A NOTE ON THE AUTOMORPHISM GROUPS OF JOHNSON GRAPHS

S. MORTEZA MIRAFZAL

ABSTRACT. The Johnson graph \( J(n, i) \) is defined as the graph whose vertex set is the set of all \( i \)-element subsets of \( \{1, \ldots, n\} \), and two vertices are adjacent whenever the cardinality of their intersection is equal to \( i-1 \). In Ramras and Donovan [SIAM J. Discrete Math, 25(1): 267-270, 2011], it is proved that if \( n \neq 2i \), then the automorphism group of \( J(n, i) \) is isomorphic with the group \( \text{Sym}(n) \) and it is conjectured that if \( n = 2i \), then the automorphism group of \( J(n, i) \) is isomorphic with the group \( \text{Sym}(n) \times \mathbb{Z}_2 \).

In this paper, we will find these results by different methods. We will prove the conjecture in the affirmative.

Keywords : Johnson graph, Line graph, Automorphism group
AMS subject classifications. 05C25, 05C69, 94C15

1. Introduction

Johnson graphs arise from the association schemes with the same name. They are defined as follows.

Given \( n, m \in \mathbb{N} \) with \( m \leq n - 1 \), the Johnson graph \( J(n, m) \) is defined by:

(1) The vertex set is the set of all subsets of \( I = \{1, 2, \ldots, n\} \) with cardinality exactly \( m \).

(2) Two vertices are adjacent if and only if the cardinality of their intersection is equal to \( m - 1 \).

The Johnson graph \( J(n, m) \) is a vertex transitive graph [7]. It follows from the definition that for \( m = 1 \), the Johnson graph \( J(n, 1) \) is the complete graph \( K_n \). For \( m = 2 \) the Johnson graph \( J(n, 2) \) is the line graph of the complete graph on \( n \) vertices, also known as the triangular graph \( T(n) \). Thus, for instance, \( J(5, 2) \) is the complement of the Petersen graph, displayed in Figure 1, and in general, \( J(n, 2) \) is the complement of the Kneser graph \( K(n, 2) \).
We know that complementation of subsets $M \mapsto M^c$ induces an isomorphism $J(n, m) \cong J(n, n - m)$, hence we may assume without loss of generality that $m \leq \frac{n}{2}$. This graph has been studied by various authors and some of the recent papers are [1,4,6,8,9,14]. In this paper we determine the automorphism group $\text{Aut}(J(n, m))$, for $6 \leq n$ and $m \leq \frac{n}{2}$. Actually, the automorphism group of $J(n, m)$ for both the $n = 2m$ and $n \neq 2m$ cases was already determined in [8], but the proof given there uses heavy group-theoretic machinery. The main result of [14] was to provide a proof for the $n \neq 2m$ case that uses only elementary group theory, the proof is based on an analysis of the clique structure of the graph. In [14] the authors leave the $n = 2m$ case open but make a conjecture for this case. Also in [6] the conjecture is resolved in the affirmative by providing a proof that again uses only elementary group theory. We will again find these results by different methods which we believe are also elementary.

2. Preliminaries

In this paper, a graph $\Gamma = (V, E)$ is considered as an undirected simple graph where $V = V(\Gamma)$ is the vertex-set and $E = E(\Gamma)$ is the edge-set. For all the terminology and notation not defined here, we follow [3, 7, 14].

The graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ are called isomorphic, if there is a bijection $\alpha : V_1 \rightarrow V_2$ such that $\{a, b\} \in E_1$ if and only if $\{\alpha(a), \alpha(b)\} \in E_2$ for all $a, b \in V_1$. In such a case the bijection $\alpha$ is called an isomorphism. An automorphism of a graph $\Gamma$ is an isomorphism of $\Gamma$ with itself. The set of
A NOTE ON THE AUTOMORPHISM GROUPS OF JOHNSON GRAPHS

Aut\text{omorphisms of } \Gamma \text{ with the operation of composition of functions is a group, called the automorphism group of } \Gamma \text{ and denoted by } Aut(\Gamma). \text{ The group of all permutations of a set } V \text{ is denoted by } Sym(V) \text{ or just } Sym(n) \text{ when } |V| = n. \text{ A permutation group } G \text{ on } V \text{ is a subgroup of } Sym(V). \text{ In this case we say that } G \text{ act on } V. \text{ If } \Gamma \text{ is a graph with vertex-set } V, \text{ then we can view each automorphism as a permutation of } V, \text{ and so } Aut(\Gamma) \text{ is a permutation group. Let } G \text{ acts on } V, \text{ we say that } G \text{ is transitive (or } G \text{ acts transitively on } V), \text{ if there is just one orbit. This means that given any two elements } u \text{ and } v \text{ of } V, \text{ there is an element } \beta \text{ of } G \text{ such that } \beta(u) = v. \text{ The graph } \Gamma \text{ is called vertex transitive, if } Aut(\Gamma) \text{ acts transitively on } V(\Gamma). \text{ The action of } Aut(\Gamma) \text{ on } V(\Gamma) \text{ induces an action on } E(\Gamma), \text{ by the rule } \beta\{x,y\} = \{\beta(x), \beta(y)\}, \beta \in Aut(\Gamma), \text{ and } \Gamma \text{ is called edge transitive if this action is transitive. The graph } \Gamma \text{ is called distance-transitive if for all vertices } u,v,x,y, \text{ of } \Gamma \text{ such that } d(u,v) = d(x,y), \text{ there exists some } g \in Aut(\Gamma) \text{ satisfying } g(u) = x \text{ and } g(v) = y. \text{ It is clear that a distance transitive graph is a symmetric graph. The Johnson graph } J(n,m) \text{ is an example of a distance-transitive graph [3]. For } v \in V(\Gamma) \text{ and } G = Aut(\Gamma), \text{ the stabilizer subgroup } G_v \text{ is the subgroup of } G \text{ containing of all automorphisms which fix } v. \text{ In the vertex transitive case all stabilizer subgroups } G_v \text{ are conjugate in } G, \text{ and consequently isomorphic, in this case, the index of } G_v \text{ in } G \text{ is given by the equation, } |G : G_v| = \frac{|G|}{|G_v|} = |V(\Gamma)|. \text{ If each stabilizer } G_v \text{ is the identity group, then no element of } G, \text{ except the identity, fixes any vertex, and we say that } G \text{ act semiregularly on } V. \text{ We say that } G \text{ act regularly on } V \text{ if and only if } G \text{ acts transitively and semiregularly on } V, \text{ and in this case we have } |V| = |G|. \text{ Although, in most situations it is difficult to determine the automorphism group of a graph } \Gamma \text{ and how it acts on the vertex set of } \Gamma, \text{ there are various papers in the literature, and some of the recent works appear in the references [6,8,10,11,12,13,14,16].} \text{ 3. Main results}}

Let } \Gamma \text{ be a connected graph with diameter } D \text{ and } x \text{ be a vertex of } \Gamma. \text{ Let } \Gamma_i = \Gamma_i(x) \text{ be the set of vertices of } \Gamma \text{ at distance } i \text{ from } x. \text{ Thus } \Gamma_0 = \{x\}, \text{ and } \Gamma_1 = N(x) \text{ is the set of vertices which are adjacent to the vertex } x. \text{ Therefore, } V(\Gamma) \text{ is partitioned into the disjoint subsets } \Gamma_0(x),...,\Gamma_D(x). \text{ Let } v,w \in \Gamma \text{ and}
$d(v, w)$ denotes the distance between the vertices $v$ and $w$ in the graph $\Gamma$. It is an easy task to show that for any two vertices $v, w$ of $J(n, m)$, $d(u, v) = k$ if and only if $|v \cap w| = m - k$ (when regarding $v, w$ as $m$-sets).

**Proposition 3.1.** Let $\Gamma = J(n, m)$, $n \geq 6$, $3 \leq m \leq \frac{n}{2}$. Let $x \in V(\Gamma)$, $\Gamma_i = \Gamma_i(x)$ and $v \in \Gamma_i$. Then we have:

$$\bigcap_{w \in \Gamma_{i-1} \cap N(v)} (N(w) \cap \Gamma_i) = \{v\}$$

**Proof.** A proof of this result appeared in [14], but for the sake of completeness and since we need our proof in the sequel, we offer a proof which is slightly different from that. It is clear that $v \in \bigcap_{w \in \Gamma_{i-1} \cap N(v)} (N(w) \cap \Gamma_i)$. Let $x = \{x_1, ..., x_m\}$. We can assume that $v = \{x_1, ..., x_{m-i}, y_1, ..., y_t\}$ where $I = \{x_1, ..., x_m, y_1, ..., y_{n-m}\} = \{1, 2, ..., n\}$. If $w \in \Gamma_{i-1} \cap N(v)$, then $|w \cap v| = m - 1$ and $|w \cap x| = m - i + 1$ and thus

$$w = w_{rt} = (v - y_t) \cup \{x_r\} = \{x_1, ..., x_{m-i}, x_r, y_1, ..., y_{t-1}, y_{t+1}, ..., y_i\}$$

where $m - i + 1 \leq r \leq m$, $1 \leq t \leq i$.

We show that if $u \in \Gamma_i$ and $u \neq v$ and $u$ is adjacent to some $w_{rt}$, then there is some $w_{pq}$ such that $u$ is not adjacent to $w_{pq}$.

If $v \neq u \in \Gamma_i$ is adjacent to $w_{rt}$, then $u$ has one of the following forms,

$$ (*) \quad u_1 = \{x_1, ..., x_{m-i}, y_1, ..., y_{t-1}, y_{t+1}, ..., y_i, y_a\}; \quad i < a \leq n - m$$

$$ (** ) \quad u_2 = \{x_1, ..., x_{j-1}, x_{j+1}, ..., x_{m-i}, x_r, y_1, ..., y_{t-1}, y_{t+1}, ..., y_i, y_b\}$$

where $b \in \{i + 1, ..., n - m\} \cup \{t\}$, $1 \leq j \leq m - i$.

In the case $(*)$, $u_1$ is not adjacent to $w_{r(t-1)}$ (or $w_{r(t+1)}$), because $x_r, y_t \in w_{r(t-1)}$ but $x_r, y_t \notin u_1$.

In the case $(**)$, if $y_b = y_t$, say,

$$u_2 = \{x_1, ..., x_{j-1}, x_{j+1}, ..., x_{m-i}, x_r, y_1, ..., y_{t-1}, y_{t+1}, ..., y_i\}$$

then $u_2$ is not adjacent to $w_{st}(s \neq r)$ because $x_j, x_s \in w_{st}$ and $x_j, x_s \notin u_2$. Also in the case $(**)$, if $y_b \neq y_t$, then we have

$$u_2 = \{x_1, ..., x_{j-1}, x_{j+1}, ..., x_{m-i}, x_r, y_1, ..., y_{t-1}, y_{t+1}, ..., y_i, y_b\}$$

where $b \in \{i + 1, ..., n - m\}$. But in this case $u_2$ is not adjacent to $w_{s(t-1)}$ (or $w_{s(t+1)}$) because $x_j, y_t \in w_{r(t-1)}$ and $x_j, y_t \notin u_2$.

Our discussion shows that if $v \neq u \in \Gamma_i$, then $u \notin \bigcap_{w \in N(v) \cap \Gamma_{i-1}} (N(w) \cap \Gamma_i)$ and thus we have $\bigcap_{w \in N(v) \cap \Gamma_{i-1}} (N(w) \cap \Gamma_i) = \{v\}$. 

\[\Box\]
Let $\Gamma$ be a graph, then the line graph $L(\Gamma)$ of the graph $\Gamma$ is constructed by taking the edges of $\Gamma$ as vertices of $L(\Gamma)$, and joining two vertices in $L(\Gamma)$ whenever the corresponding edges in $\Gamma$ have a common vertex. There is an important relation between $Aut(\Gamma)$ and $Aut(L(\Gamma))$. Indeed, we have the following result [3, chapter 15] which obtained by Whitney [17]. Although, the proof of this result is tedious but uses elementary facts in graph theory and group theory, and it is not difficult (see [2, chapter 13] for a proof).

**Theorem 3.2.** The mapping $\theta : Aut(\Gamma) \to Aut(L(\Gamma))$ defined by;

$$\theta(g)\{u,v\} = \{g(u), g(v)\}, \ g \in Aut(\Gamma), \ \{u,v\} \in E(\Gamma)$$

is a group homomorphism and in fact we have;

(i) $\theta$ is a monomorphism provided $\Gamma \neq K_2$;

(ii) $\theta$ is an epimorphism provided $\Gamma$ is not $K_4$, $K_4$ with one edge deleted, or $K_4$ with two edges deleted.

For example, by using the above fact, we can obtain the following result.

**Proposition 3.3.** The automorphism group of Johnson graph $J(n,2)$ is isomorphic with the symmetric group $Sym(n)$.

**Proof.** We know that the Johnson graph $J(n,2)$ is the line graph of the complete graph on $n$ vertices, namely, $J(n,2) \cong L(K_n)$. Thus, we have $Aut(J(n,2)) \cong Aut(L(K_n))$. By Theorem 3.2. it follows that $Aut(L(K_n)) \cong Aut(K_n)$. Now, since $Aut(K_n) \cong Sym(n)$, then we have $Aut(J(n,2)) \cong Sym(n)$. $\Box$

We now try to prove that the above result is true not only for the case $m = 2$ in $J(n,m)$ but also for every possible $m$.

**Proposition 3.4.** Let $v$ be a vertex of the Johnson graph $J(n,m)$. Then, $\Gamma_1 = < N(v) >$, the induced subgraph of $N(v)$ in $J(n,m)$, is isomorphic with $L(K_{m,n-m})$, where $K_{m,n-m}$ is the complete bipartite graph with partitions of orders $m$ and $n - m$.

**Proof.** Let $I = \{1,2,...,n\}$, $v = \{x_1,...,x_m\}$ and $w = v^c = \{y_1,...,y_{n-m}\}$ be the complement of the subset $v$ in $I$. Let $x_{ij} = v - \{x_j\} \cup \{y_i\}, 1 \leq j \leq m$, $1 \leq i \leq n - m$. Then,

$$N(v) = \{x_{ij}|1 \leq j \leq m, 1 \leq i \leq n - m\}$$
In $\Gamma_1 = \langle N(v) \rangle$ two vertices $x_{ij}$ $x_{rs}$ are adjacent if and only if $i = r$ or $j = s$. In fact, $\{x_1, ..., x_{j-1}, y_i, x_{j+1}, ..., x_m\}$ and $\{x_1, ..., x_{s-1}, y_r, x_{s+1}, ..., x_m\}$ have $m - 1$ element(s) in common if and only if $x_i = x_r$ or $y_j = y_s$. Let $X = \{v_1, ..., v_m\}$ and $Y = \{w_1, ..., w_{n-m}\}$ where $X \cap Y = \emptyset$. We know that the complete bipartite graph $K_{m,n-m}$ is the graph with vertex set $X \cup Y$, and edge set $E = \{\{v_i, w_j\}, 1 \leq i \leq m, 1 \leq j \leq n - m\}$. Then $L(K_{n,n-m})$ is the graph with vertex set $V(L(K_{n,n-m})) = E$ in which vertices $\{v_i, w_j\}$ and $\{v_r, w_s\}$ are adjacent if and only if $v_i = v_r$ or $w_j = w_s$. Now it is an easy task to show that the mapping

$$\phi : L(K_{n,n-m}) \rightarrow \Gamma_1 = \langle N(v) \rangle, \; \phi(v_i, w_j) = x_{ij} = v - \{x_i\} \cup \{y_j\}$$

is a graph isomorphism.

\[\Box\]

**Theorem 3.5.** Let $\Gamma = J(n, m)$, $n \geq 4$, $2 \leq m \leq \frac{n}{2}$. If $n \neq 2m$, then $\text{Aut}(\Gamma) \cong \text{Sym}(n)$. If $n = 2m$, then $\text{Aut}(\Gamma) \cong \text{Sym}(n) \times \mathbb{Z}_2$.

**Proof.** Let $G = \text{Aut}(\Gamma)$. Let $x \in V = V(\Gamma)$, and $G_x = \{f \in G | f(x) = x\}$ be the stabilizer subgroup of the vertex $x$ in $\Gamma$. Let $\langle N(x) \rangle = \Gamma_1$ be the induced subgroup of $N(x)$ in $\Gamma$. If $f \in G_x$ then $f|_{N(x)}$, the restriction of $f$ to $N(x)$ is an automorphism of $\Gamma_1$. We define $\varphi : G_x \rightarrow \text{Aut}(\Gamma_1)$ by the rule $\varphi(f) = f|_{N(x)}$. It is an easy task to show that $\varphi$ is a group homomorphism.

We show that $\ker(\varphi)$ is the identity group. If $f \in \ker(\varphi)$, then $f(x) = x$ and $f(w) = w$ for every $w \in N(x)$. Let $D$ be the diameter of $\Gamma = J(n, m)$ and $\Gamma_i$ be the set of vertices of $\Gamma$ at distance $i$ from the vertex $x$. Then $V = V(\Gamma) = \bigcup_{i=0}^{D} \Gamma_i$. We prove by induction on $i$ that $f(u) = u$ for every $u \in \Gamma_i$. Let $d(u, x)$ be the distance of the vertex $u$ from $x$. If $d(u, x) = 1$, then $u \in \Gamma_1$ and we have $f(u) = u$. Assume $f(u) = u$ when $d(u, x) = i - 1$. If $d(u, x) = i$, then by proposition 3.1. $\{u\} = \cap_{w \in \Gamma_{i-1} \cap N(u)} N(w)$, and therefore $f(u) = \cap_{w \in \Gamma_{i-1} \cap N(u)} N(f(w))$. Since, $w \in \Gamma_{i-1}$, then $d(w, x) = i - 1$, and hence $f(w) = w$. Thus,

$$f(u) = \cap_{w \in \Gamma_{i-1} \cap N(u)} N(f(w)) = \cap_{w \in \Gamma_{i-1} \cap N(u)} N(w) = u$$

Thus, $\ker(\varphi) = \{1\}$. On the other hand,

$$\frac{G_v}{\ker(\varphi)} \cong \varphi(G_v) \leq \text{Aut}(\Gamma_1) = \text{Aut}(\langle N(x) \rangle)$$

and thus $G_v \cong \varphi(G_v) \leq \text{Aut}(\Gamma_1)$.
and hence $|G_v| \leq |\text{Aut}(\Gamma_1)|$.

By Proposition 3.4, $\Gamma_1 \cong L(K_{m,n-m})$, thus $\text{Aut}(\Gamma_1) \cong \text{Aut}(L(K_{m,n-m}))$.

Since by Theorem 3.2, $\text{Aut}(L(K_{m,n-m})) \cong \text{Aut}(K_{m,n-m})$, hence $|\text{Aut}(\Gamma_1)| = |\text{Aut}(K_{m,n-m})|$, and therefore $|G_v| \leq |\text{Aut}(K_{m,n-m})|$. Note that if $P = K_{s,t}$ is a complete bipartite graph, then for $t \neq s$ we have $|\text{Aut}(P)| = s!t!$, and for $t = s$ we have $|\text{Aut}(P)| = 2(s!)^2$ [3, chapter17].

Since $\Gamma = J(n,m)$ is a vertex transitive graph, we have $|V(\Gamma)| = \frac{|G|}{|\text{Aut}(\Gamma)|}$, thus $|G| = |G_v||V(\Gamma)| \leq |\text{Aut}(K_{m,n-m})|^n$. Now, if $n \neq 2m$, then we have

$$|G| \leq (m!(n-m)!n!) = n!$$

and if $n = 2m$, then we have

$$(* \quad |G| \leq |\text{Aut}(K_{m,n-m})|^{2m}, \text{and hence } |G| \leq 2m!^2 \frac{(2m)!}{(m!)^2} = 2(2m)!$$

We know that If $\theta \in \text{Sym}(I)$ where $I = \{1, 2, \ldots, n\}$, then

$$f_\theta : V(\Gamma) \rightarrow V(\Gamma), f_\theta(\{x_1, \ldots, x_m\}) = \{\theta(x_1), \ldots, \theta(x_m)\}$$

is an automorphism of $\Gamma$ and the mapping $\psi : \text{Sym}(I) \rightarrow \text{Aut}(\Gamma)$, defined by the rule $\psi(\theta) = f_\theta$ is an injection.

Now if $n \neq 2m$, since $|G| = |\text{Aut}(\Gamma)| \leq n!$, we conclude that $\psi$ is a bijection, and hence $\text{Aut}(\Gamma) \cong \text{Sym}(I)$.

If $n = 2m$, then $\Gamma = J(n,m) = J(2m,m)$ and the set $\{f_\theta | \theta \in \text{Sym}(2m)\} = H$, is a subgroup of $\text{Aut}(\Gamma)$. It is an easy task to show that the mapping $\alpha : V(\Gamma) \rightarrow V(\Gamma), \alpha(v) = v^c$ where $v^c$ is the complement of the set $v$ in $I$, is also an automorphism of $\Gamma$, say, $\alpha \in G = \text{Aut}(\Gamma)$.

We show that $\alpha \notin H$. If $\alpha \in H$, then there is a $\theta \in \text{Sym}(I)$ such that $f_\theta = \alpha$. Since $o(\alpha) = 2$ ($o(\alpha) =$ order of $\alpha$), then $o(f_\theta) = o(\theta) = 2$. We assert that $\theta$ has no fixed points, say, $\theta(x) \neq x$, for every $x \in I$. In fact, if $x \in I$, and $\theta(x) = x$, then for the $m$-set $v = \{x, y_1, \ldots, y_{m-1}\} \subseteq I$, we have

$$f_\theta(v) = \{\theta(x), \theta(y_1), \ldots, \theta(y_{m-1})\} = \{x, \theta(y_1), \ldots, \theta(y_{m-1})\}$$

hence $x \in f_\theta(v) \cap v$, and therefore $f_\theta(v) \neq v^c = \alpha(v)$ which is a contradiction. Therefore, $\theta$ has a form such as $\theta = (x_1, y_1)\ldots(x_m, y_m)$ where $(x, y_i)$ is
a transposition of $\text{Sym}(I)$. Now, for the $m$-set $v = \{x_1, y_1, x_2, ..., x_{m-1}\}$ we have

$$\alpha(v) = f_\theta(v) = \{\theta(x_1), \theta(y_1), \theta(x_{m-1})\} = \{y_1, x_1, ..., \theta(x_{m-1})\}$$

and thus $x_1, y_1 \in f_\theta(v) \cap v$, hence $f_\theta(v) \neq v^c = \alpha(v)$, which is a contradiction.

We assert that for every $\theta \in \text{Sym}(I)$, we have $f_\theta \alpha = \alpha f_\theta$. In fact, if $v = \{x_1, ..., x_m\}$ is an $m$-subset of $I$, then there are $y_j \in I, 1 \leq j \leq m$, such that $I = \{x_1, ..., x_m, y_1, ..., y_m\}$. Now we have

$$f_\theta \alpha(v) = f_\theta \{y_1, ..., y_m\} = \{\theta(y_1), ..., \theta(y_m)\}$$

On the other hand, we have

$$\alpha f_\theta(v) = \alpha \{\theta(x_1), ..., \theta(x_m)\} = \{\theta(y_1), ..., \theta(y_m)\}$$

because $I = \theta(I) = \{\theta(x_1), ..., \theta(x_m), \theta(y_1), ..., \theta(y_m)\}$. Consequently, $f_\theta \alpha(v) = \alpha f_\theta(v)$. We now deduce that $f_\theta \alpha = \alpha f_\theta$.

Note that if $X$ is a group and $Y, Z$ are subgroups of $X$, then the subset $YZ = \{yz \mid y \in Y, z \in Z\}$ is a subgroup in $X$ if and only if $YZ = ZY$. According to this fact, we conclude that $H < \alpha >$ is a subgroup of $G$.

Since $\alpha \notin H$ and $o(\alpha) = 2$, then $H < \alpha >$ is a subgroup of $G$ of order

$$\frac{|H|\langle \alpha \rangle}{|H < \alpha >|} = 2|H| = 2((2m)!!)$$

Now, since by (*) $|G| \leq 2((2m)!!)$, then $G = H < \alpha >$. On the other hand, since $f_\theta \alpha = \alpha f_\theta$, for every $\theta \in \text{Sym}(I)$, then $H$ and $\langle \alpha \rangle$ are normal subgroups of $G$. Thus, $G$ is a direct product of two groups $H$ and $\langle \alpha \rangle$, namely, we have $G = H \times \langle \alpha \rangle \cong \text{Sym}(2m) \times \mathbb{Z}_2$.

\[\square\]

References

[1] Brian Alspach. Johnson graphs are Hamilton-connected. Ars Mathematica Contemporanea 6 (2013), 21-23.
A NOTE ON THE AUTOMORPHISM GROUPS OF JOHNSON GRAPHS

[2] Mehdy Behzad, Gary Chartrand, Introduction to the Theory of Graphs, Allyn and Bacon Inc, 1971.
[3] N. L. Biggs, Algebraic Graph Theory (Second edition), Cambridge Mathematical Library (Cambridge University Press, Cambridge, 1993).
[4] Cannon, Andrew D.; Bamberg, John; Praeger, Cheryl E. A classification of the strongly regular generalized Johnson graphs. Ann. Comb. 16 (2012), no. 3, 489-506.
[5] J. D. Dixon, B. Mortimer, Permutation Groups, Graduate Texts in Mathematics 163, Springer-Verlag, New York, 1996.
[6] Ashwin Ganesan, On the automorphism group of a Johnson graph, arXiv:1412.5055v1 [math.CO], Dec 2014.
[7] C. Godsil, G. Royle, Algebraic Graph Theory, Springer.
[8] G. A. Jones. Automorphisms and regular embeddings of merged Johnson graphs. European Journal of Combinatorics, 26:417-435, 2005. (2001).
[9] Krebs, Mike; Shaheen, Anthony. On the spectra of Johnson graphs. Electron. J. Linear Algebra 17 (2008), 154-167.
[10] S. Morteza Mirafzal. On the symmetries of some classes of recursive circulant graphs, Transactions on Combinatorics, Volume 3, Issue 1, March 2014, 1-6.
[11] S. Morteza Mirafzal. On the automorphism groups of regular hyperstars and folded hyperstars, Ars Comb. 123, 75-86 (2015).
[12] S. Morteza Mirafzal. Some other algebraic properties of folded hypercubes, Ars Comb. 124, 153-159 (2016).
[13] S. Morteza Mirafzal. More odd graph theory from another point of view. Discrete Mathematics, 2017, http://dx.doi.org/10.1016/j.disc.2017.08.032.
[14] M. Ramras and E. Donovan. The automorphism group of a Johnson graph. SIAM Journal on Discrete Mathematics, 25(1):267-270, 2011.
[15] Rotman, J. J., An Introduction to the Theory of Groups, 4th ed., Springer-Verlag, New York, 1995.
[16] Yi Wang, Yan-Quan Feng, Jin-Xin Zhou. Automorphism Group of the Varietal Hypercube Graph, Graphs and Combinatorics, DOI 10.1007/s00373-017-1827-y, 2017.
[17] H. Whitney, Congruent graphs and the connectivity of graphs, Amer. J. Math, 1932, 150-168.

Department of Mathematics, Lorestan University, Khoramabad, Iran
E-mail address: smorteza.mirafzal@yahoo.com
E-mail address: mirafzal.m@lu.ac.ir