The Energy-Momentum Tensor in Fulling-Rindler Vacuum

Renaud Parentani*

The Racah Institute of Physics, The Hebrew University of Jerusalem, Jerusalem 91904, ISRAEL

Abstract. The energy density in Fulling-Rindler vacuum, which is known to be negative ”everywhere” is shown to be positive and singular on the horizons in such a fashion as to guarantee the positivity of the total energy. The mechanism of compensation is displayed in detail.

ULB preprint ULB-TH-15/92

* on leave from: Université Libre de Bruxelles, Service de Physique Théorique. Campus Plaine, C.P. 225, Bd. du Triomphe, B1050 Belgium
1. Introduction

In connection with the problem of the radiation emitted and absorbed by an accelerating observer [1, 2, 3, 10, 14], an interesting conceptual problem has arisen. What is the behavior of the energy density in Fulling-Rindler (FR) vacuum?

We remind the reader that Minkovski vacuum presents itself as a thermal distribution of FR quanta within the quadrant containing the accelerating trajectory [1, 4, 5]. One often describes excitation of the detector in terms of absorption of these FR quanta so the above question naturally arises (i.e. what would the state look like when all FR quanta are absorbed?).

Existent calculations [6, 7] in the literature give the result that the energy density in FR vacuum is negative. This has been interpreted as the absence of the thermal FR quanta which characterize Minkovski vacuum. When this density is integrated over all space, the total energy in FR vacuum would be therefore less than that of Minkovski vacuum. But this later is the ground state.

In this paper, this dilemma is solved. We show that the energy density is singular and positive in such a fashion that the total (integrated) energy is positive.

Our method proceeds through the introduction of an ultra violet regulator (\(\varepsilon\)) which permits the evaluation of the energy density close and on the horizon. As \(\varepsilon \to 0\), the energy density tends to a distribution. For all points off the horizon, this density is the negative value previously obtained. Yet its integral is positive.

We note in passing that the same behaviour exists in Boulware vacuum near the Schwarzschild horizon [6, 7], as well as near the horizons of the static patch in de Sitter space, in the state of vacuum defined by the static system.

2. The FR Vacuum and Minkovski Vacuum.

For simplicity we work in two space time dimensions with a massless scalar field \(\phi\). It is then particularly useful to work in light cone coordinates to benefit from the conformal invariance of this massless theory. First we introduce \(U\) and \(V\) defined by:

\[
U = t - z \\
V = t + z
\]

where \(t\) and \(z\) are the usual Minkovski coordinates. We define also \(u_R\) and \(v_R\) (and \(u_L, v_L\)) by:

\[
\begin{align*}
  u_R &= -\theta(-U) \frac{1}{a} \ln(-aU) \\
  v_R &= \theta(V) \frac{1}{a} \ln(aV) \\
  u_L &= \theta(U) \frac{1}{a} \ln(aU) \\
  v_L &= -\theta(-V) \frac{1}{a} \ln(-aV)
\end{align*}
\]

where \(a\) has the dimension of a \([\text{length}]^{-1}\) and where \(R(L)\) indicates that only the right (left) hand side of the Minkovski plane is covered by those coordinates.
In the first quadrant (i.e. $|t| < z > 0$), $u_R$ and $v_R$ are related to the usual Rindler coordinates $(\tau, \rho)$ by:

\[
\begin{align*}
(u_R + v_R)/2 &= \tau = (1/a)\arctanh(t/z) \\
e^{(v_R - u_R)} &= a^2 \rho^2 = a^2(z^2 - t^2)
\end{align*}
\]  

(4)

$\tau$ is the proper time of a system following the trajectory $\rho = 1/a$.

Let us now obtain the FR and the usual Minkovski vacua through analysis of the solutions of $\Box \phi = 0$, which in any lightcone set (which we denote generically by $u, v$) reads $\partial_u \partial_v \phi = 0$. The general solution is thus $\phi = \phi(u) + \phi(v)$ and we may restrict the analysis to say, the $u$-part of the field only, the $v$-part being treated in an identical way.

The Minkovski normalised wave-functions, solutions of $i \partial_U \phi_k = k \phi_k$, are given by:

\[
\phi_k(U) = \frac{1}{(4\pi k)^{1/2}} e^{-ikU}
\]  

(5)

which, together with their complex conjugates, constitute a complete set if $k \in [0, \infty]$. They define the operators $a_k$ which annihilate the Minkovski vacuum.

\[
a_k \equiv \langle \phi_k | \phi \rangle = \int_{-\infty}^{+\infty} dU \phi_k^* i \partial_U \phi
\]  

(6)

whereupon

\[
\phi(U) = \int_0^\infty dk \left( a_k \phi_k + a_k^\dagger \phi_k^* \right)
\]  

(7)

\[
a_k \mid 0, \text{Mink} \rangle = 0
\]

Similarly, the FR wave functions $\phi_{\lambda,R(L)}$ are solutions of

\[
i \partial_{u_{R(L)}} \phi_{\lambda,R(L)} = i a \mid U \mid \partial_U \phi_{\lambda,R(L)} = \lambda \phi_{\lambda,R(L)}
\]  

(8)

and given by:

\[
\phi_{\lambda,R} = \frac{1}{(4\pi \lambda)^{1/2}} e^{-i\lambda u_R} \theta(-U) = \frac{\theta(-U)}{(4\pi \lambda)^{1/2}} (-aU)^{i\lambda/a}
\]

\[
\phi_{\lambda,L} = \frac{1}{(4\pi \lambda)^{1/2}} e^{-i\lambda u_L} \theta(U) = \frac{\theta(U)}{(4\pi \lambda)^{1/2}} (aU)^{-i\lambda/a}
\]  

(9)

They are normalized and constitute another complete set if $\lambda \in [0, \infty]$. They define the FR operators $b_{\lambda,R(L)}$ ($b_{\lambda,R(L)} = \langle \phi_{\lambda,R(L)} | \phi \rangle$ see(6)). Those operators define the FR vacuum:

\[
b_{\lambda,R} \mid 0, R \rangle \mid 0, L \rangle = b_{\lambda',L} \mid 0, R \rangle \mid 0, L \rangle = 0 \quad \forall \lambda, \lambda'
\]  

(10)
It is particularly useful to introduce a third complete set \( \{ \phi_{\lambda,M} \}, \lambda \in [-\infty, +\infty] \) \[1, 8\], which is also composed of eigenmodes of \( U \partial_U \) (see (8)), but built from positive energy Minkovski modes (5) only, thus of the form

\[
\phi_{\lambda,M} = \int_0^\infty dk \, \alpha_{\lambda,k} \phi_k (U)
\]

They are normed functions if

\[
\alpha_{\lambda,k} = \frac{1}{(2\pi k)^{1/2}} \left( \frac{k}{a} \right)^{-i\lambda/a}
\]

The associated annihilation operators are given by

\[
a_{\lambda} = \int_0^\infty dk \, \alpha_{\lambda,k}^* a_k = \langle \phi_{\lambda,M} | \phi \rangle
\]

and hence annihilate the Minkovski vacuum \( |0, \text{Mink} \rangle \) (7).

The Bogoliubov coefficients giving the overlap between \( \phi_{\lambda,M} \) and \( \phi_{\lambda,R(L)} \), are obtained by the direct evaluation of (11) with \( \alpha_{\lambda,k} \) given by (12). One gets:

\[
\phi_{\lambda,M} (U) = \frac{1}{(8\pi^2 a)^{1/2}} \Gamma (-i\lambda/a) \left[ \theta (-U) (-aU)^{i\lambda/a} e^{\pi \lambda/2a} + \theta (U) (aU)^{i\lambda/a} e^{-\pi \lambda/2a} \right]
\]

\[
= \left[ \Gamma (-i\lambda/a) e^{\pi \lambda/2a} \right] \phi_{\lambda,R} + \left[ \Gamma (-i\lambda/a) e^{-\pi \lambda/2a} \right] \phi_{\lambda,L} \quad \text{(for } \lambda > 0) \]

\[
= \left[ \Gamma (-i\lambda/a) e^{\pi \lambda/2a} \right] \phi_{-\lambda,R}^* + \left[ \Gamma (-i\lambda/a) e^{-\pi \lambda/2a} \right] \phi_{-\lambda,L} \quad \text{(for } \lambda < 0) \]

The coefficients of the inverse transformation are readily checked to have the same absolute value as the coefficients of the initial one as they appear in (14) and (15). This is a simple consequence of the absence of frequency mixing (i.e. the transformation is diagonal in \( \lambda \)). Thus, the relation between \( b_{\lambda,R(L)} \) and \( a_{\lambda}, a_{-\lambda}^\dagger \) is

\[
b_{\lambda,R} = \alpha_{\lambda} a_{\lambda} + \beta_{\lambda} a_{-\lambda}^\dagger \quad (\lambda > 0)
\]

\[
b_{\lambda,L} = \alpha_{\lambda} a_{-\lambda} + \beta_{\lambda} a_{\lambda}^\dagger \quad (\lambda > 0)
\]

where
\[
\alpha_\lambda = \beta_\lambda e^{\pi \lambda/a} = \left(1 - e^{-2\pi \lambda/a}\right)^{-1} = \left|\frac{\Gamma(-i\lambda/a) e^{\pi \lambda/2a}}{(2a\lambda)^{1/2}}\right|
\]

where \(\alpha_\lambda\) and \(\beta_\lambda\) are defined to be real by a redefinition of the irrelevant absolute phase of the waves \(\phi_{\lambda,M}\). Hence, the unitary relation between \(|0, \text{Mink}\rangle\) and \(|0, R\rangle|0, L\rangle\) is given by [8]:

\[
|0, \text{Mink}\rangle = U|0, R\rangle|0, L\rangle = \frac{1}{Z} \Pi_{\lambda>0} e^{-\pi \lambda/ab_\lambda \dagger R b_\lambda \dagger L} |0, R\rangle|0, L\rangle \tag{19}
\]

\[
|0, R\rangle|0, L\rangle = U^{-1}|0, \text{Mink}\rangle = \frac{1}{Z} \Pi_{\lambda>0} e^{+\pi \lambda/a a_\lambda \dagger a_{-\lambda} \dagger} |0, \text{Mink}\rangle \tag{20}
\]

where

\[
U^{-1} = U^\dagger = \Pi_{\lambda>0} e^{\text{arctanh}(e^{-\pi \lambda/a}) (a_\lambda \dagger a_{-\lambda} \dagger - a_{-\lambda} a_{-\lambda})} \tag{21}
\]

\[
Z^{-1} = \Pi_{\lambda>0} (1/\alpha_\lambda) = \langle \text{Mink}, 0| (|0, R\rangle|0, L\rangle) \tag{22}
\]

Equation (19) is the familiar rewriting of the Minkowski vacuum as a superposition of FR states containing excited pairs. This rewriting insures that the expectation value of any operator localized in the first quadrant is a thermal average. Equation (20) is the less familiar expression of the FR vacuum in terms of Minkowski pair excitations (weighted by the same factor \(\beta_\lambda/\alpha_\lambda = e^{-\pi \lambda/a}\)). As a corollary, the total energy of FR vacuum is greater than the energy of Minkovski vacuum (renormalized to zero).

In the next section we shall see how eqs. (19) and (20) encode the properties of the energy momentum tensor in the FR vacuum.

**3. The energy-momentum tensor in FR vacuum.**

The "\(UU\)" component of the energy-momentum tensor in FR vacuum is given by

\[
T_{UU}^{RL} = \langle L, 0| (\partial_{U \Phi})^2 |0, R\rangle|0, L\rangle - \langle Mink, 0| (\partial_{U \Phi})^2 |0, Mink\rangle \tag{23}
\]

when the negative term defines the subtraction, here purely Minkovski. Using the decomposition (20), and expressing \(\phi\) in terms of \(\phi_{\lambda,M}\) (11), we obtain:

\[
T_{UU}^{RL} = 2 \int_{-\infty}^{+\infty} d\lambda \left[ \beta_{\lambda|^2}^2 |\partial_{U \phi_{\lambda,M}}|^2 - \alpha_{|\lambda|} |\beta_{|\lambda|}^2 Re (\partial_{U \phi_{\lambda,M}}) \partial_{U \phi_{-\lambda,M}}| \right] \tag{24}
\]

Then

\[
\int dU T_{UU}^{RL} = 2 \int dU \int_{-\infty}^{+\infty} d\lambda \beta_{\lambda|^2}^2 |\partial_{U \phi_{\lambda,M}}|^2 > 0 \tag{25}
\]
because the integral of the second term of (24) vanishes due to the orthogonality between $\phi_{\lambda,M}$ and $\phi^*_{-\lambda,M}$. Equation (25) confirms the positivity of the total mean energy in FR vacuum.

Before calculation of (24), we first recall the usual expression for $T_{UU}^{RL}$:

$$\frac{T_{UU}^{RL}}{1} = -\frac{1}{48\pi U^2}$$  \hspace{1cm} (26)

For purposes of comparison with (24), it is instructive to recall the two standard derivations of (26). The first method [9] makes use of the Wightman functions $G^+$ defined by:

$$G^+_{RL}(U,U') = \langle L,0|\langle R,0|\phi(U)\phi(U')|0,R\rangle|0,L\rangle = -\frac{1}{8\pi} \left[ \theta(U)\theta(U') + \theta(-U)\theta(-U') \right] \ln|\ln(U/U')|^2$$

$$G^+_M(U,U') = \langle \text{Mink},0|\phi(U)\phi(U')|0,\text{Mink}\rangle = -\frac{1}{8\pi} \ln|U-U'|^2$$

So that

$$T_{UU}^{RL} = \lim_{U' \to U} \partial_U \partial_{U'} [G^+_{RL}(U,U') - G^+_M(U,U')]$$

\begin{equation}
= (26)
\end{equation}

where the last equality is obtained by first taking the derivatives and then taking the coincidence limit $U' \to U$ of the difference.

The second method [7] is based on Eqs. (23) and (19). The subtraction is then the removal of the energy of the FR quanta present in Minkovski vacuum. Explicitly,

$$T_{UU} = -2 \int_0^\infty d\lambda \frac{\beta^2}{\alpha} \left[ |\partial_U \phi_{\lambda,R}|^2 + |\partial_U \phi_{\lambda,L}|^2 \right]$$

\begin{equation}
= (28)
\end{equation}

The term in $\alpha \beta \lambda$ (see (24)) is absent because $[\phi_{\lambda,R}(U)\phi_{\lambda,L}(U) = 0 \forall U \neq 0]$. One verifies that (28) gives once more (26) because

$$\int_0^\infty \frac{d\lambda \lambda}{a} e^{2\pi \lambda/a - 1} = \frac{1}{24}$$

The dilemma is clear; if one accepts that (26) is valid everywhere one finds oneself in contradiction with (25).

To prove that (25) is compatible with the value of the density (26), let us go back to (24) and evaluate $\partial_U \phi_{\lambda,M}$ with care. One has (see (11) and (12)):
\[
i \partial_U \phi_{\lambda,M} \equiv \int_0^{\infty} \frac{dk}{(2\pi)^1/2} \left( \frac{k}{a} \right)^{i\lambda/a} e^{-ikU}
\]
\[
= \lim_{\varepsilon \to 0} \frac{\Gamma(-i\lambda/a - 1)}{(2\pi)^1/2} \theta(U) (aU - i\varepsilon)^{-i\lambda/a - 1} e^{-\pi\lambda/2a}
\]
\[
+ \lim_{\varepsilon \to 0} \frac{\Gamma(-i\lambda/a - 1)}{(2\pi)^1/2} \theta(-U) (-aU + i\varepsilon)^{-i\lambda/a - 1} e^{\pi\lambda/2a}
\]

(29)

It is necessary to introduce the regulator \(\varepsilon\) to define the integral. Using this expression in (25) and grouping together the contribution from \(\lambda\) and \(-\lambda\) yields

\[
T_{UU}^{RL} = \lim_{\varepsilon \to 0} \int_0^{\infty} \frac{d\lambda}{2\pi} \left[ \beta^2 (\alpha^2 + \beta^2) \frac{1}{U^2 + \varepsilon^2} - 2\alpha^2 \beta^2 \frac{Re\left(1/U - i\varepsilon\right)^2}{1} \right]
\]

(30)

First, for \(U \neq 0\), one may send \(\varepsilon \to 0\), thereby recovering (26) because \(\alpha^2 (\alpha^2 + \beta^2) - 2\alpha^2 \beta^2 = -\beta^2\). Thus the dominant part of the energy density comes from the interference term \(\alpha \beta\) in (24).

Secondly, if one integrates (30) on \(U\) from \(-\infty\) to \(+\infty\) one cannot send \(\varepsilon \to 0\) because of the singularity at \(U = 0\). So we perform the integral at fixed \(\varepsilon\) and we verify that the \(\alpha \beta\) term integrates to zero:

\[
\int_{-\infty}^{+\infty} dU \ Re\left(1/U - i\varepsilon\right) = 0
\]

Thus we recover the positivity of the total energy [(25) with \(|\partial_U \phi_{\lambda,M}|^2\) explicitized by the use of (29)]. Moreover, the total energy \(\int_{-\infty}^{+\infty} dUT_{UU}\) diverges as

\[
\int_{-\infty}^{+\infty} dU T_{UU} = \lim_{\varepsilon \to 0} \int_0^{\infty} \frac{d\lambda}{a^2} \beta^2 (\alpha^2 + \beta^2) \int_0^{\infty} dZ \frac{2Z}{Z^2 + 1}
\]

(31)

This divergence is due to the high \(k\) behaviour of \(\alpha_{\lambda,k}\) [see (11)(12): the decreasing character in \(k\) is too weak to insure the convergence of the total energy].

Finally let us mention that the energy content of each FR quantum is divergent. To verify this, we compute \(T_{UU}\) in the state \(b^\dagger_{\lambda,R} |0, Mink\rangle\), (call it \(T_{UU}^{\lambda,R}\)). Using (16) and (29), one gets

\[
T_{UU}^{\lambda,R} = \alpha^2 |\partial_U \phi_{\lambda,M}|^2
\]
\[
= \lim_{\varepsilon \to 0} \frac{1}{4\pi} \frac{\lambda}{\alpha^2 \beta^2 \alpha^2 (U^2 + \varepsilon^2)} \left[ \theta(U) \alpha^2 + \theta(-U) \beta^2 \right]
\]

(32)

This quantity integrated over all \(U\) diverges like \(1/\varepsilon\) as well (see (31)).
The implications of those results in the case of the accelerating detector \[2, 3, 10\] or the case of the collapsing black hole \[11,12,13,6\] (or eternal black hole; see \[15\] where static solutions which include the back reaction of a quantum field, in the form \(26\), are constructed) are presently under investigation.

**Note added in proof.** After this manuscript was completed, our attention was called to the recent article of Unruh \[14\] in which was examined the transcient behaviour of radiation as it is put into interaction with an accelerating detector \[3\]. Unruh has shown that similar singular behaviour (on the horizon) arises, in this case as well.

**Acknowledgements**

We would like to thank Ph. Spindel for a helpful discussion about the definition of the gamma functions and R. Brout for interesting comments as well as a careful reading of the manuscript. I would also thank the people of the Racah Institute of Physics of the Hebrew University of Jerusalem for their warm hospitality.
References

1. W.G. Unruh, Phys. Rev. D14 287 (1976).
2. W.G. Unruh and R.M. Wald, Phys Rev. D25 942 (1982).
3. P. Grove, Class. Quantum Grav. 3 801 (1986); D.J. Raine, D.W. Scianna, P. Grove, Proc. R. Soc. Lond. A 435 (1991).
4. S. Fulling, Phys. Rev. D7 2850 (1973).
5. P.C.W. Davies, J. Phys. A: Gen. Phys. 8 609 (1975).
6. N.D. Birrel and P.C.W. Davies, Quantum field in curved space, Cambridge University Press (1982).
7. B.S. De Witt. Chap. 14 in: General Relativity. An Einstein Centenary Survey, eds. S.W. Hawking and W. Israël, Cambridge University Press.
8. R.J. Hughes, CERN preprint TH-3670 (1983).
9. P.C. Davies and S. Fulling, Proc. R. Soc. London A345 59 (1977).
10. S. Massar, R. Parentani and R. Brout, ULB preprint TH 03/92 to be published in Class. Quantum Grav.
11. S.W. Hawking “Particle creation by black holes” in Quantum Gravity (Calrendon Press, Oxford 1975) p. 226.
12. P.C.W. Davies, S.A. Fulling and W.G. Unruh, Phys. Rev. D13 2720 (1976).
13. R. Parentani and R. Brout, Int. J. of Mod. Phys. D1 169 (1992).
14. W.G. Unruh, Phys. Rev. D46 3271 (1992).
15. L. Susskind and L. Thorlacius, Stanford University preprint SU-ITP-92-12 (1992). hepth @ xxx/9203054.