The formation of compact objects at finite temperatures in a dark-matter-candidate self-gravitating bosonic system

Akhilesh Kumar Verma, Rahul Pandit, and Marc E. Brachet

1 Centre for Condensed Matter Theory, Department of Physics, Indian Institute of Science, Bangalore 560012, India
2 Laboratoire de Physique de l’Ecole Normale Supérieure, ENS, Université PSL, CNRS, Sorbonne Université Université de Paris, F-75005 Paris, France
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We study self-gravitating bosonic systems, candidates for dark-matter halos, by carrying out a suite of direct numerical simulations (DNSs) designed to investigate the formation of finite-temperature, compact objects in the three-dimensional (3D) Fourier-truncated Gross-Pitaevskii-Poisson equation (GPPE). This truncation allows us to explore the collapse and fluctuations of compact objects, which form at both zero temperature and finite temperature. We show that the statistically steady state of the GPPE, in the large-time limit and for the system sizes we study, can also be obtained efficiently by tuning the temperature in an auxiliary stochastic Ginzburg-Landau-Poisson equation (SGLPE). We show that, over a wide range of model parameters, this system undergoes a thermally driven first-order transition from a collapsed, compact, Bose-Einstein condensate (BEC) to a tenuous Bose gas without condensation. By a suitable choice of initial conditions in the GPPE, we also obtain a binary condensate that comprises a pair of collapsed objects rotating around their center of mass.

Gravitational effects are important on stellar scales; it might also be possible to mimic such effects in laboratory Bose-Einstein condensates [1] and thus emulate gravitationally bound, condensed, assemblies of bosons, which are candidates for dark-matter halos [2–5]. Although many experiments have been carried out to establish the identity of dark matter, there is still no unambiguous dark-matter candidate. For many years the front runners are candidates for dark-matter halos [2–5]. Although these studies by using Fourier-truncated GP model, in which this truncation generates a classical-field model [15–18], we define the dispersion relation [20, 21] or by defining a Newtonian cosmological constant [22]. By linearizing Eq. (1) around the Jeans instability, for wave numbers $k < k_3 = \sqrt{G n_0 / m + k^2 g n_0 / m + k^4 (\hbar / 2 m)^2}$, which displays a low-$k$ Jeans instability, for wave numbers $k < k_3 = \sqrt{G n_0 / m + k^2 g n_0 / m + k^4 (\hbar / 2 m)^2}$. In the absence of gravity ($G = 0$), we identify the speed of sound $c = \sqrt{\pi n_0 / m}$ and the coherence length $\xi = \sqrt{\hbar / 2 \pi n_0 m}$. Units relevant to

\[ i \hbar \partial_t \psi = -\frac{\hbar^2}{2m} \nabla^2 \psi + \left[ G \Phi + g |\psi|^2 \right] \psi, \]

\[ \nabla^2 \Phi = |\psi|^2 - \langle |\psi|^2 \rangle, \]

where $m$ is the mass of the bosons, $n = |\psi|^2$ their number density, $\Phi$ is the gravitational potential field, and $G = 4 \pi G_N m^2$ ($G_N$ is Newton's constant), and $g = 4 \pi a \hbar^2 / m$, with $a$ the s-wave scattering length. The subtraction of the mean density $\langle |\psi|^2 \rangle$ can be justified either by taking into account the cosmological expansion [20, 21] or by defining a Newtonian cosmological constant [22]. By linearizing Eq. (1) around the constant $|\psi|^2 = n_0$, we obtain the dispersion relation $\omega(k) = \sqrt{-G n_0 / m + k^2 g n_0 / m + k^4 (\hbar / 2 m)^2}$, which displays a low-$k$ Jeans instability, for wave numbers $k < k_3 = \sqrt{G g / 2} \left[ \left( 1 + \sqrt{1 + \frac{G \hbar^2}{mg n_0} / m} \right) / 2 \right]^{1/2}$. In the absence of gravity ($G = 0$), we identify the speed of sound $c = \sqrt{\pi n_0 / m}$ and the coherence length $\xi = \sqrt{\hbar / 2 \pi n_0 m}$. Units relevant to

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FIG. 1: Columns (1) – (3): Ten-level contour plots of $|\psi(x,t)|^2$, at representative times: SGLPE ($T = 0$) (top row, run R1); SGLPE (second row, run R2); GPPE (third row, run R3); GPPE (fourth row, run R4); and 256$^3$ GPPE (fifth row, run R5) [the videos V1-V5 (Supplemental Material [19]) show, respectively, the complete spatiotemporal evolution for these cases].

Column (4): Plots of the scaled radius of gyration $R = \frac{1}{2} \sqrt{\int_{V} \rho(r) r^2 dr / \int_{V} \rho(r) dr}$ (blue) and the scaled gravitational energy $E_{grav} / E_a$ (red) versus the scaled time $t/ (\xi / v)$ for the different runs, where $E_a = 2^{5/4} \pi^4 (G/g)^{1/2}$.

Astrophysics are discussed in [14].

We solve the GPPE (1) by using the 3D Fourier pseudospectral method, with the 2/3-rule for dealiasing [17, 18, 23]: We expand the 2$\pi$ periodic wave function as $\psi(x) = \sum_{k \in \mathbb{Z}^3} \hat{\psi}_k \exp(i k \cdot x)$ and then we truncate it spectrally, by setting $\hat{\psi}_k \equiv 0$ for $|k| > k_{max}$, with $k_{max} = \lfloor N/3 \rfloor$, where $N$ is the resolution and $\lfloor \cdot \rfloor$ denotes the integer part [15, 16]. If we introduce the Galerkin projector $P_G$ [in Fourier space $P_G[\hat{\psi}_k] = \theta(k_{max} - |k|) \hat{\psi}_k$ with $\theta(\cdot)$ the Heaviside function], the Fourier-truncated GPPE becomes

$$i\hbar \frac{\partial \psi}{\partial t} = P_G[-\frac{\hbar^2}{2m} \nabla^2 \psi + P_G[(G\nabla^{-2} + g)|\psi|^2] \psi].$$

Equation (2) conserves exactly the number of particles.
Table I: Dimensionless variables: in the second column of the table we use quantities from the first column; $G_N$ denotes Newton’s constant and $G = 4\pi G_N m^2$.

| $M_a = \frac{\hbar}{\sqrt{G_N m a}}$ | $M = \frac{M\sqrt{G g}}{4\pi \hbar^2}$ |
|--------------------------------|--------------------------------|
| $R_a = \sqrt{\frac{a h^2}{m^3 G N}}$ | $\frac{R}{\bar{R}_a} = \frac{R}{\sqrt{g/G}}$ |
| $a_Q = \frac{h^2}{G_N M^2 m}$ | $\frac{a}{a_Q} = \frac{M^2 G g}{(4\pi \hbar)^2}$ |
| $R_Q = \frac{h^2}{G_N m^2 M}$ | $\frac{R}{\bar{R}_Q} = \frac{M G R}{4\pi \hbar^2}$ |

$N = \int d^3x|\psi|^2$ and the energy $E = E_{kq} + E_{int} + E_G$, where $E_{kq} = \frac{\hbar^2}{2m} \int d^3x|\nabla \psi|^2$, $E_{int} = \frac{g}{2} \int d^3x|\mathcal{P}_G|\psi|^2|\nabla^2 |\mathcal{P}_G|\psi|^2|$, and $E_G = E_{Grav} = \frac{G}{2} \int d^3x |\mathcal{P}_G|\psi|^2 \nabla^2 |\mathcal{P}_G|\psi|^2$. If we use the 2/3-rule for dealiasing, then the momentum $\mathbf{P} = \frac{\hbar}{m} \int d^3x (\nabla \psi^* - \nabla \psi)$, where the over-bar denotes complex conjugation, is also conserved.

This spectral truncation generates a classical-field model that allows us to study finite-T effects in the GPPE (Refs. [16–18] for the GP case). We show that this spectrally truncated GPPE can describe dynamical effects and, at the same time, yield thermalized states, which we can obtain both by the thermalization of the long-time GPPE dynamics or directly, by using the SGLPE

$$\hbar \frac{\partial \psi}{\partial t} = \mathcal{P}_G \left[ \frac{\hbar^2}{2m} \nabla^2 \psi + \mu \psi - \mathcal{P}_G \left[ (G \nabla^2 - g) |\psi|^2 \psi \right] \right] + \sqrt{\frac{2\hbar}{\beta}} \mathcal{P}_G \left[ \zeta(x,t) \right],$$

where the zero-mean, Gaussian white noise $\zeta(x,t)$ has the variance $\langle \zeta(x,t) \zeta(x',t') \rangle = \delta(t-t') \delta(x-x')$, with $\beta = \frac{1}{\sqrt{\hbar}}$ the inverse temperature; we tune the chemical potential $\mu$ at each time step to conserve the total number of particles $N$. The finite-T SGLPE dynamics does not describe any physical evolution; but it converges more rapidly, than does the GPPE dynamics, toward a thermalized state (Refs. [16–18] for the GP case). Clearly, the SGLPE leads to a state with a given temperature; but the GPPE yields a state with a given energy.

Our pseudospectral DNSs of the GPPE [1] and SGLPE [2] use a cubical computational domain that is $(2\pi)^3$ periodic; we normalise $\psi$ such that $N = (2\pi)^3$, and use units with $\hbar = 1$ and $m = 1$. We list the parameters for different runs in Tables II and III.

In columns (1) – (3) of Fig. 1 we show ten-level contour plots of $|\psi(x,t)|^2$ to illustrate, at representative times, the spatial organization of $|\psi(x,t)|^2$ that we obtain via the SGLPE ($T = 0$) (top row, run R1), the SGLPE (second row, run R2), and three GPPE runs (third row, run R3; fourth row, run R4; fifth row, run R5); the videos V1-V5 (Supplemental Material [19]) show, respectively, the complete spatiotemporal evolution of $|\psi(x,t)|^2$ for these five runs. In column (4) of Fig. 1 we give, for these runs, plots of the scaled radius of gyration $R/L = \frac{1}{L} \sqrt{\frac{1}{V} \int_{V} \rho(r) r^2 dr}$ (blue curves) and the scaled gravitational energy $E_{grav}/E_a$ (red curves) versus the scaled time $t/(\xi/v)$, where $E_a = 2\pi^4 (G/g)^{3/2}$. If we tune $T$ in the SGLPE [2], it yields a statistically steady state whose properties (like $R/L$ and $E_{grav}/E_a$) are close to their counterparts in the thermalized state of the GPPE (e.g., by comparing rows (2) and (3) in column (4) of Fig. 1). We see that the final state of the SGLPE has nearly the same energy and radius as the corresponding statistically steady GPPE state. Furthermore, convergence to this thermalized state is more rapid in the (canonical) SGLPE than in the (microcanonical) GPPE. To validate our DNSs, we have checked explicitly (Supplemental Material [19]) that, at zero temperature, our results agree with those of the $T = 0$ study of Refs. [10–13].

Table II: This table shows the representative runs for which we give plots in Fig. 1.
$k_{\text{B}}T/E_a$ (often referred to as heating and cooling runs in statistical mechanics). In these SGLPE runs, we use the final steady-state configuration for $\psi(x)$, from the previous temperature, as the initial condition at the next temperature; clearly, there is significant hysteresis at the first-order transition from the - collapsed BEC to the non-collapsed state. In the right side panels of Fig. 3 we show ten-level contour plots of $|\psi(x)|^2$ and the associated spectra $|\psi(k)|^2$ to illustrate, at representative points on heating and cooling curves in the hysteresis plot, the real-space density distribution and the $k$-space density spectra ($k_{\text{B}}T/E_a = 2.7 \times 10^{-5}$ and $k_{\text{B}}T/E_a = 3.62 \times 10^{-5}$ in the top panels A and B, respectively, and $k_{\text{B}}T/E_a = 2.3 \times 10^{-5}$ and $k_{\text{B}}T/E_a = 3.16 \times 10^{-5}$ in the bottom panels C and D, respectively). From these density distributions and spectra, we conclude that our system undergoes a first-order transition from a collapsed BEC to a tenuous, non-condensed assembly. The panel of figures at the very bottom of Fig. 3 show that this first-order transition occurs at $g = 0$ too. We expect that this also occurs when $g < 0$, which is the appropriate parameter range for axion stars [13, 14]; the $g < 0$ case requires a quintic nonlinearity in Eq. 1 for stability; finite-temperature effects can be studied for this case by using the methods that we have described above (as we will show in future work).

Do our Fourier-truncated GPPE yield only single collapsed objects? No. We now show, for the illustrative parameter values $g = 20$, $G = 1000$, and $128^3$ collocation points, that this GPPE can also yield long-lived states with temporal oscillations. In particular, by using an initial condition with two rotating spherical compact objects, the truncated GPPE dynamics yields a rotating binary system, which we depict at representative times in Fig. 4 and in the Video V6 in the Supplementary Material [19] (this also gives the initial data).

Our study goes well beyond earlier studies [24, 25] that employ the Thomas-Fermi approximation and include finite-temperature effects at the level of a non-interacting Bose gas (with $g = 0$). We have shown that the truncated GPPE can be used effectively to study the gravitational collapse of an assembly of weakly interacting bosons at finite temperature. Our study of this collapse shows that there is a clear, thermally driven first-order transition from a non-condensed bosonic gas, at high $T$, to a condensed BEC phase at low $T$; this transition should be contrasted with the continuous BEC transition in the weakly interacting Bose gas, which is described by the GPE, in the absence of gravitation. Furthermore, we have shown that gravitationally bound, rotating binary objects can be obtained in our GPPE simulations. Therefore, our work opens up the possibility of carrying out detailed finite-temperature studies of self-gravitating bosonic systems, which are potentially relevant for studies of dark-matter candidates like boson stars and axions.

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FIG. 2: The plots of (a) the scaled radius of gyration \( (R/R_Q) \) versus scaled scattering length \( (a/a_Q) \) for the GPPE Runs A1-A30, (b) the scaled mass \( (M/M_a) \) versus the scaled radius of gyration \( (R/R_a) \) for GPPE Runs B1-B30 on a log-log scale, (c) the scaled energy components \( (E_{kq}/E_a, E_{int}/E_a, E_G/E_a) \), and the total energy \( (E/E_a) \) versus the scaled temperature \( k_B T/E_a \).

FIG. 3: Top left panel: plots of the dimensionless radius \( (R/L) \) versus the dimensionless temperature \( (k_B T/E_a) \), for heating (red) and cooling (green) runs showing a hysteresis loop. We show ten-level contour plots of \( |\psi(x)|^2 \) and the associated spectra \( |\psi(k)|^2 \) to illustrate, at representative points on heating and cooling curves in the hysteresis plot, the real-space density distribution and the \( k \)-space density spectra \( (k_B T/E_a = 2.7 \times 10^{-5} \text{ and } k_B T/E_a = 3.62 \times 10^{-5} \text{ in panels A and B of top panels, respectively, and } k_B T/E_a = 2.3 \times 10^{-5} \text{ and } k_B T/E_a = 3.16 \times 10^{-5} \text{ in panels C and D of bottom panels, respectively}) \). The analogs of these plots, for the case \( g = 0 \), are shown in the panels at the very bottom. In the bottom left panel we use \( |E_G| \) at \( T = 0 \) to make the temperature dimensionless.

FIG. 4: Ten-level contour plots of the \( |\psi(x, t)|^2 \) (a) at \( t = 0.018 \), (b) at \( t = 0.025 \), and (c) at \( t = 0.03 \), for the initial condition for \( \psi(x, t) \) given in Eq.(1) of the Supplementary Material [19], showing rotating binary system (see video V6 in supplemental material).