Asymptotics of Hermite-Padé rational approximants for two analytic functions with separated pairs of branch points (case of genus 0)

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Abstract
We investigate the asymptotic behavior for type II Hermite-Padé approximation to two functions, where each function has two branch points and the pairs of branch points are separated. We give a classification of the cases such that the limiting counting measures for the poles of the Hermite-Padé approximants are described by an algebraic function of order 3 and genus 0. This situation gives rise to a vector-potential equilibrium problem for measures \( \lambda, \mu_1, \) and \( \mu_2, \) and the poles of the common denominator are asymptotically distributed like \( \lambda/2. \) We also work out the strong asymptotics for the corresponding Hermite-Padé approximants by using a \( 3 \times 3 \) Riemann-Hilbert problem that characterizes this Hermite-Padé approximation problem.

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1 Introduction

1.1 Definition of Hermite-Padé approximants and general statement of the problem

Let \( \vec{f} = (f_1, \ldots, f_p) \) be a vector of Laurent series near infinity

\[
f_j(z) = \sum_{k=0}^{\infty} f_{j,k} z^k, \quad j = 1, \ldots, p.
\]

(1.1)

The Hermite-Padé rational approximants (of type II)

\[
\pi_{\vec{n}} = \left( \frac{Q_{\vec{n}}^{(1)}}{P_{\vec{n}}}, \ldots, \frac{Q_{\vec{n}}^{(p)}}{P_{\vec{n}}} \right)
\]

for the vector \( \vec{f} \) and multi-index \( \vec{n} = (n_1, \ldots, n_p) \in \mathbb{N}^p \) are defined by

\[
\text{deg } P_{\vec{n}} \leq |\vec{n}| = n_1 + \cdots + n_p,
\]

\[
f_j(z)P_{\vec{n}}(z) - Q_{\vec{n}}^{(j)}(z) = R_{\vec{n}}^{(j)}(z) = O\left(\frac{1}{z^{n_j+1}}\right), \quad z \to \infty,
\]

(1.2)

where the \( Q_{\vec{n}}^{(j)} \) are polynomials, for \( j = 1, \ldots, p \). This definition is equivalent to a homogeneous linear system of equations for the coefficients of the polynomial \( P_{\vec{n}} \). This system always has a solution, but the solution is not necessarily unique. In the case of uniqueness (up to a multiplicative constant) and in case any non-trivial solution has full degree \( \vec{n} \), the multi-index \( \vec{n} \) is called normal and the polynomial \( P_{\vec{n}} \) can be normalized as monic

\[
P_{\vec{n}}(z) = \prod_{k=1}^{|\vec{n}|} (z - z_{k,\vec{n}}).
\]

The Hermite-Padé approximants \( \pi_{\vec{n}} \) provide the best local (near infinity) simultaneous rational approximation of the vector \( (f_1, \ldots, f_p) \) of Laurent series (1.1). The construction (1.2) was introduced by Hermite [42] in connection with his proof of the transcendence of \( e \). See the papers [53, 62, 19, 9, 4, 73] and the monograph [58] for more details.

In this paper we study the asymptotic behavior of the diagonal Hermite-Padé approximants \( (\vec{n} = (n, n)) \) for two functions \( f_1 \) and \( f_2 \) with branch points at the points \( A_1 = \{a_1, b_1\} \) and \( A_2 = \{a_2, b_2\} \) respectively. We say that

\[
f_j \in \mathcal{A}(\mathbb{C} \setminus A_j), \quad A_j = \{a_j, b_j\}
\]

(1.3)

if the Laurent expansion (1.1) is convergent in a neighborhood of infinity and has an analytic continuation along any path in \( \mathbb{C} \setminus A_j \). A typical example is the function

\[
f_j(z) = \log((z - a_j)/(z - b_j)).
\]

Actually the class of functions we allow is larger and we give a more precise definition in Section 2. We shall assume that the pairs of branch points do not coincide, i.e., \( A_1 \neq A_2 \), although they might have non-empty intersection. Our goal is
To determine the limiting distribution of the zeros of the common denominator \( P_{\vec{n}} \), which are the poles of the Hermite-Padé approximants:

\[
\nu_{P_{\vec{n}}} = \frac{1}{2n} \sum_{k=1}^{2n} \delta(z - z_{k,\vec{n}}) \xrightarrow{\gamma} 0 \quad n \to \infty.
\]  

(1.4)

To obtain asymptotic formulas for the Hermite-Padé polynomials \( P_{\vec{n}} \) and the functions \( R^{(j)}_{\vec{n}} \).

To prove convergence theorems for the Hermite-Padé approximants

\[
\lim_{n \to \infty} \frac{Q^{(j)}_{\vec{n}}(z)}{P_{\vec{n}}(z)} = f_j(z), \quad z \in \Omega_j, \ j = 1, 2,
\]

and to describe the domains of convergence \( \Omega_1 \) and \( \Omega_2 \), depending on the location of the points \( a_1, a_2, b_1, b_2 \in \mathbb{C} \).

For the situation under consideration \((p = 2 \text{ and } \vec{n} = (n,n))\) we will use the notation

\[
\pi_{\vec{n}} = \pi_n, \quad P_{\vec{n}} = P_n, \quad Q^{(j)}_{\vec{n}} = Q^{(j)}_n, \quad R^{(j)}_{\vec{n}} = R^{(j)}_n, \quad j = 1, 2,
\]

and we assume that \( P_{\vec{n}} \) is monic. The rigorous definition of the classes of functions under consideration and the statement of the results of this paper will be presented in the next section. In the following subsections of this introduction we give a brief historical review of the analytic aspects of the Padé and Hermite-Padé approximants in order to introduce some basic notions and problems. Then we conclude the introduction with a general description of the results in this paper.

1.2 Padé approximants (analytic aspect). Motivation for Hermite-Padé analysis

The special case of Hermite-Padé approximation (for \( p = 1 \) i.e., the best local rational approximation to one function near infinity) corresponds to Padé approximation. When the coefficients of the Laurent series are the moments of a positive measure \( \mu \) supported on \( \mathbb{R} \)

\[
f(z) = \sum_{k=0}^{\infty} \frac{c_k}{z^{k+1}} = \int_{\mathbb{R}} \frac{d\mu(x)}{z-x}, \quad c_k = \int_{\mathbb{R}} x^k d\mu(x),
\]

(1.5)

then the denominators \( P_n \) of the Padé approximants are polynomials orthogonal to the powers \( \leq n - 1 \) with respect to \( \mu \):

\[
\int_{\mathbb{R}} P_n(x) x^k d\mu(x) = 0, \quad k = 0, 1, \ldots, n - 1.
\]

(1.6)

In the general case when we are dealing with general coefficients in \( \mathbb{C} \) the orthogonality relations become non-Hermitian or complex.
The analytic theory of Padé approximants for the real case (1.3) is based on remarkable classical results. These include Markov’s theorem [54] on the locally uniform convergence outside the convex hull $\Delta$ of the compact support of $\mu$:

$$\lim_{n \to \infty} \pi_n(z) = \int \frac{d\mu(x)}{z - x}, \quad z \in \overline{C} \setminus \Delta,$$

where $K$ is compact, and the theory of Bernstein and Szegö [13, 72] on the strong asymptotics of orthogonal polynomials satisfying (1.6).

The analytic theory for the complex case, particularly for functions $f \in A(\mathbb{C} \setminus A)$, where $A$ is a finite set of points in $\mathbb{C}$, started to be developed not so long ago. Nuttall [59] has put forward the important relation between the maximal domain of analyticity for the analytic function $f$ and the domain of convergence of the diagonal Padé approximants. The Padé approximants, which are single valued rational functions, approximate a holomorphic branch of the analytic function in the domain of their convergence. At the same time most of the poles of the rational approximants tend to the boundary of the domain of convergence and the support of their limiting distribution models the cuts which make the function $f$ single valued. Nuttall has conjectured (and proved for some important special cases [59, 63, 60, 38]) that these cuts have a minimal logarithmic capacity among all cuts converting the function to a single valued branch. Thus the domain of convergence corresponds to the maximal (in the sense of minimal boundary) domain of holomorphicity for the analytic function $f \in A(\mathbb{C} \setminus A)$. The complete proof of Nuttall’s conjecture (even in a more general setting where the set $A$ has capacity 0) was obtained by Stahl. In a series of papers [66, 67] he proved:

- Existence of a domain $\Omega^*$ such that $f$ is holomorphic in $\Omega^*$ ($f \in H(\Omega^*)$) and the boundary $\Delta = \partial \Omega^*$ has the property

$$\text{cap} \Delta = \min_{\partial \Omega: f \in H(\Omega)} \text{cap} \partial \Omega.$$

- The weak limit of the pole counting measure (1.4)

$$\nu_{P_n} \to \lambda, \quad \text{supp} \lambda = \Delta,$$

and weak asymptotics for the denominators of the Padé approximants

$$\lim_{n \to \infty} \log |P_n(z)| = -V^\lambda(z), \quad z \in \Omega^*,$$

where $V^\lambda$ is the logarithmic potential

$$V^\lambda(z) = \int \log \frac{1}{|z - t|} d\lambda(t)$$

of an extremal measure $\lambda$ minimizing the energy functional

$$I(\lambda) = \iint \log \frac{1}{|x - t|} d\lambda(t) d\lambda(x) = \min_{\text{supp} \mu \subset \Delta, \mu(\Delta) = 1} I(\mu)$$

among all probability measures on $\Delta$. 

5
The extremal measure possesses the equilibrium properties

\[
\begin{cases}
  V^\lambda = \text{const.} \\
  \partial V^\lambda / \partial n = \partial V^\lambda / \partial n_\pm \quad \text{a.e. on } \Delta,
\end{cases}
\] (1.7)

where \( \partial / \partial n_{\pm} \) denotes the normal derivatives on \( \Delta \) (which is a finite union of analytic arcs). The conditions (1.7) characterize the measure \( \lambda \) and its support \( \Delta \). These results lead to Stahl’s main convergence theorem

\[
\lim_{n \to \infty} \pi_n(z) = f(z), \quad z \in \Omega^* \text{ in capacity,}
\]

where the convergence in capacity is defined in the same manner as convergence in measure.

The generalization of these notions and results from Padé approximation to Hermite-Padé approximation is a very difficult and challenging problem. As we will see, the geometry of the domains of convergence and the extremal compact sets where the poles of the Hermite-Padé approximants accumulate is much more diverse and complicated. The analytic techniques capable of proving the asymptotics and convergence results require a significant development in comparison with the methods appropriate for Padé approximation and orthogonal polynomials. These circumstances give a good motivation and direction for the development of the analytic aspect of Hermite-Padé approximation.

1.3 Short survey of asymptotic results for general classes of Hermite-Padé approximants

Perhaps one of the first results on the asymptotics of Hermite-Padé polynomials was obtained by Kalyagin [44]. For a special class of multiple orthogonal polynomials \( P_{n,n}^{\alpha,\beta,\gamma}(x) = x^{2n} + \cdots \) generalizing the Jacobi polynomials:

\[
\int_{\Delta_j} P_{n,n}^{\alpha,\beta,\gamma}(x) x^k w(x) \, dx = 0, \quad k = 0, 1, \ldots, n - 1, \ j = 1, 2,
\]

with \( \Delta_1 = [-1, 0], \ \Delta_2 = [0, 1] \) and weight function

\[
w(x) = |x + 1|^\alpha |x|^\gamma |x - 1|^\beta, \quad x \in \Delta_j, \ j = 1, 2,
\]

and for the corresponding functions of the second kind

\[
R_n^{(j)}(z) = \frac{1}{2\pi i} \int_{\Delta_j} \frac{P_{n,n}^{\alpha,\beta,\gamma}(x)}{x - z} w(x) \, dx, \quad j = 1, 2,
\] (1.8)

he proved the strong asymptotics (Szegő type asymptotics) as \( n \to \infty \):

\[
P_{n,n}^{\alpha,\beta,\gamma}(z) = C_0^{-n} \Phi_0^{-n}(z) [F_0(z) + o(1)], \quad z \in \Omega_0 = \overline{\mathbb{C}} \setminus (\Delta_1 \cup \Delta_2),
\]
\[
R_n^{(j)}(z) = C_j^{-n} \Phi_j^{-n}(z) [F_j(z) + o(1)], \quad z \in \Omega_j = \overline{\mathbb{C}} \setminus \Delta_j, \ j = 1, 2,
\]
and the corresponding formulas on the intervals $\Delta_j$ ($j = 1, 2$), where the convergence is uniform on compact subsets of the indicated domains. Here

$$F_\ell, 1/F_\ell \in H(\Omega_{\ell}), \quad \ell = 0, 1, 2,$$

depend on $\alpha, \beta, \gamma$ and they are analogs of the Szegő function. The main terms $\Phi_0, \Phi_1, \Phi_2$ of the asymptotics are the single valued branches of an algebraic function:

$$\Phi^3(z) + 3\Phi^2(z) + \left(3 - \frac{27}{4}z^2\right)\Phi(z) - 1 = 0,$$

where the branch points are at $A = \{-1, 0, 1\}$. We note that $\Phi$ is independent of $\alpha, \beta, \gamma$ and depends on the supports $\Delta_1, \Delta_2$ of the weight, i.e., on the set $A$. The function $\Phi$ is a rational function on the three-sheeted Riemann surface

$$\mathcal{R}(A) = \mathcal{R}_0 \cup \mathcal{R}_1 \cup \mathcal{R}_2$$

obtained by gluing the sheets $\mathcal{R}_j = \mathbb{C} \setminus \Delta_j$ ($j = 1, 2$) to the sheet $\mathcal{R}_0 = \mathbb{C} \setminus (\Delta_1 \cup \Delta_2)$ so that the upper and lower sides of the cuts on two neighboring sheets are identified. The function $\Phi$ is defined (up to a multiplicative constant) by its divisor (set of poles and zeros),

$$\Phi(z) = \begin{cases} \frac{1}{C_0z^2} + \cdots, & z \to \infty^{(0)}, \\ \frac{z}{C_j} + \cdots, & z \to \infty^{(j)}, \quad j = 1, 2, \end{cases}$$

and the normalization is chosen so that $C_0C_1C_2 = 1$. The Riemann surface $\mathcal{R}$ and rational functions on it play an important role for the asymptotic analysis of Hermite-Padé approximants.

Another important notion of vector potential equilibrium was introduced by Gonchar and Rakhmanov in [39] (see also [40, 41]). Let $\{\Delta_1, \ldots, \Delta_p\}$ be a collection of compact sets in $\mathbb{C}$ and let $D = (d_{i,j})_{i,j=1}^p$ be a real symmetric nonsingular positive definite matrix. An additional condition on $D$ to be compatible with $(\Delta_1, \ldots, \Delta_p)$ is that $d_{i,j} \geq 0$ whenever $\Delta_i \cap \Delta_j \neq \emptyset$. For a vector of measures

$$\vec{\mu} = (\mu_1, \ldots, \mu_p), \quad \text{supp } \mu_j \subset \Delta_j, \quad j = 1, \ldots, p,$$

the energy functional $I(\vec{\mu})$ is defined as

$$I(\vec{\mu}) = \sum_{i=1}^p \sum_{j=1}^p d_{i,j}I(\mu_i, \mu_j),$$

where $I(\mu_i, \mu_j)$ is the mutual energy of two scalar measures

$$I(\mu_i, \mu_j) = \int_{\Delta_i} \int_{\Delta_j} \log \frac{1}{|x-t|} d\mu_i(x) d\mu_j(t).$$

The extremal vector measure $\vec{\lambda}$, minimizing the energy functional (1.11) among all $\vec{\mu}$ where all $\mu_j$ are probability measures possesses the equilibrium properties

$$U_{\gamma_j}(x) = \sum_{i=1}^p d_{ij}V_{\lambda_i}(x) \begin{cases} = \kappa_j, & x \in \text{supp } \lambda_j = \Delta_j^*, \\ \geq \kappa_j, & x \in \Delta_j \setminus \Delta_j^*. \end{cases}$$
Here the vector \( \vec{U} = (U_1^\lambda, \ldots, U_p^\lambda) \) is called the vector potential of the vector valued measure \( \vec{\lambda} \) with respect to the interaction matrix \( D \).

In the paper \[39\] Gonchar and Rakhmanov investigated the Hermite-Padé approximants \((1.2)\) for the system of Markov-type functions \((1.5)\)

\[
f_j(z) = \hat{\mu}_j(z) = \int_{\Delta_j} \frac{d\mu_j(x)}{z-x}, \quad \mu'_j > 0 \text{ on } \Delta_j \subset \mathbb{R}, \quad j = 1, \ldots, p, \tag{1.13}
\]

where \( \Delta_j \) \((j = 1, \ldots, p)\) are non-overlapping intervals

\[
\hat{\Delta}_i \cap \hat{\Delta}_j = \emptyset, \quad i \neq j,
\]

where \( \hat{\Delta} \) denotes the interior of the interval \( \Delta \). They proved the weak asymptotics for the common denominator \( P_{\vec{n}} \) where \( \vec{n} = (n, \ldots, n) \) of the Hermite-Padé approximants as \( n \to \infty \):

\[
\nu_{P_{\vec{n}}} \to \frac{1}{p} \sum_{j=1}^{p} \lambda_j,
\]

\[
\frac{1}{n} \log |P_{\vec{n}}(z)| \to - \sum_{j=1}^{p} V^{\lambda_j}(z), \quad z \in \mathbb{C} \setminus \sum_{j=1}^{p} \Delta_j^*,
\]

where \( \lambda_j \) \((j = 1, \ldots, p)\) are the components of the extremal (equilibrium) vector measure with matrix of interaction

\[
d_{j,j} = 2, \quad d_{i,j} = 1, \quad 1 \leq i \neq j \leq p. \tag{1.15}
\]

The potentials of the components of the equilibrium measure (after normalization) can be harmonically continued through the intervals \( \Delta_j^* \) forming the Riemann surface (as in \((1.9)\) where the index \( j \) runs from 1 to \( p \) and the cuts on the sheets join the endpoints of the supports \( \Delta_j^* \)). This fact had been noticed in \[8\]. Thus the notion of rational function \((1.10)\) on \( \mathbb{R} \) and vector equilibrium problem \((1.12)\) are equivalent and they are related by

\[
\exp(-V^{\lambda_j}(z)) = \left| \frac{\Phi_j(z)}{C_j} \right|, \quad j = 1, \ldots, p, \quad \exp \left( \sum_{j=1}^{p} V^{\lambda_j}(z) \right) = \left| \frac{\Phi_0(z)}{C_0} \right|,
\]

where the normalization constants \( C_1, \ldots, C_p \) and the equilibrium constants \( \kappa_1, \ldots, \kappa_p \) are connected by a linear system of equations. The following convergence theorem was proved in \[39\]:

\[
\lim_{n \to \infty} \frac{Q_{\vec{n}}^{(j)}(z)}{P_{\vec{n}}(z)} = \begin{cases} 
\int_{\Delta_j} \frac{d\mu_j(x)}{z-x}, & z \in \Omega_j^*, \\
\infty, & z \in (\mathbb{C} \setminus \Delta) \setminus \Omega_j^*, \end{cases} \quad j = 1, \ldots, p,
\]

where

\[
\Omega_j^* = \{ z : |\Phi_j(z)| > |\Phi_0(z)| \}.
\]

Note that, in view of \((1.10)\), \( \Omega_j^* \neq \emptyset \). The existence of the non-empty domain \( \mathbb{C} \setminus \Omega_j^* \) depends on the input geometry, i.e., on the size and the location of the \( \Delta_j \) \((j = 1, \ldots, p)\). Thus the results in \[39\] show that there are two new phenomena for the asymptotic behavior of Hermite-Padé approximants as compared to Padé approximants:
1. The components $\Delta_j^*$ of the support of the pole counting measure do not correspond to cuts making the functions $f_j$ ($j = 1, \ldots, p$) holomorphic. We call this the *pushing effect*: it might happen that $\Delta_j \setminus \Delta_j^* \neq \emptyset$ for some $j \in \{1, \ldots, p\}$.

2. The appearance of domains of divergence inside the domain of holomorphicity of $f_j$: it might happen that $(\mathbb{C} \setminus \Delta_j) \setminus \Omega_j^* \neq \emptyset$ for some $j \in \{1, \ldots, p\}$.

These two phenomena are related by

$$\Delta_j \setminus \Delta_j^* \neq \emptyset \nRightarrow (\mathbb{C} \setminus \Delta_j) \setminus \Omega_j^* \neq \emptyset.$$ 

The system (1.13)–(1.14) was introduced in 1919 by Angelesco [2] as a system for which all the multi-indices of the Hermite-Padé approximants are normal, and this system was later rediscovered in [55]. Another system of Markov-type functions (1.5) with normal diagonal multi-indices for the Hermite-Padé approximants was introduced by Nikishin in [56]. A system (1.13) is a Nikishin system of order $p$ if $\Delta_j = \Delta$ for $j = 1, \ldots, p$ and $d\mu_j/d\mu_1$ ($j = 2, \ldots, p$) have analytic continuation from $\Delta$ and form a Nikishin system of order $p - 1$ with respect to another interval $F$ for which $F \cap \Delta = \emptyset$. The asymptotic behavior of the denominators of the Hermite-Padé approximants for a Nikishin system is similar to the behavior of Padé approximants in the sense that the two phenomena for Angelesco systems do not appear (see [56, 57, 20, 34, 68]). However, for a Nikishin system a new effect appears for the functions of the second kind (1.8)

$$R_n^{(j)}(z) = P_n(z)f_j(z) - Q_n^{(j)}(z), \quad j = 2, \ldots, p.$$ (1.16)

They have extra zeros which accumulate on the interval $F$ and are dense on this interval as $n \to \infty$. These are extra interpolation points for the Hermite-Padé approximants, apart from the interpolation condition at $\infty$.

To conclude this survey of the results for the real case, i.e., Hermite-Padé approximants for a vector of Markov type functions, we mention the recent paper [41] on mixed Angelesco-Nikishin systems defined by a graph-tree, and the papers [3, 5] where the strong Szegő-type asymptotics of the Hermite-Padé polynomials for Angelesco systems and Nikishin systems was obtained.

The analytic theory of Hermite-Padé approximants for the complex case has been initiated by Nuttall. In the two pioneering papers [12, 61] of 1981 he obtained some asymptotic formulas for Hermite-Padé approximants to functions with separated complex branch points [12] (a complex analog of an Angelesco system) and to functions meromorphic on the same Riemann surface [61] (i.e., functions with the same set of branch points, like a Nikishin system for the real case). The results of [12] were verified by some heuristic considerations and numerical experiments, and the paper [61] contains rigorous theorems. In his fundamental work [62] of 1984, Nuttall made an attempt to formulate a general conjecture about the asymptotic behavior, as $n \to \infty$, of the diagonal $\vec{n} = (n, n, \ldots, n)$ Hermite-Padé polynomials. On the basis of his conjecture lies a $(p + 1)$-sheeted Riemann surface like (1.9) which depends on the set of $p$ functions which are being approximated. He showed how to determine this Riemann surface for some special classes of functions, but the general case was left as an open problem. Nevertheless, assuming the existence of the appropriate Riemann surface $\mathfrak{R}$, he conjectured that the strong asymptotics can
be described by solutions of some boundary value problem on $\mathcal{R}$. For the main term of the asymptotics one would have

$$|P_n(z)|^{1/n} \to |\Phi^{-1}(z)| = \exp \left( -\operatorname{Re} G(z) \right), \quad n \to \infty,$$

(1.17)

where $G$ is an Abelian integral of the third kind with logarithmic poles at $\infty^{(\ell)}$, $(\ell = 0,1,\ldots,p)$ with residues

$$G(z) = \begin{cases} 
-p \log z + \mathcal{O}(1), & z \to \infty^{(0)}, \\
\log z + \mathcal{O}(1), & z \to \infty^{(j)}, \quad j = 1,\ldots,p,
\end{cases}$$

(1.18)

and elsewhere $G$ is analytic in the local variable. If the genus of $\mathcal{R}$ is greater than zero, then an additional condition on $G$ is imposed: all the periods of $G$ are purely imaginary. Such a function is unique up to an additive constant and $\operatorname{Re} G(z)$ is a single valued function on $\mathcal{R}$ (see, for example, [68]). We note that the condition $\operatorname{genus}(\mathcal{R}) = 0$ implies single-valuedness of $\Phi$ in (1.17), therefore $\Phi$ is a rational function on $\mathcal{R}$, uniquely (up to a multiplicative normalization) defined on $\mathcal{R}$ by its divisor (1.10).

After Nuttall’s results there were practically no other rigorous results for the complex case of Hermite-Padé approximants. One of the reasons was the absence of suitable techniques for the analysis of strong asymptotics of non-Hermitian (complex) orthogonal polynomials which could be adopted to Hermite-Padé approximation. There was recently substantial progress in proving new results for the strong asymptotics of orthogonal polynomials by means of a matrix-valued Riemann-Hilbert method. The method is based on the reformulation of the definition (1.6) of the orthogonal polynomials in terms of a $2 \times 2$ matrix-valued Riemann-Hilbert problem (due to Fokas, Its, and Kitaev [37, 36]) and the steepest descent analysis of this Riemann-Hilbert problem for $n \to \infty$ (due to Deift and Zhou [31]). This method was initially designed to study asymptotics for integrable PDEs and was later applied to prove asymptotic results for polynomials orthogonal on the real axis with respect to real valued analytic weights, including varying weights (depending on $n$) [14, 26, 27, 28, 29, 30, 46, 47] and related questions from random matrix theory. It has later been noticed [6, 14, 43] that the method also works for the non-Hermitian orthogonality in the complex plane with respect to complex valued weights. In [74] multiple orthogonality for Hermite-Padé polynomials was reformulated in terms of a $(p+1) \times (p+1)$ matrix-valued Riemann-Hilbert problem.

In this survey subsection we introduced the main players for the asymptotics of the complex case of Hermite-Padé approximation in the historical order of their appearance on the scene. They are

- An appropriate $(p+1)$-sheeted Riemann surface $\mathcal{R}$.
- Vector potential equilibrium problems.
- A standard Abelian integral (1.18) and rational functions on $\mathcal{R}$.
- Analysis of a $(p+1) \times (p+1)$ matrix-valued Riemann-Hilbert problem as a method for proving the asymptotic results.

1except for the special case of Hermite-Padé approximation to the $e^z$ [18, 33, 35, 48, 49, 50, 69, 70, 71, 75, 76].
Concluding this survey of results which have important and direct influence on this paper, we would like to mention the papers [6, 7, 10, 16, 17, 25, 50, 51, 52] which have been written during the work on the present paper and in which some of the ideas and methods elaborated here have already been implemented.

1.4 General description of the limiting behavior of Hermite-Padé approximants

The analysis of numerical computations of the zeros of the polynomials $P_{n}$, which are the common denominators for the Hermite-Padé approximants to functions of the class $\textbf{I.3}$, shows that the support for the limiting zero distribution has a different geometry, depending on the position of the branch points. In Figures 1.1–1.5 we present the results of the computations for the functions

$$f_{j}(z) = \log \left( \frac{z - a_{j}}{z - b_{j}} \right), \quad j = 1, 2.$$ 

There are several typical patterns for the support. We describe those that are related to a Riemann surface of genus zero.

**Case I.** When the pairs $\{a_{1}, b_{1}\}$ and $\{a_{2}, b_{2}\}$ are ‘far away’ from each other, the zeros of $P_{n}$ accumulate on two disjoint arcs $\Delta_{1}$ and $\Delta_{2}$, which are joining the branch points $a_{1}$ and $b_{1}$ for the function $f_{1}$ and the branch points $a_{2}$ and $b_{2}$ of the function $f_{2}$, respectively (Figure 1.1). Each of the arcs $\Delta_{1}$ and $\Delta_{2}$ accumulates half of the zeros of $P_{n}$.

![Figure 1.1: $a_{1} = 0$, $b_{1} = i$, $a_{2} = 2$, $b_{2} = (3 + 3i)/2$ (case I). The zeros of $P_{n,n}$ accumulate on two disjoint arcs $\Delta_{1}$ and $\Delta_{2}$ that connect the branch points. The zeros of $P_{20,20}$ are indicated by $\circ$.](image-url)
Case II. When the pairs \( \{a_1, b_1\} \) and \( \{a_2, b_2\} \) are closer to each other, then it may happen that the zeros of \( P_{n,n} \) accumulate on a set \( \Delta \) that connects all four branch points \( a_1, b_1, a_2 \) and \( b_2 \) as shown in Figure 1.2. In this case the support can not be split into two separate pieces of equal mass.

Figure 1.2: \( a_1 = i, b_1 = 0, a_2 = 1/2 + i, b_2 = 1/2 \), (case II). The zeros of \( P_{n,n} \) accumulate on a contour \( \Delta \) that connects all four branch points. The zeros of \( P_{20,20} \) are indicated by \( \circ \).

Case III. For certain configurations of pairs \( \{a_1, b_1\} \) and \( \{a_2, b_2\} \), the zeros of \( P_{n,n} \) accumulate on a set \( \Delta_0 \) that consists of two disjoint arcs \( \Delta_1^* \) and \( \Delta_2 \) (as in case I), but contrary to the case I, the arcs are not joining all branch points, see Figure 1.3. One of the branch points \( (b_1 \text{ in Figure 1.3}) \) does not belong to \( \Delta_0 \). The arc \( \Delta_1^* \) connects \( a_1 \) with a point \( b^* \) which is different from \( b_1 \). The other arc \( \Delta_2 \) connects \( a_2 \) with \( b_2 \). Both arcs accumulate half of the zeros of \( P_{n,n} \).

Figure 1.3: \( a_1 = 0, b_1 = 15/8, a_2 = 2 + i/4, b_2 = 2 - i/8 \) (case III). The zeros of \( P_{n,n} \) accumulate on two disjoint arcs \( \Delta_1^* \) and \( \Delta_2 \). The branch point \( b_1 \) is not contained in \( \Delta_1^* \). The zeros of \( P_{20,20} \) are indicated by \( \circ \).
The above three cases concern situations where the four branch points \(a_j, b_j\) are all distinct. The final two cases deal with situations where \(\{a_1, b_1\}\) and \(\{a_2, b_2\}\) have a common point, and we take it so that
\[
b_1 = b_2 = b.
\]

**Case IV.** The pairs \(\{a_1, b\}\) and \(\{a_2, b\}\) are such that the zeros of \(P_{\vec{n}}\) accumulate on a set \(\Delta\) that connects all three branch points.

**Case V.** Certain positions of the pairs \(\{a_1, b\}\) and \(\{a_2, b\}\) are such that the zeros of \(P_{\vec{n}}\) accumulate on a set \(\Delta_0\) that consists of two disjoint arcs \(\Delta_1^*\) and \(\Delta_2\) as in case III. Thus, \(\Delta_2\) connects \(a_2\) with \(b\), but the other arc \(\Delta_1^*\) connects \(a_1\) with a point \(b^*\) which is different from \(b\), and which does not belong to \(\Delta_2\). Some of this situations can be realized as limiting cases of case III where the point \(b_1\) tends to \(b_2\). These cases are not part of the case V, but rather belong to case III. The case V contains the situations that cannot be realized as limiting cases of case III.

There are also ‘higher genus cases’ such as the one shown in Figure 1.6. Here the zeros of \(P_{\vec{n}}\) accumulate on a set \(\Delta\) consisting of three disjoint arcs. Each of the branch points is contained in \(\Delta\), but only two of them (in the figure it is \(a_2\) and \(b_2\)) are on the same arc \(\Delta_2\). This arc accumulates half of the zeros of \(P_{\vec{n}}\). The other half are on the two remaining arcs. We will not treat the higher genus cases in this paper.
Figure 1.6: $a_1 = i, b_1 = -1, a_2 = 1 + i/2, b_2 = 1 - i/2$ (higher genus case).

There are also critical cases when there is a transition from one case to another case, e.g., the critical case where the two arcs $\Delta_1$ and $\Delta_2$ have one common point, which is the transition from case I to case II.

The cases I and III present a complex generalization of an Angelesco system (1.13)–(1.14). As we will show, the limiting distribution of the poles of the Hermite-Padé approximants for these cases is the equilibrium measure of the vector potential extremal problem (1.11)–(1.12) with the Angelesco interaction matrix (1.15), and the support $\Delta := \Delta_1 \cup \Delta_2$ of this measure is characterized by a vector analog of Stahl’s symmetry property

$$\frac{\partial U_j^x}{\partial n_+} = \frac{\partial U_j^x}{\partial n_-} \text{ on } \Delta_j, \quad j = 1, 2.$$ 

The cases II, IV and V have features of a Nikishin system because of the presence of the common piece and we cannot separate $\Delta$ into two disjoint pieces. For these cases a new curve $E$ appears on which the finite zeros of the function of the second kind (i.e., extra interpolation points) accumulate and an extremal vector potential problem describes both the limiting distribution of the poles of the Hermite-Padé approximants and the extra interpolation points.

The three-sheeted Riemann surface which is appropriate for the cases I, III and V has four branch points at the end points of the components of the support of the vector equilibrium measure (for case I they are the branch points of the functions which are being approximated). Thus this Riemann surface has genus zero. The three-sheeted Riemann surfaces for the cases II and IV have a more complicated sheet structure but their genus is also zero.
1.5 Objectives, structure, main results, and tools of the paper

In this paper we prove the asymptotic formulas and the convergence theorem of the Hermite-Padé approximants for the cases when the appropriate Riemann surface is of genus zero. In a forthcoming paper we plan to describe the cases when the appropriate Riemann surfaces are of genus one and two. In this paper we will handle cases I, II, III, IV, V that give rise to a genus zero Riemann surface as described above.

Rigorous definitions of the class of functions which we are approximating corresponding to the geometrical cases and statements of the results will be presented in the next section. The rest of the paper is devoted to the proofs of the theorems. Here we present a general description of the obtained results.

1.5.1 Geometry of the problem

Our approach to the geometry of the problem, i.e., finding the appropriate Riemann surface, classification of the geometrical cases, etc., is based on the algebraic function $h$ defined by

$$h^3(z) - 3 \frac{P_2(z)}{\Pi_4(z)} h(z) + 2 \frac{P_1(z)}{\Pi_4(z)} = 0,$$

(1.19)

where the polynomial $\Pi_4$ is defined by the input parameters (the branch points of the functions we are approximating)

$$\Pi_4(z) = (z - a_1)(z - b_1)(z - a_2)(z - b_2),$$

and the polynomials $P_2$ and $P_1$ contain two parameters

$$P_2(z) = z^2 - d_1 z + d_2,$n
$$P_1(z) = z - c,$n
$$d_1 = \frac{2c + a_1 + b_1 + a_2 + b_2}{3},$$

(1.20)

which can be determined by the input data $a_1, a_2, b_1, b_2$, using the information about the geometrical case to which the input data belongs. The function $h$ has three branches $h_0, h_1, h_2$ which we fix at infinity as

$$\begin{cases} h_0(z) = \frac{2}{z} + \cdots, \\ h_j(z) = \frac{1}{z} + \cdots, & j = 1, 2. \end{cases}$$

(1.21)

The function $h$ is a rational function on its Riemann surface $\mathcal{R}$. It has

- **four poles** at the points on $\mathcal{R}$ for which the projection $\pi : \mathcal{R} \to \overline{\mathbb{C}}$ are the points $\{a_1, a_2, b_1, b_2\}$:

$$h(\xi) = \infty \quad \Rightarrow \quad \xi \in \mathcal{R} : \pi(\xi) \in \{a_1, a_2, b_1, b_2\};$$

(1.22)

- **three zeros** at infinity on each sheet and **one zero** at the point for which the projection is the parameter $c$ from (1.20):

$$h(\xi) = 0 \quad \Rightarrow \quad \begin{cases} \xi = \infty^{(i)}, & i = 0, 1, 2, \\ \xi \in \mathcal{R} : \pi(\xi) = c. \end{cases}$$

(1.23)
If the zero at \(c\) cancels a pole from \(\{a_1, a_2, b_1, b_2\}\), for example when \(c = b_1\), then the equation (1.19) for the algebraic function \(h\) reduces to
\[
h^3(z) - \frac{3(z - d)}{(z - a_1)(z - a_2)(z - b_2)} h(z) + \frac{2}{(z - a_1)(z - a_2)(z - b_2)} = 0, \tag{1.24}
\]
where
\[
d = \frac{a_1 + a_2 + b_2}{3}.
\]
The discriminant \(D\) of the equation (1.24) is
\[
D = \frac{(z - d)^3 - (z - a_1)(z - a_2)(z - b_2)}{(z - a_1)(z - a_2)(z - b_2)},
\]
hence the function \(h\) from (1.24) has branch points at \(a_1, a_2, b_2\) (the poles of \(h\)) and at the point
\[
b^* = \frac{a_1a_2b_2 - d^3}{a_1a_2 + a_1b_2 + a_2b_2 - 3d^2}. \tag{1.25}
\]
The Riemann surface corresponding to equation (1.24) has genus zero. This Riemann surface was considered for the first time by Nuttall [62, 12] by means of another equation, and independently by Kalyagin [44, 8]. In our work this Riemann surface will be an appropriate \(R\) for the geometrical cases III and V (for Case V the functions we are approximating have a joint branch point, i.e., \(b_1 = b_2 = b\) in (1.3)).

For the geometrical Case I the appropriate Riemann surface is defined by the general equation (1.19) where the two unknown parameters in (1.20) are determined by the two conditions

1. the genus of \(R\) is zero,
2. the monodromy condition
\[
h_0 \in H(C \setminus (\tilde{\Delta}_1 \cup \tilde{\Delta}_2)), \quad h_j \in H(C \setminus \tilde{\Delta}_j), \quad j = 1, 2, \quad \tilde{\Delta}_1 \cap \tilde{\Delta}_2 = \emptyset.
\]

Here \(\tilde{\Delta}_j\) is an arbitrary Jordan arc joining the points \(a_j\) and \(b_j\) \((j = 1, 2)\). The discriminant \(D\) of the equation (1.19) is
\[
D = \frac{\tilde{D}}{\Pi_4^3}, \quad \tilde{D} = P_2^3 - \Pi_4P_1^2,
\]
where the polynomial \(\tilde{D}\) has degree 4 due to (1.20). Condition 1 above implies that
\[
\tilde{D}(z) = \text{const} (z - z_1)^2(z - z_2)^2, \tag{1.26}
\]
which gives a system of two algebraic equations (of high order) for the determination of the two unknown parameters in (1.20). This system of algebraic equations has several solutions with different monodromy properties, for example, \(h_0 \in H(C \setminus (\tilde{\Delta}_1 \cup \tilde{\Delta}_2)), \quad h_1 \in H(C \setminus (\tilde{\Delta}_1 \cup \tilde{\Delta}_2)), \quad h_2 \in H(C \setminus (\tilde{\Delta}_1 \cup \tilde{\Delta}_1))\), hence condition 2 above chooses the right solution. We will not work with this system of algebraic equations because it is too cumbersome. Instead we introduce in the next section a substitution for (1.20) which automatically fulfills (1.26).

Thus the geometrical part of the problem consists of the determination of the algebraic function \(h\) starting from the input branch points \(A_j = \{a_j, b_j\}\), \((j = 1, 2)\). As we already mentioned above, we characterize in this paper the position of \(A_1\) and \(A_2\) which guarantee that the algebraic curve (1.19) is of genus zero.
1.5.2 Standard functions for the asymptotics

Since the geometrical analysis gives us the algebraic function \( h \) and its Riemann surface \( \mathcal{R} \), we can define the functions which allow us to state the asymptotical results. The Abelian integral, see (1.18), is defined as

\[
G(\xi) = \int_{a_1}^{\xi} h(z) \, dz, \quad \xi \in \mathcal{R},
\]

and the function

\[
\Phi(\xi) = \exp G(\xi), \quad \xi \in \mathcal{R}
\]

is a single valued (rational) function on \( \mathcal{R} \), which is a consequence of the fact that the genus of \( \mathcal{R} \) is zero. The local selection of the branches of the algebraic function \( h \) in (1.21) gives us the local definition of the branches for the algebraic function \( \Phi \) (see (1.10)):

\[
\Phi_0(z) = \frac{1}{C_0} z^2 + \cdots, \quad \Phi_j(z) = \frac{z}{C_j} + \cdots, \quad j = 1, 2, \quad z \to \infty,
\]

\[
C_1 C_2 C_0 = 1, \quad C_1 > 0.
\]

Using the union of the analytic curves defined by

\[
\Gamma = \{ z \in \mathbb{C} : |\Phi_\ell(z)| = |\Phi_k(z)|, \text{ for some } 0 \leq \ell < k \leq 2 \},
\]

we define the holomorphic branches of \( \Phi \) (and respectively \( h \)) globally:

\[
\Phi_\ell \in H(\mathbb{C} \setminus \gamma_\ell), \quad \gamma_\ell \subset \Gamma, \quad \ell = 0, 1, 2.
\]

Details will be given in the next section.

Thus the appropriate Riemann surface \( \mathcal{R} = \mathcal{R}_0 \cup \mathcal{R}_1 \cup \mathcal{R}_2 \) can be realized as three sheets of the extended complex plane cut along the contours \( \gamma_\ell \):

\[
\mathcal{R}_\ell = \mathbb{C} \setminus \gamma_\ell,
\]

and pasted through

\[
\mathcal{R}_\ell \cap \mathcal{R}_k = \gamma_{\ell,k}, \quad \ell \neq k, \quad \ell, k = 0, 1, 2,
\]

so that the upper and the lower sides of the cuts on two neighboring sheets are identified.

1.5.3 Asymptotic results and convergence

Using the global definition of the branches of the standard algebraic functions \( h \) and \( \Phi \) (depending on the geometrical case) we can sketch our asymptotic results. The limiting distribution (1.24) of the zeros of the diagonal \( \vec{n} = (n, n) \) Hermite-Padé denominator \( P_{\vec{n}} \) and a strong asymptotic formula can be written in a unique way for all geometrical cases under consideration. In the present paper we prove that

\[
\nu_{P_{\vec{n}}} \to \frac{1}{2} \lambda, \quad n \to \infty,
\]

where the real valued measure \( \lambda \), which is of total mass two, is given by

\[
d\lambda(z) = \frac{1}{2\pi i} [(h_0)_+ - (h_0)_-] \, dz, \quad z \in \gamma_0,
\]

(1.29)
where the subscript + or − as usual denotes the limiting value of the function taken from the left or right, respectively, when traversing the contour according to the orientation determined by the complex line element $dz$. The strong asymptotics for $P_{\vec{n}}$ is

$$P_{\vec{n}}(z) = C_{0}^{-n} \Phi_{0}^{-n}(z) (F_{0}(z) + O(1/n)), \quad z \in \overline{\mathbb{C}} \setminus \gamma_{0},$$

where $F_{0} \in H(\overline{\mathbb{C}} \setminus \gamma_{0})$ is an analog of the Szegö function defined by mean of a certain boundary value problem (see details in the next section).

The asymptotic formulas for the functions of the second kind (1.8) and (1.16) essentially depend on the geometrical case. For the cases I and III we prove that

$$R_{\vec{n}}^{(j)}(z) = C_{j}^{-n} \Phi_{j}^{-n}(F_{j}(z) + O(1/n)), \quad z \in \overline{\mathbb{C}} \setminus \gamma_{j}, \quad j = 1, 2, \quad (1.30)$$

for certain functions $F_{j} \in H(\overline{\mathbb{C}} \setminus \gamma_{j}) (j = 1, 2)$. For the case II, IV and V the answer for the functions of the second kind is more involved. A general feature for these cases is that the asymptotic formula (1.30) remains valid in a neighborhood of $\infty$ which is smaller than $\overline{\mathbb{C}} \setminus \gamma_{j}, j = 1, 2$, and around the $\gamma_{j} (j = 1, 2)$ there may appear a domain where one of the main terms of the asymptotics of $\{\Phi_{1}, \Phi_{2}\}$ changes from one to the other, so as a result the functions of the second kind on the boundary of this domain have an oscillatory asymptotic behavior which leads to an accumulation of its zeros there.

From (1.16) we have that

$$f_{j}(z) - \frac{Q_{\vec{n}}^{(j)}(z)}{P_{\vec{n}}(z)} = \frac{R_{\vec{n}}^{(j)}(z)}{P_{\vec{n}}(z)},$$

hence our asymptotic results give a complete picture of the convergence of the Hermite-Padé approximants with a description of the possible regions of divergence and the sets of the accumulation points of the extra interpolation points.

### 1.5.4 Tools

To prove the asymptotic results we start with a $3 \times 3$ matrix-valued Riemann-Hilbert boundary value problem characterizing Hermite-Padé approximants for two functions, which was proposed in [74]. Then, using the information about the geometry of the problem, we develop the steepest descent method of Deift and Zhou [31] which was already successfully used for $2 \times 2$ matrix-valued Riemann-Hilbert problems.

We can say that for the geometrical cases I and III the jump matrices in the initial Riemann-Hilbert problem have a block structure and therefore most of the steps of the asymptotic analysis of the solution of the $3 \times 3$ matrix-valued Riemann-Hilbert problem can be reduced to the $2 \times 2$ problem, which has already been developed. As a new feature we like to mention the procedure of finding the explicit solutions for the matrix Riemann-Hilbert problem with non-varying jumps (i.e., jumps which do not depend on $n$), reducing the matrix problem to a boundary value problem on the corresponding Riemann surface.

For the geometrical cases II, IV and V the jump matrices of the Riemann-Hilbert problem do not possess this block structure. We introduce a new decomposition of the jump matrix to block structure jump matrices (which can be treated by the traditional local decomposition) and a jump matrix with exponentially growing non-diagonal terms. Nevertheless, due to the analyticity of the solution of the Riemann-Hilbert problem, the contour on which this growing jump occurs can be moved in the domain where the growing
terms of the jump become exponentially decaying. We call this new decomposition a *global opening of the lenses*. This new procedure brings about new curves which initially have not been present in the statement of the problem. For example, we will discover the analytic curves on which the extra interpolation points accumulate. We also like to mention a wonderful picture: when the contours of the jumps make their global movement they may meet each other or one may pass through another and the corresponding jumps interact: we can see interference, transparent penetration or even annihilation.

## 2 Rigorous definitions and statements of the results

### 2.1 Class of functions and reformulation of the Hermite-Padé approximation problem

Now we will be more precise in the definition of the class of functions (1.3) we are approximating and we state a matrix Riemann-Hilbert problem as a reformulation of the Hermite-Padé approximation problem.

Let $a$ and $b$ be points in the complex plane and let $\Delta$ be a Jordan rectifiable arc joining $a$ and $b$:

$$a, b \in \mathbb{C}, \quad \Delta = \{\Delta(t) : t \in [0, 1]\}, \quad \Delta(0) = a, \Delta(1) = b. \quad (2.1)$$

We will consider functions $f$ of the form

$$f(z) = \frac{1}{2\pi i} \int_{\Delta} \frac{w(\xi)}{\xi - z} d\xi, \quad (2.2)$$

where $w$ is some “nice” function on $\Delta$ as specialized in the following definition. Note that

$$w(\xi) = f_+(\xi) - f_-(\xi), \quad \xi \in \Delta, \quad (2.3)$$

where $f_{\pm}$ denote the boundary values of $f$ from the left and right using the orientation on $\Delta$ from $a$ and $b$.

**Definition 2.1.** Let $\alpha, \beta > -1$ and let $\Omega$ be a domain containing the arc $\Delta$ joining $a$ and $b$. Then we say that the function $f$ given in (2.2) belongs to the class $\mathcal{A}(a, \alpha; b, \beta; \Omega)$, if the function $w$ given in (2.3) satisfies

$$\begin{cases} 
  w(\xi) = w_0(\xi)(a - \xi)^\alpha(b - \xi)^\beta, & \alpha, \beta > -1, \\
  w_0, 1/w_0 \in H(\Omega). 
\end{cases} \quad (2.5)$$

Here we fix a holomorphic branch on $\Delta \subset \Omega$ of $(a - z)^\alpha(z - b)^\beta$. Thus a function $f \in \mathcal{A}((a, \alpha; b, \beta; \Omega)$ has analytic continuation across the arc $\Delta$ into the domain $\Omega$ situated on the next sheet of the Riemann surface of $f$. Moreover in the representation (2.2) of

---

2The phenomenon of global opening of lenses was first discovered during the research leading to this paper. It turns out to be a general feature for higher order RH problems and was also used in the papers [7, 25].
For functions $f_1$ and $f_2$ satisfying (2.2)–(2.3), the definition of the Hermite-Padé approximants (1.2) is equivalent (by Cauchy’s theorem) to the determination of the multiple orthogonal polynomial $P_{n_1,n_2}$ which satisfies

$$
\int_{\Delta_j} P_{n_1,n_2}(\xi) \xi^k w_j(\xi) \, d\xi = 0, \quad k = 0, 1, \ldots, n_j - 1, \quad j = 1, 2,
$$

and the functions of the second kind are given by

$$
R^{(j)}_{n_1,n_2}(z) = \frac{1}{2\pi i} \int_{\Delta_j} P_{n_1,n_2}(\xi) w_j(\xi) \xi^{-z} \, d\xi, \quad j = 1, 2.
$$

Therefore the error of the approximation is

$$
f_j(z) - Q^{(j)}_{n_1,n_2}(z) = \frac{R^{(j)}_{n_1,n_2}(z)}{P_{n_1,n_2}(z)}. \quad (2.8)
$$

We note that the right hand side of (2.8) is independent of the normalization of $P_{n_1,n_2}$. We will fix the normalization of $P_{n_1,n_2}$ by choosing it to be a monic polynomial of degree $n_1 + n_2$. Then

$$
P_{n_1,n_2}(z) = z^{n_1+n_2} + \ldots, \quad (2.9)
$$

and

$$
\begin{cases}
R^{(j)}_{n_1,n_2}(z) = \frac{1 + O(1/z)}{m^{(j)}_{n_1,n_2} z^{n_j}} z \to \infty \\
\frac{1}{m^{(j)}_{n_1,n_2}} = -\frac{1}{2\pi i} \int_{\Delta_j} P_{n_1,n_2}(\xi) \xi^{n_j} w_j(\xi) \, d\xi.
\end{cases} \quad (2.10)
$$

However, the existence of $P_{n_1,n_2}$ satisfying (2.6) and (2.9) is not guaranteed: we recall from the introduction that this is only guaranteed for normal multi-indices.

For normal multi-indices the Hermite-Padé polynomials $P_{n_1,n_2}$ and the functions of the second kind (2.7) can also be defined by means of a matrix-valued Riemann-Hilbert problem [74]. We state the problem for the diagonal multi-indices (the case of our interest)

$$
\vec{n} = (n, n), \quad \vec{n}_1 = (n - 1, n), \quad \vec{n}_2 = (n, n - 1). \quad (2.11)
$$

If $\vec{n}, \vec{n}_1, \vec{n}_2$ are normal indices, then the $3 \times 3$ matrix-valued analytic function

$$
Y(z) = \begin{pmatrix}
P_{\vec{n}}(z) & R_{\vec{n}}^{(1)}(z) & R_{\vec{n}}^{(2)}(z) \\
m_1 P_{\vec{n}_1}(z) & m_1 R_{\vec{n}_1}^{(1)}(z) & m_1 R_{\vec{n}_1}^{(2)}(z) \\
m_2 P_{\vec{n}_2}(z) & m_2 R_{\vec{n}_2}^{(1)}(z) & m_2 R_{\vec{n}_2}^{(2)}(z)
\end{pmatrix}, \quad z \in \mathbb{C} \setminus \Delta_0, \quad \Delta_0 = \Delta_1 \cup \Delta_2, \quad (2.12)
$$

with $m_1 = m^{(1)}_{\vec{n}_1}$ and $m_2 = m^{(2)}_{\vec{n}_2}$ (see (2.10)), is a solution of the following Riemann-Hilbert
problem:

1. $Y \in H^{3 \times 3}(\mathbb{C} \setminus \Delta_0)$ with boundary values $Y_{\pm}$ on $\Delta_0$.

2. Jump relation on $\Delta_0$

   \[ Y_+(\xi) = Y_-(\xi)W(\xi), \quad \xi \in \Delta_0. \]  

3. Normalization near infinity:

   \[
   Y(z) = \left[ I + \mathcal{O}\left(\frac{1}{z}\right) \right] \begin{pmatrix}
   z^{2n} & 0 & 0 \\
   0 & z^{-n} & 0 \\
   0 & 0 & z^{-n}
   \end{pmatrix}, \quad z \to \infty.
   \]  

Here the jump matrix is

\[
W(\xi) = \begin{pmatrix}
1 & \omega_1(\xi) & \omega_2(\xi) \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]  

where we assume that

- $w_1 = 0$ on $\Delta_2 \setminus \Delta_1$
- $w_2 = 0$ on $\Delta_1 \setminus \Delta_2,$

and $I$ is the $3 \times 3$ identity matrix. If we suppose in addition that the entries of the jump matrix \(W(\xi)\) belong to the class \((2.5),\) then the matrix Riemann-Hilbert boundary value problem \((2.13),\) supplemented with conditions at the endpoints of $\Delta_0 = \Delta_1 \cup \Delta_2,$ has a unique solution. The endpoint conditions when $z \to a_1$ are

\[
Y(z) = \begin{cases}
\mathcal{O}(1) & \mathcal{O}(\log |z - a_1|) & \mathcal{O}(1), \\
\mathcal{O}(1) & \mathcal{O}(\log |z - a_1|) & \mathcal{O}(1), \\
\mathcal{O}(1) & \mathcal{O}(\log |z - a_1|) & \mathcal{O}(1)
\end{cases}, \quad \alpha_1 > 0,
\]

\[
Y(z) = \begin{cases}
\mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(1) \\
\mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(1) \\
\mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(1)
\end{cases}, \quad \alpha_1 = 0,
\]

\[
Y(z) = \begin{cases}
\mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(1) \\
\mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(1) \\
\mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(1)
\end{cases}, \quad -1 < \alpha_1 < 0,
\]

Here $\alpha_1$ is assumed to be an endpoint of $\Delta_1$ and not an endpoint of $\Delta_2.$ If $a_1$ coincides with an endpoint of $\Delta_2$ (e.g., $a_1 = a_2$), then we will assume that

\[
a_1 = a_2 \Rightarrow \alpha_1 = \alpha_2,
\]

\[
(2.16)
\]
and the endpoint conditions as $z \to a_1$ become

$$Y(z) = \begin{cases} 
(\Theta(1) \quad \Theta(|z - a_1|^{\alpha_1}) \quad \Theta(|z - a_1|^{\alpha_1})), & -1 < \alpha_1 = \alpha_2 < 0, \\
(\Theta(1) \quad \Theta(\log|z - a_1|) \quad \Theta(\log|z - a_1|)), & \alpha_1 = \alpha_2 = 0, \\
(\Theta(1) \quad \Theta(\log|z - a_1|) \quad \Theta(\log|z - a_1|)), & \alpha_1 = \alpha_2 > 0, \\
(\Theta(1) \quad \Theta(1) \quad \Theta(1)), & \alpha_1 = \alpha_2 < 0.
\end{cases}$$

When $z$ approaches an endpoint other than $a_1$ then the endpoint conditions are similar to (2.15)–(2.17). We then have

**Proposition 2.2.** Suppose the Riemann-Hilbert boundary value problem (2.13) has the matrix jump $W$ given by (2.14) with entries satisfying (2.5) and endpoint conditions (2.15) or (2.16)–(2.17) (and similar conditions at the other endpoints). If this Riemann-Hilbert problem has a solution, then this solution is unique and its entries (2.12) are the Hermite-Padé polynomials (multiple orthogonal polynomials) (2.6) and the functions of the second kind (2.7) corresponding to the normal index $\bar{n} = (n, n)$ of the Hermite-Padé approximants for the functions

$$f_j \in \mathcal{A}(a_j, \alpha_j; b_j, \beta_j; \Omega_j), \quad j = 1, 2.$$  

*Proof.* See [74] for the relation between this $3 \times 3$ matrix-valued Riemann-Hilbert problem and multiple orthogonal polynomials. The uniqueness when the endpoints conditions are imposed follows in a similar way as in [47].

Our goal is to find a solution of the problem (2.13)–(2.17) for large $n$. This would give the normality of the Hermite-Padé approximants for large $n$ and their asymptotics as $n \to \infty$. For this purpose we apply a steepest descent method for asymptotic analysis (as $n \to \infty$) of the matrix Riemann-Hilbert problem. One of the main ingredients of this method consists in finding a location for the arcs $\Delta_1$ and $\Delta_2$ within the domains $\Omega_1$ and $\Omega_2$ such that the jump matrices in the Riemann-Hilbert problem admit a factorization as a product of matrices which tend exponentially fast (as $n \to \infty$) to the identity matrix outside $\Delta = \Delta_1 \cup \Delta_2$ and a matrix independent of $n$. The location of these arcs depends on the position of the points $\{a_1, a_2, b_1, b_2\}$. Thus the form of the asymptotics of the solution of the Riemann-Hilbert boundary value problem and the method of obtaining it depend strongly on the position of the points $\{a_1, b_1, a_2, b_2\}$.

We will assume throughout that the domains $\Omega_1$ and $\Omega_2$ are such that they contain the "optimal" arcs $\Delta_1$ and $\Delta_2$. We also assume that the domains are large enough so that the deformations we need to do in the course of the steepest descent analysis can be performed within $\Omega_1$ and $\Omega_2$. This is in particular important for the cases II, IV, and V where one of the steps involves global deformation of contours.
2.2 Geometry of the problem. Cases I and II

In this section we define a class of positions of the points

\[ A := \{a_1, b_1; a_2, b_2\} \]  \hspace{1cm} (2.19)

which we call the **geometrical cases I and II**

\[ A \in I \cup II. \]

This class characterizes analytic functions \((2.4)\)

\[ f_j \in A(a_j, \alpha_j; b_j, \beta_j, \Omega_j), \quad j = 1, 2, \] \hspace{1cm} (2.20)

whose Hermite-Padé asymptotics are described by means of algebraic functions of the third order with the only branch points \(a_1, b_1, a_2, b_2\). We define the two classes I and II and we fix for the input data (2.19) the global branches of the algebraic functions needed for the presentation of the asymptotic results and we introduce measures for the description of the limiting behavior of the zeros of the Hermite-Padé polynomials.

2.2.1 New coordinates for the input data

The algebraic function \(h\) (see (1.19)) associated with the input data (2.19) satisfies the equation

\[ h^3(z) - 3 \frac{P_2(z)}{\Pi_4(z)} h(z) + 2 \frac{P_1(z)}{\Pi_4(z)} = 0, \] \hspace{1cm} (2.21)

and plays a key role for the classification of the geometrical cases. We recall from the introduction that \(\Pi_4(z) = (z-a_1)(z-b_1)(z-a_2)(z-b_2)\) and the two unknown parameters of the monic polynomials \(P_1\) and \(P_2\), of degree one and two respectively (see (1.20)), have to satisfy — for the cases under consideration in this paper — the condition that the genus of \(h\) is zero. It follows from this that the discriminant \(D\) of the equation (2.21)

\[ D = \frac{\tilde{D}}{\Pi_4^3(z)}, \quad \tilde{D} = P_2^3 - \Pi_4 P_1^2, \]

has zeros of even multiplicity, i.e., (2.19) are the only branch points of \(h\). As we already mentioned in the introduction, this gives us two algebraic equations (of rather high order) for the determination of the unknown parameters of \(P_1\) and \(P_2\). In practice it is better to consider the algebraic function \(h\) with a special class of coefficients in (2.21) satisfying the condition that the zeros of the discriminant of \(h\) have multiplicity two. In other words, starting from known coefficients in (2.21), we will obtain the set \(A\) in (2.19) as the set of zeros for the polynomial

\[ \Pi_4(z) = (z-a_1)(z-b_1)(z-a_2)(z-b_2). \] \hspace{1cm} (2.22)

To realize this, we introduce new coordinates \((k, p, s, c)\) in \(\mathbb{C}^4\) for the input data. The coefficients in (2.21) are given by the following
Proposition 2.3. Suppose we are given
\[ B = (k, p, s, c) \in \mathbb{C}^4, \]  
(2.23)
and we define
\[ P_1(z) = z - c, \quad P_2(z) = P_1^2(z) + 2pP_1(z) + s^2. \]  
(2.24)
Then
\[ \Pi_4(z) = \frac{P_2^3(z) - D(z)}{P_1^2(z)}, \quad \text{with} \quad D(z) = (kP_1^2(z) + 3psP_1(z) + s^3)^2, \]
is a polynomial of degree 4 in the variable \( z \):
\[ \Pi_4 = P_1^4 + 6pP_1^3 + (3s^2 + 12p^2 - k^2)P_1^2 + (8p^3 + 12s^2p - 6psk)P_1 + 3s^4 + 3p^2s^2 - 2s^3k. \]  
(2.25)
The proof of Proposition 2.3 is given in Subsection 3.1.1.

So given \( k, p, s, c \) we construct the polynomials \( P_1, P_2, \) and \( \Pi_4 \) as in (2.24) and (2.25) and we use these polynomials in the definition (2.21) of the algebraic function \( h \). Then it follows from Proposition 2.3 that \( D \) is a perfect square, and (2.21) has branch points at the zeros of \( \Pi_4 \) only, which we denote by \( a_1, b_1, a_2, b_2 \) (taken in some order) and we assume that they are all distinct. This defines a mapping
\[ B = (k, p, s, c) \mapsto A = \{a_1, b_1, a_2, b_2\} \]  
(2.26)
Thus, if we use the coordinates \( B \) in (2.23), then we can determine both the set \( A \) of four points in (2.19) and all the coefficients of the equation (2.21) of the algebraic function \( h \). Generically, this function \( h \) has genus zero, that is, with the exception of a lower dimensional subset of the four dimensional \((k, p, s, c)\)-space. However, this class also contains functions of higher genus (for example, functions with branch points of the third order). These degenerate examples are not connected with our task and we exclude them from our considerations.

2.2.2 The function \( \Phi \) and the contour \( \Gamma \)

Since the algebraic function \( h \) has genus zero, the exponential of the Abelian integral
\[ \Phi = \exp \left( \int h(z) \, dz \right) \]  
(2.27)
is also an algebraic function with the same Riemann surface as \( h \). The equation (2.27) defines \( \Phi \) up to a multiplicative constant.

Proposition 2.4. For a suitable normalization we have that the function \( \Phi \) satisfies the equation
\[ \Phi^3(z) + q_1(z)\Phi^2(z) + q_2(z)\Phi(z) + q_0 = 0, \]
where \( q_j \) are polynomials of degree \( \leq j \) \((j = 0, 1, 2)\) given by
\[ q_1(z) = -6\sqrt{3}pP_1(z) + \frac{1}{\sqrt{3}} \left[ (k - 2\sqrt{3}p)\kappa_+ + (k + 2\sqrt{3}p)\kappa_- \right], \]
\[ q_2(z) = -\kappa_+\kappa_- \left( P_1^2(z) + 4pP_1(z) + \frac{9s^2 - 4k^2 + 36p^2}{9} \right), \]  
(2.28)
\[ q_0 = \frac{2\sqrt{3}}{243} (3s - 2k)^2 \kappa_+^2 \kappa_-^2, \]
where \( P_1(z) = z - c \) and
\[
\kappa_{\pm} = 3s + k \pm 3\sqrt{3}p. \tag{2.29}
\]

The proof of Proposition is in Subsection 3.1.1.

We assume from now on that \( k, p, \) and \( s \) are such that
\[
q_0 \neq 0
\]
This also implies that \( \kappa_+ \neq 0 \) and \( \kappa_- \neq 0 \).

The function \( \Phi \) is defined in (2.27) up to a multiplicative constant. In what follows we often choose a normalization such that the product of all three branches of \( \Phi \) equals one:
\[
\Phi_0 \Phi_1 \Phi_2 = 1. \tag{2.30}
\]
Therefore the function \( \Phi \) with normalization (2.30) is related to the function \( \Phi \) from Proposition 2.4 by
\[
\Phi(z) = -\frac{\Phi(z)}{q_0^{1/3}},
\]
where we recall our assumption that \( q_0 \neq 0 \). Taking into account the explicit expression (2.28) for the equation above, we see that the three branches of the normalized function \( \Phi \) behave at infinity as
\[
\begin{cases}
\Phi_j(z) = \frac{z}{C_j} + \cdots, & j = 1, 2, \\
\Phi_0(z) = \frac{1}{C_0 z^2} + \cdots & z \to \infty,
\end{cases} \tag{2.31}
\]
where
\[
C_1 = -\frac{q_0^{1/3}}{\kappa_+}, \quad C_2 = \frac{q_0^{1/3}}{\kappa_-}, \quad C_0 = \frac{1}{C_1 C_2}.
\]
From the equation for \( \Phi \) we can obtain a parametrization of the curve
\[
\Gamma = \{ z : |\Phi_j(z)| = |\Phi_k(z)| \text{ for some } 0 \leq j < k \leq 2 \} \tag{2.32}
\]
in terms of the function
\[
J(\nu, z) = \nu^3 + A(z)\nu^2 + B(z)\nu + C(z) \tag{2.33}
\]
where
\[
A(z) = \frac{3q_0 - q_1(z)q_2(z)}{q_0}, \quad B(z) = \frac{q_0 q_1^2(z) + q_2^2(z) - 5q_0 q_1(z)q_2(z) + 3q_0^2}{q_0^2}, \quad C(z) = \frac{2q_0 q_1^3(z) - q_1^2(z)q_2^2(z) + 2q_2^3(z) - 4q_0 q_1(z)q_2(z) + q_0^2}{q_0^2} \tag{2.34}
\]
and \( q_0, q_1, q_2 \) are the coefficients (2.28) of the equation for \( \Phi \).

**Proposition 2.5.** The set \( \Gamma \) given in (2.32) for the function \( \Phi \) given in (2.27) satisfies
\[
\Gamma = \{ z : J(\nu, z) = \nu^3 + A(z)\nu^2 + B(z)\nu + C(z) = 0 \text{ for some } \nu \in [-2, 2] \}, \tag{2.35}
\]
where \( A, B, \) and \( C \) are given by (2.34).

The proof of Proposition 2.5 is also given in Subsection 3.1.1.
2.2.3 Structure of $\Gamma$. Definition of cases I and II

Here we shall use the global structure of $\Gamma$ to define the geometrical cases I and II. Since the polynomial $J(\nu, z)$ in (2.33) is of degree 6 in the variable $z$, the contour $\Gamma$ consists of six trajectories $z_j(\nu)$ parametrized by $\nu \in [-2, 2]$. When $\nu = 2$ we have

$$J(2, z) = -\frac{\Pi_4(z)[P_1(z)(3s + k) + 9sp]^2}{q_0^2}.$$ 

These trajectories start from the points $a_1, b_1, a_2, b_2$ and two trajectories start from the point

$$\alpha = c + \frac{9sp}{3s + k}.$$ 

Here we assume that $\alpha$ is different from $a_j, b_j, j = 1, 2$. The case when $\alpha$ coincides with one of the $a_j, b_j, j = 1, 2$ will be treated in the next subsection (see the case III). We denote by

$$\gamma_{a_1}, \gamma_{b_1}, \gamma_{a_2}, \gamma_{b_2}, \gamma_{a_1}, \gamma_{a_2}$$

these trajectories which are then continuously extended as $\nu$ decreases from 2, see Figure 2.1. When $\nu = -2$ we have

$$J(-2, z) = \frac{[q_0 - q_1(z)q_2(z)]^2}{q_0^2} = \text{const } (z - \beta_1)^2(z - \beta_2)^2(z - \beta_3)^2. \quad (2.37)$$

These six trajectories therefore meet pairwise at the points $\beta_1, \beta_2, \beta_3$, which are the zeros of $q_0 - q_1q_2$ (see Figure 2.2).

Figure 2.1: Start ($\nu = 2$) of the trajectories of $\Gamma$

Figure 2.2: Finish ($\nu = -2$) of the trajectories of $\Gamma$

**Definition 2.6.** We say that the set of points $\{a_1, b_1, a_2, b_2\}$ belongs to the geometrical cases I or II

$$A = \{a_1, b_1, a_2, b_2\} \in I \cup II \quad (2.38)$$

if there exist coordinates $B = (k, p, s, c)$ which are mapped by (2.26) to $A$ such that
1. the algebraic function \( z(\nu) \) defined by the equation \( J(\nu, z) = 0 \), see (2.33), has no branch points on \((-2, 2)\), i.e., that by analytic continuation the trajectories (2.36) are defined globally for \( \nu \in [-2, 2] \)

\[
\gamma_a(\nu), \gamma_b(\nu), \gamma_a(\nu), \gamma_b(\nu), \gamma_a(\nu), \gamma_b(\nu), \nu \in [-2, 2].
\]

2. When \( \nu = -2 \) we have

\[
\gamma_a(-2) = \gamma_b(-2), \quad \gamma_a(-2) = \gamma_b(-2).
\]

Now, for \( A \in I \cup II \) we can define two arcs in \( \mathbb{C} \)

\[
\gamma_j = \gamma_{a_j} \cup \gamma_{b_j}, \quad j = 1, 2,
\]

(2.39) each connecting \( a_j \) and \( b_j \) \( j = 1, 2 \), and a closed analytic curve

\[
\gamma_\alpha := \gamma_{a_1} \cup \gamma_{a_2}.
\]

**Definition 2.7.** Given \( A \in I \cup II \), we say that (see Figures 2.3 and 2.4)

1. \( A \in I \) if \( \gamma_1 \cap \gamma_2 = \emptyset \).

2. \( A \in II \) if \( \gamma_1 \) and \( \gamma_2 \) have two points of intersection

\[
\gamma_1 \cap \gamma_2 = \{c_1, c_2\}.
\]

In the case II we assume that the branch points \( a_j, b_j \) and the intersection points \( c_j \) are labeled as shown in Figure 2.4. That is, if we follow the curve \( \gamma_j \) starting at \( a_j \) \( (j = 1, 2) \) then we first meet \( c_1 \) and then \( c_2 \).
2.2.4 Riemann surface for case I. Definition of the global branches for the algebraic functions \( h \) and \( \Phi \)

We denote for the case I (see Figure 2.3)

\[
\Delta_j := \delta_j := \gamma_j, \quad j = 1, 2, \quad \Delta_0 := \Delta_1 \cup \Delta_2.
\]  

(2.40)

**Definition 2.8.** For a set of points \( A \in I \) the corresponding Riemann surface

\[
\mathfrak{R}(A) = \overline{\mathfrak{R}_0 \cup \mathfrak{R}_1 \cup \mathfrak{R}_2}
\]

(2.41)

is formed by glueing the sheets of the complex plane, cut along the arcs \( \Delta_j \) in (2.39)

\[
\mathfrak{R}_j = \overline{\mathbb{C} \setminus \Delta_j}, \quad j = 1, 2,
\]

so that the positive (negative) side of the cut on one sheet is identified with the negative (positive) side on the neighboring sheet as in Figure 2.5. The curves are oriented from \( a_j \) to \( b_j \) and the positive side is on the left, the negative side on the right of the oriented curve.

![Figure 2.5: Riemann surface for case I](image)

The function \( h \) in (2.21) is therefore a rational function on the Riemann surface \( \mathfrak{R}(A) \) in (2.41) with divisor (1.22)–(1.23), that is, \( h \) has simple poles at the branch points \( a_j, b_j \), simple zeros at the points at infinity on each of the three sheets, and a fourth simple pole at a value whose projection is the parameter \( c \). The local definition of the branches of \( h \) at infinity (1.21) can be extended globally as

\[
h_0 \in H(\overline{\mathbb{C} \setminus (\Delta_1 \cup \Delta_2)}),
\]

\[
h_j \in H(\overline{\mathbb{C} \setminus \Delta_j}), \quad j = 1, 2.
\]

The function \( \Phi \) in (2.27) is also a rational function on \( \mathfrak{R}(A) \) with a double zero at the point at infinity on the sheet \( \mathfrak{R}_0 \) and a simple pole at the points at infinity on the other
two sheets, see (2.31), and normalization (2.30). The local definition of the branches of \( \Phi \) at infinity (2.31) can be extended globally as

\[ \Phi_j \in H(\mathbb{C} \setminus \Delta_j), \quad j = 0, 1, 2. \]  

(2.42)

The contour \( \Gamma \) can now be written as a union \( \Gamma = \Gamma_{0,1} \cup \Gamma_{0,2} \cup \Gamma_{1,2} \) where

\[ \Gamma_{j,k} = \{ z \in \mathbb{C} \mid |\Phi_j(z)| = |\Phi_k(z)| \}, \quad 0 \leq j < k \leq 2. \]  

(2.43)

In (2.43) it is understood that for \( z \) on one of the cuts \( \Delta_1 \) and \( \Delta_2 \) we should take limiting values. If equality holds on one side of the cut then equality holds on the other side as well.

The three branches \( \Phi_j, j = 0, 1, 2, \) also determine a number of regions in the complex plane

\[ \Omega_{j,k,\ell} = \{ z \in \mathbb{C} : |\Phi_j(z)| < |\Phi_k(z)| < |\Phi_\ell(z)| \}, \quad j, k, \ell = 0, 1, 2. \]  

(2.44)

We also define

\[ \Omega_{j,k} = \{ z \in \mathbb{C} : |\Phi_j(z)| < |\Phi_k(z)| \}, \quad j, k = 0, 1, 2. \]  

(2.45)

The contours \( \Gamma_{j,k} \) and the regions \( \Omega_{j,k,\ell} \) give a partition of the complex plane.

![Diagram](image.png)

Figure 2.6: Possible partitions of \( \mathbb{C} \) by \( \Gamma \) for case I

**Proposition 2.9.** For \( A \in I \) we have

1. There are open sets \( U_j \) such that

\[ \Delta_j \subset U_j, \quad U_j \setminus \Delta_j \subset \Omega_{0,j}. \]

2. There exist sets \( A \in I \) such that (see Figure 2.6)

\[ \Omega_{1,0} \neq \emptyset \quad \text{or} \quad \Omega_{2,0} \neq \emptyset. \]

Figure 2.6 illustrates possible partitions of the complex plane by the regions \( \Omega_{j,k,\ell} \) and the contours \( \Gamma_{j,k} \). In the left-most figure we have that \( \Omega_{1,0} \) and \( \Omega_{2,0} \) are both empty. In the other two figures the set \( \Omega_{1,0} \) is non-empty and it contains part of \( \Delta_2 \) (as in the middle figure) or all of \( \Delta_2 \) (as in right-most figure) in its interior.

Now we introduce measures which will describe the limiting behavior of the Hermite-Padé approximants.
Theorem 2.10. For $A$ in case I we have that the jump of the function $h_0$ on $\Delta_0 = \Delta_1 \cup \Delta_2$ produces a positive measure $\lambda$ of total mass 2

$$d\lambda(\xi) = \frac{1}{2\pi i} \left( (h_0)_+ - (h_0)_- \right) d\xi, \quad \xi \in \Delta_0.$$  (2.46)

The measure $\lambda$ consists of two measures of mass 1 with densities (with respect to complex line element $d\xi$)

$$\lambda'_j(\xi) = \frac{1}{2\pi i} \left( (h_0)_+ - (h_0)_- \right) \bigg|_{\Delta_j}, \quad (2.47)$$

each supported on $\Delta_j$,

$$\lambda = \begin{cases} 
\lambda_1 & \text{on } \Delta_1, \\
\lambda_2 & \text{on } \Delta_2,
\end{cases}$$

and

$$\lambda'_j(\xi) = \frac{m_j(\xi)}{\sqrt{(\xi - a_j)(\xi - b_j)}}, \quad j = 1, 2,$$

where $m_j$ is analytic on $\Delta_j$ for $j = 1, 2$.

The proof of Proposition 2.9 and Theorem 2.10 is in Subsection 3.1.2.

2.2.5 Riemann surface for the case II. Definition of the global branches for the algebraic functions $h$ and $\Phi$

In the case II we have $\gamma_1 \cap \gamma_2 = \{c_1, c_2\}$. We assume that the points $a_j, b_j$ and $c_j$ are labelled as shown in Figure 2.4. We use $\gamma_{a_j,c_1}$ to denote the part of the arc $\gamma_j$ between $a_j$ and $c_1$ and similarly for $\gamma_{b_j,c_2}$. Then we denote for the case II

$$\tilde{\Delta}_j := \gamma_{a_j,c_1} \cup \gamma_{b_j,c_2}, \quad j = 1, 2, \quad \begin{cases} 
E_1 := \gamma_2 \setminus \tilde{\Delta}_2, \\
E_2 := \gamma_1 \setminus \tilde{\Delta}_1.
\end{cases} \quad (2.48)$$

The arcs $E_1$ and $E_2$ form a boundary of a lens-shaped domain $G$

$$\partial G := E_1 \cup E_2.$$  (2.49)

Note that the analytic curve $\gamma_\alpha$ has to pass through the points $c_1, c_2$ and that it divides the domain $G$ into two parts (otherwise it would give a contradiction with the maximum principle for harmonic functions). We set

$$\Delta_{1,2} := \gamma_\alpha \cap G.$$  (2.50)

Finally, we denote

$$\Delta_j := \tilde{\Delta}_j \cup \Delta_{1,2}, \quad \Delta_0 := \Delta_1 \cup \Delta_2 = \tilde{\Delta}_1 \cup \Delta_{1,2} \cup \tilde{\Delta}_2.$$  (2.51)

Definition 2.11. For a set of points $A \in \Pi$ the corresponding Riemann surface

$$\mathfrak{R}(A) := \mathfrak{R}_0 \cup \mathfrak{R}_1 \cup \mathfrak{R}_2.$$
is formed by glueing the sheets of the cut complex plane

\[
\begin{align*}
\mathcal{R}_1 &= \mathbb{C} \setminus (\Delta_1 \cup E_1), \\
\mathcal{R}_2 &= \mathbb{C} \setminus (\widetilde{\Delta}_2 \cup E_1) = \mathbb{C} \setminus \gamma_2
\end{align*}
\] (2.52)

to the sheet

\[
\mathcal{R}_0 := \mathbb{C} \setminus \Delta_0 = \mathbb{C} \setminus (\Delta_1 \cup \widetilde{\Delta}_2),
\]

and along \( E_1 \) the sheets \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) are glued to each other (see Figure 2.7).

![Figure 2.7: Riemann surface for case II](image)

**Remark 2.12.**

1. The defined Riemann surface possesses a certain non-symmetry with respect to the pairs \( \{a_1, b_1\} \) and \( \{a_2, b_2\} \). We also can use a dual \( \mathcal{R} \) given by

\[
\begin{align*}
\mathcal{R}_0 &:= \mathbb{C} \setminus \Delta_0, \\
\mathcal{R}_2 &:= \mathbb{C} \setminus (\Delta_2 \cup E_2), \\
\mathcal{R}_1 &:= \mathbb{C} \setminus \gamma_1.
\end{align*}
\]

2. Although all three sheets are glued together at the points \( c_1, c_2 \), it can easily be checked that these points are not branch points of \( \mathcal{R} \).

3. Note that the \( \mathcal{R}_1 \) sheet is a disconnected set. It consists of two components: a domain \( G_1 \) bounded by \( E_1 \) and \( \Delta_{1,2} \)

\[
\partial G_1 := E_1 \cup \Delta_{1,2},
\]

and the domain \( \mathbb{C} \setminus (\overline{G}_1 \cup \widetilde{\Delta}_1) \).

The structure of the sheets (2.52) defines the global branches of the functions \( h \) and \( \Phi \):

\[
\begin{align*}
h_0, \Phi_0 &\in H(\mathbb{C} \setminus \Delta_0), \\
h_1, \Phi_1 &\in H(\mathbb{C} \setminus (\widetilde{\Delta}_1 \cup \overline{G}_1)) \cup H(G_1), \\
h_2, \Phi_2 &\in H(\mathbb{C} \setminus (\widetilde{\Delta}_2 \cup E_1)).
\end{align*}
\] (2.53)
More precisely, in the domains
\[ \mathbb{C} \setminus \Delta_0, \quad \mathbb{C} \setminus (\tilde{\Delta}_1 \cup G_1), \quad \mathbb{C} \setminus (\tilde{\Delta}_2 \cup E_1), \]
the branches
\[ (h_0, \Phi_0), \quad (h_1, \Phi_1), \quad (h_2, \Phi_2) \]
are respectively the result of analytic continuation of (1.21) and (2.31) from point at infinity and the branches \((h_1, \Phi_1)\) in \(G_1\) are the result of analytic continuation of \((h_0, \Phi_0)\) through \(\Delta_{1,2}\) or \((h_2, \Phi_2)\) through \(E_1\). Using the continuity of the global branches of \(\Phi\) along \(\gamma_1, \gamma_2, \gamma_\alpha\) and the maximum principle, we obtain a partition of \(\mathbb{C}\) by \(\Gamma\) into domains \(\Omega_{j,k,\ell}\) (see (2.44) for the definition), as is shown in Figure 2.8.

Figure 2.8: Partition of \(\mathbb{C}\) by \(\Gamma\) into \(\Omega_{j,k,\ell}\). The case II

**Proposition 2.13.** If \(A \in \Pi\) then the contour \(\Gamma = \Gamma_{0,1} \cup \Gamma_{0,2} \cup \Gamma_{1,2}\) defined in (2.32)–(2.33) has the following structure:
\[
\begin{align*}
\Gamma_{0,1} & := \{ z : |\Phi_0(z)| = |\Phi_1(z)| \} = \Delta_1, \\
\Gamma_{0,2} & := \{ z : |\Phi_0(z)| = |\Phi_2(z)| \} = \tilde{\Delta}_2, \\
\Gamma_{1,2} & := \{ z : |\Phi_1(z)| = |\Phi_2(z)| \} = E_1 \cup E_2 \cup (\gamma_\alpha \setminus \Delta_{1,2}),
\end{align*}
\]
and for the domains \(\Omega_{j,k,\ell}\) we have (see Figure 2.8)
\[
\begin{align*}
\partial \Omega_{0,1,2} = \gamma_\alpha \cup \tilde{\Delta}_1 \cup E_1 \cup E_2, \\
\partial \Omega_{0,2,1} = (\gamma_\alpha \setminus \Delta_{1,2}) \cup \tilde{\Delta}_2 \cup E_2.
\end{align*}
\] (2.54)

The proof of Proposition 2.13 is in Subsection 3.1.3, as is the proof of the following Theorem 2.14 which introduces the measures.

**Theorem 2.14.** For \(A \in \Pi\) we have:
1. The jump of $h_0$ on $\Delta_0$ produces a positive measure $\lambda$ of total mass 2

$$\frac{1}{2\pi i} \left( h_{0+}(\xi) - h_{0-}(\xi) \right) d\xi =: d\lambda(\xi), \quad \xi \in \Delta_0.$$ 

The measure $\lambda$ consists of two measures $\lambda_1$ and $\tilde{\lambda}_2$ supported on $\Delta_1$ and $\tilde{\Delta}_2$

$$\lambda = \begin{cases} 
\lambda_1 & \text{on } \Delta_1, \\
\tilde{\lambda}_2 & \text{on } \tilde{\Delta}_2,
\end{cases}$$

with respective densities

$$\lambda_1'(\xi) = \frac{m_1(\xi)}{\sqrt{(\xi - a_1)(\xi - b_1)}}, \quad m_1 \in H(\tilde{\Delta}_1) \cap H(\Delta_1),$$

$$\tilde{\lambda}_2'(\xi) = \frac{m_2(\xi)}{\sqrt{(\xi - a_2)(\xi - b_2)}}, \quad m_2 \in H(\tilde{\Delta}_2).$$

2. The jump of $h_1$ on $E_1$ produces a positive measure $\mu_1$

$$\frac{1}{2\pi i} \left( h_{1+}(\xi) - h_{1-}(\xi) \right) d\xi =: d\mu_1(\xi), \quad \xi \in E_1$$

and $\mu_1' \in H(E_1)$.

3. There are connections among the total masses of these measures:

$$|\lambda_1| + |\tilde{\lambda}_2| = 2, \quad |\lambda_1| - |\mu_1| = 1.$$ 

Thus the Riemann surface for the case II produces a system of three positive measures

$$\{\lambda_1, \tilde{\lambda}_2, \mu_1\}, \quad \begin{cases} 
supp(\lambda_1) = \Delta_1, \\
supp(\tilde{\lambda}_2) = \tilde{\Delta}_2, \\
supp(\mu_1) = E_1,
\end{cases} \quad \begin{cases} 
|\lambda_1| + |\tilde{\lambda}_2| = 2, \\
|\lambda_1| - |\mu_1| = 1.
\end{cases}$$

If we consider the dual Riemann surface (see Remark 2.12 item 1), then we arrive at a dual system of three positive measures

$$\{\lambda_2, \tilde{\lambda}_1, \mu_2\}, \quad \begin{cases} 
supp(\lambda_2) = \Delta_2, \\
supp(\tilde{\lambda}_1) = \tilde{\Delta}_1, \\
supp(\mu_2) = E_2,
\end{cases} \quad \begin{cases} 
|\lambda_2| + |\tilde{\lambda}_1| = 2, \\
|\lambda_2| - |\mu_2| = 1, 
\end{cases}$$

and we have

$$\lambda_1 + \tilde{\lambda}_2 = \lambda_2 + \tilde{\lambda}_1 = \lambda.$$
2.3 Geometry of the problem. Case III

Recall that the geometrical case III is such that the zeros of $P_{ni}$ accumulate on two disjoint arcs which do not contain all the branch points. We assume that the branch points are numbered such that one arc connects $a_2$ and $b_2$ and such that the other arc connects $a_1$ and $b^*$ with $b^* \neq b_1$, see Figure 1.3. We again start from the algebraic function $h$ in (2.21), however we now use its reduced form (1.24). Without loss of generality for the set $A = \{a_1, b_1; a_2, b_2\}$ we assume that

$$|a_1 - b_1| \geq |a_2 - b_2|, \quad \text{dist}(a_1, [a_2, b_2]) \geq \text{dist}(b_1, [a_2, b_2]).$$

(2.56)

For a set $A$ with conditions (2.56) we set

$$A' = \{a_1; a_2, b_2\}$$

(2.57)

and associate with the triple $A'$ the algebraic function (1.24):

$$h^3 - 3 \frac{z - a_1 + a_2 + b_2}{(z - a_1)(z - a_2)(z - b_2)} h + \frac{2}{(z - a_1)(z - a_2)(z - b_2)} = 0. \quad (2.58)$$

We recall from (1.24) that $h$ in (2.58) has four branch points at

$$a_1, a_2, b_2, b^* = \frac{a_1a_2b_2 - \left(\frac{a_1 + a_2 + b_2}{3}\right)^3}{a_1a_2 + a_1b_2 + a_2b_2 - 3 \left(\frac{a_1 + a_2 + b_2}{3}\right)^3}. \quad (2.59)$$

We do not need to use the coordinates (2.23) now since we already have the explicit expressions in terms of the input data $A'$ for the coefficients of the equation (2.58). However, in order to avoid too cumbersome expressions we set, without loss of generality,

$$a_1 := -1, \quad b_2 := 0, \quad a_2 := a, \quad (2.60)$$

and $a \neq -1, a \neq 0$.

**Proposition 2.15.** The exponential function of the Abelian integral

$$\Phi = \exp \left( \int h(z) \, dz \right)$$

of the function $h$ in (2.58) is, up to a multiplicative constant, an algebraic function satisfying the equation

$$\Phi^3 + q_1(z)\Phi^2 + q_2(z)\Phi + q_0 = 0, \quad (2.61)$$

where $\deg q_j \leq j$ for $j = 0, 1, 2$ and the $q_j$ are rational functions of $a_1, a_2, b_2$. If we take into account (2.60) then we have

$$q_1(z) = z(a - 1)(a^2 + \frac{5a}{2} + 1) + \frac{a(a^2 + 4a + 1)}{2},$$

$$q_2(z) = -\kappa \left(\frac{27z^2}{4} - \frac{9}{2}(a - 1)z - \frac{1}{4}(a^2 + 10a + 1)\right),$$

$$q_0 = \kappa^2$$

where $\kappa = a^2(a + 1)^2/4.$

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The proof of Proposition 2.15 is in Subsection 3.1.4.

Now substitute the $q_0, q_1, q_2$ from (2.62) into the expressions (2.34) for the coefficients $A, B, C$ (see Proposition 2.5) of

$$J(\nu, z) = \nu^3 + A(z)\nu^2 + B(z)\nu + C(z).$$

Then as in Proposition 2.5, we obtain a parametrization of the contour $\Gamma$

$$\Gamma := \{ z : |\Phi_j(z)| = |\Phi_k(z)| \text{ for some } 0 \leq j < k \leq 2 \}$$

as the union of six trajectories. When $\nu = 2$ the three trajectories $\gamma_{a_1}, \gamma_{a_2}, \gamma_{b_2}$ start from the points $a_1, a_2, b_2$ and the other three trajectories $\gamma_{b^*}^{(j)} (j = 1, 2, 3)$ start from the point $b^*$ (see Figure 2.9). When $\nu = -2$ these trajectories meet pairwise at the points $\beta_1, \beta_2, \beta_2$ given by (2.37) (as in Figure 2.2).

**Figure 2.9**: Start ($\nu = 2$) of the trajectories of $\Gamma$ for Case III

**Definition 2.16.** We call the triple $A'$ in (2.57) acceptable for case III if

1. The algebraic function $z(\nu)$ defined by (2.63) has no branch points on $(-2, 2)$;

2. When $\nu = -2$ we have

$$\gamma_{a_2}(-2) = \gamma_{b_2}(-2), \quad \gamma_{a_1}(-2) = \gamma_{b^*}^{(j)}(-2) \text{ for some } j \in \{1, 2, 3\}.$$

   If part 2 of the definition is satisfied we assume without loss of generality that the trajectories starting from $b^*$ are numbered such that

$$\gamma_{a_1}(-2) = \gamma_{b^*}^{(1)}(-2).$$

It follows from Definition 2.16 that the trajectories $\gamma_{a_2}, \gamma_{b_2}, \gamma_{a_1}, \gamma_{b^*}^{(1)}$ for an acceptable triple $A'$ are defined globally for $\nu \in [-2, 2]$ and we can define the arcs joining the branch points (2.59) as

$$\Delta_1^* := \gamma_{a_1} \cup \gamma_{b^*}^{(1)}, \quad \Delta_2 := \gamma_{a_2} \cup \gamma_{b_2}, \quad \Delta_0 := \Delta_1^* \cup \Delta_2.$$  \hspace{1cm} (2.64)

Then we define for an acceptable triple $A'$ the Riemann surface as in subsection 2.2.4 (see (2.41))

$$R^*(A') = R_0^* \cup R_1^* \cup R_2^*,$$  \hspace{1cm} (2.65)
with three sheets
\[ \mathcal{R}_0^* = \mathbb{C} \setminus (\Delta_1^* \cup \Delta_2), \quad \mathcal{R}_1^* = \mathbb{C} \setminus \Delta_1^*, \quad \mathcal{R}_2^* = \mathbb{C} \setminus \Delta_2, \]
and the global branches of the algebraic functions \( h \) and \( \Phi \), defined by (2.58) and (2.61), are
\[ h_0, \Phi_0 \in H(\mathbb{C} \setminus (\Delta_1^* \cup \Delta_2)), \quad h_1, \Phi_1 \in H(\mathbb{C} \setminus \Delta_1^*), \quad h_2, \Phi_2 \in H(\mathbb{C} \setminus \Delta_2). \]  
(2.66)

Possible partitions of \( \mathbb{C} \) by \( \Gamma \) into the domains \( \Omega_{j,k,l} \) are shown in Figure 2.10. We have

**Proposition 2.17.**

1. There exists an open set \( U_2 \) such that
\[ \Delta_2 \subset U_2, \quad U_2 \setminus \Delta_2 \subset \Omega_{0,2} \]
and there exists an open set \( U_1 \) such that
\[ \Delta_1^* \setminus \{b^*\} \subset U_1, \quad U_1 \setminus \Delta_1^* \subset \Omega_{0,1}. \]

2. For any acceptable triple \( A' \) we have
\[ \Omega_{1,0} \neq \emptyset \quad \text{and} \quad b^* \in \partial \Omega_{0,1}. \]

In addition we have that \( \Omega_{1,0} \) is connected.

Moreover the trajectories \( \gamma_{b^*}^{(j)} \) (\( j = 1, 2, 3 \)) start from \( b^* \) at an angle \( 2\pi/3 \) and split a neighborhood of \( b^* \) into two domains belonging to \( \Omega_{0,1} \) (the boundaries of the domain contain \( \Delta_1^* \)) and one domain belonging to \( \Omega_{1,0} \) (the boundary contains \( \gamma_{b^*}^{(2)} \) and \( \gamma_{b^*}^{(3)} \)). See Figure 2.10.

Now we introduce the measures.

**Theorem 2.18.** For an acceptable triple \( A' \) we have that the jump of the branch \( h_0 \) on \( \Delta = \Delta_1^* \cup \Delta_2 \) produces a positive measure \( \lambda \) on \( \gamma \) of total mass 2. The measure consists of two measures of mass one:
\[ \lambda'(\xi) = \begin{cases} \lambda_1'(\xi) = \frac{1}{2\pi i}[(h_0)_+ (\xi) - (h_0)_- (\xi)], & \xi \in \Delta_1^* \\ \lambda_2'(\xi) = \frac{1}{2\pi i}[(h_0)_+ (\xi) - (h_0)_- (\xi)], & \xi \in \Delta_2 \end{cases} \]
and
\[ \lambda_1'(\xi) = \sqrt{\frac{\xi - b^*}{\xi - a_1}} m_1(\xi), \quad \lambda_2'(\xi) = \frac{m_2(\xi)}{\sqrt{(\xi - a_2)(\xi - b_2)}}, \]
with \( m_1 \) analytic in a neighborhood of \( \Delta_1^* \) and \( m_2 \) analytic in a neighborhood of \( \Delta_2 \).

The proofs of Proposition 2.17 and Theorem 2.18 are in Subsection 3.1.4.
Definition 2.19. We say that a set of points $A := \{a_1, b_1; a_2, b_2\}$ satisfying (2.56) belongs to the geometrical Case III $A \in III$ if

1. The triple $A' = \{a_1; a_2, b_2\}$ is acceptable.
2. $b_1 \in \Omega_{1,0}$.

For $A \in III$ we assign to $A$ the Riemann surface $\mathcal{R}^*$ and the algebraic functions $h$ and $\Phi$ corresponding to its triple $A'$, i.e., we use (2.58), (2.61), and (2.65).

2.4 Geometry of the problem. Common branch point: cases IV and V

In this section we present a complete classification of the geometry of the Hermite-Padé approximation problem for the two functions (1.3) with a common branch point, i.e.,

$$f_j \in \mathcal{A}(\mathbb{C} \setminus \{a_j, b\}), \quad a_1 \neq a_2.$$  

i.e.,

$$A = \{a_1, b; a_2, b\}.$$  

Again, as in the case III we associate with the triple

$$A' = \{a_1; a_2, b\}$$  

(2.67)

the algebraic function $h$ defined in (2.58), and the Abelian exponential $\Phi$ given by (2.61). There are three possibilities for the position of the points $A$:

1. The triple $A'$ in (2.67) is acceptable (see Definition 2.16) and $b \in \overline{\Omega}_{1,0}$. Then $A$ can be realized as a limiting situation of case III and we say $A \in III$,

(see top figure of Figure 2.11).

2. The triple $A'$ is acceptable but $b \notin \overline{\Omega}_{1,0}$ (see Figure 2.11 in the middle).

3. The triple $A'$ is not acceptable (see bottom figure of Figure 2.11).
Figure 2.11: The subclasses III, V and IV for a common branch point

**Definition 2.20.** We call the subclasses 2 and 3 above the cases V and IV respectively.

As in the previous subsection (see (2.60)) we set, without loss of generality,

\[ a_1 = -1, \quad b = 0, \quad a_2 := a, \quad |a| < 1. \]

In order to give the explicit description of these three subclasses we define three regions \( D_j \) of the disc \( D = \{a : |a| < 1\} \) as follows

\[
\begin{align*}
\{a \in D_1 &\iff A = \{-1,0; a,0\} \in \text{III}, \\
\{a \in D_2 &\iff A \in \text{V}, \\
\{a \in D_3 &\iff A \in \text{IV}. \\
\end{align*}
\]

(2.68)

We have

**Theorem 2.21.**
1. Two branches of the algebraic curve

\[ a^4 + (4 - 4\nu)a^3 + (22 - 8\nu)a^2 + (4 - 4\nu)a + 1 = 0 \]

lying in \( D \) for \( \nu \in [-2, 2] \) form the boundary \( \partial D_1 \) of the domain \( D_1 \).

2. The algebraic curve

\[
\hat{P}(a, \nu) := 16a^{12} + 96a^{11} + (336 - 108\nu)a^{10} + (800 - 540\nu)a^9 + (2169 - 1404\nu)a^8 \\
+ (4932 - 2376\nu)a^7 + (6630 - 2808\nu)a^6 + (4932 - 2376\nu)a^5 \\
+ (2169 - 1404\nu)a^4 + (800 - 540\nu)a^3 + (336 - 108\nu)a^2 + 96a + 16 = 0,
\]

which for \( \nu = 2 \) is factorized as

\[(a - 1)^4(2a + 1)^4(a + 2)^4 = 0,\]

has six branches in \( D \) for \( \nu \in [-2, 2] \). These six branches start (for \( \nu = 2 \)) from the points 1 (two branches) and \(-1/2\) (four branches) and give the outer boundary \( \partial D_2^{\text{out}} \) of the region \( D_2 \). The boundary \( \partial D_2 \) therefore consists of

\[ \partial D_2 = \partial D_1 \cup \partial D_2^{\text{out}}. \]

3. The region \( D_3 \) is the open set bounded by \( \partial D_2^{\text{out}} \) and \( \{|a| = 1\} \).

The proof of Theorem 2.21 is in section 3.1.5.

![Figure 2.12: The regions \( D_1, D_2, D_3 \) and their boundaries](image-url)
Now we have to assign the Riemann surface to these subclasses and we have to define the global branches of the algebraic functions $h$ and $\Phi$. The subclass III is already considered (see subsection 2.3). For the subclass V we again use (as in III) all the attributes of the acceptable triple $A'$, i.e., the Riemann surface $R^*(A')$ in (2.65) and the global selection of the branches (2.66). For the subclass V we can also make item 2 of Proposition 2.17 more precise.

**Proposition 2.22.** For $A \in V$ we have (see Figure 2.11, second picture) that $b \in \overline{\Omega}_{0,2,1}$.

The proof of Proposition 2.22 is in section 3.1.5.

We denote the connected component of $\Omega_{0,2,1}$ containing $b$ by $G_1$ and its boundary by $E_1$. Also we set (see (2.64))

$$\Delta_{1,2} := \Delta_2 \cap G_1.$$ 

For the subclass V we use the measures $\lambda_1$, $\lambda_2$ introduced in Theorem 2.18. We set

$$\tilde{\lambda}_1 := \lambda_1, \quad \lambda_{1,2} := \lambda_2 \bigg|_{\Delta_{1,2}}, \quad \lambda_2 := \lambda_2,$$

where $\lambda_1$ and $\lambda_2$ at the right hand sides are from (2.66). In addition to these measures we shall use the balayage of the measure $\lambda_{1,2}$ on the contour $E_1$. We denote this measure by $\mu_1$:

$$V^{\mu_1} = V^{\lambda_{1,2}}, \quad \text{on } E_1.$$ 

The last case IV requires special treatment. We start with a characterization of the contour $\Gamma = \{z : |\Phi_j(z)| = |\Phi_k(z)|, j \neq k; j, k = 0, 1, 2\}$ for a non-acceptable triple $A'$. Consider the alternative to Definition 2.16 of an acceptable triple (see Figure 2.13)

**Definition 2.23.**

A. If the algebraic function $z(\nu)$ has a branch point on $(-2, 2)$ (see (2.63)) then the triple $A'$ is called critical.

B. ‘The triple $A'$ is called strictly non-acceptable when

1. The first condition in Definition 2.16 still holds.
2. For $\nu = 2$ we have (maybe after renumbering of trajectories $\gamma_{b^*}^{(j)}$)

$$\gamma_{a_1}(-2) = \gamma_{b^*}^{(1)}(-2), \quad \gamma_{a_2}(-2) = \gamma_{b^*}^{(2)}(-2), \quad \gamma_{b}(-2) = \gamma_{b^*}^{(3)}(-2).$$

From the definition we derive (see Figure 2.13)

**Proposition 2.24.**

A. For a critical triple one of the trajectories $\gamma_{\alpha}, \alpha \in \{a_2, b\}$ meets the trajectory $\gamma_{b^*}^{(2)}$ or $\gamma_{b^*}^{(3)}$, i.e.,

$$\exists \nu_m \in [-2, 2), \exists \alpha \in \{a_2, b_2\}, \exists j \in \{2, 3\} : \gamma_{\alpha}(\nu_m) = \gamma_{b^*}^{(j)}(\nu_m).$$

For a critical triple the meeting point of the trajectories coincides with $b^*$

$$\gamma_{\alpha}(\nu_m) = \gamma_{b^*}^{(j)}(\nu_m) = b^*.$$
For a strictly non-acceptable triple the arcs
\[ \gamma_1 = \gamma_{a_1,c} \cup \gamma_{c,b^*,1}, \quad \gamma_2 = \gamma_{a_2,c} \cup \gamma_{c,b^*,2}, \quad \gamma_3 = \gamma_{b,c} \cup \gamma_{c,b^*,3} \] (2.70)
intersect at one point \( c \)
\[ \{c\} = \gamma_1 \cap \gamma_2 \cap \gamma_3 \neq \emptyset. \]
The proof of this proposition is also in Subsection 3.1.5.

Figure 2.13: Non-acceptable triples: left is \( A \)-critical, right is \( B \)-strictly non-acceptable

For strictly non-acceptable triples we now assign a Riemann surface and define the global branches for the algebraic functions \( h \) and \( \Phi \). We have that each arc from (2.70) is divided into two parts by the point \( c \):
\[ \gamma_1 = \gamma_{a_1,c} \cup \gamma_{c,b^*,1}, \quad \gamma_2 = \gamma_{a_2,c} \cup \gamma_{c,b^*,2}, \quad \gamma_3 = \gamma_{b,c} \cup \gamma_{c,b^*,3}. \] (2.71)

We take three sheets of the complex plane cut as indicated (see Figure 2.14):
\[ R_0^* = \mathbb{C} \setminus (\gamma_{a_1,c} \cup \gamma_{a_2,c} \cup \gamma_{b,c}), \]
\[ R_1^* = \mathbb{C} \setminus \gamma_1, \]
\[ R_2^* = \mathbb{C} \setminus (\gamma_{c,b^*,1} \cup \gamma_{a_2,c} \cup \gamma_{b,c}), \] (2.72)
and then we glue them together so that they form a Riemann surface of genus zero:
\[ R^*(A') = R_0^* \cup R_1^* \cup R_2^*, \]
which we assign to the strictly non-acceptable triple \( A' \). The algebraic functions \( h \) and \( \Phi \) for the set of parameters \( A' = \{a_1; a_2, b\} \) are single valued rational functions on \( R^*(A') \). The structure of the three sheets (2.72) defines the global branches of \( h \) and \( \Phi \):
\[ h_0, \Phi_0 \in H(\mathbb{C} \setminus (\gamma_{a_1,c} \cup \gamma_{a_2,c} \cup \gamma_{b,c})), \]
\[ h_1, \Phi_1 \in H(\mathbb{C} \setminus \gamma_1), \]
\[ h_2, \Phi_2 \in H(\mathbb{C} \setminus (\gamma_{c,b^*,1} \cup \gamma_{a_2,c} \cup \gamma_{b,c})). \] (2.73)

We denote
\[ \Delta_1 := \gamma_{a_1,c}, \quad \Delta_2 := \gamma_{a_2,c}, \quad \Delta_{1,2} := \gamma_{c,b}, \]
and
\[ \Delta_1 := \bar{\Delta}_1 \cup \Delta_{1,2}, \quad \Delta_2 := \bar{\Delta}_2 \cup \Delta_{1,2}, \quad \Delta_0 := \Delta_1 \cup \Delta_2. \] (2.74)
We also denote
\[ E_2 := \gamma_1 \setminus \bar{\Delta}_1, \quad E_1 := \gamma_2 \setminus \bar{\Delta}_2, \quad E_0 := \gamma_3 \setminus \Delta_{1,2}. \] (2.75)
Figure 2.14: Sheet structure of the Riemann surface for a strictly non-acceptable triple (case IV)

**Proposition 2.25.** The contour $\Gamma$ (see (2.32) – (2.33)) has the following structure:

$$
\begin{align*}
\Gamma_{0,1} & := \{ z : |\Phi_0(z)| = |\Phi_1(z)| \} = \tilde{\Delta}_1, \\
\Gamma_{0,2} & := \{ z : |\Phi_0(z)| = |\Phi_2(z)| \} = \Delta_2, \\
\Gamma_{1,2} & := \{ z : |\Phi_1(z)| = |\Phi_2(z)| \} = \bigcup_{\ell=1}^{3} \gamma_{b^*,c}^{(\ell)} = E_0 \cup E_1 \cup E_2,
\end{align*}
$$

and for the domains $\Omega_{j,kt}$ we have (see Figure 2.11, third picture)

$$
\Omega_{0,2,1} = G \cup \tilde{G},
$$

where the components $G$ and $\tilde{G}$ are defined by their boundaries

$$
\partial G = E_1 \cup E_2, \quad \partial \tilde{G} = E_0 \cup E_2,
$$

and

$$
\Omega_{0,1,2} = \mathbb{C} \setminus \overline{\Omega_{0,2,1}}.
$$

Now we introduce the measures.

**Theorem 2.26.** For $A \in IV$ we have

1. The jump of $h_0$ on $\Delta_0$ produces a positive measure $\lambda$ of total mass 2

$$
\frac{1}{2\pi i} \left( h_{0+}(\xi) - h_{0-}(\xi) \right) d\xi =: d\lambda(\xi), \quad \xi \in \Delta_0.
$$
The measure $\lambda$ consists of two measures $\tilde{\lambda}_1$ and $\lambda_2$, with $|\tilde{\lambda}_1| + |\lambda_2| = 2$, supported on $\tilde{\Delta}_1$ and $\Delta_2$ respectively

$$
\lambda := \begin{cases} 
\tilde{\lambda}_1 & \text{on } \tilde{\Delta}_1, \\
\lambda_2 & \text{on } \Delta_2,
\end{cases}
$$

and

$$
\tilde{\lambda}_1(\xi) = \frac{m_1(\xi)}{\sqrt{\xi - a_1}}, \quad m_1 \in H(\tilde{\Delta}_1),
$$

$$
\lambda_2(\xi) = \frac{m_2(\xi)}{\sqrt{(\xi - a_2)(\xi - b)}}, \quad m_2 \in H(\Delta_2).
$$

2. The jump of $h_2$ on $E_2$ produces a positive measure $\mu_2$

$$
\frac{1}{3\pi i} \left( h_{2+}(\xi) - h_{2-}(\xi) \right) d\xi =: d\mu_2(\xi), \quad \xi \in E_2,
$$

and

$$
\mu'_2(\xi) = \sqrt{\xi - b^*} m_3(\xi), \quad m_3 \in H(E_2).
$$

3. There are connections among the total masses of these measures:

$$
|\tilde{\lambda}_1| + |\lambda_2| = 2, \quad |\lambda_2| - |\mu_2| = 1.
$$

We also denote

$$
\lambda_{1,2} := \lambda|_{\Delta_{1,2}}, \quad \lambda_1 := \lambda|_{\Delta_1}, \quad \tilde{\lambda}_2 := \lambda|_{\Delta_2}.
$$

The proof of Proposition 2.25 and Theorem 2.26 is given in Subsection 3.1.3.

Remark 2.27. [on critical cases] There are several cases of location of the branch points $A = \{a_1, b; a_2, b\}$ which are the limits of the cases described above. For example, the case (considered by Kalyagin in [44]) when

$$
A_0 := \{-1, 0; 1, 0\}
$$

is the limit of the class I

$$
A_{-\epsilon, \epsilon} := \{-1, -\epsilon; 1, \epsilon\}, \quad \text{as } \epsilon \to 0,
$$

and at the same time it is the limit of the class III

$$
A_{0, \epsilon} = \{-1, 0; 1, \epsilon\}, \quad \text{as } \epsilon \to 0.
$$

Also, the critical triple is both the limit of the strictly non-acceptable triple (case IV) and of the case V. We call these cases the critical cases. For the critical cases the assignment of the Riemann surface and the definition of the global branches for the algebraic functions $h$ and $\Phi$ is carried out by passing to the limit in the non-critical cases described above. Even though one of the main ingredients of our analysis, namely the geometry of the problem, is clear for the critical cases, we do not consider the Hermite-Padé asymptotics for these cases in this paper. (In what follows we exclude the critical triples from the
class IV.) The point is that the rigorous proof of the asymptotics for these cases requires a non-standard local Riemann-Hilbert analysis.

Certain critical cases of $2 \times 2$ matrix-valued Riemann-Hilbert problems have been studied recently and it was found that the local Riemann-Hilbert analysis can be constructed in an explicit way with the isomonodromy characterization of certain Painlevé transcendents, see [15, 21, 22, 23, 24, 32, 43]. Similar constructions can be expected to apply to some of the critical cases related to this work, but there are other critical cases as well which are special for $3 \times 3$ matrix-valued Riemann-Hilbert problems and which do not occur for $2 \times 2$ problems.

2.5 Weak asymptotics, convergence and vector potential problems

In this section we formulate corollaries from the strong asymptotics of the Hermite-Padé polynomials regarding the weak asymptotics of the poles of the Hermite-Padé approximants and their convergence. The weak results follow from the strong asymptotic formulas (Theorems 2.35 and 2.36) but they are stated here since to present these results we only need the basic functions which we have defined during the description of the geometry of the problem. The results will be different for each geometrical case.

2.5.1 Weak convergence

We recall that we consider the Hermite-Padé approximants for two functions (2.19), see Definition 2.1 and convention (2.16):

$$f_j \in A(a_j, \alpha_j; b_j, \beta_j; \Omega_j), \quad j = 1, 2,$$

where the domains $\Omega_j$ for the analytic continuation of the weight functions $w_{0,j}$, defining $f_j$ (see (2.5) and (2.2)), depend on the location of the branch points

$$A = \{a_1, b_1; a_2, b_2\}.$$

We assume $\Omega_j$ is such that:

1) $\Omega_j \supset \Delta_j$, \quad $j = 1, 2$, \quad for $A \in I$

2) $\Omega_j \supset \Delta_j \cup G$, \quad $j = 1, 2$, \quad for $A \in II$, \quad (2.80)

3) \begin{align*}
\Omega_1 & \supset \Delta^*_1, & \text{for } A \in III, \\
\Omega_2 & \supset \Delta_2, & \end{align*}

4) $\Omega_j \supset \Delta_j \cup G$, \quad $j = 1, 2$, \quad for $A \in IV$.

5) \begin{align*}
\Omega_1 & \supset \Delta^*_1 \cup G_1, & \text{for } A \in V \\
\Omega_2 & \supset \Delta_2 \cup G_1, & \end{align*}

We recall that the definition of the classes I, II, III, IV, and V is given in Definitions 2.7, 2.19, 2.20 and the corresponding arcs $\Delta_1$, $\Delta^*_1$, $\Delta_2$ and sets $G_1$, $G$ are defined in (2.40), (2.51), (2.49), (2.64), Proposition 2.24, (2.74), and (2.77).

The weak limit of the counting measures $\nu_{P_n}$, which have equal mass $1/(2n)$ at the poles of the Hermite-Padé approximants (see (1.4)), has a universal character.
Theorem 2.28. Suppose that $A$ belongs to one of the cases I, II, III, IV, or V. Then the poles of the Hermite-Padé approximants \((1.2)\) for the functions in \((2.79)\)–\((2.80)\) have a weak limit

$$\nu_{P_n} \to \lambda/2, \quad n \to \infty,$$

where the limiting measure $\lambda$ is defined in Theorems 2.10, 2.14, 2.18, 2.26, depending on the geometrical class.

Now we state a result about the finite zeros of the functions of the second kind \((1.2)\)

$$R_n^{(j)} := f_j P_n - Q_n^{(j)}, \quad j = 1, 2.$$ 

These zeros represent the extra interpolation points. We use the notation $\nu_{R_n^{(j)}}$ for the counting measures with equal mass $1/n$ at the finite zeros of $R_n^{(j)}$.

Theorem 2.29. Consider $R_n^{(j)}$ for the functions \((2.79)\) where $\Omega_j$ is as in \((2.80)\).

1. When $A \in \text{I} \cup \text{III}$ there are no extra interpolation points for $n$ large enough.
2. When $A \in \text{II} \cup \text{IV}$ we have

$$\nu_{R_n^{(j)}} \to \mu_j, \quad j = 1, 2.$$ 

3. When $A \in \text{V}$ the function $R_n^{(2)}$ has no finite zeros for $n$ large enough and

$$\nu_{R_n^{(1)}} \to \mu_1.$$ 

In 2 and 3 the limiting measures $\mu_1, \mu_2$ are defined in Theorems 2.14, 2.26 and in \((2.55)\), \((2.63)\), depending on the geometrical class.

The next theorem describes the $n$th root asymptotics of the error term

$$f_j - \pi_n^{(j)} = \frac{R_n^{(j)}}{P_n}, \quad \pi_n = \frac{Q_n^{(j)}}{P_n}, \quad j = 1, 2,$$

and the convergence of the Hermite-Padé approximants \((1.2)\).

Theorem 2.30. Consider the Hermite-Padé approximants for the functions \((2.79)\) with $\Omega_j$ as in \((2.80)\).

1. When $A \in \text{I}$ we have

$$|f_j - \pi_n^{(j)}|^{1/n} \to \frac{\Phi_0}{\Phi_j},$$

uniformly on compact subsets of $\mathbb{C} \setminus \Delta_j$, $j = 1, 2$, and therefore for the Hermite-Padé approximants $\pi_n^{(j)}$ we have

$$\pi_n^{(j)} \to f_j, \quad \text{ on } \Omega_{0,j},$$

$$|\pi_n^{(j)} - f_j| \to \infty, \quad \text{ on } \mathbb{C} \setminus (\Delta_j \cup \Omega_{0,j}), \quad j = 1, 2,$$

with geometric rate. Furthermore (see Figure 2.6) there exists an $A \in \text{I}$ such that $\Omega_{1,0} \neq \emptyset$ or $\Omega_{2,0} \neq \emptyset$. 

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2. When $A \in \text{III}$ we have

$$|f_j - \pi_n^{(j)}|^{1/n} \to \left| \frac{\Phi_0}{\Phi_j} \right|$$

uniformly on compact subsets of $\overline{C} \setminus \Delta_j^*$ when $j = 1$ and $\overline{C} \setminus \Delta_2$ when $j = 2$, and therefore, with the convention (2.56), we have

$$\pi_n^{(1)} \to f_1 \quad \text{on } \overline{C} \setminus (\Delta_1^* \cup \Omega_{1,0}),$$

$$|\pi_n^{(1)} - f_1| \to \infty \quad \text{on } \Omega_{1,0},$$

$$\pi_n^{(2)} \to f_2 \quad \text{on } \overline{C} \setminus \Delta_2.$$

Furthermore (see Figure 2.10) $\Omega_{1,0} \neq \emptyset$.

3. When $A \in \text{V}$ we have (uniformly on compact subsets of the indicated sets)

$$|f_1 - \pi_n^{(1)}|^{1/n} \to \begin{cases} 
\left| \frac{\Phi_0}{\Phi_1} \right| & \text{on } (\overline{C} \setminus \Delta_1^*) \setminus G_1 \\
\left| \frac{\Phi_0}{\Phi_2} \right| & \text{on } G_1 \setminus \Delta_2
\end{cases}$$

and therefore (see Figure 2.11)

$$\pi_n^{(1)} \to f_1 \quad \text{on } \overline{C} \setminus (\Delta_1^* \cup \Omega_{1,0}),$$

$$|\pi_n^{(1)} - f_1| \to \infty \quad \text{on } \Omega_{1,0},$$

where $\Omega_{1,0} \neq \emptyset$, and

$$\pi_n^{(2)} \to f_2 \quad \text{on } \overline{C} \setminus \Delta_2.$$

4. When $A \in \text{IV}$ we have (uniformly on compact subsets of the indicated sets)

$$|f_1 - \pi_n^{(1)}|^{1/n} \to \begin{cases} 
\left| \frac{\Phi_0}{\Phi_1} \right| & \text{on } (\overline{C} \setminus \Delta_1) \setminus G \\
\left| \frac{\Phi_0}{\Phi_2} \right| & \text{on } G \setminus \Delta_{1,2}
\end{cases}$$

$$|f_2 - \pi_n^{(2)}|^{1/n} \to \left| \frac{\Phi_0}{\Phi_2} \right| \quad \text{on } \overline{C} \setminus (\Delta_2 \cup E_2)$$

and therefore we have

$$\pi_n^{(j)} \to f_j \quad \text{on } \overline{C} \setminus \Delta_j, \quad j = 1, 2.$$

5. When $A \in \text{II}$ we have (uniformly on compact subsets of the indicated sets)

$$|f_1 - \pi_n^{(1)}|^{1/n} \to \left| \frac{\Phi_0}{\Phi_1} \right| \quad \text{on } \overline{C} \setminus (\Delta_1 \cup E_1),$$
\[ |f_2 - \pi^{(2)}_n|^{1/n} \rightarrow \begin{cases} \Phi_0 \\ \Phi_2 \end{cases} \text{ on } (\mathbb{C} \setminus \Delta_2) \setminus G \]

and therefore we have
\[ \pi_n^{(j)} \rightarrow f_j \text{ on } \mathbb{C} \setminus \Delta_j, \quad j = 1, 2. \]

We recall that the definition of the branches of the algebraic function \( \{ \Phi_0, \Phi_1, \Phi_2 \} \) and the domains of divergence \( \Omega_{j,0} \), \( j = 1, 2 \), depending on the geometrical case is given in (2.45), (2.27)–(2.28), (2.42), (2.53), (2.66), and (2.73).

### 2.5.2 Vector equilibrium problem

Concluding this section we state a universal **vector equilibrium problem** for the logarithmic potentials of the measures \( \lambda = \lambda_1 + \lambda_2, \mu_1, \mu_2 \). These measures were introduced in Theorems 2.10, 2.14, 2.18, and 2.26 for the description of the weak limits of the poles and of the extra interpolation points (see Theorems 2.28 and 2.29).

**Theorem 2.31.** Suppose that
\[ A \in \text{I} \cup \text{III} \cup \text{II} \cup \text{V} \cup \text{IV}, \]
then

1. a) There exist piecewise analytic arcs \( \Delta_1, \Delta_2 \) which make the functions \( f_1, f_2 \) in (2.79) holomorphic
\[ f_j \in H(\mathbb{C} \setminus \Delta_j), \quad j = 1, 2, \]
and a piecewise analytic contour \( E \) containing the common part of \( \Delta_1 \) and \( \Delta_2 \)
\[ \Delta_{1,2} := \Delta_1 \cap \Delta_2, \quad (\Delta_{1,2} = \emptyset \Rightarrow E = \emptyset). \]

b) There exists a triple of measures \( (\lambda_1, \tilde{\lambda}_2, \mu_1) \) with supports (we use the notation \( S(\mu) = \text{supp}(\mu) \))
\[ S(\lambda_1) \subset \Delta_1, \quad S(\tilde{\lambda}_2) \subset \bar{\Delta}_2 := \Delta_2 \setminus \Delta_{1,2}, \quad S(\mu_1) \subset E, \]
and with the relations on their total mass
\[ \begin{cases} |\lambda_1| + |\tilde{\lambda}_2| = 2 \\ |\lambda_1| - |\mu_1| = 1. \end{cases} \]

c) This triple of measures possesses the following equilibrium relations with some constants \( \kappa_1 \) and \( \tilde{\kappa}_2 \):
\[ U_1 := 2V^{\lambda_1} + V^{\tilde{\lambda}_2} - V^{\mu_1} \]
\[ \begin{cases} = \kappa_1, & \text{on } S(\lambda_1), \\ \geq \kappa_1, & \text{on } \Delta_1, \end{cases} \]
\[ U_2 := V^{\lambda_1} + 2V^{\tilde{\lambda}_2} + V^{\mu_1} \]
\[ \begin{cases} = \tilde{\kappa}_2, & \text{on } S(\tilde{\lambda}_2), \\ \geq \tilde{\kappa}_2, & \text{on } \bar{\Delta}_2, \end{cases} \]
\[ U_3 := -V^{\lambda_1} + V^{\tilde{\lambda}_2} + 2V^{\mu_1} \]
\[ \begin{cases} = \tilde{\kappa}_2 - \kappa_1, & \text{on } S(\mu_1), \\ \geq \tilde{\kappa}_2 - \kappa_1, & \text{on } E. \end{cases} \]
d) The supports of the measures possess the following symmetry relations

\[
\begin{align*}
\frac{\partial U_1}{\partial n_+} &= \frac{\partial U_1}{\partial n_-}, \quad \text{on } S(\lambda_1), \\
\frac{\partial U_2}{\partial n_+} &= \frac{\partial U_2}{\partial n_-}, \quad \text{on } S(\tilde{\lambda}_2), \\
\frac{\partial U_3}{\partial n_+} &= \frac{\partial U_3}{\partial n_-}, \quad \text{on } S(\mu_1),
\end{align*}
\]

(2.84)

where \(\partial /\partial n_{\pm}\) denotes the normal derivatives on the respective contours.

II. There is also a dual problem regarding the triple \((\lambda_2, \tilde{\lambda}_1, \mu_2)\) which can be obtained from the problem I.a)–I.d) by interchanging the indices 1 and 2.

III. The equilibrium measures \((\lambda_1, \tilde{\lambda}_2, \mu_1)\) and \((\lambda_2, \tilde{\lambda}_1, \mu_2)\) are related as follows

\[
\begin{align*}
\lambda &= \lambda_1 + \tilde{\lambda}_2 = \lambda_2 + \tilde{\lambda}_1, \quad S(\lambda) \subset \Delta_0 := \Delta_1 \cup \Delta_2, \\
\mu &= \mu_1 + \mu_2, \quad S(\mu_1) \cup S(\mu_2) = E, \\
V^\mu \bigm|_E &= V^{\lambda_{1,2}} \bigm|_E, \quad \lambda_{1,2} := \lambda \bigg|_{\Delta_{1,2}}.
\end{align*}
\]

IV. The measure \(\lambda/2\) is the weak limit (2.81) of the poles of the Hermite-Padé approximants of the functions (2.79), and the measures \(\mu_1\) and \(\mu_2\) are the weak limits (2.82) of the extra interpolation points.

The proof of Theorem 2.30 is given in Section 3.2.

We now describe the basic notions of the equilibrium problems I and II of Theorem 2.30

\[
\begin{align*}
\{\Delta_1, \tilde{\Delta}_2, E \} & \quad \{\Delta_2, \tilde{\Delta}_1, E \} \\
\{\lambda_1, \tilde{\lambda}_2, \mu_1\} & \quad \{\lambda_2, \tilde{\lambda}_1, \mu_2\}
\end{align*}
\]

in conformity with the geometrical cases.

Remark 2.32.

1. In the case I we have \(\Delta_{1,2} = \emptyset\) and therefore

\[
\begin{align*}
\Delta_j &= \tilde{\Delta}_j, & j &= 1, 2, \\
E &= \emptyset
\end{align*}
\]

(2.85)

\[
\begin{align*}
\lambda_j &= \tilde{\lambda}_j, & \mu_j &= 0, \quad S(\lambda_j) = \Delta_j, & j &= 1, 2,
\end{align*}
\]

and in the systems of the equilibrium and symmetry relations (2.83)–(2.84) only the first two relations are needed.

2. In the case III, we set

\[
\Delta_1 = \Delta^*_1 \cup \delta^*_1, \quad \delta^*_1 \subset \Omega_{1,0}
\]

(2.86)

where \(\delta^*_1\) is an arbitrary rectifiable arc in \(\Omega_{1,0,2}\) joining the points \(b^*\) and \(b_1\) (see (2.45) and Figure 2.10). For this case we have the same setting (2.85) as for the case I with the minor difference that \(S(\lambda_1) = \Delta^*_1 \subset \Delta_1\).
3. We note that for the cases I and III the equilibrium relations (2.83) reduce to the equilibrium relations for an Angelesco system (1.12), (1.15) in the complex plane.

4. In the case V we set (see Proposition 2.24)

\[ \Delta_1 := \widetilde{\Delta}_1 \cup \delta_1 \cup \Delta_{1,2}, \quad \widetilde{\Delta}_j := \Delta_j \setminus \Delta_{1,2}, \quad j = 1, 2, \]

where \( \delta_1 \) is an arbitrary rectifiable arc in \( \Omega_{1,0,2} \) and joining the corresponding end points of the arcs \( \Delta_1 \) and \( \Delta_{1,2} \). Here we have

\[ E = E_1 \neq \emptyset. \]

If we consider the dual problem (see II of Theorem 2.31) for the triple

\[ \{ \Delta_2, \widetilde{\Delta}_1, E \} \]

\[ \{ \lambda_2, \lambda_1, \mu_2 \} \]

then the solution of the Angelesco equilibrium problem for the case III

\[ \begin{cases} 2V^{\lambda_2} + V^{\lambda_1} = \kappa_2, & \text{on } \Delta_2, \\ V^{\lambda_2} + 2V^{\lambda_1} \geq \tilde{\kappa}_1, & \text{on } \Delta_1 (= \kappa_1 \text{ on } \Delta_1^*) \end{cases} \]

provides the solution of the dual equilibrium problem with \( \mu_2 = 0 \), because (see Figure 2.11, second picture)

\[ V^{\lambda_1} - \tilde{\kappa}_1 = V^{\lambda_2} - \kappa_2, \quad \text{on } E. \]

In order to obtain the measure \( \mu_1 \) we have to take the balayage of the measure

\[ \lambda_{1,2} := \lambda_2|_{\Delta_{1,2}} \]

on the contour \( E \) or consider the problem (2.83)–(2.84) for the triple

\[ \{ \Delta_1, \widetilde{\Delta}_2, E \} \]

\[ \{ \lambda_1, \lambda_2, \mu_1 \}. \]

5. In the cases IV and II the contours for the equilibrium problems are defined in (2.74)–(2.75) and (2.48)–(2.51). In these cases (see Theorem 2.14 and 2.26) there are no degeneracies of the components of the equilibrium problem (2.83) and the dual problem.

2.6 Szegő functions for the Hermite-Padé polynomials

The Szegő function is an important ingredient of the strong asymptotic formulas for orthogonal polynomials on an interval of the real line and on the unit circle. In this section we introduce a generalization of the Szegő function which we need for the presentation of the strong asymptotics of the Hermite-Padé polynomials. We shall define the Szegő
function as a solution of a certain boundary value problem on the three-sheeted Riemann surface

\[ \mathcal{R} := \bigcup_{j=0}^{2} \mathcal{R}_j. \]

Before we state the boundary value problem we need some preparation. We fix the cuts \( \delta_{j,k} \) (Jordan arcs in the complex plane or a union of Jordan arcs) where the sheets \( \mathcal{R}_j \) and \( \mathcal{R}_k \) are glued together

\[ \mathcal{R}_\ell := \mathbb{C} \setminus (\delta_{\ell,j} \cup \delta_{\ell,k}), \quad \ell \notin \{k, j\}, \ k \neq j, \ \ell, k, j = 0, 1, 2. \]

The two sides of the cuts \( (\delta_{\ell,j} \cup \delta_{\ell,k})^{(\pm)} \) form a boundary of the sheet \( \mathcal{R}_\ell \) (Jordan curves of the Riemann surface \( \mathcal{R} \))

\[ \partial \mathcal{R}_\ell := \partial \mathcal{R}_{\ell,j} \cup \partial \mathcal{R}_{\ell,k}, \quad \partial \mathcal{R}_{\ell,j,k} := \mathcal{R}_\ell \cap \mathcal{R}_k = \delta_{\ell,k}^{(+) \cup \delta_{\ell,k}^{(-)}}. \]

Thus

\[ \mathcal{R} := \left( \bigcup_{j=0}^{2} \mathcal{R}_j \right) \cup \partial \mathcal{R}, \quad \partial \mathcal{R} := \partial \mathcal{R}_{0,1} \cup \partial \mathcal{R}_{0,2} \cup \partial \mathcal{R}_{1,2}. \]

We fix the orientation of the union of the Jordan curves \( \partial \mathcal{R} \) such that

\[ \partial \mathcal{R}_{j,k}^{(+)} \subset \mathcal{R}_j, \quad \partial \mathcal{R}_{j,k}^{(-)} \subset \mathcal{R}_k, \quad j > k, \ j, k = 0, 1, 2. \]

We also fix an orientation of the Jordan arcs \( \delta_{j,k} \). Our Riemann surfaces have four branch points at \( \{a_1, b_1, a_2, b_2\} \), with a possible replacement of \( b_1 \) by \( b^* \) for the cases III, IV and V. We have

1. \( \{a_j, b_j\} \subset \delta_{0,j}, \ j = 1, 2 \) for \( A \in I \cup II \),
2. \( \{a_1, b^*\} \subset \delta_{0,1}, \ \{a_2, b\} \subset \delta_{0,2} \) for \( A \in III \cup V \),
3. \( a_1 \in \delta_{0,1}, \ b^* \in \delta_{1,2}, \ \{a_2, b\} \subset \delta_{0,2} \) for \( A \in IV \).

Let \( \omega_j(z) \) be the branch of the function

\[ \omega_j^2(z) = (z - a_j)(z - b_j), \quad \omega_j(z) = z + \cdots, \quad z \to \infty, \quad j = 1, 2. \tag{2.87} \]

For the cases III, IV and V we replace \( b_1 \) by \( b^* \) in \( (2.87) \). From the statement of the problem (see Section \( 2.1 \)) and from the definitions of the geometrical cases (Sections \( 2.2-2.5 \)) it follows that the weight functions \( w_1 \) and \( w_2 \) are defined on the arcs \( \delta_{0,1} \) and \( \delta_{0,2} \).

We set

\[ \tilde{w}_j := i w_j \omega_j, \quad \text{on} \ \delta_{0,j}, \quad j = 1, 2. \tag{2.88} \]

We assume that the orientation of the arcs \( \delta_{k,j} \) is fixed so that (see, for example, Figure \( 2.7 \)) each \( \tilde{w}_j \) has an analytic continuation on \( \delta_{1,2} \) (when \( \delta_{1,2} \neq \emptyset \)). Then we define

\[ \tilde{w} := \begin{cases} \tilde{w}_1, & \text{on} \ \delta_{0,1}, \\ \tilde{w}_2, & \text{on} \ \delta_{0,2}, \\ \tilde{w}_2/\tilde{w}_1, & \text{on} \ \delta_{1,2}. \end{cases} \tag{2.89} \]
We note that, depending on the position of the points \( A \), we have
\[
A \in I \cup III \cup V \Rightarrow \delta_{1,2} = \emptyset, \\
A \in II \cup IV \Rightarrow \delta_{1,2} \neq \emptyset.
\]

Finally we duplicate the weight function \( \tilde{w} \) on both sides of \( \delta_{j,k}^{(\pm)} \), i.e., we define \( \tilde{w} \) on \( \partial\mathcal{R} \)
\[
\tilde{w}_+ = \tilde{w}_- \quad \text{on} \quad \delta_{k,j} \Rightarrow \tilde{w} \quad \text{on} \quad \partial\mathcal{R}.
\] (2.90)

Now we formulate the boundary value problem on \( \mathcal{R} \) whose solution defines the desired generalization of the Szegő function. We are looking for a piecewise holomorphic function \( \mathcal{F} \) on \( \mathcal{R} \) such that

\[
\begin{align*}
1. & \quad \mathcal{F} \in H(\mathcal{R} \setminus \partial\mathcal{R}), \\
2a. & \quad \exists \mathcal{F}_\pm \in C(\partial\mathcal{R} \setminus \{a_1, b_1, a_2, b_2\}) : \mathcal{F}_+ = \mathcal{F}_- \tilde{w} \quad \text{on} \quad \partial\mathcal{R} \\
2b. & \quad |\mathcal{F}_2(z)\tilde{w}(z)| = O(1), \quad z \to A \\
3. & \quad \mathcal{F}(\infty(0))\mathcal{F}(\infty(1))\mathcal{F}(\infty(2)) = 1.
\end{align*}
\] (2.91)

**Remark 2.33.** The maximum principle implies that if \( \mathcal{F} \) from (2.91) exists, then

a) For every \( z \in \overline{\mathbb{C}} \)
\[
\mathcal{F}(z(0))\mathcal{F}(z(1))\mathcal{F}(z(2)) = 1.
\] (2.92)

b) \( \mathcal{F} \) is unique.

Here \( z(j) := \pi^{-1}(z) \), where \( \pi : \mathcal{R} \to \overline{\mathbb{C}} \) is the projection of the sheets of \( \mathcal{R} \) onto \( \overline{\mathbb{C}} \).

Due to (2.90) the index of the boundary value problem (2.91) is equal to zero. This gives the existence of the solution of (2.91). In order to write an expression for the solution of (2.91) we use the meromorphic (Cauchy) differential on \( \mathcal{R} \)
\[
d\xi(z), \quad z \in \mathcal{R}, \xi \in \mathcal{R},
\]
which has simple poles at the points \( \xi \in \mathcal{R} \) and at the points \( \tilde{\xi} \in \mathcal{R} \) where \( \tilde{\xi} \) has the same projection on \( \overline{\mathbb{C}} \) as \( \xi \), i.e., \( \pi(\xi) = \pi(\tilde{\xi}), \xi \neq \tilde{\xi} : \)
\[
M_\xi'(z) \to \infty, \quad z \to \{\xi, \tilde{\xi}(k), \tilde{\xi}(\ell)\},
\]
and the residues at the poles are
\[
\text{res}_{z=\xi} M_\xi = 2, \quad \text{res}_{z=\xi} M_\xi' = -1, \quad \pi(\xi) = \pi(\tilde{\xi}), \xi \neq \tilde{\xi}.
\]

Then Cauchy’s theorem on \( \mathcal{R} \) and (2.92) give
\[
\frac{1}{2\pi i} \int_{\partial\mathcal{R}} \log \tilde{w}(z) d\xi(z) = 2 \log \mathcal{F}(\xi) - \log \mathcal{F}(\tilde{\xi}(k)) - \log \mathcal{F}(\tilde{\xi}(\ell)) = 3 \log \mathcal{F}(\xi).
\]

Summarising we have proved

**Theorem 2.34.** There exists a unique solution of the boundary value problem (2.91), which is given by
\[
\mathcal{F}(\xi) = \exp \left( \frac{1}{6\pi i} \int_{\partial\mathcal{R}} \log \tilde{w}(z) d\xi(z) \right), \quad \xi \in \mathcal{R} \setminus \partial\mathcal{R}.
\]

Later on we shall use the notation for the branches of the multivalued function \( \mathcal{F} \)
\[
F_\ell(z) := \mathcal{F}(z(\ell)), \quad \ell = 0, 1, 2,
\] (2.93)

where \( z \in \overline{\mathbb{C}} \setminus (\delta_{\ell,j} \cup \delta_{\ell,k}), \ell \notin \{k, j\}, k \neq j, \) and \( \ell, k, j = 0, 1, 2. \)
2.7 Strong asymptotics

We can now state the main results of this paper on the strong asymptotics of the Hermite-Padé approximants. The formulas for strong asymptotics contain the functions $\Phi := \{\Phi_0, \Phi_1, \Phi_2\}$ from (1.28) and the Szegő functions $F := \{F_0, F_1, F_2\}$ from (2.91) and (2.93). These functions are defined by the Riemann surface $\mathfrak{R}$ of the corresponding geometrical cases (defined in Sections 2.2 - 2.5).

The strong asymptotics of the denominators $P_n$ of the Hermite-Padé approximants have a universal character.

**Theorem 2.35.** Suppose that

$$A = \{a_1, b_1; a_2, b_2\} \in I \cup II \cup III \cup IV \cup V.$$  

Then the denominators of the Hermite-Padé approximants (1.2) for the functions $f_j$, with branch points $\{a_1, b_1, a_2, b_2\}$ (see (2.79) - (2.80)) have the following asymptotic formulas as $n \to \infty$, which hold uniformly on compact subsets of the indicated sets:

$$P_n(z) = \frac{F_0(\infty)}{F_0(z)}(C_0\Phi_0(z))^{-n}(1 + O(1/n)), \quad z \in \mathbb{C} \setminus (\delta_{0,1} \cup \delta_{0,2}),$$

and

$$P_n(z) = \left[\left(\frac{F_0(\infty)}{F_0(z)}(C_0\Phi_0(z))^{-n}\right) + \left(\frac{F_0(\infty)}{F_0(z)}(C_0\Phi_0(z))^{-n}\right)\right](1 + o(1)), \quad z \in (\delta_{0,1} \cup \delta_{0,2}) \setminus A.$$

We recall that we have

$$\begin{cases} 
\delta_{0,j} = \Delta_j, & j = 1, 2, \quad \text{for} \ A \in I, \\
\delta_{0,1} = \Delta^*_1, \delta_{0,2} = \Delta_2, & \text{for} \ A \in III \cup V, \\
\delta_{0,1} = \Delta_1, \delta_{0,2} = \Delta_2, & \text{for} \ A \in II, \\
\delta_{0,1} = \tilde{\Delta}_1, \delta_{0,2} = \Delta_2, & \text{for} \ A \in IV.
\end{cases}$$

(2.94)

The strong asymptotics for the functions of the second kind $R_n^{(j)}$, $j = 1, 2$, have different forms, depending on the geometrical case.

**Theorem 2.36.** For the asymptotics, as $n \to \infty$, of the functions of the second kind $R_n^{(j)}$, given by (2.71), of the Hermite-Padé approximants for the functions $f_j$ with branch points $\{a_1, b_1, a_2, b_2\}$ in (2.79) - (2.80), we have

1. When $A \in I \cup III$ we have

$$R_n^{(j)}(z) = -\frac{F_0(\infty)}{\omega_j(z)F_j(z)}(C_0\Phi_j(z))^{-n}(1 + O(1/n)), \quad z \in \mathbb{C} \setminus \delta_{0,j},$$

$$R_n^{(j)}(z) = \left(-\frac{F_0(\infty)}{\omega_j(z)F_j(z)}(C_0\Phi_j(z))^{-n}(1 + O(1/n))\right)_\pm, \quad z \in \delta_{0,j} \setminus A,$$

for $j = 1, 2$. Here the $\delta_{0,j}$ are as in (2.94).
2. When \( A \in V \) we have

\[
R_n^{(2)}(z) = -\frac{F_0(\infty)}{\omega_2(z)F_2(z)}(C_0\Phi_2(z))^{-n}(1 + \mathcal{O}(1/n)), \quad z \in \overline{\mathbb{C}} \setminus \Delta_2,
\]

\[
R_{n \pm}^{(2)}(z) = \left( -\frac{F_0(\infty)}{\omega_2(z)F_2(z)}(C_0\Phi_j(z))^{-n}(1 + \mathcal{O}(1/n)) \right)_\pm, \quad z \in \Delta_2 \setminus A,
\]

and

\[
R_n^{(1)}(z) = -\frac{F_0(\infty)}{\omega_1(z)F_1(z)}(C_0\Phi_1(z))^{-n}(1 + \mathcal{O}(1/n)), \quad z \in (\overline{\mathbb{C}} \setminus \Delta_1^*) \setminus \overline{G}_1,
\]

\[
R_n^{(1)}(z) = -\frac{F_0(\infty)}{\omega_1(z)F_1(z)}(C_0\Phi_1(z))^{-n}\frac{w_1(z)}{w_2(z)}(1 + \mathcal{O}(1/n)), \quad z \in G_1 \setminus \Delta_{1,2},
\]

\[
R_{n \pm}^{(1)}(z) = \left( -\frac{F_0(\infty)}{\omega_1(z)F_1(z)}(C_0\Phi_1(z))^{-n} \right)_\pm (1 + \mathcal{O}(1/n)), \quad z \in \Delta_1^* \setminus A,
\]

\[
R_{n \pm}^{(1)}(z) = \left( -\frac{F_0(\infty)}{\omega_1(z)F_1(z)}(C_0\Phi_1(z))^{-n} \right)_\pm\frac{w_1(z)}{w_2(z)}, \quad z \in \Delta_{1,2} \setminus A,
\]

\[
R_n^{(1)}(z) = \frac{F_0(\infty)}{C_0^n} \left( -\frac{1}{\omega_1(z)F_1(z)\Phi_1^n(z)} - \frac{1}{w_1(z)} \right) (1 + \mathcal{O}(1/n)), \quad z \in \partial G_1 \setminus A.
\]

3. When \( A \in IV \) we have, with \( \delta_{0,2} := \Delta_2 \cup E_2 \)

\[
R_n^{(2)}(z) = -\frac{F_0(\infty)}{\omega_2(z)F_2(z)}(C_0\Phi_2(z))^{-n}(1 + \mathcal{O}(1/n)), \quad z \in \overline{\mathbb{C}} \setminus \delta_{0,2},
\]

\[
R_{n \pm}^{(2)} = \left( -\frac{F_0(\infty)}{\omega_2(z)F_2(z)}(C_0\Phi_2(z))^{-n} \right)_\pm (1 + \mathcal{O}(1/n)), \quad z \in \Delta_2 \setminus A,
\]

\[
R_n^{(2)}(z) = \frac{F_0(\infty)}{C_0^n} \left( -\frac{1}{(\omega_2(z)F_2(z)\Phi_2^n(z))_+} - \frac{1}{(\omega_2(z)F_2(z)\Phi_2^n(z))_-} \right) (1 + \mathcal{O}(1/n)), \quad z \in E_2 \setminus \{b^*\},
\]

and with \( G \) such that \( \partial G = E_1 \cup E_2 \)

\[
R_n^{(1)}(z) = -\frac{F_0(\infty)}{\omega_1(z)F_1(z)}(C_0\Phi_1(z))^{-n}(1 + \mathcal{O}(1/n)), \quad z \in (\overline{\mathbb{C}} \setminus \overline{\Delta}_1) \setminus \overline{G},
\]

\[
R_n^{(1)}(z) = -\frac{F_0(\infty)}{\omega_1(z)F_1(z)}(C_0\Phi_1(z))^{-n}\frac{w_1(z)}{w_2(z)}(1 + \mathcal{O}(1/n)), \quad z \in G \setminus \Delta_{1,2},
\]

\[
R_{n \pm}^{(1)}(z) = \left( -\frac{F_0(\infty)}{\omega_1(z)F_1(z)}(C_0\Phi_1(z))^{-n} \right)_\pm (1 + \mathcal{O}(1/n)), \quad z \in \overline{\Delta}_1,
\]

\[
R_{n \pm}^{(1)}(z) = \left( -\frac{F_0(\infty)}{\omega_2(z)F_2(z)}(C_0\Phi_2(z))^{-n} \right)_\pm\frac{w_1(z)}{w_2(z)}, \quad z \in \Delta_{1,2},
\]

\[
R_n^{(1)}(z) = \frac{F_0(\infty)}{C_0^n} \left( -\frac{1}{\omega_1(z)F_1(z)\Phi_1^n(z)} - \frac{1}{w_1(z)} \right) (1 + \mathcal{O}(1/n)), \quad z \in E_1 \setminus \{b^*\},
\]

\[
R_n^{(1)}(z) = \frac{F_0(\infty)}{C_0^n} \left( 1 + \mathcal{O}(1/n) \right) + w_1(z) w_2(z) - \frac{F_0(\infty)}{C_0^n} \left( 1 + \mathcal{O}(1/n) \right) \frac{w_1(z)}{w_2(z)}, \quad z \in E_2 \setminus \{b^*\}.
\]
4. When \( A \in \Pi \) we have the same asymptotic formulas as in 3) (the case \( A \in IV \)) if we interchange the indices 1 and 2 and replace \( \{ \alpha \} \) by \( \emptyset \).

The proof of the strong asymptotics formulas is given in Section 4. It follows from a steepest descent analysis of the Riemann-Hilbert problem (2.13).

3 Proof of the geometrical results and equilibrium properties

3.1 Proof of the geometrical results

Here we present the proofs of the propositions and the theorems stated in subsections 2.2–2.4. We start with some results of general character.

3.1.1 Proof of Propositions 2.3, 2.4 and 2.5

Proof.

1. The parametrization of the algebraic curve (2.21) of order 3 and genus 0, given in Proposition 2.3 can be checked directly. Indeed, substituting the polynomial coefficients (2.24) and (2.25) into the equation (2.21) for the function \( h \) and computing the discriminant

\[
D := \frac{P_2^3 - \Pi_4 P_1^2}{\Pi_4^3},
\]

we obtain that the polynomial

\[
P_2^3 - \Pi_4 P_1^2 = \tilde{D} = (kP_1^2 + 3P_1 ps + s^3)^2
\]

is a complete square and therefore its zeros are not square root branch points of the algebraic function \( h \) (the Puiseux series at these points have integer exponents or cubic roots in some degenerate cases). Thus (in the non-degenerate cases) the only branch points of the function \( h \) are the four zeros of the polynomial \( \Pi_4 \). The Riemann-Hurwitz formula (for a function of the third order with four branch points) implies that \( h \) has genus 0.

2. To find the coefficients (2.28) for the equation of the algebraic function \( \Phi \) in Proposition 2.4 we proceed as follows. We make a linear change of variable \( z \to P_1 := z - c \) and substitute the series

\[
h = 1 + \sum_{k=1}^{\infty} \frac{c_k}{P_1^k} \]

into the equation (2.21) to find two sets of coefficients \( \{ c_k^{(1)} \} \) and \( \{ c_k^{(2)} \} \). Thus, we obtain the coefficients of the expansion at \( P_1 = \infty \) for the three branches of \( h \)

\[
1 + \sum_{k=1}^{\infty} \frac{c_k^{(1)}}{P_1^k}, \quad 1 + \sum_{k=1}^{\infty} \frac{c_k^{(2)}}{P_1^k}, \quad -2 - \sum_{k=1}^{\infty} \frac{c_k^{(1)} + c_k^{(2)}}{P_1^k}.
\]
Then, see (2.27), we integrate these three series, take the exponential function of the results and again expand it into a power series around \( P_1 = \infty \)

\[
\tilde{\Phi}_1 = P_1 + \sum_{k=0}^{\infty} \frac{d_k^{(1)}}{P_1^k}, \quad \tilde{\Phi}_2 = P_1 + \sum_{k=0}^{\infty} \frac{d_k^{(2)}}{P_1^k}, \quad \tilde{\Phi}_0 = \sum_{k=2}^{\infty} \frac{d_k^{(0)}}{P_1^k}.
\]

Next, we find the constants \( m_\ell, \ell = 0, 1, 2 \), such that the power series

\[
\sum_{\ell=0}^{2} m_\ell \tilde{\Phi}_\ell
\]

is a polynomial. In this way we obtain the series around \( P_1 = \infty \) for the branches of the function \( \Phi \)

\[
\Phi_\ell = m_\ell \tilde{\Phi}_\ell.
\]

Finally, we use the Vieta relations to obtain the coefficients (2.28) from the coefficients of the series (3.1).

3. To obtain the algebraic parametrization of the curve \( \Gamma \) in Proposition 2.5 we note that the coefficients of the polynomial

\[
J(\nu) := \left( \Phi_0(z) - \Phi_1(z) - \nu \right) \left( \Phi_0(z) + \Phi_2(z) - \nu \right) \left( \Phi_1(z) + \Phi_2(z) - \nu \right)
\]

are symmetric functions with respect to \( \Phi_0, \Phi_1, \Phi_2 \). Representing these symmetric functions by means of the basic symmetric functions defined by the coefficients of the algebraic equation for \( \Phi \) we arrive at (2.33)–(2.34).

\[\square\]

3.1.2 Proof of the geometric results for the case I

Next we prove the results concerning the geometrical case I.

Proof of Theorem 2.10 and Proposition 2.9 First we notice that (2.46) defines a charge of total mass 2. Indeed,

\[
\int_{\Delta_0} d\lambda = \frac{1}{2\pi i} \int_{\Delta_0} \left( h_{0+}(\xi) - h_{0-}(\xi) \right) d\xi = \frac{1}{2\pi i} \int_{\partial \Delta_0} h(\xi) d\xi,
\]

and by Cauchy’s theorem and (1.21) we have

\[
\int_{\Delta_0} d\lambda = 2.
\]

Analogously we have

\[
\int_{\Delta_j} d\lambda_j = 1, \quad j = 1, 2.
\]

Then we have to prove that the charges \( \lambda_j \) are positive measures. From the definition (2.27) of \( \Phi \) we have

\[
h = (\log |\Phi| + i \arg \Phi)', \quad h_0 - h_j = \left( \log \frac{\Phi_0}{\Phi_j} \right) + i \arg \frac{\Phi_0}{\Phi_j}'.
\]
Recalling (2.40) and (2.42) we have (see the notation in (2.45))
\[ \Delta_j = \Gamma_{0,j} \iff |\Phi_0| = |\Phi_j| \text{ on } \Delta_j = \gamma_j. \]

This relation clearly holds locally (in a neighborhood of the endpoints of \( \Delta_j \)) and since Definition 2.6 and 2.7 of the case I and since the arcs \( \gamma_j \) do not intersect the cuts of \( \mathcal{R} \), it holds also globally on the whole of \( \gamma_j \). Therefore
\[
\lambda'_j(\xi) = \frac{1}{2\pi} \frac{\partial}{\partial \tau} \arg \frac{\Phi_0(\xi)}{\Phi_j(\xi)}, \quad \xi \in \Delta_j (= \Gamma_{0,j} = \gamma_j). \tag{3.3}
\]

We now recall the parametrization of \( \Gamma = \Gamma_{0,1} \cup \Gamma_{0,2} \cup \Gamma_{1,2} \) see (2.32)
\[ \Gamma := \{ z : J(\nu, z) = 0, \nu \in [-2, 2] \}. \]

Since the polynomial \( J(\nu, z) \) has the form (3.2), the parameter \( \nu \) is related to the argument of \( \Phi_0/\Phi_j \) by
\[ \cos \left( \arg \frac{\Phi_0}{\Phi_j} \right) = \frac{\nu}{2}. \]

Hence, when \( \xi \) goes along \( \gamma_a \) from \( \alpha_j \) to \( \beta_j = \gamma_a(-2) = \gamma_b(-2), \) i.e., when \( \nu \) changes from 2 to \(-2\), the argument of \( \Phi_0/\Phi_j \) monotonically changes from 0 to \( \pi \) and when \( \xi \) continues to go from \( \beta_j \) to \( b_j \), the argument of \( \Phi_0/\Phi_j \) monotonically changes from \( \pi \) to \( 2\pi \) (a change of the argument here from \( \pi \) to 0 would contradict the total variation of the argument, which follows from the argument principle applied to the function \( \Phi_0/\Phi_j \)). Thus, the argument in (3.3) is a strictly monotone function and therefore, with the proper orientation of \( \gamma_j \),
\[ \lambda'_j(\xi) > 0, \quad \xi \in \Delta_j, \quad j = 1, 2. \]

The behavior of \( \lambda'_j \) at the endpoints of \( \Delta \) follows from the fact that the function \( h \) has poles (see (1.22)) at the points \( a_1, b_1, a_2, b_2 \). This proves Theorem 2.10.

The first part of Proposition 2.9 follows from the Cauchy-Riemann relations and (3.3)
\[ -\frac{\partial}{\partial n} \log \left| \frac{\Phi_0}{\Phi_j} \right| = \frac{\partial}{\partial \tau} \arg \frac{\Phi_0}{\Phi_j} > 0. \]

The fact that for some \( A \in I \) the set \( \Omega_{0,1} \) is nonempty can be verified by checking some particular examples of \( A \), see Figure 2.6 second and third picture.

\[
3.1.3 \text{ Proof of the geometric results for the case II}
\]

The following statements (Proposition 2.13 and Theorem 2.14 are related to the geometrical case II.

**Proof of Proposition 2.13.** For the geometrical case II (unlike for the case I) the analytic arcs \( \gamma_1 \) and \( \gamma_2 \) (i.e., the trajectories forming the contour \( \Gamma \) in (2.32)) do not coincide with the piecewise analytic arcs \( \Delta_1 \) and \( \Delta_2 \) (i.e., the cuts which make the \( f_1 \) and \( f_2 \) holomorphic) and they do not coincide with the cuts \( \delta_{0,1} \) and \( \delta_{0,2} \) in (1.27) of the sheets of \( \mathcal{R} \)
\[ \gamma_j \neq \Delta_j, \quad \gamma_j \neq \delta_{0,j}, \quad j = 1, 2. \]
Nevertheless, locally (in a neighborhood of the branch points of \( \mathcal{R} \)) and therefore globally (as long as the trajectories going from the branch points do not touch other cuts of \( \mathcal{R} \)) we have, with the notation (2.48) and (2.45),

\[
\tilde{\Delta}_j := \gamma_j \cap \Delta_j \subset \Gamma_{0,j} \iff |\Phi_0| = |\Phi_j| \quad \text{on } \tilde{\Delta}_j, \quad j = 1, 2. \tag{3.4}
\]

Then the trajectory \( \gamma_1 \) passes through the cut \( \delta_{0,2} \) and the branch \( \Phi_0 \) goes over to \( \Phi_2 \). Thus we have

\[
\gamma_1 \cap E_2 = E_2 \subset \Gamma_{1,2}.
\]

Analogously,

\[
\gamma_2 \cap E_1 = E_1 \subset \Gamma_{1,2},
\]

i.e.,

\[
|\Phi_1| = |\Phi_2| \quad \text{on } E_1 \cup E_2. \tag{3.5}
\]

It remains to determine which branches of \( \Phi \) have the same modulus on \( \Delta_{1,2} \). The curve \( \gamma_\alpha \) encloses one pair of the branch points and with the convention (2.56) this is \( \{a_2, b_2\} \). Indeed, it cannot enclose one or three branch points because that would contradict the compactness of \( \mathcal{R} \), and it cannot enclose four branch points because that would contradict the maximum principle. The fact that it encloses the pair \( \{a_2, b_2\} \) follows from the definition of the case I. Thus, we can join infinity and a point of \( \tilde{\Delta}_1 \) by a path without crossing \( \gamma_\alpha \setminus \Delta_{1,2} \), and at the same time any path joining infinity and a point of \( \tilde{\Delta}_2 \) has to cross \( \gamma_\alpha \setminus \Delta_{1,2} \). Taking into account that in a neighborhood of infinity the branch \( \Phi_0 \) has the smallest modulus and (3.4), we conclude that

\[
\gamma_\alpha \setminus \Delta_{1,2} \subset \Gamma_{1,2},
\]

and moreover we have that the neighborhood of infinity bounded by the parts of \( \Gamma \) belongs to \( \Omega_{0,1,2} \). Also, the connected domain bounded by the parts of \( \Gamma \) and containing \( \{a_2, b_2\} \) belongs to \( \Omega_{0,2,1} \). Then the trajectory \( \gamma_\alpha \) passes through the cut \( \delta_{0,2} \) and the branch \( \Phi_2 \) goes over to \( \Phi_0 \). Thus we have

\[
\gamma_\alpha \subset \Delta_{1,2} = \Delta_{1,2} \subset \Gamma_{1,0}. \tag{3.6}
\]

To complete the proof of the proposition we notice that (3.5) and (3.6) imply (see the notation (2.49))

\[
G \subset \Omega_{0,1,2}.
\]

Proof of Theorem 2.14.

1. From Proposition 2.13 it follows that

\[
\overline{\Omega_{0,1,2} \cup \Omega_{0,2,1}} = \overline{\mathbb{C}}.
\]

Hence the modulus \( |\Phi_0| \) is the minimal among all the moduli of all solutions of the equation for \( \Phi \). For a solution of the equation for \( \Phi \) one has

\[
\log |\Phi| \in \text{Harm}(\mathbb{C} \setminus \{a_1, b_1, a_2, b_2\})
\]
hence the function \( \log |\Phi_0| \) is a superharmonic function in \( \mathbb{C} \setminus \{a_1, b_1, a_2, b_2\} \), and by the Riesz decomposition theorem, see e.g. [64, Theorem II.3.1],

\[
\log |\Phi_0(z)| = V^\lambda(z) - \log |c_0|, \tag{3.7}
\]

see (2.31), where \( \lambda \) is a positive measure. The total mass of this measure is equal to 2 due to (2.31) and this measure is supported on \( \Delta_0 = \Delta_1 \cup \tilde{\Delta}_2 \) because \( \log |\Phi_0| \in \text{Harm}(\mathbb{C} \setminus \Delta) \). The relation (3.7) can also be obtained by taking the real part of the primitives of both sides of the Cauchy integral formula

\[
\frac{\Phi_0'(z)}{\Phi_0(z)} = \frac{1}{2\pi i} \int_{\Delta_0} \frac{d\xi}{\xi - z} \left[ \left( \frac{\Phi_0'(\xi)}{\Phi_0(\xi)} \right)_+ - \left( \frac{\Phi_0'(\xi)}{\Phi_0(\xi)} \right)_- \right].
\]

Since \( \Phi_0'/\Phi_0 = h \) and taking into account the pole type singularities of \( h \) at the branch points, we obtain part 1) of the theorem.

2. Since the jump of the function \( h_0 \) over \( \Delta_1 \cup \tilde{\Delta}_2 \) produces the positive measure \( \lambda \), the jump of the function \( h_1 \) over \( \Delta_1 \) and the jump of the function \( h_2 \) over \( \tilde{\Delta}_2 \) produce measures with negative sign, i.e., \( -\lambda_1 \) and \( -\tilde{\lambda}_2 \). Repeating the arguments of the proof of Theorem 2.10 we see that the jump function \( h_2 \) over \( \tilde{\Delta}_2 \cup E_1 \) produces a measure with negative sign

\[-\mu_1 - \tilde{\lambda}_2, \quad \text{supp}(\mu_1) = E_1, \quad \text{supp}(\tilde{\lambda}_2) = \tilde{\Delta}_2,
\]

with total mass equal to \(-1\)

\[-|\mu_1| - |\tilde{\lambda}_2| = -1.
\]

The measure \( \mu_1 \) with positive sign can also be obtained by means of the jump of the function \( h_1 \) over \( E_1 \). This proves part 2) of the theorem.

3. Since the total mass of the charge produced by the jump function \( h_1 \) over \( \Delta_1 \cup E_1 \) is equal to \(-1\), see (1.21)

\[|\mu_1| - |\lambda_1| = -1,
\]

we obtain the last part of the theorem.

\[\square\]

### 3.1.4 Proof of the geometric results for the case III

The following statements (Proposition 2.15, 2.17, and Theorem 2.18) are related to the geometrical case III. Proposition 2.15 is just a more explicit form (relating to III) of the general result from Proposition 2.4. The equation (2.61)–(2.62) in Proposition 2.15 is well known (see [8]: the equivalent form of the algebraic equation for the Riemann surface of the case III goes back to Nuttall [62, 12]). The proofs of Theorem 2.18 and Proposition 2.17 are a repetition of the proofs of the corresponding results for the case I in Theorem 2.10 and Proposition 2.9. A minor change in Theorem 2.18 is the behavior of the weight \( \lambda'_1 \) at the point \( b^* \). This change is due to the regularity at \( b^* \) of the function \( h \) given by (2.58), unlike in the case \( A \in I \). Another minor change in Proposition 2.17 is that the domain \( \Omega_{1,0} \) is always present for case III; for the case I it exists just for some
$A \in I$. To prove this fact we notice that all trajectories $\gamma_b^{(j)}$, $j = 1, 2, 3$ belong to $\Gamma_{0,1}$, see the notation in (2.45). Indeed, the multiplicity of the root in (2.63) and (3.2) gives that all $\gamma_b^{(j)}$ belong to the same $\Gamma_{k,\ell}$, but $\gamma_b^{(1)} \subset \Gamma_{0,1}$, see Definition 2.16-2. Thus, from the outside the Jordan curve $J$

$$J \subset \gamma_b^{(2)} \cup \gamma_b^{(3)} \subset \Gamma_{0,1}, \quad b^* \in J,$$

is a part of the boundary of $\Omega_{0,1,2}$ and inside it contains a least a part of $\gamma_2 \subset \Gamma_{0,2}$, otherwise it would be in contradiction with the maximum principle, and therefore $\Omega_{1,0,2}$ is always inside $J$ (see Figure 2.10).

3.1.5 Proof of the geometric results for the cases IV and V

As we already mentioned, Theorem 2.21 gives a complete classification of the geometry for the problem with three branch points

Proof of Theorem 2.21 and Propositions 2.22, 2.24.

1. According to the definition (2.68) of the domains $D_j$, $j = 1, 2, 3$, if a point $a$ belongs to the boundary between the domains $D_1$ and $D_2$, then the trajectory $\{\gamma_b^{(j)}(\nu) : \nu \in (-2, 2)\}$ has to pass through the point $b = 0$ (see Figure 2.10). To find this set of points $a$ we take the explicit form of the equation (2.33), substituting in (2.34) the explicit expressions (2.62):

$$J(\nu, z) = 0$$

and set $z = 0$. As a result we have

$$J(\nu, z) = \frac{(\nu - 2)(a^4 + 4(1 - \nu)a^3 + (22 - 8\nu)a^2 + 4(1 - \nu)a + 1)^2}{16a^2(a + 1)^4}.$$ 

If we set the numerator equal to zero, then we obtain the algebraic parametrization of $D_1$.

2. The boundary between $D_2$ and $D_3$ is formed by the critical triples (see Definition 2.23). To find the set of points $a$ such that the triple $\{-1, 0, a\}$ is critical, we can proceed in two ways. The first way is just to substitute

$$z = b^* = \frac{(a - 1)^3}{9(a^2 + a + 1)}$$

from (2.59) in the explicit form of the equation (2.33). As a result we get

$$J(\nu, b^*) = \frac{(\nu - 2)\bar{P}(a, \nu)^2}{11664a^4(a + 1)^6(a + 1)^4} = 0,$$

where $\bar{P}(a, \nu)$ is the same polynomial as in part 2) of Theorem 2.21. Thus we obtained the characterization of the critical triples for which for some $\nu \in (-2, 2)$ the trajectories of $\Gamma$ pass through $b^*$. The second way is to follow Definition 2.23 and to find out when the algebraic curve $J(\nu, z)$ has branch points for $\nu \in (-2, 2)$. For this purpose we compute the discriminant of $J(\nu, z)$ and as a result we find

$$\bar{D}_J = C(\nu - 2)^3(\nu + 2)^3P(a, \nu)^2\bar{P}(a, \nu)^3,$$
where $C$ is a constant and $P(a, \nu)$ is a polynomial for which the zeros do not produce branch points (because $P$ appears with a square). The zeros of the polynomial $\tilde{P}(a, \nu)$ are the branch points and the local analysis $(z_j - z_i)^2 \simeq (\nu - \nu_0)^3$ shows that the trajectories at these branch points behave like

$$z = c_0 + c_2(\nu - \nu_0) \pm c_3(\nu - \nu_0)^{3/2},$$

i.e., at these branch points the trajectories meet each other at a zero angle. This proves Theorem 2.21.

3. Now we complete the proof of Proposition 2.24 (part A of this proposition has just been proven). It remains to prove that for a strictly non-acceptable triple the analytic arcs $\gamma_1$, $\gamma_2$, and $\gamma_3$ intersect at one point $c$. First we notice that these arcs cannot join the point $b^*$ with the points $a_1, a_2, b$ without intersection, because otherwise we get a contradiction with the irreducibility of $\Phi$. Indeed, $\Phi$ has a quadratic branch point at $b^*$ and therefore along all three trajectories $\gamma_i(b^*)$ ($i = 1, 2, 3$) we have the equality of the modulus of the same two branches and these trajectories therefore give equality of the values of these branches to all other branch points $a_1, b_1, b$. The same reason shows that $\gamma_1$, $\gamma_2$, and $\gamma_3$ cannot have only pairwise intersections. Thus for this triple we must have a set of points $Z$ where the three arcs intersect and where all three branches have the same modulus

$$Z := \{z : |\Phi_0(z)| = |\Phi_1(z)| = |\Phi_2(z)|\}.$$  

If we take the normalization (2.30) into account, then we have for these points the prescribed values of $\Phi$

$$Z := \{z : \{\Phi_0(z), \Phi_1(z), \Phi_2(z)\} = \{1, e^{2\pi i/3}, e^{4\pi i/3}\}\}. $$

From the equation (2.61) for $\Phi$ it then follows that there exist at most two such points. For an acceptable triple we may have all possibilities; see Figure 2.10 where $Z = \emptyset$ in the first picture, $Z$ has one point in the second picture, and $Z$ has two points in the third picture. For a non-acceptable triple we have that $Z$ contains one point; as we already explained $Z = \emptyset$ contradicts the irreducibility of $\Phi$ and two points in $Z$ leads to a contradiction with the maximum principle. This proves Proposition 2.24.

4. Proposition 2.22 follows immediately from Definition 2.20 ($b \in \Omega_{0,1}$) and the fact that $b \in \Gamma_{0,2}$.

The last set of statements (Proposition 2.24 and Theorem 2.26) is related to the geometrical case IV. Their proofs go along the same lines as the proofs of the corresponding statements for the geometrical case II. One minor change should be taken into account, namely the regular behavior of the function $h$ at the branch point $b^*$. It causes a change of the local behavior of the measures near this point.
3.2 Proof of the vector equilibrium properties for the weak limits of the Hermite-Padé approximants

The weak asymptotic formulas and the convergence theorems for the Hermite-Padé approximants (Theorems 2.28, 2.29, and 2.30) are direct consequences of the strong asymptotic formulas (Theorems 2.35 and 2.36) which will be proven in the next section. In this section we concentrate on the verification of the vector potential problem related with the weak asymptotics.

Proof of Theorem 2.31. In order to prove the theorem we have to verify the statements of the theorem for each geometrical case of the position of the branch points $A := \{a_1, b_1; a_2, b_2\}$. We shall try to do this in a unique way. The proof will be given in the following way.

1. First, we consider the real part of the Abelian integral (1.18), (1.27)

$$g(\xi) := \text{Re } G(\xi), \quad \xi \in \mathcal{R},$$

where $\mathcal{R}$ is an arbitrary algebraic Riemann surface of order 3, and we introduce universal global branches for $g := \{g_0, g_1, g_2\}$. These global branches of $g$ define a sheet structure for $\mathcal{R}$. This universal sheet structure for $\mathcal{R}$ goes back to Nuttall [62] and is different from what we use for the definition of our geometrical cases.

2. Second, we define two universal measures $\Lambda$ and $M$ supported on the new cuts of $\mathcal{R}$. These measures have total mass $|\Lambda| = 2$ and $|M| = 1$ and they satisfy the vector equilibrium property (1.12) with the matrix of interaction

$$\left( d_{ij} \right)_{i,j=1,2} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

(3.10)

The vector potential problem (1.12) with (3.10) was introduced by Nikishin [57, 58].

3. Finally, we perform a reglueing of the Riemann surface and make a correspondence between the universal sheet structure of $\mathcal{R}$ and the specific sheet structure $\{\mathcal{R}_0, \mathcal{R}_1, \mathcal{R}_2\}$ which we have defined for each geometrical case. In practice this procedure gives us the relation (by means of some balayages) between the measures $(\Lambda, M)$ and $(\lambda, \mu)$ and transforms the universal vector potential problem with the Nikishin matrix of interaction (3.10) to the potential problem (2.83) in the theorem.

We emphasize that the potential problem of Theorem 2.9 also has a universal character, i.e., it does not depend on the geometrical case under consideration. Although the problem (2.83) is more sophisticated in comparison with the potential problem (1.12) with (3.10), this problem has an advantage because it is formulated in terms of the functions $f_1, f_2$ which we are approximating (i.e., in terms of the branch points and the cuts which make the functions holomorphic) and not in terms of $\mathcal{R}$. We also mention that the procedure to construct the proper $\mathcal{R}$ starting from the functions $f_1, f_2$ is the most difficult step in proving the Nuttall conjectures (see [62, Section 3.4]) and this was in general not solved before.
We start the proof with an arbitrary algebraic \( \mathfrak{R} \) of the third order and consider the real part of the Abelian integral (1.27), which is a single-valued (up to an additive constant) function on \( \mathfrak{R} \), with the only singularities at the points at infinity

\[
\begin{align*}
\{ & g_0(\xi) = -2 \log |\xi| + c_0 + \cdots, \quad \xi \to \infty^{(0)}, \\
& g_j(\xi) = \log |\xi| + c_j + \cdots, \quad \xi \to \infty^{(j)}, \quad j = 1, 2, \quad \text{(3.11)}
\end{align*}
\]

and

\[
g(\xi) \neq \infty, \quad \xi \in \mathfrak{R} \setminus \pi^{-1}(\infty).
\]

This local definition of the branches of \( g(\xi) \) can be continued on \( \mathbb{C} \) as follows

\[
\forall z \in \mathbb{C}: g_0(z) \leq g_1(z) \leq g_2(z). \quad \text{(3.12)}
\]

From (3.11) and the maximum principle for harmonic functions we have

\[
g_0(z) + g_1(z) + g_2(z) = c_0 + c_1 + c_2 = 0, \quad \forall z \in \mathbb{C}, \quad \text{(3.13)}
\]

and by (3.13) we have then fixed the additive constant for \( g \). Thus we have two piecewise analytic curves

\[
\Gamma_\Lambda := \{ z \in \mathbb{C} : g_0(z) = g_1(z) \} \quad \text{(3.14)}
\]

and

\[
\Gamma_M := \{ z \in \overline{\mathbb{C}} : g_1(z) = g_2(z) \}.
\]

which define the Nuttall sheet structure of \( \mathfrak{R} \)

\[
\tilde{\mathfrak{R}}_0 := \overline{\mathbb{C}} \setminus \Gamma_\Lambda, \quad \tilde{\mathfrak{R}}_1 := \overline{\mathbb{C}} \setminus (\Gamma_\Lambda \cup \Lambda_M), \quad \tilde{\mathfrak{R}}_2 := \overline{\mathbb{C}} \setminus \Gamma_M. \quad \text{(3.15)}
\]

We shall denote the Riemann surface with the sheet structure (3.15) by \( \tilde{\mathfrak{R}} \):

\[
\tilde{\mathfrak{R}} := \bigcup_{j=0}^{2} \mathfrak{R}_j.
\]

Then, like in the proof of Theorem 2.14, we have by (3.12) that \( g_0 \) is a superharmonic function in \( \mathbb{C} \) and a harmonic function in \( \mathbb{C} \setminus \Gamma_\Lambda \), and \( g_2 \) is a subharmonic function in \( \mathbb{C} \) and a harmonic function in \( \overline{\mathbb{C}} \setminus \Gamma_M \). Therefore, by the Riesz decomposition theorem there exist positive measures \( \Lambda \) and \( M \) such that

\[
\begin{align*}
\{ & g_0(z) = V^\Lambda(z) + c_0, \quad \text{supp}(\Lambda) = \Gamma_\Lambda, \\
& g_2(z) = -V^M(z) + c_2, \quad \text{supp}(M) = \Gamma_M. \quad \text{(3.16)}
\end{align*}
\]

From (3.11) we have for the total mass of \( \Lambda \) and \( M \)

\[
|\Lambda| = 2, \quad |M| = 1,
\]

and (3.13) - (3.14) gives the Nikishin vector equilibrium relations

\[
\begin{align*}
\{ & 2V^\Lambda(z) - V^M(z) = -2c_0 - c_2, \quad \text{on } \Gamma_\Lambda, \\
& -V^\Lambda(z) + 2V^M(z) = 2c_2 + c_0, \quad \text{on } \Gamma_M. \quad \text{(3.17)}
\end{align*}
\]
We also note that, in the notation we used earlier, we have for $|z|$ large enough,

$$g_\ell(z) = \log |\Phi_\ell(z)|,$$

(3.18)

however, the domains where the branches $\Phi_\ell$ are single-valued (which we defined earlier) are different from the domains where the branches $g_\ell$ are single-valued. We define the second set of global branches for the algebraic functions $h = \{\tilde{h}_0, \tilde{h}_1, \tilde{h}_2\}$ and $\Phi = \{\tilde{\Phi}_0, \tilde{\Phi}_1, \tilde{\Phi}_2\}$ in accordance with the Nuttall sheet structure (3.15):

$$\tilde{h}_0, \tilde{\Phi}_0 \in H(\mathbb{C} \setminus \Gamma_\Lambda), \quad \tilde{h}_1, \tilde{\Phi}_1 \in H(\mathbb{C} \setminus (\Gamma_\Lambda \cup \Gamma_M)), \quad \tilde{h}_2, \tilde{\Phi}_2 \in H(\mathbb{C} \setminus \Gamma_M).$$

Then we have

$$
\begin{cases}
    d\Lambda(z) = \frac{1}{2\pi i} (\tilde{h}_0^+(z) - \tilde{h}_0^-(z)) \, dz, & z \in \Gamma_\Lambda, \\
    -dM(z) = \frac{1}{2\pi i} (\tilde{h}_2^+(z) - \tilde{h}_2^-(z)) \, dz, & z \in \Gamma_M.
\end{cases}
$$

(3.19)

Now we perform the following *reglueing* procedure to transform the Nuttall Riemann surface $\tilde{R}$ (3.15) into the Riemann surfaces with sheet structure defined in Subsections 2.2–2.4, for which we use the notation $R$. We note that

$$\Gamma_\Lambda \cup \Gamma_M = \Gamma,$$

where $\Gamma$ is defined in (2.32)–(2.34) and does not depend on the sheet structure of $R$. The union of the cuts of the sheets $\{\tilde{R}_0, \tilde{R}_1, \tilde{R}_2\}$ coincides with $\Gamma$:

$$\bigcup_{\ell,k=0}^2 \tilde{\delta}_{\ell,k} = \Gamma,$$

and at the same time the union of the cuts of $\{R_0, R_1, R_2\}$ occupies just a subset of $\Gamma$:

$$\bigcup_{\ell,k=0}^2 \delta_{\ell,k} \subset \Gamma.$$

To transform $R$ into $\tilde{R}$ we consider the regions $\Omega_{j,k,\ell}$, $j \neq k \neq \ell$ pairwise, $j, k, \ell = 0, 1, 2$, see (2.44). The contour $\Gamma$ divides $\mathbb{C}$ into these regions. Then we take the components of $\Omega_{j,k,\ell}$ with indices $(j, k, \ell) \neq (0, 1, 2)$, lift them to $\pi^{-1}(\Omega_{j,k,\ell})$ on $R$, cut $R$ along the boundaries $\partial(\pi^{-1}(\Omega_{j,k,\ell}))$, and interchange these pieces of $R$ by reglueing them such that

$$
\pi_j^{-1}(\Omega_{j,k,\ell}) \leftrightarrow \pi_0^{-1}(\Omega_{j,k,\ell}), \\
\pi_k^{-1}(\Omega_{j,k,\ell}) \leftrightarrow \pi_1^{-1}(\Omega_{j,k,\ell}), \\
\pi_\ell^{-1}(\Omega_{j,k,\ell}) \leftrightarrow \pi_2^{-1}(\Omega_{j,k,\ell}).
$$

As a result we obtain $\tilde{R}$ with Nuttall’s structure of the sheets. During this procedure some extra cuts have appeared. The transformation of $\tilde{R}$ to $R$ works in the reverse order and some cuts may disappear. Since $\infty \in \Omega_{0,1,2}$ for all $R$ under consideration, this procedure provides balayages of some parts of the measures $\lambda$ and $\mu$ to parts of the measures $\Lambda$ and $M$. This allows to transform the potential problem (3.17) to the potential problem (2.83).

We now apply the transformation $\tilde{R} \to R$ to prove the theorem. We start with the cases I and III. Of course, for these cases it is easy to prove the theorem directly (see
Remark 2.32, parts 1 and 3). However, for methodological reasons we use here the more sophisticated general procedure. We consider a generic case of the classes I and III (see Figure 2.6 second picture, and Figure 2.10 second picture). For this case all the regions $\Omega_{0,1,2}$, $\Omega_{0,2,1}$, and $\Omega_{1,0,2}$ are present. It follows from Propositions 2.9 and 2.17 that the contours $\Gamma_L$ and $\Gamma_M$ are (see Figure 3.1)

$$
\Gamma_L := \Gamma_{0,1} \cup \gamma_{c,b_2}, \quad \Gamma_M := \Gamma_{1,2} \cup \gamma_{c,a_2}.
$$

Thus, the sheets $\{\tilde{R}_0, \tilde{R}_1, \tilde{R}_2\}$ are as shown in Figure 3.2. If we re-glue the sheets $\tilde{R}_0$ and $\tilde{R}_1$ along $\pi^{-1}(\Gamma_{0,1} \setminus \gamma_{a_1,b_1})$, interchange the domains $\pi_0^{-1}(\Omega_{1,0,2})$ and $\pi_1^{-1}(\Omega_{1,0,2})$, and then glue $\tilde{R}_0$ and $\tilde{R}_1$ along $\pi^{-1}(\Gamma_{0,1} \setminus \gamma_{a_1,b_1})$, and $\tilde{R}_1$ and $\tilde{R}_2$ along $\pi^{-1}(\gamma_{a_2,c})$, we see that $\pi^{-1}(\gamma_{a_2,c})$ disappears from the first sheet and moves to the zero sheet. If we do the same with the sheets $\tilde{R}_1$ and $\tilde{R}_2$ and the domains $\pi_1^{-1}(\Omega_{0,2,1})$ and $\pi_2^{-1}(\Omega_{0,2,1})$, then we obtain the Riemann surface $\mathcal{R}$ from Figure 2.5.

Decomposing the measure $\Lambda$ we have, due to (3.19) and the structure of $\tilde{R}$,

$$
\Lambda = \lambda_1 + \lambda_{c,b_2} + \mu_{0,1},
$$

where

$$
\lambda_{c,b_2} := \lambda\big|_{\gamma_{c,b_2}}, \quad \mu_{0,1} := \Lambda\big|_{\Gamma_{0,1} \setminus \Delta_1},
$$

and $\lambda_1, \lambda_2$ are from Theorem 2.10. Analogously, we have

$$
M = \lambda_{c,a_2} + \mu_{1,2},
$$

where

$$
\lambda_{c,a_2} := \lambda_2\big|_{\gamma_{c,a_2}}, \quad \mu_{1,2} := M\big|_{\Gamma_{1,2}}.
$$

The equality

$$
g_0 = \log|\Phi_0|,
$$

see (3.18), is valid in the neighborhood of infinity bounded by $\Gamma$. We have

$$
\log|\Phi_0| = V^\lambda - \log|\mathcal{C}_0|
$$

due to the definition of $\lambda$ in (2.46) and the relations between $h$ and $\Phi$, see (2.27), (2.30), and (2.31). Therefore in the neighborhood of infinity bounded by $\Gamma$, we have

$$
V^{\lambda_1} + V^{\lambda_{c,b_2}} + V^{\mu_{0,1}} + c_0 = V^{\lambda_1} + V^{\lambda_2} - \log|\mathcal{C}_0|,
$$
and we conclude that
\[ c_0 = - \log |C_0|, \]
and the measure \( \mu_{0,1} \) is the balayage of the measure \( \lambda_{c,a_2} \) from \( \gamma_{c,a_2} \) to \( \Gamma_{0,1} \setminus \Delta_1 \), i.e.,
\[ V^{\mu_{0,1}} = V^{\lambda_{c,a_2}}, \quad \text{on } \Gamma_{0,1} \setminus \Delta_1 \text{ and outside.} \quad (3.22) \]
Analogously we have in the neighborhood of infinity bounded by \( \Gamma_M \)
\[ g_2 = \log |\Phi_2| \]
and
\[ \log |\Phi_2| = -V^{\lambda_2} - \log |C_2|. \]
Therefore in this domain we have
\[ V^{\lambda_{c,a_2}} + V^{\mu_{1,2}} = V^{\lambda_2}, \quad c_2 = - \log |C_2|, \]
and we obtain another balayage relation
\[ V^{\mu_{1,2}} = V^{\lambda_{c,b_2}}, \quad \text{on } \Gamma_{1,2} \text{ and outside.} \quad (3.23) \]
In addition to the balayage relations (3.22)–(3.23), which are valid outside \( \Gamma_{0,1} \setminus \Delta_1 \) and \( \Gamma_{1,2} \), we need relations between the potentials inside these curves. Let us consider the first relation of the Nikishin equilibrium problem (3.17) on the curve \( \Gamma_{0,1} \setminus \Delta_1 \). If we substitute the decompositions (3.20)–(3.21) into (3.17) and use the balayages (3.22)–(3.23) we obtain
\[ 2V^{\lambda_1} + V^{\mu_{0,1}} + V^{\lambda_{c,b_2}} = -2c_0 - c_2, \quad \text{on } \Gamma_{0,1} \setminus \Delta_1. \quad (3.24) \]
The left hand side of (3.24) is a harmonic function inside the domain bounded by $\Gamma_{0,1} \setminus \Delta_1$. Hence by the maximum principle for harmonic functions, the relation (3.24) is valid in this domain. In the same way we obtain from the second relation in (3.17) that

$$V^{\mu_1} + V^{\mu_2} - V^{\nu_1} = 2c_2 + c_0, \quad \text{on } \Gamma_{1,2} \text{ and inside.} \quad (3.25)$$

Now it is easy, for the cases under consideration, to derive the equilibrium relations of Theorem 2.31 from the equilibrium relations (3.17) using (3.22)–(3.25). The first relation of (3.17) considered on $\Delta_1$ gives us, using (3.20)–(3.21) and (3.22)–(3.23),

$$2V^{\nu_1} + V^{\nu_2} = -2c_0 - c_2 =: \kappa_1, \quad \text{on } \Delta_1.$$  

The first relation of (3.17) on $\gamma_{c,b_2}$ (see Figure 3.1) gives us, using also (3.25),

$$2V^{\nu_2} + V^{\nu_1} = -2c_0 - c_2 - (2c_2 + c_0) = -3(c_0 + c_2) =: \kappa_2, \quad \text{on } \gamma_{c,b_2}.$$  

We get the same relation on $\gamma_{a_2, c}$ from the second relation of (3.17), from (3.23) and with the help of (3.24), which is also valid on $\gamma_{a_2, c}$ as we have seen. Thus, from the general Nikishin potential problem (3.17) we obtain the Angelesco potential problem (1.12), (1.15)

$$\begin{cases} 
2V^{\nu_1} + V^{\nu_2} = \kappa_1, & \text{on } \Delta_1, \\
V^{\nu_1} + 2V^{\nu_2} = \kappa_2, & \text{on } \Delta_2. 
\end{cases}$$

Therefore for the cases I, III, and V (see Remark 2.32, 1–4) the equilibrium relations are verified. We emphasize the importance of the balayages (3.22)–(3.23) for the description of the weak limits of the extra interpolation points in the case V (see Remark 2.32-4).

Now we apply the same approach to verify the equilibrium relations (2.7) for the remaining geometrical cases II and IV. For the case II Proposition 2.13 gives (see Figure 2.8 and (2.48), (2.51))

$$\begin{align*}
\Gamma_{\Lambda} &= \Delta_1 \cup \tilde{\Delta}_2, \\
\Gamma_{M} &= E_1 \cup E_2 \cup E_0, \quad E_0 := \gamma_{\alpha} \setminus \Delta_{1,2},
\end{align*}$$

and the corresponding measures (3.16) and (3.19) are given by

$$\begin{align*}
\Lambda &= \lambda_1 + \tilde{\lambda}_2, \\
M &= \mu_1 + \mu_2 + \mu_0, \quad \mu_2 := M\bigg|_{E_2}, \quad \mu_0 := M\bigg|_{E_0},
\end{align*}$$

where the measures

$$\begin{align*}
\lambda_1 &= \Lambda\bigg|_{\Delta_1}, \\
\tilde{\lambda}_2 &= \Lambda\bigg|_{\tilde{\Delta}_2}, \\
\mu_1 &= M\bigg|_{E_1}
\end{align*}$$

are defined in Theorem 2.14. The reglueing of the Riemann surface $\tilde{\mathcal{R}}$ gives the balayage relation

$$V^{\tilde{\lambda}_2} = V^{\mu_2} + V^{\mu_0}, \quad \text{on } E_2 \cup E_0 \text{ and outside.} \quad (3.26)$$

The second equilibrium relation of (3.17), the balayage (3.26) and the maximum principle give

$$2V^{\nu_1} + V^{\mu_2} + V^{\mu_0} - V^{\lambda_1} = 2c_2 + c_0, \quad \text{on } E_2 \cup E_0 \text{ and inside.} \quad (3.27)$$
The first equilibrium relation in (3.17) on $\Delta_1$ and (3.26) give the first equilibrium relation in (2.83)

$$2V^\mu_1 + 2V^\lambda_2 - V^\mu_1 - V^\mu_2 - V^\mu_0 = 2V^\lambda_1 + V^\lambda_2 - V^\mu_1 = -2c_0 - c_2 := \kappa_1,$$

on $\Delta_1$.

If we take the first relation in (3.17) on $\tilde{\Delta}_2$ and use (3.27), then we find the second relation in (2.83)

$$2V^\mu_1 + 2V^\lambda_2 - V^\mu_1 - V^\mu_2 - V^\mu_0 = V^\lambda_1 + 2V^\lambda_2 + V^\mu_1 = \kappa_1 + 2c_2 + c_0 := \tilde{\kappa}_2,$$

on $\tilde{\Delta}_2$.

Finally, the second relation in (3.17) on $E = E_1 \cup E_2$ and the balayage (3.26) lead to the third relation of (2.83)

$$2V^\mu_1 + 2V^\mu_2 + 2V^\mu_0 - V^\lambda_1 - V^\lambda_2 = 2V^\mu_1 + V^\lambda_2 - V^\lambda_1 = 2c_2 + c_0 = \tilde{\kappa}_2 - \kappa_1,$$

on $E$.

Thus the equilibrium relations (2.83) for the case II are verified.

If we verify the equilibrium relations for the last case IV, then we will also get the dual equilibrium problem (see II and III of Theorem 2.31). For the case IV Proposition 2.25 (see (2.74) and (2.75)) gives

$$\Gamma_\Lambda = \Delta_1 \cup \tilde{\Delta}_2 = \Delta_1 \cup \Delta_{1,2} \cup \tilde{\Delta}_2,$$

$$\Gamma_M = E_0 \cup E_1 \cup E_2.$$

The Nuttall sheet structure of $\tilde{\mathcal{R}}$ is shown in Figure 3.3. It defines the global branches (3.16) for the real part of the Abelian integral (3.11)

$$\begin{cases}
g_0 = V^\lambda_1 + \lambda_2 + \lambda_{1,2} + c_0, \\
g_1 = -V^\lambda_1 + \lambda_2 + \lambda_{1,2} + V^\mu_1 + \mu_2 + \mu_0 - c_0 - c_2, \\
g_2 = -V^\mu_1 + \mu_2 + \mu_0,
\end{cases}$$

where the measures

$$\lambda_1 = M\big|_{\Delta_1}, \quad \lambda_2 = M\big|_{\Delta_2}, \quad \lambda_{1,2} = M\big|_{\Delta_{1,2}}, \quad \mu_2 = M\big|_{E_2},$$

are the same as in Theorem 2.26 and

$$\mu_1 := M\big|_{E_1}, \quad \mu_0 := M\big|_{E_0}.$$

Equating

$$\begin{cases}
g_0 = g_1, \quad \text{on } \Gamma_\Lambda, \\
g_1 = g_2, \quad \text{on } \Gamma_M,
\end{cases}$$

gives the Nikishin equilibrium (3.7). Observe that (3.7) has just two equilibrium relations since the 0-sheet and the second sheet do not intersect.
Figure 3.3: The Nuttall sheet structure of \( \tilde{R} \) (the case IV)

In order to obtain (2.83) we re-glue the Riemann surface shown in Figure 3.3. We interchange the domains bounded by \( E_0 \cup E_2 \) on the sheets 1 and 2 and as a result the cut along \( E_0 \cup E_2 \) between the sheets 1 and 2 disappears and we arrive at the Riemann surface \( \mathcal{R} \) shown in Figure 3.4.

The new sheet structure defines new global branches for the real part of the Abelian integral (3.11)

\[
\begin{align*}
\tilde{g}_0 &= V^{\lambda_1+\lambda_2+\lambda_1,2} + c_0, \\
\tilde{g}_1 &= -V^{\lambda_1+\lambda_1,2} + V^{\mu_1} - c_0 - c_2, \\
\tilde{g}_2 &= -V^{\lambda_2} - V^{\mu_1} + c_2.
\end{align*}
\]

Equating \( \tilde{g}_0 = \tilde{g}_1 \) on \( \Delta_1 \), where \( \Delta_1 \) is the cut joining the sheet 0 and 1, we have

\[
V^{\lambda_1} + V^{\lambda_2} + V^{\lambda_1,2} + c_0 = -V^{\lambda_1} - V^{\lambda_1,2} + V^{\mu_1} - c_0 - c_2, \quad \text{on } \Delta_1,
\]

which gives the first relation in (2.83):

\[
2V^{\lambda_1} + V^{\lambda_2} - V^{\mu_1} = -2c_0 - c_2 := \kappa_1, \quad \text{on } \Delta_1.
\]
Equating $\tilde{g}_0 = \tilde{g}_2$ on $\tilde{\Delta}_2$ gives the second relation in (2.83):

$$V^{\lambda_1} + 2V^{\lambda_2} + V^{\mu_2} = c_2 - c_0 =: \kappa_2, \quad \text{on} \quad \tilde{\Delta}_2.$$  

Finally, equating $\tilde{g}_1 = \tilde{g}_2$ on $E_1$ gives the third relation in (2.83):

$$-V^{\lambda_1} + 2V^{\mu_1} = 2c_2 + c_0 = \kappa_2 - \kappa_1, \quad \text{on} \quad E_1.$$  

Thus the equilibrium relations (2.83) are verified for the case IV.

To derive the dual equilibrium problem we re-glue the Riemann surface shown in Figure 3.3 by interchanging the domains bounded by $E_0 \cup E_1$ on the sheets 1 and 2, and as a result the cut along $E_0 \cup E_1$ between the sheets 1 and 2 disappears and we arrive at the Riemann surface shown in Figure 2.14. The new sheet structure defines new global branches for the real part of the Abelian integral (3.11):

$$g =: \begin{cases} 
\hat{g}_0 = V^{\lambda_1} + V^{\lambda_2} + c_0, \\
\hat{g}_1 = -V^{\lambda_1} - V^{\mu_2} - c_0 - c_2, \\
\hat{g}_2 = -V^{\lambda_2} + V^{\mu_2} + c_2. 
\end{cases}$$
Equating \( \hat{g}_0 = \hat{g}_1 \) on \( \tilde{\Delta}_1 \) (see Figure 2.14) gives the first relation for the dual equilibrium problem
\[
2V^{\lambda_1} + V^{\lambda_2} + V^{\mu_2} = \kappa_1, \quad \text{on } \tilde{\Delta}_1.
\]
Equating \( \hat{g}_0 = \hat{g}_2 \) on \( \Delta_2 \) gives the second relation
\[
V^{\lambda_1} + 2V^{\lambda_2} - V^{\mu_2} = \kappa_2, \quad \text{on } \Delta_2,
\]
and \( \hat{g}_1 = \hat{g}_2 \) on \( E_2 \) gives
\[
V^{\lambda_1} - V^{\lambda_2} + 2V^{\mu_2} = \kappa_1 - \kappa_2, \quad \text{on } E_2.
\]
Thus the equilibrium problem for the case IV (see II and III of Theorem 2.31) is verified. The dual equilibrium problem can be verified in a similar way for the other geometrical cases.

To conclude the proof of Theorem 2.31 we verify the symmetry property (2.84). We show how to do this for the geometrical case I; one can use a similar reasoning for the other cases. The equilibrium relations for the case I have the form
\[
\begin{align*}
U_1 &:= 2V^{\lambda_1} + V^{\lambda_2} = \kappa_1, \quad \text{on } \Delta_1, \\
U_2 &:= V^{\lambda_1} + 2V^{\lambda_2} = \kappa_2, \quad \text{on } \Delta_2.
\end{align*}
\]
For the harmonic conjugate \( \tilde{U}_j \) one has
\[
\frac{\partial \tilde{U}_j}{\partial n} = \frac{\partial U_j}{\partial \tau} = 0, \quad \text{on } \Delta_j, \quad j = 1, 2,
\]
hence one has to verify for \( U_j = U_j + i\tilde{U}_j \) that
\[
\frac{\partial}{\partial n_+}U_j(z) = \frac{\partial}{\partial n_-}U_j(z), \quad z \in \Delta_j \setminus \{a_j, b_j\}, \quad j = 1, 2.
\]
This relation holds because the function \( U_j \) is analytic on \( \Delta_j \setminus \{a_j, b_j\} \) (because \( U_j = \log \Phi_0/\Phi_j + \text{const} \)), and therefore its derivative does not depend on the direction. The theorem is now proved.

4 Asymptotic analysis of the matrix Riemann-Hilbert problem for the Hermite-Padé polynomials

In this section we prove the results about the strong asymptotics of the Hermite-Padé polynomials, i.e., Theorem 2.35 and Theorem 2.36. We demonstrate here the detailed proof for the case II. The proof for the case I is just a simplified version of the proof for the case II and the most delightful pieces of the proof degenerate. The same is true for the cases III and V which are both a simplified version of the proof for the case IV, and the proof for the case IV is just a repetition of the proof for the case II with minor differences in the local analysis of the asymptotics in the neighborhood of the branch point \( b^* \). For these reasons we present the proof for the generic case II in detail and give a sketch of the proofs for the other cases.
For the proof we use the steepest descent method of Deift and Zhou (see [31]). The method consists of several consecutive transformations

\[ Y \mapsto Z \mapsto \tilde{Z} \mapsto \hat{Z} \mapsto \check{Z} \]

of the matrix Riemann-Hilbert problem (2.13) for the matrix function \( Y \) to the Riemann-Hilbert problem

\[
\begin{cases}
\hat{Z} \in H^{3 \times 3}(\mathbb{C} \setminus \check{\Sigma}), \\
\exists \hat{Z}_\pm \in C(\check{\Sigma}) : \hat{Z}_+ = \hat{Z}_-(I + \mathcal{O}(1/n)), \text{ uniformly on } \check{\Sigma} \text{ as } n \to \infty, \\
\hat{Z}(z) = I + \mathcal{O}(1/z), & z \to \infty.
\end{cases}
\]

It is known that for large \( n \), the problem (4.1) has a solution which satisfies

\[ \check{Z} = I + \mathcal{O}(1/n), \text{ uniformly in } \mathbb{C} \setminus \check{\Sigma} \text{ as } n \to \infty. \]

Thus, after making the inverse transformations from \( \check{Z} \) to \( Y \) we obtain the existence of the solution \( Y \) of the Riemann-Hilbert problem (2.13) for large \( n \) and asymptotics for \( n \to \infty \) for its components.

### 4.1 Normalization of the Riemann-Hilbert problem at infinity and decomposition of the jumps

The goal of the first step is to transform the Riemann-Hilbert problem (2.13)–(2.17) such that the solution of the new problem has the same normalization at infinity as the solution of (4.1). For this purpose we use the algebraic function \( \Phi = \{ \Phi_0, \Phi_1, \Phi_2 \} \) in (2.27)–(2.31). We denote

\[
S := \begin{pmatrix} \Phi_0^n & 0 & 0 \\ 0 & \Phi_1^n & 0 \\ 0 & 0 & \Phi_2^n \end{pmatrix}, \quad C := \begin{pmatrix} C_0^n & 0 & 0 \\ 0 & C_1^n & 0 \\ 0 & 0 & C_2^n \end{pmatrix},
\]

where \( C_0, C_1, C_2 \) are the inverses of the leading coefficients of the expansion of \( \Phi_0, \Phi_1, \Phi_2 \) near infinity, see (2.31). If we set

\[ Z := CYS, \]

then, because of (2.13) and (2.31)

\[ Z(z) = I + \mathcal{O}\left(\frac{1}{z}\right), \quad z \to \infty, \]

and the piecewise analytic matrix function \( Z \) satisfies the following jump condition:

\[ Z_+ = Z_- J, \quad \text{on } \tilde{\Delta}_1 \cup \Delta_{1,2} \cup \tilde{\Delta}_2 \cup E_1 =: \Sigma, \]

(see Figures 2.8 and 4.1), and for the jump matrix \( J \) we have from (2.14) that \( J := S^{-1}WS_+ \) so that

\[
J := \begin{cases}
J(w_1, w_2), & \text{on } \Delta_{1,2}, \\
J(w_1, 0), & \text{on } \tilde{\Delta}_1, \\
J(0, w_2), & \text{on } \tilde{\Delta}_2, \\
J(0, 0), & \text{on } E_1,
\end{cases}
\]
where we set

\[
J(w_1, w_2) := \begin{pmatrix}
\Phi^n_{0+} & \frac{\Phi^n_{1+}}{\Phi^n_{0-}} w_1 & \frac{\Phi^n_{2+}}{\Phi^n_{0-}} w_2 \\
0 & \frac{\Phi^n_{1+}}{\Phi^n_{0-}} & 0 \\
0 & 0 & \frac{\Phi^n_{2+}}{\Phi^n_{0-}}
\end{pmatrix}.
\]  

(4.7)

Note that, in comparison with \( Y \), the RH problem for \( Z \) has a new contour \( E_1 \) where \( Z \) is discontinuous, due to the jumps in the functions \( \Phi_j \).

![Jump contours for the Riemann-Hilbert problem for \( Z \) and \( \widetilde{Z} \) (normalization and the global lens)](image)

Figure 4.1: Jump contours for the Riemann-Hilbert problem for \( Z \) and \( \widetilde{Z} \) (normalization and the global lens)

Now we analyze the jump matrices (4.6)–(4.7). We start with the usual decomposition of the jumps on \( \widetilde{\Delta}_1 \) and \( \widetilde{\Delta}_2 \), which are commonly used in the analysis of \( 2 \times 2 \) matrix-valued Riemann-Hilbert problems. If we define

\[
D_1 := \begin{pmatrix}
1 & 0 & 0 \\
\frac{\Phi^n_{0}}{\Phi^n_{1} w_1} & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad W_1 := \begin{pmatrix}
0 & w_1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

\[
W_1 := \begin{pmatrix}
0 & 0 & w_2 \\
0 & 1 & 0 \\
-1 & 0 & 0
\end{pmatrix},
\]

(4.8)
then we have, since $\Phi_{0\pm} = \Phi_{1\mp}$ on $\tilde{\Delta}_1$,

$$J = J(w_1, 0) = \begin{pmatrix} \Phi_0^n / \Phi_1^n & w_1 & 0 \\ 0 & \Phi_0^n / \Phi_1^n & 0 \\ 0 & 0 & 1 \end{pmatrix} = D_{1-}W_1D_{1+}, \quad \text{on } \tilde{\Delta}_1, \quad (4.9)$$

and similarly, since $\Phi_{0\pm} = \Phi_{2\mp}$ on $\tilde{\Delta}_2$,

$$J = J(0, w_2) = D_{2+}W_2D_{2-}, \quad \text{on } \tilde{\Delta}_2. \quad (4.10)$$

The following decompositions of the jumps on $\Delta_{1,2}$ and $E_1$ were not used before in the analysis of Riemann-Hilbert problems. If we define

$$D_{2,1} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\Phi_2^n w_2 / \Phi_1^n w_1 & 0 \\ 0 & 0 & 1 & \end{pmatrix}, \quad W_{1,2} := \begin{pmatrix} 1 & 0 & 0 & w_2 / w_1 \\ 0 & 0 & -w_1 / w_2 & 0 \\ 0 & w_1 / w_2 & 0 & \end{pmatrix}, \quad (4.12)$$

then we observe that

$$J = J(w_1, w_2) = D_{2,1-}J(w_1, 0)D_{2,1+}^{-1}, \quad \text{on } \Delta_{1,2}, \quad (4.11)$$

i.e., after the decomposition (4.11) the jump has a block structure and $J(w_1, 0)$ can then be decomposed again in the usual way as in (4.9), since $\Phi_{0\pm} = \Phi_{1\mp}$ holds also on $\Delta_{1,2}$. If we define

$$D_{1,2} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \Phi_1^n w_1 / \Phi_2^n w_2 & 0 & 0 \\ 0 & 0 & \Phi_2^n w_2 / \Phi_1^n w_1 & 0 \end{pmatrix}, \quad W_{1,2} := \begin{pmatrix} 1 & 0 & 0 & \Phi_2^n w_2 / \Phi_1^n w_1 \\ 0 & 0 & -w_1 / w_2 & 0 \\ 0 & w_1 / w_2 & 0 & \end{pmatrix}, \quad (4.12)$$

then we have on $E_1$ (since $\Phi_{1\pm} = \Phi_{2\mp}$)

$$J = J(0, 0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \Phi_1^n w_1 / \Phi_2^n w_2 & 0 & 0 \\ 0 & 0 & \Phi_2^n w_2 / \Phi_1^n w_1 & 0 \end{pmatrix} = D_{1,2-} \begin{pmatrix} 1 & 0 & 0 & \Phi_2^n w_2 / \Phi_1^n w_1 \\ 0 & 0 & -w_1 / w_2 & 0 \\ 0 & w_1 / w_2 & 0 & \end{pmatrix} W_{1,2} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & \Phi_2^n w_2 / \Phi_1^n w_1 & 1 & \end{pmatrix} D_{2,1+}^{-1}.$$ 

Since

$$\begin{pmatrix} 1 & \Phi_2^n w_2 / \Phi_1^n w_1 \\ 0 & w_1 / w_2 \\ 0 & w_2 / w_1 \end{pmatrix} - \begin{pmatrix} 0 & -w_2 / w_1 \\ 0 & w_1 / w_2 \\ 0 & w_2 / w_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & \Phi_2^n w_2 / \Phi_1^n w_1 & 1 & \end{pmatrix} \quad \text{on } E_1,$$

we have

$$J = J(0, 0) = D_{1,2-}W_{1,2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \Phi_1^n w_1 / \Phi_2^n w_2 & 1 \end{pmatrix} D_{2,1+}^{-1}, \quad \text{on } E_1.$$
and we obtain
\[ J = J(0, 0) = D_{1,2} - W_{1,2} D_{1,2+}^{-1} D_{2,1+}, \quad \text{on } E_1. \]  
(4.13)

We also point out the commutation relations
\[ D_1 D_2 = D_2 D_1, \quad D_1 D_{1,2} = D_{1,2} D_1. \]  
(4.14)

### 4.2 Opening a global lens (in preparation of opening local lenses)

The goal of the second step is to transform the Riemann-Hilbert problem (4.4)–(4.7) for the function \( Z \) in (4.3) to a Riemann-Hilbert problem with new jumps which can be decomposed in the usual way by (4.9)–(4.10) to jumps which do not depend on \( n \) or to jumps which tend to the identity matrix as \( n \to \infty \).

We fix a neighborhood \( O_{c_j} \) of the point \( c_j \) (\( j = 1, 2 \)) (see Figure 4.1) and denote
\[ \tilde{\Delta}_2(c_j) := \tilde{\Delta}_2 \cap O_{c_j}. \]

We join the endpoints (different from \( c_j \)) of \( \tilde{\Delta}_2(c_j) \), which we denote by \( c_j^+ \), by an arc \( E_2^+ \) such that
\[ E_2^+ \subset \Omega_{2,1}, \]  
(4.15)

where we used the notation (2.45). We denote by \( G^+ \) the domain bounded by
\[ \partial G^+ := E_1 \cup \tilde{\Delta}_2^{(c_1)} \cup \tilde{\Delta}_2^{(c_2)} \cup E_2^+. \]  
(see Figures 2.9 and 4.1). We define
\[ \tilde{Z} := \begin{cases} Z D_{2,1}, & \text{in } G^+, \\ Z, & \text{in } \mathbb{C} \setminus G^+. \end{cases} \]  
(4.16)

This piecewise analytic function \( \tilde{Z} \) has a jump
\[ \tilde{Z}_+ = \tilde{Z}_- J, \quad \text{on } \Sigma \cup E_2^+ =: \tilde{\Sigma}^+, \]  
(4.17)

where for \( \tilde{J} \) on \( \tilde{\Sigma}^+ \setminus (\Delta_{1,2} \cup E_1 \cup E_2^+ \cup \tilde{\Delta}_2^{(c_1)} \cup \tilde{\Delta}_2^{(c_2)}) \), see (4.16),
\[ \tilde{J} = J = \begin{cases} J(w_1, 0), & \text{on } \tilde{\Delta}_1 \cup (\tilde{\Delta}_2 \setminus \bigcup_{j=1}^2 \tilde{\Delta}_2^{(c_j)}). \end{cases} \]  
(4.18)

On other parts of \( \tilde{\Sigma} \) the jump is changed to
\[ \tilde{J} = \begin{cases} J(0, w_2) D_{2,1}, & \text{on } \tilde{\Delta}_2^{(c_1)} \cup \tilde{\Delta}_2^{(c_2)}, \\ D_{2,1}^{-1}, & \text{on } E_2^+. \end{cases} \]  
(4.19)

On \( E_1 \) we obtain, using (4.13),
\[ \tilde{Z}_+ = Z_+ D_{2,1+} = Z_- (D_{1,2} - W_{1,2} D_{1,2+} D_{2,1+}^{-1}) D_{2,1+} = \tilde{Z}_- D_{1,2} - W_{1,2} D_{1,2+}, \]
and therefore we have
\[ \tilde{J} = D_{1,2} - W_{1,2}, \quad \text{on } E_1. \]  
(4.20)
On $\Delta_{1,2}$ we have, using (4.11),
\[ \tilde{Z}_+ = Z_+D_{2,1+} = Z_-(D_{2,1-J}(w_1,0)D_{2,1+})D_{2,1+} = \tilde{Z}_-J(w_1,0), \]
and therefore we have
\[ \tilde{J} = J(w_1,0), \quad \text{on } \Delta_{1,2}. \quad (4.21) \]
Thus the desired form of the jumps (4.18)–(4.21) is indeed obtained after the second transformation.

### 4.3 Opening local lenses

The goal of the third step is to transform the Riemann-Hilbert problem (4.17)–(4.21) for the function $\tilde{Z}$ from (4.16) to a Riemann-Hilbert problem with jumps which do not depend on $n$ or which tend to the identity matrix as $n \to \infty$. We start with some notation. In addition to the points $c_j^+ := \partial O_{c_j} \cap \tilde{\Delta}_2$, $j = 1, 2$, (see Figures 4.2, 4.3) we denote
\[ c_j^- := \partial O_{c_j} \cap E_1, \quad \tilde{c}_j^+ := \partial O_{\tilde{c}_j} \cap \Delta_{1,2}, \quad \tilde{c}_j^- := \partial O_{\tilde{c}_j} \cap \tilde{\Delta}_1, \quad j = 1, 2. \]
We define a local perturbation of the arcs $\tilde{\Delta}_2$ joining the points $a_2$ with $\tilde{c}_1^+$ and $b_2$ with $\tilde{c}_2^+$ by Jordan arcs $\gamma_{a_2,\tilde{c}_1^+}$ and $\gamma_{b_2,\tilde{c}_2^+}$
\[ \tilde{\Delta}_2^\pm := \gamma_{a_2,\tilde{c}_1^+} \cup \gamma_{b_2,\tilde{c}_2^+}. \]
Analogously we define (see Figure 4.2)
\[ \tilde{\Delta}_1^\pm := \gamma_{a_1,\tilde{c}_1^+} \cup \gamma_{b_1,\tilde{c}_2^+}, \quad \Delta_{1,2}^\pm := \gamma_{\tilde{c}_1^+,\tilde{c}_2^+}, \quad E_1^\pm := \gamma_{\tilde{c}_1^+,\tilde{c}_2^+}. \]

Around the piecewise analytic arc $\tilde{\Delta}_1 \cup \Delta_{1,2}$ we now define the lens shaped domains $T_1^\pm$ and $T_1^-$ bounded by
\[ \partial T_1^\pm := \Delta_1 \cup (\tilde{\Delta}_1^\pm \cup \Delta_{1,2}^\pm), \]
and
\[ T_1 := T_1^+ \cup T_1^- \cup \Delta_1. \]
Analogously, around the analytic arc $\tilde{\Delta}_2$ we define domains $T_2^\pm$ bounded by
\[ \partial T_2^\pm := \tilde{\Delta}_2 \cup \tilde{\Delta}_2^\pm, \]
and
\[ T_2 := T_2^+ \cup T_2^- \cup \tilde{\Delta}_2. \]
Also, around $E_1$ we define the domains $T_{E_1}^\pm$
\[ \partial T_{E_1}^\pm := E_1 \cup E_1^\pm, \]
and
\[ T_{E_1} := T_{E_1}^+ \cup T_{E_1}^- \cup E_1. \]
Now we can transform the Riemann-Hilbert problem (4.17)–(4.21). We define (see Figure 4.3),

\[
\tilde{Z} := \begin{cases} 
\tilde{Z}D_{1}^{-1}, & \text{in } T_{1}^{+} \setminus T_{2}, \\
\tilde{Z}D_{1}, & \text{in } T_{1}^{-} \setminus T_{E_{1}}, \\
\tilde{Z}D_{2}^{-1}, & \text{in } T_{2}^{+} \setminus T_{1}, \\
\tilde{Z}D_{2}, & \text{in } T_{2}^{-} \setminus T_{1}, \\
\tilde{Z}D_{1,2}^{-1}, & \text{in } T_{E}^{+} \setminus T_{1}^{-}, \\
\tilde{Z}D_{1,2}, & \text{in } T_{E}^{-} \setminus T_{1}^{+}, \\
\tilde{Z}D_{1}^{-1}D_{2}, & \text{in } T_{1}^{+} \cap T_{2}^{-}, \\
\tilde{Z}D_{1}^{-1}D_{2}^{-1}, & \text{in } T_{1}^{+} \cap T_{2}^{+}, \\
\tilde{Z}D_{1,2}D_{1,2}, & \text{in } T_{1}^{-} \cap T_{E_{1}^{-}}, \\
\tilde{Z}D_{1,2}^{-1}, & \text{in } T_{1}^{-} \cap T_{E_{1}^{+}}, \\
\tilde{Z}, & \text{in } \mathbb{C} \setminus (T_{1} \cup T_{2} \cup T_{E_{1}}).
\end{cases}
\]  

(4.22)

The piecewise analytic matrix function \(\tilde{Z}\) has a jump (see (4.17))

\[
\tilde{Z}_{+} = \tilde{Z}_{-}\tilde{J}, \quad \text{on } \tilde{\Sigma}^{+} \cup \partial T_{1} \cup \partial T_{2} \cup \partial T_{E_{1}} =: \Sigma.
\]  

(4.23)

We now describe the explicit form of the matrix function \(\tilde{J}\) on the different parts of \(\tilde{\Sigma}\). From (4.20) we have on \(E_{1} \setminus T_{1}\)

\[
\tilde{Z}_{+} = \tilde{Z}_{-}D_{1,2}^{-1} = \tilde{Z}_{-}D_{1,2}W_{1,2}D_{1,2}^{-1}D_{1,2}^{-1} = \tilde{Z}_{-}W_{1,2}, \quad \text{on } E_{1} \setminus T_{1}.
\]

The same relation holds on \(E_{1} \cap T_{1}\) since \(D_{1}^{-} = D_{1}^{-}\) on \(E_{1}\). From (4.18), (4.21) and (4.9)
we have on $\Delta_1 \setminus (T_2 \cup T_{E_1})$

$\tilde{Z}_+ = \tilde{Z} D_{1+}^{-1} D_{2+} = \tilde{Z} (D_1 - D_{1,2}D_{1,-}^{-1} D_{1+}^{-1})D_{1+}^{-1} D_{2+} = \tilde{Z}_- D_{1,2}^{-1} W_1 D_{2+}$. 

The remarkable fact is that this relation in fact holds on $\Delta_1 \cap (T_2 \cup T_{E_1})$ as well. Indeed, on $\tilde{\Delta}_1 \cap (T_2 \cup T_{E_1})$ we have (see Figure 4.3)

$\tilde{Z}_+ = \tilde{Z}_1 D_{1+}^{-1} D_{2+} = \tilde{Z}_- (D_1 - D_{1,2}D_{1,-}^{-1} D_{1+}^{-1})D_{1+}^{-1} D_{2+} = \tilde{Z}_- D_{1,2}^{-1} W_1 D_{2+}$. 

Note that

$D_{1,2}^{-1} W_1 D_{2+} = W_1$, on $\tilde{\Delta}_1$, and therefore

$\tilde{Z}_+ = \tilde{Z}_- W_1$, on $\tilde{\Delta}_1 \cap (T_2 \cup T_{E_1})$. 

Analogously, since $D_{1,2} W_1 D_{2+}^{-1} = W_1$, on $\Delta_{1,2}$, we have

$\tilde{Z}_+ = \tilde{Z}_- D_{1+}^{-1} D_{2+}^{-1} = \tilde{Z}_- (D_1 - D_{1,2}W_1 D_{2+}^{-1})D_{1+}^{-1} D_{2+}^{-1} = \tilde{Z}_- W_1$, on $\tilde{\Delta}_{1,2} \cap (T_2 \cup T_{E_1})$.
In the same way we have on \( \tilde{\Delta}_2 \setminus T_1 \), see (4.10);
\[
\hat{Z}_+ = \tilde{Z}_+ D_{2+}^{-1} = \tilde{Z}_- D_{-2+} W_2 D_{2+}^{-1} = \tilde{Z}_- W_2, \quad \text{on } \tilde{\Delta}_2 \setminus T_1,
\]
and on \( \tilde{\Delta}_2 \cap T_1 \), see (4.18) and (4.10),
\[
\hat{Z}_+ = \tilde{Z}_+ D_1^{-1} D_2^{-1} = \tilde{Z}_- (D_2 W_2 D_{2+1}) D_{1+}^{-1} D_2^{-1}.
\]
Using the commutation relations (4.14) and the observation that
\[
D_1 W_2 D_{2+} D_{2+1}^{-1} D_2^{-1} = W_2, \quad \text{on } \tilde{\Delta}_2,
\]
we arrive at
\[
\hat{Z}_+ = \tilde{Z}_- D_1^{-1} D_2 = \tilde{Z}_- W_2, \quad \text{on } \tilde{\Delta}_2 \cap T_1.
\]
Summarizing, we have for the jump matrix \( \hat{J} \) in (4.23)
\[
\hat{J} = \begin{cases} 
D_1, & \text{on } \Delta_1^\pm, \\
D_2, & \text{on } \tilde{\Delta}_2^\pm, \\
D_{1,2}, & \text{on } E_1^\pm, \\
D_{2,1}, & \text{on } E_2^+, \\
W_1, & \text{on } \Delta_1, \\
W_2, & \text{on } \tilde{\Delta}_2, \\
W_{1,2}, & \text{on } E_1. 
\end{cases}
\] (4.24)

We also note that when we go around the point \( c_j \) (\( j = 1, 2 \)), then the total jump of \( \hat{Z} \) is
\[
W_2 W_1^{-1} W_{1,2}^{-1} W_1 = I,
\]
i.e., the intersection points \( c_1 \) and \( c_2 \) are not branch points of \( \hat{Z} \). Note also that the matrices \( W_1, W_2, W_{1,2} \) do not depend on \( n \) and due to Proposition 2.13 (see Figure 2.8) and (4.15) the matrices \( D_1, D_2, D_{1,2}, D_{2,1} \) tend to the identity matrix \( I \) on compact sets of the domains of their definition in (4.24). Therefore we achieved the goal of the third transformation.

### 4.4 Parametrix away from the branch points

The goal of the fourth step is to construct a solution for the model Riemann-Hilbert problem with jumps that do not depend on \( n \), so that later on we can eliminate these jumps in (4.24). We are looking for a function \( X \) for which
\[
\begin{cases} 
X \in H^{3 \times 3}(\mathbb{C} \setminus \Sigma), \\
X_+ = X_+ \hat{W}, & \text{on } \Sigma, \\
X(z) = I + O(1/z), & z \to \infty, 
\end{cases}
\] (4.25)
where \( \Sigma := \Delta_1 \cup \tilde{\Delta}_2 \cup E_1 \) (see Figure 4.1), and
\[
\hat{W} := \begin{cases} 
W_1, & \text{on } \Delta_1, \\
W_2, & \text{on } \tilde{\Delta}_2, \\
W_{1,2}, & \text{on } E_1. 
\end{cases}
\] (4.26)
see the notation in (4.8) and (4.12). The function $X$ should also satisfy the endpoint condition (2.15)–(2.17). We used the notation $\Sigma^0 := \Sigma \setminus \{a_1, b_1, a_2, b_2\}$ in (4.25).

To construct the solution of (4.25)–(4.26) we use the Szegö functions (see subsection 2.6), i.e., the solution of the following system of scalar boundary value problems, see (2.91), (2.93),

\[
\begin{align*}
1) & \quad \begin{cases}
F_0 \in H(\overline{C} \setminus (\Delta_1 \cup \tilde{\Delta}_2)), \\
F_1 \in H(\overline{C} \setminus (E_1 \cup \Delta_1)), \\
F_2 \in H(\overline{C} \setminus (E_1 \cup \tilde{\Delta}_2)),
\end{cases} \\
2) & \quad \begin{cases}
F_{0\pm} = iF_{1\pm}\omega_1 - w_1, & \text{on } \Delta_1 \setminus \{a_1, b_1\} =: \tilde{\Delta}_1, \\
F_{0\pm} = iF_{2\pm}\omega_2 - w_2, & \text{on } \tilde{\Delta}_2 \setminus \{a_2, b_2\} =: \tilde{\tilde{\Delta}}_2, \\
F_{1\pm} = F_{2\pm} \frac{\omega_2 - w_2}{\omega_1 - w_1}, & \text{on } E_1,
\end{cases} \\
3) & \quad \begin{cases}
\text{normalization condition: } F_0 F_1 F_2 = 1 \text{ in } \overline{C}, \\
\text{local behavior (2.91)-2b.}
\end{cases}
\end{align*}
\]

Here $\omega_j$ is the branch of $\omega_j^2(z) = (z - a_j)(z - b_j)$, see (2.87). The solution of this system of boundary value problems exists and is given in Theorem 2.3. We define

\[
F(z) := \text{diag}(F_0(z), F_1(z), F_2(z)), \quad F_\infty := F(\infty),
\]

and transform the Riemann-Hilbert problem (4.25)–(4.26) to the Riemann-Hilbert problem for the function

\[
\tilde{X} := F_{\infty}^{-1} X F.
\]

We have

\[
\begin{align*}
\tilde{X} & \in H(3 \times 3)(\overline{C} \setminus \Sigma^0), \\
\tilde{X} & = \tilde{X}_- H, \quad \text{on } \Sigma^0, \\
\tilde{X}(z) & = I + O(1/z), \quad z \to \infty,
\end{align*}
\]

where the jump

\[
\tilde{H} = F_{\infty}^{-1} \tilde{W} F_+,
\]

due to (4.27)-2, is

\[
\tilde{H} := \begin{cases}
0 & F_{1+} \omega_1 - w_1 & 0 \\
F_{0+} & 0 & 0 \\
-F_0 & 0 & 0 \\
F_{1-} & 0 & 1
\end{cases} = \begin{cases}
0 & 1 & 0 \\
i\omega_1 & 0 & 0 \\
0 & 0 & 1
\end{cases}, \quad \text{on } \tilde{\Delta}_1, \\
0 & 0 & 1 \\
i\omega_2 & 0 & 0 \\
0 & 1 & 0 \\
-\omega_2 & 0 & 0 \\
1 & 0 & 0 \\
0 & -\omega_1 & 0 \\
0 & 0 & \omega_1 \\
\omega_2 & 0 & \omega_1 \\
\omega_1 & 0 & 0
\end{cases} \begin{cases}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{cases}, \quad \text{on } \tilde{\tilde{\Delta}}_2, \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & \omega_1 & 0 \\
0 & 0 & \omega_1 \\
0 & 0 & 0 \\
0 & 0 & \omega_1 \\
0 & 0 & 0
\end{cases} \begin{cases}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{cases}, \quad \text{on } E_1.
\]
For the solution of the Riemann-Hilbert problem \((4.28)–(4.29)\) we take the Riemann surface \(\mathcal{R}\) from Definition \([2.11]\) and define on \(\mathcal{R}\) the rational function \(X_j\) with one pole (and one zero)

\[
X_j := (X_j^{(0)}, X_j^{(1)}, X_j^{(2)}) \in \mathcal{M}(\mathcal{R}) : \begin{cases} 
X_j(\xi) = O(1/\xi), & \xi \to \infty^{(0)}, \\
X_j(\xi) = -\xi + \cdots, & \xi \to \infty^{(j)}, \end{cases} \quad j = 1, 2. \tag{4.30}
\]

Then the function

\[
\tilde{X} = \begin{pmatrix}
\tilde{x}_{0,0} & \tilde{x}_{0,1} & \tilde{x}_{0,2} \\
\tilde{x}_{1,0} & \tilde{x}_{1,1} & \tilde{x}_{1,2} \\
\tilde{x}_{2,0} & \tilde{x}_{2,1} & \tilde{x}_{2,2}
\end{pmatrix} = \begin{pmatrix}
1 & -1/\omega_1 & -1/\omega_2 \\
i\omega_1 - \tilde{x}_{0,1} & -1/\omega_1 - \tilde{x}_{0,2} & -1/\omega_2 - \tilde{x}_{0,2} \\
i\omega_1 - \tilde{x}_{1,1} & -1/\omega_1 - \tilde{x}_{1,2} & -1/\omega_2 - \tilde{x}_{1,2} \\
i\omega_1 - \tilde{x}_{2,1} & -1/\omega_1 - \tilde{x}_{2,2} & -1/\omega_2 - \tilde{x}_{2,2}
\end{pmatrix} \tag{4.31}
\]

is the solution of the Riemann-Hilbert problem \((4.28)–(4.29)\). Indeed, the normalization at infinity in \((4.28)\) clearly holds because of \((4.30)\). To verify the jump condition in \((4.28)–(4.29)\) we check the relation

\[
\tilde{X}_+ = \tilde{X}_- H
\]

on the different parts of \(\mathfrak{C}\). Substituting here \((4.29)\) we need to check that

\[
\begin{pmatrix}
\tilde{x}_{0,0}+ & \tilde{x}_{0,1}+ & \tilde{x}_{0,2}+ \\
\tilde{x}_{1,0}+ & \tilde{x}_{1,1}+ & \tilde{x}_{1,2}+ \\
\tilde{x}_{2,0}+ & \tilde{x}_{2,1}+ & \tilde{x}_{2,2}+
\end{pmatrix} = \begin{pmatrix}
\tilde{x}_{0,0}- & \tilde{x}_{0,1}- & \tilde{x}_{0,2}- \\
\tilde{x}_{1,0}- & \tilde{x}_{1,1}- & \tilde{x}_{1,2}- \\
\tilde{x}_{2,0}- & \tilde{x}_{2,1}- & \tilde{x}_{2,2}-
\end{pmatrix}, \quad \text{on } \mathfrak{C}, \tag{4.32}
\]

If we substitute the expression \((4.31)\) for \(\tilde{x}_{k,\ell}\) \((k, \ell = 0, 1, 2)\) into \((4.32)\), then we indeed see that \((4.32)\) holds identically. Thus the jump condition in \((4.28)–(4.29)\) holds and \(\tilde{X}\) in \((4.31)\) is the solution of the Riemann-Hilbert problem \((4.28)–(4.29)\). Finally we have from \((2.93)\) and \((4.30)\) that

\[
X := F_{\infty} \tilde{X} F = \begin{pmatrix}
F_0(\infty) & -1/\omega_1 F_1(\infty) & -1/\omega_2 F_0(\infty) \\
i\omega_1 X_1^{(0)} F_0(\infty) & -1/\omega_1 X_1^{(1)} F_1(\infty) & -1/\omega_2 X_1^{(2)} F_0(\infty) \\
i\omega_2 X_2^{(0)} F_0(\infty) & -1/\omega_1 X_2^{(1)} F_1(\infty) & -1/\omega_2 X_2^{(2)} F_2(\infty)
\end{pmatrix} \tag{4.33}
\]

is the desired solution of the model Riemann-Hilbert problem \((4.25)–(4.26)\).

**Remark 4.1.** We recall that in Theorem \([2.34]\) we presented the solution of the scalar boundary value problem \((2.91)\) on the Riemann surface by means of the Cauchy integral

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with a meromorphic differential on $\mathfrak{R}$. There is another way to solve the scalar boundary value problem (2.91); see [7] for a similar approach. The idea is the following. Since $\mathfrak{R}$ has genus zero, it is conformally equivalent to $\mathbb{C}$. Let $\zeta$ be a conformal mapping

$$\zeta(\mathfrak{R}) = \mathbb{C}, \quad \zeta(\infty^{(0)}) = \infty.$$ 

We consider $\mathfrak{R}$ from Definition 2.11 (see Figure 2.7). The image of $\mathfrak{R}$ is given in Figure 4.4. Analyzing the jumps around the points $\zeta(c_j)$, see the notation (2.88), we see that the special form of the jumps (2.89) provides the existence of a continuous solution in the neighborhood of the points $c_j$ ($j = 1, 2$).

![Figure 4.4: Conformal mapping of $\mathfrak{R}$ to $\mathbb{C}$ (case II)](image)

### 4.5 Local parametrices

The function $X$ is not a good approximation near the branch points $A := \{a_1, b_1, a_2, b_2\}$. We need a local analysis near each of the branch points. The goal of the fifth step is to find a matrix function $U$ in a neighborhood $O_e$ of each branch point $e \in A$ which asymptotically (as $n \to \infty$) matches $X$ on the boundary $\partial O_e$ and which has the same jumps as $\tilde{Z}$. This function is the solution of the following Riemann-Hilbert problem: if $e_j$ denotes $a_j$ or $b_j$ ($j = 1, 2$), then we want to find a function $U_{e_j}$ such that

$$\begin{align*}
1) \quad & U_{e_j} \in H^{3\times3}(O_{e_j} \setminus (\hat{\Delta}_j \cup \hat{\Delta}_j^+ \cup \hat{\Delta}_j^-)), \\
2) \quad & U_{e_j+} = U_{e_j-} \tilde{J} \quad \text{on } \hat{\Sigma}_{e_j} := O_{e_j} \cap (\hat{\Delta}_j \cup \hat{\Delta}_j^+ \cup \hat{\Delta}_j^-) : \quad (4.34) \\
3) \quad & U_{e_j} = (I + O(1/n))X \quad \text{uniformly on } \partial O_{e_j} \text{ as } n \to \infty,
\end{align*}$$

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and the elements of the matrix $U_{e_j}(z)$ have the same limiting behavior as the elements of $\hat{\mathcal{G}}(z)$ as $z \to e_j$.

We recall the explicit expressions of the jump matrix $\hat{J}$, see (4.24) and (4.8),

\[
\hat{J} = \begin{cases}
\left( \begin{array}{ccc} 1 & 0 & 0 \\ \frac{\Phi_0^n}{\Phi_1^n} & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) & \text{on } \Delta_1^{(\pm)}, \\
\left( \begin{array}{ccc} 0 & w_1 & 0 \\ -\frac{1}{w_1} & 0 & 0 \\ 0 & 0 & 1 \end{array} \right) & \text{on } \Delta_1, \\
\left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\Phi_0^n}{\Phi_2^n} & 0 & 1 \end{array} \right) & \text{on } \Delta_2^{(\pm)}, \\
\left( \begin{array}{ccc} 0 & 0 & w_2 \\ 0 & 1 & 0 \\ -\frac{1}{w_2} & 0 & 0 \end{array} \right) & \text{on } \Delta_2.
\end{cases}
\]

We see that the jump matrices related to a branch point from $A$ have a $2 \times 2$ block structure. The solution of the Riemann-Hilbert problem (4.34) in the $2 \times 2$ case, with the local behavior of the equilibrium measure as in (2.54), was obtained in [47]. Thus, using the results in [47], it is easy to write down the explicit form of the solution for the local Riemann-Hilbert problem (4.34). This solution is

\[
U_{e_j} = E_{e_j} V_{e_j} A_{e_j}, \quad j = 1, 2, \tag{4.35}
\]

where the matrices $A_{e_j}$ are

\[
A_{e_1} := \text{diag} \left( \left[ \left( \frac{\Phi_0}{\Phi_1} \right)^{n/2} \tilde{w}_{e_1} \right]^{-1}, \left( \frac{\Phi_0}{\Phi_1} \right)^{n/2} \tilde{w}_{e_1}, 1 \right),
\]

\[
A_{e_2} := \text{diag} \left( \left[ \left( \frac{\Phi_0}{\Phi_2} \right)^{n/2} \tilde{w}_{e_2} \right]^{-1}, 1, \left( \frac{\Phi_0}{\Phi_2} \right)^{n/2} \tilde{w}_{e_2} \right),
\]

and, see (2.35),

\[
\tilde{w}_{a_j}(\xi) = \left[ w_{0,j}(\xi)(b_j - \xi)^{\beta_j}(a_j - \xi)^{\alpha_j} \right]^{1/2},
\]

\[
\tilde{w}_{b_j}(\xi) = \left[ w_{0,j}(\xi)(\xi - b_j)^{\beta_j}(\xi - a_j)^{\alpha_j} \right]^{1/2}.
\]

The branch of the square roots above is chosen such that

\[
\tilde{w}_{e_j} + \tilde{w}_{e_j} = w_j, \quad \text{on } \Delta_j \cap O_{e_j}, \quad e_j \in \{a_j, b_j\}, \quad j = 1, 2.
\]
The matrices $E_{ej}$ in (4.33) are

\[
E_{a1} := \frac{1}{2}X\text{diag}(\hat{w}_{a1}, \hat{w}_{a1}^{-1}, 1) \begin{pmatrix} 1 & i & 0 \\ i & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{diag} \left( \frac{1}{\sqrt{\pi n\varphi_1}}, \frac{1}{\sqrt{\pi n\varphi_1}}, 1 \right),
\]

\[
E_{b1} := \frac{1}{2}X\text{diag}(\hat{w}_{b1}, \hat{w}_{b1}^{-1}, 1) \begin{pmatrix} 1 & -i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{diag} \left( \frac{1}{\sqrt{\pi n\varphi_1}}, \frac{1}{\sqrt{\pi n\varphi_1}}, 1 \right),
\]

\[
E_{a2} := \frac{1}{2}X\text{diag}(\hat{w}_{a2}, 1, \hat{w}_{a2}^{-1}) \begin{pmatrix} 1 & i & 0 \\ i & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{diag} \left( \frac{1}{\sqrt{\pi n\varphi_2}}, 1, \frac{1}{\sqrt{\pi n\varphi_2}} \right),
\]

\[
E_{b2} := \frac{1}{2}X\text{diag}(\hat{w}_{b2}, 1, \hat{w}_{b2}^{-1}) \begin{pmatrix} 1 & -i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{diag} \left( \frac{1}{\sqrt{\pi n\varphi_2}}, 1, \frac{1}{\sqrt{\pi n\varphi_2}} \right).
\]

Here $X$ is given by (4.33) and we use the notation

\[
\varphi_j := \log \left( \frac{\Phi_0}{\Phi_j} \right).
\]

To present the explicit expression for the matrices $V_{ej}$ in (4.35) we follow [47] and introduce the matrices

\[
\Psi_{a1} := \begin{pmatrix} I_{a1} \left( -\frac{n_{\varphi_1}}{2} \right) & -\frac{i}{\pi} K_{\alpha_1} \left( -\frac{n_{\varphi_1}}{2} \right) & 0 \\ n\pi i\varphi_1 I'_{a1} \left( -\frac{n_{\varphi_1}}{2} \right) & -\frac{i}{\pi} n\varphi_1 K'_{\alpha_1} \left( -\frac{n_{\varphi_1}}{2} \right) & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

\[
\Psi_{b1} := \begin{pmatrix} I_{\beta_1} \left( \frac{n_{\varphi_1}}{2} \right) & \frac{i}{\pi} K_{\beta_1} \left( \frac{n_{\varphi_1}}{2} \right) & 0 \\ n\pi i\varphi_1 I'_{\beta_1} \left( \frac{n_{\varphi_1}}{2} \right) & -\frac{i}{\pi} n\varphi_1 K'_{\beta_1} \left( \frac{n_{\varphi_1}}{2} \right) & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

\[
\Psi_{a2} := \begin{pmatrix} I_{a2} \left( -\frac{n_{\varphi_2}}{2} \right) & 0 & -\frac{i}{\pi} K_{\alpha_2} \left( -\frac{n_{\varphi_2}}{2} \right) \\ 0 & 1 & 0 \\ n\pi i\varphi_1 I'_{a2} \left( -\frac{n_{\varphi_2}}{2} \right) & n\varphi_2 K'_{\alpha_2} \left( -\frac{n_{\varphi_2}}{2} \right) & 0 \end{pmatrix},
\]

\[
\Psi_{b2} := \begin{pmatrix} I_{\beta_2} \left( \frac{n_{\varphi_2}}{2} \right) & 0 & \frac{i}{\pi} K_{\beta_2} \left( \frac{n_{\varphi_2}}{2} \right) \\ 0 & 1 & 0 \\ -n\pi i\varphi_2 I'_{\beta_2} \left( \frac{n_{\varphi_2}}{2} \right) & n\varphi_2 K'_{\beta_2} \left( \frac{n_{\varphi_2}}{2} \right) & 0 \end{pmatrix},
\]

where $I_{\alpha}$ and $K_{\alpha}$ are the modified Bessel functions of order $\alpha$ (see [1]).

We denote by $O_{ej}^{(\pm)}$ the sector domain bounded by

\[
\partial O_{ej}^{(\pm)} := \partial O_{ej} \cup (\bar{\Delta}_j \cap O_{ej}) \cup (\Delta_j^+ \cap O_{ej}).
\]

Analogously, $O_{ej}^{(-)}$ has the boundaries

\[
\partial O_{ej}^{(-)} := \partial O_{ej} \cup (\bar{\Delta}_j \cap O_{ej}) \cup (\Delta_j^- \cap O_{ej}),
\]

and

\[
O_{ej}^{(\pm)} := O_{ej} \setminus (O_{ej}^{(\pm)} \cup O_{ej}^{(-)}).
\]

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The matrices $V_j$ in (4.35) are

$$V_{a_1} := \begin{cases} \Psi_{a_1}, & \text{in } O_{a_1}^{(s)} , \\ \left( \begin{array}{ccc} 1 & 0 & 0 \\ -e^{-\alpha_1 \pi i} & 1 & 0 \\ 0 & 0 & 1 \end{array} \right), & \text{in } O_{a_1}^{(+)}, \\ \Psi_{a_1} \left( e^{\alpha_1 \pi i} \right) & \text{in } O_{a_1}^{(-)} , \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{cases}, \quad V_{b_1} := \begin{cases} \Psi_{b_1}, & \text{in } O_{b_1}^{(s)} , \\ \left( \begin{array}{ccc} 1 & 0 & 0 \\ e^{\beta_1 \pi i} & 1 & 0 \\ 0 & 0 & 1 \end{array} \right), & \text{in } O_{b_1}^{(+)}, \\ \Psi_{b_1} \left( e^{-\beta_1 \pi i} \right) & \text{in } O_{b_1}^{(-)} , \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{cases}$$

$$V_{a_2} := \begin{cases} \Psi_{a_2}, & \text{in } O_{a_2}^{(s)} , \\ \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -e^{-\alpha_2 \pi i} & 0 & 1 \end{array} \right), & \text{in } O_{a_2}^{(+)}, \\ \Psi_{a_2} \left( e^{\alpha_2 \pi i} \right) & \text{in } O_{a_2}^{(-)} , \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{cases}, \quad V_{b_2} := \begin{cases} \Psi_{b_2}, & \text{in } O_{b_2}^{(s)} , \\ \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ e^{\beta_2 \pi i} \right) & \text{in } O_{b_2}^{(+)}, \\ \Psi_{b_2} \left( e^{-\beta_2 \pi i} \right) & \text{in } O_{b_2}^{(-)} , \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{cases}$$

This gives all the ingredients for the solution (4.35) of the local Riemann-Hilbert problem (4.34).

### 4.6 Final transformation. Asymptotic formulas

We now finish the transformation of the original Riemann-Hilbert problem (2.13)–(2.17) to the Riemann-Hilbert problem (4.1). The final transformation is

$$\tilde{Z} := \begin{cases} \tilde{Z}X^{-1}, & \text{in } \overline{\mathbb{C}} \setminus \bigcup_{j=1}^2 (O_{a_j} \cup O_{b_j}), \text{ (away from the branch points)}, \\ \tilde{Z}U_{e_j}^{-1}, & \text{in } O_{e_j}, e_j \in \{a_j, b_j\}, j = 1, 2, \text{ (near the branch points)}. \end{cases} \quad (4.36)$$

We have on $\Delta_1 \cup \Delta_2 \cup E_1$

$$\tilde{Z}_+ = \tilde{Z}_-JX^{-1} = \tilde{Z}_-X^{-1}(X_-JX_+^{-1}).$$

From (4.25)–(4.26) we have

$$X_+JX_+^{-1} = X_-\tilde{W}(X_-\tilde{W})^{-1},$$

so that

$$\tilde{Z}_+ = \tilde{Z}_- , \quad \text{on } (\Delta_1 \cup \Delta_2 \cup E_1) \setminus O_{e_1}.$$ 

Analogously, see (4.34), we have

$$\tilde{Z}_+ = \tilde{Z}_- , \quad \text{on } (\Delta_1 \cup \Delta_2 \cup E_1) \setminus O_{e_2}.$$ 

Thus, denoting (see Figure 4.5)

$$\tilde{\Sigma} := \Delta_1^+ \cup \Delta_2^+ \cup \Delta_{1,2}^+ \cup \Delta_1^+ \cup \Delta_2^+ \cup \bigcup_{e_j \in A} \partial O_{e_j},$$

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we have

\[
\begin{align*}
1) \quad & \tilde{Z} \in H^{(3 \times 3)}(\mathbb{C} \setminus \Sigma), \\
2) \quad & \tilde{Z}_+ = \tilde{Z}_- \tilde{J}, \quad \text{on } \Sigma, \\
3) \quad & \tilde{Z}(z) = I + \mathcal{O}(1/z), \quad z \to \infty.
\end{align*}
\tag{4.37}
\]

Figure 4.5: The contour $\tilde{\Sigma}$ for the final transformation

Since as $n \to \infty$

\[
\tilde{J} = I + \mathcal{O}(1/n), \quad \text{uniformly on } \bigcup_{e_j \in A} \partial O_{e_j},
\]

and for some $C > 0$

\[
\tilde{J} = I + \mathcal{O}(e^{-Cn}), \quad \text{uniformly on } \tilde{\Sigma} \setminus \bigcup_{e_j \in A} \partial O_{e_j},
\]

we can conclude that the solution of the problem (4.37) exists for $n$ large enough, and

\[
\tilde{Z} = I + \mathcal{O}(1/n),
\tag{4.38}
\]

uniformly for $z \in \overline{\mathbb{C}}$.

Now we can return through the sequence of transformations (4.36), (4.22), (4.16), (4.3),

\[
\tilde{Z} \to \tilde{Z} \to \tilde{Z} \to Z \to Y
\tag{4.39}
\]

and we can conclude that for large $n$ the solution of the Riemann-Hilbert problem (2.13) exists, i.e., the Hermite-Padé approximants exists and the indices are normal, and we get the asymptotic formulas of Theorems 2.35 and 2.36. To obtain these formulas we note that in the transformation (4.39) it is sufficient to keep track of the first row of the
matrices. We start with the asymptotics on compact sets in \( \overline{C} \setminus \bar{\Sigma} \), where \( \bar{\Sigma} := \Sigma \cup E_2 \), see (4.5). Comparing the inverse and direct transformation in (4.39), we have

\[
\tilde{Z} = (I + O(1/n))X, \quad \text{on } \overline{C} \setminus \bar{\Sigma},
\]

and

\[
\tilde{Z} = Z = \begin{pmatrix} C_n^\alpha \Phi_0^n P_n & C_n^\alpha \Phi_1^n R_n^{(1)} & C_n^\alpha \Phi_2^n R_n^{(2)} \\ * & * & * \\ * & * & * \end{pmatrix}, \quad \text{in } (\overline{C} \setminus \overline{G}) \setminus (\tilde{\Delta}_1 \cup \tilde{\Delta}_2),
\]

\[
\tilde{Z} = ZD_{2,1} = \begin{pmatrix} C_n^\alpha \Phi_0^n P_n & C_n^\alpha \Phi_1^n R_n^{(1)} & C_n^\alpha \Phi_2^n R_n^{(2)} - C_n^\alpha \Phi_2^n R_n^{(1)} w_2/w_1 \\ * & * & * \\ * & * & * \end{pmatrix}, \quad \text{in } G \setminus \Delta_{1.2}.
\]

This gives

\[
P_n = \frac{X_{0.0}}{C_0^\alpha \Phi_0^n} (1 + O(1/n)), \quad \text{on } \overline{C} \setminus \bar{\Sigma}, \quad (4.40)
\]

\[
R_n^{(1)} = \frac{X_{0.1}}{C_0^\alpha \Phi_1^n} (1 + O(1/n)), \quad \text{on } \overline{C} \setminus \bar{\Sigma}, \quad (4.41)
\]

\[
R_n^{(2)} = \begin{cases} \frac{X_{0.2}}{C_0^\alpha \Phi_2^n} (1 + O(1/n)), & \text{on } (\overline{C} \setminus \overline{G}) \setminus (\tilde{\Delta}_1 \cup \tilde{\Delta}_2), \\ \left( \frac{X_{0.2}}{C_0^\alpha \Phi_2^n} + \frac{X_{0.1}}{C_0^\alpha \Phi_1^n} \right), & \text{on } G \setminus \Delta_{1.2}. \end{cases} \quad (4.42)
\]

The asymptotic formulas (4.40)–(4.42) are valid uniformly on compact subsets of the indicated sets. If we substitute (4.33) in these formulas, then we arrive at the formulas of Theorems 2.35 and 2.36 for \( \tilde{\Sigma} \setminus \{a_1, b_1, a_2, b_2\} \). Since \( X_{0.0}, \Phi_0 \in H(\overline{C} \setminus (\tilde{\Delta}_1 \cup \tilde{\Delta}_2 \cup \Delta_{1.2})) \), the asymptotic formula (4.40) holds on compact subsets in \( \overline{C} \setminus (\tilde{\Delta}_1 \cup \tilde{\Delta}_2 \cup \Delta_{1.2}) \) and therefore in particular on \( E_1 \cup E_2 \). For the same reason the asymptotic formula (4.41) holds on compact subsets of \( \overline{C} \setminus \Sigma \), hence in particular on \( E_2 \), and the second asymptotic formula in (4.42) holds on \( E_2 \). Thus the asymptotic behavior on \( E_2 \) is verified.

Now consider the asymptotics on \( E_1 \). On \( E_1 \) we have, see (4.25)–(4.26),

\[
X_{0.1+} = X_{0.2-} \frac{w_1}{w_2}, \quad \Phi_{1+} = \Phi_{2-}, \quad \text{on } E_1,
\]

hence we have from (4.42)

\[
R_n^{(2)} = \frac{X_{0.2-}}{C_0^\alpha \Phi_2^n} (1 + O(1/n)) = \frac{X_{0.1+}}{C_0^\alpha \Phi_1^n} \frac{w_2}{w_1} (1 + O(1/n)), \quad \text{on } E_1. \quad (4.43)
\]

In \( T_{E_1} \), see (4.22), we have

\[
\tilde{Z} = \begin{cases} ZD_{2,1}D_{1,2}^{-1}, & \text{in } T_{E_1}^{(+)} \\ ZD_{1,2}, & \text{in } T_{E_1}^{(-)} \end{cases},
\]

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and therefore

\[
\hat{Z} = \begin{cases}
\left( \frac{C_0^n\Phi_0^nP_n}{C_0^n\Phi_0^n}, \frac{C_0^n\Phi_0^n}{w_1}R_n^{(2)}, C_0^n\Phi_2^nR_n^{(2)} - C_0^n\Phi_2^n\frac{w_2}{w_1}R_n^{(1)} \right), & \text{on } T_{E_1}^{(+)};
\end{cases}
\]

On the other hand we have

\[
\hat{Z} = (I + O(1/n))X, \quad \text{in } \mathbb{C} \setminus \bigcup_{j=1}^{2}(O_{a_j} \cup O_{b_j}),
\]

hence using (4.33) we arrive at

\[
R_n^{(1)} = \left( \frac{X_{0,1+} - X_{0,2+}w_1}{C_0^n\Phi_1^n + w_2} \right) (1 + O(1/n))
= \left( \frac{X_{0,2-}}{C_0^n\Phi_2^n} + \frac{X_{0,1-}}{C_0^n\Phi_2^n} \right) (1 + O(1/n)), \quad \text{on } E_1.
\]

Thus the asymptotics on \(E_1\) are also verified.

Now consider the asymptotics on \(\tilde{\Delta}_j \quad (j = 1, 2)\). In \(T_j\), away from \(c_1\) and \(c_2\), we have

\[
\tilde{Z} D_j^{-1} = \begin{cases}
\left( \frac{C_0^n\Phi_0^nP_n}{C_0^n\Phi_0^n}, \frac{C_0^n\Phi_0^n}{w_j}R_n^{(j)}, C_0^n\Phi_1^nR_n^{(1)} - C_0^n\Phi_2^nR_n^{(2)} \right), & \text{on } T_j^{(+)};
\end{cases}
\]

\[
\tilde{Z} D_j = \begin{cases}
\left( \frac{C_0^n\Phi_0^nP_n}{C_0^n\Phi_0^n}, \frac{C_0^n\Phi_0^n}{w_j}R_n^{(j)}, C_0^n\Phi_1^nR_n^{(1)} - C_0^n\Phi_2^nR_n^{(2)} \right), & \text{on } T_j^{(-)}.
\end{cases}
\]

Taking the limiting value we obtain

\[
R_n^{(1)} = \frac{X_{0,1+}}{C_0^n\Phi_1^n}(1 + O(1/n)), \quad R_n^{(2)} = \frac{X_{0,2+}}{C_0^n\Phi_2^n}(1 + O(1/n)), \quad \text{on } \tilde{\Delta}_j, j = 1, 2. \quad (4.44)
\]

Using these asymptotics we arrive at

\[
P_n = \left( \frac{X_{0,0+}}{C_0^n\Phi_0^n} + \frac{X_{0,j+}}{w_jC_0^n\Phi_j^n} \right) (1 + O(1/n))
= \left( \frac{X_{0,0-}}{C_0^n\Phi_0^n} - \frac{X_{0,j-}}{w_jC_0^n\Phi_j^n} \right) (1 + O(1/n))
= \left[ \left( \frac{X_{0,0}}{C_0^n\Phi_0^n} \right)_+ + \left( \frac{X_{0,0}}{C_0^n\Phi_0^n} \right)_- \right] (1 + O(1/n)), \quad \text{on } \tilde{\Delta}_k, j = 1, 2. \quad (4.45)
\]
Finally we consider the asymptotics on $\Delta_{1,2}$. On $\Delta_{1,2}^{(\pm)}$ we have the limiting values

$$\hat{Z}_\pm = (\tilde{Z} D_1^{-1})_\pm = \begin{pmatrix} C_0^m \Phi_0^n P_n - C_0^m \Phi_0^n R_n^{(1)} & C_0^m \Phi_1^n R_n^{(1)} & C_0^m \Phi_2^n R_n^{(2)} - C_0^m w_2 \Phi_2^n R_n^{(1)} \\ * & * & * \end{pmatrix}.$$ 

From this we have that the asymptotics (4.45) for $P_n$ and (4.44) for $R_n^{(1)}$ are also valid on $\Delta_{1,2}$. For $R_n^{(2)}$ we have

$$C_0^m \Phi_2^n R_n^{(2)} - C_0^m w_2 \Phi_2^n w_1 C_0^m \Phi_1^n = X_{0,1} \pm (1 + O(1/n)), \quad \text{in } \Delta_{1,2}.$$ 

Taking into account that $|\Phi_0| = |\Phi_1| < |\Phi_2|$ in $\Delta_{1,2}$ (Proposition 2.13) we obtain

$$R_n^{(2)} = \frac{w_2}{w_1} \frac{X_{0,1} \pm}{C_0^m \Phi_1^n} (1 + O(1/n)), \quad \text{in } \Delta_{1,2}. \quad (4.46)$$

It remains to observe that the limiting values of the asymptotic formulas (4.43–4.46) along $\tilde{\Sigma}$ at the points $c_1$ and $c_2$ coincide.

This proves Theorems 2.35 and 2.36 for the case II.

**Remark 4.2.** We can get formulas for the local asymptotics for $P_n$ and $R_n^{(j)}$ ($j = 1, 2$) from (4.38) by using the transformations (4.39) in $O_{e_j}$, with $e_j \in \{a_j, b_j\}$. These formulas are similar to the corresponding formulas from [47] and contain Bessel functions.

### 4.7 Sketch of the proof for the other geometrical cases

**Case I**

For this case $\Delta_1 \cap \Delta_2 = \emptyset$, hence we do not need the second transformation (4.13), i.e., we do not need to open a global lens. All the other steps are as in the proof for the case II. We just have to use for $\{\Phi_0, \Phi_1, \Phi_2\}$ the algebraic function (2.27), (2.42) and we have to drop some parts of the proof which involve the intersection $\Delta_1 \cap \Delta_2$.

**Case IV**

This is also a generic case. All the steps of the proof above are present for this case. The proof now uses the Riemann surface (2.72) and for the normalization we use the branches (2.73) of the function $\Phi$:

$$Z := CY S,$$

where we use the notation (4.12). The second step is to open the global lens around $\Delta_{1,2}$ in the domain $G^+$ with boundary $\partial G^{(+)} = E_2 \cup E_1^+$ (see Figure 4.6 and the notation (2.75)) and we obtain

$$\tilde{Z} := \begin{cases} Z D_{1,2}, & \text{in } G^+, \\ Z, & \text{in } \overline{\mathbb{C}} \setminus G^+. \end{cases}$$

Here we use the notation (4.12). The function $\tilde{Z}$ has the jump

$$\tilde{Z}_+ = \tilde{Z}_- J, \quad \text{on } \Delta_1 \cup \Delta_2 \cup E_2 \cup E_1^+,$$

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where
\[ \tilde{J} := D_{1,2}, \quad \text{on } E_1^+ \]
and, see (4.8),
\[ \tilde{J} := \begin{cases} D_1-W_1D_{1+}, & \text{on } \tilde{\Delta}_1, \\ D_2-W_2D_{2+}, & \text{on } \Delta_2 := \tilde{\Delta}_2 \cup \Delta_{1,2}. \end{cases} \tag{4.47} \]

If we use the analog of (4.13)
\[ J(0,0) = D_{2,1}W_{1,2}D_{1,2}^{-1}, \quad \text{on } E_2, \]
then we have
\[ \tilde{J} := D_{2,1}W_{1,2}D_{2,1+}, \quad \text{on } E_2. \tag{4.48} \]

![Figure 4.6: The global lens (case IV)](image)

The third step is the opening of the local lenses around the arcs \( \Delta_1, \tilde{\Delta}_2, E_2 \) (see Section 4.3). The decompositions (4.47)–(4.48) give as a result a new function \( \hat{Z} \) for which the jumps on these arcs do not depend on \( n \)
\[ \hat{Z}_+ = \hat{Z}_- \tilde{J}, \]
with
\[ \tilde{J} = W := \begin{cases} W_1, & \text{on } \tilde{\Delta}_1, \\ W_2, & \text{on } \Delta_2, \\ W_{1,2}, & \text{on } E_2, \end{cases} \]
and on the lenses \( \tilde{\Delta}_1^\pm, \Delta_{1,2}^\pm, \Delta_2^\pm, E_2^\pm \) (excluding the points \( a_1, a_2, b, b^* \)) the jump \( \tilde{J} \) tends to the identity matrix as \( n \to \infty \) (due to Proposition 2.25).

The fourth step (the parametrix away from the branch points) is the solution of the Riemann-Hilbert problem for the function \( X \) with the jump \( W \). This is just a repetition of Section 4.4 but exchanging \( E_1 \) and \( E_2 \) and with another Riemann surface \( \mathfrak{R} \), namely (2.72) (see Figure 2.14).

The fifth step (local parametrices) has a new feature which is not present in the case II. The solutions of the local matrix Riemann-Hilbert problems (4.34) for the function \( U_e \), with \( e \in \{ a_1, a_2, b \} \), are the same as for the case II (see (4.35)). However, around the branch point \( b^* \) the solution is different. Because of Theorem 2.26-2, the solution
is represented by means of Airy functions. The local Riemann-Hilbert problem in the neighborhood of \( b^* \) is

\[
\begin{align*}
1) \quad & U_{b^*} \in H^{(3 \times 3)}(O_{b^*} \setminus \hat{\Sigma}_{b^*}), \\
2) \quad & U_{b^*+} - U_{b^*-} = \hat{J}, \quad \text{on } \hat{\Sigma}_{b^*}, \\
3) \quad & U_{b^*} = (I + \mathcal{O}(1/n))X, \quad \text{uniformly on } \partial O_{b^*} \text{ as } n \to \infty,
\end{align*}
\]

where (see Figure 4.7)

\[\hat{\Sigma}_{b^*} := O_{b^*} \cap (E_2 \cup E_2^+ \cup E_2^- \cup E_1^+),\]

and, see (4.8) and (4.12)

\[\hat{J} := \begin{cases} 
D_{2,1}, & \text{on } E_2^+ \cap O_{b^*}, \\
W_{1,2}, & \text{on } E_2 \cap O_{b^*}, \\
D_{1,2}, & \text{on } E_1^+ \cap O_{b^*}.
\end{cases}\]

Figure 4.7: Local analysis around \( b^* \)

Observe the block structure of this matrix Riemann-Hilbert problem, so that we can use the known solution for the \( 2 \times 2 \) case. Following [31, 26] we have

\[
U_{b^*} = E_{b^*} V_{b^*} A_{b^*},
\]

where

\[
A_{b^*} := \text{diag} \left( 1, \left( \frac{\Phi_1^n}{\Phi_2^n} \frac{w_1}{w_2} \right)^{-1/2}, \left( \frac{\Phi_1^n}{\Phi_2^n} \frac{w_1}{w_2} \right)^{1/2} \right),
\]

\[
E_{b^*} := -\sqrt{\pi} e^{i\pi/6} X \text{ diag} \left( 1, \left( \frac{w_1}{w_2} \right)^{-1/2}, \left( \frac{w_1}{w_2} \right)^{1/2} \right)
\times \begin{pmatrix}
2 e^{-i\pi/6} & 0 & 0 \\
0 & 1 & 1 \\
0 & -i & i
\end{pmatrix}
\text{ diag} \left( 1, \left( \frac{3n}{2} \varphi_{1,2} \right)^{-1/6}, \left( \frac{3n}{2} \varphi_{1,2} \right)^{1/6} \right),
\]
and
\[ \varphi_{1,2} := \log \frac{\Phi_1}{\Phi_2}. \]

To present \( V_{b^*} \) we use the notation
\[
\Psi_{b^*} := \begin{pmatrix}
1 & 0 & 0 \\
0 & \text{Ai} \left( \frac{3n}{2} \varphi_{1,2} \right)^{2/3} & \text{Ai} \left( \frac{2}{3n} \varphi_{1,2} \right)^{2/3} \\
0 & \text{Ai}' \left( \frac{3n}{2} \varphi_{1,2} \right)^{2/3} & \frac{2}{3n} \text{Ai}' \left( \frac{2}{3n} \varphi_{1,2} \right)^{2/3}
\end{pmatrix} \tilde{\sigma},
\]

\[
\tilde{\Psi}_{b^*} := \begin{pmatrix}
1 & 0 & 0 \\
0 & \text{Ai} \left( \frac{3n}{2} \varphi_{1,2} \right)^{2/3} & \frac{2}{3n} \text{Ai} \left( \frac{2}{3n} \varphi_{1,2} \right)^{2/3} \\
0 & \text{Ai}' \left( \frac{3n}{2} \varphi_{1,2} \right)^{2/3} & -\frac{2}{3n} \text{Ai}' \left( \frac{2}{3n} \varphi_{1,2} \right)^{2/3}
\end{pmatrix} \tilde{\sigma},
\]

where \( \text{Ai} \) is the usual Airy function and
\[
\tilde{\sigma} := \text{diag} \left( 1, e^{i\pi/6}, e^{-i\pi/6} \right), \quad \epsilon_3 = e^{2\pi i/3}.
\]

The matrix \( V_{b^*} \) is then given by

\[
V_{b^*} := \begin{cases}
\Psi_{b^*}, & \text{in } S(E_1^+, E_2^+), \\
\Psi_{b^*} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, & \text{in } S(E_2^+, E_2), \\
\tilde{\Psi}_{b^*} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, & \text{in } S(E_2, E_2^-), \\
\tilde{\tilde{\Psi}}_{b^*}, & \text{in } S(E_2^-, E_1^+),
\end{cases}
\]

where \( S(\gamma_1, \gamma_2) \) is the domain in \( O_{b^*} \) bounded by \( \gamma_1 \) and \( \gamma_2 \) (see Figure 4.7).

**Case V**

In this case (in comparison with the case IV) the branch point \( b^* \) moves on the 0-sheet of \( \mathfrak{R} \) (see Figure 2.11 and (2.65)). As a result the arc \( E_2 \) degenerates and after the normalization (4.3) of the initial Riemann-Hilbert problem (2.13) we have a new jump on \( \delta_1^* \) instead of a jump on \( E_2 \) (see Remark 2.32-4),

\[
Z_+ = Z_- D_1^*, \quad \text{on } \delta_1^* \subset \Omega_{1,0,2},
\]

where
\[
D_1^* := \begin{pmatrix} 1 & \left( \frac{\Phi_1}{\Phi_0} \right)^n w_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

which tends to the identity matrix as \( n \to \infty \). In this case we open a global lens around \( \Delta_{1,2} \) in the domain bounded by \( E_1^+ \) and then, in the usual way, we open local lenses.
around \( \Delta_2 \) and \( \tilde{\Delta}_1 \). The model Riemann-Hilbert problem (the parametrix away from the branch points) for the function \( X \) for this case is identical with the corresponding problem of case I. Since \( b^* \) is now on the 0-sheet of \( \mathcal{R} \), the local Riemann-Hilbert problem around \( b^* \) is modified. Instead of the jump (4.50) in (4.49) we have the jump

\[
\hat{J} = \begin{cases} 
D_1, & \text{on } \Delta_1^{(\pm)} \cap O_{b^*}, \\
W_1, & \text{on } \Delta_1^* \cap O_{b^*}, \\
D_{1*}, & \text{on } \delta_1^* \cap O_{b^*}, 
\end{cases}
\]

Thus the Airy function solution given in (4.51) for the problem (4.49), (4.52) should be modified slightly: the non-trivial \( 2 \times 2 \) block now is in the left-upper corner (instead of the right-lower corner in the case IV) and instead of \((\Phi_1, \Phi_2, w_1/w_2)\) we now have \((\Phi_0, \Phi_1, w_1)\); compare (4.50) and (4.52). For the solution of an identical local \( 3 \times 3 \) Riemann-Hilbert problem, see [50, 48, 49, 7, 51, 52].

Case III

The proof for this case is just a repetition of the proof for the case V, with one simplification. Since \( \Delta_{1,2} = \emptyset \) and \( \Delta_1 = \Delta_1^* \cup \delta_1 \) for this case, we do not need to open the global lens. Thus we are in the same situation as in the case I. The only difference is the solution of the local Riemann-Hilbert problem around \( b^* \), which is the same as for the case V.

5 Conclusion

In this conclusion we highlight the main results of this paper. They are

1. The classification of the sets \( A := \{a_1, b_1; a_2, b_2\} \) such that the limiting counting measures for the poles and interpolation points (1.4) of the Hermite-Padé approximants (1.2)

\[
\nu_n^{(j)} := \frac{Q_n^{(j)}}{P_n}, \quad R_n^{(j)} := P_n f_j - Q_n^{(j)},
\]

for functions (2.4)

\[
f_j \in A(a_j, \alpha_j; b_j, \beta_j; \Omega),
\]

are described by an algebraic function \( h \) of order 3 and genus 0, see (1.19), which give rise to measures \( \lambda, \mu_1, \) and \( \mu_2 \) such that

\[
\nu_n \xrightarrow{*} \lambda/2, \quad \nu_n^{(j)} \xrightarrow{*} \mu_j, \quad j = 1, 2.
\]

2. A universal vector-potential equilibrium problem (2.83) for these limiting measures.

3. Strong asymptotics for the corresponding Hermite-Padé approximants.

More precisely
In the definition of the **classes of sets** \( A \) we used a distinction in the formation of the system of curves

\[
\Gamma := \{ z \in \mathbb{C} : \text{Re} \int h_j(z) \, dz = \text{Re} \int h_k(z) \, dz, \ j \neq k, j, k = 0, 1, 2 \} \tag{5.2}
\]

with a certain normalization of the primitives. Since the genus of \( h \) is 0 the set \( \Gamma \) admits an algebraic parametrization. It is formed by the trajectories

\[
\Gamma = \bigcup_{\ell=1}^6 \gamma_\ell, \quad \gamma_\ell := z_\ell(\eta), \quad \ell = 1, \ldots, 6, \ \eta \in [-2, 2],
\]

given by the branches of an algebraic function \( z(\eta) \) of order six, when \( \eta \) runs from 2 to \(-2\). The peculiarities of the behavior of these trajectories define different classes of \( A \). Thus, starting from the input data \( A \) (i.e., the branch points \( a_1, b_1, a_2, b_2 \)) we have a finite number (because the genus of \( h \) is 0) of algebraic functions \( h \) satisfying

\[
h^3 - 3 \frac{P_2(z)}{\Pi_4(z)} h + 2 \frac{P_1(z)}{\Pi_4(z)} = 0, \tag{5.3}
\]

with \( \Pi_4(z) = (z - a_1)(z - b_1)(z - a_2)(z - b_2) \). From the coefficients of the equation \( (5.3) \) we get explicit expressions of the coefficients of the equation for the function \( z(\eta) \). Then, observing the behavior of the branches \( z(\eta) \) when \( \eta \) runs from 2 to \(-2\), we can conclude to which class \( A \) belongs. Depending on the class we define (globally in \( \mathbb{C} \)) the branches of the algebraic function \( h := \{ h_0, h_1, h_2 \} \) and the algebraic function

\[
\Phi := \exp \left( \int h(z) \, dz \right) = \{ \Phi_0, \Phi_1, \Phi_2 \},
\]

i.e., we define the sheet structure of the Riemann surface \( \mathfrak{R} \) of the function \( h \). Finally, the jumps of the function \( h \) on certain parts (depending on the geometrical case) of the contour \( \Gamma \) given in (5.2), give the densities of the limiting measures \( \lambda \) and \( \mu_1, \mu_2 \) in (5.1). In particular

\[
d\lambda(\xi) = \frac{1}{2\pi i} (h_{0+}(\xi) - h_{0-}(\xi)) \, d\xi, \quad \xi \in \Delta_0 \subset \Gamma,
\]

where \( \Delta_0 \) is a union of cuts which form the boundary of the domain of analyticity of the branch \( h_0 \), i.e., \( \Delta_0 \) are the cuts of the 0-sheet \( \mathfrak{R}_0 \) of the Riemann surface \( \mathfrak{R} \).

The **vector-potential equilibrium problem** (Theorem 2.31) does not depend on the geometrical class of \( A \) (universality). The equilibrium relations of this problem are considered on cuts \( \Delta_1 \) and \( \Delta_2 \) joining the pairs \((a_j, b_j)\), \( j = 1, 2 \). In addition, if \( \Delta_1 \cap \Delta_2 = \Delta_{1,2} \neq \emptyset \), an extra equilibrium relation is imposed on a curve \( E \) containing \( \Delta_{1,2} \), see (2.83). This equilibrium problem reduces to the known vector-potential problem for an Angelesco system when \( \Delta_{1,2} = \emptyset \). The cut \( \Delta_j \) makes the function \( f_j \) holomorphic. Thus the sets \( \Delta_1, \Delta_2, \Delta_{1,2}, E \) are natural input data for the vector-potential problem and we expect that the limiting measures (5.1) of the Hermite-Padé approximants hold for a wider class of functions. We also recall that
we impose in this paper an extra analyticity condition on \((f_1, f_2)\) when \(\Delta_{1,2} \neq \emptyset\): we require that the ratio of the jumps of \(f_1\) and \(f_2\) on \(\Delta_{1,2}\)
\[
u(\xi) := \frac{f_{1+} - f_{1-}(\xi)}{f_{2+} - f_{2-}(\xi)}, \quad \xi \in \Delta_{1,2}\]
(5.4)
has a holomorphic (meromorphic) continuation from \(\Delta_{1,2}\) to the domain \(\overline{\mathbb{C}}\)
\[u \in H(\overline{\mathbb{C}}), \quad \partial G = E. \quad (5.5)\]
The analyticity condition (5.4) gives a link of our vector-potential problem (2.83) with the known equilibrium problem for a Nikishin system.

- For the derivation of the strong asymptotics for the multiple orthogonal polynomials \(P_n\) and for their functions of the second kind \(R_n^{(j)}\) \((j = 1, 2)\), we use a \(3 \times 3\) matrix-valued Riemann-Hilbert problem. The increase of the order of the matrix functions (in comparison with \(2 \times 2\) matrix-valued Riemann-Hilbert problems for the usual orthogonal polynomials) brings new features into the standard Riemann-Hilbert technology. One of these features is a new decomposition of the matrix jump on \(\Delta_{1,2} \neq \emptyset\) which implies the opening of a global lens containing the domain \(G\), see (5.5). This procedure introduces a new effect (in comparison with usual orthogonal polynomials) of oscillatory asymptotics on some curves \(\subset E \in \mathbb{C} \setminus (\Delta_1 \cup \Delta_2)\) for the functions of the second kind \(R_n^{(j)}\) and, as a result, there is an accumulation of zeros of \(R_n^{(j)}\) on \(E\).

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