NULL CONTROLLABILITY FOR DISTRIBUTED SYSTEMS WITH TIME-VARYING CONSTRAINT AND APPLICATIONS TO PARABOLIC-LIKE EQUATIONS

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ABSTRACT. We consider the null controllability problem for linear systems of the form \( y'(t) = Ay(t) + Bu(t) \) on a Hilbert space \( Y \). We suppose that the control operator \( B \) is bounded from the control space \( U \) to a larger extrapolation space containing \( Y \). The control \( u \) is constrained to lie in a time-varying bounded subset \( \Gamma(t) \subset U \). From a general existence result based on a selection theorem, we obtain various properties on local and global constrained null controllability. The existence of the time optimal control is established in a general framework. When the constraint set \( \Gamma(t) \) contains the origin in its interior at each \( t > 0 \), the local constrained property turns out to be equivalent to a weighted dual observability inequality of \( L^1 \) type with respect to the time variable. We treat also the problem of determining a steering control for general constraint sets \( \Gamma(t) \) in nonsmooth convex analysis context. Applications to the heat equation are treated for distributed and boundary controls under the assumptions that \( \Gamma(t) \) is a closed ball centered at the origin and its radius is time-varying.

1. Introduction, preliminary results.

1.1. Problem formulation and reference to the literature. In this paper, we treat the problem of null controllability for linear infinite dimensional distributed control systems such as

\[
\begin{align*}
  y'(t) &= Ay(t) + Bu(t), \\
  y(0) &= y_0.
\end{align*}
\]

We consider the situation where, at each instant \( t > 0 \), the control \( u \) is constrained to take on values in a preassigned subset \( \Gamma(t) \) of the control space \( U \) and the following issues will be studied: (a) Does there exist a constrained control \( u \) steering the system to the origin at some time \( T > 0 \)? (b) Assuming that such a control exists, can we characterize it by specific properties? (c) Otherwise, can we characterize an appropriate control which steers the system (1) as close as possible to 0 at time \( T \)? Such problems are naturally motivated by real world applications involving actuators with amplitude and rate limitations and challenging related issues have not been investigated clearly so far, even for the case where the time-varying

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constraints subsets are reduced to a time invariant one $\Gamma$. Let us recall some facts about the time-invariant case. Classical results are by now well known in the finite dimensional case. They depend on the assumptions on $\Gamma$ and they assert that controllability is equivalent, in a certain sense, to unconstrained controllability plus an additional condition due to the constraint (16). A major advance on the subject has been made in (35) where another point of view is presented. Here a finite dimensional selection theorem (12) is used and it leads to integral conditions in terms of the nontrivial solutions of the adjoint equation. In infinite dimension, we mention [1], [27] and [38] where various references about the pioneering attacks on the constrained controllability problem for special control systems are presented. In the abstract setting, constrained controllability of infinite dimensional systems has been treated, to the best of our knowledge, first in [37]. In this reference, the necessary and sufficient conditions for local controllability which have been obtained can be considered as an extension to the infinite-dimensional case of classical results, proved in the finite-dimensional context. Other abstract results are available in [1], [27] and [29]. In [1] and [27] the results which are obtained are similar to those established in [35] for multivariable systems with general constraints. In [29] the case where the control is contained in a prescribed ball centered at the origin is treated from a time optimal control view point. While the references above deal with state and control spaces which are supposed to be reflexive Banach spaces, some attention has been addressed to the delicate question where this assumption does not hold. See, for instance, [40]. Moreover, in the monograph [13] similar subtle questions on constrained controllability are treated in the context of time optimal and norm optimal problems for parabolic-like systems. In this setting, it is clearly pointed out the fact that some bounds on the control lead unavoidably to non-reflexive Banach spaces. Note that in all these references, it is supposed that the control operator $B$ is bounded. To the best of our knowledge, [6], [43] and [44] are the first works dealing with the question of constrained controllability with unbounded control operator $B$. While the general study in [43] and [44] is concerned with bang-bang control issues and the applications are focused on parabolic-like systems, the results in [6] are established in a more general framework and they are applied to various hyperbolic-like systems. Moreover, in [6] a general convex variational approach is performed in order to build a steering control. This approach has been already used in [5] when the control operator $B$ is bounded. Our aim is to treat the constrained null controllability with eventually unbounded control operator in a general framework. As we shall see in the applications, such systems with $B$ unbounded appear naturally when we model boundary control. Such a project has already been initiated in [6] and the additional contribution of the present paper is twofold. On the one hand, here the contributions are more general at an abstract level and we itemize them as follows.

- At each instant $t > 0$, the constraint control set $\Gamma(t)$ is general and time dependent. Beside the null controllability question, the time optimal control issue is discussed in a general framework. In addition, interesting questions such as null controllability from a measurable subset in $(0, T)$ and other kinds of time optimal control questions can be deduced from our general approach.
- The usual assumption that $0 \in \Gamma(t)$ is dispensed with. This enables us to treat the important case where the control of a system is realizable in only one direction. See, for instance [38]. As prototypical example of such a situation,
we mention the practical case where the controls are supposed to be positive ([33]).

- Under the assumption that \( \Gamma(t) \) contains 0 in its interior, we establish in the time-varying constraint context the fact that the constrained controllability can be reduced to some kind of weighed observability relative to the uncontrolled adjoint system.

- Exploiting the properties dealing with the constrained null controllability, we present some methods characterizing the appropriate control solution of one of the problems (a), (b), and (c) stated above. Note that the study of the existence of such a control has independent interest. By using nonsmooth convex analysis tools, we shall see that for general time-varying constraint sets \( \Gamma(t) \), such a control satisfies a maximum principle property. We consider also the associated time optimal control question in a general setting. Furthermore, when \( \Gamma(t) \) contains 0 in its interior, the maximum principle property turns out to be a general time-varying bang-bang property.

On the other hand, the applications treated in the present paper are drastically different from the ones given in [6] as follows.

- While the applications in [6] concern hyperbolic-like equations such as wave and beams equations, here we focus our study to the heat equation with distributed and boundary control.

- In our general approach, we recover and extend similar results which are mainly established in the case where \( \Gamma(t) \) coincides with a closed ball centered at the origin and having (eventually) time-varying radius.

The plan of the paper is as follows. In the remaining part of this section, we introduce our general abstract framework. The notion of admissible unbounded control operator \( B \) is given. The corresponding notion of admissible observation operator which is needed for its adjoint \( B^* \) is deduced by duality. We recall also the main results on unconstrained null controllability given in observability inequality form and we precise the various notions relative to the constrained null controllability. Moreover, we present a variational characterization of any control steering the system from a given initial state to the origin. In section 2 we treat Problem (a) and we extend the existing results to general time-varying constraint sets \( \Gamma(t) \) with unbounded admissible control operator. The problems (b) and (c) are addressed in the third section. The fourth section deals with the application of the results obtained in the preceding section to specific partial differential equations (PDEs) including heat equation with distributed and boundary control. In the last section, some further comments and concluding remarks are provided.

1.2. **Basic concepts and auxiliary results.** Let \( Y, U \) be two Hilbert spaces denoting the state space and the control space respectively. Also, let \( A \) be the infinitesimal generator of a linear \( C_0 \)-semigroup on \( Y \) denoted by \( (e^{tA})_{t \geq 0} \). Finally, let \( B \) denote the linear control operator which may be unbounded in the sense that it is bounded from \( U \) into a space larger than the state space \( Y \) as follows. We define the space \( Y_1 \) to be \( D(A) \) with the norm \( \| \cdot \|_1 \) given by

\[
\|y\|_1 = \| (\beta I - A)y \|_Y ,
\]

for some \( \beta \in \rho(A) \), the resolvent set of \( A \), and the space \( Y_{-1} \) to be the completion of \( Y \) with respect to the norm

\[
\|y\|_{-1} = \| (\beta I - A)^{-1}y \|_Y .
\]
It is easy to verify that the norm $\| \cdot \|_1$ is equivalent to the graph norm on $D(A)$. Using the graph norm topology in $D(A^*)$, we can as well define the analogue spaces $(Y')_1$ and $(Y')_{-1}$. By duality, we can check that ([32], Remark 2.1)

$$D(A)' = (Y_1)' = (Y)'_{-1}, \quad (Y_{-1})' = (Y')_1 = D(A^*).$$

Exploiting these facts, we note that the duality product $\langle \cdot, \cdot \rangle_{Y', Y}$ can be expressed by the duality product $\langle \cdot, \cdot \rangle_{(Y')_1, Y_{-1}}$ as follows

$$\langle y', y \rangle_{Y', Y} = \langle (\overline{\beta}I - A^*)^{-1}y', (\beta I - A)y \rangle_{(Y')_1, Y_{-1}}, \quad y \in Y, \ y' \in Y',$$

and

$$\langle y', y \rangle_{(Y')_1, Y_{-1}} = \langle (\overline{\beta}I - A^*)y', (\beta I - A)^{-1}y \rangle_{Y', Y}, \quad y \in Y_{-1}, \ y' \in (Y')_1,$$

for some $\beta \in \rho(A)$, where $\overline{\beta}$ stands for the conjugate of the complex number $\beta$. It is clear that both duality pairings coincide when $y \in Y_1$ and $y' \in (Y')_1$ so that they both will be denoted by $\langle \cdot, \cdot \rangle$ throughout this paper.

We consider the abstract control system (1) where $u \in L^p_{loc}(0, \infty; U)$, $1 < p < \infty$, $y_0 \in Y$ and $B$ is bounded from $U$ to $D(A^*)$. Let $y(\cdot; y_0, u)$ denote the solution of (1). Considering formally the variation of constant formula

$$y(t; y_0, u) = e^{tA}y_0 + \int_0^t e^{(t-s)A}Bu(s)ds, \quad (4)$$

we introduce for any $T > 0$ the operator $L_T$ defined by

$$L_Tu = \int_0^T e^{(T-t)A}Bu(t)dt. \quad (5)$$

It is clear that $L_T \in \mathcal{L}(L^p(0, T; U), D(A^*))$. Throughout this paper, we suppose that the control operator $B$ is $p$-admissible in the sense that it satisfies for some (and hence any) $T > 0$ and for any $u \in L^p(0, T; U)$ one of the following equivalent admissibility conditions

$$L_Tu = \int_0^T e^{(T-t)A}Bu(t)dt \in Y, \quad (6)$$

$$\|L_Tu\|_Y = \left\| \int_0^T e^{(T-t)A}Bu(t)dt \right\|_Y \leq C_T \|u\|_{L^p(0, T; U)} \quad (7)$$

for some positive constant $C_T$. It is well known that (6) or (7) ensures the existence of a unique solution for (1) in the class $y \in C(0, T; Y)$ ([48]). Moreover, we get from [48] and [49] the fact that whenever (6) holds for any $u \in L^1(0, T; U)$, then the control operator $B$ is necessarily bounded from $U$ to $Y$. On the other hand, identifying $U$ with its dual, we have $B^* \in \mathcal{L}(D(A^*), U)$. Let $1 < q < \infty$ be the conjugate exponent of $p$ satisfying

$$\frac{1}{p} + \frac{1}{q} = 1. \quad (8)$$

Then, considering $B^*$ as observation operator, we introduce the dual $q$-admissibility condition ([49])

$$\int_0^T \|B^* e^{tA^*} \varphi_0\|_U^q dt \leq C_T \|\varphi_0\|_{q'}, \quad \text{for all } \varphi_0 \in D(A^*), \quad (9)$$
for some positive constant $C_T$. Note that the output function
\[ z(t) = B^* e^{tA^*} \varphi_0, \quad t > 0, \quad (10) \]
is well defined for $\varphi_0 \in D(A^*)$. The admissibility condition (9) implies that for any $T > 0$, the expression in (10), considered as a function of the variable $\varphi_0$, can be extended to a linear continuous operator from $Y'$ to $L^q(0; T; U)$. It is well known that the $p$-admissibility of $B$ as control operator is equivalent to the dual $q$-admissibility of $B^*$ as observation operator ([49]). Moreover, it is easy to see that the adjoint operator $L_T^*: D(A^*) \to L^q(0; T; U)$ is given by
\[ L_T^* \varphi_0(t) = B^* e^{(T-t)A^*} \varphi_0, \quad 0 < t < T. \quad (11) \]

Then, (11) means that $L_T^*$ admits a continuous extension, still denoted by $L_T^*$, from $Y'$ to $L^q(0; T; U)$. In particular, given a prescribed time $T > 0$, we shall be concerned with the final state
\[ y(T; y_0, u) = e^{TA}y_0 + \int_0^T e^{(T-t)A} Bu(t) dt \]
and the following space of reachable states from $y_0$ in time $T$
\[ R(y_0, T, p) = \{ y(T; y_0, u) : u \in L^p(0; T; U) \}. \quad (12) \]

Then the notion of unconstrained null controllability is defined as follows.

**Definition 1.1.** The system (1) (or the pair $(A, B)$) is $p$-null controllable in time $T$ if given any initial state $y_0$, the set of reachable states $R(T, y_0, p)$ contains 0.

Combining the admissibility of the operators $B$ and $B^*$ as control and observation operators respectively, we state the following variational formulation of the fact that the control $u \in L^p_{\text{loc}}(0, \infty; U)$ steers the system to the origin. See, for instance, [6]. The origin is reached from $y_0$ in time $T$ by a control $u \in L^p(0; T; U)$ if and only if,
\[ \int_0^T \left\langle u(t), B^* e^{(T-t)A^*} \varphi_0 \right\rangle_U dt + \left\langle e^{TA^*} \varphi_0, y_0 \right\rangle_U = 0 \quad \text{for all } \varphi_0 \in Y'. \quad (13) \]

The following remark precises the fact that the notion of controllability can be related to an observability inequality. We shall obtain a similar property in the context of constrained control.

**Remark 1.** It is well known that the $p$-null controllability is equivalent to the following final state $L^q$-observability inequality. See ([10], Theorem 3.25 (d), p. 71) and ([11], Theorem 2.3)
\[ \int_0^T \left\| B^* e^{(T-t)A^*} \varphi_0 \right\|_U^q dt \geq c_q \left\| e^{TA^*} \varphi_0 \right\|_{Y'}^q \quad \text{for all } \varphi_0 \in Y', \quad (14) \]
for some positive constant $c_q$.

Let us precise the framework of constrained controllability related to the family of constraint subsets $\Gamma(t)$ introduced above. We suppose that this family satisfies the assumption (H$_\Gamma$) specified by
- $\Gamma(t)$ is closed convex for each $t > 0$.
- there exists a closed bounded subset $\Gamma_\infty$ in $U$ such that $\Gamma(t) \subset \Gamma_\infty$ for all $t > 0$. 

Let \( \overline{\Gamma}_p \) denote the set of admissible control functions \( u \) in \( L^p_{loc}((0, \infty); U) \) such that \( u(t) \in \Gamma(t) \) almost everywhere in \((0, \infty)\). Also, \( \overline{\Gamma}_p(T) \) will denote the restriction of \( \overline{\Gamma}_p \) to the interval \([0, T] \). Any control function in \( \overline{\Gamma}_p \) or \( \overline{\Gamma}_p(T) \) will be referred to as \( (\Gamma(\cdot), p) \)-admissible control. We introduce the following set of reachable states with \( (\Gamma(\cdot), p) \)-admissible controls

\[
R_T(T, y_0, p) = \left\{ y(T; y_0, u) : u \in \overline{\Gamma}_p(T) \right\}.
\]  

(15)

Then we define the constrained null controllability notions as follows.

**Definition 1.2.** The system (1) is \( (\Gamma(\cdot), p) \)-null controllable in time \( T \) at \( y_0 \) if there exists a control \( u \in \overline{\Gamma}_p(T) \) such that \( y(T; y_0, u) = 0 \).

**Definition 1.3.** The system (1) is locally \( (\Gamma(\cdot), p) \)-null controllable in time \( T \) if there exists an open subset \( V \subset Y \), containing the origin, such that (1) is \( (\Gamma(\cdot), p) \)-null controllable in time \( T \) at any \( y_0 \in V \).

Following the framework used in the finite dimensional systems literature, we can enlarge the notion of \( (\Gamma(\cdot), p) \)-null controllability by varying the final time \( T \).

**Definition 1.4.** The system (1) is said to be globally \( (\Gamma(\cdot), p) \)-null controllable in finite time if for any initial state \( y_0 \in Y \), there exists a final time \( T = T(y_0) \) in which the system (1) is \( (\Gamma(\cdot), p) \)-null controllable at \( y_0 \).

The following result gives basic properties fulfilled by the set of admissible controls \( \overline{\Gamma}_p(T) \) and the corresponding set of reachable states \( R_T(T, y_0, p) \). These properties will be used throughout this paper. Note that the case \( p = \infty \) can be seen as a degenerate one for which the conclusions of the lemma are not verified.

**Lemma 1.5.** Suppose that \( (H_T) \) holds. Then \( \overline{\Gamma}_p(T) \) and \( R_T(T, y_0, p) \) are weakly compact.

**Proof.** Clearly \( (H_T) \) implies that \( \overline{\Gamma}_p(T) \) is convex and bounded. Also, \( \overline{\Gamma}_p(T) \) is strongly closed. Indeed, Let \( \{f_n\} \) be sequence in \( \overline{\Gamma}_p(T) \) such that \( f_n \to f \) strongly in \( L^p(0, T; U) \). It is well known that, for some subsequence still denoted by \( \{f_n\} \), we have \( f_n(t) \to f(t) \) almost everywhere in \((0, T) \). Since \( f_n(t) \in \Gamma(t) \) almost everywhere in \((0, T) \) and \( \Gamma(t) \) is closed, it follows that \( f_n(t) \in \Gamma(t) \) almost everywhere in \((0, T) \) so that \( f \in \overline{\Gamma}_p(T) \). Moreover, as a convex subset, \( \overline{\Gamma}_p(T) \) is weakly closed. Then the weak compactness property for \( \overline{\Gamma}_p(T) \) is a ready consequence of the facts that \( 1 < p < \infty \) and \( L^p(0, T; U) \) is a reflexive Banach space. On the other hand, the \( p \)-admissibility condition of the control operator \( B \) formulated by (7) implies that \( L_T \) given by (5) is linear continuous from \( L^p(0, T; U) \) to \( Y \). Thus, \( L_T \) is also linear continuous with respect to the corresponding weak topologies, which yields the weak compactness of \( L_T(\overline{\Gamma}_p(T)) \) and its translate \( R_T(T, y_0, p) \).

\[ \square \]

2. Null controllability with time-varying constraint.

2.1. Preliminaries, measurable selection theorem. We recall some basic ideas on calculus of measurable maps in the context defined by the time-varying constraint subsets \( \{ \Gamma(t) \}_{t>0} \). To this end, we shall treat these subsets as a set-valued map denoted by \( \Gamma(\cdot) : (0, \infty) \rightharpoonup U \). If such a map has closed images, then it is said to be measurable if the inverse of each open set in \( U \) is a measurable set in \((0, \infty)\). Then
we introduce for each $t > 0$ the support function of $\Gamma(t)$ defined by $S_T(t,.) : U \to \mathbb{R}$ which for any $u \in U$ is given by

$$S_T(t,u) = \sup_{v \in \Gamma(t)} \langle u,v \rangle_U. \quad (16)$$

The resulting function $S_T(.,.) : (0,\infty) \times U \to \mathbb{R}$ possesses the following properties.

**Lemma 2.1.** Suppose that the set-valued map $\Gamma(.)$ satisfies the assumption $(H_T)$. Then the function $S_T$ is a Carathéodory map on $(0,\infty) \times U$ in the sense that for each $t > 0$, the function $S_T(t,.) : u \mapsto S_T(t,u)$ is continuous on $U$ and for each $u \in U$, the function $S_T(.,u) : t \mapsto S_T(t,u)$ is measurable on $(0,\infty)$. Moreover, for any measurable function $w : (0,\infty) \to U$, the function defined by $t \mapsto S_T(t,w(t))$ is measurable on $(0,\infty)$.

*Proof.* Since $\Gamma(t)$ is closed convex for each $t > 0$, it follows that $\Gamma(.)$ is a measurable set-valued function on $(0,\infty)$ ([4], Theorem 8.2.2, p. 311). Furthermore, by using Theorem 8.2.14 in ([4], p. 318), the support function $S_T(.,u)$ is measurable for any $u \in U$. On the other hand, we deduce from $(H_T)$ that $S_T(t,.)$ is convex and there exists a non-empty open set in $U$ over which it is bounded above so that it is locally Lipschitz for any $t > 0$ ([12], p. 12). Thus, $S_T(.,.)$ is Carathéodory map on $(0,\infty) \times U$. Hence, by virtue of Lemma 8.2.3 in ([4], p. 311), it follows that for every measurable function $w : (0,\infty) \to U$, the map $t \mapsto S_T(t,w(t))$ is also measurable on $(0,\infty)$. This completes the proof of the lemma. \qed

Taking into account the admissibility condition (9), the function

$$t \mapsto h(t) = S_T \left( t, B^* e^{(T-t)A^*} \varphi_0 \right) \quad (17)$$

is well defined in $L^1_{loc}(0,\infty)$ for any $\varphi_0 \in Y'$. We shall use a general version of the selection theorem in the infinite dimensional context. Note that the selection theorem stated in [2] deals with finite dimensional spaces and it has been exploited for constrained controllability purposes in [35]. Many versions of the selection theorem are disseminated in various references. Below we shall present the one which will be convenient in the context of unbounded admissible control operator.

**Lemma 2.2.** Suppose that $\Gamma(.)$ satisfies the assumption $(H_T)$. Then, for any $\varphi_0 \in Y'$, there exists a measurable function $\hat{v}$ such that $\hat{v} \in \Gamma_T(p)$ and

$$S_T \left( t, B^* e^{(T-t)A^*} \varphi_0 \right) = \langle \hat{v}(t), B^* e^{(T-t)A^*} \varphi_0 \rangle_U \quad a.e \ on \ (0,T). \quad (18)$$

*Proof.* Let us consider the functional $g : (0,\infty) \times U \to \mathbb{R}$ given by

$$g(t,u) = \langle u, B^* e^{(T-t)A^*} \varphi_0 \rangle_U. \quad (19)$$

For each $u \in U$, the real valued function $t \mapsto g(t,u)$ is measurable. Furthermore, by the admissibility condition (9) we have after extension to $Y'$, $B^* e^{(T-t)A^*} \varphi_0 \in U$ for almost all $t > 0$ so that the function $u \mapsto g(t,u)$ is continuous on $U$. Hence, the function $g$ is a Carathéodory map. On the other hand, the function $h$ introduced in (17) is a measurable function and, at each instant $t > 0$, $\Gamma(t)$ is closed, bounded and convex so that it is also weakly compact in $U$. Moreover, since, for almost all $0 < t < T$, the linear functional

$$u \mapsto \langle u, B^* e^{(T-t)A^*} \varphi_0 \rangle_U$$
is continuous on \( U \) with respect to the weak topology, it follows that
\[
h(t) = \left\langle u_t, B^* e^{(T-t)A^*} \varphi_0 \right\rangle_U,
\]
for some \( u_t \in \Gamma(t) \). Thus, \( h(t) \in g(t, \Gamma(t)) \) for almost all \( t \geq 0 \). By using Filippov’s theorem in ([4], p. 316), it follows that there exists a measurable selection \( \tilde{v}(t) \in \Gamma(t) \) such that
\[
h(t) = g(t, \tilde{v}(t)) \text{ a.e on } (0, T).
\]
This yields (18). Obviously, \( \tilde{v} \in \tilde{\Gamma}_T(p) \) and this completes the proof of the lemma.

2.2. Null controllability under general time-varying constraint sets. The measurable selection theorem formulated in Lemma 2.2 enables us to establish the following \((\Gamma(.), p)\)-null controllability result under general varying constraint sets \( \Gamma(t), t > 0 \). Our formulation is based on the functional \( J_{(\cdot)}(\cdot; T, y_0) : Y' \to \mathbb{R} \) defined by
\[
J_{(\cdot)}(\varphi_0; y_0, T) = \int_0^T S_T \left(t, B^* e^{(T-t)A^*} \varphi_0 \right) dt + \left\langle e^{TA^*} \varphi_0, y_0 \right\rangle.
\]

**Theorem 2.3.** Let \( \Gamma : (0, \infty) \to U \) be a multi-valued function satisfying (H\(\Gamma\)). Suppose that for any \( t > 0 \), \( Bu = 0 \) for some \( u \in \Gamma(t) \). Then, the system (1) is \((\Gamma(.), p)\)-null controllable in time \( T \) at \( y_0 \) if and only if
\[
J_{(\cdot)}(\varphi_0; y_0, T) \geq 0
\]
for all \( \varphi_0 \in Y' \).

**Proof.** From Lemma 1.5 it follows that \( R_T(y_0, T, p) \) is a closed bounded convex subset of \( Y \). On the other hand, the system (1) is \((\Gamma(.), p)\)-null controllable at \( y_0 \) in time \( T \) if and only if \( 0 \in R_T(y_0, T, p) \). Then we use a corollary of Hahn-Banach separation theorem stated in terms of inequality between appropriate support functionals. Let \( S_R : Y \to \mathbb{R} \) and \( S_0 : Y \to \mathbb{R} \) denote the support functionals associated to \( R_T(y_0, T, p) \) and \( \{0\} \) respectively. Reducing the problem to the inclusion \( \{0\} \subset R_T(y_0, T, p) \) and using ([9], Proposition 2.42, p. 43), we obtain that the system (1.1) is \((\Gamma(.), p)\)-null controllable at \( y_0 \) in time \( T \) if and only if for any \( \varphi_0 \in Y' \)
\[
S_R(\varphi_0) \geq S_0(\varphi_0).
\]
This inequality means that
\[
\left\langle \varphi_0, e^{TA} y_0 \right\rangle + \sup_{u_0 \in \tilde{\Gamma}_T(p)} \left\{ \varphi_0, \int_0^T e^{(T-t)A} Bu(t) dt \right\} \geq 0,
\]
or, equivalently,
\[
\sup_{u_0 \in \tilde{\Gamma}_T(p)} \left\{ \varphi_0, \int_0^T e^{(T-t)A} Bu(t) dt \right\} + \left\langle e^{TA^*} \varphi_0, y_0 \right\rangle \geq 0.
\]
It follows that the proof of the theorem is complete provided that
\[
\int_0^T S_T \left(t, B^* e^{(T-t)A^*} \varphi_0 \right) dt = \sup_{u_0 \in \tilde{\Gamma}_T(p)} \left\{ \varphi_0, \int_0^T e^{(T-t)A} Bu(t) dt \right\}.
\]
By using the admissibility conditions (7), (8) and (9), the equality (25) amounts to

\[ \int_0^T S_T \left( t, B^* e^{(T-t)A^*} \varphi_0 \right) dt = \sup_{u \in \Gamma_T(p)} \left\{ \int_0^T \left\langle B^* e^{(T-t)A^*} \varphi_0, u(t) \right\rangle_U dt \right\} \]  

(26)

In order to establish (26), let us consider the functional \( F_T : L^p(0, T; U) \rightarrow \mathbb{R} \) given by

\[ F_T(u) = \int_0^T \left\langle B^* e^{(T-t)A^*} \varphi_0, u(t) \right\rangle_U dt. \]  

(27)

The admissibility condition (9) implies that \( F_T \) is linear continuous on \( L^p(0, T; U) \). Since \( \Gamma_T(p) \) is weakly compact in \( L^p(0, T; U) \), \( F_T \) attains its supremum on \( \Gamma_T(p) \) so that

\[ \sup_{u \in \Gamma_T(p)} \left\{ \int_0^T \left\langle B^* e^{(T-t)A^*} \varphi_0, u(t) \right\rangle_U dt \right\} = \int_0^T \left\langle B^* e^{(T-t)A^*} \varphi_0, u^0(t) \right\rangle_U dt, \]  

(28)

for some \( u^0 \in \Gamma_T(p) \). Furthermore, clearly we have

\[ \int_0^T S_T \left( t, B^* e^{(T-t)A^*} \varphi_0 \right) dt \geq \int_0^T \left\langle B^* e^{(T-t)A^*} \varphi_0, u^0(t) \right\rangle_U dt, \]

so that

\[ \int_0^T S_T \left( t, B^* e^{(T-t)A^*} \varphi_0 \right) dt \geq \sup_{u \in \Gamma_T(p)} \left\{ \int_0^T \left\langle B^* e^{(T-t)A^*} \varphi_0, u(t) \right\rangle_U dt \right\}. \]

On the other hand, Lemma 2.2 yields for some \( \tilde{v} \in \Gamma_T(p) \)

\[ \int_0^T S_T \left( t, B^* e^{(T-t)A^*} \varphi_0 \right) dt = \int_0^T \left\langle \tilde{v}(t), B^* e^{(T-t)A^*} \varphi_0 \right\rangle_U dt \]

\[ \leq \int_0^T \left\langle u^0(t), B^* e^{(T-t)A^*} \varphi_0 \right\rangle_U dt. \]

From (28) it follows that

\[ \int_0^T S_T \left( t, B^* e^{(T-t)A^*} \varphi_0 \right) dt \leq \sup_{u \in \Gamma_T(p)} \left\{ \int_0^T \left\langle B^* e^{(T-t)A^*} \varphi_0, u(t) \right\rangle_U dt \right\}. \]

Thus (26) is verified and this completes the proof of the theorem. \( \square \)

**Remark 2.** (i) Given any subset \( \Lambda \subset Y' \) containing the origin in its interior, it is easy to see by a positive homogeneity argument that the system (1) is \( (\Gamma(\cdot), p) \)-null controllable at \( y_0 \) in time \( T \) if and only if

\[ \inf_{\varphi_0 \in \Lambda} J_{\Gamma(\cdot)}(\varphi_0; y_0, T) = 0. \]

Let us anticipate by considering the particular case where \( \Lambda \) coincides with the closed unit ball in \( Y' \) defined by

\[ \Phi_1 = \{ \varphi_0 \in Y' : \| \varphi_0 \|_{Y'} \leq 1 \}. \]  

(29)

It follows that the system (1) is \( (\Gamma(\cdot), p) \)-null controllable at \( y_0 \) in time \( T \) if and only if

\[ \min_{\varphi_0 \in \Phi_1} J_{\Gamma(\cdot)}(\varphi_0; y_0, T) = 0. \]  

(30)
This characterization will play a crucial role for various questions related to the steering control.

(ii) By virtue of (5) and (11), the characterization property in Theorem 2.3 can be expressed by

\[ \int_0^T S_T(t, L_T^{*} \varphi_0(t)) \, dt + \left\langle e^{T A^*} \varphi_0, y_0 \right\rangle \geq 0 \text{ for all } \varphi_0 \in Y'. \quad (31) \]

**Corollary 1.** Suppose that the assumptions of Theorem 2.3 hold. Then the system (1) is locally \((\Gamma(\cdot), p)\)-null controllable in time \(T > 0\) if and only if for some \(c > 0\),

\[ \int_0^T S_T \left( t, B^* e^{(T-t) A^*} \varphi_0 \right) \, dt \geq c \left\| e^{T A^*} \varphi_0 \right\|_Y, \quad (32) \]

for all \(\varphi_0 \in Y'\).

**Proof.** From Theorem 2.3, the system (1) is locally \((\Gamma(\cdot), p)\)-null controllable in time \(T > 0\) if and only if for some \(r > 0\) and any initial state satisfying \(\|y_0\|_Y \leq r\), we have

\[ \int_0^T S_T \left( t, B^* e^{(T-t) A^*} \varphi_0 \right) \, dt + \left\langle \varphi_0, e^{T A} y_0 \right\rangle \geq 0, \forall \varphi_0 \in Y', \]

or, equivalently, for any \(\varphi_0 \in Y'\)

\[ \int_0^T S_T \left( t, B^* e^{(T-t) A^*} \varphi_0 \right) \, dt \geq \left\langle e^{T A^*} \varphi_0, y_0 \right\rangle, \]

for all \(y_0 \in Y\) satisfying \(\|y_0\|_Y \leq r\). Clearly, this amounts to

\[ \int_0^T S_T \left( t, B^* e^{(T-t) A^*} \varphi_0 \right) \, dt \geq r \left\| e^{T A^*} \varphi_0 \right\|_Y. \]

\(\square\)

**Remark 3.** From the proof, it turns out that the constant \(c\) in (32) can have the following interpretation. It can be seen as the radius \(r\) of a closed ball in \(Y\) centered at the origin and contained in the set of initial states which can be steered to the origin in time \(T\) by \((\Gamma(\cdot), p)\)-admissible controls.

Let us consider a useful situation where global \((\Gamma(\cdot), p)\)-null controllability holds whenever the system is locally \((\Gamma(\cdot), p)\)-null controllable. Recall that the semigroup \((e^{t A})_{t \geq 0}\) is said to be stable if for any \(z \in Y\), \(\|e^{t A} z\|_Y \rightarrow 0\) as \(t \rightarrow +\infty\).

**Theorem 2.4.** Suppose that the assumptions of Theorem 2.3 hold and the semigroup \((e^{t A})_{t \geq 0}\) is stable. Assume also that \(0 \in \Gamma(t)\) for all \(t > 0\) and the family \((\Gamma(t))_{t > 0}\) is increasing. If the system (1) is locally \((\Gamma(\cdot), p)\)-null controllable in some time \(T > 0\), then it is also globally \((\Gamma(\cdot), p)\)-null controllable.

**Proof.** If the system (1) is locally \((\Gamma(\cdot), p)\)-null controllable in some time \(T > 0\), then for some positive constant \(\delta > 0\), the system is \((\Gamma(\cdot), p)\)-null controllable at \(y_0\) in time \(T\) whenever \(\|y_0\| < \delta\). Then given an arbitrary initial state \(y_0 \in Y\), let \(T_1 = T_1(y_0) > 0\) be such that \(\|e^{T_1 A} y_0\| < \delta\) and consider the steering control \(v\) such that \(y(T; e^{T_1 A} y_0, v) = 0\). Let us introduce the control given by

\[ u(t) = \begin{cases} 
0 & \text{if } 0 < t < T_1, \\
v(t - T_1) & \text{if } T_1 \leq t \leq T_1 + T.
\end{cases} \quad (33) \]
Proof. Assume that (34) holds for some \( T > 0 \). Let us introduce the set of instants given by

\[ T = \left\{ t \in (0, T_f) : \min_{\varphi_0 \in \Phi_1} J_{\Gamma, t} (\varphi_0; y_0, T) = 0 \right\}. \]

Clearly \( T \) is a nonempty set and it has 0 as lower bound so that its infimum \( T^* := \inf T \) satisfies \( T^* \geq 0 \). We shall see later on that actually we have \( T^* > 0 \). Let \( (T_n)_{n \geq 1} \) a decreasing sequence in \( T \) such that \( T_n \to T^* \) as \( n \to \infty \). For a given \( \varphi_0 \in \Phi_1 \), we deduce from the proof of Lemma 2.2 that for any \( n \geq 1 \) there exists \( \tilde{\varphi}_n \in \Gamma_{T_n}(p) \) such that

\[ S_{\Gamma} \left( t, B^* e^{(T_n-t)A^*} \varphi_0 \right) = \left\langle \tilde{\varphi}_n(t), B^* e^{(T_n-t)A^*} \varphi_0 \right\rangle_U \quad \text{a.e. on } (0, T_n). \]

From

\[ J_{\Gamma, t} (\varphi_0; y_0, T_n) \geq 0 \]

we get for all \( n \geq 1 \)

\[ \int_0^{T_n} \left\langle \tilde{\varphi}_n(t), B^* e^{(T_n-t)A^*} \varphi_0 \right\rangle_U dt + \left\langle e^{T_n A^*} \varphi_0, y_0 \right\rangle \geq 0. \]

Then we consider the decomposition

\[ \int_0^{T_n} \left\langle \tilde{\varphi}_n(t), B^* e^{(T_n-t)A^*} \varphi_0 \right\rangle_U dt = \int_0^{T^*} \left\langle \tilde{\varphi}_n(t), B^* e^{(T_n-t)A^*} \varphi_0 \right\rangle_U dt + \int_{T^*}^{T_n} \left\langle \tilde{\varphi}_n(t), B^* e^{(T_n-t)A^*} \varphi_0 \right\rangle_U dt. \]
and we denote by $v^*_n$ the restriction of $\tilde{v}_n$ to the interval $(0, T^*)$. Since $v^*_n \in \overline{\Gamma}_{T^*}(p)$ for any $n \geq 1$, it follows by using Lemma 1.5 that for some subsequence still denoted by $\{v^*_n\}_n$, $v^*_n \rightarrow v^*$ weakly in $L^p(0, T^*; U)$ for some $(\Gamma, p)$-admissible control $v^* \in \overline{\Gamma}_{T^*}(p)$. On the other hand, we have

$$
\int_0^{T^*} \left\langle v^*_n(t), B^*e^{(T_n-t)A^*} \varphi_0 \right\rangle_U \, dt - \int_0^{T^*} \left\langle v^*(t), B^*e^{(T^*-t)A^*} \varphi_0 \right\rangle_U \, dt
$$

$$
= \int_0^{T^*} \left\langle v^*_n(t), B^* \left( e^{(T_n-t)A^*} \varphi_0 - e^{(T^*-t)A^*} \varphi_0 \right) \right\rangle_U \, dt
$$

$$
+ \int_0^{T^*} \left\langle v^*_n(t) - v^*(t), B^*e^{(T^*-t)A^*} \varphi_0 \right\rangle_U \, dt
$$

The $q$-admissibility of the observation operator $B^*$ implies that as $n \rightarrow \infty$,

$$
\int_0^{T^*} \left\langle v^*_n(t) - v^*(t), B^*e^{(T^*-t)A^*} \varphi_0 \right\rangle_U \, dt \rightarrow 0.
$$

Furthermore, we have

$$
\int_0^{T^*} \left\langle v^*_n(t), B^* \left( e^{(T_n-t)A^*} \varphi_0 - e^{(T^*-t)A^*} \varphi_0 \right) \right\rangle_U \, dt
$$

$$
= \int_0^{T^*} \left\langle v^*_n(t), B^*e^{(T^*-t)A^*} \left( e^{(T_n-t)A^*} \varphi_0 - \varphi_0 \right) \right\rangle_U \, dt.
$$

Combined with the extended $q$-admissibility condition (9) this equality gives

$$
\left| \int_0^{T^*} \left\langle v^*_n(t), B^*e^{(T^*-t)A^*} \left( e^{(T_n-t)A^*} \varphi_0 - \varphi_0 \right) \right\rangle_U \, dt \right|
$$

$$
\leq \int_0^{T^*} \left| \left\langle v^*_n(t), B^*e^{(T^*-t)A^*} \left( e^{(T_n-t)A^*} \varphi_0 - \varphi_0 \right) \right\rangle_U \right| \, dt
$$

$$
\leq \|v^*_n\|_{L^p(0, T^*; U)} \left( \int_0^{T^*} \left\| B^*e^{(T^*-t)A^*} \left( e^{(T_n-t)A^*} \varphi_0 - \varphi_0 \right) \right\|_U^q \, dt \right)^{\frac{1}{q}}
$$

$$
\leq C \|v^*_n\|_{L^p(0, T^*; U)} \|e^{(T_n-t)A^*} \varphi_0 - \varphi_0\|_Y^q,
$$

for some positive constant $C$. Since the sequence $\{v^*_n\}_n$ is bounded and $\|e^{(T_n-t)A^*} \varphi_0 - \varphi_0\|_Y \rightarrow 0$, it follows that as $n \rightarrow \infty$,

$$
\int_0^{T^*} \left\langle v^*_n(t), B^*e^{(T_n-t)A^*} \varphi_0 \right\rangle_U \, dt \rightarrow \int_0^{T^*} \left\langle v^*(t), B^*e^{(T^*-t)A^*} \varphi_0 \right\rangle_U \, dt. \quad (39)
$$

Moreover,

$$
\int_T^{T_n} \left\langle \tilde{v}_n(t), B^*e^{(T_n-t)A^*} \varphi_0 \right\rangle_U \, dt \leq \|\tilde{v}_n\|_{L^p(T^*, T_n; U)} \left( \int_0^{T_n} \left\| B^*e^{(T_n-t)A^*} \varphi_0 \right\|_U^q \, dt \right)^{\frac{1}{q}}
$$

and an elementary change of variable gives

$$
\int_0^{T_n} \left\| B^*e^{(T_n-t)A^*} \varphi_0 \right\|_U^q \, dt = \int_0^{T_n} \left\| B^*e^{tA^*} \varphi_0 \right\|_U^q \, dt
$$

$$
\leq \int_0^{T_1} \left\| B^*e^{tA^*} \varphi_0 \right\|_U^q \, dt.
$$
Hence for any $n \geq 1$ we obtain
\[
\left| \int_{T^n}^T \left\langle \tilde{v}_n(t), B^* e^{(T_n-t)A^*} \varphi_0 \right\rangle_U dt \right| \leq \|\tilde{v}_n\|_{L^p(T^n, T_n; U)} \left( \int_0^{T_1} \left\| B^* e^{tA^*} \varphi_0 \right\|_U^q dt \right)^{\frac{1}{q}}
\]
\[
\leq C \|\tilde{v}_n\|_{L^p(T^n, T_n; U)} \|\varphi_0\|_{Y'}
\]
for some positive constant $C$. By construction and following the assumption $(H_T)$, there exists clearly some positive constant $C_\infty$ such that for any $n \geq 1$
\[
\tilde{v}_n(t) \leq C_\infty \text{ a.e on } (0, T_n)
\]
so that
\[
\|\tilde{v}_n\|_{L^p(T^n, T_n; U)} \leq C_\infty |T_n - T^*|^\frac{1}{p}.
\]
Consequently, taking into account (38) and (39) we deduce that as $n \to \infty$
\[
J_{\Gamma(t)}(\varphi_0; y_0, T_n) = \int_0^{T_n} \left\langle \tilde{v}_n(t), B^* e^{(T_n-t)A^*} \varphi_0 \right\rangle_U dt + \left\langle e^{T_nA^*} \varphi_0, y_0 \right\rangle
\]
\[
\to \int_0^T \left\langle v^*(t), B^* e^{(T^*-t)A^*} \varphi_0 \right\rangle_U dt + \left\langle e^{T^*A^*} \varphi_0, y_0 \right\rangle
\]
so that by using (36) we get
\[
J_{\Gamma(t)}(\varphi_0; y_0, T^*) \geq \int_0^T \left\langle v^*(t), B^* e^{(T^*-t)A^*} \varphi_0 \right\rangle_U dt + \left\langle e^{T^*A^*} \varphi_0, y_0 \right\rangle \geq 0.
\]
Furthermore, the inequality $T^* > 0$ holds since otherwise we get the excluded trivial case $e^{T^*A}y_0 = 0$ resulting from
\[
\left\langle \varphi_0, e^{T^*A}y_0 \right\rangle \geq 0 \text{ for all } \varphi_0 \in \Phi_1.
\]
The converse part of the theorem is obvious. This completes the proof of the theorem. \hfill \square

2.4. **The case $0 \in \text{int}(\Gamma(t))$.** Let us examine more closely the special case which arises under the strengthened assumption $0 \in \text{int}(\Gamma(t))$ in the prototypical situation where at each instant $t > 0$, $\Gamma(t)$ is a closed ball centered at the origin. To this end, we introduce
\[
L^\infty_+(0, \infty) = \{ m \in L^\infty(0, \infty) : m(t) > 0 \text{ a.e on } (0, \infty) \},
\]
and we define for a given function $m \in L^\infty_+(0, \infty)$ the time-varying constraint control set denoted by $\Gamma_{m(t)}$ and given by
\[
\Gamma_{m(t)} = \{ u \in U : \|u\|_U \leq m(t) \}.
\]
Then clearly $\Gamma_{m(t)}$ satisfies $(H_T)$ and, for each $t > 0$, the analogue of the corresponding support function in (17), denoted by $S_{\Gamma_{m(t)}}$, is
\[
S_{\Gamma_{m(t)}}(u) = S_{\Gamma_{m(t)}}(t, u) = m(t) \|u\|_U \text{ for all } u \in U.
\]
Hence the corresponding functional $J_{\Gamma_{m(t)}}$ is given by
\[
J_{\Gamma_{m(t)}}(\varphi_0; y_0, T) = \int_0^T m(t) \left\| B^* e^{(T-t)A^*} \varphi_0 \right\|_U dt + \left\langle e^{TA^*} \varphi_0, y_0 \right\rangle.
\]
Hence, the results concerned with $\Gamma_{m(t)}$-null controllability can be stated as follows.
Corollary 2. The system (1) is locally $(\Gamma_{m(\cdot), p})$-null controllable in time $T > 0$ if and only if for some $c_1 > 0$

$$\int_0^T m(t) \left\| B^* e^{(T-t)A^*} \varphi_0 \right\|_U \, dt \geq c_1 \left\| e^{TA^*} \varphi_0 \right\|_Y,$$

(45)

for all $\varphi_0 \in Y'$.

Let us introduce for any $r > 0$ the closed ball

$$\Gamma_r = \{ u \in U : \| u \|_U \leq r \}.$$  

(46)

In the case where at each instant $t > 0$, $\Gamma(t)$ contains $\Gamma_r$ for some $r > 0$, then the assumption (H$\Gamma$) implies that there exists $R > 0$ such that

$$\Gamma_r \subset \Gamma(t) \subset \Gamma_R,$$

(47)

so that at any $t > 0$

$$S_{\Gamma_r}(u) = r \| u \|_U \leq S_{\Gamma(t,u)} \leq S_{\Gamma_R}(u) = R \| u \|_U,$$

(48)

for all $u \in U$. Then we get easily from Corollary 2 that for $(\Gamma(\cdot), p)$-null controllability question, the structure of the sets $\{ \Gamma(t) \}_{t>0}$ will not matter. Hence, our time-varying $(\Gamma(\cdot), p)$-null controllability problem can be reduced to a canonical one in which the constraint set coincides with the closed unit ball $\Gamma_1$. Moreover, it will be interesting to reformulate the results above in this situation. Indeed, in this case, the support function of the canonical constraint set $\Gamma_1$ has the more familiar expression

$$S_{\Gamma_1}(u) = \| u \|_U$$

(49)

so that the corresponding functional $J_{\Gamma_1}$ is given by

$$J_{\Gamma_1}(\varphi_0; y_0, T) = \int_0^T \left\| B^* e^{(T-t)A^*} \varphi_0 \right\|_U \, dt + \left\langle e^{TA^*} \varphi_0, y_0 \right\rangle.$$  

(50)

Hence, the results concerned with $(\Gamma_1, p)$-null controllability can be stated as follows.

Corollary 3. The system (1) is locally $(\Gamma_1, p)$-null controllable in time $T > 0$ if and only if for some $c_1 > 0$

$$\int_0^T \left\| B^* e^{tA^*} \varphi_0 \right\|_U \, dt \geq c_1 \left\| e^{T A^*} \varphi_0 \right\|_Y,$$

(51)

for all $\varphi_0 \in Y'$.

Remark 4. The inequality (51) is well known as final state $L^1(0,T)$-observability property for the system (1). Hence we conclude from Corollary 3 that final state $L^1(0,T)$-observability and local $(\Gamma_1, p)$-null controllability in time $T$ are equivalent. Similarly, considering the inequality (45), the local $(\Gamma_{m(\cdot), p})$-null controllability can be viewed as equivalent to a weighted final state $L^1(0,T)$-observability with respect to the weight $m$.

Remark 5. If we set

$$m_\infty = \| m \|_{L^\infty(0,\infty)},$$

(52)

then, the system (1.1) is locally $(\Gamma_{m_\infty}, p)$-null controllable in time $T > 0$ provided that it is locally $(\Gamma_{m(\cdot), p})$-null controllable in time $T$. Moreover, taking into account $\Gamma_{m(\cdot)} \subset \Gamma_{m_\infty}$ (47), (48) and Corollary 2, it follows that whenever the system (1) is locally $(\Gamma_{m(\cdot), p})$-null controllable in time $T$, then it is also locally $(\Gamma_r, p)$-null controllable in time $T$ for any $r > 0$. 


2.5. **Null controllability from measurable set in time.** Here we shall consider an interesting situation where, beside the time-varying control constraint sets \( \Gamma(t) \), the constrained control is activated only over a measurable subset \( E \subset (0, T) \) with measure \( \mu(E) > 0 \). Hence we are led to consider the system

\[
\begin{align*}
y'(t) &= Ay(t) + B\chi_Eu(t), \\
y(0) &= y_0,
\end{align*}
\]

where \( \chi_E \) denotes the characteristic function of \( E \). Then it follows that the constrained null controllability for (53) can be reduced to the one for (1) under appropriate time-varying control constraint sets denoted by \( \Gamma^E(t) \). Indeed, it is easy to see that if we set

\[
\Gamma^E(t) = \begin{cases} 
\Gamma(t) & \text{if } t \in E, \\
\{0\} & \text{if } t \notin E,
\end{cases}
\]

then the resulting constrained null controllability of the system (53) amounts to the \((\Gamma^E(.), p)\)-null controllability of the system (1). We can proceed as in subsections 2.1 and 2.2, by introducing for each \( t > 0 \) the corresponding support function of \( \Gamma^E(t) \) given by \( S_{\Gamma^E}(t, u) : U \to \mathbb{R} \)

\[
S_{\Gamma^E}(t, u) = \sup_{v \in \Gamma^E(t)} \langle u, v \rangle_U.
\]

Then the corresponding functional \( J_{\Gamma^E(.)}(\varphi_0; y_0, T) : Y' \to \mathbb{R} \) is clearly defined by

\[
J_{\Gamma^E(.)}(\varphi_0; y_0, T) = \int_E S_{\Gamma^E(t)} \left( t, B^* e^{(T-t)A^*} \varphi_0 \right) dt + \left\langle e^{TA^*} \varphi_0, y_0 \right\rangle.
\]

From Theorem 2.3 and Corollary 1 we get

**Corollary 4.** Suppose that the set-valued map \( \Gamma^E : (0, \infty) \to U \) satisfies \((H_\Gamma)\). Suppose that for any \( t > 0 \), \( Bu = 0 \) for some \( u \in \Gamma^E(t) \). Then, the system (1) is \((\Gamma^E(.), p)\)-null controllable in time \( T \) at \( y_0 \) if and only if

\[
J_{\Gamma^E(.)}(\varphi_0; y_0, T) \geq 0
\]

for all \( \varphi_0 \in Y' \).

**Corollary 5.** Suppose that the assumptions of Corollary 4 hold. Then the system (1) is locally \((\Gamma^E(.), p)\)-null controllable in time \( T > 0 \) if and only if for some \( c > 0 \),

\[
\int_E S_{\Gamma} \left( t, B^* e^{(T-t)A^*} \varphi_0 \right) dt \geq c \left\| e^{TA^*} \varphi_0 \right\|_{Y'},
\]

for all \( \varphi_0 \in Y' \).

**Remark 6.** We can as well treat the case where the control from a measurable set in time \( E \) is contained in closed balls with time-varying radius so that at each instant \( t \in E \), we have \( u(t) \in \Gamma_{m(t)} \) with \( \Gamma_{m(t)} \) defined by (41) and (42). Hence, we are led to introduce for \( m \in L^\infty_t (0, \infty) \) the associated function \( m_E \) defined by

\[
m_E(t) = \begin{cases} 
m(t) & \text{if } t \in E, \\
0 & \text{if } t \notin E.
\end{cases}
\]

Then the resulting constrained null controllability of the system (53) amounts to the \((\Gamma_{m_E(.)}, p)-null controllability of the system (1). In particular, the system (1) is locally \((\Gamma_{m_E(.)}, p)-null controllable in time \( T > 0 \) if and only if for some \( c_1 > 0 \)

\[
\int_0^T m_E(t) \left\| B^* e^{(T-t)A^*} \varphi_0 \right\|_U dt = \int_E m(t) \left\| B^* e^{(T-t)A^*} \varphi_0 \right\|_U dt
\]
\[ \geq c_1 \left\| e^{TA^* \varphi_0} \right\|_Y, \]  

for all \( \varphi_0 \in Y' \).

**Remark 7.** Beside the time optimal control problem introduced in subsection 2.3, there exists another one in the situation where we try to reach the origin at a fixed time \( T > 0 \) while delaying the activation of the constrained control until some instant \( 0 < \tau < T \). Hence the system under study would have the form

\[
\begin{cases}
    y'(t) = Ay(t) + B\chi_{(\tau,T)}u(t), \\
    y(0) = y_0,
\end{cases}
\]

where \( \chi_{(\tau,T)} \) denotes the characteristic function of the subinterval \( (\tau,T) \). The system (61) has the form (53) if we set \( E = (\tau,T) \). There is a kind of time optimal control problem whose aim is to delay initiation of active constrained control in (61) as late as possible, such that the corresponding solution reaches the origin by a fixed ending time \( T \). Considering the associated functional given by

\[
J_{T(\tau,T)}(\varphi_0; y_0, T) = \int_{\tau}^{T} S_T \left( t, B^* e^{(T-t)A^*} \varphi_0 \right) dt + \left\langle e^{TA^* \varphi_0}, y_0 \right\rangle, \]

then by proceeding as in Theorem 2.5, it is easy to establish that this second version of time optimal control problem admits a solution if and only if there exists some initial activating time \( 0 < \tau_0 < T \) such that

\[
\min_{\varphi_0 \in \Phi_1} \left\{ J_{T(\tau_0,T)}(\varphi_0; y_0, T) : \varphi_0 \in \Phi_1 \right\} = 0. \]

Moreover, the optimal time, denoted by \( \tau^* \), is given by

\[
\tau^* = \max \left\{ \tau \in [0,T) : \min_{\varphi_0 \in \Phi_1} J_{T(\tau,T)}(\varphi_0; y_0, T) = 0 \right\}. \]

3. Steering control.

3.1. Preliminaries. Beside the results on constrained null controllability, we shall examine the problem of characterizing the appropriate control steering \( y_0 \) to the origin. We note that, assuming that such a control exists, a classical strategy has consisted of reducing the question to the two following problems. The first one is the time optimal control problem already introduced in section 2. The second one is the so-called norm optimal control which consists of determining the steering control which minimizes an appropriate norm. To best of our knowledge, at general abstract level, the solutions obtained for these problems concern the case where the control operator \( B \) is bounded. See \[13\] and \[14\]. The case where \( B \) is unbounded has been treated mainly in the context of boundary control problems for heat equations with \( L^\infty \) setting with respect to the time variable. See, for instance, \[22\], \[23\] and \[34\]. Here, we shall not consider directly these issues. Instead, we shall consider the case where \( B \) is (eventually) unbounded and, taking into account from the results stated in section 2 the fact that the existence of the steering control depends closely on the initial state \( y_0 \) and the related final time \( T = T(y_0) \), we shall formulate our problem in the following general form:

**(P)** Given a final time \( T > 0 \), characterize any control steering the system (1) as close as possible to 0 in time \( T \).

We shall treat this problem by using a variational method based on minimizing an appropriate convex functional. Having in mind the existing literature which
makes use of dual convex arguments and introduces backward adjoint system, it
turns out that the null control problem is reduced to the minimization of a dual
conjugate functional with respect to the final condition of the adjoint state. The
form of this functional is closely related to the type of system under study. The
null control problem is reduced to the minimization of a dual

$$J_H(\varphi_0; y_0, T) = \int_0^T K_H \left( B^* e^{(T-t)A^*} \varphi_0 \right) dt + \left( e^{T A^*} \varphi_0, y_0 \right),$$

(65)

where $K_H : U \to \mathbb{R}^+$ is convex and lower semicontinuous. It has been established
that the steering control can be obtained in constructive way provided that $K_H$ is
differentiable, locally Lipschitz and satisfies $K'(U) \subset \Gamma$. This result covers even
the case of unconstrained null controllability where $\Gamma = U$. See [5] and [6] for
details. Inspired by the treatment of the null controllability issue performed in [46]
in the unconstrained control context for the heat equation, it seems that, in the
context of parabolic-like equations, the most convenient functional valid even in the
time-varying constraint sets can be defined by $J_P(\varphi_0; y_0, T) : Y' \to \mathbb{R}$

$$J_P(\varphi_0; y_0, T) = K_P \left( \int_0^T S_T \left( t, B^* e^{(T-t)A^*} \varphi_0 \right) dt \right) + \left( e^{T A^*} \varphi_0, y_0 \right),$$

(66)

where $K_P : \mathbb{R}^+ \to \mathbb{R}^+$ is an appropriate convex function. In particular, following
the approach initiated in [35] for multivariable systems, a natural choice for a given
general time-varying control constraint sets $\Gamma(t)$ will consist of taking $K_P$ as the
identity function so that $J_P(\varphi_0; y_0, T)$ coincides with the functional $\tilde{J}_\Gamma(\varphi_0; y_0, T)$
already defined in (22). Note that an immediate difficulty arises because of the lack
of differentiability of $J_{\Gamma(t)}(\varphi_0; y_0, T)$ resulting from the sup-operation present in the
support function $S_T(\varphi_0, y_0)$. Here, using nonsmooth analysis tools, we shall extend
the results in [6] and [35] to the case of time-varying constraints $\Gamma(t)$ by minimizing
$J_{\Gamma(t)}(\varphi_0; y_0, T)$ on the closed unit ball $\Phi_1 \subset Y'$ introduced in (29). Moreover, we shall
carefully the case of the constraints $\Gamma_{m(t)}$ defined by (41) and (42) in the
context of parabolic-like equations. Roughly speaking, by such a context, we mean
that the following assumptions hold:

$$(H_P^1)$$ (Regularizing property) For each $\varphi_0 \in Y'$, we have $e^{tA^*} \varphi_0 \in D(A^*)$ for all $t > 0$ and the function $t \mapsto e^{t A^*} \varphi_0$ is continuous from $(0, \infty)$ to $D(A^*)$.

$$(H_P^2)$$ (Backward uniqueness property) $e^{T A^*}$ is one to one for any $T > 0$.

$$(H_P^3)$$ (Final time weak observability) For any measurable subset $E \subset (0, T)$ with
positive measure, we have for each $\varphi_0 \in Y'$

$$B^* e^{T A^*} \varphi_0 = 0 \text{ on } E \Rightarrow e^{T A^*} \varphi_0 = 0.$$
domain of \( f \), denoted \( \text{dom} \ f \), is the set

\[
\text{dom} \ f = \{ x \in X : f(x) < \infty \}.
\]

The function \( f \) is called proper when \( \text{dom} \ f \neq \emptyset \).

Let \( f \) be a proper function and let \( x \in \text{dom} \ f \). An element \( \zeta \) of \( X' \) is called a subgradient of \( f \) at \( x \) (in the sense of convex analysis) if it satisfies the following subgradient inequality:

\[
f(y) - f(x) \geq \langle \zeta, y - x \rangle_{X',X} \quad \text{for all } y \in X.
\]

The (eventually empty) set of all subgradients of \( f \) at \( x \) is denoted by \( \partial f(x) \), and referred to as the subdifferential of \( f \) at \( x \). Note that if \( f \) is convex and \( x \in \text{dom} \ f \) is a point of continuity of \( f \), then \( \partial f(x) \) is nonempty and weakly compact ([9], p. 62).

Let \( P \) be a convex subset of \( X \). The tangent cone to \( P \) at a point \( x \in P \), denoted \( T_P(x) \), consists of all points \( v \in X \) expressible in the form

\[
v = \lim_{n \to \infty} \frac{x_n - x}{t_n},
\]

where \( (x_n)_n \) is a sequence in \( P \) converging to \( x \) and \( (t_n)_n \) is a positive sequence decreasing to 0. The normal cone to \( P \) at \( x \in P \), denoted \( N_P(x) \), is the subset of the dual space \( X' \) defined by

\[
N_P(x) = \left\{ \zeta \in X' : \langle \zeta, v \rangle_{X',X} \leq 0 \quad \forall v \in T_P(x) \right\}.
\]

It can be shown that the normal cone can be characterized by ([3], Proposition 4, p. 168)

\[
N_P(x) = \left\{ \zeta \in X' : \langle \zeta, x \rangle_{X',X} = \sup_{v \in P} \langle \zeta, v \rangle_{X',X} \right\}.
\]

The indicator function of \( P \) is the function \( I_P : X \to \mathbb{R} \cup \{+\infty\} \) which has value 0 on \( P \) and \( +\infty \) elsewhere. Let us note that when \( P \) is a convex subset of \( X \), then \( \partial I_P(x) = N_P(x) \); that is, the subdifferential of the indicator function is the normal cone ([9], p. 61).

The conjugate function \( f^* : X' \to \mathbb{R} \cup \{+\infty\} \) of \( f \) is defined by

\[
f^*(\zeta) = \sup_{x \in X} \langle \zeta, x \rangle_{X',X} - f(x).
\]

If \( g : X' \to \mathbb{R} \cup \{+\infty\} \) is a proper function, its conjugate \( g^* : X \to \mathbb{R} \cup \{+\infty\} \) is defined by

\[
g^*(x) = \sup_{\zeta \in X'} \langle \zeta, x \rangle_{X',X} - g(\zeta).
\]

For any nonempty closed convex subset \( P \) of \( X' \), we have \( S_P^* = I_P^* \); that is, the conjugate function of its support function \( S_P \) is given by its indicator function ([9], p. 70). When we take \( g \) to be \( f^* \), then we obtain the biconjugate of \( f \), namely the function \( f^{**} : X \to \mathbb{R} \cup \{+\infty\} \) defined as follows (when \( f^* \) is proper):

\[
f^{**}(x) = \sup_{\zeta \in X'} \langle \zeta, x \rangle_{X',X} - f^*(\zeta).
\]

Given a proper function \( f : X \to \mathbb{R} \cup \{+\infty\} \), it is well known that \( f \) is convex and lower semicontinuous, if and only if \( f^* \) is proper and \( f = f^{**} \) ([9], p. 69). Moreover, for such a function, we have the subdifferential inversion formula ([31], p. 35)

\[
\zeta \in \partial f(x) \iff x \in \partial f^*(\zeta).
\]
3.2. Steering control in nonsmooth analysis setting. The following lemma gives an identification of the subdifferential of $J_{\Gamma(\cdot)}(\cdot; y_0, T)$.

**Lemma 3.1.** For any $y_0 \in Y$, $T > 0$, the subdifferential of $J_{\Gamma(\cdot)}(\cdot; y_0, T)$ at any $\varphi_0 \in Y'$ consists of $\psi_0 \in Y$ such as

$$\psi_0 = e^{TA}y_0 + \int_0^T e^{(T-t)A}Bu(t)dt$$  \hspace{1cm} (74)

for some $(\Gamma(\cdot), p)$-admissible control $u \in \Gamma_T(p)$ satisfying the maximum principle property

$$\max_{\mu \in \Gamma(t)} \left\langle \mu, B^*e^{(T-t)A^*} \varphi_0 \right\rangle_U = \left\langle u(t), B^*e^{(T-t)A^*} \varphi_0 \right\rangle_U \text{ a.e on } (0, T).$$  \hspace{1cm} (75)

**Proof.** Since $J_{\Gamma(\cdot)}(\cdot; y_0, T)$ is the sum of finite convex functions, $\psi_0 \in \partial J_{\Gamma(\cdot)}(\cdot; y_0, T)$ if and only if ([9], Theorem 4.10, p. 63)

$$\psi_0 \in \partial \left( \int_0^T S_{\Gamma}(t, B^*e^{(T-t)A^*} \varphi_0) dt \right) + \partial \left( \left\langle \varphi_0, e^{TA}y_0 \right\rangle \right),$$

which is equivalent to

$$\psi_0 - e^{TA}y_0 \in \partial \left( \int_0^T S_{\Gamma}(t, B^*e^{(T-t)A^*} \varphi_0) dt \right).$$  \hspace{1cm} (76)

In order to precise the subdifferential above, we introduce the functional $R : L^q(0, T; U) \to \mathbb{R}$ given by

$$R(v) = \int_0^T S_{\Gamma}(t, v(t))dt.$$  \hspace{1cm} (77)

Then by applying the result in Theorem 23 of [31], page 62, it follows that $u \in \partial R(v)$ if and only if,

$$u(t) \in \partial S_{\Gamma}(t, v(t)) \text{ a.e on } (0, T),$$  \hspace{1cm} (78)

or, equivalently, by using (73)

$$v(t) \in \partial I_{\Gamma(t)}(u(t)) = N_{\Gamma(t)}(u(t)) \text{ a.e on } (0, T).$$  \hspace{1cm} (79)

Hence (70) yields

$$\max_{\mu \in \Gamma(t)} \left\langle \mu, v(t) \right\rangle_U = \left\langle u(t), v(t) \right\rangle_U \text{ a.e on } (0, T).$$  \hspace{1cm} (80)

On the other hand, we note that

$$\int_0^T S_{\Gamma}(t, B^*e^{(T-t)A^*} \varphi_0) dt = \int_0^T S_{\Gamma}(t, L_T^* \varphi_0(t)) dt$$

$$= R(L_T^* \varphi_0),$$

and by using ([9], Theorem 4.13, p. 64), we obtain

$$\partial (R(L_T^* \varphi_0)) = L_T \partial R(L_T^* \varphi_0).$$

Hence, $\psi_0 - e^{TA}y_0$ has the form

$$\psi_0 - e^{TA}y_0 = L_T u,$$

for some $u \in \partial R(L_T^* \varphi_0)$. Taking into account (80), $u$ satisfies the maximum principle property (75). This completes the proof of the lemma. \qed
Theorem 3.2. Suppose that \(1\) closed unit ball \(\Phi\) that this fact is true whenever for almost all \(t\) system (1) as close as possible to 0 at time \(T\),

\[
\text{Remark 8. In principle, the subdifferential of } J_{\Gamma(\cdot)}(\cdot; y_0, T) \text{ is a multi-valued function. In the constraint case } \Gamma_{m(\cdot)}, \text{ the subdifferential of } J_{\Gamma_{m(\cdot)}(\cdot; y_0, T)} \text{ at any } \varphi_0 \in Y' \text{ consists of } \psi_0 \in Y \text{ satisfying (74) with}
\]

\[
u(t) = \begin{cases} 
 m(t) \frac{B^* e^{(T-t)A^*} \varphi_0}{\|B^* e^{(T-t)A^*} \varphi_0\|_U} & \text{if } B^* e^{(T-t)A^*} \varphi_0 \neq 0, \\
 \Gamma_{m(t)} \text{ otherwise}, 
\end{cases}
\]

for almost all \(t \in (0, T)\). Hence, if

\[B^* e^{(T-t)A^*} \varphi_0 \neq 0 \text{ a.e on } (0, T),\]

then \(\psi_0\) is uniquely determined by (74) and (81) so that the subdifferential of \(J_{\Gamma_{m(\cdot)}(\cdot; y_0, T)}\) will correspond to the gradient \(J'_{\Gamma_{m(\cdot)}(\cdot; y_0, T)}\). It is easy to check that this fact is true whenever \(\varphi_0 \neq 0\) and the system (1) is a parabolic-like equation satisfying the assumptions \((H^T_1), (H^T_2)\) and \((H^T_3)\).

The following theorem gives a strategy guaranteeing a control which steers the system (1) as close as possible to 0 at time \(T\) by minimizing \(J_{\Gamma(\cdot)}(\cdot; T, y_0)\) over the closed unit ball \(\Phi_1\) in \(Y'\).

**Theorem 3.2.** Suppose that \(\tilde{\varphi}_0 \in Y'\) minimizes \(J_{\Gamma(\cdot)}(\cdot; y_0, T)\) on \(\Phi_1\). Then any solution \(\tilde{u}\) to the problem \((P)\) satisfies

\[
\max_{\mu \in \Gamma(t)} \left\{ \mu, B^* e^{(T-t)A^*} \tilde{\varphi}_0 \right\}_U = \left\langle \tilde{u}(t), B^* e^{(T-t)A^*} \tilde{\varphi}_0 \right\rangle_U \text{ a.e on } (0, T).
\]

Moreover, the corresponding final state at time \(T\) given by

\[
y(T; y_0, \tilde{u}) = e^{TA} y_0 + \int_0^T e^{(T-t)A} B \tilde{u}(t) dt,
\]

satisfies \(y(T; y_0, \tilde{u}) \in \partial J_{\Gamma(\cdot)}(\tilde{\varphi}_0; y_0, T)\).

**Proof.** We introduce two functionals \(F : L^p(0, T; U) \to \mathbb{R}\) and \(G : Y \to \mathbb{R}\) given by

\[
F(u) = \begin{cases} 
 0 & \text{if } u \in \tilde{\Gamma}_T(u), \\
 +\infty & \text{otherwise},
\end{cases}
\]

\[G(y) = \left\| e^{TA} y_0 + y \right\|_Y.\]

Then the problem \((P)\) can be reformulated as

\[
\inf_{u \in \tilde{\Gamma}_T(u)} \left\| e^{TA} y_0 + L_T(u) \right\|_Y = \inf_{u \in L^p(0, T; U)} F(u) - G(L_T(u)).
\]

We now apply Rockafellar’s extension duality theory in the sense of Fenchel ([12], chap. 3) and [30]. The functionals \(F\) and \(G\) are, respectively, proper convex and concave functions. Moreover, for any \(u \in \tilde{\Gamma}_T(p)\), \(F\) is finite at \(u\) and \(G\) is continuous at \(L_T(u)\). Hence the problem (84) is “stably set” so that the duality method implies that the problem

\[
\min_{\varphi_0 \in \Phi_1} J_{\Gamma(\cdot)}(\varphi_0; y_0, T)
\]

is dual to (86) in the following sense

\[
\inf_{u \in L^p(0, T; U)} (F(u) - G(L_T(u))) + \min_{\varphi_0 \in \Phi_1} J_{\Gamma(\cdot)}(\varphi_0; y_0, T) = 0.
\]

To this end, we have to express \(J_{\Gamma(\cdot)}(\varphi_0; y_0, T)\) over \(\Phi_1\) as appropriate combination of the conjugate functions \(F^*\) and \(G^*\). For each \(\varphi_0 \in \Phi_1\) we have

\[
F^*(L_T \varphi_0) = \sup_{u \in L^p(0, T; U)} \left\langle L_T \varphi_0, u \right\rangle - F(u)
\]
By using (11) and (28) we obtain
\[ F^*(L_T^* \varphi_0) = \int_0^T S_\Gamma(t, L_T^* \varphi_0(t)) \, dt. \] (89)

On the other hand, the conjugate function of \( G \) over \( \Phi_1 \) is given by
\[ G^*(\varphi_0) = \sup_{y \in Y} \langle \varphi_0, y \rangle - \| e^{TA} y_0 + y \|_Y \]
\[ = - \langle \varphi_0, e^{TA} y_0 \rangle. \] (90)

Indeed, for any \( \varphi_0 \in \Phi_1 \)
\[ - \langle \varphi_0, e^{TA} y_0 \rangle - \langle \varphi_0, y \rangle + \| e^{TA} y_0 + y \|_Y = \| e^{TA} y_0 + y \|_Y - \langle \varphi_0, e^{TA} y_0 + y \rangle \]
\[ \geq \| e^{TA} y_0 + y \|_Y - \| \varphi_0 \|_Y \| e^{TA} y_0 + y \|_Y \]
\[ \geq 0 \text{ for all } y \in Y. \]

It follows that
\[ \min_{\varphi_0 \in \Phi_1} J_{1,t} (\varphi_0; y_0, T) = \min_{\varphi_0 \in \Phi_1} \{ F^*(L_T^* \varphi_0) - G^*(\varphi_0) \}. \] (91)

By applying Theorem 3 of [30] and using the facts that \( F \) and \( G \) coincide with their respective biconjugates
\[ \min_{\varphi_0 \in \Phi_1} \{ F^*(L_T^* \varphi_0) - G^*(\varphi_0) \} = \max_{u \in L^p(0,T;U)} G^*(L_T(u)) - F^**(u) \]
\[ = \max_{u \in L^p(0,T;U)} G(L_T(u)) - F(u) \]
\[ = - \min_{u \in L^p(0,T;U)} F(u) - G(L_T(u)) \]
so that (88) holds. Furthermore, the “extremality condition” in [30] implies a necessary condition which must be satisfied by all solution pairs \( \tilde{\varphi}_0 \), solving (87) and \( \tilde{u} \) solving (86). This condition would read
\[ L_T^*(\tilde{\varphi}_0) \in \partial F(\tilde{u}). \] (92)

Since \( F \) coincides with the indicator function of \( \overline{\Gamma}_T(p) \), the condition (92) means that the normal cone of \( \overline{\Gamma}_T(p) \) at \( \tilde{u} \), denoted by \( N(\tilde{u}) \), contains the function \( L_T^*(\tilde{\varphi}_0) \). This amounts to
\[ \int_0^T \langle \tilde{u}(t), B^* e^{(T-t)A^*} \tilde{\varphi}_0 \rangle_U \, dt = \int_0^T \sup_{\mu \in \Gamma(t)} \langle \mu, B^* e^{(T-t)A^*} \tilde{\varphi}_0 \rangle_U \, dt. \]

This is possible only if \( \mu = \tilde{u}(t) \) achieves the supremum of \( \langle \mu, B^* e^{(T-t)A^*} \tilde{\varphi}_0 \rangle \) for almost all \( t \in [0,T] \). Equivalently, we must have (82). The remaining part of the proof is a ready consequence of Lemma 3.1. \( \square \)

**Remark 9.** Let us examine some consequences of Theorem 3.2 in the parabolic context defined by the assumptions \( (H_1^P), (H_2^P) \) and \( (H_3^P) \). Since the initial state \( y_0 \in Y \) satisfies \( y_0 \neq 0 \), whenever the system (1.1) is \( (\Gamma(.), p) \)-null controllable at \( y_0 \) in time \( T \), the null control can not be a solution to the problem (P). Moreover, from Theorem 2.5, we have
\[ \min_{\varphi_0 \in Y} J_{1,t} (\varphi_0; y_0, T) = \min_{\varphi_0 \in \Phi_1} J_{1,t} (\varphi_0; y_0, T) = 0. \] (93)
We note by using Theorem 3.2 that the obvious minimizer  $\hat{\varphi}_0 = 0$ is useless since it leads to a steering control function characterized by the trivial fact
\[ \hat{u}(t) \in \Gamma(t). \]  

(94)

However, when the control constraint sets are the closed balls $\Gamma_{m(t)}$ given by (41) and (42), we can obtain an abstract time-varying bang-bang property verified by any steering control.

**Theorem 3.3.** Suppose that the assumptions $(H_1^P)$, $(H_2^P)$ and $(H_3^P)$ hold.

(i) If the system (1) is not $(\Gamma_{m(.)}, p)$-null controllable at $y_0$ in time $T$, then the steering control $\hat{u}$ solution to Problem $(P)$ under the constraints $\Gamma_{m(.)}$ satisfies the following time-varying bang-bang property
\[ \|\hat{u}(t)\|_U = m(t) \]  

a.e on $(0, T)$.

(ii) Assume also that the system (1) is locally $(\Gamma_1, p)$-null controllable over each subset $E \subset (0, T)$ of positive measure and the function $m(.)$ is nonincreasing on $(0, \infty)$. If the system (1) is $(\Gamma_{m(.)}, p)$-null controllable at $y_0$ in time $T$, then the minimal time control $u^*$ corresponding to the minimal time $0 < T^* \leq T$ satisfies the time-varying bang-bang property (95) a.e on $(0, T^*)$.

**Proof.** (i) If the system (1) is not $(\Gamma_{m(.)}, p)$-null controllable at $y_0$ in time $T$, then any minimizer  $\hat{\varphi}_0$ would satisfy $J_{\Gamma_{1}}(\hat{\varphi}_0; y_0, T) < 0$ and $\hat{\varphi}_0 \neq 0$. Taking into account $(H_2^P)$ and $(H_3^P)$, it follows that
\[ B^* e^{(T-t)A^*} \hat{\varphi}_0 \neq 0, \text{ a.e on } (0, T) \]  

so that (81) and Theorem 3.2 imply that any steering control satisfies (95).

(ii) If the system (1) is $(\Gamma_{m(.)}, p)$-null controllable at $y_0$ in time $T$, then the existence of the minimal time control $u^*$ and the associated minimal time $0 < T^* \leq T$ is guaranteed by Theorem 2.5 so that the corresponding solution, denoted by $y^*$, satisfies $y^*(T^*; y_0, u^*) = 0$. By contradiction, suppose that for some $\epsilon > 0$ and some subset $E \subset (0, T^*)$ with positive measure, denoted by $|E|$, we have
\[ \|u^*(t)\|_U \leq m(t) - \epsilon \text{ for all } t \in E. \]  

(97)

Let $d_0 = \frac{|E|}{T}$ and $\hat{E} = E \cap (d_0, T^*)$. Moreover, taking into account (48) and Corollary 3, it follows that the system (1) is locally $(\Gamma_{\epsilon}, p)$-null controllable in time $T$ over $\hat{E}$. Hence, for $0 < t_0 < d_0$ small enough, there exists a $(\Gamma_{\epsilon}, p)$-admissible control $u \in \Gamma_p(T^*)$ such that the solution of the system
\[ \left\{ \begin{array}{l} y'(t) = Ay(t) + Bu\chi_E(t), \\ y(t_0) = y_0 - y^*(t_0), \end{array} \right. \]  

(98)

can be steered to the origin at time $T^*$ so that $y(T^*; y(t_0) = y_0 - y^*(t_0), u\chi_{\hat{E}}) = 0$. Moreover, if we set $\hat{u} = u^* + u\chi_{\hat{E}}$ and $z = y^* + y$ then we have
\[ z'(t) = Az(t) + B\hat{u}(t), \quad t_0 < t < T^*, \]  

and
\[ z(t_0) = y^*(t_0) + y(t_0) = y^*(t_0) + y_0 - y^*(t_0) = y_0. \]
On the other hand, letting \( \tilde{u}(t) = \tilde{u}(t + t_0) \) and \( \tilde{y}(t) = z(t + t_0) \) for \( 0 < t < T^* - t_0 \), then since the function \( m(\cdot) \) is nonincreasing, it follows that \( \tilde{u} \) is \((\Gamma_m, p)-\)admissible and we obtain that

\[
\begin{aligned}
\begin{cases}
\tilde{y}'(t) = A\tilde{y}(t) + B\tilde{u}(t), \\
\tilde{y}(0) = y_0,
\end{cases}
\end{aligned}
\]

(99)

and \( \tilde{y}(T^* - t_0; y_0, \tilde{u}) = 0 \). This contradicts the time minimality of \( T^* \) with respect to the \((\Gamma_m, p)-null\) controllability for the system (1). This completes the proof of the theorem.

\[\square\]

4. Applications. In this section, we present some applications of the abstract results stated above to systems governed by parabolic partial differential equations (PDEs) by treating the heat equation. Here, we shall treat both distributed and boundary control cases. In what follows, \( \Omega \subset \mathbb{R}^N \) is an open bounded domain with sufficiently smooth boundary \( \partial \Omega \), \( \omega \) and \( \gamma \) denote non-empty open subsets of \( \Omega \) and \( \partial \Omega \) respectively. Moreover, we shall regard \( L^2(\omega) \) and \( L^2(\gamma) \) as subspaces of \( L^2(\Omega) \) and \( L^2(\partial \Omega) \) respectively. We denote by \( \chi_\omega \) and \( \chi_\gamma \) their respective characteristic functions. Along this section, we shall still denote by \( \Delta \) the Dirichlet Laplacian operator defined by

\[
D(\Delta) = H^2(\Omega) \cap H^1_0(\Omega).
\]

(100)

We shall make use of the well-known identification \( D((-\Delta)^{\frac{1}{2}}) \equiv H^1_0(\Omega) \). See, for instance, [39], p. 93). We shall also denote both the scalar product \( \langle \cdot, \cdot \rangle_{L^2(\Omega)} \) and the duality product \( \langle \cdot, \cdot \rangle_{H^1_0(\Omega), H^{-1}(\Omega)} \) by \( \langle \cdot, \cdot \rangle \). Furthermore, the solution of the adjoint equation given by the backward heat equation

\[
\begin{aligned}
\begin{cases}
\varphi' = -\Delta \varphi & \text{in } \Omega \times (0, \infty), \\
\varphi = 0 & \text{on } \partial \Omega \times (0, \infty), \\
\varphi(x, T) = \varphi_0(x) & \text{in } \Omega,
\end{cases}
\end{aligned}
\]

(101)

will be denoted \( e^{(T-t)\Delta} \varphi_0 \).

Example 4.1. (Heat equation with distributed control) We consider the heat equation given by

\[
\begin{aligned}
\begin{cases}
y' = \Delta y + u \chi_\omega & \text{in } \Omega \times (0, \infty), \\
y = 0 & \text{on } \partial \Omega \times (0, \infty), \\
y(x, 0) = y_0(x) & \text{in } \Omega.
\end{cases}
\end{aligned}
\]

(102)

This system has the form (1) if we set \( Y = L^2(\Omega), U = L^2(\omega) \) and consider as operator \( A \) the Dirichlet Laplacian. Moreover, as appropriate control operator \( B : L^2(\omega) \to L^2(\Omega) \), we can set

\[
Bu = \chi_\omega u.
\]

(103)

We shall be concerned with the time-varying saturation constraint specified by the set

\[
\Gamma^\omega_m(t) = \left\{ u \in L^2(\omega) : \|u\|_{L^2(\omega)} \leq m(t) \right\},
\]

(104)

where \( m \in L^\infty_2(0, \infty) \) has been introduced by (41). Since the control operator \( B \) is bounded, it is obviously 2-admissible with respect to the state space \( L^2(\Omega) \). Besides \((H^1_T)\) and \((H^2_T)\), the assumption \((H^3_T)\) is a ready consequence of the following observability inequality established in [2] and [28]. For any measurable subset \( E \subset (0, T) \) with positive measure, there exists some positive constant \( c_0 \) such that the solution of the backward equation (101) satisfies

\[
\int_{\omega \times E} \left| e^{(T-t)\Delta} \varphi_0 \right| dx dt \geq c_0 \left\| e^{T\Delta} \varphi_0 \right\|_{L^2(\Omega)}
\]

(105)
for all $\varphi_0 \in L^2(\Omega)$. Moreover, (105) yields

$$\int_E \|e^{(T-t)\Delta} \varphi_0\|_{L^2(\omega)} dt \geq c_1 \|e^{T\Delta} \varphi_0\|_{L^2(\Omega)}$$

(106)

for some positive constant $c_1$. It follows that the system (102) is locally ($\Gamma_1,2$)-null controllable in time $T$ over $E$. We define the analogue of the functional in (22) corresponding to the constraint control sets $\Gamma_m(\cdot;y_0,T): L^2(\Omega) \to \mathbb{R}$ and

$$J_m(\varphi_0; y_0, T) = \int_0^T m(t) \|e^{(T-t)\Delta} \varphi_0\|_{L^2(\omega)} dt + \langle e^{T\Delta} \varphi_0, y_0 \rangle.$$  

(107)

Theorem 3.2 implies that any control $u$ solution to the problem (P) with respect to the state space $L^2(\Omega)$ would satisfy

$$u(t) = \begin{cases} 
 m(t) \chi_\omega e^{(T-t)\Delta} \hat{\varphi}_0 & \text{if } \chi_\omega e^{(T-t)\Delta} \hat{\varphi}_0 \neq 0, \\
 \|e^{(T-t)\Delta} \varphi_0\|_{L^2(\omega)} & \text{otherwise}, 
\end{cases}$$

(108)

for almost all $t \in (0,T)$, where $\hat{\varphi}$ satisfies

$$J_m(\hat{\varphi}_0; y_0, T) = \min_{\|\varphi_0\|_{L^2(\Omega)} \leq 1} J_m(\varphi_0; y_0, T).$$

(109)

It turns out that, due to the parabolic context, the set of instants $0 < t < T$ for which

$$e^{(T-t)\Delta} \hat{\varphi}_0 = 0 \text{ on } \omega,$$

is of null measure whenever $\hat{\varphi}_0 \neq 0$. Hence by using Theorem 3.3, we obtain the following facts which are similar to the time-varying bang-bang properties established in [8].

- If the system (102) is not ($\Gamma_m,2$)-null controllable at $y_0$ in time $T$, then the steering control $\hat{u}$ solution to Problem (P) under the constraints $\Gamma_m$ satisfies the following time-varying bang-bang property

$$\|\hat{u}(t)\|_{L^2(\omega)} = m(t)$$

(110)

a.e on $(0,T)$.

- If the system (102) is ($\Gamma_m,2$)-null controllable at $y_0$ and the function $m(.)$ is nonincreasing on $(0,\infty)$, then the minimal time control $u^*$ corresponding to the minimal time $0 < T^* \leq T$ satisfies the time-varying bang-bang property (110) a.e on $(0,T^*)$.

Example 4.2. (Heat equation with Dirichlet boundary control) We consider the heat equation with boundary control given by

$$\begin{cases} 
 y' = \Delta y \text{ in } \Omega \times (0,\infty), \\
 y = \chi_\omega u \text{ on } \partial \Omega \times (0,\infty), \\
 y(x,0) = y_0(x) \text{ in } \Omega. 
\end{cases}$$

(111)

We introduce also the Dirichlet map $D$ defined by

$$D : L^2(\partial \Omega) \to L^2(\Omega), \ v \mapsto Dv = h,$$

(112)

$$\begin{cases} 
 \Delta h = 0 \text{ in } \Omega, \\
 h = u \text{ on } \partial \Omega. 
\end{cases}$$

(113)
Then the system (111) can be modeled as an abstract equation similar to (1) and given by ([39], Proposition 10.7.1, p. 342)

\[ y'(t) = Ay(t) + \Delta D\chi_{\gamma}u(t), \quad (114) \]

Moreover, we recall from elliptic theory in ([18], Chap. 2) that \( D \) is continuous so that the control operator can be defined by \( B : U = L^2(\gamma) \to D(\Delta)^{\prime} \) with

\[ Bu = \Delta D\chi_{\gamma}u. \quad (115) \]

Hence, this system has the form (1) if we choose \( U = L^2(\gamma) \) as control space. However, it is well known that there exists \( u \in L^2(0, T; L^2(\gamma)) \) for which \( y(T; y_0, u) \notin L^2(\Omega) \) so that the control operator \( B \) in (111) is not 2-admissible if we consider \( L^2(\Omega) \) as state space. See, for instance, ([17], p. 202). In order to precise the appropriate state space and the corresponding admissibility condition, we consider two possible situations. The first one is based on the following estimate established in [47]. For some positive constant \( C > 0 \), we have for each \( t > 0 \)

\[ \|\Delta e^{t\Delta}Du\|_{L^2(\Omega)} \leq Ct^{-\frac{3}{2}} \|u\|_{L^2(\gamma)} \quad (116) \]

for all \( u \in L^2(\gamma) \). It follows that \( y(T; y_0, u) \in L^2(\Omega) \) for all \( u \in L^p(0, T; L^2(\gamma)) \) whenever \( p > 4 \). Hence the operator defined by (115) is \( p \)-admissible with respect to the state space \( Y = L^2(\Omega) \) for any \( p > 4 \). In other words, we can preserve \( L^2(\Omega) \) as state space by smoothing the space of control functions. The second admissibility condition uses the fact that the solution of (111) satisfies \( y \in C(0, T; H^{-1}(\Omega)) \) for any \( u \in L^2(0, T; L^2(\gamma)) \). See ([39], p. 343). Hence the operator \( B \) defined by (115) is 2-admissible with respect to the state space \( Y = H^{-1}(\Omega) \) so that we can use \( L^2(0, T; L^2(\gamma)) \) as control functions space with respect to the larger state space \( H^{-1}(\Omega) \). In the sequel, we shall be concerned with this case. In order to precise the corresponding dual 2-admissibility, we note that the operator

\[ L_T : u \in L^2(0, T; L^2(\gamma)) \to y(T; 0, u) \in H^{-1}(\Omega) \]

is linear continuous. Its adjoint \( L^*_T : H_0^1(\Omega) \to L^2(0, T; L^2(\gamma)) \) is defined by

\[ \langle L_Tu, \varphi_0 \rangle = \int_0^T \langle (L^*_T\varphi_0)(t), u(t) \rangle_{L^2(\gamma)} dt \]

\[ = \langle y(T; 0, u), \varphi_0 \rangle. \]

Consider for \( \varphi_0 \in H_0^1(\Omega) \) the solution of the adjoint equation (101). If we compute the duality product \( \langle ., . \rangle \) of (101) with \( y(., 0, u) \) and apply the Green’s formula, we obtain by integrating by parts

\[ \langle y(T; 0, u), \varphi_0 \rangle = \int_0^T \langle u(t), \partial_\nu \left( e^{(T-t)\Delta} \varphi_0 \right) \rangle_{L^2(\gamma)} dt, \]

where \( \partial_\nu \) denotes the outward normal derivative to \( \partial \Omega \). Hence, \( L^*_T \) has the form

\[ L^*_T\varphi_0(t) = (AD\chi_{\gamma})^*e^{(T-t)\Delta} \varphi_0 = \chi_{\gamma}\partial_\nu \left( e^{(T-t)\Delta} \varphi_0 \right). \quad (117) \]

As for the constrained null controllability question, let

\[ \Gamma_m(t) = \left\{ u \in L^2(\gamma) : \|u\|_{L^2(\gamma)} \leq m(t) \right\} \quad (118) \]

be the time-varying constraint control sets for the system (111). Then beside \( (H^p_T) \) and \( (H^p_3) \), the assumption \( (H^p_5) \) is a ready consequence of the following observability inequality established in [2]. For any measurable subset \( E \subset (0, T) \) with
positive measure, there exists some positive constant \(c_0\) such that the solution of the backward equation (101) satisfies
\[
\int_{\gamma \times E} \| \partial_\nu e^{(T-t)\Delta} \varphi_0 \| \, d\sigma dt \geq c_0 \| e^{T\Delta} \varphi_0 \|_{L^2(\Omega)}
\]  
(119)

for all \(\varphi_0 \in L^2(\Omega)\). Note that it follows from Corollary 3 and Corollary 5 that the system (111) is locally \((\Gamma, 2)\)-null controllable over each subset \(E \subset (0,T)\) with respect to the state space \(H^{-1}(\Omega)\). In fact, an analogue of this property holds true for the extended state space \(H^{-1}(\Omega)\).

**Proposition 1.** The system (111) is locally \((\Gamma, 2)\)-null controllable over each subset \(E \subset (0,T)\) of positive measure in time \(T\) with respect to the state space \(H^{-1}(\Omega)\).

Proof. Given an arbitrary initial state \(y_0 \in H^{-1}(\Omega)\), it is well-known that \(\tilde{y}_0 := e^{\frac{\partial T}{2}\Delta} y_0 \in L^2(\Omega)\). Let \(E \subset (0,T)\) be a subset of positive measure. The system (111) is locally \((\Gamma, p)\)-null controllable over \(E \cap (0, \frac{T}{2})\) in time \(\frac{T}{2}\) with respect to the space \(L^2(\Omega)\) so that for some positive constant \(\delta > 0\), the system is \((\Gamma, p)\)-null controllable over \(E\) at \(\tilde{y}_0\) in time \(\frac{T}{2}\) whenever \(\| \tilde{y}_0 \| < \delta\). Let \(v\) be the appropriate steering control such that \(y(\frac{T}{2}; \tilde{y}_0, v) = 0\) and consider the control given by
\[
u(t) = \begin{cases}
0 & \text{if } t \in E \cap (0, \frac{T}{2}), \\
v(t - \frac{T}{2}) & \text{if } t \in E \cap (\frac{T}{2}, T).
\end{cases}
\]
(120)

It is easy to check that \(y(T; y_0, u) = 0\). On the other hand, taking into account that for some \(C > 0\) ([26], Theorem 6.13, p. 74)
\[
\| \tilde{y}_0 \|_{L^2(\Omega)} = \left\| e^{\frac{\partial T}{2}\Delta} y_0 \right\|_{L^2(\Omega)} \leq C \| y_0 \|_{H^{-1}(\Omega)},
\]
(121)

there exists \(\delta > 0\) such that
\[
\| y_0 \|_{H^{-1}(\Omega)} < \delta \Rightarrow \| \tilde{y}_0 \|_{L^2(\Omega)} < \tilde{\delta}.
\]
(122)

This implies that the system (111) is locally \((\Gamma, 2)\)-null controllable over \(E\) with respect to the state space \(H^{-1}(\Omega)\) and the proof of the proposition is complete. \(\square\)

We define the analogue of the functional in (22) corresponding to the constraint control set \(\Gamma_{m()}\) by \(J^\gamma_{m()}(\cdot; T, y_0) : H^1_0(\Omega) \to \mathbb{R}\) and
\[
J^\gamma_{m()}(\varphi_0; T, y_0) = \int_0^T m(t) \left\| \partial_\nu \left( e^{(T-t)\Delta} \varphi_0 \right) \right\|_{L^2(\gamma)} dt + \left\langle e^{T\Delta} \varphi_0, y_0 \right\rangle.
\]
(123)

It follows from Theorem 3.2 that any control \(u\) solution to the problem (P) would satisfy
\[
u(t) = \begin{cases}
m(t)\chi_{\gamma} \partial_\nu \left( e^{(T-t)\Delta} \varphi_0 \right) & \text{if } \chi_{\gamma} \partial_\nu \left( e^{(T-t)\Delta} \varphi_0 \right) \neq 0, \\
\Gamma^\gamma_{m(t)} & \text{otherwise},
\end{cases}
\]
(124)

for almost all \(t \in (0,T)\) where \(\varphi_0\) satisfies
\[
J^\gamma_{m()}(\varphi_0; y_0, T) = \min_{\| \varphi_0 \|_{m()} \leq 1} J^\gamma_{m()}(\varphi_0; y_0, T).
\]
(125)

As in the case of distributed control, the parabolic context implies that the set of time instants \(0 < t < T\) for which
\[
\partial_\nu \left( e^{(T-t)\Delta} \varphi_0 \right) = 0 \text{ on } \gamma,
\]
(126)
is of null measure whenever $\tilde{\phi}_0 \neq 0$. Hence by using Theorem 3.3, we obtain the following similar facts:

(i) If the system (111) is not $(\Gamma_{m(\cdot)}, 2)$-null controllable at $y_0$ in time $T$, then the steering control $\tilde{u}$ solution to Problem (P) under the constraints $\Gamma_{m(\cdot)}$ satisfies the following time-varying bang-bang property

$$\|\tilde{u}(t)\|_{L^2(\gamma)} = m(t)$$

(127) a.e on $(0,T)$.

(ii) If the system (111) is both $(\Gamma_{m(\cdot)}, 2)$-null controllable at $y_0$ and the function $m(.)$ is nonincreasing on $(0, \infty)$, then the minimal time control $u^*$ corresponding to the minimal time $0 < T^* \leq T$ satisfies the time-varying bang-bang property (127) a.e on $(0, T^*)$.

5. Further comments and concluding remarks. This article has provided a general variational approach to constrained null controllability in an abstract setting. Despite the fact that the applications has been focused on heat systems, we conjecture that other potential candidates to these applications are more general parabolic equations and systems, Stokes-like and linearized Boussinesq-like systems, delayed systems. See, for instance, the applications of our method to hyperbolic-like systems under time-invariant constraints developed in [6]. Furthermore, interesting problems could be considered in connection with the results and methods developed in this paper. Among them we mention the following ones.

- An interesting issue would consist on treating the null controllability problem under constraints on the control and the state of the system. We mention particularly other constraint sets of positivity type which are interesting from a practical point of view. In the absence of control saturation, these questions have been treated in [19] and [21] for heat systems. The case of finite dimensional systems is also considered in [20].

- The results given in this paper are theoretical and it would be interesting to address the question of building efficient numerical algorithms to compute the steering control. If we consider the parabolic-like systems modeled by the heat equation, we note that recent developments indicate that the null controllability issue is difficult and still largely open even in the unconstrained control case ([24], [25]). Recall that such an analysis leads to the necessity of distinguishing two different methods, the continuous and the discrete ones. In the continuous one, after characterizing the steering control of the system, the emphasis is placed on building efficient numerical methods to approximate it. In the discrete one, one analyzes the null controllability of discrete models obtained after discretizing the state equation by suitable numerical methods and their possible convergence towards the steering control when the mesh-size parameters tend to zero. See, for instance, [45]. Moreover, it may happen that one has null controllability in a very general space but not in a classical space of sufficiently smooth functions, then the numerical approximation will necessarily develop singularities ([7], [15]). This phenomenon, unavoidable even in the finite dimensional systems and whatever be the numerical approximation used, strongly deteriorates the efficiency of the algorithms so that smoothing remedies should be used ([25], [36]).

- The variational characterization of the constrained steering control can be addressed in a semilinear setting. This is still to be performed.
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