Yet another comment on four-dimensional Einstein-Gauss-Bonnet gravity

Julio Arrechea,1 Adrià Delhom,2 and Alejandro Jiménez-Canc3

1Instituto de Astrofísica de Andalucía (IAA-CSIC), Glorieta de la Astronomía, Granada, Spain
2Departamento de Física Teórica and IFIC, Centro Mixto Universidad de Valencia - CSIC. Universidad de Valencia, Burjassot-46100, Valencia, Spain
3Departamento de Física Teórica y del Cosmos and CAFPE, Universidad de Granada, 18071 Granada, Spain

(Dated: April 28, 2020)

In this comment we clarify several aspects concerning four-dimensional Einstein-Gauss-Bonnet gravity. We argue that the limiting procedure outlined in [Phys. Rev. Lett. 124, 081301 (2020)] is not well defined as it generally produces divergent terms in the four dimensional field equations. Potential ways to circumvent this issue are discussed, alongside some remarks regarding specific solutions of the theory. We prove that, although linear perturbations are well behaved around maximally symmetric backgrounds, the equations for second order perturbations are ill-defined even around a Minkowskian background. Additionally, we perform a detailed analysis of the spherically symmetric solutions, and find that the central curvature singularity can be reached within a finite proper time.

I. INTRODUCTION

In has been recently claimed [1] that there exists a theory of gravitation in four spacetime dimensions which fulfills all the assumptions of the Lovelock theorem [2] yet not its conclusions. This was done by formulating the Einstein-Gauss-Bonnet (EGB) theory in an arbitrary dimension \( D \) with a coupling constant for the Gauss-Bonnet term re-scaled by a \( 1/(D-4) \) factor, as defined by the following action

\[
S = \int d^D x \sqrt{|g|} \left[ -\Lambda_0 + \frac{M_P^2}{2} R + \frac{\alpha}{D-4} \mathcal{G} \right]. \tag{1}
\]

Here \( \Lambda_0 \) is a cosmological constant term, \( R \) is the Ricci scalar and \( \mathcal{G} \) the Gauss-Bonnet (GB) term. As it is well known, the GB term is a topological invariant only in \( D = 4 \), not in higher dimensions, thus generally yielding a non-trivial contribution to the field equations in arbitrary \( D \). In [1] it is claimed that the contribution of the Gauss-Bonnet (GB) term to the equations of motion is always proportional to a \( (D-4) \) factor, which in principle compensates the divergence introduced in the coupling constant, thus allowing for a well defined \( D \to 4 \) limit at the level of the field equations. Thus the authors of [1] argue that a non-trivial correction to General Relativity due to the GB term in [1] remains even in \( D = 4 \).

Since the above action is one of the celebrated Lovelock actions in arbitrary \( D \), it is stated in [1] that all the assumptions of Lovelock theorem hold, although the resulting field equations do violate the conclusions of the Lovelock theorem. This is accomplished by defining a 4-dimensional diffeomorphism-invariant theory satisfying the metricity condition and having second order field equations which differ from those of General Relativity (GR). The authors of [1] then proceed to show the consequences of these modifications to GR in some scenarios with a high degree of symmetry. The interest in this theory, which we will refer to as \( D \to 4 \) Einstein-Gauss-Bonnet (D4EGB for short), has recently emerged strongly boosted. In particular there has been an ongoing discussion [3–10] regarding the nature and/or the well-definedness of D4EGB. In this comment we also elaborate in this direction.

Let us now draw the attention of the reader towards some subtleties on the definition of D4EGB and in some of the solutions to its field equations. It was claimed in [1] that the contribution of the GB term to the field equations (and not just its trace) is proportional to \( (D-4) \), and that this would imply the GB contribution to the field equations vanishes in four spacetime dimensions. The authors of [1] then consider a coupling constant with a \( 1/(D-4) \) factor that would regularise the otherwise vanishing GB contribution, now yield a finite correction to the four dimensional field equations. We will show, in agreement with [11], that besides a term proportional to \( (D-4) \), the GB term contributes to the field equations with an additional part from which no power of \( (D-4) \) can be factorized, but which nonetheless vanishes identically in \( D = 4 \).

Regarding tensor perturbations in D4EGB we will reproduce the results of [1] for linear perturbations around a maximally symmetric background. This allows to find that the theory only propagates a massless graviton and that the corrections to GR provided by the regularized GB term only enter through a global \( \alpha \)–dependent factor multiplying the linear perturbation equations in GR. Nonetheless we will see that the field equations describing second order perturbations contain ill-defined terms that diverge as \( 1/(D-4) \) in the \( D \to 4 \) limit even around a Minkowskian background. This result suggests that D4EGB is not perturbatively well-defined. Furthermore, we will argue that unless one is looking for solutions with enough symmetry so as to render a specific combination
of Weyl tensors vanishing in arbitrary dimensions, the term that is not proportional to \((D - 4)\) spoils the well-definedness of the full D4EGB field equations, suggesting that the theory is not well defined at the non-perturbative level either.

Finally, we will comment on the geometries presented in [1] as the \(D \to 4\) limit of the spherically symmetric solutions for EGB theory in \(D \geq 5\) found in [12]. We will see that the claim made in [1] that no particle can reach the central curvature singularity in a finite proper time within these geometries does not hold for freely-falling particles with vanishing angular momentum. Furthermore, we will show that the regularized D4EGB field equations are not well defined in spherically symmetric spacetimes unless the contribution which is not proportional to \((D - 4)\) is artificially stripped away from the field equations. Also, in the case one decides to remove this term, we will argue that the spherically symmetric geometries presented in [1] are not solutions of the remaining field equations in \(D = 4\).

II. THE \(D \to 4\) PROCEDURE

Let us first comment on whether the \(D \to 4\) limit taken in [1] corresponds to a well-defined continuous process. To that end, consider the \(k\)-th order Lovelock term in an arbitrary dimension \(D\)

\[
S^{(k)} = \int R^{a_1 a_2} \cdots \wedge R_{a_{2k-1} a_{2k}} \ast (e_{a_1} \wedge \ldots \wedge e_{a_{2k}})
\]

\[
= \frac{(2k)!}{2^k} \int \frac{\sqrt{|g|}}{dDx} \right|^{\mu_1 \mu_2 \ldots \mu_{2k-1} \mu_{2k}} \right|_{\nu_1 \nu_2 \ldots \nu_{2k-1} \nu_{2k}} (D - 2k)(D - 2k - 1)! J_{ac}^{(k)} e^c,
\]

where Greek indices refer to a coordinate basis and Latin indices to a frame in which the metric is the Minkowski metric \((g_{ab} = \eta_{ab})\). As also noted in [11] for the Gauss-Bonnet term \((k = 2)\), if we analyze the problem in different form notation, when varying the action with respect to the coframe \(e^a\) we find

\[
\ast \frac{\delta S^{(k)}}{\delta e^a} = (D - 2k)(D - 2k - 1)! J_{ac}^{(k)} e^c,
\]

where \(J_{ac}^{(k)}\) is a regular tensor built from combinations of the Riemann tensor that differ for each \(k\). The second factor comes from the contraction of two Levi-Civita symbols. Therefore, it is of combinatorial nature: it essentially has to do with the counting of the number of possible antisymmetric permutations of a bunch of indices. Notice that this counting process is not a continuous process in which the number of indices being counted (or equivalently the dimension) can take any value, but it ought to be an integer one. Indeed, for \(D\) to be valid, \(D\) must be greater than \(2k\) because a \((-1)!\) cannot arise from counting possible permutations. Since [3] is not valid for \(D = 2k\), it cannot be stated that the factor \((D - 2k)\) is the responsible for the vanishing of [3] in \(D = 2k\). The reason under its vanishing can actually be traced back to the properties of \(2k\)-forms in \(2k\) dimensions [13]: by explicitly writing the Hodge star operator in [2], in the critical dimension we obtain

\[
\ast (e_{a_1} \wedge \ldots \wedge e_{a_{2k}})^{(D - 2k)} = F e_{a_1 \ldots a_{2k}},
\]

where \(e_{a_1 \ldots a_{2k}}\) is the Levi-Civita tensor associated to the Minkowski metric and \(F\) is a non-zero constant that depends on \(k\). As a consequence of this, and the well-known fact that the curvature factors in the action do not contribute (via spin connection) to the dynamics in Lovelock theories [14], the vielbein equations of motion are identically satisfied. Observe that this is no longer true if \(D > 2k\), since, in that case, the Hodge dual of \(e_{a_1} \wedge \ldots \wedge e_{a_{2k}}\) is not a 0-form and gives a non-trivial contribution to the equation of motion of the vielbein.

It is clarifying as well to consider [3] as a metric variation, i.e. avoiding the differential form notation and working directly with the metric components. Then, for a general \(k\)-th order Lovelock term in an arbitrary dimension \(D \geq 2k\), its variation with respect to the metric is not proportional to \((D - 4)\), but rather of the form

\[
\frac{1}{\sqrt{|g|}} \frac{\delta S^{(k)}}{\delta g_{\mu \nu}} = (D - 2k)A_{\mu \nu} + W_{\mu \nu},
\]

where no \(D - 2k\) factor can be extracted from \(W_{\mu \nu}\). For instance, the 1st order Lovelock term (the Einstein-Hilbert action) leads to \(A_{\mu \nu} = 0\) and \(W_{\mu \nu} = g_{\mu \nu}\), which vanishes in \(D = 2\). Analogously, by decomposing the Riemann tensor into its irreducible pieces (see e.g. [15]), the 2nd order Lovelock term, i.e. the Gauss-Bonnet term, leads to

\[
A_{\mu \nu}^{GB} = \frac{D - 3}{(D - 2)^2} \left[ \frac{2D}{D - 1} R_{\mu \nu} R - \frac{4(D - 2)}{D - 3} R^\lambda \mu \nu R^\lambda \right],
\]

\[
-4 R_{\rho \mu} R_{\nu \rho} + 2 g_{\mu \nu} R_{\rho \lambda} R^{\rho \lambda} - \frac{D + 2}{2(D - 1)} g_{\mu \nu} R^2
\]

\[
= 2 \left[ C_{\mu \rho \lambda \sigma} C_{\nu \rho \lambda \sigma} - \frac{1}{4} g_{\mu \nu} C_{\tau \rho \lambda \sigma} C^{\tau \rho \lambda \sigma} \right],
\]

where we have introduced the Weyl tensor \(C_{\mu \rho \lambda \sigma}\). Therefore, the field equations given by [1] in arbitrary dimension are

\[
G_{\mu \nu} + \frac{1}{M_p^2} \Lambda g_{\mu \nu} + \frac{2\alpha}{M_p^2} \left( A_{\mu \nu}^{GB} + W_{\mu \nu}^{GB} \right) \frac{D - 4}{D} = 0.
\]

[1] The calculations have been checked with xAct [16] and we leave the notebook as supplementary material for anyone to check it.

[2] Since the trace of \(W_{\mu \nu}^{GB}\) is proportional to \((D - 4)\), the divergence disappears from the trace of the equation of motion, although this factorization cannot be made in the full equation.
Hence, with the regularization made in [1], i.e. evaluating $D = 4$ after calculating the equations of motion in arbitrary $D$, the $A_{\mu \nu}^{\text{GB}}$ term indeed provides a finite non-trivial correction to the Einstein field equations if the coupling constant of the GB term is $\alpha/(D - 4)$. However, in this case, the $W_{\mu \nu}^{\text{GB}}$ term will be ill defined in arbitrary backgrounds, where $W_{\mu \nu}^{\text{GB}} \neq 0$, since it does not go to zero as $(D - 4)$. Indeed, the reason for $W_{\mu \nu}^{\text{GB}}$ to vanish in $D = 4$ is that the Riemann tensor loses independent components as one lowers the dimension and, in $D = 4$, this loss of components implies that $W_{\mu \nu}^{\text{GB}}$ necessarily vanishes by algebraic reasons, analogously to what happens to the Einstein tensor in $D = 2$. In other words, the reason for these expressions to be zero in certain dimensions is that they are algebraic identities fulfilled by the curvatures of all metrics in the critical dimension, as opposed to functional identities at which one could arrive by a continuous limiting process given a suitable topology. A somewhat simpler example of the fact that the vanishing of the GB variation is due to algebraic reasons is provided by Galileon or interacting massive vector field theories. There, it can be seen that due to the Cayley-Hamilton theorem, the interaction Lagrangians of a given order $k$ identically vanishes for dimensions higher than the critical dimension associated to $k$ [17].

Let us also mention that the authors of [1] appeal to an analogy between their method and the method of dimensional regularization commonly employed in quantum field theory. The dimensional regularization method allows to extract the divergent and finite contributions from integrals that are divergent in $D = 4$ but non-divergent for higher $D$. It consists on considering the analytic continuation of such integrals to the complex plane as a function of the complexified dimension $D$, and then taking the limit $D \to 4$ in a manner that allows to separate the divergent and finite contributions of the integrals. A key aspect that ensures the well-definedness of dimensional regularization as an analytic continuation is that the regularized integrals are scalar functions which have no algebraic structure sensitive to the number of dimensions of the space they are defined in. Note however that this is not the case for the Gauss-Bonnet term, which has a non-trivial tensorial structure that is not well defined for non-integer dimensions. Thus, although the process of dimensional regularization can be defined by using the smooth $D \to 4$ limit of the appropriate analytic continuation of the scalar integrals, this fails to be a continuous limiting process when the quantities involved have a non-trivial algebraic structure, such as tensors or $p$-forms do.

III. PERTURBATIONS AROUND MAXIMALLY SYMMETRIC BACKGROUNDS

Despite the above considerations, we acknowledge that even though the regularization method proposed in [1] will not work in general, it suffices for finding solutions that satisfy enough symmetries so as to render the $W_{\mu \nu}^{\text{GB}}$ identically zero in arbitrary dimension. Thus, by symmetry-reducing the action before enforcing $D = 4$, we can get rid of the problematic $W_{\mu \nu}^{\text{GB}}$ term and arrive to well-defined equations of motion. This is the case, for instance, of all conformally flat geometries, which have an identically vanishing Weyl tensor in $D \geq 4$, thus satisfying the desired property that $W_{\mu \nu}^{\text{GB}} = 0$ in $D \geq 4$ which makes the $D \to 4$ limit of the (symmetry-reduced) D4EGB field equations [8] well defined. Maximally symmetric geometries or FLRW spacetimes are conformally flat, and therefore the regularized D4EGB equations are well defined in such situations. Let us analyse the maximally symmetric solutions of [8] studied in [1]. In these geometries, the Riemann tensor is given by

$$R_{\mu \nu}{}^{\rho \sigma} = \frac{\Lambda}{M_p^2(D - 1)} \left( \delta_\mu^\rho \delta_\nu^\sigma - \delta_\mu^\sigma \delta_\nu^\rho \right),$$

and $W_{\mu \nu}^{\text{GB}}$ vanishes in arbitrary dimension as explained above. In this case, the variation of the GB term is indeed proportional to $(D - 4)$ and, therefore, after this symmetry-reduction of the action [1], the field equations [8] read

$$G_{\mu \nu} + \frac{1}{M_p^2} A_0 g_{\mu \nu} + 2\alpha M_p^{-2} A_{\mu \nu}^{\text{GB}} = 0,$$

where $A_{\mu \nu}^{\text{GB}}$ provides a regular, $\alpha-$dependent correction to GR. Although these properties will be shared by any conformally flat solution, one should bear in mind that arbitrary perturbations around these backgrounds will be sensitive to the ill-defined contributions that come from the $W_{\mu \nu}^{\text{GB}}$ dependence of the full D4EGB field equations [8].

It is worth noticing, though, that the ill-defined corrections which enter the equations of motion through the $\alpha W_{\mu \nu}^{\text{GB}}/(D - 4)$ term do not contribute to linear order in perturbation theory around a maximally symmetric background. Thus, presumably, this is the reason why these ill-defined contributions were not noticed in [1], where only linear perturbations were considered. Nonetheless, the divergent terms related to $W_{\mu \nu}^{\text{GB}}$ will enter the perturbations at second order.

To show this, let us consider a general perturbation around a maximally symmetric background by splitting the full metric as

$$g_{\mu \nu} = \bar{g}_{\mu \nu} + \epsilon h_{\mu \nu}$$

where $\bar{g}_{\mu \nu}$ is a maximally symmetric solution of [8]. Therefore, the left hand side of [8] can be written as a perturbative series in $\epsilon$ of the form

$$E^{(0)}_{\mu \nu} + \epsilon E^{(1)}_{\mu \nu} + \epsilon^2 E^{(2)}_{\mu \nu} \ldots,$$

3Typically the tensorial structures within the integrals are extracted from them by employing Lorentz-covariance arguments, and therefore the integral to regularise is always a scalar function.
where $E^{(0)}_{\mu \nu} = 0$ are the background field equations, $E^{(1)}_{\mu \nu} = 0$ are the equations for linear perturbations, and so on. Using the zeroth-order equation, the linear perturbations in $D$ dimensions and around a maximally symmetric background are described by a

$$\left(1 + \frac{4(D-3)}{D-1} \frac{\alpha D}{M^2} \right) \left[ \nabla^\mu h^\rho_{\mu \nu} + \nabla^\rho h^\mu_{\nu \rho} - \nabla^\rho h_{\mu \nu} - \nabla^\mu \nabla^\nu h + \delta^\mu_{\nu} (\nabla^\sigma h_{\sigma \rho} - \nabla^\rho h_{\sigma \nu}) - \frac{\Lambda}{M^2} (\delta^\mu_{\nu} h - 2h_{\mu \nu}) \right] = 0,$$

(13)

where $h \equiv h^\sigma_{\sigma}$ and the indices have been raised with $\tilde{g}^\mu_{\nu}$. By inspection, we can see that this equation is regular in $D = 4$. Furthermore, as noted in [1], the equation governing linear perturbations (13) is essentially that of GR, although multiplied by an overall factor that depends on $\alpha$. Let us now go to quadratic order in the perturbations. For our purpose it will be sufficient to consider quadratic perturbations around a Minkowskian background. By using the zeroth- and first-order perturbation equations, and enforcing a vanishing background curvature $\Lambda = 0$, we can write the second order perturbation equations $E^{(2)}_{\mu \nu} = 0$ as

$$0 = [\text{GR terms of } \mathcal{O}(h^2)]_{\mu \nu} + \frac{\alpha}{M^2 (D-4)} \left\{ -2 \nabla_\gamma \nabla_\alpha h_{\beta \gamma} \nabla^\gamma \nabla^\beta h_{\mu \alpha} + 2 \nabla_\gamma \nabla_\beta h_{\alpha \beta} \nabla^\gamma \nabla^\beta h_{\mu \alpha} + 2 \nabla_\gamma \nabla^\beta h_{\mu \alpha} \nabla \nabla_\alpha h_{\beta \gamma} - 2 \nabla_\gamma \nabla^\beta h_{\mu \alpha} \nabla_\gamma \nabla_\beta h_{\alpha \gamma} + 2 \nabla_\gamma \nabla^\beta h_{\mu \alpha} \nabla_\beta h_{\alpha \gamma} - 2 \nabla_\gamma \nabla^\beta h_{\mu \alpha} \nabla_\beta h_{\gamma \alpha} - 2 \nabla_\gamma \nabla^\beta h_{\mu \alpha} \nabla_\gamma h_{\beta \alpha} + 2 \nabla_\gamma \nabla^\beta h_{\mu \alpha} \nabla_\gamma h_{\beta \gamma} \right\}

(14)

In view of the above expression it becomes clear that, even around a flat background, the $W^\text{GB}_{\mu \nu}$ piece of (8) contributes to the second order perturbation equations with a term that is ill-defined in $D = 4$. These results provide a clear example which suggests that the D4EGB field equations [8] are generally ill-defined. This results are somewhat in the line to those found in [4], where it was seen that the amplitudes of GB in the $D \rightarrow 4$ limit correspond to those of a scalar-tensor theory where the scalar is infinitely strongly coupled. Hence, they concluded that this new pathological degree of freedom would only show up beyond linear order in perturbations.

IV. AN ACTION FOR THE REGULARIZED EQUATIONS?

We have seen that unless the field equations [8] are stripped away of the $W^\text{GB}_{\mu \nu}$ term after taking the variation of the D4EGB action [11], they will be, in general, ill-defined. Let us now comment on the possibility of finding a diffeomorphism-invariant action whose field equations

4 Although [11] and the equations for linear perturbations in [11] differ by the ordering of the covariant derivatives of the $\nabla_\rho h^\mu_{\nu \rho}$ term and the sign in the mass term, our equation [8] coincides with those in e.g. [15] for linearized perturbations around a maximally symmetric background. In any case the difference is not physically relevant, as can be seen by choosing a particular gauge.

5 See supplementary material.

6 Note that we have preserved the covariant derivatives in [14] since our result is not restricted to a particular coordinate choice.

7 Even though the $D \rightarrow 4$ process, if understood as a limit, will have the same conceptual problems described in section [11] in this case might be swept under the rug since the $1/(D-4)$ dependence actually disappears from the field equations.

8 This is due to the Bianchi identity under diffeomorphisms, see e.g. [15].
Together with the fact that the variation with respect to the metric of any diffeomorphism-invariant action is identically divergence-free, the above result implies that the term $W^\mu\nu_{\text{GB}}$ does not come from an action that is a scalar under diffeomorphisms. Consequently, there does not exist any term that can be added to the action \([1]\) to cancel the $W^\mu\nu_{\text{GB}}$ contribution in the D4EGB field equations \([8]\) without spoiling its diffeomorphism-invariance. Other authors have proposed alternative ways to regularize the action \([1]\), generally leading to a scalar-tensor theory of the Horndeski family \([3,6]\), thus leaving the Lovelock theorem intact.

We thus conclude that no diffeomorphism-invariant action can give the desired field equations \((10)\) in $D \geq 4$. Nevertheless, nothing prevents the existence of a non-diffeomorphism-invariant action having \((10)\) as its field equations. Should it be possible to find such a theory, however, the absence of diffeomorphism invariance would potentially unleash the well known pathologies that occur in massive gravity \((\text{see e.g.} \ [15,19])\), thus propagating a Boulware-Deser ghost \([20]\).

V. GEODESIC ANALYSIS OF THE SPHERICALLY SYMMETRIC SOLUTIONS

In addition to maximally symmetric and FLRW spacetimes, spherically symmetric solutions of D4EGB were also considered in \([1]\), where it was stated that they are described by the 4-dimensional metric

$$ds^2 = A_{\pm}(r)dt^2 - A_{\pm}^{-1}(r)dr^2 - r^2d\Omega_2^2,$$  \((18)\)

where $A_{\pm}(r)$ has the form

$$A_{\pm}(r) = 1 + \frac{r^2}{32\pi\alpha G} \left[ 1 \pm \sqrt{1 + \frac{128\pi\alpha G^2 M}{r^3}} \right].$$  \((19)\)

First of all let us point out that $D$-dimensional spherically symmetric geometries described by metrics of the form \((19)\)

$$ds^2 = A(r)dt^2 - A^{-1}(r)dr^2 - r^2d\Omega_{D-2}^2,$$  \((20)\)

do not in general satisfy that $W^\mu\nu_{\text{GB}} = 0$ in arbitrary $D \geq 4$. To see this, it suffices to restrict us to the 5-dimensional case, where the condition for $W^\mu\nu_{\text{GB}}$ to vanish is

$$r^2\frac{d^2A}{dr^2} - 2r\frac{dA}{dr} + 2A - 2 = 0.$$  \((21)\)

This only happens for the particular case $A = 1 + C_1 r + C_2 r^2$ where $C_i$ are integration constants. This suggests that \([19]\) cannot be regarded as a solution of the D4EGB field equations, given that \([8]\) is not well-defined for $D$-dimensional spherically symmetric metrics \([20]\) in the $D \rightarrow 4$ limit. Indeed, as the authors of \([1]\) explain, the 4-dimensional spherically symmetric geometries \([19]\) were obtained by first re-scaling $\alpha$ with a $1/(D - 4)$ factor in the solutions obtained in \([12]\) for EGB in $D \geq 5$ and then taking the $D \rightarrow 4$ limit, instead of solving the $D \rightarrow 4$ limit of \([3]\).

Nevertheless, it could be that the spherically symmetric geometries of \([1]\) are solutions of \([19]\), namely the field equations \([8]\) after being stripped away of the pathological $W^\mu\nu_{\text{GB}}$ term. In the supplementary material, it can be seen that \((10)\) has four different branches of solutions for $\alpha > 0$. Two of them are exactly the Schwarzschild and Schwarzschild-(anti-)de Sitter

$$A_1 = 1 - \frac{2GM}{r},$$

$$A_2 = 1 + \frac{r^2}{16\pi\alpha G} - \frac{2GM}{r}.$$  \((22)\)

and the other two cannot be solved analytically, though their asymptotic behavior near the origin can be seen to be $A \cong r^{-3 - 2\sqrt{3}} + O(r^0)$. Thus these solutions can neither be the ones found in \([1]\), although they approach the Schwarzschild and Schwarzschild-(anti-)de Sitter solutions at spatial infinity.

Let us now turn to the behavior of the spherically symmetric geometries presented in \([1]\). As noted in \([1]\), the $\alpha < 0$ branch of the above solution is not well defined for values of the radial coordinate below $r < (-128\pi\alpha G^2 M)^{-1/3}$, so their analysis focused on the $\alpha > 0$ branches, showing that the above metric describes solutions which behave asymptotically as Schwarzschild or Schwarzschild-de Sitter solutions by choosing the negative and positive signs respectively.

Concerning the former branch of solutions, it was shown in \([1]\) that its causal structure (namely, the presence or absence of event horizons) depends on the ratio between the mass parameter $M$ and a new mass scale $M_* = \sqrt{16\pi\alpha G}$ that characterizes the D4EGB corrections to GR. From \([18]\) and \([19]\) it can be shown that the $g_{tt}$ component of the metric vanishes at the spherical surfaces

$$r_{\pm} = GM \left[ 1 \pm \sqrt{1 - \left( \frac{M_*}{M} \right)^2} \right].$$  \((23)\)

In view of this expression it comes clear that solutions have no horizons for the $M < M_*$ case, outer and inner horizons if $M > M_*$ and one degenerate horizon if $M = M_*$. Interestingly, the mass scale $M_*$ plays a role similar to that of the electric (and magnetic) charges in the Reissner-Nordström spacetime, with the exception that, in this case, the origin of such contributions comes exclusively from the gravitational field. The effect of the Gauss-Bonnet terms is that of making gravity repulsive at short distances, the magnitude of this repulsion being dictated by the strength of the GB coupling $\alpha$.

Regarding the presence of singularities in the solutions, we see that despite the metric components \([19]\) being...
finite at the origin

$$A(r) = 1 - \sqrt{\frac{2M}{GM^2}} r^{1/2} + O(r^{3/2}), \quad (24)$$

curvature invariants diverge as $R \propto r^{-3/2}$, $R_{\mu\nu}R^{\mu\nu} \sim R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \propto r^{-3}$. In [1] it is argued that an observer could never reach this curvature singularity given the repulsive effect of gravity at short distances. This would imply that the spacetime described by (18) is complete in the sense that no (classical) physical observer ever reaches the curvature singularity at $r = 0$ in a finite proper time. Nonetheless, there was no explicit proof in [1] showing that this was indeed the case. We thus proceed to answer precisely the following question: does any (classical) physical observer reach the curvature singularity of (18) in a finite proper time? To answer that question, it suffices to study the sub-class of radial freely-falling (classical) observers, described by time-like geodesics. We will also consider radial null geodesics for completeness.

Consider the geodesic equation in the equatorial plane $\theta = \pi/2$ for the metric (18),

$$\frac{dr}{d\tau} = E^2 - V_{\text{eff}}(r), \quad (25)$$

with

$$V_{\text{eff}}(r) = A(r) \left( \frac{L^2}{r^2} - \kappa \right), \quad (26)$$

and where $\tau$ is the proper time of the observer that moves along the solution $r(\tau)$. Here, $\kappa$ takes the values $\{-1, 1, 0\}$ for space-like, time-like and null geodesics respectively. $E$ and $L$ are constants of motion associated with time-translation and rotational symmetries respectively. It will suffice for our purpose to analyse radial geodesics, characterized by $L = 0$. Firstly note that since photon trajectories are insensitive to the value of $A(r)$ in a spacetime described by any metric of the form (18), the trajectories stay the same as in GR. The solution to (25) for time-like geodesics is plotted in fig. 1 for the cases with different causal structures. There, it can be seen that infalling massive particles starting in a region well beyond the Schwarzschild radius (where the spacetime is effectively the same as in GR) reach the curvature singularity at $r = 0$ in a finite proper time (no matter what its initial velocity is). Notice that, as can be seen in fig. 2 the deviations from the GR trajectories are not relevant until the particle is at $r \approx r_S$. An asymptotic analysis of the geodesic equation reveals that, while in GR the curvature singularity at $r = 0$ is reached with infinite velocity $dr/d\tau|_{\text{GR}} \propto r^{-1/2} + O(r^0)$, the geodesics described by (18) reach it with finite velocity

$$\left(\frac{dr}{d\tau}\right)^2|_{\text{D4EGB}} = E^2 - 1 + \sqrt{2M/M^2} r^{1/2} + O(r^{3/2}). \quad (27)$$

It is interesting to note that in the case that the infalling particle starts at rest, no matter what its initial position is, it will reach the singularity with zero velocity (characterized by $E^2 = 1$): attractive and repulsive effects compensate each other along the trajectory of the particle. The above proves that the statement made in [1] that particles cannot reach the central singularity in spacetimes described by (18) is not correct, as the singularity is reached in finite affine parameter. Therefore,

---

9Since spacetime is spherically symmetric, geodesics will lie in a plane, which can be chosen as the equatorial one in suitable coordinates. See e.g. [21] for details on the derivation of the geodesic equation and [22] for the completeness analysis.
the hope that these solutions avoid the singularity problem is cast into serious doubt. Furthermore, the authors of [1] also claim that under a realistic stellar collapse, matter would stop before reaching the singularity. This of course must be verified by a self-consistent analysis of the dynamical collapsing geometry, as was done in [23], revealing that the singularity indeed forms and gets covered by a horizon. Furthermore, the authors of [23] also found that if the collapse is modelled à la Oppenheimer-Snyder, where dust is initially at rest, matter reaches the singularity with zero velocity, in agreement with our results.

We also note that, even if geodesic observers did never arrive at the singularity, the usual problems regarding curvature singularities would still remain: quantum corrections would be expected to become non-perturbative near the singularity and the background could not be treated classically anymore. However, the solutions would be classically singularity free in this case.

**FINAL REMARKS**

Here we have looked upon the idea of providing corrections to four dimensional General Relativity by means of the Gauss-Bonnet term analysed in devised in [24] and recently revisited in [1]. We have shown that this idea cannot be implemented for the Gauss-Bonnet term (k-th order Lovelock) term in four (2k) spacetime dimensions by means of the procedure considered in [1] without encountering inconsistencies. When considering solutions with a high degree of symmetry, such as maximally symmetric or general conformally flat solutions, this issue gets concealed at the level of the equations of motion due to the fact that the problematic terms $W_{\mu\nu}^{GB}$ in [8] vanish for arbitrary $D$ in these scenarios. Indeed we have shown that, when considering perturbations around a Minkowskian (or any maximally symmetric) background beyond linear order, such inconsistencies are immediately unveiled. This is also aligned with the conclusions at which the authors of [1] arrived by analysing the GB amplitudes.

Regarding the spherically symmetric geometries presented in [1], we showed that they do not attain the required degree of symmetry as to make $W_{\mu\nu}^{GB}$ vanish in arbitrary dimension and thus bypass the pathologies encountered in the field equations (8). By artificially removing the divergent $W_{\mu\nu}^{GB}$ term from (8), we encountered four spherically symmetric solutions, none of which coincides with those presented in [1]. Moreover, a geodesic analysis of the geometries from [1] contradicts the observation about the singularity being unreachable by any observer in finite proper time.

The idea of extracting a non-trivial corrections to the dynamics of a theory from topological terms by considering a divergent coupling constant is indeed very appealing, since its range of applicability extends far beyond gravitational contexts. For instance, it might serve to introduce parity-violating effects in Yang-Mills theories through the corresponding $F\tilde{F}$ terms that are topological in four dimensions. Indeed, a similar idea has been seen to lead to well-defined theories in the context of Weyl geometry [25–27]. It could thus be interesting to explore various possibilities in this direction.

**ACKNOWLEDGEMENTS**

The authors are very grateful to Carlos Barceló, Jose Beltrán Jiménez, Bert Janssen, Gonzalo J. Olmo and Jorge Zanelli for their useful comments and interesting discussions. AJC and AD are supported by PhD contracts of the program FPU 2015 with references FPU15/02864 and FPU15/05406 (Spanish Ministry of Economy and Competitiveness), respectively. This work is supported by the Spanish Projects No. FIS2017-84440-C2-1-P (MINECO/FEDER, EU) and FIS2016-78198-P (MINECO), the Project No. H2020-MSCA-RISE-2017 Grant No. FunFiCO-777740, Project No. SEJ1/2017/042 (Generalitat Valenciana), the Consolider Program CPANPHY-1205388, and the Severo Ochoa Grant No. SEV-2014-0398 (Spain). JA acknowledges financial support from the Spanish Government through the project FIS2017-86497-C2-2-P (with FEDER contribution), and from the State Agency for Research of the Spanish MCIU through the “Center of Excellence Severo Ochoa” award to the Instituto de Astrofísica de Andalucía (SEV-2017-0709).

[1] D. Glavan and C. Lin, *Einstein-Gauss-Bonnet Gravity in Four-Dimensional Spacetime*, Phys. Rev. Lett. 124 (2020) 081301.

[2] D. Lovelock, *The four-dimensionality of space and the Einstein tensor*, J. Math. Phys. 13 (1972) 874–876.

[3] H. Lu and Yi Pang, *Horndeski gravity as $d \to 4$ limit of Gauss-Bonnet* (2020), arXiv:2004.09066 [hep-ex].

[4] J. Bonifacio, K. Hinterbichler, and L. A. Johnson, *Amplitudes and 4D Gauss-Bonnet Theory* (2020), arXiv:2004.10716 [hep-th].

[5] P. G. S. Fernandes, P. Carrilho, T. Clifton, and D. J. Mulryne, *Derivation of Regularized Field Equations for the Einstein-Gauss-Bonnet Theory in Four Dimensions* (2020), arXiv:2004.08362 [gr-qc].

[6] R. A. Hennigar, D. Kubiznak, R. B. Mann, and C. Pollack, *On Taking the $D \to 4$ limit of Gauss-Bonnet Gravity: Theory and Solutions* (2020), arXiv:2004.09472 [gr-qc].

[7] S. X. Tian and Z. H. Zhu, *Comment on “Einstein-Gauss-Bonnet Gravity in Four-Dimensional Spacetime”* (2020), arXiv:2004.09954 [gr-qc].
[8] T. Kobayashi, Effective scalar-tensor description of regularized Lovelock gravity in four dimensions (2020), arXiv:2003.12771 [gr-qc].

[9] S. Mahapatra, A note on the total action of 4D Gauss-Bonnet theory (2020), arXiv:2004.09214 [gr-qc].

[10] W. Y. Ai, A note on the novel 4D Einstein-Gauss-Bonnet gravity (2020), arXiv:2004.02858 [gr-qc].

[11] M. Gurses, T. Cagri Sisman, and B. Tekin, Is there a novel Einstein-Gauss-Bonnet theory in four dimensions? (2020), arXiv:2004.03390 [gr-qc].

[12] D. G. Boulware and S. Deser, String Generated Gravity Models, Phys. Rev. Lett. 55 (1985) 2656.

[13] J. Zanelli (private communication, 2020).

[14] A. Mardones and J. Zanelli, Lovelock-Cartan theory of gravity, Class. Quan. Grav. 8 (1991) 1545–1558.

[15] T. Ortín, Gravity and strings, Cambridge Univ. Press, 2004.

[16] J. M. Martín-García et al., xAct: Efficient tensor computer algebra for the Wolfram Language.

[17] J. Beltrán Jiménez, L. Heisenberg, Derivative self-interactions for a massive vector field, Phys. Lett. B 757 (2016) 405–411.

[18] K. Hinterbichler, Theoretical Aspects of Massive Gravity, Rev. Mod. Phys. 84 (2012) 671–710.

[19] C. de Rham, Massive Gravity, Living Rev. Rel. 17 (2014) 7.

[20] D. G. Boulware and S. Deser, Can gravitation have a finite range?, Phys. Rev. D 6 (1972) 3368–3382.

[21] R. M. Wald, General Relativity, Chicago Univ. Pr., Chicago, USA, 1984.

[22] G. J. Olmo, D. Rubiera-Garcia, and A. Sanchez-Puente, Geodesic completeness in a wormhole spacetime with horizons, Phys. Rev. D 92 (2015) 044047.

[23] D. Malafarina, B. Toshmatov, and N. Dadhich, Dust collapse in 4D Einstein-Gauss-Bonnet gravity (2020), arXiv:2004.07089 [gr-qc].

[24] Y. Tomozawa, Quantum corrections to gravity (2011), arXiv:1107.1424 [gr-qc].

[25] J. Beltrán Jiménez and T. S. Koivisto, Extended Gauss-Bonnet gravities in Weyl geometry, Class. Quant. Grav. 31 (2014) 135002.

[26] J. Beltrán Jiménez and T. S. Koivisto, Spacetimes with vector distortion: Inflation from generalised Weyl geometry, Phys. Lett. B 756 (2016) 400–404.

[27] J. Beltrán Jiménez, L. Heisenberg and T. S. Koivisto, Cosmology for quadratic gravity in generalized Weyl geometry, J. Cosmol. Astropart. Phys. 2016 (2016), 046.