INVERSE INITIAL PROBLEM FOR FRACTIONAL REACTION-DIFFUSION EQUATION WITH NONLINEARITIES

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Abstract. The initial inverse problem of finding solutions and their initial values \( t = 0 \) appearing in a general class of fractional reaction-diffusion equations from the knowledge of solutions at the final time \( t = T \). Our work focuses on the existence and regularity of mild solutions in two cases:

- The first case: The nonlinearity is globally Lipschitz and uniformly bounded which plays important roles in PDE theories, and especially in numerical analysis.
- The second case: The nonlinearity is locally critical which widely arises from the Navier-Stokes, Schrodinger, Burgers, Allen-Cahn, Ginzburg-Landau equations, etc.

Our solutions are local-in-time and are derived via fixed point arguments in suitable functional spaces. The key idea is to combine the theories of Mittag-Leffler functions and fractional Sobolev embeddings. To firm the effectiveness of our methods, we finally apply our main results to time fractional Navier-Stokes and Allen-Cahn equations.

Keywords: Riemann-Liouville fractional derivative, Nonlocally differential operator, Time diffusion equation, Existence and Regularity, Navier-Stokes equation, Allen-Cahn equation.

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Contents

1. Introduction
2. Notations
3. Well-posedness with globally Lipschitz nonlinearities
   3.1. The case \( \varphi \in \mathcal{D}(A) \)
   3.2. The case \( \varphi \in \mathcal{D}(A^\zeta) \) for some \( 0 < \zeta < 1 \)
4. Well-posedness with critical nonlinearities and its applications
   4.1. Well-posedness
   4.2. Applications to time fractional Navier-Stokes equations
   4.3. Applications to time fractional Allen-Cahn equations
Appendix
References

1. Introduction

Let \( \Omega \) be a \( C^2 \) bounded open set of \( \mathbb{R}^k \), \( k \geq 2 \), and let \( \mathcal{A} \) be a symmetric and uniformly elliptic operator on \( \Omega \) defined by

\[
\mathcal{A}f(x) = -\sum_{i=1}^{k} \frac{\partial}{\partial x_i} \left( \sum_{j=1}^{k} a_{ij}(x) \frac{\partial}{\partial x_j} f(x) \right) + b(x)f(x), \quad x \in \Omega,
\]

where \( a_{ij} \in C^1(\overline{\Omega}) \), \( b \in C(\overline{\Omega}; [0, +\infty)) \), and \( a_{ij} = a_{ji}, 1 \leq i, j \leq k \). We assume that there exists a constant \( c_0 > 0 \) such that, for \( x \in \overline{\Omega}, y = (y_1, y_2, ..., y_k) \in \mathbb{R}^k \),

\[
\sum_{1 \leq i, j \leq k} y_i a_{ij}(x) y_j \geq c_0 |y|^2.
\]

In this paper, we consider the following time-fractional diffusion equation

\[
\begin{aligned}
\frac{\partial}{\partial t} u(x,t) &= -\frac{\partial^{1-\alpha}}{\partial t} \mathcal{A}u(x,t) + G(x, t, u(x,t)), & x \in \Omega, \; 0 < t \leq T, \\
u(x, t) &= 0, & x \in \partial \Omega, 0 < t < T,
\end{aligned}
\] (1)
where $\alpha \in (0, 1)$ and $\partial_t^{1-\alpha}$ is the Riemann-Liouville fractional derivative of order $1 - \alpha$ given by
\[
\frac{\partial_t^{1-\alpha}}{\partial t} f(t) := \frac{1}{\Gamma(\alpha)} \frac{d}{dt} \left( \int_0^t (t - s)^{\alpha-1} f(s) ds \right).
\]
Here $\Omega$ is a sufficiently smooth boundary in $\mathbb{R}^n$, $n \geq 1$, and $G$ is a nonlinear source function which appears in some physical phenomena. We study solutions of equations (1) subjected to the final condition
\[
u(x, T) = \varphi(x), \quad x \in \Omega.
\]
If $\alpha = 1$, then the fractional derivative $\partial_t^{1-\alpha}A/\partial t$ coincides with $A$ and Problem (1)-(2) becomes the classical parabolic equation which can be transformed into the integral equation
\[
u(x, t) = e^{(T-t)A} \varphi(x) - \int_t^T e^{(\mu-t)A} G(x, \mu, u(x, \mu)) d\mu.
\]
If $\varphi \in L^2(\Omega)$ and $G$ is globally Lipschitz, then the ill-posedness of this problem on $C([0, T]; L^2(\Omega))$ has been showed, e.g., see [10]. If $0 < \alpha < 1$, the Problem (1)-(2) is called an inverse initial problem (or final value problem) for time-fractional reaction-diffusion equation.

Recall that time-fractional parabolic partial differential equations have been successfully applied in a wide variety of research areas, including physics [3], finance [4, 5, 6, 8], hydrology [9, 59]. Note also that denoting by $\partial_t^\alpha$ the Caputo fractional derivative operator of order $\alpha$, the first equation of (1) with Riemann-Liouville derivative can be transformed into the following form
\[
\frac{\partial_t^{\alpha}}{\partial t} u(x, t) = -Au(x, t) + I_t^{1-\alpha} G(x, t, u(x, t)),
\]
where $I_t^{1-\alpha}$ is the fractional integral of the order $1 - \alpha$ defined by
\[
I_t^{1-\alpha} v(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t - s)^{-\alpha} v(s) ds.
\]
Let us observe that many mathematical studies have been devoted to the analysis of linear PDEs with time-fractional derivative. Without being exhaustive, we can mention the works of [20, 50, 51, 24, 27, 52, 43, 29, 31, 44, 45] dealing with fractional diffusion equations with time-fractional derivatives, in the Caputo sense, in different contexts. We mention also the works of [33, 34, 35, 36, 37, 50] devoted to the regularity of solutions of such equations. Note that the presence of a nonlinear term increases considerably the difficulty of these problems, and so studying the regularity of solutions of nonlinear fractional PDEs is a challenge. In [48], the authors considered the existence, uniqueness, and regularity of a solution to the nonlinear time-fractional Fokker-Planck equations. Y. Kian and M. Yamamoto [56] gave the existence of a local solution to a semilinear fractional wave equation. M. Warma et al. [52] investigated the existence and regularity of a global solution of a nonlinear time-fractional wave equation. B. Ahmad et al. [36] studied the existence and regularity of a solution to the time-fractional diffusion equations in Banach space. F. Camilli et al. [45] investigated the existence, uniqueness and regularity properties of a classical solution to the time-fractional Hamilton-Jacobi equation arising in Mean Field Games theory. Y. Giga et al. [27] established the existence, stability and some regularity results of a viscosity solution of the Hamilton-Jacobi equation with Caputos time-fractional derivative. Very recently, Y. Giga et al. [28] introduce a discrete scheme for second-order fully nonlinear parabolic PDEs with Caputo’s time-fractional derivatives.

Note that the initial value problem for (1) has a long history and has attracted much attention among the mathematical community. Both linear and nonlinear equations have been considered. In the linear case, corresponding to $G = G(x, t)$, the work [5] is one of the first works devoted to the fractional diffusion equations similar to Problem (1). The study of such equations received much attention such as [39, 12]. Note also that models with a singular kernel at the origin are well-known and arise in the heat conduction and viscoelasticity as well as other models (see [13, 14, 15]). We also mention some works that devoted to the study of (1) can be found in [15, 14] as well as in [53], where the regularity of solutions of a time-fractional diffusion equation similar to (1) has been considered. Numerical solutions of the alternative representation of Problem (1) have been studied in [54, 55, 60, 49, 53, 64]. In contrast to the linear case, nonlinear equations of the form Problem (1) with $G = G(x, t, u)$ have received less attention. We can mention results like [19], where the authors studied the local well-posedness for the Cauchy problem of a semilinear fractional diffusion equation. Very recently, the work [5] considered a
family of nonlinear Volterra equations which is a generalization of model (1). The existence of a local mild solution to the problem and the continuous dependence concerning initial data have been discussed.

Despite the numerous papers devoted to the study of the direct problem as mentioned above, as far as we know, the inverse initial problems (called terminal value problem, or backward problem) for fractional diffusion-wave equations have not been investigated widely. However, backward problems for fractional diffusion equations have an important role in some practical areas, where the recovery of the previous distribution of physical diffusion processes plays an important role in several areas. For instance, some engineering problems consist in determining the previous data of the physical field from its present state. Backward problems for time-fractional diffusions with Caputo derivative can be found in some recent works like [61, 62, 41, 42].

This paper aims to study the inverse initial value problem (1)-(2) for nonlinear fractional reaction-diffusion with respect to $0 < \alpha < 1$. We are particularly interested in the theory of existence, uniqueness, and regularity of solutions. Our analysis is motivated by the necessity for a rigorous study of a numerical scheme to approximate solutions in several applications. In our knowledge, regularity results on inverse initial value problems (final value problems) for fractional diffusion equations are still limited. We also emphasize that, to the best of our knowledge, there are no results devoted to the analysis of Problem (1)-(2).

In what follows, we briefly present some difficulties that appear in the study of Problem (1)-(2):

- A substitution cannot transfer the problem into the respective initial value problem since the Riemann-Liouville fractional derivative is locally defined. Indeed, for $0 < \alpha < 1$,
  \[
  \frac{\partial^{1-\alpha}}{\partial s} Au(x, s) \bigg|_{s=T-t} \neq \frac{\partial^{1-\alpha}}{\partial t} Au(x, T-t),
  \]
- It is well-known that the Grönwall’s inequality is a powerful tool to prove the well-posedness of initial value problems. Unfortunately, applying the Grönwall’s inequality is not fitting to deal with inverse initial value problems containing non-locally fractional derivatives (see [8] and [12]). Besides, one cannot convert Problem (1)-(2) into a same form as [8].
- The operators $E^{-1}_{\alpha,1}(-T^\alpha A)$ and $E^{-1}_{\alpha,1}(-T^\alpha A)E_{\alpha,1}(-t^\alpha A)$, $t \geq 0$, which appear in the precise representation (see [12]) of solutions is weaker than $E_{\alpha,1}(-t^\alpha A)$ in the following sense:
  \[
  E_{\alpha,1}(-t^\alpha A)(L^2(\Omega)) \subset C([0,T];L^2(\Omega)),
  \]
and for fixed $0 \leq s < 1$,
  \[
  \left\{ \begin{array}{l}
  E^{-1}_{\alpha,1}(-T^\alpha A)(W^{2s,2}(\Omega)) \not\subset L^2(\Omega), \\
  E^{-1}_{\alpha,1}(-T^\alpha A)E_{\alpha,1}(-t^\alpha A)(W^{2s,2}(\Omega)) \not\subset L^\infty(0,T;L^2(\Omega)).
  \end{array} \right.
  \]

In addition, the representation (12) of solutions of Problem (1)-(2) is more complex than the respective initial problem [8].

This paper is organized as follows. In Section 2, we present some basic definitions and the setting for our work. Moreover, we obtain a precise representation of solutions by using Mittag-Leffler operators. In Section 3, we investigate the well-posedness, and regularity for Problem (1)-(2) under globally Lipschitz assumptions imposed on the source term. Furthermore, we also discuss regularity results for the mild solution. This section contains two main results. The first one is obtained by using the Banach fixed point theorem in the strongly final data case $\phi \in D(A)$, where $A = A$ acts on $L^2(\Omega)$ with domain $D(A) = H^1_0(\Omega) \cap H^2(\Omega)$. The regularity results for the solution and its derivative of the first order are studied by basing on some Sobolev embeddings. The second result concerns the problem in the weakly final data case $\phi \in D(A^\zeta)$ with some $0 < \zeta < 1$. An important ingredient in the proof of this result arises from an applying of Picard iterations. The difficulty here comes from finding where solutions belong. In Section 4, we investigate the existence of a mild solution under a class of critical nonlinearities. As we know, nonlinear PDEs with critical nonlinearities are an interesting topic. We can mention this in Y. Giga [29] and A. N. Carvalho et al [67] and references therein. The study for critical nonlinearities is certainly difficult, and it requires using embeddings between the Sobolev spaces and the Hilbert scales spaces associated with fractional powers of $A$. In this section, we use the Banach fixed point argument in some suitable spaces. The main idea is to prove that the nonlinearity $G$ maps from $W^{12/\zeta, q} \to W^{11, \eta}(\Omega)$ with suitable numbers $l_1 > 0$, $l_2 < 0$, $q \geq 1$, and then establish the existence of a mild solution in the space $C^\alpha([0,T];W^{11, \eta}(\Omega))$. The main difficulty comes from the choice of involved parameters, especially,
the choice of $l_1, l_2, q$. Finally, to firm the effectiveness of our methods, we give some applications to fractional Navier-Stokes and Allen-Cahn equations.

2. Notations

For a given Banach space $X$, we define by $L^\infty(0, T; X)$ the space of all essentially bounded measurable functions $f$ on $[0, T]$ endowed with the norm $\|f\|_{L^\infty(0, T; X)} := \text{ess sup}_{0 \leq t \leq T} \|f(t)\|_X$. For $0 < \sigma < 1$, we denote by $\mathcal{C}^\sigma([0, T]; X)$ the subspace of $C([0, T]; X)$ containing all Holder continuous functions with exponent $\sigma$ with the norm

$$\|f\|_{\mathcal{C}^\sigma([0, T]; X)} := \sup_{0 \leq t \leq T} \|f(t)\|_X + \sup_{0 < t_1 < t_2 \leq T} \frac{\|f(t_1) - f(t_2)\|_X}{|t_1 - t_2|^{\sigma}}.$$

For $0 < \sigma_1, \sigma_2 < 1$ and $0 < \sigma_1 + \sigma_2 < 1$, we denote by $\mathcal{F}^{\sigma_1, \sigma_2}((0, T]; X)$ the space of all functions $f$ in $C((0, T]; X)$ satisfying

$$\|f\|_{\mathcal{F}^{\sigma_1, \sigma_2}((0, T]; X)} := \sup_{0 < t \leq T} t^{\sigma_1} \|f(t)\|_X + \sup_{0 < t_1 < t_2 \leq T} t_1^{\sigma_1 + \sigma_2} \frac{\|f(t_1) - f(t_2)\|_X}{|t_1 - t_2|^{\sigma_2}} < \infty.$$

For $X_1, X_2$ two Banach spaces, we denote by $\mathcal{L}(X_1, X_2)$ the space of all bounded linear operators $S$ from $X_1$ to $X_2$ endowed with the norm

$$\|S\|_{\mathcal{L}(X_1, X_2)} = \sup_{f \in X_1, f \neq 0} \frac{\|Sf\|_{X_2}}{\|f\|_{X_1}}.$$

To study Problem (11), we consider the operator $A = A$ acting on $L^2(\Omega)$ with domain $\mathcal{D}(A) = H_0^1(\Omega) \cap H^2(\Omega)$. There exist the sequences $\{\lambda_n\}_{n \geq 1}$ and $\{\phi_n\}_{n \geq 1} \subset \mathcal{D}(A)$ which are the eigenvalues and eigenvectors of $A$ respectively. As we known that $\{\lambda_n\}_{n \geq 1}$ is positive, non-decreasing and $\lim_{n \to \infty} \lambda_n = \infty$.

Moreover, $A\phi_n = \lambda_n \phi_n$. So, we can define the fractional power operator $A^\beta, \beta \geq 0$, by

$$A^\beta f(x) := \sum_{n=1}^{\infty} \phi_n(x) \lambda_n^\beta \langle f, \phi_n \rangle, \quad (5)$$

whereupon $\langle, \rangle$ denotes the usual bracket of $L^2(\Omega)$. The domain of $A^\beta$ is defined by

$$\mathcal{D}(A^\beta) := \left\{ f \in L^2(\Omega) : \|f\|_{\mathcal{D}(A^\beta)}^2 := \|A^\beta f\|_2^2 = \sum_{n=1}^{\infty} \lambda_n^2 \langle f, \phi_n \rangle^2 < \infty \right\},$$

in which $\|\|_2$ is the usual norm of $L^2(\Omega)$. For $\beta = 0$, $\mathcal{D}(A^0) = L^2(\Omega)$. We identify the dual space $[L^2(\Omega)]^* = L^2(\Omega)$, and so that we can set $\mathcal{D}(A^{-\beta}) = [\mathcal{D}(A^\beta)]^*$ with

$$A^{-\beta} f(x) := \sum_{n=1}^{\infty} \phi_n(x) \lambda_n^{-\beta} \langle f, \phi_n \rangle_{-\beta}, \quad \|f\|^2_{\mathcal{D}(A^{-\beta})} := \sum_{n=1}^{\infty} \lambda_n^{-2\beta} \langle f, \phi_n \rangle_{-\beta}^2, \quad (6)$$

where $\langle, \rangle_{-\beta}$ is the duality bracket between $\mathcal{D}(A^\beta)$ and $\mathcal{D}(A^{-\beta})$.

In what follows, we would find a representation of solutions of Problem (11)-(2) basing on some special operators. For this purpose, we first consider the initial value problem

$$\begin{cases}
\frac{\partial}{\partial t} u(x, t) = -\frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}} Au(x, t) + G(x, t, u(x, t)), & x \in \Omega, \ 0 < t \leq T, \\
u(x, t) = 0, & x \in \partial \Omega, 0 < t < T, \\
u(x, 0) = u_0(x), & x \in \Omega.
\end{cases} \quad (7)$$

There are several methods which help to find a representation for solutions of Problem (7), see e.g. [6, 11, 18, 19, 20] and therein. More specifically, the Laplace transform method can be referred in [17, 20]. We refer to the method in [6] where solutions of Problem (7) represented by

$$u(x, t) = E_{\alpha, 1}(-t^\alpha A)u_0(x) + \int_0^t E_{\alpha, 1}(-(t - \mu)^\alpha A)G(x, \mu, u(\mu, \mu))d\mu. \quad (8)$$

Here, the operator $E_{\alpha, 1}(-t^\alpha A)$ is a Mittag-Leffler operator associated with $A$ defined by

$$E_{\alpha, 1}(-t^\alpha A)v(x) := \int_0^\infty M_\alpha(z) e^{-zt^\alpha A}v(x)dz,$$
where $M_\alpha(z) := \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(1-\alpha(n+1))}$ is the Wright-type function. For more details on the operator $E_{\alpha,1}(-t^\alpha A)$, we refer to [20]. In view of the expansion $v = \sum_{j=1}^{\infty} v_je_j$, $v_j = \langle v, e_j \rangle$, and the Mittag-Leffler function

$$E_{\alpha,1}(-z) := \int_0^\infty M_\alpha(\omega)e^{-z\omega}d\omega = \sum_{n=1}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)},$$

$z \in \mathbb{R}$, the operator $E_{\alpha,1}(-t^\alpha A)$ can be expressed as follows

$$E_{\alpha,1}(-t^\alpha A)v(x) = \sum_{n=1}^{\infty} E_{\alpha,1}(-t^\alpha \lambda_n) v_n \phi_n(x).$$

According to Equation (9) and the substitution $t$ by $T$, we derive a constrain between $u(., T) = \varphi$ and $u_0$. In order to find a representation of solutions of Problem (11)-(12), we would recovery $u_0$ from $\varphi$. Hence, this motivates us to find the inverse operator of the Mittag-Leffler operator $E_{\alpha,1}(-t^\alpha A)$. Base on the Hilbert spaces analysis, we define the operator

$$E_{\alpha,1}^{-1}(-t^\alpha A)v(x) := \sum_{n=1}^{\infty} \frac{1}{E_{\alpha,1}(-t^\alpha \lambda_n)} v_n \phi_n(x).$$

Then, it is obvious to see that the operators $E_{\alpha,1}(-t^\alpha A)$ and $E_{\alpha,1}^{-1}(-t^\alpha A)$ are the inverse operators of each other. Hence, we obtain a representation of solutions of Problem (11)-(12) as

$$u(x, t) = E_{\alpha,1}^{-1}(-T^\alpha A)E_{\alpha,1}(-t^\alpha A)\varphi(x) + \int_0^t E_{\alpha,1}(-t - \mu)^\alpha A)G(x, \mu, u(x, \mu))d\mu - \int_0^T E_{\alpha,1}^{-1}(-T^\alpha A)E_{\alpha,1}(-t^\alpha A)E_{\alpha,1}(-T - \mu)^\alpha A)G(x, \mu, u(x, \mu))d\mu. \quad (12)$$

Now, we give a definition of mild solutions of Problem (11)-(12).

**Definition 2.1.** If a function $u$ in $L^\infty(0, T; L^2(\Omega))$ satisfies Equation (12), then it is said to be a mild solution of Problem (11)-(12).

### 3. Well-posedness with globally Lipschitz nonlinearities

In this section, we present the existence, uniqueness and regularity of a mild solution of Problem (11)-(12) which are based on a Lipschitz assumption on the source function $G$.

(G1) $G(0) = 0$, and there exists a constant $K > 0$ such that, for all $0 \leq t \leq T$ and $v_1, v_2 \in L^2(\Omega)$,

$$\|G(., t, v_1) - G(., t, v_2)\| \leq K \|v_1 - v_2\|.$$  

(G2) $G(0) = 0$, and there exists a constant $\tilde{K} > 0$ such that, for all $0 \leq t_1, t_2 \leq T$, $v_1, v_2 \in L^2(\Omega)$,

$$\|G(., t_1, v_1) - G(., t_2, v_2)\| \leq \tilde{K} (\|t_1 - t_2\| + \|v_1 - v_2\|).$$

For $\beta > 0$, we note that $\langle f_1, f_2 \rangle_{-\beta, \beta} = \langle f_1, f_2 \rangle$, if $f_1 \in L^2(\Omega), f_2 \in \mathcal{D}(A^\beta)$, see [11, 12, 17]. Consequently, it holds $\langle \phi_m, \phi_n \rangle_{-\beta, \beta} = \langle \phi_m, \phi_n \rangle = \delta_{mn}$, where $\delta_{mn}$ is the Kronecker delta. Therefore, in view of the definitions (10), (11), the operators $E_{\alpha,1}(-t^\alpha A)$, $E_{\alpha,1}^{-1}(-T^\alpha A)$ are communicative with each others, i.e.,

$$E_{\alpha,1}^{-1}(-T^\alpha A)E_{\alpha,1}(-t^\alpha A) = E_{\alpha,1}(-t^\alpha A)E_{\alpha,1}^{-1}(-T^\alpha A).$$

To deal with the problem, it is necessary to construct some more useful properties of the solution operators. Due to the properties of the Mittag-Leffler functions, we can obtain some properties of the Mittag-Leffler operators $E_{\alpha,1}(-t^\alpha A)$, $E_{\alpha,1}^{-1}(-T^\alpha A)$ associating with the elliptic operator $A$ and its inverse which will be given in the next lemmas.
3.1. The case $\varphi \in \mathcal{D}(A)$. In this section, we will establish existence and uniqueness of a mild solution of Problem (1)-(2). A prediction of solution space can be shown by setting $G = 0$ which corresponds to the homogeneous-solution

$$ u(x, t) = E_{\alpha,1}^{-1}(-T^\alpha A)E_{\alpha,1}(-t^\alpha A)\varphi(x). $$

By assuming that $\varphi \in \mathcal{D}(A)$, and applying consecutively Part c of Lemma 4.7 with respect to $\beta_1 = 0$, Part a with respect to $\beta_1 = 0$, $\beta_2 = 1$, we obtain

$$ \text{ess sup}_{0 \leq t \leq T} \|u(\cdot, t)\| \leq \theta_1(\lambda_1^{-1} + T^\alpha)\|E_{\alpha,1}(-t^\alpha A)\varphi\| \leq \theta_1\theta_2(\lambda_1^{-1} + T^\alpha)\|\varphi\|_{\mathcal{D}(A)}. $$

Hence, $u \in L^\infty(0, T; L^2(\Omega)).$ This suggests us to find solutions in $L^\infty(0, T; L^2(\Omega))$ under the Lipschitz assumptions (G1), (G2). For more convenience, we use notation $a \oplus b$ to denote the sum $\frac{1}{a} + \frac{1}{b}$, and introduce the following sets

$$ V^{2, \oplus}_{\infty} = \{(a; b) : a > 1, b > 1, a \oplus b = \infty\}, $$

$$ V^{2, \nu, \infty} = \{(a; b) : a > 1, b > 1, a \oplus b = \infty\}, $$

$$ V^\alpha_{\alpha, \nu} = \{(a; b; c) : a > 1, c > 1, 0 < b \leq \frac{a - \alpha}{\alpha}, 2a < c < 2\}, $$

$$ V^\alpha_{\alpha, \nu, \infty} = \{(a; b; c; d) : a > 1, b > 1, d > 1, 0 < c \leq \frac{a - \alpha}{\alpha}, 2a < d < 2\}. $$

**Theorem 3.1.** Let the numbers $p, q, r, s$ be satisfied that $(r; s) \in V^{2, \oplus}_{\infty}$, $(p; q; r) \in V^3$. If $\varphi \in \mathcal{D}(A)$, and $C_0 := \theta_2 KT + \theta_1\theta_2(\lambda_1^{-1} + T^\alpha)\lambda_1^{-1} < 1/2$, $\lambda_1 = 1 - \alpha$, then Problem (4)-(5) has a unique mild solution $u$ which belongs to $L^\infty(0, T; L^2(\Omega)) \cap L^{\frac{\alpha}{\beta}}(0, T; D(A^\frac{\alpha}{\beta})).$

Moreover, there exists a positive constant $C_1$ such that

$$ \|u\|_{L^\infty(0, T; L^2(\Omega))} + \|u\|_{L^{\frac{\alpha}{\beta}}(0, T; D(A^\frac{\alpha}{\beta}))} \leq C_1\|\varphi\|_{\mathcal{D}(A)}. $$

**Proof.** This proof is divided into the following steps. First, we prove that if Problem (1)-(2) has a mild solution $u$ in $L^\infty(0, T; L^2(\Omega))$, then it must be bounded to

$$ u \in L^{\frac{\alpha}{\beta}}(0, T; D(A^\frac{\alpha}{\beta})) \cap F^{\frac{\alpha}{\beta}}((0, T); D(A^\frac{\alpha}{\beta})). $$

Secondly, we prove that Problem (1)-(2) accepts a unique mild solution in $L^\infty(0, T; L^2(\Omega))$. Finally, we will obtain the inequality (15).

Let us consider the first step. Assume that $u \in L^\infty(0, T; L^2(\Omega))$ is a mild solution of Problem (1)-(2), i.e., it satisfies (12). We are going to show that $u \in F^{\frac{\alpha}{\beta}}((0, T); D(A^\frac{\alpha}{\beta}))$. For $t, h$ satisfying $0 < t < t + h \leq T$, $h \leq \Lambda$, Equation (12) leads to

$$ u(x, t + h) - u(x, t) = \Pi^{\alpha, h}_{E,A}(t)E_{\alpha,1}^{-1}(-T^\alpha A)\varphi(x) + \int_0^{t+h} \left[ E_{\alpha,1}(-t + h - \mu^\alpha A)G(x, \mu, u(x, \mu))d\mu \right]_{\gamma_{\alpha,1,\beta}(\varphi)(x,t)} $$

$$ + \int_0^t \Pi^{\alpha, h}_{E,A}(t - \mu)G(x, \mu, u(x, \mu))d\mu + \int_0^t \Pi^{\alpha, h}_{E,A}(t)E_{\alpha,1}^{-1}(-T^\alpha A)E_{\alpha,1}(-t^\alpha A)G(x, \mu, u(x, \mu))d\mu, $$

where $\Pi^{\alpha, h}_{E,A}(t) := E_{\alpha,1}(-t + h^\alpha A) - E_{\alpha,1}(-t^\alpha A)$. By applying Lemma 4.2 and the fundamental theorem of calculus, we see that

$$ E_{\alpha,1}(-t + h^\alpha \lambda_n) - E_{\alpha,1}(-t^\alpha \lambda_n) = -\int_t^{t+h} \lambda_n^\alpha e^{-\lambda_n^\alpha t}E_{\alpha,\alpha}(-\lambda_n^\alpha)dt. $$

Here, we have $|E_{\alpha,\beta}(-\lambda_n^\alpha)| \leq \theta_2(1 + \nu^{\alpha \lambda_n^\alpha})^{-1}$ by Lemma 4.5. Since $(r; s) \in V^{2, \oplus}_{\infty}$, it holds that $\frac{1}{a + \beta} \in (\frac{1}{2}; 1)$. Hence, we find that

$$ (1 + \nu^{\alpha \lambda_n^\alpha})^{-1} \leq (1 + \nu^{2\alpha \lambda_n^\alpha})^{-\frac{1}{1+\beta}} \leq \nu^{\frac{1}{1+\beta} - \frac{\alpha}{\beta}} \lambda_n^{\frac{1}{1+\beta} \alpha}. $$


This implies the following estimate
\[ |E_{α,1}(-(t + h)^αλ_n) - E_{α,1}(-(t - h)^αλ_n)| ≤ θ_2 \int_t^{t+h} λ_n^α ν^{−\frac{α}{2}} - 1 dv, \]
which associated with (10), and the Parseval’s identity yields
\[
\left\| Π_{E,α}^h(t) E_{α,1}(-(T - α)A)φ \right\|_{D(A^+)} ≤ θ_2 \left\| E_{α,1}(-(T - α)A)φ \right\|_1 \int_t^{t+h} ν^{−\frac{α}{2}} - 1 dv. (17)
\]
The integral in the right hand side of (17) is equal to \( \frac{r}{α} τ^{\frac{α}{2}} - 1 \). Therefore, we obtain
\[
\left\| \mathcal{Y}_{α,1}(h)(φ)(\cdot,t) \right\|_{D(A^+)} ≤ \frac{θ_2 r}{α} \left( λ_1^{-1} + Tα \right) t^{−α} h^{α} \varphi \| \varphi \|_{D(A)}, (18)
\]
where we have used Part c of Lemma 17 for \( β_1 = 0 \), the inequality \( (t + h)^{\frac{α}{2}} - 1 ≤ h^{\frac{α}{2}} \) as \( (r,s) ∈ \mathbf{V}_{1}^{2}, \frac{r}{α} ∈ (0,1) \), and the inequality \( (t + h)^{\frac{α}{2}} ≥ t^{\frac{α}{2}} h^{\frac{α}{2}} - h^{\frac{α}{2}} \) as \( h > 0 \). The assumption \( (p; ρ; r) ∈ \mathbf{V}_{α}^{3} \) implies \( \frac{r}{α} ∈ (\frac{1}{2}, 1) \). So, by applying Part b of Lemma 17 for \( β_1 = β_2 = \frac{1}{2} \), we obtain
\[
\left\| \mathcal{Y}_{α,2}(h)(u)(\cdot,t) \right\|_{D(A^+)} ≤ \theta_2 \int_t^{t+h} \left\| E_{α,1}(-(t + h - μ)^α A)G(\cdot, μ, u(\cdot, μ)) \right\|_{D(A^+)} dμ
\]
\[
≤ \frac{θ_2 r}{α} \int_t^{t+h} (t + h - μ)^{−\frac{α}{2}} \| G(\cdot, μ, u(\cdot, μ)) \|_{D(A)} dμ.
\]
We note that \( \| G(\cdot, μ, u(\cdot, μ)) \|_{L^\infty(0,T;L^2(Ω))} \) since \( u ∈ L^\infty(0,T;L^2(Ω)) \), and \( G \) satisfies the Lipschitz assumption (G1). This together with the latter inequality that
\[
\left\| \mathcal{Y}_{α,2}(h)(u)(\cdot,t) \right\|_{D(A^+)} ≤ \frac{θ_2 r}{α} \int_t^{t+h} \left( t + h - μ \right)^{−\frac{α}{2}} \| G(\cdot, μ, u(\cdot, μ)) \|_{D(A^+)} dμ
\]
\[
(19)
\]
where we note \( r > α \) as \( (p; ρ; r) ∈ \mathbf{V}_{α}^{3} \), and \( h^{1−\frac{α}{2}} ≤ Λ^2 h^{\frac{α}{2}} \) as \( (r,s) ∈ \mathbf{V}_{α}^{2}. \) By using the same way as (18), we derive the following estimate
\[
\left\| Π_{E,α}^h(t - μ)G(\cdot, μ, u(\cdot, μ)) \right\|_{D(A^+)} ≤ \frac{θ_2 r}{α} \left( t + h - μ \right)^{−\frac{α}{2}} \| G(\cdot, μ, u(\cdot, μ)) \|, (20)
\]
which directly leads to
\[
\left\| \mathcal{Y}_{α,3}(h)(u)(\cdot,t) \right\|_{D(A^+)} ≤ \frac{θ_2 r}{α} \int_0^t \frac{h^{\frac{α}{2}}}{(t + h - μ)^{−\frac{α}{2}}} \| G(\cdot, μ, u(\cdot, μ)) \| dμ.
\]
(21)
The quantity \( \mathcal{Y}_{α,4}(u) \) can be estimated by using a same argument as (20). Applying Part c, and Part b of Lemma 17 consecutively gives that
\[
\left\| \mathcal{Y}_{α,4}(h)(u)(\cdot,t) \right\|_{D(A^+)} ≤ \int_0^T \left\| Π_{E,α}^h(t) E_{α,1}(-(T - α)A)E_{α,1}(-(T - μ)^α A)G(\cdot, μ, u(\cdot, μ)) \right\|_{D(A^+)} dμ
\]
\[
≤ \frac{θ_2 r}{α} \frac{h^{\frac{α}{2}}}{t^{−α}(t + h)^{−\frac{α}{2}}} \int_0^T \left\| E_{α,1}(-(T - α)A)E_{α,1}(-(T - μ)^α A)G(\cdot, μ, u(\cdot, μ)) \right\|_{D(A)} dμ
\]
\[
≤ \frac{θ_2 r}{α} \frac{θ_1 (λ_1^{-1} + Tα)}{t^{−α}(t + h)^{−\frac{α}{2}}} \int_0^T \left\| E_{α,1}(-(T - μ)^α A)G(\cdot, μ, u(\cdot, μ)) \right\|_{D(A)} dμ
\]
\[
≤ \frac{θ_2 r}{α} \frac{θ_1 (λ_1^{-1} + Tα)}{t^{−α}(t + h)^{−\frac{α}{2}}} \int_0^T \left( T - μ \right)^{−α} \| G(\cdot, μ, u(\cdot, μ)) \| dμ.
\]
(22)
By associating the Lipschitz assumption (G1), and \( (t + h - μ)^{\frac{α}{2}} ≥ (t - μ)^{\frac{α}{2}} \) as \( h > 0 \), we derive
\[
\int_0^t \frac{1}{(t - μ)^{\frac{α}{2}}(t + h - μ)^{−\frac{α}{2}}} \| G(\cdot, μ, u(\cdot, μ)) \| dμ ≤ \frac{r}{r - 2α} K \| u \|_{L^\infty(0,T;L^2(Ω))} T^{1−\frac{2α}{2}} h^{\frac{α}{2}},
\]
where we note \( r > 2α \) as \( (p; ρ; r) ∈ \mathbf{V}_{α}^{3}. \) This together with \( \frac{r}{α} > 1 > 0 \), as \( (p; ρ; r) ∈ \mathbf{V}_{α}^{3} \) again, imply
\[
\left\| \mathcal{Y}_{α,3}(h)(u)(\cdot,t) \right\|_{D(A^+)} ≤ \frac{θ_2 r}{α} \frac{θ_1 (λ_1^{-1} + Tα)}{t^{−α}(t + h)^{−\frac{α}{2}}} \int_0^T \left( T - μ \right)^{−α} \| G(\cdot, μ, u(\cdot, μ)) \| dμ.
\]
(23)
where we have used the fact that $h^\frac{p}{\alpha} \leq A(\frac{p}{\alpha} - 1)\alpha h^\frac{p}{\alpha}$. Similarly, the right hand side of equation 23 can be treated as $\frac{t^n h^\frac{p}{\alpha}}{\alpha}$ by noting $(t + h)^n \geq t^n h^\frac{n}{\alpha}$ again. Indeed, one can see that

$$\left\| Y_{\alpha,4}(u)(x,t) \right\|_{D(A^\frac{p}{\alpha})} \leq \frac{\theta_2^p r^\alpha \theta_1(\lambda_1^{-1} + T^\alpha)}{\alpha} T^\frac{\alpha}{2} K \left\| u \right\|_{L^\infty(0,T;L^2(\Omega))} t^{-\alpha} h^\frac{p}{\alpha}. \quad (24)$$

By combining the estimates 13, 10, 23, 24, we deduce that $u$ belongs to $C((0,T); D(A^\frac{p}{\alpha}))$, and there exists a positive constant $\omega_1$ which does not depend on $x$, $t$, $h$ such that

$$t^n \left\| u((t+h) - u(t)) \right\|_{D(A^\frac{p}{\alpha})} \leq \omega_1 \left( \left\| \varphi \right\|_{D(A)} + \left\| u \right\|_{L^\infty(0,T;L^2(\Omega))} \right). \quad (25)$$

Now, we proceed to estimate $\left\| u(t) \right\|_{D(A^\frac{p}{\alpha})}$, whereupon

$$u(x,t) = E_{\alpha,1}^{-1}(-T^\alpha A)E_{\alpha,1}(-t^\alpha A)\varphi(x) + \int_0^T E_{\alpha,1}(-(t+\mu)^\alpha A)G(x,\mu,u(\mu))d\mu \quad (26)$$

By applying consecutively Part c of Lemma 47 with $\beta_1 = \frac{1}{p}$, and Part b of Lemma 47 with $\beta_1 = 1 + \frac{1}{p}$, $\beta_2 = \frac{1}{p} \in (0;1)$, where $p > 1$ as $(p;\rho;r) \in V_\alpha$, we derive

$$t^\frac{p}{\alpha} \left\| Z_{\alpha,1}(\varphi)(.) \right\| \leq t^\frac{p}{\alpha} \left\| E_{\alpha,1}(-t^\alpha A)\varphi \right\|_{D(A^\frac{p}{\alpha})} \leq \theta_1 \left\| E_{\alpha,1}(\lambda_1^{-1} + T^\alpha) \right\|_{D(A^\frac{p}{\alpha})} \quad (27)$$

In addition, using Part b of Lemma 47 for $\beta_1 = \beta_2 = \frac{1}{p} \in (0;1)$, we get

$$t^\frac{p}{\alpha} \left\| Z_{\alpha,2}(u)(.) \right\| \leq t^\frac{p}{\alpha} \int_0^t \left\| E_{\alpha,1}(-(\mu)^\alpha A)G(.) \right\|_{D(A^\frac{p}{\alpha})} d\mu \quad (28)$$

which implies the estimate

$$t^\frac{p}{\alpha} \left\| Z_{\alpha,3}(u)(.) \right\| \leq \theta_2 t^\frac{p}{\alpha} \left\| G(.) \right\|_{D(A^\frac{p}{\alpha})} d\mu, \quad (29)$$

In addition, the following chain of inequalities is obtained by applying consecutively Lemma 47 three times. Firstly, we apply Part c for $\beta_1 = \frac{1}{p}$. Secondly, we apply Part b for $\beta_1 = 1 + \frac{1}{p}$, $\beta_2 = \frac{1}{p} \in (0;1)$. Thirdly, we apply Part b for $\beta_1 = \beta_2 = 1$. These are presented as follows

$$t^\frac{p}{\alpha} \left\| Z_{\alpha,3}(u)(.) \right\| \leq \theta_1 t^\frac{p}{\alpha} \left\| G(.) \right\|_{D(A^\frac{p}{\alpha})} d\mu, \quad (30)$$
We see that $t \mapsto t^{-\frac{1}{2}}$ belongs to $L^{\frac{2}{2^-\rho}}(0, T)$ as $\rho$ satisfies the assumption $(\rho; \rho; r) \in V^3_\alpha$. This implies $u$ belongs to $L^{\frac{2}{2^-\rho}}(0, T; D(\mathcal{A}^\frac{1}{\alpha}))$, and there exists a constant $\omega_3 > 0$ such that

$$
\|u\|_{L^{\frac{2}{2^-\rho}}(0, T; D(\mathcal{A}^\frac{1}{\alpha}))} \leq \omega_3 \left( \|\varphi\|_{D(\mathcal{A})} + \|u\|_{L^\infty(0, T; L^2(\Omega))} \right). \tag{31}
$$

On the other hand, the estimate (31) holds for all $p > 1$, hence, it also holds for $p = r$. Combining this with the estimate (25), we obtain

$$
u \in L^{\frac{2}{2^-\rho}}(0, T; D(\mathcal{A}^\frac{1}{\alpha})), \quad \mathcal{F}^{\frac{1}{\alpha}, \frac{1}{2}} (0, T; D(\mathcal{A}^\frac{1}{\alpha})).
$$

Moreover, there exists a constant $\omega_4 > 0$ such that

$$t^\frac{1}{\alpha} \|u(., t)\|_{D(\mathcal{A}^\frac{1}{\alpha})} + \frac{\|u(., t + h) - u(., t)\|_{D(\mathcal{A}^\frac{1}{\alpha})}}{h^\frac{1}{\alpha}} \leq \omega_4 \left( \|\varphi\|_{D(\mathcal{A})} + \|u\|_{L^\infty(0, T; L^2(\Omega))} \right),$$

i.e.,

$$u \in L^{\frac{2}{2^-\rho}}(0, T; D(\mathcal{A}^\frac{1}{\alpha})), \quad \mathcal{F}^{\frac{1}{\alpha}, \frac{1}{2}} (0, T; D(\mathcal{A}^\frac{1}{\alpha})). \tag{32}
$$

By the above arguments, the proof will be completed by showing that Problem (11-12) has a unique mild solution $u$ in $L^\infty(0, T; L^2(\Omega))$, and then the inequality (15). Here, we will apply the contraction mapping principle. Let us denote $\mathcal{R} := \frac{\theta_1 \theta_2 (\lambda_1^{-1} + T^\alpha)}{1 - c_0} \|\varphi\|_{D(\mathcal{A})}$, and the ball $B_\mathcal{R} := \{ v \in L^\infty(0, T; L^2(\Omega)) : \|v\|_{L^\infty(0, T; L^2(\Omega))} \leq \mathcal{R} \}$. On $B_\mathcal{R}$, we define the mapping

$$\mathcal{H} v(x, t) := Z_{\alpha, 1}(\varphi)(x, t) + Z_{\alpha, 2}(v)(x, t) + Z_{\alpha, 3}(v)(x, t). \tag{33}
$$

We now prove $\mathcal{H}$ is well-defined on $B_\mathcal{R}$. For this purpose, we will apply Lemma (12) and the Lipschitz condition (G1). Indeed, for $v \in B_\mathcal{R}$, one can check the following estimates for three terms of the right hand side of (33). The first one is obtained similarly as (13). The second one is derived by applying Part a of Lemma (14) with respect to $\beta_1 = 0$, and then using the Lipschitz assumption (G1) as

$$\|Z_{\alpha, 2}(v)(., t, \mu)\| \leq \theta_2 \int_0^t \|G(\mu, v(., \mu))\| d\mu \leq \theta_2 KT \mathcal{R}. \tag{34}
$$

The last one can be derived by using Lemma (14) three times as follows. We firstly apply Part c for $\beta_1 = 0$. Then, we secondly apply Part a for $\beta_1 = 0$. Finally, we apply Part b for $\beta_1 = \beta_2 = 1$, i.e.,

$$\|Z_{\alpha, 3}(v)(., t, \mu)\| \leq \theta_1 (\lambda_1^{-1} + T^\alpha) \int_0^T \|E_{\alpha, 1}(-t^\alpha A)E_{\alpha, 1}(-(T - \mu)^\alpha A)G(\mu, v(., \mu))\|_{D(\mathcal{A})} d\mu
$$

$$\leq \theta_1 \theta_2 (\lambda_1^{-1} + T^\alpha)^2 \int_0^T \|E_{\alpha, 1}(-(T - \mu)^\alpha A)G(\mu, v(., \mu))\|^2_{D(\mathcal{A})} d\mu
$$

$$\leq \theta_1 \theta_2^2 (\lambda_1^{-1} + T^\alpha)^3 \int_0^T \|G(\mu, v(., \mu))\|^2 d\mu
$$

$$\leq \theta_1 \theta_2^2 (\lambda_1^{-1} + T^\alpha)K\mathcal{A}^{-1}T^\alpha \mathcal{R}. \tag{35}
$$

The estimates (14), (33), (34) associated with (33) to allow estimate

$$\|\mathcal{H} v(., t)\| \leq \theta_1 \theta_2 (\lambda_1^{-1} + T^\alpha) \|\varphi\|_{D(\mathcal{A})} + \left( \theta_2 KT + \theta_1 \theta_2^2 (\lambda_1^{-1} + T^\alpha)K\mathcal{A}^{-1}T^\alpha \right) \mathcal{R}. \tag{36}
$$

Hence, $\|\mathcal{H} v(., t)\| \leq \mathcal{R}$ by the definition of $\mathcal{R}$. This leads to $\|\mathcal{H} v\|_{L^\infty(0, T; L^2(\Omega))} \leq \mathcal{R}$, i.e., the mapping $\mathcal{H}$ is well-defined on $B_\mathcal{R}$. Now, we prove $\mathcal{H}$ is a contraction mapping. It follows from Equation (33) that

$$\mathcal{H} v_1(x, t) - \mathcal{H} v_2(x, t) = \int_0^T E_{\alpha, 1}(-(t - \mu)^\alpha A) [G(x, \mu, v_1(x, \mu)) - G(x, \mu, v_2(x, \mu))] d\mu
$$

$$- \int_0^T E_{\alpha, 1}(-(T - \mu)^\alpha A)E_{\alpha, 1}(-(T - \mu)^\alpha A) [G(x, \mu, v_1(x, \mu)) - G(x, \mu, v_2(x, \mu))] d\mu,
$$

for $v_1, v_2 \in B_\mathcal{R}$. Therefore, the assumption $C_0 = \theta_2 KT + \theta_1 \theta_2^2 (\lambda_1^{-1} + T^\alpha)K\mathcal{A}^{-1}T^\alpha < \frac{1}{2}$ combined with a similar argument as proof of (36) to allow the estimate

$$\|\mathcal{H} v_1 - \mathcal{H} v_2\|_{L^\infty(0, T; L^2(\Omega))} \leq \frac{1}{2} \|v_1 - v_2\|_{L^\infty(0, T; L^2(\Omega))}. \tag{37}
$$
We then deduce that $\mathcal{H}$ is a contraction mapping on $\mathcal{B}_R$. Hence, by the contraction mapping principle, there exists a unique fixed point $u$ of $\mathcal{H}$ in $\mathcal{B}_R$, namely $\mathcal{H}u = u$. This means that $u$ is the unique mild solution of Problem (1)-(2). Moreover, there exists a constant $\omega_3 > 0$ such that

$$\|u\|_{L^\infty((0,T,L^2(\Omega))} \leq \omega_3 \|\varphi\|_{\mathcal{C}(A)^+}.$$  \hfill (38)

The proof is finally completed by taking the estimate (31), (32), and (38) together. \hfill \Box

**Remark 3.1.** Applying (15), one can derive the following bounds for the mild solution of Problem (1)-(2). Moreover, there exists a constant $a$ of Lemma 4.7, we derive the following estimate

$$\lim_{t \to 0} \left( \int_0^t \|\varphi\|_{\mathcal{C}(A)^+} \right) = 0$$

**Remark 3.2.** In Theorem 3.1, we have just considered the continuity of the mild solution on $[0,T]$, without considering the initial time. Indeed, Equation (12) is not defined for $t = 0$, namely, we cannot substitute $t = 0$ to obtain $u(.,0)$. Hence, it is necessary to recover the initial value $u(.,0)$ in $\mathcal{C}(A)^+$.

It is useful to note that the Sobolev embedding $L^2(\Omega) \hookrightarrow D(A^{-\frac{1}{2}})$ holds. This means there exists a positive constant $\omega_0 > 0$ satisfying, for all $f \in L^2(\Omega)$,

$$\|f\|_{D(A^{-\frac{1}{2}})} \leq \omega_0 \|f\|_{L^2(\Omega)}.$$  \hfill (39)

The following lemma will be useful for the recovery $u(.,0)$ by considering the limit $\lim_{h \to 0^+} u(.,h)$.

**Lemma 3.2.** For $q > 1$, $\varphi \in D(A)$, and $G$ satisfy the Lipschitz assumption (G1). Then, we have

$\lim_{h \to 0^+} Z_{\alpha,1}(\varphi)(.,h) = E_{\alpha,1}^{-1}(-T^\alpha A)\varphi$, \quad $\lim_{h \to 0^+} Z_{\alpha,2}(u)(.,h) = 0$,

$\lim_{h \to 0^+} Z_{\alpha,3}(u)(.,h) = \int_0^T E_{\alpha,1}^{-1}(-T^\alpha A)E_{\alpha,1}(-(T - \mu^\alpha)A)G(\mu,u(.,\mu))d\mu$, in the sense of the space $D(A^{-\frac{1}{2}})$.

**Proof.** We firstly recall that $|E_{\alpha,1}(\mu^\alpha \lambda_0)| \leq \theta_1(1 + \mu^\alpha \lambda_0^2)^{-1}$. Thus, it follows from $\frac{2\lambda_0}{q} \in (0; \frac{1}{2})$, as $q > 1$, that $|E_{\alpha,1}(\mu^\alpha \lambda_0)| \leq \theta_2 \mu^{-\alpha \frac{q-1}{q}} \lambda_0^\frac{1}{q}$. This combined with (16) allow the estimate

$$\left\| \Pi_{E_{\alpha,1}^{\alpha,h}(0)} E_{\alpha,1}^{-1}(T^\alpha A) \varphi \right\|_{D(A^{-\frac{1}{2}})} \leq \theta_2 \left\| E_{\alpha,1}^{-1}(T^\alpha A) \varphi \right\| \int_0^h \mu^{-\frac{1}{2}} d\mu.$$  \hfill (40)

Hence, letting $\beta_1 = 0$ in Part c of Lemma 4.7 gives

$$\left\| Z_{\alpha,1}(\varphi)(.,h) - E_{\alpha,1}^{-1}(T^\alpha A)\varphi \right\|_{D(A^{-\frac{1}{2}})} \leq \theta_2 \left( \lambda_0^{-1} + T^\alpha \right) \left\| \varphi \right\|_{\mathcal{C}(A)^+}.$$  \hfill (41)

This implies $\lim_{h \to 0^+} Z_{\alpha,1}(\varphi)(.,h) = E_{\alpha,1}^{-1}(T^\alpha A)\varphi$ in the sense of the space $D(A^{-\frac{1}{2}})$. Moreover, by using the inequality (39), noting $\|G(\mu,u(.,\mu))\| \leq K \|\varphi\|_{\mathcal{C}(A)^+}$ as Theorem 3.1, and letting $\beta_1 = 0$ in Part a of Lemma 4.7, we derive the following estimate

$$\left\| Z_{\alpha,2}(u)(.,h) \right\|_{D(A^{-\frac{1}{2}})} \leq \omega_0 \frac{1}{h} \theta_2 K \|\varphi\|_{\mathcal{C}(A)^+}.$$  \hfill (42)

This implies $\lim_{h \to 0^+} Z_{\alpha,2}(u)(.,h) = 0$ in the sense of the space $D(A^{-\frac{1}{2}})$. Let us show the last limit. By using the inequality (40), one can see that

$$\left\| \int_0^T \Pi_{E_{\alpha,1}^{\alpha,h}(0)} E_{\alpha,1}^{-1}(T^\alpha A)E_{\alpha,1}(-(T - \mu^\alpha)A)G(\mu,u(.,\mu))d\mu \right\|_{D(A^{-\frac{1}{2}})} \leq \frac{\theta_2 q}{\alpha} h^{-\frac{1}{2}} \int_0^T \left\| E_{\alpha,1}^{-1}(T^\alpha A)E_{\alpha,1}(-(T - \mu^\alpha)A)G(\mu,u(.,\mu)) \right\| d\mu.$$  \hfill (43)
The integrand of the above integral can be estimated by using Part c, and Part b of Lemma 4.7 as
\[ \|E_{\alpha,1}^{-1}(-T^\alpha A)E_{\alpha,1}(-(T - \mu)^\alpha A)G(\cdot, \mu, u(\cdot, \mu))\| \leq \theta_1(\lambda_1^{-1} + T^\alpha)\theta_2(T - \mu)^{-\alpha}KC_1\|\varphi\|_{D(A)}. \]
Hence, by denoting \( \omega_6 := \frac{\theta_2 q}{\alpha} \theta_1(\lambda_1^{-1} + T^\alpha)\hat{\gamma}^{-1}T^\alpha KC_1 \), we get
\[
\left\| Z_{\alpha,3}(u)(., h) - \int_0^T E_{\alpha,1}^{-1}(-T^\alpha A)E_{\alpha,1}(-(T - \mu)^\alpha A)G(\cdot, \mu, u(\cdot, \mu))d\mu \right\|_{D(A^{-\frac{1}{2}})} \leq \omega_6\|\varphi\|_{D(A)}h^{\frac{\alpha}{2}}.
\]
Therefore, the lastev estimate is obtained by letting \( h \) tends to 0.

Due to the above lemma, it is relevant to define the initial value by \( u(., 0) = \lim_{t\to 0^+} u(., t), \) namely
\[ u(., 0) := E_{\alpha,1}^{-1}(-T^\alpha A)\varphi - \int_0^T E_{\alpha,1}^{-1}(-T^\alpha A)E_{\alpha,1}(-(T - \mu)^\alpha A)G(\cdot, \mu, u(\cdot, \mu))d\mu \quad (41) \]
in \( D(A^{-\frac{1}{2}}) \). Next, we will establish the continuity of the mild solution \( u \) on the whole interval \( [0; T] \).

**Theorem 3.3.** Let the numbers \( p, q, \rho, s \) satisfy \( (r; s) \in V_1^{25}, \) \( (p; q; r) \in V_2^4 \), and \( \varphi, C_0 \) satisfy Theorem 3.1 Then, the mild solution \( u \) of Problem 3.1 belongs to \( L^\infty(0; T; L^q(\Omega)) \cap L^\infty(0; T; D(A^{\frac{1}{2}})) \cap F^{\frac{3}{q} - \frac{1}{2}}((0; T]; D(A^{\frac{1}{2}})) \cap C^p((0; T]; D(A^{\frac{1}{2}})). \)
Moreover, there exists a positive constant \( C_2 \) such that \( \|u\|_{C^p([0; T]; D(A^{\frac{1}{2}}))} \leq C_2\|\varphi\|_{D(A)} \), i.e.,
\[
\sup_{0 \leq t \leq T} \left\| u(., t) \right\|_{D(A^{\frac{1}{2}})} + \sup_{0 \leq t < h \leq T} \frac{\|u(., t + h) - u(., t)\|_{D(A^{\frac{1}{2}})}}{h^{\frac{\alpha}{2}}} \leq C_2\|\varphi\|_{D(A)} \quad (42)
\]

**Proof.** We note that \( (p; \rho; r) \in V_3^3 \) as \( (p; q; r) \in V_2^4 \). Hence, the assumptions of this theorem also fulfills Theorem 3.1 Therefore, it is sufficient to prove that \( u \in C^p([0; T]; D(A^{\frac{1}{2}})) \). Firstly, we estimate \( \|u(., t)\|_{D(A^{\frac{1}{2}})} \). Here, we also use the notations \( Z_{\alpha,j}, 1 \leq j \leq 3 \), in the proof of Theorem 3.1. Note that \( (p; q; r) \in V_2^4 \) includes \( q > 1 \). Hence, by using the inequality (39), and Lemma 4.7 we derive
\[
\left\| Z_{\alpha,1}(\varphi)(., t) \right\|_{D(A^{\frac{1}{2}})} \leq \omega_0, -\frac{\theta_1 \theta_2}{\alpha}(\lambda_1^{-1} + T^\alpha)\|\varphi\|_{D(A)},
\]
\[
\left\| Z_{\alpha,2}(u)(., t) \right\|_{D(A^{\frac{1}{2}})} \leq \omega_0, -\frac{\theta_1 \theta_2}{\alpha}TKC_1\|\varphi\|_{D(A)},
\]
where have used that \( \|G(\cdot, \mu, u(\cdot, \mu))\| \leq KC_1\|\varphi\|_{D(A)} \). Moreover, the following estimate holds
\[
\left\| Z_{\alpha,3}(u)(., t) \right\|_{D(A^{\frac{1}{2}})} = \left\| \int_0^T E_{\alpha,1}^{-1}(-T^\alpha A)E_{\alpha,1}(-(T - \mu)^\alpha A)G(\cdot, \mu, u(\cdot, \mu))d\mu \right\|_{D(A^{\frac{1}{2}})} \leq \omega_0, -\frac{\theta_1 \theta_2}{\alpha}TKC_1\|\varphi\|_{D(A)}.
\]
This estimate implies
\[
\left\| Z_{\alpha,3}(u)(., t) \right\|_{D(A^{\frac{1}{2}})} \leq \omega_0, -\frac{\theta_1 \theta_2}{\alpha}(\lambda_1^{-1} + T^\alpha)\hat{\gamma}^{-1}T^\alpha KC_1\|\varphi\|_{D(A)}.
\]
Therefore, there exists a constant \( \omega_7 > 0 \) such that, for all \( t \in [0; T] \),
\[
\|u(., t)\|_{D(A^{\frac{1}{2}})} \leq \|Z_{\alpha,1}(\varphi)(., t)\|_{D(A^{\frac{1}{2}})} + \sum_{j=1,2,3} \|Z_{\alpha,j}(u)(., t)\|_{D(A^{\frac{1}{2}})} \leq \omega_7\|\varphi\|_{D(A)}. \quad (43)
\]
Secondly, we consider \( \|u(., t + h) - u(., t)\|_{D(A^{\frac{1}{2}})} \), for \( t, h \) satisfying that \( 0 \leq t < t + h \leq T \), and \( h \leq \Lambda \). Here, the notations \( Y_{\alpha,j,b}, 1 \leq j \leq 4 \), in the proof of Theorem 3.3 will be used again. In a similar way to Lemma 4.2 we deduce that
\[
\|Y_{\alpha,1,b}(\varphi)(., t)\|_{D(A^{\frac{1}{2}})} = \|\Pi_{E,A}(t)E_{\alpha,1}^{-1}(-T^\alpha A)\varphi\|_{D(A^{\frac{1}{2}})} \leq \frac{\theta_1 \theta_2}{\alpha}(\lambda_1^{-1} + T^\alpha)\|\varphi\|_{D(A)}h^{\frac{\alpha}{2}},
\]
where \((t + h)\frac{\dot{\tau}}{s} - t\frac{\dot{\tau}}{s}\leq h\frac{\dot{\tau}}{s}\) as \((p; q; r) \in \mathbf{V}_a^4\) and \(\frac{\dot{\tau}}{s} \in (0, 1)\). In addition, applying (39) and Part a of Lemma 3.2, we derive the following estimate

\[
\|\mathcal{Y}_{\alpha,2,h}(u)(.,t)\|_{D(A^{-\frac{1}{2}})} \leq \omega_0 -\frac{\theta_2}{\alpha} \int_t^{t+h} \|G(.,\mu,u(.,\mu))\| \, d\mu \leq \omega_0 -\frac{\theta_2}{\alpha} K\|\varphi\|_{D(A)} A^{1-\frac{\alpha}{\dot{\tau}}}.\]

Moreover, repeating the arguments of Lemma 3.2, we obtain

\[
\|\mathcal{Y}_{\alpha,3,h}(u)(.,t)\|_{D(A^{-\frac{1}{2}})} = \left\|\int_0^t \Pi_{E;A}(t-\mu)G(\cdot,\mu,u(\cdot,\mu))d\mu\right\|_{D(A^{-\frac{1}{2}})} \leq \frac{\theta_2 q}{\alpha} T K\|\varphi\|_{D(A)},
\]

\[
\|\mathcal{Y}_{\alpha,4,h}(u)(.,t)\|_{D(A^{-\frac{1}{2}})} = \left\|\int_0^T \Pi_{E;A}(t)E_{\alpha,1}(-T^\alpha A)E_{\alpha,1}(-\mu^\alpha A)G(\cdot,\mu,u(\cdot,\mu))d\mu\right\|_{D(A^{-\frac{1}{2}})} \leq \frac{\theta_2 q}{\alpha} t \lambda_1^{-1} + T^\alpha \lambda^{-1} T K\|\varphi\|_{D(A)} h^\frac{\alpha}{\dot{\tau}}.
\]

Due to the identity

\[
u(.,t+h) - u(.,t) = \mathcal{Y}_{\alpha,1,h}(\varphi)(.,t) + \sum_{2 \leq j \leq 4} \mathcal{Y}_{\alpha,j,h}(u)(.,t),
\]

and the above arguments, there exists a constant \(\omega_s > 0\) which does not depend \(x, t, h\) such that

\[
\|u(.,t+h) - u(.,t)\|_{D(A^{-\frac{1}{2}})} \leq \omega_s \|\varphi\|_{D(A)} h^\frac{\alpha}{\dot{\tau}}.
\]

Hence, putting the estimates (33), (41), Lemma 3.2 together gives \(u \in C^\frac{\alpha}{\dot{\tau}}([0,T];D(A^{-\frac{1}{2}}))\) and

\[
\sup_{0 \leq t \leq T} \|u(.,t)\|_{D(A^{-\frac{1}{2}})} + \sup_{0 \leq t \leq T|h|^\frac{\alpha}{\dot{\tau}} \|u(.,t)\|_{D(A^{-\frac{1}{2}})} \leq (\omega_7 + \omega_8)\|\varphi\|_{D(A)},
\]

The proof is completed by letting \(C_2 = \omega_7 + \omega_8\). \(\square\)

In the next result, we will investigate the existence of the fractional derivative \(\frac{\partial}{\partial t}\) of the mild solution \(u\). For this purpose, we define \(\mathcal{P}_N\), the projection from \(L^2(\Omega)\) to \(\text{span}\{\phi_n\}_{1 \leq n \leq N}\) by

\[
\mathcal{P}_N f(x) := \sum_{n=1}^{N} (f,\phi_n)\phi_n(x).
\]

In view of Lemma 3.5, we have \(\frac{\partial}{\partial t}E_{\alpha,1}(-t^\alpha \lambda_n) = -t^\alpha \lambda_n E_{\alpha,1}(-t^\alpha \lambda_n)\), for \(t > 0\). Therefore, noting that the projection \(\mathcal{P}_N\) has finite range, we deduce from Equation (12) that

\[
\frac{\partial}{\partial t}\mathcal{P}_N u(.,t) = \underbrace{-t^\alpha \lambda_n E_{\alpha,1}(-t^\alpha A)E_{\alpha,1}(-t^\alpha A)\mathcal{P}_N\varphi}_{O_{\alpha,1,\mathcal{P}_N}(\varphi)(.,t)}
\]

\[
+ \underbrace{\mathcal{P}_N G(.,t,u(.,t))}_{O_{\alpha,2,\mathcal{P}_N}(u)(.,t)} - \underbrace{\int_0^t (t-\mu)^\alpha \lambda_n E_{\alpha,1}(-\mu^\alpha A)\mathcal{P}_N G(\cdot,\mu,u(\cdot,\mu))d\mu}_{O_{\alpha,3,\mathcal{P}_N}(u)(.,t)}
\]

\[
+ \underbrace{\int_0^T t^\alpha \lambda_n E_{\alpha,1}(-t^\alpha A)E_{\alpha,1}(-t^\alpha A)\mathcal{P}_N G(\cdot,\mu,u(\cdot,\mu))d\mu}_{O_{\alpha,4,\mathcal{P}_N}(u)(.,t)}.
\]

In the following theorem, we will show the existence of the derivative \(\frac{\partial}{\partial t}u(.,t)\) in a suitable space by studying the limit \(\lim_{N \to \infty} \frac{\partial}{\partial t}\mathcal{P}_N u(.,t)\).

**Theorem 3.4.** Let the numbers \(p, q, \hat{p}, \hat{q}, \rho, \hat{\rho}, r, s, \hat{r}, \hat{s}\) satisfy the following condition

\[(p; q; r), (\hat{p}; \hat{q}; \hat{r}; \hat{s}) \in \mathbf{V}_a^4, \quad (r; s), (\hat{p}; \hat{q}) \in \mathbf{V}_1^{2, \oplus}, \quad (\hat{r}; \hat{s}) \in \mathbf{V}_1^{2, \ominus},\]

(\(\mathbf{V}_a^4, \mathbf{V}_1^{2, \ominus}\) are spaces of vector fields and \(\mathbf{V}_1^{2, \oplus}\) is a space of vector fields with \(2\)-dimensional vector valued entries).
\(\varphi, C_0, G\) satisfy Theorem 3.1, and \(G\) satisfies the Lipschitz assumption (G2). Then, the mild solution \(u\) of Problem (1)-(2) satisfies \(u \in L^p_{\infty,\beta}(0, T; \mathcal{D}(A^{-\frac{\beta}{2}}))\), and there exists a positive constant \(C_3\) such that
\[
\left\| \frac{\partial}{\partial t} u \right\|_{L^p_{\infty,\beta}(0, T; \mathcal{D}(A^{-\frac{\beta}{2}}))} \leq C_3 \|\varphi\|_{\mathcal{D}(A)}.
\]
(47)

If \(\alpha > \frac{1}{2}\), and \(\beta < \frac{\alpha}{1 - \alpha}\), then
\[
\left\| \frac{\partial}{\partial t} u \right\|_{L^p_{\infty,\beta}(0, T; \mathcal{D}(A^{-\frac{\beta}{2}}))} \leq C_3 \|\varphi\|_{\mathcal{D}(A)}.
\]
(48)

Proof. Firstly, we will prove that: given \(0 < \varepsilon < T\), and \(t \in [\varepsilon; T]\), the sequences \(\{\mathcal{O}_{\alpha,j,\nu}\}_{N \geq 1}\) and \(\mathcal{P}_{M,N}\) are uniformly convergent in \(D(A^{-\frac{\beta}{2}})\). For any natural numbers \(M, N\), \(M > N > 1\), we denote by \(\mathcal{P}_{M,N} := \mathcal{P}_{M} - \mathcal{P}_{N}\). Applying Part c of Lemma 17 for \(\beta_1 = 1\), and then Part d of Lemma 17 for \(\beta_1 = 1 + \frac{1}{q}, \beta_2 = \frac{1}{2q} \in (0, \frac{1}{2})\) as \((\hat{p}; \hat{q}) \in V^{2, \beta}_{1}\), we obtain

\[
\|\mathcal{O}_{\alpha,1,\nu}(\varphi,(.),t)\|_{\mathcal{D}(A^{-\frac{\beta}{2}})} = t^{-\alpha-1}\|E_{\alpha,1}(-t^\alpha A)E_{\alpha,1}(-\hat{t}^\alpha A)\mathcal{P}_{M,N}\mathcal{P}_{\nu,\nu}\|_{\mathcal{D}(A^{-\frac{\beta}{2}})} \\
\leq \theta_1(t_1^{-1} + T^\alpha)\|E_{\alpha,1}(-\hat{t}^\alpha A)\mathcal{P}_{M,N}\mathcal{P}_{\nu,\nu}\|_{\mathcal{D}(A^{-\frac{\beta}{2}})} \\
\leq \theta_1\theta_2(t_1^{-1} + T^\alpha)\|E_{\alpha,1}(-\hat{t}^\alpha A)\mathcal{P}_{M,N}\mathcal{P}_{\nu,\nu}\|_{\mathcal{D}(A^{-\frac{\beta}{2}})},
\]
(49)

where \((\hat{p}; \hat{q}) \in V^{2, \beta}_{1}\). In addition, letting \(\beta_1 = \frac{1}{q}, \beta_2 = \frac{1}{2q}\) in Part d of Lemma 17 gives

\[
\|\mathcal{O}_{\alpha,1,\nu}(u,(.),t)\|_{\mathcal{D}(A^{-\frac{\beta}{2}})} \leq \int_0^t (t - \mu)^{-\alpha-1}\|E_{\alpha,1}(-\hat{t}^\alpha A)\mathcal{P}_{M,N}\mathcal{P}_{\nu,\nu}\mathcal{P}_{\mu,\mu}\|_{\mathcal{D}(A^{-\frac{\beta}{2}})} d\mu
\]
(50)

The quantity \(\mathcal{O}_{\alpha,1,\nu}(u,(.),t)\) can be estimated by using the same arguments as estimates of \(\mathcal{O}_{\alpha,1,\nu}(\varphi,(.),t)\), and \(\mathcal{O}_{\alpha,1,\nu}(u,(.),t)\) in [19, 49]. Indeed, one can check the following chain of the inequalities

\[
\|\mathcal{O}_{\alpha,1,\nu}(u,(.),t)\|_{\mathcal{D}(A^{-\frac{\beta}{2}})} \leq \theta_1(t_1^{-1} + T^\alpha)\int_0^T t^{-\alpha-1}\|E_{\alpha,1}(-\hat{t}^\alpha A)\mathcal{P}_{M,N}\mathcal{P}_{\nu,\nu}\mathcal{P}_{\mu,\mu}\|_{\mathcal{D}(A^{-\frac{\beta}{2}})} d\mu
\]
(51)

Since \(\varphi \in \mathcal{D}(A)\), we have \(\lim_{M,N \to \infty} \|\mathcal{P}_{M,N}\varphi\|_{\mathcal{D}(A)} = 0\). As a consequence of 19, the sequence \(\{\mathcal{O}_{\alpha,1,\nu}(\varphi,(.),t)\}\) is a Cauchy sequence in \(\mathcal{D}(A^{-\frac{\beta}{2}})\), and uniformly converges on \([\varepsilon; T]\). On the other hand, we have \(\|\mathcal{P}_{M,N}\mathcal{P}_{\nu,\nu}\mathcal{P}_{\mu,\mu}\| \leq K\mathcal{C}_1 \|\varphi\|_{\mathcal{D}(A)}\), for each \(0 < \mu < T\), and it follows

\[
\lim_{M,N \to \infty} \|\mathcal{P}_{M,N}\mathcal{P}_{\nu,\nu}\mathcal{P}_{\mu,\mu}\| = 0.
\]

Since the functions \((t - \mu)^{-\alpha-1}, (T - \mu)^{-\alpha}\) of the variable \(\mu\) are integrable, the dominated convergence theorem yields that

\[
\lim_{M,N \to \infty} \int_0^T (t - \mu)^{-\alpha-1}\|\mathcal{P}_{M,N}\mathcal{P}_{\nu,\nu}\mathcal{P}_{\mu,\mu}\| d\mu = 0,
\]
for \( t \in [\varepsilon; T] \). Hence, the sequences \( \{O_{\alpha,2,\mathcal{P}_N}(u)(\cdot, t)\}, \{O_{\alpha,3,\mathcal{P}_N}(u)(\cdot, t)\} \) are Cauchy sequence in \( \mathcal{D}(A^{-\frac{\alpha}{2}}) \), and uniformly converge on \([\varepsilon; T]\).

According to the above arguments and the Sobolev embedding \( L^2(\Omega) \hookrightarrow \mathcal{D}(A^{-\frac{\alpha}{2}}) \), the derivative \( \partial_{tt} u(\cdot, t) \) exists in the space \( \mathcal{D}(A^{-\frac{\alpha}{2}}) \) for \( t > 0 \), and there exists a constant \( \omega_9 > 0 \) satisfying

\[
\left\| \frac{\partial}{\partial t} u(\cdot, t) \right\|_{\mathcal{D}(A^{-\frac{\alpha}{2}})} = \left\| -O_{\alpha,1,\mathcal{P}_\infty}(\varphi)(\cdot, t) - O_{\alpha,2,\mathcal{P}_\infty}(u)(\cdot, t) + O_{\alpha,3,\mathcal{P}_\infty}(u)(\cdot, t) + G(\cdot, t, u(\cdot, t)) \right\|_{\mathcal{D}(A^{-\frac{\alpha}{2}})}
\leq \omega_9 t^{\frac{\alpha}{2}-1} \left\| \varphi \right\|_{\mathcal{D}(A)} + \omega_9 \int_0^t (t - \mu)^{\frac{\alpha}{2}-1} \left\| G(\cdot, \mu, u(\cdot, \mu)) \right\| d\mu
\nonumber
\]

\[
+ \omega_9 \left\| G(\cdot, t, u(\cdot, t)) \right\| + \omega_9 t^{\frac{\alpha}{2}-1} \int_0^t (T - \mu)^{-\alpha} \left\| G(\cdot, \mu, u(\cdot, \mu)) \right\| d\mu,
\]

where we denote \( O_{\alpha,1,\mathcal{P}_\infty} = \lim_{N \to \infty} O_{\alpha,1,\mathcal{P}_N}, \ 1 \leq j \leq 3 \). This associated with the estimate \( \|G(\cdot, t, u(\cdot, t))\| \leq \hat{K}_{\mathcal{C}1} \|\varphi\|_{\mathcal{D}(A)} \), for all \( t > 0 \), implies that there exists a constant \( \omega_{10} > 0 \) such that

\[
\left\| \frac{\partial}{\partial t} u(\cdot, t) \right\|_{\mathcal{D}(A^{-\frac{\alpha}{2}})} \leq \omega_{10} t^{\frac{\alpha}{2}-1} \left\| \varphi \right\|_{\mathcal{D}(A)}.
\] (52)

Here, it is obvious that the function \( t^{\frac{\alpha}{2}-1} \) clearly belongs to \( L^{\infty,1-\frac{\alpha}{2}}(0, T) \) as \((\hat{\nu}; \hat{\varphi}; \hat{\tau}) \in \mathcal{V}_1^{\beta}\). Hence, we deduce that \( \frac{\partial}{\partial t} u \in L^{1-\frac{\alpha}{2}}(0, T; \mathcal{D}(A^{-\frac{\alpha}{2}})) \), and the inequality (52) is proved.

Next, we consider \( \alpha > \frac{1}{2} \) and prove that \( \frac{\partial}{\partial t} u \in L^{1-\frac{\alpha}{2}}(0, T; \mathcal{D}(A^{-\frac{\alpha}{2}})) \). Let \( 0 < t < t + h \leq T \), and \( h \leq \Lambda \). Thanks to the differentiation formula \( \frac{d}{dv}(v^{\alpha-1} E_{\alpha,\alpha}(-v^\alpha)) = v^{\alpha-2} E_{\alpha,\alpha-1}(-v^\alpha) \), for \( \alpha > 0, \\lambda > 0 \), see Lemma [4.7] we have

\[
O_{\alpha,1,\mathcal{P}_\infty}(\varphi)(\cdot, t + h) - O_{\alpha,1,\mathcal{P}_\infty}(\varphi)(\cdot, t) = \int_t^{t+h} v^{\alpha-2} E_{\alpha,\alpha-1}(-v^\alpha) E_{1,1}^{-1}(-T^\alpha) \varphi dv.
\] (53)

By applying Part d of Lemma [4.7] for \( \beta_2 = \frac{1}{\sigma} \in (0; 1) \) as \((\hat{\nu}; \hat{\varphi}; \hat{\tau}) \in \mathcal{V}_1^{\beta}\), one get

\[
\left\| O_{\alpha,1,\mathcal{P}_\infty}(\varphi)(\cdot, t + h) - O_{\alpha,1,\mathcal{P}_\infty}(\varphi)(\cdot, t) \right\|_{\mathcal{D}(A^{-\frac{\alpha}{2}})} \leq \int_t^{t+h} v^{\alpha-2} \left\| E_{\alpha,\alpha-1}(-v^\alpha) E_{1,1}^{-1}(-T^\alpha) \varphi \right\|_{\mathcal{D}(A^{-\frac{\alpha}{2}})} dv
\leq \theta_2 \int_t^{t+h} v^{\alpha-2} \left\| E_{1,1}^{-1}(-T^\alpha) \varphi \right\| dv.
\] (54)

Since \((\hat{\nu}; \hat{\varphi}; \hat{\tau}) \in \mathcal{V}_1^{\beta}\), we get \( 1 - \frac{\alpha}{2} \in (0; 1), \) and \( \frac{\alpha}{2} - \frac{\alpha}{2} = 0, \ 1 - \frac{2\alpha}{2} + \frac{\alpha}{2} \geq 0. \) This ensures the inequalities \( \left| (t+h)^{1-\frac{\alpha}{2}} - t^{1-\frac{\alpha}{2}} \right| \leq h^{1-\frac{\alpha}{2}}, \) and \( (t+h)^{1-\frac{\alpha}{2}} \geq \hat{\nu}^{1-\frac{\alpha}{2}} h^{1-\frac{\alpha}{2}} + \hat{\tau}^{1-\frac{\alpha}{2}}. \) Therefore, we deduce from (52) that

\[
\left\| O_{\alpha,1,\mathcal{P}_\infty}(\varphi)(\cdot, t + h) - O_{\alpha,1,\mathcal{P}_\infty}(\varphi)(\cdot, t) \right\|_{\mathcal{D}(A^{-\frac{\alpha}{2}})} \leq \theta_1 \theta_2 (\hat{\nu}^{1-\frac{\alpha}{2}} + T^\alpha) t^{1-\frac{\alpha}{2}} \int_t^{t+h} v^{\alpha-2} \left\| \varphi \right\|_{\mathcal{D}(A)} dv
\leq \frac{\hat{\nu}}{\hat{\tau} - \alpha} \theta_1 \theta_2 (\hat{\nu}^{1-\frac{\alpha}{2}} + T^\alpha) t^{1-\frac{\alpha}{2}} \frac{h^{1-\frac{\alpha}{2}}}{(t + h)^{1-\frac{\alpha}{2}}} \left\| \varphi \right\|_{\mathcal{D}(A)} \hat{\nu}^{\frac{\alpha}{2}} - \frac{\hat{\tau}}{\hat{\tau} - \alpha} \theta_1 \theta_2 (\hat{\nu}^{1-\frac{\alpha}{2}} + T^\alpha) \left\| \varphi \right\|_{\mathcal{D}(A)} h^{\frac{\alpha}{2}} - \frac{\hat{\tau}}{\hat{\tau} - \alpha} \theta_1 \theta_2 (\hat{\nu}^{1-\frac{\alpha}{2}} + T^\alpha) \left\| \varphi \right\|_{\mathcal{D}(A)} h^{\frac{\alpha}{2}}.
\]
Now, let us estimate $O_{\alpha,2,p_\infty}(u)(., t + h) - O_{\alpha,2,p_\infty}(u)(., t)$. Similarly as \([53]\), we have

\[
O_{\alpha,2,p_\infty}(u)(., t + h) - O_{\alpha,2,p_\infty}(u)(., t) = \left[ \int_0^t \int_{t-\mu}^{t+h-\mu} \nu^{\alpha-2} A E_{\alpha,\alpha-1}(-\nu^\alpha A) G(., \mu, u(., \mu)) d\nu d\mu \right]_{O_{\alpha,2,p_\infty}(u)(., t)}^{O_{\alpha,2,p_\infty}(u)(., t + h)} \]

\[
+ \int_{t}^{t+h} (t + h - \mu)^{\alpha-1} A E_{\alpha,\alpha}(-t + h - \mu)^{\alpha} G(., \mu, u(., \mu)) d\mu .
\]

The first integral is estimated by

\[
\left\| O_{\alpha,2,p_\infty}^{(1)}(u)(., t) \right\|_{D\left( A^{-\frac{\alpha}{\alpha}} \right)} \leq \int_0^t \int_{t-\mu}^{t+h-\mu} \nu^{\alpha-2} \left\| E_{\alpha,\alpha-1}(-\nu^\alpha A) G(., \mu, u(., \mu)) \right\|_{D\left( A^{-\frac{\alpha}{\alpha}} \right)} d\nu d\mu \\
\leq \theta_2 \int_0^t \int_{t-\mu}^{t+h-\mu} \nu^{\frac{\alpha-2}{\alpha}} \left\| G(., \mu, u(., \mu)) \right\|_{D(A)} d\nu d\mu \\
\leq \frac{\theta_2}{\bar{\gamma} - \alpha} \int_0^t \left[ (t - \mu)^{\frac{\alpha}{\alpha}} - (t - \mu)^{\frac{\alpha}{\alpha}} \right] d\mu \tilde{K} \mathcal{C}_1 \| \varphi \|_{D(A)} .
\]

where we also note that \( |(t + h - \mu)^{\frac{\alpha}{\alpha}} - (t - \mu)^{\frac{\alpha}{\alpha}} | \leq h^{\frac{\alpha}{\alpha}} \), and \( \bar{\gamma} \odot \hat{\alpha} = 1 \). This associated with the inequality \( (t + h - \mu)^{\frac{\alpha}{\alpha}} \geq (t - \mu)^{\frac{\alpha}{\alpha}} h^{\frac{\alpha}{\alpha}} + \hat{\alpha} \) implies that

\[
\left\| O_{\alpha,2,p_\infty}^{(1)}(u)(., t) \right\|_{D\left( A^{-\frac{\alpha}{\alpha}} \right)} \leq \frac{\theta_2}{(\bar{\gamma} - \alpha) \alpha} \left[ \tilde{h}^{\frac{\alpha}{\alpha}} \mathcal{C}_1 \| \varphi \|_{D(A)} .
\]

In addition, applying Part b of Lemma \([1,7]\) with respect to \( \beta_1 = \beta_2 = \frac{1}{\hat{\alpha}} \in (0, 1) \) leads to the following estimates for the second integral

\[
\left\| O_{\alpha,2,p_\infty}^{(2)}(u)(., t) \right\|_{D\left( A^{-\frac{\alpha}{\alpha}} \right)} \leq \int_t^{t+h} (t + h - \mu)^{\alpha-1} \left\| E_{\alpha,\alpha}(-t + h - \mu)^{\alpha} G(., \mu, u(., \mu)) \right\|_{D\left( A^{-\frac{\alpha}{\alpha}} \right)} d\mu \\
\leq \theta_2 \int_t^{t+h} (t + h - \mu)^{\frac{\alpha}{\alpha}-1} \left\| G(., \mu, u(., \mu)) \right\|_{D(A)} d\mu \leq \theta_2 \tilde{K} \mathcal{C}_1 \| \varphi \|_{D(A)} \tilde{h}^{\frac{\alpha}{\alpha}} \mathcal{C}_1 \| \varphi \|_{D(A)} .
\]

According to the above estimates, there exists a constant \( \omega_{11} > 0 \) satisfying

\[
\left\| O_{\alpha,2,p_\infty}(u)(., t + h) - O_{\alpha,2,p_\infty}(u)(., t) \right\|_{D\left( A^{-\frac{\alpha}{\alpha}} \right)} \leq \omega_{11} \left\| \varphi \|_{D(A)} \tilde{h}^{\frac{\alpha}{\alpha}} \mathcal{C}_1 \right\|
\]

Next, we will estimate $O_{\alpha,3,p_\infty}(u)(., t + h) - O_{\alpha,3,p_\infty}(u)(., t)$ by using the same techniques as the above estimates. Indeed, in the same way as \([53]\), we find

\[
O_{\alpha,3,p_\infty}(u)(., t + h) - O_{\alpha,3,p_\infty}(u)(., t) = \int_0^T \int_{t}^{t+h} \nu^{\alpha-2} A E_{\alpha,\alpha-1}(-\nu^\alpha A) E_{\alpha,1}(-T^\alpha A) E_{\alpha,1}(-T^\alpha A) G(., \mu, u(., \mu)) d\nu d\mu ,
\]

\]

15
which implies the following chain of inequalities

\[
\left\| \mathcal{O}_{\alpha,3,p_{\infty}}(u)(t+h) - \mathcal{O}_{\alpha,3,p_{\infty}}(u)(t) \right\|_{D(\mathcal{A}^{-\frac{1}{2}})} \\
\leq \int_0^T \int_t^{t+h} \nu^{\alpha-2} \left\| E_{\alpha,-1}(-\nu^\alpha A) E_{\alpha,-1}(-T^\alpha A) E_{\alpha,1}(-T^\alpha A) G(\cdot,\mu, u, \mu) \right\| \, dv \, d\mu \\
\leq \theta_2 \int_0^T \int_t^{t+h} \nu^{\alpha-2} \left\| E_{\alpha,-1}(-\nu^\alpha A) E_{\alpha,1}(-T^\alpha A) G(\cdot,\mu, u, \mu) \right\| \, dv \, d\mu \\
\leq \theta_1 \theta_2^2 (\lambda_1^{-1} + T^\alpha) \int_0^T \int_t^{t+h} \nu^{\alpha-2} \left\| G(\cdot,\mu, u, \mu) \right\| \, dv \, d\mu \\
\leq \omega_1 \left\| \varphi \right\|_{D(\mathcal{A})} t^{-(1-\frac{\alpha}{2})} \tilde{h}^{\frac{\alpha}{2} - \frac{\alpha}{2}},
\]

where we denote by \( \omega_1 = \theta_1 \theta_2^2 (\lambda_1^{-1} + T^\alpha) \tilde{\mathcal{K}} c_1 \frac{\tilde{\varphi}}{\nu} \). We arrive at

\[
t^{1-\frac{\alpha}{2}} \left\| \mathcal{O}_{\alpha,3,p_{\infty}}(u)(t+h) - \mathcal{O}_{\alpha,3,p_{\infty}}(u)(t) \right\|_{D(\mathcal{A}^{-\frac{1}{2}})} \leq \omega_1 \left\| \varphi \right\|_{D(\mathcal{A})} h^{\frac{\alpha}{2} - \frac{\alpha}{2}}.
\]

Moreover, similarly as \( \varphi \), the Sobolev embedding \( L^2(\Omega) \hookrightarrow D(\mathcal{A}^{-\frac{1}{2}}) \) yields that there exists a positive constant \( \omega_{0,-\frac{1}{2}} \) such that

\[
\left\| G(\cdot, t+h, u(\cdot,t+h)) - G(\cdot, t, u(\cdot,t)) \right\|_{D(\mathcal{A}^{-\frac{1}{2}})} \\
\leq \tilde{\mathcal{K}} \omega_{0,-\frac{1}{2}} \left( h + \left\| \mathcal{Y}_{\alpha,1,h}(\varphi)(\cdot, t) \right\| + \sum_{2 \leq j \leq 4} \left\| \mathcal{Y}_{\alpha,2,h}(u)(\cdot, t) \right\| \right),
\]

where we have used the Lipschitz assumption (G2). We note that \( 2\alpha < \frac{\alpha}{1-\alpha} \) as \( \alpha > \frac{1}{2} \), and \( 1 < 2\alpha < \tilde{r} < \min \left( \frac{\alpha}{1-\alpha}, \frac{1}{2} \right) \) as \( (\tilde{p}; \tilde{q}; \tilde{r}) \in \mathcal{V}^\alpha_3, (\tilde{r}; \tilde{q}) \in \mathcal{V}^{2,\alpha}_{1,\infty} \), and \( \tilde{r} < \frac{1}{(1-\alpha)(\alpha-1)} \). This ensures that \( \frac{1}{2} < \frac{1}{2\alpha} + \frac{1}{\tilde{r}} < \frac{1}{2\alpha} + \frac{1}{2} = \frac{1}{2} + \frac{1}{2} \left( \frac{1}{\alpha} - \frac{1}{\tilde{r}} \right) \) < 1,

i.e., \( \frac{1}{2\alpha} + \frac{1}{\tilde{r}} \in \left( \frac{1}{2}; 1 \right) \), and so that \( \left| E_{\alpha,\alpha}(-\nu^\alpha \lambda_n) \right| \leq \theta_2 \nu^{\frac{\alpha}{2} - \frac{1}{2}} \lambda_n^{\frac{1}{2} - \frac{\alpha}{2}} \). Therefore, in a same way as \( \mathcal{Y}_{\alpha,1,h} \), \( \mathcal{Y}_{\alpha,2,h} \) shows

\[
\left| E_{\alpha,1}(-(t+h)^\alpha \lambda_n) - E_{\alpha,1}(-T^\alpha \lambda_n) \right| \leq \theta_2 \lambda_1^{\frac{1}{2} - \frac{\alpha}{2}} \int_t^{t+h} \nu^{\alpha-2} \, dv,
\]

where \( \frac{1}{2} - \frac{\alpha}{2} < 0 \). We get

\[
\left\| \mathcal{Y}_{\alpha,1,h}(\varphi)(\cdot, t) \right\| = \left\| \Pi^\alpha_{E_{\alpha,1}(T^\alpha A)} E_{\alpha,1}(-T^\alpha A) \varphi \right\| \leq \frac{\theta_2 \tilde{r}}{\rho - \alpha} \lambda_1^{\frac{1}{2} - \frac{\alpha}{2}} \left( \frac{h^{1-\frac{\alpha}{2}}}{(l-\alpha)(1-\frac{\alpha}{2})^{1-\frac{\alpha}{2}}} \right) \left\| E_{\alpha,1}(-T^\alpha A) \varphi \right\| \\
\leq \frac{\theta_2 \tilde{r}}{\rho - \alpha} \lambda_1^{\frac{1}{2} - \frac{\alpha}{2}} \left( \frac{T^{\alpha-1} \tilde{h}^{\alpha-\frac{\alpha}{2}}}{\nu} \right) \theta_1 (\lambda_1^{-1} + T^\alpha) \left\| \varphi \right\|_{D(\mathcal{A})}.
\]

On the other hand, an estimate for \( \mathcal{Y}_{\alpha,2,h}(u)(\cdot, t) \) on \( L^2(\Omega) \) can be obtained as follows

\[
\left\| \mathcal{Y}_{\alpha,2,h}(u)(\cdot, t) \right\| \leq \theta_2 \tilde{\mathcal{K}} c_1 \left\| \varphi \right\|_{D(\mathcal{A})} h \leq \theta_2 \tilde{\mathcal{K}} c_1 \left\| \varphi \right\|_{D(\mathcal{A})} \Lambda^{1-\frac{\alpha}{2}} \tilde{h}^{\frac{\alpha}{2} - \frac{\alpha}{2}}.
\]

Now, by using the same techniques as \( \mathcal{Y}_{\alpha,1,h} \), and \( \mathcal{Y}_{\alpha,2,h} \), we derive

\[
\left\| \mathcal{Y}_{\alpha,3,h}(u)(\cdot, t) \right\| \leq \frac{\theta_2 \tilde{r}}{\rho - \alpha} \lambda_1^{\frac{1}{2} - \frac{\alpha}{2}} \int_t^T (t-\mu)^{\frac{\alpha}{2} - 1} \tilde{h}^{\frac{\alpha}{2} - \frac{\alpha}{2}} \left\| G(\cdot, \mu, u(\cdot, \mu)) \right\| \, d\mu \\
\leq \frac{\theta_2 \tilde{r}}{\rho - \alpha} \lambda_1^{\frac{1}{2} - \frac{\alpha}{2}} \tilde{T}^{\alpha - 1} \tilde{h}^{\frac{\alpha}{2} - \frac{\alpha}{2}} \tilde{\mathcal{K}} c_1 \left\| \varphi \right\|_{D(\mathcal{A})},
\]

\[
\end{align*}
\]
where we note that \((t + h - \mu)^{1 - \frac{p}{q}} \geq (t - \mu)^{\frac{p}{q}} h^{1 - \frac{p}{q} + \frac{q}{q}}\), and
\[
\|\mathcal{Y}_{\alpha,h}(u)\| \leq \frac{\theta_2 r^p}{T - \alpha} \lambda_1 \left( \frac{p}{q} \right)^{\frac{q}{q}} \int_0^T \|E_{\alpha,1}^{-1}(-T^\alpha A)E_{\alpha,1}(-T^\alpha A)G(.\mu, u(.\mu))\|d\mu \\
\leq \frac{\theta_2 r^p}{T - \alpha} \theta_1(T^{1 - \alpha} + \alpha) \lambda_1 \left( \frac{p}{q} \right)^{\frac{q}{q}} t^{1 - \frac{p}{q} + \frac{q}{q}} \int_0^T (T - (T - \mu)^\alpha \|G(.\mu, u(.\mu))\|d\mu \\
\leq \frac{\theta_2 r^p}{T - \alpha} \theta_1(T^{1 - \alpha} + \alpha) \lambda_1 \left( \frac{p}{q} \right)^{\frac{q}{q}} \alpha \lambda_1 T^{-\alpha} \hat{K} C_1 \|\phi\|_{D(A)}.
\] (60)

It follows from (55), (57), (58), (59), (60) that there exists a positive constant \(\omega_{13}\) satisfying
\[
\|G(.t + h, u(.t + h)) - G(.\mu, u(.\mu))\|_{D(A - \hat{A})} \leq \omega_{13} h^{1 - \frac{p}{q} + \frac{q}{q}}.
\]

We recall that the estimate (62) holds for all \(\hat{p} > 1\). Thus, it also holds for \(\hat{p} = \hat{r}\), i.e., there exists a positive constant \(\omega_{14}\) such that \(t^{1 - \frac{p}{q} + \frac{q}{q}} \|\partial_t u(.\mu, t)\|_{D(A - \hat{A})} \leq \omega_{14} \|\phi\|_{D(A)}\). This combined with the estimates for \(\mathcal{O}_{1,\infty}(u)(.t + h) - \mathcal{O}_{1,\infty}(\phi)(.t + h)\), and
\[
\|\mathcal{O}_{1,\infty}(u)(.t + h) - \mathcal{O}_{1,\infty}(\phi)(.t + h)\|
\]
implies that there exists a constant \(\omega_{15} > 0\) such that
\[
t^{1 - \frac{p}{q} + \frac{q}{q}} \|\partial_t u(.\mu, t)\|_{D(A - \hat{A})} + t^{1 - \frac{p}{q}} \|\partial_t u(.\mu, t) - \partial_t u(.\mu, t)\|_{D(A - \hat{A})} \leq \omega_{15} \|\phi\|_{D(A)}.
\] (61)

Hence, \(u \in L^{1 - \frac{p}{q} + \frac{q}{q}}((0, T); D(A - \hat{A}))\). The inequality (62) is proved by combing (62) and (61). This completes the proof. \(\square\)

3.2. The case \(\phi \in D(A^\alpha)\) for some \(0 < \zeta < 1\). In the previous section, we have proved the existence of a mild solution of Problem (1)-(2) in \(L^\infty(0, T; L^2(\Omega))\) according to the final value data \(\phi \in D(A)\). In this section, we will investigate solutions of the problem when \(\phi \in D(A^\alpha)\) for some \(0 < \zeta < 1\). At the first glance, one can see that Problem (1)-(2) has no solution in the space \(L^\infty(0, T; L^2(\Omega))\). Indeed, by using the inequalities (101) and the same techniques as Lemma (67), one can check that
\[
\|E_{\alpha,1}^{-1}(-T^\alpha A)\| \geq \theta_2 T^\alpha \|\psi\|_{D(A)} \text{ for } \psi \in D(A).
\]

We note that the function \(t \mapsto E_{\alpha,1}(-t^\alpha \lambda)\) is decreasing, and \(E_{\alpha,1}(0) = 1\). This combined with (63) imply the following inequality
\[
\text{ess sup}_{0 \leq t \leq T} \|\psi(.\mu, t)\| \geq \left( \sum_{n=1}^\infty \frac{E_{\alpha,1}^2(0)\phi_n^2}{E_{\alpha,1}^{-1}(-T^\alpha \lambda_n)} \right)^{1/2} = \|E_{\alpha,1}^{-1}(-T^\alpha A)\| \geq \theta_2 T^\alpha \|\psi\|_{D(A)}.
\]

The above computation shows that if \(\phi \notin D(A)\), then \(u \notin L^\infty(0, T; L^2(\Omega))\). However, in the following theorem, we will show the existence and uniqueness of a mild solution of Problem (1)-(2) in the space \(L^{\rho_0}(0, T; L^2(\Omega))\), for some suitable \(\rho_0 \geq 1\), whereupon the mild solution \(u\) is bounded by a power function, i.e.,
\[
\|u(.\mu, t)\| \lesssim t^{\rho - \alpha}, \text{ for some } \rho > 1, \text{ and for all } 0 < t \leq T.
\]

Here the notation \(\lesssim\) is used to ignore some constants which do not depend on \((\alpha, T)\). For \((a; b; c)\) and \((a_{-}; b_{-}; c_{-})\) in \(V_{\alpha,\infty}^\circ\), we use notation \((a; b; c) \preceq (a_{-}; b_{-}; c_{-})\) if \(a < a_{-}\). In addition, for given \((\rho^*; q^*; \rho^*) \in V_{\alpha,\infty}^\circ\), we denote by \(\omega_{\rho^*} = \rho^*\) the constant in the Sobolev embedding \(L^2(\Omega) \hookrightarrow D(A^{-\hat{r}})\) as (39), and by \(B(p^*, q^*) := B \left(1 - \frac{p^*}{q^*} - 1 - \frac{\alpha}{q}\right)\), where \(B\) is the Beta function.
Theorem 3.5. Given \((p^*; q^*; \rho^*), (p^*_n; q^*_n; \rho^*_n) \in V^{3,\infty}_{\alpha,1}\) satisfying \((p^*; q^*; \rho^*) \preceq (p^*_n; q^*_n; \rho^*_n). Assume that \(C_0(\alpha) := \left[ \theta_1 \theta_2 (\lambda_1^{-1} + T^\alpha) + \theta_2 \omega_0 - \frac{T^{\frac{\alpha}{2}}}{-1} \right] \) \(K T^{\alpha} B(p^*, q^*) < 1.\) If \(\phi\) belongs to \(D(A^{\frac{-\alpha}{2}})\), \(G\) satisfies the Lipschitz assumption (G1), then Problem (1)–(3) has a unique mild solution \(u\) belonging to \(L^{\frac{-\alpha}{2}} - \rho^*(0, T; L^2(\Omega)) \cap L^{\frac{-\alpha}{2}} - \rho_n^*(0, T; D(A^{\frac{-\alpha}{2}} + \frac{1}{2})).\)

Proof. We first establish existence and uniqueness of a mild solution in \(L^{\frac{-\alpha}{2}} - \rho^*(0, T; L^2(\Omega)) \cap L^{\frac{-\alpha}{2}} - \rho_n^*(0, T; D(A^{\frac{-\alpha}{2}} + \frac{1}{2})).\) We define the Picard sequence \(\{\xi_n\}\) by

\[
\begin{cases}
\xi_1(t) := \varphi, \\
\xi_n(t) := Z_{\alpha,1}(\varphi)(t) + Z_{\alpha,2}(\xi_{n-1})(t) + Z_{\alpha,3}(\xi_{n-1})(t).
\end{cases}
\]  

(62)

Here the expression \(Z_{\alpha,j}, j = 1, 2, 3,\) are given by \(26.\) In what follows, we will prove \(\{\xi_n\}\) is a Cauchy sequence in \(L^{\frac{-\alpha}{2}}(0, T; L^2(\Omega))\) by proving inductively that there exists a positive constant \(M^*_\alpha(\alpha),\) which does not depend on \((x, t, n)\) such that

\[
\|\xi_n(t)\| \leq M^*_\alpha(\alpha) T^{\frac{\alpha}{2}} \|\varphi\|_{D(A^{\frac{-\alpha}{2}})},
\]  

(63)

for all \(0 < t \leq T,\) and \(n \geq 1.\) By applying consecutively Part c, and Part b of Lemma \(44.\) one get

\[
\|Z_{\alpha,1}(\varphi)(t)\| = \left\| E_{\alpha,1}(-T^\alpha A) E_{\alpha,1}(-t^\alpha A) \varphi \right\| \leq \theta_1 \theta_2 (\lambda_1^{-1} + T^\alpha) T^{\frac{\alpha}{2}} \|\varphi\|_{D(A^{\frac{-\alpha}{2}})},
\]  

(64)

where we note that \(p^* \oplus q^* = 1.\) In addition, the Sobolev embedding \(L^2(\Omega) \hookrightarrow D(A^{\frac{-\alpha}{2}})\) yields that there exists a constant \(\omega_0 - \frac{T^{\alpha}}{2} > 0\) satisfying

\[
\|G(\cdot, \cdot, \xi_{n-1}(\cdot, \cdot))\|_{D(A^{\frac{-\alpha}{2}})} \leq \omega_0 - \frac{T^{\alpha}}{2} \|G(\cdot, \cdot, \xi_{n-1}(\cdot, \cdot))\|
\]

for all \(n \geq 2,\) and \(0 < \mu < T.\) Hence, Part b of Lemma \(17.\) yields the estimate

\[
\|Z_{\alpha,2}(\xi_{n-1})(t)\| \leq \theta_2 \omega_0 - \frac{T^{\alpha}}{2} \int^t_0 (t - \mu) - \frac{T^{\alpha}}{2} \|G(\cdot, \cdot, \xi_{n-1}(\cdot, \cdot))\| d\mu
\]

\[
\leq \theta_2 \omega_0 - \frac{T^{\alpha}}{2} K \int^t_0 (t - \mu) - \frac{T^{\alpha}}{2} \|\xi_{n-1}(\cdot, \cdot)\| d\mu.
\]  

(65)

The last term \(Z_{\alpha,3}(\xi_{n-1})\) is estimate as follows

\[
\left\| Z_{\alpha,3}(\xi_{n-1})(t) \right\| \leq \theta_1 \theta_2 (\lambda_1^{-1} + T^\alpha) T^{\frac{\alpha}{2}} \int^T_0 \left\| E_{\alpha,1}(-(T - \mu)^{\alpha} A) G(\cdot, \cdot, \xi_{n-1}(\cdot, \cdot)) \right\|_{D(A^{\frac{-\alpha}{2}})} d\mu
\]

\[
\leq \theta_1 \theta_2^2 (\lambda_1^{-1} + T^\alpha) T^{\frac{\alpha}{2}} \int^T_0 (T - \mu) - \frac{T^{\alpha}}{2} \left\| G(\cdot, \cdot, \xi_{n-1}(\cdot, \cdot)) \right\|_{D(A^{\frac{-\alpha}{2}})} d\mu,
\]

and so that

\[
\left\| Z_{\alpha,3}(\xi_{n-1})(t) \right\| \leq \theta_1 \theta_2^2 (\lambda_1^{-1} + T^\alpha) T^{\frac{\alpha}{2}} K \int^T_0 (T - \mu) - \frac{T^{\alpha}}{2} \|\xi_{n-1}(\cdot, \cdot)\| d\mu.
\]  

(66)

Let \(M^*_\alpha(\alpha) \geq \omega_0 - \frac{T^{\alpha}}{2} T^{\frac{\alpha}{2}},\) then \(\|\varphi\| \leq M^*_\alpha(\alpha) T^{\frac{\alpha}{2}} \|\varphi\|_{D(A^{\frac{-\alpha}{2}})}\). Thus, the inequality \(63\) holds for \(n = 1.\) We assume that \(63\) holds for \(n = n^* - 1 \geq 1.\) Then,

\[
\int^t_0 (t - \mu) - \frac{T^{\alpha}}{2} \left\| \xi_{n-1}(\cdot, \cdot) \right\|_{D(A^{\frac{-\alpha}{2}})} d\mu \leq M^*_\alpha(\alpha) \|\varphi\|_{D(A^{\frac{-\alpha}{2}})} \int^t_0 (t - \mu) - \frac{T^{\alpha}}{2} \mu - \frac{T^{\alpha}}{2} d\mu.
\]  

(67)

Combining \(64, 65, 66, 67,\) and the definition \(62,\) we find

\[
\|\xi_n(t)\| \leq \|\xi_1(\varphi)(t)\| + \sum_{i=2,3} \left\| Z_{\alpha,i}(\xi_{n-1})(t) \right\| \leq \left[ \omega_1 + C_0(\alpha) M^*_\alpha(\alpha) \right] \|\varphi\|_{D(A^{\frac{-\alpha}{2}})} T^{\frac{\alpha}{2}},
\]  

(68)
where \( \omega_{16} := \theta_1 \theta_2 (\lambda_1^{-1} + T^\alpha) + \omega_0, -\frac{1}{p^*} T^\frac{\alpha}{p^*} \). By the above arguments, it is sufficient to take the constant \( \mathcal{M}^T_{\mu}(\alpha) \geq \omega_{16} (1 - C_0^\alpha(\alpha))^{-1} \), which does not depend on \((x,t,n)\), then the inequality \((63)\) holds for \( n = n^* \). Therefore, \((63)\) is proved completely by the induction method.

Next, we will prove \( \{ \Xi_n \} \) is a Cauchy sequence in the Banach space \( L^{\frac{p^*}{p^*}}(0,T;L^2(\Omega)) \). Thanks to the inequality \((63)\), we derive
\[
\| \Xi_{m-1}(\cdot,t) - \Xi_{n-1}(\cdot,t) \| \leq 2 \mathcal{M}^T_{\mu}(\alpha) \| \varphi \|_{D(A^\frac{1}{p^*})} t^{-\frac{\alpha}{p^*}}.
\]
Noting that \( \Xi_{m}(\cdot,t) - \Xi_{n}(\cdot,t) = \sum_{i=0}^{\infty} \mathcal{Z}_{\alpha,i}(\Xi_{m-1}(\cdot,t) - \Xi_{n-1}(\cdot,t)) \), and applying arguments similar to those used for \( (65) \), we obtain the iterative estimate
\[
\| \Xi_{m}(\cdot,t) - \Xi_{n}(\cdot,t) \| \leq 2 (C_0^\alpha(\alpha))^{\min(m,n)} \mathcal{M}^T_{\mu}(\alpha) \| \varphi \|_{D(A^\frac{1}{p^*})} t^{-\frac{\alpha}{p^*}},
\]
which implies the existence of a positive constant \( \omega_{17} \) such that
\[
\| \Xi_m(\cdot,t) - \Xi_n(\cdot,t) \|_{L^{\frac{p^*}{p^*}}(0,T;L^2(\Omega))} \leq \omega_{17} (C_0^\alpha(\alpha))^{\min(m,n)} \| \varphi \|_{D(A^\frac{1}{p^*})}.
\]
Therefore, we deduce that \( \{ \Xi_n \} \) is a Cauchy and convergent sequence in \( L^{\frac{p^*}{p^*}}(0,T;L^2(\Omega)) \). Let \( u^* = \lim_{n \to \infty} \Xi_n \) in the sense of \( L^{\frac{p^*}{p^*}}(0,T;L^2(\Omega)) \). Then, we have
\[
\lim_{n \to \infty} \int_0^t E_{\alpha,1}(-(t - \mu)^\alpha \theta) G(\cdot,\mu,\Xi_n(\cdot,\mu))d\mu = \int_0^t E_{\alpha,1}(-(t - \mu)^\alpha \theta) G(\cdot,\mu,u^*(\cdot,\mu))d\mu,
\]
\[
\lim_{n \to \infty} \int_0^T E_{\alpha,1}^{-1}(-T^\alpha \theta) E_{\alpha,1}(-(t - \mu)^\alpha \theta) G(\cdot,\mu,\Xi_n(\cdot,\mu))d\mu
\]
\[
= \int_0^T E_{\alpha,1}^{-1}(-T^\alpha \theta) E_{\alpha,1}(-(t - \mu)^\alpha \theta) G(\cdot,\mu,u^*(\cdot,\mu))d\mu.
\]
Combining this with \((62)\), we can conclude that \( u^* \) satisfies Equation \((12)\) and it is a mild solution of Problem \((1) \) in \( L^{\frac{p^*}{p^*}}(0,T;L^2(\Omega)) \). Moreover, it follows from \((63)\) that
\[
\| u^*(\cdot,t) \|_{L^{\frac{p^*}{p^*}}(0,T;L^2(\Omega))} \leq \omega_{18} \| \varphi \|_{D(A^\frac{1}{p^*})},
\]
for some positive constant \( \omega_{18} > 0 \).

We now improve the spatial regularity of the mild solution \( u^* \) by considering the fractional derivative \( A^\frac{p^*}{p^*} - \frac{\alpha}{p^*} \) of the mild solution \( u^* \). Here \( 0 < \frac{1}{p^*} - \frac{1}{p_\alpha} < 1 \) since \((p^*;q^*;\rho^*)\), \((p_\alpha^*;q_\alpha^*;\rho_\alpha^*)\) belong to \( V_{\alpha,1}^\alpha \) and \((p^*;q^*;\rho^*) \subset (p_\alpha^*;q_\alpha^*;\rho_\alpha^*)\). Using Part c, and Part b of Lemma \((4.1)\) and the regularity \((69)\) on \( L^2(\Omega) \) gives that
\[
\| \mathcal{Z}_{\alpha,1}(\varphi)(\cdot,t) \|_{D(A^\frac{p^*}{p^*} - \frac{\alpha}{p^*})} \leq \theta_1 \theta_2 (\lambda_1^{-1} + T^\alpha) t^{-\frac{\alpha}{p^*}} \| \varphi \|_{D(A^\frac{1}{p^*})},
\]
\[
\| \mathcal{Z}_{\alpha,2}(u^*)(\cdot,t) \|_{D(A^\frac{p^*}{p^*} - \frac{\alpha}{p^*})} \leq \theta_2 K \mathcal{M}^T_{\mu}(\alpha) \| \varphi \|_{D(A^\frac{1}{p^*})} \int_0^t (t - \mu)^{-\frac{\alpha}{p^*} - \frac{\alpha}{p_\alpha^*}} \| \varphi \|_{D(A^\frac{1}{p_\alpha^*})} \mu^{-\frac{\alpha}{p_\alpha^*}} d\mu
\]
\[
\leq \theta_2 K \mathcal{M}^T_{\mu}(\alpha) \| \varphi \|_{D(A^\frac{1}{p_\alpha^*})} T^{\alpha + \frac{\alpha}{p^*}} B \left( 1 + \frac{\alpha}{p^*} - \frac{\alpha}{p_\alpha^*}, 1 - \frac{\alpha}{q^*} \right) t^{-\frac{\alpha}{p^*}},
\]
where we note that \( p^* \oplus q^* = 1, p_\alpha^* \oplus q_\alpha^* = 1 \). Furthermore, the following estimate is allowed by using Part b of Lemma \((4.1)\) and
\[
\| \mathcal{Z}_{\alpha,3}(u^*)(\cdot,t) \|_{D(A^\frac{p^*}{p^*} - \frac{\alpha}{p^*})}
\]
\[
\leq \theta_1 \theta_2 (\lambda_1^{-1} + T^\alpha) t^{-\frac{\alpha}{p^*}} \int_0^t \| E_{\alpha,1}(-(T - \mu)^\alpha \theta) G(\cdot,\mu,u^*(\cdot,\mu)) \|_{D(A^\frac{1}{p^*})} d\mu
\]
\[
\leq \theta_1 \theta_2 (\lambda_1^{-1} + T^\alpha) \theta_2 K \mathcal{M}^T_{\mu}(\alpha) \| \varphi \|_{D(A^\frac{1}{p^*})} t^{-\frac{\alpha}{p^*}} \int_0^T (T - \mu)^{-\frac{\alpha}{p^*} - \frac{\alpha}{p_\alpha^*}} \mu^{-\frac{\alpha}{p_\alpha^*}} d\mu
\]
\[
\leq \theta_1 \theta_2 (\lambda_1^{-1} + T^\alpha) \theta_2 K \mathcal{M}^T_{\mu}(\alpha) \| \varphi \|_{D(A^\frac{1}{p^*})} t^{-\frac{\alpha}{p^*}} T^{\alpha + \frac{\alpha}{p^*}} B \left( 1 - \frac{\alpha}{p^*}; 1 - \frac{\alpha}{q^*} \right).
\]
We deduce from (19), (21), (22) and the fact that \( u^*(x, t) = Z_{\alpha, 1}(\varphi)(x, t) + \sum_{i=2, 3} Z_{\alpha, i}(u^*)(x, t) \), there exists a positive constant \( \omega_{19} > 0 \) satisfying
\[
\| u^*(., t) \|_{D(\mathcal{A}^\frac{\rho}{q} \cdot \mathcal{P}^\frac{q}{p})} \leq \omega_{19} \| \varphi \|_{D(\mathcal{A}^\frac{\rho}{q} \cdot \mathcal{P}^\frac{q}{p})} t^{-\frac{q}{p}}.
\]
Hence, \( u^* \) belongs to \( L^\frac{q}{p} \stackrel{\rho}{\rightarrow} \mathcal{P}^\frac{q}{p} \mathcal{P} \mathcal{L} \omega (0, T; D(\mathcal{A}^\frac{\rho}{q} \cdot \mathcal{P}^\frac{q}{p})) \), and satisfies the estimate, for some \( \omega_{20} > 0 \),
\[
\| u^* \|_{L^\frac{q}{p} \stackrel{\rho}{\rightarrow} \mathcal{P}^\frac{q}{p} \mathcal{L} \omega (0, T; D(\mathcal{A}^\frac{\rho}{q} \cdot \mathcal{P}^\frac{q}{p}))} \leq \omega_{20} \| \varphi \|_{D(\mathcal{A}^\frac{\rho}{q} \cdot \mathcal{P}^\frac{q}{p})}.
\]
This completes the proof of the theorem. \( \square \)

4. WELL-POSEDNESS WITH CRITICAL NONLINEARITIES AND ITS APPLICATIONS

4.1. Well-posedness. In this section, we present the existence, uniqueness and regularity of a mild solution of Problem (1)-(2) under critical nonlinearities which will be defined in Theorem 4.2. Before stating the existence of a mild solution, we present the following Lemma which defines the couple of solution of Problem (1)-(2) under critical nonlinearities which will be defined in Theorem 4.2. Before

Let \( \omega \in (0, 1) \) satisfies the estimate, gradient estimate for the solution.

Proof of Lemma 4.1. Firstly, we give some useful explanations for the numbers \( l_1, l_2 \). The assumption \( \frac{|q-2|}{4q} k \leq 1 \) ensures that the interval \([0, l^1]\) is not empty. Moreover, \( l^1 \leq 0 \) holds since \( l_1 \in [0, l^1] \), and so that the interval \([l^1, 0]\) is not empty. Moreover, one can check that
\[
\left( \frac{1}{2} l_1 + \frac{q-2}{4q} k 1_{q \geq 2} \right) - \left( \frac{1}{2} l_2 + \frac{q-2}{4q} k 1_{q < 2} \right) - 1 = \frac{1}{2} \left( l_1 - 2 + \frac{|q-2|}{2q} k \right) - \frac{1}{2} l_2 = \frac{1}{2} (l^1 - l_2).
\]
Hence, we deduce from the inequality \( l_2 \geq l^1 \) that
\[
\left( \frac{1}{2} l_1 + \frac{q-2}{4q} k 1_{q \geq 2} \right) \leq \left( \frac{1}{2} l_2 + \frac{q-2}{4q} k 1_{q < 2} \right) + 1,
\]
and therefore there exist real numbers \( l'_1, l'_2 \) such that
\[
\left( \frac{1}{2} l_1 + \frac{q-2}{4q} k 1_{q \geq 2} \right) \leq l'_1 \leq (l'_2 + 1) \leq \left( \frac{1}{2} l_2 + \frac{q-2}{4q} k 1_{q < 2} \right) + 1,
\]
which invokes \( 0 \leq l'_1 - l'_2 \leq 1 \) and
\[
l'_2 \leq \left( \frac{1}{2} l_2 + \frac{q-2}{4q} k 1_{q < 2} \right) \leq \frac{1}{2} l_2 \leq 0 \leq \frac{1}{2} l_1 \leq \left( \frac{1}{2} l_1 + \frac{q-2}{4q} k 1_{q \geq 2} \right) \leq l'_1.
\]
This completes the proof. \( \square \)

Thanks to Lemma 4.2, we obtain the following Theorem which shows the existence of a mild solution in \( C^\alpha([0, T]; W^{l, q}(\Omega)) \), and establishes \( L^r \) estimate, gradient estimate for the solution.

Theorem 4.2. Assume that \( k, q, l_1, l_2, l'_1, l'_2 \) are determined by Lemma 4.1 and \( q > 0 \). Let the nonlinearity
\[
G : \Omega \times (0, T) \times W^{l_1, q}(\Omega) \rightarrow W^{l_2, q}(\Omega)
\]
satisfies \( G(0, \cdot, v) = 0 \). Assume that there exists a non-negative function \( L \) on \((0, T)\) such that
\[
\| G(., t, v_1) - G(., t, v_2) \|_{W^{l_2, q}(\Omega)} \leq L(t) \left( 1 + \| v_1 \|_{W^{l_1, q}(\Omega)}^q + \| v_2 \|_{W^{l_1, q}(\Omega)}^q \right) \| v_1 - v_2 \|_{W^{l_1, q}(\Omega)},
\]
\]
for all \(t \in (0, T)\), \(v_1, v_2 \in W^{2; q}(\Omega)\), and there exists \(\delta > 0\) such that \(t^{1-(1+\nu)\alpha}L(t) \leq L_0t^\delta\). If \(\varphi \in \mathcal{D}(A^{1+\frac{1}{2}\nu})\), then there exists \(T\) small enough such that Problem (1)-(2) has a unique mild solution \(u \in C^\alpha((0, T]; W^{1, q}(\Omega))\) and
\[
\|u(. , t)\|_{W^{1, q}(\Omega)} \leq \|\varphi\|_{D(A^{1+\frac{1}{2}\nu})}.
\] (73)

**Proof.** Based on Lemma 4.4, we start by considering some useful Sobolev embeddings. The first one is given by \(\mathcal{D}(A^{1}) \hookrightarrow W^{2q; 2}(\Omega)\) as \(l_1 \geq 0\). Besides, in the case \(1 < q < 2\) we have \(W^{2q; 2}(\Omega) \hookrightarrow W^{1, q}(\Omega)\) as \(2l_1 \geq 1\), and in the case \(q \geq 2\) this embedding holds since \(\left(\frac{1}{2l_1} + \frac{q}{4q}k_1\right) \leq l_1'\). We obtain \(W^{2q; 2}(\Omega) \hookrightarrow W^{1, q}(\Omega)\), and so
\[
\mathcal{D}(A^1) \hookrightarrow W^{2q; 2}(\Omega) \hookrightarrow W^{1, q}(\Omega),
\] (74)

which yields that there exists \(D_1 > 0\) such that \(\|v\|_{W^{1, q}(\Omega)} \leq D_1\|v\|_{\mathcal{D}(A^1)}\) for all \(v \in \mathcal{D}(A^1)\).

On the other hand, if \(q \geq 2\) then \(W^{2q; 2}(\Omega) \hookrightarrow W^{2q; 2}(\Omega)\) holds since \(l_2 \geq 2l_2'\), and if \(1 < q < 2\) then this embedding also holds since \(l_2' \leq \left(\frac{1}{2l_2} + \frac{q}{4q}k_1\right)\). Namely, we derive \(W^{2q; 2}(\Omega) \hookrightarrow W^{2q; 2}(\Omega)\). Furthermore, \(W^{2q; 2}(\Omega) \hookrightarrow \mathcal{D}(A^2)\) as \(l_2' \leq 0\). Thus,
\[
W^{2q; 2}(\Omega) \hookrightarrow W^{2q; 2}(\Omega) \hookrightarrow \mathcal{D}(A^2),
\] (75)

which means that there exists \(D_2 > 0\) such that \(\|v\|_{\mathcal{D}(A^2)} \leq D_2\|v\|_{W^{2q; 2}(\Omega)}\) for all \(v \in W^{2q; 2}(\Omega)\).

Now, let us define the mapping \(\mathcal{H} : C^\alpha((0, T]; W^{1, q}(\Omega)) \rightarrow C^\alpha((0, T]; W^{1, q}(\Omega))\), which is also formulated in (38). Then, we prove the existence of a mild solution to Problem (1)-(2) by employing the Banach contraction principle. In what follows, we denote by
\[
\mathcal{U}_R = \{v \in C^\alpha((0, T]; W^{1, q}(\Omega)) : \|v\|_{C^\alpha((0, T]; W^{1, q}(\Omega))} \leq R\},
\]
and divide the proof into the following parts.

**Part 1:** Proving \(\mathcal{H}v \in C^\alpha((0, T]; W^{1, q}(\Omega))\) for \(v \in C^\alpha((0, T]; W^{1, q}(\Omega))\). This part will be broken into the two following sub-parts.

**Part 1A:** Deriving the supremum \(\sup_{0 < t \leq T} \|\mathcal{H}v(. , t)\|_{W^{1, q}(\Omega)}\) for \(v \in C^\alpha((0, T]; W^{1, q}(\Omega))\). In what follows, we will estimate the \(W^{1, q}(\Omega)\)-norms of the quantities \(Z_{\alpha, j}, j = 1, 2, 3\), which are given by (28). In order to estimate the first term \(Z_{\alpha, 1}\), we will jointly use the embedding \(\mathcal{D}(A^1) \hookrightarrow W^{1, q}(\Omega)\) in (27), and Parts a, b of Lemma 4.4 with \(\beta = 1\) as follows

\[
\|Z_{\alpha, 1}(\varphi)(., t)\|_{W^{1, q}(\Omega)} \leq D_1\|Z_{\alpha, 1}(\varphi)(., t)\|_{\mathcal{D}(A^1)} \leq D_1\|E_{\alpha, 1}(-T^\alpha A)E_{\alpha, 1}(-t^\alpha A)\varphi\|_{\mathcal{D}(A^1)}
\leq D_1\|E_{\alpha, 1}(-T^\alpha A)\varphi\|_{\mathcal{D}(A^1)} + D_1\|E_{\alpha, 1}(-t^\alpha A)\varphi\|_{\mathcal{D}(A^1)}
\leq D_1\|E_{\alpha, 1}(-T^\alpha A)\varphi\|_{\mathcal{D}(A^1)} + D_1\|E_{\alpha, 1}(-t^\alpha A)\varphi\|_{\mathcal{D}(A^1)}
\leq D_1\|E_{\alpha, 1}(-T^\alpha A)\varphi\|_{\mathcal{D}(A^1)} + D_1\|E_{\alpha, 1}(-t^\alpha A)\varphi\|_{\mathcal{D}(A^1)}.
\] (76)

where the embedding \(\mathcal{D}(A^1) \hookrightarrow \mathcal{D}(A^1)\) holds since \(l_1' \leq \left(\frac{1}{2l_1} + \frac{q}{4q}k_1\right) + 1 \leq 1 + \frac{1}{2}l_1 = \frac{1}{2}l_1 + \frac{q}{4q}k_1\), and \(\nu \geq 1\) as \(\frac{q}{4q}k_1 > 0\). In order to estimate the second term \(Z_{\alpha, 2}\), we will simultaneously apply the embeddings (25), (23) and Part b of Lemma 4.4 with \(\beta = l_1' - l_2' \in [0, 1]\). Indeed,

\[
\|Z_{\alpha, 2}(v)(., t)\|_{W^{1, q}(\Omega)} \leq \int_0^t \|E_{\alpha, 1}(-T^\alpha A)G(., \mu, v(., \mu))\|_{W^{1, q}(\Omega)}d\mu
\leq D_1\int_0^T \|E_{\alpha, 1}(-T^\alpha A)G(., \mu, v(., \mu))\|_{\mathcal{D}(A^1)}d\mu
\leq D_1\|E_{\alpha, 1}(-T^\alpha A)\varphi\|_{\mathcal{D}(A^1)} + D_1\|E_{\alpha, 1}(-t^\alpha A)\varphi\|_{\mathcal{D}(A^1)}
\leq D_1\|E_{\alpha, 1}(-T^\alpha A)\varphi\|_{\mathcal{D}(A^1)} + D_1\|E_{\alpha, 1}(-t^\alpha A)\varphi\|_{\mathcal{D}(A^1)}.
\] (77)
where we have used $W^{1,2,q}(\Omega) \hookrightarrow D(A_l^{1/2})$ in (76). The above integral can be estimated by using the assumption of the critical nonlinearity $G$. We also note that the norm $\|v(\cdot, \mu)\|_{W^{1,1,1}(\Omega)}$ is bounded by $R\mu^{-\alpha}$ as $v \in U_R \subset C^\infty((0, T]; W^{1,1,1}(\Omega))$. These properties lead us to the estimate

$$\|Z_{\alpha,2}(v)(\cdot, t)\|_{W^{1,1,1}(\Omega)} \leq D_3 L_0 \int_0^t (t - \mu)^{-\alpha(t_1' - t_2')} L(\mu) \left(1 + \|v(\cdot, \mu)\|_{W^{1,1,1}(\Omega)}^\alpha\right) \|v(\cdot, \mu)\|_{W^{1,1,1}(\Omega)} \, d\mu,$$

$$\leq D_3 L_0 \int_0^t (t - \mu)^{-\alpha(t_1' - t_2')} L(\mu) \left(1 + \mu^{-\alpha} R^\theta\right) \mu^{-\alpha} \, d\mu,$$

$$\leq D_3 L_0 (T^{\alpha_0} + R^\theta) \int_0^t \mu^{-\alpha(t_1' - t_2')} \mu^{-\alpha(t_1' - t_2')} \, d\mu,$$

where $D_3 := D_1 \theta_2 D_2$. We now recall that $L$ satisfies the assumption $\mu^{-\alpha(t_1' - t_2')} \mu^{-\alpha(t_1' - t_2')} < L_0 \mu^{-\alpha(t_1' - t_2')}$. Hence, we have

$$\int_0^t (t - \mu)^{-\alpha(t_1' - t_2')} \mu^{-\alpha(t_1' - t_2')} \mu^{-\alpha(t_1' - t_2')} \, d\mu = B(1 - \alpha(t_1' - t_2'); \delta) \mu^{-\alpha(t_1' - t_2')} < \infty.$$

It follows

$$\|Z_{\alpha,2}(v)(\cdot, t)\|_{W^{1,1,1}(\Omega)} \leq D_3 D_4 L_0 (T^{\alpha_0} + R^\theta) \mathcal{R} t^{-\alpha},$$

(77) where $D_4 := B(1 - \alpha(t_1' - t_2'); \delta) T^{\alpha(t_1' - t_2')}$ and $\mathcal{R} t^{-\alpha}$.

In order to estimate the second term $Z_{\alpha,3}$, we will combine the estimates for the terms $Z_{\alpha,1}$ and $Z_{\alpha,2}$ again. Indeed, we have

$$\|Z_{\alpha,3}(v)(\cdot, t)\|_{W^{1,1,1}(\Omega)} = \left\|Z_{\alpha,1} \left(\int_0^T \mathcal{E}_{\alpha,1}(-(T - \mu)^{\alpha} A) G(\cdot, \mu, v(\cdot, \mu)) \, d\mu\right)\right\|_{W^{1,1,1}(\Omega)}$$

$$\leq D_1 \theta_1 (\lambda_1^{-1} + T^\alpha) \theta_2 t^{-\alpha} \left\|\int_0^T \mathcal{E}_{\alpha,1}(-(T - \mu)^{\alpha} A) G(\cdot, \mu, v(\cdot, \mu)) \, d\mu\right\|_{D(A_l^{1/2})}$$

$$\leq D_1 \theta_1 (\lambda_1^{-1} + T^\alpha) \theta_2 t^{-\alpha} D_3 \int_0^T (t - \mu)^{-\alpha(t_1' - t_2')} \|G(\cdot, \mu, v(\cdot, \mu))\|_{W^{1,1,1}(\Omega)} \, d\mu.$$

Therefore, denoting $D_5 := D_1 \theta_1 (\lambda_1^{-1} + T^\alpha) \theta_2 D_3$ and using again the critical property of the nonlinearity $G$, we obtain

$$\|Z_{\alpha,3}(v)(\cdot, t)\|_{W^{1,1,1}(\Omega)} \leq D_3 D_4 L_0 (T^{\alpha_0} + R^\theta) \mathcal{R} t^{-\alpha}.$$ (78)

Employing the estimates (69), (71), (78) gives that

$$t^a \|Hv(\cdot, t)\|_{W^{1,1,1}(\Omega)} \leq aR^{1+\theta} + bR + \tilde{c} \left\|\varphi\right\|_{D(A_l^{1/2} + T^\alpha)}$$

(79) where $a := (D_3 + D_4) D_3 L_0$, $b := (D_3 + D_4) D_4 L_0 T^{\alpha_0}$, and $\tilde{c} := D_1 \theta_1 \theta_2 (\lambda_1^{-1} + T^\alpha)$.

**Part 1B:** Proving that $Hv \in C((0, T]; W^{1,1,1}(\Omega))$ for $v \in C^\infty((0, T]; W^{1,1,1}(\Omega))$. We use again the notations $\mathcal{Y}_{\alpha,3, h}$ introduced in Theorem 3.1. Firstly, the embeddings $D(A_l^{1/2}) \hookrightarrow W^{1,1,1}(\Omega)$ in (74), and the estimate $\|E_{\alpha,0}(\alpha^\lambda)\| \lesssim \left(1 + \nu^{2\alpha} - \alpha L_0^2 \right)^{-1} \lesssim \nu^{-\alpha_0 - \alpha L_0^2} \mu^{-\alpha(t_1' - t_2')} + \frac{1}{\nu}$ as Lemma 6.3 can be combined to get

$$\|\mathcal{Y}_{\alpha,1,h}(\varphi)(\cdot, t)\|_{W^{1,1,1}(\Omega)} = \left\|\Pi_{E, A}^{h, h}(t) E_{\alpha,1}^{-(T^\alpha A)} \varphi\right\|_{W^{1,1,1}(\Omega)}$$

$$\lesssim \left\|\Pi_{E, A}^{h, h}(t) E_{\alpha,1}^{-(T^\alpha A)} \varphi\right\|_{D(A_l^{1/2})} \int_0^t \mu^{\alpha(t_1' - t_2')} \, d\nu$$

$$\leq \left\|\varphi\right\|_{D(A_l^{1/2})} \int_0^t \mu^{\alpha(t_1' - t_2')} \, d\nu,$$

whereupon the above right hand sides obviously converge to zero as $t > 0$ and $h$ approaches zero from the right. Now, the estimate for $\mathcal{Y}_{\alpha,2,h}$ can be established as Part 1A. Besides, we observe from $\alpha(t_1' - t_2') \leq \alpha < 1$ that there exists $\epsilon > 0$ such that $\alpha(t_1' - t_2') + \epsilon < 1$. Hence, the embeddings
\( \mathcal{D}(A_1^\alpha) \hookrightarrow W^{1, q}(\Omega) \), and \( W^{1, q}(\Omega) \hookrightarrow \mathcal{D}(A_1^\alpha) \), in \((\mathbb{R}^n)\), and the critical property of the nonlinearity \( G \) imply
\[
\| Y_{a, b, h}(u)(t) \|_{W^{1, q}(\Omega)} \lesssim \int_t^{t+h} (t + h - \mu)^{-\alpha(l'_3 - l'_2)} \| G(\cdot, \mu, u(\cdot, \mu)) \|_{W^{1, q}(\Omega)} d\mu,
\]
where we note that the powers inside of the latter right hand side are greater than \(-1\). Next, we will estimate \( Y_{a, 3, h} \). Indeed, one can see \( |E_{a, \alpha}(-\nu^\alpha \lambda_n)| \lesssim \nu^{-\alpha+\alpha l'_1 - \frac{1}{2} + \frac{1}{l'_2}} \) by using Lemma 4.5 which infers that
\[
\| Y_{a, 3, h}(\varphi)(\cdot, t) \|_{W^{1, q}(\Omega)} \lesssim \int_0^t \| \Pi_{E, A}^{t, h}(t - \mu)G(\cdot, \mu, u(\cdot, \mu)) \|_{\mathcal{D}(A_1^\alpha)} d\mu,
\]
and then
\[
\| Y_{a, 3, h}(\varphi)(\cdot, t) \|_{W^{1, q}(\Omega)} \lesssim \left( \frac{1}{l'_3 - l'_2} - \frac{1}{l'_3 - l'_2} \right) + h(t + h)\delta^{-\alpha(l'_3 - l'_2)} - \epsilon.
\]
Combining the above estimates, we find
\[
\| Y_{a, 3, h}(\varphi)(\cdot, t) \|_{W^{1, q}(\Omega)} \lesssim \int_0^T (T - \mu)^{-\alpha} \mu^{\delta} d\mu \int_t^{t+h} \nu^{-\alpha(l'_3 - l'_2) - 1} d\nu,
\]
where the above right hand sides clearly converge to zero as \( t > 0 \) and \( h \to 0^+ \). Summarily, we conclude that \( \mathcal{H} \in C((0, T]; W^{1, q}(\Omega)) \).

**Part 2:** Proving the problem has a unique solution in \( C^\alpha((0, T]; W^{1, q}(\Omega)) \) and deriving the inequality \((73)\). Firstly, we will show there exists \( \mathcal{R}_0 > 0 \) such that \( \mathcal{H} \) maps \( \mathcal{U}_{\mathcal{R}_0} \) into itself. Let us denote by \( c := \varpi \| \varphi \|_{H_{A_1^\alpha}(\alpha)} \). Then, it can be observed from \( a, b \) (which are defined in Part 1) is increasing \( T \) that
\[
b < \frac{\varrho}{1 + \varrho}, \quad \text{and} \quad a < \left( \frac{\varrho}{c} \right)^{\varrho} \left( \frac{1 - b}{1 + \varrho} \right)^{1 + \varrho}
\]
if \( T \) is small enough. And therewith,
\[
b < 1, \quad \text{and} \quad \left( 1 - b \right)^{1 + 1/\varrho} > \frac{ca^{1/\varrho}(1 + \varrho)^{1+1/\varrho}}{\varrho}.
\]
Hence, using notation \( \mathcal{R}_* := (1/(a(1 + \varrho)))^{1/\varrho} \) deduces
\[
\frac{\varrho(1 - b)}{1 + \varrho} \mathcal{R}_* = \frac{\varrho}{1 + \varrho} \left( \frac{1}{a(1 + \varrho)} \right)^{1/\varrho} \left( 1 - b \right)^{1 + 1/\varrho} > \frac{\varrho}{1 + \varrho} \left( \frac{1}{a(1 + \varrho)} \right)^{1/\varrho} \frac{ca^{1/\varrho}(1 + \varrho)^{1+1/\varrho}}{\varrho} = c.
\]
Now, we denote by \( \mathcal{E}(\mathcal{R}) := a\mathcal{R}^{1+\varrho} + b\mathcal{R} + c \) and consider it as a function of the variable \( \mathcal{R} \) on \((0, \mathcal{R}_*)\), then \( t^\alpha \| \mathcal{H}v(\cdot, t) \|_{W^{1, q}(\Omega)} \leq \mathcal{E}(\mathcal{R}) \), for all \( 0 < t \leq T \). Let us consider the function \( \mathcal{E}(\mathcal{R}) := \mathcal{E}(\mathcal{R}) - \mathcal{R} \)
which is continuous on \((0, R_*)\), and
\[
\tilde{\mathcal{E}}(0)\tilde{\mathcal{E}}(R) = e^{aR^*\mathbb{R}_* + b\mathbb{R}_* - \mathbb{R}_* + c}
\]
\[
= c\left(\frac{1 - b}{a(1 + \varrho)}\mathbb{R}_* + b\mathbb{R}_* - \mathbb{R}_* + c\right) = c\left(\frac{\varrho(1 - b)}{1 + \varrho}\mathbb{R}_*\right) < 0. \tag{81}
\]
Therefore, the equation \(\tilde{\mathcal{E}}(R) = 0\) has at least solution \(R_0 \in (0, R_*)\), i.e., \(\mathcal{E}(R_0) = R_0\). This infers the estimate \(t^n\|\tilde{H}(\cdot, t)\|_{W^{1, \infty}(\Omega)} \leq R_0\), and so that \(\|\tilde{H}(\cdot, t)\|_{C^0((0, T); W^{1, \infty}(\Omega))} \leq R_0\) by taking the supremum on \((0, T)\). We deduce that the mapping \(\mathcal{H}\) actually maps the ball \(U_{R_0}\) into itself.

Secondly, we will prove \(\mathcal{H}\) is a contraction mapping. Let \(v_1, v_2\) be contained in the ball \(U_{R_0}\), then repeating the arguments used for proving \((79)\), one can show that
\[
\|\mathcal{H}v_1 - \mathcal{H}v_2\|_{C^0((0, T); W^{1, \infty}(\Omega))} \leq (aR_0^* + b)\|v_1 - v_2\|_{C^0((0, T); W^{1, \infty}(\Omega))} \\
\leq \left(\frac{1}{1 + \varrho} + b\right)\|v_1 - v_2\|_{C^0((0, T); W^{1, \infty}(\Omega))}. \tag{82}
\]
It follows that \(\mathcal{H}\) is a contraction mapping since \(\frac{1}{1 + \varrho} + b < \frac{1}{1 + \varrho} + \frac{\varrho}{1 + \varrho} = 1\). Henceforth, \(\mathcal{H}\) possesses a fixed point \(u \in U_{R_0} \subset C^0((0, T); W^{1, \infty}(\Omega))\), namely \(\mathcal{H}u = u\) in \(C^0((0, T); W^{1, \infty}(\Omega))\) and \(u\) is a mild solution to Problem \((1)-(2)\). Moreover, the typically applying the same establishments as proof of \((79)\) also combined with the argument \((82)\) to allow the estimate
\[
\|u\|_{C^0((0, T); W^{1, \infty}(\Omega))} \leq \left(\frac{1}{1 + \varrho} + b\right)\|u\|_{C^0((0, T); W^{1, \infty}(\Omega))} + \tilde{c}\|\varphi\|_{D'(A^{1/2}; L^2)}, \tag{83}
\]
which infers that \((\varrho/(1 + \varrho) - b)\|u\|_{C^0((0, T); W^{1, \infty}(\Omega))} \leq \tilde{c}\|\varphi\|_{D'(A^{1/2}; L^2)}\), namely, \((79)\) is proved. \(\Box\)

### 4.2. Applications to time fractional Navier-Stokes equations

In this subsection, we discuss about an applications of our methods given in Subsection 4.1 to a motion of viscous fluid substances in physics. Let us consider the final value problem of finding the flow velocity \(u = [u_j]_{k \times 1}\) of a fluid for the following time fractional Navier-Stokes equations
\[
\begin{align*}
\frac{\partial u}{\partial t} + b(t)(u, \nabla)u - \frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}}\Delta u &= -\nabla p, \quad (x, t) \in \Omega \times (0, T), \\
\nabla \cdot u &= 0, \quad (x, t) \in \Omega \times (0, T), \\
u &= 0, \quad (x, t) \in \partial \Omega \times (0, T), \\
u(\cdot, T) &= \varphi, \quad x \in \Omega,
\end{align*}
\]
where \(0 < \alpha \leq 1, \Omega\) is a \(C^2\) bounded open set of \(\mathbb{R}^k\) \((k \geq 2)\), \(\text{div} \, u = \sum_{j=1}^k \nabla_j u_j\) is the divergence of \(u\), the notation \(\text{div} \, u\) \(\cdot \text{div} \, u\) is defined by
\[
(u, \nabla)u := \left(\left[u_j\right]_{k \times 1} \left[\nabla_j\right]_{1 \times k}\right) \left[u_j\right]_{k \times 1}, \quad \nabla_j := \frac{\partial}{\partial x_j},
\]
and \(p\) is the pressure, \(\nabla p = [\nabla_j p]_{k \times 1}\). Note that the order of operations in \((83)\) is very important. More specifically, the operation in the brackets must be done first. In the case \(\alpha = 1\), the first equation of \((83)\) becomes the classical Navier-Stokes equation
\[
\frac{\partial u}{\partial t} + b(t)(u, \nabla)u - \Delta u = -\nabla p,
\]
which has been studied by many authors such as Y. Giga \([25]\), H. Iwashita \([68]\), E.S. Titi et al \([93]\). In order to study Problem \((54)\), we now recall some analysis on the Stokes operator. For each \(1 < r < \infty\), let us set
\[
X_r = \text{Closure of } \left\{u \in C_0^\infty(\Omega) : \text{div} \, u = 0 \right\} \text{ in } \left[L^r(\Omega)\right]^k, \\
G_r = \left\{\nabla p : p \in W^{1,r}(\Omega)\right\}.
\]
Then, according to the Helmholtz decomposition, we have the direct sum \(\left[L^r(\Omega)\right]^k = X_r \oplus G_r\). Therefore, we can consider the continuous projection \(P_r\) (called the Helmholtz projection) from \(\left[L^r(\Omega)\right]^k\) to \(X_r\).
Now, we let $-\Delta$ be the negative Laplace operator on the domain $D(-\Delta) = \{ v \in [W^{2,r}(\Omega)]^k : v|_{\partial\Omega} = 0 \}$, then

$$P_r(-\Delta) : X_r \cap D(-\Delta) \rightarrow X_r$$

is called the Stokes operator, e.g. see p.201 [25] or [24]. Furthermore, applying the Stokes operator on both sides of (54) deduces the following equations with eliminating the pressure term

$$\left\{ \begin{array}{l}
\frac{\partial u}{\partial t} + b(t)P_r((u, \nabla)u) = -\frac{\partial^{1-\alpha}}{\partial t^\alpha}P_r((-\Delta)u), \\
\text{div } u = 0, \\
u|_{\partial \Omega} = 0, \\
u(., T) = \varphi.
\end{array} \right. \tag{86}$$

By getting rid of the vector fields, a simple comparison between Problem (11-14) and Problem (84) yields that the second one is a special case of the first one with respect to the operator $A = P_r(-\Delta)$ and the nonlinearity $G = -P_r((u, \nabla)u)$. We also note that denoting by $\partial^\alpha$ the Caputo fractional derivative operator of order $\alpha$ can transforms Problem (84) into the form

$$\frac{\partial^\alpha}{\partial t^\alpha}u + Au = T^{-\alpha}_r F(u, \nabla u), \tag{87}$$

where $F(u; \nabla u) := -b(t)P_r((u, \nabla)u)$, and $T^\alpha_r$ is the fractional integral of the order $1 - \alpha$. To apply our method in Subsection 4.1, we recall the following properties of the spectrum of the Stokes operator in the case $r = 2$, that is, $A$ is an unbounded, self-adjoint, and positive operator in $X_2$ and has a compact inverse, e.g. see Section 6 in [70]. Then, it possesses eigenvalues and eigenfunctions as what mentioned in Section 2. Now let us define the following sets which corresponds to some cases of the dimension $n$

$$\Theta_{k,q}^{(1)} := \left\{ (k, q) : 2 \leq k \leq 3, \frac{3k}{k + 3} < q < \frac{k}{k - 3} \right\},$$

$$\Theta_{k,q}^{(2)} := \left\{ (k, q) : 4 \leq n \leq 5, \frac{3k}{k + 3} < q < \frac{k}{k - 3} \right\}.$$

In the following theorem, we give some results on existence and regularity of a mild solution of Problem (54), where we also introduce a gradient estimate of the solution.

**Theorem 4.3.** Given $k$ and $q$ defined by $\Theta_{k,q} := \Theta_{k,q}^{(1)} \cup \Theta_{k,q}^{(2)}$. Let the numbers $l_1, l_2$ be given by

$$l_1 \in \left[ \frac{1}{2} + \frac{k}{2q}, l_1^* \right], \quad l_2 \in (l_1^*, 0),$$

whereupon $l_1^* := \min \{ l_1^*, k/q \}$, $l_1^* := \max \{ l_1^*, -k/q \}$, and $l_1$, $l_1^*$ are formulated as Theorem 4.2. Assume that $\varphi \in D((P_r(-\Delta))^{1+\frac{3}{2}l_2})$, and $|b(t)| \lesssim t^\gamma_0$ with some $\gamma_0 > 2\alpha - 1$. If $T$ is small enough, then Problem (54) has a unique mild solution $u \in C^\alpha((0, T]; [W^{1,q}(\Omega)]^k)$ such that

$$\|u\|_{L^\infty([0, T]; [W^{1,q}(\Omega)]^k)}^k + \|\nabla u\|_{L^\infty([0, T]; [W^{1,q}(\Omega)]^k)}^k \lesssim t^{-\alpha}\|\varphi\|_{D((P_r(-\Delta))^{1+\frac{3}{2}l_2})}.$$

**Proof.** To establish the ensuing estimates of the nonlinearity, we will present some interpretations of the given numbers. Firstly, we note that if $k = 5$ then $q$ belongs to the interval $(15/8, 5/2)$, and so that $|q - 2|k/4q$ is bounded by $1/4$. Therefore, the condition $|q - 2|k/4q < 1$ holds true. Secondly, we will prove that $1/2 + k/(2q) < l_1$. Since $k/(k - 3)$ is always less than $k$ if $(k, q) \in \Theta_{k,q}^{(2)}$, we claim that $q$ is always less than $k$ for all $(k, q) \in \Theta_{k,q}^{(1)} \cup \Theta_{k,q}^{(2)}$. This ensures that $1/2 + k/(2q) < k/q$. Hence, in order to show $1/2 + k/(2q) < l_1^*$, we will show that $1/2 + k/(2q) < l_1$. Indeed, if $(k, q) \in \Theta_{k,q}^{(1)}$, then

$$l_1 = \left( \frac{1}{2} + \frac{k}{2q} \right) = \frac{3}{2} + \left( \frac{q - 2}{2q} - \frac{k}{2q} \right) 1_{1<q<2} + \left( \frac{2 - q}{2q} - \frac{k}{2q} \right) 1_{q \geq 2} = \frac{3}{2} + \frac{q - 3}{2q} 1_{1<q<2} + \frac{1 - q}{2q} 1_{q \geq 2} \geq \frac{3}{2} + \frac{3k}{6k/(k + 3)} - 3 1_{1<q<2} + \frac{1 - k}{2k} 1_{q \geq 2} = \frac{3}{2} - \frac{3}{2} 1_{1<q<2} - \left( \frac{1}{2} 1_{k=3} + 1_{k=3} \right) 1_{q \geq 2} \geq 0.$$
If \((k, q) \in \Theta_{k, q}^{(2)}\) then employing the above techniques shows that

\[
l^1 - \left( \frac{1}{2} + \frac{k}{2q} \right) = \frac{3}{2} \left[ \frac{q - 3}{2q} k_{1 < q < 2} + \frac{1 - q}{2q} k_{q \geq 2} \right] \\
> \frac{3}{2} \left[ \frac{3k/(k + 3) - 3}{6k/(k + 3)} k_{1 < q < 2} + \frac{1 - k/(k - 3)}{2k/(k - 3)} k_{q \geq 2} \right] \\
= \frac{3}{2} \left[ 3k_{1 < q < 2} - \frac{3}{2} k_{q \geq 2} \right] \geq 0.
\]

Thus, we conclude that \(1/2 + k/(2q) < l^1\). The above analysis consequently indicates that the interval \([1/2 + k/(2q), l^1]\) is not empty and that the interval \((l^1, 0)\) is also not empty. Hence, the given assumptions on the numbers are relevant.

Now, we prove that the nonlinearity \(G(u) = -b(t) P_r ((u, \nabla) u)\) satisfies the assumption of Theorem 4.2, where we note that

\[
\begin{align*}
\|v - w\|_{L^\frac{kq}{kq-1}(\Omega)}^k &= \left( \sum_{i=1}^k \|v_i - w_i\|_{L^\frac{kq}{kq-1}(\Omega)}^{\frac{kq}{kq-1}} \right)^k, \\
\|
abla v\|_{L^\frac{kq}{kq+2}(\Omega)}^k &= \left\{ \sum_{j=1}^k \sum_{i=1}^k \|
abla v_{ij}\|_{L^\frac{kq}{kq+2}(\Omega)}^{\frac{kq}{kq+2}} \right\}^k.
\end{align*}
\]

By the definition of the space \(X_{\frac{kq}{kq-1}}\) one has the following embedding \(X_{\frac{kq}{kq-1}} \hookrightarrow [W^{l_2, q}(\Omega)]^k\), since Sobolev embedding \(L^\frac{kq}{kq-1}(\Omega) \hookrightarrow W^{l_2, q}(\Omega)\), where we recall that \(-k/q < l_2 < 0\). Therefore, gathering the above Sobolev embedding and the Helmholtz projection together derives that

\[
\begin{align*}
\|P_r ((v, \nabla)v) - P_r ((w, \nabla)w)\|_{[W^{l_2, q}(\Omega)]^k} &
\lesssim b(t) \left\| P_r \left\{ [v_j]_{k \times 1} [\nabla j]_{1 \times k} [v_j - w_j]_{k \times 1} \right\} \right\|_{X_{\frac{kq}{kq-1}}} \\
+ b(t) \left\| P_r \left\{ [v_j - w_j]_{k \times 1} [\nabla j]_{1 \times k} [v_j]_{k \times 1} \right\} \right\|_{X_{\frac{kq}{kq-1}}} \\
\lesssim b(t) \left\| \left( [v_j]_{k \times 1} [\nabla j]_{1 \times k} \right) [v_j - w_j]_{k \times 1} \right\|_{L^\frac{kq}{kq+2}(\Omega)^k} \\
+ b(t) \left\| \left( [v_j - w_j]_{k \times 1} [\nabla j]_{1 \times k} \right) [w_j]_{k \times 1} \right\|_{L^\frac{kq}{kq+2}(\Omega)^k}.
\end{align*}
\]

It is useful to detailedly calculate the above quantities. Indeed, some computations performs the following results which help to estimate the terms of the right hand side of (89).

\[
\begin{align*}
\|U_1\|_{L^\frac{kq}{kq+2}(\Omega)}^k &= \left( \sum_{j=1}^k \sum_{i=1}^k (v_i - w_i) \nabla j v_j \right)_{L^\frac{kq}{kq+2}(\Omega)}^{\frac{kq}{kq+2}}, \\
\|U_2\|_{L^\frac{kq}{kq+2}(\Omega)}^k &= \left( \sum_{j=1}^k \sum_{i=1}^k v_i \nabla j (v_j - w_j) \right)_{L^\frac{kq}{kq+2}(\Omega)}^{\frac{kq}{kq+2}}.
\end{align*}
\]
In the next step we will try to bound the two last right hand sides by the norms of \(v, w, \nabla v\) and \(\nabla w\) in suitable space. Applying the discrete Hölder inequality we obtain the following inequalities

\[
\left\| U_1 \right\|_{L^k(\Omega)}^k \lesssim \left( \sum_{i=1}^k \left| v_i - w_i \right| \left\| \frac{kq - q}{kq - q_i} \right\|_{L^{\frac{kq}{kq - q_i}}(\Omega)} \right)^{k - q} \times \left\{ \sum_{j=1}^k \left( \left\| \nabla_i w_j \right\|_{L^{\frac{kq}{kq - q_i}}(\Omega)} \right) \right\}^{\frac{k - q}{kq}} \times \left\{ \sum_{j=1}^k \left\| \nabla_i w_j \right\|_{L^{\frac{kq}{kq - q_i}}(\Omega)} \right\}^{\frac{k - q}{kq}} ,
\]

(90)

where the latter factor can be estimated by also using the discrete Hölder inequality as follows

\[
\left\{ \sum_{j=1}^k \left( \sum_{i=1}^k \left\| \nabla_i w_j \right\|_{L^{\frac{kq}{kq - q_i}}(\Omega)} \right) \right\}^{\frac{k - q}{kq}} \lesssim \left\{ \sum_{j=1}^k \left( \sum_{i=1}^k \left\| \nabla_i w_j \right\|_{L^{\frac{kq}{kq - q_i}}(\Omega)} \right) \right\}^{\frac{k - q}{kq}}.
\]

So we directly obtain the first one of the following estimates

\[
\begin{align*}
\left\| U_1 \right\|_{L^{\frac{kq}{kq - q_i}}(\Omega)}^k & \lesssim \left\| v - w \right\|_{L^{\frac{kq}{kq - q_i}}(\Omega)} \left\| \nabla w \right\|_{L^{\frac{kq}{kq - q_i}}(\Omega)}^k , \\
\left\| U_2 \right\|_{L^{\frac{kq}{kq - q_i}}(\Omega)}^k & \lesssim \left\| v \right\|_{L^{\frac{kq}{kq - q_i}}(\Omega)} \left\| \nabla (v - w) \right\|_{L^{\frac{kq}{kq - q_i}}(\Omega)}^k ,
\end{align*}
\]

(91)

where the second one can be established by using arguments similar to the first one. Next, we will use the following embeddings in the ensuing estimates. Indeed, since \(0 \leq l_1 < k/q\) as \(l_1 \in [1/2 + k/(2q), l_1^*)\), we have

\[
\left[ W^{l_1, q}(\Omega) \right]^k \hookrightarrow L^{\frac{kq}{kq - q_i}}(\Omega).
\]

(92)

Furthermore, since \(l_1\) is given in the interval \([1/2 + k/(2q), l_1^*)\), and \(l_2 < 0\), one can see that

\[
l_1 - 1 = \left( \frac{k}{q} - \frac{k}{kq/(ql_1 - q_l_2)} \right) + 2l_1 - 1 - l_2 - \frac{k}{q} \\
\geq \left( \frac{k}{q} - \frac{k}{kq/(ql_1 - q_l_2)} \right) + 2 \left( \frac{1}{2} + \frac{k}{2q} \right) - 1 - l_2 - \frac{k}{q} \geq \left( \frac{k}{q} - \frac{k}{kq/(ql_1 - q_l_2)} \right) .
\]

Hence, it holds that

\[
\left[ W^{l_1, q}(\Omega) \right]^k \hookrightarrow W^{1, \frac{kq}{q_l_1 - q}}(\Omega)^k ,
\]

(93)

and, combining the embeddings (92) and (93) with the estimates (91), (91), one can deduce that there exists \(L_* > 0\) such that

\[
\left\| P_\epsilon((v, \nabla)v) - P_\epsilon((w, \nabla)w) \right\|_{W^{l_1, q}(\Omega)}^k \\
\leq L_b(t) \left\| v - w \right\|_{W^{l_1, q}(\Omega)}^k \left( \left\| v \right\|_{W^{l_1, q}(\Omega)}^k + \left\| w \right\|_{W^{l_1, q}(\Omega)}^k \right) .
\]

(94)

We note from Theorem 4.2 that \(q = 1\) and \(L(t) = L_0 b(t)\) in this problem. Since \(\gamma_0 > 2\alpha - 1\) as the assumption of \(b(t)\), one can find \(\delta > 0\) small enough such that \(\gamma_0 + 1 - 2\alpha - \delta > 0\). Therefore, letting \(L_0 = T^{\gamma_0 + 1 - 2\alpha - \delta} L_*\) implies that the function \(t^{1-(1+p)a} L(t) \leq L_0 t^B\). Succinctly stating that the nonlinearity \(G(u)\) satisfies the assumption of Theorem 4.2.

Therefore, by using the same methods in Theorem 4.2, we can conclude that Problem (84) has a unique mild solution \(u \in C^0((0, T]; [W^{l_1, q}(\Omega)]^k)\) if \(T\) is small enough. Finally, it is necessary to establish a gradient estimate for the solution. Indeed, the embedding (93) also indicates that

\[
\left\| \nabla u \right\|_{L^{\frac{kq}{kq - q_i}}(\Omega)}^k \lesssim \left\| u \right\|_{W^{l_1, q}(\Omega)}^k \lesssim t^{-\alpha} \left\| \varphi \right\|_D \left( (P_\epsilon(-\Delta))^{1 + \frac{B}{2}} \right) ,
\]

(95)
where the second estimate follows from the inequality (73). In addition, by using the embedding (22), we see that

$$\|u\|_{L^{\frac{k}{1+\delta}}(\Omega)} \lesssim \|u\|_{W^{1,q}(\Omega)} \lesssim t^{-\alpha} \|\varphi\|_{D((-\Delta)^{1+\frac{3k}{2q}})}.$$

The desired estimate is derived by taking (22) and (72) together. This eventually completes the proof. □

**Remark 4.2.** By the verbose of this article, we do not present results for the case $k \geq 6$, which can be obtained similarly. Moreover, if $\alpha \in (0,1/2)$ then by letting $\delta = 1 - 2\alpha > 0$ in Theorem 4.4, the assumption $t^{1-(1+\alpha)\delta}L(t) \leq L_0 t^{k}$ is equivalent to $L(t) \lesssim 1$, where we recall that $q = 1$, see (73). These explanations mean that, in the case $\alpha \in (0,1/2)$, our method is also available to deal with the following problem

$$\begin{cases}
\frac{\partial u}{\partial t} + (u, \nabla)u - \nabla p = 0, & (x, t) \in \Omega \times (0,T), \\
\text{div} u = 0, & (x, t) \in \Omega \times (0,T), \\
u = 0, & (x, t) \in \partial \Omega \times (0,T), \\
u(., T) = \varphi. & x \in \Omega,
\end{cases}$$

4.3. Applications to time fractional Allen-Cahn equations. In this subsection, we apply our results in Subsection 4.1 to establish the existence of a mild solution to time fractional Allen-Cahn equation. The Allen-Cahn equation is well-known and arise in mathematical biology, quantum mechanics and plasma physics (see for instance [65] [66]). The mentioned fractional derivatives here will be considered in Caputo’s sense. More precisely, we study the final value problem of finding $u = u(x,t)$ satisfying

$$\begin{cases}
\frac{\partial u}{\partial t} - \nabla^\alpha \Delta u + b(t)W'(u) = 0, & (x, t) \in \Omega \times (0,T), \\
u = 0, & (x, t) \in \partial \Omega \times (0,T), \\
u(., T) = \varphi. & x \in \Omega,
\end{cases}$$

where $0 < \alpha \leq 1$, $\Omega$ is a $C^2$ bounded open set of $\mathbb{R}^k$ ($k \geq 2$). Here, the quantity $W(u) := (1 - u^2)^2/4$ refers to a bi-stable and balanced. Here the derivative $W'(u) := d (-1 - u^2)^2/4/du$ is equal to $u - u^3$ and refers to a double-well potential. We introduce the sets

$$\begin{align*}
\Phi_{k,q}^{(1)} := \{(n,q): k = 2, \frac{4}{3} < q < \frac{16}{9}\}, \\
\Phi_{k,q}^{(2)} := \{(n,q): k = 3, \frac{12}{7} < q < 4\}.
\end{align*}$$

In the next theorem, we establish the existence and regularity of a mild solution of Problem (97) in the case of $2 \leq k \leq 3$.

**Theorem 4.4.** Given $k$ and $q$ defined by $\Phi_{k,q} := \Phi_{k,q}^{(1)} \cup \Phi_{k,q}^{(2)}$. Let the numbers $l_1, l_2$ satisfy the condition

$$l_1 \in \left[\frac{3k}{2q} - \frac{1}{2} \frac{k}{q}, l_2 \in \left(l_1, 3l_1 - \frac{3k}{q}\right),
\right]$$

where $l_1^1 := \max\{l_1, -k/q\}$, and $l_1^2, l_1^3$ are given by Lemma 4.4. Assume that $\varphi \in D((-\Delta)^{1+\frac{3k}{2q}})$, and $|b(t)| \lesssim t^{\gamma_1}$ for some $\gamma_1 > 3\alpha - 1$. If $T$ is small enough, then Problem (97) has a unique mild solution $u \in C^\alpha([0,T]; W^{1,q}(\Omega))$ such that

$$L^F \text{ estimate: } \|u\|_{L^\frac{k}{1+\delta}(\Omega)} \lesssim t^{-\alpha} \|\varphi\|_{D((-\Delta)^{1+\frac{3k}{2q}})},$$

$$\text{Double-well potential estimate: } \|W'(u)\|_{W^{1,q}(\Omega)} \lesssim t^{-3\alpha} \|\varphi\|_{D((-\Delta)^{1+\frac{3k}{2q}})}.$$
Proof. The existence of a mild solution as stated in this theorem will be based on Theorem 1.7 by verifying all assumptions of Theorem 1.2. Then, we will establish suitable estimates for the solution and its gradient. It is also useful to estimate the double-well potential, bi-stable and balanced. Firstly, we present some interpretations of the given numbers as follows. We will show that $k/q < t^1$ by considering the cases $1 < q < 2$ or $q \geq 2$. Indeed, by some direct computations, it holds

$$t^1 - \frac{k}{q} = 2 - \frac{k}{q} + \frac{q-2}{2q}k_{1<q<2} + \frac{2-q}{2q}k_{q\geq2}$$

$$= \left(\frac{4+k}{2} - \frac{2k}{q}\right)_{1<q<2} + \left(\frac{4-k}{2} - \frac{2k}{q}\right)_{q\geq2}$$

$$= \left(\frac{3q-4}{q}k_{q=2} + \frac{7q-12}{2q}k_{q=3}\right)_{1<q<2} + \frac{1}{2}k_{q=q}k_{q=2} > 0.$$  

This implies that $3k/(2q) - (1/2)t^1$ is less than $k/q$. Hence, the interval $[3k/(2q) - (1/2)t^1, k/q]$ is not empty. Besides, by assumption $l_1 > 3k/(2q) - (1/2)t^1$, we also have $t^1 < 3l_1 - 3k/q$. Secondly, we show that $t^1 < 5k/(3q)$ as follows

$$t^1 - \frac{5k}{3q} = 2 - \frac{5k}{3q} + \frac{q-2}{2q}k_{1<q<2} + \frac{2-q}{2q}k_{q\geq2}$$

$$= \left(\frac{4+k}{2} - \frac{8k}{3q}\right)_{1<q<2} + \left(\frac{4-k}{2} - \frac{2k}{3q}\right)_{q\geq2}$$

$$= \left(\frac{9q-16}{3q}k_{q=2} + \frac{7q-12}{2q}k_{q=3}\right)_{1<q<2} + \frac{q-4}{2q}k_{q=q}k_{q=2} \geq 0.$$  

Therefore, by using assumptions $l_1 > 3k/(2q) - (1/2)t^1$ and estimate $t^1 \leq 5k/(3q)$, we then have $l_1 > 3k/(2q) - 5k/(6q) = 2k/(3q)$, which means that $3l_1 - 3k/q$ is greater than $-k/q$. Thus, $l_2^1$ is less than $3l_1 - 3k/q$, and so that the interval $(l_1^1, 3l_1 - 3k/q]$ is not empty.

We also note that $3k/(2q) - (1/2)t^1 > 0$ as $(1/2)t^1 < 5k/(6q) < 3k/(2q)$, which infers the inclusion $[3k/(2q) - (1/2)t^1, k/q) \subset [0, t^1]$. In addition, the number $3l_1 - 3k/q$ is clearly non-positive, and therefore the inclusion $(l_1^1, 3l_1 - 3k/q) \subset [t^1, 0]$ holds. Summarily, the number $l_1$ and $l_2$ satisfy Theorem 1.2.

Let us denote by $G(u)$ the product of $b(t)$ and the double-well potential $\frac{d}{dt} (\frac{1}{2}(1 - u^2)^2)$. We will show that it satisfies Theorem 1.2. Let $v_1, v_2$ belong to $C^\infty([0, T]; W^{1, q}(\Omega))$. It is obvious that

$$\frac{d}{dt} \left(\frac{1}{4}(1 - v_1^2)^2\right) - \frac{d}{dt} \left(\frac{1}{4}(1 - v_2^2)^2\right) = (v_1 - v_2)(v_1^2 + v_2^2 + v_1v_2 - 1),$$

which deduces $|G(v_1) - G(v_2)| \lesssim b(t)|v_1 - v_2|(1 + v_1^2 + v_2^2)$. This associated with an application of the Sobolev embedding $L^{\frac{6q}{6q-5}}(\Omega) \hookrightarrow W^{1, q}(\Omega)$, as $-\frac{k}{q} \leq l_2 < 0$, and the Hölder inequality yields

$$\|G(v_1) - G(v_2)\|_{W^{1, q}(\Omega)} \lesssim \|b(t)\| \|v_1 - v_2\|(1 + v_1^2 + v_2^2)$$

$$\lesssim \|b(t)\| \|v_1 - v_2\| L^{\frac{6q}{6q-5}}(\Omega) \left(1 + \|v_1\|^2_{L^{\frac{6q}{6q-5}}(\Omega)} + \|v_2\|^2_{L^{\frac{6q}{6q-5}}(\Omega)}\right)$$

$$\lesssim \|b(t)\| \|v_1 - v_2\| L^{\frac{6q}{6q-5}}(\Omega) \left(1 + \|v_1\|^2_{L^{\frac{6q}{6q-5}}(\Omega)} + \|v_2\|^2_{L^{\frac{6q}{6q-5}}(\Omega)}\right).$$

Here, in the last line of the above estimates, we have used the embedding

$L^{\frac{kq}{kq-5}}(\Omega) \hookrightarrow L^{\frac{6q}{6q-5}}(\Omega),$ since $\frac{kq}{kq-5} \geq \frac{3kq}{kq-5} = 3l_1 - l_2 \geq 3k/q$.

Therefore, there exists $L_2 > 0$ such that

$$\|G(v_1) - G(v_2)\|_{W^{1, q}(\Omega)} \leq L_2 b(t) \|v_1 - v_2\|_{W^{1, q}(\Omega)} \left(1 + \|v_1\|^2_{W^{1, q}(\Omega)} + \|v_2\|^2_{W^{1, q}(\Omega)}\right),$$

(99)

where we note that the embedding $W^{1, q}(\Omega) \hookrightarrow L^{\frac{6q}{6q-5}}(\Omega)$, as $0 < l_1 < \frac{k}{q}$ holds true. We conclude that the double-well potential satisfies Theorem 1.2 with respect to $\rho = 2$. We note that $\gamma_1 > 3\alpha - 1$, so there exists $\delta > 0$ small enough such that $\gamma_1 + 1 - 3\alpha - \delta > 0$. Henceforth, by letting $L_0 = T_{\gamma_1 + 1 - 3\alpha - \delta} L_2$ and $L(t) = L_0 b(t)$, we obtain $t^{1-(1+\delta)\alpha} L(t) \leq L_0 t^\alpha$. Summarily, we claim that $G(u)$ satisfies the...
assumptions of Theorem 4.2. Thus, Theorem 4.2 deduces that Problem (97) has a unique mild solution \( u \in C^\alpha((0,T];W^{1,\theta}(\Omega)) \) as \( T \) small enough. Moreover, the solution also satisfies

\[
\|u\|_{L^\frac{2}{\alpha}(\Omega)} \lesssim \|u\|_{W^{1,\alpha}(\Omega)} \lesssim t^{-\alpha}\|\varphi\|_{D(-\Delta)^{1+\frac{\alpha}{2}}}. \tag{100}
\]

Finally, we will prove the double-well potential and gradient estimates. In fact, by estimate (100) for the nonlinearity \( G \) and boundedness (100), we have

\[
\|W'(u)\|_{W^{1,\theta}(\Omega)} \lesssim \|u\|_{W^{1,\alpha}(\Omega)}(1+\|u\|_{W^{1,\alpha}(\Omega)}^2) \lesssim t^{-\alpha}\|\varphi\|_{D(-\Delta)^{1+\frac{\alpha}{2}}} \lesssim t^{-3\alpha}\|\varphi\|_{D(-\Delta)^{1+\frac{\alpha}{2}}}. \]

On the other hand, since \( l_1 > 3k/(2q) - 5k/(6q) = 2k/(3q) \), we can deduce that \( l_1 \geq 1 \) if \( k = 3 \) and \( 12/7 < q \leq 2 \). Hence, it holds that

\[
W^{1,\theta}(\Omega) \hookrightarrow W^{1,\frac{kq}{kq/(k - q)(1 + q)l_1 + q)}(\Omega), \quad \text{since} \quad l_1 - 1 = \frac{k}{q} - \frac{kq}{kq/(k - q)(1 + q)l_1 + q}.
\]

The gradient estimate is so obtained by combining this embedding and the boundedness (100).

\[\Box\]

**APPENDIX**

We give some useful lemmas exhibiting important properties of the Mittag-Leffler functions. These lemmas will be useful for the construction of solution to our problem. Here, the more general Mittag-Leffler function \( E_{\alpha,\gamma}(z) \) is defined by

\[
E_{\alpha,\gamma}(z) := \sum_{n=1}^{\infty} \frac{z^n}{\Gamma(\alpha n + \gamma)}, \quad z \in \mathbb{R}, \alpha > 0, \beta \in \mathbb{R}.
\]

**Lemma 4.5. (Theorem 1.6, [2])** Given \( 0 < \alpha < 1, z \leq 0 \), there exist positive constants \( \theta_1, \theta_2 \) depending only on \( \alpha \) such that

\[
\frac{\theta_1}{1 + |z|^2} \leq |E_{\alpha,1}(z)| \leq \frac{\theta_2}{1 + |z|^2}, \quad |E_{\alpha,\alpha}(z)| \leq \min \left\{ \frac{\theta_2}{1 + |z|^2}, \frac{\theta_2}{1 + |z|^2} \right\}. \tag{101}
\]

**Lemma 4.6. (Lemma 3.2, [17])** Given \( 0 < \alpha < 1, \lambda > 0 \) and \( t > 0 \). The following identities hold

\[
\frac{d}{dt}E_{\alpha,1}(-t^\alpha \lambda) = -\lambda t^{\alpha-1}E_{\alpha,\alpha}(-t^\alpha \lambda), \quad \frac{d}{dt}(t^{\alpha-1}E_{\alpha,\alpha}(-t^\alpha \lambda)) = t^{\alpha-2}E_{\alpha,\alpha-1}(-t^\alpha \lambda).
\]

In the following lemma, we present the basic boundedness of the solution operators which are applied to establish almost results in this paper.

**Lemma 4.7.** Given \( \gamma \in \mathbb{R} \). Let \( \beta_1, \beta_2 \) be real numbers such that \( 0 \leq \beta_2 \leq 1 \).

a) \( \|E_{\alpha,\gamma}(-t^\alpha A)\|_{\mathcal{L}(D(A^{\beta_1});D(A^{\beta_2}))} \leq \theta_2 \), for \( t \geq 0 \).

b) \( \|E_{\alpha,\gamma}(-t^\alpha A)\|_{\mathcal{L}(D(A^{\beta_1-\gamma});D(A^{\beta_2}))} \leq \theta_2 t^{-\alpha \beta_2} \), for \( t > 0 \).

c) \( \|E_{\alpha,\gamma}^{-1}(-t^\alpha A)\|_{\mathcal{L}(D(A^{\beta_1+\gamma});D(A^{\beta_2}))} \leq \theta_1 (\lambda_1^{-1} + t^\alpha) \), for \( t > 0 \).

d) \( \|E_{\alpha,\alpha}(-t^\alpha A)\|_{\mathcal{L}(D(A^{\beta_1-\gamma});D(A^{\beta_2}))} \leq \theta_2 t^{-2 \alpha \beta_2} \), for \( t > 0 \).

**Proof of Lemma 4.7.** We firstly consider the case \( \beta_1 \geq 0 \). In view of (10), we have, for \( \psi_n = \langle \psi, \phi_n \rangle \),

\[
\|E_{\alpha,1}(-t^\alpha A)\psi\|_{D(A^{\beta_1})} = \left[ \sum_{n=1}^{\infty} \lambda_n^{2\beta_1} E_{\alpha,1}^2(-t^\alpha \lambda_n) \psi_n^2 \right]^{1/2} \leq \theta_2 \|A^{\beta_1} \psi\|_{D(A^{\beta_1})}, \tag{102}
\]

where we have used that \( |E_{\alpha,1}(-t^\alpha \lambda_n)| \leq \theta_2 \) as Lemma 101. Now let us consider the case \( \beta_1 < 0 \), we note that \( \langle \phi_n, \phi_m \rangle_{\beta_1,\beta_1} = \delta_{mn}, \) see [14, 17]. Thus,

\[
A^{\beta_1} E_{\alpha,1}(-t^\alpha A) \psi = \sum_{n=1}^{\infty} \phi_n A^{\beta_1} \left[ \sum_{m=1}^{\infty} E_{\alpha,1}(-t^\alpha \lambda_m) \psi_m \phi_n, \phi_n \right] \leq \sum_{n=1}^{\infty} \lambda_n^{2 \beta_1} E_{\alpha,1}(-t^\alpha \lambda_n) \psi_n \phi_n, \phi_n,
\]

and so that we can apply the same estimate as (102). Namely, Part a is proved. Similarly, the following arguments also include both cases \( \beta_1 \geq 0 \) and \( \beta_1 < 0 \). We secondly prove Part b. For \( \beta_1 \geq 0 \), combining the expansion (10) and Lemma 101 shows

\[
\|E_{\alpha,1}(-t^\alpha A)\psi\|_{D(A^{\beta_1})} = \left[ \sum_{n=1}^{\infty} \lambda_n^{2 \beta_1} E_{\alpha,1}^2(-t^\alpha \lambda_n) \psi_n^2 \right]^{1/2} \leq \theta_2 \left[ \sum_{n=1}^{\infty} \lambda_n^{2 \beta_1} (1 + t^\alpha \lambda_n)^{-2} \psi_n^2 \right]^{1/2}.
\]
It deduces from $0 \leq \beta_2 \leq 1$ that $(1 + t^\alpha \lambda_n)^{-1} \leq t^{-\alpha \beta_2} \lambda_n^{-\beta_2}$. Hence,

$$
\|E_{\alpha,1}(-t^\alpha A)\psi\|_{D(A^{\beta_1})} \leq \theta_2 \left[ \sum_{n=1}^{\infty} \lambda_n^{2\beta_1} t^{-2\alpha \beta_2} \lambda_n^{-2\beta_2} \psi_n^2 \right]^{1/2} = \theta_2 t^{-\alpha \beta_2} \|\psi\|_{D(A^{\beta_1-\beta_2})}.
$$

This proves Part b. Finally, we proceed to prove Part c. In view of (111), we have

$$
\|E_{\alpha,1}(-t^\alpha A)\psi\|_{D(A^{\alpha_1})} = \left[ \sum_{n=1}^{\infty} \lambda_n^{2\beta_1} E_{\alpha,1}^{-2}(-t^\alpha \lambda_n) \psi_n^2 \right]^{1/2} \leq \theta_1 \left[ \sum_{n=1}^{\infty} \lambda_n^{2\beta_1} (1 + t^\alpha \lambda_n)^2 \psi_n^2 \right]^{1/2},
$$

where Lemma (111) has been used. This associated with $1 + t^\alpha \lambda_n \leq (\lambda_n^{-1} + t^\alpha) \lambda_n$ to allow Part c. Finally, Part d is proved as

$$
\|E_{\alpha,\alpha}(-t^\alpha A)\psi\| = \left[ \sum_{n=1}^{\infty} E_{\alpha,\alpha}^2(-t^\alpha \lambda_n) \psi_n^2 \right]^{1/2} \leq \theta_2 \left[ \sum_{n=1}^{\infty} \left( \frac{1}{1 + t^2 \alpha \lambda_n^2} \right) \psi_n^2 \right]^{1/2}.
$$

Therefore this Part d is proved by noting that $1 + t^2 \alpha \lambda_n^2 \geq (1 + t^2 \alpha \lambda_n^2)^{\beta_2} \geq t^2 \alpha \lambda_n^{2\beta_2}$. \hfill \Box

References

[1] Kai Diethelm; *The analysis of fractional differential equations*, Springer, Berlin, 2010.
[2] I. Podlubny; *Fractional differential equations*, Academic Press, London, 1999.
[3] Stefan G. Samko, Anatoly A. Kilbas and Oleg I. Marichev; *Fractional integrals and derivatives*, Theory and Applications, Gordon and Breach Science, Naukai Tekhnika, Minsk (1987).
[4] R. Courant, D. Hilbert; *Methods of Mathematical Physics*, Vol. 1, Interscience, New York, 1953.
[5] H. Hirata and C. Miao; *A nonlinear parabolic equation backward in time: regularization with new error estimates*, Nonlinear Anal. 73 (2010), no. 6, 1842–1852.
[6] A. Viana; *A local theory for a fractional reaction-diffusion equation*, Commun. Contemp. Math., (2018), S0219199718500335
[7] W.R. Schneider, and W. Wyss, (1989). *Fractional diffusion and wave equations*. Journal of Mathematical Physics, 30(1), 134–144.
[8] A. Viana; *Fractional diffusion and wave equations*. Journal of Mathematical Physics, 30(1), 134–144.
[9] N.H. Tuan, D.D. Trong; *A nonlinear parabolic equation backward in time: regularization with new error estimates*. Nonlinear Anal. 73 (2010), no. 6, 1842–1852.
[10] H. Koebbe, C. Miao; *Space-time estimates of linear flow and application to some nonlinear integro-differential equations corresponding to fractional-order time derivative*, Adv. Differential Equations, (2002), no. 2, 217–236.
[11] A. N. Kochubei; *Cauchy problem for fractional diffusion-wave equations with variable coefficients*, Appl. Anal. 93 (2014), no. 10, 2211–2242.
[12] S. Carillo, V. Valente and G. Vergara Caffarelli; *A linear viscoelasticity problem with a singular memory*. J. Math. Anal. Appl., 382 (2011), 426–447.
[13] S. Carillo, V. Valente and G. Vergara Caffarelli; *Heat conduction with memory: a singular kernel problem*. Evol. Equ. Control Theory 3 (2014), no. 3, 399–410.
[58] A. Hanyga, Wave propagation in media with singular memory, Math. Comput. Model. 34 (2001) 1399-1421. MR1868410 (2002m:74028)
[59] R. Hilfer (ed.), Applications of Fractional Calculus in Physics, World Scientific, Singapore (2000). MR1890104 (2002d:00009)
[60] K. Mustapha, An L1 approximation for a fractional reaction-diffusion equation, a second-order error analysis over time-graded meshes, https://arxiv.org/pdf/1909.06739.pdf
[61] B. Kaltenbacher, W. Rundell On an inverse potential problem for a fractional reaction-diffusion equation, Inverse Problems, Volume 35, Number 6, 2019.
[62] B. Kaltenbacher, W. Rundell Regularization of a backward parabolic equation by fractional operators Inverse Probl. Imaging 13 (2019), no. 2, 401–430
[63] H. Liao, D. Li, and J. Zhang, Sharp error estimate of the nonuniform L1 formula for linear reaction-subdiffusion equations, SIAM J. Numer. Anal., 56 (2018), 1112-1133.
[64] H. Liao, W. McLean, and J. Zhang, A discrete Gronwall inequality with applications to numerical schemes for subdiffusion problems, SIAM J. Numer. Anal., 57 (2019), 218-237.
[65] Q. Du, J. Yang, Z. Zhou, Time-Fractional Allen-Cahn Equations: Analysis and Numerical Methods, https://arxiv.org/pdf/1906.06584.pdf
[66] H. Liu, A. Cheng, H. Wang, J. Zhao, Time-fractional Allen-Cahn and Cahn-Hilliard phase-field models and their numerical investigation, Comput. Math. Appl. 76 (2018), no. 8, 1876–1892.
[67] M.J. Arrieta, A.N. Carvalho, Abstract parabolic problems with critical nonlinearities and applications to Navier-Stokes and heat equations. Trans. Am. Math. Soc. 352, 285-310 (1999).
[68] H. Iwashita, Lq -Lr estimates for solutions of the nonstationary Stokes equations in an exterior domain and the Navier-Stokes initial value problems in Lq spaces Math. Ann. 285 (1989), no. 2, 265-288
[69] C. Cao, E.S. Titi, Regularity criteria for the three-dimensional Navier-Stokes equations. Indiana Univ. Math. J. 57 (2008), no. 6, 2643-2661
[70] C. Foias, R. Temam, O. Manley, R. Rosa, Navier Stokes equations and Turbulence, Cambridge University Press, 2001.

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