New Explicit Good Linear Sum-Rank-Metric Codes

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Abstract—Sum-rank-metric codes have wide applications in universal error correction, multishot network coding, space-time coding and the construction of partial-MDS codes for repair in distributed storage. Fundamental properties of sum-rank-metric codes have been studied and some explicit or probabilistic constructions of good sum-rank-metric codes have been proposed. In this paper we give three simple constructions of explicit linear sum-rank-metric codes. In finite length regime, numerous larger linear sum-rank-metric codes with the same minimum sum-rank distances as the previously constructed codes can be derived from our constructions. For example several better linear sum-rank-metric codes over \( F_q \) with small block sizes and the matrix size \( 2 \times 2 \) are constructed for \( q = 2, 3, 4 \) by applying our construction to the presently known best linear codes. Asymptotically our constructed sum-rank-metric codes are close to the Gilbert-Varshamov-like bound on sum-rank-metric codes for some parameters. Finally we construct a linear MSRD code over an arbitrary finite field \( F_q \) with various square matrix sizes \( n_1, n_2, \ldots, n_t \) satisfying \( n_1 \geq n_2^2 + 1 + \cdots + n_t^2 \), \( i = 1, 2, \ldots, t-1 \), for any given minimum sum-rank distance. There is no restriction on the block lengths \( t \) and parameters \( N = n_1 + \cdots + n_t \) of these linear MSRD codes from the sizes of the fields \( F_q \).

Index Terms—Sum-rank-metric code, Singleton-like bound, Gilbert-Varshamov-like bound, MSRD code.

I. INTRODUCTION

For a vector \( a \in F_q^n \), the Hamming weight \( wt_H(a) \) of \( a \) is the number of non-zero coordinate positions. The Hamming distance \( d_H(a, b) \) between two vectors \( a \) and \( b \) is defined as \( wt_H(a-b) \). For a code \( C \subseteq F_q^n \), its Hamming distance is the minimum of Hamming distances \( d_H(a, b) \) between any two different codewords \( a \) and \( b \) in \( C \),

\[
d_H = \min_{a \neq b} \{ d_H(a, b) : a, b \in C \}.
\]

It is well-known that the Hamming distance of a linear code \( C \) is the minimum Hamming weight of its non-zero codewords. The theory of Hamming metric error-correcting codes has been extensively studied and numerous constructions have been proposed, see e.g. [18]. For a linear \([n, k, d_H]_q \) code, the Singleton bound asserts \( d_H \leq n-k+1 \). When equality holds, this code is called a maximal distance separable (MDS) code. The main conjecture of MDS codes claims that the length of an MDS code over \( F_q \) is at most \( q+1 \), except some special cases. In [1] the main conjecture of MDS codes was proved for codes over prime fields.

In this paper the repetition code in the Hamming metric \( C = \{(c_1, \ldots, c_n) : c_1 = \cdots = c_n \} \) over some finite field will be used. For each nonzero codeword in this code, the Hamming weight is exactly \( n \).

The rank-metric on the space \( F_q^{(m,n)} \) of size \( m \times n \) matrices over \( F_q \) is defined by the ranks of matrices, \( d_r(A, B) = \text{rank}(A - B) \). The minimum rank-distance of a code \( C \subseteq F_q^{(m,n)} \) is

\[
d_r(C) = \min_{A \neq B} \{ d_r(A, B) : A, B \in C \}.
\]

The rate of this code \( C \) is \( \text{rate}(C) = \frac{\log_q |C|}{mn} \). For a code \( C \subseteq F_q^{(m,n)} \) with the minimum rank distance \( d_r(C) \geq d \), it is well-known that the number of codewords in \( C \) is upper bounded by \( q^\text{max\{m,n\} (\text{min\{m,n\}} - d + 1) \) , see [11]. A code attaining this bound is called a maximal rank distance (MRD) code. The Gabidulin code \( Gab(n, v) \subseteq F_q^{(n,v)} \) is a linear \( F_q \)-linear mappings on \( F_q^n \) defined by \( q \)-polynomials \( a_0x + a_1x^q + \cdots + a_tx^{qt} \), where \( a_0, a_1, \ldots, a_t \in F_q \) are arbitrary elements in \( F_q \), see [11]. The rank-distance of the Gabidulin code is at least \( n-t \) since there are at most \( v^t \) roots in \( F_q \) for each such \( q \)-polynomial. There are \( v^{q^{t(q+1)}} \) such \( q \)-polynomials. Hence the size of the Gabidulin code is \( q^{n(q+1)} \) and it is an MRD code. Let \( h \) be a non-negative integer and \( \phi : F_q^h \rightarrow F_q^{(h,k)} \) be a \( F_q \)-linear embedding. Then

\[
a_0\phi(x)^{q^h} + a_1\phi(x)^{q^h-1} + \cdots + a_t\phi(x)^{q^h} + a_0\phi(x)
\]

is a \( F_q \)-linear mapping from \( F_q^k \) to \( F_q^{(k+h)} \), where \( a_i \in F_q^{(k+h)} \) for \( i = 0, 1, \ldots, t \). It is clear that the dimension of the kernel of any such mapping is at most \( t \). Then the rank-metric code consisting of all such linear mappings is an MRD code with rank distance \( k-t \) and size \( q^{(k+h)(k+1)} \) elements. MRD codes have been widely used in previous constructions of constant dimension subspace codes, see [10]. We refer to [2], [3], [13] for recent results on rank-metric codes and [16] and [30] for recent results on MRD codes.

Sum-rank-metric codes have applications in multishot network coding, see [23], [29], [33], and [43], space-time coding, see [39], and coding for distributed storage see [6], [21], and [24]. For fundamental properties and constructions of sum-rank-metric codes, we refer to [4], [5], [7], [21], [24], [25], [27], [28], and [34]. Now we recall some basic concepts and results for sum-rank-metric codes in [4]. Let \( n_1 \leq m_3 \) be 21 positive integers satisfying \( m_1 \geq m_2 \cdots \geq m_4 \). Set
Let \( N = n_1 + \cdots + n_t \). Then
\[
F_q^{(n_1,m_1),\ldots,(n_t,m_t)} = F_q^{n_1 \times m_1} \oplus \cdots \oplus F_q^{n_t \times m_t}
\]
be the set of all \( x = (x_1, \ldots, x_t) \), where \( x_i \in F_q^{n_i \times m_i}, i = 1, \ldots, t \), is a \( n_i \times m_i \) matrix over \( F_q \). We call \( n_i \times m_i, i = 1, \ldots, t \), matrix sizes of sum-rank-metric codes. Set
\[
w_{sr}(x_1, \ldots, x_t) = \text{rank}(x_1) + \cdots + \text{rank}(x_t)
\]
and
\[
d_{sr}(x, y) = w_{sr}(x - y),
\]
for \( x, y \in F_q^{(n_1,m_1),\ldots,(n_t,m_t)} \). This is indeed a metric on \( F_q^{(n_1,m_1),\ldots,(n_t,m_t)} \).

**Definition 1.1:** A sum-rank-metric code \( C \subseteq F_q^{(n_1,m_1),\ldots,(n_t,m_t)} \) is a subset of the finite metric space \( F_q^{(n_1,m_1),\ldots,(n_t,m_t)} \). Its minimum sum-rank distance is defined by
\[
d_{sr}(C) = \min_{x \neq y, x, y \in C} d_{sr}(x, y).
\]
The code rate of \( C \) is \( R_{sr} = \frac{\log_2 |C|}{\sum_i n_i m_i} \). The relative distance is \( \delta_{sr} = \frac{d_{sr}}{n t} \).

The basic goal of sum-rank-metric coding is to construct good sum-rank-metric codes with large cardinalities and large minimum sum-rank distances. For some basic upper bounds on sizes of sum-rank-metric codes, we refer to [4, Section III & IV].

The following several special cases of parameters are important. For \( t = 1 \), this is the rank-metric code case. For \( m_1 = \cdots = m_t = m \) and \( n_1 = \cdots = n_t = n \), this is the \( t \)-sum-rank-metric code over \( F_{q^m} \) with the code length \( N = n t \). For \( m = n = 1 \), this is the Hamming metric error-correcting code case. Hence the sum-rank-metric is a generalization and combination of the Hamming metric and the rank-metric.

A Singleton-like bound for the sum-rank-metric was proposed in [4] and [24]. The general form Theorem III.2 in [4] is as follows. The minimum sum-rank distance \( d \) can be written uniquely as the form
\[
d_{sr} = \sum_{i=1}^{t} n_i + \delta + 1 \quad \text{where} \quad 0 \leq \delta \leq n_j - 1,
\]
then
\[
|C| \leq q^{m t (N-d_{sr}+1)}.
\]
The code attaining this bound is called a maximal sum-rank-metric (MSRD) code. When \( m_1 = \cdots = m_t = m \), this bound is of the form
\[
|C| \leq q^{mn (N-d_{sr}+1)}.
\]
We call the difference \( mn (N-d_{sr}+1) - \log_q |C| \) the *defect* of the sum-rank-metric code \( C \).

When \( t \leq q - 1 \) and \( N \leq (q-1)m \), MSRD codes attaining the above Singleton-like bound were constructed in [21] and [28]. They are called linearized Reed-Solomon codes, we also refer to [31] for the further results. When \( t = q \), it was proved in [4] Example VI.9, MSRD codes may not exist for some minimum sum-rank distance. In [4] the maximal block lengths of MSRD codes are upper bounded in Theorem VI.12. In several other cases, for example, when the minimum sum-metric distance is 2 or \( N \), MSRD codes exist for all parameters, they were constructed in Section VII of [4]. In [26] more linear MSRD codes with the same matrix size defined over smaller fields were constructed by extended Moore matrices.

One-weight sum-rank-metric codes were studied and constructed in [22] and [32]. Generalized sum-rank-weights were defined for sum-rank-metric codes via optimal antcodes in [7]. It was proved in [7] that the generalized sum-rank-weights of an MSRD code is determined by its block size, matrix size, dimension and distance parameters, as that of the generalized Hamming weights of an MDS code, see [38] and [45]. MSRD codes have applications in space-time coding, see e.g. [39], maximally recoverable LRC codes and partial-MDS codes, see e.g. [6] and [24].

Sum-rank BCH codes of the matrix size \( n_1 = \cdots = n_t = n \), \( m_1 = \cdots = m_t = m \) were proposed and studied in [25] by the deep algebraic method. These sum-rank-metric codes are linear over \( F_{q^m} \). There is a designed distance of such a sum-rank BCH code such that the minimum sum-rank distance is always greater than or equal to the designed distance. On the other hand, the dimension of these sum-rank BCH codes is lower bounded in [25] Theorem 9. Many sum-rank-metric codes with parameters \( n = m = 2 \) and \( q = 2 \) were constructed in Tables V, VI and VII of [25]. It will be shown in Section III and the Appendix, many of our constructed sum-rank-metric codes are larger than these sum-rank BCH codes constructed in [25] of the same minimum sum-rank distances.

The volume of radius \( r \) in the sum-rank-metric is
\[
\text{vol}(B_{sr}(F_q^{(n_1,m_1),\ldots,(n_t,m_t)})) = \sum_{s=0}^{r} \left( \sum_{s_1+s_2+\cdots+s_r=s} \prod_{i=1}^{t} \left( \frac{n_i}{s_i} \right) \prod_{j=0}^{s_i-1} \left( q^{m_i-j} - q^j \right) \right),
\]
we refer to Lemma III.5 in [4]. In the case \( n_1 = \cdots = n_t = n_t, m_1 = \cdots = m_t = m_t \), set
\[
f(z) = \sum_{i=0}^{n} \left( \begin{array}{c} n_i \\ i \end{array} \right) \prod_{j=0}^{i-1} (q^m - q^j)^{z^j},
\]
and
\[
H_{sr}(\rho) = \frac{1}{m n} \min_{z \in \mathbb{C}(0,1)} \log_q \left( \frac{f(z)}{z^\rho} \right),
\]
where \( \rho \) is a positive real number satisfying \( 0 < \rho < n \). Then from [4] Theorem IV.9, when \( n, m, \rho < n \) are fixed,
\[
\lim_{t \to \infty} \frac{\log_q \left( \text{vol}(B_{sr}(F_q^{(n_1,m_1),\ldots,(n_t,m_t)})) \right)}{t} = H_{sr}(\rho).
\]
This is an entropy function for the sum-rank. From Lemma 2 in [34] we have
\[
H_{sr}(\rho) \geq \frac{m n - n - \rho}{m n} - \frac{1}{\log_q \gamma},
\]
where \( \gamma_q = \prod_{i=1}^{r} (1 - q^{-i})^{-1}, \) for example \( \gamma_2 \approx 3.463, \gamma_3 \approx 1.785 \) and \( \gamma_4 \approx 1.452 \). Non-list-decodability of Gabidulin codes and linearized Reed-Solomon codes as the sum-rank code generalization of Gabidulin codes were studied in [37] and [42], which are closely related to the above entropy function for the sum-rank.

We now recall the sum-rank-metric Gilbert-Varshamov-like bound given in [34] and [36] for the case \( n_1 = \cdots = n_t = n \) and \( m_1 = \cdots = m_t = m \).
Asymptotic Gilbert-Varshamov-Like Bound: For fixed positive integers \(n\) and \(m\), and positive integers \(t, N = nt\), positive real number \(R_{sr}\) and \(\delta_{sr} = \frac{d}{N}\) and \(2 < d \leq N\) satisfying
\[
R_{sr} \leq \delta_{sr}^2 \frac{n}{m} - \delta_{sr}(1 + \frac{n}{m} + \frac{2}{N}) + 1 + \frac{n}{N} + \frac{n}{N^2} \sum_{i=1}^{N-1} \frac{\log_2(1 + \frac{1}{i}) + \log_2(\delta_{sr}, N - 1)}{N} - \frac{\log_2(\gamma)}{nm},
\]
there exists a linear sum-metric code of rate \(R_{sr}\) and the relative minimum sum-rank distance at least \(\delta_{sr}\). When \(m = \xi n\) goes to the infinity and \(m \in \omega(\log_2(t))\), where \(\xi\) is a fixed constant, then
\[
R_{sr} \sim \delta_{sr}^2 - \delta_{sr}(1 + \frac{1}{\xi}) + 1.
\]

It was proved in [34] that random linear sum-rank-metric codes attain the Gilbert-Varshamov-like bound with high probability.

In this paper we give three constructions of linear sum-rank-metric codes. Our constructions are based by combining Hamming metric codes and \(q\)-polynomial representations of rank-metric codes. These constructions works for various block lengths and matrix sizes. Then many explicit good linear sum-rank-metric codes are constructed from our second construction and the presently known best codes over \(F_4\) and \(F_9\) from [14]. The linear sum-rank-metric codes constructed in this paper are good in the sense that our constructed codes are asymptotically good linear sum-rank-metric codes which are close to the Gilbert-Varshamov like bound are also given. Sum-rank-metric codes constructed in this paper is clearly \(F_q\)-linear, not always \(F_{q^n}\)-linear as in the sense of [21] and [25].

There have been several constructions of sum-rank-metric MSRD codes of various matrix sizes, we refer to [4], [17], and [26]. In previous papers on constructions of sum-rank-metric codes [4], [5], [7], [21], [23], [25], [31], [32], the main focus is on the matrix size case \(n_1 = n_2 = \cdots = n_t, m_1 = m_2 = \cdots = m_t\). MSRD codes over a fixed field with an arbitrary block length and the matrix size \(n_1 \times m_1, n_t \neq m_t\), were constructed in [26, Subsection 4.5]. In Section V we construct new linear MSRD codes over an arbitrary finite field \(F_q\) with square matrix sizes \(n_1 \times n_1, n_2 \times n_2, \ldots, n_t \times n_t\), where \(n_1, n_2, \ldots, n_t\) are \(t\) positive integers satisfying \(n_i \geq n_{i+1} + \cdots + n_t\), for \(i = 1, 2, \ldots, t-1\), and any given minimum sum-rank distance \(d_{sr}\). There is no restriction on the length of the code from the size \(q\) of the finite field. Our result illustrates that the theory of sum-rank-metric codes with various square matrix sizes is basically different with the theory of sum-rank-metric codes with the same square matrix size. This is also quite different to the essence of the main conjecture on MDS codes in the Hamming metric. On the other hand, comparing with constructed non-trivial optimal LRC codes, quantum MDS codes and entanglement-assisted quantum MDS codes attaining the Singleton bound in [9], [15], and [35], code lengths are bounded by some \(O(q^n)\), the block lengths \(t\) and the parameters \(N = n_1 + \cdots + n_t\) of MSRD codes with various square matrix sizes, have no relation with the field \(F_q\).

II. EXPLICIT CONSTRUCTIONS OF SUM-RANK-METRIC CODES

In this section we give our first and second explicit constructions of linear sum-rank-metric codes. The matrix size is restricted to the case \(n_1 = \cdots = n_t = n = m_1 = \cdots = m_t\). The construction can be generalized to the matrix size case \(n_1 = n_2 = \cdots = n_t, n_i \leq m_i, i = 1, 2, \ldots, t\) directly.

A. Construction I: One Hamming Metric Code Over the Large Field

The Gabidulin code \(Gab(n, v)\) in \(F_q^{(n, n)}\) of the minimum rank distance \(n - v\) and cardinality \(q^{n(v+1)}\) is identified with the set of all \(q\)-polynomials \(a_0 x + a_1 x^q + \cdots + a_v x^{q^v}\), where \(a_0, a_1, \ldots, a_v\) are arbitrary elements in \(F_q\). Hence \((a_0, a_1, \ldots, a_v)\) can be considered as elements in the finite field \(F_q^{n(v+1)}\). Let \(C\) be a linear \([t, w, d]_{q^{n(v+1)}}\) code over this finite field \(F_q^{n(v+1)}\), with the dimension \(w\) and minimum Hamming distance \(d\). Then we have a linear sum-rank-metric code \(SR(C)\) consisting of all \((c_0, c_1, c_2)\) where \(c_i = (c_{0i}, c_{1i}, \ldots, c_{ti})\), \(c_{ij} \in F_{q^v}\), are considered as a \(q\)-polynomial \(c_{0i} + c_{1i} x^q + \cdots + c_{ti} x^{q^v}\) associated with the codeword in \(Gab(n, v)\). This code \(SR(C)\) is \(F_{q^n}\)-linear.

Remark 2.1: From the coordinate form of the code \(SR(C)\), the above construction is similar to the construction for convolutional codes in [29].

The following result is obvious.

Proposition 2.1: The dimension over \(F_q\) of the above linear sum-rank-metric code \(SR(C)\) is \(u n (v + 1)\) and the minimum rank-sum distance of \(SR(C)\) is at least \(d(n - v)\). Hence the code rate of \(SR(C)\) is \(R_{sr}(SR(C)) = \frac{u n (v + 1)}{tn^2} = \frac{w}{tn} = R(C) \cdot \frac{v+1}{n}\) and the relative minimum sum-rank distance is at least \(d(n - v) = \delta(C) \cdot 2 - \frac{1}{n}\), where \(R(C)\) and \(\delta(C)\) are the rate and the relative minimum Hamming distance of the code \(C\). Therefore for fixed \(n\) and \(v\), asymptotically good linear sum-rank-metric codes with positive rate and positive relative minimum sum-rank distance can be constructed from asymptotically good linear codes in the Hamming metric. Assume that \(n\) is even, \(v = n - 2\), and by using the algebraic geometry code sequence satisfying \(R + \delta \geq 1 - \frac{1}{q^{n(v+1)} - 1}\), see [41], we can get asymptotically good sum-rank-metric code sequence with the rate
\[
R_{sr} = \frac{n - 1}{n} \cdot R
\]
and the relative minimum sum-rank distance at least
\[
\delta_{sr} \geq \frac{2}{n} \left(1 - R - \frac{1}{q^{n(v+1)} - 1}\right).
\]
Then we have
\[
R_{sr} + 2\delta_{sr} - \delta_{sr}^2 \geq \frac{4}{n} + \frac{n - 5}{n} R - \frac{4}{n^2} (1 - R)^2.
\]
When \(n\) is large and \(R\) is close to 1, the rate and the relative minimum sum-rank distance is close to the Gilbert-Varshamov-like bound
\[
R_{sr} \sim \delta_{sr} - 2\delta_{sr} + 1,
\]
in [34], or see the bound cited in Section I.
Remark 2.2: Comparing with the previous probabilistic construction of sum-rank-metric codes attaining the Gilbert-Varshamov-like bound in [34], the codes in the above construction is given with the help of algebraic geometry codes attaining the Tsfasman-Vlădut-Zink bound. Then these sum-rank-metric codes can be constructed by a low-complexity polynomial-time algorithm. We refer to [40] for a low-complexity polynomial-time algorithm constructing algebraic geometry codes attaining the Tsfasman-Vlădut-Zink bound.

B. Construction 2: Several Hamming Metric Codes Over the Small Field

A modification of the construction 1 makes the resulted sum-rank-metric codes larger. The first linear $[t, k_0, w_0]_{q^n}$ code $C_0 \subseteq F^t_q$ corresponds to $a_i^0$ in the $q$-polynomials $a_i^0x + a_i^1x^q + \cdots + a_i^{v-1}x^{q^{v-1}}$ in the $i$-th copy of Gabidulin code $Gab(n, v)$ at the $i$-th block position, for $i = 1, \ldots, t$. The second linear $[t, k_1, w_1]_{q^n}$ code $C_1 \subseteq F^t_q$ corresponds to $a_i^1$ in the $q$-polynomials $a_i^0x + a_i^1x^q + \cdots + a_i^{v-1}x^{q^{v-1}}$ in the $i$-th copy of Gabidulin code $Gab(n, v)$ at the $i$-th block position, $i = 1, 2, \ldots, t$. For the $(v+1)$-th linear $[t, k_v, w_v]_{q^n}$ code $C_v$ corresponds to $a_i^v$ in the $q$-polynomials $a_i^0x + a_i^1x^q + \cdots + a_i^{v-1}x^{q^{v-1}}$ in the $(v+1)$-th copy of Gabidulin code $Gab(n, v)$ at the $i$-th block position, $i = 1, 2, \ldots, t$. The sum-rank-metric code $SR(C_0, \ldots, C_v)$ consisting of algebraic geometry codes as follows,

$$SR(C_0, \ldots, C_v) = \{(a_0^0x + a_1^1x^q + \cdots + a_v^{v-1}x^{q^{v-1}}), \ldots, (a_0^0, \ldots, a_v^0) \in C_0, \ldots, (a_0^1, \ldots, a_v^1) \in C_v\}.$$

This is a linear (over $F_q$, not $F_{q^n}$) sum-rank-metric code with the minimum sum-rank distance at least $\min\{w_0n, w_1(n-1), \ldots, w_v(n-v)\}$. It is easy to verify the linear independence, therefore the dimension is

$$\dim_{F_q}(SR(C_0, \ldots, C_v)) = n(k_0 + \cdots + k_v).$$

Theorem 2.1: Let $C_i \subseteq F^t_q$ be a linear $[t, k_i, w_i]_{q^n}$ code, for $i = 0, 1, \ldots, v$. Then $SR(C_0, \ldots, C_v)$ is a block length $t$ and matrix size $n \times n$ linear sum-rank-metric code over $F_q$ of the dimension $n(k_0 + \cdots + k_v)$. The minimum sum-rank distance of $SR(C_0, \ldots, C_v)$ is at least $\min\{w_0n, w_1(n-1), \ldots, w_v(n-v)\}$.

Proof: The dimension can be calculated directly. The lower bound on the minimum sum-rank distance is from the formation of $q$-polynomials in this code.

Remark 2.3: The main difference between the construction 2 and the construction 1 is as follows. When several codes over $F_{q^n}$ are used in Theorem 2.1 to construct a linear sum-rankmetric code $w_0, w_1, \ldots, w_v$, $w_0n, w_1(n-1) = \cdots = w,v(n-v)$, or these $v+1$ numbers $w_0n, w_1(n-1), \ldots, w_v(n-v)$ are close. Therefore the first dimensions of the first several codes $C_0, \ldots, C_i$ can be larger, since their minimum Hamming distances are smaller. This property makes the sum-rank-metric codes in the construction 2 larger.

C. Sum-Rank-Metric Codes From Hamming Metric BCH Codes

From Theorem 2.1, it is natural to use several Hamming metric BCH codes over $F_{q^n}$ of length $q^{un} - 1$ to construct sum-rank-metric codes of the block length $q^{un} - 1$. Based on some previous calculations of dimensions of BCH codes in [20] and [44], we get the lower bound on dimensions of these sum-rank-metric codes. It is interesting to notice that the parameters of these linear sum-rank-metric codes from Hamming metric BCH codes can be compared with the parameters of these sum-rank BCH codes developed by the deep algebraic method in [25]. As shown in some examples below, some linear sum-rank-metric codes from BCH codes applied in Theorem 2.1 have almost the same dimensions as the smaller codes in [25]. However the construction in Theorem 2.1 is direct and simple and can be applied to arbitrary Hamming metric codes, not only restricted to BCH codes.

Theorem 2.2: Let $q$ be a prime power and $u \geq 4$ be an even positive integer, $u_0, \ldots, u_v$, $v \leq n-1$, be positive integers satisfying $2 \leq u_i < q^{\frac{2u}{u-1}} + 1$, $i = 0, 1, \ldots, v$. Then we have a block size $q^{un} - 1$ and matrix size $n \times n$ linear sum-rank-metric code $C$. The dimension of this linear sum-rank-metric code is at least $\dim_{F_q}(C) = n(\sum_{i=0}^{v}(q^{un} - 1 - u(u_i - 1 - [\frac{u_i-1}{q^u}])))$, and the minimum sum-rank distance is at least $\min\{w_0n, w_1(n-1), \ldots, w_v(n-v)\}$.

Proof: We take primitive BCH codes over $F_{q^n}$ of length $q^{un} - 1$ and the designed distance $u_0, u_1, \ldots, u_v$. The conclusion in the case 1 of Theorem 17 in [20] asserts that there is a BCH code $C_{q^n-1, q^n-1}$ of the designed distance $u_i$ and the dimension $q^{un} - 1 - u(u_i - 1 - [\frac{u_i-1}{q^u}])$. We take $v \leq n-1$ such BCH codes over $F_{q^n}$ as codes $C_0, \ldots, C_v$ in Theorem 2.1. The conclusion follows immediately.

However to compare with the $F_{q^n}$-linear sum-rank BCH codes constructed in [25], we can use primitive BCH codes over $F_q$ of length $q^un - 1$ and designed distances $u_0, \ldots, u_v$. These codes can be considered as linear codes over $F_{q^n}$. Then we have the following the sum-rank-metric codes as follows from Theorem 17 in [20] about BCH codes over $F_{q^n}$.

Corollary 2.1: Let $q$ be a prime power and $u \geq 4$ be an even positive integer, $u_0, \ldots, u_v$, $v \leq n-1$ be positive integers satisfying $2 \leq u_i < q^2 + 1$, $i = 0, 1, \ldots, v$. Then we have a block length $q^un - 1$ and matrix size $n \times n$ linear sum-rank-metric code $C$ of the dimension $\dim_{F_q}(C) = n(\sum_{i=0}^{v}(q^{un} - 1 - u(u_i - 1 - [\frac{u_i-1}{q^u}])))$, the minimum sum-rank distance of this sum-rank-metric code is at least $\min\{w_0n, w_1(n-1), \ldots, w_v(n-v)\}$.

For $q = 2$, $n = 2$, $u = 6$, $u_0 = 2$, $u_1 = 4$, this is a sum-rank-metric code of the block size 63, with the dimension $\dim_{F_2}(C) = 2(63 - 6) + (63 - 6 - 2) = 2 \cdot 108$, and minimum sum-rank distance 4. Table VI, page 5166 of [25] there are two linear sum-rank-metric codes of the block size 63 with minimum sum-rank distance 4, the smaller code has the dimension lower bounded by 2 · 108 and the larger code has the dimension lower bounded by 2 · 112. Hence our code have the same parameters as the smaller one constructed in [25] by the deep algebraic method. For $q = 2$, $n = 2$, $u = 6$, $u_0 = 3$, $u_1 = 6$, this is a sum-rank-metric code of the block size 63, with the dimension $\dim_{F_2}(C) = 2((63 - 6) + (63 - 6 - 3)) = 2 \cdot 102$, and minimum sum-rank distance 6. For $q = 2$, $n = 2$, $u = 6$, $u_0 = 5$, $u_1 = 10$, this is a linear sum-rank-metric code of the block size 63, with the dimension $\dim_{F_2}(C) = 2((63 - 12) + (63 - 6 - 5)) = 2 \cdot 84$, and minimum
sum-rank distance 10. In Table VI, page 5166 of [25], these two codes have the same parameters as the smaller codes constructed in [25].

**Theorem 2.3:** Let $q$ be a prime power and $u \geq 5$ be an odd positive integer, $u_0, u_1, \ldots, u_v, v \leq n-1$, be positive integers satisfying $2 \leq u_i \leq q^{\frac{n-1}{2}}+1$, $i=0,1,\ldots,v$. Then we have a block length $q^{nu}-1$ and matrix size $n \times n$ linear sum-rank-metric code $C$ of the dimension $\text{dim}_{F_q}(C) = n(\sum_{i=0}^v(q^u - 1 - u(u_i - 1 - \left\lfloor \frac{u_i - 1}{q} \right\rfloor)))$. The minimum sum-rank distance of this sum-rank-metric code is at least $\min\{u_0n, u_1(n-1), \ldots, u_v(n-v)\}$.

**Proof:** We take primitive BCH codes over $\mathbb{F}_{q^n}$ of length $q^{nu}-1$ and designed distance $u_0, u_1, \ldots, u_v$. The conclusion in the case 1 of Theorem 18 in [20] asserts that there is a BCH code over $\mathbb{F}_{q^n}$ with the designed distance $u_i$, and the dimension $q^{nu}-1 - u(u_i - 1 - \left\lfloor \frac{u_i - 1}{q} \right\rfloor)$. Then the conclusion follows from the construction Theorem 2.1 from $v+1$ such BCH codes over $\mathbb{F}_{q^n}$.

Form the conclusion in the case 1 of Theorem 18 in [20] about BCH codes over $\mathbb{F}_q$, we get the following result.

**Corollary 2.2:** Let $q$ be a prime power and $u \geq 5$ be an odd positive integer, $u_0, u_1, \ldots, u_v$ be positive integers satisfying $2 \leq u_i \leq q^{\frac{n-1}{2}}+1$, $i=0,1,\ldots,v$. Then we have a block length $q^{nu}-1$ and matrix size $n \times n$ linear sum-rank-metric code $C$ of the dimension $\text{dim}_{F_q}(C) = n(\sum_{i=0}^v(q^u - 1 - u(u_i - 1 - \left\lfloor \frac{u_i - 1}{q} \right\rfloor)))$. The minimum sum-rank distance of this sum-rank-metric code is at least $\min\{u_0n, u_1(n-1), \ldots, u_v(n-v)\}$.

For $q=2$, $n=2$, $v=5$, the block length is 31, we can construct a sum-rank-metric code with the dimension $\text{dim}_{F_2}(C) = 2((31-5) + (2 (31-5) - 3)) = 2 \cdot 42$, and minimum sum-rank distance 6, a sum-rank-metric code with the dimension $\text{dim}_{F_2}(C) = 2((31-10) + (31 - 5 \cdot 4)) = 2 \cdot 32$, and minimum sum-rank distance 8, and a sum-rank-metric code with the dimension $\text{dim}_{F_2}(C) = 2((31-10) + (31 - 25)) = 2 \cdot 27$, and minimum sum-rank distance 10. In Table V, page 5165 of [25] for each pair of above dimensions and minimum sum-rank distances, there are two linear sum-rank-metric codes, the smaller one has the same dimension and the minimum sum-rank distance as our code.

From the result in [44], we have the following linear sum-rank-metric codes of the matrix size $n \times n$.

**Theorem 2.4:** Let $q$ be a prime power, $\lambda$ and $n$ be two positive integers satisfying $\lambda q^n - 1$, $u$ be an odd positive integer, $u_0, u_1, \ldots, u_v, v \leq n-1$ be $v+1$ positive integers satisfying $1 \leq u_i - 1 \leq 2^{\frac{n-1}{2x}}$, $i=0,1,\ldots,v$. Then we have a block length $q^{nu}-1$ and matrix size $n \times n$ linear sum-rank-metric code $C$. The dimension of this sum-rank-metric code is $\text{dim}_{F_q}(C) = n(\sum_{i=0}^v(q^u - 1 - u(u_i - 1 - \left\lfloor \frac{u_i - 1}{q} \right\rfloor)))$ and the minimum sum-rank distance of $C$ is at least $\min\{u_0n, u_1(n-1), \ldots, u_v(n-v)\}$.

**Proof:** Theorem 1 in page 4701 of [44] asserts that there is a BCH code over $\mathbb{F}_{q^n}$ with designed distance $u_i$ and dimension $q^{nu}-1 - u(u_i - 1 - \left\lfloor \frac{u_i - 1}{q} \right\rfloor)$. Then applying Theorem 2.1 to $v+1$ such BCH codes we get the conclusion.

For $q=2$, $n=2$, $\lambda=3$, a block length 341 binary linear sum-rank-metric code of the dimension 607 can be constructed from Theorem 3.3. The minimum sum-rank distance is at least 14. There is no such sum-rank-metric code of the block length 341 constructed in [25]. Notice that all previous results about BCH codes in the references of [20] and [44] of various lengths can be applied in our construction Theorem 2.1 to get linear sum-rank-metric codes. It is obvious that our construction Theorem 2.1 is simpler than the deep algebraic method in [25].

**Remark 2.4:** BCH codes are not the presently known best linear codes for many parameters when compared to table [14]. The above linear sum-rank-metric codes constructed from BCH codes can be improved significantly by applying codes [14] in our construction 2. These linear sum-rank-metric codes from codes [14] applied in the construction 2 will be give in Section III and the Appendix. We can see that many of them are larger than the linear sum-rank-metric codes constructed in [25].

**D. Sum-Rank-Metric Codes From Algebraic Geometry Codes**

Since we can use arbitrary error-correcting codes $(C_0, C_1, \ldots, C_n)$ in the Hamming metric in our construction 2, it is natural to use the Reed-Solomon code and its generalization algebraic geometry codes to construct good linear sum-rank-metric codes. The advantage of this construction is the flexibility of block lengths of sum-rank-metric codes.

Let $F_q$ be an arbitrary finite field, $P_1, \ldots, P_n$ be $n \leq q$ elements in $F_q$. The Reed-Solomon code $RS(n,k)$ is defined by

$$RS(n,k) = \{(f(P_1), \ldots, f(P_n)) : f \in F_q[x], \deg(f) \leq k-1\}.$$ 

This is an MDS $[n,k,n-k+1]_q$ linear code attaining the Singleton bound $d_H \leq n-k+1$, since a degree $\deg(f) \leq k-1$ nonzero polynomial has at most $k-1$ roots.

**Theorem 2.5:** Let $q$ be a prime power, $1 \leq t \leq q^2$ and $1 \leq k < t$ be two positive integers satisfying $t-k$ is odd. Then we have a block length $t$ and matrix size $2 \times 2$ linear sum-rank-metric code $C$. The dimension of $C$ is $\text{dim}_{F_q}(C) = t + 3k + 1$ and the minimum sum-rank distance of this code $C$ is at least $t-k+1$.

**Proof:** Applying Theorem 2.1 to the Reed-Solomon $[t,k,t-k+1]_q$ code and the Reed-Solomon $[t,t+k+1,k+1]_q$ code, we get the linear sum-rank-metric code.

Notice that linearized Reed-Solomon codes (MSRD) in [21] can be constructed only if $t \leq q-1$. In Theorem 4.1, $t \leq q^2$, the defect is $2(t-t-k+1)-1 = t-k-1$.

Reed-Solomon codes can be generalized to algebraic geometry codes, we refer to [18, Chapter 13]. Let $X$ be an absolutely irreducible non-singular genus $g$ curve defined over $F_q$ with $n$ rational points. Then a linear algebraic geometry $[n,m-g+1,\geq n-m]_q$ code can be constructed, where $m$ is the degree of the rational divisor satisfying $2g-2 < m < n$.

Reed-Solomon codes are just algebraic geometry codes over the genus 0 curve. One achievement of the theory of algebraic geometry codes is the sequence of algebraic-geometric codes over $F_{q^2}$ satisfying the Tsfasman-Vlăduţ-Zink bound

$$R + \delta \geq 1 - \frac{1}{q-1}.$$
which is exceeding the Gilbert-Varshamov bound when \( q \geq 7 \). We refer to [41] for the detail.

We restrict to the case \( n_1 = \cdots = n_t = m_1 = \cdots = n \), then length \( t \) algebraic-geometric codes over \( \mathbb{F}_{q^n} \) are used to construct sum-rank-metric codes as follows.

**Theorem 2.6:** Let \( X \) be an absolutely irreducible non-singular genus \( g \) curve defined over \( \mathbb{F}_{q^n} \) with \( N + 1 \) rational points (over \( \mathbb{F}_{q^n} \)). Let \( u_0, \ldots, u_v \) be \( v \) positive integers satisfying \( 2q - 2 < u_i < N, i = 0, \ldots, v \). Then we have a block length \( N \) and matrix size \( n \times n \) linear rank-sum-rank-metric code \( C \) over \( \mathbb{F}_q \). The dimension of this code \( \dim_{\mathbb{F}_q}(C) = n \left( \sum_{i=0}^v (u_i - g + 1) \right) \) and the minimum sum-rank-distance of \( C \) is at least \( \min \{ (N - u_0)n, (N - u_1)(n - 1), \ldots, (N - u_v)(n - v) \} \).

From the curves over \( \mathbb{F}_4 \) in [12] some sum-rank-metric codes over \( \mathbb{F}_2 \) of the matrix size \( 2 \times 2 \) can be constructed as follows. For example we take a genus 50 curve over \( \mathbb{F}_4 \) with 91 rational points, then a block length 90 and matrix size \( 2 \times 2 \) linear sum-rank-metric code of the dimension 140 is constructed. The minimum sum-rank-distance is at least 8. From algebraic geometry codes over a genus 12 curve over \( \mathbb{F}_8 \) with 49 rational points, see [12], a block length 48 and matrix size \( 2 \times 2 \) linear sum-rank-metric code of the dimension 89 is constructed. The minimum sum-rank-distance is at least 12.

### III. Linear Sum-Rank-Metric Codes From the Presently Known Best Codes

Thanks to the table of the presently known best linear codes in [14], numerous good small block length linear sum-rank-metric codes over \( \mathbb{F}_q \) of the matrix size \( 2 \times 2 \), for \( q = 2, 3, 4 \), can be constructed explicitly from Theorem 2.1. We give tables to list some linear sum-rank-metric codes constructed from Theorem 2.1 and the presently known best Hamming metric codes of [14] in this section and the Appendix. Many of constructed sum-rank-metric codes have larger dimensions when compared with codes constructed in [25]. It is clear that many of our constructed codes from Theorem 2.1 are close to the Singleton-like bound. For example when the block length is 7 we have the following table of linear sum-rank-metric codes.

The following table lists the block length 31, the matrix size \( 2 \times 2 \), binary linear sum-rank-metric codes. Comparing with Table V in [25] our codes are larger. Many of our codes from Theorem 2.1 are close to the Singleton-like bound.

More good linear sum-rank-metric codes from the codes in [14] applied in the construction 2 are given in the Appendix.

**Remark 3.1:** However it should be indicated that the true dimensions and the true minimum sum-rank distances were not calculated explicitly in [25]. The entries from [25] in Table I, II and the tables in the Appendix are lower bounds of dimensions and distances.

### IV. Asymptotically Good Sum-Rank-Metric Codes Close to the GV-Like Bound

In this section let \( q \) be a fixed prime power and the square matrix size parameter \( n \) will be a sufficiently large even positive integer. Asymptotically good sequences of \( \mathbb{F}_q \)-linear sum-rank-metric codes over \( \mathbb{F}_q \) close to the Gilbert-Varshamov-like bound in [34] are presented. In Theorem 2.1, let \( v \leq n - 1 \) be a fixed positive integer, we need a linear \( [n(n-1)\cdots(n-v)w_j, k_i, n(n-1)\cdots(n-v)w_j]_{q^v} \) code for \( i = 0, 1, \ldots, v \), where \( w_j \) be a sequence of positive integers going to the infinity. From Theorem 2.6 about sum-rank-metric codes from algebraic geometry codes over \( \mathbb{F}_{q^n} \), we take a family of algebraic curves \( \{C_i\}_{j=1,2,\ldots} \) over \( \mathbb{F}_{q^n} \), with the genus \( \{g_j\}_{j=1,2,\ldots} \) and \( \{N(C_i)\}_{i=1,2,\ldots} \) rational points satisfying \( N(C_i) - 1 \geq n(n-1) \cdot (n-v)w_j \) rational points, see [41]. Then we have

\[
\frac{k_i}{n(n-1)\cdots(n-v)w_j} \geq 1 - \frac{1}{n-I - q^v - 1},
\]

from the Tsfasman-Vlădut-Zink bound. Therefore the minimum sum-rank-distance is at least \( n(n-1)\cdots(n-v)w_j \). The relative minimum sum-rank-distance is at least

\[
\delta_{sr} \geq \frac{1}{n}.
\]

The dimension over \( \mathbb{F}_q \) is \( n \sum_{i=0}^{v} k_i \) and the code rate of this sum-rank-metric code is at least

\[
R_{sr} \geq \frac{1}{n} \left( v + 1 - \sum_{i=0}^{v} \frac{1}{n - i} \cdot \frac{v + 1 - q^v - 1}{q^v - 1} \right).
\]
Hence
\[ R_{sr} + 2\delta_{sr} - \delta_{sr}^2 \geq \frac{1}{n} \left( v + 3 - \sum_{i=0}^{v} \frac{1}{n-i} - \frac{1}{q^{n/2}-1} - \frac{1}{n} \right). \]

When \( v = n - 2 \), \( R \) and \( \delta \) satisfy the following
\[ R_{sr} + 2\delta_{sr} \geq 1 + \frac{1}{n} - \frac{1}{n} \left( \frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{2} \right) - \frac{n-1}{q^{n/2}-1}. \]

**Theorem 4.1:** For any fixed square prime power \( q \) and positive integers \( n \) and \( v \) satisfying \( v \leq n-1 \), we have a sequence of \( F_q \)-linear sum-rank-metric codes with positive code rate \( R_{sr} \) and positive relative minimum sum-rank distance \( \delta_{sr} \), satisfying
\[ R_{sr} + 2\delta_{sr} - \delta_{sr}^2 \geq \frac{1}{n} \left( v + 3 - \sum_{i=0}^{v} \frac{1}{n-i} - \frac{1}{q^{n/2}-1} - \frac{1}{n} \right). \]

From the asymptotical Gilbert-Varshamov-like bound on sum-rank-metric codes in [34], or see Section I, we have the following result.

**Corollary 4.1:** When \( q \) is fixed and \( n \) is sufficiently large, sequences of asymptotically good \( F_q \)-linear sum-rank-metric codes close to the Gilbert-Varshamov-like bound can be constructed.

**Proof:** We consider the harmonic series
\[ \lim_{n \to \infty} \left( 1 + \frac{1}{2} + \cdots + \frac{1}{n} \right) = \gamma, \]
where \( \gamma = 0.577215 \ldots \) is the Euler-Mascheroni constant. Then we have
\[ R_{sr} + 2\delta_{sr} - \delta_{sr}^2 \geq 1 - \frac{1}{q^{n/2}-1} - \frac{\ln n}{n}, \]
from Theorem 4.1. It is clear that \( \lim_{n \to \infty} \frac{\ln n}{n} = 0 \) and the above sequence of linear sum-rank-metric codes has their \( R_{sr} \) and \( \delta_{sr} \) close to the GV-like bound
\[ R_{sr} \sim \delta_{sr}^2 - 2\delta_{sr} + 1, \]
when \( n \) is sufficiently large.

**Remark 4.1:** The only non-explicitness of these codes in Theorem 4.1 is from the fact that the algebraic geometry code sequence achieving the Tsfasman-Vlăduţ-Zink bound has not been constructed explicitly. However these codes in Corollary 4.1 can be constructed by a low-complexity polynomial time algorithm, see [40]. It seems that both construction 1 and 2 are not sufficient to construct a sequence of sum-rank-metric codes achieving or exceeding the GV-like bound.

V. LINEAR MSRD CODES

A. Block Size Two MSRD Codes

In this section we first give linear MSRD codes of block size \( t = 2 \) for the convenience of understanding. First of all the matrix space \( M_{n \times n}(F_q) \) is identified with all \( q \)-polynomials \( a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} \), where \( a_0, a_1, \ldots, a_{n-1} \in F_q \).

**Theorem 5.1:** Let \( n_1 \) and \( n_2 \) be two positive integers satisfying \( n_1 \geq n_2^2 \). Then a linear MSRD code \( C \) with the block size 2, matrix sizes \( n_1 \times n_1, n_2 \times n_2 \) over an arbitrary field \( F_q \) and any given minimum sum-rank distance can be constructed explicitly.

**Proof:** We discuss two cases. The first case is \( d_{sr} \leq n_1 - 1 \). Then the Singleton-like bound is \( q^{n_1(n_1-d_{sr}+1)+n_2^2} \). The first part of \( C \) is consisting of \( q^{n_1(n_1-d_{sr}+1)} \)-codewords of the form \( (a_0 x + \cdots + a_{n_1-d_{sr}} x^{n_1-d_{sr}}) , 0) \), where \( a_0, a_1, \ldots, a_{n_1-d_{sr}} \) are \( n - d_{sr} + 1 \) arbitrary elements on the field \( F_q^{n_1} \), and the \( q \)-polynomial is understood as a matrix in \( F_q^{n_1(n_1)} \). The second matrix is the all-zero matrix. It is clear that the first part is a linear code with the minimum sum-rank distance at least \( n_1 - (n_1 - d_{sr}) = d_{sr} \).

The second part corresponds to \( q^{n_2^2} \) codewords. First of all we decompose the \( F_q \)-linear space \( F_q^{n_2^2} \cdot x^{n_2^2-d_{sr}+1} \) as the direct sum of \( n_2 \) linear subspaces \( V_1, \ldots, V_{n_2} \), of the dimension \( \dim F_q(V_i) = n_2, i = 1, \ldots, n_2 \). This is guaranteed from the condition \( n_1 \geq n_2^2 \). Then each of these subparts satisfies the second part is a dimension \( n_2 \) code consisting of all codewords of the form \( (a_{n-d_{sr}+1} + x^{d_{sr}+1}) b_1 x^{d_{sr}}, \ldots, b_{n_2} x^{d_{sr}}, 0) \), where \( a_{n-d_{sr}+1} \in V_i, b_i \in F_q^{n_2} \), and moreover they are the same after a suitable \( F_q \) linear space isomorphism of \( V_i \) to \( F_q^{n_2} \). Therefore the dimension of each subpart is \( n_2 \). The second part is the direct sum of all these \( n_2 \) subparts and it is obvious the dimension is \( n_2^2 \). The minimum sum-rank distance is at least \( n_1 - (n_1 - d_{sr}) + 1 = d_{sr} \), since \( a_0 - d_{sr} + 1 \) and \( b_i \) are nonzero for a nonzero codeword.

When \( d_{sr} = n_1 + d, 0 \leq d \leq n_2 - 1 \), the Singleton-like bound is \( q^{n_2(n_2-d+1)} \). The linear code \( C \) is the direct sum of \( n_2 - d + 1 \) subcodes of the dimension \( n_2 \). As in the first case, we decompose the \( F_q \)-linear space \( F_q^{n_2^2} \cdot x^{d_{sr}} \) as the direct sum of \( n_2 \) linear subspaces \( V_1, \ldots, V_{n_2} \), of the dimension \( \dim F_q(V_i) = n_2, i = 1, \ldots, n_2 \). This is guaranteed from the condition \( n_1 \geq n_2^2 \). Then each of these \( n_2 - d + 1 \) linear subcodes is consisting of codewords of the form \( (a_{n-d+1} x^{d+1}) b_1 x^{d_{sr}}, \ldots, b_{n_2} x^{d_{sr}}, 0) \), where \( a_0 \in V_i, b_1 \in F_q^{n_2} \) for \( j = 1, \ldots, n_2 - d + 1 \), and \( a_0 \) and \( b_j \) are the same with a suitable linear isomorphism of \( V_j \) with \( F_q^{n_2} \). This is a dimension \( n_2 \) linear subspace. It is easy to verify that the minimum sum-rank distance is at least \( n_1 + n_2 - (n_2 - d) = n_1 + d = d_{sr} \).

It is easy to verify that all these linear subcodes are linear independent. Hence the conclusion is proved.

B. MSRD Code Construction

In this section we give our explicit construction of linear MSRD codes of various square matrix sizes. This is a generalization of the construction of the block length two case in the previous subsection. From the following it is clear that the constructed code attains the Singleton-like bound.

**Theorem 5.2:** Let \( n_1 > n_2 > \cdots > n_t \) be \( t \) positive integers. Let \( d_{sr} = \sum_{j=1}^{t} n_j + d \) where \( j \in \{1, \ldots, t\} \) and \( 0 \leq d \leq n_t - 1 \) be the unique representation of the minimum sum-rank distance. Suppose that \( n_1, \ldots, n_t \) and \( d_{sr} \) satisfy
1) \( n_j - 1 \geq n_j (n_j - d + 1) + n_{j+1}^2 + \cdots + n_{t}^2; \)
2) \( n_j \geq n_{j+1}^2 + \cdots + n_{t}^2; \)

Then a linear MSRD code with \( q^{n_j(n_j-d+1)+\sum_{j=1}^{t} n_j^2} \) codewords over an arbitrary field \( F_q \) can be constructed explicitly. The minimum sum-rank distance of this MSRD code is \( d_{sr} \).
Proof: In each block position we have linearly independent $F_q$-linear mappings $x^0, x^2, \ldots, x^{q^n-1}$ over $F_q$, for $i = 1, \ldots, t$. In our construction, many copies of repetition codes over $F_q$, $i = j, j+1, \ldots, t$, are used. When each such repetition code is used, new $x^q$'s in some block position are introduced, so the linearly independence is guaranteed. It is important that coefficients of some $x^q$'s at different block positions are not zero for a nonzero codeword.

For each $x^q$ at the $i$-th block position, the set of all coefficients is the field $F_q$. This is a $F_q$-linear space of the dimension $n_q$, therefore the linear space of all coefficients of $x^q$ at the $i$-th position, can be decomposed to the direct sum of $(n_j - d + 1) + n_{j+1} + \cdots + n_t$. $F_q$-linear subspaces $V_{ij}$, $i = 1, 2, \ldots, j-1$ and $w = 1, 2, \ldots, n_j - d + 1 + n_{j+1} + \cdots + n_t$, of dimensions $n_j, n_{j+1}, \ldots, n_t$, are not zero for a nonzero codeword.

If $i \leq j - 1$, this is guaranteed from the condition $n_j - 1 \leq n_j(n_j - d + 1) + n_{j+1} + \cdots + n_t$.

The dimension in the Singleton-like bound for sum-rank-metric codes is $n_j(n_j - d + 1) + n_{j+1} + \cdots + n_t$. The first term $n_j(n_j - d + 1)$ comes from $x^q, x^2, \ldots, x^{q^n-1}$ at the $j$-th block position. We use $n_j - d + 1$ copies of length $j$ repetition code as in the proof of Theorem 2.1. Then for a nonzero codeword $c = (c_1, \ldots, c_j)$ in this repetition code, $c_1, \ldots, c_j$ are not zero. The dimension $n_j$ linear subspace of $F_q$ of the coefficient of $x^q$ at the $1$st block position is used for $c_1$ with a suitable base of $F_q$, $i = 1$, therefore the $n_j$ linear subspace of $F_q$ of the coefficient of $x^q$ at the $(j-1)$-th block position is used for $c_{j-1}$ with a suitable base of $F_q$, $i = 1$, $j = 1, 2, \ldots, n_j - d + 1 + n_{j+1} + \cdots + n_t$, the coefficient of $x^q, x^2, \ldots, x^{q^n-1}$ at the $j$-th block position is used for $c_j$, $j = 1, 2, \ldots, n_j - 1$, and the sum-rank at $j$-th block positions, zero $q$-polynomials are used. The $q$-polynomials of these $n_j$ copies of codewords are as follows, $M_i = \{ (c_1)^i x^0, \ldots, (c_{j-1})^i x^0, (c_j)^i x^{q^n-j} + 0, \ldots, 0, (c_{j+1})^i x^q, \ldots, 0 \}$, where $i = 0, 1, \ldots, n_j$, $c_1, \ldots, c_{j-1}$ are suitable images of $c_1, \ldots, c_j$ in the $F_q$ space of the coefficients, $(c_j)^i$ is the suitable image in $F_q$ space of coefficients. Then we have $q^{n_j(n_j - d + 1)}$ codewords in the constructed sum-rank-metric code from these $n_j$ copies of the repetition code. The minimum sum-rank distance is at least $n_1 + \cdots + n_{j-1} + (n_j - n_j - d + 1) + 1 = n_1 + \cdots + n_j - 1 + d = d_{sr}$, since $c_1, \ldots, c_{j-1}$ are not zero for a nonzero codeword in the repetition code.

The above constructed sum-rank-metric code is $F_q$-linear. For two different codewords $x_1$ and $x_2$ in the above code, the difference $x_1 - x_2$ has the rank at least $n_i$ at the $i$-th block position, for $i = 1, \ldots, t - 1$, and the sum-rank at least $d$, at $j, j+1, \ldots, t$-th block positions. Then the minimum sum-rank distance is at least $n_1 + \cdots + n_{j-1} + d = d_{sr}$. The conclusion is proved.

Corollary 5.1: Let $n_1, n_2, \ldots, n_t$ be $t$ positive integers satisfying $n_i \geq n_i^{2} + \cdots + n_t^2$ for $i = 1, 2, \ldots, t - 1$. Then a linear MSRD code over an arbitrary finite field $F_q$ with matrix sizes $n_1 \times n_1, \ldots, n_t \times n_t$, and any given minimum sum-rank distance can be constructed explicitly.

Remark 5.1: As in the two constructions of sum-rank-metric codes in Section II, the construction of linear MSRD codes in this section is a combination of the Hamming metric codes and the $q$-polynomial representation of rank-metric codes. The main point of the construction is as follows. From the condition $n_j \geq n_j^{2} + \cdots + n_t^2$, $j = 1, \ldots, t - 1$, there are sufficiently many $q$-polynomials with suitable degrees to construct a code attaining the Singleton-like bound.

VI. CONCLUSION

In this paper three simple constructions of linear sum-rank-metric codes from the combination of Hamming metric codes and $q$-polynomial representations of rank-metric codes are proposed. Numerous good linear sum-rank-metric codes over $F_q$, $q = 2, 3, 4$, have been given. Many of these linear sum-rank-metric codes from the presently known best Hamming
metric codes have larger dimensions when compared with previous codes of the same minimum sum-rank distances. Asymptotically good sum-rank-metric code sequences close to the Gilbert-Varshamov-like bound are also presented. These asymptotically good sequences of sum-eank-metric codes can be constructed by a polynomial-time algorithm. Explicit linear MSRD codes with square matrix sizes \( n_1 \times n_1, \ldots, n_t \times n_t \) satisfying \( n_i \geq n_{i+1}^2 + \cdots + n_t^2 \) for \( i = 1, 2, \ldots, t-1 \) over an arbitrary finite field, are constructed for all possible minimum sum-rank distances. We show that the decoding of binary linear sum-rank-metric codes constructed in this paper can be reduced to the fast decodings in the Hamming metric in our paper [8].

### Table IV

| \( d_{sr} \) | Dimension | Singleton |
|-------------|-----------|-----------|
| 4           | 2.29      | 2.31      |
| 5           | 2.25      | 2.30      |
| 6           | 2.24      | 2.29      |
| 7           | 2.24      | 2.28      |
| 8           | 2.21      | 2.27      |
| 9           | 2.20      | 2.26      |
| 10          | 2.16      | 2.25      |
| 11          | 2.14      | 2.24      |
| 12          | 2.13      | 2.23      |
| 13          | 2.11      | 2.22      |
| 14          | 2.10      | 2.21      |
| 15          | 2.9       | 2.20      |
| 16          | 2.8       | 2.19      |
| 17          | 2.7       | 2.18      |

### Table V

| \( d_{sr} \) | Dimension | Table VII, [25] | Singleton |
|-------------|-----------|-----------------|-----------|
| 4           | 2.119     | 2.112           | 2.123     |
| 5           | 2.114     | 2.108           | 2.122     |
| 6           | 2.112     | 2.106           | 2.121     |
| 7           | 2.107     | 2.100           | 2.120     |
| 8           | 2.106     | none            | 2.119     |
| 9           | 2.101     | none            | 2.118     |
| 10          | 2.091     | 2.088           | 2.117     |
| 11          | 2.096     | none            | 2.116     |
| 12          | 2.095     | none            | 2.115     |
| 13          | 2.088     | none            | 2.114     |
| 14          | 2.087     | 2.070           | 2.113     |
| 15          | 2.084     | none            | 2.112     |
| 16          | 2.082     | none            | 2.111     |
| 17          | 2.077     | none            | 2.110     |
| 18          | 2.077     | none            | 2.109     |
| 19          | 2.072     | none            | 2.108     |
| 20          | 2.071     | none            | 2.107     |
| 21          | 2.070     | none            | 2.106     |
| 22          | 2.069     | 2.052           | 2.105     |
| 23          | 2.067     | none            | 2.104     |
| 24          | 2.066     | none            | 2.103     |
| 25          | 2.058     | none            | 2.102     |
| 26          | 2.058     | none            | 2.101     |
| 27          | 2.056     | none            | 2.100     |
| 28          | 2.055     | none            | 2.099     |
| 29          | 2.051     | none            | 2.098     |
| 30          | 2.050     | 2.028           | 2.097     |
| 31          | 2.047     | none            | 2.096     |
| 32          | 2.046     | none            | 2.095     |
| 33          | 2.037     | 2.016           | 2.089     |
| 34          | 2.029     | 2.018           | 2.081     |
| 35          | 2.020     | 2.018           | 2.073     |

### Table VI

| \( d_{sr} \) | Dimension | Table VII, [25] | Singleton |
|-------------|-----------|-----------------|-----------|
| 4           | 2.246     | 2.238           | 2.251     |
| 5           | 2.240     | 2.233           | 2.250     |
| 6           | 2.236     | 2.231           | 2.249     |
| 7           | 2.231     | 2.224           | 2.248     |
| 10          | 2.221     | 2.210           | 2.245     |
| 14          | 2.202     | 2.189           | 2.241     |
| 22          | 2.172     | 2.154           | 2.233     |
| 30          | 2.146     | 2.112           | 2.225     |
| 38          | 2.121     | 2.086           | 2.217     |
| 46          | 2.106     | 2.070           | 2.209     |
| 54          | 2.091     | 2.042           | 2.201     |
| 62          | 2.077     | 2.028           | 2.193     |

### Table VII

| \( d_{sr} \) | Dimension | Singleton |
|-------------|-----------|-----------|
| 4           | 2.057     | 2.059     |
| 5           | 2.051     | 2.058     |
| 6           | 2.052     | 2.057     |
| 7           | 2.049     | 2.056     |
| 8           | 2.047     | 2.055     |
| 9           | 2.044     | 2.054     |
| 10          | 2.042     | 2.053     |
| 11          | 2.040     | 2.052     |
| 12          | 2.039     | 2.051     |
| 13          | 2.036     | 2.050     |
| 14          | 2.035     | 2.049     |
| 15          | 2.032     | 2.048     |
| 16          | 2.031     | 2.047     |
| 17          | 2.029     | 2.046     |
| 18          | 2.028     | 2.045     |
| 19          | 2.025     | 2.044     |
| 20          | 2.024     | 2.043     |
| 21          | 2.022     | 2.042     |
| 22          | 2.021     | 2.041     |
| 23          | 2.020     | 2.040     |
| 24          | 2.019     | 2.039     |
| 25          | 2.018     | 2.038     |
| 26          | 2.017     | 2.037     |
| 27          | 2.015     | 2.036     |
| 28          | 2.014     | 2.035     |
| 29          | 2.013     | 2.034     |
| 30          | 2.013     | 2.033     |

### Appendix

We list more small block size linear sum-rank-metric codes constructed from Theorem 2.1 and the presently known best Hamming metric codes of [14]. These codes are compared with the codes constructed in [25] and the Singleton-like bound. It is clear that most of our codes are larger than these previously constructed codes of the same sum-rank distances. Some of our codes are close to the Singleton-like bound.

Notice that from Theorem 2.1 and the presently known best linear codes over \( \mathbb{F}_4 \) in [14], our constructed linear sum-rank-metric codes have arbitrary block lengths \( t \leq 256 \). Therefore much more linear sum-rank-metric codes with relative good parameters can be obtained. Comparing with sum-rank-metric codes constructed in [25], our codes have more flexibilities of their parameters. For example the following table lists linear sum-rank-metric codes over \( \mathbb{F}_2 \) of the block size 17 and the matrix size \( 2 \times 2 \).

In the following table, the block length \( t = 63 \) and the matrix size \( 2 \times 2 \) binary linear sum-rank-metric codes are listed...
and compared with Table VI of [25]. It is obvious that our codes are larger and closer to the Singleton-like bound.

In the following table, the block length \( t = 127 \) and the matrix size \( 2 \times 2 \) linear binary sum-rank-metric codes are listed and compared with Table VII of [25]. It is obvious that our codes are larger and closer to the Singleton-like bound.

Numerous good small block length linear sum-rank-metric codes over \( \mathbb{F}_3 \) of the matrix size \( n = m = 2 \) can be constructed from the presently known best small linear codes over \( \mathbb{F}_9 \) in [14]. In the following tables, we give some such linear rank-sum-metric codes of the block length \( t = 31 \). No previous code can be compared. Many of our codes are close to the Singleton-like bound.

Similarly many good small block length linear sum-rank-metric codes over \( \mathbb{F}_4 \) of the matrix size \( n = m = 2 \) can be constructed from small length linear codes over \( \mathbb{F}_8 \), considered as linear codes over \( \mathbb{F}_16 \). In the following tables, we give some block length 21 linear rank-sum-metric codes over \( \mathbb{F}_4 \) of the matrix size \( 2 \times 2 \). Many of our codes are close to the Singleton-like bound.

There have been few known linear binary sum-rank-metric codes of the matrix size \( 3 \times 3 \) in the literature. We give some such codes from Theorem 2.1 and the presently known best linear codes over \( \mathbb{F}_8 \) in [14]. Many of our constructed codes are close to the Singleton-like bound.

More binary linear sum-rank-metric codes from the presently known best codes in [14] and Construction 2 are listed in [19]. More sum-rank-metric codes from quaternary BCH codes, quaternary Goppa codes constructed in [8] are also listed in [19] for the convenience of readers. New explicit sum-rank-metric codes are welcomed to be included in the webpage [19].

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