 & \text{for } s \leq -\tau; \\
G_s & \text{for } -\tau \leq s \leq \tau; \\
F_{s-2\tau}, & \text{for } \tau \leq s \leq 3\tau; \\
H, & \text{for } s \geq 3\tau;
\end{cases}
$$

is a compact homotopy from the zero function to $H$.

Now consider the family of contractible Hamiltonian loops $\varrho_{s,t} = \phi_{\tilde{H}_s}^t \circ (\phi_{G_s}^t)^{-1}$.

For each value of $s$, $\varrho_{s,t}$ is a loop based at the identity, and

$$
\varrho_{s,t} = \begin{cases} 
id, & \text{for } s \leq \tau; \\
\phi_{F_{s-2\tau}}^t \circ (\phi_{G}^t)^{-1}, & \text{for } \tau \leq s \leq 3\tau; \\
\varrho_t, & \text{for } s \geq 3\tau.
\end{cases}
$$

From $\varrho_{s,t}$ we obtain two families of normalized Hamiltonian, $A_s$ and $B_s$, defined by

$$
\partial_s(\varrho_{s,t}(p)) = X_{A_s}(\varrho_{s,t}(p))
$$

and

$$
\partial_t(\varrho_{s,t}(p)) = X_{B_s}(\varrho_{s,t}(p)).
$$

The standard composition formula for Hamiltonian flows implies that

$$
B_s = \tilde{H}_s - G_s \circ \varrho_{s,t}^{-1},
$$

where $(G_s \circ \varrho_{s,t}^{-1})(t, p) = G_s(t, \varrho_{s,t}^{-1}(p))$. From [13], we also have the following useful relation

$$
(12) \quad \partial_s B_s - \partial_t A_s + \{B_s, A_s\} = 0.
$$

Define

$$
\tilde{\mathcal{H}} = (\tilde{H}_s, \tilde{K}_s, \tilde{J}_s) = (\tilde{H}_s, A_s + K_s \circ \varrho_{s,t}^{-1}, d\varrho_{s,t} \circ J_s \circ d(\varrho_{s,t}^{-1})).
$$

It is easy to verify that $\tilde{H}$ is a homotopy triple for $(H, J_H)$. We claim that the map $\Psi_{HG}^L$ defined on $\mathcal{L}(x; G)$ by

$$
\Psi_{HG}^L(u)(s,t) = \varrho_{s,t}(u(s,t))
$$

is a bijection onto $\mathcal{L}(\Psi_{HG}(x); \tilde{\mathcal{H}})$. It suffices to prove that for $\tilde{u} = \Psi_{HG}^L(u)$ we have

$$
\partial_s \tilde{u} - X_{\tilde{K}_s}(\tilde{u}) + \tilde{J}_s(\tilde{u})(\partial_t \tilde{u} - X_{\tilde{H}_s}(\tilde{u})) = 0.
$$
This follows from the simple computation

\[
\partial_s \tilde{u} + \tilde{J}_s(\tilde{u}) \partial_t \tilde{u} = d(g_{s,t}) \partial_s u + X_{A_s}(\tilde{u}) + \tilde{J}_s(\tilde{u}) (d(g_{s,t}) \partial_t u + X_{B_s}(\tilde{u}))
\]

\[
= d(g_{s,t}) (\partial_s u + J_s(u) \partial_t u) + X_{A_s}(\tilde{u}) + \tilde{J}_s(\tilde{u}) X_{B_s}(\tilde{u})
\]

\[
= X_{K_s \circ \varrho_{s,t}^{-1}}(\tilde{u}) + \tilde{J}_s(\tilde{u}) (d(g_{s,t}) X_{G_s}(u)) + X_{A_s}(\tilde{u}) + \tilde{J}_s(\tilde{u}) X_{B_s}(\tilde{u})
\]

\[
= X_{K_s \circ \varrho_{s,t}^{-1}}(\tilde{u}) + \tilde{J}_s(\tilde{u}) X_{G_s \circ \varrho_{s,t}^{-1}}(\tilde{u}) + X_{A_s}(\tilde{u}) + \tilde{J}_s(\tilde{u}) X_{B_s}(\tilde{u})
\]

The additional facts that

\[
\mu_{\text{CZ}}(x, u) = \mu_{\text{CZ}}(\Psi_{HG}(x), \Psi_{HG}^C(u)) \text{ and } \mathcal{A}_G(x, u) = \mathcal{A}_H(\Psi_{HG}(x), \Psi_{HG}^C(u)),
\]

are well known from the work of Seidel in [Se]. (Another proof can be found in [Sch].) \(\square\)

**Proposition 2.7.** For \(\mathcal{G}\) and \(\tilde{\mathcal{H}}\) as in Proposition 2.6,

\[
|||\kappa(\tilde{\mathcal{H}})|||^\pm = |||\kappa(\mathcal{G})|||^\pm.
\]

**Proof.** The formulas \(\tilde{H}_s = B_s + G_s \circ \varrho_{s,t}^{-1}\) and \(\tilde{K}_s = A_s + K_s \circ \varrho_{s,t}^{-1}\), together with Banyaga’s formula, (12), imply that

\[
(13) \quad \kappa(\tilde{\mathcal{H}}) = \kappa(\mathcal{G}) \circ \varrho_{s,t}^{-1} + \{B_s, K_s \circ \varrho_{s,t}^{-1}\} + dG_s[\partial_s(\varrho_{s,t}^{-1})] + \{G_s \circ \varrho_{s,t}^{-1}, A_s\},
\]

where

\[
\kappa(\mathcal{G}) \circ \varrho_{s,t}^{-1} = \partial_s G_s \circ \varrho_{s,t}^{-1} - \partial_t K_s \circ \varrho_{s,t}^{-1} + \{G_s, K_s\} \circ \varrho_{s,t}^{-1}.
\]

The second term on the right side of (13) vanishes since \(B_s = 0\) for \(s \in [-\tau, \tau]\) and \(K_s = 0\) for \(|s| > \tau\). We claim that the last two terms on the right side of (13) cancel. For \(s \leq \tau\) and \(s \geq 3\tau\), both terms vanish since \(\varrho_{s,t}\) is independent of \(s\) in these ranges. For \(s \in [\tau, 3\lambda]\), we have

\[
dG_s(\partial_s(\varrho_{s,t}^{-1})) + \{G_s \circ \varrho_{s,t}^{-1}, A_s\} = dG_s(\varrho_{s,t}^{-1}) + \{G \circ \varrho_{s,t}^{-1}, A_s\}
\]

\[
= dG_s(\varrho_{s,t}^{-1}) + d(\varrho_{s,t}^{-1})(X_{A_s})
\]

\[
= dG_s(\varrho_{s,t}^{-1}) + d(\varrho_{s,t}^{-1})[\partial_s(\varrho_{s,t})(\varrho_{s,t}^{-1})]
\]

\[
= 0.
\]

The final equality follows from the identity

\[
0 = \partial_s(\varrho_{s,t}^{-1} \circ \varrho_{s,t}(p)) = \partial_s(\varrho_{s,t}^{-1})(\varrho_{s,t}(p)) + d(\varrho_{s,t}^{-1})[\partial_s(\varrho_{s,t})(\varrho_{s,t}^{-1})].
\]

Hence, we have \(\kappa(\tilde{\mathcal{H}}) = \kappa(\mathcal{G}) \circ \varrho_{s,t}^{-1}\) which implies Proposition 2.7. \(\square\)
3. Proof of Theorem 1.1

With the machinery of the previous section in hand, the proof of Theorem 1.1 is almost immediate. Assume that $H$ is a Floer Hamiltonian such that $\|H\| < \hbar$. If $\phi^t_H$ does not minimize the positive Hofer length in its homotopy class then we can find a Hamiltonian $G$ in $C^\infty_0(\phi^t_H)$ such that $\|G\|^+ < \|H\|^+$. Fix a family $J_G$ in $J_{S^1}(M,\omega)$ and let

$$J_H = d(\phi^t_H \circ (\phi^t_G)^{-1}) \circ J_G \circ d(\phi^t_G \circ (\phi^t_H)^{-1}).$$

For the Hamiltonian data $(H, J_H)$, we consider the cap data $H_G = (\tilde{H}_G, H)$. Here, $\tilde{H}$ is a linear homotopy triple for $(H, J_H)$ as in Example 2.1 and $\tilde{H}_G$ is the homotopy triple obtained by applying Proposition 2.6 to a linear homotopy triple $\tilde{G}$ for $(G, J_G)$.

Proposition 2.7 implies

$$|||\kappa(H_G)||| = |||\kappa(\tilde{H}_G)|||^+ + |||\kappa(\tilde{H})|||^- = |||\kappa(\tilde{G})|||^+ + |||\kappa(\tilde{H})|||^- = \|G\|^+ + \|H\|^- < \hbar.$$  

Hence, our data satisfies conditions A1 and A2 and can therefore be chosen to satisfy A3, as well. Applying Proposition 2.5 we obtain at least $rk(\mathcal{H}(M;\mathbb{Z}))$ elements $x_j$ of $\mathcal{P}(H)$ which admit spanning discs $u_j$ such that

$$-n \leq \mu_{C^2}(x_j, u_j) \leq n$$

and

$$-\|H\|^- = -|||\kappa(\tilde{H})|||^- \leq \mathcal{A}_H(x_j, u_j) \leq |||\kappa(\tilde{H}_G)|||^+ = \|G\|^+ < \|H\|^+.$$  

4. Proof of Theorem 1.2

Let $H$ be a possibly degenerate Hamiltonian such that $\|H\| < \hbar$ and $\phi^t_H$ does not minimize the positive Hofer length in its homotopy class. To detect the desired 1-periodic orbit $y$ and spanning disc $w$ such that

$$-\|H\|^- \leq \mathcal{A}_H(y, w) < \|H\|^+,$$

we need to analyze a sequence of Floer caps. However, to obtain both the lower and upper bounds for the action we first need a slight generalization of the machinery from Section 2.
4.1. Preparations. For this discussion we assume that $H$ is a Floer Hamiltonian. The required generalization begins (and essentially ends) with the type of homotopy triples, $\mathcal{H} = (H_s, K_s, J_s)$, we associate to the Hamiltonian data $(H, J)$. We now allow $H_s$ to be a compact homotopy from any constant function $c$ to $H$. The notions of curvature, cap data $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R)$, and left and right Floer caps are defined in the same way as before.

Central orbits and central pairs are also defined as in Section 2. Only our action and energy bounds must be adjusted. (The bounds on the Conley-Zehnder indices are unchanged.) Equations (4) and (5) become

(14) $0 \leq E(u) = c_L - \mathcal{A}_H(x, u) + \int_{\mathbb{R} \times S^1} \kappa(\mathcal{H}_L)(s, t, u) \, ds \, dt,$

and

(15) $0 \leq E(v) = \mathcal{A}_H(x, \bar{v}) - c_R - \int_{\mathbb{R} \times S^1} \kappa(\mathcal{H}_R)(s, t, \bar{v}) \, ds \, dt,$

where $c_L$ and $c_R$ are the constant functions coming from $\mathcal{H}_L$ and $\mathcal{H}_R$, respectively.

If $x$ is central for $\mathcal{H}$ and $(u, v)$ is a central pair for $x$, we now have action bounds

(16) $-|||\kappa(\mathcal{H}_R)||| - c_R \leq \mathcal{A}_H(x, \bar{v}) = \mathcal{A}_H(x, u) \leq |||\kappa(\mathcal{H}_L)||| + c_L.$

For a central pair $(u, v)$, the previous uniform energy bound (7) becomes

(17) $E(u), E(v) \leq |||\kappa(\mathcal{H})||| + c_L - c_R.$

If we replace assumption $A_2$ with the assumption that $|||\kappa(\mathcal{H})||| + c_L - c_R < \hbar$, then Propositions 2.4 and 2.5 hold, for the adjusted action bounds coming from (16). In particular, for every Morse-Smale pair $(f, g)$ one can construct a $\mathbb{Z}$-module homomorphism

$\Phi_\mathcal{H}: \text{CM}(f) \to \text{CM}(f)$

which is chain homotopic to the identity, and this implies the existence of at least $rk(\mathcal{H}(M; \mathbb{Z}))$ orbits in $\mathcal{P}(H)$ which are central with respect to $\mathcal{H}$.

The point of this modification is that we can now use decreasing linear homotopy triples of the form

$\mathcal{H} = (- (1 - \eta(s)) ||H||^- + \eta(s) H, 0, J_s).$

Here, $\eta$ is the same function as in Example 2.1 and $c = -||H||^-$. The curvature norms for such triples are

(18) $|||\kappa(\mathcal{H})|||^+ = ||H||$ and $|||\kappa(\mathcal{H})|||^-= 0.$

Our proof of Theorem 1.2 will rely on a special property acquired by Floer caps defined using deceasing linear homotopy triples (see Lemma 4.1).
4.2. **The proof.** Returning to the setting of Theorem 1.2, we have \( \|H\| < \hbar \) and \( \|H\|^+ > \rho^+ (\phi_H^t) \). Choose a \( G \) in \( C^\infty_0 (\phi_H^t) \) such that \( \|G\|^+ < \|H\|^+ \).

For some \( \epsilon > 0 \) we then have

\[
\|G\|^+ < \|H\|^+ - 2\epsilon.
\]

Fix a family of almost complex structures \( J_H \) in \( J_{S^1}(M, \omega) \). To apply the Floer theoretic tools developed above, we consider a sequence of Hamiltonian data \((H_k, J_k)\) such that each \( H_k \) is a Floer Hamiltonian and the sequence \((H_k, J_k)\) converges to \((H, J_H)\) in the \( C^\infty\)-topology.

We may assume that \( \|H_k\| < \hbar \) for all \( k \). We may also assume that for each \( k \) there is a Hamiltonian \( G_k \) in \( C_0^\infty [\phi_G^t] \) such that \( \|G_k\|^+ < \|H_k\|^+ - \epsilon \).

To see this, consider the Hamiltonian loop \( \phi_H^t \circ (\phi_H^t)^{-1} \) which converges to zero in the \( C^\infty\)-topology. The path \( \phi_H^t \circ (\phi_H^t)^{-1} \circ \phi_G^t \) is homotopic to \( \phi_H^t \), relative its endpoints, and is generated by the function \( G_k = F_k + G \circ (\phi_G^t)^{-1} \).

The functions \( G_k \) clearly converge to \( G \) and hence satisfy

\[
\|G_k\|^+ \leq \|H_k\|^+ - \epsilon
\]

for sufficiently large \( k \), as desired.

We now proceed as in the proof of Theorem 1.1. Let \( \mathcal{H}_k \) be a decreasing linear homotopy triple for \((H_k, J_k)\). Choose the families of almost complex structures \( J_{k,s} \) in the \( \mathcal{H}_k \) so that they converge to \( J_s \) in \( J_{S^1}(M, \omega) \). Then \( \mathcal{H}_k \) converges to

\[
\mathcal{H} = (- (1 - \eta(s))) \|H\|^- + \eta(s) H, 0, J_s)
\]

in the \( C^\infty\)-topology. By (15) and (18), the action of any \( x \) with respect to a right Floer cap \( v \in R(x; \mathcal{H}_k) \) satisfies

\[
A_{H_k} (x, v) \geq - \|\kappa(\mathcal{H}_k)\|^- - \|H_k\|^- = - \|H_k\|^-. \tag{20}
\]

Let \( G_k \) be a decreasing linear homotopy triple for \( G_k \). As in Proposition 2.6, one can construct from \( G_k \) a homotopy triple \( \mathcal{H}_{G_k} \) for \((H_k, J_k)\) and an action-preserving bijection between the corresponding spaces of left Floer caps. Hence, by (14) and (18), the action of any \( x \) with respect to any left Floer cap \( u \in L(x; \mathcal{H}_{G_k}) \) satisfies

\[
A_{H_k} (x, u) \leq \|\kappa(G_k)\|^+ - \|G_k\|^+ = \|G_k\|^+. \tag{21}
\]

For each \( k \) we then consider the homotopy data

\[
\mathbf{H}_k = (\mathcal{H}_{G_k}, \mathcal{H}_k).
\]
Arguing as in Proposition 2.7 one can show that $G_k$ and $\tilde{H}_{G_k}$ have the same curvature norms. Hence,
\[
|||\kappa(H_k)||| + c_{L,k} - c_{R,k} = \left(|||\kappa(\tilde{H}_{G_k})||| + |||\kappa(H_k)|||\right) - ||G_k|| - ||H_k|| = ||G_k|| + ||H_k|| = ||H_k||. 
\]

For a Morse-Smale pair $(f,g)$ we can then construct, for each $k$, a chain map
\[
\Phi_{H_k} = \Phi_{R,k} \circ p_k \circ \Phi_{L,k} : \text{CM}(f) \to \text{CM}(f)
\]
which is chain homotopic to the identity. This detects at least $rk(H(M;\mathbb{Z}))$ central orbits with respect to $H_k$. If $x$ is one of these orbits with a central pair $(u,v)$, then (19),(20) and (21) imply that
\[
-\|H_k\| \leq A_{H_k}(x, v) = A_{H_k}(x, u) \leq \|H_k\| + \epsilon.
\]

For simplicity, let us choose the Morse function $f$ so that it has a unique local (and hence global) maximum at $Q \in M$. Standard arguments imply that $Q$ is a nonexact cycle of degree $2n$ in the Morse complex $(\text{CM}_*(f),\partial_g)$, and that $\Phi_{H_k}(Q) = Q$. Let
\[
X_k = p_k \circ \Phi_{L,k}(Q).
\]
By the construction of $\Phi_{H_k}$, $X_k$ is a finite sum of the form
\[
X_k = \sum n_j^k x_j^k
\]
where the $n_j^k$ are nonzero integers and the $x_j^k$ are central periodic orbits of $H_k$ with respect to $H_k$. Since $X_k$ gets mapped to $Q$ under $\Phi_{R,k}$, the moduli space
\[
\mathcal{R}_0(X_k, Q; H_k, f) = \bigcup_j \mathcal{R}_0(x_j^k, Q; \tilde{H}_k, f),
\]
which determine $\Phi_{R,k}(X_k)$, must be nonempty. Choose a $(v_k, \sigma_k)$ in $\mathcal{R}_0(X_k, Q; H_k, f)$ for each $k$. The caps $v_k$ belongs to $\mathcal{R}(x_j^k, H_k)$ for some $x_j^k$ in $\mathcal{P}(H)$ which appears in $X_k$ with a nonzero coefficient. Moreover, each $v_k$ is part of a central pair for $x_j^k$, and so by (22) we have
\[
-\|H_k\| \leq A_{H_k}(x_j^k, v_k) \leq \|H_k\| + \epsilon.
\]

By (17) and the computation above, we also have the uniform bound
\[
E(v_k) \leq |||\kappa(H_k)||| + c_{L,k} - c_{R,k} < h.
\]
Since the $H_k$ converge to $H$, the energy bound (24) implies that there is a subsequence of the $v_k$ (which we still denote by $v_k$) that converges to a solution $v : \mathbb{R} \times S^1 \to M$ of
\[
\partial_s v + \tilde{J}_s(v)(\partial_t v - X_{\tilde{H}_s}(v)) = 0,
\]
for \( \overrightarrow{H}_s = -(1 - \eta(s))\|H\|^- + \eta(s)H \). The limit \( v \) also satisfies the energy bound \( E(v) < h \). Hence, \( \lim_{s \to +\infty} v(s, t) \) exists and is equal to some point \( P \) in \( M \). There is also a sequence \( s_j \to -\infty \) such that \( v(s_j, t) \) converges to some \( y(t) \in \mathcal{P}(H) \).

For \( s \in \mathbb{R} \), we set
\[
y^{[s]}(t) = v(s, t)
\]
and
\[
\overrightarrow{v}^{[s]} = \overrightarrow{v}|_{(-\infty, -s]}.
\]
Note that \( \overrightarrow{v}^{[s]} \) is a spanning disc for \( y^{[s]} \). In this notation, \( y^{[s]} \to y \) as \( j \to \infty \), and for large values of \( j \) the discs \( \overrightarrow{v}^{[s_j]} \) can be easily extended to a spanning disc \( w \) for \( y \in \mathcal{P}(H) \) in a fixed homotopy class. It remains to show that the action of \( y \) with respect to \( w \) is less than \( \|H\|^+ \) and greater than or equal to \( -\|H\|^- \). Since the extensions of the \( \overrightarrow{v}^{[s_j]} \) can be made arbitrarily small for sufficiently large \( j \), it suffices to prove that for all \( j \) we have
\[
-\|H\|^- \leq A_H(y^{[s_j]}, \overrightarrow{v}^{[s_j]}) \leq \|H\|^+ - \epsilon.
\]

By definition,
\[
y^{[s_j]}(t) = \lim_{k \to \infty} v_k(s_j, t) = \lim_{k \to \infty} y^{[s_j]}_k(t).
\]
Hence,
\[
A_H(y^{[s_j]}, \overrightarrow{v}^{[s_j]}) = \lim_{k \to \infty} A_{H_k}(y^{[s_j]}_k, \overrightarrow{v}^{[s_j]}_k).
\]

Lemma 4.1. The function \( s \mapsto A_{H_k}(y^{[s]}_k, \overrightarrow{v}^{[s]}_k) \) is nonincreasing.

Proof. As a right Floer cap for \( H_k \), \( v_k \) is a solution of the equation
\[
\partial_s v_k + J_{k,s}(v_k)(\partial_t v_k - X_{H_{k,s}}(v_k)) = 0
\]
for
\[
\overrightarrow{H}_{k,s} = -(1 - \eta(-s))\|H_k\|^- + \eta(-s)H_k.
\]
Let \( b > a \). Then
\[
0 \leq \int_0^1 \int_a^b \omega(\partial_s v_k, \overrightarrow{J}_{k,s}(v_k)(\partial_s v_k)) \, ds \, dt
\]
\[
= \int_0^1 \int_a^b \omega(\partial_s v_k, \partial_t v_k) - \omega(\partial_s v_k, X_{H_{k,s}}) \, ds \, dt
\]
\[
= A(y^{[b]}_k, \overrightarrow{v}^{[b]}_k) - A(y^{[a]}_k, \overrightarrow{v}^{[a]}_k) - \int_0^1 \int_a^b dH_{k,s}(\partial_s v_k) \, ds \, dt
\]
\[
= A(y^{[b]}_k, \overrightarrow{v}^{[b]}_k) - A(y^{[a]}_k, \overrightarrow{v}^{[a]}_k) - \int_0^1 \int_a^b \left( \partial_s(\overrightarrow{H}_{k,s}(v_k)) - \partial_s H_{k,s}(v_k) \right) \, ds \, dt
\]
\[
= A_{H_k}(y^{[a]}_k, \overrightarrow{v}^{[a]}_k) - A_{H_k}(y^{[b]}_k, \overrightarrow{v}^{[b]}_k) + \int_0^1 \int_a^b \partial_s \overrightarrow{H}_{k,s}(v_k) \, ds \, dt.
\]

\footnote{The negative limit of \( v \) may not exist since the orbits of \( H \) may be degenerate.}
Since
\[ \partial_s \overline{H}_{k,s} = \dot{\eta}(-s)(-\|H_k\| - H_k) \leq 0 \]
we have
\[ A_{H_k}(y_k^{[a]}, \overline{v}_k^{[a]}) \geq A_{H_k}(y_k^{[b]}, \overline{v}_k^{[b]}). \]
\[ \square \]
Lemma 4.1 together with (23), implies that
\[ A_{H_k}(y_k^{[s_j]}, \overline{v}_k^{[s_j]}) \leq \lim_{j \to \infty} A_{H_k}(y_k^{[s_j]}, \overline{v}_k^{[s_j]}) = \int_0^1 H_k(t, P) dt = -\|H_k\|^- \]
and so
\[ A_{H}(y^{[s_j]}, \overline{v}^{[s_j]}) = \lim_{k \to \infty} A_{H_k}(y_k^{[s]}, \overline{v}_k^{[s]}) \geq -\|H\|^- . \]
This proves (25) and the proof of Theorem 1.2 is complete.

5. PROOF OF THEOREM 1.4

Let \( H \) be a quasi-autonomous Floer Hamiltonian on a spherically rational symplectic manifold \((M, \omega)\), such that \( \|H\| < r(M, \omega) \). We also assume that \( \phi_{H}^{t} \) does not minimize \( \rho_{H} \) in its homotopy class. Hence, \( \phi_{H}^{t} \) does not minimize at least one of the one-sided seminorms. We consider the case when it does not minimize \( \rho^{+} \). The other case can be dealt with in similar way (see Corollary 6.1).

By Theorem 1.1, there are at least \( r(k) (H(M; \mathbb{Z})) \) contractible 1-periodic orbits \( x_j \) of \( H \) which admit spanning disks \( u_j \) such that the action values \( A_{H}(x_j, u_j) \) lie in the interval \([-\|H\|^{-}, \|H\|^{+}] \). Let \( P \) be a fixed global maximum of \( H \). It is a constant 1-periodic orbit of \( H \) since \( dH_t(P) = 0 \) for all \( t \in [0, 1] \). We claim that \( P \) is not one of the orbits detected by Theorem 1.1.
Any spanning disk $u$ for $P$ represents an element $[u]$ in $\pi_2(M)$ and so
\[
A_H(P,u) = \int_0^1 H(P) \, dt - \int_{D^2} u^*\omega = \|H\|^+ - \omega([u]).
\]
If $\omega([u]) \leq 0$, then have $A_H(P,u) \geq \|H\|^+$. Otherwise we have $\omega([u]) \geq r(M,\omega) - \|H\|^+ + \|H\|^-$ and $A_H(P,u) < -\|H\|^-$. In either case, $P$ does not admit a spanning disc with action in $[-\|H\|^-,\|H\|^+]$ and so does not contribute to the count of orbits detected in Theorem 1.1. Hence, there are at least $rk(H(M;\mathbb{Z})) + 1 \geq SB(M) + 1$ elements of $\mathcal{P}(H)$.

The proofs of the Arnold conjecture for compact symplectic manifolds in [Fl1, Fl2, HS, FO, LiT], imply that the number of elements in $\mathcal{P}(H)$ is at least $SB(M)$. Moreover, they imply that the cardinality of $\mathcal{P}(H)$ is equal to $SB(M)$ modulo 2. In particular, the various versions of Floer homology in these works are shown to be isomorphic to the Morse homology of $M$, with suitable coefficient rings. Hence, the number of generators of the Floer complex has the same parity as the number of critical points of a Morse function. By the strong Morse inequalities this parity is equal to the parity of $SB(M)$. Therefore, since we have detected at least $SB(M) + 1$ orbits in $\mathcal{P}(H)$, there must be at least $SB(M) + 2$.

6. Proof of Theorem 1.9

Suppose that $H$ is Floer Hamiltonian and that $\tilde{\rho}((\phi_{H}^t)) < h$. We may assume that $\epsilon^\pm > 0$ satisfy
\[
\epsilon^+ + \epsilon^- < h - \tilde{\rho}((\phi_{H}^t)).
\]
Choose Hamiltonians $G$ and $F$ in $C^\infty_0([\phi_{H}^t])$ such that
\[
\|G\|^+ < \rho^+((\phi_{H}^t)) + \epsilon^+,
\]
and
\[
\|F\|^- < \rho^-((\phi_{H}^t)) + \epsilon^-.
\]
Fix a family $J_G$ in $J_{S^1}(M,\omega)$ and set
\[
J_H = d(q_{tH}^{HC}) \circ J_G \circ d((q_{tH}^{HC})^{-1})
\]
and
\[
J_F = d(q_{tF}^{FC}) \circ J_G \circ d((q_{tF}^{FC})^{-1}),
\]
for $q_{tH}^{HC} = \phi_{H}^t \circ (\phi_{G}^t)^{-1}$ and $q_{tF}^{FC} = \phi_{F}^t \circ (\phi_{G}^t)^{-1}$. We then consider the cap data $\tilde{H}_{GF} = (\tilde{H}_G, \tilde{H}_F)$ for the Hamiltonian data $(H, J_H)$, where $\tilde{H}_G$ and $\tilde{H}_F$ are the homotopy triples obtained by applying Proposition 2.6 to linear homotopy triples $\tilde{G}$ and $\tilde{F}$ for $(G, J_G)$ and $(F, J_F)$, respectively. Proposition
2.7 together with our choice of \( G \) and \( F \), implies that
\[
|||(\kappa(\tilde{H}_{GF}))||| = |||(\kappa(\tilde{H}_G))||| + |||(\kappa(\tilde{H}_F))|||\]
\[
= |||(\kappa(G))||| + |||(\kappa(F))||| - \hbar.<
\]
Hence, we can again apply Proposition 2.5 to obtain at least \( r_k(H(M;\mathbb{Z})) \) elements \( x_j \) of \( \mathcal{P}(H) \) which admit spanning discs \( u_j \) such that
\[
-n \leq \mu_{CZ}(x_j, u_j) \leq n
\]
and
\[
-\rho^-(|\phi_H^t|) - \epsilon^- \leq -\|F\|^- \leq A_H(x_j, u_j) \leq \|G\|^+ \leq \rho^+(|\phi_H^t|) + \epsilon^+.
\]

Corollary 6.1 (Theorem 1.1 for \( \rho^- \)). Suppose \( H \) is a Floer Hamiltonian such that \( \|H\| < \hbar \) and \( \phi_H^t \) does not minimize the negative Hofer length in its homotopy class. There are at least \( r_k(H(M;\mathbb{Z})) \) elements \( x_j \) of \( \mathcal{P}(H) \) which admit spanning discs \( u_j \) such that
\[
-n \leq \mu_{CZ}(x_j, u_j) \leq n
\]
and
\[
-\|H\|^- < A_H(x_j, u_j) \leq \|H\|^+.
\]

Proof. Let \( F \) be a Hamiltonian in \( C_0^\infty(|\phi_H^t|) \) such that \( \|F\|^- < \|H\|^- \). Choose \( \epsilon^+ = 0 \) and \( G = H \), and repeat the proof of Theorem 1.9. \( \square \)

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