On Selfadjoint Subspace of One-Speed Boltzmann Operator

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1 Introduction

It is well-known [1, 2] that a nonself-adjoint operator in a Hilbert space can be represented as an orthogonal sum of a self-adjoint one, and an operator having no reducing subspaces on which it induces a self-adjoint operator. A natural question about operators arising in applications is whether the first (selfadjoint) component in this sum is trivial, that is, whether the operator is completely nonself-adjoint. For differential Schr"odinger operators this question was studied earlier [4] and is related to the unique continuation property for solutions. In this note we study complete nonself-adjointness for one-speed Boltzmann operator [3] arising in the theory of neutron transport in a medium with multiplication.

The main result of the paper – theorem 2 – is that the selfadjoint subspace is non-trivial for any Boltzmann operator with polynomial collision integral if the multiplication coefficient has a lattice of gaps in the support of arbitrarily small width, that is, if the coefficient vanishes on an $\varepsilon$-neighborhood of the set $a\mathbb{Z}$ for some $a, \varepsilon > 0$. On the other hand, the operator of the isotropic problem turns out to be completely nonself-adjoint if the multiplication coefficient is non-zero on a semi-axis (proposition 3). For anisotropic problem we give an example (corollary 2) showing that under an appropriate

∗The was partially suppported by INTAS Grant 05-100008-7883 and RFBR Grant 06-01-00249.
choice of the collision integral the operator may turn out to be completely nonself-adjoint for any non-vanishing multiplication coefficient. Finally, for the three-dimensional Boltzmann operator we establish non-triviality of the selfadjoint subspace for any non-zero multiplication coefficient (theorem 3).

Let us describe the structure of the paper. Proposition 1 gives a version of the abstract theorem on decomposition of an operator in the sum of selfadjoint and completely nonself-adjoint ones convenient for our purposes. A close assertion in terms of the resolvent is contained in [2]. Theorem 2 is proved by a direct construction of a non-zero function lying in the selfadjoint subspace. It occupies sections 3 and 4. The same problem for the three-dimensional Boltzmann operator is studied in section 5.

The authors are indebted to P. Kargaev for a useful discussion.

The following notation is used throughout:

- If \( \{ S_i \}_{i \in I} \) is a family of subsets of a Hilbert space, then \( \bigvee_{i \in I} S_i \) is the closure of the linear span of the set \( \bigcup_{i \in I} S_i \).

- If \( f \in L^2(\mathbb{R}) \), then \( \hat{f} \in L^2(\mathbb{R}) \) is the Fourier transform of \( f \):
  \[
  \hat{f}(p) \overset{\text{def}}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipx} f(x) \, dx.
  \]

- \( H^2_{\pm} \) — the Hardy classes of analytic functions in the upper and lower half planes, respectively.

- The abbreviation a. e. refers to the Lebesgue measure on \( \mathbb{R} \). For a measurable function \( f \) on \( \mathbb{R} \) the notation \( \text{supp} f \) stands for the set \( \{ x \in \mathbb{R} : f(x) \neq 0 \} \) defined up to a set of zero measure.

2 Definitions and Preliminaries

The Boltzmann operator acts in the space \( L^2(\mathbb{R} \times [-1, 1]) \) of functions \( u = u(x, \mu) \) \( (x \in \mathbb{R}, \mu \in [-1, 1]) \) endowed with the standard Lebesgue measure, by the formula:
\[(Lu)(x, \mu) = i\mu \left( \partial_x u \right)(x, \mu) + i \sum_{\ell=1}^n c_\ell(x) \varphi_\ell(\mu) \int_{-1}^1 u(x, \mu') \overline{\varphi_\ell(\mu')} \, d\mu'. \quad (1)\]

Here the functions \(c_\ell \in L^\infty(\mathbb{R}), \varphi_\ell \in L^\infty[-1, 1], \ell = 1, \ldots, n\) are known parameters of the problem. The functions \(c_\ell\) are assumed to be real-valued. The case \(n = 1, \varphi_1 \equiv 1\) (isotropic scattering) is of special interest. In this situation the function \(c_1\) is called the local multiplication coefficient, and the index 1 is omitted. Without loss of generality, one assumes throughout that the functions \(\{\varphi_\ell\}_{\ell=1}^n\) are linearly independent.

Under the conditions imposed the operator \(L\) is the sum of the operator \(L_0 = i\mu \partial_x\), selfadjoint on its natural domain, and a bounded one (see [3, 5] for details). According to the non-stationary Boltzmann equation, if \(u^t\) is the particle density at time \(t\), then \(u^t = e^{itL}u^0\). Notice that the literature on the Boltzmann operator uses for it an expression different from (1) by the factor \(i\), because of a different definition of the exponential function. The evolution operator \(u^0 \mapsto u^t\) in our notation coincides with the standard one.

Let \(D\) be an operator in a Hilbert space \(H\) of the form \(D = A + iK\), where \(A\) is selfadjoint, and \(K\) is selfadjoint and bounded.

**Definition 1.** The subspace \(H_0 \subset H\) is the selfadjoint subspace of the operator \(D\), if

1. \(H_0\) reduces \(D\), and the restriction \(D\vert_{H_0}\) is a selfadjoint operator in \(H_0\);
2. any reducing subspace \(H'\) of the operator \(D\) such that \(D\vert_{H'}\) is a selfadjoint operator in \(H'\) is contained in \(H_0\).

An operator \(D\) is called completely nonself-adjoint if its selfadjoint subspace is trivial.

**Proposition 1.** The orthogonal complement of the selfadjoint subspace of the operator \(D\) coincides with the subspace

\[H_1 \overset{\text{def}}{=} \bigvee_{t \in \mathbb{R}} e^{iAt} \text{Ran } K.\]

\(^1\)that is, the orthogonal projection on \(H_0\) in \(H\) preserves the domain \(D\) of the operator \(D\), and \(Df \in H_0, D^*f \in H_0\) for all \(f \in H_0 \cap D\).
Proof. Let $H_0$ be the selfadjoint subspace of the operator $D$. By definition, the subspace $H_1$, and hence $H_1^\perp$, reduces the operator $A$. Since, obviously, $H_1^\perp \subset \text{Ker } K$ we obtain from this that $H_1$ and $H_1^\perp$ are reducing subspaces of the operator $D$ such that the restriction of $D$ to $H_1^\perp$ is a selfadjoint operator. Thus, $H_1^\perp \subset H_0$. Let us show that $H_1 \subset H_0^\perp$. Indeed, the subspace $H_0^\perp$ reduces $D$, and therefore the operator $A = (D + D^*)/2$ as well. This means, in particular, that $e^{itA}\text{Ran } K \subset H_0^\perp$ for all real $t$, since $H_0 \subset \text{Ker } K$. \hfill \qed

In what follows we are going to use the fact that the selfadjoint subspace $H_0$ is reducing for the operator $A$ as well, and the selfadjoint part of $L$ coincides with the restriction of $A$ to $H_0$.

For the Boltzmann operator $(\Pi) A = L_0 = i\mu \partial_x$, and a straightforward calculation gives

$$(e^{itA}f)(x,\mu) = f(x - \mu t, \mu).$$

For a function $\xi \in L^\infty(\mathbb{R})$ let us denote by $D_\xi$ the set of compactly supported functions $h \in L^2(\mathbb{R})$ vanishing outside $\text{supp } \xi$.

Corollary 1. A function $f \in L^2(\mathbb{R} \times [-1,1])$ belongs to the selfadjoint subspace of the operator $(\Pi)$ if and only if

$$\int_{[-1,1] \times \mathbb{R}} f(x - \mu t, \mu) \overline{\varphi_\ell(\mu)} h(x) \, d\mu dx = 0$$

for all $\ell = 1, \ldots, n$, $t \in \mathbb{R}$, and $h \in D_{c\ell}$.

Thus, we have to find out if there exists a non-zero function $f$ satisfying the condition (3). Let us first explain on the formal level the method we use. For simplicity, let the function $c$ be the indicator of an interval $I$, and $\varphi \equiv 1$. Then the condition of the lemma means that

$$\int_{-1}^{1} f(x - \mu t, \mu) \, d\mu = 0$$

for all $t \in \mathbb{R}$ and $x \in I$. We will search for the function $f$ in the form

$$f(x, \mu) = \int_{\mathbb{R}} e^{i\mu q} u(q, \mu) \, dq.$$
Substituting and interchanging the order of integrations, we obtain:

$$0 = \int_{\mathbb{R}} dq \, e^{-iqt} \int_{-1}^{1} e^{\frac{xq}{\mu}} u(q, \mu) \, d\mu.$$ 

Since this equality is an identity in $t$, the inner integral must be zero for all $q$. After the change of variable $p = 1/\mu$ in this integral we arrive at the following uniqueness problem for the Fourier transform: is there a nonzero function $v(q, p)$ such that its Fourier transform in the second variable $\mathcal{F}v$ vanishes at all points of the form $(q, -xq)$, $q \in \mathbb{R}$, $x \in I$. A rigorous argument requires analysis of certain integral transforms of the Fourier type, definitions and elementary properties of which are given in the next section.

3 Integral Transforms

Let $\omega \subset \mathbb{R}$ be a compact interval. We set the notation for certain classes of functions of variables $q \in \mathbb{R}$ and $\mu \in [-1, 1]$ and transforms between them:

- $H = L^2(\mathbb{R} \times [-1, 1])$.
- $C_\omega$ — the linear set of functions $u \in C^\infty(\mathbb{R} \times [-1, 1])$ vanishing for $q \notin \omega$.
- $H_{\omega}$ — the subspace in $L^2(\mathbb{R} \times [-1, 1], |\mu| \, dq \, d\mu)$ of functions vanishing for $q \notin \omega$.
- $\Phi$ and $\Phi^*$ — unitary mutually inverse operators
  
  $$\Phi : H \to L^2(\mathbb{R} \times [-1, 1], |\mu| \, dq \, d\mu),$$
  
  $$\Phi^* : L^2(\mathbb{R} \times [-1, 1], |\mu| \, dq \, d\mu) \to H,$$

  defined on finite smooth functions by formulae

  $$(\Phi f)(q, \mu) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ixq\mu} f(x, \mu) \, dx,$$

  $$(\Phi^* u)(x, \mu) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\frac{xq}{\mu}} u(q, \mu) \, dq.$$
Remark 1. It is obvious that $\Phi(-i\mu \partial_x) \Phi^*$ is the operator of multiplication by the independent variable $q$ in $L^2(\mathbb{R} \times [-1, 1], |\mu| dq \, d\mu)$.

Let us denote by $\mathcal{F}$ the unitary operator $\mathcal{F} : L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)$ of the Fourier transform in the second variable:

$$(\mathcal{F}u)(q, p) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-isp} u(q, s) \, ds.$$ 

We are going to use the change of variables

$$(Jv)(q, s) := \begin{cases} 
    s^{-2}v(q, s^{-1}), & |s| > 1 \\
    0, & |s| < 1,
\end{cases}$$

which defines an isometry $J : L^2(\mathbb{R} \times [-1, 1], |\mu| dq \, d\mu) \rightarrow L^2(\mathbb{R}^2, |p| dq \, dp)$.

For any smooth finite function $v$ defined in the strip $\mathbb{R} \times [-1, 1]$ let

$$(\Psi v)(q, x) = \frac{1}{\sqrt{2\pi}} \int_{|p|>1} e^{ixqp} p^{-2} v \left( q, \frac{1}{p} \right) \, dp.$$ 

(4)

Lemma 1. For any closed interval $\omega$ not containing $0$, the transform defined by the formula (4) on $C^\circ \omega$ is extended to a bounded operator $\Psi$ from $H_\omega$ to $L^2(\mathbb{R}^2)$ acting by the following formula:

$$(\Psi v)(q, x) = (\mathcal{F}Jv)(q, -xq), \ v \in H_\omega.$$ 

(5)

Proof. By definition (1), the equality (5) is satisfied for all $v \in C^\circ \omega$, and the Fourier transform $\mathcal{F}$ in it can be understood classically. Then, for any $g \in L^2(\mathbb{R}^2)$ vanishing when $|q| < a := \text{dist}(0, \omega)$, we have:

$$\int_{\mathbb{R}^2} |(\mathcal{F}g)(q, -xq)|^2 \, dq \, dx = \int_{\mathbb{R}} dq \frac{1}{|q|} \int_{\mathbb{R}} |(\mathcal{F}g)(q, x)|^2 \, dx \leq \frac{1}{a} \int_{\mathbb{R}^2} |g(q, x)|^2 \, dx \, dq.$$ 

Thus, the map $\Psi$ is a composition of the bounded operator $J : H_\omega \rightarrow L^2(\mathbb{R}^2)$ and the map $g \mapsto (\mathcal{F}g)(q, -xq)$, which is a bounded operator from the subspace $JH_\omega \subset L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$. Since the linear set $C^\circ \omega$ is dense in $H_\omega$, it follows that the map $\Psi$ defines a bounded operator. Simultaneously, we have proved (5).
Lemma 2. Let $\omega$ be a closed interval not containing 0, and let $\varphi \in L^\infty(-1, 1)$. Then for any compactly supported function $h \in L^2(\mathbb{R})$ and any $u \in H_\omega$ the following equality is satisfied for all $t \in \mathbb{R}$:

$$
\int_{[-1,1] \times \mathbb{R}} d\mu \, dx \, h(x) \varphi(\mu)(\Phi^* u)(x - \mu t, \mu) = \int_{-\infty}^\infty dq \, e^{-iqt} \langle \Psi(u\varphi)(q, \cdot), h \rangle_{L^2(\mathbb{R})}.
$$

(6)

Proof. Since $\tilde{C}_\omega$ is dense in $H_\omega$, it is enough to prove (6) for arbitrary function $u \in \tilde{C}_\omega$. We have:

$$
\int_{-1}^1 \varphi(\mu)(\Phi^* u)(x - \mu t, \mu) d\mu = \frac{1}{\sqrt{2\pi}} \int_{-1}^1 d\mu \varphi(\mu) \int_{-\infty}^\infty e^{i(x-\mu t)q} u(q, \mu) dq =
$$

$$
= \int_{-\infty}^\infty dq \, e^{-iqt} \frac{1}{\sqrt{2\pi}} \int_{-1}^1 e^{i\frac{2\pi q}{\mu} u(q, \mu)} \varphi(\mu) d\mu = \int_{-\infty}^\infty dq \, e^{-iqt} (\tilde{\Psi}(u\varphi))(q, x).
$$

Multiplying this equality by $\overline{h}$ and integrating in $x$, we obtain (6). The interchange of integrations in $x$ and $q$ in the right hand side is possible because of the function $h$ having a compact support. \qed

4 Conditions of Complete Nonself-Adjointness

In what follows, the functions $\varphi_\ell \in L^\infty[-1, 1]$ from the definition of the Boltzmann operator are supposed to be extended by zero to the whole of the real line.

Theorem 1. Let the function $F(q, p) \in L^2(\mathbb{R}^2, |p|dqdp)$ satisfy the following conditions:

$$
F(q, p) = 0 \text{ for } |p| < 1;
$$

(7)

$$
\int_{\mathbb{R}} e^{ipxq} F(q, p) \varphi_\ell(\frac{1}{p}) dp = 0
$$

(8)

for all $\ell = 1, \ldots, n$, and a.e. $q \in \mathbb{R}$, $x \in \text{supp} c_\ell$. Then the vector

$$
f = \Phi^* u,
$$

$$
u(x, \mu) = \mu^{-2} F(q, \mu^{-1}), \quad |\mu| < 1,
$$

7
belongs to the selfadjoint subspace $H_0$ of the operator (1). The mapping $F \mapsto f$ defines an isomorphism of the subspace $X \subset L^2(\mathbb{R}^2, |p|dqdp)$ singled out by conditions (7) and (8), and the space $H_0$. In particular, the space $H_0$ is non-zero if, and only if, there exists a non-zero function $F$ satisfying conditions (7) and (8).

**Remark 2.** The equality (8) is understood as a condition of vanishing of the Fourier transform in the second variable of the function $F(q,p)\varphi_\ell(p^{-1})$ lying in the space $L^2(\mathbb{R}^2)$, on the set $M_\ell \equiv \{(q,-xq) : q \in \mathbb{R}, x \in \text{supp} c_\ell\}$ of positive planar Lebesgue measure.

**Proof.** Substituting $p = \mu^{-1}$, we immediately verify that the map $F \mapsto f$, defined in the theorem, is an isometry from $X$ to $H$. Let us show that $f \in H_0$ for any $F$ from the dense in $X$ linear set of functions $F \in X$ such that $F(q,p) = 0$ when $q \notin \omega$ for some closed interval $\omega = \omega(F)$ not containing 0. For such $F$’s the function $v = u\varphi_\ell$ obeys the equality (5):

$$
\Psi[u\varphi_\ell](q, x) = F(q, p, \varphi_\ell(p^{-1}))(q, -xq).
$$

(10)

By assumption (8), the right hand side vanishes on the set $\{(q, x) : q \in \mathbb{R}, x \in \text{supp} c_\ell\}$, and thus $\Psi[u\varphi_\ell](q, x)h(x)$ is identically zero for any function $h \in L^2(\mathbb{R})$ supported on $\text{supp} c_\ell$. Applying the identity (6), we conclude from this that

$$
\int_{[-1,1] \times \mathbb{R}} d\mu dx h(x)\varphi_\ell(\mu)(\Phi^*u)(x-\mu t, \mu) = 0
$$

for all $t \in \mathbb{R}$ and $h \in D_{c_\ell}$, that is, $f$ satisfies the condition of corollary (1).

It remains to check that the range of the map $F \mapsto f$ is the whole of $H_0$. Let $f_0 \in H$ be a vector of the form $f_0 = P_\omega g$, where $g \in H_0$, $\omega$ is a closed interval not containing point 0, and $P_\omega$ is the spectral projection of $L_0$ corresponding to the interval $\omega$. Then, $f_0 \in H_0$ since the subspace $H_0$ reduces the operator $L_0$, and hence any of his spectral projections. We shall show that $f_0$ lies in the range of the constructed isometry from $X$ to $H$.

Let $u = \Phi f_0$, and let $F(q,p) = p^{-2}u(q,p^{-1})$ for $q \in \omega$ and $|p| > 1$, $F(q,p) = 0$ for any other $(q,p) \in \mathbb{R}^2$. By construction, the function $F$ belongs to $L^2(\mathbb{R}^2, |p|dqdp)$ and satisfies (7) and (9) with $f = f_0$. Notice that the function $u$ vanishes when $q \notin \omega$ since, according to remark (1) (which is assumed to be true), the operator of multiplication by the $q$ variable. Thus, the function $u \in H_\omega$, lemma (2) applies to it because $0 \notin \omega$, and the equality (10) holds true. As
the function \( f_0 \) belongs to \( H_0 \), and hence obeys condition (2), the left hand side in (6) vanishes for the \( u \) under consideration for all \( t \in \mathbb{R} \) and \( h \in D_{c\ell}, \ell = 1, \ldots, n \). By uniqueness of the Fourier transform it follows that

\[
\langle \Psi[U \phi \ell](q, \cdot), h \rangle_{L^2(\mathbb{R})} = 0
\]

for a.e. \( q \in \mathbb{R} \) and all \( h \in D_{c\ell} \). The arbitrariness of \( h \) implies that the function \( \Psi[U \phi \ell](q, x) \) vanishes on \( \mathbb{R} \times \text{supp} \ c_{\ell} \). Then the right hand side in (10) also vanishes on \( \mathbb{R} \times \text{supp} \ c_{\ell} \), that is, condition (8) is satisfied. It remains to notice that the set of vectors \( f_0 \) of the form under consideration is dense in \( H_0 \) since the operator \( L_0 \) is absolutely continuous.

**Theorem 2.** Let the function \( c(x) \) be bounded, and let there be \( a, \varepsilon > 0 \) such that \( c(x) = 0 \) for \( |x - x_0 - aj| < \varepsilon, j \in \mathbb{Z} \), with some \( x_0 \in \mathbb{R} \), and let all the functions \( \varphi_{\ell}(\mu), 1 \leq \ell \leq n \), be polynomials. Then the selfadjoint subspace \( H_0 \) of the Boltzmann operator

\[
L = i\mu \partial_x + ic(x) \sum_{\ell=1}^{n} \varphi_{\ell}(\mu) \int_{-1}^{1} \cdot \overline{\varphi_{\ell}(\mu')} d\mu'
\]  

(11)

is non-trivial, and, moreover, the restriction of the selfadjoint part of the operator \( L \) to its spectral subspace corresponding to the interval \([−\pi/a, π/a]\) has Lebesgue spectrum of infinite multiplicity\(^2\).

**Proof.** Without loss of generality one can assume that \( x_0 = 0 \) and \( n = 1 + \max_{\ell} \deg \varphi_{\ell} \). Let us search for a function \( F \), satisfying the conditions of theorem 1 in the form \( F(q, p) = \chi(q)f(pq) \), where \( \chi \) is an arbitrary bounded function on the real axis such that \( \text{supp} \ \chi = [-b, b] \) for some positive \( b < \pi/a \), and \( \chi(q)/q \in L^2 \). The conditions of Theorem 1 will be met if the function \( f \in L^2(\mathbb{R}, |p|dp) \) obeys the following requirements:

(i) \( f(p) = 0 \) for \( |p| \leq b \);

(ii) \( \text{supp} \ f p^{-j} \) is contained in the \( \varepsilon \)-vicinity of the set \( a\mathbb{Z} \) for all \( j, 0 \leq j \leq n - 1 \).

We are going to use the following observation: let \( h \in L^2_{\text{loc}}(\mathbb{R}) \) be an arbitrary \( 2\pi \)-periodic function, and \( \omega \) be a smooth function on the real line

\(^2\)This means that the restriction is unitarily equivalent to an orthogonal sum of infinitely many copies of the operator of multiplication by the independent variable in \( L^2 \) over this interval.
supported on an interval \((-\delta, \delta), \delta > 0\). Then the function \(\xi = h\hat{\omega}\), obviously, belongs to \(L^2(\mathbb{R}, |p|dp)\), and its Fourier transform vanishes outside the \(\delta\)-vicinity of \(\mathbb{Z}\). This observation follows from elementary properties of convolution since \(h = \hat{\rho}\) where \(d\rho = \sum_j \rho_j \delta(x - j)\) is the discrete measure with masses being the Fourier coefficients \(\rho_j\) of the restriction of the function \(h\) to a period.

Fix an arbitrary nonzero function \(h\), satisfying the conditions above and such that \(h(p) = 0\) for \(|p| \leq \pi - \nu, \nu > 0\). Let \(\delta = \varepsilon/a\), choose an arbitrary nonzero function \(\omega_0 \in C_0^\infty(\mathbb{R})\) supported on \((-\delta, \delta)\) and define the corresponding function \(\xi\) setting \(\omega = \omega_0^{(n)}\). Define \(f(p) = \xi(ap)\). By construction, conditions (i) and (ii) hold true for the function \(f\) for all \(\nu > 0\) small enough. Fix such a \(\nu\) and let:

\[
u(q, \mu) = \mu^{-2} F \left( q, \frac{1}{\mu} \right).
\]

By theorem 1 the nonzero function \(g \equiv \Phi^* u\) belongs to \(H_0\), and the non-triviality of the subspace \(H_0\) is proved.

To establish the assertion about the multiplicity of the spectrum notice that, as follows from remark \(\Pi\) the restriction of \(L_0\) to its reducing subspace generated by the function \(g\) is unitarily equivalent to the operator of multiplication by the independent variable in the space \(L^2\) over the support of \(\chi\), that is, in \(L^2(-b, b)\). Each choice of the function \(h\) in the construction above then corresponds to a reducing subspace, and if continuous functions \(h_j, j = 1, \ldots, N, N < \infty\), are mutually linearly independent, then so are the corresponding reducing subspaces \(Y_j \subset H_0\). Indeed, the last assertion means that for any finite \(M\), any \(h_j\) satisfying the conditions above, and any \(\chi_j \in L^2(-b, b), j \leq N\), the following implication is true:

\[
\sum_{1}^{M} h_j(pq)\hat{\omega}(pq)\chi_j(q) \equiv 0 \Rightarrow \chi_j(q) \equiv 0 \forall j \leq N,
\]

which is easily verified by induction. It is then enough to choose an arbitrary \(c \neq 0\) such that \(\hat{\omega}(c) \neq 0\), and \(h_j(c) \neq 0\) for at least one \(j\), and let \(p = c/q\).

Thus, we have proved that for any \(b < \pi/a\) there is a reducing subspace in \(H_0\) such that the restriction of the operator to it has Lebesgue spectrum of infinite multiplicity on \([-b, b]\), hence the same is true of \(b = \pi/a\). Since the operator \(L_0\) is absolutely continuous, it follows that the restriction of \(L\) to \(H_0\) possesses the same property.
Remark 3. The proof of theorem 2 is constructive – nonzero vectors from $H_0$ were found explicitly.

Sometimes it is possible to say more about the spectrum of the selfadjoint part.

Proposition 2. Let the function $c(x)$ be compactly supported, and let all the functions $\varphi_\ell(\mu)$, $1 \leq \ell \leq n$, be polynomials. Then the selfadjoint part of the operator $L$ of the form (11) is unitarily equivalent to an orthogonal sum of infinitely many copies of the operator of multiplication by the independent variable in $L^2(\mathbb{R})$.

Proof. Let us first consider the case $n = 1$, $\varphi_1 \equiv 1$. Let $I$ be an arbitrary closed interval, $\chi(q)$ its indicator function. We shall search for the function $F(q, p)$ in the form of the product $\chi(q)f(p)$ where $f \in L^2(\mathbb{R}, |p|dp)$ is a function vanishing on $[-1, 1]$ and such that $\hat{f}$ vanishes on an interval $[-M, M]$. It is clear that for $M$ large enough such a function $F$ obeys all the conditions of theorem [1]. A supply of functions $f$ with the desired properties is provided by the following lemma.

Lemma 3. Let $\alpha > 0$, and let $\rho(z)$ be an arbitrary nonzero function analytic in the plane cut along a compact interval $J \subset (-\infty, -1]$ and such that $\rho(z) = O(|z|^{-2})$ when $|z| \to \infty$ uniformly in $\arg z$, and the restrictions of $\rho(z)$ to $\mathbb{C}_+$ and $\mathbb{C}_-$ belong to $H^2_+$ and $H^2_-$, respectively. Define the function

$$
\phi_\alpha(z) = \exp \left[ 2i\alpha \left( -\frac{z}{2} + \frac{1}{1 - \sqrt{\frac{z-1}{z+1}}} \right) \right] \rho(z),
$$

(12)

where the branch of the square root is chosen so that $\phi_\alpha(z)$ be analytic in the plane cut along the rays $(-\infty, -1] \cup [1, +\infty)$, and $\Im \sqrt{\frac{z-1}{z+1}} > 0$. Let $f_\alpha^\pm$ be the boundary values of the function $\phi_\alpha$ on the real axis in the sense of the Hardy classes.

Then the (obviously, nonzero) function

$$
f_\alpha(x) \overset{\text{def}}{=} f_\alpha^+(x) - f_\alpha^-(x)
$$

obeys:

\textsuperscript{3}Theorem 2 in the situation under consideration only ensures the existence of the spectrum in a vicinity of 0.
1. $f_{\alpha} \in L^2(\mathbb{R}, |p|dp)$;

2. $f_{\alpha}(x) = 0$ for $|x| < 1$;

3. $\hat{f}_{\alpha}(p) = 0$ for $|p| \leq \alpha$.

Proof. Property 2 is obvious. Since the boundary values of the exponent in (12) on the real axis have the modulus $\leq 1$ for the given choice of the square root brunch, the inclusion $f_{\alpha}^{\pm} \in L^2(\mathbb{R}, |p|dp)$ is immediate from the assumptions about the function $\rho(z)$. It remains to check the property 3.

The following asymptotics hold for $|z| \to \infty$ in each of the halfplanes $\mathbb{C}_\pm$ uniformly in $\arg z$:

$$1 - \frac{1}{\sqrt{z+1}} = 1 - (1 - \frac{1}{z} + O\left(\frac{1}{z^2}\right)) = z + O(1), \text{ for } \text{Im } z > 0;$$

$$1 - \frac{1}{\sqrt{z+1}} = \frac{1}{1 + (1 - \frac{1}{z} + O\left(\frac{1}{z^2}\right))} = O(1), \text{ for } \text{Im } z < 0.$$

Therefore for $|z| \to \infty$ we have:

$$\exp\left[2i\alpha \left(\frac{-z}{2} + \frac{1}{1 - \sqrt{z+1}}\right)\right] = \exp \left( i\alpha z \text{ sign}(\text{Im } z) + O(1) \right).$$

Thus, the restrictions of the functions $e^{\pm i\alpha z} \phi_{\alpha}$ to the halfplanes $\mathbb{C}_\pm$ are in $H^2_\pm$, respectively. By the Paley-Wiener theorem this implies that $\hat{f}_{\alpha}^{\pm}(p) = 0$ when $\pm p \leq \alpha$, hence $\hat{f}_{\alpha}(p) = 0$ for $|p| \leq \alpha$.

For the function $\rho$ in this lemma one can take, for instance, the branch of the function $\ln^n \left(\frac{z + a}{z + b}\right)$, $1 < b < a$, $n \geq 2$, analytic in the plane cut along the interval $[-a, -b]$, fixed by the condition $\text{Im } \ln \frac{z+a}{z+b}|_{z=0} = 0$. 

End of proof of proposition 2. Let $\alpha$ be a number such that $|qx| < \alpha$ for all $q \in I$, $x \in \text{supp } c$, $f_{\alpha} \in L^2(\mathbb{R}, |p|dp)$ an arbitrary function vanishing on $[-1, 1]$ and such that $\hat{f}_a$ vanishes on the interval $[-\alpha, \alpha]$. Let $F(q,p) = \chi(q) f_{\alpha}(p)$. Define a vector $g \in H_0$ via the function $F$ in the same way as in the proof of theorem 2. The restriction of the operator $L$ to its reducing subspace $Y = Y(f_{\alpha})$, generated by the vector $g$, is unitarily equivalent to the
operator of multiplication by the independent variable in the space $L^2(I)$, and if functions $f_{\alpha,j}$, $j = 1, \ldots, n < \infty$, are mutually linear independent, then so are the corresponding subspaces $\{Y(f_{\alpha,j})\}$. The assertion of the proposition now follows from this and the fact that the linear space of functions $f_{\alpha}$ constructed in lemma 3 is infinite-dimensional.

The general case ($n \neq 1$) is considered in a similar way, we only require additionally the function $\rho$ in lemma 3 to have a zero of order $n - 1$ at the point 0. If this requirement is satisfied, the Fourier transforms of $f p^{-j}$ vanish on $[-\alpha, \alpha]$ for all $j \leq n - 1$, and the proof proceeds as above. \hfill \blacksquare

Commentary to the proof of theorem 2. The question if there exists a nonzero function $f \in L^2(\mathbb{R})$ such that the restrictions $f|_S = 0$ and $\hat{f}|_{\Sigma} = 0$ for a given interval $S$ and a set $\Sigma \subset \mathbb{R}$ is known as the Beurling problem and has been studied for a long time [7]. For instance, the Amrein-Berthier theorem [7] establishes the existence of such functions if the set $\Sigma$ has finite measure, the Kargaev theorem [8] – in a situation generalizing theorem 2 to the case of gaps narrowing at infinity. These results are, however, not immediately applicable to the problem under consideration, when the function $f$ is subject to an additional condition of square summability with the growing weight $|p|$. To use them, one would have to smoothen up the functions constructed which would lead to assertions close to theorem 2 and proposition 2, obtained here by elementary methods.

The selfadjoint subspace found in theorem 2 is quite large, and it is natural to ask if there is much else. On this is the following

Remark 4. Results in paper [9] show that in the isotropic problem the essential spectrum of the restriction of the operator $L$ to $H^1_0$ coincides with the real line if the function $c$ is compactly supported, and $c(x) \geq 0$ a. e.

In the direction opposite to theorem 2 the following simple assertion holds.

Proposition 3. Let the function $c \in L^\infty(\mathbb{R})$ be such that $c(x) \neq 0$ a. e. on a semi-axis. Then the Boltzmann operator

$$L = i\mu \partial_x + ic(x) \int_{-1}^{1} \cdot d\mu'$$

is completely nonself-adjoint.
Proof. Without loss of generality one can assume that $c(x) \neq 0$ for a. e. $x > 0$. Suppose that the selfadjoint subspace $H_0 \neq \{0\}$. Then by theorem 1 (see (5)) there exists a nonzero function $F(q,p) \in L^2(\mathbb{R}^2)$ such that for a. e. $q > 0$ we have: $(F^*F)(q,x) = 0$ for a. e. $x > 0$. By the Paley-Wiener theorem this implies that $F(q,\cdot) \in H^2_+$ for a. e. $q > 0$, and, since $F(q,p) = 0$ for $|p| < 1$, by properties of the Hardy classes it follows that the function $F(q,\cdot) = 0$ for a. e. $q > 0$. Similarly, one considers the case $q < 0$. We thus obtain that $F$ is the zero function, a contradiction.

The following proposition is aimed at clarifying the main result of theorem 2. The operator in a strip of half-width dealt in it has possibly no physical relevance.

Proposition 4. Let $\varphi \in L^\infty(0,1)$, supp $\varphi = [0,1]$; $c \in L^\infty(\mathbb{R})$, and let $L$ be an operator in the Hilbert space $H = L^2(\mathbb{R} \times [0,1])$ defined by the formula

$$
(Lu)(x,\mu) = i\mu (\partial_x u)(x,\mu) + ic(x)\varphi(\mu) \int_0^1 u(x,\mu')\overline{\varphi(\mu')} d\mu'
$$

on a natural domain of its real part $L_0 = i\mu\partial_x$. Then the operator $L$ is completely nonself-adjoint if $c \neq 0$.

Proof. Arguing as in the proof of theorem 1, it is easy to see that the operator defined by (13) is completely nonself-adjoint if any function $F(q,p) \in L^2(\mathbb{R}^2,|p|dqdp)$, satisfying the condition (5) and such that $F(q,p) = 0$ for $p < 1$, vanishes identically. The condition (5) means that for a. e. $q \in \mathbb{R}$ the Fourier transform in the second variable of the function $G(q,p) = F(q,p)\overline{\varphi(p)}$ vanishes on a set of positive measure. On the other hand, $G(q,p) = 0$ for $p < 1$. By properties of the Hardy classes this implies that the function $G(q,\cdot) \equiv 0$ for a. e. $q \in \mathbb{R}$, and thus $F \equiv 0$.

A similar assertion holds for the strip $\mathbb{R} \times [-1,0]$. Considering the orthogonal sum, we obtain the following

Corollary 2. Let the functions $\varphi_{1,2} \in L^\infty(-1,1)$ be such that supp $\varphi_1 = [0,1]$, supp $\varphi_2 = [-1,0]$. Then the Boltzmann operator of the form

$$
L = i\mu\partial_x + ic_1(x)\varphi_1(\mu) \int_{-1}^1 \overline{\varphi_1(\mu')} d\mu' + ic_2(x)\varphi_2(\mu) \int_{-1}^1 \overline{\varphi_2(\mu')} d\mu'
$$

on a natural domain of its real part $L_0 = i\mu\partial_x$. Then the operator $L$ is completely nonself-adjoint if $c_1, c_2 \neq 0$.
is completely nonself-adjoint if neither of the functions $c_1, c_2$ vanishes identically.

Thus, in the anisotropic case the Boltzmann operator may turn out to be completely nonself-adjoint for perturbations having arbitrarily small support.

## 5 Three Dimensional Boltzmann Operator

Let $S^2 = \{ s \in \mathbb{R}^3 : |s| = 1 \}$. The 3D Boltzmann operator acts in the space $L^2(\mathbb{R}^3 \times S^2)$ of functions $u = u(x, \mu) \ (x \in \mathbb{R}^3, \mu \in S^2)$ by the formula:

$$(Lu)(x, \mu) = i\mu (\nabla_x u)(x, \mu) + i \sum_{\ell=1}^{n} c_\ell(x) \phi_\ell(\mu) \int_{S^2} u(x, \mu') \overline{\phi_\ell(\mu')} \, dS(\mu'). \quad (14)$$

Here $c_\ell \in L^\infty(\mathbb{R}^3)$ and $\phi_\ell \in L^2(S^2)$, $\ell = 1, \ldots, n$, are known functions. The operator $L$ is a bounded perturbation of the operator $L_0 = i\mu \nabla_x$ selfadjoint on its natural domain.

**Theorem 3.** The selfadjoint subspace of the Boltzmann operator (14) is non-trivial.

**Proof.** Let $U : H \to H$ be the Fourier transform in the $x$ variable. Let $\hat{L} = ULU^*$, $\hat{L}_0 = UL_0U^*$ etc.

As in the 1D case, a vector $u$ belongs to the selfadjoint subspace of the operator (14) if, and only if

$$\int_{S^2} v(x - \mu t, \mu) \overline{\phi_\ell(\mu)} \, dS(\mu) = 0$$

for all $\ell = 1, \ldots, n$, $t \in \mathbb{R}$, and a.e. $x \in \text{supp} \ c_\ell$. It is easy to see that this equality is satisfied if $\hat{v} := Uv$ obeys

$$\int_{S^2} \exp \left( it\langle p, \mu \rangle_{\mathbb{R}^3} \right) \hat{v}(p, \mu) \overline{\phi_\ell(\mu)} \, dS(\mu) = 0 \quad (15)$$

for a.e. $p \in \mathbb{R}^3$ and all $t \in \mathbb{R}$, $\ell = 1, \ldots, n$.

For each $p \in \mathbb{R}^3$ define the spherical coordinates $(\psi_p, \theta_p)$ on the sphere $S^2$ of the $\mu$ variable choosing the polar axis aimed along the vector $p$. Here $\theta_p$ and
\( \psi_p \) are the azimuthal and precession angles, respectively. Then, obviously, any smooth function \( u \in L^2(\mathbb{R}^3 \times S^2) \) such that

\[
\int_{-\pi}^{\pi} u(p, \mu(\psi_p, \theta_p)) \varphi_\ell(\mu(\psi_p, \theta_p)) \, d\psi_p = 0
\]

for a. e. \( p \in \mathbb{R}^3, \theta_p \in [\frac{-\pi}{2}, \frac{\pi}{2}], \ell = 1, ..., n \), satisfies (15), and hence \( U^*u \) belongs to the subspace \( H_0 \).

**Remark 5.** The reducing subspace of the selfadjoint part of the operator constructed in the course of the proof, is, in general, a proper subspace in \( H_0 \).
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