A new formula for conserved charges of Lovelock gravity in AdS spacetimes and its generalization

Jun-Jin Peng\textsuperscript{1,2,*}  Hui-Fa Liu\textsuperscript{1†}

\textsuperscript{1}School of Physics and Electronic Science, Guizhou Normal University, Guiyang, Guizhou 550001, People’s Republic of China;
\textsuperscript{2}Guizhou Provincial Key Laboratory of Radio Astronomy and Data Processing, Guizhou Normal University, Guiyang, Guizhou 550001, People’s Republic of China

Abstract

Within the framework of the Lovelock gravity theory, we propose a new rank-four divergenceless tensor consisting of the Riemann curvature tensor and inheriting its algebraic symmetry characters. Such a tensor can be adopted to define conserved charges of the Lovelock gravity theory in asymptotically anti-de Sitter (AdS) spacetimes. Besides, inspired with the case of the Lovelock gravity, we put forward another general fourth-rank tensor in the context of an arbitrary diffeomorphism invariant theory of gravity described by the Lagrangian constructed out of the curvature tensor. On basis of the newly-constructed tensor, we further suggest a Komar-like formula for the conserved charges of this generic gravity theory.

\textsuperscript{*}corresponding author: pengjjph@163.com
\textsuperscript{†}hfalin@163.com
1 Introduction

As is well-known, Lovelock (or referred to as Lanczos-Lovelock) gravity \cite{1} is the most natural higher-derivative extension of general relativity to higher dimensions of spacetimes. Specifically, apart from the cosmological constant, as well as the Einstein-Hilbert term, the Lagrangians describing the Lovelock gravity theory involve quadratic and higher-order polynomials of the curvature tensor with the degree relying on the dimensions of spacetimes. In spite of this, the equations of motion comprise merely up to second-order derivatives of the metric tensor. Such behavior of the field equation enables the Lovelock gravity to overcome the so-called Ostrogradsky instability, which is a linear instability existing in the Hamiltonian associated with a non-degenerate Lagrangian containing time derivative terms higher than the first order \cite{39}.

In comparison with general relativity, due to the existence of the higher power curvature terms, the Lovelock gravity exhibits a series of peculiar features. See for instance thermodynamical aspects in references \cite{2,3,4,5} and references therein. See for instance geometrical aspects in references \cite{9,11,12} as well. As a consequence, the Lovelock gravity has attracted an increasingly widespread attention since 1990s. Particularly, some solutions have been found in works \cite{6,7,8,9,10,11,12,13}. In order to interpret these solutions, such as the first law of thermodynamics and other thermodynamic properties, an important question desired to address is to find proper approaches to define the conserved charges of the Lovelock gravity. Till now, some research has been devoted to this question from different perspectives \cite{13,14,15,16,17,18,19,20,21,22,23,24} and several methods have been proposed. Despite all this, within the present paper, motivated by the Abbott-Deser-Tekin (ADT) formalism \cite{27,28,29} and the covariant phase space approach \cite{36,37,38} put forward by Wald and his collaborators, we plan to provide another effective formula to calculate the conserved charges of the Lovelock gravities in asymptotically anti-de Sitter (AdS) spacetime. Apart from this, we attempt to extend it to generic diffeomorphism invariant theories of gravity built out of the curvature tensor, with the Lagrangians given by Eq. \eqref{4.1}, that is, \( \mathcal{L}_{Riem} = \sqrt{-g} L_{Riem}(g^{\mu\nu}, R_{\alpha\beta\rho\sigma}) \).

Quite recently, within the framework of the Einstein gravity theory described by the Einstein-Hilbert-\(\Lambda\) Lagrangian, a fourth-rank conserved tensor that inherits the symmetries of the Riemann curvature tensor was suggested in works \cite{25,26}. By following the ADT approach \cite{27,28,29}, then it was adopted to define the conserved charges of the Einstein
gravity in asymptotically AdS spacetime. In fact, if this conserved tensor is reformulated into the form involving the generalized Kronecker delta, it will be demonstrated below that it can be modified straightforwardly as a generic divergenceless tensor in the context of the Lovelock gravity, which preserves the symmetries of the Riemann curvature tensor as well. In terms of the perturbation of such a tensor about the AdS background, we are going to propose another formula for the conserved charges of the Lovelock gravity theory in asymptotically AdS spacetimes.

Subsequently, inspired with the specific example on the Lovelock gravity, we shall go further and take into consideration of the definition for the conserved charges of the diffeomorphism invariant theory of gravity, described by the Lagrangian $\mathcal{L}_{\text{Riem}}$ in Eq. (4.1). Our goal is to find a proper potential being of the similar form as the Noether potential to define the conserved charges of such a gravity theory in asymptotically AdS spacetimes. The resulted definition is required to coincide with that via the ADT formalism or the covariant phase space method. To achieve this, like in the case for the Lovelock gravity, we are going to construct a general rank-four tensor inheriting the algebraic symmetry properties of the Riemann curvature tensor. With help of the newly-constructed tensor, then a 2-form Komar-like potential associated with a Killing vector can be defined. The perturbation for the potential on the AdS background further gives rise to a Komar-like formula for the conserved charges of the general covariant gravity theory with AdS asymptotics. By contrast, it will be showed that our formula for conserved charges is successful in matching the one defined through the ADT method.

Throughout this paper, we will adopt the following notations and conventions. We work in the case where the cosmological constant $\Lambda$ is negative ($\Lambda < 0$). Nonetheless, we expect our results can be generalized to spacetimes with a de Sitter ($\Lambda > 0$) asymptotic. We make use of units in which both the gravitational constant $G$ and the speed of light in vacuum $c$ are set equal to one, namely, $G = c = 1$. The positive integer $D$ denotes spacetime dimensions. The generalized Kronecker delta $\delta^{\mu_1 \ldots \mu_k}_{\nu_1 \ldots \nu_k}$ is given by $\delta^{\mu_1 \ldots \mu_k}_{\nu_1 \ldots \nu_k} = k! \delta^{[\mu_1 \ldots \mu_k]}_{\nu_1 \ldots \nu_k}$. For instance, when $k = 2$, $\delta^{\mu \nu}_{\rho \sigma} = 2\delta^{(\mu}_{\rho} \delta^{\nu)}_{\sigma} = \delta^{\mu}_{\rho} \delta^{\nu}_{\sigma} - \delta^{\nu}_{\rho} \delta^{\mu}_{\sigma}$. The objects with a bar such as the covariant derivative $\bar{\nabla}$ or the metric tensor $\bar{g}_{\mu \nu}$ represent the ones defined with respect to the AdS background metric $\bar{g}_{\mu \nu}$. For example, $\bar{R}_{\mu \nu \rho \sigma} = R_{\mu \nu \rho \sigma}(g_{\alpha \beta} \rightarrow \bar{g}_{\alpha \beta})$, and $P^{\mu \nu \rho \sigma}_{(0)} = 2\bar{g}^{[\mu \nu} \bar{g}^{\rho \sigma]}$ is the counterpart of the rank-four tensor $P^{\mu \nu \rho \sigma}_{(0)} = 2g^{[\mu \nu} g^{\rho \sigma]}$ on the AdS background.

The layout of this paper goes as follows. In section 2 a new rank-four conserved tensor will be constructed and utilized to define the conserved charges of the Lovelock gravity
theories in the AdS spacetime. Section 3 is devoted to the comparison between the conserved charges built from that newly-constructed tensor and the ones via the ADT formalism. In section 4, motivated by the case of the Lovelock gravity, we are going to put forward another two generic tensors to give a definition of the conserved charges for general covariant gravity theories built out of the curvature tensor.

2 The divergenceless tensor $P^{\mu\nu\rho\sigma}_{(i)}$ and the conserved charges of the Lovelock gravities

In this section, we are going to propose a new rank-4 divergence-free tensor $P^{\mu\nu\rho\sigma}_{(i)}$, which depends on the Riemann curvature tensor and preserves its symmetries. Then such a tensor will be applied to define the conserved charges of various solutions with the AdS asymptotic in the framework of Lovelock gravities.

Let us start with the $D$-dimensional Einstein gravity described by the well-known Einstein-Hilbert-Λ Lagrangian

$$
L_{EH} = \sqrt{-g} (R - 2\Lambda) = \frac{1}{2} \sqrt{-g} (R_{\mu\nu}^\rho\sigma \delta_{\rho\sigma}^{\mu\nu} - 4\Lambda) .
$$

(2.1)

Accordingly, the action for the Einstein gravity is of the form $S_{EH} = (16\pi)^{-1} \int L_{EH} dx^D$. The derivative of the scalar $L_{EH} = L_{EH}/\sqrt{-g}$ with respect to the Riemann tensor gives rise to

$$
P^{\mu\nu}_{(0)\rho\sigma} = 2 \frac{\partial L_{EH}}{\partial R_{\mu\nu}^{\rho\sigma}} = \delta^{\mu\nu}_{\rho\sigma},
$$

or

$$
P^{\mu\nu}_{(0)\rho\sigma} = 2g^{\delta\mu}\delta^{\nu\rho} g^{\sigma\sigma}. 
$$

(2.2)

It is easy to verify that the tensor $P^{\mu\nu\rho\sigma}_{(0)}$ possesses the same index symmetries as the Riemann tensor and it is divergence-free, namely, $\nabla_{\mu} P^{\mu\nu\rho\sigma}_{(0)} = 0$. In works [23, 26], by making use of $P^{\mu\nu}_{(0)\rho\sigma}$, together with a divergence-free tensor $P^{\mu\nu}_{(1)\rho\sigma}$, given by

$$
P^{\mu\nu}_{(1)\rho\sigma} = R^{\mu\nu}_{\rho\sigma} - 4R^{[\mu}_{\rho} g^{\nu]}_{\sigma]} + \frac{1}{2} R_{\rho\sigma}^{\mu\nu} = \frac{1}{4} R^{\alpha\beta,\gamma\delta}_{\rho\sigma} g^{\lambda\mu}_{\alpha\beta} g^{\lambda\sigma}_{\rho\sigma},
$$

(2.3)

the authors proposed a new conserved tensor

$$
P^{\mu\nu\rho\sigma} = P^{\mu\nu\rho\sigma}_{(1)} - \frac{(D - 2)(D - 3)}{2} P^{\mu\nu\rho\sigma}_{(0)},
$$

(2.4)
to define the conserved charges of the Einstein gravity theory in asymptotically AdS spacetimes by following the ordinary ADT approach \[27, 28, 29\]. In Eq. (2.4) and what follows, the constant parameter $\ell$ is related to the cosmological constant $\Lambda$ through the relation,

$$\ell = \frac{2\Lambda}{(D-1)(D-2)}. \tag{2.5}$$

Actually, motivated by the form of $P_{\mu\nu(1)}^{\rho\sigma}$ expressed by the last equality in Eq. (2.4), the conserved rank-4 tensor $P_{\mu\nu\rho\sigma}^{(i)}$ can be naturally generalized to the Lovelock gravity theory. This will be explicitly demonstrated in the remainder of this section.

Now, we consider the Lovelock-type Lagrangians $L_{(i)}^{(i)}$ $(0 \leq i \leq [D/2 - 1])$ in a $D$-dimensional spacetime with the metric $g_{\mu\nu}$, taking the form [1]

$$L_{(i)}^{(i)} = \sqrt{-g} \left( L_{(i)}^{(i)} - 2\tilde{\Lambda} \right),$$

$$L_{(i)}^{(i)} = \frac{1}{4^i(i + 1)} \delta^{\gamma_1\lambda_1 \cdots \gamma_{i+1}\lambda_{i+1}}_{\alpha_1\beta_1 \cdots \alpha_i\beta_i} \prod_{r=1}^{i+1} R_{\alpha_r\beta_r}^{\gamma_r\lambda_r}, \tag{2.6}$$

where the constant parameter $\tilde{\Lambda}$ depends on the cosmological constant $\Lambda$, since it is assumed that the Lagrangian $L_{(i)}^{(i)}$ allows the existence of (asymptotically) AdS solutions. Particularly, if $i = 0$ and $\tilde{\Lambda} = 2\Lambda$, $L_{(i)}^{(0)} = 2L_{EH}$. Through the derivative of the scalar $L_{(i)}^{(i)}$ with respect to the Riemann tensor, we are able to define a fourth-rank tensor $P_{\mu\nu(1)}^{\rho\sigma}$ as [3]

$$P_{\mu\nu(1)}^{\rho\sigma} = \frac{\partial L_{(i)}^{(i)}}{\partial R_{\mu\nu(1)}^{\sigma\rho}} = \frac{1}{4^i} R_{\alpha_1\beta_1}^{\gamma_1\lambda_1} \cdots R_{\alpha_i\beta_i}^{\gamma_i\lambda_i} \delta^{\gamma_1\lambda_1 \cdots \gamma_{i+1}\lambda_{i+1}}_{\alpha_1\beta_1 \cdots \alpha_i\beta_i} \delta^{\gamma_1\lambda_1 \cdots \gamma_{i+1}\lambda_{i+1}}_{\alpha_1\beta_1 \cdots \alpha_i\beta_i} \cdot \tag{2.7}$$

One can check that $P_{\mu\nu(1)}^{\rho\sigma}$ inherits the following symmetries of the Riemann curvature tensor:

$$P_{\mu\nu(1)}^{\rho\sigma} = -P_{\mu\nu(1)}^{\rho\sigma}, \quad P_{\mu\nu(1)}^{\rho\sigma} = -P_{\mu\nu(1)}^{\sigma\rho}, \quad P_{\mu\nu(1)}^{\rho\sigma} = P_{\mu\nu(1)}^{\sigma\mu}, \quad P_{\mu\nu(1)}^{\rho\sigma} = 0. \tag{2.8}$$

Apart from this, it satisfies the divergence-free equation $\nabla_{\mu} P_{\mu\nu(1)}^{\rho\sigma} = 0 = \nabla_{(i)} P_{\mu\nu(1)}^{\rho\sigma}$, as well as $\nabla_{\mu} P_{\mu\nu(1)}^{\rho\sigma} = 0 = \nabla_{(i)} P_{\mu\nu(1)}^{\rho\sigma}$, arising from the fact that

$$\nabla_{\mu} P_{\mu\nu(1)}^{\rho\sigma} = \frac{i}{4^i} \nabla_{\mu} R_{\alpha_1\beta_1}^{\gamma_1\lambda_1} \cdots R_{\alpha_i\beta_i}^{\gamma_i\lambda_i} \delta^{\gamma_1\lambda_1 \cdots \gamma_{i+1}\lambda_{i+1}}_{\alpha_1\beta_1 \cdots \alpha_i\beta_i} \delta^{\gamma_1\lambda_1 \cdots \gamma_{i+1}\lambda_{i+1}}_{\alpha_1\beta_1 \cdots \alpha_i\beta_i} = 0.$$
To obtain the second equality, we have made use of the Bianchi identity $\nabla [\lambda R_{\mu\nu\rho\sigma}] = 0$. For more information on the tensor $P_{\mu\nu\rho\sigma}^{(i)}$, for instance, see the Refs. [33, 34]. By letting $P_{\mu\nu\rho\sigma}^{(i)}$ subtract a term proportional to the tensor $P_{\mu\nu\rho\sigma}^{(0)}$ rather than to its counterpart $\bar{P}_{\mu\nu\rho\sigma}^{(0)}$ on the AdS spacetime, where $\bar{P}_{\mu\nu\rho\sigma}^{(0)} = 2\bar{g}^{[\mu}[\bar{g}^{\rho\nu]}$ with the help of the metric tensor $\bar{g}_{\mu\nu}$ for the AdS spacetime, we put forward a divergenceless rank-four conserved tensor $P_{\mu\nu\rho\sigma}^{(i)}$, being of the form

$$P_{\mu\nu\rho\sigma}^{(i)} = P_{\mu\nu\rho\sigma}^{(0)} - \lambda_{(i)} P_{\mu\nu\rho\sigma}^{(0)}, \quad \lambda_{(i)} = \left( \frac{\ell}{2} \right)^i \frac{(D - 2)!}{(D - 2i - 2)!}. \quad (2.9)$$

For the $D$-dimensional AdS spacetime endowed with the Riemann curvature tensor

$$\bar{R}_{\mu\nu\rho\sigma} = \delta_{\mu\nu} \bar{g}_{\rho\sigma}, \quad (2.10)$$

it is observed that $\bar{P}_{\mu\nu\rho\sigma}^{(i)} = P_{\mu\nu\rho\sigma}^{(i)} (g_{\alpha\beta} \rightarrow \bar{g}_{\alpha\beta})$, giving rise to that $\bar{P}_{\mu\nu\rho\sigma}^{(i)} = P_{\mu\nu\rho\sigma}^{(i)}$, identically vanishes. This plays a key role in the construction of the formula for the conserved charges. Particularly, when $i = 1$, $P_{\mu\nu\rho\sigma}^{(i)}$ becomes the tensor $P_{\mu\nu\rho\sigma}$ given by Eq. (2.4). To understand more on the tensor $P_{\mu\nu\rho\sigma}^{(i)}$, some remarks will be made in the next section.

Next, assumed that $\xi^\mu$ is a Killing vector of the spacetime with the metric $g_{\mu\nu}$, we follow the covariant phase space method [36, 37, 38] to propose a 2-form superpotential $K_{\mu\nu}^{(i)}$ as

$$K_{\mu\nu}^{(i)} = P_{\mu\nu\rho\sigma}^{(i)} \nabla_{\rho} \xi_{\sigma} = K_{\mu\nu}^{(1)} - 2 \left( \frac{\ell}{2} \right)^i \frac{(D - 2)!}{(D - 2i - 2)!} \nabla_{[\mu} \xi_{\nu]}, \quad (2.11)$$

where the skew-symmetric tensor $K_{\mu\nu}^{(1)}$ is the Noether potential. Perturbing $K_{\mu\nu}^{(i)}$ on the AdS spacetime, we arrive at

$$\delta K_{\mu\nu}^{(i+1)} = \left( \delta P_{\mu\nu\rho\sigma}^{(i+1)} \right) \nabla_{\rho} \bar{\xi}_{\sigma} + \bar{P}_{\rho(\nu}^{(i+1)} \delta (\nabla_{\rho} \xi_{\sigma})$$

$$= \left( \delta P_{\mu\nu\rho\sigma}^{(i+1)} \right) \nabla_{\rho} \bar{\xi}_{\sigma}$$

$$= (i + 1) \ell^2 \frac{(D - 4)!}{(D - 2i - 4)!} \delta K_{\mu\nu}^{(1)}, \quad (2.12)$$

where $\bar{\xi}_{\sigma}$ is demanded to be the Killing vector of the AdS background $\bar{g}_{\mu\nu}$. To obtain the second equality in Eq. (2.12), we have used $\bar{P}_{\rho(\nu}^{(i+1)} \delta (\nabla_{\rho} \xi_{\sigma}) = 0$. When $i = 1$, the quantity $\delta K_{\mu\nu}^{(1)}$ can be written as [25, 26]

$$\delta K_{\mu\nu}^{(1)} = \left( \delta P_{\mu\nu\rho\sigma}^{(1)} \right) \nabla_{\rho} \bar{\xi}_{\sigma}$$

$$= -2\ell (D - 3) \bar{Q}_{\mu\nu}^{(1)} + \nabla_{\gamma} \bar{U}_{(1)}^{\gamma\mu\nu}, \quad (2.13)$$

6
with the 2-form $\bar{Q}_{\mu\nu}^{EH}$ given by [27, 28, 29]

$$\bar{Q}_{\mu\nu}^{EH} = \frac{1}{2} \delta_{(0)\rho\sigma}^{\mu\nu} \left( h_\lambda^{[\rho} \nabla^{\sigma] \xi_\lambda + \xi_\lambda \nabla^{[\rho} h_\lambda^{\sigma]} \right) + \frac{1}{4} h_{(0)\rho\sigma}^{\mu\nu} \bar{\nabla}_{\rho} \bar{\nabla}_{\sigma} \xi_{\lambda} h_\lambda^{\mu},$$  \hspace{1cm} (2.14)

as well as the 3-form $\bar{U}^{(1)\mu\nu}$ taking the form

$$\bar{U}^{(1)\mu\nu} = \frac{1}{2} \delta^{\alpha\beta} \lambda_{\rho\sigma} \left( \bar{\nabla}_{\alpha} h_\beta^{\lambda} \right) \bar{\nabla}_{\rho} \bar{\nabla}_{\sigma}. \hspace{1cm} (2.15)$$

In Eqs. (2.14) and (2.15), $\delta_{(0)\rho\sigma}^{\mu\nu}$ and the perturbation $h_{\mu\nu} = \delta g_{\mu\nu}$ of the metric $g_{\mu\nu}$ is defined through

$$h_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}. \hspace{1cm} (2.16)$$

We shall see that the 2-form $\bar{Q}_{\mu\nu}^{EH}$ is just the ADT potential of the Einstein gravity theory described by the Lagrangian (2.1) in the following section.

Finally, supposed that there exists a subregion given by a $(D-1)$-dimensional hypersurface $\Sigma$ with the boundary $\partial \Sigma$, we are able to make use of the 2-form $\delta K_{(i+1)}^{\mu\nu}$ to define the conserved charge $Q_{(i)}$ associated with the Lovelock gravity with Lagrangian (2.6) in this subregion. The surface charge $Q_{(i)}$ takes the following form

$$Q_{(i)} = -\frac{1}{8\pi(i+1)(D-2i-3)\ell} \int_{\partial \Sigma} \delta K_{(i+1)}^{\mu\nu} d\Sigma_{\mu\nu}$$

$$= -\frac{\ell^{i-1}}{2^{i+3}\pi (D-2i-3)!} \int_{\partial \Sigma} \delta K_{(1)}^{\mu\nu} d\Sigma_{\mu\nu}$$

$$= -\frac{\ell^{i}}{2^{i-1}(D-2i-3)!} Q_{EH}, \hspace{1cm} (2.17)$$

where the conserved charge $Q_{EH}$ can be regarded as the ADT one for the Einstein gravity theory [19, 28, 30, 31], presented by

$$Q_{EH} = \frac{1}{8\pi} \int_{\partial \Sigma} \bar{Q}_{\mu\nu}^{EH} d\Sigma_{\mu\nu}. \hspace{1cm} (2.18)$$

What is more, let us take into consideration of the conserved charges for the full Lovelock Lagrangian $L_L$, having the form

$$L_L = \sqrt{-g} \sum_{i=0}^{[D/2-1]} c_i L_L^{(i)} - 2\Lambda \sqrt{-g}, \hspace{1cm} (2.19)$$

\(^1\)To match the ordinary AdS background metric, it is reasonable to set up the coordinate system $\{x^{\mu}\} = \{t, r, x^i\} (i = 1, 2, \cdots, D - 2)$, where $r$ is the radial coordinate. Thus, as usual, $\partial \Sigma$ denotes the surface with $t = \text{const}$ and $r = \infty$ when the conserved charge $Q_{(i)}$ represents the mass or the angular momentum. As an example, see the explicit calculations on the conserved charges of black holes in the works [31, 35].
where \( c_i \)'s are arbitrary coupling constants in principle. In terms of the formula (2.17), the definition \( Q_L \) for the conserved charges associated with the Lagrangian (2.19) in asymptotically AdS spacetime can be presented by

\[
Q_L = \sum_{i=0}^{[D/2-1]} c_i Q_{(i)} = Q_{EH} \sum_{i=0}^{[D/2-1]} \frac{c_i \ell^i}{2^{i-1} (D-2i-3)!},
\]

(2.20)

which coincides with the corresponding result recently obtained through the so-called field-theoretical approach in [18]. As a significant application, the formula (2.17) can be utilized to compute the conserved charges of solutions in the Lovelock gravity theories, for instance, the ones in the references [6, 7, 8, 9, 10, 11, 12, 13]. Besides, inspired with the works [40, 41, 42], the formula (2.17) may be modified properly to derive the entropy and the first law of the black hole solutions, particularly in the framework of the Lovelock gravities formulated in terms of orthonormal coframes.

What is more, for the Lovelock gravity theory described by the Lagrangian (2.19), apart from the ordinary AdS spacetime, this theory can admit maximally-symmetric AdS spacetimes with the Riemann curvature tensor \( R_{\rho\sigma}{}^{\mu\nu} = -\ell_{eff}^{-2} \delta_{\rho\sigma}^{\mu\nu} \), where the effective AdS radius \( \ell_{eff} \), solved from the field equation, depends on the coupling constants \( c_i \)'s [16, 17]. Particularly, under some conditions with respect to the coupling constants, there exist degenerate AdS spacetimes on which the linear perturbation to the expression for the equation of motion disappears. As a consequence, when such spacetimes are chosen as the reference background, the ADT approach yields vanishing charges. This is attributed to the fact that the conserved ADT current, proportional to the linear perturbation of the expression for the equation of motion, vanishes identically. The formula (2.20) encounters the same outcome since we shall see below that the potentials involved in this formula coincide with the ones via the ADT approach. Regarding to this, we conclude that the ADT approach, as well as the method in this paper, breaks down to the Lovelock theories with a vacuum degeneracy. By contrast, in the works [16, 17], a strategy has been worked out to derive a conserved charge expression for the asymptotically AdS solutions having degenerate vacua in the Lovelock gravities. The resulted charges are dependent of the degeneracy of the theories.
3 Comparison with the ADT formalism

In the present section, the formula (2.17) for the conserved charges will be compared with the one via the ADT formalism [27, 28, 29].

In terms of the works [19, 30, 31], the perturbation on an arbitrary background gives rise to the (off-shell) ADT potential $Q_{\mu\nu}^{(i)}$ associated to the Lovelock Lagrangian (2.6), being of the form

$$Q_{\mu\nu}^{(i)} = \frac{1}{\sqrt{-g}} \delta \left( \sqrt{-g} K_{\mu\nu}^{(i)} \right) - \xi^{[\mu} \Theta_{\nu]}^{(i)}$$

$$= \delta (P_{\mu\nu}^{\rho\sigma} \nabla_\rho \xi_\sigma) + \frac{1}{2} g^{\alpha\beta} (\delta g_{\alpha\beta}) P_{\mu\nu}^{\rho\sigma} \nabla_\rho \xi_\sigma$$

$$- 2 \xi^{[\mu} P^{\nu]}_{\rho\sigma} \nabla_\rho \delta g_{\sigma\lambda}. \quad (3.1)$$

Here the surface term $\Theta_{\mu}^{(i)} = 2 P_{\mu\nu}^{\rho\sigma} \nabla_\sigma \delta g_{\rho\nu}$, and the variation of the Noether potential $K_{\mu\nu}^{(i)} = P_{\mu\nu}^{\rho\sigma} \nabla_\rho \xi_\sigma$ can be further expressed as

$$\delta K_{\mu\nu}^{(i)} = \frac{2i}{4} \varepsilon^\omega \left( \nabla^\alpha h_\alpha^\beta \right) \delta^{\gamma_1 \lambda \lambda_1 \gamma_1 \gamma_1 \mu \mu \nu \nu} R_{\gamma_1 \lambda r}^{\alpha_1 \beta_1 \gamma_1} R_{\rho_1 \lambda r}^{\alpha_1 \beta_1 \gamma_1} = \frac{i}{4} \delta R_{\gamma_1 \lambda r}^{\alpha_1 \beta_1 \gamma_1} R_{\rho_1 \lambda r}^{\alpha_1 \beta_1 \gamma_1}$$

$$+ P_{\mu\nu}^{\rho\sigma} \delta (\nabla_\rho \xi_\sigma) + \nabla_\gamma U_{\gamma\mu\nu}^{(i)} \quad (3.2)$$

with the 3-form $U_{\gamma\mu\nu}^{(i)}$ given by

$$U_{\gamma\mu\nu}^{(i)} = \frac{2i}{4} \left( \nabla^\alpha h_\alpha^\beta \right) \left( \nabla^\rho \xi_\rho \right) \delta^{\gamma_1 \lambda \lambda_1 \gamma_1 \gamma_1 \mu \mu \nu \nu} R_{\gamma_1 \lambda r}^{\alpha_1 \beta_1 \gamma_1} R_{\rho_1 \lambda r}^{\alpha_1 \beta_1 \gamma_1} = \frac{i}{4} \delta R_{\gamma_1 \lambda r}^{\alpha_1 \beta_1 \gamma_1} R_{\rho_1 \lambda r}^{\alpha_1 \beta_1 \gamma_1} \quad (3.3)$$

In Eqs. (3.2) and (3.3), $h_\rho^\mu$ stands for that $h_\rho^\mu = g^{\mu\rho} \delta g_{\rho\nu}$. It is worth noting that the ADT potential $Q_{\gamma\mu\nu}^{(i)}$ can be also obtained via the well-known covariant phase space approach [36, 37, 38] put forward by Wald and his collaborators (for example, see the works [2, 3, 14, 15]).

In particular, taking into account the $i = 1$ case of Eq. (3.2), one obtains

$$\delta (P_{\mu\nu}^{\rho\sigma} \nabla_\rho \xi_\sigma) = P_{\mu\nu}^{\rho\sigma} \delta (\nabla_\rho \xi_\sigma) - \frac{1}{4} \delta R_{\gamma_1 \lambda r}^{\alpha_1 \beta_1 \gamma_1} \delta^{\gamma_1 \lambda \lambda_1 \gamma_1 \gamma_1 \mu \mu \nu \nu} R_{\gamma_1 \lambda r}^{\alpha_1 \beta_1 \gamma_1} R_{\rho_1 \lambda r}^{\alpha_1 \beta_1 \gamma_1}$$

$$+ \frac{1}{2} R_{\rho_1 \lambda r}^{\alpha_1 \beta_1 \gamma_1} \delta^{\gamma_1 \lambda \lambda_1 \gamma_1 \gamma_1 \mu \mu \nu \nu} \nabla_\rho \delta g_{\sigma\lambda} + \frac{1}{2} \nabla_\gamma \left[ \delta^{\lambda \mu \nu \alpha \beta \rho} R_{\alpha \beta \rho}^{\lambda \mu \nu} \nabla_\rho h_\alpha^\beta \right]. \quad (3.4)$$

This equation naturally yields the 2-form $\delta K_{\mu\nu}^{(i)}$ in Eq. (2.13) when the reference background is the AdS metric $g_{\mu\nu}$. 


Let us pay attention to the ADT potential \( Q_{(i)}^{\mu
u} \) on the fixed AdS reference background with the Riemann curvature tensor \( g_{\mu
u\rho\sigma} \). In such a case, \( Q_{(i)}^{\mu
u} \) turns into

\[
Q_{(i)}^{\mu\nu} = \left( \frac{\ell}{2} \right)^{i-1} \frac{(D-4)!}{(D-2i-2)!} \left[ (D-3)(D-2i-2)\ell \bar{Q}_{EH}^{\mu\nu} + i\nabla \gamma^{\mu\nu}_{(1)} \right].
\]

(3.5)

The above equation leads to that \( \bar{Q}_{EH}^{\mu\nu} = \bar{Q}_{(0)}^{\mu\nu}/2 \), verifying the statement in the previous section that \( \bar{Q}_{EH}^{\mu\nu} \) is the ADT potential for the Einstein gravity theory. According to Eqs. (2.12) and (3.5), the relation between \( \bar{Q}_{(i)}^{\mu\nu} \) and \( \delta K^{\mu\nu}_{(i+1)} \) is read off as

\[
\bar{Q}_{(i)}^{\mu\nu} = \frac{\ell^{i-1}}{2^i(D-3)!} \frac{(D-2)!}{(D-2i-2)!} \bar{U}^{\gamma\mu\nu}_{(1)} - \frac{\delta K^{\mu\nu}_{(i+1)}}{(i+1)(D-2i-3)\ell}.
\]

(3.6)

This demonstrates that the ADT potential \( \bar{Q}_{(i)}^{\mu\nu} \) is equivalent with the superpotential \( \delta K^{\mu\nu}_{(i+1)} \). Therefore, in the framework of the Lovelock gravity, one can conclude that the ADT formula is consistent with the formula \( (2.17) \). Furthermore, the equivalence between \( \bar{Q}_{(i)}^{\mu\nu} \) and \( \delta K^{\mu\nu}_{(i+1)} \) demonstrates that the subtracted term related to \( P^{\mu\nu\rho\sigma}_{(0)} \) in Eq. (2.9) compensates the contribution from the surface term \( \Theta^{(i)} \).

Inspired with the relationship between \( \bar{Q}_{(i)}^{\mu\nu} \) and \( \delta K^{\mu\nu}_{(i+1)} \), we make some remarks on the divergence-free tensor \( P^{\mu\nu\rho\sigma}_{(i)} \), given by Eq. (2.20). Firstly, we emphasize that the subtracted term \( 2\lambda(\theta^{(i)}g^{(i)}g^{(i)})\) in \( P^{\mu\nu\rho\sigma}_{(i)} \) can not be naively replaced with the one \( P^{\mu\nu\rho\sigma}_{(0)} = 2\lambda(\theta^{(i)}g^{(i)}g^{(i)}) \) associated to the AdS background metric, that is to say, the tensor \( P^{\mu\nu\rho\sigma}_{(i)} \) in \( P^{\mu\nu\rho\sigma}_{(i)} \) can not be changed as its counterpart \( P^{\mu\nu\rho\sigma}_{(0)} \) upon the reference background. Otherwise, \( \delta ( \bar{P}^{\mu\nu\rho\sigma}_{(i)} \nabla_{\rho} \bar{\xi}_{\sigma} ) = \delta ( P^{\mu\nu\rho\sigma}_{(0)} \nabla_{\rho} \bar{\xi}_{\sigma} ) + 4h^{\lambda\mu} \nabla_{\lambda} \bar{\xi}_{\sigma} \) brings an additional subtracted term \( 4\lambda(\theta^{(i)}g^{(i)}g^{(i)}) \) to the potential \( \delta K^{\mu\nu}_{(i+1)} \), rendering it inconsistent with \( \bar{Q}_{(i)}^{\mu\nu} \). Secondly, the tensor \( P^{\mu\nu\rho\sigma}_{(i)} \) is not uniquely qualified to the construction of the potential \( K^{\mu\nu}_{(i)} \) in Eq. (2.11).

As a matter of fact, through the linear combination, the tensor \( P^{\mu\nu\rho\sigma}_{(i)} \) can be generalized as

\[
\bar{P}^{\mu\nu\rho\sigma}_{(i+1)} = \sum_{j=0}^{i} \beta_j P^{\mu\nu\rho\sigma}_{(j+1)},
\]

(3.7)

with the constant parameters \( \beta_j \)'s constrained by

\[
\sum_{j=0}^{i} \beta_j (j+1)\ell^j \frac{(D-4)!}{(D-2j-4)!} = \frac{(i+1)\ell^i}{2^i} \frac{(D-4)!}{(D-2i-4)!}.
\]

(3.8)

\footnote{We thank the anonymous referee for pointing out this.}
On basis of \( \mathcal{P}_{(i+1)}^{\mu\nu\rho\sigma} \), the potential \( \mathcal{K}_{(i)}^{\mu\nu} \) is reexpressed as \( \mathcal{K}_{(i+1)}^{\mu\nu} = \mathcal{P}_{(i+1)}^{\mu\nu\rho\sigma} \nabla_\rho \xi_\sigma \). In spite of this, in order to guarantee that the definition of the conserved charges matches the one via the ADT formalism and the covariant phase space method in a straightforward and simple manner, then it appears natural to adopt \( \mathcal{P}_{(i)}^{\mu\nu\rho\sigma} \) rather than its linear combinations to define the conserved charges. Thirdly, the vanishing of \( \bar{\mathcal{P}}_{(i)}^{\mu\nu\rho\sigma} \) yields \( \bar{\mathcal{K}}_{(i)}^{\mu\nu} = 0 \), further giving rise to that

\[
\delta \mathcal{K}_{(i)}^{\mu\nu} = \mathcal{K}_{(i)}^{\mu\nu} - \bar{\mathcal{K}}_{(i)}^{\mu\nu} = \mathcal{K}_{(i)}^{\mu\nu}.
\]

As a consequence, the perturbation of the potential \( \delta \mathcal{K}_{(i+1)}^{\mu\nu} \), involved in the formula (2.17) for the conserved charges, can be replaced by \( \mathcal{K}_{(i+1)}^{\mu\nu} \). To this point, one is able to see that the formula (2.17) for the conserved charges of the Lovelock gravity theory has the advantage of avoiding the perturbation for the potential. Thus it is much simpler than the one via the ADT approach.

4 The formula of conserved charges for generic diffeomorphism invariant theories of gravity

In this section, inspired with the tensor \( \mathcal{P}_{(i)}^{\mu\nu\rho\sigma} \), we are going to put forward another two generic rank-4 tensors and make use of each of them to define the conserved charges of an arbitrary gravity theory that is constructed from the curvature tensor in asymptotically AdS spacetimes.

We take into consideration of the general gravity theory described by the Lagrangian

\[
\mathcal{L}_{\text{Riem}} = \sqrt{-g} L_{\text{Riem}},
\]

where the scalar \( L_{\text{Riem}} \) is treated as a functional built from the Riemann curvature tensor \( R_{\alpha\beta\rho\sigma} \) in combination with the metric tensor \( g^{\mu\nu} \), that is,

\[
\mathcal{L}_{\text{Riem}} = \sqrt{-g} L_{\text{Riem}}(g^{\mu\nu}, R_{\alpha\beta\rho\sigma}).
\] (4.1)

Apparently, \( \mathcal{L}_{\text{Riem}} \) covers the Lovelock-type Lagrangians. Like in [2], through the derivative of the scalar \( L_{\text{Riem}} \) with respect to the Riemann tensor \( R_{\mu\nu\rho\sigma} \), we are able to define a tensor \( P_{R}^{\mu\nu\rho\sigma} \) as

\[
P_{R}^{\mu\nu\rho\sigma} = \frac{\partial L_{\text{Riem}}}{\partial R_{\mu\nu\rho\sigma}}.
\] (4.2)

\( P_{R}^{\mu\nu\rho\sigma} \) exhibits the same symmetries as the Riemann tensor, namely, \( P_{R}^{\mu\nu\rho\sigma} = -P_{R}^{\nu\mu\rho\sigma} = -P_{R}^{\mu\rho\sigma\nu} \) and \( P_{R}^{\mu\nu\rho\sigma} = P_{R}^{\rho\sigma\mu\nu} \). Nevertheless, unlike the conserved tensor \( \mathcal{P}_{(i)}^{\mu\nu\rho\sigma} \), here the
tensor $P^\mu_\nu^\rho_\sigma$ could be divergence-free or not. Besides, it is supposed that the value of $P^\mu_\nu^\rho_\sigma$ on the background AdS spacetime fulfills

$$
\bar{P}^\mu_\nu^\rho_\sigma = P^\mu_\nu^\rho_\sigma (g_{\alpha\beta} \rightarrow \bar{g}_{\alpha\beta}) = k \delta^\mu_\nu,
$$

where $k$ is a certain constant parameter. It should be pointed out that the condition (4.3) can be always guaranteed. This is attributed to the fact that $P^\mu_\nu^\rho_\sigma$ can be expressed as the linear combination of the terms having the general form

$$
(g^{\cdots\cdot} R^{\cdots\cdot\cdot\cdot\cdot\cdot\cdot})^\alpha_\beta^\gamma_\lambda \delta^\mu_\nu^\rho_\sigma^\delta_\alpha^\beta^\gamma^\lambda + ([\mu\nu] \leftrightarrow [\rho\sigma]).
$$

As a consequence, when $g^\mu_\nu = \bar{g}^\mu_\nu$ and $R^\mu_\nu^\rho_\sigma = 2 \ell \bar{g}^\mu_\nu \bar{g}^\rho_\sigma$, the above expression (4.4) must be proportional to the generalized Kronecker delta $\delta^\mu_\nu^\rho_\sigma$. With the help of the rank-4 tensor $P^\mu_\nu^\rho_\sigma$, we put forward another fourth-rank tensor $P^\mu_\nu^\rho_\sigma$, being of the form

$$
P^\mu_\nu^\rho_\sigma = P^\mu_\nu^\rho_\sigma - k \delta^\mu_\nu^\rho_\sigma - \frac{k}{(D-3)\ell} P^{\mu_\nu}_R^{(1)} \rho_\sigma.
$$

By contrast with the tensor $P^\mu_\nu^\rho_\sigma$, obviously, here $\bar{P}^\mu_\nu^\rho_\sigma = P^\mu_\nu^\rho_\sigma (g_{\alpha\beta} \rightarrow \bar{g}_{\alpha\beta})$ still identically disappears, and $P^\mu_\nu^\rho_\sigma$ inherits the symmetries of the Riemann curvature tensor as well. In particular, for the Einstein gravity, $P^\mu_\nu^\rho_\sigma = -2(D-3)\ell^{-1} P^{\alpha_\beta}_R \rho_\sigma$ since $P^\mu_\nu^\rho_\sigma = \delta^\mu_\nu^\rho_\sigma/2$ and $k = 1/2$. To this point, the tensor $P^\mu_\nu^\rho_\sigma$ could be regarded as the generalization of $P^{\mu_\nu}_R^{(1)} \rho_\sigma$ in the context of the generic gravity theories described by the Lagrangian (4.1).

In parallel with the situation of the Lovelock gravity theory, motivated by the Noether potential $K^\mu_\nu^\rho_\sigma$ via the covariant phase space method [36, 37, 38], that is,

$$
K^\mu_\nu^\rho_\sigma = P^\mu_\nu^\rho_\sigma \nabla_\rho \xi_\sigma - 2 \xi_\sigma \nabla_\rho P^\mu_\nu^\rho_\sigma,
$$

we make use of the tensor $P^\mu_\nu^\rho_\sigma$ to replace $P^\mu_\nu^\rho_\sigma$ in the above equation to introduce a similar superpotential $K^\mu_\nu^\rho_\sigma$, which is given by

$$
K^\mu_\nu^\rho_\sigma = P^\mu_\nu^\rho_\sigma \nabla_\rho \xi_\sigma - 2 \xi_\sigma \nabla_\rho P^\mu_\nu^\rho_\sigma.
$$

Alternately, it is convenient to express $K^\mu_\nu^\rho_\sigma$ as

$$
K^\mu_\nu^\rho_\sigma = K^\mu_\nu^\rho_\sigma - \frac{6k}{(D-3)\ell} R^{[\rho_\sigma} \nabla_\rho \xi^\nu] + k(D-4)\nabla_\rho \xi^\nu.
$$

In some sense, the superpotential $K^\mu_\nu^\rho_\sigma$ can be called as Komar-like potential if it is interpreted as the generalization of the ordinary Komar potential $P^{\mu_\nu}_R^{(0)} \nabla_\rho \xi^\sigma$ associated with
the Einstein gravity theory. According to Eq. (3.4), the perturbation of $K_R^{\mu\nu}$ on the AdS reference background leads to

$$\delta K_R^{\mu\nu} = \bar{Q}_R^{\mu\nu} - \frac{k}{(D-3)\ell} \bar{\nabla}_\gamma U^{(1)}_{\gamma\mu\nu},$$

$$\bar{Q}_R^{\mu\nu} = (\delta P_R^{\mu\nu})_\rho\sigma \bar{\nabla}_\rho \bar{\xi}_\sigma - 2\bar{\xi}_\rho \bar{P}_R^{\mu\nu} \nabla_\rho \bar{\xi}_\sigma - 2\bar{\xi}_\mu \bar{P}_R^{\mu\nu} \nabla_\sigma \bar{h}_{\rho\lambda}.$$  \hspace{1cm} (4.9)

As before, by following the works [30, 31], one can verify that the 2-form $\bar{Q}_R^{\mu\nu}$ is the (off-shell) ADT potential on a fixed AdS background, and it also coincides with the results in [32]. What is more, in terms of Eq. (4.4), one finds that $\delta P_R^{\mu\nu}$ could be generally expressed as

$$\delta P_R^{\mu\nu} = \lambda_1 \delta R^{\mu\nu}_\rho\sigma + \lambda_2 \bar{R}_{\rho}[^{(\mu}_{\rho} \nu_{\sigma}]} + \lambda_3 \delta R^{\mu\nu} \delta^{\rho\sigma} + \lambda_4 h^{(\mu}_{\rho} \nu_{\sigma)},$$  \hspace{1cm} (4.10)

where $\lambda_i$'s are constant parameters and the concrete expressions for $\delta R^{\mu\nu}_\rho\sigma$, $\delta R_{\rho}^{\mu\nu}$ and $\delta R$ were given in the appendix of Ref. [31]. For instance, when $\lambda_1 = 1$, $\lambda_2 = -4$, $\lambda_3 = 1/2$ and $\lambda_4 = 0$, one obtains

$$\delta P_R^{(1)\rho\sigma} = \delta R^{\mu\nu}_\rho\sigma - 4\delta R_{\rho}[^{(\mu}_{\rho} \nu_{\sigma}]} + \frac{1}{2} \delta R \delta^{\rho\sigma}.$$  \hspace{1cm} (4.11)

In terms of $\delta K_R^{\mu\nu}$, when the dimension $D > 3$, a formula for the conserved charges of the gravity theory described by the Lagrangian (4.1) can be proposed as a Komar-like integral

$$Q_{Riem} = \frac{1}{8\pi} \int_{\partial \Sigma} \delta K_R^{\mu\nu} d\Sigma_{\mu\nu},$$  \hspace{1cm} (4.12)

which is able to be completely determined by Eqs. (4.10) and (4.11). Apart from $P_R^{\mu\nu\rho\sigma}$, another fourth-rank tensor $\tilde{P}_R^{\mu\nu\rho\sigma}$, given by

$$\tilde{P}_R^{\mu\nu\rho\sigma} = \frac{1}{2} \left( P_R^{\mu\nu\rho\sigma} + P_R^{\alpha\beta} P^{\alpha\beta}_{(1)\rho\sigma} - 2(D-3)\ell \delta_R^{\rho\sigma} \right),$$  \hspace{1cm} (4.13)

can also be adopted to define the conserved charges. To see this clearly, by introducing the 2-form

$$\tilde{K}_R^{\mu\nu} = \tilde{P}_R^{\mu\nu\rho\sigma} \nabla_\rho \bar{\xi}_\sigma + \frac{1}{(D-3)\ell} \bar{\xi}_\rho \nabla_\rho \tilde{P}_R^{\mu\nu\rho\sigma},$$  \hspace{1cm} (4.14)

whose perturbation on the AdS background yields

$$\delta \tilde{K}_R^{\mu\nu} = -2(D-3)\ell \tilde{Q}_R^{\mu\nu} + 2k \bar{\nabla}_\gamma U^{(1)}_{\gamma\mu\nu},$$  \hspace{1cm} (4.15)

\textsuperscript{3}If $\tilde{P}_R^{\mu\nu\rho\sigma}$ is not required to exhibit all the algebraic symmetry properties for the Riemann tensor, it can be alternatively defined as $\tilde{P}_R^{\mu\nu\rho\sigma} \rightarrow P_R^{\mu\nu\rho\sigma} P^{\alpha\beta}_{(1)\rho\sigma} - 2(D-3)\ell \delta_R^{\rho\sigma}$. Unlike $P_R^{\mu\nu\rho\sigma}$, here $\tilde{P}_R^{\mu\nu\rho\sigma} = P^{\mu\nu\rho\sigma}$ for the Einstein gravity.
the formula (4.12) is rewritten as
\[ Q_{\text{Riem}} = -\frac{1}{16\pi(D-3)\ell} \int_{\partial\Sigma} \delta \tilde{K}^\mu_\nu \, d\Sigma_{\mu\nu}. \] (4.16)

As an example to demonstrate the conserved quantity \( Q_{\text{Riem}} \), substituting
\[ P_{R(\rho\sigma)} = P_{(i)(\rho\sigma)}, \quad k = \lambda_{(i)} = \left( \frac{\ell}{2} \right) \frac{(D-2)!}{(D-2i-2)!} \]
into the formula (4.12) or (4.16), one obtains the conserved charge \( Q_{(i)} \) for the Lovelock-type Lagrangian (2.6), which has been presented in Eq. (2.17).

A remark is in order here. By contrast with the conventional ADT formalism, our definition for the conserved charges at least has the following merits. First, calculations on the potential become more operable. It straightforwardly arises from the perturbation of \( K^\mu_\nu \) on the AdS background. Second, because of the vanishing of the tensor \( P_{R(\rho\sigma)} \) on the background spacetimes, apart from the variation of \( K^\mu_\nu \), the 2-form \( K^\mu_\nu \) itself can be directly adopted to enter into the surface integral. As a consequence, all the computations with respect to the perturbation of the gravitational field are avoidable, extremely simplifying the formula. Third, the potential \( K^\mu_\nu \) takes a closer form to that through the well-known covariant phase space method. Thus, the formulation in the present work gives assistance to built a connection between the ADT approach and the covariant phase space method. Actually, in comparison with the Iyer-Wald potential given by the covariant phase space method, the quantity \( \delta (K^\mu_\nu - K^\mu_\nu) \) just compensates the contribution from the surface term, rendering \( \delta K^\mu_\nu \) equivalent to the Iyer-Wald potential. Furthermore, with the guidance of such an equivalence, we arrive at the conclusion that the ADT potential is equivalent with the one via the covariant phase space method in the context of the generic gravities endowed with the Lagrangian (4.1). However, an obvious demerit of the formula (4.12) for the conserved charges is that it does not incorporate the contribution for the matter fields, which is of great importance, particularly to the conserved charges of Gödel-type black holes [35]. This deserves to be dealt with in future.

5 Summary

In the present paper, within the framework of the Lovelock gravity theory, we propose a new rank-four divergenceless tensor \( P_{R(\rho\sigma)} \) given in Eq. (2.9). This tensor is dependent of the Riemann curvature tensor and preserves its all algebraic symmetries. In terms of the
tensor $\mathcal{P}_{(i)}^{\mu \nu \rho \sigma}$, a 2-form $\mathcal{K}_{(i)}^{\mu \nu}$ associated with an arbitrary Killing vector $\xi^\mu$ is constructed. The perturbation of $\mathcal{K}_{(i+1)}^{\mu \nu}$ on the AdS background serves as an ideal superpotential in the definition for the conserved charges of the Lovelock gravity theories in asymptotically AdS spacetimes, and it further give rises to the formula (2.17). Subsequently, inspired by the tensor $\mathcal{P}_{(i)}^{\mu \nu \rho \sigma}$, a fourth-rank tensor $\mathcal{P}_{R}^{\mu \nu \rho \sigma}$ (or $\tilde{\mathcal{P}}_{R}^{\mu \nu \rho \sigma}$) is constructed in the context of the general diffeomorphism invariant theories of gravity with the Lagrangian (4.1). In parallel, with the help of this tensor, as well as its perturbation upon the AdS background, we have put forward the definition for the conserved charges $Q_{Riem}$ in Eq. (4.12) or (4.16), which is applicable to an arbitrary gravity theory depending merely on the curvature tensor. What is more, we have clarified that all the newly-constructed formulas of the conserved charges are equivalent with the ones via the ADT method and the covariant phase space approach.

Acknowledgments

We would like to thank the anonymous referees for their valuable suggestions and comments. This work was supported by the Natural Science Foundation of China under Grant Nos. 11865006 and 11505036. It was also partially supported by the Technology Department of Guizhou province Fund under Grant Nos. [2018]5769.

References

[1] D. Lovelock, The Einstein tensor and its generalizations, *J. Math. Phys.* 12, 498 (1971).

[2] T. Padmanabhan and D. Kothawala, Lanczos-Lovelock models of gravity, *Phys. Rept.* 531, 115 (2013).

[3] T. Padmanabhan, *Gravitation, Foundations and Frontiers*, (Cambridge University Press, Cambridge, 2010).

[4] S. Chakraborty, Lanczos-Lovelock gravity from a thermodynamic perspective, *J. High Energy Phys.* 08, 029 (2015).

[5] N. Dadhich, J.M. Pons and K. Prabhu, Thermodynamical universality of the Lovelock black holes, *Gen. Rel. Grav.* 44, 2595 (2012).
[6] R. Aros and M. Estrada, Regular black holes and its thermodynamics in Lovelock gravity, *Eur. Phys. J. C* **79**, 259 (2019); K. Meng, Hairy black holes of Lovelock-Born-Infeld-scalar gravity, *Phys. Lett. B* **784**, 56 (2018); N. Farhangkhah, Topologically nontrivial black holes in Lovelock-Born-Infeld gravity, *Phys. Rev. D* **97**, 084031 (2018); S.H. Hendi, S. Panahiyan and B. Eslam Panah, Extended phase space of Black Holes in Lovelock gravity with nonlinear electrodynamics, *PTEP* **2015**, 103E01 (2015); M.H. Dehghani, N. Alinejadi and S.H. Hendi, Topological black holes in Lovelock-Born-Infeld gravity, *Phys. Rev. D* **77**, 104025 (2008).

[7] J.M. Toledo and V.B. Bezerra, Black holes with a cloud of strings in pure Lovelock gravity, *Eur. Phys. J. C* **79**, 117 (2019).

[8] M. Cvetic, X.H. Feng, H. Lu and C.N. Pope, Rotating solutions in critical Lovelock gravities, *Phys. Lett. B* **765**, 181 (2017); R.H. Yue, D.C. Zou, T.Y. Yu, P. Li and Z.Y. Yang, Slowly rotating charged black holes in anti-de Sitter third order Lovelock gravity, *Gen. Rel. Grav.* **43**, 2103 (2011).

[9] N. Dadhich, On Lovelock vacuum solution, *Math. Today* **26**, 37 (2011).

[10] R.G. Cai , L.M. Cao and N. Ohta, Black holes without mass and entropy in lovelock gravity, *Phys. Rev. D* **81**, 024018 (2010); R.G. Cai and N. Ohta, Black holes in pure lovelock gravities, *Phys. Rev. D* **74**, 064001 (2006).

[11] M. Banados, C. Teitelboim and J. Zanelli, Dimensionally continued black holes, *Phys. Rev. D* **49**, 975 (1994).

[12] N. Dadhich, J.M. Pons and K. Prabhu, On the static Lovelock black holes, *Gen. Rel. Grav.* **45**, 1131 (2013).

[13] G. Kofinas and R. Olea, Universal regularization prescription for Lovelock AdS gravity, *J. High Energy Phys.* **11**, 069 (2007).

[14] E. Frodden and D. Hidalgo, Surface charges toolkit for gravity, arXiv:1911.07264 [hep-th].

[15] J. Kastikainen, Quasi-local energy and ADM mass in pure Lovelock gravity, *Class. Quantum Grav.* **37**, 025001 (2020).
[16] G. Arenas-Henriquez, R.B. Mann, O. Miskovic and R. Olea, Mass in Lovelock unique vacuum gravity theories, *Phys. Rev. D* **100**, 064038 (2019).

[17] G. Arenas-Henriquez, O. Miskovic and R. Olea, Vacuum degeneracy and conformal mass in Lovelock AdS gravity, *J. High Energy Phys.* **1711**, 128 (2017).

[18] A.N. Petrov, Field-theoretical construction of currents and superpotentials in Lovelock gravity, *Class. Quantum Grav.* **36**, 235021 (2019).

[19] K. Bhattacharya and B.R. Majhi, Abbott-Deser-Tekin like conserved quantities in Lanczos-Lovelock gravity: beyond Killing diffeomorphisms, *Class. Quantum Grav.* **36**, 065009 (2019).

[20] S. Chakraborty and N. Dadhich, Brown-York quasilocal energy in Lanczos-Lovelock gravity and black hole horizons, *J. High Energy Phys.* **1512**, 003 (2015).

[21] E. Gravanis, Conserved charges in (Lovelock) gravity in first order formalism, *Phys. Rev. D* **81**, 084013 (2010).

[22] D. Kastor, Komar integrals in higher (and lower) derivative gravity, *Class. Quantum Grav.* **25**, 175007 (2008).

[23] M.H. Dehghani, N. Bostani and A. Sheykhi, Counterterm method in Lovelock theory and horizonless solutions in dimensionally continued gravity, *Phys. Rev. D* **73**, 104013 (2006); D. Astefanesei, N. Banerjee and S. Dutta, (Un)attractor black holes in higher derivative AdS gravity, *J. High Energy Phys.* **0811**, 070 (2008); Y. Brihaye and E. Radu, Black objects in the Einstein-Gauss-Bonnet theory with negative cosmological constant and the boundary counterterm method, *J. High Energy Phys.* **0809**, 006 (2008).

[24] T. Jacobson and R.C. Myers, Black hole entropy and higher curvature interactions, *Phys. Rev. Lett.* **70**, 3684 (1993).

[25] E. Altas and B. Tekin, New approach to conserved charges of generic gravity in AdS spacetimes, *Phys. Rev. D* **99**, 044016 (2019).

[26] E. Altas and B. Tekin, Conserved charges in AdS: A new formula, *Phys. Rev. D* **99**, 044026 (2019).
[27] L. F. Abbott and S. Deser, Stability of gravity with a cosmological constant, *Nucl. Phys. B* **195**, 76 (1982); L. F. Abbott and S. Deser, Charge definition in non-abelian gauge theories, *Phys. Lett. B* **116**, 259 (1982).

[28] S. Deser and B. Tekin, Gravitational Energy in Quadratic Curvature Gravities, *Phys. Rev. Lett.* **89**, 101101 (2002); S. Deser and B. Tekin, Energy in Generic Higher Curvature Gravity Theories, *Phys. Rev. D* **67**, 084009 (2003).

[29] S. Deser and B. Tekin, New energy definition for higher curvature gravities, *Phys. Rev. D* **75**, 084032 (2007).

[30] W. Kim, S. Kulkarni and S.H. Yi, Quasi-Local conserved charges in covariant theory of gravity, *Phys. Rev. Lett.* **111**, 081101 (2013).

[31] J.J. Peng, Conserved charges of black holes in Weyl and Einstein-Gauss-Bonnet gravities, *Eur. Phys. J. C* **74**, 3156 (2014).

[32] A.J. Amsel and D. Gorbonos, Wald-like formula for energy, *Phys. Rev. D* **87**, 024032 (2013); C. Senturk, T.C. Sisman and B. Tekin, Energy and angular momentum in generic F(Riemann) theories, *Phys. Rev. D* **86**, 124030 (2012).

[33] X.O. Camanho and N. Dadhich, On Lovelock analogs of the Riemann tensor, *Eur. Phys. J. C* **76**, 149 (2016); N. Dadhich, Characterization of the Lovelock gravity by Bianchi derivative, *Pramana* **74**, 875 (2010).

[34] D. Kastor, The Riemann-Lovelock curvature tensor, *Class. Quantum Grav.* **29**, 155007 (2012).

[35] J.J. Peng, Mass and angular momentum of charged rotating Gödel black holes in five-dimensional minimal supergravity, *Int. J. Mod. Phys. A* **34**, 1950217 (2019).

[36] J. Lee and R. M. Wald, Local symmetries and constraints, *J. Math. Phys.* **31**, 725 (1990).

[37] V. Iyer and R.M. Wald, Some properties of the Noether charge and a proposal for dynamical black hole entropy, *Phys. Rev. D* **50**, 846 (1994).

[38] R.M. Wald and A. Zoupas, A general definition of ‘conserved quantities’ in general relativity and other theories of gravity, *Phys. Rev. D* **61**, 084027 (2000).
[39] R.P. Woodard, Ostrogradsky’s theorem on Hamiltonian instability, *Scholarpedia* **10**, 32243 (2015).

[40] T. Jacobson and A. Mohd, Black hole entropy and Lorentz-diffeomorphism Noether charge, *Phys. Rev. D* **92**, 124010 (2015).

[41] K. Prabhu, The First Law of Black Hole Mechanics for Fields with Internal Gauge Freedom, *Class. Quantum Grav.* **34**, 035011 (2017).

[42] A. Mishra, S. Chakraborty, A. Ghosh and S. Sarkar, On the physical process first law for dynamical black holes, *J. High Energy Phys.* **09**, 034 (2018).