Many-particle confinement by constructed disorder and quantum computing

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Abstract. Many-particle confinement (localization) is studied for a 1D system of spinless fermions with nearest-neighbor hopping and interaction, or equivalently, for an anisotropic Heisenberg spin-1/2 chain. This system is frequently used to model quantum computers with perpetually coupled qubits. We construct a bounded sequence of site energies that leads to strong single-particle confinement of all states on individual sites. We show that this sequence also leads to a confinement of all many-particle states in an infinite system for a time that scales as a high power of the reciprocal hopping integral. The confinement is achieved for strong interaction between the particles while keeping the overall bandwidth of site energies comparatively small. The results show viability of quantum computing with time-independent qubit coupling.

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1. Introduction

Spatial localization of excitations is central to quantum control and measurement. It is often required both for addressing excitations individually and for determining their states. Localization is particularly important for quantum computing. A quantum computer (QC) is usually thought of as a system of physical objects that can be modeled by two-level systems, qubits. The objects can be individually accessed to perform quantum operations. They also provide a natural physical basis for measurement. In many proposed implementations of QCs the qubit-qubit interaction is not turned off. Generally, the interaction leads to hopping of excitations between the qubits. The hopping is not related to relaxation, but they both may be devastating for QC operation. Therefore eliminating excitation hopping, or at least significantly slowing down the hopping rate, is a prerequisite for quantum computation.

The problem of localization has a long history in physics, most notably in condensed matter physics, where localization is due to broken translational symmetry. Here it is usually discussed for particles. The relation to the case of qubit excitations is particularly simple for one-dimensional systems. The presence of a particle on site $n$ corresponds to the $n$th qubit being excited, and the particle energy is the excitation energy. Much work has been done on single-particle localization. One of the best known examples is Anderson localization due to random disorder, which has been studied in depth and by now is well understood. It has been demonstrated also, both theoretically and experimentally, that single-particle localization occurs also in a different type of systems, where the symmetry breaking is due to an incommensurate periodic potential.

The work on localization in systems with many excitations is more recent. It has demonstrated that the particle-particle interaction can significantly affect localization. We note that numerical studies of this problem are not very helpful, because the Hilbert space of a many-particle system is exponentially large, with the exponent proportional to the number of particles, and therefore only systems with a small number of particles can be simulated with classical computers.

In the present paper we are interested in a specific formulation of the problem of localization. We study a many-particle (or equivalently, many-excitation) system and consider localization of all states in such system. More than that, bearing in mind applications to quantum control and quantum computing, we study strong on-site localization, where the wave function of each particle is mostly confined to one site. This is a stronger requirement than exponential decay of the wave functions at large distances. Formally it corresponds to the localization length being less than the intersite distance, for all states. We call such localization on-site confinement or simply confinement, for brevity.

Many-particle confinement does not arise in a disordered system with bounded random site energies. Indeed, consider a chain where particles occupy $N$ sites. For short-range hopping an $N$-particle state is directly coupled to $\sim N$ other $N$-particle
states. With probability $\propto N$ one of them will be in resonance with the initial state, provided the site energies are uniformly distributed over a finite-width band. This means that the energy difference between the two states will be less or of the order of the intersite hopping integral $J$ (we set $\hbar = 1$). Resonating states are hybridized over time $\sim J^{-1}$.

In contrast to random disorder, in a QC the disorder can often be constructed at will, because site excitation energies can be individually controlled. This makes it possible to look at many-particle localization from a new perspective. Not only is “localization by construction” necessary for quantum computing, but in itself it is an exciting area of applications of QCs, which is of special interest to condensed-matter physics. This is because the very possibility of localization of all stationary many-particle states is an open question, as the distance between the energy levels is exponentially small for a large number of particles.

In what follows we extend the previous work [17] to show that, for an infinite 1D system, many-particle confinement can be obtained by constructing a sequence of site energies. The proposed sequence has a comparatively narrow bandwidth. This is physically reasonable and is also important for applications in quantum computing, because the qubit tuning range is limited. Additionally, in a QC a smaller bandwidth is desirable for a higher speed of quantum gate operations.

Rather than studying the stationary states, we put emphasis on the localization lifetime, i.e., the time during which all many-particle states remain confined to the initially occupied sites. We show that this lifetime can largely exceed $J^{-1}$. We also find that, for moderately long sections of the constructed energy sequence, all stationary many-particle states can be strongly localized.

The idea of our approach is to map the problem of the lifetime of localized states on the problem of many-particle scattering. We first construct site energies that lead to strong confinement of all single-particle states. Then delocalization of many-particle states corresponds to resonant scattering in which, as a result of the interaction, particles move away from the initially occupied sites. For real scattering to happen, the final configuration must be close in energy to the initial configuration. The lifetime of a state is proportional to the inverse of the matrix element of hopping onto a resonating state.

A large localization lifetime is obtained if resonant scattering between many-particle configurations requires several virtual single-particle transitions via nonresonant states. The amplitudes of such transitions are small. The lifetime is determined by their minimal number, which we call the “order” of the transition. When it is limited to 4, it is sufficient to consider transitions that involve no more than 4 particles. The approach developed in this paper allows us to obtain long lifetime not for comparatively weak [17] but rather for strong interparticle coupling. As we show, this can be done without a significant increase of the bandwidth of site energies compared to the weak-coupling case. The strong-coupling confinement is of interest, in particular, for some models of QC’s, like a QC based on electrons on helium, where the dimensionless coupling parameter $\Delta$ may be as large as $\sim 10$ [18].
The paper is organized as follows. In Section II a qualitative picture of localization is provided. Section III contains a description of the effective localizing sequence \[17\] and a discussion of single-particle localization. In Sec. IV the localization lifetime is introduced and many-particle scattering is described. In Section V the bands of combination energies for many-particle transitions are analyzed and the conditions for eliminating resonances for transitions up to 4th order are found. Section VI contains a brief discussion of the results.

2. Localization by constructed disorder: preliminaries

Finding the ways of localizing excitations has been appreciated as an important problem of quantum computing. A natural formulation here is to map a QC onto a system of interacting spins, with \( S = 1/2 \). The spins are in a magnetic field, and the qubit excitation energy is the spin Zeeman energy. Several approaches to solving the localization problem have been put forward. An interesting idea substantiated by extensive numerical calculations is to arrange the Zeeman (site) energies into a ladder, so that none of them is in resonance with each other \[19\]. The bandwidth of site energies in this case linearly increases with the size of the system, and localization was studied for the Ising interaction between the spins. Another interesting idea is to create interaction-free subspaces \[20\]. The qubits that form these subspaces are separated from each other by the isolators, which are qubits prepared in a special state. The bandwidth of site energies remains finite in this case even for an infinite system, and the spin-spin interaction is not limited to the Ising one. However, the preparation requires that initially the interaction between the qubits is turned off.

We propose a different approach, largely inspired by the results on Anderson localization in condensed-matter physics. We show that, by constructing site disorder, confinement can be achieved for a finite energy bandwidth and for the nearest-neighbor interaction that has a general form and is never turned off.

Constructing a sequence of site energies \( \varepsilon_n \) that leads to single-particle (single-excitation) confinement is quite straightforward. If all \( \varepsilon_n \) are the same, as in Fig. 1(a), intersite hopping leads to formation of a band of fully delocalized states, with energy bandwidth proportional to the hopping integral \( J \).

The first step towards localization is to detune neighboring sites from each other by an energy \( h \gg J \), see Fig. 1(b). In this case nearest neighbor hopping becomes nonresonant. However, this is insufficient for localization, even though in quantum computing literature there are claims to the opposite. Indeed, a particle can make a virtual transition to a neighboring site and from there to the resonating next nearest site. The corresponding transition matrix element is \( J^2/h \). Therefore the states will be again band-like, with two bands of width \( \propto J^2/h \), which are centered at \( \pm h/2 \).

A natural next step is to detune the energies of 2nd neighbors. The detuning in this case should again exceed the matrix element of effective hopping. However, since the latter is \( \sim J^2/h \ll J \), the detuning can be smaller than \( h \) by a factor \( \sim J/h \), see
Figure 1. The successive detuning of site energies $\varepsilon_n$ in order to achieve single-particle localization in a chain. (a) All energies are the same, $\varepsilon_n = 0$. (b) Nearest neighbors are detuned from each other by an energy $h$. (c) Second, fourth, and higher-order neighbors are detuned from each other, with the detuning that becomes smaller and smaller with the increasing inter-site distance. The data in (b) and (c) correspond to the energy sequence $\mathcal{E}$ with $\alpha = 0$ and 0.1, respectively.

Fig. 1(c). At the next step it is necessary to detune 4th neighbors, but here the detuning can be smaller by an extra factor $J/h$. The process should be continued. It leads to a geometric-series type level detuning, with a finite overall bandwidth of the energy spectrum. A specific example is discussed below.

A significantly more complicated situation arises for a many-particle system. Here, one should tune away from each other not only site energies, but also their combinations. This is necessary in order to eliminate resonant multi-particle transitions. The total number of states for $N$ particles on an $L$-site long chain is $L!/N!(L-N)! \propto 4^N$ for $N = L/2$, and therefore for large $N$ the distance between the energy levels is exponentially small. This makes the problem of eliminating resonances hard. Nevertheless, as mentioned above, one can eliminate resonances that require at least a certain minimal number of virtual transitions to nonresonant states, leading to a long time over which all many-particle configurations remain confined to their sites.

3. Localization of stationary states

3.1. One-parameter sequence of site energies

A 1D spin system or a qubit chain can be mapped onto a system of spinless fermions via the Jordan-Wigner transformation $[21]$. The Hamiltonian of the fermions is

$$H = H_0 + H_i, \quad (1)$$

$$H_0 = \sum_n \varepsilon_n a_n^\dagger a_n + \frac{1}{2} J \sum_n (a_n^\dagger a_{n+1} + a_{n+1}^\dagger a_n),$$

$$H_i = J \Delta \sum_n a_{n+1}^\dagger a_{n+1} a_n.$$  

Here, $a_{n}^\dagger$, $a_n$ are the fermion creation and annihilation operators. The site fermion energies $\varepsilon_n$ are the Zeeman energies of the spins or the excitation energies of the qubits.

The spin-spin interaction leads to two effects, in terms of fermions. One is the intersite fermion hopping, with the hopping integral $J$. It is determined by the $XY$ spin-spin interaction $S_n^x S_{n+1}^x + S_n^y S_{n+1}^y$ or, in the case of qubits, by the interaction between the qubit transition dipoles. Fermion hopping is a single-particle effect: the $XY$ spin-spin interaction does not lead to interaction between the fermions. A different role is played
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by the Ising spin-spin interaction $S_n^z S_{n\pm 1}^z$, or, in the case of qubits, by the change of the transition energy of a qubit depending on whether or not the neighboring qubit is excited. It leads to the interaction of fermions, which is described by the Hamiltonian $H_i$. The parameter $\Delta$ characterizes the anisotropy of the spin-spin interaction: for the isotropic Heisenberg model $\Delta = 1$. For concreteness we set $J, \Delta > 0$.

A simple bounded sequence of site energies that implements the construction described in Sec. 2 for a semi-infinite chain with $n \geq 1$ has the form \[\varepsilon_n = \frac{1}{2} h \left[ (-1)^n - \sum_{k=2}^{n+1} (-1)^{\lfloor n/k \rfloor} \alpha^{k-1} \right], \tag{2}\]

where $\lfloor \cdot \rfloor$ is the integer part.

Besides the energy scale $h$, sequence (2) is characterized by one dimensionless parameter $\alpha < 1$. The overall energy bandwidth is $W \lesssim \frac{h}{1 - \alpha}$. For $\alpha \ll 1$ the energies \[\text{form two subbands centered at } \pm \frac{h}{2}.\]

Numerically, the subbands remain well separated for $\alpha < 0.4$.

3.2. Single-particle localization

The expression (2) has a special symmetry: as $n$ increases, the coefficients at any given power $\alpha^q$ are repeated with period $2(q + 1)$. This symmetry was used to obtain an analytical expression for the amplitude $K_n(m)$ of a single-particle transition from site $n$ to site $n + m$ \[K_n(m) = \alpha^{-m} (J/2h)^m. \tag{3}\]

The exponent $\nu$ is determined by $\lim_{m \to \infty} m^{-1} \log[K_n(m)]$. It is not the reciprocal average decay length: it varies from site to site, $\nu \equiv \nu(n)$, as shown in Fig. 2.

![Figure 2](image_url)

**Figure 2.** The exponent $\nu = \nu(n)$ of the $\alpha$ dependence of the transition amplitude $K_n(m)$ \[\text{as a function of site } n.\]

The values of $\nu$ for the same site but for positive and negative $m$ are different but lie within the same bounds.

The values of $\nu$ are bounded to a narrow region centered at $\nu = 1$, with $0.89 < \nu < 1.19$. For estimates one can use $\nu = 1$, i.e.,

$$K_n(m) \approx K^m, \quad K = J/2\alpha h. \tag{4}$$
Along with decay “forward”, i.e., for $m > 0$, we have studied decay “backward”, i.e., for $m < 0$. Because of the symmetry of the sequence we expect that the decay backward should be also quasi-exponential in $m$, and the bounds on $\nu$ should be the same as for positive $m$. The numerical data in Fig. 2 substantiate this conjecture. Therefore in the estimates given below we use Eq. (4) in the form $K_n(m) \approx K^{|m|}$, which applies to both positive and negative $m$.

The characteristic decay length of the transition amplitude is $1/|\ln K|$. For $K \ll 1$, i.e. $J \ll 2\alpha h$, all single-particle states are strongly localized and essentially confined to single sites, with small tails on neighboring sites.

### 3.3. Inverse participation ratio

The extent to which stationary states in a system are confined can be conveniently characterized by the inverse participation ratio (IPR). For an $N$-particle eigenstate $|\psi_{N\lambda}\rangle$ ($\lambda$ enumerates the eigenstates) it is given by

$$I_{N\lambda} = \frac{1}{\left( \sum_{n_1<...<n_N} |\langle \Phi_{n_1...n_N}|\psi_{N\lambda}\rangle|^4 \right)^{-1}},$$

(5)

where $|\Phi_{n_1...n_N}\rangle$ is an on-site $N$-particle wave function (quantum register) with particles on sites $n_1, ..., n_N$.

If the stationary state $|\psi_{N\lambda}\rangle$ is fully localized, all terms in the sum (5) are equal to 0 except one, which is equal to 1, and $I_{N\lambda} = 1$. In the opposite case of delocalized eigenstates all matrix elements in the sum (5) are small, and $I_{N\lambda} \gg 1$ for a large system. The on-site confinement that we are interested in corresponds to $I_{N\lambda}$ being close to 1 for all states $\lambda$.

Figure 3. The maximum inverse participation ratio $I_{N\max}$ for one particle ($N = 1$) in an $L$-site chain. The energies $\epsilon_n$ are given by Eq. (2), and $h/J = 20$. The data refer to the segments of the chain with $1 \leq n \leq L$. The plateau and sharp isolated peaks in the region $0.03 \lesssim \alpha < 0.1$ result primarily from pairwise hybridization of states. An example of the peak for $L = 300$ is shown in the inset on the linear in $\alpha$ scale. In the whole region $0.1 < \alpha \lesssim 0.4$ the IPR $I_{1\max} \approx 1.003$, which indicates strong confinement of all stationary states. The asymptotic localization threshold $\alpha = J/2h$ is shown by the dashed line.
In Fig. 3 we show the results of the numerical calculation of the maximal IPR $I_{N \text{max}} = \max_\lambda I_{N\lambda}$ for energy sequence (2) for one particle, $N = 1$. In agreement with Eqs. (3), (4), all single-particle states are strongly localized for $\alpha \geq 0.1$, where $K \leq 0.25$. It is seen from Fig. 3 that $I_{N \text{max}}$ sharply drops down for $\alpha \approx 0.03$, which is close to the asymptotic value $J/2h = 0.025$ given by the condition $K = 1$. For $0.03 \lesssim \alpha \lesssim 0.05$ the IPR is close to 2 for all $L$, which indicates that only a few on-site states are appreciably hybridized; for example, $I_{1\lambda} = 2$ if two on-site states are hybridized with equal weight in the wave function $|\psi_{1\lambda}\rangle$.

Depending on the chain length, pairwise hybridization also occasionally occurs for particular values of $\alpha$ in the range $0.05 \lesssim \alpha < 0.1$. An example is the peak for $L = 300$ shown in the inset of Fig. 3. It is due to hybridization of the states on sites 296 and 300 [17]. The hybridization happens because the interaction $J a_n^\dagger a_{n+1}$ shifts the site energies by $\sim J^2/h$. The shift is different for the state on the boundary, since it has less neighbors. In the particular case of the chain with $L = 300$ a simple calculation shows that this difference leads to resonance between the aforementioned on-site states. For the chosen $J/2h$ in the region $\alpha \lesssim 0.06$ the wave functions have longer tails than what follows from the asymptotic expression (3), and therefore the states away from the boundary are still affected by it. This leads to the peak of $I_{N \text{max}}$ for $L = 500$, $\alpha = 0.056$ seen in Fig. 3. Changing $L$ in the region $\alpha \lesssim 0.06$ leads to disappearance of some peaks and onset of other peaks, with hybridization of different states. It is important, however, that for $0.4 \gtrsim \alpha \gtrsim 0.1$ these peaks disappear and all states are strongly localized.

We have also studied [17] a chain of length $L = 12$ with $N = L/2$ excitations (where the number of states is maximal, for a given $L$). For the section of the chain $1 \leq n \leq L$, the maximal IPR $I_{6 \text{max}} \leq 1.02$ for $h/J = 20$, in a broad range $0.2 \lesssim \alpha \lesssim 0.4$, except for sharp isolated peaks for certain $\alpha$. This indicates strong confinement of all many-particle states. However, the IPR for large $L$ and $N$ could not be studied because of the large number of states.

### 4. The localization lifetime

Confining a many-particle system is much more complicated than a single-particle one. It requires eliminating all possible many-particle combinational resonances $\varepsilon_{n_1} + \ldots + \varepsilon_{n_k} \approx \varepsilon_{m_1} + \ldots + \varepsilon_{m_l}$. Therefore we approach the problem of strong many-particle confinement from a different perspective. Rather than studying stationary states, we will consider the time over which interacting particles leave the sites where they were placed initially. We call this time the localization lifetime, $t_{\text{loc}}$.

The decay of on-site states is not exponential, at least for not too long times, and the time $t_{\text{loc}}$ is not the decay time. On-site states evolve primarily due to resonant transitions to other on-site states with the same energy. The transition rates between different resonating many-particle states vary strongly provided the single-particle states are all strongly localized. First there occurs hybridization of a given on-site many-particle state with the resonating state that is most strongly coupled to it. Then these states may
further hybridize with other resonating states, but this may take much longer. For our purpose, the localization lifetime is the time of the first hybridization. In other words, $t_{\text{loc}}$ is the reciprocal maximal rate of a transition to a resonating on-site state.

An important characteristic of the dynamics of quantum systems is the coherence time $\sim t_{\text{coh}}$. In any real system $t_{\text{coh}}$ is finite due to thermal fluctuations, decay into excitations of the medium, and fluctuations induced by external noise. We are interested in strong coherent confinement, in which case $t_{\text{loc}}$ calculated in the neglect of decoherence is $\gtrsim t_{\text{coh}}$. We note that, in fact, decoherence can increase the time particles spend on their sites by suppressing coherent quantum transitions; however, this suppression is outside the scope of the present paper.

In systems proposed for quantum computing $J$ and $J\Delta$ characterize the rate of two-qubit operations, and typically the number of operations that can be performed over the coherence time is $\lesssim 10^5$. Therefore, for practical purposes, in a quantum computer excitations will be localized provided $t_{\text{loc}}$ exceeds $J^{-1}$ or $(J\Delta)^{-1}$ by a factor $10^5$. This condition must be satisfied for all many-excitation states and should apply to an infinite system for the quantum computer to be scalable.

4.1. Many-particle scattering

Even though the intersite transitions themselves are single-particle, resonant many-particle transitions may happen even where single-particle resonances are eliminated. This is a consequence of the energy exchange between interacting particles and also the energy change due to the interaction. The interaction (II) is nearest-neighbor two-particle. Therefore a many-particle transition results from a sequence of single-particle intersite transitions and scattering of particles on neighboring sites. In what follows we assume that, even though the particle-particle coupling may be strong, $\Delta \gg 1$, still $\Delta \ll h/J$. Then nonresonant many-particle effects can be considered by perturbation theory.

A simple example of a Feynman diagram that describes a two-particle transition is shown in Fig. 4. In this example particles located initially on sites $(n, n+3)$ first make two single-particle transitions $(n, n+3) \rightarrow (n, n+2) \rightarrow (n, n+1)$, then experience scattering by each other and exchange their energies, and then make two more single-particle transitions $(n, n+1) \rightarrow (n, n+2) \rightarrow (n-1, n+2)$. The overall sequence of transitions will be resonant if $\varepsilon_n + \varepsilon_{n+3} \approx \varepsilon_{n-1} + \varepsilon_{n+2}$.

An alternative approach to the analysis of many-particle scattering is based on diagonalizing the single-particle part of the Hamiltonian (II) (II). Since all single-particle states are strongly confined, the resulting states are very close to on-site states. However, the particle-particle interaction acquires a small off-diagonal long-range part. It is clear in this approach that it is the interaction that leads to resonant many-particle scattering.

From inspection of the diagram in Fig. 4 it is obvious that, for a many-particle transition to occur, the particles have to be brought to neighboring sites. This requires single-particle transitions via nonresonant sites. For a two-particle resonant transition
Figure 4. A diagram for a two-particle transition from sites \((n, n+3)\) to \((n-1, n+2)\). The crosses indicate single-particle transitions to neighboring sites, and the dashed line indicates scattering of two particles located on neighboring sites.

\((k_4, k_3) \leftrightarrow (k_1, k_2)\) (we choose for concreteness \(k_3 > k_4\) and \(k_2 > k_1\)) the minimal number of the needed single-particle transitions is given by the parameter

\[
\varpi = \min_p (|k_1 - p| + |k_2 - p - 1| + |k_3 - p - 1| + |k_4 - p|).
\]

(6)

This is easy to see: the particles move from the initially occupied sites to neighboring sites \((p, p+1)\) and from there to the final sites; the minimum over \(p\) in Eq. (6) is taken so as to minimize the total distance involved in the transition.

For large \(\varpi\), the overall amplitude of the involved single-particle transitions is \(K^{\varpi}\), according to Eq. (4). Since the two-particle interaction energy is \(J\Delta\), the localization lifetime of the state \((k_4, k_3)\) with respect to a transition to the state \((k_1, k_2)\) as given by the reciprocal transition matrix element is

\[
t_{\text{loc}} \approx (J\Delta K^{\varpi})^{-1}, \quad \varpi \gg 1.
\]

(7)

This estimate is made assuming that the transition involves only one particle-particle collision. The role of multiple collisions will be studied in a separate paper [22].

We are interested in the localization lifetime with respect to all resonant transitions and for all states. Then, from Eq. (7), \(\varpi\) should exceed a certain minimal value \(\varpi_{\text{min}}\) for all resonant transitions. This means that all resonances separated by less than \(\varpi_{\text{min}}\) steps must be eliminated.

In what follows we will show explicitly how, in an infinite system, all resonances with \(\varpi \leq 4\) can be eliminated for moderately large \(h/J \ll 10^2\) even for \(\Delta\) as large as \(\Delta \sim 10\). Then, with \(K \sim 0.1\), the single-collision estimate (7) leads to \(t_{\text{loc}} \gtrsim 10^5 (J\Delta)^{-1}\).

The important simplification from considering resonances with \(\varpi \leq 4\) is that the analysis can be limited to transitions in which only two, three, and four particles change their sites. We leave the analysis of three- and four-particle transitions for a separate paper [22]; in the present paper we will only give its result and concentrate on eliminating resonances related to two-particle transitions.

The change of the on-site energy in a two-particle transition \((k_4, k_3) \leftrightarrow (k_1, k_2)\) is

\[
\delta \varepsilon = |\varepsilon_{k_1} + \varepsilon_{k_2} - \varepsilon_{k_3} - \varepsilon_{k_4}|.
\]

(8)

To avoid resonances, this energy difference should largely exceed the transition matrix element. However, this is not enough, because of the change of the interaction energy that may accompany the transition. To the lowest order in \(J/h\) it is determined by the
change of the number of nearest neighbors of the hopping particles. The analysis of diagrams of the type shown in Fig. 4 shows that, for $\kappa \leq 4$, this number may remain unchanged or change by one or two, in which case the interaction energy will change by $J\Delta$ or $2J\Delta$.

In the previous work [17] we studied the case $\Delta \lesssim 1$. In this case it was sufficient to open a gap at $\delta \varepsilon = 0$ with width $> 2J\Delta$. However, numerically the gap was relatively narrow, it was limited to $\delta \varepsilon / h \sim 0.01$. Therefore a direct application of the results to the case of large $\Delta$ would require using very large $h/J$, which is undesirable. In the next Section we develop an alternative approach that applies in the strong-interaction case.

5. The spectrum of two-particle transition energies

The overall structure of the differences of site energies for two-particle transitions $\delta \varepsilon$ is fairly simple. If $\alpha = 0$ in Eq. (2), we have $\varepsilon_n = \pm h/2$. Then from Eq. (5) $\delta \varepsilon$ takes on three values, $\delta \varepsilon = 0, h, 2h$. With increasing $\alpha$ each of these values splits into a band. For small $\alpha$ the bandwidth is $\propto \alpha h$ and the bands are well separated.

On the other hand, the change of the interaction energy in a two-particle transition is close either to zero or $J\Delta$ or $2J\Delta$, as explained above. To avoid resonances related to this energy change, it should lie within the gaps of $\delta \varepsilon$.

A natural way to obtain strong confinement while keeping the bandwidth $h$ of site energies moderately large is the following. All transitions where the interaction energy change is close to zero should have $\delta \varepsilon$ that is larger than this change but may be still much smaller than $h$, see below. On the other hand, all transitions where the interaction energy change is $J\Delta$ or $2J\Delta$ should have $\delta \varepsilon$ in the gap centered at $h/2$. Then the ratio $J\Delta/h$ does not have to be extremely small. Even though this is not the only option, we found that it allows using not too small values of $\alpha$, which is advantageous in terms of robustness with respect to errors in the site energies. The inequality $J\Delta/h < 1/2$ imposes a much less restrictive condition on the bandwidth $h$ than the requirement that $2J\Delta$ is smaller than the gap of $\delta \varepsilon$ at zero energy [17].

We denote the widths of the gaps of $\delta \varepsilon$ at energies $0, J\Delta,$ and $2J\Delta$ by $\delta_0, \delta_1,$ and $\delta_2,$ respectively. These widths should exceed the matrix elements of the appropriate transitions. They should also be larger than the energy renormalization due to (possible) occupation of the second neighbors of the sites involved in a transition. The renormalization can be obtained by noticing that the energy change is due to making a virtual transition on a neighboring site, interacting with the particle on the next neighboring site, and making a transition back. This gives the energy shift $\sim J\Delta(J/2h)^2$. Therefore we need

$$\delta_{0,1,2} \gg J^3\Delta/(2h)^2.$$
5.1. Zero-energy gap

Meeting the condition (9) for the zero-energy gap $\delta_0$ requires a modification of the site energy sequence (2). The most “dangerous”, for $\kappa \leq 4$, is the resonance in $\delta\varepsilon$ associated with a transition $(n, n + 1) \leftrightarrow (n - 1, n + 2)$ [17]. For this transition $\kappa = 2$, whereas the energy difference is $\delta\varepsilon \sim \alpha^2 \hbar$ with $\xi \geq 4$ for all $n = 6k - 1$. This is a very small $\delta\varepsilon$ for $\kappa = 2$, because the transition matrix element is comparatively large in this case. In addition we have $\delta\varepsilon \sim \alpha^{n-1} \hbar$ when $n$ and $n + 2$ are prime numbers, which leads to $\delta_0 \to 0$ in an infinite chain.

A simple way to eliminate this resonance and to open a gap $\delta_0$ is to modify energy sequence (2) by shifting the energy on each 6th site, 

$$
\varepsilon_{md}^n = \varepsilon_n + (h/2)\alpha' \quad \text{for} \quad n = 6k,
$$

while keeping it unchanged on other sites, $\varepsilon_{md}^n = \varepsilon_n$ for $n \neq 6k$ [17]. The parameter $\alpha'$ should be chosen in such a way as to avoid creating new resonances.

All $\kappa \leq 4$ transitions with small $\delta\varepsilon$ can be found by modifying the diagram in Fig. 4. The rule is simply that the numbers of even and odd sites should be the same at the input and output [22]. The result coincides with what was obtained earlier [17] using a different technique. The zero-energy gap for the sequence (10) is $\delta_0 \gtrsim \alpha^3 \hbar$ for $\alpha \gg \alpha' \gg \alpha^3$. This gap is illustrated in Fig. 5 where we plot the change in modified site energies

$$
\delta\varepsilon_{md}^n = |\varepsilon_{md}^{k_1} + \varepsilon_{md}^{k_2} - \varepsilon_{md}^{k_3} - \varepsilon_{md}^{k_4}|
$$

for all $\kappa \leq 4$ transitions in which one of the sites $k_{1,2,3,4}$ is $n$.

![Figure 5](image)

**Figure 5.** The low-energy parts of the two-particle energy differences $\delta\varepsilon_{md}^n$ (11) for the transitions with $\kappa \leq 4$ in which one of the involved particles is on the $n$th site ($n > 2$). The left and right panels refer to the energy sequence (2), (10) with $\alpha = 0.25, \alpha' = 0.22$ and $\alpha = 0.1, \alpha' = 1/15$, respectively. In agreement with the analytical estimates, the zero-energy gap $\delta_0/\hbar \sim \alpha^3$.

With the appropriately chosen $\alpha'$, the condition on $\alpha$ imposed by Eq. (9) takes the form

$$
\alpha^3 \gg 2(J/2\hbar)^3 \Delta.
$$

(12)

It can be seen that, when Eq. (9) holds, not only is the many-particle renormalization of site energies small, but also the matrix elements of $\kappa \leq 4$ transitions turn out to be
We will be interested in the gap for all \( \varkappa \leq 4 \). To leading order in \( \varkappa \), the bands at energies 0 and \( h \) are bounded by \( (2\alpha + \varkappa)/2h \). The distance from \( \alpha h \) to the band edges should significantly exceed the transition matrix elements and the energy renormalization \( J\Delta(J/2h)^2 \); these quantities are small compared to \( \alpha h \).

5.2. High-energy gap

We now discuss the gap in the two-particle transition energies \( \delta\varepsilon_n^{\text{md}} \) centered at \( h/2 \). We will be interested in the gap for all \( n \) and for all transitions with \( \varkappa \leq 4 \). The gap is determined by the widths of the energy bands of \( \delta\varepsilon_n^{\text{md}} \) centered at \( \delta\varepsilon_n^{\text{md}} = 0 \) and \( \delta\varepsilon_n^{\text{md}} = h \) for the energy sequence \( \{2, 10\} \). Because the sequence is regular, the single-energy distribution has no tails and the bands have sharp boundaries.

It is seen from Eqs. \( \{2, 10\} \) that, in a two-particle transition \( (k_4, k_3) \leftrightarrow (k_1, k_2) \), the terms in \( \varepsilon_{k_1}, \ldots, \varepsilon_{k_4} \) that are linear in \( \alpha \) may all add up or pairwise compensate each other. The sum of all of them can be equal therefore to \( \pm 2\alpha h, \pm \alpha h \), or 0. Similarly, the term \( \alpha' h/2 \) can be added or subtracted. Note that, for \( \varkappa \leq 4 \), only one term \( \alpha \) can contribute to \( \delta\varepsilon_n^{\text{md}} \). As a result, the width of the band of \( \delta\varepsilon_n^{\text{md}} \) at \( \delta\varepsilon_n^{\text{md}} = 0 \) is \( h(2\alpha + \alpha'/2) \), whereas the width of the band at \( \delta\varepsilon_n^{\text{md}} = h \) is \( h(4\alpha + \alpha') \).

The changes of the site energies in 3-particle transitions with \( \varkappa \leq 4 \)

\[
|\varepsilon_{k_1}^{\text{md}} + \varepsilon_{k_2}^{\text{md}} - \varepsilon_{k_3}^{\text{md}} - \varepsilon_{k_4}^{\text{md}} - \varepsilon_{k_5}^{\text{md}} - \varepsilon_{k_6}^{\text{md}}|
\]

also form bands centered at 0, \( h \) and \( 2h \). To leading order in \( \alpha, \alpha' \), the bands at energies 0 and \( h \) are bounded by \( (2\alpha + \alpha')h \) and \( h(1 \pm 3\alpha \pm \alpha') \). Interestingly, four-particle transitions with \( \varkappa \leq 4 \) do not change these bounds \( \{22\} \).

From this analysis, the gap in the changes of site energies of all \( \varkappa \leq 4 \) transitions \( \delta\varepsilon_{\varkappa \leq 4} \) is given by the condition

\[
2\alpha + \alpha' < h^{-1}(\delta\varepsilon_{\varkappa \leq 4}) < 1 - 3\alpha - \alpha'.
\] (13)

Higher-order terms in \( \alpha \) can be taken into account by replacing \( \alpha \to \alpha/(1 - \alpha) \) in Eq. \( \{13\} \). A direct analysis of the terms \( \sim \alpha^2 \) shows that this gives a stronger inequality than what actually follows from Eqs. \( \{2, 10\} \).

The conditions for eliminating resonant transitions with \( \varkappa \leq 4 \) in an infinite chain are given by Eq. \( \{13\} \) in which \( h^{-1}(\delta\varepsilon_{\varkappa \leq 4}) \) is replaced by \( J\Delta \) and \( 2J\Delta \). The distance from \( J\Delta, 2J\Delta \) to the band edges should significantly exceed the transition matrix elements and the energy renormalization \( J\Delta(J/2h)^2 \); these quantities are small compared to \( \alpha h \).
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Figure 6. The maximal inverse participation ratio $I_{6 \text{ max}}$ for a system of 6 particles on a 12-site chain. The site energies $\varepsilon_n^{\text{md}}$ are given by Eqs. (2), (10) with $\alpha' = 1/15$. The left and right panels corresponds to the sections of the chain with $415 \leq n \leq 426$ and $600 \leq n \leq 611$, respectively. The coupling parameter is $\Delta = 9$, and $h/J = 30$. A broad region of $\alpha$ centered at $\alpha = 0.1$ where the IPR is extremely close to 1 is seen in all sections of the chain that were tested.

The inequalities (12), (13) show that the range of the parameters $h/J, \Delta$, and $\alpha$ where two-particle resonances with $\nu \leq 4$ are eliminated is fairly large. One may expect that, in this range, stationary many-particle states will be mostly confined to their sites, at least for a not too long section of the chain. This can be probed by calculating the many-particle IPR for a strongly coupled system. The results of these calculations for modified energy sequence (10) are shown in Fig. 6. It is seen from this figure that, in a broad range of $\alpha$ where the localization lifetime is expected from Eqs. (12), (13) to be long, the IPR is extremely close to one. This shows that all stationary many-particle states are strongly confined.

6. Conclusions

In this paper we studied strong single- and many-particle confinement. We have shown that the previously proposed energy sequence (2) leads to strong confinement of all single-particle states. The decay of the wave functions is quasi-exponential in both directions in the chain, with the same bounds on the decay length.

We have also considered the problem of many-particle confinement. Here we studied the localization lifetime $t_{\text{loc}}$ during which all particles in an infinite chain remain on their sites. Of central interest was an extension of the results on long localization lifetime (17) to the case of strong particle-particle coupling, $\Delta \gg 1$. We map the problem of many-particle transitions onto the problem of resonant scattering. A natural characteristic in the analysis of scattering is the difference between the initial and final site energies $\delta \varepsilon$. Large $t_{\text{loc}}$ compared to the reciprocal hopping integral $J^{-1}$ was obtained earlier (17) when the zero-energy gap $\delta_0$ in $\delta \varepsilon$ for all transitions with up to 4 single-particle moves exceeded twice the coupling energy $2J\Delta$. For $\Delta \gg 1$ this would require a very large width of the band of site energies.

We have shown that, for large $\Delta$, a better option may be to decrease the bandwidths of the site energies of second neighbors in the chain, which leads to separation and narrowing of bands of $\delta \varepsilon$. Then, if $J\Delta$ and $2J\Delta$ are within the interband gaps of $\delta \varepsilon$, the inequalities (12), (13) show that the range of the parameters $h/J, \Delta$, and $\alpha$ where two-particle resonances with $\nu \leq 4$ are eliminated is fairly large. One may expect that, in this range, stationary many-particle states will be mostly confined to their sites, at least for a not too long section of the chain. This can be probed by calculating the many-particle IPR for a strongly coupled system. The results of these calculations for modified energy sequence (10) are shown in Fig. 6. It is seen from this figure that, in a broad range of $\alpha$ where the localization lifetime is expected from Eqs. (12), (13) to be long, the IPR is extremely close to one. This shows that all stationary many-particle states are strongly confined.

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resonant transitions are eliminated. As a result, the lifetime $t_{\text{loc}}$ scales as a high power of the reciprocal hopping integral, $t_{\text{loc}} \propto (J\Delta)^{-1}(2\alpha h/J)^6$ for transitions that involve one particle-particle collision. The overall bandwidth of site energies required for obtaining a long localization lifetime is parametrically smaller than the one given by the condition $\delta_0 > 2J\Delta$. The major limitation on the bandwidth is that it should substantially exceed the coupling energy. The results bear on many-electron conductivity in condensed-matter systems and on quantum computing with perpetually coupled qubits. They demonstrate the possibility to avoid delocalization of excitations in a QC without refocusing.

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