Quantum field theory as a problem of resummation

(Short guide to using summability methods)

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Prague, July 1991

*Thesis submitted in partial fulfillment of the requirements for obtaining the degree of ‘Candidatus Scientarium’ (the Czechoslovak equivalent of PhD).
1 Introduction

1.1 Divergences of perturbation theory

Very common situation which one encounters in physics is a lack of exact solutions. If $Z(g)$ is some physical quantity considered as a function of a “coupling constant” $g$ then the only what we have frequently at disposal are few first terms of perturbation theory,

$$Z(g) = a_0 + a_1 g + a_2 g^2 + \ldots + a_n g^n.$$ 

Unfortunately, without any regularization this expansion is badly defined in quantum field theory. In fact, in a formal perturbation expansion, the coefficients $a_n$ are divergent and thus undefined. This is first kind of divergences which, in renormalizable quantum field theories, can be cured by using a suitable renormalization prescription which gives a physically plausible recipe how to go from undefined unrenormalized coefficients $a_n$ to the renormalized ones. We shall denote the renormalized coefficients by $a^{Rn}_n$. This means that by renormalization one obtains the perturbation series in a renormalized coupling constant $g_R$,

$$Z^R(g_R) = a^R_0 + a^R_1 g_R + a^R_2 g_R^2 + \ldots + a^R_n g_R^n,$$

with well defined coefficients $a^{Rn}_n$. In general, however, still one kind of divergences remains - it is the divergence of the perturbation series as whole. Indeed, if we do not consider the Yukawa model in two dimensions, fermionic field theories with UV cutoff where, due to the Pauli exclusion principle, the radius of convergence of perturbation theory for normalized (connected) Green’s function is nonvanishing, as well as lattice gauge theories in the strong coupling region, perturbation theory of many other physical models, especially without cutoffs and in the weak coupling region, is known to be divergent. The latter divergence is the main subject of the thesis. One has to know how to deal with such divergences, what is their origin, their physical interpretation and significance if one wishes to have logically consistent understanding of quantum field theories.

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1 Throughout the paper we shall call expansion parameter as a coupling. It may, however, correspond to the inverse of physical coupling in the case of strong coupling expansion, to $1/N$ in the case of $1/N$ expansion, etc.

2 including superrenormalizable ones
In the case that a perturbation theory is asymptotic then it is no problem to deal with it in the region of small coupling (if we are only interested in perturbative phenomena). This is the case of asymptotically free theories at very large momenta or weak coupled theories at low momenta, respectively. If we are interested in nonperturbative phenomena then we stuck with the problem to give a meaning to divergent series. If the series is asymptotic (see section [3]) then by optimalizing the bound on the rest term (4) one gets the optimal number of terms to be kept. By keeping either more or less terms results in that our prediction based on the perturbation theory is getting worse. This optimalization of the bound on the rest term is for example essential to prove majority of our results. Indeed, our proofs follow mostly the same scheme and are in this sense a bit boring (sometimes were also boring for the author himself): using the Euler-Maclaurin sum formula for a given sum, its asymptotic evaluation by the saddle point technique, and optimalization of the rest term. Nevertheless the scheme turns out to be quite efficient.

In field theory the perturbation series cannot be truncated painless. When truncated perturbation theory becomes to depend on unphysical (renormalization scheme dependent) parameters. This freedom may be again used to optimalize the perturbative prediction and to extract a scheme independent result [4]. Thus if a perturbation series is known to be asymptotic and divergent some time one can get a reliable prediction by keeping only finite number of its terms. However to establish the number of terms to be kept one needs to know the large order behaviour of perturbation theory. Information about large orders ceases to be only perturbative information and, in fact, it is a nonperturbative one. The fact that a presence of nonperturbative solutions can be seen by the large order behaviour of the perturbation theory could seem to be a bit surprising at a first moment but perturbation theory, if (whenever it is possible) suitably modified, is known to give even a good, nonperturbative approximation [5]. As recently shown in [6] large orders of perturbation theory also determine multiparticle cross sections at asymptotically high energies. In order that this information be useful one needs a suitable summability method. This is the aim of this thesis to provide a selfconsistent treatment of new result of the author on regular analytic summability method which have been published in his several articles [7].

If one goes back to the history one finds that investigation of convergence
of perturbation series in physical models was initiated by Dyson [12], who considered the case of QED. More rigorous studies of the problem have been then done by a number of authors. The divergence of the perturbation series of the scalar bosonic \(\lambda \phi^4\) model was proved by [13]. More general polynomial bosonic field models and their correct large order behaviour have been studied, e.g., by [14, 15]. Perturbation theory for the Yukawa interaction (without cutoff’s) in \(\ell\) dimensions \((2 < \ell < 4)\) among fermions was shown to be divergent in [16]. So far the divergence of the perturbation theory has been also proved for many quantities in other models of quantum mechanics and quantum field theory including the energy-levels of an arbitrary anharmonic oscillator [17, 18], the ground state energy of some nonpolynomial oscillator [19], one-loop effective lagrangian of QED [20], Green’s functions in QCD [21] and recently also some amplitudes of the bosonic string theory [22].

As for the explicit large order behaviour majority of results are based on the Lipatov argument [14], which gives in general the large order behaviour (for bosonic field theories) of the following type,

\[
|a_n| \leq A a^n n^\alpha (1 + o(n))^n n!.
\]  

(1)

The bound was obtained by a formal steepest descent method in the Euclidean path integral formulation of quantum field theory for a variety of models [14, 15, 16, 23, 27]. Unfortunately, the method is based on some additional assumptions which have not been proved [23] (cf. [24]). The result, obtained by a phase-space expansion confirms the Lipatov analysis in \(< 4\) dimensions, but in \(4\) dimensions renormalization was shown to disturb the Lipatov behaviour in such a way that the constant \(a = (3/2\pi^2)\) should be replaced by \(-\beta_1/2\), where \(\beta_1(N) = (N + 8)/(2\pi^2)\) is the first nonzero coefficient of the \(\beta\)-function. Note that the ratio \(a/\beta_1\) tends to 0 as \(N \to \infty\) [25, 26].

Rigorous large order behaviour of the massless \(\phi^4\) theory has been studied in [27], but only for an UV-cutoffed theory. The rigorous large order behaviour has been only established in the case of a \(N\) component massive \(\lambda \phi^4\) model. It is worthwhile to mention that the large order behaviour of bosonic string theory also differs from the Lipatov behaviour. Here the \(n\)-th order of perturbation theory is of magnitude \(2n)!\) in place of \(n!\) in (bosonic) field theories. For further details see [28, 29, 30, 31] and references therein.

\[\text{For quantum mechanical models as well as for models where horn-shaped singularities are absent works rather well.} \]
It is worth remembering that the factorial growth of perturbation theory was proved especially for bosonic theories while inclusion of fermions slow the divergence down (see, e.g., [2, 10]). Moreover, sometimes the factorial growth is an artefact of the approximation used and a more careful analysis may give even a convergent result [32]. Similar divergences one also encounters in general relativity in connection with an existence of the de Witt integral [33].

Due to the field theoretical approach to critical phenomena [34] initiated by Wilson [35] analogous divergences inevitably appear in the study of variety of statistical systems. As for quantum spin systems with finite interaction the situation with regard to convergence is a bit better, at least in the high-temperature region which corresponds to the strong coupling region of lattice field theories [3, 36]. This is due to the fact that lattice models with finite interaction can be transformed on a polymer system [37]. Afterwords, by using general results on polymer expansions [39, 40], one can derive convergent cluster expansions for them [3, 37, 38]. The convergence of cluster expansions allows to deduce upper and lower bounds on expectations of various types of observables such as Wilson loops, 't Hooft loops and others from bounds on polymer activities. The convergence also implies exponential decay of correlations as well as that above some temperature (coupling) there is no phase transition. This expansion is not, however, a power series expansion in general. In contrast to continuum field theory where one has general arguments for its large order behaviour [14] there is no general argument how single terms of the cluster expansion should behave. In regard to convergent expansions in field theory and statistical systems see also [41, 42, 43] (and references therein). There a convergent perturbation expansion is obtained if, instead of the expansion in renormalized coupling constant, the expansion in powers of the running coupling constant is used.

For complexity we note that some quantities of topological origin in superrenormalizable theories, like topological mass term in planar QED with the Chern-Simons term, can have finite perturbation series expansion with all coefficients but the first two being zero. This is ensured by the so-called non-renormalization theorem [11].
1.2 Other perturbative techniques

In the last two decades a hard work has been done in developing perturbation theory where perturbation parameter is rank of the symmetry group of an underlying model. This approach was originally developed by ’t Hooft [45]. He also noted a connection between the $1/N$ expansion and dual models [46]. If a connection were solidly established, many of the leading mysteries of QCD would be solved. The dual model has built into it confinement, with quarks at the end of a string. Also, a clear connection between QCD and the dual model would mean that the problem of dynamical mass generation had been solved, since the dual model certainly has a mass scale (the Regge slope).

To elucidate the main features of the approach let us consider the familiar Hamiltonian of the hydrogen atom:

$$H = \frac{p^2}{2m} - \frac{e^2}{r}.$$  

Since for $e^2 = 0$ we can solve this problem exactly - it is simply the problem of the motion of a free particle - one’s first hope might be to try to understand the hydrogen atom in the small $e^2$ regime by treating the interaction term, $-e^2/r$, as a perturbation. This hope is frustrated because in the hydrogen atom $e^2$ is not really a relevant parameter. It can be eliminated from the problem by redefining the scale of distances.

After a rescaling $r \rightarrow tr$, $p \rightarrow p/t$, with $t = 1/me^2$, the Hamiltonian becomes

$$H = (me^4) \left[ \frac{p^2}{r} - \frac{1}{r} \right],$$

and one sees that the “coupling constant” $e^2$ appears only in the overall factor $me^4$ which serves merely to define the overall scale of energies and which could be absorbed in a rescaling of the time coordinate. Therefore, except for the overall scale of lengths and times, the physics of the hydrogen atom - and atomic and molecular physics in general - is independent of $e^2$, and perturbation theory in $e^2$ is meaningless.

Atom and molecules can be described by the reduced Hamiltonian with $e^2$ scaled out. For the hydrogen atom the reduced Hamiltonian is

$$H = \frac{p^2}{2} - \frac{1}{r}.$$
The reduced Hamiltonian contains no evident free parameter. However, without a free parameter there is no perturbation expansion. Without a perturbation expansion what we can do? Suppose, since this is the case of QCD and of many other analogous problems that we were unable to diagonalize the Hamiltonian exactly, and then even a computer solution were impractical. To make progress, we must make an expansion of some kind. Since there is no obvious expansion parameter we must find a hidden one. To find a hidden expansion parameter, we may treat as a free, variable parameter a quantity that one usually regards as given and fixed.

For instance, we may take a cue from spectacular developments in the last years in critical phenomena. After decades in which the study of critical phenomena was frustrated by the absence of an expansion parameter, Wilson and Fisher suggested that to introduce an expansion parameter, one should regard the number of spatial dimensions not as a fixed number, three, but as a variable parameter \( \varepsilon \). They showed that critical phenomena are simple in four dimensions and that in \( 4 - \varepsilon \) dimensions critical phenomena can be understood by perturbation theory in \( \varepsilon \). Similar ideas were used by Bender at al. to study of \( \lambda \phi^4 \) theory and some aspects of stochastic quantization [48].

How, by analogy, can we create an expansion parameter in our problem? Instead of studying atomic physics in three dimensions, where it possesses an \( O(3) \) rotation symmetry we may consider atomic physics in \( N \) dimensions, so that the symmetry is \( O(N) \). One can show that atomic physics simplifies as \( N \to \infty \) and can be solved for large \( N \) by expansion in powers of \( 1/N \) [49].

Although this expansion in atomic physics is not very useful at \( N = 3 \) (\( N \) must apparently be at least six or seven for the \( 1/N \) expansion to give good result), illustrates the main features of the approach. In QCD, as in atomic physics, the coupling constant can be scaled out of the problem and the basic difficulty is the same as in atomic physics - the seeming absence of an expansion parameter. For further details see [50].

So far the \( 1/N \) expansion has been successfully used to solve the Gross-Neveu model [51], in the study of two-dimensional Yang-Mills theory [52], \( U(\infty) \) lattice gauge theory [53], the generalized two-dimensional \( U(N) \) Thirring model [54], and the \( O(N) \)-symmetric sigma model [55]. In last years results of \( 1/N \) expansion have been also successfully applied to two dimensional quantum gravity (see [54] and references therein), as well as to quantum an-
tiferromagnets in connection with high-temperature superconductivity [56].

As for the study of planar field theories ($N \to \infty$ limit) see [57, 58]. The $1/N$ expansion can be improved by a suitable summability method just as conventional perturbation theory [59].

Unfortunately, for Yang-Mills theory, the $N = \infty$ limit does not seem to be exactly soluble, even though it is significantly simpler. Nonperturbative approaches to Yang-Mills theory are sorely needed in particle physics. There is not much hope that the Yang-Mills theory itself is exactly soluble, so one has to look for another theory which is “close enough” to it. Then Yang-Mills theory can be studied by an expansion around this soluble theory. As a general rule the more symmetry a theory has the more easily it can be solved.

Can we enlarge the symmetry group further and thus find a soluble theory? The answer is “yes”. Consider Yang-Mills theory with structure group $G$ on a spacetime $X \times R$. In the gauge $A_o = 0$ the structure group is the set $XG$ of all smooth maps from $X$ to $G$. Let us consider $G = U(N)$. When we take the $N \to \infty$ limit in the usual way, we consider an infinite sequence of Yang-Mills theories with structure groups

$$U(1) \subset U(2) \subset \ldots U(N) \subset U(N+1) \subset \ldots U(\infty).$$

The gauge group of Yang-Mills theory in the $N = \infty$ limit is then $XU(\infty)$, which contains $XU(N)$ for any $N$. But there exists a group (the universal gauge group) $U_2$ which contains $XU(N)$ not only for all $N$, but also for all $X$ [60].

There is too early to say whether this approach will be successful, but it by no means provides a useful step to find an exactly soluble theory for perturbation theory calculations.

1.3 Asymptotic expansions and strong asymptotic conditions

As it was discussed above perturbation series in quantum theory are mostly divergent and can have at best the meaning of asymptotic series. *Asymptoticity* means in general that the perturbation theory cannot determine the solution uniquely. More precisely given an arbitrary sequence $\{a_n\}_{n=0}^{\infty}$ of complex numbers and an arbitrary sector-like domain $D$, there exists for
some $\varepsilon > 0$ function $f(z)$, which is regular in $D_\varepsilon := D \cap \{ z \mid |z| < \varepsilon \}$ and

$$\lim_{z \to 0, z \in D_\varepsilon} (f(z) - a_o - \ldots - a_n z^n)/z^{n+1} = a_{n+1}$$

exists, or equivalently

$$f(z) \sim \sum_{n=0}^{\infty} a_n z^n \quad (z \to 0, z \in D_\varepsilon), \quad (2)$$

i.e., $\sum_{n=0}^{\infty} a_n z^n$ is an asymptotic series of $f(z)$ in the region $D_\varepsilon$ [61, 62]. In general there are infinitely many functions with the above properties.

Asymptotic series may be divergent as well as convergent (note that a convergent asymptotic expansion does not prevent a function from being singular at the origin). To deal with such series one has to look for a maximal region $D$ of the complex coupling constant plane in which the asymptotic expansion is uniform. In order that such series determine its sum $f(z)$ uniquely this series has to satisfy some additional conditions, so called strong asymptotic conditions (SAC). SAC are then relations which put into the balance the shape of the region $D$, or analyticity in the complex coupling constant plane, with some bound on the rest term $R_N(z)$,

$$f(z) = \sum_{n=0}^{N-1} a_n z^n + R_N(z), \quad (3)$$

in such a way that only one function $f(z)$ can satisfy them. SAC are conditions which by definition prevent appearance of nonperturbative terms like $\exp(-A/g)$ in perturbation theory (or a typical “tunnelling” like amplitudes), i.e., terms whose perturbation expansion is identically zero. Indeed, to be unseen in the standard perturbation theory the (non-zero) contribution of such a term should have an asymptotic series which is identically zero. Such are for instance all nontrivial minima $A$ of action $S$ for which $S(A) < \infty$, like instantons in non-abelian theories [54] or kinks in the double-well anharmonic potential [55, 56], etc. Violation of SAC then indicates a presence of nonperturbative effects in a theory and instability of its ground state. It was emphasized that the violation of SAC is a more serious problem of a theory than the divergence itself, therefore the violation of SAC leads to ambiguity of perturbation theory predictions [11, 63]. Of course, this ambiguity depends on the behaviour of the rest term $R_N(z)$ in (3) for $N \to \infty$. As it has
been exposed above, optimization of the bound may determine an optimal number $N_0$ of terms to be kept and the value of $R_{N_0}(z)$ then determines a maximal error by which the exact solution differs from the perturbative one even in the case that the series violates SAC and does not determine a unique solution. Sometimes ambiguity of the perturbation theory predictions, e.g., in bosonic string model are highly desirable, because “if perturbative string theory were make sense, string theory would have nothing to do with physical reality” [22]. This is because there are many features of perturbative treatment of string theory which are not shared by the real world. When SAC are fulfilled then the coefficients $a_n$ of perturbation theory determine $f(z)$ uniquely. To obtain this function an appropriate analytic regular summability method can be used [62]. We say that a summability method $\sigma$ is regular if and only if

$$\sigma(\sum_{n=0}^{\infty} a_n z^n) = \sum_{n=0}^{\infty} a_n z^n,$$

whenever the r.h.s. of (4) converges. Analogously, we say that a summability method $\sigma$ is analytic if for every power series $\sum a_n z^n$ with nonzero radius of convergence

$$\sigma(\sum_{n=0}^{\infty} a_n z^n) = f(z),$$

whenever the l.h.s. of (5) exists ($f(z)$ now being an analytic continuation of $\sum a_n z^n$). The moment constant summability methods as defined by [62] are automatically regular, although not analytic in general. We confine ourselves to the class of analytic moment constant summability method (AMCSM) [62], one member of which is the well-known Borel summability method, frequently used in physics (see [6] for a recent review).

To illustrate the main features of the AMCSM let us consider a very simple example of the series $\sum_{n=0}^{\infty} z^n$. Using the identity

$$\frac{1}{n!} \int_{0}^{\infty} e^{-t^n} dt = 1,$$

it is always true that

$$\sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{0}^{\infty} e^{-t^n} dt,$$
within the radius of convergence of $\sum_{n=0}^{\infty} z^n$, i.e., within the disc $\|z\| < 1$. However within the disc of convergence one can always interchange summation and integration. Thus,

$$\sum_{n=0}^{\infty} z^n = \int_{0}^{\infty} e^{-t} \left( \sum_{n=0}^{\infty} \frac{(zt)^n}{n!} \right) dt.$$  \hspace{1cm} (6)

Here, the r.h.s. of (6) is nothing but the Borel sum of $\sum_{n=0}^{\infty} z^n$. In this case it can be calculated explicitly,

$$\int_{0}^{\infty} e^{-t} \frac{(zt)^n}{n!} dt = \int_{0}^{\infty} e^{-t} e^{zt} dt = \int_{0}^{\infty} e^{-t(1-z)} dt = \frac{1}{1-z},$$  \hspace{1cm} (7)

whenever $\text{Re} \ z < 1$. Thus, we have obtained the expression which converges for $\text{Re} \ z < 1$, i.e., outside the disc of convergence of $\sum_{n=0}^{\infty} z^n$ and gives there an analytic continuation of $\sum_{n=0}^{\infty} z^n$ (see Fig. 4).

1.4 The Borel summability method

1.4.1 The Nevanlinna theorem

A main mathematical tool for investigation whether a series is Borel summable or not provides the Nevanlinna theorem [67, 68]:

**Theorem 1**: Let $f(z)$ be analytic in the circle $C_R := \{ z \mid \text{Re}1/z > 1/R \}$, continuous up to the boundary, and satisfy there the estimates

$$f(z) = \sum_{k=0}^{N-1} a_k z^k + R_N(z),$$

with

$$|R_N(z)| \leq A \sigma^N N! \ |z|^N,$$  \hspace{1cm} (8)

uniformly in $N$ and in $z \in \bar{C}_R$. Then $B(t)$,

$$B(t) = \sum_{n=0}^{\infty} \frac{a_n t^n}{n!};$$

10
converges for \(| t | < 1/\sigma\) and has an analytic continuation to the striplike region \(S_\sigma := \{ t \mid \text{dist}(t, R_+) < 1/\sigma\}\), satisfying the bound
\[
| B(t) | \leq Ke^{t|t|/R}
\] (9)
uniformly in every \(S_{\sigma'}\) with \(\sigma' > \sigma\). Furthermore, \(f\) can be represented by the absolutely convergent integral
\[
f(z) = (1/z) \int_0^\infty e^{-t/z} B(t) dt
\] (10)
for any \(z \in C_R\).

Conversely, if \(B(t)\) is a function analytic in \(S_{\sigma''} (\sigma'' < \sigma)\) and satisfying there the bound (9), the function \(f(z)\) defined by (10) is analytic in \(C_R\) and satisfies (8) [with \(a_n = B^{(n)}(t) |_{t=o}\)] uniformly in every \(C_{R'}\) with \(R' < R\).

1.4.2 Application to physical models

The reason why the Borel transform has been so successfully used in physics is due to the fact that the path integral can be rewritten in the Borel form [21, 39]. Indeed, consider a euclidean functional integral which has been rescaled as in a loop expansion
\[
G(g^2) = \int [dA] \exp[-S(A)/g^2].
\]
Functional integrals are the \(N \to \infty\) limit of integrals like
\[
g^{-N} \mathcal{N} \int \prod_{i=0}^N dx_i \exp[-S(x_i)/g^2],
\]
where \(\mathcal{N}\) is the usual normalizing factor. Using the Cauchy integral formula this can be rewritten as
\[
\mathcal{N} \int \prod_{i=0}^N dx_i \oint_C \frac{dz}{2\pi i} \exp(-z/g^2) \frac{\Gamma(N/2 + 1)}{[S(x_i) - z]^{N/2+1}},
\]
where \(C\) is a contour encircling the range of \(S\), which is normalized to run from 0 to \(\infty\). An interchange of integrations is allowable for finite \(N\) and
reasonable actions yielding

\[ G(g^2) = \oint_C \frac{dz}{2\pi i} \exp(-z/g^2)B_N(z), \]
\[ B_N(z) = \mathcal{N} \int \prod_{i=0}^{N} dx_i \frac{\Gamma(N/2 + 1)}{|S(x_i) - z|^{N/2+1}}. \]  

(11)

A final step is to take functional integration limit, \( B(z) = \lim_{N \to \infty} B_N(z) \). Note that since denominator is raised to power \( N/2 + 1 \), naive power counting guarantees convergence even when \( S \) is only quadratic (as in free theory). On the other hand the coefficients \( a_n \) of the perturbation series in quantum models do not grow faster than \( A\sigma^n \Gamma(\alpha n + \beta) \), where \( \Gamma \) is the usual gamma function and the constants \( A, \sigma, \alpha \) and \( \beta \) do not depend on \( n \), what is just the case which can be dealt with the Borel method.

The Borel summability of the perturbation series of the physical quantities has been studied by many authors. In quantum theory it was firstly used by [70] to show that one-loop effective lagrangian of QED exactly calculated by Schwinger in the proper-time formalism [71] is for the vanishing electric field nothing but the Borel sum of the perturbation series (see also [72]). The Borel summability of the perturbation series has been also proved in the case of the energy-levels of an anharmonic oscillator in any finite dimension [17, 18, 73], the resonances of the Stark effect on a Hydrogen-like atom [72, 74], in the case of the Zeemann effect [75], for some nonpolynomial oscillators [19] and for the planar asymptotically free massive field theory for sufficiently small coupling [57]. The method was also used for a rigorous perturbative construction of the massive euclidean Gross-Neveu model [2]. In combination with conformal mapping or with the Padé approximation the Borel summability has been successfully used to compute the critical coefficients of the statistical models [76], the zeros of the beta function [77], the ground state energy of the anharmonic oscillator at infinite coupling limit [78], to compute the beta function of the massive \( \lambda \phi^4 \) model [79] or to sum dominant coupling constant logarithms in \( QED_3 \) [80], etc.

1.4.3 Structure of the Borel transform

In physically realistic models the structure of the Borel transform is very involved. The obstacles for the Borel summing of a perturbation series appear as singularities of the Borel transform on the real positive axis. These
properties of the Borel transform can be deduced as follows. The integral for \( B_N(z) \) is very much like a Feynman parameter integral. Thus the Landau conditions may be applied to ask when \( B_N(z) \) is singular [81]. The Landau conditions have a direct interpretation in terms of solutions to the euclidean equations of motion: \( B_N(z) \) may be singular for \( z \) such that there exist \( x_i^s \) for which

\[
\begin{align*}
z & = S(x_i^s), \\
0 & = \frac{\partial S}{\partial x_i} \bigg|_{x_i=x_i^s}.
\end{align*}
\]  

(12)

This means \( B_N(z) \) may be singular for \( z \) equal to the action of a solution to the classical euclidean equations of motion (discretized). This is true for complex as well as real-valued solutions. Such solutions are generally called pseudoparticles or instantons. It is widely supposed that in the large-\( N \) limit the solutions to the discretized equations approach a limit which is identified with the solution to the continuum equations of motion.

In general, attached to each singularity will be a branch cut. Because \( B(z) \) is integrated along a contour \( C \) encircling the positive real axis, it is convenient for real, positive singular points to have their branch cuts running along the positive real axis.

The only singularities on the first sheet are on the positive real axis. These can arise from real solutions with real positive action. When no instantons are present the only singularity on the non-negative real \( z \) axis is at \( z = 0 \), the singularity arising from the vacuum solution. There is a cut extending from this singularity to positive infinity. When instantons occur, there will be another contributions to the Borel sum other than those visible in perturbations about vacuum. The instanton singularities will have branch cuts attached to them. The total discontinuity of \( B(z) \) will be a sum of discontinuities across branch cuts.

In renormalizable (not superrenormalizable) theory there are rather persusasing arguments that apart from instantons another singularities can appear which go under generic name of renormalon singularities [21, 82, 83]. A toy, finite dimensional model where renormalon singularities appear has been investigated in [84] in the context of Wilson’s renormalization group acting in the space of hamiltonians of the cutoff system. The renormalon singularities have been shown to be related to the presence of a marginal direction at the fixed point of the renormalization group besides the relevant or irrelevant ones, or to the resonance in the dynamical system language. Despite
The suggested structure of the Borel transform of QED and QCD is shown in Fig. 1 and 2. Note that the position of renormalon depends on the sign of the first nonvanishing coefficient of the corresponding $\beta$-function. While in an asymptotic free theory UV-renormalons are harmless (lie on the negative real axis) in nonasymptotic free theory like QED renormalons prevent the Borel summability (lie on the positive real axis). As for QED see also [85]. (We would like to stress out that the calculations which led to the renormalon [21, 82] was only performed for specific subclass of all graphs, which contribute to the given order of a theory, and may only serve as indication but not at all as a proof).

1.4.4 Shortcomings of the Borel summability method

Unfortunately, the use of the Borel summability method is sometimes very limited. Nonsummability means that perturbation theory violates strong asymptotic conditions (SAC), or, what is equivalent that perturbation the-
Figure 2: Borel $z$ plane for QCD. The circles denote IR divergences that might vanish or become unimportant in colour-free channels.

theory predictions can not be unambiguous. This happens due to presence of nonperturbative effects which cause perturbation theory around the trivial minimum to be unreliable. Nonperturbative effects do appear and destabilize a trial (classical) ground state in models having classically degenerate ground state \[86\] which is the case, e.g., of the double well potential \[65, 66, 87\], gauge field theories \[64, 65, 66\], as well as of the heterotic string and superstring \[88\]. However, SAC can be also violated in the case of perturbation theory around a stable nonperturbative ground state (after the nonperturbative effects have removed the degeneracy of the ground state \[65\]), as has been shown in the case of the double-well potential \[87\]. The physical meaning of the non Borel summability is illustrated on some simple models in \[89, 91\].

The next shortcomings of the Borel summability method is connected with its analytic properties. Indeed whenever the Borel sum exists then it defines a function regular in some sector-like neighbourhood of the origin (the convergence of the Borel integral need not be absolute!) \[92\]. On the other hand a maximal region of analyticity of realistic field theories in the coupling constant plane may be very small. For instance, there are strong arguments that two-point Green’s functions of four dimensional renormalizable massless field theories are only analytic in a horn-shaped region: a wedge bisected

\[4\]I thank J. Magnen for acquainting me with this reference.
Figure 3: Expected form of the maximal region of analyticity in the coupling constant complex plane of four dimensional renormalizable field theories.

by the real axis and bounded above and below by circles which are tangent to the real axis at the origin [21, 93, 94] (see Fig. 3). Hence the maximal region of analyticity has a cusp, i.e., zero opening angle at the origin. The appearance of a horn-shaped region of analyticity is in fact a nonperturbative result which we have about a theory. It is not seen in a perturbation theory and arises if the analytic structure of the (response) Green functions for complex momenta \( p^2 \) is combined with asymptotic freedom. This is the essence of 't Hooft’s argument [21, 94] which relates the standard momentum space branch points for timelike \( p^2 > 0 \) to singularities in the complex plane of a suitably chosen coupling constant by solving the Callan-Symanzik equation (for generalizations to more general couplings see [25]). In fact, the use of the Callan-Symanzik equation is not crucial in 't Hooft’s argument and it goes through even if a theoretical evidence of the asymptotic freedom is replaced by an experimental evidence (if available) of the asymptotic freedom, i.e., that a measurable coupling \( g^2(\mu) \) behaves like

\[
\frac{\mu d}{d\mu} g^2(\mu) = -\beta_1 g^4(\mu) + \mathcal{O}(g^6(\mu)),
\]  

(13)
on a sufficiently high mass scale $\mu$. Note that in this case we always can write an analogue of the equation (13). Provided that such information is available one gets rid of the problem of definition of a suitable coupling and the horn-shaped region appears as UV effect once one assumes that at high momenta, measurable quantities are only functions of the ratio $p^2/\mu_0^2$, $\mu_0$ being an integration constant of (13). Analytic structure of response functions is by itself a nonperturbative information about a theory which arises when very general theoretical principles of unitarity and causality are combined with experimental data. Hence, the appearance of horn-shaped regions is in fact a very general phenomenon resulting whenever the following points are satisfied or available:

1) unitarity
2) causality
3) asymptotic freedom (either theoretical or experimental evidence)
4) experimental data (which determine the position of thresholds and branch points in the momentum plane).

Some other arguments for producing singularities in the $g$-plane are not at all related to branch points in the momentum plane, and are due to the separatrix of the differential equation for the running coupling constant, i.e., to the line in the $g$-plane which separates the asymptotically free region of initial values $g$ from the non-asymptotically one \cite{96,97}. If the Callan-Symanzik procedure for calculating the asymptotic behaviour of one-particle-irreducible Green’s functions in the deep Euclidean region \cite{98} is correct then such horn-shaped regions of analyticity is a typical UV effect of all four dimensional renormalizable field theories. What is interesting is that a horn-shaped region is also the maximal region of analyticity in the complex coupling constant plane of energy levels of a simple quantum mechanical model - the anharmonic oscillator in the massless limit \cite{94}.

If one imposes UV cutoff then two- and four-point Green’s functions of a four dimensional massless field theory can be analytic in a disc as was exemplified in a rigorous construction of the critical $\phi^4$ theory \cite{97}. However, if one removes this UV cutoff the disc should shrink to zero. Recently it was shown that some other quantities, like $\beta$-function (defined perturbatively using the BPHZ renormalization with subtractions at vanishing external momentum) of the massive $\phi^4$ model without UV cutoff may be Borel summable and hence analytic in a disc \cite{79}. For the sake of completeness we recall that in
the case of $\phi^4_4$ with $d < 4$ there is no horn of singularities [94]. As for the non-gauge field theories some arguments in favour of ambiguity of renormalized $\phi^4_4$ field theory has been given due to $1/N$ expansion [93].

The Lipatov bound (1) is compatible with SAC only for a regions of the asymptotic type $(0, \eta)$ (a is the same for all quantities in a theory [14, 15, 16, 89, 91]). As we have mentioned above a general rigorous justification of the Lipatov method has not been established so far. However further investigations would not change the fact that perturbation theory of four dimensional renormalizable field theories violates SAC. If the large order behaviour of the model in the limit of vanishing UV-cutoff will be proved to violate (66) then this will allow for nonperturbative solutions to the theory and can sheds new light on the triviality problem of the above theories [100, 101]. To get reliable perturbation theory one must find a nonperturbative ground state and expand around it [102]. In the case of the double well potential also some other techniques were advocated [103]. However, we cannot exclude the situation that the coefficients $a_n$ of perturbation theory will obey the bound (66) in this model. Provided it is so than no nonperturbative solutions can occur and the Borel transform should be an entire function, what contradicts expected singular structure of the Borel transform. However, it may well happen that there is no reason to look at the Borel transform of the four dimensional renormalizable massless field theory, because in this theory convergence of the Borel transform contradicts the observed horn-shaped region of analyticity [4, 22], in distinction from the lower dimensional case [90].
2 Goals and methods of the thesis

2.1 Goals

The main goal of the thesis is to overcome some shortcomings of the Borel summability method. Let us summarize the shortcomings which have been solved by the author.

A) the Borel method cannot cope with the horn-shaped singularity, exhibited in the massless QCD [21], in the “massless” limit of the anharmonic oscillator [22], and in some other models. (For recent status of the problem see [20] and references therein.)

B) In the regular case, i.e., when the series on the r.h.s. of (2) has a nonzero radius of convergence, it can in general be analytically continued onto a region of the complex plane which is larger than the region of \( z \) for which the Borel sum exists. If the analytic continuation \( f(z) \) is not an entire function, the Borel sum may not exist for all \( z \) from the Mittag-Leffler (principal) star of \( f(z) \) [62] (further MLS(\( f \))), because always some sector-like domain has to be discarded from the complex plane [4, 81].

To illustrate point (B) note that the standard Borel method sums the series \( \sum_{n=0}^{\infty} z^n \) only in the complex halfplane \( \text{Re } z < 1 \) but the analytic continuation \( 1/(1 - z) \) of the series exists in the whole complex plane except the point \( z = 1 \). We should like to demonstrate in simple examples that the Borel summability method does not have any privileged role and that it is just an ordinary method among a continuous variety of other analytic moment constant summability methods (AMCSM). It should be used when a divergent series has to be summed up to a function regular in a disc \( K(0, R) \). Provided the sum of a given divergent series is known to be analytic in some other region then the corresponding AMCSM compatible with this analyticity requirement has to be used.

To generalize known SAC to horn-shaped regions which may be of some physical importance is another goal of this thesis. The well-known example
of SAC provides the Nevanlinna theorem [63, 67]. These conditions are, however, applicable only when sum of a given perturbation series is to be analytic in a disc $K(0, R) := \{z \mid \text{Re}(1/z) > 1/R \}$ tangent to the imaginary axis at the origin (modulo mapping $z \rightarrow z^{1/\alpha}$). On the other hand, we have seen that many theories are known to be analytic in a horn-shaped region only. Without the validity of SAC one cannot prove that a statement which is valid order by order in the perturbation theory is a property of the full theory. One such example is, e.g., the Callan-Symanzik procedure where certain mass insertion terms are ignored and assumed small even though one has been able to prove this fact order by order. It was found [93] that the Borel summability or SAC given in [62, 63, 67] cannot help in justifying the Callan-Symanzik procedure just because of the horn-shaped singularities. In [93] it was stated that the horn-shaped singularity puts the Callan-Symanzik assumption about the mass insertion term in doubt. Our point of view is that there is no controversy between the horn-shaped singularity and the Callan-Symanzik procedure since one has to use SAC appropriate to a given analyticity region.

We shall also give a further improvement of the SAC as discussed in
and show that a product of functions obeying these SAC also obeys the same SAC. The knowledge about the relation of SAC and summability of perturbation theory is important in order to understand, e.g., the recent result on divergency of the bosonic string perturbation theory \cite{22} and frequently quoted in connection with two dimensional quantum gravity since there is no care taken about analyticity to which the perturbation theory should be summed up, and physical consequences are simply drawn from Borel nonsummability. Similar "negligence" of analytic properties also occurs, e.g., in \cite{13,33} or \cite{29}. Fortunately, the series treated in \cite{15,22,33} can be dealt by Lemmas \ref{A} and \ref{B} (see below). As for \cite{29} the results of this paper can be valid provided a fixed-angle elastic scattering amplitude of bosonic string $A(s,\phi;g)$ (see also \cite{30}) is analytic in a region $D$ of the sheeted complex plane with an opening angle $\Theta \geq \pi$ since the resummed amplitude $A_{\text{resum}}$ is such \cite{18,67}.

SAC discussed below may provide a tool to decide whether such ambiguity (degeneracy of the ground state of an underlying theory) is actually present. However, some further work has to be accomplished in order to gain the maximal domain of the analyticity, establish a rigorous large order behaviour or to give a suitable lower bound on the coefficients of the perturbation theory.

### 2.2 Methods

To solve the above shortcomings (A) and (B) of the Borel method we use theory of complex variables \cite{81}, asymptotic series \cite{104,105}, and divergent series \cite{61,62}. We shall start with the so-called regular case, i.e., when the series on the r.h.s. of (2) has nonzero radius of convergence. Then, by virtue of Theorem 2, our method will sum $\sum_{n=0}^{\infty} z^n$ to $1/(1-z)$ in the whole complex plane except the ray $[1,\infty)$ (=MLS[$1/(1-z)$]) in contrast to the Borel method (see Fig. 4).

In the next we shall consider moment constant summability method with the moment sequence $\{\mu(n)\}_{n=0}^{\infty}$,

$$\mu(n) := \int_{0}^{\infty} \exp(-\exp t) \, t^n \, dt,$$

i.e., with the moments in the Stieltjes form. Firstly we shall prove that the moment constant method solves the above difficulties (A) and (B) of
the Borel method, and, in the next, we shall give a whole family of such methods. Indeed, one of our main results is that if \( f(z) \) is the principal branch of an analytic function regular at the origin, where \( f(z) = \sum a_n z^n \), then the integral \( I(z_0) \),

\[
I(z_0) = \int_0^\infty \exp(-\exp t) \sum_{n=o}^\infty a_n \frac{(z_0 t)^n}{\mu(n)} dt,
\]

converges if and only if \( z_0 \in MLS(f) \). If \( z_0 \in MLS(f) \) then

\[ f(z_0) = I(z_0). \]

Thus, the method is *analytic* and *regular*. The convergence is absolute and uniform in any bounded subset of \( MLS(f) \) with nonzero distance from the boundary of \( MLS(f) \), and one can also differentiate inside the sign of integration. As for the property (B) the method is analogous to the Lindelöf one \[8\]. In contrast to the Borel method the method does not only see beyond singularities and may be used for localization of critical points of some physical models. We shall also show that the issues of the Borel summability method consist in that it is based on the asymptotic properties of the Mittag-Leffler function approaching zero in the complement of some sector with nonzero opening angle only \[12, 106\], while the new method \((14)\) is based on an entire function which approaches zero in all radial directions except for one. In an application to the Rayleigh-Schrödinger perturbation theory, theory of linear operators in Banach and Hilbert spaces is used \[109\].
3 New results

3.1 Regular case

From some general considerations we expect that if a moment constant summability method exists solving problems (A) and (B), then the moment sequence has to grow like $(\ln n)^n$ (see Remark 5). However, the problem of finding a nondecreasing function $\chi(t)$,

$$\mu(n) = \int_0^\infty t^n d\chi(t),$$

for a given moment sequence $\{\mu(n)\}_{n=0}^\infty$ (the Stieltjes problem [92, 119]) is very complicated. Throughout the paper we shall only consider absolutely continuous functions $\chi(t)$. Instead of a moment sequence, we shall prefer to choose a suitable weight function. Such a weight function could be $\exp(-\exp t)$. The reasons are that the moment constant summability method with the moment sequence $\{\mu(n)\}_{n=0}^\infty$ where $\mu(n)$ are given by (14) plainly belongs to the moment constant summability methods discussed in [62], and as such it is regular. The next reason is yielded by the following Lemma.

Lemma 1: The function

$$\mu(s) := \int_0^\infty \exp(-\exp t)t^s dt$$

(15)

1) has a meromorphic extension onto the whole complex plane with simple poles on the negative real axis;
2) its asymptotic behaviour for $s$ tending to infinity, $|\arg s| < \pi$, is governed by the saddle point only and

$$\mu(s) \sim [2\pi w(s)w'(s)]^{1/2} \exp[-\exp w(s) + s \ln w(s)].$$

(16)

The saddle point $w(s)$ is given in the complex plane by the equation

$$(\exp t)t = s.$$  

(17)

3) The function $\mu(s)$ exhibits no zero for $\Re s > -1$.  

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Before the proof we give some comments on the solutions of (17) for $s$ complex. In this case we shall always assume that the $s$-complex plane is cut along either the real positive either the real negative axis. Correspondingly a sheet in the $w$-complex plane is chosen which is bounded either by the real positive axis from below and by a curve $\phi + \arctan(\phi/w) = 2\pi$ when $Re w \geq 0$, $\phi + \arctan(\phi/w) = \pi$ when $Re w \leq 0$, from above, $\phi$ being $Im w$,

$0 \leq Im w < 2\pi$. ; either by the curves $\phi + \arctan(\phi/w) = \pi$ when $Re w \geq 0$, $\phi > 0$, and $\phi + \arctan(\phi/w) = 0$ when $Re w \leq 0$, $\phi < 0$, $-\pi < Im w < \pi$.

\textit{Proof}: 1) The integral on the r.h.s. of (13) converges absolutely in the whole complex halfplane $Res > -1$ and defines there an analytic function. To perform an analytic continuation we make use of the standard Cauchy procedure. Being an entire function, $\exp(-\exp t)$ possesses a power series expansion,

\[
\exp(-\exp t) = \sum_{n=0}^{\infty} c_n t^n,
\]

with infinite radius of convergence. Then splitting the integration in (13) into two intervals, one obtains

\[
\mu(s) = \sum_{n=0}^{\infty} \frac{c_n t_0^{n+s+1}}{(n + s + 1)} + \int_{t_0}^{\infty} \exp(-\exp t)t^s \, dt,
\]

(18)

where the r.h.s. of (18) is defined for all $s$ except for $s = -n - 1$, thereby providing an analytic continuation of the integral on the r.h.s. of (13) onto the whole complex plane after one sets $t_0 = 1$.

2) For $s$ tending to infinity and $|\arg s| \leq \pi/2$ the asymptotic behaviour of $\mu(s)$ can be calculated directly using the saddle point technique on the integral on the r.h.s. of (13). The contribution $V_s$ of the saddle point (17) is calculated in the standard manner \[104, 105\]. One rewrites integrand of (13) as an exponential function $\exp h(t)$, where

\[
h(t) := -e^t + s \ln t,
\]

(19)

and looks for critical points of $h(t)$. This means that one solves the equation

\[
h'(t) = -e^t + s/t = 0.
\]

(20)
Any solution of (20) is called the saddle point and its contribution to (15) is given by the formula

\[ V_s = \left( \frac{-2\pi}{h''(t_s)} \right)^{1/2} e^{h(t_s)}, \quad (21) \]

where \( t_s \) denotes the value of a given saddle point. Under some additional conditions one can prove that for \( s \to \infty \) the value of integral (15) is completely determined by the contributions of saddle points [104, 105]. In our case one finds that (20) has a unique solution in the cut complex plane, which we shall denote by \( w(s) \). Since \( w(s) \exp(w(s)) = s \) one finds that

\[ w'(s) = \frac{1}{e^{w(s)}(w(s) + 1)}, \quad w(s)w'(s) = -\frac{1}{h''(w(s))}. \quad (22) \]

Thus the contribution of the saddle point \( w(s) \) is given by

\[ V_s = (2\pi w(s)w'(s))^{1/2} \exp(-e^{w(s)} + w(s) \ln s) \]

and the asymptotic behaviour of \( \mu(s) \) is completely governed by it [104, 105], so that

\[ \mu(s) \sim (2\pi w(s)w'(s))^{1/2}[\exp(-e^{w(s)} + w(s) \ln s)]. \]

For \( s \to \infty \) and \( |\arg s| > \pi/2 \) the situation is quite different, because the contribution \( V_e \) of the end point of the integral on the r.h.s. of (18) [104, 105]

\[ V_e = -t_o^s \exp(-\exp t_o) / s \quad (s \to \infty), \]

is far from being negligible and in fact is greater than the contribution \( V_s \) of the saddle point which approaches zero when \( s \) tends to infinity, and \( \pi/2 < |\arg s| < \pi \). However, we can prove that the contribution \( V_e \) is nothing but the sum on the r.h.s. of (18) in the large \( s \) limit with opposite sign, so that these contributions cancel each other and only the contribution of the saddle point \( V_s \) survives. In fact,

\[ \sum_{n=0}^{\infty} c_n \frac{t_o^{n+s+1}}{n + s + 1} = (t_o^{s+1}/s) \sum_{n=0}^{\infty} c_n \frac{t_o^n}{1 + (n + 1)/s}. \quad (23) \]

The sum on the r.h.s. of (23) converges uniformly in \( s, 0 \leq |\arg s| \leq \pi - \varepsilon, \)

\[ \sum_{n=0}^{\infty} c_n \frac{t_o^n}{1 + (n + 1)/s} \leq (1/\delta) \sum_{n=0}^{\infty} c_n t_o^n < \infty, \]

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where $\delta$ is some constant which does not depend on $s$, so we have

$$\lim_{s \to \infty} \sum_{n=0}^{\infty} c_n t_o^n/[1 + (n + 1)/s] = \sum_{n=0}^{\infty} c_n t_o^n = \exp(-\exp t_o),$$

and

$$\sum_{n=0}^{\infty} c_n t_o^{n+s+1}/(n + s + 1) \sim -V_e \quad (s \to \infty).$$

To prove 3) suppose that for some $s_o$,

$$\mu(s_o) = \int_{-\infty}^{\infty} \exp[-\exp(\exp t) + (s_o + 1)t]dt = 0.$$

However, this implies that for any $\delta$,

$$\int_{-\infty}^{\infty} \exp\{-\exp[\exp(t + \delta)] + (s_o + 1)t\} \quad dt = 0,$$

what means on the other hand that

$$\int_{-\infty}^{\infty} \exp[-\exp(\exp t) + (s_o + 2)t][-\exp(\exp t)] \quad dt = 0.$$

Repeating this procedure $m$-times one gets that

$$I(m) = \int_{-\infty}^{\infty} \exp[-\exp t + mt + (s_o + m) \ln t]A^{-m}[A(-1+d/dA)]^{m-1}(-A) \quad dt = 0,$$

where $A = \exp t$. Note that the term $D(t)$,

$$D(t) := A^{-m}[A(-1+d/dA)]^{m-1}(-A),$$

is bounded on $t \in (0, \infty)$ and nonzero for sufficiently large $t$. On the other side, provided $m$ is sufficiently large we may use an asymptotic formula to evaluate $I(m)$. Thus,

$$I(m) \sim [2\pi u'/u(1+1)]^{1/2} \exp[-\exp u + mu + (s_o + m) \ln u]D(u) = 0,$$

since $u(m)$ is the solution of the equation

$$-e^t + m + (s_o + m)/t = 0,$$

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which behaves like

\[ u(m) \sim \ln[m + (s_o + m)/\ln m] \quad (m \to \infty). \]

\[ \diamond \]

If one uses the approximate solution to the saddle point equation (17)

\[ w(s) = \ln(s/\ln s) + \ln \ln s/\ln s + O[(\ln \ln s/\ln s)^2] \quad (s \to \infty), \quad (24) \]

then one finds the following approximate asymptotic behaviour of \( \mu(s) \),

\[ \mu(s) \sim (2\pi s/\ln s)^{1/2} [\exp - (s/\ln s)][\ln(s/\ln s)]^s \quad (s \to \infty), \quad (25) \]

Some words about a numerical implementation of the method. Note that the method can be implemented numerically as well, since the solution to (17) is very quickly determined by the following recursive formula:

\[ w(n) = \ln(n/\ln(n/\ln(n/\ldots))), \quad (26) \]

and the asymptotic formula (16) very accurately reproduces the value of the integral (15) (in fact, up to ten orders by a very simple algorithm). One also finds that and when \( s \) tends to infinity and \( |\arg s| < \pi \) the relation (24), \( w'(s) \sim 1/s - 1/sw(s) \), still holds.

Remark 1 : Note that the proof of Lemma 1 can be directly adapted to the case of the weight function \( \exp[-e_k(t)] \), where \( e_k(t) \) is the \( k \)-fold exponential function,

\[ e_k(t) := \exp(\exp(...(\exp t)...), \]

\( k \) being an arbitrary non-negative integer. The saddle point is now given by the equation

\[ e_k(t)e_{k-1}(t)...e^t = s. \]

In the case of \( k = 0 \) one obtains a simpler proof of the asymptotic properties of the gamma function than the one usually known \[106\]. For \( k \geq 2 \) one obtains another moment constant summability method \( \mu_k \) solving the problem (B) (see below). They give nothing new in the regular case, but they play quite an important role in the singular case. Considering the integral on the r.h.s. of (13) as the Mellin transform, one could adapt Lemma 1 to study
the Mellin transform of some class of entire functions.

Now, our main problem is the determination of asymptotic properties of the moment function $F(t)$, defined by

$$ F(t) := \sum_{n=0}^{\infty} \frac{t^n}{\mu(n)}. \quad (27) $$

The reason is that for $|z| < 1$

$$ \int_{o}^{\infty} \exp(-\exp t) F(tz) \, dt = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \quad (28) $$

due to the regularity of the $\mu$-method [24]. Because of Lemma [1] the function $F(t)$ is an entire function. One expects that

$$ \max_{|t|=const} \ln |F(t)| \sim \exp(\text{const} \exp |t|) \quad (t \to \infty) $$

[103, 107]. If $F(t)$ grew no faster than $\exp |t|^A$ when $t$ approaches infinity, and $|\arg t| > 0$, where $A$ is a constant, then (28) should converge absolutely for every $z \in MLS[1/(1-z)]$, the Mittag-Leffler star of the Cauchy kernel $1/(1-z)$ providing in such way an analytic continuation of $\sum_{n=0}^{\infty} z^n$ from the unit disc onto the whole $MLS[1/(1-z)]$.

Let $f(z)$ be the principal branch of an analytic function regular at origin. Then

$$ f(z) = (1/2\pi i) \oint_{C} \frac{f(u)}{u(1-z/u)} \, du = (1/2\pi i) \oint_{C} \frac{f(u)}{u} \, du \int_{o}^{\infty} \exp(-\exp t) F(tz/u) \, dt \quad (29) $$

for every simple contour $C$ such that no singularity of $f(z)$ lies on $C$ or inside it. If in addition the contour $C$ is such that $\{z/u \mid u \in C\} \subset MLS[1/(1-z)]$, then we can invert the order of the integrations in (29), to obtain

$$ f(z) = \int_{o}^{\infty} \exp(-\exp t) \sum_{n=0}^{\infty} \frac{(tz)^n}{\mu(n)} \left( \frac{1}{2\pi i} \int_{C} \frac{f(u)}{u^{n+1}} \, du \right) \, dt $$

$$ = \int_{o}^{\infty} \exp(-\exp t) \sum_{n=0}^{\infty} a_n \frac{(tz)^n}{\mu(n)} \, dt, \quad (30) $$
$a_n$ being the Taylor coefficients of $f(z)$ at origin. It is a short exercise to show that the set of $z$ for which the $\mu$-sum (30) exists is just $\MLS(f)$. Hence to show that the $\mu$-method provides an analytic continuation of the Taylor series of $f(z)$ at origin onto the whole $\MLS(f)$ we have to prove that $F(t)$, as defined by (27), does not grow faster than $\exp(|t|A)$ when $t$ tends to infinity and $|\arg t| > 0$. One can also deal with the case where some finite number of singularities of $f(z)$, but no branch point, are elements of $C_o$ (interior domain with respect to $C$) provided the condition $\{z/u | u \in C\} \subset MLS[1/(1-z)]$ is satisfied. The result is

$$f(z) = -\sum_s' \text{Res}[f(u), z_s]/(z_s - z) + \int_0^\infty \exp(-\exp t) \sum_{n=0}^\infty \tilde{a}_n (zt)^n \mu(n) dt, \quad (31)$$

where

$$\tilde{a}_n = a_n + \sum_s' \text{Res}[f(u), z_s]/z_s^{n+1}, \quad (32)$$

and $\sum_s'$ in (31,32) runs over all singularities $z_s$ of $f(z)$, which are elements of $C_o$ (see also the next section). As an exercise one can justify the relations (31,32) for the Cauchy kernel $1/(1-z)$, where $\tilde{a}_n = 0$ so that the integral on the r.h.s. of (31) vanishes.

Let us turn now to the study of asymptotic properties of the function $F(t)$.

**Lemma 2** Let $\{\mu(n)\}_{n=0}^\infty$ be the Stieltjes moment sequence generated by the measure $\exp(-\exp t) dt$ (14). Then the function $F(t)$, defined by (27),

$$F(t) = \sum_{n=0}^\infty \frac{t^n}{\mu(n)},$$

is an entire function with the following asymptotic behaviour at infinity:

1) For $|\text{Im} t| \leq \pi/2$ the asymptotic behaviour is determined by the saddle point $s = (\exp t)t$, of the Euler-Maclaurin integral representation of $F(z)$,

$$F(t) = \int_\sigma^\infty \frac{e^{sint}}{\mu(s)} ds + \mathcal{O}(|t|\sigma) \quad (t \to \infty),$$

$\sigma$ being some constant, $-1 < \sigma < 0$. Therefore

$$F(t) \sim \exp \{\exp t + t + \ln(t + 1)\} \quad (t \to \infty).$$
2) For \( |\text{Im} t| > \pi/2 \), the asymptotic behaviour of \( F(z) \) is still determined by the saddle point whenever its contribution prevails the contribution of the end point of integration. Otherwise, as well as for \( |\text{Im} t| > \pi \),

\[
|F(t)| \leq O(|t|^\sigma) \quad (t \to \infty).
\]

Proof: By Lemma 1 \( \mu(s) \) exhibits no zero in the right complex half-plane. Then by virtue of the Euler-Maclaurin sum formula \([104, 105]\) we arrive at the ensuing integral representation of \( F(t) \),

\[
F(t) = \int_{\sigma}^{\infty} e^{\text{Re} t} \frac{\mu(s)}{\mu(s)} ds + O(|t|^\sigma) \quad (t \to \infty),
\]

(33)

where \( \sigma \) is some noninteger from the interval \((-1, 0)\). To find an asymptotic behaviour of \( F(t) \) at infinity we shall use the standard saddle point method as in the proof of Lemma 1. In the present case one does not know an explicit form of the function \( h(s) \) \((19)\) which determines the position of saddle points. Nevertheless the position of the saddle point of \((33)\) can be found exactly for sufficiently large \( t \). As one expects and as we shall show in a moment, function \( h(s) \), and thus equation for the saddle point is determined by the asymptotic behaviour of \( \mu(s) \), provided \( t \) is sufficiently large. Let as above \( w = w(s) \) be the solution of \((17)\), i.e.,

\[
w(s) \exp[w(s)] = s.
\]

(34)

Due to Lemma 1 the asymptotic behaviour of \( \mu(s) \) at infinity is

\[
\mu(s) \sim [2\pi w(s)w'(s)]^{1/2} \exp\{-\exp[w(s)] + s \ln w(s)\}.
\]

(35)

This in turn determines an asymptotic form of the function \( h(s) \),

\[
h(s) = \exp[w(s)] - s \ln w(s) + s \ln t,
\]

(36)

and hence a solution of the equation

\[
h'(s) = e^w w'(s) - \ln w - sw'(s)/w(s) + \ln t = 0.
\]

(37)

By using the relation \((17)\) the last equation is essentially reduced to

\[
h'(s) = -\ln w + \ln t = 0,
\]

(38)
and the saddle point is \( t = w(s) \). From (38) one finds that \( h''(s) = -w'(s)/w(s) \) at the saddle point. Finally, from (21) one obtains the contribution \( V_s \) of the saddle point,

\[
V_s := \left( \frac{2\pi w(s)}{w'(s)} \cdot \frac{1}{2\pi w(s)w'(s)} \right)^{1/2} e^{\psi^t} = [w'(s(t))]^{-1} e^{\psi^t} := e^{\omega(t)},
\]

where

\[
\omega(t) := \exp t + t + \ln(t + 1).
\]

If \( |\text{Im} t| \leq \pi/2 \) then one can justify that the asymptotic behaviour of \( F(t) \) is determined by the contribution of the saddle point only. If \( \pi/2 < |\text{Im} t| < \pi \) then the asymptotic behaviour is still determined by the contribution of the saddle point provided it prevails the contribution of the end point of the integral (33). For example this happens if one stays on special curves which approximates \( y = \pi/2 \; (y = -\pi/2) \) from above (below) if \( x \to \infty \) (see also below section 3.3). Otherwise, and for \( |\text{Im} t| > \pi \) as well, the contribution of the end point \( s = \sigma \) of the integral (33) prevails and \( |F(z)| \) tends to zero. If \( |\text{Im} t| \geq \pi \) the contour of the steepest descent does not exist since the saddle point does not lie on the first sheet of the multivalued function \( w(s) \). Another point of view is that we do not have any saddle point of the integral (33) since \( w(s) \) in (35) is only defined for \( |\arg s| < \pi \) for which \( |\text{Im} w(s)| < \pi \).

Hence, the relation (38) cannot be satisfied for such \( t \). In both cases, however, one can show that the asymptotic behaviour of \( F(t) \) is determined by the end point \( s = \sigma \) of the integral (33), i.e., \( F(t) \sim O(|t|^\sigma) \) for \( t \) tending to infinity and \( |\text{Im} t| \geq \pi \). For \( |\arg t| > 0 \) this can be exemplified independently as follows. By virtue of the analytic properties of \( \mu(s) \) and its behaviour at infinity (25) one can rotate the contour of integration from the real axis to the ray \((\sigma, \sigma + i\infty)\), where \( q = \text{sign} (\arg t) \). Let \( \arg t > 0 \). Then the integral \( I \),

\[
I = \int_{\sigma}^{\sigma+i\infty} \frac{e^{s\ln t}}{\mu(s)} ds,
\]

can be majorized in the subsequent manner,

\[
|I| \leq e^{\sigma \ln |t|} \int_{\sigma}^{\sigma+i\infty} \frac{e^{-y \arg t}}{|\mu(s)|} |ds| \leq 2 \text{const} e^{\sigma \ln |t|},
\]

where \( s = \sigma + iy \), because on the ray \((\sigma, \sigma + i\infty)\)

\[
|1/\mu(s)| \sim \exp[\text{const} (y/\ln y)] \quad (y \to \infty).
\]
Thus $I$ decays at worst like $|t|^\sigma$ when $t$ tends to infinity and the proof is finished. ◇

**Remark 2**: Note that the position of the saddle point is

$$w(s) = t$$

also in the case of Stieltjes moment sequence $\{\mu_k(n)\}_{n=0}^{\infty}$ generated by $\exp[e_k(t)]$, where $k$ is an arbitrary nonnegative integer. Now, $w(s)$ is the solution of (34) with $\exp w$ replaced by $(d/dw)e_k(w)$ (see Remark 1). The contribution of the saddle point is up to the Gaussian integral around the saddle point

$$V_s = \exp[e_k(w)] \quad (t \to \infty).$$

Analogously as in the above case of $k = 1$, $F_k(t)$, defined by

$$F_k(t) = \sum_{n=0}^{\infty} \frac{t^n}{\mu_k(n)},$$

where $k \geq 2$, is also polynomially bounded when $t$ tends to infinity and $|\arg t| > 0$ because of

$$|1/\mu_k(s)| \sim \exp\{\text{const}[y/\ln y...\ln_k(y)\ln_k(y)]\} \quad (y \to \infty),$$

on the ray $(\sigma, \sigma + i\infty)$, $\ln_k(y)$ being the $k$-fold logarithm. It means that such methods also solve the problem (B)! The principal difference between the Borel ($k = 0$) and $\mu_k$-methods with $k \geq 1$ consists in the fact that for the Borel method the integral on the r.h.s. of (11) may not converge if $|\arg t| > 0$. ◇

The main result of this section is the following Theorem.

**Theorem 2**: Let $f(z)$ be the principal branch of an analytic function regular at origin,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$
Then the integral $I(z_0)$,

$$I(z_0) = \int_0^\infty \exp(-\exp t) \sum_{n=0}^{\infty} a_n \frac{(z_0 t)^n}{\mu(n)} \, dt,$$

(43)

converges if and only if $z_0 \in MLS(f)$, where $\mu(n)$ are defined by (14). If $z_0 \in MLS(f)$ then

$$f(z_0) = I(z_0).$$

The convergence is absolute and uniform in any bounded subset of $MLS(f)$ with nonzero distance from the boundary of $MLS(f)$ and we can differentiate inside the sign of integration,

$$f'(z) = \int_0^\infty \exp(-\exp t) \sum_{n=1}^{\infty} n a_n \frac{z^{n-1} t^n}{\mu(n)} \, dt.$$

Provided $f(z)$ has no branch point on or inside $C$, then the relations (31,32) hold.

Proof: Note that the integrand of (43) is an entire function, so that it is well defined. The result follows immediately from Lemma 2 and the relations (29,30). ◇

Remark 3: $MLS(f)$ is invariant with respect to differentiation, i.e., $MLS[f(z)] = MLS[f'(z)]$. If $f(z)$ is as in Theorem 4, then $f'(z)$ also satisfies the conditions of this theorem, and we have another representation of $f'(z)$,

$$f'(z) = \int_0^\infty \exp(-\exp t) \sum_{n=1}^{\infty} n a_n \frac{(zt)^{n-1}}{\mu(n-1)} \, dt,$$

i.e., the moments $\mu(n)$ can be in some sense shifted. ◇

Let $f(z)$ be as in Theorem 2 and regular on the real positive axis, for instance. Then Theorem 2 enables us to give an analytic continuation of the Taylor series of $f(z)$ at origin on the whole axis. In fact the following Corollary holds.
Corollary 1: Let
1) $f(z)$ be the principal branch of an analytic function regular at origin;
2) $f(z)$ be regular on the real positive axis.

Then
\[
f(x) = \int_0^\infty \exp(-\exp t) \sum_{n=0}^\infty a_n \frac{(xt)^n}{\mu(n)} \, dt
\]
for $x \in [0, \infty)$. The integral converges absolutely and one can differentiate inside the sign of integration.
3.2 Some new results on the Borel summability method

In this section we wish to prove an analogue of Theorem 3 for the Borel method. Some results are exposed in [81], but a detailed study of the region of convergence of the Borel integral is missing. The same role as the function $F(t)$ has in the $\mu$-method discussed above plays the generalized Mittag-Leffler function in the Borel method. As we have mentioned above the Borel method is one member of the moment constant summability methods. Its weight function is $(1/\alpha)t^{(\beta/\alpha)-1}\exp(-t^{1/\alpha})$, and the moments of the Borel method are in general

$$(1/\alpha) \int_0^\infty t^{(\beta/\alpha)-1}e^{-t}t^n\,dt = \Gamma(\alpha n + \beta),$$

where $\Gamma$ is the usual gamma function, and $\alpha, \beta$ are positive constants, $0 < \alpha \leq 2$ (note that the Borel method with $\alpha = \beta = 1$ can be viewed as a special case $\mu_0$ of the $\mu_k$-methods discussed in Remark 1). The generalized Mittag-Leffler function is the function

$$E_{\alpha,\beta}(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(\alpha n + \beta)}$$

[106]. We wish to show that the problems (A) and (B) of the Borel method are results of the fact that $E_{\alpha,\beta}(z)$ is unbounded in some sector-like domain with nonzero opening angle. Indeed, its asymptotic properties when $z$ approaches infinity are as follows [52, 106]:

a) $$E_{\alpha,\beta}(z) \sim \sum_{n=1}^\infty z^{-n}/\Gamma(\beta - \alpha n) \quad if \quad |\arg(-z)| < (1 - \alpha/2)\pi,$$

b) $$E_{\alpha,\beta}(z) \sim (1/\alpha) \sum_m t_m^{1-\beta}e^{t_m} \quad if \quad |\arg z| \leq \pi \alpha/2,$$

where $t_m = z^{1/\alpha}e^{2\pi im/\alpha}$, and the sum runs over all $m$ such that $-\pi \alpha/2 \leq \arg z + 2\pi m \leq \pi \alpha/2$. Because of these asymptotic properties the integral

$$\int_0^\infty t^{\beta-1}e^{-t}E_{\alpha,\beta}(zt^\alpha)\,dt,$$ (44)
which is for \(|z|<1\) equal to \(1/(1 - z)\), does not converge in the whole MLS\([1/(1 - z)]\) but only in some domain \(B_\alpha[1/(1 - z)]\),

\[
B_\alpha[1/(1 - z)] := \{z \mid \text{Re}(z)^{1/\alpha} = r^{1/\alpha}[\cos(\theta/\alpha)] < 1\}. \tag{45}
\]

Note that \(B_\alpha[1/(1 - z)]\) does not depend on \(\beta\). Due to the relations (29,30) such a representation of the Cauchy kernel determines the representation of an arbitrary function \(f(z)\) regular at origin. Let \(C(z_o)\) be a contour given by the relation

\[
r = r_o\{\cos((\theta - \theta_o)/\alpha)\} \quad \text{where} \quad |\theta - \theta_o| < \pi\alpha/2. \tag{46}
\]

Let us draw the contour \(C(z_s)\) for each singularity \(z_s\) of \(f(z)\) and let us discard from the complex plane the domain \(C_o(z_s)\) closed up by the contour, for which the sign = in (46) is replaced by >. Then the notation \(B_\alpha(f)\) is adopted for the simply connected region of the complex plane containing the origin. By the construction \(B_\alpha(f)\) is a starlike region containing, in any case, the disc of convergence of the Taylor series of \(f(z)\) around the origin.

**Theorem 3**: Let \(f(z)\) be the principal branch of an analytic function, regular at origin,

\[
f(z) = \sum_{n=0}^{\infty} a_n z^n.
\]

Then the integral \(I(z_o)\),

\[
I(z_o) = \int_0^\infty t^{\beta-1}e^{-t} \sum_{n=0}^{\infty} a_n \frac{(z_o t)^n}{\Gamma(\beta + \alpha n)} \, dt, \tag{47}
\]

converges if and only if \(z_o \in B_\alpha(f)\). If \(z_o \in B_\alpha(f)\) then

\[
f(z_o) = I(z_o).
\]

The convergence is absolute and uniform on any bounded subset of \(B_\alpha(f)\) with nonzero distance from the boundary of \(B_\alpha(f)\). For the first derivative \(f'(z)\) of \(f(z)\) the following representation

\[
f'(z) = \int_0^\infty t^{\beta-1}e^{-t} \sum_{n=0}^{\infty} n a_n \frac{z^{n-1} t^{\alpha n}}{\Gamma(\beta + \alpha n)} \, dt \tag{48}
\]

holds.
Proof: Proof of Theorem is nothing but a slight modification of the proof of Theorem 2 to the case of another Stieltjes moment sequence. The double integral on the r.h.s. of (29) converges if and only if for a given $z_0$ a contour $C$ exists such that $\{z_0/u \mid u \in C \} \subset B_0[1/(1 - z)]$. Interchanging the order of integrations in (29) one obtains the relation (47) if and only if:

a) the origin and $z_0$ are elements of $C_o$;

b) no singularity $z_s$ of $f(z)$ is an element of $C_o \cup C$;

c) $\forall u \in C$, where $u = \rho \exp(i\phi)$ and such that $|\theta_o - \phi| < \pi \alpha/2$, the inequality

$$Re \left( z_o/u \right)^{1/\alpha} = \left( r_o/\rho \right)^{1/\alpha} \cos [\theta_o - \phi]/\alpha \leq 1 - \delta$$

holds, where $\delta$ is a positive constant.

The first two conditions are obvious; the last one follows from the fact that the integral (44) converges if and only if

$$Re \ z^{1/\alpha} \leq 1 - \delta.$$ 

One can easily check that a set of all $z_o$ such that a contour $C$ with the above properties exists is just the region $B_\alpha(f)$. In fact, for all $z \in B_\alpha(f)$ such a contour exists. If $z \notin B_\alpha(f)$, then inevitably a singularity $z_s$ of $f(z)$ exists such that

$$r \geq r_s \{ \cos [(\theta - \theta_s)/\alpha] \}^{-\alpha} \quad \text{when} \quad |\theta - \theta_s| < \pi \alpha/2.$$ 

This, however, contradicts to the properties b) and c) under which

$$r_s > r \{ \cos [(\theta - \theta_s)/\alpha] \}^{-\alpha} > \rho \quad \text{if} \quad |\theta - \theta_s| < \pi \alpha/2.$$ 

The relation (48) then follows immediately from the absolute convergence and the fact that under the conditions of the Theorem the integrand of (47) is an entire function. ◊

Remark 4: Like MLS$(f)$ the domain $B_\alpha(f)$ is also invariant under differentiation, $B_\alpha(f') = B_\alpha(f)$, so that inserting the first derivative $f'(z)$ in place of $f(z)$ in Theorem 3 one obtains another integral representation of $f'(z)$, like in the previous section,

$$f'(z) = \int_0^\infty t^{\beta - 1} e^{-t} \sum_{n=1}^\infty n a_n \frac{(zt^\alpha)^{n-1}}{\Gamma[\beta + \alpha(n - 1)]} dt.$$ ◊
Remark 5: Note that whenever $\alpha < \alpha'$, then $B_{\alpha}(f) \subset B_{\alpha'}(f)$. For $\alpha \to 0$ the domain $B_{\alpha}(f)$ approaches $MLS(f)$. However, the limit

$$\lim_{\alpha \to 0} \Gamma(\beta + \alpha n) = \Gamma(\beta)$$

is trivial (does not depend on $n$). This fact provides one of arguments for the moment sequence of the moment summability methods summing the Taylor series of $f(z)$ in the whole $MLS(f)$ to grow like $(\ln n)^n$, because

$$\mu(n) \sim (\ln n)^n = o[\Gamma(\beta + \alpha n)] \quad (n \to \infty),$$

for all positive $\alpha$ and $\beta$. Another way is to consider the weight function for the Borel moments which contains a factor $\exp(-t^{1/\alpha})$ and its behaviour when $\alpha \to 0_+$. ♦

Theorem 3 shows that in the regular case the Borel summation is not only the Laplace transform (cf. [108]). The integral (47) in the Laplace form,

$$I'(z_0) = \left(1/z_0\right)^{1/\alpha} \int_{z_0}^{\infty} t^{\beta-1} \exp(-t/z_0^{1/\alpha}) \sum_{n=0}^{\infty} a_n \frac{t^{\alpha n}}{\Gamma(\beta + \alpha n)} dt,$$

has a different region of convergence but one is an analytic continuation of the other. In fact, adopt the notation $L(z)$ ($L^\alpha(z)$) for the simple contour (the interior domain with respect to it), where $L(z)$ is parametrized by $u := \rho \exp(i\phi)$ in the following way (see Fig. 5):

a) If $|\theta - \phi| \leq |\theta - \phi_o| < \pi \alpha/2$, where $z = r \exp(i\theta)$, then

$$\rho = r \{\cos[(\theta - \phi)/\alpha]\},$$

(50) $\pm \phi_o$ being determined by the equation

$$\rho = r \left[(\cos(\theta - \phi)/\alpha)^\alpha = \varepsilon,\right.$$

where $\varepsilon$ is strictly less than radius of convergence of the Taylor series at the origin of the function under consideration;

b) If $|\theta - \phi| > |\theta - \phi_o|$, $\pm \phi_o$ being the same as above, then

$$\rho = \varepsilon.$$

(51)

The following theorem holds.
Theorem 4: Let \( f(z) \) be the principal branch of an analytic function regular at origin. Denote by \( D_\alpha(f) \) the domain
\[
D_\alpha(f) := \bigcup_{z \in B_\alpha(f)} L^\alpha(z),
\]
where \( L^\alpha(z) \) is defined by \((50),(51)\). Thus for any \( z \in D_\alpha(f) \), \( z = r \exp(i\theta) \), there exists \( z^* = r^* \exp(i\theta^*) \in B_\alpha(f) \) such that \( z \in L^\alpha(z^*) \). Let \( z_o \) be \( z_o := z \exp(-i\theta^*) \).

Then the integral
\[
I'(z) := (1/z_o)^{1/\alpha} \int_0^\infty t^{\beta-1} \exp(-t/z_o^{1/\alpha}) a_n (e^{i\theta^* t^\alpha})^n / \Gamma(\beta + \alpha n) \, dt
\]
converges (outside the Borel polygon!) and equals to \( f(z) \).

Proof: Suppose the corresponding \( z^* \in B_\alpha(f) \) has been found. Then the integral
\[
\int_0^\infty t^{\beta-1} e^{-t} \sum_{n=o}^\infty a_n \frac{(z^* t^\alpha)^n}{\Gamma(\beta + \alpha n)} \, dt
\]
converges. After the real substitution \( t \to tr^{1/\alpha} \) one gets the integral in the Laplace form,
\[
(1/r)^{1/\alpha} \int_0^\infty t^{\beta-1} \exp(-t/r^{1/\alpha}) \sum_{n=o}^\infty a_n \frac{(e^{i\theta^*} t^\alpha)^n}{\Gamma(\beta + \alpha n)} \, dt,
\]
which converges \( \forall w \in C_o \) (the complex plane of \( r \)) such that
\[
Re \, w^{1/\alpha} > r^{1/\alpha},
\]
or, what is the same,
\[
w < r [\cos(\theta/\alpha)]^\alpha.
\]
But the complex plane \( C_o \) is nothing but the rotated \( C \), i.e., \( C_o = \exp(-i\theta^*) C \).
\[\Diamond\]

Let us give an example to illustrate the Theorem. Consider the function \( f(z) = 1/(1 - z) \) and the standard Borel method (\( \alpha = \beta = 1 \)). Now, \( B_1(f) \)
is the complex halfplane $\text{Re} \, z < 1$, and $D_1(f) \equiv \text{MLS}(f)$. Let us calculate, e.g., $f(2 + 3i)$. As the corresponding $z^*$ we choose $z^* = 8i$. Hence $\theta^* = \pi/2$ so we have $z_0 = 3 - 2i$ (see Fig. 5). According to (53)

$$
\begin{align*}
&f(2 + 3i) = (3 - 2i)^{-1} \int_0^\infty \exp\left[-t/(3 - 2i)\right] \sum_{n=0}^\infty \frac{(it)^n}{n!} \, dt \\
&= (3 - 2i)^{-1} \int_0^\infty \exp\left\{-t \left(\frac{1}{3 - 2i} - i\right)\right\} \, dt = -\frac{1}{1 + 3i}.
\end{align*}
$$

Theorem 4 shows the principal obstruction for the Borel method to be convergent in the whole MLS of function under consideration. It is due to the fact that whenever the Borel sum exists for some $z_0$, then it also exists for $z \in L^0(z_0)$ and defines an analytic function there.

At the end of this section we wish to show that the Borel method can represent a function regular on the real positive axis only under assumption of its regularity in a larger sector-like domain.
Corollary 2: Let $f(z)$ be regular in a sector-like domain $S_\gamma$, $S_\gamma := \{ z \mid r > 0 \text{ and } |\theta| < \gamma \}$. Then the integral

$$\frac{1}{z^{1/\alpha}} \int_0^\infty t^{\beta - 1} \exp[-t/z^{1/\alpha}] \sum_{n=0}^\infty a_n \frac{t^{\alpha n}}{\Gamma(\beta + \alpha n)} \, dt$$

converges for $z \in (0, \infty)$ if and only if $\alpha < 2\gamma/\pi$. If $\alpha < 2\gamma/\pi$, then the integral converges on each sector $S_\eta$, $S_\eta := \{ z \mid r > 0 \text{ and } |\theta| < \eta \}$, where $\eta < \pi\alpha/2$.

Proof: Only under the condition $\alpha < 2\gamma/\alpha$ the real positive axis belongs to $B_\alpha(f)$. ♦

Note that all results of the section can be generalized to the case where some singularities of $f(z)$, but no branch point, lie in $C_0$. Note also that for the class of functions satisfying the hypotheses of the Nevanlinna theorem the Borel transform is analytic in a strip including the real positive axis. So, applying Corollary 1 one directly finds an expression for the Borel transform on the whole axis (in terms of $a'_n$).

As for the generalization of the strong asymptotic condition one can see that by replacing the factor $N!$ in the above bound of $R_N(z)$ by $\mu_k(N)$ one obtains uniqueness theorems for horn-shaped regions, $k$ characterizing the sharpness of a horn. The proof is simply performed by combining conformal mappings with the Phragmen-Lindelöf theorems. 

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3.3 Singular case

As it has been shown above the moment constant summability method with the moment sequence \( \{ \mu(n) \}_{n=0}^{\infty} \),

\[
\mu(n) := \int_0^{\infty} \exp(-\exp t)t^n dt.
\] (54)
solves the shortcoming (B) of the Borel method. In this subsection we shall show that this method can be used for the horn-shaped singularity as well.

We shall only deal with the horn \( H_R \) (see Fig. 6), defined as

\[
H_R := \{ z \mid \Re \omega(1/z) > \omega(1/R) \},
\] (55)

where

\[
\omega(z) := \ln F(z), \quad F(z) := \sum_{n=0}^{\infty} \frac{z^n}{\mu(n)},
\] (56)
i.e., roughly speaking, with the region of the asymptotic type \((1, 1)\) [63, 67, 68].

This can be confirmed as follows. By Lemma 2 we know that \( \Re \omega(z) \) tends to \(-\infty\) for \( z \to \infty \) and \( \Im z > \pi \). On the other hand, within the strip \( \Im z \leq \pi \), \( \omega(z) \) behaves according to (40) as \( \omega(z) = e^z + z + \ln(z + 1) \) provided \( \Re \omega(z) \) does not decrease too fast. Thus within the strip the equation \( \Re \omega(z) = \text{const} \) which defines the separatrix of the asymptotic behaviour of \( F(z) \) is nothing but

\[
\Re \omega(z) = e^x \cos y + x + \ln |x + iy + 1| = c,
\] (57)
c being some real constant. The Eq. (57) shows that \( \partial H_R^{-1} \) is symmetric under \( y \to \pm y \). Hence we can restrict our considerations to the right upper quadrant. In order that \( \Re \omega(z) \) be a constant the exponential term in (57) forces \( y \) to move very rapidly in the region \( \pi/2 < |y| < \pi \) and then to approach very close \( |y_o| = \pi/2 \) from above if \( x \) tends to infinity. One anticipates that the boundary \( \partial H_R \) of \( H_R \) encloses in some sense the strip \( \Im z \leq \pi/2 \).

Indeed, asymptotically, for \( x \to \infty \), the boundary of \( \partial H_R^{-1} \) is approximated from below and above by the curves

\[
y_{\pm}(x) = \left[ \frac{\pi}{2} + e^{-x} \left( x + \ln |x + 1 + i \left( \frac{\pi}{2} + e^{-x}(x \pm 2 \ln(x + 1)) \right)| - c \right) \right],
\] (58)
Figure 6: The horn $H_R$ (a) and its image $H_R^{-1}$ (b) under the mapping $z \to 1/z$.

with $x \in [x_o, \infty)$, $x_o > 0$, since

$$\Re \omega(z)|_{y=y_{\pm}(x)} = e^x \cos y_{\pm}(x) + x + \ln |x + 1 + iy_{\pm}|$$

$$\sim c + a_{\pm} \ln \pi \ln (x + 1) \frac{1}{2(x + 1)^2} e^{-x}, \quad (x \to \infty)$$

(59)

where $a_+ = -1$ and $a_- = 3$. The above considerations confirm our expectations that the boundary of $\partial H_R$ approximates the origin with zero slope, since according to (58,59),

$$\bar{y} \sim \pm \frac{\pi}{2} \bar{r}^2 \quad (\bar{r} \to \infty),$$

(60)

where the bared parameters correspond to the mapping $z \to \bar{z} = 1/z$.

From (57) it follows that

$$\frac{\partial y}{\partial x} = \left( e^x \cos y + 1 + \frac{x + 1}{(x + 1)^2 + y^2} \right) / (e^x \sin y)$$

(61)
diverges for \( y \to 0 \) as expected. The point \( x_o \) at which \( y(x_o) = 0 \) satisfies approximately the equation

\[
(x_o + 1)e^{x_o} = e^c
\]  
(62)

and the point \( x_m \) at which \( y(x) \) takes on its maximum is given by the equation,

\[
[(x_m + 1)^2 + y_m^2]^{1/2}e^{x_m} = e^{c+1}.
\]  
(63)

Both (62) and (63) are valid asymptotically for \( c \) sufficiently large. Thus, at this limit,

\[
\frac{x_o + 1}{[(x_m + 1)^2 + y_m^2]^{1/2}} e^{x_o-x_m} = e^{c-c-1} = e^{-1},
\]  
(64)

i.e., as \( x_o \to \infty (\Leftrightarrow c \to \infty) \) then \( x_m \to x_o + 1 \). Therefore, by the triangle inequality,

\[
\left| \oint_{\partial H_R} \ldots \frac{dz}{z} \right| \leq 2\pi \max(\ldots) + \int_o^R |\ldots| dr/r,
\]  
(65)

where \( \ldots \) stands for any continuous function bounded on the strip \( |\text{Im } y| \leq \pi \) and \( x \in [x_o, \infty) \), \( x_o \) being a positive real number.

Transition to more general horns of the type \((1, \eta)\) is simply accomplished by mapping \( z \to z/\eta \). The theorems could be also modified for a region of the asymptotic type \((k, \eta)\) with \( k > 1 \). The situation here is, however, more involved. The main result of this subsection is the Nevanlinna-like theorem for the horn \( H_R \) as follows.

**Theorem 5 :** Let \( f(z) \) be analytic in the horn-shaped region \( H_R := \{ \text{Re } \omega(1/z) > \omega(1/R) \} \), continuous up to the boundary, and satisfy there the estimates

\[
f(z) = \sum_{k=0}^{N-1} a_k z^k + R_N(z)
\]

with

\[
| R_N(z) | \leq A\mu(N) | z |^N
\]  
(66)

uniformly in \( N \) and \( z \in \bar{H}_R \).

Then

\[
M(t) := \sum_{n=0}^{\infty} a_n \frac{t^n}{\mu(n)}
\]  
(67)
converges for \(| t | < 1\), and has an analytic continuation to the striplike region \(S_1 = \{ t \mid \text{dist}(t, R_+) < 1\}\), satisfying the bound
\[
| M(t) | \leq K \exp[\exp(| t | /R)]
\] (68)
uniformly in every \(S_\kappa\) with \(\kappa > 1\). The analytic continuation of \(M(t)\) for \(t \in (1, \infty)\) is given as follows,
\[
M(t) = \frac{1}{2\pi i} \oint_{\partial H_R} F(t/z)f(z)dz/z .
\] (69)

Furthermore \(f\) can be represented by the absolutely convergent integral
\[
f(x) = \int_0^\infty \exp(-\exp t)M(tx) \, dt ,
\] (70)
for any \(x \in (0, R)\).

The proof of the theorem is rather complicated. Therefore, it is divided into several lemmas. However, an importance of \(F(t)\) is already seen from the subsequent argument. If a function \(f(z)\) is analytic in the horn \(H_R\) and continuous in \(H_R\), then for any \(x \in H_R \cap R\),
\[
f(x) = \frac{1}{2\pi i} \oint_{\partial H_R} f(z)/(z-x)dz
\]
\[
= \frac{1}{2\pi i} \oint_{\partial H_R} f(z)dz/z \int_0^\infty \exp(-\exp t) F(tx/z) \, dt
\] (71)
Note that such a representation of \(f(z)\) is impossible by the Borel method. Note also that unlike the disc \(C_R\) in the Nevanlinna theorem the horn \(H_R\) is not a star-like region anymore. This is in general the main difference between the regions of the asymptotic type \((0, \eta)\) and \((k, \eta)\) with \(k \geq 1\). This difference means that with moment constant summability methods one cannot recover \(f(z)\) from its asymptotic series in the whole horn \(H_R\) but only for \(z \in H_R \cap R\). Physically this is not, however, a problem, since we are expanding in real parameters (couplings).

Henceforth we shall follow Sokal’s strategy of the proof of the Nevanlinna theorem \([8]\). Lemma 2 provides us with an integral representation of the monomials \(t^n/\mu(n)\) for any positive \(n\) and \(t \geq 1\),
\[
\frac{1}{2\pi i} \oint_{\partial H_R} F(t/z)z^n dz/z = \frac{t^n}{\mu(n)},
\] (72)
where the integral is taken counterclockwise along the boundary of \( H_R \). The formula will be used to express \( M(t) \) (an analogue of \( B(t) \)) in terms of \( f(z) \).

To find the minimal domain of analyticity of \( M(t) \) we shall need a bound on \( F^{(n)}(z_0) \) for \( n \) tending to infinity (Lemma 3). Finally, after the Lemma 4 we shall give all the lemmas together and complete the proof of Theorem 5.

**Lemma 3** : For any \( z_o \) such that \( \text{Re} \ z_o > 0 \) and \( n > 0 \),

\[
|F^{(n)}(z_o)| \leq \text{const} \left( n! / \mu(n) \right) \exp \left\{ \frac{nx_o}{w(n)+1} \left( 1 + \frac{1}{w(n)} \right) - w(n) + x_o - 1 + O(w(n)w'(n)) \right\},
\]

(73)

where \( x_o = \text{Re} \ z_o \).

**Proof** : Firstly,

\[
F^{(n)}(z_o) / n! = \frac{1}{2\pi i} \oint_C F(z_o + z) / z^{n+1} dz = \frac{1}{2\pi i} \oint_C \exp[\omega(z + z_o) - n \ln z] dz,
\]

(74)

where \( C \) is a simple contour enclosing the origin. Due to the properties of \( \omega(z) \) (see (40) and Lemma 2) if \( |\text{Im} \ z_o| > \pi \) and if one takes \( C \) to be a contour of radius 1 then one can show that \( |F^{(n)}(z_o)| \to 0 \). Hence without any restriction we can confine ourselves to the region where \( |\text{Im} \ z_o| \leq \pi \). In this region we shall evaluate the integral on the r.h.s of (74) by the saddle point technique. The saddle point of the integral is a critical point of the function

\[
h(z) := e^{z_o + z} + z_o + z + \ln(z_o + z + 1) - n \ln z,
\]

(75)

i.e., a solution to the equation

\[
h'(z) = e^{z_o + z} + 1/(z_o + z + 1) - n/z = 0.
\]

(76)

By direct comparison of (76) with the defining equation for \( w(s) \) (17), \( w(s) e^{w(s)} = s \), one anticipates that for \( n \) sufficiently large the solution \( v(n) \) of (77) will be “very close” to \( w(s) \). Therefore we shall look for the solution \( v(n) \) of (77) in the form

\[
v(n) = w(n) - z_o - \delta(n),
\]

(77)
where $\delta(n)$ is an unknown function to be determined. Such parametrization will be shown to be justified as at the end of our calculation we shall obtain an asymptotic expansion of $\delta(n)$ for $n \to \infty$, according to which the dominant term $-z_o/w(n) \sim -z_o/\ln n \to 0$. To prove this we shall slightly rewrite Eq.(76),

$$v(n) \left(e^{z_o+v(n)} + 1 + \frac{1}{v(n) + z_o + 1}\right) = n, \quad (78)$$

and take logarithms of both its sides. Upon the substitution $v(n) = w(n) - z_o - \delta$ and some manipulations (after one has expanded small terms in logarithms) one gets,

$$\ln w - \frac{z_o}{w} - \frac{\delta}{w} + w - \delta + \frac{e^\delta}{e^w} + \frac{e^\delta}{e^w(w - z_o - \delta)} \sim \ln n. \quad (79)$$

By using the defining relation (17) for $w(s)$, $\ln w(n) + w(n) = n$, and subsequent multiplication of both sides of the last Eq. by $we^{-\delta}/(w + 1)$ one obtains,

$$\left(\delta + \frac{z_o}{w + 1}\right)e^{-\delta} \sim \left(1 + \frac{1}{w} + \frac{z_o + \delta}{w^2}\right)\frac{we^{-w}}{w + 1}. \quad (80)$$

By using (22) one finds that the last fraction on the r.h.s. of (80) is nothing but $w(n)w'(n)$. To get $\delta(n)$ in an explicit form one again takes logarithm of both sides of (80),

$$-\delta + \ln \left(\delta + \frac{z_o}{w + 1}\right) \sim \ln(ww') + \frac{w + z_o + \delta}{w^2}. \quad (81)$$

Dominant terms for $n \to \infty$ in (81) are logarithms. For a moment we shall parametrize $\delta + z_o/(w + 1) = ww' + \epsilon(n)ww'$, where $\epsilon(n)$ is assumed to be a small number for sufficiently large $n$. Under this assumption one gets by expanding small terms in logarithm on the l.h.s. of (81) that

$$\epsilon(n) \sim \frac{1}{w(n)} - \frac{z_o}{w(n) + 1} \quad (n \to \infty). \quad (82)$$

Thus, the assumption that $\epsilon(n)$ is small for $n \to \infty$ is justified and one arrives at the following expression for $\delta(n)$,

$$\delta(n) = -\frac{z_o}{w + 1} + w(n)w'(n)\left(1 + \frac{1}{w(n)} - \frac{z_o}{w(n) + 1}\right) + O[(ww')^2] \quad (n \to \infty). \quad (83)$$
This is one of the most important relations to prove Theorem 3. The asymptotic expansion (83) justifies the statement made at the beginning of our calculations that \( \delta(n) = \mathcal{O}(1/\ln n) \to 0 \) provided \( n \) tends to infinity.

Finally, by using (83), one arrives at the following expression for the solution \( v(n) \) of (76) in the leading order in \( n \),

\[
v(n) = w(n) - z_o + \frac{z_o}{w(n) + 1} + w(n)w'(n) + \mathcal{O}[w'(n)] \quad (n \to \infty). \tag{84}
\]

Note that \( z_o + v(n) \) approaches the real positive axis when \( n \to \infty \), and a part of the contour of the steepest descent nearby the saddle point is approximately a segment of the circle centered at \( z_o \). After the saddle point evaluation of the integral one finds that

\[
F^{(n)}(z_o)/n! \sim [v(n)v'(n)/2\pi]^{1/2} \exp[\omega(z_o + v(n)) - n \ln v(n)].
\]

Now we can calculate the behaviour of \( n(\ln w(n) - \ln v(n)) \) and \( e^{z_o + v(n)} - e^{w(n)} \) for \( n \to \infty \). Both expressions are needed to establish the bound (73) on \( F^{(n)}(z_o) \).

\[
n(\ln w(n) - \ln v(n)) = n \ln \frac{w}{v} = n \ln \frac{e^{z_o + v(n)} + 1 + 1/v(n)}{e^{w(n)}} =
\]

\[
n \ln \left( e^{-\delta} + e^{-w} + \frac{e^{-w}}{w - z_o - \delta} \right) \sim n \left[ (e^{-\delta} - 1) + e^{-w} + \frac{e^{-w}}{w - z_o - \delta(n)} \right] =
\]

\[
n[-\delta(n) + \mathcal{O}(e^{-w(n)})] = n \left[ \frac{z_o}{w(n) + 1} - w(n)w'(n) - \mathcal{O}(w'(n)) \right]. \tag{85}
\]

\[
e^{z_o + v(n)} - e^{w(n)} = e^{w(n)}(e^{-\delta} - 1) = e^{w(n)} \left[ \frac{z_o}{w(n) + 1} - w(n)w'(n) - \mathcal{O}(w') \right]
\]

\[
= \frac{n}{w(n)} \left[ \frac{z_o}{w(n) + 1} - w(n)w'(n) - \mathcal{O}(w'(n)) \right]. \tag{86}
\]

Thus, when we use that \( e^{w(n)} = n/w(n) \),

\[
\frac{F^{(n)}(z_o)\mu(n)}{n!} = (vv'w')^{1/2}
\]
\[
\exp \left\{ e^{v+z_o} + v + z_o + \ln(v + z_o + 1) - n \ln v - e^w + n \ln w \right\} = \\
\exp \left\{ \frac{n z_o}{w(n) + 1} \left( 1 + \frac{1}{w(n)} \right) - w(n) - nw(n)w'(n) \left( 1 + \frac{2}{w(n)} \right) + z_o + O(w(n)w'(n)) \right\}
\]

\[(n \to \infty). \tag{87}\]

The factor \(e^{-w(n)}\) in (87) is produced by \((vv'ww')^{1/2}\) as one can check by using (22) and (84). \(\Diamond\)

**Lemma 4**: Let \(x \in \mathbb{R}\) and \(z \in \mathbb{C}\) be independent variables and let \(J_x(z)\) be the integral

\[
J_x(z) := \int_o^1 \exp[-\exp(t/x)]F(tz)dt = \sum_{n=0}^{\infty} \frac{(xz)^n}{\mu(n)} \int_o^{1/x} \exp(-\exp t) t^n dt. \tag{88}
\]

Then we have

\[
\frac{1}{2\pi i} \oint_{\partial H_R} J_x(1/z)z^n dz/z = \frac{1}{\mu(n)} \int_o^1 \exp(-\exp t/x) t^n dt,
\]

where the integral is taken counterclockwise along the boundary \(\partial H_R\) of \(H_R\).

**Proof**: The essence of the proof is to show that the integration (88) of \(F(tz)\) does not spoil the asymptotic behaviour of \(F(z)\) too much. Note that \(x^n \int_o^{1/x} \exp(-\exp t) t^n dt\) behaves like \(\exp[-\exp(1/x)]/(nx)\) provided \(n\) is sufficiently large (the asymptotic is given by the end point of integration here). Thus one anticipates that \(x\) will only produce an overall factor to the asymptotic behaviour of \(J_x(z)\) when \(z\) tends to infinity. This can be confirmed as in Lemma 2. One uses the Euler-Maclaurin sum formula and finds that for \(|Imz| \leq \pi\) the asymptotic behaviour is governed by a saddle point (provided one stays on an appropriate curve for \(\pi/2 \leq |Imz| \leq \pi\) (see the proof of Lemma 2)), which is here determined by the equation

\[
\ln w(s) + 1/s = \ln z.
\]

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Thus, the contribution $V_s$ of the saddle point is

$$V_s = \frac{(z + 1)}{x} \exp\{\exp z[\exp(-\exp(-z)/z)] + 1\} \quad (z \to \infty).$$

By analogy with Lemma 2 one proves that for $z$ tending to infinity and $|\text{Im}z| > \pi/2 + \varepsilon$, $\varepsilon > 0$, function $J_s(z)$ tends to zero like $F(z)$ does. The proof of the statement of the lemma is then trivial. It amounts to using the Cauchy integral formula. Note that the integral from $J_s(z)/z$ along a segment of $\partial H_R^{-1}$, which starts at infinity and terminates at some $z_o$ on the contour, converges even absolutely. ◦

### 3.3.1 Proof of Theorem 5

i) Under the hypotheses of Theorem 5 one easily proves that the series (67) converges for $|t| < 1$. Let us consider the integral $d(t)$,

$$d(t) := \frac{1}{2\pi i} \oint_{\partial H_R} F(t/z)f(z)dz/z,$$  

where $t \geq 1$. Two remarks are in order. In contrast to the Nevanlinna theorem one cannot use the integral on the r.h.s. of (89) for $0 < t < 1$, as it is not possible to satisfy both conditions that the contour of integration in (72) be the contour which tends to zero on the boundary of $tH_r$ for some $r$, and at the same time lie in $H_R$. Whenever $t \notin R$ the integral is identically zero (by virtue of Lemma 2). Hence, it cannot yield an analytic continuation of $d(t)$ for $t \notin R$.

From the properties of $F(z)$, it is immediately seen that the integral on the r.h.s. of (89) converges for $t > 1$ absolutely and uniformly on any closed subset of $(1, \infty)$. For $t = 1$ the integral converges by virtue of the Abel-Dirichlet lemma. Hence, $d(t)$ is a $C^\infty$ function on the interval $(1, \infty)$ and possesses the right derivatives at the point $t = 1$.

To prove that the series (67) converges at $t = 1$, we make use of Lemma 3 and rewrite $d(1)$ as follows,

$$d(1) = \sum_{k=0}^{N-1} \frac{a_k}{\mu(k)} + \frac{1}{2\pi i} \oint_{\partial H_R} F(1/z)R_N(z)dz/z.$$  

(90)
As the integration on the r.h.s. of (90) runs along the boundary of \( H_R \) on which \( \text{Re} \omega(1/z) = \text{const} = \omega(1/R) \) due to (65) the integral can be bounded from above as follows,

\[
\left| \frac{1}{2\pi i} \oint_{\partial H_R} F(1/z) R_N(z) \frac{dz}{z} \right| \leq \frac{A}{\pi} e^{\omega(1/R)} \left( \mu(N) \int_{\partial H_R^+} \exp(N \ln r) |dz|/r \right) \]

\[
\leq 2A e^{\omega(1/R)} \mu(N) \exp(-N \ln(1/R)) \left( 1 + \frac{R}{N} \right). \quad (91)
\]

Here \( \partial H_R^+ \) means that we integrate along \( \partial H_R \) in the first quadrant. Optimization of the bound on the r.h.s. of (6) then amounts to finding a solution \( R = R(N) \) of the equation,

\[
h(R) := -N R + \exp(1/R) + 1 + \frac{1}{1 + 1/R} = 0. \quad (92)
\]

If one compares (92) with (76) then one finds that (92) is nothing but (76) with \( z_o = 0 \). Therefore the bound on the r.h.s. is optimized by the choice \( R = 1/v(N) \). After distorting the contour of integration up \( \partial H_R \) with \( R = 1/v(N) \) and using (85,86) we have,

\[
\left| \frac{1}{2\pi i} \oint_{\partial H_R} F(1/z) R_N(z) \frac{dz}{z} \right| \leq \frac{2}{\pi} A e^{\omega(1/R)} \mu(N) \left( 1 + \frac{R}{N} \right) \left( -N w(1/R) \right) \left( N w'(1/R) \right)
\]

\[
\sim 2A e^{-3w(N)/2} \longrightarrow 0 \quad (N \to \infty), \quad (93)
\]

where we have used that \( N w'(N) \to 1 \) if \( N \to \infty \). The term \(-w(N)/2\) has its origin in the Gaussian prefactor \((2\pi w N)^{1/2}\) of the asymptotic of \( \mu(N) \). Thus, \( M(1) = d(1) \). The same is also true for derivatives, i.e., \( M(n)(1_+) = d^{(n)}(1_+) \) for any \( n \in \mathbb{N} \). Indeed, if \( f(z) \) satisfies SAC then the same SAC will also satisfy its derivatives \( f^{(n)}(z) \) (may be in a horn \( H_R \) with different \( R \)). So

\[
d^{(n)}(1+) = \sum_{k=0}^{N-1} c_k/\mu(k) + \frac{1}{2\pi i} \oint_{\partial H_R} F(1/z) R_N^{(n)}(z) \frac{dz}{z},
\]

where \( c_k = (k+n)(k+n-1)...(k+1)a_{k+n} \), and the integral can be estimated in the same manner as above.
To determine the minimal region of analyticity we express \(d^{(n)}(t)\) as follows:

\[
d^{(n)}(t) = n!a_n/\mu(n) + \frac{1}{2\pi i} \oint_{\partial H_R} F^{(n)}(t/z) R_{n+1}(z) z^{-n} dz/z.
\]

Now, using Lemma 3, one finds that

\[
|d^{(n)}(t)| \leq \text{const} F(t/R) n! \exp\left[\frac{t/R}{n/\ln n + n/\ln 2 n} + 2 \ln w(n) + O(1/w(n))\right]. \tag{94}
\]

One may justify that

\[
\sum_{n=o}^{\infty} d^{(n)}(t_1)(t - t_1)^n/n! = \sum_{n=o}^{\infty} d^{(n)}(t_2)(t - t_2)^n/n!,
\]

whenever \(1 < t_1 \leq t \leq t_2\), and \(|t - t_i| < 1\), where \(i = 1, 2\).

Thus \(M(t)\)

\[
M(t) = \begin{cases} 
\sum_{n=o}^{\infty} a_n t^n/\mu(n) & -1 < t \leq 1, \\
\frac{1}{x} \int_{o}^{\infty} \exp[- \exp(t/x)] M(t) dt, & t \geq 1;
\end{cases}
\]

is shown to be analytic at least in the striplike region \(S_1\) (see Fig. 7). Bound (68) on \(M(t)\) follows immediately from the relation (94).

\(\text{i)}\) To prove the relation \(70\), let us consider the integral

\[
\frac{1}{x} \int_{o}^{\infty} \exp[- \exp(t/x)] M(t) dt,
\]

for any \(x \in (0, R)\). The integral,

\[
\int_{1}^{\infty} \exp[- \exp(t/x)] \frac{1}{2\pi i} \oint_{\partial H_R} f(z) F(t/z) \frac{dz}{z} dt,
\]

is absolutely convergent. Therefore, the r.h.s. of \(70\) (the integral \(75\)) can be recast into the form,

\[
f(x) = \frac{1}{2\pi i x} \oint_{\partial H_R} J_x(1/z) f(z) \frac{dz}{z}
\]

\[
+ \frac{1}{x} \sum_{n=o}^{\infty} \frac{a_n}{\mu(n)} \int_{o}^{1} \exp[- \exp(t/x)] t^n dt. \tag{96}
\]
Figure 7: Minimal region of analyticity of $M(t)$.

Now we shall show that the last two terms cancel each other. Indeed, using Lemma 4, one finds

$$\frac{1}{2\pi i} \oint_{\partial H_R} J_x(1/z) f(z) dz/z = \sum_{n=0}^{N-1} \frac{a_n}{\mu(n)} \int_0^1 \exp[-\exp(t/x)] t^n dt$$

$$+ \frac{1}{2\pi i} \oint_{\partial H_R} J_x(1/z) R_N(z) dz/z .$$

To show that the last integral vanishes in the limit $N \to \infty$ one optimizes the bound on the integral. To do this one chooses similarly like in (6-93), the contour of integration to be $\partial H_{1/\nu(N)}$. Then, in virtue of (17,24,34),

$$\left| \frac{1}{2\pi i} \oint_{\partial H_R} J_x(1/z) R_N(z) dz/z \right| \leq \frac{2A}{x} e^{-3w(N)/2} \to 0 \quad (N \to 0).$$

Corollary 3: Let
1) $M(t) := \sum_{n=0}^{\infty} a_n t^n / \mu(n)$ has nonzero radius of convergence and can be analytically extended on the real positive axis;

2) $|M(t)| \leq \text{const} \exp(\exp(t/R))$ on the axis.

Then

$$f(x) := \frac{1}{x} \int_{0}^{\infty} \exp(-\exp(t/x)) M(t) dt$$

converges absolutely for $x \in (0, R)$ and

$$f(x) \sim \sum_{n=0}^{\infty} a_n x^n \quad (x \to 0_+).$$

Proof: Use the Laplace method to evaluate the integral. \copyright
3.4 Comparison of the new summability method with the Borel method

Using a recent generalization of the Borel summability method [9] and the Nevanlinna theorem [10] one can easily find series which violate SAC in a horn-shaped region $H_R$ but they are Borel summable (see Example 1 below). On the other hand one can find series which are formally Borel non-summable but satisfy SAC in a suitable kidney-shaped region (see Example 2). This exemplifies the statement in the Introduction that not the Borel (non)summability but the validity (violation) of SAC prevents (indicates) the presence of nonperturbative effects which may in some cases destabilize a ground state. The first shows a series which violates SAC if summed up to a function analytic in the horn $H_R$, while it satisfies them if summed up to a function regular in a disc tangent to imaginary axis at the origin.

The main method for constructing them is to use the new AMCSM based on the momenta (14) as a method for localization of singularities in the complex plane. Indeed, this method has a nice and rather feasible advantage as compared with the Borel one - it does not only see beyond singularities. The problem of finding singularities of a function $f(z)$ which is defined by a convergent power series is then reduced to the problem of finding the asymptotic behaviour of its moment transform. To find the asymptotic behaviour of the moment transform the Euler-Maclaurin formula will be frequently used. It reduces the above problem to the problem of finding the asymptotic behaviour of an integral. By our general results [9, 10] we know that in general the $\mu$-transform may have dangerous asymptotic behaviour (i.e., which may prevent summability of its Taylor expansion) at worst on several (possibly infinite) radial rays. This should be contrasted with the Borel method - any singularity of $f(z)$ results in that the Borel transform of its Taylor series grows up very strongly (such that the Borel integral diverges) not only on the radial ray which passes through the singularity but also in the whole sector bisected by this ray and with the opening angle $\pi$.

Example 1 Let $S_1$ be a power series $\sum_{o}^{\infty} a_n g^n$ with $a_n = (-1)^n \nu^2(n).$ Then $S_1$ is Borel summable and satisfies SAC in the disc $K(0,R)$ but is not $\mu$-summable and violates SAC in the horn $H_R.$
Proof: The Borel transform of $S_1$ is an entire function and hence it is obviously analytic on a strip $S_{\rho} := \{ t \mid \text{dist}(t, R_+) < 1/\rho \}$ for any $\rho$, which is the first hypothesis of the Nevanlinna theorem [67, 9, 10]. Now we use the Euler-Maclaurin sum formula [105],

$$B(t) = \int_{\sigma}^{\infty} e^{s(\ln t + i\pi)} \frac{\mu^2(s)}{\Gamma(s + 1)} ds + O(|t^\sigma|) \quad (t \to \infty),$$

where $\sigma$ is a real number, $-1 < \sigma < 0$. One finds that a saddle point of the Euler-Maclaurin integral is at $s/w^2(s) = -t$. Thus the asymptotic of the integral is dominated by a contribution of the end point and we arrive at the result

$$B(t) \sim O(|t|^\sigma) \quad (t \to \infty),$$

in any sector with nonzero opening angle $|\arg t| < \pi/2$. Therefore the hypotheses of the Nevanlinna theorem are satisfied, $S_1$ is Borel summable, and its Borel sum satisfies SAC in the disc $K(0, R)$.

On the other hand, the generalized $\mu$-transform $M(t)$,

$$M(t) := \sum_{o} a_n \frac{t^n}{\mu(n)} = \sum_{o} (-1)^n t^n \mu(n),$$

is undefined since it has zero radius of convergence, i.e., the summability method for the horn-shaped region $H_{R}$ in this case fails. \(\diamondsuit\)

---

**Example 2** Let us consider the series $S_2 = \sum_{o} a_n g^n$ with

$$a_n = (-1)^n \Gamma(\alpha n + 1),$$

where $\alpha = 1 + \varepsilon$, $1 > \varepsilon > 0$, and $\Gamma$ is the usual gamma function.

Then $S_2$ cannot be summed up to a function maximal region the analyticity of which is a disc tangent to the imaginary axis at the origin. Hence it is not Borel summable and violates SAC in this region. Nevertheless, it can be summed up to a function whose minimal region of analyticity contains a kidney shaped region $K(\lambda, R)$,

$$K(\lambda, R) = K^\lambda(0, R) := \{ z \mid z^{1/\lambda} \in K(0, R) \},$$

having an opening angle $\Theta = \lambda \pi$ with $\lambda \geq \alpha$, $K(0, R)$ being the disc $\text{Re}(1/z) > 1/R$, i.e., $S_2$ is summable and satisfies SAC in this region.
Eventually, we shall give an example just opposite to Example 1.

**Example 3** Let \( S_3 \) be a power series with coefficients \( a_n = a^n \mu(n) \), where \( a = |a| e^{i\theta} \) is a complex number with \( 0 < \theta < \pi/2 \). Then the series \( S_3 \) is \( \mu \)-summable but not Borel summable.

**Proof**: The Borel transform of \( S_3 \) is an entire function. The position of a saddle point of its Euler-Maclaurin integral representation is given by equation

\[
w = \ln(aue^{i(\phi+\theta)}),
\]

where \( t = u e^{i\phi} \), which means that the contribution \( V_s \) of the saddle point is

\[
V_s \approx \exp[au \exp(i(\phi + \theta)) (\ln(au) - 1 + i(\phi + \theta))] \quad (u \to \infty).
\]

Hence, the Borel integral diverges whenever \( \theta < \pi/2 \). On the other hand \( \mu \)-transform \( M(t) \) can be calculated exactly,

\[
M(t) = \sum_{n=0}^{\infty} \frac{a_n}{\mu(n)} t^n = \frac{1}{1 - at},
\]

and \( S_3 \) is transparently \( \mu \)-summable (see [9, 10]).

If \( 3\pi/2 \geq \theta \geq \pi/2 \) then one can show that

\[
\int_0^\infty e^{-t} \sum_{n=0}^{\infty} \frac{a_n}{n!} (tz)^n dt = \int_0^\infty e^{-e^t} \sum_{n=0}^{\infty} \frac{a_n}{\mu(n)} (tz)^n dt,
\]

since the real positive axis belongs to the Borel polygon of \( M(z) \).

The above examples demonstrate that there exist series which being Borel summable are not \( \mu \)-summable and vice versa. They also show that \( \mu \)-method [9, 10], in combination with the Euler-Maclaurin sum formula, is a powerful tool for looking for singularities of analytic continuation of convergent power series in the complex plane, since in contrast to the Borel method \( \mu \)-method does not only see beyond singularities.
3.5 Derivation of strong asymptotic conditions for variety of regions

To generalize SAC let us start with some definitions.

**Definition 1** Let $S$ be the halfplane $\Re z > 0$, and $C_R$ a disc centered at the origin with radius $R$. Denote $\bar{S} := S \setminus C_R$, which will be called the base region. A domain $D$ of the complex plane $C$ will be called of asymptotic type $(k, \eta)$, $k \geq 1$, if there exist $R > e^{k-1}$ such that $D = 1/[\eta \ln_k(\bar{S})]$, where $e_k$ is the $k$-fold exponential, $e_0(x) := x$, $e_k(x) := \exp(e_{k-1}(x))$, and $\ln_k$ is the $k$-fold logarithm, $\ln_1(z) := \ln z$, $\ln_k(z) := \ln(\ln_{k-1}(z))$. A domain $D$ of the complex plane will be called of asymptotic type $(0, \eta)$ if $D = 1/\bar{S}^\eta$, where $\bar{S}^\eta := \{z | z^{1/\eta} \in \bar{S}\}$. $\diamond$

To establish SAC for these regions we shall start with the regions of asymptotic type $(1, \eta)$. The main result of this subsection is the following theorem.

**Theorem 6** Let $f(z)$

1) be regular in a region $D$ of the asymptotic type $(1, \eta)$ and continuous in its closure;

2) have in $D$ an asymptotic expansion

$$f(z) = \sum_{n=0}^{N-1} a_n z^n + R_N(z).$$

(97)

If $R_N(z)$ satisfies the bound

$$|R_N(z)| \leq A \rho^N |z|^N \exp[-\exp w(N) + N \ln w(N)]$$

(98)

with $\rho \leq \eta$, uniformly in $z \in \bar{D}$ and $N$, $w(s)$ being the solution of $w(s) \exp[w(s)] = s$ [9, 10], then the asymptotic series (97) determines $f(z)$ uniquely. The condition (98) is strong in the following sense: if it is known that (98) is only fulfilled with some $\rho > \eta$, uniformly in $z \in \bar{D}$ and $N$, then there exists a nonzero function with the trivial asymptotic expansion and the asymptotic
series (97) does not determine \( f(z) \) uniquely.

Note that \( \exp\{-\exp(w(N) + N \ln w(N))\} \) is, up to the factor \( [2\pi w(N)w'(N)]^{1/2} \) which arises from the Gaussian integration around a saddle point, precisely the asymptotic for large \( N \) of the moment function \( \mu(N) \) (see (17) above). For regions of, roughly speaking, asymptotic type \((1, \eta)\) SAC have been established in [10] in connection with a summability method for the horn-shaped region \( H_R \). Unfortunately, generalization of this proof to more sharper horns seems to be cumbersome. A rather brief derivation of SAC based on a slight modification of the Phragmen-Lindelöf theorem and which can be straightforward generalized to sharper horn-shaped regions has been given in [11] and is briefly repeated in Appendix. One has only to replace \( w(s) \) by \( w_k(s) \), \( w_k(s) \) now being the solution of

\[
 w(s) \exp\{w(s) + ... + e_{k-2}[w(s)] + e_{k-1}[w(s)]\} = s
\]

(see [11]). One can also consider other types of asymptotic regions which may be obtained, e.g., by more involved combinations of scaling \( z \to z/\eta \) and conformal mappings like \( z \to z^{1/\alpha} \) of \( \bar{S} \), as well as some other base regions, etc. In general, however, to any given asymptotic region such a moment sequence \( \tilde{\mu}_k(n) \) will correspond that fulfilling a bound like (98) will ensure SAC in the region. We used the above definition of the asymptotic types since the maximal region of analyticity of four dimensional renormalizable massless field theories has been suggested to be at best just a region of the asymptotic type \((1, \gamma)\), where \( \gamma \) depends on the first two coefficients \( \beta_1, \beta_2 \) of the \( \beta \)-function and on the definition of the coupling [21, 94, 96, 97]. This means that probably there is no reason to look at the Borel transform of the four dimensional renormalizable field theories since the convergence of the Borel integral contradicts the horn-shaped region of analyticity [1, 10, 12] any way that the Borel transform on the real positive axis is defined (e.g., in a distributional sense [110]). In connection with Theorem 4 this means that these theories (without UV cutoff) cannot satisfy SAC since their analyticity in the complex coupling constant plane is not compatible with the divergence of order \((n!)^\varepsilon\) of their perturbation theory no matter how small \( \varepsilon > 0 \) is.

\(^{5}\)For 't Hooft’s coupling, \( \gamma \) is just \( |\beta_1| \).
Now we want to show an important property of the class \( \mathcal{K} \) of function which obey the hypotheses of Theorem 5 and 6. Using asymptotic behaviour of \( w_k(s) \) and the method of \( [10] \) one finds that the SAC we are given preserve nonlinear perturbation conditions such as unitarity of the Feynman series. Indeed, the following Lemma holds.

**Lemma 5**: The class of function \( \mathcal{K} \) is closed under product, i.e., if \( f_1(z) \) and \( f_2(z) \) are two functions from \( \mathcal{K} \), then \( g(z) := f_1(z)f_2(z) \in \mathcal{K} \).

**Proof**: To prove the lemma it is sufficient to prove that

\[
\lim_{N \to \infty} \sum_{k=0}^{N} \frac{\mu(N-k)\mu(k)}{\mu(N)} < \infty. \tag{99}
\]

From the Lemma 1 one can derive that if \( N \) is sufficiently large then for \( q < N/2 \),

\[
\frac{\mu(N-q)}{\mu(N)} \sim \exp[-q \ln(N-q) - (q + q^2)/(N \ln N)].
\]

Thus, provided one choses a fixed \( j \) such that the asymptotic formula to estimate \( \mu(s) \) for \( s \geq j \) can be used,

\[
\sum_{k=0}^{N} \frac{\mu(N-k)\mu(k)}{\mu(N)} \leq 2 \sum_{k=0}^{\lfloor \frac{N}{2} \rfloor} \frac{\mu(N-k)\mu(k)}{\mu(N)}
\]

\[
\leq \text{const} \sum_{k \geq j} \exp \left\{ -\ln N \left( \frac{\ln k}{\ln \ln N} + k \left( 1 - \frac{\ln \ln k}{\ln \ln N} \right) \right) \right\}.
\]

The last bound gives transparently convergent series in the limit \( N \to \infty \), since its last term behaves like \( 1/[N \exp(N \ln 2/\ln N)] \).

The next two lemmas deal with the important case of perturbation series with equal sign and alternating sign of coefficients \( [15, 22, 29, 30, 89] \). To formulate the lemmas let us firstly consider a general analyticity region \( D \).
to which corresponds momenta $\tilde{\mu}_k(n)$. Then one can meet the subsequent situations:

\begin{align*}
a) & \forall t > 0 : a_n t^n / \tilde{\mu}_k(n) \to \infty \quad (n \to \infty); \\
b) & \forall t > 0 : a_n t^n / \tilde{\mu}_k(n) \to 0 \quad (n \to \infty); \\
c) & \exists t \neq 0 : a_n t^n / \tilde{\mu}_k(n) \to K \neq 0 \quad (n \to \infty).
\end{align*}

**Lemma 6** Let $S$ be a divergent power series with equal sign coefficients. Then $S$ violates SAC in any analyticity region $D$.

**Proof**: In case a) the moment constant transform $\tilde{M}_k(t)$,

$$
\tilde{M}_k(t) := \sum_{n=0}^{\infty} a_n t^n / \tilde{\mu}_k(n),
$$

(100)

does not exist. In case b) the moment constant transform is an entire function and hence defined for all complex $t$. It is clear that the maximum modulus of $\tilde{M}_k(t)$ for $|t| \leq x$ is just $\tilde{M}_k(x)$. From simple relation between the Taylor series coefficients and the maximum modulus growth of entire functions based on the Cauchy integral formula [105] one can prove that the generalized moment constant sum does not exist, since the integral

$$
\int_{\alpha}^{\infty} \exp(-e_k(t)) \tilde{M}_k(zt^\rho) dt
$$

(101)
diverges for all real $z > 0$. The same is also true in case c) since $\tilde{M}_k(t)$ is singular on the real positive axis. ◇

**Lemma 7** Let $S$ be a divergent power series with alternating sign regular coefficients $a_n$, i.e., there exist an analytic function $a(s)$ in the complex half-plane $\text{Re } s > \sigma$ such that $a_n = (-1)^n a(n)$. Let $a(s)$ have parametrization $a(s) = \exp(s \ln(b(s)))$ with $s b'(s)/b(s) \sim O(1)$ (or $o(1)$) when $s \to \infty$. Let $D$ be the analyticity domain to which the momenta $\tilde{\mu}_k(n)$ correspond. Then in cases b) and c) $S$ satisfies SAC in $D$.

**Proof**: In case b) the moment constant transform $\tilde{M}_k(z)$ is an entire function so the requirement of analyticity of $\tilde{M}_k(z)$ to sum $S$ is satisfied.
Stieltjes moment function $\tilde{\mu}_k(s)$ can be shown to be nonzero in the complex halfplane $\text{Re } s > \sigma'$ for $\sigma'$ sufficiently large \cite{9}. So, we can again use the Euler-Maclaurin sum formula to find asymptotic behaviour of $\tilde{M}_k(z)$ for $z \to \infty$. A contribution of the saddle point is proportional to

$$\exp\{e_k[\tilde{w}_k(s(z))]\} \approx \exp[e_k(-z)] \sim O(1) \quad (z \to \infty),$$

where $s(z)$ is determined by the saddle point equation $\tilde{w}_k(s) \exp[-a'(s)/a(s)] = -z := -re^{i\phi}$. However, the saddle point does not lie on the principal sheet and the contour of integration cannot probably be deformed in such a way that the asymptotic behaviour of the Euler-Maclaurin integral be governed by it. Instead of that the contour of integration can be deformed in such a way that the asymptotic behaviour of the integral can be shown to be governed by the end point of integration \cite{105}. In any case, however,

$$\tilde{M}_k(z) \sim O(|z|^{\tilde{\sigma}}) \quad (z \to \infty), \quad (102)$$

where $\tilde{\sigma} = \max\{\sigma, \sigma'\}$, i.e., the generalized moment constant sum \eqref{101} exists. In case c) $\tilde{M}_k(z)$ is a meromorphic function regular on the real positive axis. To see this one may apply $\mu$-method \cite{3,10} on the series \eqref{100}. The $\mu$-transform of $\tilde{M}_k(z)$ is an entire function with alternating sign regular coefficients and this case can be treated as case b). Using the Euler-Maclaurin integral representation of the $\mu$-transform $\tilde{M}_k(z)$ one finds that $\mu$-sum converges $\forall t \geq 0$ even in a sector and is absolutely bounded therein. Therefore $\tilde{\mu}_k$-sum also exists in this sector and satisfies SAC in $D$. ◊
4 Applications

4.1 Summability methods and the Rayleigh-Schrödinger perturbation theory

4.1.1 Preliminaries

As some example of application of the summability methods we shall consider the Rayleigh-Schrödinger perturbation theory \[108, 109\]. We shall restrict ourselves to the case of relatively bounded perturbations \[109\]. Let \(T\) and \(A\) be operators with the same domain space \(X\) (but not necessarily with the same range space) such that \(D(T) \subset D(A)\) and

\[
\|Au\| \leq a\|u\| + b\|Tu\|, \quad u \in D(T),
\]

(103)

where \(a\) and \(b\) are nonnegative constants. Then we shall say that \(A\) is relatively bounded with respect to \(T\) or simply \(T\)-bounded. The smallest lower bound \(b_o\) of all possible constants \(b\) in (103) will be called the relative bound of \(A\) with respect to \(T\) or simply the \(T\)-bound of \(A\). If \(b\) is chosen very close to \(b_o\), the other constant \(a\) will in general have to be chosen very large; thus it is in general impossible to set \(b = b_o\) in (103). Obviously a bounded operator \(A\) is \(T\)-bounded for any \(T\) with \(D(T) \subset D(A)\), with \(T\)-bound equal to zero.

To consider relatively bounded perturbations is very natural since closedness, bounded invertibility, selfadjointness, as well as some other properties are stable under relatively bounded perturbation \[109\].

Lemma 8 Let \(T\) and \(A\) be operators from \(X\) to \(Y\), and let \(A\) be \(T\)-bounded with \(T\)-bound smaller than 1. Then \(S := T + A\) is closable if and only if \(T\) is closable; in this case the closures of \(T\) and \(S\) have the same domain. In particular \(S\) is closed if and only if \(T\) is.

Proof is rather simple. We shall use the treatment exposed, e.g., in \[109\]. In the inequality (103) we may assume that \(b < 1\). Hence

\[
-a\|u\| + (1-b)\|Tu\| \leq \|Su\| \leq a\|u\| + (1+b)\|Tu\|,
\]

(104)

for \(D(T)\). Let us recall that \(T\) is closable if and only if \(u_n \in D(T)\), \(u_n \to 0\) and \(Tu_n \to v\) imply \(v = 0\). Applying the second equality of (104) to \(u\) replaced
by $u_n - u_m$, we see that a $T$ convergent sequence $\{u_n\}$ (that is a convergent sequence $\{u_n\}$ for which $Tu_n$ is also convergent) is also $S$-convergent. Similarly from the first inequality an $S$-convergent sequence $\{u_n\}$ is $T$-convergent. If $\{u_n\}$ is $S$-convergent to 0, it is $T$-convergent to 0 so that $Tu_n \to 0$ if $T$ is closable; then it follows from the second inequality of (104) that $Su_n \to 0$, which shows that $S$ is closable. Similarly, $T$ is closable if $S$ is. ◇

Let $T$ and $A$ be operators from $X$ to $Y$, $A$ being $T$ bounded (103). If $T^{-1}$ exists and is a bounded operator from $Y$ to $X$ then $AT^{-1}$ is an operator on $Y$ to $Y$ and is bounded by

$$
\|AT^{-1}v\| \leq a\|T^{-1}v\| + b\|v\| \leq (a\|T^{-1}\| + b)\|v\|.
$$

If

$$
a\|T^{-1}\| + b < 1 \quad (105)
$$

one can also show stability of bounded invertibility under relatively bounded perturbation. Indeed if (105) is valid then $S = T + A$ is automatically closed by Lemma 8. One has

$$
S = T + A = (1 + AT^{-1})T.
$$

and thus

$$
\|S^{-1}\| \leq \frac{\|T^{-1}\|}{1 - a\|T^{-1}\| - b}.
$$

In the Hilbert space one can show that selfadjointness is also stable under relatively bounded perturbations. If $T$ is selfadjoint and $A$ is symmetric and $T$ bounded with $T$-bound smaller than 1, then $T + A$ is also selfadjoint. In particular $T + A$ is selfadjoint if $A$ is bounded and symmetric with $D(A) \supset D(T)$ (109). Note that the assumption that the bound be smaller than 1 cannot be dropped in general. If $T$ is unbounded and selfadjoint, and $A = -T$, then $T + A$ is a proper restriction of the operator 0 and is not selfadjoint (109).

An important example of a relatively bounded perturbation provides the Schrödinger operator in the 3-dimensional euclidean space $R^3$ for a system of $s$ particles interacting with each other by the Coulomb forces. In this case the formal Schrödinger operator is the $3s$-dimensional Laplacian $-\Delta$ and the
perturbation $V(x)$ has the form

$$V(x) = \sum_{j=1}^{s} e_j r_j + \sum_{j<k} e_{jk} r_{jk}, \quad (106)$$

where $e_j$ and $e_{jk}$ are constants and

$$r_j = \left( x_{3j-2}^2 + x_{3j-1}^2 + x_{3j}^2 \right)^{1/2},$$
$$r_{jk} = \left[ (x_{3j-2} - x_{3k-2})^2 + (x_{3j-1} - x_{3k-1})^2 + (x_{3j} - x_{3k})^2 \right]^{1/2}.$$  

It can be proved that the minimal operator $\hat{T}$ constructed from the formal operator $-\Delta$ is essentially self-adjoint with the self-adjoint closure $H_0$. If $V$ denotes the maximal multiplication operator $V(x)$ then it can be shown that $V$ is relatively bounded with respect to $H_0$ as well as to $\hat{T}$ with relative bound equal to zero \[109\].

### 4.1.2 Setting up the problem

Henceforth we restrict ourselves to the ensuing problem. Let $H_0$ be a closed linear operator acting in a Banach space and let $V$ be another closed linear operator acting in this space, which is relatively bounded with respect to $H_0$. In addition let the bound satisfy $\|T\|$ with $T$ replaced by $H_0$. We shall be interested in the resolvent operator $R_\lambda(z) := (H_0 + \lambda V - z)^{-1}$ defined for $z$ not in the spectrum of $H_\lambda := H_0 + \lambda V$ ($\lambda \in C$). Usual perturbation theory starts from the identity

$$R_\lambda(z) = R_0(z)[1 + \lambda VR_0(z)]^{-1} \quad (107)$$

by developing the geometric series

$$[1 + \lambda VR_0(z)]^{-1} = \sum_{n=0}^{\infty} (-\lambda)^n [VR_0(z)]^n. \quad (108)$$

Under our assumptions, $VR_0(z)$ is a bounded operator, so this series will converge in norm for $\|\lambda VR_0(z)\| < 1$. If $V$ is bounded then the condition can be replaced by $\|\lambda V\| < d(z)$, where $d(z)$ is the distance of $z$ from the spectrum of $H_0$.

Let us suppose that there exists a finite system $\Sigma'(H_0)$ of eigenvalues of $H_0$, separated from the rest of the spectrum of $H_0$ by a closed curve $\Theta$
encircling $\Sigma'(H_o)$. Then there is a convex region $\Lambda$ of the complex plane, containing the origin, such that for all $\lambda \in \Lambda$ the spectrum $\Sigma(H_\lambda)$ of $H_\lambda = H_o + \lambda V$ is likewise separated by $\Theta$ into a part $\Sigma'(H_\lambda)$ and a remainder. The eigenvalues in $\Sigma'(H_\lambda)$ are analytic in $\lambda$ with only algebraic singularities \cite{109}. The usual Rayleigh-Schrödinger perturbation theory is obtained by substituting (107-108) in the expression for $P_\lambda$,

$$P_\lambda = \frac{1}{2\pi i} \oint_\Gamma R_\lambda(z) dz,$$

the projection operator onto the subspace associated with the eigenvalues $\varepsilon_\lambda$ branching off from $\varepsilon_o$ in $\Sigma'(H_o)$, where $\Gamma$ is a contour encircling $\varepsilon_\lambda$ but no other points of $\Sigma'(H_\lambda)$ (see Fig. 8). In the physical language if the contour $\Gamma$ encloses all energy levels below the Fermi surface then $P_\lambda$ is nothing but the Fermi projector or density matrix at zero temperature, respectively.

To extend the region of convergence some summability method can be used \cite{108, 111}. Reeken used the Borel summability with $\beta = 1$. As we have shown, $\mu$-method has a bigger region of convergence, so that an application of the method will provide a further extension of the Rayleigh-Schrödinger
perturbation theory. Let $A$ be a bounded operator acting in a Banach space. Consider an operator-valued function $f(z)$,

$$f(z) = \frac{1}{(1 + zA)}.$$

Because of the Theorem 2 (see also [9]), it is true that

$$f(z) = \frac{1}{(1 + zA)} = \int_{o}^{\infty} \exp(\exp t) \sum_{n=0}^{\infty} \frac{(-zAt)^n}{\mu(n)} \, dt \quad (110)$$

for $z \in MLS(f)$. The r.h.s. of (110) can be defined by means of the uniform convergence of Theorem 2 as a limit of entire functions. Due to Theorem 2 for any compact set $K \subset MLS(f)$ and each $\varepsilon$ there is $t_o > 0$ and an integer $N$ such that

$$\left| f(z) - \sum_{n=0}^{N} (z^n/a_n) \int_{o}^{t_o} e^{-e^t} t^n \, dt \right| < \varepsilon$$

uniformly in $z \in K$. Thus

$$\int_{o}^{\infty} e^{-e^t} \sum_{n=0}^{\infty} \frac{(-ztA)^n}{\mu(n)} \, dt = \lim_{t_o \to \infty} \lim_{N \to \infty} \sum_{n=0}^{N} (-zA)^n \int_{o}^{t_o} e^{-e^t} t^n \, dt$$

as a limit of entire functions. Any operator valued function can be defined on its MLS in this manner. Thus this provides an alternative to the Dunford-Schwartz integral. To define an operator valued holomorphic function $f(z)$ on $MLS(f)$ as a limit of entire functions one can also use some other AMCSM [8, 62].

The singularities of $f(z)$ are just such $z$ that $-1/z$ is from the spectrum $\Sigma(A)$ of $A$. Therefore the r.h.s. of (110) will converge for $z = 1$ if there are no singularities on the segment $[0, 1]$, or, if $-1/z \in [-\infty, -1] \notin \Sigma(A)$, respectively. So, the following Corollary holds.

**Corollary 4**: Let $A$ be a bounded operator acting in a Banach space. Let $\Sigma(A)$ be the spectrum of $A$. If $\Sigma(A)$ is contained in the complement of $(-\infty, -1]$, then

$$\frac{1}{(1 + A)} = \int_{o}^{\infty} \exp(-\exp t) \sum_{n=0}^{\infty} \frac{(-At)^n}{\mu(n)} \, dt.$$
Note that the domain of application of the representation (110) strongly depends on the sign of the parameter $\lambda$.

Now, to extend the region of convergence of the Rayleigh-Schrödinger perturbation theory one has to locate the spectrum of $\lambda V R_o(z)$. The spectrum of $\lambda V R_o(z)$ for $z \notin \Sigma(H_o)$ is the set of all $\xi = 0$ such that $z \in \Sigma[H(-\lambda/\mu)]$; $\xi = 0$ may or may not belong to the spectrum $[111]$. To show this one uses identity

$$[\lambda V R_o(z) - \mu]^{-1} = (-1/\mu)(H_o - z)[H_o + (\lambda/\mu)V - z]^{-1},$$

for $\mu \neq 0$. If $z \in \Sigma(H(-\lambda/\mu))$ the second factor on the r.h.s. is not defined. Thus $\mu \in \Sigma(\lambda V R_o(z))$. If $z \notin \Sigma(H(-\lambda/\mu))$ then the second factor is defined and its product with the unbounded operator $H_o - z$ is a bounded operator on the whole space.

From the above discussion and Corollary 4 we have the Lemma 9 which converts the problem of convergence of the $\mu$-sum for a given $\lambda$ to the problem whether $z$ belongs to the spectrum of some class of operators $H_\nu$ or not.

**Lemma 9**: Let $H_o$ and $V$ be as above. Then

$$[1 + \lambda V R_o(z)]^{-1} = \int_o^\infty \exp(-\exp t) \sum_{n=0}^\infty [-\lambda V R_o(z) t]^n / \mu(n) dt$$

for all $z$ which are not in the spectrum of $H_\nu$ for $\nu \in \{\lambda u | u \in [0, 1]\}$. ◇

Note that if $\lambda$ is a real number, as usually happens in physical applications, then one has only to consider a real parametric family of Hamiltonians in contrast to [111]. As a direct consequence of Lemma 8 and (109) one arrives at the following statement:
Corollary 5: If for $\lambda \in \Lambda$ no one of the branches emanating from $\varepsilon_0$, since the perturbation parameter $\nu$ varies on the straight line segment $[0, \lambda]$, crosses any other branches starting from other eigenvalues in $\Sigma'(H_0)$, then a contour $\Gamma$ exists encircling $\varepsilon_0$ and $\varepsilon_\lambda$ but no other eigenvalues (see Fig. 8) such that

$$P_\lambda = \int_0^\infty \exp(-e^t) \sum_{n=0}^\infty R_n \frac{(-\lambda t)^n}{\mu(n)} dt,$$

where $R_n$ is the residue of $R_o(z)VR_o(z)...VR_o(z)$ at $\varepsilon_0$. ♣

4.2 Derivative analyticity relations

Finally, we shall consider the problem of derivative analyticity relations (DAR) [112, 113]. Let us briefly sketch the problem. The real and imaginary parts of the forward scattering amplitude $F(E)$ in high-energy physics (or in optics) are related by a dispersion relation of the form

$$\text{Re } F(E)/E = \frac{2E}{\pi} \text{ v.p. } \int_{E_o}^{\infty} \frac{\text{Im } F(E)}{E(E^2 - E_o^2)} dE,$$

where the pole terms and subtraction constants are for simplicity omitted. A major “shortcoming” of the dispersion relation is that one has to know the imaginary part $\text{Im } F(E)$ on the whole infinite integration interval to obtain the real part $\text{Re } F(E)$ at a given point. The problem of DAR is that of relating the real and imaginary part of $F(E)$ at the same point. To cope with it in the general case is quite difficult. After the substitution $x = \ln E$, $x_o = \ln E_o$, $k = \ln E^*$ and $f(x) = \text{Im } F(E)/E$ the integral on the r.h.s. of (111) takes the form,

$$\frac{1}{\pi} \text{ v.p. } \int_{k-x_o}^{\infty} \frac{f(x + x_o)}{\sinh(x)} dx.$$

The solution of the problem of DAR is only known in a special case where $f(x)$ can be analytically extended to an entire function [113, 114]. Using the results of Section 3 one can establish the DAR for more general class of function $f(x)$. In fact, under the assumption of regularity of $f(z)$ on the real positive axis one arrives at the following representation of the dispersion
integral,

$$\frac{1}{\pi} v.p. \int_{k-x_o}^{\infty} dx \int_{0}^{\infty} dt \exp(-\exp t) \sum_{n=0}^{\infty} f^{(n)}(x_o) \frac{(xt)^n}{n! \mu(n) \sinh(x)}.$$ \hspace{1cm} (112)

because of Corollary [1]. If $f(x)$ can be extended to an entire function, then $\sum_{n=0}^{\infty} a_n f^{(n)}(x_o)$ converges, where

$$a_n = \int_{k-x_o}^{\infty} dx \int_{0}^{\infty} dt \exp(-\exp t) \frac{(xt)^n}{n! \mu(n) \sinh(x)}.$$ 

One may then integrate term by term in (112), reproducing the previous results [113, 114]. For the references and recent status of the problem see also [115]. A different question is that of the practical use of the extension because the real positive axis is often the only place where a scattering amplitude is singular.
5 Conclusion

We have found a family of moment constant summability methods $\mu_k$ which provide for $k \geq 1$ an analytic continuation of a function regular at the origin onto its whole Mittag-Leffler’s star, in contrast to the Borel method (see Remark 2). These methods are intimately connected with the Borel one, since for $\alpha = \beta = 1$ the Borel method is nothing but the $\mu_\sigma$-method.

The methods discussed above can be successfully applied to convergent as well as divergent perturbation series which diverge like $(\ln n)^n$ or slower. One may encounter such a situation when regularizing a theory on a lattice [3, 117] or using a cut-off in space time or momentum space [4, 37, 116]. Of some interest is also an expansion in the “artificial” parameter proposed recently by Bender et. al. in the $\lambda\phi_4^4$ theory and some other models [18], and which seems to be convergent. When some additional information is at disposal or provided one deals with truncated series our method can be made more powerful by combining it with a conformal mapping or the Padé approximation [76, 77, 78, 118]. An interesting question is to compare a numerical efficiency of the $\mu_k$-methods with the Padé or some other methods [119]. So far we did not consider the possible generalization of the $\mu_k$-methods to more variables [120]. Of some interest is also the question whether the Wynn $\varepsilon$-algorithm for calculating the Borel integral could be adapted to the $\mu_k$-methods [121], and whether the inverse of a $\mu_k$-summable function is $\mu_k$-summable [122].

It is obvious that the use of the above methods is not confined to perturbation theory where the expansion parameter is a coupling. The method can be applied to the $1/N$ expansion [50] or to the $\varepsilon$-expansion [47] as well.

We have shown that the above given summability method solves the problems (A) and (B), i.e., in the regular case provides an analytic continuation on the whole Mittag-Leffler (principal) star and can deal with the horn-shaped singularity as well. To our knowledge it is the first summability method having these properties. We have discussed advantages and shortcomings of the method, applications to the Rayleigh-Schrödinger perturbation theory which cover the very important case of the Schrödinger operator in $\mathbb{R}^3$ for a system of $s$ particles interacting with each other by the Coulomb forces [108, 109], and the derivative analyticity relations [113].

Like any analytic regular summability method our method may also have a wide domain of applications. We shall not give a list of them because the
reader can easily judge whether it is interesting for him or not.

An open question still remains how the summability properties (both Borel’s and ours) are transported in equations such as the Dyson-Schwinger equations for Green’s functions [123] and the problem of summing a perturbation series which violates SAC. To solve the last problem one has however to have more (nonperturbative) informations to pick up a physically plausible solution.

One could also establish analogous results by other slightly modified Stieltjes moments of the form

\[ \mu(n) := \int_{0}^{\infty} \exp(t - \exp t)t^n dt, \]

etc. In this case one needs only to replace \( \omega(z) \) by \( \tilde{\omega}(z) = \exp z + \ln(z + 1) \) and \( H_R \) by \( \tilde{H}_R := \{ \Re \tilde{\omega}(1/z) > \tilde{\omega}(1/R) \} \). Note that in contrast to the previous case the boundary \( \partial \tilde{H}_R^{-1} \) of \( \tilde{H}_R^{-1} \) approaches the straight line \( \Im z = \pi/2 \) from below (see Fig. 6).

We hope that a sufficient number of examples have been given to illustrate that to draw physical conclusions from the Borel (non)summability without knowing the region of analyticity to which a power series expansion should be summed up may sometimes be very dangerous. We have also shown that to prove the Callan-Symanzik assumption about the mass insertion terms one needs generalized SAC, derived in the paper, which applies to horn-shaped regions. We then were able to formulate such conditions for a whole variety of horn-shaped regions.

Finally, our method gives a generalization of [63] since Theorem 5 together with Lemma 5 provide a summability mechanism which apart from invariance conditions and linear covariances preserves also nonlinear perturbative conditions such as the unitarity of the Feynman series. We note that Theorem 4 gives a generalization of SAC in regards to [63, 67, 68]. Indeed, by the theorem there cannot exist a function which

i) is analytic in the horn \( H_R \) and continuous up to the boundary;
ii) possesses there the asymptotic expansion (66) which has equal sign coefficients \( a_n \) for \( n \geq n_o \).

We would like to point out that assertions of the type that a quantity \( Q_1 \) equals another quantity \( Q_2 \) to all orders of perturbation theory are very
vague unless the SAC are shown to be valid. These SAC provide a complete
generalization of Simon’s work \[63\]. We have given arguments that four
dimensional field theories (without UV cutoff) violate SAC and that probably
there is no reason to look at the Borel transform of these theories.

\section{Acknowledgements}

First of all I should like to thank my parents for kind attention and sup-
port. I should also like to acknowledged fruitful discussions with members of
our department. I am indebted to J. Fischer for continuous interest in my
work and valuable comments, P. Hořava for discussions on variety of physical
problems, J. Chýla, P. Kolář, S. Nemeček, and J. Rameš for their help with
computer facilities, and Mrs. M. Boušková for drawing pictures.

I take opportunity to thank J. Fuka for discussions about entire functions
which help me in derivation of the properties of the new summability method
in regular case as well as V. Šverák for many stimulating discussions which
help me to derive the analogue of the Nevanlinna theorem for horn-shaped
regions. Discussions with C. Klimčík on various aspects of field theories
are also greatfully acknowledged. I thank J. Magnen, G.’t Hooft, and A.S.
Wightman for encouraging correspondence, CPT, Ecole Polytechnique at
Palaisseau for their warm hospitality and fruitful discussions, E. Seiler for
kind hospitality in München and discussions, as well as A. Smith Albion for
reading a relevant part of the thesis.
7 Appendix

Theorems 3.5.1-4 of the Phragmen-Lindelöf type [105] can be easily modified to regions of the asymptotic type \( (1, \eta) \) (and also to other regions), i.e., if \( F(z) \)

i) is analytic in a region \( D \) of the asymptotic type \( (1, \eta) \), and continuous in its closure \( \bar{D} \);

ii) \( |F(z)| \leq M \exp[-\delta |\exp(1/\eta z)|] \quad z \in \bar{D}, \)

\( \delta \) being an arbitrary positive constant, then \( F(z) \equiv 0 \).

So, if there exist two functions \( f_1(z) \) and \( f_2(z) \) satisfying the hypotheses of the theorem then their difference \( g(z) = f_1(z) - f_2(z) \) obeys the bound

\[ |g(z)| \leq A\rho^n |z|^n \exp[-\exp w(n) + n \ln w(n)] \]

uniformly in \( z \in \bar{D} \) and \( n \). Now, choosing \( n = [w^{-1}(1/\rho r)] = [s(1/\rho r)] \) to optimize the bound, where \([b]\) is now entire part of \( b \) and \( r = |z| \), one gets that

\[ |g(z)| \leq A \exp[-\exp(1/\rho r)] \quad z \in \bar{D}, \]

for region \( D \) of the asymptotic type \( (1, \eta) \) with \( \rho \leq \eta \). By virtue of ii) then \( g(z) \equiv 0 \).

On the other hand, whenever the bound (98) is relaxed, i.e., if in a domain \( D \) of the asymptotic type \( (1, \eta) \) a bound

\[ |R_N(z)| \leq A\rho^N |z|^N \exp[-\exp w(N) + N \ln w(N)] \quad \forall z \in D, \]

is allowed, with \( \rho > \eta \), then for any \( \tilde{\rho} \) such that \( \rho > \tilde{\rho} > \eta \) the function \( \exp(-\exp(1/\tilde{\rho} z)) \) satisfies the hypotheses of the theorem. In fact,

\[
\sup_{z \in D} |z^{-N} \exp(-\exp(1/\tilde{\rho} z))| = \tilde{\rho}^N \exp(\max_{z \in \overline{D}} \text{Re}[N \ln z - \exp z]) \sim \tilde{\rho}^N \exp[-\exp w(N) + N \ln w(N)].
\]

However, this nontrivial function has trivial asymptotic series in \( D \). ♦

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