κ-deformed Poincaré algebras and quantum Clifford-Hopf algebras

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The Minkowski spacetime quantum Clifford algebra structure associated with the conformal group and the Clifford-Hopf alternative κ-deformed quantum Poincaré algebra is investigated in the Atiyah-Bott-Shapiro mod 8 theorem context. The resulting algebra is equivalent to the deformed anti-de Sitter algebra \( U_q(so(3,2)) \), when the associated Clifford-Hopf algebra is taken into account, together with the associated quantum Clifford algebra and a (not braided) deformation of the periodicity Atiyah-Bott-Shapiro theorem.

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I. INTRODUCTION

Conformal symmetry represents the fundamental spacetime symmetry, and it contains the Poincaré and de Sitter geometries as particular cases, besides describing massless particles and field symmetries. In order to investigate modifications of the relativistic kinematics at sufficiently high energy, quantum deformations of the Poincaré algebra were introduced and followed by the doubly special relativity (DSR), which contains two observer-independent parameters — the light velocity and the Planck length. The DSR framework coincides with the algebraic structure of the Poincaré algebra κ-deformation, where the deformation parameter κ is related to the Planck mass. The DSR formalism can indeed be introduced using the quantum κ-Poincaré algebra, which presents a deformation parameter κ of dimension of mass. One of the basic physical predictions of DSR, with the κ-Poincaré algebra as the symmetry algebra of the theory, is the existence of an observer-independent fundamental mass scale. In addition, in \( D = 4 \) Poincaré algebra there is a type of quantum deformation with the mass-like parameter κ. The introduction of the deformation parameter κ leads to the arising of the fundamental mass on fundamental geometrical level, and the corresponding deformations of conformal algebras introduce an original case of quantum deformations of Lie algebras, generalizing the nonstandard deformation of \( sl(2) \). Moreover, deformations of relativistic symmetries in the framework of quantum groups have also been considered, and κ-deformations have brought wide applications in physics. It has introduced a mass-like deformation parameter κ related to the Planck mass as well as to quantum gravity corrections. In particular, in the framework of Lorentz-invariance violation, modifications to the fermionic particle equation of motion were introduced, and aspects on deformed dilation transformations and some prominent applications were investigated.

A recent classification of deformed Poincaré groups has been presented based on the Lorentz group deformations. In connection with deformed Minkowski spaces, the quantum deformed κ-Poincaré algebra can thus be obtained through a nonstandard contraction of the deformed anti-de Sitter algebra \( U_q(so(3,2)) \).

In this paper we formulate the κ-Poincaré algebra as a quantum Clifford-Hopf algebra, using the Wick isomorphism that relates quantum Clifford algebras to their respective standard Clifford algebras. The main aspects of quantum Clifford algebras are reviewed in Section III, where we point out some developments by Hestenes, Oziewicz, Lounesto, Ablamowicz, and Fauser. Quantum Clifford algebras have been widely investigated, relating the \( \mathbb{Z}_n \)-graded Clifford algebra structure to \( q \)-quantization. Some physical systems are regarded, and to explore the formal point of view, see, e.g.,. Also, the \( q \)-symmetry and Hecke algebras can be described within the quantum Clifford algebra context. Fauser asserts that this structure should play a major role in the discussion of the
Yang-Baxter equation, the knot theory, the link invariants and in other related fields which are crucial for the physics of integrable systems in statistical physics\cite{58,59}, where in some cases bivectors satisfy minimal polynomial equations of the Hecke type\cite{53}. In addition, there are other germane applications concerning this formalism, for instance the structure theory of Clifford algebras over arbitrary rings\cite{16}, and the arithmetic theory of Arf invariants and the Brauer-Wall groups. It was shown that due to central extensions the ungraded bivector Lie algebras turn into Kac-Moody and Virasoro algebras and, as it is also shown in\cite{13}, to some $q$-deformed algebras. Automorphisms generated by non-isotropic vectors can give rise to infinite dimensional Coxeter groups\cite{47}, affine Weyl groups, connected to $\mathbb{Z}_n$-graded quantum Clifford algebras.

Motivated by these considerations, the main aim of this paper is to evince the Clifford-Hopf character associated with quantum $\kappa$-deformed Poincaré algebras, as a consequence of a specific deformation of the conformal Clifford algebra $\mathcal{C}_\kappa$ into its associated quantum Clifford algebra. Once the algebra of conformal transformations is derived, comprised solely in terms of the real vectors in Minkowski spacetime, the related periodicity theorem of quantum Clifford algebras is considered, regarding the deformed tensor product that is not braided by construction, in full compliance with the Hopf underlying algebraic structure of quantum Clifford algebras.

This paper is organized as follows: after presenting some algebraic preliminaries in Section II, in Section III we briefly review some introductory aspects on quantum Clifford-Hopf algebras, in particular their $\mathbb{Z}_2$-graded co-commutative structure, and the algebraic sector accomplished by the quantum Clifford algebras. Also the Wick isomorphism is introduced. In Section IV the spacetime algebra $\mathcal{C}_1$ and the algebras $\mathcal{C}_4$ and $\mathcal{C}_2$ are briefly introduced, in order to show the well known isomorphism between the associated Spin group and the fourfold covering of the special conformal transformations group. All the conformal maps are recalled in terms of the spacetime algebra and the Atiyah-Bott-Shapiro periodicity theorem. The algebraic aspects of the conformal transformations are deeply investigated, where the Lie algebra of the associated groups is reviewed, together with the fact that their elements are 2-forms, and the Poincaré algebra is obtained in this context, also reviewing some general aspects. Finally in Section V the $\kappa$-Poincaré algebra is described as a quantum Clifford-Hopf algebra, together with its Lie algebra character, where the conformal group and an alternative $\kappa$-deformed Poincaré algebra are evinced solely in terms of elements of the spacetime algebra $\mathcal{C}_1$. The conformal transformations and $\kappa$-deformed Poincaré algebras, and a quantum $\kappa$-deformed Poincaré symmetry are formulated together with the respective Hopf algebra relations, in the context of quantum Clifford algebras and accomplished by their associated Wick isomorphism.

II. PRELIMINARIES

Let $V$ be a finite $n$-dimensional real vector space and $V^*$ denotes its dual. We consider the tensor algebra $\mathcal{T}^k(V)$ from which we restrict our attention to the space $\Lambda(V) = \bigoplus_{k=0}^\infty \Lambda^k(V)$ of multivectors over $V$. $\Lambda^k(V)$ denotes the space of antisymmetric $k$-tensors, isomorphic to the $k$-forms vector space. Given $\psi \in \Lambda(V)$, $\check{\psi}$ denotes the reversion, an algebra anti-automorphism given by $\check{\psi} = (-1)^{[k]/2} \psi$ ([k] denotes the integer part of $k$). $\hat{\psi}$ denotes the main automorphism or graded involution, given by $\hat{\psi} = (-1)^k \psi$. The conjugation is defined as the reversion followed by the main automorphism. If $V$ is endowed with a non-degenerate, symmetric, bilinear map $g : V^* \times V^* \rightarrow \mathbb{R}$, it is possible to extend $g$ to $\Lambda(V)$. Given $\psi = u^1 \wedge \cdots \wedge u^k$ and $\phi = v^1 \wedge \cdots \wedge v^l$, for $u^i, v^j \in V^*$, one defines $g(\psi, \phi) = \det(g(u^i, v^j))$ if $k = l$ and $g(\psi, \phi) = 0$ if $k \neq l$. The projection of a multivector $\psi = \psi_0 + \psi_1 + \cdots + \psi_n$, $\psi_k \in \Lambda^k(V)$, on its $p$-vector part is given by $\langle \psi \rangle_p = \psi_p$. Given $\psi, \phi, \xi \in \Lambda(V)$, the left contraction is defined implicitly by $g(\psi_\phi, \xi) = g(\check{\psi} \wedge \xi)$. For $a \in \mathbb{R}$, it follows that $\check{\psi} = 0$. Given $\psi \in \Lambda(V)$, the Leibniz rule $\check{\psi}(\psi \wedge \phi) = (\psi \check{\phi}) \wedge \phi + \psi \wedge (\check{\phi} \phi)$ holds. The right contraction is analogously defined $g(\psi_\phi, \xi) = g(\phi, \psi \wedge \xi)$ and its associated Leibniz rule $(\check{\psi} \phi) \wedge \psi = \check{\phi} \wedge (\phi \psi) + (\check{\phi} \psi) \wedge \phi$ holds. Both contractions are related by $\check{\psi} \psi = -\check{\psi} \psi$. The Clifford product between $w \in V$ and $\psi \in \Lambda(V)$ is given by $w \psi = w \wedge \psi + w \check{\psi}$. The Grassmann algebra $\Lambda(V, g)$ endowed with the Clifford product is denoted by $\mathcal{Cl}(V, g)$ or $\mathcal{Cl}_p$, the Clifford algebra associated with $V \simeq \mathbb{R}^{p,q}$, $p + q = n$. In what follows $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ denote respectively the real, complex and quaternionic fields. The vector space $\Lambda_k(V)$ denotes the space of the $k$-vectors.

III. CLIFFORD-HOPF ALGEBRAS AND QUANTUM CLIFFORD ALGEBRAS

The Clifford-Hopf algebra can be introduced when a compatible co-algebra structure and an antipode endow the standard Clifford algebra structure. Denoting the unit map embedding the real or complex field $\mathbb{K}$ into the algebra with $\eta : \mathbb{K} \rightarrow \mathcal{C}(V, g)$, the co-algebra structure is then given by a coproduct $\Delta : \mathcal{C}(V, g) \rightarrow \mathcal{C}(V, g) \otimes \mathcal{C}(V, g)$ and a co-unit $\epsilon : \mathcal{C}(V, g) \rightarrow \mathbb{K}$, which arise naturally when there is a functorial dualization of the algebra structure\cite{51,77,54}. The compatibility of the algebra and the co-algebra structures requires the co-product and the co-unit to be algebra homomorphisms: $\epsilon(\text{Id}) = 1$, $\epsilon(\phi \psi) = \epsilon(\psi) \epsilon(\phi)$ — the juxtaposition denoting the Clifford product — and
also the co-product must satisfy
\[ \Delta(\text{Id}) = \text{Id} \otimes \text{Id}, \quad \Delta(\psi) = \psi \otimes \text{Id} + \text{Id} \otimes \psi, \quad \Delta(\psi \phi) = \Delta(\psi) \Delta(\phi). \] (1)

Define also the antipode \( S : \mathcal{A}^p(V) \to \mathcal{A}^p(V) \) as \( S(\psi_p) = (-1)^p \psi_p \), where \( \psi_p \in \mathcal{A}^p(V) \). The antipode can also be extended by linearity to \( \mathcal{A}(V) \) and can be identified to the graded involution of the underlying Grassmann algebra. Denoting the Clifford application by \( \gamma : V \to \mathcal{C}(V,g) \) — satisfying \( \gamma(u)\gamma(v) + \gamma(v)\gamma(u) = 2g(u,v), \forall u,v \in V \) — the 6-tuple \((\Lambda(V), g, \gamma, \epsilon, \Delta, S)\) is the Clifford-Hopf algebra, which is unique up to the isomorphism given by the universality property \([57]\). From the definitions above all axioms for a Hopf algebra are satisfied, together with the antipode axioms \( \eta \circ (S \otimes \text{Id}) = \eta \circ \epsilon \) and \( \eta \circ (\text{Id} \otimes S) = \eta \circ \epsilon \). Since \( S^2 = \text{Id} \), the Clifford-Hopf algebra is \( \mathbb{Z}_2 \)-graded co-commutative. For details, see, e.g., \([44, 45, 58, 59]\).

Now we turn to the construction of quantum Clifford algebras, comprehensively discussed in, e.g., \([58, 59]\). It is well known that standard Clifford algebras are constructed on a quadratic space \((V, B)\), where \( g \) is a symmetric, bilinear map \( g : V^* \times V^* \to \mathbb{K} \). Let \( B : V^* \times V^* \to \mathbb{K} \) be an arbitrary bilinear not necessarily symmetric form, in such a way that \((V, B)\) is a reflexive space \([59]\). In order to define the associated quantum Clifford algebra, given \( u, v, w \in V \), take the exterior algebra \( \Lambda(V) \), and using the isomorphism \( \tau : V \to V^* \) that induces \( B \) by the expression \( \tau(u)(v) := B(u, v) \), the (left) contraction is given by \( u_B \psi = B(u, v) \), satisfying \( u_B(u \wedge v) = (w_B u) \wedge v + \psi \wedge (w_B u) \), and \( (u \wedge v)_B w = u_B (v_B w) \). Quantum Clifford algebras are denoted as \( \mathcal{C}(V, B) \) and have been investigated in \([58, 59]\).

Let now \( B = g + A \), where \( A = \frac{1}{2}(B - B^T) \). Denote \( u_A \psi = A(u, v) \) and \( u_B \psi = B(u, v) \). The \( B \)-dependent Clifford product \( uv \) can be split as \( u_B v = u_B w + u \wedge v \) or \( uv = u_A \psi + u \wedge v \), where \( u \wedge v = u \wedge v + A(u, v) = u \wedge v + u_B \psi \). Indeed, regarding the product between \( u \in V \) and an element \( \psi \in \mathcal{C}(V, B) \) it follows that
\[
\frac{u_B}{B} \psi = u_B \psi + u \wedge \psi = u_B \psi + u_A \psi + u \wedge \psi = u_B \psi + u \wedge \psi
\]
\[
(2)
\]

The Wick theorem asserts that \( \mathcal{C}(V, B) \simeq \mathcal{C}(V, g) \) as \( \mathbb{Z}_2 \)-graded Clifford algebras \([58, 59]\), although the algebras \( \mathcal{C}(V, B) \) and \( \mathcal{C}(V, g) \) are not isomorphic with respect to the \( \mathbb{Z}_m \)-grading induced from the exterior algebra underlying structure, which is not preserved when the process of deformation \( \mathcal{C}(V, g) \to \mathcal{C}(V, B) \) is taken into account. It is possible to express every antisymmetric bilinear form as \( A(u, v) := F_g(u \wedge v) \), where \( F \in \Lambda^2(V) \) is appropriately chosen. Defining the outer exponential of \( F \in \Lambda^2(V) \) as \( e^F := 1 + F + \frac{1}{2!} F \wedge F + \cdot \cdot \cdot + \frac{1}{n!} \wedge^n F + \cdot \cdot \cdot \wedge^n \) which is a finite series when \( \dim V \) is finite, the Wick isomorphism is given by
\[
\mathcal{C}(V, B) = \phi^{-1}(\mathcal{C}(V, g)) = e^F \wedge \mathcal{C}(V, g) \wedge e^F = \mathcal{C}(V, g), ( )_A
\]
\[
(3)
\]
where \( (\mathcal{C}(V, g), ( )_A) \) indicates that the \( \mathbb{Z}_m \) grading arising from the exterior algebra underlying structure is now associated with the \& exterior product in Eq.\([2]\), and not with the original \& exterior product. The relation between the \& and the \&-grading is given by \( u \wedge \psi = A(u, v) + u \wedge \psi \), showing that the \( \mathbb{Z}_m \)-grading is not preserved from the Wirteis isomorphism when \( V = \mathbb{R}^p,q \), consider the Atiyah-Bott-Shapiro mod 8 index theorem, in particular the case \( \mathcal{C}_{p,q} \simeq \mathcal{C}_{p-1,q-1} \otimes \mathcal{C}_{l,1,1} \). We briefly recall this construction obtained, e.g., in \([58, 59]\). According to the \( V \) can be split orthogonally with respect to \( g \), as \( V = \mathbb{R}^p,q = N_{p-1,q-1} \perp M_{1,1} \). If one applies the Clifford map \( \gamma : \mathbb{R}^p,q \to \mathcal{C}_{p,q} \) and defines its natural restrictions \( \gamma' : N_{p-1,q-1} \to \mathcal{C}_{p-1,q-1}, \gamma'' : M_{1,1} \to \mathcal{C}_{l,1,1} \), the periodicity theorem \( \mathcal{C}_{p,q} \simeq \mathcal{C}_{p-1,q-1} \otimes \mathcal{C}_{l,1,1} \) is obtained \([60, 62]\). Using the obvious notation \( \mathcal{C}(\mathbb{R}^p,q) = \mathcal{C}_{p,q}(g) \) and introducing the restrictions of the Wick isomorphism \( \phi^{-1}|_N \) and \( \phi^{-1}|_M \), (here \( N = N_{p-1,q-1} \) and \( M = M_{1,1} \)), the splitting of \( \mathcal{C}_{p,q}(B) \) is given by the following Theorem \([58, 59]\).
\[
\mathcal{C}_{p,q}(B) = \phi^{-1}(\mathcal{C}_{p,q}(g)) = \phi^{-1}[\mathcal{C}_{p-1,q-1}(g|N) \otimes \mathcal{C}_{l,1,1}(g|M)] = \mathcal{C}_{p-1,q-1}(B|N)(\phi^{-1}(\mathcal{C}_{l,1,1}(B|M))
\]
\[
(4)
\]

The last tensor product \( \otimes \phi^{-1} \) is not braided \([58, 59]\). This expression will be used to obtain the \( \kappa \)-Poincaré algebra as a deformation of a subgroup of the conformal transformations. Note also that the 6-tuple \((\Lambda(V), B, \gamma, \epsilon, \Delta, S)\) is called the quantum Clifford-Hopf algebra.

IV. CONFORMAL MAPS IN THE CLIFFORD ALGEBRA ARENA

In this Section we briefly review some results presented in \([17, 21]\). Let \( \{\gamma_0, \gamma_1, \gamma_2, \gamma_3\} \) denote an orthonormal frame field in a Lorentzian 4-dimensional spacetime, satisfying \( \gamma_\mu \gamma_\nu = \frac{1}{2}(g_{\mu\nu} + \gamma_\nu \gamma_\mu) = g_{\mu\nu} \). The pseudoscalar \( \gamma_5 := \gamma_{0123} \) satisfies \( (\gamma_5)^2 = -1 \) and \( \gamma_\mu \gamma_5 = -\gamma_5 \gamma_\mu \).
The algebras $C\ell_{2,4}$ and $C\ell_{4,1}$ are crucial to consider conformal transformations, i.e., it can be shown that inversions, involutions, dilatations, translations, rotations, transvections, and divergions can be expressed in terms of the Clifford algebra over the space $\mathbb{R}^{4,1}$.

The groups $\operatorname{Spin}(2,4) = \{ R \in C\ell_{2,4} \mid RR^\ast = 1 \}$ and $\operatorname{Spin}(4,2) = \{ D \in C\ell_{4,1} \mid DD^\ast = 1 \}$ can also be defined. The inclusion $\operatorname{Spin}(2,4) \hookrightarrow C\ell_{2,4} \cong C \otimes \mathbb{C} \otimes \mathbb{C} \otimes C\ell_{1,3}$ follows from the definition. In particular, the Spin groups are useful in the definition of conformal transformations. Given the quadratic space $\mathbb{R}^{p,q}$, consider the injective map $\kappa : \mathbb{R}^{p,q} \rightarrow \mathbb{R}^{p+1,q+1}$ given by $x \mapsto (x, x \cdot x, 1).$ The image of $\mathbb{R}^{p,q}$ is a subset of the quadric $Q \rightarrow \mathbb{R}^{p+1,q+1},$ described by the equation $x \cdot x - \lambda \mu = 0,$ the so-called Klein absolute. The map $\kappa$ induces an injective map from $Q$ in the projective space $\mathbb{R}^{p+1,q+1}.$ Besides, $Q$ is compact and defined as the conformal compactification $\mathbb{P}^{p,q}$ of $\mathbb{R}^{p,q},$ homeomorphic to $(S^p \times S^q)/\mathbb{Z}_2.$ In the very particular case where $p = 0$ and $q = n,$ the quadric is homeomorphic to the $n$-sphere $S^n,$ the compactification of $\mathbb{R}^n$ via the addition of a point at infinity. There also exists an injective map $s : \mathbb{R} \oplus \mathbb{R}^3 \rightarrow \mathbb{R} \oplus \mathbb{R}^{4,1}$ defined by $v \mapsto s(v) = \begin{pmatrix} v \bar{v} \end{pmatrix}.$ The following theorem is introduced by Porteous:

**Theorem** (i) the map $\kappa : \mathbb{R}^{p,q} \rightarrow \mathbb{R}^{p+1,q+1};$ $x \mapsto (x, x \cdot x, 1),$ is an isometry. (ii) the map $\pi : Q \rightarrow \mathbb{R}^{p,q};$ $(x, \lambda, \mu) \mapsto x/\mu$ defined where $\lambda \neq 0$ is conformal. (iii) if $U : \mathbb{R}^{p+1,q+1} \rightarrow \mathbb{R}^{p+1,q+1}$ is an orthogonal map, the map $\Omega = \pi \circ U \circ \kappa : \mathbb{R}^{p,q} \rightarrow \mathbb{R}^{p,q}$ is conformal.

The application $\Omega$ maps conformal spheres onto conformal spheres. A quasi-sphere is a submanifold of $\mathbb{R}^{p,q},$ defined by the equation $a \cdot x + b \cdot x + c = 0,$ $a, c \in \mathbb{R},$ $b \in \mathbb{R}^{p,q}.$ From the assertion (iii) of the theorem above, we see that $\pm U$ induce the same conformal transformation in $\mathbb{R}^{p,q}.$ The conformal group is defined as $\operatorname{Conf}(p, q) \approx O(p+1, q+1)/\mathbb{Z}_2.$ The component of $\operatorname{Conf}(1,3)$ connected to the identity $\operatorname{Conf}(1,3)$ denotes the Möbius group of $\mathbb{R}^{1,3}.$ Besides, $\operatorname{SConf}(1,3)$ denotes the component connected to the identity, time-preserving and future-pointing.

Now, consider the basis $\{ e_A \}_{A=0}^5$ of $\mathbb{R}^{2,4}$ that satisfies the relations $e_0^2 = e_1^2 = e_2^2 = e_3^2 = e_4^2 = -1,$ $e_A \cdot e_B = 0,$ $(A \neq B).$ Consider also the quadratic space $\mathbb{R}^{4,1}$, with basis $\{ E_A \}_{A=0}^5$, where

$$E_0^2 = -1, \quad E_1^2 = E_2^2 = E_3^2 = 1, \quad E_A \cdot E_B = 0 \quad (A \neq B).$$

(5)

The $\{ E_A \}$ can be obtained from $\{ e_A \}$ by the isomorphism

$$\xi : C\ell_{4,1} \rightarrow \Lambda_2(\mathbb{R}^{2,4})$$

$$E_A \mapsto \xi(E_A) = \varepsilon_A e_5.$$  

(6)

Given a vector $\alpha = \alpha^A e_A \in \mathbb{R}^{2,4},$ we obtain a paravecator $b = \alpha^A E_A = \alpha^A \varepsilon_A e_5 + \alpha^5 \varepsilon_5 \in \mathbb{R} \oplus \mathbb{R}^{4,1}.$ From the periodicity theorem for this case $C\ell_{4,1} \cong C\ell_{1,1} \otimes C\ell_{3,0}$ and it is possible to express an element of $C\ell_{4,1}$ as a $2 \times 2$ matrix with entries in $C\ell_{3,0}$.

A homomorphism $\vartheta : C\ell_{4,1} \rightarrow C\ell_{3,0}$ is defined as $E_i \mapsto \vartheta(E_i) = E_i E_0 E_4 \equiv e_i.$ It can be seen that $e_i^2 = 1,$ $E_i = e_i E_4 E_0$ and $E_4 = E_+ + E_-.$ $E_0 = E_+ - E_-,$ where $E_{\pm} = \frac{1}{2}(E_4 \pm E_0).$ Then,

$$b = \alpha^5 + (\alpha^4 + \alpha^4) E_+ + (\alpha^4 - \alpha^0) E_- + \alpha^0 e_i E_4 E_0.$$  

(7)

If we choose $E_4$ and $E_0$ to be represented by $E_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$ $E_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ consequently we have $E_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$ $E_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$ $E_i E_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and then the paravecator $b \in \mathbb{R} \oplus \mathbb{R}^{4,1} \mapsto C\ell_{4,1}$ in Eq. (7) is represented by $b = \begin{pmatrix} \alpha^5 + \alpha^0 e_i \\ \alpha^4 - \alpha^0 \\ \alpha^4 - \alpha^0 \\ \alpha^0 - \alpha^0 e_i \end{pmatrix}.$

The vector $\alpha \in \mathbb{R}^{2,4}$ is in the Klein absolute, i.e., $\alpha^2 = 0 \iff b\bar{b} = 0,$ since $\alpha^2 = \alpha^A \alpha^A = \alpha^2 e_5 \alpha = \varepsilon_5 \varepsilon_5 \alpha = b\bar{b}.$ We denote $\lambda = \alpha^4 - \alpha^0,$ $\mu = \alpha^4 + \alpha^0.$ Using the matrix representation of $b\bar{b},$ the entry $(b\bar{b})_{11}$ of the matrix is given by

$$\begin{pmatrix} b\bar{b} \end{pmatrix}_{11} = x\bar{x} - \lambda \mu = 0, \quad \text{where} \quad x := (\alpha^5 + \alpha^0 e_i) \in \mathbb{R} \oplus \mathbb{R}^3 \mapsto C\ell_{4,0}.$$  

(8)

If we fix $\mu = 1,$ consequently $\lambda = x\bar{x} \in \mathbb{R}.$ This choice does correspond to a projective description, and $b \in \mathbb{R} \oplus \mathbb{R}^{4,1}$ can be represented as

$$b = \begin{pmatrix} x & \lambda \\ \mu & \bar{x} \end{pmatrix} = \begin{pmatrix} x & x\bar{x} \\ 1 & \bar{x} \end{pmatrix}.$$  

Footnote 1: which can be quasi-spheres or hyperplanes.
From Eq. (8) it follows that \((a^5 + a^i e_i)(a^5 - a^i e_i) = (a^4 - a^0)(a^4 + a^0)\), or similarly \((a^5)^2 + (a^0)^2 - (a^1)^2 - (a^2)^2 - (a^3)^2 - (a^4)^2 = 0\), which is the Klein absolute.

Now, the Möbius transformations in Minkowski spacetime can be derived. The matrix \(g = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\) is in the group \(\text{Spin}^+(2, 4)\) if, and only if, its entries \(a, b, c, d \in \mathbb{C}l_{3,0}\) satisfy the conditions \[17, 62\]

\[
\begin{align*}
(i) & \quad a\bar{a}, b\bar{b}, c\bar{c}, d\bar{d} \in \mathbb{R}, \\
(ii) & \quad a\bar{b}, a\bar{d} \in \mathbb{R} \oplus \mathbb{R}^3, \\
(iii) & \quad av\bar{c} + c\bar{v}a, \\& v \in \mathbb{R} \oplus \mathbb{R}^3, \\
(iv) & \quad avd + c\bar{v}b \in \mathbb{R} \oplus \mathbb{R}^3, \\
& \quad \forall v \in \mathbb{R} \oplus \mathbb{R}^3, \\
(v) & \quad a\bar{c} = c\bar{a}, \\
& \quad b\bar{d} = d\bar{b}, \\
(vi) & \quad ad - bc = 1. \\
\end{align*}
\]

Conditions (i), (ii), (iii), (iv) are equivalent to the condition \(\tilde{\sigma}(g)(b) := gb\bar{g} \in \mathbb{R} \oplus \mathbb{R}^{4,1}\), \(\forall b \in \mathbb{R} \oplus \mathbb{R}^{4,1}\), where \(\tilde{\sigma} : \text{Spin}^+(2, 4) \rightarrow \text{SO}^+(2, 4)\) is the twisted adjoint representation. For details, see, e.g., \[17, 18\]. Conditions (v), (vi) express \(gg = 1\), for all \(g \in \text{Spin}^+(2, 4)\).

We have just seen that a paravector \(b = (a^i b^i) \in \mathbb{R} \oplus \mathbb{R}^{4,1}\) is represented as \(b = \begin{pmatrix} x \\ \mu \end{pmatrix} = \begin{pmatrix} x^i \\ \mu^i \end{pmatrix}\) where \(x^i = (a^i x^i) = (a^i x^i)\), elements of \(\mathbb{C}l_{4,1} \simeq \mathbb{C} \otimes \mathbb{C}l_{1,3}\). The rotation of \(b \in \mathbb{R} \oplus \mathbb{R}^{4,1}\) is performed by the use of the twisted adjoint representation \(\tilde{\sigma} : \text{Spin}^+(2, 4) \rightarrow \text{SO}^+(2, 4)\), defined as \(\tilde{\sigma}(g)(b) = gb\bar{g}^{-1} = gb\bar{g}\), i.e., the adjoint representation of \(\text{Spin}^+(2, 4)\):

\[
\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} x & \lambda \\ \mu & \bar{x} \end{pmatrix} \begin{pmatrix} a & \bar{c} \\ b & \bar{d} \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} \Delta & \bar{x}' \lambda' \\ \mu & \bar{x}' \end{pmatrix} \begin{pmatrix} a & \bar{c} \\ b & \bar{d} \end{pmatrix},
\]

which is conformal \[16, 65\]. From the isomorphisms \(\mathbb{C}l_{4,1} \simeq \mathbb{C} \otimes \mathbb{C}l_{1,3}\), elements of \(\text{Spin}^+(2, 4)\) are in the Dirac algebra \(\mathbb{C} \otimes \mathbb{C}l_{1,3}\). The conformal maps are expressed by the adjoint representation of \(\text{Spin}^+(2, 4)\), by the matrices presented at Table I \[14, 16, 62, 63\]. This index-free geometric formulation allows to trivially generalize the conformal maps of \(\mathbb{R}^{1,3}\) to the ones of \(\mathbb{R}^{p,q}\), if the periodicity theorem of Clifford algebras is used.

Elements of \(\text{Spin}^+(2, 4)\) induce the orthochronous Möbius transformations. The isomorphisms \(\text{Conf}(1,3) \simeq O(2,4)/\mathbb{Z}_2 \simeq \text{Pin}(2,4)/\{1, \pm 1, \pm i\}\) are constructed in \[14\] and consequently their respective special subgroups are related by \(\text{SConf}(1,3) \simeq \text{SO}^+(2,4)/\mathbb{Z}_2 \simeq \text{Spin}^+(2,4)/\{1, \pm 1, \pm i\}\). The sequence \(\text{Spin}^+(2,4) \simeq \text{SU}(2,2) \simeq \text{SO}^+(2,4)\) is explicitly constructed in \[15\].

Now, using the well known result that the Lie algebra \(\mathfrak{spin}(p,q)\) is the algebra 2-covectors of \(\mathbb{R}^{p,q}\), i.e., \((\Lambda_q(\mathbb{R}^{p,q}), [\cdot, \cdot]) = \mathfrak{spin}(p,q)\), the Lie algebra of the conformal group and the alternative \(\kappa\)-deformed Poincaré algebra can be used. In what follows the Lie algebra associated with the conformal group is briefly reviewed. The basis \(\{E_A\}\) defined by Eq. (9) obviously satisfies Eqs. (6). An isomorphism between \(\mathbb{C}l_{4,1}\) and \(\mathbb{C} \otimes \mathbb{C}l_{1,3}\) is defined, denoting \(i = \gamma_{0123} = E_{01234}\), by

\[
E_0 \rightarrow -i\gamma_0, \quad E_1 \rightarrow -i\gamma_1, \quad E_2 \rightarrow -i\gamma_2, \quad E_3 \rightarrow -i\gamma_3, \quad E_4 \rightarrow -i\gamma_{0123}.
\]
The Lie algebra of $\text{spin}_+(2,4)$ is generated by $\Lambda^2(\mathbb{R}^{1,3})$ and its relation to the group $\text{Conf}(1,3)$ is investigated now. The generators of $\text{Conf}(1,3)$ are defined by:

$$P_\mu = \frac{i}{2}(\varepsilon_\mu \varepsilon_5 + \varepsilon_\mu \varepsilon_4), \quad K_\mu = -i(\varepsilon_\mu \varepsilon_5 - \varepsilon_\mu \varepsilon_4), \quad D = -\frac{1}{2} \varepsilon_2 \varepsilon_5, \quad M_{\mu \nu} = \frac{i}{2} \varepsilon_\nu \varepsilon_\mu.$$ 

Using (11), the generators of $\text{Conf}(1,3)$ are expressed in terms of the $\{\gamma_\mu\} \in \mathcal{Cl}_{1,3}$ as [17]:

$$P_\mu = \frac{1}{2}(\gamma_\mu + i\gamma_5), \quad K_\mu = -\frac{1}{2}(\gamma_\mu - i\gamma_5), \quad D = \frac{1}{2} i\gamma_5, \quad M_{\mu \nu} = \frac{1}{2}(\gamma_\mu \wedge \gamma_\nu).$$

They satisfy the following relations:

$$[P_\mu, P_\nu] = 0, \quad [K_\mu, K_\nu] = 0, \quad [M_{\mu \nu}, D] = 0,$$

$$[M_{\mu \nu}, P_\lambda] = -(g_{\mu \lambda} P_\nu - g_{\nu \lambda} P_\mu), \quad [M_{\mu \nu}, K_\lambda] = -(g_{\mu \lambda} K_\nu - g_{\nu \lambda} K_\mu),$$

$$[M_{\mu \nu}, M_{\sigma \rho}] = g_{\nu \sigma} M_{\mu \rho} - g_{\nu \rho} M_{\mu \sigma} - g_{\mu \sigma} M_{\nu \rho} - g_{\mu \rho} M_{\nu \sigma},$$

$$[P_\mu, K_\nu] = 2(g_{\mu \nu} D - M_{\mu \nu}), \quad [P_\mu, D] = P_\mu, \quad [K_\mu, D] = -K_\mu.$$ (13)

The commutation relations above are invariant under substitution $P_\mu \rightarrow -K_\mu$, $K_\mu \rightarrow -P_\mu$ and $D \rightarrow -D$. All the above relations are derived essentially from the periodicity theorem that implements the isomorphism $\mathcal{Cl}_{2,4} \simeq \mathcal{Cl}_{1,1} \otimes \mathcal{Cl}_{1,3}$ and also the isomorphism $\mathcal{Cl}^2_{2,4} \simeq \mathcal{Cl}_{4,1}$. In the next Section the deformation of the periodicity theorem will be used to derive the $\kappa$-Poincaré algebra.

V. THE $\kappa$-POINCARÉ ALGEBRA AS A QUANTUM CLIFFORD-HOPF ALGEBRA

Usually the standard real $\kappa$-Poincaré algebra can be obtained via the commutators of $\mathcal{U}_\phi(\mathfrak{so}(3,2))$ and subsequently performing the quantum de-Sitter contraction. Here the aim is to construct an algebra that is equivalent to the deformed anti-de Sitter algebra $\mathcal{U}_\phi(\mathfrak{so}(3,2))$.

Starting from Eqs. (12), they express Eqs. (13) in terms of elements in $\mathcal{Cl}_{1,3}$. Take the group $\text{spin}_+(2,4) = \text{spin}_+(2,4)(g)$, and use the Wick isomorphism $\phi(\text{spin}_+(2,4)) = \text{spin}_+(2,4)(B)$. There is a $F \in \Lambda^2(\mathbb{R}^{1,3})$ such that the relations (13) can be deformed, and using Eqs. (12) and the expression $e^{-\epsilon} \psi e^\epsilon$ that defines the Wick isomorphism on each $\psi \in \mathcal{Cl}_{1,3}$, the $\kappa$-Poincaré algebra can be obtained by a suitable particular construction of the non braided tensor product, in the periodicity theorem of quantum Clifford algebras. Regarding the deformed tensor product that is not braided, we also obtain a Hopf-algebraic structure arising from quantum Clifford algebras. The formalism accomplished by Eqs. (13) can be turned in the expressions defining the $\kappa$-Poincaré algebra, if the not braided tensor product in Eq. (4) is appropriately chosen.

Denoting $K_\pm = K_1 \pm iK_2 := M_{10} \pm iM_{30}$, $K_3 = M_{30}$, $M_\pm = M_{23} \pm iM_{31}$, and $P_\pm = P_2 + iP_1$, and using Eqs. (12) the $\kappa$-Poincaré algebraic sector is presented by the following commutation relations:

$$[P_\mu, P_\nu] = 0 = [M_{ij}, P_\nu], \quad [\varepsilon_{ijk} M_{ij}, P_\nu] = i \varepsilon_{i\rho} P_\rho,$$

$$[K_3, P_\nu] = \frac{i}{2} \gamma_3(1 + i\gamma_5), \quad [K_\pm, P_\nu] = \frac{1}{2}(\mp \gamma_2 + i\gamma_1)(1 + i\gamma_5),$$

$$[K_\pm, P_\nu] = \mp i \sinh\left(\frac{\gamma_0 + i\gamma_0 \gamma_5}{\kappa}\right) \pm \frac{1}{2\kappa} \gamma_3(1 + i\gamma_5),$$

$$[K_\pm, P_\nu] = i \sinh\left(\frac{\gamma_0 + i\gamma_0 \gamma_5}{\kappa}\right) - \frac{i}{2\kappa} \gamma_3(1 + i\gamma_5), \quad \left[K_\pm, P_3\right] = \mp \frac{1}{2\kappa} \gamma_3(\gamma_2 \mp i\gamma_1)(1 \pm i\gamma_5),$$

$$[M_+, M_-] = \frac{1}{2} \gamma_1 \wedge \gamma_2, \quad \left[M_{12}, M_\pm\right] = \mp \frac{1}{2} \gamma_3(\gamma_1 \pm i\gamma_2),$$

$$\left[K_+, K_-\right] = -\gamma_1 \wedge \gamma_2 \cosh\left(\frac{\gamma_0 + i\gamma_0 \gamma_5}{\kappa}\right) - \sinh\left(\frac{\gamma_0 + i\gamma_0 \gamma_5}{\kappa}\right),$$

$$\left[K_\pm, K_3\right] = \pm 1 \pm \frac{\gamma_0}{4\kappa}(1 + i\gamma_5)\gamma_3(\gamma_2 - i\gamma_1) + \frac{1}{8\kappa}((i + 1)\gamma_3 \wedge \gamma_0(\gamma_2 + i\gamma_1)(1 + i\gamma_5)), $$

$$\left[M_\pm, K_\pm\right] = \mp \frac{1}{8\kappa} \gamma_3(1 + i\gamma_5)(1 \mp 1)(1 + i\gamma_5), \quad [M_{12}, K_3] = 0, \quad [M_{12}, K_\pm] = \mp \frac{1}{2}(\gamma_1 \pm i\gamma_2) \wedge \gamma_0.
\[ [M_\pm, K_\mp] = \left( 1 + \gamma_3 + \frac{i}{8\kappa} (1 + 1)(1 - \gamma_1\gamma_2) (1 + i\gamma_5) + \frac{1}{4}(1 + \gamma_1 \wedge 2 \gamma_2) \right) (1 + i\gamma_5) + \frac{1}{4}(1 + \gamma_1 \wedge 2 \gamma_2) \gamma_3 (1 + i\gamma_5) \]

\[ [M_\pm, K_\mp] = \left( 1 + \gamma_3 + \frac{i}{8\kappa} (1 + 1)(1 - \gamma_1\gamma_2) (1 + i\gamma_5) + \frac{1}{4}(1 + \gamma_1 \wedge 2 \gamma_2) \right) (1 + i\gamma_5) + \frac{1}{4}(1 + \gamma_1 \wedge 2 \gamma_2) \gamma_3 (1 + i\gamma_5) . \]

The coalgebra sector associated with the above Lie algebra is given by the antipodes

\[ S(M_{ij}) = -M_{ij}, \quad S(P_\mu) = -P_\mu, \quad S(K_3) = \frac{1}{2} \gamma_3 (1 - i\gamma_5) + \frac{i}{2\kappa} \gamma_3 (1 + i\gamma_5) + \frac{\gamma_3}{2\kappa} \]

\[ S(K_\pm) = \frac{1}{2} (\gamma_1 + i\gamma_2) \wedge \gamma_0 \pm \frac{1}{2\kappa} (\gamma_0 \mp i\gamma_1) \]

and by the coproducts

\[ \Delta(M_{ij}) = \frac{1}{2} (\gamma_1 \wedge \gamma_2) \otimes 1 + 1 \otimes (\gamma_i \wedge \gamma_j) \]

\[ \Delta(K_3) = -\frac{\gamma_3}{2} (1 - i\gamma_5) \otimes \left( 1 + \frac{\gamma_0}{2\kappa} (1 + i\gamma_5) \right) + \left( 1 - \frac{\gamma_0}{2\kappa} (1 + i\gamma_5) \right) \otimes \frac{\gamma_3}{2} (1 - i\gamma_5) + \frac{1}{4\kappa} \left( 1 - \frac{\gamma_0}{2\kappa} (1 + i\gamma_5) \right) (\gamma_2 \wedge \gamma_3 \otimes 2 \gamma_2 (1 + i\gamma_5)) \]

\[ \Delta(K_\pm) = \frac{1}{2} (\gamma_1 \pm i\gamma_2) \wedge \gamma_0 \otimes \left( 1 + \frac{\gamma_0}{2\kappa} (1 + i\gamma_5) \right) + \left( 1 - \frac{\gamma_0}{2\kappa} (1 + i\gamma_5) \right) \otimes \frac{\gamma_3}{2} (1 - i\gamma_5) + \frac{1}{2\kappa} \left( 1 - \frac{\gamma_0}{2\kappa} (1 + i\gamma_5) \right) \gamma_0 \otimes \left( \gamma_2 \mp 1 \gamma_1 \right) \gamma_3 \wedge (\gamma_1 \wedge \gamma_2) (1 + i\gamma_5) - \left( 1 - \frac{\gamma_0}{2\kappa} (1 + i\gamma_5) \right) \frac{1}{2} (\gamma_2 \mp 1 \gamma_1) \otimes \gamma_2 \mp i\gamma_1 ) \]

\[ \Delta(P_\mu) = \frac{1}{2} (\gamma_1 (1 + i\gamma_5) \otimes (1 + \gamma_0 (1 + i\gamma_5)) + (1 + \gamma_0 (1 + i\gamma_5)) \otimes (1 + i\gamma_5)) \]

Using Eqs. (12) and defining

\[ \hat{K}_3 = \frac{1}{2} \gamma_3 - \frac{i}{16\kappa} \gamma_3 (1 - 4i + \gamma_12) (1 + i\gamma_5), \quad \hat{K}_\pm = \frac{1}{2} \left( i\gamma_0 + \frac{1}{2\kappa} (\pm 1 - \frac{i}{4}) \right) \frac{1}{2} (1 + i\gamma_5) (\gamma_1 \pm i\gamma_2), \]

these generators satisfy, besides Eq. (13), the following commutation relations:

\[ [M_{ij}, \hat{K}_k] = i\epsilon_{jkl} \hat{K}_l, \quad [\hat{K}_k, P_\mu] = \frac{i}{2} \gamma_k (1 + i\gamma_5), \quad [\hat{K}_j, P_k] = i\kappa g_{kj} \sinh \left( \frac{\gamma_0 + i\gamma_5 \gamma_5}{\kappa} \right) \]

\[ [\hat{K}_j, \hat{K}_k] = -i\gamma_j \wedge \gamma_k \cosh \left( \frac{\gamma_0 + i\gamma_5 \gamma_5}{\kappa} \right) - \frac{1}{4\kappa^2} \epsilon_{pq} \gamma_5^r \gamma_5^p \gamma_5^q \wedge \gamma^q, \]

A $\kappa$-deformation of the Poincaré algebra is obtained, and it coincides with $\mathcal{E}$. It can be forthwith shown that those equations satisfy all the well known $\kappa$-Poincaré algebra commutation relations given in $\mathcal{A}$. The respective coalgebra sector is described by the coproducts

\[ \Delta(M_{ij}) = \frac{1}{2} (\gamma_1 \wedge \gamma_2) \otimes 1 + 1 \otimes (\gamma_i \wedge \gamma_j), \quad \Delta(P_\mu) = \frac{1}{2} (\gamma_0 (1 + i\gamma_5) \otimes 1 + 1 \otimes \gamma_0 (1 + i\gamma_5)) \]

\[ \Delta(\hat{K}_i) = -\frac{1}{4} (1 + (\gamma_1 + \gamma_0) (1 - i\gamma_5)) \otimes 1 + 1 \otimes \gamma_0 (1 + i\gamma_5) ) \]

\[ + \frac{1}{2\kappa} \epsilon_{ijkl} [\hat{K}_j (1 + i\gamma_5) \otimes \gamma_1 \gamma_3 (1 + 2\kappa^{-1} \gamma_0 (1 + i\gamma_5)) + \gamma_1 \gamma_3 (1 - 2\kappa^{-1} \gamma_0 (1 + i\gamma_5)) \otimes \gamma_3 (1 + i\gamma_5)] \]

\[ \Delta(P_\mu) = \frac{1}{2} (\gamma_1 (1 + i\gamma_5) \otimes (1 + \gamma_0 (1 + i\gamma_5)) + (1 + \gamma_0 (1 + i\gamma_5)) \otimes \gamma_0 (1 + i\gamma_5) \]

The counits are given by $\epsilon(M_{\mu\nu}) = \epsilon(P_\mu) = \epsilon(\hat{K}_i) = 0$, and the antipodes are given by $S(M_{\mu\nu}) = \gamma_{\mu\nu}, \ S(P_\mu) = P_\mu, \ S(\hat{K}_i) = \gamma_i (1 + \gamma_5) + \frac{\gamma_0}{2\kappa} (1 - i\gamma_5))$. These results can be compared with $\mathcal{B}$.

Now the quantum $\kappa$-deformed Poincaré symmetries are formulated in modified bicrossproduct basis with classical Lorentz subalgebra, together with the respective Hopf algebra relations. A realization of the algebraic sector of a
quantum $\kappa$-deformed Poincaré algebra can be obtained via generators $M_{\mu\nu}, K_i, P_\mu$ that satisfy the following relations (besides Eq. (13)):

$$[M_{\mu\nu}, K_\lambda] = -(g_{\mu\lambda}K_\nu - g_{\nu\lambda}K_\mu), \quad [K_j, P_0] = -i\gamma_j \left(1 - \gamma_0 \frac{1 - i\gamma_5}{2\kappa}\right) \frac{1}{2}(1 + i\gamma_5)$$

$$[P_k, K_j] = ig_{kj}\gamma_0 \left(\frac{1 + i\gamma_5}{2}\right)(1 - 4\kappa^{-1}) + 2\kappa^{-1}\gamma_0 \left(1 - \frac{1 + i\gamma_5}{4\kappa}\right)\gamma_j\gamma_k\gamma_5$$

The commutation relations involving the rotations $M_{ij}$ and the time translation $P_0$ are classical, the momenta commute, and quantum deformation appears only when the boosts are involved. The respective coalgebra sector is described by the coproducts

$$\Delta(M_{ij}) = \frac{1}{2}(\gamma_i\gamma_j \otimes 1 + 1 \otimes \gamma_i\gamma_j), \quad \Delta(P_0) = \frac{1}{2}(\gamma_0(1 + i\gamma_5) \otimes 1 + 1 \otimes \gamma_0(1 + i\gamma_5))$$

$$\Delta(K_i) = -\frac{1}{2}(\gamma_i(1 + i\gamma_5) \otimes 1 + (4\kappa^{-1})(1 - \gamma_0(1 + i\gamma_5)) \otimes \gamma_i(1 + i\gamma_5)) + \frac{1}{2\kappa}\epsilon_{ijk}\gamma_j(1 + i\gamma_5) \otimes \gamma_i\gamma_j$$

$$\Delta(P_i) = \frac{1}{2}(\gamma_i(1 + i\gamma_5) \otimes (1 + \gamma_0(1 + i\gamma_5)) + (1 + \gamma_0(1 + i\gamma_5) \otimes \gamma_i(1 + i\gamma_5)).$$

In addition, the Wick isomorphism can also be applied to the elements exhibited in Table I, also letting the Wick isomorphism deformation accomplishment explicit.

VI. CONCLUDING REMARKS AND OUTLOOKS

In this paper we have been concerned to reveal the algebraic aspects of quantum Clifford algebras as a natural arena to construct a $\kappa$-deformed Poincaré algebra from the original Poincaré algebra, using the Wick theorem. After reviewing the algebra of conformal transformations and expressing it in terms of the spacetime algebra and the Atiyah-Bott-Shapiro periodicity theorem, some algebraic and coalgebraic aspects associated to conformal maps have been conceived in the context of quantum Clifford algebras. In this scenario, the Lie algebra associated with the conformal group and an alternative $\kappa$-deformed Poincaré algebra accrue solely in terms of elements of the spacetime algebra $\mathcal{C}\ell_{1,3}$, without any reference to representations. The $\kappa$-deformed Poincaré algebra is represented in terms of quantum Clifford algebras, and the coalgebraic sector is obtained. A possible perspective is to investigate our previous results in this context.

As the quantum $\kappa$-Poincaré algebra corresponds to DSR in the same way as the standard Poincaré algebra is related to special relativity, it has been argued that the knowledge of this quantum algebra is fundamental in the DSR theory, since the $\kappa$-Poincaré algebra is a quantum nonlinear algebra, and it can generate nonlinear transformations among momenta. Here we have presented another distinct basis in terms of the generators of spacetime algebra $\mathcal{C}\ell_{1,3}$, for which the construction is motivated by the results obtained in [8]. Formally, distinct basis of the algebra related by analytical mappings of momenta are completely equivalent, and the existence of physical equivalence remains an open problem. For more details and examples see, e.g., [28].

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