Complete Correlation, Detection loophole and Bell’s Theorem

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Abstract

Two new formulations of Bell’s theorem are given here. First, we consider a definite set of two entangled photons with only two polarization directions, for which Bell’s locality assumption is violated for the case of perfect correlation. Then, using a different approach, we prove an efficient Bell-type inequality which is violated by some quantum mechanical predictions, independent of the efficiency factors.
1 Introduction

Since its inception, Bell’s theorem [1] has gone through many conceptual and physical developments. However, even now, there are important aspects of this theorem, which still demand more deliberation. Here, we are going to focus on two such topics with both conceptual and practical significance.

This paper has two different themes. The first theme is about a demonstration of Bell’s theorem for the case of complete correlation, when the two remote analyzers are along the same direction for a given pair of entangled particles. For the case of two entangled photons\(^1\), we are dealing with a situation in which for different sub-ensembles, the polarizations of each entangled photon are measured simultaneously along different directions. For example, to show the violation of Clauser-Holt-Shimony-Horne (CHSH) inequality [2], one should measure the correlation functions \(C(\hat{a}, \hat{b}), C(\hat{a}', \hat{b})\), \(C(\hat{a}, \hat{b}')\) and \(C(\hat{a}', \hat{b}')\) along four pairs of directions \((\hat{a}, \hat{b}), (\hat{a}', \hat{b}), (\hat{a}, \hat{b}')\) and \((\hat{a}', \hat{b}')\), respectively. At the same time, we should assume a certain definition of *non-contextuality* along with *locality* in a hidden-variable theory where the polarization of each photon along a definite direction is supposed to be the same in all different contexts, including different polarization directions of the remote analyzers. However, one may propose a contextual local hidden-variable theory in which the value of each observable comes about as the effect of the system-apparatus local interaction [3]. Furthermore, it is possible to assume local common causes which may not be the same in different contexts. Now, the question arises as to why (in a conceptual sense, not merely algebraically) we cannot use a definite sub-ensemble of particle pairs having only two polarization directions, to prove the impossibility of getting quantum mechanical results by using local realism. And even going further, would it be possible to show any inconsistency for the case of perfect correlation? This situation leads to a stronger version of Bell’s theorem which its consequences should be scrutinized.

On the other hand, it is now apparent that there has not been any experimental confirmation of the Bell experiments without having a loophole [4]. As a result, people have attempted to derive more efficient inequalities (for example see [5]) or have performed new experiments to overcome the detection loophole [6]. This is the main theme of the second part of the present paper. In the photonic experiments, the most difficulty is due to the inefficiency of detectors which makes it impossible to measure the polarization states of all photon pairs. This is the so-called detection-efficiency loophole.

\(^1\)Hereafter, we always consider the photonic version of Bell’s theorem in which a source emits pairs of entangled photons with parallel linear polarizations.
which P. Grangier describes as “Achill’s heel of experimental tests of Bell’s inequalities” [7]. Consequently, some auxiliary assumptions are introduced for demonstrating the experimental violation of Bell’s inequality. Among them, the most famous one is the fair sampling assumption which means that undetected photons do not change the statistical results of the experiment [8]. Nevertheless, this loophole is still with us and Bell experiments are still being done.

Based on these considerations, in the first part of the paper, we prove a new version of Bell’s theorem for a definite sub-ensemble of correlated photons. Then, we show that a stochastic local hidden-variable (SLHV) theory cannot reproduce the quantum mechanical predictions for the case of perfect correlation. The fact that the inconsistency is present in the state of complete correlation has special significance.

In the second part of the paper, we provide more details about a new inequality (similar to the Clauser-Horne (CH) inequality [9]) which has been recently suggested as a short note [10]. We show that in a real photonic experiment, if we consider the inefficiency of the measuring instruments, our proposed inequality is violated by quantum mechanical results, independent of the inefficiencies. This makes a good ground for doing photonic experiments with more conclusive results.

2 Bell’s Theorem for The Case of Complete Correlation

Consider an ideal Bell photonic experiment in which the entangled photons are described by the state $|\psi\rangle = \frac{1}{\sqrt{2}} (|H\rangle_1 |H\rangle_2 + |V\rangle_1 |V\rangle_2)$ where, e.g., $|H\rangle_1 (|V\rangle_1)$ stands for the horizontal (vertical) polarization state of the first photon. Thus, the entangled photons have parallel linear polarizations. In such experiment, if the first photon passes through a filter with orientation $\hat{a}$, its mate crosses its corresponding filter in the same direction.

Now, consider a SLHV theory, where $\lambda$ represent all hidden variables which belong to a space $\Lambda$. At the level of hidden variables, the probability of registering the result $r$ for the polarization of the first photon along $\hat{a}$ is represented by $p^{(1)}_r(\hat{a}, \lambda)$, where $r$ is equal to +1 or −1, depending on whether it is passing through its filter or not. Similarly, the probability of registering the value $q$ for the polarization measurement of the second photon along $\hat{b}$ is represented by $p^{(2)}_q(\hat{b}, \lambda)$. The average value of the outcomes of the polarization of the photon $k$ ($k = 1, 2$) along $\hat{u}_k$ is given by:
\[ \epsilon^{(k)}(\tilde{u}_k, \lambda) = \sum_{j=\pm 1} j p_j^{(k)}(\tilde{u}_k, \lambda) \]  

(1)

where \( \tilde{u}_1 = \hat{a} \) and \( \tilde{u}_2 = \hat{b} \). From this relation, it is clear that

\[ | \epsilon^{(k)}(\tilde{u}_k, \lambda) | \leq 1 \]  

(2)

Using Bell’s locality assumption, the average value of the outcomes of polarizations of two photons along \( \hat{a} \) and \( \hat{b} \) is:

\[ \epsilon^{(12)}(\hat{a}, \hat{b}, \lambda) = \sum_{r,q=\pm 1} rq p_r^{(1)}(\hat{a}, \lambda) p_q^{(2)}(\hat{b}, \lambda) = \epsilon^{(1)}(\hat{a}, \lambda)\epsilon^{(2)}(\hat{b}, \lambda) \]  

(3)

This relation originates from the fact that we have assumed there is statistical independence at the level of hidden variables and that any common characteristic of the two photons in the past is represented by common causes \( \lambda \). Correspondingly, the relation between the correlation of the two polarizations at the experimental level, with the average of the product of two polarizations at the hidden-variable level, is given by:

\[ C(\hat{a}, \hat{b}) = \int_{\Lambda} \epsilon^{(12)}(\hat{a}, \hat{b}, \lambda) \rho(\lambda)d\lambda \]  

(4)

where \( \rho(\lambda) \) is the normalized distribution of the hidden variables \( \lambda \). From the relation (4), one can conclude that

\[ | C(\hat{a}, \hat{b}) | \leq \int_{\Lambda} | \epsilon^{(12)}(\hat{a}, \hat{b}, \lambda) | \rho(\lambda)d\lambda \]  

(5)

But, \( | \epsilon^{(12)}(\hat{a}, \hat{b}, \lambda) | \leq 1 \). Thus,

\[ | C(\hat{a}, \hat{b}) | \leq \int_{\Lambda} | \epsilon^{(12)}(\hat{a}, \hat{b}, \lambda) | \rho(\lambda)d\lambda \leq 1 \]  

(6)

Now, consider an ideal experiment in which \( \hat{a} = \hat{b} \). In this case, we have a perfect correlation and one expects that \( | C(\hat{a}, \hat{a}) | \) to be equal to one. Then, the inequality in (6) turns into the following equality:

\[ \int_{\Lambda} \left[ | \epsilon^{(12)}(\hat{a}, \hat{a}, \lambda) | - 1 \right] \rho(\lambda)d\lambda = 0 \]  

(7)

which means that \( | \epsilon^{(12)}(\hat{a}, \hat{a}, \lambda) | \) should be equal to one. But, assuming Bell’s locality condition in (3), one gets:

\[ | \epsilon^{(1)}(\hat{a}, \lambda) | | \epsilon^{(2)}(\hat{a}, \lambda) | = 1 \quad \forall \hat{a}, \lambda \]  

(8)
Thus, for any $\hat{a}$ and $\lambda$ we must have $| \epsilon^{(1)}(\hat{a}, \lambda) | = 1$ which is a restrictive form of the relation (2). This is in contradiction with the general definition of a SLHV Theory.

Also, the context dependence of the hidden variables is irrelevant here. The assumption of contextuality is always introduced when there are three or more observables. Here, only two observables are involved: An observable is assigned to the polarization of the first photon along $\hat{a}$ and the other observable to the polarization of the second photon along the same direction. In the usual formulation of Bell’s theorem, a third observable could be assigned, e.g., to the polarization of the first photon along $\hat{a}' \neq \hat{a}$ in a different measuring setup. So, one has to consider different contexts of measuring setups, and whether the value attributed to a quantity in different contexts is the same or not. But, in this approach, the polarization of each photon is measured only in a single experimental context and the assumption of contextuality cannot be the reason behind the indicated contradiction.

One could simply show that the same conclusion is achieved by considering any of the four entangled Bell states for both photonic and spin half particles\(^2\). Thus, we can conclude that:

\[ \text{In a SLHV theory, Bell’s locality condition does not coherently hold for a definite measuring context of pairs of entangled particles, described by any of the four Bell states, even for the case of complete correlation.} \]

This is a new version of Bell’s theorem for the state of complete correlation, which we call hereafter as BTCC. It is, however, important to notice that the relation (8) sounds legitimate in a deterministic local hidden-variable (DLHV) theory. To see the reason, we define $A(\hat{a}, \lambda) = \pm 1$ and $B(\hat{b}, \lambda) = \pm 1$ as the first and the second photons’ polarizations along $\hat{a}$ and $\hat{b}$, respectively, in a DLHV theory. Subsequently, one can write Bell’s locality condition as a similar relation to (3). This means that $\epsilon_{DET}^{(12)}(\hat{a}, \hat{b}, \lambda) = A(\hat{a}, \lambda)B(\hat{b}, \lambda)$ where $\epsilon^{(1)}(\hat{a}, \lambda)$ ($\epsilon^{(2)}(\hat{b}, \lambda)$) is replaced by $A(\hat{a}, \lambda)$ ($B(\hat{b}, \lambda)$) in (3) and $| \epsilon_{DET}^{(12)}(\hat{a}, \hat{b}, \lambda) | = 1$ [11]. Then, if one follows the same discussion as above for a DLHV theory, one will reach the conclusion that $| A(\hat{a}, \lambda) |$ should be equal to one. This result is trivial, however, because $A(\hat{a}, \lambda)$ accepts only the values $+1$ and $-1$. Thus, BTCC does not include DLHV theories.

Even for those local hidden-variable theories in which $\lambda$ uniquely determine the polarization state of one of the particles (e.g., if only for the first

\(^2\)For example, for spin half particles, the four Bell states are usually denoted by $| \psi^\pm \rangle = \frac{1}{\sqrt{2}}(| z^+ \rangle_1 | z^- \rangle_2 \pm | z^- \rangle_1 | z^+ \rangle_2)$ and $| \phi^\pm \rangle = \frac{1}{\sqrt{2}}(| z^+ \rangle_1 | z^+ \rangle_2 \pm | z^- \rangle_1 | z^- \rangle_2)$ where, e.g., $| z^+ \rangle_1$ ($| z^- \rangle_1$) is the spin up (spin down) state of particle 1 along $z$-direction.
particle we have $\epsilon^{(1)}(\hat{a}, \lambda) = A(\hat{a}, \lambda) = \pm 1$, BTCC cannot be concluded. Any appearance of the deterministic behavior in $\lambda$ for either of the particles would break down the above argument. So, it seems that a fundamental discrepancy between the stochastic and the deterministic local hidden-variables theories exists which reveals itself when the complete correlation is taken into account: The DLHV theories are more robust to fall into contradiction than the SLHV ones. This is in contrast to the common belief that the stochastic nature of hidden variables makes the scope of Bell’s theorem much broader, because determinism seems to be a particular case of a probabilistic formulation (at least, relationally), when each probability reaches the determinate values of zero and one.

There is also another reason which strengthens this claim: No Bell photonic experiment performed so far can prove the incongruity of the DLHV theories. It is because the auxiliary assumptions imposed to see the violation of Bell’s inequalities have statistical character and their meaning is obscure in a deterministic approach. For example, you may consider the fair sampling assumption when it is used in CHSH inequality. This assumption means that unrecorded data do not have a weighty role in calculating the polarization correlations of the entangled photons. More precisely, this means that there are alternative effective correlation functions such as

$$C_{\text{eff}}(\hat{a}, \hat{b}) = \int_{\lambda} \epsilon^{(12)}_{\text{eff}}(\hat{a}, \hat{b}, \lambda) \rho(\lambda) d\lambda$$

(9)

which are similarly bounded by CHSH inequality whenever the hidden average value $\epsilon^{(12)}_{\text{eff}}(\hat{a}, \hat{b}, \lambda)$ as well as the other similar expressions along different directions satisfy the CHSH inequality [12]. (The value of $\epsilon^{(12)}_{\text{eff}}(\hat{a}, \hat{b}, \lambda)$ for a given $\lambda$ is determined by the kind of the hidden-variable model which is used.) Therefore, according to the fair sampling assumption, we should have:

$$|C_{\text{eff}}(\hat{a}, \hat{b}) + C_{\text{eff}}(\hat{a}', \hat{b}) + C_{\text{eff}}(\hat{a}', \hat{b}') - C_{\text{eff}}(\hat{a}, \hat{b}')| \leq 2$$

(10)

where $C_{\text{eff}}(\hat{a}, \hat{b})$, e.g., is defined as:

$$C_{\text{eff}}(\hat{a}, \hat{b}) = \frac{N_{++}^{(12)}(\hat{a}, \hat{b}) + N_{--}^{(12)}(\hat{a}, \hat{b}) - N_{+-}^{(12)}(\hat{a}, \hat{b}) - N_{-+}^{(12)}(\hat{a}, \hat{b})}{N_{++}^{(12)}(\hat{a}, \hat{b}) + N_{--}^{(12)}(\hat{a}, \hat{b}) + N_{+-}^{(12)}(\hat{a}, \hat{b}) + N_{-+}^{(12)}(\hat{a}, \hat{b})}$$

(11)

in terms of the number of joint detections $N_{rq}^{(12)}(\hat{a}, \hat{b})$ for the results $r, q = \pm 1$. A convenient way for realizing of the performance of the fair sampling assumption is as follows. Using the predictions of a SLHV model, we should first obtain a relation for $\epsilon^{(12)}(\hat{a}, \hat{b}, \lambda)$, which includes the non-detection results too. Then, the value of $\epsilon^{(12)}(\hat{a}, \hat{b}, \lambda)$ should be renormalized on the basis of
the detected results. This is \( \epsilon_{\text{eff}}^{(12)}(\hat{a}, \hat{b}, \lambda) \). Subsequently, we should check to see if the inequality (10) can be obtained on the basis of relation (9) where the effective correlation functions in (10) are defined as (11). Of course, it is a well known fact that many SLHV models have been suggested which could deny the assumption of fair sampling (for example see [13]).

Similarly, one may define \( \epsilon_{\text{eff,DET}}^{(12)}(\hat{a}, \hat{b}, \lambda) = A_{\text{eff}}(\hat{a}, \lambda)B_{\text{eff}}(\hat{b}, \lambda) \) for a DLHV theory. But, what is the meaning of the efficient determinate value \( A_{\text{eff}}(\hat{a}, \lambda) \) (and/or \( B_{\text{eff}}(\hat{b}, \lambda) \))? If an efficient determinate value means that only detected results should be involved, its meaning improperly reduces to the case of an ideal experiment. On the other side, there is no alternative definition consistent with what we expect to mean for \( C_{\text{eff}}(\hat{a}, \hat{b}) \) at the experimental level. Hence, the meaning of the fair sampling assumption is completely vague in DLHV theories.

In summary, for two reasons, the formulation of BTCC provides more evidence for the insufficiency of a SLHV theory. The first reason is that such a formulation excludes any context dependence of the stochastic hidden variables, because the context is the same for all the photon pairs. It presents a stronger inconsistency of the SLHV theories, also because it includes the case of complete (anti)correlation. Secondly, it reveals a fundamental discrepancy between the SLHV and the DLHV theories, which is concealed in the usual formulations of Bell’s theorem.

Nevertheless, the problem of doing an actual Bell experiment is still with us. If one can prove a more efficient inequality (as a type of CH or CHSH inequalities) in which no auxiliary assumption is used at the level of hidden-variables, its violation in real experiments can weaken the possibility of both the SLHV and the DLHV theories. This is the main theme of the next section.

3 An Extended CH Inequality

When one faces the detection loophole in Bell experiments, a natural question may arise as to why the violation of the Bell inequalities are so sensitive to the efficiency factors in real examinations. This subject is so important that nowadays some people believe that there may be some (unknown) physical constraints in nature which could prevent us from doing perfect experiments [14]. Nevertheless, since the beginning of quantum mechanics, there

\[ \text{It should be emphasized, however, that any derivation of the Bell inequalities based on the original formulation of Bell’s theorem ignores the possibility of contextual hidden variables, because it should be always assumed that in different measuring setups, the same hidden variables should be considered.} \]
were many peculiar features of the quantum particles (e.g., the wave-particle duality of massive particles, tunneling effect and so on) which have been confirmed in experiments, even with not perfect instruments. Hence, what is specific in the correlation of two spatially separated particles which makes us so skeptical? If there is nothing special, there should be a way to find a loophole-free Bell-type inequality which could be easily violated in experiments.

To find a solution for this old problem, let us consider a Bell photonic experiment in which the linear polarizations of the photon are measured along \((\hat{a}, \hat{b}), (\hat{a}, \hat{b}'), (\hat{a}', \hat{b}),\) and \((\hat{a}', \hat{b}').\) Now, we define a function \(g_{rq}\) as:

\[
g_{rq}(\hat{a}, \hat{b}, \hat{a}', \hat{b}', \lambda) = p^1_r(\hat{a}, \lambda) \left[p^2_q(\hat{b}, \lambda) - p^2_q(\hat{b}', \lambda)\right] + p^1_r(\hat{a}', \lambda) \left[p^2_q(\hat{b}, \lambda) + p^2_q(\hat{b}', \lambda)\right] - p^1_r(\hat{a}', \lambda) p^2_r(\hat{a}', \lambda) - p^1_q(\hat{b}, \lambda) p^2_q(\hat{b}, \lambda) \tag{12}
\]

where the probability functions \(p^1_r\) and \(p^2_q\) \((r, q = \pm 1)\) are defined as before. Since, \(g_{rq}\) is a linear function of the single-particle probabilities \(p^1_r(\hat{a}, \lambda), p^2_q(\hat{b}, \lambda), p^1_r(\hat{a}', \lambda), p^2_q(\hat{b}', \lambda), p^2_r(\hat{a}', \lambda)\) and \(p^1_q(\hat{b}, \lambda),\) its upper and lower bounds are determined by the limits of these variables. Now, in an ideal experiment, if we consider all sixteen possible combinations of zero and one for the functions \(p^1_r(\hat{a}, \lambda), p^2_q(\hat{b}, \lambda), p^1_r(\hat{a}', \lambda)\) and \(p^2_q(\hat{b}', \lambda),\) when \(p^1_r(\hat{a}', \lambda)\) is zero or one, \(p^2_q(\hat{a}', \lambda)\) should be also zero or one and the same holds for \(p^2_q(\hat{b}, \lambda)\) and \(p^1_q(\hat{b}, \lambda).\) This is due to the fact that the photon pairs have parallel polarizations. Thus, if the first photon passes through an analyzer with the polarization direction \(\hat{a}'\) (i.e., \(p^1_r(\hat{a}', \lambda) = 1\)), the second photon also passes through an analyzer with the same direction of polarization (i.e., \(p^2_q(\hat{a}', \lambda) = 1\)) and vice versa. In an actual experiment, however, the situation is a little different: The upper limit of relation (12) is not necessarily equal to zero in non-ideal experiments. To explain this fact, let us define the sum of the detection probabilities as:

\[
\sum_{j=\pm 1} p^j_k(\hat{x}_k, \lambda) = \alpha^k(\hat{x}_k, \lambda) = 1 - p^0_k(\hat{x}_k, \lambda) \tag{13}
\]

where \(\hat{x}_1 = \hat{a}, \hat{a}'\) or \(\hat{b}, \hat{b}'\) or \(\hat{a}'\) and \(k = 1, 2.\) The function \(p^0_k(\hat{x}_k, \lambda)\) denotes a non-detection probability for the \(k\)th photon with the polarization along \(\hat{x}_k.\) So, \(\alpha^k(\hat{x}_k, \lambda)\) is a measure of inefficiencies at the level of hidden-variables. Table 1 shows the possible values of the lower and upper limits of the single-particle probabilities in terms of the inefficiency measures defined in (13). For more convenience, we have denoted \(\alpha^1(\hat{a}, \lambda) \equiv \alpha_1, \alpha^1(\hat{a}', \lambda) \equiv \alpha_2, \alpha^2(\hat{a}, \lambda) \equiv \alpha_3, \) and \(\alpha^2(\hat{a}', \lambda) \equiv \alpha_4.\)
\( \alpha_1', \alpha^{(2)}(\hat{a}', \lambda) \equiv \alpha_1' \), \( \alpha^{(2)}(\hat{b}, \lambda) \equiv \beta_2 \), \( \alpha^{(2)}(\hat{b}', \lambda) \equiv \beta_2' \) and \( \alpha^{(1)}(\hat{b}, \lambda) \equiv \beta_1 \). We exclude here the possibilities of either \( p_r^{(1)}(\hat{a}', \lambda) \) \( p_q^{(2)}(\hat{b}, \lambda) \) or \( p_r^{(2)}(\hat{a}', \lambda) \) \( p_q^{(1)}(\hat{b}, \lambda) \) being zero but not both. This situation is in conflict with the assumption of parallel polarization, if one takes into account the ideal limits of table 1. The lower and upper limits of \( g_{rq} \) are then classified in table 2.

| Rows | \( p_r^{(1)}(\hat{a}, \lambda) \) | \( p_q^{(2)}(\hat{b}, \lambda) \) | \( p_r^{(1)}(\hat{a}', \lambda) \) | \( p_q^{(2)}(\hat{b}', \lambda) \) | \( p_r^{(2)}(\hat{a}', \lambda) \) | \( p_q^{(1)}(\hat{b}, \lambda) \) |
|------|----------------|----------------|----------------|----------------|----------------|----------------|
| 1    | 0              | 0              | 0              | 0              | 0              | 0              |
| 2    | \( \alpha_1 \) | 0              | 0              | 0              | 0              | \( \beta_1 \) |
| 3    | 0              | \( \beta_2 \) | 0              | 0              | 0              | \( \beta_1 \) |
| 4    | 0              | 0              | \( \alpha_1' \) | 0              | \( \alpha_2' \) | 0              |
| 5    | 0              | 0              | 0              | \( \beta_2' \) | 0              | 0              |
| 6    | 0              | 0              | \( \alpha_1' \) | \( \beta_2' \) | \( \alpha_2' \) | 0              |
| 7    | 0              | \( \beta_2 \) | 0              | \( \beta_2' \) | 0              | \( \beta_1 \) |
| 8    | \( \alpha_1 \) | 0              | 0              | \( \beta_2' \) | 0              | 0              |
| 9    | 0              | \( \beta_2 \) | \( \alpha_1' \) | 0              | \( \alpha_2' \) | \( \beta_1 \) |
| 10   | \( \alpha_1 \) | 0              | \( \alpha_1' \) | 0              | \( \alpha_2' \) | 0              |
| 11   | \( \alpha_1 \) | \( \beta_2 \) | 0              | 0              | 0              | \( \beta_1 \) |
| 12   | 0              | \( \beta_2 \) | \( \alpha_1' \) | \( \beta_2' \) | \( \alpha_2' \) | \( \beta_1 \) |
| 13   | \( \alpha_1 \) | 0              | \( \alpha_1' \) | \( \beta_2' \) | \( \alpha_2' \) | 0              |
| 14   | \( \alpha_1 \) | \( \beta_2 \) | 0              | \( \beta_2' \) | 0              | \( \beta_1 \) |
| 15   | \( \alpha_1 \) | \( \beta_2 \) | \( \alpha_1' \) | 0              | \( \alpha_2' \) | \( \beta_1 \) |
| 16   | \( \alpha_1 \) | \( \beta_2 \) | \( \alpha_1' \) | \( \beta_2' \) | \( \alpha_2' \) | \( \beta_1 \) |

Table 1: The possible values of single-particle probabilities in an actual experiment.
Table 2: The limits of $g_{rq}$.

It is now apparent that the upper limit of $g_{rq}$ can exceed zero in the rows 6, 11, 12, 15 and 16 in table 2. But, it is possible to keep the zero limit at the experimental level as we explain below. Let us first average the rows 6, 11, 12, 15 and 16 in table 2 over the space $\Lambda$. According to the definition of the inefficiency measures in (13), for example, we get for row 11:

$$
\int_{\Lambda} \left[ \alpha^{(1)}(\hat{a}, \lambda) - \alpha^{(1)}(\hat{b}, \lambda) \right] \alpha^{(2)}(\hat{b}, \lambda) \rho(\lambda) d\lambda
$$

$$
= \int_{\Lambda} \sum_{r,q = \pm 1} \left[ p_r^{(1)}(\hat{a}, \lambda) - p_r^{(1)}(\hat{b}, \lambda) \right] p_q^{(2)}(\hat{b}, \lambda) \rho(\lambda) d\lambda
$$

$$
= \sum_{r,q} \left[ P_{rq}^{(12)}(\hat{a}, \hat{b}) - P_{rq}^{(12)}(\hat{b}, \hat{b}) \right]
$$

(14)

where, e.g., $P_{rq}^{(12)}(\hat{a}, \hat{b})$ is the joint probability for getting the results $r$ and $q$ for the first and second photons along $\hat{a}$ and $\hat{b}$, respectively in the experiment, and $\rho(\lambda)$ is a probability density in space $\Lambda$. One can easily show that the right hand side of relation (14) is equal to $(1 - P_{00}^{(2)}(\hat{b})) \left( P_{0}^{(1)}(\hat{b}) - P_{0}^{(1)}(\hat{a}) \right)$, assuming that the joint probability of non-detection is factorizable, i.e., $P_{00}^{(12)} = P_{0}^{(1)} P_{0}^{(2)}$. It is now obvious that if one assumes that the non-detection probabilities $P_{0}^{(1)}(\hat{b})$ and $P_{0}^{(1)}(\hat{a})$ are equal, the relation (14) will become zero.
This means in turn that the non-detection probabilities for the first particle are independent of the polarization direction. The same calculations for rows 6, 12, 15 and 16 in table 2 will show that if the non-detection probabilities for both of the particles are assumed to be independent of the polarization direction, the average value of each of those rows will be equal to zero. For the other rows in table 2, the same argument shows that the average values lie in the $[-1, 0]$ interval. So, one can obtain the following inequality:

$$- 1 \leq G_{rq}(\hat{a}, \hat{b}, \hat{a}', \hat{b}') \leq 0$$

(15)

where $G_{rq}(\hat{a}, \hat{b}, \hat{a}', \hat{b}') = \int_{\Lambda} g_{rq}(\hat{a}, \hat{b}, \hat{a}', \hat{b}', \lambda) \rho(\lambda) d\lambda$ and $g_{rq}(\hat{a}, \hat{b}, \hat{a}', \hat{b}', \lambda)$ was defined in (12). Hence, $G_{rq}(\hat{a}, \hat{b}, \hat{a}', \hat{b}')$ is equal to:

$$G_{rq}(\hat{a}, \hat{b}, \hat{a}', \hat{b}') = P_{rq}^{(12)}(\hat{a}, \hat{b}) - P_{rq}^{(12)}(\hat{a}, \hat{b}') + P_{rq}^{(12)}(\hat{a}', \hat{b}) - P_{rq}^{(12)}(\hat{a}', \hat{b}')$$

(16)

where in deriving (16), we have used Bell’s locality assumption. For example, we have used relations such as:

$$P_{rq}^{(12)}(\hat{a}, \hat{b}) = \int_{\Lambda} p_{r}^{(1)}(\hat{a}, \lambda) p_{q}^{(2)}(\hat{b}, \lambda) \rho(\lambda) d\lambda$$

(17)

Evidently, in reaching the inequality (15) the following assumption is vital:

**A- The experimental probabilities of non-detection for each of the particles are independent of the polarization directions in each wing of a Bell-type photonic experiment.**

The validity of assumption A can be checked in the experiments. This can be done by checking if the practical efficiency parameters change, when one changes the polarization directions in analyzers in many trials of a Bell experiment. This is the main difference between A and the other auxiliary assumptions used before, because the assumption A is testable. It is not introduced at the hidden-variable level. For comparison, e.g., it is constructive to remember the CHSH auxiliary assumption which is the same as A except for the term experimental [2, 15].

The inequality (15) can be tested in non-ideal experiments. Here, we define $P_{rq}^{(12)}(\hat{a}, \hat{b})$ as

$$P_{rq}^{(12)}(\hat{a}, \hat{b}) = \frac{N_{rq}^{(12)}(\hat{a}, \hat{b})}{N_{tot}^{(12)}}$$

(18)
where \( N_{rq}^{(12)}(\hat{a}, \hat{b}) \) is the number of photon pairs whose polarization measurements, for photons 1 and 2 along \( \hat{a} \) and \( \hat{b} \) has yielded \( r \) and \( q \), respectively, and \( N_{tot}^{(12)} \) is the total number of photons emitted from the source. If we define other probability functions in (16) similar to (18), after some algebra we get from (15):

\[
\frac{N_{rq}^{(12)}(\hat{a}, \hat{b}) - N_{rq}^{(12)}(\hat{a}, \hat{b}') + N_{rq}^{(12)}(\hat{a}', \hat{b}) + N_{rq}^{(12)}(\hat{a}', \hat{b}')} {N_{rr}^{(12)}(\hat{a}', \hat{a}') + N_{qq}^{(12)}(\hat{b}, \hat{b})} \leq 1 \tag{19}
\]

where \( N_{tot}^{(12)} \) is eliminated. In this inequality we only deal with recorded events. Of course, the CH inequality too, leads to a relation similar to (19). But, the important thing is that in real experiments, where we are dealing with inefficient detection, quantum mechanical predictions do not violate the CH inequality [9, 15]. However, the inequality (15) can be violated by quantum mechanical predictions. For example, in accordance with the predictions of quantum mechanics, one may consider the joint probability \( P_{rq,QM}^{(12)}(\hat{a}, \hat{b}) \) as:

\[
P_{rq,QM}^{(12)}(\hat{a}, \hat{b}) \approx \frac{1}{4} \eta_{overall} \left[ 1 + rqF \cos 2(\hat{a} - \hat{b}) \right] \tag{20}
\]

where \( \eta_{overall} \) is the overall efficiency of the devices (apart from the efficiencies of the analyzers which are assumed to be approximately perfect) and \( F \) is a measure of the correlation of the two photons. Then, by substituting (20) and relations similar to it in (16) and choosing \( r = q = +1 \), we get

\[
G_{++,QM}(\varphi) \approx \frac{1}{4} \eta_{overall} F [3 \cos \varphi - \cos 3\varphi - 2] \tag{21}
\]

where \( \frac{\varphi}{2} = |\hat{a} - \hat{b}| = |\hat{a}' - \hat{b}'| \) and \( \frac{2\varphi}{3} = |\hat{a} - \hat{b}'| \). Considering the zero limit in the inequality (15), it is then straightforward to show that the inequality can be violated for certain ranges of \( \varphi \) independent of the efficiency factors [10].

Our extended CH inequality is an instance of a set of inequalities in which a similar argument can be used to show the inconsistency. For example, one can define the following function instead of \( g_{rq} \) in (12):

\[
f_{rq}(\hat{a}, \hat{b}, \hat{a}', \lambda) = -p_{r}^{(1)}(\hat{a}, \lambda) p_{q}^{(2)}(\hat{b}, \lambda) \\
+ p_{r}^{(1)}(\hat{a}', \lambda) \left[ p_{q}^{(2)}(\hat{b}, \lambda) + p_{r}^{(2)}(\hat{a}, \lambda) - p_{r}^{(2)}(\hat{a}', \lambda) \right] \tag{22}
\]

Then, one can use the same assumption A to prove the following inequality which can be tested in the experiments:
\[-1 \leq -P_{rq}^{(12)}(\hat{a}, \hat{b}) + P_{rq}^{(12)}(\hat{a}', \hat{b}) + P_{rr}^{(12)}(\hat{a}', \hat{a}) - P_{rr}^{(12)}(\hat{a}', \hat{a}') \leq 0 \quad (23)\]

For some definite angles (e.g., $|\hat{a}' - \hat{b}| = |\hat{a} - \hat{a}'| = \frac{\theta}{2}$, $|\hat{a} - \hat{b}| = \theta = \frac{\pi}{3}$), this is violated by quantum mechanical predictions. However, it is an open problem whether one can reduce the experimental settings (e.g., from three in (23) to two or even one definite setting), while preserving the same limit of zero.

4 Conclusion

The meaning of “entanglement” and its empirical verification (as it appears in Bell’s theorem) demand still more elucidation. Here, we first formulated a different version of Bell’s theorem for two entangled photons in the case of complete correlation (called BTCC). In the original derivations of Bell’s inequality, one has to introduce the local hidden variables either counterfactually or noncontextually. According to BTCC, however, the measuring context is the same for all the photon pairs because it involves only the case of perfect (anti)correlation. Thus, BTCC illustrates a stronger inconsistency of the SLHV theories. Afterwards, an important discrepancy between the SLHV theories on one hand and the DLHV theories on the other hand appears which is beyond the scope of Bell’s theorem. Hence, any demonstration of the inconsistency of the local deterministic hidden variables calls for more strictness.

We also proved an extension of the CH inequality which indeed improves the earlier efforts for demonstrating the violation of the Bell inequalities in real experiments. The violation of our proposed inequality is independent of the efficiency factors. But a crucial assumption is essential here which can be tested in the experiments. This opens the door for a more reasonable realization of the experimental results in the Bell photonic experiments.

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