Spacetime in the Ultimate Theory

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If we assume that there is the ultimate theory at all, how should the concept of the spacetime be formulated there? The following essay is my consideration on such a question. The use of mathematical expressions is suppressed as long as possible. Any criticism on my opinion is welcome.

§1. Motivation

The universe consists of its container, spacetime, and its contents, substance. The substance which we experience daily is very much different from its intrinsic constituents, elementary particles. On the contrary, the spacetime at the microscopic level is not significantly different from the spacetime which we experience daily. Of course, the relativity theory clarified that space and time must be treated not separately but in a unified way as a 4-dimensional manifold, but it did not suggest any essentially new aspect in the microscopic structure of the spacetime.

Historically, in Heisenberg’s quantum mechanics, the coordinates of the configuration space was formulated not as real numbers (c-numbers) but as operators (q-numbers). However, in Schrödinger’s wave mechanics, describing the same contents, the spatial coordinates are formulated as c-numbers by representing their conjugate momenta by differential operators. The concept of the c-number spacetime was carried over to quantum field theory by promoting the wave functions to field operators. But quantum field theory is most naturally formulated in the Heisenberg picture but not in the Schrödinger picture. The most fundamental quantities in quantum field theory are quantum fields, which are the operator-valued (generalized) functions of the c-number spacetime.

The present standard theory of elementary particles is completely describable by quantum field theory in the 4-dimensional spacetime. Within this framework, there is no necessity for requiring any change for the concept of the c-number 4-dimensional spacetime. Never-

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theless, most of particle physicists believe that this notion of the “c-number 4-dimensional spacetime” should be extended in some sense. The main reasons for this belief may be as follows.

[A] Because the standard theory is renormalizable, the physical S-matrix can be written down so as to be free of divergences. The theory, however, involves many adjustable parameters and the distribution of their actual magnitudes cannot be explained by the theory at all. Hence, one hopes to resolve this difficulty by extending the concept of the spacetime: Those parameters might become calculable by making the unrenormalized quantities finite, or at least, one could naturally understand the hierarchy of the actual values of those parameters.

[B] The action for quantum field theory is constructed so as to satisfy the requirement of various symmetries. Thus the requirement of symmetries is the principle governing the construction of the theory; therefore, the theory having larger symmetries would be regarded as more favorable as the ultimate theory. Since the raison d’être of internal symmetries is not clear, it is more preferable that there are some intrinsic connection with the spacetime symmetry. Unfortunately, in the ordinary 4-dimensional spacetime, admissible spacetime symmetries are quite limited; hence one wishes to find the large symmetry unifying both 4-dimensional spacetime symmetry and internal symmetries by extending the concept of the spacetime itself.

[C] Owing to the coherence of theories, the gravitational field, which was the spacetime metric in classical gravity, must be quantized. Hence, in quantum gravity, the spacetime metric becomes a q-number. Of course, this fact is different from the proposition that the spacetime itself becomes non-commutative, but it means that the concept of “two events separated by less than the Planck length” cannot be well-defined at least in the geometrical sense. Therefore, the ordinary structure of the conventional 4-dimensional spacetime is not naively extendable to the Planck-scale world.

The above three problems may be mutually related, but it will be too fantastic to believe that they can simultaneously be resolved to achieve the ultimate theory, just as the superstring people dream. It is more reasonable to attack each problem separately by focusing on particular points. In what follows, I will not discuss the superstring theory, as I criticized it elsewhere.

§2. Discrete spacetime

As is well known, the ultraviolet-divergence difficulty arises from the behavior of the theory in infinitesimally small spacetime distances. Therefore, it is natural to suppose that if the spacetime is discrete, then the theory will be utterly free of ultraviolet divergences. Various kinds of discrete-spacetime models have been proposed so far. But none of them has brought any promising achievement.

The essential difficulty of the discrete spacetime is how to formulate the structure of
the spacetime. Without using continuity, it is impossible even to define the dimensionality of the spacetime as an invariant under the point transformations. It is almost inevitable to introduce a background continuum in order to formulate the theory by which quantitative calculations are possible. Once, however, one introduces a background continuum, it immediately becomes the fundamental spacetime, because everything quantitative must be defined in reference to it.

Thus, although the idea of the discrete spacetime is quite attractive, it is always destined to falling into a theory of the continuous spacetime.

§3. Higher-dimensional spacetime

It is the cheapest idea to extend the spacetime to the one having a dimensionality higher than 4. Nevertheless, recently, it has become quite fashionable to investigate the theory of extra dimensions. The grounds for considering it seem to be the excuse that any possibility is worth investigating as far as it is not completely denied by experimental evidences and the expectation that it might be possible to resolve the hierarchy problem stated in [A] by introducing new parameters in the extra-dimension space. It is quite disgusting to revive the old Kaluza-Klein theory without any essentially new idea for resolving the fundamental problem of how to expel the extra dimensions from the physical world. Furthermore, the higher-dimensional theory makes ultraviolet-divergence difficulty out of control; the extra-dimension people merely hope, without showing its ground, that the divergence problem might be resolved by a non-perturbative treatment.

The dimensionality of the spacetime where we live is undoubtedly 4. There is no other observation more manifest than this fact. The extra dimensions, whose number is denoted by \( N \), are qualitatively different from the physical spacetime dimensions from the outset. Recently, the superstring people have proposed the hypothesis that all fields other than the gravity are confined to a soliton-like (according to their claim) object called the “D-brane”, but it would be quite difficult to formulate such an idea within the framework of the \((4+N)\)-dimensional quantum field theory. A theory is worth being called a \((4+N)\)-dimensional one if and only if its fundamental action is \((4+N)\)-dimensionally symmetric. But such a theory remains \((4+N)\)-dimensionally symmetric unless one breaks it artificially. In order to make the \(N\)-dimensional extra space invisible, the extra-dimension people are forced to make it round into a tiny one by hand. This procedure implies that they are actually considering the 4-dimensional spacetime accompanied with an \(N\)-dimensional internal space. If so, they should honestly claim that the extra dimensions are internal. Then there is no logical basis for adopting a \((4+N)\)-dimensionally symmetric action. It is quite non-scientific to pretend as if such an action were a privileged one according to the symmetry principle.

There may be the objection that the extra dimensions are made round not by hand but “spontaneously”. Certainly, there is no “proof” of the no-go theorem stating that the desired spontaneous compactification can never take place. But if they assert the possibility of such compactification, the responsibility of verifying it must be attributed to those who assert it. That is, the extra-dimension people should construct at least one model in which the
spontaneous compactification of the extra dimensions takes place in a natural way. I cannot believe that such a model can be constructed without greatly changing the framework of the conventional quantum field theory, because the situation encountered here is qualitatively different from the ordinary spontaneous breakdown of symmetry. Since the dimensionality is discrete, it is quite unreasonable to characterize the spontaneous breakdown by such a continuous parameter as energy. But the most difficult problem to be shown is not how to compactify the extra dimensions but how to realize a trivial product of the physical spacetime and the compactified space. That is, they must show the reason why the structure of the compactified space does not vary point by point in the physical spacetime. The extra-dimension people might obtain some favorable results in the hierarchy problem, but they should recognize that the cost of their achievement is deseparately high.

It is quite questionable to change the spacetime structure for the purpose of resolving such a quantitative problem as the hierarchy problem. I believe that the introduction of a higher-dimensional spacetime is too simple-minded to be on the right way to the correct theory.

§4. Supersymmetry

If a higher-dimensional spacetime is introduced straightforwardly, it becomes almost inevitably necessary to discriminate the extra dimensions by hand. If so, it is more preferable to consider the possibility that such discrimination exists in the built-in form. For example, one may suppose that the extra dimensions are not real numbers but Grassmann numbers. This possibility is realized as the supersymmetric theories; in particular, SUSY is the theory obtained by supersymmetrizing the Poincaré symmetry. It is the most beautiful theory which unifies the spacetime symmetry and internal symmetries as is discussed in [B]. Unfortunately, however, it seems that NATURE does not adopt SUSY as the fundamental principle. The reasons for believing so are as follows.

(1) Because SUSY is the unique symmetry which nontrivially includes the Poincaré symmetry, it has been believed that SUSY would be a privileged symmetry. But the uniqueness is a statement concerning the symmetry for the physical S-matrix but not the symmetry for the fundamental action, which is the most important object for constructing the theory. Furthermore, evidently SUSY is not realized in the actual physical S-matrix. Thus, there is no reason for believing that SUSY is a privileged symmetry. Later I will discuss this point in more detail.

(2) The experimental evidences clearly show that the actual mass spectra of elementary particles are not consistent with the unbroken SUSY. Hence if SUSY is assumed to be a true symmetry, it must be spontaneously broken. Then there must exist the Nambu-Goldstone fermion, but its existence is completely denied experimentally. I am not certain whether or not the “super-Higgs mechanism”, by which the Nambu-Goldstone fermion is absorbed into the gravitino in the framework of supergravity, can be well formulated. But even if it can be well formulated, such explanation for the absence of the Nambu-Goldstone fermion implies
that the conventional approximation of neglecting gravity in particle physics is violated already in the low-energy region. Therefore, if one insists on the super-Higgs mechanism, such a standpoint implies to assert that the great success made so far in particle physics is no more than mere accidental luck, because one has admitted the possibility that quantum gravity may have significant contributions in the low-energy region.

(3) In spite of the fact that so many elementary particles have been found experimentally, utterly no superpartner-like particles predicted by SUSY are discovered. It is quite unnatural to assume that all superpartners have a mass so large that no present-day accelerators can produce them without exception. I believe that NATURE does not plan a “perfect crime”. If SUSY is really a physical symmetry, something should indicate its existence in the low-energy region, just as the muon demonstrated the existence of the second generation in the early stage of particle physics.

If NATURE does not adopt SUSY, supergravity and superstring must be abandoned, and many people researching them will become greatly embarrassed. But, of course, their hope on the validity of SUSY does not guarantee its actual existence. I believe that scientists must always be humble for NATURE. People should not so much adhere to SUSY but investigate other possibilities.

As is well known, in quantum theories of gauge fields and the gravitational field, one defines the physical subspace by setting up the Kugo-Ojima subsidiary condition in order to avoid the appearance of negative-norm states. That is, the physical states are defined as the states which are annihilated by the BRS generator. Then the physical S-matrix, namely, the S-submatrix between the physical states becomes unitary (modulo zero-norm states). Accordingly, when the states predicted by a particular symmetry are unphysical, they do not appear in the physical S-matrix even if that symmetry is unbroken. In the framework of indefinite-metric quantum field theory, therefore, many unbroken symmetries are available without predicting the existence of extra physical particles, so that one can construct the action having a large (super)symmetry which may unify the spacetime symmetry and internal symmetries without contradicting the no-go theorem for the extension of the Poincaré symmetry other than SUSY.

§5. Quantized spacetime

As pointed out in [C], two events separated by less than the Planck scale cannot be well defined in quantum gravity. Accordingly, one must not expect that the geometrical structure of the ordinary spacetime remains to exist in the region less than the Planck scale. This fact motivated for many researchers to consider the quantization of the spacetime.

Thinking carefully, however, this logic has a big jump. While what quantum gravity claims is the proposition that the ordinary geometrical relation loses its meaning for two (very near) spacetime points, the quantization of the spacetime asserts that nontrivial commutation relations are set up between the components of only one spacetime point. There is neither logical nor physical ground for introducing such non-commutativity between the
components of a spacetime point. The quantized spacetime is totally irrelevant to quantum gravity and the uncertainty between the components of a spacetime point has no inevitable connection with the Planck length. If such connection should have been derived, it would be a consequence of an assumption introduced tacitly.

Indeed, when Snyder proposed a theory of quantizing the spacetime half a century ago, he did not care about quantum gravity at all. His purpose was to resolve the divergence problem of quantum field theory. Recently, the quantized spacetime has been revived owing to the fashion of the non-commutative geometry. But the quantized spacetime seems to be investigated in favor of the mathematical interest rather than the physical requirement.

In most cases, what is called the quantized-spacetime quantum field theory is what is obtained by merely transcribing \( x^\mu \) from c-numbers to q-numbers but keeping the framework of the conventional quantum field theory unchanged. The commutation relations for \( x^\mu \) are set up artificially without any logical connection with the fundamental principle of quantum field theory. Indeed, in spite of the fact that, except for the 2-dimensional case, the commutator \([x^\mu, x^\nu]\) cannot be defined Lorentz covariantly as far as it is a c-number, what is adopted as the action is “what becomes Lorentz invariant if the non-commutative \( x^\mu \) is regarded as commutative”. Once the Lorentz invariance is violated in the commutator, the theory is no longer Lorentz invariant so that the action may be Lorentz non-invariant in the same order of magnitude. Nevertheless, the quantized-spacetime people do not consider the possibilities of intrinsically non-invariant actions at all. I believe that the genuine quantized-spacetime theory, if any, should have some organic relationship between the action and the q-number property of \( x^\mu \), and it should be derived from the fundamental principle in a unified way.

In considering the quantized-spacetime theory constructed artificially as stated above, one must be very careful about identifying the quantity which has happened to be written \( x^\mu \) with the quantized spacetime. In general, there are infinitely many q-number quantities which correspond to the macroscopic spacetime where we live. There is no reasonable criterion to select the genuine quantized spacetime from them in such an artificially constructed theory. If there is a c-number candidate among them (for instance, if \([p_\mu, p_\nu] = 0\), where \( p_\mu \) denotes the momentum canonically conjugate to \( x^\mu \), and if “\( x^\mu \) minus a certain function of \( p_\nu \)” is commutative with anything), it is meaningless, at least for the purpose of constructing a fundamental theory, to adopt a non-commutative one as \( x^\mu \); it is no more than the quantized-spacetime people’s wishful thinking.

Suppose that the system of \( x^\mu \) and \( p_\nu \) is algebraically closed. If one does not wish to encounter the operator-ordering problem on the right-hand sides of commutation relations, the commutators \([x^\mu, x^\nu], [x^\mu, p_\nu]\), and \([p_\mu, p_\nu]\) should be linear with respect to \( x^\mu \) and \( p_\nu \) (Furthermore, dependence on \( x^\mu \) is forbidden by translational invariance.). In order to make it possible to have a more general quantized spacetime, therefore, it is preferable to introduce a new set of c-number parameters and express \( x^\mu \) and \( p_\nu \) in terms of those parameters and their differential operators. In this formulation, however, the space defined by those parameters plays the role of the fundamental spacetime rather than a mere mathematical tool. That is, such a theory should be regarded as a kind of the higher-dimensional-spacetime
model, though it is unnecessary to make round the extra dimensions by hand in contrast to the model discussed in §2. Instead, the real problem to be clarified is on what principle the expression for $x^\mu$ in terms of the parameters is derived. I cannot imagine that it is derivable by a kind of the action principle.

I emphasize that the quantized spacetime has neither logical ground for its indispensability nor the principle on which it is based. Hence the calculations concerning it often become no more than mathematical exercises. Of course, one may claim that the quantized spacetime is an approximation of string theory, but such an assertion implies that the quantized-spacetime theory itself is not a candidate of the ultimate theory.

§6. Geometrical structure

General relativity reduced gravity to geometry. Also in quantum theory, many geometrical concepts, such as solitons, monopoles, instantons, etc., have been introduced. Particularly, in quantum field theory, the notions of topological geometry have been regarded as important in connection with the problem of anomaly. Recently, this tendency has become more and more intense, keeping pace with the fashion of the path-integral approach. This is because, since the path-integral formalism directly gives the Green functions from a c-number action, classical notions can easily be incorporated into the framework of quantized theory.

Although certainly the path-integral approach is very convenient for calculations, one should not forget the fact that the unitarity of the physical S-matrix cannot be proved without the help of the operator formalism. While only the variation of the action is relevant (according to the action principle) in the operator formalism, the action itself is regarded as a meaningful quantity in the path-integral formalism. Indeed, the value of the action is often connected with a topological invariant. It should be re-examined more critically, without believing its relevance a priori, whether or not such a quantity is really physically meaningful.

If the spacetime coordinates are written $x^\mu$ universally over the whole spacetime, one is necessarily considering a topologically trivial manifold. In order to describe a more general manifold, it is necessary to decompose it into patches, and after introducing the coordinate system to each patch, one should join together those patches consistently. The quantity $x^\mu$ is merely the name of a point on the manifold, and it varies patch by patch. Therefore, if the theory is formulated by universally using $x^\mu$ as the spacetime coordinates, most people regard such a formulation as a method devoid of generality.

This way of thinking is correct if one is discussing a model of quantum mechanics. In this case, the manifold is given from the outset, and it is usually connived why a topologically nontrivial manifold has become relevant to the problem; maybe it is what is set up by experimentalists. I believe, however, that the same is no longer true in quantum field theory. There, $x^\mu$ is a microscopic quantity involved only in the quantum fields but not the macroscopic spacetime which we directly experience. The quantum-field-theoretical structure of the spacetime is governed by the spacetime symmetry of the action, such as the
Poincaré symmetry. It is the fundamental symmetry as a physical principle, which can never be modified by the experimentalist’s will. I believe that the fundamental theory should not start with the following proposition: “There existed a manifold at the beginning”.

Some of the cosmology people claim that, because the big-bang expanding universe is the unique actual spacetime, it is more adequate to quantize the theory in the expanding universe rather than to do in the Minkowski space. If they insist on such a standpoint, however, they should not a priori postulate that the universe is homogeneous and isotropic, but solve the Einstein equation rigorously by substituting the actual distribution of the matter into its right-hand side. Of course, it is impossible to work out this task. In general, it is unreasonable to formulate the fundamental theory of physics by taking the historically accidental facts as the premise for constructing it.

When one wishes to formulate quantum gravity geometrically by means of the path-integral formalism, there arises such a notion as the “sum over all possible manifolds”. But this notion is quite ambiguous. Furthermore, it is not clear whether or not the metric signature, such as Lorentzian or Euclidean, is prescribed beforehand. If the metric signature is fixed to a particular one from the outset, it contradicts with the conventional definition of the functional integral over the metric field $g_{\mu\nu}(x)$. Of course, a manifold of Lorentzian metric cannot be obtained from that of Euclidean metric by the so-called “Wick rotation”. I think that there is no logical basis for the expectation that quantum gravity can be constructed by means of the “sum over all possible manifolds”.

Such a geometrical notion as the manifold can be well-defined only if the neighborhoods of each point are set up. The concept of the “neighborhood system” is classical. Indeed, it is impossible to introduce such a topological concept as the neighborhood consistently logically prior to the determination of the metric signature of the manifold. That is, in a topologically well-defined manifold, the relation between any two points is always definite, but not uncertain as is suggested by quantum gravity. It is quite questionable, therefore, whether or not the spacetime structure implied by quantum gravity can be reproduced by summing up the exponentiated action over such topologically well-defined manifolds à la Feynman. It seems to me that one should include in the sum some curious objects which may not necessarily constitute manifolds globally.

It seems quite unnatural in the ultimate theory to adopt either a particular manifold or all possible manifolds as the spacetime structure.

§7. Spacetime structure in the ultimate theory

Human being can observe and recognize only classical objects. However, almost all physicists believe that the fundamental law of physics governing NATURE is of quantum theory. NATURE always talks in the language of quantum theory, but in order to understand it, human being must translate it into the language of classical theory by introducing a foreign notion, “probabilistic interpretation”, for the observation of quantum systems. Therefore, we do not wish to incorporate the theory of the observation into the framework of the ultimate theory. Thus, apart from the theory of the observation, it is quite natural to
expect that quantum theory is *closed* within itself without aid of classical theory, because it is impossible to believe that NATURE has utilized the notion invented by human being in order to control itself.

Thus, I want to propose the following fundamental principle of "**quantum priority**":

*In the ultimate theory, any concept of classical physics must not appear logically prior to its quantum-theoretical construction.*

The ground for proposing the above principle is as follows: If there exists a classical quantity appearing logically prior to the quantum-theoretical construction, such a quantity is necessarily brought into the theory from its outside; this fact means that the theory is not closed within itself. Accordingly, it cannot be regarded as the ultimate theory. Since the principle of quantum priority is quite natural, there may be few people who positively object it. Nevertheless, if it is applied to the discussion of the spacetime structure, it yields a very powerful restriction on the ultimate theory.

According to the above principle, *any manifold cannot be identified with the spacetime*. This is because to choose a particular manifold is justifiable only by some classical physics, as long as one does not introduce it by hand. This remark is applied even to the Minkowski space, because the existence of the light cone is evidently a consequence of a particular classical theory, *special relativity*! Thus, the ultimate theory rejects even the microcausality, which is one of the axioms adopted by the axiomatic quantum field theory.

Then there arises a question: Does it contradict the above principle to consider $x^\mu$ as the variables of quantum fields? No! This is because $x^\mu$ merely represents a set of 4 real numbers and the concept of the real-number field can be defined by the completion of the rational-number field without employing the notion of the neighborhood. Of course, there may be the objection that the Minkowski space is also a concept definable purely mathematically. But, while evidently the Minkowski space was introduced under the special reference to special relativity, the real-number field is a generic concept which can be used universally without restricting its physical applications. That is, the real-number field is not what arises as a consequence of any particular classical physics. Thus I believe that the use of real numbers does not contradict the principle of quantum priority.

The real-number field has a very special element "0". In order to guarantee that physical principles do not depend on the special property of 0, it is sufficient for the theory to be invariant under the change of $x^\mu$ by a constant $\alpha^\mu$. Therefore, it is quite natural to require the theory to be translationally invariant. Furthermore, it is preferable that the theory is constructed in such a way that the 4 real numbers $x^\mu$ ($\mu = 0, 1, 2, 3$) be not completely mutually unrelated but all linear combinations of them have the equal right. That is, it is natural to require general linear invariance. The combined symmetry of translational invariance and general linear invariance is called the **affine symmetry**. The (4-dimensional) affine transformation is a respectable transformation, because it is the unique analytic one-to-one mapping from $\mathbf{R}^4$ to $\mathbf{R}^4$. The affine symmetry may be regarded, therefore, as the most natural symmetry, as long as the theory is based on $\mathbf{R}^4$. The affine geometry was

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2The word "field" is used in the sense of *Körper*. 
born purely from the mathematical interest, that is, it is totally irrelevant to any classical theory of physics. If the ultimate theory is constructed in terms of quantum fields, it is most adequate to require the affine symmetry.

There may still arise some questions: Why are 4 real numbers $x^\mu$ ($\mu = 0, 1, 2, 3$) considered? i.e., why no more than 4? and why not complex numbers? That is, the criticism states that to consider 4 real numbers might be a consequence of classical physics. But I think that the fact that any event is assigned by 4 real numbers is the most common experience rather than classical physics. As long as we do not admit to make any by-hand procedure in the process of relating $x^\mu$ to the macroscopic spacetime, it is extremely implausible that under any other assumption one succeeds in deriving the actual macroscopic spacetime.

§8. Quantum gravity and the spacetime structure

In §7, I have discussed that, as far as one considers the quantum-field-theoretical formulation of the ultimate theory, it is most natural to require the affine symmetry for $x^\mu$. When formulating quantum gravity, the affine symmetry can be incorporated quite naturally. As is well known, although the classical Einstein gravity is invariant under general coordinate transformations, it is impossible to quantize the gravitational field without breaking such a local symmetry, that is, it is necessary to introduce a gauge-fixing term and a Faddeev-Popov ghost term in such a way that their sum becomes BRS invariant. After doing this, the largest symmetry, directly related to $x^b$, which can remain unbroken is nothing but the affine symmetry. Indeed, as long as such a particular classical metric as the Minkowski metric is not brought into the gauge-fixing term, the affine symmetry should survive in the quantum Einstein gravity.

Many people seem to assert that the gauge-fixing term is no more than a necessary evil because the physical $S$-matrix does not depend on its choice. But I point out that this assertion is based on the standpoint of giving the priority to classical theory. Quantum gravity starts with a BRS-invariant action; if the gauge-fixing terms are different, then the corresponding theories are different as quantum theories. In the classical electromagnetism, the observable quantity is the field strength $F_{\mu\nu}$ and therefore the gauge freedom of the potential $A_\mu$ is physically meaningless, but $A_\mu$ is the quantity fundamentally important in quantum theory. Indeed, such an observable phenomenon as the Aharanov-Bohm effect cannot be explained without taking $A_\mu$ into account. Although no example of the observable effect which cannot be explained without taking the gauge-fixing term into account is found as yet, the existence of the critical dimension of the string theory was found to be not irrelevant to the choice of the gauge-fixing term in the 2-dimensional quantum gravity. At any rate, hereafter, I assume that there exists uniquely the right choice of the gauge-fixing term according to the principle of quantum priority.

The de Donder gauge fixing, which may be regarded as the most natural one, can be realized if one employs the B-field formalism, where the B field $b_\mu$ is an auxiliary bosonic field appearing in the gauge-fixing term only. The BRS-invariant theory can be constructed in the most transparent way by introducing the notion of the “intrinsic BRS transformation”,

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which is the conventional BRS transformation minus its orbital part. Then carrying out
the canonical quantization, one can explicitly calculate all equal-time (anti-)commutation
relations in closed form. The quantum theory of gravity thus constructed turns out to
have a 16-dimensional supersymmetry \(^3\) based on the “16-dimensional supercoordinates”
\(\{x^\mu, b_\nu, c^\lambda, \bar{c}_\rho\}\), where \(c^\lambda\) and \(\bar{c}_\rho\) denote the Faddeev-Popov ghost and anti-ghost, respec-
tively; the outstanding beauty of this theory may be regarded as a support for the assertion
that the de Donder gauge fixing is of the right choice. There may be the objection that
in the canonical quantization the time is treated in a specially distinguished way, but such
a criticism does not apply to the theory having the affine symmetry. This is because, in
the affine-invariant theory, any linear combination of \(x^\mu\) can be used as the “time for the
canonical quantization” and the results are essentially independent of its choice.

Although the Einstein equation is written in terms of rational functions of \(g_{\mu \nu}\) (and
its derivatives), the action integral and the symmetry generators need the square root of
\(-\det g_{\mu \nu}\). Therefore, its hermiticity is not guaranteed unless certain restriction is imposed
on \(g_{\mu \nu}\). If one avoids to introduce artificial restriction, the following procedure is the unique
natural resolution: The fundamental field is the “vierbein” field \(h^a_\mu\) but not \(g_{\mu \nu}\), which
is expressed as \(g_{\mu \nu} = \eta^{ab} h^a_\mu h^b_\nu\). Here, the “internal” Minkowski metric \(\eta^{ab}\) is a quantity
totally irrelevant to \(x^\mu\) at this stage. I emphasize that the introduction of \(\eta^{ab}\) does not imply
that of the Minkowski space; it merely assigns the Lorentzian signature to \(g_{\mu \nu}\). Correspond-
ingly, one should forget about the classical interpretation, “coordinate system of the tangent
space”, of the vierbein field.

The introduction of the vierbein field has brought 6 extra degrees of freedom correspond-
ing to the local internal Lorentz transformations, but they must not bring any new physically
observable effects. To guarantee this, one introduces the gauge-fixing term for the local in-
ternal Lorentz invariance and the corresponding Faddeev-Popov ghost term so as to become
(local-Lorentz) BRS-invariant. Here, it is very important to choose the gauge-fixing term
so as to keep the general coordinate invariance and the global internal Lorentz invariance
unbroken.

It is undoubtedly true that there exist Dirac fields (more precisely, Weyl fields, according
to the electroweak theory) as the elementary-particle fields. As is well known, the generally
covariantized Dirac field can be formulated by using the vierbein. In this formulation,
however, the spinorial transformation property of the Dirac field is necessarily transferred
to the internal Lorentz freedom, and the Dirac field becomes a spacetime scalar. It is
mathematically impossible to have a spacetime spinor field as far as one considers it in the
framework of classical gravity. As is explained in the following, however, this problem is
satisfactorily resolved in the framework of quantum gravity.

In quantum gravity, the global gauge symmetries remaining after gauge fixing are the
affine symmetry and the global internal Lorentz invariance. However, those symmetries are
spontaneously broken. The reason for this is that the vierbein field has a non-vanishing
vacuum expectation value. If translational invariance is violated, it essentially means that

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\(^3\)More precisely, its superalgebra is the (8+8)-dimensional inhomogeneous orthosymplectic superalgebra
consisting of 144 generators. Of course, the affine algebra is its subalgebra.
the natural law itself would depend on a particular classical background. Since this is quite an unwelcome situation, I assume that translational invariance is not spontaneously broken. Then the vacuum expectation value, \( \langle 0 | h_\mu^a | 0 \rangle \), of the vierbein field becomes a constant matrix independent of \( x^\mu \). It must be a non-singular matrix (i.e., its determinant is non-vanishing) so as to make the spacetime non-degenerate. This is because, since \( \langle 0 | g_{\mu\nu} | 0 \rangle \) gives the spacetime metric, its determinant must be non-vanishing if the spacetime is non-degenerate; hence the determinant of \( \langle 0 | h_\mu^a | 0 \rangle \) must be also non-vanishing (at least generically). From this fact, one can easily show that all degrees of freedom of general linear invariance and internal Lorentz invariance are spontaneously broken.

Here, however, it is extremely important to note that there still remains a certain Lorentz symmetry which is not spontaneously broken. The generators of this unbroken symmetry are certain linear combinations of the antisymmetric-part generators of general linear transformations and the generators of internal Lorentz transformations with coefficients depending on \( \langle 0 | h_\mu^a | 0 \rangle \). The physical spacetime is defined by transforming \( x^\mu \) by the matrix \( \langle 0 | h_\mu^a | 0 \rangle \). Then the above unbroken Lorentz symmetry is precisely the spacetime Lorentz symmetry of particle physics. Indeed, one can confirm, by explicitly calculating the commutators between the Dirac field and the generators of the above symmetry, that the Dirac field transforms as a spacetime spinor under it.

Thus, both the physical spacetime and the Lorentz symmetry of particle physics are the secondary concepts appearing as a consequence of the spontaneous breakdown of symmetries, just as the electromagnetic \( U(1) \) symmetry is in the electroweak theory. The above consideration suggests that supergravity cannot be a fundamental theory because it is formulated under the assumption of regarding the Lorentz symmetry as a fundamental one.

As an additional remark, it should be noted that the Nambu-Goldstone boson corresponding to the symmetric-part generators of general linear transformations is nothing but the graviton. This fact guarantees the exact masslessness of the graviton.

§9. Criticism and counterargument

I have shown that quantum gravity can be satisfactorily formulated under the principle of quantum priority. According to my establishment, the set of 4 real numbers \( x^\mu \) yields the physical spacetime as a consequence of the spontaneous breakdown of the general linear invariance.

Unfortunately, however, there are many researchers who do not wish to regard the quantum Einstein gravity as (a part of) the ultimate theory. Such a standpoint may be based on the following reasons, but I wish to show that most of them are essentially groundless.

(1) There are some people who identify the covariant formulation of the quantum Einstein gravity with the covariant perturbation theory of it; since the latter is unrenormalizable, they claim that it is not a physically sensible theory. This criticism is, however, merely due to their erroneous identification; indeed, what is not adequate is to apply the covariant perturbation theory to quantum gravity. The interaction picture, on which the covariant
perturbation theory is based, can be introduced under the assumption that $g_{\mu\nu}$ can be written as a sum of a c-number metric and a quantum gravitational field of order $\sqrt{\kappa}$, where $\kappa$ denotes the Einstein gravitational constant; that is, the covariant perturbation theory can be introduced under the assumption that $g_{\mu\nu}$ tends to a c-number as $\kappa \to 0$. However, this assumption is wrong! In the BRS-formulated theory of quantum gravity, $g_{\mu\nu}$ actually tends to a q-number as $\kappa \to 0$. Therefore, the covariant formalism of the quantum Einstein gravity should be solved in the Heisenberg picture. The method for doing this has been developed, and the $\kappa \to 0$ limit of $g_{\mu\nu}$ was explicitly calculated. Of course, it is unclear as yet whether or not the divergence problem can be resolved in the Heisenberg-picture approach. But, at any rate, I believe that it is very unreasonable to reject the quantum Einstein gravity in such an uncertain situation.

(2) In the quantum Einstein gravity, it is impossible to resolve the problems stated in [A] and [B], and therefore it is not the unified theory. But why should all problems be solved simultaneously? I think that there is no reason why the problem [A] is inevitably connected with the problem of the spacetime structure and quantum gravity. As for the problem [B], for example, there is a possibility of supersymmetrizing the local internal Lorentz symmetry only. This possibility was investigated in detail; the conclusion of this supersymmetric theory is that the admissible gauge symmetry of particle physics is the chiral $SO(N) \times SO(N)$.

(3) Some of the cosmology people believe that quantum gravity is meaningless if the beginning of the universe cannot be discussed by it. But the problem of the beginning of the universe must include the quantum observation theory in the case in which there is no external observer. Therefore, it is a problem totally out of control; and indeed they themselves do not seriously discuss the problem of the probabilistic interpretation of quantum theory in studying the early universe. Although the cosmology people prefer to adopt geometrical approach, I believe that the theory of quantum gravity should be formulated so as to be mathematically coherent to the operator formalism of gauge theory, because the latter has been well established in particle physics.

(4) Some of the superstring people believe that it is fruitless to consider any other theory because the superstring theory is the unique candidate of the ultimate theory. I believe, however, that such an assertion is too much prejudiced, because there are several fundamental difficulties in the superstring theory and nothing of them have yet been resolved. I will not discuss them here, but I wish to point out that there are absolutely neither theoretical nor experimental evidences which justify the huge extension of the theoretical framework done in the superstring theory.

In conclusion, I would like to state the following aphorism:

*Do not seriously take geometrical and classical images into account in constructing the ultimate theory.*

Since the grammar of the language spoken by NATURE is mathematics, the mathemat-
ical coherence should be the key to the ultimate theory.

I hope that more researchers of the fundamental physics recognize the possibility that quantum Einstein gravity may already provide the natural framework of the ultimate theory.

References

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