Reflected BSDE with a Constraint and its applications in incomplete market*

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Abstract. In this paper, we study a type of reflected BSDE with a constraint and introduce a new kind of nonlinear expectation via BSDE with a constraint and prove the Doob-Meyer decomposition with respect to the super(sub)martingale introduced by this nonlinear expectation. Then we an application on the pricing of American options in incomplete market.

Keywords: Reflected backward stochastic differential equation, backward stochastic differential equation with a constraint, nonlinear expectation, nonlinear Doob-Meyer decomposition, American option in incomplete market.

1 Introduction

El Karoui, Kapoudjian, Pardoux, Peng and Quenez (1997) studied the problem of BSDE (backward stochastic differential equation) with reflecting barrier, which is, a standard BSDE with an additional continuous, increasing process in this equation to keep the solution above a certain given continuous boundary process. This increasing process must be chosen in certain minimal way, i.e. an integral condition, called Skorokhod reflecting condition (cf. [43]), is satisfied. The advantage of introducing the above Skorokhod condition is that it possesses a very interesting coercive structure which permits us to obtain many useful properties such

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as uniqueness, continuous dependence and other kind of regularities. It turns out to be a powerful tool to obtain the regularity properties of the corresponding solutions of PDE with obstacle such as free boundary PDE. Recently, this Skorokhod condition is generalized to the case where the barrier $L$ is an $L^2$-process in [38].

An important application of the constrained BSDE is the pricing of contingent claims in an incomplete market, where the portfolios of an asset is constrained in a given subset. In this case the solution $(y, z)$ of the corresponding reflected BSDE must remain in this subset. In the pricing of American options in the incomplete market, the related BSDE is a reflected BSDE with constrained portfolios. This problem was studied by Karaztas and Kou (cf. [25]). They required a condition that the constraint should be a convex subset, the coefficient of the corresponding BSDE was also assumed to be a linear, or at least a concave function. This limitation is mainly due to the duality method applied as a main approach in that paper.

The main conditions of our paper is: $g$ is a Lipschitz function and the constraint $\Gamma_t(\omega), t \in [0, T]$ is a non-empty closed set. The existence of such smallest $\Gamma$-constrained supersolution of BSDE with coefficient $g$ is obtained in [35]. An interesting point of view is that this supersolution is, in fact, the solution of the BSDE with a singular coefficient $g_{\Gamma}$ defined by

$$g_{\Gamma}(t, y, z) = g(t, y, z)1_{\Gamma_t}(y, z) + (+\infty) \cdot 1_{\Gamma^C_t}(y, z).$$

(see Remark 7.1 in appendix for details). One main result of this paper, is the existence and uniqueness of reflected BSDEs with this singular coefficient $g_{\Gamma}$ and we provide the related generalized Skorokhod reflecting condition. Since our coefficient $g$ as well as our constraint $\Gamma$ need not to be concave or convex, the results of our paper provide a wide space of freedom to treat different types of situations. Typically, in the situation of differential games, the coefficients is neither convex nor concave (see [20], [21] and [23]).

Recent developments of continuous time finance requires a nonlinear version of time consistent expectation. In 1997, the first author has introduced a Brownian filtration $(\mathcal{F}_t)_{t \geq 0}$ consistent nonlinear expectation

$$\mathcal{E}^g[X] : X \in L^2(\Omega, \mathcal{F}_T, P) \rightarrow \mathbb{R}$$
call $g$-expectation, which is defined by $y_0^X$, where $(y_t^X, z_t^X)_{0 \leq t \leq T}$ is the solution of the BSDE with a given coefficient $g(t, y, z)$ and terminal condition $X$. Here we assume $g$ satisfies Lipschitz condition in $(y, z)$ as well as $g(t, y, 0) \equiv 0$. When $g$ is a linear function in $(y, z)$, this $g$-expectation $\mathcal{E}^g[\cdot]$ is just a Girsanov transformation. But it becomes a nonlinear functional once $g$ is nonlinear in $(y, z)$, i.e., $\mathcal{E}^g[\cdot]$ is a constant preserving monotonic and nonlinear functional defined on $L^2(\Omega, \mathcal{F}_T, P)$.

Recently a profound link between super-replication, risk measures (cf. [1], [18]) nonlinear expectations have being explored (cf. [3], [12], [37]). We hope that the results of this paper will be proved to be useful in this direction. We also refer to [13], [4], [32], [5], [14], [2], [31], [24] for interesting research works in this domain.

To do researches for incomplete financial market, similarly as the above $g$-expectation, we can also define the corresponding $g_{\Gamma}$-expectation the smallest solution of BSDE with $g_{\Gamma}$ as well the corresponding $g_{\Gamma}$-supermartingales and submartingales. We shall prove a $g_{\Gamma}$-supermartingale decomposition theorem, which is a nonlinear version of Doob–Meyer
decomposition theorem. We point out that for the \( g \)-\( \Gamma \)-submartingale decomposition can not be obtained by the above mentioned \( g \)-\( \Gamma \)-supermartingale decomposition. We shall obtain this decomposition theory in a quite different way.

This paper is organized as follows. In the next section we list our main notations and main conditions required. In Section 3 we present the definition and some properties of \( g \)-\( \Gamma \)-expectation, with applications. In section 4, we prove the results and proofs of the existence and uniqueness of reflected BSDE with constraints. After introducing the definitions of \( g \)-\( \Gamma \)-martingale and \( g \)-\( \Gamma \)-super(sub)martingale, we prove the nonlinear Doob-Meyer’s type decomposition theorem corresponding to \( g \)-\( \Gamma \)-super(sub)martingale in section 5. Then we give an application of reflected BSDE with constraints: pricing of American option in incomplete market in section 6. At last some useful results are presented in appendix.

2 \( g \)-\( \Gamma \)-solution: the smallest \( g \)-supersolution of BSDE with constraint \( \Gamma \)

Let \( (\Omega, \mathcal{F}, P) \) be a probability space, and \( B = (B^1, B^2, \cdots, B^d)^T \) be a \( d \)-dimensional Brownian motion defined on \([0, \infty)\). We denote \( \mathcal{F}_t = \sigma\{\{B_s; 0 \leq s \leq t\} \cup \mathcal{N}\} \), where \( \mathcal{N} \) is the collection of all \( P \)-null sets of \( \mathcal{F} \). The Euclidean norm of an element \( x \in \mathbb{R}^m \) is denoted by \(|x|\). We also need the following notations, for \( p \in [1, \infty) \):

- \( L^p(\mathcal{F}_t; \mathbb{R}^m) := \{\mathbb{R}^m\text{-valued } \mathcal{F}_t\text{-measurable random variables } X \text{ s.t. } E[|X|^p] < \infty\}; \)
- \( L^p_{\mathcal{F}}(0, t; \mathbb{R}^m) := \{\mathbb{R}^m\text{-valued and } \mathcal{F}_t\text{-progressively measurable processes } \varphi \text{ defined on } [0, t], \text{ s.t. } E\int_0^t |\varphi_s|^p ds < \infty\}; \)
- \( D^p_{\mathcal{F}}(0, t; \mathbb{R}^m) := \{\mathbb{R}^m\text{-valued and RCLL } \mathcal{F}_t\text{-progressively measurable processes } \varphi \text{ defined on } [0, t], \text{ s.t. } E[\sup_{0 \leq s \leq t} |\varphi_s|^p] < \infty\}; \)
- \( A^p_{\mathcal{F}}(0, t) := \{\text{increasing processes } A \text{ in } D^p_{\mathcal{F}}(0, t; \mathbb{R}) \text{ with } A(0) = 0\}. \)

When \( m = 1 \), they are simplified as \( L^p(\mathcal{F}_t) \), \( L^p_{\mathcal{F}}(0, t) \) and \( D^p_{\mathcal{F}}(0, t) \), respectively. We mainly interest the case of \( p = 2 \). In this section, we consider BSDE on the interval \([0, T]\), with a fixed \( T > 0 \).

We consider a function

\[ g(\omega, t, y, z) : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \]

which always plays the role of the coefficient of our BSDE. \( g \) satisfies the following assumption: there exists a constant \( \mu > 0 \), such that, for each \( y, y' \in \mathbb{R} \) and \( z, z' \in \mathbb{R}^d \), we have

\[
\begin{align*}
(i) & \quad g(\cdot, y, z) \in L^2_{\mathcal{F}}(0, T); \\
(ii) & \quad |g(t, \omega, y, z) - g(t, \omega, y', z')| \leq \mu(|y - y'| + |z - z'|), \quad dP \times dt \text{ a.s.} \tag{1}
\end{align*}
\]
Our constraint is described by $\Gamma(t, \omega) : \Omega \times [0, T] \to C(\mathbb{R} \times \mathbb{R}^d)$, where $C(\mathbb{R} \times \mathbb{R}^d)$ is the collection of all closed non-empty subsets of $\mathbb{R} \times \mathbb{R}^d$, $\Gamma(t, \omega)$, which is $\mathcal{F}_t$-adapted, namely,

\begin{align}
(\mathrm{i}) \quad (y, z) \in \Gamma(t, \omega) \text{ iff } d_{\Gamma(t, \omega)}(y, z) = 0, \ t \in [0, T], \text{ a.s.}; \\
(\mathrm{ii}) \quad d_{\Gamma(t)}(y, z) \text{ is } \mathcal{F}_t-\text{adapted process, for each } (y, z) \in \mathbb{R} \times \mathbb{R}^d,
\end{align}

where $d_{\Gamma}(\cdot, \cdot)$ is a distant function from $(y, z)$ to $\Gamma$: for $t \in [0, T]$,

$$d_{\Gamma}(y, z) := \inf_{(y', z') \in \Gamma_t} (|y - y'|^2 + |z - z'|^2)^{1/2} + 1.$$ 

d_{\Gamma}(y, z)$ is a Lipschitz function: for each $y, y'$ in $\mathbb{R}$ and $z, z'$ in $\mathbb{R}^d$, we always have

$$|d_{\Gamma}(y, z) - d_{\Gamma}(y', z')| \leq (|y - y'|^2 + |z - z'|^2)^{1/2}.$$ 

**Remark 2.1.** The constraint discussed in [BE] is

$$\Gamma_t(\omega) = \{(y, z) \in \mathbb{R}^{1+d} : \Phi(\omega, t, y, z) = 0\}. \tag{3}$$

Here $\Phi(\omega, t, y, z) : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \to [0, \infty)$ is a given nonnegative measurable function, and satisfies integrability condition and Lipschitz condition. In this paper we always consider the case

$$\Phi(t, y, z) = d_{\Gamma}(y, z).$$

We are then within the framework of super(sub)solution of BSDE of the following type:

**Definition 2.1.** ($g$-super(sub)solution, cf. El Karoui, Peng and Quenez (1997) [IC] and Peng (1999) [BE]) A process $y \in D^2_{\mathcal{F}}(0, T)$ is called a $g$–supersolution (resp. $g$–subsolution), if there exist a predictable process $z \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^d)$ and an increasing RCLL process $A \in A^2_{\mathcal{F}}(0, T)$ (resp. $K \in A^2_{\mathcal{F}}(0, T)$), such that $t \in [0, T]$,

\begin{align}
y_t &= y_T + \int_t^T g(s, y_s, z_s)ds + A_T - A_t - \int_t^T z_s dB_s, \tag{4} \\
&\quad \text{(resp. } y_t = y_T + \int_t^T g(s, y_s, z_s)ds - (K_T - K_t) - \int_t^T z_s dB_s.)
\end{align}

Here $z$ and $A$ (resp. $K$) are called the martingale part and increasing part, respectively. $y$ is called a $g$–solution if $A_t = K_t = 0$, for $t \in [0, T]$. $y$ is called a $\Gamma$–constrained $g$–supersolution if $y$ and its corresponding martingale part $z$ satisfy

$$(y_t, z_t) \in \Gamma_t, \text{ (or } d_{\Gamma}(y_t, z_t) = 0), \text{ } dP \times dt \text{ a.s. in } \Omega \times [0, T], \tag{5}$$

**Remark 2.2.** We observe that, if $y \in D^2_{\mathcal{F}}(0, T)$ is a $g$–supersolution or $g$–subsolution, then the pair $(z, A)$ in (4) are uniquely determined since the martingale part $z$ is uniquely determined. Occasionally, we also call $(y, z, A)$ a $g$–supersolution or $g$–subsolution.

By [BE], (see Appendix Theorem ??), if there exists at least one $\Gamma$–constrained $g$–supersolution, then the smallest $\Gamma$–constrained $g$–supersolution exists. In fact, a $\Gamma$-constraint $g$-supersolution can be considered as a solution of the BSDE with a singular coefficient $g_\Gamma$ defined by

$$g_\Gamma(t, y, z) = g(t, y, z)1_{\Gamma_t}(y, z) + (+\infty) \cdot 1_{\Gamma_t^c}(y, z).$$

So we define the smallest $\Gamma$–constrained $g$–supersolution by $g_\Gamma$–solution.
Definition 2.2. \((g_{\Gamma}-\text{solution})\) \(y\) or \((y_t, z_t, A_t)_{0 \leq t \leq T}\) is called \(g_{\Gamma}\)-solution on \([0, T]\) with a given terminal condition \(X\) if it is the smallest \(\Gamma\)-constrained \(g\)-supersolution with \(y_T = X\):

\[
y_t = X + \int_t^T g(s, y_s, z_s) ds + A_T - A_t - \int_t^T z_s dB_s,
\]

\[
d_{\Gamma}(y_t, z_t) = 0, \quad dP \times dt \text{ a.s. in } \Omega \times [0, T], \quad dA_t \geq 0, \quad t \in [0, T].
\]

In other words, if there exists another triple \((y', z', A')\) satisfying (6), then \(y'_t \geq y_t\), for \(t \in [0, T]\).

Remark 2.3. The above definition does not imply that the increasing process \(A\) is also the smallest one, i.e. for another triple \((\bar{y}, \bar{z}, \bar{A})\) satisfying (6), we may have \(A_t \geq \bar{A}_t\).

An example is as following.

Example 2.1. Consider the case when \([0, T] = [0, 2], X = 0, g = 0\) and \(\Gamma_t = \{(y, z) : y \geq 1_{[0,1]}(t)\}\). So the \(g_{\Gamma}\)-solution of this equation is the solution of reflected BSDE with lower barrier \(1_{[0,1]}(t)\). It’s easy to see that the smallest solution is \(y_t = 1_{[0,1]}(t)\) with \(z_t = 0\), \(A_t = 1_{[1,2]}(t)\). Obviously \(\bar{y}_t = 1_{[0,2]}(t)\) with \(\bar{z}_t = 0\), \(\bar{A}_t = 1_{\{t=2\}}(t)\) is another \(\Gamma\)-constrained \(g\)-supersolution with the same terminal condition \(y'_T = 0\). However we have \(A_t > \bar{A}_t\) on the interval \([1, 2]\).

3 Nonlinear Expectation: \(g_{\Gamma}\)-expectation and its properties

In this section we first introduce a new type of \(\mathcal{F}\)-consistent nonlinear expectations via \(g_{\Gamma}\)-solutions, then we study the properties of this nonlinear expectations. At last an application for risk measure in the incomplete market is concerned. We assume: there exists a large enough constant \(C_0\) such that for \(\forall y \geq C_0\)

\[
g(t, y, 0) \leq C_0 + \mu |y|, \quad \text{and} \quad (y, 0) \in \Gamma_t;
\]

and the terminal conditions to be in the following linear subspace of \(L^2(\mathcal{F}_T)\):

\[
L^2_{+, \infty}(\mathcal{F}_T) := \{\xi \in L^2(\mathcal{F}_T), \xi^+ \in L^\infty(\mathcal{F}_T)\}.
\]

Proposition 3.1. We assume (1), (2) and (7) hold. Then for each \(X \in L^2_{+, \infty}(\mathcal{F}_T)\), the \(g_{\Gamma}\)-solution with terminal condition \(y_T = X\) exists. Furthermore, we have \(y_t \in L^2_{+, \infty}(\mathcal{F}_t)\), for \(t \in [0, T]\).

Proof. We consider

\[
y_0(t) = (\|X^+\|_\infty \vee C_0)e^{\mu(T-t)} + C_0(T-t) + (X - \|X^+\|_\infty \vee C_0)1_{\{t=T\}}.
\]

It is the solution of the following backward equation:

\[
y_0(t) = X + \int_t^T (C_0 + \mu |y_0(s)|) ds + A^0(T) - A^0(t),
\]
where $A^0$ is an increasing process: $A^0(t) := (\|X^+\|_{\infty} \vee C_0 - X) 1_{t=T}$. Meanwhile $y_0(\cdot)$ can be expressed as:

$$y_0(t) = X + \int_t^T g(s, y_0(s), 0)ds + \int_t^T [c + \mu |y_0(s)| - g(s, y_0(s), 0)]ds + A^0(T) - A^0(t).$$

Thus the triple defined on $[0, T]$ by

$$(y_1(t), z_1(t), A_1(t)) := (y_0(t), 0, \int_0^t [c + \mu |y_0(s)| - g(s, y_0(s), 0)]ds + A^0(t))$$

is a $\Gamma$–constrained $g$–supersolution with $y_1(T) = X$. According to Theorem 7.1 in appendix, the $g_T$–solution with $y(T) = X$ exists. We also have $(y_t)^+ \in L^\infty(F_T)$ since $y_t \leq y_1(t) = y_0(t)$. □

We now introduce the notion of $g_T$–expectation:

**Definition 3.1.** We assume that for each $0 \leq t \leq T < \infty$, $g(t, 0, 0) = 0$ and $(0, 0) \in \Gamma_t$, assumptions (1), (2) and (7) hold. Then consider $X \in L^2_{\infty, \infty}(F_T)$, let $(y, z, A)$ be the $g_T$–solution defined on $[0, T]$ with terminal condition $y_T = X$. We define $E_{t,T}^g[X] := y_t$. The system

$$E_{t,T}^g[\cdot] : L^2_{\infty, \infty}(F_T) \rightarrow L^2_{\infty, \infty}(F_t), \quad 0 \leq t \leq T < \infty$$

is called $g_T$–expectation.

**Remark 3.1.** Under assumptions (1), (2) and (7), proposition 3.1 guarantees the existence of $g_T$–expectation.

We have

**Proposition 3.2.** A $g_T$–expectation is an $F$–consistent expectation, i.e., it satisfies the followings: for each $0 \leq t \leq T < \infty$ and $X, X' \in L^2_{\infty, \infty}(F_T)$,

(A1) Monotonic property: $E_{t,T}^g[X] \leq E_{t,T}^g[X']$, if $X \leq X'$;

(A2) Self-preserving: $E_{t,T}^g[X] = X$;

(A3) Time consistency: $E_{s,t}^g[E_{t,T}^g[X]] = E_{s,T}^g[X], \quad 0 \leq s \leq t \leq T$;

(A4) 1-0 law: $1_D E_{t,T}^g[X] = E_{t,T}^g[1_D X], \quad \forall D \in F_t$.

**Proof.** (A1) is a direct consequence of the comparison theorem 7.2 of the $g_T$–solution. (A2) is obvious. For (A3), it is easy the check that, if $(y_s)_{0 \leq s \leq T}$ is the $g_T$–solution on $[0, T]$ with $y_T = X$, then $(y_s)_{0 \leq s \leq t}$ is also the $g_T$–solution on $[0, t]$ with the fixed terminal condition $y_t$.

To prove (A4), we multiply $1_D$ to two sides of the equation, for $t \leq s \leq T$, since $g(s, 0, 0) \equiv 0$, and $d_{\Gamma_s}(0, 0) \equiv 0$, we have

$$1_D y_s = 1_D X + \int_s^T g(r, 1_D y_r, 1_D z_r)dr + 1_D A_T - 1_D A_s - \int_s^T 1_D z_r dB_r, \quad d_{\Gamma_s}(1_D y_s, 1_D z_s) \equiv 0.$$

Thus it is obvious that $(1_D y_s, 1_D z_s)_{0 \leq s \leq T}$ must be the $g_T$–solution on $[s, T]$ with $y_T 1_D$ as the terminal condition, which implies (A4). □

Moreover, by the comparison theorem for $g_T$–solution, we have
Proposition 3.3. Under assumptions (1), (2) and (7), for each \( t \leq T < \infty \) and \( X \in \mathbf{L}^2_{+, \infty}(\mathcal{F}_T) \), if \( \Gamma^1_t \geq \Gamma^2_t \) and \( g^1(t, y, z) \leq g^2(t, y, z) \),

then \( \mathcal{E}^{g^1}_{t, T}[X] \leq \mathcal{E}^{g^2}_{t, T}[X] \).

Now we study some properties of \( g_\Gamma \)-expectation associated with dynamic risk measure, such as constant preserving property, positive homogenous property, convex property, sub-linear property, constant translation invariant property and subadditive property. And in the following of this section, we always assume that assumptions (1) and (2) hold.

Proposition 3.4 (positive homogenous and convexity). If \( g(t, y, 0) = 0 \), and \( \mathbb{R} \times \{0\} \subset \Gamma_t \), \( t \in [0, T] \), then \( g_\Gamma \)-expectation is conditional constant preserving,

\[
\mathcal{E}^{g_\Gamma}_{t, T}[X] = X, \quad \text{for } X \in \mathbf{L}^2_{+, \infty}(\mathcal{F}_t).
\]

Specially, for \( C \in \mathbb{R} \), \( \mathcal{E}^{g_\Gamma}_{t, T}[C] = C \).

Proof. For \( X \in \mathbf{L}^1_{+, \infty}(\mathcal{F}_t) \), it is easy to check that \( (y_t, z_t, A_t) \equiv (X, 0, 0) \) is the \( g_\Gamma \)-solution of constraint BSDE associated to \( (X, g, \Gamma) \), in view of \( g(t, y, 0) = 0 \), and \( \mathbb{R} \times \{0\} \subset \Gamma_t \), \( t \in [0, T] \). So the result follows.

Proposition 3.5. Set \( g(t, 0, 0) = 0 \) and \( (0, 0) \in \Gamma_t \) hold for each \( 0 \leq t \leq T < \infty \),

(i) under assumption (7), the nonlinear \( \mathcal{F} \)-consistent expectation, \( g_\Gamma \)-expectation is positive homogenous, i.e.

\[
\mathcal{E}^{g_\Gamma}_{t, T}[cX] = c\mathcal{E}^{g_\Gamma}_{t, T}[X], \quad \text{for } c > 0, X \in \mathbf{L}^2_{+, \infty}(\mathcal{F}_T),
\]

if \( g \) is positive homogenous in \( (y, z) \) and \( \Gamma_t \) is a cone for \( t \in [0, T] \), i.e. if \( (y, z) \in \Gamma_t \), then for \( c > 0 \), \( (cy, cz) \in \Gamma_t \);

(ii) under assumption (7), if \( g \) and \( \Gamma \) are convex in \( (y, z) \), then \( g_\Gamma \)-expectation is convex,

\[
\mathcal{E}^{g_\Gamma}_{t, T}[\alpha X_1 + (1 - \alpha) X_2] \leq \alpha \mathcal{E}^{g_\Gamma}_{t, T}[X_1] + (1 - \alpha) \mathcal{E}^{g_\Gamma}_{t, T}[X_2], \quad \text{for } \alpha \in [0, 1], X_1, X_2 \in \mathbf{L}^2_{+, \infty}(\mathcal{F}_T).
\]

Proof. (i) It is easy to see that \( cX \in \mathbf{L}^2_{+, \infty}(\mathcal{F}_T) \), with \( c > 0 \), if and only if \( X \in \mathbf{L}^2_{+, \infty}(\mathcal{F}_T) \). Let \( (y, z, A) \) be the \( g_\Gamma \)-solution defined on \([t, T]\) with terminal condition \( y_T = X \), i.e. for \( t \leq s \leq T \),

\[
\begin{align*}
y_s &= X + \int_s^T g(r, y_r, z_r)dr + A_T - A_s - \int_s^T z_r dB_r, \\
d_{\Gamma_s}(y_s, z_s) &= 0, \quad \text{a.s., a.e.}
\end{align*}
\]

Since \( g \) is homogeneous and \( \Gamma \) is a cone, we have, for \( c > 0 \), \((cy, cz) \in \Gamma_s \), a.s.a.e. and

\[
\begin{align*}
cy_s &= cX + c \int_s^T g(r, y_r, z_r)dr + cA_T - cA_s - c \int_s^T z_r dB_r \\
&= cX + \int_s^T g(r, cy_r, cz_r)dr + cA_T - cA_s - c \int_s^T z_r dB_r,
\end{align*}
\]
It is obvious that \((cy, cz, cA)\) is the \(g_t\)-solution with terminal condition \(cX\), i.e. \(\mathcal{E}_{i,t}^{gr}[cX] = cy_t = c\mathcal{E}_{i,t}^{gr}[X]\).

(ii) Since \(X_1, X_2 \in L^2_{t,\infty}(\mathcal{F}_T)\), and \(\alpha \in [0, 1]\), so \(\alpha X_1 + (1-\alpha)X_2 \in L^2_{t,\infty}(\mathcal{F}_T)\). We denote \(\mathcal{E}_{i,t}^{gr}[\alpha X_1 + (1-\alpha)X_2] = y_t\), which is the \(g_t\)-solution of BSDE\((g, \Gamma)\) on \([t, T]\), with terminal condition \(\alpha X_1 + (1-\alpha)X_2\), i.e. for \(t \leq s \leq T\),

\[
y_s = \alpha X_1 + (1-\alpha)X_2 + \int_s^T g(r, y_r, z_r)dr + A_T - A_s - \int_s^T z_r dB_r,
\]

\[
d_{\Gamma_s}(y_s, z_s) = 0, \text{ a.s.e.}
\]

Set \(\mathcal{E}_{i,t}^{gr}[X_1] = y^1_t\) and \(\mathcal{E}_{i,t}^{gr}[X_2] = y^2_t\), where for \(i = 1, 2\), \((y^i, z^i, A^i)\) is the \(g_t\)-solution of BSDE with terminal value \(X^i\), associated to \((g, \Gamma)\), i.e.

\[
y^i_s = X^i + \int_s^T g(r, y^i_r, z^i_r)dr + A^i_T - A^i_s - \int_s^T z^i_r dB_r, \quad d_{\Gamma_s}(y^i_t, z^i_t) = 0, \text{ a.s.e.}
\]

Then we know that the convex combination \((\alpha y^1 + (1-\alpha)y^2, \alpha z^1 + (1-\alpha)z^2, \alpha A^1 + (1-\alpha)A^2)\) is a \(g\)-supersolution of BSDE with terminal value \(\alpha X_1 + (1-\alpha)X_2\) and coefficient \(\tilde{g}\), where

\[
\tilde{g}(s, y, z) = \alpha g(r, y^1_r, z^1_r) + (1-\alpha)g(s, \frac{1}{1-\alpha}(y - \alpha y^1_s), \frac{1}{1-\alpha}(z - \alpha z^1_s)).
\]

Moreover Since \(\Gamma_s\) is convex for \(s \in [t, T]\), \((\alpha y^1 + (1-\alpha)y^2, \alpha z^1 + (1-\alpha)z^2) \in \Gamma_s\), a.s. a.e.

Notice that \(g\) is a convex function, we have

\[
\tilde{g}(s, y_s, z_s) = \alpha g(s, y^1_s, z^1_s) + (1-\alpha)g(s, \frac{1}{1-\alpha}(y_s - \alpha y^1_s), \frac{1}{1-\alpha}(z_s - \alpha z^1_s))
\]

\[
\geq g(s, y_s, z_s).
\]

By comparison theorem, and remember that \(y_t\) is the \(g_t\)-solution, then

\[
\mathcal{E}_{i,t}^{gr}[\alpha X_1 + (1-\alpha)X_2] = y_t \leq \alpha y^1_t + (1-\alpha)y^2_t = \alpha \mathcal{E}_{i,t}^{gr}[X_1] + (1-\alpha)\mathcal{E}_{i,t}^{gr}[X_2].
\]

\(\square\)

**Corollary 3.1.** [Sublinear] Let \(g(t, 0, 0) = 0\) and \((0, 0) \in \Gamma_t\) hold, for each \(0 \leq t \leq T < \infty\).

If \(g\) is sublinear in \((y, z)\), i.e. \(g\) is homogenous and subadditive in \((y, z)\), which implies for \(c > 0\), \((y, z)\) and \((y', z')\) in \(\mathbb{R}^{1+d}\),

\[
g(t, cy, cz) = cg(t, y, z) \text{ and } g(t, y + y', z + z') \leq g(t, y, z) + g(t, y', z'),
\]

and \(\Gamma_t\) is a convex cone for \(t \in [0, T]\), then \(g_t\)-expectation is sublinear.

**Proof.** Since sublinearity is equivalent to convexity plus positive homogeneity, the thesis follows from Proposition 3.5 \(\square\)

**Proposition 3.6** (constant translation invariant). For each \(0 \leq t \leq T < \infty\),

(i) if \(g\) and \(\Gamma\) only depend on \(z\), \(g(t, z)\) is bounded and \(0 \in \Gamma_t\), then \(g_t\)-expectation is translation invariant,

\[
\mathcal{E}_{i,t}^{gr}[X + \eta] = \mathcal{E}_{i,t}^{gr}[X] + \eta, \text{ for } \eta \in L^2_{t,\infty}(\mathcal{F}_t), X \in L^2_{t,\infty}(\mathcal{F}_T);
\]
(ii) if \( g(t, y, z) = g_1(t, z) + ay \) with \( g_1(t, z) \) is bounded and \( \Gamma \) only depends on \( z \), with \( 0 \in \Gamma_t \), then \( g_t \)-expectation is constant invariant with discount factor \( e^{a(T-t)} \),

\[
\mathcal{E}^g_{t,T}[X + \eta] = \mathcal{E}^g_{t,T}[X] + \eta e^{a(T-t)}, \quad \text{for } \eta \in \mathbf{L}^2_{+,\infty}(\mathcal{F}_t), X \in \mathbf{L}^2_{+,\infty}(\mathcal{F}_T).
\]

Proof. Obviously (7) is satisfied under the assumption (i) and (ii).

(i) Obviously \( X + \eta \in \mathbf{L}^2_{+,\infty}(\mathcal{F}_T) \). By the definition of \( g_t \)-expectation, we know that 

\[
\mathcal{E}^g_{t,T}[X] := y_t, \quad \text{where } (y, z, A) \text{ is the } g_t \text{-solution of constraint BSDE}(X, g, \Gamma) \text{ on } [t, T].
\]

So for \( s \in [t, T] \),

\[
y_s + \eta = X + \eta + \int_s^T g(r, z_r) dr + A_T - A_s - \int_s^T z_r dB_r,
\]

\[
d_{\Gamma_s}(z_s) = 0, \quad \text{a.s.a.e.}
\]

It follows that \( \mathcal{E}^g_{t,T}[X + \eta] = y_t + \eta = \mathcal{E}^g_{t,T}[X] + \eta \).

(ii) By the definition of \( g_t \)-expectation, we know that 

\[
\mathcal{E}^g_{t,T}[X] := y_t, \quad \text{where } (y, z, A) \text{ is the } g_t \text{-solution on } [t, T] \text{ with terminal condition } y_T = X.
\]

Since \( \int_s^T \eta e^{a(T-r)} dr = \eta \int_s^T d(-e^{a(T-r)}) = -\eta + \eta e^{a(T-s)} \), we get

\[
y_s + \eta e^{a(T-s)} = X + \eta + \int_s^T [g_1(r, z_r) + a(y_r + \eta e^{a(T-r)})] dr + A_T - A_s - \int_s^T z_r dB_r
\]

\[
= X + \eta + \int_s^T g(r, y_r + \eta e^{a(T-r)}, z_r) dr + A_T - A_s - \int_s^T z_r dB_r.
\]

Notice that we still have \( d_{\Gamma_s}(z_s) = 0 \), a.s.a.e. And it is easy to check that \( (y, z, A) \) is the 

\( g_t \)-solution. Then \( y_s + \eta e^{a(T-s)} \) is the \( g_t \)-solution of constraint BSDE\((X + \eta, g, \Gamma)\), i.e.

\[
\mathcal{E}^g_{t,T}[X + \eta] = y_t + \eta e^{a(T-t)} = \mathcal{E}^g_{t,T}[X] + \eta e^{a(T-t)}.
\]

\[
\square
\]

As we know from Rosazza [12], we can use \( g \)-expectation to describe risk measure dynamically. However in incomplete market, since portfolio is constraint, risk of a financial position must increase. This indicates us to use our \( g_t \)-expectation to study dynamic risk measure in incomplete market.

Example 3.1 (Risk measure with no-shortselling constraint). Set \( \Gamma \) only depends on \( z \), with \( \Gamma_t = \mathbb{R}^d_+ \), and \( g \) is Lipschitz in \((y, z)\), then for a financial position \( X \in \mathbf{L}^2_{+,\infty}(\mathcal{F}_T) \) define a dynamic risk measure:

\[
\rho_t(X) = \mathcal{E}^g_{t,T}[-X].
\]

Thanks to Proposition 3.3, 3.6 and Corollary 3.7 we have

- \( \rho_t(\cdot) \) is a dynamic convex time-consistent risk measure, if \( g \) is convex in \((y, z)\).
- \( \rho_t(\cdot) \) is a dynamic coherent time-consistent risk measure, if \( g \) only depends on \( z \) and is sublinear in \( z \).
- \( \rho_t(\cdot) \) is a dynamic sublinear time-consistent risk measure, if \( g \) is sublinear in \((y, z)\).
If we define another dynamic risk measure $\bar{\rho}_t$, for a financial position $X \in L^2_{+,\infty}(\mathcal{F}_T)$, by

$$\bar{\rho}_t(X) = \mathcal{E}_g[-X|\mathcal{F}_t].$$

Here $\mathcal{E}_g[\cdot|\mathcal{F}_t]$ is a $g$-expectation, (cf. [37]). By comparison theorem for BSDE, we can easily get

$$\rho_t(X) \geq \bar{\rho}_t(X),$$

which implies that in the market with no-shortselling constraint, for same financial position, we need more money to cover its risk.

4 $g_\Gamma$–reflected BSDEs

Before we go further to study more properties of $g_\Gamma$-expectation, we change our attentions to $g_\Gamma$–reflected BSDEs, which will play important roles in further research.

4.1 Existence of $g_\Gamma$–reflected BSDEs

In this section we consider the smallest $g$–supersolution with constraint $\Gamma$ and a lower (resp. upper) reflecting obstacle $L$ (resp. $U$). We assume that the two reflected obstacles $L$ and $U$ are $\mathcal{F}_t$-adapted processes satisfying

$$L, U \in L^2_{\mathcal{F}}(0, T) \quad \text{and} \quad \text{ess sup}_{0 \leq t \leq T} L_t^+, \text{ess sup}_{0 \leq t \leq T} U_t^- \in L^2(\mathcal{F}_T). \quad (9)$$

Here we focus on the constraint $\Gamma$ which does not depend on $y$, only depends on $z$, i.e. $\Gamma(t, \omega) : \Omega \times [0, T] \to C(\mathbb{R}^d)$, where $C(\mathbb{R}^d)$ is the collection of all closed non-empty subsets of $\mathbb{R}^d$ and $\Gamma(t, \omega)$ is $\mathcal{F}_t$–adapted. In fact, this condition of $\Gamma$ is not an essential difficulty in following proofs in this section. We can easily generalize the results to the case when also depends on $y$.

First let us introduce the definition of $g_\Gamma$–reflected solutions:

**Definition 4.1.** A $g_\Gamma$–reflected solution with a lower obstacle $L$ is a quadruple of processes $(y, z, A, \bar{A})$ satisfying

**(i)** $(y, z, A, \bar{A}) \in D^2_{\mathcal{F}}(0, T) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^d) \times (A^2_{\mathcal{F}}(0, T))^2$ verifies

$$y_t = X + \int_t^T g(s, y_s, z_s)ds + A_T - A_t + \bar{A}_T - \bar{A}_t - \int_t^T z_sdB_s, \quad (10)$$

$$d_{\Gamma_t}(z_t) = 0, \quad dP \times dt \text{ a.s.}$$

**(ii)** $y_t \geq L_t$ and the generalized Skorokhod reflecting condition is satisfied: for each $L^* \in D^2_{\mathcal{F}}(0, T)$ such that $y_t \geq L^*_t \geq L_t$, $dP \times dt$-a.s., we have

$$\int_0^T (y_{s-} - L^*_{s-})d\bar{A}_s = 0, \quad a.s., \quad (11)$$

**(iii)** $y$ is the smallest one, i.e., for any quadruple $(y^*, z^*, A^*, \bar{A}^*)$ satisfying (i) and (ii), we have

$$y_t \leq y^*_t, \quad \forall t \in [0, T], \text{ a. s.}$$
Here we use two increasing processes $A$, $\bar{A}$ to push $y$ in order to keep the solution $(y, z)$ staying in constraint $\Gamma$ and upper the barrier $L$ respectively. More precisely, the role of $A$ is to keep the process $z$ staying in the given constraint $\Gamma$, while $\bar{A}$ acts only when $y$ tends to cross downwards the barrier $L$.

Our first main result in this section is:

**Theorem 4.1.** Suppose (1), (2) and (9) hold. For a given terminal condition $X \in L^2(\mathcal{F}_T)$, we assume that there exists a triple $(y^*, z^*, A^*) \in D^2_F(0, T) \times L^2_F(0, T) \times A^2_F(0, T)$, such that $dA^* \geq 0$ and following hold

$$y^*_t = X + \int_t^T g(s, y^*_s, z^*_s)ds + (A^*_T - A^*_t) - \int_t^T z^*_s dB_s,$$

$$y^*_t, z^*_t \in \Gamma_t \cap \{(L_t, \infty) \times \mathbb{R}^d\}, \quad dP \times dt\text{-a.s.}$$

Then there exists the $g_T$–reflected solution $(y, z, A, \bar{A})$ with the barrier $L$ of Definition 4.1.

**Remark 4.1.** This theorem can be generalized to the case when $\Gamma$ also depends on yeasly.

The smallest $g_T$–reflected solution with a upper obstacle $U$ is relatively more complicated than the case of the lower obstacle.

**Definition 4.2.** The $g_T$–reflected solution with an upper obstacle $U$ is a quadruple of processes $(y, z, A, K)$ satisfying

(i) $(y, z, A, K) \in D^2_F(0, T) \times L^2_F(0, T; \mathbb{R}^d) \times (A^2_F(0, T))^2$ with $dA \geq 0$ and $dK \geq 0$ verifies

$$y_t = X + \int_t^T g(s, y_s, z_s)ds + A_T - A_t - (K_T - K_t) - \int_t^T z_s dB_s,$$

$$d_{\Gamma_t}(z_t) = 0, \quad dP \times dt\text{-a.s.} \quad \mathcal{V}_{[0,T]}[A - K] = \mathcal{V}_{[0,T]}[A + K],$$

where $\mathcal{V}_{[0,T]}(\varphi)$ denotes the total variation of a process $\varphi$ on $[0, T]$.

(ii) $y_t \leq U, \quad dP \times dt\text{-a.s.}$, the generalized Skorohod reflecting condition is satisfied:

$$\int_0^T (U^*_t - y_t) dK_t = 0, \quad a.s., \text{ for any } U^* \in D^2_F(0, T), \text{ s.t. } y_t \geq U^*_t \geq U_t, \quad dP \times dt\text{-a.s.}$$

(iii) For any other quadruple $(y^*, z^*, A^*, K^*)$ satisfying (i) and (ii), we have

$$y_t \leq y^*_t, \quad 0 \leq t \leq T, \quad a.s.$$

Like increasing processes of the solution of $g_T$–reflecting solution with one lower barrier, here increasing processes $A$ and $K$ function separately. The role of $dA$ is to keep $z$ staying in the domain $\Gamma_t$, and $dK$ increases only when process $y_t$ tends to cross upwards the upper barrier $U$.

**Remark 4.2.** The formula $\mathcal{V}_{[0,T]}[A - K] = \mathcal{V}_{[0,T]}[A + K]$ in (13) implies that they never act at same time. This helps us to separate two increasing processes completely. And the proof of theorem 4.2 in subsection 3.3 shows that $A$ and $K$ are just the limit of the corresponding terms in penalization equations.
Then we have the existence of the $g_t$–reflected solution with an upper obstacle $U$:

**Theorem 4.2.** Assume that (1) holds for $g$ and (2) holds for the constraint $\Gamma$, $U$ is a $\mathcal{F}_t$-adapted RCLL process satisfying (2). For a given terminal condition $X \in L^2(\mathcal{F}_T)$, the $g_t$–reflected solution $(y, z, A, K)$ with upper obstacle $U$ of Definition 4.2 (i)-(iii) exists.

**Remark 4.3.** For general case when $\Gamma$ depends on $y$, satisfying (2), theorem 4.2 holds under the assumption of the existence of a special solution, i.e. there exists a quadruple $(y^*, z^*, A^*, K^*) \in D_{F}(0, T) \times L^2_{F}(0, T; \mathbb{R}^d) \times (A^2_{F}(0, T))^2$, s.t. $dA^*_t \geq 0$, $dK^*_t \geq 0$ and

$$y^*_t = X + \int_t^T g(s, y^*_s, z^*_s)ds + (A^*_T - A^*_t) - (K^*_T - K^*_t) - \int_t^T z^*_sdB_s,$$

$$d\Gamma_t(y^*_t, z^*_t) = 0, \text{ a.s. a.e.}$$

$$y^*_0 \leq U_t, \quad \int_0^T (y^*_t - U_t^-)dK^*_t = 0, \text{ a.s..}$$

This assumption is not easy to verify for general case. While if $\Gamma_t = [L_t, +\infty)$, it turns out to be a reflected BSDE with two barriers $L$ and $U$, then refer to [33], we know that assumption (14) can be changed to another sufficient condition: there exists a semimartingale $X$, such that $L \leq X \leq U$, $P$-a.s. a.e., which guarantee the existence of a special solution.

The proofs of Theorem 4.1 and Theorem 4.2 are given in the following subsections.

### 4.2 Existence of $g_t$-reflected BSDE with a lower barrier: Proof of Theorem 4.1

We prove theorem 4.1 by an approximation procedure. For $m, n \in \mathbb{N}$, we consider the penalization equations,

$$y_{t}^{m,n} = X + \int_t^T g(s, y_{s}^{m,n}, z_{s}^{m,n})ds + m \int_t^T d\Gamma_s(y_{s}^{m,n}, z_{s}^{m,n})ds,$$

$$\quad + n \int_t^T (L_s - y_{s}^{m,n})^+ ds - \int_t^T z_{s}^{m,n}dB_s.$$  

Define $A_{t}^{m,n} = m \int_0^t d\Gamma_s(y_{s}^{m,n}, z_{s}^{m,n})ds$ and $\bar{A}_{t}^{m,n} = n \int_0^t (L_s - y_{s}^{m,n})^+ ds$. We have the following estimate.

**Lemma 4.1.** There exists a constant $C \in \mathbb{R}$ independent of $m$ and $n$, such that

$$E[\sup_{0 \leq s \leq T} (y_{s}^{m,n})^2] + E \int_0^T |z_{s}^{m,n}|^2 ds + E[\int_0^T (A_{T}^{m,n} + \bar{A}_{T}^{m,n})^2] \leq C.$$  

**Proof.** Set $m = n = 0$, then we get a classical BSDE

$$y_{t}^{0,0} = X + \int_t^T g(s, y_{s}^{0,0}, z_{s}^{0,0})ds - \int_t^T z_{s}^{0,0}dB_s.$$
For \((y^*, z^*, A^*)\) given in (12), we have \(d_T(s, y^*_s, z^*_s) = 0\) and \((L_s - y^*_s)^+ = 0\), thus

\[
  y^*_t = X + \int_t^T g(s, y^*_s, z^*_s) ds + m \int_t^T d_T(s, y^*_s, z^*_s) ds + n \int_t^T (L_s - y^*_s)^+ ds \\
  + (A^*_T - A^*_t) - \int_t^T z^*_s dB_s,
\]

By comparison theorem, it follows \(y^*_t \geq y^{m,n}_t \geq y^{0,0}_t\), \(0 \leq t \leq T\). So we have for some constant \(C\) independent of \(m\) and \(n\),

\[
  E[\sup_{0 \leq t \leq T} (y^{m,n}_t)^2] \leq \max\{E[\sup_{0 \leq t \leq T} (y^*_t)^2], E[\sup_{0 \leq t \leq T} (y^{0,0}_t)^2]\} \leq C. \quad (17)
\]

Then applying Itô’s formula to \(|y^{m,n}_t|^2\) and taking expectation, we get

\[
  E[|y^{m,n}_t|^2] + E[\int_t^T |z^{m,n}_s|^2 ds] \\
  \leq E[X^2] + E[\int_t^T g^2(s, 0, 0) ds] + (2\mu + \mu^2) E[\int_t^T |y^{m,n}_s|^2 ds] + \frac{1}{2} E[\int_t^T |z^{m,n}_s|^2 ds] \\
  + 13C E[\sup_{0 \leq t \leq T} (y^{m,n}_t)^2] + \alpha E[(A^{m,n}_T - A^{m,n}_t + \bar{A}^{m,n}_T - \bar{A}^{m,n}_t)^2],
\]

where \(\alpha \in \mathbb{R}\) to be chosen later. Since \(A^{m,n}_t\) and \(\bar{A}^{m,n}_t\) are increasing processes, so

\[
  E\int_0^T |z^{m,n}_s|^2 ds \leq C + \alpha E[(A^{m,n}_T + \bar{A}^{m,n}_T)^2]. \quad (18)
\]

While rewrite (15) in the following form

\[
  A^{m,n}_T + \bar{A}^{m,n}_T = y^{0,n}_0 - X - \int_0^T g(s, y^{m,n}_s, z^{m,n}_s) ds + \int_0^T z^{m,n}_s dB_s,
\]

then take square and expectation on both sides, we get

\[
  E[(A^{m,n}_T + \bar{A}^{m,n}_T)^2] \leq 4E|(y^{0,n}_0)^2| + 4E[X^2] + 16TE\int_0^T g^2(s, 0, 0) ds \\
  + 16\mu^2TE\int_0^T |y^{m,n}_s|^2 ds + (16\mu^2T + 4)E\int_0^T |z^{m,n}_s|^2 ds,
\]

we then have

\[
  E[(A^{m,n}_T + \bar{A}^{m,n}_T)^2] \leq C + (16\mu^2T + 4)E\int_0^T |z^{m,n}_s|^2 ds. \quad (19)
\]

Compare (18) and (19), set \(\alpha = \frac{1}{32\mu^2T + 8}\), we deduce (16). \(\square\)

**Proof of Theorem 4.1.** In (15), we fix \(m \in \mathbb{N}\), and set

\[
  g^m(t, y, z) := (g + md_T)(t, y, z).
\]
This is a Lipschitz function. It follows from theorem 4.1 in [38] that, as \( n \to \infty \), with \( 16 \) the triple \((y^{m,n}, z^{m,n}, A^{m,n})\) converges to \((y^m, z^m, \overline{A}^m)\) in \( D^2_F(0, T) \times L^2_F(0, T) \times A^2_F(0, T) \), which is the solution of the following reflected BSDE whose coefficient is \( g^m \):

\[
y_t^m = X + \int_t^T (g + m d_{\tau_n}) (s, y^m_s, z^m_s) ds + \overline{A}^m_T - A^m_t - \int_t^T z^m_s dB_s,
\]

\[
y_t^m \geq L_t, \quad \int_0^T (y_t^m - L^*_t) d\overline{A}^m_t = 0,
\]

for each \( L^* \in D^2_F(0, T) \), such that \( y \geq L^* \geq L, dP \times dt \) a.s..

We denote \( A^m_t = m \int_0^t d\tau_n(z^m_s) ds \). By \( 16 \) we have the following estimate:

\[
E[\sup_{0 \leq t \leq T} (y^m_t)^2] + E \int_0^T |z^m_s|^2 ds + E[(A^m_T + \overline{A}^m_T)^2] \leq C.
\]

Then by comparison theorem \( 7, 3 \) for reflected BSDEs, we have \( y^m_t \leq y^{m+1}_t, \overline{A}^m_t \geq \overline{A}^{m+1}_t \) and \( d\overline{A}^m_t \geq d\overline{A}^{m+1}_t \) on \([0, T] \). Thus, when \( m \to \infty \), \( y^m_t \not\to y_t, \overline{A}^m_t \not\to \overline{A}_t \) in \( L^2(\mathcal{F}_t) \), for each \( t \in [0, T] \). Thanks to Fatou’s lemma, we get \( E[\sup_{0 \leq t \leq T} |y^m_t|^2] < \infty \), and thus \( y^m \to y \) in \( L^2_F(0, T) \) in view of dominate convergence theorem. Since \( \overline{A}^m \) is RCLL, we can not directly apply the monotonic limit theorem, Theorem 2.1 in [35]. However it is easy to know that the limit \( y \) can be written in the following form

\[
y_t = y_0 - \int_0^t g^0_s ds - A_t - \overline{A}_t + \int_0^t z_s dB_s,
\]

where \( z \) and \( g^0 \) (resp. \( A_t \)) are the weak limit of \( z^m \) and \( g^m \) (resp. \( A^m_t \)) in \( L^2(\mathcal{F}_t) \). By Lemma 2.2 in [35], we know that \( y \) is RCLL. We then apply Itô’s rule to \( |y_t^m - y_t|^2 \) on interval \([\sigma, \tau]\), with stopping times \( 0 \leq \sigma \leq \tau \leq T \). It follows that

\[
E[y^m_{\sigma} - y_{\sigma}]^2 + E \int_\sigma^\tau |z^m_s - z_s|^2 ds
\]

\[
= E[y^m_{\tau} - y_{\tau}] + E \sum_{t \in [\sigma, \tau]} [(\Delta A_t)^2 - (\overline{A}_t - \overline{A}_t)^2] - 2E \int_\sigma^\tau (y^m_s - y_s)(g^m_s - g^0_s) ds
\]

\[
+ 2E \int_{[\sigma, \tau]} (y^m_s - y_s) dA_s + 2E \int_{[\sigma, \tau]} (y^m_s - y_s) d\overline{A}_s + 2E \int_{[\sigma, \tau]} (y^m_s - y_s) d(A^m_s - \overline{A}_s).
\]

Since \( E \int_{[\sigma, \tau]} (y^m_s - y_s) dA_s \leq 0 \) and \( E \int_{[\sigma, \tau]} (y^m_s - y_s) d(\overline{A}^m_s - \overline{A}_s) \leq 0 \), so we get

\[
E \int_{[\sigma, \tau]} |z^m_s - z_s|^2 ds \leq E[y^m_{\tau} - y_{\tau}] + E \sum_{t \in [\sigma, \tau]} (\Delta A_t)^2 + 2E \int_{[\sigma, \tau]} |y^m_s - y_s| |g^m_s - g^0_s| ds
\]

\[
+ 2E \int_{[\sigma, \tau]} |y^m_s - y_s| dA_s.
\]

Now we are in the same situation as in the proof of the monotonic limit theorem (cf. [35], Proof of Theorem 2.1). We then can follow the proof and get \( z^m \to z \) strongly in \( L^p_F(0, T) \), for \( p < 2 \).
From the Lipschitz property of $g$, we deduce that $(y, z, A, \overline{A})$ verify the equation

$$y_t = X + \int_t^T g(s, y_s, z_s)ds + A_T - A_t + \overline{A}_T - \overline{A}_t - \int_t^T z_sdB_s.$$ 

The estimate $E[(A_T^m)^2] \leq C$ implies $E[(\int_0^T d\Gamma_s(z_s^m)ds)^2] \leq \frac{C}{m^2}$, thus

$$E[\int_0^T d\Gamma_s(z_s)ds] = 0, \text{ or } d\Gamma_t(z_t) \equiv 0, dP \times dt - a.s..$$

It remains to prove that $(y, A)$ satisfies condition (ii) of Definition 36, i.e., $y \geq L$ and

$$\int_0^T (y_t - L_t^*)d\overline{A}_t = 0. \quad (21)$$

By $y^m \geq L$ we have $y \geq L$ and, for each $L^* \in D_Y^2(0, T)$ such that $y \geq L^* \geq L$,

$$\int_0^T (y_t - y_t^m \land L_t^*)d\overline{A}_t = \int_0^T (y_t - y_t^m)d\overline{A}_t + \int_0^T (y_t^m - y_t^m \land L_t^*)d\overline{A}_t^m + \int_0^T (y_t^m - y_t^m \land L_t^*)d(A_t - \overline{A}_t^m).$$

As $m \to \infty$, the first term on the right side tends to zero due to Lebesgue domination theorem. The second term is null because of (20) and since $y^m \geq y^m \land L^* \geq L$. For the third term we have

$$E[\int_0^T (y_t^m - y_t^m \land L_t^*)d(A_t - \overline{A}_t^m)] \leq E[\sup_{t \in [0, T]} |y_t^m - y_t^m \land L_t^*| (A_T^m - A_T)]$$

$$\leq E[\sup_{t \in [0, T]} |y_t^m - y_t^m \land L_t^*|^2]^{1/2} E[(A_T^m - A_T)^2]^{1/2}$$

which converges also to zero since $E[(A_T^m - A_T)^2] \land 0$. Thus the left hand term must tend to zero. This with $y^m \land L^* \not\land L^*$ yields (21).

We now prove (iii). Consider a quadruple $(y^*, z^*, A^*, \overline{A}^*)$ which satisfies (i) and (ii). Since $d\Gamma_s(y_s^*, z_s^*) \equiv 0$, we have

$$y_t^* = X + \int_t^T g(s, y_s^*, z_s^*)ds + m \int_t^T d\Gamma_s(y_s^*, z_s^*)ds + A_T^* - A_t^* + \overline{A}_T^* - \overline{A}_t^* - \int_t^T z_sdB_s.$$ 

By comparison theorem 7.3 it follows that $y^* \geq y^m$, for all $m$. Thus (iii) holds.

\[ \square \]

**Remark 4.4.** If $L$ is continuous or only has positive jumps ($L_{t-} \leq L_t$), then $\overline{A}$ is a continuous process. In this case, in (24), $\overline{A}$ are continuous, and $\overline{A}_t^m \geq \overline{A}_t^{n+1}$, $d\overline{A}_t^m \geq d\overline{A}_t^{n+1}$, $0 \leq t \leq T$, with $E[(\overline{A}_T^m)^2] \leq C$. Then $\overline{A}_t \land \overline{A}_t$, $0 \leq t \leq T$. Moreover

$$0 \leq \overline{A}_t - \overline{A}_t \leq \overline{A}_T - \overline{A}_T.$$ 

Thus we have uniform convergence:

$$E[\sup_{0 \leq t \leq T} (\overline{A}_t^m - \overline{A}_t)^2] \leq E[(\overline{A}_T^m - \overline{A}_T)^2] \rightarrow 0, \text{ as } n \to \infty.$$
4.3 Some convergence results of $g_T$-reflected solution with a lower barrier

As we know, the reflected BSDE can be considered as a special kind of constraint BSDE, with $\Gamma_t = [L_t, +\infty) \times \mathbb{R}$. If we put two constraint together, i.e. set $\bar{\Gamma}_t = \Gamma_t \cap [L_t, +\infty)$, then the penalization equation becomes the following one: for $n \in \mathbb{N}$

$$y_{t,n} = X + \int_t^T g(s, y_{s,n}^n, z_{s,n}^n) ds + n \int_t^T d\bar{F}_s(y_{s,n}^n, z_{s,n}^n) ds - \int_t^T z_{s,n}^n dB_s$$

(22)

$$= X + \int_t^T g(s, y_{s,n}^n, z_{s,n}^n) ds + n \int_t^T d\bar{F}_s(y_{s,n}^n, z_{s,n}^n) ds + n \int_t^T (L_s - y_{s,n}^n)^+ ds$$

$$- \int_t^T z_{s,n}^n dB_s.$$

Setting $\hat{A}_t^n = n \int_0^t d\bar{F}_s(y_{s,n}^n, z_{s,n}^n) ds$, with monotonic limit theorem in [35], we know that let $n \to \infty$, $(y_{t,n}^n, z_{t,n}^n, A_{t,n}^n)$ converges to $(\hat{y}, \hat{z}, \hat{A}) \in L^2(0, T) \times L^2(0, T; \mathbb{R}^d) \times A^2(0, T)$, where

$$\hat{y}_t = X + \int_t^T g(s, \hat{y}_s, \hat{z}_s) ds + \hat{A}_T - \hat{A}_t - \int_t^T \hat{z}_s dB_s.$$

Then we have

Proposition 4.1. The two limits are equal in the following sense:

$$y_t = \hat{y}_t, z_t = \hat{z}_t, A_t = \hat{A}_t.$$

Before we give the proof of this proposition, we consider another way to prove the convergence by the penalization equations given by (15), i.e. first let $m \to \infty$, then let $n \to \infty$, while in former subsection, we get the $g_T$-reflected solution $(y, z, \overline{A})$ of Definition 4.1 by first letting $n \to \infty$, then letting $m \to \infty$. As $m \to \infty$, we get that the triple $(y_{t,m}^m, z_{t,m}^m, A_{t,m}^m)$ converges to $(y_t^m, z_t^m, A_t^m) \in D^2(0, T) \times L^2(0, T; \mathbb{R}^d) \times A^2(0, T)$, which is the solution of constraint BSDE with coefficient $g^n = g + n(L_t - y)^+$:

$$y_{t}^n = X + \int_t^T g(s, y_{s}^n, z_{s}^n) ds + A_T^n - A_t^n + n \int_t^T (L_s - y_{s}^n)^+ ds - \int_t^T z_{s}^n dB_s, \quad (z_{t}^n) \in \Gamma_t, \quad dP \times dt \text{-a.s.,} \quad dA^n \geq 0.$$

Define $\overline{A}_t^n = n \int_0^t (L_s - y_{s}^n)^+ ds$. With same method in former subsection, we can prove that as $n \to \infty$, $(y_t^n, z_t^n, A_t^n, \overline{A}_t^n)$ converges to $(\tilde{y}, \tilde{z}, \tilde{A}, \overline{A}_t)$ where

$$\tilde{y}_t = X + \int_t^T g(s, \tilde{y}_s, \tilde{z}_s) ds + \tilde{A}_T - \tilde{A}_t + \overline{A}_T - \overline{A}_t - \int_t^T \tilde{z}_s dB_s.$$

Then we have

Proposition 4.2. The two limits are equal, in the following sense,

$$y_t = \tilde{y}_t, z_t = \tilde{z}_t \text{ and } A_t + \overline{A}_t = \tilde{A}_t + \overline{A}_t, 0 \leq t \leq T.$$
Proof. By comparison theorem for \((15)\) and \((20)\), we have \(y_t^{m,n} \leq y_t^m\), which follows \(y_t^n \leq y_t\), when letting \(m \to \infty\). Then let \(n \to \infty\), we get \(\tilde{y}_t \leq y_t\). Symmetrically compare \((15)\) and \((23)\), \(y_t^{n,m} \leq y_t^n\), let \(n \to \infty\), we get \(y_t^n \leq \tilde{y}_t\), then as \(m \to \infty\), it follows \(y_t \leq \tilde{y}_t\). So \(y_t = \tilde{y}_t, 0 \leq t \leq T\). The rest follows easily. \(\square\)

Proof of proposition 4.1 For \(m \leq n\), by comparison theorem for \((15)\) and \((22)\), we have \(y_t^{m,n} \leq y_t^{n,n}\). Let \(n \to \infty\), then \(m \to \infty\), we get

\[
y_t \leq \tilde{y}_t
\]

Similarly, for \(m \geq n\), using again comparison theorem, we have \(y_t^{m,n} \geq y_t^{n,n}\). First let \(m \to \infty\), then \(n \to \infty\), it follows

\[
\tilde{y}_t \geq \tilde{y}_t
\]

With proposition 4.2 we obtain \(y_t = \tilde{y}_t = \tilde{y}_t\). Other equalities follow easily. \(\square\)

These results show that for \(g_t\)-reflected BSDE with a lower barrier, we can get its solution via penalisation equations by different convergence method. No matter letting \(m \to \infty\) first or letting \(n \to \infty\) first, even considering dialogue sequence \((m = n)\), the limits we get are the same. By \((22)\) and monotonic limit theorem in \([35]\), we get \(g_T\)-solution \(\tilde{y}\) directly, increasing process \(\tilde{A}\) is to keep \((y, z)\) stay in \(\tilde{\Gamma}\), but we do not know any further property. But the \(g_t\)-reflected solution, i.e. definition \(36\), permits us to have a decomposition of \(\tilde{A}\), with \(\tilde{A} = A + \overline{A}\), where \(\overline{A}\) serves for \(y_t\) to get \(y_t \geq L_t\) and \(A\) serves for \(z_t\) to keep \(z_t \in \Gamma_t\), \(dP \times dt\)-a.s.. And this property plays an important role when we study the American option in incomplete market.

Remark 4.5. Proposition 4.1 is still true if we consider the more general case \(\Gamma\) could depend on \(y\), which satisfies \((2)\). Moreover we can generalize the constraint of reflecting with a lower barrier \(L\) by another general constraint \(\Lambda(t, \omega)\) which satisfies \((2)\), and Proposition 4.1 still holds.

4.4 Existence of \(g_t\)-reflected solution with an upper barrier: Proof of Theorem 4.2

For each \(n \in \mathbb{N}\), we consider the solution \((y^n, z^n, K^n) \in D^2_T(0, T) \times L^2_T(0, T; \mathbb{R}^d) \times A^2_T(0, T)\) of the following reflected BSDE with the coefficient \(g^n = (g + nd\Gamma_t)(t, y, z)\) and the upper reflecting obstacle \(U\):

\[
y^n_t = X + \int_t^T (g + nd\Gamma_t)(s, y^n_s, z^n_s)ds - (K^n_T - K^n_t) - \int_t^T z^n_s dB_s, \tag{24}
\]

\[
y^n \leq U, \; dP \times dt\text{-a.s.}, \; dK \geq 0, \; \text{and} \; \int_0^T (U^*_t - y^n_t) dK_t^n = 0,
\]

\(\forall U^* \in D^2_T(0, T), \) such that \(y^n \leq U^* \leq U\) \(dP \times dt\text{-a.s.}\).

Since \(g^n\) is Lipschitz with respect to \((y, z)\), this equation has a unique solution. We denote

\[
A^n_t = n \int_0^t d\Gamma_s(y^n_s, z^n_s)ds.
\]

Before to prove the a priori estimation for \((y^n, z^n, A^n, K^n)\), we need the following lemma.
Lemma 4.2. For \( X \in L^2(\mathcal{F}_T) \), there exists a quadruple \((y^*, z^*, A^*, K^*)\) \( \in D^2_T(0,T) \times L^2_T(0,T;\mathbb{R}^d) \times (A^2_T(0,T))^2 \) satisfies

\[
y^*_t = X + \int_t^T g(s, y^*_s, z^*_s) ds + (A^*_T - A^*_t) - (K^*_T - K^*_t) - \int_t^T z^*_s dB_s, \tag{25}
\]

\[
dt(z^*_t) = 0, dP \times dt\text{-a.s. and } y^*_t \leq U_t, \quad \int_0^T (y^*_t - U^*_t) dK^*_t = 0, \text{ a.s.}
\]

\[\forall U^* \in D^2_T(0,T), \text{ such that } y^* \leq U^* \leq U \ dP \times dt\text{-a.s.}\]

Proof. Fix a process \( \sigma_t \in L^2_T(0,T;\mathbb{R}^d) \) satisfying \( \sigma_t \in \Gamma_t, t \in [0,T] \). We consider a forward SDE with an upper barrier \( U_t \)

\[
dx_t = -g(t, x_t, \sigma_t) dt - d\overline{A}_t + \sigma_t dB_t, \quad x_0 = 1 \wedge U_0.
\]

Here \( \overline{A} \) is a process in \( A^2_T(0,T) \), such that \( x_t \leq U_t \), a.s. a.e. Set

\[y^*_t = x_t, z^*_t = \sigma_t, A^*_t = \overline{A}_t + (X_T - X)^+ \mathbbm{1}_{\{t=T\}}, K^*_t = (X_T - X)^+ \mathbbm{1}_{\{t=T\}}.\]

Then this quadruple is just the one we need. \( \square \)

Lemma 4.3. We have the following estimates: there exists a constant \( C > 0 \), independent of \( n \), such that

\[
E[ \sup_{0 \leq t \leq T} (y^n_t)^2 ] + E \int_0^T |z^n_t|^2 ds + E[(A^n_T)^2] + E[(K^n_T)^2] \leq C. \tag{26}
\]

Proof. Consider the following reflected BSDE with \( U \) as its upper reflecting obstacle,

\[
y^0_t = Y_T + \int_t^T g(s, y^0_s, z^0_s) ds - (K^0_T - K^0_t) - \int_t^T z^0_s dB_s, \quad t \in [0,T],
\]

\[
y^0_t \leq U_t, dK^0_t \geq 0, \quad \int_0^T (y^0_t - U^*_t) dK^0_t = 0.
\]

\[\forall U^* \in D^2_T(0,T), \text{ such that } y^0 \leq U^* \leq U \ dP \times dt\text{-a.s.}\]

This equation has a unique solution \((y^0, z^0, K^0) \in D^2_T(0,T) \times L^2_T(0,T;\mathbb{R}^d) \times A^2_T(0,T)\). By comparison theorem of reflected BSDEs \( y^n \geq y^0 \).

On the other hand, from proposition \( \ref{proposition} \) there exists \((y^*, z^*, A^*, K^*)\) satisfying

\[
y^*_t = Y_T + \int_t^T (g + ndt) (s, y^*_s, z^*_s) ds + (A^*_T - A^*_t) - (K^*_T - K^*_t) - \int_t^T z^*_s dB_s,
\]

\[
y^*_t \leq U_t, \quad \int_0^T (y^*_t - U^*_t) dK^*_t = 0, \text{ a.s.}
\]

It follows from the comparison theorem \( \ref{comparison} \) for reflected BSDEs that for each \( n \in \mathbb{N} \), we have \( y^n_t \leq y^*_t, K^n_t \leq K^*_t \) and \( dK^n_t \leq dK^*_t, t \in [0,T] \). Thus there exists a constant \( C > 0 \), independent of \( n \), such that

\[
E[ \sup_{0 \leq t \leq T} (y^n_t)^2 ] \leq E[ \sup_{0 \leq t \leq T} ((y^0_t)^2 + (y^*_t)^2)] \leq C. \tag{27}
\]
and
\[ E[(K^n_T)^2] \leq E[(K^n_T)^2] \leq C. \] (28)

To estimate \((z^n, A^n)\), we apply Itô’s formula to \(|y^n_t|^2\) then get
\[
E[|y^n_0|^2] + E\left[ \int_0^T |z^n_s|^2 \, ds \right] \leq E[Y^n_T] + E\left[ \int_0^T (g(s, 0, 0))^2 \, ds \right] + (2\mu + \mu^2) \int_0^T |y^n_s|^2 \, ds + \frac{1}{2} E\left[ \int_0^T |z^n_s|^2 \, ds \right]
\]
\[
+ \left( \frac{1}{\alpha} + 1 \right) E[\sup_{0 \leq t \leq T} (y^n_t)^2] + \alpha E[(A^n_T)^2] + E[(K^n_T)^2],
\]
where \(\alpha\) is a positive constant to be chosen later. This with the above two estimates (27) and (28) yields
\[
E\left[ \int_0^T |z^n_s|^2 \, ds \right] \leq C + \alpha E[(A^n_T)^2].
\] (29)

On the other hand, again by (24),
\[
A^n_T = y^n_0 - y^n_T - \int_0^T g(s, y^n_s, z^n_s) \, ds + K^n_T - \int_0^T z^n_s \, dB_s.
\] (30)

Thus
\[
E[(A^n_T)^2] \leq 5E[(y^n_0)^2] + (y^n_T)^2 + (K^n_T)^2 + 15TE\left[ \int_0^T (g(s, 0, 0))^2 \, ds \right]
\]
\[
+15\mu^2TE\int_0^T |y^n_s|^2 \, ds + (15\mu^2T + 5)E\left[ \int_0^T |z^n_s|^2 \, ds \right],
\]
then
\[
E[(A^n_T)^2] \leq C + (15\mu^2T + 5)E\left[ \int_0^T |z^n_s|^2 \, ds \right].
\] (31)

With (29), setting \(\alpha = \frac{1}{30\mu^2T + 10}\), we finally obtain (26). \(\square\)

**Proof of Theorem 4.2.** In (24), since \(g^n(t, y, z) \leq g^{n+1}(t, y, z)\), by comparison theorem 7.3 for reflected BSDEs, \(y^0 \leq y^n \leq y^{n+1} \leq y^*\). Thus \(\{y^n\}_{n=1}^{\infty}\) increasingly converges to \(y\) as \(n \to \infty\), and
\[
E[\sup_{0 \leq t \leq T} (y_t)^2] \leq C.
\]

We also have
\[
\lim_{n \to \infty} E\left[ \int_0^T |y^n_t - y_t|^2 \, dt \right] = 0.
\]

Moreover from comparison theorem 7.3 we have \(K^n_t \leq K^{n+1}_t \leq K^*_t\) and \(dK^n_t \leq dK^{n+1}_t \leq dK^*_t, 0 \leq t \leq T\). It follows that \(\{K^n\}_{n=1}^{\infty}\) increasingly converges to an increasing process \(K \in A^{2}_{F}(0, T)\) with \(E[(K_T)^2] \leq C\). Moreover \(A^n\) are continuous increasing processes with \(E[(A^n_T)^2] \leq C\). From (29), there exists a process \(z \in L^{2}_{F}(0, T; \mathbb{R}^d)\), such that \(z^n \to z\) weakly in \(L^{2}_{F}(0, T; \mathbb{R}^d)\).
Now the conditions of the generalized monotonic limit theorem, Theorem 3.1 in [38] are satisfied. Then we have \( z^n \rightarrow z \) strongly in \( L^2_T(0, T; \mathbb{R}^d) \), for \( p < 2 \). With the Lipschitz condition of \( g \), the limit \( y \in D^2(0, T) \) can be written as

\[
y_t = X + \int_t^T g(s, y_s, z_s) ds + (A_T - A_t) - (K_T - K_t) - \int_t^T z_s dB_s,
\]

where, for each \( t \), \( A^*_n \rightarrow A_t \) weakly in \( L^2(F_t) \), \( K^n_t \rightarrow K_t \) strongly in \( L^2(F_t) \). \( A, K \in A^2_F(0, T) \) are increasing processes.

From \( E[(A^n_T)^2] = E[(n \int_0^T d_{\Gamma^n}(y^n_s, z^n_s) ds)^2] \leq C \), it follows that

\[
E[(\int_0^T d_{\Gamma^n}(y^n_s, z^n_s) ds)^2] \leq \frac{C}{n^2},
\]

while \( d_{\Gamma^n}(y^n_s, z^n_s) \geq 0 \), we get that \( \int_0^T d_{\Gamma^n}(y^n_s, z^n_s) ds \rightarrow 0 \), as \( n \rightarrow \infty \). With the Lipschitz property of \( d_{\Gamma^n}(y, z) \) and the convergence of \( y^n \) and \( z^n \), we deduce that

\[
d_{\Gamma^n}(y_t, z_t) = 0 \; \text{d}P \times \text{dt-a.s.}
\]

Now we consider (ii). From \( y^n \geq U \) we have \( y \geq U \), with

\[
\int_0^T (y^n_t - U^n_t) dK^n_t = 0, \forall U^n \in D^2_T(0, T), \text{ s.t. } U \geq U^n \geq y^n.
\]

Now, for each \( U^* \in D^2_T(0, T) \), s.t. \( U \geq U^* \geq y \), since \( y \geq y^n \), thus \( U \geq U^* \geq y^n \)

\[
\int_0^T (y^n_t - U^n_t) dK^n_t = 0 \Rightarrow \int_0^T (y_t - U^*_t) dK^n_t = 0.
\]

Recall that \( dK^n_t \leq dK_t \), and \( K^n_T \nearrow K_T \) in \( L^2(F_T) \), then

\[
0 \leq \int_0^T (U^*_t - y_t) d(K_t - K^n_t) \leq \sup_{t \in [0, T]} (U^*_t - y_t) \cdot [K_T - K^n_T],
\]

and with the estimate of \( y \) and (iii), it follows (ii) of Definition [1.2] holds.

We now prove (iii). In fact, for any other quadruple \((\overline{y}, \overline{z}, \overline{A}, \overline{K}) \in D^2_T(0, T) \times L^2_T(0, T; \mathbb{R}^d) \times (A^2(0, T))^2\) satisfying

\[
\overline{y}_t = X + \int_t^T g(s, \overline{y}_s, \overline{z}_s) ds + \overline{A}_T - \overline{A}_t - (\overline{K}_T - \overline{K}_t) - \int_t^T \overline{z}_s dB_s,
\]

\[
d_{\Gamma^n}(\overline{y}_t, \overline{z}_t) = 0, \quad d\overline{A} \geq 0, \quad d\overline{K} \geq 0,
\]

\[
\overline{y}_t \leq U^*_t, \quad \int_0^T (U^*_t - \overline{y}_t) d\overline{K}_t = 0, \text{ a.s.,}
\]

for any \( U^* \in D^2_T(0, T) \), such that \( \overline{y} \leq U^* \leq U \). Then it also satisfies

\[
\overline{y}_t = X + \int_t^T g(s, \overline{y}_s, \overline{z}_s) ds + n \int_t^T d_{\Gamma^n}(\overline{y}_s, \overline{z}_s) ds + \overline{A}_T - \overline{A}_t - (\overline{K}_T - \overline{K}_t) - \int_t^T \overline{z}_s dB_s.
\]
Compare it to (24), we have $y \geq y^n$, and $K \geq K^n$. Let $n \to \infty$, it follows
\[ y_t \geq y_t, \quad K_t \geq K_t, \quad \forall t \in [0, T], \quad \text{a.s..} \] (32)
So $y$ is the smallest process satisfying Definition 4.2 (i) and (ii).

It remains to prove the relation of the total variation in (13) holds. In fact, if it is not the case, set $\tilde{V}_t = V_{[0,t]}(A + K)$, then we define Jordan decomposition:
\[ \tilde{A}_t = \frac{1}{2}(\tilde{V}_t + A_t - K_t), \quad \tilde{K}_t = \frac{1}{2}(\tilde{V}_t - A_t + K_t). \]
With $d\tilde{K}_t = \frac{1}{2}(d\tilde{V}_t - dA_t + dK_t) \leq dK_t$ we have, for each $U^* \in D^2(0, T)$ with $U \geq U^* \geq y$,
\[ 0 \leq \int_0^T (y_t - U^*_t) d\tilde{K}_t \leq \int_0^T (y_t - U^*_t) dK_t = 0. \]
But in considering the second inequality of (32), we have $\tilde{K} \geq K$, which draws a contradiction. This completes the proof. \[ \square \]

**Remark 4.6.** From the smallest property of $y - K$, it is the $g^K$-solution with terminal condition $X - K_T$, where $g^K(t, y, z) = g(t, y + K_t, z)$.

**Remark 4.7.** If $U$ is continuous (or satisfies $U_{t-} \geq U_t$), then $K$ is a continuous process. In fact, by [15], the solution $y^n$ of (24) as well as the reflecting process $K^n$ are continuous. This with $K^n \leq K^n + 1$ and $dK^n \leq dK$ yields
\[ 0 \leq K_t - K^n_t \leq K_T - K^n_T, \]
and thus
\[ E[ \sup_{0 \leq t \leq T} (K_t - K^n_t)^2 ] \leq E[(K_T - K^n_T)^2] \to 0. \]
It follows that $K^n$ converges uniformly to $K$ on $[0, T]$. Thus $K$ is continuous.

### 5 $g^\Gamma$-super(sub)martingales and its Doob-Meyer’s type decomposition theorems

Now we introduce the definitions of $g^\Gamma$-martingale, $g^\Gamma$-supremartingale and $g^\Gamma$-submartingale, by $g^\Gamma$-expectation introduced in section 3. Suppose that $g$ satisfies (7), $g(t, 0, 0) = 0$ and $(0, 0) \in \Gamma_t$.

**Definition 5.1.** A process $Y \in D^2_{\mathcal{F}}(0, T)$ is called a $g^\Gamma$-supremartingale (resp. $g^\Gamma$-submartingale) on $[0, T]$, if for stopping times $\sigma, \tau$ valued in $[0, T]$, with $\sigma \leq \tau$, we have $Y_\tau \in L^2_{\mathcal{F}_{\tau}}(\mathcal{F}_{\tau})$ and
\[ \mathcal{E}^g_{\sigma, \tau}[Y_\tau] \leq Y_\sigma, \quad \text{resp.} \quad \geq Y_\sigma. \]

It is called a $g^\Gamma$-martingale if it is both a $g^\Gamma$-supremartingale and $g^\Gamma$-submartingale.
The nonlineare Doob-Meyer’s type decomposition theorem for $g$-super(sub)martingale in [35] plays an important role in theory of $g$-expectation. For $g_r$-super(sub)martingale, we have also Doob-Meyer’s type decomposition theorem. In fact, in [39], we have proved the decomposition for $g_r$-supermartingale, partly for $g_r$-submartingale. For completeness of this paper, we still present the proofs. And these proofs are important applications of $g_r$-reflecting solutions.

### 5.1 $g_r$-supermartingale decomposition theorem

In this section, we study the Doob-Meyer’s type decomposition theorem for $g_r$-supermartingale. Before present the main result, we first give a useful property of $g_r$-supermartingale:

**Proposition 5.1.** A process $Y$ is a $g_r$-supermartingale on $[0, T]$, if and only if for all $m \geq 0$, it is a $(g + md_r)$-supermartingale on $[0, T]$.

**Proof.** We fix $t \in [0, T]$, and set $y_s^t = \mathcal{E}_{s,t}^g[Y_t]$, $0 \leq s \leq t$. Let $(y^r, z^r, A^r)$ be the $g_r$-solution on $[0, t]$:

$$ y_s^t = Y_t + \int_s^t g(r, y_r^t, z_r^t)dr + A_t^r - A_s^r - \int_s^t z_r^t dB_r, $$

$$ d_{\Gamma_s}(y_s^t, z_s^t) = 0, \quad s \in [0, t]. $$

Consider the following penalization equation

$$ y_s^{t,m} = Y_t + \int_s^t g(r, y_r^{t,m}, z_r^{t,m})dr + m \int_s^t d_{\Gamma_r}(y_r^{t,m}, z_r^{t,m})dr - \int_s^t z_r^{t,m} dB_r, $$

We observe that the above $(y^r, z^r, A^r)$ also satisfies

$$ y_s^t = Y_t + \int_s^t g(r, y_r^t, z_r^t)dr + m \int_s^t d_{\Gamma_r}(y_r^t, z_r^t)dr + A_t^r - A_s^r - \int_s^t z_r^t dB_r. $$

From comparison theorem, we get $y^{t,m} \leq y^t$ on $[0, t]$. Thus

$$ \mathcal{E}_{s,t}^{g + md_r}[Y_t] \leq \mathcal{E}_{s,t}^{g_r}[Y_t] \leq Y_s, \quad \forall m \geq 0. $$

It follows that $Y$ is a $(g + md_r)$-supermartingale on $[0, T]$. Conversely, if for each $m \geq 0$, $Y$ is a $(g + md_r)$-supermartingale on $[0, T]$ i.e. $\mathcal{E}_{s,t}^{g + md_r}[Y_t] = y_s^{t,m} \leq Y_s$. When we let $m \to \infty$, by the monotonic limit theorem in [35], $y_s^{t,m}$ converges to $\mathcal{E}_{s,t}^{g_r}[Y_t]$, which is the $g_r$-solution. We thus have $\mathcal{E}_{s,t}^{g_r}[Y_t] \leq Y_s$, on $[0, T]$. This implies that $Y$ is a $g_r$-supermartingale.

We have the following $g_r$-supermartingale decomposition theorem.

**Theorem 5.1.** Let $Y$ be a right continuous $g_r$-supermartingale on $[0, T]$. Then there exists a unique RCLL increasing process $A \in \mathcal{A}_F^Z(0, T)$, such that $Y$ is a $g_r$-supersolution, namely,

$$ y_t = Y_T + \int_t^T g(s, y_s, z_s)ds + A_T - A_t - \int_t^T z_s dB_s, $$

$$ d_{\Gamma_t}(y_t, z_t) = 0, \quad a.e. \quad a.s.. $$
Proof. For each fixed $m \geq 0$, we consider the solution $(y^m, z^m, A^m) \in D^2_T(0, T) \times L^2_T(0, T; \mathbb{R}^d) \times A^2_T(0, T)$ of the following reflected BSDE, with the $g_T$–supermartingale as the lower obstacle:

\begin{equation}
\begin{aligned}
y_t^m &= Y_T + \int_t^T g^m(s, y_s^m, z_s^m)ds + A_T^m - A_t^m - \int_t^T z_s^m dB_s, \\
dA_t^m &\geq 0, \quad y_t^m \leq Y_t, \quad \int_0^T (Y_t - y_t^m)dA_t^m = 0.
\end{aligned}
\end{equation}

where $g^m(t, y, z) := (g + md_T)(t, y, z)$. By Proposition 5.1, this $g_T$–supermartingale is also a $g^m$–supermartingale for each $m$. It follows from the $g$–supermartigale decomposition theorem (see [35]) that $y_t^m \equiv Y_t$. Thus $z^m$ is invariant in $m$: $z^m \equiv Z \in L^2_T(0, T; \mathbb{R}^d)$ and the above equation (33) can be written

$$Y_t = Y_T + \int_t^T (g + md_T)(s, Y_s, Z_s)ds + A_T^m - A_t^m - \int_t^T Z_s dB_s.$$ 

Consequently, for all $m \geq 0$, notice that $A^m$ is a positive process, we have

$$0 \leq m \int_0^T d\Gamma(s, Y_s, Z_s)ds$$

$$\leq \left[ Y_0 - Y_T + \int_0^T Z_s dB_s - \int_0^T g(s, Y_s, Z_s)ds \right]^+ \in L^2(\mathcal{F}_T).$$

From this it follows immediately $\int_0^T d\Gamma(s, Y_s, Z_s)ds = 0$. Thus $A^m$ is also invariant in $m$: $A^m = A \in D^2_T(0, T)$. We thus complete the proof. \Box

5.2 $g_T$–submartingale decomposition theorem

We now consider the decomposition theorem of a given $g_T$–submartingale $Y \in D^2_T(0, T)$. In [39], we have proved a $g_T$–submartingale decomposition theorem under assumptions of $Y_{t-} \geq Y_t$ with

(H) There exists a quadruple $(y^*, z^*, A^*, K^*) \in D^2_T(0, T) \times L^2_T(0, T) \times (A^2_T(0, T))^2$, satisfying

\begin{equation}
\begin{aligned}
y_t^* &= Y_T + \int_t^T g(s, y_s^*, z_s^*)ds + (A_T^* - A_t^*) - (K_T^* - K_t^*) - \int_t^T z_s^* dB_s, \\
d\Gamma_t(y_t^*, z_t^*) &= 0, \text{ a.s. a.e.} \\
y_t^* &\leq Y_t, \quad \int_0^T (y_{t-}^* - Y_{t-})dK_t^* = 0, \text{ a.s.}
\end{aligned}
\end{equation}

Remark 5.1. A necessary condition for (H) holding is $\Gamma_t \cap (-\infty, Y_t] \times \mathbb{R}^d \neq \emptyset$.

Here we partly generalize this result and try to get rid of assumption (H).

Theorem 5.2. Assume $\Gamma_t$ only depends on $z$. Let $Y \in D^2_T(0, T)$ be a $g_T$-submartingale on $[0, T]$ and for stopping times $\sigma$, $\tau$ valued in $[0, T]$, with $\sigma \leq \tau$, such that

$$\mathcal{E}^\mathbb{P}_{\sigma, \tau}[Y_\tau - (Y_\tau - Y_{\tau-})^+] \geq Y_\sigma.$$ (34)
Then there exists a unique continuous increasing process \( K \in A_\Gamma^2(0, T) \), such that the triple \((Y - K, Z, A) \in D_\Gamma^2(0, T) \times L_\Gamma^2(0, T; \mathbb{R}^d) \times A_\Gamma^2(0, T)\) is the \( g^K \Gamma \)-solution with terminal condition \( Y_T - K_T \), i.e. for \( t \in [0, T] \),

\[
Y_t - K_t = Y_T - K_T + \int_t^T g^K(s, Y_s - K_s, Z_s)ds + (A_T - A_t) - \int_t^T Z_s dB_s,
\]

\[
Z_t \in \Gamma_t, \quad dP \times dt \text{-a.s.}
\]

where

\[
g^K(t, y, z) := g(t, y + K_t, z), \quad (t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d.
\]

**Proof.** Consider the BSDE\((Y_T, g_T)\) with reflecting upper obstacle \( Y \). From Theorem 4.2, we know that there exists a quadruple \((y, Z, A, K) \in D_\Gamma^2(0, T) \times L_\Gamma^2(0, T; \mathbb{R}^d) \times (A_\Gamma^2(0, T))^2\)

\[
y_t = Y_T + \int_t^T g(s, Y_s, Z_s)ds + A_T - A_t - (K_T - K_t) - \int_t^T Z_s dB_s, \tag{35}
\]

\[
Z_t \in \Gamma_t, \quad dP \times dt \text{-a.s.}, \quad dA \geq 0, \quad dK \geq 0, \quad \mathcal{V}_{[0,t]}[A - K] = \mathcal{V}_{[0,t]}[A + K],
\]

\[
y_t \leq Y_t, \quad \int_0^T (y_s - Y_s) dK_s = 0, \text{ a.s.}
\]

We want to prove that \( y \equiv Y \). It is sufficient to prove \( y_t \geq Y_t \). For each \( \delta > 0 \), we define stopping times

\[
\sigma^\delta := \inf\{t \geq 0 : y_t \leq Y_t - \delta\} \wedge T,
\]

\[
\tau := \inf\{t \geq 0 : y_t \geq Y_t\}.
\]

If \( P(\sigma^\delta < T) = 0 \) for all \( \delta > 0 \), the proof is done; if it is not such case, there exists a \( \delta > 0 \), such that \( P(\sigma^\delta < T) > 0 \). So we have \( \sigma^\delta < \tau \leq T \). Since \( y \) and \( Y \) are RCLL, \( y_{\sigma^\delta} \leq Y_{\sigma^\delta} - \delta \) and \( y_\tau \geq Y_\tau \). So \( y_\tau = Y_\tau \). By the integral equality in (35), we get \( K_\tau = K_{\sigma^\delta} \).

Since \( \mathcal{V}_{[0,t]}[A - K] = \mathcal{V}_{[0,t]}[A + K] \), \( \Delta A_\tau \cdot \Delta K_\tau = 0 \). From the integral equality in (35), we know that \((y_{\tau -} - Y_{\tau -})(K_s - K_{\tau -}) = 0 \). So if \( \Delta K_\tau \neq 0 \), then \( y_{\tau -} = Y_{\tau -} \) and \( \Delta A_\tau = 0 \), which implies \( \Delta K_\tau = (y_{\tau -} - Y_{\tau -})^+ = (Y_\tau - Y_{\tau -})^+ \).

Define

\[
\nabla_t = Y_t 1_{[0,\tau]}(t) + (Y_\tau - Y_{\tau -})^+ 1_{[\tau, T]}(t),
\]

\[
\bar{g}(t, y, z) = g(t, y, z) 1_{[0,\tau]}(t),
\]

\[
\nabla_t = \Gamma_t 1_{[0,\tau]} + R^{1 \times d} 1_{[\tau, T]}(t).
\]

Then for \( 0 \leq s \leq t \leq T \), \( E^{\mathcal{F}_s}_{\sigma^\delta} Y_T = E^{g^{\mathcal{F}_s}}_{\sigma^\delta} Y_T - (Y_\tau - Y_{\tau -})^+ 1_{[\tau, T]}(t) \).

Consider two stopping times \( 0 \leq \sigma_1 \leq \sigma_2 \leq T \), then with (34) we have

\[
E^{\nabla_{\sigma_2}}_{\sigma_1, \sigma_2}[Y_{\sigma_2}] = E^{g^\nabla_{\sigma_2}}_{\sigma_1, \sigma_2}[Y_{\sigma_2} - (Y_\tau - Y_{\tau -})^+ 1_{[\tau, T]}(\sigma_2)] \geq E^{g_{\sigma_1, \sigma_2}}_{\sigma_1, \sigma_2}[Y_{\sigma_2} - (Y_{\sigma_2 -} - Y_{\sigma_2 -})^+] \geq Y_{\sigma_1 \wedge T} \geq \nabla_{\sigma_1}.\]
So $\overline{Y}$ is a $\overline{\mathfrak{Y}}$-submartingale. Define
\[
\overline{g}_t = y_t 1_{[0,\tau]}(t) + (y_t - (y_t - y_{t-}^+)) 1_{[\tau,T]}(t),
\]
\[
\overline{K}_t = K_t 1_{[0,\tau]}(t) + K_{\tau-} 1_{[\tau,T]}(t),
\]
\[
\overline{A}_t = A_t 1_{[0,\tau]}(t) + A_t 1_{(\tau,T]}(t), \quad \overline{Z}_t = Z_t 1_{[0,\tau]}(t).
\]

Then for $0 \leq t \leq T$, we have
\[
\begin{align*}
\overline{y}_t &= \overline{Y}_T + \int_t^T g(s, \overline{g}_s, \overline{z}_s) ds + (\overline{A}_T - \overline{A}_t) - (\overline{K}_T - \overline{K}_t) - \int_t^T \overline{Z}_s dB_s, \\
\overline{Z}_t &\in \Gamma_t, dP \times dt\text{-a.s., } d\overline{A} \geq 0, \quad d\overline{K} \geq 0, \quad \mathcal{V}_{[0,t]}[\overline{A} - \overline{K}] = \mathcal{V}_{[0,t]}[\overline{A} + \overline{K}], \\
\overline{y}_t &\leq \overline{Y}_t, \quad \int_0^T (\overline{y}_{s-} - \overline{y}_{s-}) d\overline{K}_s = 0.
\end{align*}
\]

Notice that $\overline{K}_\tau = K_{\tau-} = K_{\sigma^\delta} = \overline{K}_{\sigma^\delta}$, with $\overline{y}_\tau = y_\tau - \Delta K_\tau = Y_\tau - \Delta K_\tau = \overline{Y}_\tau$, we get
\[
\begin{align*}
\overline{y}_{\sigma^\delta} &= \overline{Y}_T + \int_0^\tau g(s, \overline{y}_s, \overline{z}_s) ds + (\overline{A}_\tau - \overline{A}_{\sigma^\delta}) - \int_0^\tau \overline{Z}_s dB_s, \\
(\overline{y}_{\sigma^\delta}, \overline{Z}_{\sigma^\delta}) &\in \Gamma_t, \quad d\overline{A} \geq 0.
\end{align*}
\]

Since $y + K$ is the $g_{\T}$-solution of constraint BSDE $(Y_\tau, g, \Gamma)$ on $[\sigma^\delta, \tau]$, $\overline{y}$ is the $g_{\T}$-solution of constraint BSDE $(\overline{Y}_\tau, \overline{g}, \overline{\Gamma})$ on $[\sigma^\delta, \tau]$. So with the fact that $\overline{Y}$ is a $\overline{\mathfrak{Y}}$-submartingale, we get
\[
\overline{y}_{\sigma^\delta} = \mathcal{E}_{\sigma^\delta, \tau}^g[\overline{y}_\tau] = \mathcal{E}_{\sigma^\delta, \tau}^g[\overline{\mathfrak{Y}}_\tau] \geq \overline{y}_{\sigma^\delta}.
\]

But $\overline{y}_{\sigma^\delta} = y_{\sigma^\delta}$, $\overline{y}_{\sigma^\delta} = Y_{\sigma^\delta}$, this introduces a contradiction. \(\square\)

**Remark 5.2.** This result can easily cover the decomposition theorem in [32] when $\Gamma$ does not depend on $y$. In fact, from the condition $Y_{\tau-} \geq Y_\tau$, we know that $(Y_\tau - Y_{\tau-})^+ = 0$, so $\mathcal{E}_{s,t}^{g_{\T}}[Y_\tau - (Y_\tau - Y_{\tau-})^+] = \mathcal{E}_{s,t}^{g_{\T}}[Y_\tau] \geq Y_s$.

**Remark 5.3.** We can prove the same result for general case when $\Gamma$ also depends on $y$ under the assumption (H). This assumption is not easy to verify. However it is required by the existence of $g_{\T}$-reflected solution associated to $(Y_\tau, g, \Gamma)$ with an upper barrier $Y$, when the constraint $\Gamma$ depends on $y$.

Although in Theorem 5.2 we remove assumption (H), sometimes the assumption (34) in is not easy either. In the following result, we do not need to assume (34), but we need more assumptions on $g$.

**Theorem 5.3.** Let $Y \in D^2_T(0,T)$ be a $g_{\T}$-submartingale on $[0,T]$. Suppose $g$ and $\Gamma$ do not depend on $y$, $g(t,0) = 0$ and $0 \in \Gamma_t$. Then there exists a unique continuous increasing process $K$ with $E[K_T^2] < \infty$, such that the same decomposition result of theorem 5.2 holds.

**Proof.** As in the proof of theorem 5.2, we consider the BSDE $(Y_\tau, g_{\T})$ with reflecting upper obstacle $Y$. From Theorem 1.2 we know that there exists a quadruple $(y, Z, A, K) \in \mathcal{S}$.
\[ D_2^2(0, T) \times L_2^2(0, T; \mathbb{R}^d) \times (A_2^2(0, T))^2 \] such that

\[
y_t = Y_T + \int_t^T g(s, Z_s)ds + A_T - A_t - (K_T - K_t) - \int_t^T Z_s dB_s, \tag{36}
\]

\[
Z_t \in \Gamma_t, dP \times dt-a.s., \quad dA \geq 0, \quad dK \geq 0, \quad \mathcal{V}_{[0,t]}[A-K] = \mathcal{V}_{[0,t]}[A+K],
\]

\[
y_t \leq Y_t, \quad \int_0^T (y_s - Z_s) dK_s = 0, \text{ a.s.}
\]

As before we want to prove that \( y \equiv Y \). It suffices to prove \( y_t \geq Y_t \). For each \( \delta > 0 \), define stopping times

\[
\sigma^\delta : = \inf \{ t, y_t \leq Y_t - \delta \} \wedge T,
\]

\[
\tau^\delta : = \inf \{ t \geq \sigma^\delta : y_t \geq Y_t - \delta/2 \}.
\]

If \( P(\sigma^\delta < T) = 0 \) for all \( \delta > 0 \), the proof is done; if it is not, there exists a \( \delta > 0 \), such that \( P(\sigma^\delta < T) > 0 \). So we have \( \sigma^\delta < \tau^\delta \leq T \). Since \( y \) and \( Y \) are RCLL, \( y_{\sigma^\delta} \leq Y_{\sigma^\delta} - \delta \) and \( y_{\tau^\delta} \geq Y_{\tau^\delta} - \delta/2 \). By the integral equality in (36), we get \( K_{\tau^\delta} = K_{\sigma^\delta} \).

Thanks to proposition 3.6-(i), we know that \( \mathcal{E}^{\nu}[\cdot] \) has translation invariant property. So \( \mathcal{E}^{\nu}_{\sigma^\delta, \tau^\delta}[Y_{\tau^\delta} - \delta/2] = \mathcal{E}^{\nu}_{\sigma^\delta, \tau^\delta}[Y_{\sigma^\delta}] - \delta/2 \).

While on the interval \( [\sigma^\delta, \tau^\delta] \),

\[
y_{\sigma^\delta} = y_{\tau^\delta} + \int_{\sigma^\delta}^{\tau^\delta} g(s, Z_s)ds + A_{\tau^\delta} - A_{\sigma^\delta} - \int_{\sigma^\delta}^{\tau^\delta} Z_s dB_s,
\]

\[
Z_t \in \Gamma_t, dP \times dt-a.s., \quad dA \geq 0.
\]

So we have

\[
y_{\sigma^\delta} = \mathcal{E}^{\nu}_{\sigma^\delta, \tau^\delta}[y_{\tau^\delta}] \geq \mathcal{E}^{\nu}_{\sigma^\delta, \tau^\delta}[Y_{\tau^\delta} - \delta/2] = \mathcal{E}^{\nu}_{\sigma^\delta, \tau^\delta}[Y_{\tau^\delta}] - \delta/2 \geq Y_{\sigma^\delta} - \delta/2.
\]

This introduces a contradiction. So result follows. \( \square \)

**Remark 5.4.** By proposition 3.6-(ii), we can prove the same results, for the case when \( g(t, y, z) = g_1(t, z) + ay \) with \( g_1(t, z) \) is bounded, and \( \Gamma \) only depends on \( z \), with \( 0 \in \Gamma_t \).

### 6 Applications of \( g_\Gamma \)-reflected BSDEs: American option pricing in incomplete market

We follow the idea of El Karoui et al.(1997, [16]). Consider the strategy wealth portfolio \( (Y_t, \pi_t) \) as a pair of adapted processes in \( L_2^2(0, T) \times L_2^2(0, T; \mathbb{R}^d) \) which satisfy the following BSDE

\[-dY_t = g(t, Y_t, \pi_t)dt - \pi_t^\sigma dB_t,
\]

where \( g \) is \( \mathbb{R} \)-valued, convex with respect to \( (y, \pi) \), and satisfy Lipschitz condition (1). We suppose that the volatility matrix \( \sigma \) of \( n \) risky assets is invertible and \( (\sigma_t)^{-1} \) is bounded.
In complete market, we are concerned with the problem of pricing an American contingent claim at each time \( t \), which consists of the selection of a stopping time \( \tau \in \mathcal{T}_t \) (the set of stopping times valued in \([t, T]\)) and a payoff \( S_\tau \) on exercise if \( \tau < T \) and \( \xi \) if \( \tau = T \). Here \((S_t)\) is a continuous process satisfying \( E[\sup_s(S_t^+)^2] < \infty \). Set
\[
\tilde{S}_s = \xi 1_{\{s = T\}} + S_s 1_{\{s < T\}}.
\]
Then the price of the American contingent claim \((\tilde{S}_s, 0 \leq s \leq T)\) at time \( t \) is given by
\[
Y_t = \text{ess sup}_{\tau \in \mathcal{T}_t} Y_t(\tau, \tilde{S}_\tau).
\]
Moreover the price \((Y_t, 0 \leq t \leq T)\) corresponds to the unique solution of the reflected BSDE associated with terminal condition \( \xi \), generator \( g \) and obstacle \( S \), i.e. there exists \((\pi_t) \in \mathbf{L}^2(0, T; \mathbb{R}^d)\) and \((A_t)\) an increasing continuous process with \( A_0 = 0 \) such that
\[
-dY_t = g(s, Y_t, \pi_t)ds + dA_t - \pi_t^\top \sigma dB_t, Y_T = \xi,
\]
\[
Y_t \geq S_t, \quad 0 \leq t \leq T, \quad \int_0^T (Y_t - S_t)dA_t = 0.
\]
Furthermore, the stopping time \( D_t = \inf(t \leq s \leq T | Y_s = S_s) \land T \) is optimal, that is
\[
Y_t = Y_t(D_t, \tilde{S}_{D_t}).
\]

Now we consider in the incomplete market, i.e. there is a constraint on portfolio \( \pi_t \in \Gamma_t \), where \( \Gamma_t \) is a closed subset of \( \mathbb{R}^d \), how to price the American contingent claim \((\tilde{S}_s, 0 \leq s \leq T)\). Lucky, with the results in former sections, we have the following results:

**Theorem 6.1.** If \( \xi \) is attainable, i.e. there exists a couple \((Y', \pi')\) with \( \pi' \in \Gamma_t \), \( t \)-a.e. which replicate \( \xi \), then the price process \( Y \) of American option in the incomplete market is the \( g_{\Gamma} \)-solution reflected by the lower obstacle \( L \), i.e. there exist a process \( \pi_t \in \Gamma_t \), \( dP \times dt \)-a.s., and increasing continuous processes \( A \) and \( \bar{A} \), such that
\[
Y_t = \xi + \int_t^T g(s, Y_s, \pi_s)ds + A_T - A_t + \bar{A}_T - \bar{A}_t - \int_t^T \pi_s^\top \sigma_s dB_s, \quad (37)
\]
\[
Y_t \geq S_t, \quad 0 \leq t \leq T, \quad \int_0^T (Y_t - S_t)dA_t = 0.
\]

Furthermore, the stopping time \( D_t = \inf(t \leq s \leq T | Y_s = S_s) \land T \) is still optimal.

**Sketch of the proof.** Thanks to the results of [16] and [44], we know that the method of auxiliary market in [7] and [8] is equivalent to the penalization equations associated to \((\xi, f + nd_{\Gamma_t}, S)\), then let \( n \to \infty \), we may get the price. By theorem 4.1 since \( \xi \) is attainable, the result follows. □
6.1 Some examples of American call option

We study the American call option, set 

\[ S_t = (X_t - k)^+ \]

\[ \xi = (X_T - k)^+ \]

where \( X \) is the price of underlying stock and \( k \) is the strike price. More precisely, \( X \) is the solution of

\[ X_t = x_0 + \int_0^t \mu_s X_s ds + \int_0^t \sigma_s X_s dB_s. \]  

(38)

Correspondingly, in (37) \( g \) is a linear function

\[ g(t, y, \pi) = -r_t y - (\mu_t - r_t) \pi^T \sigma_t. \]

Proposition 6.1. If \( \xi \) is attainable, then the maturity time of American call option in incomplete market is still \( T \).

Proof. We have that \( Y_t \geq Y_0 t \geq 0 \), \( A_t \leq A_0 t \), \( t \in [0, T] \), comparing (37) and \( Y_0 \), where \( Y_0 \) is the price process of American call option without constraint, which satisfies a reflected BSDE

\[ Y_t^0 = \xi + \int_t^T g(s, Y_s^0, \pi_s^0) ds + \overline{A}_T - \overline{A}_t - \int_t^T (\pi_s^0)^T \sigma_s dB_s, \]

\[ Y_t^0 \geq S_t, \quad \int_0^T (Y_t^0 - S_t) d\overline{A}_t = 0. \]

Since American call option always exercises at terminal time \( T \), which implies \( \overline{A}_t = 0 \) and \( D_t = T \), where \( D_t = \inf(t \leq s \leq T | Y_s^0 = S_s) \wedge T \). So we have \( Y_t^0 > S_t \) on \([0, T]\). It follows that \( Y_t \geq Y_t^0 > S_t \) on \([0, T]\) and \( \overline{A}_t \leq \overline{A}_t^0 = 0, t \in [0, T] \). Then \( D_t = T \). \( \square \)

From this proposition, we know that there is no difference between the American call option and European call option even in incomplete market.

Example 6.1. No short-selling: In this case \( \Gamma_t = [0, \infty), \) for \( t \in [0, T] \). We set \( d = 1 \). By the proposition 6.1 and Example 7.1 in [8], the price process of the American call option takes same value as European call option. This means that the constraint \( K = [0, \infty) \) does not make any difference.

In fact, we have a more general result.

Proposition 6.2. Consider the constraint \( \Gamma_t = [0, \infty), \) for \( t \in [0, T] \). If \( \xi = \Phi(X_T), \)

\( S_t = l(X_t) \)

where \( \Phi, l : \mathbb{R} \to \mathbb{R} \) are both increasing in \( x \), and \( \sigma \) satisfies the uniformly elliptic condition, then the price process \( Y \) takes same value as in complete market, i.e. the constraint \( \Gamma \) does not influence the price.

Proof. It is sufficient to prove that \( \pi_t \geq 0 \), where \( (\overline{Y}, \pi, \overline{A}) \) is the solution of following reflected BSDE

\[ \overline{Y}_t = \Phi(X_T) + \int_t^T g(s, \overline{Y}_s, \pi_s) ds + \overline{A}_T - \overline{A}_t - \int_t^T \pi_s^T \sigma_s dB_s, \]

\[ \overline{Y}_t \geq l(X_t), \quad \int_0^T (\overline{Y}_t - l(X_t)) d\overline{A}_t = 0. \]
We put \((X_t^s, Y_t^s, \pi_t^s, \overline{A}_t^s)_{t \leq s \leq T}\) under Markovian framework. Define
\[
u(t, x) = Y_t^x,
\]
then by [15], we know that \(\nu\) is the viscosity solution of the PDE with an obstacle \(l\),
\[
\min\{\nu(t, x) - l(x), -\frac{\partial \nu}{\partial t} - \mathcal{L} \nu - g(t, x, u, \nabla u\sigma)\} = 0,
\]
\[
\nu(T, x) = \Phi(x),
\]
where \(\mathcal{L} = \frac{1}{2}(\sigma_\tau)^2 \frac{\partial^2}{\partial x^2} + \mu \frac{\partial}{\partial x}\). Since \((\pi_t^x)\tau \in \nabla u\sigma(r, X_t^x)\), and \(\sigma\) is uniformly elliptic, we only need to prove that \(\nabla u(t, x)\) is non-negative. Indeed, it is easy to obtain by comparison theorem. For \(x_1, x_2 \in \mathbb{R}\), with \(x_1 \geq x_2, X_t^{s, x_1} \geq X_t^{s, x_2}\). It follows that \(\Phi(X_t^{s, x_1}) \geq \Phi(X_t^{s, x_2})\) and \(l(X_t^{s, x_1}) \geq l(X_t^{s, x_2})\) in view of assumptions. By comparison theorem of BSDE, \(Y_t^{x, x_1} \geq Y_t^{x, x_2}\), which implies \(u(t, x_1) \geq u(t, x_2)\). So \(\nabla u(t, x) \geq 0\), it follows that \(\pi_t^x \geq 0\). □

### 6.2 Some examples of American put option

In this case, we set \(S_t = (k - X_t)^+, \xi = (k - X_T)^+\), where \(X\) is the price of underlying stock as in (38) and \(k\) is the strike price. Similarly to proposition 6.2, we have

**Proposition 6.3.** Consider the constraint \(\Gamma_t = (-(\infty, 0], for t \in [0, T]. If \(\xi = \Phi(X_T), S_t = l(X_t)\), where \(\Phi, l: \mathbb{R} \to \mathbb{R}\) are both decreasing functions, and \(\sigma\) satisfies uniformly elliptic condition, then the price process \(Y\) takes same value as in complete market, i.e. the constraint \(\Gamma\) has no influence on price process.

**Proof.** Similar to the proof of proposition 6.2, it is sufficient to prove that \(\pi_t \leq 0\). With the help of viscosity solution, we get the result. □

**Example 6.2.** No borrowing: \(\Gamma_t = (-\infty, Y_t]\). Obviously, \(Y_t \geq 0\), in view of \(Y_t \geq S_t \geq 0\). So \(\Gamma_t \supset (-\infty, 0]\), by proposition 6.3, we know that the price process \(Y\) takes same value as in complete market. This means that to replicate an American put option, we don’t need to borrow money.

**Example 6.3.** No short-selling: \(\Gamma_t = [0, \infty), for t \in [0, T]. Then the pricing process \(Y\) with hedging \(\pi\) satisfying
\[
Y_t = \xi + \int_t^T g(s, Y_s, \pi_s)ds + A_T - A_t + \overline{A}_T - \overline{A}_t - \int_t^T \pi^s \sigma_s dB_s,
\]
\[
Y_t \geq S_t, 0 \leq t \leq T, \int_0^T (Y_t - S_t)d\overline{A}_t = 0, \pi_t \geq 0, t-a.e.
\]

Notice that \(S_t = (k - X_t)^+ < k\). So the \(\pi\)-solution of the above equation is
\[
Y_t = \begin{cases} 
  k, & t \in [0, T) \\
  (k - X_T)^+, & t = T 
\end{cases}, \quad \pi_t = 0,
\]
\[
A_t = \begin{cases} 
  k \int_0^t r_s ds, & t \in [0, T) \\
  k \int_0^T r_s ds + k - (k - X_T)^+, & t = T 
\end{cases}, \quad \overline{A}_t = 0.
\]
In particular, $Y_0 = k$, which is the price of American put option under ‘no short-selling’ constraint.

7 Appendix

In appendix, we recall some results of $g_\Gamma$-solution in [35], and proves some comparison results of $g_\Gamma$-solution. In [35], $\Gamma$ is defined as

$$\Gamma_t(\omega) = \{(y, z) \in \mathbb{R}^{1+d} : \Phi(\omega, t, y, z) = 0\},$$

where $\Phi$ is a nonnegative, measurable Lipschitz function and $\Phi(\cdot, y, z) \in L^2_F(0, T)$, for $(y, z) \in \mathbb{R} \times \mathbb{R}^d$. Under the following assumption, the existence of the smallest solution is proved.

The following theorem of the existence of the smallest solution was obtained in [35].

**Theorem 7.1.** Suppose that the function $g$ satisfies [7] and the constraint $\Gamma$ satisfies [2]. We assume that there is at least one $\Gamma$–constrained $g$–supersolution $y' \in D^1_F(0, T)$:

$$y'_t = X' + \int_t^T g(s, y'_s, z_s) ds + A'_t - A'_t - \int_t^T z'_s dB_s, \quad (40)$$

$$A \in A^2_F(0, T), (y'_t, z'_t) \in \Gamma_t, t \in [0, T], \text{ a.s. a.e.} \quad (41)$$

Then, for each $X \in L^2(F_T)$ with $X \leq X'$, a.s., there exists the $g_\Gamma$-solution $y \in D^2_F(0, T)$ with the terminal condition $y_T = X$ (defined in Definition [24]). Moreover, $g_\Gamma$-solution is the limit of a sequence of $g^n$–solutions $y^n_t$ with $g^n = g + nd_\Gamma$, where

$$y^n_t = X + \int_t^T (g + nd_\Gamma)(s, y^n_s, z^n_s) ds - \int_t^T z^n_s dB_s, \quad (42)$$

with the convergence in the following sense:

$$y^n_t \nearrow y_t, \text{ with } \lim_{n \to \infty} E[|y^n_t - y_t|^2] = 0, \quad \lim_{n \to \infty} E \int_0^T |z_t - z^n_t|^p dt = 0, \quad (43)$$

$$A^n_t = \int_0^t (g + nd_\Gamma)(s, y^n_s, z^n_s) ds \to A_t \text{ weakly in } L^2(F_t), \quad (44)$$

where $z$ and $A$ are corresponding martingale part and increasing part of $y$, respectively.

**Proof.** By the comparison theorem of BSDE, $y^n_t \leq y^{n+1}_t \leq y'_t$. It follows that there exists a $y \leq y'$ such that, for each $t \in [0, T]$,

$$y^1_t \leq y^n_t \nearrow y_t \leq y'_t. \quad (45)$$

Consequently, there exists a constant $C > 0$, independent of $n$, such that

$$E[\sup_{0 \leq t \leq T} (y^n_t)^2] \leq C \quad \text{and} \quad E[\sup_{0 \leq t \leq T} (y^n_t)^2] \leq C. \quad (46)$$

Thanks to the monotonic limit Theorem 2.1 in [35], we can pass limit on both sides of BSDE (41) and obtain

$$y_t = X + \int_t^T g(s, y_s, z_s) ds + A_T - A_t - \int_t^T z_s dB_s. \quad (47)$$

On the other hand, by $E[(A^n_T)^2] = n^2 E[(\int_0^T d\Gamma_t(y^n_s, z^n_s) ds)^2] \leq C$, we have $d\Gamma_t(y_t, z_t) \equiv 0$. \qed
Remark 7.1. From the approximation \[47\] it is clear that, as \(n\) tends to \(\infty\), the coefficient \(g + nd_t\) tends to a singular coefficient \(g_t\) defined by
\[
gr_t(t, y, z) := g(t, y, z)1_{\Gamma_1}(y, z) + \infty \times 1_{\Gamma_2}(y, z).
\]
Thus, in the above theorem, the \(g_t\)-solution is also the solution of BSDE with singular coefficient \(g_t\).

Remark 7.2. If the constraint \(\Gamma\) is of the following form \(\Gamma_t = (-\infty, U_t] \times \mathbb{R}^d\), where \(U_t \in L^2(\mathcal{F}_t)\), then the smallest \(\Gamma\)-constrained \(g\)-supersolution solution with terminal condition \(y_T = X\) exists, if and only if \(d_{\Gamma_t}(Y_t, Z_t) \equiv 0\), a.s. a.e., where \((Y, Z)\) is the solution of the BSDE
\[
-dY_t = g(t, Y_t, Z_t)dt - Z_tdB_t, \quad t \in [0, T], \quad Y_T = X.
\]
This follows easily by comparison theorem.

We also have

**Theorem 7.2** (Comparison Theorem of \(g_t\)-solution). We assume that \(g^1, g^2\) satisfy \[7\] and \(\Gamma^1, \Gamma^2\) satisfy \[2\]. And suppose that \(\forall (t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d\),
\[
X^1 \leq X^2, g^1(t, y, z) \leq g^2(t, y, z), \Gamma^1_t \supseteq \Gamma^2_t,
\]
(44)
For \(i = 1, 2\), Let \(Y^i \in D^2_{\mathcal{F}}(0, T)\) be the \(g^i\)-solution with terminal condition \(Y^i_T = X^i\). Then we have
\[
Y^1_t \leq Y^2_t, \quad \text{for} \ t \in [0, T], \ a.s.
\]

**Proof.** Consider the penalization equations for the two constrained BSDE: for \(n \in \mathbb{N}\)
\[
y^{1, n}_t = X^1 + \int_t^T g^{1, n}(s, y^{1, n}_s, z^{1, n}_s)ds - \int_t^T z^{1, n}_s dB_s,
\]
\[
y^{2, n}_t = X^2 + \int_t^T g^{2, n}(s, y^{2, n}_s, z^{2, n}_s)ds - \int_t^T z^{2, n}_s dB_s,
\]
(45)
where
\[
g^{1, n}(t, y, z) = g^1(t, y, z) + nd_{\Gamma^1_t}(y, z),
\]
\[
g^{2, n}(t, y, z) = g^2(t, y, z) + nd_{\Gamma^2_t}(y, z).
\]
From \(44\) we have \(g^{1, n}(t, y, z) \leq g^{2, n}(t, y, z)\). It follows from the classical comparison theorem of BSDE that \(y^{1, n}_t \leq y^{2, n}_t\). While as \(n \to \infty\), \(y^{1, n} \nearrow y^1\) and \(y^{2, n} \nearrow y^2\), where \(y^1, y^2\) are the \(g_t\)-solutions of the BSDEs respectively. It follows that \(y^1_t \leq y^2_t\), \(0 \leq t \leq T\). \(\square\)

The comparison theorem is a powerful tool and useful concept in BSDE Theorem (cf. \[16\]). Here let us recall the main theorem of reflected BSDE and related comparison theorem for the case of lower obstacle \(L\). We do not repeat the case for the upper obstacle since it is essentially the same. This result, obtained in \[38\], is a generalized version of \[13\, 19\, 29\] for the part of existence, and \[22\] for the part of comparison theorem.
Theorem 7.3 (Reflected BSDE and related Comparison Theory). We assume that the coefficient $g$ satisfies Lipschitz condition (1) and the lower obstacle $L$ satisfies (9). Then, for each $X \in L^2(F_T)$ with $X \geq L_T$ there exists a unique triple $(y, z, A) \in D^2_{\mathcal{F}}(0, T) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^d) \times (D^2_{\mathcal{F}}(0, T))$, where $A$ is an increasing process, such that
\[ y_t = X + \int_t^T g(s, y_s, z_s)ds + A_T - A_t - \int_t^T z_s dB_s \]
and the generalized Skorokhod reflecting condition is satisfied: for each $L^* \in D^2_{\mathcal{F}}(0, T)$ such that $y_t \geq L^*_t \geq L_t$, dP x dt a.s., we have
\[ \int_0^T (y_s - L^*_s) d\overline{A}_s = 0, \text{ a.s.,} \]
Moreover, if a coefficient $g'$ an obstacle $L'$ and terminal condition $X'$ satisfy the same condition as $g$, $L$ and $X$ with for $\forall (t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$,
\[ X' \leq X, g'(t, y, z) \leq g(t, y, z), L'_t \leq L_t, \text{ dP x dt - a.s.,} \]
and if the triple $(y', z', A')$ is the corresponding reflected solution, then we have
\[ Y'_t \leq Y_t, \quad \forall \ t \in [0, T], \text{ a.s.} \]
and for each $0 \leq s \leq t \leq T$,
\[ A'_t \leq A_t, \quad A'_t - A'_s \leq A_t - A_s. \]

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