SYMMETRIC CRYSTALS
AND
LLTA TYPE CONJECTURES FOR THE AFFINE HECKE ALGEBRAS
OF TYPE B

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ABSTRACT. In the previous paper [EK1], we formulated a conjecture on the relations
between certain classes of irreducible representations of affine Hecke algebras of type B
and symmetric crystals for $\mathfrak{g}_{\infty}$. In the first half of this paper (sections 2 and 3), we give
a survey of the LLTA type theorem of the affine Hecke algebra of type $A$. In the latter
half (sections 4, 5 and 6), we review the construction of the symmetric crystals and the
LLTA type conjectures for the affine Hecke algebra of type $B$.

1. INTRODUCTION

1.1. The Lascoux-Leclerc-Thibon-Ariki theory connects the representation theory of the
affine Hecke algebra of type $A$ with representations of the affine quantum enveloping algebra
of type $A$. Recently, we presented the notion of symmetric crystals and conjectured that
certain classes of irreducible representations of the affine Hecke algebras of type $B$ are
described by symmetric crystals for $\mathfrak{g}_{\infty}$ or $A_{(1)}^\infty$ ([EK1]). In this paper, we review the
LLTA-theory for the affine Hecke algebra of type $A$, the symmetric crystals, and then our
conjectures for the affine Hecke algebra of type $B$. For the sake of simplicity, we restrict
ourselves in this note to the case where the parameters of the affine Hecke algebras are not
a root of unity.

This paper is organized as follows. In part I (sections 2 and 3), we review the LLTA-
thery for the affine Hecke algebra of type $A$. In section 2, we recall the representation
theory of $U_q(\mathfrak{g}_{\infty})$, especially the PBW basis, the crystal basis and the global basis. In
section 3, we recall the representation theory of the affine Hecke algebra of type $A$ and
state the LLTA-type theorems. In part II (sections 4, 5 and 6), we explain symmetric
crystals for $\mathfrak{g}_{\infty}$ and the LLTA type conjectures for the affine Hecke algebras of type $B$.
In section 4, we recall the construction of symmetric crystals based on [EK1] and state
the conjecture of existence of the crystal basis and the global basis. In section 5, we
explain a combinatorial realization of the symmetric crystals for $\mathfrak{g}_{\infty}$ by using the PBW
type basis and the $\theta$-restricted multisegments. This section is a new additional part to
the announcement [EK1]. The details will appear in [EK2]. In section 6, we explain the
representation theory of the affine Hecke algebra of type $B$ and state our LLTA-
type conjectures for the affine Hecke algebra of type $B$. We add proofs of lemmas and
propositions in [EK1, section 3.4].

1.2. Let us recall the LLTA-theory for the affine Hecke algebra of type $A$.

The representation theory of quantum enveloping algebras and the representation theory
of affine Hecke algebras have developed independently. G. Lusztig [L] constructed the PBW
type basis and canonical basis of $U_q^{-}(\mathfrak{g})$ for the $A, D, E$ cases. The second author [Kas]

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defined the crystal basis $B(\infty)$ and the (lower and upper) global bases \( \{ G^{low}(b) \}_{b \in B(\infty)} \), \( \{ G^{up}(b) \}_{b \in B(\infty)} \) of $U_q^-(\mathfrak{g})$. The lower global basis coincides with Lusztig's canonical basis. On the other hand, A. V. Zelevinsky [Z] gave a parametrization of the irreducible representations of the affine Hecke algebra of type $A$ by using multisegments. Chriss-Ginzburg [CG] and Kazhdan-Lusztig [KL] constructed all the irreducible representations of the affine Hecke algebras in geometric methods.

Lascoux-Leclerc-Thibon conjectured in [LLT] that certain composition multiplicities (called the decomposition numbers) of the Hecke algebra of type $A$ can be written by the transition matrices (specialized at $q = 1$) between the upper global basis and a standard basis of the level 1 fundamental representation of $U_q(\widehat{\mathfrak{sl}}_t)$. In [A], S. Ariki generalized and solved the conjecture for the cyclotomic Hecke algebra and the affine Hecke algebra of type $A$ by using the geometric representation theory of the affine Hecke algebra of type $A$. In [GV], I. Grojnowski and M. Vazirani proved the multiplicity-one results for the socle of certain restriction functors and the cosocle of certain induction functors on the category of the finite-dimensional representations of the affine Hecke algebras $\mathcal{H}^A$ of type $A$. By using these functors, Grojnowski ([G]) gave the crystal structure on the set of irreducible modules over the affine Hecke algebras $\mathcal{H}^A$ of type $A$. In [V], Vazirani combinatorially constructed the crystal operators on the set of multisegments and proved the compatibility between her actions and Grojnowski's actions.

For $p \in \mathbb{C}^*$, let $\mathcal{H}^A_n(p)$ be the affine Hecke algebra of type $A$ of degree $n$ generated by $T_i$ ($1 \leq i \leq n-1$) and $X_j^{\pm 1}$ ($1 \leq j \leq n$). For a subset $J$ of $\mathbb{C}^*$, we say that a finite-dimensional $\mathcal{H}^A_n$-module is of type $J$ if all the eigenvalues of $X_j$ ($1 \leq j \leq n$) belong to $J$. We can prove that in order to study the irreducible modules over the affine Hecke algebras of type $A$, it is enough to treat those of type $J$ for an orbit $J$ with respect to the $\mathbb{Z}$-action on $\mathbb{C}^*$ generated by $a \mapsto ap^2$ (see Lemma 3.3). For a $\mathbb{Z}$-orbit $J$, let $K_J(\mathcal{H}^A_n)$ be the Grothendieck group of the abelian category of finite-dimensional $\mathcal{H}^A_n$-modules of type $J$, and $K^A_n = \oplus_{n \geq 0} K_J(\mathcal{H}^A_n)$. The LLTA-theory gives the following correspondence between the notions in the representation theory of a quantum enveloping algebra $U_q(\mathfrak{gl}_\infty)$ and the ones in the representation theory of affine Hecke algebras of type $A$.

| the quantum enveloping algebra $U_q(\mathfrak{g}_\infty)$ | the affine Hecke algebra of type $A$ $\mathcal{H}^A_n(p)$ ($n \geq 0$) |
|---|---|
| $U^-_q(\mathfrak{g}_\infty)$ | $K_J^A = \oplus_{n \geq 0} K_J(\mathcal{H}^A_n(p))$ |
| $e'_a, f_a$ | certain restrictions $e_a$ and inductions $f_a$ |
| the crystal basis $B(\infty)$ | $\mathcal{M} = \{ \text{the multisegments} \}$ |
| the upper global basis $\{ G^{up}(b) \}_{b \in B(\infty)}$ | the irreducible modules $\{ L_b \}_{b \in B(\infty)}$ |
| the modified root operators $\bar{e}_a, \bar{f}_a$ | $\bar{e}_a = \text{soc}(e_a), \bar{f}_a = \text{cosoc}(f_a)$ |
| $\bar{e}_a L_b = L_{\bar{a}b}, \bar{f}_a L_b = L_{\bar{f}_a b}$ | the standard modules $\{ M(b) \}_{b \in B(\infty)}$ |

**FIGURE 1.** Lascoux-Leclerc-Thibon-Ariki correspondence in type $A$.

The additive group $K_J^A$ has a structure of Hopf algebra by the restriction and the induction. The set $J$ may be regarded as a Dynkin diagram with $J$ as the set of vertices.
and with edges between $a \in J$ and $ap^2$. Let $g_J$ be the associated Lie algebra, and $\mathfrak{g}_J$ the unipotent Lie subalgebra. Hence $g_J$ is isomorphic to $\mathfrak{g}_\infty$ if $p$ has an infinite order. Let $U_J$ be the group associated to $g_J$. Then $\mathbb{C} \otimes K^\mathfrak{g}_J$ is isomorphic to the algebra $\mathcal{O}(U_J)$ of regular functions on $U_J$. Let $U_q(g_J)$ be the associated quantized enveloping algebra. Then $U_q^+ (g_J)$ has a crystal basis $B(\infty)$ and an upper global basis $\{G^\text{up}(b)\}_{b \in B(\infty)}$. By specializing $\bigoplus \mathbb{C}[q, q^{-1}]G^\text{up}(b)$ at $q = 1$, we obtain $\mathcal{O}(U_J)$. Then the LLTA-theory says that the elements associated to the irreducible $\mathcal{H}^A$-modules correspond to the image of the upper global basis. Namely, each $b \in B(\infty)$, an irreducible $\mathcal{H}^A$-module $L_b$ is associated and we have

\[ [e_a L_b : L_{b'}] = e_{a, b'} |_{q = 1}, \quad [f_a L_b : L_{b'}] = f_{a, b'} |_{q = 1}. \]

Here $[e_a L_b : L_{b'}]$ and $[f_a L_b : L_{b'}]$ are the composition multiplicities of $L_{b'}$ of $e_a L_b$ and $f_a L_b$ in $K^\mathfrak{g}_J$. (For the definition of the functors $e_a$ and $f_a$ for $a \in J$, see Definition 3.4.) The Laurent polynomials $e_{a, b', b}$ and $f_{a, b', b}$ are defined by

\[ e_a G^\text{up}(b) = \sum_{b' \in B(\infty)} e_{a, b', b} G^\text{up}(b'), \quad f_a G^\text{up}(b) = \sum_{b' \in B(\infty)} f_{a, b', b} G^\text{up}(b'). \]

1.3. Let us explain our analogous conjectures for the affine Hecke algebras of type $B$.

For $p_0, p_1 \in \mathbb{C}^*$, let $\mathcal{H}_B^B(p_0, p_1)$ be the affine Hecke algebra of type $B$ generated by $T_i$ ($0 \leq i \leq n - 1$) and $X_j$ ($1 \leq j \leq n$). The representation theory of $\mathcal{H}_B^B(p_0, p_1)$ is studied by V. M. Kats and K. Kato. In [M], V. M. Kats defined certain restriction functors $E_a$ and the induction functors $F_a$ on the category of the finite-dimensional representations of the affine Hecke algebras of type $B$, which are analogous to G. Grojnowski-Vaisman's construction, and proved the multiplicity-one results (see sections 6.3 and 6.4). On the other hand, S. Kato obtained in [Kat] a geometric parametrization of the irreducible representations of the affine Hecke algebra $\mathcal{H}_B^B(p_0, p_1)$, which is an analogue to geometric methods of Kazhdan-Lusztig and Chriss-Ginzburg.

We say that a finite-dimensional $\mathcal{H}_B^B$-module is of type $J \subset \mathbb{C}^*$ if all the eigenvalues of $X_j$ ($1 \leq j \leq n$) belong to $J$. Let us consider the $\mathbb{Z} \times \mathbb{Z}_2$-action on $\mathbb{C}^*$ generated by $a \mapsto ap^2$ and $a \mapsto a^{-1}$. We can prove that in order to study $\mathcal{H}_B^B$-modules, it is enough to study irreducible modules of type $J$ for a $\mathbb{Z} \times \mathbb{Z}_2$-orbit $J$ in $\mathbb{C}^*$ such that $J$ is a $\mathbb{Z}$-orbit or $J$ contains one of $\pm 1, \pm p_0$ (see Proposition 6.4). Let $I = \mathbb{Z}_{\text{odd}}$ be the set of odd integers. In this paper, we consider the case $J = \{p_k^j \mid k \in I \}$ such that $\pm 1, \pm p_0 \notin J$. Let $K_I(\mathcal{H}_B^B)$ be the Grothendieck group of the abelian category of finite-dimensional representations of $\mathcal{H}_B^B(p_0, p_1)$ of type $J$.

Let $\alpha_a$ ($a \in J$) be the simple roots with

\[ (\alpha_a, \alpha_b) = \begin{cases} 2 & \text{if } a = b, \\ -1 & \text{if } b = ap_1^{\pm 2}, \\ 0 & \text{otherwise.} \end{cases} \]

Then the corresponding Lie algebra is $\mathfrak{g}_\infty$. Let $\theta$ be the involution of $J$ given by $\theta(a) = a^{-1}$. In sections 4 and 5, we introduce the ring $B_{\theta}(\mathfrak{g}_\infty)$ and the $B_{\theta}(\mathfrak{g}_\infty)$-module $V_{\theta}(0)$. They are analogues of the reduced $q$-analogue $B_{\theta}(\mathfrak{g}_\infty)$ generated by $e_a$ and $f_a$, and the $B_{\theta}(\mathfrak{g}_\infty)$-module $U_{\theta}^-(\mathfrak{g}_\infty)$. We can prove that $V_{\theta}(0)$ has the PBW type basis $\{P_{\theta}(b)\}_{b \in B_{\theta}(0)}$, the crystal basis $(L_{\theta}(0), B_{\theta}(0))$, the lower global basis $\{G_{\theta}^{\text{low}}(b)\}_{b \in B_{\theta}(0)}$ and the upper global basis $\{G_{\theta}^{\text{up}}(b)\}_{b \in B_{\theta}(0)}$. Moreover we can combinatorially describe the crystal structure by using the $\theta$-restricted multisegments.

We conjecture that the irreducible $\mathcal{H}^B$-modules of type $J$ are parametrized by $B_{\theta}(0)$ and if $L_b$ is an irreducible $\mathcal{H}^B$-module associated to $b \in B_{\theta}(0)$, then we have $E_a L_b = L_{\bar{g}_a b}$.
\( F_aL_b = L_{\bar{F}_a,b} \) and \([E_aL_b : L_{b'}] = E_{a,b,b'|q=1}, [F_aL_b : L_{b'}] = F_{a,b,b'|q=1}. \) (For the definition of the functors \( E_a, F_a, \bar{E}_a \) and \( \bar{F}_a \) for \( a \in J \), see Definition 6.5.) Here the Laurent polynomials \( E_{a,b,b'} \) and \( F_{a,b,b'} \) are defined by

\[
E_aG_{\theta}^{up}(b) = \sum_{b' \in B_{\theta}(0)} E_{a,b,b'}G_{\theta}^{up}(b') ,
\]

\[
F_aG_{\theta}^{up}(b) = \sum_{b' \in B_{\theta}(0)} F_{a,b,b'}G_{\theta}^{up}(b').
\]

\( \text{FIGURE 2. Conjectural correspondence in type B} \)

| the quantum enveloping algebra \( U_q(\mathfrak{g}_{\infty}) \) with \( \theta \) | the affine Hecke algebra of type B \( H^B_n(p_0, p_1) \) (\( n \geq 0 \)) |
|---|---|
| \( V_\theta(0) = U_q^{-}(\mathfrak{g}_{\infty})/\sum_i U_q^{-}(\mathfrak{g}_{\infty})(f_i - f_\theta(i)) \) | \( K^B_q = \oplus_{n \geq 0} K_J(H^B_n(p_0, p_1)) \) |
| \( E_a, F_a \) \( \text{the crystal basis } B_\theta(0) \) | \( \mathcal{M}_\theta = \{ \text{the } \theta\text{-restricted multisegments} \} \) |
| \( \text{the upper global basis } \{G_{\theta}^{up}(b)\}_{b \in B_\theta(0)} \) | \( \text{the irreducible modules } \{L_{b}\}_{b \in B_\theta(0)} \) |
| \( \bar{E}_a, \bar{F}_a \) \( \text{the modified root operators} \) | \( \bar{E}_a = \text{soc}(E_a), \bar{F}_a = \text{cosoc}(F_a) \) |
| \( \text{the PBW basis } \{P_\theta(b)\}_{b \in B_\theta(0)} \) | \( \text{the standard modules} \) |

**Part I. Review on Lascoux-Leclerc-Thibon-Ariki Theory**

2. REPRESENTATION THEORY OF \( U_q(\mathfrak{g}_{\infty}) \)

2.1. Quantized universal enveloping algebras and its reduced \( q \)-analogues. We shall recall the quantized universal enveloping algebra \( U_q(\mathfrak{g}) \). Let \( I \) be an index set (for simple roots), and \( Q \) the free \( \mathbb{Z} \)-module with a basis \( \{\alpha_i\}_{i \in I} \). Let \( (\cdot, \cdot): Q \times Q \rightarrow \mathbb{Z} \) be a symmetric bilinear form such that \( (\alpha_i, \alpha_i)/2 \in \mathbb{Z} > 0 \) for any \( i \) and \( (\alpha_i^\vee, \alpha_j) \in \mathbb{Z}_{\leq 0} \) for \( i \neq j \) where \( \alpha_i^\vee := 2\alpha_i/(\alpha_i, \alpha_i) \).

Let \( q \) be an indeterminate and set \( K := \mathbb{Q}(q) \). We define its subrings \( A_0, A_\infty \) and \( A \) as follows.

\[
A_0 = \{ f \in K \mid f \text{ is regular at } q = 0 \},
\]

\[
A_\infty = \{ f \in K \mid f \text{ is regular at } q = \infty \},
\]

\[
A = \mathbb{Q}[q, q^{-1}].
\]

**Definition 2.1.** The quantized universal enveloping algebra \( U_q(\mathfrak{g}) \) is the \( K \)-algebra generated by elements \( e_i, f_i \) and invertible elements \( t_i \) \( (i \in I) \) with the following defining relations.

1. The \( t_i \)'s commute with each other.
2. \( t_i e_i t_i^{-1} = q^{(\alpha_i, \alpha_i)} e_i \) and \( t_j f_j t_j^{-1} = q^{-(\alpha_j, \alpha_i)} f_i \) for any \( i, j \in I \).
3. \( [e_i, f_j] = \delta_{ij} t_i - t_i^{-1} \) for \( i, j \in I \). Here \( q_i := q^{(\alpha_i, \alpha_i)/2} \).
4. (Serre relation) For \( i \neq j \),

\[
\sum_{k=0}^{b} (-1)^k e_i^{(k)} e_j e_i^{(b-k)} = 0, \quad \sum_{k=0}^{b} (-1)^k f_i^{(k)} f_j f_i^{(b-k)} = 0.
\]
Here \( b = 1 - (\alpha_i^\vee, \alpha_j) \) and
\[
e_i^{(k)} = e_i^k/[k]_i! , \quad f_i^{(k)} = f_i^k/[k]_i! , \quad [k]_i = (q_i^k - q_i^{-k})/(q_i - q_i^{-1}) , \quad [k]_i! = [1]_i \cdots [k]_i.
\]
Let us denote by \( U_q^-(g) \) the subalgebra of \( U_q(g) \) generated by the \( f_i \)'s.
Let \( e_i' \) and \( e_i^{*} \) be the operators on \( U_q^-(g) \) defined by
\[
[e_i, a] = \frac{e_i^{*}a - a e_i^{'}}{q_i - q_i^{-1}} (a \in U_q^-(g)).
\]
These operators satisfy the following formulas similar to derivations:
\[
e_i' ab = e_i(a)b + (e_i^{*}a)(\text{Ad}(t_i)b),
\]
\[
e_i^{*}(ab) = ae_i^{*}b + (e_i^{*}a)(\text{Ad}(t_i)b).
\]
The algebra \( U_q^-(g) \) has a unique symmetric bilinear form \((\cdot, \cdot)\) such that \((1, 1) = 1\) and \((e_i^{*}a, b) = (a, f_i b)\) for any \( a, b \in U_q^-(g) \).
It is non-degenerate and satisfies \((e_i^{*}a, b) = (a, f_i b)\).

Definition 2.2. The reduced \( q \)-analogue \( B(g) \) is the \( \mathbb{Q}(q) \)-algebra generated by \( e_i^{*} \) and \( f_i \).

2.2. Review on crystal bases and global bases. Since \( e_i' \) and \( f_i \) satisfy the \( q \)-boson relation, any element \( a \in U_q^-(g) \) can be written uniquely as
\[
a = \sum_{n \geq 0} f_i^{(n)} a_n \quad \text{with} \quad e_i a_n = 0.
\]
Here \( f_i^{(n)} = \frac{f_i^n}{[n]_i!} \).

Definition 2.3. We define the modified root operators \( \tilde{e}_i \) and \( \tilde{f}_i \) on \( U_q^-(g) \) by
\[
\tilde{e}_i a = \sum_{n \geq 1} f_i^{(n-1)} a_n , \quad \tilde{f}_i a = \sum_{n \geq 0} f_i^{(n+1)} a_n.
\]

Theorem 2.4 ([Kas]). We define
\[
L(\infty) = \sum_{\ell \geq 0, i_1, \ldots, i_\ell \in I} A_0 \tilde{f}_{i_1} \cdots \tilde{f}_{i_\ell} \cdot 1 \subset U_q^-(g),
\]
\[
B(\infty) = \left\{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_\ell} \cdot 1 \mod qL(\infty) \mid \ell \geq 0, i_1, \ldots, i_\ell \in I \right\} \subset L(\infty)/qL(\infty).
\]
Then we have
(i) \( \tilde{e}_i L(\infty) \subset L(\infty) \) and \( \tilde{f}_i L(\infty) \subset L(\infty) \),
(ii) \( B(\infty) \) is a basis of \( L(\infty)/qL(\infty) \),
(iii) \( \tilde{f}_i B(\infty) \subset B(\infty) \) and \( \tilde{e}_i B(\infty) \subset B(\infty) \cup \{0\} \).
We call \((L(\infty), B(\infty))\) the crystal basis of \( U_q^-(g) \).

Let \(-\) be the automorphism of \( K \) sending \( q \) to \( q^{-1} \). Then \( A_0 \) coincides with \( A_\infty \).
Let \( V \) be a vector space over \( K \), \( L_0 \) an \( A \)-submodule of \( V \), \( L_\infty \) an \( A_\infty \)-submodule, and \( V_A \) an \( A \)-submodule. Set \( E : = L_0 \cap L_\infty \cap V_A \).
Definition 2.5 ([Kas]). We say that $(L_0, L_\infty, V_A)$ is balanced if each of $L_0$, $L_\infty$ and $V_A$ generates $V$ as a $K$-vector space, and if one of the following equivalent conditions is satisfied.

(i) $E \rightarrow L_0/qL_0$ is an isomorphism,
(ii) $E \rightarrow L_\infty/q^{-1}L_\infty$ is an isomorphism,
(iii) $(L_0 \cap V_A) \oplus (q^{-1}L_\infty \cap V_A) \rightarrow V_A$ is an isomorphism.
(iv) $A_0 \otimes_q E \rightarrow L_0$, $A_\infty \otimes_q E \rightarrow L_\infty$, $A \otimes_q E \rightarrow V_A$ and $K \otimes_q E \rightarrow V$ are isomorphisms.

Let $\gamma$ be the ring automorphism of $U_q(\mathfrak{g})$ sending $q$, $t_i$, $e_i$, $f_i$ to $q^{-1}$, $t_i^{-1}$, $e_i$, $f_i$. Let $U_q(\mathfrak{g})_A$ be the $A$-subalgebra of $U_q(\mathfrak{g})$ generated by $e_i^{(n)}$, $f_i^{(n)}$ and $t_i$. Similarly we define $U_q^-(\mathfrak{g})_A$.

Theorem 2.6. $(L(\infty), \overline{L(\infty)}, U_q^-(\mathfrak{g})_A)$ is balanced.

Let $G^{\mathrm{low}} : L(\infty)/qL(\infty) \rightarrow \overline{L(\infty)} \cap U_q^-(\mathfrak{g})_A$ be the inverse of $E \rightarrow L(\infty)/qL(\infty)$. Then $\{G^{\mathrm{low}}(b) \mid b \in B(\infty)\}$ forms a basis of $U_q^-(\mathfrak{g})$. We call it a (lower) global basis. It is first introduced by G. Lusztig ([L]) under the name of "canonical basis" for the $A$, $D$, $E$ cases.

Definition 2.7. Let $\{G^{\mathrm{up}}(b) \mid b \in B(\infty)\}$ be the dual basis of $\{G^{\mathrm{low}}(b) \mid b \in B(\infty)\}$ with respect to the inner product $(\cdot, \cdot)$. We call it the upper global basis of $U_q^-(\mathfrak{g})$.

2.3. Review on the PBW basis. In the sequel, we set $I = \mathbb{Z}_{\text{odd}}$ and

$$(\alpha_i, \alpha_j) = \begin{cases} 2 & \text{for } i = j, \\ -1 & \text{for } j = i \pm 2, \\ 0 & \text{otherwise}, \end{cases}$$

and we consider the corresponding quantum group $U_q(\mathfrak{g}_\infty)$. In this case, we can parametrize the crystal basis $B(\infty)$ by the multisegments. We shall recall this parametrization and the PBW basis.

Definition 2.8. For $i, j \in I$ such that $i \leq j$, we define a segment $\langle i, j \rangle$ as the interval $[i, j] \subset \mathbb{Z}_{\text{odd}}$. A multisegment is a formal finite sum of segments:

$$m = \sum_{i \leq j} m_{ij} \langle i, j \rangle$$

with $m_{ij} \in \mathbb{Z}_{\geq 0}$. If $m_{ij} > 0$, we sometimes say that $\langle i, j \rangle$ appears in $m$. We denote sometimes $\langle i \rangle$ for $\langle i, i \rangle$. We denote by $\mathcal{M}$ the set of multisegments. We denote by $\emptyset$ the zero element (or the empty multisegment) of $\mathcal{M}$.

Definition 2.9. For two segments $\langle i_1, j_1 \rangle$ and $\langle i_2, j_2 \rangle$, we define the ordering $\geq_{\text{PBW}}$ by the following:

$$\langle i_1, j_1 \rangle \geq_{\text{PBW}} \langle i_2, j_2 \rangle \iff \begin{cases} j_1 > j_2 \\ j_1 = j_2 \text{ and } i_1 \geq i_2. \end{cases}$$

We call this ordering the PBW ordering.

Example 2.10. We have $\langle 1, 1 \rangle >_{\text{PBW}} \langle -1, 1 \rangle >_{\text{PBW}} \langle -1, -1 \rangle$. 


Definition 2.11. We define the element $P(m) \in U_q^{-}(gl_{\infty})$ indexed by a multisegment $m$ as follows:

(1) for a segment $\langle i, j \rangle$, we define the element $\langle i, j \rangle \in U_q^{-}(gl_{\infty})$ inductively by
\[
\langle i, i \rangle = f_{i},
\]
\[
\langle i, j \rangle = \langle i, j - 2 \rangle \langle j, j \rangle - q(j, j) \langle i, j - 2 \rangle,
\]

(2) for a multisegment $m = \sum_{i \leq j} m_{ij} \langle i, j \rangle$, we define
\[
P(m) = \prod \langle i, j \rangle^{(m_{ij})}.
\]

Here the product $\prod$ is taken over segments appearing in $m$ from large to small with respect to the PBW ordering. The element $\langle i, j \rangle^{(m_{ij})}$ is the divided power of $\langle i, j \rangle$ i.e.
\[
\langle i, j \rangle^{(m_{ij})} = \frac{1}{[m_{ij}]!} \langle i, j \rangle^{m_{ij}}.
\]

Set $\text{wt}(P(m)) = -\sum_{i \leq j} m_{ij} \alpha_{ij}$.

Theorem 2.12 ([IL]). The set of elements $\{P(m) \mid m \in \mathcal{M}\}$ is a basis of the $K$-vector space $U_q^{-}(gl_{\infty})$. Moreover this is a basis of the $\mathcal{A}$-module $U_q^{-}(gl_{\infty})_{\mathcal{A}}$. We call this basis the PBW basis of $U_q^{-}(gl_{\infty})$.

Definition 2.13. For two segments $\langle i_1, j_1 \rangle$ and $\langle i_2, j_2 \rangle$, we define the ordering $\geq_{\text{cry}}$ by the following:
\[
\langle i_1, j_1 \rangle \geq_{\text{cry}} \langle i_2, j_2 \rangle \iff \left\{\begin{array}{ll}
    j_1 > j_2 & \\
    j_1 = j_2 \text{ and } i_1 \leq i_2.
  \end{array}\right.
\]

We call this ordering the crystal ordering. For $m = \sum_{i \leq j} m_{ij} \langle i, j \rangle \in \mathcal{M}$ and and $m' = \sum_{i \leq j} m'_{ij} \langle i, j \rangle \in \mathcal{M}$, we define $m' < m$ if there exists a segment $\langle i_0, j_0 \rangle$ such that $m'_{i_0, j_0} < m_{i_0, j_0}$ and $m'_{i,j} = m_{i,j}$ for any $\langle i, j \rangle >_{\text{cry}} \langle i_0, j_0 \rangle$.

Example 2.14. The crystal ordering is different from the PBW ordering. For example, we have $\langle -1, 1 \rangle >_{\text{cry}} \langle 1, 1 \rangle >_{\text{cry}} \langle -1, -1 \rangle$, while we have $\langle 1, 1 \rangle >_{PBW} \langle -1, 1 \rangle >_{PBW} \langle -1, -1 \rangle$.

Definition 2.15. We define the crystal structure on $\mathcal{M}$ as follows: for $m = \sum m_{i,j} \langle i, j \rangle \in \mathcal{M}$ and $i \in I$, set $A_{k}^{(i)}(m) = \sum_{k' \geq k} (m_{i,k'} - m_{i+2,k'+2})$ for $k \geq i$. Define $\varepsilon_{i}(m)$ as

\[
\text{max} \left\{ A_{k}^{(i)}(m) \mid k \geq i \right\} \geq 0.
\]

(i) If $\varepsilon_{i}(m) = 0$, then define $\tilde{\varepsilon}_{i}(m) = 0$. If $\varepsilon_{i}(m) > 0$, let $k_{e}$ be the largest $k \geq i$ such that $\varepsilon_{i}(m) = A_{k}^{(i)}(m)$ and define $\tilde{\varepsilon}_{i}(m) = m - \langle i, k_{e} \rangle + \delta_{k_{e} > i}(i + 2, k_{e})$.

(ii) Let $k_{f}$ be the smallest $k \geq i$ such that $\varepsilon_{i}(m) = A_{k}^{(i)}(m)$ and define $\tilde{f}_{i}(m) = m - \delta_{k_{f} < i}(i + 2, k_{f}) + \langle i, k_{f} \rangle$.

Remark 2.16. For $i \in I$, the actions of the operators $\tilde{\varepsilon}_{i}$ and $\tilde{f}_{i}$ on $m \in \mathcal{M}$ are also described by the following algorithm:

Step 1. Arrange the segments in $m$ in the crystal ordering.

Step 2. For each segment $\langle i, j \rangle$, write $-$, and for each segment $\langle i + 2, j \rangle$, write $+$.

Step 3. In the resulting sequence of $+$ and $-$, delete a subsequence of the form $++$ and keep on deleting until no such subsequence remains.
Then we obtain a sequence of the form $- - \cdots - + + \cdots +$.

(1) $\epsilon_i(m)$ is the total number of $-$ in the resulting sequence.

(2) $f_i(m)$ is given as follows:
   (a) If the leftmost $+$ corresponds to a segment $(i + 2, j)$, then replace it with $(i, j)$.
   (b) If no $+$ exists, add a segment $(i, i)$ to $m$.

(3) $\tilde{e}_i(m)$ is given as follows:
   (a) If the rightmost $-$ corresponds to a segment $(i, j)$, then replace it with $(i + 2, j)$.
   (b) If no $-$ exists, then $\tilde{e}_i(m) = 0$.

**Theorem 2.17.**

(i) $L(\infty) = \bigoplus_{m \in \mathcal{M}} A_0 P(m)$.

(ii) $B(\infty) = \{P(m) \mod qL(\infty) \mid m \in \mathcal{M}\}$.

(iii) We have
   \[ \tilde{e}_i P(m) \equiv P(\tilde{e}_i(m)) \mod qL(\infty), \]
   \[ \tilde{f}_i P(m) \equiv P(\tilde{f}_i(m)) \mod qL(\infty). \]

   Note that $\tilde{e}_i$ and $\tilde{f}_i$ in the left-hand-side is the modified root operators.

(iv) We have the expansion
   \[ \overline{P(m)} \in P(m) + \sum_{m' \prec m} A P(m'). \]

Therefore we can index the crystal basis by multisegments. By this theorem we can easily see by a standard argument that $(L(\infty), \overline{L(\infty)}, U_q^- (\mathfrak{g})_A)$ is balanced, and there exists a unique $G^{\text{low}}(m) \in L(\infty) \cap U_q^- (\mathfrak{g})_A$ such that $\overline{G^{\text{low}}(m)} = G^{\text{low}}(m)$ and $G^{\text{low}}(m) \equiv P(m) \mod qL(\infty)$. The basis $\{G^{\text{low}}(m)\}_{m \in \mathcal{M}}$ is a lower global basis.

3. **Representation Theory of $\mathcal{H}_n^A$ and the Lascloux-Leclerc-Thibon-Ariki Theory**

3.1. The affine Hecke algebra of type A.

**Definition 3.1.** For $p \in \mathbb{C}^*$, the affine Hecke algebra $\mathcal{H}_n^A$ of type $A$ is a $\mathbb{C}$-algebra generated by

\[ T_1, \ldots, T_{n-1}, X_1^\pm, \ldots, X_n^\pm \]

satisfying the following defining relations:

(1) $X_i X_j = X_j X_i$ for any $1 \leq i, j \leq n$.

(2) [The braid relations of type A]
   \[ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \quad (1 \leq i \leq n-2), \]
   \[ T_i T_j = T_j T_i \quad (|i-j| > 1). \]

(3) [The Hecke relations]
   \[ (T_i - p)(T_i + p^{-1}) = 0 \quad (1 \leq i \leq n-1). \]

(4) [The Bernstein-Lusztig relations]
   \[ T_i X_j = X_j T_i \quad (1 \leq i \leq n-1), \]
   \[ T_i X_j = X_j T_i \quad (j \neq i, i+1). \]

Since we can embed $\mathcal{H}_n^A$ into $\mathcal{H}_{n+m}^A$ by $T_i \mapsto T_{i+m}$ ($1 \leq i \leq n-1$), $X_j \mapsto X_{m+j}$ ($1 \leq j \leq m$), we consider $\mathcal{H}_m^A \otimes \mathcal{H}_n^A$ as a subalgebra of $\mathcal{H}_{n+m}^A$.
Definition 3.2. For a finite-dimensional \( \mathcal{H}_{n}^{A} \)-module \( M \), let

\[
M = \bigoplus_{a \in (\mathbb{C}^{*})^{n}} M_{a}
\]

be the generalized eigenspace decomposition with respect to \( X_{1}, \ldots, X_{n} \). Here

\[
M_{a} := \{ u \in M \mid (X_{i} - a_{i})^{N}u = 0 \text{ for any } 1 \leq i \leq n \text{ and } N \gg 0 \}
\]

for \( a = (a_{1}, \ldots, a_{n}) \in (\mathbb{C}^{*})^{n} \).

1. We say that \( M \) is of type \( J \) if all the eigenvalues of \( X_{1}, \ldots, X_{n} \) belong to \( J \subset \mathbb{C}^{*} \).
2. Put

\[
K_{J}^{A} := \bigoplus_{n \geq 0} K_{J,n}^{A}.
\]

Here \( K_{J}^{A} \) is the Grothendieck group of the abelian category of finite-dimensional \( \mathcal{H}_{n}^{A} \)-modules of type \( J \).

3. The group \( \mathbb{Z} \) acts on \( \mathbb{C}^{*} \) by \( \mathbb{Z} \ni n \mapsto a \mapsto ap^{2n} \).

Lemma 3.3. Let \( J_{1} \) and \( J_{2} \) be \( \mathbb{Z} \)-invariant subsets in \( \mathbb{C}^{*} \) such that \( J_{1} \cap J_{2} = \emptyset \).

1. If \( M \) is an irreducible \( \mathcal{H}_{n}^{A} \)-module of type \( J_{1} \) and \( N \) is an irreducible \( \mathcal{H}_{n}^{A} \)-module of type \( J_{2} \), then \( \text{Ind}_{\mathcal{H}_{n}^{A}}^{\mathcal{H}_{n+m}^{A}}(M \otimes N) \) is irreducible of type \( J_{1} \cup J_{2} \).
2. Conversely, if \( L \) is an irreducible \( \mathcal{H}_{n}^{A} \)-module of type \( J_{1} \cup J_{2} \), then there exist \( m \) (\( 0 \leq m \leq n \)), an irreducible \( \mathcal{H}_{m}^{A} \)-module \( M \) of type \( J_{1} \) and an irreducible \( \mathcal{H}_{n-m}^{A} \)-module \( N \) of type \( J_{2} \) such that \( L \) is isomorphic to \( \text{Ind}_{\mathcal{H}_{m}^{A}}^{\mathcal{H}_{n}^{A}}(M \otimes N) \).

Hence in order to study the irreducible modules over the affine Hecke algebras of type \( A \), it is enough to treat the irreducible modules of type \( J \) for an orbit \( J \) with respect to the \( \mathbb{Z} \)-action on \( \mathbb{C}^{*} \).

3.2. The \( a \)-restriction and the \( a \)-induction. For a \( \mathbb{C} \)-algebra \( A \), let us denote by \( A - \text{mod}^{\text{fd}} \) the abelian category of finite-dimensional \( A \)-modules.

Definition 3.4. For \( a \in \mathbb{C}^{*} \), let us define the functors

\[
e_{a} : \mathcal{H}_{n}^{A} - \text{mod}^{\text{fd}} \rightarrow \mathcal{H}_{n-1}^{A} - \text{mod}^{\text{fd}}, \quad f_{a} : \mathcal{H}_{n}^{A} - \text{mod}^{\text{fd}} \rightarrow \mathcal{H}_{n+1}^{A} - \text{mod}^{\text{fd}}
\]

by: \( e_{a}M \) is the generalized \( a \)-eigenspace of \( M \) with respect to the action of \( X_{n} \), and

\[
f_{a}M := \text{Ind}_{\mathcal{H}_{n}^{A}}^{\mathcal{H}_{n+1}^{A}}(M \otimes \langle a \rangle),
\]

where \( \langle a \rangle \) is the 1-dimensional representation of \( \mathbb{C}[X_{n+1}^{\pm 1}] \) defined by \( X_{n+1} \mapsto a \).

Moreover, put

\[
\overline{e}_{a}M := \text{soc} e_{a}M, \quad \overline{f}_{a}M := \text{cosoc} f_{a}M
\]

for \( a \in \mathbb{C}^{*} \). Here the socle is the maximal semisimple submodule and the cosocle is the maximal semisimple quotient module.

Theorem 3.5 (Grojnowski-Vazirani [GV]). Suppose \( M \) is irreducible. Then \( \overline{f}_{a}M \) is irreducible, and \( \overline{e}_{a}M \) is irreducible or 0 for any \( a \in \mathbb{C}^{*} \).
3.3. LLTA type theorems for the affine Hecke algebra of type A. In this subsection, we consider the case
\[ J = \{ p^k \mid k \in \mathbb{Z}_{\text{odd}} \}, \]
and suppose \( p \) is not a root of unity. For short, we shall write \( e_i, \tilde{e}_i, f_i \) and \( \tilde{f}_i \) for \( e_{p^i}, \tilde{e}_{p^i}, f_{p^i} \) and \( \tilde{f}_{p^i} \), respectively.

The LLTA type theorem for the affine Hecke algebra of type A consists of two parts. First is a labeling of finite-dimensional irreducible \( \mathcal{H}^{A} \)-modules by the crystal \( B(\infty) \). Second is a description of some composition multiplicities by using the upper global basis.

**Theorem 3.6 (Vazirani [V]).** There are complete representatives
\[ \{ L_b \mid b \in B(\infty) \} \]
of the finite-dimensional irreducible \( \mathcal{H}^{A} \)-modules of type \( J \) such that
\[ \tilde{e}_i L_b = L_{\tilde{e}_i b}, \quad \tilde{f}_i L_b = L_{\tilde{f}_i b} \]
for any \( i \in I \).

**Theorem 3.7 (Ariki [A]).** For \( i \in I = \mathbb{Z}_{\text{odd}} \), let us define \( e_{i,b,b'}, f_{i,b,b'} \in \mathbb{C}[q, q^{-1}] \) by the coefficients of the expansions:
\[ e_i^r G^{up}(b) = \sum_{b' \in B(\infty)} e_{i,b,b'} G^{up}(b'), \quad f_i^r G^{up}(b) = \sum_{b' \in B(\infty)} f_{i,b,b'} G^{up}(b'). \]

Then \[ [e_i L_b : L_{b'}] = e_{i,b,b'}|_{q=1}, \quad [f_i L_b : L_{b'}] = f_{i,b,b'}|_{q=1}. \]
Here \([M : N]\) is the composition multiplicity of \( N \) in \( M \) on \( K^A \).

**Part II. The Symmetric Crystals and some LLTA Type Conjectures for Affine Hecke Algebra of Type B**

4. GENERAL DEFINITIONS AND CONJECTURES FOR SYMMETRIC CRYSTALS

We note the followings in subsection 2.1. Let \( \theta \) be an automorphism of \( I \) such that \( \theta^2 = \text{id} \) and \((\alpha_{\theta(i)}, \alpha_{\theta(j)}) = (\alpha_i, \alpha_j)\). Hence it extends to an automorphism of the root lattice \( Q \) by \( \theta(\alpha_i) = \alpha_{\theta(i)} \), and induces an automorphism of \( U_q(\mathfrak{g}) \).

**Definition 4.1.** Let \( B_{\theta}(\mathfrak{g}) \) be the \( K \)-algebra generated by \( E_i, F_i \), and invertible elements \( T_i \) \((i \in I)\) satisfying the following defining relations:

(i) the \( T_i \)'s commute with each other,
(ii) \( T_{\theta(i)} = T_i \) for any \( i \),
(iii) \( T_i E_j T_i^{-1} = q^{(\alpha_i, \alpha_{\theta(j)})} E_j \) and \( T_i F_j T_i^{-1} = q^{(\alpha_i, \alpha_{\theta(j)}, -\alpha_i)} F_j \) for \( i, j \in I \),
(iv) \( E_i F_j = q^{-(\alpha_i, \alpha_j)} F_j E_i + (\delta_{ij} + \delta_{\theta(i), j}) T_i \) for \( i, j \in I \),
(v) the \( E_i \)'s and the \( F_i \)'s satisfy the \( q \)-Serre relations.

We set \( E_i^{(n)} = E_i^n/[n]_i! \) and \( F_i^{(n)} = F_i^n/[n]_i! \).

**Proposition 4.2.** Let \( \lambda \in P_+ := \{ \lambda \in \text{Hom}(Q, \mathbb{Q}) \mid \langle \alpha^\vee_i, \lambda \rangle \in \mathbb{Z}_{\geq 0} \text{ for any } i \in I \} \) be a dominant integral weight such that \( \theta(\lambda) = \lambda \).

(i) There exists a \( B_{\theta}(\mathfrak{g}) \)-module \( V_{\theta}(\lambda) \) generated by a non-zero vector \( \phi_\lambda \) such that
(a) \( E_i \phi_\lambda = 0 \) for any \( i \in I \),
(b) \( T_i \phi_\lambda = q^{(\alpha_i, \lambda)} \phi_\lambda \) for any \( i \in I \),
(c) \( \{ u \in V_{\theta}(\lambda) \mid E_i u = 0 \text{ for any } i \in I \} = K \phi_\lambda \).
Moreover such a $V_\theta(\lambda)$ is irreducible and unique up to an isomorphism.

(ii) there exists a unique symmetric bilinear form $(\cdot, \cdot)$ on $V_\theta(\lambda)$ such that $(\phi_\lambda, \phi_\lambda) = 1$ and $(E_i u, v) = (u, F_i v)$ for any $i \in I$ and $u, v \in V_\theta(\lambda)$, and it is non-degenerate.

(iii) There exists an endomorphism $- \psi$ of $V_\theta(\lambda)$ such that $\bar{\phi}_\lambda = \phi_\lambda$ and $\bar{\alpha} v = \bar{\bar{\alpha}} v$, $\bar{F}_i v = \bar{F}_i \bar{v}$ for any $a \in K$ and $v \in V_\theta(\lambda)$.

The pair $(B_\theta(g), V_\theta(\lambda))$ is an analogue of $(\mathcal{B}(g), U_q^-(g))$. Such a $V_\theta(\lambda)$ is constructed as follows. Let $U_q^- (g) \phi_\lambda$ and $U_q^- (g) \phi'_\lambda$ be a copy of a free $U_q^- (g)$-module. We give the structure of a $B_\theta(g)$-module on them as follows: for any $i \in I$ and $a \in U_q^- (g)$

\[
\begin{align*}
&T_i(a \phi'_\lambda) = q^{(\alpha_i, \lambda)}(Ad(t_i t_\theta(\lambda)) a) \phi'_\lambda, \\
&E_i(a \phi'_\lambda) = (\epsilon_i a + q^{(\alpha_i, \lambda)} Ad(t_i) (c_i^\theta(\lambda) a)) \phi'_\lambda, \\
&F_i(a \phi'_\lambda) = (f_i a) \phi'_\lambda
\end{align*}
\]

and

\[
\begin{align*}
&T_i(a \phi''_\lambda) = q^{(\alpha_i, \lambda)}(Ad(t_i t_\theta(\lambda)) a) \phi''_\lambda, \\
&E_i(a \phi''_\lambda) = (\epsilon_i a) \phi''_\lambda, \\
&F_i(a \phi''_\lambda) = (f_i a + q^{(\alpha_i, \lambda)} Ad(t_i) a) \phi''_\lambda.
\end{align*}
\]

Then there exists a unique $B_\theta(g)$-linear morphism $\psi: U_q^- (g) \phi_\lambda \rightarrow U_q^- (g) \phi'_\lambda$ sending $\phi_\lambda$ to $\phi'_\lambda$. Its image $\psi(U_q^- (g) \phi'_\lambda)$ is $V_\theta(\lambda)$.

Hereafter we assume further that

there is no $i \in I$ such that $\theta(i) = i$.

We conjecture that $V_\theta(\lambda)$ has a crystal basis. This means the following. Since $E_i$ and $F_i$ satisfy the $q$-boson relation $E_i F_i = q^{-(\alpha_i, \alpha_i)} F_i E_i + 1$, we define the modified root operators:

$\tilde{E}_i(u) = \sum_{n \geq 1} F_i^{(n-1)} u_n$ and $\tilde{F}_i(u) = \sum_{n \geq 0} F_i^{(n+1)} u_n$,

when writing $u = \sum_{n \geq 0} F_i^{(n)} u_n$ with $E_i u_n = 0$. Let $L_\theta(\lambda)$ be the $A_0$-submodule of $V_\theta(\lambda)$ generated by $\tilde{F}_{i_1} \cdots \tilde{F}_{i_{\ell}} \phi_\lambda$ ($\ell \geq 0$ and $i_1, \ldots, i_{\ell} \in I$), and let $B_\theta(\lambda)$ be the subset

$\{ \tilde{F}_{i_1} \cdots \tilde{F}_{i_{\ell}} \phi_\lambda \text{ mod } q L_\theta(\lambda) \mid \ell \geq 0, i_1, \ldots, i_{\ell} \in I \}$

of $L_\theta(\lambda)/q L_\theta(\lambda)$.

**Conjecture 4.3.** Let $\lambda$ be a dominant integral weight such that $\theta(\lambda) = \lambda$.

1. $\tilde{F}_{i_1} L_\theta(\lambda) \subset L_\theta(\lambda)$ and $\tilde{E}_{i_1} L_\theta(\lambda) \subset L_\theta(\lambda)$,
2. $B_\theta(\lambda)$ is a basis of $L_\theta(\lambda)/q L_\theta(\lambda)$,
3. $\tilde{F}_i B_\theta(\lambda) \subset B_\theta(\lambda)$, and $\tilde{E}_i B_\theta(\lambda) \subset B_\theta(\lambda) \cup \{0\}$,
4. $\tilde{F}_i \tilde{E}_i(b) = b$ for any $b \in B_\theta(\lambda)$ such that $\tilde{E}_i b \neq 0$, and $\tilde{E}_i \tilde{F}_i(b) = b$ for any $b \in B_\theta(\lambda)$.

Moreover we conjecture that $V_\theta(\lambda)$ has a global crystal basis. Namely we have

**Conjecture 4.4.** $(L_\theta(\lambda), \overline{L_\theta(\lambda)}$, $V_\theta(\lambda)_{\text{low}}^{\lambda})$ is balanced. Here $V_\theta(\lambda)_{\text{low}}^{\lambda} := U_q^- (g) \phi_\lambda$.

The dual version is as follows. As in [Kas], we have

**Lemma 4.5.** Assume Conjecture 4.3. Then we have

(i) $L_\theta(\lambda) = \{ v \in V_\theta(\lambda) \mid (L_\theta(\lambda), v) \subset A_0 \}$,
(ii) Let $(\cdot, \cdot)_0$ be the $C$-valued symmetric bilinear form on $L_\theta(\lambda)/q L_\theta(\lambda)$ induced by $(\cdot, \cdot)$. Then $B_\theta(\lambda)$ is an orthonormal basis with respect to $(\cdot, \cdot)_0$. 
Let us denote by $V_{\theta}\left(\lambda\right)^{\text{up}}_{\mathbf{A}}$ the dual space $\{v \in V_{\theta}\left(\lambda\right) \mid (V_{\theta}\left(\lambda\right)_{\mathbf{A}}^{\text{low}}, v) \in \mathbf{A}\}$. Then Conjecture 4.4 is equivalent to the following conjecture.

**Conjecture 4.6.** $(L_{\theta}\left(\lambda\right), c(L_{\theta}\left(\lambda\right)), V_{\theta}\left(\lambda\right)_{\mathbf{A}}^{\text{up}})$ is balanced.

Here $c$ is a unique endomorphism of $V_{\theta}\left(\lambda\right)$ such that $c(\phi_{\lambda}) = \phi_{\lambda}$ and $c(\alpha v) = \overline{\alpha}c(v)$ for any $\alpha \in \mathbb{K}$ and $v \in V_{\theta}\left(\lambda\right)$. We have $(c(v'), v) = (\overline{v'}, v)$ for any $v, v' \in V_{\theta}\left(\lambda\right)$.

Note that $V_{\theta}\left(\lambda\right)_{\mathbf{A}}^{\text{up}}$ is the largest $\mathbf{A}$-submodule $M$ of $V_{\theta}\left(\lambda\right)$ such that $M$ is invariant by the $E_{i}^{(n)'}\mathrm{s}$ and $M \cap \mathbb{K}\phi_{\lambda} = \mathbb{A}\phi_{\lambda}$.

By Conjecture 4.6, $L_{\theta}\left(\lambda\right) \cap c(L_{\theta}\left(\lambda\right)) \cap V_{\theta}(0)^{\text{up}} \rightarrow L_{\theta}\left(\lambda\right)/qL_{\theta}\left(\lambda\right)$ is an isomorphism, let $G_{\theta}^{\text{up}}$ be its inverse. Then $\{G_{\theta}^{\text{up}}(b)\}_{b \in B_{\theta}(\lambda)}$ is a basis of $V_{\theta}\left(\lambda\right)$, which we call the upper global basis of $V_{\theta}\left(\lambda\right)$.

### 5. Symmetric Crystals for $\mathfrak{g}l_{\infty}$

In this section, we consider the case $\mathfrak{g} = \mathfrak{g}l_{\infty}$ and the Dynkin involution $\theta$ of $I$ defined by $\theta(i) = -i$ for $i \in I = \mathbb{Z}_{\text{odd}}$.

We set $\bar{V}_{\theta}(0) := B_{\theta}(\mathfrak{g})/(\sum_{i}B_{\theta}(\mathfrak{g})E_i + \sum_{i}B_{\theta}(\mathfrak{g})(F_i - F_{\theta(i)})) \simeq U_{q}^{-}(\mathfrak{g}_{\infty})/\sum_{i}U_{q}^{-}(\mathfrak{g}_{\infty})(f_i - f_{\theta(i)})$.

Since $F_{i}\phi_{\lambda}' = (f_{i} + f_{\theta(i)})\phi_{\lambda}' = F_{\theta(i)}\phi_{\lambda}'$, we have an epimorphism

$$\bar{V}_{\theta}(0) \rightarrow V_{\theta}(0).$$

(5.1)

It is in fact an isomorphism (see Theorem 5.9).

#### 5.1. $\theta$-restricted multisegments.

**Definition 5.1.** If a multisegment $\mathbf{m}$ has the form

$$\mathbf{m} = \sum_{-j \leq i \leq j} m_{ij} \langle i, j \rangle,$$

we call $\mathbf{m}$ a $\theta$-restricted multisegment. We denote by $\mathcal{M}_{\theta}$ the set of $\theta$-restricted multisegments.

**Definition 5.2.** For a $\theta$-restricted segment $\langle i, j \rangle$, we define its modified divided power by

$$\langle i, j \rangle^{[m]} = \begin{cases} \langle i, j \rangle^{(m)} = \frac{1}{[m]!}\langle i, j \rangle^{m} & (i \neq -j), \\ \frac{1}{\prod_{\nu=1}^{m}[2\nu]} \langle -j, j \rangle^{m} & (i = -j). \end{cases}$$

**Definition 5.3.** For $\mathbf{m} \in \mathcal{M}_{\theta}$, we define the elements $P_{\theta}(\mathbf{m}) \in U_{q}^{-}(\mathfrak{g}) \subset B_{\theta}(\mathfrak{g})$ by

$$P_{\theta}(\mathbf{m}) = \prod_{\langle i, j \rangle \in \mathbf{m}} \langle i, j \rangle^{[m_{ij}]}.$$
Here the product $\prod_{j \leq i \leq j}$ is taken over the segments appearing in $m$ from large to small with respect to the PBW-ordering.

5.2. Crystal structure on $M_{\theta}$.

**Definition 5.4.** Suppose $k > 0$. For a $\theta$-restricted multisegment $m = \sum m_{i,j}(i,j)$, we set

$$\varepsilon_{-k}(m) = \max \{ A_{\ell}^{(-k)}(m) | \ell \geq -k \},$$

where

\[ A_{\ell}^{(-k)}(m) = \sum_{\ell \geq \ell}(m_{-k,\ell} - m_{-k+2,\ell+2}) \text{ for } \ell > k, \]
\[ A_{k}^{(-k)}(m) = \sum_{\ell > k}(m_{-k,\ell} - m_{-k+2,\ell}) + 2m_{-k,k} + \delta \quad (m_{-k+2,k} \text{ is odd}), \]
\[ A_{j}^{(-k)}(m) = \sum_{\ell > k}(m_{-k,\ell} - m_{-k+2,\ell}) + 2m_{-k,k} - 2m_{-k+2,k} - 2 \sum m_{i,k} - \sum m_{i,k-2} < k < j \leq k - 2. \]

(i) Let $n_{f}$ be the smallest $\ell \geq -k + 2$, with respect to the ordering \ldots > k + 2 > \cdots > k - 2, such that $\varepsilon_{-k}(m) = A_{\ell}^{(-k)}(m)$. We define

$$\bar{E}_{-k}(m) = \begin{cases} m - \langle -k + 2, n_{f} \rangle + \langle -k, n_{f} \rangle & \text{if } n_{f} > k, \\ m - \langle -k + 2, k \rangle + \langle -k, k \rangle & \text{if } n_{f} = k \text{ and } m_{-k+2,k} \text{ is odd}, \\ m - \delta_{n_{f} \neq 1}(\langle -k + 2, k - 2 \rangle + \langle -k + 2, k \rangle) & \text{if } n_{f} = k \text{ and } m_{-k+2,k} \text{ is even}, \\ m - \delta_{n_{f} \neq 2}(n_{f} + 2, k - 2) + \langle n_{f} + 2, k \rangle & \text{if } -k + 2 \leq n_{f} \leq k - 2. \end{cases}$$

(ii) If $\varepsilon_{-k}(m) = 0$, then $\bar{E}_{-k}(m) = 0$. If $\varepsilon_{-k}(m) > 0$, then let $n_{e}$ be the largest $\ell \geq -k + 2$, with respect to the above ordering, such that $\varepsilon_{-k}(m) = A_{\ell}^{(-k)}(m)$. We define

$$\tilde{E}_{-k}(m) = \begin{cases} m - \langle -k, n_{e} \rangle + \langle -k + 2, n_{e} \rangle & \text{if } n_{e} > k, \\ m - \langle -k, k \rangle + \langle -k + 2, k \rangle & \text{if } n_{e} = k \text{ and } m_{-k+2,k} \text{ is even}, \\ m - \delta_{n_{e} \neq 1}(\langle -k + 2, k - 2 \rangle - \langle -k + 2, k \rangle) & \text{if } n_{e} = k \text{ and } m_{-k+2,k} \text{ is odd}, \\ m - \langle n_{e} + 2, k \rangle + \delta_{n_{e} \neq 2}(n_{e} + 2, k - 2) & \text{if } -k + 2 \leq n_{e} \leq k - 2. \end{cases}$$

**Remark 5.5.** For $0 < k \in I$, the actions of $\bar{E}_{-k}$ and $\tilde{E}_{-k}$ on $m \in M_{\theta}$ are described by the following algorithm.

**Step 1.** Arrange segments in $m$ of the form $\langle -k, j \rangle (j \geq k), \langle -k + 2, j \rangle (j \geq k - 2, 0), \langle i, k \rangle (-k \leq i \leq k), \langle i, k - 2 \rangle (-k + 2 \leq i \leq k - 2)$ in the order

\[ \ldots, \langle -k, k + 2 \rangle, \langle -k + 2, k + 2 \rangle, \langle -k, k \rangle, \langle -k + 2, k \rangle, \langle -k + 2, k - 2 \rangle, \langle -k + 4, k \rangle, \langle -k + 4, k - 2 \rangle, \ldots, \langle k - 2, k \rangle, \langle k - 2, k - 2 \rangle, \langle k \rangle. \]

**Step 2.** Write signatures for each segment appearing in $m$ by the following rules.

(i) If a segment is not $\langle -k + 2, k \rangle$, then

* For $\langle -k, k \rangle$, write $--$,
* For $\langle -k, j \rangle$ with $j > k$, write $-$,
* For $\langle -k + 2, k - 2 \rangle$ with $k > 1$, write $++$,
* For $\langle -k + 2, j \rangle$ with $j > k$, write $+$,
* For $\langle j, k \rangle$ if $-k < j \leq k$, write $-$,
* For $\langle j, -k \rangle$ if $-k + 2 < j \leq k - 2$, write $+$,
• If otherwise, write no signature.

(ii) For segments \( m_{-k+2, k}(-k+2, k) \), if \( m_{-k+2, k} \) is even, then write no signature, and if \( m_{-k+2, k} \) is odd, then write a sequence \(-+\).

Step 3. In the resulting sequence of \(+\) and \(\) -, delete a subsequence of the form \(+--\) and keep on deleting until no such subsequence remains.

Then we obtain a sequence of the form 
\(- - \cdots - + \cdots +\).

(1) \( \varepsilon_{-k}(m) \) is given as the total number of \(-\) in the resulting sequence.

(2) \( \tilde{F}_{-k}(m) \) is given as follows:

(i) if the leftmost \(+\) corresponds to a segment \((-k+2, j) (j > k)\), then replace the segment with \((-k, j)\),
(ii) if the leftmost \(+\) corresponds to a segment \((j, k-2)\), then replace the segment with \((j, k)\),
(iii) if the leftmost \(+\) corresponds to segment \((-k+2, k)^{m_{-k+2, k}}\), then replace one of the segments with \((-k, k)\),
(iv) if no \(+\) exists, add a segment \((k, k)\) to \(m\).

(3) \( \tilde{E}_{-k}(m) \) is given as follows:

(i) if the rightmost \(-\) corresponds to a segment \((-k, j)\), then replace the segment with \((-k+2, j)\),
(ii) if the rightmost \(-\) corresponds to a segment \((j, k) (j \neq -k+2)\), then replace the segment with \((j, k-2)\),
(iii) if the rightmost \(-\) corresponds to segments \( m_{-k+2, k}(-k+2, k) \), then replace one of the segment with \((-k+2, k-2)\),
(iv) if no \(-\) exists, then \( \tilde{E}_{-k}(m) = 0 \).

Definition 5.6. For \( k \in I_{>0} \), we define \( \tilde{F}_{k}, \tilde{E}_{k} \) and \( \varepsilon_{k} \) by the same rule as in Definition 2.15 for \( f_{k} \) and \( \varepsilon_{k} \).

Theorem 5.7. By \( \tilde{F}_{k}, \tilde{E}_{k}, \varepsilon_{k} (k \in I) \), \( M_{\theta} \) is a crystal, in the sense that, for any \( k \in I \), we have

(i) \( \tilde{F}_{k}M_{\theta} \subset M_{\theta} \) and \( \tilde{E}_{k}M_{\theta} \subset M_{\theta} \cup \{0\} \),
(ii) \( \tilde{F}_{k}\tilde{E}_{k}(m) = m \) if \( \tilde{E}_{k}(m) \neq 0 \), and \( \tilde{E}_{k} \circ \tilde{F}_{k} = \text{id} \),
(iii) \( \varepsilon_{k}(m) = \max\{ n \geq 0 \mid \tilde{E}^{n}(m) \neq 0 \} \) < \( \infty \) for any \( m \in M_{\theta} \).

Example 5.8. (1) We shall write \( \{a, b\} \) for \( a(-1, 1) + b(1) \). The following diagram is the part of the crystal graph of \( B_{\theta}(0) \) that concerns only the 1-arrows and the (-1)-arrows.

\[
\phi \begin{array}{c}
\frac{1}{-1} \downarrow \{0, 1\} \\
\{0, 2\} \xrightarrow{1} \{0, 3\} \\
{1, 0} \xrightarrow{1} \{1, 1\} \\
{1, 2} \xrightarrow{1} \{1, 3\} \\
\{2, 0\} \xrightarrow{1} \{2, 1\}
\end{array}
\]

Especially the part of (-1)-arrows is the following diagram.

\[
\begin{array}{c}
\{0, 2n\} \xrightarrow{-1} \{0, 2n + 1\} \\
\{1, 2n\} \xrightarrow{-1} \{1, 2n + 1\} \\
\{2, 2n\} \xrightarrow{-1} \cdots
\end{array}
\]
(2) The following diagram is the part of the crystal graph of $B_{\vartheta}(0)$ that concerns only the $(-1)$-arrows and the $(-3)$-arrows. This diagram is isomorphic as a graph to the crystal graph of $A_{2}$.

(3) Here is the part of the crystal graph of $B_{\vartheta}(0)$ that concerns only the $n$-arrows and the $(-n)$-arrows for an odd integer $n \geq 3$:

\[
\phi \xrightarrow{n} \langle n \rangle \xrightarrow{n} 2\langle n \rangle \xrightarrow{n} 3\langle n \rangle \xrightarrow{n} 4(n) \cdots
\]

5.3. Main Theorem. We write $\phi$ for the generator $\phi_{0}$ of $V_{\vartheta}(0)$, for short.

Theorem 5.9. (i) The morphism

\[
\tilde{V}_{\vartheta}(0) = U_{q}^{-}(g) / \sum_{k \in I} U_{q}^{-}(g)(f_{k} - f_{-k}) \rightarrow V_{\vartheta}(0)
\]

is an isomorphism.

(ii) $\{P_{\vartheta}(m)\phi\}_{m \in M_{\vartheta}}$ is a basis of the $K$-vector space $V_{\vartheta}(0)$.

(iii) Set

\[
L_{\vartheta}(0) := \sum_{\ell \geq 0, i_{1}, \ldots, i_{\ell} \in I} A_{0} \tilde{F}_{i_{1}} \cdots \tilde{F}_{i_{\ell}} \phi \subset V_{\vartheta}(0),
\]

\[
B_{\vartheta}(0) = \left\{ \tilde{F}_{i_{1}} \cdots \tilde{F}_{i_{\ell}} \phi \mod qL_{\vartheta}(0) \mid \ell \geq 0, i_{1}, \ldots, i_{\ell} \in I \right\}
\]

Then, $B_{\vartheta}(0)$ is a basis of $L_{\vartheta}(0)/qL_{\vartheta}(0)$ and $(L_{\vartheta}(0), B_{\vartheta}(0))$ is a crystal basis of $V_{\vartheta}(0)$, and the crystal structure coincide with the one of $M_{\vartheta}$.

(iv) More precisely, we have
(a) \( L_\theta(0) = \sum_{m \in M_\theta} A_0 P_\theta(m) \phi \).
(b) \( B_\theta(0) = \{ P_\theta(m) \phi \mod qL_\theta(0) \mid m \in M_\theta \} \).
(c) For any \( k \in I \) and \( m \in M_\theta \), we have
(1) \( \tilde{E}_k P_\theta(m) \phi \equiv P_\theta(\tilde{E}_k m) \phi \mod qL_\theta(0) \),
(2) \( \tilde{F}_k P_\theta(m) \phi \equiv P_\theta(\tilde{F}_k m) \phi \mod qL_\theta(0) \), where we understand \( P_\theta(0) = 0 \).
(3) \( \tilde{E}_k \phi \) is in \( qL_\theta(0) \) if and only if \( n > \varepsilon_k(m) \).

5.3.1. Global Basis of \( V_\theta(0) \).
Recall that \( A = \mathbb{Q}[q, q^{-1}] \), and \( V_\theta(0)_A = U_q^{-}(\mathfrak{g}_{\infty})_A \).

Lemma 5.10. (i) \( V_\theta(0)_A = \bigoplus_{m \in M_\theta} A P_\theta(m) \phi \).
(ii) For \( m \in M_\theta \),
\[ \overline{P_\theta(m) \phi} \in P_\theta(m) \phi + \sum_{n < m, cry} A P_\theta(n) \phi. \]

By the above lemma, we obtain the following theorem.

Theorem 5.11. (i) \( (L_\theta(0), \overline{L_\theta(0)}) V_\theta(0)_A \) is balanced.
(ii) For any \( m \in M_\theta \), there exists an \( \iota \)-unique \( G_\theta^{low}(m) \in L_\theta(0) \cap V_\theta(0)_A \) such that
\[ \overline{G_\theta^{low}(m)} = G_\theta^{low}(m) \equiv P_\theta(m) \phi \mod qL_\theta(0). \]

6. REPRESENTATION THEORY OF \( \mathcal{H}_n^B \) AND LLTA TYPE CONJECTURES
6.1. The affine Hecke algebra of type B.

Definition 6.1. For \( p_0, p_1 \in \mathbb{C}^* \), the affine Hecke algebra \( \mathcal{H}_n^B \) of type B is a \( \mathbb{C} \)-algebra generated by
\[ T_0, T_1, \ldots, T_{n-1}, X_1^{\pm 1}, \ldots, X_n^{\pm 1} \]
satisfying the following defining relations:
(i) \( X_i X_j = X_j X_i \) for any \( 1 \leq i, j \leq n \).
(ii) [The braid relations of type B]
\[ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \quad (1 \leq i \leq n-2), \]
\[ T_i T_j = T_j T_i \quad (|i-j| > 1). \]
(iii) [The Hecke relations]
\[ (T_0 - p_0)(T_0 + p_0^{-1}) = 0, \quad (T_i - p_1)(T_i + p_1^{-1}) = 0 \quad (1 \leq i \leq n - 1). \]
(iv) [The Bernstein-Lusztig relations]
\[ T_0 X_i^{-1} T_0 = X_i, \]
\[ T_i X_i T_i = X_{i+1}, \quad (1 \leq i \leq n - 1), \]
\[ T_i X_j = X_j T_i \quad (j \neq i, i + 1). \]

Note that the subalgebra generated by \( T_i \) (\( 1 \leq i \leq n - 1 \)) and \( X_j^{\pm 1} \) (\( 1 \leq j \leq n \)) is isomorphic to the affine Hecke algebra \( \mathcal{H}_n^A \).

We assume that \( p_0, p_1 \in \mathbb{C}^* \) satisfy
\[ p_0^2 \neq 1, \quad p_1^2 \neq 1. \]

Let us denote by \( \text{Pol}_n \) the Laurent polynomial ring \( \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] \), and by \( \overline{\text{Pol}}_n \) its quotient field \( \mathbb{C}(X_1, \ldots, X_n) \). Then \( \mathcal{H}_n^B \) is isomorphic to the tensor product of \( \text{Pol}_n \) and
the subalgebra generated by the \( T_i \)’s that is isomorphic to the Hecke algebra of type \( B_n \). We have

\[
T_i a = (s_i a) T_i + (p_i - p_i^{-1}) \frac{a - s_i a}{1 - X^{-\alpha_i^\vee}}
\]

for \( a \in \mathcal{P}_n \).

Here \( p_i = p_1 \) (\( 1 < i < n \)), and \( X^{-\alpha_i^\vee} = X_i^{-2} \) (\( i = 0 \)) and \( X^{-\alpha_i^\vee} = X_i X_{i+1}^{-1} \) (\( 1 \leq i < n \)). The \( s_i \)’s are the Weyl group action on \( \mathcal{P}_n \): \( (s_i a)(X_1, \ldots, X_n) = a(X_1^{-1}, X_2, \ldots, X_n) \) for \( i = 0 \) and \( (s_i a)(X_1, \ldots, X_n) = a(X_1, \ldots, X_{i+1}, X_i, \ldots, X_n) \) for \( 1 \leq i < n \).

Note that \( \mathcal{H}_{n}^{B} = \mathbb{C} \) for \( n = 0 \).

The algebra \( \mathcal{H}_{n}^{B} \) acts faithfully on \( \mathcal{H}_{n}^{B}/\sum_{i=0}^{n-1} \mathcal{H}_{n}^{B}(T_i - p_i) \simeq \mathcal{P}_n \). Set 

\[
\varphi_i = (1 - X^{-\alpha_i^\vee}) T_i - (p_i - p_i^{-1}) \in \mathcal{H}_{n}^{B}
\]

and 

\[
\tilde{\varphi}_i = (p_i^{-1} - p_i X^{-\alpha_i^\vee})^{-1} \varphi_i \in \mathcal{P}_n \otimes \mathcal{H}_{n}^{B}.
\]

Then the action of \( \tilde{\varphi}_i \) on \( \mathcal{P}_n \) coincides with \( s_i \). They are called intertwiners.

### 6.2. Block decomposition of \( \mathcal{H}_{n}^{B} \)-mod

For \( n, m \geq 0 \), set

\[
F_{n,m} := \mathbb{C}[X_1^\pm 1, \ldots, X_{n+m}^\pm 1, D^{-1}],
\]

where 

\[
D := \prod_{1 \leq i < j \leq n + m} (X_i - p_i^2 X_j)(X_i - p_i^{-2} X_j)(X_i - p_i^2 X_j^{-1})(X_i - p_i^{-2} X_j^{-1})(X_i - X_j)(X_i - X_j^{-1}).
\]

Then we can embed \( \mathcal{H}_n^B \) into \( \mathcal{H}_{n+m}^B \otimes \mathcal{P}_n \cdot F_{n,m} \) by

\[
T_0 \mapsto \tilde{\varphi}_n \cdots \tilde{\varphi}_1 T_0 \tilde{\varphi}_1 \cdots \tilde{\varphi}_n, \quad T_i \mapsto X_{i+n} (1 \leq i < m).
\]

Its image commute with \( \mathcal{H}_{n+m}^B \subset \mathcal{H}_{n+m}^B \). Hence \( \mathcal{H}_{n+m}^B \otimes \mathcal{P}_n F_{n,m} \) is a right \( \mathcal{H}_{n+m}^B \otimes \mathcal{H}_{n+m}^B \) module. Note that \( (\mathcal{H}_{n}^{B} \otimes \mathcal{H}_{m}^{B}) \otimes \mathcal{P}_n F_{n,m} = F_{n,m} \otimes \mathcal{P}_n (\mathcal{H}_{n}^{B} \otimes \mathcal{H}_{m}^{B}) \) is an algebra.

**Lemma 6.2.** \( \mathcal{H}_{n+m}^A \otimes (\mathcal{H}_{n}^{B} \otimes \mathcal{H}_{m}^{B}) \mathcal{P}_n F_{n,m} \rightarrow^\sim \mathcal{H}_{n+m}^B \mathcal{P}_n F_{n,m} \).

**Proof.** Let \( W_n^A \) and \( W_n^B \) be the finite Weyl group of type \( A \) and \( B \). Note that \( |W_n^A| : |W_n^B| : |W_n^A|/|W_n^B| : |W_n^A| = |W_n^B| \). Hence the both sides are free modules of rank \( |W_n^B| \) over \( F_{n,m} \). We prove that the map is surjective.

For short, we denote the image of \( \mathcal{H}_{n+m}^A \otimes (\mathcal{H}_{n}^{B} \otimes \mathcal{H}_{m}^{B}) \mathcal{P}_n F_{n,m} \) by \( \mathcal{H}_{n,m}^{loc} \subset \mathcal{H}_{n+m}^B \mathcal{P}_n F_{n,m} \). Note that \( \varphi_i \cdots \varphi_n \in \mathcal{H}_{n,m}^{loc} \mathcal{P}_n F_{n,m} \) for \( 1 \leq i \leq n \).

First, we have \( \varphi_i \cdots \varphi_1 T_0 \varphi_1 \cdots \varphi_n \in \mathcal{H}_{n}^{B} \mathcal{P}_n F_{n,m} \). Since \( (\varphi_i \cdots \varphi_1)^{-1} = \varphi_1 \cdots \varphi_i \in \mathcal{H}_{n+m}^{loc} \mathcal{P}_n F_{n,m} \), we have \( \varphi_i \cdots \varphi_1 T_0 \varphi_1 \cdots \varphi_n \in \mathcal{H}_{n}^{loc} \).

Second, note that

\[
T_i = (\varphi_i(p_i^{-1} - p_i X_i^{-1} X_{i+1}^{-1}) - (p_i - p_i^{-1})X_i^{-1} X_{i+1}^{-1}) \frac{1}{1 - X_i^{-1} X_{i+1}} (1 \leq i < n).
\]

If \( T_0 T_1 \cdots T_{i-1} \varphi_i \cdots \varphi_n \in \mathcal{H}_{n,m}^{loc} \) then \( T_0 T_1 \cdots T_i \varphi_i \cdots \varphi_n \in \mathcal{H}_{n,m}^{loc} \) for \( 1 \leq i < n \). Indeed, we have

\[
T_0 \cdots T_i \varphi_i \cdots \varphi_n = T_0 \cdots T_{i-1} \varphi_i \cdots \varphi_n \frac{(p_i^{-1} - p_i X_i^{-1} X_{i+1})(1 - X_i^{-1} X_{i+1}^{-1})^{-1}}{(p_i - p_i^{-1}) T_0 \cdots T_{i-1} \varphi_i \cdots \varphi_n X_i X_{i+1}(1 - X_i^{-1} X_{i+1}^{-1})^{-1}}
\]

and

\[
T_0 \cdots T_{i-1} \varphi_i \cdots \varphi_n = \varphi_{i+1} \cdots \varphi_n T_0 \cdots T_{i-1} \in \mathcal{H}_{n+m}^A \mathcal{P}_n F_{n,m} \mathcal{H}_{n}^{B}.
\]

Therefore \( T_0 T_1 \cdots T_i \in \mathcal{H}_{n,m}^{loc} \). Hence \( T_0 T_1 \cdots T_i \in \mathcal{H}_{n,m}^{loc} \) (\( 1 \leq i < n + m \)). Indeed, if \( i < n \), then \( T_0 T_1 \cdots T_i \in \mathcal{H}_{n}^{B} \). If \( n \leq i \), then \( T_0 T_1 \cdots T_i \in \mathcal{H}_{n,m}^{loc} \) and \( T_{n+1} \cdots T_i \in \mathcal{H}_{B} \).
Finally, we prove the surjectivity by the induction on $m$. Note that
\begin{align*}
\mathcal{H}_{n+m} &= \sum_{i=1}^{n+m} T_i T_{i+1} \cdots T_{n+m-1} \mathcal{H}_{n+m-1} + \sum_{i=0}^{n+m-1} T_i \cdots T_1 T_0 T_1 \cdots T_{n+m-1} \mathcal{H}_{n+m-1}
\end{align*}
and $T_i T_{i+1} \cdots T_{n+m-1} \in \mathcal{H}_{n+m-1}^A$. Furthermore, $\mathcal{H}_{n+m-1}^B \subseteq \mathcal{H}_{n+m-1}^{\text{loc}}$ by the induction hypothesis. Thus it is sufficient to prove that $T_0 \mathcal{H}_{n+m}^A \subseteq \mathcal{H}_{n,m}^{1\text{o}c}$. Here, $\mathcal{H}_{n+m}^A$ is the subalgebra of $\mathcal{H}_{n+m}$ generated by $T_1, \ldots, T_{n+m-1}$. This follows from
\begin{align*}
\mathcal{H}_{n+m}^A &= \sum_{i=0}^{n+m-1} \langle T_2, \ldots, T_{n+m-1} \rangle T_1 T_2 \cdots T_i \\
\text{and } T_0 T_1 \cdots T_i \in \mathcal{H}_{n,m}^{\text{loc}}. \quad \square
\end{align*}

**Definition 6.3.** For a finite-dimensional $\mathcal{H}^B$-module $M$, let
\begin{align*}
M &= \bigoplus_{a \in \mathbb{C}^n} M_a
\end{align*}
be the generalized eigenspace decomposition with respect to $X_1, \ldots, X_n$:
\begin{align*}
M_a := \{ u \in M \mid (X_i - a_i)u = 0 \text{ for any } 1 \leq i \leq n \text{ and } N \gg 0 \}
\end{align*}
for $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$.

1. We say that $M$ is of type $J$ if all the eigenvalues of $X_1, \ldots, X_n$ belong to $J \subseteq \mathbb{C}^*$. Put
\begin{align*}
K^B_J := \bigoplus_{n\geq 0} K_{J,n}^B.
\end{align*}

Here $K^B_J$ is the Grothendieck group of the abelian category of finite-dimensional $\mathcal{H}^B$-modules of type $J$.

2. The semi-direct product group $\mathbb{Z} \times \mathbb{Z} = \mathbb{Z} \times \{1, -1\}$ acts on $\mathbb{C}^*$ by $(n, \epsilon) : a \mapsto a^{\epsilon} p_1^{2n}$.

3. Let $J_1$ and $J_2$ be $\mathbb{Z} \times \mathbb{Z}$-invariant subsets of $\mathbb{C}^*$ such that $J_1 \cap J_2 = \emptyset$. Then for an $\mathcal{H}^B$-module $N$ of type $J_1$, and an $\mathcal{H}^B$-module $M$ of type $J_2$, the action of $\mathbb{P} \mathcal{O}_{n+m}$ on $N \otimes M$ extends to an action of $\mathcal{F}_{n+m}$. We set
\begin{align*}
N \otimes M := (\mathcal{H}^B_{n+m} \otimes \mathbb{P} \mathcal{O}_{n+m} \mathcal{F}_{n,m}) \otimes (\mathcal{H}^B_{n} \otimes \mathcal{H}^B_{m}) \otimes \mathbb{P} \mathcal{O}_{n+m} \mathcal{F}_{n,m} (N \otimes M).
\end{align*}

By the lemma above, $N \otimes M$ is isomorphic to $\text{Ind}^\mathcal{H}^A_{\mathcal{H}^{n+m}_{m}} (N \otimes M)$ as an $\mathcal{H}^A_{n+m}$-module.

**Proposition 6.4.** Let $J_1$ and $J_2$ be $\mathbb{Z} \times \mathbb{Z}$-invariant subsets of $\mathbb{C}^*$ such that $J_1 \cap J_2 = \emptyset$.

1. Let $N$ be an irreducible $\mathcal{H}^B$-module of type $J_1$ and $M$ an irreducible $\mathcal{H}^B_{m}$-module of type $J_2$. Then $N \otimes M$ is an irreducible $\mathcal{H}^B_{n+m}$-module of type $J_1 \cup J_2$.

2. Conversely if $L$ is an irreducible $\mathcal{H}^B_{n+m}$-module of type $J_1 \cup J_2$, then there exist an integer $m$ ($0 \leq m \leq n$), an irreducible $\mathcal{H}^B_{m}$-module of type $J_1$ and an irreducible $\mathcal{H}^B_{n-m}$-module of type $J_2$ such that $L \simeq N \otimes M$.

3. Assume that a $\mathbb{Z} \times \mathbb{Z}$-orbit $J$ decomposes into $J = J_+ \cup J_-$ where $J_\pm$ are $\mathbb{Z}$-orbits and $J_- = (J_+)^{-1}$. Assume that $\pm 1, \pm p_0 \not\in J$. Then for any irreducible $\mathcal{H}^B_{m}$-module $L$ of type $J$, there exists an irreducible $\mathcal{H}^A_{n}$-module $M$ such that $L \simeq \text{Ind}^\mathcal{H}^B_{\mathcal{H}^A_{n}} M$.

**Proof.** (1) Let $(N \otimes M)_{J_1, J_2}$ be the generalized eigenspace, where the eigenvalues of $X_i$ ($1 \leq i \leq n$) are in $J_1$ and the eigenvalues of $X_j$ ($n < j \leq n+m$) are in $J_2$. Then $(N \otimes M)_{J_1, J_2} = N \otimes M$ by $J_1 \cap J_2 = \emptyset$ by the above lemma and the shuffle lemma (e.g. [G, Lemma 5.5]). Suppose there exists non-zero $\mathcal{H}^B_{n+m}$-submodule $S$ in $N \otimes M$. Then $S_{J_1, J_2} \neq 0$. 


as an \( \mathcal{H}_B \otimes \mathcal{H}_m \)-module. Hence \( S_{J_1J_2} = N \otimes M \) by the irreducibility of \( N \otimes M \) as an \( \mathcal{H}_n \otimes \mathcal{H}_m \)-module. We obtain \( S = N \circ M \).

(2) For an irreducible \( \mathcal{H}_m \)-module \( L \), the \( \mathcal{H}_m \otimes \mathcal{H}_m \)-module \( L_{J_1J_2} \) does not vanish for some \( m \). Take an irreducible \( \mathcal{H}_m \otimes \mathcal{H}_m \)-submodule \( S \) in \( L \). Then there exist an irreducible \( \mathcal{H}_m \)-module \( N \) of type \( J_1 \) and an irreducible \( \mathcal{H}_m \)-module \( M \) of type \( J_2 \) such that \( S = N \circ M \). Hence there exists a surjective homomorphism \( \text{Ind}(N \otimes M) = N \circ M \rightarrow L \). Since \( N \circ M \) is irreducible, this is an isomorphism.

(3) See [M, Section 6].

Hence in order to study \( \mathcal{H}_B \)-modules, it is enough to study irreducible modules of type \( J \) for a \( \mathbb{Z} \times \mathbb{Z}_2 \)-orbit \( J \) in \( \mathbb{C}^* \) such that \( J \) is a \( \mathbb{Z} \)-orbit or \( J \) contains one of \( \pm 1, \pm p_0 \).

### 6.3. The a-restriction and a-induction.

**Definition 6.5.** For \( a \in \mathbb{C}^* \) and a finite-dimensional \( \mathcal{H}_n \)-module \( M \), let us define the functors

\[
E_a : \mathcal{H}_n \rightarrow \mathcal{H}_{n-1}, \quad F_a : \mathcal{H}_n \rightarrow \mathcal{H}_{n+1},
\]

by:

\[
E_a M := \text{Ind}_{\mathcal{H}_n}^{\mathcal{H}_{n+1}} (M \otimes (a)),
\]

where \( (a) \) is the 1-dimensional representation of \( \mathbb{C}[X_{n+1}] \) defined by \( X_{n+1} \mapsto a \).

Define

\[
\tilde{E}_a M := \text{soc} E_a M, \quad \tilde{F}_a M := \text{cosoc} F_a M
\]

for \( a \in \mathbb{C}^* \).

**Theorem 6.6** (Miemietz [M]). Suppose \( M \) is irreducible. Then \( \tilde{F}_a M \) is irreducible and \( \tilde{E}_a M \) is irreducible or 0 for any \( a \in \mathbb{C}^* \setminus \{\pm 1\} \).

### 6.4. LLTA type conjectures for type B.

Now we take the case

\[
J = \{p^k \mid k \in \mathbb{Z}_{\text{o.d.}}\}.
\]

Assume that any of \( \pm 1 \) and \( \pm p_0 \) is not contained in \( J \). For short, we shall write \( E_i, \tilde{E}_i, F_i \)
and \( \tilde{F}_i \) for \( E_{p^i}, \tilde{E}_{p^i}, F_{p^i} \) and \( \tilde{F}_{p^i} \), respectively.

**Conjecture 6.7.** (1) There are complete representatives

\[
\{L_b \mid b \in \mathbb{B}(0)\}
\]

of the finite-dimensional irreducible \( \mathcal{H}_B \)-modules of type \( J \) such that

\[
\tilde{E}_i L_b = L_{\tilde{E}_i b}, \quad \tilde{F}_i L_b = L_{\tilde{F}_i b}
\]

for any \( i \in I := \mathbb{Z}_{\text{o.d.}} \).

(2) For any \( i \in \mathbb{Z}_{\text{o.d.}} \), let us define \( E_{i,b,b'}, F_{i,b,b'} \in \mathbb{C}[q, q^{-1}] \) by the coefficients of the following expansions:

\[
E_i G_{\theta}^{\text{up}}(b) = \sum_{b' \in \mathbb{B}(0)} E_{i,b,b'} G_{\theta}^{\text{up}}(b'), \quad F_i G_{\theta}^{\text{up}}(b) = \sum_{b' \in \mathbb{B}(0)} F_{i,b,b'} G_{\theta}^{\text{up}}(b').
\]

Then

\[
[E_i L_b : L_b'] = E_{i,b,b'}|_{q=1}, \quad [F_i L_b : L_b'] = F_{i,b,b'}|_{q=1}.
\]

Here \( [M : N] \) is the composition multiplicity of \( N \) in \( M \) on \( K_J^B \).
Remark 6.8. There is a one-to-one correspondence between the above index set $B_0(0)$ and Syu Kato’s parametrization ([Kat]) of irreducible representations of $\mathcal{H}_n^B$ of type $J$.

Remark 6.9. (i) For conjectures for other $\mathbb{Z} \rtimes \mathbb{Z}_2$-orbits $J$, see [EK1].
(ii) Similar conjectures for type $D$ are presented by the second author and Vanessa Miemietz ([KM]).

Errata to "Symmetric crystals and affine Hecke algebras of type B, Proc. Japan Acad., 82, no. 8, 2006, 131–136":
(i) In Conjecture 3.8, $\lambda = \Lambda_{p_0} + \Lambda_{p_0}^{-1}$ should be read as $\lambda = \sum_{a \in A} \Lambda_a$, where $A = I \cap \{p_0, p_0^{-1}, -p_0, -p_0^{-1}\}$. We thank S. Ariki who informed us that the original conjecture is false.
(ii) In the two diagrams of $B_0(\lambda)$ at the end of §2, $\lambda$ should be 0.
(iii) Throughout the paper, $A_{\epsilon}^{(2)}$ should be read as $A_{\epsilon}^{(1)}$.

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