Validity of Harris criterion for two-dimensional quantum spin systems with quenched disorder

Jhao-Hong Peng,1 L.-W. Huang,1 D.-R. Tan,1 and F.-J. Jiang1,2

1Department of Physics, National Taiwan Normal University, 88, Sec.4, Ting-Chou Rd., Taipei 116, Taiwan

Inspired by the recent results regarding whether the Harris criterion is valid for quantum spin systems, we have simulated a two-dimensional spin-1/2 Heisenberg model on the square lattice with a specific kind of quenched disorder using the quantum Monte Carlo (QMC) calculations. In particular, the considered quenched disorder has a tunable parameter $0 \leq p \leq 1$ which can be considered as a measure of randomness. Interestingly, when the magnitude of $p$ increases from 0 to 0.9, at the associated quantum phase transitions the numerical value of the correlation length exponent $\nu$ grows from a number compatible with the $O(3)$ result 0.7112(5) to a number slightly greater than 1. In other words, by varying $p$, $\nu$ can reach an outcome between 0.7112(5) and 1 (or greater). Furthermore, among the studied values of $p$, all the associated $\nu$ violate the Harris criterion except the one corresponding to $p = 0.9$. Considering the form of the employed disorder here, the above described scenario should remain true for other randomness if it is based on the similar idea as the one used in this study. This is indeed confirmed by our preliminary results stemming from investigating another disorder distribution.

PACS numbers:

Introduction — Studying the effects resulting from disorder has always been one of the major topics in both theoretical and experimental physics [1–15]. This is because the presence of disorder such as impurities may lead to extraordinary properties and phases of materials. In particular, the appearance of these exotic characteristics are due to the mutual influence between the quantum fluctuations and disorder. Understanding the relevance of disorder at quantum phase transitions also continues to attract a lot of attention. This is especially true considering the recent development regarding under what conditions will the celebrated Harris criterion be valid [16–23]. In other words, it is not clearly at all that with what specific features of a disorder distribution will a new universality class emerge at the studied quantum phase transition.

For a phase transition, there are three possible scenarios when disorder is present. Here we will focus on those related to the Harris criterion. The Harris criterion was originally derived for classical systems and its statement is as follows. For a $D$-dimensional classical system with disorder, the correlation length exponent $\nu$ must satisfy the inequality $\nu \geq 2/D$. If the $\nu$ of a clean model does not fulfill this inequality, then when disorder is introduced (into the clean model), a new universality class should obtain so that the described inequality is realized, assuming the phase transition remains well defined. Later the criterion was generalized to more generic situations including certain quantum systems. We would like to emphasize the fact that for a $d$-dimensional quantum system with quenched disorder, since the disorder is employed in the spatial dimension, the dimensionality $D$ appearing in the inequality is $d$, not $d + 1$ despite the quantum system can be mapped to a $d + 1$-dimensional classical system.

While the validity of Harris criterion is beyond doubt for classical models, the case of quantum spin systems is much more complicated. Particularly, at the moment only the outcome related to the two-dimensional ($2d$) spin-1/2 Heisenberg model on a bilayer square lattice with bond dilution satisfies the Harris criterion [19–23]. Other kinds of quenched disorder, including the configurational disorder considered in Ref. [10] as well as the one introduced in Ref. [15], the resulting calculations always indicate the Harris criterion is violated. Furthermore, the obtained values of the correlation length exponent $\nu$ remain the same as that of their clean counterparts. To summarize, whether the celebrated Harris criterion is valid for quantum spin systems is more involved than anticipated.

Inspired by such an indecisive answer regarding the applicability of Harris criterion for quantum spin systems, in this study we have carried out a large scale quantum Monte Carlo (QMC) calculations for a two-dimensional spin-1/2 Heisenberg model on the square lattice, starting from the clean herringbone model and then introducing a specific kind of quenched disorder into the clean system. In particular, the employed randomness distribution has a tunable parameter $p$ (Which can take values from 0 to 1 and can be considered as a measure of randomness) so that one can investigate the impact of this parameter on the effectiveness of Harris criterion for the studied model.

Remarkably, our QMC data indicate that as the magnitude of $p$ increases gradually from 0 to 0.9, the numerical value of $\nu$ grows from its $O(3)$ value 0.7112(5) [24–31] to a result slightly greater than 1. In other words, by varying $p$, the corresponding $\nu$ for the disordered systems studied in this investigation can reach outcomes lie between 0.7112(5) and 1. Moreover, the $\nu$ resulting from the considered values of $p$ all violate the Harris criterion.
except the one related to \( p = 0.9 \). Our preliminary study of another disorder distribution following the similar idea as that introduced above leads to the same conclusion, namely \( \nu \) can take a value between 0.7112(5) and 1 as well. The results demonstrated here indicate that a better understanding of the Harris criterion from a theoretical point of view is on request.

Microscopic models and observables — The Hamiltonian of the studied 2d (clean) herringbone spin-1/2 Heisenberg model on the square lattice is given by

\[
H = \sum_{ij} J_{ij} \vec{S}_i \cdot \vec{S}_j + \sum_{i'j'} J'_{i'j'} \vec{S}_{i'} \cdot \vec{S}_{j'},
\]

where in Eq. (1) \( J_{ij} \) (which is set to 1 for any pair of \((i, j)\) here) and \( J'_{i'j'} \) are the antiferromagnetic couplings (bonds) connecting nearest neighbor spins \((ij)\) and \((i'j')\) located at sites of the considered underlying square lattice, respectively, and \( \vec{S}_i \) is the spin-1/2 operator at site \( i \). The left panel of figure 1 demonstrates the typical clean herringbone models studied in the literature. The quenched disorder introduced into the (clean) system is based on the one employed in Ref. [15]. Specifically, for every bold bond in the left panel of fig. 1 its antiferromagnetic strength \( J' \) takes the value of \( 1 + (g - 1)(1 + p) \) or \( 1 + (g - 1)(1 - p) \) with equal probability. Here \( g > 1 \) and \( 0 \leq p \leq 1 \). As pointed out in Ref. [15], the average and difference of \( J' \) for these two types of bold bonds are given by \( g \) and \( 2p(g - 1) \), respectively. Moreover, \( p \) can be considered as a measure of disorder of the studied system. In our study, several values of \( p \) are chosen and for each of them, the corresponding phase transition is induced by tuning \( g := J' \).

To carry out the proposed investigation, particularly to examine the validity of Harris criterion for the considered disordered model, the observables first Binder ratio \( Q_1 \) and second Binder ratio \( Q_2 \), which are defined by

\[
Q_1 = \frac{\langle m_s^2 \rangle^2}{\langle m_s^4 \rangle},
\]

and

\[
Q_2 = \frac{\langle (m_s^2) \rangle^2}{\langle m_s^4 \rangle},
\]

respectively, are measured in our calculations. The staggered magnetization density \( m_s^z \) on a square lattice with linear box size \( L \) appearing above is given by \( m_s^z = \frac{1}{L^2} \sum_i (-1)^{v_1 + v_2} S_i^z \) with \( S_i^z \) being the third component of the spin-1/2 operator \( \vec{S}_i \) at site \( i \). Observable such as the one associated with spin stiffness \( \rho_s \) has the following finite-size scaling expression close to the critical region [26, 27]

\[
\rho_s L^z = (1 + b_0 L^{-\omega}) f(t L^{1/\nu}),
\]

where \( \nu \) and \( \omega \) are the correlation length and the confluent exponents, respectively, \( z \) is the dynamic critical exponent, \( b_0 \) is some constant, and \( f \) is a smooth function of its argument \( t L^{1/\nu} \). To apply finite-size scaling for observable associated with \( \rho_s \), one firstly needs to determine \( z \). Due to this, \( Q_1 \) and \( Q_2 \) are chosen as the relevant physical quantities for our investigation because their expected finite-size scaling formulas [27]

\[
Q_i = (1 + b_i L^{-\omega}) f_i(t L^{1/\nu}), \quad i \in \{1, 2\}
\]

do not contain the dynamic critical exponent \( z \). Such a strategy, namely using \( Q_1 \) and \( Q_2 \) in our study, dramatically eliminates the computational complexity.

The numerical results — For each of the studied \( p \), to investigate the \( g \) dependence of the correlation length ex-
component $\nu$ associated with it, we have carried out a large-scale QMC simulation using the stochastic series expansion (SSE) algorithm with very efficient operator-loop update $[33, 34]$. Furthermore, to obtain ground state properties in an efficient manner, the $\beta$-doubling scheme described in Ref. [3] is used in our simulations. Several hundred disordered configurations, each with its own random seed, are generated for every considered set of parameters. It is also important to notice that potentially there are two kinds of uncertainties for the used observables, namely the one from Monte Carlo (MC) simulations and the one from disorder averaging. We have carried out many trial simulations and have reached the conclusion that with the MC sweeps employed in this study, the resulting errors of the considered quantities are indeed dominated by the disordered sample-to-sample fluctuation.

As already mentioned in the previous section, we have focused on the observables $Q_1$ and $Q_2$. Several data sets regarding convergence in $\beta$, as well as the final ground states values of $Q_1$ and $Q_2$ are presented in Figs. 2 and 6. More data of $Q_1$ and $Q_2$ are shown in the supplemental material.

The method used for the determination of $\nu$ (and $g_c$ as well) is the Bayesian analysis which is a rigorous mathematical approach. It is demonstrated in Refs. $[36, 37]$ that the critical exponents calculated using the Bayesian analysis agree quantitatively with those determined by the conventional fits using the idea of finite-size scaling. Here we follow the methods outlined in Ref. [38]. Moreover, we have carried out many trial computations and have arrived at the same conclusion as those in Refs. $[36, 57]$, namely the results obtained from the Bayesian analysis and the conventional finite-size scaling fits are consistent with each other quantitatively.

The model considered for the Bayesian analysis is the expected finite-size scaling equations for $Q_1$ and $Q_2$ at a second order phase transition. Specifically, the explicit expression of the model for the Bayesian analysis is given by

$$\frac{1 + a_0 L^{-\nu}}{(1 + a_2 t L^{1/\nu} + a_3 (t L^{1/\nu})^2 + ...)}.$$  \hspace{1cm} (6)

Here $a_i$ for $i = 0, 1, 2, ...$ are some constants and $t = \frac{\omega - 1}{g_c}$. Moreover, this ansatz with up to third, fourth and fifth order in $t L^{1/\nu}$ are employed in the calculations of estimating the desired physical quantities $\nu$ and $g_c$, with the data of $Q_1$ and $Q_2$. Some constraints, such as the range of $g$ considered and the values of $\omega$ obtained, are taken in account in the procedure of analysis as well.

Table I summarizes the final quoted values of $\nu$ and $g_c$ for all the considered $p$. This table is based on the results of each individual $p$ obtained from the Bayesian analysis. Detailed outcomes of some values of $p$ are listed in the supplemental material.

For the clean model, the averaged $g_c$ and $\nu$ are given by 2.4981(2) and 0.703(5), respectively. The calculated $g_c$ is in nice agreement with the known results in the literature.

### Table I: Results of $\nu$ and $g_c$ obtained from the Bayesian analysis (and resampling).

| $P$ | $\nu$     | $g_c$    |
|-----|-----------|----------|
| 0.0 | 0.703(5)  | 2.4981(2)|
| 0.1 | 0.701(5)  | 2.5056(2)|
| 0.2 | 0.724(5)  | 2.5310(3)|
| 0.3 | 0.754(6)  | 2.5734(7)|
| 0.4 | 0.781(7)  | 2.6388(11)|
| 0.5 | 0.816(9)  | 2.7401(7)|
| 0.6 | 0.843(11) | 2.8945(12)|
| 0.9 | 1.04(3)   | 4.828(18)|

FIG. 3: $Q_1$ (top panel, $p = 0$) and $Q_2$ (bottom panel, $p = 0.3$) of various $L$ as functions of $g$ for the considered Herringbone models studied here. The dashed lines are added to guide the eye.
The determined $\nu$ for $p = 0$ is slightly smaller in magnitude than the expected $O(3)$ value 0.7112(5). The largest $L$ used in the simulations conducted here is $L = 48$. As a result, the small deviation between 0.703(5) found here and 0.7112(5) can be easily accounted for by the cubic term introduced in Ref. [35], which will lead to anomalous large finite-size correction.

Table I also implies that $g_c$ grows with $p$. What’s the most remarkable outcome shown in table I is that, as the magnitude of $p$ rises, the corresponding $\nu$ calculated increases in size gradually from that of $p = 0$ as well. Particularly for $p \geq 0.3$ and $p \leq 0.6$, the obtained $\nu$ from Bayesian analysis are all statistically larger than 0.7112(5), but smaller than 1.0. In addition, for $p = 0.9$, the associated $\nu$ is around 1.0 with which the Harris criterion $\nu \geq 2/d$ is satisfied. To summarize, the outcomes of our investigation, as shown in table I indicate that for each employed $p$ such that $0.3 \leq p < 0.9$, the resulting associated correlation length exponent neither stays as the $O(3)$ value $\nu = 0.7112(5)$ nor satisfies the Harris criterion $\nu \geq 2/d = 1$. Moreover, for $p = 0.9$, the Harris criterion is fulfilled. From the considered quantum spin system, we arrive at a scenario regarding the connection between $\nu$ and quenched disorder with a tunable randomness strength not known before in the literature.

It should be pointed out that the $\nu$ determined from $Q_1$ differ from that related to $Q_2$ slightly. We attribute this to corrections not taken into account in the analysis. Despite this, it is without doubt that both $Q_1$ and $Q_2$ will lead to the scenario described above.

Based on the explicit expression of the disorder taken into account here, one expects that the obtained scenario should still be valid for other randomness distributions using the similar idea as the one investigated above. Motivated by this intuitive thought, apart from simulating the disordered system introduced previously, we have considered a quenched disorder for the plaquette model (The right hand side panel of fig. 1). Specifically, each of the bold bonds takes the antiferrogenetic strength of $(1 + K)J_c$ and $(1 - K)J_c$ with probability $P$ and $1 - P$, respectively. Here $0 < P < 1$ and we have used $K = 0.5$. In addition, the $J_c$ appearing above is given by 1.8230 which is the critical point of the clean plaquette model. With such a set up, $P$ is the tunable variable for this model. The resulting $Q_1$ and $Q_2$ of this new model with the new type of quenched disorder are shown in the supplemental material. Moreover, by applying typical fits with the conventional finite-size scaling equations to $Q_1$ and $Q_2$, we arrive at $\nu = 0.79(2)$ (and the critical point $P_c = 0.5520(16)$). This number $\nu = 0.79(2)$ is without doubt statistically different from both 0.7112(5) and 1.

By changing $D$ continuously, it is anticipated that the corresponding $\nu$ will vary in a gradual manner. This is of high similarity to the scenario associated with the Herringerbone model found earlier in this study.

Discussions and Conclusions — According to Ref. [39], based on the procedure of scaling employed in this study, the Harris criterion should be $\nu_{FS} \geq 2/d$, where the subscript FS stands for finite size. In particular, Ref. [39] also demonstrates that the bulk correlation length $\nu$ can be obtained by a modified procedure and may violate the Harris criterion. Considering the facts that most of the outcomes obtained in this study violate $\nu_{FS} \geq 2/d$ and the variation among all the $g_c(p)$ determined here is not small, the observed scenario cannot be easily accounted for by the arguments in Ref. [39]. Results of some studies such as Refs. [10, 40] imply that the values of $\nu$ (or $\nu_{FS}$) calculated do not depend on the disorder strength. The scenario found in this study clearly is different from this and other established ones in the literature.

We would like to re-emphasize the following points. First of all, the $\nu_{FS}$ obtained here for $p \geq 0.3$ all violate the Harris criterion $\nu_{FS} \geq 2/d$ except for $p = 0.9$. Secondly, for any given $p$, the phase transition is due to dimerization, namely two nearest neighboring spins (on the lattice) form a singlet. Hence, theoretically it is anticipated that for two close by values of $p$, say $p = 0.3$ and $p = 0.4$, the calculated results of $\nu$ should be close to each other or even consistent within statistical error. As can be seen from table I, this is not the case. In particular, the difference of the $g_c$ between $p = 0.0$ and $p = 0.3$ is only around 3 percent, yet a new critical exponent $\nu$ emerges for $p = 0.3$. It is interesting as well to notice that while in [10] the $g_c$ of the random plaquette model varies from that of its clean counterpart by four percent, both models have the same $O(3)$ exponent $\nu = 0.7112(5)$. Therefore, closeness the critical point of a disordered system from that of its clean analogue, which may be interpreted as the statement “remains well defined” in the Harris criterion, is not crucial for the appearance of a new universality class.

Interestingly, based on the outcomes found here, it is likely that for the considered model with the designed quenched disorder, the largest value of $\nu$ one can obtain should be around 1.1 to 1.2. This number agrees with the results calculated by simulating quantum spin model on the bilayer lattice with bond dilution [19, 21]. It is plausible that for a disordered spin-1/2 system, whenever the associated $\nu$ fulfills the Harris criterion, its value is in the range of 1 to 1.2.

In conclusion, in order to obtain a theoretical explanation for the exotic scenario observed in this study, a detailed exploration of the relevant theory than what’s been accomplished for the Harris criterion is required.

This study is partially supported by MOST of Taiwan.
In this section, we list the values of $\nu$ and $g_c$ obtained from analysis with various conditions for some of the considered $p$. Specially, we will briefly describe how these outcomes are obtained from the relevant data. The method used in obtaining the corresponding results of $\nu$ and $g_c$ from the data of $Q_1$ and $Q_2$ is the Bayesian analysis. In particular, the initial values of $\nu$ and $g_c$ determined from the analysis are the Maximum a posteriori estimates (Which is defined as the result of having the least value of $\chi^2$/DOF in this study due to the flat prior). In addition, the associated uncertainties are the standard deviations of the resulting probability distributions. For each $p$ and a fixed $L_m$ ($L_m$ is the smallest box size used in the analysis), several examinations are performed using Bayesian method with various range of $g$. Moreover, resampling with respect to these outcomes obtained by considering various range of $g$ are conducted, using the related uncertainties as the Gaussian noises. The $\nu$ and $g_c$ finally presented in the following tables and used to calculate the outcomes of table 1 in the main text, are the means and standard deviations of the results calculated from the resampling procedures described above.

**SUPPLEMENTAL MATERIAL TO “VALIDITY OF HARRIS CRITERION FOR TWO-DIMENSIONAL QUANTUM SPIN SYSTEMS WITH QUENCHED DISORDER”**

Details of the results from Bayesian analysis for the Herringbone model

[1] D. S. Fisher, Phys. Rev. B 50, 3799 (1994).
[2] O. Vajk, P. Mang, M. Greven, P. Gehring, and J. Lynn, Science 295, 1691 (2002).
[3] A. W. Sandvik, Phys. Rev. B 66, 024418 (2002).
[4] G. A. Csáthy, J. D. Reppy, and M. H. W. Chan, Phys. Rev. Lett. 91, 235301 (2003).
[5] Y.-C. Lin, R. Mélén, H. Riegier, and F. Iglói, Phys. Rev. B 68, 024424 (2003).
[6] Y.-C. Lin, H. Riegier, N. Lafloréncie, and F. Iglói, Phys. Rev. B 74, 024427 (2006).
[7] N. Lafloréncie, S. Wessel, A. Läuchli, and H. Riegier, Phys. Rev. B 73, 060403(R) (2006).
[8] Thomas Vojta, J. Phys. A 39, R143-R205 (2006)
[9] T. Vojta, J. Low Temp. Phys. 161, 299 (2010).
[10] Dao-Xin Yao, Jonas Gustafsson, E. W. Carlson, and Anders W. Sandvik, Physical Review B, 82, 172409 (2010).
[11] P. Carretta, G. Prando, S. Sanna, R. De Renzi, C. Decorce, and P. Berthet, Phys. Rev. B 83, 180411(R) (2011)
[12] R. Yu, C. F. Miclea, F. Weickert, R. Movshovich, A. Paduan-Filho, V. S. Zapf, and T. Roscilde, Phys. Rev. B 86, 134421 (2012).
[13] R. Yu, L. Yin, N. S. Sullivan, J. S. Xia, C. Huan, A. Paduan-Filho, N. F. Oliveira Jr, S. Haas, A. Steppeke, C. F. Miclea, F. Weickert, R. Movshovich, E.-D. Mun, B. L. Scott, V. S. Zapf, T. Roscilde, and A. Kitaev, Nature 489, 379 (2012).
[14] Thomas Vojta, AIP Conference Proceedings 1550, 188 (2013).
[15] Nvsen Ma, Anders W. Sandvik, and Dao-Xin Yao, Phys. Rev. B 90, 104425 (2014).
[16] A. B. Harris, J. Phys. C 7, 1671 (1974).
[17] J. T. Chayes, L. Chayes, D. S. Fisher, and T. Spencer, Phys. Rev. Lett. 57, 2999 (1986).
[18] O. Motrunich, S.C. Mau, D.A. Huse, and D.S. Fisher, Phys. Rev. B 61, 1160 (2000).
[19] A. W. Sandvik, Phys. Rev. Lett. 89, 177201 (2002).
[20] O. P. Vajk and M. Greven, Phys. Rev. Lett. 89, 177202 (2002).
[21] R. Sknepnek, T. Vojta, and M. Vojta, Phys. Rev. Lett. 93, 097201 (2004).
[22] Rong Yu, Tommaso Roscilde, and Stephan Haas, Phys. Rev. Lett. 94, 197204 (2005)
[23] A. W. Sandvik, Phys. Rev. Lett. 96, 207201 (2006).
[24] Nigel Goldenfeld, Lectures On Phase Transitions And The Renormalization Group (Frontiers in Physics) (Addison-Wesley, 1992).
[25] M. Campostrini, M. Hasenbusch, A. Pelissetto, P. Rossi, and E. Vicari, Phys. Rev. B 65, 144520 (2002).
[26] Andrea Pelissetto and Ettore Vicari, Physics Reports 368 (2002) 549-727.
[27] L. Wang, K. S. D. Beach, and A. W. Sandvik, Phys. Rev. B 73, 014431 (2006).
[28] A. F. Albuquerque, M. Troyer, J. Oitmaa, Phys. Rev. B 78, 127202 (2008).
[29] S. Wenzel and W. Janke, Phys. Rev. B 79, 014410 (2009).
[30] Lincoln D. Carr, Understanding Quantum Phase Transitions (Condensed Matter Physics) (CRC Press, 2010).
[31] S. Sachdev, Quantum Phase Transitions (Cambridge University Press, Cambridge, 2nd edition, 2011).
TABLE II: Results of $\nu$ and $g_c$ obtained from Bayesian analysis (and resampling) for the clean Herringbone model. The second and third columns are the minimum box size and the order of polynomials in $t L^{1/\nu}$ (including subleading correction) used in the analysis, respectively. The $\nu$ and $g_c$ shown in the 4th (5th) and 6th (7th) columns are determined from $Q_1$ ($Q_2$).

| $P$ | $L_m$ | $t L^{1/\nu}$ | $\nu$ | $g_c$ | $g_c$ |
|-----|-------|----------------|-------|-------|-------|
| 0.0 | 8     | 3              | 0.709(1) | 0.698(1) | 2.4985(1) | 2.4980(1) |
| 0.0 | 8     | 4              | 0.709(1) | 0.699(1) | 2.4985(1) | 2.4980(1) |
| 0.0 | 8     | 5              | 0.709(1) | 0.699(1) | 2.4985(1) | 2.4980(1) |
| 0.0 | 12    | 3              | 0.707(1) | 0.699(1) | 2.4982(1) | 2.4980(1) |
| 0.0 | 12    | 4              | 0.707(1) | 0.699(1) | 2.4982(1) | 2.4980(1) |
| 0.0 | 12    | 5              | 0.707(1) | 0.699(1) | 2.4982(1) | 2.4980(1) |
| 0.0 | 16    | 3              | 0.706(1) | 0.700(1) | 2.4980(1) | 2.4980(1) |
| 0.0 | 16    | 4              | 0.706(1) | 0.700(1) | 2.4980(1) | 2.4980(1) |
| 0.0 | 16    | 5              | 0.707(1) | 0.700(1) | 2.4980(1) | 2.4980(1) |

TABLE III: Results of $\nu$ and $g_c$ obtained from Bayesian analysis (and resampling) for the disordered Herringbone model with $p = 0.2$. The second and third columns are the minimum box size and the order of polynomials in $t L^{1/\nu}$ (including subleading correction) used in the analysis, respectively. The $\nu$ and $g_c$ shown in the 4th (5th) and 6th (7th) columns are determined from $Q_1$ ($Q_2$).

| $P$ | $L_m$ | $t L^{1/\nu}$ | $\nu$ | $g_c$ | $g_c$ |
|-----|-------|----------------|-------|-------|-------|
| 0.2 | 8     | 3              | 0.730(3) | 0.721(3) | 2.5314(3) | 2.5306(2) |
| 0.2 | 8     | 4              | 0.730(3) | 0.721(3) | 2.5314(3) | 2.5306(2) |
| 0.2 | 8     | 5              | 0.730(3) | 0.721(3) | 2.5314(3) | 2.5306(2) |
| 0.2 | 12    | 3              | 0.727(3) | 0.719(3) | 2.5310(4) | 2.5309(3) |
| 0.2 | 12    | 4              | 0.728(3) | 0.719(3) | 2.5311(4) | 2.5310(3) |
| 0.2 | 12    | 5              | 0.728(3) | 0.719(3) | 2.5309(4) | 2.5310(3) |
| 0.2 | 16    | 3              | 0.727(3) | 0.719(3) | 2.5309(4) | 2.5312(4) |
| 0.2 | 16    | 4              | 0.726(3) | 0.718(3) | 2.5309(4) | 2.5311(4) |
| 0.2 | 16    | 5              | 0.727(3) | 0.719(3) | 2.5310(4) | 2.5310(4) |

TABLE IV: Results of $\nu$ and $g_c$ obtained from Bayesian analysis (and resampling) for the disordered Herringbone model with $p = 0.3$. The second and third columns are the minimum box size and the order of polynomials in $t L^{1/\nu}$ (including subleading correction) used in the analysis, respectively. The $\nu$ and $g_c$ shown in the 4th (5th) and 6th (7th) columns are determined from $Q_1$ ($Q_2$).

| $P$ | $L_m$ | $t L^{1/\nu}$ | $\nu$ | $g_c$ | $g_c$ |
|-----|-------|----------------|-------|-------|-------|
| 0.3 | 8     | 3              | 0.757(6) | 0.745(6) | 2.5744(7) | 2.5738(5) |
| 0.3 | 8     | 4              | 0.758(6) | 0.748(6) | 2.5744(7) | 2.5739(5) |
| 0.3 | 8     | 5              | 0.756(6) | 0.745(6) | 2.5741(7) | 2.5737(5) |
| 0.3 | 12    | 3              | 0.759(6) | 0.748(6) | 2.5729(6) | 2.5730(5) |
| 0.3 | 12    | 4              | 0.759(6) | 0.750(6) | 2.5726(5) | 2.5729(5) |
| 0.3 | 12    | 5              | 0.762(6) | 0.751(6) | 2.5725(5) | 2.5727(5) |
| 0.3 | 16    | 3              | 0.760(6) | 0.751(6) | 2.5730(5) | 2.5739(6) |
| 0.3 | 16    | 4              | 0.763(6) | 0.751(6) | 2.5730(5) | 2.5739(6) |
| 0.3 | 16    | 5              | 0.761(6) | 0.752(6) | 2.5726(5) | 2.5739(6) |

TABLE V: Results of $\nu$ and $g_c$ obtained from Bayesian analysis (and resampling) for the disordered Herringbone model with $p = 0.4$. The second and third columns are the minimum box size and the order of polynomials in $t L^{1/\nu}$ (including subleading correction) used in the analysis, respectively. The $\nu$ and $g_c$ shown in the 4th (5th) and 6th (7th) columns are determined from $Q_1$ ($Q_2$).

| $P$ | $L_m$ | $t L^{1/\nu}$ | $\nu$ | $g_c$ | $g_c$ |
|-----|-------|----------------|-------|-------|-------|
| 0.4 | 8     | 3              | 0.787(5) | 0.773(4) | 2.6401(8) | 2.6402(7) |
| 0.4 | 8     | 4              | 0.785(5) | 0.772(5) | 2.6396(8) | 2.6402(7) |
| 0.4 | 8     | 5              | 0.780(5) | 0.767(5) | 2.6403(8) | 2.6404(7) |
| 0.4 | 12    | 3              | 0.790(5) | 0.778(5) | 2.6378(5) | 2.6384(6) |
| 0.4 | 12    | 4              | 0.789(5) | 0.777(5) | 2.6375(5) | 2.6382(5) |
| 0.4 | 12    | 5              | 0.783(5) | 0.773(5) | 2.6379(5) | 2.6385(5) |
| 0.4 | 16    | 3              | 0.791(5) | 0.780(5) | 2.6379(5) | 2.6386(6) |
| 0.4 | 16    | 4              | 0.790(5) | 0.780(5) | 2.6378(6) | 2.6386(6) |
| 0.4 | 16    | 5              | 0.784(5) | 0.778(5) | 2.6382(6) | 2.6388(6) |
| P  | L_{\text{m}} | tL^{1/\nu} | \nu | \nu | g_{c} | g_{c} |
|----|------------|-------------|-----|-----|------|------|
| 0.5 | 8       | 3           | 0.833(7) | 0.816(6) | 2.7397(9) | 2.7403(8) |
| 0.5 | 8       | 4           | 0.824(7) | 0.809(7) | 2.7403(10) | 2.7406(9) |
| 0.5 | 8       | 5           | 0.818(7) | 0.806(7) | 2.7400(10) | 2.7408(9) |
| 0.5 | 12      | 3           | 0.832(7) | 0.815(6) | 2.7393(7) | 2.7405(9) |
| 0.5 | 12      | 4           | 0.818(7) | 0.808(7) | 2.7398(8) | 2.7409(9) |
| 0.5 | 12      | 5           | 0.818(7) | 0.805(7) | 2.7397(8) | 2.7407(9) |
| 0.5 | 16      | 3           | 0.826(8) | 0.811(7) | 2.7389(8) | 2.7406(9) |
| 0.5 | 16      | 4           | 0.817(7) | 0.805(7) | 2.7398(8) | 2.7410(9) |
| 0.5 | 16      | 5           | 0.814(8) | 0.806(7) | 2.7392(8) | 2.7404(9) |

TABLE VI: Results of $\nu$ and $g_{c}$ obtained from Bayesian analysis (and resampling) for the disordered Herringbone model with $p = 0.5$. The second and third columns are the minimum box size and the order of polynomials in $tL^{1/\nu}$ (including subleading correction) used in the analysis, respectively. The $\nu$ and $g_{c}$ shown in the 4th (5th) and 6th (7th) columns are determined from $Q_{1}$ ($Q_{2}$).

| P  | L_{\text{m}} | tL^{1/\nu} | \nu | \nu | g_{c} | g_{c} |
|----|------------|-------------|-----|-----|------|------|
| 0.9 | 8       | 3           | 1.086(9) | 1.039(7) | 4.854(14) | 4.836(14) |
| 0.9 | 8       | 4           | 1.090(10) | 1.045(8) | 4.839(15) | 4.824(13) |
| 0.9 | 8       | 5           | 1.083(10) | 1.043(8) | 4.833(15) | 4.822(14) |
| 0.9 | 12      | 3           | 1.042(9) | 1.021(9) | 4.823(14) | 4.852(17) |
| 0.9 | 12      | 4           | 1.042(9) | 1.021(8) | 4.807(11) | 4.832(16) |
| 0.9 | 12      | 5           | 1.040(9) | 1.021(9) | 4.812(14) | 4.828(17) |
| 0.9 | 16      | 3           | 1.023(10) | 1.013(9) | 4.803(13) | 4.855(17) |
| 0.9 | 16      | 4           | 1.022(9) | 1.011(9) | 4.802(13) | 4.840(17) |
| 0.9 | 16      | 5           | 1.024(10) | 1.010(9) | 4.807(14) | 4.841(19) |

TABLE VII: Results of $\nu$ and $g_{c}$ obtained from Bayesian analysis (and resampling) for the disordered Herringbone model with $p = 0.9$. The second and third columns are the minimum box size and the order of polynomials in $tL^{1/\nu}$ (including subleading correction) used in the analysis, respectively. The $\nu$ and $g_{c}$ shown in the 4th (5th) and 6th (7th) columns are determined from $Q_{1}$ ($Q_{2}$).

**More Data of $Q_{1}$ and $Q_{2}$**

In this section, more data for the disordered Herringbone and plaquette models are presented.
FIG. 5: $Q_1$ (top panel) and $Q_2$ (bottom panel) of various $L$ as functions of $P$ for the considered disordered plaquette model studied here. The dashed lines are added to guide the eye.