LIPSCHITZ CONTINUITY RESULTS FOR A CLASS OF OBSTACLE PROBLEMS

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Abstract. We prove Lipschitz continuity results for solutions to a class of obstacle problems under standard growth conditions of \( p \)-type, \( p \geq 2 \). The main novelty is the use of a linearization technique going back to [28] in order to interpret our constrained minimizer as a solution to a nonlinear elliptic equation, with a bounded right hand side. This lead us to start a Moser iteration scheme which provides the \( L^{\infty} \) bound for the gradient. The application of a recent higher differentiability result [24] allows us to simplify the procedure of the identification of the Radon measure in the linearization technique employed in [32]. To our knowledge, this is the first result for non-autonomous functionals with standard growth conditions in the direction of the Lipschitz regularity.

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1. Introduction

The aim of this paper is the study of the Lipschitz continuity of the solutions \( u \in W^{1,p}(\Omega) \) to variational obstacle problems of the form

\[
\min \left\{ \int_{\Omega} f(x, Dw) : w \in K_{\psi,\Psi}(\Omega) \right\}.
\]

The function \( \psi : \Omega \to [-\infty, +\infty) \), called obstacle, belongs to the Sobolev space \( W^{1,p}(\Omega) \) and the class \( K_{\psi,\Psi}(\Omega) \) is defined as follows, for a given \( \Psi \in W^{1,p}(\Omega) \) providing the boundary value

\[
K_{\psi,\Psi}(\Omega) := \left\{ w \in W^{1,p}(\Omega) : w(x) \geq \psi(x) \text{ a.e. in } \Omega \text{ and } w|_{\partial\Omega} = \Psi|_{\partial\Omega} \right\}.
\]

In the following, we shall assume that \( K_{\psi,\Psi}(\Omega) \) is not empty.

The study of the regularity theory for obstacle problems is a classical topic in Calculus of Variations and Partial Differential Equations. The first results concerning obstacle problems date back to the sixties to the pioneering work by G. Stampacchia [50] and G. Fichera [26]. The well known fact that solutions to the obstacle problem cannot be of class \( C^2 \)

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independently of the regularity of the obstacle led to the origin of the concept of weak solution and to the theory of variational inequalities, after the fundamental work of J.L. Lions and G. Stampacchia [39]. These problems can generally be solved by means of nowadays classical methods of functional analysis; in this respect the question to establish conditions so that weak solutions can be actually classical ones is of crucial importance (see [5]). For more details we refer to the following monographs [1], [10], [27], [37], [49].

The regularity of solutions to the obstacle problems is influenced by the one of the obstacle: in the linear case, for instance, obstacle and solutions share the same regularity ([6], [9], [37]), while in the nonlinear setting the situation is more involved. This generated along the years an intense research activity aimed to establish the regularity of the solution compared with the regularity assumed for the obstacle. In particular many results concern the Hölder continuity of solutions to the obstacle problem when the obstacle itself is Hölder continuous: see for instance [40], [11], [17], [18], [20], [7], [10]; see also [12], [38], [32], [3], [19], [21], [37], [2], [15], [1], [1], [24].

As far as we know, such analysis has not been carried out in the direction of obtaining Lipschitz regularity results. The reason may be found in the difficulty of choosing the proper test function, in order to start the iteration process, necessary to get the desired regularity; this nowadays classical technique ([45]) has been adapted along the years in several situations, also in the case of non standard growth conditions, see also for instance [41], [42], [43], [44].

In [42] Fuchs and Mingione considered constrained local minimizers of elliptic variational integrals, assuming for the Lagrangian $f$ nearly linear growth. In this respect, they were able to prove several regularity results; the one we are mainly interested in is the Lipschitz continuity of these constrained minimizers. This step has been achieved through a linearization technique, going back to [28], later refined in [16], see also [29], [30], [31].

The main idea of the linearization approach is the following: the constrained minimizer is interpreted as a solution to an elliptic equation with a bounded right hand side, which is obtained after the identification of a suitable Radon measure. At this point the main obstruction is overcome, and higher regularity for the solution to the obstacle problem can be shown.

We would like to remark that in [42] a lot of effort has been employed to identify the Radon measure and the authors explicitly say that this procedure could be significantly simplified if we would have a priori proved higher differentiability for local minimizers of the obstacle problem.

However this is exactly the case for our specific situation: in a very recent paper [24] Eleuteri and Passarelli di Napoli were able to establish the higher differentiability of integer and fractional order of the solutions to a class of obstacle problems (involving $p$–harmonic operators) assuming that the gradient of the obstacle possesses an extra (integer or fractional) differentiability property. We will make use of this result to simplify the procedure outlined in [42] and obtain the desired result.

The paper is organized as follows: Section 2 contains some notations and the statement of the main result; Section 3 contains the detailed construction of the linearization procedure while the conclusion comes with Section 4 where the a priori estimate is presented.
We would like to mention that, in order to let the linearization technique be the main focus of our work, we assumed standard growth conditions for the lagrangians of our integral functionals and \( L^\infty \) dependence with respect to the \( x \)-variable. In a similar way, we assume on the gradient of the obstacle \( W^{1,\infty} \) regularity. We expect that future improvements can be obtained both in the direction of considering non-standard growth conditions (see for instance the recent papers [23], [13], [15]), for which a counterpart of [24] would be needed in the nonstandard setting, see [33]) and in the direction of weakening the assumptions on the obstacle and on the partial map \( x \mapsto D\xi f(x,\xi) \). We plan to continue our research in these directions.

2. Notations and statement of the main result

For a domain \( \Omega \subset \mathbb{R}^n, n \geq 2 \), and functions \( u : \Omega \to \mathbb{R} \), we consider the variational integral

\[
\mathcal{F}(u) := \int_{\Omega} f(x, Du) \, dx,
\]

where \( f : \Omega \times \mathbb{R}^n \to [0, +\infty) \) is a Carathéodory function which is convex and of class \( C^2 \) with respect to the second variable. Moreover, we suppose that there exist two positive constants \( \lambda, \Lambda \) such that, for all \( \mu, \xi \in \mathbb{R}^n, \mu = \mu_i, \xi = \xi_i, i = 1, 2, \ldots, n, \) a.e. in \( \Omega \) and for \( p \geq 2 \)

\[
\lambda (1 + |\xi|^2)^\frac{p-2}{2} |\mu|^2 \leq \sum_{i,j} f_{\xi_i\xi_j}(x, \xi) \mu_i \mu_j, \tag{2.2}
\]

\[
|f_{\xi_i\xi_j}(x, \xi)| \leq \Lambda (1 + |\xi|^2)^\frac{p-2}{2}, \tag{2.3}
\]

\[
|f_{\xi x}(x, \xi)| \leq \Lambda (1 + |\xi|^2)^\frac{p-1}{2}. \tag{2.4}
\]

Throughout the paper we will denote by \( B_\rho \) and \( B_R \) balls of radii respectively \( \rho \) and \( R \) (with \( \rho < R \)) compactly contained in \( \Omega \) and with the same center, let us say \( x_0 \in \Omega \).

Our main result reads as follows

**Theorem 2.1.** Let \( u \in K_{\psi, \Psi}(\Omega) \) be a solution to the obstacle problem \( (1.1) \), under the assumptions (2.2)-(2.4). If \( \psi \in W^{2,\infty}_{\text{loc}}(\Omega) \), then \( u \in W^{1,\infty}_{\text{loc}}(\Omega) \) and the following estimate

\[
\| (1 + |Du|^2)^\frac{1}{2} \|_{L^\infty(B_\rho)} \leq \frac{C}{(R - \rho)^\beta} \left( \int_{B_R} (1 + |Du|^2)^\frac{p}{2} \, dx \right)^\frac{1}{p} \tag{2.5}
\]

holds for every \( 0 < \rho < R \) and with positive constants \( C \) and \( \beta \) depending on \( n, p, \lambda, \Lambda, R, \rho \) and the local bounds for \( \| D\psi \|_{W^{1,\infty}} \).

**Remark 2.2.** Due to the local nature of our main result, we do not assume further regularity on the boundary datum \( \Psi \).

3. Proof of Theorem 2.1: the linearization procedure

Consider a smooth function \( h_\varepsilon : (0, \infty) \to [0, 1] \) such that \( h'(s) \leq 0 \) for all \( s \in (0, \infty) \) and

\[
h(s) = \begin{cases}
1 & \text{for } s \leq \varepsilon, \\
0 & \text{for } s \geq 2\varepsilon.
\end{cases}
\]
We remark that \( u \in W^{1,p}(\Omega) \) is a solution to the obstacle problem in \( K_{\psi, \Psi}(\Omega) \) if and only if \( u \in K_{\psi, \Psi} \) solves the following variational inequality

\[
\int_{\Omega} D\xi f(x,Du) \cdot D(\varphi - u) \, dx \geq 0,
\]

for all \( \varphi \in K_{\psi, \Psi} \). In particular the function

\[
\varphi = u + t \cdot \eta \cdot h_\varepsilon(u - \psi)
\]

with \( \eta \in C^1_0(\Omega), \eta \geq 0 \) and \( 0 < t << 1 \), is such that \( \varphi \in K_{\psi, \Psi} \). Using \( \varphi \) as test function, in the variational inequality (3.1) gives

\[
\int_{\Omega} D\xi f(x,Du) \cdot D(\eta h_\varepsilon(u - \psi)) \, dx \geq 0 \quad \forall \eta \in C^1_0(\Omega).
\]

Since \( \eta \mapsto L(\eta) = \int_{\Omega} D\xi f(x,Du) \cdot D(\eta h_\varepsilon(u - \psi)) \, dx \) is a bounded positive linear functional, by the Riesz representation theorem there exists a nonnegative measure \( \lambda_\varepsilon \) such that

\[
\int_{\Omega} D\xi f(x,Du) \cdot D(\eta h_\varepsilon(u - \psi)) \, dx = \int_{\Omega} \eta \, d\lambda_\varepsilon \quad \forall \eta \in C^1_0(\Omega).
\]

We show that the measure \( \lambda_\varepsilon \) is independent from \( \varepsilon \): indeed let \( \varepsilon < \varepsilon' \). The variation

\[
\varphi_+ = u + t \cdot \eta \cdot (h_\varepsilon - h_{\varepsilon'})(u - \psi)
\]

is admissible for all \( \eta \in C^1_0(\Omega) \) and \( 0 < t << 1 \). This gives

\[
\int_{\Omega} D\xi f(x,Du) \cdot D(\eta h_\varepsilon(u - \psi)) \, dx \geq 0 \quad \forall \eta \in C^1_0(\Omega).
\]

It’s easy to see that \( (h_\varepsilon - h_{\varepsilon'})(s) \) is supported on \( [\varepsilon, 2\varepsilon'] \), which implies

\[
\int_{\{\varepsilon < u - \psi < 2\varepsilon'\}} D\xi f(x,Du) \cdot D(\eta h_\varepsilon(u - \psi)) \, dx \geq 0 \quad \forall \eta \in C^1_0(\Omega).
\]

Then, for \( t < \frac{\varepsilon}{2} \), also the variation

\[
\varphi_- = u - t \cdot \eta \cdot (h_\varepsilon - h_{\varepsilon'})(u - \psi)
\]

is an admissible test function; by the same argument we get

\[
\int_{\{\varepsilon < u - \psi < 2\varepsilon'\}} D\xi f(x,Du) \cdot D(\eta h_\varepsilon(u - \psi)) \, dx \leq 0 \quad \forall \eta \in C^1_0(\Omega);
\]

together with the previous result, this implies

\[
\int_{\Omega} D\xi f(x,Du) \cdot D(\eta h_\varepsilon(u - \psi)) \, dx = 0 \quad \forall \eta \in C^1_0(\Omega),
\]

that is equivalent to

\[
\int_{\Omega} \eta \, d\lambda_\varepsilon = \int_{\Omega} \eta \, d\lambda_{\varepsilon'} \quad \forall \eta \in C^1_0(\Omega).
\]

Now we can obtain equality between measures by a standard approximation argument, let \( \phi \in C^\infty_0(\Omega) \) be a smooth kernel with \( \phi \geq 0 \) and \( \int_{\Omega} \phi \, dx = 1 \), consider the corresponding
family of mollifiers \((\phi_s)_{s>0}\) and any compact set \(K\) such that \(K \subset \Omega\). For every \(s > 0\) such that \(s < \text{dist}(K, \Omega)\) we take in equation \((3.2)\)
\[
\eta = \chi_K \ast \phi_s
\]

letting \(s \searrow 0\) gives
\[
\int_K d\lambda_\varepsilon = \int_K d\lambda_\varepsilon',
\]
in conclusion \(\lambda_\varepsilon = \lambda_\varepsilon'\).

At this point we can rewrite our representation equation without the \(\varepsilon\) dependence on the measure
\[
\int_\Omega D_\xi f(x, Du) \cdot D(\eta h_\varepsilon(u - \psi)) \, dx = \int_\Omega \eta \, d\lambda \quad \forall \eta \in C_0^1(\Omega).
\]

By Theorem 2.1 in [24], that holds in particular under the assumptions \((2.2)-(2.4)\), we have
\[
V_p(Du) := (1 + |Du|^2)^{\frac{p-2}{2}} Du \in W^{1,2}_{\text{loc}}(\Omega),
\]

therefore we can integrate by parts and get
\[
- \int_\Omega \text{div}(D_\xi f(x, Du)) \eta h_\varepsilon(u - \psi) \, dx = \int_\Omega \eta \, d\lambda \quad \forall \eta \in C_0^1(\Omega).
\]

Now, in order to identify the measure \(\lambda\), we may pass to the limit as \(\varepsilon \searrow 0\)
\[
- \int_\Omega \text{div}(D_\xi f(x, Du)) \chi_{[u=\psi]} \eta \, dx = \int_\Omega \eta \, d\lambda \quad \forall \eta \in C_0^1(\Omega). \quad (3.3)
\]

Let us introduce
\[
g := \text{div}(D_\xi f(x, Du)) \chi_{[u=\psi]}; \quad (3.4)
\]

then the equation above takes the form:
\[
- \int_\Omega g \eta \, dx = \int_\Omega \eta \, d\lambda \quad \forall \eta \in C_0^1(\Omega).
\]

Now observing that
\[
\int_\Omega D_\xi f(x, Du) \cdot D(\eta(1 - h_\varepsilon)(u - \psi)) \, dx = 0 \quad \forall \eta \in C_0^1(\Omega)
\]
since \((1 - h_\varepsilon)(s)\) has support \([\varepsilon, +\infty)\), combining our results we get
\[
\int_\Omega D_\xi f(x, Du) \cdot D\eta \, dx = - \int_\Omega g \eta \, dx \quad \forall \eta \in C_0^1(\Omega). \quad (3.5)
\]

We are left to obtain an \(L^\infty\) estimate for \(g\): since \(Du = D\psi\) a.e. on the contact set, by \((2.3)\) and \((2.4)\) and the assumption \(D\psi \in W^{1,\infty}_{\text{loc}}(\Omega; \mathbb{R}^n)\), we have
\[
|g| = |\text{div}(D_\xi f(x, Du)) \chi_{[u=\psi]}| = |\text{div}(D_\xi f(x, D\psi))| \leq \sum_{k=1}^n |f_{\xi_k x_k}(x, D\psi)| + \sum_{k,i=1}^n |f_{\xi_k \xi_i}(x, D\psi)\psi_{x_k x_i}|
\]
\[
\leq \Lambda(1 + |D\psi|^2)^{\frac{p-1}{2}} + \Lambda(1 + |D\psi|^2)^{\frac{p-2}{2}} |D^2\psi|
\]

that is \(g \in L^\infty_{\text{loc}}(\Omega)\).
4. Proof of Theorem 2.1: a priori estimate

Our starting point is now (3.3). We make use of the supplementary assumption $u \in W^{1,\infty}_\text{loc}(\Omega)$, which is needed in order to let (3.3) to be satisfied; this assumption will be removed by means of the approximation procedure. By the standard techniques of the difference quotients (see for instance [35], [36]) we get

$$u \in W^{2,2}_\text{loc}(\Omega) \cap (1 + |Du|^2) \frac{d^2}{dx^2} |D^2u|^2 \in L^1_\text{loc}(\Omega) \quad (4.1)$$

so that the “second variation” system holds

$$\int_\Omega \left( \sum_{i,j=1}^n f_{xi,j}(x, Du) u_{x,x_i} D_{x_i} \varphi \, dx + \sum_{i=1}^n f_{xi,x_i}(x, Du) D_{x_i} \varphi \right) = \int_\Omega g_{x_i} \varphi \, dx, \quad (4.2)$$

for all $s = 1, \ldots, n$ and for all $\varphi \in W^{1,p}_0(\Omega)$; here $g$ is the function which has been introduced in (3.3). We fix $0 < \rho < R$ with $B_R$ compactly contained in $\Omega$ and we choose $\eta \in C^1_0(\Omega)$ such that $0 \leq \eta \leq 1$, $\eta \equiv 1$ on $B_{2\rho}$, $\eta \equiv 0$ outside $B_R$ and $|D\eta| \leq \frac{C}{(R-\rho)}$. We test (4.2) with $\varphi = \eta^2 (1 + |Du|^2)^\gamma u_{x,s}$, for some $\gamma \geq 0$ (which is possible due to our a priori assumption (4.1)) so that

$$D_{x_i} \varphi = 2\eta\eta_{x_i} (1 + |Du|^2)^\gamma u_{x,s} + 2\eta^2 \gamma (1 + |Du|^2)^{\gamma-1} |Du| D_{x_i}(|Du|) u_{x,s} + \eta^2 (1 + |Du|^2)^\gamma u_{x,x_i}$$

Inserting in (4.2) we get:

$$0 = \int_\Omega \sum_{i,j=1}^n f_{xi,j}(x, Du) u_{x,x_i} 2\eta\eta_{x_i} (1 + |Du|^2)^\gamma u_{x,s} \, dx + \int_\Omega \sum_{i,j=1}^n f_{xi,j}(x, Du) u_{x,x_i} \eta^2 (1 + |Du|^2)^\gamma u_{x,x_i} \, dx + \int_\Omega \sum_{i=1}^n f_{xi,x_i}(x, Du) 2\eta^2 \gamma (1 + |Du|^2)^{\gamma-1} |Du| D_{x_i}(|Du|) u_{x,s} \, dx$$

$$+ \int_\Omega \sum_{i=1}^n f_{xi,x_i}(x, Du) 2\eta^2 \gamma u_{x,x_i} \, dx + \int_\Omega \sum_{i=1}^n f_{xi,x_i}(x, Du) 2\eta^2 \gamma (1 + |Du|^2)^{\gamma-1} |Du| D_{x_i}(|Du|) u_{x,s} \, dx$$

$$- \int_\Omega g\eta_2 (1 + |Du|^2) \gamma u_{x,s} \, dx$$

$$- \int_\Omega g\eta^2 \gamma (1 + |Du|^2)^{\gamma-1} |Du| D_{x_i}(|Du|) u_{x,s} \, dx$$

$$- \int_\Omega g\eta^2 (1 + |Du|^2)^\gamma u_{x,x_i} \, dx$$

$$=: I_{1,s} + I_{2,s} + I_{3,s} + I_{4,s} + I_{5,s} + I_{6,s} + I_{7,s} + I_{8,s} + I_{9,s}.$$
We sum in the previous equation all terms with respect to $s$ from 1 to $n$, and we denote by $I_1 - I_9$ the corresponding integrals. In the sequel constants will be denoted by $C$, regardless their actual value. Only the relevant dependencies will be highlighted.

By the Cauchy-Schwarz inequality, the Young inequality and (2.3), we have

\[
|I_1| = \left| \int_\Omega 2\eta (1 + |Du|^2)^{\gamma} \sum_{i,j,s=1}^{n} f_{\xi_i,\xi_j}(x, Du) u_{x,s,x} \eta_{x_i} u_{x_s} \, dx \right|
\]

\[
= \left| \int_\Omega 2\eta (1 + |Du|^2)^{\gamma} \times \sum_{s=1}^{n} \left( \sum_{i,j=1}^{n} f_{\xi_i,\xi_j}(x, Du) \eta_{x_i} \eta_{x_j} u_{x_s}^2 \right)^{1/2} \left( \sum_{i,j=1}^{n} f_{\xi_i,\xi_j}(x, Du) u_{x_i,x_s} u_{x_j,x_s} \right)^{1/2} \, dx \right|
\]

\[
\leq \int_\Omega 2\eta (1 + |Du|^2)^{\gamma} \times \left\{ \sum_{i,j,s=1}^{n} f_{\xi_i,\xi_j}(x, Du) \eta_{x_i} \eta_{x_j} u_{x_s}^2 \right\}^{1/2} \left\{ \sum_{i,j,s=1}^{n} f_{\xi_i,\xi_j}(x, Du) u_{x_i,x_s} u_{x_j,x_s} \right\}^{1/2} \, dx
\]

\[
\leq C \int_\Omega (1 + |Du|^2)^{\gamma} \sum_{i,j,s=1}^{n} f_{\xi_i,\xi_j}(x, Du) \eta_{x_i} \eta_{x_j} u_{x_s}^2 \, dx
\]

\[
+ \frac{1}{4} \int_\Omega \eta^2 (1 + |Du|^2)^{\gamma} \sum_{i,j,s=1}^{n} f_{\xi_i,\xi_j}(x, Du) u_{x_i,x_s} u_{x_j,x_s} \, dx
\]

\[
\leq C(\Lambda) \int_\Omega \Lambda (1 + |Du|^2)^{\frac{p-2}{2} + \gamma} \sum_{i,j,s=1}^{n} \eta_{x_i} \eta_{x_j} u_{x_s}^2 \, dx
\]

\[
+ \frac{1}{4} \int_\Omega \eta^2 (1 + |Du|^2)^{\gamma} \sum_{i,j,s=1}^{n} f_{\xi_i,\xi_j}(x, Du) u_{x_i,x_s} u_{x_j,x_s} \, dx
\]

\[
\leq C(n, \Lambda) \int_\Omega |D\eta|^2 (1 + |Du|^2)^{\frac{p-2}{2} + \gamma} \sum_{s=1}^{n} u_{x_s}^2 \, dx
\]

\[
+ \frac{1}{4} \int_\Omega \eta^2 (1 + |Du|^2)^{\gamma} \sum_{i,j,s=1}^{n} f_{\xi_i,\xi_j}(x, Du) u_{x_i,x_s} u_{x_j,x_s} \, dx
\]

\[
\leq C \int_\Omega |D\eta|^2 (1 + |Du|^2)^{p/2 + \gamma} \, dx
\]

\[
+ \frac{1}{4} \int_\Omega \eta^2 (1 + |Du|^2)^{\gamma} \sum_{i,j,s=1}^{n} f_{\xi_i,\xi_j}(x, Du) u_{x_i,x_s} u_{x_j,x_s} \, dx.
\]
On the other hand, using (2.2) and the fact that \( D_x (|Du|) |Du| = \sum_{k=1}^{n} u_{x_j x_k} u_{x_k} \), we can estimate the term \( I_3 \) as follows:

\[
|I_3| = \int_{\Omega} \sum_{i,j,s=1}^{n} f_{\xi_i \xi_j}(x, Du) u_{x_j x_s} \left[ 2 \eta^2 \gamma (1 + |Du|^2)^{\gamma - 1} D_{x_i} (|Du|) |Du| \right] u_{x_s} \, dx \\
\geq 2 \gamma \int_{\Omega} \eta^2 |Du|^{2 \gamma - 1} \sum_{i,j,s=1}^{n} f_{\xi_i \xi_j}(x, Du) D_{x_i} (|Du|) u_{x_j x_s} \, dx \\
= 2 \gamma \int_{\Omega} \eta^2 |Du|^{2 \gamma - 1} \sum_{i,j=1}^{n} f_{\xi_i \xi_j}(x, Du) D_{x_i} (|Du|) \left( \sum_{s=1}^{n} u_{x_j x_s} u_{x_s} \right) \, dx \\
= 2 \gamma \int_{\Omega} \eta^2 |Du|^{2 \gamma} \sum_{i,j=1}^{n} f_{\xi_i \xi_j}(x, Du) D_{x_i} (|Du|) D_{x_j} (|Du|) \, dx \\
\geq 2 \gamma \int_{\Omega} \eta^2 |Du|^{2 \gamma} |D(|Du|)|^2 (1 + |Du|^2)^{\frac{p - 2}{2} - \gamma} \, dx \geq 0.
\]

We can estimate the fourth and the fifth term by the Cauchy-Schwarz and the Young inequalities, together with (2.4), as follows

\[
|I_4| = 2 \int_{\Omega} \eta (1 + |Du|^2)^{\gamma} \sum_{i,s=1}^{n} f_{\xi_i x_s}(x, Du) \eta_{x_i} u_{x_s} \, dx \\
\leq 2 \Lambda \int_{\Omega} \eta (1 + |Du|^2)^{\gamma + \frac{n}{p - 1}} \sum_{i,s=1}^{n} |\eta_{x_i} u_{x_s}| \, dx \\
\leq C(\Lambda) \int_{\Omega} \eta |D\eta| |Du| (1 + |Du|^2)^{\gamma + \frac{n}{p - 1}} \, dx \\
\leq C \int (\eta^2 + |D\eta|^2)(1 + |Du|^2)^{\gamma + \frac{n}{p - 1}} \, dx,
\]

and also:

\[
|I_5| = \left| \int_{\Omega} \eta^2 (1 + |Du|^2)^{\gamma} \sum_{i,s=1}^{n} f_{\xi_i x_s}(x, Du) u_{x_j x_s} \, dx \right| \\
\leq \Lambda \int_{\Omega} \eta^2 (1 + |Du|^2)^{\gamma + \frac{n}{p - 1}} \sum_{i,s=1}^{n} u_{x_j x_s} \, dx \\
= \Lambda \int_{\Omega} \eta^2 (1 + |Du|^2)^{\gamma + \frac{n}{p - 1}} |D^2 u| \, dx \\
\leq \frac{1}{4} \int_{\Omega} \eta^2 (1 + |Du|^2)^{\frac{p - 2}{2} + \gamma} |D^2 u|^2 \, dx + C \int_{\Omega} \eta^2 (1 + |Du|^2)^{\frac{p}{2} + \gamma} \, dx.
\]
Finally, by similar arguments, the Cauchy-Schwarz inequality, \( \text{(2.3)} \) and \( |D(|D u|)| \leq |D^2 u| \) give

\[
|I_6| = 2 \gamma \int \sum_{i,s=1}^{n} f_{\xi_i x_s}(x, D u) \eta^2(1 + |D u|^2)^{\gamma-1} |D u| D_{x_i}(|D u|) u_{x_s} \, dx
\]

\[
= 2 \gamma \int \eta^2(1 + |D u|^2)^{\gamma-1} |D u| \sum_{i,s=1}^{n} f_{\xi_i x_s}(x, D u) D_{x_i}(|D u|) u_{x_s} \, dx
\]

\[
\leq 2 \gamma \int \eta^2(1 + |D u|^2)^{\gamma-\frac{1}{2}+\frac{\gamma-1}{2}} \sum_{i,s=1}^{n} D_{x_i}(|D u|) u_{x_s} \, dx
\]

\[
\leq 2 \gamma \Lambda \int \eta^2(1 + |D u|^2)^{\gamma-\frac{1}{2}+\frac{\gamma-1}{2}} \sum_{i,s=1}^{n} D_{x_i}(|D u|) u_{x_s} \, dx
\]

\[
\leq 2 n \gamma \Lambda \int \eta^2(1 + |D u|^2)^{\gamma-\frac{1}{2}+\frac{\gamma-1}{2}} |D(|D u|)||D u| \, dx
\]

\[
\leq 2 n \gamma \Lambda \int \eta^2(1 + |D u|^2)^{\gamma+\frac{\gamma-1}{2}} |D^2 u| \, dx
\]

\[
\leq \frac{1}{4} \int \eta^2|D^2 u|^2(1 + |D u|^2)^{\frac{\gamma-1}{2}+\gamma} \, dx + C \gamma^2 \int \eta^2(1 + |D u|^2)^{\frac{\gamma-1}{2}+\gamma} \, dx,
\]

where the constant \( C \) depends only on \( n, p, \Lambda \) but it is independent of \( \gamma \).

Let us now deal with the terms containing the function \( g \). We use the bound

\[
\|g\|_{L_\infty^\infty(\Omega)} \leq C,
\]

established in Section \( \text{3} \) here the constant \( C \) is independent of \( n \) and \( \gamma \).

We first have

\[
|I_7| \leq \|g\|_{L_\infty(B_R)} \int_\Omega |D u|(1 + |D u|^2)^{\gamma} |D u| \, dx
\]

\[
\leq C \int_\Omega |D u|^2(1 + |D u|^2)^{\gamma+\gamma/2} \, dx.
\]

Arguing similarly as in the estimate of \( I_6 \) we deduce

\[
|I_8| \leq 2 \gamma \int |g| \eta^2(1 + |D u|^2)^{\gamma-1} |D(|D u|)| |D u|^2 \, dx
\]

\[
\leq \frac{1}{4} \int \eta^2|D^2 u|^2(1 + |D u|^2)^{\frac{\gamma-1}{2}+\gamma} \, dx + C \gamma^2 \|g\|_{L_\infty^\infty(B_R)}^2 \int_\Omega \eta^2(1 + |D u|^2)^{\gamma+\gamma} \, dx.
\]

Finally

\[
|I_9| \leq \frac{1}{4} \int \eta^2|D^2 u|^2(1 + |D u|^2)^{\frac{\gamma-1}{2}+\gamma} \, dx + C \|g\|_{L_\infty^\infty(B_R)}^2 \int_\Omega \eta^2(1 + |D u|^2)^{\gamma+\gamma} \, dx.
\]

We remark that in the estimates of the terms \( I_7 - I_9 \) we used in an essential way the fact \((1 + |D u|^2) \geq 1\). It is possible to overcome this limitation and dealing also with the degenerate
case by proceeding as in [23].

Summing up and using (5.2) we obtain
\[
\int_{\Omega} \eta^2 (1 + |Du|^2)^{\frac{n-2}{2} + \gamma} |D^2u|^2 \, dx \leq C (1 + \gamma^2) \int_{\Omega} (\eta^2 + |D\eta|^2) (1 + |Du|^2)^{\frac{n}{2} + \gamma} \, dx,
\]
for any $0 < \rho < R$, where the constant $C$ depends on $\lambda, \Lambda, n, p$ but is independent of $\gamma$. On the other hand
\[
\int_{\Omega} |D \left[ \eta (1 + |Du|^2)^{\frac{n}{2} + \gamma} \right] |^2 \, dx \\
\leq \int_{\Omega} |D\eta|^2 (1 + |Du|^2)^{\frac{n}{2} + \gamma} \, dx + \int_{\Omega} \eta^2 (1 + |Du|^2)^{\frac{n}{2} + \gamma} |Du|^2 \, dx \\
\leq C (1 + \gamma^2) \int_{\Omega} (\eta^2 + |D\eta|^2) (1 + |Du|^2)^{\frac{n}{2} + \gamma} \, dx.
\]

Now, let $2^* = \frac{2n}{n-2}$ for $n > 2$ while $2^*$ equal to any fixed real number greater than 2, if $n = 2$. By the Sobolev’s inequality there exists a constant $C$ such that
\[
\left( \int_{\Omega} \eta^2 (1 + |Du|^2)^{\frac{n}{2} + \gamma}^2 \, dx \right)^{\frac{2}{n}} \leq C \int_{\Omega} |D \left[ \eta (1 + |Du|^2)^{\frac{n}{2} + \gamma} \right] |^2 \, dx \\
\leq C (1 + \gamma^2) \int_{\Omega} (\eta^2 + |D\eta|^2) (1 + |Du|^2)^{\frac{n}{2} + \gamma} \, dx.
\]

5. **Proof of Theorem 2.1**: an iteration scheme

By the definition of $\eta$ we obtain
\[
\left( \int_{B_{R_k}} (1 + |Du|^2)^{\frac{n}{2} + \gamma} \, dx \right)^{\frac{2}{n}} \leq \frac{C(1 + \gamma^2)}{(R - \rho)^2} \int_{B_R} (1 + |Du|^2)^{\frac{n}{2} + \gamma} \, dx,
\]
for any $B_{R_k}$ and $B_R$ compactly contained in $\Omega$ with $\rho < R$ and the same center. Let us define the sequence $(\gamma_k)_k$ by induction as follows
\[
\gamma_1 = 0; \quad \gamma_{k+1} = \left( \gamma_k + \frac{p}{2} \right) \frac{2^*}{2} - \frac{p}{2}, \quad \forall k \geq 1.
\]
If $(\gamma_k)_k$ is the sequence defined in (5.2), then the following representation formula holds
\[
\gamma_k = \frac{p}{2} \left( \left( \frac{2^*}{2} \right)^{k-1} - 1 \right),
\]
the proof easily follows by induction.

Fix two rays $R_0$ and $\rho_0$ such that $0 < \rho_0 < R_0 < \rho_0 + 1$, let us define $R_k = \rho_0 + (R_0 - \rho_0)2^{-k}$ for every $k \geq 1$. Since $R_{k+1} < R_k \forall k \geq 1$, we are allowed to insert $\rho = R_{k+1}$ and $R = R_k$ in estimate (5.1), we also take $\gamma = \gamma_k$, where $(\gamma_k)_k$ has been defined in (5.2). With these choices, by (5.1) we obtain
\[
\left( \int_{B_{R_{k+1}}} (1 + |Du|^2)^{\frac{n}{2} + \gamma_k} \, dx \right)^{\frac{2}{n}} \leq \frac{C(1 + \gamma_k^2)}{(R_k - R_{k+1})^2} \int_{B_{R_k}} (1 + |Du|^2)^{\frac{n}{2} + \gamma_k} \, dx
\]
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\[ \frac{C(1 + \gamma_k^2)4^{k+1}}{(R_0 - \rho_0)^2} \int_{B_0} (1 + |Du|^2)^{\frac{p}{2} + \gamma_k} \, dx, \]  

(5.4)

for every \( k \geq 1 \). Since the constant \( C \) was independent of \( \gamma \), it is clear that now \( C \) is independent on \( k \). Let us define

\[ A_k = \left( \int_{B_{R_k}} (1 + |Du|^2)^{\frac{p}{2} + \gamma_k} \, dx \right)^{\frac{1}{\frac{p}{2} + \gamma_k}}, \quad \forall k \geq 1. \]  

(5.5)

The estimate (5.4) directly gives us an inductive estimate on \( (A_k)_k \). By definition (5.2) of \( \gamma_k \), and by (5.4) we have

\[ A_{k+1} = \left( \int_{B_{R_{k+1}}} (1 + |Du|^2)^{\frac{p}{2} + \gamma_{k+1}} \, dx \right)^{\frac{1}{\frac{p}{2} + \gamma_{k+1}}}, \]  

(5.6)

for every \( k \geq 1 \). By iterating (5.6) we obtain

\[ A_{k+1} \leq A_1 \cdot \prod_{i=1}^{k} \left[ \frac{C(1 + \gamma_i^2)4^{i+1}}{(R_0 - \rho_0)^2} \right]^{\frac{1}{\frac{p}{2} + \gamma_i}} \leq C \cdot A_1 \cdot (R_0 - \rho_0)^{-\sum_{i=1}^{\infty} 1/(p/2 + \gamma_i)}, \]  

(5.7)

for every \( k \geq 1 \). Here we used the fact that \( \prod_{i=1}^{k} \left[ C(1 + \gamma_i^2)4^{i+1} \right]^{\frac{1}{\frac{p}{2} + \gamma_i}} \) is bounded, indeed

\[ \prod_{i=1}^{\infty} \left[ C(1 + \gamma_i^2)4^{i+1} \right]^{\frac{1}{\frac{p}{2} + \gamma_i}} = \exp \left( \sum_{i=1}^{\infty} \frac{\ln(C(1 + \gamma_i^2)4^{i+1})}{p + 2\gamma_i} \right) < +\infty, \]

since the series is convergent by formula (5.3). Since \( \rho_0 < R_k < R_0 \), by exploiting \( A_1 \) in (5.7), for every \( k \geq 2 \) we deduce that

\[ \left( \int_{B_{\rho_0}} (1 + |Du|^2)^{\frac{p}{2} + 2\gamma_k} \, dx \right)^{\frac{1}{\frac{p}{2} + 2\gamma_k}} \leq \frac{C}{(R_0 - \rho_0)^{\beta}} \left( \int_{B_{\rho_0}} (1 + |Du|^2)^{\frac{p}{2}} \, dx \right)^{\frac{1}{\frac{p}{2}}}, \]  

(5.8)

where we defined

\[ \beta := \sum_{i=1}^{\infty} \frac{1}{\frac{p}{2} + \gamma_i} = \frac{2^*}{p(2^*/2) - p}. \]

Since \( p + 2\gamma_k \to +\infty \), as \( k \to +\infty \) it is clear that the left hand side of (5.8) converges to the essential supremum of \( (1 + |Du|^2)^{\frac{p}{2}} \) in \( B_{\rho_0} \). Thus, provided the further regularity that
we will remove in the next section with an approximation procedure, we conclude the proof of Theorem 2.1.

6. Proof of Theorem 2.1: approximation

First of all we state an approximation theorem for $f$ through a suitable sequence of regular functions. Let $B$ be the unit ball of $\mathbb{R}^n$ centered in the origin and consider a positive decreasing sequence $\varepsilon_\ell \to 0$. Let $\phi \in C^\infty_0(B)$ be a smooth symmetric kernel with $\phi \geq 0$ and $\int_B \phi \, dx = 1$.

Let us define
\[
f_\ell(x, \xi) = \int_{B \times B} \phi(y) \phi(\eta) f(x + \varepsilon_\ell y, \xi + \varepsilon_\ell \eta) \, d\eta \, dy,
\]
and consequently
\[
f_{\ell k}(x, \xi) = f_\ell(x, \xi) + \frac{1}{k} (1 + |\xi|^2)^{\frac{p}{2}}.
\]

The proof of the following theorem follows from similar arguments as in [23].

**Theorem 6.1.** Let $f$ satisfy the growth conditions (2.2), (2.3), (2.4), and let $f(x, \xi)$ be a convex $C^2$ function in its second argument. Then the sequence $f_{\ell k} : \Omega \times \mathbb{R}^n \to [0, +\infty)$ defined in (6.2), convex in the last variable is such that $f_{\ell k} \to f$ pointwise as $(\ell, k) \to +\infty$ for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^n$. Moreover, $f_{\ell k} \to f$ uniformly in $\Omega_0 \times K$ for every $\Omega_0 \subset\subset \Omega$ and $K$ compact set of $\mathbb{R}^n$. Moreover:

- There exist constants $M_1, M_2 > 0$, independent of $k, \ell$ such that
  \[M_1 (1 + |\xi|^2)^{\frac{p}{2}} \leq f_{\ell k}(x, \xi) \leq M_2 (1 + |\xi|^2)^{\frac{p}{2}} \text{ for a.e. } x \in \Omega, \text{ for all } \xi \in \mathbb{R}^n,
  \]
- there exist constants $\lambda_1, \Lambda_1 > 0$ such that for all $(x, \xi) \in \Omega \times \mathbb{R}^n$ and $\mu \in \mathbb{R}^n$
  \[
  \lambda_1 (1 + |\xi|^2)^{\frac{p}{2}} |\mu|^2 \leq \sum_{i,j} f_{\ell k}^{\ell k}(x, \xi) \mu_i \mu_j,
  \]
  and
  \[
  |f_{\ell k}^{\ell k}(x, \xi)| \leq \Lambda_1 (1 + |\xi|^2)^{\frac{p}{2}}.
  \]
- There exists a constant $\Lambda_2 > 0$ such that for all $(x, \xi) \in \Omega \times \mathbb{R}^n$
  \[
  |f_{\ell k}^{\ell k}(x, \xi)| \leq \Lambda_2 (1 + |\xi|^2)^{\frac{p}{4}}.
  \]

Let $u \in W^{1,p}(\Omega)$ denote a local minimizer of the problem
\[
\min \left\{ \int_{\Omega} f(x, Dw) : w \in K_{\psi, \psi}(\Omega) \right\}.
\]

Fix $x_0 \in \Omega$ and a radius $R > 0$ such that $B_R(x_0) \subset\subset \Omega$. Consider the following variational problem
\[
\inf \left\{ \int_{B_R} f_{\ell k}(x, Dv) \, dx, \ v \in W^{1,p}_0(B_R) + u, \ v \geq \psi \text{ a.e. in } \Omega \right\}.
\]
where \( f^{\ell k} \) have been defined in (6.2). By classical lower semicontinuity arguments, there exist \( v^{\ell k} \in W^{1,p}_0(B_R) + u \) solution to problem (6.8). By the growth conditions (6.3) and the minimality of \( v^{\ell k} \), we get

\[
\int_{B_R} |Dv^{\ell k}|^p \, dx \leq \int_{B_R} (1 + |Dv^{\ell k}|^2)^{\frac{p}{2}} \, dx
\]

\[
\leq C \int_{B_R} f^{\ell k}(x, Dv^{\ell k}) \, dx \leq C \int_{B_R} f^{\ell k}(x, Du) \, dx
\]

\[
= C \int_{B_R} f^\ell(x, Du) \, dx + \frac{C}{k} \int_{B_R} (1 + |Du|^2)^{\frac{p}{2}} \, dx
\]

\[
\leq C \int_{B_R} (1 + |Du|^2)^{\frac{p}{2}} \, dx.
\]

Note that the right hand side of the previous estimate does not depend on \( \ell \), thus \( \|Dv^{\ell k}\|_{L^p(B_R)} \) is bounded for every fixed \( k \). Since \( W^{1,p}_0(B_R) \) is weakly compact, up to passing to a subsequence, we can assume there exist \( v^k \in u + W^{1,p}_0(B_R) \) such that

\[
v^{\ell k} \xrightarrow{\ell \to \infty} v^k \text{ weakly in } W^{1,p}_0(B_R) + u,
\]

where, for every \( k \geq 1 \), the limit function \( v^k \) still belongs to \( \mathcal{K}_{\psi, \Psi} \) since this set is weakly closed. Moreover, by classical convolution properties \( f^{\ell}(x, Du) \to f(x, Du) \) pointwise a.e. in \( B_R \) as \( \ell \to +\infty \), and again by growth conditions (6.3)

\[
\int_{B_R} f^\ell(x, Du) \, dx \leq C \int_{B_R} (1 + |Du|^2)^{\frac{p}{2}} \, dx,
\]

for every \( \ell \geq 1 \). Thus by the Dominated Convergence Theorem we deduce

\[
\lim_{\ell \to \infty} \int_{B_R} |Dv^{\ell k}|^p \, dx \leq C \int_{B_R} f(x, Du) \, dx + \frac{C}{k} \int_{B_R} (1 + |Du|^2)^{\frac{p}{2}} \, dx.
\]

Now, in force of Theorem 6.1 we know that \( f^{\ell k} \) satisfies the growth conditions needed to apply our a priori estimate for every \( \ell, k \). Thus, applying the a priori estimate to the solutions to (6.8), using the minimality of \( v^{\ell k} \)

\[
\|Dv^{\ell k}\|_{L^\infty(B_R)} \leq \tilde{C} \left( \int_{B_R} (1 + |Dv^{\ell k}|^2)^{\frac{p}{2}} \, dx \right)^{\frac{1}{p}}
\]

\[
\leq \tilde{C} \left( \int_{B_R} f^{\ell k}(x, Dv^{\ell k}) \, dx \right)^{\frac{1}{p}}
\]

\[
\leq \tilde{C} \left( \int_{B_R} f^\ell(x, Du) \, dx \right)^{\frac{1}{p}}
\]

\[
\leq \tilde{C} \left( \int_{B_R} (1 + |Du|^2)^{\frac{p}{2}} \, dx \right)^{\frac{1}{p}},
\]
where the constant \( \tilde{C} = \frac{C}{(R-\rho)^\beta} \) is independent of \( \ell, k \). Therefore, for every fixed \( k \), the sequence \( \| Dv^{\ell k} \|_{L^\infty(B_R)} \) is bounded. Since weak limits are unique, we conclude that
\[
v^{\ell k} \overset{\ell \to \infty}{\rightharpoonup} v^k \text{ weakly in } W^{1,\infty}_{\text{loc}}(B_R).
\] (6.12)

By estimate (6.11)
\[
\| Dv^k \|_{L^p(B_R)} \leq \liminf_{\ell \to \infty} \| Dv^{\ell k} \|_{L^p(B_R)} \leq C \left( \int_{B_R} f(x, Du) \, dx + \frac{1}{k} \int_{B_R} (1 + |Du|^2)^{\frac{p}{2}} \, dx \right)^{\frac{1}{p}} \leq C \left( \int_{B_R} f(x, Du) + (1 + |Du|^2)^{\frac{p}{2}} \, dx \right)^{\frac{1}{p}},
\] (6.13)
for every \( k \geq 1 \). Moreover, for any \( \rho \) such that \( 0 < \rho < R \), we have
\[
\| Dv^k \|_{L^\infty(B_\rho)} \leq \liminf_{\ell \to \infty} \| Dv^{\ell k} \|_{L^\infty(B_\rho)} \leq \tilde{C} \left( \int_{B_R} (1 + |Du|^2)^{\frac{p}{2}} \, dx \right)^{\frac{1}{p}}.
\] (6.14)

Thus we can deduce that there exists \( \bar{v} \in u + W^{1,p}_0(B_R) \) such that, up to subsequences \( v^k \to \bar{v} \) weakly in \( W^{1,p}_0(B_R) \) and \( v^k \to \bar{v} \) weakly star in \( W^{1,\infty}_{\text{loc}}(B_R) \) and the limit function \( \bar{v} \) still belongs to \( K_{\psi, \Psi} \) since this set is weakly closed. Since estimate (6.14) holds for every \( k \geq 1 \), for any \( 0 < \rho < R \) we obtain
\[
\| D\bar{v} \|_{L^\infty(B_\rho)} \leq \frac{C}{(R-\rho)^\beta} \left( \int_{B_R} (1 + |Du|^2)^{\frac{p}{2}} \, dx \right)^{\frac{1}{p}}.
\] (6.15)

Hence combining lower semicontinuity, the minimality of \( v^{\ell k} \) for problem (6.8), and properties of mollification we obtain
\[
\int_{B_\rho} f(x, Du) \, dx \leq \liminf_{k \to \infty} \int_{B_\rho} f(x, Du^k) \, dx \leq \liminf_{k \to \infty} \liminf_{\ell \to \infty} \int_{B_\rho} f(x, Du^{\ell k}) \, dx \leq \liminf_{k \to \infty} \liminf_{\ell \to \infty} \left( \int_{B_R} f^{\ell}(x, Du^{\ell k}) + \frac{1}{k} (1 + |Du^{\ell k}|^2)^{\frac{p}{2}} \, dx \right) = \liminf_{k \to \infty} \liminf_{\ell \to \infty} \int_{B_R} f^{\ell k}(x, Du^{\ell k}) \, dx.
\]
\[ \leq \liminf_{k \to \infty} \liminf_{\ell \to \infty} \int_{B_R} f^{\ell k}(x, Du) \, dx \]
\[ = \liminf_{k \to \infty} \liminf_{\ell \to \infty} \left( \int_{B_R} f^\ell(x, Du) \, dx + \frac{1}{k} \int_{B_R} (1 + |Du|^2)^{\frac{p}{2}} \, dx \right) \]
\[ = \liminf_{k \to \infty} \left( \int_{B_R} f(x, Du) \, dx + \frac{1}{k} \int_{B_R} (1 + |Du|^2)^{\frac{p}{2}} \, dx \right) \]
\[ = \int_{B_R} f(x, Du) \, dx, \]
for every \( 0 < \rho < R \). Then, letting \( \rho \to R \) in the previous inequality, we get
\[ \int_{B_R} f(x, D\bar{v}) \, dx \leq \int_{B_R} f(x, Du) \, dx. \] 
Therefore, \( u \) and \( \bar{v} \) are two solutions to problem \((6.7)\). We claim that \( u = \bar{v} \) on \( B_R \). Indeed, if we suppose that \( u \neq v \) choose any \( \theta \in (0, 1) \) and define \( v_\theta = \theta \bar{v} + (1 - \theta)u \), since \( f(x, \xi) \) is strictly convex in its second argument we have
\[ \int_{B_R} f(x, Dv_\theta) \, dx < \theta \int_{B_R} f(x, D\bar{v}) \, dx + (1 - \theta) \int_{B_R} f(x, Du) \, dx \]
\[ = \int_{B_R} f(x, Du) \, dx, \]
contradicting the minimality of \( u \). Thus, regularity is preserved through the limit procedure, and this concludes the proof of Theorem 2.1 because we have removed the further regularity assumptions we needed to obtain the result.

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