Some applications of André-Quillen homology to classes of arithmetic rings

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Abstract

We compute the first André-Quillen homology modules for the simple over-rings of integrally closed domains and study an ideal theoretic condition arising from the vanishing of $H_1$.

André-Quillen (co)homology is known to be a powerful tool in characterizing various classes of rings or morphisms between noetherian rings. Regular and complete intersection local rings, regular, (formally) smooth or complete intersection morphisms can be characterized with the help of this theory (see André (1974) and Brezuleanu et al. (1993)). Classes of arithmetical integral domains, such as Prüfer domains, have also been characterized in this way (see Planas-Vilanova (1996)).

Let $A$ be an integrally closed domain with quotient field $K$, $0 \neq a, b \in A$ and $B = A[a/b]$. In section 1, we compute $H_0(A, B, B)$ and $H_1(A, B, B)$, and we describe $H_2(A, B, B)$ (Theorems 1.2 and 1.12). In particular, we show that $H_1(A, B, B) = 0$ if and only if $a^2A \cap b^2A = (aA \cap bA)^2$.

In section 2, we investigate this condition in its own. We say that $D$ is a $\star$-domain if $a^2D \cap b^2D = (aD \cap bD)^2$ for every $a, b \in D$. The locally GCD domains are typical examples of $\star$-domains. In Proposition 2.3 we characterize the $\star$-pseudo-valuation domains. In Corollary 2.7 we show that a two-generated domain (e.g. a quadratic extension of $\mathbb{Z}$) is a $\star$-domain if and only if it is Dedekind. Finally, in Proposition 2.9 we prove that the local class group of a Krull $\star$-domain has no element of order two.

Throughout this paper all rings are commutative. For any undefined notation or terminology, the reader is referred to André (1974) and Gilmer (1972).
1 Homological results

In Theorem 1.2 we compute the first two André-Quillen homology modules for a simple over-ring of an integrally closed domain. By an over-ring of an integral domain $A$, we mean any intermediate ring between $A$ and its quotient field. We need the following well-known result, cf. Gilmer (1972, Corollary 34.9).

Lemma 1.1 Let $A$ be an integrally closed domain with quotient field $K$ and let $0 \neq f \in K[X]$. Then

$$fK[X] \cap A[X] = fFA[X]$$

where $F = \{d \in K \mid df \in A[X]\}$.

Theorem 1.2 Let $A$ be an integrally closed domain with quotient field $K$, $0 \neq a, b \in A$ and $B = A[a/b]$. Let $I$ be the kernel of the $A$-algebra morphism $\pi : R \to B$ sending $X$ to $a/b$, where $R = A[X]$. We identify $B$ with $R/I$. Then

(i) $\Omega_{B/A} \cong \frac{B}{(bA :_A a)B} \cong \frac{abA}{(aA + bA)(aA \cap bA)} \otimes_A R$

(ii) $H_1(A, B, B) \cong \frac{b^2A \cap b^2A}{(aA \cap bA)^2} \otimes_A R$.

Proof. (i) Set $c = a/b$. Since $A$ is integrally closed, Lemma 1.1 shows that

$$I = (X - c)K[X] \cap R = (X - c)(A :_A c)R = (X - c)(bA :_A a)R. \tag{1}$$

In particular, $I = (X - c)R \cap R$. Since $\Omega_{R/A} \otimes_R B \cong B$, the Jacobi-Zariski sequence induced by $A \hookrightarrow R \xrightarrow{\pi} B$ is

$$0 \to H_1(A, B, B) \to I/I^2 \xrightarrow{\delta} B \to \Omega_{B/A} \to 0. \tag{2}$$

Let $q \in I$. Then $q = (X - c)r$ for some $r \in R$. Denoting the derivative of a polynomial $h \in R$ by $h'$, we get

$$b\delta(q + I^2) = \delta(bq + I^2) = (bq)'(c) = br(c)$$

so

$$\delta(q + I^2) = r(c) \text{ where } r = \frac{q}{X - c}. \tag{3}$$

Hence

$$\text{Im}(\delta) = (bA :_A a)B.$$

So

$$\Omega_{B/A} \cong B/\text{Im}(\delta) \cong \frac{B}{(bA :_A a)B} \cong \frac{R}{(X - c)(bA :_A a)R + (bA :_A a)R} \cong \frac{R}{(R + Rc)(bA :_A a)} \cong \frac{abR}{(aR + bR)(aR \cap bR)} \cong \frac{abA}{(aA + bA)(aA \cap bA)} \otimes_A R.$$
(ii). Now let \( f \in I \). By (3), \( \delta(f + I^2) = 0 \) if and only if \( f \in (X - c)I \). So, by (2),
\[
H_1(A, B, B) \cong \ker(\delta) \cong \frac{(X - c)I \cap I}{I^2}.
\] (4)

By (1), we get
\[
(X - c)I \cap I = (X - c)^2 K[X] \cap (X - c)R \cap R = (X - c)^2 K[X] \cap R.
\] (5)

Using Lemma 1.1, we get
\[
(X - c)^2 K[X] \cap R = (X - c)^2 (A[X] :_K (X - c)^2) R = (X - c)^2 (b^2 A :_A a^2) R.
\] (6)

Indeed, \( A :_A c^2 \subseteq A :_A c \), because \( A \) is integrally closed. So
\[
A[X] :_K (X - c)^2 = A :_K (1, 2c, c^2) = A :_A c^2 = b^2 A :_A a^2.
\]

Combining (4), (5), (6) and (1), and taking account that \( aA \cap bA = a(bA :_A a) \), we get
\[
H_1(A, B, B) \cong \frac{(X - c)^2 (b^2 A :_A a^2) R}{(X - c)^2 (bA :_A a^2) R} \cong \frac{a^2 A \cap b^2 A}{(aA \cap bA)^2} \otimes_A R.
\]

\[\bullet\]

**Remark 1.3** a) Note that \( \Omega_{B/A} = B/(bA :_A a)B \) is a flat \( B \)-module only when it is zero, that is, \( (bA :_A a)B = B \).

b) The fact that the \( R \)-modules
\[
\frac{abA}{(aA + bA)(aA \cap bA)} \otimes_A R \quad \text{and} \quad \frac{a^2 A \cap b^2 A}{(aA \cap bA)^2} \otimes_A R
\]
are \( B \)-modules, that is, they are annihilated by \( I \), can be proved directly as follows. Note that \( I \subseteq (bA :_A a)R + (aA :_A b)R \). By symmetry, it suffices to see that both modules are annihilated by \( bA :_A a \). For the first module this is clear because \( (bA :_A a)ab \subseteq (aA + bA)(aA \cap bA) \). As \( A \) is integrally closed, \( a^2 A \cap b^2 A \subseteq a^2 A \cap abA \). Hence \( (a^2 A \cap b^2 A)(bA :_A a) \subseteq (a^2 A \cap abA)(bA :_A a) = (aA \cap bA)^2 \), so the second module is annihilated by \( I \).

**Corollary 1.4** Let \( A \) be an integrally closed domain, \( 0 \neq a, b \in A \) and \( B = A[a/b] \).

i) The following conditions are equivalent:

a) \( \Omega_{B/A} = 0 \);

b) \( aA + bA \) is an invertible ideal of \( A \);

c) \( B = (bA :_A a)B \);

d) \( H_n(A, B, E) = 0 \) for every \( B \)-module \( E \) and for every \( n \geq 0 \).

ii) Consequently, the following conditions are equivalent:

a) \( A \) is a Prüfer domain;

b) \( \Omega_{A[y]/A} = 0 \) for every \( y \in K \);

c) \( H_n(A, C, E) = 0 \) for every over-ring \( C \) of \( A \), every \( C \)-module \( E \) and any \( n \geq 0 \).
Proof: i). The equivalence of a), b) and c) follows directly from Theorem 1.2 (see also Smith (1979, Theorem 1)). The implication d) ⇒ a) is obvious. To complete, we prove that b) implies d). Assume that \( aA + bA \) is an invertible ideal of \( A \) and let \( B \) be an over-ring of \( A, E \) an \( B \)-module and \( n \geq 0 \). We claim that \( H_n(A, B, E) = 0 \). Localizing (André, 1974, Corollaries 4.59 and 5.27), we may assume that \( A \) is a quasi-local domain. It follows that \( aA + bA \) is a principal, so it is generated by \( a \) or \( b \). Hence \( B \) is a localization of \( A \), so \( H_n(A, B, E) = 0 \) cf. André (1974, Corollary 5.25).

ii). A domain \( A \) is a Prüfer domain if and only if every nonzero two-generated ideal of \( A \) is invertible (cf. Gilmer (1972, Theorem 22.1)), so ii) follows from i).

Somewhat in the same vein, note that in Planas-Vilanova (1996) it was shown that \( D \) is a Prüfer domain if and only if \( H_2(D, D/I, D/I) = 0 \) for each (three-generated) ideal \( I \) of \( D \).

Corollary 1.5 If \( A \) is an integrally closed domain, \( 0 \neq a, b \in A \) and \( B = A[a/b] \), then \( H_1(A, B, B) = 0 \) if and only if \( a^2A \cap b^2A = (aA \cap bA)^2 \). In particular, if \( aA \cap bA \) is a flat ideal, then \( H_1(A, B, B) = 0 \).

Proof: The first assertion follows directly from Theorem 1.2. To complete, assume that the ideal \( J = aA \cap bA \) is \( A \)-flat. Then

\[
J^2 = (aA \cap bA)J = aJ \cap bJ = a^2 \cap abA \cap b^2A.
\]

Since \( A \) is integrally closed,

\[
a^2A \cap abA \cap b^2A = a^2A \cap b^2A,
\]

because from \( x \in a^2 \cap b^2A \) it follows that \((x/ab)^2 \in A \). Thus \( J^2 = a^2 \cap b^2A \), so the first part applies.

Remark 1.6 a) Let \( D \) be an integrally closed domain such that

\[
(aD \cap bD)(cD \cap dD) = acD \cap adD \cap bcD \cap bdD, \ \forall a, b, c, d \in D
\]

(see Zafrullah (1987)). The preceding proof shows that \( H_1(D, E, E) = 0 \) for each simple over-ring \( E \) of \( D \).

b) By Corollary 1.5, we see that if \( a, b, c, d \in A \) are nonzero elements of the integrally closed domain \( A \) and \( A[a/b] = A[c/d], \) then \( a^2A \cap b^2A = (aA \cap bA)^2 \) if and only if \( c^2A \cap d^2A = (cA \cap dA)^2 \).

Remark 1.7 Theorem 1.2 is no longer true if \( A \) is not integrally closed. Indeed, if \( A = \mathbb{Z}[i\sqrt{3}] \) and \( B = \mathbb{A}[(1+i\sqrt{3})/2], \) then it is easy to see that \( \Omega_{B/A} = 0 \) but the ideal \((1+i\sqrt{3}, 2)A \) is not invertible.
Example 1.8 Consider the integrally closed domains $A = \mathbb{Z}[X]$ and $B = \mathbb{Z}[X/2] = A[X/2]$. By Theorem 1.2 and Corollary 1.5

$$\Omega_{B/A} \cong B/(2A :_A X)B \cong B/2B$$

and $H_1(A, B, B) = 0$.

Example 1.9 Let $K$ be a field and $A = K + xL[x]$ where $L = K(y)$ and $x, y$ are indeterminates over $K$. By Anderson et al. (1991, Theorem 2.7) $A$ is integrally closed. Consider the ring $B = A[yx/x] = K[y] + xL[x]$. An easy computation shows that

$$yxA \cap xA = x^2L[x]$$

and

$$(yx)^2A \cap x^2A = x^3L[x].$$

By Theorem 1.2 we obtain

$$H_1(A, B, B) \cong (yx)^2A \cap x^2A \otimes_A B \cong \frac{x^3L[x]}{x^4L[x]} \otimes_A B \cong L \otimes_A B \cong L \otimes_K K[y] \cong L[z]$$

where $z$ is an indeterminate over $L$ and the $B$-module structure on $L$ is given by the ring morphism $B \to L$ sending $f(x, y)$ into $f(0, y)$. Hence $H_1(A, B, B)$ is not finitely generated as a $B$-module. Similarly, we get

$$\Omega_{B/A} \cong B/xL[x] \cong K[y].$$

Remark 1.10 Let $A$ be a domain, $a, b \in A$ nonzero elements and let $B = A[a/b]$. Then $H^0(A, B, B) = \text{Der}_A(B, B) = 0$. Indeed, if $D \in \text{Der}_A(B, B)$, then

$$0 = D(a) = D(b(a/b)) = bD(a/b),$$

so $D(a/b) = 0$, whence $D = 0$.

Theorem 1.11 Let $A$ be an integrally closed domain, $0 \neq a, b \in A$ and $B = A[a/b]$. If $a^2A \cap b^2A = (aA \cap bA)^2$, then $H^1(A, B, B) = ((bA :_A a)B)^{-1}/B$.

Proof. As $a^2A \cap b^2A = (aA \cap bA)^2$, Theorem 1.2 gives that $H_1(A, B, B) = 0$. Then $H^1(A, B, B) = \text{Ext}^1_B(\Omega_{B/A}, B)$, cf. André (1974, Lemma 3.19). By Theorem 1.2 we have the exact sequence

$$0 \to (bA :_A a)B \to B \to \Omega_{B/A} \to 0$$

which gives the exact sequence

$$0 \to \text{Hom}_B(B, B) \to \text{Hom}_B((bA :_A a)B, B) \to \text{Ext}^1_B(\Omega_{B/A}, B) \to 0.$$
For the next result we use the following notation. Let \( C \subseteq D \) be an extension of domains, and \( J \) an ideal of \( D \). Consider the canonical \( C \)-module epimorphism \( \pi_J : S_2(cJ) \to J^2 \) where \( S_2(cJ) \) denotes the degree-two part of the symmetric algebra of \( J \). Let \( \alpha \) be a nonzero element of \( D \). It is not hard to see \( \ker(\pi_J) \) is \( C \)-isomorphic to \( \ker(\pi_{\alpha J}) \). Call the ideal \( J \) \( C \)-syzygetic (Planas-Vilanova, 1996), if \( \pi_J \) is an isomorphism.

**Theorem 1.12** Consider the setup in Theorem 1.2. Then

\[
H_2(A, B, B) \cong W \otimes_A R
\]

where \( W \) is the kernel of the canonical map \( S_2((aA \cap bA) \to (aA \cap bA)^2) \). In particular, \( H_2(A, B, B) = 0 \) if and only if \( aA \cap bA \) is a syzygetic ideal of \( A \).

**Proof.** We use the notations of the proof of Theorem 1.2. In the Jacobi-Zariski sequence induced by \( A \hookrightarrow R \xrightarrow{\pi} B \)

\[
H_2(A, R, B) \to H_2(A, B, B) \to H_2(R, B, B) \to H_1(A, R, B)
\]

the extreme terms are zero, so \( H_2(A, B, B) \cong H_2(R, B, B) \). Since \( B = R/I \), we have \( H_2(R, B, B) \cong \ker(\pi_I) \), where \( \pi_I \) is the canonical map \( S_2(I) \to I^2 \). By the proof of Theorem 1.2, \( I = (X - c)(bA : aA)R \). As noted in the paragraph preceding Theorem 1.12, \( \pi_I \) is \( R \)-isomorphic to \( W \otimes_A R \) where \( W \) is the kernel of the canonical map \( S_2(aA \cap bA) \to (aA \cap bA)^2 \).

**Example 1.13** Consider the setup of Example 1.9. We have \( yxA \cap xA = x^2L[x] \). By Theorem 1.12 and the paragraph preceding, it follows that

\[
H_2(A, B, B) \cong W \otimes_A R
\]

where \( W \) is the kernel of the canonical map \( S_2(A[x]) \to L[x] \).

**Corollary 1.14** In the setup in Theorem 1.2 assume that \( aA \cap bA = abA \) and let \( E \) be any \( B \)-module. Then:

a) \( \Omega_B/A \cong B/bB \);

b) \( H_1(A, B, E) \cong 0 : E b \);

c) \( H_2(A, B, E) = 0 \);

d) \( H^1(A, B, E) \cong E/bE \);

e) \( H^2(A, B, E) = 0 \).

**Proof.** a) Follows at once from Theorem 1.2.

b),c) From Theorems 1.2 and 1.12 it follows that \( H_1(A, B, B) = H_2(A, B, B) = 0 \). Now by André (1974, Lemma 3.19) we have

\[
H_i(A, B, E) \cong \text{Tor}_i^B(\Omega_{B/A}, E) \cong \text{Tor}_i^B(B/bB, E), i = 1, 2.
\]

From the exact sequence

\[
0 \to bB \to B \to B/bB \to 0
\]
we obtain the exact sequence
\[ 0 \to \text{Tor}_1^B(B/bB, E) \to bB \otimes E \to B \otimes E \to B/bB \otimes E \to 0 \]
and this gives b). As for c), it follows from the fact that \( \text{fd}_B(\Omega_{B/A}) \leq 1 \).

d) Again by André (1974, Lemma 3.19), we have that
\[ H^i(A, B, E) \cong \text{Ext}_B^i(\Omega_{B/A}, E), \ i = 1, 2. \]
Now everything follows from the exact sequence
\[ 0 \to \text{Hom}(B/bB, E) \to \text{Hom}(B, E) \to \text{Hom}(bB, E) \to \text{Ext}_B^1(B/bB, E) \to 0. \]
Note that for \( E = A \) the assertion follows also from Theorem 1.11.

e) We have \( \text{pd}(\Omega_{B/A}) \leq 1 \).

\[ \text{Remark 1.15} \] If \( A \) is a GCD domain and \( B \) is a simple over-ring of \( A \), then it follows that there exists some element \( b \in B \) such that the same formulas as in the preceding corollary hold.

\[ \text{2 Ideal theoretic results} \]

In this section, we consider the condition in Corollary 1.5 in its own. We give the following ad-hoc definition.

\[ \text{Definition 2.1} \] Let \( D \) be a domain with quotient field \( K \). Call a domain \( D \) a \( \star \)-domain if \( a^2D \cap b^2D = (aD \cap bD)^2 \) (equivalently, \((bD :_D a)^2 = b^2D :_D a^2\)) for every \( a, b \in D \).

\[ \text{Remark 2.2} \] a) The condition of being a \( \star \)-domain is clearly local. Hence, a locally GCD domain is a \( \star \)-domain.

b) A \( \star \)-domain is 2-root closed, that is, \( D \) contains every element \( x \in K \) such that \( x^2 \in D \). Indeed, if \( 0 \neq a, b \in D \) such that \((a/b)^2 \in D \), then \( D = b^2D :_D a^2 = (bD :_D a)^2 \), so \( a/b \in D \). In particular, \( A[X^2, X^3] \) is not a \( \star \)-domain, for any domain \( A \).

c) While it is easy to see that \( \mathbb{Z}[i\sqrt{3}] \) is 2-root closed, \( \mathbb{Z}[i\sqrt{3}] \) is not a \( \star \)-domain. Indeed, let \( m \) be the maximal ideal \((2, 1 + i\sqrt{3})\) and set \( E = \mathbb{Z}[i\sqrt{3}] \). Then \( m(1 + i\sqrt{3}) \subseteq (2) \), so \((2) :_E (1 + i\sqrt{3}) = m \). Similarly, we get
\[ (2)^2 :_E (1 + i\sqrt{3})^2 = (2) :_E (1 - i\sqrt{3}) = m. \]
Hence
\[ (2)^2 :_E (1 + i\sqrt{3})^2 = ((2) :_E (1 + i\sqrt{3}))^2. \]
See also Corollary 2.7 for a more general assertion.

d) A 2-root closed domain \( D \) such that \( aD \cap bD \) is a flat ideal for each \( a, b \in D \) is a \( \star \)-domain. Indeed, let \( 0 \neq a, b \in D \). Since \( D \) is a 2-root closed domain, it follows that \( abD \subseteq a^2 \cap b^2D \). Now we may repeat the argument given in the proof of Corollary 1.5.

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According to Anderson and Dobbs (1980), a quasi-local domain $(D, \mathfrak{m})$ with quotient field $K$ is called a pseudo-valuation domain (PVD), if $x, y \in K$ and $xy \in \mathfrak{m}$ imply $x \in \mathfrak{m}$ or $y \in \mathfrak{m}$. Clearly, a valuation domain is a PVD. Next, we characterize the $\star$ PVDs (for other equivalent assertions see Zafrullah (1987, Theorem 4.5)).

**Proposition 2.3** Let $(D, \mathfrak{m})$ be a PVD which is not a valuation domain. Then $D$ is a $\star$-domain if and only if $D$ is 2-root closed and $\mathfrak{m} = \mathfrak{m}^2$.

**Proof:** By part b) of Remark 2.4, we may suppose that $D$ is 2-root closed. Let $aD$ and $bD$ be two incomparable principal ideals of $D$. Then $a^2D$ and $b^2D$ are also incomparable (by the 2-root closedness). By Anderson and Dobbs (1980, Prop. 1.4), we get $bD :_D a = b^2 D :_D a^2 = \mathfrak{m}$. Hence $D$ is a $\star$-domain if and only if $(bD :_D a)^2 = b^2 D :_D a^2$ for any incomparable ideals $aD$ and $bD$, if and only if $\mathfrak{m} = \mathfrak{m}^2$.

Now let $(V, \mathfrak{m})$ be a valuation domain with residue field $L, k$ a proper subfield of $L$ and $D$ the pre-image of $k$ in $V$. By Hedstrom and Houston (1978, Proposition 2.6), $D$ is a PVD which is not a valuation domain. Applying the preceding proposition, we get that $D$ is a $\star$-domain if and only if $\mathfrak{m} = \mathfrak{m}^2$ and $k$ is 2-root closed in $L$ (i.e. $x \in L$ and $x^2 \in k \Rightarrow x \in k$).

According to Sally and Vasconcelos (1974), an ideal $I$ of a quasi-local domain $(D, \mathfrak{m})$ is said to be stable if $I^2 = aI$ for some $a \in I$.

**Remark 2.4** Let $(D, \mathfrak{m})$ be a quasi-local domain and $I$ an ideal of $D$. As shown in the proof of Sally and Vasconcelos (1974, Theorem 3.4), if $I$ and $I^2$ are generated by two elements, then $I$ is stable. For the convenience of the reader we repeat the proof here.

Let $a, b$ generate $I$. As $I^2 = (a^2, ab, b^2)$ is two-generated, one of these 3 generators is superfluous. If $I^2 = (a^2, ab)$, then $I^2 = aI$, while if $I^2 = (b^2, ab)$, then $I^2 = bI$. Assume that $ab \in (a^2, b^2)$ and $I^2$ is not equal to $aI$ or $bI$. Then $ab = ra^2 + sb^2$ with $r, s \in \mathfrak{m}$. Changing the pair $(a, b)$ to $(a - b, b)$, we get $I^2 = aI$.

**Lemma 2.5** Let $(D, \mathfrak{m})$ be a quasi-local $\star$-domain. If $\mathfrak{m}$ is stable, then $\mathfrak{m}$ is principal.

**Proof:** Assume that $\mathfrak{m}$ is not principal. As $\mathfrak{m}$ is stable, $\mathfrak{m}^2 = a\mathfrak{m}$ for some $a \in \mathfrak{m}$. Take $b \in \mathfrak{m}, b \notin aD$. Then

$$mb \subseteq m^2 = a\mathfrak{m} \subseteq aD.$$ 

Hence $aD :_D b = \mathfrak{m}$, because $b \notin aD$. From $m^2 = a\mathfrak{m}$ we get $m^3 = a^2\mathfrak{m}$. Then

$$mb^2 \subseteq m^3 = a^2\mathfrak{m} \subseteq a^2D.$$ 

Hence $a^2D :_D b^2 = \mathfrak{m}$, because $D$ being a 2-root closed domain implies $b^2 \notin a^2D$. Since $D$ is a $\star$-domain, we get

$$\mathfrak{m} = a^2D :_D b^2 = (aD :_D b)^2 = m^2 \subseteq aD.$$
So \( m = aD \), a contradiction. 

**Proposition 2.6** Let \( D \) be a locally Noetherian domain such that for each maximal ideal \( m \), \( mD_m \) and \((mD_m)^2\) are generated by two elements. Then \( D \) is a \( \ast \)-domain if and only if \( D \) is almost Dedekind.

**Proof:** Apply the lemma and the paragraph preceding it, and use the fact that a local domain with principal maximal ideal is a DVR. 

**Corollary 2.7** Let \( A \) be domain whose ideals are two-generated (e.g. a quadratic extension of \( \mathbb{Z} \)). Then \( D \) is a \( \ast \)-domain if and only if \( D \) is Dedekind.

We close by giving two results concerning Krull \( \ast \)-domains.

**Proposition 2.8** A Krull domain \( D \) is a \( \ast \)-domain if and only if the square of every divisorial ideal of \( D \) is also divisorial.

**Proof:** Assume that \( D \) is a Krull domain and let \( X^1(D) \) be the set of all height-one primes of \( D \). Denote the divisorial closure of a nonzero ideal \( I \) by \( I_v \). Let \( 0 \neq a, b \in D \). By Fossum (1973, Proposition 5.9), 

\[
((aD \cap bD)^2)_v = \bigcap_{p \in X^1(D)} (aD \cap bD)^2D_p = \bigcap_{p \in X^1(D)} (a^2D_p \cap b^2D_p) = a^2D \cap b^2D
\]

because each \( D_p \) is a DVR. So \( D \) is a \( \ast \)-domain if and only if \( (aD \cap bD)^2 \) is a divisorial ideal for each \( 0 \neq a, b \in D \).

Now let \( I \) be an arbitrary nonzero divisorial ideal. By Fossum (1973, Proposition 5.11), \( I^{-1} = (c, d)_v \) for some \( c, d \in I^{-1} \). So 

\[
I = I_v = (I^{-1})^{-1} = (c, d)^{-1} = c^{-1}D \cap d^{-1}D = p^{-1}(aD \cap bD)
\]

for some \( a, b, p \in D \setminus \{0\} \). Hence \( I^2 \) is divisorial if and only if \( (aD \cap bD)^2 \) is divisorial. The assertion follows. 

Let \( D \) be a Krull domain and \( Div(D) \) its group of divisorial fractional ideals under the \( \ast \)-multiplication (Fossum, 1973, Proposition 3.4). The **local class group** \( G(D) \) of \( D \) is the factor group \( Div(D) \) modulo the subgroup of invertible fractional ideals (Bouvier and Zafrullah, 1988). By Fossum (1973, Corollary 18.15), a Krull domain has zero local class group if and only if it is locally factorial (hence it is a \( \ast \)-domain).

**Corollary 2.9** If \( D \) is a Krull \( \ast \)-domain, then \( G(D) \) has no element of order two.

**Proof:** Let \( I \) be a divisorial ideal of \( D \) such that \( (I^2)_v \) is invertible. By the preceding proposition \( I^2 = (I^2)_v \), so \( I \) is invertible.

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REFERENCES

Anderson, D.D., Anderson, D.F., Zafrullah, M. (1991). Rings between $D[X]$ and $K[X]$, *Houston J. Math.* 17:109-128.

Anderson, D.F., Dobbs, D.E. (1980). Pairs of rings with the same spectrum, *Canad J. Math.* 32:362-384.

André, M. (1974). Homologie des algèbres commutatives, *Springer, Berlin.*

Bouvier, A., Zafrullah, M. (1988). On some class group of an integral domain, *Bull. Soc. Math. Greece* 29:45-59.

Brezuleanu, A., Radu, N., Dumitrescu, T. (1993). Local Algebras, *Ed. Universității București, București.*

Fossum, R. (1973). The Ideal Class Group of a Krull domain, *Springer, New York.*

Gilmer, R. (1972). Multiplicative Ideal Theory, *Marcel Dekker, New York.*

Hedstrom, J.R., Houston, E.G. (1978). Pseudo-valuation Domains, *Pacific J. Math.* 75:137-147.

Planas-Vilanova, F. (1996). Rings of weak dimension one and syzygetic ideals, *Proc. Amer. Math. Soc.* 124:3015-3017.

Smith, W.W. (1979). Invertible ideals and overrings, *Houston J. Math.* 5:141-153.

Sally, J., Vasconcelos, W. (1974). Stable rings, *J. Pure Appl. Algebra* 4:319-336.

Zafrullah, M. (1987). On a property of pre-Schreier domains, *Comm. Algebra* 15:1895-1920.