WALL-CROSSINGS IN TORIC GROMOV–WITTEN THEORY II:
LOCAL EXAMPLES

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Abstract. In this paper we analyze six examples of birational transformations between toric orbifolds: three crepant resolutions, two crepant partial resolutions, and a flop. We study the effect of these transformations on genus-zero Gromov–Witten invariants, proving the Coates–Corti–Iritani–Tseng/Ruan form of the Crepant Resolution Conjecture in each case. Our results suggest that this form of the Crepant Resolution Conjecture may also hold for more general crepant birational transformations. They also suggest that Ruan’s original Crepant Resolution Conjecture should be modified, by including appropriate “quantum corrections”, and that there is no straightforward generalization of either Ruan’s original Conjecture or the Cohomological Crepant Resolution Conjecture to the case of crepant partial resolutions. Our methods are based on mirror symmetry for toric orbifolds.

1. Introduction

Suppose that $\mathcal{X}$ is an algebraic orbifold and that $\mathcal{Y}$ is an orbifold or algebraic variety which is birational to $\mathcal{X}$. It is natural to try to understand the relationship between the quantum cohomology of $\mathcal{X}$ and that of $\mathcal{Y}$. In this paper we analyze six examples of this situation — three crepant resolutions, two crepant partial resolutions, and a flop — which together exhibit some of the range of phenomena which can occur. Our methods are based on mirror symmetry for toric orbifolds.

Small quantum cohomology is a family of algebras depending on so-called quantum parameters. The quantum parameters $u_1, \ldots, u_s$ for $\mathcal{X}$ correspond to a choice of basis for $H^2(\mathcal{X}; \mathbb{Q})$, which we take to be primitive integer vectors on the rays of the Kähler cone for $\mathcal{X}$; the quantum parameters $q_1, \ldots, q_r$ for $\mathcal{Y}$ correspond, similarly, to a choice of basis for $H^2(\mathcal{Y}; \mathbb{Q})$. If $\mathcal{Y} \to \mathcal{X}$ is a crepant resolution (or partial resolution) of the coarse moduli space $\mathcal{X}$ then there is a natural embedding $j : H^2(\mathcal{X}; \mathbb{Q}) \to H^2(\mathcal{Y}; \mathbb{Q})$ which identifies the Kähler cone for $\mathcal{X}$ with a face of the Kähler cone for $\mathcal{Y}$. The embedding $j$ does not in general identify the integer lattices in $H^2(\mathcal{X}; \mathbb{Q})$ and $H^2(\mathcal{Y}; \mathbb{Q})$, but nonetheless we can choose bases such that $q_i \leftrightarrow u_i^r$, $1 \leq i \leq s$, for some positive rational numbers $r_i$.

An influential conjecture of Ruan asserts that if $\mathcal{Y} \to \mathcal{X}$ is a crepant resolution then there are roots of unity $\omega_i$, $1 \leq i \leq r$, and a choice of path of analytic continuation such that the algebra obtained from the small quantum cohomology of $\mathcal{Y}$ by analytic continuation in the parameters $q_i$ followed by the change of variables

$$q_i = \begin{cases} \omega_i u_i^1 & 1 \leq i \leq s \\ \omega_i & s < i \leq r \end{cases}$$

is isomorphic to the small quantum cohomology of $\mathcal{X}$. One consequence of this is the Cohomological Crepant Resolution Conjecture (CCRC) \cite{Coates}, which asserts that the algebra obtained from the small quantum cohomology of $\mathcal{Y}$ by analytic continuation in the parameters $q_i$ followed by the change of variables

$$q_i = \begin{cases} 0 & 1 \leq i \leq s \\ \omega_i & s < i \leq r \end{cases}$$

is isomorphic to the Chen–Ruan orbifold cohomology algebra of $\mathcal{X}$. An extension of Ruan’s Conjecture proposed by Bryan–Graber \cite{BryanGraber} asserts that if $\mathcal{X}$ satisfies a Hard Lefschetz condition on Chen–Ruan cohomology (a condition whose necessity was first suggested in \cite{BryanGraber}) then the big quantum cohomology algebras of $\mathcal{X}$ and $\mathcal{Y}$ coincide, after analytic continuation in the $q_i$ and the change of variables \eqref{eq:change_of_variables}, via a linear isomorphism which identifies the orbifold Poincaré pairing on $\mathcal{X}$ with the Poincaré pairing on $\mathcal{Y}$.

These conjectures have been verified in a number of examples \cite{BryanGraber, BryanGraber2, BryanGraber3, BryanGraber4, BryanGraber5, BryanGraber6, BryanGraber7, BryanGraber8}. In recent joint work with Corti, Iritani, and Tseng \cite{Coates2} we proposed a rather different picture of the relationship between the Gromov–Witten theory of $\mathcal{X}$ and that of $\mathcal{Y}$. Our conjecture was phrased in terms of Givental’s symplectic formalism \cite{Givental1, Givental2}. Genus-zero Gromov–Witten invariants of $\mathcal{X}$ (and respectively

\footnote{Similar ideas occurred in unpublished work of Ruan; an expository account can be found in \cite{Ruan}.}
We prove the Coates–Corti–Iritani–Tseng/Ruan Crepant Resolution Conjecture in each case. This has restrictive conditions such that after analytic continuation we have we conjectured the existence of a linear symplectic isomorphism mirror Landau–Ginzburg model, the variation of semi-infinite Hodge structure \[5, 39, 40\] associated to it, modest goals, and is meant to illustrate four points. First, these questions are in his discussion, it is natural to ask: “what is the point of this paper?” The discussion here has quite and the mirror theorem for toric Deligne–Mumford stacks \[21\]. Since all of our examples are included ruling out every possible choice of path of analytic continuation and all choices of roots of unity, but I expect that wherever there is a “?” in this table, the corresponding conjecture fails to hold, so that for example the original form of Ruan’s Conjecture fails in Example IV and the modified form of Ruan’s Conjecture fails in Example V. I expect also that the conclusion of the Bryan–Graber Conjecture fails to hold in every case except Example II. It is difficult to prove these assertions, as this would involve to hold in every case except Example II. It is difficult to prove these assertions, as this would involve

Thus we get a “quantum corrected” version of Ruan’s original conjecture. In this paper we consider six examples:

| Example | Conjecture |
|---------|------------|
| I       | ✓          |
| II      | ✓          |
| III     | ✓          |
| IV      | ✓          |
| V       | ✓          |
| VI      | ✓          |

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In forthcoming work, Iritani will prove our form of the Crepant Resolution Conjecture for all crepant birational transformations between toric Deligne–Mumford stacks. His method uses the full force of the mirror Landau–Ginzburg model, the variation of semi-infinite Hodge structure \[5, 39, 40\] associated to it, and the mirror theorem for toric Deligne–Mumford stacks \[21\]. Since all of our examples are included in his discussion, it is natural to ask: “what is the point of this paper?” The discussion here has quite modest goals, and is meant to illustrate four points. Firstly, these questions are not difficult. If \(X\) is a toric orbifold \(X\) and \(Y \to X\) is a crepant resolution then the relationship between the quantum cohomology of \(X\) and that of \(Y\) can be determined systematically, using well-understood methods from toric mirror symmetry. Secondly, our form of the Crepant Resolution Conjecture may also hold, without significant change, for more general crepant birational transformations: we see this here for two crepant partial resolutions and a flop. Thirdly, the method of proof described here also applies without change to the more general crepant toric situation. Finally, it seems likely that no naïve modification of Ruan’s original conjecture holds true; we discuss this further in the next paragraph. Along the way, we will see two things which were perhaps already obvious: that Givental-style mirror theorems are well-adapted to

\[\mathcal{L} \subset \mathcal{H}_{\mathcal{Y}}\). As \(\mathcal{L}_X\) and \(\mathcal{L}_Y\) are germs of submanifolds it makes sense to analytically continue them, and we conjectured the existence of a linear symplectic isomorphism \(U : \mathcal{H}_X \to \mathcal{H}_Y\) satisfying some quite restrictive conditions such that after analytic continuation we have \(U(\mathcal{L}_X) = \mathcal{L}_Y\). We also proved our conjecture when \(X\) is one of the weighted projective spaces \(\mathbb{P}(1,1,2)\) or \(\mathbb{P}(1,1,1,3)\) and \(Y \to X\) is a crepant resolution.

Our conjecture has consequences for quantum cohomology: it implies the Bryan–Graber Conjecture, the Cohomological Crepant Resolution Conjecture, and a modified version of Ruan’s Conjecture, each with the caveat that we must allow the quantities \(\omega_i\) to be arbitrary constants rather than roots of unity. (In the examples below the \(\omega_i\) turn out to be roots of unity and so the caveat disappears; Iritani has suggested an attractive conceptual reason for this to be true in general \[39\].) The modified version of Ruan’s Conjecture has an additional hypothesis, that \(X\) be semi-positive, and replaces the change of variables \(u_i = f_i(u_1, \ldots, u_s)\) where

\[
f_i(u_1, \ldots, u_r) = \begin{cases} \omega_i u_i^{r_i} + \text{higher order terms in } u_1, \ldots, u_r & 1 \leq i \leq s \\ \omega_i + \text{higher order terms in } u_1, \ldots, u_r & s < i \leq r. \end{cases}
\]

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the analysis of toric birational transformations, and that the methods of [18] are applicable to the (local) Calabi–Yau examples which are of greatest interest to physicists [3].

The original conjectures of Ruan and of Bryan–Graber have an attractive simplicity, and one might therefore ask whether our formulation of the Crepant Resolution Conjecture is unnecessarily complicated and whether some simpler statement holds [14]. The examples below constitute some evidence that the answer to these questions is “no”. In Example IV below we see that quantum corrections to Ruan’s original conjecture are probably necessary, and in Example V the situation is even worse: there is probably not even a generalization of the Cohomological Crepant Resolution Conjecture to partial resolutions which involves only small (rather than big) quantum cohomology. This is related to the absence of a Divisor Equation for degree-two classes from the twisted sectors, and is discussed further in Section 8.

Conventions. We will assume that the reader is familiar with the Gromov–Witten theory of orbifolds. This theory is constructed in [12] [15] [16]: a rapid overview can be found in [22, Section 2]. We work in the algebraic category, so for us “orbifold” means “smooth algebraic Deligne–Mumford stack over \( \mathbb{C} \).” All of our examples are non-compact, but they carry the action of a torus \( T = \mathbb{C}^\times \) such that the \( T \)-fixed locus is compact. We therefore work throughout with \( T \)-equivariant Gromov–Witten invariants, which in this setting behave much as the Gromov–Witten invariants of compact orbifolds (see e.g. [11]), and with \( T \)-equivariant Chen–Ruan orbifold cohomology. We always take the product of \( T \)-equivariant Chen–Ruan classes using the Chen–Ruan product; when we want to emphasize this, we will write the product as \( \otimes_{\text{CR}} \). The degree of a Chen–Ruan class always means its orbifold or age-shifted degree.

An expository account of our Crepant Resolution Conjecture and its consequences can be found in [25]. The reader should take care when comparing the discussion in this paper with those in [11] [25], as here we measure the degrees of orbifold curves using a basis of degree-two cohomology classes chosen as above, whereas there the authors use a so-called positive basis for \( H_\ast \). Our choice of degree conventions fits well with toric geometry, and this will be important below, but we pay a price for our choice: the presence of the rational numbers \( r \) fits well with toric geometry, and this will be important below, but we pay a price for our choice: the presence of the rational numbers \( r \) described above.

Outline of the Paper. In Section 2 we fix notation and give a precise description of the conjecture which we will prove. In Section 3 we collect various preparatory results, as well as describing how our conjecture implies versions of Ruan’s Conjecture, the Bryan–Graber Conjecture, and the Cohomological Crepant Resolution Conjecture. Examples I–VI are in Sections 4–9 respectively, and we conclude with an Appendix in which we compute various genus-zero Gromov–Witten invariants.

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2. Statement of the Conjecture

In this section we give a precise statement of the conjecture that we will prove. Before we do so, we describe our general setup and fix notation.

General Setup. Let \( \mathcal{X} \) be a Gorenstein orbifold with projective coarse moduli space \( X \) and let \( \pi: Y \to X \) be a crepant resolution. Assume that \( \mathcal{X}, X, \) and \( Y \) carry actions of a torus \( T = \mathbb{C}^\times \) such that both \( \pi \) and the structure map \( \mathcal{X} \to X \) are \( T \)-equivariant and such that the \( T \)-fixed loci on \( \mathcal{X} \) and \( Y \) are compact. Let \( \mathbb{C}[\lambda] \) denote the \( T \)-equivariant cohomology of a point, where \( \lambda \) is the first Chern class of the line bundle \( O(1) \to \mathbb{C}P^\infty \), and let \( \mathbb{C}(\lambda) \) be its field of fractions. Write \( H(\mathcal{X}) := H^\ast_{\text{CR,CR}}(\mathcal{X}; \mathbb{C}) \otimes \mathbb{C}(\lambda) \) for the localized \( T \)-equivariant Chen–Ruan orbifold cohomology of \( \mathcal{X} \), and \( H(Y) := H^\ast_{\text{CR}}(Y; \mathbb{C}) \otimes \mathbb{C}(\lambda) \) for the localized \( T \)-equivariant cohomology of \( Y \). We work throughout over the field \( \mathbb{C}(\lambda) \). The \( \mathbb{C}(\lambda) \)-vector spaces \( H(\mathcal{X}) \) and \( H(Y) \) carry non-degenerate inner products, given by

\[
(\alpha, \beta)_X := \int_{\mathcal{X}^T} i^\ast(\alpha \cup I^\ast \beta) e(N_{\mathcal{X}^T/\mathcal{X}})
\quad\text{and}\quad
(\alpha, \beta)_Y := \int_{Y^T} j^\ast(\alpha \cup \beta) e(N_{Y^T/Y})
\]

where \( I \) is the canonical involution on the inertia stack \( \mathcal{I}X \) of \( \mathcal{X} \); \( i : \mathcal{I}X^T \to \mathcal{I}X \) and \( j : Y^T \to Y \) are the inclusions of the \( T \)-fixed loci in \( \mathcal{I}X \) and \( Y \) respectively; \( N_{\mathcal{X}^T/\mathcal{X}} \) and \( N_{Y^T/Y} \) are the normal bundles to the \( T \)-fixed loci; and \( e \) is the \( T \)-equivariant Euler class. Note that the \( T \)-equivariant Euler classes are invertible over \( \mathbb{C}(\lambda) \).
The Symplectic Vector Space. In what follows write $Z$ for either $X$ or $Y$, and write $Z$ for the coarse moduli space of $Z$ (i.e. for either $X$ or $Y$). Introduce the symplectic vector space

$$\mathcal{H}_Z := H(Z) \otimes \mathbb{C}(z^{-1})$$

the vector space

$$\Omega_Z(f, g) := \text{Res}_{z=0} \left( f(-z), g(z) \right) dz$$

the symplectic form

and set $\mathcal{H}_Z := H(Z) \otimes \mathbb{C}[z]$, $\mathcal{H}_Z := z^{-1}H(Z) \otimes \mathbb{C}[z^{-1}]$. The polarization $\mathcal{H}_Z = \mathcal{H}_Z^+ \oplus \mathcal{H}_Z^-$ identifies $\mathcal{H}_Z$ with the cotangent bundle $T^*\mathcal{H}_Z$. We regard $\mathcal{H}_Z$ as a graded vector space where $\deg z = 2$.

Degrees and Novikov Variables. Fix a basis $\omega_1, \ldots, \omega_s$ for $H^2(X; \mathbb{Q})$ consisting of primitive integer vectors on the rays of the Kähler cone for $X$, and a basis $\omega'_1, \ldots, \omega'_s$ for $H^2(Y; \mathbb{Q})$ consisting of primitive integer vectors on the rays of the Kähler cone for $Y$. Note that $H^2(X; \mathbb{Q})$ is canonically isomorphic to $H^2(X; \mathbb{Q})$, so we can regard $\omega_1, \ldots, \omega_s$ as cohomology classes on $X$, and in our situation we can always insist that $\pi^*\omega_i = r_i\omega'_i$, $1 \leq i \leq s$, for some rational numbers $r_i$. We measure the degrees of orbifold curves using the bases $\omega_i$ and $\omega'_i$. Recall that a stable map $f : C \to Z$ from an orbifold curve to $Z$ has a well-defined degree in the free part

$$H_2(Z; \mathbb{Z})_{\text{free}} := H_2(Z; \mathbb{Z})/H_2(Z; \mathbb{Z})_{\text{tors}}$$

of $H_2(Z; \mathbb{Z})$; we write Eff$(Z) \subset H_2(Z; \mathbb{Z})_{\text{free}}$ for the set of degrees of stable maps from orbifold curves to $Z$. Given an element $d \in \text{Eff}(Z)$, set $d_i = \langle d, \omega_i \rangle$ if $Z = X$ and $d_i = \langle d, \omega'_i \rangle$ if $Z = Y$. Note that the $d_i$ here are in general rational numbers. Define $Q^d := Q_1^d \cdots Q_s^d$ where $d \in \text{Eff}(X)$ and $Q^d := Q_1^d \cdots Q_s^d$, where $d' \in \text{Eff}(Y)$. Here $Q_1, Q_2, \ldots$ are formal variables called Novikov variables; the number of Novikov variables associated with $Z$ is $b_2(Z)$, the second Betti number of $Z$.

Bases and Darboux Co-ordinates. We fix $\mathbb{C}(\lambda)$-bases $\phi_0, \ldots, \phi_N$ and $\phi^0, \ldots, \phi^N$ for $H(X)$ such that

(a) $\phi_0$ is the identity element $1_X \in H(X)$;
(b) $\phi_1, \phi_2, \ldots, \phi_s$ are lifts to $T$-equivariant cohomology of $\omega_1, \omega_2, \ldots, \omega_s$;
(c) $(\phi_1, \phi^1)_X = \delta_i^j$;
and $\mathbb{C}(\lambda)$-bases $\varphi_0, \ldots, \varphi_N$ and $\varphi^0, \ldots, \varphi^N$ for $H(Y)$ such that

(d) $\varphi_0$ is the identity element $1_Y \in H(Y)$;
(e) $\varphi_1, \varphi_2, \ldots, \varphi_s$ are lifts to $T$-equivariant cohomology of $\omega'_1, \omega'_2, \ldots, \omega'_s$;
(f) $(\varphi_1, \varphi^1)_Y = \delta_i^j$.

Conditions (b) and (e) here will be useful below when we discuss the Divisor Equation. Write

$$\Phi_i = \begin{cases} \phi_i & \text{if } Z = X \\ \varphi_i & \text{if } Z = Y \end{cases} \quad \text{and} \quad \Phi^i = \begin{cases} \phi^i & \text{if } Z = X \\ \varphi^i & \text{if } Z = Y \end{cases}$$

Then

$$\sum_{k \geq 0} q^k \Phi_{\alpha} z^k + \sum_{l \geq 0} p_{\beta, l} \Phi^l (-z)^{-1-l}$$

gives a Darboux co-ordinate system $\{q_{n,k}, p_{n,l}\}$ on $\mathcal{H}_Z$; here and henceforth we use the summation convention on Greek indices, summing repeated Greek (but not Roman) indices over the range $0, 1, \ldots, N$.

Gromov–Witten Invariants. We use correlator notation for $T$-equivariant Gromov–Witten invariants of $Z$, writing

$$\langle \alpha_1 \psi_1^1, \ldots, \alpha_n \psi_1^n \rangle^Z_{0,n,d} = \int_{[Z_{0,n,d}]^{vir}} \prod_{k=1}^n \text{ev}^*_k(\alpha_k) \cdot \psi_k^{i_k}$$

where $\alpha_1, \ldots, \alpha_n$ are elements of $H(Z)$ and $i_1, \ldots, i_n$ are non-negative integers. The cohomology classes $\psi_1, \ldots, \psi_n$ here are the first Chern classes of the universal cotangent line bundles on the moduli space $Z_{0,n,d}$ of genus-zero $n$-pointed stable maps to $Z$ of degree $d \in \text{Eff}(Z)$. The integral denotes the cap product with the $T$-equivariant virtual fundamental class of $Z_{0,n,d}$; we discuss this further in the next paragraph. The right-hand side of equation (3) is defined in §8.3 of [2], where it is denoted $\langle \tau_{i_1} (\alpha_1), \ldots, \tau_{i_n} (\alpha_n) \rangle_{0,d}$; our choice of notation allows compact expressions for many important quantities, such as

$$\left\langle \frac{\alpha}{z - \psi} \right\rangle^Z_{0,1,d}$$

as correlators are multilinear in their entries.
Twisted Gromov–Witten Invariants. In the examples we consider below, \( \mathcal{Z} \) will be the total space of a concave vector bundle \( \mathcal{E} \) over a compact orbifold (or manifold) \( \mathcal{B} \), and the \( T \)-action on \( \mathcal{Z} \) will rotate the fibers of \( \mathcal{E} \) and cover the trivial action on \( \mathcal{B} \). That \( \mathcal{E} \) is concave means that \( H^0(\mathcal{C}, f^*\mathcal{E}) = 0 \) for all stable maps \( f : \mathcal{C} \to \mathcal{B} \) of non-zero degree. This implies that stable maps to \( \mathcal{E} \) of non-zero degree all land in the zero section and so, for \( d \neq 0 \), the moduli space \( \mathcal{Z}_{0,n,d} \) coincides as a scheme with \( \mathcal{B}_{0,n,d} \). The natural obstruction theories on \( \mathcal{Z}_{0,n,d} \) and \( \mathcal{B}_{0,n,d} \) differ, though, and the \( T \)-equivariant virtual fundamental classes satisfy

\[
[\mathcal{Z}_{0,n,d}]^{\text{vir}} = [\mathcal{B}_{0,n,d}]^{\text{vir}} \cap e(\text{Ob}_0, n, d)
\]

where \( e \) is the \( T \)-equivariant Euler class and \( \text{Ob}_0, n, d \) is the vector bundle over \( \mathcal{B}_{0,n,d} \) with fiber at a stable map \( f : \mathcal{C} \to \mathcal{B} \) equal to \( H^1(\mathcal{C}, f^*\mathcal{E}) \). Thus

\[
\int_{[\mathcal{Z}_{0,n,d}]^{\text{vir}}} (\cdots) = \int_{[\mathcal{B}_{0,n,d}]^{\text{vir}}} (\cdots) \cup e(\text{Ob}_0, n, d).
\]

This means that Gromov–Witten invariants of \( \mathcal{Z} \) coincide with twisted Gromov–Witten invariants \( [19, 23] \) of \( \mathcal{B} \) where the twisting characteristic class is the inverse \( T \)-equivariant Euler class \( e^{-1} \) and the twisting bundle is \( \mathcal{E} \): this is explained in detail in \([19]\). Results of \([19]\) allow us to compute these twisted Gromov–Witten invariants in terms of the ordinary Gromov–Witten invariants of \( \mathcal{B} \), a fact which we exploit repeatedly below.

In the exceptional case \( d = 0 \), the moduli space \( \mathcal{Z}_{0,n,d} \) is non-compact and so we need to say what we mean by the integral in \([3]\). Since \( \mathcal{Z}_{0,n,d} \) carries a \( T \)-action with compact fixed set, we can define the integral using the virtual localization formula of Graber–Pandharipande \([35]\); note that we could do this in the case \( d \neq 0 \), too, and this would reproduce the definition which we just gave.

Gromov–Witten Potentials. The genus-zero Gromov–Witten potential \( F^0_\mathcal{Z} \) is a generating function for certain genus-zero Gromov–Witten invariants of \( \mathcal{Z} \). It is a formal power series in variables \( \tau^a \), \( 0 \leq a \leq N \), and the Novikov variables \( Q_i \), \( 1 \leq i \leq b_2(\mathcal{Z}) \), defined by

\[
F^0_\mathcal{Z} = \sum_{n \geq 0} \sum_{\tau^a \in \text{Eff}(\mathcal{Z})} \frac{Q^n}{n!} \langle \tau^1, \ldots, \tau^n \rangle^{\mathcal{Z}}_{0,n,d}
\]

where \( \tau = \tau^a \phi_a \). Since correlators are multilinear, the expression \( \langle \tau, \tau, \ldots, \tau \rangle^{\mathcal{Z}}_{0,n,d} \) expands into a polynomial in the variables \( \tau^a \). The second summation here is over the set \( \text{Eff}(\mathcal{Z}) \) of degrees of maps from orbifold curves to \( \mathcal{Z} \).

The genus-zero descendant potential \( F^0_{\mathcal{Z}} \) is a generating function for all genus-zero Gromov–Witten invariants of \( \mathcal{Z} \). It is a formal power series in variables \( t_k^a \), \( 0 \leq a \leq N \), \( 0 \leq k < \infty \), and the Novikov variables \( Q_i \), \( 1 \leq i \leq b_2(\mathcal{Z}) \), defined by

\[
F^0_{\mathcal{Z}} = \sum_{n \geq 0} \sum_{t^a_k \in \text{Eff}(\mathcal{Z})} \sum_{t_k^a < \infty} \frac{Q^n}{n!} \langle t_k^1 \psi^{k_1}, \ldots, t_k^a \psi^{k_a} \rangle^{\mathcal{Z}}_{0,n,d}
\]

where \( t_k = t_k^a \phi_a \). The expression \( \langle t_k^1 \psi^{k_1}, \ldots, t_k^a \psi^{k_a} \rangle^{\mathcal{Z}}_{0,n,d} \) here expands, by multilinearity again, into a polynomial in the variables \( t_k^a \).

Analytic Continuation. Let us call the coefficient in \( F^0_\mathcal{Z} \) of any monomial \( \tau^{a_1} \cdots \tau^{a_n} \) a coefficient series of \( F^0_\mathcal{Z} \), and call the coefficient in \( F^0_\mathcal{Z} \) of any monomial \( t_{k_1}^{a_1} \cdots t_{k_a}^{a_a} \) a coefficient series of \( F^0_\mathcal{Z} \). Each coefficient series is a formal power series in the Novikov variables \( Q_i \), \( 1 \leq i \leq b_2(\mathcal{Z}) \). All of the examples we consider below satisfy:

(A) each coefficient series of \( F^0_\mathcal{Z} \) converges in a neighbourhood of \( Q_1 = Q_2 = \cdots = 0 \) to an analytic function of the \( Q_i \); and

(B) the coefficient series of \( F^0_\mathcal{Z} \) admit simultaneous analytic continuation to a neighbourhood of \( Q_1 = Q_2 = \cdots = 1 \).

Condition (A) implies that each coefficient series of \( F^0_\mathcal{Z} \) converges in a neighbourhood of \( Q_1 = Q_2 = \cdots = 0 \) to an analytic function of the \( Q_i \), and condition (B) implies that the coefficient series of \( F^0_\mathcal{Z} \) also admit simultaneous analytic continuation to a neighbourhood of \( Q_1 = Q_2 = \cdots = 1 \); see \([24]\) Appendix for discussion of a closely-related point.

In what follows we will assume that a simultaneous analytic continuation of the coefficient series has been chosen, and will set \( Q_1 = Q_2 = \cdots = 1 \) throughout. Thus we regard the genus-zero Gromov–Witten
potential as a formal power series
\[
F^0_Z = \sum_{n \geq 0} \sum_{d \in \text{Eff}(Z)} \frac{1}{n!} \langle \tau, \tau, \ldots, \tau \rangle^n_{0,n,d} \tag{6}
\]
in the variables \( \tau^n, 0 \leq a \leq N \), and we regard the genus-zero descendant potential as a formal power series
\[
F^0_Z = \sum_{n \geq 0} \sum_{0 \leq k_n < \infty} \sum_{d \in \text{Eff}(Z)} \frac{1}{n!} \langle t_k \psi^{k_1}, \ldots, t_k \psi^{k_n} \rangle^n_{0,n,d} \tag{7}
\]
in the variables \( t_k^a, 0 \leq a \leq N, 0 \leq k < \infty \).

**The Divisor Equation.** The reader might worry that by suppressing Novikov variables — *i.e.* by setting \( Q_1 = Q_2 = \cdots = 1 \) — we have lost some information about the degrees of curves. This is not the case. We will discuss this for the case \( Z = Y \); the case \( Z = \mathcal{X} \) is entirely analogous. Recall that our basis \( \varphi_0, \ldots, \varphi_N \) for \( H(Y) \) was chosen so that \( \varphi_1, \ldots, \varphi_r \) is a lift to \( T \)-equivariant cohomology of the basis \( \omega'_1, \ldots, \omega'_r \) for \( H^2(Y; \mathbb{C}) \) with which we measure the degrees of curves. Then, writing
\[
\tau = \tau^0 \varphi, \quad \tau_{\text{rest}} = \tau^0 \varphi_0 + \tau^r \varphi_{r+1} + \tau^r \varphi_{r+2} + \cdots + \tau^N \varphi_N,
\]
the Divisor Equation\(^2\) gives
\[
F^0_Y = \frac{1}{6} (\tau \cup \tau, \tau) + \sum_{n \geq 0} \sum_{d \neq 0} \frac{e^d \tau^i \cdots e^d \tau^r}{n!} \langle \tau_{\text{rest}} \rangle^n_{0,n,d} \tag{8}
\]
and so the substitution
\[
e^\tau \mapsto Q_\epsilon e^\tau, \quad 1 \leq i \leq r,
\]
turns (6) into (8). The story for the descendant potential \( F^0_Y \) is a little more complicated — it is discussed, for instance, in [15, Remark 5.3] — but the upshot is the same: the Divisor Equation allows us to recover (8) from (7).

**The Lagrangian Submanifold-Germ.** Following Givental [23, 33, 34] we encode all genus-zero Gromov–Witten invariants of \( Z \) via the formal germ of a Lagrangian submanifold of \( \mathcal{H}^\tau_Z \), defined as follows. Regard the genus-zero descendant potential \( F^0_Z \) as the formal germ of a function on \( \mathcal{H}^\tau_Z \) via the change of variables
\[
q^a_k = \begin{cases} e^0 - 1 & \text{if } (a, k) = (0, 1) \\ e^k & \text{otherwise.} \end{cases}
\]
This change of variables is called the dilaton shift. The variables \( q^a_k \) here are the Darboux co-ordinates from (2), so a general point on \( \mathcal{H}^\tau_Z \) is \( \sum_{k \geq 0} q^a_k \Phi^a z^k \). The dilaton shift makes \( F^0_Z \) into the formal germ at \( -z \) of a function on \( \mathcal{H}^\tau_Z \). The graph of the differential of \( F^0_Z \) therefore defines the formal germ of a submanifold of \( \mathcal{H}_Z \cong T^* \mathcal{H}_Z^\tau \), defined by the equations
\[
p^a_k = \frac{\partial F^0_Z}{\partial q^a_k} \quad 0 \leq a \leq N, 0 \leq k < \infty. \tag{8}
\]
We denote this Lagrangian submanifold-germ by \( \mathcal{L}_Z \).

\(^2\)This is the identity
\[
\langle \alpha_1 \psi^{i_1}, \ldots, \alpha_n \psi^{i_n}, \gamma \rangle^n_{0,n+1,d} + \sum_{j_1 \leq j_2 \leq \gamma, i_j > 0} \langle \alpha_1 \psi^{i_1}, \ldots, \alpha_{j-1} \psi^{i_{j-1}}, (\gamma \cup \alpha_j) \psi^{i_j} \rangle^n_{0,n,d} \tag{8}
\]
where \( \gamma \in H^2(Z; \mathbb{C}) \) and either \( d \neq 0 \) or \( n \geq 3 \).
More Analytic Continuation. In what follows we will need to analytically continue the submanifold-germ $L_Z$, and to analytically continue $L_Z$ we will analytically continue each partial derivative $\frac{\partial x^n}{\partial q_k}$ in the variable $t_0^n$, $0 \leq a \leq b_2(Z)$. The partial derivative $\frac{\partial x^n}{\partial q_k}$ is a formal power series in the variables $t_1^n$, $0 \leq b \leq N$, $0 \leq l < \infty$, so we write it in the form

$$\sum_{l} f_l t^l$$

$t^l$ a monomial in the variables $t_1^n$ with $b > b_2(Z)$ or $l > 0$, $f_l$ a formal power series in the variables $t_0^n$, $0 \leq a \leq b_2(Z)$, and then analytically continue each $f_l$.

The Crepant Resolution Conjecture. We are now ready to state the conjecture.

Conjecture 2.1. There is a degree-preserving $\mathbb{C}(\langle z^{-1}\rangle)$-linear symplectic isomorphism $\mathbb{U} : \mathcal{H}_X \rightarrow \mathcal{H}_Y$ and a choice of analytic continuations of $L_X$ and $L_Y$ such that $\mathbb{U}(L_X) = L_Y$. Furthermore, $\mathbb{U}$ satisfies:

(a) $\mathbb{U}(1_X) = 1_Y + O(z^{-1})$;
(b) $\mathbb{U} \circ (\rho \cup_{cn}) = (\pi^* \rho \cup) \circ \mathbb{U}$ for every untwisted degree-two class $\rho \in H^2(X, \mathbb{C})$;
(c) $\mathbb{U}(H_X^0) = H_Y$.

This is a slight modification of a conjecture due to Coates, Corti, Iritani, and Tseng [15]; very similar ideas occurred, simultaneously and independently, in unpublished work of Ruan. An expository account of the conjecture and its consequences can be found in [25].

3. General Theory

In this section we describe various aspects of Givental’s symplectic formalism which we will need below, as well as stating some consequences of Conjecture 2.1.3

Big and Small $J$-Functions. Let $\tau = \tau^a \Phi_n$. The big $J$-function of $Z$ is

$$J_Z^{\text{big}}(\tau, z) := z + \tau + \sum_{n \geq 0} \sum_{d \in \text{Eff}(Z)} \frac{1}{n!} \left( \tau, \tau, \ldots, \tau, \frac{\Phi_n}{z - \psi} \right)_{0,n+1,d}^Z \Phi^n.$$

It is a formal family of elements of $H_Z$ — in other words, $J_Z^{\text{big}}$ is a formal power series in the variables $\tau^a$, $0 \leq a \leq N$, which takes values in $H_Z$. By writing out the equations (8) defining $L_Z$, it is easy to see that $J_Z(\tau, -z)$ is the unique family of elements of $L_Z$ of the form $-z + \tau + O(z^{-1})$.

Take $Z = Y$ and restrict the parameter $\tau$ in the big $J$-function to the locus $\tau = \tau^1 \varphi_1 + \cdots + \tau^r \varphi_r$. Then the Divisor Equation gives that $J_Z^{\text{big}}(\tau^1 \varphi_1 + \cdots + \tau^r \varphi_r, z)$ is equal to

$$ze^{\varphi_1/\psi_1} \cdots e^{\varphi_r/\psi_r} \left( 1 + \sum_{d \in \text{Eff}(Y)} e^{d_1 \tau_1} \cdots e^{d_r \tau_r} \left( \frac{\varphi}{z - \psi} \right)^Y_{0,1,d} \Phi^e \right).$$

Making the change of variables $q_i = e^{\varphi_i}$, $1 \leq i \leq r$, we define the small $J$-function of $Y$ to be

$$J_Y(q, z) := z q_1^{\varphi_1/\psi_1} \cdots q_r^{\varphi_r/\psi_r} \left( 1 + \sum_{d \in \text{Eff}(Y)} q_1^{d_1} \cdots q_r^{d_r} \left( \frac{\varphi}{z - \psi} \right)^Y_{0,1,d} \Phi^e \right).$$

In examples below we will see that this converges, in a domain where each $|q_i|$ is sufficiently small, to a multi-valued analytic function of $q_1, \ldots, q_r$ which takes values in $H_Y$. The multi-valuedness comes from the factors $q_i^{\varphi_i/\psi} := \exp(\varphi_i \log(q_i)/z)$. We have $J_Y(q, -z) \in H_Y$ for all $q$ in the domain of convergence of $J_Y$.

Similarly, take $Z = X$ and restrict the parameter $\tau$ in the big $J$-function to the locus $\tau = \tau^1 \phi_1 + \cdots + \tau^a \phi_a$. Then the Divisor Equation gives that

$$J_X^{\text{big}}(\tau^1 \phi_1 + \cdots + \tau^a \phi_a, z) = z e^{\varphi_1/\psi_1} \cdots e^{\varphi_r/\psi_r} \left( 1 + \sum_{d \in \text{Eff}(X)} e^{d_1 \tau_1} \cdots e^{d_r \tau_r} \left( \frac{\phi}{z - \psi} \right)^X_{0,1,d} \phi^e \right).$$

These variables correspond to basis elements of $H(Z)$ with degree at most 2.
Making the change of variables \( u_i = e^{r_i} \), \( 1 \leq i \leq s \), we define the \textit{small \( J \)-function} of \( \mathcal{X} \) to be

\[
J_{\mathcal{X}}(u, z) := z^{\frac{\phi_1}{z} \cdot \cdots \cdot \frac{\phi_s}{z}} \left( 1 + \sum_{d \in \text{Eff}(\mathcal{X})} u_1^{d_1} \cdots u_s^{d_s} \frac{\phi_e}{z^2(z - \psi)} \right)^{\mathcal{X}_{0,1,d}} \phi^e.
\]

(11)

In the examples below this converges, in a domain where each \( |u_i| \) is sufficiently small, to a multi-valued analytic function of \( u_1, \ldots, u_s \) which takes values in \( \mathcal{H}_\mathcal{X} \). We have \( J_{\mathcal{X}}(u, -z) \in \mathcal{L}_\mathcal{X} \) for all \( u \) in the domain of convergence of \( J_{\mathcal{X}} \).

**Three Consequences of Conjecture 2.1.** Recall that the \( \mathbb{T} \)-equivariant small quantum cohomology of \( \mathcal{X} \) is a family of algebra structures on \( H(\mathcal{X}) \) parametrized by \( u_1, \ldots, u_s \), defined by

\[
\phi_\alpha \cdot \phi_\beta = \sum_{d \in \text{Eff}(\mathcal{X})} u_1^{d_1} \cdots u_s^{d_s} \left( \frac{\phi_e}{z^2(z - \psi)} \right)^{\mathcal{X}_{0,1,d}} \phi^e.
\]

(12)

The \( \mathbb{T} \)-equivariant small quantum cohomology of \( \mathcal{Y} \) is a family of algebra structures on \( H(\mathcal{Y}) \) parametrized by \( q_1, \ldots, q_r \), defined by

\[
\varphi_\alpha \cdot \varphi_\beta = \sum_{d \in \text{Eff}(\mathcal{Y})} q_1^{d_1} \cdots q_r^{d_r} \left( \frac{\varphi_e}{z^2(z - \psi)} \right)^{\mathcal{Y}_{0,1,d}} \varphi^e.
\]

(13)

For the remainder of this subsection, assume that:

- Conjecture 2.1 holds;
- the symplectic transformation \( \mathbb{U} \) remains well-defined in the non-equivariant limit \( \lambda \to 0 \);
- \( \mathcal{X} \) is semi-positive.\(^4\)

Three consequences of Conjecture 2.1 are then as follows: these are proved\(^4\) in [25]. Define the class \( c \in H(\mathcal{Y}) \) by

\[
\mathbb{U}(1_{\mathcal{X}}) = 1_{\mathcal{Y}} - cz^{-1} + O(z^{-2}),
\]

and write

\[
c = c^1 \varphi_1 + \cdots + c^r \varphi_r + d\lambda,
\]

where \( c^1, \ldots, c^r, d \in \mathbb{C}; \) such an equality exists because \( c \) has degree 2. Then:

**Corollary 3.1.** The algebra obtained from \( \mathcal{Y} \) by analytic continuation\(^5\) in the parameters \( q_{r+1}, \ldots, q_r \) (if necessary) followed by the substitution

\[
q_i = \begin{cases} 
0 & 1 \leq i \leq s \\
e^{r_i} & s < i \leq r
\end{cases}
\]

is isomorphic to the Chen–Ruan orbifold cohomology algebra of \( \mathcal{X} \), via an isomorphism which sends \( \alpha \in H^2(\mathcal{X}; \mathbb{C}) \subset H(\mathcal{X}) \) to \( \pi^* \alpha \in H(\mathcal{Y}) \).

This is a version of Ruan’s Cohomological Crepant Resolution Conjecture [47].

Define elements \( b_c \in H(\mathcal{Y}), \) \( 0 \leq c \leq N \), by \( b_c = 0 \) if \( \deg \phi_c \leq 2 \) and

\[
\mathbb{U}(\phi_c^{1-\frac{1}{2}\deg \phi_c}) = b_c + O(z^{-1})
\]

otherwise. Define power series \( f^1, \ldots, f^r, g \in \mathbb{C}[u_1, \ldots, u_s] \) by

\[
f^1 \varphi_1 + \cdots + f^r \varphi_r + g \lambda = \sum_{d \in \text{Eff}(\mathcal{X})} u_1^{d_1} \cdots u_s^{d_s} b_c; \quad (15)
\]

such an equality exists because each class \( b_c \) has degree 2. Recall the definition of the rational numbers \( r_i, \) \( 1 \leq i \leq s \), from Section 2. Then:

\[3 - \dim_{\mathbb{C}} \mathcal{Z} \leq \langle c_1(T \mathcal{Z}), d \rangle < 0\]

All Fano and Calabi–Yau orbifolds are semi-positive, as are all orbifold curves, surfaces, and 3-folds. In particular, all the orbifolds that we consider in the examples below are semi-positive.

\[\text{This is not, strictly speaking, true: the} \ T \text{-equivariant version of the Crepant Resolution Conjecture is not treated in [25]. It is straightforward to check, however, that the arguments given there also prove the results stated here. The key point is that } \mathbb{U} \text{ has a non-equivariant limit, and so only non-negative powers of } \lambda \text{ can occur.}\]

\[\text{The analytic continuation of the small quantum product here is induced by the analytic continuation of } \mathcal{L}_\mathcal{Y}. \text{ This is explained in [25].}\]
Corollary 3.2. The algebra obtained from the small quantum cohomology algebra of \( Y \) by analytic continuation\(^6\) in the parameters \( q_{s+1}, \ldots, q_r \) (if necessary) followed by the substitution

\[
q_i = \begin{cases} 
e^{-f_i}q_i^{-1} & 1 \leq i \leq s \\ e^{c_i+f^i} & s < i \leq r 
\end{cases}
\]

is isomorphic to the small quantum cohomology algebra of \( X \), via an isomorphism which sends \( \alpha \in H^2(X; \mathbb{C}) \subset H(X) \) to \( \pi^*\alpha \in H(Y) \).

This is a “quantum-corrected” version of Ruan’s Crepant Resolution Conjecture.

Suppose now that the matrix entries of \( \mathbb{U} \) contain only non-positive powers of \( z \), so that the limit \( \mathbb{U}_\infty := \lim_{z \to -\infty} \mathbb{U} \) exists. Then \( \mathbb{U}_\infty : H(X) \to H(Y) \) is a degree-preserving linear isometry such that \( \mathbb{U}_\infty(1_X) = 1_Y \) and that \( \mathbb{U}_\infty \circ (\rho_{\text{CR}}) = \pi^*\rho \cup \mathbb{U}_\infty \) for all \( \rho \in H^2(X; \mathbb{C}) \). Furthermore:

Corollary 3.3. The map \( \mathbb{U}_\infty \) gives an isomorphism between the \( T \)-equivariant small quantum cohomology algebra of \( X \) and the algebra obtained from the \( T \)-equivariant small quantum cohomology of \( Y \) by analytic continuation in the parameters \( q_{s+1}, \ldots, q_r \) (if necessary) followed by the substitution

\[
q_i = \begin{cases} e^{c_i}u_i & 1 \leq i \leq s \\ e^{c_i} & s < i \leq r. 
\end{cases}
\]

This Corollary also holds with “small quantum cohomology” replaced by “big quantum cohomology”, but we will not pursue this. The conclusion here is a slightly modified version of the Crepant Resolution Conjecture due to Bryan–Graber\([11]\).

Three Results Which We Will Need. We next record three results which we will need below. Part (a) follows from the String Equation: this is explained in e.g.\([24]\). Part (b) is a reconstruction result for Gromov–Witten invariants — it says that all genus-zero Gromov–Witten invariants can be uniquely reconstructed from the one-point descendants \( \{\Phi_{\alpha}^{(k)}\}_{0,1,d} \). Part (c) is a generalization of part (b). One can prove (b) and (c) by repeated application of the WDVV equations and the Topological Recursion Relations. Since there does not seem to be an appropriate reference for this in the generality we need (\( T \)-equivariant, orbifolds, Calabi–Yau, etc.) we will give a proof elsewhere\([24]\); results along similar lines can be found in \([6,27,38,41,42,46]\).

Proposition 3.4.

(a) The submanifold-germ \( \mathcal{L}_Z \subset \mathcal{H}_Z \) is closed under multiplication by \( \exp(a\lambda/z) \) for any \( a \in \mathbb{C} \).

(b) If \( Z \) is semi-positive and the Chen–Ruan orbifold cohomology algebra of \( Z \) is generated by \( H^2(Z; \mathbb{C}) \) then the submanifold-germ \( \mathcal{L}_Z \) can be uniquely reconstructed from the small \( J \)-function \( J_Z(q; z) \).

(c) If \( Z \) is semi-positive and \( H^2_{\text{gen}} \subset H^2_{CR}(Z; \mathbb{C}) \) is a subspace such that the Chen–Ruan orbifold cohomology algebra of \( Z \) is generated by \( H^2_{\text{gen}} \) then the submanifold-germ \( \mathcal{L}_Z \) can be uniquely reconstructed from the restriction of the big \( J \)-function \( J_Z^{\text{big}}(\tau, z) \) to the locus \( \tau \in H^2_{\text{gen}} \).

It is easy to check that in all the examples we consider below, the Chen–Ruan cohomology algebra of \( Z \) is generated in degree 2.

Computing Twisted Gromov–Witten Invariants. As discussed above, in our examples \( Z \) will be the total space of a concave vector bundle \( \mathcal{E} \) over a compact orbifold \( \mathcal{B} \), and the \( T \)-action on \( Z \) will be the canonical \( \mathbb{C}^\times \)-action which rotates the fibers of \( \mathcal{E} \) and covers the trivial action on \( \mathcal{B} \). In this situation \( \text{Eff}(Z) \) is canonically isomorphic to \( \text{Eff}(\mathcal{B}) \) and \( H(Z) \) is canonically isomorphic to \( H(\mathcal{B}) := H_{\text{CR}}^*(\mathcal{B}; \mathbb{C}) \otimes \mathbb{C}(\lambda) \). Our bases \( \{\Phi_{\alpha}\} \) and \( \{\Phi^a\} \) for \( H(\mathcal{Z}) \) determine bases for \( H(\mathcal{B}) \), which we also denote by \( \{\Phi_{\alpha}\} \) and \( \{\Phi^a\} \). Gromov–Witten invariants of \( Z \) coincide with Gromov–Witten invariants of \( \mathcal{Z} \) twisted, in the sense of\([19,23]\), by the \( T \)-equivariant inverse Euler class \( e^{-1} \) and the vector bundle \( \mathcal{E} \).

Results in \([19]\) allow the calculation of twisted Gromov–Witten invariants in a quite general setting. We will need three special cases of these results, as follows. Each of these special cases determines a family of elements \( q \mapsto I_Z(q, -z) \) of elements of \( \mathcal{L}_Z \); in each case this family \( I_Z(q, z) \) is an appropriate hypergeometric modification of the small \( J \)-function \( J_B(q, z) \) of \( \mathcal{B} \).

Theorem 3.5. Suppose that \( \mathcal{E} \to \mathcal{B} \) is a concave line bundle. Let \( \rho \) denote the first Chern class of \( \mathcal{E} \), regarded as an element of \( T \)-equivariant Chen–Ruan cohomology \( H(\mathcal{B}) \) via the canonical
inclusion \( H^\bullet(B; \mathbb{C}) \hookrightarrow H^\bullet_{cr}(B; \mathbb{C}) \), and set
\[
M_E(d) := \prod_{k; (\rho, d) < b, \frac{\text{frac}(b)}{\text{frac}(\rho, d)}} (\lambda + \rho + b z)
\]
where \( d \in \text{Eff}(B) \) and \( \text{frac}(r) \) denotes the fractional part of \( r \). Let \( k = b_2(B) \), so that the small \( J \)-function of \( B \) is
\[
J_B(q, z) = z \prod_{i=1}^k \frac{q_i}{z} \left( 1 + \sum_{d \in \text{Eff}(B)} q_1^{d_1} \cdots q_k^{d_k} \left( \frac{\Phi_\rho}{z - \psi} \right)^B \Phi^e_{\rho, 0, 1, d} \right).
\]
Then
\[
I_Z(q, z) := z \prod_{i=1}^k \frac{q_i}{z} \left( 1 + \sum_{d \in \text{Eff}(B)} q_1^{d_1} \cdots q_k^{d_k} M_E(d) \left( \frac{\Phi_\rho}{z - \psi} \right)^B \Phi^e_{\rho, 0, 1, d} \right)
\]
satisfies \( I_Z(q, -z) \in \mathcal{L}_Z \) for all \( q \) in the domain of convergence of \( I_Z \).

**Proof.** Theorem 4.6 in [19] concerns a Lagrangian submanifold-germ \( L^\text{tw} \) which encodes twisted Gromov-Witten invariants: in our situation, \( \mathcal{L}^\text{tw} = \mathcal{L}_Z \). The Theorem gives a formula for a formal family \( \tau \mapsto I^\text{tw}(\tau, -z) \) of elements of \( \mathcal{L}^\text{tw} \), as follows. Let \( I \) be a set which indexes the components of the inertia stack \( \mathcal{I}B \) of \( B \), and let \( 0 \in \mathcal{I} \) be the index of the distinguished component \( B \subset \mathcal{I}B \). One decomposes the big \( J \)-function of \( B \) as a sum
\[
J_B^{\text{big}}(\tau, z) = \sum_{\theta \in \text{NETT}(B)} J_{\theta}(\tau, z)
\]
of contributions from stable maps of different topological types; here \( \text{NETT}(B) \) is the set of topological types. The topological type of a degree-\( d \) stable map \( f : C \to B \) from a genus-\( g \) orbifold curve with \( n \) marked points is the triple \((g, d, S)\), where \( S = (i_1, \ldots, i_n) \) is the ordered \( n \)-tuple of elements of \( I \) indexing the components of \( \mathcal{I}B \) picked out by the marked points. Then
\[
I^\text{tw}(\tau, z) := \sum_{\theta \in \text{NETT}(B)} M_\theta(z) \cdot J_\theta(\tau, z)
\]
where \( M_\theta(z) \) is a modification factor defined in §4.2 of [19].

If we set \( \tau = \tau^1 \Phi_1 + \cdots + \tau^k \Phi_k \) then \( J_\theta(\tau, z) \) vanishes unless the topological type \( \theta \) is of the form \((0, d, S)\) where \( S = (0, 0, \ldots, 0, i) \) for some \( i \in \mathcal{I} \); this is because the classes \( \Phi_i, 1 \leq i \leq k \) are supported on the distinguished component \( B \) of \( \mathcal{I}B \). In this case the modification factor \( M_\theta(z) \) depends only on \( d \) and is equal to \( M_E(d) \). Also,
\[
J_B^{\text{big}}(\tau^1 \Phi_1 + \cdots + \tau^k \Phi_k, z) = z e^{\tau^1 \Phi_1/z} \cdots e^{\tau^k \Phi_k/z} \left( 1 + \sum_{d \in \text{Eff}(B)} e^{d_1 \tau^1} \cdots e^{d_k \tau^k} \left( \frac{\Phi_\rho}{z - \psi} \right)^B \Phi^e_{\rho, 0, 1, d} \right)
\]
and it follows that \( I^\text{tw}(\tau^1 \Phi_1 + \cdots + \tau^k \Phi_k, z) \) is equal to
\[
z e^{\tau^1 \Phi_1/z} \cdots e^{\tau^k \Phi_k/z} \left( 1 + \sum_{d \in \text{Eff}(B)} e^{d_1 \tau^1} \cdots e^{d_k \tau^k} M_E(d) \left( \frac{\Phi_\rho}{z - \psi} \right)^B \Phi^e_{\rho, 0, 1, d} \right).
\]
Making the change of variables \( q_i = e^{\tau^i}, 1 \leq i \leq k \), we conclude that \( I_Z(q, -z) \in \mathcal{L}_Z \) for all \( q \) such that the series defining \( I_Z \) converges.

Exactly the same argument proves:

**Theorem 3.6.** If \( \mathcal{E} = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_m \) is the direct sum of convex line bundles,
\[
M_{\mathcal{E}}(d) := \prod_{1 \leq i \leq m} M_{\mathcal{E}_i}(d),
\]
and \( I_Z(q, z) \) is defined exactly as in [19] then \( I_Z(q, -z) \in \mathcal{L}_Z \) for all \( q \) in the domain of convergence of \( I_Z \).

The final special case which we need is where \( Z \) is the total space of a direct sum of line bundles \( \mathcal{E} = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_m \) over \( B = BZ_m \). Components of the inertia stack of \( BZ_m \) are indexed by fractions \( k/n, 0 \leq k < n \): the component indexed by \( k/n \) corresponds to the element \( [k] \in \mathbb{Z}_n \). Let \( 1_{k/n} \in H(B) \) denote the orbifold cohomology class which restricts to the unit class on the component of the inertia stack indexed by \( k/n \) and restricts to zero on the other components. The set \( \{1_{k/n} : 0 \leq k < n\} \) forms a basis for \( H(B) \); as \( H(B) \) and \( H(Z) \) are canonically isomorphic it determines a basis for \( H(Z) \) as well.
Theorem 3.7. Let $Z$ be the total space of the direct sum of line bundles $E = E_1 \oplus \cdots \oplus E_m$ over $B = BZ_n$. Let $e_i$ be the integer such that $E_i$ is given by the character $[k] \mapsto \exp(2\pi i k n)$. Then $0 \leq e_i < n$. Let
\[ P_{i,k} := \left\{ b : \frac{b}{n} = \frac{-e_k}{n}, -\frac{e_k}{n} < b \leq 0 \right\} \]
and
\[ I_Z(x, z) := \sum_{k \geq 0} x^k \prod_{i=1}^m \prod_{b \in P_{i,k}} \left( \frac{z_i}{n} \lambda + bz \right) \frac{1}{k! z^k}. \]
Then $x \mapsto I_Z(x, -z)$ is a formal family of elements of $L_Z$.

Proof. We argue as in the proof of Theorem 3.5. If we decompose the big $J$-function of $B$ as a sum
\[ J_B^{\text{big}}(\tau, z) = \sum_{\theta \in \text{NETT}(B)} J_\theta(\tau, z) \]
of contributions from stable maps of different topological types and set
\[ I^{\text{tw}}(\tau, z) := \sum_{\theta \in \text{NETT}(B)} M_\theta(z) \cdot J_\theta(\tau, z) \]
where $M_\theta(z)$ is defined in [19 §4.2] then $\tau \mapsto I^{\text{tw}}(\tau, -z)$ defines a formal family of elements of $L^{\text{tw}} = L_Z$. Proposition 6.1 in [19] gives an explicit formula for the big $J$-function of $B = BZ_n$, and we see from this that if $\tau = x1_{\mathbb{R}}$ then $J_\theta(\tau, z)$ vanishes unless the topological type $\theta$ is $(0, 0, S)$ with
\[ S = \left( \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}, \frac{m-k}{n} \right). \]
In this case,
\[ J_\theta(\tau, z) = \frac{x^k}{k! z^k} \frac{1}{k! z^k} \quad \text{and} \quad M_\theta(z) = \prod_{i=1}^m \prod_{b \in P_{i,k}} \left( \frac{z_i}{n} \lambda + bz \right). \]
Thus $x \mapsto I_Z(x, -z)$ is a formal family of elements of $L_Z$. \hfill \Box

4. Example 1: $X = [\mathbb{C}^3/\mathbb{Z}_3], \ Y = K_{\mathbb{P}^2}$

Let $X$ be the orbifold $[\mathbb{C}^3/\mathbb{Z}_3]$ where $\mathbb{Z}_3$ acts on $\mathbb{C}^3$ with weights $(1, 1, 1)$. The coarse moduli space $X$ of $X$ is the quotient\footnote{This is Miles Reid’s notation [39].} singularity $\frac{1}{3}(1, 1, 1)$, and the crepant resolution $Y$ of $X$ is the canonical bundle $K_{\mathbb{P}^2}$.

Toric Geometry. The space $Y$ is the toric variety corresponding to a fan with rays
\[
\begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}, \quad
\begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix}, \quad
\begin{pmatrix}
-1 \\
0 \\
3
\end{pmatrix}, \quad
\begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}.
\]
this fan is a cone over the picture in the plane $x + y + z = 1$ shown in Figure 1. We can construct $Y$ as

Figure 1: The fans for $X$ and $Y$ (respectively) are the cones over these pictures in the plane $x + y + z = 1$. 

\[ \text{Figure 1: The fans for } X \text{ and } Y \text{ (respectively) are the cones over these pictures in the plane } x + y + z = 1. \]
a GIT quotient, following \textit{e.g.} \cite{1}, by considering the exact sequence

$$
\begin{array}{c}
0 \\
\overset{\begin{pmatrix} 1 \\ 1 \\ 1 \\ -3 \end{pmatrix}}{\longrightarrow} \\
\overset{\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 1 \end{pmatrix}}{\longrightarrow} \\
\overset{\tau x \quad \tau y \quad \tau z \quad \tau^{-3} w}{\longrightarrow} \\
\longrightarrow 0.
\end{array}
$$

This shows that $Y$ is a quotient $\mathbb{C}^4/\mathbb{C}^\times$, where $\tau \in \mathbb{C}^\times$ acts on $\mathbb{C}^4$ as

$$
\begin{pmatrix}
x \\
y \\
z \\
w
\end{pmatrix} \mapsto
\begin{pmatrix}
\tau x \\
\tau y \\
\tau z \\
\tau^{-3} w
\end{pmatrix}.
$$

Dualizing \cite{18} gives

$$
\begin{array}{c}
0 \\
\overset{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 3 \\ 0 & 0 & 1 \end{pmatrix}}{\longrightarrow} \\
\overset{\begin{pmatrix} 1 & 1 & -3 \end{pmatrix}}{\longrightarrow} \\
\longrightarrow 0
\end{array}
$$

where the right-hand entry is $H^2(Y; \mathbb{Z})$ and the columns of the right-hand matrix give the four toric divisors in $Y$. If we draw this picture in $H^2(Y; \mathbb{R})$ then it gives the chamber decomposition for the GIT problem (Figure 2 below); this chamber decomposition is also known as the \textit{secondary fan} for $Y$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{secondary_fan.png}
\caption{The secondary fan for $Y = K_{\mathbb{P}^2}$}
\end{figure}

Each chamber in the secondary fan corresponds to a fan $\Sigma$ which is a triangulation of the rays \cite{17}; a cone $\sigma$ is in $\Sigma$ if and only if the co-ordinate subspace corresponding to the complement of $\sigma$ covers the chosen chamber. The fans are shown in Figure 1. For $\xi$ in the left-hand chamber the GIT quotient $\mathbb{C}^4/\xi \mathbb{C}^\times$ gives $X$; we delete the locus $w = 0$ from $\mathbb{C}^4$ and then take the quotient by the action \cite{19}. For $\xi$ in the right-hand chamber we have $\mathbb{C}^4/\xi \mathbb{C}^\times = Y$; we delete the locus $(x, y, z) = (0, 0, 0)$ from $\mathbb{C}^4$ and then take the quotient by \cite{19}. For $\xi = 0$ the quotient $\mathbb{C}^4/\xi \mathbb{C}^\times$ gives the coarse moduli space $X$. Moving from the right-hand chamber into the “wall” $\xi = 0$ gives the resolution map $Y \to X$; this sends

$$
\begin{pmatrix}
x \\
y \\
z \\
w
\end{pmatrix} \in \mathbb{C}^4/\xi \mathbb{C}^\times
$$
to

$$
\begin{pmatrix}
xw^{1/3} \\
yw^{1/3} \\
zw^{1/3}
\end{pmatrix} \in \mathbb{C}^3/\mathbb{Z}_3
$$

where $[A]$ denotes class of $A$ in the appropriate quotient.

\textbf{The $T$-Action.} Consider the action of $T = \mathbb{C}^\times$ on $\mathbb{C}^4$ such that $\alpha \in T$ acts as

$$
\begin{pmatrix}
x \\
y \\
z \\
w
\end{pmatrix} \mapsto
\begin{pmatrix}
x \\
y \\
z \\
\alpha w
\end{pmatrix}.
$$

This action descends to give $T$-actions on $X$, $Y$, and $X$. The induced action on $X$ is

$$
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix} \mapsto
\begin{pmatrix}
\alpha^{1/3} x \\
\alpha^{1/3} y \\
\alpha^{1/3} z
\end{pmatrix}.
$$

The induced action on $Y$ is the canonical $\mathbb{C}^\times$-action on the line bundle $K_{\mathbb{P}^2} \to \mathbb{P}^2$; it covers the trivial action on $\mathbb{P}^2$. The diagram

$$
\begin{array}{c}
\mathcal{X} \\
\downarrow \\
X
\end{array}
\begin{array}{c}
\downarrow \\
Y
\end{array}
$$

is $T$-equivariant.
Bases for Everything. We have
\[ r := \text{rank } H^2(Y; \mathbb{C}) = 1, \quad s := \text{rank } H^2(\mathcal{X}; \mathbb{C}) = 0. \]
Let \( p \) be the first Chern class of the line bundle \( \mathcal{O}(1) \to \mathbb{P}^2 \), pulled back to \( Y = K_{\mathbb{P}^2} \) via the projection \( K_{\mathbb{P}^2} \to \mathbb{P}^2 \). The class \( p \) has a canonical lift to \( T \)-equivariant cohomology, which we also denote by \( p \), and
\[ H(Y) = \mathbb{C}(\lambda)[p]/(p^3). \]
We set
\[ \varphi_0 = 1, \quad \varphi_1 = p, \quad \varphi_2 = p^2, \]
so that
\[ \varphi^0 = \lambda p^2, \quad \varphi^1 = \lambda p - 3p^2, \quad \varphi^2 = \lambda - 3p. \]
The components of the inertia stack of \( \mathcal{X} \) are indexed by elements of \( \mathbb{Z}_3 \). Let \( 1_{k/3} \in H(\mathcal{X}) \) denote the orbifold cohomology class which restricts to the unit class on the inertia component indexed by \( [k] \in \mathbb{Z}_3 \) and restricts to zero on the other components. Set
\[ \phi_0 = 1_0, \quad \phi_1 = 1_{1/3}, \quad \phi_2 = 1_{2/3}, \]
so that
\[ \phi^0 = \frac{\lambda^3}{p} 1_0, \quad \phi^1 = 31_{2/3}, \quad \phi^2 = 31_{1/3}. \]

**Step 1: A Family of Elements of \( \mathcal{L}_Y \).** Consider
\[ I_Y(y, z) := \sum_{d \geq 0} \frac{\Gamma(1 + \frac{z}{2})^3}{\Gamma(1 + \frac{p}{z} + d)^3} \frac{\Gamma(1 + \frac{\lambda - 3p}{2})}{\Gamma(1 + \frac{\lambda - 3p}{z} - 3d)} y^{d + p/z}. \]
This series converges in the region \( \{ y \in \mathbb{C} : 0 < |y| < \frac{1}{27} \} \) to a multi-valued analytic function of \( y \) which takes values in \( H_Y \). We have
\[ I_Y(y, z) = \sum_{d \geq 0} \prod_{-3d < m \leq 0} (\lambda - 3p + mz)^3 \frac{y^d}{\prod_{0 < m \leq d} (p + mz)^3} y^{d + p/z}. \]

**Proposition 4.1.**
\[ I_Y(y, -z) \in \mathcal{L}_Y \quad \text{for all } y \text{ such that } 0 < |y| < \frac{1}{27}. \]

**Proof.** We are in the situation of Theorem 3.3 with \( B = \mathbb{P}^2 \) and \( \mathcal{E} = \mathcal{O}(-3) \). Givental has proved 30 that the small \( J \)-function of \( \mathbb{P}^2 \) is
\[ J_{\mathbb{P}^2}(q, z) = z q^{p/z} \sum_{d \geq 0} \frac{q^d}{\prod_{0 < m \leq d} (p + mz)^3}, \]
and it follows (by comparing with the statement of Theorem 3.3) that
\[ \left\langle \Phi^\epsilon \right|_{\frac{q^2}{z - \psi}, 0, 1, d} \Phi^\epsilon_{\epsilon} = \frac{1}{\prod_{0 < m \leq d} (p + mz)^3} \quad \text{whenever } d > 0. \]

Theorem 3.3 thus implies that \( I_Y(y, -z) \in \mathcal{L}_Y \) for all \( y \) in the domain of convergence of \( I_Y \), as claimed. \( \square \)

**Step 2: \( I_Y \) Determines \( \mathcal{L}_Y \).** We have:

**Corollary 4.2.**
\[ J_Y(q, z) = e^{q f(y)/z} I_Y(y, z) \quad \text{where} \quad q = y \exp \left( 3f(y) \right), \]
\[ f(y) = \sum_{d > 0} \frac{(3d-1)!}{(3d)!} (-y)^d. \]
Proof. We have
\[ I_Y(y, z) = z + p \log y - (\lambda - 3p) f(y) + O(z^{-1}). \]

Applying Propositions 3.4 and 4.1, we see that
\[ y \mapsto e^{-\lambda f(y)/z} I_Y(y, -z), \quad 0 < |y| < \frac{1}{27}, \]
is a family of elements of \( \mathcal{L}_Y \). But
\[ e^{-\lambda f(y)/z} I_Y(y, -z) = -z + p \log q + O(z^{-1}), \]
where \( q \) is defined above, and the unique family of elements of \( \mathcal{L}_Y \) of this form is \( q \mapsto J_Y(q, -z). \)

As \( I_Y(y, z) \) is multivalued-analytic and the change of variables \( y \mapsto q \) is analytic, we conclude that the series defining \( J_Y(q, z) \) converges, when \( |q| \) is sufficiently small, to a multivalued analytic function of \( q \). Furthermore, as the small \( J \)-function \( J_Y(q, z) \) determines \( \mathcal{L}_Y \) (Proposition 3.4(b)), it follows that \( \mathcal{L}_Y \) is uniquely determined by the fact that \( y \mapsto I_Y(y, -z) \) is a family of elements of \( \mathcal{L}_Y \).

Aside: Computing Gromov–Witten Invariants of \( Y \). As is well-known, Corollary 4.2 determines many genus-zero Gromov–Witten invariants of \( Y \). The inverse to the change of variables \( y \mapsto q \) is
\[ y = q + 6q^2 + 9q^3 + 56q^4 - 300q^5 + \ldots \]
Substituting this into the equality
\[ z q^{p/z} \left( 1 + \sum_{d \geq 0} q^d \left( \frac{\varphi_x}{z(z - \psi)} \right)_{0,1,d} \right)^{Y} = z e^{\lambda f(y)/z} \sum_{d \geq 0} \prod_{-3d < m \leq 0} (\lambda - 3d p + mz) \prod_{0 < m \leq d} (p + mz)^{3} y^{d+p/z} \]
and comparing coefficients of \( q \), one finds that
\[ \left\langle \frac{\varphi^\alpha}{z - \psi} \right\rangle_{Y} = -\frac{9p^2}{z} + o(\lambda), \quad \left\langle \frac{\varphi^\alpha}{z - \psi} \right\rangle_{0,1,2} = \frac{135p^2}{4z} + o(\lambda), \]
\[ \left\langle \frac{\varphi^\alpha}{z - \psi} \right\rangle_{Y} = -\frac{244p^2}{z} + o(\lambda), \quad \left\langle \frac{\varphi^\alpha}{z - \psi} \right\rangle_{0,1,4} = \frac{36999p^2}{16z} + o(\lambda), \]
and so on, where \( o(\lambda) \) denotes terms containing strictly positive powers of \( \lambda \). Taking the non-equivariant limit \( \lambda \to 0 \) yields the local Gromov–Witten invariants \( K_d \) calculated in \cite{[17]} 2.2:
\[ \langle p \rangle_{0,1,1}^Y = 3, \quad \langle p \rangle_{0,1,2}^Y = \frac{45}{4}, \quad \langle p \rangle_{0,1,4}^Y = \frac{12333}{16}, \]
and therefore, using the Divisor Equation, we find
\[
\begin{array}{c|c|c|c|c|c|}
 d & 1 & 2 & 3 & 4 & \ldots \\
 K_d & 3 & -\frac{45}{8} & \frac{244}{9} & -\frac{12333}{64} & \ldots \\
\end{array}
\]

Step 3: A Family of Elements of \( \mathcal{L}_X \). Let
\[ I_X(x, z) := z x^{-\lambda/z} \sum_{l \geq 0} x^l \prod_{b \geq 0} (\frac{a}{b} - bz)^3 \mathbf{1}_{\text{frac}(\frac{a}{b})}. \tag{23} \]
This converges, in the region \( |x| < 27 \), to an analytic function which takes values in \( \mathcal{H}_X \). Theorem 3.7 and Proposition 3.4(a) imply that \( I_X(x, -z) \in \mathcal{L}_X \) for all \( x \) such that \( |x| < 27 \).

Step 4: \( I_X \) Determines \( \mathcal{L}_X \). We have:

Corollary 4.3.
\[ J_X^{\text{big}}(r^1 \mathbf{1}_{1/3}, z) = x^{\lambda/z} I_X(x, z) \]
where
\[ r^1 = \sum_{m \geq 0} (-1)^m \frac{x^{3m+1} (m+\frac{1}{3})^3}{(3m+1)!} \]
Proof. On the one hand, we know that $x^{-λ/z}I_\mathcal{X}(x, -z) \in \mathcal{L}_\mathcal{X}$, and on the other hand we know that

$$x^{-λ/z}I_\mathcal{X}(x, -z) = -z + \left( \sum_{m \geq 0} (-1)^m \frac{x^{3m+1}}{(3m+1)!} \frac{\Gamma(m + \frac{1}{3})^3}{\Gamma(\frac{1}{3})^3} \right) 1_{1/3} + O(z^{-1}).$$

As the unique family of elements of $\mathcal{L}_\mathcal{X}$ of the form $-z + \tau^3 1_{1/3} + O(z^{-1})$ is $\tau^3 \mapsto J^\text{big}_\mathcal{X}((1_{1/3}, -z)$, the result follows.

Since $x^{λ/z}I_\mathcal{X}(x, z)$ and the change of variables $x \mapsto \tau^3$ are analytic, this implies that $J^\text{big}_\mathcal{X}((1_{1/3}, z)$ depends analytically on $\tau^3$ in some region where $|\tau^3|$ is sufficiently small. It also, via Proposition 3.4c, shows that $\mathcal{L}_\mathcal{X}$ is uniquely determined by the fact that $x \mapsto I_\mathcal{X}(x, -z)$ is a family of elements of $\mathcal{L}_\mathcal{X}$.

Aside: Computing Gromov–Witten Invariants of $\mathcal{X}$. Just as we did for $\mathcal{Y}$, one can use Corollary [13] to compute genus-zero Gromov–Witten invariants of $\mathcal{X}$. This calculation is carried out in [9, §6.3]; it verifies some of the predictions made by Aganagic, Bouchard, and Klemm [3].

**Step 5: The $B$-model Moduli Space and the Picard–Fuchs System.** The $B$-model moduli space $\mathcal{M}_B$ is the toric orbifold corresponding to the secondary fan for $\mathcal{Y}$. It has two co-ordinate patches, one

$$\begin{array}{c}
\text{Figure 3: The secondary fan for } \mathcal{Y} = K_{\mathbb{P}^2}.
\end{array}$$

for each chamber. Let $x$ be the co-ordinate corresponding to the left-hand chamber (recall that this chamber gives rise to $\mathcal{X}$) and let $y$ be the co-ordinate corresponding to the right-hand chamber (recall that this chamber gives $Y$). The co-ordinate patches are related by

$$y = x^{-3} \quad (24)$$

and it follows that $\mathcal{M}_B$ is the weighted projective space $\mathbb{P}(1, 3)$. The space $\mathcal{M}_B$ is called the $B$-model moduli space as it is the base of the Landau–Ginzburg model (“the $B$-model”) which corresponds to the quantum cohomology of $Y$ (“the $A$-model”) under mirror symmetry: see e.g. [31, 38].

We regard $I_\mathcal{X}(x, z)$ as a function on the co-ordinate patch corresponding to $\mathcal{X}$ and $I_\mathcal{Y}(y, z)$ as a function on the co-ordinate patch corresponding to $\mathcal{Y}$. Writing

$$I_\mathcal{Y}(y, z) = I^0_Y \phi_0 + I^1_Y \phi_1 + I^2_Y \varphi_2,$$

the components $\{I^j_Y : j = 0, 1, 2\}$, which are functions of $y$, $\lambda$, and $z$, form a basis of solutions to the differential equation

$$D_y^3f = y(\lambda - 3D_y)(\lambda - 3D_y - z)(\lambda - 3D_y - 2z)f, \quad D_y = zy\frac{\partial}{\partial y}. \quad (25)$$

Writing

$$I_\mathcal{X}(x, z) = I^0_X \phi_0 + I^1_X \phi_1 + I^2_X \phi_2,$$

the components $\{I^j_X : j = 0, 1, 2\}$, which are functions of $x$, $\lambda$, and $z$, form a basis of solutions to the differential equation

$$D_x^3f = -27x^{-3}(\lambda + D_x)(\lambda + D_x - z)(\lambda + D_x - 2z)f, \quad D_x = zx\frac{\partial}{\partial x}. \quad (26)$$

Recall that the functions $I^j_X$ are defined in a region where $|y|$ is small. The change of variables $\lambda \mapsto \lambda^{-1}$ turns (25) into (26). This implies that if we analytically continue the functions $I^j_Y$ to a region where $|y|$ is large (and hence $|x|$ is small), and then write the analytic continuations $\bar{I}^j_Y$ in terms of the co-ordinate $x$, then $\{\bar{I}^j_Y(x, z) : j = 0, 1, 2\}$ will satisfy (26). We have a basis of solutions to (26), given by the components $\bar{I}^j_X(x, z)$ of $I_\mathcal{X}$, and so

$$\begin{align*}
\begin{pmatrix}
\bar{I}^0_X(x, z) \\
\bar{I}^1_X(x, z) \\
\bar{I}^2_X(x, z)
\end{pmatrix} & = M(\lambda, z) \begin{pmatrix} I^0_Y(x, z) \\ I^1_Y(x, z) \\ I^2_Y(x, z) \end{pmatrix}, \quad (27)
\end{align*}$$

The equation (25) is the Picard–Fuchs equation associated to the Landau–Ginzburg mirror to $\mathcal{Y}$. The fact that the quantum cohomology of $\mathcal{Y}$ can be determined from this Picard–Fuchs equation has been proved many times from many different points of view: see e.g. [17, 28, 32, 43].
for some $3 \times 3$ matrix $M$ which is independent of $x$ and $y$ (and hence depends only on $\lambda$ and $z$). The matrix $M(\lambda, -z)$ defines the $\mathbb{C}(\mathbb{Z}^{-1})$-linear symplectic transformation $U : \mathcal{H}_X \to \mathcal{H}_Y$ which we seek. It remains to calculate the analytic continuations and to determine the matrix $M$.

**Step 6: Analytic Continuation.** To compute the analytic continuation of $I_Y(y, z)$ we use the Mellin–Barnes method. Good references for this are [13; 18, Appendix; 37]. First, take the expression (20) for $I_Y$ and apply the identity $\Gamma(x)\Gamma(1-x) = \pi/\sin(\pi x)$ until each factor $\Gamma(a + bd)$ which occurs has $b > 0$:

$$I_Y(y, z) = -\Theta_Y \sum_{d \geq 0} \frac{\Gamma(3d - \frac{\lambda - 3p}{z})}{\Gamma(1 + \frac{p}{z} + d)} (-1)^d y^{d+p/z}$$

(28)

where

$$\Theta_Y = \pi^{-1} z \Gamma(1 + \frac{p}{z})^3 \Gamma(1 + \frac{\lambda - 3p}{z}) \sin \left(\pi \left[\frac{\lambda - 3p}{z}\right]\right).$$

Then, in view of [37, Lemma 3.3], consider the contour integral

$$\int_C \Theta_Y \frac{\Gamma(3s - \frac{\lambda - 3p}{z})\Gamma(s)\Gamma(1-s)}{\Gamma(1 + \frac{p}{z} + s)^3} q^{s+p/z}$$

(29)

where the contour of integration $C$ is chosen as in Figure 4. The integral (29) is defined and analytic throughout the region $|\arg(q)| < \pi$. For $|q| < \frac{1}{2\pi}$ we can close the contour to the right, and (29) is then equal to the sum of residues (28). For $|q| > \frac{1}{2\pi}$ we can close the contour to the left, and then (29) is equal to the sum of residues at

$$s = -1 - n, \quad n \geq 0,$$

and

$$s = \frac{\lambda - 3p}{3z} - \frac{n}{3}, \quad n \geq 0.$$

The residues at $s = -1 - n$, $n \geq 0$, vanish in $H(Y)$ as they are divisible by $p^3$. Thus the analytic continuation $\widetilde{I}_Y$ of $I_Y$ is equal to the sum of the remaining residues:

$$\Theta_Y \sum_{n \geq 0} \frac{(-1)^n}{3n!} \frac{\pi}{\sin \left(\pi \left[\frac{\lambda - 3p}{3z}\right]\right)} \frac{1}{\Gamma(1 + \frac{\lambda}{3} - \frac{n}{3})} y^{\lambda/3z-n/3}.$$

Writing this in terms of the co-ordinate $x$, we find that the analytic continuation $\widetilde{I}_Y(x, -z)$ is equal to

$$-z x^{\lambda/z} \sum_{n \geq 0} \frac{(-x)^n}{3n!} \frac{\sin \left(\pi \left[\frac{\lambda - 3p}{3z}\right]\right)}{\sin \left(\pi \left[\frac{\lambda - 3p}{3z} + \frac{n}{3}\right]\right)} \frac{\Gamma(1 - \frac{p}{z})^3}{\Gamma(1 - \frac{\lambda}{3z} - \frac{n}{3}) \Gamma(1 - \frac{\lambda - 3p}{3z})}.$$

(30)
Step 7: Compute the Symplectic Transformation. Our final step is to compute the linear symplectic transformation $U : H_X \to H_Y$ represented by the matrix $M(\lambda, -z)$. We have $U(I_X(x, -z)) = I_Y(x, -z)$, and

$$I_X(x, -z) = -z x^{\lambda/2} \left(1 - \frac{x}{z} 1_{1/3} + \frac{x^2}{2z^2} 1_{2/3} + O(x^3) \right).$$

(31)

As the transformation $U$ does not depend on $x$, we can compute it by equating powers of $x$ in (30) and (31):

$$U(1_0) = \frac{1}{3} \sin \left( \pi \left[ \frac{\lambda - 3p}{3z} \right] \right) \frac{\Gamma(1 - \frac{2}{z})^3}{\Gamma(1 - \frac{\lambda - 3p}{z})} \Gamma(1 - \frac{\lambda - 3p}{z})$$

$$U(1_{1/3}) = \frac{z}{3} \sin \left( \pi \left[ \frac{\lambda - 3p}{3z} \right] \right) \frac{\Gamma(1 - \frac{2}{z})^3}{\Gamma(1 - \frac{\lambda - 3p}{z})} \Gamma(1 - \frac{\lambda - 3p}{z})$$

$$U(1_{2/3}) = \frac{z^2}{3} \sin \left( \pi \left[ \frac{\lambda - 3p}{3z} \right] \right) \frac{\Gamma(1 - \frac{2}{z})^3}{\Gamma(1 - \frac{\lambda - 3p}{z})} \Gamma(1 - \frac{\lambda - 3p}{z}).$$

The matrix $M$ of $U$ does not have a simple form, but in the non-equivariant limit it becomes

$$\left( \begin{array}{ccc} 1 & 0 & 0 \\ -\frac{2\pi z}{\Gamma(\frac{3}{2})} & \frac{2\pi z}{\Gamma(\frac{3}{2})} & \frac{2\pi z}{\Gamma(\frac{3}{2})} \\ -\frac{\pi z^2}{2\pi} & -\frac{\pi z^2}{2\pi} & -\frac{\pi z^2}{2\pi} \end{array} \right).$$

(32)

From this point of view it is not obvious a priori that $U$ is a symplectomorphism, or that it satisfies conditions (a) and (c) in Conjecture 2.1 — this is one advantage of the more sophisticated approach taken in [15, 40] — but now that we have an explicit expression for $U$ it is easy to check these things.

**Theorem 4.4** (The Crepant Resolution Conjecture for $[\mathbb{C}^3/\mathbb{Z}_3]$). Conjecture 2.1 holds for $X = [\mathbb{C}^3/\mathbb{Z}_3]$, $Y = K_{\mathbb{P}^2}$.

**Proof.** It remains only to check that, after analytic continuation, $U$ maps $\mathcal{L}_X$ to $\mathcal{L}_Y$. But $U$ was constructed so as to map $I_X$ to the analytic continuation of $I_Y$, and $\mathcal{L}_X$ (respectively $\mathcal{L}_Y$) is uniquely determined by the fact that $x \mapsto I_X(x, -z)$ is a family of elements of $\mathcal{L}_X$ (respectively that $y \mapsto I_Y(y, -z)$ is a family of elements of $\mathcal{L}_Y$). Thus $U$ maps $\mathcal{L}_X$ to the analytic continuation of $\mathcal{L}_Y$. □

**Corollary 4.5** (The Cohomological Crepant Resolution Conjecture for $[\mathbb{C}^3/\mathbb{Z}_3]$). The algebra obtained from the $T$-equivariant small quantum cohomology algebra of $Y = K_{\mathbb{P}^2}$ by analytic continuation in the parameter $q_1$ followed by the specialization $q_1 = 1$ is isomorphic to the $T$-equivariant Chen–Ruan orbifold cohomology of $X = [\mathbb{C}^3/\mathbb{Z}_3]$.

**Proof.** The quantity $c^1$ defined in (14) is zero. Now apply Corollary 3.1 □

**Remark.** The symplectic transformation (32) with $z = 1$ looks similar to the symplectic transformation computed by Aganagic–Bouchard–Klemm in [3], but it is not the same. It would be interesting to understand the source of the discrepancy.

5. **Example II**: $X = K_{\mathbb{P}(1,1,2)}$, $Y = K_{\mathbb{P}^2}$

In this example we take $X := K_{\mathbb{P}(1,1,2)}$ to be the total space of the canonical bundle of the weighted projective space $\mathbb{P}(1,1,2)$ and $Y := K_{\mathbb{P}^2}$ to be the total space of the canonical bundle of the Hirzebruch surface $\mathbb{P}_2$. We use exactly the same methods as before.

**Toric Geometry.** Consider the action of $(\mathbb{C}^\times)^2$ on $\mathbb{C}^5$ such that $(s, t) \in (\mathbb{C}^\times)^2$ acts as

$$\begin{pmatrix} x \\ y \\ z \\ u \\ v \end{pmatrix} \mapsto \begin{pmatrix} tx \\ ty \\ sz \\ st^{-2} u \\ s^{-2} v \end{pmatrix}.$$  

(33)

The secondary fan is:
Figure 5: The secondary fan for $Y = K_{\mathbb{P}^2}$

where the roman numerals label the different chambers. There is an exact sequence:

$$0 \to \mathbb{Z}^2 \to \mathbb{Z}^5 \to \mathbb{Z}^3 \to 0,$$

and so each chamber in the secondary fan corresponds to a toric orbifold with fan equal to some triangulation of the rays

$$\left(\begin{array}{c} 1 \\ -1 \\ 0 \end{array}\right), \left(\begin{array}{c} 0 \\ 2 \\ -1 \end{array}\right), \left(\begin{array}{c} 0 \\ 0 \\ 2 \end{array}\right), \left(\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right).$$

These fans are cones over the following pictures in the plane $x + y + z = 1$:

Figure 6: The fans corresponding to chambers I–IV (respectively) are the cones over these pictures in the plane $x + y + z = 1$

The toric orbifold corresponding to a chamber $C$ in the secondary fan is the GIT quotient $C^5/\xi(C^\times)^2$, $\xi \in C$. This is produced by deleting an appropriate union of co-ordinate subspaces from $C^5$ and then taking the quotient by the action $\xi$. When $C$ is chamber I, the corresponding toric orbifold is the canonical bundle $K_{\mathbb{P}^2}$; chamber II gives rise to the canonical bundle $K_{\mathbb{P}(1,1,2)}$; chamber III gives the orbifold $[C^3/Z_4]$, where $Z_4$ acts on $C^3$ with weights $(1,1,2)$; and chamber IV gives a quotient by $Z_2$ of the total space of the vector bundle $\mathcal{O} \oplus \mathcal{O}(-2) \to \mathbb{P}^1$.

| chamber | locus to delete | quotient |
|---------|----------------|----------|
| I       | $\{x = y = 0\} \cup \{z = u = 0\}$ | $Y = K_{\mathbb{P}^2}$ |
| II      | $\{u = 0\} \cup \{x = y = z = 0\}$ | $\mathcal{X} = K_{\mathbb{P}(1,1,2)}$ |
| III     | $\{u = 0\} \cup \{v = 0\}$ | $[C^3/Z_4]$ |
| IV      | $\{v = 0\} \cup \{x = y = 0\}$ | $[(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))/Z_2]$ |

Table 1: The different GIT quotients given by the secondary fan for $Y = K_{\mathbb{P}^2}$
In this section we study the crepant resolution

\[
\begin{align*}
K_{\mathbb{P}_2} & \quad K_{\mathbb{P}(1,1,2)} \\
\downarrow & \quad \downarrow \\
X & \quad [\mathbb{C}^3/\mathbb{Z}_4]
\end{align*}
\]

induced by moving from chamber I to chamber II. In the next section we consider the crepant partial resolution

\[
\begin{align*}
K_{\mathbb{P}(1,1,2)} & \quad [\mathbb{C}^3/\mathbb{Z}_4] \\
\downarrow & \\
\mathbb{C}^3/\mathbb{Z}_4 &
\end{align*}
\]

obtained by moving from chamber II to chamber III. We will not discuss chamber IV at all.

**The \( T \)-Action.** The action of \( T = \mathbb{C}^\times \) on \( \mathbb{C}^5 \) such that \( \alpha \in T \) maps

\[
\begin{pmatrix}
x \\
y \\
z \\
u \\
v
\end{pmatrix}
\mapsto
\begin{pmatrix}
x \\
y \\
z \\
u \\
u \alpha v
\end{pmatrix}
\]

descends to give actions of \( T \) on \( \mathcal{X}, \mathcal{X}, \) and \( Y \). The induced actions on \( \mathcal{X} \) and \( Y \) are respectively the canonical \( \mathbb{C}^\times \)-actions on the line bundles \( K_{\mathbb{P}_2} \to \mathbb{P}_2 \) and \( K_{\mathbb{P}(1,1,2)} \to \mathbb{P}(1,1,2) \); they cover the trivial actions on respectively \( \mathbb{P}_2 \) and \( \mathbb{P}(1,1,2) \). The crepant resolution \( \mathcal{X} \) is \( T \)-equivariant.

**Bases for Everything.** We have

\[ r := \text{rank} \ H^2(Y; \mathbb{C}) = 2, \quad s := \text{rank} \ H^2(\mathcal{X}; \mathbb{C}) = 1. \]

Let \( p_1, p_2 \in H(Y) \) denote the \( T \)-equivariant Poincaré-duals to the divisors \( \{z = 0\} \) and \( \{x = 0\} \) respectively. Then

\[ H(Y) = \mathbb{C}(\lambda)[p_1, p_2]/\langle p_2^3, p_1(p_1 - 2p_2) \rangle. \]

We set

\[ \varphi_0 = 1, \quad \varphi_1 = p_1, \quad \varphi_2 = p_2, \quad \varphi_3 = p_1p_2, \]

so that

\[ \varphi^0 = \lambda p_1p_2, \quad \varphi^1 = \lambda p_2 - 2p_1p_2, \quad \varphi^2 = \lambda p_1 - 2\lambda p_2, \quad \varphi^3 = \lambda - 2p_1. \]

The inertia stack \( \mathcal{I} \mathcal{X} \) of \( \mathcal{X} \) is the disjoint union \( \mathcal{X}_0 \coprod \mathcal{X}_1/2 \), where \( \mathcal{X}_f \) is the component of the inertia stack corresponding to the fixed locus of the element \( (1, e^{2\pi i f/\sqrt{\mathbb{C}}}) \in (\mathbb{C}^\times)^2 \). We have \( \mathcal{X}_0 = K_{\mathbb{P}(1,1,2)} \) and \( \mathcal{X}_1/2 = [\mathbb{C}/\mathbb{Z}_2] \). Define \( 1_f \in H(\mathcal{X}) \) to be the class which restricts to the unit class on the component \( \mathcal{X}_f \) and restricts to zero on the other component, and let \( p \in H(\mathcal{X}) \) denote the first Chern class of the line bundle \( \mathcal{O}(1) \to \mathbb{P}(1,1,2) \), pulled back to \( \mathcal{X} = K_{\mathbb{P}(1,1,2)} \) via the natural projection and then regarded as an element of Chen–Ruan orbifold cohomology via the inclusion \( \mathcal{X} \to \mathcal{I} \mathcal{X} \). Let

\[ \phi_0 = 1_0, \quad \phi_1 = p, \quad \phi_2 = p^2, \quad \phi_3 = 1_{1/2}, \]

so that

\[ \phi^0 = 2\lambda p^2, \quad \phi^1 = 2\lambda p - 8p^2, \quad \phi^2 = 2\lambda 1_0 - 8p, \quad \phi^3 = 2\lambda 1_{1/2}, \]

and \( r_1 = \frac{1}{2} \).
Step 1: A Family of Elements of $\mathcal{L}_Y$. Consider

$$I_Y(y_1, y_2, z) := z \sum_{k,l \geq 0} \frac{\Gamma(1 + \frac{p_1}{z})^2}{\Gamma(1 + \frac{p_2}{z} + l)^2} \frac{\Gamma(1 + \frac{p_1}{z} + k)}{\Gamma(1 + \frac{p_2}{z} + k - 2l)} \times \frac{\Gamma(1 + \frac{\lambda - 2m}{z})}{\Gamma(1 + \frac{\lambda - 2m}{z} - 2k)} y_1^{k + p_1/z} y_2^{l + p_2/z}. \quad (35)$$

This series converges, in a region where $|y_1|$ and $|y_2|$ are sufficiently small, to a multi-valued analytic function of $(y_1, y_2)$ which takes values in $\mathcal{H}_Y$. We have

$$I_Y(y_1, y_2, z) = z \sum_{k,l \geq 0} \prod_{1 \leq m \leq (p_2 + m z)^2} \prod_{1 \leq m \leq k (p_1 + m z)} \prod_{m \leq k - 2l (p_1 - 2 p_2 + m z)} y_1^{k + p_1/z} y_2^{l + p_2/z}. \quad (36)$$

Note that all but finitely many terms in the two infinite products here cancel.

**Proposition 5.1.**

$$I_Y(y_1, y_2, -z) \in \mathcal{L}_Y \quad \text{for all } (y_1, y_2) \text{ in the domain of convergence of } I_Y.$$

**Proof.** We combine Theorem 3.5, which tells us how to modify the small $J$-function of $\mathbb{F}_2$, with Theorem 0.1 in [31], which tells us how to compute the small $J$-function of $\mathbb{F}_2$. In detail, this goes as follows. We apply Theorem 3.5 with $B = \mathbb{F}_2$ and $C = \mathcal{K}_2$. Note that $c_1(K_2) = -2 p_1$. Theorem 3.5 implies that if

$$M_k := \prod_{-2k < m \leq 0} (\lambda - 2 p_1 + m z)$$

and

$$I_Y(y_1, q_2, z) := z q_1^{p_1/z} q_2^{p_2/z} \left( 1 + \sum_{k,l \geq 0} M_k \sum_{0 \leq 0} \frac{\varphi_e}{z(z - \psi)} \right) \left( \frac{\varphi_e}{z(z - \psi)} \right)_{0,1,k_{l+1},l_{l+2}} \quad (37)$$

then $(q_1, q_2) \mapsto I_Y(q_1, q_2, -z)$ is a family of elements of $\mathcal{L}_Y$.

Givental has shown [31] Theorem 0.1 that the small $J$-function of $\mathbb{F}_2$,

$$J_{\mathbb{F}_2}(q_1, q_2, z) = z q_1^{p_1/z} q_2^{p_2/z} \left( 1 + \sum_{k,l \geq 0} \left( \frac{\varphi_e}{z(z - \psi)} \right)_{0,1,k_{l+1},l_{l+2}} \right),$$

coincides with

$$I_{\mathbb{F}_2}(w_1, w_2, z) = z w_1^{p_1/z} w_2^{p_2/z} \left( 1 + \sum_{k,l \geq 0} \left( \frac{\varphi_e}{z(z - \psi)} \right)_{0,1,k_{l+1},l_{l+2}} \right) \prod_{1 \leq m \leq (p_2 + m z)^2} \prod_{1 \leq m \leq k (p_1 + m z)} \prod_{m \leq k - 2l (p_1 - 2 p_2 + m z)} \prod_{m \leq k - 2l (p_1 - 2 p_2 + m z)} w_1^{k_{l+1}} w_2^{l_{l+2}} \prod_{m \leq 0 (p_1 - 2 p_2 + m z)}$$

after the change of variables

$$q_1 = w_1 \exp \left( -f(w_2) \right), \quad q_2 = w_2 \exp \left( 2f(w_2) \right),$$

where

$$f(x) = \sum_{l \geq 0} \frac{(2l - 1) !}{(l !)^2} x^l.$$ 

The inverse change of variables is

$$w_1 = q_1 \exp \left( F(q_2) \right), \quad w_2 = q_2 \exp \left( -2F(q_2) \right),$$

for some function $F$ and so, from the equality

$$J_{\mathbb{F}_2}(q_1, q_2, z) = I_{\mathbb{F}_2}(w_1, w_2, z),$$

we deduce that

$$1 + \sum_{k,l \geq 0} q_1^{k_{l+1}} q_2^{l_{l+2}} \left( \frac{\varphi_e}{z(z - \psi)} \right)_{0,1,k_{l+1},l_{l+2}} \exp \left( \frac{2p_2}{z} F(q_2) \right) \sum_{k,l \geq 0} \left( \frac{\varphi_e}{z(z - \psi)} \right)_{0,1,k_{l+1},l_{l+2}} \prod_{1 \leq m \leq (p_2 + m z)^2} \prod_{1 \leq m \leq k (p_1 + m z)} \prod_{m \leq k - 2l (p_1 - 2 p_2 + m z)} \prod_{m \leq k - 2l (p_1 - 2 p_2 + m z)} q_1^{k_{l+1}} q_2^{l_{l+2}} \exp \left( (k - 2l)F(q_2) \right) \prod_{m \leq 0 (p_1 - 2 p_2 + m z)}$$

In this example it is easy to compute a closed form for $F$, but typically this is not the case.
Extracting the coefficient of $q_1^k$ here and substituting it into (37) gives
\[
I_Y(q_1, q_2, z) = z q_1^{p_1/z} q_2^{p_2/z} \exp \left( \frac{p_1 - 2p_2}{z} F(q_2) \right) \sum_{k,l \geq 0} \frac{\prod_{1 \leq m \leq l} (p_2 + mz)^2 \prod_{1 \leq m \leq k} (p_1 + mz)}{\prod_{m \leq k-2l} (p_1 - 2p_2 + mz)} q_1^k q_2^l \exp ((k - 2l) F(q_2)).
\]

Now setting
\[
y_1 = q_1 \exp (F(q_2)), \quad y_2 = q_2 \exp (-2F(q_2)),
\]
we see that $I_Y(q_1, q_2, z) = I_Y(y_1, y_2, z)$. It follows that $I_Y(y_1, y_2, -z) \in \mathcal{L}_Y$ for all $(y_1, y_2)$ in the domain of convergence of $I_Y$.

**Step 2: $I_Y$ Determines $\mathcal{L}_Y$.** We have:

**Corollary 5.2.**

\[
J_Y(q_1, q_2, z) = e^{\lambda g(y_1, y_2)/z} I_Y(y_1, y_2, z)
\]

where:
\[
q_1 = y_1 \exp \left( 2g(y_1, y_2) - f(y_2) \right), \quad q_2 = y_2 \exp \left( 2f(y_2) \right),
\]
\[
f(y_2) = \sum_{l > 0} \frac{(2l - 1)!}{(l!)^2} y_2^l, \quad g(y_1, y_2) = \sum_{0 < k \leq \infty} \frac{(2k - 1)!}{(l!)^2 k!(k-2l)!} y_1^k y_2^l.
\]

**Proof.** We argue exactly as in Corollary 4.2. Note that
\[
I_Y(y_1, y_2, z) = z + p_1 \left[ \log y_1 - f(y_2) + 2g(y_1, y_2) \right] + p_2 \left[ \log y_2 + 2f(y_2) \right] - \lambda g(y_1, y_2) + O(z^{-1}).
\]

It follows from Propositions 3.4 and 4.1 that
\[
y \mapsto e^{-\lambda g(y_1, y_2)/z} I_Y(y_1, y_2, -z)
\]
is a family of elements of $\mathcal{L}_Y$. But
\[
e^{-\lambda g(y_1, y_2)/z} I_Y(y_1, y_2, -z) = -z + p_1 \log q_1 + p_2 \log q_2 + O(z^{-1}),
\]
where $q_1$ and $q_2$ are defined above, and the unique family of elements of $\mathcal{L}_Y$ of this form is $(q_1, q_2) \mapsto J_Y(q_1, q_2, -z)$. $\square$

It follows, as before, that the series defining $J_Y(q_1, q_2, z)$ converges to a multivalued analytic function when $|q_1|$ and $|q_2|$ are sufficiently small. Proposition 5.3 implies that $\mathcal{L}_Y$ is uniquely determined by the fact that $(y_1, y_2) \mapsto I_Y(y_1, y_2, -z)$ is a family of elements of $\mathcal{L}_Y$.

**Aside: Computing Gromov–Witten Invariants of $Y$.** As in the previous example, one can invert the change of variables $(y_1, y_2) \mapsto (q_1, q_2)$ and read off genus-zero Gromov–Witten invariants of $Y$. In the non-equivariant limit $\lambda \to 0$ this reproduces the results of the $B$-model calculation of Chiang–Klemm–Yau–Zaslow: see [17] Table 11 and the discussion thereafter.

**Step 3: A Family of Elements of $\mathcal{L}_X$.** Let
\[
I_X(x, z) := \sum_{d,d \geq 0, \ 2d \in \mathbb{Z}} x^{d+p/z} \frac{\Gamma(1 - \frac{1}{2} \frac{\lambda}{z} - d + \frac{p}{z})}{\Gamma(1 + \frac{p}{z} + d)} \frac{\Gamma(1 + \frac{\lambda}{z} - p/2) \Gamma(1 + \frac{\lambda}{z} - p/2 + 2d)}{\Gamma(1 + \frac{\lambda}{z} - p/2)} \frac{1}{2^{\text{frac}(d)}}
\]

This converges, in the region $\{x \in \mathbb{C} : 0 < |x| < \frac{1}{16}\}$, to a multivalued analytic function which takes values in $\mathcal{H}_X$. We have
\[
I_X(x, z) = \sum_{d,d \geq 0, \ 2d \in \mathbb{Z}} x^{d+p/z} \prod_{-4d \leq m \leq 0} \prod_{b \leq c \leq d, \ \text{frac}(b) = \text{frac}(c)} (p + mz) \frac{1}{\text{frac}(d)}
\]

**Proposition 5.3.**

\[
I_X(x, -z) \in \mathcal{L}_X \quad \text{for all } x \text{ such that } 0 < |x| < \frac{1}{64}.
\]

**Proof.** Argue exactly as in Propositions 4.1 and 5.1 combining Theorem 5.3 with [22] Theorem 1.7. Theorem 5.3 here tells us how to modify the small $J$-function of $\mathbb{P}(1, 1, 2)$ and Theorem 1.7 in [22] tells us how to compute the small $J$-function of $\mathbb{P}(1, 1, 2)$. $\square$
Step 4: $I_X$ Determines $\mathcal{L}_X$. We have:

**Corollary 5.4.**

$$J_X(q, z) = e^{\lambda h(x)/x} I_X(x, z)$$

where

$$q = x \exp \left(4h(x)\right)$$

$$h(x) = \sum_{n>0} \frac{(4n-1)!}{(n!)^2(2n)!} x^n.$$  

**Proof.** Argue exactly as in Corollaries 4.2 and 5.2.  

This implies that the series defining $J_X(q, z)$ converges, for $|q|$ sufficiently small, to a multivalued analytic function of $q$. It also implies, via Proposition 3.4b, that $\mathcal{L}_X$ is uniquely determined by the fact that $x \mapsto I_X(x, -z)$ is a family of elements of $\mathcal{L}_X$.

**Aside: Computing Gromov–Witten Invariants of $\mathcal{X}$.** We can, as before, use Corollary 5.4 to compute genus-zero Gromov–Witten invariants of $\mathcal{X}$. We do this in Appendix A.

Step 5: The $B$-model Moduli Space and the Picard–Fuchs System. The $B$-model moduli space $\mathcal{M}_B$ here is the toric orbifold corresponding to the secondary fan for $Y$ (Figure 5). It has four co-ordinate patches, one for each chamber. We will concentrate on the co-ordinate patches corresponding to chambers I and II. The co-ordinates $(y_1, y_2)$ coming from chamber I are dual respectively to $p_1$ and $p_2$; the co-ordinates $(\hat{y}_1, \hat{y}_2)$ from chamber II are dual respectively to $p_1$ and $p_1 - 2p_2$. We have

$$y_1 = \hat{y}_1 \hat{y}_2 \quad \hat{y}_1 = y_1^{1/2}$$

$$y_2 = \hat{y}_2^2 \quad \hat{y}_2 = y_2^{-1/2}.$$  

(40)

We regard $I_Y(y_1, y_2, z)$ as a function on the co-ordinate patch corresponding to chamber I. Writing

$$I_Y(y, z) = I_Y^0 \varphi_0 + I_Y^1 \varphi_1 + I_Y^2 \varphi_2 + I_Y^3 \varphi_3,$$

the components $\{I_Y^j : j = 0, 1, 2, 3\}$, which are functions of $y_1, y_2, \lambda$, and $z$, form a basis of solutions to the system of differential equations

$$D_1(D_1 - 2D_2)f = y_1(\lambda - 2D_1)(\lambda - 2D_1 - z)f$$

$$D_2^2 f = y_2(D_1 - 2D_2)(D_1 - 2D_2 - z)f.$$  

(41)

where $D_1 = zy_1 \partial_{\lambda}$ and $D_2 = zy_2 \partial_{\lambda}$.

We regard $I_X(x, z)$ as a function on the sublocus $(\hat{y}_1, \hat{y}_2) = (x^{1/2}, 0)$ of the co-ordinate patch corresponding to chamber II. (The choice of square root causes no ambiguity here, as the locus $\hat{y}_2 = 0$ in the orbifold $\mathcal{M}_B$ has automorphism group $\mathbb{Z}_2$.) Writing

$$I_X(x, z) = I_X^0 \phi_0 + I_X^1 \phi_1 + I_X^2 \phi_2 + I_X^3 \phi_3,$$

the components $\{I_X^j : j = 0, 1, 2, 3\}$, which are functions of $x, \lambda$, and $z$, form a basis of solutions to the differential equation

$$D_x^2(2D_x)(2D_x - z)f = \left[ x \prod_{m=0}^3 (\lambda - 4D_x - mz) \right] f.$$  

(42)

where $D_x = zx \partial_{\lambda}$.

Restricting the system of differential equations (41) to the locus $(\hat{y}_1, \hat{y}_2) = (x^{1/2}, 0)$ gives the differential equation (42). Thus if we analytically continue $I_Y(y_1, y_2, z)$ to a region where $|y_2|$ is large, write the analytic continuation $\bar{I}_Y$ in terms of the co-ordinates $(\hat{y}_1, \hat{y}_2)$, and then set $\hat{y}_1 = x^{1/2}$, $\hat{y}_2 = 0$ then the components $\bar{I}_Y^j$ of $\bar{I}_Y$ will satisfy the differential equation (42). The components $\bar{I}_X^j$ of $\bar{I}_X$ give a basis of solutions to (42), so

$$\begin{pmatrix} \bar{I}_Y^0(x^{1/2}, 0, z) \\ \bar{I}_Y^1(x^{1/2}, 0, z) \\ \bar{I}_Y^2(x^{1/2}, 0, z) \\ \bar{I}_Y^3(x^{1/2}, 0, z) \end{pmatrix} = M(\lambda, z) \begin{pmatrix} I_X^0(x, z) \\ I_X^1(x, z) \\ I_X^2(x, z) \\ I_X^3(x, z) \end{pmatrix}$$  

(43)

for some $3 \times 3$ matrix $M$ which is independent of $x$ (and hence depends only on $\lambda$ and $z$). The matrix $M(\lambda, z)$ defines the $\mathbb{C}((z^{-1}))$-linear symplectic transformation $U : \mathcal{H}_X \rightarrow \mathcal{H}_Y$ which we seek. To determine it, we first calculate the analytic continuation $\bar{I}_Y$.  

22
Step 6: Analytic Continuation. To calculate \( \tilde{I}_Y \) we, for each \( k \geq 0 \), extract the coefficient of \( q_1^k \) from (35) and then analytically continue it to a region where \( |y_2| \) is large, using the Mellin–Barnes method described in Section II. The result is:

\[
\tilde{I}_Y(y_1, y_2, z) = z \sum_{k,n \geq 0} (-1)^{k-n} \frac{\sin \left( \pi \left[ \frac{p_1-2p_2}{2} \right] \right)}{2n!} \frac{\sin \left( \pi \left[ \frac{p_1-2p_2}{2} + k \right] \right)}{\Gamma \left( 1 + \frac{p_1}{2} \right)^2} \times \frac{\Gamma \left( 1 + \frac{p_1}{2} \right)^2}{\Gamma \left( 1 + \frac{p_1}{2} + k \right)} \Gamma \left( 1 + \frac{p_1-2p_2}{2} \right) \Gamma \left( 1 + \frac{p_1-2p_2}{2} + k \right) \Gamma \left( 1 + \frac{p_1-2p_2}{2} + k \right) \Gamma \left( 1 + \frac{p_1-2p_2}{2} - k \right).
\]

Thus

\[
\tilde{I}_Y(x^{1/2}, 0, -z) = -z x^{-p_2/2} \sum_{k \geq 0} (-x)^{k/2} \frac{\sin \left( \pi \left[ \frac{p_1-2p_2}{2} \right] \right)}{2n!} \frac{\sin \left( \pi \left[ \frac{p_1-2p_2}{2} + k \right] \right)}{\Gamma \left( 1 + \frac{p_1}{2} \right)^2} \times \frac{\Gamma \left( 1 + \frac{p_1}{2} \right)^2}{\Gamma \left( 1 + \frac{p_1}{2} + k \right)} \Gamma \left( 1 + \frac{p_1-2p_2}{2} \right) \Gamma \left( 1 + \frac{p_1-2p_2}{2} + k \right) \Gamma \left( 1 + \frac{p_1-2p_2}{2} - k \right). \tag{44}
\]

Step 7: Compute the Symplectic Transformation. Recall that \( U(I_{\mathcal{X}}(x, -z)) = \tilde{I}_Y(x^{1/2}, 0, -z) \) and that

\[
I_{\mathcal{X}}(x, -z) = -z x^{-p_2/2} \left( I_0 - x^{1/2} \frac{4\lambda(\lambda + z)}{z^3} I_{1/2} + \cdots \right). \tag{45}
\]

We compute \( U \) by comparing powers of \( x^a (\log x)^b \) in (44) and (45). This gives:

\[
\begin{align*}
U(I_0) &= (-1)^{p_1-2p_2}, \\
U(p) &= \frac{p_1}{2}, \\
U(p^2) &= \frac{p_1p_2}{2}, \\
U(I_{1/2}) &= (-1)^{p_1-2p_2} (-1)^{1/2} \frac{p_1 - 2p_2}{2}.
\end{align*}
\]

Note that the expression (44) simplifies significantly when evaluated in the algebra \( H(Y) \), and in particular the dependence of \( U \) on \( \lambda \) cancels. The matrix of \( U \) is

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
\frac{\pi \sqrt{-1}}{2} & 1 & 0 & \frac{\sqrt{-1}}{2} \\
-\frac{\pi \sqrt{-1}}{2} & 0 & 0 & -\frac{\sqrt{-1}}{2} \\
\frac{z^2}{4\pi^2} & 0 & 1 & \frac{z}{\pi^2}
\end{pmatrix}.
\]

**Theorem 5.5** (The Crepant Resolution Conjecture for \( K_{\mathbb{P}(1,1,2)} \)). *Conjecture 2.1 holds for \( \mathcal{X} = K_{\mathbb{P}(1,1,2)} \), \( Y = K_{\mathbb{P}_2} \).*

**Proof.** Argue exactly as in the proof of Theorem 4.4 □

**Corollary 5.6** (The Bryan–Graber Conjecture for \( K_{\mathbb{P}(1,1,2)} \)). *The \( \mathbb{C}(\lambda) \)-linear map \( U : H(\mathcal{X}) \to H(Y) \) given by

\[
\begin{align*}
U_\infty(I_0) &= 1_Y, \\
U_\infty(p) &= \frac{p_1}{2}, \\
U_\infty(p^2) &= \frac{p_1p_2}{2}, \\
U_\infty(I_{1/2}) &= \frac{\sqrt{-1}}{2}(p_1 - 2p_2)
\end{align*}
\]

induces an algebra isomorphism between the small quantum cohomology of \( \mathcal{X} \) and the algebra obtained from the small quantum cohomology of \( Y \) by analytic continuation in the parameter \( q_2 \) followed by the substitution \( q_1 = -u_1^{1/2} \sqrt{-1} \), \( q_2 = -1 \).*

**Proof.** Apply Corollary 3.3 □
6. Example III: \( \mathcal{X} = \mathbb{C}^3 / \mathbb{Z}_4 \), \( Y = \mathbb{P}(1,1,2) \)

We next consider an example of a crepant partial resolution. Let \( \mathcal{X} \) be the orbifold \( \mathbb{C}^3 / \mathbb{Z}_4 \) where \( \mathbb{Z}_4 \) acts on \( \mathbb{C}^3 \) with weights \((1,1,2)\). The coarse moduli space \( X \) of \( \mathcal{X} \) is the quotient singularity \( \frac{1}{4}(1,1,2) \), and a crepant partial resolution \( Y \) of \( X \) is the canonical bundle \( K_{\mathbb{P}(1,1,2)} \). We make the obvious modifications to our general setup, replacing the vector space \( H \) with \( H(Y) := H_{\ast, K_{\mathbb{P}(1,1,2)}}(Y; \mathbb{C}) \otimes \mathbb{C}(\lambda) \) and writing \( Y \) for the coarse moduli space of \( \mathcal{X} \). In this section we omit some details and all proofs, as the argument is completely parallel to that in Section 3.

**Toric Geometry.** Consider the action of \( \mathbb{C}^\times \) on \( \mathbb{C}^4 \) such that \( s \in \mathbb{C}^\times \) acts as

\[
\begin{pmatrix}
x \\
y \\
z \\
w
\end{pmatrix}
\mapsto
\begin{pmatrix}
x \\
y \\
z \\
w
\end{pmatrix}
\begin{pmatrix}
& s & & \\
& & s & \\
& s^2 & & \\
& s^4 & & \\
\end{pmatrix},
\]

The secondary fan is:

![Figure 7: The secondary fan for \( \mathcal{Y} = \mathbb{P}(1,1,2) \)](image)

For \( \xi \) in the right-hand chamber, the GIT quotient \( \mathbb{C}^4 /_\xi \mathbb{C}^\times \) gives \( \mathcal{Y} \); for \( \xi \) in the left-hand chamber, \( \mathbb{C}^4 /_\xi \mathbb{C}^\times \) gives \( \mathcal{X} \).

**The \( T \)-Action.** The action of \( T = \mathbb{C}^\times \) on \( \mathbb{C}^4 \) such that \( \alpha \in T \) acts as

\[
\begin{pmatrix}
x \\
y \\
z \\
w
\end{pmatrix}
\mapsto
\begin{pmatrix}
x \\
y \\
z \\
\alpha w
\end{pmatrix},
\]

descends to give \( T \)-actions on \( \mathcal{X} \), \( X \), and \( \mathcal{Y} \). The induced action on \( \mathcal{X} \) is

\[
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
\mapsto
\begin{pmatrix}
\alpha^{1/4} x \\
\alpha^{1/4} y \\
\alpha^{1/2} z
\end{pmatrix},
\]

and the induced action on \( \mathcal{Y} \) is the canonical \( \mathbb{C}^\times \)-action on the line bundle \( K_{\mathbb{P}(1,1,2)} \to \mathbb{P}(1,1,2) \). The crepant partial resolution

\[
K_{\mathbb{P}(1,1,2)} \to \mathbb{C}^3 / \mathbb{Z}_4
\]

is \( T \)-equivariant; the left-hand map here collapses the zero section.

**Bases for Everything.** We have

\[
r := \text{rank } H^2(\mathcal{Y}; \mathbb{C}) = 1, \quad s := \text{rank } H^2(\mathcal{X}; \mathbb{C}) = 0.
\]

Fix bases for \( H(\mathcal{Y}) \) exactly as in the previous section:

\[
\begin{align*}
\varphi_0 &= 1_0, & \varphi_1 &= p, & \varphi_2 &= p^2, & \varphi_3 &= 1_{1/2}, \\
\varphi_0^0 &= 2 \lambda p^2, & \varphi_1^1 &= 2 \lambda p - 8 p^2, & \varphi_2^2 &= 2 \lambda 1_0 - 8 p, & \varphi_3^3 &= 2 \lambda 1_{1/2}.
\end{align*}
\]

The components of the inertia stack of \( \mathcal{X} \) are indexed by elements of \( \mathbb{Z}_4 \). Let \( 1_k / 4 \in H(\mathcal{X}) \) denote the orbifold cohomology class which restricts to the unit class on the inertia component indexed by \([k] \in \mathbb{Z}_4\) and restricts to zero on the other components. Let

\[
\begin{align*}
\phi_0 &= 1_0, & \phi_1 &= 1_{1/4}, & \phi_2 &= 1_{1/2}, & \phi_3 &= 1_{3/4}.
\end{align*}
\]
where

\[ D = \frac{\lambda}{m} 1_0, \quad \phi^1 = 41_{3/4}, \quad \phi^2 = 2\lambda 1_{1/2}, \quad \phi^3 = 41_{1/4}. \]

**Step 1: A Family of Elements of \( L_Y \).** Let

\[
I_Y(y, z) := z \sum_{d,d \geq 0, 2d \in \mathbb{Z}} y^{d+p/z} \prod_{b, 0 < b < \frac{d}{z}, \frac{b}{\text{frac}(b)} = \text{frac}(d)} (\lambda - b2z) \prod_{b, 0 \leq b < \frac{d}{z}, \frac{b}{\text{frac}(b)} = \text{frac}(d)} (p + bz) \prod_{1 \leq m \leq 2d(2p + mz)} 1_{\text{frac}(d)}. \tag{47}
\]

Proposition \[ 5.3 \] shows that \( y \mapsto I_Y(y, -z) \) gives a family of elements of \( L_Y \).

**Step 2: \( I_Y \) Determines \( L_Y \).** We showed in Step 4 of Section \[ 5 \] that \( L_Y \) is uniquely determined by the fact that \( y \mapsto I_Y(y, -z) \) is a family of elements of \( L_Y \).

**Step 3: A Family of Elements of \( L_X \).** Let

\[
I_X(x, z) := z x^{-\lambda/z} \sum_{l \geq 0} x^{l/z} \prod_{b, 0 \leq b < \frac{4}{z}, \frac{b}{\text{frac}(b)} = \frac{4}{z}} (\lambda_1 - b2z) \prod_{b, 0 \leq b < \frac{4}{z}, \frac{b}{\text{frac}(b)} = \frac{4}{z}} (\lambda_2 - b2z) 1_{\text{frac}(4)}. \tag{48}
\]

This converges, in a region where \( |x| \) is small, to a multivalued analytic function which takes values in \( H_X \). Theorems \[ 6.7 \] and Proposition \[ 8.4 \] imply that \( I_X(x, -z) \in L_X \) for all \( x \) in the domain of convergence of \( I_X \).

**Step 4: \( I_X \) Determines \( L_X \).** We have:

\[
J^0_X \, (\tau^1 1_{1/4}, z) = x^{\lambda/z} I_X(x, z) \quad \text{where} \quad \tau^1 = \sum_{m \geq 0} x^{4m+1} \frac{\Gamma(\frac{4}{3})^2}{(4m+1)!} \frac{\Gamma(\frac{4}{3})}{(\frac{4}{3} - m)^2} \Gamma(\frac{4}{3} - 2m). \tag{49}
\]

Proposition \[ 8.4 \] shows that \( L_X \) is uniquely determined by the fact that \( x \mapsto I_X(x, -z) \) is a family of elements of \( L_X \). In Appendix \[ A \] we use Corollary \[ 6.1 \] to compute genus-zero Gromov–Witten invariants of \( X \); our results verify predictions made by Brini and Tanzini \[ 9 \] on the basis of a correspondence between Gromov–Witten theory and certain five-dimensional gauge theories.

**Step 5: The \( B \)-model Moduli Space and the Picard–Fuchs System.** The \( B \)-model moduli space \( M_B \) here has two co-ordinate patches, one for each chamber in the secondary fan. Let \( x \) be the co-ordinate corresponding to the left-hand chamber (this is the chamber that gives rise to \( X \)) and let \( y \) be the co-ordinate corresponding to the right-hand chamber (this chamber gives \( Y \)). The co-ordinate patches are related by

\[
y = x^{-4} \tag{49}
\]

and so \( M_B \) is the weighted projective space \( \mathbb{P}(1, 4) \).

We regard \( I_X(x, z) \) as a function on the co-ordinate patch corresponding to \( X \) and \( I_Y(y, z) \) as a function on the co-ordinate patch corresponding to \( Y \). The components of \( I_Y(y, z) \) form a basis of solutions to the differential equation

\[
D_y^2(2D_y)(2D_y - z)f = \left[ y \prod_{m=0}^3 (\lambda - 4D_y - mz) \right] f \tag{50}
\]

where \( D_y = zy \frac{\partial}{\partial y} \) (c.f. equation \[ 12 \]). The components of \( I_X \) form a basis of solutions to the differential equation

\[
(-1)^3 D_x^2 (\lambda - 1 D_x) (\lambda - 1 D_x - z)f = \left[ x^{-4} \prod_{m=0}^3 (\lambda + D_x - mz) \right] f \tag{51}
\]

where \( D_x = zx \frac{\partial}{\partial x} \).

The change of variables \[ 19 \] turns \[ 60 \] into \[ 61 \] so, as before, if \( \tilde{I}_Y(x, z) \) denotes the analytic continuation of \( I_Y \) to the region where \(|y|\) is large then there exists a \( \mathbb{C}(\{z^{-1}\}) \)-linear map \( \mathbb{U} : H_X \to H_Y \) such that \( \mathbb{U}(I_X(x, -z)) = \tilde{I}_Y(x, -z) \). This map \( \mathbb{U} \) is the linear symplectomorphism which we seek.
Step 6: Analytic Continuation. Using the Mellin–Barnes method as before, but treating the coefficients of $1_0$ and $1_{1/2}$ in (47) separately, we find that

$$
\tilde{I}_Y(x, z) = z^{x-\lambda/2} \sum_{n \geq 0} x^n \left[ -\frac{1}{4n!} \frac{\sin \left( \frac{\pi (4\lambda z)}{4} \right)}{\sin \left( \frac{\pi (4\lambda z)}{4} \right)} \right] \times
\frac{\Gamma(1+\frac{2\lambda}{z})}{\Gamma(1+\frac{\lambda}{z} - \frac{3}{4})} \frac{\Gamma(1+\frac{\lambda}{z} - \frac{3}{4})}{\Gamma(1+\frac{\lambda}{z} - \frac{3}{4})} \Gamma(1+\frac{\lambda}{z} - \frac{3}{4}) 1_{1/2}.
$$

Step 7: Compute the Symplectic Transformation. By comparing powers of $x$ in the equality $\mathcal{U}(I_X(x, z)) = \tilde{I}_Y(x, z)$, we find that:

$$
\mathcal{U}(1_0) = \left( -\frac{1}{4} + \frac{\lambda + \frac{1}{4}}{4} \right) \sin \left( \frac{\pi (4\lambda z)}{4} \right) \Gamma \left( 1 - \frac{\lambda}{z} \right) \Gamma \left( 1 - \frac{\lambda}{z} \right) 1_{1/2},
$$

$$
\mathcal{U}(1_{1/4}) = \left( -\frac{1}{4} + \frac{\lambda + \frac{1}{4}}{4} \right) \sin \left( \frac{\pi (4\lambda z)}{4} \right) \Gamma \left( 1 - \frac{\lambda}{z} \right) \Gamma \left( 1 - \frac{\lambda}{z} \right) 1_{1/2},
$$

$$
\mathcal{U}(1_{1/2}) = \left( -\frac{1}{4} + \frac{\lambda + \frac{1}{4}}{4} \right) \sin \left( \frac{\pi (4\lambda z)}{4} \right) \Gamma \left( 1 - \frac{\lambda}{z} \right) \Gamma \left( 1 - \frac{\lambda}{z} \right) 1_{1/2},
$$

$$
\mathcal{U}(1_{3/4}) = \left( -\frac{1}{4} + \frac{\lambda + \frac{1}{4}}{4} \right) \sin \left( \frac{\pi (4\lambda z)}{4} \right) \Gamma \left( 1 - \frac{\lambda}{z} \right) \Gamma \left( 1 - \frac{\lambda}{z} \right) 1_{1/2}.
$$

Note that

$$
\mathcal{U}(1_0) = \left( 1 + \frac{\lambda}{2} \frac{\sqrt{-\pi}}{\pi} \right) 1_0 - \frac{\pi}{\pi} \frac{\sqrt{-\pi}}{\pi} p + O(z^{-2}).
$$

The matrix $M$ of $\mathcal{U}$ does not have a simple form, but in the non-equivariant limit it becomes

$$
\begin{pmatrix}
1 & -\frac{\sqrt{-\pi}}{\pi} & 0 & 0 \\
-\frac{\sqrt{-\pi}}{\pi} & \frac{\sqrt{2\pi}}{\sqrt{-\pi}} & \sqrt{-\pi} & -\frac{\pi}{\pi} \\
0 & -\frac{\sqrt{-\pi}}{\pi} & \frac{\sqrt{2\pi}}{\sqrt{-\pi}} & \frac{\sqrt{2\pi}}{\sqrt{-\pi}} \\
0 & 0 & 1 & 0
\end{pmatrix}.
$$

Conclusions. We have shown that Conjecture 2.1 holds, exactly as stated, for the crepant partial resolution $\mathcal{Y} \to X$.

Theorem 6.2 (A “Crepant Partial Resolution Conjecture” for $[\mathbb{C}^3/\mathbb{Z}_4]$). Conjecture 2.1 holds for $\mathcal{X} = [\mathbb{C}^3/\mathbb{Z}_4]$, $\mathcal{Y} = K_{\mathbb{Z}(1,1,2)}$. □

This gives a Ruan-style “Cohomological Crepant Partial Resolution Conjecture”:

Corollary 6.3. The algebra obtained from the $T$-equivariant small quantum cohomology algebra of $\mathcal{Y} = K_{\mathbb{Z}(1,1,2)}$ by analytic continuation in the parameter $q_1$ followed by the specialization $q_1 = -1$ is isomorphic to the $T$-equivariant Chen–Ruan orbifold cohomology of $\mathcal{X} = [\mathbb{C}^3/\mathbb{Z}_4]$.

Proof. Corollary 3.1 can be generalized to treat crepant partial resolutions. The result follows from this and from equation (54), which shows that $c^1 = \pi \sqrt{-1}$. □

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Corollary 6.4 (The Crepant Resolution Conjecture for \([\mathbb{C}^3/\mathbb{Z}_4]\)). Conjecture 2.1 holds for \(X = [\mathbb{C}^3/\mathbb{Z}_4]\), \(Y = K_{\mathbb{P}_2}\).

Proof. Take the symplectic transformation \(U\) to be the composition of those from Theorems 6.2 and 5.5. \(\Box\)

Let \(U\) be the symplectic transformation from Corollary 6.4. Condition (b) in Conjecture 2.1 makes it easy to compute \(U\): one essentially just needs to make the substitutions

\[ p \rightarrow \frac{p_1 + z^2 - 2p_2}{2}, \quad 1_0 \rightarrow (-1)^{p_1 - 2p_2}(-1)^{1/2}p_1 - 2p_2, \]

in (59). The resulting transformation, after setting \(z = 1\), agrees with that calculated by Brini–Tanzini in [9]. We have

\[ U(1_{3/4}) = z \frac{(1 - \sqrt{-1})\sqrt{\pi}}{4\Gamma(\frac{1}{4})^2} (\lambda - 2p_2) + \text{lower-order terms in } z, \]

so we do not expect the Bryan–Graber conjecture to hold here. But

\[ U(1_0) = 1_{K_{\mathbb{P}_2}} + \frac{\lambda \sqrt{-1}}{4z} - \frac{\sqrt{-1}p_2}{z} + O(z^{-2}), \]

so we have

Corollary 6.5 (The Cohomological Crepant Resolution Conjecture for \([\mathbb{C}^3/\mathbb{Z}_4]\)). The algebra obtained from the \(T\)-equivariant small quantum cohomology algebra of \(Y = K_{\mathbb{P}_2}\) by analytic continuation in the parameters \(q_1, q_2\) followed by the specialization \(q_1 = 1, q_2 = -1\) is isomorphic to the \(T\)-equivariant Chen–Ruan orbifold cohomology of \(X = [\mathbb{C}^3/\mathbb{Z}_4]\).

7. Example IV: \(X = K_{\mathbb{P}(1,1,3)}\)

Let us now consider the case where \(X := K_{\mathbb{P}(1,1,3)}\) is the canonical bundle of the weighted projective space \(\mathbb{P}(1,1,3)\) and \(Y \rightarrow X\) is the toric crepant resolution of the coarse moduli space of \(X\). We can treat this example using essentially the same methods as before, so we present our results as a series of exercises for the reader.

Toric Geometry. Consider the action of \((\mathbb{C}^\times)^2\) on \(\mathbb{C}^5\) such that \((s, t) \in (\mathbb{C}^\times)^2\) acts as

\[
\begin{pmatrix}
 x \\
 y \\
 z \\
 u \\
 v
\end{pmatrix} \mapsto \begin{pmatrix}
 t x \\
 t y \\
 s z \\
 st^{-3} u \\
 s^{-2} t v
\end{pmatrix}.
\]

The secondary fan is:

![Secondary Fan](image)

Exercise 7.1. Show that choosing \(\xi\) to lie in chamber II gives \(\mathbb{C}^4/\xi(\mathbb{C}^\times)^2 \cong X\), and choosing \(\xi\) to lie in chamber I gives \(\mathbb{C}^4/\xi(\mathbb{C}^\times)^2 \cong Y\).

Note that, unlike all the other examples considered in this paper, the non-compact toric variety \(Y\) is not presented as the total space of a vector bundle.
The $T$-Action. The action of $T = \mathbb{C}^\times$ on $\mathbb{C}^5$ such that $\alpha \in T$ maps

\[
\begin{pmatrix}
  x \\
  y \\
  z \\
  u \\
  v
\end{pmatrix}
\mapsto
\begin{pmatrix}
  x \\
  y \\
  z \\
  u \\
  \alpha v
\end{pmatrix}
\]

descends to give actions of $T$ on $X$, $X$, and $Y$, and the crepant resolution

\[
\begin{align*}
Y &\quad \to X \\
\end{align*}
\]

is $T$-equivariant.

Bases for Everything. We have

\[
 \begin{align*}
 r &:= \text{rank } H^2(Y; \mathbb{C}) = 2, \\
 s &:= \text{rank } H^2(X; \mathbb{C}) = 1.
 \end{align*}
\]

Let $p_1, p_2 \in H(Y)$ denote the $T$-equivariant Poincaré-duals to the divisors $\{z = 0\}$ and $\{x = 0\}$ respectively, so that

\[
 H(Y) = \mathbb{C}(\lambda)[p_1, p_2]/(p_2^2(\lambda + p_2 - 2p_1), p_1(p_1 - 3p_2), p_1^2 p_2).
\]

Set

\[
\varphi_0 = 1, \quad \varphi_1 = p_1, \quad \varphi_2 = p_2, \quad \varphi_3 = p_1 p_2, \quad \varphi_4 = p_2^2.
\]

Write the inertia stack $\mathcal{I}Y$ of $X$ as the disjoint union $X_0 \coprod X_{1/3} \coprod X_{2/3}$, where $X_f$ is the component of the inertia stack corresponding to the fixed locus of the element $(1, e^{2\pi i \sqrt{-1}}) \in (\mathbb{C}^\times)^2$. We have $X_0 = K_{\mathbb{C}(1,3)}$ and $X_{1/3} = X_{2/3} = K_{\mathbb{C}(2,3)}$. Define $1_f \in H(X)$ to be the class which restricts to the unit class on the component $X_f$ and restricts to zero on the other components, and let $p_0 \in H(X)$ denote the first Chern class of the line bundle $O(1) \to \mathbb{P}(1,1,3)$, pulled back to $K_{\mathbb{C}(1,3)}$ via the natural projection and then regarded as an element of Chen–Ruan cohomology via the inclusion $X = X_0 \to \mathcal{I}X$. Set

\[
\phi_0 = 1_0, \quad \phi_1 = p, \quad \phi_2 = p^2, \quad \phi_3 = 1_{1/3}, \quad \phi_4 = 1_{2/3},
\]

so that $r_1 = \frac{1}{3}$.

Characterising $\mathcal{L}_Y$. Let

\[
I_Y(y_1, y_2, z) := z \sum_{k,l \geq 0} \frac{\Gamma(1 + \frac{p_1}{z} + k) \Gamma(1 + \frac{p_2}{z} + l) \Gamma(1 + \frac{p_1 - 3p_2}{z} + k - 3l)}{\Gamma(1 + \frac{p_1}{z} + k) \Gamma(1 + \frac{p_2}{z} + l) \Gamma(1 + \frac{p_1 - 3p_2}{z} + k - 3l)} \times
\]

\[
\frac{\Gamma(1 + \frac{\lambda - 2p_1 + p_2}{z})}{\Gamma(1 + \frac{\lambda - 2p_1 + p_2}{z} - 2k + l)} y_1^{k + p_1/z} y_2^{l + p_2/z}. \tag{56}
\]

Claim 7.1 (cf. Section 5 Step 1.).

\[
I_Y(y_1, y_2, -z) \in \mathcal{L}_Y \quad \text{for all } (y_1, y_2) \text{ in the domain of convergence of } I_Y.
\]

The Claim can be proved using the argument which proves Theorem 0.1 in [31]. Theorem 0.1 as stated only applies to compact toric varieties, but the argument which proves it works for the non-compact toric variety $Y$ as well. The reader who would prefer not to check this should wait for the full generality of [21].

Exercise 7.2. (cf. Section 5 Step 2.)

(a) Check that the series \[56\] converges, in a region where $|y_1|$ and $|y_2|$ are sufficiently small, to a multi-valued analytic function of $(y_1, y_2)$ which takes values in $H_Y$.

(b) Use Claim 7.2 to produce an expression for the small $J$-function $J_Y(q_1, q_2, z)$.

(c) Show that the series \[56\] defining $J_Y(q_1, q_2, z)$ converges (to a multi-valued analytic function) when $|q_1|$ and $|q_2|$ are sufficiently small.

(d) Show that $\mathcal{L}_Y$ is uniquely determined by the fact that $(y_1, y_2) \mapsto I_Y(y_1, y_2, -z)$ is a family of elements of $\mathcal{L}_Y$. 

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Characterising $\mathcal{L}_x$. Let
\[
I_x(x_1, x_2, z) := z \sum_{d,e \geq 0, e \geq 0, 3d \in \mathbb{Z}} \sum_{3e \in \mathbb{Z}} x_1^{3d+3p/z} x_2^{3e} \prod_{b, b \leq 0} \frac{b}{b-\text{frac}(d-e)} (p + bz)^2 \prod_{b, b \leq d-e} \frac{b}{b-\text{frac}(d-e)} (p + bz)^2 \times \prod_{1 \leq m \leq 3d} (3p + mz) 1_{\text{frac}(e-d)}. \tag{57}
\]

Claim 7.2 (cf. Section 5 Step 3.).
\[
I_x(x_1, x_2, -z) \in \mathcal{L}_x \quad \text{for all } (x_1, x_2) \text{ in the domain of convergence of } I_x.
\]

The methods of this paper will only prove that $I_x(x, 0, -z) \in \mathcal{L}_x$ for all $x$ such that $(x, 0)$ lies in the domain of convergence of $I_x$. But this example requires the stronger result, which will follow from [21].

Exercise 7.3. (cf. Section 5 Step 4.)
(a) Check that the series (57) converges, in a region where $|x_1|$ and $|x_2|$ are sufficiently small, to a multivalued analytic function which takes values in $\mathcal{H}_x$.
(b) Show that
\[
I_x(x_1, x_2, z) = z \log x_1 - g(x_1, x_2)(x_2 - 5p)1_{1/3} + O(z^{-1})
\]
for appropriate power series $g(x_1, x_2)$ and $h(x_1, x_2)$, and deduce that
\[
J^\log_x(r, z) = e^g(x_1, x_2)/z I_x(x_1, x_2, z)
\]
where
\[
q = x_1^3 \exp(5g(x_1, x_2)), \quad r = h(x_1, x_2) \tag{58}
\]
(c) Show that the series $J^\log_x(r, z) = e^g(x_1, x_2)/z I_x(x_1, x_2, z)$ converges, in a region where $|q|$ and $|r|$ are sufficiently small, to a multivalued analytic function of $q$ and $r$ which takes values in $\mathcal{H}_x$.

Exercise 7.4. (calculating some Gromov–Witten invariants of $X$)
(a) Calculate the first few terms of the power series inverse to the “mirror map” (58).
(b) Deduce that
\[
\begin{align*}
(11/3)^x_{0,1,1/3} &= -2, \\
(11/3)^x_{0,1,4/3} &= -13/18, \\
(11/3,11/3)^x_{0,1,2/3} &= -7/18, \\
(11/3,11/3,11/3)^x_{0,1,0} &= 1/3, \\
(11/3,11/3,11/3,11/3)^x_{0,1,1/3,1/3} &= -2/27,
\end{align*}
\]
and so on.

The Picard–Fuchs System. Once again, define the $B$-model moduli space $\mathcal{M}_B$ to be the toric orbifold corresponding to the secondary fan for $Y$ (Figure 3). Each chamber of the secondary fan gives a co-ordinate patch on $\mathcal{M}_B$: the co-ordinates $(y_1, y_2)$ coming from chamber I are dual respectively to $p_1$ and $p_2$, and the co-ordinates $(x_1, x_2)$ from chamber II are dual respectively to $p_1$ and $p_1 - 3p_2$. These co-ordinate patches are related by
\[
\begin{align*}
y_1 &= x_1 x_2, & x_1 &= y_1 y_2^{1/3}, \\
y_2 &= x_2^{-3}, & x_2 &= y_2^{-1/3}. \tag{60}
\end{align*}
\]
We regard $I_Y(y_1, y_2, z)$ as a function on the co-ordinate patch corresponding to chamber I and $I_x(x_1, x_2, z)$ as a function on the co-ordinate patch corresponding to chamber II. Let
\[
D_{x_1} = z x_1 \frac{\partial}{\partial x_1}, \quad D_{x_2} = z x_2 \frac{\partial}{\partial x_2}, \quad D_{y_1} = z y_1 \frac{\partial}{\partial y_1}, \quad D_{y_2} = z y_2 \frac{\partial}{\partial y_2}.
\]

Exercise 7.5. (cf. Section 5 Step 5.)
Exercise 7.6. (cf Section 5, Steps 6 and 7.)

(a) Show, using the Mellin–Barnes method, that:

\[
\tilde{I}_Y(x_1, x_2, z) = z \sum_{k, n \geq 0} \frac{(-1)^{n+k}}{3n!} \frac{\sin \left( \pi \left[ \frac{p_1 - 3p_2}{3} \right] \right)}{\sin \left( \pi \left[ \frac{p_1 - 3p_2 + k - n}{3} \right] \right)} \frac{\Gamma \left( 1 + \frac{p_2}{3} \right)^2 \Gamma \left( 1 + \frac{p_1 - 3p_2 + k - n}{3} \right)^2 \times \Gamma \left( 1 + \frac{p_2}{3} \right)^2 \Gamma \left( 1 + \frac{p_1 - 3p_2 + k - n}{3} \right)^2}{\Gamma \left( 1 + \frac{3n + p_1 - 3p_2}{3} \right) \Gamma \left( 1 + \frac{3n + p_1 - 3p_2 + k - n}{3} \right) \Gamma \left( 1 - \frac{3n + p_1 - 3p_2}{3} \right) \Gamma \left( 1 - \frac{3n + p_1 - 3p_2 + k - n}{3} \right)} x_1^{k+n} x_2^n. \tag{61}
\]

(b) By comparing coefficients of \(x_1^k x_2^n (\log x_1)^c\) in (61) and (62), compute the symplectic transformation \(U\). The non-equivariant limit \(\lim_{z \to 0} U\) has matrix:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & 0 & 0 & \frac{2\pi}{3\sqrt{3} \Gamma \left( \frac{1}{3} \right)} \\
0 & 0 & \frac{1}{3} & 0 & 0 & \frac{2\pi}{3\sqrt{3} \Gamma \left( \frac{1}{3} \right)} \\
\frac{\pi^2}{9\pi^2} & 0 & \frac{1}{3} & 0 & 0 & \frac{2\pi}{3\sqrt{3} \Gamma \left( \frac{1}{3} \right)} \\
\frac{\pi^2}{9\pi^2} & 0 & 0 & \frac{1}{3} & 0 & \frac{2\pi}{3\sqrt{3} \Gamma \left( \frac{1}{3} \right)} \\
\frac{\pi^2}{9\pi^2} & 0 & 0 & 0 & \frac{1}{3} & \frac{2\pi}{3\sqrt{3} \Gamma \left( \frac{1}{3} \right)} \\
\end{pmatrix}. \tag{62}
\]

(c) Prove:

**Theorem 7.3** (The Crepant Resolution Conjecture for \(K_{P(1,1,3)}\)). *Conjecture [24] holds for \(X = K_{P(1,1,3)}\) and \(Y\) its crepant resolution.*

Conclusions. Having proved the Crepant Resolution Conjecture in this case, we can now extract information about small quantum cohomology using Corollary 5.2. When we do this, we find that the quantum corrections to Ruan’s conjecture do not vanish:

**Corollary 7.4.** Let \(X = K_{P(1,1,3)}\) and let \(Y \to X\) be the crepant resolution of the coarse moduli space of \(X\). There is a power series

\[
f(u) = \frac{2\pi}{\sqrt{3} \Gamma \left( \frac{1}{3} \right)} \left( -2u^{1/3} + \frac{3757}{648}u^{4/3} + \ldots \right)
\]

such that the algebra obtained from the small quantum cohomology algebra of \(Y\) by analytic continuation in the parameter \(q_2\) followed by the substitution

\[
q_i = \begin{cases} 
eq f(u) & i = 1 \\
eq 3f(u) & i = 2 
\end{cases}
\]

is isomorphic to the small quantum cohomology algebra of \(X\), via an isomorphism which sends \(p \in H(X)\) to \(\frac{2}{3}p_1 \in H(Y)\).
Proof. This is Corollary 3.2. The quantities $c_1$ and $c_2$ defined in (121) are zero, and the power series $f(u)$ comes from equations (15) and (57).

8. Example V: $\mathcal{X} = [\mathbb{C}^3/\mathbb{Z}_5]$, $\mathcal{Y} = K_{\mathbb{P}(1,1,3)}$

Consider now the crepant partial resolution of $\mathcal{X} = [\mathbb{C}^3/\mathbb{Z}_5]$ by $\mathcal{Y} = K_{\mathbb{P}(1,1,3)}$. We can treat this using exactly the same methods as before, and we omit all details.

![Diagram](https://via.placeholder.com/150)

Figure 9: The secondary fan for $\mathcal{Y} = K_{\mathbb{P}(1,1,3)}$

The secondary fan is shown in Figure 9, the $B$-model moduli space $\mathcal{M}_B$ is $\mathbb{P}(1,5)$, and the $I$-functions are

$$I_\mathcal{X}(x_1,x_2,z) := z \sum_{k,l \geq 0} \frac{x_1^k x_2^l}{k!l!} z^{k+l+1} \prod_{b:0 \leq b < \frac{k+l+2l}{2l}, \frac{b}{b+1}} (\frac{b}{b+1} - b)^2 \prod_{b:0 \leq b < \frac{k+l+1}{2l}, \frac{b}{b+1}} (\frac{b}{b+1} - b) \frac{1}{\Gamma(\frac{k+l+2l}{2l})}$$

(c.f. [19] Theorem 4.6 and Proposition 6.1) and

$$I_\mathcal{Y}(y_1,y_2,z) := z \sum_{d,d \geq 0, d \in \mathbb{Z}} \sum_{e \geq 0} \frac{z^{3d+3p/e}}{(3e)!} \frac{y_1^d y_2^e}{\prod_{d \leq d - e, \frac{b}{b+1}} (p + b)^2 \prod_{d \leq d - e, \frac{b}{b+1}} (p + b)^2} \prod_{b:0 \leq b < \frac{d + p}{b+1}} (\lambda - 5p + b) \frac{1}{\Gamma(\frac{d + p}{b+1})}$$

(c.f. Section 7 above). Use the bases

$$\phi_0 = 1_0, \quad \phi_1 = 1_{1/5}, \quad \phi_2 = 1_{2/5}, \quad \phi_3 = 1_{3/5}, \quad \phi_4 = 1_{4/5}$$

for $H(\mathcal{X})$ and

$$\varphi_0 = 1_0, \quad \varphi_1 = p, \quad \varphi_2 = p^2, \quad \varphi_3 = 1_{1/3}, \quad \varphi_4 = 1_{2/3}$$

for $H(\mathcal{Y})$: see Sections 6 and 7 for the notation. The Mellin–Barnes method produces a linear symplectomorphism $U : \mathcal{H}_\mathcal{X} \to \mathcal{H}_\mathcal{Y}$ with matrix given, in the non-equivariant limit $\lambda \to 0$, by

$$\begin{pmatrix}
1 & 0 & 0 & 0 \\
\frac{\pi}{\sqrt{2}} & \frac{\pi}{\sqrt{2}} & \frac{\pi}{\sqrt{2}} & \frac{\pi}{\sqrt{2}} \\
\frac{\sqrt{2} \pi^2}{3} & \frac{\sqrt{2} \pi^2}{3} & \frac{\sqrt{2} \pi^2}{3} & \frac{\sqrt{2} \pi^2}{3} \\
\frac{\pi}{\sqrt{2}} & \frac{\pi}{\sqrt{2}} & \frac{\pi}{\sqrt{2}} & \frac{\pi}{\sqrt{2}}
\end{pmatrix}.$$
The secondary fan is the twisted sector cohomology. It seems also that any such generalization will no longer involve only roots of unity. Solutions cannot be phrased in terms of small quantum cohomology alone: it must involve big quantum cohomology. Resolution Conjecture (and hence also any generalization of Ruan’s Conjecture) to crepant partial resolutions cannot be phrased in terms of small quantum cohomology alone: it must involve big quantum cohomology.

Conclusions. In light of this, it seems likely that any generalization of the Cohomological Crepant Resolution Conjecture and hence also any generalization of Ruan’s Conjecture) to crepant partial resolutions cannot be phrased in terms of small quantum cohomology alone: it must involve big quantum cohomology. It seems also that any such generalization will no longer involve only roots of unity.

9. Example VI: A Toric Flop

Finally, consider the action of $\mathbb{C}^\times$ on $\mathbb{C}^5$ such that $s \in \mathbb{C}^\times$ acts as
\[
\begin{pmatrix}
x \\
y \\
z \\
u \\
v
\end{pmatrix} \mapsto \begin{pmatrix} sx \\
sy \\
sz \\
s^{-1}u \\
s^{-2}v
\end{pmatrix}.
\]

The secondary fan is:

\begin{center}
\begin{tikzpicture}
\node (1) at (0,0) {$5$};
\node (2) at (1,0) {$4$};
\node (3) at (2,0) {$1,2,3$};
\end{tikzpicture}
\end{center}

Figure 10: The secondary fan for a toric flop

For $\xi$ in the right-hand chamber of the secondary fan, the GIT quotient $Y := \mathbb{C}^5/\xi \mathbb{C}^\times$ is the total space of the vector bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-2) \to \mathbb{P}^2$. For $\xi$ in the left-hand chamber, the GIT quotient $\mathcal{X} := \mathbb{C}^5/\xi \mathbb{C}^\times$ is the total space of $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}(1,2)$. The birational transformation $Y \dashrightarrow \mathcal{X}$ induced by moving from the right-hand chamber to the left-hand chamber is a flop [20].

To treat this example, we need to make some changes to our general setup (described in Section 2), but the required modifications are obvious and so we make them without comment. As we have not yet discussed a birational transformation of this type, we once again give some details of the calculation: the reader will see that our methods apply here too without significant change.

Bases and I-Functions. We have
\[
r := \text{rank } H^2(Y; \mathbb{C}) = 1, \quad s := \text{rank } H^2(\mathcal{X}; \mathbb{C}) = 1.
\]

The action of $T = \mathbb{C}^\times$ on $\mathbb{C}^5$ such that $\alpha \in T$ acts as
\[
\begin{pmatrix}
x \\
y \\
z \\
u \\
v
\end{pmatrix} \mapsto \begin{pmatrix} \alpha x \\
\alpha y \\
\alpha z \\
u \\
v
\end{pmatrix}
\]

induces actions of $T$ on $Y$ and $\mathcal{X}$, and the flop $Y \dashrightarrow \mathcal{X}$ is $T$-equivariant. Let $p$ be the canonical $T$-equivariant lift of the first Chern class of the line bundle $\mathcal{O}(1) \to \mathbb{P}^2$, so that
\[
H(Y) = \mathbb{C}(\lambda)[p]/(p^3).
\]
We use the basis
\[ \varphi_0 = 1, \quad \varphi_1 = p, \quad \varphi_2 = p^2 \]
for \( H(Y) \). The inertia stack of \( \mathcal{X} \) is the disjoint union \( \mathcal{X}_0 \bigsqcup \mathcal{X}_{1/2} \), where \( \mathcal{X}_0 = \mathcal{X} \) and \( \mathcal{X}_{1/2} = B\mathbb{Z}_2 \). Let \( 1_f \in H(\mathcal{X}) \) denote the class which restricts to the unit class on the component \( \mathcal{X}_f \) and restricts to zero on the other component, and let \( p \in H(\mathcal{X}) \) denote the canonical \( T \)-equivariant lift of the first Chern class of the line bundle \( O(1) \to \mathbb{P}(1,2) \), pulled back to \( \mathcal{X} \) via the natural projection \( \mathcal{X} \to \mathbb{P}(1,2) \) and then regarded as an element of Chen–Ruan cohomology via the inclusion \( \mathcal{X} = \mathcal{X}_0 \to \mathcal{X} \). We use the basis
\[ \phi_0 = 1_0, \quad \phi_1 = p, \quad \phi_2 = 1_{1/2} \]
for \( H(\mathcal{X}) \).

Let
\[ I_Y(x, y) = z \sum_{d \geq 0} \frac{\prod_{-2d < m \leq 0}(2\lambda - 2p + mz) \prod_{-d < m \leq 0}(\lambda - p + mz)}{\prod_{0 < m \leq d}(p + mz)^3} y^{d + p/z}, \]
and let
\[ I_X(x, z) = z x^{-\lambda/z} \sum_{d \geq 0, 2d \geq \mathbb{Z}} x^{d + p/z} \frac{\prod_{b : -d < b \geq 0, \frac{b}{d} = \frac{\lambda}{p}} (\lambda - p + mz)^3}{\prod_{b : 0 \leq b \leq d, \frac{b}{d} = \frac{\lambda}{p}} (p + bz) \prod_{1 \leq m \leq 2d}(2p + mz)} 1_{\lambda(p - d)} \]
Arguing exactly as before yields:

**Proposition 9.1.** We have \( I_Y(y, -z) \in \mathcal{L}_Y \) for all \( y \) such that \( 0 < |y| < \frac{1}{d} \), and \( I_X(x, -z) \in \mathcal{L}_X \) for all \( x \) such that \( |x| < 4 \).

Furthermore, as
\[ x^{-\lambda/z} I_X(x, -z) = -z + p \log x + O(z^{-1}) \quad \text{and} \quad I_Y(y, -z) = -z + p \log y + O(z^{-1}) \]
we conclude that:

**Corollary 9.2.**
\[ J_X(u, z) = x^{\lambda/z} I_X(u, z) \quad \text{and} \quad J_Y(q, z) = I_Y(q, z). \]
Note that the mirror maps here are trivial.

It follows that the Lagrangian submanifold-germs \( \mathcal{L}_X \) and \( \mathcal{L}_Y \) are uniquely determined by Proposition 9.1.

**The B-model Moduli Space and Analytic Continuation.** The \( B \)-model moduli space \( \mathcal{M}_B \) here is \( \mathbb{P}^1 \); it has a co-ordinate patch (with co-ordinate \( x \)) corresponding to \( \mathcal{X} \) and a co-ordinate patch (with co-ordinate \( y \)) corresponding to \( Y \), related by \( y = x^{-1} \). Regard \( I_X(x, z) \) as a function on the co-ordinate patch corresponding to \( \mathcal{X} \) and \( I_Y(y, z) \) as a function on the co-ordinate patch corresponding to \( Y \), and denote by \( \bar{I}_Y(x, z) \) the analytic continuation of \( I_Y \) to a neighbourhood of \( x = 0 \). As before, both \( I_X \) and \( \bar{I}_Y \) have components which form a basis of solutions to the Picard–Fuchs differential equation
\[ -xD^3f = (\lambda + D)(2\lambda + 2D)(2\lambda + 2D - D)f, \quad D = \frac{\partial}{\partial x}. \]
It follows that there exists a \( \mathbb{C}((z^{-1})) \)-linear isomorphism \( U : \mathcal{H}_X \to \mathcal{H}_Y \) such that \( U(I_X(x, -z)) = \bar{I}_Y(x, -z) \). This is the linear symplectomorphism that we seek.

The Mellin–Barnes method gives
\[ \bar{I}_Y(x, z) = z x^{-\lambda/z} \sum_{k \geq 0} \frac{x^k}{k!(2k + 1)!} \frac{\Gamma(1 + \frac{\lambda}{z})^3}{\Gamma(1 + \frac{\lambda}{z} - k - \frac{1}{2})} \Gamma(-k - \frac{1}{2}) \Gamma(1 + \frac{2\lambda - 2z}{z}) \Gamma(1 + \frac{\lambda - p}{z}) \sin \left( \frac{\lambda e}{z} \right) \sin \left( \frac{\lambda e}{z} \right) \]
\[ - z x^{-\lambda/z} \sum_{k \geq 0} \frac{x^k}{k!(2k + 1)!} \frac{\Gamma(1 + \frac{\lambda}{z})^3}{\Gamma(1 + \frac{\lambda}{z} - k - \frac{1}{2})} \Gamma(1 + \frac{2\lambda - 2z}{z}) \Gamma(1 + \frac{\lambda - p}{z}) \sin \left( \frac{\lambda e}{z} \right) \sin \left( \frac{\lambda e}{z} \right) \]
\[ = \left( H_{2k} + \frac{H_{2k}^{(3/2)}}{\pi} \frac{3\pi}{2} \frac{\gamma}{\sin \left( \frac{\lambda e}{z} \right)} \right), \]
where $\gamma$ is Euler’s constant, $\psi(z)$ is the logarithmic derivative of $\Gamma(z)$, and $H_k$ is the $k$th harmonic number. Thus

$$U(1_0) = -\frac{\Gamma(1 - \frac{2\lambda}{p})^3}{\Gamma(1 - \frac{2\lambda}{p})^3} \Gamma(1 - \frac{2\lambda}{p}) \sin \left(\frac{\pi}{\lambda} \right) \left(\frac{3\lambda}{2} + \frac{3}{2} \psi(1 - \frac{2\lambda}{p}) - \frac{3}{2} \cot \left(\frac{\pi}{\lambda} \right) \right),$$

$$U(p) = -\frac{\pi}{2} \frac{\Gamma(1 - \frac{2\lambda}{p})^3}{\Gamma(1 - \frac{2\lambda}{p})^3} \Gamma(1 - \frac{2\lambda}{p}) \sin \left(\frac{\pi}{\lambda} \right),$$

$$U(1_{1/2}) = -\frac{z^2}{4} \frac{\Gamma(1 - \frac{2\lambda}{p})^3}{\Gamma(1 - \frac{2\lambda}{p})^3} \Gamma(1 - \frac{2\lambda}{p}) \Gamma(1 - \frac{2\lambda}{p}) \sin \left(\frac{\pi}{\lambda} \right) \sin \left(\frac{\pi}{\lambda} \right).$$

Note that

$$U(1_0) = 1 + O(z^{-2}),$$
$$U(p) = (\lambda - p) + O(z^{-2}),$$
$$U(1_{1/2}) = (\lambda - p)^2 + O(z^{-1}).$$

In the non-equivariant limit $\lambda \to 0$, our expressions for $U$ simplify:

$$U(1_0) \to 1 - \frac{\pi^2 p^2}{32\pi^2}, \quad U(p) \to -p, \quad U(1_{1/2}) \to p^2.$$

**Theorem 9.3 (A “Flop Conjecture” for $X$ and $Y$).** There is a choice of analytic continuations of $L_X$ and $L_Y$ such that, after analytic continuation, $U(L_X) = L_Y$. Furthermore $U : \mathcal{H}_X \to \mathcal{H}_Y$ is a degree-preserving $\mathbb{C}(z^{-1})$-linear symplectic isomorphism which satisfies

(a) $U(1_X) = 1_Y + O(z^{-1});$
(b) $U(p) = (\lambda - p) + O(z^{-2});$
(c) $U(1_{1/2}) = (\lambda - p)^2 + O(z^{-1}).$

**Proof.** Argue as in the proof of Theorem 4.4. \qed

The transformation $U$ does not satisfy any condition analogous to property (b) in Conjecture 2.1, but we should not expect this. Property (b) arises from the fact that $U$ intertwines certain monodromies (let us call them the relevant monodromies) of the system of Picard–Fuchs equations coming from mirror symmetry: see [18] Proposition 4.7. In the case of toric crepant resolutions the relevant monodromies generate $H^2(X)$, but for general toric crepant birational transformations this is not the case. The Mellin–Barnes method will always produce a transformation $U$ which intertwines the relevant monodromies, but in the case at hand this is vacuously true as the set of relevant monodromies is empty. For a general flop

$$\begin{array}{cc}
X & Y \\
p_1 \downarrow & \downarrow p_2 \\
\gamma & Z
\end{array}$$

it is reasonable to expect that property (b) should be replaced by the assertion

$$U \circ (p_1^* \alpha \cup_{ch} \ast \gamma) = (p_2^* \alpha \cup_{ch} \ast \gamma) \circ U \quad \text{for all } \alpha \in H^2(Z; \mathbb{C});$$

this condition is also vacuous here.

**Corollary 9.4 (A Ruan/Bryan–Graber-style Flop Conjecture).** The $\mathbb{C}(\lambda)$-linear map $U_{\infty} : H(X) \to H(Y)$ given by

$$U_{\infty}(1_0) = 1, \quad U_{\infty}(p) = (\lambda - p), \quad U_{\infty}(1_{1/2}) = (\lambda - p)^2,$$

induces an algebra isomorphism between the small quantum cohomology of $X$ and the algebra obtained from the small quantum cohomology of $Y$ by analytic continuation in the quantum parameter $q$ followed by the substitution $u = q^{-1}$. \qed

**Proof.** Look at equation (64), and then apply the discussion in [24 §9].
Genus–Zero Gromov–Witten Invariants of $[\mathbb{C}^3/\mathbb{Z}_4]$ and $K_{\mathbb{P}(1,1,2)}$

Genus–Zero Gromov–Witten Invariants of $[\mathbb{C}^3/\mathbb{Z}_4]$. Set

$$A_{n,m} = \left[ \frac{n}{1/4, \ldots, 1/4, 1_{1/2, \ldots, 1/2}} \right]_{0,n+m,0}^{[\mathbb{C}^3/\mathbb{Z}_4]}$$

$$B_{n,m} = \left[ \frac{n}{1/4, \ldots, 1/4, 1_{1/2, \ldots, 1/2, 1_{3/4}}} \right]_{0,n+m+1,0}^{[\mathbb{C}^3/\mathbb{Z}_4]}$$

Then:

| $n$ | $m=0$ | $m=1$ | $m=2$ | $m=3$ | $m=4$ | $m=5$ | $m=6$ | $m=7$ | $m=8$ | $m=9$ | $m=10$ |
|-----|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $n=0$ | 0 | 0 | 0 | 0 | $-\frac{1}{8}$ | 0 | 0 | 0 | $-\frac{9}{64}$ | 0 | 0 |
| $n=1$ | 0 | 0 | $\frac{1}{4}$ | 0 | 0 | 0 | $\frac{7}{128}$ | 0 | 0 | 0 | 0 |
| $n=2$ | 0 | 0 | 0 | 0 | $-\frac{1}{32}$ | 0 | 0 | 0 | $-\frac{147}{1024}$ | 0 | 0 |
| $n=3$ | 0 | 0 | $\frac{1}{32}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $n=4$ | $-\frac{1}{16}$ | 0 | 0 | 0 | $-\frac{11}{256}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $n=5$ | 0 | 0 | $\frac{1}{32}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $n=6$ | $-\frac{1}{16}$ | 0 | 0 | 0 | $-\frac{147}{1024}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $n=7$ | 0 | 0 | $\frac{87}{1024}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $n=8$ | $-\frac{1}{8}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $n=9$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $n=10$ | $-\frac{17}{128}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table 2: The values of $A_{n,m}$ for $n + m \leq 10$.

| $n$ | $m=0$ | $m=1$ | $m=2$ | $m=3$ | $m=4$ | $m=5$ | $m=6$ | $m=7$ | $m=8$ | $m=9$ | $m=10$ |
|-----|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $n=0$ | 0 | 0 | 0 | 0 | 0 | $-\frac{5\lambda}{128}$ | 0 | 0 | 0 | $\frac{865\lambda}{2048}$ | 0 |
| $n=1$ | 0 | 0 | $\frac{\lambda}{128}$ | 0 | 0 | 0 | $\frac{17\lambda}{1024}$ | 0 | 0 | 0 | 0 |
| $n=2$ | 0 | $-\frac{\lambda}{32}$ | 0 | 0 | 0 | $-\frac{41\lambda}{1024}$ | 0 | 0 | 0 | 0 | 0 |
| $n=3$ | 0 | 0 | 0 | $\frac{5\lambda}{256}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $n=4$ | 0 | $-\frac{\lambda}{64}$ | 0 | 0 | 0 | $-\frac{487\lambda}{4096}$ | 0 | 0 | 0 | 0 | 0 |
| $n=5$ | 0 | 0 | 0 | $\frac{201\lambda}{4096}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $n=6$ | 0 | $-\frac{\lambda}{32}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $n=7$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $n=8$ | 0 | $-\frac{17\lambda}{128}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $n=9$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table 3: The values of $B_{n,m}$ for $n + m \leq 9$.

The first rows of these tables can be produced, as in Section 4, using the fact that the $I$-function $[48]$ gives a family $x \mapsto x^{-\lambda/2} I_X(x, -z)$ of elements of $L_{[\mathbb{C}^3/\mathbb{Z}_4]}$. The rest of the tables can be produced in the same way, using the fact that

$$I'(x, z) := z \sum_{k, l \geq 0} \frac{x^{k,l}}{k!l!z^{k+l}} \prod_{b \in \mathcal{I}, b < \frac{k+l}{2}} (\frac{\lambda}{\lambda})^2 \prod_{b \in \mathcal{I}, b < \frac{k+l}{2}} (\frac{\lambda}{\lambda})^2 \frac{1}{\frac{b}{b+\frac{\lambda}} + \frac{2\lambda}{\lambda}}$$

gives a family $x \mapsto I'(x, z)$ of elements of $L_{[\mathbb{C}^3/\mathbb{Z}_4]}$: this is an immediate consequence of [19] Theorem 4.6 and Proposition 6.1.
Genus-Zero Gromov–Witten Invariants of $K_{\mathbb{P}(1,1,2)}$. Set

$$a_d = \langle K_{\mathbb{P}(1,1,2)}^{0,0,d} \rangle, \quad b_d = \langle p^2 K_{\mathbb{P}(1,1,2)}^{0,1,d} \rangle, \quad c_d = \langle 1/2 K_{\mathbb{P}(1,1,2)}^{0,1,d} \rangle,$$

where $a_d$ is the correlator with no insertions. Applying Corollary 5.4 just as in Section 3 gives:

| $d$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ |
|-----|-----|-----|-----|-----|-----|-----|
| $a_d$ | $-7/2$ | $-16$ | $-25$ | $-47$ | $-128$ | $25$ |
| $b_d$ | $11\lambda$ | $525\lambda$ | $6152\lambda$ | $1146765\lambda$ | $53395261\lambda$ | $51550873\lambda$ |

Table 4: The values of $a_d$ and $b_d$ for $d \leq 6$.

| $d$ | $1/2$ | $3/2$ | $5/2$ | $7/2$ | $9/2$ | $11/2$ |
|-----|------|------|------|------|------|------|
| $c_d$ | $-2$ | $52/9$ | $-2092/25$ | $-83004/49$ | $-3554552/81$ | $-154984300/121$ | $-6835086702/169$ |

Table 5: The values of $c_d$ for $d \leq 7$.

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