A Super-Exponential Decaying Property of Odd-Dimensional Wave Scattered by an Obstacle

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January 21, 2011

Abstract

We examine an inverse backscattering property of wave motion imposed by an obstacle. We show that if the wave propagator decays super-exponentially along the back-scattered geodesics, then the involved scatterer must be trivial. In particular, if the fundamental solution decays super-exponentially some time after $t = 0$, it vanishes for all time. We use finite speed of propagation in this article.

1 Introduction and main results

Let $\Gamma$ be an embedded hypersurface in $\mathbb{R}^n$ with odd $n \geq 3$ such that

$$\mathbb{R}^n \setminus \Gamma = \Omega \cup \mathcal{O},$$

where both $\mathcal{O}$ and $\Omega$ are open. We denote $\mathcal{O}$ as an obstacle and $\Omega$ as its exterior.

We consider the following exterior problem. Let $u \in H^2(\Omega)$ be the solution of

$$-\Delta u + qu = 0 \text{ in } \Omega$$

with the boundary operator $\gamma$ on $\Gamma$ given either by

$$\gamma u = \left( \partial/\partial \nu \right) u \text{ or } \gamma u = u.$$

We assume that $q$ is real-valued function uniformly Hölder continuous on $\Omega \cup \Gamma$ with a compact support.

Let us use $A(x, D_x) := H$ outside $\mathcal{O}$ satisfying boundary condition (1.3) to review Irvii’s theory [3, 4]. Let us investigate the relation of the following three PDEs in this paper. Let $u(x, y, t)$ be the solution of

$$
\begin{cases}
P(x, D_x, D_t)u(x, y, t) := (D^2_t - A(x, D_x))u(x, y, t) = 0; \\
u(x, y, t)|_{t=0} = 0, \quad u_t(x, y, t)|_{t=0} = \delta(x-y).
\end{cases}
$$

with boundary condition (1.3); let $u_0(x, y, t)$ be the solution of

$$
\begin{cases}
P(x, D_x, D_t)u_0(x, y, t) = 0; \\
u_0(x, y, t)|_{t=0} = 0, \quad u_0(x, y, t)|_{t=0} = \delta(x-y);
\end{cases}
$$

let $u_1(x, y, t)$ be the solution of

$$
\begin{cases}
P(x, D_x, D_t)u_1(x, y, t) = 0; \\
u_1(x, y, t)|_{t=0} = u_1(x, y, t)|_{t=0} = 0, \quad B_{\pm}u_1(x, y, t) = -B_{\pm}u_0(x, y, t),
\end{cases}
$$

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where \( B_{-v} = v|_T \) and \( B_{+v} = \frac{\partial}{\partial n} v|_T \). In operator form, \( u(x,y,t) \) is the Schwartz kernel of \( \sin \frac{t \sqrt{H}}{\sqrt{H}} \). We see that \( u_1(x,y,t) = u(x,y,t) - u_0(x,y,t) \).

We have to remind ourselves of the relation of (1.13) and (1.6) to the following Cauchy problem for the wave equation: let \( U(x,y,t) \) be the solution of

\[
\begin{align*}
P(x,D_x,D_t)U(x,y,t) := (D_t^2 - A(x,D_x))U(x,y,t) &= 0; \\
U(x,y,t)|_{t=0} &= \delta(x-y), \quad U_t(x,y,t)|_{t=0} = 0,
\end{align*}
\]

with either boundary condition (1.3). Let \( U_0(x,y,t) \) be the solution of

\[
\begin{align*}
P(x,D_x,D_t)U_0(x,y,t) = 0; \\
U_0(x,y,t)|_{t=0} = \delta(x-y), \quad U_0(t,x,y,t)|_{t=0} = 0.
\end{align*}
\]

We define

\[
U_1(t,x,y) := U(t,x,y) - U_0(t,x,y).
\]

Formally, we write \( U(t) \) or \( \cos t \sqrt{H} \) as the solution operator. In terms of Fourier transform, \( \cos t \sqrt{H} \) and \( \sin t \sqrt{H} \) differs by \( \lambda \). \( \lambda \) is the frequency variable. In short time, the wave trace is supported in a 3-ball. That is the manifold the analysis in this paper is carried out. See section 2 for a picture.

In Sá Barreto and Zworski [8, 9], they try to answer the following question: let \( E(t,x,y) \) be the fundamental solution of the following perturbed wave equation

\[
\begin{align*}
(D_t^2 - P)E(t,x,y) &= 0; \\
E(0,x,y) &= 0; \\
\partial_t E(0,x,y) &= \delta(x-y),
\end{align*}
\]

where \( P \) is an elliptic self-adjoint operator in \( L^2(\mathbb{R}^n, \sqrt{g}) \) defined as

\[
P := -\frac{1}{\sqrt{g}} \sum_{i,j=1}^n \partial_{x_i} \sqrt{gg^{i,j}} \partial_{x_j} + V, \quad \bar{g} = \det(g_{ij}), \quad (g^{ij}) = (g_{ij})^{-1},
\]

where \( V, g_{ij} \) are among smooth functions with bounded derivatives, \( C^\infty(\mathbb{R}^n) \), such that \( V(x) \) and \( |g_{ij} - \delta_{ij}| \) decays super-exponentially. Does a super-exponentially decaying fundamental solution \( E(t,x,y) \) of (1.10) inside its characteristic cone implies the solution \( E(t,x,y) \) actually vanishes there? They give affirmative answer there. Is there a similar property valid for an obstacle scatterer?

On the other hand, the relation between the wave decaying speed and the location of the poles from the meromorphic continuation of the Green’s function in \( \mathbb{C} \) is classical. For instance, let \( u(x,t) \) be the solution of the Cauchy problem:

\[
\begin{align*}
D_t^2 u(x,t) + Hu(x,t) &= \varphi(x,t); \\
u(x,0) &= f(x); \\
D_t u(x,0) &= g(x),
\end{align*}
\]

where \( H := -\Delta + q \) and the regularity condition on \( q, \varphi, f, g \) are specified as in Thoe [16]. Suppose that \( H \) has neither \( L^2 \)-discrete spectrum nor \( L^2 \)-embedded spectrum at 0. Then, it is shown in Thoe [16] sec.3] that the solution \( u(x,t) \) behaves like \( O(e^{-\gamma t}) \), \( \gamma \), as \( t \to \infty \), in such a fashion that \( \gamma \) is any positive number less than the minimal distance of the poles of its Green’s function from the real axis in \( \mathbb{C} \). Is the statement valid for an obstacle scatterer?

We state the main result in this paper as

**Theorem 1.1** Let \( U_1(t,x,y) \) be described as in (1.12). If for all \( x \in \mathbb{R}^n, n \geq 3 \), odd, and for all \( N \in \mathbb{N} \) and for some constants \( T, C \) such that for all \( |t| > T > 0 \),

\[
|U_1(t,x,x)| \leq Ce^{-N|t|},
\]

then \( \mathcal{O} = \phi \).

Comparing the assumption (1.13) with [8, 9], the constant \( C \) in (1.13) is independent of \( x \). In particular, letting \( N \) goes to infinity, we have \( U_1(x,x,t) \equiv 0 \) for \( |t| > T \). In this case, we will show the scatterer \( \mathcal{O} \) is void. We have \( U_1(x,x,t) = U(x,x,t) - U_0(x,x,t) \equiv 0 \) by the uniqueness of the wave equation (1.8), \( U_1(x,x,t) \) is entirely zero for all \( t \). Hence, under assumption (1.13), finite speed of propagation makes our analysis here behaves as a compact case.
2 Spectral analysis

We start with this paper with the meromorphic extension theory for resolvent operator

\[(H - \lambda^2)^{-1} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n).\]

It is well-known in the literature. We refer to Sjöstrand and Zworski [14] that

\[R(\lambda) := (H - \lambda^2)^{-1} : L^2(\mathbb{R}^n) \to \mathcal{H}^2(\mathbb{R}^n), \exists \lambda > 0, \lambda^2 \notin \sigma(H),\]  

(2.1)

meromorphically extends to

\[R(\lambda) : L^2_{\text{comp}}(\mathbb{R}^n) \to \mathcal{H}^2_{\text{loc}}(\mathbb{R}^n),\]  

(2.2)

where for odd \(n\), \(\lambda\) is defined over \(\mathbb{C}\); for even \(n\), \(\lambda\) is defined over \(\Lambda\), the logarithmic plane. We refer Zworski for [19] [18] for a scattering theory. Let \(\chi_1, \chi_2 \in C^\infty_0(\mathbb{R}^n)\) such that \(\chi_2 \chi_1 = \chi_1\) and \(\chi_1\) covers \(\mathcal{D}\). (2.2) is equivalent to

\[\chi_2 R(\lambda) \chi_1 : L^2(\mathbb{R}^n) \to \mathcal{H}^2(\mathbb{R}^n),\]  

(2.3)

as a meromorphic family of operators. Such an extension theory does not depend on the cutoff functions \(\chi_i\), \(i = 1, 2\). That means we can shrink the domain of the resolvent and enlarge the image space to make \(R(\lambda)\) defined as a meromorphic family of operators. Alternatively, we let \(\chi \in C^\infty(\mathbb{R}^n; [0, 1])\) be a local cutoff function such that \(\chi_i\), \(i = 1, 2\), has supports in \(\text{supp}(\chi)\). We alternatively say

\[\chi R(\lambda) \chi : L^2(\mathbb{R}^n) \to \mathcal{H}^2(\mathbb{R}^n)\]  

meromorphically.

(2.4)

From [14], we know \(\chi R(\lambda) \chi\) in a black box perturbation can be written as

\[\chi R(\lambda) \chi = \chi \{Q_0(\lambda) \chi + Q_1(\lambda_0) \chi\} (I + K(\lambda, \lambda_0))^{-1}.\]  

(2.5)

Without any possible confusion, we still use \(R(\lambda)\) as the extended cutoff resolvent.

The poles are of finite rank and their multiplicity

\[m_{\lambda_0}(R) := \text{rank} \int_{\partial D(\lambda_0, \epsilon)} R(\lambda) \lambda d\lambda.\]  

(2.6)

In addition to lemmas above, we may take the resolvent kernel \(R(\lambda, x, y)\) as the Green’s function of the elliptic problem \((H - \lambda^2)u = 0\) with boundary condition [13] which is well-known in the setting as Shenk and Thoe [11] Theorem 5.1] and Thoe [10]. In particular,

Lemma 2.1. Let \(\Gamma = \Gamma_1 \cup \Gamma_2\) where \(\Gamma_1\) and \(\Gamma_2\) are two disjoint subsets of \(\Gamma\). Then the elliptic problem [12] and [13] has the Green’s function \(G(x, y, \lambda)\) such that (i) \(G(x, y, \lambda)\) is an outgoing solution of

\[\begin{cases} (-\Delta_x + q(x) - \lambda^2)G(x, y, \lambda) = 0, \forall x \in \Omega \setminus \{y\}; \\
(\partial_{\nu_x} - \sigma(x))G(x, y, \lambda) = 0, \forall x \in \Gamma_1; \\
G(x, y, \lambda) = 0, \forall x \in \Gamma_2, \end{cases}\]  

(2.7)

such that

\[u(y) = \int_{\Gamma} [u(x) \frac{\partial}{\partial \nu_x} G(x, y, \lambda) - G(x, y, \lambda) \frac{\partial}{\partial \nu_x} u(x)] dS_x + \int_{\Omega} G(x, y, \lambda)(-\Delta_x + q(x) - \lambda^2)u(x) dx,\]  

(2.8)

\(\forall y \in \Omega\) and for all outgoing function \(u \in \mathcal{H}^2_{\text{loc}}(\Omega) \cap C^{2+\alpha}(\Omega)\)

(ii) For fixed \(y \in \Omega\), the map

\[G(\lambda, x, y) : \mathbb{F} \setminus \text{resonances} \to C^1(\Omega \cup \Gamma) \cap C^{2+\alpha}(\Omega)\]  

(2.9)

is continuous; for fixed \(y \in \Omega\), the map

\[G(\lambda, x, y) : \mathbb{F} \to C^1(\Omega \cup \Gamma) \cap C^{2+\alpha}(\Omega)\]  

(2.10)

is meromorphic.
From (2.5), we can define a scattering amplitude for black box formalism.

\[ A(\lambda) := C_n \lambda^{-2} \mathbb{E} \eta(-\lambda)[\Delta_0, \chi^2] R(\lambda)[\Delta_0, \chi]^2 \mathbb{E} \eta(\lambda), \]  

(2.11)

where

\[ \mathbb{E}(\lambda) : L^2(S^{n-1}) \to C^\infty(\mathbb{R}^n) \; u(\omega) \mapsto C_n \lambda^{\frac{n+1}{2}} \int_{S^{n-1}} u(\omega) e^{i\lambda \cdot \omega} d\omega, \]

where \( C_n \) is a constant depending on \( n \). \( A(\lambda) \) actually comes from the radiation pattern of \( R(\lambda) [-\Delta_0, \chi] e^{i(\cdot, \omega)}. \)

Scattering theory happens mostly on the continuous spectrum. The scattering behavior of the solution is represented by the spectral integration over the continuous spectrum. It is standard in spectral analysis that

\[ U_1(t) = \int_{\mathbb{R}^+} e^{-i \lambda t} \{ R(\lambda) - R_0(\lambda) - R(-\lambda) + R_0(-\lambda) \} d\lambda^2 \]

\[ + \Pi(0) + 2 \sum_{\lambda_j > 0} \Pi(\lambda_j). \]  

(2.12)

The first term on the right hand side of (2.13) comes from the spectral measure

\[ dE_\lambda := \{ R(\lambda) - R_0(\lambda) - R(-\lambda) + R_0(-\lambda) \} d\lambda^2 \]

integrating along the continuous spectrum, the second one from the possible embedded eigenvalue on the continuous spectrum and the third one is from eigenvalues. Each discrete eigenspace are finite dimensional. Hence, there exists a canonical kernel \( \Pi(\lambda_j, x, y) \) for each \( \lambda_j \). The kernel of resolvent is understood as the Green’s function described in Lemma 2.1 extended as a \( L^2 \)-kernel.

In more general setting, \( R(\lambda) - R_0(\lambda) \in \mathcal{D}'(\Omega \times \Omega) \) by Schwartz kernel theorem. Another mathematical treatment on this Schwartz kernel of \( R(\lambda) \) is to see it as a Green’s operator in sense of Ivrii [3, (1.3)] which is an oscillatory integral. Hence, the following identity holds in \( \mathcal{D}'(\Omega \times \Omega), \)

\[ U_1(t, x, y) = \int_{\mathbb{R}^+} e^{-i \lambda t} \{ R(\lambda, x, y) - R_0(\lambda, x, y) - R(-\lambda, x, y) + R_0(-\lambda, x, y) \} d\lambda^2 \]

\[ + \Pi(0, x, y) + 2 \sum_{\lambda_j > 0} \Pi(\lambda_j, x, y). \]  

(2.13)

\( U_1(t) \) also has a distributional trace. See Zworski [18].

To continue our scattering theory, we define

\[ s(\lambda) := \det S(\lambda) \]  

(2.14)

where \( S(\lambda) \) is the relative scattering matrix in sense of Zworski [18, 19]. Let us define \( \sigma(\lambda) := \frac{i}{2\pi} \log s(\lambda) \). By functional analysis,

\[ \sigma'(\lambda) = \frac{i}{2\pi} \frac{s'(\lambda)}{s(\lambda)}. \]  

(2.15)

We have a Weierstrass product form for \( s(\lambda) \). Let \( m_\mu(R) \) be the multiplicity of \( R(\lambda) \) near \( \lambda = \mu \). We have

\[ P(\lambda) := \prod_{\{\mu: \text{resonances}\}} E(\frac{\lambda}{\mu}, [m]^m(\lambda)), \]  

(2.16)

where

\[ E(z, p) := (1 - z) \exp(1 + \cdots + \frac{z^p}{p}). \]  

(2.17)

Most important of all,

\[ s(\lambda) = e^{\sigma(\lambda)} \frac{P(-\lambda)}{P(\lambda)}. \]  

(2.18)
The scattering determinant grows outside its resonances/poles like
\[
|s(\lambda)| \leq C e^{C|\lambda|^n}, \ C \text{ constants.} \tag{2.19}
\]
See Vodev \[17\] for a black box formalism. \(g(\lambda)\) is a polynomial of order at most \(n\). See Vodev \[17\]. In particular,
\[
\sigma'(\lambda) = \frac{i}{2\pi} g'(\lambda) + \frac{i}{2\pi} \sum_j \frac{1}{\lambda + \lambda_j} - \frac{1}{\lambda - \lambda_j} + Q_{\lambda}(\lambda_j) - Q_{\lambda}(-\lambda_j), \tag{2.20}
\]
where
\[
Q_{\lambda}(\lambda_j) := \left( \frac{1}{\lambda_j} \right)(1 + \left( \frac{\lambda}{\lambda_j} \right) + \cdots + \left( \frac{\lambda}{\lambda_j} \right)^{[m]-1}), \tag{2.21}
\]
which is also a polynomial in \(\lambda\) provided \(\{\lambda_j\} \neq 0\). Furthermore, for \(\lambda \in 0i + \mathbb{R}\),
\[
2\lambda Tr\{R(\lambda) - R_0(\lambda) - R(-\lambda) + R_0(-\lambda)\} = \sigma'(\lambda). \tag{2.22}
\]
This is the proof of Birman-Krein theorem in black box formalism setting. See \[18, 19\]. In this case,
\[
\int_{\mathbb{R}} e^{\lambda t} TrU_1(t) dt = \sigma'(\lambda) + m_0(\lambda) \delta(\lambda^2) + 2 \sum_{\lambda_j > 0} m_{\lambda_j}(\lambda) \delta_0(\lambda^2 - \lambda_j^2). \tag{2.23}
\]
Furthermore, when \(n \geq 3, 0\) is neither a resonance nor an eigenvalue of \(R(\lambda)\). To see this, we use the asymptotic behavior of \(\sigma'(\lambda)\) near zero. From \[2.11\] by Zworski’s theory \[19\], we can show
\[
\sigma'(\lambda) = \lambda^{n-3} f(\lambda) \text{ as } \lambda \to 0^+, \text{ where } f \text{ is smooth near } \lambda = 0. \tag{2.24}
\]
In addition, the self-adjoint operator \(H\) has no \(L^2\)-eigenvalue by standard argument in spectral analysis. See \[4\]. Therefore, \[2.20\] becomes
\[
\sigma'(\lambda) = \int_{\mathbb{R}} e^{\lambda t} TrU_1(t) dt. \tag{2.25}
\]
Using the theorem assumption, the kernel of \(U_1(t, x, x)\) is super-exponentially decaying for \(|t| > T\), then \[2.23\] is in sense of Laplace transform in \(\mathbb{C}\) and
\[
\int_{\mathbb{R}} e^{\lambda t} TrU_1(t) dt \text{ converges and is entire in } \mathbb{C}. \tag{2.26}
\]
Accordingly,

**Lemma 2.2** Under theorem assumption \[16\], the inverse Laplace transform \(TrU_1(t)\) is unique and the two-phased \(\sigma'(\lambda)\) has no resonance in \(\mathbb{C}\).

We recall Zworski \[19\] Theorem 4] and the remark thereafter, for any \(\gamma > 0 \text{ and } k\),
\[
\left| \frac{\partial}{\partial t} \right|^k (TrU_1(t) - \sum_{\lambda \leq \gamma \log |\lambda|} m(\lambda) e^{i|t|\lambda}) \leq C_k t^{-n+2-k}, \ t > t_k > \frac{n+k}{\gamma}. \tag{2.27}
\]
The right hand side decays polynomially when \(n\) is even; exponentially for odd \(n\). Since there is no resonance, the summation over all of the resonance is void. Consequently, letting \(t \to 0\) which means \(\gamma \to \infty\), we conclude that the \(C^\infty\)-singularity support is \(\{0\}\). We can say more.

**Proposition 2.3** Under the Theorem 1.1 assumption, the Fourier-Laplace transform \(\int_{\mathbb{R}} e^{\lambda t} U_1(t, x, x) dt\) in \[2.24\] can be extended as a Fourier-Laplace transform over \(\mathbb{C}\) if \(n\) is odd. In particular, \[2.24\] depends only on the short-time behavior of \(U_1(t, x, x)\):
\[
\int_{-\infty}^{\infty} e^{\lambda t} U_1(t, x, x) dt = \int_{-\infty}^{\infty} e^{\lambda t} U_1(t, x, x) \rho_1(t) dt + \text{a rapidly decreasing function}, \tag{2.28}
\]
for some cutoff function \(\rho_1(t)\) supported at \(t = 0\).
Proof We divide the Fourier transform on $U_1(t, x, y)$ into three time intervals:

$$
\langle \int_{-\infty}^{\infty} e^{i\lambda t} U_1(t, x, x) dt, \varphi(\lambda) \rangle := \langle \int_{-\infty}^{\infty} e^{i\lambda t} U_1(t, x, x) \rho_1(t) dt + \int_{-\infty}^{\infty} e^{i\lambda t} U_1(t, x, x) \rho_2(t) dt + \int_{-\infty}^{\infty} e^{i\lambda t} U_1(t, x, x) \rho_3(t) dt, \varphi(\lambda) \rangle
$$

$$
:= \langle I_1 + I_2 + I_3, \varphi(\lambda) \rangle,
$$

where $\rho_i \in C^\infty(\mathbb{R})$, $i = 1, 2, 3$. Let $\rho_1, \rho_3 \in C^\infty(\mathbb{R})$ be two positive cutoff functions such that $\rho_1$ has small compact support at $t = 0$ and $\rho_3$ has support outside $(-T, T)$, for some $T > 0$ as given by (1.13). We also assume $\rho_1(t) = 1$ near $0$ and $\rho_3(t) = 1$ near $t = \infty$. We take $\rho_2(t) = 1 - \rho_1(t) - \rho_3(t)$ to be of compact support. We take $(-T, T) \subset \text{supp}(\rho_1 + \rho_2)$. This is a partition of unity.

Using Paley-Wiener’s theorem for $I_1$,

$$
|\int_{-\infty}^{\infty} e^{i\lambda t} U_1(t, x, x) \rho_1(t) dt| \leq C(1 + |\lambda|)^N e^{h(3\lambda)},
$$

for some $N \in \mathbb{N}$ and for some constant $C$. $h$ is the support function of $U_1(t, x, x) \rho_1(t)$. We just keep $I_1$. $N$ will be specified by Ivrii’s result [3, 4]. $I_1$ is holomorphic in $\mathbb{C}$.

We apply Paley-Wiener’s theorem to $I_2$. By (2.27), $U_1(t, x, x) \rho_2(t)$ is a smooth function with compact support. By construction $\rho_2(t)$ is the union of two cutoff functions. In this case,

$$
|I_2(\lambda)| \leq C_N (1 + |\lambda|)^{-N}, \forall N \in \mathbb{N}, \text{whenever } \lambda \in 0i + \mathbb{R}.
$$

This is a rapidly decreasing term.

For $I_3$, we use the theorem assumption by letting $N \to \infty$ in (1.13). We obtain $|I_3(\lambda)| \equiv 0$.

Therefore, the oscillatory integral

$$
\int_{\mathbb{R}} e^{i\lambda t} U_1(t, x, y) dt = I_1(\lambda),
$$

mod a rapidly decreasing term. Q.E.D.

For such a short-time wave trace, we can embed the support of the influenced set into a torus. This is finite speed of propagation.

3 A proof on Theorem 1.1

We now prove Theorem 1.1. We start with

$$
I_1(\lambda) := \int_{-\infty}^{\infty} e^{i\lambda t} U_1(t, x, x) \rho_1(t) dt.
$$

From (2.30), we have

$$
|I_1(\lambda)| \leq C(1 + |\lambda|)^N h|3\lambda|,
$$

for some $N \in \mathbb{N}$ and for some constant $C$. $h$ is the support function of $U_1(t, x, x) \rho_1(t)$. That means we may choose supp$(\rho_1(t))$ small such that

$$
|I_1(\lambda)| \leq Ce^{\delta|\lambda|}, \text{for some constant } C \text{ for any } \delta > 0.
$$

$I_1(\lambda)$ is entire function in of order 1 of minimal type. Moreover, we look at the local behavior of the wave trace. According to Ivrii [4], as $\lambda \to 0i \pm \infty$,

$$
I_1(\lambda) \sim a_1(x)|\lambda|^{n-2} + a_2(x)\lambda|\lambda|^{n-3} + \cdots,
$$

where $\omega_{n-1}$ is $n-1$-sphere volume and $\int_{\Gamma} a_1(x) dS_x = \pm \frac{1}{4}(2\pi)^{-n+1} \omega_{n-1} \text{volume}(\Gamma)$. The sign depends on the boundary conditions. We refer to [4] Theorem 2.1 and the remark thereafter for the Fourier transform of the short-time wave trace. We also disregard the rapidly decreasing term from Proposition 2.3. We need a Phragmén-Lindelöf type of lemma.
**Lemma 3.1** Let \( f(z), \ z = x + iy, \) be an entire function of exponential type \( \sigma \) such that \(|f(x)| \leq C(1 + |x|^d), \ d \in \mathbb{N}, \) then \( f(z)e^{-\sigma|y|} \leq C_1(1 + |z|^d). \) In particular, when \( \sigma = 0, \) then \( f(z) \) is a polynomial of degree no greater than \( d. \)

**Proof** This is stated in B. Ya. Levin’s book [5, p.39]. □

Hence,

**Lemma 3.2** \( I_1(\lambda) \) is a polynomial of degree no greater than \( n - 2. \)

In this case,

\[
\partial^{n-1}_\lambda I_1(\lambda) = \int_{-\infty}^{\infty} e^{\lambda t} (it)^{n-1} U_1(t, x, x) \rho_1(t) dt = 0, \forall \lambda \in \mathbb{C}. \tag{3.5}
\]

Moreover, Fourier inversion formula tells us

\[
t^{n-1}U_1(t, x, x) \equiv 0 \text{ in } \mathcal{D}'((\delta, \delta)), \forall \delta > 0. \tag{3.6}
\]

Since \( n \) is odd, (3.4) and (3.6) implies

\[
t^{n-1}\left\{ \frac{a_1(x)}{t^{n-1}} + a_2(x) (t^{n-3}) + \cdots + \text{constant term} + \cdots \right\} \equiv 0 \text{ in } \mathcal{D}'((\delta, \delta)). \tag{3.7}
\]

By distribution theory at \( t = 0, \)

\[a_1(x) = 0, \forall x \in \Omega.\]

In particular, \( \int_{\Gamma} \zeta(x)a_1(x) dS_x = 0, \forall \zeta \in C_0^\infty(\mathbb{R}^n_x, [0, 1]). \) In this case, \( \Gamma = \phi. \) Theorem 1.1 is proved.

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