Convergence to a self-normalized G-Brownian motion

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Abstract

G-Brownian motion has a very rich and interesting new structure which nontrivially generalizes the classical one. Its quadratic variation process is also a continuous process with independent and stationary increments. We prove a self-normalized functional central limit theorem for independent and identically distributed random variables under the sub-linear expectation with the limit process being a G-Brownian motion self-normalized by its quadratic variation. To prove the self-normalized central limit theorem, we also establish a new Donsker’s invariance principle with the limit process being a generalized G-Brownian motion.

Keywords: sub-linear expectation; G-Brownian motion; central limit theorem; invariance principle; self-normalization

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1 Introduction

Let \{X_n; n \geq 1\} be a sequence of independent and identically distributed random variables on a probability space \((\Omega, \mathcal{F}, P)\). Set \(S_n = \sum_{j=1}^{n} X_j\). Suppose \(EX_1 = 0\) and \(EX_1^2 = \sigma^2 > 0\). The well-known central limit theorem says that

\[
\frac{S_n}{\sqrt{n}} \xrightarrow{d} N(0, \sigma^2),
\] (1.1)

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or equivalently, for any bounded continuous function \(\psi(x)\),

\[
E\left[\psi\left(\frac{S_n}{\sqrt{n}}\right)\right] \to E[\psi(\xi)], \tag{1.2}
\]

where \(\xi \sim N(0, \sigma^2)\) is a normal random variable. If the normalization factor \(\sqrt{n}\) is replaced by \(\sqrt{V_n}\) where \(V_n = \sum_{j=1}^{n} X_j^2\), then

\[
\frac{S_n}{\sqrt{V_n}} \xrightarrow{d} N(0, 1). \tag{1.3}
\]

Giné, Götze and Mason (1997) proved that (1.3) holds if and only if \(EX_1 = 0\) and

\[
\lim_{x \to \infty} \frac{x^2 P(|X_1| \geq x)}{EX_1^2 I\{|X_1| \leq x\}} = 0. \tag{1.4}
\]

The result (1.3) is refereed to as the self-normalized central limit theorem. The purpose of this paper is to establish the self-normalized central limit theorem under the sub-linear expectation.

The sub-linear expectation or called and G-expectation is a nonlinear expectation advancing the notions of backward stochastic differential equations, g-expectations, and provides a flexible framework to model non-additive probability problems and the volatility uncertainty in finance. Peng (2006, 2008a, 2008b) introduced a general framework of the sub-linear expectation of random variables and introduced the notions of G-normal random variable, G-Brownian motion, independent and identically distributed random variables etc under the sub-linear expectations. The construction of sub-linear expectations on the space of continuous paths and discrete time paths can also be founded in Yan et al (2012) and Nutz and Handel (2013). For basic properties of the sub-linear expectations, one can refer to Peng (2008b, 2009, 2010a, etc). For stochastic calculus and stochastic differential equations with respect to a G-Brownian motion, one can refer Li and Peng (2011), Hu, et al (2014a,b) etc and a book of Peng (2010a).

The central limit theorem under the sub-linear expectation was first established by Peng (2008b). It says that (1.2) remains true when the expectation \(E\) is replaced by a sub-linear expectation \(\hat{E}\) if \(\{X_n; n \geq 1\}\) are independent and identically distributed under \(\hat{E}\), i.e.,

\[
\frac{S_n}{\sqrt{n}} \xrightarrow{d} \xi \text{ under } \hat{E}, \tag{1.5}
\]
where $\xi$ is a G-normal random variable.

In the classical case, when $E[X_1^2]$ is finite, (1.3) follows from the central limit theorem (1.1) immediately by Slutsky’s lemma and the fact that

$$\frac{V_n}{n} \rightarrow^P \sigma^2.$$  

The later is due to the law of large numbers. In the framework of the sub-linear expectation, $\frac{V_n}{n}$ no longer converges to a constant. The self-normalized central limit theorem can not follow from the central limit theorem (1.5) directly. In this paper, we will prove that

$$\frac{S_n}{\sqrt{V_n}} \xrightarrow{d} \frac{W_1}{\sqrt{\langle W \rangle_t}} \text{ under } \mathbb{E},$$

(1.6)

where $W_t$ is a G-Brownian motion and $\langle W \rangle_t$ is its quadratic variation process. A very interesting phenomenon of G-Brownian motion is that its quadratic variation process is also a continuous process with independent and stationary increments, and thus can be still regarded as a Brownian motion. When the sub-linear expectation $\mathbb{E}$ reduces to a linear one, $W_t$ is the classical Brownian motion with $W_1 \sim N(0, \sigma^2)$ and $\langle W \rangle_t = t\sigma^2$, and then (1.6) is just (1.3). Our main results on the self-normalized central limit theorem will be given in Section 3 where the process of the self-normalized partial sums $S_{[nt]}/\sqrt{V_n}$ is proved to converge to a self-normalized G-Brownian motion $W_t/\sqrt{\langle W \rangle_t}$. We also consider the case that the second moments of $X_i$s are infinite and obtain the self-normalized central limit theorem under a condition similar to (1.4). In the next section, we state basic settings in a sub-linear expectation space including, capacity, independence, identical distribution, G-Brownian motion etc. One can skip this section if he/she is familiar with these concepts. To prove the self-normalized central limit theorem, we establish a new Donsker’s invariance principle in Section 4 with the limit process being a generalized G-Brownian motion. The proof is given in the last section.

2 Basic Settings

We use the framework and notations of Peng (2008b). Let $(\Omega, \mathcal{F})$ be a given measurable space and let $\mathcal{H}$ be a linear space of real functions defined on $(\Omega, \mathcal{F})$ such that
if $X_1, \ldots, X_n \in \mathcal{H}$ then $\varphi(X_1, \ldots, X_n) \in \mathcal{H}$ for each $\varphi \in C_b(\mathbb{R}^n) \cup C_{l,Lip}(\mathbb{R}^n)$, where $C_b(\mathbb{R}^n)$ denote the space of all bounded continuous functions and $C_{l,Lip}(\mathbb{R}^n)$ denotes the linear space of (local Lipschitz) functions $\varphi$ satisfying

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)|x - y|, \quad \forall x, y \in \mathbb{R}^n,$$

for some $C > 0, m \in \mathbb{N}$ depending on $\varphi$.

\(\mathcal{H}\) is considered as a space of “random variables”. In this case we denote $X \in \mathcal{H}$. Further, we let $C_{b,Lip}(\mathbb{R}^n)$ denote the space of all bounded and Lipschitz functions on $\mathbb{R}^n$.

2.1 Sub-linear expectation and capacity

**Definition 2.1** A sub-linear expectation $\mathbb{E}$ on $\mathcal{H}$ is a functional $\mathbb{E} : \mathcal{H} \to \mathbb{R}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have

(a) **Monotonicity:** If $X \geq Y$ then $\mathbb{E}[X] \geq \mathbb{E}[Y]$;

(b) **Constant preserving:** $\mathbb{E}[c] = c$;

(c) **Sub-additivity:** $\mathbb{E}[X + Y] \leq \mathbb{E}[X] + \mathbb{E}[Y]$ whenever $\mathbb{E}[X] + \mathbb{E}[Y]$ is not of the form $+\infty - \infty$ or $-\infty + \infty$;

(d) **Positive homogeneity:** $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X], \lambda \geq 0$.

Here $\overline{\mathbb{R}} = [-\infty, \infty]$. The triple $(\Omega, \mathcal{H}, \mathbb{E})$ is called a sub-linear expectation space. Give a sub-linear expectation $\mathbb{E}$, let us denote the conjugate expectation $\mathbb{E}$ of $\mathbb{E}$ by

$$\mathbb{E}[X] := -\mathbb{E}[-X], \quad \forall X \in \mathcal{H}.$$ 

Next, we introduce the capacities corresponding to the sub-linear expectations.

Let $\mathcal{G} \subset \mathcal{F}$. A function $V : \mathcal{G} \to [0, 1]$ is called a capacity if

$$V(\emptyset) = 0, \quad V(\Omega) = 1 \quad \text{and} \quad V(A) \leq V(B) \quad \forall A \subset B, \quad A, B \in \mathcal{G}.$$ 

It is called to be sub-additive if $V(A \cup B) \leq V(A) + V(B)$ for all $A, B \in \mathcal{G}$ with $A \cup B \in \mathcal{G}$. 

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Let \((\Omega, \mathcal{H}, \widehat{E})\) be a sub-linear space, and \(\widehat{E}\) be the conjugate expectation of \(\widehat{E}\). It is natural to define the capacity of a set \(A\) to be the sub-linear expectation of the indicator function \(I_A\) of \(A\). However, \(I_A\) may be not in \(\mathcal{H}\). So, we denote a pair \((\mathbb{V}, \mathbb{V})\) of capacities by

\[
\mathbb{V}(A) := \inf\{\widehat{E}[\xi] : I_A \leq \xi, \xi \in \mathcal{H}\}, \quad \forall A \in \mathcal{F},
\]

where \(A^c\) is the complement set of \(A\). Then \(\mathbb{V}\) is sub-additive and

\[
\mathbb{V}(A) = \widehat{E}[I_A], \quad \mathbb{V}(A) = \widehat{E}[I_A], \quad \text{if } I_A \in \mathcal{H}
\]

\[
\widehat{E}[f] \leq \mathbb{V}(A) \leq \widehat{E}[g], \quad \widehat{E}[f] \leq \mathbb{V}(A) \leq \widehat{E}[g], \quad \text{if } f \leq I_A \leq g, f, g \in \mathcal{H}.
\]

Further, we define an extension of \(\widehat{E}^*\) of \(\widehat{E}\) by

\[
\widehat{E}^*[X] = \inf\{\widehat{E}[Y] : X \leq Y, Y \in \mathcal{H}\}, \quad \forall X : \Omega \to \mathbb{R},
\]

where \(\inf\emptyset = +\infty\). Then

\[
\widehat{E}^*[X] = \widehat{E}[X] \quad \text{if } X \in \mathcal{H}, \quad \forall \mathbb{V}(A) = \widehat{E}^*[I_A],
\]

\[
\widehat{E}[f] \leq \widehat{E}^*[X] \leq \widehat{E}[g] \quad \text{if } f \leq X \leq g, f, g \in \mathcal{H}.
\]

### 2.2 Independence and distribution

**Definition 2.2** (Peng (2006, 2008b))

(i) **(Identical distribution)** Let \(X_1\) and \(X_2\) be two \(n\)-dimensional random vectors defined respectively in sub-linear expectation spaces \((\Omega_1, \mathcal{H}_1, \widehat{E}_1)\) and \((\Omega_2, \mathcal{H}_2, \widehat{E}_2)\). They are called identically distributed, denoted by \(X_1 \overset{d}{=} X_2\) if

\[
\widehat{E}_1[\varphi(X_1)] = \widehat{E}_2[\varphi(X_2)], \quad \forall \varphi \in C_{l,\text{Lip}}(\mathbb{R}^n),
\]

whenever the sub-expectations are finite. A sequence \(\{X_n; n \geq 1\}\) of random variables is said to be identically distributed if \(X_i \overset{d}{=} X_1\) for each \(i \geq 1\).

(ii) **(Independence)** In a sub-linear expectation space \((\Omega, \mathcal{H}, \widehat{E})\), a random vector \(Y = (Y_1, \ldots, Y_n), Y_i \in \mathcal{H}\) is said to be independent to another random vector \(X = (X_1, \ldots, X_m), X_i \in \mathcal{H}\) under \(\widehat{E}\) if for each test function \(\varphi \in C_{l,\text{Lip}}(\mathbb{R}^m \times \mathbb{R}^n)\) we have

\[
\widehat{E}[\varphi(X, Y)] = \widehat{E}[\widehat{E}[\varphi(x, Y)]|_{x=X}],
\]

whenever \(\varphi(x) := \widehat{E}[|\varphi(x, Y)|] < \infty\) for all \(x\) and \(\widehat{E}[|\varphi(X)|] < \infty\).
(iii) **IID random variables** A sequence of random variables \( \{X_n; n \geq 1\} \) is said to be independent and identically distributed (IID), if \( X_i \overset{d}{=} X_1 \) and \( X_{i+1} \) is independent to \( (X_1, \ldots, X_i) \) for each \( i \geq 1 \).

### 2.3 G-normal distribution, G-Brownian motion and its quadratic variation

Let \( 0 < \varphi \leq \sigma < \infty \) and \( G(\alpha) = \frac{1}{2}(\varphi^2 \alpha^+ - \sigma^2 \alpha^-) \). \( X \) is call a normal \( N(0, [\varphi^2, \sigma^2]) \) distributed random variable (write \( X \sim N(0, [\varphi^2, \sigma^2]) \)) under \( \widehat{\mathbb{E}} \), if for any bounded Lipschitz function \( \varphi \), the function \( u(x,t) = \widehat{\mathbb{E}}[\varphi (x + \sqrt{t}X)] \) \( (x \in \mathbb{R}, t \geq 0) \) is the unique viscosity solution of the following heat equation:

\[
\partial_t u - G (\partial_{xx}^2 u) = 0, \quad u(0,x) = \varphi(x).
\]

Let \( C[0,1] \) be a function space of continuous functions on \( [0,1] \) equipped with the super-norm \( \|x\| = \sup_{0 \leq t \leq 1} |x(t)| \) and \( C_b(C[0,1]) \) is the set of bounded continuous functions \( h(x) : C[0,1] \to \mathbb{R} \). The modulus of the continuity of an element \( x \in C[0,1] \) is defined by

\[
\omega_h(x) = \sup_{|t-s| < \delta} |x(t) - x(s)|.
\]

It is showed that there is a sub-linear expectation space \((\widetilde{\Omega}, \widetilde{\mathcal{H}}, \widetilde{\mathbb{E}})\) with \( \widetilde{\Omega} = C[0,1] \) and \( C_b(C[0,1]) \subset \widetilde{\mathcal{H}} \) such that \((\widetilde{\mathcal{H}}, \widetilde{\mathbb{E}}[\| \cdot \|])\) is a Banach space, and the canonical process \( W(t)(\omega) = \omega_t(\omega \in \widetilde{\Omega}) \) is a G-Brownian motion with \( W(1) \sim N(0, [\varphi^2, \sigma^2]) \) under \( \widetilde{\mathbb{E}} \), i.e., for all \( 0 \leq t_1 < \ldots < t_n \leq 1, \varphi \in C_{l, lip}(\mathbb{R}^n) \),

\[
\widetilde{\mathbb{E}}[\varphi(W(t_1), \ldots, W(t_{n-1}), W(t_n) - W(t_{n-1}))] = \widetilde{\mathbb{E}}[\psi(W(t_1), \ldots, W(t_{n-1}))],
\]

where \( \psi(x_1, \ldots, x_{n-1}) = \widetilde{\mathbb{E}}[\varphi(x_1, \ldots, x_{n-1}, \sqrt{t_n - t_{n-1}}W(1))] \) (c.f. Peng (2006, 2008a, 2010), Denis, Hu and Peng (2011)).

The quadratic variation process of a G-Brownian motion \( W \) is defined by

\[
\langle W \rangle_t = \lim_{\|\Pi^N_t\| \to 0} \sum_{j=1}^{N-1} (W(t_j^N) - W(t_{j-1}^N))^2 = W^2(t) - 2 \int_0^t W(t)dW(t),
\]

where \( \Pi^N_t = \{t_0^N, t_1^N, \ldots, t_N^N\} \) is a partition of \( [0,t] \) and \( \|\Pi^N_t\| = \max_j |t_j^N - t_{j-1}^N| \), and
the limit is taken in $L_2$, i.e.,

$$
\lim_{\|\Pi_N\|\to 0} \hat{\mathbb{E}} \left[ \left( \sum_{j=1}^{N-1} (W(t_j^N) - W(t_{j-1}^N))^2 - \langle W \rangle_t \right)^2 \right] = 0.
$$

The quadratic variation process $\langle W \rangle_t$ is also a continuous process with independent and stationary increments. For the properties and the distribution of the quadratic variation process, one can refer to a book of Peng (2010a).

Denis, Hu and Peng (2011) showed the following representation of the G-Brownian motion (c.f, Theorem 52).

**Lemma 2.1** Let $(\Omega, \mathcal{F}, P)$ be a probability measure space and $\{B(t)\}_{t \geq 0}$ is a $P$-Brownian motion. Then for all bounded continuous function $\varphi : C_b[0, 1] \to \mathbb{R}$,

$$
\hat{\mathbb{E}} \left[ \varphi(W(\cdot)) \right] = \sup_{\theta \in \Theta} \mathbb{E}_P \left[ \varphi(W_\theta(\cdot)) \right], \quad W_\theta(t) = \int_0^t \theta(s) dB(s),
$$

where

$$
\Theta = \{ \theta : \theta(t) \text{ is } \mathcal{F}_t\text{-adapted process such that } \sigma \leq \theta(t) \leq \sigma \},
$$

$$
\mathcal{F}_t = \sigma \{ B(s) : 0 \leq s \leq t \} \vee \mathcal{N}, \quad \mathcal{N} \text{ is the collection of } P\text{-null subsets.}
$$

In the sequel of this paper, the sequences $\{X_n; n \geq 1\}$, $\{Y_n; n \geq 1\}$ etc of the random variables are considered in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$. Without specification, we suppose that $\{X_n; n \geq 1\}$ is a sequence of independent and identically distributed random variables in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ with $\hat{\mathbb{E}}[X_1] = \hat{\mathbb{E}}[X_1] = 0$, $\hat{\mathbb{E}}[X_2] = \sigma^2$ and $\hat{\mathbb{E}}[X_2^2] = \sigma^2$. Denote $S_0^X = 0$, $S_n^X = \sum_{k=1}^n X_k$, $V_0 = 0$, $V_n = \sum_{k=1}^n X_k^2$. And suppose that $(\hat{\Omega}, \hat{\mathcal{H}}, \hat{\mathbb{E}})$ is a sub-linear expectation space which is rich enough such that there is a G-Brownian motion $W(t)$ with $W(1) \sim N(0, [\sigma^2, \sigma^2])$. We denote a pair of capacities corresponding to the sub-linear expectation $\hat{\mathbb{E}}$ by $(\hat{\mathbb{V}}, \hat{\mathbb{V}})$, and the extension of $\hat{\mathbb{E}}$ by $\hat{\mathbb{E}}^*$. 
3 Main results

We consider the convergence of the process $S_{[nt]}$. Because it is not in $C[0, 1]$, it needs to be modified. Define the $C[0, 1]$-valued random variable $\tilde{S}_n^X(\cdot)$ by setting

$$\tilde{S}_n^X(t) = \begin{cases} 
\sum_{j=1}^{k} X_j, & \text{if } t = k/n \ (k = 0, 1, \ldots, n); \\
\text{extended by linear interpolation in each interval} \\
[k-1]^{-1}, kn^{-1}.
\end{cases}$$

Then $\tilde{S}_n^X(t) = S_{[nt]}^X + (nt - [nt])X_{[nt]+1}$. Here $[nt]$ is the largest integer less than or equal to $nt$. Zhang (2015) obtained the functional central limit theorem as follows.

**Theorem A** Suppose $\hat{E}[X_i^2 - b] \to 0$ as $b \to \infty$. Then for all bounded continuous function $\varphi : C[0, 1] \to \mathbb{R}$,

$$\hat{E} \left[ \varphi \left( \frac{\tilde{S}_n^X}{\sqrt{n}} \right) \right] \to \tilde{E} \left[ \varphi \left( W(\cdot) \right) \right]. \quad (3.1)$$

Replacing the normalization factor $\sqrt{n}$ by $\sqrt{V_n}$, we obtain the self-normalized process of partial sums:

$$W_n(t) = \frac{\tilde{S}_n^X(t)}{\sqrt{V_n}},$$

where $\theta$ is defined to be 0. Our main result is the following self-normalized functional central limit theorem (FCLT).

**Theorem 3.1** Suppose $\hat{E}[X_i^2 - b] \to 0$ as $b \to \infty$. Then for all bounded continuous function $\varphi : C[0, 1] \to \mathbb{R}$,

$$\hat{E}^* \left[ \varphi \left( W_n(\cdot) \right) \right] \to \tilde{E} \left[ \varphi \left( \frac{W(\cdot)}{\sqrt{\langle W \rangle_1}} \right) \right]. \quad (3.2)$$

In particular, for all bounded continuous function $\varphi : \mathbb{R} \to \mathbb{R}$,

$$\hat{E}^* \left[ \varphi \left( \frac{S_n^X}{\sqrt{V_n}} \right) \right] \to \tilde{E} \left[ \varphi \left( \frac{W(1)}{\sqrt{\langle W \rangle_1}} \right) \right] \quad (3.3)$$

$$= \sup_{\theta \in \Theta} E_p \left[ \varphi \left( \frac{\int_0^1 \theta(s)dB(s)}{\sqrt{\int_0^1 \theta^2(s)ds}} \right) \right].$$

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Remark 3.1 It is obvious that
\[ \hat{E} \left[ \varphi \left( \frac{W(\cdot)}{\sqrt{\langle W \rangle}} \right) \right] \geq E_P \left[ \varphi (B(\cdot)) \right]. \]

An interesting problem is how to estimate the upper bounds of the expectations on the right hands of (3.2) and (3.3).

Further, \( \frac{W(t)}{\sqrt{\langle W \rangle}} \) is a G-Brownian motion with \( \hat{W}(1) \sim N(0, [r^{-2}, 1]) \), \( r^2 = \sigma^2 / \sigma' \).

For the classical self-normalized central limit theorem, Giné, Götze and Mason (1997) showed that the finiteness of the second moments can be relaxed to the condition (1.4). Csörgő, Szyszkowicz and Wang (2003) proved the self-normalized functional central limit theorem under (1.4). The next theorem gives a similar result under the sub-linear expectation and is an extension of Theorem 3.1.

**Theorem 3.2** Let \( \{X_n; n \geq 1\} \) be a sequence of independent and identically distributed random variables in the sub-linear expectation space \( (\Omega, \mathcal{H}, \hat{E}) \) with \( \hat{E}[X_1] = \hat{E}[X_1] = 0 \). Denote \( l(x) = \hat{E}[X_1^2 \land x^2] \). Suppose

(I) \( x^2 \hat{V}(|X_1| \geq x) = o(l(x)) \) as \( x \to \infty \);

(II) \( \lim_{x \to \infty} \frac{\hat{E}[X_1^2 \land x^2]}{\hat{E}[X_1^2 \land x^2]} = r^2 < \infty \);

(III) \( \hat{E}((|X_1| - c)^+) \to 0 \) as \( c \to \infty \).

Then the conclusions of Theorem 3.1 remain true with \( W(t) \) being a G-Brownian motion such that \( W(1) \sim N(0, [r^{-2}, 1]) \).

**Remark 3.2** The conditions (III) implies that \( \hat{E}((|X_1| - x)^+) \leq \int_x^\infty \hat{V}(|X_1| \geq y)dy \) and \( = o(x^{-1}l(x)) \) if the condition (I) is satisfied. When \( \hat{E} \) is a continuous sub-linear expectation, then for any random variable \( Y \) we have \( \hat{E}[|Y|] \leq \int_0^\infty \hat{V}(|Y| \geq y)dy \) and so the condition (III) can be removed.

4 Invariance principle

To prove Theorems 3.1 and 3.2, we will prove a new Donsker’s invariance principle. Let \( \{(X_i, Y_i); i \geq 1\} \) be a sequence of independent and identically distributed random
vectors in the sub-linear expectation space \((Ω, H, \hat{E})\) with \(\hat{E}[X_1] = \hat{E}[-X_1] = 0\), 
\(\hat{E}[X_1^2] = \sigma^2\), \(\hat{E}[X_1] = \mu\). Denote 
\[G(p, q) = \hat{E}\left[\frac{1}{2} p X_1^2 + q Y_1\right], \quad p, q \in \mathbb{R}.\] (4.1)

Let \(ξ\) be a G-normal distributed random variable, \(η\) be a maximal distributed random variable such that the distribution of \((ξ, η)\) is characterized by the following parabolic partial differential equation (PDE for short) defined on \([0, \infty) \times \mathbb{R} \times \mathbb{R}\):

\[\partial_t u - G(\partial_y u, \partial_{xx}^2 u) = 0, \quad (4.2)\]
i.e., if for any bounded Lipschitz function \(ϕ(x, y) : \mathbb{R}^2 \to \mathbb{R}\), the function \(u(x, y, t) = \hat{E}\left[ϕ(x + \sqrt{t}ξ, y + tη)\right] (x, y \in \mathbb{R}, t \geq 0)\) is the unique viscosity solution of the PDE (4.2) with Cauchy condition \(u|_{t=0} = ϕ\). Further, let \(B_t\) and \(b_t\) be two random processes such that the distribution of the process \((B_t, b_t)\) is characterized by

(i) \(B_0 = 0, b_0 = 0;\)

(ii) for any \(0 \leq t_1 \leq \ldots \leq t_k \leq s \leq t + s\), \((B_{s+t} - B_s, b_{s+t} - b_s)\) is independent to \((B_{t_j}, b_{t_j}), j = 1, \ldots, k\), in sense that, for any \(ϕ \in C_{l, Lip}(\mathbb{R}^{2(k+1)}),\)

\[\hat{E}\left[ϕ((B_{t_1}, b_{t_1}), \ldots, (B_{t_k}, b_{t_k}), (B_{s+t} - B_s, b_{s+t} - b_s))\right] = \hat{E}\left[ψ((B_{t_1}, b_{t_1}), \ldots, (B_{t_k}, b_{t_k}))\right]\] (4.3)

where

\[ψ((x_1, y_1), \ldots, (x_k, y_k)) = \hat{E}\left[ϕ((x_1, y_1), \ldots, (x_k, y_k), (B_{s+t} - B_s, b_{s+t} - b_s))\right];\]

(iii) for any \(t, s > 0\), \((B_{s+t} - B_s, b_{s+t} - b_s) \sim (B_t, b_t)\) under \(\hat{E};\)

(iv) for any \(t > 0\), \((B_t, b_t) \sim (\sqrt{t}B_1, tb_1)\) under \(\hat{E};\)

(v) the distribution of \((B_1, b_1)\) is characterized by the PDE (4.2).

It is easily seen that \(B_t\) is a G-Brownian motion with \(B_1 \sim N(0, [\sigma^2, \sigma^2])\), and \((B_t, b_t)\) is a generalized G-Brownian motion introduced by Peng (2010a). The existence of the generalized G-Brownian motion can be found in Peng (2010a).
Theorem 4.1 Suppose $\hat{E}[(X_1^2 - b^2)] \to 0$ and $\hat{E}[(|Y_1| - b^2)] \to 0$ as $b \to \infty$. Let

$$\hat{W}_n(t) = \left( \frac{\hat{S}_n^X(t)}{\sqrt{n}}, \frac{\hat{S}_n^Y(t)}{n} \right).$$

Then for any bounded continuous function $\varphi : C[0, 1] \times C[0, 1] \to \mathbb{R}$,

$$\lim_{n \to \infty} \hat{E} \left[ \varphi \left( \hat{W}_n(\cdot) \right) \right] = \hat{E} \left[ \varphi (B, b) \right]. \quad (4.4)$$

Further, let $p \geq 2$, $q \geq 1$, and assume $\hat{E}[|X_1|^p] < \infty$, $\hat{E}[|Y_1|^q] < \infty$. Then for all continuous function $\varphi : C[0, 1] \times C[0, 1] \to \mathbb{R}$ with $|\varphi(x, y)| \leq C(1 + \|x\|^p + \|y\|^q)$,

$$\lim_{n \to \infty} \hat{E}^* \left[ \varphi \left( \hat{W}_n(\cdot) \right) \right] = \hat{E} \left[ \varphi (B, b) \right]. \quad (4.5)$$

Here $\|x\| = \sup_{0 \leq t \leq 1} |x(t)|$ for $x \in C[0, 1]$.

Remark 4.1 When $X_k$ and $Y_k$ are random vectors in $\mathbb{R}^d$ with $\hat{E}[X_k] = \hat{E}[-X_k] = 0$, $\hat{E}[|X_1|^2 - b^2] \to 0$ and $\hat{E}[|Y_1|^2 - b^2] \to 0$ as $b \to \infty$. Then the function $G$ in (4.7) becomes

$$G(p, A) = \hat{E} \left[ \frac{1}{2} \langle AX_1, X_1 \rangle + \langle p, Y_1 \rangle \right], \quad p \in \mathbb{R}^d, A \in \mathbb{S}(d),$$

where $\mathbb{S}(d)$ is the collection of all $d \times d$ symmetric matrices. The conclusion of Theorem 4.1 remains true with the distribution of $(B_1, b_1)$ being characterized by the following parabolic partial differential equation defined on $[0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$:

$$\partial_t u - G(D_y u, D_{xx}^2 u) = 0, \quad u|_{t=0} = \varphi,$$

where $D_y = (\partial_{y_i})_{i=1}^n$ and $D_{xx}^2 = (\partial_{x_i x_j}^2)_{i,j=1}^d$.

Remark 4.2 As a conclusion of Theorem 4.1, we have

$$\hat{E} \left[ \varphi \left( \frac{\hat{S}_n^X}{\sqrt{n}}, \frac{\hat{S}_n^Y}{n} \right) \right] \to \hat{E} \left[ \varphi (B_1, b_1) \right], \quad \varphi \in C_0(\mathbb{R}^2).$$

This is proved by Peng (2010a) under the conditions $\hat{E}[|X_1|^{2+\delta}] < \infty$ and $\hat{E}[|Y_1|^{1+\delta}] < \infty$ (c.f., Theorem 3.6 and Remark 3.8 therein).

Before the proof, we need several lemmas. For random vectors $X_n$ in $(\Omega, \mathcal{F}, \hat{E})$ and $X$ in $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\hat{E}})$, we write $X_n \overset{d}{\to} X$ if

$$\hat{E}[\varphi(X_n)] \to \hat{E}[\varphi(X)].$$
for any bounded continuous \( \varphi \). Write \( X_n \xrightarrow{\mathcal{V}} x \) if \( \mathbb{V}(\|X_n - x\| \geq \epsilon) \to 0 \) for any \( \epsilon > 0 \).

\{X_n\} is called to be uniformly integrable if

\[
\lim_{b \to \infty} \limsup_{n \to \infty} \mathbb{E}[\|X_n\| - b]^+] = 0.
\]

The following three lemmas are obvious.

**Lemma 4.1** If \( X_n \xrightarrow{d} X \) and \( \varphi \) is a continuous function, then \( \varphi(X_n) \xrightarrow{d} \varphi(X) \).

**Lemma 4.2** (Slutsky’s Lemma) Suppose \( X_n \xrightarrow{d} X \), \( Y_n \xrightarrow{V} y \), \( \eta_n \xrightarrow{V} a \), where \( a \) is a constant and \( y \) is a constant vector, and \( \mathbb{V}(\|X\| > \lambda) \to 0 \) as \( \lambda \to \infty \). Then \( (X_n, Y_n, \eta_n) \xrightarrow{d} (X, y, a) \), and as a result, \( \eta_nX_n + Y_n \xrightarrow{d} aX + y \).

**Remark 4.3** Suppose \( X_n \xrightarrow{d} X \). Then \( \mathbb{V}(\|X\| > \lambda) \to 0 \) as \( \lambda \to \infty \) is equivalent to that \( \{X_n; n \geq 1\} \) is tight, i.e.,

\[
\lim_{\lambda \to \infty} \limsup_{n \to \infty} \mathbb{V}(\|X_n\| > \lambda) = 0,
\]

because for all \( \epsilon > 0 \), one can define a continuous function \( \varphi(x) \) such that \( I\{x > \lambda + \epsilon\} \leq \varphi(x) \leq I\{x > \lambda\} \) and so

\[
\mathbb{V}(\|X\| > \lambda + \epsilon) \leq \mathbb{E}[\varphi(\|X\|)] = \mathbb{E}[\varphi(\|X_n\|)] \leq \limsup_{n \to \infty} \mathbb{V}(\|X_n\| > \lambda),
\]

\[
\limsup_{n \to \infty} \mathbb{V}(\|X_n\| > \lambda + \epsilon) \leq \lim_{n \to \infty} \mathbb{E}[\varphi(\|X_n\|)] = \mathbb{E}[\varphi(\|X\|)] \leq \mathbb{V}(\|X\| > \lambda).
\]

**Lemma 4.3** Suppose \( X_n \xrightarrow{d} X \).

(a) If \( \{X_n\} \) is uniformly integrable and \( \mathbb{E}[(\|X\| - b)^+] \to 0 \) as \( b \to \infty \), then

\[
\mathbb{E}[X_n] \to \mathbb{E}[X]. \tag{4.6}
\]

(b) If \( \sup_n \mathbb{E}[\|X_n\|^q] < \infty \) and \( \mathbb{E}[\|X\|^q] < \infty \) for some \( q > 1 \), then \( \mathbb{E}[X_n] \to \mathbb{E}[X] \).

The following lemma is proved by Zhang (2015).

**Lemma 4.4** Suppose that \( X_n \xrightarrow{d} X \), \( Y_n \xrightarrow{d} Y \), \( Y_n \) is independent to \( X_n \) under \( \mathbb{E} \) and \( \mathbb{V}(\|X\| > \lambda) \to 0 \) and \( \mathbb{V}(\|Y\| > \lambda) \to 0 \) as \( \lambda \to \infty \). Then \( (X_n, Y_n) \xrightarrow{d} (X, Y) \), where \( X = X \), \( Y = Y \) and \( Y \) is independent to \( X \) under \( \mathbb{E} \).
The next lemma is on the Rosenthal type inequalities due to Zhang (2014).

**Lemma 4.5** Let \( \{X_1, \ldots, X_n\} \) be a sequence of independent random variables in \((\Omega, \mathcal{F}, \hat{E})\).

(a) Suppose \( p \geq 2 \). Then

\[
\hat{E} \left[ \max_{k \leq n} |S_k|^p \right] \leq C_p \left\{ \sum_{k=1}^{n} \hat{E}[|X_k|^p] + \left( \sum_{k=1}^{n} \hat{E}[|X_k|^2] \right)^{p/2} \right. \\
+ \left. \left( \sum_{k=1}^{n} \left[ (\hat{E}[X_k])^- + (\hat{E}[X_k])^+ \right] \right)^p \right\}. \tag{4.7}
\]

(b) Suppose \( \hat{E}[X_k] \leq 0, k = 1, \ldots, n \). Then

\[
\hat{E} \left[ \max_{k \leq n} (S_n - S_k)^p \right] \leq 2^{2-p} \sum_{k=1}^{n} \hat{E}[|X_k|^p], \quad \text{for } 1 \leq p \leq 2 \tag{4.8}
\]

and

\[
\hat{E} \left[ \max_{k \leq n} (S_n - S_k)^p \right] \leq C_p \left\{ \sum_{k=1}^{n} \hat{E}[|X_k|^p] + \left( \sum_{k=1}^{n} \hat{E}[|X_k|^2] \right)^{p/2} \right\}
\leq C_p n^{p/2-1} \sum_{k=1}^{n} \hat{E}[|X_k|^p], \quad \text{for } p \geq 2. \tag{4.9}
\]

**Lemma 4.6** Suppose \( \hat{E}[X_1] = \hat{E}[-X_1] = 0 \) and \( \hat{E}[X_1^2] < \infty \). Let \( X_{n,k} = (-\sqrt{n}) \vee X_k \land \sqrt{n}, \hat{X}_{n,k} = X_k - X_{n,k}, \hat{S}_{n,k}^X = \sum_{j=1}^{k} X_{n,j} \) and \( \hat{S}_{n,k} = \sum_{j=1}^{k} \hat{X}_{n,j}, k = 1, \ldots, n \).

Then

\[
\hat{E} \left[ \max_{k \leq n} \left| \frac{\hat{S}_{n,k}^X}{\sqrt{n}} \right|^q \right] \leq C_q, \quad \text{for all } q \geq 2,
\]

and

\[
\lim_{n \to \infty} \hat{E} \left[ \max_{k \leq n} \left| \frac{\hat{S}_{n,k}^X}{\sqrt{n}} \right|^p \right] = 0
\]

whenever \( \hat{E}[|X_1|^p - b^+] \to 0 \) as \( b \to \infty \) if \( p = 2 \), and \( \hat{E}[|X_1|^p] < \infty \) if \( p > 2 \).

**Proof.** Note \( \hat{E}[X_1] = \hat{E}[X_1] = 0 \). So, \( |\hat{E}[\hat{X}_{n,1}]| = |\hat{E}[X_1] - \hat{E}[\hat{X}_{n,1}]| \leq \hat{E}[|\hat{X}_{n,1}|] \leq \hat{E}[|X_1|^2 - n^+]n^{-1/2} \) and \( |\hat{E}[\hat{X}_{n,1}]| = |\hat{E}[X_1] - \hat{E}[\hat{X}_{n,1}]| \leq \hat{E}[|\hat{X}_{n,1}|] \leq \hat{E}[|X_1|^2 -
\[ n^+ n^{-1/2}. \] By Rosenthal’s inequality (c.f, (4.7)),
\[
\hat{E}\left[ \max_{k \leq n} \left| S_{n,k}^X \right|^q \right] \leq C_p \left\{ n\hat{E}[| \mathcal{X}_{n,1} |^q] + \left( n\hat{E}[| \mathcal{X}_{n,1} |^2] \right)^{q/2} \right. \\
+ \left( n\left[ (\hat{E}[\mathcal{X}_{n,1}])^- + (\hat{E}[\mathcal{X}_{n,1}])^+ \right] \right)^q \right\} \\
\leq C_q \left\{ nn^{q/2-1} \hat{E}[|X_1|^2] + n^{q/2} \left( \hat{E}[|X_1|^q] \right)^{q/2} + \left( nn^{-1/2} \hat{E}[(X_1^2 - n)] \right)^q \right\} \\
\leq C_q n^{q/2} \left\{ \hat{E}[|X_1|^2] + \left( \hat{E}[|X_1|^q] \right)^q \right\}, \text{ for all } q \geq 2
\]
and
\[
\hat{E}\left[ \max_{k \leq n} \left| S_{n,k}^X \right|^p \right] \leq C_p \left\{ n\hat{E}[| \mathcal{X}_{n,1} |^p] + \left( n\hat{E}[| \mathcal{X}_{n,1} |^2] \right)^{p/2} \right. \\
+ \left( n\left[ (\hat{E}[\mathcal{X}_{n,1}])^- + (\hat{E}[\mathcal{X}_{n,1}])^+ \right] \right)^p \right\} \\
\leq C_p \left\{ n\hat{E}\left[ (|X_1|^p - n^{p/2})^+ \right] + n^{p/2} \left( \hat{E}[(X_1^2 - n)] \right)^{p/2} \right. \\
+ n^{p/2} \left( \hat{E}[(X_1^2 - n)] \right)^p \right\}, \text{ for } p \geq 2.
\]
The proof is completed. □

**Lemma 4.7** (a) Suppose \( p \geq 2, \) \( \hat{E}[X_1] = \hat{E}[-X_1] = 0, \) \( \hat{E}[(X_1^2 - b)^+] \to 0 \) as \( b \to \infty \)
and \( \hat{E}[|X_1|^p] < \infty. \) Then
\[
\left\{ \max_{k \leq n} \left| \frac{S_{n,k}^X}{\sqrt{n}} \right|^p \right\}_{n=1}^\infty \text{ is uniformly integrable and so is tight.}
\]
(b) Suppose \( p \geq 1, \) \( \hat{E}[(|Y_1| - b)^+] \to 0 \) as \( b \to \infty, \) and \( \hat{E}[|Y_1|^p] < \infty. \) Then
\[
\left\{ \max_{k \leq n} \left| \frac{S_{n,k}^Y}{n} \right|^p \right\}_{n=1}^\infty \text{ is uniformly integrable and so is tight.}
\]

**Proof.** (a) follows from Lemma 4.5. (b) is obvious by noting
\[
\hat{E} \left[ \left( \left( \frac{\max_{k \leq n} |S_{k,n}^Y|}{n} - b \right)^+ \right)^p \right] \leq \hat{E} \left[ \left( \frac{\sum_{k=1}^n |Y_k| - b}{n} \right)^+ \right]^p \\
\leq C_p \left( \sum_{k=1}^p \hat{E}[(|Y_k| - b)^+] \right)^p \\
+ C_p \hat{E} \left[ \left( \sum_{k=1}^p \{(|Y_k| - b)^+ - \hat{E}[(|Y_k| - b)^+] \} \right)^+ \right]^{n^p} \\
\leq C_p \left( \hat{E}[(|Y_1| - b)^+] \right)^p + C_p \left( n^{-p/2} + n^{1-p} \right) \hat{E}[(|Y_1|^p - b)^+] \\
\]
by the Rosenthal type inequalities (4.8) and (4.9). □
Lemma 4.8 Suppose \( \hat{E}[|Y_1| - b]^+ \to 0 \) as \( b \to \infty \). Then for any \( \epsilon > 0 \),

\[
\forall \left( \frac{S_n Y}{n} > \hat{E}[Y_1] + \epsilon \right) \to 0 \quad \text{and} \quad \forall \left( \frac{S_n Y}{n} < \hat{E}[Y_1] - \epsilon \right) \to 0.
\]

Proof. Let \( Y_{k,b} = (-b) \vee Y_k \wedge b \), \( S_{n,1} = \sum_{k=1}^n Y_{k,b} \) and \( S_{n,2} = S_n^Y - S_{n,1} \). Note \( \hat{E}[Y_{1,b}] \to \hat{E}[Y_1] \) as \( b \to \infty \). Suppose \( |\hat{E}[Y_{1,b}] - \hat{E}[Y_1]| < \epsilon/4 \). Then by Kolmogorov’s inequality (c.f. (4.8)),

\[
\forall \left( \frac{S_{n,1}}{n} > \hat{E}[Y_1] + \epsilon/2 \right) \leq \forall \left( \frac{S_{n,1}}{n} > \hat{E}[Y_{k,b}] + \epsilon/4 \right)
\]

\[
\leq \frac{16}{n^2} \hat{E} \left[ \left( \sum_{k=1}^n (Y_{k,b} - \hat{E}[Y_{k,b}]) \right)^2 \right]
\]

\[
\leq \frac{32}{n^2} \hat{E} \left[ (Y_{k,b} - \hat{E}[Y_{k,b}])^2 \right] \leq \frac{32(2b)^2}{n^2} \to 0.
\]

On the other hand,

\[
\forall \left( \frac{S_{n,2}}{n} > \epsilon/2 \right) \leq \frac{2}{n} \sum_{k=1}^n \hat{E}|Y_k - Y_{k,b}| \leq \frac{2}{\epsilon} \hat{E}[|Y_1| - b]^+ \to 0 \quad \text{as} \quad b \to \infty.
\]

It follows that

\[
\forall \left( \frac{S_n Y}{n} > \hat{E}[Y_1] + \epsilon \right) \to 0.
\]

By considering \( \{-Y_k\} \) instead, we have

\[
\forall \left( \frac{S_n Y}{n} < \hat{E}[Y_1] - \epsilon \right) = \forall \left( \frac{-S_n Y}{n} > \hat{E}[-Y_1] + \epsilon \right) \to 0. \quad \square
\]

Proof of Theorem 4.1 We first show the tightness of \( \tilde{W}_n \). It is easily seen that

\[
w_\delta \left( \frac{\tilde{S}_n Y(\cdot)}{n} \right) \leq 2\delta b + \frac{\sum_{k=1}^n (|Y_k| - b)^+}{n}.
\]

It follows that for any \( \epsilon > 0 \), if \( \delta < \epsilon/(4b) \), then

\[
\sup_n \forall \left( w_\delta \left( \frac{\tilde{S}_n Y(\cdot)}{n} \right) \geq \epsilon \right) \leq \sup_n \forall \left( \sum_{k=1}^n (|Y_k| - b)^+ \geq n\frac{\epsilon}{2} \right) \leq \frac{2}{\epsilon} \hat{E} \left[ (|Y_1| - b)^+ \right].
\]

Letting \( \delta \to 0 \) and then \( b \to \infty \) yields

\[
\sup_n \forall \left( w_\delta \left( \frac{\tilde{S}_n Y(\cdot)}{n} \right) \geq \epsilon \right) \to 0 \quad \text{as} \quad \delta \to 0.
\]
For any $\eta > 0$, we choose $\delta_k \downarrow 0$ such that, if

$$A_k = \left\{ x : \omega_{\delta_k}(x) < \frac{1}{k} \right\},$$

then $\sup_n \mathbb{V}\left(\tilde{S}_n^Y(\cdot)/n \in A_k^c\right) \leq \eta/2^{k+1}$. Let $A = \{ x : |x(0)| \leq a \}$, $K_2 = A \cap_{k=1}^\infty A_k$. Then by the Arzelà-Ascoli theorem, $K_2 \subset C_b(C[0,1])$ is compact. It is obvious that $\{\tilde{S}_n^Y(\cdot)/n \notin A\} = \emptyset$ since $\tilde{S}_n^Y(0)/n = 0$. Next, we show that

$$\mathbb{V}(\tilde{S}_n^Y(\cdot)/n \in K^c) \leq \mathbb{V}(\tilde{S}_n^Y(\cdot)/n \in A^c) + \sum_{k=1}^\infty \mathbb{V}(\tilde{S}_n^Y(\cdot)/n \in A_k^c).$$

Note that when $\delta < 1/(2n)$,

$$\omega_{\delta}(\tilde{S}_n^Y(\cdot)/n) \leq 2n|t - s| \max_{i \leq n}|Y_i|/n \leq 2\delta \max_{i \leq n}|Y_i|.$$ 

Choose a $k_0$ such that $\delta_k < 1/(2Mk)$ for $k \geq k_0$. Then on the event $E = \{ \max_{i \leq n}|Y_i| \leq M \}$, $\{\tilde{S}_n^Y(\cdot)/n \in A_k^c\} = \emptyset$ for $k \geq k_0$. So, by the (finite) sub-additivity of $\mathbb{V}$,

$$\mathbb{V}(E \cap \{\tilde{S}_n^Y(\cdot)/n \in K^c\}) \leq \mathbb{V}(E \cap \{\tilde{S}_n^Y(\cdot)/n \in A^c\}) + \sum_{k=1}^{k_0} \mathbb{V}(E \cap \{\tilde{S}_n^Y(\cdot)/n \in A_k^c\})$$

$$\leq \mathbb{V}(\tilde{S}_n^Y(\cdot)/n \in A^c) + \sum_{k=1}^\infty \mathbb{V}(\tilde{S}_n^Y(\cdot)/n \in A_k^c).$$

On the other hand,

$$\mathbb{V}(E^c) \leq \frac{\hat{\mathbb{E}}[\max_{i \leq n}|Y_i|]}{M} \leq \frac{n\hat{\mathbb{E}}[|Y_1|]}{M}.$$ 

It follows that

$$\mathbb{V}(\tilde{S}_n^Y(\cdot)/n \in K_2^c) \leq \mathbb{V}(\tilde{S}_n^Y(\cdot)/n \in A^c) + \sum_{k=1}^\infty \mathbb{V}(\tilde{S}_n^Y(\cdot)/n \in A_k^c) + \frac{n\hat{\mathbb{E}}[|Y_1|]}{M}.$$ 

Letting $M \to \infty$ yields

$$\mathbb{V}(\tilde{S}_n^Y(\cdot)/n \in K_2^c) \leq \mathbb{V}(\tilde{S}_n^Y(\cdot)/n \in A^c) + \sum_{k=1}^\infty \mathbb{V}(\tilde{S}_n^Y(\cdot)/n \in A_k^c)$$

$$< 0 + \sum_{k=1}^\infty \frac{\eta}{2^{k+1}} < \frac{\eta}{2}.$$ 

We conclude that for any $\eta > 0$, there exists a compact $K_2 \subset C_b(C[0,1])$ such that

$$\sup_n \hat{\mathbb{E}}^n\left[ I \left\{ \frac{\tilde{S}_n^Y(\cdot)}{n} \notin K_2 \right\} \right] = \sup_n \mathbb{V}\left\{ \frac{\tilde{S}_n^Y(\cdot)}{n} \notin K_2 \right\} < \eta/2. \quad (4.10)$$
Next, we show that for any \( \eta > 0 \), there exists a compact \( K_1 \subset C_b(C[0,1]) \) such that
\[
\sup_n \mathbb{E}^* \left[ I \left\{ \frac{S_n^X(\cdot)}{\sqrt{n}} \not\in K_1 \right\} \right] = \sup_n \mathbb{V} \left\{ \frac{S_n^X(\cdot)}{\sqrt{n}} \not\in K_1 \right\} < \eta/2. \tag{4.11}
\]

Similar to (4.10), it is sufficient to show that
\[
\sup_n \mathbb{V} \left( w_\delta \left( \frac{S_n^X(\cdot)}{\sqrt{n}} \right) \geq \epsilon \right) \rightarrow 0 \text{ as } \delta \rightarrow 0. \tag{4.12}
\]

With the same argument of Billingsley (1968, Pages 56-59, c.f., (8.12)), for large \( n \),
\[
\mathbb{V} \left( w_\delta \left( \frac{S_n^X(\cdot)}{\sqrt{n}} \right) \geq 3\epsilon \right) \leq \frac{2}{\delta} \mathbb{V} \left( \max_{i \leq [n\delta]} |S_i^X| \geq \epsilon \right) \leq \frac{4}{\epsilon^2} \mathbb{E} \left( \max_{i \leq [n\delta]} \frac{|S_i^X|^2}{\sqrt{n}} - \frac{\epsilon^2}{2\delta} \right). \]

It follows that
\[
\lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} \sup \mathbb{V} \left( w_\delta \left( \frac{S_n^X(\cdot)}{\sqrt{n}} \right) \geq 3\epsilon \right) = 0
\]
by Lemma 4.7 (a) where \( p = 2 \). On the other hand, for fixed \( n \), if \( \delta < 1/(2n) \) then
\[
\omega_\delta(S_n^X(\cdot)/n) \leq 2n|t-s| \max_{i \leq n} |X_i|/\sqrt{n} \leq 2\delta \sqrt{n} \max_{i \leq n} |X_i|.
\]
It follows that
\[
\lim_{\delta \rightarrow 0} \mathbb{V} \left( w_\delta \left( \frac{S_n^X(\cdot)}{\sqrt{n}} \right) \geq \epsilon \right) = 0
\]
for each \( n \). It follows that (4.12) holds.

Now, by combing (4.10) and (4.11) we obtain the tightness of \( \tilde{W}_n \) as follows.
\[
\sup_n \mathbb{E}^* \left[ I \left\{ \tilde{W}_n(\cdot) \not\in K_1 \times K_2 \right\} \right] < \eta. \tag{4.13}
\]

Define \( \widehat{E}_n \) by
\[
\widehat{E}_n[\varphi] = \mathbb{E} \left[ \varphi(\tilde{W}_n(\cdot)) \right], \quad \varphi \in C_b(C[0,1] \times C[0,1]).
\]

Then the sequence of sub-linear expectations \( \{\widehat{E}_n\}_{n=1}^\infty \) is tight by (4.13). By Theorem 9 of Peng (2010b), \( \{\widehat{E}_n\}_{n=1}^\infty \) is weakly compact, namely, for each subsequence \( \{\tilde{E}_{n_k}\}_{k=1}^\infty \), \( n_k \rightarrow \infty \), there exists a further subsequence \( \{\tilde{E}_{m_j}\}_{j=1}^\infty \subset \{\tilde{E}_{n_k}\}_{k=1}^\infty \), \( m_j \rightarrow \infty \), such that, for each \( \varphi \in C_b(C[0,1] \times C[0,1]) \), \( \{\tilde{E}_{m_j} [\varphi]\} \) is a Cauchy sequence. Define \( \mathbb{F} [\cdot] \) by
\[
\mathbb{F} [\varphi] = \lim_{j \rightarrow \infty} \tilde{E}_{m_j} [\varphi], \quad \varphi \in C_b(C[0,1] \times C[0,1]).
\]
Let $\overline{\Omega} = C[0,1] \times C[0,1]$, and $(\xi_t, \eta_t)$ be the the canonical process $\xi_t(\omega) = \omega_t^{(1)}$, $\eta_t(\omega) = \omega_t^{(2)}$ $(\omega = (\omega^{(1)}, \omega^{(2)}) \in \overline{\Omega})$. Then

$$\mathbb{E} \left[ \varphi(\overline{W}_{m_j}(\cdot)) \right] \to \mathbb{F}[\varphi(\xi, \eta)], \quad \varphi \in C_b(C[0,1] \times C[0,1]).$$ \hspace{1cm} (4.14)

The topological completion of $C_b(\overline{\Omega})$ under the Banach norm $\mathbb{F}[\| \cdot \|]$ is denoted by $L_\mathbb{F}(\overline{\Omega})$. $\mathbb{F}[]$ can be extended uniquely to a sub-linear expectation on $L_\mathbb{F}(\overline{\Omega})$.

Next, it is sufficient to show that $(\xi_t, \eta_t)$ defined on the sub-linear space $(\overline{\Omega}, L_\mathbb{F}(\overline{\Omega}), \mathbb{F})$ satisfies (i)-(v) and so $(\xi, \eta) \overset{d}{=} (B, b)$, which means that the limit distribution of any subsequence of $\overline{W}_n(\cdot)$ is uniquely determined.

(i) is obvious.

Let $0 \leq t_1 \leq \ldots \leq t_k \leq s \leq t+s$. By (4.14), for any bounded continuous function $\varphi : \mathbb{R}^{2(k+1)} \to \mathbb{R}$ we have

$$\mathbb{E} \left[ \varphi(\overline{W}_{m_j}(t_1), \ldots, \overline{W}_{m_j}(t_k), \overline{W}_{m_j}(s+t) - \overline{W}_{m_j}(s)) \right] \to \mathbb{F} \left[ \varphi((\xi_{t_1}, \eta_{t_1}), \ldots, (\xi_{t_k}, \eta_{t_k}), (\xi_{s+t} - \xi_s, \eta_{s+t} - \eta_s)) \right].$$

Note

$$\sup_{0 \leq t \leq 1} \frac{|\tilde{S}_n^X(t) - S_{[n]}^X|}{\sqrt{n}} \leq \max_{k \leq n} \frac{|X_k|}{\sqrt{n}} \rightarrow 0,$$

$$\sup_{0 \leq t \leq 1} \frac{|\tilde{S}_n^Y(t) - S_{[n]}^Y|}{n} \leq \max_{k \leq n} \frac{|Y_k|}{n} \rightarrow 0.$$ 

It follows that by Lemmas 4.2 and 4.7,

$$\mathbb{E} \left[ \varphi \left( \left( \frac{S_{[n]}^X}{m_j}, \frac{S_{[n]}^Y}{m_j} \right), \ldots, \left( \frac{S_{[n]}^X}{m_j}, \frac{S_{[n]}^Y}{m_j} \right), \left( \frac{S_{[n]}^X - S_{[n]}^X}{m_j}, \frac{S_{[n]}^Y - S_{[n]}^Y}{m_j} \right) \right) \right]$$

$$\to \mathbb{F} \left[ \varphi\left( (\xi_{t_1}, \eta_{t_1}), \ldots, (\xi_{t_k}, \eta_{t_k}), (\xi_{s+t} - \xi_s, \eta_{s+t} - \eta_s) \right) \right].$$ \hspace{1cm} (4.15)

In particular,

$$\left( \frac{S_{[n]}^X - S_{[n]}^X}{m_j}, \frac{S_{[n]}^Y - S_{[n]}^Y}{m_j} \right) \overset{d}{=} \left( \frac{S_{[n]}^X - S_{[n]}^X}{m_j}, \frac{S_{[n]}^Y - S_{[n]}^Y}{m_j} \right)$$

$$\overset{d}{\to} (\xi_{s+t} - \xi_s, \eta_{s+t} - \eta_s).$$

It follows that

$$\left( \frac{S_{[n]}^X}{m_j}, \frac{S_{[n]}^Y}{m_j} \right) \overset{d}{\to} (\xi_{s+t} - \xi_s, \eta_{s+t} - \eta_s).$$ \hspace{1cm} (4.16)
Hence,
\[ \mathbb{F} [\phi(\xi_{s+t} - \xi_s, \eta_{s+t} - \eta_s)] = \mathbb{F} [\phi(\xi_t, \eta_t)] \quad \text{for all } \phi \in C_b(\mathbb{R}^2). \quad (4.17) \]

Next, we show that
\[ \mathbb{F}[|\xi_{s+t} - \xi_s|^p] \leq C_p t^{p/2} \quad \text{and} \quad \mathbb{F}[|\eta_{s+t} - \eta_s|^p] \leq C_p t^p, \quad \text{for all } p \geq 2 \text{ and } t, s \geq 0. \quad (4.18) \]

By Lemma 4.8
\[ \tilde{V}(t\mu - \epsilon \leq \eta_{s+t} - \eta_s \leq t\mu + \epsilon) = 1 \quad \text{for all } \epsilon > 0. \quad (4.19) \]

It follows that
\[ \mathbb{F}[|\eta_{s+t} - \eta_s|^p] \leq t^p |\tilde{E}[|Y_t|]|^p. \]

On the other hand, let \( S_{n,k}^X \) and \( \tilde{S}_{n,k}^X \) be defined as in Lemma 4.6. Then \( S_k^X = S_{n,k}^X + \tilde{S}_{n,k}^X \). By (4.16) and Lemmas 4.6 and 4.2

\[ \frac{S_{[m,t],[m,t]}}{\sqrt{M_j}} \xrightarrow{d} \xi_{s+t} - \xi_s \quad \text{and} \quad \tilde{E} \left[ \left| \frac{S_{[m,t],[m,t]}}{\sqrt{M_j}} \right|^p \right] \leq C_p t^{p/2}, \quad p \geq 2. \]

It follows that
\[ \mathbb{F} \left[ |\xi_{s+t} - \xi_s|^p \wedge b \right] = \lim_{n \to \infty} \tilde{E} \left[ \left| \frac{S_{[m,t],[m,t]}}{\sqrt{M_j}} \right|^p \wedge b \right] \leq C_p t^{p/2}, \quad \text{for any } b > 0. \]

Hence,
\[ \mathbb{F} \left[ |\xi_{s+t} - \xi_s|^p \right] = \lim_{b \to \infty} \mathbb{F} \left[ |\xi_{s+t} - \xi_s|^p \wedge b \right] \leq C_p t^{p/2} \]

by the completeness of \((\tilde{\Omega}, L_{\mathcal{F}}(\tilde{\Omega}), \mathbb{F})\). (4.18) is proved.

Now, note that \((X_i, Y_i), i = 1, 2, \ldots, \) are independent and identically distributed. By (4.15) and Lemma 4.4 it is easily seen that \((\xi_s, \eta_s)\) satisfies (4.3) for \( \phi \in C_b(\mathbb{R}^{2(k+1)}) \).

Note that, by (4.18), the random variables concerned in (4.3) and (4.17) have finite moments of each order. The function space \( C_b(\mathbb{R}^{2(k+1)}) \) and \( C_b(\mathbb{R}^2) \) can be extended to \( C_{l,Lip}(\mathbb{R}^{2(k+1)}) \) and \( C_{l,Lip}(\mathbb{R}^2) \), respectively, by elemental arguments. So, (ii) and (iii) is proved.

For (iv) and (v), we let \( \phi : \mathbb{R}^2 \to \mathbb{R} \) be a bounded Lipschitz function and consider
\[ u(x, y, t) = \mathbb{F} [\phi(x + \xi_t, y + \eta_t)]. \]
It is sufficient to show that $u$ is a viscosity solution of the PDE (4.2). In fact, due to the uniqueness of the viscosity solution, we will have

$$
\mathbb{F} \left[ \varphi(x + \xi_t, y + \eta_t) \right] = \mathbb{E} \left[ \varphi(x + \sqrt{t} \xi, y + t\eta) \right], \quad \varphi \in C_{b,Lip}(\mathbb{R}^2).
$$

Taking $x = 0$ and $y = 0$ yields (iv) and (v).

To verify the PDE (4.2), firstly it is easily seen that

$$
\mathbb{E} \left[ \frac{q}{2} \left( \frac{S^X_{[nt]}}{\sqrt{n}} \right)^2 + p \frac{S^Y_{[nt]}}{n} \right] = \frac{[nt]}{n} \mathbb{E} \left[ \frac{q}{2} \left( \frac{S^X_{[nt]}}{\sqrt{[nt]}} \right)^2 + p \frac{S^Y_{[nt]}}{[nt]} \right] = \frac{[nt]}{n} G(p, q).
$$

Note that $\left\{ \frac{q}{2} \left( \frac{S^X_{[nt]}}{\sqrt{n}} \right)^2 + p \frac{S^Y_{[nt]}}{n} \right\}$ is uniformly integrable by Lemma 4.7. By Lemma 4.3 we conclude that

$$
\mathbb{F} \left[ \frac{q}{2} \xi_t^2 + p \eta_t \right] = \lim_{m_i \to \infty} \mathbb{E} \left[ \frac{q}{2} \left( \frac{S^X_{[m_i,t]}}{\sqrt{m_i}} \right)^2 + p \frac{S^Y_{[m_i,t]}}{m_i} \right] = tG(p, q).
$$

Also, it is easy to verify that $|u(x, y, t) - u(\overline{x}, \overline{y}, t)| \leq C(|x - \overline{x}| + |y - \overline{y}|)$, $|u(x, y, t) - u(x, y, s)| \leq C\sqrt{|t - s|}$ by Lipschitz continuity of $\varphi$, and

$$
u(x, y, t) = \mathbb{E} \left[ \varphi(x + \xi_s + \xi_t - \xi_s, y + \eta_s + \eta_t - \eta_s) \right] = \mathbb{E} \left[ \mathbb{E} \left[ \varphi(x + \overline{x} + \xi_t - \xi_s, y + \overline{y} + \eta_t - \eta_s) \right] | (\overline{x}, \overline{y}) = (\xi_s, \eta_s) \right] = \mathbb{E} \left[ u(x + \xi_s, y + \eta_s, t - s) \right], \quad 0 \leq s \leq t.
$$

Let $\psi(\cdot, \cdot, \cdot) \in C^3_{b}(\mathbb{R}, \mathbb{R}, [0, 1])$ be a smooth function with $\psi \geq u$ and $\psi(x, y, t) = u(x, y, t)$. Then

$$
0 = \mathbb{E} \left[ u(x + \xi_s, y + \eta_s, t - s) - u(x, y, t) \right] \leq \mathbb{E} \left[ \psi(x + \xi_s, y + \eta_s, t - s) - \psi(x, y, t) \right]
$$

$$
= \mathbb{E} \left[ \partial_x \psi(x, y, t) \xi_s + \frac{1}{2} \partial_{xx} \psi(x, y, t) \xi_s^2 + \partial_y \psi(x, y, t) \eta_s - \partial_t \psi(x, y, t)s + I_s \right]
$$

$$
\leq \mathbb{E} \left[ \partial_x \psi(x, y, t) \xi_s + \frac{1}{2} \partial_{xx} \psi(x, y, t) \xi_s^2 + \partial_y \psi(x, y, t) \eta_s - \partial_t \psi(x, y, t)s \right] + \mathbb{E} \left[ |I_s| \right]
$$

$$
= \mathbb{E} \left[ \frac{1}{2} \partial_{xx} \psi(x, y, t) \xi_s^2 + \partial_y \psi(x, y, t) \eta_s \right] - \partial_t \psi(x, y, t)s + \mathbb{E} \left[ |I_s| \right]
$$

$$
= sG(\partial_y \psi(x, y, t), \partial_{xx} \psi(x, y, t)) - s\partial_t \psi(x, y, t) + \mathbb{E} \left[ |I_s| \right],
$$

where

$$
|I_s| \leq C(|\xi_s|^3 + |\eta_s|^2 + s^2).
$$
By (4.18), we have \( \mathbb{F}[|I_n|] \leq C(s^{3/2}+s^2+s^2) = o(s) \). It follows that \( [\partial_t \psi - G(\partial_y \psi, \partial^2_{xx})](x, y, t) \leq 0 \). Thus \( u \) is a viscosity subsolution of (4.2). Similarly we can prove that \( u \) is a viscosity supersolution of (4.2). Hence (4.4) is proved.

As for (4.5), let \( \varphi : C[0, 1] \times C[0, 1] \to \mathbb{R} \) be a continuous function with \( |\varphi(x, y)| \leq C_0(1 + \|x\|^p + \|y\|^q) \). For \( \lambda > 4C_0 \), let \( \varphi(x, y) = (-\lambda) \vee (\varphi(x, y) \wedge \lambda) \in C_b(C[0, 1]) \).

It is easily seen that \( \varphi(x, y) = \varphi(x, y) \) if \( |\varphi(x, y)| \leq \lambda \). If \( |\varphi(x, y)| > \lambda \), then

\[
|\varphi(x, y) - \varphi(x, y)| = |\varphi(x, y)| - \lambda \leq C_0(1 + \|x\|^p + \|y\|^q) - \lambda
\leq C_0 \left\{ \left( \|x\|^p - \lambda/(4C_0) \right)^+ + \left( \|y\|^q - \lambda/(4C_0) \right)^+ \right\}.
\]

Hence

\[
|\varphi(x, y) - \varphi(x, y)| \leq C_0 \left\{ \left( \|x\|^p - \lambda/(4C_0) \right)^+ + \left( \|y\|^q - \lambda/(4C_0) \right)^+ \right\}.
\]

It follows that

\[
\lim_{\lambda \to \infty} \lim_{n \to \infty} \ \inf_{S \in \Omega_n} \left[ \mathbb{E}^s \left[ \varphi \left( \mathbb{W}_n(\cdot) \right) \right] - \mathbb{E} \left[ \varphi \left( \mathbb{W}_n(\cdot) \right) \right] \right]
\leq \lim_{\lambda \to \infty} \lim_{n \to \infty} C_0 \left\{ \mathbb{E} \left[ \left( \max_{k \leq n} \left| S_k^x \right|^p - \frac{\lambda}{4C_0} \right)^+ \right] + \mathbb{E} \left[ \left( \max_{k \leq n} \left| S_k^y \right|^q - \frac{\lambda}{4C_0} \right)^+ \right] \right\}
= 0,
\]

by Lemma 4.7. On the other hand, by (4.4),

\[
\lim_{n \to \infty} \mathbb{E} \left[ \varphi (\mathbb{W}_n(\cdot)) \right] = \mathbb{E} \left[ \varphi (B, b.) \right] \to \mathbb{E} \left[ \varphi (B, b.) \right] \quad \text{as} \quad \lambda \to \infty.
\]

(4.5) is proved, and the proof Theorem 4.1 is now completed. \( \square \)

**Proof of Remark 4.1.** When \( X_k \) and \( Y_k \) are \( d \)-dimensional random vectors, the tightness (4.13) of \( \mathbb{W}_n(\cdot) \) also follows because each sequence of the components of the vector \( \mathbb{W}_n(\cdot) \) is tight. Also, (4.18) remains true because each component has this property. On the other hand, it follows that

\[
\mathbb{F} \left[ \frac{1}{2} \left< A \xi_t, \xi_t \right> + \left< p, \eta_t \right> \right] = \lim_{m_j \to \infty} \mathbb{E} \left[ \frac{1}{2} \left< A \frac{S^X_{[m_j,t]}}{\sqrt{m_j}}, \frac{S^X_{[m_j,t]}}{\sqrt{m_j}} \right> + \left< p, \frac{S^Y_{[m_j,t]}}{m_j} \right> \right]
= \lim_{m_j \to \infty} \frac{[m_j,t]}{m_j} G(p, A) = tG(p, A).
\]

The remainder proof is the same as that of Theorem 4.1. \( \square \)
5 Proof of the self-normalized FCLTs

Let $Y_k = X_k^2$. The function $G(p, q)$ in (4.1) becomes

$$G(p, q) = \hat{E}\left[\left(\frac{q}{2} + p\right)X_1^2\right] = \left(\frac{q}{2} + p\right)^+\sigma^2 - \left(\frac{q}{2} + p\right)^-\sigma^2, \quad p, q \in \mathbb{R}.$$ 

Then the process $(B_t, b_t)$ in (4.4) and the process $(W(t), \langle W \rangle_t)$ are identically distributed.

In fact, note

$$\langle W \rangle_{t+s} - \langle W \rangle_t = (W(t + s) - W(t))^2 - 2 \int_0^s (W(t + x) - W(t))d(W(t + x) - W(t)).$$

It is easy to verify that $(W(t), \langle W \rangle_t)$ satisfies (i)-(iv) for $(B, b)$. It remains to show that $(B_1, b_1) \overset{d}= (W(1), \langle W \rangle_1)$. Let \( \{X_n; n \geq 1\} \) be a sequence of independent and identically distributed random variables with $X_1 \overset{d}= W(1)$. Then by Theorem 4.1,

$$\left(\frac{\sum_{k=1}^n X_k}{\sqrt{n}}, \frac{\sum_{k=1}^n X_k^2}{n}\right) \overset{d}= (B_1, b_1).$$

On the other hand, let $t_k = \frac{k}{n}$. Then

$$\left(\frac{\sum_{k=1}^n X_k}{\sqrt{n}}, \frac{\sum_{k=1}^n X_k^2}{n}\right) \overset{d}= \left(W(1), \sum_{k=1}^n (W(t_k) - W(t_{k-1}))^2\right) \overset{L^2}\to (W(1), \langle W \rangle_1).$$

Hence $(B, b) \overset{d}= (W(\cdot), \langle W \rangle(\cdot))$. We conclude the following proposition from Theorem 4.1.

**Proposition 5.1** Suppose $\hat{E}[(X_1^2 - b)^+] \to 0$ as $b \to \infty$. Then for all bounded continuous function $\psi : C[0, 1] \times C[0, 1] \to \mathbb{R},$

$$\hat{E} \left[ \psi \left( \frac{\tilde{S}_n^X(\cdot)}{\sqrt{n}}, \frac{\tilde{V}_n(\cdot)}{n} \right) \right] \to \tilde{E} \left[ \psi \left( W(\cdot), \langle W \rangle(\cdot) \right) \right],$$

where $\tilde{V}_n(t) = V_{[nt]} + (nt - [nt])X_{[nt]+1}^2$, and in particular, for all bounded continuous function $\psi : C[0, 1] \times \mathbb{R} \to \mathbb{R},$

$$\hat{E} \left[ \psi \left( \frac{\tilde{S}_n^X(\cdot)}{\sqrt{n}}, \frac{V_n}{n} \right) \right] \to \tilde{E} \left[ \psi \left( W(\cdot), \langle W \rangle_1 \right) \right]. \quad (5.1)$$
Now, we begin the proof of Theorem 3.1. Let \( a = \sigma^2/2 \) and \( b = 2\sigma^2 \). According to (4.19), we have \( \tilde{V}(\bar{a}^2 - \epsilon < \langle W \rangle_1 < \bar{a}^2 + \epsilon) = 1 \) for all \( \epsilon > 0 \). Let \( \varphi: C[0, 1] \to \mathbb{R} \) be a bounded continuous function. Define
\[
\psi(x, y) = \varphi \left( \frac{x(\cdot)}{\sqrt{a \lor (V_n/n) \land b}} \right), \quad x(\cdot) \in C[0, 1], \; y \in \mathbb{R}.
\]
Then \( \psi: C[0, 1] \times \mathbb{R} \to \mathbb{R} \) is a bounded continuous function. Hence by Proposition 3.1
\[
\mathbb{E} \left[ \varphi \left( \frac{\tilde{S}_n^X(\cdot)/\sqrt{n}}{\sqrt{a \lor (V_n/n) \land b}} \right) \right] \to \mathbb{E} \left[ \varphi \left( \frac{W(\cdot)}{\sqrt{\langle (W) \rangle_1}} \right) \right].
\]
On the other hand,
\[
\limsup_{n \to \infty} \mathbb{E}^{\ast} \left[ \varphi \left( \frac{\tilde{S}_n^X(\cdot)/\sqrt{n}}{\sqrt{V_n/n}} \right) \right] - \mathbb{E} \left[ \varphi \left( \frac{\tilde{S}_n^X(\cdot)/\sqrt{n}}{\sqrt{a \lor (V_n/n) \land b}} \right) \right] \leq C \limsup_{n \to \infty} \mathbb{V} (V_n/n \notin (a, b)) \leq C \tilde{V} (\langle W \rangle_1 \geq 3\sigma^2/2) + C \tilde{V} (\langle W \rangle_1 \leq 2\sigma^2/3) = 0.
\]
It follows that
\[
\mathbb{E}^{\ast} \left[ \varphi \left( \frac{\tilde{S}_n^X(\cdot)/\sqrt{n}}{\sqrt{V_n}} \right) \right] \to \mathbb{E} \left[ \varphi \left( \frac{W(\cdot)}{\sqrt{\langle (W) \rangle_1}} \right) \right].
\]
The proof is now completed. \( \square \)

**Proof of Theorem 3.2** Firstly, note that
\[
\mathbb{E}[X_1^2 \land x^2] \leq \mathbb{E}[X_1^2 \land (kx)^2] \leq \mathbb{E}[X_1^2 \land x^2] + k^2 x^2 \mathbb{V}(|X_1| > x), \quad k \geq 1,
\]
\[
\mathbb{E}[|X_1|^r \land x^r] \leq \mathbb{E}[|X_1|^r \land (\delta x)^r] + \mathbb{E}[(\delta x)^r \lor |X_1|^r \land x^r] \leq \delta^{-2} x^{-2} l(\delta x) + x^r \mathbb{V}(|X_1| \geq \delta x), \quad 0 < \delta < 1, \; r > 2.
\]
The condition (I) implies that \( l(x) \) is slowly varying as \( x \to \infty \) and
\[
\mathbb{E}[|X_1|^r \land x^r] = o(x^{r-2} l(x)), \quad r > 2.
\]
Further
\[
\frac{\mathbb{E}^{\ast}[X_1^2 I\{|X_1| \leq x\}]}{l(x)} \to 1,
\]
\[
C_{\mathbb{V}}(|X_1|^r I\{|X_1| \geq x\}) = \int_{x^r}^{\infty} \mathbb{V}(|X_1|^r \geq y) dy = o(x^{2-r} l(x)), \quad 0 < r < 2.
\]
If the conditions (I) and (III) are satisfied, then
\[ \hat{E}[(|X_1| - x)^+] \leq \hat{E}^{*}[|X_1|I\{|X| \geq x\}] \leq C_{\mathcal{V}}\{|X_1|I\{|X_1| \geq x\}\} = o(x^{-1}l(x)). \]

Now, let \( d_n = \inf \{x : x^{-2}l(x) = t^{-1}\} \). Then \( nl(d_n) = d_n^2 \). Similar to Theorem 3.1, it is sufficient to show that for all bounded continuous function \( \psi : C[0,1] \times C[0,1] \rightarrow \mathbb{R} \),
\[ \hat{E}\left[ \psi\left( \frac{\hat{S}_n(\cdot)}{d_n}, \frac{\hat{V}_n(\cdot)}{d_n^2} \right) \right] \rightarrow \hat{E}[\psi(W(\cdot), \langle W \rangle)] \text{ with } W(1) \sim N(0, [r^{-2}, 1]). \]

Let \( \overline{X}_k = \overline{X}_{k,n} = (-d_n) \vee X_k \wedge d_n, \overline{S}_k = \sum_{i=1}^{k} \overline{X}_i, \overline{V}_k = \sum_{i=1}^{k} \overline{X}_i^2 \). Denote \( \overline{S}_n(t) = \overline{S}_{[nt]} + (nt - [nt])\overline{X}_{[nt]+1} \) and \( \overline{V}_n(t) = \overline{V}_{[nt]} + (nt - [nt])\overline{X}_{[nt]+1}^2 \). Note
\[ \forall (X_k \neq \overline{X}_k \text{ for some } k \leq n) \leq n\mathbb{V}(\{|X_1| \geq d_n\}) = n \cdot o\left( \frac{l(d_n)}{d_n^2} \right) = o(1). \]

It is sufficient to show that for all bounded continuous function \( \psi : C[0,1] \times C[0,1] \rightarrow \mathbb{R} \),
\[ \hat{E}\left[ \psi\left( \frac{\overline{S}_n(\cdot)}{d_n}, \frac{\overline{V}_n(\cdot)}{d_n^2} \right) \right] \rightarrow \hat{E}[\psi(W(\cdot), \langle W \rangle)]. \]

Following the line of the proof of Theorem 4.1, we need only to show that

(a) For any \( 0 < t \leq 1 \),
\[ \limsup_{n \to \infty} \hat{E}\left[ \max_{k \leq [nt]} \left| \frac{\overline{S}_k}{d_n} \right|^p \right] \leq C_p t^{p/2}, \quad \limsup_{n \to \infty} \hat{E}\left[ \max_{k \leq [nt]} \left| \frac{\overline{V}_k}{d_n^2} \right|^p \right] \leq C_p t^p, \quad \forall p \geq 2; \]

(b) For any \( 0 < t \leq 1 \),
\[ \lim_{n \to \infty} \hat{E}\left[ \frac{q}{2} \left( \frac{\overline{S}_{[nt]}}{d_n} \right)^2 + \frac{p}{2} \frac{\overline{V}_{[nt]}}{d_n^2} \right] = tG(p, q), \]
where
\[ G(p, q) = \left( \frac{q}{2} + p \right)^+ - r^{-2} \left( \frac{q}{2} + p \right)^-; \]

(c)
\[ \max_{k \leq n} \frac{|X_k|}{d_n} \xrightarrow{v} 0. \]

In fact, (a) implies the tightness of \( \left( \frac{\overline{S}_n(\cdot)}{d_n}, \frac{\overline{V}_n(\cdot)}{d_n^2} \right) \) and (4.18), and (b) implies the distribution of the limit process is uniquely determined.
Firstly, (c) is obvious since
\[
\forall \left( \max_{k \leq n} |X_k| \geq \epsilon d_n \right) \leq n \forall \left( |X_1| \geq \epsilon d_n \right) = o(1)n \frac{l(\epsilon d_n)}{\epsilon^2 d_n^2} = o(1)n \frac{l(d_n)}{d_n^2} = o(1).
\]
As for (a), by the Rosenthal type inequality (4.7),
\[
\mathbb{E} \left[ \max_{k \leq \lfloor nt \rfloor} \left| \overline{S}_k \right|^p \right] \leq C_p d_n^{-p} \left\{ [nt] \mathbb{E} \left[ |X|_1^p \wedge d_n^p \right] + \left( [nt] \mathbb{E} \left[ |X|_1^2 \wedge d_n^2 \right] \right)^{p/2} \right.
\]
\[
\left. + \left( [nt] \mathbb{E} \left[ (-d_n) \vee X_1 \wedge d_n \right] \right)^p + [nt] \mathbb{E} \left[ (|X_1| - d_n) \right]^p \right\}
\leq C_p d_n^{-p} \left\{ [nt] \mathbb{E} \left[ |X|_1^p \wedge d_n^p \right] + \left( [nt] \mathbb{E} \left[ |X|_1^2 \wedge d_n^2 \right] \right)^{p/2} \right.
\]
\[
\left. + \left( [nt] \mathbb{E} \left[ |X_1| - d_n \right]^p \right) \right\}
\leq o(1) \left( \frac{d_n}{d_n^2} \right)^{p/2} \left( \frac{n l(d_n)}{d_n^2} \right)^{p/2} + o(1) \left( [nt] \mathbb{E} \left[ |X|_1^2 \wedge d_n^2 \right] \right)^p \leq C_p t^{p/2} + o(1), \quad (5.2)
\]
and similarly,
\[
\mathbb{E} \left[ \max_{k \leq \lfloor nt \rfloor} \left| \overline{V}_k \right|^p \right] \leq C_p d_n^{-2p} \left\{ [nt] \mathbb{E} \left[ |X|_1^{2p} \wedge d_n^{2p} \right] + \left( [nt] \mathbb{E} \left[ |X|_1^4 \wedge d_n^4 \right] \right)^{p/2} \right.
\]
\[
\left. + \left( [nt] \mathbb{E} \left[ X_1^2 \wedge d_n^2 \right] + [nt] \mathbb{E} \left[ X_1^2 \wedge d_n^2 \right] \right)^p \right\}
\leq o(1) + C_p \left( [nt] \mathbb{E} \left[ |X|_1^2 \wedge d_n^2 \right] \right)^p \leq C_p t^p + o(1).
\]
Thus (a) follows.

As for (b), note
\[
\frac{q}{2} \left( \frac{\overline{S}_{\lfloor nt \rfloor}}{d_n} \right)^2 + p \frac{\overline{V}_{\lfloor nt \rfloor}}{d_n^2} = \left( \frac{q}{2} + p \right) \frac{\overline{V}_{\lfloor nt \rfloor}}{d_n^2} + q \sum_{k=1}^{\lfloor nt \rfloor} \overline{S}_{k-1} \overline{X}_k.
\]
By (5.2),
\[
\mathbb{E} \left[ \sum_{k=1}^{\lfloor nt \rfloor} \overline{S}_{k-1} \overline{X}_k \right] \leq \sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E} \left[ \overline{S}_{k-1} \overline{X}_k \right]
\leq \sum_{k=1}^{\lfloor nt \rfloor} \left\{ \mathbb{E} \left[ (\overline{S}_{k-1})^+ \right] \mathbb{E} \left[ \overline{X}_k \right] - \mathbb{E} \left[ (\overline{S}_{k-1})^- \right] \mathbb{E} \left[ \overline{X}_k \right] \right\}
\leq \sum_{k=1}^{\lfloor nt \rfloor} \left( \mathbb{E} \left[ |\overline{S}_{k-1}|^2 \right] \right)^{1/2} \mathbb{E} \left[ (|X|_1 - d_n)^+ \right]
= O \left( \frac{n l(d_n)}{d_n^2} \right) \cdot n \mathbb{E} \left[ (|X|_1 - d_n)^+ \right]
= O(d_n) \cdot n \cdot o \left( \frac{l(d_n)}{d_n^2} \right) = o(d_n^2),
\]
and similarly,
\[ \hat{E} \left[ - \sum_{k=1}^{[nt]-1} S_{k-1} X_k \right] = o(d_n^2). \]

On the other hand,
\[ \frac{\hat{E}[V_{[nt]}]}{d_n^2} = \frac{[nt]\hat{E}[X_t^2 \wedge d_n^2]}{d_n^2} = \frac{[nt]nl(d_n)}{n} \xrightarrow{n \to \infty} t \]
and
\[ \frac{\hat{E}[V_{[nt]}]}{d_n^2} = \frac{[nt]\hat{E}[X_t^2 \wedge d_n^2]}{d_n^2} = \frac{[nt]\hat{E}[X_t^2 \wedge d_n^2]}{\hat{E}[X_t^2 \wedge d_n^2]} \rightarrow t^{r-2}. \]

Hence we conclude that
\[ \hat{E} \left[ \frac{q}{2} \left( \frac{S_{[nt]}}{d_n} \right)^2 + \frac{V_{[nt]}}{d_n^2} \right] = \hat{E} \left[ \left( \frac{q}{2} + p \right) \frac{V_{[nt]}}{d_n^2} \right] + o(1) \]
\[ = t \left[ \left( \frac{q}{2} + p \right)^+ - r^{-2} \left( \frac{q}{2} + p \right)^- \right] + o(1). \]

Thus (b) is verified, and the proof is completed. \( \square \)

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