The maximum stellar mass, star-cluster formation and composite stellar populations

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ABSTRACT
We demonstrate that the mass of the most massive star in a cluster correlates non-trivially with the cluster mass. A simple algorithm according to which a cluster is filled up with stars that are chosen randomly from the standard IMF but sorted with increasing mass yields an excellent description of the observational data. Algorithms based on random sampling from the IMF without sorted adding are ruled out with a confidence larger than 0.9999. A physical explanation of this would be that a cluster forms by more-massive stars being consecutively added until the resulting feedback energy suffices to revert cloud contraction and stops further star formation. This has important implications for composite populations. For example, $10^3$ clusters of mass $10^5 M_\odot$ will not produce the same IMF as one cluster with a mass of $10^6 M_\odot$. It also supports the notion that the integrated galaxial IMF (IGIMF) should be steeper than the stellar IMF and that it should vary with the star-formation rate of a galaxy.

Key words: stars: formation – stars: luminosity function, mass function – galaxies: star clusters – galaxies: evolution – galaxies: stellar content – Galaxy: stellar content

1 INTRODUCTION
The insight that clustered star formation may be the dominant mode for star formation has grown over the last years. The form of the true distribution of stellar masses within these clusters, of the stellar initial mass function (IMF), has been a subject of debate for a long time. The evolution of the stars, unresolved binaries, and the dynamical evolution of the clusters complicates the observational efforts to extract the IMF. Unfortunately the most promising objects, very young stellar clusters (age $< 3$ Myr), are often still embedded in their natal cloud - again aggravating observations.

Nevertheless the distribution of stars in young clusters seems to be fairly well described by a multi power-law function with a slope or index ($\alpha$) of 2.35 (the so-called 'Salpeter' value) for stars with a mass larger than 0.5 $M_\odot$ (Kroupa 2001). The initial mass function (IMF),
\begin{equation}
\xi(m) \propto m^{-\alpha},
\end{equation}
where $\xi(m) dm$ is the number of stars in the mass interval $m, m + dm$. Several observations find the Salpeter value ($\alpha = 2.35$) for a large variety of conditions. The hypothesis of an invariant, multi-power law form (Kroupa et al. 1993; Kroupa 2001, 2002). As pointed out by Scalo (1998, 2005) Galactic-field and cluster stars below 1 $M_\odot$ and also represents young populations above 1 $M_\odot$ (Kroupa et al. 1993; Kroupa 2001, 2002). We refer to this form as the standard or canonical stellar IMF because this form fits the luminosity function of Galactic-field and cluster stars below 1 $M_\odot$ and also represents young populations above 1 $M_\odot$ (Kroupa et al. 1993; Kroupa 2001, 2002). As pointed out by Scalo (1998, 2003) though, significant uncertainties remain in the determination of the IMF to the point that the case can also be made that a single form of the IMF may not exist. In view of this, the ansatz made here and elsewhere is to propose the hypothesis of an invariant standard or canonical IMF (eq. 2) and test if the variation of the observed IMF can be understood to be the result of astrophysical effects (obscuration, stellar evolution, stellar multiplicity), dynamical effects (mass segregation, stellar evaporation and ejections), stochastic effects (finite $N$-sampling from the IMF) and the construction of composite populations (addition of many different clusters).

Similarly, the embedded cluster mass function (ECMF)

\begin{equation}
\alpha_0 = +0.30, \quad 0.01 \leq m/M_\odot < 0.08,
\alpha_1 = +1.30, \quad 0.08 \leq m/M_\odot < 0.50,
\alpha_2 = +2.35, \quad 0.50 \leq m/M_\odot < 1.00,
\alpha_3 = +2.35, \quad 1.00 \leq m/M_\odot.
\end{equation}
has been found to be well-described by at least one power-law,
\[ \xi_{\text{ecl}}(M_{\text{ecl}}) \propto M_{\text{ecl}}^{-\beta}, \]  
where \( dN_{\text{ecl}} = \xi_{\text{ecl}}(M_{\text{ecl}}) \, dM_{\text{ecl}} \) is the number of embedded clusters in the mass interval \( M_{\text{ecl}}, M_{\text{ecl}} + dM_{\text{ecl}} \) and \( M_{\text{ecl}} \) is the cluster mass in stars. The observational evidence points to a possibly universal form of the ECMF: Lada & Lada (2003) find a slope \( \beta = 2 \) in the solar neighbourhood for clusters with masses between 50 and 1000 \( M_\odot \), while Hunter et al. (2003) find \( 2 \lesssim \beta \lesssim 2.4 \) for \( 10^3 \lesssim M_{\text{ecl}}/M_\odot \lesssim 10^4 \) in the SMC and LMC, and Zhang & Fall (1999) find 1.95 ± 0.03 for \( 10^4 \lesssim M_{\text{ecl}}/M_\odot \lesssim 10^5 \) in the Antennae galaxies. Weidner, Kroupa & Larsen (2004) discovered that \( \beta = 2.35 \) best reproduces the observed correlation between the brightest young cluster and the galaxy-wide star-formation-rate for a large sample of late-type galaxies.

As already mentioned by Vanbeveren (1982) and discussed in more detail by Kroupa & Weidner (2003), the composite or integrated galaxial stellar IMF (IGIMF) is obtained by summing up the stellar IMFs contributed by all the star clusters that formed over the age of a galaxy, 
\[ \xi_{\text{IGIMF}}(m; t) = \int_{M_{\text{ecl,min}}}^{M_{\text{ecl,max}}} \xi(m \leq m_{\text{max}}(M_{\text{ecl}})) \cdot \xi_{\text{ecl}}(M_{\text{ecl}}) \, dM_{\text{ecl}}, \]  
where \( \xi_{\text{ecl}}(M_{\text{ecl}}) \) is the ECMF and \( \xi(m \leq m_{\text{max}}(M_{\text{ecl}})) \) is the stellar IMF in a particular cluster within which the maximal mass of a star is \( m_{\text{max}} \). \( M_{\text{ecl,min}} = \frac{5}{M_\odot} \) (Taurus-Auriga-type “clusters”) is the minimal cluster mass, while the maximal cluster mass, \( M_{\text{ecl,max}} \), depends on the galaxy-wide star-formation rate, SFR (Weidner, Kroupa & Larsen 2004).

A critical function entering this description is thus \( m_{\text{max}}(M_{\text{ecl}}) \). Assuming the stellar IMF to be a continuous distribution function, this mass of the most massive star in an embedded cluster with the total mass \( M_{\text{ecl}} \) in stars is given by
\[ 1 = \int_{m_{\text{max}}}^{m_{\text{max}}(M_{\text{ecl}})} \xi(m) \, dm, \]  
with
\[ M_{\text{ecl}} = \int_{m_{\text{low}}}^{m_{\text{max}}} m \, \xi(m) \, dm, \]  
where since there exists exactly one most massive star in each cluster, and neglecting statistical variations. Here \( m_{\text{low}} \approx 0.01 \, M_\odot \) is the minimal fragmentation mass and \( m_{\text{max}} \approx 150 \, M_\odot \) is the measured maximal stellar mass limit (Weidner & Kroupa 2003; Figer 2003; Oev & Clarke 2003). On combining eqs. 5 and 4 the function
\[ m_{\text{max}} = m_{\text{max}}(M_{\text{ecl}}) \]  
is quantified by Weidner & Kroupa (2004) and in § 2. This is the analytical (“ana”) maximum-stellar-mass–cluster-mass relation which incorporates the fundamental stellar upper mass limit (noted by the leading superscript “\( \text{a} \)” of \( m_{\text{max}} \)) = 150 \( M_\odot \). Later on other maximum-stellar-mass–cluster-mass relations are indicated by different superscripts: “ran”, “con” and “sort” for the different Monte-Carlo sampling methods (see § 2.3 and also “u” for the case without a fundamental stellar upper mass limit.

Weidner & Kroupa (2004) infer that a fundamental upper stellar mass limit, \( m_{\text{max}} \approx 150 \, M_\odot \), appears to exist above which stars do not occur, unless \( \alpha_3 \gtrsim 2.8 \), in which case no conclusions can be drawn based on the expected number of massive stars. As reviewed by Kroupa & Weidner (2003), the existence of such a stellar upper mass limit has been further substantiated by Figer (2003) and Oev & Clarke (2003) for a range of star clusters and OB associations.

We thus have, for each \( M_{\text{ecl}} \), the maximal stellar mass, \( m_{\text{max}}(M_{\text{ecl}}) \leq m_{\text{max}} \), and with this information eq. 3 can be evaluated to compute the IGIMF. Kroupa & Weidner (2003) find the IGIMF, when evaluated to the highest cluster masses, to be significantly steeper than the stellar IMF, and Weidner & Kroupa (2005) extend the analysis to a time varying ECMF by noting that \( M_{\text{ecl,max}} \) increases with the star-formation rate of a galaxy. They show the IGIMF to be not only steeper than the stellar IMF, but also to depend on galaxy type. The implications of these findings are rather significant for the supernova rate (Weidner & Kroupa 2005) and for the chemical evolution of galaxies (Kroupa, Weidner & Kroupa 2006).

But these results remain not without a challenge. Elmegreen (2004) argues that there is no evidence of a relation \( m_{\text{max}} = m_{\text{max}}(M_{\text{ecl}}) \leq m_{\text{max}} \). This relation implies that many small, low-mass, star-forming events will not have the same combined IMF as one major SF event of the same mass. Thus, according to Kroupa & Weidner (2005), 10^5 clusters each with a mass of 20 \( M_\odot \) would provide a combined IMF that differs from that of one cluster with a mass of \( 2 \times 10^6 \, M_\odot \) by being underrepresented in stars with a mass above about \( 5 \approx 1 \, M_\odot \). The contrary, often voiced view is that stellar masses sample the IMF purely statistically such that 10^5 stars in a mass above about \( 5 \approx 1 \, M_\odot \). This relation implies that many small, low-mass, star-forming events will not have the same combined IMF as one major SF event of the same mass.
Larson (1982) compared the properties of molecular clouds with the spatial distribution of the associated stellar population. He found that more massive and dense clouds favour the formation of massive stars and fitted the following formula to the observations,

$$m_{\text{max}} = 0.33 M_{\text{cloud}}^{0.43}.$$  

(8)

He re-evaluated (Larson 2003) this equation with more recent data and applied it directly to the cluster mass instead of the cloud mass,

$$m_{\text{max}} = 1.2 M_{\text{ecl}}^{0.45}.$$  

(9)

This correlation is plotted as a dotted line in Figs. 1 and 7. Elmegreen (1983) proposed a model for the origin of bound galactic clusters where the luminosity of the stars from a Miller-Scalo-IMF overcomes the binding energy of a molecular cloud core. The star-formation efficiency then discriminates between bound clusters and OB associations. He derived an analytical estimate for a relation between the maximal star mass and the cluster mass from statistical considerations regarding the appearance of stars with various masses from the Miller-Scalo-IMF,

$$M_{\text{ecl}} = \left( \frac{C_2}{A_2} \right)^{1/2} \left( \frac{m_{\text{max}} - A_3}{C_2} \right)^{1/2} e^{C_1 - C_2} dt,$$  

(10)

with

$$F(x) = \left( \frac{2\pi}{x} \right)^{1/2} \int_{-\infty}^{x} e^{-t^2/2} dt,$$  

(11)

and $A_1 = (2C_1)^{1/2}$, $A_2 = \ln 10(C_2 + \ln 10/4C_1) = -1.064$, $A_3 = C_2 + \ln 10/2C_1 = 0.065$, $C_1 = 1.09$ and $C_2 = -0.99$. In Figs. 1 and 7 this relation is shown as a short-dashed line.

On the other hand, using a single-power-law IMF with a Salpeter (1955) slope, Elmegreen (2000) solved eqs. 8 and 9 assuming $m_{\text{max}} = \infty$, with the result,

$$M_{\text{ecl}} = 3 \times 10^5 \left( \frac{m_{\text{max}}}{100 M_\odot} \right)^{1.35} M_\odot,$$  

(12)

which is shown as a long-dashed line in Figs. 1 and 7. Bonnell, Bate & Vine (2003) and Bonnell, Vine & Bate (2004) numerically studied star-formation in clusters using their smooth-particle-hydrodynamics (SPH) code. Here a turbulent molecular cloud fragments hierarchically into smaller subunits. When the density of a lump gets higher than a critical value, it is replaced by a so-called ‘sink’ particle which only lets matter in but not out. These sink particles form the final stellar cluster by interactions and mergers and are called stars at the end of the simulation. This hierarchical cluster formation scenario leads to the relation,

$$m_{\text{max}} \propto M_{\text{ecl}}^{3/2},$$  

(13)

and is shown as a short-dashed-dotted line in Figs. 1 and 7. There eq. 13 is normalised to $m_{\text{max}} = 27$ for a cluster of $M_{\text{ecl}} = 580 M_\odot$ (Bonnell, Bate & Vine 2003; Bonnell, Vine & Bate 2004). It should be noted here that these simulations do not include magnetic fields, stellar feedback and stellar mergers, all of which are believed to be of great importance for star-formation.

Weidner & Kroupa (2004) started with similar assumptions as Elmegreen (2000) but included a physical upper limit for the stellar mass, $m_{\text{max}} = 150 M_\odot$, while solving eqs. 8 and 9. Due to $m_{\text{max}}$, and using the standard multi-power law IMF (eq. 4), the result cannot be written out analytically but the equations have to be solved numerically. The result is plotted as a thick solid line in Figs. 1 and 7.

In a broader statistical analysis, Oey & Clarke (2003) calculated the probabilities that the observed upper mass limits in several clusters and OB associations in the MW, LMC and SMC come from a sample with a fundamental upper mass limit, $m_{\text{max}}$, or not. They conclude that an upper mass limit between 120 and 200 $M_\odot$ is the most likely explanation for the observed maximum masses. In order to do so, they calculated the expectation value for the maximum mass if a number $N$ of stars is randomly sampled from an IMF. This yields the equation,

$$\langle m_{\text{max}} \rangle = m_{\text{max}} - \int_0^{m_{\text{max}}} \left[ \int_0^{M_{\text{ecl}}} \xi(m) dm \right]^N dM_{\text{ecl}}.$$

(14)

Integrating this numerically yields the long-dash-short-dashed line in Figs. 1 and 7 assuming $m_{\text{max}} = 150 M_\odot$.

2.2 Monte-Carlo experiments

All the above mentioned investigations suggest that the cluster mass indeed appears to have a limiting influence on the stellar masses within it. However, it would be undisputed...
that a stochastic sampling effect from the IMF must be present when stars form. To investigate the possible existence of a maximal stellar mass in clusters statistically and to confirm or rule out if such a relation is purely the result of the random selection from an IMF, three Monte-Carlo experiments are conducted:

- pure random sampling (random sampling),
- mass-constrained random sampling (constrained sampling),
- mass-constrained random sampling with sorted adding (sorted sampling).
For each, two possibilities are probed: stars are sampled from the IMF without a maximal mass \( m_{\text{max}} = \infty \), or their masses are limited by \( m_{\text{max}} = 150 \, M_\odot \).

### 2.2.1 Random sampling

For the random sampling experiment, \( 2.5 \times 10^7 \) clusters are randomly taken from a cluster distribution with a power-law index \( \beta_N = 2.35 \). The clusters contain between 12 stars and \( 2.7 \times 10^7 \) stars. The relevant distribution function is the embedded-cluster star-number function (ECSNF),

\[
dN_{\text{ecl}} \propto N_\text{stars}^{\beta_N},
\]

which is the number of clusters containing \( N \in [N', N' + dN'] \) stars. Each cluster is then filled with \( N \) stars randomly from the standard IMF (eq. 2) without a mass limit, or by imposing the physical stellar mass limit, \( m \leq 150 \, M_\odot \).

The stellar masses are then summed to get the cluster mass, \( M_{\text{ecl}} \). Note that such a cluster distribution gives an embedded cluster mass function (ECMF) that is virtually identical to eq. 3 (Fig. 2). The resulting distribution of maximum stellar masses is plotted in the \( m_{\text{max}}, M_{\text{ecl}} \) plane (Fig. 3) as contour lines, to show the overall distribution (for a more detailed discussion see § 2.2.4).

As in this method the higher cluster masses are only very scarcely sampled, a second method is used to evaluate the mean of maximal masses in more detail. In order to do so the cluster star numbers \( N = 12 \) to \( 2.7 \times 10^7 \) are divided into 10 logarithmically equally-spaced values. Each of these numbers is then filled 10000 times with stars form the IMF, keeping only the mass of the most massive star for each cluster. The mean maximum mass for every bin then defines,

\[
\overline{m}_{\text{max}}(M_{\text{ecl}}),
\]

with a limit of \( m_{\text{max}} = 150 \, M_\odot \) and

\[
\overline{m}_{\text{max}}(M_{\text{ecl}}),
\]

in the unlimited case.

### 2.2.2 Constrained sampling

In this case \( 2.5 \times 10^7 \) clusters are randomly taken from the ECMF (eq. 3) between 5 \( M_\odot \) (the minimal, Taurus-Auriga-type, star-forming “cluster” counting \( \approx 15 \) stars)

\[1\] For practical reasons of numerical computation \( m_{\text{max}} \) is adopted to be \( m_{\text{max}} = 10^6 \, M_\odot \) in the unlimited case.

10^6 \, M_\odot \) (an approximate maximum mass for a single stellar population that consists of one metallicity and age) [1] (solid line). The slopes are virtually the same (\( \beta_N = \beta \)).

Only for very small cluster masses does the solid line deviate due to the underlying standard IMF because small-\( N \) clusters can nevertheless have masses \( > 10 \, M_\odot \) if their constituent stars happen to be massive. This accounts for the turn down below 10 \( M_\odot \) and the surplus in the range 10-150 \( M_\odot \).

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Note that \( \beta_N \approx \beta \) because the ECSNF and the ECMF only differ by a nearly-constant average stellar mass (Fig. 2). Then stars are taken randomly from the standard IMF and added until they reach or just surpass the respective cluster mass, \( M_{\text{ecl}} \). Afterwards the clusters are searched for their maximum stellar mass (plotted as contours in Fig. 4 see § 2.2.4 for discussion).

Again the sampling of high cluster masses is very poor. Therefore, the cluster masses from 5 \( M_\odot \) to \( 10^6 \, M_\odot \) are divided into 10 logarithmically equally-spaced values. Then each of these 10 cluster masses is filled 10000 times with stars form the IMF until their combined mass reach or just surpass the cluster mass and only the maximum star mass is recorded. The average \( m_{\text{max}} \) values for each of the 10 cluster masses define the relation,

\[
\overline{m}_{\text{max}}(M_{\text{ecl}}),
\]

\[
\overline{m}_{\text{max}}(M_{\text{ecl}}),
\]

in the unlimited case.

### 2.2.3 Sorted sampling

For the third method again \( 2.5 \times 10^7 \) clusters are randomly taken from the ECMF (eq. 3) between 5 \( M_\odot \) and \( 10^6 \, M_\odot \) and with \( \beta = 2.35 \). However, this time the number \( N \) of stars which are to populate the cluster is estimated from...
$N = \frac{M_{\text{ecl}}}{m_{\text{av}}}$, where $m_{\text{av}} = 0.36 M_\odot$ is the average stellar mass for the standard IMF (eq. 2) between 0.01 $M_\odot$ and 150 $M_\odot$. These stars are added to give $M_{\text{ecl,ran}}$,

$$M_{\text{ecl,ran}} = \sum_{N} m_i,$$

such that $m_i \leq m_{i+1}$. If $M_{\text{ran}} < M_{\text{ecl}}$ in this first step, an additional number of stars, $\Delta N$, is picked randomly from the IMF, where $\Delta N = (M_{\text{ecl}} - M_{\text{ecl,ran}}) / m_{\text{av}}$. Again these stars are added to obtain an improved estimate of the desired cluster mass,

$$* M_{\text{ecl,ran}} = \sum_{N+\Delta N} m_i,$$

again with $m_i \leq m_{i+1}$. When $* M_{\text{ecl,ran}}$ surpasses $M_{\text{ecl}}$, it is checked whether the sum is closer to $M_{\text{ecl}}$ when the last star (the most massive one) is subtracted or not. If $* M_{\text{ecl,ran}}$ is closer to $M_{\text{ecl}}$ with the last star this is the most massive one for the cluster, otherwise it is the second last star. This procedure is followed and repeated until all clusters from the ECMF are ‘filled’. The contour plots of the most massive star for each cluster are shown in Fig. 4 and discussed in § 2.2.2.

Again, for a more detailed analysis, 10 cluster masses are generated like in § 2.2.2 but filled with stars in the sorted way described above. The mean over every of the 10 cluster masses then yields the relation

$$\bar{m}_{\text{max}}^\text{sort}(M_{\text{ecl}}),$$

(20)

### 2.2.4 Comparison of the samplings

In order to exemplify the differences between the three sampling methods the following gedanken experiment may be considered:

A sample of 10 stars consists of 9 stars with 5 $M_\odot$ and one with 11 $M_\odot$. For the random sampling, this would be a cluster with $M_{\text{ecl,ran}} = 56$ $M_\odot$, with $m_{\text{max,ran}} = 11$ $M_\odot$.

If, for the sorted sampling, the aimed-at cluster mass is 50 $M_\odot$, the actual cluster mass would be 45 $M_\odot$ ($= 9 \times 5$), because 45 is closer to 50 than 56, and $m_{\text{max,sort}}$ would be 5 $M_\odot$. In the case of constrained sampling the order would be important. If the aimed-at cluster mass is 50 $M_\odot$ and the 11 $M_\odot$ star is among the first 9 stars, $M_{\text{max,con}}$ would be 51 ($\approx 8 \times 5 + 11$) and $m_{\text{max,con}} = 11$ $M_\odot$. But if the 11 $M_\odot$ star is the tenth star, $M_{\text{ecl,con}}$ would be 45 $M_\odot$ and $m_{\text{max,con}} = 5$ $M_\odot$ as in the sorted-sampling case because, 45, rather than 56, is closer to 50 $M_\odot$.

Figs. 3 to 5 plot the contour lines of the most massive stars of all the simulated clusters for random sampling (Fig. 3), constrained sampling (Fig. 4) and sorted sampling (Fig. 5), all with the physical limit $m_{\text{max}} = 150$ $M_\odot$.

As can be seen from Fig. 4 the random sample occupies the whole parameter space between the two extremes, which are either that nearly the whole cluster consists only of low-mass stars, or a single star accounts for the entire cluster mass ($m_{\text{max}} = M_{\text{ecl}}$). Such one-star-clusters would correspond to freak star-formation events such as is envisaged for a variable gas-equation-of-state-star-formation theory (Li, Mac Low & Klessen 2000). For sorted sampling the clusters are shifted more towards smaller stellar masses and never touch this line. While the constrained sampling lies in between these two extremes and still populates the parameter space up to the identity relation.

Note that $m_{\text{max}}^\text{ana}(M_{\text{ecl}})$, shown in Figs. 3 to 5 (thick dotted line in Fig. 3), is nearly identical to the semi-analytical estimate $m_{\text{max}}^\text{ana}(M_{\text{ecl}})$ (eq. 17) thick solid line in Figs. 3 to 5. The slight deviations are probably due to the stochastic element in the process to decide which is the most massive star in the sorted-Monte-Carlo experiment (see § 2.2.2).

The non-smoothness of the contour lines in the upper right edge of the figures is an effect of low-number sampling. Here only the outer-most contour line is present - indicating that there are only single events in this region of parameter space obtained with $2.5 \times 10^7$ clusters, which have together

![Figure 3. Maximal mass of stars versus cluster mass. The contour lines show how often a certain combination of cluster mass and maximal star mass occurs. The further-out they lie (towards the upper right) the smaller is the number of clusters with this combination. They are spaced logarithmically such that the outer-most one represents a single cluster with a certain mass and maximal star ($\log_{10} N_{\text{ecl}} = 0$), eg. $M_{\text{ecl}} = 10^5 M_\odot$, $m_{\text{max}} \approx 100 M_\odot$. The inner-most one (near $M_{\text{ecl}} = 7 M_\odot$, $m_{\text{max}} \approx 5 M_\odot$) stands for $\log_{10} N_{\text{ecl}} = 4.5$ clusters with this maximal star mass. The lines in-between are in steps of 0.5 dex. Here the contour lines are the result of the Monte-Carlo simulations with random sampling. The semi-analytic result (eq. 16), $m_{\text{max}}^\text{ana}(M_{\text{ecl}})$, is the thick solid line. Mean values are shown as the thick dashed line for random sampling ($m_{\text{max}}^\text{ran}(M_{\text{ecl}})$, eq. 18) as the thin dashed line for constrained sampling ($m_{\text{max}}^{\text{con}}(M_{\text{ecl}})$, eq. 19) and as the thin dotted line for sorted sampling ($m_{\text{max}}^{\text{sort}}(M_{\text{ecl}})$, eq. 20). The identity relation $m_{\text{max}} = M_{\text{ecl}}$ is plotted as a thin solid line.](image-url)
Therefore the result should agree with the numerically integrated (semi-analytical) one.

Another difference between the samplings lies in the average mean stellar masses in the clusters. To determine these, cluster masses from $5 \, M_\odot$ to $10^6 \, M_\odot$ are divided in 10 logarithmically equally-spaced values and each is filled 10000 times with stars from the IMF in the three different ways described in §2.2.1, 2.2.2, and 2.2.3. The mean stellar mass for each cluster is calculated by

$$\overline{m} = \frac{1}{N} \sum_{i=1}^{N} m_i,$$

where $N$ is the number of stars in each cluster. Then for each cluster mass, $M_{\text{ecl}}$ ($j = 1...10$), the 'mean of means' is computed by

$$\overline{m}_j(M_{\text{ecl}}) = \frac{1}{N_{\text{ecl}}} \sum_{i=1}^{N_{\text{ecl}}} \overline{m}_i,$$

with $N_{\text{ecl}} = 10000$.

The different average mean stellar masses are shown in Fig. 6. For the random sampling the mean is always about $0.36 \, M_\odot$, as expected for the canonical IMF between 0.01 and 150 $M_\odot$. The other sampling methods have lower means for light clusters which rise up to $0.36 \, M_\odot$ for more massive ones. This is a result of the limit which low-mass clusters impose on their stellar content. For random sampling only the number of stars determines the cluster mass, in contrast to reality where the natal cloud-mass and the star-formation-efficiency rule the final cluster mass.

Also shown in Fig. 6 is the average mean stellar mass for the analytical model (eq. 4), which is given by

$$\overline{m} = \frac{\int_{m_{\text{low}}}^{m_{\text{mix}}}(M_{\text{ecl}})}{\int_{m_{\text{low}}}^{m_{\text{mix}}}(M_{\text{ecl}})} m \xi dm = \frac{\int_{m_{\text{low}}}^{m_{\text{mix}}}(M_{\text{ecl}})}{\int_{m_{\text{low}}}^{m_{\text{mix}}}(M_{\text{ecl}})} \xi dm,$$

where $\xi(m)$ is eq. 2.

### 2.2.5 Comparison with observational data

Table 4 shows a non-exhaustive compilation of cluster masses and upper stellar masses for a number of MW and LMC clusters (see appendix A for more details). These data are shown as dots with error-bars in Figs. 7, 8, and 9. The result of a cluster formation simulation by Bonnell, Bate & Vine (2003) is shown as a large triangle.

Fig. 4 compares the mean $m_{\text{mix}}$ values of the Monte-Carlo experiments together with the observations from Tab. 1, the semi-analytical result of Weidner & Kroupa (2004) and the different results of the previous' studies (§2.2.1). The “unlimited” mean values are marked with a “u”, while the limited ones ($m_{\text{mix}} = 150 \, M_\odot$) with an “l”.

The mean values of the “u” cases are all clearly distinct from the observations and not regarded further, while for the “l” cases the mean values are in reasonable agreement with the observations. But especially for the analytical model (eq. 7), which is given by

$$m_{\text{mix}} = m_{\text{ecl}}\xi dm,$$

where $\xi(m)$ is eq. 2.

![Figure 4](image-url)  
**Figure 4.** Like Fig. 3 but this time the contour lines are the result of the Monte-Carlo simulations with mass-constrained sampling, and $\overline{m}_{\text{mix}}(M_{\text{ecl}})$ is shown as a thick line, while $\overline{m}_{\text{mix}}(M_{\text{ecl}})$ and $\overline{m}_{\text{mix}}(M_{\text{ecl}})$ are thin. $\overline{m}^\text{ana}_{\text{mix}}$ is the thick solid line.

![Figure 5](image-url)  
**Figure 5.** Like Fig. 3 and 4 but this time the contour lines are the result of the Monte-Carlo simulations with sorted sampling with $\overline{m}_{\text{mix}}(M_{\text{ecl}})$ as a thick line.

about 600 $\cdot 10^6$ stars. This number of clusters comprises the computational limit of the available hardware.

The agreement of the mean value $\overline{m}_{\text{mix}}(M_{\text{ecl}})$ and the semi-analytic result of Weidner & Kroupa (2004), $\overline{m}_{\text{mix}}(M_{\text{ecl}})$, is in principal not surprising. The method of sorted adding of stellar masses in order to get the cluster mass corresponds to a Monte-Carlo integration of eqs. 4 and 5.
Taken this into account leads to the semi-analytical (eq. 7) result, but in doing so they over-predict the observed upper mass limit for stars near 150 $M_\odot$. The solid line shows the relation described by eq. 23. Note that the long-dashed line lies above the solid curve because the maximal stellar mass in the sorted-sampling algorithm is systematically higher than the analytical result (Fig. 7 to 9 below) for cluster masses below $10^5 M_\odot$.

2.3 Ageing of the stars

To see if stellar evolution together with constrained sampling can mimic the effect of the sorted sampling, stars with masses between 2 and 14 $M_\odot$ are evolved with the single stellar evolution (SSE) package from Hurley, Pols & Tout (2000), while for more-massive stars fitting formulae derived from stellar evolution models computed by Schaller et al. (1992) are used (see Appendix C for the detailed fitting functions). For this purpose $1 \times 10^3$ clusters are chosen from an ECMF with $\beta = 2.35$ and then aged for 1, 2, 2.5, 3 and 3.5 Myr. The evolved stellar masses are added after excluding neutron stars and black holes to give $M_{\text{ ecl}}$ and then searched for the most massive star in each cluster.

In Fig. 6 the effect of this ageing is shown. Within the first 2.5 Myr the mean values of the sorted sampling and the constrained sampling algorithm are closer to the observations, making a distinction between constrained and sorted sampling not possible.

### Table 1. Empirical cluster masses, maximal star masses within these clusters and cluster ages from the literature.

| Designation | $M_{\text{ ecl}}$ [$M_\odot$] | $m_{\text{ max obs}}$ [$M_\odot$] | age [Myr] | Source |
|-------------|------------------|------------------|-----------|--------|
| Tau-Aur     | 25 $\pm$ 15      | 2.2 $\pm$ 0.2    | 1-2       | (1)    |
| Ser SVS2    | 30 $\pm$ 15      | 2.2 $\pm$ 0.2    | 2         | (2)    |
| NGC1333     | 80 $\pm$ 30      | 5 $\pm$ 1.0      | 1-3       | (3)    |
| $\rho$ Oph  | 100 $\pm$ 50     | 8 $\pm$ 1.0      | 0.1-1     | (4)    |
| IC348       | 109 $\pm$ 20     | 6 $\pm$ 1.0      | 1.3       | (5)    |
| NGC2024     | 225 $\pm$ 30     | 20 $\pm$ 4       | 0.5       | (6)    |
| $\sigma$ Ori| 225 $\pm$ 30     | 20 $\pm$ 4       | 2.5       | (6)    |
| Mon R2      | 259 $\pm$ 60     | 10 $\pm$ 1       | 0-3       | (7)    |
| NGC2264     | 355 $\pm$ 50     | 25 $\pm$ 5       | 3.1       | (8)    |
| NGC6530     | 815 $\pm$ 115    | 20 $\pm$ 4       | 2.3       | (9)    |
| Ber 86      | 1500 $\pm$ 500   | 40 $\pm$ 8       | 2-3       | (10)   |
| M42         | 2200 $\pm$ 300   | 45 $\pm$ 5       | <1        | (11)   |
| NGC2244     | 6240 $\pm$ 124   | 70 $\pm$ 14      | 1.9       | (12)   |
| NGC6611     | 2.10$^5$ $\pm$ 10000 | 85 $\pm$ 15 | 1.3 $\pm$ 0.3 | (13) |
| Tr 14/16    | 4.3$^2$ $^{10^{-2}}$ $^{3}$$^{0.5}^{+2}$$^{0.4}$ | 120 $\pm$ 15 | <3       | (14)   |
| Arches      | 5$^2$ $^{10^{-2}}$ $^{3.5}$$^{0.5}^{+2}$$^{0.4}$ | 135 $\pm$ 15 | 2.5       | (15)   |
| R136        | 1$^2$ $^{10^{-3}}$ $^{1.5}$$^{0.5}^{+1.5}$$^{0.5}$ | 145 $\pm$ 10 | 1-2       | (16)   |

$m \leq 50 M_\odot$ are evolved with the single stellar evolution (SSE) package from Hurley, Pols & Tout (2000), while for more-massive stars fitting formulae derived from stellar evolution models computed by Schaller et al. (1992) are used (see Appendix C for the detailed fitting functions). For this purpose $1 \times 10^3$ clusters are chosen from an ECMF with $\beta = 2.35$ and then aged for 1, 2, 2.5, 3 and 3.5 Myr. The evolved stellar masses are added after excluding neutron stars and black holes to give $M_{\text{ ecl}}$ and then searched for the most massive star in each cluster.

In Fig. 9 the effect of this ageing is shown. Within the first 2.5 Myr the mean values of the sorted sampling and the constrained sampling algorithms are clearly distinct. The lines change due to mass loss of the heavy stars, which amounts to about 40 to 50% of the initial stellar mass, according to the models used (see Appendix C for a comparison of different models). After 2.5 Myr the massive stars start to explode as supernovae and the constrained sampled clusters ($\overline{m}_{\text{ max obs}}(M_{\text{ ecl}})$) move closer to the sorted-sampled ones ($\overline{m}_{\text{ max obs}}(M_{\text{ ecl}})$) for $M_{\text{ ecl}} \geq 10^3 M_\odot$. After 1 Myr and for clusters with $M_{\text{ ecl}} \geq 10^3 M_\odot$, ageing shifts $\overline{m}_{\text{ max obs}}(M_{\text{ ecl}})$ closer to the observations, making a distinction between constrained and sorted sampling not possible. Nevertheless, the
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Figure 7. The thick solid line shows the dependence of the maximal star mass on the cluster mass for $\alpha_3 = 2.35$ from the semi-analytical model $l_{\text{max,ana}}(M_{\text{ecl}})$, eq. 7. The thick short-dashed line shows the mean maximum stellar mass for sorted sampling $l_{\text{max,sort}}(M_{\text{ecl}})$, see §2.2.3. The long-dashed lines are mass-constrained random-sampling $l_{\text{max,con}}(M_{\text{ecl}})$, see §2.2.2 results with an upper mass limit of $10^6 M_\odot$ (straight line) and $150 M_\odot$ (curved line). Pure random sampling models $l_{\text{max,ran}}(M_{\text{ecl}})$, see §2.2.1) are printed as dot-dashed lines. The curved one is sampled to $m_{\text{max}} = 150 M_\odot$ while the straight one assumes $m_{\text{max}} = 10^6 M_\odot$. The thin solid line shows the identity relation, were a “cluster” consists only of one star. The dots with error bars are observed clusters (see Tab. I), while the triangle is a result from the star-formation simulation with an SPH code (Bonnell, Bate & Vine 2003).

Figure 8. The thick solid line shows $l_{\text{max,ana}}(M_{\text{ecl}})$ for $\alpha_3 = 2.35$ (eq. 7). The thick dashed lines are $l_{\text{max,sort}}(M_{\text{ecl}})$ for $\alpha_3 = 2.35$ (upper line) and 2.70 (lower line). The dot-dashed lines are constrained sampling Monte-Carlo results, $l_{\text{max,con}}(M_{\text{ecl}})$, with three different slopes ($\alpha_3 = 2.35, 2.70$ and 3.00 above $1 M_\odot$) of the input stellar IMF. The thin solid line shows the identity relation. The dots with error bars are observed clusters (Tab. I), while the triangle is a result from the star-formation simulation with an SPH code (Bonnell, Bate & Vine 2003).

Figure 9. As Fig. 4 but the mean curves include ageing by 1, 2, 2.5, 3 and 3.5 Myr. The stars in the Monte-Carlo-simulations are subject to stellar evolution according to the SSE package by Hurley, Pols & Tout (2000) and our own extensions for stars $\geq 50 M_\odot$ which includes not only finite life-times but also stellar mass-loss. The thick dashed lines are for clusters which are constructed using sorted sampling, while the dot-dashed lines are for constrained sampling. Note that $l_{\text{max,con}}(M_{\text{ecl}}) = l_{\text{max,sort}}(M_{\text{ecl}})$ for ages $\geq 3$ Myr and $M_{\text{ecl}} \geq 10^3 M_\odot$.

2.4 Statistical analysis

In order to evaluate the statistical significance of the differences between the Monte-Carlo simulations and the observations, two statistical tests are applied. The statistical tests are only applied to the zero-age Monte-Carlo samples, as the PD and ZAMS masses for low-mass stars do not differ observational data for $M_{\text{ecl}} < 10^3 M_\odot$ agree much better with sorted than with constrained sampling for all ages.

It must be noted here that the observed stellar masses ($m_{\text{max,obs}}$) in Tab. I are a mixture of present-day (PD) masses and zero-age main-sequence (ZAMS) masses. For stars below roughly 50 $M_\odot$ this is not critical, as for them mass-loss is not dominant during the first 3 Myr. But the three most-massive clusters have ZAMS maximal stellar masses and can therefore not be compared with the aged tracks in Fig. 4. Our previous (§2.2.2) comparison of the data with zero-age main-sequence isochrones is thus justified.
The probabilities that the observed masses of the most-massive stars, \( m_{\text{maxobs}} \), in the clusters are drawn from the three different Monte-Carlo samples are calculated. In order to do so, the distribution of \( m_{\text{max}} \) around each observed cluster mass, \( \mu(m_{\text{max}} : M_{\text{cl}}) \), is examined for the three samplings, whereby we only use those that have a maximal mass, \( m_{\text{max}} = 150 M_{\odot} \). If the mean value, \( \bar{m}_{\text{max}}(M_{\text{cl}}) \), of the Monte-Carlo distribution, \( \mu(m_{\text{max}}) \), for a cluster mass is higher than the observed maximal mass, \( m_{\text{maxobs}} \), then the distribution is integrated from the lower limit, \( m_{\text{low}} \), to \( m_{\text{maxobs}} \) and divided by the integral from \( m_{\text{low}} \) to \( \bar{m}_{\text{max}} \),

\[
P(m_{\text{max}} \leq m_{\text{maxobs}}) = \frac{\int_{m_{\text{low}}}^{m_{\text{maxobs}}} \mu dm}{\int_{m_{\text{low}}}^{\bar{m}_{\text{max}}(M_{\text{cl}})} \mu dm}. \tag{24}
\]

This is the probability of observing a maximum mass \( m_{\text{max}} \) as small as or smaller than \( m_{\text{maxobs}} \).

If, on the other hand, the mean value is smaller than \( m_{\text{maxobs}} \), the integral is taken from \( m_{\text{maxobs}} \) to the upper mass limit, \( m_{\text{max}} \), and is divided by the integral from the mean value, \( \bar{m}_{\text{max}}(M_{\text{cl}}) \), to \( m_{\text{max}} \),

\[
P(m_{\text{max}} \geq m_{\text{maxobs}}) = \frac{\int_{m_{\text{maxobs}}}^{m_{\text{max}}} \mu dm}{\int_{m_{\text{maxobs}}}^{\bar{m}_{\text{max}}(M_{\text{cl}})} \mu dm}. \tag{25}
\]

This is the probability of observing an \( m_{\text{max}} \) as large as or larger than \( m_{\text{maxobs}} \). Together, eqs. \( 24 \) and \( 25 \) are the probability that the observed masses of the most-massive stars, \( m_{\text{maxobs}} \), in the clusters are drawn from the three Monte-Carlo samples.

To obtain a completely different statistic on the correspondence between data and theory, we also apply a Wilcoxon signed-rank test (Bhattacharyya & Johnson 1977) to determine how significant the differences between the data and the \( \bar{m}_{\text{max}} \) are. For this test the differences of the data points and the mean values are calculated,

\[
\Delta m_l = m_{\text{maxobs}, l} - \bar{m}_{\text{max}} \quad \text{for a given } M_{\text{cl}, l}, \quad \tag{26}
\]

and then ranked according to their absolute value. Afterwards, only the positive-signed ranks are added and this sum is then cross-checked with tabulated values (Bhattacharyya & Johnson 1977) in order to get the probability \( P \) that the null hypothesis (data points and the \( \bar{m}_{\text{max}} \) relations are the same within the uncertainties) is true. In the case of random sampling (§ 4) and constrained sampling (§ 5), \( P = 0.00014 \), while \( P = 0.0458 \) for sorted sampling (§ 6).

Thus, both tests taken together suggest strongly that sorted-sampling best describes the empirical data. The physical interpretation of this result is discussed in § 4.

### Table 2. Probabilities that the \( m_{\text{maxobs}} \) are from one of the three Monte-Carlo samples.

| Cluster   | random sampling | constrained sampling | sorted sampling |
|-----------|-----------------|----------------------|-----------------|
| Tau-Aur   | 25.0            | 36.2                 | 45.4            |
| Ser SVS2  | 12.5            | 33.1                 | 41.5            |
| NGC1333   | 24.7            | 29.4                 | 41.1            |
| ρ Oph     | 31.5            | 60.6                 | 81.7            |
| IC348     | 9.2             | 38.2                 | 52.5            |
| NGC2024   | 69.1            | 86.0                 | 76.4            |
| σ Ori     | 69.1            | 86.0                 | 76.4            |
| Mon R2    | 25.8            | 27.1                 | 37.5            |
| NGC2264   | 57.6            | 81.9                 | 84.7            |
| NGC6530   | 27.2            | 32.4                 | 44.3            |
| Ber 86    | 39.3            | 53.9                 | 76.3            |
| M42       | 43.9            | 53.6                 | 75.5            |
| NGC2244   | 34.1            | 40.4                 | 68.0            |
| NGC6611   | 6.4             | 5.6                  | 32.2            |
| Tr 14/16  | 42.4            | 44.8                 | 96.6            |
| Arches    | 75.0            | 88.2                 | 67.4            |
| R136      | 65.7            | 82.3                 | 42.2            |

\( \bar{m}_{\text{ran, con, sort}} \) - Maximum-stellar-mass–cluster-mass-relation

### 3 THE MONTE-CARLO SIMULATIONS OF THE IGIMF

We have thus seen that the observational data strongly favour a particular \( m_{\text{max}}(M_{\text{cl}}) \) relation, namely the \( \bar{m}_{\text{max}} \approx m_{\text{ana}} \) relation. This has profound implications for composite stellar populations.

Fig. 12 shows the result of the semi-analytic approach by Kroupa & Weidner (2003) assuming \( \beta = 2.35 \) for the ECMF. The stellar IMF in each cluster has, in all cases, the standard or canonical three-part power-law form (eq. 2). For the minimum “cluster” mass, \( M_{\text{cl,min}} = 5 M_{\odot} \) (a dozen stars), and for the maximal cluster mass, \( M_{\text{cl,max}} = 10^6 M_{\odot} \), are used. The power-law index \( \alpha_{\text{IGIMF}} \) of the resulting semi-analytical IGIMF is well approximated by \( \alpha_{\text{IGIMF}} = 3.00 \) for \( m \approx 1 M_{\odot} \). In Kroupa & Weidner (2003) we already noted that this is probably the reason why the Galactic-field-IMF deduced by Scalo (1986) (\( \alpha_3 \approx 2.7 \)) is steeper than the canonical stellar IMF.

We now apply the Monte-Carlo experiments introduced above to the computation of the IGIMF (eq. 4). The resulting IGIMF is constructed by mass-binning all stars in all clusters. It is shown as a long dashed line in Fig. 12 for constrained sampling and as a short dashed line for sorted sampling. Additionally the ‘input’ standard stellar IMF (solid line) and the semi-analytical IGIMF from Fig. 12 (dotted
3.1 Different ECMF

Following-on from our discussion in Weidner & Kroupa (2005) we explore how a different ECMF affects the IGIMF. Any ECMF with $\beta > 2.35$ will increase the steepening of the IGIMF. However, below about 50 or 100 $M_\odot$ the ECMF is poorly defined observationally (Lada & Lada 2003), and we consider here the implication of a flatter ECMF at low masses while keeping $\beta = 2.35$ for $M_{\text{ecl}} > 50$ or 100 $M_\odot$.

4 DISCUSSION AND CONCLUSIONS

In this contribution a number of Monte-Carlo experiments are conducted in order to constrain the relation between the maximal mass a star can have in a cluster and the mass of the cluster, and to further study the IMF of composite populations, i.e. the integrated galaxial IMF.

We consider three possible ways of forming clusters: (1) Completely randomly. Clusters are filled randomly with stars and then their masses, $M_{\text{ecl}}$, are calculated (random sampling). (2) Cluster-masses are picked from an ECMF
Figure 12. Solid line: Canonical stellar IMF, $\xi(m)$, in logarithmic units and given by the standard three-part power-law form (eq. 2), which has $\alpha_3 = 2.35$ for $m > 0.5 M_\odot$. Dotted line: semi-analytical $\xi_{IGIMF}(m)$ for $\beta = 2.35$. The IMFs are scaled to have the same number of objects in the mass interval $0.01 - 1.0 M_\odot$. Note the turn down near $m_{\text{max}} = 150 M_\odot$ which comes from taking the fundamental upper mass limit explicitly into account (Weidner & Kroupa 2004). Two lines with slopes $\alpha_{\text{line}} = 2.35$ and $\alpha_{\text{line}} = 3.00$ are indicated.

Figure 13. Solid line: Standard stellar IMF with $\alpha_3 = 2.35$ for $m > 1 M_\odot$ (same as in Fig. 12). Dotted line: IGIMF resulting from the semi-analytical approach with $\beta = 2.35$ (as in Fig. 12). Short dashed line: IGIMF obtained from sorted sampling. Long dashed line: IGIMF produced by constrained sampling of stars. As the IMF below $1 M_\odot$ does not change, only the region above $1 M_\odot$ is plotted here.

Figure 14. As Fig. 13 (all lines from Fig. 13 are solid) but in addition the results for two different ECMFs are plotted as dotted and dashed lines. In the dotted case the slope of the ECMF is $\beta_1 = 1$ for clusters below $50 M_\odot$. In the dashed case $\beta_1 = 1$ below $100 M_\odot$. In both cases $\beta_2 = 2.35$ for clusters above these limits. As the IMF below $1 M_\odot$ does not change, only the region above $1 M_\odot$ is shown. The respective models assuming sorted sampling are always steeper than those assuming constrained sampling.

and used as a constrain in constructing the stellar content of each cluster (constrained sampling). (3) Cluster-masses are picked from an ECMF, and the clusters are then filled with stars by randomly selecting from the canonical IMF, sorting the stellar masses in ascending order and constraining their sum to be the cluster mass (sorted sampling). In all cases (1-3), the most massive star, $m_{\text{max}}$, in each cluster is found, and the average or expectation value, $m_{\text{max}}$, is calculated for the ensemble of clusters near $M_{\text{ecf}}$ to give the relations $l, u_{\text{max}}$, con, sort $(M_{\text{ecf}})$, where “l, u” refers to models with or without a physical stellar mass limit of $150 M_\odot$.

The most important and surprising result is that the sorted-sampling algorithm best represents the observational data of young ($\lesssim 3$ Myr) clusters. Constrained and random sampling do not fit the observations.

That our sorted-sampling algorithm for making stars fits the observational maximal-stellar-mass—star-cluster-mass data so well would appear to imply that clusters form in an organised fashion. The physical interpretation of the algorithm (i.e. of the Monte-Carlo integration) is that as a pre-cluster core contracts under self gravity the gas densities increases and local density fluctuations in the turbulent medium lead to low-mass star formation, perhaps similar to what is seen in Taurus-Auriga. As the contraction proceeds and before feedback from young stars begins to disrupt the cloud, star-formation activity increases in further density fluctuations with larger amplitudes thereby forming more massive stars. The process stops when the most massive stars that have just formed supply sufficient feedback energy to disrupt the cloud (Elmegreen 1983). Thus, less-massive pre-cluster cloud-cores would “die” at
a lower maximum stellar mass than more massive cores. But in all cases stellar masses are limited by the physical maximum mass, \( m \leq m_{\text{max}}(M_{\text{cl}}) \leq m_{\text{max}}. \) This scenario is nicely consistent with the hydrodynamic cluster formation calculations presented by \( \text{Bonnell, Bate & Vine (2003)} \) and \( \text{Bonnell, Vine & Bat (2004)}. \) We note here that \( \text{Bonnell, Vine & Bat (2004)} \) found their theoretical clusters to form hierarchically from smaller sub-clusters, and together with continued competitive accretion this leads to the relation \( m_{\text{max}} \propto M_{\text{cl}}^{2/3} \) (eq. 13) in excellent agreement with our compilation of observational data for clusters with masses below \( M_{\text{cl}} = 4000 M_\odot. \) While this agreement is stunning, the detailed outcome of the currently available SPH modelling in terms of stellar multiplicities is not right \( \text{(Goodwin & Kroupa 2003)}, \) and feedback that ultimately dominates the process of star-formation, given the generally low star-formation efficiencies observed in cluster-forming volumes \( \text{(Lada & Lada 2003)}, \) is not yet incorporated in the modelling.

Stellar evolution is the major caveat here. But the comparison of different models (see Appendix C) shows a general agreement of the lifetimes and relevant parameters (mass, \( T_\text{eff} \) and luminosity) for the models considered here. Therefore, not different models but an intrinsically steeper IMF \( (\alpha > 2.8) \) could shift the expectation values for random and constrained sampling into the observed regime. Such a steep IMF may be possible if it is masked by unresolved multiple stars, something we are investigating now.

Furthermore, the Monte-Carlo experiments ascertain the results of \( \text{Kroupa & Weidner (2003)}, \) and \( \text{Weidner & Kroupa (2005)} \) regarding the steep IGIMF, especially so if sorted sampling is used. In the constrained sampling case the IGIMF slope is still steeper than the input IMF but less steep than with sorted sampling. But it should be noted here that a very recent result by \( \text{Elmegreen \& Scala (2006)} \) shows that it is also possible to interpret PDFM variations falsely, as IMF variations when the SFR is assumed to be constant when in reality being burst-like. This result has not yet been implemented in our description of the IGIMF.

In summary:

- There exists a well-defined relation, \( m_{\text{max}} = m_{\text{max}}(M_{\text{cl}}), \) between the most-massive star in a cluster and the cluster mass. The conjecture that a cluster consists of stars randomly picked from an invariant IMF between 0.01 and 150 \( M_\odot \) would therefore appear to be wrong.
- Star clusters appear to form in an ordered fashion, starting with the lowest-mass stars until feedback is able to outweigh the gravitationally-induced formation process.
- IGIMFs must always be steeper for \( m > 1 M_\odot \) than the stellar IMF that results from a local star-formation event.

It will be important to further test the results presented here on the \( m_{\text{max}}(M_{\text{cl}}) \) relation by compiling larger observational samples of young clusters. As this contribution has shown, it appears that the \( m_{\text{max}}(M_{\text{cl}}) \) relation would be rather fundamental to galactic and extragalactic astrophysics.

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Maximum-stellar-mass–cluster-mass-relation

Table A1. Observed mass ranges and the corresponding mean masses for those clusters for which we need to derive cluster masses.

| Cluster   | number of stars observed | % of all stars | total number of stars | mass range $[M_\odot]$ | $m_{\text{mean}}$ $[M_\odot]$ | $M_{\text{ecl}}$ $[M_\odot]$ |
|-----------|--------------------------|---------------|-----------------------|------------------------|-------------------------------|------------------|
| Tau-Aur   | 100                      | 92.8          | 108                   | 0.02 - 2.2             | 0.236                         | 25.5             |
| IC348     | 241                      | 62.5          | 386                   | 0.08 - 6.0             | 0.282                         | 109              |
| Mon R2    | 475                      | 55.2          | 861                   | 0.1 - 10.0             | 0.301                         | 259              |
| NGC2264   | 600                      | 55.3          | 1086                  | 0.1 - 25.0             | 0.327                         | 355              |
| Ber 86    | 340                      | 7.7           | 4421                  | 0.8 - 40.0             | 0.338                         | 1500             |
| NGC2244   | 400                      | 2.2           | 17937                 | 2.0 - 70.0             | 0.348                         | 6240             |
| NGC6611   | 362                      | 0.6           | 56563                 | 5.0 - 85.0             | 0.352                         | 20900            |
| Tr 14/16  | 768                      | 0.6           | 120000                | 5.0 - 120.0            | 0.357                         | 43000            |

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APPENDIX A: CLUSTER AND MAXIMAL STAR MASS DETERMINATION

The masses of the clusters in Tab. 1 are acquired as follows:

In the case of NGC1333, NGC2024, NGC6530, M42, \( \rho \) Ophichundi, \( \sigma \) Orionis, Serpens SVS2, Arches and R136 the authors of the corresponding papers provide the required mass estimates. The masses for NGC2244, NGC2264, NGC6611, IC348, Monoceros R2, Taurus-Auriga, Berkley 86 and Trumpler 14/16 are calculated by determining the fraction (given as a percentage in Tab. A1) of observed stars in comparison with a canonical IMF from 0.01\( M_\odot \) up to the observed upper mass limit. With the fraction of all stars the total number of stars in the cluster is estimated by dividing the observed number of stars by the fraction. The total mass of the cluster, \( M_{\text{ecl}} \), is then calculated by multiplying the total number of stars with the mean stellar mass, \( m_{\text{mean}} \), for the canonical IMF from 0.01\( M_\odot \) to the observed upper mass limit. The relevant values are shown in Tab. A1.

For the maximal stellar masses in these clusters the values within the papers are used whenever possible, which is for NGC2244, NGC2264, NGC6611, M42, \( \rho \) Ophichundi, \( \sigma \) Orionis, Monoceros R2, Berkley 86, Trumpler 14/16, Arches and R136. In the other cases (NGC1333, NGC2024, NGC6530, IC348, Serpens SVS2, Taurus-Auriga) mass estimates are derived from the spectral types of the most luminous members using the spectral-type mass-relationship from Cox (2000).

APPENDIX B: FITTING FORMULAE FOR MASSIVE STAR EVOLUTION

Because the Hurley, Pols & Tout (2000) single stellar evolution (SSE) package is only calibrated for stellar models up to 50\( M_\odot \), additional fitting formulae have been developed.
for more-massive stars. Based on the Schaller et al. (1992) models for 60, 85 and 120 $M_\odot$, functions for $m(t)$, $L(t)$ and $T_{\text{eff}}(t)$ have been obtained:

**B1 Mass Evolution**

As long as the age, $t$ (in Myr), of the star is below $\tau_m$, the main-sequence life-time, the mass-evolution can be described according to,

$$m(t) = m_{\text{ini}} \cdot e^{-\left(\frac{t}{\tau_m}\right)^2}.$$  \hspace{1cm} (B1)

For $\tau_m < t \leq \tau_m + dt$

$$m(t) = \frac{a_2}{dt} \cdot t + b_1.$$  \hspace{1cm} (B2)

For both the parameters are

$$\tau_m = \frac{10^6}{1 + \left(\frac{m_{aa}}{m_{ab}} - \frac{m_{ab}}{m_{ac}}\right)},$$

$$a_1 = \frac{1}{(m_{aa} - \frac{m_{ab}}{m_{ac}}) + m_{ac}},$$

$$a_2 = m_{\text{ini}} \cdot (1 - f_1),$$

$$f_1 = \left(\frac{m_{\text{ini}}}{m_{\text{fin}}}- f_0\right),$$

$$b_1 = a_2 \cdot \left(1 + \frac{1}{\frac{\tau_m}{a_2}}\right),$$

and constants

$$m_{aa} = 48.0,$$

$$m_{ab} = 24.7,$$

$$m_{ac} = 3.15,$$

$$dt = 0.42,$$

$$f_m = 3.523808 \cdot 10^{-3},$$

$$f_b = 6.190428 \cdot 10^{-3},$$

$$t_{af} = 23.5,$$

$$t_{am} = 1.25 \cdot 10^7,$$

$$t_{ab} = 2.5 \cdot 10^6.$$  

When $t$ is larger than $\tau_m + dt$ the star is considered dead. No remnant mass, $T_{\text{eff}}$, or luminosity is assigned. The resulting curves for a 120, 85, 60 and 50 $M_\odot$ star (solid lines) in comparison with the model data (dotted lines) are plotted in Fig. B1.

**B2 Effective Temperature Evolution**

For $t \leq \tau_m$ the equation

$$T_{\text{eff}}(t) = T_{\text{eff,ini}} \cdot e^{-\left(\frac{t}{\tau_m}\right)^2},$$  \hspace{1cm} (B3)

adequately captures the evolution. For $\tau_m < t \leq \tau_m + dt$,

$$T_{\text{eff}}(t) = f_2 \cdot t + (T_{\text{low}} - (f_2 \cdot \tau_m)),$$  \hspace{1cm} (B4)

is used.

The parameters are

$$T_{\text{eff,ini}} = \left\{\begin{array}{ll}
T_{\text{ini},1} \cdot m_{\text{ini}} + T_{\text{ib},1} & m_{\text{ini}} \geq 60M_\odot, \\
T_{\text{ini},2} \cdot m_{\text{ini}} + T_{\text{ib},2} & m_{\text{ini}} < 60M_\odot,
\end{array}\right.$$  

and the constants

$$T_{\text{ini},1} = 4.3 \cdot 10^{-3},$$

$$T_{\text{ib},1} = 4.425,$$

$$T_{\text{ini},2} = 7.3333 \cdot 10^{-4},$$

$$T_{\text{ib},2} = 4.64,$$

$$b_2 = 9.0,$$

$$tta_1 = \frac{T_{\text{peak}} - T_{\text{low}}}{dt},$$

$$T_{\text{low}} = T_{\text{eff,ini}} \cdot e^{-\left(\frac{\tau_m}{a_2}\right)^2},$$

and the constants

$$T_{\text{ini},1} = 4.3 \cdot 10^{-3},$$

$$T_{\text{ib},1} = 4.425,$$

$$T_{\text{ini},2} = 7.3333 \cdot 10^{-4},$$

$$T_{\text{ib},2} = 4.64,$$

$$b_2 = 9.0,$$

$$tta_1 = \frac{T_{\text{peak}} - T_{\text{low}}}{dt},$$

$$T_{\text{low}} = T_{\text{eff,ini}} \cdot e^{-\left(\frac{\tau_m}{a_2}\right)^2},$$

and the constants

$$T_{\text{ini},1} = 4.3 \cdot 10^{-3},$$

$$T_{\text{ib},1} = 4.425,$$

$$T_{\text{ini},2} = 7.3333 \cdot 10^{-4},$$

$$T_{\text{ib},2} = 4.64,$$

$$b_2 = 9.0,$$

$$tta_1 = \frac{T_{\text{peak}} - T_{\text{low}}}{dt},$$

$$T_{\text{low}} = T_{\text{eff,ini}} \cdot e^{-\left(\frac{\tau_m}{a_2}\right)^2},$$

The resulting $T_{\text{eff}}$ fitting curves are plotted in Fig. B2 for the same masses as in Fig. B1.

**B3 Luminosity Evolution**

The luminosity evolution is divided into three parts. For $t \leq \tau_m$,

$$L(t) = \left(\frac{L_{\text{jump}} - L_{\text{ini}}}{\tau_m}\right) \cdot t + L_{\text{ini}}.$$  \hspace{1cm} (B5)

For $\tau_m < t < \tau_b$

$$L(t) = L_{\text{jump}} + (L_{\text{peak}} - L_{\text{jump}}) \cdot \sin(\omega \cdot (t - \tau_m)).$$  \hspace{1cm} (B6)

And finally for $\tau_b < t \leq (\tau_m + dt)$
Figure B2. Effective temperature evolution for massive stars. The line styles are as in Fig. B1. As the $T_{\text{eff}}$ for massive stars are rather similar, the lines in the plot have been shifted upwards as indicated in the plot.

\[ L(t) = a_4 \cdot t + b_3. \]  

Here the parameters are

\[ L_{\text{ini}} = \frac{m_{\text{ini}}}{M_L}, \]
\[ M_L = m_{\text{ini}} \cdot m_{\text{ini}} + m_b, \]
\[ L_{\text{jump}} = \exp \left[ \left( \frac{L_{\text{jump}}}{L_{\text{ini}}} \right) \right] + L_{\text{jc}}, \]
\[ \tau_b = 1 \cdot 10^{-6}, \]
\[ \tau_{bb} = \frac{1}{\tau_{bb} - \left( \frac{m_{\text{ini}}}{m_b} \right)}, \]
\[ L_{\text{peak}} = L_{\text{pa}} \cdot \log_{10} \left( \frac{m_{\text{ini}}}{L_{\text{pe}}} \right) + L_{\text{pc}}, \]
\[ a_4 = \frac{L_{\text{jump}} - L_{\text{low}}}{\tau_{m} + dt - \tau_b}, \]
\[ b_3 = L_{\text{low}} - (a_4 \cdot (\tau_{m} + dt)). \]

The constants are

\[ m_{\text{ini}} = 0.145590532, \]
\[ m_b = 1.722994092, \]
\[ L_{\text{je}} = 9.0 \cdot 10^{-3}, \]
\[ L_{\text{pf}} = 2.5, \]
\[ L_{\text{jc}} = 3.95, \]
\[ \tau_{bb} = 4.6296293 \cdot 10^{-7}, \]
\[ \tau_{bm} = 1.1111 \cdot 10^{-5}, \]
\[ L_{\text{fin}} = 15.7, \]
\[ L_{\text{pa}} = 1.35, \]
\[ L_{\text{pc}} = 4.215, \]
\[ L_{\text{low}} = 5.2. \]

Figure B3. Luminosity evolution for massive stars. The line styles are as in Fig. B1.

Appendix C: Comparison of Different Models

Several different sets of theoretical models for stellar evolution of massive stars exist. Figs. C1, C2, and C3 compare the mass, $T_{\text{eff}}$ and luminosity evolution of three different sets of models (Schaller et al. 1992; Meynet & Maeder 2003; Hurley, Pols & Tout 2000). These models agree qualitatively on the compared stellar properties but show minor differences in the details.
Figure C1. Stellar mass evolution for massive stars. The solid lines are the Geneva models \citep{schaller1992}, while the dashed lines are the Geneva models with rotation \citep{meynet2003}. Thick dashed line: no rotation, medium dashed line: 300 km/s, thin dashed line: 500 km/s, only for $m = 60 \, M_\odot$ and the results from the SSE package \citep{hurley2000} are shown as dotted lines.

Figure C2. Effective temperature evolution for massive stars. The line styles are as in Fig. C1. As the $T_{\text{eff}}$ for massive stars are rather similar, the lines in the plot have been shifted (as indicated in the plot) in the following way: the upper-most by +1.0 dex and the second one by +0.5 dex. The lowest plot has not been shifted.

Figure C3. Luminosity evolution for massive stars. The line styles are as in Fig. C1.