A Toy Model for the Time–Frequency Structure of Fast Radio Bursts: Implications for the CHIME/FRB Burst Dichotomy

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Abstract

We introduce a toy model for the time–frequency structure of fast radio bursts, in which the observed emission is produced as a narrowly peaked intrinsic spectral energy distribution sweeps down in frequency across the instrumental bandpass as a power law in time. Though originally motivated by emission models that invoke a relativistic shock, the model could in principle apply to a wider range of emission scenarios. We quantify the burst’s detectability using the frequency bandwidth over which most of its signal-to-noise ratio is accumulated. We demonstrate that, by varying just a single parameter of the toy model—the power-law index $\beta$ of the frequency drift rate—one can transform a long (and hence preferentially time-resolved) burst with a narrow time-integrated spectrum into a shorter burst with a broad power-law time-integrated spectrum. We suggest that source-to-source diversity in the value of $\beta$ could generate the dichotomy between burst duration and frequency-bandwidth recently found by CHIME/FRB. In shock models, the value of $\beta$ is related to the radial density profile of the external medium, which, in light of the preferentially longer duration of bursts from repeating sources, may point to diversity in the external environments surrounding repeating versus one-off FRB sources. 

Unified Astronomy Thesaurus concepts: Radio transient sources (2008)

1. Introduction

CHIME/FRB reported in their first catalog paper (CHIME/FRB Collaboration et al. 2021; Pleunis et al. 2021) an apparent dichotomy in the observed properties of fast radio bursts (FRBs), which supports earlier suggestions based on smaller sample sizes (e.g., Scholz et al. 2016; CHIME/FRB Collaboration et al. 2019; Hashimoto et al. 2020). In particular, they find that FRBs with longer durations are observed to (1) possess narrower time-integrated spectra that often peak in the middle of the instrumental bandpass, (2) preferentially be associated with FRB sources thus far observed to repeat. By contrast, shorter bursts possess power-law-like spectra covering most or all of the instrumental bandpass and are more frequently associated with nonrepeating one-off bursts (i.e., sources not yet observed to repeat). Furthermore, while temporally resolved bursts from repeaters exhibit narrow $\lesssim 100$ MHz spectral structures (“subbursts”) that often drift downwards in frequency with time (e.g., Gajjar et al. 2018; Michilli et al. 2018; CHIME/FRB Collaboration et al. 2019; Hessels et al. 2019; Josephy et al. 2019), this behavior is seen less frequently in the nonrepeating sources.

On the other hand, the distribution of dispersion measures (DMs) of the repeating and nonrepeating populations are broadly similar (CHIME/FRB Collaboration et al. 2021), consistent with a similar range of source distances (though DM is an imperfect proxy for distance, given that it receives contributions from the Milky Way, intergalactic medium, host galaxy, and potentially also from the immediate environment of the FRB source). Instrumental effects can in some cases distort the observed burst properties (e.g., broadband bursts that occur in the instrumental side lobes may exhibit artificially narrow spectra), but the dichotomy observed by CHIME/FRB was argued to be robust to these contaminants (Pleunis et al. 2021). If intrinsic, this difference in repeating and nonrepeating classes could point to fundamental diversity in the FRB central engines and/or emission mechanisms.

A large number of FRB models have been proposed in the literature (e.g., Cordes & Chatterjee 2019; Platt et al. 2019; Petroff et al. 2019, 2021 for reviews), but at a basic level many of these models attribute the radio emission to a sudden release of energy from a stellar mass compact object, such as a pulsar, magnetar, or accreting black hole (e.g., Falcke & Rezzolla 2014; Lyubarsky 2014; Fuller & Ott 2015; Beloborodov 2017; Ioka & Zhang 2020; Sridhar et al. 2021). The emission mechanism could be synchrotron maser emission at a relativistic magnetized shock (Lyubarsky 2014; Beloborodov 2017; Metzger et al. 2019; Plotnikov & Sironi 2019), fast magnetosonic waves generated by magnetic reconnection (Philippov et al. 2019; Lyubarsky 2019, 2020), or curvature radiation resulting from charge starvation in a neutron star magnetosphere (Kumar et al. 2017; Lu et al. 2020), among other possibilities (e.g., Wadiasingh & Timokhin 2019; Kumar & Bošnjak 2020; Lyutikov 2021). However, one feature potentially common to a wide range of models is a power-law evolution of the burst properties relative to some zero-point
(corresponding to the onset of the energy release), which creates the observed emission as the peak of the SED of the sources sweeps down across the instrumental bandpass. For example, in the synchrotron maser shock model (e.g., Lyubarsky 2014; Beloborodov 2017; Margalit et al. 2020; Sironi et al. 2021) the FRB emission drops in luminosity and frequency as the shock decelerates and probes lower densities at larger radii from the central engine (Metzger et al. 2019; Margalit et al. 2019). Magnetospheric models for FRB emission could also predict power-law decay in the burst characteristics—for instance, as a result of radius-to-frequency mapping as the emission region moves outward in the magnetosphere (e.g., Lyutikov 2020).

In this paper, we introduce a simple, physically motivated “toy” model for the time–frequency structure of FRB emission, which we demonstrate is consistent with many of their observed features. We then explore to what extent this burst structure could unify the dichotomy in FRB properties observed by CHIME/FRB (Pleunis et al. 2021). As we will show, varying just one parameter of the model—the power-law index of the frequency drift rate—can transform a long and hence preferentially time-resolved burst into a (preferentially unresolved) short burst. Insofar that the same drift-rate parameter determines the frequency bandpass over which the burst’s signal-to-noise (S/N) is accumulated, this will act to preferentially narrow the observed time-integrated spectra of longer bursts in an S/N-limited sample, consistent with the CHIME/FRB dichotomy (Pleunis et al. 2021). If confirmed by future work, this interpretation for the duration/spectral-bandwidth dichotomy would further refine the question of what distinguishes repeating from nonrepeating FRB sources into one of how the toy model parameters map onto the diversity of the FRB central engines or their activity levels.

This paper is organized as follows. In Section 2, we introduce the toy model. In Section 3, we connect the model parameters to various FRB observables. Section 4 discusses implications of the model for burst detectability and for interpreting the CHIME/FRB dichotomy. Section 5 summarizes our conclusions.

2. Toy Model of FRB Emission

We introduce a parameterized model for the time–frequency structure of FRB emission (see Figure 1 for a schematic illustration and Table 1 for an enumeration of the model parameters). Though originally motivated by shock scenarios, in which the free parameters are set by the properties of the ejecta from the central engine and of the external upstream medium, this model can in principle accommodate a wider range of emission mechanisms.

2.1. Single-component Burst Model

The flux density of a simple, single-component FRB evolves in frequency \( \nu \) and time \( t \) according to

\[
F(\nu, t) = F(t) \exp \left[ -\frac{(\nu - \nu(t))^2}{\Delta \nu(t)^2} \right],
\]

(1)

where \( \nu(t) \) and \( \Delta \nu(t) \) are the central frequency and intrinsic bandwidth, respectively, and the normalization \( F(t) \) is a function defined below. Our particular choice of a Gaussian shape for the spectral “envelope” is somewhat arbitrary; however, the qualitative conclusions to follow will hold for more general spectral shapes insofar as they are narrowly peaked (i.e., \( \Delta \nu \ll \nu \)), consistent with time-resolved FRB observations (e.g., Kumar et al. 2021).

Motivated by the observed downward drifting of frequency structure in FRB bursts (e.g., Hessels et al. 2019), the central frequency is assumed to decay as a power law in time,

\[
\nu(c) = \nu_0 \left( \frac{t - \tau}{\tau} \right)^{-\beta},
\]

(2)

where \( \tau \) is a singular time (usually corresponding to the point of energy release from the central engine) and \( \nu_0 \) the central frequency at some fiducial time \( \tau \) after \( \tau \). Equations (1) and (2) cannot be valid at all times, since in general this would lead to a divergence in the total burst energy. To alleviate this, we assume the burst emission starts after a time \( t = \tau + \tau \) at which point \( \nu_c(t) = \nu_0 \). For example, in shock scenarios, the start time \( \tau \) could correspond to when the relativistic ejecta from the central engine has swept up a sufficient mass from the external medium to appreciably decelerate (Appendix). Beniamini & Kumar (2020) explore a similar setup, from which they explore how the measured burst duration at a given frequency is related to the total energy emitted in that band.

Rewriting Equation (2) in terms of a new time variable, now measured starting at \( \tau + \tau \), we find

\[
\nu_c(t) = \nu_0 \left( 1 + \frac{t}{\tau} \right)^{-\beta}.
\]

(3)

As discussed further below, there are two cases to consider (left and central panels of Figure 2). When the initial frequency \( \nu_0 \) is above the instrument bandpass, then a burst will contribute flux across the entire bandpass as \( \nu_c \) sweeps downward through it. By contrast, when \( \nu_0 \) is within the instrument bandpass, the observed emission starts at \( \nu_0 \) and is restricted to frequencies \( \nu < \nu_0 \). Although bursts with extremely sharp mid-band onsets (such as the example shown in the middle panel of Figure 2) are not common, the observer onset will be smoother if some emission occurs also at times \( t \approx \tau \) prior to that captured by the toy model and once instrumental smoothing effects are accounted for (see Figure 7).
The intrinsic bandwidth of the emission envelope can in principle also evolve independently of the central frequency:

\[
\frac{\Delta \nu_c}{\nu_c} = \chi \left(1 + \frac{t}{t_0}\right)^{-\mu}.
\]

In most of the examples presented throughout this paper, we will take \( \mu = 0 \), and hence \( \Delta \nu_c/\nu_c = \chi = \text{constant} \), where \( \chi \ll 1 \) to reproduce the narrowband spectra of time-resolved bursts (e.g., Farah et al. 2018; Michilli et al. 2018; Hessels et al. 2019; Day et al. 2020; Kumar et al. 2021).

The frequency-integrated total flux is also assumed to exhibit a power-law temporal decay,

\[
\int_0^\infty F_\nu d\nu \approx F_0 \left(1 + \frac{t}{t_0}\right)^{-\alpha},
\]

with a different index \( \alpha \), where \( F_0 \) is a constant and (using Equation (1))

\[
\int_0^\infty F_\nu d\nu \approx \frac{\Delta \nu_c(t)}{2} \cdot F(t).
\]

From Equations (4)–(6), we deduce that

\[
F(t) \approx \frac{2}{\sqrt{\pi}} \frac{F_0}{\chi \nu_0} \left(1 + \frac{t}{t_0}\right)^{(\beta - \mu - \alpha)/3}.
\]

Observed FRB spectra, as revealed by high-time-resolution studies, are often extremely complex (e.g., Farah et al. 2018; Michilli et al. 2018; Cho et al. 2020; Day et al. 2020; Nimmo et al. 2021) and do not conform in detail to the model description above, which is smooth in time and frequency. However, it is possible that the observed fine-scale time–frequency structures are imprinted by propagation effects, such as plasma lensing (e.g., Cordes et al. 2017; Simard & Ravi 2020; Beniamini et al. 2022; Sobacchi et al. 2021), rather than being an intrinsic property of the emission. In particular, the dissolution of the FRB light curve into a train of “subbursts” (often roughly equally spaced in frequency)—suggestive of interference fringes—may well result from such effects (see also right panel of Figure 2). Nonetheless, there are at least several bursts in which heretofore considered propagation effects were argued unlikely to be the result of the observed strong spectro-temporal-spectral variability (Beniamini & Kumar 2020; Lu et al. 2022).

We do not attempt to model propagation effects in this paper. Rather, our goal is to describe the overall “envelope” of the burst’s time structure, under the (admittedly strong) assumption that, in spite of any propagation-induced modulation, key features of the intrinsic emission are preserved and can be described within the framework of the toy model. This assumption would be violated if, for instance, key features of the burst emission, such as the downward frequency evolution, are also the result of propagation effects (e.g., additional dispersion-induced delay close to the source; Tuntsov et al. 2021).

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**Table 1**

Summary of Burst Observables (for \( \mu = 0 \) and \( \nu_0 \geq \nu_{\text{obs}} \))

| Burst Property | Controlling Parameters | Equation |
|----------------|------------------------|---------|
| Onset time     | \( t \)                | (2)     |
| Burst duration, \( t_{\text{FRB}} \) | \( t_0, \beta, \nu_0 \) | (11)    |
| Frequency drift rate, \( \dot{\nu}_c \) | \( t_0, \beta, \nu_0 \) | (10)    |
| Total fluence, \( E \) | \( F_0, \alpha, t_0, \beta, \nu_0 \) | (16)    |
| Power-law index of time-integrated SED, \( \gamma \) | \( \alpha, \beta \) | (15)    |
| Time-resolved instantaneous flux bandwidth, \( \Delta \nu_c \) | \( \chi, \nu_0 \) | (4)     |
| Time-integrated maximum S/N bandwidth, \( \Delta \nu_{\text{obs}} \) | \( F_0, \alpha, t_0, \beta, \nu_0 \) | (21), (22) |

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**Figure 2.** Synthetic time–frequency (“waterfall”) plots of \( F_\nu(t, \nu) \) for an FRB characterized by \( \alpha = 1, \beta = 0.5, \chi = 0.1, t_0 = 1 \text{ ms} \). The left panel shows an example in which the burst starts above the observing band \( (\nu_0 > \nu_{\text{obs}}) \), while the middle panel shows an example where the burst starts in-band \( (\nu_{\text{min}} < \nu_0 < \nu_{\text{obs}}) \). The right panel shows a burst model otherwise identical to that in the left panel, but for which we have applied a periodic Gaussian filter to the signal, mimicking possible propagation effects, which acts to break the signal into distinct subbursts.
2.1.1. Model Parameters in Physical Emission Models

Table 2 and the Appendix summarize the toy model parameters \( \{ \alpha, \beta \} \) predicted for synchrotron maser emission from a relativistic shock propagating into a stationary or subrelativistically expanding external medium (Metzger et al. 2019). For the typically considered case in which emission from the shock is observed on a timescale longer than that of the energy release from the central engine \( t \gtrsim t_\beta \), the scenario predicts \( \alpha \approx 1 \) and \( \beta \approx 0.2 - 0.7 \), depending on the power-law index of the radial density profile of the upstream medium, \( n(r) \propto r^{-k} (k \approx 0 - 2) \).

However, a wider range of \( \{ \alpha, \beta \} \) values are permitted under other assumptions within the shock model, such as a relativistically expanding upstream medium (Beloborodov 2020; Sridhar et al. 2021b). For example, for internal shocks in the accelerating relativistic wind of a binary neutron star merger (Sridhar et al. 2021b):

\[
\alpha = \frac{4m - 21}{2(m - 5)} \in [1, 1.75]; \quad \beta = \frac{6m - 27}{4(m - 5)} \in [1.9, 3],
\]

where \( M \propto a^{-m} \) is the (theoretically uncertain) time-dependent mass-loss rate of the wind as a function of the shrinking binary semimajor axis \( a \) and \( m \in [5.5, 7] \).

Magnetospheric models, in which the frequency drift is set by the radius-to-frequency mapping, typically predict \( \beta \approx 1 \) (e.g., Lyutikov 2020), as long as the magnetosphere is slowly rotating, and for a wide range of dependencies of \( v_\nu \) on the dipole magnetic field. Faster (relativistically) rotating magnetospheres, result in a drift rate that can deviate significantly from a power-law dependence and can also vary significantly between different bursts from the same engine.

2.2. Multicomponent Burst Model

Although our main focus in this paper is on the simple burst model presented in Section 2.1, possible generalizations to more complex light curves should be kept in mind. In particular, FRB light curves composed of multiple separate light curve peaks can be accommodated within the same framework as above by summing multiple Gaussian components with distinct values of \( F_\nu(t) \) and \( v_{\nu,i} \) (each obeying Equations (3) and (7)) for the same values of \( \alpha \) and \( \beta \), but different initial frequencies \( v_\nu = v_{\nu,i} \), viz.

\[
F_\nu(t) = \sum_i F_i(t) \exp \left[ -\frac{(\nu - v_{\nu,i}(t))^2}{\Delta v_{\nu,i}(t)^2} \right].
\]

As each individual frequency component \( v_{\nu,i} \) sweeps through the observational bandpass, it will generate its own peak in the light curve. Such a burst structure could be motivated, for instance, in shock models if harmonics of the synchrotron maser emission generate distinct peaks in the burst’s SED (e.g., Babul & Sironi 2020), which appear as distinct burst components as each harmonic peak crosses the instrumental bandpass. In such cases, the separation between two sub-components increases as the burst width increases (see Figure 3 for an example).

Table 2

| Case | \( t \lesssim t_\beta \) | \( t \gtrsim t_\beta \) | \( t \lesssim t_\beta \) | \( t \gtrsim t_\beta \) |
|------|-----------------|-----------------|-----------------|-----------------|
| Bare shock | \( t \lesssim t_\beta \) | \( 2 \frac{6a_0}{c} \frac{3}{k=0} = 0.25 \) | \( 2 \frac{6a_0}{c} \frac{3}{k=0} = 0.38 \) | \( 6 \frac{3k}{k=0} = 0.19 \) | \( - \frac{28}{2.7k} - 14 \) |
| ICS absorbed | \( t \lesssim t_\beta \) | \( 2 \frac{6a_0}{c} \frac{3}{k=0} = 0.06 \) | \( 6 \frac{3k}{k=0} = 1.16 \) | \( \cdots = -0.29 \) |
| \( \nu_{\nu, \Delta \nu_{\nu} \propto \nu_{\nu, m}} \) | \( t \lesssim t_\beta \) | \( 2 \frac{6a_0}{c} \frac{3}{k=0} = 0.22 \) | \( 6 \frac{3k}{k=0} = -5 \) | \( - \frac{28}{2.7k} - 14 \) |

where \( \beta = \alpha^{-1} - \alpha^{-1} \) and \( \gamma = \alpha^{-1} - 1 \).

3. Burst Observables

We now describe how the hypothesized time–frequency structure (Equation (1)) of the burst maps onto observed FRB properties. In what follows, we consider the emission as detected by an idealized telescope with sensitivity in the frequency range \( \nu \in [\nu_{\min}, \nu_{\max}], \) with a central frequency \( \nu_{\text{obs}} = (\nu_{\min} + \nu_{\max})/2 \) and total bandwidth \( \Delta \nu_{\text{obs}} = \nu_{\max} - \nu_{\min} \lesssim \nu_{\text{obs}}. \) When necessary to specify, the instrumental time resolution is taken to be \( \Delta t_{\text{res}} \approx 1 \) ms for CHIME/FRB.

Figure 4 shows contour plots depicting the result of a direct numerical evaluation of the various burst properties described below, as a function of the model power-law indices \( \alpha \) and \( \beta. \) Many of the results in Figure 4 have been normalized to analytic estimates derived below. In this example, we assume the emission begins above the top of the instrumental bandpass \( \nu_0 = 2\nu_{\max}; \) similar to the example shown in the left panel of Figure 2, though qualitatively similar results would follow if \( \nu_0 \) were to instead occur within the instrumental bandpass (as expected in particular for longer bursts that concentrate their \( S/N \) in a narrow frequency range about \( \nu_0 \) – see Section 4).

3.1. Frequency Drift Rate

Evolution of the burst frequency structure occurs as \( \nu_\nu \) crosses down through the observing band. The drift rate as measured around frequency \( \nu_\nu \) is given by

\[
\nu_\nu = \frac{\beta \nu_\nu}{\nu_0} \left( \frac{\nu_0}{\nu_\nu} \right)^{1/\beta} \approx \frac{\beta \nu_\nu}{\nu_0} \left( \frac{\nu_{\text{obs}}}{\nu_{\nu}} \right)^{(\beta+1)/\beta},
\]

where in the second line we have taken \( \nu_\nu \approx \nu_{\text{obs}} \) at the center of the bandpass.

For values of \( \beta > 0, \) the drift rate is negative (“sad trombone”) and its magnitude increases with radio frequency \( \nu_{\text{obs}} \) consistent with observations of the well-studied repeating sources FRB 121102 (Hessels et al. 2019; Josephy et al. 2019; Caleb et al. 2020; CHIME/FRB Collaboration et al. 2020a) and FRB 180916 (Sand et al. 2021). Fitting the drift rates of distinct bursts from FRB 121102 measured across a range of radio frequencies from 0.6 GHz to 6.5 GHz, Caleb et al. (2020) find a linear dependence \( \nu \propto \nu_{\text{obs}} \) (see also Josephy et al. 2019), which from Equation (10) would require \( \beta \gg 1. \) However, large scatter is observed in the drift rate at a fixed observing
frequency (e.g., 1.4 GHz) for different bursts from a single repeating source. Given that drift rates have not, to our knowledge, been measured simultaneously for the same burst at different frequencies, the value of $\beta$ cannot yet be precisely constrained.

As shown in Figure 4(e), for $\alpha \sim 1$, Equation (10) does a good job of reproducing the measured drift rate, as measured at the epoch of half accumulated fluence. However, for larger (smaller) values of $\alpha$, the burst fluence peaks at the top (bottom) of the instrumental bandpass (Section 3.3) and hence the drift rate $\dot{\nu}_c \propto \nu^{\alpha/2}$ defined this way becomes very sensitive to $\beta$ for $\beta \ll 1$.

3.2. Burst Duration

One commonly used definition of the duration of a burst is the timescale over which the majority of its fluence is received. This is roughly given by the time required for the (e.g., Gaussian) emission envelope to sweep down across the instrument bandpass $\Delta \nu_{\text{obs}}$.

$$t_{\text{FRB}} \sim \frac{\Delta \nu_{\text{obs}}}{|\dot{\nu}_c|} \approx \frac{t_0}{\beta} \left( \frac{\Delta \nu_{\text{obs}}}{\nu_{\text{obs}}} \right) \left( \frac{\nu_{\text{obs}}}{\nu_0} \right)^{-1/\beta}, \text{ single burst},$$

(11)

where we have used Equation (10). As we shall discuss in Section 4, the burst duration is shorter than Equation (11) (typically by a factor $\sim \beta$ when $\beta \ll 1$) if it is defined as the time to accumulate most of the $S/N$ instead of the fluence.

On the other hand, for more complex bursts generated by multiple downward-drifting components in the SED (Equation (9)) spaced over a total frequency range $\sim N \nu_c$, where $N \gtrsim 1^\circ$, the total burst duration (as defined from the first to last peak of comparable fluence) can be considerably longer:

$$t_{\text{FRB}} \sim \frac{\Delta \nu_{\text{obs}}}{|\dot{\nu}_c|} \approx \frac{t_0}{\beta} \left( \frac{\Delta \nu_{\text{obs}}}{\nu_{\text{obs}}} \right) \left( \frac{\nu_{\text{obs}}}{\nu_0} \right)^{-1/\beta}, \text{ multipeaked burst}. \quad (12)$$

Note that, in a multicomponent burst, the total burst duration scales with the model parameters in the same way as the temporal separation between the individual light curve peaks (Figure 3). This behavior is consistent with observations by CHIME/FRB (Pleunis et al. 2021), provided the subbursts arise from distinct spectral components drifting down across the instrumental bandpass (instead of a single-component burst “broken apart” by propagation effects).

As shown in Figure 4(a), Equation (11) does a reasonable job of reproducing the duration over which 90% of the burst fluence is accumulated. To order of magnitude, we have $t_{\text{FRB}} \approx t_0/\beta$ for bursts that begin within or just above the instrumental bandpass (i.e., $\nu_0 \sim \nu_{\text{obs}}$). However, for bursts that instead start far above the observing frequency (i.e., $\nu_0 \gg \nu_{\text{obs}}$), the burst duration $t_{\text{FRB}}$ can be $\gg t_0$ for $\beta \ll 1$. Also note that the duration of a given burst $\propto \nu_{\text{obs}}^{-1/\beta}$ is significantly shorter if the burst is observed at higher radio frequencies (even once accounting for the effects of scattering during propagation to Earth), consistent with observations (Michilli et al. 2018; Li et al. 2021).

The burst duration and frequency drift rate can be combined into a dimensionless quantity:

$$\sigma \equiv \frac{\dot{\nu}_c t_{\text{FRB}}}{\nu_{\text{obs}}}. \quad (13)$$

Based on a sample of repeating FRBs with measured frequency drifts, Margalit et al. (2019) found values $\sigma \approx 0.1$–0.4 (what they define as “$\beta$”). Likewise, Chamma et al. (2020) find $\sigma \sim 0.1$ for three separate repeater sources (see their Figure 1). As shown in Figure 4(f), this would appear to favor a parameter...
space around $\alpha \approx 0$ and $\beta \lesssim 0.2$ if the total burst duration is that of a single burst. However, in cases when $t_{\text{FRB}}$ represents the total duration of a train of closely spaced burst components (Equation (12)), then the predicted value of $\sigma$ is correspondingly larger than shown in Figure 4 by a factor $N \gtrsim \text{few}$ and the permitted parameter space of $\{\alpha, \beta\}$ expands considerably. Furthermore, the true burst duration entering Equation (13) will be shorter than estimated in Figure 4 (and hence $\sigma$ overestimated) if it is defined as the time over which the majority of the burst’s S/N is accumulated instead of its fluence (Figure 5).

3.3. Time-integrated Spectral Energy Distribution

If the instrumental time resolution is sufficiently poor, $\Delta t_{\text{res}} \gg t_{\text{FRB}}$, it is not possible to measure the spectral peak
sweeping down across the observing bandpass, because the latter occurs on a time \( t_{\text{B}} \sim \Delta \nu_{\text{obs}} / \nu_c \) shorter than \( \Delta t_{\text{res}} \). In such cases, only the time-integrated flux, or spectral energy distribution (SED), can be measured. For a single burst, this can be estimated as

\[
\frac{dE}{d\nu} \approx \int_{0}^{\infty} F_{\nu} \frac{\Delta \nu_c}{\nu_c} dt \approx F(\nu_c) \frac{\Delta \nu_c}{\nu_c} \approx \frac{2}{\sqrt{\pi}} \frac{F_{\nu_0}}{\nu_0} \left( \frac{\nu}{\nu_0} \right)^{\alpha - 1 - \frac{1}{\beta}},
\]

where we have used Equation (7). For temporally unresolved bursts, the instrument thus measures a power-law spectrum:

\[
\frac{dE}{d\nu} \propto \nu^{\gamma}; \quad \gamma \equiv \frac{(\alpha - \beta - 1)}{\beta}.
\]

This power-law SED is confirmed by direct integration of Equation (1) in Figure 4(c), which shows that \( \gamma \approx -1 \) obtains across the parameter space straddling \( \alpha \approx 1 \). This predicts \( \nu (dE_{\nu}/d\nu) \sim \text{const.} \), i.e., approximately equal fluence per logarithmic frequency interval, which improves the likelihood of multiband FRB detections (e.g., Majid et al. 2020; Pearlman et al. 2020). Based on a sample of 23 ASKAP bursts, Macquart et al. (2019) found a mean power-law index \( \gamma = -1.5^{+0.2}_{-0.3} \) from 23 ASKAP FRBs (shown as a blue dotted line in Figure 4), which according to Equation (15) would imply \( \alpha \approx 1 - 0.5\beta \leq 1 \).

On the other hand, for \( \alpha > 1 (\alpha < 1) \) the value of \( \gamma \) is much greater than (much less than) \(-1\), in particular for low values of \( \beta \leq 1 \). This same parameter range acts to concentrate the burst fluence over a narrow frequency range \( \Delta \nu_c < \Delta \nu_{\text{obs}} \) near the top or bottom of the instrumental bandpass (Figure 4(d)); see below), respectively, or around \( \nu_0 \) when the burst starts within the instrumental bandpass \( \nu_0 < \nu_{\text{max}} \).

For a complex burst composed of multiple downward-drifting frequency components (Equation (9)), the time-integrated SED is just the sum of the contributions from each component. If each burst component shares the same parameter values (i.e., \( \alpha, \beta \)), then the time-integrated spectrum will remain a power law with the same index \( \gamma \) (Equation (15)).

### 3.4. Burst Fluence

The total fluence of the burst is obtained by integrating Equation (14) across the relevant frequency range

\[
E = \int_{\nu_{\text{min}}}^{\nu_{\text{max}}} \frac{dE}{d\nu} d\nu \approx \frac{2}{\sqrt{\pi}} \frac{\nu_0}{\nu_{\text{max}}} \left( \frac{\nu_{\text{max}}^{(\alpha - 1)/\beta}}{\nu_0^{(\alpha - 1)/\beta}} - \frac{\nu_{\text{min}}^{(\alpha - 1)/\beta}}{\nu_0^{(\alpha - 1)/\beta}} \right)
\]

where the maximum observed frequency,

\[
\nu_{\text{max}}' = \min[\nu_{\text{max}}, \nu_0],
\]

is the top of the bandpass \( \nu_{\text{max}}' = \nu_{\text{max}} \) if the burst starts above the bandpass (i.e., \( \nu_0 > \nu_{\text{max}} \)) or \( \nu_{\text{max}} = \nu_0 \) if the burst starts within the bandpass (i.e., \( \nu_{\text{min}} < \nu_0 < \nu_{\text{max}} \)). As shown in Figure 4(b), the naive estimate \( E \approx F_{\nu_{\text{B}}} \nu_{\text{B}} \) is accurate only around \( \alpha \approx 1 \). Figure 4(d) shows the spectral bandwidth, \( \Delta \nu_{\nu_0} \), over which half the burst fluence is accumulated, normalized to the instrumental bandpass \( \Delta \nu_{\text{obs}} \).

### 3.5. Summary of Burst Properties

In summary, for \( \mu = 0 \) and \( \chi \ll 1 \) (intrinsic narrow spectra), the FRB model is described by six free parameters: \( \bar{t}, t_0, \nu_0, F_0, \alpha, \beta \). The singular time \( \bar{t} \) is related trivially to the epoch marking the beginning of the burst and hence is not normally considered a parameter of intrinsic interest (except in relation to other bursts). For the typical case in which \( \nu_0 \) is within or just above the observing bandpass, the parameters \( \beta \) and \( t_0 \) control the burst drift rate and the burst duration. Along with \( \beta \), the value of \( \alpha \) determines the power-law slope of the time-integrated SED, \( \gamma \) (Equation (15)). The burst fluence \( E \) mainly depends on the product of \( F_{\nu_0} \), though the values of \( \alpha \) and \( \beta \) also enter for \( \alpha \approx 1 \). A burst with more complex time structure can be built up by adding together many individual bursts (corresponding to individual narrow features in the intrinsic SED), in general with different values for \( \{\bar{t}, t_0, \alpha, \beta\} \) but potentially the same values for \( \{\nu_0, F_0\} \).

### 4. Burst Detectability and the CHIME/FRB Dichotomy

With the connection between the toy model and FRB observables in place, we now move to describe which of the same parameters control the detectability of a burst and describe how this could shape or bias existing FRB samples.

#### 4.1. Signal-to-noise Ratio

The CHIME/FRB burst sample is defined largely by a threshold on the burst S/N (CHIME/FRB Collaboration et al. 2021;
see their Figure 24), as are other FRB surveys (e.g., ASKAP, Shannon et al. 2018; see Keane & Petroff 2015 for a general discussion). To predict the measured bandpass of a burst, we must calculate over what frequency range the burst S/N will be accumulated within the framework of the toy model.

For an instrument sensitivity (system equivalent flux density) \( S_{\text{sys}} \), the burst S/N is given by the radiometer equation. Within a single frequency channel of bandwidth \( \Delta \nu_{\text{ch}} \) and centroid \( \nu_c \), the S/N is

\[
\frac{S}{N} = \frac{\Delta E_{\nu_c}}{\nu_{\text{ch}}^2 \Delta \nu_{\text{obs},i} S_{\text{sys}}} ,
\]

where \( \Delta E_{\nu_c} \) is the burst fluence in the channel and \( \Delta \nu_{\text{obs},i} \sim \max(\Delta \nu_{\text{res, FRB},i}) \) is the observed burst duration, which is the greater of the true duration the burst takes to cross that channel \( \nu_{\text{ch}} \). For time-resolved bursts with \( \nu_{\text{ch}} < \nu_c \), the total burst S/N is then obtained by summing over all \( N = \Delta \nu_{\text{obs},i}/\Delta \nu_{\text{ch}} \) channels:

\[
\frac{S}{N} = \frac{1}{\sqrt{N}} S_{\text{sys}}^{-1} S/N_i = \frac{1}{\sqrt{\nu_{\text{ch}} \Delta \nu_{\text{obs,i}}}} \sum_{i=1}^{N} \frac{\Delta E_{\nu_c}}{\Delta \nu_{\text{ch}}} \Delta \nu_{\text{obs},i}^{1/2} \Delta \nu_c d E_{\nu_c} \Delta \nu_{\text{obs}}(\nu)^{-1/2} d \nu .
\]

For the analytic toy model, this can be expressed more explicitly using \( d E_{\nu_c}/d \nu \) (Equation (14)) to write

\[
\frac{S}{N} = \frac{2}{\sqrt{\pi}} \frac{F_0 f_0}{\beta} S_{\text{sys}} \frac{1}{\Delta \nu_{\text{obs}}} \left[ \int_{\nu_{\text{max}}}^{\nu_{\text{min}}} \left( \frac{\nu}{\nu_0} \right)^{a-1} \frac{1}{\sqrt{\Delta \nu_{\text{obs}}(\nu)}} d \nu \right]^{1/2} .
\]

For time-resolved bursts with \( \nu_{\text{FRB}} > \nu_{\text{res, FRB}} \), we have \( \Delta \nu_{\text{obs}} = \nu_{\text{FRB}} - \nu_{\text{ch}} \sim (\nu_0/\beta)(\nu/\nu_0)^{-1/2} \), and hence

\[
\frac{S}{N} = \frac{2}{\sqrt{\pi}} \frac{F_0 f_0}{\beta} S_{\text{sys}} \frac{1}{\Delta \nu_{\text{obs}}} \left[ \int_{\nu_{\text{max}}}^{\nu_{\text{min}}} \left( \frac{\nu}{\nu_0} \right)^{a-1} \frac{1}{\sqrt{\Delta \nu_{\text{obs}}(\nu)}} d \nu \right]^{-1/2} \left[ \left( \frac{\nu_{\text{max}}}{\nu_0} \right)^{a+1} - \left( \frac{\nu_{\text{min}}}{\nu_0} \right)^{a+1} \right] ,
\]

where in the final line we have made use of Equation (16). Thus, for temporally unresolved bursts, detectability is proportional to the burst fluence. By contrast, for time-resolved bursts, the S/N is smaller by a factor \( \sim (\Delta \nu_{\text{res}}/\nu_{\text{FRB}})^{1/2} < 1 \) for typical parameters.

### 4.2. Duration/Bandwidth Dichotomy

Equations (21) and (22) reveal several important features. First, for typical values of \( \alpha > 0.5 \) (\( \alpha > 1 \)) the burst S/N is peaked at high frequencies in the case of resolved (unresolved) bursts. Second, the “sharpness” of this high-frequency bias depends sensitively on the value of \( \beta \): low values of \( \beta \ll 1 \) imply that the burst’s cumulative S/N is strongly concentrated at the highest frequencies. The red dashed contours in Figure 5 indicate the fractional bandwidth over which \( \geq 50\% \) of the total burst S/N is accumulated in the space of \( \{\alpha, \beta\} \) for the same burst parameters used in Figure 4.

Even given a perfect instrument sensitive to all frequencies emitted by the burst \( \nu \in [\nu_{\text{0}}, \nu_0] \), half of the burst S/N would be accumulated in a bandpass \( \Delta \nu_{\text{S/N}} \) no greater than

\[
\frac{\Delta \nu_{\text{S/N}}}{\nu_0} \approx 1 - \left( \frac{1}{2} \right)^{\frac{3\beta}{2}} \approx \frac{1.4\beta}{2\alpha - 1} ,
\]

where we have assumed \( \alpha > 1/2 \) and \( \alpha > 1 \), in the resolved and unresolved cases, respectively. The value of \( \Delta \nu_{\text{S/N}} \) estimated above is the maximum S/N bandwidth because the derivation of Equation (23) implicitly assumes an infinitely narrow intrinsic spectral peak; in reality, the spectral bandwidth cannot be smaller than \( \Delta \nu_{\text{ch}}/\nu_0 \sim \gamma \) (e.g., the Gaussian width in Equation (2)).

The fact that low values of \( \beta \) give rise to narrower S/N-limited spectral bandpasses \( \Delta \nu_{\text{S/N}} \approx \beta \) may have important implications for the dichotomy between short- and long-duration FRBs identified by CHIME/FRB (Pleunis et al. 2021). In particular, replacing \( \Delta \nu_{\text{obs}} \) with \( \Delta \nu_{\text{S/N}} \sim \beta \nu_0 \) (Equation (23) in the \( \beta \ll \alpha \ll 1 \) limit) in our estimate of \( \nu_{\text{FRB}} \) (Equation (11)), we obtain the approximate relationship:

\[
\beta \propto \ln \left( \frac{\nu_0}{\nu_{\text{obs}}} \right) \Rightarrow \nu_{\text{FRB}} \propto t_0 \exp \left[ \frac{\nu_0}{\Delta \nu_{\text{S/N}}} \right] .
\]

This expression demonstrates that the same values of \( \beta \ll 1 \) that give rise to long bursts \( (\nu_{\text{FRB}} \gg t_0) \); see also Figure 4) also correspond to those with narrowband S/N \( (\Delta \nu_{\text{S/N}} \ll \nu_0) \). If such a burst were to begin within the instrumental bandpass (i.e., \( \nu_0 < \nu_{\text{max}} \)), it would generate a long-duration burst with a time-integrated spectrum peaked within the band around \( \nu = \nu_0 \). Note that this would remain true even if the burst fluence does not fall off too abruptly with frequency (e.g., \( \gamma \approx 1.5 \)), because the S/N is much more strongly dependent on frequency \( (S/N \propto \nu\gamma+1+1/(2\beta)) \). Furthermore, if such a low- \( \beta \) burst were instead to start well above the instrumental bandpass \( (\nu_0 \gg \nu_{\text{max}}) \), it would likely go undetected (see discussion below).

Alternatively, the opposite regime, where \( \beta \) is not too small (\( \beta \approx 1 \)) implies bursts with shorter durations (Equation (11)) that accumulate significant S/N across a wider bandpass extending to frequencies \( \ll \nu_0 \) (i.e., \( \Delta \nu_{\text{S/N}} \gg \nu_0 \)). Such bursts, even if they begin well above the instrumental bandpass, can be detected as they sweep down through it and accumulate S/N.
across a wide range of frequencies, resulting in a short (preferentially unresolved) burst with a broadband (power-law) spectrum across the entire band.

We illustrate the effects just described by means of two figures. Figure 5 shows an example of how, in the space of \([\alpha, \beta]\), bursts with longer duration (solid black contours) also possess narrower S/N bandwidth (red dashed contours).

Figure 7 shows several example synthetic waterfall plots, with different rows corresponding to bursts with otherwise identical parameters but for three values of \(\beta = 0.1, 0.5, 1\). The left column shows the signal with zero background noise and infinite time resolution, similar to those shown earlier in Figure 2. The right column shows the same bursts as viewed at a coarser 1 ms time resolution; the colors here represent flux density averaged over the coarse \(t - \nu\) grid. Furthermore, in the right column, we also added a time- and frequency-independent white noise to the signal in order to mimic the observed data. The S/N thus achieved, at each frequency channel, is presented in the left subpanel. We quantify the effect of the added noise with the parameters \(\nu S/N,50\) (horizontal dashed line) and \(t S/N,50\) (vertical dashed line), which denote the frequency and time over which half the S/N has been accumulated. As expected, the S/N evolves rapidly with frequency—accumulating most of the S/N at the top of the band (i.e., large \(\nu S/N,50\) and small \(t S/N,50\))—for smaller \(\beta \sim 0.1\), and vice versa. In accordance with Equations (21) and (22), we see that, for smaller values of \(\beta \sim 0.1\), the burst is more readily time-resolved and its spectrum would appear to “linger” around a single frequency, while in the \(\beta = 1\) case, the burst duration would be hardly resolved, and the spectrum would be broad with an S/N spread evenly over the bandpass.

In summary, even if all FRBs in nature share a similar distribution of \(t_0\), variation of the parameter \(\beta\) would act to divide an S/N-limited sample into long time-resolved bursts with narrow spectra \((\beta < 1)\) and shorter unresolved bursts with broadband spectra \((\beta \gg 1)\). Figure 6 illustrates this dichotomy schematically. Confirmation that this effect contributes to the duration–spectral bandwidth relationship identified by CHIME/FRB (Pleunis et al. 2021) would require a more direct modeling of their data within the framework of the toy model.

We have focused on the effects of varying \(\beta\) above, though in principle other parameters of the model could also enter into shaping the burst dichotomy. Although the burst duration is proportional to \(t_0\), the frequency bandwidth of the burst S/N is insensitive to its value. Likewise, while the S/N bandwidth is sensitive to \(\alpha\) for large \(\alpha\), its value is constrained to obey \(\alpha \approx 1 − 0.5\beta \lesssim 1\) to match the time-integrated spectral index \(\gamma \approx −1.5\) found by Macquart et al. (2019).

How likely is it that a burst detected in an S/N-limited sample will begin its evolution within the instrumental bandpass (i.e., \(\nu_0 < \nu_{\text{max}}\)), as opposed to above it? The answer to this question is also related to the bandpass over which the burst S/N is accumulated. If the S/N is accumulated over only a narrow fraction of the instrument’s bandwidth \((\Delta \nu/S/N \ll \Delta \nu_{\text{obs}}, \beta \ll 1)\) (top row of Figure 7), then—assuming that FRB sources in nature sample from a flat distribution of \(\nu_0\) values spanning many decades in frequency—a burst observed by an S/N-limited survey is just as likely to start at any location within the instrumental bandpass \((\nu_{\text{min}} < \nu_0 < \nu_{\text{max}})\) as it would marginally above it \((\nu_{\text{max}} < \nu_0 < \nu_{\text{max}} + \Delta \nu_{\text{obs}}/2)\). This is again consistent with FRB observations that show longer bursts with narrower time-averaged spectra do not always start at the top of the band, while the shorter bursts more frequently spread their S/N across a wider range of frequencies (and thus began their downward frequency evolution starting above the instrumental bandpass).

5. Conclusions

We have presented a toy model for the time–frequency structure of FRBs. The key feature is a narrowly peaked SED whose peak frequency and luminosity scale as power laws with respect to a singular point in time (corresponding, e.g., to the onset of energy release from the central engine). The model can naturally accommodate many observed FRB properties, including: narrow spectral bandwidths of time-resolved bursts and broad power-law spectra of unresolved bursts; downward frequency evolution of burst structure across the burst duration; tendency for shorter intrinsic burst durations at higher radio frequencies; and larger temporal separation between separate burst components for bursts of total longer bursts (Figure 3). Our model is independent of the FRB emission mechanism, though decelerating relativistic shocks (e.g., Metzger et al. 2019) or radius–frequency mapping in a neutron star magnetosphere (e.g., Lyutikov 2020) offer possible physical realizations.

We have shown that burst-to-burst variation in a single parameter of the model—the power-law index of the frequency drift rate \(\beta\)—naturally generates a dichotomy in burst properties: narrow, slowly drifting spectral peaks for temporally resolved longer bursts, and broader power-law spectra for unresolved shorter bursts, similar to the dichotomy in the CHIME/FRB sample (CHIME/FRB Collaboration et al. 2021). This separation arises because bursts accumulate their S/N over a narrower frequency bandpass when the intrinsic SED drifts downward in time more slowly. In cases when \(\beta\) is sufficiently large, all the burst S/N is accumulated at roughly the initial
frequency $\nu_0$ and the burst may appear at a single frequency in the middle of the observing band that does not evolve in time (e.g., FRB 20190515D, a narrowband Gaussian burst centered around 550 MHz; see Figure 3 of Pleunis et al. 2021).

Another consequence of the intrinsic nature of the frequency drifting of our model is that it could in principle be degenerate with that contributed by the burst DM, in particular for low-S/N bursts where the hallmark $\propto \nu^{-2}$ frequency dependence of the DM-induced time delay would not serve as a clear discriminant. Such degeneracy could in principle result in small systematic uncertainties in the DM or the scattering measure, though this remains to be quantified.
Assuming the FRB duration/spectral-bandwidth dichotomy can be attributed to a continuous variation of burst parameters (e.g., $\beta$), this does not itself address why bursts from repeating and nonrepeating FRB sources would preferentially exhibit these different parameters (e.g., smaller and larger values of $\beta$, respectively). We cannot deduce any compelling reason why bursts with narrower spectra but otherwise similar burst properties would preferentially have multiple bursts detected and therefore be categorized as repeaters. This suggests the required diversity in $\beta$ may lie with intrinsic differences in the central engines or environments between repeaters and nonrepeaters.

In scenarios for FRB emission arising from a relativistic shock propagating into a subrelativistically expanding upstream medium (Metzger et al. 2019), the value of $\beta$ is connected to the density profile of the upstream medium $n \propto r^{-k}$. For example, in the “bare shock ($t \geq \delta t$)” scenario reviewed in the Appendix (Table 2), one predicts

$$k = \frac{8 \beta - 3}{2 \beta},$$

such that $\beta \in [0.3, 1]$ maps into $k \in [-1, 2.5]$. All else being equal, longer/narrowband bursts ($\beta$) would arise from shocks entering a flat or rising density profile ($k \leq 0$), while shorter/broadband bursts ($\beta$) would arise from shocks entering a wind-type medium ($k \geq 2$). If the upstream medium is produced by the ejecta shell of a previous burst or bursts (e.g., Metzger et al. 2019), then it would not be surprising for more frequently repeating FRB sources to be surrounded by a different external medium than those from sources that repeat less frequently. Alternatively, in shock scenarios in which the upstream medium is that of a relativistic wind (e.g., of a magnetar, as in Beloborodov 2020, or a binary neutron star merger, as in Sridhar et al. 2021b), the typical values of $\beta$ may be larger $\sim 2$–$3$ (Equation (8)), and hence bursts produced by this mechanism may preferentially be of shorter duration (for an otherwise similar physical size of the central engine, $\sim t_\nu$).

Two of the best-studied repeating FRB sources, FRB 180916 and FRB 121102, have exhibited periodicity in the burst arrival times of 16 and 160 days, respectively, (CHIME/FRB Collaboration et al. 2020b; Rajwade et al. 2020), in which windows of FRB activity are separated by long “dead” periods with little or no burst activity. This periodic behavior may be related to rotation or precession of the central engine (e.g., Beniamini et al. 2019; Levin et al. 2020; Lyutikov et al. 2020; Sridhar et al. 2021a). If the value of $\beta$ is related to a systematic way to the burst environment, we could expect a systematic variation in $\beta$ (and hence of the burst duration or spectral bandwidth) across the active phase window.

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### Appendix

#### Toy Model Parameters for Shock Emission Scenario

This section briefly reviews the predicted values of the toy model parameters $\{\alpha, \beta\}$ within a model in which FRB emission is generated by synchrotron maser emission from a relativistic magnetized shock (e.g., Lyubarsky 2014; Beloborodov 2017) in the case of a stationary upstream medium or one which is expanding subrelativistically (Metzger et al. 2019).

Consider an impulsive injection of energy over time $\delta t$ that generates a shock propagating into a stationary external medium with a radial density profile $n \propto r^{-3}$. The shock will decelerate in a self-similar manner (Blandford & McKee 1976), such that the Lorentz factor, density, and kinetic luminosity of the shock at observer time $t$ after the injection, obey

$$\Gamma \propto \frac{\Gamma^{-3/2}}{t^{-1/2}}, \quad t < \delta t$$

(A1)

$$n \propto \frac{n^{-3/2}}{t^{-1/3}}, \quad t > \delta t$$

(A2)

$$L_{sh} \propto \frac{L_{sh}^{-3/2}}{t^{-1}}, \quad t > \delta t$$

(A3)

where separate scalings are given for $t < \delta t$ and $t > \delta t$.

For a fixed magnetization of the upstream medium, the synchrotron maser emission will peak at a frequency (e.g., Plotnikov & Sironi 2019)

$$\nu_{pk} \approx \nu_{pk}^\prime \propto \Gamma^{-3/2}\nu_k \approx \frac{\Gamma^{-1/2}}{k = 0} \approx \Gamma^{-3/8}, \quad t > \delta t$$

(A4)

However, due to induced Compton scattering (ICS), only frequency $\nu > \nu_{pk}$ is able to escape from the upstream, where (Metzger et al. 2019; Margalit et al. 2020)

$$\nu_{max} \propto \Gamma^{-3/8}\nu_k \approx \frac{\Gamma^{-1/4}}{k = 0} \approx \Gamma^{-3/8}, \quad t > \delta t$$

(A5)

The total power of the synchrotron maser emission (most of which emerges around $\nu_{pk}$) scales with the power of the shock, $L_{sh}$. Assuming the synchrotron maser emission possesses a power-law spectrum $\nu_{max} \propto \nu^{-3-\kappa}$ for $\nu \gtrsim \nu_{pk}$, where $\kappa \approx 4$ is estimated from the results of particle-in-cell shock simulations (e.g., Plotnikov & Sironi 2019), then the observed radio luminosity (at the marginally thin frequency $\nu_{max}$) obeys (e.g., Margalit et al. 2020)

$$L_{FRB} \propto L_{sh, pk} \nu_{pk}^3 \nu_{pk} \nu_{max}^{-3-\kappa} \propto L_{sh} \nu_{pk}^3 \nu_{max}^{-4},$$

(A6)

and hence in the case of ICS-absorbed emission, we have

$$L_{FRB} \propto \frac{\Gamma^{-1/4} \nu_{max}^{-3/2}}{k = 0} \approx \nu_{max}^{-3/2} \frac{\nu_{max}^{-3/2}}{k = 0} \approx t^{-3/2}, \quad t > \delta t$$

(A7)

There are two possibilities to consider, as summarized in Table 2. If the intrinsic narrow spectra of the FRB emission
results from a combination of a decaying spectrum of the maser emission toward higher frequencies and an attenuation due to induced Compton scattering at low frequencies ("ICS Absorbed" model; Metzger et al. 2019), then both $\nu_c$ and $\Delta \nu_c$ should scale $\propto \nu_{\max}$ (Equation (A5)). On the other hand, if the observed spectral peak is instead a narrow feature intrinsic to the maser emission (e.g., Babul & Sironi 2020) emitted into an otherwise transparent upstream ("Bare Shock" model), then we instead expect $\Delta \nu_c$, $\nu_c \propto \nu_{pk}$ (Equation (A4)).

As far as the luminosity normalization (Equation (5)), the simplest version of the ICS Absorbed model predicts $\int F_{\nu} d\nu \propto L_{sh} (\nu_{pk} / \nu_{\max})$ (Equation (A7)). By contrast, in the Bare Shock model, one instead will have $\int F_{\nu} d\nu \propto L_{sh} \propto t^{-1}$ for $t \gtrsim \delta t$ (Equation (A3)), i.e., $\alpha = 1$ independent of $k$. The luminosity decay is shallower ($\alpha \approx 0$) at early times when the reverse shock is still crossing through the ejecta shell ($t < \delta t$).

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