Cross ratios on boundaries of symmetric spaces and Euclidean buildings

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Abstract: We generalize the natural cross ratio on the ideal boundary of a rank one symmetric spaces, or even CAT(−1) space, to higher rank symmetric spaces and (non-locally compact) Euclidean buildings - we obtain vector valued cross ratios defined on simplices of the building at infinity. We show several properties of those cross ratios; for example that (under some restrictions) periods of hyperbolic isometries give back the translation vector. In addition, we show that cross ratio preserving maps on the chamber set are induced by isometries and vice versa - motivating that the cross ratios bring the geometry of the symmetric space/Euclidean building to the boundary.

1 Introduction

Cross ratios on boundaries are a crucial tool in hyperbolic geometry and more general negatively curved spaces. In this paper we show that we can generalize these cross ratios to (the non-positively curved) symmetric spaces of higher rank and thick Euclidean buildings with many of the properties of the cross ratio still valid.

On the boundary $\partial_\infty \mathbb{H}^2$ of the hyperbolic plane $\mathbb{H}^2$ there is naturally a cross ratio defined by $\text{cr}_{\partial_\infty \mathbb{H}^2}(z_1, z_2, z_3, z_4) = \frac{z_1 - z_2}{z_1 - z_4} \frac{z_3 - z_4}{z_3 - z_2}$ when considering $\mathbb{H}^2$ in the upper half space model, i.e. $\partial_\infty \mathbb{H}^2 = \mathbb{R} \cup \{\infty\}$. This cross ratio plays an essential role in hyperbolic geometry. For example it characterizes the isometry group by its boundary action and therefore allows to study the geometry of the space from its boundary - an important tool in hyperbolic geometry.

The absolute value of this cross ratio can be generalized in a way broader context, namely CAT(−1) spaces [Bon95]: Let $\partial_\infty Y$ be the ideal boundary of a CAT(−1) space $Y$, $x, y \in \partial_\infty Y$ and $o \in Y$. Then one can define a Gromov product $(\cdot|\cdot)_o : \partial_\infty Y^2 \to [0, \infty]$ by $(x|y)_o = \lim_{t \to \infty} t - \frac{1}{2} d(\gamma_{ox}(t), \gamma_{oy}(t))$, where $\gamma_{ox}, \gamma_{oy}$ are the unique unit speed geodesics from $o$ to $x, y$, respectively. Then a multiplicative cross ratio $\text{cr}_{\partial_\infty Y} : A \subset \partial_\infty Y^4 \to [0, \infty]$ is defined by $\text{cr}_{\partial_\infty Y}(x, y, z, w) := \exp(-(x|y)_o - (z|w)_o + (x|w)_o + (z|y)_o)$ for all $(x, y, z, w) \in A$.

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\(\partial_{\infty}Y^4\) with no entry occurring three or four times. As the notation suggests one can show that the cross ratio is independent of the base point. By construction \(cr_{\partial_{\infty}Y}\) has several symmetries with respect to \((\mathbb{R},+\cdot)\). The logarithm of the above cross ratio \(\log(cr_{\partial_{\infty}Y}) : A \to [-\infty, \infty]\) is called an additive cross ratio, as the symmetries are with respect to \((\mathbb{R},+\cdot)\). In analogy to the hyperbolic plane, maps \(f : \partial_{\infty}Y \to \partial_{\infty}Y\) that leave \(cr_{\partial_{\infty}Y}\) under the diagonal action invariant are called Moebius maps. It follows from the definition of the cross ratio together with the basepoint independence that isometries are Moebius maps when restricted to the boundary.

The cross ratios \(cr_{\partial_{\infty}Y}\) and Moebius maps have been proven to be very useful in hyperbolic geometry. For example Bourdon [Bon96] has shown that Moebius maps of rank one symmetric spaces extend uniquely to isometric embeddings of the interior, and with this he gave a new proof of Hamenstädt’s ‘entropy against curvature’ theorem [Ham90]. Otal [Ota90] has (implicitly) shown that Moebius bijections on boundaries of universal covers of closed negatively-curved surfaces can be uniquely extended to isometries; which yields that marked length spectrum rigidity holds for those manifolds - a prominent conjecture formulated in [BK85]. We want to point out that in general it is not known that Moebius maps always extend to isometries and it seems to be very difficult to answer - for negatively curved manifolds this would be equivalent to marked length spectrum rigidity; see e.g. [Bis15].

Moreover, there is a close relation between the cross ratio on the boundary of the universal cover of a closed negatively curved manifold and the quasi-conformal structure on the boundary, and to dynamical properties of the geodesic flow; see e.g. [Led95].

On the boundary \(\partial_{\infty}\tilde{S}\) of the universal cover of a closed surface \(S\) there are many other cross ratios, besides the above constructed one, that parametrize classical objects associated to the surface; such as simple closed curves, measured laminations, points of Teichmüller space [Bon88], Hitchin representations [Lab07] and positively ratioed representations [MZ17] - to name a few.

This prominence and importance of cross ratios in negative curvature motivates us to ask if such objects also exists for non-positively curved spaces and how much information about the geometry they carry.

There is already some work done in this context. In [CM17], see also [CCM18], a coarse cross ratio for arbitrary CAT(0) spaces on some subset of the boundary has been constructed. For CAT(0) cube complexes there is a cross ratio on the Roller boundary constructed in [BFIM], using essentially the combinatorial structure of the space. In both cases Moebius (respectively quasi-Moebius) bijections are connected to isometries (respectively quasi-
In this paper we will construct cross ratios for symmetric spaces and Euclidean buildings, which will generalize the cross ratios of CAT(−1) spaces. There is little need to explain the importance of symmetric spaces in differential geometry and related areas. However, we want to point out that there is currently an active field of research on symmetric spaces going on; namely Anosov representations and subgroups (e.g. [Lab06], [KL14], [GW12] and many more). Anosov representations are representations of hyperbolic groups, for example surface groups, into semi-simple Lie groups which come with an equivariant boundary map satisfying some contracting/expanding properties. They yield a class of very well behaved discrete subgroups which in many situations carry a lot of geometric information (e.g. [CG05]). The natural boundary maps Anosov representations come with can be used to pullback the cross ratios that we construct to get cross ratios on the boundary of the group. This has been done for representation into SL(n, R) in [Lab07] using an ad-hoc definition of a cross ratio and this is in the spirit of [MZ17]; and we hope that the vector valued cross ratio we analyze here allows for further applications in this area.

Euclidean buildings arise in many different areas of mathematics. See [Ji12] for an overview of some applications. Probably most prominently they arise in the study of algebraic groups and geometric group theory; they have also been a crucial tool in the proof of quasi-isometric rigidity of symmetric spaces [KL97] (extending Mostow-Prasad rigidity) - to name a few.

We want to construct (generalized) cross ratios for symmetric spaces or thick (non-locally compact) Euclidean buildings similar as for CAT(−1) spaces. For CAT(0) spaces the Gromov product as above is still well defined. However, the cross ratio might be only defined on very small sets and carry little information; as for example for the Euclidean plane. Denote by M either a symmetric space or a thick Euclidean building. Then we will use the building at infinity $\Delta_{\infty} M$ of those spaces to extract some subset of the ideal boundary on which the Gromov product and cross ratio will be generically defined and well behaved. More precisely, to any point in the ideal boundary $\partial_{\infty} M$ we can associate a type, i.e. we have a map $\text{typ} : \partial_{\infty} M \to \sigma$ with $\sigma$ the closed fundamental chamber of the Weyl (actually Coxeter) group. Then one can show that to each type $\xi \in \sigma$ there is a unique type $\iota \xi \in \sigma$ such that elements in $\text{typ}^{-1}(\xi)$ can be generically joined by a geodesic to elements in $\text{typ}^{-1}(\iota \xi)$ - for symmetric spaces generically means that there is a dense and open subset in the product. This yields that the Gromov product $(\cdot | \cdot)_o$ for any $o \in M$ restricted to $\text{typ}^{-1}(\xi) \times \text{typ}^{-1}(\iota \xi)$ is generically finite and hence we get a generically defined additive cross ratio on $(\text{typ}^{-1}(\xi) \times \text{typ}^{-1}(\iota \xi))^2$ in the same way as for CAT(−1) spaces. While the definition requires a base point, one can show that the cross ratio is independent of the choice. We denote this cross ratio by $\text{cr}_\xi$. 


Let \( \tau \) be a face of the simplex \( \sigma \) and let \( \xi \in \text{int}(\tau) \), the interior of \( \tau \). Moreover, denote by \( \text{Flag}_\tau(M) \subset \Delta_\infty M \) the set of simplices of the building at infinity of type \( \tau \) - in case of symmetric spaces those are exactly the Furstenberg boundaries. Then one can naturally identify type \( \text{typ}^{-1}(\xi) \) with \( \text{Flag}_\tau(M) \) and in the same way type \( \text{typ}^{-1}(i\xi) \) with \( \text{Flag}_{i\tau}(M) \). In this way we get a cross ratio \( \text{cr}_\xi : \mathcal{A}_\tau \subset (\text{Flag}_\tau(M) \times \text{Flag}_{i\tau}(M))^2 \to [-\infty, \infty] \) which has by construction similar symmetries as the one on CAT(−1) spaces - for \( \mathcal{A}_\tau \) see equation (2.1), for the symmetries see equation (3.1). In general there are less symmetries as in the CAT(−1) situations, since the Gromov product is not symmetric.

It is immediate by construction that we are not getting one cross ratio on the set \( \mathcal{A}_\tau \), but a whole collection parametrized by \( \xi \in \text{int}(\tau) \). We show that we can put together this collection of cross ratios to a single vector valued cross ratio with the same symmetries. In case of a symmetric space \( X \), denote by \( G = Iso_0(X) \), be \( \text{Lie}(G) = g = \mathfrak{t} + \mathfrak{p} \) the Cartan decomposition and \( \mathfrak{a} \) a maximal abelian subspace of \( \mathfrak{p} \) together with the Weyl group \( W \) induced by the restricted roots. In case of an Euclidean building \( E \) denote by \( (\mathfrak{a}, W) \) the Coxeter complex over which the building is modeled. In both cases let \( \mathfrak{a}^+ \) be the positive sector in \( \mathfrak{a} \) and \( \mathfrak{a}_1 \) the unit sphere. Then \( \sigma = \mathfrak{a}^+ \cap \mathfrak{a}_1 \).

Denote by \( \mathfrak{a}_\tau \) the unique subspace containing the face \( \tau \) of \( \sigma \). Then the vector valued cross ratio \( \text{cr}_\tau \) with respect to the face \( \tau \) of \( \sigma \) takes values in \( \mathfrak{a}_\tau \subset \mathfrak{a} \). In particular, if considering the chamber set of the building at infinity of a symmetric space or Euclidean building \( M \), i.e. \( \text{Flag}_\tau(M) \), we get a cross ratio (possibly) taking values in all of \( \mathfrak{a} \) (union \( \pm \infty \)).

It seems natural to consider vector valued cross ratios. On one hand, consider a hyperbolic element \( g \in G \) in an irreducible symmetric space \( X \) not of type \( A_n, D_{2n+1} \) for \( n \geq 2 \) or \( E_6 \) and \( G \) as above. Let \( g^+ \in \text{Flag}_\tau(X) \) be the attractive and repulsive fixed points. Then the so called \( \text{period} \) \( \text{cr}_\tau(g^-, g^+, x, g_\tau^+ x) \) gives exactly the translation vector of \( g \) along the unique maximal flat joining \( g^- \) and \( g^+ \) (for generic, but not all \( x \in \text{Flag}_\tau(X) \)). On the other hand, there is a nice geometric interpretation of the vector valued cross ratio; in case of the chamber set \( \text{Flag}_\tau(X) \) of a symmetric space \( X \) this reads as follows: Let \( x_1, x_2, y_1, y_2 \in \text{Flag}_\tau(X) \) such that \( x_i \) is opposite of \( y_j \) for \( i, j = 1, 2 \). Let \( B_{x_1} = \text{stab}(x_1) \), i.e. \( B_{x_1} \) is a Borel subgroup of \( G \), and let \( N_x \), be the nilpotent (or horospherical) subgroup of \( B_{x_1} \); set in the same way \( B_{y_1} \) and \( N_{y_1} \). Let \( n_1 \in N_{y_1} \) be the unique element with \( n_1 x_1 \cdot y_2 = y_1 \) (the horospherical subgroup acts simply transitive on the opposite chambers); and define in the same way \( n_2 \in N_{y_2} \) through \( n_2 x_2 \cdot y_1 = y_2 \) as well as \( n_{y_1} \in N_{y_1} \) by \( n_{y_1} x_1 = x_2, n_{y_2} x_2 = x_1 \). Moreover, denote by \( F(x_1, y_1) \) the unique maximal flat joining \( x_1 \) and \( y_1 \), and let \( o \in F(x_1, y_1) \). Then the vector valued difference of \( o \) and \( n_x, n_{y_1}, n_{x_2}, n_{y_2}, o \in F(x_1, y_1) \) equals \( 2\text{cr}_\tau(x_1, y_1, x_2, y_2) \) - we remark that there is a natural identification of \( F(x_1, y_1) \) with a such that \( o \) is identified with 0. Similar geometric interpretations hold for Euclidean buildings and all vector valued cross ratios.
Let $M_1, M_2$ be either two symmetric spaces or two thick Euclidean buildings. Let $\sigma_1, \sigma_2$ be the according fundamental chambers of the two spaces and let $\xi_i \in \text{int}(\sigma_i)$ be two types. Let $f : \text{Flag}_{\sigma_1}(M_1) \rightarrow \text{Flag}_{\sigma_2}(M_2)$ be a surjective map. If $\text{cr}_{\xi_1}(x, y, z, w) = \text{cr}_{\xi_2}(f(x), f(y), f(z), f(w))$ for all $(x, y, z, w) \in A_{\sigma_1}$ then $f$ is called a $\xi_1$-Moebius bijection. If $\text{cr}_{\sigma_1}(x, y, z, w) = \text{cr}_{\sigma_2}(f(x), f(y), f(z), f(w))$ for all $(x, y, z, w) \in A_{\sigma_1}$ then $f$ is called a $\sigma_1$-Moebius bijection. Moreover we call a locally compact Euclidean building with discrete translation group a combinatorial Euclidean building. Then we can show the following theorems:

**Theorem A.** Let $M_1, M_2$ be either symmetric spaces or thick combinatorial Euclidean buildings and $\xi_1 \in \text{int}(\sigma_1)$. If $M_1, M_2$ are irreducible, then every $\xi_1$-Moebius bijection $f : \text{Flag}_{\sigma_1}(M_1) \rightarrow \text{Flag}_{\sigma_2}(M_2)$ can be extended to an isometry $F : M_1 \rightarrow M_2$. If none of the spaces is a Euclidean cone over a spherical building, then this extension is unique. If $M_1, M_2$ are reducible, one can rescale the metric of $M_1$ on irreducible factors - denote this space by $\hat{M}_1$ - such that $f$ can be extended to an isometry $F : M_1 \rightarrow M_2$.

**Theorem B.** Let $E_1, E_2$ be thick Euclidean buildings. Then for every $\sigma_1$-Moebius bijection $f$ one can rescale the metric of $E_1$ on irreducible factors - denote this space by $\hat{E}_1$ - such that $f : \text{Flag}_{\sigma}(E_1) \rightarrow \text{Flag}_{\sigma}(E_2)$ can be extended to an isometry $F : \hat{E}_1 \rightarrow E_2$. If none of the irreducible factors is a Euclidean cone over a spherical building, then $f$ can be extended to an isometry $F : E_1 \rightarrow E_2$ (without rescaling the metric).

We remark that essentially by definition of the cross ratio every isometry gives rise to a Moebius bijection. Then these theorems show that the cross ratios - at least for the chamber set of the building at infinity - carry a lot of the geometric information of the space, as the characterize isometries by their boundary action. In this spirit we hope that those cross ratios will be a valuable tool in the studies of symmetric spaces and Euclidean buildings.

We want to refer the reader to section 5 to slightly more results in this spirit, e.g. when we get a one-to-one correspondence of Moebius bijections and isometries, and also an analysis in which situations the rescaling of the metric is really necessary.

Concerning the proofs of those theorems: It is essential that a Moebius bijection splits as a product of Moebius bijections of irreducible factors; and that Moebius bijections can be extended to building isomorphisms. For rank one symmetric spaces and trees it is already known that Moebius bijections extend to isometries; for irreducible thick combinatorial Euclidean buildings it is enough that Moebius maps are restrictions of building isomorphisms to the chamber set. For the cases of symmetric spaces and (general) thick Euclidean buildings, we derive additional properties of the building map, using the cross ratio. Those properties will allow us to use theorems (essentially due to Tits) showing that the according maps can be extended to isometries.
In the preliminaries we try to recall those facts about symmetric spaces and Euclidean buildings (of the huge theory available) that will be relevant for us in the ongoing. We assume the reader to be familiar with those objects, but still briefly mention most facts used. Moreover, we show some basic lemmas we need later on.

In section 3 we define \( \mathbb{R} \)-valued cross ratios and show basic properties. We illustrate the objects with two examples: the product of two copies of the hyperbolic plane and the symmetric space \( \text{SL}(n, \mathbb{R})/\text{SO}(n, \mathbb{R}) \).

In section 4 we show that the collections of \( \mathbb{R} \)-valued cross ratios fit together to a single vector valued cross ratio. We motivate that this is the natural object, by showing that under some assumptions periods of hyperbolic elements give rise to the translation vector of the latter. Moreover, we give a geometric interpretation for the vector valued cross ratios.

In the last section, section 5, we show that Moebius maps, i.e. cross ratio preserving maps, on the chamber set extend to isometries. When considering symmetric spaces and combinatorial Euclidean buildings it is enough to consider a \( \mathbb{R} \)-valued cross ratio. When considering general Euclidean buildings we need to consider the vector valued cross ratio.

Related Work: In [Kim10] essentially the same \( \mathbb{R} \)-valued cross ratio as in Definition 3.6 has been constructed. However, few of further properties have been shown. Labourie [Lab07] has given one of the cross ratios in Example 3.13 ad-hoc and used it as tool to understand Hitchin representations. Moreover, Martone and Zhang [MZ17] have constructed cross ratios on boundaries of surface groups, which in particular for \( \text{SL}(n, \mathbb{R}) \)-Hitchin representations coincide with the pullback under the boundary map of some of the cross ratios in Example 3.13.

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## 2 Preliminaries

We use the notation that \( M \) is either a symmetric spaces of non-compact type or a thick Euclidean building, \( X \) is a symmetric space of non-compact type and \( E \) is a thick Euclidean building.

Reference for symmetric spaces of non-compact type are for example [Ebe96], [BGS85]; for Euclidean buildings we refer to [KW14], [Par00], [Tit86] and also [KL97]. We will use the definition due to [Tit86], which is equivalent to the axioms in [KW14] and [Par00], while the definition in [KL97] would additionally assume metrically completeness.
In the case of a symmetric space when writing affine apartment we mean a maximal flat.

We begin with recalling several facts about symmetric spaces and Euclidean buildings - mainly those relevant for us in the ongoing. Then we prove some basic lemmas we need later on.

Coxeter complex and spherical buildings [Ron09], [AB08]: Let $W$ be a finite Coxeter group and $S$ the standard set of generators consisting of involutions. Then $W$ can be realized as a reflection group along hyperplanes in $\mathbb{R}^r$ with $r = |S|$. The hyperplanes decompose $\mathbb{R}^r$ and the unit sphere $S^{r-1}$ into (cones over) simplical cells. The maximal, i.e. $r$-dimensional, closed cells in $\mathbb{R}^r$ are called Weyl sectors. Lower dimensional cells will be called conical cells. The maximal, i.e. $r-1$-dimensional, closed simplical cells in $S^{r-1}$ are called Weyl chambers. The set $S$ corresponds to exactly the hyperplanes bounding a Weyl sector. This Weyl sector will be called the positive sector, the corresponding chamber in $S^{r-1}$ will be called positive chamber. We can give each simplex adjacent to the positive chamber or positive sector a different label. Then the action of $W$ on the simplical complex induces a unique labeling for all simplices. A fixed label will be called type.

In this paper we refer to $(\mathbb{R}^r, W)$ as the Coxeter complex and to $(S^{r-1}, W)$ as the spherical Coxeter complex.

A spherical building is a simplical complex $B$ together with a collection of subcomplexes $\text{Apt}(B)$, called apartments, which are isomorphic to a fixed spherical Coxeter complex $(S^{r-1}, W)$, such that the following holds:

1. For any two simplices $a,b \in B$ there is an apartment $A \in \text{Apt}(B)$ with $a,b \in A$.
2. If $A,A'$ are apartments containing the simplices $a,b$, then there is a type preserving simplical isomorphism $A \to A'$ fixing $a,b$.

We say that the building is modeled over the spherical Coxeter complex $(S^{r-1}, W)$.

A spherical building is called thick if each non-maximal simplex is contained in at least three chambers. A (spherical) Coxeter complex is called irreducible if the Coxeter group can not be written as a product $W = W_1 \times W_2$ of two nontrivial Coxeter groups. A spherical building is called irreducible if the spherical Coxeter complex over which it is modeled is irreducible. If a building $B$ is reducible, i.e. modeled over the spherical Coxeter complex $W_1 \times W_2$, then it can be written as the spherical join of two buildings, i.e. $B = B_1 \circ B_2$ for two spherical buildings $B_1, B_2$ modeled over $W_1, W_2$ and $\circ$ being the spherical join [KL97, Sc.3.3].

Given a simplex $x \in B$ with $B$ a thick spherical building. We denote by $\text{Res}(x) := \{ y \in B \mid x \preceq y \}$ and call this the residue of $x$. Let $A$ be an apartment containing $x$, i.e. a Coxeter complex containing $x$. Let $W$ be the Coxeter group of $A$ and denote by $W_x$ the stabilizer of $x$ under $W$. If $x$ is
not a chamber then Res(x) is itself a spherical building modeled over the Coxeter complex to $W_z$ \cite[3.12]{Tit74}.

**Euclidean buildings** \cite{KWL14, Par00, Tit86, KL97}: Let $\hat{W}$ be an affine Coxeter group, i.e. $\hat{W}$ can be realized as a subgroup of the isometry group of $\mathbb{R}^r$ and can be decomposed as a semi-direct product $\hat{W} = W \ltimes T_W$, where $W$ is a finite reflection group and $T_W < \mathbb{R}^r$ is a co-bounded subgroup of translations. Here we assume $r = |S|$, where $S$ is the standard generating set of $W$. Moreover, let $(E,d)$ be a metric space. A chart is an isometric embedding $\phi : \mathbb{R}^r \rightarrow E$, and its image is called affine apartment; the image of a Weyl sectors and conical cells are again called Weyl sectors and conical cells. Two charts $\phi, \psi$ are called $\hat{W}$-compatible if $Y = \phi^{-1} \psi(\mathbb{R}^r)$ is convex in the Euclidean sense and if there is an element $w \in \hat{W}$ such that $\psi \circ w|_Y = \phi|_Y$. A metric space $E$ together with a collection of charts $\mathcal{C}$, called apartment system, is called a Euclidean building (with Coxeter group $\hat{W}$) if it has the following properties:

1. For all $\phi \in \mathcal{C}$ and $w \in \hat{W}$, the composition $\phi \circ w$ is in $\mathcal{C}$.
2. Any two points $p, q \in E$ are contained in some affine apartment.
3. The charts are $\hat{W}$-compatible.
4. If $a, b \in E$ are Weyl sectors, then there exists an affine apartment $A$ such that the intersections $A \cap a$ and $A \cap b$ contain Weyl sectors.
5. If $A$ is an affine apartment and $p \in A$ a point, then there is a $1$-Lipschitz retraction $\rho : E \rightarrow A$ with $d(p,q) = d(p,\rho(q))$ for all $q \in E$.

From this properties it follows that the metric space $E$ is necessarily CAT(0). The dimension of $\mathbb{R}^r$ is called the rank of $E$, i.e. $rk(E) = r$. While the definition depends on a fixed set of affine apartments, there is always a unique maximal set of affine apartments, called the complete apartment system. A set is an affine apartment in the complete apartment system if and only if it is isometric to $\mathbb{R}^r$. In the ongoing we will always consider $E$ with its complete apartment system. If the subgroup of translations $T_W$ is discrete and $E$ is locally compact we call $E$ a combinatorial Euclidean building.

**The ideal boundary and Busemann functions** \cite[Ch.8]{BH99}: We recall here several properties valid for CAT(0) spaces; hence for Euclidean buildings and symmetric spaces.

We denote by $\partial_\infty M$ the ideal boundary, i.e. the equivalence classes of geodesic rays - here equivalence means finite Hausdorff distance. Equipped with the cone topology $\partial_\infty M$ is naturally a topological space. For every $o \in M$ and every $x \in \partial_\infty M$ we denote by $\gamma_{ox}$ the unique unit-speed geodesic ray joining $o$ to $x$, i.e. $\gamma_{ox}(0) = o$ and $\gamma_{ox}$ in the class of $x$. For $o, p, q \in M$ the Gromov product on $M$ is defined by $(p|q)_o = \frac{1}{2}(d(o,p) + d(o,q) − d(p,q))$. Let $o \in M$ and $x, y \in \partial_\infty M$. Then $(\cdot | \cdot)_o : \partial_\infty M \times \partial_\infty M \rightarrow [0,\infty]$, the Gromov
product on the ideal boundary with respect to $o$, is given by

$$(x|y)_o = \lim_{t \to \infty} (\gamma_{ox}(t)|\gamma_{oy}(t))_o = \lim_{t \to \infty} t - \frac{1}{2}d(\gamma_{ox}(t)|\gamma_{oy}(t)).$$

We remark that the convexity of the distance functions guarantees the existence of the limit in $[0, \infty]$.

Given $x \in \partial_\infty M$ the Busemann function with respect to $x$, which will be denoted by $b_x : M \times M \to (-\infty, \infty)$, is defined by

$$b_x(o,p) = \lim_{t \to \infty} d(o,\gamma_{px}(t)) - d(p,\gamma_{px}(t)) = \lim_{t \to \infty} d(o,\gamma_{px}(t)) - t.$$ 

It holds that $-d(o,p) \leq b_x(o,p) = b_x(p,o) \leq d(o,p)$ and $b_x(o,p) + b_x(p,q) = b_x(o,q)$ for $o,p,q \in M$. Moreover, it follows directly that $b_x(o,\gamma_{ox}(s)) = s$ for all $s \geq 0$ and for all $s \in \mathbb{R}$ if $\gamma_{ox}$ is extended bi-infinitely.

An easy argument in Euclidean geometry yields that the level sets of Busemann functions in $\mathbb{R}^n$ with respect to $x$ in the boundary sphere are affine hyperplanes orthogonal to the direction $x$. In general Busemann level sets with respect to one coordinate are called horospheres and the collection of horospheres is independent of the choice of the other coordinate.

The isometry group $\text{Iso}(M)$ acts naturally by homeomorphisms on $\partial_\infty M$, since they map equivalence classes of geodesic rays to equivalence classes of geodesic rays. Moreover, by construction of the Busemann function, it follows $b_x(o,p) = b_x(g \cdot o,g \cdot p)$ for every $g \in \text{Iso}(M)$.

**Symmetric spaces** [Ebe96, Ch.2]: Let $X$ be a symmetric space. We will always assume that $X$ is of non-compact type. In particular $X$ is a Hadamard manifold and therefore CAT(0). We denote by $G = \text{Iso}_0(X)$, i.e. the connected component of the identity of the isometry group.

Let $g = \text{Lie}(G)$ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the Cartan decomposition. If we fix a maximal flat $F$ in $X$ together with a basepoint $o \in F$, we get the identification $T_oM \cong \mathfrak{p}$. This identification yields $T_oF \cong \mathfrak{a}$ where $\mathfrak{a}$ a maximal abelian subspace of $\mathfrak{p}$. The restricted root system of $g$ with respect to $\mathfrak{a}$ defines hyperplanes in $\mathfrak{a}$ - namely the zero sets of the restricted roots. The Weyl group $W$ of $X$ is the group generated by the reflections along those hyperplanes with respect to the metric that $\mathfrak{a}$ inherits from $T_oF \subset T_oX$. Hence we can associate to $X$ a Coxeter complex $(\mathfrak{a},W)$. Let $a_1$ be the unit sphere in $\mathfrak{a}$, then we also get a spherical Coxeter complex $(a_1,W)$. It is well known that up to isometry the Coxeter complex is independent of the choices. We fix a Weyl sector in $\mathfrak{a}$ which we denote by $\mathfrak{a}^+$ and call positive sector. Then $a_1^+$ will be called the positive chamber.\textsuperscript{2} The rank of $X$ is the usual rank and equals $\text{rk}(X) = \dim \mathfrak{a}$. To keep the notation consistent with buildings we will call maximal flats in $X$ affine apartments.

\textsuperscript{2}Usually $\mathfrak{a}^+$ is called positive Weyl chamber. However, as we will consider Euclidean buildings and symmetric spaces at the same time and we want to distinguish between spherical chambers and cones, we change the usual notation.
Let \( M \) now be either a symmetric space or a Euclidean building. To keep notation simple, we will denote by \((a, W)\) also the Coxeter complex over which a Euclidean building is modeled. Moreover, \( a_1 \) is the unit sphere in \( a \) and hence \((a_1, W)\) a spherical Coxeter complex. We fix a positive Weyl sector \( a^+ \subset a \) and the according positive chamber \( a_1^+ = a_1 \cap a^+ \). Let \( S \) denote the generating set of \( W \) consisting of reflections along the walls of \( a^+ \). By definition we have \( \text{rk}(M) = \dim a \).

The ideal boundary \( \partial_\infty M \) carries naturally the structure of a spherical building \( \Delta_\infty M \) modeled over the spherical Coxeter complex \((a_1, W)\). The building \( \Delta_\infty M \) will be called the building at infinity.

For a Euclidean building \( E \) the building at infinity arises as follows: Let \( A \subset E \) be an affine apartment. Then \( A \) being the image of \((a, W)\) under a chart implies that \( A \) is decomposed into conical cells. Each conical cell defines a simplex in \( \partial_\infty E \) by taking the geodesic rays contained in the cell for all times. One can show that two conical cells define the same set in \( \partial_\infty E \) if and only if they have finite Hausdorff distance. In the latter case we say the conical cells are equivalent. Therefore, taking all conical cells in \( E \) modulo the equivalence relation we get a simplicial structure on \( \partial_\infty E \), which can be shown to be a spherical building over the spherical Coxeter complex \((a_1, W)\).

Apartments in \( \Delta_\infty M \) correspond to the ideal boundaries of affine apartments of \( M \). It is well known that \( \Delta_\infty X \) is a thick building. We call \( E \) a thick Euclidean building if \( \Delta_\infty E \) is thick. If the rank of \( E \) is one, i.e. \( E \) is a tree we call \( E \) thick if \( \partial_\infty E \) contains at least three points, i.e. \( E \neq \mathbb{R} \).

In particular the following important property holds: To every two points \( p, q \in M \cup \partial_\infty M \) we find an affine apartment \( A \) in \( M \) such that \( p, q \in A \cup \partial_\infty A \). We say that \( A \) joins \( p \) and \( q \).

Given two affine apartments \( A, A' \) in a Euclidean building \( E \) that have a common chamber at infinity, i.e. \( c \in \Delta_\infty E \) such that \( c \subset \partial_\infty A \) and \( c \subset \partial_\infty A' \). Then the intersection \( A \cap A' \) contains a Weyl sector with boundary \( c \). Such a Weyl sector is called a common subsector of \( A \) and \( A' \).

The type map \([KL97, \text{Sc.} 4.2.1],[KL-P, \text{Sc.} 2.4]\): To the visual boundary \( \partial_\infty M \) with the building structure \( \Delta_\infty M \) there is naturally a type map \( \text{typ} : \partial_\infty M \to a_1^+ \) associated. Given \( x \in \partial_\infty M \) there is a chamber \( c_x \in \Delta_\infty M \) with \( x \in c_x \) and an affine apartment \( A \) with \( c_x \subset \partial_\infty A \). Then this yields an isometry from \( c_x \) to \( a_1^+ \) with respect to the Tits metric on \( c_x \) and the angular metric.
on $a^+_1$. In this way we can assign to each element of $\partial_\infty M$ an unique element of $a^+_1$. It can be shown that the image is independent of the chamber and the apartment chosen, hence we get a well defined map $\text{typ} : \partial_\infty M \to a^+_1$. The type map is consistent with the types of the spherical building $\Delta_\infty M$, i.e. two simplices of $\Delta_\infty M$ are of the same type if and only if they are mapped to the same face of $a^+_1$ under $\text{typ}$. Hence we also call the faces of $a^+_1$ types ($a^+_1$ will be a face of itself). When speaking of types we denote $\sigma = a^+_1$, i.e. a simplex of $\Delta_\infty M$ is a chamber if and only if it is of type $\sigma$. Faces of $\sigma$ will usually be denoted by $\tau$. The set of simplices in $\Delta_\infty M$ of type $\tau$ will be denoted by $\text{Flag}_\tau(M)$, or just by $\text{Flag}_\tau$ if $M$ is clear out of the context and will be called flag space. If we consider chambers we denote this by $\text{Flag}_\sigma$ and call it full flag space.

We (ambiguously) call elements in $\xi \in \sigma = a^+_1$ types. However, out of the context it is clear if an element or a simplex is meant. We denote by $\text{int}(\tau)$ the interior of a simplex (and set the interior of a point to be the point itself). Given a simplex in $x \in \text{Flag}_\tau$ and $\xi \in \tau$, we denote by $x_\xi$ the unique point in $\in \partial_\infty M$ of type $\xi$.

Any isometry $F : M_1 \to M_2$ between either two symmetric spaces or two thick Euclidean buildings induces a building isomorphism $F_\infty : \Delta_\infty M_1 \to \Delta_\infty M_2$. The map $F_\infty$ is in general not type preserving. However, that $M_1, M_2$ are isometric implies that they are modeled over the same Coxeter complex and hence have the same fundamental chamber $\sigma$. Then we can associate to $F$ a type map $F_\sigma : \sigma \to \sigma$ such that $\text{typ}(F_\infty(x)) = F_\sigma(\text{typ}(x))$ for every $x \in \partial_\infty M_1$ and $F_\sigma$ is a isometry with respect to the angular metric. Moreover, we have that $F(\text{Flag}_\sigma(M_1)) = \text{Flag}_{F_\sigma(\tau)}(M_2)$.

THE G-ACTION AND FLAG MANIFOLDS [Ebe96 Ch.3], [KLP Sc.2.4]: Let $X$ be a symmetric space and $G = \text{Iso}_0(X)$. Then the cone topology on $\partial_\infty X$ induces a topology on $\Delta_\infty X$ such that all flag spaces are compact. Moreover, given $x \in \text{Flag}_\sigma(X)$ be $P_\sigma = \text{stab}(x)$. Then we can identify $\text{Flag}_\sigma(X) \simeq G/P_\sigma$ with the identification being $G$-equivariant and homeomorphic; the group $P_\sigma$ is a parabolic subgroup of $G$ (every parabolic subgroup arises as the stabilizer of an element of $\Delta_\infty X$) and $G/P_\sigma$ is equipped with the quotient topology of the topological group $G$. Moreover, the identification $\text{Flag}_\sigma(X) \simeq G/P_\sigma$ gives a smooth structure on $\text{Flag}_\tau(X)$ making it a compact connected manifold. The spaces $G/P_\sigma$ are called Furstenberg boundaries or flag manifolds, motivating our notion of flag space.

Let $K$ be a maximal compact subgroup of $G$. Then already $K$ acts transitive on the flag manifolds and given $x \in \text{Flag}_\tau$ we can identify $K$-equivariant and homeomorphically $\text{Flag}_\tau(X) \simeq K/K_x$ while $K_x = \text{stab}_{K}(x)$.

Moreover, we remark that the $G$-action is type preserving, i.e. $g_\sigma = \text{id}$ for all $g \in G$.

THE OPPOSITION INVOLUTION: A important map for us will be the opposition involution $\iota : a \to a$, which is given by $\iota = -\text{id} \circ w_0$ with $w_0 \in W$. 

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the maximal element of the Coxeter group with respect to the generating set \( S \). If \( W \) is an irreducible Weyl group, then \( \iota = \text{id} \) if and only if \( W \) is not of type \( A_n \) with \( n \geq 2 \), \( D_{2n+1} \) with \( n \geq 2 \) or \( E_6 \) \cite[2.39]{Hum74}. Moreover, we remark that we can restrict \( \iota : \mathfrak{a}^*_1 \to \mathfrak{a}^*_1 \) and that \( \iota \) is an isometry with respect to the angular metric.

Opposite simplices \cite[Sc.2.2, 2.4]{KLP}: There is a natural notion of \textit{opposition} in spherical buildings. This corresponds to the following: Let \( x, y \in \Delta_\infty M \) and let \( A_\infty \) be an apartment in \( \Delta_\infty M \) such that \( x, y \in A_\infty \). Since \( A_\infty \) can be identified with the unit sphere \( \mathfrak{a}_1 \), there is a natural map \( \text{id}: A_\infty \to A_\infty \). Then \( x \) is \textit{opposite} of \( y \), denoted by \( x \text{ op } y \), if and only if \( x = -\text{id}(y) \). The action of the spherical Coxeter group \( W \) leaves the type invariant. Therefore, assume for the moment that \( W \) is modeled in \( A_\infty \) and \( x \) is a face of the positive chamber. Denote by \( w_0 : A_\infty \to A_\infty \) the maximal element of \( W \). Then \( w_0(y) \) is a face of the positive chamber and of the same type as \( y \) and hence \( y \) is of type \( -\text{id} \circ w_0(x) = \iota x \). In particular all simplices opposite of elements in \( \text{Flag}_x \) are contained in \( \text{Flag}_{\iota x} \).

We denote for later use

\[ \mathcal{A}_\tau := \{(x_1, y_1, x_2, y_2) \in (\text{Flag}_x \times \text{Flag}_{\iota x})^2 \mid x_i \text{ op } y_i \text{ or } x_i \text{ op } y_j, i, j = 1, 2, i \neq j \} \]

\[ \mathcal{A}_\tau^\text{op} := \{(x_1, y_1, x_2, y_2) \in (\text{Flag}_x \times \text{Flag}_{\iota x})^2 \mid x_1, x_2 \text{ op } y_1, y_2 \subset \mathcal{A}_\tau \} \tag{2.1} \]

Opposition of simplices has the following important connection to bi-infinite geodesics: Let \( z_1, z_2 \in \partial_\infty M \) and \( A \subset M \) an affine apartment with \( z_1, z_2 \in \partial_\infty A \). Then one can show that there exists a bi-infinite geodesics joining \( z_1 \) and \( z_2 \) if and only if there exists one in \( A \). From Euclidean geometry it follows that the \( z_i \) can be joined by a bi-infinite geodesic in \( A \) if and only if \( z_1 = -\text{id}(z_2) \) with \( -\text{id} : \partial_\infty A \to \partial_\infty A \) as before. This can easily be seen to be equivalent to the unique simplices \( \tau_{z_i} \in \Delta_\infty M \) containing the \( z_i \) in its interior being opposite, i.e. \( \tau_{z_1} \text{ op } \tau_{z_2} \), and \( \text{typ}(z_1) = \text{typ}(z_2) \).

We will call points \( z_1, z_2 \in \partial_\infty M \) \textit{opposite} if they can be joined by a bi-infinite geodesic and denote this also by \( z_1 \text{ op } z_2 \). Given \((x, y) \in \text{Flag}_x \times \text{Flag}_{\iota x} \) with \( x \text{ op } y \), for every \( \xi \in \tau \) it follows that \( x_\xi \) is opposite to \( y_\xi \).

Symmetric spaces, Langlands decomposition \cite[Sc.2.17]{Ebe96}, \cite[Sc.2.10]{KLP}: In case of a symmetric space \( X \), given \( x \in \text{Flag}_x(X) \), the set of simplices opposite to \( x \) coincides is an open and dense subset of \( \text{Flag}_{\iota x}(X) \) (which can be deduced from the Bruhat decomposition of \( G/P \)). Moreover, for \((x, y) \in \text{Flag}_x(X) \times \text{Flag}_{\iota x}(X) \) we have \( x \text{ op } y \) if and only if the pair is in the unique open and dense \( G \)-orbit in \( \text{Flag}_x(X) \times \text{Flag}_{\iota x}(X) \).

In particular, it follows in this case that \( \mathcal{A}_\tau \) and \( \mathcal{A}_\tau^\text{op} \) are open and dense subsets of \((\text{Flag}_x \times \text{Flag}_{\iota x})^2 \).

Every parabolic subgroup \( P_x \) has a natural decomposition \( P_x = K_xA_xN_x \) called the \textit{Langlands decomposition}, where \( K_x \) is compact and \( N_x \) is nilpotent. The group \( N_x \) is called \textit{horospherical subgroup} and is unique, while \( K_x \)
and $A_\tau$ are not. The horospherical subgroup has several important properties. It leaves the Busemann function with respect to $x_\xi \in \text{Flag}_\tau(X)$ invariant, i.e. $b_{x_\xi}(o,p) = b_{x_\xi}(n_\tau,p)$ for all $n \in N_x$ and $\xi \in \tau$. Given a geodesic ray $\gamma_{x_\xi}$ with endpoint in $x \in \partial_{\infty}X$, we have $d(\gamma_{x_\xi}(t), n\cdot \gamma_{x_\xi}(t)) \to 0$ for $t \to \infty$ for all $n \in N_x$. Moreover, $N_x$ acts simply transitive on the simplices opposite to $x$. If $x$ is a chamber, i.e. $x \in \text{Flag}_\tau(M)$, then $N_x$ acts simply transitive on the maximal flats containing $x$ in its boundary.

Parallel sets [EBEG90, Sc.2.11, 2.20], [KLP, Sc.2.4], [KL97, Sc.4.8]: Let $(x,y) \in \text{Flag}_\tau \times \text{Flag}_\tau$ with $x \parallel y$ and denote by $\text{int}(\tau)$ the interior of $\tau$. Moreover, be $\xi \in \text{int}(\tau)$. Then the parallel set with respect to $x,y$ denoted by $P(x,y)$ is the set of all points that lie on a bi-infinite geodesic joining $x_\xi$ to $y_\xi$.

The parallel sets split metrically as products, i.e. $P(x,y) \simeq F_{xy} \times CS(x,y)$, where $F_{xy}$ is an isometrically embedded $\mathbb{R}^n$ such that $x,y \in \partial_{\infty}F_{xy}$ and $x,y$ are simplices of maximal dimension in the sphere $\partial_{\infty}F_{xy}$ - in particular $n - 1$ equals the dimension of the spherical simplices $x,y$. Then it follows that the parallel set is independent of the choice of type $\xi \in \text{int}(\tau)$, i.e. for each type $\xi \in \text{int}(\tau)$ geodesics in $M$ joining $x_\xi,y_\xi$ are of the form $(\gamma_{x_\xi y_\xi}(t), p)$ with $\gamma_{x_\xi y_\xi}$ a geodesic in $F_{xy}$ joining $x_\xi,y_\xi$ and $p$ is a point in $CS(x,y)$.

The space $CS(x,y)$ is called cross section. In case of a symmetric space $X$ the cross section is itself a symmetric space without Euclidean de Rham factors, in case of a Euclidean building the cross section is again a Euclidean building. In both cases the rank is given by $\text{rk}(CS(x,y)) = \text{rk}(M) - \dim F_{xy}$.

Let $\tau$ be a face of $\sigma = a_1$. Then be $a_\tau$ the subspace of $a$ defined by $\tau$, i.e. the smallest subspace of $a$ containing $\tau$ and $0$. Let $\xi_1,\ldots,\xi_k$ be the corners of the spherical simplex $\tau$. Then $a_\tau = \text{span}_{i=1,\ldots,k} \xi_i$. It is immediate that we can also identify $P(x,y) \simeq a_\tau \times CS(x,y)$. We can additionally impose that this identification is in such a way that $x \simeq \partial\infty(a_\tau \cap a^+)$.

Lemma 2.1. Let $(x,y) \in \text{Flag}_\tau \times \text{Flag}_\tau$ with $x \parallel y$ and be $p,q \in P(x,y)$. Let $\pi : P(x,y) \simeq a_\tau \times CS(x,y) \to a_\tau$ be the projection to the first coordinate. Then for each $\xi \in \tau$ we have that $b_{x_\xi}(p,q) = (b_{x_\xi}|_{a_\tau}(\pi(p),\pi(q)))$, i.e. the Busemann function is independent of the second coordinate of the product.

Proof. Let $\gamma_{q_{x_\xi}}$ denote the geodesic ray from $q$ to $x_\xi$. Moreover, be $q = (q_1,q_2)$ under the identification $P(x,y) \simeq a_\tau \times CS(x,y)$. Then we have that $\gamma_{q_{x_\xi}} \simeq (\gamma_{q_1 x_\xi},q_2)$ where $\gamma_{q_1 x_\xi}$ is the geodesic ray in $a_\tau$ from $q_1$ to $x_\xi$. Using that metrically $P(x,y) \simeq a_\tau \times CS(x,y)$ and $p = (p_1,p_2)$ we derive $d(p,\gamma_{q_{x_\xi}}(t)) = \sqrt{d(p_1,\gamma_{q_1 x_\xi}(t))^2 + d(p_2,q_2)^2}$. If we set $K_2 := d(p_2,q_2)^2$, then $b_{x_\xi}(p,q) = \lim_{t \to \infty} \sqrt{d(p_1,\gamma_{q_1 x_\xi}(t))^2 + K_2} - t$. As $p_1,\gamma_{q_1 x_\xi}(t) \in a_\tau$, it reduces to Euclidean geometry; hence $d(p_1,\gamma_{q_1 x_\xi}(t)) = \sqrt{b_{x_\xi}(p_1,\gamma_{q_1 x_\xi}(t))^2 + K_1}$ with $K_1$ the squared distance from $p_1$ to the (now) bi-infinite geodesic $\gamma_{q_1 x_\xi}$.\hfill \qed

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We call this map (horospherical) retract hyperplanes orthogonal to the direction in its boundary, which implies that they have a common subsector. Hence

\[ b_{x_\xi}(p_1, q_1, t) = t + b_{x_\xi}(p_1, q_1). \]

Using a substitution \( t = s^{-1} \) and a Taylor series for the root expression below yields

\[ b_{x_\xi}(p, q) = \lim_{t \to \infty} \sqrt{(t + b_{x_\xi}(p_1, q_1))^2 + K_1 + K_2 - t} = \lim_{s \to 0} \sqrt{1 + 2sb_{x_\xi}(p_1, q_1) + s^2(b_{x_\xi}(p_1, q_1)^2 + K_1 + K_2) - 1} = b_{x_\xi}(p_1, q_1). \]

\[ \square \]

We will also need the following lemma.

**Lemma 2.2.** Let \((x, y) \in \Flag_x \times \Flag_{x, \tau}\) with \(x \preceq y\) and \(\xi \in \tau\). Moreover be \(p_1, p_2 \in P(x, y)\). Then \(b_{x_\xi}(p_1, p_2) = -b_{y_\xi}(p_1, p_2)\).

**Proof.** Let \(\gamma_i, i = 1, 2\) be bi-infinite geodesics such that \(\gamma_i(0) = p_i, \gamma_i(+\infty) = x_\xi\) and \(\gamma_i(-\infty) = y_\xi\), which exists by assumption. The \(\gamma_i\) are parallel and denote by \(C\) their distance. Then the Flat Strip Theorem (see e.g. [BH99]) implies that the convex hull of \(\gamma_1(\mathbb{R}) \cup \gamma_2(\mathbb{R})\) is isometric to a flat strip \(\mathbb{R} \times [0, C] \subset \mathbb{R}^2\) with \(\gamma_i\) identified with \(\mathbb{R} \times 0, \mathbb{R} \times C\) respectively.

It follows that the level sets of the Busemann function \(b_{x_\xi}(), \gamma_1\) in \(\mathbb{R} \times [0, C]\) are given by hyperplanes orthogonal to \(\gamma_i\), i.e. are of the form \(s \times [0, C]\) and the same holds for \(b_{y_\xi}(), \gamma_i\). In addition, it follows directly from the definition that \(b_{x_\xi}(p_1, p_2)_{\gamma_1} = -b_{y_\xi}(p_1, p_2)_{\gamma_i}\). Then the claim is direct consequence.

**Retracts [Par00]**: Lastly we need to introduce the notion of retracts of \(M\) to affine apartments with respect to chambers at infinity. For the construction we will distinguish between Euclidean buildings and symmetric spaces.

Let \(E\) be a Euclidean building. Let \(A \subset E\) be an affine apartment and \(x \in \partial_\infty A\) a chamber of the building at infinity. Then there exists a \(1\)-Lipschitz map \(\rho_{x, A} : E \to A\) which is an isometry when restricted to any affine apartment \(A'\) with \(x \in \partial_\infty A'\), i.e. any affine apartment that contains the chamber \(x\) in its boundary, and the identity on \(A\) [Par00, Prop.1.20]. We call this map (horospherical) retract with respect to \(x\). Horospherical retracts have the following important property:

**Lemma 2.3.** Let \(\rho_{x, A} : E \to A\) be a horospherical retract with respect to \(x \in \Flag_\sigma(E)\). Then \(b_{x_\xi}(o, p) = b_{x_\xi}(\rho_{x, A}(o), p) = b_{x_\xi}(o, \rho_{x, A}(p))\) for all \(o, p \in E\) and \(\xi \in \sigma\).

**Proof.** To \(o \in E\) there exists an affine apartment \(A_o\) containing \(o\) and \(x \in \partial_\infty A_o\). As mentioned, the horospheres with respect to \(x_\xi\) in \(A_o\) are hyperplanes orthogonal to the direction \(x_\xi\).

By construction, the two affine apartments \(A, A_o\) have the same chamber in its boundary, which implies that they have a common subsector. Hence
$\rho_{x,A}$ is the identity on the non-empty intersection $A \cap A_o$. Moreover, $\rho_{x,A}$ is an isometry when restricted to $A_o$. Since $\rho_{x,A}$ leaves each horospheres intersecting $A \cap A_o$ invariant it has to map the level set of $b_{x\xi}(\cdot, p)$ in $A_o$ to the corresponding level set in $A$. The other equality follows for example form the symmetry $b_{x\xi}(o, p) = -b_{x\xi}(p, o)$.

Let $X$ be a symmetric space, $A \subset X$ be a maximal flat (an affine apartment for us) and $x \in \partial_{\infty} A$ a chamber at infinity. To any $o \in X$ there exists a unique maximal flat $A_o$ with $o \in A_o$ and $x \in \partial_{\infty} A_o$. Then we define $\rho_{x,A}(o) := n_{x,A_o} \cdot o$ for $n_{x,A_o}$ the unique element in $N_x$ that maps $A_o$ to $A$. Again we call $\rho_{x,A} : X \rightarrow A$ (horospherical) retraction.

For later reference: To every affine apartment $A \subset M$ and chamber $x \in \partial_{\infty} A$ we have a well defined map $\rho_{x,A} : M \rightarrow A$ such that

$$b_{x\xi}(o, p) = b_{x\xi}(\rho_{x,A}(o), p) = b_{x\xi}(o, \rho_{x,A}(p))$$

(2.2)

for all $o, p \in M$ and $\xi \in \sigma$. Moreover, it is known that two opposite chambers $x, y \in \text{Flag}_\sigma$ are contained in an unique apartment $A_{\infty}$ of $\Delta_{\infty} M$ and this corresponds to an unique affine apartment $A_{xy} \subset M$. Hence to $x, y \in \text{Flag}_\sigma$ with $x \text{ op } y$ we set $\rho_{x,y} := \rho_{x,A_{xy}}$.

**Lemma 2.4.** Let $x, y \in \text{Flag}_\sigma$ with $x \text{ op } y$ and $o \in M$. Then for all $\xi \in \tau$ we have that $\rho_{c_x,c_y}(\gamma_{ox\xi}(t))$ is a geodesic in $P(x, y)$, where $c_x, c_y \in \text{Flag}_\sigma$ such that $x$ is a face of $c_x$, $y$ is a face of $c_y$ and $c_x \text{ op } c_y$.

We remark that $x \text{ op } y$ implies that such $c_x, c_y \in \text{Flag}_\sigma$ always exist. Namely, take an apartment containing $x$ and $y$. Take $c_x \in \text{Flag}_\sigma$ such that $x$ is a face of $c_x$. Take $c_y \in \text{Flag}_\sigma$ the unique opposite chamber in the apartment. Then $x \text{ op } y$ implies that $y$ is a face of $c_y$.

**Proof.** For a symmetric space $X$ this follows since $\rho_{c_x,c_y}$ is the same element of $G$ for all points $\gamma_{ox\xi}(t)$ and that $G < \text{Iso}(X)$. Hence $\rho_{c_x,c_y}(\gamma_{ox\xi}(t))$ is the image of a geodesic under an isometry.

Consider a Euclidean building $E$. Denote by $A_{xy}$ the unique affine apartment joining $c_x$ and $c_y$. Let $A$ be an affine apartment containing $o$ and $c_x \subset \partial_{\infty} A$. Then it follows that $\gamma_{ox\xi}(t) \in A$ for all $t \in \mathbb{R}_+$. As $\rho_{c_x,c_y}$ is an isometry on affine apartments containing $c_x$, it follows that $\rho_{c_x,c_y}(\gamma_{ox\xi}(t)) \subset A_{xy}$ is the image of a geodesic under an isometry. Since one of the endpoints is $x\xi$, we can extend the geodesic in $A_{xy}$ uniquely to a bi-infinite geodesic joining $x\xi$ and $y\xi$. Thus $\rho_{c_x,c_y}(\gamma_{ox\xi}(t)) \subset P(x, y)$.

### 3 Cross ratios

Let $M$ be a symmetric space of non-compact type or a thick Euclidean building. Let $\sigma$ be the fundamental chamber of the associated spherical
Coxeter complex and \( \tau \) a face of \( \sigma \). For any type \( \xi \in \sigma \) such that \( \xi \in \text{int}(\tau) \) and any \( o \in M \) we define a Gromov product \((\cdot,\cdot)_{\alpha,\xi} : \text{Flag}_{\sigma,\tau}(M) \times \text{Flag}_{\sigma,\tau}(M) \to [0,\infty) \) with base-point \( o \) by

\[
(x|y)_{\alpha,\xi} := \lim_{t \to \infty} t - \frac{1}{2} d(\gamma_{\alpha,\xi}(t), \gamma_{\alpha,\xi}(t))
\]

for \((x,y) \in \text{Flag}_{\sigma,\tau}(M) \times \text{Flag}_{\sigma,\tau}(M)\) and \( \gamma_{\alpha,\xi}(t), \gamma_{\alpha,\xi}(t) \) the unit speed geodesics from \( o \) to \( x_{\xi}, y_{\xi} \), respectively. Using this we define the (additive) cross ratio \( \text{cr}_{\alpha,\xi} : \mathcal{A}_{\sigma} \to [-\infty, \infty) \) with respect to \((o,\xi)\) by

\[
\text{cr}_{\alpha,\xi}(x_1, y_1, x_2, y_2) := -(x_1| y_1)_{\alpha,\xi} - (x_2| y_2)_{\alpha,\xi} + (x_1| y_2)_{\alpha,\xi} + (x_2| y_1)_{\alpha,\xi}
\]

where \( \mathcal{A}_{\sigma} \) is the set of quadrupels \((x_1, y_1, x_2, y_2) \in (\text{Flag}_{\sigma,\tau}(M) \times \text{Flag}_{\sigma,\tau}(M))^2 \) as in equation (2.1). If \( \xi \in \text{int}(\tau) \), we also denote \( \mathcal{A}_{\xi} := \mathcal{A}_{\sigma} \). By definition \( \text{cr}_{\alpha,\xi} \) has the following symmetries, whenever all factors are defined\(^3\)

\[
\begin{align*}
\text{cr}_{\alpha,\xi}(x_1, y_1, x_2, y_2) &= -\text{cr}_{\alpha,\xi}(x_1, y_2, x_2, y_1) = -\text{cr}_{\alpha,\xi}(x_2, y_1, x_1, y_2) \\
\text{cr}_{\alpha,\xi}(x_1, y_1, x_2, y_2) &= \text{cr}_{\alpha,\xi}(x_1, y_1, w_2) + \text{cr}_{\alpha,\xi}(w_1, x_2, y_2) \\
\text{cr}_{\alpha,\xi}(x_1, y_1, x_2, y_2) &= \text{cr}_{\alpha,\xi}(x_1, y_2, x_2, v) + \text{cr}_{\alpha,\xi}(x_1, v, x_2, y_2).
\end{align*}
\]

The last two symmetries are called cocycle identities.

**Notation:** Let \( \tau \) be face of \( \sigma \) and be \( \xi \in \partial \tau \). Then for \((x,y) \in \text{Flag}_{\sigma,\tau} \times \text{Flag}_{\sigma,\tau} \) we drop the projection maps in the Gromov product (and in the cross ratio) for notational reasons, i.e. \((x|y)_{\alpha,\xi} := (\pi_{v}(x), \pi_{\xi}(y))_{\alpha,\xi} \), where \(\pi_{\xi}\) is the face of \( \tau \) containing \( \xi \) in its interior and \(\pi_{\xi} : \text{Flag}_{\sigma,\tau} \to \text{Flag}_{\sigma,\tau}^{\xi} \), \(\pi_{\xi} : \text{Flag}_{\sigma,\tau} \to \text{Flag}_{\sigma,\tau} \) are the obvious projection maps.

**Proposition 3.1.** Let \( M \) be a symmetric space or thick Euclidean building, \( o \in M \), \((x,y) \in \text{Flag}_{\sigma,\tau} \times \text{Flag}_{\sigma,\tau} \) with \( x \varphi y \) and \( c_x, c_y \in \text{Flag}_{\sigma,\tau} \) such that \( x \) is a face of \( c_x \), \( y \) is a face of \( c_y \) and \( c_x \varphi c_y \). Then for every \( \xi \in \tau \)

\[
(x|y)_{\alpha,\xi} = \frac{1}{2} b_{\xi}(o,\rho_{c_y},c_x(o)) = \frac{1}{2} b_{\xi}(o,\rho_{c_x},c_y(o)).
\]

**Proof.** In case of a symmetric space let \( N_x \) be the horospherical subgroup of \( P_x = \text{stab}(x) \) and be \( n_x(o,y) \in N_x \) the unique element such that \( n_x(o,y) \cdot o \in P(x,y) \): Extend \( \gamma_{\alpha,\xi} \) bi-infinitely and let \( z \in \text{Flag}_{\sigma,\tau} \) be such that \( \gamma_{\alpha,\xi}(-\infty) \in z \). Then \( n_x(o,y) \in N_x \) is the unique element with \( n_x(o,y)(z) = y \). By construction we have \( n_x(o,y) \cdot o \in P(x,y) \).

We define in the same way \( n_y(o,x) \in N_y \) and set \( \gamma_{xy}(t) := n_x(o,y) \cdot \gamma_{\alpha,\xi}(t) \) and \( \gamma_{xy}(t) := n_x(o,y) \cdot \gamma_{\alpha,\xi}(t) \). Then \( \gamma_{xy}, \gamma_{yx} \) are geodesics in \( P(x,y) \) with the same (un-ordered) end points. Hence they are parallel. Moreover, \( n_x(o,y) \in N_x \) implies that \( d(\gamma_{\alpha,\xi}(t),\gamma_{xy}(t)) \to 0 \) for \( t \to \infty \) and similarly \( d(\gamma_{\alpha,\xi}(t),\gamma_{yx}(t)) \to 0 \).

\(^3\)These are the symmetries that for example the cross ratios in [Lab07] have.
The triangle inequality yields that $(x|y)_{o,ξ} = \lim_{t \to \infty} t - \frac{1}{2}d(γ_{xy}(t), γ_{yx}(t))$. By construction $γ_{xy}, γ_{yx}$ are parallel geodesics; hence by the Flat Strip Theorem (see e.g. [13H99]) the distance $d(γ_{xy}(t), γ_{yx}(t))$ decomposes into a part parallel to the geodesics and the distance of the images of the geodesics, which is a constant and will be denoted by $C$.

The part parallel to the geodesics is $b_{x_ξ}(γ_{yx}(t), γ_{xy}(t))$ - or equally $b_{y_ξ}(γ_{xy}(t), γ_{yx}(t))$. Using that we have geodesics asymptotic to $x_ξ$ we derive that $b_{x_ξ}(γ_{yx}(t), γ_{xy}(t)) = 2t + b_{x_ξ}(γ_{yx}(0), γ_{xy}(0))$. Altogether

$$
(x|y)_{o,ξ} = \lim_{t \to \infty} t - \frac{1}{2}d (γ_{oxξ}(t), γ_{oyξ}(t)) = \lim_{t \to \infty} t - \frac{1}{2}d(γ_{xy}(t), γ_{yx}(t))
$$

$$
= \lim_{t \to \infty} t - \frac{1}{2}(\sqrt{(2t + b_{x_ξ}(γ_{yx}(0), γ_{xy}(0)))^2 + C^2})
$$

$$
= -\frac{1}{2}b_{x_ξ}(γ_{yx}(0), γ_{xy}(0)) = \frac{1}{2}b_{x_ξ}(γ_{yx}(0), γ_{xy}(0)),
$$

while the second to last equality follows using Taylor series at $s = 0$ after substituting $s = t^{-1}$ (see also the calculations in example (3.7)).

In case of a Euclidean building $E$, let $A_o$ be an affine apartment containing $γ_{oxξ}(t)$, let $d_x ∈ \text{Flag}_{σ}$ be such that $d_x ∈ ∂_∞ A_o$ and $x ∈ d_x$. Moreover, be $d_y ∈ \text{Flag}_{σ}$ a chamber opposite to $d_x$ such that $y$ is a face of $d_y$ and let $A_{xy}$ be the unique affine apartment that $d_x$ and $d_y$ define.

Then the affine apartments $A_o$ and $A_{xy}$ have a common subsector. Hence there exists $T_x ≥ 0$ such that for $t ≥ T_x$ the geodesic $γ_{oxξ}(t)$ is parallel to a geodesic $γ_{xy}$ in the subsector - denote the distance of the geodesic rays by $C_x$. Extend $γ_{xy}$ bi-infinite in $A_{xy}$ such that it is in the same horosphere with respect to $x_ξ$ as $γ_{oxξ}(t)$ for all (positive) time. That $γ_{xy}$ is in $A_{xy}$ with one endpoint being $x_ξ$ implies that $γ_{xy}$ joins $x_ξ$ and $y_ξ$ and hence $γ_{xy} ∈ P(x, y)$.

In the same way we construct $γ_{yx} ∈ P(x, y)$ to $γ_{oyξ}$ such that those geodesics are parallel for $t ≥ T_y$ - denote the distance by $C_y$. Since $γ_{xy}, γ_{yx}$ join the same points at infinity, they are parallel - denote the distance by $C_0$. Then the triangle inequality together with the Flat Strip theorem yields for $t ≥ \max\{T_x, T_y\}$ that $d(γ_{oxξ}(2t), γ_{oyξ}(2t))$ is smaller or equal than

$$
d(γ_{oxξ}(2t), γ_{xy}(t)) + d(γ_{xy}(t), γ_{yx}(t)) + d(γ_{yx}(t), γ_{oyξ}(2t))
$$

$$
= \sqrt{t^2 - C_x^2} + \sqrt{b_{x_ξ}(γ_{yx}(0), γ_{xy}(0))^2 + C_y^2} + \sqrt{t^2 - C_y^2}
$$

Since $γ_{xy}$ and $γ_{yx}$ are are asymptotic to $x_ξ$, we derive that $b_{x_ξ}(γ_{yx}(t), γ_{xy}(t))) = 2t + b_{x_ξ}(γ_{yx}(0), γ_{xy}(0))$. Therefore

$$
(x|y)_{o,ξ} ≥ \lim_{t \to \infty} 2t - \frac{1}{2}(\sqrt{t^2 - C_x^2} + \sqrt{(2t + b_{x_ξ}(γ_{yx}(0), γ_{xy}(0)))^2 + C_y^2} + \sqrt{t^2 - C_y^2}).
$$

We substitute $t = \frac{1}{s}$. Then a Taylor expansions for the root expressions at $s = 0$ yields that $(x|y)_{o,ξ} ≥ -\frac{1}{s}b_{x_ξ}(γ_{yx}(0), γ_{xy}(0)) = \frac{1}{s}b_{x_ξ}(γ_{xy}(0), γ_{yx}(0))$. 17
We claim that \( \lim_{t \to \infty} b_{x_t}(\gamma_{xy}(t), \gamma_{xy}(t)) - b_{x_t}(\gamma_{ox}(t), \gamma_{ax}(t)) = 0 \): By construction \( b_{x_t}(\gamma_{xy}(t), \gamma_{xy}(t)) = 0 \). Therefore it is enough to show that \( \lim_{t \to \infty} b_{x_t}(\gamma_{ox}(t), \gamma_{xy}(t)) = 0 \), as Busemann functions satisfy \( b_z(p,q) + b_z(q,o) = b_z(p,o) \).

By construction we have that the geodesic \( \gamma_{xy} \) joins \( x_t \) and \( y_t \). Therefore \( b_{x_t}(\gamma_{ox}(t), \gamma_{xy}(t)) = \lim_{s \to \infty} d(\gamma_{ox}(t), \gamma_{xy}(t-s)) - s \). Moreover, \( d(\gamma_{ox}(t), \gamma_{xy}(t-s)) \leq d(\gamma_{ox}(t), \gamma_{xy}(T_y)) + |t - s - T_y| \).

Applying the Flat Strip Theorem with an according Taylor expansion as before, we derive that \( \lim_{t \to \infty} d(\gamma_{ox}(t), \gamma_{xy}(T_y)) - t \to -T_y \). In particular
\[
\lim_{t \to \infty} b_{x_t}(\gamma_{ox}(t), \gamma_{xy}(t)) \leq \lim_{t \to \infty} (\lim_{s \to \infty} d(\gamma_{ox}(t), \gamma_{xy}(T_y)) - t + s + T_y - s) = 0.
\]

It follows from the definition of Busemann functions that if \( q \in M \) lies on a bi-infinite geodesics joining \( z, w \in \partial M \), then \( b_z(p,q) + b_w(p,q) \geq 0 \). Hence we derive \( b_{x_t}(\gamma_{ox}(t), \gamma_{xy}(t)) + b_{y_t}(\gamma_{ox}(t), \gamma_{xy}(t)) \geq 0 \). By construction \( b_{x_t}(\gamma_{xy}(t), \gamma_{ox}(t)) = 0 \). Thus \( b_{x_t}(\gamma_{ox}(t), \gamma_{xy}(t)) \geq 0 \), which yields the claim.

We have \( d(\gamma_{ox}(t), \gamma_{xy}(t)) \geq b_{x_t}(\gamma_{ox}(t), \gamma_{xy}(t)) \to b_{x_t}(\gamma_{xy}(t), \gamma_{xy}(t)) \), for \( t \to \infty \). Thus
\[
(x|y)_{ox} \leq \lim_{t \to \infty} \frac{1}{2} b_{x_t}(\gamma_{xy}(t), \gamma_{xy}(t)) = \frac{1}{2} b_{x_t}(\gamma_{xy}(0), \gamma_{xy}(0))
\]
Altogether \( (x|y)_{ox} = \frac{1}{2} b_{x_t}(\gamma_{xy}(0), \gamma_{xy}(0)) \).

Consider a symmetric space or a Euclidean building \( M \) and let \( \gamma_{xy}, \gamma_{yz} \) be the accordingly constructed geodesics. Then \( b_{x_t}(\gamma_{xy}(0), \gamma_{xy}(0)) = 0 \) while \( \gamma_{ox}(0) = o \) and also \( b_{y_t}(\gamma_{xy}(0), o) = 0 \). For notational reasons set \( \rho_x := \rho_{xy,ox} \) and \( \rho_y := \rho_{xy,oy} \). Then \( \rho_x(o), \gamma_{xy}(0) \in P(x,y) \). Together with equation \((2.2)\) and Lemma \((2.2)\) this yields
\[
b_{x_t}(\gamma_{xy}(0), \gamma_{xy}(0)) = b_{x_t}(\gamma_{xy}(0), \rho_x(o)) + b_{x_t}(\rho_x(o), \rho_y(o)) + b_{x_t}(\rho_y(o), \gamma_{xy}(0)) \\
= b_{x_t}(o, \rho_y(o)) - b_{y_t}(\rho_y(o), \gamma_{xy}(0)) = b_{x_t}(o, \rho_y(o))
\]
In a similar way it follows also \( b_{x_t}(\gamma_{xy}(0), \gamma_{xy}(0)) = b_{y_t}(o, \rho_x(o)) \). Finally, \( (x|y)_{ox} = \frac{1}{2} b_{x_t}(\gamma_{xy}(0), \gamma_{xy}(0)) \) implies the claim. \( \square \)

**Corollary 3.2.** Let \( (x,y) \in \text{Flag}_r \times \text{Flag}_{r'} \) and \( o \in M \). Then \( (x|y)_{ox} = \infty \iff x \not\sim y \).

**Proof.** Let \( (x,y) \in \text{Flag}_r \times \text{Flag}_{r'} \) be such that \( x \not\sim y \). Let \( A \) be an affine apartment containing \( x, y \) in its boundary. Let \( p \in A \) and \( \gamma_{px}, \gamma_{py} \) be the unit speed geodesics joining \( p \) to \( x_t, y_t \), respectively. A straightforward argument in Euclidean geometry yields that \( d(\gamma_{px}(t), \gamma_{py}(t)) = 2at \) with
Proposition 3.4. The cross ratio is generically defined.

Given \( \gamma_{px}(t) = \gamma_{py}(t) \) and hence \( \alpha < 1 \), i.e. \( (x|y)_p = \infty \).

Now let \( \gamma_{ox}, \gamma_{oy,\xi} \) be the unit speed geodesics joining \( o \) to \( x_\xi, y_\xi \), respectively. Since \( \gamma_{ox} \) and \( \gamma_{px,\xi} \) define the same point in the ideal boundary, we can derive - by the convexity of the distance functions along geodesics in non-positive curvature - that \( d(\gamma_{ox}(t), \gamma_{px}(t)) \leq d(o, p) \) for all \( t \geq 0 \). Thus

\[
(x|y)_{o,\xi} = \lim_{t \to \infty} t - \frac{1}{2}d(\gamma_{ox}(t), \gamma_{oy,\xi}(t)) \geq \lim_{t \to \infty} t - \frac{1}{2}d(\gamma_{px}(t), \gamma_{p\xi,\xi}(t)) - d(o, p) = \infty.
\]

Let \( (x, y) \in \text{Flag}_\tau \times \text{Flag}_{\tau^\ast} \) be such that \( x \vartriangleright y \). Then by the above proposition \( (x|y)_{o,\xi} = \frac{1}{2}b_{x}(o, \rho_{c_{\xi}, c_y}(o)) \leq d(o, \rho_{c_{\xi}, c_y}(o)) \), i.e. \( (x|y)_{o,\xi} < \infty \).

**Remark 3.3.** The above corollary implies that \( A_\xi \) is the maximal domain of definition for \( c_{r,\xi} \). As mentioned, in case of a symmetric space \( X \), the set \( A_\xi \) is an open and dense subset of \( (\text{Flag}_{\tau}(X) \times \text{Flag}_{\tau^\ast}(X))^2 \), i.e. the cross ratio is generically defined.

**Proposition 3.4.** Let \( o, \hat{o} \in M \), \( (x, y) \in \text{Flag}_\tau \times \text{Flag}_{\tau^\ast} \) and \( \xi \in \tau \). Then

\[
(x|y)_{o,\xi} = (x|y)_{\hat{o},\xi} = \frac{1}{2}b_{x}(o, \hat{o}) + \frac{1}{2}b_{y}(o, \hat{o}).
\]

**Proof.** If \( x \vartriangleright y \), then by the above corollary \( (x|y)_{o,\xi} = \infty = (x|y)_{\hat{o},\xi} \).

If \( x \vartriangleright y \), let \( \rho_{x,y}, \rho_{y,x} \) be any horospherical retracts as in Proposition 3.3. Then

\[
b_{x}(o, \rho_{y,x}(o)) = b_{x}(o, \hat{o}) + b_{x}(\hat{o}, \rho_{y,x}(\hat{o})) + b_{x}(\rho_{y,x}(\hat{o}), \rho_{y,x}(o)).
\]

By construction \( \rho_{x,y}(o), \rho_{y,x}(\hat{o}) \in P(x, y) \). Moreover \( x, y \) are opposite and hence by Lemma 2.2 and equation (2.2)

\[
b_{x}(\rho_{y,x}(\hat{o}), \rho_{y,x}(o)) = -b_{y}(\rho_{y,x}(\hat{o}), \rho_{y,x}(o)) = -b_{y}(\hat{o}, o) = b_{y}(o, \hat{o}).
\]

Together with Proposition 3.3, the claim follows.

**Proposition 3.5.** Let \( o, \hat{o} \in M \). Then \( c_{r,\xi}(x_1, y_1, x_2, y_2) = c_{r,\xi}(x_1, y_1, x_2, y_2) \) for all \( (x_1, y_1, x_2, y_2) \in A_\xi \).

**Proof.** Plugging in the above proposition in the definitions of \( c_{r,\xi} \) and \( c_{r,\xi} \) yields directly the result.

**Definition 3.6.** Given \( (x_1, y_1, x_2, y_2) \in A_\xi \), we define the cross ratio with respect to \( \xi \in \sigma \) to be \( c_{r,\xi}(x_1, y_1, x_2, y_2) = c_{r,\xi}(x_1, y_1, x_2, y_2) \) for some \( o \in M \).
Example 3.7. (see also [Kim10]) Consider the symmetric space \( X = \mathbb{H}^2 \times \mathbb{H}^2 \), where \( \mathbb{H}^2 \) is the hyperbolic plane. The ideal boundary \( \partial_{\infty}(\mathbb{H}^2 \times \mathbb{H}^2) \) can be identified with \( S^1 \times S^1 \times [0, \omega] \) - this is in such a way that the unit-speed geodesic ray from a base-point \((o_1, o_2) \in \mathbb{H}^2 \times \mathbb{H}^2 \) to the point in \((x_1, x_2, \alpha) \in S^1 \times S^1 \times [0, \omega] \) \( \in \partial_{\infty}(\mathbb{H}^2 \times \mathbb{H}^2) \) is given by \((\gamma_{o_1x_1}(\cos(\alpha)t), \gamma_{o_2x_2}(\sin(\alpha)t))\).

The types are exactly determined by the angle \( \alpha \) and the opposition involution equals the identity. In particular every type is self opposite.

Fix \( o = (o_1, o_2) \in \mathbb{H}^2 \times \mathbb{H}^2 \) and \( x = (x_1, x_2, \alpha), y = (y_1, y_2, \alpha) \in \partial_{\infty}(\mathbb{H}^2 \times \mathbb{H}^2) \) and set \( \gamma_1 := \gamma_{o_1x_1}, \gamma_1 := \gamma_{o_1y_1}, \gamma_2 := \gamma_{o_2x_2} \) and \( \gamma_2 := \gamma_{o_2y_2} \). Then

\[
(x|y)_{o,\alpha} = \lim_{t \to \infty} t - \frac{1}{2} \sqrt{|\gamma_1(\cos(\alpha)t)\gamma_1(\cos(\alpha)t)|^2 + |\gamma_2(\sin(\alpha)t)\gamma_2(\sin(\alpha)t)|^2}.
\]

Using \( \lim_{t \to \infty} |\gamma_1(\cos(\alpha)t)\gamma_1(\cos(\alpha)t)| - 2 \cos(\alpha)t = -2(x_1|y_1)_{o_1}, \) if \( \alpha \neq \frac{\pi}{2} \)

\[
(x|y)_{o,\alpha} = \lim_{t \to \infty} t - \sqrt{(-x_1|y_1)_{o_1} + \cos(\alpha)t}^2 + (-x_2|y_2)_{o_2} + \sin(\alpha)t)^2
= \lim_{t \to \infty} t - \sqrt{2t(\cos(\alpha)(x_1|y_1)_{o_1} + \sin(\alpha)(x_2|y_2)_{o_2}) + (x_1|y_1)_{o_1} + (x_2|y_2)_{o_2}^2}.
\]

We substitute \( t = \frac{s}{s} \). Then a Taylor expansion for the root expression at \( s = 0 \) yields that

\[
(x|y)_{o,\alpha} = \lim_{s \to 0} \frac{1}{s} (1 - (1 - s(\cos(\alpha)(x_1|y_1)_{o_1} + \sin(\alpha)(x_2|y_2)_{o_2}) + o(s)) - \cos(\alpha)(x_1|y_1)_{o_1} + \sin(\alpha)(x_2|y_2)_{o_2}.
\]

Therefore \( \cr_{\alpha} = \cos(\alpha) \log |\cr_{\partial_{\infty}\mathbb{H}^2}| + \sin(\alpha) \log |\cr_{\partial_{\infty}\mathbb{H}^2}|, \) where \( \cr_{\partial_{\infty}\mathbb{H}^2} \) is the usual multiplicative cross ratio on \( \partial_{\infty}\mathbb{H}^2 \).

Lemma 3.8. Let \( X \) be a symmetric space. Then for every \( o \in X \) the Gromov product \( (\cdot|\cdot)_{o,\xi} : \text{Flag}_{\rightarrow}(X) \times \text{Flag}_{\rightarrow}(X) \to [0, \infty] \) is continuous. In particular also \( \cr_{\xi} \) is continuous.

Proof. Since \( \text{Flag}_{\rightarrow}(X) \), \( \text{Flag}_{\rightarrow}(X) \) are manifolds it is enough to consider sequential continuity. Therefore let \( (x, y) \in \text{Flag}_{\rightarrow}(X) \times \text{Flag}_{\rightarrow}(X) \) and let \( x_i \to x \) and \( y_i \to y \).

If \( x \not\equiv y \), we have \( (x|y)_{o,\xi} = \infty \). We set \( (x|y)_{o,\xi}(t) := (\gamma_{o\alpha}(t)\gamma_{o\alpha}(t))_o \) with Gromov product on the right hand side the usual Gromov product on the metric space \( (X, d) \). As \( X \) is non-positively curved, the function \( t \mapsto (x|y)_{o,\xi}(t) \) is monotone increasing. Let \( C > 0 \) be given. Then there is \( t_C \in \mathbb{R}^+ \) such that \( (x|y)_{o,\xi}(t_C) \geq C + 2 \). Since the topology on \( \text{Flag}_{\rightarrow}(X) \) is induced by the cone topology, we have that \( (x_i)_{\xi} \to x_{\xi} \) in the cone topology and similarly for \( y_i \) and \( y \). Hence we find \( L \in \mathbb{N} \) such that \( d(\gamma_{o\alpha}(t_C), \gamma_{o\alpha}(t_C)) < 1 \) and \( d(\gamma_{o\alpha}(y_i)_{\xi}(t_C), \gamma_{o\alpha}(y_{\xi})(t_C)) < 1 \) for all \( i \geq L \). Hence by the triangle inequality \( (x_i|y_j)_{o,\xi}(t_C) > (x|y)_{o,\xi}(t_C) - 2 > C \) for all \( i, j \geq L \). As \( C \) was arbitrary, this yields \( \lim_{i,j \to \infty} (x_i|y_j)_{o,\xi} = \infty \) - which proves continuity for \( x \not\equiv y \).
Assume \( x \circ y \). Let \( K = \text{stab}_G(o) \). We know that \( K \) acts transitively on \( \text{Flag}_r(X) \) and we have a \( K \)-equivariant and homeomorphic identification \( \text{Flag}_r(X) \cong K/K_x \). Therefore \( x_i \rightarrow x \) implies that we find \( k_i \in K \) such that \( k_ix_i = x \) and \( k_i \rightarrow e \in G \). Now, \( x \circ y \) and opposition being an open condition, together with \( y_i \rightarrow y \) and \( k_i \rightarrow e \), imply that there exists \( L \in \mathbb{N} \) such that \( k_iy_j \circ x \) for all \( i, j \geq L \). Thus there exists a unique \( n_{ij} \in \mathbb{N} \) such that \( n_{ij}x_iy_j = y \) for \( i, j \geq L \). From \( k_i \rightarrow e \) and \( y_j \rightarrow y \) it follows \( n_{ij} \rightarrow e \in G \) for \( i, j \rightarrow \infty \). We set \( g_{ij} := n_{ij}k_i \) and by construction \( g_{ij} \circ e = g_{ij}x_i = x \). Hence \( (x_i|y_j)_{o,\xi} = (x|y)_{o,\xi} \). Proposition 3.11 and \( g_{ij} \rightarrow e \) yield that \( (x_i|y_j)_{o,\xi} \rightarrow (x|y)_{o,\xi} \).

**Lemma 3.9.** Let \( (x, y) \in \text{Flag}_r \times \text{Flag}_{r,\tau} \) and \( x \circ y \). Moreover, let \( \xi_i \in \tau \) be a sequence with \( \xi_i \rightarrow \xi_0 \in \tau \). Then \( (x|y)_{o,\xi_i} \rightarrow (x|y)_{o,\xi_0} \). In particular, \( cr_{\xi_i}(x, y, z, w) \rightarrow cr_{\xi_0}(x, y, z, w) \) for all \( (x, y, z, w) \in A^2_\tau \).

**Proof.** Let \( c_x, c_y \in \text{Flag}_r \) such that \( c_x \circ c_y \) is a face of \( c_x \) and \( y \) is a face of \( c_y \). Then Proposition 3.11 and equation (2.2) imply \((x|y)_{o,\xi} = \frac{1}{2}b_{x_i}(\rho_{c_x,c_y}(o),\rho_{c_y,c_x}(o)) \) for all \( \xi \in \tau \). Denote \( p_x := \rho_{c_x,c_y}(o) \), \( p_y := \rho_{c_y,c_x}(o) \) and by \( A_{xy} \) the unique affine apartment with \( c_x, c_y \in \partial_\infty A_{xy} \).

Every affine apartment can be isometrically identified with \( \mathbb{R}^r \) where \( r \) is the rank of \( M \). We identify \( A_{xy} \) with \( \mathbb{R}^r \) such that \( 0 \sim p_x \). Let \( v_\xi \in A_{xy} \rightarrow \mathbb{R}^r \) be of norm one and such that the line from \( 0 \) through \( v_\xi \) is the geodesic ray in \( A_{xy} \) from \( p_x \) to \( x_\xi \). Then Euclidean geometry yields that \( b_{x_\xi}(p_x, p_y) = \langle v_\xi, p_y \rangle \). In particular, we get

\[(x|y)_{o,\xi_i} = \frac{1}{2} \langle v_\xi, p_y \rangle.
\]

Moreover \( \xi_i \rightarrow \xi_0 \) implies that \( v_\xi_i \rightarrow v_\xi_0 \) and hence the claim follows.

**Remark 3.10.** The assumption of opposition in the above lemma is needed, since there are \( (x, y) \in \text{Flag}_r \times \text{Flag}_{r,\tau} \) with \( x \circ y \) but there are faces \( x_0 \) of \( x \) and \( y_0 \) of \( y \) with \( x_0 \circ y_0 \). Then if \( \xi \in \text{int}(\tau) \) converge to \( \xi_0 \) such that \( \xi_0 \in \text{int}(\tau_0) \) and \( \tau_0 \) is the type of \( x_0 \), we get \( (x|y)_{o,\xi_i} \rightarrow (x_0|y_0)_{o,\xi_0} \) (as the latter is finite).

We remind that any isometry \( F : M_1 \rightarrow M_2 \) induces a building isomorphism \( F_\infty : \Delta_\infty M_1 \rightarrow \Delta_\infty M_2 \) together with a type map \( F_\sigma : \sigma_1 \rightarrow \sigma_2 \) with the property that \( F(\text{Flag}_r(M_1)) = \text{Flag}_{F_r(\tau)}(M_2) \).

**Proposition 3.11.** Let \( F : M_1 \rightarrow M_2 \) be an isometry between either symmetric spaces or thick Euclidean buildings, \( F_\infty : \Delta_\infty M_1 \rightarrow \Delta_\infty M_2 \) the induced building isomorphism and \( \xi \in \sigma_1 \). Then

\[cr_{\xi_1}(x_1, y_1, x_2, y_2) = cr_{F_\sigma(\xi_1)}(F_\infty(x_1), F_\infty(y_1), F_\infty(x_2), F_\infty(y_2))\]

for all \( (x_1, y_1, x_2, y_2) \in A_{\xi_1} \). Equivalently, \( cr_{\xi_1} = F_\infty^{*}cr_{F_\sigma(\xi_1)} \) with \( F_\infty^{*} \) denoting the pullback under \( F_\infty \).
Proof. Let \( \xi_1 \in \text{int}(\tau) \) and \((x, y) \in \text{Flag}_\sigma(M_1) \times \text{Flag}_{\sigma^*}(M_1) \). Since the Gromov product \( (\cdot)_\sigma,\xi_1 \) is defined in terms of a limit of distances involving unit speed geodesics and isometries leave those invariant, it follows that \( (x | y)_{\sigma,\xi_1} = (F_\infty(x) | F_\infty(y))_{F_\sigma(o),F_\sigma(\xi_1)} \). Hence Corollary 3.1 implies that if \((x_1, y_1, x_2, y_2) \in \mathcal{A}_{\xi_1} \), then \( (F_\infty(x_1), F_\infty(y_1), F_\infty(x_2), F_\infty(y_2)) \in \mathcal{A}_{F_\sigma(\xi_1)} \). Finally, \( \text{cr}_{\xi_1} = \text{cr}_{\sigma,\xi_1} = F^*_\sigma \text{cr}_{F(o),F_\sigma(\xi_1)} = F^*_\sigma \text{cr}_{F_\sigma(\xi_1)} \) by Proposition 3.3. \( \square \)

**Corollary 3.12.** Let \( g \in \text{Iso}(M) \) and \( \xi_0 \) be the center of gravity of \( \sigma \) with respect to the angular metric. Then \( \text{cr}_{\xi_0} = g^* \text{cr}_{\xi_0} \). In case of a symmetric space \( X \) and \( g \in G \) we have \( \text{cr}_{\xi,X} = g^* \text{cr}_{\xi,X} \) for all \( \xi \in \sigma \).

**Proof.** For the center of gravity \( \xi_0 \in \sigma \) we have \( g_\sigma(\xi_0) = \xi_0 \) for all \( g \in \text{Iso}(M) \), as \( g_\sigma: \sigma \to \sigma \) is an isometry with respect to the angular metric. Then the first claim follows. In case of a symmetric space \( g \in G \), we know \( g_\sigma = id_\sigma \), which implies the second claim. \( \square \)

**Example 3.13.** We want to determine the Gromov products and cross ratios of the symmetric spaces \( X(n) := SL(n, \mathbb{R})/SO(n, \mathbb{R}) \). For a deeper description of the symmetric space \( X(n) \) see [Ebe96].

The ideal boundary \( \partial_\infty X(n) \) can be identified with eigenvalue flag pairs \((\lambda, F)\), where \( F = (V_1, \ldots, V_l) \) is a flag in \( \mathbb{R}^n \), i.e. the \( V_i \) are subspaces of \( \mathbb{R}^n \) with \( V_i \not\subseteq V_{i+1} \), \( V_l = \mathbb{R}^n \), and \( \lambda = (\lambda_1, \ldots, \lambda_l) \in \mathbb{R}^l \) such that \( \lambda_i > \lambda_{i+1} \), \( \sum_{i=1}^l m_i \lambda_i = 0 \) for \( m_i = \dim V_i - \dim V_{i-1} \) and \( \sum_{i=1}^l m_i \lambda_i^2 = 1 \). In particular, \( 2 \leq l \leq n \).

The action of \( g \in SL(n, \mathbb{R}) \) on an eigenvalue flag pair is given by \( g \cdot (\lambda, F) = (\lambda, g \cdot F) \), where \( g \cdot (V_1, \ldots, V_l) = (g \cdot V_1, \ldots, g \cdot V_l) \) and \( F = (V_1, \ldots, V_l) \).

The “eigenvalues” \( \lambda \) in the eigenvalue flag pairs \((\lambda, F)\) determine the type of any point in the ideal boundary. A point in the boundary is contained in the interior of a chamber if and only if \( l = n \) which means \( \lambda \) consists of \( n \) different ”eigenvalues”; equivalently \( m_i = \dim V_i - \dim V_{i-1} = 1 \) for all \( i \). The action of the opposition involution is given by \( i(\lambda_1, \ldots, \lambda_l) = (-\lambda_1, \ldots, -\lambda_l) \).

The space of flags \( F = (V_1, \ldots, V_n) \), i.e. \( m_i = \dim V_i - \dim V_{i-1} = 1 \), is called the space of full flags and equals \( \text{Flag}_\sigma(X(n)) \).

Let \( V = (V_1, \ldots, V_l), Y = (Y_1, \ldots, Y_l) \in \text{Flag}_\sigma \) and \( W = (W_1, \ldots, W_l), Z = (Z_1, \ldots, Z_l) \in \text{Flag}_{\sigma^*} \). Let \( i_j = \dim V_j \). Then fix a basis \((v_1, \ldots, v_n)\) such that \( V_j = \text{span}\{v_1, \ldots, v_{i_j}\} \). In the same way we fix basis \((w_1, \ldots, w_n)\) and \((z_1, \ldots, z_n)\) for \( W, Y, Z \), respectively. Additionally, fix an identification \( \wedge^n \mathbb{R}^n \cong \mathbb{R} \). We set \( V_j \wedge W_{l-j} := v_1 \wedge \ldots \wedge v_{i_j} \wedge w_1 \wedge \ldots \wedge w_{n-i_j} \) - this is such that \( W_{l-j} = \text{span}\{w_1, \ldots, w_{n-i_j}\} \) - and in the same way for the other flags. Then the term

\[
\frac{V_j \wedge W_{l-j} \cdot Y_j \wedge Z_{l-j}}{V_j \wedge Z_{l-j} \cdot Y_j \wedge W_{l-j}}
\]

can be shown to be independent of all choices for all \( j = 1, \ldots, l-1 \) - compare e.g. [MZT7].
Let \( V, W, Y, Z \) be as before and \( \lambda = (\lambda_1, \ldots, \lambda_l) \) a type with \( \lambda \in \text{int}(\tau) \). Then

\[
\text{cr}_\lambda(V, W, Y, Z) = \prod_{j=1}^{l-1} (\lambda_j - \lambda_{j+1}) \log \left( \frac{V \wedge W \wedge Y \wedge Z}{V_j \wedge Z_{l-j}} \right),
\]

using the above conventions - see the appendix for a proof. We remark that some specific of those cross ratios are known already and have been used for analyzing Hitchin representations (see e.g. [Lab07], [MZ1 7]).

Let \( M = M_1 \times \ldots \times M_k \) be a product of either symmetric spaces or Euclidean buildings. Then the building at infinity \( \Delta_\infty M \) is the spherical join of the buildings \( \Delta_\infty M_i \) [KL97, Sc.4.3]. In particular, the Weyl chamber \( \sigma \) decomposes as a spherical join \( \sigma = \sigma_1 \circ \ldots \circ \sigma_k \). Hence we get a surjective map

\[
\pi : \sigma_1 \times \ldots \times \sigma_k \times S^+_k \rightarrow \sigma,
\]

where \( S^+_k := \{ \mu = (\mu_1, \ldots, \mu_k) \in [0,1]^k \mid \sum_{i=1}^k \mu_i^2 = 1 \} \). We remark that \( \pi \) is in general not injective, since it is independent of the exact choice of the type \( \xi \in \sigma_i \) if \( \mu_i = 0 \).

Let \( \xi = \pi(\xi_1, \ldots, \xi_k, \mu) \) with \( \mu = (\mu_1, \ldots, \mu_k) \in S^+_k \) and let \( x = (x_1, \ldots, x_k) \in \text{Flag}_\tau(M) \cong \text{Flag}_{\tau_1}(M_1) \times \ldots \times \text{Flag}_{\tau_k}(M_k) \) such that \( \xi \in \text{int}(\tau) \) and \( \xi_i \in \text{int}(\tau_i) \). For simplicity we assume \( \mu_i \neq 0 \) for all \( 1 \leq i \leq k \) - if some \( \mu_i = 0 \) essentially the same formula holds, but the factor \( \text{Flag}_{\tau_i}(M_i) \) is not apparent in the decomposition of \( \text{Flag}_\tau(M) \).

We remark that the unit-speed geodesic from some point \( (o_1, \ldots, o_k) \in M \) to \( x \) is of the form \( (\gamma_{o_1 x_{\xi_1}}(\mu_1 t), \ldots, \gamma_{o_k x_{\xi_k}}(\mu_k t)) \), where \( \gamma_{o_i x_{\xi_i}} \) denote the unit speed geodesics in the factors \( M_i \) joining \( o_i \) to \( (x_i)_{\xi_i} \) - cp. also Example 3.7.

Let \( y = (y_1, \ldots, y_k) \in \text{Flag}_{\tau\tau}(M) \cong \text{Flag}_{\tau_1}(M_1) \times \ldots \times \text{Flag}_{\tau_k}(M_k) \) and be \( x \) and \( \xi \) as above. Then similar calculations as in Example 3.7 yield that

\[
(x | y)_{(o_1, \ldots, o_k), \pi(\xi_1, \ldots, \xi_k, \mu)} = \mu_1(x_1 | y_1)_{o_1, \xi_1} + \ldots + \mu_k(x_k | y_k)_{o_k, \xi_k}.
\]

**Proposition 3.14.** Notations as before. Moreover, \( z \in \text{Flag}_\tau(M) \), \( w \in \text{Flag}_{\tau\tau}(M) \). Then

\[
\text{cr}_\pi(\xi_1, \ldots, \xi_k, \mu)(x, y, z, w) = \mu_1 \text{cr}_{\xi_1}(x_1, y_1, z_1, w_1) + \ldots + \mu_k \text{cr}_{\xi_k}(x_k, y_k, z_k, w_k)
\]

for \( (x, y, z, w) \in A_\pi(\xi_1, \ldots, \xi_k, \mu) \).

\( ^4 \)Actually we would have a spherical join instead of the product. However, we can naturally identify a simplex in a join with the product of the simplices in the different factors - and that is what we do here for simplicity.
Let $X$ be a reducible symmetric space of non-compact type and $\Phi : X \to X_1 \times \cdots \times X_k$ the isometry coming from the de Rham decomposition. We recall that every isometry induces an isometry with respect to the angular metric of types, i.e. we get from the isometry $\Phi$ the isometry $\Phi_{\sigma} : \sigma \to \sigma_1 \circ \cdots \circ \sigma_k$.

Moreover, $\Phi$ induces a building isomorphism $\Phi_{\infty} : \Delta_{\infty} X \to \Delta_{\infty} (X_1 \times \cdots \times X_k)$ and hence we get a map $\Phi_{\tau} : \text{Flag}_\tau (X) \to \text{Flag}_{\tau_1} (X_1) \times \cdots \times \text{Flag}_{\tau_k} (X_k)$, where $\Phi_{\sigma}(\tau) = \tau_1 \circ \cdots \circ \tau_k$.

**Corollary 3.15.** Let $X$ be a reducible symmetric space of non-compact type and $\Phi : X \to X_1 \times \cdots \times X_k$ the isometry coming from the de Rham decomposition and $\Phi_{\tau}, \Phi_{\sigma}$ be the maps induced by $\Phi$ as above. Then for $\xi \in \tau$ we have

$$\text{cr}_\xi = (\Phi_{\tau} \times \Phi_{\tau})^* (\mu_1 \text{cr}_{\xi_1} + \cdots + \mu_k \text{cr}_{\xi_k})$$

for $(\xi_1, \ldots, \xi_k, \mu) \in \sigma_1 \times \cdots \times \sigma_k \times S_k^+$ and $\mu = (\mu_1, \ldots, \mu_k)$ with $\pi(\xi_1, \ldots, \xi_k, \mu) = \Phi_{\sigma}(\xi)$ - for the map $\pi$ see equation (3.4).

**Proof.** Since the Gromov product was defined in terms of distances and $\Phi$ is an isometry, it is straightforward to show that for $\xi \in \tau$, $(x, y) \in \text{Flag}_\tau (X) \times \text{Flag}_{\tau} (X)$ and $o \in M$ we have that

$$(x|y)_{o, \xi} = (\Phi_{\tau}(x)|\Phi_{\tau}(y))_{\Phi(o), \Phi_{\sigma}(\xi)}.$$ 

Hence $\text{cr}_\xi = (\Phi_{\tau} \times \Phi_{\tau})^* \text{cr}_{\Phi_{\sigma}(\xi)}$ and therefore the claim follows from the proposition above. \qed

## 4 Vector valued cross ratios

So far, we have constructed families of cross ratios on subsets of the spaces $(\text{Flag}_\tau \times \text{Flag}_{\tau})^2$ which are parametrized by $\xi \in \text{int}(\tau)$. In this section we show that such a family gives rise to a single vector valued cross ratio containing all the information of the family. The vector valued cross ratio has the same symmetries as the usual cross ratios (cp. equations (3.1)) justifying the name cross ratio.

We remind that $\sigma = \mathfrak{a}_1^\perp$; hence every type can be viewed as vector in $\mathfrak{a}$ of norm one.

**Lemma 4.1.** Let $\tau$ be a face of $\sigma$ and $\xi_0, \xi_1, \ldots, \xi_j \in \tau$ such that there exist $a_i \in \mathbb{R}$ with $\xi_0 = \sum_{i=1}^j a_i \xi_i$. Then for $(x, y) \in \text{Flag}_\tau \times \text{Flag}_{\tau}$ with $x \text{ op } y$ we have

$$(x|y)_{o, \xi_0} = \sum_{i=1}^j a_i (x|y)_{o, \xi_i}.$$ 

In particular it follows that $\text{cr}_{\xi_0}(x, y, z, w) = \sum_{i=1}^j a_i \text{cr}_{\xi_i}(x, y, z, w)$ for all $(x, y, z, w) \in \mathcal{A}_\tau^\text{op}$.
Proof. Let $c_x, c_y \in \text{Flag}_\sigma$ such that $c_x \text{ op } c_y$, $x$ is a face of $c_x$ and $y$ is a face of $c_y$. We recall the notation of the proof of Lemma 3.9. We denote $p_x := \rho_{c_x, c_y}(o)$, $p_y := \rho_{c_y, c_x}(o)$ and by $A_{xy}$ the unique apartment with $c_x, c_y \subset \partial_\infty A_{xy}$. Moreover, let $A_{xy} \cong \mathbb{R}^r$ such that $p_x \simeq 0$, in particular $A_{xy}$ inherits a inner product. Let $v_\xi \in A_{xy} \cong \mathbb{R}^r$ be of norm one and such that the line from $p_x = 0$ through $v_\xi$ is the geodesic ray in $A_{xy}$ from $p_x$ to $x_\xi$. Then we know from equation (3.3) that $(x|y)_{o, \xi} = \frac{1}{2}(v_\xi, p_y)$.

By the definition of the $v_\xi$, it is immediate that $v_{\xi_0} = \sum_{i=1}^r a_i v_{\xi_i}$, where we have the addition inherited to $A_{xy}$ under the identification with $\mathbb{R}^r$ such that $p_x \simeq 0$. Hence

$$(x|y)_{o, \xi_0} = \frac{1}{2}(v_{\xi_0}, p_y) = \sum_{i=1}^r \frac{1}{2} a_i(v_{\xi_i}, p_y) = \sum_{i=1}^r a_i(x|y)_{o, \xi_i}.$$ 

\[\square\]

Let $\xi_1, \ldots, \xi_r$ be the corners of $\sigma = a_+^r$. Then every subset $J \subset \{1, \ldots, r\}$ defines a simplex in $\sigma$, i.e. a face $\tau$ of $\sigma$. In the same way every simplex $\tau \subset \sigma$ gives a subset $J_\tau \subset \{1, \ldots, r\}$.

Given a simplex $\tau$ we recall that $a_\tau = \text{span}_{j \in J_\tau} \xi_j \subset a$. Moreover, we define $\alpha_j^\tau \in a_\tau$ for $j \in J_\tau$ by $(\alpha_j^\tau, \xi_i) = \delta_{ij}$ for all $i \in J_\tau$ - the gives well defined vectors, as the $\xi_i$ with $i \in J_\tau$ form a basis of $a_\tau$. We remind that $a$ was naturally equipped with an inner product.

The $\xi_j$ correspond to normalized fundamental weights of the root system and the $\alpha_j^\tau$ to possibly rescaled roots.

**Definition 4.2.** Let $\tau$ be a face of $\sigma$ and $J_\tau$, $\alpha_j^\tau$ as above. Then we define a (vector valued) cross ratio $cr_\tau : A_\tau \rightarrow a_\tau \cup \{\pm \infty\}$ by

$$cr_\tau(x, y, z, w) := \sum_{i \in J_\tau} cr_{\xi_i}(x, y, z, w)\alpha_i^\tau.$$ 

Here we set $cr_\tau(x, y, z, w) := -\infty$ if $x \not\equiv y$ or $z \not\equiv w$ and $cr_\tau(x, y, z, w) := \infty$ if $x \equiv y$ or $z \equiv w$.

It is straightforward to see that $cr_\tau$ has the same symmetries as in equations (3.1), where the addition is now in the vector space $a_\tau$.

The vector valued cross ratio contains the full information of the collection of cross ratios form the previous section:

**Lemma 4.3.** Let $\xi \in \text{int}(\tau)$. If $(x, y, z, w) \in A_\tau^\mathbb{P}$, then

$$(cr_\tau(x, y, z, w), \xi) = cr_\xi(x, y, z, w).$$

If $(x, y, z, w) \in A_\tau \backslash A_\tau^\mathbb{P}$, then $cr_\tau(x, y, z, w) = \pm \infty = cr_\xi(x, y, z, w).$
Proof. If \((x, y, z, w) \in \mathcal{A}_r \setminus \mathcal{A}_r^{\Omega}\), then the equality is immediate. Hence assume \((x, y, z, w) \in \mathcal{A}_r^{\Omega}\). Then

\[
\langle \delta, \xi \rangle = \sum_{i \in J_r} \mathrm{cr}_\tau(x, y, z, w)(\alpha_i^r, \xi).
\]

Since \(\langle \alpha_i^r, \xi \rangle = \delta_{ij}\) for all \(i \in J_r\), we derive that \(\langle \sum_{i \in J_r} \alpha_i^r, \xi \rangle \langle \alpha_i^r, \alpha_j^r \rangle = \langle \xi, \alpha_j^r \rangle\) for all \(j \in J_r\). Moreover, it is immediate that the \(\alpha_j^r\) form a base of \(\mathfrak{a}_r\). Thus we get that \(\sum_{i \in J_r} \langle \alpha_i^r, \xi \rangle \xi_i = \xi\). Therefore Lemma 4.1 implies \(\sum_{i \in J_r} \langle \alpha_i^r, \xi \rangle \mathrm{cr}_\tau(x, y, z, w) = \mathrm{cr}_\xi(x, y, z, w)\).

We remark that the above lemma also holds for \(\xi \in \partial \tau\) as long as \((x, y, z, w) \in \mathcal{A}_r^{\Omega}\), but does not hold for general \((x, y, z, w) \in \mathcal{A}_r\) - in this case \(\mathrm{cr}_\tau(x, y, z, w)\) might be finite while \(\mathrm{cr}_\tau(x, y, z, w)\) is not (compare Remark 3.10).

The following corollary captures the topological properties of \(\mathrm{cr}_\tau\) in case of symmetric spaces. It is an immediate consequence of the lemma above and Lemma 3.8.

**Corollary 4.4.** Let \(X\) be a symmetric space. The map \(\mathrm{cr}_\tau\) restricted to \(\mathcal{A}_r^{\Omega}\) is continuous and for all \(\xi \in \text{int}(\tau)\) the map \((\mathrm{cr}_\tau(\cdot), \xi) : \mathcal{A}_r \to \mathbb{R} \cup \{\pm \infty\}\) is continuous.

**Lemma 4.5.** Let \(\pi_\tau : \mathfrak{a} \to \mathfrak{a}_r\) be the orthogonal projection and let \((x, y, z, w) \in \mathcal{A}_r^{\Omega}\). Then \(\mathrm{cr}_\tau(x, y, z, w) = \pi_\tau(\mathrm{cr}_\sigma(x, y, z, w))\).

**Proof.** We claim that \(\alpha_i^\tau = \pi_\tau(\alpha_i^\sigma)\) for all \(i \in J_r\): We can decompose \(\alpha_i^\tau = \pi_\tau(\alpha_i^\sigma) + \alpha_i^\tau\) with \(\alpha_i^\tau\) orthogonal to \(\mathfrak{a}_r\). Now, let \(i, j \in J_r\) and \(i \neq j\). Then

\[
0 = \left< \xi_i, \alpha_j^\sigma \right> = \left< \xi_i, \pi_\tau(\alpha_j^\sigma) \right> + \left< \xi_i, \alpha_j^\tau \right>.
\]

By definition of \(\alpha_j^\tau\) it follows \(\left< \xi_i, \alpha_j^\tau \right> = 0\) and hence \(\left< \xi_i, \pi_\tau(\alpha_j^\sigma) \right> = 0\). In the same way it follows for \(i \in J_r\) that

\[
1 = \left< \xi_i, \alpha_j^\sigma \right> = \left< \xi_i, \pi_\tau(\alpha_j^\sigma) \right> + \left< \xi_i, \alpha_j^\tau \right> = \left< \xi_i, \pi_\tau(\alpha_j^\sigma) \right>.
\]

Hence \(\left< \xi_i, \pi_\tau(\alpha_j^\sigma) \right> = \delta_{ij}\) for all \(i, j \in J_r\). This yields \(\alpha_i^\tau = \pi_\tau(\alpha_i^\sigma)\).

We know that \(\langle \alpha_j^\sigma, \xi_i \rangle = 0\) for all \(j \in \{1, \ldots, r\} \setminus J_r\) and \(i \in J_r\), i.e. all those \(\alpha_j^\sigma\) are orthogonal to \(\mathfrak{a}_r\). Thus \(\pi_\tau(\alpha_j^\tau) = 0\). Altogether we get

\[
\pi_\tau(\mathrm{cr}_\sigma(x, y, z, w)) = \sum_{i=1}^r \mathrm{cr}_\xi(x, y, z, w) \pi_\tau(\alpha_i^\sigma) = \sum_{i \in J_r} \mathrm{cr}_\xi(x, y, z, w) \alpha_i^r,
\]

where the right hand side equals \(\mathrm{cr}_\tau(x, y, z, w)\) by definition.

**Translation vectors and periods**

We assume for this section that \(\tau\) is self-opposite, i.e. \(\tau = \iota \tau\). Moreover denote by \(\text{Iso}_c(M)\) the subgroup of \(\text{Iso}(M)\) such that \(g_\sigma = \text{id}\) for all \(g \in \text{Iso}_c(M)\) - in particular \(G = \text{Iso}_c(X)\) for a symmetric space \(X\). Let \(g \in \text{Iso}_c(M)\)
Iso\(_e(M)\) such that \(g\) stabilizes two points \(g^+ \in \text{Flag}_\tau\) with \(g^- \circ \text{op} g^+\). Since \(g\) is an isometry, it maps every geodesic connecting points of the interior of \(g^-\) and \(g^+\) to another geodesic connecting the same points. In particular \(g\) stabilizes \(P(g^-, g^+)\) set-wise.

In the preliminaries we have seen that \(P(g^-, g^+)\) splits as a product \(a_\tau \times CS(g^-, g^+)\) such that \(g^\pm\) are identified with the positive and negative, respectively; maximal dimensional simplices in \(a_\tau\), i.e. \(g^+ \simeq \partial_\infty a^+\) where \(a^+_\tau := a_\tau \cap a^+\). Therefore, \(g\) being an isometry of \(a_\tau\) stabilizing each boundary point, yields that \(g\) acts as a translation on \(a_\tau\). Let \(\ell^\pm_g\) denote the translation vector.

**Proposition 4.6.** Let \(g \in Iso_e(M)\) such that \(g^+ \in \text{Flag}_\tau\) with \(g^- \circ \text{op} g^+\) are stabilized by \(g\). Let \(\ell^\pm_g\) denote the translation vector along the first factor of \(P(g^-, g^+) \simeq a_\tau \times CS(g^-, g^+)\). Then \(\text{cr}_\tau(g^-, g \cdot x, g^+, x) = \frac{1}{2}(\ell^+_g + \ell^-_g)\), for any \(x \in \text{Flag}_\tau\) with \(x \circ \text{op} g^+\).

**Proof.** We remark that \(\text{cr}_\tau(g^-, g \cdot x, g^+, x)\) is independent of the choice of \(x \circ \text{op} g^+\). This follows from the symmetries of \(\text{cr}_\tau\) together with Proposition 3.11. Therefore, we fix one \(x \in \text{Flag}_\tau\) with \(x \circ \text{op} g^+\).

Let \(o \in P(g^-, g^+)\) and \(\xi_i\) with \(i \in J_\tau\) the corners of \(\tau\). By assumption \(x \circ \text{op} g^+\) and hence \(g \cdot x \circ \text{op} g^+\). Then Proposition 3.11 yields

\[
(g^+|x)_{o,\xi_i} = (g^+|x)_{o,\xi_i} = (g^+|x)_{o,\xi_i} + \frac{1}{2} b_{\xi_i}^g (g^+ \cdot o, o) + \frac{1}{2} b_{\xi_i}^g (g^+ \cdot o, o).
\]

Moreover, we have \(b_{\xi_i}^g (g^+ \cdot o, o) = b_{\xi_i}^g (o, g \cdot o)\). If we plug this in the definition of \(\text{cr}_{\xi_i}\), several terms cancel and we get \(\text{cr}_{\xi_i}(g^-, g \cdot x, g^+, x) = \frac{1}{2} b_{\xi_i}^g (o, g \cdot o) - \frac{1}{2} b_{\xi_i}^g (o, g \cdot o)\). Since \(o, g \cdot o \in P(g^-, g^+)\) and \(g_{\xi_i}^+ \in g^+\) is the point opposite to \(g_{\xi_i}^- \in g^-\), Lemma 2.2 implies \(b_{\xi_i}^- (o, g \cdot o) = -b_{\xi_i}^+ (o, g \cdot o)\). In particular \(\text{cr}_{\xi_i}(g^-, g \cdot x, g^+, x) = \frac{1}{2} b_{\xi_i}^g (o, g \cdot o) + \frac{1}{2} b_{\xi_i}^g (o, g \cdot o)\).

Since \(o\) was arbitrary in \(P(g^-, g^+)\) we can assume that its first coordinate under the identification \(P(g^-, g^+) \simeq a_\tau \times CS(g^-, g^+)\) is \(0 \in a_\tau\). Moreover, we can use Lemma 2.2 to see that only the first factor matters for the Busemann functions \(b_{\xi_i}^g, b_{\xi_i}^g\). As \(g\) acts as a translation on \(a_\tau\), we have that \(g \cdot 0 = \ell^+_g\). Hence \(b_{\xi_i}^g (o, g \cdot o) = (\xi_i, \ell^+_g)\) (cp. the argumentation around equation (3.3)).

By assumption \(\tau = \iota \tau\), hence \(\iota\) restricts to an isometry \(\iota : a_\tau \rightarrow a_\tau\). Together with \(\iota^2 = id\), this yields \((\iota \xi_i, \ell^+_g) = (\xi_i, \iota \ell^+_g)\). Altogether we derive

\[
\text{cr}_\tau(g^-, g \cdot x, g^+, x) = \sum_{i \in J_\tau} \frac{1}{2} ((\xi_i, \ell^+_g) + (\xi_i, \ell^+_g)) \xi_i^7.
\]

It is an immediate consequence that \((\iota \text{cr}_\tau(g^-, g \cdot x, g^+, x), \xi_i) = \frac{1}{2} ((\xi_i, \ell^+_g) + (\xi_i, \iota \ell^+_g))\) for all \(i \in J_\tau\). Since the \(\xi_i\) with \(i \in J_\tau\) form a basis of \(\tau\), it follows that \(\text{cr}_\tau(g^-, g \cdot x, g^+, x) = \frac{1}{2}(\ell^+_g + \iota \ell^+_g)\).
Let $g \in Iso_c(M)$ be as before. Then the term $cr_{\tau}(g^-, g \cdot x, g^*, x)$ is also called period - in analogy to rank one spaces. In particular, the periods give rise to the translation vector of the first factor of the parallel set if $\iota = \text{id}$.

**Geometric interpretation of the cross ratio**

In this section we give an explicit geometric interpretation of the vector valued cross ratio $cr_{\tau}$.

Let $x, z \in \text{Flag}_\tau$ and $y, w \in \text{Flag}_\tau$ with $x, z \parallel y, w$. Pick $c_x, c_z, d_y, d_w \in \text{Flag}_\sigma$ such that $x$ is a face of $c_x$ and accordingly the other chambers and that $c_x, c_z \parallel d_y, d_w$. Then we use the following notations for the horospherical retracts $\rho_x := \rho_{c_x, d_y}, \rho_w := \rho_{d_w, c_z}, \rho_z := \rho_{c_z, d_y}$ and $\rho_y := \rho_{d_y, c_z}$.

**Lemma 4.7.** Let $(x, y, z, w) \in A_{2p}$ and let $\rho_x, \rho_w, \rho_z$ and $\rho_y$ as above. Moreover, be $o$ in the unique affine apartment joining $c_x$ and $d_y$. Then for all $i \in J_\tau$

$$2cr_{\xi_i}(x, y, z, w) = b_{\xi_i}(o, \rho_x \rho_w \rho_z \rho_y(o)).$$

**Proof.** Denote by $A_{xy}$ the unique affine apartment joining $c_x$ and $d_y$. Then $\rho_{d_y, c_z}$ restricted to $A_{xy}$ is the identity, i.e. $\rho_{d_y, c_z}(o) = o$. Therefore Proposition 3.1 implies that $2(x|y)_{o, \xi_i} = b_{\xi_i}(o, o) = 0$.

By definition $\rho_y(o)$ is contained in the unique affine apartment joining $c_z$ and $d_y$. Then in the same way it follows that $(z|y)_{\rho_y(o), \xi_i} = 0$. Moreover, equation (2.2) yields $b_{\xi_i}(o, \rho_y(o)) = b_{\xi_i}(o, o) = 0$.

We can use Proposition 3.4 and again equation (2.2) to derive that $2(z|y)_{\rho_y(o), \xi_i} = b_{\xi_i}(o, \rho_y(o)) + b_{\xi_i}(o, \rho_y(o)) = b_{\xi_i}(o, \rho_z \rho_y(o))$.

In a very similar way we get

$$2(z|w)_{o, \xi_i} = b_{\xi_i}(o, \rho_z \rho_y(o)) + b_{\xi_i}(o, \rho_z \rho_y(o))$$

$$2(x|w)_{o, \xi_i} = b_{\xi_i}(o, \rho_z \rho_y(o)) + b_{\xi_i}(o, \rho_z \rho_y(o)).$$

Using that $cr_{\xi_i}(x, y, z, w) = -(x|y)_{o, \xi_i} - (z|w)_{o, \xi_i} + (x|w)_{o, \xi_i} + (z|y)_{o, \xi_i}$, we get $2cr_{\xi_i}(x, y, z, w) = b_{\xi_i}(o, \rho_z \rho_y(o))$. 

**Proposition 4.8.** Let $\rho_x, \rho_w, \rho_z$ and $\rho_y$ as before. Let $o$ be in the unique affine apartment joining $c_x, d_y$ such that we have under the identification $P(x, y) = \mathfrak{a}_* \times CS(x, y)$ that $\pi(o) = 0 \in \mathfrak{a}_*$, where $\pi$ is the projection to the first factor (also assume $x \simeq \mathfrak{a}_*^+)$. Then

$$2cr_{\tau}(x, y, z, w) = \pi(\rho_x \rho_w \rho_z \rho_y(o)).$$

**Proof.** By construction we have that $o, \rho_x \rho_w \rho_z \rho_y(o)$ are in the unique affine apartment joining $c_x$ and $d_y$. Then by Lemma 2.1 and from similar arguments as around equation (3.3) we can derive that $b_{\xi_i}(o, \rho_x \rho_w \rho_z \rho_y(o)) =$
\[ \langle \xi_i, \pi(\rho_x\rho_y\rho_z\rho_y(o)) \rangle \] for all \( i \in J^r \). Together with Lemma \ref{lem:cross-ratio} and the definition of \( \text{cr}_r \) we get
\[
2\text{cr}_r(x, y, z, w) = \sum_{i \in J^r} \langle \xi_i, \pi(\rho_x\rho_y\rho_z\rho_y(o)) \rangle \alpha_i^r.
\]
The \( \xi_i \in \mathfrak{a}_r \) for \( i \in J^r \) form a basis of \( \mathfrak{a}_r \). Moreover, for all \( i \in J^r \) we have that \( (2\text{cr}_r(x, y, z, w), \xi_i) = \langle \xi_i, \pi(\rho_x\rho_y\rho_z\rho_y(o)) \rangle \). Thus it follows that
\[
2\text{cr}_r(x, y, z, w) = \pi(\rho_x\rho_y\rho_z\rho_y(o)).
\]

## 5 Cross ratio preserving maps

We assume in this section that \( \tau \) is self opposite, i.e. \( \tau = \iota \tau \).

**Definition 5.1.** Let \( M_i, i = 1, 2 \) be either both symmetric spaces or thick Euclidean buildings. A map \( f : \text{Flag}_{\tau_1}(M_1) \rightarrow \text{Flag}_{\tau_2}(M_2) \) is called \( \xi_1 \)-Moebius map (or cross ratio preserving) if there exists \( \xi_i \in \text{int}(\tau_i) \) such that
\[
\text{cr}_{\xi_i}(x, y, z, w) = \text{cr}_{\xi_i}(f(x), f(y), f(z), f(w))
\]
for all \( (x, y, z, w) \in \mathcal{A}_{\tau_1} \). In particular we assume that \( f(\mathcal{A}_{\tau_1}) \subset \mathcal{A}_{\tau_2} \).

If \( f \) is a \( \xi_1 \)-Moebius map with respect to \( \xi_1, \xi_2 \), we also denote this by \( \text{cr}_{\xi_1} = f^* \text{cr}_{\xi_2} \). If \( \xi_1 \) is clear out of the context, we sometimes call \( f \) just Moebius map. Moreover, for any map \( f : \text{Flag}_{\tau_1}(M_1) \rightarrow \text{Flag}_{\tau_2}(M_2) \) we denote \( f^* \text{cr}_{\xi_2}(x, y, z, w) := \text{cr}_{\xi_2}(f(x), f(y), f(z), f(w)) \) for \( x, y, z, w \in \text{Flag}_{\tau_1}(M_1) \).

** Lemma 5.2.** Let \( x, y \in \text{Flag}_\tau \). Then there exists \( z \in \text{Flag}_\tau \) with \( z \text{ op } x, y \).

**Proof.** We take \( c_x, c_y \in \text{Flag}_\sigma \) such that \( x \) is a face of \( c_x \) and \( y \) is a face of \( c_y \). Then there exists \( c_z \in \text{Flag}_\sigma \) with \( c_z \text{ op } c_x, c_y \). Be \( z \) the face of \( c_z \) which is of type \( \tau \). Then \( z \in \text{Flag}_\tau \) with \( z \text{ op } x, y \).

**Lemma 5.3.** Let \( f : \text{Flag}_{\tau_1}(M_1) \rightarrow \text{Flag}_{\tau_2}(M_2) \) be a \( \xi_1 \)-Moebius map. Then for \( x, y \in \text{Flag}_{\tau_1}(M_1) \) we have that \( x \text{ op } y \) if and only if \( f(x) \text{ op } f(y) \).

**Proof.** Let \( x, y \in \text{Flag}_{\tau_1}(M_1) \) be given. Choose \( z_1, z_2, z_3 \in \text{Flag}_{\tau_1}(M_1) \) such that \( z_3 \text{ op } x; z_2 \text{ op } y, z_3 \) and \( z_1 \text{ op } x, z_2 \). By Corollary \ref{cor:cross-ratio}(2)
\[
\text{cr}_{\xi_1}(x, y, z_2, z_3) = r, \quad \text{cr}_{\xi_1}(x, z_1, z_2, z_3) = \pm \infty,
\]
i.e. \( x \text{ op } y \iff r = -\infty \). Since \( \text{cr}_{\xi_1} = f^* \text{cr}_{\xi_2} \), we derive \( f(z_1) \text{ op } f(z_2) \) and thus \( f(x) \text{ op } f(y) \iff r = -\infty \). In particular \( f(x) \text{ op } f(y) \iff x \text{ op } y \).

A map \( f : \text{Flag}_{\tau_1}(M_1) \rightarrow \text{Flag}_{\tau_2}(M_2) \) such that for all \( x, y \in \text{Flag}_{\tau_1}(M_1) \) it holds that \( x \text{ op } y \) if and only if \( f(x) \text{ op } f(y) \) is called opposition preserving.
Lemma 5.4. Let $f : \text{Flag}_{\gamma_1}(M_1) \to \text{Flag}_{\gamma_2}(M_2)$ be a $\xi_1$-Moebius map. Then $f$ is injective.

Proof. Assume there exist $x \neq y \in \text{Flag}_{\gamma_1}(M_1)$ with $f(x) = f(y)$. Take $a \in \text{Flag}_{\gamma_1}(M_1)$ with $a \varnothing x$ and $a \varnothing y$: For example take an apartment which contains $x$ and $y$. Take a opposite of $x$ in this apartment. Then $x \neq y$ implies that $a \varnothing y$ - opposite points are unique in apartments.

In addition, choose $z, w \in \text{Flag}_{\gamma_1}(M_1)$ such that $z \varnothing a$ and $w \varnothing z, x$. Then $\text{cr}_{\xi_1}(x, a, z, w) \neq \pm \infty$ and $\text{cr}_{\xi}(y, a, z, w) = -\infty$ or is not defined. However

$$\text{cr}_{\xi_1}(x, a, z, w) = f^* \text{cr}_{\xi_2}(x, a, z, w) = f^* \text{cr}_{\xi_2}(y, a, z, w) = \text{cr}_{\xi_1}(y, a, z, w),$$

which contradicts $\text{cr}_{\xi_1}(x, a, z, w) \neq \text{cr}_{\xi_1}(y, a, z, w)$. Thus $f(x) \neq f(y)$ if $x \neq y$. \hfill \Box

Definition 5.5. A surjective $\xi_1$-Moebius map is called a $\xi_1$-Moebius bijection.

When restricting to the full flag space we can apply the following result due to Abramenko and van Maldeghem.\(^5\)

Proposition 5.6. (Corollary 5.2 of [AV00]) Let $f : \text{Flag}_{\sigma}(M_1) \to \text{Flag}_{\sigma}(M_2)$ be a surjective automorphism that preserves opposition. Then $f$ extends in an unique way to an automorphism of the building $f : \Delta_\infty M_1 \to \Delta_\infty M_2$.

Lemma 5.7. Let $B = B_1 \circ \ldots \circ B_k$ and $B' = B'_1 \circ \ldots \circ B'_l$ be joins of irreducible thick spherical buildings. Moreover, be $f : B \to B'$ a building isomorphism. Then $k = k'$ and there exists a permutation $s$ on $k$ numbers such that $f = f_1 \times \ldots \times f_k$ with $f_i : B_i \to B'_{i(s)}$ building isomorphisms.

Proof. That $f$ is a building isomorphism implies that $B$ and $B'$ are modeled over the same spherical Coxeter complex, i.e. over the Coxeter group $W = W_1 \times \ldots \times W_k$, where $W_i$ are irreducible Coxeter groups. The irreducibility of the buildings $B_i, B'_i$ yields then that $k = k'$.

Assume without loss of generality that $|W_1| \leq |W_i|$ for all $i = 1, \ldots, k$. Let $x_1$ be a chamber in $B_1$. Then $x_1$ is a simplex in $B$. We know that $\text{Res}(x_1)$ is a spherical building over the spherical Coxeter complex to $W_2 \times \ldots \times W_k$. As $f$ is a building isomorphism, we derive that $f(\text{Res}(x_1)) = \text{Res}(f(x_1))$ is a spherical building over $W_2 \times \ldots \times W_k$. If $f(x_1)$ would not correspond to a chamber in an irreducible factor $B'_i$, then there would be a subgroup $W'$ of $W$ isomorphic to $W_2 \times \ldots \times W_k$ such that the projection of $W'$ to each $W_i$ is non-trivial (as $W_1$ is minimal). This would yield a decomposition of $W$.

\(^5\)We remark that every spherical building is 2-spherical as in the notation of [AV00]. Moreover, the buildings at infinity of symmetric spaces and thick Euclidean building are thick - hence we can apply their result.
$W_2 \times \ldots \times W_k$ into $k$ Coxeter groups, which contradicts the irreducibility of the factors. In particular, up to reordering $\text{Res}(f(x_1))$ is a building over $W_1 \times W_3 \times \ldots \times W_k$ and $W_1$ is isomorphic to $W_2$. Thus $f(x_1) = y_2$ for a chamber $y_2 \in B'_2$. Since $f$ is a building isomorphism it maps all simplices of the same type as $x_1$ to simplices of the same type as $y_2$ i.e. it maps the chambers of $B_1$ to chambers of $B'_2$. In particular, $f$ induces a building isomorphism $f_1 = f|_{B_1} : B_1 \to B'_2$ ($B_1$ is naturally a subset of $B$, namely the set of simplices of $B$ fully contained in $B_1$) and thus $f = f_1 \times f_0$ for a building isomorphism $f_0 : B_2 \circ \ldots \circ B_k \to B'_2 \circ B'_3 \circ \ldots \circ B'_r$. A straight forward induction yields the result. □

We remark that multiplying the metric of a space $M$ by some positive constant $\alpha$, yields that the Gromov product on $\text{Flag}_r(\alpha M)$ is given by $(\cdot | \cdot)_{\text{Flag}_r(M)} = \alpha (\cdot | \cdot)_{\text{Flag}_r(M)}$ and hence also $\text{cr}_{\xi,\alpha M} = \alpha \text{cr}_{\xi,M}$. Moreover, there is a natural identification of $\text{Flag}_r(\alpha M)$ with $\text{Flag}_r(M)$.

**Lemma 5.8.** Let $M = M^1_1 \times \ldots \times M^k_k$ be products of either irreducible symmetric spaces or irreducible thick Euclidean buildings and $f : \text{Flag}_r(M_1) \to \text{Flag}_r(M_2)$ a $\xi_1$-Moebius bijection. Then there exists a permutation $s$ on $k$ numbers such that $f = f_1 \times \ldots \times f_k$ with $f_i : \text{Flag}_r(M^i_1) \to \text{Flag}_r(M^i_2)$ a $\xi_1$-Moebius bijection and $M^i_1$ is the space $M^i_1$ with its metric rescaled (for the types $\xi_1$ see the proof).

**Proof.** Let $f : \Delta_{\infty} M_1 \to \Delta_{\infty} M_2$ be the building isomorphism from Proposition 5.6 From Lemma 5.7 we get a permutation $s$ on $k$ letters and building isomorphisms $f_i : \Delta_{\infty} M^i_1 \to \Delta_{\infty} M^i_2$ such that

$$f = f_1 \times \ldots \times f_k : \Delta_{\infty} M^1_1 \circ \ldots \circ \Delta_{\infty} M^k_k \to \Delta_{\infty} M^1_2 \circ \ldots \circ \Delta_{\infty} M^k_k.$$ 

Moreover, we know from Proposition 3.14 that $\text{cr}_{\xi_i} = \mu^1_i \text{cr}_{\xi_i} + \ldots + \mu^k_i \text{cr}_{\xi_k}$ with $\xi_i \in \sigma_i$ for $i = 1, 2$ and $j = 1, \ldots, k$ and $\mu_i \in S^+_k$ such that $\xi_i = \pi_i(\xi^1_i, \ldots, \xi^k_i, \mu_i)$ with $\pi_i$ as in the proposition (the numbers in the exponent are for indexing, not powers). Fix $(x_0, y_0, z_0, w_0) \in \text{Flag}_{\sigma_1}(M^1_2) \circ \ldots \circ \text{Flag}_{\sigma_k}(M^1_k)$ with $x_0, z_0 \in \mathcal{A}_{\pi_1}$ we get

$$\begin{align*}
\mu^1_i \text{cr}_{\xi_i}(x_1, y_1, z_1, w_1) + (\mu^2_i \text{cr}_{\xi_2} + \ldots + \mu^k_i \text{cr}_{\xi_5})(x_0, y_0, z_0, w_0) \\
= \mu^s_1 \text{cr}_{\xi_1}(x_1, y_1, z_1, w_1) + f_0^s(\mu^s_2 \text{cr}_{\xi_2} + \ldots + \mu^s_5 \text{cr}_{\xi_5})(x_0, y_0, z_0, w_0)
\end{align*}$$

with $f_0 = f_2 \times \ldots \times f_k$. The equality also holds when we replace $(x_0, y_0, z_0, w_0)$ with $(z_0, y_0, x_0, w_0)$. Moreover, we have $(\mu^2_i \text{cr}_{\xi_2} + \ldots + \mu^k_i \text{cr}_{\xi_5})(x_0, y_0, z_0, w_0) = -(\mu^2_i \text{cr}_{\xi_2} + \ldots + \mu^k_i \text{cr}_{\xi_5})(z_0, y_0, x_0, w_0)$. Hence we derive that $\mu^1_i \text{cr}_{\xi_i}(x_1, y_1, z_1, w_1) = \mu^s_1 \text{cr}_{\xi_1}(x_1, y_1, z_1, w_1)$. As $(x_1, y_1, z_1, w_1)$ was arbitrary in $\mathcal{A}_{\pi_1}$ we get

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\( \mu_1^1 \xi_{\mathcal{c}_1} = \mu_2^{s(1)} f_1^* \xi\). In the same way it follows for all \( i = 1, \ldots, k \) that
\( \mu_1^1 \xi_{\mathcal{c}_1} = \mu_2^{s(i)} f_1^* \xi\).

If we rescale the metric on \( M_1^i \) by \( \mu_2^{s(i)}/\mu_1^1 \) - denote this space by \( \hat{M}_1^i \) - then \( f_1 : \Delta_\infty \hat{M}_1^i \to \Delta_\infty M_2^{s(i)} \) restricts to a Moebius bijection on the chamber sets, i.e. we get a Moebius bijection \( f_1 : \text{Flag}_\sigma(\hat{M}_1^i) \to \text{Flag}_\sigma(M_2^{s(i)}) \).

We will need the following fact:

**Theorem 5.9.** ([BS17]) Let \( T_1, T_2 \) be geodesically complete trees such that \( |\partial_\infty T_i| \geq 3 \). Then every isometry from \( T_1 \) to \( T_2 \) restricted to the boundary is a Moebius bijection and every Moebius bijection \( f : \partial_\infty T_1 \to \partial_\infty T_2 \) can be uniquely extended to an isometry.

Let \( T \) be a rank one thick Euclidean building; in particular \( T \) is a tree. Then every geodesic segment in \( T \) lies in an affine apartment, i.e. in a bi-infinite geodesic. This means that \( T \) is geodesically complete (in the notation of [BS17]). Moreover, by definition of thickness for rank one Euclidean buildings we have that \( |\partial_\infty T| \geq 3 \).

We remark that \( \text{rk}(T) = 1 \) implies that the positive chamber of the Coxeter complex \( \sigma_T \) consists of a single point. Thus \( \Delta_\infty T = \text{Flag}_\sigma(T) = \partial_\infty T \). Hence there is a unique Gromov product \( (\cdot, \cdot)_{\sigma_T} \) for any \( \sigma_T \in T \) on \( \partial_\infty T^2 \) and a unique cross ratio \( \text{cr}_T \) on \( \mathcal{A}_T \subset \partial_\infty T^4 \).

**Proposition 5.10.** Let \( E_1, E_2 \) be irreducible thick combinatorial Euclidean buildings. Then every Moebius bijection \( f : \text{Flag}_\sigma(E_1) \to \text{Flag}_\sigma(E_2) \) is the restriction of an isometry \( F : E_1 \to E_2 \) to the boundary where \( E_1 \) is \( E_2 \) with its metric rescaled. If \( E_1 \) is not the cone over a spherical building, then \( F \) is unique.

**Proof.** If the rank is one, then the result follows from the theorem above.

If the rank is two, Struyve has shown in [Str16] that every isometry between \( \partial_\infty E_1 \) and \( \partial_\infty E_2 \) with respect to the Tits metric is induced by an isometry after rescaling the metric on \( E_1 \). The isometry is unique if \( E_1 \) is not the cone over a spherical building. We know that \( f \) induces a building isomorphism \( f : \Delta_\infty E_1 \to \Delta_\infty E_2 \) and this yields an isometry \( f : \partial_\infty E_1 \to \partial_\infty E_2 \) with respect to the Tits metric when viewing simplices as subset of \( \partial_\infty E_i \). Hence we can apply the result of Struyve.

**Corollary 5.11.** Let \( E_1 \) and \( E_2 \) be combinatorial Euclidean buildings and let \( f : \text{Flag}_\sigma(E_1) \to \text{Flag}_\sigma(E_2) \) Moebius bijection. Then one can rescale the metric of \( E_1 \) on irreducible factors - denote this space by \( \hat{E}_1 \) - such that \( f \) is the restriction of an isometry \( F : \hat{E}_1 \to E_2 \) to the boundary. If none of the irreducible factors is a cone over a spherical building the isometry \( F \) is unique.
Proof. This follows from Lemma 5.8 together with the proposition above.

5.1 Symmetric spaces

We want to show that the above corollaries hold in a similar way for symmetric spaces. Therefore we essentially only need to show that Moebius bijections are homeomorphisms. Hence we analyze some topological properties of Moebius bijections for the case of symmetric spaces.

In this section we only consider symmetric spaces $X$. For $r \in \mathbb{R}$, $x_2, y_1, y_2 \in \text{Flag}_r(X)$ and $\xi \in \text{int}(\tau)$ we define

$$B^+_{r,\xi}(y_1, x_2, y_2) := \{ x_1 \in \text{Flag}_r(X) \mid (x_1, y_1, x_2, y_2) \in A_\xi, \ c_r(x_1, y_1, x_2, y_2) > r \},$$

$$B^-_{r,\xi}(y_1, x_2, y_2) := \{ x_1 \in \text{Flag}_r(X) \mid (x_1, y_1, x_2, y_2) \in A_\xi, \ c_r(x_1, y_1, x_2, y_2) < r \}.$$

Those sets are open by the continuity of $c_r$ and the fact that $A_\xi$ is open. However, it can happen that they are empty - which holds if $x_2 \not\in \text{Flag}_r y_1, y_2$.

**Proposition 5.12.** Let $X$ be a symmetric space. The sets $B^\pm_{r,\xi}(y_1, x_2, y_2)$ varying over all $r \in \mathbb{R}$ and all $x_2, y_1, y_2 \in \text{Flag}_r$ form a subbase of the topology on $\text{Flag}_r(X)$.

**Proof.** As mentioned, those sets are open. Thus we need to show that any open neighborhood $U$ of a point $x \in \text{Flag}_r(X)$ contains an open neighborhood $V$ won by finite intersection and arbitrary union of sets of the form $B^\pm_{r,\xi}(y_1, x_2, y_2)$.

Let $x \in \text{Flag}_r(X)$ and let any neighborhood $U$ of $x$ be given. We set $K := \text{Flag}_r \setminus U$. Then $K$ is compact and $x \notin K$.

For any $a \in K$, choose $y_a \in \text{Flag}_r(X)$ such that $y_a \not\in \text{Flag}_r a$ and $y_a \not\in \text{Flag}_r x$. In addition, choose $w_a, z_a \in \text{Flag}_r(X)$ such that $w_a \not\in \text{Flag}_r a, x$ and $z_a \not\in \text{Flag}_r y_a, w_a$. This yields $c_r(x, y_a, z_a, w_a) = -\infty$ and $c_{\xi}(a, y_a, z_a, w_a) > r_a$ for some $r_a \in \mathbb{R}$ and hence $x \in B^-_{r_a,\xi}(y_a, z_a, w_a), x \notin B^+_{r_a,\xi}(y_a, z_a, w_a), a \in B^\pm_{r_a,\xi}(y_a, z_a, w_a)$.

Varying over all $a \in K$ the sets $B^\pm_{r_a,\xi}(y_a, z_a, w_a)$ cover $K$ and by compactness we find a finite number of points $a_i \in K, i = 1, \ldots, l$ such that the according sets already cover $K$. We set $V := \cap_{a_i = 1, \ldots, l} B^-_{r_a,\xi}(y_{a_i}, z_{a_i}, w_{a_i})$. As a finite intersection of open sets, $V$ is open. Furthermore, $x \in V$ and hence $V$ is non-empty. By construction $V \subset K^c$ and hence $V \subset U$.

**Lemma 5.13.** Let $f : \text{Flag}_{\tau_1}(X_1) \rightarrow \text{Flag}_{\tau_2}(X_2)$ be a $\xi_1$-Moebius bijection. Then $f$ is a homeomorphism.

**Proof.** Since $f$ leaves the cross ratio invariant and is a bijection, it is immediate that $f(B^\pm_{r,\xi}(y, z, w)) = B^\pm_{r,\xi_2}(f(y), f(z), f(w))$. This means that $f$ yields a bijection of subbases of the topology and hence $f$ is a homeomorphism.
As mentioned, for a symmetric space $X$ the boundary $\text{Flag}_\tau(X)$ can be identified homeomorphically with $G/P_x$ for $P_x = \text{stab}(x)$ and $x \in \text{Flag}_\tau(X)$. Hence $\text{Flag}_\tau(X)$ can be given the structure of compact connected manifold (without boundary) - inherited from $G/P_x$. Using this there is a different way to characterize Moebius bijections captured in the following lemma.

**Lemma 5.14.** Let $X_1, X_2$ be symmetric spaces such that $\dim \text{Flag}_{\tau_1}(X_1) = \dim \text{Flag}_{\tau_2}(X_2)$ and $f : \text{Flag}_{\tau_1}(X_1) \to \text{Flag}_{\tau_2}(X_2)$ be a continuous $\xi_1$-Moebius map. Then $f$ is a homeomorphism, in particular $f$ is a $\xi_1$-Moebius bijection.

**Proof.** Since $f$ is a $\xi_1$-Moebius map and hence injective, we know that $f : \text{Flag}_{\tau_1}(X_1) \to \text{Im}(f)$ is a bijection, with $\text{Im}(f)$ denoting the image. Moreover, $f^*cr_{\xi_2} = cr_{\xi_1}$ implies $f(B_\tau^{-\xi_1}(y, z, w)) = B_\tau^{-\xi_2}(f(y), f(z), f(w)) \cap \text{Im}(f)$. Then Proposition 5.12 yields that $f$ maps a subbase of the topology on $\text{Flag}_{\tau_1}(X_1)$ into a subbase of the topology on $\text{Im}(f)$ equipped with the subset topology. Hence $f : \text{Flag}_{\tau_1}(X_1) \to \text{Im}(f)$ is open and therefore a homeomorphism.

We derive that $\text{Im}(f)$ is compact connected submanifold of $\text{Flag}_{\tau_2}(X_2)$ of the same dimension. However, $\text{Flag}_{\tau_2}(X_2)$ is a compact connected manifold without boundary and hence the only such submanifold is $\text{Flag}_{\tau_2}(X_2)$ itself, i.e. $\text{Im}(f) = \text{Flag}_{\tau_2}(X_2)$ - which proves the claim. □

**Theorem 5.15.** Let $X_1, X_2$ be symmetric spaces of non-compact type of rank at least two with no rank one de Rham factors and let $f : \text{Flag}_{\sigma}(X_1) \to \text{Flag}_{\sigma}(X_2)$ be a $\xi_1$-Moebius bijection. Then one can multiply the metric of $X_1$ by positive constants on de Rham factors - denote this space by $\hat{X}_1$ - such that $f$ is the restriction of an unique isometry $F : \hat{X}_1 \to X_2$ to $\text{Flag}_{\sigma}(X_1)$.

**Proof.** We know that a $\xi_1$-Moebius bijection $f : \text{Flag}_{\sigma}(X_1) \to \text{Flag}_{\sigma}(X_2)$ can uniquely be extended to a building isomorphism $f : \Delta_\infty \hat{X}_1 \to \Delta_\infty X_2$. Moreover, $f$ is a homeomorphism on the chamber sets $\text{Flag}_{\sigma}(X_1)$ by Lemma 5.13 Then for such maps the result is known [Ebe96 Sc.3.9]. □

Actually all we need for the above result is that $f : \text{Flag}_{\sigma}(X_1) \to \text{Flag}_{\sigma}(X_2)$ is opposition preserving and a homeomorphism. However, when dealing also with rank one factors we really need Moebius maps.

**Corollary 5.16.** Let $X_1$ and $X_2$ be symmetric spaces of non-compact type and let $f : \text{Flag}_{\sigma}(X_1) \to \text{Flag}_{\sigma}(X_2)$ be a Moebius bijection. Then one can rescale the metric of $X_1$ on de Rham factors - denote this space by $\hat{X}_1$ - such that $f$ is the restriction of an unique isometry $F : \hat{X}_1 \to X_2$ to the boundary.

**Proof.** This follows from Lemma 5.14 together with the theorem above and the fact that Moebius bijections of rank one symmetric spaces can be uniquely extended to isometries. For the latter result see [Bou96]. □
One should be able to derive the same (or at least similar) topological properties of the cross ratios and Moebius maps for Bruhat-Tits buildings, i.e. combinatorial Euclidean building associated to semi-simple algebraic groups over non-archimedian local field with finite residue field. However, since in this case Proposition 5.10 and Corollary 5.11 already yield the main result, we have not included these topological properties for simplicity.

5.2 Rescaling on irreducible factors

In this generality it is not possible to drop the scaling on the irreducible factors in the Corollaries 5.11, 5.16 and Theorem 5.15. For example consider the following situation: Let \( M_0 \) be a symmetric space or a combinatorial Euclidean building. We set \( M_1 := \mu_1^{-1} M_0 \), \( M_2 := \mu_2^{-1} M_0 \) for \( \mu_i > 0 \) with \( \mu_1^2 + \mu_2^2 = 1 \) and \( M := M_1 \times M_2 \). Here \( M_1 = \mu_i^{-1} M_0 \) means we take the space \( M_0 \) with its metric multiplied by \( \mu_i^{-1} \). Moreover, we define \( f : \text{Flag}_{\sigma}(M) \rightarrow \text{Flag}_{\sigma}(M) \) by \( f(x, y) := (y, x) \). Let \( \xi \in \text{int}(\sigma_0) \) and \( \sigma_0 \) the fundamental of the space \( M_0 \). Consider the cross ratio \( \text{cr}_{\pi((\xi, \xi, (\mu_1, \mu_2))) \cdot M} = \mu_1 \text{cr}_{\xi,M_1} + \mu_2 \text{cr}_{\xi,M_2} \) - cp. Proposition 3.14. As mentioned, we have \( \mu_1 \text{cr}_{\xi,M_1} = \mu_2 \text{cr}_{\xi,M_2} \) and hence \( f \) is a \( \pi(\xi, \xi, (\mu_1, \mu_2)) \)-Moebius bijection.

It is immediate to see that \( f \) is induced by a map \( F := F_1 \times F_2 : M_1 \times M_2 \rightarrow M_2 \times M_1 \) such that \( F_i : \text{Flag}_{\sigma}(M_i) \rightarrow \text{Flag}_{\sigma}(M_j) \), \( i \neq j \) is the identity (under the natural identification with \( \text{Flag}_{\sigma}(M_0) \)). As \( F \) and hence the \( F_i \) shall be isometries, it follows that \( F(p, q) = (q, p) \) and clearly \( F \) is an isometry only after rescaling on de Rham factors.

Let \( M_1 \) be a symmetric space or a combinatorial Euclidean building and assume that the image of \( \text{cr}_{\sigma,M_1} \) lies not in a proper subspace of \( a_{M_1} \). Then the above situation is essentially the only possibility where rescaling can appear:

Let \( M_1, M_2 \) be irreducible - actually one of them being irreducible would be enough. In addition, be \( f : \text{Flag}_{\sigma}(M_1) \rightarrow \text{Flag}_{\sigma}(M_2) \) a \( \xi \)-Moebius bijection, i.e. \( \text{cr}_{\xi_1} = f^* \text{cr}_{\xi_2} \). Then we know that we can rescale the metric on \( M_1 \) by some positive number \( \mu_1 \), such that \( f \) is induced by an isometry \( F : \mu_1 M_1 \rightarrow M_2 \). Thus Proposition implies 5.11 \( f^* \text{cr}_{\xi_2} = \text{cr}_{\xi_1,\mu_1 M_1} = \mu_1 \text{cr}_{\xi_1,M_1} \) for \( \xi_1 \in \sigma_1 \) with \( F_{\sigma}(\xi_1') = \xi_2 \).

However, it follows from the assumption on \( \text{cr}_{\sigma,M_1} \) together with Lemma 4.3 that \( \text{cr}_{\xi} \neq \alpha \text{cr}_{\xi'} \) for \( \xi \neq \xi' \in \sigma_1 \) and any \( \alpha \in \mathbb{R} \). Therefore \( \text{cr}_{\xi_1,M_1} = f^* \text{cr}_{\xi_2} = \mu_1 \text{cr}_{\xi_1,M_1} \) implies \( \xi_1 = \xi_1' \) and \( \mu_1 = 1 \) - in particular \( f \) is induced by an isometry without rescaling the metric.

We remark that for symmetric spaces with \( \iota = \text{id} \) the image of \( \text{cr}_{\sigma} \) is all of \( a \). This follows from the fact that every vector of \( a \) can be realized as a translation vector of a hyperbolic element in \( G \). Then the periods of those elements in \( G \) are exactly those translation vectors, as seen in Proposition 4.6. Hence the above discussion applies.
Corollary 5.17. Let $M$ either be a symmetric space or a combinatorial Euclidean building with none of the irreducible factors being a cone over a spherical building. In addition, assume that the image of $cr_\sigma$ is not contained in a proper subspace of $\mathfrak{a}$. Let $\xi_0 \in \sigma$ be the center of gravity of $\sigma$. Then there is a one-to-one correspondence between $Iso(M)$ and $\xi_0$-Moebius bijections.

Proof. Let $g \in Iso(M)$ and $g_\sigma : \sigma \to \sigma$ the induced map. Then $g_\sigma$ is an isometry with respect to the angular metric, hence $g_\sigma$ stabilizes the center of gravity $\xi_0$ of $\sigma$. Therefore Proposition 3.11 yields a $\xi_0$-Moebius bijections for each $g \in Iso(M)$.

On the other hand, by Corollaries 5.11 and 5.16 we know that each $\xi_0$-Moebius bijections is induced by a unique isometry - after possible rescaling on irreducible factors. However, following the above discussion we can exclude rescaling of the metric:

Let $f$ be a $\xi_0$-Moebius bijection and let $f = f_1 \times \ldots \times f_k$ be the decomposition on irreducible factors $M_1, \ldots, M_k$ as in Lemma 5.8. Assume w.l.o.g. that $f_1 : \text{Flag}_{\xi_1}(M_1) \to \text{Flag}_{\xi_2}(M_2)$, i.e. $M_1, M_2$ are isometric after possibly rescaling the metric. From Proposition 5.11 we know $cr_{\xi_0} = \mu_1 cr_{\xi_1, M_1} + \mu_2 cr_{\xi_2, M_2} + \ldots + \mu_k cr_{\xi_k, M_k}$. However, $\xi_0 \in \sigma$ being the center of gravity of $\sigma$ and $M_1, M_2$ isometric after possibly rescaling the metric implies $\mu_1 = \mu_2$ and $\xi_1 \simeq \xi_2$. Then $f_1$ is $\xi_1$-Moebius bijection between irreducible spaces. From the above discussion it follows that it is induced by an isometry without rescaling the metrics. The same argument implies the result for all $f_i$ and hence the claim follows.

5.3 General Euclidean buildings

In this section we consider general Euclidean buildings, i.e. in particular non-locally compact ones. The goal is again to show that Moebius bijections are induced by isometries. However, now we will need the vector valued cross ratio $cr_\sigma$ to derive such a result.

Let $E$ be a thick Euclidean building considered with the complete apartment system. Let $x \in \text{Flag}_{\sigma}(E)$ and $y \in \text{Flag}_{\sigma \tau}(E)$ with $x \circ \text{o} \ y$ and $\tau$ is a codimension 1 face of $\sigma$ - in this case $x,y$ are called panels of the building $\Delta_{\sigma \tau}$. Then metrically we have the splitting $P(x,y) = a_\tau \times CS(x,y)$, where $CS(x,y)$ is a Euclidean building of rank $rk(E) - \dim a_\tau = 1$, i.e. $CS(x,y)$ is an $\mathbb{R}$-tree. This tree is called wall tree and will be denoted by $T_{xy}$. One can show that the isomorphism type of $T_{xy}$ does not depend on the choice of $y \in \text{Flag}_{\tau}$ with $y \circ \text{o} \ x$ [KW14]; hence the isomorphism class of $T_{xy}$ will be denoted by $T_x$.

We recall that the residue of an element $z \in \Delta_{\sigma \tau}$ is defined by $\text{Res}(z) = \{ w \in \Delta_{\sigma \tau} \mid z \not\subset w \}$. In case of a panel $x \in \Delta_{\sigma \tau}$ we have that $\text{Res}(x)$ consists of all the chambers in $\Delta_{\sigma \tau}$ containing $x$.

It is known that one can naturally identify $\text{Res}(x) \simeq \partial_{\sigma \tau}$. For convenience of the reader we describe this identification: Fix $y \circ \text{o} \ x$. Then
Let \( o \in P(x, y) \). Then one can identify the chambers in \( \text{Res}(x) \) with (specific) Weyl sectors in \( P(x, y) \) with tip \( o \) [Par00, Cor. 1.9.]. Pick \( o \in P(x, y) \) such that we can identify \( P(x, y) \cong a_r \times T_{xy} \), \( o \cong (0, o_T) \) and \( x \cong \partial_\infty (a_r \cap a^*) \) - i.e. \( x \) corresponds to the positive chamber in \( a_r \). Recall that \( a_r^+ = a_r \cap a^* \). Then the affine apartments in \( P(x, y) \cong a_r \times T_{xy} \) containing \( o \) are of the form \( a_r \times \gamma \), where \( \gamma \) is bi-infinite geodesic ray in \( T_{xy} \) passing through \( o_T \) (those are easily seen to be isometric to \( \mathbb{R}^\gamma \)). By definition every Weyl sector is contained in an affine apartment; hence we can derive that every Weyl sector with tip \( o \) and boundary chamber \( c \in \text{Res}(x) \) is contained in \( a_r^+ \times \gamma_{o_Tz} \) where \( \gamma_{o_Tz} \) is a geodesic in \( T_{xy} \) from \( o_T \) to a boundary point \( z \in \partial_\infty T_{xy} \). This yields a one-to-one correspondence of \( \text{Res}(x) \) with geodesic rays emanating from \( o_T \). As those rays are in one-to-one correspondence with \( \partial_\infty T_{xy} \), we get \( \text{Res}(x) \cong \partial_\infty T_x \) as claimed.

**Remark 5.18.** It follows that for \( z \in \partial_\infty T_{xy}, c \in \text{Res}(x) \) and \( d \in \text{Res}(y) \) we have that \( z \cong c \) and \( z \cong d \) under \( \text{Res}(x) \cong \partial_\infty T_{xy}, \text{Res}(y) \cong \partial_\infty T_{xy} \) respectively if and only if the Weyl sectors with tip \( o \cong (0, o_T) \) defining \( c,d \) are contained in \( a_r^+ \times \gamma_{o_Tz}, a_r^- \times \gamma_{o_Tz} \), respectively.

By definition \( \text{Res}(x) \) is the set of chambers that contain \( x \). Hence there is a unique corner \( \xi_x \) of \( \sigma \) such that \( c_{\xi_x} \notin x \) for every chamber \( c \in \text{Res}(x) \). In the same way we get a type from \( y \) and it is immediate that this type equals \( \tau_\xi \) - following for example from the fact that \( x \in \text{Flag}_r \) implies that \( y \in \text{Flag}_{\tau_\xi} \).

**Lemma 5.19.** Let \( x,y \) be opposite panels in \( \Delta_\infty E \) and \( T_{xy} \) the associated tree. Let \( z_c, z_d \in \partial_\infty T_{xy}, c \in \text{Res}(x) \) such that \( c \cong z_c \) under \( \text{Res}(x) \cong \partial_\infty T_{xy} \) and \( d \in \text{Res}(y) \) such that \( d \cong z_d \) under \( \text{Res}(y) \cong \partial_\infty T_{xy} \). Then \([c|d]_{o, \xi_x} = \sin(\alpha)(z_c|z_d)_{o_T} \) where \( o \cong (0, o_T) \) under \( P(x, y) \cong a_r \times T_{xy} \) and \( \alpha \in (0, \pi) \) does only depend on \( \sigma \) and the type of \( x \).

**Proof.** Let \( \gamma_c, \gamma_d \) be the geodesics in \( P(x, y) \) from \( o \) to \( c_{\xi_x} \) and \( d_{\xi_x} \), respectively. The splitting \( P(x, y) \cong a_r \times T_{xy} \) gives geodesics \( \gamma_x, \gamma_y \) in \( a_r \) emanating from \( 0 \) and \( \gamma_{zc}, \gamma_{zd} \) in \( T_{xy} \) emanating from \( o_T \) such that \( \gamma_c(t) = (\gamma_x(t), \gamma_{zc}(t)) \) and \( \gamma_d(t) = (\gamma_y(t), \gamma_{zd}(t)) \) - while \( \gamma_c, \gamma_d \) are unit speed, the geodesics \( \gamma_x, \gamma_y, \gamma_{zc} \) and \( \gamma_{zd} \) are not. It is clear that the geodesics \( \gamma_x, \gamma_y \) do not depend on the choice of \( c,d \) and are in opposite directions (since the \( \gamma_c, \gamma_d \) are): The geodesics \( \gamma_c, \gamma_d \) are along those corners of Weyl sectors that are not contained in \( a_r \). Since Weyl sectors are isometric to convex subsets of \( \mathbb{R}^\gamma \), it reduces to Euclidean geometry: for example \( \gamma_x \) is the geodesic in \( a_r \) from \( 0 \) to \( \pi_{\tau_x}(\xi_x) \), where \( \pi_{\tau_x} \) is the orthogonal projection from \( \sigma \) to \( \tau_x \) and \( \tau_x \) is the type of \( x \). In particular we have \( d(\gamma_x(t), \gamma_y(t)) = 2t \).

Let from now on \( \gamma_x, \gamma_y, \gamma_{zc} \) and \( \gamma_{zd} \) be the geodesics as above but now parametrized such that they are unit speed. Let \( \alpha \) be the angle of \( \xi_x \) and \( \pi_{\tau_x}(\xi_x) \). Then we have \( \gamma_c(t) = (\gamma_x(\cos(\alpha)t), \gamma_{zc}(\sin(\alpha)t)) \). Basic facts of
The cross ratio on Corollary 5.20. trees imply that \( d(\gamma_x(t), \gamma_y(t)) = 2t - 2(z_c|z_d)_{o_T} \) for \( t \geq (z_c|z_d)_{o_T} \) - see e.g. [BS17]. Together with \( d(\gamma_x(t), \gamma_y(t)) = 2t \) we get

\[
\left(\frac{c}{d}\right)_{\alpha, \xi} = \lim_{t \to \infty} t - \frac{1}{2} \sqrt{4 \cos^2(\alpha) t^2 + (2 \sin(\alpha) t - 2(z_c|z_d)_{o_T})^2}
= \lim_{t \to \infty} t - \sqrt{t^2 - 2t \sin(\alpha)(z_c|z_d)_{o_T} + (z_c|z_d)^2_{o_T}} = \sin(\alpha)(z_c|z_d)_{o_T},
\]

while the last equality follows from a Taylor series in the same way as we have seen several times before. \( \square \)

**Corollary 5.20.** The cross ratio on \( \partial_\infty T_{xy} \) is given by \( c_{T_{xy}}(z_1, w_1, z_2, w_2) = \sin(\alpha) c_{\xi, \sigma}(c_1, d_1, c_2, d_2) \) where \( \xi \in \sigma \) is the corner not contained in \( \tau_x \), the type of \( x \), \( \alpha \) is the angle between \( \xi_x \) and \( \tau_x \), \( c_i \approx z_i \) under \( \text{Res}(x) \approx \partial_\infty T_{xy} \) and \( d_i \approx w_i \) under \( \text{Res}(y) \approx \partial_\infty T_{xy} \).

The thickness of \( E \) means that \( \Delta_\infty E \) is thick and therefore we have for every panel \( x \) that \( |\partial_\infty T_x| \geq 3 \) (as \( \text{Res}(x) \approx \partial_\infty T_x \)), i.e. \( T_x \) is thick and geodesically complete. Therefore Theorem 5.9 implies that the whole isometry class \( T_x \) has a natural cross ratio \( c_{T_x} \).

**Definition 5.21.** Let \( E_1, E_2 \) be thick irreducible Euclidean buildings. A building isomorphism \( \phi : \Delta_\infty E_1 \to \Delta_\infty E_2 \) is called tree-preserving or ecological, if for every panel \( x \in \Delta_\infty E_1 \) we have that \( \phi|_{\text{Res}(x)} : \text{Res}(x) \to \text{Res}(\phi(x)) \) is induced by an isometry \( \phi_x : T_x \to T_{\phi(x)} \) - i.e. \( (\phi_x)|_{\partial_\infty T_x} \approx \phi|_{\text{Res}(x)} \) under the identification \( \text{Res}(x) \approx \partial_\infty T_x \).

**Theorem 5.22.** ([Tits, Thm 2]) Let \( E_1, E_2 \) be thick irreducible Euclidean buildings and \( \phi : \Delta_\infty E_1 \to \Delta_\infty E_2 \) an ecological isomorphism. Then \( \phi \) extends to an isomorphism, i.e. an isometry after possibly rescaling the metric on \( E_1 \).

In a similar way as before, we call a surjective map \( f : \text{Flag}_{\sigma_1}(E_1) \to \text{Flag}_{\sigma_2}(E_2) \) such that \( c_{\sigma_1}(x, y, z, w) = f^* c_{\sigma_2}(x, y, z, w) \) for all \( (x, y, z, w) \in A_{\sigma_1} \) a \( \sigma_1 \)-Moebius bijection. We remark that to identify the image of \( c_{\sigma_1} \), with the one of \( c_{\sigma_2} \) it is already necessary that \( E_1 \) and \( E_2 \) are modeled over the same spherical Coxeter complex, i.e. \( \sigma_1 \approx \sigma_2 =: \sigma \).

It is immediate that such a map is a \( \xi_0 \)-Moebius map, for \( \xi_0 \) the center of gravity of \( \sigma \). We assume \( f \) to be surjective, hence \( f \) is a \( \xi_0 \)-Moebius bijection and therefore can be extended uniquely to a building isomorphism \( f : \Delta_\infty E_1 \to \Delta_\infty E_2 \) by Proposition 5.6.

We recall that the affine Weyl group \( \hat{W} = W \rtimes T_W \) of the Coxeter complex over which a Euclidean building is defined gives a collection of hyperplanes, namely the hyperplanes of the finite reflection group \( W \) together with all its translates under \( T_W \). Each hyperplane defines two half spaces which we call **affine half apartments**. The image of an affine half apartment under a chart map is again called **affine half apartment**.
In spherical buildings the hyperplanes associated to the spherical Coxeter group define walls in apartments and those walls separate the apartments in two halves, called half apartments. One can show that the boundary of an affine half apartment $H \subset E$ defines a half apartment in $H_\infty \subset \Delta_\infty E$ and to every half apartment in $H_\infty \subset \Delta_\infty E$ we find an affine half apartment $H \subset E$ which has $H_\infty$ as its boundary.

Now, let $f : \Delta_\infty E_1 \to \Delta_\infty E_2$ be a building isomorphism and let $x, y$ be opposite panels. Then the identifications $\partial_\infty T_{xy} \simeq \text{Res}(x)$ and $\partial_\infty T_{zy} \simeq \text{Res}(y)$ together with the maps $f|_{\text{Res}(x)} : \text{Res}(x) \to \text{Res}(f(x)), f|_{\text{Res}(y)} : \text{Res}(y) \to \text{Res}(f(y))$ induce two maps $f_x, f_y : \partial_\infty T_{xy} \to \partial_\infty T_{f(x)f(y)}$.

**Lemma 5.23.** Notations as above, i.e. $x, y$ are opposite panels and $f_x, f_y : \partial_\infty T_{xy} \to \partial_\infty T_{f(x)f(y)}$ the maps induced by $f|_{\text{Res}(x)} : \text{Res}(x) \to \text{Res}(f(x)), f|_{\text{Res}(y)} : \text{Res}(y) \to \text{Res}(f(y))$. Then $f_x = f_y$.

**Proof.** Let $z \in \partial_\infty T_{xy}$, i.e. $z$ is an equivalence class of geodesic rays. Every ray $\gamma_z$ in the class starting at a branching point defines an affine half apartment $a_{\gamma_z}$ in $E_1$ and thus (the equivalence class of rays) defines a half apartment $H_\infty \subset \Delta_\infty E_1$. Then it follows form Remark 5.18 that $c \simeq z$ with $c \in \text{Res}(x)$ if and only if $c$ is contained in the half apartment $H_\infty$ and in the same way $d \simeq z$ with $d \in \text{Res}(y)$ if and only if $d$ is contained in the half apartment $H_\infty$. By assumption, $f$ is a building isomorphism, i.e. $f(H_\infty) \subset \Delta_\infty E_2$ is a half apartment with $f(x), f(y) \in f(H_\infty)$. The metric splitting $P(f(x), f(y)) = a_{\gamma_x} \times T_{f(x)f(y)}$ yields that we find an affine half apartment $a_{\gamma_w}$ with $\gamma_w$ a geodesic ray in $T_{f(x)f(y)}$ and boundary point $w \in \partial_\infty T_{f(x)f(y)}$ such that the boundary of this affine half apartment is exactly $f(H_\infty)$. By definition $f(c), f(d) \in f(H_\infty)$. Hence from Remark 5.18 we get that $f(c) \simeq w \simeq f(d)$. Therefore $f_x(z) = w$ and $f_y(z) = w$. $\square$

**Theorem 5.24.** Let $E_1, E_2$ be thick irreducible Euclidean buildings. Let $f : \text{Flag}_\sigma(E_1) \to \text{Flag}_\sigma(E_2)$ be a $\sigma$-Moebius bijection. Then the induced isomorphism $f : \Delta_\infty E_1 \to \Delta_\infty E_2$ is ecological and hence can be extended to an isomorphism $F : E_1 \to E_2$, i.e. an isometry after possibly rescaling the metric on $E_1$

**Proof.** What we need to show is, given a panel $x \in \Delta_\infty E_1$, the induced map $f_x : \partial_\infty T_x \to \partial_\infty T_{f(x)}$ is the restriction of an isometry. This implies that $f$ is ecological and therefore by the Theorem of Tits induced by an isomorphism.

We fix $y$ opposite to $x$ to get a tree $T_{xy}$ in the class of $T_x$. Since we are considering isometry classes of trees, it is enough to show that $f_{xy} : \partial_\infty T_{xy} \to \partial_\infty T_{f(x)f(y)}$ is induced by an isometry.

From Corollary 5.20 we derive for $z_1, w_1, z_2, w_2 \in \partial_\infty T_{xy}$ and $c_1, c_2 \in \text{Res}(x), d_1, d_2 \in \text{Res}(y)$ with $z_i \simeq c_i, w_i \simeq d_i$ that there is some $\alpha \in (0, \pi)$ with $\text{cr}_{T_{xy}}(z_1, w_1, z_2, w_2) = \sin(\alpha)\text{cr}_{\xi}(c_1, c_2, d_1, d_2) = \sin(\alpha)f^*\text{cr}_{\xi}(c_1, d_1, c_2, d_2)$,
while the last equality follows from $f$ being a $\sigma$-Möbius bijection. By construction $f_{xy} \circ \partial_\infty T_{xy} \simeq \Res(x) \to \partial_\infty T_{f(x)} \simeq f(x)$ is defined in the way that $f(e_1) \simeq f_{xy}(z_1)$ under $\partial_\infty T_{f(x)} \simeq f(x)$ and similar for $c_2$. In light of Lemma 5.23 we have that $f(d_i) \simeq f_{xy}(w_i)$. Applying again Corollary 5.20 this yields that $\sin(\alpha) \circ \cr_\xi \circ (c_1, d_1, c_2, d_2) = f_{xy} \circ \cr_{f(x)} \simeq (z_1, w_1, z_2, w_2)$ - we remark that the $\alpha$ is the same as before as the simplices $\sigma_1$ and $\sigma_2$ coincide. Hence $f_{xy}$ is a Möbius bijection. Since $T_{xy}$ is a geodesically complete tree and the thickness of $E_1$ implies that $|\partial_\infty T_{xy}| \geq 3$ we can apply Theorem 5.9 to derive that $f_{xy}$ is induced by an isometry.

\[\square\]

Corollary 5.25. Let $E_1$ and $E_2$ be thick Euclidean buildings and let $f : \Flag_\sigma(E_1) \to \Flag_\sigma(E_2)$ be a $\sigma$-Möbius bijection. Then we can rescale the metric on the irreducible factors of $E_1$ - denote this space by $E_1'$ - such that $f$ is the restriction of an isometry $F : E_1' \to E_2$ to the boundary.

Proof. From Lemma 4.3 we know that for every type $\xi \in \sigma$ we have that $f^* \cr_\xi = \cr_\xi$, as $f$ is a $\sigma$-Möbius bijection. Let $\sigma = \sigma_1 \circ \ldots \circ \sigma_k$ be the decomposition of $\sigma$ corresponding to the decomposition of $E_1$ into irreducible factors - the decompositions coincide as both buildings are thick and modeled over the same spherical Coxeter complex. Moreover, be $f = f_1 \times \ldots \times f_k$ the decomposition from Lemma 5.8. Then $f^* \cr_\xi = \cr_\xi$ for all $\xi \in \sigma$ implies that each $f_i$ is a $\sigma_i$-Möbius bijection. Thus the above theorem yields the claim.

\[\square\]

Corollary 5.26. Let $E_1, E_2$ be thick irreducible Euclidean buildings. Moreover, assume that there exists a wall tree $T_x$ for a panel $x \in \Delta_\infty E_1$ which has more than one branching point. Let $f : \Flag_\sigma(E_1) \to \Flag_\sigma(E_2)$ be a $\sigma$-Möbius bijection. Then $f$ can be extended to an isometry $F : E_1 \to E_2$ (without rescaling the metric).

Moreover, if $E_1$ is not a Euclidean cone over a spherical building then every wall tree has more than one branching point.

Proof. From Theorem 5.24 we know that we can rescale the metric by some $\mu \in \mathbb{R}$ such that $f$ is induced by an isometry $F : \mu E_1 \to E_2$, where $\mu E_1$ is $E_1$ with the metric rescaled by $\mu$. Let $x \in \Delta_\infty E_1$ be a panel such that the wall tree $T_x$ has more than one branching point. Then clearly the wall tree of $x \in \Delta_\infty \mu E_1$ is $\mu T_x$. Let $f_x : \partial_\infty T_x \to \partial_\infty T_{f(x)}$ be the induced map from $f$ on the wall tree. Since $F$ restricted to the boundary is $f$, the map induced from $F$ on $\partial_\infty \mu T_x$ equals $f_x$. Therefore we have $\cr_{T_x} = f_x^* \cr_{T_{f(x)}} = \mu \cr_{T_x} = \mu\cr_{T_x}$ (the first equality follows from $f$ being a $\sigma$-Möbius bijection, the second from $f_x = F_{\partial_\infty \mu T_x}$).

By assumption $T_x$ has two branching points. The distance of those two points can be given in terms of the cross ratio - i.e. let $p, q \in T_x$ be the branching points, then there exist $z_1, z_2, w_1, w_2 \in \partial_\infty T_x$ such that $d(p, q) = \ldots$
we derive from $\text{cr}_{T_\sigma}(z_1, w_1, z_2, w_2)$ [BS17 Lem 4.2]. Since this distance $d(p, q)$ is non-zero, we derive from $\text{cr}_{T_\sigma}(z_1, w_1, z_2, w_2) = \mu \text{cr}_{T_\sigma}(z_1, w_1, z_2, w_2)$ that $\mu = 1$. Hence $F$ is an isometry without rescaling the metric on $E_1$.

The second claim is a direct consequence of Proposition 4.21. and 4.26 in [KW14].

The second claim of Theorem [4] follows now from the fact that every $\sigma$-Moebius bijection splits as a product of $\sigma$-Moebius bijections on irreducible factors, as in the proof of Corollary 5.24. The corollary above implies that those $\sigma$-Moebius bijections induce isometries without the need of rescaling.

6 Appendix

Here, we determine the cross ratio for the symmetric spaces $X(n) := \text{SL}(n, \mathbb{R})/\text{SO}(n, \mathbb{R})$. We will use the notation as in Example 3.13.

The map $g \cdot \text{SO}(n, \mathbb{R}) \mapsto gg^t$ yields an identification of $X(n)$ with

$$P_n = \{ A \in \text{Mat}(n \times n, \mathbb{R}) | A = A^t \wedge \det(A) = 1 \wedge A \text{ is positive definite} \}.$$ 

The action of $g \in \text{SL}(n, \mathbb{R})$ on $A \in P_n$ is given by $g \cdot A = gAg^t$. By the definition of the cross ratio, it will be enough to determine $(\cdot | \cdot)_{I_n, \lambda}$ with $I_n$ being the identity matrix in $P_n$ and $\lambda = (\lambda_1, \ldots, \lambda_n)$ be identified with some type.

We begin with considering types $\lambda = (\lambda_1, \ldots, \lambda_n) \in \text{int}(\sigma)$: Let $S = (V_1, \ldots, V_n)$ with $V_i = \text{span}\{e_1, \ldots, e_i\}$ denote the standard flag - the $e_i$ being the standard base of $\mathbb{R}^n$. We know that $K = \text{SO}(n, \mathbb{R})$ acts transitively on $\text{Flag}_\sigma$, hence we need to determine $(k_1S|k_2S)_{I_n, \lambda}$ for $k_1, k_2 \in \text{SO}(n, \mathbb{R})$.

**Lemma 6.1.** Notations as before and in Example 3.13. Then for any $h, k \in \text{SO}(n, \mathbb{R})$

$$(kS|hS)_{I_n, \lambda} = n \sum_{j=1}^{n-1} (\lambda_{j+1} - \lambda_j) \log |\det(\hat{k}_1 | \cdots | \hat{k}_j | \hat{h}_1 | \cdots | \hat{h}_{n-j})|$$

with $\hat{h}_i$ denoting the $i$-th column of the matrix $h$ and accordingly $\hat{k}_i$.

**Proof.** Since $(\cdot | \cdot)_{I_n, \lambda}$ is invariant under the action of $\text{SO}(n, \mathbb{R})$, we have $(kS|hS)_{I_n, \lambda} = (h^{-1}kS|hS)_{I_n, \lambda}$, i.e. it reduces to determine $(kS|hS)_{I_n, \lambda}$ or in the same way $(S|kS)_{I_n, \lambda}$ for any $k \in \text{SO}(n, \mathbb{R})$.

Proposition 3.1 implies that $(S|kS)_{I_n, \lambda} = \frac{1}{N_{S_\lambda}} b_{S_\lambda}(I_n, n_{kS}(I_n, S) \cdot I_n)$, where $S_\lambda$ is point in the ideal boundary $\partial_{\infty}X(n)$ determined by the eigenvalue flag pair $(\lambda, S)$ and $n_{kS}(I_n, S) \in N_{kS}$, i.e. the element in the horospherical subgroup to $kS$ such that $n_{kS}(I_n, S) \cdot I_n \in P(kS, S)$.

We want to determine $n_{kS}(I_n, S) \cdot I_n$. First, we remark that $N_{kS} = kN_S k^{-1} = kN_S k^t$. The group $N_S$ consists of upper triangular matrices with ones on the diagonal.
Let $W$ be the regular flag opposite to $S$ in the flat containing $I_n$, i.e. $W = (V_1^*, \ldots, V_n^*)$ with $V_i^* = \text{span}\{e_n, \ldots, e_{n-i+1}\}$. Let

$$k_{w_0} = \begin{pmatrix}
-1 \\
1 \\
\vdots \\
1
\end{pmatrix} \in \text{SO}(n, \mathbb{R}).$$

Then $W = k_{w_0}S$. Since any $k \in \text{SO}(n, \mathbb{R})$ stabilizes $I_n$, the maximal flat through $kS$ and $I_n$ is the unique maximal flat (i.e. affine apartment) that joins $kS$ and $kW = k k_{w_0}S$. This yields $n k_S(I_n, S) = n k_S(k k_{w_0}S, S)$ - here $n k_S(k k_{w_0}S, S) \in N_k S$ is the unique element mapping $k k_{w_0}S$ to $S$.

Therefore we are looking for $n_S \in N_S$ such that $k n_S k_{w_0} k W = S$, i.e. $k n_S k_{w_0} \in \text{stab}(S)$. This is equivalent to $k n_S k_{w_0}$ being upper triangular. The elements in $N_S$ are upper triangular matrices with ones on the diagonal. Hence $n_S$ is such that

$$n_S k_{w_0} = \begin{pmatrix}
* & \cdots & * & -1 \\
\vdots & \ddots & \vdots & \vdots \\
* & \cdots & 1 & \vdots \\
1 & \cdots & \cdots & 1
\end{pmatrix} \quad \text{and} \quad - k_{1, n} \quad \in \quad \left(\begin{array}{cc}
\begin{array}{cccc}
-1 \\
1 \\
\vdots
\end{array} & - k_1 \\
\vdots & \vdots
\end{array}\right) n_S k_{w_0} = \begin{pmatrix}
* & \cdots & * \\
\vdots & \ddots & \vdots \\
* & \cdots & 1
\end{pmatrix}.$$

It is straightforward to check that the $j$-th column of $n_S k_{w_0}$ is given by

$$n_S k_{w_0} = \begin{pmatrix}
k_1, n-j+1 \\
k_1, n-j+2 \\
\vdots \\
k_1, n \\
k_j, n+1-j \\
k_j, n+1-j+1
\end{pmatrix} \begin{pmatrix}
- k_1 \\
\vdots \\
- k_n
\end{pmatrix} \begin{pmatrix}
1 \\
0 \\
\vdots \\
0
\end{pmatrix}, \quad \text{(6.1)}$$

and the $n$-th column equals $- \sum_{i=1}^n a_{i,1} k_i$. This yields

$$A := n k_S = \begin{pmatrix}
- k_1 \\
\vdots \\
- k_n
\end{pmatrix} \left(\begin{array}{cccc}
l_{n+1,1} k_1 \\
l_{n+1,2} k_1 \\
\vdots \\
l_{n+1,n} k_1 \\
l_{n+1,n+1-j}
\end{array}\right) = \begin{pmatrix}
1 \\
0 \\
\vdots \\
0
\end{pmatrix}, \quad \begin{pmatrix}
- k_1 \\
\vdots \\
- k_n
\end{pmatrix} \begin{pmatrix}
1 \\
0 \\
\vdots \\
0
\end{pmatrix} = \begin{pmatrix}
-1 \\
0 \\
\vdots \\
0
\end{pmatrix}.$$

Therefore, $n k_S(I_n, S) \cdot I_n = (k n_S k^t) \cdot I_n = k n_S n^t k^t = A A^t$. The Busemann function on $X(n)$ is well known - see Lemmata 2.4, 2.5 in [Hat95]. Namely, for $p \in P_n$ we have $b_{\Delta_j}(p, I_n) = n \log(\prod_{j=1}^{n-1} (\det \Delta_j(p))^{\lambda_{n-j} - \lambda_{n-1-j}})$, where $\Delta_j(p)$ is the lower right $j \times j$-minor of $p$ - e.g. $\Delta_1(p) = p_{n,n}$ or $\Delta_2(p) = \begin{pmatrix}
p_{n-1,n} & p_{n-1,n-1} \\
p_{n,n-1} & p_{n,n}
\end{pmatrix}$. This gives

$$(S|kS)_{I_n, \lambda} = \frac{n}{2} \sum_{j=1}^{n-1} (\lambda_{n+1-j} - \lambda_{n-j}) \log \det(\Delta_j(A A^t)).$$
We are left with determining $\det \Delta_j(AA^t)$ for all $1 \leq j \leq n$. Consider the matrix $(J)_{i,j} = \delta_{i,n+1-j}$ with $\delta_{i,j}$ being Kronecker’s delta. Then

$$AA^t = AJJA^t = \begin{pmatrix} a_{1,n} & \cdots & a_{1,1} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,1} \end{pmatrix} \begin{pmatrix} a_{1,n} \\ \vdots \\ a_{n,1} \end{pmatrix}$$

and by the structure of this product we get that $\Delta_j(AA^t) = \Delta_j(AJ)\Delta_j(JA^t)$. This yields $\det \Delta_j(AA^t) = \det \Delta_j(AJ) \det \Delta_j(JA^t) = a_{n,1}^2 \cdots a_{n+1-j,n}^2$.

If we apply Cramer’s rule to equation (6.1) we get

$$a_{j,n+1-j} = (-1)^{j+1} \det \begin{pmatrix} k_{1,n-j+2} & \cdots & k_{j-1,n-j+2} \\ \vdots & \ddots & \vdots \\ k_{1,n} & \cdots & k_{j-1,n} \end{pmatrix} / \det \begin{pmatrix} k_{1,n-j+1} & \cdots & k_{j,n-j+1} \\ \vdots & \ddots & \vdots \\ k_{1,n} & \cdots & k_{j,n} \end{pmatrix},$$

for $j \geq 2$ and $a_{1,n} = k_{1,n}^{-1}$. This implies

$$\det \Delta_{n-j}(AA^t) = a_{n,1}^2 \cdots a_{n+1-n-j}^2 = \det \left( \begin{pmatrix} I_{n-j} \\ 0 \end{pmatrix}_{k_i} \right) = \det(e_1 | \cdots | e_{n-j} | k_1 | \cdots | k_j)^2$$

with $I_{n-j}$ the $n-j \times n-j$ identity matrix, $k_i$ denoting the $i$-th row of $k$ and $e_i$ the $i$-th standard vector. In particular

$$(kS|S)_{l_n,\lambda} = (S|k^tS)_{l_n,\lambda} = n \sum_{j=1}^{n-1} (\lambda_{n+1-j} - \lambda_{n-j}) \log |\det(e_1 | \cdots | e_{n-j} | k_1 | \cdots | k_n)|,$$

with $k_i$ the $i$-th column of $k \in SO(n, \mathbb{R})$.

Let $k, h \in SO(n, \mathbb{R})$. Then the $i$-th column of $h^{-1}k$ is given by $h^{-1}k \cdot e_i = h^{-1}k_i$. Then $\det(h_1 | \cdots | h_{n-j}) = \det(h_1 | \cdots | h_{n-j})$. With this and the remark at the beginning of the proof we finally get that

$$(kS|hS)_{l_n,\lambda} = n \sum_{j=1}^{n-1} (\lambda_{n+1-j} - \lambda_{n-j}) \log |\det(h_1 | \cdots | h_{n-j})|,$$

hence

$$(kS|hS)_{l_n,\lambda} = n \sum_{j=1}^{n-1} (\lambda_{n+1-j} - \lambda_{n-j}) \log |\det(h_1 | \cdots | h_{n-j})|.$$

Now let $S_\tau$ be a non-full standard flag, i.e. $S_\tau = (V_i, \ldots, V_i)$ with $\tau = (i_1, \ldots, i_l)$, $i_j \in \{1, \ldots, n\}$ such that $i_l = n$, $i_j < i_m$ for $1 \leq j < m \leq l \leq n$, and $V_{i_j} = \text{span}\{e_i, e_{i+1}, \ldots, e_{i_l}\}$. Let $S_{\tau^*}$ be the standard opposite flag to $S_\tau$, i.e. $S_{\tau^*} = (V_i^*, \ldots, V_i^*)$ with $V_i^* = \text{span}\{e_n, e_{n-1}, \ldots, e_{i_j+1}\}$. Furthermore, be $\lambda = (\lambda_1, \ldots, \lambda_l) \in \mathbb{R}^l$ such that $\lambda_j > \lambda_{j+1}$, $\sum_{j=1}^{l} m_j \lambda_j = 0$ for $m_j = \dim V_{i_j} - \dim V_{i_{j-1}}$ if $j > 1$, $m_1 = \dim V_{i_1}$ and $\sum_{j=1}^{l} m_j \lambda_j^2 = 1$.
Lemma 6.2. Notations as before, in particular let $\lambda = (\lambda_1, \ldots, \lambda_l)$ be a type and $S_\tau$, $S_\iota$ the associated standard flags. Then

$$(kS_\tau|hS_\iota)_{I_n,\lambda} = n \sum_{j=1}^{l-1} (\lambda_{j+1} - \lambda_j) \log |\det(\hat{k}_1 | \cdots | \hat{k}_j | \hat{h}_1 | \cdots | \hat{h}_{n-1})|$$

with $\hat{h}_i$ denoting the $i$-th column of the matrix $h$ and accordingly $\hat{k}_i$.

Proof. This is a direct consequence of the lemma above together with Lemma 3.9. \qed

Proposition 6.3. Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ be a type, and $\tau$ such that $\lambda \in \text{int}(\tau)$. Let $V = (V_1, \ldots, V_l), Y = (Y_1, \ldots, Y_l) \in \text{Flag}_\tau$ and $W = (W_1, \ldots, W_l), Z = (Z_1, \ldots, Z_l) \in \text{Flag}_\iota$. Then

$$\text{cr}_\lambda(V, W, Y, Z) = n \sum_{j=1}^{l-1} (\lambda_j - \lambda_{j+1}) \log \left(\frac{V_j \land W_{i-j} \cdot Y_j \land Z_{i-j}}{V_j \land Z_{i-j} \cdot Y_j \land W_{i-j}}\right),$$

using the above conventions.

Proof. As mentioned in Example 3.13, the term is independent of the choices made.

By the transitivity of the $\text{SO}(n, \mathbb{R})$ action, we know that every flag $V \in \text{Flag}_\tau$ can be written as $hS_\tau$ for $S_\tau \in \text{Flag}_\tau$ the standard flag and some $k \in \text{SO}(n, \mathbb{R})$. Then the columns $\hat{k}_i$ are such that $V_j = \text{span}\{\hat{k}_1, \ldots, \hat{k}_i\}$. In the same way every flag $W \in \text{Flag}_\iota$ can be written as $hS_\iota$ for $S_\iota \in \text{Flag}_\iota$ the standard flag and some $k, h \in \text{SO}(n, \mathbb{R})$.

We fix the identification $\wedge^n \mathbb{R}^n \simeq \det$. Let $k, h \in \text{SO}(n, \mathbb{R})$ such that $V = kS_\tau$ and $W = hS_\tau$. Then $|V_j \land W_{i-j}| = |\det(\hat{k}_1 | \cdots | \hat{k}_j | \hat{h}_1 | \cdots | \hat{h}_{n-1})|$. Therefore the claim follows from the lemma above. \qed

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