Lyapunov-type inequality and solution for a fractional differential equation

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Abstract
In this paper, we consider the linear fractional differential equation

\[
\begin{cases}
\frac{\partial^\nu}{\partial t^\nu} u(t) + q(t)u(t) = 0, & t \in (0,1), \nu \in (1,2], \\
u(0) = \delta u(1), & u'(0) = \gamma u'(1).
\end{cases}
\]

By obtaining the Green’s function we derive the Lyapunov-type inequality for such a boundary value problem. Furthermore, we use the contraction mapping theorem to study the existence of a unique solution for the corresponding nonlinear problem.

MSC: 34A08

Keywords: Fractional differential equation; Green’s function; Lyapunov-type inequality; Contraction mapping theorem

1 Introduction
In 1907, Lyapunov [1] stated the following outstanding result.

Theorem 1.1 ([1]) If the boundary value problem (BVP)

\[
\begin{cases}
y''(t) + q(t)y(t) = 0, & a < t < b, \\
y(a) = y(b) = 0,
\end{cases}
\]

has a nontrivial solution, then we have the following Lyapunov inequality:

\[
\int_a^b|q(s)|\,ds > \frac{4}{b-a}.
\]

Inequality (1) is very useful in various problems related to differential equations. Since the appearance of Lyapunov’s fundamental paper [1], many improvements and generalizations of inequality (1) for integer-order (second- and higher-order) BVPs have appeared in the literature; we refer the reader to the summary reference by Tiryaki [2].

Recently, the studies on Lyapunov’s inequality for fractional boundary value problem (FBVP) have begun, in which fractional derivatives (Riemann–Liouville derivative \(_{a}^{R}D_{t}^\nu\) or
Caputo derivative $^{C}_a D^{\nu} y$ are used instead of the classical ordinary derivative. Such a work was initiated by Ferreira [3] in 2013, who obtained a Lyapunov inequality for the following differential equation with Riemann–Liouville fractional derivative:

$$
\left( {^{R}_a D^{\nu} y}(t) + q(t)y(t) \right) = 0, \quad a < t < b, \quad 1 < \nu \leq 2,
$$

subject to the boundary value condition

$$
y(a) = y(b) = 0.
$$

Next, in 2014, Ferreira [4] obtained a Lyapunov inequality for the following differential equation with Caputo fractional derivative:

$$
\left( {^{C}_a D^{\nu} y}(t) + q(t)y(t) \right) = 0, \quad a < t < b, \quad 1 < \nu \leq 2,
$$

subject to boundary value condition (3).

After [3] and [4], many results appeared in the literature; we refer the reader to [5–10], where Lyapunov or Lyapunov-type inequalities are obtained for fractional differential equation subject various boundary value conditions such as

$$
y'(a) = y'(b) = y'(c) = 0, \quad a < b, c \in [a, b];
$$

$$
y(a) = y'(a) = 0, \quad y'(b) = \beta y'(\xi);
$$

$$
y(a) = y'(a) = y''(a) = y''(b) = 0;
$$

$$
y(a) = y'(a) = y(b) = 0.
$$

Inspired by the works mentioned, in this paper, we aim to investigate the Lyapunov-type inequality for the following fractional differential equations:

$$
\begin{cases}
^{C}_0 D^{\nu} u(t) + q(t)u(t) = 0, & t \in (0, 1), \quad \nu \in (1, 2], \\
u(0) = \delta u(1), & u'(0) = \gamma u'(1),
\end{cases}
$$

where $\delta$ and $\gamma$ are real numbers, and $q(t) \in L(0, 1)$ is not identically zero on any compact subinterval of $(0, 1)$. Furthermore, we obtain the existence of a solution for the corresponding nonlinear problem:

$$
\begin{cases}
^{C}_0 D^{\nu} u(t) + q(t)f(u(t)) = 0, & t \in (0, 1), \quad \nu \in (1, 2], \\
u(0) = \delta u(1), & u'(0) = \gamma u'(1).
\end{cases}
$$

BVP (6) was recently studied in [11], but we should point out that only the case of $\delta > 1$ and $0 < \gamma < 1$ was considered in [11]. In this paper, we give a comprehensive discussion on parameters $\delta$ and $\gamma$.

### 2 Preliminaries and lemmas

For convenience, we present some definitions and lemmas from fractional calculus theory in the sense of Riemann–Liouville and Caputo.
\textbf{Definition 2.1} \cite{12} Let $\Gamma(v) = \int_0^\infty t^{v-1}e^{-t} \, dt$, $v > 0$, be the gamma function. Then the Riemann–Liouville fractional integral of order $v$ for $y(t)$ is defined as
\[
\left(\mathcal{I}_t^v y\right)(t) := \frac{1}{\Gamma(v)} \int_a^t (t-s)^{v-1} y(s) \, ds, \quad t \in [a, b].
\]

\textbf{Definition 2.2} \cite{12} Let $\nu > 0$ and $n = \lfloor \nu \rfloor + 1$, where $\lfloor \nu \rfloor$ denotes the integer part of a number $\nu$. Then the Caputo fractional derivative of order $\nu$ for $y(t)$ is defined as
\[
\left(\mathcal{D}_t^\nu y\right)(t) := \frac{1}{\Gamma(n-\nu)} \int_a^t \frac{y^{(n)}(s)}{(t-s)^{\nu+1-n}} \, ds, \quad t \in [a, b].
\]

By Definitions 2.1 and 2.2 we have
\[
\left(\mathcal{I}_t^\nu \left(\mathcal{D}_t^\nu y\right)\right)(t) + \left(\mathcal{D}_t^\nu \left(y(t)u(t)\right)\right)(t) = 0.
\]

\textbf{Lemma 2.1} A function $u(t)$ is a solution of the boundary value problem (5) if and only if $u(t)$ satisfies
\[
u(t) = \int_0^1 G(t,s)q(s)u(s) \, ds,
\]
where
\[
G(t,s) = \frac{1}{\Gamma(v)} \begin{cases} 
(1-v)\frac{\nu(1-\nu)t}{(1-\nu)(1-s)}(1-s)^{v-2} - \frac{1}{1-\nu}(1-s)^{v-1} - (t-s)^{v-1}, \\
0 \leq s \leq t \leq 1, \\
(1-v)\frac{\nu(1-\nu)t}{(1-\nu)(1-s)}(1-s)^{v-2} - \frac{1}{1-\nu}(1-s)^{v-1}, \\
0 \leq t \leq s \leq 1.
\end{cases}
\]

\textbf{Proof} Let $u(t)$ be a solution of (5). Then
\[
\left(\mathcal{I}_t^\nu \left(\mathcal{D}_t^\nu u\right)\right)(t) + \left(\mathcal{D}_t^\nu \left(q(t)u(t)\right)\right)(t) = 0.
\]
By (7) we obtain
\[
u(t) = C_1 + C_2 t - \frac{1}{\Gamma(v)} \int_0^t (t-s)^{v-1} q(s)u(s) \, ds.
\]
Considering $u(0) = \delta u(1)$, we have
\[
C_1 = \delta C_1 + \delta C_2 - \frac{\delta}{\Gamma(v)} \int_0^1 (1-s)^{v-1} q(s)u(s) \, ds;
\]
considering $u'(0) = \gamma u'(1)$, we have
\[
C_2 = \frac{\gamma(v-1)}{\Gamma(v)(v-1)} \int_0^1 (1-s)^{v-2} q(s)u(s) \, ds,
\]

(10)
and thus we get

\[
C_1 = \frac{\delta y(v - 1)}{\Gamma(v)(1 - \delta)(y - 1)} \int_0^1 (1 - s)^{v - 2} q(s)u(s) \, ds
- \frac{\delta}{\Gamma(v)(1 - \delta)} \int_0^1 (1 - s)^{v - 1} q(s)u(s) \, ds.
\]

(11)

Substituting (10) and (11) into (9), we obtain

\[
u(t) = \frac{\delta y(v - 1)}{\Gamma(v)(1 - \delta)(y - 1)} \int_0^1 (1 - s)^{v - 2} q(s)u(s) \, ds
- \frac{\delta}{\Gamma(v)(1 - \delta)} \int_0^1 (1 - s)^{v - 1} q(s)u(s) \, ds
+ \frac{\gamma(v - 1)t}{\Gamma(v)(y - 1)} \int_0^1 (1 - s)^{v - 2} q(s)u(s) \, ds - \frac{1}{\Gamma(v)} \int_0^t (t - s)^{v - 1} q(s)u(s) \, ds
= \int_0^1 G(t,s)q(s)u(s) \, ds,
\]

where \(G(t,s)\) is the Green's function:

\[
G(t,s) = \frac{1}{\Gamma(v)} \begin{cases} 
(1 - v) \frac{\gamma(1 - v) + \nu}{\Gamma(v)(1 - \delta)(1 - \gamma)} (1 - s)^{v - 2} - \frac{\delta}{1 - \delta} (1 - s)^{v - 1} - (t - s)^{v - 1}, & 0 \leq s \leq t \leq 1, \\
(1 - v) \frac{\gamma(1 - v) + \nu}{\Gamma(v)(1 - \delta)(1 - \gamma)} (1 - s)^{v - 2} - \frac{\delta}{1 - \delta} (1 - s)^{v - 1}, & 0 \leq t \leq s \leq 1.
\end{cases}
\]

Lemma 2.2 When \(\delta \in (0, 1)\) and \(\gamma \in (0, 1)\), Green's function \(G(t,s)\) satisfies the following properties:

(i) \(G(t,s) \leq 0, (t,s) \in [0,1] \times [0,1]\);  
(ii) \(\max_{0 \leq s \leq 1} |G(t,s)| = -G(1,s) = \frac{(1 - \delta)^{v - 2}}{\Gamma(v)(1 - \delta)(1 - \gamma)} [\gamma(v - 1) + (1 - \gamma)(1 - s)] \), for \(s \in [0,1]\),  
(iii) \(\int_0^1 |G(t,s)| \, ds \leq \frac{\gamma(v - 1)}{\Gamma(v)(1 - \gamma)\Gamma(v)(1 - \delta)}\).

Proof (i) \(G(t,s) \leq 0\) is obvious since \(\delta \in (0, 1)\) and \(\gamma \in (0, 1)\).

(ii) For \(s \in [0,1]\) and \(t \in [s,1]\), we have

\[
G'(t,s) = \frac{1 - v}{\Gamma(v)} \left[ \frac{\gamma}{1 - \gamma} (1 - s)^{v - 2} + (t - s)^{v - 2} \right] \leq 0,
\]

which means

\[
G(1,s) \leq G(t,s) \leq G(s,s) \leq 0, \quad s \leq t \leq 1;
\]

for \(t \in [0,s]\), we have

\[
G'(t,s) = \frac{\gamma(1 - v)}{\Gamma(v)(1 - \gamma)} (1 - s)^{v - 2} \leq 0,
\]

which means

\[
G(s,s) \leq G(t,s) \leq G(0,s) \leq 0, \quad 0 \leq t \leq s.
\]

(13)
Inequalities (12) and (13) show that, for \( s \in [0, 1] \),

\[
G(1, s) \leq G(t, s) \leq G(0, s) \leq 0, \quad 0 \leq t \leq 1.
\]

Therefore, for \( s \in [0, 1] \),

\[
\max_{0 \leq t \leq 1} \left| G(t, s) \right| = -G(1, s) = \frac{1}{\Gamma(v)} \left[ \frac{(v-1)\gamma}{(1-\delta)(1-\gamma)(1-s)^{v-2}} + \frac{(1-s)^{v-1}}{1-\delta} \right]
\]

\[
= \frac{(1-s)^{v-2}}{\Gamma(v)(1-\delta)(1-\gamma)} \left[ \gamma(v-1) + (1-\gamma)(1-s) \right].
\]

(iii) By (ii) we have

\[
\int_0^1 \left| G(t, s) \right| ds \leq \int_0^1 -G(1, s) ds
\]

\[
= \int_0^1 \frac{(1-s)^{v-2}}{\Gamma(v)(1-\delta)(1-\gamma)} \left[ \gamma(v-1) + (1-\gamma)(1-s) \right] ds = \frac{\gamma(v-1) + 1}{\Gamma(v+1)(1-\gamma)(1-\gamma)}.
\]

**Lemma 2.3** When \( \delta \in (1, +\infty) \) and \( \gamma \in (0, 1) \), Green's function \( G(t, s) \) satisfies the following properties:

(i) \( G(t, s) \geq 0 \), \((t, s) \in [0, 1] \times [0, 1] \);

(ii) \( \max_{0 \leq t \leq 1} \left| G(t, s) \right| = G(0, s) = \frac{\delta(1-s)^{v-2}}{\Gamma(v)(\delta-1)(1-\gamma)} \left[ \gamma(v-1) + (1-\gamma)(1-s) \right] \) for \( s \in [0, 1] \),

(iii) \( \int_0^1 \left| G(t, s) \right| ds \leq \frac{\delta(v+1-\gamma)}{\Gamma(v+1)(\delta-1)(1-\gamma)} \).

**Proof** (i) When \( 0 \leq t \leq s \leq 1 \),

\[
G(t, s) = (v-1) \frac{\delta \gamma (1-t) + \gamma t}{\Gamma(v)(\delta-1)(1-\gamma)} (1-s)^{v-2} + \frac{\delta}{\Gamma(v)(\delta-1)} (1-s)^{v-1} \geq 0.
\]

When \( 0 \leq s \leq t \leq 1 \),

\[
G(t, s) = (v-1) \frac{\delta \gamma (1-t) + \gamma t}{\Gamma(v)(\delta-1)(1-\gamma)} (1-s)^{v-2} + \frac{1}{\Gamma(v)} \left[ \frac{1}{\delta-1} + 1 - \left( \frac{t-s}{1-s} \right)^{v-1} \right]
\]

\[
\geq 0.
\]

(ii) For \( s \in [0, 1] \) and \( t \in [s, 1] \), we have

\[
G'(t, s) = \frac{1-v}{\Gamma(v)} \left[ \frac{\gamma}{1-\gamma} (1-s)^{v-2} + (t-s)^{v-2} \right] \leq 0,
\]

which means

\[
0 \leq G(1, s) \leq G(t, s) \leq G(s, s), \quad s \leq t \leq 1.
\]
For $t \in [0,s]$, we have

$$G'(t,s) = \frac{\gamma(1-v)(1-s)^{v-2}}{(1-\gamma)\Gamma(v)} \leq 0,$$

which means

$$G(s,s) \leq G(t,s) \leq G(0,s), \quad 0 \leq t \leq s. \quad (15)$$

Inequalities (14) and (15) show us that, for $s \in [0,1]$,

"max_{0 \leq t \leq 1} \left| G(t,s) \right| = G(0,s) = \frac{\delta(1-s)^{v-2}}{\Gamma(\delta)(\delta-1)(1-\gamma)} \left[ (v-1) + (1-\gamma)(1-s) \right]."

(iii) From (ii) we have

$$\int_0^1 \left| G(t,s) \right| ds \leq \int_0^1 G(0,s) ds$$

$$= \int_0^1 \frac{\delta(1-s)^{v-2}}{\Gamma(\delta)(\delta-1)(1-\gamma)} \left[ (v-1) + (1-\gamma)(1-s) \right] ds$$

$$= \frac{\delta(\nu+1-\gamma)}{\Gamma(\nu+1)(\delta-1)(1-\gamma)}. \quad \square$$

**Lemma 2.4** When $\delta \in (0,1)$ and $\gamma \in (1,1 + \frac{(v-1)\delta}{\nu v})$, Green's function $G(t,s)$ satisfies the following properties:

(i) $G(t,s) \succeq 0, (t,s) \in [0,1] \times [0,1]$;

(ii) $\max_{0 \leq t \leq 1} \left| G(t,s) \right| = G(1,s) = \frac{1}{\Gamma(\nu)(1-s)\Gamma(\nu-1)} \left[ (v-1) - (\nu-1)(1-s) \right]$ for $s \in [0,1]$;

(iii) $\int_0^1 \left| G(t,s) \right| ds \leq \frac{\gamma(v-1)+1}{\Gamma(\nu+1)(\nu-1)\Gamma(\nu-1)}$.

**Proof** We first prove (i) and (ii). For $s \in [0,1]$, when $t \in [0,s]$,

$$G'_v(t,s) = \frac{(v-1)(1-s)^{v-2}}{\Gamma(v)(\nu-1)} \geq 0,$$

which means that, for $s \in [0,1]$,

$$G(0,s) \leq G(t,s) \leq G(s,s), \quad t \in [0,s]. \quad (16)$$

When $t \in [s,1]$,

$$G'_v(t,s) = \frac{(v-1)}{\Gamma(v)} \left[ \frac{\nu}{\nu-1} (1-s)^{v-2} - (t-s)^{v-2} \right],$$

$$G''_v(t,s) = \frac{1}{\Gamma(v)} (v-1)(2-v)(t-s)^{v-3} \geq 0. \quad (17)$$

Letting $G'_v(t,s) = 0$, we get $t^* = (\frac{\nu}{\nu-1})^{\frac{1}{v-2}} (1-s) + s \in [s,1]$. Combining with (17), for $s \in [0,1]$, we have

$$G(t^*,s) \leq G(t,s) \leq G(s,s), \quad t \in [s,t^*], \quad (18)$$

$$G(t^*,s) \leq G(t,s) \leq G(1,s), \quad t \in [t^*,1]. \quad (19)$$
Inequalities (16), (18), and (19) show us that, for \( s \in [0, 1] \),

\[
\max_{0 \leq t \leq 1} |G(t,s)| = \max \left\{ |G(0,s)|, |G(s,s)|, |G(t^*,s)|, |G(1,s)| \right\}.
\]

Now we prove \( G(0,s), G(s,s), G(t^*,s), \) and \( G(1,s) \) are all nonnegative.

For \( G(0,s) \), we have

\[
G(0,s) = \frac{\delta \gamma(v-1)}{\Gamma(v)} \left( \frac{1}{1-\delta}(\gamma-1)(1-s)^{v-2} - \frac{\delta}{1-\delta}(1-s)^{v-1} \right),
\]

\[
G'_0(s) = \frac{\delta(v-1)(1-s)^{v-3}}{(1-\delta)\Gamma(v)} \left( (1-s) + \frac{\gamma(2-v)}{\gamma-1} \right) \geq 0,
\]

which means that \( G(0,s) \) is increasing for \( s \in [0, 1] \). Considering that \( \gamma < \frac{1}{2-v} \) in case of \( 1 < \gamma \leq 1 + \frac{(v-1)\delta}{2-v} \) and \( 0 < \delta < 1 \), we have, for \( s \in [0, 1] \),

\[
G(0,s) \geq G(0,0) = \frac{\delta}{\Gamma(v)(1-\delta)(\gamma-1)} \left[ 1 - \gamma(2-v) \right] > 0.
\]  \hspace{1cm} (20)

Inequalities (16) and (20) show that \( G(s,s) > 0 \).

For \( G(t^*,s) \), we have

\[
G(t^*,s) = \frac{(1-s)^{v-2}}{\Gamma(v)} \left\{ \frac{\delta \gamma(v-1)}{(\gamma-1)(1-\delta)} + \left( 2-v \right) \left( \frac{\gamma}{\gamma-1} \right)^{\frac{v-1}{v}} + \frac{\delta}{1-\delta} \right\} (s-1)
\]

\[
+ \frac{\gamma(v-1)}{\gamma-1} s
\]

\[
= \frac{(1-s)^{v-2}}{\Gamma(v)} g(s),
\]

where

\[
g(s) = \frac{\delta \gamma(v-1)}{(\gamma-1)(1-\delta)} + \left( 2-v \right) \left( \frac{\gamma}{\gamma-1} \right)^{\frac{v-1}{v}} + \frac{\delta}{1-\delta} (s-1) + \frac{\gamma(v-1)}{\gamma-1} s, \quad s \in [0, 1].
\]

Obviously, \( g(s) \) is increasing on \([0, 1]\), and thus

\[
g(s) \geq g(0) = \frac{\delta \gamma(v-1)}{(\gamma-1)(1-\delta)} - (2-v) \left( \frac{\gamma}{\gamma-1} \right)^{\frac{v-1}{v}} - \frac{\delta}{1-\delta}, \quad s \in [0, 1].
\]  \hspace{1cm} (22)

Let \( k(t) = \frac{\delta(v-1)}{1-\delta} t - (2-v)t^{\frac{v-1}{v}} - \frac{\delta}{1-\delta}, \ t \in [1, +\infty) \). Then

\[
k'(t) = \frac{\delta(v-1)}{1-\delta} + (v-1)t^{\frac{v-2}{v}} \geq 0,
\]  \hspace{1cm} (23)

which means that \( k(t) \) is increasing in \([1, +\infty)\). Letting \( k(t_0) = 0 \), we get

\[
t_0 = \frac{(1-\delta)(2-v)t_0^{\frac{v-1}{v}} + \delta}{\delta(v-1)}
\]

and

\[
t_0 - 1 = \frac{(1-\delta)(2-v)t_0^{\frac{v-1}{v}} + \delta(2-v)}{\delta(v-1)} > 0.
\]
Then
\[
\frac{t_0}{t_0 - 1} = 1 + \frac{1}{t_0 - 1}
= 1 + \frac{(v - 1)\delta}{\delta(2 - v) + (1 - \delta)(2 - v)t_0^{1-v}}
> 1 + \frac{(v - 1)\delta}{2 - v} \geq \gamma,
\]
that is, \(t_0 < \frac{\gamma}{2-\gamma}\). By (23) we obtain
\[
g(0) = k(\frac{\gamma}{2-\gamma}) \geq k(t_0) = 0,\]
and thus
\[
g(s) \geq g(0) \geq 0. \tag{24}
\]
From (21) and (24) it follows that \(G(t^*, s) \geq 0\).

Since \(G(t^*, s) \geq 0\), by (19) it follows that \(G(1, s) \geq 0\)

Above all, we conclude that
\[
G(t, s) \geq 0, \quad (t, s) \in [0, 1] \times [0, 1]
\]
and
\[
\max_{0 \leq t \leq 1} |G(t, s)| = \max \{G(s, s), G(1, s)\}.
\]

Since \(\gamma < \frac{1}{2-\gamma}\) in the case of \(1 < \gamma \leq 1 + \frac{(v-1)\delta}{2-v}\) and \(0 < \delta < 1\), we get
\[
G(s, s) - G(1, s) = \frac{(1 - v)(1-s)^{v-2}}{\Gamma(v)} \left[ \frac{\delta \gamma (1-s) + \gamma s}{(1-\gamma)(1-\delta)} - \frac{\gamma}{(1-\gamma)(1-\delta)} + (1-s)^{v-1} \right]
= \frac{(1-s)^{v-1}}{\Gamma(v)(\gamma-1)} \left[(2-v)\gamma - 1\right] \leq 0,
\]
so
\[
\max_{0 \leq t \leq 1} |G(t, s)| = G(1, s) = \frac{(1-s)^{v-2}}{\Gamma(v)(1-\delta)(\gamma-1)} \left[ \gamma(v-1) - (\gamma-1)(1-s) \right].
\]

(iii) By (ii) we have
\[
\int_0^1 |G(t, s)| \, ds \leq \int_0^1 G(1, s) \, ds
= \int_0^1 \frac{(1-s)^{v-2}}{\Gamma(v)(1-\delta)(\gamma-1)} \left[ \gamma(v-1) - (\gamma-1)(1-s) \right] \, ds
= \frac{\gamma(v-1) + 1}{\Gamma(v+1)(1-\delta)(\gamma-1)}.
\]

**Lemma 2.5** When \(\delta \in (1, +\infty)\) and \(\gamma \in (1, \frac{1}{2-\gamma})\), Green’s function \(G(t, s)\) satisfies the following properties:

(i) \(G(t, s) \leq 0, \quad (t, s) \in [0, 1] \times [0, 1]\);
(ii) For $s \in [0, 1]$, 
\[
\max_{0 \leq t \leq 1} |G(t, s)| 
\leq \frac{(1 - s)^{v-2}}{\Gamma(v)(\delta - 1)(\gamma - 1)} \times \left\{ \delta \gamma (v - 1) - \left[ \delta (\gamma - 1) - \gamma (2 - v)(\delta - 1) \left( \frac{\gamma - 1}{\gamma} \right)^{\frac{1}{\gamma}} \right] (1 - s) \right\};
\]

(iii) \[
\int_0^1 |G(t, s)| \, ds \leq \frac{1}{\Gamma(v)(\delta - 1)(\gamma - 1)} \left\{ \delta [1 + \gamma (v - 1)] + (\delta - 1) \gamma (2 - v)(\frac{\gamma - 1}{\gamma})^{\frac{1}{\gamma}} \right\}.
\]

**Proof** We first prove (i) and (ii). For $s \in [0, 1]$ and $t \in [0, s]$,
\[
G_t'(t, s) = \frac{\gamma (v - 1)(1 - s)^{v-2}}{(\gamma - 1)\Gamma(v)} \geq 0,
\]
which means
\[
G(0, s) \leq G(t, s) \leq G(s, s), \quad t \in [0, s]. \tag{25}
\]

When $t \in [s, 1]$,
\[
G_t'(t, s) = \frac{(v - 1)}{\Gamma(v)} \left[ \frac{\gamma}{\gamma - 1} (1 - s)^{v-2} - (t - s)^{v-2} \right],
\]
\[
G''_t(t, s) = \frac{1}{\Gamma(v)} (v - 1)(2 - v)(t - s)^{v-3} \geq 0.
\]

Letting $G_t'(t, s) = 0$, we get $t^* = (\frac{v - 1}{\gamma - 1})^{\frac{1}{v - 2}} (1 - s) + s \in [s, 1]$. Combining (26), for $s \in [0, 1]$, we have
\[
G(t^*, s) \leq G(t, s) \leq G(s, s), \quad t \in [s, t^*], \tag{27}
\]
\[
G(t^*, s) \leq G(t, s) \leq G(1, s), \quad t \in [t^*, 1]. \tag{28}
\]

Inequalities (25), (27), and (28) show that, for $s \in [0, 1]$,
\[
\max_{0 \leq t \leq 1} |G(t, s)| = \max \left\{ |G(0, s)|, |G(s, s)|, |G(t^*, s)|, |G(1, s)| \right\}.
\]

We now prove that $G(0, s), G(s, s), G(t^*, s), G(1, s)$ are all nonpositive.

For $G(s, s)$, we have
\[
G(s, s) = \frac{(1 - s)^{v-2}}{\Gamma(v)(\delta - 1)(\gamma - 1)} \left\{ [(2 - v)\gamma - 1] \delta + [(v - 1)\delta \gamma - (\gamma - 1)\delta - (v - 1)\gamma] s \right\}
\]
\[
= \frac{(1 - s)^{v-2}}{\Gamma(v)(\delta - 1)(\gamma - 1)} L(s),
\]
where
\[
L(s) = [(2 - v)\gamma - 1] \delta + [(v - 1)\delta \gamma - (\gamma - 1)\delta - (v - 1)\gamma] s, \quad s \in [0, 1].
\]
We have

\[ L(0) = \delta [(2 - v)\gamma - 1] \leq 0, \quad L(1) = -(v - 1)\gamma < 0, \]

which means that \( L(s) \leq 0, s \in [0, 1] \), and therefore

\[ G(s, s) \leq 0, \quad s \in [0, 1]. \]

Inequalities \( G(0, s) \leq 0 \) and \( G(t^*, s) \leq 0 \) follow from (25), (27), and \( G(s, s) \leq 0 \).

For \( G(1, s) \), we have

\[
G(1, s) = \frac{1}{\Gamma(\nu)} \left[ \frac{\gamma(\nu - 1)}{(1 - \delta)(\gamma - 1)} (1 - s)^{\nu - 2} - \frac{1}{1 - \delta} (1 - s)^{\nu - 1} \right],
\]

\[
G'(1, s) = \frac{-(\nu - 1)(1 - s)^{\nu - 3}}{\Gamma(\nu)(\delta - 1)(\gamma - 1)} [(2 - v)\gamma + (\gamma - 1)(1 - s)] \leq 0,
\]

and thus, for \( s \in [0, 1] \),

\[
G(1, s) \leq G(1, 0) = \frac{1}{\Gamma(\nu)(\delta - 1)(\gamma - 1)} [\gamma(2 - v) - 1] \leq 0.
\]

Above all, we get that \( G(t, s) \leq 0 \) and, for \( s \in [0, 1] \),

\[
\max_{0 \leq t \leq 1} |G(t, s)| = \max \{-G(0, s), -G(t^*, s)\}.
\]

We can easily compute that

\[
-G(t^*, s) = \frac{(1 - s)^{\nu - 2}}{\Gamma(\nu)} h_1(s),
\]

where

\[
h_1(s) = \left\{ \frac{\delta[1 - \gamma(2 - v)]}{(\delta - 1)(\gamma - 1)} + (2 - v) \left( \frac{\gamma}{\gamma - 1} \right)^{\frac{\nu - 1}{\nu}} \right\} s,
\]

and

\[
-G(0, s) = \frac{(1 - s)^{\nu - 2}}{\Gamma(\nu)} h_2(s),
\]

where

\[
h_2(s) = \frac{\delta[1 - \gamma(2 - v)]}{(\delta - 1)(\gamma - 1)} + \frac{\delta}{\delta - 1} s.
\]
Obviously,

\[ h_1(0) = \frac{\delta[1 - \gamma(2 - \nu)]}{(\delta - 1)(\gamma - 1)} + (2 - \nu) \left( \frac{\gamma - 1}{\nu - 1} \right)^{\frac{1}{\nu - 2}} > \frac{\delta[1 - \gamma(2 - \nu)]}{(\delta - 1)(\gamma - 1)} = h_2(0) > 0, \]

\[ h_2(1) = \frac{\delta\nu(\nu - 1)}{(\delta - 1)(\gamma - 1)} > \frac{\gamma(\nu - 1)}{(\delta - 1)(\gamma - 1)} = h_1(1) > 0. \]

So, if we make a line \( H(s) \) through \((0, h_1(0))\) and \((1, h_2(1))\), that is,

\[ H(s) = \frac{\delta[1 - \gamma(2 - \nu)]}{(\delta - 1)(\gamma - 1)} + (2 - \nu) \left( \frac{\gamma - 1}{\nu - 1} \right)^{\frac{1}{\nu - 2}} + \left[ \frac{\delta}{\delta - 1} - (2 - \nu) \left( \frac{\gamma - 1}{\nu - 1} \right)^{\frac{1}{\nu - 2}} \right] s, \]

then we have

\[ 0 \leq h_1(s), h_2(s) \leq H(s), \quad s \in [0, 1]. \]

Therefore, for \( s \in [0, 1] \),

\[
\max_{0 \leq t \leq 1} \left| G(t,s) \right| = \max \left\{ -G(0,s), -G(t^*,s) \right\} \\
= \frac{(1-s)^{\nu-2}}{\Gamma(\nu)} \max \{ h_1(s), h_2(s) \} \\
\leq \frac{(1-s)^{\nu-2}}{\Gamma(\nu)} H(s) \\
= \frac{(1-s)^{\nu-2}}{\Gamma(\nu)(\delta - 1)(\gamma - 1)} \left\{ \delta\nu(\nu - 1) - \delta(\nu - 1) - \gamma(2 - \nu)(\delta - 1) \left( \frac{\gamma - 1}{\nu - 1} \right)^{\frac{1}{\nu - 2}} \right\} (1-s),
\]

(iii) easily follows by (ii):

\[
\int_0^1 |G(t,s)| \, ds \\
\leq \int_0^1 \max_{0 \leq t \leq 1} |G(t,s)| \, ds \\
\leq \int_0^1 \frac{(1-s)^{\nu-2}}{\Gamma(\nu)(\delta - 1)(\gamma - 1)} \\
\times \left\{ \delta\nu(\nu - 1) - \delta(\nu - 1) - \gamma(2 - \nu)(\delta - 1) \left( \frac{\gamma - 1}{\nu - 1} \right)^{\frac{1}{\nu - 2}} \right\} (1-s) \, ds \\
= \frac{1}{\Gamma(\nu + 1)(\delta - 1)(\gamma - 1)} \left\{ \delta[1 + \gamma(\nu - 1)] + (2 - \nu)(\delta - 1)\gamma \left( \frac{\gamma - 1}{\nu - 1} \right)^{\frac{1}{\nu - 2}} \right\}. \]

3 Main result

Theorem 3.1 Suppose the boundary value problem (5) has a nonzero solution \( u(t) \).

(i) If \( \delta \in (0,1) \) and \( \gamma \in (0,1) \), then

\[
\int_0^1 (1-s)^{\nu-2} \left[ \gamma(\nu - 1) + (1 - \gamma)(1-s) \right] q(s) \, ds > \Gamma(\nu)(1 - \delta)(1 - \gamma);
\]

\[
\int_0^1 (1-s)^{\nu-2} \left[ \gamma(\nu - 1) + (1 - \gamma)(1-s) \right] q(s) \, ds > \Gamma(\nu)(1 - \delta)(1 - \gamma);
\]
(ii) If $\delta \in (1, +\infty)$ and $\gamma \in (0, 1)$, then
\[
\int_0^1 (1-s)^\gamma \left( \frac{\Gamma(\nu)(\delta-1)(1-\gamma)}{\delta} \right) ds.
\]

(iii) If $\delta \in (0, 1)$ and $\gamma \in (1, 1 + \frac{(\nu-1)\delta}{2-\nu})$, then
\[
\int_0^1 (1-s)^\gamma \left( \frac{\Gamma(\nu)(1-\delta)(\gamma-1)}{\delta} \right) ds.
\]

(iv) if $\delta \in (1, +\infty)$ and $\gamma \in (1, 1 - \frac{2-\nu}{\nu})$, then
\[
\int_0^1 (1-s)^\gamma \left( \frac{\Gamma(\nu)(1-\delta)(\gamma-1)}{\delta} \right) ds.
\]

Proof Let $u(t)$ be a nonzero solution of the boundary value problem (5). By Lemma 2.1 we have
\[
u(t) = \int_0^1 G(t, s) q(s) u(s) ds.
\]

Let $m = \max_{t \in [0, 1]} |u(t)|$. Then
\[
u(t) = \int_0^1 G(t, s) |q(s)| |u(s)| ds \leq m \int_0^1 G(t, s) |q(s)| ds.
\]

Next, since $|G(t, s)||q(s)| \leq \max_{0 \leq t \leq 1} |G(t, s)||q(s)|$, but $|G(t, s)||q(s)| \neq \max_{0 \leq t \leq 1} |G(t, s)| \times |q(s)|$, we have
\[
\int_0^1 G(t, s) |q(s)| ds < \int_0^1 \max_{0 \leq t \leq 1} G(t, s) |q(s)| ds,
\]

which means
\[
|u(t)| < m \int_0^1 \max_{0 \leq t \leq 1} G(t, s) |q(s)| ds,
\]

that is,
\[
1 < \int_0^1 \max_{0 \leq t \leq 1} G(t, s) |q(s)| ds.
\]

By Theorem 3.1 we have the following conclusions.
Theorem 3.2
(i) when \( \delta \in (0, 1) \) and \( \gamma \in (0, 1) \), if
\[
\int_0^1 (1 - s)^{\nu - 2} \left[ \nu (v - 1) + (1 - \gamma)(1 - s) \right] |q(s)| \, ds \leq \Gamma(\nu)(1 - \delta)(1 - \gamma),
\]
then the boundary value problem (5) has no nonzero solution.
(ii) when \( \delta \in (1, +\infty) \) and \( \gamma \in (0, 1) \), if
\[
\int_0^1 (1 - s)^{\nu - 2} \left[ \nu (v - 1) + (1 - \gamma)(1 - s) \right] |q(s)| \, ds \leq \frac{\Gamma(\nu)(\delta - 1)(1 - \gamma)}{\delta},
\]
then the boundary value problem (5) has no nonzero solution.
(iii) when \( \delta \in (0, 1) \) and \( \gamma \in (1, \frac{\nu - 1}{2 - \nu}) \), if
\[
\int_0^1 (1 - s)^{\nu - 2} \left[ \nu (v - 1) - (\gamma - 1)(1 - s) \right] |q(s)| \, ds \leq \Gamma(\nu)(1 - \delta)(\gamma - 1),
\]
then the boundary value problem (5) has no nonzero solution.
(iv) when \( \delta \in (1, +\infty) \) and \( \gamma \in (1, \frac{1}{2 - \nu}) \), if
\[
\int_0^1 (1 - s)^{\nu - 2} \left\{ \delta \nu (v - 1)
\right.
- \left[ \delta (\gamma - 1) - \gamma (2 - \nu)(\delta - 1) \left( \frac{\nu - 1}{\gamma} \right)^{\frac{1}{\gamma - 1}} \right] (1 - s) \right\} |q(s)| \, ds
\leq \Gamma(\nu)(\delta - 1)(\gamma - 1),
\]
then the boundary value problem (5) has no nonzero solution.

Now we consider the existence of solutions to the following nonlinear boundary value problem:
\[
\begin{cases}
C_0^\mathcal{D}_\nu u(t) + f(t, u(t)) = 0, \\
u(0) = \delta u(1), \quad u'(0) = \gamma u'(1).
\end{cases}
\]  
\( (30) \)

**Theorem 3.3** Let \( f : [0, 1] \times \mathbb{R} \to \mathbb{R} \) be continuous and satisfy the following Lipschitz condition with Lipschitz constant \( L \):
\[
|f(t, u_1) - f(t, u_2)| \leq L|u_1 - u_2|
\]  
for all \((t, u_1), (t, u_2) \in [0, 1] \times \mathbb{R} \). If
\[
L < \begin{cases}
\frac{\Gamma(\nu + 1)(1 - \delta)(1 - \gamma)}{\gamma(\nu - 1) + 1}, & \delta \in (0, 1), \gamma \in (0, 1), \\
\frac{\Gamma(\nu + 1)(\delta - 1)(1 - \gamma)}{\delta(\nu + 1 - \gamma)}, & \delta \in (1, +\infty), \gamma \in (0, 1), \\
\frac{\Gamma(\nu + 1)(1 - \delta)(1 - \gamma)}{\gamma(\nu - 1) + 1}, & \delta \in (0, 1), \gamma \in (1, 1 + \frac{\nu - 1}{2 - \nu}), \\
\frac{\Gamma(\nu + 1)(\delta - 1)(1 - \gamma)}{\delta(1 + \gamma - 1) + (\delta - 1)(1 - \gamma)^{\frac{1}{\gamma - 1}}}, & \delta \in (1, +\infty), \gamma \in (1, \frac{1}{2 - \nu}),
\end{cases}
\]  
\( (32) \)
then (30) has a unique solution.
Proof Let $E$ be the Banach space $C[0,1]$ with norm $\| u \| = \max_{t \in [0,1]} |u(t)|$.

By Lemma 2.1, $u \in E$ is a solution of (30) if and only if it satisfies the integral equation

$$u(t) = \int_0^1 G(t,s)f(s,u(s)) \, ds.$$

Define the operator $T : E \to E$ by

$$Tu(t) = \int_0^1 G(t,s)f(s,u(s)) \, ds.$$

Then $T$ is completely continuous. We claim that $T$ has a unique fixed point in $E$. In fact, for any $u_1, u_2 \in E$, we have

$$\| Tu_1(t) - Tu_2(t) \| \leq \int_0^1 |G(t,s)||f(s,u_1(s)) - f(s,u_2(s))| \, ds$$

$$\leq L \int_0^1 |G(t,s)||u_1(s) - u_2(s)| \, ds$$

$$\leq L \int_0^1 |G(t,s)| \, ds \| u_1 - u_2 \|. \tag{33}$$

Substituting of Lemma 2.2(iii), Lemma 2.3(iii), Lemma 2.4(iii), and Lemma 2.5(iii) into (33), we conclude that $T$ is a contraction mapping and thus obtain the desired result. □

4 Conclusion

In this paper, we study a linear fractional differential equation. Firstly, by obtaining the Green’s function we derive a Lyapunov-type inequality for such a boundary value problem. Furthermore, we use the contraction mapping theorem to study the existence of a unique solution for the corresponding nonlinear problem.

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