How ‘Complex’ is the Dirac Equation?

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Abstract

A representation of the Lorentz group is given in terms of $4 \times 4$ matrices defined over a simple non-division algebra. The transformation properties of the corresponding four component spinor are studied, and shown to be equivalent to the transformation properties of the usual complex Dirac spinor. As an application, we show that there exists an algebra of automorphisms of the complex Dirac spinor that leave the transformation properties of its eight real components invariant under any given Lorentz transformation. Interestingly, the representation of the Lorentz group presented here has a natural embedding in a cover of $\text{SO}(3,3)$ instead of the conformal symmetry $\text{SO}(2,4)$. 
1 Introduction

This article is motivated by the simple observation that the transformation properties of the eight real components of a complex Dirac spinor under a Lorentz transformation may be concisely formulated without any explicit reference to complex-valued quantities. This is accomplished by a representation of the Lorentz group using $4 \times 4$ matrices defined over a simple non-division algebra. After studying how this new representation is related to the usual complex one, we establish an automorphism symmetry of the complex Dirac spinor. We also discuss natural embeddings of this new representation into a maximal group, which turns out to be SO(3,3), and thus not equal to the usual conformal symmetry SO(2,4).

To begin, we revisit the familiar Lie algebra of the Lorentz group $O(1,3)$.

2 The Lorentz Algebra

2.1 A Complex Representation

Under Lorentz transformations, the complex Dirac 4-spinor $\Psi_C$ transforms as follows [1]:

$$\Psi_C \rightarrow \begin{pmatrix} e^{\frac{i}{2} \sigma \cdot (\theta - i \phi)} & 0 \\ 0 & e^{\frac{i}{2} \sigma \cdot (\theta + i \phi)} \end{pmatrix} \cdot \Psi_C,$$

where $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ represents the well known Pauli spin matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

The three real parameters $\theta = (\theta_1, \theta_2, \theta_3)$ correspond to the generators for spatial rotations, while $\phi = (\phi_1, \phi_2, \phi_3)$ represents Lorentz boosts along each of the coordinate axes. There are thus six real numbers parameterizing a given element in the Lorentz group.

Let us now introduce the six matrices $E_i$ and $F_i$, $i = 1, 2, 3$, by writing

$$E_1 = \frac{1}{2} \begin{pmatrix} \sigma_x & 0 \\ 0 & -\sigma_x \end{pmatrix}, \quad E_2 = -\frac{1}{2} \begin{pmatrix} \sigma_y & 0 \\ 0 & \sigma_y \end{pmatrix}, \quad E_3 = \frac{1}{2} \begin{pmatrix} \sigma_z & 0 \\ 0 & -\sigma_z \end{pmatrix},$$

$$F_1 = \frac{1}{2} \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix}, \quad F_2 = \frac{1}{2} \begin{pmatrix} \sigma_y & 0 \\ 0 & -\sigma_y \end{pmatrix}, \quad F_3 = \frac{1}{2} \begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix}.$$  

Then the Lorentz transformation [1] may be written as follows:

$$\Psi_C \rightarrow \exp \left( \phi_1 E_1 - \theta_2 E_2 + \phi_3 E_3 + \theta_1 F_1 + \phi_2 F_2 + \theta_3 F_3 \right) \cdot \Psi_C.$$
It is a straightforward exercise to check that the matrices $E_i$ and $F_i$ defined in (3) satisfy the following commutation relations:

\[
\begin{align*}
[E_1, E_2] &= E_3 & [F_1, F_2] &= -E_3 & [E_1, F_2] &= F_3 & [F_1, E_2] &= F_3 \\
[E_2, E_3] &= E_1 & [F_2, F_3] &= -E_1 & [E_2, F_3] &= F_1 & [F_2, E_3] &= F_1 \\
[E_3, E_1] &= -E_2 & [F_3, F_1] &= E_2 & [E_3, F_1] &= -F_2 & [F_3, E_1] &= -F_2
\end{align*}
\]

All other commutators vanish. Abstractly, these relations define the Lie algebra of the Lorentz group $O(1,3)$, and the matrices $E_i$ and $F_i$ defined by (3) correspond to a complex representation of this algebra.

### 2.2 Another Representation

Our goal in this section is to present an explicit representation of the Lorentz algebra (5) in terms of $4 \times 4$ matrices defined over a non-division algebra. The non-division algebra will be discussed next.

#### 2.2.1 The Semi-Complex Number System

We consider numbers of the form

\[ x + yj, \tag{6} \]

where $x$ and $y$ are real numbers, and $j$ is a commuting element satisfying the relation

\[ j^2 = 1. \tag{7} \]

The set of all such numbers forms a simple Clifford algebra, which will be denoted by the symbol $\mathbb{D}$, and called the ‘semi-complex number system’. Addition, subtraction, and multiplication are defined in the obvious way:

\[
\begin{align*}
(x_1 + y_1j) \pm (x_2 + y_2j) &= (x_1 \pm x_2) + j(y_1 \pm y_2), \\
(x_1 + y_1j) \cdot (x_2 + y_2j) &= (x_1x_2 + y_1y_2) + j(x_1y_2 + y_1x_2).
\end{align*}
\]

Given any semi-complex number $w = x + yj$, we define the ‘semi-complex conjugate’ of $w$, written $\overline{w}$, to be

\[ \overline{w} = x - yj. \tag{10} \]

It is easy to check the following; for any $w_1, w_2 \in \mathbb{D}$, we have

\[
\begin{align*}
\overline{w_1 + w_2} &= \overline{w_1} + \overline{w_2}, \tag{11} \\
\overline{w_1 \cdot w_2} &= \overline{w_1} \cdot \overline{w_2}. \tag{12}
\end{align*}
\]
We also have the identity
\[ \overline{w} \cdot w = x^2 - y^2 \]  
for any semi-complex number \( w = x + jy \). Thus \( \overline{w} \cdot w \) is always real, although unlike the complex number system, it may take negative values.

At this point, it is convenient to define the ‘modulus squared’ of \( w \), written \( |w|^2 \), as
\[ |w|^2 = \overline{w} \cdot w. \]  
A nice consequence of these definitions is that for any semi-complex numbers \( w_1, w_2 \in D \), we have
\[ |w_1 \cdot w_2|^2 = |w_1|^2 \cdot |w_2|^2. \]  
Now observe that if \( |w|^2 \) doesn’t vanish, the quantity
\[ w^{-1} = \frac{1}{|w|^2} \cdot \overline{w} \]  
is a well-defined unique inverse for \( w \). So \( w \in D \) fails to have an inverse if and only if \( |w|^2 = x^2 - y^2 = 0 \). The semi-complex number system is therefore a non-division algebra.

2.2.2 The Semi-Complex Unitary Groups

Suppose \( H \) is an \( n \times n \) matrix defined over \( D \). Then \( H^\dagger \) will denote the \( n \times n \) matrix which is obtained by transposing \( H \), and then conjugating each of the entries: \( H^\dagger = \overline{H}^T \). We say \( H \) is Hermitian with respect to \( D \) if \( H^\dagger = H \), and anti-Hermitian if \( H^\dagger = -H \).

Note that if \( H \) is an \( n \times n \) Hermitian matrix over \( D \), then \( U = e^{jH} \) has the property
\[ U^\dagger \cdot U = U \cdot U^\dagger = 1. \]  
The set of all \( n \times n \) matrices over \( D \) satisfying the constraint (17) forms a group, which we will denote as \( U(n, D) \), and call the ‘unitary group of \( n \times n \) semi-complex matrices’. The ‘special unitary’ subgroup \( SU(n, D) \) will be defined as all elements \( U \in U(n, D) \) satisfying the additional constraint
\[ \det U = 1. \]  
Note that the semi-complex unitary groups we have defined above may be isomorphic to well known non-compact groups that are usually defined over the complex number
field. For example, the semi-complex group $SU(2, D)$ is isomorphic to the complex group $SU(1,1)$ by virtue of the identification

$$
\begin{pmatrix}
  a_1 + ia_2 & b_1 + ib_2 \\
  b_1 - ib_2 & a_1 - ia_2
\end{pmatrix} \leftrightarrow \begin{pmatrix}
  a_1 + jb_1 & -a_2 + jb_2 \\
  a_2 + jb_2 & a_1 - jb_1
\end{pmatrix},
$$

(19)

where the four real parameters $a_1, a_2, b_1$ and $b_2$ satisfy the constraint $a_1^2 + a_2^2 - b_1^2 - b_2^2 = 1$.

2.2.3 A Semi-Complex Representation

As promised, we will give an explicit representation of the Lorentz algebra (5) in terms of matrices defined over $D$. First, we define three $2 \times 2$ matrices $\tau = (\tau_1, \tau_2, \tau_3)$ by writing

$$
\tau_1 = \begin{pmatrix}
  0 & 1 \\
  1 & 0
\end{pmatrix}, \quad \tau_2 = \begin{pmatrix}
  0 & -j \\
  j & 0
\end{pmatrix}, \quad \tau_3 = \begin{pmatrix}
  1 & 0 \\
  0 & -1
\end{pmatrix}.
$$

(20)

These matrices satisfy the following commutation relations:

$$
[\tau_1, \tau_2] = 2j\tau_3 \quad [\tau_2, \tau_3] = 2j\tau_1 \quad [\tau_3, \tau_1] = -2j\tau_2
$$

(21)

Now redefine the matrices $E_i$ and $F_i$, $i = 1, 2, 3$, by setting

$$
E_i = \frac{j}{2} \begin{pmatrix}
  \tau_i & 0 \\
  0 & \tau_i
\end{pmatrix}, \quad F_i = \frac{1}{2} \begin{pmatrix}
  0 & \tau_i \\
  -\tau_i & 0
\end{pmatrix}, \quad i = 1, 2, 3.
$$

(22)

The $4 \times 4$ semi-complex matrices $E_i$ and $F_i$ defined above are anti-Hermitian with respect to $D$, and satisfy the Lorentz algebra (5). We may therefore introduce a semi-complex 4-component spinor $\Psi_D$ transforming as follows under Lorentz transformations:

$$
\Psi_D \rightarrow \exp \left( \phi_1 E_1 - \theta_2 E_2 + \phi_3 E_3 + \theta_1 F_1 + \phi_2 F_2 + \theta_3 F_3 \right) \cdot \Psi_D,
$$

(23)

which is evidently the semi-complex analogue of transformation (4). Note that the transformation (23) has the form $\Psi_D \rightarrow U \cdot \Psi_D$, where $U \in SU(4, D)$, since the generators $E_i$ and $F_i$ are traceless and anti-Hermitian with respect to $D$. Thus the Lorentz group is a subgroup of the semi-complex group $SU(4, D)$.

In the next section, we discuss a relation between the complex Dirac spinor $\Psi_C$, and the 4-component semi-complex spinor $\Psi_D$ defined above.

3 Equivalence of Spinor Representations
3.1 An Equivalence

Consider an infinitesimal Lorentz transformation of the complex Dirac spinor,

\[ \Psi_C \rightarrow \exp (\phi_1 E_1 - \theta_2 E_2 + \phi_3 E_3 + \theta_1 F_1 + \phi_2 F_2 + \theta_3 F_3) \cdot \Psi_C, \quad (24) \]

where

\[ \Psi_C = \begin{pmatrix} x_1 + iy_1 \\ x_2 + iy_2 \\ x_3 + iy_3 \\ x_4 + iy_4 \end{pmatrix}. \quad (25) \]

In terms of the eight real variables \( x_i \) and \( y_i, \ i = 1, 2, 3, 4 \), an infinitesimal transformation of the form (24) is equivalent to the following eight real transformations:

\[
\begin{align*}
    x_1 & \rightarrow x_1 + \frac{\phi_1}{2} \cdot x_2 + \frac{\theta_2}{2} \cdot x_2 + \frac{\phi_3}{2} \cdot x_1 - \frac{\theta_1}{2} \cdot y_2 + \frac{\phi_2}{2} \cdot y_2 - \frac{\theta_3}{2} \cdot y_1 \\
    x_2 & \rightarrow x_2 + \frac{\phi_1}{2} \cdot x_1 - \frac{\theta_2}{2} \cdot x_1 - \frac{\phi_3}{2} \cdot x_2 - \frac{\theta_1}{2} \cdot y_1 - \frac{\phi_2}{2} \cdot y_1 + \frac{\theta_3}{2} \cdot y_2 \\
    x_3 & \rightarrow x_3 - \frac{\phi_1}{2} \cdot x_4 + \frac{\theta_2}{2} \cdot x_4 - \frac{\phi_3}{2} \cdot x_3 - \frac{\theta_1}{2} \cdot y_3 - \frac{\phi_2}{2} \cdot y_3 + \frac{\theta_3}{2} \cdot y_4 \\
    x_4 & \rightarrow x_4 - \frac{\phi_1}{2} \cdot x_3 + \frac{\theta_2}{2} \cdot x_3 + \frac{\phi_3}{2} \cdot x_4 - \frac{\theta_1}{2} \cdot y_4 + \frac{\phi_2}{2} \cdot y_4 + \frac{\theta_3}{2} \cdot y_3 \\
    y_1 & \rightarrow y_1 + \frac{\phi_1}{2} \cdot y_2 + \frac{\theta_2}{2} \cdot y_2 + \frac{\phi_3}{2} \cdot y_1 + \frac{\theta_1}{2} \cdot x_2 - \frac{\phi_2}{2} \cdot x_2 + \frac{\theta_3}{2} \cdot x_1 \\
    y_2 & \rightarrow y_2 + \frac{\phi_1}{2} \cdot y_1 - \frac{\theta_2}{2} \cdot y_1 - \frac{\phi_3}{2} \cdot y_2 + \frac{\theta_1}{2} \cdot x_1 + \frac{\phi_2}{2} \cdot x_1 - \frac{\theta_3}{2} \cdot x_2 \\
    y_3 & \rightarrow y_3 - \frac{\phi_1}{2} \cdot y_4 + \frac{\theta_2}{2} \cdot y_4 + \frac{\phi_3}{2} \cdot y_3 + \frac{\theta_1}{2} \cdot x_4 + \frac{\phi_2}{2} \cdot x_4 + \frac{\theta_3}{2} \cdot x_3 \\
    y_4 & \rightarrow y_4 - \frac{\phi_1}{2} \cdot y_3 + \frac{\theta_2}{2} \cdot y_3 + \frac{\phi_3}{2} \cdot y_4 + \frac{\theta_1}{2} \cdot x_3 - \frac{\phi_2}{2} \cdot x_3 - \frac{\theta_3}{2} \cdot x_4 \quad (26)
\end{align*}
\]

where we have used the representation specified by (3). Now consider the corresponding infinitesimal Lorentz transformation of the semi-complex spinor \( \Psi_D \),

\[ \Psi_D \rightarrow \exp (\phi_1 E_1 - \theta_2 E_2 + \phi_3 E_3 + \theta_1 F_1 + \phi_2 F_2 + \theta_3 F_3) \cdot \Psi_D, \quad (27) \]

where

\[ \Psi_D = \begin{pmatrix} a_1 + jb_1 \\ a_2 + jb_2 \\ a_3 + jb_3 \\ a_4 + jb_4 \end{pmatrix}. \quad (28) \]
In terms of the eight real variables \( a_i \) and \( b_i, \ i = 1, 2, 3, 4 \), this infinitesimal transformation is equivalent to the following eight real transformations:

\[
\begin{align*}
    a_1 & \rightarrow a_1 + \frac{\phi_1}{2} \cdot b_2 + \frac{\theta_2}{2} \cdot a_2 + \frac{\phi_3}{2} \cdot b_1 + \frac{\theta_1}{2} \cdot a_4 - \frac{\phi_2}{2} \cdot b_4 + \frac{\theta_3}{2} \cdot a_3 \\
    a_2 & \rightarrow a_2 + \frac{\phi_1}{2} \cdot b_1 - \frac{\theta_2}{2} \cdot a_1 - \frac{\phi_3}{2} \cdot b_2 + \frac{\theta_1}{2} \cdot a_3 + \frac{\phi_2}{2} \cdot b_3 - \frac{\theta_3}{2} \cdot a_4 \\
    a_3 & \rightarrow a_3 + \frac{\phi_1}{2} \cdot b_4 + \frac{\theta_2}{2} \cdot a_4 + \frac{\phi_3}{2} \cdot b_3 - \frac{\theta_1}{2} \cdot a_2 + \frac{\phi_2}{2} \cdot b_2 - \frac{\theta_3}{2} \cdot a_1 \\
    a_4 & \rightarrow a_4 + \frac{\phi_1}{2} \cdot b_3 - \frac{\theta_2}{2} \cdot a_3 - \frac{\phi_3}{2} \cdot b_4 - \frac{\theta_1}{2} \cdot a_1 - \frac{\phi_2}{2} \cdot b_1 + \frac{\theta_3}{2} \cdot a_2 \\
    b_1 & \rightarrow b_1 + \frac{\phi_1}{2} \cdot a_2 + \frac{\theta_2}{2} \cdot b_2 + \frac{\phi_3}{2} \cdot a_1 + \frac{\theta_1}{2} \cdot b_4 - \frac{\phi_2}{2} \cdot a_4 + \frac{\theta_3}{2} \cdot b_3 \\
    b_2 & \rightarrow b_2 + \frac{\phi_1}{2} \cdot a_1 - \frac{\theta_2}{2} \cdot b_1 - \frac{\phi_3}{2} \cdot a_2 + \frac{\theta_1}{2} \cdot b_3 + \frac{\phi_2}{2} \cdot a_3 - \frac{\theta_3}{2} \cdot b_4 \\
    b_3 & \rightarrow b_3 + \frac{\phi_1}{2} \cdot a_4 + \frac{\theta_2}{2} \cdot b_4 + \frac{\phi_3}{2} \cdot a_3 - \frac{\theta_1}{2} \cdot b_2 + \frac{\phi_2}{2} \cdot a_2 - \frac{\theta_3}{2} \cdot b_1 \\
    b_4 & \rightarrow b_4 + \frac{\phi_1}{2} \cdot a_3 - \frac{\theta_2}{2} \cdot b_3 - \frac{\phi_3}{2} \cdot a_4 - \frac{\theta_1}{2} \cdot b_1 - \frac{\phi_2}{2} \cdot a_1 + \frac{\theta_3}{2} \cdot b_2
\end{align*}
\]

(29)

where we have used the representation specified by the semi-complex matrices (22).

It is now straightforward to check that the infinitesimal transformations (20) and (29) for the complex and semi-complex spinors respectively are equivalent if we make the following identifications\(^1\):

\[
\begin{align*}
    a_1 & \leftrightarrow \frac{1}{\sqrt{2}}(y_1 + y_3) & a_2 & \leftrightarrow \frac{1}{\sqrt{2}}(y_2 + y_4) & a_3 & \leftrightarrow \frac{1}{\sqrt{2}}(x_1 + x_3) & a_4 & \leftrightarrow \frac{1}{\sqrt{2}}(x_2 + x_4) \\
    b_1 & \leftrightarrow \frac{1}{\sqrt{2}}(y_1 - y_3) & b_2 & \leftrightarrow \frac{1}{\sqrt{2}}(y_2 - y_4) & b_3 & \leftrightarrow \frac{1}{\sqrt{2}}(x_1 - x_3) & b_4 & \leftrightarrow \frac{1}{\sqrt{2}}(x_2 - x_4)
\end{align*}
\]

(30)

In particular, we have the identification

\[
\begin{aligned}
    (I) \quad \Psi_C &= \begin{pmatrix} x_1 + iy_1 \\ x_2 + iy_2 \\ x_3 + iy_3 \\ x_4 + iy_4 \end{pmatrix} \quad \leftrightarrow \quad \Psi_D = \frac{1}{\sqrt{2}} \begin{pmatrix} (y_1 + y_3) + j(y_1 - y_3) \\ (y_2 + y_4) + j(y_2 - y_4) \\ (x_1 + x_3) + j(x_1 - x_3) \\ (x_2 + x_4) + j(x_2 - x_4) \end{pmatrix},
\end{aligned}
\]

(31)

which establishes an exact equivalence between a complex Lorentz transformation [Eqn(4)] acting on the Dirac 4-spinor \( \Psi_C \), and the corresponding Lorentz transformation [Eqn(23)] acting on a semi-complex 4-spinor \( \Psi_D \).

It turns out that the equivalence specified by the identification (31) is not unique. There are additional identifications that render the complex and semi-complex Lorentz

\(^1\)The factor of \( 1/\sqrt{2} \) is arbitrary, and introduced for later convenience.
transformations equivalent, and we list three more below:

\[(II) \begin{pmatrix} x_1 + iy_1 \\ x_2 + iy_2 \\ x_3 + iy_3 \\ x_4 + iy_4 \end{pmatrix} \leftrightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} -(y_2 + y_4) + j(y_2 - y_4) \\ (y_1 + y_3) - j(y_1 - y_3) \\ (x_2 + x_4) - j(x_2 - x_4) \\ -(x_1 + x_3) + j(x_1 - x_3) \end{pmatrix}, \quad (32)\]

\[(III) \begin{pmatrix} x_1 + iy_1 \\ x_2 + iy_2 \\ x_3 + iy_3 \\ x_4 + iy_4 \end{pmatrix} \leftrightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} -(x_1 + x_3) - j(x_1 - x_3) \\ -(x_2 + x_4) - j(x_2 - x_4) \\ (y_1 + y_3) + j(y_1 - y_3) \\ (y_2 + y_4) + j(y_2 - y_4) \end{pmatrix}, \quad (33)\]

and

\[(IV) \begin{pmatrix} x_1 + iy_1 \\ x_2 + iy_2 \\ x_3 + iy_3 \\ x_4 + iy_4 \end{pmatrix} \leftrightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} -(x_2 + x_4) + j(x_2 - x_4) \\ (x_1 + x_3) - j(x_1 - x_3) \\ -(y_2 + y_4) + j(y_2 - y_4) \\ (y_1 + y_3) - j(y_1 - y_3) \end{pmatrix}, \quad (34)\]

Four more identifications may be obtained by a simple ‘reflection’ procedure; simply multiply each semi-complex spinor appearing in identifications (I),(II),(III) and (IV) above by the semi-complex variable $j$. This has the effect of interchanging the ‘real’ and ‘imaginary’ parts of each semi-complex component in the spinor.

### 3.2 Parity

Under parity, the Dirac 4-spinor $\Psi_C$ transforms as follows [1]:

$$\Psi_C \rightarrow \begin{pmatrix} 0 & 1_{2 \times 2} \\ 1_{2 \times 2} & 0 \end{pmatrix} \cdot \Psi_C,$$  \quad (35)$$

or, in terms of the eight real components $x_i$ and $y_i$, $i = 1, 2, 3, 4$, of the Dirac 4-spinor $\Psi_C$ specified by (25), we have

\[x_1 \rightarrow x_3 \quad x_2 \rightarrow x_4 \quad x_3 \rightarrow x_1 \quad x_4 \rightarrow x_2 \]
\[y_1 \rightarrow y_3 \quad y_2 \rightarrow y_4 \quad y_3 \rightarrow y_1 \quad y_4 \rightarrow y_2 \quad (36)\]

According to the identifications (I),(II),(III) and (IV) of Section 3.1, a parity transformation on $\Psi_C$ corresponds to conjugating $\Psi_D$. Thus, $\Psi_D \rightarrow \Psi_D^*$ under parity for these identifications. The ‘reflected’ forms of these identifications induces the transformation $\Psi_D \rightarrow j\Psi_D^*$ under parity. Semi-complex conjugation is therefore closely related to the parity symmetry.

\[\Psi_D^* \text{ denotes taking the semi-complex conjugate of each element in } \Psi_D.\]
4 The Automorphism Algebra of the Dirac Spinor

The existence of distinct equivalences between the complex and semi-complex spinors permits one to construct automorphisms of the complex Dirac spinor that leave the transformation properties of its eight real components intact under Lorentz transformations.

In order to investigate the algebra underlying the set of all possible automorphisms, it is convenient to change our current basis to the so-called ‘standard representation’ of the Lorentz group \([1]\). The Dirac 4-spinor \(\Psi_{SR}^C\) in the standard representation is related to the original 4-spinor \(\Psi_C\) according to the relation

\[
\Psi_C^{SR} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \Psi_C.
\] (37)

The identifications (I)-(IV) stated in Section 3.1 are now equivalent to the following identifications:

\[
(I)' \quad \Psi_D = \begin{pmatrix} a_1 + \text{j}b_1 \\ a_2 + \text{j}b_2 \\ a_3 + \text{j}b_3 \\ a_4 + \text{j}b_4 \end{pmatrix} \leftrightarrow \Psi_C^{SR} = \begin{pmatrix} a_3 + \text{i}a_1 \\ a_4 + \text{i}a_2 \\ b_3 + \text{i}b_1 \\ b_4 + \text{i}b_2 \end{pmatrix} \] (38)

\[
(II)' \quad \Psi_D = \begin{pmatrix} a_1 + \text{j}b_1 \\ a_2 + \text{j}b_2 \\ a_3 + \text{j}b_3 \\ a_4 + \text{j}b_4 \end{pmatrix} \leftrightarrow \Psi_C^{SR} = \begin{pmatrix} -a_4 + \text{i}a_2 \\ a_3 - \text{i}a_1 \\ b_4 - \text{i}b_2 \\ -b_3 + \text{i}b_1 \end{pmatrix} \] (39)

\[
(III)' \quad \Psi_D = \begin{pmatrix} a_1 + \text{j}b_1 \\ a_2 + \text{j}b_2 \\ a_3 + \text{j}b_3 \\ a_4 + \text{j}b_4 \end{pmatrix} \leftrightarrow \Psi_C^{SR} = \begin{pmatrix} -a_1 + \text{i}a_3 \\ -a_2 + \text{i}a_4 \\ -b_1 + \text{i}b_3 \\ -b_2 + \text{i}b_4 \end{pmatrix} \] (40)

\[
(IV)' \quad \Psi_D = \begin{pmatrix} a_1 + \text{j}b_1 \\ a_2 + \text{j}b_2 \\ a_3 + \text{j}b_3 \\ a_4 + \text{j}b_4 \end{pmatrix} \leftrightarrow \Psi_C^{SR} = \begin{pmatrix} a_2 + \text{i}a_4 \\ -a_1 - \text{i}a_3 \\ -b_2 - \text{i}b_4 \\ b_1 + \text{i}b_3 \end{pmatrix} \] (41)

In addition, we have four more which correspond to the ‘reflected’ form of the above identifications, and are obtained by interchanging the real and imaginary parts of the semi-complex components of \(\Psi_D\):

\[
(V)' \quad \Psi_D = \begin{pmatrix} a_1 + \text{j}b_1 \\ a_2 + \text{j}b_2 \\ a_3 + \text{j}b_3 \\ a_4 + \text{j}b_4 \end{pmatrix} \leftrightarrow \Psi_C^{SR} = \begin{pmatrix} b_3 + \text{i}b_1 \\ b_4 + \text{i}b_2 \\ a_3 + \text{i}a_1 \\ a_4 + \text{i}a_2 \end{pmatrix} \] (42)
\[(VI)ʼ \quad \Psi_D = \begin{pmatrix} a_1 + jb_1 \\ a_2 + jb_2 \\ a_3 + jb_3 \\ a_4 + jb_4 \end{pmatrix} \leftrightarrow \Psi_{C}^{SR} = \begin{pmatrix} -b_4 + ib_2 \\ b_3 - ib_1 \\ a_4 - ia_2 \\ -a_3 + ia_1 \end{pmatrix} \quad (43)\]

\[(VII)ʼ \quad \Psi_D = \begin{pmatrix} a_1 + jb_1 \\ a_2 + jb_2 \\ a_3 + jb_3 \\ a_4 + jb_4 \end{pmatrix} \leftrightarrow \Psi_{C}^{SR} = \begin{pmatrix} -b_1 + ib_3 \\ -b_2 + ib_4 \\ -a_1 + ia_3 \\ -a_2 + ia_4 \end{pmatrix} \quad (44)\]

\[(VIII)ʼ \quad \Psi_D = \begin{pmatrix} a_1 + jb_1 \\ a_2 + jb_2 \\ a_3 + jb_3 \\ a_4 + jb_4 \end{pmatrix} \leftrightarrow \Psi_{C}^{SR} = \begin{pmatrix} b_2 + ib_4 \\ -b_1 - ib_3 \\ -a_2 - ia_4 \\ a_1 + ia_3 \end{pmatrix} \quad (45)\]

Recall what these identifications mean; namely, under any given semi-complex Lorentz transformation [Eqn(23)] of \(\Psi_D\), the eight real components \(a_i\) and \(b_i\) \((i = 1, 2, 3, 4)\) transform in exactly the same way as the eight real components \(a_i\) and \(b_i\) that appear in the (eight) complex spinors \(\Psi_{C}^{SR}\) listed above, after being acted on by the corresponding complex Lorentz transformation\(^3\) [Eqn(4)].

We now define an operator \(\rho_{II}\) which takes the complex spinor \(\Psi_{C}^{SR}\) in the identification (I)ʼ above and maps it to the complex spinor \(\Psi_{C}^{SR}\) in the identification (II)ʼ. Thus \(\rho_{II}\) is defined by

\[\rho_{II} \cdot \begin{pmatrix} x_1 + iy_1 \\ x_2 + iy_2 \\ x_3 + iy_3 \\ x_4 + iy_4 \end{pmatrix} = \begin{pmatrix} -x_2 + iy_2 \\ x_1 - iy_1 \\ x_4 - iy_4 \\ -x_3 + iy_3 \end{pmatrix},\]

for any real variables \(x_i\) and \(y_i\). Similarly, we may construct the operators \(\rho_{III}, \rho_{IV}, \ldots, \rho_{VIII}\), whose explicit form we omit for brevity.

If we let \(\mathcal{V}(\Psi_{C}^{SR})\) denote the eight-dimensional vector space formed by all real linear combinations of complex 4-spinors, then the linear map \(\rho_{II}\), for example, is an automorphism of \(\mathcal{V}(\Psi_{C}^{SR})\). In particular, the transformation properties of the eight real components of \(\Psi_{C}^{SR}\) under a Lorentz transformation is identical to the transformation properties of the transformed spinor \(\rho_{II}(\Psi_{C}^{SR})\) under the same Lorentz transformation. One can show that the set of eight operators

\[\{1, \rho_{II}, \rho_{III}, \ldots, \rho_{VIII}\}\]

generate an eight dimensional closed algebra with respect to the real numbers.. The subset \(\{1, \rho_{II}, \rho_{III}, \rho_{IV}\}\), for example, generates the algebra of quaternions.

\(^3\) We assume the \(E_i\)’s and \(F_i\)’s are now in the standard representation.
One may also consider all commutators of the seven elements $\rho_{II}, \rho_{III}, \ldots, \rho_{VIII}$. These turn out to generate a Lie algebra that is isomorphic to $SU(2) \times SU(2) \times U(1)$. The $SU(2) \times SU(2)$ part is a Lorentz symmetry. The $U(1)$ factor is intriguing.

5 Discussion

In this work, we constructed a representation of the six-dimensional Lorentz group in terms of $4 \times 4$ generating matrices defined over a non-division algebra. This algebra was referred to as the ‘semi-complex’ number system.

The transformation properties of the corresponding ‘semi-complex 4-spinor’ was shown to be equivalent to the transformation properties of the usual complex Dirac spinor, after making an appropriate identification of components. The non-uniqueness of this identification led to an automorphism algebra defined on the vector space of Dirac spinors. These automorphisms have the property of preserving the transformation properties of the eight-real components of a 4-spinor in any given Lorentz frame. Properties of this algebra were studied.

It is interesting to note that the semi-complex representation of the Lorentz group turns out to be a subgroup of $SU(4,D)$. This group is fifteen dimensional, and so a reasonable guess is that it is isomorphic to the conformal group $SU(2,2)$, since we know $SU(2,D) \cong SU(1,1)$. However, this is not the case. By noting that the generators of $SU(4,D)$ are $4 \times 4$ anti-Hermitian matrices with respect to $D$, and that the semi-complex variable $j$ may be mapped to the element 1 or $-1$, one can show that the Lie algebra of $SU(4,D)$ is isomorphic to $SL(4,R)$. But we also know $SL(4,R) \cong SO(3,3)$ [4]. Thus the semi-complex Lorentz group is naturally embedded in a group with $SO(3,3)$ symmetry, rather than the conformal symmetry $SO(2,4)$.

Thus, from the viewpoint of naturally embedding the Lorentz symmetry into some larger group, the semi-complex and complex representations stand apart. We leave the physics of $SU(4,D)$ as an intriguing topic yet to be studied.

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