Spanning trees and spanning Eulerian subgraphs with small degrees. II

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Abstract

Let $G$ be a connected graph with $X \subseteq V(G)$ and with the spanning forest $F$. Let $\lambda \in [0, 1]$ be a real number and let $\eta : X \to (\lambda, \infty)$ be a real function. In this paper, we show that if for all $S \subseteq X$, $\omega(G \setminus S) \leq \sum_{v \in S} (\eta(v) - 2) + 2 - \lambda e_G(S) + 1$, then $G$ has a spanning tree $T$ containing $F$ such that for each vertex $v \in X$, $d_T(v) \leq \lceil \eta(v) - \lambda \rceil + \max\{0, d_F(v) - 1\}$, where $\omega(G \setminus S)$ denotes the number of components of $G \setminus S$ and $e_G(S)$ denotes the number of edges of $G$ with both ends in $S$. This is an improvement of several results and the condition is best possible. Next, we also investigate an extension for this result and deduce that every $k$-edge-connected graph $G$ has a spanning subgraph $H$ containing $m$ edge-disjoint spanning trees such that for each vertex $v$, $d_H(v) \leq \lceil m k (d_G(v) - 2m) \rceil + 2m$, where $k \geq 2m$; also if $G$ contains $k$ edge-disjoint spanning trees, then $H$ can be found such that for each vertex $v$, $d_H(v) \leq \lceil m k (d_G(v) - m) \rceil + m$, where $k \geq m$. Finally, we show that strongly 2-tough graphs, including $(3 + 1/2)$-tough graphs of order at least three, have spanning Eulerian subgraphs whose degrees lie in the set $\{2, 4\}$. In addition, we show that every 1-tough graph has spanning closed walk meeting each vertex at most 2 times and prove a long-standing conjecture due to Jackson and Wormald (1990).

Keywords:
Spanning tree; spanning Eulerian; spanning closed walk; connected factor; toughness; total excess.

1 Introduction

In this article, all graphs have no loop, but multiple edges are allowed and a simple graph is a graph without multiple edges. Let $G$ be a graph. The vertex set, the edge set, the maximum degree, and the number of components of $G$ are denoted by $V(G)$, $E(G)$, $\Delta(G)$, and $\omega(G)$, respectively. The degree $d_G(v)$ of a vertex $v$ is the number of edges of $G$ incident to $v$. The set of edges of $G$ that are incident to $v$ is denoted by $E_G(v)$. We denote by $d_G(C)$ the number of edges of $G$ with exactly one end in $V(C)$, where $C$ is a subgraph of $G$. For a set $X \subseteq V(G)$, we denote by $G[X]$ the induced subgraph of $G$ with the vertex set $X$ containing precisely those edges of $G$ whose ends lie in $X$. Let $g$ and $f$ be two nonnegative integer-valued functions on...
A spanning tree $T$ is called spanning $f$-tree, if for each vertex $v$, $d_T(v) \leq f(v)$. Likewise, one can define a spanning $f$-forest. A $(g,f)$-factor of $G$ is a spanning subgraph $H$ such that for each vertex $v$, $g(v) \leq d_H(v) \leq f(v)$. An $f$-walk (trail) in a graph refers to a spanning closed walk (trail) meeting each vertex $v$ at most $f(v)$ times. For a spanning subgraph $H$ with the integer-valued function $h$ on $V(H)$, the total excess of $H$ from $h$ is defined as follows:

$$te(H,h) = \sum_{v \in V(H)} \max\{0, d_H(v) - h(v)\}.$$ 

According to this definition, $te(H,h) = 0$ if and only if for each vertex $v$, $d_H(v) \leq h(v)$. For a set $A$ of integers, an $A$-factor is a spanning subgraph with vertex degrees in $A$. Let $F$ be a spanning subgraph of $G$. The graph obtained from $G$ by contracting any component of $F$ is denoted by $G/F$. A component of $F$ is said to be trivial, if it consists of only one vertex. Likewise, $F$ is said to be trivial, if it has no edge.

A vertex set $S$ of a graph $G$ is called independent, if there is no edge of $G$ connecting vertices in $S$. For a vertex $v$, denote by $d_G(v,F)$ the number of edges of $G$ that are incident to $v$ and whose ends of each of them lie in different components of $F$. Let $S \subseteq V(G)$. The graph obtained from $G$ by removing all vertices of $S$ is denoted by $G \setminus S$. Denote by $G \setminus [S,F]$ the graph obtained from $G$ by removing all edges incident to the vertices of $S$ except the edges of $F$. Note that while the vertices of $S$ are deleted in $G \setminus S$, no vertices are removed in $G \setminus [S,F]$. Denote by $e_G(S)$ the number of edges of $G$ with both ends in $S$. Furthermore, the number of edges of $G$ with both ends in $S$ joining different components of $F$ is denoted by $e_G(S,F)$. Let $P$ be a partition of $V(G)$. Denote by $e_G(P)$ the number of edges of $G$ whose ends lie in different parts of $P$. The graph obtained from $G$ by contracting all vertex sets of $P$ is denoted by $G/P$. Note that the edge set of $G/P$ can be considered as an edge subset of $E(G)$. A graph is called $K_{1,n}$-free, if it has no induced subgraph isomorphic to the complete bipartite graph $K_{1,n}$. A graph $G$ is called $m$-tree-connected, if it has $m$ edge-disjoint spanning trees. In addition, an $m$-tree-connected graph $G$ is called minimally $m$-tree-connected, if $|E(G)| = m(|V(G)| - 1). In other words, for any edge $e$ of $G$, the graph $G \setminus e$ is not $m$-tree-connected. The vertex set of any graph $G$ can be expressed uniquely (up to order) as a disjoint union of vertex sets of some induced $m$-tree-connected subgraphs. These subgraphs are called the $m$-tree-connected components of $G$. For a graph $G$, we define the parameter $\Omega_m(G) = |P| - \frac{1}{m}e_G(P)$ to measure tree-connectivity, where $P$ is the unique partition of $V(G)$ obtained from the $m$-tree-connected components of $G$. Note that $\Omega_1(G)$ is the same number of components of $G$, while $\Omega_m(G)$ is less or equal than the number of $m$-tree-connected components of $G$. In Subsection 6.1, we will show that $\omega(G) = \Omega_1(G) \leq \Omega_2(G) \leq \cdots \leq |V(G)|$ and also $G$ is $m$-tree-connected if and only if $\Omega_m(G) = 1$. The definition implies that the null graph $K_0$ with no vertices is not $m$-tree-connected and $\Omega_m(K_0) = 0$. In this paper, we assume that all graphs are nonnull, except for the graphs that obtained by removing vertices. We say that a graph $F$ is $m$-critical, if whose $m$-tree-connected components are minimal. We will show that every $m$-tree-connected graph $H$ containing the $m$-critical graph $F$ with the minimum number of edges is minimally $m$-tree-connected. Moreover, it is not hard to check that $\Omega_m(F) > \Omega_m(F \setminus e)$, for any edge $e$ of $F$. Clearly, 1-critical graphs are forests. Let $t$ be a positive real number, a graph $G$ is said to be $t$-tough, if $\omega(G \setminus S) \leq \max\{1, \frac{1}{t}|S|\}$ for all $S \subseteq V(G)$. Likewise, $G$ is said to be $m$-strongly $t$-tough, if $\Omega_m(G \setminus S) \leq \max\{1, \frac{1}{t}|S|\}$ for all $S \subseteq V(G)$. We will show that tough enough graphs with sufficiently large order are also $m$-strongly tough enough. For convenience,
we abbreviate the term ‘2-strongly’ to strongly, and for notational simplicity, we write Ω(G) for Ω2(G). Throughout this article, all integer variables $k$ and $m$ are positive.

In 1976 Frank and Gyárfás investigated spanning trees with bounded degrees in terms of directed graphs. A special case of their result can be shown as the following theorem.

**Theorem 1.1.** (\cite{frank1976}) Let $G$ be a graph. If for all $S \subseteq V(G)$, $\omega(G \setminus S) \leq \sum_{v \in S} (f(v) - 1) + 1 - e_G(S)$, then $G$ has a spanning tree $T$ such that for each vertex $v$, $d_T(v) \leq f(v)$, where $f$ is a positive integer-valued function on $V(G)$.

In 1989 Win \cite{win1989} established a result related to spanning trees and toughness of graphs, and Ellingham, Nam, and Voss (2002) generalized it as the following. Former, Ellingham and Zha (2000) \cite{ellingham2000} found the following fact for constant function form.

**Theorem 1.2.** (\cite{win1989}) Let $G$ be a connected graph with the spanning forest $F$. If for all $S \subseteq V(G)$, $\omega(G \setminus S) \leq \sum_{v \in S} (f(v) - 2) + 2$, then $G$ has a spanning tree $T$ containing $F$ such that for each vertex $v$,

\[
d_T(v) \leq \begin{cases} f(v) + d_F(v), & \text{if } c \geq 1; \\ f(v) + d_F(v) - 1, & \text{if } c \geq 2, \end{cases}
\]

where every component of $F$ contains at least $c$ vertices and $f$ is a positive integer-valued function on $V(G)$.

Liu and Xu (1998) and Ellingham, Nam, and Voss (2002) independently investigated spanning trees with bounded degrees in highly edge-connected graphs and found the following theorem.

**Theorem 1.3.** (\cite{liu1998, ellingham2002}) Every $k$-edge-connected simple graph $G$ has a spanning tree $T$ such that for each vertex $v$, $d_T(v) \leq \lceil \frac{d_G(v)}{k} \rceil + 2$.

Recently, the present author (2015) refined Theorem 1.3 and concluded the next theorems.

**Theorem 1.4.** (\cite{towards}) Every $k$-edge-connected graph $G$ has a spanning tree $T$ such that for each vertex $v$, $d_T(v) \leq \lceil \frac{d_G(v) - 2}{k} \rceil + 2$.

**Theorem 1.5.** (\cite{towards}) Every $k$-tree-connected graph $G$ has a spanning tree $T$ such that for each vertex $v$, $d_T(v) \leq \lceil \frac{d_G(v) - 1}{k} \rceil + 1$.

In this paper, we improve Theorems 1.1, 1.2, 1.4, and 1.5 as the following stronger version, where the special cases $\lambda = \{0, 1, 2/k, 1/k\}$ can conclude them (not necessarily directly). It also gives a number of new applications on connected factors.

**Theorem 1.6.** Let $G$ be a connected graph with $X \subseteq V(G)$ and with the spanning forest $F$. Let $\lambda \in [0, 1]$ be a real number and let $\eta : X \to (\lambda, \infty)$ be a real function. If for all $S \subseteq X$, $\omega(G \setminus S) \leq \sum_{v \in S} (\eta(v) - \lambda)$...
2) + 2 − \lambda(e_G(S) + 1), then G has a spanning tree T containing F such that for each v ∈ X, d_T(v) ≤ \lceil \eta(v) − \lambda \rceil + \max\{0, d_F(v) − 1\}.

Jackson and Wormald (1990) [17] conjectured that every \( \frac{1}{n-1} \)-tough graph with \( n \geq 2 \) has an \( n \)-walk. They also observed that this conjecture is true for \( \frac{1}{n-2} \)-tough graphs, when \( n \geq 3 \). In 2000 Ellingham and Zha [11] proved the remaining case \( n = 2 \) for 4-tough graphs and triangle-free 3-tough graphs. In Section 5, we prove this conjecture completely and provides the following stronger version.

**Theorem 1.7.** Let \( G \) be a graph with the positive integer-valued function \( f \) on \( V(G) \). If for all \( S \subseteq V(G) \), \( \omega(G \setminus S) \leq \sum_{v \in S} (f(v) - 1) + 1 \), then \( G \) has an \( f \)-walk passing through the edges of a given arbitrary matching.

In Section 6, we make the next theorem, by investigating bounded degree minimally \( m \)-tree-connected spanning subgraphs, with the arguments more complicated than Theorem 1.6. As an application, it can help us to strengthen Theorems 1.4 and 1.5 toward this concept as mentioned in the abstract. Finally, we present a common generalization for the following theorem, the above-mentioned theorem, and also a recent result due to Ozeki (2015) [32]. Owing to its complicated form, we postpone it until Section 8.

**Theorem 1.8.** Let \( G \) be an \( m \)-tree-connected graph with \( X \subseteq V(G) \). Let \( \lambda \in [0, 1/m] \) be a real number and let \( \eta : X \rightarrow (m\lambda + \frac{m-1}{m}, \infty) \) be a real function. If for all \( S \subseteq X \), \( \Omega_m(G \setminus S) \leq \sum_{v \in S} (\eta(v) - 2) + 2 - \lambda(e_G(S) + m) \), then \( G \) has an \( m \)-tree-connected spanning subgraph \( H \) such that for each \( v \in X \), \( d_H(v) \leq \lceil m\eta(v) - m^2\lambda \rceil \).

In [16] it was remarked that Theorem 1.5 can reduce the needed edge-connectivity of the main results in [3, 35]. Alternatively, in this paper we show that every \( k \)-tree-connected bipartite graph \( G \) with one partite set \( A \) has an \( m \)-tree-connected spanning subgraph \( H \) such that for each \( v \in A \), \( d_H(v) \leq \lceil \frac{m}{k}d_G(v) \rceil \), where \( k \geq m \). Fortunately, by reviewing the proof of the above-mentioned papers, we find out one can use this result to reduce the needed edge-connectivities further down. For instance, it can reduce the edge-connectivity of the following theorem down to 75 with exactly the same proof.

**Theorem 1.9.** ([3]) Every 191-edge-connected simple graph has an edge-decomposition into tree \( Y \) if and only if its size is divisible by 4, where \( Y \) is the unique tree with degree sequence \((1, 1, 1, 2, 3)\).

In 1973 Chvátal [8] conjectured that there exists a positive real number \( t_0 \) such that every \( t_0 \)-tough graph of order at least three admits a Hamiltonian cycle (1-trail). In 2000 Bauer, Broersma, and Veldman [4] showed that (strongly) 2-tough graphs of may have no Hamiltonian cycles. In Section 9, we show that strongly 2-tough graphs, including \((3 + 1/2)\)-tough graphs of order at least three, have 2-trails. More generally, we form the following result from Theorem 1.8. Moreover, we show that higher toughness can guarantee the of spanning closed trails meeting each vertex \( r \) or \( r + 1 \) times.

**Theorem 1.10.** Let \( G \) be a 2-tree-connected graph with the positive integer-valued function \( f \) on \( V(G) \). If for all \( S \subseteq V(G) \), \( \Omega(G \setminus S) \leq \sum_{v \in S} (f(v) - \frac{3}{2}) + 2 \), then \( G \) has an \( f \)-trail.
2 Preliminary result

Here, we state the following fundamental theorem which was implicitly studied in [28] and provides an improvement for Theorem 1 in [10]. We shall apply it to prove Theorem 1.6, while Theorem 1 in [10] can alternatively be applied with minor modifications.

Theorem 2.1. ([10, 28]) Let $G$ be a connected graph with the spanning forest $F$ and let $h$ be an integer-valued function on $V(G)$. If $T$ is a spanning tree of $G$ containing $F$ with the minimum total excess from $h + d_F$, then there exists a subset $S$ of $V(G)$ with the following properties:

1. $\omega(G \setminus [S, F]) = \omega(T \setminus [S, F])$.
2. $S \supseteq \{v \in V(G) : d_T(v) > h(v) + d_F(v)\}$.
3. For each vertex $v$ of $S$, $d_T(v) \geq h(v) + d_F(v)$.

Proof. Define $V_0 = \emptyset$ and $V_1 = \{v \in V(T) : d_T(v) > h(v) + d_F(v)\}$. For any $S \subseteq V(G)$ and $u \in V(G) \setminus S$, let $\mathcal{A}(S, u)$ be the set of all spanning trees $T'$ of $G$ containing $F$ such that $d_T(v) \leq h(v) + d_F(v)$ for all $v \in V(G) \setminus V_1$, and also $T'$ and $T$ have the same edges, except for some of the edges of $G$ whose ends are in $V(C) \setminus S$, where $C$ is the component of $T \setminus [S, F]$ containing $u$. Now, for each integer $n$ with $n \geq 2$, recursively define $V_n$ as follows:

$$V_n = V_{n-1} \cup \{v \in V(G) \setminus V_{n-1} : d_{T'}(v) \geq h(v) + d_F(v), \text{ for all } T' \in \mathcal{A}(V_{n-1}, v)\}.$$ 

Now, we prove the following claim.

Claim. Let $x$ and $y$ be two vertices in different components of $T \setminus [V_{n-1}, F]$. If $xy \in E(G) \setminus E(T)$, then $x \in V_n$ or $y \in V_n$.

Proof of claim. By induction on $n$. For $n = 1$, the proof is clear. Assume that the claim is true for $n - 1$. Now we prove it for $n$. Suppose otherwise that $x$ and $y$ are in different components of $T \setminus [V_{n-1}, F]$, respectively, with the vertex sets $X$ and $Y$, $xy \in E(G) \setminus E(F)$, and $x, y \notin V_n$. Since $x, y \notin V_n$, there exist $T_x \in \mathcal{A}(V_{n-1}, x)$ and $T_y \in \mathcal{A}(V_{n-1}, y)$ with $d_{T_x}(x) < h(x) + d_F(x)$ and $d_{T_y}(y) < h(y) + d_F(y)$. By the induction hypothesis, $x$ and $y$ are in the same component of $T \setminus [V_{n-2}, F]$. Let $P$ be the unique path connecting $x$ and $y$ in $T$. Notice that the vertices of $P$ lie in the same component of $T \setminus [V_{n-2}, F]$. Pick $e \in E(P) \setminus E(F)$ such that $e$ is incident to a vertex $z \in V_{n-1} \setminus V_{n-2}$. Now, let $T'$ be the spanning tree of $G$ with $E(T') = E(T) - e + xy - E(T[X]) + E(T_x[X]) - E(T[Y]) + E(T_y[Y])$.

If $n \geq 3$, then it is not hard to see that $d_{T'}(z) < d_T(z) \leq h(z) + d_F(z)$ and $T'$ lies in $\mathcal{A}(V_{n-2}, z)$. Since $z \in V_{n-1}$, we arrive at a contradiction. For the case $n = 2$, since $z \in V_1$, it is easy to see that $h(z) + d_F(z) \leq d_{T'}(z) < d_T(z)$ and $te(T', h + d_F) < te(T, h + d_F)$, which is again a contradiction. Hence the claim holds.
Obviously, there exists a positive integer \( n \) with \( V_1 \subseteq \cdots \subseteq V_{n-1} = V_n \). Put \( S = V_n \). Since \( S \supseteq V_1 \), Condition 2 clearly holds. For each \( v \in V_i \setminus V_{i-1} \) with \( i \geq 2 \), we have \( T \in \mathcal{A}(V_{i-1}, v) \) and so \( d_T(v) \geq h(v) + d_F(v) \). This establishes Condition 3. Because \( S = V_n \), the previous claim implies Condition 1 and completes the proof. \( \square \)

3 Spanning \( ([\eta - \lambda] + d_F - 1) \)-trees

The following lemma establishes a simple but important property of forests.

**Lemma 3.1.** Let \( T \) be a forest with the spanning forest \( F \). If \( S \subseteq V(T) \) and \( F = T \setminus E(F) \), then
\[
\sum_{v \in S} d_F(v) = \omega(T \setminus [S, F]) - \omega(T) + e_F(S).
\]

**Proof.** By induction on the number of edges of \( F \) which are incident to the vertices in \( S \). If there is no edge of \( F \) incident to a vertex in \( S \), then the proof is clear. Now, suppose that there exists an edge \( e = uu' \in E(F) \) with \( |S \cap \{u, u'\}| \geq 1 \). Hence

1. \( \omega(T) = \omega(T \setminus e) - 1 \),
2. \( \omega(T \setminus [S, F]) = \omega((T \setminus e) \setminus [S, F]) \),
3. \( e_F(S) = e_{F \setminus e}(S) + |S \cap \{u, u'\}| - 1 \),
4. \( \sum_{v \in S} d_F(v) = \sum_{v \in S} d_{F \setminus e}(v) + |S \cap \{u, u'\}| \).

Therefore, by the induction hypothesis on \( T \setminus e \) with the spanning forest \( F \) the lemma holds. \( \square \)

The following theorem is essential in this section.

**Theorem 3.2.** Let \( G \) be a connected graph with \( X \subseteq V(G) \) and with the spanning forest \( F \). Let \( \lambda \in [0, 1] \) be a real number and let \( \eta: X \to (\lambda, \infty) \) be a real function. If for all \( S \subseteq X \),
\[
\omega(G \setminus [S, F]) < 1 + \sum_{v \in S} (\eta(v) - 2) + 2 - \lambda(e_G(S, F) + 1),
\]
then \( G \) has a spanning tree \( T \) containing \( F \) such that for each \( v \in X \), \( d_T(v) \leq [\eta(v) - \lambda] + d_F(v) - 1 \).

**Proof.** For each vertex \( v \), define
\[
h(v) = \begin{cases} d_G(v) + 1, & \text{if } v \notin X; \\ [\eta(v) - \lambda] - 1, & \text{if } v \in X. \end{cases}
\]
Let $T$ be a spanning tree of $G$ containing $F$ with the minimum total excess from $h + d_F$. Define $S$ to be a subset of $V(G)$ with the properties described in Theorem 2.1. If $S$ is empty, then $te(T, h + d_F) = 0$ and the theorem clearly holds. So, suppose $S$ is nonempty. Obviously, $S \subseteq X$. Put $\mathcal{F} = T \setminus E(F)$. By Lemma 3.1,
\[
\sum_{v \in S} h(v) + te(T, h + d_F) = \sum_{v \in S} d_F(v) = \omega(T \setminus [S, F]) - \omega(T) + e_F(S),
\]
and so
\[
\sum_{v \in S} h(v) + te(T, h + d_F) = \omega(G \setminus [S, F]) - 1 + e_F(S).
\] (1)
Also, by the assumption, we have
\[
\omega(G \setminus [S, F]) - 1 + e_F(S) < 1 + \sum_{v \in S} (\eta(v) - 2) - \lambda e_G(S, F) + 1 + e_F(S) + 1.
\] (2)
Since $e_F(S) \leq e_G(S, F)$ and $e_F(S) \leq |S| - 1$,
\[
- \lambda e_G(S, F) + 1 + e_F(S) + 1 \leq -\lambda e_F(S) + 1 + e_F(S) + 1 \leq (1 - \lambda)|S|.
\] (3)
Therefore, Relations (1), (2), and (3) can conclude that
\[
\sum_{v \in S} h(v) + te(T, h + d_F) \leq \omega(G \setminus [S, F]) - 1 + e_F(S) < 1 + \sum_{v \in S} (\eta(v) - \lambda - 1).
\]
On the other hand, by the definition of $h(v)$,
\[
\sum_{v \in S} (\eta(v) - \lambda - 1 - h(v)) \leq 0.
\]
Hence $te(T, h + d_F) = 0$ and the theorem holds.

3.1 Graphs with high essential edge-connectivity

The following lemma provides two upper bounds on $\omega(G \setminus [S, F])$ depending on two parameters of connectivity of $G/F$ and $d_G(v, F)$ of the vertices $v$ in $S$.

Lemma 3.3. Let $G$ be a graph with the spanning forest $F$ and let $S \subseteq V(G)$. Then
\[
\omega(G \setminus [S, F]) \leq \begin{cases} 
\sum_{v \in S} \left(\frac{d_G(v,F)}{k} + 1\right) - \frac{2}{k} e_G(S,F), & \text{if } G/F \text{ is } k\text{-edge-connected and } S \neq \emptyset; \\
\sum_{v \in S} \frac{d_G(v,F)}{k} + 1 - \frac{1}{k} e_G(S,F), & \text{if } G/F \text{ is } k\text{-tree-connected}.
\end{cases}
\]

Proof. First, assume that $G/F$ is $k$-edge-connected and $S$ is nonempty. Thus there are at least $k(\omega(G \setminus [S, F]) - |S|)$ edges of $G$ with exactly one end in $S$ joining different components of $G \setminus [S, F]$, because $S$ is nonempty and there are at least $\omega(G \setminus [S, F]) - |S|$ components of $G \setminus [S, F]$ without any vertex of $S$. On the other hand, there are $\sum_{v \in S} d_G(v, F) - 2e_G(S, F)$ edges of $G$ with exactly one end in $S$ joining different components of $F$. Hence we have
\[
k(\omega(G \setminus [S, F]) - |S|) \leq \sum_{v \in S} d_G(v, F) - 2e_G(S, F).
\]
Next, assume that $G/F$ is $k$-tree-connected. Thus there are at least $k(\omega(G \setminus [S, F]) - 1)$ of edges of $G$ with at least one end in $S$ joining different components of $G \setminus [S, F]$. On the other hand, there are $\sum_{v \in S} d_G(v, F) - e_G(S, F)$ edges of $G$ with at least one end in $S$ joining different components of $F$. Hence we have

$$k(\omega(G \setminus [S, F]) - 1) \leq \sum_{v \in S} d_G(v, F) - e_G(S, F).$$

These inequalities complete the proof. \hfill \Box

The following theorem generalizes Theorems 1.4 and 1.5.

**Theorem 3.4.** Let $G$ be a graph with the spanning forest $F$. Then $G$ has a spanning tree $T$ containing $F$ such that for each vertex $v$,

$$d_T(v) \leq \begin{cases} \left\lfloor \frac{d_G(v) - d_F(v) - 2}{k} \right\rfloor + d_F(v) + 2, & \text{if } G/F \text{ is } k\text{-edge-connected;} \\ \left\lfloor \frac{d_G(v) - d_F(v) - 1}{k} \right\rfloor + d_F(v) + 1, & \text{if } G/F \text{ is } k\text{-tree-connected.} \end{cases}$$

Furthermore, for a given arbitrary vertex $u$ the upper can be reduced to $\lfloor (d_G(u) - d_F(u))/k \rfloor + d_F(u)$.

**Proof.** We may assume that $k \geq 2$, as the assertions trivially hold when $k = 1$. Let $S \subseteq V(G)$. If $G/F$ is $k$-edge-connected and $S \neq \emptyset$, then by Lemma 3.3, we have

$$\omega(G \setminus [S, F]) \leq \sum_{v \in S} \left( \frac{d_G(v, F)}{k} + 1 \right) - 2 \frac{k - 1}{k} e_G(S, F) \leq \frac{k - 1}{k} + \sum_{v \in S} (\eta(v) - 2) + 2 - \frac{2}{k} (e_G(S, F) + 1),$$

where $\eta(u) = \frac{d_G(u) - d_F(u) + 3}{k}$ and $\eta(v) = \frac{d_G(v) - d_F(v)}{k} + 3$ for all $v \in V(G) \setminus u$. If $G$ is $k$-tree-connected, then by Lemma 3.3, we also have

$$\omega(G \setminus [S, F]) \leq \sum_{v \in S} \frac{d_G(v, F)}{k} + 1 - \frac{1}{k} e_G(S, F) \leq \frac{k - 1}{k} + \sum_{v \in S} (\eta(v) - 2) + 2 - \frac{1}{k} (e_G(S, F) + 1),$$

where $\eta(u) = \frac{d_G(u) - d_F(u) + 2}{k}$ and $\eta(v) = \frac{d_G(v) - d_F(v)}{k} + 2$ for all $v \in V(G) \setminus u$. Thus the assertions follow from Theorem 3.2 for $\lambda \in \{2/k, 1/k\}$. Note that $0 < \lambda \leq 1$ and $\left\lfloor \frac{d_G(u) - d_F(u)}{k} \right\rfloor = \left\lfloor \frac{d_G(u) - d_F(u) + 1}{k} \right\rfloor - 1$. \hfill \Box

### 3.2 A necessary and sufficient condition

The following theorem provides a necessary and sufficient condition for the existence of a spanning tree with the described properties.

**Theorem 3.5.** Let $G$ be a graph with the spanning forest $F$ and let $X \subseteq V(G)$ with $e_G(X, F) = 0$. Then $G$ has a spanning tree $T$ containing $F$ such that for each $v \in X$, $d_T(v) \leq h(v) + d_F(v)$, if and only if for all $S \subseteq X$, $\omega(G \setminus [S, F]) \leq \sum_{v \in S} h(v) + 1$, where $h$ is a nonnegative integer-valued function on $X$. 

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Proof. Assume that $G$ has a spanning tree $T$ containing $F$ such that for each $v \in X$, $d_T(v) \leq h(v) + d_F(v)$. Put $\mathcal{F} = T \setminus E(F)$ and let $S \subseteq X$. According to the assumption on $X$, one can conclude that $e_\mathcal{F}(S) = 0$. Since for each $v \in S$, $d_\mathcal{F}(v) \leq h(v)$, and $\omega(T) = 1$, with respect to Lemma 3.1, $\omega(G \setminus [S, F]) \leq \omega(T \setminus [S, F]) = \sum_{v \in S} d_\mathcal{F}(v) + 1 \leq \sum_{v \in S} h(v) + 1$. To prove the converse, one can apply Theorem 3.2 with $\lambda = 1$. Note that $G$ is connected, because $\omega(G \setminus [\emptyset, F]) \leq 1$. \hfill \Box

Corollary 3.6.[(14), see Page 5 in [33]) Let $G$ be a graph with the independent set $X$. Then $G$ has a spanning tree $T$ such that for each $v \in X$, $d_T(v) \leq f(v)$, if and only if $\omega(G \setminus S) \leq \sum_{v \in S} (f(v) - 1) + 1$ for all $S \subseteq X$, where $f$ is a positive integer-valued function on $X$.

Proof. Apply this fact $\omega(G \setminus [S, F]) = \omega(G \setminus S) + |S|$ when $F$ is the trivial spanning forest. \hfill \Box

4 Spanning $([\eta - \lambda] + \max\{0, d_F - 1\})$-trees

In this section, the notation $\omega(G \setminus S)$ plays an essential role instead of $\omega(G \setminus [S, F])$. In order to prove the next theorem, we need the next lemma that provides a relationship between $\omega(G \setminus S)$ and $\omega(G \setminus [S, F])$.

Lemma 4.1. Let $G$ be a graph with the spanning forest $F$. If $S \subseteq V(G)$ then 
$$
\omega(G \setminus [S, F]) \leq \omega(G \setminus S) + |\{v \in S : d_F(v) = 0\}| + \frac{1}{c-1} e_F(S),
$$
where every non-trivial component of $F$ contains at least $c$ vertices with $c \geq 2$.

Proof. Note that every component of $F$ of order $i$ whose vertices entirely lie in the set $S$ has exactly $i - 1$ edges with both ends in $S$. For each $i$ with $i \geq 1$, let $t_i$ be the number of components of $F$ of order $i$ whose vertices entirely lie in the set $S$. Clearly, $t_1 = |\{v \in S : d_F(v) = 0\}|$. Since $\omega(G \setminus [S, F]) \leq \omega(G \setminus S) + t_1 + \sum_{i \leq 1} t_i$ and $\sum_{i \leq 1} (i - 1)t_i \leq e_F(S)$, the lemma holds. \hfill \Box

The following theorem is essential in this section.

Theorem 4.2. Let $G$ be a connected graph with $X \subseteq V(G)$ and with the spanning forest $F$. Let $\lambda \in [0, 1]$ be a real number and let $\eta : X \rightarrow (\lambda, \infty)$ be a real function. If for all $S \subseteq X$,
$$
\omega(G \setminus S) < 1 + \sum_{v \in S} (\eta(v) - 2) + 2 - \lambda(e_G(S) + 1),
$$
then $G$ has a spanning tree $T$ containing $F$ such that for each $v \in X$, $d_T(v) \leq [\eta(v) - \lambda] + \max\{0, d_F(v) - 1\}$.

Proof. For each vertex $v$, define
$$
h(v) = \begin{cases} 
d_G(v) + 1, & \text{if } v \notin X; \\
[\eta(v) - \lambda], & \text{if } v \in X \text{ and } d_F(v) = 0; \\
[\eta(v) - \lambda] - 1, & \text{if } v \in X \text{ and } d_F(v) \neq 0. 
\end{cases}
$$
Let $T$ be a spanning tree of $G$ containing $F$ with the minimum total excess from $h + d_F$. Define $S$ to be a subset of $V(G)$ with the properties described in Theorem 2.1. If $S$ is empty, then $te(T, h + d_F) = 0$ and the theorem clearly holds. So, suppose $S$ is nonempty. Obviously, $S \subseteq X$. Put $F = T \setminus E(F)$. By Lemma 3.1,

$$
\sum_{v \in S} h(v) + te(T, h + d_F) = \sum_{v \in S} d_F(v) = \omega(T \setminus [S, F]) - \omega(T) + e_F(S),
$$

and so

$$
\sum_{v \in S} h(v) + te(T, h + d_F) = \omega(G \setminus [S, F]) - 1 + e_F(S).
$$

Since $e_F(S) + e_F(S) = e_T(S)$, Lemma 4.1 in the special case $c = 2$ can deduce that

$$
\sum_{v \in S} h(v) - |\{v \in S : d_F(v) = 0\}| + te(T, h + d_F) \leq \omega(G \setminus S) - 1 + e_T(S). \tag{4}
$$

Also, by the assumption, we have

$$
\omega(G \setminus S) - 1 + e_T(S) < 1 + \sum_{v \in S} (\eta(v) - 2) - \lambda e_G(S) + 1 + e_T(S) + 1. \tag{5}
$$

Since $e_T(S) \leq e_G(S)$ and $e_T(S) \leq |S| - 1,$

$$
- \lambda e_G(S) + 1 + e_T(S) + 1 \leq -\lambda(e_T(S) + 1) + e_T(S) + 1 \leq (1 - \lambda)|S|. \tag{6}
$$

Therefore, Relations (4), (5), and (6) can conclude that

$$
\sum_{v \in S} h(v) - |\{v \in S : d_F(v) = 0\}| + te(T, h + d_F) \leq \omega(G \setminus S) - 1 + e_T(S) < 1 + \sum_{v \in S} (\eta(v) - \lambda - 1).
$$

On the other hand, by the definition of $h(v)$,

$$
\sum_{v \in S} (\eta(v) - \lambda - 1 - h(v)) + |\{v \in S : d_F(v) = 0\}| \leq 0.
$$

Hence $te(T, h + d_F) = 0$ and the theorem holds. \qed

**Corollary 4.3.** Let $G$ be a connected graph with the independent set $X \subseteq V(G)$ and with the spanning forest $F$. Let $\eta : X \to (0, \infty)$ be a real function. If for all $S \subseteq X$,

$$
\omega(G \setminus S) < 1 + \sum_{v \in S} (\eta(v) - 1) + 1,
$$

then every spanning forest $F$ can be extended to a spanning tree $T$ such that for each $v \in X$, $d_T(v) \leq [\eta(v)] + \max\{0, d_F(v) - 1\}$.

**Proof.** Apply Theorem 4.2 with $\lambda = 1$ and with replacing $\eta + 1$ instead of $\eta$. Note that $e_G(S) = 0$ for all $S \subseteq X$. \qed
### 4.1 Graphs with high edge-connectivity

A special case of Lemmas 3.3 is restated as the following lemma, since $\omega(G \setminus [S, F]) = \omega(G \setminus S) + |S|$ when $F$ is the trivial spanning forest.

**Lemma 4.4.** Let $G$ be a graph with $S \subseteq V(G)$. Then

$$\omega(G \setminus S) \leq \begin{cases} \sum_{v \in S} \frac{dg(v)}{k} - \frac{2}{k}e_G(S), & \text{if } G \text{ is } k\text{-edge-connected and } S \neq \emptyset; \\ \sum_{v \in S} \left(\frac{dg(v)}{k} - 1\right) + 1 - \frac{1}{k}e_G(S), & \text{if } G \text{ is } k\text{-tree-connected}. \end{cases}$$

Another generalization of Theorems 1.4 and 1.5 is given in the next theorem.

**Theorem 4.5.** Let $G$ be a graph with $X \subseteq V(G)$. Then every spanning forest $F$ can be extended to a spanning tree $T$ such that for each $v \in X$,

$$d_T(v) \leq \begin{cases} \left\lfloor \frac{dg(v)-2}{k}\right\rfloor + 2 + \max\{0, d_F(v) - 1\}, & \text{if } G \text{ is } k\text{-edge-connected}; \\ \left\lfloor \frac{dg(v)}{k}\right\rfloor + 1 + \max\{0, d_F(v) - 1\}, & \text{if } G \text{ is } k\text{-tree-connected}; \\ \left\lfloor \frac{dg(v)}{k}\right\rfloor + 1 + \max\{0, d_F(v) - 1\}, & \text{if } G \text{ is } k\text{-edge-connected and } X \text{ is independent}; \\ \left\lfloor \frac{dg(v)}{k}\right\rfloor + \max\{0, d_F(v) - 1\}, & \text{if } G \text{ is } k\text{-tree-connected and } X \text{ is independent}. \end{cases}$$

Furthermore, for a given arbitrary vertex $u$ the upper can be reduced to $\lfloor d_G(u)/k\rfloor + \max\{0, d_F(u) - 1\}$.

**Proof.** We may assume that $k \geq 2$, as the assertions trivially hold when $k = 1$. Let $S \subseteq V(G)$. If $G$ is $k$-edge-connected and $S \neq \emptyset$, then by Lemma 4.4, we have

$$\omega(G \setminus S) \leq \sum_{v \in S} \frac{dg(v)}{k} - \frac{2}{k}e_G(S) \leq \frac{k-1}{k} + \sum_{v \in S} (\eta(v) - 2) + 2 - \frac{2}{k}(e_G(S) + 1),$$

where $\eta(u) = \frac{dg(u)+2}{k} - \frac{k-1}{k}$ and $\eta(v) = \frac{dg(v)}{k} + 2$ for all $v \in V(G) \setminus u$. If $G$ is $k$-tree-connected, then by Lemma 4.4, we also have

$$\omega(G \setminus S) \leq \sum_{v \in S} \left(\frac{dg(v)}{k} - 1\right) + 1 - \frac{1}{k}e_G(S) \leq \frac{k-1}{k} + \sum_{v \in S} (\eta(v) - 2) + 2 - \frac{1}{k}(e_G(S) + 1),$$

where $\eta(u) = \frac{dg(u)+1}{k} - \frac{k-1}{k}$ and $\eta(v) = \frac{dg(v)}{k} + 1$ for all $v \in V(G) \setminus u$. Thus the first two assertions follow from Theorem 4.2 for $\lambda \in \{2/k, 1/k\}$. The second two assertions can similarly be proved using Corollary 4.3. \(\square\)

**Corollary 4.6.** If $G$ is a $(r-2)$-edge-connected $r$-regular graph with $r \geq 3$, then every $(g', f')$-factor can be extended to a connected $(g', f' + 2)$-factor, where $g'$ is a nonnegative integer-valued function on $V(G)$ and $f'$ is a positive integer-valued function on $V(G)$.
Proof. Let $H$ be a $(g', f')$-factor of $G$, and let $F$ be a spanning forest of $H$ with the same vertex components. Extend $F$ to a spanning tree $T$ such that for each vertex $v$, $d_T(v) \leq 3 + \max\{0, d_F(v) - 1\}$. Since $E(T) \cap E(H) = E(F)$, $d_{T \cup H}(v) \leq 3 + \max\{0, d_H(v) - 1\}$, for all vertices $v$. Since $f'(v)$ is positive, we have $d_{T \cup H}(v) \leq 3 + f'(v) - 1$, whether $d_H(v) = 0$ or not. Thus $T \cup H$ is the desired connected factor. For an arbitrary vertex $u$, we can also have $d_{T \cup H}(u) \leq f(u) + f'(u)$. □

4.2 $K_{1,n}$-free simple graphs and $t$-tough graphs with $0 < t \leq 1$

In this subsection, we devote a stronger version to Theorem 1.2 that provides slight improvements for two known results which were discovered or rediscovered with a new proof in [10].

Theorem 4.7. Let $G$ be a connected graph. If for all $S \subseteq V(G)$, $\omega(G \setminus S) \leq \sum_{v \in S} (f(v) - 2) + 2$, then every spanning forest $F$ can be extended to a spanning tree $T$ such that for each vertex $v$, $d_T(v) \leq f(v) + \max\{0, d_F(v) - 1\}$, where $f$ is a positive integer-valued function on $V(G)$.

Proof. Apply Theorem 4.2 with $\eta = f$ and with $\lambda = 0$. □

Ellingham, Nam, and Voss [10] discovered the following result, when $g'$ is a positive function.

Corollary 4.8. Let $G$ be a connected graph. If for all $S \subseteq V(G)$, $\omega(G \setminus S) \leq \sum_{v \in S} (f(v) - 2) + 2$, then every $(g', f')$-factor can be extended to a connected $(g', f' + f - 1)$-factor, where $g'$ is a nonnegative integer-valued function on $V(G)$, and $f'$ and $f$ are positive integer-valued functions on $V(G)$.

Proof. Let $H$ be a $(g', f')$-factor of $G$, and let $F$ be a spanning forest of $H$ with the same vertex components. Extend $F$ to a spanning tree $T$ such that for each vertex $v$, $d_T(v) \leq f(v) + \max\{0, d_F(v) - 1\}$. Since $E(T) \cap E(H) = E(F)$, $d_{T \cup H}(v) \leq f(v) + \max\{0, d_H(v) - 1\}$, for all vertices $v$. Since $f'(v)$ is positive, we have $d_{T \cup H}(v) \leq f(v) + f'(v) - 1$, whether $d_H(v) = 0$ or not. Thus $T \cup H$ is the desired connected factor. □

Corollary 4.9. Let $G$ be a connected graph. If for all $S \subseteq V(G)$, $\omega(G \setminus S) \leq \sum_{v \in S} (f(v) - 2) + 2$, then $G$ has a spanning tree $T$ containing a given arbitrary matching such that for each vertex $v$, $d_T(v) \leq f(v)$, where $f$ is a positive integer-valued function on $V(G)$.

Lemma 4.10. ([10]) If $G$ is a connected $K_{1,n}$-free simple graph with $n \geq 3$, then $\omega(G \setminus S) \leq (n - 2)|S| + 1$ for all $S \subseteq V(G)$.

Xu, Liu, and Tokuda [41] discovered the following result, when $g'$ is a positive function.

Corollary 4.11. If $G$ is a connected $K_{1,n}$-free simple graph with $n \geq 3$, then every $(g', f')$-factor can be extended to a connected $(g', f' + n - 1)$-factor, where $g'$ is a nonnegative integer-valued function on $V(G)$ and $f'$ is a positive integer-valued function on $V(G)$. 

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4.3 Connected factors, matchings, and spanning trees

In this section, we devote a stronger version to Theorem 1 in [38] that provides some relationships between connected factors, matchings, and spanning trees.

**Theorem 4.12.** Let $G$ be a graph with the factor $F$ and spanning tree $T$. If $M$ is a matching of $F$ having exactly one edge of every non-trivial component of $F$ incident to some of the non-cut vertices of $F$, then $G$ has a connected factor $H$ containing $E(F) \setminus M$ such that for each vertex $v$,

$$d_F(v) \leq d_H(v) \leq d_T(v) + \max\{0, d_F(v) - 1\}.$$  

**Proof.** We may assume that $G = T \cup F$. Set $M = \{x_1y_1, \ldots, x_ty_t\}$, $X = \{x_i : 1 \leq i \leq t\}$, and $Y = \{y_i : 1 \leq i \leq t\}$. Assume that each $x_i \in X$ is not a cut vertex in $F$. Let $H$ be a connected factor of $G$ containing $E(F) \setminus M$ and let $T'$ be a spanning tree of $H$ such that the following conditions hold: (i) if $v \in V(H)$ and $d_H(v) = d_T(v) + d_F(v)$, then $E_T'(v) \cap E_F(v) = \emptyset$, (ii) if $x_i \in X$ and $x_iy_i \notin E(H)$, then $E_T'(x_i) \cap E_F(x_i) = \emptyset$.

Note that $G$ and $T$ are natural candidates for $H$ and $T'$. Consider $H$ with the minimum $|E(H)|$. We claim that $d_H(v) \leq d_T(v) + \max\{0, d_F(v) - 1\}$, for all vertices $v$. Suppose, by way of contradiction, that $d_H(u) = d_T(u) + d_F(u)$ and $d_F(u) > 0$, for some vertex $u$. Let $C$ be the non-trivial component of $F$ containing $u$ and let $x_iy_i$ be the single edge in $M \cap E(C)$. First, assume that $d_H(v) = d_T(v) + d_F(v)$, for all vertices $v$ in $V(C)$. Since $d_H(x_i) = d_G(x_i)$, we have $x_iy_i \notin E(H)$ and also item (i) implies that $E_T'(x_i) \cap E_F(x_i) = \emptyset$ and $x_iy_i \notin E(T')$. In this case, define $H' = H - x_iy_i$ and $T' = T'$. Next, assume that $d_H(v) < d_T(v) + d_F(v)$, for some vertex $v$ in $V(C)$. If $x_iy_i \in E(H)$, take $ab$ to be an edge on a $vu$-path in the connected graph $C$ such that $d_H(a) < d_T(a) + d_F(a)$ and $d_H(b) = d_T(b) + d_F(b)$. If $x_iy_i \notin E(H)$, take $ab$ to be an edge on a $y_iu$-path in the connected graph $C - x_i$ such that $d_H(a) < d_T(a) + d_F(a)$ and $d_H(b) = d_T(b) + d_F(b)$. Since $d_H(b) = d_G(b)$, we have $ab \in E(H)$ and also item (i) implies that $E_T'(b) \cap E_F(b) = \emptyset$ and $ab \notin E(T')$. Thus there is an edge $bc \in E(T')$ such that $T' - bc + ab$ is connected. In this case, define $H' = H - bc$ and $T'' = T' - bc + ab$. Note that $d_H(b) < d_G(b)$. Since $E_T'(b) \cap E_F(b) = \emptyset$, we have $bc \notin E(F)$ and so the graph $H'$ contains $E(F) \setminus M$. In both cases, it is not hard to check that $H'$ and $T''$ have the desired properties of $H$ and $T'$, while $|E(H')| < |E(H)|$, which is impossible and so the claim holds.

Now, among all such connected factors, consider $H$ with the maximum $|E(H) \cap M|$. We are going to prove that $d_H(v) \geq d_F(v)$, for all vertices $v$. If $v$ is a vertex that is not incident to the edges in $M \setminus E(H)$, then we obviously have $d_H(v) \geq d_{F \setminus M}(v) = d_F(v)$. If $x_i \in X$ and $x_iy_i \notin E(H)$, then by item (ii), $E_T'(x_i) \cap E_F(x_i) = \emptyset$ and so $d_H(x_i) = d_{F \setminus M}(v) + d_T'(x_i) \geq d_F(x_i)$. Suppose, by way of contradiction, that there is a vertex $y_i \in Y$ with $x_iy_i \notin E(H)$ such that $d_H(y_i) \leq d_F(y_i) - 1$. By item (ii), $E_T'(x_i) \cap E_F(x_i) = \emptyset$ and so $x_iy_i \notin E(T')$. Thus there is an edge $x_iz \in E(T')$ such that $T' - x_iz + x_iy_i$ is connected. Define
\[ H' = H - x_i z + x_i y_i \] and \[ T'' = T' - x_i z + x_i y_i. \] Note that \( d_{H'}(x_i) < d_G(x_i). \) Since \( E_{T'}(x_i) \cap E_F(x_i) = \emptyset, \) we have \( x_i z \notin E(F) \) and so the graph \( H' \) contains \( E(F) \setminus M. \) It is not hard to check that \( H' \) and \( T'' \) have the desired properties of \( H \) and \( T', \) while \( |E(H')| = |E(H)| \) and \( |E(H') \cap M| > |E(H) \cap M|, \) which is again impossible. Hence the theorem holds.

The next corollary can develop a result due to Rivera-Campo [34], who gave a sufficient condition for the existence of a spanning tree with bounded maximum degree containing a given arbitrary matching.

**Corollary 4.13.** If every matching of a graph \( G \) can be extended to a spanning \( f \)-tree, then every \((g', f')\)-factor can also be extended to a connected \((g', f' + f - 1)\)-factor, where \( g' \) is a nonnegative integer-valued function on \( V(G), \) and \( f' \) and \( f \) are positive integer-valued functions on \( V(G). \)

**Proof.** Let \( F \) be a \((g', f')\)-factor of \( G \) and consider \( M \) as a matching of \( F \) having exactly one edge of every non-trivial component of \( F \) incident to non-cut vertices of \( F. \) By the assumption, the graph \( G \) has a spanning \( f \)-tree \( T \) containing \( M. \) Theorem 4.12 implies that \( T \cup F \) has a connected factor \( H' \) containing \( E(F) \setminus M \) such that for each vertex \( v, \) \( d_F(v) \leq d_{H'}(v) \leq d_T(v) + \max\{0, d_F(v) - 1\}. \) Define \( H = H' \cup M. \) Since \( M \subseteq E(T) \cap E(F), \) for each vertex \( v, \) we still have \( d_H(v) \leq d_T(v) + \max\{0, d_F(v) - 1\}, \) whether \( v \) is incident to an edge in \( M \) or not. It is not hard to check that \( H \) is the desired connected \((g', f' + f - 1)\)-factor we are looking for.

Tokuda, Xu, and Wang [38] discovered the following result, when \( g' \) is a positive function.

**Corollary 4.14.** Let \( G \) be a graph. If \( G \) contains a \((g', f')\)-factor and a spanning \( f \)-tree, then \( G \) has a connected \((g', f' + f - 1)\)-factor, where \( g' \) is a nonnegative integer-valued function on \( V(G), \) and \( f' \) and \( f \) are positive integer-valued functions on \( V(G). \)

## 5 Applications to spanning closed walks

Our aim in this section is to prove a long-standing conjecture due to Jackson and Wormald [17] with a stronger version. Before doing so, we state some results on spanning parity forests.

### 5.1 Spanning parity \( f \)-forests

In 1985 Amahashi [1] introduced a criterion for the existence of a spanning odd forest with bounded maximum degree. Later, Yuting and Kano (1988) generalized it by establishing the following theorem. We denote below by \( odd(G) \) the number of components of \( G \) with odd order.
Theorem 5.1. ([43]) Let $G$ be a graph and let $f$ be an odd positive integer-valued function on $V(G)$. Then $G$ has a spanning $f$-forest with odd degrees if and only if for all $S \subseteq V(G)$,

$$\text{odd}(G \setminus S) \leq \sum_{v \in S} f(v).$$

Kano, Katona, and Szabó (2009) studied a more general version for Theorem 5.1 which gives a criterion for the existence of parity $f$-forests. We denote below by $\text{odd}_f(G)$ the number of components of $G$ with odd number of vertices $v$ with $f(v)$ odd.

Theorem 5.2. ([21]) Let $G$ be a graph and let $f$ be a nonnegative integer-valued function on $V(G)$. Then $G$ has a spanning $f$-forest $F$ such that for each vertex $v$, $d_F(v)$ and $f(v)$ have the same parity, if and only if for all $S \subseteq V(G)$,

$$\text{odd}_f(G \setminus S) \leq \sum_{v \in S} f(v).$$

In the following, we present some corollaries of Theorem 5.2 which will be used several times in this paper.

Corollary 5.3. Let $G$ be a graph with the positive integer-valued function $f$ on $V(G)$, and let $Q \subseteq V(G)$, where $|Q|$ is even. If for all $S \subseteq V(G)$,

$$\omega(G \setminus S) \leq \sum_{v \in S} (f(v) - 1) + 1,$$

then $G$ has a spanning $f$-forest $F$ such that for each vertex $v$, $d_F(v)$ is odd if and only if $v \in Q$.

Proof. For each $v$, define $f'(v)$ to be either $f(v)$ or $f(v) - 1$ such that $f'(v)$ is odd if and only if $v \in Q$. Let $S \subseteq V(G)$. By the assumption, $\text{odd}_{f'}(G \setminus S) \leq \omega(G \setminus S) \leq \sum_{v \in S} f'(v) + 1$. Clearly, $\sum_{v \in V(G)} f'(v)$ is even. It is easy to check that $\text{odd}_{f'}(G \setminus S) + \sum_{v \in S} f'(v)$ and $\sum_{v \in V(G)} f'(v)$ have the same parity and so $\text{odd}_{f'}(G \setminus S)$ and $\sum_{v \in S} f'(v)$ have the same parity. Thus $\text{odd}_{f'}(G \setminus S) \leq \sum_{v \in S} f'(v)$. By Theorem 5.2, the graph $G$ has a spanning $f'$-forest $F$ such that for each vertex $v$, $d_{F'}(v)$ and $f'(v)$ have the same parity. Hence the the proof is completed. \hfill \Box

The following result improves the upper bounds in Theorem 4.5, when the existence of parity forests are considered. The special case $k = 1$ of the this result is well-known, see [36, Lemma 1].

Corollary 5.4. Let $G$ be a graph with $Q \subseteq V(G)$, where $|Q|$ is even. Then $G$ has a spanning forest $F$ such that for each vertex $v$,

$$d_F(v) \leq \left\lfloor \frac{d_G(v)}{k} \right\rfloor + 1, \quad \text{if } G \text{ is } k\text{-edge-connected};$$

$$\left\lfloor \frac{d_G(v)}{k} \right\rfloor, \quad \text{if } G \text{ is } k\text{-tree-connected},$$

and also $d_F(v)$ is odd if and only if $v \in Q$.

Proof. Apply Lemma 4.4 and Corollary 5.3. \hfill \Box
5.2 Jackson-Wormald Conjecture is true

The following theorem gives a sufficient condition for the existence of \( f \)-walks passing through the edges of a given arbitrary matching. Note that if a graph admits an \( f \)-walk passing through the edges of a given matching, then that graph must have a spanning \((f+1)\)-tree containing the same matching. To prove this, apply Theorem 4.5 on the 2-edge-connected Eulerian graph which can be obtained from that spanning closed walk.

**Theorem 5.5.** Let \( G \) be a graph and let \( f \) be a positive integer-valued function on \( V(G) \). If for all \( S \subseteq V(G) \),
\[
\omega(G \setminus S) \leq \sum_{v \in S} (f(v) - 1) + 1,
\]
then \( G \) has an \( f \)-walk passing through the edges of a given arbitrary matching.

**Proof.** The graph \( G \) must automatically be connected, because \( \omega(G \setminus \emptyset) = 1 \). Thus by Theorem 4.7, the graph \( G \) has a spanning \((f+1)\)-tree \( T \) containing a given arbitrary matching. By Corollary 5.3, the graph \( G \) contains a spanning \( f \)-forest \( F \) such that for each vertex \( v \), \( d_F(v) \) and \( d_T(v) \) have the same parity. Add a copy of \( F \) to \( T \) and call the resulting connected graph \( H \). For each vertex \( v \), \( d_H(v) = d_T(v) + d_F(v) \leq 2f(v) + 1 \). Therefore, the graph \( H \) admits an \( f \)-trail and so \( G \) admits an \( f \)-walk. Note that if the bound on \( \omega(G \setminus S) \) pushed up by one, this method can only guarantee the existence of a spanning \( f \)-walk not necessarily closed. \( \square \)

The following corollary confirms Conjecture 2.1 in [17]. Note that there are infinitely many graphs with toughness approaching \( \frac{1}{n-5/8} \) having no \( n \)-walks, which were constructed by Ellingham and Zha [11].

**Corollary 5.6.** Every \( \frac{1}{(n-1)} \)-tough graph with \( n \geq 2 \) admits an \( n \)-walk.

The next result improves Theorem 4.2 in [17] and implies Corollary 3.1 in [22]. Note that there are infinitely many \( k \)-connected \( K_{1,n} \)-free simple graphs with \( k \geq 2 \) and \( n \geq 3 \) having no \( \lceil \frac{n-1}{k} \rceil \)-walks, which were constructed by Jin and Li [19].

**Corollary 5.7.** Every \( k \)-connected \( K_{1,n} \)-free simple graph with \( n \geq 3 \) has an \( (\lceil \frac{n-1}{k} \rceil + 1) \)-walk.

**Proof.** We repeat the proof of Theorem 4.2 in [17]. Let \( S \) be a nonempty subset of \( V(G) \). Since \( G \) is \( k \)-connected, every component of \( G \setminus S \) is joined to at least \( k \) vertices in \( S \). Since \( G \) is \( K_{1,n} \)-free, every vertex of \( S \) is joined to at most \( n-1 \) components of \( G \setminus S \). Hence \( \omega(G \setminus S)k \leq (n-1)S \). Thus the corollary immediately follows from Corollary 5.6 with replacing \( \lceil (n-1)/k \rceil + 1 \) instead of \( n \). \( \square \)

**Corollary 5.8.**([17]) Every connected \( K_{1,n} \)-free simple graph with \( n \geq 3 \) has an \( (n-1) \)-walk.
Proof. Apply Lemma 4.10 and Theorem 5.5.

The next result confirms Conjecture 23 in [10]. Note that there are infinitely many \(r\)-edge-connected \(r\)-regular simple graphs with \(r \geq 3\) having no 1-walks, which were constructed by Meredith [25].

**Corollary 5.9.** Every \(r\)-edge-connected \(r\)-regular graph admits a 2-walk.

**Proof.** Apply Lemma 4.4 and Corollary 5.6. □

6 Highly tree-connected spanning subgraphs with small degrees

6.1 Basic tools

In this subsection, we present some basic tools for working with tree-connected graphs. We begin with the following well-known result which gives a criterion for a graph to have \(m\) edge-disjoint spanning trees.

**Theorem 6.1.** (Nash-Williams [27] and Tutte [39]) A graph \(G\) is \(m\)-tree-connected if and only if for every partition \(P\) of \(V(G)\), \(e_G(P) \geq m(|P| - 1)\).

For every vertex \(v\) of a graph \(G\), consider an induced \(m\)-tree-connected subgraph of \(G\) containing \(v\) with the maximal order. It is known that these subgraphs are unique and decompose the vertex set of \(G\) [7]. In fact, these subgraphs are the \(m\)-tree-connected components of \(G\) that already introduced in the Introduction. The following observation simply shows that these subgraphs are well-defined.

**Observation 6.2.** Let \(G\) be a graph with \(X \subseteq V(G)\) and \(Y \subseteq V(G)\). If \(G[X]\) and \(G[Y]\) are \(m\)-tree-connected and \(X \cap Y \neq \emptyset\), then \(G[X \cup Y]\) is also \(m\)-tree-connected.

**Proof.** Let \(P\) be a partition of \(X \cup Y\). Define \(P_1\) and \(P_2\) to be the partitions of \(X\) and \(Y\) with

\[
P_1 = \{A \cap X : A \in P \text{ and } A \cap X \neq \emptyset\} \text{ and } P_2 = \{A \in P : A \cap X = \emptyset\} \cup \{Y \cap \bigcup_{A \cap X \neq \emptyset} A\}.
\]

Since \(|P_1| + (|P_2| - 1) = |P|\), by Theorem 6.1, we have

\[
e_{G[X \cup Y]}(P) \geq e_{G[X]}(P_1) + e_{G[Y]}(P_2) \geq m(|P_1| - 1) + m(|P_2| - 1) = m(|P| - 1).
\]

Again, by applying Theorem 6.1, the graph \(G[X \cup Y]\) must be \(m\)-tree-connected. □

The next observation presents a simple way for deducing tree-connectivity of a graph from whose special spanning subgraphs and whose contractions.
Observation 6.3. Let $G$ be a graph with $X \subseteq V(G)$. If $G[X]$ and $G/X$ are $m$-tree-connected, then $G$ itself is $m$-tree-connected.

Proof. It is enough to apply the same argument in the proof of Observation 6.2, by setting $Y = V(G)$. Note that we again have $e_{G[Y]}(P_2) \geq m(|P_2| - 1)$, since $G/X$ is $m$-tree-connected. □

The following theorem is a valuable tool for finding a pair of edges such that replacing them preserves tree-connectivity of a given spanning subgraph.

Theorem 6.4. Let $G$ be a graph with the $m$-tree-connected spanning subgraph $H$, and let $M$ be a nonempty edge subset of $E(H)$. If a given edge $e' \in E(G) \setminus E(H)$ joins different $m$-tree-connected components of $H \setminus M$, then there is an edge $e$ belonging to $M$ such that $H - e + e'$ is still $m$-tree-connected.

Proof. We proceed by induction on $|M|$. Assume first that $M = \{e\}$. Suppose, by way of contradiction, that $H - e + e'$ is not $m$-tree-connected. Consequently, by Theorem 6.1, there is a partition $P$ of $V(H')$ such that $e_{H'}(P) < m(|P| - 1)$, where $H' = H - e + e'$. On the other hand, by the assumption, we have $e_{H}(P) \geq m(|P| - 1)$. These inequalities imply that $e$ joins different parts of $P$, both ends of $e'$ lie in the same part $C$ of $P$, and also $e_{H}(P) = m(|P| - 1)$. To obtain a contradiction, it suffices to show that $H[C]$ is $m$-tree-connected, which implies that both ends of $e'$ lie in the same $m$-tree-connected components of $H - e$. For this purpose, one can apply Theorem 6.1 and use the following inequalities

$$e_{H[C]}(P) \geq e_{H}(P') - e_{H}(P) \geq m(|P'| - 1) - m(|P| - 1) = m(|P| - 1),$$

where $\mathcal{P}$ is an arbitrary partition of $C$ and $P'$ is a new partition of $V(H)$ with $P' = (P - C) \cup \mathcal{P}$. Now, assume that $|M| \geq 2$. Pick $e \in M$. If $e'$ joins different $m$-tree-connected components of $H - (M \setminus e)$, then the theorem follows by induction. Suppose that both ends of $e'$ lie in the same $m$-tree-connected component of $H - (M \setminus e)$ with the vertex set $C$. By the assumption, both ends of $e$ must also lie in $C$ and moreover $e'$ joins different $m$-tree-connected components of $H[C] - e$. By applying induction to $H[C]$, the graph $H[C] - e + e'$ must be $m$-tree-connected. Thus by Observation 6.3, $H - e + e'$ is also $m$-tree-connected. Hence the theorem holds. □

Observation 6.5. Let $G$ be a graph with the $m$-critical spanning subgraph $F$. If $H$ is an $m$-tree-connected spanning subgraph of $G$ containing $F$ with the minimum number of edges, then $H$ is minimally $m$-tree-connected.

Proof. Let $e$ be an edge of $H$ joining different $m$-tree-connected components of $F$. If $H - e$ is $m$-tree-connected, then we must have $e \in E(F)$ and so by Theorem 6.4, there is an edge $e' \in E(H - e) \setminus E(F)$ such that the graph $(H - e) + e - e'$ is $m$-tree-connected, which is impossible. Therefore, the graph obtained from $H$ by contracting $m$-tree-connected components of $F$ is minimally $m$-tree-connected. Since every $m$-tree-connected component of $F$ is minimally $m$-tree-connected, the graph $H$ itself is minimally $m$-tree-connected. □
Observation 6.6. For any graph $G$ with the spanning subgraph $H$, we have $\Omega_m(G) \leq \Omega_m(H)$. Furthermore, the equality holds if and only if every edge of $E(G) \setminus E(H)$ whose ends lie in the same $m$-tree-connected component of $H$.

Proof. We may assume that $H = G - e$ for an edge $e$. If $e$ joins different $m$-tree-connected components of $G$, then we obviously have $\Omega_m(H) = \Omega_m(G) + \frac{1}{m}$. So, suppose both ends of $e$ lie in the same $m$-tree-connected component of $G$ with the vertex set $X$. If $G[X]\setminus e$ is also $m$-tree-connected, then $\Omega_m(H) = \Omega_m(G)$. Assume that $G[X]\setminus e$ is not $m$-tree-connected. Thus by Theorem 6.1, there is a partition $P$ of $X$ such that $e_{G[X]\setminus e}(P) = m(|P| - 1) - 1$. In this case, we also have $\Omega_m(H) = \Omega_m(G) + \frac{1}{m}$. \hfill $\square$

The following proposition establishes an important property of minimally $m$-tree-connected graphs.

Proposition 6.7. If $H$ is a minimally $m$-tree-connected graph and $S \subseteq V(H)$, then

$$\Omega_m(H \setminus S) = \sum_{v \in S} \left( \frac{d_H(v)}{m} - 1 \right) + 1 - \frac{1}{m} e_H(S).$$

Proof. Let $P$ be the partition of $V(H) \setminus S$ obtained from the $m$-tree-connected components of $H \setminus S$. Obviously, $e_H(P \cup \{v : v \in S\}) = \sum_{v \in S} d_H(v) - e_H(S) + e_{H \setminus S}(P)$. Since $|E(H)| = m(|V(H) - 1|)$ and for any $C \in P$, $e_H(C) \geq m(|C| - 1)$, one can easily check that $m(|P| - |S| - 1) = e_H(P \cup \{v : v \in S\})$. Therefore, we must have $m|P| - e_{H \setminus S}(P) = \sum_{v \in S} (d_H(v) - m) + m - e_H(S)$, which can complete the proof. \hfill $\square$

The following lemma gives useful information about the existence of non-trivial $m$-tree-connected components and is similar to a result in [37, Section 2].

Lemma 6.8 ([42]). Every graph $G$ of order at least two containing at least $m(|V(G)| - 1)$ edges has an $m$-tree-connected subgraph with at least two vertices.

Proof. The proof is by induction on $|V(G)|$. For $|V(G)| = 2$, the proof is clear. Assume $|V(G)| \geq 3$. Suppose the lemma is false. By Theorem 6.1, there exists a partition $P$ of $V(G)$ such that $e_G(P) < m(|P| - 1)$. By induction hypothesis, for every $C \in P$, we have $e_G(C) \leq m(|C| - 1)$, whether $|C| = 1$ or not. Therefore,

$$m(|V(G)| - 1) \leq |E(G)| = e_G(P) + \sum_{C \in P} e_G(C) < m(|P| - 1) + m \sum_{C \in P} (|C| - 1) \leq m(|V(G)| - 1).$$

This result is a contradiction, as desired. \hfill $\square$

The following result describes a relationship between tree-connectivity measures of graphs.
**Theorem 6.9.** For every graph $G$, we have 
\[ \Omega_1(G) \leq \Omega_2(G) \leq \ldots \leq |V(G)|. \]
Furthermore, $G$ is $m$-tree-connected if and only if $\Omega_m(G) = 1$.

**Proof.** Let $P$ and $P'$ be the partitions of $V(G)$ obtained from the $m$-tree-connected and $m'$-tree-connected components of $G$, where $m \leq m'$. If $G$ is $m$-tree-connected, then we have $|P| = 1$ and so $e_G(P) = 0$ and $\Omega_m(G) = 1$. Oppositely, if $G$ is not $m$-tree-connected, then by Lemma 6.8, $e_G(P) < m(|P| - 1)$ and hence $\Omega_m(G) > 1$. For every $C \in P$, define $P'_C$ to be the partition of $C$ obtained from the vertex sets of $P'$. By applying Lemma 6.8 to the graph $G[C]$, we have $e_G[C](P'_C) \leq m'(|P'_C| - 1)$, whether $|P'_C| = 1$ or not. Thus 
\[
e_G(P') - e_G(P) = \sum_{C \in P} e_G[C](P'_C) \leq m' \sum_{C \in P} (|P'_C| - 1) = m'(|P'| - |P|).
\]

Therefore, 
\[ \Omega_m(G) = |P| - \frac{1}{m}e_G(P) \leq |P| - \frac{1}{m'}e_G(P) \leq |P'| - \frac{1}{m'}e_G(P') = \Omega_{m'}(G) \]
This equality can complete the proof. \qed

### 6.2 Structures of $m$-tree-connected spanning subgraphs with the minimum total excess

Here, we state following fundamental theorem, which gives much information about $m$-tree-connected spanning subgraphs with the minimum total excess. In Section 7, we present a stronger version for this result with a proof, but we feel that it helpful to state the proof of this special case before the general version.

**Theorem 6.10.** Let $G$ be an $m$-tree-connected graph and let $h$ be an integer-valued function on $V(G)$. If $H$ is a minimally $m$-tree-connected spanning subgraph of $G$ with the minimum total excess from $h$, then there exists a subset $S$ of $V(G)$ with the following properties:

1. $\Omega_m(G \setminus S) = \Omega_m(H \setminus S)$.
2. $S \supseteq \{ v \in V(G) : d_H(v) > h(v) \}$.
3. For each vertex $v$ of $S$, $d_H(v) \geq h(v)$.

**Proof.** Define $V_0 = \emptyset$ and $V_1 = \{ v \in V(H) : d_H(v) > h(v) \}$. For any $S \subseteq V(G)$ and $u \in V(G) \setminus S$, let $A(S, u)$ be the set of all minimally $m$-tree-connected spanning subgraphs $H'$ of $G$ such that $d_{H'}(v) \leq h(v)$ for all $v \in V(G) \setminus V_1$, and $H'$ and $H$ have the same edges, except for some of the edges of $G$ whose ends are in $X$, where $H[X]$ is the $m$-tree-connected component of $H \setminus S$ containing $u$. Note that the graphs $H'[X]$ must be $m$-tree-connected. Now, for each integer $n$ with $n \geq 2$, recursively define $V_n$ as follows:
\[ V_n = V_{n-1} \cup \{ v \in V(G) \setminus V_{n-1} : d_{H'}(v) \geq h(v), \text{ for all } H' \in A(V_{n-1}, v) \}. \]
Now, we prove the following claim.

**Claim.** Let \( x \) and \( y \) be two vertices in different \( m \)-tree-connected components of \( H \setminus V_{n-1} \). If \( xy \in E(G) \setminus E(H) \), then \( x \in V_n \) or \( y \in V_n \).

**Proof of Claim.** By induction on \( n \). For \( n = 1 \), the proof is clear. Assume that the claim is true for \( n-1 \). Now we prove it for \( n \). Suppose otherwise that vertices \( x \) and \( y \) are in different \( m \)-tree-connected components of \( H \setminus V_{n-1} \), respectively, with the vertex sets \( X \) and \( Y \), \( xy \in E(G) \setminus E(H) \), and \( x, y \notin V_n \). Since \( x, y \notin V_n \), there exist \( H_x \in \mathcal{A}(V_{n-1}, x) \) and \( H_y \in \mathcal{A}(V_{n-1}, y) \) with \( d_{H_x}(x) < h(x) \) and \( d_{H_y}(y) < h(y) \). By the induction hypothesis, \( x \) and \( y \) are in the same \( m \)-tree-connected component of \( H \setminus V_{n-2} \) with the vertex set \( Z \) so that \( X \cup Y \subseteq Z \). Hence we must have \( Z \cap V_{n-1} \neq \emptyset \) and so by Theorem 6.4, there exists an edge \( zz' \) of \( H[Z] \) such that \( z \in Z \cap V_{n-1} \) and \( H[Z] - zz' + xy \) is \( m \)-tree-connected. Now, let \( H' \) be the spanning subgraph of \( G \) with 
\[
E(H') = E(H) - zz' + xy - E(H[X]) + E(H_x[X]) - E(H[Y]) + E(H_y[Y]).
\]
By repeatedly applying Observation 6.3, one can easily check that \( H' \) is \( m \)-tree-connected. For each \( v \in V(H') \), we have 
\[
d_{H'}(v) = \begin{cases} 
  d_{H_x}(v), & \text{if } v \in X \setminus \{ x, z' \}; \\
  d_{H_y}(v), & \text{if } v \in Y \setminus \{ y, z' \}; \\
  d_H(v), & \text{if } v \notin X \cup Y \cup \{ z, z' \}.
\end{cases}
\]
If \( n \geq 3 \), then it is not hard to see that \( d_{H'}(z) < d_H(z) \leq h(z) \) and \( H' \) lies in \( \mathcal{A}(V_{n-2}, z) \). Since \( z \in V_{n-1} \setminus V_{n-2} \), we arrive at a contradiction. For the case \( n = 2 \), since \( z \in V_1 \), it is easy to see that \( h(z) \leq d_{H'}(z) < d_H(z) \) and \( te(H', h) < te(H, h) \), which is again a contradiction. Hence the claim holds.

Obviously, there exists a positive integer \( n \) such that and \( V_1 \subseteq \cdots \subseteq V_{n-1} = V_n \). Put \( S = V_n \). Since \( S \supseteq V_1 \), Condition 2 clearly holds. For each \( v \in V_i \setminus V_{i-1} \) with \( i \geq 2 \), we have \( H \in \mathcal{A}(V_{i-1}, v) \) and so \( d_H(v) \geq h(v) \). This establishes Condition 3. Because \( S = V_n \), the previous claim implies Condition 1 and completes the proof. \( \square \)

### 6.3 Sufficient conditions depending on tree-connectivity measures

The following theorem is essential in this section.

**Theorem 6.11.** Let \( G \) be an \( m \)-tree-connected graph with \( X \subseteq V(G) \). Let \( \lambda \in [0, 1/m] \) be a real number and let \( \eta : X \to (m\lambda + \frac{m-1}{m}, \infty) \) be a real function on \( X \). If for all \( S \subseteq X \),
\[
\Omega_m(G \setminus S) < \frac{1}{m} + \sum_{v \in S} (\eta(v) - 2) + 2 - \lambda(c_G(S) + m),
\]
then \( G \) has an \( m \)-tree-connected spanning subgraph \( H \) such that for each \( v \in X \),
\[
d_H(v) \leq \lceil mn\eta(v) - m^2\lambda \rceil.
\]
Proof. For each vertex \( v \), define

\[
    h(v) = \begin{cases} 
        d_G(v) + 1, & \text{if } v \notin X; \\
        \lceil m\eta(v) - m^2\lambda \rceil, & \text{if } v \in X.
    \end{cases}
\]

Let \( H \) be a minimally \( m \)-tree-connected spanning subgraph of \( G \) with the minimum total excess from \( h \). Define \( S \) to be a subset of \( V(G) \) with the properties described in Theorem 6.10. If \( S \) is empty, then \( te(H, h) = 0 \) and the theorem clearly holds. So, suppose \( S \) is nonempty. Obviously, \( S \subseteq X \). By Proposition 6.7,

\[
    \sum_{v \in S} h(v) + te(H, h) = \sum_{v \in S} d_H(v) = m\Omega_m(H \setminus S) + m|S| - m + e_H(S).
\]

and so

\[
    \sum_{v \in S} h(v) + te(H, h) = m\Omega_m(G \setminus S) + m|S| - m + e_H(S). \tag{7}
\]

Also, by the assumption, we have

\[
    m\Omega_m(G \setminus S) + m|S| - m + e_H(S) < 1 + \sum_{v \in S} (m\eta(v) - m) - m\lambda(e_G(S) + m) + e_H(S) + m. \tag{8}
\]

Since \( e_H(S) \leq e_G(S) \) and \( e_H(S) \leq m(|S| - 1) \),

\[
    -m\lambda(e_G(S) + m) + e_H(S) + m \leq -m\lambda(e_H(S) + m) + e_H(S) + m \leq (1 - m\lambda)m|S|. \tag{9}
\]

Therefore, Relations (7), (8), and (9) can conclude that

\[
    \sum_{v \in S} h(v) + te(H, h) < 1 + \sum_{v \in S} (m\eta(v) - m^2\lambda).
\]

On the other hand, by the definition of \( h(v) \),

\[
    \sum_{v \in S} (m\eta(v) - m^2\lambda - h(v)) \leq 0.
\]

Hence \( te(H, h) = 0 \) and the theorem holds. \( \square \)

When we consider independent sets \( X \), the theorem becomes simpler as the following result.

**Corollary 6.12.** Let \( G \) be an \( m \)-tree-connected graph with the independent set \( X \subseteq V(G) \). Let \( \eta : X \rightarrow (\frac{m-1}{m}, \infty) \) be a real function. If for all \( S \subseteq X \),

\[
    \Omega_m(G \setminus S) < \frac{1}{m} + \sum_{v \in S} (\eta(v) - 1) + 1,
\]

then \( G \) has an \( m \)-tree-connected spanning subgraph \( H \) such that for each \( v \in X \), \( d_H(v) \leq \lceil m\eta(v) \rceil \).

**Proof.** Apply Theorem 6.11 with \( \lambda = 1/m \) and with replacing \( \eta + 1 \) instead of \( \eta \). Note that \( e_G(S) = 0 \) for all \( S \subseteq X \). \( \square \)
The next corollary gives a sufficient condition, similar to the toughness condition, that guarantees the existence of a highly tree-connected spanning subgraph with bounded maximum degree.

**Corollary 6.13.** Let $G$ be an $m$-tree-connected graph and let $n$ be a positive integer. If for all $S \subseteq V(G)$,
\[ \Omega_m(G \setminus S) \leq \frac{n}{m}|S| + 2, \]
then $G$ has an $m$-tree-connected spanning subgraph $H$ such that $\Delta(H) \leq 2m + n$.

**Proof.** Apply Theorem 6.11 with $\lambda = 0$ and $\eta(v) = 2 + n/m$. \qed

### 6.4 Graphs with high edge-connectivity

Highly edge-connected graphs are natural candidates for graphs satisfying the assumptions of Theorem 6.11. We examine them in this subsection, beginning with the following extended version of Lemma 4.4.

**Lemma 6.14.** Let $G$ be a graph with $S \subseteq V(G)$. Then
\[ \Omega_m(G \setminus S) \leq \begin{cases} \sum_{v \in S} \frac{d_G(v)}{k} - \frac{2}{k} e_G(S), & \text{if } G \text{ is } k\text{-edge-connected, } k \geq 2m, \text{ and } S \neq \emptyset; \\ \sum_{v \in S} \left( \frac{d_G(v)}{k} - 1 \right) + 1 - \frac{1}{k} e_G(S), & \text{if } G \text{ is } k\text{-tree-connected and } k \geq m. \end{cases} \]

**Proof.** Let $P$ be the partition of $V(G) \setminus S$ obtained from the $m$-tree-connected components of $G \setminus S$. Obviously, we have
\[ e_G(P \cup \{v : v \in S\}) = \sum_{v \in S} d_G(v) - e_G(S) + e_{G \setminus S}(P). \]

If $G$ is $k$-edge-connected and $S \neq \emptyset$, then there are at least $k$ edges of $G$ with exactly one end in $C$, for any $C \in P$. Thus $e_G(P \cup \{v : v \in S\}) \geq k|P| - e_{G \setminus S}(P) + e_G(S)$ and so if $k \geq 2m$, then
\[ \Omega_m(G \setminus S) = |P| - \frac{1}{m} e_{G \setminus S}(P) \leq |P| - \frac{2}{k} e_{G \setminus S}(P) \leq \sum_{v \in S} \frac{d_G(v)}{k} - \frac{2}{k} e_G(S). \]

When $G$ is $k$-tree-connected, we have $e_G(P \cup \{v : v \in S\}) \geq k(|P| + |S| - 1)$ and so if $k \geq m$, then
\[ \Omega_m(G \setminus S) = |P| - \frac{1}{m} e_{G \setminus S}(P) \leq |P| - \frac{1}{k} e_{G \setminus S}(P) \leq \sum_{v \in S} \left( \frac{d_G(v)}{k} - 1 \right) + 1 - \frac{1}{k} e_G(S). \]

These inequalities complete the proof. \qed

Now, we are ready to strengthen Theorems 1.4 and 1.5 as mentioned in the abstract.
**Theorem 6.15.** Let $G$ be a graph with $X \subseteq V(G)$. Then $G$ has an $m$-tree-connected spanning subgraph $H$ such that for each $v \in X$,

$$d_H(v) \leq \begin{cases} \left\lceil \frac{n}{k} (d_G(v) - 2m) \right\rceil + 2m, & \text{if } G \text{ is } k\text{-edge-connected and } k \geq 2m; \\ \left\lceil \frac{n}{k} (d_G(v) - m) \right\rceil + m, & \text{if } G \text{ is } k\text{-tree-connected and } k \geq m; \\ \left\lceil \frac{n}{k} d_G(v) \right\rceil + m, & \text{if } G \text{ is } k\text{-edge-connected, } k \geq 2m, \text{ and } X \text{ is independent}; \\ \left\lceil \frac{n}{k} d_G(v) \right\rceil, & \text{if } G \text{ is } k\text{-tree-connected, } k \geq m, \text{ and } X \text{ is independent.} \end{cases}$$

Furthermore, for a given arbitrary vertex $u$ the upper can be reduced to $\left\lfloor \frac{n}{k} d_G(u) \right\rfloor$.

**Proof.** Let $S \subseteq V(G)$. If $G$ is $k$-edge-connected, $k \geq 2m$, and $S \neq \emptyset$, then by Lemma 6.14, we have

$$\Omega_m(G \setminus S) \leq \sum_{v \in S} \left(\frac{d_G(v)}{k} - 1\right) + \frac{1}{k} e_G(S) \leq \frac{k-1}{km} + \sum_{v \in S} (\eta(v) - 2) + 2 - \frac{2}{k} (e_G(S) + m),$$

where $\eta(u) = \frac{d_G(u) + 2m}{k} - \frac{k-1}{km}$ and $\eta(v) = \frac{d_G(v)}{k} + 2$ for any $v \in V(G) \setminus u$. If $G$ is $k$-tree-connected and $k \geq m$, then by Lemma 6.14, we also have

$$\Omega_m(G \setminus S) \leq \sum_{v \in S} \left(\frac{d_G(v)}{k} - 1\right) + \frac{1}{k} e_G(S) \leq \frac{k-1}{km} + \sum_{v \in S} (\eta(v) - 2) + 2 - \frac{1}{k} (e_G(S) + m),$$

where $\eta(u) = \frac{d_G(u) + m}{k} - \frac{k-1}{km}$ and $\eta(v) = \frac{d_G(v)}{k} + 1$ for any $v \in V(G) \setminus u$. Thus the first two assertions follow from Theorem 6.11 for $\lambda \in \{2/k, 1/k\}$. The second two assertions can similarly be proved using Corollary 6.12. \qed

**Corollary 6.16.** Every $2m$-edge-connected graph $G$ has an $m$-tree-connected spanning subgraph $H$ such that for each vertex $v$,

$$d_H(v) \leq \left\lceil \frac{d_G(v)}{2} \right\rceil + m.$$ 

Furthermore, for a given arbitrary vertex $u$ the upper can be reduced to $\left\lfloor \frac{d_G(u)}{2} \right\rfloor$.

**Proof.** Apply Theorem 6.15 with $k = 2m$. \qed

In the following, we shall give two simpler proofs for Corollary 6.16 inspired by the proofs that introduced in [2, 23, 35] for the special case $m = 1$. For this purpose, we need some well-known results. Note that the first one was also implicitly appeared in [5].

**Theorem 6.17.** (Nash-Williams [26], see Theorem 2.1 in [2]) Every $2m$-edge-connected graph $G$ has an $m$-arc-strong orientation such that for each vertex $v$, $\lfloor d_G(v)/2 \rfloor \leq d_G^+(v) \leq \lceil d_G(v)/2 \rceil$.

**Theorem 6.18.** (Edmonds [9]) Let $G$ be a directed graph with $u \in V(G)$. If $d_G^+(X) \geq m$ for all $X \subseteq V(G)$ with $u \in X$, then $G$ has an $m$-tree-connected spanning subgraph $H$ such that $d_H^+(u) = 0$, and $d_H^-(v) = m$ for all $v \in V(G) \setminus u$, where $d_G^+(X)$ denotes the number of outgoing edges in $G$ from $X$. 

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**Second proof of Corollary 6.16.** Consider an \( m \)-arc-strong orientation for \( G \) with the properties stated in Theorem 6.17. We may assume that the out-degree of \( u \) is equal to \( |d_G(u)/2| \); otherwise, we reverse the orientation of \( G \). Take \( H \) to be an \( m \)-tree-connected spanning subgraph of \( G \) with the properties stated in Theorem 6.18. For each vertex \( v \), we have \( d_H(v) = d_H^+(v) + d_H^-(v) \leq d_G^+(v) + d_G^-(v) \leq \lfloor d_G(v)/2 \rfloor + m \). In particular, \( d_H(u) \leq d_G^+(u) + d_G^-(u) \leq \lfloor d_G(u)/2 \rfloor \). Thus \( H \) is the desired spanning subgraph we are looking for. \( \square \)

**Remark 6.19.** Note that an alternative proof for Corollary 6.12 can also be provided as with the second proof of Corollary 6.16, using a combination of Edmonds’ Theorem [9] and a special case of Theorem 1 in [15].

**Theorem 6.20.** (Mader [24], see Section 3 in [29]) Let \( G \) be a \( 2m \)-edge-connected graph with \( z \in V(G) \). If \( d_G(z) \geq 2m + 2 \), then there are two edges \( xz \) and \( yz \) incident to \( z \) such that after removing them, and inserting a new edge \( xy \) for the case \( x \neq y \), the resulting graph is still \( 2m \)-edge-connected.

**Third proof of Corollary 6.16.** By induction on the sum of all \( d_G(v) - 2m - 1 \) taken over all vertices \( v \) with \( d_G(v) \geq 2m + 2 \). First suppose that \( \Delta(G) \leq 2m + 1 \). Let \( M \subseteq E_G(u) \) be an edge set of size \( m \) or \( m + 1 \) with respect to \( d_G(u) = 2m \) or \( d_G(u) = 2m + 1 \). We claim that \( G \setminus M \) is \( m \)-tree-connected and so the theorem obviously holds by setting \( H = G \setminus M \). Otherwise, Theorem 6.1 implies that there is a partition \( P \) of \( V(G) \) such that \( m(|P| - 1) > e_{G \setminus M}(P) \geq e_G(P) - |M| \geq m|P| - |M| \). This implies that the edges of \( M \) join different parts of \( P \), \( |M| = m + 1 \), \( d_G(u) = 2m + 1 \), and \( d_G(C) = 2m \) for all \( C \in P \), where \( d_G(C) \) denotes the number of edges of \( G \) with exactly one end in \( C \). It is not hard to check that \( (E_G[U, \overline{U}] \cup E_G(u)) \setminus M \) forms an edge cut of size \( 2m - 1 \) for \( G \), which is contradiction, where \( u \in U \subset P \) and \( E_G[U, \overline{U}] \) denotes the set of edges of \( G \) with exactly one end in \( U \).

Now, suppose that there is a vertex \( z \) with \( d_G(z) \geq 2m + 2 \). By Theorem 6.20, there are two edges \( xz \) and \( yz \) incident to \( z \) such that after removing them, and inserting a new edge \( xy \) for the case \( x \neq y \), the resulting graph \( G' \) is still \( 2m \)-edge-connected. By the induction hypothesis, the graph \( G' \) has a spanning subgraph \( H' \) containing \( m \) edge-disjoint spanning trees \( T_1, \ldots, T_m \) such that \( d_{H'}(u) \leq \lfloor d_G(u)/2 \rfloor \) and for each vertex \( v \) with \( v \neq u \), \( d_{H'}(v) \leq \lfloor d_G(v)/2 \rfloor + m \). If \( xy \notin E(T_1 \cup \cdots \cup T_m) \), then the theorem clearly holds. Thus we may assume that \( xy \in E(T_1) \) and \( z \) and \( x \) lie in the same component of \( T_1 - xy \). Define \( T'_1 = T_1 - xy + yz \). It is easy to see that \( T'_1 \) is connected and \( T'_1 \cup T_2 \cup \cdots \cup T_m \) is the desired spanning subgraph of \( G \) we are looking for. \( \square \)

### 6.5 Tough enough graphs

As we already observed, \( m \)-strongly tough enough graphs are tough enough. In this subsection, we shall prove the converse statement and examine tough enough graphs for Corollary 6.13. To do that, we need the following two lemmas.
Lemma 6.21. Let $G$ be a graph. If $S$ is a vertex subset of $V(G)$ with the maximum $\Omega_m(G \setminus S) - |S|/m$ and with the maximal $|S|$, then every component of $G \setminus S$ is $m$-tree-connected or has maximum degree at most $m$.

Proof. Let $v$ be a vertex of $G \setminus S$ such that $d_{G \setminus S}(v) \geq m + 1$ and $v$ is incident to an edge joining different $m$-tree-connected components of $G \setminus S$. Take $S' = S \cup \{v\}$. It is not difficult to check that $\Omega_m(G \setminus S') - |S'|/m \geq \Omega_m(G \setminus S) + |S|/m$, whether $v$ lies in a non-trivial $m$-tree-connected component of $G \setminus S$ or $v$ itself is an $m$-tree-connected component of $G \setminus S$. By the maximality of $S$, the graph $G \setminus S$ has no such vertices and hence the proof is completed.

Lemma 6.22. (Brooks [6]) If $G$ is a connected graph with maximum degree at most $\Delta$, then the vertices of $G$ can be colored with $\Delta$ colors such that any two adjacent vertices admit different colors, unless $G$ is the complete graph of order $\Delta + 1$ or it is an odd cycle when $\Delta = 2$.

Now, we are ready to prove the main result of this subsection.

Theorem 6.23. Every $(m^2 + m - 1)$-tough graph $G$ of order at least $(m^2 + 3m + 2)/2$ is $m$-strongly $m$-tough.

Proof. For convenience, we write $k$ for $m^2 + m - 1$. Let $S \subseteq V(G)$ with properties described in Lemma 6.21 so that every component of $G \setminus S$ is $m$-tree-connected or has maximum degree at most $m$. If $V(G) \leq k + 1$, then $G$ must be complete and hence $|V(C)| \leq m + 1$ and $|S| \geq |V(G)| - m - 1 \geq m(m + 1)/2$. Obviously,

$$\Omega_m(G \setminus S) = |V(C)| - \frac{1}{m}|E(C)| \leq \frac{m + 1}{2} \leq \frac{1}{m}|S|.$$  

Now, assume that $|V(G)| \geq k + 2$ and $m \geq 2$. Denote by $r$ the number of components of $G \setminus S$ which are $m$-tree-connected. Take $C$ to be the union of all components of $G \setminus S$ which are not $m$-tree-connected. Note that $\Delta(C) \leq m$. If $|V(C)| = 0$, then we have $\Omega_m(G \setminus S) = \omega(G \setminus S)$ and the proof is completed. We may assume that $|V(C)| \geq 1$. By Lemma 6.22, it is easy to check that the graph $C$ has an independent set $X$ of size at least $\frac{1}{m}|V(C)| - \frac{1}{m}c$, where $c$ denotes the number of cycle components of $C$ of odd order when $m = 2$ and denotes the number of components of $C$ which are the complete graph of order $m + 1$ when $m \geq 3$. We may assume that $|X| \geq 2$ when $C$ is not connected. Let $S' \subseteq V(C) \setminus X$ such that $\frac{1}{m}|V(C)| - \frac{1}{m}c \leq |X| \leq \omega(C \setminus S')$ and $|S'| \leq \frac{m - 1}{m}|V(C)| - \frac{m - 1}{m}c$. This implies that

$$\frac{1}{m}|V(C)| - \frac{1}{m}c + r \leq |X| + r \leq \omega(C \setminus S') + r = \omega(G \setminus S \cup S'),$$  

If $\omega(G \setminus S \cup S') = 1$, then $C$ must be connected, $|V(C)| \leq m + 1$, and $r = 0$. In this case, we have

$$\Omega_m(G \setminus S) = |V(C)| - \frac{1}{m}|E(C)| \leq |V(C)| - \frac{1}{m}(|V(C)| - 1) \leq \frac{|V(G)| - |V(C)|}{m} = \frac{1}{m}|S|.$$  

If $\omega(G \setminus S \cup S') > 1$, then by the assumption,

$$\omega(G \setminus S \cup S') \leq \frac{1}{k}(|S| + |S'|) \leq \frac{1}{k}|S| + \frac{m - 1}{km}|V(C)| - \frac{m - 1}{km}c,$$  

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Therefore, Relations (10) and (11) can conclude that

\[ |V(C)| - c + r \leq \frac{m}{k - m + 1}|S|, \]

which implies

\[ \Omega_m(G \setminus S) = |V(C)| - \frac{1}{m}|E(C)| + r \leq |V(C)| - c + r \leq \frac{m}{k - m + 1}|S| = \frac{1}{m}|S|. \]

Hence the theorem holds. \qed

The following corollary gives a sufficient toughness condition for the existence of an \( m \)-tree-connected spanning subgraph with maximum degree at most \( 2m + 1 \). It is natural to ask whether higher toughness can guarantee the existence of an \( m \)-tree-connected spanning subgraph with maximum degree at most \( 2m \). The special case \( m = 1 \) of this question verifies Chvátal’s Conjecture [8] for Hamiltonian paths and has not yet been settled.

**Corollary 6.24.** Every \((m^2 + m - 1)\)-tough graph \( G \) of order at least \( 2m \) has an \( m \)-tree-connected spanning subgraph \( H \) with \( \Delta(H) \leq 2m + 1 \).

**Proof.** For the case \(|V(G)| < (m^2 + 3m + 2)/2\), the graph \( G \) must be complete and the proof is straightforward. Note that \( \omega(G \setminus S) = 1 \), for all \( S \subseteq V(G) \) with \( |S| = |V(G)| - 2 \). For the case \(|V(G)| \geq (m^2 + 3m + 2)/2\), apply Theorem 9.5 and Corollary 6.13 with \( n = 1 \). \qed

7 Highly tree-connected spanning subgraphs with bounded degrees

In this section, we shall develop Theorem 6.11 in two ways. The first one generalizes Theorem 3.2 and the second one generalizes Theorem 4.2. We will improve the second one for tough enough graphs, as well. Before doing so, we establish the following promised generalization of Theorems 2.1 and 6.10.

**Theorem 7.1.** Let \( G \) be an \( m \)-tree-connected graph with the spanning subgraph \( F \) and let \( h \) be an integer-valued function on \( V(G) \). If \( H \) is an \( m \)-tree-connected spanning subgraph of \( G \) containing \( F \) with the minimum total excess from \( h + d_F \), then there exists a subset \( S \) of \( V(G) \) with the following properties:

1. \( \Omega_m(G \setminus [S, F]) = \Omega_m(H \setminus [S, F]) \).
2. \( S \supseteq \{v \in V(G) : d_H(v) > h(v) + d_F(v)\} \).
3. For each vertex \( v \) of \( S \), \( d_H(v) \geq h(v) + d_F(v) \).
Proof. Define $V_0 = \emptyset$ and $V_1 = \{ v \in V(H) : d_H(v) > h(v) + d_F(v) \}$. For any $S \subseteq V(G)$ and $u \in V(G) \setminus S$, let $\Delta(S, u)$ be the set of all $m$-tree-connected spanning subgraphs $H'$ of $G$ containing $F$ such that $d_{H'}(v) \leq h(v) + d_F(v)$ for all $v \in V(G) \setminus V_1$, $H'[X]$ is $m$-tree-connected, and $H'$ and $H$ have the same edges, except for some of the edges of $G$ whose ends are in $X$, where $H[X]$ is the $m$-tree-connected component of $H \setminus [S, F]$ containing $u$. Now, for each integer $n$ with $n \geq 2$, recursively define $V_n$ as follows:

$$ V_n = V_{n-1} \cup \{ v \in V(G) \setminus V_{n-1} : d_{H'}(v) \geq h(v) + d_F(v), \text{ for all } H' \in \Delta(V_{n-1}, v) \}. $$

Now, we prove the following claim.

Claim. Let $x$ and $y$ be two vertices in different $m$-tree-connected components of $H \setminus [V_{n-1}, F]$. If $xy \in E(G) \setminus E(H)$, then $x \in V_n$ or $y \in V_n$.

Proof of Claim. By induction on $n$. For $n = 1$, the proof is clear. Assume that the claim is true for $n - 1$. Now we prove it for $n$. Suppose otherwise that vertices $x$ and $y$ are in different $m$-tree-connected components of $H \setminus [V_{n-1}, F]$, respectively, with the vertex sets $X$ and $Y$, $xy \in E(G) \setminus E(H)$, and $x, y \notin V_n$. Since $x, y \notin V_n$, there exist $H_x \in \Delta(V_{n-1}, x)$ and $H_y \in \Delta(V_{n-1}, y)$ with $d_{H_x}(x) < h(x) + d_F(x)$ and $d_{H_y}(y) < h(y) + d_F(y)$. By the induction hypothesis, $x$ and $y$ are in the same $m$-tree-connected component of $H \setminus [V_{n-2}, F]$ with the vertex set $Z$ such that $X \cup Y \subseteq Z$. Let $M$ be the nonempty set of edges of $H[Z] \setminus E(F)$ incident to the vertices in $V_{n-1} \setminus V_{n-2}$ whose ends lie in different $m$-tree-connected components of $H[Z] \setminus [Z \cap V_{n-1}, F]$. By Theorem 6.4, there exists an edge $zz' \in M$ with $z \in Z \cap V_{n-1}$ such that $H[Z] - zz' + xy$ is $m$-tree-connected. Now, let $H'$ be the spanning subgraph of $G$ containing $F$ with

$$ E(H') = E(H) - zz' + xy - E(H[X]) + E(H_x[X]) - E(H[Y]) + E(H_y[Y]). $$

By repeatedly applying Observation 6.3, one can easily check that $H'$ is $m$-tree-connected. For each $v \in V(H')$, we have

$$ d_{H'}(v) \leq \begin{cases} 
    d_{H_x}(v) + 1, & \text{if } v \in \{x, y\}; \\
    d_{H_x}(v), & \text{if } v = z'; \\
    d_{H_y}(v), & \text{if } v \in Y \setminus \{y, z'\}; \\
    d_{H_y}(v), & \text{if } v \notin X \cup Y \cup \{z, z'\}. 
\end{cases} \quad \text{and} \quad d_{H'}(v) = \begin{cases} 
    d_{H_x}(v), & \text{if } v \in X \setminus \{x, z'\}; \\
    d_{H_x}(v), & \text{if } v \in Y \setminus \{y, z'\}; \\
    d_{H_y}(v), & \text{if } v \notin X \cup Y \cup \{z, z'\}. 
\end{cases} $$

If $n \geq 3$, then it is not hard to see that $d_{H'}(z) < d_H(z) \leq h(z) + d_F(z)$ and $H'$ lies in $\Delta(V_{n-2}, z)$. Since $z \in V_{n-1} \setminus V_{n-2}$, we arrive at a contradiction. For the case $n = 2$, since $z \in V_1$, it is easy to see that $h(z) + d_F(z) \leq d_{H'}(z) < d_H(z)$ and $te(H', h + d_F) < te(H, h + d_F)$, which is again a contradiction. Hence the claim holds.

Obviously, there exists a positive integer $n$ such that and $V_1 \subseteq \cdots \subseteq V_{n-1} = V_n$. Put $S = V_n$. Since $S \supseteq V_1$, Condition 2 clearly holds. For each $v \in V_i \setminus V_i$ with $i \geq 2$, we have $H \in \Delta(V_{i-1}, v)$ and so $d_H(v) \geq h(v) + d_F(v)$. This establishes Condition 3. Because $S = V_n$, the previous claim implies Condition 1 and completes the proof. \qed

In the above-mentioned theorem, we could assume that $\Omega_m(H) = \Omega_m(G)$ and choose $H$ with the minimum $te(H, h + d_F)$, whether $G$ is $m$-tree-connected or not. More precisely, the edge $e'$ in Theorem 6.4 can be
found such that $\Omega_m(H) = \Omega_m(H - e + e')$, whether $H$ is $m$-tree-connected or not. Conversely, if we assume that $te(H, h + d_F) = 0$ and choose $H$ with the minimum $\Omega_m(H)$, the next theorem can be derived, see [11, Theorem 1]. However, the above-mentioned theorem works remarkably well, we shall use the this result to get further improvement in the last subsection.

**Theorem 7.2.** Let $G$ be a graph with the spanning subgraph $F$ and let $h$ be a nonnegative integer-valued function on $V(G)$. If $H$ is a spanning subgraph of $G$ containing $F$ with $te(H, h + d_F) = 0$ and with the minimum $\Omega_m(H)$, then there exists a subset $S$ of $V(G)$ with the following properties:

1. $\Omega_m(G \setminus [S, F]) = \Omega_m(H \setminus [S, F])$.

2. For each vertex $v$ of $S$, $d_H(v) = h(v) + d_F(v)$.

**Proof.** Define $V_0 = \emptyset$. For any $S \subseteq V(G)$ and $u \in V(G) \setminus S$, let $A(S, u)$ be the set of all spanning subgraphs $H'$ of $G$ containing $F$ with $te(H', h + d_F) = 0$ such that $\Omega_m(H') = \Omega_m(H)$, $H'[X]$ is $m$-tree-connected, and $H'$ and $H$ have the same edges, except for some of the edges of $G$ whose ends are in $X$, where $H[X]$ is the $m$-tree-connected component of $H \setminus [S, F]$ containing $u$. Now, for each integer $n$ with $n \geq 2$, recursively define $V_n$ as follows:

$$V_n = V_{n-1} \cup \{ v \in V(G) \setminus V_{n-1} : d_{H'}(v) = h(v) + d_F(v), \text{ for all } H' \in A(V_{n-1}, v) \}.$$ 

Now, we prove the following claim.

**Claim.** Let $x$ and $y$ be two vertices in different $m$-tree-connected components of $H \setminus [V_{n-1}, F]$. If $xy \in E(G) \setminus E(H)$, then $x \in V_n$ or $y \in V_n$.

**Proof of Claim.** By induction on $n$. Suppose otherwise that vertices $x$ and $y$ are in different $m$-tree-connected components of $H \setminus [V_{n-1}, F]$, respectively, with the vertex sets $X$ and $Y$, $xy \in E(G) \setminus E(H)$, and $x, y \notin V_n$. Since $x, y \notin V_n$, there exist $H_x \in A(V_{n-1}, x)$ and $H_y \in A(V_{n-1}, y)$ with $d_{H_x}(x) < h(x) + d_F(x)$ and $d_{H_y}(y) < h(y) + d_F(y)$. For $n = 1$, define $H'$ to be the spanning subgraph of $G$ containing $F$ with

$$E(H') = E(H) + xy - E(H[X]) + E(H_x[X]) - E(H[Y]) + E(H_y[Y]).$$

Since the edge $xy$ joins different $m$-tree-connected components of $H$, we must have $\Omega_m(H') < \Omega_m(H)$. Since $te(H', h + d_F) = 0$, we arrive at a contradiction. Now, suppose $n \geq 2$. By the induction hypothesis, $x$ and $y$ are in the same $m$-tree-connected component of $H \setminus [V_{n-2}, F]$ with the vertex set $Z$ so that $X \cup Y \subseteq Z$. Let $M$ be the nonempty set of edges of $H[Z] \setminus E(F)$ incident to the vertices in $V_{n-1} \setminus V_{n-2}$ whose ends lie in different $m$-tree-connected components of $H[Z] \setminus [Z \cap V_{n-1}, F]$. By Theorem 6.4, there exists an edge $zz' \in M$ with $z \in Z \cap V_{n-1}$ such that $H[Z] - zz' + xy$ is $m$-tree-connected. Now, let $H'$ be the spanning subgraph of $G$ containing $F$ with

$$E(H') = E(H) - zz' + xy - E(H[X]) + E(H_x[X]) - E(H[Y]) + E(H_y[Y]).$$

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It is easy to see that the $m$-tree-connected components of $H'$ and $H$ have the same vertex sets. Since $H$ and $H'$ have the same edges joining these $m$-tree-connected components, $\Omega_m(H') = \Omega_m(H)$. For each $v \in V(H')$, we have

$$d_{H'}(v) \leq \begin{cases} d_{H_s}(v) + 1, & \text{if } v \in \{x, y\}; \\ d_H(v), & \text{if } v = z'. \end{cases}$$

and

$$d_{H'}(v) = \begin{cases} d_{H_s}(v), & \text{if } v \in X \setminus \{x, z'\}; \\ d_{H_n}(v), & \text{if } v \in Y \setminus \{y, z'\}; \\ d_H(v), & \text{if } v \notin X \cup Y \cup \{z, z'\}. \end{cases}$$

It is not hard to check that $d_{H'}(z) < d_H(z) \leq h(z) + d_F(z)$ and $H'$ lies in $\mathcal{A}(V_{n-2}, z)$. Since $z \in V_{n-1} \setminus V_{n-2}$, we arrive at a contradiction. Hence the claim holds.

Obviously, there exists a positive integer $n$ such that and $V_1 \subseteq \cdots \subseteq V_{n-1} = V_n$. Put $S = V_n$. For each $v \in V_i \setminus V_{i-1}$, we have $H \in \mathcal{A}(V_{i-1}, v)$ and so $d_H(v) = h(v) + d_F(v)$. This establishes Condition 2. Because $S = V_n$, the previous claim implies Condition 1 and completes the proof. \hfill \Box

### 7.1 The first generalization

The following lemma is a common generalization of Lemma 3.1 and Proposition 6.7.

**Lemma 7.3.** Let $H$ be an $m$-critical graph with the spanning subgraph $F$. If $S \subseteq V(H)$ and $F = H \setminus E(F)$, then

$$\sum_{v \in S} d_F(v) = m\Omega_m(H \setminus [S, F]) - m\Omega_m(H) + e_F(S).$$

**Proof.** By induction on the number of edges of $F$ which are incident to the vertices in $S$. If there is no edge of $F$ incident to a vertex in $S$, then the proof is clear. Now, suppose that there exists an edge $e = uv \in E(F)$ with $|S \cap \{u, u'\}| \geq 1$. Hence

1. $m\Omega_m(H) = m\Omega_m(H \setminus e) - 1$,
2. $\Omega_m(H \setminus [S, F]) = \Omega_m((H \setminus e) \setminus [S, F])$,
3. $e_F(S) = e_{F \setminus e}(S) + |S \cap \{u, u'\}| - 1$,
4. $\sum_{v \in S} d_F(v) = \sum_{v \in S} d_{F \setminus e}(v) + |S \cap \{u, u'\}|$.

Therefore, by the induction hypothesis on $H \setminus e$ with the spanning subgraph $F$ the lemma holds. \hfill \Box

A common generalization of Theorems 6.11 and 3.2 is given in the following theorem. We here denote by $e^m_G(S, F)$ the number of edges of $E(G) \setminus E(F)$ with both ends in $S$ joining different $m$-tree-connected components of $G \setminus [S, F]$. 

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Theorem 7.4. Let $G$ be an $m$-tree-connected graph with $X \subseteq V(G)$ and with the spanning subgraph $F$. Let $\lambda \in [0, 1/m]$ be a real number and let $\eta : X \to (m\lambda + \frac{m-1}{m}, \infty)$ be a real function. If for all $S \subseteq X$, 
\[ \Omega_m(G \setminus [S, F]) < \frac{1}{m} + \sum_{v \in S} (\eta(v) - 2) + 2 - \lambda(e^m_G(S, F) + m), \]
then $G$ has an $m$-tree-connected spanning subgraph $H$ containing $F$ such that for each $v \in X$, 
\[ d_H(v) \leq \lceil m\eta(v) - m^2\lambda \rceil + d_F(v) - m. \]

**Proof.** For each vertex $v$, define 
\[ h(v) = \begin{cases} d_G(v) + 1, & \text{if } v \notin X; \\ \lceil m\eta(v) - m^2\lambda \rceil - m, & \text{if } v \in X. \end{cases} \]
First, suppose that $F$ is $m$-critical. Let $H$ be a minimally $m$-tree-connected spanning subgraph of $G$ containing $F$ with the minimum total excess from $h + d_F$. Define $S$ to be a subset of $V(G)$ with the properties described in Theorem 7.1. If $S$ is empty, then $te(H, h + d_F) = 0$ and the theorem clearly holds. So, suppose $S$ is nonempty. Obviously, $S \subseteq X$. Put $F = H \setminus E(F)$. Thus by Lemma 7.3, 
\[ \sum_{v \in S} h(v) + te(H, h + d_F) = \sum_{v \in S} d_F(v) = m\Omega_m(H \setminus [S, F]) - m + e_F(S), \]
and so 
\[ \sum_{v \in S} h(v) + te(H, h + d_F) = m\Omega_m(G \setminus [S, F]) - m + e_F(S). \] (12)

Also, by the assumption, 
\[ m\Omega_m(G \setminus S) - m + e_F(S) < 1 + \sum_{v \in S} (m\eta(v) - 2m) - m\lambda(e^m_G(S, F) + m) + e_F(S) + m. \] (13)

Since $e_F(S) \leq e^m_G(S, F)$ and $e_F(S) \leq m(|S| - 1)$, 
\[ - m\lambda(e^m_G(S, F) + m) + e_F(S) + m \leq -m\lambda(e_F(S) + m) + e_F(S) + m \leq (1 - m\lambda)m|S|. \] (14)

Therefore, Relations (12), (13), and (14) can conclude that 
\[ \sum_{v \in S} h(v) + te(H, h + d_F) < 1 + \sum_{v \in S} (m\eta(v) - m^2\lambda - m). \]

On the other hand, by the definition of $h(v)$, 
\[ \sum_{v \in S} (m\eta(v) - m^2\lambda - m - h(v)) \leq 0 \]
Hence $te(H, h + d_F) = 0$ and the theorem holds. Now, suppose that $F$ is not $m$-critical. Remove some of the edges of the $m$-tree-connected components of $F$ until the resulting $m$-critical graph $F'$ have the same $m$-tree-connected components. Obviously, $\Omega_m(G \setminus [S, F']) = \Omega_m(G \setminus [S, F])$, for all $S \subseteq V(G)$. It is enough, now, to apply the theorem on $F'$ and finally add the edges of $E(F) \setminus E(F')$ to that explored $m$-tree-connected spanning subgraph.  
\[ \square \]
7.2 The second generalization

In this subsection, the notation \( \Omega_m(G \setminus S) \) plays an essential role instead of \( \Omega_m(G \setminus [S, F]) \). In order to prove the next theorem, we need the next lemma, which provides a relationship between \( \Omega_m(G \setminus S) \) and \( \Omega_m(G \setminus [S, F]) \).

**Lemma 7.5.** Let \( G \) be a graph with the spanning subgraph \( F \). If \( S \subseteq V(G) \) then

\[
\Omega_m(G \setminus [S, F]) \leq \Omega_m(G \setminus S) + \sum_{v \in S} \frac{1}{m} \max\{0, m - d_F(v)\} + \frac{1}{m} e_F(S).
\]

Furthermore, \( \Omega_m(G \setminus [S, F]) \leq \Omega_m(G \setminus S) + \frac{1}{m(c-1)} e_F(S) \), when every \( m \)-tree-connected component \( C \) of \( F \) contains at least \( c - \frac{1}{2m} d_F(C) \) vertices and \( c \geq 2 \).

**Proof.** Define \( P \) and \( P' \) to be the partitions of \( V(G) \) and \( V(G) \setminus S \) obtained from the \( m \)-tree-connected components of \( G \setminus [S, F] \) and \( G \setminus S \). Set \( R = \{ A \in P : A \subseteq S \}, R_1 = \{ A \in R : |A| = 1 \}, \) and \( R_2 = \{ A \in R : |A| \geq 2 \} \). It is not difficult to check that

\[
e_{G\setminus[S,F]}(P) \geq e_{G\setminus[S,F]}(P') - \sum_{A \in P \setminus R} e_{G[A,S]}(P'_{A\setminus S}) + D_F(R),
\]

where \( P'_{A\setminus S} \) denotes the partition of \( A \setminus S \) obtained from vertex sets of \( P' \), and \( D_F(R) \) denotes the number of edges of \( F \) joining different parts of \( P \) incident to vertex sets in \( R \). Thus

\[
m\Omega_m(G \setminus [S, F]) - |P| \leq m\Omega_m(G \setminus S) - \sum_{A \in P \setminus R} m\Omega_m(G[A \setminus S]) - D_F(R).
\]

Since \( \Omega_m(G[A \setminus S]) \geq 1 \), for any \( A \in P \setminus R \), we have

\[
m\Omega_m(G \setminus [S, F]) \leq m\Omega_m(G \setminus S) - m|R| - D_F(R).
\]

In the first statement, \( e_F(A) = m \), for any \( A \in R_2 \), and so

\[
m|R| - D_F(R) \leq m|R_1| + \sum_{A \in R_2} e_F(A) - \sum_{v \in R_1} d_F(v) + e_F(R_1) \leq \sum_{v \in R_1} (m - d_F(v)) + e_F(S).
\]

Therefore,

\[
m\Omega_m(G \setminus [S, F]) \leq m\Omega_m(G \setminus S) + e_F(S) + \sum_{v \in S} \max\{0, m - d_F(v)\}.
\]

In the second statement, \( |A| \geq c - \frac{1}{2m} d_F(A) \) for any \( A \in R \), and so

\[
m|R| - m \sum_{A \in R} \frac{|A| - 1}{c - 1} \leq \sum_{A \in R} \frac{1}{2} d_F(A) \leq D_F(R).
\]

Since \( e_F(A) \geq m(|A| - 1) \), for any \( A \in R \), it is easy to check that

\[
m\Omega_m(G \setminus [S, F]) \leq m\Omega_m(G \setminus S) + \frac{1}{(c - 1)} e_F(S),
\]

Hence the lemma holds. \( \square \)
A common generalization of Theorems 6.11 and 4.2 is given in the next theorem.

**Theorem 7.6.** Let $G$ be an $m$-tree-connected graph with $X \subseteq V(G)$ and with the spanning subgraph $F$. Let $\lambda \in [0, 1/m]$ be a real number and let $\eta : X \rightarrow (m\lambda + \frac{m-1}{m}, \infty)$ be a real function. If for all $S \subseteq X$,

$$
\Omega_m(G \setminus S) < \frac{1}{m} + \sum_{v \in S} (\eta(v) - 2) + 2 - \lambda(e_G(S) + m),
$$

then $G$ has an $m$-tree-connected spanning subgraph $H$ containing $F$ such that for each $v \in X$,

$$
d_H(v) \leq \left[ m\eta(v) - m^2\lambda \right] + \max\{0, d_F(v) - m\}.
$$

**Proof.** For each vertex $v$, define

$$
h(v) = \begin{cases} d_G(v) + 1, & \text{if } v \not\in X; \\ \left[ m\eta(v) - m^2\lambda \right] - \min\{m, d_F(v)\}, & \text{if } v \in X. \end{cases}
$$

First, suppose that $F$ is $m$-critical. Let $H$ be a minimally $m$-tree-connected spanning subgraph of $G$ containing $F$ with the minimum total excess from $h + d_F$. Define $S$ to be a subset of $V(G)$ with the properties described in Theorem 7.1. If $S$ is empty, then $te(H, h + d_F) = 0$ and the theorem clearly holds. So, suppose $S$ is nonempty. Obviously, $S \subseteq X$. Put $F = H \setminus E(F)$. By Lemma 7.3,

$$
\sum_{v \in S} h(v) + te(H, h + d_F) = \sum_{v \in S} d_F(v) = m\Omega_m(H \setminus [S, F]) - m + e_F(S),
$$

and so

$$
\sum_{v \in S} h(v) + te(H, h + d_F) = m\Omega_m(G \setminus [S, F]) - m + e_F(S).
$$

Since $e_F(S) + e_F(S) = e_H(S)$, Lemma 7.5 implies that

$$
\sum_{v \in S} (h(v) - \max\{0, m - d_F(v)\}) + te(H, h + d_F) \leq m\Omega_m(G \setminus S) - m + e_H(S) \tag{15}
$$

Also, by the assumption,

$$
m\Omega_m(G \setminus S) - m + e_H(S) < 1 + \sum_{v \in S} (m\eta(v) - 2m) - m\lambda(e_G(S) + m) + e_H(S) + m \tag{16}
$$

Since $e_H(S) \leq e_G(S)$ and $e_H(S) \leq m(|S| - 1),

$$
- m\lambda(e_G(S) + m) + e_H(S) + m \leq -m\lambda(e_H(S) + m) + e_H(S) + m \leq (1 - m\lambda)m|S|, \tag{17}
$$

Therefore, Relations (15), (16), and (17) can conclude that

$$
\sum_{v \in S} (h(v) - \max\{0, m - d_F(v)\}) + te(H, h + d_F) < 1 + \sum_{v \in S} (m\eta(v) - m^2\lambda - m).
$$

On the other hand, by the definition of $h(v),

$$
\sum_{v \in S} (m\eta(v) - m^2\lambda - m - h(v) + \max\{0, m - d_F(v)\}) \leq 0.
$$
Hence \( te(H, h + d_F) = 0 \) and the theorem holds. Now, suppose that \( F \) is not \( m \)-critical. Remove some of the edges of the \( m \)-tree-connected components of \( F \) until the resulting \( m \)-critical graph \( F' \) have the same \( m \)-tree-connected components. For each vertex \( v \) with \( d_F'(v) < d_F(v) \), we have \( d_F(v) \geq d_F'(v) \geq m \), since \( v \) must lie in a non-trivial \( m \)-tree-connected component of \( F' \). It is enough, now, to apply the theorem on \( F' \) and finally add the edges of \( E(F) \setminus E(F') \) to that explored \( m \)-tree-connected spanning subgraph. \( \Box \)

### 7.3 Toughness and the existence of \( m \)-tree-connected \( \{r, r + 1\} \)-factors

Our aim in this subsection is to prove that tough enough graphs of order at least \( r + 1 \) admit \( m \)-tree-connected \( \{r, r + 1\} \)-factors, when \( r \geq 2m \). For this purpose, we improve below Theorem 7.6 for \( m \)-strongly tough enough graphs which enables us to choose \( \eta(v) \) small enough, in compensation we require that the given spanning subgraph \( F \) approximately have large \( m \)-tree-connected components.

**Theorem 7.7.** Let \( G \) be an \( m \)-tree-connected graph with the spanning subgraph \( F \) which every \( m \)-tree-connected component \( C \) of \( F \) contains at least \( c - \frac{1}{2m} d_F(C) \) vertices with \( c \geq 2 \). Let \( \eta : V(G) \rightarrow [0, \infty) \) be a real function. If for all \( S \subseteq V(G) \),

\[
\Omega_m(G \setminus S) < \frac{1}{m} + \sum_{v \in S} \left( \frac{c}{2c - 2} \eta(v) - \frac{1}{c - 1} \right) + \frac{c}{c - 1},
\]

then \( G \) has an \( m \)-tree-connected spanning subgraph \( H \) containing \( F \) such that for each vertex \( v \), \( d_H(v) \leq \lceil m\eta(v) \rceil + d_F(v) \).

**Proof.** For each vertex \( v \), define \( h(v) = \lceil m\eta(v) \rceil \). First, suppose that \( F \) is \( m \)-critical. Let \( H \) be an \( m \)-critical spanning subgraph of \( G \) containing \( F \) with \( te(H, h + d_F) = 0 \) and with the minimum \( \Omega_m(H) \).

Define \( S \) to be a subset of \( V(G) \) with the properties described in Theorem 7.2. If \( S \) is empty, then \( \Omega_m(H) = \Omega_m(G) = 1 \) and the theorem clearly holds. So, suppose \( S \) is nonempty. Put \( \mathcal{F} = H \setminus E(F) \). By Lemma 7.3,

\[
\sum_{v \in S} h(v) = \sum_{v \in S} d_F(v) = m\Omega_m(H \setminus [S, F]) - m\Omega_m(H) + e_F(S),
\]

and so

\[
m\Omega_m(H) = m\Omega_m(G \setminus [S, F]) + e_F(S) - \sum_{v \in S} h(v). \tag{18}
\]

Since \( e_F(S) + e_F(S) = e_H(S) \leq m(|S| - 1) \) and \( e_F(S) \leq \frac{1}{2} \sum_{v \in S} d_F(v) = \frac{1}{2} \sum_{v \in S} h(v) \), we have

\[
e_F(S) + \frac{1}{c - 1} e_F(S) \leq \frac{1}{2} \sum_{v \in S} h(v) + \frac{1}{c - 1} \left(m|S| - m - \frac{1}{2} \sum_{v \in S} h(v) \right). \tag{19}
\]

Also, by Lemma 7.5,

\[
m\Omega_m(G \setminus [S, F]) + e_F(S) \leq m\Omega_m(G \setminus S) + e_F(S) + \frac{1}{c - 1} e_F(S). \tag{20}
\]
Therefore, Relations (18), (19), and (20) can conclude that
\[
m\Omega_m(H) \leq m\Omega_m(G \setminus S) - \frac{c}{2c - 2} \sum_{v \in S} h(v) + \frac{m|S| - m}{c - 1} < m + 1
\]
Hence \(\Omega_m(H) = 1\) and the theorem holds. Now, suppose that \(F\) is not \(m\)-critical. Remove some of the edges of the \(m\)-tree-connected components of \(F\) until the resulting \(m\)-critical graph \(F'\) have the same \(m\)-tree-connected components. For every \(m\)-tree-connected component \(C\) of \(F'\), we still have \(d_{F'}(C) = d_F(C)\). It is enough, now, to apply the theorem on \(F'\) and finally add the edges of \(E(F) \setminus E(F')\) to that explored \(m\)-tree-connected spanning subgraph.

\[\square\]

The following corollary improves Theorem 2 (iii, iv) in [10] and implies Theorem 3.5 (i) in [11].

**Corollary 7.8.** Let \(G\) be a connected graph with the spanning subgraph \(F\) which every component of it contains at least \(c\) vertices with \(c \geq 3\). Let \(h\) be a nonnegative integer-valued function on \(V(G)\). If for all \(S \subseteq V(G)\),
\[
\omega(G \setminus S) < \sum_{v \in S} \left( \frac{c}{2c - 2} h(v) - \frac{1}{c - 1} \right) + 2 + \frac{1}{c - 1},
\]
then \(G\) has a connected spanning subgraph \(H\) containing \(F\) such that for each vertex \(v\), \(d_H(v) \leq h(v) + d_F(v)\).

When we consider the special cases \(\eta(v) \leq 1/m\), the theorem becomes simpler as the following result.

**Theorem 7.9.** Let \(G\) be a graph with the spanning subgraph \(F\) which every \(m\)-tree-connected component \(C\) of \(F\) contains at least \(c - \frac{1}{2m} d_F(C)\) vertices with \(c \geq 2m + 1\). If for all \(S \subseteq V(G)\),
\[
\omega(G \setminus S) \leq \frac{c - 2m}{2m(c - 1)} |S| + 1
\]
then \(G\) has an \(m\)-tree-connected spanning subgraph \(H\) containing \(F\) such that for each vertex \(v\), \(d_H(v) \leq d_F(v) + 1\), and also \(d_H(u) = d_F(u)\) for a given arbitrary vertex \(u\).

**Proof.** Let \(G'\) be the union of \(m\) copies of \(G\) with the same vertex set. It is easy to check that \(\Omega_m(G' \setminus S) = \omega(G' \setminus S)\), for every \(S \subseteq V(G)\). Since \(\Omega_m(G' \setminus \emptyset) = 1\), the graph \(G'\) is \(m\)-tree-connected. Define \(\eta(u) = 0\) and \(\eta(v) = 1/m\) for each vertex \(v\) with \(v \neq u\). By Theorem 7.7, the graph \(G'\) has an \(m\)-tree-connected spanning subgraph \(H\) containing \(F\) such that for each vertex \(v\), \(d_H(v) \leq [m\eta(v)] + d_F(v) \leq 1 + d_F(v)\). According to the construction, the graph \(H\) must have no multiple edges of \(E(G') \setminus E(F)\). Hence \(H\) itself is a spanning subgraph of \(G\) and the proof is completed.

\[\square\]

Enomoto, Jackson, Katerinis, and Saito (1985) [12] showed that every \(r\)-tough graph \(G\) of order at least \(r + 1\) with \(r|V(G)|\) even admits an \(r\)-factor. For the case that \(r|V(G)|\) is odd, the same arguments can imply that the graph \(G\) admits a factor such that whose degrees are \(r\), except for a vertex with degree \(r + 1\). A combination of Theorem 7.9 and this result can conclude the next results.
Corollary 7.10. Every $t$-tough graph $G$ of order at least $r + 1$ has an $m$-tree-connected \{r, r + 1\}-factor, where $r \geq 2m$ and $t \geq \max\{\frac{2m}{r+1}, r\}$.

Proof. We may assume that $G$ is a $t$-tough simple graph, by deleting multiple edges from $G$ (if necessary). Let $F$ be a factor of $G$ such that each of whose vertices has degree $r$, except for at most one vertex $u$ with degree $r + 1$ [12]. Let $C$ be an $m$-tree-connected component of $F$. Since $F$ is simple, it is easy to check that $d_F(C) \geq |V(C)|(r - |V(C)|) + 1$. If $|V(C)| \leq r + 1$, then we must have

$$|V(C)| + \frac{r}{2m}d_F(C) \geq |V(C)| + (r + 1 - |V(C)|) = r + 1.$$  

By applying Theorem 7.9 with $c = r + 1$, the graph $G$ has an $m$-tree-connected spanning subgraph $H$ containing $F$ such that for each vertex $v$, $d_H(v) \leq d_F(v) + 1$, and also $d_H(u) = d_F(u)$. This implies that $H$ is an $m$-tree-connected \{r, r + 1\}-factor. □

Corollary 7.11. ([10, 11]) Every $r$-tough graph of order at least $r + 1$ admits a connected \{r, r + 1\}-factor, where $r \geq 3$.

Corollary 7.12. Every $4m^2$-tough graph of order at least $2m + 1$ has an $m$-tree-connected \{2m, 2m + 1\}-factor containing a $2m$-factor.

The following theorem gives a sufficient toughness condition for extending 2-factors with girth at least five to 2-connected \{2, 3\}-factors. Ellingham and Zha [11] proved that 2-factors with girth at least three of 4-tough graphs can be extended to connected \{2, 3\}-factors.

Theorem 7.13. Let $G$ be a graph and let $F$ be a spanning subgraph of $G$ with even degrees which every component of it contains at least $c$ vertices with $c \geq 5$. If for all $S \subseteq V(G)$,

$$\omega(G \setminus S) \leq \frac{c - 4}{4c - 4}|S| + 1,$$

then $G$ has a 2-edge-connected spanning subgraph $H$ containing $F$ such that for each vertex $v$, $d_H(v) \leq d_F(v) + 1$.

Proof. Duplicate the edges of $F$ and call the resulting graphs $G'$ and $F'$. Obviously, every 2-tree-connected component of $F'$ contains at least $c$ vertices. By Theorem 7.9, the graph $G'$ has a 2-tree-connected spanning subgraph $H'$ containing $F'$ such that for each vertex $v$, $d_{H'}(v) \leq d_{F'}(v) + 1$. Remove a copy of $F$ from $H'$ and call the resulting graph $H$. It is easy to check that for each vertex $v$, $d_H(v) \leq d_F(v) + 1$, and $H/F$ is still 2-tree-connected. By the assumption, every component of $F$ is Eulerian and consequently 2-edge-connected. Hence $H$ itself is 2-edge-connected and the proof is completed. □

Corollary 7.14. Every $16$-tough graph $G$ of girth at least five has a 2-connected \{2, 3\}-factor.

Proof. Let $F$ be a 2-factor of $G$ so that every component of it contains at least five vertices [12]. By Theorem 7.13, the graph $F$ can be extended to a 2-edge-connected \{2, 3\}-factor $H$ so that has no cut vertices. Hence the proof is completed □

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8 Total excesses from comparable functions

In 2011 Enomoto, Ohnishi, and Ota [13] established a new extension for Win’s result based on total excess. Later, Ohnishi and Ota [28] generalized Theorem 1.2 toward this concept and concluded the following theorem.

**Theorem 8.1.** ([28]) Let $G$ be a connected graph, let $t$ be a nonnegative integer, and let $h$ be an integer-valued function on $V(G)$. If for all $S \subseteq V(G)$, $\omega(G \setminus S) \leq \sum_{v \in S} (h(v) - 2) + 2 + t$, then $G$ has a spanning tree $T$ satisfying $te(T, h) \leq t$.

Recently, Ozeki (2015) refined the above-mentioned theorem by making the following theorem similar to Theorems 8 in [31]. He also gave an application for it and remarked that the condition “$h_1 \geq \cdots \geq h_p$” is necessary, in the sense that Theorem 8.2 does not hold for non-comparable functions.

**Theorem 8.2.** ([32]) Let $G$ be a connected graph and let $p$ be a positive integer. For each integer $i$ with $1 \leq i \leq p$, let $t_i$ be a nonnegative integer and let $h_i$ be an integer-valued function on $V(G)$ with $h_1 \geq \cdots \geq h_p$. If for all $S \subseteq V(G)$ and $i \in \{1, \ldots, p\}$,

$$\omega(G \setminus S) \leq \sum_{v \in S} (h_i(v) - 2) + 2 + t_i,$$

then $G$ has a spanning tree $T$ satisfying $te(T, h_i) \leq t_i$ for all $i$ with $1 \leq i \leq p$.

Motivated by Ozeki-type condition, we formulate the following strengthened version of the main result of this paper. As its proof requires only minor modifications, we shall only state the strategy of the proof in the subsequent subsection. In the following theorem, we denoted by $e_m^w(S)$ the maximum number of edges of $H$ with both ends in $S$ taken over all minimally $m$-tree-connected spanning subgraphs $H$ of $G$. It is not hard to verify that $e_m^w(S) = m(|S| - \Omega_m(G[S]))$.

**Theorem 8.3.** Let $G$ be an $m$-tree-connected graph with the spanning subgraph $F$. Let $p$ be a positive integer. For each integer $i$ with $1 \leq i \leq p$, let $t_i$ be a nonnegative integer, let $\lambda_i \in [0, 1/m]$ be a real number, and let $h_i$ be a real function on $V(G)$ with $m\eta_1 - m^2 \lambda_1 \geq \cdots \geq m\eta_p - m^2 \lambda_p$. If for all $S \subseteq V(G)$ and $i \in \{1, \ldots, p\}$,

$$\Omega_m(G \setminus S) \leq \sum_{v \in S} (\eta_i(v) - 2) + 2 - \lambda_i(e_m^w(S) + m) + \frac{t_i + 1}{m},$$

then $F$ can be extended to an $m$-tree-connected spanning subgraph $H$ satisfying $te(H, h_i) \leq t_i$ for all $i$ with $1 \leq i \leq p$, where $h_i(v) = \lceil m\eta_i(v) - m^2 \lambda_i \rceil + \max\{0, d_F(v) - m\}$ for all vertices $v$.

8.1 Strategy of the proof

Let $G$ be a graph with the spanning subgraph $H$ and take $xy \in E(G) \setminus E(H)$. Let $h$ be an integer-valued function on $V(G)$. It is easy to check that if $d_H(x) < h(x)$ and $d_H(y) < h(y)$, then $te(H + xy, h) = te(H, h)$,
and also this equality holds for any other integer-valued function \( h' \) on \( V(G) \) with \( h' \geq h \). This observation was used by Ozeki [32] to prove Theorem 8.2 with a method that decreases total excesses from comparable functions, step by step, by starting from the largest function to the smallest function. Inspired by Ozeki’s method, we now formulate the following strengthened version of Theorem 7.1.

**Theorem 8.4.** Let \( G \) be an \( m \)-tree-connected graph with the spanning subgraph \( F \). Let \( h_1, \ldots, h_q \) be \( q \) integer-valued functions on \( V(G) \) with \( h_1 \geq \cdots \geq h_q \). Define \( \Gamma_0 \) to be the set of all \( m \)-tree-connected spanning subgraphs \( H \) of \( G \) containing \( F \). For each positive integer \( n \) with \( n \leq q \), recursively define \( \Gamma_n \) to be the set of all graphs \( H \) belonging to \( \Gamma_{n-1} \) with the smallest \( te(H, h_n + d_F) \). If \( H \in \Gamma_q \), then there exists subset \( S \) of \( V(G) \) with the following properties:

1. \( \Omega_m(G \setminus [S, F]) = \Omega_m(H \setminus [S, F]) \).
2. \( S \supseteq \{ v \in V(G) : d_H(v) > h_q(v) + d_F(v) \} \).
3. For each vertex \( v \) of \( S \), \( d_H(v) \geq h_q(v) + d_F(v) \).

**Proof.** Apply the same arguments of Theorems 7.1 with replacing \( h_q(v) \) instead of \( h(v) \). \( \square \)

**Proof of Theorem 8.3.** First, define \( h_q(v) \) and \( \lambda_q \) as with \( h(v) \) and \( \lambda \) in the proof of Theorem 7.6 by replacing \( \eta_q \) instead of \( \eta \), where \( 1 \leq q \leq p \). Next, for a fixed graph \( H \in \Gamma_p \subseteq \cdots \subseteq \Gamma_1 \), show that \( te(H, h_q + d_F) \leq t_q \), for any \( q \) with \( 1 \leq q \leq p \), by repeatedly applying Theorem 8.4 and using the same arguments in the proof of Theorem 7.6. \( \square \)

**9 Applications to spanning Eulerian subgraphs**

The following theorem gives a sufficient condition for the existence of \( f \)-trails.

**Theorem 9.1.** Let \( G \) be a 2-tree-connected graph. Let \( \lambda \in [0, 1/2] \) be a real number and let \( f \) be a positive integer-valued function on \( V(G) \). If for all \( S \subseteq V(G) \),

\[
\Omega(G \setminus S) < \sum_{v \in S} (f(v) + 2\lambda - 3/2) + 5/2 + \lambda (e_G(S) + 2),
\]

then \( G \) has an \( f \)-trail.

**Proof.** By applying Theorem 6.11 for the special case \((m, \eta) = (2, f + 1/2 + 2\lambda)\), we can deduce that \( G \) has a 2-tree-connected spanning subgraph \( H \) such that for each vertex \( v \), \( d_H(v) \leq 2f(v) + 1 \). Since \( H \) has a spanning Eulerian subgraph [18], the graph \( G \) admits an \( f \)-trail. \( \square \)

The following corollary gives another sufficient condition for the existence of \( f \)-walks.
Corollary 9.2. Let \( G \) be a connected graph. Let \( \lambda \in [0,1] \) be a real number and let \( f \) be a positive integer-valued function on \( V(G) \). If for all \( S \subseteq V(G) \),
\[
\omega(G \setminus S) < \sum_{v \in S} (f(v) + \lambda - 3/2) + 5/2 - \lambda(e_G(S) + 1),
\]
then \( G \) has an \( f \)-walk.

Proof. Duplicate the edges of \( G \) and call the resulting graph \( G' \). It is easy to check that \( \Omega(G' \setminus S) = \omega(G' \setminus S) \) and \( e_{G'}(S) = 2e_G(S) \), for every \( S \subseteq V(G) \). By Theorem 9.1, where \( \lambda/2 \) plays the role of \( \lambda \), the graph \( G' \) has an \( f \)-trail which implies that \( G \) admits an \( f \)-walk. \( \square \)

By restricting our attention to independent sets, Theorem 9.1 becomes simpler and surprisingly this special case can be strengthened as the following theorem. Indeed, this version discounts the condition \( \Omega(G \setminus S) \leq \sum_{v \in S}(f(v) - 1/2) + 1 \), for all \( S \subseteq A \), when we know that \( G \) admits a spanning closed trail meeting each \( v \in A \) at most \( f(v) \) times.

Theorem 9.3. Let \( G \) be a 2-tree-connected graph with the independent set \( X \subseteq V(G) \) and let \( f \) be a positive integer-valued function on \( X \). If for every \( S \subseteq X \),
\[
\Omega(G \setminus S) \leq \sum_{v \in S}(f(v) - 1/2) + 1,
\]
or \( G \) has a spanning closed trail meeting each \( v \in S \) at most \( f(v) \) times, then \( G \) has a spanning closed trail meeting each \( v \in X \) at most \( f(v) \) times.

Proof. Let \( H \) be a 2-tree-connected spanning subgraph of \( G \) with the minimum total excess from \( h \), where \( h(v) = 2f(v) + 1 \), for each \( v \in X \), and \( h(v) = d_G(v) + 1 \), for each \( v \in V(G) \setminus X \). Define \( S \) to be a subset of \( X \) with the properties described in Theorem 6.10. If \( \Omega(G \setminus S) \leq \sum_{v \in S}(f(v) - 1/2) + 1 \), then by an argument similar to the proof of Theorem 6.11, one can conclude that \( te(H,h) = 0 \). Thus the graph \( H \) admits a spanning closed trail meeting each \( v \in X \) at most \( f(v) \) times, and so does \( G \). If \( \Omega(G \setminus S) > \sum_{v \in S}(f(v) - 1/2) + 1 \), then by the assumption \( G \) contains a spanning Eulerian subgraph \( L \) such that for each \( v \in S \), \( d_L(v) \leq 2f(v) \). We are going to replace some of the edges of \( L \) by some of the edges of \( H \) to obtain a new Eulerian graph. Define \( P \) to be the partition of \( V(G) \setminus S \) obtained from the 2-tree-connected components of \( H \setminus S \). Take \( T_1 \) and \( T_2 \) to be two edge-disjoint spanning trees of \( H \) such that \( T_1[A] \) and \( T_2[A] \) are connected, for all \( A \in P \). First, for any \( A \in P \), remove the edges of \( L[A] \) from \( L \) and replace them by the edges of \( T_1[A] \). Call the resulting graph \( L_1 \). This graph is connected, since all graphs \( T_1[A] \) together with \( L/P \) are connected, where \( A \in P \). Define \( Q \) to be the set of all vertices of \( L_1 \) with odd degrees so that \( Q \cap S = \emptyset \). Since \( L \) is Eulerian, the number of edges of \( L \) with exactly one end in \( A \) is even, and so does \( L_1 \). This implies that \( |Q \cap A| \) is even. By applying Corollary 5.4 to every graph \( T_2[A] \), one can conclude that there is a spanning forest \( F \) of \( T_2 \) such that for each vertex \( v \), \( d_F(v) \) is odd if and only if \( v \in Q \). Finally, add the edges of these forest to \( L_1 \) and call the resulting Eulerian graph \( L_2 \).
Recall that there is no edge of \( E(G) \setminus E(H) \) joining different parts of \( P \). Since \( X \) is independent, for each \( v \in X \setminus S \), we must have \( E_{L_2}(v) \subseteq E_H(v) \) and so \( d_{L_2}(v) \leq d_H(v) \leq 2f(v) + 1 \). Also, for each \( v \in S \), we have \( d_{L_2}(v) = d_L(v) \leq 2f(v) \). Thus \( G \) has a spanning closed trail meeting each \( v \in X \) at most \( f(v) \) times. \( \square \)

**Corollary 9.4.** Let \( G \) be a connected graph with the independent set \( X \subseteq V(G) \) and let \( f \) be a positive integer-valued function on \( X \). If for every \( S \subseteq X \),

\[
\omega(G \setminus S) \leq \sum_{v \in S} (f(v) - 1/2) + 1,
\]

or \( G \) has a spanning closed walk meeting each \( v \in S \) at most \( f(v) \) times, then \( G \) has a spanning closed walk meeting each \( v \in X \) at most \( f(v) \) times.

**Proof.** Duplicate the edges of \( G \) and apply Theorem 9.3 to the resulting graph. \( \square \)

### 9.1 Toughness and the existence of connected \( \{2,4\} \)-factors

The following theorem verifies a weaker version of Chvátal’s Conjecture [8] which gives a sufficient toughness condition for the existence of connected \( \{2,4\} \)-factors.

**Theorem 9.5.** Let \( G \) be a 2-tree-connected graph of order at least two. If for all \( S \subseteq V(G) \),

\[
\omega(G \setminus S) \leq \frac{2}{7} |S| + \frac{9}{7},
\]

then \( G \) admits a connected \( \{2,4\} \)-factor.

**Proof.** We repeat the proof of Theorem 6.23 in the same way with some careful estimation. For convenience, we write \( k \) for \( 3 + 1/2 \) and \( \epsilon \) for \( 9/7 \). If \( G \) has no connected \( \{2,4\} \)-factors, then by a combination of Theorem 9.1 and Lemma 6.21, there is a vertex subset \( S \) of \( V(G) \) with \( \Omega(G \setminus S) \geq |S|/2 + 5/2 \) such that every component of \( G \) is 2-tree-connected or has maximum degree at most 2. Let \( C \) be the union of all components of \( G \setminus S \) which are not 2-tree-connected. If \( |V(C)| = 0 \), then we have \( \Omega(G \setminus S) = \omega(G \setminus S) \) which is impossible. We may assume that \( |V(C)| \geq 1 \). Note that \( C \) is the union of some paths and cycles. It is easy to check that the graph \( C \) has an independent set \( X \) of size at least \( \frac{1}{2}(|V(C)| - c_3 - c_o + p_o) \), where \( c_o \) denotes the number of odd cycles of \( C \) with order at least five, \( c_3 \) denotes the number of triangles of \( C \), and \( p_o \) denotes the number of path components of \( C \) with odd order. We also denote by \( r \) the number of components of \( G \setminus S \) which are 2-tree-connected. Let \( S' \subseteq V(C) \setminus X \) such that \( \omega(C \setminus S') \geq |X| \) and \( |S'| \leq \frac{1}{2} |V(C)| - \frac{3}{2} c_3 - \frac{1}{2} c_o - p_2 - p_e \), where \( p_e \) denotes the number of path components of \( C \) with even order greater than four and \( p_2 \) denotes the number of path components of \( C \) with order two. Thus

\[
\omega(G \setminus S \cup S') \leq \frac{1}{k}(|S| + |S'|) + \epsilon \leq \frac{1}{k} |S| + \epsilon + \frac{1}{2k} |V(C)| - \frac{3}{2k} c_3 - \frac{1}{2k} c_o - \frac{1}{k} p_2 - \frac{1}{k} p_e.
\]

On the other hand,

\[
\frac{1}{2} (|V(C)| - c_3 - c_o + p_o) + r \leq \omega(C \setminus S') + r = \omega(G \setminus S \cup S').
\]
Hence the following inequality can be derived
\[
\frac{1}{2} |V(C)| + \frac{1}{k-1} p_2 + \frac{1}{k-1} p_c + \frac{k}{2k-2} p_o + r \leq \frac{1}{k-1} |S| + \frac{k}{k-1} \epsilon + \frac{k-3}{2k-2} c_3 + \frac{1}{2} c_o. \tag{21}
\]

Let us estimate \( c_o \). Note that if \( G \) has no induced odd cycles with order at least five, then \( c_o = 0 \) and we could replace \( k \) by 3 and replace \( \epsilon \) by 4/3. Define \( S'' \) to be a vertex subset of \( V(C) \) of size \( 2c_o + p_c \) containing exactly one middle vertex of any path component of \( C \) with even order greater than four and exactly two nonadjacent vertices of any odd cycle of \( C \) with order at least five. By the assumption,

\[
\omega(G \setminus S) + c_o + p_c = \omega(G \setminus S \cup S'') \leq \frac{1}{k} (|S| + |S''|) + \epsilon = \frac{1}{k} (|S| + 2c_o + p_c) + \epsilon.
\]

Since \( c_3 + c_o + p_2 + p_c \leq \omega(G \setminus S) \), we can derive the following inequality
\[
\frac{1}{2} c_o + \frac{k}{4k-4} p_2 + \frac{2k-1}{4k-4} p_c + \frac{k}{2k-2} p_o + r \leq \frac{1}{4k-4} |S| + \frac{k}{4k-4} \epsilon - \frac{k}{4k-4} c_3. \tag{22}
\]

Therefore, Relations (21) and (22) can conclude that
\[
\frac{1}{2} |V(C)| + \frac{k+4}{4k-4} p_2 + \frac{2k+3}{4k-4} p_c + \frac{k}{2k-2} p_o + r \leq \frac{1}{2} |E(C)| + \frac{5k}{4k-4} \epsilon + \frac{k-6}{4k-4} c_3,
\]
which implies
\[
\Omega(G \setminus S) = |V(C)| - \frac{1}{2} |E(C)| + r \leq \frac{1}{2} |V(C)| + \frac{1}{2} (p_2 + p_c + p_o) + r \leq \frac{5k}{4k-4} |S| + \frac{5k}{4k-4} \epsilon < \frac{1}{2} |S| + \frac{5}{2}.
\]

This is a contradiction. Hence the theorem holds. \( \square \)

**Corollary 9.6.** Every \((3 + \frac{1}{2})\)-tough graph \( G \) of order at least three admits a connected \( \{2, 4\} \)-factor.

**Proof.** For \(|V(G)| = 3\), the proof is straightforward. For \(|V(G)| \geq 4\), it is easy to see that the graph \( G \) is 4-connected and consequently 2-tree-connected. Hence the assertion follows from Theorem 9.5. \( \square \)

### 9.2 Toughness and the existence of \( m \)-tree-connected \( \{r, r+2\} \)-factors

As we already observed tough enough graphs have \( m \)-tree-connected \( \{r, r+1\} \)-factors, when \( r \geq 2m \). In this subsection, we investigate \( m \)-tree-connected \( \{r, r+2\} \)-factors in highly edge-connected graphs and tough enough graphs, when \( r \geq 4m \) or \( r = 4m - 2 \). For this purpose, we first present the following result.

**Theorem 9.7.** Let \( G \) be a graph with \( Q \subseteq V(G) \), where \(|Q| \) is even. Then \( G \) has an \( m \)-tree-connected spanning subgraph \( H \) such that for each vertex \( v \),
\[
d_H(v) \geq d_G(v) - \begin{cases} \left\lceil \frac{d_G(v) - m}{k} \right\rceil, & \text{if } G \text{ is } (k + m)\text{-tree-connected}, \\
\left\lfloor \frac{d_G(v)}{2k} \right\rfloor + 1, & \text{if } G \text{ is } (2k + 2m)\text{-edge-connected}.
\end{cases}
\]
and also \( d_H(v) \) is odd if and only if \( v \in Q \).
Proof. If $G$ is $(2k+2m)$-edge-connected, then by Corollary 6.16, the graph $G$ has a $(k+m)$-tree-connected spanning subgraph $G'$ such that for each vertex $v$, $d_{G'}(v) \leq \lceil \frac{d_G(v)}{2} \rceil + k + m$. If $G$ is $(k+m)$-tree-connected, set $G' = G$. Decompose $G'$ into two edge-disjoint spanning subgraphs $L'$ and $L$ such that $L'$ is $m$-tree-connected and $L$ is $k$-tree-connected. Note that for each vertex $v$, we have $d_L(v) = d_{G'}(v) - d_{L'}(v) \leq d_{G'}(v) - m$. Denote by $O$ the set of all vertices of $G$ with odd degree. Since $|O|$ is even, $|O \cap Q|$ and $|O \setminus Q|$ have the same parity. Since $|Q|$ is even, $|O \cap Q|$ and $|Q \setminus O|$ have the same parity. Thus $|Q'|$ must be even where $Q' = (O \setminus Q) \cup (Q \setminus O)$. By Corollary 5.4, the graph $L$ has a spanning forest $F$ such that for each vertex $v$, $d_F(v) \leq \lceil \frac{d_L(v)}{k} \rceil \leq \lceil \frac{d_G(v) - m}{k} \rceil$, and also $d_F(v)$ is odd if and only if $v \in Q'$. For the case that $G$ is $(2k+2m)$-edge-connected, the graph $F$ can be found such that for each vertex $v$, $d_F(v) \leq \lceil \frac{d_L(v)}{k} \rceil \leq \lceil \frac{d_G(v)}{2k} \rceil + 1$. It is not hard to see that $G \setminus E(F)$ is the desired spanning subgraph. \hfill \Box

Corollary 9.8. Every $(2\lceil r/6 \rceil + 2m)$-edge-connected $r$-regular graph of even order with $r \geq 4$ has an $m$-tree-connected $\{r-3, r-1\}$-factor.

Proof. Apply Theorem 9.7 with $Q = \emptyset$ when $r$ is odd and with $Q = V(G)$ when $r$ is even. \hfill \Box

Corollary 9.9. Let $G$ be an $\{r + 2, r + 3\}$-graph with $r|V(G)|$ even and $r \geq 2$. Then $G$ has an $m$-tree-connected $\{r, r + 2\}$-factor, if

$$r + 2 \leq \begin{cases} 
3k + m, & \text{when } G \text{ is } (k+m)\text{-tree-connected}, \\
4k, & \text{when } G \text{ is } (2k+2m)\text{-edge-connected}.
\end{cases}$$

Proof. Apply Theorem 9.7 with $Q = \emptyset$ when $r$ is even and with $Q = V(G)$ when $r$ is odd. \hfill \Box

Corollary 9.10. Every $t$-tough graph $G$ of order at least $r + 1$ with $r|V(G)|$ even has an $m$-tree-connected $\{r, r + 2\}$-factor, where $2k + 2m \leq r + 2 \leq 3k + m$ and $t \geq \frac{2k+2m}{r+3-2k-2m}(r + 2)$.

Proof. For $|V(G)| \leq r+2$, the graph must be complete and the proof is straightforward. For $|V(G)| \geq r+3$, it is enough to apply a combination of Corollary 7.10 and Corollary 9.9. \hfill \Box

The next result gives a sufficient toughness condition for the existence of spanning closed trails meeting each vertex $r$ or $r + 1$ times. It remains to decide whether higher toughness can guarantee the existence of spanning closed trails meeting each vertex exactly $r$ times.

Corollary 9.11. Every $(2r+2)^2$-tough graph $G$ of order at least $2r+1$ admits a connected $\{2r, 2r+2\}$-factor.

Proof. Apply Corollary 9.10 with $m = 1$. \hfill \Box
9.3 Graphs with high edge-connectivity

Recently, the present author [16] showed that every \((r - 1)\)-edge-connected \(r\)-regular graph with \(r \geq 4\) admits a connected \(\{2, 4, 6\}\)-factor. We present below a more powerful version for this result and push down the needed edge-connectivity around \(2r/3\). The special cases \(k \in \{1, 2\}\) of the following theorem was formerly found in [17, 20].

Theorem 9.12. Let \(G\) be a \(k\)-edge-connected graph.

1. If \(k \geq 4\), then \(G\) has an \(f\)-trail, where \(f(v) = \left\lceil \frac{d_G(v) + k/2 - 4}{k} \right\rceil + 1\), for each vertex \(v\).

2. If \(k \leq 3\), then \(G\) has an \(f\)-walk, where \(f(v) = \left\lceil \frac{d_G(v) - 1}{k} \right\rceil + 1\), for each vertex \(v\).

Proof. The proof of \(k = 1\) is straightforward. For \(k = 2\), apply Lemma 4.4 and Corollary 9.2 with \(\lambda = 1\). For \(k \geq 4\), apply Lemma 6.14 and Theorem 9.1 with \(\lambda = 2/k\). It remains to prove the special case \(k = 3\).

By Corollary 5.4, the graph \(G\) has a spanning forest \(F\) such that for each vertex \(v\), \(d_F(v)\) and \(d_G(v)\) have the same parity and \(d_F(v) \leq \left\lceil \frac{d_G(v)}{3} \right\rceil + 1\). Add a copy of \(F\) to \(G\) and call the resulting Eulerian graph \(G'\).

For each vertex \(v\), we have \(d_{G'}(v) \leq d_G(v) + d_F(v) \leq \left\lceil \frac{4d_G(v)}{3} \right\rceil + 1\). Since \(d_{G'}(v)\) is even, we must have \(d_{G'}(v) \leq \left\lceil \frac{4d_G(v) - 1}{3} \right\rceil + 1\), whether \(4d_G(v) \equiv 3 \pmod{4}\) or not. Since \(G'\) is 3-edge-connected and whose degrees are even, this graph is 4-edge-connected as well. By the first item, \(G'\) has a spanning Eulerian subgraph \(L\) such that for each vertex \(v\), \(\frac{1}{2}d_L(v) \leq \left\lceil \frac{d_{G'}(v)}{4} \right\rceil + 1\). Therefore, one can conclude that \(\frac{1}{2}d_L(v) \leq \left\lceil \frac{d_G(v) - 1}{3} \right\rceil + 1\). Hence \(L\) yields a spanning closed walk in \(G\) with the desired properties. \(\square\)

The following corollary partially answers Conjecture 3 in [16].

Corollary 9.13. Every \(r\)-edge-connected \(r\)-regular graph with \(4 \leq r \leq 8\) admits a connected \(\{2, 4\}\)-factor.

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