Analyzing Correlation Functions with Tesseral and Cartesian Spherical Harmonics

Pawel Danielewicz

Department of Physics and Astronomy, and National Superconducting Cyclotron Laboratory, Michigan State University, East Lansing, Michigan, 48824

Scott Pratt

Department of Physics and Astronomy,
Michigan State University,
East Lansing, Michigan, 48824

(Dated: March 31, 2022)

The dependence of inter-particle correlations on the orientation of particle relative-momentum can yield unique information on the space-time features of emission in reactions with multiparticle final states. In the present paper, the benefits of a representation and analysis of the three-dimensional correlation information in terms of surface spherical harmonics is presented. The harmonics include the standard complex tesseral harmonics and the real cartesian harmonics. Mathematical properties of the lesser-known cartesian harmonics are illuminated. The physical content of different angular harmonic components in a correlation is described. The resolving power of different final-state effects with regarding to determining angular features of emission regions is investigated. The considered final-state effects include identity interference and strong and Coulomb interactions. The correlation analysis in terms of spherical harmonics is illustrated with the cases of gaussian and blast-wave sources for proton-charged meson and baryon-baryon pairs.

I. INTRODUCTION

Measurements of low relative-velocity correlations yield access to the size and shape of relative emission sources in reactions with multiparticle final states [1, 2, 3, 4]. This information plays an especially important role in heavy-ion collisions. The duration of emission, which is linked to the nuclear equation of state, can be also inferred from those measurements. Perhaps the most startling finding, during the first years of the Relativistic Heavy Ion Collider (RHIC) program, was the disagreement between the features of experimentally determined emission sources [5, 6, 7] and predictions of hydrodynamic models [8, 9, 10, 11] incorporating an anticipated phase transition, which would yield a long emission duration. A long emission duration would have resulted in an elongated shape (\(R_{\text{out}}/R_{\text{side}} >> 1\) in the usual RHIC nomenclature) of the emission source for particles of a specific momentum. In the literature, the most exhaustively studied correlation has been that produced by interference for identical mesons. For pure interference, the correlation expressed as a function of the relative particle momentum is related to the relative emission source and predictions of hydrodynamic models [8, 9, 10, 11] incorporating an anticipated phase transition, which would yield a long emission duration. In the present paper, the benefits of a representation and analysis of the three-dimensional correlation data, and also better facilitate isolating and focusing on specific physical properties of the source. Different source parameterizations will be employed for illustration of our discussions. For example, the \(\ell = 2\) harmonic expansion coefficients, characterizing quadrupole deformation, provide information on the ratio of radii \(R_{\text{out}}/R_{\text{side}}\) in the common gaussian source representation.

The correlation, associated with the final-state effects at low relative velocities within a subsystem of the particles \(a\) and \(b\), can be linked to a source function \(S\) through the relation:

\[
\frac{d^6N_{ab}/d^3p_ad^3p_b}{(dN_{ab}/d^3p_a)(dN_{ab}/d^3p_b)} = 1 + \mathcal{R}(P, q) = \int d^3r |\phi(<q, r)|^2 S(P, r).
\]  (1)

Here, \(\mathcal{R}(P, q)\) is the deviation of the l.h.s. correlation function from unity, \(P\) is the total momentum of the pair and \(q\) is the relative momentum measured in the two-particle rest frame. The factor \(|\phi(<)|^2\) represents the relative wavefunction squared, in the two-particle rest frame, with asymptotic condition imposed on the outgoing wave. The square of the wavefunction is averaged over spins. The function \(S(P, r)\) represents the probability density that the particles \(a\) and \(b\), moving with the same velocity \(V\) corresponding to the total momentum \(P\), are emitted from the reacting system a distance \(r\) apart in their rest frame. Since at large \(q\), the emission in multiparticle final states becomes uncorrelated, i.e. \(\mathcal{R} \rightarrow 0\) while \(|\phi(<)|^2\) averages to 1 as a function of \(r\), it follows that \(S\) is normalized to 1, \(\int d^3r S(P, r) = 1\). A detailed derivation of Eq. (1) can be found in [12] (see also [14]).
The probability density of emission at the relative distance may be expressed in terms of single-particle probability densities $s$:

$$
S(P, r) = \frac{\int d^4 x_a d^4 x_b s_a (m_a P/(m_a + m_b), x_a) s_b (m_b P/(m_a + m_b), x_b) \delta(r - r_a - r_b)}{\int d^4 x_a d^4 x_b s_a (m_a P/(m_a + m_b), x_a) s_b (m_b P/(m_a + m_b), x_b)},
$$

(2)

where $s_a(p, x)$ is the density for emitting particles of type $a$ with momentum $p$ from space-time point $x$. The spatial arguments are for the frame moving at the center-of-mass velocity $V$. According to Eq. (1), the correlation data access the probability distribution in the relative rest-frame separation only. Information on emission duration can only be inferred from the influence of that duration on the relative spatial distribution $\mathcal{S}$.

Similarly, single-particle distributions $s$ can be deduced only indirectly from data (1) because of the folding in (2) with an integration over the average particle position. For a set of investigated particles, the constraining of single-particle distributions may be enhanced by an analysis of correlations (1) of all possible particle pairs.

In the following, we concentrate on what might be directly determined from data, i.e., the determination of $S(P, r)$ given measurement of $\mathcal{R}(P, q)$. Since each individual value of $P$ nominally represents a different problem, we suppress the $P$ arguments in the functions. As access to $S$ is conditioned on any deviations of $|\phi|^2$ from unity, we subtract unity from both sides of (1) arriving at

$$
\mathcal{R}(q) = \int d^3 r \left[ |\phi^{(-)}(q, r)|^2 - 1 \right] S(r) \equiv \int d^3 r \mathcal{K}(q, r) S(r),
$$

(3)

where we have utilized the normalization of $S(P, r)$. Here, both $q$ and $r$ are evaluated in a frame where $P = 0$. It is seen that $\mathcal{R}$ is related to $S$ through an integral transform where $|\phi^{(-)}(q, r)|^2 - 1$ plays the role of the kernel $K$ of transformation. The determination of $S(r)$ from $\mathcal{R}(q)$ amounts then to the inversion of the integral transform, often referred to as imaging $\mathcal{R} \rightarrow S$.

Imaging has already been extensively performed for angle-averaged correlation functions, $\mathcal{R}(q)$, which can provide angle-averaged distributions $S(r)$. It is the goal of this paper to describe how one might extract the angular information in $S(r)$ by decomposing $\mathcal{R}(q)$ in terms of surface-spherical tesseral and cartesian harmonics. A parallel effort, employing tesseral harmonics in the case of pure interference, is described in [24].

The wavefunction squared $|\phi(q, r)|^2$ depends, after summation over spins, only on the magnitudes of $q$ and $r$ and on the angle $\theta_{qr}$ between $q$ and $r$. Thus, the square is invariant under rotations. As a consequence, the coefficients of expansion in terms of spherical harmonics, for the correlation and source functions, are directly related to each other [21],

$$
\mathcal{R}_{\ell m}(q) = 4\pi \int d r^2 r^2 \mathcal{K}_{\ell}(q, r) S_{\ell m}(r).
$$

(4)

Here, the expansion coefficients and expansion for the correlation function are defined by the relations:

$$
\mathcal{R}(q) = \sqrt{4\pi} \sum_{\ell m} \mathcal{R}_{\ell m}(q) Y_{\ell m}(\Omega),
$$

$$
\mathcal{R}_{\ell m}(q) = \frac{1}{\sqrt{4\pi}} \int d\Omega q Y_{\ell m}(\Omega) \mathcal{R}(q),
$$

(5)

with analogous expressions for the source function. The partial kernel $K_{\ell}$ encodes the information from the wavefunction,

$$
K_{\ell}(q, r) \equiv \frac{1}{2} \int d\cos\theta_{qr} \left[ |\phi^{(-)}(q, r, \cos\theta_{qr})|^2 - 1 \right] P_{\ell}(\cos\theta_{qr}).
$$

(6)

According to the above, an $(\ell, m)$ coefficient of the correlation function is tied to the coefficient of the source function with the same $(\ell, m)$. Thus, the source restoration can be reduced to a series of one-dimensional inversions of the integral transformations for individual $(\ell, m)$, employing kernels for specific $\ell$. This sequence of one-dimensional inversions is computationally simpler than a single three-dimensional inversion. The angular resolution of data would determine how far the restoration can proceed in $\ell$.

One problem with the above is the emergence of complex coefficients for the real-valued correlation and source functions, lacking direct geometric interpretation. The last issue is circumvented by introducing an alternate real-valued basis in the space of spherical angles, surface-spherical cartesian harmonics $A_\ell(\Omega) [12, 23, 20]$.
Cartesian harmonics $A_ℓ$ are linear combinations of $Y_{ℓm}$ with different $m$ but the same $ℓ = ℓ_x + ℓ_y + ℓ_z$. In terms of the directional unit vector $\mathbf{n}$ pointing in the $\Omega$ direction, the cartesian harmonics are explicitly given by \[ A_ℓ(\Omega) = \sum_{0 ≤ m ≤ ℓ/2} \left( -\frac{1}{2} \right)^m \frac{(2ℓ - 2m - 1)!!}{(2ℓ - 1)!!} \frac{ℓ_x!}{(ℓ_x - 2m_x)! m_x!} \times \frac{ℓ_y!}{(ℓ_y - 2m_y)! m_y!} \frac{ℓ_z!}{(ℓ_z - 2m_z)! m_z!} n_x^{ℓ_x-2m_x} n_y^{ℓ_y-2m_y} n_z^{ℓ_z-2m_z}, \] where $m = m_x + m_y + m_z$ and $(-1)!! = 1$. The leading term of $A_ℓ$ is $n_x^{ℓ_x} n_y^{ℓ_y} n_z^{ℓ_z}$, while the subsequent terms ensure that $r^4 A_ℓ$ satisfies the Laplace equation and, thus, is a combination of the $Y_{ℓm}$ of different $m$.

The expansion in terms of cartesian harmonics are defined with the relations
\[ \mathcal{R}(q) = \sum_ℓ ℓ! \mathcal{R}_ℓ A_ℓ(Ωq) = \sum_ℓ ℓ! \mathcal{R}_ℓ q_x^{ℓ_x} q_y^{ℓ_y} q_z^{ℓ_z}, \] \[ \mathcal{R}_ℓ(q) = \frac{(2ℓ + 1)!!}{ℓ!} \int \frac{dΩ_q}{4π} A_ℓ(Ωq) \mathcal{R}(q). \]

In many situations the explicit form of $A_ℓ$ is not needed. Just the existence of the cartesian harmonics and their properties justify representing a function of the spherical angle in the second series in \[ (8). \]

Since $A_ℓ$ is a combination of $Y_{ℓm}$ with $ℓ = ℓ_x + ℓ_y + ℓ_z$, the coefficients of expansion in terms of cartesian harmonics for the source and correlation function are related to each other in the same way as the coefficients of expansion in terms of tesseral harmonics in Eq. \[ (4), \]
\[ \mathcal{R}_ℓ(q) = 4π \int dr^2 K_ℓ(q, r) S_ℓ(r). \]

In the next section, the structure of the kernel $K_ℓ$ in Eqs. \[ (4) \] and \[ (10) \] is investigated for the cases of pure identical-particle interference and for strong and Coulomb interactions within a pair. We find that all three classes of final-state effects provide significant resolving power for different $ℓ$. Section \[ (11) \] provides a detailed discussion of the properties of cartesian harmonics. The subsequent section presents a discussion of the utility of higher $ℓ$-moments and their relation to specific geometric features of the source. In particular, we show that the $ℓ = 1$ moments can reveal an offset between the probability clouds for two different species, that the $ℓ = 2$ moments can reveal the magnitude and orientation of axes in the ellipsoidal approximation for a source, while the $ℓ = 3$ moments can reveal the “boomerang” nature of a source. In Sec. \[ (11) \] the cases of gaussian and blast wave sources are studied, as examples, in quantitative detail. In the final section, we summarize our results and discuss the prospects and challenges expected in analyses employing the harmonics.

II. CORRELATION KERNELS $K_ℓ$

The ability to extract source $S$ anisotropies depends on the kernels $K_ℓ(q, r)$ defined in Eq. \[ (4). \] Data on $\mathcal{R}$ are obtained at a certain resolution in $q$ and suffer from errors. The theoretical relation \[ (10) \] involves approximations \[ (13, 27), \] in assuming that the relative momentum $q$ is small compared to the scales characterizing the rest of the system and in ignoring final-state effects other than those between the two studied particles moving slow relative to each other. If the kernels $K_ℓ$ drop rapidly as $ℓ$ increases, for typical $q$-values and for values of $r$ characteristic to the source function, there will be no chance to determine details of the shape of $S(r)$. In the following we analyze the situation for different types of final-state interactions at different $ℓ$, $q$ and $r$.

In the process of source restoration, the values of $\mathcal{R}_ℓ$ would be known at discrete values of $q$. In parallel, the source $S_ℓ(r)$ might be discretized in $r$ or decomposed in some basis. The source restoration would then amount to the inversion of a matrix $K_ℓ(q, r)$, cf. \[ (21, 28). \] An inability to extract source features might be signalled by the proximity of matrix eigenvalues to zero, resulting in an instability of the inversion. Such a situation could be encountered when extracting details of $S_ℓ$ in an $r$-region where $K_ℓ$ lacks resolving power. Practical experience in matrix-inversion strategy, in analyzing data, has been gained for $ℓ = 0$ \[ (18, 19, 21, 22, 23). \] Importantly, this strategy has revealed non-Gaussian features in $S_{ℓ=0}(r)$, which were especially significant for proton sources in intermediate energy collisions. More common in the literature than the source discretization, has been a parameterization of the source which is then fit to $\mathcal{R}$. When parameterizing the source in three-dimensions, the benefit regarding angular moments can be
in understanding a systematic of the moments, $R_{E}$ and $S_{E}$, as a function of $E$. With regard to the fit strategy, the features of $K_{\ell}(q,r)$ determine what parameters of $S$ may be potentially constrained by data.

In the first subsection we discuss kernels for the case of pure identity interference between spin-0 bosons. The two subsequent subsections are dedicated to the Coulomb and the strong interactions. Before launching into a detailed discussion of specific classes of interactions, we emphasize a general observation that a three-dimensional source $S_{E}(r)$, expanded in a Taylor series around $r = 0$, behaves as $r^{\ell}$ for small $r$. Consequently, there is not much to be discerned about the source at small $r$ for $\ell \geq 1$, particularly for large $\ell$, and the structure of $K_{\ell \geq 1}$ at small $r$ will not be overly important. On the other hand, much could be learned about the angular structure of the source at large $r$. However, when the large-$r$ region gets mapped onto small $q$, the ability to gain angular information becomes limited by the experimental resolution capabilities. Thus, in practice, the most important features of $K_{\ell \geq 1}(q,r)$ are those at moderate $q$ combined with moderate to high $r$. The naive expectation regarding strong interactions may be that when an $s$-wave dominates the interaction, there is no possibility to discern source shapes. We shall see that this is not the case.

A. Kernels for Spin-0 Boson Identity-Interference

For non-interacting identical spin-0 bosons, the squared relative wave function is

$$|\phi(q,r,\cos \theta_{qr})|^2 = 1 + \cos (2qr \cos \theta_{qr}) ,$$

(11)

where the relative momentum convention is that $q$ stands for the momentum of one of the particles in the center-of-mass frame. In this case, the three-dimensional transformation (10) becomes the cosine Fourier transform (23). Since the cosine term in (11) may be represented as the real part of $e^{iqr}$ and the latter may be expanded in Legendre polynomials and spherical Bessel functions, the $K_{\ell}$-transformations (10) and (11), with (6), emerge as spherical-Bessel transforms (see also [24]):

$$K_{\ell}(q,r) = \begin{cases} (-1)^{\ell/2} j_{\ell}(2qr), & \text{for even } \ell, \\ 0, & \text{for odd } \ell. \end{cases}$$

(12)

The properties of $j_{\ell}(2qr)$, as a function of $qr$, are as follows. At low $qr$, the function starts out in the power form, $j_{\ell}(2qr) \approx (2qr)^{\ell} / \ell !$, which is followed, for $\ell \geq 2$, by a maximum at $qr \sim \frac{\pi}{4} + \frac{1}{4}$ and by a subsequent alternation at the approximate period of $\pi$. This has the following consequences for the ability to discern the source angular components $S_{\ell}$. If $R$ is a scale characterizing $S_{\ell}$, $S_{\ell}$ will have little impact on $R_{\ell}$ if, at $\ell \geq 2$, $qR$ falls into the hole in $j_{\ell}$ at low argument values. Likewise, $S_{\ell}$ will have little impact on $R_{\ell}$, if many oscillations of $j_{\ell}$ are averaged over in the spatial integration over the source. The maximal impact is expected if $qR$ falls into the region of the first maximum in $j_{\ell}$ and, thus, most information on $S_{\ell}$ may be expected in $R_{\ell}$ at $q \sim (\ell + \frac{1}{4})/(2R)$. Otherwise, the coefficients $R_{\ell}$ are expected to behave as $q^{\ell}$ for low $q$ and vanish at large $q$.

As characteristic for Fourier transforms, $q$ and $r$ enter the kernels as a product, and the low-$r$ region of the source gets predominantly mapped onto the high-$q$ region of the correlation function. Conversely, the high-$r$ region of the source function mainly affects the low-$q$ region of the correlation function. The inverse to the cosine Fourier transform is again the cosine Fourier transform and the inverse to the spherical Bessel transform is the spherical Bessel transform. Given the Bessel function completeness relation,

$$\int_{0}^{\infty} dq q^{2} j_{\ell}(2qr) j_{\ell}(2q' r) = \frac{\pi}{16} \delta(r - r'),$$

(13)

we can introduce an inverse kernel

$$K_{\ell}^{-1}(r,q) = \begin{cases} (-1)^{\ell/2} j_{\ell}(2qr), & \text{for even } \ell, \\ 0, & \text{for odd } \ell, \end{cases}$$

(14)

and obtain

$$S_{\ell}(r) = 4\pi \int dq q^{2} K_{\ell}^{-1}(r,q) R_{E}(q).$$

(15)
FIG. 1: Kernel, $K(q, r, \cos \theta_{qr}) = |\phi(\cdot)|^2 - 1$, for classical repulsive Coulomb interaction, shown as a function of $\cos \theta_{qr}$ at a couple of values of the dimensionless parameter, $x = r_C/r$, where $r_C = 2\mu Z_1 Z_2 e^2/q^2$. The structure around $\cos \theta_{qr} = -1$ is due to the deflection of classical trajectories directed towards the center of the repulsive Coulomb interaction. As the relative momentum within the pair increases, at fixed $r$, the range of backward angles where the kernel is significant shrinks.

B. Kernels for Repulsive Coulomb Interaction

The Coulomb final-state interaction also provides means for determining $S(r)$ from $R(q)$ [29, 30]. In the Coulomb case, there is no straightforward way to invert the relation to the source function as in the case of pure interference. To understand how $S$ impacts $R$ for the Coulomb interaction, the structure of the kernel must be considered in detail.

In the classical limit of a repulsive Coulomb interaction, the kernel depends solely on $\theta_{qr}$ and on the ratio of the pair relative-energy to the pair-Coulomb-energy at emission:

$$x = \frac{Z_1 Z_2 e^2}{r} \frac{2\mu}{q^2} = \frac{r_C}{r},$$

where $\mu$ is the reduced mass and $r_C$ is the radius of the Coulomb barrier in the pair interaction. Quantal effects become important for $qr/h$ of the order of unity. In the quantal situation, the kernel also depends on the dimensionless variable $qr/h$. The Gamow-factor parameter is then given by the parameter product $\eta = x qr/2 = Z_1 Z_2 e^2/v$ and a further parameter multiplication yields $x(qr)^2 = 2r/a_0$, where $a_0 = 1/Z_1 Z_2 e^2 \mu$ is the Bohr radius.

In the classical limit, the squared wavefunction represents the ratio of initial and final spatial densities during motion to the detectors for particles of a given initial momentum, or, due to the conservation of phase-space density, the inverse ratio of the momentum densities at a fixed initial separation:

$$|\phi(\cdot)|^2(r, \cos \theta_{qr}) \to \frac{d^3 r_{\infty}}{d^3 r} \phi^2 - \frac{d^3 q_0}{d^3 q} = \frac{1}{(1 + \cos \theta_{qr} - x)^2 - x^2} \Theta(1 + \cos \theta_{qr} - 2x).$$

Here, $q_0$ represents the relative momentum of the particles at separation $r$, while $r_{\infty}$ represents a remote separation of the particles at some instant when the relative momentum has approached $q$. The r.h.s. result in (17) follows from Coulomb-trajectory considerations [29]. The implications of this result simplify [25] for $\ell = 0$ and in the high-energy ($x \ll 1$) and low-energy ($x \gg 1$) limits. Thus, for $\ell = 0$, the partial kernel is

$$K_0(q, r) = \frac{q_0^2}{q^2} \frac{dq_0}{dq} - 1 = \frac{q_0}{q} - 1 = \frac{\Theta(x)}{\sqrt{1 - x}} - 1.$$

The three-dimensional kernel $K$ is illustrated in Fig. 1 as a function of $\cos \theta_{qr}$, for two $x$ values. In the high energy limit of $x \ll 1$, the wavefunction squared $|\phi|^2$ is close to unity for most values of $\cos \theta_{qr}$. In that limit, the trajectories emerging from $r$ are practically straight lines. The exception are the trajectories aiming towards the center of interaction, that cannot penetrate the region of radius $r_C$. The trajectory directions with $\cos \theta_{qr} < -1 + 2x$ get shadowed. (The factor of 2 in front of $x$ stems from the fact that trajectories get deflected both during the motion towards and away from the barrier.) The deflected trajectories pile up primarily outside of the shadowed region. The pile-up produces an integrable singularity in the kernel as shown by the peaks in Fig. 1. At low $\ell$, the Legendre polynomials have a limited angular resolution. Given that limited resolution, for the purpose of the
angular integration in (9), the angular structure in $|\phi|^2$, in vicinity of the backward direction $\cos \theta_{qr} = -1$, may be approximated by a $\delta$-function multiplied by the integral over the structure:

$$|\phi^{(-)}(q, r, \cos \theta_{qr})|^2 \approx 1 - \frac{x}{2} \delta(1 + \cos \theta_{qr}), \quad \text{for } x \cdot \max(1, \ell) \ll 1.$$  \hspace{1cm} (19)

Since $P_\ell(-1) = (-1)^\ell$, the high-energy kernel can then be expressed as

$$K_\ell(q, r) \approx (-1)^{\ell+1} x/2, \quad \text{for } x \ll 1/\max(1, \ell) \text{ and } qr \gg \max(1, \ell).$$  \hspace{1cm} (20)

For more implications of the classical limit (17) for kernels $K_\ell$, see ref. [25]. The classical condition of the Coulomb interaction is of interest in the context of emission of intermediate-mass fragments (IMF) from low-energy central nuclear reactions [24]. The practical condition of applicability for the classical approach is that $QR \gg \max(1, \ell)$, where $Q$ is the characteristic momentum which yields $r_C \sim R$. With $Z \sim A/2$, this produces, at low $\ell$, the condition $(\ell^2 A^3 m_N R)^{3/2} \gg 2$. Given a typical value of $R \sim 5$ fm for heavy ion reactions, this amounts to $A^{3/2} \gg 5$ and the classical limit is met at the fragment mass of $A \gtrsim 6$. IMF correlations are commonly expressed in terms of reduced velocity $v_{\text{red}} = q/\mu(Z_1 + Z_2)^{1/2}$. With $Z \sim A/2$, $r_C \sim 2\ell^2/m_N v_{\text{red}}$ and $R \sim 5$ fm, the high-energy limit of $x \ll 1$ for the kernel is reached at $v_{\text{red}} \gtrsim 0.04c$.

At small $qr$ in an interacting system, quantum effects become important as phase-space delocalization produces diffraction and tunneling. The diffraction affects the angle of an emerging particle at the level of $1/qr$, smearing out the angular structure in the kernel. Given the resolution of Legendre polynomials changing with $\ell$, the diffraction effects become consequently important for $qr \ll \max(1, \ell)$, with the kernel $K_\ell$ then being reduced on average. The tunneling into the region of $x > 1$ becomes significant down to $r = 0$, for $2\eta = xqr \equiv q r_C \lesssim 1$. For singly charged particles, such as identical pions, this corresponds to $v \gtrsim 0.02c$. Unlike the classical expression (17), the Coulomb wavefunction squared

$$|\phi^{(-)}(q, r, \cos \theta_{qr})|^2 = G(\eta) |M(i\eta, 1, -iqr(1 + \cos \theta_{qr}))|^2$$  \hspace{1cm} (21)

is analytic around $r = 0$ in the argument $r(1 + \cos \theta_{qr}) \equiv r + \mathbf{r q}$. In the quantal expression,

$$G(\eta) = \frac{2\pi \eta}{e^{2\pi \eta} - 1}$$  \hspace{1cm} (22)

is the Gamow factor. The terms in the Taylor series for the squared wavefunction (21) behave as $r^\ell (1 + \cos \theta_{qr})^\ell$. Since the highest power of $\cos \theta_{qr}$ for an $\ell$'th term is $\ell$, the quantal Coulomb kernel $K_\ell$ from (9) behaves around $r = 0$ as $r^\ell$, similarly to the identity-interference kernel (12). The power behavior, associated with the quantal delocalization, holds up to $r \lesssim \min(a_0, q^{-1})$. The behavior extends past the classical Coulomb barrier when $r_C \lesssim \min(a_0, q^{-1})$, leading to a further dampening of $K_\ell$ at $\ell \geq 1$ in the barrier region.

As an example, in Fig. 2 we consider $K_\ell$ kernels for the $pK^+$ interaction. The kernels are shown for $\ell = (0-3)$, at relative momenta of $q = 15$ (left panels) and 75 MeV/c (right panels), as a function of distance $r$. Regions of higher and lower relative momenta within a correlation function $R(q)$ are, typically, used to determine, respectively, about the short and long-range features of the source $S(r)$. The solid lines, dashed lines and symbols, represent the classical Coulomb kernels, quantal Coulomb kernels and quantal kernels with inclusion of strong interactions.

We first address the Coulomb kernels at 75 MeV/c. At large $r$ in the right panels of Fig. 2, the classical Coulomb kernels $K_\ell$ are, up to a sign, independent of $\ell$ and fall off as $1/r$, consistently with Eq. (20). Differences emerge at low $r$. At $\ell \geq 1$ the different classical kernels $K_\ell$ switch sign ($\ell - 1$), number of times, cf. [29]. With the distance scale of $r_C$ being, though, a fraction of a Fermi, $r_C = 0.17$ fm at $q = 75$ MeV/c, the structures associated with the sign switching are all pushed to very small $r$. Those sign-switching structures cannot be seen in the figure as the kernels at $r < 1.5$ fm are suppressed for the sake of clarity. For the quantum kernels, given $q^{-1} = 2.6$ fm, those kernels are seen to fluctuate with $r$ around the classical kernels at $r \gg \max(1, \ell) q^{-1}$. At lower $r$ and $\ell \geq 1$ quantal effects dampen the kernels, and in particular, at very low $r$ the classical sign-switching in the kernels gets completely erased with the kernels behaving as $r^\ell$. With inclusion of the effects of strong interactions in the kernels, additional oscillations are observed with $r$ within the figure, as the asymptotic behavior of the partial wavefunctions changes for low angular momenta, producing wave-function distortions at moderate distances. We will focus on strong interactions in the next subsection.

At $q = 15$ MeV/c, the classical Coulomb radius rises to $r_C = 4.1$ fm, on one hand pushing the asymptotic high-energy region out of the range shown in Fig. 2 and, on the other hand, making some of the low-$r$ details of the classical Coulomb kernels visible within the shown range. Also, since $q^{-1} = 13.1$ fm then, much of the region for the left panels of Fig. 2 is dominated by quantal effects. The kernels with inclusion of strong interactions get to be very close to the Coulomb kernels, since the phase shifts, characterizing changes in the partial-wave asymptotic behavior, are close to zero at low $q$. 
The quantal suppression of the kernels at $qr < \ell$, progressing with $\ell$, makes it difficult to learn about the directional characteristics of a source from the particle correlations at low $q \lesssim R^{-1} = 40$ MeV/c for $R = 5$ fm. Small kernels produce small contributions to the correlation which are difficult to detect experimentally. At the other end, at high $qr$, while the kernels become independent of $\ell$, they also become small, falling off as $1/(q^2 r)$, cf. [20]. As a consequence, an optimal bracket of $q$ for accessing the directional characteristic of sources with typical size emerges, such as $q \sim (30 - 100)$ MeV/c for the $pK^+$ system.

We have seen the importance of the analyticity of $|\phi|^2$ in $r$ around $r = 0$, which implied the $r^\ell$ suppression of the $K_\ell$ ($\ell \geq 1$) kernels at low $r$. The analyticity in $q$ is of interest in the context of the analysis of correlation functions as a function of $q$. In the case of pure identity interference, the wavefunction squared [11] is analytic in $q$ which makes the kernels from [4] analytic in $q$. The resulting correlation coefficients $R_\ell$ are also analytic and behave as $q^\ell$ at small $q$. 

FIG. 2: (Color online) Kernels $K_\ell$, multiplied by $(-1)^\ell$, for $pK^+$ interactions at $q = 15$ MeV/c (left panels) and $q = 75$ MeV/c (right panels), shown as a function of $1.5$ fm < $r$ < $30$ fm, for $\ell \leq 3$. Classical and quantal Coulomb kernels and quantal kernels with inclusion of strong interactions are represented by solid lines, dashed lines and symbols, respectively. The vertical scale may change from a panel to panel to emphasize details. Kernels are suppressed at the lowest $r$ for the sake of clarity.
In contrast, because of the general dependence on the parameter
\[ \eta = \frac{Z_1 Z_2 e^2 \mu}{q}, \]  
(23)
inversely proportional to \( q \), and because of the appearance of the Gamow factor [22], the square of the Coulomb wavefunction [24] lacks analyticity in \( q \) around \( q = 0 \). Still, it may be tempting to factor out the Gamow factor, as has been done in the analysis of pion correlations, arriving, with \( |\phi|^2 / G = |M|^2 \), at a Gamow-corrected kernel. The corrected kernel remains finite at a given \( r \) and \( \theta_{qr} \), when \( q \to 0 \), but retains a dependence in this limit on \( \cos \theta_{qr} \) for any finite value of \( r \). Thus, \( |M| \) reaches a different value depending on the side from which \( q = 0 \) is reached and lacks therefore analyticity in \( q \). Correspondingly, even the Gamow-corrected correlation functions from [8] lack analyticity in \( q \). The Gamow-corrected kernels \( K_q \) tend to finite values as \( q \to 0 \), rather than behaving as \( q^4 \).

As to the high-\( q \) limit of the Coulomb kernels, while the resolving power becomes independent of \( \ell \), the accessed information on the source represents one moment only per correlation coefficient, as
\[ \mathcal{R}_{\ell \ell'}(q) \approx \frac{2\sqrt{\pi}(-1)^{\ell} Z_1 Z_2 e^2 \mu}{q^2} \left( \frac{1}{r} \frac{1}{r} A_{\ell \ell'}(\Omega) \right), \]  
(24)
\[ \mathcal{R}_{\ell}(q) \approx \frac{(-1)^{\ell} Z_1 Z_2 e^2 \mu}{q^2} \left( \frac{1}{r} A_{\ell \ell'}(\Omega) \right), \]  
(25)
where
\[ \langle F \rangle = \int d^3 r \, S(r) \, F(r). \]  
(26)
Since the classical approximation is accurate at large \( qr \), this relation is unchanged by quantum considerations. Working within the classical approximation, one might be tempted to continue the \( r c / r \) kernel expansion, producing expansion of the correlation in powers of \( 1/q^2 \). The next expansion term could give access to the moment of the source function with a factor of \( 1/r^2 \) in place of \( 1/r \). However, beyond the leading term, the expansion gets altered by quantum effects making the expectation invalid for lighter particle pairs such as \( pK^+ \).

C. Kernels for Short-Range Interactions

Strong interactions between two particles also provide leverage for extracting source information from measured correlations and are particularly effective when the strong-interaction cross-sections are large [29]. In analysis of the effects of the interaction, the two-particle scattering wavefunction may be conveniently decomposed into angular-momentum eigenstates, here with any intrinsic spins suppressed:
\[ \phi^{(-)}(q, r, \cos \theta_{qr}) = \sum_{\ell_1} a_1 R_{\ell_1}(r) P_{\ell_1}(\cos \theta_{qr}) \approx \sum_{\ell_1} e^{iqr \cos \theta_{qr}} + f^{(-)}(\pi - \theta_{qr}) e^{-iqr} |qr| \]  
(27)
The coefficients \( a_1 \) in front of the radial wavefunctions \( R_{\ell} \) are adjusted to make the large-\( r \) outgoing wave-contributions to \( \phi^{(-)} \) identical to those for a wave in the absence of short-range interactions. The explicit asymptotic form of the wavefunction, on the r.h.s., is shown for the case with no Coulomb interactions. The amplitude \( f \equiv f^{(+)} \) is the scattering amplitude associated with the more standard scattering wavefunction \( \phi^{(+)} \) where conditions are imposed onto incoming wave-contributions in the asymptotic region. The amplitude factors in the two wavefunctions are related with \( f^{(-)}(\pi - \theta_{qr}) = f^{(+)}(\pi - \theta_{qr}) \).

The three-dimensional kernel from Eq. [27] is
\[ \mathcal{K}(q, r, \cos \theta_{qr}) \equiv |\phi^{(-)}|_2 - 1 = \frac{1}{r^2} \sum_{\ell_1 \ell_2} a_{\ell_1} a_{\ell_2} R_{\ell_1}(r) R_{\ell_2}(r) P_{\ell_1}(\cos \theta_{qr}) P_{\ell_2}(\cos \theta_{qr}) - 1 \]  
(28)
Since most shape information is contained at intermediate to large \( r \), the large \( r \) limit is especially insightful and can be found from Eq. [27],
\[ \mathcal{K}(q, r, \cos \theta_{qr}) \approx \frac{1}{r^2} \frac{d\sigma}{d\Omega}(\pi - \theta_{qr}) - \frac{\text{Im} \, f(\pi - \theta_{qr})}{r} \frac{\sin[qr(1 + \cos \theta_{qr})]}{r} + \frac{\text{Re} \, f(\pi - \theta_{qr})}{r} \frac{\cos[qr(1 + \cos \theta_{qr})]}{r} \]  
(29)
\[ \rightarrow \frac{1}{r^2} \frac{d\sigma}{d\Omega}(\pi - \theta_{qr}) - \frac{\sigma}{2\pi r^2} \delta(1 + \cos \theta_{qr}). \]  
(30)
In arriving at (29) and (30), we used an expression for the scattering cross section, \(|f|^2 = d\sigma/d\Omega\), and the optical theorem, \(\text{Im} f(0) = q\sigma/4\pi\). Further, in obtaining the expression (30) we have exploited the fact that for \(qr \to \infty\), under an \(r\)-integration, the respective products in (24) approach the limits: \(qr \sin(qrx) \to \delta(x)\) and \(qr \cos(qrx) \to 0\). The result (30) has a simple geometric interpretation. The first term represents the probability that a particle aimed directly at the scatterer reflects into the direction of \(q\), while the negative term represents the shadowing of a spherical angle, \(\sigma/4\pi r^2\), around the backward direction.

The partial kernels \(K_\ell\), respectively from Eq. (28) and (30), are

\[
K_\ell(q, r) = \frac{1}{r^2} \sum_{\ell_1, \ell_2} \binom{\ell}{\ell_1, \ell_2}^2 \alpha_{\ell_1} \alpha_{\ell_2}^{*} R_{\ell_1}(r) R_{\ell_2}(r) - \delta_{\ell 0}
\]

\[
\approx \frac{(-1)^\ell}{r^2} \left[ \frac{1}{2} \int d\cos \theta P_\ell(\cos \theta) \frac{d\sigma}{d\Omega}(\theta) - \frac{\sigma}{4\pi} \right],
\]

where we have made use of the Wigner 3-j-symbol. In the asymptotic region, all \(\ell \geq 1\) kernels decrease as \(1/r^2\) and are proportional to the scattering cross section. Larger cross sections, such as associated with resonances, will lead to larger kernels at a given \(q\). Otherwise, the kernels are sensitive to the angular dependence of the cross sections. For \(q\) in the range for which correlations are typically studied, cross sections are usually fairly isotropic. Thus, the first term in the brackets on the r.h.s. of (32) decreases quickly as \(\ell\) increases, yielding

\[
K_\ell(q, r) \approx \frac{(-1)^{\ell+1} \sigma}{4\pi r^2}, \quad \text{for large } r \text{ and } \ell.
\]

Apart from the sign, this kernel is independent of \(\ell\), similarly to the Coulomb kernel (20). In analogy to the classical Coulomb results for individual \(\ell\), the asymptotic results (32) require \(qr \gg \ell\) for their validity. An additional condition for validity is that \(r/d \gg \ell\) where \(d\) is the interaction range.

Regarding the limiting short-range (33) and Coulomb (20) \(K_\ell\)-results, their independence from \(\ell\) results from the kernels being exclusively influenced by the shadowing of particles by each other. The effect of shadowing decreases with distance between the particles, but more slowly for the Coulomb than for purely short-range interaction. In the case of short-range interactions, the kernel falls off as \(\sigma/r^2\), while for the Coulomb forces the kernels fall off as \(r_C/r\), where \(r_C = 2\mu Z_1 Z_2 e^2/q^2\) is the radius of the Coulomb barrier. Since \(r_C\) explicitly decreases with \(q\), Coulomb kernels decrease monotonically with increasing \(q\), whereas for the strong interaction, the \(q\) dependence on the kernel tends to follow the energy dependence of the cross section. Strong interactions can thus provide resolving power at large \(q\), if the cross sections are large, e.g. at resonance energies.

When the two terms in the asymptotic form of the kernel (33) are integrated over full spherical angle, those terms exactly cancel, reflecting the fact that, for a short-range interaction, the particles which are shadowed for \(\cos \theta_{qr} = -1\) reappear at other angles at the same magnitude of relative momentum. Consequently, the \(\ell = 0\) coefficient for the \(1/r^2\) asymptotic fall-off in (32) vanishes, implying that the \(K_0\) kernel from (31) vanishes faster than \(1/r^2\) as \(r \to \infty\). These results may be contrasted with what is found for the Coulomb interactions. The large-\(r\) Coulomb kernel from (13) yields a finite result when integrated over the spherical angle. In direct consequence, the \(\ell = 0\) kernel is finite in the asymptotic region, Eqs. (18) and (20). The difference, between the short-range and Coulomb cases, is due to the fact that a Coulomb interaction does not just change the orientation of the relative momentum but also makes the momentum magnitude different from the asymptotic value at any finite distance from the interaction center.
At low relative momenta \( q \), \( s \)-wave scattering is likely to dominate. An important result, visible already in the asymptotic limit, Eqs. (32) and (33), is that the \( s \)-wave scattering contributes to the kernels at all \( \ell \). At the level of Eq. (31), it is seen as associated with the interference between the waves. At resonance, the radial wavefunctions \( R_{\ell} \) will be particularly large in the near zone giving rise to enhanced \( K_{\ell} \) there, including \( \ell = 0 \). At low \( q \) and finite \( \ell \), the effects of the centrifugal barrier can be important. When \( q \) is so low that the barrier is encountered outside of the strong-interaction range, \( qr < \ell + \frac{3}{2} \) at \( r > d \), the radial wavefunctions are approximately given in terms of the spherical Bessel functions, \( R_{\ell}/r \propto j_{\ell}(qr) \propto (qr)^{\ell} \). With \( R_{0} \) approaching a constant value as a function of both \( q \) and \( r \) in the region \( d < r < 1/q \), at \( q \to 0 \), given the triangle inequality for the \( \ell \)-values in the 3\( j \) symbol in (32), the strong-interaction kernels will behave as \( K_{\ell}(q,r) \propto (qr)^{\ell}(1 + b/r) \) for \( \ell \geq 1 \) in the region. The kernels will be suppressed then at low \( q \) and low \( r \) as well for larger \( \ell \). A short-range repulsion may enhance suppression at lowest \( r < d \) beyond that produced by the centrifugal barrier.

Outside of the interaction range, at \( r > d \), the wavefunctions \( R_{\ell} \) are fully determined by the respective strong-interaction phase-shifts, both in the absence and presence of Coulomb interactions. We exploit this [29] in obtaining the kernels for strong interactions, e.g., the kernels in Fig. 2. Energy derivatives additionally constrain the integrals of the wavefunctions \( R_{\ell} \) squared over the region \( r < d \) [29, 31]. This facilitates an extrapolation of the kernels down to \( r = 0 \), adequate for source resolutions that can be practically achieved, without the need to resort to potential models for the strong interaction. Still, with the strong-interaction kernels becoming model-dependent at short distances, we refrain from displaying them at \( r < 1.5 \) fm. Needless to say that, given the quark substructure of hadrons, the description in terms of wavefunction for relative motion becomes questionable at the shortest distances \( r \lesssim 0.5 \) fm. For IMFs the same becomes true at even larger distances, due to the finite size of the nuclei.

### III. SPHERICAL CARTESIAN HARMONICS

#### A. Harmonics within Spherical Coordinates

We start out with a review of basic concepts concerning surface spherical harmonics. A harmonic function is one which satisfies the Laplace equation

\[
\nabla^2 F(r) = 0. \tag{34}
\]

When using spherical coordinates, it is convenient to express harmonic functions, which are regular at the origin, in terms of tesseral harmonics \( Y_{\ell m} \),

\[
F(r) = \sqrt{4\pi} \sum_{\ell m} F_{\ell m}^* r^\ell Y_{\ell m}(\Omega), \tag{35}
\]

where the coefficients \( F_{\ell m} \) take on arbitrary values. In the summation over \( \ell \), the individual terms in (35), of the form

\[
F^{(\ell)}(r) = \sqrt{4\pi} r^\ell \sum_{m} F_{\ell m}^* Y_{\ell m}(\Omega), \tag{36}
\]

are harmonic functions of degree \( \ell \). The latter stand for homogenous functions of degree \( \ell \) in \( x, y \) and \( z \), that are harmonic. A surface spherical of degree \( \ell \) is a harmonic function of degree \( \ell \), taken on a unit sphere, \( r = 1 \):

\[
F^{(\ell)}(\Omega) = \sqrt{4\pi} \sum_{m} F_{\ell m}^* Y_{\ell m}(\Omega). \tag{37}
\]

Given the completeness relation for the tesseral spherical harmonics,

\[
\delta(\Omega - \Omega') = \frac{1}{4\pi} \sum_{\ell} (2\ell + 1) P_{\ell}(\mathbf{n} \cdot \mathbf{n}') = \sum_{\ell m} Y_{\ell m}^*(\Omega') Y_{\ell m}(\Omega), \tag{38}
\]

any function of spherical angle, and in particular \( R \) and \( S \) in Eqs. (8)-(10), can be expanded in the tesseral harmonics upon using (38) with the identity

\[
G(\Omega) = \int d\Omega' \delta(\Omega - \Omega') G(\Omega'). \tag{39}
\]

Equation (33) for the harmonic functions represents an example of the expansion, where the expansion coefficients have a specific radial dependence. Within every rank \( \ell \) of an expansion, the expanded functions are generally described
in terms of \((2\ell + 1)\) complex coefficients \(F_{\ell m}\). However, for real expanded functions, the coefficients \(F_{\ell m=0}\) must be real and, otherwise, the coefficients must satisfy \(F_{-m} = (-1)^m F_{\ell m}\). Thus, for each \(\ell\), the expansion is described in terms \((2\ell + 1)\) independent real numbers. Under rotations of the coordinate system, for a given \(\ell\), the tesseral functions transform like the components of a spherical tensor of rank \(\ell\). For a function that is independent of the coordinate choice, the expansion coefficients transform further in this fashion; the sums over \(m\) in (35), (36) and (4) represent scalar products of the spherical tensors. In fact, the r.h.s. of Eq. (35) also represents a superposition of the scalar products and, in this, represents a covariant generalization of the middle result in (35), which is seen as a superposition of the scalar products of the tensors in the case when the coordinate \(z\)-axis is oriented either along \(\mathbf{n}\) or \(\mathbf{n}'\).

### B. Cartesian Harmonics

Cartesian harmonics allow functions of the spherical angle to be expressed as superpositions of the scalar products of cartesian tensors. For real functions, the coefficients of expansion in cartesian harmonics are real, which is not necessarily the case when expanding functions in tesseral harmonics. The expansion coefficients for cartesian harmonics are generally easier to interpret than those for tesseral harmonics.

Let us consider a homogenous function of degree \(\ell\). When using cartesian coordinates, a homogenous function may be represented as

\[
F^{(\ell)}(\mathbf{r}) = \sum_{\alpha_1, \alpha_2 \ldots, \alpha_\ell} F^{(\ell)}_{\alpha_1 \alpha_2 \ldots \alpha_\ell} r_{\alpha_1} r_{\alpha_2} \ldots r_{\alpha_\ell},
\]

where \(\alpha = x, y, z\) are cartesian indices. The following remarks can be made regarding this representation in terms of cartesian coordinates. Since the product of cartesian coordinates on the r.h.s. of (40) is symmetric under the interchange of its terms, attention may be limited to the coefficients of expansion \(F_{\alpha_1 \alpha_2 \ldots \alpha_\ell}\) which are symmetric under the interchange of indices. Since the product of \(\ell\) coordinates transforms further as a cartesian tensor of rank \(\ell\), the coefficients \(F_{\alpha_1 \alpha_2 \ldots \alpha_\ell}\) must transform as a cartesian tensor of rank \(\ell\) for a function \(F^{(\ell)}\) that is independent of the choice of directions for coordinate axes. Given the symmetry of the tensorial coefficients, the r.h.s. of (40) can be conveniently rewritten by grouping terms with different numbers, \(\ell_x, \ell_y\) and \(\ell_z\), respectively, of the \(x, y\) and \(z\) coefficients, obtaining the representation such as in (8),

\[
F^{(\ell)}(\mathbf{r}) = \sum_{\ell} \frac{\ell!}{\ell_x! \ell_y! \ell_z!} F_{\ell} x^{\ell_x} y^{\ell_y} z^{\ell_z} \equiv F^\top \mathbf{r}^\ell,
\]

where \(\ell = \ell_x + \ell_y + \ell_z\). The last expression in (41) is a schematic representation for the convolution of the cartesian tensors.

On substituting the representation (40) into the Laplace equation (34), we find that the homogenous function is harmonic if and only if the cartesian tensorial coefficients are traceless (12), i.e.,

\[
\sum_{\alpha} F^{(\ell)}_{\alpha_1 \ldots \alpha_{\ell-2} \alpha \alpha} = 0,
\]

or

\[
F_{\ell_x+2, \ell_y, \ell_z} + F_{\ell_x, \ell_y+2, \ell_z} + F_{\ell_x, \ell_y, \ell_z+2} = 0,
\]

where \(\ell_x + \ell_y + \ell_z + 2 = \ell\). The components for a symmetric tensor, labelled with the values of \((\ell_x, \ell_y, \ell_z)\), are illustrated with a triangle in Fig. 4. When moving along lines parallel to the different sides of the triangle, either the value of \(\ell_x\), \(\ell_y\) or \(\ell_z\) stays constant. The tracelessness condition (43) relates components at the corners of an equilateral triangle two \(\ell\)-units on the side in the figure.

In the appendix we show that there exists a real projection operator \(\mathcal{P}\), which operates in the space of cartesian tensors of rank \(\ell\) and projects, out of any tensor, the part which is symmetric and traceless. As an operator in the space of rank \(\ell\) tensors, \(\mathcal{P}\) itself is a rank \(2\ell\) tensor, \(\ell\) times covariant and \(\ell\) times contravariant. As a projection operator, \(\mathcal{P}\) satisfies \(\mathcal{P}^2 = \mathcal{P}\) and \(\mathcal{P}^\top = \mathcal{P}\). In terms of the operator \(\mathcal{P}\), for any tensor \(T\) of rank \(\ell\), the function given by

\[
(\mathcal{P} T)^\top \mathbf{r}^\ell = T^\top \mathcal{P}^\top \mathbf{r}^\ell = T^\top \mathcal{P} \mathbf{r}^\ell
\]
is harmonic, where we follow the schematic notation. Since, the function (44) is harmonic for any tensor $T$, each of the components of $P r'$ on the r.h.s. must be harmonic. On the sphere of $r = 1$, those components define the cartesian surface spherical harmonics

$$A^{(\ell)}_{\alpha_1 \alpha_2 \ldots \alpha_\ell}(\Omega) = \sum_{\alpha'_1 \alpha'_2 \ldots \alpha'_{\ell}} P^{(\ell;\ell)}_{\alpha_1 \alpha_2 ; \alpha'_1 \alpha'_2 \ldots \alpha'_{\ell}} n_{\alpha_1} n_{\alpha_2} \ldots n_{\alpha_{\ell}},$$

(45)

where $n = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$, which can be further also written as

$$A_{\ell}(\Omega) = \sum_{\ell_1 \ell_2} \delta_{\ell_1 \ell_2} P_{\ell_1 \ell_2} n_{\ell_1} n_{\ell_2}.$$

(46)

Cartesian components of the direction vector $n$ are combinations of the tesseral rank-1 harmonics $Y_{1m}$. The product of $\ell$ components, as e.g. demonstrated by the quantal rules of angular-momentum superposition, is a superposition of tesseral harmonics of rank $\ell$ and lower, of the same evenness as $\ell$:

$$n_{\alpha_1} n_{\alpha_2} \ldots n_{\alpha_{\ell}} = \sum_{\ell' \leq \ell} c^{\ell' m' s}_{\alpha_1 \alpha_2 \ldots \alpha_{\ell}} Y_{\ell' m'}(\Omega) \equiv A^{(\ell)}_{\alpha_1 \alpha_2 \ldots \alpha_{\ell}} + \left( n_{\alpha_1} n_{\alpha_2} \ldots n_{\alpha_{\ell}} - A^{(\ell)}_{\alpha_1 \alpha_2 \ldots \alpha_{\ell}} \right).$$

(47)

Since $r' A^{(\ell)}$ are harmonic, $A^{(\ell)}$ can consist of the tesseral rank-$\ell$ harmonics only. As will be demonstrated in the appendix, the operator $P$ consists out of a symmetrization operator, i.e. the unit tensor in the space of symmetric cartesian tensors, and of operator terms which involve taking traces of one or more pairs of the cartesian indices. In acting on the the cartesian product $n^\ell$, the symmetrization operator leaves the product $n^\ell$ intact while the other operator terms in $P$ produce symmetrized terms of the form $n^{\ell - 2k} (\delta)^k$, where $k = 1, \ldots, \ell/2$. The latter terms represent superpositions of tesseral harmonics up to the rank of $\ell - 2$, of the same evenness as $\ell$. The implication is

FIG. 4: Triangular diagram representing the components of a symmetric tensor, in terms of dots labelled with the component $[\ell_x, \ell_y, \ell_z]$ values, for $\ell = \ell_x + \ell_y + \ell_z = 4$, after [24]. When moving along the lines parallel to the respective sides of the triangle, either $\ell_x$, $\ell_y$ or $\ell_z$ stays constant. The tracelessness condition (43) relates tensor components at the corners of an equilateral triangle two $\ell$-units on the side, within the diagram, such as those marked by the squares. As the set of linearly independent components of a traceless symmetric tensor, the $\ell_z = 0$ and $\ell_z = 1$ components may be chosen, marked with the thick grey lines in the figure.
TABLE I: Cartesian harmonics for $\ell \leq 4$. Other harmonics can be found by either permuting the indices, i.e. $A_{xyz} = A_{xyz}$, or by swapping indices on both sides of an equality, e.g. $x \leftrightarrow y$. Thus, given $A_{xy} = n_x^2 n_y - (1/5)n_y$, swapping the indices $y \leftrightarrow z$ yields $A_{xzz} = n_z^2 n_x - (1/5)n_x$.

$$
\begin{array}{|c|c|}
\hline
A_{1}^{(1)} & n_x \\
A_{2}^{(2)} & n_x^2 - 1/3 \\
A_{3}^{(3)} & n_x^3 \\
A_{4}^{(4)} & n_x^4 - (6/7)n_x^2 + 3/35 \\
\hline
\end{array}
$$

that the projection operator $P$ removes from $n^t$ a portion represented by the superposition of the lower rank tesseral harmonics (in parenthesis on the r.h.s. of (47)) while retaining the highest rank $\ell$ portion intact. Given the general form of $A^{(\ell)}$, from application of $P$ onto $n^t$, the cartesian harmonics may be constructed recursively [28], based on the tracelessness of the $A^{(\ell)}$ tensors within each rank, starting with $A^{(\ell=0)} = 1$:

$$
A_{\alpha_1 \ldots \alpha_\ell}^{(\ell)} = \frac{1}{\ell} \sum_{i=1}^{\ell} n_{\alpha_i} A_{\alpha_1 \ldots \alpha_{i-1} \alpha_{i+1} \ldots \alpha_\ell}^{(\ell-1)} - \frac{2}{\ell (2\ell - 1)} \sum_{1 \leq i < j \leq \ell} \sum_{\alpha} \delta_{\alpha_1 \alpha_j} n_{\alpha} A_{\alpha_1 \ldots \alpha_{i-1} \alpha_{i+1} \ldots \alpha_{j-1} \alpha_{j+1} \ldots \alpha_\ell}^{(\ell-1)}. 
$$

Expressions for cartesian harmonics of lowest rank are given in Table I and the recursion with (48), otherwise, produces the series (7). One of the features of the cartesian harmonics is that the three cartesian axes are treated equally, unlike in the case of tesseral harmonics. Within a given rank $\ell$, not all cartesian harmonics are independent. A symmetric tensor has $(\ell + 1)(\ell + 2)/2$ different components, cf. Fig. 4. The tracelessness condition (43) reduces the number of linearly independent components, including the number of independent $A^{(\ell)}$ functions, to $(2\ell + 1)$, equal to the number of $Y_{\ell m}$ functions. As a linearly independent set, one might choose the components with $\ell_x = 0$ and $\ell_z = 1$, see the figure.

In the case of $\alpha_1 = \alpha_2 = \ldots = \alpha_\ell = z$ in (47), both the cartesian component product $n^t$ and $A^{(\ell)}$ are invariant with respect to rotations around the $z$ axis. For invariance, both functions must be a superposition of $m' = 0$ tesseral harmonics only. It follows then that $A^{(\ell)}_{xzz}$ must be proportional to $Y_{\ell 0}$ or $P_{\ell}$. Since the action of the projection operator $P$ leaves the highest rank terms unchanged, the coefficient for $P_{\ell}$ in $A^{(\ell)}$ must be the same as in $n^t$. Accordingly, the respective cartesian harmonics are given by

$$
A_{z}^{(\ell)}(\Omega) = \frac{\ell}{(2\ell - 1)!!} P_{\ell}(\cos \theta).
$$

Starting with the first equality in (48), we are now in position to produce a completeness relation in terms of the cartesian harmonics. Thus, if $n^t$ is directed along the $z$-axis in (18), we can express, given (19), the Legendre polynomial in the middle term in (18) as $(2\ell - 1)!!/\ell! n^t)^{\top} A$. When rotating $n^t$ and $n$ to move $n^t$ away from the alignment with the $z$-axis, the scalar product of the tensors transforms covariantly, allowing to obtain different general representations for the $\delta$-function in spherical angle in terms of the cartesian harmonics

$$
\delta(\Omega - \Omega') = \frac{1}{4\pi} \sum_{\ell} (2\ell + 1) P_{\ell}(n \cdot n')
$$

$$
= \frac{1}{4\pi} \sum_{\ell} (2\ell + 1)!! \sum_{\alpha_1 \ldots \alpha_\ell} n_{\alpha_1} \ldots n_{\alpha_\ell} A^{(\ell)}_{\alpha_1 \ldots \alpha_\ell}(\Omega)
$$

$$
= \frac{1}{4\pi} \sum_{\ell} (2\ell + 1)!! \sum_{\alpha_1 \ldots \alpha_\ell} A^{(\ell)}_{\alpha_1 \ldots \alpha_\ell}(\Omega') A^{(\ell)}_{\alpha_1 \ldots \alpha_\ell}(\Omega)
$$

$$
= \frac{1}{4\pi} \sum_{\ell} (2\ell + 1)!! \sum_{\alpha_1 \ldots \alpha_\ell} A^{(\ell)}_{\alpha_1 \ldots \alpha_\ell}(\Omega') A^{(\ell)}_{\alpha_1 \ldots \alpha_\ell}(\Omega)
$$

$$
= \frac{1}{4\pi} \sum_{\ell} (2\ell + 1)!! \sum_{\alpha_1 \ldots \alpha_\ell} A^{(\ell)}_{\alpha_1 \ldots \alpha_\ell}(\Omega') A^{(\ell)}_{\alpha_1 \ldots \alpha_\ell}(\Omega)
$$

In obtaining the third equality in (50) and later, we use the projection properties of the operator $P$:

$$
(n^{\ell'})^{\top} A = (n^{\ell'})^{\top} P A = (P n^{\ell'})^{\top} A = (A')^{\top} A.
$$

(51)
Use of (50) in the identity (59) allows for functions of spherical angle to be expressed as a sum of cartesian harmonics, such as in (8). Independent of the orientation of the coordinate axes, the coefficients of expansion in cartesian harmonics, such as in (9), transform as components of a cartesian tensor under rotations. As the tensor is traceless, only \((2\ell + 1)\) coefficients are independent within each rank \(\ell\).

C. Operations with Cartesian Harmonics

When given some specific expansion coefficients in terms of cartesian harmonics, it may be of interest to calculate the associated cartesian moments. To facilitate that, it is useful to express products of cartesian components in terms of cartesian harmonics, which can be done through recursion starting from Eq. (7), obtaining

\[
\sum_{m}^{\ell} (2\ell - 4m + 1)!! \quad 2m (2\ell - 2m + 1)!! \quad A_{m} \quad \ell_{x}! \quad (\ell_{x} - 2m_{x})! \quad m_{x}! \quad (\ell_{y} - 2m_{y})! \quad m_{y}! \quad (\ell_{z} - 2m_{z})! \quad m_{z}! \quad A_{\ell - 2m}(\Omega) .
\]

The rank of \(A\) within each cartesian component on the r.h.s. is equal to or lower than the power of the corresponding cartesian component on the l.h.s. From (52) and (9), we find

\[
\int d^{3}r S(r) x^{\ell_{x}} y^{\ell_{y}} z^{\ell_{z}} = \frac{4\pi \ell! (2\ell + 1)!!}{(2\ell)!} \sum_{m}^{\ell} (2\ell - 4m + 1)!! \quad 2m (2\ell - 2m + 1)!! \quad A_{m} \quad \ell_{x}! \quad (\ell_{x} - 2m_{x})! \quad m_{x}! \quad (\ell_{y} - 2m_{y})! \quad m_{y}! \quad (\ell_{z} - 2m_{z})! \quad m_{z}! \quad \int_{0}^{\infty} dr r^{\ell + 2} S_{\ell - 2m}(r) ,
\]

which allows the moments to be obtained from cartesian harmonic coefficients.

As combinations of tesseral harmonics of different rank \(\ell\), the cartesian harmonics of different rank are orthogonal to one another. Due to different behavior under inversion, see Eq. (7), two cartesian harmonics are further orthogonal if any pair of their \(\ell_{i}\) subscripts differs by an odd number. Otherwise, the scalar product of harmonics is given by (20)

\[
\langle \ell | \ell' \rangle = \int \frac{d\Omega}{4\pi} A_{\ell} A_{\ell'} = \frac{1}{(2\ell + 1)!! (2\ell - 1)!!} \sum_{m}^{\ell} \left( \frac{1}{2} \right)^{m} m! (2\ell - 2m + 1)!! \quad \prod_{i=x,y,z} \frac{\ell_{i}! \ell_{i}'!}{((\ell_{i}' - \ell_{i})/2 + m_{i})! (\ell_{i}' - 2m_{i})! m_{i}!} .
\]

The anticipated symmetry between \(\ell\) and \(\ell'\) on the r.h.s. of (54) may be seen by eliminating \(m_{i}\) in favor of \(m_{i}' = m_{i} + (\ell_{i} - \ell_{i}')/2\).

Cartesian expansion coefficients may be converted into tesseral expansion coefficients and \textit{vice versa}. In terms of cartesian harmonics, the \(m \geq 0\) tesseral harmonics are given by (20)

\[
Y_{\ell m}(\Omega) = (-1)^{m} (2\ell - 1)!! \left[ \frac{2\ell + 1}{4\pi (\ell + m)! (\ell - m)!} \right]^{1/2} \sum_{k=0}^{m} \frac{m!}{(m - k)!} \frac{m!}{k!} ! A_{[k,m-k,\ell-m]}(\Omega) .
\]

The cartesian harmonics may be, on the other hand, expressed as

\[
A_{\ell}(\Omega) = \sum_{m=-\ell}^{\ell} (a_{\ell}^{m})^{*} Y_{\ell m}(\Omega) ,
\]

where (20), for \(m \geq 0\),

\[
a_{\ell}^{m} = (-1)^{m} (2\ell - 1)!! \left[ \frac{2\ell + 1}{4\pi (\ell + m)! (\ell - m)!} \right]^{1/2} \sum_{k=0}^{m} \frac{m!}{(m - k)!} \frac{m!}{k!} ! \langle \ell | k, m - k, \ell - m \rangle .
\]
TABLE II: Transformation equations between cartesian and tesseral harmonics for $\ell \leq 4$.

$Y_{10} = (1/2)\sqrt{3/\pi} A_{10}^{(1)}$
$Y_{1\pm 1} = -(i/2)\sqrt{3/2\pi} A_{1\pm 1}^{(1)} \pm (1/2)\sqrt{3/2\pi} A_{1\mp 1}^{(1)}$
$Y_{20} = (3/4)\sqrt{5/\pi} A_{20}^{(2)}$
$Y_{2\pm 1} = -(i/2)\sqrt{15/2\pi} A_{2\pm 1}^{(2)} \pm (1/2)\sqrt{15/2\pi} A_{2\mp 1}^{(2)}$
$Y_{2\pm 2} = -(1/4)\sqrt{15/2\pi} A_{2\pm 2}^{(2)} \pm (i/2)\sqrt{15/2\pi} A_{2\mp 2}^{(2)} + (1/4)\sqrt{15/2\pi} A_{2\pm 2}^{(2)}$
$Y_{30} = (5/4)\sqrt{7/\pi} A_{30}^{(3)}$
$Y_{3\pm 1} = -(5i/8)\sqrt{21/\pi} A_{3\pm 1}^{(3)} \mp (5i/8)\sqrt{21/\pi} A_{3\mp 1}^{(3)}$
$Y_{3\pm 2} = -(1/4)\sqrt{105/2\pi} A_{3\pm 2}^{(3)} \pm (i/2)\sqrt{105/2\pi} A_{3\mp 2}^{(3)} \pm (1/4)\sqrt{105/2\pi} A_{3\pm 2}^{(3)}$
$Y_{3\pm 3} = -(i/8)\sqrt{35/\pi} A_{3\pm 3}^{(3)} \pm (3i/8)\sqrt{35/\pi} A_{3\mp 3}^{(3)} \mp (3i/8)\sqrt{35/\pi} A_{3\pm 3}^{(3)} \mp (1/8)\sqrt{35/\pi} A_{3\pm 3}^{(3)}$
$Y_{40} = (105/16)\sqrt{1/\pi} A_{40}^{(4)}$
$Y_{4\pm 1} = -(21i/8)\sqrt{5/\pi} A_{4\pm 1}^{(4)} \pm (21i/8)\sqrt{5/\pi} A_{4\mp 1}^{(4)}$
$Y_{4\pm 2} = -(21i/8)\sqrt{5/\pi} A_{4\pm 2}^{(4)} \pm (21i/8)\sqrt{5/\pi} A_{4\mp 2}^{(4)}$
$Y_{4\pm 3} = -(3i/8)\sqrt{35/\pi} A_{4\pm 3}^{(4)} \pm (9i/8)\sqrt{35/\pi} A_{4\mp 3}^{(4)} - (9i/8)\sqrt{35/\pi} A_{4\pm 3}^{(4)} \mp (3i/8)\sqrt{35/\pi} A_{4\mp 3}^{(4)}$
$Y_{4\pm 4} = -(3i/8)\sqrt{35/\pi} A_{4\pm 4}^{(4)} \mp (3i/8)\sqrt{35/\pi} A_{4\mp 4}^{(4)} + (1/16)\sqrt{35/\pi} A_{4\pm 4}^{(4)}$
$A_{10}^{(1)} = 2\sqrt{3/\pi} Y_{10}$
$A_{11}^{(1)} = i\sqrt{2/3} Y_{11} + i\sqrt{2/3} Y_{1-1}$
$A_{12}^{(1)} = -\sqrt{2/3} Y_{11} - \sqrt{2/3} Y_{1-1}$
$A_{20}^{(2)} = (4/3)\sqrt{5/\pi} Y_{20}$
$A_{21}^{(2)} = i\sqrt{2/5} Y_{21} + i\sqrt{2/5} Y_{2-1}$
$A_{22}^{(2)} = -\sqrt{2/5} Y_{21} - \sqrt{2/5} Y_{2-1}$
$A_{30}^{(3)} = i\sqrt{2/15} Y_{30}$
$A_{31}^{(3)} = -\sqrt{2/15} Y_{31} - \sqrt{2/15} Y_{3-1}$
$A_{32}^{(3)} = -\sqrt{2/15} Y_{32} - (2/3)\sqrt{2/15} Y_{3-2}$
$A_{40}^{(4)} = i\sqrt{2/15} Y_{40}$
$A_{41}^{(4)} = -\sqrt{2/15} Y_{41} - i\sqrt{2/15} Y_{4-1}$
$A_{42}^{(4)} = -\sqrt{2/15} Y_{42} - i\sqrt{2/15} Y_{4-2}$
$A_{43}^{(4)} = -\sqrt{2/15} Y_{43} - (2/3)\sqrt{2/15} Y_{4-3}$
$A_{44}^{(4)} = (16/105)\sqrt{2/5} Y_{44}$
$A_{45}^{(4)} = (4i/21)\sqrt{2/5} Y_{45}$
$A_{46}^{(4)} = (i/21)\sqrt{2/5} Y_{46}$
$A_{47}^{(4)} = (i/21)\sqrt{2/5} Y_{47}$
$A_{48}^{(4)} = (i/21)\sqrt{2/5} Y_{48}$
$A_{49}^{(4)} = (i/21)\sqrt{2/5} Y_{49}$
$A_{50}^{(4)} = (i/21)\sqrt{2/5} Y_{50}$
$A_{51}^{(4)} = (i/21)\sqrt{2/5} Y_{51}$
$A_{52}^{(4)} = (i/21)\sqrt{2/5} Y_{52}$
$A_{53}^{(4)} = (i/21)\sqrt{2/5} Y_{53}$
$A_{54}^{(4)} = (i/21)\sqrt{2/5} Y_{54}$
$A_{55}^{(4)} = (i/21)\sqrt{2/5} Y_{55}$
$A_{56}^{(4)} = (i/21)\sqrt{2/5} Y_{56}$
$A_{57}^{(4)} = (i/21)\sqrt{2/5} Y_{57}$
$A_{58}^{(4)} = (i/21)\sqrt{2/5} Y_{58}$
$A_{59}^{(4)} = (i/21)\sqrt{2/5} Y_{59}$
$A_{60}^{(4)} = (i/21)\sqrt{2/5} Y_{60}$
the overlap integral $\langle \cdot | \cdot \rangle$ is given in (54) and $a^{-m} = (-1)^m \left( a^m \right)^*$. Formulas for conversion between tesseral and cartesian harmonics for $\ell \leq 4$ are given in Table II.

Many circumstances require multiplication or division of the functions of spherical angle. For example, the empirical correlation function $C = 1 + R$, as a function of spherical angle at a given $P$ and $q$, follows from an equation of the form

$$X(\Omega) = C(\Omega) B(\Omega),$$

(58)

where the l.h.s. represents the measured two-particle yield, $X(\Omega) = \frac{4\pi}{d_P^a d_P^b}$, cf. Eq. (1), while $B$ on the r.h.s. is the product of single-particle yields, $4\pi \frac{d_P^a}{d_P^b}$, and possibly of two-particle efficiency. In order to find $C(\Omega)$, one could bin the three functions in two dimensions of $\cos \theta$ and $\phi$ and obtain $C$ within each bin by dividing the value of $X$ in the bin by the value of $B$ in the bin. However, if one next wanted to expand $C$ in cartesian harmonics, the result could be distorted by residual binning effects which would disappear in the limit of fine binning, but that could be only afforded for high statistics.

An alternative approach is to explicitly solve for the coefficients $C_\ell$ given $X_\ell$ and $B_\ell$. With this, one could forego the binning in $\cos \theta$ and $\phi$. For instance, when assessing $X$ and identifying a particle pair at an angle $\Omega$, one could increment values in an array for $X_\ell$ by $[(2\ell + 1)!!/\ell!] A_\ell(\Omega)$ as described by Eq. (59). One could then carry out an analogous procedure for the mixed pairs, filtering them through two-particle efficiency. In this way, both $X_\ell$ and $B_\ell$ could be obtained in a straightforward fashion, without any binning in $\cos \theta$ and $\phi$.

Though the determination of $X_\ell$ and $B_\ell$ could be straightforward, the determination of $C_\ell$ can be more difficult. Here, we outline one possible procedure based on the expansion (8) of the functions in powers of $\hat{q}$. If we introduce

$$\gamma(\ell) = \frac{\ell}{\ell_x \ell_y \ell_z},$$

(59)

we can write

$$X(\hat{q}) = \sum_\ell \gamma(\ell) X_\ell \hat{q}_x^\ell \hat{q}_y^\ell \hat{q}_z^\ell = \sum_\ell \sum_{0 \leq \ell_i \leq \ell} \gamma(\ell - \ell_i) \gamma(\ell_i) B_{\ell - \ell_i} C_{\ell_i} \hat{q}_x^{\ell_x} \hat{q}_y^{\ell_y} \hat{q}_z^{\ell_z}.$$

(60)

If the condition of tracelessness for $\bar{C}_\ell$ is lifted, the solution to Eq. (60) ceases to be unique, because $\bar{q}_i$ are not independent due to the constraint $\hat{q}_x^2 + \hat{q}_y^2 + \hat{q}_z^2 = 1$. We will first find a solution to (60) without imposing the tracelessness condition, which will provide us with $C(\Omega)$ from which we can next find the traceless $C_\ell$.

The simplest strategy in finding a solution to (60) is to treat the components $\bar{q}_i$ as if they were independent. If we denote the solution by $\bar{C}_\ell$, we obtain

$$\gamma(\ell) X_\ell = \sum_{0 \leq \ell_i \leq \ell} \gamma(\ell - \ell_i) \gamma(\ell_i) B_{\ell - \ell_i} \bar{C}_{\ell_i},$$

(61)

which may be solved for $\bar{C}_\ell$ iteratively. The $\ell = 0$ term follows right away,

$$\bar{C}_0 = \frac{X_0}{B_0},$$

(62)

while the subsequent coefficients follow from

$$\bar{C}_\ell = \frac{X_\ell}{B_\ell} - \frac{1}{\gamma(\ell) B_\ell} \sum_{\ell' \leq \ell - 1} \gamma(\ell - \ell') \gamma(\ell') B_{\ell - \ell'} \bar{C}_{\ell'},$$

(63)

The combination of

$$C(\Omega) = \sum_\ell \gamma(\ell) \bar{C}_\ell \hat{q}_x^{\ell_x} \hat{q}_y^{\ell_y} \hat{q}_z^{\ell_z},$$

(64)

and of (9) next yields for the traceless coefficients

$$C_\ell = \frac{(2\ell + 1)!!}{\ell!} \int \frac{d\Omega}{4\pi} A_\ell(\Omega) C(\Omega) = \frac{(2\ell + 1)!!}{\ell!} \sum_{\ell} \gamma(\ell + 2\ell) \gamma(\ell + 2\ell) \bar{C}_{\ell + 2\ell},$$

(65)
with
\[ \langle \vec{l} | \vec{l'} \rangle = \int \frac{d\Omega}{4\pi} A_{\ell}(\Omega) \hat{n}_{x_{\ell}} \hat{n}_{y_{\ell}} \hat{n}_{z_{\ell}} \]
\[ = \frac{(2\ell + 1)!!}{(\ell + \ell' + 1)!!} \sum_{m=0}^{\ell} \frac{\ell'!}{(\ell'_{x} - 2m_{x})! m_{x}!} \frac{\ell''!}{(\ell'_{y} - 2m_{y})! m_{y}!} \frac{\ell''_{z}!}{(\ell'_{z} - 2m_{z})! m_{z}!} \langle \vec{l} | \vec{l'} - 2\vec{m} \rangle , \]
(66)

where in the second equality we have made use of the expansion \( \overline{\text{P}} \) for the product of direction vector components.

In practical circumstances, the sums over \( \ell \) must be cut off at some \( \ell_{\text{max}} \). This is because, on one hand, the experimental apparatus and the statistic limit the angular resolution and, on the other, the features of a reaction limit the number of \( \ell \) for which \( C_{\ell} \) may be significant. Exceeding a sensible \( \ell_{\text{max}} \) will result in coefficients dominated by noise. A likely situation is that of a suitable value of \( \ell_{\text{max}} \) higher for \( X \), \( B \), and \( C \) than for \( X_{\ell} \). This is because the anisotropies associated with measurement efficiencies, particularly at small \( q \), can be much stronger than physical anisotropies, especially when the experiment is blind in certain directions. In the latter case, the binning method would necessarily fail for small bins, while the method relying on the \( \vec{l} \)-decomposition from start could succeed when using large \( \ell_{\text{max}} \) for \( X \) and \( B \), as long as only low \( \ell \)-values were demanded of \( C \).

Strategies similar to that employed for finding \( C \) in Eq. \( \text{(55)} \) may be employed for finding the expansion coefficients for an inverse of an angular function or for a product of angular functions. In finding the inverse of \( B \) in terms of \( B_{\vec{l}} \), one would replace \( X(\Omega) \) in \( \text{(55)} \) by 1, leading to \( X_{\ell} = \delta_{00} \) in Eqs. \( \text{(52)} \) and \( \text{(53)} \). On the other hand, in looking for the product of \( B \) and \( C \) in terms of expansion coefficients, one can first obtain the coefficients
\[ \vec{X}_{\ell} = \frac{1}{\gamma(\ell)} \sum_{0 \leq \ell' \leq \ell} \gamma(\ell - \ell') \gamma(\ell') B_{\ell - \ell'} C_{\ell'} , \]
(67)

that do not meet the condition of tracelessness. Next, one can obtain traceless coefficients \( \vec{X}_{\ell} \) following such a procedure as for \( C \) in Eqs. \( \text{(55)} - \text{(56)} \). Obviously, that specific detracing procedure is a form of application of the projection operator \( \mathcal{P} \), described in the appendix, to a symmetric tensor.

We have, thus, demonstrated some practical procedures for dividing, inverting and multiplying angular functions in terms of \( \vec{l} \)-arrays without resorting to binning of the functions in \( \cos\theta \) or \( \phi \) and without performing matrix inversion operations.

IV. PRACTICAL ASPECTS OF SHAPE ANALYSIS

The expansion of correlations and sources in tesseral or cartesian harmonics brings about practical benefits. The three-dimensional dependencies on relative momentum and spatial separation of the functions are replaced by sets of one-dimensional dependencies of angular expansion coefficients. The replacing of the three-dimensional by one-dimensional dependencies allows, principally, for a more thorough representation of the correlation data and of the physical picture of those data’s implications. The three-dimensional integral relation between the correlation function and source gets replaced by a set of one-dimensional relations between the respective expansion coefficients, simplifying the task of finding the source features. The expansion coefficients for the functions, as either spherical or cartesian tensors, transform covariantly under rotations.

Of the two sets of harmonic functions for expansion, the tesseral set represents an orthonormal basis, but leads to coefficients which are complex and hard to interpret. On the other hand, the cartesian set is overcomplete, generally nonorthogonal within any rank, but yields real coefficients with relatively straightforward interpretations. The transformation properties, under rotations, are more straightforward for the cartesian than for the tesseral coefficients. We have shown, in the preceding section, how to transcribe results between the two harmonic representations.

A. Shapes in Terms of Cartesian Harmonics

Along any direction of relative momentum or separation, the correlation or source function are equal to combinations of the cartesian coefficients \( \overline{\text{P}} \). Particularly simple results follow along any of the employed cartesian axes; e.g. along the positive direction for the \( x \)-axis the correlation function is:
\[ R(q) = R^{(0)}(q) + R^{(1)}_{x}(q) + R^{(2)}_{xx}(q) + R^{(3)}_{xxx}(q) + \ldots . \]
(68)
Terms for subsequent $\ell$ refine the information on the dependence of the function on $q$, along the direction. As we have mentioned, the physics, on one hand, and statistics and apparatus, on the other, will limit the number of significant terms in the expansion, with the high-$\ell$ terms being dominated by noise [24].

For identical particles, the correlation function and accessed emission source are invariant under inversion. In such a case, the odd-$\ell$ terms in the harmonic expansion vanish. That is, generally, not the case for nonidentical particles. For instance, if the protons in a reaction are, on the average, emitted earlier than pions, the distribution $S$ of pairs in $\mathbf{r}_p - \mathbf{r}_\pi$ will be pushed out in the outward direction, away from 0. The three $\ell = 1$ cartesian moments [52], computed from $S^{(1)}_\alpha$, yield the magnitude and precise direction of the average displacement of the distribution. At individual $r$, the effects onto $S$ of different emission times and possibly of different emission locations will result in a dipole distortion representable in terms of magnitude and direction:

$$S^{(1)}_\alpha(r) = S^{(1)}(r) e^{(1)}_\alpha(r).$$

In the above, the direction vector is $e^{(1)} = [\sin \theta^{(1)} \cos \phi^{(1)}, \sin \theta^{(1)} \sin \phi^{(1)}, \cos \theta^{(1)}]$. Not only the magnitude of the angular distortion, but also the characteristic directions of the distortion can depend on the relative distance. If the distortion direction is independent of the distance, this direction will be further the same for the average displacement and the same for the correlation at any relative momentum. The latter is due to the fact that the kernel $K_\ell$ is universal within a given rank, i.e. the same for different tensor components. The $\ell = 1$ distortions are behind the effects investigated by Gelderloos et al. [32] and Lednicky et al. [33], who have compared pair emission probabilities for pairs with relative momenta directed along and opposite to the pair total momentum. The advantage of using the cartesian harmonics is the ability to determine the distortion directions in correlation, and the implication of those distortions for source distortions, without any presumption regarding the distortion direction when no possible symmetry arguments can be invoked.

Quadratic moments of the emission source are associated with the rank $\ell = 2$ and $\ell = 0$ expansion coefficients, cf. Eq. (55), with $\ell = 2$ rank giving rise to an anisotropy of the tensor out of the moments, $\langle r_{\alpha_1} r_{\alpha_2} \rangle$. A triaxial ellipsoid could be associated with those moments, or with Euler angles describing rotation from the coordinate axes to the axes of the ellipsoid, i.e. the axes along which the tensor of moments gets diagonalized. At any value of the corresponding relative variable, for either the source or the correlation function, the rank $\ell = 2$ symmetric and traceless tensor of expansion coefficients may be diagonalized and described in terms of 5 independent parameters, 2 distortions and 3 angles:

$$S^{(2)}_{\alpha_1 \alpha_2}(r) = S^{(2)}_1 e^{(2)}_{\alpha_1 \alpha_2} - (S^{(2)}_1 + S^{(2)}_3) e^{(2)}_{2 \alpha_1 \alpha_2} + S^{(2)}_3 e^{(2)}_{3 \alpha_1 \alpha_2}.$$  

(70)

The three eigenvectors $e^{(2)}_i$ in (70) form an orthonormal set. Due to the tracelessness of $S^{(2)}$, the distortions associated with the three directions add up to zero, $S^{(2)}_2 = -(S^{(2)}_1 + S^{(2)}_3)$. In terms of the Euler angles $\Phi^{(2)}$, $\Theta^{(2)}$, and $\Psi^{(2)}$, for rotation of unit vectors along coordinate axes to $\{e^{(2)}_i\}_{i=1,2,3}$, the eigenvectors may be represented as

$$e_1 = \left[\cos \Psi \cos \Phi - \cos \Theta \sin \Phi, \cos \Psi \sin \Phi, \sin \Psi \sin \Theta\right],$$

$$e_2 = \left[-\sin \Psi \cos \Phi - \cos \Theta \sin \Phi, \sin \Psi \sin \Phi, \cos \Theta \cos \Phi, \cos \Psi \sin \Theta\right],$$

$$e_3 = \left[\sin \Theta \sin \Phi, -\sin \Theta \cos \Phi, \cos \Theta \cos \Phi, \cos \Psi \sin \Theta\right],$$

(71)

where the ($2$) superscripts are suppressed. Just as the distortion values, the angles may depend on $r$. If the angles do not depend on $r$, then those angles coincide with those for the diagonalization of the tensor of quadratic moments, $\langle r_{\alpha_1} r_{\alpha_2} \rangle$, and for the diagonalization of the quadrupole coefficients of the correlation function at any $q$. The 5 independent parameters at any relative variable correspond to 5 independent expansion coefficients for rank $\ell = 2$.

With regard to the higher, $\ell > 2$, shape distortions, one way to associate the directions of axes with them, is to choose one direction, defined by $e^{(\ell)}_1$, by demanding that the distortion along that direction, equal to

$$S^{(\ell)}(\Omega) = \sum_{\alpha_1 \ldots \alpha_\ell} S^{(\ell)}_{\alpha_1 \ldots \alpha_\ell} e^{(\ell)}_{1 \alpha_1} \ldots e^{(\ell)}_{\ell \alpha_\ell},$$

(72)

is extremal, minimal or maximal [25]. Directing a coordinate axis along $e^{(\ell)}_1$, makes the two $S^{(\ell)}$-coefficients, characterized by $\ell_1 = \ell - 1$, vanish. The next axis, defined by $e^{(\ell)}_2$, may be chosen again as one extremizing the distortion, but now within the direction plane perpendicular to $e^{(\ell)}_1$. When directing the second coordinate axis along $e^{(\ell)}_2$ and the third along the direction perpendicular to the first two, we find that the choice puts to zero the $S^{(\ell)}$-coefficient characterized by $\ell_1 = 0$, $\ell_2 = \ell - 1$ and $\ell_3 = 1$. 
B. Harmonic Decomposition vs Fits in Three-Dimensions

For a given total momentum, the correlation function contains three-dimensional information pertinent to the three-dimensional structure of the source function. Effectively, there are three competing approaches for handling the information in the correlation function. In the first approach, most common till now in literature, the data is presented along chosen discrete directions of relative momentum, typically one or three. In the second approach the data is fitted to a parameterized form, e.g., a three-dimensional Gaussian, in the full three-dimensional space of relative momentum. When going beyond the simplest parameterizations in this approach, it may be difficult to assess, working in the three-dimensional space, which aspects of the source parameterization affect which particular aspects of the correlation. In the third approach, the data is expanded in harmonic functions. A set of one-dimensional functions then yields a complete representation of the three-dimensional data. Upon decomposition, the deduction of source features is reduced to a set of one-dimensional problems, with one-to-one correspondence between the coefficients of the correlation and the source. A specific coefficient of the source affects the specific coefficient of the correlation function and no other independent coefficients. Whether one were to advance the source analysis as an inversion problem, or as a problem of fitting, it should be certainly preferable to divide and conquer, i.e. approach the issue as a set one-dimensional problems.

Spherical harmonic decompositions generally carry more information than can be expressed with parameters of gaussian sources employed in the literature. The integrated nine independent source coefficients, \( S^{(\ell)} \), for \( \ell \leq 2 \), fully determine the moments \( \langle r_{\alpha_1} r_{\alpha_2} \rangle \) and \( \langle r_{\alpha} \rangle \), for all \( \alpha \)-indices, cf. Eq. (52). Those moments and the integral of \( S^{(0)} \), \( \lambda = 4\pi \int dr r^2 S^{(0)}(r) \), determine all the ten parameters that can be set for a gaussian source such as discussed in [25] or in the next section. However, the coefficients \( S^{(\ell)} \), for \( \ell \leq 2 \), may have a different dependence on \( r \) than those for a gaussian source. In addition, the \( \ell > 2 \) coefficients may further yield different characteristic moments. When coordinate axes are directed along the principal axes of the gaussian quadrupole deformation, the gaussian moments, relative to gaussian center, factorize:

\[
\langle (r_1 - \langle r_1 \rangle)^{f_1} (r_2 - \langle r_2 \rangle)^{f_2} (r_3 - \langle r_3 \rangle)^{f_3} \rangle = \langle (r_1 - \langle r_1 \rangle)^{f_1} \rangle \langle (r_2 - \langle r_2 \rangle)^{f_2} \rangle \langle (r_3 - \langle r_3 \rangle)^{f_3} \rangle .
\]

(73)

For distinguishable particles in the midrapidity region of symmetric ultrarelativistic collisions, the above implies that \( \langle r_{\text{out}} r_{\text{long}}^2 \rangle = \langle r_{\text{out}} \rangle \langle r_{\text{long}}^2 \rangle \). The latter factorization will be violated for instance for boost-invariant particle emission, such as discussed in Sec. [V]. In the case of boost-invariant emission, with a transverse expansion, the single-particle sources acquire a boomerang-type shape, as particles at lower positions along the beam axis get emitted locally earlier than particles at higher positions. The violation of the factorization above will become apparent if one e.g. observes that the boomerang shape will be preserved in the relative source when one of the emitted particles is heavy (baryon) and the other light (\( \pi \) meson). Features of a boomerang shape may be quantified with the source expansion coefficients for \( \ell = 3 \), which contribute to the moment \( \langle r_{\text{out}} r_{\text{long}}^2 \rangle \), cf. (52), that may be compared to the product \( \langle r_{\text{out}} \rangle \langle r_{\text{long}}^2 \rangle \) from \( \ell = 1 \) and \( \ell = 2 \), besides \( \ell = 0 \).

With regard to the dependence of \( S^{(\ell)} \) on \( r \), already several studies have focused on determining the non-gaussian behavior of the angle-averaged source function, \( S^{(\ell=0)}(r) \). With regard to \( \ell \geq 1 \), significant values of \( S^{(\ell)}(r) \) are principally expected only there where \( S^{(\ell=0)}(r) \) begins to fall, i.e. around the edges of the source, whereas, at small \( r \), \( S^{(0)}(r) \propto r^{\ell} \). Concerning the low-\( r \) region, in the source-correlation relation, the convolution with kernel \( K_\ell \) brings in a factor \( r^2 \). That low-\( r \) suppression generally reduces the impact of small-\( r \) \( S^{(\ell)} \) on the correlation, enhancing errors in the restored source, making it likely difficult to detect any non gaussian features there at \( \ell \geq 1 \). On the other hand, in the source tails at \( \ell \geq 1 \) in the source coefficients, the non-gaussian behavior may be actually enhanced compared to \( \ell = 0 \), both at low and high reaction energies. For instance, long-lived slowly cooling systems with emission rates falling off as \( e^{-t/\tau} \) should give rise to relative sources with an exponential fall-off along the direction that the pair total momentum has in the frame of the emitting system. For low \( \ell \), strength of the exponential tail should increase with \( \ell \) for distortion along the direction of the total momentum, Eq. (72). In fact, looking for the maximal distortion in source tail may help to locate the emitting source within the velocity space. In ultrarelativistic collisions, the emission becomes boost-invariant around the midrapidity. For a freezing-out boost-invariant thermal source, the single-particle distribution of emission points in the space-time, cf. Eq. (2), has the form

\[
s(p, r, t) \sim \delta(\tau - \tau_f) e^{-E_\perp \sqrt{1 + z^2}\tau_f/\tau} ,
\]

(74)

where \( \tau = \sqrt{t^2 - z^2} \) and \( z \) points along the beam axis. The single-particle source above falls off exponentially with an increase in \( |z| \) and so would the relative source with an increase in \( |r_{\text{long}}| \). This behavior should be particularly revealed in the \( S^{(0,0,2k)} \) coefficients of the relative source.

While it may be straightforward to compute the expansion coefficients as convolutions of the correlation with different harmonics, some caution should be exerted in selecting the range of relative momentum \( q \) for the source
investigation. The caution is required due to the fact that there exist other causes for inter-particle correlation than the final-state effects within a two-particle system. At the mathematical level, those causes can give rise to a $q$-dependence of the source $S$, assumed to be negligible in arriving at Eq. (1). The dependence might be produced by a collective expansion of strength comparable or larger than the local thermal motion in the emitting region. If the dependence is not explicitly corrected for, the analysis for source restoration should be limited to the region where the source $q$-dependence is deemed weak. In case of the final-state effects due to resonances, short-distances in $S$ are of importance, with the $q$-dependence governed there by local temperature at freeze-out, giving rise to the requirement of $q \ll \sqrt{2\pi T}$ for the source analysis. This condition is practically always satisfied for $pp$ pairs whose correlation function peaks at $q \sim 20$ MeV/c. On the other hand, for $p\pi^+$ pairs in the region of $\Delta$-resonance at $q \sim 225$ MeV/c, the condition is practically never satisfied in nuclear reactions. Correlations due to directed flow associated with the reaction plane, such as due to elliptic and sideward flows, can be partially accounted for by controlling the reaction plane in constructing the denominator in (1). Otherwise, those type of correlations tend to peak at $q$ of the order of the mean $p_\perp$ (i.e. a few hundred MeV/c for $p$ or $\pi$ in energetic collisions), reaching there values of the order of few percent. By comparison, Coulomb correlations fall off with increase of $q$, as $1/q^2$, down to about one percent for $pp$ typically at $q \sim 100$ MeV/c. That further underscores the need to limit the range of $q$ in source analysis. In ultrarelativistic collisions, jets are an additional cause for inter-particle correlations. The characteristic scale for those correlations is of the order of several hundred MeV/c and the magnitude of the order of one percent towards lower $p_\perp$. Overall, it is apparent that caution should be executed when considering small values of the range. If conclusions are to hinge on such small values, some supporting evidence should be provided to justify the assumption that the correlation originates from final-state interactions or symmetrization.

V. EXAMPLES OF EMISSION

Here, we provide examples of what may be expected in reactions regarding harmonic characterization of emission sources and of correlation functions.

A. Gaussian Sources

In the analyses of correlation functions, it is quite common to aim at a gaussian representation of the emission sources. When cast in the gaussian form, a source is expressed in terms of a minimal number of parameters needed for shape description.

Thus, a gaussian source is generally described in terms of 9 parameters, out which 3 describe the position of the source center $d$ and 6 correspond to the independent independent matrix elements of the symmetric positive shape matrix $M$:

$$S(r) = \frac{[\text{det} M]^{1/2}}{(4\pi)^{3/2}} \exp \left\{ -\frac{1}{4} \sum_{\alpha_1, \alpha_2} M_{\alpha_1 \alpha_2} (r_{\alpha_1} - d_{\alpha_1}) (r_{\alpha_2} - d_{\alpha_2}) \right\}. \quad (75)$$

A tenth parameter would describe the normalization of the source, which could vary from unity if a significant fraction of the pairs had effectively infinite separations due to weak decays and were outside the description in terms of a single gaussian. For identical particles, the accessed relative source must be symmetric under inversion, representing particle interchange. In that case $d = 0$, leaving only 6 independent shape parameters for the source.

The convenient choice for the shape parameters are three Euler angles describing an orientation of the eigenvectors $\{u_i\}$ of $M$ and three gaussian radii $R_i$ in terms of which the matrix is $M_{\alpha_1 \alpha_2} = \sum_i (1/R_i^2) u_{i \alpha_1} u_{i \alpha_2}$. The specific factor of $(1/4)$ multiplying $M$ and $(1/R^2)$ in (75) is associated with the tradition of assigning the radii to the single-particle sources $s$ in (2). However, absolute positions in a reaction are not observable; for the description of relative positions more appropriate would be a factor of $(1/2)$ yielding radii larger by a factor of $\sqrt{2}$. When coordinate axes are oriented along the gaussian shape eigenvectors, the gaussian shape takes the form:

$$S(r) = \frac{1}{(4\pi)^{3/2} R_x R_y R_z} \exp \left\{ -\frac{(x - d_x)^2}{4R_x^2} - \frac{(y - d_y)^2}{4R_y^2} - \frac{(z - d_z)^2}{4R_z^2} \right\}. \quad (76)$$

The displacement $d$ of the source for distinguishable particles, relative to $r = 0$, is of interest in both intermediate-energy and high-energy collisions [32, 33]. For instance, at low energy it might be of interest whether neutron emission preceded proton emission [32], producing a $d$ in the direction of pair total momentum, or whether intermediate-mass fragments were emitted simultaneously with the protons [32]. At high energy, one might look for evidence of strange...
The exponential \( e^S \) where \( S \) of either type, leading to corresponding expansion for the source Gaussian is quadratic in coordinates, that argument can be represented in terms of harmonics of rank \( \ell \) even when the Gaussian source is isotropic around its center. Incidentally, in a similar manner, for \( \mathbf{d} = \mathbf{0} \), the anisotropy of a Gaussian gives rise to non-zero angular moments for all even \( \ell \geq 2 \), not just \( \ell = 2 \).

For a general displaced and anisotropic Gaussian, the angular coefficients can be obtained, as a function of \( r \), through a direct angular integration. Otherwise, the power series for an exponential may be employed, in combination with the properties of spherical harmonics, to yield converging series for the coefficients. Specifically, as the argument of a source Gaussian is quadratic in coordinates, that argument can be represented in terms of harmonics of rank \( \ell \leq 2 \), leading to:

\[
S(\mathbf{r}) = c(r) e^{\Gamma(\mathbf{r})},
\]

where

\[
\Gamma(\mathbf{r}) = \sqrt{4\pi} \sum_{1 \leq \ell \leq 2} \sum_{m=0}^{\ell} \Gamma_{\ell m}(r) Y_{\ell m}(\Omega) = \sum_{1 \leq \ell \leq 2} \gamma(\ell) \Gamma_{\ell}(r) \hat{n}^\ell \hat{r}^\ell \hat{z}^\ell.
\]

The exponential \( e^{\Gamma} \) may be next expanded in the power series yielding,
where \( \langle \cdot | \cdot \rangle \) represents a Clebsch-Gordan coefficient. It follows that terms in the sum decrease rapidly with \( N \), for \( N \gg \max \ell_m |\Gamma_{\ell m}(r)| \). When aiming at the source in terms of cartesian harmonic coefficients, first the cartesian coefficients \( \bar{S}_{N\ell} \) which do not meet the tracelessness condition are found through recursion:

\[
\bar{S}_{N\ell}(r) = \frac{1}{N} \gamma(\ell) \sum_{\bar{m}} \gamma(\bar{m}) \gamma(\ell - \bar{m}) \Gamma_{\bar{m}}(r) \bar{S}_{(N-1)(\ell-\bar{m})}(r). \tag{83}
\]

In the next stage, the traceless coefficients \( S_{N\ell} \) are found following such a procedure as in Eqs. (65)–(66). Finally, the coefficients for the source may be obtained by summing over the respective series:

\[
S_{\ell m}(r) = \sum_{N=0}^{\infty} S_{N\ell m}(r), \quad \text{or} \quad S_{\ell}(r) = \sum_{N=0}^{\infty} S_{N\ell}(r), \tag{84}
\]

depending whether the tesseral or cartesian harmonics are followed. Coefficients for gaussian sources in this Section have been obtained following the recursion outlined above. Kernels for use in Eqs. (4) and (10) have been obtained using known phase-shifts.

In order to illustrate the possibility of measuring source shapes, we consider a relative source represented by an axially symmetric gaussian oriented along the beam axis with parameters \( R_0 = R_q = 4 \text{ fm} \) and \( R_z = 8 \text{ fm} \). For distinguishable particles, we additionally displace this gaussian away from the origin at \( r = 0 \) by \( \mathbf{d} = (0, 0, 4 \text{ fm}) \). Such a source may result in consequence of emission of the individual species within two single-particle sources differentially elongated along the z axis, with the source for the a species displaced by \( d \) along the z-axis relative to the source for the \( b \) species. For the chosen relative source, the anisotropies of associated correlation functions can be quite significant. Examples of the associated correlations, for different particle pairs, \( pK^+ \), \( \pi\pi^- \), \( pp \), \( pn \), \( nn \) and \( \Lambda \), are presented in Fig. 5. The correlations are shown there as a function of \( \cos \theta_{pq} \) at fixed \( q \). The observed angular dependence of the functions is exclusively due to the source anisotropy. While the correlations are stronger at low than at high \( q \), the correlations and their anisotropies are generally of the same order. In the dependence of correlations on \( \cos \theta_{qq} \) for distinguishable particles, one recognize both components odd and even in \( \cos \theta_{qq} \). The odd components are associated with the displacement \( \mathbf{d} \) in the source while even are associated with both the displacement \( \mathbf{d} \) and \( R_\perp \neq R_z \). In the correlations of identical particles, only the effects of \( R_\perp \neq R_z \) are present.

With the choice of axial symmetry for the specific source, all cartesian components with odd number or x or y indices vanish. For identical particles, this further implies vanishing of cartesian components with odd number of z indices, since there are no indices left to pair them with to yield a net even index number. For distinguishable particles, at \( \ell \leq 3 \), the only nonvanishing moments are: \( S^{(0)} \), \( S^{(1)}_x \), \( S^{(2)}_{xy} \), \( S^{(2)}_{yy} \), \( S^{(2)}_{zz} \), \( S^{(3)}_{xy} \), \( S^{(3)}_{yz} \) and \( S^{(3)}_{xz} \). Not all of those moments are independent, as \( S^{(2)}_{xx} = -\left(S^{(2)}_{yy} + S^{(2)}_{zz}\right) \) and \( S^{(3)}_{xx} = -\left(S^{(3)}_{yz} + S^{(3)}_{xz}\right) \), in consequence of the tensor tracelessness. In addition, the axial symmetry implies equality between components: \( S^{(2)}_{xx} = S^{(2)}_{yy} \) and \( S^{(3)}_{xx} = S^{(3)}_{yy} \). The left panels of Fig. 6 show the \( \ell \leq 3 \) cartesian coefficients of correlations for proton-charged meson pairs. Those coefficients which are not shown are easily generated from those which are, as e.g. \( S^{(2)}_{zz} = -S^{(2)}_{xx}/2 \), etc.

The correlations of protons with charged mesons in Fig. 6 are plotted for \( q < 80 \text{ MeV}/\text{c} \). Within that momentum range, the correlation should be dominated by Coulomb effects. For comparison purposes, also shown in the figure are the \( p-K^+ \) correlations expected in the classical limit of the Coulomb interactions. As expected following the discussions in Sec. 11, the quantum effects seriously affect the correlations at \( qR \lesssim \ell + \frac{1}{2} \), dampening out in that region the classical structures and hampering thus practical extraction of source shape information. With a smaller reduced mass in the \( \pi\pi^- \) than in the \( pK^+ \) system, the classical Coulomb hole has a smaller \( q \)-size in the \( \pi\pi^- \) system. In consequence, the quark effects have a stronger impact on the features of the \( p\pi^+ \) correlations, making the extraction of source shape harder from the \( p\pi^- \) than from the \( pK^+ \) correlations.

Although difficult, it is certainly tenable to measure correlations of the order of one percent or less. In order to exhibit better the correlations at large \( q \), Fig. 7 displays the same proton-charged meson correlations as those shown in Fig. 6 but now weighted by \( q^2 \). It is apparent that the correlations decrease as \( \ell \) increases. An important feature of the exhibited correlations is that they are smooth in \( q \) for all \( \ell \). Due to this smooth behavior, it is possible to analyze those correlations utilizing wide momentum bins. The phase space increasing as \( q^2 \) should further help to garner statistics needed for analyzing the correlations at the level of one percent or less. The practical limitation for the analysis at large \( q \) may turn out to be, though, the need to separate the correlations due to final-state effects within the pair from the possible competing correlations induced by collective flow and, at RHIC, by jets.

Correlations of elementary baryons are good candidates for source analyses, due to the large scattering lengths within the pairs which lead to strong correlation structures on the relative-momentum scale of \( q \approx 20 \text{ MeV}/\text{c} \). The right-side panels of Fig. 6 display the correlations for \( pn \), \( nn \), \( pp \), \( p\Lambda \) pairs, associated with the gaussian sources. When present, the \( \ell \geq 1 \) baryon-baryon correlations tend to be significantly stronger at most momenta of experimental interest than
FIG. 5: (Color online) Correlations associated with a relative source in the form of an axially symmetric gaussian, characterized by $R_x = R_y = 4$ fm and $R_z = 8$ fm, for different indicated particle pairs. For distinguishable particle pairs, the center of this gaussian is shifted by $d = (0, 0, 4$ fm) relative to the origin at $r = 0$. The correlations are shown as a function of cosine of the angle that the relative momentum $q$ makes with the $z$-axis, for two values of momentum magnitude, $q = 25$ MeV and $q = 55$ MeV. For the specific employed source, angular anisotropies in the correlation functions are of the same order as the functions.

the corresponding Coulomb-dominated meson-baryon correlations. The odd correlation moments vanish for the $pp$ and $nn$ pairs as the pair sources and correlations are symmetric under particle interchange. The odd-$\ell$ correlations are especially important for the $pn$ pairs emitted from intermediate-energy heavy ion collisions as these correlations tie to the issue of whether neutrons are emitted earlier than protons and, further, to the symmetry energy. Within high-energy collisions, odd-$\ell$ $p\Lambda$ correlations are of interest as they can provide information on whether the hyperons leave earlier than protons.

B. Boost-Invariant Blast-Wave Sources

Results of early pion-pion correlation measurements at RHIC have been satisfactorily described with fits within the blast wave models [37, 38]. Those fits have suggested a nearly instantaneous emission directly contradicting the formation of a long-lived mixed phase which would have resulted in a dilatory emission and in relative sources with sizes extended in the outward direction of the pair total momentum. Given the importance of those measurements and findings, it can be important to verify the conclusions on the nature of particle emission at RHIC by extracting sources from the correlations of other pairs such as involving kaons or protons.

The minimal blast-wave model used here for illustration employs four parameters describing the single-particle
Fig. 6: (Color online) Angular coefficients of the correlations associated with an axially symmetric gaussian source characterized by $R_x = R_y = 4 \, \text{fm}$ and $R_z = 8 \, \text{fm}$, for proton-charged meson pairs (symbols in the left panels) and baryon-baryon pairs (symbols in the right panels), as a function of the relative momentum $q$. For distinguishable particle pairs, the center of the gaussian source is shifted by $d = (0, 0, 4 \, \text{fm})$ relative to the origin at $r = 0$. The $pK^+$ and $p\pi^+$ correlations in the left panels tend to be dominated by the Coulomb interaction. To illustrate the degree to which quantal effects wash out the information on emission source, the left panels show also (solid lines) the $pK^+$ correlations in the limit of classical Coulomb interactions.

Sources that are boost-invariant along the beam axis:

\[
s(\mathbf{p}, \mathbf{r}, t) \propto \delta(\tau - \tau_f) \theta(r_{\text{max}} - r_\perp) m_\perp \cosh(y - \eta) \exp \left( -\frac{p^\mu u_\mu}{T} \right).
\]  

Here, $y$ and $\eta$ are, respectively, energy-momentum and space-time longitudinal rapidities. The emission is assumed to occur at one proper time $\tau_f = 10 \, \text{fm}/c$. The decoupling is assumed to be spread out over the cylindrical volume of radius $r_{\text{max}} = 13 \, \text{fm}$ and take place in the presence of a collective motion characterized by transverse rapidity $\rho$ ($u_\perp = \sin \rho$) proportional to transverse position, $\rho \propto r_\perp$, producing the maximal transverse velocity in the longitudinal center of mass of $v^\perp_{\text{max}} = \tanh \rho_{\text{max}} = 0.75 \, c$, at $r_\perp = r_{\text{max}}$. The breakup temperature is assumed to be $T = 105 \, \text{MeV}/c$.

The chosen parameters describe particle spectra at $p_\perp < 1 \, \text{GeV}/c$, as well as basic aspects of pion-pion correlations, in Au + Au collisions at $\sqrt{s_{NN}} = 130 \, \text{GeV}$. Here, our goal is not to fit data, but to use the blast-wave model to see what results might be expected for correlations of other particle pairs analyzed in terms of $\mathcal{R}^{(\ell)}(q)$.

We focus on correlations of nonidentical particles, because of the possibility for associated finite odd angular moments. The $\ell \leq 3$ relative-source moments for the $pK^+$ and $p\pi^+$ pairs moving at transverse velocity of $v_\perp = 0.6 \, c$ are shown in Fig. 8. To obtain the displayed results, emission positions $\mathbf{r}_{a,b}$ for each of the particles $a$ and $b$ had been generated within the particle rest frame following the blast-wave description. After generating several thousands
points, the number of relative positions \( r_a - r_b \) for the relative source becomes of the order of a few millions, sufficient for finding quite smooth source moments, except at the lowest \( |r_a - r_b| \). We associate the \( z \)-axis direction with the beam axis and the \( x \)-axis direction with the outward direction along the pair momentum. The angular moments for the blast-wave model are compared in Fig. 8 to the moments for a Gaussian source with the same cartesian moments, \( \langle x \rangle \), \( \langle x^2 \rangle \), \( \langle y^2 \rangle \) and \( \langle z^2 \rangle \), as the blast-wave relative source.

As elsewhere, values for the underlying three-dimensional source structure can be restored along any direction from the cartesian coefficients, in particular from those displayed in Fig. 8 exploiting tracelessness. Thus e.g. along the \( x \)-axis the source values are

\[
S = S^{(0)} \pm S_x^{(1)} + \frac{1}{2} \left( S_{xx}^{(2)} - S_{yy}^{(2)} \right) - \frac{1}{2} S_{zz}^{(2)} \pm S_{xyz}^{(3)} \pm S_{xzz}^{(3)} + \ldots ,
\]

where the upper and lower signs refer to the positive and negative directions along the axis. Within the \( xy \)-plane, at 45° to the \( x \)-axis, the source values are

\[
S = S^{(0)} \pm \frac{1}{\sqrt{2}} S_x^{(1)} - \frac{1}{2} S_{xx}^{(2)} \pm \frac{1}{2\sqrt{2}} S_{zz}^{(3)} + \ldots ,
\]

with the upper signs again referring to the positive direction along the \( x \)-axis.

Within the blast-wave model, the heavier particles end up being emitted farther out, on the average, along the outward direction, than the lighter particles. This is due to the following. In an infinite system with Hubble expansion, a particle with a given velocity would get, on the average, emitted from a spatial patch centered around the point with the collective velocity matching the particle velocity. The size of such an emission patch would depend on the particle mass, with the patch being tighter for heavier particles with lower thermal velocities and being more extended.
FIG. 8: (Color online) Angular cartesian coefficients, at $\ell \leq 3$, multiplied by $4\pi r^2$, of relative sources for $pK$ (left) and $p\pi$ (right) pairs moving out at transverse velocity of $0.6c$ relative to the beam axis. Symbols represent the results obtained in the blast-wave model. Lines represent the results from a gaussian source with the same pair c.m. cartesian moments, $\langle r_i \rangle$ and $\langle r_i r_j \rangle$, as the blast-wave source. The non-gaussian features of a blast-wave source become more apparent as the mass difference within a pair increases.

for lighter particles. Relative to the infinite system, however, in a system with a finite transverse extension, the outer portions of the emission patches would get cut off and more so for the light than for heavy particles, forcing the emission region of the light particles inwards relative to the heavy. In the past, relative displacements of emission between particles of different mass, e.g. $K\pi$, have been accessed by comparing the correlations at opposite signs of outward momentum, $q_{\text{out}} > 0$ and $q_{\text{out}} < 0$, at low values of the remaining $q$-components [39, 40]. Such a strategy discards potentially useful information on the relative displacement in the remainder of the relative momentum space.

The displacement of species within a pair is visible in Fig. 8 in the finite values of $S_x^{(1)}$. With the mass difference being larger within the $p\pi$ than $pK$ pairs, the $S_x^{(1)}$ coefficients peak farther out and reach significantly higher values for the $p\pi$ than $pK$ pairs.

Some other features of blast-wave sources become apparent in the context of comparisons to gaussian sources. When making such a comparison, a few points need to be kept in mind. In an infinite system with a Hubble-like
effects. 

identical pion pairs and it provides possibility for carrying out multidimensional analysis for other classes of final-state 

in terms of cartesian coefficients offers an alternative way of carrying out a multidimensional correlation analysis for 

transformation allowing for easy modelling with specific sources such as gaussian. The decomposition of correlations 

leaving off correlations induced by identical-particle interference. The latter are related to the source through Fourier 

In the analyses to date, Coulomb interactions have been normally corrected for in a shape-independent manner, 

analyses at RHIC have focused on identical pions. In identical-pion correlations, strong interactions play little role. 

multidimensional correlation analysis, there is little chance for a detailed understanding of the impact of dynamics on 

the forefront tasks before the correlation analysis is to decipher the mire of RHIC data. Without carrying out a 

One of the benefits of decomposing information with angular harmonics is that the correlations induced by Coulomb 

and strong interactions carry, on their own, some shape information that can be exploited in the correlation analysis. 

VI. SUMMARY 

Correlation measurements are a crucial tool for understanding the dynamics of central nuclear reactions. One of 

the forefront tasks before the correlation analysis is to decipher the mire of RHIC data. Without carrying out a 
multidimensional correlation analysis, there is little chance for a detailed understanding of the impact of dynamics on 
correlations in those reactions and for understanding of the role played by the equation of state. To date, correlation 
analyses at RHIC have focused on identical pions. In identical-pion correlations, strong interactions play little role. 

In the analyses to date, Coulomb interactions have been normally corrected for in a shape-independent manner, 

leaving off correlations induced by identical-particle interference. The latter are related to the source through Fourier 

transformation allowing for easy modelling with specific sources such as gaussian. The decomposition of correlations 
in terms of cartesian coefficients offers an alternative way of carrying out a multidimensional correlation analysis for 

identical pion pairs and it provides possibility for carrying out multidimensional analysis for other classes of final-state 
effects. 

One of the benefits of decomposing information with angular harmonics is that the correlations induced by Coulomb 

and strong interactions carry, on their own, some shape information that can be exploited in the correlation analysis. 
The representation of the correlations in terms of the cartesian or tesseral coefficients carries the full three-dimensional 

information in terms of a set of one-dimensional functions. Data represented in this fashion can be then be compared
FIG. 9: (Color online) Cartesian coefficients for the correlations of $pK^+$ (left) and $p\pi^+$ (right) pairs moving out at transverse velocity of 0.6 c relative to the beam axis. Symbols represent the results obtained in the blast-wave model, while lines represent the results from a gaussian source with the same pair c.m. cartesian moments, $\langle r_i \rangle$ and $\langle r_i r_j \rangle$, as the blast-wave source. The corresponding cartesian source coefficients have been represented in Fig. 8. As in the case of Fig. 8, differences between the blast-wave and gaussian results are more pronounced for the $p\pi^+$ than for $pK^+$ pairs.

to models visually, one harmonic at a time, providing the means to better identify which shape characteristics of the data are being described by the model.

The ability to discern information about source shapes from correlation measurements depends on the structure of the kernels, $K_\ell(q, r)$, for different spherical ranks $\ell$. In Section II it was demonstrated that the identical-particle interference, Coulomb and strong interactions all produce effective kernels. The resolving power of Coulomb interactions was shown to rise with mass and charge within the interacting pair. Strong interactions are most effective in the resonance region and, otherwise, for large cross sections. Even purely $s$ wave interactions can produce kernels of good resolving power at higher $\ell$.

While expansions employing tesseral harmonics nominally provide as powerful means for storing information in correlation functions as cartesian harmonics, they do not provide as transparent or as intuitive means for understanding the shapes as expansions done using the cartesian harmonic basis. Cartesian harmonics have a variety of useful properties which can be exploited in manipulating the angular information as was shown in Sec. III. The connection between specific aspects of the source geometry and specific angular moments has been emphasized in Sec. IV. It has
been pointed out that the average relative displacement of emission points in the c.m. frame of nonidentical particles can be assessed through the $\ell = 1$ angular moments. The quadrupole distortion, most commonly quantified in terms of the radii $R_{\text{out}}$, $R_{\text{side}}$ and $R_{\text{long}}$, can be studied with the $\ell = 2$ moments. The $\ell = 3$ moments test the more complex aspects of sources such as boomerang features.

Detailed examples of sources and correlations have been presented in Sec. V. Using gaussian sources, it has been, in particular, demonstrated that the $pK^+$ and $p\pi^+$ correlations can realistically be expected to provide shape information on emission sources. The strengths of correlations characterizing shape anisotropies can reach the magnitude of few percent. While the upcoming RHIC data sets will provide sufficient statistics to assess correlations between different particles at the level of one percent or better, the true challenge may turn out to the subtraction of competing shape anisotropies at low relative momenta, yielding correlations of the same order or larger than the proton-meson correlations, such as those produced by jets. The proton-charged meson correlations are dominated by Coulomb interactions. The strong interactions in baryon-baryon systems can also, though, provide good means for discerning shape anisotropies at low relative momenta, yielding correlations of the same order or larger than the proton-meson correlations. One common feature of all pairs and sources examined here is that the variation of the correlation with angle is comparable in magnitude to the angle-integrated correlation. Consequently, if one has sufficient statistics to measure a one-dimensional correlation, only a modest enhancement in statistics would suffice to find, or constrain in a meaningful manner, the higher angular moments. In the context of the blast-wave sources, opportunities for detecting more complex shapes have been discussed, such as boomerang shapes.

APPENDIX A: PROJECTION OPERATOR $P$

In arriving at the cartesian harmonics, an important role has been played by the cartesian projection operator $P$ which projects out the symmetric traceless portion of any cartesian tensor. As a projection operator, the operator $P$ must have the following properties:

\begin{align}
P^{(\ell\ell)}_{\alpha_1...\alpha_\ell:...\alpha'_\ell...\alpha'_\ell} &= \mathcal{P}^{(\ell\ell)}_{\alpha_1...\alpha_\ell:...\alpha'_\ell...\alpha'_\ell} \
P^{(\ell\ell)}_{\alpha_1...\alpha_\ell:...\alpha'_\ell...\alpha'_\ell} &= \sum_{\alpha_1'...\alpha_\ell'} \mathcal{P}^{(\ell\ell)}_{\alpha_1...\alpha_\ell:...\alpha'_\ell...\alpha'_\ell} P^{(\ell\ell)}_{\alpha_1'...\alpha_\ell'...\alpha'_\ell...\alpha'_\ell} \tag{A2}
\end{align}

The projection operator acting on any tensor $T$ must yield a traceless result $P T$. Since we can choose, in particular, a tensor that is finite for any single set of indices and zero otherwise, we find that the operator $P$ itself must be traceless:

\begin{equation}
\sum_{\alpha} P^{(\ell\ell)}_{\alpha_1...\alpha_{\ell-2}...\alpha:...\alpha'_\ell...\alpha'_\ell} = \sum_{\alpha} P^{(\ell\ell)}_{\alpha_1...\alpha_\ell:...\alpha'_\ell...\alpha'_\ell} \alpha_\alpha = 0 \tag{A3}
\end{equation}

We shall demonstrate existence of the operator $P$ by construction. Notably, when acting onto a symmetrized traceless tensor, the operator must reproduce the tensor. Thus, we can represent $P$ as

\begin{equation}
P^{(\ell\ell)}_{\alpha_1...\alpha_\ell:...\alpha'_\ell...\alpha'_\ell} = S^{(\ell\ell)}_{\alpha_1...\alpha_\ell:...\alpha'_\ell...\alpha'_\ell} + \Delta P^{(\ell\ell)}_{\alpha_1...\alpha_\ell:...\alpha'_\ell...\alpha'_\ell} \tag{A4}
\end{equation}

or, in the shortened notation,

\begin{equation}
P^{(\ell\ell)} = S^{(\ell\ell)} + \Delta P^{(\ell\ell)} \tag{A5}
\end{equation}

where $S$ is the symmetrized identity operator

\begin{equation}
S^{(\ell\ell)}_{\alpha_1...\alpha_\ell:...\alpha'_\ell...\alpha'_\ell} = \frac{1}{\ell!} \sum_{\sigma \in \Pi(\ell)} \delta_{\alpha_\sigma(1)} \alpha'_1 \ldots \delta_{\alpha_\sigma(\ell)} \alpha'_\ell \tag{A6}
\end{equation}

and $\Delta P$ is an operator which yields a zero when acting upon a traceless tensor. In (A6), the sum is over permutations of $\ell$ indices. Importantly, a traceless symmetric tensor of rank $\ell'$ is traceless and symmetric within any subset $\ell < \ell'$
FIG. 10: Diagrams illustrating the construction of the projection operator $P^{(\ell;\ell)}$, using the lower-rank operator $P^{(\ell-1;\ell-1)}$, represented by the rectangular block, and $\delta$-symbols, represented by lines. (a) Diagrammatic representation for the leading term $I^{P^{(\ell;\ell)}}$, Eq. (A8). (b) Diagrammatic representation for the terms resulting from evaluating the trace of $I^{P^{(\ell;\ell)}}$. The terms represented by the left diagram yield zero due to the tracelessness of $P^{(\ell-1;\ell-1)}$. (c) Diagrammatic representation for the $II^{P^{(\ell;\ell)}}$ and $III^{P^{(\ell;\ell)}}$ terms within the construct, Eqs. (A9) and (A10), respectively. (d) Representation for the nonvanishing terms from evaluating the trace of $III^{P^{(\ell;\ell)}}$. (e) Diagrammatic representation for the final $IV^{P^{(\ell;\ell)}}$ term within the construct, Eq. (A11).

of its indices. When acting on any of $\ell$ indices of a rank-$\ell'$, $\ell' > \ell$, traceless tensor, the constructed operator $P^{(\ell;\ell)}$ will reproduce that tensor and $\Delta P^{(\ell;\ell)}$ alone will annihilate it.

In our construction of $P$, we will find a graphical method similar to that employed for interactions useful. Within the method, the Kronecker $\delta_{\alpha\alpha'}$ is represented as a line joining the indices $\alpha$ and $\alpha'$. A convolution of delta symbols $\sum_{\alpha_1...\alpha_n} \delta_{\alpha_1\alpha_1} \delta_{\alpha_2\alpha_2} ... \delta_{\alpha_n\alpha_n} = \delta_{\alpha\alpha'}$ is represented by joining the line segments which produces a single line from $\alpha$ to $\alpha'$. A line which closes on itself represents a scalar factor which is the trace of the $\delta$-symbol, equal to the dimension of space, $\sum_{\alpha} \delta_{\alpha\alpha} = 3$. Examples will be provided in Fig. 10.

For $\ell = 0$ and $\ell = 1$, the operator $P$ is simply the identity operator, $P^{(0;0)} = 1$ and $P^{(1;1)}_{\alpha\alpha'} = \delta_{\alpha\alpha'}$. We shall show that, when the operator for the rank $(\ell - 1)$ tensors is known, it is possible to construct the operator with the required properties for the next rank $\ell$ tensors. Thus, the operator for any rank can be constructed by recursion. Let us assume that we know the operator for rank $(\ell - 1)$, $P^{(\ell-1;\ell-1)}$. We shall construct the next rank operator as a sum of terms, exploiting the operator $P^{(\ell-1;\ell-1)}$. The terms of different structure will be added to achieve in (A7) the properties (A1)-(A3) for $P^{(\ell;\ell)}$. In particular, only $I^P$ in our construction will contain the symmetrization operator $S$ on the r.h.s. of Eq. (A5).

Since, within each rank, the operator must, on one hand, contain a symmetrized identity operator and, on the other, be traceless, it is natural to start with $I^P$ as a tensor product of the lower-rank operator and identity, illustrated in Fig. 10(a),

$$\delta_{\alpha_{\ell} \alpha'_{\ell}} P^{(\ell-1;\ell-1)}_{\alpha_1...\alpha_{\ell-1};\alpha'_1...\alpha'_{\ell-1}},$$

(A8)
symmetrized separately in the index sets $\alpha$ and $\alpha'$. Evaluation of the trace for $\mathcal{P}_\ell$ produces terms which correspond to the trace of the lower-rank operator $\mathcal{P}^{(\ell-1;\ell-1)}$, represented by the left diagram in Fig. 10(b), and to the convolutions of the $\delta$-symbol with $\mathcal{P}^{(\ell-1;\ell-1)}$, represented by right diagram in Fig. 10(b). Terms of the first type yield zero due to the tracelessness of $\mathcal{P}^{(\ell-1;\ell-1)}$, but those of the second type generally yield a finite contribution. Aiming at the tracelessness of $\mathcal{P}^{(\ell;\ell)}$, one can next add to the construct a counter-term $\mathcal{P}_\ell$ of the form

$$- c_{II} \delta_{\alpha_{\ell-1}} \alpha_{\ell} \mathcal{P}^{(\ell-1;\ell-1)}_{\alpha_1...\alpha_{\ell-2},\alpha'_{\ell},...\alpha'_{\ell-1}},$$

(A9)

to cancel the last contribution, symmetrized separately in the indices $\alpha$ and $\alpha'$, represented by the left diagram in Fig. 10(c). The constant $c_{II}$ should be adjusted to achieve the tracelessness of $\mathcal{P}^{(\ell;\ell)}$. Taking the trace of $\mathcal{P}_\ell$ produces the same type of terms as for $\mathcal{P}_\ell$, that were represented in Fig. 10(b). Thus, an adjustment of $c_{II}$ can indeed produce a traceless sum $\mathcal{P}_\ell + \mathcal{P}^{(\ell;\ell)}$. However, the operator $\mathcal{P}^{(\ell;\ell)}$ should be symmetric under the transposition of the indices $\alpha$ and $\alpha'$, Eq. (A11). In consequence, when adding the term $\mathcal{P}_\ell$ to the construction, one also needs to add a term $\mathcal{P}^{(\ell;\ell)} = (\mathcal{P}^{(\ell;\ell)})^T$, of the form

$$- c_{IV} \delta_{\alpha_{\ell-1}} \alpha_{\ell} \mathcal{P}^{(\ell-1;\ell-1)}_{\alpha_1...\alpha_{\ell-2},\alpha'_{\ell},...\alpha'_{\ell-1}},$$

(A10)
symmetrized separately in the index sets $\alpha$ and $\alpha'$ and represented by the right diagram in Fig. 10(c). The nonvanishing terms from evaluating the trace of $\mathcal{P}^{(\ell;\ell)}$ are illustrated with the diagram in Fig. 10(d). That last contribution to the trace may be compensated for by adding the final term $\mathcal{P}_\ell$ to the construct, symmetric under transposition, of the form

$$c_{IV} \delta_{\alpha_{\ell-1}} \alpha_{\ell} \delta_{\alpha'_{\ell-1}} \alpha'_{\ell} \sum_\beta \mathcal{P}^{(\ell-1;\ell-1)}_{\alpha_1...\alpha_{\ell-2},\alpha'_{\ell-1}...\alpha'_{\ell-2}} \mathcal{P}^{(\ell-1;\ell-1)}_{\alpha_1...\alpha_{\ell-2},\alpha'_{\ell-1}...\alpha'_{\ell-2}} \delta_{\alpha'_{\ell-1}} \alpha'_{\ell},$$

(A11)
symmetrized separately in the indices $\alpha$ and $\alpha'$, and illustrated with the diagram in Fig. 10(d). Taking the trace of $\mathcal{P}_\ell$ produces, up to a factor, the same result as taking the trace of $\mathcal{P}^{(\ell;\ell)}$.

The constants $c_{II}$ and $c_{IV}$ can be now set to make the trace of $\mathcal{P}^{(\ell;\ell)}$ vanish. The net result for the operator, with the symmetrizations made explicit, is

$$\mathcal{P}^{(\ell;\ell)}_{\alpha_1...\alpha_{\ell};\alpha'_1...\alpha'_{\ell}} = \frac{1}{\ell!^2} \sum_{m,n=1}^\ell \delta_{\alpha_m \alpha_n} \mathcal{P}^{(\ell-1;\ell-1)}_{\alpha_1...\alpha_{m-1},\alpha_{m+1}...\alpha_{\ell};\alpha'_1...\alpha'_{m-1},\alpha'_{m+1}...\alpha'_{\ell}} - \frac{2}{\ell^2 (2\ell - 1)} \sum_{1 \leq m < r \leq \ell, 1 \leq n \leq \ell} \delta_{\alpha_m \alpha_n} \mathcal{P}^{(\ell-1;\ell-1)}_{\alpha_1...\alpha_{m-1},\alpha_{m+1}...\alpha_{r-1},\alpha_{r+1}...\alpha_{\ell};\alpha'_1...\alpha'_{m-1},\alpha'_{m+1}...\alpha'_{r-1},\alpha'_{r+1}...\alpha'_{\ell}} - \frac{2}{\ell^2 (2\ell - 1)} \sum_{1 \leq m \leq \ell, 1 \leq n < s \leq \ell} \delta_{\alpha_m \alpha_n} \mathcal{P}^{(\ell-1;\ell-1)}_{\alpha_1...\alpha_{m-1},\alpha_{m+1}...\alpha_{s-1},\alpha_{s+1}...\alpha_{\ell};\alpha'_1...\alpha'_{m-1},\alpha'_{m+1}...\alpha'_{s-1},\alpha'_{s+1}...\alpha'_{\ell}} \delta_{\alpha'_n \alpha'_s}$$

$$+ \frac{4}{\ell^2 (2\ell - 1)^2} \sum_{1 \leq m < r \leq \ell, 1 \leq n < s \leq \ell} \delta_{\alpha_m \alpha_n} \delta_{\alpha'_r \alpha'_s} \mathcal{P}^{(\ell-1;\ell-1)}_{\alpha_1...\alpha_{m-1},\alpha_{m+1}...\alpha_{r-1},\alpha_{r+1}...\alpha_{s-1},\alpha_{s+1}...\alpha_{\ell};\beta \alpha'_1...\alpha'_{m-1},\alpha'_{n+1}...\alpha'_{s-1},\alpha'_{s+1}...\alpha'_{r-1},\alpha'_{r+1}...\alpha'_{\ell};\alpha'_n \alpha'_s}$$

$$\times \sum_\beta \mathcal{P}^{(\ell-1;\ell-1)}_{\alpha_1...\alpha_{m-1},\alpha_{m+1}...\alpha_{r-1},\alpha_{r+1}...\alpha_{s-1},\alpha_{s+1}...\alpha_{\ell};\beta \alpha'_1...\alpha'_{m-1},\alpha'_{n+1}...\alpha'_{s-1},\alpha'_{s+1}...\alpha'_{r-1},\alpha'_{r+1}...\alpha'_{\ell};\alpha'_n \alpha'_s} \mathcal{P}^{(\ell-1;\ell-1)}_{\alpha_1...\alpha_{m-1},\alpha_{m+1}...\alpha_{r-1},\alpha_{r+1}...\alpha_{s-1},\alpha_{s+1}...\alpha_{\ell};\beta \alpha'_1...\alpha'_{m-1},\alpha'_{n+1}...\alpha'_{s-1},\alpha'_{s+1}...\alpha'_{r-1},\alpha'_{r+1}...\alpha'_{\ell};\alpha'_n \alpha'_s}.$$
In general, for $P^{(\ell;\ell)}$ at $\ell \geq 2$, the repeated recursion with Eq. (A12) produces, besides the leading symmetrized identity term, the correction terms which involve replacing $k$ pairs of Kronecker symbols in the identity, linking the covariant and contravariant indices, with the pairs of Kronecker symbols linking separately the covariant and contravariant indices, $k = 1, 2 \ldots \ell/2$. Application of both sides of Eq. (A12) for $P$ to $n'$ produces the recursion relation (48) for $A$.

ACKNOWLEDGMENTS

The authors acknowledge insightful discussions with Paul Chung, David Brown and Richard Lednicky. Support was provided by the U.S. National Science Foundation, Grant No. PHY-0555893, and by the U.S. Department of Energy, Grant No. DE-FG02-03ER41259.

[1] U. W. Heinz and B. V. Jacak, Ann. Rev. Nucl. Part. Sci. 49, 529 (1999), nucl-th/9902020.
[2] W. Bauer, C. K. Gelbke, and S. Pratt, Ann. Rev. Nucl. Part. Sci. 42, 77 (1992).
[3] U. A. Wiedemann and U. W. Heinz, Phys. Rept. 319, 145 (1999), nucl-th/9901094.
[4] B. Tomasik and U. A. Wiedemann (2002), hep-ph/0210250.
[5] C. Adler et al. (STAR), Phys. Rev. Lett. 87, 082301 (2001).
[6] K. Adcox et al. (PHENIX), Phys. Rev. Lett. 88, 192302 (2002).
[7] M. Baker et al. (PHOBOS), nucl-ex/0212009.
[8] M. Gyulassy and D. Rischke, Nucl. Phys. A 608, 479 (1996).
[9] D. Teaney, J. Laurent, and E. Shuryak, nucl-th/0110037.
[10] S. Soff, S. Bass, and A. Dumitru, Phys. Rev. Lett. 86, 3981 (2001).
[11] P. F. Kolb and U. Heinz (2003), nucl-th/0305084.
[12] J. Applequist, J. Phys. A: Math. Gen. 22, 4303 (1989).
[13] D. Anchishkin, U. W. Heinz, and P. Renk, Phys. Rev. C 57, 1428 (1998), nucl-th/9710051.
[14] P. Danielewicz and P. Schuck, Phys. Lett. B 274, 268 (1992).
[15] S. Pratt, Phys. Rev. D 33, 1314 (1986).
[16] G. F. Bertsch, Nucl. Phys. A 498, 173c (1989).
[17] T. Csorgo, J. Zimanyi, J. Bondorf, H. Heiselberg, and S. Pratt, Phys. Lett. B 241, 301 (1990).
[18] G. Verde et al., Phys. Rev. C 67, 034606 (2003), nucl-ex/0301013.
[19] G. Verde et al., Phys. Rev. C 65, 054609 (2002), nucl-ex/0112004.
[20] S. Y. Panitkin et al. (E895), Phys. Rev. Lett. 87, 112304 (2001), nucl-ex/0103011.
[21] D. A. Brown and P. Danielewicz, Phys. Rev. C 64, 014902 (2001), nucl-th/0010108.
[22] D. A. Brown, F. Wang, and P. Danielewicz, Phys. Lett. B 470, 33 (1999), nucl-th/9908033.
[23] D. A. Brown and P. Danielewicz, Phys. Lett. B 398, 252 (1997), nucl-th/9701010.
[24] D. A. Brown, A. Enokizono, M. Heffner, R. Soltz, P. Danielewicz, and S. Pratt, Phys. Rev. C 72, 054902 (2005), nucl-th/0507015.
[25] F. Danielewicz and S. Pratt, Phys. Lett. B 618, 60 (2005), nucl-th/0501003.
[26] J. Applequist, Theor. Chem. Acc 107, 103 (2002).
[27] G. F. Bertsch, P. Danielewicz, and M. Herrmann, Phys. Rev. C 49, 442 (1994), nucl-th/9309005.
[28] D. A. Brown and P. Danielewicz, Phys. Rev. C 57, 2474 (1998), nucl-th/9712066.
[29] S. Pratt and S. Petriconi, Phys. Rev. C 68, 054901 (2003), nucl-th/0305018.
[30] Y. Kim, R. de Souza, C. Gelbke, W. Gong, and S. Pratt, Phys. Rev. C 45, 387 (1992).
[31] B. K. Jennings, D. H. Boal, and J. C. Shillcock, Phys. Rev. C 33, 1303 (1986).
[32] C. J. Gelderloos et al., 75, 3082 (1995).
[33] R. Lednicky, V. Lyuboshits, B. Erazmus, and D. Nouais, Phys. Lett. B 373, 20 (1996).
[34] S. Voloshin, R. Lednicky, and S. Panitkin, Phys. Rev. Lett. 79, 4766 (1997).
[35] R. Ghetti et al., Phys. Rev. C 70, 034601 (2004).
[36] F. Wang and S. Pratt, Phys. Rev. Lett. 83, 3138 (1999), nucl-th/9907019.
[37] F. Retiere and M. A. Lisa, Phys. Rev. C 70, 044907 (2004), nucl-th/0312024.
[38] B. Tomasik (2004), nucl-th/0400074.
[39] J. Adams et al. (STAR), Phys. Rev. Lett. 91, 262302 (2003), nucl-ex/0307025.
[40] F. Retiere (STAR), Nucl. Phys. A 715, 591 (2003), nucl-ex/0212026.
[41] P. Cvitanovic, Phys. Rev. D 14, 1536 (1976).