Fractional Stokes-Einstein and Debye-Stokes-Einstein relations in a network forming liquid

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(Dated: May 7, 2006)

We study the breakdown of the Stokes-Einstein (SE) and Debye-Stokes-Einstein (DSE) relations for translational and rotational motion in a prototypical model of a network-forming liquid, the ST2 model of water. We find that the emergence of “fractional” SE and DSE relations at low temperature is ubiquitous in this system, with exponents that vary little over a range of distinct physical regimes. We also show that the same fractional SE relation is obeyed by both mobile and immobile dynamical heterogeneities of the liquid.

PACS numbers:

The translational ($D$) and rotational ($D_r$) diffusion constants of a macroscopic object in a simple liquid are well-characterized by the Stokes-Einstein (SE) relation, $D \tau / T = c$, and the Debye-Stokes-Einstein (DSE) relation, $D_r \tau / T = c_r$. In these relations, $\tau$ is a relaxation time proportional to the viscosity of the liquid, $T$ is the temperature, and the value of the constants $c$ and $c_r$ depend on the geometry of the object and the boundary conditions. While these relations were originally formulated for the diffusion of macroscopic objects, they are known to hold when the diffusing object is itself a molecule of the liquid [1, 2]. Hence the SE and DSE relations provide a simple connection between mass and momentum transport in a liquid.

As $T$ decreases and the liquid becomes increasingly viscous (assuming crystallization is avoided), numerous experiments have shown a failure of the SE relation for $T \lesssim 1.3 T_g$, where $T_g$ is the glass transition temperature [3, 4, 5, 6, 7, 8]. Specifically the ratio $D \tau / T$ is found to increase by as much as 2 or 3 orders of magnitude on cooling [5]. Translational diffusion is thus said to be enhanced relative to viscosity. When the SE relation fails, it has been empirically found that a “fractional” Stokes-Einstein (F-SE) relation, $D \sim (\tau / T)^{-\xi}$ holds for a wide range of liquids [9, 10, 11], where $0.5 \leq \xi \leq 0.95$, with values between 0.7 and 0.8 being most commonly reported. While a definitive theoretical prediction of the value of $\xi$ has been elusive, various theoretical models predict $\xi$ in the same range [12, 13, 14, 15]. The situation is more complex for the DSE equation. In the same $T$ range where the SE equation fails, the DSE equation has been experimentally found to be valid for most liquids [4, 5]. However, in other experiments [9, 16, 15] a fractional Debye-Stokes-Einstein (F-DSE) relation, $D_r \sim (\tau / T)^{-\xi_r}$, has been observed. Further complicating the situation is the fact that $D_r$ is experimentally inaccessible, and experiments must test the DSE relation using a re-orientational correlation time, which may not be simply related to $D_r$. Molecular dynamics simulations can directly evaluate $D_r$, and simulations of dumbbell molecules show a failure of the DSE relation using $D_r$ [18].

A commonly proposed explanation for the breakdown of the SE relation is the presence of dynamical heterogeneity (DH) [16, 20], i.e. spatially correlated regions of relatively high or low mobility that persist for a finite lifetime in the liquid, and that grow in size as $T$ decreases. The existence of DH has been confirmed and quantified in both experiments [4, 5] and simulations [21]. However, only a few liquid simulation studies directly probe the relationship between DH and the breakdown of the SE relation [22, 23].

In this Letter, we study the breakdown of the SE and DSE relations in simulations of the ST2 model of water [24]. The ST2 model is a tetrahedral arrangement of charges, centered within a Lennard-Jones envelope, that qualitatively reproduces a number of water’s anomalies. However, our goal is not to elucidate the properties of water specifically; indeed there are many water models more accurate than ST2. Rather, we choose this model because it is known to display a diversity of extreme liquid behavior within a single system (Fig. 1). At high density $\rho$, ST2 behaves as a typical fragile glass-forming molecular liquid. However, near $\rho = 0.93$ g/cm$^3$ and $T = 240$ K, a liquid-liquid critical point occurs in the equation of state, demarcating the onset of phase separation between a low density liquid (LDL) and a high density liquid (HDL) phase [25]. At still lower $\rho$, near 0.83 g/cm$^3$, the behavior of ST2 is dominated by the emergence of an increasingly well-structured random tetrahedral network (RTN) as $T$ decreases. The emergence of the RTN is associated with the onset of a fragile-to-strong crossover in the liquid transport properties [24]. We study each of these three distinct regimes, to test how the character of SE and DSE breakdown changes as the behavior of the liquid state itself changes. We also test if the origins of the F-SE relation can be found in the DH of the system.

Unless otherwise indicated, our data are generated

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by molecular dynamic simulations using \( N = 1728 \) molecules with a time step of 1 fs. Simulations consist of equilibration, followed by a production phase, each of which is run for the longer of 100 ps or the time needed for the mean-squared displacement per molecule \( \langle r^2 \rangle \) to reach 1 nm\(^2\) – roughly three molecular diameters. Simulations are carried out in the range 250 K \( \leq T \leq 400 \) K at intervals of 5 K and for \( 0.80 \) g/cm\(^3\) \( \leq \rho \leq 1.20 \) g/cm\(^3\) at intervals of 0.01 g/cm\(^3\). For simplicity of the figures, we only present data along three representative isochores: (i) \( \rho = 0.83 \) g/cm\(^3\), where the RTN structure dominates; (ii) \( \rho = 0.93 \) g/cm\(^3\), approximately the critical density for the liquid-liquid transition; and (iii) \( \rho = 1.03 \) g/cm\(^3\) at which the system behaves more as a simple fragile liquid. The simulations are carried out at fixed \( \rho \) and \( T \), employing the Berendsen heat bath with a time constant of 2 ps. Electrostatic interactions are truncated at 0.78 nm and the energy and pressure are corrected using the reaction field method \[27\].

We evaluate \( D \) from the asymptotic behavior of \( \langle r^2 \rangle = 6Dt \); and \( D_r \) by tracking the mean-squared angular displacement \( \langle \phi^2 \rangle \) of the molecular dipole moment vector and use the asymptotic relation \( \langle \phi^2 \rangle = 4D_r \tau \). The alpha-relaxation time \( \tau \) is defined as the time where the normalized coherent intermediate-scattering function decays to a value of \( 1/e \), evaluated at the closest discrete wave number to \( q = 18 \) nm\(^{-1}\), the approximate location of the first peak in the static structure factor. The dipole relaxation time \( \tau_2 \) is defined as the time at which the correlation function \( \langle P_2[\cos \theta(t)] \rangle \) decays to a value of \( 1/e \), where \( P_2 \) is the second Legendre polynomial and \( \theta(t) \) is the angle defined by the orientations of the dipole moment vector of a molecule at times \( t \) and zero.

To determine in which regions of the phase diagram the SE and DSE relations are valid, we show the behavior of \( D\tau/T \) and \( D_r\tau/T \) in Fig. 2. For \( T \geq 300 \) K, Figs. 2(a) and (b) show that both the SE and DSE ratios are nearly constant, as expected. The growth of the SE relation reflects the expected enhanced translational diffusion. In the same sense, we also find enhanced rotational diffusion; noting that Fig. 2(b) employs a log scale, the enhancement of rotational motion exceeds that of
translational motion. This is consistent with spin-lattice relaxation experiments on water. The \( T \) at which the SE relation breaks down is weakly \( \rho \) dependent, as shown in Fig. 1.

The deviation from the DSE relation can be seen as somewhat surprising since many (but not all) experiments have found that the DSE relation appears to be valid even when the SE relationship fails. In experimental studies, \( D_r \) cannot be directly measured. As a result, the inverse dipole relaxation time \( \tau_2^{-1} \) is frequently used as a substitute, since it can be readily measured. Since we found a deviation from the DSE relation using \( D_r \), we consider whether the using \( \tau_2 \) yields different results, as shown in Fig. 2(c). Curiously, using \( \tau_2 \) we find no \( T \) range where the DSE relation appears valid. For the system we study, the difference found using \( \tau_2 \) versus \( D_r \) may result from large reorientational motions associated with hydrogen bond switching. Whatever the cause, Fig. 2(c) illustrates that substitution of \( D_r \) with \( \tau_2 \) may not, in general, be appropriate.

To test if the SE relation is replaced by the F-SE form at low \( T \), we parametrically plot the relation between \( D \) and \( \tau/T \) in Fig. 3(a). We find that there are two distinct regions: (i) \( T \gtrsim 275 \) where we find an exponent \( \xi = 1 \), consistent with the SE relation, and (ii) lower \( T \) where we find a F-SE relation with an exponent in the commonly observed range \( 0.7 \leq \xi \leq 0.8 \). Moreover, the value of \( \xi \) is nearly independent of \( \rho \). Since the DSE relation is also clearly violated in this system, we check for a F-DSE relation in Fig. 3(b). Indeed, analogous to Fig. 3(a), we find that a high-\( T \) DSE regime is replaced at low \( T \) by a F-DSE relation with \( \xi_r \approx 0.25 \), significantly smaller than \( \xi \) for the F-SE relation. The smaller value of \( \xi_r \) compared to \( \xi \) is also observed in experiments on OTP and is consistent with our finding that the rotational motion is more dramatically enhanced than the translational diffusion. Like \( \xi \), we find that \( \xi_r \) varies weakly with \( \rho \), if at all. Hence, despite the wide variation of liquid state properties over the density range studied here, the quantitative form of the F-SE and F-DSE behaviors is strikingly uniform.

We next analyze the behavior of the ST2 model to test
for a connection between DH and F-SE behavior. We focus on the $\rho = 0.83$ g/cm$^3$ isochore, and conduct 40 independent simulations with $N = 216$ molecules at each $T$ for $255 \leq T \leq 275$ K, at 5 K intervals, again using the criterion that each run continues until $\langle r^2 \rangle = 1$ nm$^2$. The smaller system (compared to $N = 1728$) allows us to probe much longer time scales and to generate an ensemble of many runs in a computationally accessible time. As DH emerges at low $T$, each member of the ensemble at the same $T$ becomes more strongly influenced by transient mobile and immobile regions. As a consequence, some members of the ensemble take a longer time to reach the run-time criterion, and some shorter. Thus, we can study the differences between systems dominated by mobile and immobile DH subdomains without the need to classify individual molecules by their mobility.

It has been hypothesized that the SE violation is due primarily to the presence of transient mobile regions $[4,14,20]$. In this picture, a “background” of largely immobile regions obeys the normal SE behavior, while mobile regions violate the SE relation. If true, we should find that the ensembles members that are dominated by mobile regions should exhibit a prominent F-SE behavior, while the members dominated by immobile regions should tend to follow a normal SE behavior. In this case, a parametric scatter plot of $D$ against $\tau/T$ (Fig. 4) for all ensembles should show a local curvature of the data for each ensemble at a given $T$, such that the low-$D$ envelope of the data reflects SE behavior, and the high-$D$ envelope gives F-SE behavior. This is not what Fig. 4 shows. Rather, we find that all systems, both those that are relatively fast and those that are slow, all obey the same F-SE relation with $\xi \approx 0.77$. Therefore, for the ST2 model of water at $\rho = 0.83$ g/cm$^3$, both the mobile and immobile regions of the liquid deviate in the same way from the SE relation. In this sense, the violation of the SE relation is spatially homogeneous, despite the existence of DH in the system.

We note that $\xi = 0.77$ matches the prediction of the entropic barrier hopping theory of Ref. $[14]$, although Ref. $[14]$ assumes the SE relation is valid over small domains, and that FSE behavior arises from averaging over spatial variation of the domains. However, more recent work by the same authors $[30]$ observes a spatially ubiquitous FSE, as found in this work. Very recently, Ref. $[22]$ studied the impact of heterogeneous dynamics on the SE relation in the hard sphere system by identifying a subset of “hopping”mobile molecules; their results indicated that only the hopping molecules violate the SE relation, though they did not investigate F-SE behavior. It would be useful to conduct a similar analysis of our system, in which hopping can also be expected at low $T$.

Overall, our results demonstrate the remarkable robustness of the F-SE and F-DSE behaviors to variations in both the mean and local molecular environment. The fractional behavior is observed across three distinct physical regimes (fragile, critical, and network-forming), with little variation of the exponents $\xi$ and $\xi_r$, indicating an almost complete insensitivity to changes in the average liquid structure. Even the large static fluctuations associated with the approach to a second order critical point do not observably effect the manifestation of the F-SE and F-DSE relations. We also find no sensitivity of the F-SE behavior on the relative proportion of mobile and immobile dynamical heterogeneities within a given system. Both locally fast and slow regions of the system obey the same F-SE relation, so the phenomenon cannot be attributed to any one extreme subset. We emphasize that our results do not imply that F-SE behavior and DH are disconnected, since the two phenomenon emerge in the same range of $T$ $[31]$, only that the relationship is not the anticipated one. Our analysis also does not indicate a specific alternative origin for F-SE and F-DSE behavior. However, it does suggest that in a successful theory of the phenomena, F-SE and F-DSE behavior should emerge as an intrinsic behavior of the liquid state as $T \rightarrow T_g$, insensitive to the details of the structural and dynamical fluctuations occurring in the system. This is certainly consistent with the nearly ubiquitous observation of SE breakdown in glass forming liquids.

We thank J. Douglas, S. Kumar, D. Leporini, and K. Schweizer for helpful discussion. SRB and FWS thank the NSF for support under grant number DMR-0427239. PHP acknowledges the support of NSERC, CFI, AIF and the CRC Program. We thank G. Lukeman and the StFX lpcLAB for computing resources and support.

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