Explicit travelling waves and invariant algebraic curves

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Abstract
We introduce a precise definition of algebraic travelling wave solution of \(n\)-th order partial differential equations and prove that the only algebraic travelling waves solutions for the celebrated Fisher–Kolmogorov equation are the ones found in 1979 by Ablowitz and Zeppetella. This question is equivalent to study when an associated one-parameter family of planar ordinary differential systems has invariant algebraic curves.

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1. Introduction and main results

Mathematical modelling of dynamical processes in a great variety of natural phenomena leads in general to non-linear partial differential equations. There is a particular class of solutions for these non-linear equations that are of considerable interest. They are the travelling wave solutions (TWSs) [\(1, 6, 11, 13, 14, 29\)]. Such a wave is a special solution of the governing equations, that may be localized or periodic, which does not change its shape and which propagates at constant speed. In the case of linear equations the profile is usually arbitrary. In contrast, a non-linear equation will normally determine a restricted class of profiles, as the result of a balance between nonlinearity and dissipation. These waves appear in fluid dynamics [\(16, 20\)], chemical kinetics involving reactions [\(11, 21\)], mathematical biology [\(14, 26\)], lattice vibrations in solid state physics [\(24\)], plasma physics and laser theory [\(15\)], optical fibers [\(4\)], etc.
In these systems the phenomena of dispersion, dissipation, diffusion, reaction and convection are the fundamental physical common facts.

Consider general \( n \)-th order partial differential equations of the form

\[
\frac{\partial^n u}{\partial x^n} = F\left(u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \ldots, \frac{\partial^n u}{\partial x^n}, \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial x \partial t}, \ldots, \frac{\partial^n u}{\partial x^n \partial t}, \frac{\partial u}{\partial t}, \ldots, \frac{\partial^n u}{\partial t^n}\right),
\]

(1)

where \( x \) and \( t \) are real variables and \( F \) is a smooth map. The TWSs of (1) are particular solutions of the form \( u = u(x, t) = U(x - ct) \). Plugging this expression into (1) we get that \( U(s) \) has to be a solution of the \( n \)-th order ordinary differential equation

\[
U^{(n)} = F(U, U', -cU', U''', -cU''', c^2 U''', \ldots, U^{(n-1)}', -cU^{(n-1)}', \ldots, (-c)^{n-2} U^{(n-1)}, (-c)^{n-1} U^{(n)})
\]

(2)

where \( U = U(s) \) and the derivatives are taken with respect to \( s \). The parameter \( c \) is called the speed of the TWS. We introduce the following definition:

**Definition.** We will say that \( u(x, t) = U(x - ct) \) is an algebraic TWS of (1) if \( U(s) \) is a non constant function that satisfies (2) and there exists a polynomial \( p \in \mathbb{R}[z, w] \) such that \( p(U(s), U'(s)) = 0 \).

All the explicit TWSs known by the authors are algebraic when \( F \) is a polynomial. In particular, in lemma 2.1 we will prove that the class of TWSs given by \( U(s) = q(e^{\lambda s}) \) for some non-zero real number \( \lambda \) and some non-constant rational function \( q \in \mathbb{R}(z) \) are always algebraic TWSs.

Some well-known examples having TWSs with \( U(s) = q(e^{\lambda s}) \) are Fisher–Kolmogorov, Burgers and Boussinesq equations. More complicated algebraic TWSs appear for instance in the Korteweg–de Vries and the so-called improved modified Boussinesq equations. Next we list some known results for the Fisher–Kolmogorov equation. The other examples are recalled in section 2.

The celebrated Fisher–Kolmogorov reaction-diffusion equation

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1 - u).
\]

(3)

was introduced in 1937 in the classical papers [8, 19] to model the spreading of biological populations; see also [10] for some recent results. From [8, 19], it is known that travelling waves \( u = U(x - ct) \) satisfying

\[
\lim_{s \to -\infty} U(s) = 1 \quad \text{and} \quad \lim_{s \to \infty} U(s) = 0,
\]

(4)

only exist for \( c \geq 2 \). These boundary conditions are the interesting ones because they correspond to the meaningful solutions of the real problems modelled by the equation. In [2], Ablowitz and Zeppetella proved that when \( c = 5/\sqrt{6} \), the equation exhibits the algebraic TWS,

\[
U(x, t) = \frac{1}{1 + ke^{\frac{1}{\sqrt{6}} (s - \frac{x}{\sqrt{6}})}}, \quad k > 0.
\]

(5)

These explicit TWSs have been found by applying the Painlevé method; see [12] for an introduction to this method.

In fact, algebraic TWSs for (3) can exist only when an associated one-parameter family of planar systems of ordinary differential equations has invariant algebraic curves. By studying this equivalent problem we obtain the main result of this paper:
Theorem 1.1. The Fisher–Kolmogorov equation (3) has algebraic TWSs satisfying the boundary conditions (4) if and only if the speed is \( c = \frac{5}{\sqrt{6}} \) and they are the ones given in (5) found by Ablowitz and Zeppetella.

Many papers are devoted to find explicit TWSs. On the contrary, our point of view allows to characterize when a partial differential equation has explicit algebraic TWSs. To the best of our knowledge, this is the first time that a result of this type has been obtained. This approach can also be applied to study other polynomial partial differential equations, like for instance the Nagumo equation ([25, chapter 13]), the Newell–Whitehead–Segel equation ([27, 32]), the Zeldovich equation ([36]) or some of the equations considered in [12, 14, 30, 31, 35].

2. Examples of algebraic TWS

This section collects several examples of known partial differential equations with algebraic TWSs. First we prove that a especial class of TWSs are algebraic.

Lemma 2.1. If equation (1) has a TWS with

\[
U(s) = q(e^{\lambda s}),
\]

for some real number \( \lambda \neq 0 \) and some non-constant rational function \( q \), then it is an algebraic TWS.

Proof. Note that

\[
U'(s) = \lambda q'(e^{\lambda s}).
\]

Write

\[
U(s) = \frac{q_1(z)}{q_2(z)},
\]

and

\[
U'(s) = \frac{q_3(z)}{q_4(z)},
\]

with \( z = e^{\lambda s} \), for some polynomials \( q_j \in \mathbb{R}[z], j = 1, \ldots, 4 \). Next, define

\[
p(U, U') = \text{Res}(q_2(z)U - q_1(z), q_4(z)U' - q_3(z), z),
\]

where \( \text{Res}(M(z), N(z), z) \) denotes the resultant of the polynomials \( M \) and \( N \) with respect to \( z \); see [34, p 45]. Then, clearly \( p(U(s), U'(s)) = 0 \) for some polynomial \( p \), as we wanted to prove. \( \square \)

Apart from the Fisher–Kolmogorov equation, we will see that Burgers and Boussinesq equations have also TWSs of the form

\[
U(s) = q(e^{\lambda s}),
\]

for some rational function \( q \), and hence, as a consequence of lemma 2.1, algebraic TWSs.

The Burgers equation is

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} - a \frac{\partial^2 u}{\partial x^2} = 0,
\]

where \( a \neq 0 \) is an arbitrary constant. This equation appears in the modelling of acoustic and hydrodynamic waves, gas dynamics and traffic flow (see [28]) and has the one-parametric family of solutions

\[
u(x, t) = c \left( 1 - \tanh \left( \frac{c}{2a} (x - ct) \right) \right),
\]

where \( c \), the speed of the wave, is an arbitrary constant.

The Boussinesq equation is

\[
\frac{\partial^2 u}{\partial t^2} + u \frac{\partial^2 u}{\partial x^2} - a \frac{\partial^2 u}{\partial x^2} + \left( \frac{\partial u}{\partial x} \right)^2 - \frac{\partial^4 u}{\partial x^4} = 0
\]

and describes surface water waves (see [1, 16]). It has the two-parametric family of solutions

\[
\nu(x, t) = \left( 1 - 8k^2 - c^2 \right) + 12k^2 \tanh^2 (k(x - ct)),
\]

where \( k \) and \( c \) are arbitrary constants.

We remark that for the Fisher–Kolmogorov equation, algebraic TWSs only exist for a fixed value of the speed \( c \), while for the other two examples given above the speed \( c \) is arbitrary. As
we will see in remark 3.3 our approach for proving theorem 1.1 will also provide an explanation of this fact.

Next two examples show more complicated algebraic TWSs. The famous Korteweg–de Vries equation
\[
\frac{\partial u}{\partial t} - 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0
\]
appears in several domains of physics, non-linear mechanics, water waves, etc (see [1,6,18,28]).
It has the three parametric family of cnoidal waves given by
\[
u(x, t) = -\frac{c}{4k^2} + \frac{4k^2}{2m - 1} - \frac{12k^2 m}{6},
\]
where \(c, k\) and \(m\) are arbitrary parameters and \(\text{cn}(x, m)\) is the Jacobi elliptic function of elliptic modulus \(m\) that reduces to \(\cos(x)\) when \(m = 0\) and to \(1/\cosh(x)\) when \(m = 1\) (see [3]). For this example
\[
p(U, U') = -108(U')^2 + 4k^2 (2m - 1) - 12k^2 m \text{cn}^2(k(x - ct), m),
\]
The improved modified Boussinesq equation is
\[
\frac{\partial^2 u}{\partial t^2} - u \frac{\partial^2 u}{\partial x^2} - \left(\frac{\partial u}{\partial x}\right)^2 - \frac{\partial^4 u}{\partial x^2 \partial t^2} = 0.
\]
It appears in the modelling of non-linear waves in a weakly dispersive medium (see for instance [17]) and has the three-parametric family of TWSs
\[
u(x, t) = c^2 - 1 + 4c^2 k^2 - 8c^2 m k^2 + 12c^2 m k^2 \text{cn}^2(k(x - ct), m),
\]
where \(c, k\) and \(m\) are arbitrary constants, see [37]. For this case
\[
p(U, U') = 3c^2 (U')^2 + U^3 + 3(1 - c^2) U^2 + \left(48c^4 (m - m^2 - 1) k^4 + 3(1 - c^2)^3\right) U
+ 64c^6 (-1 + 2m)(m + 1)(m - 2) k^6 + 48c^4 (1 - c^2)(m - m^2 - 1) k^4 + (1 - c^2)^3 .
\]

3. Travelling waves and invariant algebraic curves

It is said that an \(n\)-dimensional differential system has an invariant algebraic curve \(C\) if this curve is invariant by the flow and moreover it is contained in the intersection of \(n - 1\) functionally independent algebraic varieties of codimension one.

Next result relates the existence of an algebraic TWS for (1) with the existence of invariant algebraic curves of an associated differential system. When \(n = 2\) the result is simply by definition of algebraic TWS.

**Proposition 3.1.** If the partial differential equation (1) has an algebraic TWS with speed \(c\) then the first order differential system
\[
\begin{align*}
y'_1 &= y_2, \\
y'_2 &= y_3, \\
\vdots &= \vdots \\
y'_{n-1} &= y_n, \\
y'_n &= G_c(y_1, y_2, \ldots, y_n),
\end{align*}
\]

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where
\[ G_c(y_1, y_2, \ldots, y_n) = F(y_1, y_2, -cy_2, y_3, -c^2 y_3, \ldots, y_n, -cy_n, \ldots, (-c)^{n-2} y_n, (-c)^{n-1} y_n). \]
has an invariant algebraic curve.

**Proof.** If equation (1) has an algebraic TWS, \( u(x, t) = U(x - ct) \), then, by definition, \( p(U(s), U'(s)) = 0 \) for some polynomial \( p \). Hence when \( n = 2 \) there is nothing to be proved. When \( n \geq 3 \), we define \( p_1 := p \) and doing successive derivatives we obtain \( n - 2 \) new polynomials \( p_j, j = 2, \ldots, n - 1 \), for which
\[ p_j(U(s), U'(s), U''(s), \ldots, U^{(j)}(s)) = 0. \]
Therefore, the vector function
\[ (y_1(s), y_2(s), \ldots, y_n(s)) = (U(s), U'(s), \ldots, U^{(n-1)}(s)) \]
is a parametric representation of a curve \( C \) in the phase space of the system (6) associated to (1). Moreover, \( C \) is an algebraic curve, because it is contained in the intersection of the \( n - 1 \) functionally independent algebraic hypersurfaces \( p_j(y_1, \ldots, y_{j+1}) = 0, j = 1, 2, \ldots, n - 1 \), as we wanted to prove.

**Remark 3.2.** If the function \( U(s) \) defining the algebraic TWS in proposition 3.1 satisfies the boundary conditions
\[ \lim_{s \to -\infty} U(s) = a \quad \text{and} \quad \lim_{s \to \infty} U(s) = b, \]
then the corresponding curve given in (7) is a homoclinic \((a = b)\) or a heteroclinic \((a \neq b)\) algebraic invariant solution of system (6) joining the critical points \((a, 0, \ldots, 0)\) and \((b, 0, \ldots, 0)\).

**Remark 3.3.** It is not difficult to prove that the systems (6) associated to Burgers and Korteweg–de Vries equations have, for any \( c \in \mathbb{R} \), all their solutions contained in algebraic curves. In particular this implies that all their travelling waves are algebraic. On the contrary, as we will see in next section, the system (6) associated to Fisher–Kolmogorov equation for \( c \geq 2 \) has only algebraic curves (passing through the critical points \((0, 0, \ldots, 0)\) and \((1, 0, \ldots, 0)\)) when \( c = 5/\sqrt{6} \). Therefore, for Fisher–Kolmogorov equation, algebraic TWSs only exist for this value of \( c \).

The problem of determining necessary conditions for the existence of algebraic invariant curves for \( n \)-dimensional polynomial differential systems goes back to the work of Poincaré. This problem has been extensively investigated in the last years for the case \( n = 2 \), see for instance [5, 7, 22] and references therein. Let us recall recall some basic results that we will need for proving theorem 1.1.

Consider a planar differential system,
\[ \begin{cases} x' = P(x, y), \\ y' = Q(x, y), \end{cases} \]
where \( P \) and \( Q \) are polynomials of degree at most \( N \), and assume that there is a polynomial \( g(x, y) \) such that the set \( \{ g(x, y) = 0 \} \) is non-empty and invariant by the flow of (9). If \( g \) is not irreducible in \( \mathbb{C}[x, y] \) then there exist several irreducible polynomials, \( \tilde{g}_j, j = 1, \ldots, k \), such that for each \( j \), the corresponding set \( \{ \tilde{g}_j(x, y) = 0 \} \) is also non-empty and invariant by the flow of the system and \( \{ g(x, y) = 0 \} = \bigcup_{j=1}^k \{ \tilde{g}_j(x, y) = 0 \} \).

For irreducible polynomials, we have the following algebraic characterization of invariant algebraic curves, which is the one that we will use in section 4, see again [5, 7, 22].
Proposition 3.4. Given an irreducible polynomial \( f(x, y) \), then \( f(x, y) = 0 \) is an invariant algebraic curve for the system (9) if and only if there exists a polynomial of degree at most \( N - 1 \), \( k(x, y) \), called the cofactor of \( f \), such that
\[
P(x, y) \frac{\partial f(x, y)}{\partial x} + Q(x, y) \frac{\partial f(x, y)}{\partial y} - k(x, y) f(x, y) = 0. \tag{10}
\]

When an invariant algebraic curve passes by an elementary critical point, in many cases, the value of the cofactor at this point can be obtained. These type of results, based on previous works of Seidenberg (\cite{33}), are proved in \cite{5}. In the next proposition, which is included in \cite[theorem 14]{5}, we state one of these cases.

Proposition 3.5. Let \( f(x, y) = 0 \) be an invariant algebraic curve of a planar system with corresponding cofactor \( k(x, y) \). Assume that it contains a critical point of the system, \((x_0, y_0)\), and that it is a hyperbolic saddle with eigenvalues \( \lambda^- < 0 < \lambda^+ \). Then \( k(x_0, y_0) \in \{\lambda^+, \lambda^-, \lambda^+ + \lambda^-\} \).

The research on the existence of invariant algebraic curve when \( n > 2 \) is only beginning, see for instance \cite{9, 23}. A characterization similar to (10) can also be used for \( n \)-dimensional systems to determine codimension one invariant algebraic varieties, see for instance \cite{23}. For finding invariant algebraic curves the cofactor becomes a \((n - 1) \times (n - 1)\) matrix of cofactors, see \cite{9}.

4. Proof of theorem 1.1

As we will readily see, theorem 1.1 is a simple corollary of next result, which is a key point of our paper.

Theorem 4.1. Consider the system
\[
\begin{align*}
x' &= -y, \\
y' &= -x - cy + x^2,
\end{align*}
\tag{11}
\]
with \( c \geq 2 \). Assume that it has an irreducible invariant algebraic curve that passes through the origin. Then \( c = 5/\sqrt{6} \) and this curve is
\[
y^2 + 2\sqrt{\frac{2}{3}}(1 - x)y + \frac{2}{3}x(1 - x)^2 = 0. \tag{12}
\]

Proof of theorem 1.1. By \cite{8, 19} we already know that TWSs for equation (3), satisfying the boundary conditions (4), only exist when \( c \geq 2 \). Moreover, by the definition of algebraic TWS, this type of solutions only exist when the planar system
\[
\begin{align*}
y'_1 &= y_2, \\
y'_2 &= -cy_2 - y_1(1 - y_1),
\end{align*}
\]
has an invariant algebraic curve \( g(y_1, y_2) = 0 \), containing the critical points \((0, 0)\) and \((1, 0)\). Furthermore, without loss of generality, we can assume that it is irreducible. Taking \( x = 1 - y_1 \) and \( y = y_2 \) we obtain system (11). Then, it should also have an irreducible invariant algebraic curve \( f(x, y) = 0 \), with \( f(0, 0) = f(1, 0) = 0 \). By theorem 4.1 we get that \( c = 5/\sqrt{6} \) and \( f \) has to be the function given in (12). Then the theorem follows by simple computations. \( \square \)

Finally, for proving theorem 4.1 we need some preliminary results. The first one collects two properties of the Gamma function. The second one gives some properties of the possible invariant algebraic curves of system (11) and their associated cofactors. We also recall that the Pochhammer symbol is \( x^{[m]} := x(x + 1)(x + 2) \cdots (x + m - 1) \)
Lemma 4.2. For \( x, y \in \mathbb{R} \) and \( p, q, m \in \mathbb{N} \),

(i) \( \sum_{j=0}^{m} \binom{m}{j} \Gamma(x + j) \Gamma(y + m - j) = \frac{\Gamma(x) \Gamma(y) \Gamma(x + y + m)}{\Gamma(x+y)} \).

(ii) \( \sum_{j=0}^{m} \binom{m}{j} (m-j) \Gamma(x+j) \Gamma(y+m-j) = \frac{m \Gamma(x) \Gamma(y) \Gamma(x+y+m)}{(x+y) \Gamma(x+y)} \).

Lemma 4.3. Let

\[ f_n(x, y) = h_n(x) y^n + h_{n-1}(x) y^{n-1} + \cdots + h_1(x) y + h_0(x) = 0 \]  

be an irreducible invariant algebraic curve of degree \( n \) of system (11). Then its cofactor \( h_k(x, y) \) must be constant, i.e. \( k(x, y) \equiv c_0 \), its degree has to be even \( n = 2m \) and

\[ h_0(x) = \left( \frac{2}{3} \right)^m x^{3m} + O(x^{3m-1}); \]

\[ h_1(x) = \frac{1}{5} \left( \frac{2}{3} \right)^m \left( 5c_0 \right)^{(\frac{2}{3})^m} + O(x^{3m-2}). \]

Proof. Since the system (11) is quadratic, by proposition 3.4 the cofactor of \( f_n(x, y) = 0 \) has to be linear, i.e. \( k(x, y) \equiv c_0 \). Then, equation (10) writes as

\[ -y \frac{\partial f_n(x, y)}{\partial x} + (-x - cy + x^3) \frac{\partial f_n(x, y)}{\partial y} = (c_0 + c_1 x + c_2 y) f_n(x, y) = 0. \]

Imposing that the higher order term in \( y \) of the above equation vanishes we get the differential equation \( c_2 h_n(x) + h'_n(x) = 0 \). Since \( h_n \) has to be a polynomial we obtain that \( c_2 = 0 \) and that \( h_n(x) \) is a constant. Hence, without loss of generality, we can assume that \( h_n(x) \equiv 1 \). Then, equality (16) is equivalent to the following set of linear differential equations

\[ h_{j-2}(x) = j x (x-1) h_j(x) - ((j-1)c + c_0 + c_1 x) h_{j-1}(x), \quad j = n + 1, n, \ldots, 2, 1; \]

where \( h_n(x) \equiv 1 \) and \( h_{n+1}(x) \equiv h_{-1}(x) \equiv 0 \).

If \( c_1 \neq 0 \), using (17) we can obtain the degrees of the functions \( h_j \). They are:

\[ \deg(h_{n-k}) = 2k, \quad k = 0, 1, \ldots, n - 1, n. \]

In particular \( \deg(h_1) = 2n - 2 \) and \( \deg(h_0) = 2n \). From (17), for \( j = 1 \), we obtain that

\[ -c_0 h_0(x) - c_1 x h_0(x) - x h_1(x) + x^2 h_1(x) = 0. \]

Studying the higher order terms in \( x \) of this equation we get that relation (18) can never be satisfied. As a consequence \( c_1 = 0 \) and so \( k(x, y) \equiv c_0 \) as we wanted to prove.

Consider now equation (17) with \( c_1 = 0 \). Assume, to arrive to a contradiction, that \( n \) is odd. Studying again the degrees of the functions \( h_j \) we get that

\[ \deg(h_{n-k}) = 3k \quad \text{and} \quad \deg(h_{n-(2k+1)}) \leq 3k + 1, \quad k = 0, 1, \ldots, (n-1)/2. \]

In particular, \( \deg(h_0) \leq (3n-1)/2 \) and \( \deg(h_1) = 3(n-1)/2 \). Again, as in the case \( c_1 \neq 0 \), the higher order terms in \( x \) corresponding to equation (18) can not be cancelled. Therefore \( n = 2m \), as we wanted to prove.

To prove (14) and (15) note that the coefficients \( h_j \) of \( f_n \) must satisfy the differential equations (17), with \( c_1 = 0 \). Arguing as in that first part of the proof of the lemma we obtain the degrees of each \( h_j \). We can write

\[ h_j(x) = a_j (2m)x^{\deg(h_j)} + O(x^{\deg(h_j)-1}), \]

where

\[ \deg(h_j) = \begin{cases} 3k - 2, & \text{when } j = 2m - (2k - 1), \\ 3k, & \text{when } j = 2m - 2k, \end{cases} \]

for \( k = 0, 1, \ldots, m \) and \( a_{2m}(2m) = 1 \). Let us determine these functions.
Plugging the above expressions in (17) we obtain that the terms \( a_j = a_j(2m) \) satisfy the following recurrences

\[
a_{2m-2k} = \frac{2m - (2k - 2)}{3k} a_{2m-(2k-2)}, \quad k = 1, 2, \ldots, m, \tag{19}
\]

\[
a_{2m-(2k+1)} = \frac{(2m - (2k - 1))a_{2m-(2k-1)} + h(2m - 2k)a_{2m-2k}}{3k + 1}, \quad k = 1, 2, \ldots, m - 1, \tag{20}
\]

where \( h(j) = -(c_0 + jc) \) and the initial conditions are \( a_{2m} = 1 \) and \( a_{2m-1} = h(2m) = -(c_0 + 2mc) \). The even terms \( a_{2j} \) can be easily obtained from (19). We get

\[
a_{2m-2j} = \left(\frac{m}{j} \right) \left(\frac{2}{3} \right)^j \tag{21}
\]

and in particular \( a_0 = (2/3)^m \) as we wanted to prove. It remains to obtain the general expression of the last odd term \( a_1 = a_1(2m) \). We take advantage of the linearity of the problem with respect to the initial condition \( a_{2m-1} \) and write

\[
a_1(2m) = -\left(\tilde{a}_1(2mc_0 + \tilde{a}_1(2m)c) \right),
\]

where \( \tilde{a}_1 \) and \( \tilde{a}_1 \) are the solution of the recurrences (19)–(20) with initial conditions \( a_{2m} = 1 \) and \( a_{2m-1} = 1 \) or \( a_{2m-1} = 2m \), respectively.

Substituting expression (21) in (20) and developing the recurrent expressions we arrive at

\[
\tilde{a}_1(2m) = \sum_{j=0}^{m-1} \left(\frac{m}{j} \right) \left(\frac{2}{3} \right)^j \prod_{k=0}^{m-j-1} (2k + 1) \prod_{k=j}^{m-1} \frac{1}{3k + 1}.
\]

\[
\tilde{a}_1(2m) = 2 \sum_{j=0}^{m} (m - j) \left(\frac{m}{j} \right) \left(\frac{2}{3} \right)^j \prod_{k=0}^{m-j-1} (2k + 1) \prod_{k=j}^{m-1} \frac{1}{3k + 1}.
\]

We introduce the following auxiliary functions

\[
\alpha(m) = \frac{\Gamma \left(\frac{1}{2} \right) \Gamma \left(\frac{1}{3} + m \right)}{\Gamma \left(\frac{1}{2} \right) \Gamma \left(\frac{1}{3} \right)}, \quad \beta(m) = \frac{\Gamma \left(\frac{1}{2} \right) \Gamma \left(\frac{2}{3} + m \right)}{\Gamma \left(\frac{2}{3} \right) \Gamma \left(\frac{1}{3} + m \right)} = \left(\frac{2}{3} \right)^m \frac{\Gamma \left(\frac{1}{2} \right) \Gamma \left(\frac{1}{3} + m \right)}{\Gamma \left(\frac{1}{3} \right)}.
\]

Let us simplify the expressions of \( \tilde{a}_1 \) and \( \tilde{a}_1 \) using the above functions and the equalities given in lemma 4.2.

\[
\tilde{a}_1(2m) = \frac{1}{\alpha(m)} \left(\frac{2}{3} \right)^m \sum_{j=0}^{m-1} \left(\frac{m}{j} \right) \Gamma \left(\frac{1}{2} + m - j \right) \Gamma \left(\frac{1}{3} + j \right)
\]

\[
= \frac{1}{\alpha(m)} \left(\frac{2}{3} \right)^m \left(\frac{\Gamma \left(\frac{1}{2} \right) \Gamma \left(\frac{1}{3} + m \right)}{\Gamma \left(\frac{1}{3} \right)} - \alpha(m) \right) = \left(\frac{2}{3} \right)^m (\beta(m) - 1).
\]

Similarly,

\[
\tilde{a}_1(2m) = \frac{2}{\alpha(m)} \left(\frac{2}{3} \right)^m \sum_{j=0}^{m} (m - j) \left(\frac{m}{j} \right) \Gamma \left(\frac{1}{2} + m - j \right) \Gamma \left(\frac{1}{3} + j \right)
\]

\[
= \frac{2}{\alpha(m)} \left(\frac{2}{3} \right)^m \frac{\Gamma \left(\frac{1}{2} \right) \Gamma \left(\frac{1}{3} + m \right)}{\Gamma \left(\frac{1}{3} \right)} m = \left(\frac{2}{3} \right)^m \frac{6}{5} \beta(m) m.
\]

Hence

\[
a_1(2m) = -\left(\frac{2}{3} \right)^m (\beta(m) - 1)c_0 + \frac{6}{5} \beta(m) mc
\]

\[
= \frac{1}{5} \left(\frac{2}{3} \right)^m \left(5c_0 - (5c_0 + 6mc) \frac{\Gamma \left(\frac{1}{2} \right) \Gamma \left(\frac{1}{3} + m \right)}{\Gamma \left(\frac{1}{3} \right)} \right),
\]

as we wanted to prove. \[\square\]
Proof of theorem 4.1. By lemma 4.3 we know that the invariant curve has constant cofactor
\( k(x, y) = c_0 \), even degree \( n = 2m \), and it can be written as in (13), with \( h_0 \) and \( h_1 \) given in (14) and (15). Using that \( h_0 \) and \( h_1 \) must satisfy (18) we get the identity
\[-c_0 h_0(x) - x h_1(x) + x^2 h_1(x) \equiv 0.\]
Since,
\[-c_0 h_0(x) - x h_1(x) + x^2 h_1(x) = -5c_0 + 6mc \left( \frac{5}{3} \right)^m \frac{(\frac{5}{3})^m}{\left( \frac{1}{3} \right)^m} x^{3m} + O\left(x^{3m-1}\right),\]
we get that
\[5c_0 + 6mc = 0. \tag{22}\]
The origin of (11) is a saddle point with eigenvalues \( \lambda^\pm = -c \pm \sqrt{c^2 + 4} \), where \( \lambda^- < 0 < \lambda^+ \). Since, by hypothesis, \( f(0, 0) = 0 \) we can apply proposition 3.5 to determine \( c_0 = k(0, 0) \). We obtain that \( c_0 \in \{ \lambda^+, \lambda^-, -c \} \). When \( c_0 = -c \), equation (22) gives \( (6m - 5) c = 0 \), which is in contradiction with the hypothesis \( c \geq 2 \). Therefore, if the system has an algebraic invariant curve under the above hypotheses, then \( c_0 \in \{ \lambda^+, \lambda^- \} \). Take \( c_0 = \lambda^\pm \). Hence, equation (22) writes as \( 6mc + 5\lambda^\pm = 0 \), or equivalently,
\[c = \mp \frac{5}{6} \frac{1}{\sqrt{m(6m - 5)}}.\]
Imposing that \( c \geq 2 \) we get that the only possibility is \( c_0 = \lambda^- \) and \( m = 1 \). Then, \( c = 5/\sqrt{6} \) and \( n = 2 \), as we wanted to prove. Finally, simple computations give (12) and the theorem follows.

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