A convex duality method for optimal liquidation with participation constraints

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Abstract

In spite of the growing consideration for optimal execution issues in the financial mathematics literature, numerical approximations of optimal trading curves are almost never discussed. In this article, we present a numerical method to approximate the optimal strategy of a trader willing to unwind a large portfolio. The method we propose is very general as it can be applied to multi-asset portfolios with any form of execution costs, including a bid-ask spread component, even when participation constraints are imposed. Our method, based on convex duality, only requires Hamiltonian functions to have $C^{1,1}$ regularity while classical methods require additional regularity and cannot be applied to all cases found in practice.

1 Introduction

When he is willing to unwind a large portfolio, a trader faces a trade-off between market risk on the one hand and market impact and execution costs on the other hand. Selling rapidly a large quantity of shares is indeed costly as it requires to take liquidity from limit order books. Selling slowly however exposes to price risk because of other market participants’ actions. If a trader makes the decision to unwind his portfolio at a given time, his decision is certainly based on market prices at that time. He needs therefore to sell fast enough so that prices do not move too much over the course of the execution process.

Optimal execution has been an important topic in the academic literature since the turn of the millenium. The above trade-off between execution costs and market impact on the one hand and price risk on the other hand has indeed been modeled in the seminal papers [2, 3] of Almgren and Chriss published in 1999 and 2001. Almgren and Chriss proposed a simple framework to solve the problem of optimally scheduling the execution process and

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this model has largely been used and extended since then, by both practitioners and academics. Initially written in the one-asset case, in discrete time, with quadratic execution costs, and with a Bachelier dynamics for the price, the model has then been considered in continuous time and generalized to allow for more realistic execution costs \(^1\) and random execution costs \(^2\). Black-Scholes dynamics for the price has also been considered – see \(^3\) – and attempts to generalize the model in other directions have been made in the one-asset case, for instance to take account of stochastic volatility and stochastic liquidity \(^4\). Discussions on the optimization criterions and their consequences on optimal strategies have also an important place in the literature (see for instance \([1], [10], [17], [21], [30]\), and the very interesting results in the case of increasing or decreasing risk aversion obtained in \([28]\)). Important extensions to multi-asset portfolios have been developed for instance by Schied and Schöneborn \([29]\). Guéant also developed a general framework for optimal liquidation in \([16]\) and used it to price block trades.

In most of these papers, the optimal strategy does not depend on the evolution of prices (see in particular \([29]\)). The outcome of models à la Almgren-Chriss is indeed an optimal trading curve, stating, before it starts, the optimal time scheduling of the execution process. This trading curve constitutes the first (strategic) layer of most execution algorithms. An execution algorithm is indeed usually made of two layers \(^2\): a strategic layer, which controls the risk with respect to a benchmark and mitigates execution costs, and a tactical layer, which seeks liquidity inside order books, through all types of orders, and across all other liquidity pools (lit or dark). The second layer has been studied in the literature more recently through new models involving limit orders (see for instance \([7, 13, 14, 22]\)) or through the study of liquidation with dark pools (see \([18, 19]\) and \([23]\)).

In this article, we focus on the first layer of execution algorithms for a multi-asset portfolio. \(^3\) We consider a Von Neumann-Morgenstern expected utility framework in the specific case of an investor with constant absolute risk aversion. We consider a general form of execution costs and the optimal strategy is characterized by a Hamiltonian system as in \([16]\) and \([29]\). Numerical approximations of the optimal strategy are briefly discussed in \([16]\) but the method presented is limited to a small class of Hamiltonian functions (strictly convex functions with \(C^2\) regularity). The method we present in this paper is more general as it only requires Hamiltonian functions to be \(C^{1,1}\). It enables to solve numerically the problem of the optimal strategy to unwind a multi-asset portfolio when bid-ask spreads are taken into account and when one adds maximum participation rates (an upper bound to the volume that can be traded, relative to market volume).

Numerical methods are very rarely discussed in the literature (one important exception is \([20]\)). In fact, the problem is rather simple in the one-asset case as the bid-ask spread plays

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\(^1\) Another strand of literature has developed following Obizhaeva and Wang paper \([25]\) – that has been a preprint since 2005. This part of the literature considers optimal execution with transient market impact.

\(^2\) See \([21]\) for a general description of execution algorithms. See also \([8]\) for another viewpoint.

\(^3\) The case of a multi-asset portfolio is also interesting for one-asset portfolio liquidation as it enables to consider a round trip on an additional asset for hedging.
no role and as most papers do not consider participation constraints. In that one-asset case, indeed, a shooting method or a Newton’s method can be used to obtain a precise numerical approximation of the optimal trading curve. In the multi-asset case, shooting methods are not anymore possible and a Newton’s method can only be used in the case of smooth Hamiltonian functions... too smooth to embed bid-ask spread or participation constraints. The method we present is based on convex duality and allows to consider all conceivable practical cases as it only requires Hamiltonian functions to be $C^{1,1}$.

We present the general framework in Section 2 and we recall existence and uniqueness results for the optimal liquidation strategy. We also provide the Hamiltonian characterization of the optimal strategy that plays a major role in our paper. In Section 3, we present our numerical method based on convex duality and we prove a convergence result. Finally, in Section 4, we present numerical examples. Proofs are presented in Appendix A.

2 Optimal liquidation: setup and classical results

2.1 Setup

We consider a portfolio of $d$ different stocks with initial quantities $q_0 = (q^1_0, \ldots, q^d_0)$. We consider the problem of unwinding this portfolio over the time window $[0, T]$.

Let us consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $(\mathcal{F}_t)_{t \in [0, T]}$ satisfying the usual conditions. We assume that all stochastic processes are defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$. We also introduce the set $\mathcal{P}(0, T)$ of progressively measurable processes defined on $[0, T]$.

Market volume processes are denoted by $(V^1_t, \ldots, V^d_t)_{t \in [0, T]}$. They are assumed to be deterministic ($\mathcal{F}_0$-measurable), positive and bounded. For each stock, we consider a maximum participation rate and we denote them by $(\rho^1_m, \ldots, \rho^d_m) \in \mathbb{R}^{+d}$. This allows to define the set of admissible liquidation strategies:

$$\mathcal{A} = \left\{(v^1_t, \ldots, v^d_t)_{t \in [0, T]} \in \mathcal{P}(0, T), \forall i, |v^i_t| \leq \rho^i_m V^i_t \text{ a.e. on } [0, T] \times \Omega, \right.$$ 

$$\forall i, \int_0^T |v^i_t| dt \in L^\infty(\Omega), \quad \forall i, \int_0^T v^i_t dt = q^i_0 \text{ a.s.} \right\}.$$

For a liquidation strategy $(v^1_t, \ldots, v^d_t)_{t \in [0, T]} \in \mathcal{A}$ – representing the velocity at which the trader sells his shares –, we denote by $(q^1_t, \ldots, q^d_t)_{t \in [0, T]}$ the process giving the state of the portfolio. It verifies:

$$\forall i, q^i_t = q^i_0 - \int_0^t v^i_s ds.$$

**Remark 1.** In other words, our hypotheses on the strategies $(v^1, \ldots, v^d)$ are simply that one cannot trade too much, relatively to market volume, and that we indeed liquidate the portfolio by time $T$. 

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Remark 2. We shall always assume that liquidation is feasible in the sense that
\[ ∀i, |\rho_0^i| \leq ρ^i_m \int_0^T V_i^i dt. \]

For each stock, we consider Brownian dynamics for the price:
\[ ∀i, dS^i_t = σ^i dW^i_t \quad σ^i > 0, \]
and we assume that the \(d\)-dimensional Brownian motion \((σ^1 W^1, \ldots, σ^d W^d)\) has a covariance matrix \(Σ\) that is not singular.

Remark 3. Considering Brownian dynamics for prices instead of Black-Scholes dynamics has almost no influence in terms of modelling, as we consider time horizons of at most a few hours. From a mathematical point of view however, the former is more practical than the latter.

The price obtained by the trader at time \(t\) for his trades on stock \(i\) is not \(S^i_t\) because of execution costs. To model these execution costs, we introduce \(d\) functions \(L^1, \ldots, L^d\) verifying the following hypotheses:

- \(∀i, L^i(0) = 0,\)
- \(∀i, L^i\) is an even function,
- \(∀i, L^i\) is increasing on \(\mathbb{R}_+\),
- \(∀i, L^i\) is strictly convex,
- \(∀i, L^i\) is asymptotically superlinear, that is:
  \[ \lim_{ρ \to +∞} \frac{L^i(ρ)}{ρ} = +∞. \]

Now, for \((v^1, \ldots, v^d) ∈ A\), we define the cash process \((X^i_t)_{t ∈ [0,T]}\) by:
\[ X_t = \int_0^t \sum_{i=1}^d \left( v^i_s S^i_s - V^i_s \left( \frac{v^i_s}{V^i_s} \right) \right) ds. \]

Remark 4. In practice, we need to cover the cases \(L^i(ρ) = \eta^i |ρ|^{1+φ^i} + ψ^i |ρ|\) for \(η^i > 0, ψ^i ≥ 0\) and \(φ^i ∈ (0,1]\). The proportional part of the function corresponds to bid-ask spread, stamp duty and/or financial transaction tax. The superlinear part is the classical execution cost component of all models à la Almgren-Chriss.

\(^4\)We do not consider permanent market impact as it plays no role in liquidation strategies – see [11] and [13].
Our objective function for \((v^1, \ldots, v^d) \in A\) is
\[
J(v^1, \ldots, v^d) = \mathbb{E} [-\exp(-\gamma X_T)],
\]
where \(\gamma > 0\) is the absolute risk aversion parameter of the trader.

In other words, our problem is to find:
\[
(v^{1*}, \ldots, v^{d*}) \in \text{argmax}_{(v^1, \ldots, v^d) \in A} J(v^1, \ldots, v^d).
\]
Adapting the results obtained in [16] and [29], we can show that there exists such an optimal liquidation strategy. We can also show that stochastic strategies cannot do better than deterministic strategies. These results, along with a characterization of the unique deterministic optimal strategy, are exhibited in the next paragraphs.

2.2 Optimal liquidation strategy

The results obtained in [16] and [29] can easily be adapted to the case considered here. First of all, one can restrict the set of liquidation strategies to deterministic processes in \(A\). We denote by \(A_{\text{det}}\) this set, which consists of the liquidation strategies in \(A\) that are \(\mathcal{F}_0\)-measurable.

Our first Proposition states that, in the case of deterministic strategies, the objective function simplifies as final wealth \(X_T\) is normally distributed.

**Proposition 2.1.** If \((v^1, \ldots, v^d) \in A_{\text{det}}\) then \(X_T\) is normally distributed and
\[
J(v^1, \ldots, v^d) = -\exp \left( -\gamma \left( q_0^0 S_0 - \sum_{i=1}^{d} \int_0^T V_i^s L_i^s \left( \frac{v_i^s}{V_i^s} \right) ds - \frac{1}{2} \gamma \int_0^T q_i^s \Sigma q_i^s ds \right) \right).
\]

A consequence of this Proposition is that the problem boils down to solving the following variational problem:

\[
(\mathcal{P}) \quad \inf_{(q^1, \ldots, q^d) \in \mathcal{C}} I(q^1, \ldots, q^d) := \sum_{i=1}^{d} \int_0^T V_i^s L_i^s \left( \frac{\dot{q}_i^s(s)}{V_i^s} \right) ds + \frac{1}{2} \gamma \int_0^T q_i^s \Sigma q_i^s ds
\]

where
\[
\mathcal{C} = \left\{ q \in W^{1,1}((0,T), \mathbb{R}^d), q(0) = q_0, q(T) = 0, \forall i, |\dot{q}_i(t)| \leq \rho_i V_i^t, \text{ a.e. in } (0,T) \right\}.
\]

Existence and uniqueness of minimizers for \(I\) are obtained using classical techniques of variational calculus and convex optimization as in [16]. Moreover, the optimal liquidation strategy is characterized by a Hamiltonian system.

**Theorem 2.1** (Existence, uniqueness and characterization of an optimal strategy). There exists a unique function \(q^* \in \mathcal{C}\) that minimizes the function \(I\) defined in problem \((\mathcal{P})\).

There exists a \(W^{1,1}\) function \(p\) such that \((q^*, p)\) is a solution of the Hamiltonian system
\[
(S_H) : \begin{cases}
\dot{p}(t) = \gamma \Sigma q(t) \\
\dot{q}_i(t) = V_i^s H_i^s(p(t)), \quad \forall i \in \{1, \ldots, d\}
\end{cases}
\]
with boundary conditions \( q(0) = q_0, \quad q(T) = 0 \), where:

\[
H^i(p) = \sup_{|\rho| \leq \rho^i_m} pp - L^i(\rho),
\]

Moreover, if a pair \((q,p)\) of \(W^{1,1}\) functions is solution of \((S_H)\), then \(q = q^*\).

**Remark 5.** Although there is uniqueness of the optimal trajectory \(q^*\), there may not be uniqueness of \(p\). This turns out to be important when it comes to numerics.

**Remark 6.** If market volume processes are continuous, it is clear that for any solution \((p,q)\) of the system \((S_H)\), \(q\) has \(C^1\) regularity and \(p\) has \(C^2\) regularity. We shall therefore approximate \(p\) instead of \(q\).

The Hamiltonian characterization \((S_H)\) is more suited to numerics than the Euler-Lagrange equation used in most papers since the execution cost functions \(L^i\)'s are not of class \(C^1\) as soon as there is a bid-ask spread component. Finding an approximation to a solution of the Hamiltonian system \((S_H)\) is the goal of this paper and we present our method in the next section. But before turning to our numerical method, let us conclude this section by stating that the best stochastic strategy cannot do better than the best deterministic strategy.

**Theorem 2.2** (Optimality of deterministic strategies).

\[
\sup_{(v^1,\ldots,v^d) \in A} E[-exp(-\gamma X_T)] = \sup_{(v^1,\ldots,v^d) \in A_{det}} E[-exp(-\gamma X_T)].
\]

### 3 Numerical method: convex duality to the rescue

#### 3.1 Preliminary remarks

The goal of this section is to present a new numerical method for approximating a solution \((p,q)\) of the Hamiltonian system \((S_H)\). More precisely, we shall approximate a solution of a discretization of this system. This discretized version \((\tilde{S}_H)\) corresponds to the optimality conditions of the discrete counterpart of the optimization problem of Section 2 (see Appendix B). Considering a time grid \(0, \Delta t, \ldots, N\Delta t = T\), we are looking for \(((q^i_n)_{1 \leq i \leq d, 0 \leq n \leq N}, (p^i_n)_{1 \leq i \leq d, 0 \leq n \leq N-1})\) satisfying:

\[
(\tilde{S}_H): \quad \begin{cases} 
    p_{n+1} = p_n + \Delta t \gamma \Sigma q_{n+1}, & 0 \leq n < N - 1 \\
    q_{n+1}^i = q_n^i + \Delta t V^i_{n+1} H^i(p_n^i), & 0 \leq n < N, \quad \forall i \in \{1, \ldots, d\} \\
    q_0 = q_0, \quad q_N = 0.
\end{cases}
\]

The first part of the system is made of linear equations. The difficulty then relates to the nonlinearity introduced by the Hamiltonian functions \(H^i\)'s. In particular, it is noteworthy that in the initial Almgren-Chriss case [2, 3] with a quadratic execution cost function, the system boiled down to a linear one.

However, in practice, a classical form for the execution cost functions \(L^i\)'s is

\[
L(\rho) = \eta|\rho|^{1+\phi} + \psi|\rho|, \quad \eta > 0, \psi \geq 0, \phi \in (0, 1].
\]
The superlinear component has been considered since the initial papers by Almgren and Chriss. In [2, 3], the authors considered the quadratic case \( \phi = 1 \) to obtain closed form solutions. However, realistic values for \( \phi \) are usually estimated to be between 0.4 and 0.8. For instance, Almgren and coauthors from Citigroup estimated \( \phi \approx 0.6 \) using a large dataset of meta-orders (see [5]). The proportional component models the influence of the bid-ask spread, of a stamp duty, or of a financial transaction tax.

The Hamiltonian function \( H \) associated to \( L(\rho) = \eta|\rho|^{1+\phi} + \psi|\rho| \) with participation constraint \( \rho_m \) is:

\[
H(p) = \sup_{|\rho| \leq \rho_m} \rho p - \eta|\rho|^{1+\phi} - \psi|\rho| = \begin{cases} 
0 & \text{if } |p| \leq \psi \\
\phi \eta \left( \frac{|p|-\psi}{\eta(1+\phi)} \right)^{1+\frac{1}{\phi}} & \text{if } \psi < |p| \leq \psi + \eta(1+\phi)\rho_m^\phi \\
(|p| - \psi)\rho_m - \eta \rho_m^{1+\phi} & \text{if } \psi + \eta(1+\phi)\rho_m^\phi < |p|
\end{cases}
\]

As \( L \) is strictly convex, \( H \) is \( C^1 \) with:

\[
H'(p) = \text{sign}(p) \min \left( \rho_m, \left( \frac{\max(|p| - \psi, 0)}{\eta(1+\phi)} \right)^{\frac{1}{\phi}} \right)
\]

This function \( H' \) is not differentiable. Consequently, Newton’s method cannot be used to find an approximate solution of the system \( (S_H) \). However the function \( H' \) is Lipschitz as \( \phi \in (0,1] \) and the method we present applies to Hamiltonian functions with regularity \( C^{1,1} \) – that is whenever \( H' \) is Lipschitz.

### 3.2 Dual problem

The problem \((P)\) we considered in Section 2 is:

\[
\inf_{(q^1, \ldots, q^d) \in C} \sum_{i=1}^d \int_0^T V_i^s L_i^s \left( \frac{\dot{q}^i(s)}{\dot{q}^i_\delta} \right) ds + \frac{1}{2} \gamma \int_0^T q(s)' \Sigma q(s) ds.
\]

The associated dual problem \((D)\) is:

\[
\inf_{(p^1, \ldots, p^d) \in W^{1,1}((0,T),\mathbb{R}^d)} J(p^1, \ldots, p^d) := \sum_{i=1}^d \int_0^T V_i^s H^s_i \left( p^i(s) \right) ds + \frac{1}{2} \gamma \int_0^T \dot{p}(s)' \Sigma^{-1} \dot{p}(s) ds + p(0)' q_0,
\]

the link between the primal variable \( q \) and the dual variable \( p \) being \( q = \frac{1}{\gamma} \Sigma^{-1} \dot{p} \).

If we consider the discrete counterpart of our model (see Appendix B), then the discrete counterpart \((\tilde{P})\) of \((P)\) is:

\[
\inf_{(q^i_n)_{0 \leq n \leq N, 1 \leq i \leq d}, q_0 = q_N = 0} \tilde{I}(q^i_n)_{0 \leq n \leq N, 1 \leq i \leq d},
\]
where
\[
\tilde{I}((q_i^j)_{0 \leq n \leq N, 1 \leq i \leq d}) = \sum_{i=1}^{d} \sum_{n=0}^{N-1} L_i \left( \frac{q_{i+1}^j - q_n^j}{V_{n+1}^i \Delta t} \right) V_{n+1}^i \Delta t + \frac{\gamma}{2} \sum_{n=0}^{N-1} q_{n+1}^j \Sigma q_{n+1} \Delta t
\]

The dual problem (\(\tilde{D}\)) associated to (\(\tilde{P}\)) is:
\[
\inf_{(p_i^j)_{0 \leq n \leq N, 1 \leq i \leq d}} \tilde{J}((p_i^j)_{0 \leq n \leq N, 1 \leq i \leq d}),
\]
where
\[
\tilde{J}((p_i^j)_{0 \leq n \leq N, 1 \leq i \leq d}) = \sum_{i=1}^{d} \sum_{n=0}^{N-1} V_{n+1}^i (p_i^j) \Delta t + \frac{1}{2\gamma \Delta t} \sum_{n=1}^{N-1} (p_n - p_{n-1}) \Sigma^{-1} (p_n - p_{n-1}) + \sum_{i=1}^{d} p_0^i q_0^i
\]

This dual problem is at the core of our numerical approximation method. The initial idea is in fact to use a gradient descent algorithm to find a minimizer of \(\tilde{J}\). However, since
\[
\nabla J(p) = -\frac{1}{\gamma} \Sigma^{-1} \tilde{p} + \begin{pmatrix} V^1 H^{1f}(p^1) \\ \vdots \\ V^d H^{df}(p^d) \end{pmatrix},
\]
considering a simple gradient descent is equivalent to using an explicit scheme to approximate a solution of the PDE
\[
\partial_\theta p - \frac{1}{\gamma} \Sigma^{-1} \partial^2 p + \begin{pmatrix} V^1 H^{1f}(p^1) \\ \vdots \\ V^d H^{df}(p^d) \end{pmatrix} = 0
\]
with Neumann boundary conditions \(\dot{p}(\theta, 0) = \gamma \Sigma q_0\) and \(\dot{p}(\theta, T) = 0\), and with an initial condition at \(\theta = 0\).

Consequently, a simple gradient descent would require a very small step in order to converge to a minimum. The methodology we propose consists instead of considering a semi-implicit gradient descent.

### 3.3 Semi-implicit gradient descent

The method we propose is inspired from a gradient descent. However, we consider an implicit scheme for what would correspond to the heat operator in continuous time.

The idea is in fact to decompose \(\tilde{J}\) as \(\tilde{J}_1 + \tilde{J}_2\) where
\[
\tilde{J}_1((p_i^j)_{0 \leq n \leq N, 1 \leq i \leq d}) = \frac{1}{2\gamma \Delta t} \sum_{n=1}^{N-1} (p_n - p_{n-1}) \Sigma^{-1} (p_n - p_{n-1})
\]
and
\[
\tilde{J}_2((p_i^j)_{0 \leq n \leq N, 1 \leq i \leq d}) = \sum_{i=1}^{d} \sum_{n=0}^{N-1} V_{n+1}^i (p_i^j) \Delta t + \sum_{i=1}^{d} p_0^i q_0^i
\]
Then, starting from an initial guess \( p^0 = (p^{1,0}_0, \ldots, p^{d,0}_0, \ldots, p^{1,0}_{N-1}, \ldots, p^{d,0}_{N-1})' \), we compute \( p^{k+1} \) from \( p^k \) by:

\[
p^{k+1} = p^k - \frac{\Delta \theta}{\Delta t} \left( \nabla \tilde{J}_1(p^{k+1}) + \nabla \tilde{J}_2(p^k) \right),
\]

where \( \Delta \theta > 0 \) is to be chosen to guarantee convergence of the sequence \( (p^k)_k \).

In other words, we propose the following semi-implicit gradient descent:

- Start with an initial guess \( (p^{i,0}_n)_{0 \leq n < N, 1 \leq i \leq d} \)
- For \( k \geq 0 \) and \( \Delta \theta > 0 \), define recursively \( (p^{i,k+1}_n)_{0 \leq n < N, 1 \leq i \leq d} \) from \( (p^{i,k}_n)_{0 \leq n < N, 1 \leq i \leq d} \) by:

\[
\frac{p^{i,k+1}_n - p^{i,k}_n}{\Delta \theta} - \frac{1}{\gamma \Sigma^{-1}} \frac{p^{i,k+1}_{n+1} - 2p^{i,k+1}_n + p^{i,k+1}_{n-1}}{\Delta t^2} + \begin{pmatrix} V^1 H^1'(p^{1,k}_n) \\ \vdots \\ V^d H^d'(p^{d,k}_n) \end{pmatrix} = 0, \quad 0 \leq n < N,
\]

where, by convention, we define:

\[
p^{i,-1}_n = p^{i,0}_n - \Delta t \gamma \Sigma q_0, \quad p^{i,0}_N = p^{i,0}_{N-1}
\]

As the method is semi-implicit, we need first to state that there is no issue with the recursive definition. For that purpose, we start with a straightforward Lemma:

**Lemma 3.1.** For \( p = (p^{1}_0, \ldots, p^{d}_0, \ldots, p^{1}_{N-1}, \ldots, p^{d}_{N-1})' \),

\[
\nabla \tilde{J}_1(p) = \frac{1}{\gamma \Delta t} M \otimes \Sigma^{-1} p
\]

where the \( N \times N \) matrix \( M \) is defined by:

\[
M = \begin{pmatrix}
1 & -1 & 0 \\
-1 & 2 & -1 \\
\vdots & \vdots & \ddots \\
-1 & 2 & -1 \\
0 & -1 & 1
\end{pmatrix}.
\]

Then, the next Proposition states that our method is indeed well defined.

**Proposition 3.1.** The sequence \( (p^k)_k \) is well defined.

Now, our goal is to prove that for sufficiently small values of \( \Delta \theta \), the sequence \( (p^k)_k \) converges. Then, the optimal trajectory \( q^* \) will also be obtained as a limit using the relation between the dual variable \( p \) and the primal variable \( q \). For that purpose, we start with a straightforward Lemma.

\(^5 p^{i,k}_n \) denotes the \( \mathbb{R}^d \) column vector \( (p^{1,k}_n, \ldots, p^{d,k}_n)' \).
Lemma 3.2. Assume that the Hamiltonian functions $H_i$’s are $C^{1,1}$, then $\nabla \tilde{J}_2$ is a Lipschitz function with

$$\|\nabla \tilde{J}_2\|_{\text{Lip}} \leq \Delta t \sup_{i,n} V_{n+1}^i \|H_i'\|_{\text{Lip}}.$$  

We can now state our convergence result.

Theorem 3.1 (Convergence of the semi-implicit gradient descent). Assume that the Hamiltonian functions $H_i$’s are $C^{1,1}$, and let us consider $K$ such that $\|\nabla \tilde{J}_2\|_{\text{Lip}} \leq K \Delta t$.

Then, if $\Delta \theta < \frac{2}{K}$, $(p^k)_k$ converges towards a minimum of $\tilde{J}$ and

$$\forall n \in \{1, \ldots, N - 1\}, \quad q^*_n = \left( q^*_n, \ldots, q^{d*}_n \right)' = \lim_{k \to +\infty} \frac{1}{\gamma \Delta t} I_N \otimes \Sigma^{-1} \left( p^i_n - p^i_{n-1} \right).$$

This theorem proves that our method works better than a simple gradient descent, as $\Delta \theta$ can be chosen independently of $\Delta t$. Also, it does not require to consider second derivatives of Hamiltonian functions contrary to a Newton’s scheme. Before turning the to the practical use of our method, let us notice that the bound $\frac{2}{K}$ can be made explicit in the case of execution costs of the form $L(\rho) = \eta |\rho|^{1+\phi} + \psi |\rho|$

Remark 7. If $\forall i, L^i(\rho) = \eta^i |\rho|^{1+\phi^i} + \psi^i |\rho|$, then we can consider

$$K = \sup_{i,n} V_{n+1}^i \frac{1}{\eta^i \phi^i (1+\phi^i)} \rho_m^{1-\phi^i}$$

In practice, convergence may occur for values of $\Delta \theta$ above $\frac{2}{K}$.

4 Practical examples

4.1 Preliminary remarks

In practice, it may be convenient to diagonalize $\Sigma$ to simplify computations in the semi-implicit gradient descent. If we write indeed $\Sigma = QDQ^{-1}$ the spectral decomposition of $\Sigma$, the variable $y^k$ defined by $y^k_n = Q^{-1} p^k_n$ verifies:

$$\frac{y^k_{n+1} - y^k_n}{\Delta \theta} - \frac{1}{\gamma} D^{-1} y^k_{n+1} - \frac{2y^k_{n+1} + y^k_{n-1}}{\Delta t^2} + Q^{-1} \left( V^{1H^{1'}} \left( (Qy^k_n)^1 \right) \right) = 0, \quad 0 \leq n < N,$$

where, by convention, we define:

$$y^k_{-1} = y^k_0 - \Delta t \gamma DQ^{-1} q_0, \quad y^k_N = y^k_{N-1}.$$

This formulation enables indeed to consider the discrete heat operator on each stock independently.
We now turn to the practical use of our method with specific examples. We shall not proceed to comparative statics as the role of the parameters have been described in many papers (see for instance [6, 16]). Instead we focus on specific cases where the constraint on the participation rate is binding, or where bid-ask spread plays a role.

### 4.2 Examples in the one-asset case

To examplify the use of our method, we consider first the liquidation of a portfolio with a single stock. The parameters of the stock are the following:

- Stock price $S_0 = 75$,
- Volatility 20%, corresponding to $\sigma = 0.9375$,
- Market volume is assumed to be constant over the day with $V_t = 2000000$,
- The execution cost function is $L(\rho) = \eta |\rho|^{1+\phi} + \psi|\rho|$, with $\eta = 0.045$, $\phi = 0.5$ and $\psi = 0.0081$.

![Figure 1: Liquidation with different values of the maximum participation rate $\rho_m$. Risk aversion: $\gamma = 4.10^{-7}$.](image)

We consider the liquidation of $q_0 = 300000$ shares (that is 15% of market daily volume) over one day ($T = 1$), with three different figures for the maximum participation rate. In the first

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6The figures are inspired from the French stock Sanofi.
case, we set \( \rho_m = 60\% \) and since the constraint is never binding, this corresponds to setting no constraint at all on participation rate. The two other cases correspond respectively to \( \rho_m = 40\% \) and \( \rho_m = 20\% \). The results for the optimal liquidation strategy are shown on Figure 1. We see that, as expected, the more we constrain the participation rate, the slower the execution process. In particular, we see that the constraint is binding in the two cases \( \rho_m = 40\% \) and \( \rho_m = 20\% \), as the liquidation process starts in straight line (the slope of the line corresponding to the maximum participation rate).

4.3 Examples in the multi-asset case

The above examples were in the case of a one-asset portfolio. We now turn to three different cases involving several assets. In the first case, we liquidate a long portfolio of 2 positively correlated assets. In the second case, we liquidate a portfolio with a long position in the first asset and a short position in the second asset, the two assets being still positively correlated. The third case corresponds to the liquidation of a one-asset portfolio when a round trip on another (correlated) asset is allowed, in order to hedge risk.

We consider that the first asset is as above:

- Stock price \( S^1_0 = 75 \),
- Volatility 20\%, corresponding to \( \sigma^1 = 0.94 \),
- Market volume is assumed to be constant over the day with \( V^1_t = 2000000 \),
- The execution cost function is \( L^1(\rho) = \eta^1|\rho|^{1+\phi^1} + \psi^1|\rho| \), with \( \eta^1 = 0.045 \), \( \phi^1 = 0.5 \) and \( \psi^1 = 0.0081 \).

For the first two cases we illustrate, the second asset we consider has the following characteristics:

- Stock price \( S^2_0 = 50 \),
- Volatility 17\%, corresponding to \( \sigma^2 = 0.53 \),
- Market volume is assumed to be constant over the day with \( V^2_t = 4500000 \),
- The execution cost function is \( L^2(\rho) = \eta^2|\rho|^{1+\phi^2} + \psi^2|\rho| \), with \( \eta^2 = 0.0255 \), \( \phi^2 = 0.5 \) and \( \psi^2 = 0.005 \).

Correlation between the two assets is assumed to be 0.5.

For the first example, we assume that \( q^1_0 = 300000 \) and \( q^2_0 = 675000 \), that is 15\% of the daily market volume of each asset. Also, maximum participation rates are assumed to be \( \rho^1_m = \rho^2_m = 40\% \).

\footnote{The figures are inspired from the French stock Total.}
The results are shown on Figure 2. As a benchmark, we plotted the optimal strategy for the liquidation of the portfolio with Asset 1 only. We see that the presence of the two assets in the portfolio accelerates the liquidation of Asset 1. Since the two assets are positively correlated, price risk is increased by the presence of Asset 2. Therefore, it is natural that the liquidation of Asset 1 occurs faster in the presence of Asset 2. It is also interesting to notice that the velocity at which liquidation occurs when the full portfolio is considered would be even more important without participation contraints. The constraint on Asset 1 is indeed binding over the first 10% of the time window in the two-asset case.

![Figure 2: Liquidation of a 2-asset long portfolio. Risk aversion: $\gamma = 4.10^{-7}$](image)

The opposite case where liquidation is slower in the multi-asset case than in the one-asset case may correspond either to the liquidation of a long/short portfolio with two positively correlated assets or to the liquidation of a long-only or short-only portfolio with two negatively correlated assets.

To illustrate the former, we consider the same assets as above but the portfolio is $q_1^0 = 300000$ and $q_2^0 = -675000$. Also, maximum participation rates are assumed to be $\rho_m^1 = \rho_m^2 = 30\%$.

The results are shown on Figure 3. As expected, because price risk is reduced by the presence of the second asset, the liquidation of Asset 2 is slower in the two-asset case than in the one-asset case. It is however noteworthy that the constraint on Asset 1 is binding at the very begining in both cases.
The third case we consider is a pure hedging case, the trader does not have an initial position on the Asset 2, but this second asset will be used for hedging purpose. We consider the case where Asset 2 is much more liquid than Asset 1:

- Stock price $S_2^0 = 50$,
- Volatility 17%, corresponding to $\sigma^2 = 0.53$,
- Market volume is assumed to be constant over the day with $V_t^2 = 10000000$,
- The execution cost function is $L^2(\rho) = \eta^2|\rho|^{1+\phi^2} + \psi^2|\rho|$, with $\eta^2 = 0.002$, $\phi^2 = 0.5$ and $\psi^2 = 0.001$.

Correlation between the two asset prices is assumed to be 0.5. As far as maximum participation rates are concerned, we took $\rho_{m1}^1 = 50\%$ and $\rho_{m2}^2$ large enough for the constraint not to be binding.

The results are shown on Figure 4. As a benchmark, we also plotted in dotted line the theoretical hedging curve $q^2 = -\rho\sigma^2S_0^1S_0$, had there be no execution costs. In fact, in order to avoid paying too much for the round trip, the trader restricts its round trip by staying on a plateau (this is link to the proportional term). We also see, as expected, that the execution process is slowed down thanks to the hedging instrument.
Figure 4: Liquidation of a one-asset portfolio with and without hedge. Risk aversion: $\gamma = 1.10^{-6}$. The left axis corresponds to Asset 1, the right axis corresponds to Asset 2.

Appendix A: Proofs

**Proof of Proposition 2.1**

After integrating by parts in the definition of $X_T$, we obtain:

$$X_T = q_0 S_0 - \int_0^T V^i_s L^i \left( \frac{v^i_s}{V^i_s} \right) ds + \int_0^T \sigma^i_s q^i_s dW^i_s.$$ 

Hence, if $(v^1, \ldots, v^d) \in \mathcal{A}_{\text{det}}$, $(q^1, \ldots, q^d)$ is deterministic and $X_T$ is Gaussian. Therefore $J(v^1, \ldots, v^d) = \mathbb{E} \left[ - \exp(-\gamma X_T) \right]$ can be computed in closed form as we know the Laplace transform of a Gaussian variable:

$$J(v^1, \ldots, v^d) = -\exp \left( -\gamma \left( q_0 S_0 - \sum_{i=1}^d \int_0^T V^i_s L^i \left( \frac{v^i_s}{V^i_s} \right) ds - \frac{1}{2} \gamma \int_0^T q^i_s \Sigma q^i_s ds \right) \right).$$

**Proof of Theorem 2.1**
Existence can be obtained using the same method as in Theorem 2.1 of [13].

Uniqueness comes straightforwardly from the fact that the quadratic form \( x \mapsto x'\Sigma x \) is a strictly convex function.

Now, coming to the Hamiltonian characterization, we consider the generalized functions:

\[
\mathcal{L}^i(\rho) = \begin{cases} 
L^i(\rho) & \text{if } |\rho| \leq \rho_m^i, \forall i \\
+\infty & \text{if } |\rho| > \rho_m^i
\end{cases}
\]

Then, the problem \( (P) \) is equivalent to

\[
\inf_{(q^1,\ldots,q^d) \in \tilde{\mathcal{C}}} \sum_{i=1}^d \int_0^T V_s^i \mathcal{L}^i \left( \frac{\dot{q}^i(s)}{V_s^i} \right) ds + \frac{1}{2} \gamma \int_0^T q(s)'\Sigma q(s) ds
\]

where

\[
\tilde{\mathcal{C}} = \left\{ q \in W^{1,1}((0,T),\mathbb{R}^d), q(0) = q_0, q(T) = 0 \right\}.
\]

Applying Theorem 6 of [26] and its corollary to this problem, we obtain the Hamiltonian characterization stated in Theorem 2.1.

**Proof of Theorem 2.2**

The proof is exactly the same as in [16] and [29].

**Proof of Proposition 3.1**

Using Lemma 3.1 we have:

\[
p^{k+1} - p^k = -\frac{\Delta \theta}{\gamma \Delta t^2} M \otimes \Sigma^{-1} p^{k+1} - \frac{\Delta \theta}{\Delta t} \nabla \tilde{J}_2(p^k).
\]

Hence, to prove that \( p^{k+1} \) is uniquely defined from \( p^k \), we need to prove that the matrix

\[
I_{Nd} + \frac{\Delta \theta}{\gamma \Delta t^2} M \otimes \Sigma^{-1}
\]

is invertible. For that purpose, let us write \( \Sigma = QDQ^{-1} \) the spectral decomposition of \( \Sigma \), \( D \) being diagonal with positive coefficients. Notice then that

\[
I_{Nd} + \frac{\Delta \theta}{\gamma \Delta t^2} M \otimes D^{-1}
\]

is a strictly diagonally dominant matrix and hence it is invertible. Therefore,

\[
I_{Nd} + \frac{\Delta \theta}{\gamma \Delta t^2} M \otimes \Sigma^{-1} = (I_N \otimes Q) \left( I_{Nd} + \frac{\Delta \theta}{\gamma \Delta t^2} M \otimes D^{-1} \right) (I_N \otimes Q^{-1})
\]

\(^8\)The proof is even simplified by the presence of participation constraints that enables to avoid the use of Dunford-Pettis Theorem.
is invertible.

\[ \text{Proof of Theorem 3.1} \]

To improve readability, let us introduce
\[ A = \frac{1}{\gamma} M \otimes \Sigma^{-1} \quad \text{and} \quad B = I_{Nd} + \frac{\Delta \theta}{\Delta t^2} A. \]

Then, using Lemma 3.1 straightforward computations give:
\[ p^{k+1} - p^k = -\frac{\Delta \theta}{\Delta t} B^{-1} \nabla \tilde{J}(p^k). \]

Now, we decompose \( \tilde{J}(p^{k+1}) - \tilde{J}(p^k) \) as \( \tilde{J}_1(p^{k+1}) - \tilde{J}_1(p^k) + \tilde{J}_2(p^{k+1}) - \tilde{J}_2(p^k) \).

- For the first part, we have
  \[ \tilde{J}_1(p^{k+1}) - \tilde{J}_1(p^k) = \nabla \tilde{J}_1(p^k)'(p^{k+1} - p^k) + \frac{1}{2}(p^{k+1} - p^k)' \frac{A}{\Delta t}(p^{k+1} - p^k) \]
  \[ = -\frac{\Delta \theta}{\Delta t} \nabla \tilde{J}_1(p^k)' B^{-1} \nabla \tilde{J}(p^k) + \frac{1}{2} \left( \frac{\Delta \theta}{\Delta t} \right)^2 \nabla \tilde{J}(p^k)'B^{-1} \frac{A}{\Delta t} B^{-1} \nabla \tilde{J}(p^k). \]

- For the second part, we have
  \[ \tilde{J}_2(p^{k+1}) - \tilde{J}_2(p^k) \leq \nabla \tilde{J}_2(p^k)'(p^{k+1} - p^k) + \frac{1}{2} K \Delta t \| p^{k+1} - p^k \|^2 \]
  \[ \leq -\frac{\Delta \theta}{\Delta t} \nabla \tilde{J}_2(p^k)' B^{-1} \nabla \tilde{J}(p^k) + \frac{1}{2} K \Delta t \left( \frac{\Delta \theta}{\Delta t} \right)^2 \nabla \tilde{J}(p^k)' B^{-2} \nabla \tilde{J}(p^k). \]

Summing, we get:
\[ \tilde{J}(p^{k+1}) - \tilde{J}(p^k) \leq -\frac{\Delta \theta}{\Delta t} \nabla \tilde{J}(p^k)' B^{-1} \nabla \tilde{J}(p^k) + \frac{1}{2} \left( \frac{\Delta \theta}{\Delta t} \right)^2 \nabla \tilde{J}(p^k)' B^{-1} \left( K \Delta t I_{Nd} + \frac{A}{\Delta t} \right) B^{-1} \nabla \tilde{J}(p^k) \]
\[ \leq -\frac{\Delta \theta}{\Delta t} \nabla \tilde{J}(p^k)' B^{-1} \left( B - \frac{1}{2} \frac{\Delta \theta}{\Delta t} \left( K \Delta t I_{Nd} + \frac{A}{\Delta t} \right) \right) B^{-1} \nabla \tilde{J}(p^k). \]

Hence, writing
\[ \frac{\Delta \theta}{\Delta t} \nabla \tilde{J}(p^k)' B^{-1} R B^{-1} \nabla \tilde{J}(p^k) \leq \tilde{J}(p^k) - \tilde{J}(p^{k+1}) \]
and summing over \( k \), we obtain for \( \kappa \in \mathbb{N} \):
\[ \frac{\Delta \theta}{\Delta t} \sum_{k=0}^{\kappa} \nabla \tilde{J}(p^k)' B^{-1} R B^{-1} \nabla \tilde{J}(p^k) \leq \tilde{J}(p^0) - \inf \tilde{J}. \]

Now,
\[ R = B - \frac{1}{2} \Delta \theta \left( K \Delta t I_{Nd} + \frac{A}{\Delta t} \right) = \left( 1 - \frac{1}{2} K \Delta \theta \right) I_{Nd} + \frac{1}{2} \frac{\Delta \theta}{\Delta t^2} A \]

is a positive-definite matrix for \( \Delta \theta < \frac{2}{K} \).

Therefore, the series of positive terms \( \sum_k \nabla \tilde{J}(p^k)' B^{-1} \nabla \tilde{J}(p^k) \) is convergent.

As \( R \) and \( B \) are positive-definite matrices, we can conclude that the series \( \sum_k \| \nabla \tilde{J}(p^k) \|_2^2 \) is also convergent.

Now, since \( p^{k+1} - p^k = -\Delta \theta B^{-1} \nabla \tilde{J}(p^k) \), the series \( \sum_k \| p^{k+1} - p^k \|_2^2 \) is convergent and we can conclude that the sequence \( (p^k)_k \) converges toward a vector \( p^* \) such that \( \nabla \tilde{J}(p^*) = 0 \).

Now, if we define \( q^*_0 = q_0, q^*_N = 0, \) and \( \forall n \in \{1, \ldots, N-1\}, \)

\[ q^*_n = \left( q^1_n, \ldots, q^d_n \right)' = \frac{1}{\gamma \Delta t} I_N \otimes \Sigma^{-1} \left( p^*_n - p^*_n - 1 \right), \]

it is straightforward to verify that \( (p^*, q^*) \) is a solution of \( (\tilde{S}_H) \).

**Appendix B: Discrete model**

This appendix is dedicated to the discrete counterpart of our model. We consider for that purpose that time is divided into slices of length \( \Delta t \) and we denote by \( t_0 = 0 \leq \ldots \leq t_N = n \Delta t \leq \ldots \leq t_N = T \) the relevant sequence of times for our discrete model.

For \( i \in \{1, \ldots, d\} \), we denote by \( v^i_{n+1} \Delta t \) the number of shares of stock \( i \) sold by the trader between \( t_n \) and \( t_{n+1} \). As a consequence, the state of the portfolio \( q = (q^1, \ldots, q^d) \) is given by

\[ \forall i, q^i_{n+1} = q^i_n - v^i_{n+1} \Delta t, \quad 0 \leq n < N. \]

Price processes are modeled by:

\[ S^i_{n+1} = S^i_n + \sigma^i \sqrt{\Delta t} \epsilon^i_{n+1}, \]

where \( (\sigma^1 \epsilon^1_n, \ldots, \sigma^d \epsilon^d_n)_n \) are i.i.d. \( N(0, \Sigma) \) random variables.

The amount of cash obtained by the trader for the \( v^i_{n+1} \Delta t \) shares of stock \( i \) he sold over \( (t_n, t_{n+1}] \) depends on \( v^i_{n+1} \Delta t \) itself and on the market volume for stock \( i \) over \( (t_n, t_{n+1}] \), assumed to be \( V^i_{n+1} \Delta t \).

The resulting dynamics for the cash account is:

\[ X_{n+1} = X_n + \sum_{i=1}^d \left( v^i_{n+1} S^i_n - L^i \left( \frac{v^i_{n+1}}{V^i_{n+1}} \right) V^i_{n+1} \Delta t \right), \quad X_0 = 0. \]
The optimization criterion we consider is:

\[ \mathbb{E}[\exp(-\gamma X_N)]. \]

As above, \( X_N \) can be computed:

\[
X_N = \sum_{i=1}^{d} \left( q_i^0 S_0^i + \sigma^i \sqrt{\Delta t} \sum_{n=0}^{N-1} q_{n+1}^i \frac{\epsilon_{n+1}}{V_{n+1}} - \sum_{n=0}^{N-1} L^i \left( \frac{v_{n+1}^i}{V_{n+1}} \right) V_{n+1} \Delta t \right).
\]

Hence, \( X_N \) is a Gaussian variable with mean

\[
\sum_{i=1}^{d} \left( q_i^0 S_0^i - \sum_{n=0}^{N-1} L^i \left( \frac{v_{n+1}^i}{V_{n+1}} \right) V_{n+1} \Delta t \right)
\]

and variance

\[
\Delta t \sum_{n=0}^{N-1} q_{n+1}^i \Sigma q_{n+1}.
\]

Therefore,

\[
\mathbb{E}[\exp(-\gamma X_N)] = -\exp \left( -\gamma \left( \sum_{i=1}^{d} \left( q_i^0 S_0^i - \sum_{n=0}^{N-1} L^i \left( \frac{v_{n+1}^i}{V_{n+1}} \right) V_{n+1} \Delta t \right) - \frac{\gamma}{2} \Delta t \sum_{n=0}^{N-1} q_{n+1}^i \Sigma q_{n+1} \right) \right),
\]

and the problem boils down to minimizing

\[
\sum_{i=1}^{d} \sum_{n=0}^{N-1} \left( L^i \left( \frac{q_{n+1}^i - q_{n+1}^i}{V_{n+1}} \right) V_{n+1} \Delta t \right) \right) + \frac{\gamma}{2} \Delta t \sum_{n=0}^{N-1} q_{n+1}^i \Sigma q_{n+1} \Delta t,
\]

over

\[
\{(q^1, \ldots, q^d) \in (\mathbb{R}^{N+1})^d, \forall i, q_0^i = q_0^i, q_N^i = 0\}.
\]

Now, using the parity of execution functions, this is exactly problem \((\tilde{P})\) and the associated Hamiltonian system is:

\[
(\tilde{S}_H^i) : \begin{cases}
p_{n+1} = p_n + \Delta t \gamma \Sigma q_{n+1}, & 0 \leq n < N - 1 \\
q_{n+1}^i = q_n^i + \Delta t V_{n+1}^i H^{i*}(p_{n}^i), & 0 \leq n < N, \forall i \in \{1, \ldots, d\}
\end{cases}
\]

\[q_0 = q_0^i, \quad q_N = 0.\]

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