GAUSSIAN FLUCTUATIONS OF YOUNG DIAGRAMS AND STRUCTURE CONSTANTS OF JACK CHARACTERS

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ABSTRACT. In this paper, we consider a deformation of Plancherel measure linked to Jack polynomials. Our main result is the description of the first and second-order asymptotics of the bulk of a random Young diagram under this distribution, which extends celebrated results of Vershik-Kerov and Logan-Shepp (for the first order asymptotics) and Kerov (for the second order asymptotics). This gives more evidence of the connection with Gaussian $\beta$-ensemble, already suggested by some work of Matsumoto.

Our main tool is a polynomiality result for the structure constant of some quantities that we call Jack characters, recently introduced by Lassalle. We believe that this result is also interested in itself and we give several other applications of it.

1. INTRODUCTION

1.1. Jack deformation of Plancherel measure and random matrices. Random partitions occur in mathematics and physics in a wide variety of contexts, in particular in the Gromov-Witten and Seiberg-Witten theories, see [Oko03] for an introduction to the subject. Another aspect which makes attractive the study of random partitions is the link with random matrices. Indeed, some classical models of random matrices have random partition counterparts, which display the same kind of asymptotic behaviour.

In this paper, we consider the following probability measure on the set of partitions (or equivalently, Young diagrams) of size $n$:

$$P_n^{(\alpha)}(\lambda) = \frac{\alpha^n n!}{j_{\lambda}^{(\alpha)}},$$

where $j_{\lambda}^{(\alpha)}$ is a deformation of the square of the hook products:

$$j_{\lambda}^{(\alpha)} = \prod_{\square \in \lambda} \left( \alpha a(\square) + \ell(\square) + 1 \right) \left( \alpha a(\square) + \ell(\square) + \alpha \right).$$

Here, $a(\square) := \lambda_j - i$ and $\ell(\square) := \lambda'_i - j$ are respectively the arm and leg length of the box $\square = (i, j)$ (the same definitions as in [Mac95], Chapter I)).

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When $\alpha = 1$, the measure $P_n^{(\alpha)}$ specializes to the well-known Plancherel measure for the symmetric groups. In general, it is called Jack deformation of Plancherel measure (or Jack measure for short), because of its connection with the celebrated Jack polynomials that we shall explain later. It has appeared recently in several research papers [BO05, Ful04, Mat08, Ols10, Mat10] and is presented as an important area of research in Okounkov’s survey on random partitions [Oko03, Section 3.3].

Recall that Plancherel measure has a combinatorial interpretation: it is the push-forward of the uniform measure on permutations by Robinson-Schensted algorithm (we keep only the common shape of the tableaux in the output of RS algorithm). A similar description holds for Jack measure for $\alpha = 2$ (and $\alpha = 1/2$ by symmetry): it is the push-forward of the uniform measure on fixed point free involutions by RS algorithm (in this case, the resulting diagram has always only even parts and we divide each part by 2) – see [Mat08, Section 3].

Thus Jack measure can be seen as an interpolation between these two combinatorially relevant models of random partitions.

1.1.1. $\alpha = 1$ case: Plancherel measure and GUE ensemble. There is a strong connection between the Plancherel measure and the Gaussian unitary ensemble (called GUE) in random matrix theory. The Gaussian unitary ensemble is a random Hermitian matrix with independent normal entries. The probability density function for the eigenvalues of that matrix (of the size $d \times d$) is proportional to the weight

$$e^{-\beta/2} \sum x_i^2 \prod_{i<j \leq d} (x_i - x_j)^\beta$$

with $\beta = 2$ (see [AGZ10]). Looking for the rescaled spectral measure of the GUE ensemble

$$\mu_d^{(2)} := \frac{1}{d} (\delta_{x_1} + \cdots + \delta_{x_d}),$$

where $x_1 \geq \cdots \geq x_d$ are eigenvalues, the famous Wigner law states that, as $d \to \infty$, the spectral measure tends almost surely to a semicircular law, i.e. to a probability measure $\mu_{SC} := \frac{2}{\pi} \sqrt{1-x^2} 1_{[-2,2]}(x) dx$ supported on the interval $[-2,2]$ (see [AGZ10]). The second order asymptotics is also known and one can observe Gaussian fluctuations around the limiting distribution (see [Joh98]). Informally speaking, looking at the rescaled spectral measure of GUE as a generalized function

$$\mu_d^{(2)}(x) = \frac{1}{d} (\delta_{x-x_1} + \cdots + \delta_{x-x_d})$$

we have that

$$\mu_d^{(2)}(x) \sim \mu_{SC}(x) + \frac{1}{d} \Delta^{(2)}(x),$$

as $d \to \infty$, where $\Delta^{(2)}(x)$ is a Gaussian process on the interval $[-2,2]$ with values in the space of generalized functions $(C^\infty(\mathbb{R}))'$. It was discovered that a similar phenomenon holds for the Plancherel measure. The first order asymptotics for the Plancherel measure was found by Vershik and
Kerov \([KV77]\) and, independently, by Logan and Shepp \([LS77]\). They noticed that a Young diagram \(\lambda\) can be encoded by a continuous piecewise-affine \(\omega(\lambda)\) function from \(\mathbb{R}\) to \(\mathbb{R}\): this encoding is represented on Figure 1 and formally defined in Section 2.4. Then they proved that for appropriately scaled Young diagrams \(\lambda(n)\) of size \(n\) distributed according to the Plancherel measure (the bar encodes the scaling), one has the convergence in probability
\[
\sup_{x \in \mathbb{R}} |\omega(\lambda(n))(x) - \Omega(x)| \xrightarrow{(P)} 0
\]
where the limit shape \(\Omega\) is given by
\[
\Omega(x) = \begin{cases} 
| x | & \text{if } |x| \geq 2; \\
\frac{1}{2} \left( x \cdot \arcsin \frac{x}{2} + \sqrt{4 - x^2} \right) & \text{otherwise}.
\end{cases}
\]

There is a strong connection between this limit shape and the semicircular law \(\mu_{S-C}\), namely the so-called transition measure of the continuous Young diagram \(\Omega\) is the semicircular law \(\mu_{S-C}\) — see \([Bia98, Section 1.2]\).

The problem of the second order asymptotics was stated as an open question in late seventies and was solved by Kerov \([Ker93a]\) who proved that, exactly as in the GUE case, the fluctuations around the limit shape are Gaussian. Informally Kerov’s result can be presented as follows
\[
\omega(\lambda(n))(x) \sim \Omega(x) + \frac{2}{\sqrt{n}} \Delta^{(1)}_{\infty}(x)
\]
where \(\Delta^{(1)}_{\infty}\) is the Gaussian process on \([-2, 2]\) with values in \((C^\infty(\mathbb{R}))'\) defined by:
\[
\Delta^{(1)}_{\infty}(2 \cos(\theta)) = \frac{1}{\pi} \sum_{k=2}^{\infty} \frac{\xi_k}{\sqrt{k}} \sin(k\theta).
\]

The detailed proof of this remarkable theorem can be found in \([IO02]\). Although they are not equal, the Gaussian processes \(\Delta^{(1)}_{\infty}\) (which describe bulk fluctuations...
of Young diagram under Plancherel measure) and $\Delta^{(2)}$ (which describe bulk fluctuations of GUE random matrices) have quite similar definition.

But the similarity between these two objects does not only take place at the level of bulk fluctuations but also "at the edge". To be more precise, it was proved that the distribution of finitely many (properly scaled) first rows of random Young diagrams (with respect to the Plancherel measure) is the same as the distribution of the same number of (properly scaled) largest eigenvalues of the GUE ensemble, as $n \to \infty$ (see [BDJ99, Oko00, BOO00, Joh01a, Joh01b, BDR01] for details).

1.1.2. General $\alpha$-case and Gaussian $\beta$-ensembles. There are two famous analogues of the GUE ensembles in random matrix theory: Gaussian orthogonal ensembles (GOE) and Gaussian symplectic ensembles (GSE) (see [Meh04]). The GOE ensemble (resp. GSE ensemble) is a random real symmetric matrix (resp. complex self-adjoint quaternion matrix) with independent normal entries (with mean 0 and well-chosen variance). The density function for the distribution of eigenvalues of GOE (GSE, respectively) is, up to normalization, the function given by (2) with parameter $\beta = 1$ ($\beta = 4$, respectively). Therefore, it is tempting to introduce Gaussian $\beta$-ensemble ($\mathbf{G}\beta\mathbf{E}$, also called $\beta$-Hermite ensemble) that has a distribution density function proportional to (2) for any positive real value of $\beta$.

The $\mathbf{G}\beta\mathbf{E}$ ensembles are well-studied objects. For the first order of asymptotic behaviour of the spectral measure

$$\mu_d^{(\beta)} := \frac{1}{d} \left( \frac{\delta_{\beta x_1}}{x_1} + \cdots + \frac{\delta_{\beta x_d}}{x_d} \right),$$

where $x_1 \geq \cdots \geq x_d$ are eigenvalues of the $\mathbf{G}\beta\mathbf{E}$ of size $d \times d$, Johansson [Joh98] showed that the Wigner law holds, i.e., as $d \to \infty$ then

$$\mu_d^{(\beta)} \to \mu_{SC}$$

almost surely. A central limit theorem for the $\mathbf{G}\beta\mathbf{E}$ was proved by Dumitriu and Edelman [DE06]. Here, we can observe Gaussian fluctuations around the limit shape, similarly to GUE case. However, one can see a very interesting phenomenon: the Gaussian process that describes the second order asymptotic is translated by a deterministic shift, which disappears for $\beta = 2$ (see [DE06] for details).

A natural question is to find a discrete counterpart for $\mathbf{G}\beta\mathbf{E}$. Some results of Matsumoto [Mat08] suggest that a good candidate for such probability measure on the set of Young diagrams is Jack measure given by (1), where the relation between parameters $\alpha$ and $\beta$ is given by $\beta = \frac{2}{\alpha}$. Matsumoto was studying a restriction of Jack measure to the set of Young diagrams of size $n$ with at most $d$ rows. His main result states that the joint distribution of that suitably normalized $d$ rows is the same as the joint distribution of the eigenvalues of $d$-dimensional traceless $\mathbf{G}\beta\mathbf{E}$ with $\beta = \frac{2}{\alpha}$, as $d$ is fixed and $n \to \infty$.

1.2. Main result. The main result of this paper is the description of the first and second-order asymptotic of the bulk of Young diagrams under Jack measure.
First, we prove a law of large numbers. If \( \lambda \) is a Young diagram of size \( n \), we denote by \( A_\alpha(\lambda) \) the (generalized) Young diagram obtained from \( \lambda \) by a horizontal stretching of ratio \( \sqrt{\alpha/n} \) and a vertical stretching of ratio \( 1/\sqrt{n\alpha} \) (a formal definition of generalized Young diagrams is given in Section 2.3). We will prove in Section 6 the following result.

**Theorem 1.1.** For each \( n \), let \( \lambda(n) \) be a random Young diagram of size \( n \) distributed according to Jack measure. Then

\[
\lim_{n \to \infty} \sup_{x \in \mathbb{R}} \left| \omega(A_\alpha(\lambda(n))) (x) - \Omega(x) \right| = 0
\]

in probability.

Note that the limiting curve is exactly the same as in the case \( \alpha = 1 \).

Then we establish a central limit theorem. Informally, it can be presented as follows.

\[
\omega(A_\alpha(\lambda(n))) (x) \sim \Omega(x) + \frac{2}{\sqrt{n}} \Delta_\alpha^\infty(x),
\]

where \( \Delta_\alpha^\infty \) is the Gaussian process on \([-2,2]\) with values in \((C^\infty(\mathbb{R}))'\) defined by:

\[
\Delta_\alpha^\infty(2 \cos(\theta)) = \frac{1}{\pi} \sum_{k=2}^{\infty} \frac{\Xi_k}{\sqrt{k}} \sin(k\theta) - \gamma/4 + \gamma \theta/2\pi.
\]

Here, and throughout the paper, \( \gamma \) is the difference \( 1/\sqrt{\alpha} - \sqrt{\alpha} \). The formal version of this result is stated and proved in Section 8 while the explanation for the informal reformulation is given in Section 8.3. Note that the random part of the second-order asymptotics is independent of \( \alpha \), while a deterministic term proportional to \( \gamma \) (and, hence, vanishing for \( \alpha = 1 \)) appears.

Here again, the similarity with G\( \beta \)E ensemble is striking. Indeed, for the bulk of the spectral measure of a G\( \beta \)E ensemble, we also have the following phenomena:

- the first-order asymptotics is independent of \( \alpha \) – see equation (4);
- the second-order asymptotics is the sum of two terms: a random one and a deterministic one. Moreover, the quotient of the deterministic one over the random one is proportional to \( \gamma \) (see [DE06, Theorem 1.2]).

Therefore our result is a new hint for the deep connection between Jack measure and the G\( \beta \)E ensemble.

### 1.3. Jack polynomials and Jack measure.

To explain our intermediate results and the main steps of the proof, we first need to review the connection between Jack measure and Jack polynomials.

#### 1.3.1. Jack polynomials.

In a seminal paper [Jac71], Jack introduced a family of symmetric functions \( J^{(\alpha)}_\lambda \) depending on an additional parameter \( \alpha \). These functions are now called Jack polynomials. For some special values of \( \alpha \), they coincide with some established families of symmetric functions. Namely, up to multiplicative constants, for \( \alpha = 1 \) Jack polynomials coincide with Schur polynomials, for
For $\alpha = 2$ they coincide with zonal polynomials, for $\alpha = \frac{1}{2}$ they coincide with symplectic zonal polynomials, for $\alpha = 0$ we recover the elementary symmetric functions and finally their highest degree component in $\alpha$ are the monomial symmetric functions. Moreover, some other specializations appear in different contexts: the case $\alpha = 1/k$, where $k$ is an integer, has been considered by Kadell in relation with generalizations of Selberg’s integral [Kad97]. In addition, Jack polynomials for $\alpha = -(k + 1)/(r + 1)$ verify some interesting annihilation conditions [FJMM02] and this property makes them useful in some statistical physics models.

Over the time it has been shown that several results concerning Schur and zonal polynomials can be generalized in a rather natural way to Jack polynomials (Section (VI.10) of Macdonald’s book [Mac95] gives a few results of this kind), therefore Jack polynomials can be viewed as a natural interpolation between several interesting families of symmetric functions.

1.3.2. A characterization of Jack measure. Expanding Jack symmetric function $J^{(\alpha)}(\lambda)$ in power-sum symmetric basis the coefficients $\theta^{(\alpha)}_\rho(\lambda)$ are defined simply by the following equation

\[
J^{(\alpha)}(\lambda) = \sum_{\rho, |\rho| = |\lambda|} \theta^{(\alpha)}_\rho(\lambda) p_\rho.
\]

In the case $\alpha = 1$, Jack polynomials specialize to

\[
J^{(1)}(\lambda) = \frac{n!}{\dim(\lambda)} s_\lambda,
\]

where $s_\lambda$ is the Schur function and $\dim(\lambda)$ the dimension of the symmetric group representation associated to $\lambda$. Hence, using Frobenius formula [Mac95], we can express $\theta^{(1)}_\rho(\lambda)$ in terms of irreducible character values of the symmetric group. Namely

\[
\theta^{(1)}_\rho(\lambda) = \frac{n!}{z_\rho \dim(\lambda)} \chi^\lambda_\rho,
\]

where $\chi^\lambda_\rho$ is the character of the irreducible representation indexed by $\lambda$ evaluated on a permutation of cycle-type $\rho$. By analogy to this, we use in the general case the terminology Jack characters (while they do not have any representation theoretical interpretation, they share a lot of property with characters of symmetric groups).

The following property, which corresponds to the case $\pi = (1^n)$ of [Mat10, Equation (8.4)], characterizes Jack measure:

\[
\mathbb{E}_{\mathbb{P}^{(\alpha)}_n}(\theta_\rho(\lambda)) = \delta_{\rho,1^n},
\]

where $\lambda$ is a random Young diagram with $n$ boxes distributed according to $\mathbb{P}^{(\alpha)}_n$.

1.3.3. A central limit theorem for Jack characters. As in the case $\alpha = 1$, an important intermediate result, which may be of independent interest, is an algebraic central limit theorem. Namely we prove that Jack characters for hooks (i.e. $\rho$ is a hook) have joint Gaussian fluctuations.
Theorem 1.2. Choose a sequence \((\Xi_k)_{k=2,3,...}\) of independent standard Gaussian random variables. Let \((\lambda_n(n))_{n\geq 1}\) be a sequence of random Young diagrams, \(\lambda(n)\) having size \(n\) and being distributed according to Jack measure. Define
\[
W_k(\lambda) = \sqrt{k} \frac{\theta^{(\alpha)}_{(k,1^{n-k})}(\lambda_{(n)})}{n^{k/2}}.
\]
Then, as \(n \to \infty\), we have:
\[
(W_k)_{k=2,3,...} \xrightarrow{d} (\Xi_k)_{k=2,3,...},
\]
where \(\xrightarrow{d}\) means convergence in distribution of the finite-dimensional law.

In the case \(\alpha = 1\), this theorem was proved independently by Kerov [IO02] and Hora [Hor98]. With the method developed here, one can even give an upper bound on the speed of convergence of the distribution function:

Theorem 1.3. We use the same notation as in Theorem 1.2. Then, for any integer \(d \geq 2\) and real numbers \(x_2, \cdot \cdot \cdot, x_d\),
\[
|\mathbb{P}(W_2 \leq x_2, \cdot \cdot \cdot, W_d \leq x_d) - \mathbb{P}(\Xi_2 \leq x_2, \cdot \cdot \cdot, \Xi_d \leq x_d)| = O(n^{-1/4}),
\]
where the constant hidden in the Landau symbol \(O\) is uniform on \(x_2, \cdot \cdot \cdot, x_d\), but depends on \(d\).

The case of \(W_2\) has already been established by Fulman [Ful04]. Some ideas from this paper are fundamental here, as explained below.

The bound on the speed of convergence for the vector of random variables \((W_2, \cdot \cdot \cdot, W_d)\) is also new in the case \(\alpha = 1\), even if, for this case, all ingredients were already in the literature.

Remark. We are not able to describe the fluctuations of Jack character \(\theta^{(\alpha)}_{\rho}\) when \(\rho\) is not a hook. This is discussed in Subsection 7.7.

1.4. Ingredients of the proof. We shall now say a word on the proof of our main result and compare it to the one of Ivanov, Kerov and Olshanski.

Remark. While beautiful and elementary, Hora’s proof of the central limit theorem for characters in the case \(\alpha = 1\) seems very hard to generalized to any value of \(\alpha\) as it relies from the beginning on the representation-theoretical background.

1.4.1. Polynomial functions. A central idea in the paper [IO02] is to consider characters as functions on all Young diagrams by defining:
\[
\text{Ch}_{\mu}^{(1)}(\lambda) = \begin{cases} 
n(n-1) \cdot \cdot \cdot (n-k+1) \frac{\lambda^{\alpha}}{\min(\lambda)} & \text{if } |\lambda| \geq |\mu| \\
0 & \text{if } |\lambda| < |\mu|.
\end{cases}
\]
In Sections 1-4 of paper [IO02], the authors prove that the functions \(\text{Ch}_{\mu}^{(1)}\) span linearly a subalgebra of the algebra of functions on all Young diagrams, give several equivalent descriptions of this subalgebra and describe combinatorially the product \(\text{Ch}_{\mu}^{(1)} \text{Ch}_{\nu}^{(1)}\) (\(\text{Ch}_{\mu}^{(1)}\) is denoted \(p_{\mu}^{\#}\) in [IO02]). This subalgebra is called the algebra
of polynomial functions on Young diagrams (see also \cite{KO94}) and denoted here $\Lambda_\star^{(1)}$.

In the general $\alpha$-case, one can define a deformation of the function above as follows: for an integer partition $\mu$, denote $|\mu|$ its size, $\ell(\mu)$ its length, $m_i(\mu)$ its number of parts of $\mu$ equal to $i$ and $z_\mu$ the standard numerical factor $\prod_i i^{m_i(\mu)}m_i(\mu)!$. We define

$$\text{Ch}^{(\alpha)}_{\mu}(\lambda) = \left\{ \begin{array}{ll} \alpha^{\frac{|\mu|-\ell(\mu)}{2}} \frac{|\lambda|-|\mu|+m_1(\mu)}{m_1(\mu)} z_\mu \theta^{(\alpha)}_{\mu,1,|\lambda|-|\mu|}(\lambda). & \text{if } |\lambda| \geq |\mu|; \\
0 & \text{if } |\lambda| < |\mu|. \end{array} \right.$$  

While Jack characters have been studied for a long time, the idea, due to Lassalle, to look at them as a function of $\lambda$ as above is quite recent \cite{Las08, Las09}. Among other things, he proved that, as in the case $\alpha = 1$, the functions $\text{Ch}^{(\alpha)}_{\mu}$ span linearly a subalgebra of functions on all Young diagrams, which has a nice characterization: we present these results in Section 2, see in particular Proposition 2.9. This subalgebra is called the algebra of $\alpha$-polynomial functions on Young diagrams (see also \cite{KO94}) and denoted here $\Lambda_\star^{(\alpha)}$.

As a function on all Young diagram, $\text{Ch}^{(\alpha)}_{\mu}$ can be restricted to diagrams of size $n$ and hence considered as a random variable in our problem. Its expectation is straightforward to compute:

$$\mathbb{E}_{\mathbb{P}_n}^{(\alpha)}(\text{Ch}^{(\alpha)}_{\mu}(\lambda)) = \begin{cases} n(n-1) \cdots (n-k+1) & \text{if } \mu = 1^k \text{ for some } k \leq n, \\ 0 & \text{otherwise}. \end{cases} $$  

1.4.2. Moment method and structure constants. The idea in paper \cite{IO02} is to use the method of moments and thus to compute asymptotically (for $h \geq 1$)

$$\mathbb{E}_{\mathbb{P}_n}^{(1)}(\text{Ch}^{(1)}_{(k)}(\lambda(n))^h).$$

Recall that the algebra $\Lambda_\star^{(1)}$ has a linear basis given by the family of normalized characters $\left( \text{Ch}^{(1)}_{\mu(\lambda(n))} \right)_\mu$. As the expectation of $\text{Ch}^{(1)}_{\mu}(\lambda(n))$ is particularly simple – see equation (7) –, one can compute expectation of $\left( \text{Ch}^{(1)}_{(k)} \right)^h$ by expanding it on the basis $\text{Ch}^{(1)}_{(k)}$ of $\Lambda_\star^{(1)}$.

To do this, the authors of \cite{IO02} need to understand how a product $\text{Ch}^{(1)}_{\mu} \text{Ch}^{(1)}_{\rho}$ expands on the $\text{Ch}^{(1)}_{(k)}$ basis, that is to study the structure constants of this basis. They provide a combinatorial description of these structure constant \cite[Proposition 4.5]{IO02}. Unfortunately, this combinatorial description relies on the representation-theoretical interpretation of $\theta^{(1)}_{\rho}(\lambda)$ and has a priori no extension to a general value of $\alpha$.

To overcome this difficulty, we prove that the structure constants of the $\text{Ch}^{(\alpha)}_{\mu}$ basis depends polynomially on the auxiliary parameter $\gamma = \frac{1-\alpha}{\sqrt{\alpha}}$. This is a highly non-trivial result and has other interesting applications than the study of large Young diagrams under Jack measure. Therefore, we think that it may be of independent interest and present it in details in Section 1.5 as our second main result.
Our polynomiality result for structure constants (Theorem 1.4) allows us to show that some properties proved combinatorially in the case $\alpha = 1$ still holds in the general $\alpha$-case (we will also rely on the case $\alpha = 2$, which also has some combinatorial background, and use polynomial interpretation). Unfortunately, our result does not give a good estimate of all moments of $Ch_{(k)}^{(\alpha)}$ and we have to use another ingredient in our proof: the multivariate Stein’s method.

1.4.3. Multivariate Stein’s method and Fulman’s construction. Stein’s method is a classical method in probability to prove convergence in distribution towards Gaussian or Poisson distribution, together with bounds on the speed of convergence; see the monograph of Stein [Ste86]. To use it, one needs to construct an exchangeable pair for the relevant random variable. But, when this pair is constructed, one can prove Gaussian fluctuations, using only bounds on (mixed conditional) moments of order at most 4 (while the moment method requires control on moments of all order).

In the framework of Jack characters, an exchangeable pair has already been built by Fulman to prove its fluctuation result for $Ch_{(2)}^{(\alpha)}$. The same construction works for $Ch_{(k)}^{(\alpha)}$, but the analysis of the first moments becomes more tricky, requires new ideas and heavily relies on our polynomiality result for structure constants.

Let us note that, unlike Fulman’s result, our result is a result of convergence in distribution of vectors of random variables. Therefore we need to use a multivariate analog of Stein’s classical theorem. The one recently established by Reinert and Röllin [RR09] turns out to be suitable for our purpose.

Remark. As mentioned earlier, Stein’s method gives, in addition to the convergence in distribution towards a Gaussian vector, some bounds on the speed of convergence. However, it does not allow to prove the moment convergence.

1.5. Second main result: polynomiality of structure constant of Jack characters. It follows from the work of Lassalle – see Proposition 2.9 – that the functions $Ch_{\mu}^{(\alpha)}$ span linearly the algebra of $\alpha$-polynomial functions denoted by $\Lambda_{+}^{(\alpha)}$ (when $\mu$ runs over integer partitions of all sizes). Hence, there exist some real numbers $g_{\mu,\nu,\pi}^{(\alpha)}$, depending on $\alpha$ such that

$$(8) \quad Ch_{\mu}^{(\alpha)} \cdot Ch_{\nu}^{(\alpha)} = \sum_{\pi \text{ partition of any size}} g_{\mu,\nu,\pi}^{(\alpha)} Ch_{\pi}^{(\alpha)}.$$  

These numbers are often called structure constants of the basis $(Ch_{\mu}^{(\alpha)})$. It is a worthy goal to understand them, because they describe the multiplicative structure of the algebra.
Our second main result is a polynomiality result for these structure constants with precise bounds on the degree: denote

\[ n_1(\mu) = |\mu| + \ell(\mu), \]
\[ n_2(\mu) = |\mu| - \ell(\mu), \]
\[ n_3(\mu) = |\mu| - \ell(\mu) + m_1(\mu). \]

Then we have:

**Theorem 1.4.** Fix three partitions \( \mu, \nu \) and \( \pi \). The structure constant \( g^{(\alpha)}_{\mu,\nu;\pi} \) is a polynomial in \( \gamma = \frac{1 - \alpha}{\sqrt{\alpha}} \) with rational coefficients and of degree (at most)

\[ \min_{i=1,2,3} n_i(\mu) + n_i(\nu) - n_i(\pi). \]

Moreover, if \( n_1(\mu) + n_1(\nu) - n_1(\pi) \) is even (respectively, odd), it is an even (respectively, odd) polynomial.

1.5.1. **Other applications of the second main result.** We have found that Theorem [1.4] can be applied to several different problems from the literature.

- Our theorem contains a fifty-year old result from Farahat and Higman, stating that the structure constants of the symmetric group algebra behave polynomially in \( n \) [FH59].
- More recently, in analogy with Farahat and Higman’s result, Aker and Can [AC12] have studied the structure constants in the Hecke algebra of the pair \( (S_{2n}, H_n) \). Unfortunately, there is a hole in the proof of their main result [Can12].
  
  Theorem 1.4 implies also an analog of Farahat and Higman’s result in this context: up to some explicit normalization factor, structure constants also behave polynomially in \( n \). The same result has been proved independently by Tout [Tou13], after the release of an early version of this paper.
- In their article [GJ96a], using Jack polynomials, Goulden and Jackson have defined an interpolation between the structure constants of both algebras. By construction, these quantities are rational functions in \( \alpha \) but they conjectured that they are in fact polynomials in \( \alpha - 1 \) with non-negative integer coefficients having some combinatorial interpretation [GJ96a, Section 4]. Here, we prove that they are polynomials in \( \alpha \) (or equivalently in \( \alpha - 1 \)) with rational coefficients. Unfortunately, we are not able (and our method does not seem suitable) to prove either the integrity or the positivity of the coefficients.
- We are also able to prove two conjectures of Matsumoto [Mat10, Section 9], arising in the context of matrix integrals (Section B.3).
- We can give a short proof of a recent result of Vassilieva [Vas13] which generalizes a famous result of Dénes [Dén59] for the number of minimal factorizations of the cycle in the symmetric group.

The link between our main result and the first two items is presented in Section 5 while the connection with the last three items is explained in Appendix B.
1.5.2. Tool: Kerov’s polynomial for Jack characters. Let us now say a word about the proof of our second main result.

The algebra $\Lambda^{(\alpha)}$, linearly spanned by the functions $\text{Ch}^{(\alpha)}$, admits also some interesting algebraic basis: for example the basis of free cumulants $(R_{k}^{(\alpha)})_{k \geq 2}$ – see Section [2]. Thus $\text{Ch}^{(\alpha)}$ writes uniquely as a polynomial in free cumulants. As it was first considered by Kerov in the case $\alpha = 1$, it is usually termed Kerov’s polynomial (or Kerov’s expansion to avoid repetition of the word polynomial).

In [Las09], Lassalle has described an inductive algorithm to compute the coefficients of this expansion. In this paper, by a careful analysis of Lassalle’s algorithm, we obtain some polynomiality results (with several bounds on the degree) for these coefficients: see Propositions [3.7], [3.10] and [3.13].

Clearly, writing some functions in a multiplicative basis may help to understand how to multiply them and we can deduce Theorem [1.4] from these results on Kerov’s polynomials.

While inappropriate to obtain close formulas, this way of studying structure constants is, as far as we know, original. Usually, results of structure constants are obtained using their combinatorial description of via representation theory tools [FH59, GJ96b, IK99, GS98, Tou13].

To finish this paragraph, let us mention that there is an appealing positivity conjecture on Kerov’s polynomials for Jack characters [Las09, Conjecture 1.2]. While we can not solve this conjecture, our analysis of Lassalle’s algorithm gives some partial result: we prove in general the polynomiality of the coefficients and are able to compute a few specific values that were conjectured by Lassalle (see Appendix A).

Another interesting application of our result on Kerov’s polynomials is a new proof of the polynomial dependence of Jack polynomials in term of Jack parameter $\alpha$. This is presented in Section [3.6].

1.6. An open problem: edge fluctuations of Jack measure. A natural question on asymptotics of Jack measure is the behavior of the first few rows of the Young diagrams, called edge fluctuations. Our law of large number on the bulk on Young diagrams implies that, for any fixed positive integer $k$ and real number $C < 1$,

$$\mathbb{P} \left[ \frac{(\lambda(n))_k}{\sqrt{n}} \leq \frac{2C}{\sqrt{\alpha}} \right] \to 0,$$

while Lemma [6.5] tells us that $(\lambda(n))_k/\sqrt{n}$ exceeds $(2\varepsilon)/\sqrt{\alpha}$ with exponentially small probability.

A natural conjecture, considering the case $\alpha = 1$ where the edge fluctuations are well described (see [Oko00, BOO00] and references therein) and the link with $\beta$ ensemble would be the following: for each integer $k \geq 1$, the quantity $(\lambda(n))_k/\sqrt{n}$ converges in probability towards $2/\sqrt{\alpha}$ and the joint vector

$$\left[ n^{1/3} \left( \frac{(\lambda(n))_j}{\sqrt{n}} - \frac{2}{\sqrt{\alpha}} \right) \right]_{1 \leq j \leq k}$$
converges in law towards the $\beta$-Tracy-Widom distribution, which has been introduced and studied in [RRV11] to study edge fluctuations of $\beta$-ensemble. Naturally, a similar conjecture can be formulated for the lengths of the first columns of the Young diagram $\lambda_{(n)}$.

These conjectures hold true for the first row/column in the case $\alpha = 1/2$ and $\alpha = 2$, thanks to the interpretation using Robinson-Schensted on random fixed point free involutions: see [BR01].

We have not made computer experiments to confirm this conjecture and let this problem wide open for future research.

1.7. Outline of the paper. The paper is organized as follows. Section 2 gives all definitions and background on Jack characters, free cumulants and Kerov polynomials. In Section 3 we prove the polynomiality of the coefficients of Kerov’s polynomials, with bounds on the degrees. Then our second main result (that is the polynomiality of structure constants, with precise bound on the degree) is proved in Section 4. Section 5 presents technical statements on structure constants, that will be used in the analysis of large Young diagrams. The last three sections deal with convergence results for large Young diagrams: Section 6 presents the first-order asymptotics. Section 7 gives the central limit theorem for Jack characters and Section 8 establishes the Gaussian fluctuations of large random Young diagrams around the limit shape.

Appendices are devoted to partial answers or some solutions to questions from the literature.

2. Jack characters and Kerov polynomials

2.1. Polynomial functions on the set of Young diagrams. The ring $\Lambda^{(1)}_*$ of polynomial functions on the set of Young diagrams (briefly: the ring of polynomial functions) has been introduced by Kerov and Olshanski in order to study irreducible character values of symmetric groups [KO94].

The first characterization of $\Lambda^{(1)}_*$, that we shall use as definition, is the following.

Definition 2.1. A function $F$ on the set of all Young diagrams belongs to $\Lambda^{(1)}_*$ if there exists a collection of polynomials $(F_h \in \mathbb{Q}[\lambda_1, \ldots, \lambda_h])_{h>0}$ such that

- for a diagram $\lambda = (\lambda_1, \ldots, \lambda_h)$ of length $h$, one has $F(\lambda) = F_h(\lambda_1, \ldots, \lambda_h)$;
- each $F_h$ is symmetric in variables $\lambda_1 - 1, \lambda_2 - 2, \ldots, \lambda_h - h$;
- the compatibility relation
  \[ F_{h+1}(\lambda_1, \ldots, \lambda_h, 0) = F_h(\lambda_1, \ldots, \lambda_h) \]
  holds true for all values of $h$.

The ring $\Lambda^{(1)}_*$ as defined above is sometimes called the ring of shifted symmetric functions in $\lambda_1, \lambda_2, \ldots$. It was first considered by Knop and Sahi [KS96] in a more general context. While this is not obvious, one can prove that $\text{Ch}^{(1)}_{\mu}$ belongs to $\Lambda^{(1)}_*$. In fact, one has more [KO94, Section 3].
**Proposition 2.2.** When $\mu$ runs over all partitions, the family $(\text{Ch}_\mu^{(1)})$ forms a linear basis of $\Lambda^{(1)}_\ast$.

An equivalent description of $\Lambda^{(1)}_\ast$ can be given using Kerov’s interlacing coordinates of a Young diagram. Recall that the content of a box of a Young diagram is $j - i$, where $j$ is its column index and $i$ its row index and, more generally, the content of a point of a plane is the difference of its $x$-coordinate and its $y$-coordinate. We denote by $\mathbb{I}_\lambda$ the sets of contents of the inner corners of $\lambda$, that is corners, at which a box could be added to $\lambda$ to obtain a new diagram of size $|\lambda| + 1$. Similarly, the set $\mathbb{O}_\lambda$ is defined as the contents of the outer corners, that is corners at which a box can be removed from $\lambda$ to obtain a new diagram of size $|\lambda| - 1$. An example is given on Figure 2 (we use the French convention to draw Young diagrams).

If $k$ is a positive integer, one can consider the power-sum symmetric function $p_k$, evaluated on the difference of alphabets $\mathbb{I}_\lambda - \mathbb{O}_\lambda$. By definition, it is a function on Young diagrams given by:

$$\lambda \mapsto p_k(\mathbb{I}_\lambda - \mathbb{O}_\lambda) := \sum_{i \in \mathbb{I}_\lambda} i^k - \sum_{o \in \mathbb{O}_\lambda} o^k.$$ 

It can easily be seen that, for any Young diagram, $p_1(\mathbb{I}_\lambda - \mathbb{O}_\lambda) = 0$. As any symmetric function can be written (uniquely) in terms of $p_k$, we can define $f(\mathbb{I}_\lambda - \mathbb{O}_\lambda)$ for any symmetric function $f$ as follows: if $f = \sum_\rho a_\rho p_{\rho_1} \cdots p_{\rho_\ell}$, then by definition

$$f(\mathbb{I}_\lambda - \mathbb{O}_\lambda) = \sum_\rho a_\rho p_{\rho_1}(\mathbb{I}_\lambda - \mathbb{O}_\lambda) \cdots p_{\rho_\ell}(\mathbb{I}_\lambda - \mathbb{O}_\lambda).$$

With this notion of symmetric functions evaluated on difference of alphabets, the ring $\Lambda^{(1)}_\ast$ admits the following equivalent description [IO02, Corollary 2.8].

**Proposition 2.3.** The functions $(\lambda \mapsto p_k(\mathbb{I}_\lambda - \mathbb{O}_\lambda))_{k \geq 2}$ form an algebraic basis of $\Lambda^{(1)}_\ast$.

In other terms, any function $F$ in $\Lambda^{(1)}_\ast$ is equal to $\lambda \mapsto f(\mathbb{I}_\lambda - \mathbb{O}_\lambda)$, for some symmetric function $f$. This symmetric function $f$ is unique up to addition of a multiple of $p_1$. 

**Figure 2.** A Young diagram with its inner and outer corners (marked respectively with $i$ and $o$).
2.2. Transition measure and free cumulants. Kerov \[Ker93b\] introduced the notion of transition measure of a Young diagram. This is a probability measure \(\mu_\lambda\) on the real line \(\mathbb{R}\) which is associated to \(\lambda\) and is defined by its Cauchy transform

\[
G_{\mu_\lambda}(z) = \int_{\mathbb{R}} \frac{d\mu_\lambda(x)}{z-x} = \prod_{o \in O_\lambda} \frac{z-o}{z-\gamma} \prod_{i \in I_\lambda} \frac{z-i}{z-\delta).
\]

In particular, transition measure is supported on \(I_\lambda\). Besides, its moment generating series is given by

\[
\sum_{k \geq 0} M_{(1)}^k(\lambda) t_k := \frac{1}{t} G_{\mu_\lambda}(1/t) = \prod_{o \in O_\lambda} \frac{1-ot}{1-it} \prod_{i \in I_\lambda} \frac{1-it}{1-it},
\]

where \(M_{(1)}^k(\lambda) := \int_{\mathbb{R}} x^k d\mu_\lambda(x)\) is the \(k\)-th moment of \(\mu_\lambda\). It is easily seen that, for any diagram, \(M_{(1)}^0(\lambda) = 1\) and \(M_{(1)}^1(\lambda) = 0\). This generating series can be rewritten as

\[
\sum_{k \geq 0} M_{(1)}^k(\lambda) t_k = \exp \left( \sum_{i \in I_\lambda} \sum_{k \geq 1} \frac{i^k}{k} t^k - \sum_{o \in O_\lambda} \sum_{k \geq 1} \frac{o^k}{k} t^k \right) = \exp \left( \sum_{k \geq 1} \frac{1}{k} p_k(I_\lambda - O_\lambda) t^k \right).
\]

This implies that \(M_{(1)}^k(\lambda) = h_k(I_\lambda - O_\lambda)\), where \(h_k\) is the complete symmetric function of degree \(k\); see [Mac95, page 25].

**Corollary 2.4.** The family \((M_{(1)}^k)_{k \geq 2}\) forms an algebraic basis of \(\Lambda_{(1)}^*\).

We will also be interested in free cumulants \(R_{(1)}^k(\lambda)\) of the transition measure \(\mu_\lambda\). They are defined by their generating series

\[
K_\lambda(t) = t^{-1} + \sum_{k \geq 1} R_{(1)}^k(\lambda) t^{k-1},
\]

where \(K_\lambda\) is the (formal) compositional inverse of \(G_{\mu_\lambda}\). The fact that \(M_{(1)}^1(\lambda) = 0\) implies that either \(R_{(1)}^1(\lambda) = 0\) (for all diagrams \(\lambda\)).

As explained by Lassalle [Las09, Section 5], they can be expressed as

\[
R_{(1)}^k(\lambda) = e^*_k(I_\lambda - O_\lambda)
\]

for some homogeneous symmetric function \(e^*_k\) of degree \(k\). Functions \(e^*_k\) form an algebraic basis of symmetric functions, hence we have the following corollary of Proposition 2.3.

**Corollary 2.5.** \((R_{(1)}^k)_{k \geq 2}\) is an algebraic basis of ring of polynomial functions on the set of Young diagrams.
Remark. Free cumulants are classical objects in free probability theory [Voi86, Spe94], but considering them outside this context may seem strange at first sight. The relevance of free cumulants of the transition measure of Young diagrams first appeared in the work of Biane [Bia98] and they have played an important role in asymptotic representation theory since then.

2.3. Generalized Young diagrams. The second description of $\Lambda^{(1)}_\ast$ is interesting because it shows that the value of a polynomial function is defined on more general objects than just Young diagrams.

**Definition 2.6.** A generalized Young diagram is a broken line going from a point $(0, y)$ on the $y$-axis to a point $(x, 0)$ on the $x$-axis such that every piece is either a horizontal segment from left to right or a vertical segment from top to bottom. Any Young diagram can be seen as such a broken line: just consider its border. The notions of inner and outer corners can be easily adapted to generalized Young diagrams, as well as the sets $I_L$ and $O_L$ of their contents. It is illustrated in Figure 3.

Note also that the relation $p_1(I_L - O_L) = 0$ holds for generalized Young diagrams as well.

Any polynomial function $F$ on the set of Young diagrams corresponds to the function

$$\lambda \mapsto f(I_\lambda - O_\lambda)$$

for some symmetric function $f$. Recall that $f$ is uniquely determined up to addition of a multiple of $p_1$. Thus, $F$ can be canonically extended to generalized Young diagrams by setting

$$F(L) = f(I_L - O_L).$$

As the relation $p_1(I_L - O_L) = 0$ holds, this is well-defined, i.e. it does not depend on the choice of $f$. We will be in particular interested in the following generalized Young diagrams. Let $\lambda$ be a (generalized) Young diagram and $s$ and $t$ two positive real numbers. We denote $T_{s,t}(\lambda)$ the generalized Young diagram obtained from $\lambda$ by stretching it horizontally by a factor $s$ and vertically by a factor $t$ in French convention (see Figure 3).
\[ \lambda \rightarrow T_{2, \frac{1}{2}}(\lambda) \]

**Figure 4.** Example of a Young diagram \( \lambda \) on the left and a stretched Young diagram \( T_{2, \frac{1}{2}}(\lambda) \) on the right.

Figure 4. These anisotropic Young diagrams have been introduced by Kerov in [Ker00a].

In the case \( s = t \), we denote by \( D_s(\lambda) := T_{s,s}(\lambda) \) the diagram obtained from \( \lambda \) by applying a homothetic transformation of ratio \( s \) and we will call it dilated Young diagram. In the case \( s = t - 1 = \sqrt{\alpha} \) for some \( \alpha \in \mathbb{R}_+ \) we denote by \( A_\alpha(\lambda) := T_{\sqrt{\alpha}, \sqrt{\alpha}-1}(\lambda) \) the diagram obtained from \( \lambda \) by stretching it horizontally by a factor \( \sqrt{\alpha} \) and vertically by a factor \( \sqrt{\alpha} - 1 \). We call it \( \alpha \)-anisotropic Young diagram. It is easy to check that the sets \( I_{D_s}(\lambda) \) and \( O_{D_s}(\lambda) \) are obtained from \( I_\lambda \) and \( O_\lambda \) by multiplying all values by \( s \). In particular, if \( F \) is a polynomial function such that the corresponding symmetric function \( f \) is homogeneous of degree \( d \), then

\[ \lambda \mapsto F(D_s(\lambda)) = f(I_{D_s}(\lambda) - O_{D_s}(\lambda)) = s^d f(I_\lambda - O_\lambda) = s^d F(\lambda) \]

is also a polynomial function. Finally, for any fixed \( s > 0 \), \( F \) is a polynomial function if and only if \( \lambda \mapsto F(D_s(\lambda)) \) is a polynomial function.

**2.4. Continuous Young diagrams.** A generalized Young diagram can also be seen as a function on the real line. Indeed, if one rotates the zigzag line counterclockwise by 45° and scale it by a factor \( \sqrt{2} \) (so that the new z-coordinate corresponds to contents), then it can be seen as the graph of a piecewise affine continuous function with slope \( \pm 1 \). We denote this function \( \omega(\lambda) \). This definition is illustrated on Figure 1. It is very useful to state convergence results for Young diagrams.

Note that the limiting function \( \Omega \) corresponds neither to a real Young diagram, nor to a generalized Young diagram. Therefore, it is natural to work with even more general objects than generalized Young diagrams, i.e. **continuous Young diagrams**.

**Definition 2.7.** We say that a function \( \omega : \mathbb{R} \rightarrow \mathbb{R} \) is a continuous Young diagram if:

- \( \omega \) is Lipshitz continuous function constant 1, i.e. for any \( x_1, x_2 \in \mathbb{R} \)
  \[ |\omega(x_1) - \omega(x_2)| \leq |x_1 - x_2|; \]
- \( \omega(x) - |x| \) is compactly supported.
There is a natural extension for the definitions of transition measure and evaluation of polynomial functions for continuous Young diagrams, see [Bia98, Section 1.2]. However, in this paper, we only need to know that the free cumulants of the transition measure of $\Omega$ are

$$R_k(\Omega) = \begin{cases} 1 & \text{if } k = 2; \\ 0 & \text{if } k > 2. \end{cases}$$

This was established by Biane [Bia01, Section 3.1].

2.5. $\alpha$-polynomial functions.

Definition 2.8. We say that $F$ is an $\alpha$-polynomial function on the set of (continuous) Young diagrams if

$$\lambda \mapsto F(T_{\alpha^{-1}, 1}(\lambda))$$

is a polynomial function. The set of $\alpha$-polynomial functions is an algebra which will be denoted by $\Lambda_\alpha$.\end{definition}

Using Definition 2.1, this means that the polynomial $F(\alpha^{-1}\lambda_1, \ldots, \alpha^{-1}\lambda_h)$ is symmetric in $\lambda_1 - 1, \ldots, \lambda_h - h$. Equivalently (by a change of variables), $F$ is symmetric in $\alpha\lambda_1 - 1, \ldots, \alpha\lambda_h - h$ or in

$$\lambda_1 - \frac{1}{\alpha}, \ldots, \lambda_h - \frac{h}{\alpha}.$$

The last characterization is the definition of what is usually called an $\alpha$-shifted symmetric function [OO97, Las08].

It would be equivalent to ask in the definition of $\alpha$-polynomial functions that

$$\lambda \mapsto F(A_{\alpha^{-1}, 1}(\lambda))$$

is a polynomial function, where $A_{\alpha^{-1}, 1}(\lambda) = T_{\sqrt{\alpha^{-1}}, \sqrt{\alpha}}(\lambda)$ is an $\alpha$-anisotropic Young diagram. Indeed, $T_{\sqrt{\alpha^{-1}}, \sqrt{\alpha}}(\lambda)$ is a dilatation of $T_{\alpha^{-1}, 1}(\lambda)$ and the property of being polynomial is invariant by dilatation of the argument.

Therefore, the $\alpha$-anisotropic moments and free cumulants defined by

$$M_k^{(\alpha)}(\lambda) := M_k^{(1)}(A_\alpha(\lambda)),$$
$$R_k^{(\alpha)}(\lambda) := R_k^{(1)}(A_\alpha(\lambda)),$$

are $\alpha$-polynomial. Moreover, the families $(M_k^{(\alpha)})_{k \geq 2}$ and $(R_k^{(\alpha)})_{k \geq 2}$ are algebraic bases of the algebra $\Lambda^{(\alpha)}$ of $\alpha$-polynomial functions.

The following property is due to Lassalle (under a slightly different form).

**Proposition 2.9.** When $\mu$ runs over all partitions, Jack characters $\text{Ch}_\mu^{(\alpha)}$ form a linear basis of the algebra of $\alpha$-polynomial functions.
Proof. In [Las08 Section 3], Lassalle builds a linear isomorphism $\lambda \mapsto \lambda^\#$ between symmetric functions and $\alpha$-shifted symmetric functions. Then he shows [Las08 Proposition 2] that, for any two partitions $\lambda$ and $\mu$ with $|\lambda| \geq |\mu|$, 

$$\alpha^{(|\mu|-\ell(\mu))/2}p_\mu^\#(\lambda) = \text{Ch}_\mu^{(\alpha)}(\lambda).$$

It is straightforward to check, that, for $|\lambda| < |\mu|$, both sides of equality above are equal to 0. Hence, as functions on all Young diagrams, $\text{Ch}_\mu^{(\alpha)}$ is equal, up to a scalar multiple, to $p_\mu^\#$. The facts that $p_\mu$ is a basis of the symmetric function ring and $f \mapsto f^\#$ a linear isomorphism conclude the proof. □

In particular, they are $\alpha$-polynomial functions and can be expressed in terms of the algebraic bases above.

**Proposition 2.10.** Let $\mu$ be a partition and $\alpha > 0$ a fixed real number. There exist unique polynomials $L_\mu^{(\alpha)}$ and $K_\mu^{(\alpha)}$ such that, for every $\lambda$,

$$\text{Ch}_\mu^{(\alpha)}(\lambda) = L_\mu^{(\alpha)}\left(M_2^{(\alpha)}(\lambda), M_3^{(\alpha)}(\lambda), \ldots \right);$$

$$\text{Ch}_\mu^{(\alpha)}(\lambda) = K_\mu^{(\alpha)}\left(R_2^{(\alpha)}(\lambda), R_3^{(\alpha)}(\lambda), \ldots \right).$$

The polynomials $K_\mu^{(\alpha)}$ have been introduced by Kerov in the case $\alpha = 1$ [Ker00b, Bia03] and by Lassalle in the general case [Las09] and they are called Kerov polynomials. Once again, our normalizations are different from his. We will explain this choice later.

From now on, when it does not create any confusion, we suppress the superscript $(\alpha)$.

We present a few examples of polynomials $L_\mu$ and $K_\mu$ (in particular the case of a one-part partition $\mu$ of length lower than 6). This data has been computed using the one given in [Las09 page 2230]. Recall that we set $\gamma := \frac{1-\alpha}{\sqrt{\alpha}}$.

$$L_{(1)} = M_2,$$
$$L_{(2)} = M_3 + \gamma M_2,$$
$$L_{(3)} = M_4 - 2M_2^2 + 3\gamma M_3 + (1 + 2\gamma^2)M_2,$$
$$L_{(4)} = M_5 - 5M_3M_2 + 6\gamma M_4 - 11\gamma M_2^2 + (5 + 11\gamma^2)M_3 + (7\gamma + 6\gamma^3)M_2,$$
$$L_{(5)} = M_6 - 6M_4M_2 - 3M_2^3 + 7M_3^2 + 10\gamma M_5 - 45\gamma M_3M_2 + (15 + 35\gamma^2)M_4 - (25 + 60\gamma^2)M_2^2$$
$$+ (55\gamma + 50\gamma^3)M_3 + (8 + 46\gamma^2 + 24\gamma^4)M_2,$$
$$L_{(2,2)} = M_3^2 + 2\gamma M_3M_2 - 4M_4 + (\gamma^2 + 6)M_2^2 - 10\gamma M_3 - (6\gamma^2 + 2)M_2,$$
Similarly,

\[ K(1) = R_2, \]
\[ K(2) = R_3 + \gamma R_2, \]
\[ K(3) = R_4 + 3\gamma R_3 + (1 + 2\gamma^2)R_2, \]
\[ K(4) = R_5 + 6\gamma R_4 + \gamma R_2^2 + (5 + 11\gamma^2)R_3 + (7\gamma + 6\gamma^3)R_2, \]
\[ K(5) = R_6 + 10\gamma R_5 + 5\gamma R_3 R_2 + (15 + 35\gamma^2)R_4 + (5 + 10\gamma^2)R_2^2 \]
\[ + (55\gamma + 50\gamma^3)R_3 + (8 + 46\gamma^2 + 24\gamma^4)R_2, \]
\[ K(2,2) = R_3^2 + 2\gamma R_3 R_2 - 4R_4 + (\gamma^2 - 2)R_2^2 - 10\gamma R_3 - (6\gamma^2 + 2)R_2. \]

A few striking facts appear on these examples. First, all coefficients are polynomials in the auxiliary parameter \( \gamma \): we prove this fact in the next section with explicit bounds on the degrees. Besides, for one part partition, polynomials \( K_r \) have non-negative coefficients. We are unfortunately unable to prove this statement, which is a more precise version of [Las09, Conjecture 1.2]. A similar conjecture holds for several part partitions, see also [Las09, Conjecture 1.2].

3. Polynomiality in Kerov’s expansion

3.1. Notations. As in the previous sections, most of our objects are indexed by integer partitions. Therefore it will be useful to use some short notations for small modifications (adding or removing a box or a part) of partitions. We denote by \( \mu \cup (r) \) (\( \mu \setminus (r) \)), respectively the partition obtained from \( \mu \) by adding (deleting, respectively) one part equal to \( r \). We denote by \( \mu_{\downarrow r} = \mu \setminus (r) \cup (r - 1) \) the partition obtained by removing one box in a row of size \( r \). The reader might wonder what do \( \mu \setminus (r) \) and \( \mu_{\downarrow r} \) mean if \( \mu \) does not have a part equal to \( r \): we will not use these notations in this context. Finally, if \( o \) is an outer corner of \( \lambda \), we denote by \( \lambda^o \) the diagram obtained from \( \lambda \) by adding a box at place \( o \).

3.2. How to compute Jack character polynomials? Unfortunately, the argument given above to prove the existence of \( L_\mu \) and \( K_\mu \) is not effective. M. Lassalle [Las09] gave an algorithm for computing \( K_\mu \) by induction over \( \mu \). In this section we present a slightly simpler version of this algorithm which allows to compute \( L_\mu \) instead of \( K_\mu \).

One of the base ingredients is the following formula, which corresponds to [Las09, Proposition 8.3].

**Proposition 3.1.** Let \( k \geq 2 \), \( \lambda \) be a Young diagram and \( o = (x, y) \) an outer corner of \( \lambda \).

\[
M_k(\lambda^o) - M_k(\lambda) = \sum_{\substack{r \geq 1, s, t \geq 0, \\ 2r + s + t \leq k}} z^{k-2r-s-t} \binom{k-t-1}{2r+s-1} \binom{r+s-1}{s} \left( \frac{-\beta}{\sqrt{\alpha}} \right)^s M_t(\lambda),
\]
where $\beta = 1 - \alpha$ and $z_o = \sqrt{\alpha x - \alpha^{-1} y}$ is the anisotropic content of the corner corresponding to $o$ in the $\alpha$-anisotropic diagram $A_\alpha(\lambda)$.

**Proof.** As mentioned above, this is exactly [Las09, Proposition 8.3]. To help the reader, we compare our notations to Lassalle's ones (we use boldface to refer to his notations):

\begin{align*}
M_k(\lambda^o) & = \alpha^{k/2} M_k(\lambda^{(i)}); \\
M_t(\lambda) & = \alpha^{t/2} M_t(\lambda); \\
z_o & = \sqrt{\alpha} \cdot x_i.
\end{align*}

Note that the quantity $\gamma = \beta \sqrt{\alpha}$ plays a particular role in the above formula.

For any partition $\rho$ we define $M_\rho(\lambda^o) := \prod_i M_{\rho_i}(\lambda^o)$ by multiplicativity. The above proposition implies immediately the following corollary:

**Corollary 3.2.** For any partition $\rho$, any diagram $\lambda$ and any outer corner $o$ of $\lambda$,

\[ M_\rho(\lambda^o) = M_\rho(\lambda) + \sum_{g,h \geq 0, \pi \vdash h} b^\rho_{g,\pi}(\gamma) z_o^g M_\pi(\lambda), \]

where $b^\rho_{g,\pi}(\gamma)$ is a polynomial in $\gamma$.

**Proof.** The case when $\rho$ consists of only one part is a direct consequence of Proposition 3.1 (one even has an explicit expression for $b^\rho_{g,\pi}(\gamma)$ in this case). The general case follows by multiplication. \qed

This corollary is an analogue of Equation (8.1) in [Las09].

Let $\mu$ be a partition. By definition of $L_\mu$, there exist some numbers $a^\mu_\rho$ (depending on $\alpha$) such that, for any Young diagram $\lambda$,

\[ \text{Ch}_\mu(\lambda) = \sum_\rho a^\mu_\rho M_\rho(\lambda). \]

Using Corollary 3.2 we can compute

\[ \text{Ch}_\mu(\lambda^o) = \sum_\rho a^\mu_\rho M_\rho(\lambda^o) \]

\[ = \text{Ch}_\mu(\lambda) + \sum_\rho a^\mu_\rho \left( \sum_{g,h \geq 0, \pi \vdash h} b^\rho_{g,\pi}(\gamma) z_o^g M_\pi(\lambda) \right). \]

The second ingredient of Lassalle’s algorithm is a linear identity between the values of Jack character evaluated on different diagrams. We denote by $c_o(\lambda)$ the probability of the corner $o$ in the transition measure $\mu_{A_\alpha}(\lambda)$, so that

\[ M_k(\lambda) = \sum_o c_o(\lambda) z_o^k. \]
In particular,
\begin{align}
\sum_o c_o(\lambda) &= 1 \\
\sum_o c_o(\lambda) z_o &= 0
\end{align}

Then we have [Las09, Equation (3.6)] the following proposition.

**Proposition 3.3.** For any (continuous) Young diagram \(\lambda\) and any partition \(\mu\)
\[
\sum_{o \in \mathcal{O}_\lambda} c_o(\lambda) \text{Ch}_{\mu}(\lambda^o) = m_1(\mu) \text{Ch}_{\mu \setminus 1}(\lambda) + \text{Ch}_\mu(\lambda),
\]
\[
\sum_{o \in \mathcal{O}_\lambda} c_o(\lambda) z_o \text{Ch}_{\mu}(\lambda^o) = \sum_{r \geq 2} rm_r(\mu) \text{Ch}_{\mu \setminus r}(\lambda).
\]

**Proof.** It is an exercise to adapt Equations (3.6) of [Las09] to our notations. □

Using Equations (10), (11), (12) and (13) together with Proposition 3.3, we obtain the following equalities between functions on the set of (continuous) Young diagrams: for any partition \(\mu\),
\[
\sum_{\rho} a_\rho^\mu \left( \sum_{g,h \geq 0, \pi \vdash h} b_{g,\pi}^\rho(\gamma) M_{\pi} M_g \right) = m_1(\mu) \text{Ch}_{\mu \setminus 1},
\]
\[
\sum_{\rho} a_\rho^\mu \left( \sum_{g,h \geq 0, \pi \vdash h} b_{g,\pi}^\rho(\gamma) M_{\pi} M_{g+1} \right) = \sum_{r \geq 2} r \cdot m_r(\mu) \text{Ch}_{\mu \setminus r}.
\]

Fix some partition \(\tau\). We can identify the coefficient of a given monomial \(M_\tau\) in the above equations. This gives us two linear equations which will be denoted by (A\(\tau\)) and (B\(\tau\)):
\[
(\text{A}\_\tau) \quad \sum_{\rho} a_\rho^\mu \left( \sum_{g,h \geq 0, \pi \vdash h} b_{g,\pi}^\rho(\gamma) \right) = m_1(\mu) a_\tau^\mu \setminus 1,
\]
\[
(\text{B}\_\tau) \quad \sum_{\rho} a_\rho^\mu \left( \sum_{g,h \geq 0, \pi \vdash h} b_{g,\pi}^\rho(\gamma) \right) = \sum_{r \geq 2} r \cdot m_r(\mu) a_\tau^\mu \setminus r.
\]

Now assume that, for some partition \(\mu\), we can compute \(L_\nu\) for all partitions \(\nu\) of size smaller than \(|\mu|\). Then the equations (A\(\tau\)) and (B\(\tau\)) can be interpreted as a linear system, where the variables are the coefficients \(a_\rho^\mu\).

This is a finite system of linear equations (indeed, \(a_\rho^\mu = 0\) as soon as \(|\rho| \geq |\mu| + \ell(\mu)\) [Las09 Proposition 9.2 (ii)]). As explained by M. Lassalle, the system obtained that way has a unique solution (we shall see another explanation of that
in the next paragraph) and thus, one can compute the coefficients $a_{\rho}^\mu$ by induction over $|\mu|$.

3.3. A triangular subsystem. In the previous section we explained how to determine the coefficients $a_{\rho}^\mu$ (where $\rho$ runs over partitions without parts equal to 1) of $L_\mu$ as the solution of an overdetemined linear system of equations. In this section, we extract from this system a triangular subsystem.

We will need an order on all partitions: let us define $<_1$ as follows:

$$\rho <_1 \rho' \iff \begin{cases} |\rho| < |\rho'|; \\
|\rho| = |\rho'| \text{ and } \ell(\rho) > \ell(\rho'); \\
|\rho| = |\rho'|, \, \ell(\rho) = \ell(\rho') \text{ and } \min(\rho) > \min(\rho'). \end{cases}$$

We say that an equation involves a variable if its coefficient is non-zero.

**Lemma 3.4.** Let $\rho$ be a partition and $q = \min(\rho)$ its smallest part.

- If $q = 2$, set $\tau = \rho \setminus (2)$. Then Equation (A_\tau) involves the variable $a_{\rho}^\mu$ and involves some of the variables $a_{\rho'}^\mu$ for $\rho' > \rho$ (and no other variables $a_{\rho}^\mu$).
- If $q > 2$, set $\tau = \rho \setminus 2$. Then Equation (B_\tau) involves the variable $a_{\rho}^\mu$ and some of the variables $a_{\rho'}^\mu$ for $\rho' > \rho$ (and no other variables $a_{\rho}^\mu$).

**Proof.** We can refine Corollary 3.2 as follows: for any partition $\rho$, any diagram $\lambda$ and any outer corner $o$ of $\lambda$,

$$M_\rho(\lambda^o) = M_\rho(\lambda) + \sum_{i \leq \ell(\rho)} M_{\rho \setminus \rho_i} \left( \sum_{g = \rho_i - 2 - t} \sum_{t \geq 0} (\rho_i - t - 1)M_t(\lambda) z_o^g \right) + \sum_{\pi, g \neq 0} b_{g, \pi}^\rho(\gamma)M_\pi(\lambda) z_o^g.$$  \hspace{1cm} (14)

Indeed, it is true for $\rho = (k)$ and follows directly for any $\rho$ by multiplication. The right-hand side is a linear combination of $M_\pi z_o^g$ with

$$|\pi| + g \leq |\rho| - 2.$$  

Moreover equality occurs only if $\pi \cup (g)$ is obtained from $\rho$ by choosing a part, removing 2 to this part and splitting it in two (possibly empty) parts.

Let us consider the first statement of the lemma. Fix a partition $\rho$ with a smallest part equal to 2, that is $\rho = \tau \cup (2)$ for some $\tau$. Let us determine which variables $a_{\rho'}^\mu$ appear in the left-hand side of Equation (A_\tau). In other terms, we want to determine for which $\rho'$, the difference $M_{\rho'}(\lambda^o) - M_\rho(\lambda)$ contains some term $M_\pi z_o^g$, for which $\tau = \pi \cup (g)$. As explained above, a necessary condition is the inequality $|\tau| = |\pi| + g \leq |\rho'| - 2$, i.e. $|\rho'| \geq |\tau| + 2$. Moreover, if $|\rho'| = |\tau| + 2$, then $\tau$ must be obtained from $\rho'$ by removing 2 from some part and splitting it into two. In particular, $\rho'$ cannot be longer than $\tau$, unless both new parts are empty. This happens only if the split part of $\rho'$ was 2, that is if $\rho' = \tau \cup (2)$. We have proved that (A_\tau) can involve $a_{\rho'}^\mu$ only if $\rho' = \rho$ or $\rho' > \rho$ (either $\rho'$ has a bigger size than...
if there is equality, splitting it into two. One of the two new parts is always non-empty, thus to the smallest part (assume $q > 2$) and $\tau = \rho_\downarrow q$. The variables $a^\mu_{\rho'}$ can appear in the equation $(B_\tau)$ only if $|\tau| = |\rho'| + g + 1 \leq |\rho'| - 1$, i.e. $|\rho'| \geq |\tau| + 1$. Moreover, if there is equality, $\tau$ must be obtained from $\rho'$ by removing 1 from some part and splitting it into two. One of the two new parts is always non-empty, thus $\rho'$ is at most as long as $\tau$. If they have the same length, it means that $\tau$ is obtained from $\rho'$ by shortening a part. If this part is equal to $q$, then $\rho' = \rho$. Otherwise $\rho'$ contains a part $q - 1$ and thus $\rho' >_1 \rho$ (they have same size and same length). Finally, we have proved that $(B_\tau)$ can involve $a^\mu_{\rho'}$ only if $\rho' >_1 \rho$. Once again, the coefficient of $a^\mu_{\rho'}$ in $(B_\tau)$ is easy to compute: it is equal to $(q - 1)m_q(\rho)$ and, hence, non-zero. \hfill \Box

The first interesting consequence is the following.

**Corollary 3.5.** The coefficient $a^\mu_\rho$ is a polynomial in $\gamma$ with rational coefficients. The same is true for the coefficients of Kerov’s polynomials $K_\mu$.

**Proof.** We proceed by induction over $|\mu|$. The quantities $a^\mu_\rho$ are the solution of a triangular linear system, whose right-hand side is a vector of $a^{\mu'}_{\rho''}$ with $|\mu'| < |\mu|$. By induction hypothesis, the right-hand side belongs to $\mathbb{Q}[\gamma]$. The coefficients $b^\rho_{g,\pi}(\gamma)$ of the system also belong to $\mathbb{Q}[\gamma]$. Moreover, the diagonal coefficients of the system (given in the proof above) are invertible in $\mathbb{Q}[\gamma]$, hence the solution is also in $\mathbb{Q}[\gamma]$.

For the second statement, it is enough to say that each $M_k$ is a polynomial in the $R_k$’s with integer coefficients. \hfill \Box

**Remark.** Lemma 3.4 does not hold for Lassalle’s system of equation [Las09, Equations (9.1) and (9.2)], which computes recursively $K_\mu$.

### 3.4. A first bound on the degree

Recall that $(M_k)_{k \geq 2}$ is an algebraic basis of the ring $\Lambda^{(a)}_\ast$ of $a$-polynomial functions on Young diagrams. Hence, we can define a gradation on $\Lambda^{(a)}_\ast$ by choosing arbitrarily the degree of each of the generators $M_k$. In this section, we do the following natural choice:

$$\text{deg}_1(M_k) = k \quad \text{for } k \geq 2.$$ 

Our goal is to obtain a bound on the degree of the polynomial $a^\mu_\rho \in \mathbb{Q}[\gamma]$. We begin by the following lemma concerning the polynomials $b^\rho_{g,\pi}(\gamma)$.

**Notational convention.** To emphasize the difference with gradations on $\Lambda^{(a)}_\ast$, we denote throughout the paper degrees of polynomials in $\gamma$ by $\text{deg}_\gamma$.

**Lemma 3.6.** Let $\rho$ and $\pi$ be two partitions and $g \geq 0$ be an integer. One has

$$\text{deg}_\gamma(b^\rho_{g,\pi}(\gamma)) \leq \text{deg}_1(M_\rho) - \text{deg}_1(M_{\pi \cup (g)}) - 2.$$ 

Moreover, if the right-hand side is an even (resp. odd) number, then $b^\rho_{g,\pi}(\gamma)$ is an even (resp. odd) polynomial.
Proof: By Proposition 3.1, $M_k(\lambda^\circ)$ can be written as a linear combination of terms of the form $b(\gamma) M_{\pi} =_{\gamma} 0$. We define the pre-degree (with respect to $\deg_{\gamma}$) of such a term to be the quantity $\deg_{\gamma}(b) + |\pi| + g$. This degree is multiplicative. Then,

$$M_k(\lambda^\circ) = M_k(\lambda) + \text{ terms of degree smaller or equal to } k - 2.$$ 

By multiplying this kind of expressions we obtain that

$$M_\rho(\lambda^\circ) = M_\rho(\lambda) + \text{ terms of degree smaller or equal to } |\rho| - 2,$$

which corresponds to our bound on the degree. The parity also follows immediately from the one-part case by multiplication. \qed

This yields the following result.

**Proposition 3.7.** The coefficient $a_\rho^\mu$ of $M_\rho$ in Jack character polynomial $L_\mu$ is a polynomial in $\gamma$ of degree smaller or equal to $|\mu| + \ell(\mu) - |\rho|$. Moreover, it has the same parity as the integer $|\mu| + \ell(\mu) - |\rho|$.

The same is true for $K_\mu$.

Proof: We proceed by induction over $(\mu, \rho)$. The base case $\mu = (1)$ is trivial as $L_{(1)} = M_2$. Fix two partitions $\mu$ and $\rho$. We assume that our result holds for any pair $(\mu', \rho')$ with $|\mu'| < |\mu|$ or $|\mu'| = |\mu|$ and $\rho' > 1 \rho$.

It may seem strange to assume that the result holds for $\rho' > 1 \rho$. We are indeed doing some kind of descending induction. This is possible because, for a given $\mu$, the number of partitions $\rho$ we shall consider is finite: indeed, $a_\rho^\mu = 0$ as soon as $|\rho| \geq |\mu| + \ell(\mu)$ [Las09, Proposition 9.2 (ii)]. The same remark holds for most proofs in this section.

Let us first consider the case when $\rho = \tau \cup (2)$ contains a part equal to 2. By Lemma 3.4, Equation ($A_\tau$) can be written as:

$$m_2(\rho) \cdot a_\rho^\mu = m_1(\mu) a_\rho^\mu \setminus 1 - \sum_{\pi, g, \rho' \in \gamma} \sum_{\rho' > 1 \rho} b_{g, \pi}^\rho(\gamma) a_\rho^\mu.$$

The first term on the right-hand-side is by convention equal to 0 if $\mu$ does not contain any part equal to 1. If $\mu$ contains a part equal to 1, as $|\mu \setminus 1|$ is smaller than $|\mu|$, by induction hypothesis $a_{\rho'}^\mu \setminus 1$ is a polynomial of degree at most

$$|\mu \setminus 1| + \ell(\mu \setminus 1) - |\tau| = |\mu| - 1 + \ell(\mu) - 1 - (|\rho| - 2) = |\mu| + \ell(\mu) - |\rho|.$$

As $\rho' > 1 \rho$, we can also apply the induction hypothesis to each summand of the second term: $a_{\rho'}^\mu$ is polynomial of degree at most $|\mu| + \ell(\mu) - |\rho'|$. But using Lemma 3.6, $b_{g, \pi}^\rho(\gamma)$ has degree at most $|\rho'| - |\tau \cup (g)| - 2$. Hence the degree of the product is bounded by

$$|\mu| + \ell(\mu) - (|\tau \cup (g)| + 2) = |\mu| + \ell(\mu) - |\rho|.$$

The last equality comes from the fact that $\tau \cup (g) = \tau = \rho \setminus 2$. 


The proof of the case when the smallest part of $\rho$ is $q > 2$ is similar. We use Equation $(B_\tau)$ for $\tau = \rho_{i,q}$, which takes the form:

\[(q - 1)m_q(\rho) \cdot a_{\rho}^\mu = \sum_{r \geq 2} r \cdot m_r(\mu) a_{\rho}^{\mu,r} - \sum_{\pi \cup (g+1) = \tau} \sum_{\rho' > 1} b_{\rho,g,\pi}(\gamma) a_{\rho'}^\mu.
\]

Note that $|\mu_{i,r}| < |\mu|$, therefore by induction hypothesis $a_{\rho,i,r}^\mu$ is a polynomial in $\gamma$ of degree at most

\[|\mu_{i,r}| + \ell(\mu_{i,r}) - |\tau| = |\mu| - 1 + \ell(\mu) - (|\rho| - 1) = |\mu| + \ell(\mu) - |\rho|.
\]

For the second summand, the argument is the same as before, except that here the equality $|\pi \cup (g)| + 2 = |\rho|$ comes from the fact that $|\tau| = |\rho| - 1$ and $|\tau| = |\pi \cup (g+1)| = |\pi \cup (g)| + 1$.

The parity is obtained the same way.

**Corollary 3.8.** For any partition $\mu$ one has:

\[\deg_1(Ch_\mu) = |\mu| + \ell(\mu).
\]

Moreover

\[Ch_\mu = \prod_i R_{\mu_i,1} + \text{lower degree terms with respect to } \deg_1.
\]

**Proof.** By Proposition 3.7, $Ch_\mu$ has at most degree $|\mu| + \ell(\mu)$ (this has also been proved by Lassalle [Las09, Proposition 9.2 (ii)]) and its component of degree $|\mu| + \ell(\mu)$ does not depend on $\alpha$. Hence the result follows as this dominant term is known in the case $\alpha = 1$ (see for example [Sni06, Theorem 4.9]).

### 3.5. A second bound on degrees

For some purposes the bound on the degree of $a_\rho^\mu$ given by Proposition 3.7 is not strong enough. In this section we give another bound which is related to another gradation of $\Lambda^{(\alpha)}$ defined by:

\[\deg_2(M_k) = k - 2 \quad \text{for } k \geq 2.
\]

One has the following analogue of Lemma 3.6.

**Lemma 3.9.** Let $\rho$ and $\pi$ be two partitions and $g \geq 0$ an integer. Then

\[
\begin{align*}
\deg_2(b_{g,\pi}^\rho(\gamma)) &\leq \deg_2(M_{\rho}) - \deg_2(M_{\pi \cup (g)}), \\
\deg_2(b_{g,\pi}^\rho(\gamma)) &\leq \deg_2(M_{\rho}) - \deg_2(M_{\pi \cup (g+1)}) - 1.
\end{align*}
\]

**Proof.** The proof is similar to the one of Lemma 3.6. We define the pre-degree (with respect to $\deg_2$) of an expression of the form $b(\gamma) M_{\pi} z_{\rho}^g$ to be $\deg_2(b(\gamma)) + \deg_2(M_{\pi}) + g$. By Proposition 3.11 the pre-degree of $M_k(\lambda^\alpha)$ is equal to $k - 2$. Note that this pre-degree is multiplicative. Then $M_{\rho}(\lambda^\alpha)$ has pre-degree $|\rho| - 2\ell(\rho) = \deg_2(M_{\rho})$. The lemma follows because of the following inequalities: for $g \geq 0$,

\[
\begin{align*}
\deg_2(M_{\pi \cup (g)}) &\leq \deg_2(M_{\pi}) + g; \\
\deg_2(M_{\pi \cup (g+1)}) &\leq \deg_2(M_{\pi}) + g - 1.
\end{align*}
\]
Note that in the first inequality the difference between the right hand side and the left hand side is equal to 2, unless \( g = 0 \); in that case we have an equality. In the second inequality, the case \( g = 0 \) is obvious as \( M_{\pi \cup (1)} = 0 \) and hence its degree is \(-\infty\) by convention. In all other cases, we have an equality. □

We deduce from this lemma a new bound on the degree of \( a^\mu_\rho \).

**Proposition 3.10.** The coefficient \( a^\mu_\rho \) of \( M_\rho \) in Jack character polynomial \( L^\mu \) is a polynomial in \( \gamma \) of degree smaller or equal to \(|\mu| - \ell(\mu) - (|\rho| - 2\ell(\rho))\).

*Proof.* It is a straightforward exercise to adapt the proof of Proposition 3.7. We use Lemma 3.9 instead of Lemma 3.6 and \(|\rho| \) has to be replaced by \(|\rho| - 2\ell(\rho))\). □

As an immediate consequence we have

**Corollary 3.11.** For any partition \( \mu \) one has:

\[
\deg_2(\text{Ch}^\mu) = |\mu| - \ell(\mu),
\]

and the top degree part does not depend on \( \alpha \).

Note that Proposition 3.10 is neither weaker nor stronger than Proposition 3.7. But it is sometimes more appropriate, as we shall see in the next section.

**Remark.** The top degree part of \( \text{Ch}_\mu \) for \( \deg_2 \) does not admit an explicit expression as for \( \deg_1 \). One can however compute its linear terms in free cumulants, see [FG13, Section 3].

### 3.6. Polynomiality of \( \theta_\mu(\lambda) \)

In this section we prove that \( \theta_\mu(\lambda) \) is a polynomial with rational coefficients in \( \alpha \). This simple statement does not follow directly from the definition of Jack polynomials and had been open for twenty years. It was then proved by Lapointe and Vinet [LV95], who also proved the integrality of the coefficients in the monomial basis. Short after that, this result was completed by a positivity result from Knop and Sahi [KS97].

Using the material of this Section, we can find a new proof of the polynomiality in \( \alpha \). Integrity and positivity seem unfortunately impossible to obtain via this method.

First, we consider the dependence of \( M_k(\lambda) \) on \( \alpha \).

**Lemma 3.12.** Let \( k \geq 2 \) be an integer and \( \lambda \) a partition. Then \( \sqrt{\alpha}^{k-2} M_k(\lambda) \) is a polynomial in \( \alpha \) with integer coefficients.

*Proof.* We use induction over \( |\lambda| \) and \( k \). Proposition 3.1 can be rewritten as

\[
\sqrt{\alpha}^{k-2} M_k(\lambda^\circ) - \sqrt{\alpha}^{k-2} M_k(\lambda) = \sum_{r \geq 1, s, t \geq 0, 2r+s+t \leq k} \alpha^r (\sqrt{\alpha z_0})^{k-2r-s-t} \binom{k-t-1}{2r+s-1} \binom{r+s-1}{s} (\alpha - 1)^s \sqrt{\alpha}^{t-2} M_t(\lambda).
\]

Note that \( \sqrt{\alpha z_0} = \alpha x - y \) is a polynomial in \( \alpha \) with integer coefficients. Hence the induction is immediate. □
Now we write, for $\mu, \lambda \vdash n$,

$$z_\mu \theta_\mu(\lambda) = \alpha \frac{|\mu| - \ell(\mu)}{2} \sum_{\rho} a_\rho^\mu M_\rho(\lambda)$$

$${\text{Ch}}_\mu(\lambda) = \alpha \frac{|\mu| - \ell(\mu)}{2} \sum_{\rho} a_\rho^\mu \left( \prod_{i \leq \ell(\rho)} \sqrt{\alpha^\rho - 2M_\rho(\lambda)} \right).$$

The quantities $\alpha \frac{|\mu| - \ell(\mu)}{2} a_\rho^\mu$ and $\sqrt{\alpha^\rho - 2M_\rho(\lambda)}$ are polynomials in $\alpha$ (by Proposition 3.10 and Lemma 3.12), hence $\theta_\mu(\lambda)$ is a polynomial in $\alpha$.

3.7. Yet another gradation and bound on degrees. The gradation introduced in Section 3.5 is suitable for some purposes (as we have seen in the previous section), but it has the unpleasant aspect that all homogeneous spaces have an infinite dimension. In particular, Proposition 3.10 does not give any information on the maximal power of $M_2$ which can appear in $L_\mu$. In this section we propose a way to avoid this difficulty, little bit technical but will be quite useful in the next section.

We define a new algebraic basis of $\Lambda_*(\alpha)$ by:

$$M'_2 = M_2,$$

$$M'_k = M_k - (-\gamma)^{k-2} M_2$$

for $k \geq 3$.

We also consider the gradation defined by:

$${\text{deg}}_3(M'_2) = 1, \quad {\text{deg}}_3(M'_k) = k - 2$$

for $k \geq 3$,

so that $${\text{deg}}_3(M'_\rho) = |\rho| - 2\ell(\rho) + m_2(\rho)$$. Obviously, there exists a polynomial $L'_\mu$ such that

$${\text{Ch}}_\mu = L'_\mu(M'_2, M'_3, \ldots).$$

For example, one has:

$$L'_{(2,2)} = (M'_3)^2 + 6(M'_2)^2 - 4M'_4 - 10\gamma M'_3 - 2M'_2.$$

We denote by $(a')^\mu_\rho$ the coefficient of $M'_\rho$ in $L'_\mu$. Then, one has the following result.

**Proposition 3.13.** The coefficient $(a')^\mu_\rho$ is a polynomial in $\gamma$ of degree at most $|\mu| - \ell(\mu) + m_1(\mu) - (|\rho| - 2\ell(\rho) + m_2(\rho))$.

**Remark.** The analogous result is not true for $a_\rho^\mu$, as it can be seen on the case $\mu = (2,2)$.

The algorithm to compute the coefficient $(a')^\mu_\rho$ is the same as for $a_\rho^\mu$ and the proof of the bound on degrees is similar to those of Propositions 3.7 and 3.10. Let us give some details.
First, one can rewrite Proposition 3.1 in terms of the quantities $M'_k$:

\[
(M'_k(\lambda^o) - M'_k(\lambda)) = M_k(\lambda^o) - M_k(\lambda) - (-\gamma)^{k-2} \\
= \sum_{r,s,t \geq 0, 2r+s+t \leq k, (r,s,t) \neq (1,k-2,0)} z_o^{k-2r-s-t} \binom{k-t-1}{2r+s-1} (r+s-1) \\
\cdot (-\gamma)^s (M'_p(\lambda) + (-\gamma)^{t-2} M'_2(\lambda)).
\]

Please note that the term $(-\gamma)^{k-2}$ corresponding to $(r,s,t) = (1,k-2,0)$ does not belong to the sum any more. By multiplication, there exist some polynomials $(b')_{g,\pi}(\gamma)$ such that

\[
M'_p(\lambda^o) = M'_p(\lambda) + \sum_{g,\pi} (b')_{g,\pi}(\gamma) z_o^g M'_\pi(\lambda).
\]

Using Equation (10) and Proposition 3.3 we obtain the following equalities:

\[
(A') \quad \sum_{\rho} (a')^\mu_{\rho} \left( \sum_{g,h \geq 0, \pi \vdash h} (b')_{g,\pi}(\gamma) M'_\pi M_g \right) = m_1(\mu) \text{Ch}_{\mu \setminus 1},
\]

\[
(B') \quad \sum_{\rho} (a')^\mu_{\rho} \left( \sum_{g,h \geq 0, \pi \vdash h} (b')_{g,\pi}(\gamma) M'_\pi M_{g+1} \right) = \sum_{r \geq 2} r \cdot m_r(\mu) \text{Ch}_{\mu \setminus r}.
\]

Plugging $M_g = M'_g + (-\gamma)^{g-2} M'_2$ in these equations and identifying the coefficient of $M'_2$ on both sides, we obtain the following system:

\[
(A'_r) \quad \sum_{\rho} (a')^\mu_{\rho} \left( \sum_{\pi \vdash (g+1)} (b')_{g,\pi}(\gamma) + \sum_{g \geq 2, \pi \vdash (2)} (-\gamma)^{g-2} (b')_{g,\pi}(\gamma) \right) = m_1(\mu)(a')^\mu_{\rho \setminus 1},
\]

\[
(B'_r) \quad \sum_{\rho} (a')^\mu_{\rho} \left( \sum_{\pi \vdash (g+1)} (b')_{g,\pi}(\gamma) + \sum_{g \geq 2, \pi \vdash (2)} (-\gamma)^{g-1} (b')_{g,\pi}(\gamma) \right) = \sum_{r \geq 2} r \cdot m_r(\mu)(a')^\mu_{\rho \setminus r},
\]

It is easy to check that Lemma 3.4 still holds for this system.

The next step is to give a bound on the degree of $(b')_{g,\pi}(\gamma)$. 

Lemma 3.14.
\[
\deg_{\gamma}(b_{g,\pi}(\gamma)) \leq \deg_{3}(M_{\rho}') - \deg_{3}(M_{\pi}') - \max(g, 1).
\]

Proof. Let us call pre-degree (with respect to $\deg_{3}$) of an expression of the form $b(\gamma)\ M_{\rho}'\ z_{g}^{|\gamma|}$ the quantity $\deg_{\gamma}(b) + \deg_{3}(M_{\rho}') + g$. It is multiplicative. Clearly, $M_{k}'(\lambda^{o})$ has pre-degree $\max(k - 2, 1)$ (see Equation (15)), thus $M_{\rho}'(\lambda^{o})$ has pre-degree $\deg_{3}(M_{\rho}')$, which finishes the proof of the case $g = 1$. For $g = 0$, one has to look at the term which does not involve $z_{0}$. It is easy to check on Equation (15) (here, it is crucial to use $M'$ and not $M$) that
\[
M_{k}'(\lambda^{o})|_{z_{0}=0} = M_{k}'(\lambda)|_{z_{0}=0} + \text{(terms of pre-degree } k - 3).\]

Hence by multiplication,
\[
M_{\rho}'(\lambda^{o})|_{z_{0}=0} = M_{\rho}'(\lambda)|_{z_{0}=0} + \text{(terms of pre-degree } \deg_{3}(M_{\rho}') - 1),
\]
which finishes the proof of the lemma. \qed

We have now all the tools to prove Proposition 3.13 by induction. As usual, we first consider the case where $\rho = \tau \cup (2)$ has the smallest part equal to 2. Then Equation (A_{\tau}) can be written as:
\[
m_{2}(\rho) \cdot (a')_{\rho}^{\mu} = m_{1}(\mu)(a')_{\tau}^{\mu}|_{1} - \sum_{\pi \cup (g) = \tau} \sum_{\rho' > 1\rho} (b')_{g,\pi}(\gamma)(a')_{\rho'}^{\mu} - \sum_{\gamma \rho = 2, \pi \cup (2) = \tau} \sum_{\rho' > 1\rho} (-\gamma)^{g-2}(b')_{g,\pi}(\gamma)(a')_{\rho'}^{\mu}.
\]

With arguments similar to the ones used previously, the first two terms are polynomials in $\gamma$ of degree at most $|\mu| - \ell(\mu) + m_{1}(\mu) - \deg_{3}(M_{\rho}')$. Let us focus on the last summand. By induction hypothesis $(a')_{\rho}^{\mu}$ is a polynomial of degree $|\mu| - \ell(\mu) + m_{1}(\mu) - \deg_{3}(M_{\rho}')$. By Lemma 3.14 $(b')_{g,\pi}(\gamma)$ has degree equal to $\deg_{3}(M_{\rho}') - \deg_{3}(M_{\pi}') - g$. Hence the product of these two terms with $(-\gamma)^{g-2}$ has degree at most
\[
|\mu| - \ell(\mu) + m_{1}(\mu) - (\deg_{3}(M_{\pi}') - 2) = |\mu| - \ell(\mu) + m_{1}(\mu) - \deg_{3}(M_{\rho}').
\]

The equality comes from the fact that $\rho = \tau \cup (2) = \pi \cup (2, 2)$.

Finally one obtains that $(a')_{\rho}^{\mu}$ has degree at most $|\mu| - \ell(\mu) + m_{1}(\mu) - \deg_{3}(M_{\rho}')$.

The case when $\rho$ has no parts equal to 2 is similar. \qed

Corollary 3.15. For any partition $\mu$ one has:
\[
\deg_{3}(Ch_{\mu}) = |\mu| - \ell(\mu) + m_{1}(\mu),
\]
and the top degree part does not depend on $\alpha$.

Remark. The top degree part of $Ch_{\mu}$ for $\deg_{3}$ has, as far as we know, no close expression.
3.8. Gradations and characters. In the previous sections we have defined three different gradations. The elements of our favorite basis \((Ch_\mu)\) are not homogeneous, but have the following nice property: if we denote

\[ V^d_i = \{ x \in \Lambda^{(\alpha)} : \deg_i(x) \leq d \} \]

then, for \(i = 1\) or \(i = 3\), each \(V^d_i\) is spanned linearly by the functions \(Ch_\mu\) that it contains (this comes from a direct dimension argument). This simple observation will be useful later.

The same argument cannot be used for \(i = 2\), as the spaces \(V^d_2\) are all infinite dimensional.

**Remark.** The functions \(\deg_i\), for \(i = 1, 3\), define some gradations and hence some filtrations on \(\Lambda^{(\alpha)}\). These filtrations were known in the cases \(\alpha = 1, 2\); see [IO02, Fér12, Tou13].

In fact, Ivanov and Olshanski [IO02, Proposition 4.9] give many more filtrations for \(\alpha = 1\), but we have not been able to prove that they hold for general \(\alpha\). In particular, the filtration

\[ \deg(Ch_\mu) = |\mu| + m_1(\mu) \]

is central in their analysis of fluctuations of random Young diagrams. Unfortunately, we are unable to prove that (16) still defines a filtration in the general \(\alpha\)-case. We leave this as an open question. Would we be able to positively answer it, we could use a moment method to obtain our fluctuation results (without the bound on the speed of convergence).

4. Polynomiality of structure constants of Jack characters

4.1. Structure constants are polynomials in \(\gamma\). In this section we are going to prove our main result for the structure constants of the algebra \(\Lambda^{(\alpha)}\) of \(\alpha\)-polynomial functions which was stated as Theorem 1.4.

**Proof of Theorem 1.4.** First observe that for each \(i \in \{1, 2, 3\}\) one has:

\[ n_i(\mu) = \deg_i(Ch_\mu), \]

hence our bound on the degree of structure constants can be equivalently formulated using three gradations introduced in Section 3.

Let us consider the bound involving \(\deg_1\) (the case of \(\deg_3\) is similar). We know (Proposition 3.7) that

\[ Ch_\mu = \sum_\rho a^\mu_\rho M_\rho, \]

where each \(a^\mu_\rho\) is a polynomial in \(\gamma\) of degree \(\deg_1(Ch_\mu) - \deg_1(M_\rho)\). Hence, we have

\[ Ch_\mu \cdot Ch_\nu = \sum_\rho b^\mu_\nu_\rho M_\rho, \]
where each $b_{\rho}^{\mu,\nu}$ is a polynomial in $\gamma$ of degree $\deg_1(\text{Ch}_\mu) + \deg_1(\text{Ch}_\nu) - \deg_1(M_\rho)$. In particular $\text{Ch}_\mu \cdot \text{Ch}_\nu$ has degree at most $\deg_1(\text{Ch}_\mu) + \deg_1(\text{Ch}_\nu)$ and hence, thanks to the remark of Section 3.8, $g_{\mu,\nu;\pi} = 0$ whenever

$$\deg_1(\text{Ch}_\mu) + \deg_1(\text{Ch}_\nu) < \deg_1(\text{Ch}_\pi).$$

The structure constants are obtained by solving the linear system:

$$(S) \quad \sum_{\tau} a_{\rho}^{\tau} g_{\mu,\nu;\tau} = b_{\rho}^{\mu,\nu},$$

where $\mu$ and $\nu$ are fixed and $\rho$ runs over all partitions without parts equal to 1; the variables are $g_{\mu,\nu;\tau}$.

We will prove our statement by induction over

$$\deg_1(\text{Ch}_\mu) + \deg_1(\text{Ch}_\nu) - \deg_1(\text{Ch}_\pi).$$

If this quantity is equal to $-1$, the coefficient $g_{\mu,\nu;\tau}$ is equal to 0 and the statement is true. Note that $a_{\rho}^{\tau}$ vanishes as soon as $\deg_1(\text{Ch}_\tau) < \deg_1(M_\rho)$. We fix a partition $\pi$ and we suppose that for all partitions $\tau$ bigger than $\pi$ (in the sense that $\deg_1(\text{Ch}_\tau) > \deg_1(\text{Ch}_\pi)$), the degree of $g_{\mu,\nu;\tau}$ is bounded from above by $\deg_1(\text{Ch}_\mu) + \deg_1(\text{Ch}_\nu) - \deg_1(\text{Ch}_\pi)$. Then from $(S)$ we extract a subsystem

$$(S') \quad \sum_{\tau, \deg_1(\text{Ch}_\tau) = \deg_1(\text{Ch}_\pi)} a_{\rho}^{\tau} g_{\mu,\nu;\tau} = b_{\rho}^{\mu,\nu} - \sum_{\tau, \deg_1(\text{Ch}_\tau) > \deg_1(\text{Ch}_\pi)} a_{\rho}^{\tau} g_{\mu,\nu;\tau},$$

where $\rho$ runs over partitions such that $\deg_1(M_\rho) = \deg_1(\text{Ch}_\pi)$. The variables are $g_{\mu,\nu;\tau}$ for $\tau$ with $\deg_1(\text{Ch}_\tau) = \deg_1(\text{Ch}_\pi)$. This system is invertible (because $(\text{Ch}_\pi)$ is a basis of $\Lambda_\star(\mathfrak{a})$) and the coefficients are rational numbers (by Proposition 3.7). Besides, all terms on the right-hand side are polynomials in $\gamma$ of degree at most $\deg_1(\text{Ch}_\mu) + \deg_1(\text{Ch}_\nu) - \deg_1(\text{Ch}_\pi)$ which finishes the proof.

The proof of the parity follows in the same way.

The bound involving $\deg_2$ is obtained in a slightly different way. First note that if $\mu$ and $\nu$ do not have any parts equal to 1, this bound is weaker that the one with $\deg_3$. Hence, it holds in this case. Then the general case follows, using the fact that $\text{Ch}_{\mu,\nu}(1) = (|\lambda| - |\mu|) \text{Ch}_\mu$.

4.2. Projection on functions on Young diagrams of size $n$. Recall that $\Lambda_\star(\alpha)$ is a subalgebra of the algebra of functions on all Young diagrams. The latter has a natural projection map $\varphi_n$ onto $\mathcal{F}(\mathcal{Y}_n, \mathbb{Q})$, the algebra of functions on Young diagrams of size $n$. Note that, as Jack symmetric functions $J_\lambda$ form a basis of the symmetric function ring, the functions $(\theta_\mu)_{\mu \vdash n}$ form a basis of $\mathcal{F}(\mathcal{Y}_n, \mathbb{Q})$ (see [Fer12 Proposition 4.1]).

We consider the structure constants $c_{\mu,\nu;\pi}$ of $\mathcal{F}(\mathcal{Y}_n, \mathbb{Q})$ with basis $(\theta_\mu)_{\mu \vdash n}$, that is the numbers uniquely defined by:

$$\text{for all } \lambda \vdash n, \quad \theta_\mu(\lambda) \cdot \theta_\nu(\lambda) = \sum_{\pi \vdash n} c_{\mu,\nu;\pi} \theta_\pi(\lambda).$$
Note that $c_{\mu,\nu;\pi}$ depends on $\alpha$, but, according to our convention, we omit the superscript, when it does not bring any confusion. It is important to keep in mind that the $c$’s are indexed by triples of partitions of the same size, while the $g$’s are indexed by any triple of partitions.

It turns out that the quantities $c_{\mu,\nu;\pi}$ can be expressed in terms of the quantities $g_{\mu,\nu;\tau}$. To explain that, for any partition $\mu$, let us denote $\tilde{\mu}$ the partition obtained by erasing all parts equal to 1. Fix two partitions $\mu$ and $\nu$ of the same integer $n$; then

$$\text{Ch}_\tilde{\mu} \cdot \text{Ch}_\tilde{\nu} = \sum_{\tau} g_{\tilde{\mu},\tilde{\nu};\tau} \text{Ch}_\tau.$$ 

But using the definition of Ch, this implies that, for all $\lambda \vdash n$, one has:

$$\alpha^{-\frac{|\mu| - \ell(\mu)}{2}} z_{\tilde{\mu}} \theta_\mu(\lambda) \cdot \alpha^{-\frac{|\nu| - \ell(\nu)}{2}} z_{\tilde{\nu}} \theta_\nu(\lambda) = \sum_{|\tau| \leq n} g_{\tilde{\mu},\tilde{\nu};\tau} \alpha^{-\frac{|\mu| - \ell(\mu)}{2}} z_{\tau} \left( \frac{n - |\tau| + m_1(\tau)}{m_1(\tau)} \right)^{\theta_\tau 1^{n-|\tau|}(\lambda)}. $$

Every partition $\tau$ with $|\tau| \leq n$ can be written uniquely as $\tilde{\pi} 1^i$ where $\pi$ is a partition of $n$ and $i \leq m_1(\pi)$. Denoting

$$d(\mu, \nu; \pi) = |\mu| - \ell(\mu) + |\nu| - \ell(\nu) - (|\pi| - \ell(\pi)), $$

one has

$$\theta_\mu(\lambda) \cdot \theta_\nu(\lambda) = \frac{\alpha^{d(\mu, \nu; \pi)/2}}{z_{\tilde{\mu}} z_{\tilde{\nu}}} \sum_{\pi \vdash n} \left( \sum_{0 \leq i \leq m_1(\pi)} g_{\tilde{\mu},\tilde{\nu};\tilde{\pi} 1^i} \cdot z_{\tilde{\pi} 1^i} \cdot i! \cdot \left( \frac{n - |\tilde{\pi}|}{i} \right) \right) \theta_\pi(\lambda).$$

As this is true for all partitions $\lambda$ of $n$, one can use Equation (17) to obtain:

$$c_{\mu,\nu;\pi} = \frac{\alpha^{d(\mu,\nu;\pi)/2}}{z_{\tilde{\mu}} z_{\tilde{\nu}}} \sum_{0 \leq i \leq m_1(\pi)} g_{\tilde{\mu},\tilde{\nu};\tilde{\pi} 1^i} \cdot i! \cdot \left( \frac{n - |\tilde{\pi}|}{i} \right).$$

Using Theorem 1.4 with $\deg_3$, we know that $g_{\tilde{\mu},\tilde{\nu};\tilde{\pi} 1^i}$ is a polynomial of degree at most $d(\mu, \nu; \pi) - i$. We have thus prove the following result:

**Proposition 4.1.** Let $\mu$, $\nu$ and $\pi$ be three partitions without parts equal to 1. Then, 

$$\alpha^{-d(\mu,\nu;\pi)/2} c_{\mu 1^n-|\mu|,\nu 1^n-|\nu|,\pi 1^n-|\pi|}$$

is a polynomial in $n$ and $\gamma$ with rational coefficients, of total degree at most $d(\mu, \nu; \pi)$.

Moreover, seen as a polynomial in $\gamma$, it has the same parity as $d(\mu, \nu; \pi)$.

**Corollary 4.2.** The quantity $c_{\mu 1^n-|\mu|,\nu 1^n-|\nu|,\pi 1^n-|\pi|}$ is a polynomial in $n$ and $\alpha$. Moreover, it has degree at most $d(\mu, \nu; \pi)$ in $n$ and at most $d(\mu, \nu; \pi)$ in $\alpha$ (the total degree may be bigger).

Applications of this statements are given in appendix: see Sections B.1 and B.2.
5. Special values of $\alpha$ and polynomial interpolation

5.1. Case $\alpha = 1$: symmetric group algebra. In the case $\alpha = 1$, the structure constants considered in the previous section are linked with the symmetric group algebras. Let $S_n$ denote the symmetric group of size $n$, i.e. the group of permutations of the set $[n] := \{1, \ldots, n\}$. Recall that the cycle-type of a permutation $\sigma \in S_n$ is the integer partition $\mu \vdash n$ obtained by sorting the lengths of the cycles of $\sigma$. We consider the group algebra $\mathbb{Q}[S_n]$ of $S_n$ over the rational field $\mathbb{Q}$. Its center $Z(\mathbb{Q}[S_n])$ is spanned linearly by the conjugacy classes, that is the elements $C_\mu = \sum_{\sigma \in S_n, \text{cycle-type}(\sigma) = \mu} \sigma$. 

By a classical result of Frobenius (see [Fro00] or [Mac95, (I,7.8)]), for any $\lambda \vdash n$,

$$\text{Tr} \rho^\lambda(C_\mu) \quad \text{dimension of } \rho^\lambda = \theta^{(1)}_\mu(\lambda),$$

where $\rho^\lambda$ is the irreducible representation of the symmetric group associated to the Young diagram $\lambda$. In other words: $\theta^{(1)}_\mu$ is the image of $C_\mu$ by the abstract Fourier transform, which is an algebra morphism. Hence, the structure constants of the algebra $Z(\mathbb{Q}[S_n])$ with the basis $(C_\mu)_{\mu \vdash n}$ coincide with $c^{(1)}_{\mu, \nu; \pi}$.

These structure constants have been widely studied in the last fifty years in algebra and combinatorics (they count some families of graphs embedded in orientable surfaces). A famous result in this topic is due to Farahat and Higman [FH59, Theorem 2.2]: the quantity $c^{(1)}_{\mu_1 n - |\mu|; \nu_1 n - |\nu|; \pi_1 n - |\pi|}$ is a polynomial in $n$. Note that this is a consequence of Proposition 4.1.

Besides, structure constants of the center of $Z(\mathbb{Q}[S_n])$ have a well-known and obvious combinatorial interpretation.

**Lemma 5.1.** Fix a permutation of permutation $\sigma$ of cycle-type $\pi$. Then $c^{(1)}_{\mu, \nu; \pi}$ is the number of pairs of permutations $(\sigma_1, \sigma_2)$, such that $\sigma_1$ has cycle-type $\mu$, $\sigma_2$ has cycle-type $\nu$ and $\sigma_1 \cdot \sigma_2 = \sigma$.

This can be used to compute $c^{(1)}_{\mu, \nu; \pi}$ in some particular cases, that will be useful later.

**Lemma 5.2.** We have the following identities:

1. $c^{(1)}_{\mu, \nu; 1^n} = 0$ for any $\mu \neq \nu$;
2. $c^{(1)}_{(k^n), (k^n); (k^n)} = \binom{k+n}{k} (k-1)!$.

**Proof.** Consider the first item. The only permutation $\pi$ of cycle type $(1^n)$ is $\sigma = \text{id}$. Hence, the condition $\sigma_1 \cdot \sigma_2 = \sigma$ corresponds to $\sigma_2 = \sigma^{-1}$ and in particular $\sigma_1$ and $\sigma_2$ must have the same cycle-type.

Consider now the second item. As before, we must choose $\sigma = \text{id}$. As $\sigma_2 = \sigma^{-1}$ has always the same cycle-type as $\sigma_1$, the coefficient $c^{(1)}_{(k1^n), (k1^n); (k^n)}$ is simply...
the number of permutations of $k+n$ of type $(k1^n)$. This is well-known [Sag01, equation (1.2)] to be
\[
\frac{(k+n)!}{z_{(k1^n)}} = \binom{k+n}{k}(k-1)!.
\]
\[\square\]

Remark. The quantities $g_{\mu,\nu}^{(1)}$ also have a direct combinatorial interpretation in terms of partial permutations, see [IK99].

5.2. Case $\alpha = 2$: Hecke algebra of $(S_{2n}, H_n)$. An analogous interpretation of the structure constants is available in the case $\alpha = 2$. We explain it here, following the development given in [GJ96b].

We can view the elements of the symmetric group $S_{2n}$ as permutations of the following set: \{1, \overline{1}, \ldots, n, \overline{n}\}. A subgroup, denoted by $H_n$ and called hyperoctahedral group, is formed by permutations $\sigma$ such that $\sigma(i) = \sigma(i)$ for $i \in \{1, \ldots, n\}$, where by convention $\overline{j} = j$. We consider the subalgebra $\mathbb{Q}[H_n \backslash S_{2n}/H_n]$ of the elements which are invariant by multiplication on the left or on the right by any element of $H_n$; in other words
\[x \in \mathbb{Q}[H_n \backslash S_{2n}/H_n] \iff hxh' = x \text{ for all } h, h' \in H_n.\]

A non-trivial result is that this algebra is commutative.

The equivalence classes for the relation $x \sim hxh'$ (for $x \in S_{2n}$ and $h, h' \in H_n$) are called double-coset. They are naturally indexed by partitions of $n$, see [Mac95, (VII,2)]. We denote by $C^{(2)}_{\mu} \in \mathbb{Q}[H_n \backslash S_{2n}/H_n]$ the sum of all elements in the double coset corresponding to $\mu$. The family $(C^{(2)}_{\mu})_{\mu \vdash n}$ is a basis of $\mathbb{Q}[H_n \backslash S_{2n}/H_n]$.

One can show (see [GJ96b, Equation (3) and (5)]) that there exist some orthogonal idempotents $E_\lambda$ such that:
\[C^{(2)}_{\mu} = 2^n n! \sum_{\lambda \vdash n} \theta^{(2)}_{\mu}(\lambda) E_\lambda.\]

Hence, one has
\[C^{(2)}_{\mu} \cdot C^{(2)}_{\nu} = (2^n n!)^2 \sum_{\lambda \vdash n} \theta^{(2)}_{\mu}(\lambda) \theta^{(2)}_{\nu}(\lambda) E_\lambda = (2^n n!) \sum_{\pi \vdash n} \theta^{(2)}_{\mu,\nu}(\pi) C^{(2)}_{\pi}.
\]

Hence, the structure constants $h_{\mu,\nu,\pi}$ of the algebra $\mathbb{Q}[H_n \backslash S_{2n}/H_n]$ for the basis $(C^{(2)}_{\mu})_{\mu \vdash n}$ are, up to a factor $2^n n!$, the same as the ones of the algebra $\mathcal{F}(Y_n, \mathbb{Q})$ with the basis $(\theta^{(2)}_{\mu})_{\mu \vdash n}$.

In particular, Proposition 4.1 implies the following, which is an analog of Farhurat and Higman result (a combinatorial proof of the polynomiality has recently been given by O. Tout in [Tou13]).
Proposition 5.3. Let $\mu$, $\nu$ and $\pi$ be partitions without parts equal to 1. The renormalized structure constant of the algebra $\mathbb{Q}[H_n \setminus \mathfrak{S}_{2n}/H_n]$ 
\[
\frac{h_{\mu 1^n - |\mu|, \nu 1^n - |\nu|, \pi 1^n - |\pi|}}{n! 2^n \sqrt{2}}
\]
is a polynomial in $n$ of degree at most $d(\mu, \nu, \pi)$. Moreover, it has the coefficient of $n^{d(\mu, \nu, \pi)}$ the same as in $c_{\mu 1^n - |\mu|, \nu 1^n - |\nu|, \pi 1^n - |\pi|}(1)$. In particular,
• When $|\mu| - \ell(\mu) + |\nu| - \ell(\nu) = |\pi| - \ell(\pi)$, one has 
\[
\frac{h_{\mu 1^n - |\mu|, \nu 1^n - |\nu|, \pi 1^n - |\pi|}}{n! 2^n} = c_{\mu 1^n - |\mu|, \nu 1^n - |\nu|, \pi 1^n - |\pi|}(1)
\]
and this quantity is independent of $n$.
• When $|\mu| - \ell(\mu) + |\nu| - \ell(\nu) = |\pi| - \ell(\pi) - 1$, 
\[
\frac{h_{\mu 1^n - |\mu|, \nu 1^n - |\nu|, \pi 1^n - |\pi|}}{n! 2^n}
\]
is independent of $n$.

Proof. The claim that the renormalized structure constant mentioned above is a polynomial and the bound on its degree follow from Proposition 4.1 specialized to $\alpha = 2$. The dominant coefficient is a polynomial in $\gamma$ of degree 0, so it is the same for $\gamma \in \{0, -1/\sqrt{2}\}$, that is $\alpha \in \{1, 2\}$. The first item follows immediately.

In the second item, we consider the case $d(\mu, \nu, \pi) = 1$. So,
\[
\frac{h_{\mu 1^n - |\mu|, \nu 1^n - |\nu|, \pi 1^n - |\pi|}}{n! 2^n \sqrt{2}}
\]
is an affine function of $n$ with the same linear coefficient as $c_{\mu 1^n - |\mu|, \nu 1^n - |\nu|, \pi 1^n - |\pi|}(1)$. But the latter is identically equal to 0. Indeed, the fact that the sign of permutations is a group morphism implies that 
\[
c_{\mu 1^n - |\mu|, \nu 1^n - |\nu|, \pi 1^n - |\pi|}(1) = 0
\]
whenever $d(\mu, \nu, \pi)$ is odd. □

Goulden and Jackson [GJ96b] described the coefficients $h_{\mu, \nu, \pi}$ combinatorially. Let $\mathcal{F}_S$ be the set of all (perfect) matchings on a set $S$. For $F_1, \ldots, F_k \in \mathcal{F}_S$, let $G(F_1, \ldots, F_k)$ be the multigraph with vertex-set $S$ whose edges are formed by the pairs in $F_1, \ldots, F_k$. The components of $G(F_1, F_2)$ are even cycles. Let the list of their lengths in weakly decreasing order be $(2\theta_1, 2\theta_2, \ldots) = 2\theta$, and define $\Lambda$ by $\Lambda(F_1, F_2) = \theta$. Let $\mathcal{F}_n$ denote the set of all matchings on the set $\{1, 2, \ldots, 2n\}$.

Lemma 5.4 ([GJ96b] Lemma 2.2.). Let $F_1, F_2$ be two fixed matchings in $\mathcal{F}_n$ such that $\Lambda(F_1, F_2) = \pi$, where $\pi \vdash n$. Then, for any $\mu, \nu \vdash n$ we have 
\[
h_{\mu, \nu, \pi} = 2^n n! |\{F_3 \in \mathcal{F}_n : \Lambda(F_1, F_3) = \mu, \Lambda(F_2, F_3) = \nu\}|.
\]
In particular, 
\[
c_{\mu, \nu, \pi}(2) = |\{F_3 \in \mathcal{F}_n : \Lambda(F_1, F_3) = \mu, \Lambda(F_2, F_3) = \nu\}|.$
From this lemma one can evaluate some special cases of structure constants which will be helpful in the next subsection.

**Lemma 5.5.** We have the following identities:

1. \( c_{\mu,\nu}^{(2)} = 0 \) for any \( \mu \neq \nu \);
2. \( c_{(k+n), (k+n)}^{(2)} = \binom{k+n}{k} 2^{k-1}(k-1)! \).

**Proof.** The first item is immediate from Lemma 5.4, since whenever \( F_1 \) and \( F_2 \) are matchings such that \( \Lambda(F_1, F_2) = 1^n \) then clearly \( F_1 = F_2 \).

Let \( F_1, F_2 \) be two fixed matchings such that \( \Lambda(F_1, F_2) = (1^{k+n}) \) (hence \( F_1 = F_2 \)). We are looking for the number of matchings \( F_3 \) such that \( \Lambda(F_1, F_3) = \Lambda(F_2, F_3) = (k^n) \). Of course, as \( F_1 = F_2 \), it is the number of matchings \( F_3 \) with \( \Lambda(F_1, F_3) = (k^n) \). This number does not depend on \( F_1 \). Using \cite[Lemma 2.4]{FS11}, we know that the numbers of pairs \((F, F_3)\) with \( \Lambda(F, F_3) = (k^n) \) is given by

\[
\frac{(2n + 2k)!}{z(k^n)^{2n+1}}.
\]

As there are \( \frac{(2n+2k)!}{(2n+1)(n+k)!} \) matchings in \( F_{n+k} \), that is possible values for \( F \), for a fixed \( F_1 \), the number of matchings \( F_3 \) with \( \Lambda(F_1, F_3) = (k^n) \) is

\[
\frac{2^{n+k}(n + k)!}{z(k^n)^{2n+1}} = \binom{k+n}{k} 2^{k-1}(k-1)!.
\]

\[ \square \]

### 5.3. A technical lemma obtained by polynomial interpolation.

In order to study asymptotics of large Young diagrams we need to understand some special cases of structure constants. Here, we are going to present some technical, but very useful lemma about them:

**Lemma 5.6.** We have the following identities:

1. \( g_{\mu,\nu;\rho} = \delta_{\rho,\mu,\nu} \) for \( |\rho| + \ell(\rho) \geq |\mu| + \ell(\mu) + |\nu| + \ell(\nu) \);
2. \( g_{\mu,\nu;1^k} = 0 \) for \( \mu \neq \nu \) and \( 2k \geq |\mu| + \ell(\mu) + |\nu| + \ell(\nu) - 2 \);
3. \( g_{(k), (k);1^k} = k \);
4. \( g_{(k), (l);\rho} = 0 \) for \( |\rho| + \ell(\rho) = k + l + 1 \).

**Proof.** Let \( x(\mu, \nu; \rho) := |\mu| + \ell(\mu) + |\nu| + \ell(\nu) - (|\rho| + \ell(\rho)) \).

By Theorem 1.4 we know that \( g_{\mu,\nu;\rho} \) is a polynomial in \( \gamma \) of degree less or equal than \( x(\mu, \nu; \rho) \) and even or odd, depending whether \( x(\mu, \nu; \rho) \) is even or odd. Hence, if one wants to prove that for some particular partitions, \( g_{\mu,\nu;\rho} \) is identically equal to some constant \( c \), it is enough to prove that:

- \( g_{\mu,\nu;\rho}^{(1)} = c \) in the case \( x(\mu, \nu; \rho) = 0 \);
- \( g_{\mu,\nu;\rho}^{(2)} = c \) in the case \( x(\mu, \nu; \rho) = 1 \) (necessarily, \( c = 0 \) in this case);
- \( g_{\mu,\nu;\rho}^{(3)} = g_{\mu,\nu;\rho}^{(2)} = c \) in the case \( x(\mu, \nu; \rho) = 2 \).
Applying this idea, we see that the first item holds true, since this is true for \( \alpha = 1 \) (IO02 Proposition 4.9).

Consider now the second item. In this case, \( x(\mu, \nu; \rho) \leq 2 \), hence we need to prove that \( g^{(1)}_{\mu, \nu; 1} = g^{(2)}_{\mu, \nu; 1} = 0 \). We know from Lemma 5.2 (1) that \( c^{(1)}_{\pi, \rho; 1} = 0 \) for any pair of different partitions \( \pi, \rho \vdash n \). It means that for any sufficiently big \( n \), thanks to (18), we have the following equation:

\[
0 = \sum_{0 \leq i \leq k+1} g^{(1)}_{\tilde{\pi}, \tilde{\rho}; (1^i)} \cdot i! \cdot \binom{n}{i},
\]

hence

\[
g^{(1)}_{\tilde{\pi}, \tilde{\rho}; (1^i)} = 0.
\]

The same holds for \( g^{(2)} \) using Lemma 5.5. This finishes the case where \( \mu \) and \( \nu \) have no parts equal to 1.

The general case follows, since for any \( \lambda \vdash n \) one has

\[
\text{Ch}_{\mu}(\lambda) = (n - |\bar{\mu}|) m_{1(\mu)} \text{Ch}_{\bar{\mu}}(\lambda).
\]

In order to prove the third item, we need to prove the equalities

\[
g^{(1)}_{(k), (k); 1^k} = k = g^{(2)}_{(k), (k); 1^k} \text{ (as } x((k), (k); 1^k) = 2). \]

Here, we use Equation (18) again. We have that

\[
c^{(1)}_{(k+1), (k+1); (1^k+n)} = \frac{1}{k^2} \sum_{0 \leq i \leq k} g^{(1)}_{(k), (k); (1^i)} i! \binom{k+n}{i}.
\]

By Lemma 5.2 (2) one has that

\[
c^{(1)}_{(k+1), (k+1); (1^k+n)} = \binom{k+n}{k} (k-1)!.
\]

It gives us

\[
\binom{k+n}{k} (k-1)! = \frac{1}{k^2} \sum_{0 \leq i \leq k} g^{(1)}_{(k), (k); (1^i)} i! \binom{k+n}{i}
\]

and since both sides of the equation are polynomials in \( n \), the equation \( g^{(1)}_{(k), (k); (1^k)} = k \) follows. The same proof works for \( \alpha = 2 \), using Lemma 5.5 (2).

Finally, let us prove the last item. As \( x((k), (l), \rho) = 1 \) in this case, it is enough to prove that \( g^{(2)}_{(k), (l), \rho} = 0 \). Here, we shall use a different approach. It is proved in [FS11 Theorem 5.3] that the coefficient of

\[
\left(R^{(2)}_2\right)^s_2 \left(R^{(2)}_3\right)^s_3 \cdots
\]

in Kerov’s expansion of \( \text{Ch}^{(2)}_{(k)} \text{Ch}^{(2)}_{(l)} - \text{Ch}^{(2)}_{(k, l)} \) is, up to some constant factor, the number of maps with 2 faces, \( k + l \) edges, \( 2s_2 + 3s_3 + \cdots \) vertices and some
additional properties (the details of which will not be important here). But, using the theory of Euler characteristic, such maps may exist only if
\[
2 - (k + l) + 2s_2 + 3s_3 + \cdots \leq 2,
\]
that is
\[
2s_2 + 3s_3 + \cdots \leq k + l.
\]
This implies that
\[
\deg_1 \left( \text{Ch}^{(2)}_{(k)} \text{Ch}^{(2)}_{(l)} - \text{Ch}^{(2)}_{(k,l)} \right) = k + l.
\]
Expanding it in the Jack characters basis one has
\[
\text{Ch}^{(2)}_{(k)} \text{Ch}^{(2)}_{(l)} = \text{Ch}^{(2)}_{(k,l)} + \sum_{|\rho| + \ell(\rho) \leq k + l} g^{(2)}_{(k),(l);\rho} \text{Ch}^{(2)}_{\rho},
\]
which finishes the proof. □

**Remark.** The equality \(g^{(1)}_{(k),(k);(1^k)} = k\) was also established by Kerov, Ivanov and Olshanski [IO02, Proposition 4.12.], using their combinatorial interpretation for \(g^{(1)}_{\mu,\nu;\rho}\).

### 6. Jack Measure: Law of Large Numbers

The purpose of this Section is to prove Theorem 1.1. As in [IO02], the key point is to prove the convergence of polynomial functions.

#### 6.1. Convergence of polynomial functions

Let us recall the equation (7) which expresses an expectation of Jack characters with respect to Jack measure:
\[
\mathbb{E}_{P_n^{(\alpha)}}(\text{Ch}^{(\alpha)}_{\mu}) = \begin{cases} 
  n(n-1) \cdots (n-k+1) & \text{if } \mu = 1^k \text{ for some } k \leq n, \\
  0 & \text{otherwise}.
\end{cases}
\]
As \(\text{Ch}_{\mu}\) is a linear basis of \(\Lambda^{(\alpha)}\), it implies the following lemma (which is an analogue of [Ols10, Theorem 5.5] with another gradation).

**Lemma 6.1.** Let \(F\) be an \(\alpha\)-polynomial function. Then \(\mathbb{E}_{P_n^{(\alpha)}}(F)\) is a polynomial in \(n\) of degree at most \(\deg_1(F)/2\).

**Proof.** It is enough to verify this lemma on the basis \(\text{Ch}_{\mu}\) because of the remark in Section 3.8. But in this case \(\mathbb{E}_{P_n^{(\alpha)}}(F)\) is explicit (see formula (7)) and the lemma is obvious (recall that \(\deg_1(\text{Ch}_{\mu}) = |\mu| + \ell(\mu)\), see Section 3.4). □

Informally, smaller terms for \(\deg_1\) are asymptotically negligible. We can now prove the following weak convergence result:

**Proposition 6.2.** Let \((\lambda(n))_{n \geq 1}\) be a sequence of random partitions distributed with Jack measure. For any \(1\)-polynomial function \(F \in \Lambda^{(1)}_\ast\), when \(n \to \infty\), one has
\[
F \left( D_{1/\sqrt{n}}(A_\alpha(\lambda(n))) \right) \xrightarrow{P_n^{(\alpha)}} F(\Omega),
\]
where \( \overline{P}^{(\alpha)} \to \) means convergence in probability.

**Proof.** As \((R_k^{(1)})_{k \geq 2}\) is an algebraic basis of \( \Lambda^{(1)} \), it is enough to prove the proposition for any \( R_k^{(1)} \).

Let \( \mu \) be partition. By Corollary 3.8

\[
\prod_{i \leq \ell(\mu)} R_{\mu_i + 1} = \text{Ch}_\mu + \text{terms of degree at most} \left| \mu \right| + \ell(\mu) - 1 \text{ with respect to } \text{deg}_1.
\]

Together with Lemma 6.1 and the formula (7) for \( \mathbb{E}_{\overline{P}^{(\alpha)}}(\text{Ch}_\mu) \), this implies:

\[
\mathbb{E}_{\overline{P}^{(\alpha)}} \left( \prod_{i \leq \ell(\mu)} R_{\mu_i + 1} \right) = \begin{cases} \frac{n(n - 1) \cdots (n - k + 1) + O(n^{k-1})}{o(n^{\frac{\left| \mu \right| + \ell(\mu)}{2}})} & \text{if } \mu = 1^k \text{ for some } k; \\
o(1) & \text{otherwise.} \end{cases}
\]

In particular

\[
\mathbb{E}_{\overline{P}^{(\alpha)}}(R_k(D_{1/\sqrt{n}}(\lambda(n)))) = \frac{1}{n^{k/2}} \mathbb{E}_{\overline{P}^{(\alpha)}}(R_k) = \delta_{k,2} + O \left( \frac{1}{\sqrt{n}} \right),
\]

\[
\text{Var}_{\overline{P}^{(\alpha)}}(R_k(D_{1/\sqrt{n}}(\lambda(n)))) = \frac{1}{n^k} \left( \mathbb{E}_{\overline{P}^{(\alpha)}}((R_k)^2) - \mathbb{E}_{\overline{P}^{(\alpha)}}(R_k)^2 \right) = O \left( \frac{1}{n} \right).
\]

Thus, for each \( k \), \( R_k(D_{1/\sqrt{n}}(\lambda(n))) \) converges in probability towards \( \delta_{k,2} \). But, by definition

\[
R_k(D_{1/\sqrt{n}}(\lambda(n))) = R_k^{(1)} \left( D_{1/\sqrt{n}}(A_\alpha(\lambda(n))) \right)
\]

and \((\delta_{k,2})_{k \geq 2}\) is the sequence of free cumulants of the continuous diagram \( \Omega \) (see [Bia01, Section 3.1]), i.e.

\[
\delta_{k,2} = R_k^{(1)}(\Omega).
\]

**Shape convergence.** The last ingredient to prove Theorem 1.1 is the following technical lemma, proved by Fulman [Ful04, Lemma 6.6]:

**Lemma 6.3.** Suppose that \( \alpha > 0 \). Then

1. \[
\mathbb{P}_n^{(\alpha)} \left( \lambda_1 \geq 2e \sqrt{\frac{n}{\alpha}} \right) \leq \alpha n^{2-e} \sqrt{\frac{\pi}{n}},
\]

2. \[
\mathbb{P}_n^{(\alpha)} \left( \lambda_1' \geq 2e \sqrt{n/\alpha} \right) \leq \frac{n^2}{\alpha} 4^{-e \sqrt{n/\alpha}}.
\]

In particular

\[
\lim_{n \to \infty} \mathbb{P}_n^{(\alpha)} \left( \left\lfloor - \frac{\lambda_1'}{\sqrt{n}} \right\rfloor = \left\lfloor -2e \sqrt{\frac{n}{\alpha}} \frac{2e}{\sqrt{\alpha}} \right\rfloor \right) = 1.
\]

**End of proof of Theorem 1.1.** It follows from Proposition 6.2 and Lemma 6.3 by the same argument as the one given in [IO02, Theorem 5.5].
7. JACK MEASURE: CENTRAL LIMIT THEOREM FOR JACK CHARACTERS

In this section we prove the central limit theorem for Jack characters (Theorem 1.2) and the bound of the speed of convergence in this theorem (Theorem 1.3).

7.1. Multivariate Stein’s method. As explained in introduction, our main tool will be a multivariate analog of the so-called Stein’s method due to Reinert and Röllin [RR09]. For any discrete random variables $W,W^*$ with values in $\mathbb{R}^d$, we say that the pair $(W,W^*)$ is exchangeable if for any $w_1,w_2 \in \mathbb{R}^d$ one has $\mathbb{P}(W = w_1, W^* = w_2) = \mathbb{P}(W = w_2, W^* = w_1)$. Let $\mathbb{E}^W(\cdot)$ denote for the conditional expected value given $W$. The theorem of Reinert and Röllin is the following [RR09, Theorem 2.1]:

**Theorem 7.1** (multivariate Stein’s theorem). Let $(W,W^*)$ be an exchangeable pair of $\mathbb{R}^d$-valued random variables such that $\mathbb{E}(W) = 0$ and $\mathbb{E}(WW^t) = \Sigma$, where $\Sigma \in M_{d \times d}(\mathbb{R})$ is symmetric and positive definite matrix. Suppose that $\mathbb{E}W(W^* - W) = -\Lambda W$, where $\Lambda \in M_{d \times d}(\mathbb{R})$ is invertible. Then, if $Z$ has $d$-dimensional standard normal distribution, we have for every three times differentiable function $h: \mathbb{R}^d \to \mathbb{R}$,

\[
\left| \mathbb{E}h(W) - \mathbb{E}h(\Sigma^{1/2}Z) \right| \leq \frac{|h|_2^2}{4} A + \frac{|h|_3^3}{12} B,
\]

where, using the notation $\lambda^{(i)} := \sum_{1 \leq m \leq d} |(\Lambda^{-1})_{m,i}|$,

\[
|h|_n = \sup_{i_1,\ldots,i_n} \left\| \frac{\partial^n}{\partial x_{i_1} \cdots \partial x_{i_n}} h \right\|,
\]

\[
A = \sum_{1 \leq i,j \leq d} \lambda^{(i)} \sqrt{\text{Var} \mathbb{E}^W(W_i^* - W_i)(W_j^* - W_j)},
\]

\[
B = \sum_{1 \leq i,j,k \leq d} \lambda^{(i)} \mathbb{E}\left| (W_i^* - W_i)(W_j^* - W_j)(W_k^* - W_k) \right|.
\]

Let $d$ be a positive integer. For $k \geq 2$, as in the statement of Theorem 1.2 we define the following function of Young diagrams of size $n$:

\[
W_k = n^{-k/2} \sqrt{k} \Theta^{(a)}(k,1_{n-k}) = n^{-k/2} \sqrt{k}^{-1} \text{Ch}(k).
\]

It can be seen as a random variable on the probability space of Young diagrams of size $n$ endowed with Jack measure. We also consider the corresponding random vector

\[
\tilde{W}_d = (W_2,\ldots,W_{d+1}).
\]

Theorem 1.2 states that $\tilde{W}_d$ converges in distribution towards a vector of independent random variables. Therefore we would like to apply the theorem above to this $d$-plet of random variables. In the next sections, we shall construct an exchangeable pair and check the hypothesis of Theorem 7.1.
7.2. **An exchangeable pair.** The first step consists in building a \(d\)-uple \(\tilde{W}^*_d\), such that \((\tilde{W}_d, \tilde{W}^*_d)\) is an exchangeable pair. The construction that we will describe here is due to Fulman [Ful04] and works for any tuple of random variables under Jack measure.

His construction uses Markov chains, so let us begin by fixing some terminology. Let \(X\) be a finite set. A Markov chain \(M\) on \(X\) is the data of transition probability \(M(x, y)\) indexed by pairs of elements of \(X\) with

\[
M(x, y) \geq 0 \quad \text{and} \quad \sum_{y \in X} M(x, y) = 1.
\]

If \(x\) is a random element of \(X\) distributed with probability \(P\), then, applying once the Markov chain \(M\), we obtain by definition a random element \(y\) of \(X\), defined on the same probability space as \(x\), whose conditional distribution is given by:

\[
P(y = y_0 | x = x_0) = M(x_0, y_0).
\]

Using the notation above, the Markov chain \(M\) is termed **reversible** with respect to \(P\) if the distribution of \((x, y)\) is the same as the distribution of \((y, x)\), or equivalently, for any \(x_0, y_0\) in \(X\),

\[
P(\{x_0\})M(x_0, y_0) = P(\{y_0\})M(y_0, x_0).
\]

Reversible Markov chains can be used to construct exchangeable pairs as follows. Let \(M\) be a reversible Markov chain on a finite set \(X\) with respect to a probability measure \(P\). Consider also a \(R^d\)-valued function \(W\) on \(X\). We consider a random element \(x\) distributed with respect to \(P\) and \(y\) obtained by applying the Markov chain \(M\) to \(P\). Then, directly from the definition, one sees that \((W(x), W(y))\) is exchangeable.

So, to construct an exchangeable pair for \(\tilde{W}_d\), it is enough to construct a reversible Markov chain with respect to Jack measure. We present now Fulman’s construction of such Markov chain.

Let \(\tau \vdash n - 1\) and \(\lambda \vdash n\), and let \(C_{\lambda/\tau} (R_{\lambda/\tau}, \text{respectively})\) be the union of columns (rows, respectively) of \(\lambda\) that intersect \(\lambda/\tau\). We define

\[
\phi^{(\alpha)}(\lambda/\tau) = \prod_{\square \in C_{\lambda/\tau} \setminus R_{\lambda/\tau}} (\alpha a_\lambda(\square) + \ell_\lambda(\square) + 1)(\alpha a_{\tau}(\square) + \ell_{\tau}(\square) + \alpha).
\]

Let

\[
c_\lambda^{(\alpha)} = \prod_{\square \in \lambda} (\alpha a(\square) + \ell(\square) + 1)
\]

and

\[
(c_\lambda^{(\alpha)})^{(\alpha)} = \prod_{\square \in \lambda} (\alpha a(\square) + \ell(\square) + \alpha).
\]

We recall that \(j^{(\alpha)}_\lambda = c_\lambda^{(\alpha)}(c_\lambda^{(\alpha)})^{(\alpha)}\). For \(\lambda, \rho \vdash n\) we define two functions:

\[
M^{(\alpha)}(\lambda, \rho) = \frac{(c_\lambda^{(\alpha)})^{(\alpha)}}{n! c_\rho^{(\alpha)}} \sum_{\tau \vdash n - 1} \frac{\phi^{(\alpha)}(\lambda/\tau)\phi^{(\alpha)}(\rho/\tau)c_\tau^{(\alpha)}}{(c_\tau^{(\alpha)})^{(\alpha)}}
\]
and

\[
L^{(\alpha)}(\lambda, \rho) = \frac{1}{\alpha^n n!} \sum_{\mu \vdash n} (z_\mu)^2 \alpha^{2\ell(\mu)} \theta_\mu(\lambda) \theta_\mu(\rho) \theta_\mu((n - 1, 1)).
\]

As explained by Fulman [Ful04], both \(M^{(\alpha)}\) and \(L^{(\alpha)}\) are defined to be a deformation of a certain Markov chain which is reversible with respect to Plancherel measure. Roughly speaking, this Markov chain remove one box from a given Young diagram with certain probability and add another box with some probability to obtain a new Young diagram of the same size as the one from which we started. Fulman [Ful04] proved the following:

**Proposition 7.2.** [Ful04, Section 4]

1. If \(\rho \neq \lambda\) then
   \[L^{(\alpha)}(\lambda, \rho) = \frac{\alpha(n - 1) + 1}{\alpha(n - 1)} M^{(\alpha)}(\lambda, \rho);\]

2. Let \(\lambda \vdash n\). Then
   \[
   \sum_{\rho \vdash n} L^{(\alpha)}(\lambda, \rho) = \sum_{\rho \vdash n} M^{(\alpha)}(\lambda, \rho) = 1;
   \]

3. \(L^{(\alpha)}\) (hence \(M^{(\alpha)}\) as well) is reversible with respect to Jack measure.

For more details about this construction we refer to Fulman [Ful04].

### 7.3. Checking hypotheses.

Recall that we have defined
\[W_k = n^{-k/2} \sqrt{k^{-1}} Ch(k)\]
and the random vector \(\tilde{W}_d = (W_2, \ldots, W_{d+1})\).

Let \(\lambda\) be a random partition distributed according to Jack measure, and \(\lambda^* (\lambda', \) respectively) be obtained from \(\lambda\) by applying the Markov chain \(M^{(\alpha)} (L^{(\alpha)}, \) respectively). By a small abuse of notations, set \(\tilde{W}_d := \tilde{W}_d(\lambda), W_d^\ast := \tilde{W}_d(\lambda^*)\) and \(\tilde{W}_d' := \tilde{W}_d(\lambda')\). We shall prove now, that for any \(d \in \mathbb{N}\), the pair \((\tilde{W}_d, W_d^\ast)\) of random vectors satisfies conditions of Theorem 7.1.

Let us begin by two technical known statements about Jack polynomials.

**Lemma 7.3.** (1) [Mac95] Page 382

\[
\sum_{\rho \vdash n} \frac{\theta_\mu(\rho) \theta_\nu(\rho)}{\lambda^{(\alpha)}} = \frac{\delta_{\mu, \nu}}{z_\mu \alpha^{\ell(\mu)}};
\]

(2) [Sta89] Page 107

\[
\theta_\mu((n - 1, 1)) = \frac{\alpha^{n-\ell(\mu)} n! (\alpha(n - 1) + 1) m_1(\mu) - n}{z_\mu \alpha(n - 1)}.
\]

Recall that \((E_n^{(\alpha)})\tilde{W}_d\) denotes the conditional expectation given \(\tilde{W}_d\). Similarly, we denote by \((E_n^{(\alpha)})\lambda\) the conditional expectation given \(\lambda\).
Proposition 7.4. Let \( d \in \mathbb{N} \) and let \( 2 \leq k \leq d + 1 \). Then, one has
\[
(\mathbb{E}_n^{(\alpha)})^{\lambda}(W_k^*) = (\mathbb{E}_n^{(\alpha)})^{\lambda}(W_k^*),
\]
\[
(\mathbb{E}_n^{(\alpha)})^{\lambda}(W_k^*) = \left(1 - \frac{k}{n}\right)W_k;
\]
\[
(\mathbb{E}_n^{(\alpha)})^{\lambda}(W_k^*) = \left(1 - \frac{k(\alpha(n - 1) + 1)}{\alpha n(n - 1)}\right)W_k.
\]

Proof. By definition,
\[
(\mathbb{E}_n^{(\alpha)})^{\lambda}(W_k^*) = \sum_{\lambda^*} M(\lambda, \lambda^*)W_k^{\lambda^*}.
\]

By [Ful04, Proposition 6.2.], we have that \((\theta_\mu(\lambda))_{\lambda^*} \) is an eigenvector of \( M^{(\alpha)} \) with eigenvalue
\[
d_\mu := 1 + \frac{\alpha(n - 1)}{\alpha(n - 1) + 1} \left(\frac{z_\mu}{\alpha^{\ell(\mu)n}!}\theta_\mu((n - 1), 1) - 1\right).
\]

As \((W_k^*(\lambda))_{\lambda^*} \) is a multiple of \((\theta_\mu(\lambda))_{\lambda^*} \), it is also an eigenvector of \( M^{(\alpha)} \), with the same eigenvalue. Hence,
\[
(\mathbb{E}_n^{(\alpha)})^{\lambda}(W_k^*) = d_{(k, 1^{n-k})}W_k^{\lambda}.
\]

The expression of \( c_\mu \) can be evaluated using Lemma 7.3-(2):
\[
d_\mu = \frac{(\alpha(n - 1) + 1)m_1(\mu)}{n(\alpha(n - 1) + 1)} = \frac{m_1(\mu)}{n}.
\]

For \( \mu = (k, 1^{n-k}) \), we get \( d_{(k, 1^{n-k})} = 1 - \frac{k}{n} \), which finishes the proof of the second equality of the first statement.

In particular, we see that \((\mathbb{E}_n^{(\alpha)})^{\lambda}(W_k^*) \) depends only on \( W_k \) and hence on \( \tilde{W}_d \).

As, conversely, \( \tilde{W}_d \) is determined by \( \lambda \), one has
\[
(\mathbb{E}_n^{(\alpha)})^{\lambda}(W_k^*) = (\mathbb{E}_n^{(\alpha)})^{\lambda}(W_k^*).
\]

A similar proof yields the second statement of the proposition. \( \square \)

Corollary 7.5. Let \( d \in \mathbb{N} \). Then
\[
(\mathbb{E}_n^{(\alpha)})^{\lambda}(\tilde{W}_d^* - \tilde{W}_d) = -\Lambda\tilde{W}_d,
\]
with \( \Lambda_{i,j} = \delta_{i,j} \frac{i+1}{n} \). In particular, with the notation of Theorem 7.7
\[
\lambda(i) = \frac{n}{i + 1}.
\]

Proof. It is a straightforward consequence of Proposition 7.4. \( \square \)

Consider now \( \Sigma = \mathbb{E}_n^{(\alpha)}(\tilde{W}_d\tilde{W}_d^*) \) as in the statement of Theorem 7.1. This matrix is symmetric by definition, but one has to check that it is positive definite.

For a matrix \( A \), denote \( \|A\| := \max_{i,j} |A_{i,j}| \) the supremum norm on matrices.

Proposition 7.6. There exists a constant \( A_{d,\alpha} \) which depends only on \( d \) and \( \alpha \) such that for any \( n \geq A_{d,\alpha} \) the matrix \( \Sigma \) is positive definite. Moreover,
\[
\|\Sigma^{1/2} - \text{Id}\| = O(n^{-1/2}).
\]
Proof. Strictly from the definition we have that

$$\Sigma_{i,j} = \mathbb{E}_n^{(\alpha)} \left( \frac{1}{\sqrt{1 + 1/\sqrt{1 + 1/n(i+j+2)/2}}} \, \text{Ch}_{(i+1) \text{ Ch}_{(j+1)}} \right)$$

$$= \frac{1}{\sqrt{1 + 1/\sqrt{1 + 1/n(i+j+2)/2}}} \sum_k g(i+1)(j+1); \mathbb{E}_n^{(\alpha)}(\text{Ch}_k).$$

Since

$$\mathbb{E}_n^{(\alpha)}(\text{Ch}_\mu) = \begin{cases} (n)_k & \text{if } \mu = 1^k \text{ for some } k \leq n, \\ 0 & \text{otherwise}, \end{cases}$$

where \((n)_k := n(n-1) \cdots (n-k+1)\), we have that

$$\Sigma_{i,j} = \frac{1}{\sqrt{1 + 1/\sqrt{1 + 1/n(i+j+2)/2}}} \sum_k g(i+1)(j+1); (n)_k,$$

and using items (1), (2) and (3) of Lemma 7.6 we have that

$$\Sigma_{i,j} = \delta_{i,j} + O(n^{-1}).$$

In other terms,

$$||\Sigma - \text{Id}|| = O(n^{-1}).$$

As the set of positive definite matrix is an open set of the space of symmetric matrices, this implies that \(\Sigma\) is positive definite for \(n\) big enough.

Besides, the application \(A \mapsto \sqrt{A}\) is differentiable on this open set, which implies the bound \(||\Sigma^{1/2} - \text{Id}|| = O(n^{-1/2}).\)

\(\square\)

7.4. Error term. Since we have checked that the pair \((\tilde{W}_d, \tilde{W}_d^+\)) of the random vectors satisfies the assumptions of Theorem 7.1, we would like to estimate the error term in that theorem. In order to prove that the random vector \(\tilde{W}_d\) is asymptotically Gaussian, we need to show that quantities \(A\) and \(B\) from Theorem 7.1 vanish as \(n \to \infty\). This section is devoted to making these calculations.

Lemma 7.7. The following inequality holds:

$$\text{Var}_n^{(\alpha)} \left( \left( \mathbb{E}_n^{(\alpha)} \right)^{\tilde{W}_d}(W_i - W_i^*)(W_j - W_j^*) \right) \leq \frac{1}{ijn^2 + j+2}$$

$$\times \left( \sum_{\mu_1, \mu_2; \mu} H_{\mu_1, \mu_2; \mu}^{ij} \mathbb{E}_n^{(\alpha)}(\text{Ch}_{\mu})(\lambda) - (i+j)^2 \left( \sum_{\mu} g(i)(j); \mathbb{E}_n^{(\alpha)}(\text{Ch}_{\mu}) \right)^2 \right),$$

where

$$H_{\mu_1, \mu_2; \mu}^{ij} := (i+j - |\mu_1| + m_1(\mu_1))(i+j - |\mu_2| + m_1(\mu_2))g(i)(j); \mu_1 g(i)(j); \mu_2 g(i)(j); \mu_2.$$

Proof. Following Fulman [Ful04 Proof of Proposition 6.4], from Jensen’s inequality for conditional expectations, the fact that \(\tilde{W}_d\) is determined by \(\lambda\) implies that

$$\mathbb{E}_n^{(\alpha)} \left( \left( \mathbb{E}_n^{(\alpha)} \right)^{\tilde{W}_d}(W_i - W_i^*)(W_j - W_j^*) \right) \leq \mathbb{E}_n^{(\alpha)} \left( \left( \mathbb{E}_n^{(\alpha)} \right)^{\lambda}(W_i - W_i^*)(W_j - W_j^*) \right)^2.$$
We have, by Proposition 7.2, that

\begin{equation}
(\mathbb{E}_n^{(\alpha)})^\lambda(W_i^*-W_i)(W_j^*-W_j) = \frac{\alpha(n-1)}{\alpha(n-1)+1}(\mathbb{E}_n^{(\alpha)})^\lambda(W_i^*-W_i)(W_j^*-W_j) \\
= \frac{\alpha(n-1)}{\alpha(n-1)+1} \left((\mathbb{E}_n^{(\alpha)})^\lambda(W_i^*W_j^*) - (\mathbb{E}_n^{(\alpha)})^\lambda(W_i^*)W_j - (\mathbb{E}_n^{(\alpha)})^\lambda(W_j^*)W_i + W_iW_j \right) \\
= \frac{\alpha(n-1)}{\alpha(n-1)+1} \left((i+j)(\alpha(n-1)+1)-1\right) W_iW_j,
\end{equation}

where the last equality follows from Proposition 7.4. Strictly from the definition of $L^{(\alpha)}$, one has

\begin{equation}
(\alpha^\mu) \mathbb{E}_n^{(\alpha)}(W_i^*W_j^*) = \sum_{\mu \vdash \mu} \theta_\mu(\lambda) \theta_\mu(n-1,1) \frac{(z_\mu)^2 \alpha^{2\ell(\mu)}}{\alpha^\mu n!} \\
\times \frac{1}{\sqrt{ijn^{i+j/2}}} \sum_{|\tau| \leq i+j+2} g(i,j) : \tau \sum_{\rho \vdash \mu} Ch_\tau(\rho) \theta_\rho(\mu) \frac{\sum_{j_\rho(\alpha)}}{1}. \tag{26}
\end{equation}

We may assume $n \geq i+j$. Recall that $Ch_\tau(\rho)$ is a multiple of $\theta_{\tau 1^{n-|\tau|}}(\rho)$ and, hence, using Lemma 7.3 (1), only terms corresponding to $\mu = \tau 1^{n-|\tau|}$ survives and we get:

\begin{equation}
(\mathbb{E}_n^{(\alpha)})^\lambda(W_i^*W_j^*) = \frac{1}{\sqrt{ijn^{i+j/2}}} \sum_{\tau} g(i,j) : \tau \sum_{\rho \vdash \mu} Ch_\tau(\rho) \theta_\rho(\mu) \frac{\sum_{j_\rho(\alpha)}}{1}. \tag{27}
\end{equation}

We now apply Lemma 7.3 (2):

\begin{equation}
(\mathbb{E}_n^{(\alpha)})^\lambda(W_i^*W_j^*) = \frac{1}{\sqrt{ijn^{i+j/2}}} \sum_{\mu} g(i,j): \mu \ Ch_\mu(\lambda) \times \frac{(\alpha(n-1)+1)(n-|\mu|+m_1(\mu)) - n}{\alpha n(n-1)}. \tag{28}
\end{equation}

Moreover, strictly from the definition, one has

\begin{equation}
W_i(\lambda)W_j(\lambda) = \frac{1}{\sqrt{ijn^{i+j/2}}} \sum_{\mu} g(i,j) : \mu \ Ch_\mu(\lambda). \tag{29}
\end{equation}

One can substitute two above equations to the equation (25) and simplifies it to obtain

\begin{equation}
(\mathbb{E}_n^{(\alpha)})^\lambda(W_i^* - W_i)(W_j^* - W_j) = \frac{1}{\sqrt{ijn^{i+j/2}}} \sum_{\mu} g(i,j) : \mu \ Ch_\mu(\lambda). \tag{30}
\end{equation}

In the formula above, the summation runs over integer partitions $\mu$. But $g(i,j) : \mu = 0$ for $|\mu| + \ell(\mu) > i+j$, unless $\mu = (i,j)$ (Lemma 5.6 (4)). In the latter case ($\mu = (i,j)$), the numerical factor $i+j - |\mu| + m_1(\mu)$ vanishes. Finally, summation in equation (28) can be restricted to partitions $\mu$ with $|\mu| + \ell(\mu) \leq i+j$. 

where the first equality comes from the fact that $\ast W E 46 M. DOI ˛ EGA AND V. FÉRAY

Proof. (31) $E_n(\alpha) \left( (E_n(\alpha) \ast (W_i^* - W_i)(W_j^* - W_j) \right)^2

= \frac{1}{ijn+2} \sum_{|\mu|+\ell(\mu) \leq i+j} (i + j - |\mu| + m_1(\mu)) g(i,j;\mu) \text{Ch}_\mu(\lambda) \right)^2

= \frac{1}{ijn+2} \sum_{|\mu_1|+\ell(\mu_1) \leq i+j, |\mu_2|+\ell(\mu_2) \leq i+j} H_{\mu_1,\mu_2;\mu}(\text{Ch}_\mu(\lambda)),

where $H_{\mu_1,\mu_2;\mu}$ is defined as in the statement of the lemma. Besides,

(30) $\left( (E_n(\alpha) \tilde{W}_d(W_i^* - W_i)(W_j^* - W_j) \right)^2

= (2E_n(\alpha)(W_i W_j) - E_n(\alpha) \left( (E_n(\alpha) \tilde{W}_d(W_i^*)E_n(\alpha)(W_j) \right) - E_n(\alpha) \left( (E_n(\alpha) \tilde{W}_d(W_j^*)E_n(\alpha)(W_i) \right) \right)^2

= \left( \frac{i+j}{n} E_n(\alpha)(W_i W_j) \right)^2 \frac{(i+j)^2}{ijn+2} \left( \sum_{\mu} g(i,j;\mu) E_n(\alpha)(\text{Ch}_\mu) \right)^2,

where the first equality comes from the fact that $W_i$ has the same distribution as $W_i^*$, and the second equality follows from Proposition 7.4. We finish the proof by the following inequality:

(31) $\text{Var}_n(\alpha) \left( (E_n(\alpha) \tilde{W}_d(W_i^* - W_i)(W_j - W_j^*) \right)

= E_n(\alpha) \left( (E_n(\alpha) \tilde{W}_d(W_i^* - W_i)(W_j - W_j^*) \right)^2 - \left( E_n(\alpha) \left( (E_n(\alpha) \tilde{W}_d(W_i^* - W_i)(W_j - W_j^*) \right) \right)^2

\leq E_n(\alpha) \left( (E_n(\alpha) \lambda(W_i^* - W_i)(W_j^* - W_j) \right)^2 - \left( E_n(\alpha) \left( (E_n(\alpha) \tilde{W}_d(W_i^* - W_i)(W_j^* - W_j) \right) \right)^2.

\square

Proposition 7.8. Let $d \in \mathbb{N}$ and let

$$A = \sum_{2 \leq i,j \leq d+1} \frac{n}{i} \sqrt{\text{Var}_n(\alpha) \left( (E_n(\alpha) \tilde{W}_d(W_i^* - W_i)(W_j - W_j^*) \right.}.$$.

Then $A = O(n^{-1/2})$.

Proof. By Lemma 7.7 we need to estimate the following sum:

$$\left( \sum_{|\mu_1|+\ell(\mu_1) \leq i+j, |\mu_2|+\ell(\mu_2) \leq i+j} H_{\mu_1,\mu_2;\mu}(\text{Ch}_\mu(\lambda)) - (i+j)^2 \left( \sum_{\mu} g(i,j;\mu) E_n(\alpha)(\text{Ch}_\mu) \right)^2 \right)^{1/2}.$$
Since
\[ E_{F_n}(\text{Ch}_\mu) = \begin{cases} (n)_k & \text{if } \mu = 1^k \text{ for some } k \leq n, \\ 0 & \text{otherwise}, \end{cases} \]
we, in fact, have to bound
\[ (32) \]
\[ \left( \sum_{\mu_1, \mu_2 \geq 0} H^{(i,j)}_{\mu_1, \mu_2; (1^l)}(n)_l - (i + j)^2 \sum_{l,k} g(i,i),(j,j);(1^k)(n)_l(n)_k \right)^{1/2}. \]

Let us consider the second sum, which is simpler. Suppose first that \( i \neq j \). Then, by Lemma 5.6 (2), summands corresponding to \( l \geq (i + j)/2 \) or \( k \geq (i + j)/2 \) vanish and the sum is \( O(n^{i+j-1}) \).

Consider now the case \( i = j \) (recall that \( i, j \geq 2 \)). We use this time Lemma 5.6 (1): summands corresponding to \( l \geq i + 1 \) or \( k \geq i + 1 \) vanish. Therefore there are no summands of order \( n^{i+j+1} \). The unique summand of order \( n^{i+j} \) (corresponding to \( l = k = i \)) is \( i^2(n)_i^2 \) by Lemma 5.6 (3).

Finally, we have that
\[ (33) \]
\[ (i + j)^2 \sum_{l,k} g(i,i),(j,j);(1^k)(n)_l(n)_k = \delta_{i,j} 4 i^4 n^{i+j} + O(n^{i+j-1}). \]

Let us describe the terms corresponding to \( l \geq i + j \) in the first sum of (32). By Lemma 5.6 (1), \( g_{\mu_1, \mu_2; (1^l)} \), and hence \( H_{\mu_1, \mu_2; (1^l)} \), vanishes unless
\[ |\mu_1| + \ell(\mu_1) + |\mu_2| + \ell(\mu_2) \geq 2l \geq 2(i + j). \]

Comparing it with the conditions under the first summation symbol of (32), we have, in fact, equalities instead of inequalities above, i.e.
\[ l = i + j = |\mu_1| + \ell(\mu_1) = |\mu_2| + \ell(\mu_2). \]

Moreover, in this case, by Lemma 5.6 (1) we have that \( g_{\mu_1, \mu_2; (1^{i+j})} = 0 \) unless \( \mu_1 \cup \mu_2 = 1^{i+j} \), which means that \( \mu_1 = \mu_2 = (1^{i+j}/2) \) (in particular, \( i + j \) must be even). Then, by Lemma 5.6 (2), one has that \( g(i,i),(j,j);(1^{i+j/2}) = 0 \) unless \( i = j \). Besides, using the definition of \( H_{\mu_1, \mu_2; \mu} \) and Lemma 5.6 (3), one has that
\[ H_{(1^l), (1^k); (1^{2l})} = 4 i^4. \]

Concluding,
\[ (34) \]
\[ \sum_{\mu_1, \mu_2, \mu} H^{(i,j)}_{\mu_1, \mu_2; (1^l)}(n)_l = \delta_{i,j} 4 i^4 n^{i+j} + O(n^{i+j-1}). \]

Comparing equations (33) and (34), it gives
\[ (35) \]
\[ \left( \sum_{\mu_1, \mu_2, \mu} H^{(i,j)}_{\mu_1, \mu_2; (1^l)}(n)_l - (i + j)^2 \sum_{l,k} g(i,i),(j,j);(1^k)(n)_l(n)_k \right)^{1/2} = O(n^{(i+j-1)/2}), \]
which implies that
\[ A = \sum_{2 \leq i,j \leq d+1} \frac{1}{i\sqrt{1 + j^2} n^{(i+j)/2}} O(n^{(i+j-1)/2}) = O(n^{-1/2}), \]
which finishes the proof. \qed

**Lemma 7.9.** For any \( k \geq 2 \), and \( \lambda \vdash n \) there exists \( B_{\alpha,k} \in \mathbb{R} \), which depends only on \( k \) and \( \alpha \) such that
\[ |M_k(\lambda)| \leq B_{\alpha,k} \max(\lambda_1, \lambda'_1)^k. \]

**Proof.** This is an immediate consequence of the fact that
\[ M_k(\lambda) = h_k(\mathbb{Q}_\alpha(\lambda) - \mathbb{I}_\alpha(\lambda)). \]

**Proposition 7.10.** Let \( d \in \mathbb{N} \) and let
\[ \beta := \sum_{2 \leq i,j,k \leq d+1} n^k |(W^*_i - W_i)(W^*_j - W_j)(W^*_k - W_k)|. \]
Then \( B := E^{(\alpha)}_n(\beta) = O(n^{-1/2}). \)

**Proof.** From the definition of \( M^{(\alpha)} \) it is clear that \( \lambda^* \) is obtained from \( \lambda \) by removing a box from the diagram of \( \lambda \) and reattaching it somewhere. It means that \( \lambda = \tau^{(\alpha_1)} \) and \( \lambda^* = \tau^{(\alpha_2)} \) for some \( \tau \vdash n - 1 \). It implies that
\[ |W^*_k - W_k| \leq \sqrt{k^{-1}} n^{-k/2} \left| \mathrm{Ch}_k(\tau^{(\alpha_1)}) - \mathrm{Ch}_k(\tau^{(\alpha_2)}) \right|. \]

By equation (10), the right hand side of the above inequality is equal to
\[ \sqrt{k^{-1}} n^{-k/2} \left| \sum_\rho a^{(k)}_\rho \left( \sum_{g,h \geq 0, \pi = \lambda} b^\rho_{g,h}(\gamma)^* M_\gamma(\tau) \left( z^g_{\tau_1} - z^g_{\tau_2} \right) \right) \right|, \]
where \( |\pi| \leq |\rho| - g - 2 \). By Proposition 3.7, we know that \( a^{(k)}_\rho = 0 \) for \( |\rho| > k + 1 \), hence \( a^{(k)}_\rho b^\rho_{g,h}(\gamma)^* = 0 \) for \( |\pi| > k + g - 1 \). It gives us, thanks to Lemma 7.9, that there exists some \( C_{\alpha,k} \in \mathbb{R} \) which depends only on \( k \) and \( \alpha \) such that
\[ |W^*_k - W_k| \leq n^{-k/2} C_{\alpha,k} \max(\lambda_1, \lambda'_1)^{k-1}. \]

If \( \lambda_1 \leq 2e\sqrt{\frac{\alpha}{n}} \) and \( \lambda'_1 \leq 2e\sqrt{\frac{n}{\alpha}} \), then
\[ |(W^*_i - W_i)(W^*_j - W_j)(W^*_k - W_k)| \leq |(W^*_i - W_i)||(W^*_j - W_j)||(W^*_k - W_k)| = O(n^{-3/2}) \]
and \( \beta = O(n^{-1/2}) \). If not, then
\[ |(W^*_i - W_i)(W^*_j - W_j)(W^*_k - W_k)| \leq |(W^*_i - W_i)||(W^*_j - W_j)||(W^*_k - W_k)| = O(n^{(i+j+k)/2-3}) \]
and \( \beta = O(n^{(i+j+k)/2-\frac{1}{2}}) \). By Lemma 6.3 the probability that the second case occurs is exponentially small. Hence the bound of the first case holds in expectation. It finishes the proof. □

7.5. **Proof of the central limit theorem.** Now, since we have checked, that all necessary conditions required by Theorem 7.1 are satisfied, we are ready to prove our main result from this section:

**Proof of Theorem 1.2** Let \( \tilde{\Xi}_d = (\Xi_2, \ldots, \Xi_{d+1}) \). In order to show that

\[
(\frac{\text{Ch}(\xi)}{\sqrt{K_n^{1/2^n}}} )_{k=2,3,\ldots}^d \xrightarrow{d} (\Xi_k)_{k=2,3,\ldots}
\]

as \( n \to \infty \), it is enough to show, that for all \( d \in \mathbb{N} \) and for any smooth function \( h \) on \( \mathbb{R}^d \), with all derivatives bounded, one has, as \( n \to \infty \):

\[
\left| \mathbb{E}(\alpha)^n h(\tilde{W}_d) - \mathbb{E} h(\tilde{\Xi}_d) \right| \to 0.
\]

Fix a positive integer \( d \) and a function \( h : \mathbb{R}^d \to \mathbb{R} \) as above. Let \( \Sigma = (\mathbb{E}^{(\alpha)}(\tilde{W}_d \tilde{W}_d^*)) \). As \( h \) has its first derivative bounded, one has

\[
\left| \mathbb{E} \left( h(\tilde{\Xi}_d) - h(\Sigma^{1/2} \tilde{\Xi}_d) \right) \right| \leq |h|_1 \cdot d \cdot \| \Sigma^{1/2} \| \cdot \mathbb{E}(\|\tilde{\Xi}_d\|).
\]

But \( |h|_1 \) and \( \mathbb{E}^{(\alpha)}(\|\tilde{\Xi}_d\|) \) are fixed finite numbers, while \( \| \Sigma^{1/2} \| \) is \( O(n^{-1/2}) \) by Proposition 7.6. Hence,

\[
\lim_{n \to \infty} \left| \mathbb{E}(h(\tilde{\Xi}_d)) - \mathbb{E}(h(\Sigma^{1/2} \tilde{\Xi}_d)) \right| = 0.
\]

By Corollary 7.5 and Proposition 7.6 we know, that the pair \( (\tilde{W}_d, \tilde{W}_d^*) \) satisfies all necessary conditions of Theorem 7.1. Using this theorem, we get

\[
\left| \mathbb{E}^{(\alpha)} h(\tilde{W}_d) - \mathbb{E}^{(\alpha)} h(\Sigma^{1/2} \tilde{\Xi}_d) \right| \leq |h|_2 \frac{A}{4} + |h|_3 \frac{B}{12},
\]

where

\[
A = \sum_{2 \leq i, j \leq d+1} \frac{n}{i} \sqrt{\text{Var}_{n}^{(\alpha)}} \left( (\mathbb{E}^{(\alpha)})(\tilde{W}_i - W_i^*)(W_j - W_j^*) \right)
\]

and

\[
B = \sum_{2 \leq i, j, k \leq d+1} \frac{n}{i} \mathbb{E}^{(\alpha)}((W_i^* - W_i)(W_j^* - W_j)(W_k^* - W_k)).
\]

Propositions 7.8 and 7.10 imply that

\[
\left| \mathbb{E}^{(\alpha)} h(\tilde{W}_d) - \mathbb{E} h(\Sigma^{1/2} \tilde{\Xi}_d) \right| = O(n^{-1/2}).
\]

Together with equation (38), it finishes the proof. □
7.6. **Speed in convergence.** In this Section, one shall use [RR09, Corollary 3.1], which gives an estimate for

$$\left| E_n(\alpha) h(\tilde{W}_d) - E h(\tilde{\Xi}_d) \right|$$

for non-smooth test functions $h$. In particular, we shall consider functions $h$ in the set $\mathcal{H}$ of indicator functions of convex sets. We have the following result, which is stronger than Theorem 1.3.

**Theorem 7.11.** For any integer $d \geq 2$, we have

$$\sup_{h \in \mathcal{H}} |E h(\tilde{W}_d) - E h(\tilde{\Xi}_d)| = O(n^{-1/4}).$$

**Proof.** Fix an integer $d \geq 2$ and consider the exchangeable pair $(\tilde{W}_d, \tilde{W}_d^*)$ defined as in the previous sections.

As shown in Section 7.3, this exchangeable pair fulfills the condition of [RR09, Theorem 2.1]. Besides, as mentioned in [RR09, Section 3], the set $\mathcal{H}$ of functions fulfills conditions $(C_1), (C_2)$ and $(C_3)$ (with $a = 2\sqrt{d}$) from this paper. Therefore, one can apply [RR09, Corollary 3.1] and we find that there exists a constant $\zeta = \zeta(d)$ such that

$$\sup_{h \in \mathcal{H}} |E h(\tilde{W}_d) - E h(\tilde{\Xi}_d)| \leq \zeta \left( \frac{A' \log(1/T')}{2} + \frac{B'}{2\sqrt{T'}} + 2\sqrt{dT'} \right),$$

where we define

$$\hat{\lambda}(i) = \sum_{m=1}^{d} |(\Sigma^{-1/2} \Lambda^{-1} \Sigma^{1/2})_{m,i}|,$$

$$A' = \sum_{i,j} \hat{\lambda}(i) \sqrt{\sum_{k,l} \Sigma_{i,k}^{-1/2} \Sigma_{j,l}^{-1/2} \text{Var} E \tilde{W}_d(W_k' - W_k)(W_l' - W_l)},$$

$$B' = \sum_{i,j,k} \hat{\lambda}(i) E \left( \sum_{r,s,t} \Sigma_{i,r}^{-1/2} \Sigma_{j,s}^{-1/2} \Sigma_{k,t}^{-1/2} (W_r' - W_r)(W_s' - W_s)(W_t' - W_t) \right),$$

$$T' = \frac{1}{4d} \left( \frac{A'}{2} + \sqrt{B' \sqrt{d} + \frac{(A')^2}{4}} \right)^2.$$

Comparing with the statement in [RR09], note that in our case, there is no remaining matrix $R$, and hence $C' = 0$. Besides, for the set $\mathcal{H}$ considered here (indicator functions of convex subsets of $\mathbb{R}^d$), one can choose $a = 2\sqrt{d}$.

We shall now describe the asymptotic behaviour of the quantities above. Note that all sums appearing above have fixed number of summands since they are summed on integer less or equal to $d$.

Recall that, in our setting $\Lambda$ is the diagonal matrix $(n/(i + 1) \cdot \delta_{i,j})$. Besides $\Sigma^{-1/2}$ (well-defined for $n$ big enough) is a bounded matrix (Proposition 7.6). Hence for any $i \leq d$,

$$\hat{\lambda}(i) = O(n).$$
Consider now $A'$. It was proven in Section 7.3 (Lemma 7.7 and equation (35)) that, for any $k$ and $\ell$,

$$| \text{Var} \mathbb{E}^{\tilde{W}_d}(W'_k - W_k)(W'_\ell - W_\ell) | \leq O(n^{-3}).$$

Together with the bound on $\hat{\lambda}^{(i)}$ and the fact that $\Sigma^{-1/2}$ is bounded, this implies 

$$A' = O(n^{-1/2}).$$

Consider now $B'$. We have proved that the bound (36) holds in expectation, that is

$$\mathbb{E} \left| (W'_r - W_r)(W'_s - W_s)(W'_t - W_t) \right| \leq O(n^{-3/2}).$$

As before, the bound above on $\hat{\lambda}^{(i)}$ and the fact that $\Sigma^{-1/2}$ is bounded implies 

$$B' = O(n^{-1/2}).$$

Combining the bounds for $A'$ and $B'$, we get $T' = O(n^{-1/2})$. Strictly from the definition of $T'$, we also get $T' \geq B'$, which implies that $\frac{B'}{2\sqrt{T'}} \leq \sqrt{B'} = O(n^{-1/4})$. Plugging all these estimates in equation (39), we get the desired result. \qed

### 7.7. Fluctuations of other polynomial functions.

As $\text{Ch}_{(k)}$ is an algebraic basis of polynomial functions, Theorem 1.2 imply that any polynomial function $F$ converge, after proper renormalization, towards a polynomial evaluated in independent Gaussian variables. However, the good order of renormalization and the actual polynomial are not easy to determined explicitly as this relies on the expansion of $F$ in the $\text{Ch}_{(k)}$ basis. In particular, we are not able to do it for $\text{Ch}_\mu$, when $\mu$ is not a hook and hence we can not describe the fluctuations of these random variables as done in the case $\alpha = 1$; see [Hor98] and [IO02, Theorem 6.5].

### 8. Jack Measure: Central Limit Theorems for Young Diagrams and Transition Measures

In this section, we state formally and prove our fluctuation results for Young diagrams under Jack measure. We will also present a fluctuation for the transition measures of these diagrams.

Before we state our result, we need some preparations. First, we define

$$u_k(x) = U_k(x/2) = \sum_{0 \leq j \leq [k/2]} (-1)^j \binom{k-j}{j} x^{k-2j},$$

$$t_k(x) = 2T_k(x/2) = \sum_{0 \leq j \leq [k/2]} (-1)^j \binom{k-j}{j} x^{k-2j},$$

where $T_k$ and $U_k$ are respectively Chebyshev polynomials of the first and second kind. They can be alternatively defined by the following equations:

$$u_k(2 \cos(\theta)) = \frac{\sin((k+1)\theta)}{\sin(\theta)};$$

$$t_k(2 \cos(\theta)) = 2 \cos(k\theta).$$
It is known that \( u_k(x) \) and \( t_k(x) \) form a family of orthonormal polynomials with respect to the measures \( \frac{\sqrt{4-x^2}}{2\pi} dx \) and \( \frac{1}{\sqrt{4-x^2}} dx \), respectively, i.e.: 
\[
\int_{-2}^{2} u_k(x) u_l(x) \frac{\sqrt{4-x^2}}{2\pi} dx = \delta_{k,l};
\]
\[
\int_{-2}^{2} t_k(x) t_l(x) \frac{1}{2\pi \sqrt{4-x^2}} dx = \delta_{k,l}.
\]

The measure \( \frac{\sqrt{4-x^2}}{2\pi} dx \) supported on the interval \([-2,2]\) is called the semi-circular distribution and is denoted by the \( \mu_{S-C} \) (see Subsection 1.1).

Recall that Theorem 1.1 states that the limit shape of a scaled Young diagram \( \omega \left( D_{1/\sqrt{n}}(A_{\alpha}(\lambda(n))) \right) \) is given by \( \Omega \), where \( \lambda(n) \) is a random Young diagram with \( n \) boxes distributed according to Jack measure. Hence, in order to study fluctuations of the random Young diagrams around the limit shape, we introduce the following application from the set of Young diagrams to the space of functions from \( \mathbb{R} \) to \( \mathbb{R} \):
\[
\Delta^{(\alpha)}(\lambda)(x) := \sqrt{n} \omega \left( D_{1/\sqrt{n}}(A_{\alpha}(\lambda)) \right)(x) - \Omega(x),
\]
where \( n \) is the number of boxes of \( \lambda \).

It was shown by Kerov [Ker93a] that the transition measure (see Subsection 2.2) of the continuous Young diagram \( \Omega \) is the semi-circular distribution (see Subsection 1.1). Thus we define the following function on the set of Young diagrams with \( n \) boxes with values in the space of real signed measures:
\[
\hat{\Delta}^{(\alpha)}(\lambda) := \sqrt{n} \left( \mu \left( D_{1/\sqrt{n}}(A_{\alpha}(\lambda)) \right) - \mu_{S-C} \right).
\]

As above, \( n \) is the number of boxes of \( \lambda \). This function describes the (rescaled) difference between the transition measure of the rescaled Young diagram and the limiting semi-circular measure.

Now, we are ready to formulate the central limit theorem for the Jack measure: we use the usual notation [condition] for the indicator function of the corresponding condition.

**Theorem 8.1.** Choose a sequence \( (\Xi_k)_{k=2,3,...} \) of independent standard Gaussian random variables and let \( \lambda(n) \) be a random Young diagram of size \( n \) distributed according to Jack measure. As \( n \to \infty \), we have:

1. Central limit theorem for Young diagrams:
\[
\left( \left( u_k^{(\alpha)}(\lambda(n)) \right)_{k=1,2,...} \right) \xrightarrow{d} \left( \frac{\Xi_{k+1}}{\sqrt{k+1}} - \frac{\gamma}{k+1} \left[ k \text{ is odd} \right] \right)_{k=1,2,...},
\]
where \( u_k^{(\alpha)}(\lambda) = \int_{\mathbb{R}} u_k(x) \Delta^{(\alpha)}(\lambda)(x) \, dx \);
(2) Central limit theorem for transition measures:

\[
\left( \mathcal{L}_k^{(\alpha)}(\lambda^{(\alpha)}) \right)_{k=3,4,\ldots} \xrightarrow{d} \left( \sqrt{k-1} \Xi_{k-1} - \gamma \text{ [k is odd]} \right)_{k=2,3,\ldots},
\]

where \( t_k^{(\alpha)}(\lambda) = \int_{\mathbb{R}} t_k(x) \tilde{\Delta}^{(\alpha)}(\lambda)(dx) \).

**Remark.** Notice that, for \( \gamma = 0 \) (i.e. \( \alpha = 1 \)), this theorem specializes to Kerov’s central limit theorems for Plancherel measure [Ker93a, IO02].

8.1. Extended algebra of polynomial functions and gradations. The proof heavily relies on the proof of Kerov, Ivanov and Olshanski for the case \( \alpha = 1 \). In particular we shall compare some \( \alpha \)-polynomial functions with their counterpart for \( \alpha = 1 \). Therefore, throughout this section, we will make the dependence in \( \alpha \) explicit and use the notations \( \mathrm{Ch}_{\mu}^{(\alpha)} \), \( M_k^{(\alpha)} \), and so on. The only exception to this is \( \mathrm{Ch}_{(1)} \) as for any \( \alpha \), the function \( \mathrm{Ch}_{(1)} \) associates to a Young diagram its number of boxes.

To prove Theorem 8.1, it is convenient to extend the algebra \( \Lambda_*^{(\alpha)} \) in the same way as Ivanov and Olshanski [IO02]: we adjoin to it the square root of the element \( \mathrm{Ch}_{(1)} \) and then localize over the multiplicative family generated by \( \sqrt{\mathrm{Ch}_{(1)}} \). Let \( \left( \Lambda_*^{(\alpha)} \right)^{\text{ext}} \) denote the resulting algebra.

We also define, for a partition \( \mu \) of length \( \ell \),

\[
\widetilde{\mathrm{Ch}}_{\mu}^{(\alpha)} := \prod_{i=1}^{\ell} \mathrm{Ch}_{(\mu_i)}^{(\alpha)}.
\]

Then \( \widetilde{\mathrm{Ch}}_{\mu}^{(\alpha)} \) is a multiplicative basis of \( \Lambda_*^{(\alpha)} \), while a multiplicative basis in \( \left( \Lambda_*^{(\alpha)} \right)^{\text{ext}} \) is given by

\[
\widetilde{\mathrm{Ch}}_{\mu}^{(\alpha)}(\mathrm{Ch}_{(1)})^{m/2}, \text{ with } \widetilde{\mathrm{Ch}}_{\mu}^{(\alpha)} := \prod_{i} \mathrm{Ch}_{\mu_i}^{(\alpha)}, \quad m_1(\mu) = 0, \quad m \in \mathbb{Z}.
\]

We equip \( \left( \Lambda_*^{(\alpha)} \right)^{\text{ext}} \) with a gradation defined by

\[
\deg_4 \left( \widetilde{\mathrm{Ch}}_{\mu}^{(\alpha)}(\mathrm{Ch}_{(1)})^{m/2} \right) = |\mu| + m.
\]

Note that some elements have negative degree. Besides, for a general partition \( \mu \), one has:

\[
\deg_4 \left( \widetilde{\mathrm{Ch}}_{\mu}^{(\alpha)} \right) = \mu + m_1(\mu).
\]

In [IO02], for \( \alpha = 1 \), the authors consider a slightly different filtration on \( \left( \Lambda_*^{(1)} \right)^{\text{ext}} \), namely they define \( \deg_{\{1\}} \) as follows\footnote{In [IO02], \( \deg_{\{1\}} \) is abbreviated as \( \deg_{\dagger} \) but we shall not do that to avoid a conflict of notation with Section 3.4}: for \( \mu \) without part equal to 1 and \( m \in \mathbb{Z} \),

\[
\deg_{\{1\}} \left( \mathrm{Ch}_{\mu}^{(1)}(\mathrm{Ch}_{(1)})^{m/2} \right) = |\mu| + m.
\]
Note that, for a general partition \( \mu \), one has:

\[
\deg_{\{1\}} \left( \text{Ch}_\mu^{(1)} \right) = \mu + m_1(\mu).
\]

Let us compare \( \deg_4 \) and \( \deg_{\{1\}} \). For any integer \( d \) (positive or not), denote \( V^{\leq d} \), resp. \( V^{\leq d}_{\{1\}} \), the subspaces of elements \( x \) in \( \left( \Lambda_*^{(1)} \right)^{\text{ext}} \) with \( \deg_4(x) \leq d \), resp. \( \deg_{\{1\}}(x) \leq d \).

**Lemma 8.2.** For any integer \( d \), one has

\[
V^{\leq d} = V^{\leq d}_{\{1\}}.
\]

**Proof.** Let us first show that

\[
V^{\leq d} \cap \Lambda_*^{(1)} = V^{\leq d}_{\{1\}} \cap \Lambda_*^{(1)}.
\]

By definition, the left-hand side has basis \( (\widetilde{\text{Ch}}_\mu^{(1)} |_{\mu}^{\leq m_1(\mu)} \leq d) \), while the right-hand side has basis \( (\text{Ch}_\mu^{(1)} |_{\mu}^{\leq m_1(\mu)} \leq d) \). But, if \( |\mu| + m_1(\mu) \leq d \),

\[
\deg_{\{1\}} (\widetilde{\text{Ch}}_\mu^{(1)}) \leq \sum_{i=1}^\ell \deg_{\{1\}} (\widetilde{\text{Ch}}_{(\mu_i)}^{(1)}) = |\mu| + m_1(\mu) \leq d,
\]

which shows an inclusion between the two spaces. As they have the same dimension, (41) holds.

But now, observe that, for both filtrations, an element \( F \in \left( \Lambda_*^{(1)} \right)^{\text{ext}} \) has degree at most \( d \) if and only if it can be written as

\[
F = \text{Ch}_1^{m_1} F_1 + \text{Ch}_1^{m_1+1/2} F_2
\]

for some integer \( m \) and elements \( F_1, F_2 \) from \( \Lambda_*^{(1)} \) of degree at most \( d_1 \) and \( d_2 \) with \( 2m + d_1 \leq d \) and \( 2m + 1 + d_2 \leq d \). Hence, the lemma follows from (41). \( \square \)

**Remark.** To extend (40), it would have been natural to define

\[
\deg_{\{1\}} \left( \text{Ch}_\mu^{(\alpha)} \left( \text{Ch}_1^{(1)} \right)^{m/2} \right) = |\mu| + m.
\]

Since we are unable to show that this defines indeed a filtration of \( \left( \Lambda_*^{(\alpha)} \right)^{\text{ext}} \), we used the multiplicative family \( \widetilde{\text{Ch}}_\mu^{(\alpha)} \).
8.2. Proof of Theorem 8.1. The main part of the proof of Kerov, Ivanov and Olshanski is to prove that $u_k^{(1)}$ and $t_k^{(1)}$ are in $(\Lambda_1^{(1)})^{\text{ext}}$ and fulfill

\begin{align}
&\forall k \geq 1, \\
&u_k^{(1)} = \frac{\text{Ch}_k^{(1)}}{(k+1) \text{Ch}_k^{(1)}} + \text{terms of negative degree for deg}_4; \quad (42) \\
&t_k^{(1)} = \frac{\text{Ch}_{k-1}^{(1)}}{\text{Ch}_k^{(1)}} + \text{terms of negative degree for deg}_4. \quad (43)
\end{align}

These equations are respectively the last equations of Sections 7 and 8 in paper [IO02]. As the notations are a little bit different here, let us give a few precisions.

- The quantity $\eta_{k+1}$ in [IO02] is defined by equation (6.5) and definition 3.1.
- In [IO02], it is shown that the reminder has negative degree in the filtration $\text{deg}_{(1)}$, while here we use the gradation $\text{deg}_4$. But we have proven in Lemma 8.2 that both notions coincide on $\Lambda_1^{(1)}$.
- The identities in [IO02] are equalities of random variables, that is of functions on the set of Young diagrams of size $n$ (which is here the probability space); as they are valid for any $n$, we have in fact identities of functions on the set of Young diagrams, as claimed above.

Note that, as equalities of functions on the set of Young diagrams, one can evaluate them on continuous diagrams, in particular on $A_{A_\alpha}(\lambda)$.

Our goal is to establish similar formulas in the general $\alpha$-case. From the definition, it is straight-forward that

\begin{align}
&\forall k \geq 1, \\
&u_k^{(\alpha)}(\lambda) = u_k^{(1)}(A_{A_\alpha}(\lambda)); \quad (44) \\
&t_k^{(\alpha)}(\lambda) = t_k^{(1)}(A_{A_\alpha}(\lambda)). \quad (45)
\end{align}

Therefore, applying equations (42) and (43) on $A_{A_\alpha}(\lambda)$, we get:

\begin{align}
&u_k^{(\alpha)}(\lambda) = \frac{\text{Ch}_{k+1}^{(1)}(A_{A_\alpha}(\lambda))}{(k+1) \text{Ch}_k^{(1)}} + \text{terms of negative degree}; \quad (46) \\
&t_k^{(\alpha)}(\lambda) = \frac{\text{Ch}_{k-1}^{(1)}(A_{A_\alpha}(\lambda))}{\text{Ch}_k^{(1)}} + \text{terms of negative degree}. \quad (47)
\end{align}

Of course in general, although both quantities lie in $\Lambda_{A_\alpha}^{(\alpha)}$,

$$\text{Ch}_k^{(1)}(A_{A_\alpha}(\lambda)) \neq \text{Ch}_k^{(\alpha)}(A_{A_\alpha}(\lambda)).$$

The following lemma compares the highest degree terms of quantities above:

**Lemma 8.3.** For any integer $m \geq 1$ we have that

\begin{align}
&\forall k \geq 1, \\
&\text{Ch}_k^{(1)}(A_{A_\alpha}(\lambda)) = \text{Ch}_k^{(\alpha)}(A_{A_\alpha}(\lambda)) - \gamma \left(\text{Ch}_k^{(1)}\right)^{(k/2)} [k \text{ is even}]
+ \text{terms of degree less than } k \text{ with respect to } \text{deg}_4.
\end{align}
Proof. We know, by Corollary 3.8, that for any $k \geq 2$,
\[ \deg_1 \left( \text{Ch}_{(k)}^{(1)}(A_\alpha(\lambda)) - \text{Ch}_{(k)}^{(\alpha)}(\lambda) \right) = k. \]

Let us consider its $\tilde{\text{Ch}}_\mu$ expansion:
\[ \text{Ch}_{(k)}^{(1)}(A_\alpha(\lambda)) - \text{Ch}_{(k)}^{(\alpha)}(\lambda) = \sum_{\mu} a_{\mu}^{k} \tilde{\text{Ch}}_{\mu}. \]

Since $\deg_4(\tilde{\text{Ch}}_\mu) \leq \deg_1(\tilde{\text{Ch}}_\mu)$ with equality only for $\mu = (1^m)$ for some non-negative integer $m$, one has that
\[ \text{Ch}_{(2m+1)}^{(1)}(A_\alpha(\lambda)) = \text{Ch}_{(2m+1)}^{(\alpha)}(\lambda) \]
+ terms of degree less than $2m + 1$ with respect to $\deg_4$;
\[ \text{Ch}_{(2m)}^{(1)}(A_\alpha(\lambda)) = \text{Ch}_{(2m)}^{(\alpha)}(\lambda) - a_{(1^m)}^{2m} \left( \text{Ch}_{(1)}^{(1)} \right)^m \]
+ terms of degree less than $2m$ with respect to $\deg_4$.

But
\[ a_{(1^m)}^{2m} = [R_2^{(\alpha)}(\lambda)^m] \left( \text{Ch}_{(2m)}^{(\alpha)}(\lambda) - \text{Ch}_{(2m)}^{(1)}(A_\alpha(\lambda)) \right) = [R_2^m] \text{Ch}_{(2m)}^{(\alpha)}(\lambda) = \gamma, \]
by [Las09 Theorem 10.3] (see also Proposition A.3), which finishes the proof. □

Corollary 8.4. One has the following equalities in $\left( \Lambda_\alpha^{(\alpha)} \right)^{\text{ext}}$:
\[ u_k^{(\alpha)} = \frac{\text{Ch}_{(k+1)}^{(\alpha)}}{(k+1) \left( \text{Ch}_{(1)}^{(k+1)} \right)^{1/2}} - \frac{\gamma}{k+1} [k \text{ is odd}] + \text{terms of negative degree}; \tag{49} \]
\[ t_k^{(\alpha)} = \frac{\text{Ch}_{(k-1)}^{(\alpha)}}{(\text{Ch}_{(1)}^{(k-1)})^{1/2}} - \gamma [k \text{ is odd}] + \text{terms of negative degree}. \tag{50} \]

Proof. For any Young diagram $\lambda$, equation (49) evaluated at $\lambda$ is obtained from equation (46) and Lemma 8.3. Similarly, equation (50) is a consequence of equation (47) and Lemma 8.3. □

Now, we prove in the following lemma that elements of negative degree are asymptotically negligible.

Lemma 8.5. Let $f \in \left( \Lambda_\alpha^{(\alpha)} \right)^{\text{ext}}$ be a function of degree less than 0. Then, as $n \to \infty$
\[ f(\lambda(n)) \overset{d}{\to} 0, \]
where the distribution of $\lambda(n)$ is Jack measure of size $n$. 
Proof. It is enough to show that, as \( n \to \infty \)
\[
\hat{\text{Ch}}_{\mu}(\lambda(n)) \left( \text{Ch}_1(\lambda(n)) \right)^{-m/2} \xrightarrow{d} 0
\]
for \( |\mu| < m \), where the distribution of \( \lambda(n) \) is Jack measure of size \( n \). But this is a consequence of Theorem 1.2. Indeed, let \((\Xi_k)_{k=2,3,...}\) be a family of independent standard Gaussian random variables. Then Theorem 1.2 states that
\[
\hat{\text{Ch}}_{\mu}(\lambda(n)) n^{-|\mu|/2} \xrightarrow{d} \prod_i \sqrt{\mu_i} \Xi_{\mu_i}.
\]
As \( \text{Ch}_1(\lambda(n)) \equiv n \), this implies (51) and finishes the proof. \( \square \)

Finally, Theorem 8.1 follows from Corollary 8.4, Theorem 1.2 and Lemma 8.5.

8.3. Informal reformulation of Theorem 8.1. Choose as above a sequence of independent standard Gaussian random variables \((\Xi_k)_{k=2,3,...}\) and consider the random series
\[
\Delta^{(\alpha)}(2 \cos(\theta)) := \frac{1}{\pi} \sum_{k=2}^{\infty} \left( \frac{\Xi_k}{\sqrt{k}} - \frac{\gamma}{k} [k \text{ is even}] \right) \sin(k\theta)
\]
\[
= \frac{1}{\pi} \sum_{k=2}^{\infty} \frac{\Xi_k}{\sqrt{k}} \sin(k\theta) - \frac{\gamma}{4} + \frac{\gamma\theta}{2\pi}.
\]
This series is nowhere convergent (almost surely), but it makes sense as a generalized Gaussian process with values in the space of generalized function \((C^\infty(\mathbb{R}))'\), that is the dual of the space of infinitely differentiable functions; see [IO02, Section 9] for details.

For polynomials \( u_k(x) \), the quantity
\[
\langle u_k, \Delta^{(\alpha)} \rangle = \int_{-2}^{2} u_k(x) \Delta^{(\alpha)}(x) dx = \frac{\Xi_{k+1}}{\sqrt{k+1}} - \frac{\gamma}{k+1} [k \text{ is odd}]
\]
is exactly the limit in distribution of
\[
\langle u_k, \Delta^{(\alpha)}(\lambda(n)) \rangle.
\]
By linearity this holds for any polynomial \( P \) (instead of \( u_k \)) and also for vectors of polynomials. Hence, \( \Delta^{(\alpha)} \) can be informally seen as the limit of the random functions \( \Delta^{(\alpha)}(\lambda(n)) \), which justifies equation (5).

A similar informal statement could be given for transition measures.

APPENDIX A. KEROV POLYNOMIALS

In this Section, we answer some questions of Lassalle concerning Kerov polynomials [Las09]. The results are consequences of the result or of the method of Section 3. As Kerov polynomials are used here only as a tool, we decided to present them in Appendix.
A.1. Comparison with Lassalle’s normalizations. Recall that our normalization is different than one used by Lassalle. As in Section 3.2, we use boldface font for quantities defined in Lassalle’s paper [Las09]. Our second bound on the degree of coefficients of $K_\mu$, implies the following result, using Lassalle’s conventions

**Proposition A.1.** The coefficient $c_\mu^\rho$ of $R_\rho$ in $K_\mu$ with Lassalle’s normalization is a polynomial in $\alpha$ divisible by $\alpha^{\rho - \ell(\rho)}$.

**Proof.** Let us start by a comparison of Lassalle’s conventions with ours. If $\mu$ does not contain a part equal to $1$ then 

$$\vartheta^\lambda_\mu(\alpha) = z_\mu \theta^{(\alpha)}_{\mu,1}|\lambda| - |\mu| (\lambda),$$

so that 

$$Ch_\mu(\lambda) = \alpha - \frac{|\mu| - \ell(\mu)}{2} \vartheta^\lambda_\mu(\alpha).$$

Besides, 

$$R_k(\lambda) = \alpha^{-k/2} R_k(\lambda)$$

and 

$$\vartheta^\lambda_\mu(\alpha) = K_\mu(R_2, R_3, \cdots).$$

Finally, the coefficient $c_\mu^\rho$ of $R_\rho$ in $K_\mu$ with Lassalle’s normalization is related to the coefficient $d_\rho^\mu$ of $R_\rho$ in $K_\mu$ with our conventions by:

$$c_\mu^\rho = \alpha^{\rho - \ell(\rho)} + \frac{|\rho|}{2} d_\rho^\mu.$$

But we have shown that $d_\rho^\mu$ is a polynomial in $\gamma = \frac{1 - \alpha}{\sqrt{\alpha}}$ of degree less than $|\mu| - \ell(\mu) - (|\rho| - 2\ell(\rho))$. Thus $c_\rho^\mu$ is a polynomial in $\sqrt{\alpha}$ divisible by $\alpha^{\rho - \ell(\rho)}$. The parity of $d_\rho^\mu$ implies that $c_\rho^\mu$ is in fact a polynomial in $\alpha$. □

Lassalle had only proved in his article that these quantities were rational functions in $\alpha$. He conjectured that they are polynomials with integer coefficients [Las09 Conjecture 1.1]. Our result is weaker than this conjecture as we are not able to prove the integrity of the coefficients. However, we also proved that the polynomials are divisible by $\alpha^{\rho - \ell(\rho)}$, which fits with Lassalle’s data [Las09 Section 1], but was not mentioned by him.

A.2. Linear terms in Kerov polynomials. In this short section, we compute the top degree part of the coefficients of linear terms in Kerov polynomials. This proves a conjecture of Lassalle [Las09 page 31].

**Proposition A.2.** For any integers $k > 0$ and $k - 1 \geq i \geq 0$, we have 

$$[R_{k+1-i}]K_{(k)} = \begin{bmatrix} k \\ k - i \end{bmatrix} \gamma^i + \text{lower degree terms},$$

where $\begin{bmatrix} k \\ k - i \end{bmatrix}$ denotes the positive Stirling number of the first kind.
Proof. We recall that
\[ \text{Ch}_\mu(\lambda) = \sum_{\rho} a^\mu_{\rho} M_{\mu}(\lambda). \]
It is enough to prove that
\[ a^{(k)}_{(k+1-i)} = \binom{k}{k-i} \gamma^i + \text{lower degree terms} \]
for any positive integers \( k > 0 \) and \( k - 1 \geq i \geq 0 \). We will do it by induction over \( k \). For \( k = 1 \) we have that \( K(1) = M_2 = R_2 \) and the inductive assertion holds in this case.

Putting \( \mu = (k) \) in Equation (11) we have that
\[
\sum_{\rho} a^{(k)}_{\rho} \left( \sum_{g,h \geq 0, \pi \vdash h} b^\rho_{g,\pi}(\gamma) M_{\pi \cup (g+1)} \right) = kL_{k-1},
\]
hence
\[ \sum_{\rho} a^{(k)}_{\rho} b^\rho_{k-1-i,0}(\gamma) = ka^{(k-1)}_{(k-i)} \]
for any integer \( 0 \leq i \leq k - 1 \). We have that \( b^\rho_{k-1-i,0} = 0 \) for \( |\rho| < k + 1 - i \) by Lemma 3.6. Moreover, by Proposition 3.7 and by Lemma 3.9 we have that
\[ \deg_{\gamma}(a^{(k)}_{\rho} b^\rho_{k-1-i,0}(\gamma)) \leq k + 1 - |\rho| + |\rho| - 2\ell(\rho) - (k - 3 - i) = i - 2(\ell(\rho) - 1), \]
hence by inductive assertion we have that
\[ \sum_{k+1 \geq r \geq k+1-i} a^{(k)}_{\rho} b^{(r)}_{k-1-i,0}(\gamma) = k \binom{k-1}{k-1-i} \gamma^i + \text{lower degree terms} \]
for any integer \( 0 \leq i \leq k - 1 \). By Proposition 3.1 we know that
\[ b^{(r)}_{k-1,i,0}(\gamma) = \binom{r-1}{r-(k-i)} (-\gamma)^r - (k-i+1). \]

Putting it into Equation (52) we obtain that in order to finish the proof it is enough to prove the following identity:
\[
\sum_{0 \leq j \leq i} \binom{k-i+j}{j+1} (-1)^j \binom{k}{k-i+j} = k \binom{k-1}{k-1-i}
\]
for any integer \( 0 \leq i \leq k - 1 \).

The following proof has been communicated to us by I. Goulden. It uses the fact (see, e.g., [GJ04], Ex. 3.3.17) that Stirling numbers of the first kind are defined by
\[
\sum_{j \geq 0} \binom{k}{j} x^j = (x)^{(k)}, \quad k \geq 0,
\]
using the notation for rising factorials $(a)^{(m)} = a(a+1) \cdots (a+m-1)$ for positive integer $m$, and $(a)^{(0)} = 1$. Thus we have

$$
\sum_{0 \leq j \leq i} \binom{k-i+j}{j+1} (-1)^j \binom{k}{k-i+j}
= - \sum_{-1 \leq j \leq i} \binom{k-i+j}{k-i-1} (-1)^{j+1} \begin{bmatrix} k \\ k-i+j \end{bmatrix} + \begin{bmatrix} k \\ k-i-1 \end{bmatrix}
= - \sum_{-1 \leq j \leq i} [x^{k-i-1}](x-1)^{k-i+j} \begin{bmatrix} k \\ k-i+j \end{bmatrix} + [x^{k-i-1}](x)^{(k)}
= -[x^{k-i-1}] \left( \sum_{j \geq 0} \binom{k}{j} (x-1)^{j} - \sum_{0 \leq j \leq k-i-2} \binom{k}{j} (x-1)^{j} \right) + [x^{k-i-1}](x)^{(k)}
= -[x^{k-i-1}](x-1)^{(k)} + [x^{k-i-1}](x)^{(k)}
= [x^{k-i-1}](x)^{(k-1)} \left\{ -(x-1) + (x + k - 1) \right\}
= k \begin{bmatrix} k-1 \\ k-i-1 \end{bmatrix},
$$

for all $0 \leq i \leq k - 1$, establishing the required identity. \qed

A.3. High degree terms of Kerov polynomials for $\deg_1$. In Corollary 3.8 we have given the highest degree term of $K_\mu$ for $\deg_1$. We shall now describe the next two terms for a one-part partition $\mu = (k)$.

Let $h_\pi(\mu)$ denote for the monomial symmetric function indexed by $\pi$ evaluated in variables $\mu_1, \mu_2, \ldots$. For example,

$$
h_{12}(\mu) = \sum_{i<j} \mu_i \mu_j.
$$

We also introduce the notation $\tilde{R}_i = (i-1)R_i$ and $\tilde{R}_\mu = \prod_i R_{\tilde{R}_i}^{m_i(\mu)}$.

Proposition A.3. For $k \geq 1$, one has

$$
K(k) = R_{k+1} + \gamma \frac{k}{2} \sum_{|\mu|=k} (\ell(\mu) - 1)! \tilde{R}_\mu + \sum_{|\mu|=k-1} \left( \frac{1}{4} \binom{k+1}{3} \right) + \gamma^2 k \frac{3h_2(\mu) + 4h_{12}(\mu) + 2h_1(\mu)}{24} \ell(\mu)! \tilde{R}_\mu + \text{terms of lower degree with respect to } \deg_1.
$$

Proof. Let us write:

$$
K(k) = \sum_{\mu} c_\mu R_\mu.
$$

By Proposition 3.7, $c_\mu$ is a polynomial in $\gamma$ of degree at most $k+1 - |\mu|$, hence $c_\mu$ is a polynomial in $\gamma$ of degree at most 2 for $|\mu| \geq k-1$. Moreover, we know explicitly
how to express $K_k$ in terms of free cumulants for $\alpha \in \{\frac{1}{2}, 1, 2\}$ (which corresponds to $\gamma \in \{-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\}$). The case $\alpha = 1$ has been solved separately in papers [GR07, Sm06], while the cases $\alpha = 1/2$ and 2 follows from the combinatorial interpretation given in [FS11] and the explicit computation done in [CJ11]. □

Remark. One can notice, that the explicit formulas for $c_\mu$ with $|\mu| \geq k$ were also proved by Lassalle [Las09, Theorems 10.2 and 10.3]. Moreover, our calculations for $c_\mu$ with $|\mu| = k-1$ are consistent with Lassalle’s computer experiments [Las09, p. 2257], which provide a new evidence to Conjecture 11.2 of Lassalle [Las09].

APPENDIX B. OTHER CONSEQUENCES OF THE SECOND MAIN RESULT

We present here three consequences of our polynomiality result for structure constants for Jack characters. These results were mentioned in the introduction (Section 1.5.1), but, as they are quite independent of the rest of the paper, we present them in Appendix.

B.1. Recovering a recent result of Vassilieva. Corollary 4.2 which gives a bound on the degree in $\alpha$ of $c_\mu,\nu,\pi$, can be used to give a short proof of a recent result of Vassilieva. In the paper [Vas13], she considered the following quantity: for $\mu$ a partition of $n$, denote $r = |\mu| - \ell(\mu)$ and

$$ a_r^{\mu}(\alpha) = \sum_{\lambda \vdash n} \frac{1}{j_\lambda} \theta_\mu(\lambda) \left( (2,1^{n-2})(\lambda) \right)^r. $$

Using structure constants, we can write: for any partition $\lambda$ of $n$,

$$ (2,1^{n-2})(\lambda) = \sum_{\mu_1,\mu_2,\ldots,\mu_r \vdash n} \left( \prod_{i=1}^{r-1} c_{\mu_i,(2,1^{r-2});\mu_{i+1}} \right) \theta_{\mu^r}(\lambda). $$

Plugging this into Equation (54) and using the orthogonality relation presented in Lemma 7.3 (1): 

$$ \sum_{\lambda \vdash n} \frac{1}{j_\lambda} \theta_\mu(\lambda) \theta_\nu(\lambda) = \frac{\delta_{\mu,\nu}}{z_\mu \alpha^\ell(\mu)} $$

(see [Vas13, Section 3.3]), we get

$$ a_r^{\mu}(\alpha) = \frac{1}{z_\mu \alpha^\ell(\mu)} \sum_{\mu_1,\mu_2,\ldots,\mu_r \vdash n} \prod_{i=1}^{r-1} c_{\mu_i,(2,1^{r-2});\mu_{i+1}}. $$

From Corollary 4.2, the coefficient $c_{\mu_i,(2,1^{r-2});\mu_{i+1}}$ vanishes unless

$$ |\mu^i| - \ell(\mu^i) \leq |\mu^{i+1}| - \ell(\mu^{i+1}). $$

As $|\mu^1| - \ell(\mu^1) = 1$ and $|\mu^r| - \ell(\mu^r) = r$, for any non-zero summand in (55), one has equality in (56) for all integers $i$. But, from Corollary 4.2, equality in (56) implies that the coefficient $c_{\mu_i,(2,1^{r-2});\mu_{i+1}}$ is independent of $\alpha$. Hence, the quantity $\alpha^\ell(n) z_\mu a_r^{\mu}(\alpha)$ is independent on $\alpha$. 


In the case $\alpha = 1$, it can be interpreted as some number of minimal factorizations in the symmetric group (see [Vas13, Lemma 1] or [GJ96a, Proposition 3.1]), which has been computed by Dénes in [Dén59]:

$$z_\mu a_\mu^r(1) = \left( \frac{r}{\mu_1 - 1, \ldots, \mu_{\ell(\mu)} - 1} \right)^{\ell(\mu) - 1} \prod_{i=1}^{\ell(\mu)} \mu_i^{\mu_i - 2}.$$  

Dénes in fact considered only the case $\mu = (n)$, that is minimal factorizations of a cycle, but it can be easily proved that minimal factorizations of a product of disjoint cycles are obtained by shuffling factors of minimal factorizations of its cycles.

From the case $\alpha = 1$ and the independence on $\alpha$, we conclude immediately that

$$a_\mu^r(\alpha) = \frac{1}{\alpha^{\ell(\mu)}} z_\mu \left( \frac{r}{\mu_1 - 1, \ldots, \mu_{\ell(\mu)} - 1} \right)^{\ell(\mu) - 1} \prod_{i=1}^{\ell(\mu)} \mu_i^{\mu_i - 2},$$

which is the main result in [Vas13].

B.2. Goulden’s and Jackson’s $b$-conjecture. In this Section, we explain that our quantities $c_{\mu,\nu;\pi}$ (for a general value of the parameter $\alpha$) are the same as quantities $c_{\pi,\mu,\nu}(b)$ considered by Goulden and Jackson in [GJ96a]. As a consequence, we give a partial answer to a question raised by these authors. We use the convention that the boldface quantities refer to the notations of Goulden and Jackson [GJ96a].

To establish this connection we will need to use the $\alpha$ scalar product on the symmetric functions, for which Jack polynomials and power sums are orthogonal basis [Mac95, (VI,10)].

**Proposition B.1.** Let $\mu$, $\nu$ and $\pi$ be three partitions of the same integer $n$. Then

$$c_{\mu,\nu;\pi} = \frac{n!}{z_\pi} \frac{\alpha^{\ell(\pi)}}{\alpha^{\ell(\mu)} \alpha^{\ell(\nu)}} \sum_{\lambda \vdash n} \frac{\theta_{\pi}(\lambda) \theta_{\mu}(\lambda) \theta_{\nu}(\lambda)}{\langle J_\lambda, J_\lambda \rangle}.$$  

**Proof.** Let partitions $\mu \vdash n$ and $\nu \vdash n$ be fixed. We consider the following symmetric function:

$$F := \sum_{\lambda \vdash n} \frac{\theta_{\mu}(\lambda) \theta_{\nu}(\lambda)}{\langle J_\lambda, J_\lambda \rangle} J_\lambda.$$  

By definition of $c_{\mu,\nu;\pi}$, one has:

$$F = \sum_{\lambda \vdash n} \sum_{\pi \vdash n} c_{\mu,\nu;\pi} \left( \frac{\theta_{\pi}(\lambda)}{\langle J_\lambda, J_\lambda \rangle} J_\lambda \right).$$  

But $\theta_{\pi}(\lambda)$ is defined by

$$J_\lambda = \sum_{\pi \vdash n} \theta_{\pi}(\lambda) p_\pi.$$

As $p_\pi$ is an orthogonal basis, this implies

$$\theta_{\pi}(\lambda) = \frac{\langle J_\lambda, p_\pi \rangle}{\langle p_\pi, p_\pi \rangle}. $$
But \( J_\lambda \) is also an orthogonal basis, hence:

\[
(58) \quad p_\pi = \sum_\lambda \frac{\langle J_\lambda, p_\pi \rangle}{\langle J_\lambda, J_\lambda \rangle} J_\lambda = \frac{\theta_\pi(\lambda)}{\langle J_\lambda, J_\lambda \rangle} J_\lambda.
\]

Plugging this into (57), one has:

\[
F = \sum_{\pi \vdash n} c_{\mu, \nu; \pi} \frac{p_\pi}{\langle p_\pi, p_\pi \rangle}
\]

and thus,

\[
c_{\mu, \nu; \pi} = \langle F, p_\pi \rangle = \sum_{\lambda \vdash n} \frac{\theta_\mu(\lambda) \theta_\nu(\lambda)}{\langle J_\lambda, J_\lambda \rangle} \theta_\pi(\lambda).
\]

As \( \langle p_\pi, p_\pi \rangle = n! / z_\pi \cdot \alpha^\ell(\pi) \), we obtain the claimed formula. \( \square \)

Comparing the proposition with the definition of the connection series \( c_{\mu, \nu}(b) \) \[\text{[GJ96a, equations (1) and (5)]}\], we get that

\[
(59) \quad c_{\mu, \nu; \pi} = c_{\mu, \nu}(b).
\]

I. P. Goulden and D. M. Jackson had conjectured that they were polynomials with non-negative integer coefficients in \( b = \alpha - 1 \) (which have conjecturally a combinatorial meaning in terms of matchings ; see [GJ96a, Section 4]). Corollary 4.2 implies the following weaker statement, which was not known yet.

**Proposition B.2.** The connection series \( c_{\mu, \nu}(b) \) introduced in [GJ96a] is a polynomial in \( b \) of degree at most \( d(\mu, \nu; \pi) \).

**B.3. Symmetric functions of contents.** In this section we consider a closely related problem considered by S. Matsumoto in [Mat10, Section 8] in connection with matrix integrals. Our results allow us to prove two conjectures stated in his paper.

For a box \( \Box = (i, j) \) of a Young diagram \( \lambda \) \( (i \) is the row-index, \( j \) the column index and \( j \leq \lambda_i) \), we define its \((\alpha-)content\) as \( c(\Box) = \sqrt{\alpha(j - 1)} - \sqrt{\alpha - 1}(i - 1) \). The alphabet of the contents of \( \lambda \) is the multiset \( \mathcal{C}_\lambda = \{ c(\Box) : \Box \in \lambda \} \).

Matsumoto [Mat10, Equation (8.9)] (beware that in his paper the normalization is different than ours) showed the following remarkable result: for any partition \( \lambda \)

\[
e_k(C_\lambda) = \sum_{\mu : |\mu| = k, m_1(\mu) = 0} \frac{\text{Ch}_\mu(\lambda)}{z_\mu}.
\]

In particular, \( \lambda \mapsto e_k(C_\lambda) \) is a shifted symmetric function. Therefore for any symmetric function \( F \), the map \( \lambda \mapsto F(C_\lambda) \) is also a shifted symmetric function.
and one may wonder how it can be expressed in the Ch basis. Explicitly, we are interested in the coefficients $a_\mu(F)$ defined by:

$$F(C_\lambda) = \sum_{\mu \text{ partition}} a_\mu(F) \text{Ch}_\mu(\lambda).$$

(61)

Using the results of Section 4.1, one has the following result:

**Proposition B.3.** Let $F$ be a symmetric function of degree $d$ and $\mu$ be a partition. The coefficient $a_\mu(F)$ is a polynomial in $\gamma$ of degree at most

$$d - (|\mu| - \ell(\mu) + m_1(\mu)).$$

Proof. By (60), the proposition is true for $F = e_k$ for any $k \geq 1$. Besides, if it is true for two symmetric functions $F_1$ and $F_2$, it is clearly true for any linear combination of them. By Theorem 1.4, it is also true for $F_1 \cdot F_2$. Since the elementary symmetric functions form a basis of symmetric functions, it follows that the proposition is true for any symmetric function $F$. □

From now on, we use the convention that the boldface quantities refer to the notation of Matsumoto. The coefficients $a_\mu(F)$ are closely related to the quantities $A_\mu^{(\alpha)}(F, n)$ introduced by S. Matsumoto [Mat10]. Namely, one has the following lemma (which extends [Mat10, Lemma 8.5]):

**Lemma B.4.** Let $\mu$ be a partition. For $n \geq |\mu| + \ell(\mu)$, we denote $\pi$ for the partition $\mu + (1^{n-|\mu|})$ of $n$ obtained from $\mu$ by adding 1 to every part and adding new parts equal to 1. Then, for any homogeneous symmetric function $F$ of degree $d$, one has:

$$A_\mu^{(\alpha)}(F, n) = \alpha \frac{d - (|\pi| - \ell(\pi))}{2} \sum_{i \leq m_1(\pi)} a_{\pi 1^i}(F) z_{\pi 1^i} \left( \frac{n - |\pi|}{i} \right),$$

where $A_\mu^{(\alpha)}(F, n)$ is the quantity defined in [Mat10] Section 8.3.

Proof. If we fix the integer $n$, one may rewrite Equation (61) using the definition of Ch:

$$F(C_\lambda) = \sum_{|\nu| \leq n} a_\nu(F) \alpha \frac{|\nu| - \ell(\nu)}{2} z_\nu \left( \frac{n - |\nu| + m_1(\nu)}{m_1(\nu)} \right) \theta_{\nu 1^{n-|\nu|}}(\lambda)$$

$$= \sum_{\pi \vdash n} \theta_\pi(\lambda) \left[ \alpha \frac{|\pi| - \ell(\pi)}{2} \sum_{i \leq m_1(\pi)} a_{\pi 1^i}(F) z_{\pi 1^i} \left( \frac{n - |\pi|}{i} \right) \right].$$

The notations are the same as in Section 4.2. The second equality comes from the fact that each partition $\nu$ of size at most $n$ writes uniquely as $\pi 1^i$ where $\pi$ is a partition of $n$ and $i$ a non-negative integer smaller or equal to $m_1(\pi)$. We denote $A_\pi$ for the expression in the bracket in the equation above.
As in the proof of Proposition B.1 we shall use the Jack deformation of Hall scalar product on symmetric function.

\[ \sum_{\lambda \vdash n} F(C_\lambda) \frac{J_\lambda}{\langle J_\lambda, J_\lambda \rangle} = \sum_{\lambda, \pi \vdash n} A_\pi \theta_\pi(\lambda) \frac{J_\lambda}{\langle J_\lambda, J_\lambda \rangle} = \sum_{\pi \vdash n} A_\pi \frac{p_\pi}{\langle p_\pi, p_\pi \rangle}. \]

The last equality corresponds to (58). We deduce that

\[ A_\pi = \left\langle \sum_{\lambda \vdash n} F(C_\lambda) \frac{J_\lambda}{\langle J_\lambda, J_\lambda \rangle}, p_\pi \right\rangle = \sum_{\lambda \vdash n} F(C_\lambda) \frac{\theta_\pi(\lambda) \cdot \langle p_\pi, p_\pi \rangle}{\langle J_\lambda, J_\lambda \rangle}. \]

This formula is close to the definition of \( \mathcal{A}_{\mu}^{(\alpha)}(F, n) \) in [Mat10, paragraph 8.3]. More precisely,

\[ \mathcal{A}_{\mu}^{(\alpha)}(F, n) = \alpha^{d/2} A_\pi. \]

The only difficulty is the difference of notations. To help the reader, we provide the following dictionary. First recall that \( \langle p_\pi, p_\pi \rangle = z_\pi \alpha^{\ell(\pi)} \). Then our partition \( \pi \) corresponds to \( \mu + (1^{n-l(\mu)}) \). In particular, one has \( |\mu| = |\pi| - \ell(\pi) \) and \( z_{\mu + (1^{n-l(\mu)})} = z_\pi \). Besides, \( F(C_\lambda) \) in our paper corresponds to \( \alpha^{d/2} F(A_{\alpha, \lambda}) \) in [Mat10]. Finally, the probability \( P_n^{(\alpha)}(\lambda) \) is simply given by \( \frac{n! \alpha^n}{\langle J_\lambda, J_\lambda \rangle} \).

Proposition B.3 when translated into Matsumoto’s notation by Lemma B.4 has several interesting consequences. Recall that \( d \) is the degree of the homogeneous symmetric function \( F \).

- If \( d = |\mu| \), the only term of the sum which can be non-zero corresponds to \( i = 0 \) (by Proposition B.3). Moreover, it does not depend on \( \alpha \). Besides, the exponent of \( \alpha \) in the formula is equal to zero. Finally, \( \mathcal{A}_{\mu}^{(\alpha)}(F, n) \) does not depend neither on \( \alpha \) nor on \( n \), which proves [Mat10, Conjecture 9.2].
- If \( d = |\mu| + 1 \), there are only two terms (corresponding to \( i = 0, 1 \)) which can be non-zero in the sum. Besides, the coefficient \( a_{\pi,1} \) does not depend on \( \alpha \) because of Proposition B.3. But it is easy to prove that it is equal to \( 0 \) in the case \( \alpha = 1 \) (it comes from the combinatorial interpretation of \( \mathcal{A}_{\mu}^{(1)}(F, n) \), see [Mat10 Example 9.2]). Hence, \( a_{\pi,1} = 0 \) and only the term corresponding to \( i = 0 \) is non-zero. In particular, one can see that \( \mathcal{A}_{\mu}^{(\alpha)}(F, n) \) does not depend on \( n \), which proves [Mat10 Conjecture 9.3].
- In the general case, non-zero terms of the sum are indexed by values of \( i \) smaller or equal to \( d - |\mu| \) (by Proposition B.3). Hence \( \mathcal{A}_{\mu}^{(\alpha)}(F, n) \) is a polynomial in \( n \) of degree at most \( d - |\mu| \). This result is not stronger than the bound of S. Matsumoto on the degree of \( \mathcal{A}_{\mu}^{(\alpha)}(F, n) \) [Mat10, Theorem 8.8]. Nevertheless, it is better in some cases (as illustrated by the proofs of the conjectures above). Besides, we also give information on the dependence on \( \alpha \).
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