Data compression limit for an information source of interacting qubits

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Abstract

A system of interacting qubits can be viewed as a non-i.i.d quantum information source. A possible model of such a source is provided by a quantum spin system, in which spin-1/2 particles located at sites of a lattice interact with each other. We establish the limit for the compression of information from such a source and show that asymptotically it is given by the von Neumann entropy rate. Our result can be viewed as a quantum analogue of Shannon’s noiseless coding theorem for a class of non-i.i.d. quantum information sources.
1 Introduction

In this paper we study the issue of compression of information from a particular class of quantum information sources, formed by systems of interacting qubits [see Section 2 for details]. Our aim is to quantify the minimal physical resources necessary to store the output from such a source or to transmit it through a noiseless channel. We shall use the words message, signal and output from a source interchangeably. The parameter that we minimise is the dimension of the Hilbert space to which a typical signal can be projected (i.e., “compressed”) with high fidelity. In addition, it is expected that the interaction between qubits in the systems under consideration yields highly-entangled states; this is a motivation for the present work, even though the issue of entanglement is not discussed here.

The analysis that follows shows that the data compression limit for output from such a source is given by the von Neumann entropy rate. This result can be viewed as a quantum analogue of Shannon’s noiseless coding theorem [21] for our class of non-i.i.d quantum sources. It can be considered as an extension of Schumacher’s coding theorem [20].

Shannon’s noiseless coding theorem quantifies the extent to which one can compress the information being produced by a classical information source. A standard model of such a source is described by a sequence of random variables $X_1, X_2, \ldots, X_n$ whose values $x_1, x_2, \ldots, x_n$ represent the output of the source. For simplicity, consider random variables which take values from a finite alphabet of symbols or letters (extensions to infinite alphabets also hold). Let $X := (X_1, X_2, \ldots, X_n)$ denote the sequence of random variables representing the source and $x := (x_1, x_2, \ldots, x_n)$ the values that it takes. The source is described by a set of probabilities $p(x) := \text{Prob} (X = x)$.

An i.i.d classical source is one for which the random variables $X_1, X_2 \ldots X_n$ are independent and identically distributed. In this case

$$p(x) := p(x_1)p(x_2)\ldots p(x_n),$$

where $\{p(x)\}$ is the single symbol distribution.

The main ingredient of Shannon’s noiseless coding theorem is the Shannon entropy given by

$$H(X) := -\sum_x p(x) \log_2 p(x);$$

for an i.i.d. source this reduces to $H(X) = nH(X)$, where

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In classical information theory one encodes the signal from a source into a string of binary digits (or bits). For purposes of storage and transmission, the aim is to encode the messages with sequences that are as short as possible.
An information source (classical or quantum) has redundancy, in the sense that certain outputs occur more frequently than the rest. This fact can be used to compress the source output: data compression is achieved by assigning shorter descriptions to the most frequent outputs of the source. The compression of data from a classical information source works as follows: A compression map, $C^n$, of rate $R$ takes a sequence $x = (x_1, \ldots, x_n)$ of length $n$ to a binary string of length $\lfloor nR \rfloor$ (the symbol $\lfloor \cdot \rfloor$ denoting the integer part). A decompression map, $D^n$, takes a binary string of length $\lfloor nR \rfloor$ to a string of symbols of length $n$. The compression scheme is said to be reliable if with probability approaching one, as $n \to \infty$, $D^n(C^n(x)) = x$.

Shannon’s noiseless coding theorem indicates how well such a compression scheme works. More precisely, it asserts that for a large class of sources (i.e., stationary and ergodic), the mean length of encoded bit sequences is asymptotically given by the Shannon entropy, $H(X)$. More precisely, the data compression limit, which is the limiting number of bits per symbol, is given by the Shannon entropy rate:

$$h := \lim_{n \to \infty} \frac{1}{n} H(X_1 \ldots X_n).$$

An attempt to represent the source using fewer bits than this would result in a high probability of error when the information is decompressed. Hence a compression scheme of rate $R$ is reliable only if $R > h$.

A quantum information source is defined in this paper by a set of distinguishable quantum-mechanical states $|\psi_j\rangle$, i.e., orthonormal vectors from a given Hilbert space, and a set of corresponding probabilities $\{\kappa_j\}$. We interpret the $|\psi_j\rangle$’s as signals of the source and the $\kappa_j$’s as the probabilities with which the signals are produced. Such a definition arises naturally from the density matrix formalism where a quantum-mechanical system is described by a convex linear combination of pure states:

$$\rho = \sum_j \kappa_j |\psi_j\rangle \langle \psi_j|.$$ 

Here the $|\psi_j\rangle$’s are identified as the orthonormal eigenvectors of $\rho$. The eigenvalue of $\rho$ corresponding to $|\psi_j\rangle$ is $\kappa_j$, and we have

$$\kappa_j \geq 0 \quad \text{and} \quad \sum_j \kappa_j = 1.$$ 

More precisely, we deal with a sequence of $2^n \times 2^n$ density matrices $\rho_n$, $n \to \infty$, and relate asymptotic properties of their eigenvalues $\kappa_j^{(n)}$ to the von Neumann entropy rate. The von Neumann entropy rate is given by

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$$\rho = \sum_j \kappa_j |\psi_j\rangle \langle \psi_j|.$$
Neumann entropy of a density matrix $\rho_n$ is given by

$$S(\rho_n) = -\text{tr} \rho_n \log_2 \rho_n = - \sum_j \kappa_j^{(n)} \log_2 \kappa_j^{(n)},$$

and the von Neumann entropy rate by

$$h = \lim_{n \to \infty} \frac{1}{n} S(\rho_n).$$

A useful example is an i.i.d. case where $\rho_n$ acts on a tensor product Hilbert space $\mathcal{H}_n = \mathcal{K}^\otimes n$ and is given by

$$\rho_n = \pi^\otimes n.$$

Here $\mathcal{K}$ is a fixed Hilbert space (representing an "elementary" quantum subsystem) and $\pi$ is a density matrix acting on $\mathcal{K}$:

$$\pi = \sum_i q_i |\phi_i\rangle \langle \phi_i|.$$

The eigenvectors $|\psi_j^{(n)}\rangle$ of $\rho_n$ are tensor products

$$|\psi_j^{(n)}\rangle = |\phi_{j_1}\rangle \otimes |\phi_{j_2}\rangle \cdots \otimes |\phi_{j_n}\rangle,$$

and its eigenvalues $\kappa_j^{(n)}$ are given by

$$\kappa_j^{(n)} = q_{j_1} \cdots q_{j_n}.$$

This provides a convenient identification of label $j$ as a "classical string" $(j_1, \ldots, j_n)$ which will be emphasized by the notation $\underline{j}$ below. The von Neumann entropy is in this case $S(\pi^\otimes n) = n S(\pi)$.

In the case where $\dim \mathcal{K} = 2$, the space $\mathcal{H}_n$ represents a system of $n$ qubits. In analogy with classical data compression, it is desirable to represent typical outputs, $|\psi_j^{(n)}\rangle$, by vectors from a lower dimensional Hilbert space, thereby reducing the number of qubits needed for the source description.

In his seminal paper [20], Schumacher proved that the number of qubits necessary to represent, reliably, the signal from an i.i.d quantum information source is asymptotically given by the von Neumann entropy. More precisely, there exists a reliable compression scheme of rate $R$ only when $R > S(\pi)$ (under a suitable definition of fidelity). Schumacher’s approach was developed further in [8,1]. Extensions of Schumacher’s theorem to some classes of quantum sources with memory have been established by Petz et al (see [14,15] and references therein).

As was said before, in this paper we consider data compression for a class of quantum information sources which are modelled by a system of interacting quantum spins. This is an example of a quantum system with a strong coupling between the spins and with the environment and it does not fall into the classes of sources considered in the literature before.
Besides, we consider properties of eigenvalues $\kappa^{(n)}$ which hold asymptotically with probability one; this is a refinement of results obtained in [14, 15]. From the probabilistic point of view, our result is an analogue of the Shannon-McMillan-Breiman theorem (which is a version of the Law of large numbers), see [4].

Even though we consider so-called quantum spin systems as models of a quantum source in this paper, our results also hold for sources modelled by quantum lattice gases, where the statistics (Bose or Fermi) of the particles is taken into account.

Models of quantum information sources, based on large systems of interacting spins or particles, are being used increasingly in experiments with entanglement [9, 19], as well as in theoretical research [10, 13]. As mentioned before, our main result can be viewed as an extension of Schumacher’s coding theorem to this class of sources. Section 2 contains a mathematical description of the class of systems under consideration. In Section 3 we prove that the data compression limit for such a class is given by the von Neumann entropy rate [see (11)]. The proof of the main theorem, which yields the data compression limit, is given in Section 4.

2 Quantum spin systems

We consider a quantum-mechanical system on a $d$-dimensional lattice $\mathbb{Z}^d$, with a spin-1/2 particle attached to each site of the lattice. The particle can be either in an up-spin state (denoted by $|\uparrow\rangle$) or a down-spin state (denoted by $|\downarrow\rangle$). Hence, to each lattice site $x \in \mathbb{Z}^d$ is associated a Hilbert space $K_x$ which is isomorphic to $\mathbb{C}^2$, the single-qubit Hilbert space. For any finite subset $X \subset \mathbb{Z}^d$, the corresponding Hilbert space is given by

$$H_X = \otimes_{x \in X} K_x = (\mathbb{C}^2)^{\otimes |X|}. $$

Here, and below, $|B|$ stands for the number of elements in a finite set $B$. Furthermore, we denote by $\mathcal{A}_X$ the algebra of $2^{|X|} \times 2^{|X|}$ matrices acting in $H_X$ – the local observable algebra.

To each site $x$ of the lattice, we associate a variable $j_x \in \{1, -1\}$ such that $j_x = 1 (-1)$ when the spin at $x$ is $\uparrow (\downarrow)$. A configuration $\omega_\Lambda$ in a finite volume $\Lambda \subset \mathbb{Z}^d$ is an assignment $\{j_x, x \in \Lambda\}$ of $j_x$ to each $x \in \Lambda$; the set of configurations $\{\omega_\Lambda\}$ provides labels for a quasiclassical basis $\{|\omega_\Lambda\rangle\}$ in $H_\Lambda = (\mathbb{C}^2)^{\otimes |\Lambda|}$.

The physics of the system is described by an interaction, $\Phi = \{\Phi_X\}$, which is a map taking finite subsets $X \subset \mathbb{Z}^d$ to (self-adjoint) operators $\Phi_X$ from $\mathcal{A}_X$; see [6]. We study quantum systems that are small perturbations of classical ones. That is, we consider interactions of the form $\Phi = \Phi_0 + Q$, with

$$\Phi_X = \Phi_{0X} + Q_X, \quad (2)$$

where, for all $X$, $\Phi_{0X}$ is diagonal in the quasiclassical basis $\{|\omega_\Lambda\rangle\}$ and $Q_X$ is small in norm (see below). We will write $\Phi_0 = \{\Phi_{0X}\}$ and $Q = \{Q_X\}$.

The corresponding Hamiltonian $H_\Lambda = \sum_{X \subset \Lambda} \Phi_X$ of a system confined to a finite volume $\Lambda \subset \mathbb{Z}^d$ is written as a sum

$$H_\Lambda = H_{0\Lambda} + V_\Lambda, \quad (3)$$
where \( H_{0\Lambda} := \sum_{X \subset \Lambda} \Phi_{0X}, \) \( V_\Lambda := \sum_{X \subset \Lambda} Q_X. \) We make the following assumptions:

(i) We consider translation-invariant interactions (for details, see\(^1\)) i.e., \( \Phi_{X+a} \simeq \Phi_X \) for all finite \( X \subset \mathbb{Z}^d \) and \( a \in \mathbb{Z}^d. \) The range of the interaction is defined as the supremum of the diameters of sets \( X \) from \( \{ X \subset \mathbb{Z}^d : X \ni 0 \text{ and } \Phi_X \neq 0 \}. \) We use the \( \ell^\infty \)-diameter

\[
\text{diam } M := \max_{x,y \in M} \max_{1 \leq i \leq d} |x_i - y_i|,
\]

and consider \( \Phi \) to be of a finite range, i.e., with \( R < \infty. \)

(ii) The classical part \( \Phi_{0X} \) of \( \Phi_X \) can be considered as a real-valued function on the set of configurations \( \omega_X \) in \( X \) (i.e., an assignment \( \{ j_x, x \in X \} \)). It is convenient to think of \( \Phi_{0X} \) as a function of the infinite-volume configuration \( w \equiv \{ j_x, x \in \mathbb{Z}^d \} \), which depends on its restriction \( w_X \) only. Similarly, \( H_{0\Lambda} \) is a real-valued function of \( \omega \) depending on \( \omega_\Lambda \) only. We call an infinite-volume configuration \( \omega \equiv \{ j_x, x \in \mathbb{Z}^d \} \) periodic if \( j_x = j_{x+a(i)}, i = 1, \ldots, d, \) for all \( x \in \mathbb{Z}^d \) and a given collection of periods \( a(i) = n(i) e(i) \) where \( e(i) = (0, \ldots, 1, \ldots, 0) \in \mathbb{Z}^d \) (entry 1 at position \( i \)) and \( n(i) \) is a given integer. A periodic \( \omega \) is called a ground state configuration for \( \Phi_0 \) if

\[
\liminf_{\Lambda / \mathbb{Z}^d} \frac{1}{|\Lambda|} [H_{0\Lambda}(\omega') - H_{0\Lambda}(\omega)] \geq 0 \]

for any infinite-volume configuration \( \omega'. \)

We assume that \( \Phi_0 \) has a finite number of periodic classical ground states, \( \sigma^{(1)}, \ldots, \sigma^{(m)} \), and satisfies the so-called Peierls condition\(^2\). The latter is a condition for stability of the ground states relative to “local” perturbations. (See\(^3\) and references therein for details.)

(iii) The term \( V_\Lambda \) is a quantum perturbation (i.e., \( [H_{0\Lambda}, V_\Lambda] \neq 0 \)), with

\[
||Q_X|| \leq c_\lambda s(X)
\]

for some constant \( c \) and some \( 0 < \lambda < 1. \) Here \( s(X) \) denotes the number of sites in the smallest connected subset of the lattice containing \( X. \) We consider \( \lambda \) as the perturbation parameter.

Assumptions (i) - (iii) constitute the framework of the so-called quantum Pirogov-Sinai theory\(^4\)\(^5\)\(^6\)\(^7\)\(^8\).

We fix a boundary condition outside volume \( \Lambda, \) i.e., assume that the configuration on \( \Lambda^c := \mathbb{Z}^d \setminus \Lambda \) coincides with a fixed reference configuration \( \sigma, \) which is one of the periodic ground states \( \sigma^{(1)}, \ldots, \sigma^{(m)} \) of \( H_{0\Lambda}. \)

Since the interaction is of a finite range, the spins in \( \Lambda \) interact only with those spins in \( \Lambda^c \) that are in the enveloping volume \( \Lambda^\partial:

\[
\Lambda^\partial = \{ i \in \Lambda^c : \text{dist}(i,j) \leq R \text{ for some } j \in \Lambda \}.
\]

\(^2\) Here and below, the symbol \( \Lambda / \mathbb{Z}^d \) is used for the thermodynamical limit, taken along a sequence of growing finite volumes \( \Lambda_1 \subset \Lambda_2 \subset \ldots \subset \mathbb{Z}^d \) of “nice” shape (e.g., hypercubes \( \Lambda_n = [-n,n]^d \cap \mathbb{Z}^d \)), with \( \bigcup_n \Lambda_n = \mathbb{Z}^d. \)
Let $P^\sigma_\Lambda$ be the orthogonal projection onto the subspace $\mathcal{H}_\Lambda^\sigma \subset \mathcal{H}_{\Lambda^\cup \Lambda^\partial}$ of dimension $2^{\mid \Lambda \mid}$, spanned by states for which the configuration on $\Lambda^\partial$ is fixed to $\sigma$. Then the Hamiltonian governing the spin system in $\Lambda$ under the boundary condition $\sigma$ is given by

$$H^\sigma_\Lambda = P^\sigma_\Lambda H_{\Lambda^\cup \Lambda^\partial} P^\sigma_\Lambda \equiv \sum_{X \cap \Lambda \neq \emptyset} P^\sigma_\Lambda \Phi_X P^\sigma_\Lambda.$$

The spin system with Hamiltonian $H^\sigma_\Lambda$ can be viewed as a system of interacting spins entangled with its environment. It is considered at a finite but low temperature. Due to the interaction between spins, the density matrix cannot be written as a tensor product of the density matrices of the individual spins and hence the quantum information source is non-i.i.d. The density matrix is written in the standard Gibbsian form:

$$\rho^{\sigma, \Lambda} = e^{-\beta H^\sigma_\Lambda} \Xi^{\sigma, \Lambda},$$

where $\beta > 0$ is the inverse temperature. The denominator on the RHS of (5) is the partition function:

$$\Xi^{\sigma, \Lambda} = \text{tr}_{\mathcal{H}_\Lambda^\sigma} e^{-\beta H^\sigma_\Lambda}.$$

The expectation of an observable $A \in \mathcal{A}_{\Lambda^\cup \Lambda^\prime}$ in the Gibbs state $\rho^{\sigma, \Lambda}$ is given by

$$\langle A \rangle^{\sigma}_\Lambda \equiv \text{tr}_{\mathcal{H}_\Lambda^\sigma} \rho^{\sigma, \Lambda} A = \frac{1}{\Xi^{\sigma, \Lambda}} \text{tr}_{\mathcal{H}_\Lambda^\sigma} A \exp (-\beta H^\sigma_\Lambda).$$

Here and below, the trace is taken in the space $\mathcal{H}_\Lambda^\sigma$; for notational simplicity, the subscript $\mathcal{H}_\Lambda^\sigma$ will often be omitted. For $A := \exp(i\tau H^\sigma_\Lambda)$, where $\tau \in \mathbb{R}$, (5) yields the characteristic function for the eigenvalues of the Hamiltonian $H^\sigma_\Lambda$:

$$\varphi^{\sigma, \Lambda} (\tau) := \langle e^{i \tau H^\sigma_\Lambda} \rangle^\sigma_\Lambda.$$

The eigenvalues $\kappa^{\sigma, \Lambda}_j$ of $\rho^{\sigma, \Lambda}$ can be written as

$$\kappa^{\sigma, \Lambda}_j = \frac{1}{\Xi^{\sigma, \Lambda}} \langle \psi^{\sigma, \Lambda}_j | e^{-\beta H^\sigma_\Lambda} | \psi^{\sigma, \Lambda}_j \rangle = \frac{1}{\Xi^{\sigma, \Lambda}} \exp \left(-\beta \langle \psi^{\sigma, \Lambda}_j | H^\sigma_\Lambda | \psi^{\sigma, \Lambda}_j \rangle \right),$$

where $|\psi^{\sigma, \Lambda}_1\rangle, \ldots, |\psi^{\sigma, \Lambda}_{2^{\mid \Lambda \mid}}\rangle$ are the orthonormal eigenvectors of $\rho^{\sigma, \Lambda}$ (sometimes denoted by $\psi_1^{\sigma, \Lambda}, \ldots, \psi_{2^{\mid \Lambda \mid}}^{\sigma, \Lambda}$). The eigenvalues $\kappa^{\sigma, \Lambda}_j$ satisfy

$$\sum_j \kappa^{\sigma, \Lambda}_j = 1.$$

The von Neumann entropy of $\rho^{\sigma, \Lambda}$ is given by

$$S(\rho^{\sigma, \Lambda}) = -\text{tr} \rho^{\sigma, \Lambda} \log_2 \rho^{\sigma, \Lambda} = -\sum_j \kappa^{\sigma, \Lambda}_j \log_2 \kappa^{\sigma, \Lambda}_j.$$
The von Neumann entropy rate in this case is defined as

\[ h = \lim_{\Lambda \to \mathbb{Z}^d} \frac{S(\rho^{\sigma, \Lambda})}{|\Lambda|} \]

\[ = c_0 \lim_{\Lambda \to \mathbb{Z}^d} \text{tr} \, \rho^{\sigma, \Lambda} \left( \beta \frac{H^{\sigma}_\Lambda}{|\Lambda|} + \frac{1}{|\Lambda|} \log_e \Xi^{\sigma, \Lambda} \right) \]

\[ = \beta c_0 (g^{(\sigma)} - f) \quad (11) \]

where \( c_0 = \log_2 e \) and \( f \) and \( g^{(\sigma)} \) are standard thermodynamical functions (the free energy and the infinite volume energy per lattice site):

\[ f = \lim_{\Lambda \to \mathbb{Z}^d} \frac{-1}{\beta |\Lambda|} \log_e \Xi^{\sigma, \Lambda}; \quad (12) \]

\[ g^{(\sigma)} = \lim_{\Lambda \to \mathbb{Z}^d} \left( \frac{H^{\sigma}_\Lambda}{|\Lambda|} \right)^\Lambda. \quad (13) \]

We see that the von Neumann entropy rate \( h \) is well-defined if the above limits, (12) and (13), exist. The following theorem, proved in \([5]\), states that these limits do exist for the class of quantum spin systems under consideration.

**Proposition 1** Under the above assumptions, for \( \beta \) large and \( \lambda \) small enough, the limits (12) and (13) exist.

**Remark:** In this paper we deal with a sequence of density matrices \( \rho^{\sigma, \Lambda}, \Lambda \not\to \mathbb{Z}^d \), not generated by a single state of a quasi-local algebra (see e.g. \([3]\)). This puts us in a context different from that considered e.g. in \([11]\). Hence we need Proposition \([5]\) to guarantee the existence of the von Neumann entropy rate.

In view of (13), the eigenvalues \( \kappa^{\sigma, \Lambda}_j, \, 1 \leq j \leq 2^{|\Lambda|}, \) can be interpreted as the probabilities of the system being in the states \( |\psi^{\sigma, \Lambda}_j\rangle \). Let \( P^{\sigma, \Lambda} \) be the corresponding probability distribution and consider a random variable \( K^{\sigma, \Lambda} \) which takes a value \( \kappa^{\sigma, \Lambda}_j \) with probability \( \kappa^{\sigma, \Lambda}_j \):

\[ K^{\sigma, \Lambda}(\psi^{\sigma, \Lambda}_j) = \kappa^{\sigma, \Lambda}_j; \quad P^{\sigma, \Lambda}(K^{\sigma, \Lambda} = \kappa^{\sigma, \Lambda}_j) = \kappa^{\sigma, \Lambda}_j. \]

The data compression limit is related to asymptotical properties of random variables \( K^{\sigma, \Lambda} \) as \( \Lambda \not\to \mathbb{Z}^d \).

### 3 Data Compression Limit

The main result of the paper is the following theorem.

**Theorem 1** Under the above assumptions, for \( \beta \) large and \( \lambda \) small enough, for all \( \delta > 0 \)

\[ \lim_{\Lambda \to \mathbb{Z}^d} P^{\sigma, \Lambda} \left( \left| \frac{-1}{|\Lambda|} \log_2 K^{\sigma, \Lambda} - h \right| \leq \delta \right) = \lim_{\Lambda \to \mathbb{Z}^d} \sum_j \kappa^{\sigma, \Lambda}_j \mathbf{1} \left( \left| \frac{-1}{|\Lambda|} \log_2 \kappa^{\sigma, \Lambda}_j - h \right| \leq \delta \right) = 1, \quad (14) \]

where \( \mathbf{1}(\cdot) \) denotes an indicator function.
Note that
\[
\mathbb{E}_{P^\sigma,\Lambda} \left( -\frac{1}{|\Lambda|} \log_2 K^{\sigma,\Lambda} \right) = \sum_j \kappa_j^{\sigma,\Lambda} \left( -\frac{1}{|\Lambda|} \log_2 \kappa_j^{\sigma,\Lambda} \right) = \frac{S(\rho^{\sigma,\Lambda})}{|\Lambda|},
\]
where \(\mathbb{E}_{P^\sigma,\Lambda}(\cdot)\) denotes the expectation value with respect to the probability distribution \(P^{\sigma,\Lambda}\). Hence,
\[
\lim_{\Lambda \rightarrow \mathbb{Z}^d} \mathbb{E}_{P^\sigma,\Lambda} = h,
\]
and Theorem 4 gives a Law of large numbers for random variables \((-\log_2 K^{\sigma,\Lambda})\).

The proof of Theorem 4 is given in Section 4. Here we discuss some of its consequences.

The statement of the theorem can be alternatively expressed as follows:
\[
\forall \delta > 0 \lim_{\Lambda \rightarrow \mathbb{Z}^d} P^{\sigma,\Lambda} \left( 2^{-|\Lambda|(h+\delta)} \leq K^{\sigma,\Lambda} \leq 2^{-|\Lambda|(h-\delta)} \right) = 1.
\]
(16)

In other words, \(\forall \epsilon > 0\) and for \(\Lambda\) large enough, the eigenvalues \(\kappa_j^{\sigma,\Lambda}\) of \(\rho^{\sigma,\Lambda}\) satisfy
\[
2^{-|\Lambda|(h+\delta)} \leq \kappa_j^{\sigma,\Lambda} \leq 2^{-|\Lambda|(h-\delta)}
\]
(17)

with probability \(\geq (1-\epsilon)\). That is, the eigenstates \(|\psi_j^{\sigma,\Lambda}\rangle\) that correspond to eigenvalues \(\kappa_j^{\sigma,\Lambda}\) satisfying (17) are those which occur most frequently. We refer to them as typical states (or more precisely, \(\delta\)-typical states). Let \(\mathcal{M}_\delta^{\sigma,\Lambda}\) be the subspace spanned by such states:
\[
\mathcal{M}_\delta^{\sigma,\Lambda} := \text{span}\{ |\psi_j^{\sigma,\Lambda}\rangle : (17) \text{ holds} \}
\]
(18)

and \(|\mathcal{M}_\delta^{\sigma,\Lambda}\rangle\) denote the dimension of this subspace. The following lemma establishes the growth rate of \(|\mathcal{M}_\delta^{\sigma,\Lambda}\rangle\).

**Lemma 1** For all \(\delta > 0\)
\[
\lim_{\Lambda \rightarrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \log_2 |\mathcal{M}_\delta^{\sigma,\Lambda}| = h.
\]
(19)

**Proof:** We follow a standard information-theoretical argument (see e.g. [12]). From (10) it follows that the probability of a state being \(\delta\)-typical is at least \((1-\epsilon)\) in the limit \(\Lambda \rightarrow \mathbb{Z}^d:\)
\[
\liminf \sum_j \kappa_j^{\sigma,\Lambda} \geq 1 - \epsilon.
\]
(20)

where the sum \(\sum_j\) is over those \(j\)’s for which \(\kappa_j^{\sigma,\Lambda}\) satisfies (17), i.e., \(|\psi_j^{\sigma,\Lambda}\rangle \in \mathcal{M}_\delta^{\sigma,\Lambda}\). From (20) (and the definition (18) of the set \(\mathcal{M}_\delta^{\sigma,\Lambda}\)) we deduce that \(\forall \epsilon > 0\),
\[
1 - \epsilon \leq \sum_j \kappa_j^{\sigma,\Lambda} \leq 2^{-|\Lambda|(h-\delta)} |\mathcal{M}_\delta^{\sigma,\Lambda}|.
\]
(21)
Also, from (9) and (18) we have

\[ 2^{-|\Lambda|(h+\delta)} |\mathcal{M}_{\delta}^{\sigma,\Lambda}| \leq \sum_j \kappa_j^{\sigma,\Lambda} \leq 1. \]  

(22)

From (21) and (22) it follows that

\[ 2^{|\Lambda|(h-\delta)} \leq |\mathcal{M}_{\delta}^{\sigma,\Lambda}| \leq 2^{|\Lambda|(h+\delta)}. \]

Since this holds for all \( \epsilon > 0 \), we conclude that

\[ \lim \sup \frac{1}{|\Lambda|} \log_2 |\mathcal{M}_{\delta}^{\sigma,\Lambda}| \leq h + \delta, \quad \lim \inf \frac{1}{|\Lambda|} \log_2 |\mathcal{M}_{\delta}^{\sigma,\Lambda}| \geq h - \delta. \]  

(23)

Moreover, since \( \delta \) is arbitrary,

\[ \lim \sup \frac{1}{|\Lambda|} \log_2 |\mathcal{M}_{\delta}^{\sigma,\Lambda}| = \lim \inf \frac{1}{|\Lambda|} \log_2 |\mathcal{M}_{\delta}^{\sigma,\Lambda}|. \]

Hence \( \lim_{\Lambda \to \mathbb{Z}^d} \frac{1}{|\Lambda|} \log_2 |\mathcal{M}_{\delta}^{\sigma,\Lambda}| \) exists and is given by (19).

**Lemma 2** Consider a quantum information source described by the density matrix \( \rho^{\sigma,\Lambda} \):

\[ \rho^{\sigma,\Lambda} = \frac{e^{-\beta H_{\sigma}}}{\Xi^{\sigma,\Lambda}} = \sum_j \kappa_j^{\sigma,\Lambda} |\psi_j^{\sigma,\Lambda}\rangle \langle \psi_j^{\sigma,\Lambda}|. \]

Let \( h \) be the von Neumann entropy rate [see (14)]. If \( R > h \) then there exists a reliable compression scheme of rate \( R \).

**Proof**: Since there are at most \( 2^{|\Lambda|h} \) \( \delta \)-typical states (see Lemma 1), one requires at most \( |\Lambda|h \) qubits to uniquely identify a \( \delta \)-typical state. The data can be compressed as follows:

Map each \( \delta \)-typical state \( |\psi_j^{\sigma,\Lambda}\rangle \) to a quasiclassical state \( |x\rangle = |x_1\rangle \otimes |x_2\rangle \otimes \ldots \otimes |x_{|\Lambda|h}\rangle \), where \( x \) is a binary string of length \( |\Lambda|h \):

\[ x = (x_1, x_2, \ldots, x_{|\Lambda|h}) \in \{0, 1\}^{[|\Lambda|h]}. \]

Clearly, this can be done in a one-to-one fashion, enabling us to recover any \( \delta \)-typical state. In other words, the information contained in \( |\Lambda\) interacting qubits is compressed into \( |\Lambda|h \) non-interacting qubits, which can be later decompressed unambiguously. In the limit \( \Lambda \to \mathbb{Z}^d \) this scheme succeeds with probability one. Hence, the data compression limit, for the class of non-i.i.d. quantum information sources considered in this paper, is given by the von Neumann entropy rate \( h \).

The following lemma shows that a compression scheme of rate \( R < h \) is not reliable.
Lemma 3 Let $S_\Lambda$ be any set of eigenstates $\{|\psi_\sigma^{\sigma,\Lambda}\rangle\}$ of $\rho^{\sigma,\Lambda}$ such that

$$|S_\Lambda| = 2^{[\Lambda|R]},$$

where $R < h$ is fixed. Then for any $\epsilon > 0$ and sufficiently large $\Lambda$

$$\sum_{j \in S_\Lambda} \kappa_j^{\sigma,\Lambda} \leq \epsilon. \quad (24)$$

Proof: The LHS of (24) gives the probability that an eigenstate of $\rho^{\sigma,\Lambda}$ belongs to the set $S_\Lambda$. We can write it as a sum of the probability that a state belonging to $S_\Lambda$ is $\delta$-typical and that it is atypical:

$$\sum_{j \in S_\Lambda} \kappa_j^{\sigma,\Lambda} = \sum_{j \in S_\Lambda}^{(\delta)} \kappa_j^{\sigma,\Lambda} + \sum_{j \in S_\Lambda}^{(\delta)' \Lambda} \kappa_j^{\sigma,\Lambda}; \quad (25)$$

here the second sum on the RHS of (25) is over the atypical states in $S_\Lambda$. Choose $\delta > 0$ such that $R < h - \delta$ and $0 < \delta < \epsilon/2$. In the limit $\Lambda \rightarrow Z^d$, the probability of atypical states is negligible. By (17) the total probability of atypical states can be made $< \epsilon$. There are at most $2^{[\Lambda|R]}$ $\delta$-typical states in the set $S_\Lambda$, each with an eigenvalue $\leq 2^{-|\Lambda|([h - \delta])}$. Hence, the first term on RHS of (25) is bounded by

$$2^{-|\Lambda|([h - \delta])} 2^{[\Lambda|R]} \leq 2^{-|\Lambda|\epsilon/2},$$

which goes to zero in the limit $\Lambda \rightarrow Z^d$.

We conclude this section with a theorem giving giving the data compression limit and the limiting fidelity of the compression scheme for general (not necessarily orthogonal) decompositions of $\rho^{\sigma,\Lambda}$.

Consider any representation of the density matrix $\rho^{\sigma,\Lambda}$:

$$\rho^{\sigma,\Lambda} = \sum_i p_i^{\sigma,\Lambda} |\phi_i^{\sigma,\Lambda}\rangle \langle \phi_i^{\sigma,\Lambda}|,$$

where $|\phi_i^{\sigma,\Lambda}\rangle \in H_\Lambda^{\sigma}$ are arbitrary vectors of unit norm (not necessarily orthogonal or even linearly independent), and $p_i^{\sigma,\Lambda} \geq 0$, $\sum_i p_i^{\sigma,\Lambda} = 1$. To apply the above data compression scheme consider an orthogonal projection $\Pi : H_\Lambda^{\sigma} \rightarrow C$, where $C$ is a subspace of $H_\Lambda^{\sigma}$ such that the vectors $|\Pi \phi_i^{\sigma,\Lambda}\rangle$ are either collinear or orthogonal for different $i$ (some of them may be 0). If such a projection exists then, necessarily, the vectors spanning $C$ are eigenvectors of $\rho^{\sigma,\Lambda}$ and each non-zero vector $|\Pi \phi_i^{\sigma,\Lambda}\rangle$ is collinear to one of these eigenvectors. If we take $C$ to be the subspace $M^{\delta,\Lambda}_{\phi}$, spanned by the $\delta$-typical states $|\psi_j^{\sigma,\Lambda}\rangle$ of $\rho^{\sigma,\Lambda}$, then to each non-zero vector $|\Pi \phi_i^{\sigma,\Lambda}\rangle$ we can assign a quasiclassical state $|x\rangle$ associated with the eigenvector $|\psi_j^{\sigma,\Lambda}\rangle$ collinear to $|\Pi \phi_i^{\sigma,\Lambda}\rangle$. Here $x$ is a binary string of length $\leq \lceil \log_2 (\dim C) \rceil + 1$. In this case, the compression scheme can be represented by the two maps given below:

$$E : |\phi_i^{\sigma,\Lambda}\rangle \mapsto |\psi_j^{\sigma,\Lambda}\rangle \quad \text{where } |\psi_j^{\sigma,\Lambda}\rangle \in M^{\delta,\Lambda}_{\phi}; \quad (26)$$

$$C : |\psi_j^{\sigma,\Lambda}\rangle \mapsto |x^{(j)}\rangle \quad \text{where } x^{(j)} \in \{0, 1\}^r; \quad r \leq \lceil \log_2 (\dim C) \rceil + 1. \quad (27)$$
We use the symbols $E$ and $C$ for the maps (26) and (27) to denote encoding and compression. Note that map $C$ is one–to–one. Hence, the quasiclassical state $|\psi_j^{\sigma,\Lambda}\rangle$ can be decompressed unambiguously to yield the $\delta$–typical state $|\psi_j^{\sigma,\Lambda}\rangle$. However, map $E$ is not necessarily one–to–one. Consequently, the original vector $|\phi_i^{\sigma,\Lambda}\rangle$ cannot be recovered with certainty from the state $|\psi_j^{\sigma,\Lambda}\rangle$. Hence, we consider the following prescription for decoding the state $|\psi_j^{\sigma,\Lambda}\rangle$ (denoted by the map $D$):

$$D : |\psi_j^{\sigma,\Lambda}\rangle \mapsto |\phi_k^{\sigma,\Lambda}\rangle,$$

where $|\phi_k^{\sigma,\Lambda}\rangle$ satisfies the relation:

$$\langle \phi_k^{\sigma,\Lambda}|\psi_j^{\sigma,\Lambda}\rangle = \max_i \langle \phi_i^{\sigma,\Lambda}|\psi_j^{\sigma,\Lambda}\rangle.$$

The fidelity of such a coding–decoding scheme can be defined as:

$$F_{\Lambda} := \sum_i p_i^{\sigma,\Lambda} \langle \phi_i^{\sigma,\Lambda}|\Pi|\phi_i^{\sigma,\Lambda}\rangle. \quad (28)$$

The fidelity takes values between 0 and 1 and equals to unity only when all the states $|\Pi|\phi_i^{\sigma,\Lambda}\rangle$ are correctly decoded. In the following theorem we show that $F_{\Lambda}$ tends to unity as $\Lambda \nearrow \mathbb{Z}^d$.

**Theorem 2**

(i) Choose $C$ to be the space of $\delta$–typical states of $\rho^{\sigma,\Lambda}$:

$$C = \mathcal{M}^{\sigma,\Lambda}_\delta := \text{span}\{|\psi_j^{\sigma,\Lambda}\rangle : 2^{-|\Lambda|(h-\delta)} \geq \kappa_j^{\sigma,\Lambda} \geq 2^{-|\Lambda|(h+\delta)}\},$$

where the $|\psi_j^{\sigma,\Lambda}\rangle's$ are orthonormal eigenstates of $\rho^{\sigma,\Lambda}$ and $\kappa_j^{\sigma,\Lambda}$ are their corresponding eigenvalues. Let $\Pi$ be the orthoprojection $\mathcal{H}_\Lambda^\sigma \to C$. The fidelity $F_{\Lambda}$ of the map $\Pi$, given by (28), approaches one:

$$\lim_{\Lambda \nearrow \mathbb{Z}^d} F_{\Lambda} := \lim_{\Lambda \nearrow \mathbb{Z}^d} \sum_i p_i^{\sigma,\Lambda} \langle \phi_i^{\sigma,\Lambda}|\Pi|\phi_i^{\sigma,\Lambda}\rangle = 1.$$

(ii) If, for some subspace $\mathcal{D} \subseteq \mathcal{H}_\Lambda^\sigma$, the orthoprojection $\Pi^{\sigma,\Lambda}_\Lambda : \mathcal{H}_\Lambda^\sigma \to \mathcal{D}$ has fidelity tending to one then

$$\lim_{\Lambda \nearrow \mathbb{Z}^d} \inf \frac{1}{|\Lambda|} \log_2 \dim \mathcal{D} \geq h,$$

where $h$ is the von Neumann entropy rate.

**Proof:** To verify (i), write:

$$\sum_i p_i^{\sigma,\Lambda} \langle \phi_i^{\sigma,\Lambda}|\Pi|\phi_i^{\sigma,\Lambda}\rangle = \sum_j |\psi_j^{\sigma,\Lambda}\rangle \langle \psi_j^{\sigma,\Lambda}| \sum_i \langle \psi_j^{\sigma,\Lambda}|\phi_i^{\sigma,\Lambda}\rangle \langle \phi_i^{\sigma,\Lambda}| \Pi |\psi_j^{\sigma,\Lambda}\rangle |1(\psi_j^{\sigma,\Lambda} \in C)\rangle |\phi_i^{\sigma,\Lambda}\rangle$$

$$= \sum_j |\psi_j^{\sigma,\Lambda}\rangle \langle \psi_j^{\sigma,\Lambda}| \sum_i \langle \phi_i^{\sigma,\Lambda}|\phi_i^{\sigma,\Lambda}\rangle \langle \phi_i^{\sigma,\Lambda}| \Pi |\psi_j^{\sigma,\Lambda}\rangle |\psi_j^{\sigma,\Lambda}\rangle$$

$$= \sum_j |\psi_j^{\sigma,\Lambda}\rangle \rho^{\sigma,\Lambda} |\psi_j^{\sigma,\Lambda}\rangle |1(\psi_j^{\sigma,\Lambda} \in C)\rangle$$

$$= \sum_j \kappa_j^{\sigma,\Lambda} |1(\frac{1}{|\Lambda|} \log \kappa_j^{\sigma,\Lambda} - h| \leq \delta) \to 1,$$  \quad (29)
by Theorem [1]. Property (ii) is checked in a similar fashion.

Remark. The argument in the proof of Theorem 2 does not depend on the nature of the density matrix $\rho_\sigma^\Lambda$ or space $\mathcal{H}_\Lambda^\sigma$. In a somewhat different context, a statement similar to Theorem 2 was established in [14] (see also the references therein).

4 Proof of Theorem 1

In view of (8), eq. (14) is equivalent to

$$
\lim_{\Lambda/\mathbb{Z}^d} \sum_j \kappa_j^\Lambda \ 1 \left( |c_0 \beta \langle \psi_j^\sigma^\Lambda | H_\Lambda^\sigma | \psi_j^\sigma^\Lambda \rangle + \left( \frac{c_0}{|\Lambda|} \log_2 \Xi_{\sigma,\Lambda} - h \right) | \leq \delta \right) = 1.
$$

This fact, together with Proposition [1] and eq. (11) reduces the assertion of Theorem 1 to the following fact: $\forall \delta > 0$

$$
\lim_{\Lambda/\mathbb{Z}^d} \sum_j \kappa_j^\Lambda \ 1 \left( \left| \frac{1}{|\Lambda|} \langle \psi_j^\sigma^\Lambda | H_\Lambda^\sigma | \psi_j^\sigma^\Lambda \rangle - g^{(\sigma)} \right| \geq \frac{c_0 \delta}{\beta} \right) = 0,
$$

(30)

where $g^{(\sigma)}$ is defined through (13).

Eq. (30) is a Law of large numbers for the random variables $\langle \psi_j^\sigma^\Lambda | H_\Lambda^\sigma | \psi_j^\sigma^\Lambda \rangle$ (with respect to probability distributions $\mathcal{P}_{\sigma,\Lambda}$). In terms of characteristic functions, (30) is equivalent to the following lemma:

**Lemma 4** For $\beta$ large enough and $\lambda$ small enough, for any $t \in \mathbb{R}$ the following limit exists:

$$
\lim_{\Lambda/\mathbb{Z}^d} \varphi^{(\Lambda)}(t/|\Lambda|) = e^{itg^{(\sigma)}}
$$

(31)

where $\varphi(\cdot)$ is defined through (7) and $g^{(\sigma)}$ by (13).

**Proof:**

From (8) and (7) we have that

$$
\varphi^{\sigma,\Lambda}(t/|\Lambda|) = \langle e^{it H_\Lambda^\sigma/|\Lambda|} \rangle_{\sigma,\Lambda} = \frac{\text{tr} \left( e^{it H_\Lambda^\sigma/|\Lambda|} e^{-\beta H_\Lambda^\sigma} \right)_{\Xi_{\sigma,\Lambda}}}{\Xi_{\sigma,\Lambda}}
$$

(32)

Henceforth, we shall suppress the superscript $\sigma$ from the notation $H_\Lambda^\sigma$ and $\Xi_{\sigma,\Lambda}$.

Expanding $e^{it H_\Lambda^\sigma/|\Lambda|}$ on the RHS of (32) we obtain

$$
\varphi^{(\Lambda)}(t/|\Lambda|) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{it}{|\Lambda|} \right)^n \text{tr} \left( H_\Lambda^n e^{-\beta H_\Lambda} \right)_{\Xi_{\Lambda}}
$$

$$
= 1 + \sum_{n=1}^{\infty} T_n
$$

(33)
Let us first estimate the term \( T_1 \).

\[
T_1 = \frac{it}{|\Lambda|} \text{tr}_{H_\Lambda} \left( H_\Lambda e^{-\beta H_\Lambda} \right) / \Xi \Lambda
\]

\[
= \frac{it}{|\Lambda|} \left( \sum_{X: X \cap \Lambda \neq \emptyset} \text{tr}_{H_\Lambda} \left( \Phi_X e^{-\beta H_\Lambda} \right) \right) / \Xi \Lambda
\]

\[
= \frac{it}{|\Lambda|} \sum_{j \in \Lambda} \sum_{X: X \ni j \cap \Lambda \neq \emptyset} \frac{1}{|X|} \text{tr} \left( \Phi_X e^{-\beta H_\Lambda} \right) / \Xi \Lambda
\]

\[
= \frac{it}{|\Lambda|} \sum_{j \in \Lambda} \text{tr} \left( \Theta_j e^{-\beta H_\Lambda} \right) / \Xi \Lambda = \frac{it}{|\Lambda|} \sum_{j \in \Lambda} \left\langle \Theta_j^\Lambda \right\rangle^\sigma_{\Lambda}
\]  

(34)

where

\[
\Theta_j^\Lambda := \sum_{X: X \ni j \cap \Lambda \neq \emptyset} \frac{1}{|X|} \Phi_X.
\]  

(35)

Now

\[
\langle \Phi_X \rangle^\sigma_{\Lambda} := (\text{tr} \Phi_X e^{-\beta H_\Lambda}) / \Xi \Lambda,
\]

and

\[
|\langle \Phi_X \rangle^\sigma_{\Lambda}| \leq \|\Phi_X\|,
\]

(36)

where \( \| \cdot \| \) denotes the Hilbert-Schmidt norm of the interaction \( \Phi_X \). Let

\[
c_0 := \max_{X: X \ni \Lambda \neq \emptyset} \|\Phi_X\|.
\]  

(37)

Due to the finite range of the interaction, we have that

\[
\# \{ X \ni j | \Phi_X \neq 0, j \in \mathbb{Z}^d \} = (2^{2R})^d,
\]

for any site \( j \in \mathbb{Z}^d \). Hence,

\[
|\langle \Theta_j^\Lambda \rangle^\sigma_{\Lambda}| \leq \sum_{X: X \ni j \cap \Lambda \neq \emptyset} \frac{1}{|X|} |\langle \Phi_X \rangle^\sigma_{\Lambda}| \leq c_0 \left(2^{2R}\right)^d < \infty.
\]  

(38)

It is known that for \( \beta \) large enough and \( \lambda \) small enough, the following limit exists

\[
\langle \Phi_X \rangle^\sigma := \lim_{\Lambda / \mathbb{Z}^d} \langle \Phi_X \rangle^\sigma_{\Lambda}
\]  

(39)

and defines the infinite volume Gibbs state \([5, 2]\).

Moreover,

\[
\langle \Theta_0 \rangle^\sigma_{\Lambda} := \sum_{X: X \ni 0 \cap \Lambda \neq \emptyset} \frac{1}{|X|} \langle \Phi_X \rangle^\sigma_{\Lambda} \equiv \sum_{X: X \ni j \cap \Lambda \neq \emptyset} \frac{1}{|X|} \langle \Phi_X \rangle^\sigma_{\Lambda} \quad \forall j \in \mathbb{Z}^d.
\]  

(40)
The last equality follows from the translational invariance of the interactions.

Further, by using methods of [5] it can be shown that for $\beta$ large and $\lambda$ small enough, the following bound holds:

$$
|\langle \Phi_X \rangle^{\Lambda} - \langle \Phi_X \rangle^{\partial \Lambda}| \leq ||\Phi_X|| c(s(X)) \Gamma(\text{dist}(X, \partial \Lambda));
$$

where $\partial \Lambda$ denotes the boundary of the volume $\Lambda$, $s(X)$ is the number of sites in the smallest connected set of sites containing $X$, and the function $\Gamma(r)$, $r > 0$, obeys

$$
|\Gamma(r)| \leq \exp(-c_1 r),
$$

where $c_1 > 0$ is a constant depending on $\beta$ and $\lambda$. Note that $\Gamma$ does not depend on $\Lambda$.

Using our assumptions on $\Phi$, one can prove the following Cesaro convergence:

$$
\lim_{\Lambda \nearrow \mathbb{Z}^d} T_1 \equiv \lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{it}{|\Lambda|} \sum_{j \in \Lambda} \langle \Theta^\Lambda_j \rangle^{\Lambda} = it \langle \Theta_0 \rangle_{\mathbb{Z}^d}.
$$

To prove (43) consider $\Lambda$ to be a finite hypercubic volume $[-n,n]^d \cap \mathbb{Z}^d$ and define a subvolume $\hat{\Lambda}$ as follows:

$$
\hat{\Lambda} := \{ i \in \Lambda \mid \text{dist}(i,j) \geq \log_e l(\Lambda) \forall j \in \partial \Lambda \}.
$$

Here $l(\Lambda) = 2n + 1$ is the linear size of the volume $\Lambda$. In the limit $\Lambda \nearrow \mathbb{Z}^d$, we have:

$$
\begin{align*}
(a) \quad & \frac{|\hat{\Lambda}|}{|\Lambda|} \to 1 \\
(b) \quad & \frac{|\Lambda \setminus \hat{\Lambda}|}{|\Lambda|} \equiv \frac{|\Lambda \setminus \hat{\Lambda}|}{|\Lambda|} \to 0 \\
(c) \quad & \frac{|\partial \hat{\Lambda}|}{|\Lambda|} \to 0 \\
(d) \quad & \text{dist}(\hat{\Lambda}, \partial \Lambda) \to \infty.
\end{align*}
$$

We can write

$$
T_1 = \frac{it}{|\Lambda|} \sum_{j \in \Lambda} \langle \Theta^\Lambda_j \rangle^{\Lambda}
$$

$$
= \frac{it}{|\Lambda|} \left[ \sum_{j \in \hat{\Lambda}} \langle \Theta^\Lambda_j \rangle^{\Lambda} + \frac{it}{|\Lambda|} \sum_{j \in \Lambda \setminus \hat{\Lambda}} \langle \Theta^\Lambda_j \rangle^{\Lambda} \right].
$$

Now

$$
\lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{it}{|\Lambda|} \sum_{j \in \Lambda \setminus \hat{\Lambda}} \langle \Theta^\Lambda_j \rangle^{\Lambda} \leq \lim_{\Lambda \nearrow \mathbb{Z}^d} |t| \frac{|\Lambda \setminus \hat{\Lambda}|}{|\Lambda|} \sup_{j \in \Lambda \setminus \hat{\Lambda}} |\langle \Theta^\Lambda_j \rangle^{\Lambda}|.
$$

Hence, from (48) and (40b)

$$
\text{RHS of (47)} \leq \lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{c_0 |t| (2^R)^d |\Lambda \setminus \hat{\Lambda}|}{|\Lambda|} = 0.
$$

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Consequently, in the infinite volume limit, the second term on the RHS of (46) goes to zero, thus allowing us to concentrate on the first term alone:

\[
\lim_{\Lambda \rightarrow \mathbb{Z}^d} T_1 = \lim_{\Lambda \rightarrow \mathbb{Z}^d} \frac{it}{|\Lambda|} \sum_{j \in \Lambda} \langle \Theta_j^\Lambda \rangle^\Lambda
\]

\[
= \lim_{\Lambda \rightarrow \mathbb{Z}^d} \left[ \frac{it}{|\Lambda|} \sum_{j \in \Lambda} \left( \langle \Theta_j^\Lambda \rangle^\Lambda - \langle \Theta_0 \rangle^\mathbb{Z}^d \right) + \frac{it|\Lambda|}{|\Lambda|} \langle \Theta_0 \rangle^\mathbb{Z}^d \right]
\]

\[
= it \langle \Theta_0 \rangle^\sigma + A,
\]

where

\[
A := \lim_{\Lambda \rightarrow \mathbb{Z}^d} \frac{it}{|\Lambda|} \sum_{j \in \Lambda} \left[ \langle \Theta_j^\Lambda \rangle^\Lambda - \langle \Theta_0 \rangle^\mathbb{Z}^d \right].
\]

The last line of (49) follows from (45a). We shall prove that \( A = 0 \). Write

\[
A = \lim_{\Lambda \rightarrow \mathbb{Z}^d} \frac{it}{|\Lambda|} \sum_{j \in \Lambda} \left[ \langle \Theta_j^\Lambda \rangle^\Lambda - \langle \Theta_0 \rangle^\mathbb{Z}^d \right]
\]

\[
=: A_1 + A_2.
\]

Recall that the interaction governing the system is of a finite range \( R \). Define:

\[
\tilde{\Lambda}^{(R)}_j := \{ i \in \Lambda | \text{dist}(i, j) \leq R \}, \quad j \in \tilde{\Lambda}.
\]

Then we have

\[
|A_1| \leq \lim_{\Lambda \rightarrow \mathbb{Z}^d} \frac{|t|}{|\Lambda|} \sum_{j \in \Lambda} \sum_{X \subseteq \tilde{\Lambda}^{(R)}_j} \frac{1}{|X|} \left| \langle \Phi_X \rangle^\Lambda - \langle \Phi_X \rangle^\mathbb{Z}^d \right|
\]

Using (41) we obtain

\[
|A_1| \leq \lim_{\Lambda \rightarrow \mathbb{Z}^d} \frac{|t|}{|\Lambda|} \sum_{j \in \Lambda} \sum_{X \subseteq \tilde{\Lambda}^{(R)}_j} \frac{1}{|X|} \left| \Phi_X \right| c(s(X)) \Gamma(\text{dist}(X, \partial \Lambda)).
\]

Set:

\[
c_2 := \sup_{X \subseteq \tilde{\Lambda}^{(R)}_j} c(s(X)).
\]

We have that

\[
\# \{ X | X \subseteq \tilde{\Lambda}^{(R)}_j \} = \left( 2^{2R} \right)^d,
\]

and for \( X \subseteq \tilde{\Lambda}^{(R)}_j \),

\[
\Gamma(\text{dist}(X, \partial \Lambda)) \leq \Gamma(\text{dist}(\tilde{\Lambda}, \partial \Lambda) - R).
\]
Hence,

$$|A_1| \leq \lim_{\Lambda \to \mathbb{Z}^d} \frac{|t|}{|\Lambda|} c_0 c_2 (2R)^d \Gamma(\text{dist}(\hat{\Lambda}, \partial \Lambda) - R) = 0.$$  (52)

by (45d) and (42).

The second term on the RHS of (50) is bounded as follows:

$$|A_2| \leq \lim_{\Lambda \to \mathbb{Z}^d} \frac{|t|}{|\Lambda|} \sum_{j \in \hat{\Lambda}} \sum_{X \ni j, X \not\subset \hat{\Lambda} \cup \Lambda \partial} \frac{1}{|X|} |\langle \Phi_X \rangle^\sigma|$$

$$\leq \lim_{\Lambda \to \mathbb{Z}^d} \frac{|t|}{|\Lambda|} \sum_{j \in \hat{\Lambda}} \sum_{X \ni j, X \not\subset \hat{\Lambda}} \frac{1}{|X|} |\langle \Phi_X \rangle^\sigma|,$$

(53)

since $\hat{\Lambda} \subset \Lambda$. Now

$$\#\{X| X \ni j, X \not\subset \hat{\Lambda}, j \in \hat{\Lambda}, \Phi_X \neq 0\} = (2R)^d$$

and

$$\#\{j \in \hat{\Lambda} | \exists X \ni j, \text{such that } X \not\subset \hat{\Lambda}, \Phi_X \neq 0\} = |\text{Int}^{(R)}(\hat{\Lambda})|,$$

where

$$\text{Int}^{(R)}(\hat{\Lambda}) := \{i \in \hat{\Lambda} | \text{dist}(i, \partial \hat{\Lambda}) < R\}$$

is the $R$-interior of the volume $\hat{\Lambda}$. We have

$$|A_2| \leq \lim_{\Lambda \to \mathbb{Z}^d} \frac{|t|}{|\Lambda|} (2R)^d c_0 |\text{Int}^{(R)}(\hat{\Lambda})|.$$

However,

$$|\text{Int}^{(R)}(\hat{\Lambda})| \leq R |\partial \hat{\Lambda}|.$$

Hence,

$$|A_2| \leq \lim_{\Lambda \to \mathbb{Z}^d} \frac{|t|}{|\Lambda|} (2R)^d R c_0 (2R)^d$$

$$= 0,$$

(54)

by (45c). From (49), (50), (52) and (54) we readily get (43).

This argument admits a generalisation for the $n^{th}$ term in the expansion on the RHS of (33). We have:

$$T_n := \frac{1}{n!} \left( \frac{it}{|\Lambda|} \right)^n \text{tr} \left( H^\Lambda e^{-\beta H} \right) / \Xi^\Lambda$$

$$= \frac{1}{n!} \left( \frac{it}{|\Lambda|} \right)^n \sum_{X_1, \ldots, X_n \subset \Lambda, \partial \not\subset \Lambda} \text{tr} \left( \Phi_{X_1} \cdots \Phi_{X_n} e^{-\beta H} \right) / \Xi^\Lambda$$
\[ \begin{align*}
&= \frac{1}{n!} \left( \frac{it}{|\Lambda|} \right)^n \sum_{j_1 \ldots j_n \in \Lambda} \frac{1}{X_1 \cap \Lambda \neq \emptyset} \cdots \frac{1}{X_n \cap \Lambda \neq \emptyset} (\Phi_{X_1} \ldots \Phi_{X_n})_{\Lambda} \\
&= \frac{1}{n!} \left( \frac{it}{|\Lambda|} \right)^n \sum_{j_1 \ldots j_n \in \Lambda} (\Theta_{j_1}^\Lambda \ldots \Theta_{j_n}^\Lambda)_{\Lambda}^\sigma.
\end{align*} \]

(55)

We prove below that for each \( n \geq 2 \),

\[ \lim_{\Lambda \rightarrow \mathbb{Z}^d} T_n = \frac{\left( \frac{it}{|\Lambda|} \right)^n}{n!} (\langle \Theta_0 \rangle_{\mathbb{Z}^d})^n. \]

(56)

Define volumes \( \Lambda^{(n)} \) and \( \hat{\Lambda}^{(n)} \):

\[ \begin{align*}
\Lambda^{(n)} &= \{ (i_1, \ldots, i_n) | i_k \in \Lambda \quad 1 \leq k \leq n \}, \\
\hat{\Lambda}^{(n)} &= \{ (j_1, \ldots, j_n) | (j_1, \ldots, j_n) \in \Lambda^{(n)}, L_n \geq \ell^{(n)} \},
\end{align*} \]

(57)

where

\[ L_n \equiv L(j_1, \ldots, j_n) := \min \left\{ \min_{1 \leq k < \ell \leq n} [\text{dist}(j_k, j_\ell)], \min_{1 \leq k \leq n} [\text{dist}(j_k, \partial \Lambda)] \right\}, \]

and \( \partial \Lambda \) is the boundary of the volume \( \Lambda \). The quantity \( \ell^{(n)} \) is chosen so that

\[ \lim_{\Lambda \rightarrow \mathbb{Z}^d} \ell^{(n)} = \infty \]

(58)

and

\[ \begin{align*}
& (a) \ |\hat{\Lambda}^{(n)}| \rightarrow 1, \quad (b) \ |\Lambda^{(n)} \setminus \hat{\Lambda}^{(n)}| \rightarrow 0, \quad (c) \ |\partial \hat{\Lambda}^{(n)}| \rightarrow 0.
\end{align*} \]

(59)

[Note that \( |\Lambda^{(n)}| = |\Lambda|^n \).] Writing \( \hat{j} = (j_1, \ldots, j_n) \), we prove (56) as follows:

\[ T_n := \frac{1}{n!} \left( \frac{it}{|\Lambda|} \right)^n \left[ \sum_{\hat{j} \in \Lambda^{(n)}} (\Theta_{j_1}^\Lambda \ldots \Theta_{j_n}^\Lambda)_{\Lambda}^\sigma + \sum_{\hat{j} \in \Lambda^{(n)} \setminus \hat{\Lambda}^{(n)}} (\Theta_{j_1}^\Lambda \ldots \Theta_{j_n}^\Lambda)_{\Lambda}^\sigma \right] \\
= T_n(1) + T_n(2).
\]

(60)

Now,

\[ |\langle \Theta_{j_1}^\Lambda \ldots \Theta_{j_n}^\Lambda \rangle_{\Lambda}^\sigma| \leq \sum_{X_1 \cap \Lambda \neq \emptyset} \cdots \sum_{X_n \cap \Lambda \neq \emptyset} \frac{1}{|X_1|} \cdots \frac{1}{|X_n|} |\langle \Phi_{X_1} \ldots \Phi_{X_n} \rangle_{\Lambda}^\sigma| \]

\[ \leq \left[ \left( 2^{2R} \right)^d \right]^n \| \Phi_{X_1} \ldots \Phi_{X_n} \| \]

\[ \leq c^n_0 \left( 2^{2R} \right)^{nd}. \]

(61)
Hence,

$$\lim_{\Lambda/\mathbb{Z}^d} T_n(2) \leq \lim_{\Lambda/\mathbb{Z}^d} \frac{|\Lambda^{(n)} \setminus \hat{\Lambda}^{(n)}| |t|^n}{n! |\Lambda|} c_0^n (2^{2R})^{nd} = 0,$$

by (58c). Consequently, in the infinite volume limit, the only non-zero contribution to $T_n$ arises from the term $T_n(1)$ on the RHS of (60). This term can in turn be written as follows:

$$\lim_{\Lambda/\mathbb{Z}^d} T_n(1) \leq \frac{1}{n!} \left( \frac{it}{|\Lambda|} \right)^n \left[ \sum_{j \in \Lambda(n)} \left[ (\Theta^{\Lambda}_{j_1} \cdots \Theta^{\Lambda}_{j_n}) - (\Theta^{\Lambda}_{j_n})^n \right] + (\Theta^{\Lambda}_{0})^n \right] = \frac{(it)^n}{n!} (\Theta^{\Lambda}_{0})^n + B,$$

where

$$B := \lim_{\Lambda/\mathbb{Z}^d} T_n(2) \leq \frac{1}{n!} \left( \frac{it}{|\Lambda|} \right)^n \sum_{j \in \Lambda(n)} \left[ (\Theta^{\Lambda}_{j_1} \cdots \Theta^{\Lambda}_{j_n}) - (\Theta^{\Lambda}_{0})^n \right].$$

The first term on the RHS of the last line of (63) follows from (59a). We prove below that $B = 0$. We can write $B$ as follows:

$$B := \frac{1}{n!} \left( \frac{it}{|\Lambda|} \right)^n \sum_{j \in \Lambda(n)} \left[ \sum_{X_1 \subseteq \Lambda \cup \Lambda^0} \cdots \sum_{X_n \subseteq \Lambda \cup \Lambda^0} \frac{1}{|X_1|} \cdots \frac{1}{|X_n|} \right]
\times \left( (\Phi_{X_1} \cdots \Phi_{X_n})^\sigma - (\Phi_{X_1} \cdots \Phi_{X_n})^\sigma \right) + \sum_{X_1 \supset j_1} \cdots \sum_{X_n \supset j_n} \frac{1}{|X_1|} \cdots \frac{1}{|X_n|}
\times (\Phi_{X_1} \cdots \Phi_{X_n})^\sigma 1(X \not\subset \Lambda \cup \Lambda^0 \text{ for some } 1 \leq i \leq n)
:= B_1 + B_2.$$

By using methods of [5] it can be shown that for $\beta$ large and $\lambda$ small enough, the following bound holds:

$$\left| \langle \prod_{i=1}^n \Phi_{X_i} \rangle^\sigma - \langle \prod_{i=1}^n \Phi_{X_i} \rangle^\sigma \right| \leq \left( \prod_{i=1}^n ||\Phi_{X_i}|| \right) c(s(X_1, \ldots, X_n) \Gamma(\Delta_n),$$

where

$$\Delta_n \equiv \Delta_n(X_1, \ldots, X_n) = \min\{\text{dist}(X_i, X_j) | 1 \leq i < j \leq n\},$$

and $\Gamma(r)$ is a monotonically decreasing function of $r$, satisfying the bound (42). Further, recall that $||\Phi_{X_i}|| \leq c_0$ and let $c_3 := \sup_{X_1, \ldots, X_n \in \mathbb{Z}^d} c(s(X_1, \ldots, X_n))$. For $X_i \subseteq \Lambda$ and $X_i \supset j_i$ for $1 \leq i \leq n$,

$$\min_{1 \leq i < j \leq n} \{\text{dist}(X_i, X_j)\} \geq (\ell_{\Lambda})^n - 2R.$$
Hence,

\[ \Gamma(\Delta_n) \leq \Gamma(\ell_n^\omega - 2R), \]

and

\[ |B_1| \leq \lim_{\Lambda \sim \mathbb{Z}^d} \frac{|t|^n}{n!} \frac{\hat{\Lambda}^{(n)}}{|\Lambda^{(n)}|} c_3 (c_0)^n (2^2R)^{nd} \Gamma(\ell_n^\omega - 2R) = 0, \]

(65)

by (58) and (42). Moreover,

\[ |B_2| \leq \lim_{\Lambda \sim \mathbb{Z}^d} \frac{|t|^n}{|\Lambda^{(n)}|} \frac{1}{n!} \sum_{j \in \hat{\Lambda}^{(n)}} \sum_{\Lambda_1 \ni j_1} \cdots \sum_{\Lambda_n \ni j_n} ||\Phi_{X_1} \cdots \Phi_{X_n}|| \mathbf{1}(X_i \not\subset \Lambda \cup \Lambda^\partial \text{ for some } 1 \leq i \leq n) \]

\[ \leq \lim_{\Lambda \sim \mathbb{Z}^d} \frac{|t|^n}{|\Lambda^{(n)}|} \frac{1}{n!} \sum_{j \in \hat{\Lambda}^{(n)}} \sum_{\Lambda_1 \ni j_1} \cdots \sum_{\Lambda_n \ni j_n} ||\Phi_{X_1} \cdots \Phi_{X_n}|| \mathbf{1}(X_i \not\subset \hat{\Lambda} \text{ for some } 1 \leq i \leq n) \]

\[ \leq \lim_{\Lambda \sim \mathbb{Z}^d} \frac{|t|^n}{|\Lambda^{(n)}|} c_0^n (2^2R)^{nd} |\text{Int}^{(R)}(\hat{\Lambda}^{(n)})|. \]

(66)

In fact,

\[ \# \{ j \in \hat{\Lambda}^{(n)} | \exists X_1 \ni j_1, \ldots, X_n \ni j_n \text{ such that } X_i \not\subset \hat{\Lambda} \text{ for some } 1 \leq i \leq n \} = |\text{Int}^{(R)}(\hat{\Lambda}^{(n)})|. \]

Here, as before

\[ \text{Int}^{(R)}(\hat{\Lambda}^{(n)}) = \{ j \in \hat{\Lambda}^{(n)} | \text{dist}(j, \partial \hat{\Lambda}^{(n)}) < R \}. \]

Hence, \( |\text{Int}^{(R)}(\hat{\Lambda}^{(n)})| \leq R |\partial \hat{\Lambda}^{(n)}| \) and

\[ |B_2| \leq \lim_{\Lambda \sim \mathbb{Z}^d} \frac{|\partial \hat{\Lambda}^{(n)}|}{|\Lambda^{(n)}|} \frac{|t|^n}{n!} c_0^n R (2^2R)^{nd} = 0, \]

(67)

by (52). From (54), (55) and (57) it follows that \( B = 0 \). Hence, from (53) and (55) one obtains

\[ \lim_{\Lambda \sim \mathbb{Z}^d} T_n = \frac{(it)^n}{n!} (\langle \Theta_0 \rangle_{\mathbb{Z}^d})^n \quad \forall n \geq 1. \]

(68)

From (58) we now see, in view of Lebesgue’s dominated convergence theorem, that

\[ \lim_{\Lambda \sim \mathbb{Z}^d} \varphi^{(n)}(t/|\Lambda|) = 1 + \sum_{n=1}^{\infty} \frac{(it)^n}{n!} (\langle \Theta_0 \rangle_{\mathbb{Z}^d})^n. \]

(69)

The limiting energy density per lattice site, \( g^{(\sigma)} \), defined through (43), can be written as

\[ g^{(\sigma)} := \lim_{\Lambda \sim \mathbb{Z}^d} \langle H_{\Lambda} / |\Lambda| \rangle^{\sigma}_{\Lambda} = \lim_{\Lambda \sim \mathbb{Z}^d} \sum_{X_i : X_i \cap \Lambda \neq \emptyset} \frac{\langle \Phi_{X_i} \rangle^{\sigma}_{\Lambda}}{|\Lambda|} = \lim_{\Lambda \sim \mathbb{Z}^d} \frac{1}{|\Lambda|} \sum_{j \in \Lambda} \sum_{X_i : X_i \cap \Lambda \neq \emptyset} \frac{\langle \Phi_{X_i} \rangle^{\sigma}_{\Lambda}}{|X_i|}. \]

(70)
Since the interaction $\Phi = \{\Phi_X\}$ is assumed to be translationally invariant we can write

$$g(\sigma) = \lim_{\Lambda \rightarrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \sum_{j \in \Lambda} \sum_{\substack{X, \lambda > 0 \\lambda \cap \phi \neq \emptyset}} \frac{\langle \Phi_X \rangle_{\Lambda}^{\sigma}}{|X|} = \lim_{\Lambda \rightarrow \mathbb{Z}^d} \sum_{\substack{X, \lambda > 0 \\lambda \cap \phi \neq \emptyset}} \frac{\langle \Phi_X \rangle_{\Lambda}^{\sigma}}{|X|} = \sum_{\substack{X, \lambda > 0 \\lambda \cap \phi \neq \emptyset}} \frac{\langle \Phi_X \rangle_{\Lambda}^{\sigma}}{|X|} = \langle \Theta_0 \rangle_{\mathbb{Z}^d}. \tag{71}$$

Hence, (69) can be written as

$$\lim_{\Lambda \rightarrow \mathbb{Z}^d} \varphi^{(\Lambda)}(t/|\Lambda|) = e^{igt(\sigma)},$$

which proves Lemma 4.

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