Homology equivalences of manifolds and 
zero-in-the-spectrum examples

Shengkui Ye

December 12, 2013

Abstract

Working with group homomorphisms, a construction of manifolds is introduced to preserve homology groups. The construction gives as special cases Quillen’s plus construction with handles obtained by Hausmann, the existence of one-sided \( h \)-cobordism of Guilbault and Tinsley, the existence of homology spheres and higher-dimensional knots proved by Karvaire. We also use it to get counter-examples to the zero-in-the-spectrum conjecture found by Farber-Weinberger, and by Higson-Roe-Schick.

1 Introduction

The aim of this note is to propose a general surgery plus construction on manifolds. This is a manifold version of the generalized plus construction for CW complexes found by the author in [21]. As applications, we give a unified approach to the plus construction with handles of Hausmann [8], the \((\text{mod} \ L)\)-one-sided \( h \)-cobordism of Guilbault and Tinsley [6], the existence of homology spheres of Kervaire [15] and the existence of higher-dimensional knots of Kervaire [14]. We also use it to get some examples for the zero-in-the-spectrum conjecture found by Farber-Weinberger [4] and Higson-Roe-Schick [11]. First, we briefly review these existing works.

Let \( M \) be an \( n \)-dimensional \((n \geq 5)\) closed manifold with fundamental group \( \pi_1(M) = H \). Suppose that \( \Phi : H \rightarrow G \) is a surjective group homomorphism of finitely presented groups with the kernel \( \ker \Phi \) a perfect subgroup. Hausmann shows that Quillen’s plus construction with respect to \( \ker \Phi \) can
by made by adding finitely many two and three handles to $M \times 1 \subset M \times [0,1]$ (cf. Section 3 in Hausmann [8] and Definition of $\varphi_1$ on page 115 in Hausmann [9]). The resulting cobordism $(W; M, M')$ has $W$ and $M'$ the homotopy type of the Quillen plus construction $M^+$. In other words, the fundamental group $\pi_1(M') = \pi_1(W) = G$ and for any $\mathbb{Z}[G]$-module $N$, the inclusion map $M \hookrightarrow W$ induces homology isomorphisms $H_n(M; N) \cong H_n(W; N)$.

Guilbault and Tinsley [6] obtain a generalized manifold plus construction, as follows. Let $(W, Y)$ be a connected CW pair and $L$ a normal subgroup of $\pi_1(Y)$. The inclusion $Y \hookrightarrow W$ is called a (mod $L$)-homology equivalence if it induces an isomorphism of fundamental groups and is a homology equivalence with coefficients $\mathbb{Z}[\pi_1(Y)/L]$. A compact cobordism $(W; X, Y)$ is a (mod $L$)-one-sided $h$-cobordism if $X \hookrightarrow W$ induces a surjection of fundamental groups and $Y \hookrightarrow W$ is a (mod $L$)-homology equivalence. Let $B$ be a closed $n$-manifold $(n \geq 5)$ and $\alpha : \pi_1(B) \to G$ a surjective homomorphism onto a finitely presented group such that $\ker(\alpha)$ is strongly $L'$-perfect, i.e. $\ker(\alpha) = [\ker(\alpha), L']$ for some group $L'$, where $\ker(\alpha) \leq L' \leq \pi_1(B)$. For $L = L'/\ker(\alpha)$, Guilbault and Tinsley [6] show that there exists a (mod $L$)-one-sided $h$-cobordism $(W; B, A)$ such that $\pi_1(W) = G$ and $\ker(\pi_1(B) \to \pi_1(W)) = \ker(\alpha)$.

An $n$-dimensional homology sphere is a closed manifold $M$ having the homology groups of the $n$-sphere, i.e. $H_*(M) \cong H_*(S^n)$. Let $\pi$ be a finitely presented group and $n \geq 5$. Kervaire shows that there exists an $n$-dimensional homology sphere $M$ with $\pi_1(M) = \pi$ if and only if the homology groups satisfy $H_1(\pi; \mathbb{Z}) = H_2(\pi; \mathbb{Z}) = 0$.

For an integer $n \geq 1$, define an $n$-knot to be a differential imbedding $f : S^n \to S^{n+2}$ and the group of the $n$-knot $f$ to be $\pi_1(S^{n+2} - f(S^n))$. Let $G$ be a finitely presentable group. The weight $w(G)$ is the smallest integer $k$ such that there exists a set of $k$ elements $\alpha_1, \alpha_2, \ldots, \alpha_k \in G$ whose normal closure equals $G$. Kervaire [14] shows that a finitely presented group $G$ is the group of a $n$-knot $(n \geq 5)$ if and only if $H_1(G; \mathbb{Z}) = \mathbb{Z}$, $H_2(G; \mathbb{Z}) = 0$ and the weight of $G$ is 1.

The zero-in-the-spectrum conjecture goes back to Gromov, who asked for a closed, aspherical, connected and oriented Riemannian manifold $M$ whether there always exists some $p \geq 0$, such that zero belongs to the spectrum of the Laplace-Beltrami operator $\Delta_p$ acting on the square integrable $p$-forms on the universal covering $\tilde{M}$ of $M$. Farber and Weinberger [4] show that the conjecture is not true if the condition that $M$ is aspherical is dropped. More generally, Higson, Roe and Schick [11] show that for a finitely presented
group $G$ satisfying $H_0(G;C_r^\ast(G)) = H_1(G;C_r^\ast(G)) = H_2(G;C_r^\ast(G)) = 0$, there always exists a closed manifold $Y$ of dimension $n \ (n \geq 6)$ with $\pi_1(Y) = G$ such that $Y$ is a counterexample to the conjecture if $M$ is not required to be aspherical.

In this note, a more general construction is provided to preserve homology groups. For this, we have to introduce the notion of a finitely $G$-dense ring.

**Definition 1.1.** A finitely $G$-dense ring $(R, \phi)$ is a unital ring $R$ together with a ring homomorphism $\phi : \mathbb{Z}[G] \to R$ such that, when $R$ is regarded as a left $\mathbb{Z}[G]$-module via $\phi$, for any finitely generated right $\mathbb{Z}[G]$-module $M$, finitely generated free right $R$-module $F$ and $R$-module surjection $f : M \otimes_{\mathbb{Z}[G]} R \twoheadrightarrow F$, the module $F$ has a finite $R$-basis in $f(M \otimes 1)$.

This is an analog of $G$-dense rings defined in Ye [21]. Examples of finitely $G$-dense rings include the real reduced group $C^\ast$-algebra $C^\ast_r(G)$, the real group von Neumann algebra $N_RG$, the real Banach algebra $l^1_R(G)$, the rings $k = \mathbb{Z}/p \ (\text{prime } p)$ and $k \subseteq \mathbb{Q}$ a subring of the rationals (with trivial $G$-actions), the group ring $k[G]$, and so on.

**Conventions.** Let $\pi$ and $G$ be two groups. Suppose that $R$ is a $\mathbb{Z}[G]$-module and $BG, B\pi$ are the classifying spaces. For a group homomorphism $\alpha : \pi \to G$, we will denote by $H_\ast(G, \pi; R)$ the relative homology group $H_\ast(BG, B\pi; R)$ with coefficients $R$. All manifolds are assumed to be connected smooth manifolds, until otherwise stated.

Our main result is the following.

**Theorem 1.2.** Assume that $G$ is a finitely presented group and $(R, \phi)$ is a finitely $G$-dense ring. Let $X$ be a connected $n$-dimensional ($n \geq 5$) closed orientable manifold with fundamental group $\pi = \pi_1(X)$. Assume that $\alpha : \pi \to G$ is a group homomorphism of finitely presented groups such that the image $\alpha(\pi)$ is finitely presented and

\[
\begin{align*}
H_1(\alpha) & : \ H_1(\pi; R) \to H_1(G; R) \text{ is injective, and} \\
H_2(\alpha) & : \ H_2(\pi; R) \to H_2(G; R) \text{ is surjective.}
\end{align*}
\]

Suppose either that $R$ is a principal ideal domain or that the relative homology group $H_1(G, \pi; R)$ is a stably free $R$-module. When $2$ is not invertible in $R$, suppose that the manifold $M$ is a spin manifold. Then there exists a closed $R$-orientable manifold $Y$ with the following properties:
(i) $Y$ is obtained from $X$ by attaching 1-handles, 2-handles and 3-handles, such that

$\pi_1(Y) \cong G$ and the inclusion map $g : X \to W$, the cobordism between $X$ and $Y$, induces the same fundamental group homomorphism as $\alpha$, and

(iii) for any integer $q \geq 2$ the map $g$ induces an isomorphism of homology groups

$$g_* : H_q(X; R) \cong H_q(W; R). \quad (1)$$

Theorem 1.2 has the following applications.

(1) $\alpha$ surjective.

- When $\ker \alpha$ is perfect and $R = \mathbb{Z}[G]$ the group ring, this is Quillen’s plus construction with handles, which is obtained by Hausmann [8] and [9] (see Corollary 4.1).

- When $\ker \alpha$ is strongly $L'$-perfect and $R = \mathbb{Z}[G/L]$, Theorem 1.2 implies the existence of (mod $L$)-one-sided $h$-cobordism obtained by Guilbault and Tinsley [5, 6] (see Corollary 4.2).

(2) $\pi = 1$ ($\alpha$ injective).

- When $X = S^n$ ($n \geq 5$), $R = \mathbb{Z}$ and $G$ a superperfect group, Theorem 1.2 recovers the existence of homology spheres, which is obtained by Kervaire [15] (see Corollary 4.4).

- When $X = S^n$ ($n \geq 5$), $R = \mathbb{Z}$ and $H_1(G; \mathbb{Z}) = \mathbb{Z}$, $H_2(G; \mathbb{Z}) = 0$, Theorem 1.2 recovers the existence higher-dimensional knots, which is obtained by Kervaire [14] (see Corollary 4.5).

- When $X = S^n$ ($n \geq 6$) and $R = C^\ast_R(G)$, the theorem yields the results obtained by Farber-Weinberger [4] and Higson-Roe-Schick [11] on the zero-in-the-spectrum conjecture (see Corollary 4.7).

For Bousfield’s integral localization, Rodríguez and Scevenels [19] show that there is a topological construction that, while leaving the integral homology of a space unchanged, kills the intersection of the transfinite lower
central series of its fundamental group. Moreover, this is the maximal subgroup that can be factored out of the fundamental group without changing the integral homology of a space. As another application of Theorem 1.2 with $\alpha$ surjective and $R = \mathbb{Z}$, we obtain a manifold version of Rodríguez and Scevenels’ result.

**Corollary 1.3.** Let $n \geq 5$ and $X$ be a closed $n$-dimensional spin manifold with fundamental group $\pi$ and $N$ a normal subgroup of $\pi$. The following are equivalent.

(i) There exists a closed manifold $Y$ obtained from $X$ by adding 2-handles and 3-handles with $\pi_1(Y) = \pi/N$, such that for any $q \geq 0$ there is an isomorphism

$$H_q(Y; \mathbb{Z}) \cong H_q(X; \mathbb{Z}).$$

(ii) The group $N$ is normally generated by finitely many elements and is a relatively perfect subgroup of $\pi$, i.e. $[\pi, N] = N$.

The article is organized as follows. In Section 2, we introduce some basic facts about finitely $G$-dense rings, surgery theory, Poincaré duality with coefficients and one-sided homology cobordism. The main theorem is proved in Section 3 and some applications are given in Section 4.

## 2 Preliminary results and basic facts

### 2.1 Finitely $G$-dense rings

Recall the concept of finitely $G$-dense rings in Definition 1.1 (see also Ye [22], Definition 1). Compared with the definition of $G$-dense rings, we require all the modules in Definition 1.1 to be finitely generated. It is clear that a $G$-dense ring is finitely $G$-dense. The following lemma from Ye [22] gives some typical examples of finitely $G$-dense rings.

**Lemma 2.1** (Ye [22], Lemma 2). Finitely $G$-dense rings include the real reduced group $C^*$-algebra $C^*_r(G)$, the real group von Neumann algebra $\mathcal{N}_R G$, the real Banach algebra $l^1(R)(G)$, the rings $k = \mathbb{Z}/p$ (prime $p$) and $k \subseteq \mathbb{Q}$ a subring of the rationals (with trivial $G$-actions), the group ring $k[G]$.

Similar to Example 2.6 in Ye [21], one can show that the ring of Gaussian integers $\mathbb{Z}[i]$ is not finitely $G$-dense for the trivial group $G$. 


2.2 Basic facts on surgery

The proof of Theorem 1.2 is based on some facts in surgery theory. The following definition and lemmas can be found in Ranicki [18].

Definition 2.2. Given an $n$-manifold $M$ and an embedding $S^i \times D^{n-i} \subset M$ ($-1 \leq i \leq n$) define the $n$-manifold $M' = (M - S^i \times D^{n-i}) \cup D^{i+1} \times S^{n-i-1}$ obtained from $M$ by surgery by an $i$-surgery. The trace of the surgery on $S^i \times D^{n-i} \subset M$ is the elementary $(n+1)$-dimensional cobordism $(W; M, M')$ obtained from $M \times [0, 1]$ by attaching an $(i+1)$-handle $W = M \times [0, 1] \cup_{S^i \times D^{n-i} \times 1} D^{i+1} \times D^{n-i}$.

The following lemma gives homotopy relations between the surgery trace and the manifolds.

Lemma 2.3. Let $M, M'$ and $W$ be the manifolds defined in Definition 2.2. There are homotopy equivalences

$$M \cup e^{i+1} \simeq W \simeq M' \cup e^{n-i},$$

where attaching maps are induced by embedding of handles.

For a manifold $M$, denote by $w(M)$ the first Stiefel-Whitney class of the tangent bundle over $M$. The following lemma is Proposition 4.24 in Ranicki [18].

Lemma 2.4. Let $(W; M, M')$ be the trace of an $n$-surgery on an $m$-dimensional manifold $M$ killing $x \in H_n(M)$.

1) If $1 \leq n \leq m - 2$, then $W$ and $M'$ has the same orientation type as $M$ (which means that $M'$ is orientable if $M$ is orientable).

2) If $n = m - 1$ and $M$ is orientable, so are $M'$ and $W$.

3) If $n = m - 1$ and $M$ is nonorientable, then $M'$ is orientable if and only if $x = w(M) \in H_{m-1}(M; \mathbb{Z}_2) = H^1(M; \mathbb{Z}_2)$.

4) If $n = 0$ and $M$ is nonorientable, then so are $W$ and $M'$.

2.3 Poincaré duality with coefficients

In this subsection, we collect some facts about the Poincaré duality with coefficients. For more details, see Chapter 2 of Wall’s book [20]. Let $X$ be a finite CW complex with a universal covering space $\tilde{X}$. Denote by
Let $C_*(\tilde{X})$ be the cellular chain complex of $X$ and by $C^*(\tilde{X})$ the chain complex $\text{Hom}_{\mathbb{Z}\pi_1(X)}(C_*(\tilde{X}), \mathbb{Z}\pi_1(X))$. We call a finite CW complex $X$ a simple Poincaré complex if for some positive integer $n$ and a representative cycle $\xi \in C_n(\tilde{X}) \otimes \mathbb{Z}\pi_1(X)$ the cap product induces a chain homotopy equivalence

$$\xi \cap : C^*(\tilde{X}) \to C_{n-*}(\tilde{X})$$

and the Whitehead torsion is vanishing. Similarly, we can define Poincaré pair $(Y, X)$ by a simple homotopy equivalence

$$\xi \cap : C^*(\tilde{Y}) \to C_{n-*}(\tilde{Y}, \tilde{X}).$$

When $X$ is an $n$-dimensional closed manifold, $X$ is a simple Poincaré complex of formal dimension $n$. When $X$ is a compact manifold with boundary $\partial X$, the pair $(X, \partial X)$ is a simple Poincaré pair (cf. Theorem 2.1 on page 23 in Wall [20]).

**Lemma 2.5.** Let $M$ be an $n$-dimensional orientable compact manifold with boundary $\partial M = X \cup Y$ for closed manifolds $X$ and $Y$. Then for any integer $q \geq 0$ and any $\mathbb{Z}\pi_1(M)$-module $R$, there is an isomorphism

$$H^q(M, X; R) \to H_{n-q}(M, Y; R).$$

**Proof.** Since $X$ is a Poincaré complex and $(M, X \cup Y)$ is a Poincaré pair, the lemma can be proved by considering the long exact homology and cohomology sequences for the cofiber sequence of pairs

$$(X, \emptyset) \to (M, Y) \to (M, X \cup Y)$$

using Poincaré duality for $X$ and the pair $(M, X \cup Y)$. When $R$ is commutative, this is Theorem 3.43 of Hatcher’s textbook [7]. The proof of general case is similar. \hfill \Box

### 2.4 One-sided $R$-homology cobordism

Recall from Guilbault and Tinsley [8] that a one-sided $h$-cobordism $(W; X, Y)$ is a compact cobordism between closed manifolds such that $Y \leftrightarrow W$ is a homotopy equivalence. Motivated by this, we can define one-sided homology cobordism.
Definition 2.6. Let \((W; X, Y)\) be a compact cobordism between closed manifolds and \(R\) a \(\mathbb{Z}\pi_1(W)\)-module. We call \((W; X, Y)\) a one-sided \(R\)-homology cobordism if the inclusion \(Y \hookrightarrow W\) induces \(\pi_1(Y) \cong \pi_1(W)\) and, for any integer \(q \geq 0\) an isomorphism \(H_q(Y; R) \cong H_q(W; R)\).

The following are some easy facts.

(1) When \(R = \mathbb{Z}[[\pi_1(W)]]\), one-sided \(R\)-homology cobordism is the same as one-sided \(h\)-cobordism.

Proof. By the Whitehead theorem, the homotopy equivalence follows from the homology equivalence with coefficients \(\mathbb{Z}[[\pi_1(W)]]\) and the isomorphism of fundamental groups.

(2) Let \((W; X, Y)\) be a one-sided \(R\)-homology cobordism. For any integer \(q \geq 0\), the inclusion map induces an isomorphism \(H^q(W; R) \cong H^q(X; R)\).

Proof. This follows directly from Poincaré duality with coefficients as in the previous subsection.

(3) For a one-sided \(h\)-cobordism \((W; X, Y)\), the inclusion map \(X \hookrightarrow W\) is a Quillen’s plus construction.

Proof. Since \(Y \hookrightarrow W\) is a homotopy equivalence, we have that for any integer \(q \geq 0\), the relative cohomology group \(H^q(W, Y; \mathbb{Z}[[\pi_1(W)]])) = 0\). By Poincaré duality, the inclusion map \(X \hookrightarrow W\) induces homology isomorphism with coefficients \(\mathbb{Z}[[\pi_1(W)]]\). According to 4.3 xi in Berrick [2], the inclusion map is then a Quillen’s plus construction.

(4) Let \(R\) be a principal ideal domain and \((W; X, Y)\) a one-sided \(R\)-homology cobordism. Then for any integer \(q \geq 0\), there is an isomorphism \(H_q(X; R) \cong H_q(Y; R)\).

Proof. When the inclusion map \(Y \hookrightarrow W\) induces an \(R\)-homology equivalence, it also induces an \(R\)-cohomology equivalence by the universal coefficients theorem. By Poincaré duality, \(X\) has the same homology as \(W\), also as \(Y\).
3  Proof of Theorem [1.2]

In this section, we will prove Theorem [1.2]. First, we need some facts about finitely presented groups.

Recall that a normal subgroup $N$ of a group $\pi$ is called normally finitely generated if there exists a finite set $S \subset N$ such that $N$ is generated by elements of the form $gsg^{-1}$ for $s \in S$ and $g \in \pi$. The following lemma gives an elementary characterization of when a normal subgroup is normally finitely generated. Since it is helpful for our later argument, we present a short proof here.

**Lemma 3.1.** Let $\pi$ be a finitely presented group and $N$ a normal subgroup. Then $N$ is normally finitely generated if and only if $\pi/N$ is finitely presented.

**Proof.** The necessity of the condition is obvious. Conversely, choose a presentation of $\pi/N$ with finitely many generators and finitely many relations. Let $F_n$ be the free group with $n$ generators and $f : F_n \to \pi$ a surjection, with normally finitely generated kernel $K$. We can also assume that that the generators of $F_n$ are mapped surjectively to the generators of $\pi/N$ by the composition of $f$ with the quotient map $\pi \to \pi/N$. Here we use the fact that the condition that a group is finitely presented does not depend on the choice of a generator system (cf. Prop. 1.3 in Ohshika [17]). Since $\pi/N$ is finitely presented, $f^{-1}(N)$ is normally finitely generated. Then $N = f(f^{-1}(N))$ is normally finitely generated. \[\square\]

In order to prove Theorem [1.2] we use the following lemma, which is a more general version of Hopf’s exact sequence.

**Lemma 3.2** (Lemma 2.2 in Higson-Roe-Schick [11]). Let $G$ be a group and $V$ be a left $\mathbb{Z}[G]$-module. For any CW complex $X$ with fundamental group $G$ and universal covering space $\tilde{X}$, there is an exact sequence

$$H_2(\tilde{X}) \otimes_{\mathbb{Z}[G]} V \to H_2(X; V) \to H_2(G; V) \to 0.$$ 

**Proof of Theorem [1.2]** We construct a manifold $Y_1$ whose fundamental group $\pi_1(Y) = G$ as follows. Fix a finite presentation

$$\langle x_1, x_2, \ldots, x_k | y_1, y_2, \ldots, y_l \rangle$$
of $\alpha(\pi)$. Extend the presentation of $\alpha(\pi)$ by generators and relations to yield a presentation

$$\langle x_1, x_2, \ldots, x_k, g_1, g_2, \ldots, g_u \mid y_1, y_2, \ldots, y_l, r_1, r_2, \ldots, r_v \rangle$$

of $G$ by adding some generators and relations. For adding the generators $g_1, g_2, \ldots, g_u$, let $S^0_i$ be a copy of the 0-sphere $S^0$ and

$$f_1 : \prod_{i=1}^u S^0_i \times D^n \to X$$

be an embedding with disjoint image. Add 1-handles along $f_1$ to $X$. The resulting manifold is $X_1$ and denote by $W_1$ the surgery trace. We see that the manifold $X_1$ is actually the connected sum $X \# \bigcup_{i=1}^u S^1 \times S^{n-1}$ and can have the same orientation type as $X$. Denote by $e^j_i$ a copy of $j$-cells indexed by $i$. By Lemma 2.3 there are homotopy equivalences

$$X \cup_{i=1}^u e^1_i \simeq W_1 \simeq X_1 \cup_{i=1}^u e^n_i.$$

According to the construction, the fundamental group of $X_1$ is

$$\pi_1(X_1) = \pi \ast F(g_1, g_2, \ldots, g_u),$$

the free product of $\pi = \pi_1(X)$ and the free group of $u$ generators. By Lemma 3.1 the kernel $\ker \alpha$ is normally generated by finitely many elements $z_1, z_2, \ldots, z_p$. Denote by $S$ the finite set $\{z_1, z_2, \ldots, z_p, r_1, r_2, \ldots, r_v\}$. Choose as usual a contractible open set $U$ in $X_1$ as "base point". According to Whitney’s theorem, any element in $\pi_1(X_1)$ is represented by an embedded 1-sphere. Since the manifold $X_1$ is orientable, the normal bundle of any embedded 1-sphere is trivial. For the elements in $S$, let $S^1_{\lambda}$ be a copy of the 1-sphere $S^1$ and let

$$f_2 : \prod_{\lambda \in S} S^1_{\lambda} \times D^{n-1} \to X_1$$

be disjoint embeddings representing the corresponding elements in $\pi_1(X_1)$. Do surgery along $f_2$ by attaching 2-handles. The resulting manifold is $X_2$, and denote by $W_2$ the surgery trace. By Lemma 2.3 once again, there are homotopy equivalences

$$X_1 \cup_{\lambda \in S} e^2_{\lambda} \simeq W_2 \simeq X_2 \cup_{\lambda \in S} e^{n-1}_{\lambda}.$$

Since $n \geq 4$, the fundamental group of $X_2$ is $G$. Let $W'$ be the manifold obtained by gluing the two traces $W_1$ and $W_2$ together along $X_1$. There are
homotopy equivalences
\[ W' \simeq X \cup_{i=1}^n e_i^1 \cup_{\lambda \in S} e_\lambda^2 \simeq X_1 \cup_{i=1}^n e_i^n \cup_{\lambda \in S} e_\lambda^2 \]
\[ \simeq X_2 \cup_{i=1}^n e_i^n \cup_{\lambda \in S} e_\lambda^{n-1}. \]

This implies the fundamental group of \( W' \) is also \( G \), since \( n > 3 \).

We consider the homology groups of the pair \((W', X)\). Let \( \tilde{X} \) and \( \tilde{W}' \) be the universal covering spaces of \( X \) and \( W' \). By Lemma 3.2, there is a commutative diagram

\[
\begin{array}{cccccc}
H_2(\tilde{X}) \otimes_{\mathbb{Z}G} R & \rightarrow & H_2(\tilde{W}') \otimes_{\mathbb{Z}G} R \\
\downarrow & & \downarrow j_4 \\
H_2(X; R) & \xrightarrow{j_2} & H_2(W'; R) & \xrightarrow{j_1} & H_2(W', X; R) \rightarrow H_1(X'; R) \rightarrow H_1(W'; R) \\
\downarrow j_3 & & \downarrow j_5 & & \\
H_2(\pi; R) & \xrightarrow{\alpha_*} & H_2(G; R)
\end{array}
\]

where the middle horizontal chain is the long exact sequence of homology groups for the pair \((W', X)\) and the two vertical lines are the Hopf exact sequences as in Lemma 3.2. Notice that \( H_1(X; R) \cong \pi_1(R; R) \rightarrow H_1(W'; R) \cong H_1(G; R) \)

is injective by assumption. This implies \( j_1 : H_2(W'; R) \rightarrow H_2(W', X; R) \)

is surjective in the above diagram. Note that the map \( \alpha_* : H_2(\pi; R) \rightarrow H_2(G; R) \)

is surjective by assumption. By a diagram chase (for more details, see the proof of Theorem 1.1 in Ye [21]), there is a surjection

\[ j_1 \circ j_1 : H_2(\tilde{W}') \otimes_{\mathbb{Z}G} R \rightarrow H_2(W', X; R). \]

As \( \tilde{W}' \) is simply connected, the homology group \( H_2(\tilde{W}') \)

is isomorphic to \( \pi_2(W') \cong \pi_2(X_2) \). Notice that the homology group \( H_2(W', X; R) \)

can be taken to be a finitely generated free \( R \)-module as in the proof of Theorem

1 in Ye [21]. Since the ring \( R \) is a finitely \( G \)-dense ring in the sense of

Definition [1.1], we can find a finite set \( S' \) of elements in \( \pi_2(X_2) \) such that the image \( j_1 \circ j_1(S') \)

forms an \( R \)-basis for \( H_2(W', X; R) \). Then there are maps \( b_\lambda : S_\lambda^2 \rightarrow X_2 \) with \( \lambda \in S' \) such that for all \( q \geq 2 \), the composition of maps

\[ H_q(\cup_{\lambda \in S'} S_\lambda^2; R) \rightarrow H_q(W'; R) \rightarrow H_q(W', X; R) \]

is an isomorphism.
We construct the manifolds $Y$ and $W$ as follows. Notice that an embedded 2-sphere in a $k$-dimensional ($k \geq 5$) orientable manifold $M$ has trivial normal bundle if and only if it represents 0 in $\pi_1(SO(k)) = \mathbb{Z}/2$ through the classifying map $M \to BSO(k)$ (cf. page 45 of Milnor [16]). When 2 is invertible in the ring $R$, we can always choose the 2-spheres in $S'$ to have trivial normal bundles. When 2 is not invertible in $R$, the manifold $X$ is a spin manifold by assumption. This implies any embedded 2-sphere has a trivial normal bundle. Since $n \geq 5$, we can choose a map $f_3 : \coprod_{\lambda \in S'} S^2_\lambda \times D^{n-2} \to X_2$ as disjoint embedding, whose components represent the elements $b_\lambda$. Do surgery along $f_3$ by attaching 3-handles. Let $Y$ denote the resulting manifold and $W_3$ denote the surgery trace. Suppose that $W$ is the manifold obtained by gluing $W'$ and $W_2$ along $X_2$, which is a cobordism between $X$ and $Y$. By Lemma 2.3, there are homotopy equivalences $X_2 \cup_{\lambda \in S'} e^3_\lambda \simeq W_3 \simeq Y \cup_{\lambda \in S'} e^{n-2}_\lambda$ and $W \simeq W' \cup_{\lambda \in S'} e^3_\lambda \simeq Y \cup_{i=1}^\mu e^n_i \cup_{\lambda \in S} e^{n-1}_\lambda \cup_{\lambda \in S'} e^{n-2}_\lambda$.

By the van Kampen theorem, the fundamental group of $Y$ is still $G$, since $n > 4$. Denoting by $H_*(\_ ; R)$ the homology groups $H_*(\_ ; R)$, we have the following commutative diagram:

$$
\begin{array}{cccc}
\cdots \to H_3(\vee D^3, \vee S^2) & \to H_2(\vee S^2, pt) & \to H_2(\vee D^3, pt) & \to H_2(\vee D^3, \vee S^2) \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\cdots \to H_3(W, W') & \to H_2(W', X) & \to H_2(W, X) & \to H_2(W, W').
\end{array}
$$

By a five lemma argument and the assumption that $\alpha_* : H_1(\pi; R) \to H_1(G; R)$ is injective, for any integer $q \geq 2$, the relative homology group $H_q(W, X; R) = 0$. This shows for any integer $q \geq 2$, the homology groups $H_q(X; R) \cong H_q(W; R)$ and proves the isomorphism in (1).

**Remark 3.3.** From the proof, we can see that for some special group homomorphism $\alpha$ and coefficients $R$, the orientability or spin-ness of $X$ in Theorem 1.2 can be dropped. For example, when $\alpha$ is surjective and $\ker(\alpha) < [\pi, \pi]$, we do not need $X$ to orientable. When $\ker(\alpha)$ is perfect (or weakly $L$-perfect for some normal group), the spin-ness of $X$ can be dropped (cf. the proof of Theorem 4.1 and Theorem 5.2 in Guilbault and Tinsley [5]).
4 Applications

In this section, we give several applications of Theorem 1.2.

4.1 Surgery plus construction for manifolds

In this subsection, we get a manifold version of Quillen’s plus construction by doing surgery. The following proposition is a special case of Theorem 1.2 when \( \pi \) has a normally finitely generated normal perfect subgroup \( P \), \( G = \pi / P \) and \( R = \mathbb{Z}[G] \).

**Corollary 4.1.** Let \( n \geq 5 \) be an integer and \( M \) a closed \( n \)-dimensional spin manifold. Suppose that \( P \) is a normal perfect subgroup in \( \pi_1(M) \) normally generated by some finite elements. Then there exists a one-sided h-cobordism \((W; M, Y)\) such that \( \pi_1(W) = \pi_1(M)/P \). More precisely, there exists a closed spin manifold \( Y \) with the following properties:

(i) \( Y \) is obtained from \( M \) by attaching 2-handles and 3-handles, such that

(ii) \( \pi_1(Y) = \pi_1(M)/P \) and the inclusion map \( g : M \to W \), the cobordism between \( X \) and \( Y \), is Quillen’s plus construction inducing epimorphism \( \pi_1(M) \to \pi_1(M)/P \) of fundamental groups; and

(iii) \( Y \) has the homotopy type of Quillen’s plus construction \( M^+ \).

**Proof.** We apply Theorem 1.2. Let \( X = M \) and \( \alpha : \pi = \pi_1(M) \to \pi / P \) be the quotient map. By Hilton and Stammbach [12], there is an exact sequence

\[
H_2(\pi; \mathbb{Z}[\pi/P]) \to H_2(\pi/P; \mathbb{Z}[\pi/P]) \to \mathbb{Z}[\pi/P] \otimes_{\mathbb{Z}[\pi/P]} P_{ab} \to H_1(\pi; \mathbb{Z}[\pi/P]) \to H_1(\pi/P; \mathbb{Z}[\pi/P]) \to 0.
\]

When \( \mathbb{Z}[\pi/P] \otimes_{\mathbb{Z}[\pi/P]} P_{ab} \cong \mathbb{Z} \otimes_{\mathbb{Z}} P_{ab} = 0 \), we can see

\[
H_2(\pi; \mathbb{Z}[\pi/P]) \to H_2(\pi/P; \mathbb{Z}[\pi/P])
\]

is surjective and \( H_1(\pi; \mathbb{Z}[\pi/P]) \to H_1(\pi/P; \mathbb{Z}[\pi/P]) \) is an isomorphism. Therefore, the conditions of group homomorphism \( \alpha \) are satisfied. By Theorem 1.2, there exists a closed spin manifold \( Y \) and cobordism \((W; M, Y)\) such that for any integer \( q \geq 0 \), there is an isomorphism

\[
H_q(M; \mathbb{Z}[G]) \cong H_q(W; \mathbb{Z}[G]).
\]
This implies the inclusion map \( g : X \rightarrow W \) is the Quillen plus construction (cf. 4.3 xi in Berrick \[2\]). Therefore, for all integers \( q \), the relative cohomology groups \( H^q(W, X; \mathbb{Z}[G]) = 0 \). According to the Poincaré duality in Lemma 2.5 for each integer \( q \geq 0 \), the relative homology group \( H_q(W, Y; \mathbb{Z}[G]) = 0 \) and there is an isomorphism

\[
H_q(Y; \mathbb{Z}[G]) \cong H_q(W; \mathbb{Z}[G])
\]
as well. This means the universal covering spaces of \( Y \) and \( W \) are homology equivalent and therefore also homotopy equivalent. Since \( Y \) and \( W \) have the same fundamental group, this implies the inclusion map \( Y \rightarrow W \) is a homotopy equivalence. This finishes the proof.

\[\square\]

Corollary 4.1 was first obtained by Hausmann \[8\] (see also Hausmann \[9\] and Guilbault-Tinsley \[5, 6\]).

Recall the definition of \((\text{mod } L)\)-one-sided \( h \)-cobordism by Guilbault and Tinsley \[6\]. We see that a \((\text{mod } L)\)-one-sided \( h \)-cobordism is a one-sided \( \mathbb{Z}[\pi_1(Y)/L] \)-homology cobordism. The following result proved by Guilbault and Tinsley \[6\] is a corollary of Theorem 1.2 when \( R = \mathbb{Z}[\pi_1(Y)/L] \). Note that \( \mathbb{Z}[\pi_1(Y)/L] \) is a finitely \( \pi_1(Y) \)-dense ring.

**Corollary 4.2** (Guilbault-Tinsley \[6\], Theorem 5.2). Let \( B \) be a closed \( n \)-manifold \((n \geq 5)\) and \( \alpha : \pi_1(B) \rightarrow G \) a surjective homomorphism onto a finitely presented group such that \( \ker(\alpha) \) is strongly \( L' \)-perfect, i.e. \( \ker(\alpha) = [\ker(\alpha), L'] \) for some group \( L' \), where \( \ker(\alpha) \leq L' \leq \pi_1(B) \) and all loops representing elements in \( L' \) are orientation-preserving. Then for \( L = L'/\ker(\alpha) \), there exists a \((\text{mod } L)\)-one-sided \( h \)-cobordism \((W; B, A)\) such that \( \pi_1(W) = G \) and \( \ker(\pi_1(B) \rightarrow \pi_1(W)) = \ker(\alpha) \).

**Proof.** The proof is similar to that of Corollary 4.1. When \( \ker(\alpha') \) is \( L' \)-perfect, we have that \( \mathbb{Z}[G/L] \otimes_{\mathbb{Z}[G]} \ker(\alpha)_{\text{ab}} = 0 \). According to the 5-term exact sequence for group homology (cf. (8.2) in Hilton and Stammbach \[12\], page 202)

\[
H_2(\pi_1(B); \mathbb{Z}[G/L]) \xrightarrow{H_2(\alpha)} H_2(G; \mathbb{Z}[G/L]) \rightarrow \mathbb{Z}[G/L] \bigotimes_{\mathbb{Z}[G]} \ker(\alpha)_{\text{ab}}
\]

\[
\rightarrow H_1(\pi_1(B); k[G/L]) \xrightarrow{H_1(\alpha)} H_1(G; k[G/L]) \rightarrow 0,
\]
we can see that \( H_2(\alpha) \) is surjective and \( H_1(\alpha) \) is isomorphic. By Theorem 1.2 with \( R = \mathbb{Z}[G/L] \) and the remark followed, there exists a cobordism.
(W; A, B) such that \( \pi_1(B) = \pi_1(W) = G \) and the inclusion \( B \hookrightarrow W \) induces homology equivalence with coefficients \( R \). Considering the covering spaces of \( B \) and \( W \) with deck transformation group \( G/L \), we can see that the inclusion \( B \hookrightarrow W \) also induces a cohomology equivalence with coefficients \( R \). By Poincaré duality in Lemma 2.5, the inclusion \( A \hookrightarrow W \) induces homology equivalence with coefficients \( R \). This finishes the proof.

\[ \text{\[\square\]} \]

4.2 Surgery preserving integral homology groups

In this subsection, we study the case when the integral homology groups of a manifold are preserved by doing surgery. Corollary 1.3 is a special case of Theorem 1.2 when \( R = \mathbb{Z} \).

**Proof of Corollary 1.3.** We show that (ii) implies (i) first. By Hilton and Stammbach [12], there is an exact sequence

\[ H_2(\pi; \mathbb{Z}) \to H_2(\pi/N; \mathbb{Z}) \to N/\langle \pi, N \rangle \to H_1(\pi; \mathbb{Z}) \to H_1(\pi/N; \mathbb{Z}) \to 0. \]  

(3)

When \( N = \langle \pi, N \rangle \), we have that the map \( H_2(\pi; \mathbb{Z}) \to H_2(\pi/N) \) is surjective and \( H_1(\pi; \mathbb{Z}) \to H_1(\pi/N) \) is an isomorphism. According to Theorem 1.2 with \( R = \mathbb{Z} \), there exists a closed spin manifold \( Y \) obtained from \( X \) by adding 2-handles and 3-handles with \( \pi_1(Y) = \pi_1(X)/N \). Let \( W \) be the cobordism between \( X \) and \( Y \). Furthermore, we have for any \( q \geq 0 \) there is an isomorphism \( H_q(X; \mathbb{Z}) \cong H_q(W; \mathbb{Z}) \). According to the universal coefficients theorem, for all integers \( q \geq 0 \) the relative cohomology groups \( H^q(W, X; \mathbb{Z}) = 0 \). Therefore by Theorem 1.2 (ii), for any integer \( q \geq 0 \) there is an isomorphism

\[ H_q(Y; \mathbb{Z}) \cong H_q(X; \mathbb{Z}). \]

Conversely, suppose \( (W; X, Y) \) is a cobordism with the boundary \( X \) and \( Y \). Since \( Y \) is obtained from \( X \) by doing surgery below the middle dimension, we have isomorphisms \( H_1(Y; \mathbb{Z}) \cong H_1(W; \mathbb{Z}) \) and \( H_2(Y; \mathbb{Z}) \cong H_2(W; \mathbb{Z}) \cong H_2(X; \mathbb{Z}) \). Therefore the inclusion map \( X \to W \) induces an isomorphism

\[ H_1(\pi; \mathbb{Z}) = H_1(X; \mathbb{Z}) \cong H_1(W; \mathbb{Z}) = H_1(\pi_1(W); \mathbb{Z}). \]

According to the Hopf exact sequence (cf. Lemma 3.2), there is a commutative diagram

\[
\begin{array}{ccc}
H_2(X; \mathbb{Z}) & \to & H_2(\pi; \mathbb{Z}) \\
\downarrow & & \downarrow \\
H_2(W; \mathbb{Z}) & \to & H_2(\pi_1(Y); \mathbb{Z})
\end{array}
\]
where the left vertical map is an isomorphism. This shows that the right vertical map is an epimorphism. According to the same exact sequence (3) above, we have \( N = [\pi, N] \), which means \( N \) is relative perfect. Since \( \pi_1(X)/N = \pi_1(Y) \) is finitely presented, \( N \) is normally finitely generated by Lemma 3.1.

Corollary 1.3 is a manifold version of a result obtained by Rodríguez and Scevenels [19] for CW complexes.

### 4.3 The fundamental groups of homology manifolds

In this subsection, we study the fundamental groups of manifolds with the same homology type as a 2-connected manifold.

**Theorem 4.3.** Let \( G \) be a finitely presented group, \( R \) a subring of the rationals or the constant ring \( \mathbb{Z}/p \) (prime \( p \)) and \( n \geq 5 \) an integer. Suppose \( M \) is a 2-connected manifold of dimension \( n \). Then the following are equivalent:

(i) There exists an \( n \)-dimensional manifold \( Y \) obtained from \( M \) by adding 1-handles, 2-handles and 3-handles with \( \pi_1(Y) = G \) and for any integer \( q \geq 0 \), we have
\[
H_q(Y; R) \cong H_q(M; R);
\]

(ii) The group \( G \) is \( R \)-superperfect, i.e. \( H_1(G; R) = 0 \) and \( H_2(G; R) = 0 \).

**Proof.** We show that (i) implies (ii) first. By assumptions, we have \( H_2(M; R) = H_1(M; R) = 0 \). If some manifold \( Y \) has the homology groups with coefficients \( R \) as \( M \), then \( H_1(G; R) \cong H_1(M; R) = 0 \). According to the Hopf exact sequence in Lemma 3.2, we get \( H_2(G; R) = 0 \).

Conversely, suppose that \( G \) is a finitely presented group with \( H_2(G; R) = H_1(G; R) = 0 \). Let \( X = M, \pi = 1, \) the trivial group and \( f : \pi \to G \) the obvious group homomorphism. Note that the 2-connected manifold \( X \) is always a spin manifold and \( R \) is a principal ideal domain. According to Theorem 1.2, we get a manifold \( Y \) with \( \pi_1(Y) = G \) such that for any integer \( q \geq 0 \), there is an isomorphism \( H_q(W; R) \cong H_q(M; R) \) for the cobordism \( W \). According to the universal coefficient theorem, for all integers \( q \geq 0 \) the relative cohomology groups \( H^q(W, M; R) = 0 \). Therefore, for any integer \( q \geq 0 \), there is an isomorphism \( H_q(Y; R) \cong H_q(M; R) \) by Theorem 1.2 (ii). \( \square \)
Recall that an \( n \)-dimensional \( R \)-homology sphere is an \( n \)-dimensional manifold \( Y \) such that \( H_i(Y; R) = H_i(S^n; R) \) for any integer \( i \geq 0 \). The first part of the following result proved by Kervaire in [15] is a special case of Theorem 4.3 when \( M = S^n \) and \( R = \mathbb{Z} \).

**Corollary 4.4.** Let \( G \) be a finitely presented group and \( n \geq 5 \) be an integer. Then there exists an \( n \)-dimensional homology sphere \( Y \) with \( \pi_1(Y) = G \) if and only if \( G \) is superperfect, i.e. \( H_1(G; \mathbb{Z}) = 0 \) and \( H_2(G; \mathbb{Z}) = 0 \). Moreover, such manifolds can taken to be in the same cobordism class as \( S^n \).

Hausmann and Weinberger [10] constructed a superperfect group \( G \) for which any 4-manifold \( Y \) with \( \pi_1(Y) = G \) satisfies \( \chi(Y) > 2 \). As a consequence it follows that Theorem 4.3 and Corollary 4.4 do not extend to dimension four.

### 4.4 The fundamental groups of higher-dimensional knots

In this subsection, we study the fundamental groups of higher-dimensional knots. The following result proved by Kervaire [14] is a corollary of Theorem 1.2 (Recall the definition of weight \( w(G) \) from the Introduction).

**Corollary 4.5.** Given an integer \( n \geq 3 \), a finitely presentable group \( G \) is isomorphic to \( \pi_1(S^{n+2} - f(S^n)) \) for some differential embedding \( f : S^n \to S^{n+2} \) if and only if the first homology group \( H_1(G; \mathbb{Z}) = \mathbb{Z} \), the weight of \( G \) is 1 and the second homology group \( H_2(G; \mathbb{Z}) = 0 \).

**Proof.** Suppose for a finitely presentable group \( G \), we have \( H_1(G; \mathbb{Z}) = \mathbb{Z} \), \( H_2(G; \mathbb{Z}) = 0 \) and the weight \( w(G) = 1 \). Let \( X = S^{n+2} \), \( \pi = 1 \), the trivial group, and \( \alpha : \pi \to G \) the trivial group homomorphism. Notice that \( \alpha \) induces an injection of the first homology groups and surjection of the second homology groups. By Theorem 1.2 with \( R = \mathbb{Z} \), we get a closed manifold \( Y \) with \( \pi_1(Y) = G \), obtained from \( S^{n+2} \) by attaching 1-handles, 2-handles and 3-handles. Suppose that \( W \) is the surgery trace. By Poincaré duality, for any integer \( q \leq n + 1 \) the relative cohomology group \( H^q(W, Y) = 0 \). According to the universal coefficient theorem, we have that for each integer \( 2 \leq i \leq n + 1 \) there is an isomorphism

\[
H_i(Y) \cong H_i(X) = 0.
\]

Let \( \gamma \in G \) be an element such that \( G \) is normally generated by \( \gamma \) and \( \varphi : S^1 \to Y \) be a differential embedding representing \( \gamma \). Extend \( \varphi \) to be an embedding
\( \varphi' : S^1 \times D^{n+1} \to Y \). Do surgery to \( Y \) along \( \varphi' \) to get a manifold \( M \). It can be easily seen that \( M \) is simply connected and for each integer \( 1 \leq i \leq n \) the homology group \( H_i(M) = 0 \). Therefore, \( M \) is a \((n+2)\)-sphere by the solution of higher-dimensional Poincaré conjecture (note that all manifolds are assumed to be smooth). Let \( \phi : D^2 \times S^n \to M \) be the embedding and choose \( f = \phi(0,-) \) to be the embedding \( S^n \to M \). It can be directly checked that

\[
\pi_1(M - f(S^n)) \cong \pi_1(M - \phi(D^2 \times S^n)) \\
\cong \pi_1(Y - \varphi'(S^1 \times D^{n+1})) \cong \pi_1(Y) = G.
\]

This finishes the "if" part.

Conversely, suppose \( f : S^n \to S^{n+2} \) is a differential embedding. According to Alexander duality, we have \( H_2(S^{n+2} - f(S^n)) = 0 \) and \( H_1(S^{n+2} - f(S^n)) = H_1(G;\mathbb{Z}) = 0 \). By Hopf’s theorem in Lemma 3.2 (with the coefficient \( V = \mathbb{Z} \)), the second homology group \( H_2(G;\mathbb{Z}) = 0 \). Let \( \alpha : S^1 \to S^{n+2} - f(S^n) \) be an embedding such that \( \alpha(S^1) \) bounds a small 2-disc in \( S^{n+2} \) that intersects \( f(S^n) \) transversally at exactly one point. Then the group \( \pi_1(S^{n+2} - f(S^n)) \) is normally generated by the element represented by \( \alpha \). For more details, see the proof of Lemma 2 in Kervaire [14]. This proves the weight \( w(G) = 1 \) and finishes the proof.

Corollary 4.5 is Theorem 1 in Kervaire [14]. Similarly, we can show that Theorem 3 in [14] concerning the fundamental groups of links is also a corollary of Theorem 1.2. That is for an integer \( n \geq 3 \), a finitely presentable group \( G \) is isomorphic to \( \pi_1(S^{n+2} - L_k) \) for some \( k \) disjointly embedded \( n \)-spheres \( L_k \) if and only if \( H_1(G;\mathbb{Z}) = \mathbb{Z}^k \), \( w(G) = k \) and \( H_2(G;\mathbb{Z}) = 0 \) (cf. Theorem 3 in [14]). The proof is of the same pattern as that of Corollary 4.5 and will be left to the reader.

### 4.5 Zero-in-the-spectrum conjecture

In the notation of the Introduction, \( \text{zero not belonging to the spectrum of } \Delta = \Delta_* \) can also be expressed as the vanishing of \( H_*(M;C_*^*(\pi_1(M))) \). The following is a version of the zero-in-the-spectrum conjecture using homology. For more details, we refer the reader to the book of Lück [13].

**Conjecture 4.6.** Let \( M \) be a closed, connected, oriented and aspherical manifold with fundamental group \( \pi \). Then for some \( i \geq 0 \), \( H_i(X;C_*(\pi)) \neq 0 \).
If the condition that \( X \) is aspherical is dropped, the following corollary, which is a special case of Theorem 1.2 when \( R = C^*_R(G) \) and \( \pi = 1 \), shows the above conjecture is not true. This result is a generalization of the results obtained by Farber-Weinberger [4] and Higson-Roe-Schick [11]. Recall that the real \( C^* \)-algebra \( C^*_R(G) \) is a finitely \( G \)-dense ring.

**Corollary 4.7** (Higson-Roe-Schick [11]). Let \( G \) be a finitely presented group with the homology groups

\[
H_0(G; C^*_r(G)) = H_1(G; C^*_r(G)) = H_2(G; C^*_r(G)) = 0.
\]

For every integer \( n \geq 6 \) there is a closed manifold \( M \) of dimension \( n \) such that \( \pi_1(Y) = G \) and for each integer \( n \geq 0 \), the homology group \( H_n(Y; C^*_r(G)) = 0 \).

**Proof.** According to Proposition 4.8 in Ye [21], the vanishing of lower degree homology groups with coefficients \( C^*_r(G) \) is the same as that with coefficients \( C^*_R(G) \). Then we have

\[
H_0(G; C^*_R(G)) = H_1(G; C^*_R(G)) = H_2(G; C^*_R(G)) = 0.
\]

Note that the real \( C^* \)-algebra \( C^*_R(G) \) is a finitely \( G \)-dense ring. Let \( \alpha : \pi = 1 \to G, R = C^*_R(G) \) and \( X = S^n \) in Theorem 1.2. By the long exact sequence of homology groups

\[
\cdots \to H_1(1; C^*_R(G)) \to H_1(G; C^*_R(G)) \to H_1(G, 1; C^*_R(G)) \to H_0(1; C^*_R(G)) \to H_0(G; C^*_R(G)) \to 0,
\]

we have

\[
H_1(G, \pi; C^*_R(G)) = H_0(1; C^*_R(G)) = C^*_R(G),
\]

which is a free \( C^*_R(G) \)-module. Therefore, there exists a closed manifold \( Y \) by Theorem 1.2 such that for any integer \( 0 \leq q \leq \lfloor n/2 \rfloor \), the homology group \( H_q(Y; C^*_R(G)) = 0 \). According to the universal coefficients theorem and Poincaré duality for \( L^2 \)-homology (cf. Theorem 6.6 and Theorem 6.7 in Farber [3]), we get that for any integer \( q \geq 0 \), the homology group \( H_n(Y; C^*_R(G)) = 0 \).

\[\square\]

**Acknowledgements**
The author wants to thank Professor Wolfgang Lück for supporting from his Leibniz-Preis a visit to Hausdorff Center of Mathematics in University of Bonn from April 2011 to July 2011, when parts of this paper were written. He is also grateful to his advisor Professor A. J. Berrick for many helpful discussions.

References

[1] A.K. Bousfield, *The localization of spaces with respect to homology*, Topology 14 (1975), 133-150.

[2] A.J. Berrick, *An Approach to Algebraic K-Theory*, Pitman Research Notes in Math 56, (London 1982).

[3] M. Farber, *Homological algebra of Novikov-Shubin invariants and Morse inequalities*, GAFA 6 (1996), 628–665.

[4] M. Farber and S. Weinberger, *On the zero-in-the-spectrum conjecture*, Ann. of Math. (2) 154 (2001), 139-154.

[5] C.R. Guilbault and F.C. Tinsley, *Manifolds with non-stable fundamental groups at infinity, III*, Geometry and Topology, 10 (2006), 541-556.

[6] C.R. Guilbault and F.C. Tinsley, *Spherical alterations of handles: embedding the manifold plus construction*, Algebraic & Geometric Topology 13 (2013), 35-60.

[7] A. Hatcher, *Algebraic Topology*, Cambridge, New York, NY: Cambridge University Press, 2002.

[8] J-C. Hausmann, *Homological surgery*, Annals of Math. 104 (1976), 573-584.

[9] J-C. Hausmann, *Manifolds with a Given Homology and Fundamental Group*, Commentarii Mathematici Helvetici 53 (1978) (1), 113-134.

[10] J.-C. Hausmann and S. Weinberger, *Caractéristiques d'Euler et groupes fondamentaux des variétés de dimension 4*, Comment. Math. Helv. 60 (1985), 139-144.
[11] N. Higson, J. Roe and T. Schick, *Spaces with Vanishing $l^2$-Homology and their Fundamental Groups (after Farber and Weinberger)*, Geometriae Dedicata, 87 (2001), 335-343.

[12] P. Hilton and U. Stammbach, *A course in homological algebra*, 2nd ed.. Graduate Texts in Math., Springer, New York, 1997.

[13] W. Lück, *$L^2$-Invariants: Theory and Applications to Geometry and K-Theory*, Springer, 2002.

[14] M.A. Kervaire, *On higher dimensional knots*, in Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse) (edited by S.S.Cairns), Princeton University Press, Princeton (1965), 105-119.

[15] M. Kervaire, *Smooth homology spheres and their fundamental groups*, Transactions of the American Mathematical Society 144 (1969), 67-72.

[16] J.W. Milnor, *A procedure for killing the homotopy groups of differentiable manifolds*, Differential geometry, Proc. Symp. Pure Math. , 3 , Amer. Math. Soc. (1961) pp. 39-55.

[17] K. Ohshika, *Discrete Groups*, Translations of mathematical monographs 207, Amer. Math. Soc., 2002.

[18] A. Ranicki, *Algebraic and Geometric Surgery*, Oxford Mathematical Monograph (OUP), 2002.

[19] J.L. Rodriguez and D. Scevenels, *Homology equivalences inducing an epimorphism on the fundamental group and Quillen’s plus construction*. Proc. Amer. Math. Soc. 132 (2004), no. 3, 891-898.

[20] C.T.C. Wall, *Surgery on Compact Manifolds*. Academic Press, London, 1970. London Mathematical Society Monographs, No. 1.

[21] S. Ye, *A unified approach to the plus-construction, Bousfield localization, Moore spaces and zero-in-the-spectrum examples*. Israel Journal of Mathematics. 192 (2012), 699-717.

[22] S. Ye, Erratum to “A unified approach to the plus-construction, Bousfield localization, Moore spaces and zero-in-the-spectrum examples”. Israel Journal of Mathematics. DOI: 10.1007/s11856-012-0170-5.
Department of Mathematics, National University of Singapore, Kent Ridge 119076, Singapore.

Current address:

Mathematical Institute, University of Oxford, 24-29 St Giles’, Oxford, OX1 3LB, UK.

Email: Shengkui.Ye@maths.ox.ac.uk