Location and Weyl formula for the eigenvalues of some non self-adjoint operators

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Abstract We present a survey of some recent results concerning the location and the Weyl formula for the complex eigenvalues of two non self-adjoint operators. We study the eigenvalues of the generator $G$ of the contraction semigroup $e^{tG}$, $t \geq 0$, related to the wave equation in an unbounded domain $\Omega$ with dissipative boundary conditions on $\partial \Omega$. Also one examines the interior transmission eigenvalues (ITE) in a bounded domain $K$ obtaining a Weyl formula with remainder for the counting function $N(r)$ of complex (ITE). The analysis is based on a semi-classical approach.

1 Introduction

Let $P(x,D_x)$ be a second order differential operator with $C^\infty(K)$ real-valued coefficients in a bounded domain $K \subset \mathbb{R}^d$, $d \geq 2$, with $C^\infty$ boundary $\partial K$. Consider a boundary problem

\[
\begin{aligned}
P(x,D_x)u &= f \text{ in } K, \\
B(x,D_x)u &= g \text{ on } \partial K,
\end{aligned}
\]

where $B(x,D_x)$ is a differential operator with order less or equal to 1 and the principal symbol $P(x,\xi)$ of $P(x,D_x)$ satisfies $p(x,\xi) \geq c_0|\xi|^2$, $c_0 > 0$. Assume that there exists $0 < \varphi < \pi$ such that the problem

\[
\begin{aligned}
(P(x,D_x) - z)u &= f \text{ in } K, \\
B(x,D_x)u &= g \text{ on } \partial K,
\end{aligned}
\]

is parameter-elliptic for every $z \in \Gamma_\varphi = \{z : \arg z = \varphi\}$, $0 < |\psi| \leq \varphi$. Then following a classical result of Agranovich-Vishik [1] we can find a closed operator $A$ with

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domain $D(A) \subset H^2(K)$ related to the problem (1.1). Moreover, for every closed angle $Q = \{z \in \mathbb{C} : \alpha \leq \arg z \leq \beta\} \subset \{z \in \mathbb{C} : |\arg z| < \phi\}$ which does not contain $\mathbb{R}^+$ there exists $a_Q > 0$ such that the resolvent $(A - z)^{-1}$ exists for $z \in Q, |z| \geq a_Q$ and the operator $A$ has a discrete spectrum in $\mathbb{C}$ with eigenvalues with finite multiplicities.

Let $\{\lambda_j\}_{j=1}^\infty$ be the eigenvalues of $A$ ordered as follows

$$0 \leq |\lambda_1| \leq \ldots \leq |\lambda_m| \leq \ldots$$

In general $A$ is not a self-adjoint operator and the analysis of the asymptotics of the counting function

$$N(r) = \#\{ |\lambda_j| \leq r \} \text{ as } r \to +\infty,$$

where every eigenvalues is counted with its multiplicity, is a difficult problem. In particular, it is quite complicated to obtain a Weyl formula for $N(r)$ with a remainder and many authors obtained results which yield only the leading term of the asymptotics. On the other hand, even for parameter-elliptic boundary problems the result in [1] says that in any domain $0 < \psi < |\arg z| < \phi$ we can have only finite number eigenvalues but we could have a bigger eigenvalues-free domains. To obtain a better remainder in the Weyl formula for $N(r)$ we must obtain a eigenvalues-free region outside some parabolic neighborhood of the real axis.

On the other hand, in mathematical physics there are many problems which are not parameter-elliptic. Therefore, the results of [1] cannot be applied and the analysis of the eigenvalues-free regions must be studied by another approach.

For the spectrum of non self-adjoint operators we have three important problems:

(I) Prove the discreteness of the spectrum of $A$ in some subset $U \subset \mathbb{C}$.

(II) Find eigenvalues-free domains in $\mathbb{C}$ having the form

$$|\text{Im } z| \geq C\delta(|\text{Re } z| + 1)^\delta, \pm \text{Re } z \geq 0, 0 < \delta, < 1,$$

(III) Establish a Weyl asymptotic with remainder for the counting function

$$N(r) = cr^d + o(r^{d-k}), r \to \infty, 0 < k < 1.$$

In this survey we discuss mainly the problems (II) and (III) for two non self-adjoint operators related to the scattering theory. The problem (I) is easier to deal with and the analysis of (II) in many cases implies that $A - z$ is a Fredholm operator for $z$ in a suitable region. We apply a new semi-classical approach for both problems (II) and (III). The analysis of (II) is reduced to the invertibility of a $h$-pseudodifferential operator, while for the asymptotic of $N(r)$ one exploits in a crucial way the existence of parabolic neighborhood of the real axis containing the (ITE). The purpose of this survey is to present the recent results in [21], [22], [14], [15], [13], [6], where the above problems are investigated by the same approach. We expect that our arguments can be applied to more general non self-adjoint operators covering the case of parameter-elliptic boundary problems [1,2].
2 Two spectral problems related to the scattering theory

I. Let \( K \subset \mathbb{R}^d, d \geq 2 \), be a bounded non-empty domain and let \( \Omega = \mathbb{R}^d \setminus K \) be connected. We suppose that the boundary \( \Gamma \) of \( \Omega \) is \( C^\infty \). Consider the boundary problem

\[
\begin{aligned}
&u_t - \Delta u = 0 \text{ in } \mathbb{R}^d_+ \times \Omega, \\
&\partial_v u - \gamma(x) u_t = 0 \text{ on } \mathbb{R}^d_+ \times \Gamma,
\end{aligned}
\]

with initial data \( f = (f_1, f_2) \in H^1(\Omega) \times L^2(\Omega) = H \). Here \( \gamma(x) \) is the unit outward normal to \( x \in \Gamma \) pointing into \( \Omega \) and \( \gamma(x) \geq 0 \) is a \( C^\infty \) function on \( \Gamma \). The solution of (2.1) is given by

\[
(u, u_t) = V(t) f = e^{tG} f, \quad t \geq 0,
\]

where \( V(t) \) is a contraction semi-group in \( H \) whose generator

\[
G = \begin{pmatrix} 0 & 1 \\ \Lambda & 0 \end{pmatrix}
\]

has a domain \( D(G) \) which is the closure in the graph norm of functions \( f = (f_1, f_2) \in C^\infty_0(\mathbb{R}^d) \times C^\infty_0(\mathbb{R}^d) \) satisfying the boundary condition \( \partial_v f_1 - \gamma f_2 = 0 \) on \( \Gamma \). The spectrum of \( G \) in \( \text{Re } z < 0 \) is formed by isolated eigenvalues with finite multiplicity (see [11] for \( d \) odd and [13] for \( d \) even), while the continuous spectrum of \( G \) coincides with \( i \mathbb{R} \). Next, if \( Gf = \lambda f \) with \( f = (f_1, f_2) \neq 0 \), we get

\[
\begin{aligned}
&\left( \Lambda - \lambda^2 \right) f_1 = 0 \text{ in } \Omega, \\
&\partial_v f_1 - \gamma f_1 = 0 \text{ on } \Gamma.
\end{aligned}
\]

Thus if \( \text{Re } \lambda < 0, f \neq 0, (u(t,x), u_t(t,x)) = V(t)f = e^{\lambda t}f(x) \), then \( u(t,x) \) is a solution of (2.1) with exponentially decreasing global energy. Such solutions are called asymptotically disappearing and they perturb the inverse scattering problems. Recently, it was proved (see [5]) that if we have at least one eigenvalue \( \lambda \) of \( G \) with \( \text{Re } \lambda < 0 \), then the wave operators \( W_\pm \) related to the problem (2.1) and the Cauchy problem for the wave equation are not complete, that is \( \text{Ran } W_- \neq \text{Ran } W_+ \). Hence we cannot define the scattering operator \( S \) related to (2.1) by \( S = W_+^{-1} \circ W_- \). We may define \( S \) by another evolution operator. For problems associated to unitary groups, the associated scattering operator \( S(z) : L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{S}^{d-1}) \) satisfies the equality

\[
S^{-1}(z) = S^*(\bar{z}), \quad z \in \mathbb{C},
\]

provided that \( S(z) \) is invertible at \( z \). This implies that \( S(z) \) is invertible for \( \text{Im } z > 0 \), since \( S(z) \) and \( S^*(z) \) are analytic for \( \text{Im } z < 0 \) (see [10] for more details). For dissipative boundary problems the above relation is not true and \( S(z_0) \) may have a non trivial kernel for some \( z_0, \text{Im } z_0 > 0 \). In the case of odd dimensions \( d \) Lax and Phillips [11] proved that \( i z_0 \) is an eigenvalue of \( G \). Consequently, the analysis of the location of the eigenvalues of \( G \) is important for the inverse scattering problems.
The eigenvalues of $G$ are symmetric with respect to the real axis, so it is sufficient to examine the location of the eigenvalues whose imaginary part is nonnegative. A. Majda [12] proved that if $\sup_{x \in \Gamma} \gamma(x) < 1$, then the eigenvalues of $G$ lie in the region
\[ E_1 = \{ z \in \mathbb{C} : |\text{Re } z| \leq C_1(|\text{Im } z|^{3/4} + 1), \text{Re } z < 0 \}, \]
while if $\sup_{x \in \Gamma} \gamma(x) \geq 1$, the eigenvalues of $G$ lie in $E_1 \cup E_2$, where
\[ E_2 = \{ z \in \mathbb{C} : |\text{Im } z| \leq C_2(|\text{Re } z|^{1/2} + 1), \text{Re } z < 0 \}. \]
The case $\gamma(x) = 1$, $\forall x \in \Gamma$, is special since as it was mentioned by Majda [12] for some obstacles there are no eigenvalues of $G$. On the other hand, to our best knowledge we did not found a proof of this result in the literature. In the Appendix in [13], the case when $K = B_3 = \{ x \in \mathbb{R}^3 : |x| \leq 1 \}$ is ball and $\gamma > 0$ is a constant has been examined and it was proved that if $\gamma = 1$, there are no eigenvalues of $G$.

On the other hand, for $\gamma = \text{const} > 1$ all eigenvalues of $G$ are real and for $0 < \gamma < 1$ there are no real eigenvalues.

We will improve the above result of Majda and one examines two cases:

(\textbf{A}) : $0 < \gamma(x) < 1$, $\forall x \in \Gamma$.

(\textbf{B}) : $\gamma(x) > 1$, $\forall x \in \Gamma$.

\textbf{II.} We discuss another important spectral problem for inverse scattering leading to non self-adjoint operator. For simplicity we assume that $d$ is odd. The inhomogeneous medium in $K$ is characterized by a smooth function $n(x) > 0$ in $\bar{K}$, called \textit{contrast}. The scattering problem is related to an \textit{incident wave} $u_i$ which satisfies the equation $(\Delta + k^2)u_i = 0$ in $\mathbb{R}^d$ and the \textit{total wave} $u = u_i + u_s$ satisfies the transmission problem
\begin{equation}
\begin{aligned}
\Delta u + k^2 u &= 0 \text{ in } \mathbb{R}^d \setminus \bar{K}, \\
\Delta u + k^2 n(x) u &= 0 \text{ in } K, \\
u^+ = u^- &\text{ on } \Gamma, \\
\left( \frac{\partial u}{\partial v} \right)^+ &= \left( \frac{\partial u}{\partial v} \right)^- \text{ on } \Gamma,
\end{aligned}
\end{equation}
where $f^\pm(x) = \lim_{\epsilon \to 0} f(x \pm \epsilon v(x))$ for $x \in \Gamma$. Here $k > 0$ and the outgoing scattering wave $u_s$ satisfies the outgoing Sommerfeld radiation condition
\[ \lim_{r \to +\infty} r^{(1-d)/2} \left( \frac{\partial u_s}{\partial r} - iku_s \right) = 0 \]
uniformly with respect to $\theta = x/r \in \mathbb{S}^{d-1}$, $r = |x|$.

If the incident wave has the form $u_i = e^{ik(x, \omega)}$, $\omega \in \mathbb{S}^{d-1}$, then
\[ u_s(r \theta, k) = e^{ikr} r^{-(d-1)/2} \left( a(k, \theta, \omega) + O\left( \frac{1}{r} \right) \right), \quad r \to +\infty. \]
The function \( a(k, \theta, \omega) \) is called scattering amplitude and the far-field operator \( F(k) : L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{S}^{d-1}) \) has the form

\[
(F(k)f)(\theta) = \int_{\mathbb{S}^{d-1}} a(k, \theta, \omega) f(\omega) d\omega.
\]

Notice also that the scattering operator has the representation

\[
S(k) = I + \left( \frac{ik}{2\pi} \right)^{(d-1)/2} F(k).
\]

The inverse scattering problem of the reconstruction of \( K \) based on the linear sampling method of Colton and Kress (see \[3\]) breaks down for frequencies \( k \) such that \( F(k) \) has a non trivial kernel or co-kernel. Assume that for some \( k \in \mathbb{R}^+ \) the kernel of \( F(k) \) is not trivial and let \( F(k)f = 0, f \neq 0 \). We may consider an incident Herglotz wave

\[
u_i(x) = \int_{\mathbb{S}^{d-1}} e^{i k \langle x, \omega \rangle} f(\omega) d\omega.
\]

Then one obtains a scattering wave \( \nu_s = o(\frac{1}{r^2}) \) since the leading term

\[
\int_{\mathbb{S}^{d-1}} a(k, \theta, \omega) f(\omega) d\omega = 0
\]

vanishes. On the other hand, \((\Delta + k^2)u_t = 0\) in \( \mathbb{R}^d \setminus \bar{K} \), so the Rellich theorem implies \( u_t = 0 \) in \( \mathbb{R}^d \setminus \bar{K} \). Therefore the functions \( u = \nu_i|_K \neq 0 \) and \( w = (\nu_i + \nu_s)|_K \) satisfy the following problem

\[
\begin{cases}
\Delta u + k^2 u = 0 \quad &\text{in } K \\
\Delta w + k^2 n(x) w = 0 \quad &\text{in } K \\
u = w, \; \partial_\nu u = \partial_\nu w \quad &\text{on } \Gamma
\end{cases}
\]

and \( \lambda = k^2 \) is called interior transmission eigenvalue (ITE). The inverse statement in general is not true and we may have complex (ITE).

We consider a more general setting. For \( d \geq 2 \), a complex number \( \lambda \in \mathbb{C} \setminus \{0\} \), is called interior transmission eigenvalue (ITE) if the following problem has a non-trivial solution \( (u_1, u_2) \neq 0 \):

\[
\begin{cases}
(\nabla c_1(x) \nabla + \lambda n_1(x)) u_1 = 0 \quad &\text{in } K \\
(\nabla c_2(x) \nabla + \lambda n_2(x)) u_2 = 0 \quad &\text{in } K \\
u_1 = u_2, \; c_1 \partial_\nu u_1 = c_2 \partial_\nu u_2 \quad &\text{on } \Gamma
\end{cases}
\]

where \( c_j(x), n_j(x) \in C^\infty(\bar{K}), \; j = 1, 2 \) are strictly positive real-valued functions. For the analysis of (ITE) one imposes the condition

\[
d(x) = c_1(x)n_1(x) - c_2(x)n_2(x) \neq 0, \quad \forall x \in \Gamma.
\]
3 Dirichlet-to-Neumann map

The analysis of the eigenvalues-free domains is based on a semi-classical analysis. Let \(0 < \hbar \ll 1\) and let \(P(h) = -\hbar^2 \Delta\). Introduce the sets
\[
Z_1 = \{ z \in \mathbb{C} : \text{Re} z = 1, \hbar^{1/2 - \epsilon} \leq \text{Im} z \leq 1, 0 < \epsilon \ll 1 \}, \\
Z_2 = \{ z \in \mathbb{C} : \text{Re} z = -1, |\text{Im} z| \leq 1 \}, \\
Z_3 = \{ z \in \mathbb{C} : |\text{Re} z| \leq 1, \text{Im} z = 1 \},
\]
and consider for \(z \in Z_1 \cup Z_2 \cup Z_3\) the semi-classical problem
\[
\begin{cases}
(P(h) - z)u = 0 \text{ in } \Omega, \\
u \in H^2(\Omega), \\
u = f \text{ on } \Gamma,
\end{cases}
\]
(3.1)

We need to introduce some \(h\)-pseudo-differential operators on a manifold with boundary \(V\). We say that \(a(x, \xi; h) \in S^{k, m}_\delta(V)\) if the following conditions are satisfied:

(i) for \(|\xi| \geq L \gg 1\) we have
\[
|\partial_x^\alpha \partial_\xi^\gamma a(x, \xi; h)| \leq C_{\alpha, \gamma, L}(1 + |\xi|)^{m-|\gamma|}, \forall \alpha, \forall \gamma.
\]

(ii) for \(|\xi| \leq L\) we have
\[
|\partial_x^\alpha \partial_\xi^\gamma a(x, \xi; h)| \leq C_{\alpha, \gamma, L} \hbar^{-k-\delta(|\alpha|+|\gamma|)}, \forall \alpha, \forall \gamma.
\]

For \(a \in S^{k, m}_\delta(V)\), consider the operator
\[
\left( \text{Op}_h(a)f \right)(x) = (2\pi h)^{-d+1} \int e^{i(x-y, \xi)/h} a(x, \xi; h)f(y)dyd\xi.
\]

We have a calculus for the \(h\)-pseudo-differential operators with symbols in \(S^{k, m}_\delta\) if \(0 < \delta < 1/2\). In particular, if \(a \in S^{0, 1}_\delta, b \in S^{0, -1}_\delta\), one gets
\[
||\text{Op}_h(a)\text{Op}_h(b) - \text{Op}_h(ab)||_{L^2} \leq C \hbar^{1-2\delta}.
\]

We refer to [7] and [21] for the calculus of \(h\)-pseudo-differential operators.

Let \(D_\nu = -i\partial_\nu\), and let \(\gamma_0\) denote the trace on \(\Gamma\). It is important to construct a semi-classical parametrix for the problem (3.1) in \(Z_1 \cup Z_2 \cup Z_3\) and to find an approximation for the (exterior) semi-classical Dirichlet-to-Neumann map defined
by
\[
\mathcal{N}_{\text{cut}}(z,h) : H^s_h(\Gamma) \ni f \longrightarrow \gamma hD_xu \in H^{-1}_h(\Gamma).
\]
(3.2)

Here for \( s \in \mathbb{R} \), \( H^s_h(\Gamma) \) is the semi-classical Sobolev space with norm \( \| hD^s u \|_{L^2(\Gamma)} \).

Vodev [21] constructed a semi-classical parametrix \( \tilde{u} \) where the equation in (3.1) is satisfied for \( x \in K \). In fact the construction in [21] is made in a very small neighborhood of the boundary \( \Gamma \) and the local parametrix is a Fourier integral operator with complex phase function. By using the resolvent \( (\gamma h^2 \Delta_D - z)^{-1} \) of the Dirichlet Laplacian in \( \Omega \), one may modify the proof in [21] to obtain a parametrix in \( \Omega \) (see [13] for more details).

To describe the local parametrix, consider normal geodesic coordinates \((x_0, x')\) in a neighborhood of a fixed point \( x_0 \in \Gamma \), where \( x_1 = \text{dist}(x, \Gamma) \). Then locally the boundary \( \Gamma \) is given by \( x_1 = 0 \). Let \( \psi(x') \in C^\infty_0(\Gamma) \) be a cut-off function with support in a small neighborhood of \( x_0 \in \Gamma \) and \( \psi(x') = 1 \) in another neighborhood of \( x_0 \). Then \(-\frac{h^2}{c(x)} \nabla c(x) \nabla - z \frac{m(x)}{c(x)} \) in these coordinates has the form
\[
\mathcal{P}(z,h) = h^2 D^2 + r(x, hD_x) + q(x, hD_x) + h^2 \bar{q}(x) - zm(x).
\]

with
\[
D_{x_1} = -i \partial_{x_1}, \quad D_{x'} = -i \partial_{x'}, \quad m(x) = \frac{n(x)}{c(x)}, \quad r(x, \xi') = (R(x)\xi', \xi'), \quad q(x, \xi) = (q(x), \xi).
\]

Here \( R(x) \) is a symmetric \((d-1) \times (d-1)\) matrix with smooth real-valued entries and \( r(0, x', \xi') = r_0(x', \xi') \) is the principal symbol of the Laplace-Beltrami operator \(-\Delta_\Gamma\) on \( \Gamma \). Let
\[
\rho = \sqrt{z m(x) - r_0(x', \xi')} \in C^\infty(T^*(\Gamma))
\]
be the root of the equation \( \rho^2 + r_0(x', \xi') - zm(x) = 0 \) with \( \text{Im} \rho > 0 \). Let \( \phi(\sigma) \in C^\infty(\mathbb{R}) \) be cut-off function such that \( \phi(\sigma) = 1 \) for \( |\sigma| \leq 1 \), \( \phi(\sigma) = 0 \) for \( |\sigma| \geq 2 \). In [21] for small \( \delta_1 > 0 \) and for \( x \) close to the boundary it was constructed a parametrix
\[
\begin{align*}
\tilde{u}_{\psi}(x) &= (2\pi h)^{-d+1} \int \int e^{\frac{i}{h} \phi(x,y',\xi')} \phi \left( \frac{x}{\rho_1} \right) \\
&\times \phi \left( \frac{x_1}{\rho_1} \right) a(x, \xi', z; h) f(y') dy' d\xi', \\
\tilde{u}_{\psi} |_{x_1=0} &= \psi f,
\end{align*}
\]
(3.3)

where \( 0 < \delta_1 < 1 \) is small enough and \( \rho_1 = 1 \) if \( z \in Z_2 \cup Z_3 \), \( \rho_1 = |\rho|^3 \) if \( z \in Z_1 \). The phase \( \phi(x, y', \xi', z) \) is complex-valued and we have
\[
\phi |_{x_1=0} = -\langle x' - y', \xi' \rangle, \quad \partial_{x_1} \phi |_{x_1=0} = \rho, \quad \text{Im} \phi \geq x_1 \text{Im} \rho / 2,
\]
while \( a |_{x_1=0} = \psi(x') \). Next, \( a = \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} x_k \partial_{x_j} a_{k,j}(x', \xi', z), \quad \phi = -\langle x' - y', \xi' \rangle + \sum_{k=1}^{N-1} x_k \phi_k(x', \xi', z), \quad \phi_1 = \rho, \)
$N \gg 1$ being a large integer. The phase $\varphi$ and the amplitude $a$ are determined so that

$$e^{-\frac{1}{2\pi}}\mathcal{P}(z,h)e^{\frac{1}{2\pi}}a = x_1^N A_N(x,\xi',z;h) + h^N B_N(x,\xi',z;h),$$

where $A_N, B_N$ are smooth functions and their behavior for $|\xi'| \to \infty$ is related to negative powers of $|\rho|$. For example,

$$|\partial^\alpha_x \partial^\beta_{\xi'} \partial^\gamma_{\xi} A_N(\phi(\delta_0\rho_0(x',\xi')))| \leq C_{k,\alpha,\beta}|\rho|^{2-3N-3k-2|\alpha|-2|\beta|}.$$

Moreover, for $x_1 > 0$ the parametrix $\tilde{u}$ has a decay $\mathcal{O}(e^{-x_1 \frac{|\im|}{2|h^pn|}})$ and for $x_1 \geq |\rho|^3/\delta$ we get an estimate $\mathcal{O}(e^{-C_{n}^{|\im|/h}})$.

Consider the (interior) semi-classical Dirichlet-to-Neumann map $\mathcal{N}_{\text{int}}(z,h)f = \gamma_0 \partial_\nu u$, related to the problem

$$\begin{cases}
(-\frac{\partial^2}{n(x)}) \nabla c(x) \nabla - z)u = 0 \text{ in } K, \\
u = f \text{ on } \Gamma,
\end{cases}
$$

(3.4)

where $n(x) > 0$, $c(x) > 0$ are $C^\infty$ functions on $\Gamma$. Then we have the following

**Proposition 1** ([21]). Given $0 < \varepsilon \ll 1$, there exists $0 < h_0(\varepsilon) \ll 1$ such that for $z \in Z_1$ and $0 < h \leq h_0(\varepsilon)$ we have

$$\|\mathcal{N}_{\text{int}}(z,h)f - \text{Op}_h(\rho + h^b)f\|_{L^2(\Gamma)} \leq \frac{Ch}{\sqrt{|\im|}}\|f\|_{L^2(\Gamma)},
$$

(3.5)

where $b \in S^{0,0}_0(\Gamma)$ does not depend on $z, h$ and the function $n(x)$. Moreover, for $z \in Z_2 \cup Z_3$ the above estimate holds with $|\im|$ replaced by $1$.

With some modifications of the proof the same result remains true for unbounded domains $\mathbb{R}^d \setminus \bar{K}$ and obtain the estimate (3.5) for the semi-classical Dirichlet-to-Neumann operator $\mathcal{N}_{\text{ext}}(z,h)$ related to the problem (3.1) with $n(x) = c(x) = 1$. (see [13]).

### 4 Location of the eigenvalues of $G$

Let $u = (u_1, u_2) \neq 0$ be an eigenfunction of $G$ with eigenvalue $\lambda$, $\Re \lambda < 0$, and let $f = u_1|_\Gamma$. Then from (2.2) we deduce $(-\Delta + \lambda^2)u_1 = 0$ and $\partial_\nu u_1 - \lambda \gamma u_1 = 0$ on $\Gamma$.

Setting

$$\lambda = \frac{i\sqrt{\varepsilon}}{h}, 0 < h \ll 1,$$

for $z \in Z_1 \cup Z_2 \cup Z_3$, one obtains the problem
and define
\[
\begin{cases}
(-\hbar^2\Delta - z)u_1 = 0 \text{ in } \Omega, \\
\mathcal{M}_{ex}(z, h) f - \sqrt{\gamma} f = 0 \text{ on } \Gamma.
\end{cases}
\]

Consider the case (A) and notice that there exists \( \varepsilon > 0 \) such that
\[
0 < \varepsilon < \gamma(x) < 1 - \varepsilon, \quad \forall x \in \Gamma.
\]
We will discuss the case \( z \in Z_1 \), the case \( z \in Z_2 \cup Z_3 \) is more simple. According to Proposition 1 for \( \mathcal{M}_{ex}(z, h) \), for \( z \in Z_1 \), \( 1 - \text{Im} z \geq h^\delta \), \( \delta = 1/2 - \varepsilon \), we have
\[
\| \text{Op}_h(\rho) f - \gamma \sqrt{\gamma} f \|_{L^2(\Gamma)} \leq C \frac{h}{\sqrt{\text{Im} z}} \| f \|_{L^2(\Gamma)},
\]
while for \( z \in Z_2 \cup Z_3 \) the above estimate holds with \( |\text{Im} z| \) replaced by 1. Consider the symbol
\[
c(x', \xi', z) = \rho(x', \xi', z) - \gamma \sqrt{\gamma} = \frac{(1 - \gamma^2)z - r_0(x', \xi')}{\rho(x', \xi', z) + \gamma \sqrt{\gamma}}.
\]
We will show that \( c(x', \xi', z) \) is elliptic in a suitable class.

Clearly, \( c \) is elliptic for \( |\xi'| \) large enough. So it remains to examine the behavior of \( c \) for \( |\xi'| \leq C_0 \) and for these values of \( \xi' \) we have \( |\rho + \gamma \sqrt{\gamma}| \leq C_1 \). Introduce the set
\[
\mathcal{F} = \{ (x', \xi') : |1 - r_0(x', \xi')| \leq \frac{\varepsilon_0}{2} \}.
\]
Then \( \text{Re} \left( (1 - \gamma^2)z - r_0 \right) = 1 - r_0 - \gamma^2 \leq -\frac{\varepsilon_0^2}{2} \). If \( (x', \xi') \notin \mathcal{F} \), we get
\[
\text{Im} \left( (1 - \gamma^2)z - r_0 \right) = (1 - \gamma^2) \text{Im} z \geq (1 - \gamma^2) h^\delta \geq \varepsilon_1 h^\delta, \quad \varepsilon_1 > 0.
\]
Consequently, the symbol \( c \) is elliptic and
\[
\text{Im}(\rho + \gamma \sqrt{\gamma}) = \text{Im} \rho + \gamma \text{Im} \sqrt{\gamma} \geq Ch^\delta.
\]
Thus, for bounded \( |\xi'| \) we have \( |c| \geq C_3 h^\delta, C_3 > 0 \), while for large \( |\xi'| \) we have \( |c| \sim |\xi'| \). Introduce the function
\[
\chi(x', \xi') = \phi(\delta_0 r_0(x', \xi')), \quad 0 < \delta_0 \leq 1/2
\]
and define \( \mathcal{M}_1 := Z_1 \times \text{supp } \chi, \mathcal{M}_2 := (Z_1 \times \text{supp } (1 - \chi)) \cup ((Z_2 \cup Z_3) \times T^* \Gamma) \). Set \( \langle \xi' \rangle = (1 + |\xi'|^2)^{1/2} \). It is easy to see that for \( (z, x', \xi') \in \mathcal{M}_1, \text{Im } z \neq 0 \), we have
\[
|\partial_{x'}^\alpha \partial_{\xi'}^\beta \rho| \leq C_{\alpha, \beta} |\text{Im } z|^{1/2 - |\alpha| - |\beta|}, \quad |\alpha| + |\beta| \geq 1,
\]
while for \( (z, x', \xi') \in \mathcal{M}_2 \) we have
\[
|\rho| \leq C.
\]
\[ |\hat{\partial}_x^\alpha \hat{\partial}_t^\beta \rho| \leq C_{\alpha, \beta} |\xi|^{1-|\beta|}. \quad (4.3) \]

Thus, we conclude that \( c = (\rho - \gamma \sqrt{z}) \in S_0^{0,1} \). A similar analysis shows that \( |\text{Im} z| c^{-1} \in S_0^{0,-1} \), while for \( z \in Z_2 \cup Z_3 \) we get \( c^{-1} \in S_0^{0,-1} \). Therefore

\[ \| \text{Op}_h(c^{-1})g \|_{L^2(\Gamma)} \leq C |\text{Im} z|^{-1} \| g \|_{L^2(\Gamma)} \]

and we deduce

\[ \| \text{Op}_h(c^{-1}) \text{Op}_h(c)f \|_{L^2(\Gamma)} \leq C_1 \frac{h}{|\text{Im} z|^{3/2}} \| f \|_{L^2(\Gamma)}. \]

A more fine analysis (see [13]) shows that

\[ \| \text{Op}_h(c^{-1}) \text{Op}_h(c)f - f \|_{L^2(\Gamma)} \leq C_2 \frac{h}{|\text{Im} z|^3} \| f \|_{L^2(\Gamma)}. \]

Consequently, one concludes that

\[ \| f \|_{L^2(\Gamma)} \leq C_3 \left( h^{1-2\delta} + h^{1-\frac{\delta}{2}} \right) \| f \|_{L^2(\Gamma)}. \quad (4.4) \]

Since \( \delta = 1/2 - \varepsilon, 0 < \varepsilon \ll 1 \), from \( (4.4) \) it follows \( f = 0 \) for \( 0 < h \leq h_0(\varepsilon) \) small enough. Since \(-h^2 \Delta\) with Dirichlet boundary conditions does not have eigenvalues in \( \{ z \in \mathbb{C} : \text{Re} z < 0 \} \), one gets \( \mu_1 = 0 \). Going back to the eigenvalues and using the scaling, one obtains that in the case (A) the eigenvalues of \( G \) lie in the region

\[ \Lambda_\varepsilon = \{ z \in \mathbb{C} : |\text{Re} z| \leq C_\varepsilon(|\text{Im} z|^{1/2} + 1), \text{Re} z < 0 \}. \]

In the case (B) the above analysis works only for \( z \in Z_1 \cup Z_3 \). Indeed for \( z \in Z_1 \) we have

\[ \text{Re}((1 - \gamma^2) - r_0) \leq (1 - \gamma^2) < -\eta_0 < 0 \]

and again \( c \in S_0^{0,1}, c^{-1} \in S_0^{0,-1} \). Thus for \( z \in Z_1 \cup Z_3 \) we obtain that the eigenvalues \( \lambda = \frac{1}{2}(1 - \gamma^2) \) must lie in \( \Lambda_\varepsilon \). For \( z \in Z_2 \) the argument, exploited in the case (A), breaks down since for \( \text{Re} z = -1, \text{Im} z = 0 \) the symbol

\[ 1 + r_0(x', \xi', t') - \gamma(x') \]

is not elliptic and it may vanish for some \( (x'_0, \xi'_0) \).

Let \( z = -1 + i \text{Im} z \in Z_2 \). For such \( z \) we have an better approximation \( T(z, h) \) of the operator \( \mathcal{M}_{cd}(z, h) \) (see [21], [13]) for which we have

\[ \| \mathcal{M}_{cd}(z, h)f - T(z, h)f \|_{H^1(\Gamma)} \leq C_N h^{-s_d + N} \| f \|_{L^2(\Gamma)}, \forall N \in \mathbb{N}, \quad (4.5) \]

with \( s_d > 0 \) depending only on the dimension \( d \). Therefore, if \( f \) is related to the trace of an eigenfunction of \( G \), from the equality \( \mathcal{M}_{cd}(z, h)f - \gamma \sqrt{z}f = 0 \) on \( \Gamma \) we obtain
Next, by applying Taylor formula, we write

\[
\text{Re} \left( (T(z,h) - \gamma \sqrt{z}) f, f \right)_{L^2(\Gamma)} = \text{Re} \left( (T(-1,h) - i\gamma) f, f \right)_{L^2(\Gamma)} - \text{Im} \text{Im} \left( \left( \frac{\partial T}{\partial z}(z_t,h) - \gamma \frac{1}{2\sqrt{z_t}} \right) f, f \right)_{L^2(\Gamma)} \quad (4.6)
\]

with \( z_t = -1 + i\text{Im} z \in \mathbb{Z}_2, 0 < t < 1 \). We may replace in (4.3) the operator \( \frac{\partial T}{\partial z}(z_t,h) \) by the operator \( \text{Op}_h \left( \frac{d\nu}{dz}(z_t,h) \right) \) modulo \( \mathcal{O}(h)\|f\|_{L^2(\Gamma)}^2 \) term and a sharp analysis shows that

\[
\text{Im} \left( \left( \text{Op}_h \left( \frac{d\nu}{dz}(z_t,h) \right) - \gamma \frac{1}{2\sqrt{z_t}} \right) f, f \right)_{L^2(\Gamma)} \geq \alpha_0 \|f\|_{L^2(\Gamma)}^2, \quad \alpha_0 > 0.
\]

We refer to \([13]\) for the details of this argument. Combining (4.5) and (4.6), one estimates \( |\text{Im} z| \) and for small \( h \) and every \( N \in \mathbb{N} \), we obtain that the eigenvalues \( \lambda = \frac{i\sqrt{z}}{h} \) of \( G \) with \( z \in \mathbb{Z}_2 \) must lie in the region

\[ \mathcal{R}_N = \{ z \in \mathbb{C} : |\text{Im} z| \leq C_N (|\text{Re} z| + 1)^{-N}, \text{Re} z < 0 \}. \]

Finally, we have the following

**Theorem 1** ([13]). In the case (A) for every \( \varepsilon > 0 < \varepsilon' \ll 1 \), the eigenvalues of \( G \) lie in the region \( \Lambda_{\varepsilon'} \). In the case (B) for every \( \varepsilon > 0 < \varepsilon' \ll 1 \), and every \( N \in \mathbb{N} \) the eigenvalues of \( G \) lie in the region \( \Lambda_{\varepsilon'} \cup \mathcal{R}_N \).

For strictly convex obstacles \( K \) we have a more precise result concerning the operator \( \mathcal{N}_{\text{out}}(z,h) \) based on the construction of a semi-classical parametrix for the problem (3.1) when \( \text{Re} z = 1 \) and \( h^{1/2-\varepsilon} \geq |\text{Im} z| \geq h^{2/3} \) (see [22], [13]) or \( 0 < |\text{Im} z| \leq h^{2/3} \) (see [20]). This makes possible to improve the above result in the case (B) and to obtain the following

**Theorem 2** ([13]). In the case (B) for every \( N \in \mathbb{N} \) outside the region \( \mathcal{R}_N \) we have only finite number eigenvalues of the generator \( G \).

Moreover, we have the following

**Proposition 2** ([5]). Assume that \( d \) is odd. Then the operator \( G \) has no a sequence of eigenvalues \( \lambda_j, \text{Re} \lambda_j < 0 \) such that \( \lim_{j \to +\infty} \lambda_j = i\omega_0, \omega_0 \in \mathbb{R} \).

It is world noting that the Dirichlet-to-Neumann map can be used to establish the discreetness of the spectrum of \( G \) in \( \{ z \in \mathbb{C} : \text{Re} z < 0 \} \). We follow below the argument of [13]. For \( \text{Re} \lambda < 0 \) introduce the map

\[
\mathcal{N}(\lambda) : H^s(\Gamma) \ni f \mapsto \partial_\nu u|_{\Gamma} \in H^{s-1}(\Gamma),
\]

where \( u \) is the solution of the problem
\[
\begin{cases}
(\Delta - \lambda^2)u = 0 \text{ in } \Omega, \ u \in H^2(\Omega), \\
u = f \text{ on } \Gamma.
\end{cases}
\] (4.7)

It is well known that \( \mathcal{N}(\lambda) \) is a meromorphic function in \( \mathbb{C} \) for \( d \) odd and in the logarithmic covering of \( \mathbb{C} \) for \( d \) even and the poles of \( \mathcal{N}(\lambda) \) in \( \mathbb{C} \setminus \{0\} \) coincide with the resonances of the Dirichlet problem for the Laplacian (see for instance, [20]).

On the other hand, \( u \in H^2(\Omega) \) implies that \( u \) is \( \lambda \)–incoming in the sense of Lax and Phillips (see Chapter IV in [10]). Notice that the definition of outgoing/incoming solutions in [20] is different from that in [10] and the resonances in [20] lie in \( \text{Im } z < 0 \), while in [10] they are in the half-plan \( \text{Im } z > 0 \). Consequently, \( \mathcal{N}(\lambda) \) is analytic for \( \text{Re } \lambda < 0 \). The same is true for the Neumann problem for the \( \Delta - \lambda^2 \), hence \( \mathcal{N}^{-1}(\lambda) \) is also analytic for \( \text{Re } \lambda < 0 \) and the poles of \( \mathcal{N}^{-1}(\lambda) \) are the resonances of the Neumann problem ([19]). Therefore, the boundary condition in (2.2) may be written as follows

\[
\mathcal{N}(\lambda) \left( I - \lambda \mathcal{N}^{-1}(\lambda) \gamma \right) f_1 = 0, \ \text{Re } \lambda < 0, \ \chi \in \Gamma.
\]

The operator \( \mathcal{N}(\lambda) : L^2(\Gamma) \rightarrow H^1(\Gamma) \) is compact and Theorem 1 guarantees that there are points \( \lambda_0, \ \text{Re } \lambda_0 < 0 \), for which \( (I - \lambda_0 \mathcal{N}^{-1}(\lambda_0) \gamma) \) is invertible. Applying the analytic Fredholm theorem, we conclude that the spectrum of \( G \) in \( \{z \in \mathbb{C} : \text{Re } z < 0\} \) is formed by isolated eigenvalues with finite multiplicities.

We finish this section by a trace formula involving the operator

\[
C(\lambda) := \mathcal{N}(\lambda) - \lambda \gamma = \mathcal{N}(\lambda) \left( I - \lambda \mathcal{N}^{-1}(\lambda) \gamma \right),
\]

which is an analytic operator-valued function in \( \{z \in \mathbb{C} : \text{Re } z < 0\} \), while \( C(\lambda)^{-1} \) is meromorphic in the same domain. We wish to find a formula for the trace

\[
\text{tr} \frac{1}{2\pi i} \int_{\delta} (\lambda - G)^{-1} d\lambda,
\] (4.8)

where \( \omega \subset \{\text{Re } z < 0\} \) has as a boundary the curve \( \delta \) and \( (G - \lambda)^{-1} \) is analytic on \( \delta \). We know that \( (G - \lambda)^{-1} \) is meromorphic in \( \omega \) and if \( \lambda_0 \) is a pole of \( (G - \lambda)^{-1} \), then the multiplicity of the eigenvalue \( \lambda_0 \) of \( G \) is given by

\[
\text{mult } (\lambda_0) = \text{rank } \frac{1}{2\pi i} \int_{|\lambda - \lambda_0| = \epsilon_0} (\lambda - G)^{-1} d\lambda,
\]

with \( \epsilon_0 > 0 \) small enough. Therefore, (4.8) is equal to the number of the eigenvalues of \( G \) in \( \omega \) counted with their multiplicities.

Let \( (u, w) = (G - \lambda)^{-1}(f, g) \). Then \( w = \lambda u + f \) and setting \( q = u|_{\Gamma} \), one gets

\[
u = R_D(\lambda)(g + \lambda f) + K(\lambda)q.
\]
Here $R_D(\lambda) = (\Delta_D - \lambda^2)^{-1}$ is the resolvent of the operator $\Delta_D$ with Dirichlet boundary conditions and $K(\lambda)$ satisfies
\[
\begin{cases}
(\Delta - \lambda^2)K(\lambda) = 0 \text{ in } \Omega, \\
K(\lambda) = \text{Id on } \Gamma.
\end{cases}
\]

The boundary condition on $\Gamma$ yields
\[
\partial_\nu[R_D(\lambda)(g + \lambda f) + K(\lambda)q] - \gamma \lambda[R_D(\lambda)(g + \lambda f) + K(\lambda)q] - \gamma f = 0, \quad x \in \Gamma
\]
and the term $\gamma \lambda \left[R_D(\lambda)(g + \lambda f)\right]$ vanishes. Since $N(\lambda) = \partial_\nu K(\lambda)|_\Gamma$ is the Dirichlet-to-Neumann map, assuming that $C^{-1}(\lambda)$ is invertible, we deduce
\[
q = C^{-1}(\lambda) \left(\partial_\nu R_D(\lambda)(g + \lambda f)\right) - \gamma f.
\]

Therefore
\[
u = \left[\lambda R_D(\lambda) + K(\lambda)C^{-1}(\lambda)\lambda \partial_\nu R_D(\lambda) - \gamma\right]f + Xg,
\]
\[
\omega = Yf + \left[\lambda R_D(\lambda) + \lambda K(\lambda)C^{-1}(\lambda)\partial_\nu R_D(\lambda)\right]g,
\]
where the operators $X$ and $Y$ are not important for the calculus of the trace. Thus we are going to study the integral
\[
\text{tr} \int_{\delta} \left(2\lambda K(\lambda)C^{-1}(\lambda)\partial_\nu R_D(\lambda) - C^{-1}(\lambda)\gamma\right)d\lambda.
\]

For the first term we apply the cyclicity of the trace and the fact that
\[
\frac{\partial N}{\partial \lambda}(\lambda) = \partial_\nu \frac{\partial K}{\partial \lambda}(\lambda) = 2\lambda \partial_\nu R_D(\lambda)K(\lambda).
\]

Finally, we obtain the following

**Proposition 3 ([13]),** Let $\delta \subset \{z \in \mathbb{C} : \text{Re } \lambda < 0\}$ be a closed positively oriented curve and let $\omega$ be the domain bounded by $\delta$. Assume that $C^{-1}(\lambda)$ is meromorphic in $\omega$ without poles on $\delta$. Then
\[
\text{tr} \int_{\delta} \frac{1}{2\pi i} (\lambda - G)^{-1}d\lambda = \text{tr} \int_{\delta} \frac{1}{2\pi i} C^{-1}(\lambda) \frac{\partial C}{\partial \lambda}(\lambda)d\lambda.
\] (4.9)

The idea to write the right-hand side of (4.9) as the trace of an integral involving the product of a meromorphic function $T^{-1}(\lambda)$ and its derivative $\frac{dT}{d\lambda}(\lambda)$ going back to [19], [4] (see also Proposition 3 in the next section). We expect that in the case (B) Proposition 3 combined with the techniques in [19] will imply a Weyl formula for the eigenvalues of $G$ lying in $\mathbb{R}_N$.

We conjecture that for $N$ large enough and $\gamma(x) > 1, \forall x \in \Gamma$, the counting function
\[
N(r) = \#\{\lambda_j \in \sigma_p(G) : |\lambda_j| \leq r, \lambda_j \in \mathbb{R}_N\}
\]
Proposition 4 ([13]). We have the following concerning the case \( v_0 \) functions and constant \( \gamma \). Moreover, following the analysis in [13], we may prove that (4.10) holds for

\[ K = \begin{cases} \text{a Weyl asymptotics of all eigenvalues of } G, & \text{for ball } B_3 \end{cases} \]

This will imply a Weyl asymptotics of all eigenvalues of \( G \). Notice that for ball \( B_3 \) we have the following

**Proposition 4 ([13]).** For \( \gamma \equiv \text{const} > 1 \) and \( K = B_3 \) all eigenvalues \( \lambda_j \) of \( G \) are real and they lie in the interval \((-\infty, -\frac{1}{\gamma^2})\]. Moreover, there is an infinite number of real eigenvalues of \( G \).

Hence in this case we must study the asymptotic of \( N(r) \) for \( r \geq -\frac{1}{\gamma^2} = r_0(\gamma) \). Moreover, following the analysis in [13], we may prove that (4.10) holds for \( K = B_3 \) and constant \( \gamma \).

By a similar argument we may study the eigenvalues of the generator \( G \) of the contraction semigroup associated to Maxwell’s equations with dissipative boundary conditions

\[
\begin{align*}
\partial_t E &= \text{curl} B, \quad \partial_t B = -\text{curl} E \text{ in } \mathbb{R}^+_t \times \Omega, \\
E_{\text{tan}} - \gamma(x)(v(x) \wedge B_{\text{tan}}) &= 0 \text{ on } \mathbb{R}^+_t \times \Gamma, \\
E(0, x) &= e_0(x), \quad B(0, x) = b_0(x). \\
\end{align*}
\]

(4.11)

The solution of the problem (4.11) is given by a contraction semigroup

\[ (E, B) = V(t)f = e^{tG_b}f, \quad t \geq 0, \]

where the generator \( G_b \) has domain \( D(G_b) \) that is the closure in the graph norm of functions \( u = (v, w) \in (C^m(\mathbb{R}^3))^3 \times (C^m(\mathbb{R}^3))^3 \) satisfying the boundary condition \( v_{\text{tan}} - \gamma(v \wedge w_{\text{tan}}) = 0 \text{ on } \Gamma \). Here \( u_{\text{tan}} = u - \langle u, v \rangle v \). For Maxwell’s equations for \( 0 < \gamma(x) < 1 \) and \( \gamma(x) > 1 \) we have the same location of eigenvalues of \( G_b \). This location has been examined in [6] by a semi-classical analysis of a \( h \)-pseudo-differential system on the boundary \( \Gamma \). We have the following

**Theorem 3 ([6]).** Assume that for all \( x \in \Gamma, \gamma(x) \neq 1 \). Then for every \( 0 < \varepsilon \ll 1 \) and every \( N \in \mathbb{N} \) there are constants \( C_\varepsilon > 0 \) and \( C_N > 0 \) such that the eigenvalues of \( G_b \) lie in the region \( \Lambda_\varepsilon \cap \mathcal{R}_N \), where

\[
\Lambda_\varepsilon = \{ z \in \mathbb{C} : |\text{Re} z| \leq C_\varepsilon(|\text{Im} z|^{1/2+\varepsilon} + 1), \text{Re} z < 0 \},
\]

\[
\mathcal{R}_N = \{ z \in \mathbb{C} : |\text{Im} z| \leq C_N(|\text{Re} z| + 1)^{-N}, \text{Re} z < 0 \}.
\]

It is interesting to notice that for Maxwell’s equation if \( \gamma(x) \equiv 1, \forall x \in \Gamma \), and \( K = B_3 \) is the unit ball in \( \mathbb{R}^3 \), then \( G_b \) has no eigenvalues (see [6] for other results concerning the case \( \gamma = \text{const \ and} \ B_3 \)).
5 Location and Weyl formula for the (ITE)

To examine the location of the (ITE), set $\lambda = \frac{z}{h^2}$, $z \in Z_1 \cup Z_2 \cup Z_3$. If $\lambda$ is an (ITE) with eigenfunction $(u, w)$, consider $u|_{\Gamma} = w|_{\Gamma} = f$. Introduce the Dirichlet-to-Neumann operators $\mathcal{M}_j = \mathcal{M}_j(z, h)$, $j = 1, 2$ related to

$$\mathcal{D}_j(z, h) = -\frac{h^2}{n_j(x)} \nabla c_j(x) \nabla - \frac{c_j(x)}{n_j(x)}, j = 1, 2.$$ 

The boundary condition in the problem (2.5) implies

$$c_1\mathcal{M}_1(z, h)f - c_2\mathcal{M}_2(z, h)f = 0.$$ 

As in the Section 3, one introduces normal geodesic coordinates $(x_1, x')$ and define

$$\rho_j = \sqrt{\frac{n_j(x)}{c_j(x')} - r_0(x', x')}, j = 1, 2$$

with $\text{Im } \rho_j > 0$. Applying Proposition 1 for the operators $\mathcal{M}_j(x, h)$, we deduce

$$\|c_1\text{Op}_h(\rho_1)f - c_2\text{Op}_h(\rho_2)f\|_{L^2(\Gamma)} \leq \frac{C h}{\sqrt{\text{Im } z}} \|f\|_{L^2(\Gamma)}.$$ 

Below we discuss only the case $c_1(x) = c_2(x) \equiv 1, \forall x \in \Gamma$. Then we have a better estimate

$$\|\text{Op}_h(\rho_1)f - \text{Op}_h(\rho_2)f\|_{H^1_0(\Gamma)} \leq \frac{C h}{\sqrt{\text{Im } z}} \|f\|_{L^2(\Gamma)} \quad (5.1)$$

and we must invert the operator $\text{Op}_h(\rho_1) - \text{Op}_h(\rho_2)$. Writing

$$\rho_1 - \rho_1 = \frac{z(n_1(x') - n_2(x'))}{\rho_1 + \rho_2},$$

it is easy to see that $\rho_1 - \rho_2$ is elliptic and $(\rho_1 - \rho_2)^{-1} \in S^{0,-1}_\delta$ for $z \in Z_1$, while $(\rho_1 - \rho_2)^{-1} \in S^{0,-1}_\delta$ for $z \in Z_2 \cup Z_3$. For $\delta = 1/2 - \epsilon < 1/2$ we may use the calculus of h-pseudo-differential operators and (5.1) implies, as in Section 4, $f = 0$. The latter yields $u = w = 0$. Returning to the eigenvalues $\lambda = \frac{z}{h^2}$, we get that the (ITE) lie in the domain $\Lambda_+$ defined below. The analysis of the general case when $c_j(x)$ are not equal to 1 is more complicated and we refer to [21] for the details. Thus we have the following

**Theorem 4 ([21])**. Assume (2.6) fulfilled together with the condition

$$c_1(x) = c_2(x), \partial x c_1(x) = \partial c_2(x), \forall x \in \Gamma.$$ 

Then for every $0 < \epsilon \ll 1$ the (ITE) lie the region
A weaker result in a partial case that
\[(3.4) \quad \text{and Re} \, \lambda \leq 0, \quad \forall x \in \Gamma, \quad \text{the (ITE) lie in}
\]
\[
\Lambda_{+, \varepsilon} := \{ z \in \mathbb{C} : \text{Re} \, \lambda \geq 0, \ |\text{Im} \, \lambda| \leq C_\varepsilon (\text{Re} \, \lambda + 1)^{3/4 + \varepsilon} \}
\]
and there are only a finite number (ITE) with Re \( \lambda < 0 \). If \( c_1(x) \neq c_2(x), \forall x \in \Gamma \), the (ITE) lie in
\[
\Lambda'_{+, \varepsilon} := \{ z \in \mathbb{C} : \text{Re} \, \lambda \geq 0, \ |\text{Im} \, \lambda| \leq C_\varepsilon (\text{Re} \, \lambda + 1)^{4/5 + \varepsilon} \}.
\]
If \((c_1(x) - c_2(x))d(x) > 0, \forall x \in \Gamma \), we have only a finite number (ITE) with Re \( \lambda < 0 \). Moreover, if we assume that \((c_1(x) - c_2(x))d(x) < 0, \forall x \in \Gamma \), then for Re \( \lambda > 0 \) the (ITE) are in \( \Lambda_+ \), while for Re \( \lambda < 0 \) and every \( N \geq 1 \) there exists \( C_N > 0 \) such that (ITE) lie in
\[
\mathcal{R}_N = \{ \lambda \in \mathbb{C} : |\text{Im} \, \lambda| \leq C_N (|\text{Re} \, \lambda| + 1)^{-N}, \quad \text{Re} \, \lambda \leq 0 \}.
\]
A weaker result in a partial case \( n_1(x) \equiv 1, n_2(x) > 1 \) in \( K \) with eigenvalues-free region
\[
\{ z \in \mathbb{C} : \text{Re} \, \lambda \geq 0, \ |\text{Im} \, \lambda| \geq C (\text{Re} \, \lambda + 1)^{24/25} \}
\]
has been obtained in [8].

For strictly convex obstacles one may construct a parametrix for the problem (3.4) and Re \( z = 1, h^{1/2 - \varepsilon} \geq |\text{Im} \, z| \geq h^{1-\varepsilon} \) by using more complicated construction and exploiting the properties of the Airy function \( \text{Ai}(z) \) (see [22] for more details). This leads to the following improvement of Theorem 4.

**Theorem 5 ([22]).** Assume \( K \) strictly convex, the condition (2.6) satisfied and \( c_1(x) = c_2(x), \partial_v c_1(x) = \partial_v c_2(x), x \in \Gamma \). Then for \( \varepsilon > 0 \) the (ITE) lie in the region
\[
M_{+, \varepsilon} := \{ z \in \mathbb{C} : \text{Re} \, \lambda \geq 0, \ |\text{Im} \, \lambda| \leq C_\varepsilon (\text{Re} \, \lambda + 1)^{1/2 + \varepsilon} \}
\]
and there are only a finite number (ITE) with Re \( \lambda < 0 \).

This results is almost optimal, since for the unit ball in \( \mathbb{R}^d \) we have the following

**Theorem 6 ([15]).** Let \( K = \{ x \in \mathbb{R}^d : |x| \leq 1 \}, \ d \geq 2 \). Suppose that the functions \( c_j, n_j, j = 1, 2 \), are constants everywhere in \( K \), \( c_1 = c_2 \), and the condition (2.6) is satisfied. Then, there are no (ITE) in the region \( \mathcal{M}_{+, 0} \).

The case \( d = 1 \) and \( K = \{ x \in \mathbb{R} : |x| \leq 1 \} \) has been previously examined in [18] and [16].

Now we pass to the Weyl formula for the counting function \( N(r) \) of the (ITE) and introduce the coefficients
\[
\tau_j = \frac{\omega_d}{(2\pi)^d} \int_K \left( \frac{n_j(x)}{c_j(x)} \right)^{d/2} dx, \ j = 1, 2,
\]
where \( \omega_d \) is the volume of the unit ball in \( \mathbb{R}^d \).

In the anisotropic case \( c_1(x) = 1, n_1(x) = 1, c_2(x) \neq 1, c_2(x)n_2(x) \neq 1, \forall x \in \mathcal{R} \), the asymptotics
\[ N(r) \sim (\tau_1 + \tau_2)^d, \ r \to +\infty. \] (5.2)

has been obtained by Lakshatanov and Vainberg [9] under some additional assumptions which guarantee that the boundary problem is parameter-elliptic.

By the results of Agranovich and Vishik [11] for the closed operator \( \mathcal{A} \) related to (2.5) outside every angle \( D_\alpha = \{ z \in \mathbb{C} : |\arg z| \leq \alpha \} \), we have only a finite number of (ITE) and the following estimate holds

\[ \| (z - \mathcal{A})^{-1} \| \leq C_\alpha |z|^{-1}, \ z \notin D_\alpha, \ |z| \gg 1. \]

The authors applied directly a result of Boimanov-Kostjuchenko [2] leading to (5.2).

The isotropic case \( c_1(x) = c_2(x) = 1, \ \forall x \in \mathcal{K}, n_1(x) = 1, n_2(x) \neq 1, \ \forall x \in \Gamma \), is more difficult since the corresponding operator \( \mathcal{A} \) has domain

\[
D(\mathcal{A}) = \{ (u, w) \in L^2(K) \times L^2(K) : \Delta u \in L^2(K), \Delta v \in L^2(K), \
\quad u - w = 0, \partial \nu (u - w) = 0 \text{ on } \Gamma \}. 
\]

Thus \( D(\mathcal{A}) \) is not included in \( H^2(K) \), and the problem is not parameter-elliptic. In this case Robbiano [17] obtained (5.2) by establishing the asymptotics

\[
\sum_j \frac{1}{|\lambda_j|^p + t} = \alpha t^{-1+\frac{d}{p}} + o(t^{-1+\frac{d}{p}}), \ t \to +\infty, 
\]

where \( p \in \mathbb{N} \) is sufficiently large. An application of the Tauberian theorem of Hardy-Littlewood yields the result. By this argument one obtains a very weak estimate for the remainder which can be estimated by the principal term divided by a logarithmic factor. To get better results, it is important to take into account parabolic eigenvalues-free regions and to apply different techniques which are not based on Tauberian theorems.

**Theorem 7** ([14]). Under the condition (2.6), assume that there are no (ITE) in the region

\[
\{ \lambda \in \mathbb{C} : |\text{Im } \lambda| \geq C(|\text{Re } \lambda| + 1)^{1-\frac{1}{2}} \}, C > 0, 0 < \kappa \leq 1. \] (5.3)

Then for every \( 0 < \epsilon \ll 1 \) we have the asymptotics

\[ N(r) = (\tau_1 + \tau_2)^d + O_\epsilon(r^{-d+1/\epsilon}), \ r \to +\infty. \] (5.4)

- According to Theorem 4, for arbitrary obstacles and \( c_1(x) = c_2(x), \partial \nu c_1(x) = \partial \nu c_2(x), \forall x \in \Gamma \), we can take \( \kappa = \frac{1}{2} - \epsilon \) and we obtain a remainder \( O_\epsilon(r^{d-1/2+\epsilon}). \)

- Taking into account Theorem 5, for strictly convex obstacles we choose \( \kappa = 1 - \epsilon, \forall \epsilon. \) Consequently, we have in this case a remainder \( O_\epsilon(r^{d-1+\epsilon}). \)

- The optimal result should be to have a eigenvalues-free region with \( \kappa = 1 \) as it was proved in [15], [18], [16] for the case when \( K \) is a ball and the functions \( c_j, n_j \) are constants. However, even with \( \kappa = 1 \), to obtain an optimal remainder \( O(r^{d-1}) \) some extra work is needed and this is an interesting open problem.
The proof of Theorem 7 is long and technical. After a semi-classical scaling, the idea is to reduce the analysis of \( N(r) \) to the trace of an integral involving the product of a meromorphic function \( T^{-1} (\lambda) \) and its derivative \( \frac{dT}{d\lambda} (\lambda) \) similar to Proposition 3. Set \( Z = \{ z \in \mathbb{C}; \frac{1}{3} \leq | \text{Re} \ z | \leq 3, | \text{Im} \ z | \leq 1 \} \) and consider for \( z \in Z \) and \( 0 < h \ll 1 \) the operator

\[
hT(z/h^2) := c_1 \mathcal{N}_1(z, h) - c_2 \mathcal{N}_2(z, h),
\]

where the DN-maps \( N_j(z, h) \) are defined in the beginning of this section.

Let \( G_{j}^{(i)} \), \( j = 1, 2 \), be the Dirichlet self-adjoint realization of the operator \( L_j := -n_j^{-1} \nabla c_j \nabla \) in the space \( H_j = L^2(K, n_j(x) dx) \). Set \( \mathcal{H} = H_1 \oplus H_2 \) and let \( R(\lambda) \) be the resolvent of the transmission boundary problem. We omit in the notation \( j = 1, 2 \) and consider the operators

\[
\mathcal{N}(z, h) Op_h(1 - \chi) f = \mathcal{N}(z, h) f - y_0 D_\nu (h^2 G_\nu - z)^{-1} \frac{c}{n} \chi \nabla f,
\]

\[
F(z, h) = \mathcal{N}(z, h) - \mathcal{N}(z, h) Op_h(\chi) = \mathcal{N}(z, h) Op_h(1 - \chi) f - y_0 D_\nu (h^2 G_\nu - z)^{-1} \frac{c}{n} \nabla f,
\]

where \( \chi(x', \xi') = \Phi(\delta r_0(x', \xi')) \) with \( \Phi(\sigma) = 1 \) for \( |\sigma| \leq 1 \) and \( \Phi(\sigma) = 0 \) for \( |\sigma| \geq 2 \), while \( 0 < \delta_0 \ll 1 \) is small enough. Here \( \mathcal{N}(z, h) \) is the parametrix of the DN operator \( \mathcal{N}(z, h) Op_h(1 - \chi) \) in the domain where \( r_0(x', \xi') > \frac{1}{\delta_0} \) and \( p \) is some symbol having behavior \( \mathcal{O}(h^N) \) with all its derivatives. The number \( N \) will be taken large enough and it depends only on the parametrix construction.

The operator \( F(z, h) \) is meromorphic with values in the space of trace class operators and we denote by \( \mu_j(F(z, h)) \) its characteristic eigenvalues.

**Lemma 1.** If \( z/h^2 \) does not belong to spec \( G_D \), then for every integer \( 0 \leq m \leq N/4 \) we have

\[
\mu_j(F(z, h)) \leq \frac{C}{\delta(z, h)} \left( h_j^{1/(d-1)} \right)^{-2m}, \forall j \in \mathbb{N},
\]

where \( \delta(z, h) := \min\{ 1, \text{dist} \{ z, \text{spec} h^2 G_D \} \} > 0 \) and \( C > 0 \) depends on \( m \) and \( N \) but is independent of \( z, h, j \).

Let

\[
T(\lambda) := y_0 c_1 D_\nu K_1(\lambda) - y_0 c_2 D_\nu K_2(\lambda),
\]

where \( K_j(\lambda) f = u \), and \( u \) is the solution of the problem

\[
\begin{cases}
(L_j - \lambda) u = 0 \text{ in } K, \\
u = f \text{ on } \Gamma.
\end{cases}
\]

**Proposition 5.** Assume that \( T(\lambda)^{-1} \) is a meromorphic function with residues of finite rank. Let \( \delta \subset \mathbb{C} \) be a simple closed positively oriented curve which avoids the eigenvalues of \( G_{j}^{(i)} \), \( j = 1, 2 \), as well as the poles of \( T(\lambda)^{-1} \). Then we have the identity
\[-\text{tr}_{\mathcal{H}}(2\pi i)^{-1} \int_{\delta} R(\lambda) d\lambda = \sum_{j=1}^{2} \text{tr}_{H_j}(2\pi i)^{-1} \int_{\delta} (\lambda - G_{D}^{(j)})^{-1} d\lambda \]
\[-\text{tr}_{L^2(\Gamma)}(2\pi i)^{-1} \int_{\delta} T(\lambda)^{-1} \frac{dT(\lambda)}{d\lambda} d\lambda. \tag{5.5}\]

Let us mention that if \( R(\lambda) \) is an operator-valued meromorphic function with residues of finite rank, the multiplicity of a pole \( \lambda_k \in \mathbb{C} \) of \( R(\lambda) \) is defined by

\[
\text{mult}(\lambda_k) = -\text{rank}(2\pi i)^{-1} \int_{|\lambda - \lambda_k| = \varepsilon} R(\lambda) d\lambda, \quad 0 < \varepsilon \ll 1.
\]

On the other hand, the rank of the operator above is equal to the trace of this operator and on the left-hand side of \((5.5)\) we have the sum of the multiplicities of the (ITE) lying in the domain \( \omega_\delta \subset \mathbb{C} \) bounded by \( \delta \). Clearly, the terms with \( (\lambda - G_{D}^{(j)})^{-1} \) yield the sum of eigenvalues of \( G_{D}^{(j)} \) in \( \omega_\delta \) counted with their multiplicities.

It is possible to construct invertible, bounded operator \( E(\varepsilon, z): H_{h}^{s}(\Gamma) \to H_{h}^{s+1}(\Gamma) \) with bounded inverse \( E(\varepsilon, z)^{-1}: H_{h}^{s}(\Gamma) \to H_{h}^{s-1}(\Gamma), \forall s \in \mathbb{R} \), so that

\[
hT(z/h^2) = E^{-1}(z, h)(I + \mathcal{K}(z, h)),
\]

\[
(hT(z/h^2))^{-1} = (I + \mathcal{K}(z, h))^{-1}E(z, h)
\]

with a trace class operator

\[
\mathcal{K}(z, h) = E(z, h)(c_1 F_1(z, h) - c_2 F_2(z, h)) + \mathcal{L}(z, h).
\]

Moreover, the operators \( E(z, h), E^{-1}(z, h) \), are holomorphic with respect to \( z \) in \( Z \), while \( \mathcal{K}(z, h) \) is meromorphic operator-valued function in this region. Then

\[
\text{tr} \int_{\delta} T^{-1}(z/h^2) \frac{d}{dz} T(z/h^2) dz = \text{tr} \int_{\delta} (I + \mathcal{K}(z, h))^{-1} \frac{d}{dz} \mathcal{K}(z, h) dz.
\]

Set \( g_h(z) := \det(I + \mathcal{K}(z, h)) \) and denote by \( M_\delta(h) \) the number of the poles \( \{ \lambda_k \} \) of \( R(\lambda) \) such that \( h^2 \lambda_k \) are in \( \omega_\delta \). Similarly, we denote by \( M_\delta^{(j)}(h) \) the number of the eigenvalues \( \nu_k \) of \( G_{D}^{(j)} \) such that \( h^2 \nu_k \in \omega_\delta \). Then using the well-known formula

\[
\text{tr} \left[ (I + \mathcal{K}(x, h))^{-1} \frac{\partial \mathcal{K}(z, h)}{\partial z} \right] = \frac{\partial}{\partial z} \log \det(I + \mathcal{K}(z, h)),
\]

we get from \((5.5)\) the following

**Lemma 2.** Let \( \delta \subset Z \) be closed positively oriented curve which avoid the eigenvalues of \( h^2 G_{D}^{(j)}, j = 1, 2 \) as well as the poles of \( T(z/h^2)^{-1} \). Then we have

\[
M_\delta(h) = M_\delta^{(1)}(h) + M_\delta^{(2)}(h) + \frac{1}{2\pi i} \int_{\delta} \frac{d}{dz} \log g_h(z) dz. \tag{5.6}
\]
The leading term in (5.4) is obtained from the $M^{(1)}_{\delta}(h) + M^{(2)}_{\delta}(h)$ after a scaling. The crucial point is to examine the asymptotic of the integral involving $\log g_{\delta}(z)$. The details of this analysis are given in [14].

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