This paper proposes a general semantic framework for verifying programs with arbitrary monadic side-effects using Dijkstra monads, which we define as monad-like structures indexed by a specification monad. We prove that any monad morphism between a computational monad and a specification monad gives rise to a Dijkstra monad, which provides great flexibility for obtaining Dijkstra monads tailored to the verification task at hand. We moreover show that a large variety of specification monads can be obtained by applying monad transformers to various base specification monads, including predicate transformers and Hoare-style pre- and postconditions. For simplifying the task of defining correct monad transformers, we propose a language inspired by Moggi’s monadic metalanguage that is parameterized by a dependent type theory. We also develop a notion of Plotkin and Power’s algebraic operations for Dijkstra monads, together with a corresponding notion of effect handlers. We implement our framework in both Coq and F∗, and illustrate that it supports a wide variety of verification styles for effects such as partiality, exceptions, nondeterminism, state, and input-output.

1 INTRODUCTION

The aim of this paper is to provide a semantic framework for specifying and verifying programs with arbitrary side-effects modeled by computational monads (Moggi 1989). We base this framework on Dijkstra monads, which have proven valuable in practice for verifying effectful code (Swamy et al. 2016). A Dijkstra monad $D A w$ is a monad-like structure that classifies effectful computations returning values in $A$ and specified by $w : WA$, where $W$ is what we call a specification monad.1 A typical specification monad contains predicate transformers mapping postconditions to preconditions. For instance, for computations in the state monad $St A = S \rightarrow A \times S$, a natural specification monad is $W^{St} A = (A \times S \rightarrow P) \rightarrow (S \rightarrow P)$, mapping postconditions, which in this case are predicates on final results and states, to preconditions, which are predicates on initial states (here $P$ stands for the internal type of propositions). However, given an arbitrary monadic effect, how do we find such a specification monad? Is there a single specification monad that we can associate to each effect? If not, what are the various alternatives, and what are the constraints on this association for obtaining a proper Dijkstra monad?

A partial answer to this question was provided by the Dijkstra Monads for Free (DM4Free) approach of Ahman et al. (2017): from a computational monad defined as a term in a metalanguage called DM, a (single) canonical specification monad is automatically derived through a syntactic translation. Unfortunately, while this approach works for stateful and exceptional computations, it cannot handle several other effects, such as input-output (IO), due to various syntactic restrictions in DM.

To better understand and overcome such limitations, we make the novel observation that a computational monad in DM is essentially a monad transformer applied to the identity monad; and that the specification monad is obtained by applying this monad transformer to the continuation

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1 Prior work has used the term "Dijkstra monad" both for the indexed structure $D$ and for the index $W$ (Ahman et al. 2017; Jacobs 2014, 2015; Swamy et al. 2013, 2016). In order to prevent confusion, we use the term "Dijkstra monad" exclusively for the indexed structure $D$ and use the term "specification monad" for the index $W$. 
monad with answer type \(\mathbb{P}\), i.e., \(\text{Cont}_\mathbb{P}A = (A \rightarrow \mathbb{P}) \rightarrow \mathbb{P}\). Returning to the example of state, the monad \(W^\text{St}A\) can be obtained by applying the state monad transformer \(\text{StT} MA = S \rightarrow M(A \times S)\) to \(\text{Cont}_\mathbb{P}\). This reinterpretation of the \(\text{DM4Free}\) approach sheds light on its limitations: For a start, the class of supported computational monads is restricted to those that can be decomposed as a monad transformer applied to the identity monad. This rules out various effects such as nondeterminism or IO, for which no proper monad transformer is known (Hyland et al. 2007).

Further, obtaining both the computational and specification monad from the same monad transformer introduces a very tight coupling. In particular, \(\text{DM4Free}\) does not support the association of different specification monads to a given computational effect. For instance, the exception monad \(\text{Exc} A = A + E\) is associated by \(\text{DM4Free}\) to the specification monad \(W^\text{Exc} A = ((A + E) \rightarrow \mathbb{P}) \rightarrow \mathbb{P}\), i.e., the exception monad transformer \(\text{ExcT} MA = M(A + E)\) applied to \(\text{Cont}_\mathbb{P}\). This specification monad requires the postcondition to deal with both the success and the failure cases. While this is desirable in many cases, at times it may be more convenient to use the simpler specification monad \(\text{Cont}_\mathbb{P}\) directly, allowing exceptions to be thrown freely, without having to explicitly allow this in specifications. Likewise, for IO, one may wish to have rich specifications that depend on the history of interactions with the external world, or simpler context-free specifications in which verification conditions are as local as possible. In general, one should have the freedom to choose a specification monad that is expressive enough for the verification task at hand, but also simple enough so that the verification effort is manageable in practice.

Moreover, even for a fixed computational monad and a fixed specification monad there can be more ways to associate the two in a Dijkstra monad. For instance, to specify exceptional computations using \(\text{Cont}_\mathbb{P}\), we could allow all exceptions to be thrown freely—as explained above, which corresponds to a partial correctness interpretation—but a different choice is to prevent any exceptions from being raised at all—which corresponds to a total correctness interpretation. Similarly, for specifying nondeterministic computations, two interpretations are possible for \(\text{Cont}_\mathbb{P}\): a demonic one, in which the postcondition should hold for all possible result values (Dijkstra 1975), and an angelic one, in which the postcondition should hold for some possible result value (Floyd 1967).

The key idea of this paper is to decouple the computational monad and the specification monad: instead of insisting on deriving both from the same monad transformer as in \(\text{DM4Free}\), we consider them independently and only require that they are related by a monad morphism, i.e., a mapping between two monads that respects their monadic structure. For instance, a monad morphism from nondeterministic computations could map a finite set of possible outcomes to a predicate transformer in \((A \rightarrow \mathbb{P}) \rightarrow \mathbb{P}\). Given a finite set \(R\) of results in \(A\), and a postcondition \(\text{post} : A \rightarrow \mathbb{P}\), there are only two reasonable ways to obtain a single proposition: either take the conjunction of \(\text{post} \,\forall \,\nu \,\in \,R\) (demonic nondeterminism), or the disjunction (angelic nondeterminism). For the case of IO, in our framework we can consider at least two monad morphisms relating the IO monad to two different specification monads, \(W^\text{Fr}\) and \(W^\text{Hist}\), where \(\mathcal{E}\) is the alphabet of IO events:

\[
\begin{align*}
W^\text{Fr} X &= (X \times \mathcal{E}^* \rightarrow \mathbb{P}) \rightarrow \mathbb{P} & \text{IO} & \rightarrow & W^\text{Hist} X &= (X \times \mathcal{E}^* \rightarrow \mathbb{P}) \rightarrow (\mathcal{E}^* \rightarrow \mathbb{P})
\end{align*}
\]

While both specification monads take postconditions of the same type (predicates on the final value and the produced IO events), the produced precondition of \(W^\text{Hist} X\) has an additional argument \(\mathcal{E}^*\), which denotes the history of already performed IO events.

This paper makes the following contributions:

- We propose a new semantic framework for verifying programs with arbitrary monadic effects using Dijkstra Monads. By decoupling the computational monad from the specification monad we remove all previous restrictions on the supported computational monads. Moreover, this decoupling allows us to flexibly choose the specification monad and monad morphism most suitable for the verification task at hand. We investigate a large variety of specification monads
that are obtained by applying monad transformers to various base monads, including predicate transformers (e.g., weakest preconditions and strongest postconditions) and Hoare-style pre- and postconditions. This flexibility allows a wide range of verification styles for partiality, nondeterminism, and IO—none of which was possible with DM4Free.

We give the first general definition of Dijkstra monads as monad-like structures indexed by a specification monad. We show that any monad morphism gives rise to a Dijkstra monad, and that from any such Dijkstra monad we can recover the monad morphism. More generally, we prove that there is an adjunction between Dijkstra monads and a generalization of monad morphisms we call monadic relations, and this adjunction induces the above-mentioned equivalence.

We recast DM4Free as a special case of our general semantic framework. For this we introduce SM, a principled metalanguage for defining correct-by-construction monad transformers. SM is inspired by DM and Moggi’s monadic metalanguage, but is parameterized by a dependent type theory. We show that SM terms give rise to correct monad transformers (satisfying all the usual laws) as well as canonical monadic relations, defined from a logical relation. This allows us to reap the benefits of the DM4Free construction when it works well (e.g., state, exceptions), and to explicitly provide monad morphisms when it does not (e.g., nondeterminism, IO).

We give an account of Plotkin and Power’s algebraic operations in the setting of Dijkstra monads. We show that a monad morphism equips both its specification monad and the corresponding Dijkstra monad with algebraic operations. We also develop a general notion of effect handlers for Dijkstra monads. This allows us to both provide a uniform treatment of DM4Free’s hand-rolled examples of exception handlers, and subsume the prior work on weakest exceptional preconditions (Leino and van de Snepscheut 1994; Sekerinski 2012).

We illustrate the generality of our semantic framework by applying it to the verification of monadic programs in both Coq and F*.

Outline. The paper is organized as follows: We start by briefly reviewing the use of monads in effectful programming and the closest related approaches for reasoning about such programs (§2). We then give a gentle overview of our approach through illustrative examples (§3). After this, we dive into the technical details: First, we explain how we can obtain a wide range of specification monads by applying monad transformers defined in SM to various base specification monads (§4). Then, we show the tight and natural correspondence between Dijkstra monads and monadic relations (§5). We also study algebraic operations and effect handlers for Dijkstra monads (§6). Finally, we present our implementations of these ideas in F* and Coq (§7), before discussing related (§8) and future work (§9). The supplementary materials for this paper include: (1) verification examples and the implementation of our framework in F* (https://github.com/FStarLang/FStar/tree/guido_effects/examples/dm4all); and (2) verification examples and a formalization of SM in Coq (https://gitlab.inria.fr/kmaillar/dijkstra-monads-for-all).

2 BACKGROUND: MONADS AND MONADIC REASONING

We start by briefly reviewing the use of monads in effectful programming, as well as the closest related approaches for verifying monadic programs.

2.1 The Monad Jungle Book

Side effects are an important part of programming. They arise in a multitude of shapes, be it imperative algorithms, nondeterministic operations, potentially diverging computations, or interactions with the external world. These various effects can be uniformly captured by the algebraic gadget known as a computational monad (Benton et al. 2000; Moggi 1989). This uniform interface is provided via a type \( MA \) of computations returning values of type \( A \). A function \( \text{ret}^M : A \to MA \)
coerces a value \( v : A \) to a trivial computation, for instance seeing \( v \) as a stateful computation leaving the state untouched, and a function \( \text{bind}^M m f \) sequentially composes the monadic computations \( m : MA \) with \( f : A \rightarrow MB \), for instance threading through the state. Equations specify that \( \text{ret}^M \)
does not have any computational effect, and that sequencing is associative.

The generic monad interface \((\text{ret}^M, \text{bind}^M)\) is, however, not enough to write programs that exploit the underlying effect. To this end, each computational monad also comes with operations for causing effects. We briefly recall a few examples of computational monads and their operations:

**Partiality:** A simple model for partial computations is given by adding a new element that expresses divergence, i.e., \( \text{Div} \) \( A = A + \{\perp\} \). Returning a value \( v \) is the obvious injection, while sequencing \( m \) with \( f \) is given by applying \( f \) to \( m \) if \( m \) is a terminating value, or \( \perp \) if \( m \) was already diverging. A partial computation can diverge with the operation \( \Omega : \text{Div} \not\rightarrow \), implemented as \( \Omega = \text{inr} \perp \). While here we stick to this simple model of partiality, more complex models would also allow to define fixpoints (Altenkirch et al. 2017; McBride 2015).

**Exceptions:** A computation with exceptions of type \( E \) can be represented by the monad \( \text{Exc} A = A + E \); returning and sequencing are similar to the partiality effect above. The operation \( \text{throw} : E \not\rightarrow \text{Exc} 0 \) is defined by right injection of \( E \) in \( \text{Exc} 0 = 0 + E \).

**State:** A stateful computation can be modeled as a state passing function, i.e., \( \text{St} A = A \rightarrow A \times S \), where \( S \) is the type of the state. Returning a value \( v \) is the function \( \lambda s. \langle v, s \rangle \) that produces the value \( v \) and the unmodified state, whereas binding \( m \) to \( f \) is obtained by threading through the state, i.e., \( \lambda s. \text{let} \langle v, s' \rangle = m \in f \circ v \circ s' \). The state monad comes with operations \( \text{get} : \text{St} S = \lambda s. \langle v, s \rangle \) to retrieve the state, and \( \text{put} : S \rightarrow \text{St} 1 = \lambda s. \lambda s'. \langle *, s \rangle \) to overwrite it.

**Nondeterminism:** A nondeterministic computation can be represented by a finite set of possible outcomes, i.e. \( \text{NDet} A = \mathcal{P}_f (A) \). Returning a value \( v \) is provided by the singleton \( \{v\} \), whereas sequencing \( m \) with \( f \) amounts to forming the union \( \bigcup_{v \in m} f \circ v \). This monad comes with an operation \( \text{pick} : \text{NDet} \not\rightarrow = \{ \text{true}, \text{false} \} \), which nondeterministically chooses a boolean value, and an operation \( \text{fail} : \text{NDet} 0 = \emptyset \), which unconditionally fails. One can nondeterministically choose an element of a finite set by repeatedly applying \( \text{pick} \).

**Interactive input-output (IO):** An interactive computation with input type \( I \) and output type \( O \) can be represented by the inductively defined monad \( \text{IO} A = \mu Y.A + (I \rightarrow Y) + O \times Y \), which describes three possible kinds of computations: either return a value \( (A) \), expect to receive an input and then continue \( (I \rightarrow Y) \), or output and continue \( (O \times Y) \). Returning \( v \) is constructing a leaf, whereas sequencing \( m \) with \( f \) amounts to tree grafting: replacing each leaf with value \( a \) in \( m \) with the tree \( f a \). The operations for IO are \( \text{read} : \text{IO} I \) and \( \text{write} : O \not\rightarrow \text{IO} 1 \).

### 2.2 Reasoning About Computational Monads

Many approaches have been proposed for reasoning about effectful programs written in monadic style; we review the ones closest to our work. In an imperative setting, Hoare introduced a program logic to reason about properties of programs (Hoare 1969). The judgments of this logic are \( \text{Hoare triples} \{ \text{pre} \} \ c \{ \text{post} \} \). Intuitively, if the precondition \( \text{pre} \) is satisfied, then running the program \( c \) will leave us in a situation where \( \text{post} \) is satisfied, provided that \( c \) terminates. For imperative programs—i.e., statements changing the program’s state—\( \text{pre} \) and \( \text{post} \) are predicates over states.

Hoare’s approach can be directly adapted to the monadic setting by replacing imperative programs \( c \) with monadic computations \( m : MA \). This approach was first proposed in Hoare Type Theory (Nanevski et al. 2008b), where a Hoare monad of the form \( \text{HST} \ \text{pre} \ A \ \text{post} \) augments the state monad over \( A \) with a precondition \( \text{pre} : S \not\rightarrow \not\rightarrow \) and postcondition \( \text{post} : A \times S \not\rightarrow \not\rightarrow \). So while preconditions are still predicates over initial states, postconditions are now predicates over both
final states and results. While this approach was successfully extended to a few other effects (Delbianco and Nanevski 2013; Nanevski et al. 2008a, 2013), there is still no general story on how to define a Hoare monad or even just the shape of pre- and postconditions for an arbitrary effect.

A popular alternative to proving properties of imperative programs is Dijkstra’s weakest precondition calculus (Dijkstra 1975). The main insight of this calculus is that we can typically compute a weakest precondition \( \text{wp}(c, \text{post}) \) such that \( \text{pre} \Rightarrow \text{wp}(c, \text{post}) \) if and only if \( \{ \text{pre} \} \ c \ \{ \text{post} \} \), and therefore partly automate the verification process by reducing it to a logical decision problem. Recently, Swamy et al. (2013) observed that it was possible to adopt Dijkstra’s technique to ML programs with state and exceptions elaborated to monadic style, proposing a notion of Dijkstra monad of the form \( \text{DST} \ A \ \text{wp} \), where \( \text{wp} \) is a predicate transformer that specifies the behavior of the monadic computation. These predicate transformers are represented as functions that given a postcondition on the final state and the result or an exception of type \( E \), calculate a corresponding precondition on the initial state. Their predicate transformer type can be written as follows:

\[
\text{W}^{\text{ML}} = \frac{\text{postconditions}}{(A + E) \times S \to \mathbb{P})} \to (S \to \mathbb{P})
\]

In subsequent work, Swamy et al. (2016) extended this to programs that combine multiple sub-effects, computing more efficient weakest preconditions by using Dijkstra monads that precisely capture the actual effects of the code, instead of verifying everything using \( \text{W}^{\text{ML}} \) above. For example, pure computations are verified using a Dijkstra monad whose specifications have type:

\[
\text{W}^{\text{Pure}} A = \text{Cont}_{\mathbb{P}} A = (A \to \mathbb{P}) \to \mathbb{P},
\]

while stateful (but exception-free) computations are verified using specifications of type:

\[
\text{W}^{\text{St}} A = (A \times S \to \mathbb{P}) \to (S \to \mathbb{P}).
\]

More recently, Ahman et al.’s (2017) DM4Free work shows that these originally disparate specification monads can be derived in a unified way from computational monads in their DM metalanguage.

An important observation present in these works is that predicate transformers have a natural monadic structure. For instance, it is not hard to see that the predicate transformer type \( \text{W}^{\text{Pure}} \) is simply the continuation monad with answer type \( \mathbb{P} \), that \( \text{W}^{\text{St}} \) is the state monad transformer applied to \( \text{W}^{\text{Pure}} \), and that \( \text{W}^{\text{ML}} \) is the state and exceptions monad transformers applied to \( \text{W}^{\text{Pure}} \). It is this monadic structure that allows to easily write computations that carry their own specification, and, in the next section, we will see that it is also the basis for what we call a specification monad.

## 3 A GENTLE INTRODUCTION TO DIJKSTRA MONADS FOR ALL

In this section we introduce a few basic definitions and illustrate the main ideas of our framework on various relatively simple examples. We start from the observation that the kinds of specifications most commonly used in practice form ordered monads (§3.1). On top of this we define effect observations, as just monad morphisms between a computation and a specification monad (§3.2) and give various examples (§3.3). Finally, we explain how to use effect observations to obtain Dijkstra monads, and how to use Dijkstra monads for program verification (§3.4).

### 3.1 Specification Monads

The realization that predicate transformers form monads (Ahman et al. 2017; Jacobs 2014, 2015; Swamy et al. 2013, 2016) is the starting point to provide a uniform notion of specifications. Generalizing over prior work, we show that this is true not only for weakest precondition transformers, but also for strongest postconditions, and pairs of pre- and postconditions (see §4.1). Intuitively, elements of a specification monad can be used to specify properties of some computation, e.g., \( \text{W}^{\text{Pure}} \) can specify pure or nondeterministic computations, and \( \text{W}^{\text{St}} \) can specify stateful computations.
The specification monads we consider are ordered. Formally, a monad $W$ is ordered when $W A$ is equipped with a preorder $\leq W A$ for each type $A$ and $\text{bind}^W$ is monotonic in both arguments:

\[ \forall w_1 \leq W A, w_2 : A \rightarrow W B, (\forall x : A, w_2 x \leq W B w_2') \Rightarrow \text{bind}^W w_1 w_2 \leq W B \text{bind}^W w_1' w_2' \]

This order allows specifications to be compared as being more or less precise. For example, for the specification monads $W^{\text{Pure}}$ and $W^{\text{St}}$, the ordering is given by

\[ w_1 \leq w_2 : W^{\text{Pure}} A \iff \forall p : A \rightarrow \mathbb{P}, w_1 p \Rightarrow w_2 p \]
\[ w_1 \leq w_2 : W^{\text{St}} A \iff \forall (p : A \times S \rightarrow \mathbb{P})(s : S), w_1 p s \Rightarrow w_2 p s \]

For $W^{\text{Pure}}$ and $W^{\text{St}}$ to actually constitute ordered monads, it turns out that we need to restrict our attention to monotonic predicate transformers, i.e., those mapping (pointwise) stronger postconditions to stronger preconditions, rather than considering the full continuation monad $\text{Cont}_\mathbb{P}$ for specifications. This technical condition, quite natural from the point of view of verification, will be assumed implicitly for all the predicate transformers, and will be studied in detail in §4.1.

How can we construct specification monads? A simple and powerful way is to apply monad transformers to existing specification monads. For instance, applying the exception transformer $\text{ExcT} M A = M (A + E)$ to $W^{\text{Pure}}$ we obtain

\[ W^{\text{Exc}} A = \text{ExcT} W^{\text{Pure}} A = ((A + E) \rightarrow \mathbb{P}) \rightarrow \mathbb{P} \equiv (A \rightarrow \mathbb{P}) \rightarrow (E \rightarrow \mathbb{P}) \rightarrow \mathbb{P} \]

$W^{\text{Exc}}$ is a natural specification monad for programs that can raise exceptions, transporting a normal postcondition $A \rightarrow \mathbb{P}$ and an exceptional postcondition $E \rightarrow \mathbb{P}$ to a precondition. Further specification monads using this idea will be introduced along with the examples in §3.3.

### 3.2 Effect Observations

Now that we have a presentation of specifications as elements of a monad, we can relate computational monads to such specifications. Since an object relating computations to specifications provides a particular insight on the potential effects of the computation, they have been called effect observations (Katsumata 2014). As explained in §1, a computational monad can have effect observations into multiple specification monads, or multiple effect observations into a single specification monad. Using the exception monad $\text{Exc}$ as running example of a computational monad, we argue that monad morphisms provide a natural notion of effect observation in our setting, and we provide example monad morphisms supporting this claim. Further examples are explored in §3.3.

**Effect observations are monad morphisms.** As explained in §2.1, computations raising exceptions can be modeled by monadic expressions $m : \text{Exc} A = A + E$. A natural way to specify $m$ is to consider the specification monad $W^{\text{Exc}} A = (A + E) \rightarrow \mathbb{P}$ and to map $m$ to the predicate transformer $\theta^{\text{Exc}}(m) = \lambda p. p m : W^{\text{Exc}} A$, applying the postcondition $p$ to the computation $m$.

The mapping $\theta^{\text{Exc}} : \text{Exc} \rightarrow W^{\text{Exc}}$ relating the computational monad $\text{Exc}$ and the specification monad $W^{\text{Exc}}$ is parametric in the return type $A$ and it verifies two important properties with respect to the monadic structures of $\text{Exc}$ and $W^{\text{Exc}}$. First, a returned value is specified by itself:

\[ \theta^{\text{Exc}}(\text{ret}^{\text{Exc}} v) = \theta^{\text{Exc}}(\text{inl} v) = \lambda p. p (\text{inl} v) = \text{ret}^{W^{\text{Exc}}} v \]

and second, $\theta$ preserves the sequencing of computations:

\[ \theta^{\text{Exc}}(\text{bind}^{\text{Exc}} (\text{inl} v) f) = \theta^{\text{Exc}}(f v) = \text{bind}^{W^{\text{Exc}}} (\text{ret}^{W^{\text{Exc}}} v) (\theta^{\text{Exc}} \circ f) = \text{bind}^{W^{\text{Exc}}} \theta^{\text{Exc}}(\text{inl} v) (\theta^{\text{Exc}} \circ f) \]
\[ \theta^{\text{Exc}}(\text{bind}^{\text{Exc}} (\text{inr} e) f) = \theta^{\text{Exc}}(\text{inr} e) = \text{bind}^{W^{\text{Exc}}} \theta^{\text{Exc}}(\text{inr} e) (\theta^{\text{Exc}} \circ f) \]

These properties together prove that $\theta^{\text{Exc}}$ is a monad morphism. More importantly, they allow us to compute specifications from computations compositionally, e.g., the specification of $\text{bind}$ can be computed from the specifications of its arguments. This leads us to the following definition:
When specifying and verifying monadic programs there is generally a large variety of options we would have an unsatisfiable precondition. However, as outlined in §1, a simpler solution is possible. This effect observation gives a total correctness monad. The following examples illustrate the possibilities in such cases.

### 3.3 Examples of Effect Observations

When specifying and verifying monadic programs there is generally a large variety of options regarding both the specification monads and the effect observations. We will now consider a few of the computational monads from §2.1, and present various natural effect observations. We will now consider a few of the computational monads from §2.1, and present various natural effect observations.

**Specification monads are not canonical.** When writing a program using the exception monad, we may want to write pure sub-programs that actually do not raise exceptions. In order to make sure that these sub-programs are pure, we could use the previous specification monad and restrict ourselves to postconditions that map exceptions to false (⊥): hence raising an exception would have an unsatisfiable precondition. However, as outlined in §1, a simpler solution is possible. Taking as specification monad \( W^{\text{Pure}} A = (A \rightarrow P) \rightarrow P \), we can define the following effect observation \( \theta^\bot : \text{Exc} \rightarrow W^{\text{Pure}} \)

\[
\theta^\bot (\text{inl} \, v) = \lambda p. \, p \, v, \quad \theta^\bot (\text{inr} \, e) = \lambda p. \bot.
\]

This effect observation gives a total correctness interpretation to exceptions, which prevents them from being raised at all. Thus we have effect observations from Exc to two different specification monads \( W^{\text{Exc}} \) and \( W^{\text{Pure}} \).

**Effect observations are not canonical.** Looking closely at the effect observation \( \theta^\bot \), it is clear that we made a rather arbitrary choice when mapping an exception \( \text{inr} \, e \) to \( \bot \). Mapping \( \text{inr} \, e \) to true (\( \top \)) instead also gives us an effect observation \( \theta^\top : \text{Exc} \rightarrow W^{\text{Pure}} \). This effect observation assigns a trivial precondition to the \( \text{throw} \) operation, providing a partial correctness interpretation: given a program \( m : \text{Exc} \, A \) and a postcondition \( p : A \rightarrow P \), if \( \theta^\top (m)(p) \) is satisfiable and \( m \) evaluates to \( \text{inl} \, v \) then \( p \, v \) holds; but \( m \) may also raise any exception instead. Thus, \( \theta^\top, \theta^\bot : \text{Exc} \rightarrow W^{\text{Pure}} \) are two natural effect observations into the same specification monad. Even more generally, we can vary the choice for each exception; in fact, the effect observations from Exc into \( W^{\text{Pure}} \) are in one-to-one correspondence with functions of type \( E \rightarrow P \) (see §4.3).
possible outcomes, and a postcondition \( post : A \rightarrow \mathbb{P} \), we obtain a set \( P \) of propositions by applying \( post \) to each element of \( m \). There are then two reasonable ways to interpret this set of propositions as a proposition:

- we can take the conjunction \( \bigwedge_{p \in P} p \), which corresponds to the weakest precondition such that any outcome of \( m \) satisfies \( post \) (demonic nondeterminism); or
- we can take the disjunction \( \bigvee_{p \in P} p \), which corresponds to the weakest precondition such that some outcome of \( m \) satisfies \( post \) (angelic nondeterminism).

To see that these two choices both lead to monad morphisms \( \theta^\ast, \theta^\exists : \text{NDet} \rightarrow \text{Cont}_\mathbb{P} \), it is enough to check that taking the conjunction when \( P = \{ p \} \) is a singleton is equivalent to \( p \), and that a conjunction of conjunctions \( \bigwedge_{a \in A} \bigwedge_{p \in P_a} p \) is equivalent to a conjunction on the union of the ranges \( \bigwedge_{p \in \bigcup_{a \in A} p_a} p \) and similarly for disjunctions. Both conditions are straightforward to check.

**Interactive Input-Output.** Let us now consider programs in the IO monad from §2.1. We want to define an effect observation \( \theta : \text{IO} \rightarrow W \) to some specification monad \( W \) to be determined. A first thing to note is that since no equations constrain the \( \text{read} \) and \( \text{write} \) operations, we can specify their interpretations \( \theta(\text{read}) : W I \) and \( \forall_o : O, \theta(\text{write} o) : W 1 \) independently from each other.

Simple effect observations for IO can already be provided using the specification monad \( W^{\text{Pure}} \). The interpretation of the \( \text{write} \) operation in this simple case needs to provide a result in \( \mathbb{P} \) from an output element \( o : O \) and a postcondition \( p : 1 \rightarrow \mathbb{P} \). Besides returning a constant proposition (like for \( \theta^\ast, \theta^T \) in §3.2), a reasonable interpretation is to forget the \( \text{write} \) operation and return \( p \ast \) (where \( \ast \) is the unit value). For the definition of \( \theta(\text{read}) : (I \rightarrow \mathbb{P}) \rightarrow \mathbb{P} \), we are given a postcondition on the possible inputs \( post : I \rightarrow \mathbb{P} \) and need to build a proposition. Two canonical solutions are to take either the universal quantification \( \forall_i : I, \text{post} i \), meaning that there exists some input such that the program’s continuation satisfies the postcondition, analogously to the two modalities of evaluation logic (Moggi 1995; Pitts 1991).

To get more interesting effect observations accounting for reads and writes we can, for instance, extend \( W^{\text{Pure}} \) with ghost state (Owicki and Gries 1976) capturing the list of executed IO events.\(^2\) This is achieved by applying the state monad transformer with state type \( \mathcal{E} \) to \( W^{\text{Pure}} \), obtaining a specification monad \( W^{\text{HistST}} A = (A \times \text{list} \mathcal{E} \rightarrow \mathbb{P}) \rightarrow \text{list} \mathcal{E} \rightarrow \mathbb{P} \). We can provide interpretations of the \( \text{read} \) and \( \text{write} \) operations that access this ghost state to keep track of the history of events:

\[
\theta^{\text{HistST}}(\text{write} o) = \lambda (p : 1 \times \text{list} \mathcal{E} \rightarrow \mathbb{P})(\text{log} : \text{list} \mathcal{E}). p (\ast, (\text{Out} o)) :: \text{log} : W^{\text{HistST}}(1)
\]

\[
\theta^{\text{HistST}}(\text{read}) = \lambda (p : I \times \text{list} \mathcal{E} \rightarrow \mathbb{P})(\text{log} : \text{list} \mathcal{E}). \forall_i, p (i, (\text{In} i)) :: \text{log} : W^{\text{HistST}}(I)
\]

This specification monad is however somewhat inconvenient in that postconditions are written over the *global* history of events, instead of over the events of the expression in question. Further, one can write specifications that “shrink” the global history of events, such as \( \lambda p. \text{log} . p (\ast, []) \), which no expression satisfies. For these reasons, we introduce an *update monad* (Ahman and Uustalu 2013) variant of \( W^{\text{HistST}} \), written \( W^{\text{Hist}} \), which provides a more concise way to describe the events. In particular, in \( W^{\text{Hist}} \) the postcondition can specify only the events produced by the expression, while the precondition is still free to specify any previously-produced events, allowing us to define:

\[
\theta^{\text{Hist}}(\text{write} o) = \lambda (p : 1 \times \text{list} \mathcal{E} \rightarrow \mathbb{P})(\text{log} : \text{list} \mathcal{E}). p (\ast, (\text{Out} o)) :: \text{log} : W^{\text{Hist}}(1)
\]

\[
\theta^{\text{Hist}}(\text{read}) = \lambda (p : I \times \text{list} \mathcal{E} \rightarrow \mathbb{P})(\text{log} : \text{list} \mathcal{E}). \forall_i, p (i, (\text{In} i)) :: \text{log} : W^{\text{Hist}}(I)
\]

While \( W^{\text{Hist}} = W^{\text{HistST}} \), the two monads differ in their \( \text{ret} \) and \( \text{bind} \) functions. For instance,

\[
\text{bind}^{\text{HistST}} w f = \lambda p. \text{log} . w (\lambda (x, \text{log}') . f \times p \text{log}') \text{log}
\]

\(^2\) Importantly, the ghost state only appears in specifications and not in user programs; these still use only (stateless) IO.
\(\text{bind}^{\text{Hist}} \ w \ f = \lambda p \ \text{log}. \ w (\lambda (x, \ \text{log'}) . f \ x (\lambda (y, \ \text{log''}) . p \ (y, \ \text{log'} \ + + \text{log''})) \ (\text{log} + + \text{log''})) \ \text{log}
\)

where the former overwrites the entire history, while the latter merely updates it with new events.

While \(\text{Hist}^{\text{Hist}}\) provides a good way to reason about IO, some IO programs do not depend on the context of past interactions. We can provide an even more parsimonious way to specify and verify such programs by applying a writer transformer to \(\text{Pure}^{\text{ Hist}}\). The resulting specification monad \(\text{Fr}^{\text{Hist}}\), allows for the following interpretations:
\[
\theta^{\text{Fr}}(\text{write} \ o) = \lambda (p : \text{I} \times \text{list E} \rightarrow \text{P}). \ p \ (\langle *, \ \text{Out} \ o \rangle) \ : \ \text{Fr}^{\text{Hist}}(\text{I}) \quad (1)
\]
\[
\theta^{\text{Fr}}(\text{read}) = \lambda (p : \text{I} \times \text{list E} \rightarrow \text{P}). \ \forall i, p \ (i, [\text{In} \ i]) \ : \ \text{Fr}^{\text{Hist}}(I) \quad (2)
\]

This is in fact a special case of \(\text{Hist}^{\text{Hist}}\) where the history is taken to be \(\text{I}\) (Ahman and Uustalu 2013).

In fact, there is even more variety possible here, e.g., it is straightforward to write specifications that speak only of write events and not read events, and vice versa. It is also easy to extend this style of reasoning to combinations of IO and other effects. For instance, we can simultaneously reason about state changes and IO events by considering computations in \(\text{IO}^{\text{Hist}} A = S \rightarrow \text{IO}(A \times S)\), resulting from applying the state monad transformer to IO, together with the specification monad \(\text{Fr}^{\text{IO}^{\text{Hist}}} A = (A \times S \times \text{list E} \rightarrow \text{P}) \rightarrow S \rightarrow \text{list E} \rightarrow \text{P}\). As such, we recover the style proposed by Malecha et al. (2011), though they also cover separation logic, which we leave as future work.

Being able to choose between specification monads and effect observations allows one to keep the complexity of the specifications low when the properties are simple, yet increase it if required.

### 3.4 Recovering Dijkstra Monads

We now return to Dijkstra monads, which provide a practical and automatable verification technique in dependent type theories like \(\text{F}^*\) (Swamy et al. 2016), where they are a primitive notion, and Coq, where they can be embedded via dependent types. We explain how a Dijkstra monad can be obtained from a computational monad, a specification monad, and an effect observation relating them. Then we show how the obtained Dijkstra monads can be used for actual verification.

Let us start with stateful computations as an illustrating example, taking the computational monad \(\text{St}\), the specification monad \(\text{Fr}^{\text{St}}\), and the following effect observation:
\[
\theta^{\text{St}} : \ \text{St} \rightarrow \text{Fr}^{\text{St}}
\]
\[
\theta^{\text{St}}(m) = \lambda \text{post} \ s_0, \ \text{post} \ (m \ s_0)
\]

We begin by defining the Dijkstra monad type constructor, \(\text{ST} : (A : \text{Type}) \rightarrow \text{Fr}^{\text{St}} A \rightarrow \text{Type}\). The type \(\text{ST} A\) contains all those computations \(c : \text{St} A\) such that \(w\) "correctly specifies" \(c\). The meaning of "correctly specifies" is provided by the effect observation: we say that \(w\) correctly specifies \(c\) when \(w \leq \theta^{\text{St}}(c)\), that is, when \(w\) is weaker than (or equal to) the specification given from the effect observation. Unfolding the definitions of \(\leq\) and \(\theta^{\text{St}}\), this intuitively says that for any initial state \(s_0\) and postcondition \(\text{post} : A \times S \rightarrow \text{P}\), the precondition \(w \ \text{post} \ s_0\) computed by \(w\) is enough to ensure that \(m\) returns a value \(v : A\) and a final state \(s_1\) satisfying \(\text{post} \ (v, \ s_1)\); in other words, \(w \ \text{post} \ s_0\) implies the weakest precondition of \(m\).

The concrete definition for the type of a Dijkstra monad can vary according to the type theory in question. For instance, in our Coq development, we define it (roughly) as a dependent pair of a computation \(c : \text{St} A\) and a proof that \(c\) is correctly specified by \(w\). In \(\text{F}^*\), it is instead a primitive notion. In the rest of this section, we shall not delve into such representation details.

The Dijkstra monad \(\text{ST}\) is equipped with monad-like functions \(\text{ret}^{\text{ST}}\) and \(\text{bind}^{\text{ST}}\) whose definitions come from the computational monad \(\text{St}\), while their specifications come from the specification
monad \( W^{St} \). The general shape for the \( \text{ret} \) and \( \text{bind} \) of the obtained Dijkstra monad is:\(^3\)

\[
\begin{align*}
\text{ret}^{St} & = \text{ret}^{St} : (v : A) \rightarrow ST A (\text{ret}^{W^{St}} v) \\
\text{bind}^{St} & = \text{bind}^{St} : (c : ST A w_c) \rightarrow (f : (x : A) \rightarrow ST B (w_f x)) \\
& \rightarrow ST B (\text{bind}^{W^{St}} w_c w_f)
\end{align*}
\]

which, after unfolding the state-specific definitions becomes:

\[
\begin{align*}
\text{ret}^{St} & = \text{ret}^{St} : (v : A) \rightarrow ST A (\lambda \text{post} s_0. \text{post} \langle v, s_0 \rangle) \\
\text{bind}^{St} & = \text{bind}^{St} : (c : ST A w_c) \rightarrow (f : (x : A) \rightarrow ST B (w_f x)) \\
& \rightarrow ST B (\lambda p s_0. w_c (\lambda \langle x, s_1 \rangle. w_f x p s_1) s_0)
\end{align*}
\]

The operations of the computational monad are also reflected into the Dijkstra monad. Their specifications are simply computed by the effect observation. Namely, for an operation \( op^{St} \) of type \((x_1 : A_1) \rightarrow \cdots \rightarrow (x_n : A_n) \rightarrow ST B\), we can define:

\[
\begin{align*}
\text{op}^{St} & = \text{op}^{St} : (x_1 : A_1) \rightarrow \cdots \rightarrow (x_n : A_n) \rightarrow ST B (\text{post}(\text{op}^{St} x_1 \ldots x_n))
\end{align*}
\]

Concretely, for state, we get the following operations for the Dijkstra monad:

\[
\begin{align*}
\text{get} & : ST S (\lambda p s_0. p \langle s_0, s_0 \rangle), \\
\text{put} & : (s : S) \rightarrow ST \top (\lambda p s_0. p \langle *, s \rangle).
\end{align*}
\]

Given this refined version of the ordinary state monad, computing specifications of (non-recursive) programs becomes simply a matter of type inference. For instance, given

\[
\text{modify} (f : S \rightarrow S) = \text{bind}^{St} \text{get} (\lambda x. \text{put}(fx))
\]

we can compute the type

\[
ST \top (\text{bind}^{W^{St}} (\lambda p s_0. p \langle s_0, s_0 \rangle) (\lambda s p s_0. p \langle *, f s \rangle)) = ST \top (\lambda p s_0. p \langle *, f s_0 \rangle).
\]

which correctly describes the behavior of the program, both in terms of the returned value and of its effect on the state.

This construction is independent from how the computational monad, the specification monad, and the effect observation were obtained, and the same approach can be followed for the NDet monad coupled with any of its effect observations. We take the demonic one here, for which the \( \text{pick} \) and \( \text{fail} \) actions for the Dijkstra monad have types:

\[
\text{pick}^{ND^*} : ND^* \oplus (\lambda p. p \text{ true } \land p \text{ false}) \quad \text{fail}^{ND^*} : ND^* \oplus (\lambda p. \top)
\]

With this, we can define and verify \( F^* \) functions like the following:

\[
\begin{align*}
\text{let} \ \text{rec} & \ \text{pick}\_\text{from} ((\text{list } a) : ND a (\lambda p \rightarrow \forall x. \text{ elem } x l \Longrightarrow p x) = \\
& \ \text{match} \ l \ \text{with} \ \text{|} \ \text{|} \ \longrightarrow \ \text{fail} \ () \ | x : x s \rightarrow \text{if} \ p () \ \text{then} \ x \ \text{else} \ \text{pick}\_\text{from} \ x s) \\
\text{let} \ \text{guard} (b:\text{bool}) : ND \text{ unit} (\lambda p \rightarrow b \Longrightarrow p () = \text{if} b \ \text{then} () \ \text{else} \ \text{fail} ()
\end{align*}
\]

The first one nondeterministically chooses an element from a list, guaranteeing in its specification that the value indeed belongs to it, and the second checks that a given boolean condition holds, failing otherwise. The specification for \text{guard} ensures that the provided boolean is true in the continuation. Using these two routines, we can write and verify concise nondeterministic programs. For instance, the program below computes Pythagorean triples. Our specification simply says that every result (if any!) is a Pythagorean triple, while in the implementation we have some concrete bounds for the search:

\[
\begin{align*}
\text{let} & \ pyths () : ND (\int \& \int \& \int) (\lambda p \rightarrow \forall x y z. x \cdot x + y \cdot y = z \cdot z \Longrightarrow p (x,y,z)) = \\
& \ \text{let} \ l = [1;2;3;4;5;6;7;8;9;10] \ \text{in} \\
& \ \text{let} \ (x,y,z) = (\text{pick}\_\text{from} l, \text{pick}\_\text{from} l, \text{pick}\_\text{from} l) \ \text{in} \ \text{guard} (x \cdot x + y \cdot y = z \cdot z); (x,y,z)
\end{align*}
\]

\(^3\text{Note that if the representation of the Dijkstra monad is, say, dependent pairs, then the code here does not typecheck as-is and requires some tweaking. For this section we will assume they are defined as refinements of the computational monad, without any explicit proof terms to carry around. In our Coq implementation we use \text{Program} and \text{evars} to hide such details.}\)
For IO, the story is similar, and constructing a Dijkstra monad for it is again straightforward given the effect observation. As a first example, let us take the context-free interpretation \( \theta^\text{Fr} : \text{IO} \rightarrow W^\text{Fr} \) (1), for which the IO operations have the following interface:

\[
\begin{align*}
\text{write}^{\text{IOFr}} & : (o : O) \rightarrow \text{IOFree} \ (\lambda p. \ p \ (\ast, [\text{Out } o])) \\
\text{read}^{\text{IOFr}} & : \text{IOFree} \ (\lambda p. \ \forall (i : I), p (i, [\text{In } i]))
\end{align*}
\]

We can define and specify a program that duplicates its input (assuming an implicit coercion \( \theta \) whereby given the effect observation. As a first example, let us take the context-free interpretation

\[
\text{let } \text{print_increasing } (\lambda i : \text{Int}) = \text{IOFree } (\lambda h. \ \forall i, p (i, [\text{In } i]))
\]

However, with this specification monad, we cannot reason about the history of previous IO events. To overcome this issue, we can switch the specification monad to \( W^{\text{Hist}} \) and obtain

\[
\begin{align*}
\text{read}^{\text{IOHist}} & : \text{IOHist} \ (\lambda p. \ \forall i, p (i, [\text{In } i])) \\
\text{write}^{\text{IOHist}} & : (o : O) \rightarrow \text{IOHist} \ (\lambda p. \ p \ (\ast, [\text{Out } o]))
\end{align*}
\]

The computational part of this Dijkstra monad fully coincides with that of \( \text{IOFr} \), but the specifications are much richer. For instance, we can define the following computation:

\[
\text{mustHaveOccurred } = \lambda _. \ \text{ret}^{\text{IOHist}} * : (\text{IOHist} \ i) \rightarrow \text{IOHist} \ (\lambda p. \ p \ (\ast, [\text{Out } o]))
\]

which has no effect, yet requires that a given value \( o \) was already output before it is called. This is weakening the specification of \( \text{ret}^{\text{IOHist}} * \) (namely \( \text{ret}^{W^{\text{Hist}}} * = \lambda p. \ p \ (\ast, [\text{Out } o]) \)) to have a stronger precondition. By having this amount of access to the history one can verify that certain invariants are respected. For instance, the following program will verify successfully:

\[
\text{let } \text{print_increasing } (\lambda i : \text{Int}) = \text{IOHist } (\lambda \ p \ h \ \forall i, p (i, [\text{In } i]))
\]

The program has a “trivial” specification: it does not guarantee anything about the trace of events, nor does it put restrictions on the previous log. However, internally, the call to \( \text{mustHaveOccurred} \) has a precondition that \( i \) was already output, which can be proven from the postcondition of write \( i \).

If this write is removed, the program will (rightfully) fail to verify.

Finally, when considering the specification monad \( W^{\text{IOSt}} \), we have both state and IO operations:

\[
\begin{align*}
\text{read}^{\text{IOSt}} : \text{IOSt} \ (\lambda p, s, h. \ \forall i, p (i, [\text{In } i])) \\
\text{write}^{\text{IOSt}} : (\text{IOSt} \ i) \rightarrow \text{IOSt} \ (\lambda p, s, \text{Out } o) \\
\text{put}^{\text{IOSt}} : (s : S) \rightarrow \text{IOSt} \ (\lambda p, s, \text{Out } o)
\end{align*}
\]

where \( \text{read}^{\text{IOSt}}, \text{write}^{\text{IOSt}} \) keep state unchanged, and \( \text{get}^{\text{IOSt}}, \text{put}^{\text{IOSt}} \) do not perform any IO. With this, we can write and verify programs that combine state and IO in non-trivial ways, e.g.,

\[
\text{let } \text{do_io_then_roll_back_state }: \text{IOSt} \ (\lambda \ p \ h \ \forall i, p (i, [\text{In } i]; \text{Out } (s \ast i))) =
\]

The program mutates the state in order to compute output from input, possibly interleaved with pure computations, but eventually rolls it back to its initial value, as described in its specification.

In all three examples above, we have obtained a Dijkstra monad via the same recipe from the application of a monad transformer to the monad of predicate transformers \( \text{Cont}_\text{Fr} \). It is reminiscent of a recipe that already works well for computational monads: to get a useful

4 DEFINING SPECIFICATION MONADS

To express various verification styles, in §3 we introduced multiple examples of specification monads arising from the application of a monad transformer to the monad of predicate transformers \( \text{Cont}_\text{Fr} \).
specification monad, we can stack monad transformers on top of a basic specification monad. In this section we start by studying a few basic specification monads (§4.1). We then present our specification metalanguage SM, as a means for defining correct-by-construction monad transformers (§4.2). SM is a more principled variant of the DM language by Ahman et al. (2017), and similarly to DM, we give SM a semantics based on logical relations. Observing that not all SM terms give rise to monad transformers, we also carefully study the algebraic properties of this semantics, extracting conditions under which we are guaranteed to obtain monad transformers, and providing an explanation for the somewhat artificial syntactic restrictions in DM. Finally, we illustrate a specific way to define effect observations from monad algebras (§4.3).

4.1 Basic Specification Monads

We consider four kinds of basic specification monads, all of which are ordered in the sense of §3.1:

**Predicate monad.** Arguably the simplest way to specify a computation is to provide a postcondition on its outcomes. This can be done by considering the specification monad $P\text{red}(X) = X \to P$ (the covariant powerset monad) with order $p_1 \leq_{P\text{red}} p_2 \iff \forall x : X, p_2 x \Rightarrow p_1 x$. To specify the behavior of returning values, we can always map a value $v : X$ to the singleton predicate $\text{ret}_{P\text{red}} v = \lambda y. (y = v) : P\text{red} X$. And given a predicate $p : P\text{red}(X)$ and a function $f : X \to P\text{red}(Y)$, the predicate on $Y$ defined by $\text{bind}_{P\text{red}} p f = \lambda y. \exists x, p x \land f x y$ specifies the behavior of sequencing two computations, where the first computation outputs a value $x$ satisfying $p$ and under this assumption the second computation outputs a value satisfying $f x$. Note that while a specification $p : P\text{red} X$ provides information on the outcome of the computation, it cannot require Preconditions, so computations need to be defined independently of the logical context. In order to specify programs with non-trivial preconditions, for instance specifying that the division function $\text{div} x y$ requires $y$ to be non-zero, one must introduce more expressive specification monads.

**Pre-/postcondition monad.** To overcome this limitation, we study the monad of pre- and postconditions $P\text{prePost}(X) = P \times (X \to P)$, bundling a precondition together with a postcondition. Here the behavior of returning a value $v : X$ is specified by requiring a trivial precondition and ensuring as above a singleton predicate: $\text{ret}_{P\text{prePost}} v = (\top, \lambda y. y = v) : P\text{prePost}(X)$. And, given $p = (\text{pre}, \text{post}) : P\text{prePost}(X)$ and a function $f = \lambda x. \langle \text{pre}' x, \text{post}' x \rangle : X \to P\text{prePost}(Y)$, the sequential composition of two computations is naturally specified by defining

$$\text{bind}_{P\text{prePost}} p f = \langle (\text{pre} \land \forall x, \text{post} x \implies \text{pre}' x), \lambda y. \exists x, \text{post} x \land \text{post}' x y \rangle : P\text{prePost}(Y)$$

The resulting precondition ensures that the precondition of the first computation holds and, under the assumption that the postcondition of the first computation holds, the postcondition of the second computation holds. The resulting postcondition is then simply the conjunction of the postconditions of the two computations. The order on $P\text{prePost}$ naturally combines the pointwise backward implication order on postconditions with a forward implication order on preconditions.

We formally show that this specification monad is more expressive than the predicate monad above: Any predicate $p : P\text{red}(X)$ can be coerced to $(\top, p) : P\text{prePost}(X)$, and in the other direction, any pair $(\text{pre}, \text{post}) : P\text{prePost}(X)$ can be approximated by a predicate $\lambda x. \text{pre} \Rightarrow \text{post}(x)$, giving rise to a Galois connection, as illustrated in Figure 1. While the monad $P\text{prePost}$ has the clear advantage of being more intuitive, practical generation of efficient verification conditions has historically favored predicate transformers over using pairs of pre and postconditions (Leino 2005).

**Forward predicate transformer monad.** The predicate monad $P\text{red}$ can be extended in an alternative way. Instead of fixing a precondition as it is the case for $P\text{prePost}$, a specification can consist of a mapping from preconditions to postconditions. A particular case of such a specification is the strongest postcondition transformer. Given a precondition $\text{pre}$ and a computation $c$, the strongest postcondition of $c$ with respect to $\text{pre}$ is a predicate characterizing the possible outcomes of $c$. 


Intuitively, a forward predicate transformer on $X$ has the type $\mathbb{P} \to X \to \mathbb{P}$. However, to actually obtain a monadic structure on such predicate transformers, we have to consider the smaller type $\mathcal{S}\text{Post}_X = ((\text{pre} : \mathbb{P}) \to X \to \mathbb{P})_{\text{pre}}$, of predicate transformers that are monotonic with respect to $\text{pre}$, where $\mathbb{P}_{\text{pre}}$ is the subtype of propositions implying $\text{pre}$.

Returning a value $v : X$ is specified by the predicate transformer $\text{ret}^{\mathcal{S}\text{Post}} v = \lambda x. \text{pre} \land v = x$, and the sequential composition of two computations is specified as the predicate transformer $\text{bind}^{\mathcal{S}\text{Post}} m f = \lambda y. \exists x, f x (m x y), \text{for } m : \mathcal{S}\text{Post}_X$ and $f : X \to \mathcal{S}\text{Post}_Y$.

**Backward predicate transformer and continuation monad.** As explained in §2.2, the postconditions-to-preconditions predicate transformers can be described using the continuation monad with the return type of propositions $\mathbb{P}$, namely, $\text{Cont}_\mathbb{P} = (X \to \mathbb{P}) \to \mathbb{P}$. Specifically, elements $w : \text{Cont}_\mathbb{P}(X)$ can be seen as predicate transformers mapping postconditions $\text{post} : X \to \mathbb{P}$ to preconditions $w \text{post} : \mathbb{P}$. And pointwise implication is a natural candidate for the order:

$$w_1 \leq w_2 : \text{Cont}_\mathbb{P}(X) \iff \forall p : X \to \mathbb{P}, w_1 p \Rightarrow w_2 p$$

However, $\text{Cont}_\mathbb{P}$ is not an ordered monad with respect to this order because its $\text{bind}$ is not monotonic. In order to recover an ordered monad, we restrict our attention to the submonad $\text{MonCont}_\mathbb{P}$ of $\text{Cont}_\mathbb{P}$ containing the monotonic predicate transformers, that is those $w : \text{Cont}_\mathbb{P}(X)$ such that

$$\forall p_1, p_2 : X \to \mathbb{P}, \text{ } (\forall x : X, p_1 x \Rightarrow p_2 x) \Rightarrow w p_1 \Rightarrow w p_2$$

This restriction is natural from a verification perspective: we do expect that stronger postconditions map to stronger preconditions.

This specification monad is more expressive than the pre-/postcondition monad above (Swamy et al. 2016). Formally, a pair $(\text{pre, post}) : \mathcal{P}\text{PrePost}(X)$ maps to monotonic predicate transformers via

$$\lambda x : X. \text{pre } \land (\forall y : X, \text{post } x \Rightarrow y) : \text{MonCont}_\mathbb{P}(X)$$

and vice versa, a predicate transformer $w : \text{MonCont}_\mathbb{P}(X)$ can be approximated by the pair

$$\lambda x, \tau \cdot \lambda x. (\forall p : \mathbb{P}, w p \Rightarrow p x) : \mathcal{P}\text{PrePost}(X)$$

These two mappings define a Galois connection, as illustrated in Figure 1. Further, this Galois connection exhibits $\mathcal{P}\text{PrePost}(X)$ as the submonad of $\text{MonCont}_\mathbb{P}(X)$ on conjunctive predicate transformers, e.g., predicate transformers $w$ commuting to non-empty conjunctions/intersections.

### 4.2 Defining Monad Transformers

By a **monad transformer** (Liang et al. 1995), we formally mean a pointed endofunctor from the category of monads to itself (Lüth and Ghani 2002; Moggi 1990), that is, in more detail:

- a mapping $\mathcal{T}$ from monads to monads
- assigning functorially monad morphisms to monad morphisms
- equipped with a monad morphism $\textbf{lift}_M : M \to \mathcal{T} M$
- and such that the $\textbf{lift}_M$ is natural in $M$, that is for any monad morphism $\theta : M_1 \to M_2$,

$$\mathcal{T} \theta \circ \textbf{lift}_{M_1} = \textbf{lift}_{M_2} \circ \theta$$

Moreover, we want our monad transformers to preserve the order structure present on the monads and the lifts to be monotonic. Providing all this data and proofs represents a significant effort, that we reduce to defining a mere monad in our metalanguage. A translation from the metalanguage to the object language elaborates this monad to a monad transformer with the required properties.

![Fig. 1. Relationships between basic specification monads](image-url)
We give a few examples of such monads internal to $\text{SM}$

In order to define monad transformer-like structures, we introduce the specification metalanguage $\text{SM}$ in Figure 2. $\text{SM}$ is a close relative of Moggi’s monadic metalanguage, since it contains the same type formers: a unary type constructor $\vdash$, pairs $C_1 \times C_2$ and function type $A \rightarrow C$. A first difference is that the language is not parameterized on a set of simple base types but on a dependent type theory $\mathcal{L}$. Consequently the function types take the more general form of a dependent product $(x : A) \rightarrow C[x]$ when the domain is in $\mathcal{L}$, but are restricted to non-dependent functions $C_1 \rightarrow C_2$ when the domain is already in $\mathcal{SM}$. We write abstractions of dependent type as $\lambda x. t$, whereas we write those of non-dependent type as $\lambda^c x. t$, as also shown in the standard typing rules from Figure 3. $\text{SM}$ captures the essential elements of the metalanguage DM of Ahman et al. (2017), leaving the non-necessary parts, such as sum types, to the base language $\mathcal{L}$.

$\text{SM}$ has enough structure to define internal monads, e.g.,

- a type constructor $X : \text{Type} \vdash_{\text{SM}} C[X]$;
- terms $A : \text{Type} \vdash_{\text{SM}} \text{ret}^C : A \rightarrow C[A]$ and $A, B : \text{Type} \vdash_{\text{SM}} \text{bind}^C : (A \rightarrow C[B]) \rightarrow C[A] \rightarrow C[B]$;
- such that the monadic equations are derivable in the equational theory of $\text{SM}$. We give a few examples of such monads internal to $\text{SM}$:

**Reader** fixing a type of input $\mathcal{I}$ in $\mathcal{L}$, $\text{rd}(X : \text{Type}) = \mathcal{I} \rightarrow \vdash X$;

**Writer** fixing a type of output $\mathcal{O}$ in $\mathcal{L}$, $\text{wr}(X : \text{Type}) = \vdash (X \times \mathcal{O})$;

**Exceptions** fixing a type of exceptions $\mathcal{E}$ in $\mathcal{L}$, $\text{exc}(X : \text{Type}) = \vdash (X + \mathcal{E})$
**State** fixing a type of state $S$ in $L$, $\mathcal{S}(X : \text{Type}) = \mathcal{S} \to \mathcal{M}(X \times S)$

**Monotonic state** fixing a preorder $\succeq$ on $S$, $\mathcal{M}(\mathcal{S}(X)) = (s_0 : S) \to \mathcal{M}(X \times (s_1 : S) \times s_1 \succeq s_0)$

**Continuations** fixing an answer type $\mathcal{A}\text{ns}$ in $L$, $\text{Conv}_{\mathcal{A}\text{ns}}(X) = (X \to \mathcal{M}\mathcal{A}\text{ns}) \to \mathcal{M}\mathcal{A}\text{ns}$

The interesting part is that an internal monad in $SM$ is almost enough to define a monad-transformer on $L$. How? We define a denotation $[-]_M$ of SM types (Figure 4) and terms (provided in the supplementary material together with the equational theory) with respect to a monad $M$ that preserves the equational theory of SM, provided $L$ has extensional dependent product and pairs, so the data of a monad $C$ internal to $SM$ induces a mapping from monads to monads:

$$(M, \text{ret, bind}) \quad \mapsto \quad ([C]_M, [\text{ret}^C]_M, [\text{bind}^C]_M)$$

The key observation is that the denotation $[C]_M$ of an SM type $C$ in $L$ can be endowed with an $M$-algebra structure $\alpha^C_M : M[C]_M \to [C]_M$. This $M$-algebra structure is defined by induction on the structure of the SM type $C$, using the free one when $C = M\alpha$ and the pointwise one in all the other cases. This $M$-algebra structure allows to define a lifting function from the monad $M$ to the monad $[C]_M$ as follows:

$$1 \text{ift}^C_{M,X} : M(X) \xrightarrow{M(\text{ret}^1^{C,M})} M[C]_M(X) \xrightarrow{\alpha^C_{M,X}} [C]_M(X)$$

Since we want to define monad transformers, we still need to build a functorial action mapping monad morphism $\theta : M_1 \to M_2$ between monads $M_1, M_2$ in $L$ to a monad morphism $[C]_{M_1} \to [C]_{M_2}$. However, the denotation of the arrow $C_1 \to C_2$ does not allow for such a functorial action. In order to get an action on monad morphisms, we first build a (logical) relation between the denotations. Given $M_1, M_2$ monads in $L$ and a family of relations $R_A \subset M_1 A \times M_2 A$ indexed by types $A$, we build a relation $\{[C] \}^R_{M_1, M_2} \subset [C]_{M_1} \times [C]_{M_2}$ as follows:

$$m_1 \{[M]A\} m_2 \quad = \quad m_1 R_A m_2$$

$$(m_1, m_1') \{[C_1 \times C_2]\} (m_2, m_2') \quad = \quad m_1 \{[C_1]\} m_2 \land m_1' \{[C_2]\} m_2'$$

$$f_1 \{[(x : A) \to C]\} f_2 \quad = \quad \forall(x : A), f_1 x \{[C x]\} f_2 x$$

$$f_1 \{[C_1 \to C_2]\} f_2 \quad = \quad \forall m_1, m_2, m_1 \{[C_1]\} m_2 \Rightarrow f_1 m_1 \{[C_2]\} f_2 m_2$$

Now, when a type $C$ in SM comes with the data of an internal monad, the relational denotation $\{[C]\}_M^W$ maps not only families of relations to families of relations but preserves also the following structure that we call a monadic relation:

**Definition 2 (Monadic Relation).** A monadic relation $\mathcal{R} : M \leftrightarrow W$ between a computational monad $M$ and a specification monad $W$, consists of:

- a family of relations $R_A : M A \times W A \to \mathbb{P}$ indexed by type $A$
- such that returned values are related $(\text{ret}^M v) R_A (\text{ret}^W v)$ for any value $v : A$
- and such that sequencing of related values is related

$$m_1 R_A w_1 \quad \forall x : A, \quad (m_2 x) R^B (w_2 x) \quad \frac{}{(\text{bind}^M m_1 m_2) R^B (\text{bind}^W w_1 w_2)}$$

The simplest example of monadic relation is the graph of a monad morphism $\theta : M \to W$. Given a monadic relation, we extend the relational translation to terms and obtain the so-called fundamental lemma of logical relations.

**Theorem 1 (Fundamental Lemma of Logical Relations).** For any monads $M_1, M_2$ in $L$, monad relation $\mathcal{R} : M_1 \leftrightarrow M_2$, term $\Gamma \vdash_{SM} t : C$ and substitutions $\gamma_1 : \Gamma | M_1$ and $\gamma_2 : \Gamma | M_2$, if for all $(x : C') \in \Gamma$, $\gamma_1(x) \{[C'] \}^R_{M_1, M_2} \gamma_2(x)$ then $\Gamma \vdash_{M_1} \{[t] \}^\gamma_{M_1, M_2} \{[\gamma_2(t)] \}^\gamma_{M_2}$. 

---

This document discusses the concept of monadic relations and how they can be used to map between computational monads and specification monads. It introduces the fundamental lemma of logical relations, which is a key result in understanding how monadic relations can be used to preserve the equational theory of monads when mapping between different computational environments.
As a corollary, an internal monad $C$ in SM preserves monadic relations, the relational interpretation of $\text{ret}^C$ and $\text{bind}^C$ providing witnesses to the preservation of the monadic structure. In particular, any monad morphism $\theta : M_1 \to M_2$ defines a monadic relation $\{C\}_{M_1, M_2}^{\text{graph}(\theta)} : [C]_{M_1} \leftrightarrow [C]_{M_2}$.

With the exception of continuations, which we discuss below, in all the examples of monad internal to SM above the resulting monadic relation is actually functional, so it is a monad morphism and we can go on to define an actual monad transformer. We give syntactic necessary conditions that can be checked at the source level to ensure that a monad $C$ internal to SM induces a monad transformer. First, the occurrences of $\mathbb{M}$ in $C$ should be covariant, meaning that it cannot contain any occurrence of an arrow $C_1 \to C_2$ where $C_1$ is a type in SM. This condition ensures that the relational interpretation of $C$ with respect to a function is still a function, in particular $C$ maps monad morphism to monad morphisms, naturality following from the relational interpretation of $\text{ret}^C$ and $\text{bind}^C$. The second condition ensures that for any monad $M$, $\text{lift} : M \to [C]_M$ is natural, ensuring that the following diagram commutes for any $A, B$ and $f : A \to B$.

$$
\begin{array}{c}
\begin{array}{ccc}
MA & \xrightarrow{M(\text{ret}^C)} & M[C]_MA \\
MF & \downarrow & \downarrow M[C]_Mf \\
MB & \xrightarrow{M(\text{ret}^C)} & M[C]_MB \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{ccc}
\xrightarrow{\alpha^C_{M,A}} & & \xrightarrow{\alpha^C_{M,A}} \\
\xrightarrow{\alpha^C_{M,B}} & & \xrightarrow{\alpha^C_{M,B}} \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{ccc}
 [C]_MA & \xrightarrow{[C]_Mf} & [C]_MA \\
 \end{array}
\end{array}
$$

The left square commutes by naturality of $M(\text{ret}^C)$, however for the right square to commute $[C]_Mf = \text{bind}^C M(\text{ret}^C M \circ f)$ should be an $M$-algebra homomorphism. We can ensure it by asking that $\text{bind}^C$ maps functions to $M$-algebra homomorphism, which can be syntactically captured by a linearity condition in a modified type system for SM equipped with a stoop, that is a distinguished variable in the context such that the term is linear with respect to that variable (Egger et al. 2014; Munch-Maccagnoni 2013). We omit this refined type system here and refer to the appendix for the complete details.

To summarize, given an internal monad $C$ to SM, we obtain:

- a mapping from ordered monads to ordered monads using the denotation $[C]_{-}$;
- when $C$ is covariant, a functorial mapping from ordered monad morphisms to ordered monad morphisms using $[\cdot]_{-}$;
- a $\text{lift}^C : M \to [C]_M$ thanks to the $M$-algebra structure $\alpha^C$ on $[C]_M$, which is natural whenever $\text{bind}^C$ satisfy the linearity criterion;
- under the previous conditions, $\text{lift}^C$ is also a natural transformation from the identity functor to $[C]_{-}$ on the category of ordered monad.

The **Continuation monad pseudo-transformer**. Crucially, the internal continuation monad $\text{Cont}^{\text{Ans}}$ does not verify the conditions to define a monad transformer since it is not covariant in $\mathbb{M}$. We study this (counter-)example in deeper detail since it clarifies the prior work of Ahman et al. (2017); Jaskelioff and Moggi (2010). We can recover the computational continuation monad $\text{Cont}^{\text{Ans}}_{\text{Id}} = \text{Cont}^{\text{Ans}}$ and a specification monad $\text{Cont}^{\text{Ans}}_{\text{Conf}} = \text{Cont}^{\text{Conf}}(\text{Ans})$, but only a monadic relation between the two and not a monad morphism.

$$
\begin{array}{c}
\begin{array}{c}
\text{Cont}^{\text{Ans}}_{\text{Id}} \leftarrow \text{Cont}^{\text{Ans}}_{\text{Id}, \text{Conf}} \rightarrow \text{Cont}^{\text{Ans}}_{\text{Conf}}
\end{array}
\end{array}
$$

What are the elements related by this relation? Unfolding the definition we get that $m : \text{Cont}^{\text{Ans}}(X)$ and $w : \text{Cont}^{\text{Ans}}_{\text{Conf}}(X)$ are related if

$$
\begin{array}{c}
\begin{array}{c}
m \in \text{Cont}^{\text{Ans}}_{\text{Id}, \text{Conf}} w \\
\iff \forall (k : X \to \text{Ans})(w_k : X \to \text{Conf}(\text{Ans})), (\forall x : X, \text{ret}(k x) = w_k x) \Rightarrow \text{ret}(m k) = w w_k
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\iff \forall (k : X \to \text{Ans}), \text{ret}(m k) = w (\lambda x. \text{ret}(k x))
\end{array}
\end{array}
$$
Taking $\mathcal{A}ns = \mathbb{1}$ to illustrate this on a simpler case, the last condition reduces to $\forall p : \mathbb{P}, \lambda q. (q (k x)) p = p (m k)$.

In particular any sequence $x_0, \ldots, x_n$ induces an element $w = \lambda k p. k x_0(\ldots k x_n p) : \mathbb{P} \mathbb{A}ns_\mathbb{Cont}_\mathbb{P}(X)$. As illustrated in §3.4, Dijkstra monads can be obtained from a pair of monads $\mathbb{M}$.

4.3 Effect observations from monad algebras

There is often a simpler way to define effect observations with codomain $\mathbb{M}$-algebras $M \rightarrow \mathbb{Cont}_R$ and $M$-algebras $M R \rightarrow R$. We can extend this result to the ordered setting so that it applies to $W^{\mathbb{Pure}} = \mathbb{MonCont}_\mathbb{P}$ and $W^{\mathbb{St}}$ as submonads of continuation monads: effects observation $\theta : M \rightarrow \mathbb{MonCont}_\mathbb{P}$-algebras on $\mathbb{M}$ are in one-to-one correspondence with $\mathbb{M}$-algebra $\alpha : M \mathbb{P} \rightarrow \mathbb{P}$ that are monotonic with respect to the free lifting on $M \mathbb{P}$ of the implication order on $\mathbb{P}$. This last correspondence underlies the examples of nondeterminism in §3.3: the effect observations $\theta^\mathbb{Y}(m) = \lambda p. \alpha^\mathbb{Y} (\mathbb{NDet}(p) m)$ and $\theta^\mathbb{3}(m) = \lambda p. \alpha^\mathbb{3} (\mathbb{NDet}(p) m)$ arise from the $\mathbb{NDet}$-algebras $\alpha^\mathbb{Y}$ and $\alpha^\mathbb{3}$ taking respectively the conjunction and disjunction of a set of propositions $\mathbb{NDet}(\mathbb{P})$. Conversely, we could recover $\alpha^\mathbb{Y}$ (resp. $\alpha^\mathbb{3}$) as $\lambda m. \theta^\mathbb{Y}_p(m) \mathbb{id}_\mathbb{P}$ (resp. $\theta^\mathbb{3}$).

5 DIJKSTRA MONADS FROM EFFECT OBSERVATIONS

As illustrated in §3.4, Dijkstra monads can be obtained from a pair of monads $M, W$ and an effect observation $\theta : M \rightarrow W$. As we shall see this construction is generic and leads to a categorical equivalence between Dijkstra monads and effect observations. In this section, we introduce more formally the notion of Dijkstra monad using dependent type theory, seen as the internal language of a comprehension category (Jacobs 1993), and then build a category $\mathbb{DMon}$ of Dijkstra monads. In order to compare this notion of Dijkstra monad to the effect observations, we first introduce a category of monadic relations $\mathbb{MonRel}$ and show that there is an adjunction

$$\int \dashv \mathbb{fib} : \mathbb{MonRel} \rightarrow \mathbb{DMon}. \quad (3)$$

Intuitively, an adjunction can be seen as a weak form of equivalence between two categories, in this case $\mathbb{MonRel}$ and $\mathbb{DMon}$. When we restrict this adjunction, we obtain an actual equivalence between Dijkstra monads and effect observations. For the sake of explanation, we proceed in two steps: first, we consider Dijkstra monads and effect observations over specification monads with a discrete preorder (i.e., ordinary monads), describing the adjunction above in this situation; later, we extend that construction to general preorders, thus obtaining the full adjunction we are interested in. We write a $\leq$ exponent for categories defined over non-discrete specification monads.

Dependent type theory and comprehension categories. We work in an extensional type theory with dependent products $x : A \rightarrow B$, strong sums $x : A \times B$, an identity type $x =^A y$ for $x, y : A$ where the type $A$ is usually left implicit, a type of propositions $\mathbb{P}$, and quotients of equivalence relations. This syntax is the internal language of a comprehension category (Jacobs 1993) with enough structure and we will write $\mathcal{Type}$ for any such category. This interpretation of type theory allows us to call any object $\Gamma \in \mathcal{Type}$ a type. Working internally to $\mathcal{Type}$, a preorder is a type $A$ equipped with a binary predicate $\leq : A \times A \rightarrow \mathbb{P}$ that is reflexive and transitive; a monotonic function between preorders $A, B$ is a function $f : A \rightarrow B$ preserving the preorder. Preorders and

\(^4\)Under a mild condition that the category is democratic (Clairambault and Dybjer 2014).
monotonic functions in Type give raise to a category \(Ord\), and there is a triple adjunction
\[
Q + \Delta \dashv \mathcal{U} : Ord \rightarrow Type
\]
where \(\mathcal{U}\) forgets the preorder structure, \(\Delta\) equips a type with the discrete preorder relation, and \(Q\) quotient a preorder by the equivalence relation induced by the preorder.

As a leading example of our use of the syntax of type theory, consider the following definition of a fibered monad on Type. A (fibered) monad \(W\) consists of an assignment of a type \(WA\) for each type \(A\), and two functions \(\text{ret}^W : A \rightarrow WA\) and \(\text{bind}^W : WA \rightarrow (A \rightarrow WB) \rightarrow WB\) that are required to satisfy the usual monadic equations.

Dijkstra monads. A *Dijkstra monad* over a monad \(W\) is composed of
- for each type \(A\) and specification \(w : WA\), a type \(\mathcal{D}A\) of “computations specified by \(w\)”;
- return and bind functions specified respectively by the return and bind of \(W\)

\[
\text{ret}^D : (x : A) \rightarrow \mathcal{D}A(\text{ret}^W x), \\
\text{bind}^D : \mathcal{D}A w_1 \rightarrow ((x : A) \rightarrow \mathcal{D}B w_2(x)) \rightarrow \mathcal{D}B(\text{bind}^W w_1 w_2);
\]
- monadic equations asserting the unitality of \(\text{ret}^D\) and associativity of \(\text{bind}^D\):

\[
\text{bind}^D m \text{ret}^D = m, \\
\text{bind}^D(\text{ret}^D x) f = f x,
\]

\[
\text{bind}^D(\text{bind}^D m f) g = \text{bind}^D m (\lambda x. \text{bind}^D (f x) g)
\]
where \(m : \mathcal{D}A w, x : A, f : (x : A) \rightarrow \mathcal{D}B(w' x), g : (y : B) \rightarrow \mathcal{D}C(w'' y)\) for \(A, B, C\) any types and \(w : WA, w' : (x : A) \rightarrow WB, w'' : (y : B) \rightarrow WC\). Note that these equations depend on the monadic equations for \(W\) and would not be well-typed otherwise.

Moreover, when \(W\) is ordered, \(\mathcal{D}\) should be equipped with a *weakening* structure

\[
\text{weaken} : w_1 \leq_A w_2 \times \mathcal{D}A w_2 \rightarrow \mathcal{D}A w_1
\]
such that the following axioms hold, where we write \(w_1 \leq w_2\) for its unique witness

\[
\text{weaken}(w \leq w, m) = m, \\
\text{weaken}(w_1 \leq w_2 \leq w_3, m) = \text{weaken}(w_1 \leq w_2, \text{weaken}(w_2 \leq w_3, m)).
\]

A morphism of Dijkstra monads between \(D_1 A(w_1 : W_1 A)\) and \(D_2 A(w_2 : W_2 A)\) over the monad morphism \(\Theta^W : W_1 \rightarrow W_2\) is the data of a family of functions indexed by types \(A\) and specifications \(w_1 : W_1 A\)

\[
\Theta^D_{A_1, w_1} : D_1 A w_1 \rightarrow D_2 A(\Theta^W w_1)
\]
verifying the following axioms

\[
\Theta^D(\text{ret}^{D_1} x) = \text{ret}^{D_2} x, \\
\Theta(\text{bind}^{D_1} m f) = \text{bind}^{D_2}(\Theta^D m)(\Theta^D f).
\]

We form the category \(\mathcal{DMon}\) with pairs \((W, \mathcal{D})\) of a monad \(W\) and a Dijkstra monad \(\mathcal{D}\) on \(W\) as its objects and pairs \((\Theta^W, \Theta^D)\) of a monad morphism \(\Theta^W\) and a Dijkstra monad morphism \(\Theta^D\) over \(\Theta^W\) as its morphisms. The corresponding category \(\mathcal{DMon}^\leq\) in the ordered setting has pairs of an ordered monad and a Dijkstra monad with a weakening structure as objects, and pairs of a monotonic monad morphism, and a Dijkstra monad morphism preserving the weakening structure as morphisms.

**Monadic Relations.** Given a monadic relation \(R : M \leftrightarrow W\) between a computational monad \(M\) and a specification monad \(W\), we construct a Dijkstra monad \(\text{fib} R\) on \(W\) in the following way:

\[
(\text{fib} R) A (w : WA) = (m : MA) \times m R_A w
\]
That is \((\text{fib} R) A w\) consists of those elements of \(MA\) that are related by \(R\) to the specification \(w\). When \(R\) is the graph a monad morphism \(\theta\) (or equivalently, \(R\) is functional), \(\text{fib}(R : M \leftrightarrow W)\) maps an element \(w : WA\) to the fiber \(\theta^{-1}(w) = \{ m : MA \mid \theta(m) = w \}\).
Conversely, a Dijkstra monad $D$ over $W$ yields a monad structure on
\[ \int D A = (w : W A) \times D A w \]
and the projection of the first component is a monad morphism $\pi_1 : \int D \to W$.

In order to explain the relation between these two operations $\fib$ and $\int -$, we introduce the category $\MonRel$ of monadic relations. An object of $\MonRel$ is a pair of monads $W, M$ together with a monadic relation $R : M \leftrightarrow W$ between them. A morphism between $R^1 : M_1 \leftrightarrow W_1$ and $R^2 : M_2 \leftrightarrow W_2$ is a pair $\Theta^M, \Theta^W$ such that
\[ \forall m : MA, w : WA, m \Theta^1_A w \implies \Theta^M(m) \Theta^2_A \Theta^W(w). \] (6)

The construction $\fib$ extends to a functor on $\MonRel$ by sending a pair $(\Theta^W, \Theta^M)$ to a pair $(\Theta^W, \Theta^D)$ where $\Theta^D_{A, w}$ is the restriction of $\Theta^M$ to the appropriate domain. Conversely, $\int$ packs up a pair $(\Theta^W, \Theta^D)$ as a pair $(\Theta^W, \Theta^M)$, taking $\Theta^M(w, m) = \Theta^D_{A, w}(m)$. Since $\Theta^M$ maps the fiber over $w$ to the fiber over $\Theta^W(w)$, condition (6) is satisfied. Moreover, this gives rise to a natural bijection
\[ \MonRel(\int D, R) \cong D \Mon(\int D, \fib R) \]
that establishes the adjunction (3). We can restrict this adjunction to an equivalence by considering only those objects for which the unit (respectively the counit) of the adjunction is an isomorphism. Every Dijkstra monad $D$ is isomorphic to its image $\fib (\int D)$, whereas a monadic relation $R$ is isomorphic to $\int ((\fib R))$ if and only if it is functional, that is a monad morphism. We obtain this way an equivalence of categories between $D \Mon$ and the category of effect observations on monads with discrete preorder.

The ordered setting. In our examples from §3.4, Dijkstra monads $D A (w : W A)$ derived from an effect observation $\theta$ makes use of the order on $W$ to allow computations $m$ whose specification $\theta(m)$ is stronger than $w$. We now explain how to generalize the definition of $\fib$ to ordered monads to account for this mismatch.

Let’s first consider the case of a monad morphism $\theta : M \to W$.
\[ (\fib \theta) A (w : W A) = (m : MA) \times w \leq_A \theta(m) \] (7)
This definition coincides with (5) when the order on $W$ is discrete. Moreover, we can equip $\fib \theta$ with a weakening structure:
\[ \text{weaken}(w_1 \leq w_2, (m, w_2 \leq \theta(m))) = (m, w_1 \leq w_2 \leq \theta(m)) \]

More generally, we can do the same construction starting with an upper closed monadic relation $R : M \leftrightarrow W$, that is such that
\[ \forall m_1 \leq^M_A m_2, w_1 \leq^W_A w_2, m_1 R_A w_1 \implies m_2 R_A w_2. \]

We obtain in this way, a functor $\fib$ from the category $\MonRel^\leq$ of upper-closed monadic relations to the category $D \Mon^\leq$ of ordered Dijkstra monads with a weakening structure. We can build exactly as previously a left adjoint $\int$ to $\fib$.

However, there is still a small mismatch with the expected construction on practical examples. Indeed, starting from a monad morphism $\theta : M \to W$ with a discrete preorder on $M$, $\int (\fib \theta)$ reduces to $(\Sigma M, W, \pi_1)$ where $\Sigma M A = (w : W A) \times (m : MA) \times w \leq_A \theta_A(m)$ multiplicates all $m : MA$, providing one instance for each admissible specification $w : W A$. These copies are non-essential since the weakening structure of the Dijkstra monad $\fib \theta$ identifies them, and we can compose the adjunction $\int + \fib$ with (a lifting of) the adjunction $Q + \Delta$ to quotient these, thus finally obtaining the desired adjunction.

To summarize, we can construct Dijkstra monads with weakening out of effect observations and the other way around. Moreover, when starting from an effect observation $\theta : M \to W$, the
roundtrip effect observation \( \int (\text{fib } \theta) \) is equivalent to the initial \( \theta \), showing that we did not loose anything on the way.

**Dijkstra monads as displayed algebras.** Dijkstra monads and part of their equivalence to effect observation can be alternatively presented as an instance of a general result about displayed algebras and categories with families. Indeed there is a signature \( \Sigma^{\text{mon}} \) in the sense of Kaposi and Kovács (2019), generating a category with families of \( \Sigma^{\text{mon}} \)-algebras with monads as algebras and Dijkstra monads as displayed algebras. This category with families verifies the hypothesis of prop. 9 of (Clairambault and Dybjer 2014), thus the category of Dijkstra monads over a fixed monad \( W \) is equivalent to the slice category over \( W \), that is monad morphisms with target \( W \).

### 6 ALGEBRAIC EFFECTS AND EFFECT HANDLERS FOR DIJKSTRA MONADS

#### 6.1 Algebraic effects

Plotkin and Power (2002, 2003) observed that many computational monads, such as the ones discussed in §2.1, come with canonical operations satisfying natural equations. We now show how effect observations equip both the specification monad and the corresponding Dijkstra monad with such operations. We observed several instances of this phenomena for state, IO, and nondeterminism in §3.4, and we can now explain it in terms of algebraic effects and effect observations.

**Algebraic operations.** For any monad \( M \), an algebraic operation \( \text{op} : I \sim O \) with input (parameter) type \( I \) and output (arity) type \( O \) is a family \( \text{op}_M^I : I \times (O \rightarrow MA) \rightarrow MA \) that satisfies the following coherence law for all \( i : I, m : O \rightarrow MA \), and \( f : A \rightarrow MB \) (Plotkin and Power 2003):

\[
\text{bind}^M (\text{op}_M^I(i, m)) f = \text{op}_M^I(i, \lambda o. \text{bind}^M (m o) f) \tag{8}
\]

For \( \text{NDet} \), the two operations are \( \text{pick} : \mathbb{1} \sim \mathcal{E} \) and \( \text{fail} : \mathbb{1} \sim \mathbb{0} \). For \( \text{St} \), the operations are \( \text{get} : \mathbb{1} \sim S \) and \( \text{put} : S \sim \mathbb{1} \). Plotkin and Power also showed that such algebraic operations are in one-to-one correspondence with *generic effects*, i.e., morphisms \( \text{gen}_M^I : I \rightarrow MO \), which is often a more natural presentation for programming. For example, the generic effect corresponding to the \( \text{put} \) operation for \( \text{St} \) has type \( S \rightarrow \text{St} \mathbb{1} \). While in §2.1 we illustrated the example computational monads using generic effects, in the following we mainly use their algebraic counterparts as they are mathematically more convenient to work with. Nevertheless, they are interconvertible as follows:

\[
\text{gen}_M^I i = \text{op}_O^M (i, \lambda o. \text{ret}^M o) \quad \text{op}_A^I (i, m) = \text{bind}^M (\text{gen}_M^I i) (\lambda o. m o) \tag{9}
\]

Plotkin and Power show that a signature \( \text{Sig} \) of algebraic operations can determine many computational monads, once it is also equipped with suitable equations \( \text{Eq} \). In the following we write \( T_{\langle \text{Sig}, \text{Eq} \rangle} \), abbreviated as \( T \), for the monad corresponding to \( \text{Sig} \) and \( \text{Eq} \). Equational presentations (\( \text{Sig}, \text{Eq} \)) also allow generic reasoning about programs written using algebraic effects (Plotkin and Pretnar 2008). Dijkstra monads provide an alternative way to reason about algebraic effects.

**Effect observations.** In §5, we saw that Dijkstra monads are categorically equivalent to effect observations \( \theta : M \rightarrow W \). Since \( \theta \) is a monad morphism, it automatically transports any algebraic operations on the computation monad \( T \) to the (ordered) specification monad \( W \):

\[
\text{op}_A^W (i, w) = \mu_A^W (\theta_{WA} (\text{op}_W^I (i, \lambda o. \text{ret}^T (w o)))) \tag{10}
\]

where \( \mu^W : W \circ W \rightarrow W \) is the multiplication (or join) of \( W \), defined as \( \mu_A^W w = \text{bind}^W w (\lambda w'. w') \).

This derivation of algebraic operations is in fact a result of a more general phenomenon. Namely, given any monad morphism \( \theta : M \rightarrow W \), we get a family of \( M \)-algebras on \( W \), natural in \( A \), by

\[
\mu_A^W \circ \theta_{WA} : MWA \rightarrow WWA \rightarrow WA
\]

Furthermore, the derived algebraic operations \( \text{op}_A^W \) (resp. the derived \( M \)-algebras on \( W \)) are monotonic with respect to the free lifting of the preorder \( \leq^W_A \) on \( WA \) to \( TWA \) (resp. to \( MWA \)).
The derivation of operations on the specification monad from operations on the computational monad, via the effect observation, explains the way we were able to systematically generate (computationally natural) specifications for operations in §3.4.

**Dijkstra monads.** Finally, we show that the Dijkstra monad \(D = \text{fib} \theta\) we derived from a given effect observation \(\theta : T \rightarrow W\) in (7) also supports algebraic operations, with their computational structure given by the operations of \(T\) and their specificational structure given by the operations of \(W\) derived in (10). This completes the process of lifting operations from computational monads to Dijkstra monads that we sketched in §3.4. In detail, we define an algebraic operation for \(D\) as

\[
\begin{align*}
op_A^D & : (i : I) \rightarrow (c : (o : O) \rightarrow D A (w o) \rightarrow D A (\text{op}_A^W (i, w))) \\
op_A^D i c & = \{ \text{op}_A^T (i, \lambda o. c o) , \text{op}_A^W (i, w) \leq \theta_A (\text{op}_A^T (i, \lambda o. |o|)) \}
\end{align*}
\]

As we have defined \(\text{op}^D\) in terms of algebraic operations for \(T\) and \(W\), then it is easy to see that it also satisfies an appropriate variant of the algebraic operations coherence law (8), namely

\[
\text{bind}^D (\text{op}_A^D (i, c)) f = \text{op}_A^D (i, \lambda o. \text{bind}^D (c o) f)
\]

Finally, based on (9), we note that the generic effect corresponding to \(\text{op} : I \leadsto O\) is given by

\[
\text{gen}_A^D i = \{ \text{gen}_A^T i , \text{gen}_A^W i \leq \theta_O (\text{gen}_A^T i) \} : I \rightarrow D O (\text{gen}_A^W i)
\]

### 6.2 Effect handlers

Of course, algebraic operations and equational presentations are only one side of the algebraic effects coin. The other and increasingly more important side concerns effect handlers (Plotkin and Pretnar 2013). These are a generalization of exception handlers to arbitrary algebraic effects. They are defined by providing a concrete implementation for each of the (abstract) algebraic operations, such as `get` and `put`. As such, they enable a generic and clean programming style in which programmers write their code generically in terms of the algebraic operations (such as `get` and `put`) and then use handlers to modularly provide fit-for-purpose implementations of such generic code. Category theoretically, effect handlers denote user-defined algebras for the algebraic effect at hand.

Below we show a way in which effect handlers can be accommodated within Dijkstra monads. Compared to algebraic operations that worked “out of the box”, accommodating handlers requires the given effect observation to come equipped with a certain algebra extension structure, as explained below. This will allow us to reconstruct the weakest precondition semantics for exceptions with try/catch in the setting of Dijkstra monads ((Leino and van de Snepscheut 1994; Sekerinski 2012), and put the ad-hoc exception handling examples of Ahman et al.’s (2017) DM4Free on a general footing. However, more complex effect handlers present difficulties, as we describe below.

**Effect handling.** Following Plotkin and Pretnar (2013), we define the *handling* of \(T\) (presented by an equational presentation \((\text{Sig}, Eq)\)) into a monad \(M\) to be given by the following operation

\[
\begin{align*}
\text{handle-with}^{T,M} : T A & \rightarrow (h_{\text{op}} : I \times (O \rightarrow MB) \rightarrow MB)_{\text{op}^T \in O \in \text{Sig}} \rightarrow \text{ok}(h) \rightarrow (A \rightarrow MB) \rightarrow MB \\
\text{handle-with}^{T,M} (\text{ret}^T a) h p f & = f a \\
\text{handle-with}^{T,M} (\text{op}_A^T (i, t)) h p f & = h_{\text{op}} (i, \lambda o. \text{handle-with}^{T,M} (t o) h p f)
\end{align*}
\]

where \(\text{ok}(h)\) is a proof obligation ensuring that the operation cases \(h_{\text{op}}\) of the *handler* \(h\) satisfy the equations in \(Eq\). For brevity, we elide the details of \(\text{ok}(h)\) and leave it implicit in the later use of \(\text{handle-with}^{T,M}\), referring the reader to Ahman (2018) for more details about accommodating effect handlers and their proof obligations in dependently typed languages.

Together, \(h\) and \(p\) form a \(T\)-algebra \(\alpha_{(h,p)} : T M B \rightarrow MB\), and \(\text{handle-with}^{T,M} (-) h p f\) amounts to the induced mediating \(T\)-algebra homomorphism \(\alpha_{(h,p)} \circ T(f) : TA \rightarrow TMB \rightarrow MB\).
**Specification monads.** Based on this category theoretic view of effect handling, we can define a notion of handling any monad $M$ into some other monad $M'$:

\[
\text{handle-with}^{M, M'} : M A \to (\alpha : MM' B \to M'B) \to \text{ok}(\alpha) \to (A \to M'B) \to M'B
\]

\[
\text{handle-with}^{M, M'} m \alpha pf = (\alpha \circ M(f)) m
\]

where $\text{ok}(\alpha)$ is a proof obligation ensuring that $\alpha$ is indeed an algebra for the monad $M$. As with $\text{ok}(h)$, we leave $\text{ok}(\alpha)$ implicit in the uses of $\text{handle-with}^{M, M'}$. Below we are specifically interested in using $\text{handle-with}^{W, W'}$ when $M$ and $M'$ are specification monads because, in contrast to $T$, the structure of specification monads is not determined by the equational presentation (Sig. Eq) alone.

**Dijkstra monads.** Based on the smooth lifting of algebraic operations we described above, then when defining effect handling for the Dijkstra monad $D = \text{fib} \theta$ induced by some effect observation $\theta : T \to W$ into some other Dijkstra monad $D' = \text{fib} \theta'$ for $\theta' : M \to W'$, we would expect the computational (resp. specification) structure of handling to be given by that for $T$ (resp. $W$).

However, simply giving an effect observation $\theta$ turns out to be insufficient for handling $D$ into $D'$. Category theoretically, the problem lies in the operation cases for $W$ giving us a $T$-algebra $TW'B \to W'B$, but to use $\text{handle-with}^{W, W'}$ (which we need to define the specification of handling) we instead need a $W$-algebra $WW'B \to W'B$. To overcome this difficulty, we introduce a more refined notion of effect observation, relative to the specification monad $W'$ we are handling into.

**Definition 3 (Effect Observation with Handling).** An effect observation with handling for an ordered monad $W'$ is an effect observation $\theta : T \to W$ such that for any $T$-algebra $\alpha : TW'A \to W'A$ there is a choice of a $W$-algebra $\alpha_* : WW'A \to W'A$ that is (i) monotonic with respect to the preorders of the ordered monads $W$ and $W'$, and (ii) for which we additionally have that $\alpha_* \circ \theta_{W'A} = \alpha$.

Intuitively, the condition (ii) expresses that $\alpha_*$ extends a $T$-algebra to a $W$-algebra in a way that is identity on the $T$-algebra structure, specifically on the algebraic operations corresponding to $\alpha$.

It is worth noting that needing to turn algebras $TW'A \to W'A$ into algebras $WW'A \to W'A$ is not simply a quirk due to Dijkstra monads, but the same need arises when giving a monadic semantics to a language with algebraic effects and handlers using a monad different from $T_{\text{Sig}}$.

Using this refined notion of effect observation, we can now define handling for Dijkstra monads. Given an effect observation $\theta : T \to W$ with handling for $W'$ and another effect observation $\theta' : M \to W'$, we define the handling of $D = \text{fib} \theta$ into $D' = \text{fib} \theta'$ as the following operation

\[
\text{handle-with}^{D, D'} : D A w_1 \\
\stackrel{\text{(h)}}{\longrightarrow} (h_{\text{op}}^{W'} : I \times (O \to W'B) \to W'B)_{\text{op} : I \to O \in \text{Sig}} \\
\stackrel{\text{(h)}}{\longrightarrow} (h_{\ast}^{D'} : ((i, c) : (i : I) \times ((o : O) \to D'B(w o))) \to D'B(h_{\ast}^{\text{op}}(i, w)))_{\text{op} : I \to O \in \text{Sig}} \\
\stackrel{\text{(h)}}{\longrightarrow} (p^{W'} : \text{ok}(h_{\ast}^{W'})) \to (p^{D'} : \text{ok}(h_{\ast}^{D'})) \\
\stackrel{\text{(h)}}{\longrightarrow} ((a : A) \to D'B(w_2 a)) \to D'B(\text{handle-with}^{W, W'} w_1(a_{(h_{\ast}^{W'}, p_{W'})}a), w_2)
\]

\[
\text{handle-with}^{D, D'} c_1 h_{W} h_{\ast}^{\text{op}} p_{W'} p_{D'} c_2 = \langle \text{handle-with}^{T \cdot M} c_1 h_{\ast}^{\text{op}}(\lambda a. c_2 a), \rangle \\
\langle \text{handle-with}^{W, W'} w_1(a_{(h_{\ast}^{W'}, p_{W'})}a), w_2 \rangle \leq \theta_{D'}^{B} (\text{handle-with}^{T \cdot M} c_1 h_{\ast}^{\text{op}}(\lambda a. c_2 a))
\]

where $(a_{(h_{\ast}^{W'}, p_{W'})}a) : WW'B \to W'B$ is the extension of the $T$-algebra $\alpha_{(h_{\ast}^{W'}, p_{W'})} : TW'B \to W'B$ induced by the pair $(h_{\ast}^{W'}, p_{W'})$ to a $W$-algebra, needed to use $\text{handle-with}^{W, W'}$.

**Exception handling.** One effect observation supporting handling is $\theta^{\text{Exc}} : \text{Exc} \to W^{\text{Exc}}$ from §3.2. Plotkin and Power (2003) showed that Exc is determined by the equational presentation ($\{\text{throw}, \emptyset\}$). To model handling potentially exceptional computations into other potentially exceptional computations, as is often the case in languages with exceptions but no effect system, we take
\[ \mathcal{D} = \mathcal{D}' = \text{EXC} = \text{fib } \theta^{\text{Exc}} \text{ and observe that } \text{handle-with}^{\text{Exc}, \text{Exc}} \text{ can be simplified to } \]

\[
\text{try-catch : EXC } A . w_1 \to (h^{\text{Exc}}_{\text{throw}} : E \to \text{WExc } B) \to (h^{\text{Exc}}_{\text{throw}} : (e : E) \to \text{EXC } B (h^{\text{Exc}}_{\text{throw}} e p q))
\]

\[
\to ((\alpha : A) \to \text{EXC } B (w_2 a)) \to \text{EXC } B (\lambda p q. \; w_1 (\lambda x. \; w_2 x p q)(\lambda e. \; h^{\text{Exc}}_{\text{throw}} e p q))
\]

because \( \text{ok}(h^{\text{Exc}}_{\text{throw}}) \) and \( \text{ok}(h^{\text{Exc}}) \) hold vacuously as the equational presentation of exceptions does not contain equations, by unfolding the definition of \( \text{handle-with}^{\text{WExc}, \text{WExc}} \), and by defining the extension \( \alpha : \text{WExcWExc } B \to \text{WExc } B \) of an \( \text{Exc-algebra} \) \( \alpha : \text{ExcWExc } B \to \text{WExc } B \) as

\[
\alpha_s w = \lambda p q. w (\lambda \dot{w}. \; \alpha (\text{inl } \dot{w}) p q) (\lambda e. \; \alpha (\text{inr } e) p q) = \lambda p q. w (\lambda \dot{w}. \; \dot{w} p q) (\lambda e. \; \alpha (\text{inr } e) p q)
\]

(11)

where the second equality holds because \( \alpha \) is an \( \text{Exc-algebra} \) and thus \( \alpha (\text{ret}^{\text{Exc}} v) = \alpha (\text{inl } v) = v \).

On closer inspection, it turns out that \( \text{try-catch} \) corresponds exactly to the weakest exceptional preconditions for exception handlers studied by Leino and van den Snepscheut (1994) and Sekerinski (2012). Furthermore, with \( \text{try-catch} \) we can also put Ahman et al.’s (2017) hand-rolled \( \text{DM4Free} \) exception handlers (which they defined on a case-by-case basis using monadic reflection) to a common footing. For example, we can define their integer division example using our \( \text{try-catch} \) as

\[
\text{let } \text{div_wp} (i j) = \text{if } j = 0 \text{ then raise } \text{div_by_zero_exn} \text{ else } i / j
\]

\[
\text{let } \text{div} (i j) : \text{EXC } \text{int } (\text{div_wp } i j) = \text{try_catch } (\text{div } i j) (\lambda p q. \text{try} \to (\lambda x. \text{try} \to \lambda x. \text{try} \to \lambda x. \text{try})
\]

where \( \text{try_div} \) handles any exception (including \( \text{div_by_zero_exn} \)) by returning the integer 0.

Of course the above is not the only way to handle exceptions. Another common example in the algebraic effects literature involves handling a computation in \( T A \) to a pure computation in \( \text{Id} (A + E) \). This is of course trivial category theoretically, but in a programming language where elements of \( T \) are considered abstract, it allows one to get their hands on the values returned and exceptions thrown, analogously to Ahman et al.’s (2017) \( \text{DM4Free} \) monadic reification for exceptions. To this end, we take \( \mathcal{D} = \text{EXC} = \text{fib } \theta^{\text{Exc}} \) and \( \mathcal{D}' = \text{PURE} = \text{fib } \theta^{\text{Pure}} \), and define

\[
\text{reify : EXC } A . w \to \text{PURE } (A + E) (\lambda p. \; w (\lambda x. \; p (\text{inl } x)) (\lambda e. \; p (\text{inr } e)))
\]

\[
\text{reify } c = \text{handle-with}^{\text{Exc}, \text{PURE}} c (\lambda e. \; \text{ret}^{\text{Pure}} (\text{inr } e)) (\lambda e. \; \text{ret}^{\text{PURE}} (\text{inr } e)) (\lambda x. \; \text{ret}^{\text{PURE}} (\text{inl } x))
\]

with the algebra extension operation \( \alpha_s \) defined as above because \( \text{WPure}(A + E) \equiv \text{WExc } A \).

**Other (non-)examples.** Unfortunately, effect observations discussed in this paper other than exceptions and partiality (i.e., state, nondeterminism, and IO) do not support effect handling in the above sense. Specifically, one is unable to define the algebra extension operation \( \alpha_s \) for these effect observations. Intuitively, the problem with defining \( \alpha_s \) in those examples lies in the corresponding specification monads not revealing enough information to the postcondition about the computation being reasoned about. In particular, in order to follow the same pattern for defining \( \alpha_s \) as in (11) for exceptions, of first "running" the specification and then calling \( \alpha \) in the postcondition position, the postcondition would need to have access to the whole computation not just the returned values (e.g., as in case of nondeterminism) or to a single program trace (e.g., as in case of IO).

For IO, we in fact actually know of another specification monad for which \( \alpha_s \) can be defined, namely, the categorical coproduct of the IO and continuation monads (Hyland et al. 2007), given by

\[
\text{WIO } A = (A \to \text{F}) \times ((I \to \text{WIO } A) \to \text{F}) \times (O \times \text{WIO } A \to \text{F}) \to \text{F}
\]

Note that compared to the specification monads for IO from §3.4, the postcondition(s) of \( \text{WIO} \) have a tree-like structure that enables one to recover enough information to (recursively) define \( \alpha_s \).

There are however two major problems with using \( \text{WIO} \) as a specification monad. First, \( \text{WIO} \) is not well-defined in many categories of interest, such as Set (Hyland et al. 2007). Second, defining \( \text{WIO} \) type theoretically requires non strictly positive inductive types, which leads to inconsistency in frameworks with impredicative universes such as Coq and \( \text{P}^* \) (Coquand and Paulin 1988).
We leave investigating other effect observations that support effect handling, and alternative notions of effect handling for Dijkstra monads that do not require algebra extension, for future work. The difficulty appears to lie in the highly compositional nature of both Dijkstra monads with weakest preconditions and handlers. The general construction of handling given above does not assume any particular semantics of effects for operations in the handled computation; expecting this to be “filled in” when the handler is used. This is straightforward with a non-resumable effect like exceptions, but for a resumable effect like IO, the predicate transformer induced by the interplay between the computation and the handler is much more complex, as evidenced by the recursively defined specification monad we discussed for IO above. We suspect that a solution lies in requiring the user to specify “invariants” for the operations before the handler is applied, which describe the contract between the handler and the computation, and serve to break the circularity.

7 IMPLEMENTATION AND FORMALIZATION IN F* AND COQ

Extending Dijkstra monads in F*. We have extended the effect definition mechanism of F*, previously modelled after the DM4Free approach, with the ideas of this paper. Programmers can now define effects by providing both the computational and specification monads, along with a monad morphism between them (or, more generally, a monadic relation), which provides a high level of freedom in the choice of specifications. The F* type-checker computes weakest preconditions, since they interact nicely with both F*’s existing effect framework and the SMT solver used to discharge proof obligations. Thus, we restrict specification monads to these, but it is customary for F* users to write require and ensure clauses to provide Hoare-style pre- and post- conditions, for which we leverage the adjunction described in §4.1. This extension enabled the verification of the examples of §3.4 in F*, for which effects such as nondeterminism and IO were previously out of reach.

Dijkstra monads in Coq & verification examples. We have also embedded Dijkstra Monads in Coq, demonstrating that the concept is applicable in languages beyond F*. As with the F* implementation, programmers can supply their own computational and specification monads, with a monad morphism or relation between them. We implemented the specification monads of §4.1, in particular the construction of monad morphisms into continuation monads from algebras of §4.3, which provides a convenient way for building a specification monad and effect observation at the same time. The examples in §3.4 have also been verified in Coq.

Formalization of SM in Coq. We have formalized the SM language of §4.2 in Coq, taking Gallina as the base language L and providing an implementation of the denotation and logical relation in the discrete case. SM is implemented using higher-order abstract syntax (HOAS) for the λx.t binders and De Bruijn indices for the λ⋄x.t ones. We also provide a PHOAS (Chlipala 2008) version for simpler input of internal monads. We build the functional version of the logical relation for a covariant type C, but showing that linear terms are homomorphisms is work in progress. The operation part of monad transformers can then be derived, but not yet all the equations. We use the category library of Timany and Jacobs (2016) to define monad and monad transformers.

8 RELATED WORK

This work directly builds on the previous work on Dijkstra Monads in F* (Swamy et al. 2013, 2016), in particular the DM4Free approach (Ahman et al. 2017), which we discussed in detail in §1 and §2.2. Our generic framework has important advantages: (1) it removes the previous restrictions on the computational monad; (2) it gives much more flexibility in choosing the specification monad and effect observation; (3) it builds upon a generic dependent type theory, not on F* in particular.

Jacobs (2015) studies adjunctions between state transformers and predicate transformers at a categorical level. From this he derives a class of specification monads that are obtained from the
state monad transformer and an abstract notion of logical structures. He gives abstract conditions for the existence of such specification monads and of effect observations. Hasuo (2015) builds on the state-predicate adjunction of Jacobs to provide algebra-based effect observations (in the style of our last paragraph from §4.1) for various pairs of computation and specification monads. Our work takes inspiration from this, but provides a more concrete account focused at precisely covering the use of Dijkstra monads for program verification. In particular, we provide concrete recipes for building specification monads useful for practical verification (§4). Finally, we show that our Dijkstra monads are equivalent to the monad morphisms built in these earlier works.

Katsumata (2014) uses graded monads to provide a semantics to type-and-effect systems, introduces effect observations as monad morphisms, and constructs graded monads out of effect observations by restricting the specification monads to their value at \( 1 \). We extend his construction to Dijkstra monads, showing that they are equivalent to effect observations, and unify Katsumata’s two notions of algebraic operation. A graded monad can intuitively be seen as a non-dependent version of a Dijkstra monad (a monad-like structure indexed by a monoid rather than a monad) but providing a unifying formal account is not completely straightforward; we leave it as future work.

Katsumata (2013) gives a semantic account of Lindley and Stark (2005)’s \( \top \top \) -lifting, a generic way of lifting relations on values to relations on monadic computations, parameterised by a basic notion of relatedness at a fixed type. Monad morphisms \( MA \rightarrow ((A \rightarrow P) \rightarrow P) \), as used to generate Dijkstra Monads in §3.4 are also unary relational liftings \( (A \rightarrow P) \rightarrow (MA \rightarrow P) \), and could be generated by \( \top \top \) -lifting. Further, binary relational liftings could be used to generate monadic relations that yield Dijkstra monads by the construction in §5. In both cases, what is specifiable about the underlying computation would be controlled by the chosen basic notion of relatedness.

Rauch et al. (2016) provide a generic verification framework for first-order programs with monadic effects. Their framework is quite different from ours, even beyond their restriction to first-order programs, since their specifications are “innocent” effectful programs, which can observe the computational context (e.g., the state), but not change it. However, this introduces a tight coupling between computations and specifications, while we take advantage of effect observations to provide much bigger flexibility. In fact, we can embed their framework into ours, since their notion of weakest precondition gives rise to an effect observation.

Generic reasoning about computational monads dates back to Moggi’s (1989) seminal work that proposes an embedding of his computational metalanguage into higher-order logic. Pitts & Moggi’s evaluation logic (Moggi 1995; Pitts 1991) later introduces modalities to reason about the result(s) of computations, but not about the computational context. Plotkin and Pretnar (2008) propose a generic logic for algebraic effects that encompasses Moggi’s computational \( \lambda \)-calculus, evaluation logic, and Hennesy-Milner logic, but does not extend to Hoare logic style reasoning for state.

9 CONCLUSION AND FUTURE WORK

This work proposes a general semantic framework for verifying programs with arbitrary monadic effects using Dijkstra monads obtained from effect observations, which are monad morphisms from a computation to a specification monad. This loose coupling between the computation and the specification monad provides great flexibility in choosing the effect observation most suitable for the verification task at hand. We show that our ideas are general by applying them to both Coq and F*, and we believe that they could also be applied to other dependently typed programming languages, such as Agda (Norell 2007), Lean (de Moura et al. 2015), NuPRL (Constable et al. 1986), Cedille (Diehl et al. 2018), or even Dependent Haskell (Weirich et al. 2017).

In the future, we plan to apply our framework to more computational monads, such as probabilities (Giry 1982). We also plan to investigate richer partiality monads that support fixpoints (Altenkirch et al. 2017; McBride 2015), aiming to reconstruct from first principles F*’s primitive
We start by expliciting the denotation from SM. We plan to complete the SM development and release it as a separate library for Coq. Categorical (ordered) relative monads for our specifications, by making return and bind work on pairs of values.

Another interesting direction is extending Dijkstra monads and our semantic framework to relational reasoning, in order to obtain principled semi-automated verification techniques for properties of multiple program executions (e.g., noninterference) or of multiple programs (e.g., program equivalence). As a first step, we plan to investigate switching from (ordered) monads to (ordered) relative monads for our specifications, by making return and bind work on pairs of values.

Finally, the SM language provides a general way to obtain correct-by-construction monad transformers, which could be useful in many other settings, especially within proof assistants. We plan to complete the SM development and release it as a separate library for Coq. Categorical intuitions also provide potential principled extensions of SM, e.g., some form of refinement types.

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A APPENDIX

A.1 Dijkstra monads as displayed algebras, relation to graded monads

The framework developed by Kaposi and Kovács (2019) can be used to capture the notion of Dijkstra monad in a more concise way: they can be seen as display algebras of a signature $\Sigma_{\text{mon}}$. Concretely, Kovács proposed (in private communication) the following signature to capture Dijkstra monads:

$$
\begin{aligned}
M & : \text{Set} \Rightarrow \mathcal{U}, \\
\text{ret} & : (A : \text{Set}) \Rightarrow A \Rightarrow \mathcal{I}(M A), \\
(-)^\dagger & : (AB : \text{Set}) \Rightarrow (\Pi_A M B) \Rightarrow MA \Rightarrow \mathcal{I}(M A), \\
\text{bind-ret} & : (A : \text{Set}) \Rightarrow (m : MA) \Rightarrow \text{Id}(MA) (\text{ret}^\dagger m) m, \\
\text{ret-bind} & : (A B : \text{Set})(x : A) \Rightarrow (f : \Pi_A M B) \Rightarrow \text{Id}(M B)(f^\dagger (\text{ret} x)) (f x), \\
\text{bind-assoc} & : (A B C : \text{Set}) \Rightarrow (m : MA)(f : \Pi_A M B)(g : \Pi_B M C) \Rightarrow \text{Id}(M C)(g^\dagger (f^\dagger m)) ((\lambda x. g^\dagger (f x))^\dagger m)
\end{aligned}
$$

Here $\Pi$ is here the constructor for infinitary ($A$-indexed for any Set $A$) products. Taking models of this signature in the CwF of sets and families gives monads on Set, and unary logical predicate gives the simpler notion of Dijkstra monad. Taking models in the CwF of preordered sets, monotonic functions and fibrations of preorders gives the notion of ordered monad and the unary logical predicate should provide the notion of Dijkstra monad with weakening.

B PROOF OF MONOTONICITY OF THE DENOTATIONS FROM SM

We start by expliciting the denotation from SM to $\mathcal{L}$ on terms. We write $[t]^\gamma_M : [C]_M$ for the denotation of the term $\Gamma \vdash t : C$ with respect to a monad $M$ and substitution $\gamma : [\Gamma]^\gamma_M$. 

$$
\begin{aligned}
[t]^\gamma_M &= \text{ret}^M, \\
(t_1, t_2)^\gamma_M &= \text{bind}^M, \\
[t]^\gamma_M &= \langle [t_1]^\gamma_M, [t_2]^\gamma_M \rangle, \\
[\pi_t]^\gamma_M &= \pi_t[t]^\gamma_M \\
[x]^\gamma_M &= \gamma(x), \\
[\lambda x^C. t]^\gamma_M &= \lambda x^{[C]}_M. [t]^\gamma_M^{[x:=x]}, \\
[t_1 \cdot t_2]^\gamma_M &= [t_1]^\gamma_M \cdot [t_2]^\gamma_M \\
[\lambda x^A. t]^\gamma_M &= \lambda x^A. [t]^\gamma_M^{[x:=x]}, \\
[t u]^\gamma_M &= [t]^\gamma_M u
\end{aligned}
$$
\[ \Gamma \vdash_{\text{SM}} \text{bind} \ (\text{ret} \ x) \ f \equiv f \ x \]

\[ \Gamma \vdash_{\text{SM}} \text{bind} \ m \ \text{ret} \equiv m \]

\[ \Gamma \vdash_{\text{SM}} \text{bind} \ m \ (\lambda x. \text{bind} \ (f \ x) \ g) \equiv \text{bind} \ (\text{bind} \ m \ f) \ g \]

\[ \Gamma \vdash_{\text{SM}} \pi_i \ (t_1, \ t_2) \equiv t_i \]

\[ \Gamma \vdash_{\text{SM}} \ (\lambda x. \ t) \ u \equiv t \{u/x\} \]

\[ \Gamma \vdash_{\text{SM}} \lambda x. \ t \ x \equiv t \]

\[ \Gamma \vdash_{\text{SM}} (\lambda^\circ x. \ t_1) \ t_2 \equiv t_1 \{t_2/x\} \]

+ reflexivity, symmetry, transitivity and congruence for all term constructors

Fig. 5. Equational theory for SM

We prove the two missing lemmas in the Coq development to extend to the case when the order on the monad \( M \) used for the denotation is not discrete. From these lemmas, we obtain that applying a monad transformer defined via an internal monad in SM to a specification monad is still a specification monad and that the \text{lift} \ t \ are monotonic.

**Theorem 2 (Monotonicity of denotation).** Let \( M \) be an ordered monad, \( \Delta; \Gamma \vdash_{\text{SM}} t : C \) a term in \( \text{SM} \), \( \vdash_{\mathcal{L}} \delta : \Delta \) a substitution for the \( \mathcal{L} \) context \( \Delta_i \vdash_{\text{SM}} \gamma_i : [\Gamma]_M \) \( i=1,2 \) substitutions for the \( \text{SM} \) context \( \Gamma \) such that \( \forall (x : C_0) \in \Gamma, \gamma_i(x) \leq C_0 \gamma_2(x) \). Then \( \llbracket t \delta \gamma_1 \rrbracket_M \leq C \llbracket t \delta \gamma_2 \rrbracket_M \).

**Proof.** By induction on the typing derivation of \( t \):

**Case** \( t = \text{ret}_A : A \rightarrow \text{M} A \), by reflexivity

\[ \llbracket \text{ret}_A \rrbracket_M \delta \gamma_1 = \llbracket \text{ret}_A \rrbracket_M \leq A \rightarrow \text{M} A \]

\[ \llbracket \text{ret}_A \rrbracket_M \delta \gamma_2 = \llbracket \text{ret}_A \rrbracket_M \]

**Case** \( t = \text{bind}_{A,B} : (A \rightarrow \text{M} B) \rightarrow (\text{M} A \rightarrow \text{M} B) \), by reflexivity, that holds because \( \text{bind}^M \) is monotonic

\[ \llbracket \text{bind}_{A,B} \rrbracket_M \delta \gamma_1 = \llbracket \text{bind}_{A,B} \rrbracket_M \leq (A \rightarrow \text{M} B) \rightarrow (\text{M} A \rightarrow \text{M} B) \]

\[ \llbracket \text{bind}_{A,B} \rrbracket_M \delta \gamma_2 = \llbracket \text{bind}_{A,B} \rrbracket_M \]

**Case** \( t = (t_1, \ t_2) : A \times B \), by induction

\[ \llbracket t_1 \rrbracket_M \delta \gamma_1 \leq A \llbracket t_1 \rrbracket_M \delta \gamma_2 \]

\[ \llbracket t_2 \rrbracket_M \delta \gamma_1 \leq B \llbracket t_2 \rrbracket_M \delta \gamma_2 \]

so

\[ \llbracket (t_1, \ t_2) \rrbracket_M \delta \gamma_1 = \llbracket (t_1) \rrbracket_M \delta \gamma_1 \llbracket (t_2) \rrbracket_M \delta \gamma_2 \leq A \times B \llbracket (t_1) \rrbracket_M \delta \gamma_2 \llbracket (t_2) \rrbracket_M \delta \gamma_2 \]

**Case** \( t = \pi_i t' : A_i \), by induction and extensionality

\[ \llbracket \pi_i[t'] \rrbracket_M \delta \gamma_1 \leq A_i \llbracket t' \rrbracket_M \delta \gamma_2 \]

**Case** \( t = \lambda x. \ t : (x : A) \rightarrow C \), by induction for any \( v : A \),

\[ \llbracket t' \rrbracket_M \delta \gamma_1 = C \{v/x\} \llbracket t' \rrbracket_M \delta \gamma_2 \]

we conclude by reduction since

\[ \llbracket \lambda x. \ t \rrbracket_M \delta \gamma_1 \ v = (\lambda y. \ llbracket t' \rrbracket_M \delta \gamma_1 \ y) \ v = \llbracket t' \rrbracket_M \delta \gamma_1 \ y \]

**Case** \( t = t' \ v : C \{v/x\} \), by induction

\[ \forall v_0 : A_i \llbracket t' \rrbracket_M \delta \gamma_1 \ v_0 \leq C \{v_0/x\} \llbracket t' \rrbracket_M \delta \gamma_2 \ v_0 \]

so

\[ \llbracket t' \ v \rrbracket_M \delta \gamma_1 \ v \leq C \{v/x\} \llbracket t' \rrbracket_M \delta \gamma_2 \ v = \llbracket t' \ v \rrbracket_M \delta \gamma_2 \]
Case $t = \lambda^x. t': C_1 \rightarrow C_2$, for any $m_1 \leq t^1, m_2, y_1[x := m_1] \leq t^2, y_1[x := m_2]$ and by induction
$$\llbracket t' \rrbracket^1_M \leq C_1 \llbracket t' \rrbracket^2_M$$
and we conclude since for $i = 1, 2$
$$(\llbracket \lambda^x. t' \rrbracket^1_M)^i_m = (\lambda y_i. \llbracket t' \rrbracket^2_M)^i_{m_i} = \llbracket t' \rrbracket^i_M$$
Case $t = t_1 \times t_2 : C_2$, by induction hypothesis applied to $t_2 : C_1$,
$$\llbracket t_1 \rrbracket^1_M \llbracket t_2 \rrbracket^2_M \leq C_1 \llbracket t_1 \rrbracket^2_M \llbracket t_2 \rrbracket^2_M$$
so by induction hypothesis applied to $t_1 : C_1 \rightarrow C_2$
$$\llbracket t_1 \rrbracket^1_M \llbracket t_2 \rrbracket^2_M \leq C_2 \llbracket t_1 \rrbracket^2_M \llbracket t_2 \rrbracket^2_M$$
\[\square\]

**Theorem 3 (Monotonicity of Relational Interpretation).** Let $\Delta \vdash_{SM} C$ type, $M_1, M_2$ two ordered monads and $(\mathcal{R}_A)A$ a family of monotonic relations $R_A : M_1 A \times M_2 A \rightarrow \mathbb{P}$, that is $R_A$ is an ideal wrt the order on $M_1 A \times M_2 A$, then $\llbracket C \rrbracket^R_{M_1, M_2}$ is monotonic.

**Proof.** by induction on the derivation of $C$ :

**Case** $C = \mu A \{ \llbracket \mu A \rrbracket^R_{M_1, M_2} = R_A \}$ is monotonic by assumption

**Case** $C = C_1 \times C_2$, suppose $(m_1, m_2) \llbracket C_1 \times C_2 \rrbracket^R_{M_1, M_2} (n_1, n_2)$, $(m_1, m_2) \leq C_1 \times C_2 (m_1', m_2'), (n_1, n_2) \leq C_1 \times C_2 (n_1', n_2')$ then by induction hypothesis $m_1' \llbracket C_1 \rrbracket^R_{M_1, M_2} n_1'$ and $m_2' \llbracket C_2 \rrbracket^R_{M_1, M_2} n_2'$ so $(m_1', m_2') \llbracket C_1 \times C_2 \rrbracket^R_{M_1, M_2} (n_1', n_2')$

**Case** $C = (x : A) \rightarrow C'$, suppose $f \llbracket (x : A) \rightarrow C' \rrbracket^R_{M_1, M_2} g, f \leq (x : A) \rightarrow C' g'$ then for any $\nu : A, (f \nu) \llbracket C' \{ \nu / x \} \rrbracket^R_{M_1, M_2} (g \nu), f \nu \leq C' \{ \nu / x \} g' \nu$ so by inductive hypothesis $(f' \nu) \llbracket C' \{ \nu / x \} \rrbracket^R_{M_1, M_2} (g' \nu)$, hence $f' \llbracket (x : A) \rightarrow C' \rrbracket^R_{M_1, M_2} g'$

**Case** $C = C_1 \rightarrow C_2$, suppose $f \llbracket C_1 \rrbracket^R_{M_1, M_2} C_2 \rrbracket^R_{M_1, M_2} g, f \leq C_1 \rightarrow C_2 f'$ and $g \leq C_1 \rightarrow C_2 g'$, for any $m \llbracket C_1 \rrbracket^R_{M_1, M_2} m, (f m) \llbracket C_2 \rrbracket^R_{M_1, M_2} (g n), m \leq C_1 m$ and $n \leq C_2 n$ so if $m \leq C_2 f' m$ and $g n \leq C_2 g' n$, hence by induction hypothesis $(f' m) \llbracket C_2 \rrbracket^R_{M_1, M_2} (g' n)$

\[\square\]

**C Linear Type System for SM**

We present a type system for SM where contexts $\Gamma | \Xi$ are equipped with distinguished position $\Xi$ called a stoup. The stoup can be either empty or containing one variable of a type $C$ from SM. Linear types are a refinements of types from $\Xi$ given by the following grammar

$$L := C | C_1 \rightarrow C_2 | L_1 \times L_2 | (x : A) \rightarrow L | L_1 \rightarrow L_2$$

where $A \in Type_L$, $C, C_1, C_2 \in Type_{SM}$. In particular the linear function space $C_1 \rightarrow C_2$ should be understood as a subtype of $C_1 \rightarrow C_2$ whose denotation ought to be a set of homomorphisms with respect to the algebra structures on the denotations of its domain and codomain, thus cannot be nested.

The following theorem holds:

**Theorem 4 (Linear terms are homomorphisms).** Let $M$ be a monad, $\Gamma | \gamma : \llbracket \Gamma \rrbracket_M$, then the following diagram commutes

$$
\begin{array}{c}
M[C_1]_M \xrightarrow{\alpha_{C_1}} [C_1]_M \\
\downarrow M[t]_M \downarrow \downarrow \downarrow \downarrow \\
M[C_2]_M \xrightarrow{\alpha_{C_2}} [C_2]_M
\end{array}
$$
that in turn proves that the right square in the diagram below commutes thanks to Theorem 4:

\[
\begin{array}{ccc}
A | - \vdash_{\text{lin}} \text{ret} : A \to MA & A, B | - \vdash_{\text{lin}} \text{bind} : MA \to (A \to MB) \to MB \\
(x : C) \in \Gamma & \Gamma | \Xi \vdash_{\text{lin}} t_1 : C_1 & \Gamma | \Xi \vdash_{\text{lin}} t : C_1 \times C_2 \\
\Gamma | x : C \vdash x : C & \Gamma | - \vdash x : C & \Gamma | \Xi \vdash_{\text{lin}} \langle t_1, t_2 \rangle : C_1 \times C_2 \\
\Gamma, x : A | \Xi \vdash_{\text{SM}} \lambda x. t : (x : A) \to C & \Gamma | \Xi \vdash_{\text{SM}} \lambda x. t : C_1 \to C_2 & \Gamma | \Xi \vdash_{\text{lin}} \pi_1 t : C_i \\
\Gamma | \Xi \vdash_{\text{SM}} \lambda_2 t_2 : C_1 & \Gamma | \Xi \vdash_{\text{SM}} t_1 : C_1 \to C_2 & \Gamma | \Xi \vdash_{\text{lin}} t_2 : C_2 \\
\Gamma | - \vdash_{\text{lin}} \lambda_3 x. t : C_1 \to C_2 & \Gamma | - \vdash_{\text{lin}} t : C_1 \to C_2 & (\Xi = \Xi_1 \land \Xi_2 = \emptyset) \lor (\Xi = \Xi_2 \land \Xi_1 = \emptyset) \\
\Gamma | \Xi_1 \vdash_{\text{lin}} t_1 : t_2 : C_2
\end{array}
\]

Fig. 6. Typing rules for SM with linearity condition.

For an internal monad \(X \vdash_{\text{SM}} C\) in SM, the linearity condition on \(\text{bind}^C\) requires a derivation of

\[
A, B | - \vdash_{\text{lin}} \text{bind}^C : C\{A/X\} \to (A \to C\{B/X\}) \to C\{B/X\}
\]

from which we can derive that

\[
A, B, f : A \to B | - \vdash_{\text{lin}} \lambda^x. \text{bind}^C (f y) : C\{A/X\} \to C\{B/X\}
\]

that in turn proves that the right square in the diagram below commutes thanks to Theorem 4:

\[
\begin{array}{ccc}
MA & \xrightarrow{M(\text{ret}^C)} & M[C]_M A \\
Mf & \downarrow & \downarrow \\
MB & \xrightarrow{M(\text{ret}^B)} & M[C]_M B
\end{array}
\]

\[
\begin{array}{ccc}
\alpha^C_{A,M} & & \alpha^C_{M,A} \\
\alpha^B_{M,B} & & \alpha^C_{M,B}
\end{array}
\]

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