A SEMI-EMPIRICAL MASS-LOSS RATE IN SHORT-PERIOD CATAclySMIC VARIABLES

FEDIR V. SIROTKIN AND WOONG-TAE KIM

Department of Physics and Astronomy, FPRD, Seoul National University, Seoul 151-742, South Korea; sirotkin.f.v@gmail.com, wkim@astro.snu.ac.kr

Received 2010 May 31; accepted 2010 August 4; published 2010 September 9

ABSTRACT

The mass-loss rate of donor stars in cataclysmic variables (CVs) is of paramount importance in the evolution of short-period CVs. Observed donors are oversized in comparison with those of isolated single stars of the same mass, which is thought to be a consequence of the mass loss. Using the empirical mass–radius relation of CVs and the homologous approximation for changes in effective temperature \( T_e \), orbital period \( P \), and luminosity of the donor with the stellar radius, we find the semi-empirical mass-loss rate \( M_2 \) of CVs as a function of \( P \). The derived \( M_2 \) is at \( \sim 10^{-9.5} - 10^{-10} \, M_\odot \, yr^{-1} \) and weakly depends on \( P \) when \( P > 90 \) minutes, while it declines very rapidly toward the minimum period when \( P < 90 \) minutes, emulating the \( P-T_e \) relation. Due to strong deviation from thermal equilibrium caused by the mass loss, the semi-empirical \( M_2 \) is significantly different from and has a less-pronounced turnaround behavior with \( P \) than suggested by previous numerical models. The semi-empirical \( P-M_2 \) relation is consistent with the angular momentum loss due to gravitational wave emission and strongly suggests that CV secondaries with \( 0.075 < M_2 < 0.2 \, M_\odot \) are less than 2 Gyr old. When applied to selected eclipsing CVs, our semi-empirical mass-loss rates are in good agreement with the accretion rates derived from the effective temperatures \( T_1 \) of white dwarfs, suggesting that \( M_2 \) can be used to reliably infer \( T_2 \) from \( T_1 \). Based on the semi-empirical \( M_2 \), SDSS 1501 and 1433 systems that were previously identified as post-bounce CVs have yet to reach the minimal period.

Key words: binaries: close – binaries: eclipsing – novae, cataclysmic variables – stars: dwarf novae – stars: fundamental parameters – stars: low-mass – white dwarfs

Online-only material: color figures

1. INTRODUCTION

Cataclysmic variables (CVs) are semi-detached binaries with a white dwarf (WD) primary and a low-mass companion which transfers mass to a WD through Roche lobe overflow. The observed orbital period distribution of CVs has three important features: a sharp short-period cut-off at the orbital period of \( P_{\text{min}} \approx 76.2 \) minutes, referred to as the “minimum period;” the period gap, a dearth of systems with periods between \( P_{\text{gap}}^{-} \approx 129 \) minutes and \( P_{\text{gap}}^{+} \approx 192 \) minutes (Knigge 2006); and the “period minimum spike,” a significant accumulation of systems at \( P \approx 80–86 \) minutes (Gänsicke et al. 2009). For CVs with period above \( P_{\text{gap}}^{+} \), the dominant angular momentum loss mechanism is thought to be magnetic braking, while the gravitational radiation may be responsible for the angular momentum loss for CVs below \( P_{\text{gap}}^{-} \). In this work, we focus on CVs only with \( P < P_{\text{gap}}^{-} \).

A characteristic feature of CVs is the mass transfer from a secondary to a WD. Since the gas stream carries substantial angular momentum, it forms an accretion disk around the WD. The reaction of a CV on the mass transfer can be qualitatively understood with the help of the dimensionless parameter

\[
\tau = \frac{\tau_M}{\tau_{KH}},
\]

where \( \tau_M \) and \( \tau_{KH} \) denote the mass-loss and Kelvin–Helmholtz timescales of the donor, respectively (Paczynski 1981). At the initial phase of mass transfer, \( \tau \gg 1 \), so that the donor remains close to thermal equilibrium and thus follows the mass–radius relation (MRR) of main-sequence stars

\[
R_2 \propto M_2^{\xi_E},
\]

with the effective mass–radius exponent \( \xi_E \) close to \( 0.64_{-0.02}^{+0.02} \) (Knigge 2006). Here, the subscript “2” denotes the quantities of CV donors. Both \( R_2 \) and \( P \) of the CV decrease as the donor keeps losing mass during this early phase of CV evolution.

As the mass transfer continues, the binary shrinks due to the angular momentum loss, reducing \( \tau \). When \( \tau_M \) and \( \tau_{KH} \) become comparable to each other (\( \tau \approx 1 \), the donor goes out of thermal equilibrium. Since the surface luminosity is no longer balanced with the nuclear luminosity, the donor expands, causing \( \xi_E \) to decrease with time (King 1988). Using the Hayashi theory and homologous approximation, Stehle et al. (1996) studied the reaction of the donor to mass loss and found that \( R_2 \) increases with decreasing \( \tau \). Observations confirm this theoretical prediction, showing that the radii of donors in short-period CVs are \( \sim 5\%–50\% \) larger than those of isolated main-sequence stars (Patterson et al. 2005; Knigge 2006).

The effective mass–radius exponent \( \xi_E \) becomes smaller as \( M_2 \) decreases toward the brown dwarfs’ masses. The change in the MRR of donors slows down the temporal change in the orbital period as the donor continues losing mass. The orbital period reaches a minimum value when \( \xi_E \approx 1/3 \) at which \( dP/dM_2 = 0 \) (Rappaport et al. 1982). As \( \tau \) decreases further, \( \xi_E \) declines to \( 0.21_{-0.10}^{+0.02} \) (Knigge 2006). This in turn leads to an increase in \( P \). Such a turnaround of the orbital period at \( P_{\text{turn}} \) is referred to as the “period bounce.”

The evolutionary scenario described above suggests that the mass-loss rates \( M_2 \) (and the related thermal relaxation) of donors control the evolution of CVs. It also implies that the observed distribution of CVs versus the orbital period may be a consequence of a certain relation between \( M_2 \) and \( P \). Since \( M_2(P) \) not only provides a way to estimate the ages of CVs but can also be used to check the dominant mechanism for angular...
momentum loss, finding a reliable $P$–$M_2$ relationship is of great importance in understanding CV evolution.

There have been numerous studies to derive the $P$–$M_2$ relation; these can be categorized into two groups: purely theoretical work (e.g., Rappaport et al. 1982; Ritter 1988; Kolb & Baraffe 1999; Howell et al. 2001) and semi-empirical work (e.g., Patterson 1984; Urban & Sion 2006; Townsley & Gänsicke 2009). For the first, theoretical studies made use of stellar evolution codes combined with the analytical mass-transfer models of Ritter (1988) or others. The common results of these numerical experiments are that (1) the orbital period exhibits the turnaround behavior at $P_{\text{turn}}$ that is about $\sim 10\%$ smaller than the observed $P_{\text{min}}$ (Rappaport et al. 1982; Kolb & Baraffe 1999; Howell et al. 2001) and (2) compared to isolated stars, the secondaries in short-period CVs are oversized due to the mass loss but only by $\sim 5\%$–$20\%$, which is about a factor $\sim 1.5$–$2$ smaller than the observed CV sizes. Since the bloating of the donor depends on the efficiency of the mass loss, these results suggest either the numerical models underestimate $M_2$ above $P_{\text{turn}}$ or the efficiency of the angular momentum loss is higher than expected (Kolb & Baraffe 1999; Renvoizé et al. 2002). Despite the attempts to include the effects of tidal and rotational perturbations (Rizzolli et al. 2001; Renvoizé et al. 2002; Kolb & Baraffe 1999), additional angular momentum loss via magnetic braking (Andronov et al. 2003), accretion disk winds and magnetic propeller (see Barker 2003 for review), etc., clear explanations for the discrepancies between $P_{\text{turn}}$ and $P_{\text{min}}$ and between the empirical and numerical MRRs have yet to be found.

For the second, semi-empirical approach uses, for instance, the effective temperatures of WDs (Townsley & Bildsten 2003, 2004; Townsley & Gänsicke 2009) or accretion disk luminosities (Patterson 1984) to infer the accretion rates $M_1$. The physical basis for the former is that WDs in CVs are observed to be hotter than expected from their ages (Sion 1995), and such “overheating” of WDs may be a result of the compressional heating due to the gas piled up on their surfaces (Townsley & Bildsten 2003). For the latter method, the accretion luminosities can give direct information on $M_1$ if the disk properties are well constrained. When applied to dwarf novae (DNe) that accrete sporadically and have thermally unstable disks, $M_1$ calculated from the two methods have a different scatter for a given $P$.

The mass-transfer rates based on the accretion luminosities probably measure an accretion rate averaged over short time intervals, e.g., a few decades, which makes them quite sensitive to short-time disk variability (Townsley & Gänsicke 2009). This inevitably leads to a high dispersion in $M_1$ estimated from the disk luminosities. On the other hand, the effective temperatures of WDs are likely to give a relatively long-term averaged $M_1$, although this method may suffer from uncertainties related to unknown properties such as masses and radii of WDs as well as some auxiliary assumptions on their structures and boundary layers (Patterson 2009).

In this paper, we construct another semi-empirical relation between the mass-loss rate and the orbital period, and compare the result with those of the purely theoretical and the semi-empirical approaches mentioned above. We start from the empirical MRR of Knigge (2006) for superhumping CVs and apply the homologous approximation in order to calculate the dependencies of the effective temperature and gravothermal luminosity upon the bloating of the donor. Assuming that the observed MRR of CV donors is a consequence of the mass-loss and thermal relaxation processes, we calculate the mass-loss rates. We will show that the resulting mass-loss rates are broadly in line with those on the basis of the effective temperatures of WDs, while different from those of the purely theoretical models of Kolb & Baraffe (1999). We will also show that our results are consistent with the angular momentum loss of CVs due primarily to the emission of gravitational waves. Since radii of CV donors likely trace the long-term evolution of $M_2$ (Knigge 2006) and depend only weakly on details of the mass-transfer processes, the resulting mass-loss rates from the empirical MRR are less model-dependent and less sensitive to the short-term effects than the other semi-empirical estimates that rely on accretion activities.

The organization of this paper is as follows. In Section 2, we introduce the theoretical MRR for isolated single stars and the empirical MRR for CVs’ donors, and define the bloating factor. We then describe a method to find the effective temperature and gravothermal luminosity of donors by using the bloating factor. In Section 3, we calculate the semi-empirical mass-loss rate, construct the mass-loss history to estimate the CV ages, and show that the derived mass-loss rate is consistent with CV evolution by angular momentum loss via the emission of gravitational waves. In Section 4, we apply our results to selected eclipsing short-period CVs and show that the semi-empirical mass-loss rate is consistent with the accretion rates inferred from the effective temperatures of WDs. We also suggest a way to estimate the effective temperatures of donors assuming that mass is conserved. Finally, we summarize our results in Section 5.

2. MASS–RADIUS AND PERIOD-EFFECTIVE TEMPERATURE RELATIONS

We consider a CV consisting of a WD (or accretor) with mass $M_1$ and a secondary (or donor) with mass $M_2$, radius $R_2$, and effective temperature $T_2$. We assume that the donor fills its critical equipotential surface. In accordance with the results of Patterson et al. (2005) and Knigge (2006), we assume $M_1 = 0.75 \, M_\odot$ throughout this paper. We use an additional subscript “0” to denote the quantities of isolated, single stars in hydrostatic and thermal equilibrium to which CV secondaries will be compared. In this section, we use the empirical relationship between masses and radii of CV secondaries to calculate the bloating factor in comparison with the single stars of the same mass. We then treat the bloating factor as perturbations to the properties of single stars in order to find the orbital period, effective temperature, and gravothermal luminosity of CVs.

2.1. Mass–Radius Relations

As the MRRs of single stars that we use as the references to those of CV secondaries, we take composite relations constructed in a manner similar to Knigge (2006): the BCAH98 isochrone of Baraffe et al. (1998) for stars with $0.075 \, M_\odot < M < 0.3 \, M_\odot$, the DUSTY isochrone of Baraffe et al. (2002) for $0.06 \, M_\odot < M < 0.075 \, M_\odot$ (1500 $\lesssim T_2 \lesssim 2000$ K), and the COND isochrone of Baraffe et al. (2003) for $M < 0.06 \, M_\odot$. There are quite large uncertainties in the estimation of CV ages. Period bouncers are thought of as a final stage of CV evolution, suggesting that they are older than their “parent” CVs (see, e.g., Paczynski 1981; Rappaport et al. 1982; Kolb & Baraffe 1999). Although numerical experiments predict that CVs cross the period gap at $\sim$1 Gyr and reach $P_{\text{turn}}$ in $\sim$4 Gyr after the onset of the mass loss below the gap (e.g., Kolb & Baraffe 1999; Howell et al. 2001), temporal evolution of
CVs strongly depends on the mass-loss rate that is not well constrained observationally. To cover a wide range of CV ages, we adopt two composite isochrones with 1 Gyr and 5 Gyr ages (hereafter B1 and B5 sequences, respectively). Figure 1 plots the composite \( R_\text{2,0} \) of single stars as functions of \( M_2 \) for the B1 and B5 sequences, as dashed and solid lines, respectively. Both sequences give almost the same radii above \( 0.075 M_\odot \), while below this point the stellar radii in the B1 sequence are about \( \sim 10\% \) larger than in the B5 sequence.

As the MRR of CV secondaries, we take the relation derived by Knigge (2006) from the observed properties of superhumping CVs:

\[
\frac{R_2}{R_\odot} = \begin{cases} 
0.110 \left( \frac{M_2}{0.063 M_\odot} \right)^{0.21}, & M_2 < 0.063 M_\odot, \\
0.230 \left( \frac{M_2}{0.2 M_\odot} \right)^{0.64}, & 0.063 M_\odot \leq M_2 < 0.2 M_\odot, \\
0.299 \left( \frac{M_2}{0.2 M_\odot} \right)^{0.67}, & 0.2 M_\odot \leq M_2.
\end{cases}
\]

This empirical fit assumes \( M_1 = 0.75 M_\odot \) and incorporates in itself the additional empirical constraints that reproduce the observed locations of the period gap and the period minimum, such that \( P_\text{gap} \) occurs at \( M_2 = 0.2 \pm 0.02 M_\odot \) and \( P_\text{min} \) at \( M_2 = 0.063 \pm 0.009 M_\odot \). The intrinsic errors in \( R_2 \) for this empirical MRR are estimated to be about 2\%–3\% (Knigge 2006); we take allowance for 4\% uncertainties in \( R_2 \) throughout this work. We note a caveat that our choice of errors may underestimate real values below \( 0.075 M_\odot \) where only a few systems are observed and thus there is practically no calibrator. In addition, the masses of WDs may differ from the assumed mass of \( 0.75 M_\odot \), which can also change \( R_2 \). Equation (3) is plotted in Figure 1 as a dotted line with the shaded region denoting the associated uncertainties in \( R_2 \). Columns 1 and 2 in Table 1 give the \( M_2 \)–\( R_2 \) relation.

For comparison, Figure 1 also plots as a dot-dashed line the MRR of CVs with \( 0.03 M_\odot \leq M_2 < 0.21 M_\odot \) from the numerical models of Kolb & Baraffe (1999), which clearly has smaller radii than the empirical relation, typically by \( \sim 5\%–10\% \). This indicates that the deviation of real donors from thermal equilibrium is larger than that expected from the numerical models.

### 2.2. Bloating Factor

We define the bloating factor

\[
\alpha = \frac{R_2}{R_{2,0}},
\]

as the relative size of a CV secondary to a single star with the same mass. Figure 2 plots \( \alpha \) as dashed and solid lines for the B1 and B5 sequences, respectively, with the shaded and hatched regions representing the corresponding uncertainties. The discontinuity of \( \alpha \) at \( M_2 \geq 0.2 M_\odot \) corresponds to the period gap. Since \( P \propto \alpha^{3/2} \) (see below), the ratio of the bloating factors above and below the gap satisfies \( \alpha^{2/3}_\text{gap} \). In good agreement with the width of the gap. At the upper edge of the gap, the donors detach from their critical lobes and the mass loss stops. Since

---

1. The composite isochrones we adopt are slightly different from those in Knigge (2006) who took the 5 Gyr isochrone for donors with \( M \geq 0.075 M_\odot \) and 1 Gyr isochrone for \( M < 0.075 M_\odot \). The choice of the 1 Gyr isochrone in Knigge (2006) was to represent the numerical results of Kolb & Baraffe (1999).
explained as follows. Since stars with lower edge of the gap, they resume the mass loss and become they shrink within the gap. When the donors come out at the lower bound of the mass gap, they obtain a new equilibrium. This raises the gravothermal luminosity, \( L_K \), and causes the change of the mass loss. The donors are thought to be undersized due to the mass loss, they shrink within the gap. When the donors come out at the lower edge of the gap, they resume the mass loss and become undersized again.

Figure 2 shows that the bloating factor increases with the decreasing donor mass for \( M_2 > 0.075 M_\odot \) and then decreases with decreasing \( M_2 \) below this point. This can be qualitatively explained as follows. Since stars with \( M_2 < 0.3 M_\odot \) are fully convective and isentropic, they are well described by polytropes with index \( n = 3/2 \) (Burrows & Leibig 1993). The MRR for these polytropes is

\[
R_2 \propto K M_2^{-1/3},
\]

with \( K \) being the pressure constant (Chandrasekhar 1939), and then gives

\[
\alpha = \frac{K}{K_0}.
\]

The mass loss is capable of driving the donor out of thermal equilibrium. This raises the gravothermal luminosity, \( L_g = L_2 - L_{\text{mac}} \), where \( L_{\text{mac}} \) is the nuclear luminosity of the donor and \( L_2 \) is the surface luminosity. For fully convective stars, the change of \( K \) depends on \( L_g \) and \( \tau_{KH} \) as

\[
\frac{K}{K_0} \propto \frac{1}{\tau_{KH} L_2}
\]

(King 1988; Ritter 1996). Combining Equations (5)–(7), one obtains

\[
\frac{\dot{M}}{M} \propto - \frac{1}{\tau_{KH} L_2} + \frac{1}{\tau_M} \left( \xi_0 + \frac{1}{3} \right),
\]
where $\xi_{E0}$ is the mass–radius exponent of unperturbed, single stars. At the lower edge of the period gap, a donor remains close to thermal equilibrium so that $L_g \approx 0$ and $\xi_{E0} \approx 1$. In this case, the second term in the right-hand side of Equation (8) dominates and $\alpha/\alpha > 0$, indicating that $\alpha$ increases as the mass loss continues. When the donor reaches the minimum hydrogen burning mass (MHBM) $\approx 0.075 M_\odot$, the nuclear burning stops and donors are born as brown dwarfs, so that $L_g/L_2 = 1$. In the brown dwarf regime, on the other hand, the internal pressure is dominated by the degeneracy pressure with $\kappa$ mainly depending on the chemical composition (Burrows & Leibert 1993). In this case, $\xi_{E0}$ tends to $-1/3$ and the first term in the right-hand side of Equation (8) dominates, resulting in $\alpha/\alpha < 0$. For brown dwarf donors, the change in the equation of state partially compensates for the increasing tendency of the bloating factor due to the mass loss.

In addition to the mass loss, tidal and rotational perturbations can also change the donor size, depending on the polytropic index and the mass ratio. We assume that the applied perturbations do not change the specific entropy, which is reasonable since the stellar distortions occur in dynamical timescales. In Appendix A, we calculate the enlargement factor $\alpha_{\text{enl}}$ of a Roche lobe-filling donor as a function of $n$ for $q = 0.1$. The resulting $\alpha_{\text{enl}}$ for $n = 3/2$ polytropes against $M_2$ is plotted in Figure 2 as a dotted line. Note that the expansion of low-mass donors due to the tidal and rotational distortions is $\approx 4\%-5\%$, insensitive to the mass ratio for $q \approx 0.04$–0.4 (see also, e.g., Uryu & Eriguchi 1999; Renvoizé et al. 2002; Sirotkin & Kim 2009). It can be seen that while $\alpha_{\text{enl}}$ is, in general, considerably smaller than the observed bloating factor for stars with $0.04 M_\odot \lesssim M_2 \lesssim 0.15 M_\odot$, it is comparable to $\alpha$ when $M_2 \approx 0.2 M_\odot$ corresponding to $P_{\text{gap}}$ or when $M_2 < 0.04 M_\odot$ as donors approach $P_{\text{min}}$. This indicates that the effect of the isentropic expansion of donors on the bloating factor is non-negligible for CVs that just resume mass loss or are close to the minimum period.

2.3. Orbital Period and Effective Temperature

For a given MRR, the orbital period can be determined by the Roche lobe geometry and Kepler’s law as

$$P^2 = \frac{3\pi}{G\bar{\rho}} \frac{q}{f^3(1 + q)}, \tag{9}$$

where $\bar{\rho} = 3M_2/4\pi R_2^3$ is the mean density of the donor, $f = (2/3^{1/3})^{1/3}(1 + q)^{1/3}$ is the relative size of the donor to the orbital separation, and $q = M_2/M_1$ is the mass ratio (e.g., Kopal 1972). Equation (9) implies that the orbital period depends on the bloating of the donor as

$$P = P_0\alpha^{3/2}. \tag{10}$$

Clearly, the orbital period increases with the radius of a secondary.

Since donors below the period gap are fully convective, their luminosities are determined by the conditions at the very outermost layers (Kippenhahn et al. 1990), so that one can neglect the influence of the distortions on the central parts of a donor. To obtain the properties of the photospheric layers, we use the standard Hayashi theory. For the opacity law of the form

$$\kappa = \kappa_0 P_g^{a-1}, \tag{11}$$

with the gas pressure $P_g$, the ratio of the effective temperatures satisfies

$$\frac{T_2}{T_{2,0}} = \alpha^\gamma. \tag{12}$$

Figure 3. Dependence of the effective temperature $T_2$ of a donor on the orbital period $P$. The thick dashed and solid lines plot the semi-empirical $P$–$T_2$ relations for the B1 and B5 sequences, respectively, with the dashed and hatched regions denoting the corresponding uncertainties. The thin dashed and solid lines give the unperturbed $P_0$–$T_2$ relations for the B1 and B5 sequences, respectively. The semi-empirical $P$–$T_2$ relation of Knigge (2006) is plotted as the dot-dashed line.

(A color version of this figure is available in the online journal.)

where $\gamma = (3a - 1)/(5a + 2b + 5)$ (e.g., Kippenhahn et al. 1990; see also Stehle et al. 1996). For the realistic opacity laws applicable for low-mass main-sequence stars, $\gamma$ has a very small value. If the opacity is governed by $H^+$, for example, $a \approx 0.5$ and $b \approx 4.5$, leading to $\gamma \approx 0.03$. If the molecular absorption instead is a dominant opacity source ($a \approx 1$ and $b \approx 0$), $\gamma \approx 0.2$. Therefore, donors in short-period CVs have approximately the same effective temperatures as the corresponding isolated single stars of identical masses (e.g., Stehle et al. 1996).

To construct the semi-empirical $P$–$T_2$ relations, we first use the theoretical $M_2$–$T_{2,0}$ relation for isolated stars and calculate $P_0$ from Equation (9) by fixing $M_1 = 0.75 M_\odot$. We then use the empirical MRR and the bloating factor of donors to calculate $P$ and $T_2$ from Equations (10) and (12). The $P_0$–$T_{2,0}$ relations are plotted in Figure 3 as thin lines, while the thick lines are for the semi-empirical $P$–$T_2$ relations. The dashed and solid lines correspond to the B1 and B5 sequences, respectively. Again, the shaded and hatched regions give the corresponding uncertainties. Columns 3, 6, and 11 in Table 1 list the values of $P$ and $T_2$ against $M_2$ for both sequences. For a given mass, the expansion of a donor increases the orbital period while the effective temperature remains almost unchanged. The largest difference between $P$ and $P_0$ occurs at $T_2 \sim 2000$–$2200$ K, corresponding to $M_2 \approx 0.75 M_\odot$ where the bloating factor is maximal. The dot-dashed line plots the semi-empirical result of Knigge (2006). Because of the difference in the adopted ages of the isochrones, the result of Knigge (2006) for $M_2 > 0.075 M_\odot$ (with $T_2 > 2000$ K) and $M_2 < 0.06 M_\odot$ (with $T_2 < 1800$ K) is in agreement with our results for the B5 and B1 sequences, respectively. The donors at $P_{\text{min}}$ have mass $M_2 \sim 0.054$–$0.072 M_\odot$, corresponding to $T_2 \sim 1000$–$1800$ K and $\sim 1600$–$2100$ K for the B5 and B1.
sequences, respectively. The right choice of the stellar age is quite important in evaluating the mass-loss rate near $P_{\text{min}}$ since it depends rather sensitively on $T_2$ as $M_2 \propto T_2^4$ (Rappaport et al. 1982).

2.4. Luminosities

Combining Equation (12) with the Stefan–Boltzmann law

$$L_2 = 4\pi \sigma R_2^2 T_2^4,$$

(13)

where $\sigma$ is the Stefan–Boltzmann constant, one can show that the change in the stellar luminosity is given by

$$L_2/L_{2,0} = \alpha^{\frac{3+4\epsilon_2}{3+\epsilon_2}},$$

(14)

Although the effective temperature does not change much with the bloating factor, the additional $\alpha^2$ term in the Stefan–Boltzmann law increases $L_2/L_{2,0}$ significantly. Columns 5 and 10 in Table 1 give the values of $L_2$ for the B1 and B5 sequences, respectively.

The nuclear luminosity $L_{\text{nuc}}$ is determined by the energy produced in the central parts of a donor, so that $L_{\text{nuc}} \simeq M_2\epsilon_c$, where $\epsilon_c$ is the thermonuclear reaction rate given by

$$\epsilon_c \propto T_2^n \rho_2^{\eta-1},$$

(15)

with $T_2$ and $\rho_2$ denoting the central temperature and density, respectively. Since $T_2 \propto K_0^{2/3} \propto \alpha^{-1}$ for a monatomic ideal gas, the change in the nuclear luminosity due to the bloating factor is

$$\frac{L_{\text{nuc}}}{L_{\text{nuc},0}} = \alpha^{3-3\eta-n}.$$ 

(16)

Theoretical models for stellar structure and evolution give $n \approx 6$ and $\eta \approx 2$ (Burrows & Leibert 1993) for $T_2 \simeq 3 \times 10^6$ K and $\rho_2 \simeq 150$ g cm$^{-3}$, typical for the centers of low-mass stars.

Since $L_{2,0} = L_{\text{nuc},0}$ for isolated stars in thermal equilibrium, the gravothermal luminosity satisfies

$$\frac{L_g}{L_2} = 1 - \frac{L_{\text{nuc}}}{L_{\text{nuc},0}} \frac{L_2}{L_{2,0}}.$$ 

(17)

Thermonuclear burning stops in CV secondaries once they reach the MHBM $\approx 0.075 M_\odot$. At this point, donors are effectively born as brown dwarfs and the gravitational collapse becomes the dominant source of the stellar radiation. That is, $L_g/L_2 = 1$ for $M_2 \leq 0.075 M_\odot$.

Figure 4 plots $L_g/L_2$ versus $M_2$ for the B1 and B5 sequences as dashed and solid lines, respectively, with the uncertainties denoted by the shaded and hatched regions. Figures 2 and 4 together imply that at any given mass the contribution of the gravothermal luminosity increases with the bloating factor. If $R_2$ increases by 20%, more than 80% of the donor luminosity is generated by the release of thermal and gravitational energies.

3. MASS-LOSS RATE

Finding the theoretical mass-loss rates of CV donors is a daunting task since it involves many complicated processes including thermal and adiabatic relaxations, nuclear evolution, effects of tidal and rotational distortions, orbital angular momentum loss, etc., all of which may affect the internal structure significantly. Although the stationary model of Ritter (1988) has been used in various theoretical models of CV evolution, it relies on the parameters describing the donor atmosphere in the vicinity of the inner Lagrangian point, which are poorly known. By relating the mass-loss rate with angular momentum loss, on the other hand, the Ritter model allows us to check whether a proposed angular momentum loss mechanism is indeed responsible for CV evolution. As mentioned in Section 1, the numerical calculations that incorporate the stellar evolution and the mass-loss model of Ritter (1988) produce MRRs that differ from the empirical MRR.

3.1. Semi-empirical Mass-loss Rate

Instead of employing the Ritter model to calculate the mass-loss rate, we seek the $M_2$ that is required to reproduce the empirical MRR. We make use of the well-known relation between the effective mass–radius exponent $\xi_E$ of the donor and the mass-loss timescale:

$$\xi_E = \xi_S + \frac{7}{3} \frac{L_g}{L_2} \frac{7}{3} L_2$$ 

(18)

(Ritter 1996), where $\xi_S \approx -1/3$ is the adiabatic mass–radius exponent for fully convective stars. Equation (18) dictates how the MRR should behave in response to the mass-loss and thermal relaxation processes. When the mass transfer just commences, $L_g \approx 0$ and the donor response is adiabatic, leading to $\xi_E \approx \xi_S$. Since the ensuing deviation from the adiabatic evolution depends on how the donor reacts thermally on the mass loss, $\xi_E$ increases linearly with $\tau$ (Ritter 1996).

Combining Equations (9), (13), and (18) with the definitions of the Kelvin–Helmholtz timescale ($\tau_{\text{KH}} \equiv GM_2^2/(R_2L_2)$) and mass-loss timescale ($\tau_M \equiv M_2/|M_2|$), one obtains

$$\dot{M}_2 = \frac{1}{(\xi_E + 1/3)} \frac{56 \sigma L_g T_2^4 p^2}{243 \pi L_2^7}.$$ 

(19)
When $\xi_E = 1/3$ corresponding to the period bounce, Equation (19) recovers Equation (34) of Rappaport et al. (1982) for the mass-loss rate at the minimum period.

In Figure 5, we plot the semi-empirical $P-M_2$ relations calculated from Equation (19) for the 1 Gyr and 5 Gyr isochrones as dashed and solid lines in red color, respectively. The corresponding uncertainties are given by the shaded and hatched regions. Columns 7 and 12 of Table 1 list the values of $M_2$. The semi-empirical mass-loss rate is essentially non-stationary. For 129 minutes $>P > 90$ minutes, $M_2$ is in the range of $-10^{-5.5}$ to $-10^{-10} M_\odot$ yr$^{-1}$ and decreases relatively slowly as $M_2 \propto P^3$. Below 90 minutes, $M_2$ sharply declines toward $P_{\text{min}}$ as a result of a strong dependence of $M_2$ upon $T_2$, emulating the $P-T_2$ relation shown in Figure 3. Since the accretion rate decreases with $M_2$, this rapid decline of $M_2$ is consistent with the observed low accretion activities of CVs within the period spike identified in the Sloan Digital Sky Survey (SDSS) samples (Gänsicke et al. 2009). Note that the mass-loss rates for the B1 and B5 sequences are almost similar to each other for masses above the MHBM, suggesting that these are perhaps permanent values. In the brown dwarf regime, on the other hand, the ambiguity in the adopted stellar age leads to a large difference in the mass-loss rate, suggesting that it is necessary to use the appropriate isochrones for an accurate estimation of $M_2$ for donors with $M_2 < 0.075 M_\odot$.

For comparison, Figure 5 also plots as a dot-dashed line the $P-M_2$ relation from the numerical models of Kolb & Baraffe (1999), which shows a pronounced “boomerang” shape such that $M_2$ is nearly stationary above $P_{\text{min}}$ with $M_2 \propto P^{0.5}$. It also has $M_2$ significantly smaller for $M_2 > 0.075 M_\odot$ than our semi-empirical results. To see what causes the difference between our semi-empirical $P-M_2$ relation and the numerical result of Kolb & Baraffe (1999), we calculate $M_2$ from Equation (19) by using the MRR of Kolb & Baraffe (1999) together with the 5 Gyr isochrone. The resulting $P-M_2$ relation is plotted in Figure 5 as a dotted line, which is overall in good agreement with the numerical results of Kolb & Baraffe (1999). This suggests that smaller radii of donors in their work are a primary reason for the discrepancies between the results of the current work and Kolb & Baraffe (1999). This also proves to some extent that our approach provides a reasonable way to estimate the mass-loss rate.

From the semi-empirical $M_2$, we calculate $\tau$ for the B1 and B5 sequences, which are listed in Columns 8 and 13 of Table 1, respectively. Note that donors in CVs below $P_{\text{gap}}$ are out of thermal equilibrium with $\tau \lesssim 1$, which is the main cause of the donor expansion. Equations (17) and (19) imply that for given $M_2$ and $\xi_E$, the mass-loss rate increases more than $\alpha^2$ due mainly to changes in the orbital period and gravothermal luminosity. The resulting semi-empirical $P-M_2$ relation still has a boomerang shape, which is much less dramatic than that of Kolb & Baraffe (1999) due to the donor bloating and the associated thermal relaxation. Nevertheless, it is very difficult to detect the boomerang shape observationally because of the sharp decline of $T_2$ and $M_2$ below 90 minutes.

Once the $P-M_2$ and $P-M_2$ relations are found, it is straightforward to calculate the temporal change of the donor mass by solving $dM_2/dt = M_2$. We assume that donors are 1 Gyr old and have a fixed initial mass of $0.2 M_\odot$ when they emerge from the period gap. Since CVs with $P < 2$ hr are predominantly DNe (Ritter & Kolb 2003), we further assume that the masses of WDs are constant at $M_1 = 0.75 M_\odot$ throughout CV evolution. Figure 6 plots as dashed and solid lines the evolution of donor mass and orbital period calculated from the semi-empirical $M_2$ for the B1 and B5 sequences, respectively, as functions of time $t$ elapsed since $P_{\text{gap}}$ (i.e., $t_0 = 1$ Gyr). The dot-dashed lines show $M_2(t)$ and $P(t)$ from the mass-loss rate of Kolb & Baraffe (1999) for which $M_1 = 0.6 M_\odot$ and $M_2 = 0.21 M_\odot$ initially at $P_{\text{gap}}$. The changes in the donor mass and orbital period in the numerical model of Kolb & Baraffe (1999) are in general slower because of a lower mass-loss rate. Our semi-empirical results indicate that $M_2$ decreases from $0.2 M_\odot$ to $0.075 M_\odot$ in $\sim 1$ Gyr and that the minimum period is reached in $\sim 1$–2 Gyr after $P_{\text{gap}}$ without any essential dependence on the adopted stellar age. On the other hand, the numerical model of Kolb & Baraffe (1999) predicts that the same decrement of $M_2$ occurs in $\sim 3.5$ Gyr, while it takes the donors $\sim 4$ Gyr to reach $P_{\text{min}}$. Although uncertainties in the stellar ages make it difficult to follow $M_2(t)$ and $P(t)$ accurately below the MHBM, the close agreement between the results based on the B1 and B5 sequences suggests that the ages of CV secondaries with $0.075 M_\odot < M_2 < 0.2 M_\odot$ are less than 2 Gyr.

### 3.2. Consistency Check

Mass transfer in CVs below the period gap is considered to be driven by gravitational wave radiation (Paczynski 1981; Patterson 1984; Kolb & Baraffe 1999; Howell et al. 2001). While Equation (19) is useful to obtain $M_2$ from the empirical MRR independent of specifics of mass transfer, it does not provide any clue regarding what drives mass transfer in CVs. In this subsection, we make use of the stationary mass-transfer model of Ritter (1988) and check whether the semi-empirical $M_2$ calculated in Section 3.1 is consistent with the angular momentum loss via emission of gravitational waves.
Ritter (1988) presented a simple analytical model in which the mass transfer from a donor is treated as a stationary isothermal, subsonic flow of gas through the inner Lagrangian point. Taking allowance for the finite scale height of the donor photosphere and considering the temporal changes in the volume radii of the donor and the Roche lobe, Ritter (1988) showed that the stationary mass-transfer rate is given by

$$-\dot{M}_{2,R} = \frac{M_2}{(\xi_S - \xi_R)} \left[ \frac{7}{3} \frac{L_2}{L_\text{KH}} \frac{1}{\tau_1} - \frac{2}{\tau_J} \right],$$

where $\tau_1$ is the angular momentum loss timescale, and $\xi_R$ is the mass–radius exponent of the critical Roche lobe. Note that we neglect the nuclear-timescale term in Equation (20) since it is much longer than the other timescales involved. In the case of DNe for which the mass transfer is non-conservative, $\xi_R$ is given by

$$\xi_R = -\frac{5}{3} + \left( \frac{2}{3} + 2q \right) \frac{q}{1 + q}$$

(Soberman et al. 1997). If the gravitational wave emission is the dominant mechanism for the shrinkage of the orbital radius, $\tau_1$ can be set equal to $\tau_G$ defined as

$$\tau_G = \frac{5}{32} \frac{c^5}{G^{5/3}} \left( \frac{P}{2\pi} \right)^{8/3} \frac{(M_1 + M_2)^{1/3}}{M_1 M_2}$$

(Landau & Lifshitz 1975).

We use the empirical MRR of Knigge (2006) and solve Equation (20) with $\tau_1 = \tau_G$ for $M_{2,R}$, the resulting $P$–$M_{2,R}$ relations are plotted in Figure 5 as dashed and solid lines in green for the B1 and B5 sequences, respectively. The adopted stellar ages make a small difference in $M_{2,R}$, since the first term in the square brackets of Equation (20) is insensitive to the isochrones when $M_2 > \text{MHBM}$, while it becomes much smaller than the second term when $M_2 < \text{MHBM}$. Figure 5 shows that $M_{2,R}$ agrees, within error bars, with $M_2$ for both the B1 and B5 sequences before reaching the minimum period. This verifies that our semi-empirical mass-loss rate is consistent with the angular momentum loss due to emission of gravitational waves, at least for pre-bouncers. Beyond $P_{\text{min}}$, $M_2$ for the B1 sequence appears to match $M_{2,R}$ better than the B5 sequence, suggesting that the CV ages are presumably less than 5 Gyr.

4. DISCUSSION

Since the mass-loss rate cannot be observed directly, we are only able to compare our semi-empirical results with the accretion rates estimated by means of other indicators. For this purpose, we select 10 short-period, eclipsing CVs from Patterson et al. (2005), Littlefair et al. (2008), and Townsley & Gänsicke (2009). Table 2 represents a compilation of our sample CVs, with the parameters in Columns 2–7 taken from the mentioned papers. To our knowledge, these are the only eclipsers with $P$ below 130 minutes, for which the reliable data for the binary parameters such as $P$, $M_1$, $M_2$, and $T_1$ are all available. To infer the mass-accretion rates for the sample CVs, we use

$$T_1 = 1.7 \times 10^4 \left( \frac{\langle M_1 \rangle}{10^{-10} \frac{M_\odot}{\text{yr}}} \right)^{1/4} \left( \frac{M_1}{0.9 \frac{M_\odot}{\text{yr}}} \right),$$

where $T_1$ and $M_1$ are the effective temperature and mass of a WD and $\langle M_1 \rangle$ is the time-averaged accretion rate, taken from Townsley & Gänsicke (2009). Since the gravitational energy released by the accreted material onto the surface of a WD is deposited at shallow depths and soon radiated away, the quiescent surface luminosity should be proportional to the mass-accretion rate, leading to $T_1 \propto \langle M_1 \rangle^{1/4}$ (e.g., Townsley & Bildsten 2003, 2004). Filled circles with error bars in Figure 7(a) plot $\langle M_1 \rangle$ for the sample CVs calculated from Equation (23); these are also listed in Column 8 of Table 2. It can be seen that $\langle M_1 \rangle$ is broadly in line with our semi-empirical $M_2$ for both the B1 and B5 sequences. This suggests that our semi-empirical $P$–$M_2$ relation is realistic, provided that $-M_2 = \langle M_1 \rangle$ in these CVs during accretion disk quiescence. Townsley & Gänsicke (2009) noted that $\langle M_1 \rangle$ significantly declines toward $P_{\text{min}}$ for $P < 84$ minutes. Since the accretion rate decreases...
with decreasing $M_2$, this downward tendency of $\langle M_1 \rangle$ with $P$ is in agreement with the behavior of the semi-empirical $M_2$.

Donors in short-period CVs are very faint objects, making it difficult to measure their effective temperatures $T_2$ directly from observations, especially in the brown dwarfs regime: in our sample, only two systems (HT Cas and DV UMa) have measured $T_2$ available in the Ritter & Kolb (2003) catalog. One way to estimate $T_2$ is to use the strong dependence of $\langle T_2 \rangle$ on the quiescence phase of DNe, we combine Equations (19) and (23) to derive

$$T_2 \approx 11.594 \left( \frac{\delta_T}{P} + \frac{1/3}{P^{1/2}} \right)^{1/4} \left( \frac{L_2}{L_{\odot}} \right)^{-1/4} \left( \frac{M_1}{0.9 M_{\odot}} \right)^{-1}. \tag{24}$$

Since donors with mass below the MHBM have $L_2/L_2 = 1$, $T_2/T_1$ depends only on the orbital period and the effective mass–radius exponent. For masses above the MHBM, on the other hand, an accurate estimation of $T_2$ requires us to consider the effect of $(L_2/L_2)^{-1/4}$ term, which cannot be determined observationally.

Using the method described in Section 2.4, we calculate $L_2/L_2$ for the sample CVs listed in Table 2. For this, we take $\delta_T = 0.21 \pm 0.1$ for systems with $M_2 < 0.063 M_{\odot}$ and $\delta_T = 0.64 \pm 0.02$ for $M_2 > 0.063 M_{\odot}$ in accordance with the adopted empirical MRR. The bloating factor of each system is calculated relative to the 1 Gyr isochrone. We then calculate $T_2$ from Equation (24) and plot the results in Figure 7(b) as filled circles with error bars; these are also tabulated in Column 9 in Table 2.

![Figure 7](image-url) Comparison between our semi-analytic results (lines) and the 10 observed eclipsing CVs (filled circles) listed in Table 2 for (a) the mass-loss rate and (b) the effective temperature. The dashed and solid lines are for the B1 and B5 sequences, respectively, with the shaded and hatched regions denoting the uncertainties. In (b), $T_2$ is estimated from Equation (24) assuming that mass is conserved (i.e., $-M_2 = \langle M_1 \rangle$).

(A color version of this figure is available in the online journal.)

Table 2

Data on Eclipsing CVs ($P < 130$ minutes) with Estimated Effective Temperature of WD

| System | $P$ (min) | $M_1$ ($M_{\odot}$) | $q$ ($M_1/M_2$) | $T_2$ (K) | $K_2$ (μm) | $M_1$ ($M_{\odot}$) | $T_2$ (K) | Ref. |
|--------|-----------|---------------------|-----------------|----------|-----------|---------------------|----------|------|
| SDSS 1507 | 66.6119 | 0.910+0.070 | 0.650+0.004 | 1100+1250 | 0.057+0.004 | 0.973+0.003 | −10.78+0.33 | 1730+360 | 3 |
| SDSS 1433 | 78.1066 | 0.863+0.007 | 0.680+0.003 | 1280+950  | 0.060+0.003 | 1.018+0.003 | −10.43+0.14 | 1990+160 | 3 |
| SDSS 1501 | 81.8513 | 0.800+0.030 | 0.670+0.003 | 1250+950  | 0.055+0.003 | 1.078+0.004 | −10.31+0.19 | 1902+200 | 3 |
| SDSS 1035 | 80.2896 | 0.940+0.010 | 0.550+0.020 | 1010+1000 | 0.057+0.002 | 1.018+0.003 | −10.98+0.18 | 1370+150 | 3 |
| SDSS 1502 | 84.8298 | 0.820+0.030 | 0.109+0.030 | 1230+950  | 0.090+0.004 | 1.013+0.003 | −10.40+0.19 | 2260+300 | 3 |
| SDSS 0903 | 83.0659 | 0.966+0.030 | 0.117+0.030 | 1300+1050 | 0.111+0.004 | 0.815+0.003 | −10.58+0.19 | 2470+430 | 3 |
| SDSS 1227 | 90.6610 | 0.810+0.030 | 0.114+0.030 | 1500+1250 | 0.096+0.004 | 0.814+0.003 | −10.03+0.21 | 2690+370 | 3 |
| OY Car   | 90.8928 | 0.840+0.040 | 0.102+0.030 | 1500+2000 | 0.086+0.005 | 0.903+0.003 | −10.10+0.33 | 2550+500 | 1.2, 3 |
| HT Cas   | 106.0560 | 0.610+0.040 | 0.150+0.010 | 1400+1500 | 0.091+0.020 | 0.150+0.013 | −9.66+0.24 | 400+400 | 1.2 |
| DV UMa   | 128.2800 | 1.041+0.024 | 0.1510+0.0010 | 2000+1500 | 0.157+0.010 | 0.204+0.016 | −9.97+0.17 | 2380+470 | 1.2 |

Note. $P$ is the orbital period in minutes; $M_1$ and $M_2$ are the masses of the WD primary and the donor in the solar units, respectively; $q = M_2/M_1$ is the mass ratio; $T_1$ and $T_2$ are the effective temperatures of the WD and the donor in Kelvin, respectively; $R_2$ is the radius of the donor in the solar units; $M_2$ is the mass-loss rate in units of $M_{\odot}$ yr$^{-1}$.

References. (1) Patterson et al. 2005; (2) Townsley & Gansicke 2009; (3) Littlefair et al. 2008.
type of M4.5 ± 0.5, from the Ritter & Kolb (2003) catalog. Such a big difference in $T_2$ cannot be due to ambiguity in
the $(ξ + 1/3)^1/4$ or $(L_e/L_z)^{-1/4}$ terms since both are close to unity for this object. Townsley & Bildsten (2003)
noted that the best-constrained quantity from a measured $T_2$ is the accretion rate per unit area, $(\dot{m})_1 = \langle 1 \rangle /4\pi R_1^2$, while the subsequent estimation of $(\dot{m})_1$ from $(\dot{m})_1$ requires a certain MRR of WDs, which is poorly known for CVs. The WD mass in DV UMa is highest
among the sample, and it is highly probable that the MRR for massive WDs differs from $R_1 \propto M_1^{1.8}$ adopted by Townsley & Gänsicke (2009).

Based on the donor masses, Littlefair et al. (2008) claimed that SDSS 1035, SDSS 1501, SDSS 1507, and SDSS 1433 possess post-period minimum CVs. Although period bouncers should have low $M_2$, the precise mass of a particular secondary does not signify its evolutionary history or status (Patterson 2009). The evolutionary status of such objects can be clarified more clearly by their positions, for instance, on the $P$–$T_2$ or $P$–$M_2$ diagram where the boomerang effect can be seen. Figure 7 suggests that SDSS 1501 and SDSS 1433 among the sample of Littlefair et al. (2008) have not reached $P_{\text{min}}$ yet. This result does not depend on the stellar age, since these have $M_2 > 10^{-10.5} M_⊙$ yr$^{-1}$ for which the semi-empirical mass-loss rate is insensitive to the adopted isochrone. While the evolutionary status of SDSS 1035 is unclear, the fact that the $P$–$M_2$ dependence for the B1 sequence reconciles well with that due to gravitational wave emission indicates that it is likely a post-bounce CV.

Kolb & Baraffe (1999) argued that the tidal and rotational distortions do not change the value of $P_{\text{min}}$ appreciably because the internal structure of CV secondaries is only weakly affected by such perturbations. To show this, they compared the effective temperatures of the distorted and undistorted stars and found almost no change in $T_2$. This result is totally expected since short-period CVs are fully convective stars. As we showed in Section 2.3, the Hayashi theory gives that the reaction of $T_2$ on the bloating of a donor is small for isolated, fully convective stars. We also show in Appendix B that the influence of an isentropic expansion on $T_2$ caused by tidal and rotational distortions can be neglected in most practical cases. All of these suggest that the effective temperature does not reliably measure the amount of the stellar bloating, and consequently cannot be used as an indicator of the degree of tidal and rotational distortions.

5. SUMMARY

The mass-loss rate of donors is one of the most important parameters that govern the evolution of short-period CVs. Since the mass-loss rate is not directly observable, most previous work employed either purely numerical methods or semi-empirical approaches that use the observed properties of CV accretors rather than donors. Yet, the confirmation of the resulting mass-loss rates has to be made. In this work, we use the empirical MRR of short-period CVs constructed by Knigge (2006) as an input parameter and derive the semi-empirical relationship between the orbital period and the mass-loss rate. We assume that the bloating of CV donors is mainly caused by the mass loss and associated thermal relaxation processes. We also calculate the responses of the effective temperature and luminosities on the bloating of the CV secondaries. We discuss our semi-empirical mass-loss rate in comparison with those from the previous numerical and semi-empirical studies, and in terms of estimating the effective temperature of the donors. Our main results are summarized as follows.

1. The semi-empirical $P$–$M_2$ dependence shows that the mass-loss rate is essentially non-stationary. For $P > 90$ minutes, $M_2$ is in the range of $10^{-9.5}$–$10^{-10} M_⊙$ yr$^{-1}$ and decreases with decreasing $P$ relatively slowly as $M_2 \propto P^{-3}$. Below 90 minutes, on the other hand, $M_2$ sharply declines toward $P_{\text{min}}$ as a result of a strong dependence of $M_2$ upon $T_2$, emulating the $P$–$T_2$ relation. The $P$-dependence and amplitudes of the semi-empirical $M_2$ are significantly different from those predicted by the purely numerical models of Kolb & Baraffe (1999). The main reason for the discrepancies is small donor radii in the latter. Donors in short-period CVs are out of thermal equilibrium with $\tau > 1$. Thermal relaxation processes change the slope of the $P$–$M_2$ and $P$–$T_2$ diagrams significantly above $P_{\text{min}}$, making the boomerang shape less pronounced. Due to the sharp decline of $T_2$ and $M_2$ below 90 minutes, the observational detection of the boomerang shape on the $P$–$M_2$ and $P$–$T_2$ is highly unlikely.

2. The semi-empirical $P$–$M_2$ relation on the basis of the 1 Gyr isochrone is consistent with that estimated from the mass-transfer model of Ritter (1988) with the angular momentum carried dominantly by gravitational waves. Our semi-empirical $P$–$M_2$ relation predicts that after emerging from the period gap, donors lose mass from 0.2 $M_⊙$ to 0.075 $M_⊙$ in $\sim 1$ Gyr and reach the minimum period in $\sim 1$–2 Gyr, suggesting that the ages of CV secondaries with $0.075 M_⊙ < M < 0.2 M_⊙$ are less than 2 Gyr.

3. For 10 selected eclipsing CVs (listed in Table 2) for which the reliable observational data are available, the accretion rates inferred from the observed effective temperatures of WDs are in good agreement with our semi-analytic mass-loss rates, indicating that our semi-empirical $P$–$M_2$ relation is realistic. This also suggests that the effective temperature of a CV secondary can be estimated from the effective temperature of its WD primary through Equation (24). When analyzed on the $P$–$M_2$ or $P$–$T_2$ plane, two CVs (SDSS 1501 and SDSS 1433) that were previously considered as post-bouncers by Littlefair et al. (2008) are most likely to be systems before reaching the minimum period.

The authors are grateful to the referee, Dr. J. Patterson, for a thoughtful report. This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korean government (MEST), No. 2009-0063616.

APPENDIX A

ISENTROPIC EXPANSION/CONTRACTION OF POLYTROPIC DONORS

Even without considering the effect of mass loss, Sirotkin & Kim (2009) showed that the tidal and rotational distortions alone make donors in CVs larger by about $4%$–$8%$ for $n = 1.5$ polytropes and $2%$–$3%$ for $n = 3.5$ polytropes, depending on the mass ratio. In this appendix, we evaluate the enlargement factor for polytropes with arbitrary $0 < n < 5$.

Consider a polytropic star with index $n$, mass $M_1$, and pressure constant $K$. Its radius is $R_2$ in isolation. The MRR of the unperturbed polytropes is given by

$$K = N_{r,0} GM_2^{(n-1)/n} R_2^{(3-n)/n},$$

(A1)

where $N_{r,0}$ is the dimensionless pressure constant that varies weakly with $n$ (Chandrasekhar 1933). When it is situated in a CV, the presence of tidal and rotational perturbations
Figure 8. Changes in the volume radius of a polytrope due to the isentropic expansion under tidal and rotational perturbations against the polytropic index $n$, when the mass ratio is $q = 0.1$. A binary component expands when $n < 3$ or $n \gtrsim 3.3$, while it contracts when $3 < n \lesssim 3.3$. Note that $n = 3$ is a special case where the size of a polytrope cannot be constrained by its mass.

causes it to expand or contract, depending on $n$, by adjusting the internal structure to restore hydrostatic equilibrium. Let the new equilibrium be achieved at the volume radius $R_2$ with the corresponding dimensionless pressure constant $N_n$. Since the global internal adjustment occurs in the dynamical timescale, shorter than the thermal timescale and the mass-loss time scale, one may assume that the specific entropy of the gas inside the donor does not change much under the influence of the distortional perturbations.

For isentropic responses (i.e., $K$ is unchanged), the enlargement factor is given simply by

$$\alpha_{\text{isen}} \equiv \frac{R_2}{R_{2,0}} = \left(\frac{N_n}{N_{n,0}}\right)^{n/(n-3)}.$$  \hspace{1cm} (A2)

We use a self-consistent field method outlined in Sirotkin & Kim (2009) to construct detailed internal structure of polytropes in hydrostatic equilibrium. We then calculate $N_{n,0}$ and $N_n$ for polytropes in critical (i.e., Roche lobe-filling) configurations by varying $n$ and the mass ratio $q = M_2/M_1$. Figure 8 plots the resulting relative changes, $\Delta R_2/R_{2,0} = \alpha_{\text{isen}} - 1$, in the stellar size as a function of $n$ for $q = 0.1$. The behavior of $\alpha_{\text{isen}}$ with $M_2$ (or with the mass ratio $q$ for $M_1 = 0.75 M_\odot$) for $n = 3/2$ polytropes is plotted in Figure 2 as a dotted line. Figure 8 shows that $\Delta R_2 > 0$ for $n < 3$ or $n \gtrsim 3.3$, indicating that the perturbations make the donors bigger; the increase in the equatorial radius is larger than the decrease in the polar radius. On the other hand, critical polytropes with $3 < n \lesssim 3.3$ have $\Delta R_2 < 0$, since the perturbations decrease the polar radius much more than increasing the equatorial radius.

Notice a discontinuity at $n = 3$ for which one cannot determine the stellar enlargement factor. For these polytropes, the central density is proportional to $R_2^{-3}$ and the mass becomes independent of $R_2$, making it impossible to constrain the mass from its radius (e.g., Chandrasekhar 1933). $\Delta R_2$ diverges to either positive or negative infinity as $n \to 3$, indicating that a small change of $n$ near 3 leads to a big difference in $\Delta R_2$. This suggests that one should be cautious when treating $n = 3$ polytropes numerically. Using smoothed particle hydrodynamics (SPH) simulations, Renvoizé et al. (2002) found that the critical configurations have an enlargement factor $\sim 10\% – 12\%$ for $n = 3$ polytropes when $q = 0.05 – 1.0$. This is presumably due to some (unavoidable) numerical errors in the SPH simulations that somehow change their $n = 3$ polytropes to $n \sim 2.7 – 2.8$ ones “effectively,” resulting in $\Delta R_2/R_{2,0} = 0.10 – 0.12$.

APPENDIX B

CHANGE OF EFFECTIVE TEMPERATURE DUE TO ISENTROPIC EXPANSION

In Section 2.3, we describe how the effective temperature $T_2$ of a fully convective donor varies due to a homologous expansion using the Hayashi theory. Here, we study how $T_2$ responds to an isentropic expansion. We assume that the pressure $P_i$ in the entire stellar interiors below photosphere depends on temperature as

$$P_i = \frac{\mu R^{5/2} T^{5/2}}{\kappa M^{5/2}} K^{3/2},$$  \hspace{1cm} (B1)

where $\mu$ is the mean molecular weight and $\kappa$ is the gas constant. The photosphere is determined at the radius where the optical depth $\tau_{\text{ph}} = 2/3$. If the radiative pressure is unimportant, the photospheric pressure, $P_p$, is given by

$$P_p(\tau_{\text{ph}}) \approx \frac{2 \bar{g}}{3 \kappa},$$  \hspace{1cm} (B2)

where $\kappa$ is the opacity and $\bar{g} = GM_2/R_2^2$ is the mean surface gravity. Using a simple opacity law of the form $\kappa = \kappa_0 P_n^{-a} T^b$, we obtain

$$P_p(\tau_{\text{ph}}) \approx \left(\frac{2 \bar{g}}{3 \kappa_0} \frac{1}{T^b}\right)^{\frac{1}{1-a}}.$$  \hspace{1cm} (B3)

Since we assume that the donor is in hydrostatic equilibrium, the photospheric pressure should equal the interior pressure at the interface between the convective interiors and the radiative photospheric layer. Assuming $K = K_0$, Equations (B1) and (B3) are combined to yield

$$\frac{T_2}{T_{2,0}} = \alpha_{\text{isen}}^{1 - \frac{a}{1-a}}.$$  \hspace{1cm} (B4)

In contrast to the Hayashi theory, the effective temperature decreases due to the donor expansion. Nevertheless, $T_2/T_{2,0} = 0.99$ for $\alpha_{\text{isen}} = 1.05$, $a = 0.5$, and $b = 4.5$, typical values for CV secondaries, so that the change in $T_2$ due to the tidal and rotational distortions is still very small and can thus be neglected in practice.

REFERENCES

Andronov, N., Pinsonneault, M., & Sills, A. 2003, ApJ, 582, 358A
Baraffe, I., Chabrier, G., Allard, F., & Hauschildt, P. H. 1998, A&A, 337, 403
Baraffe, I., Chabrier, G., Allard, F., & Hauschildt, P. H. 2002, A&A, 382, 563
Baraffe, I., Chabrier, G., Barman, T. S., Allard, F., & Hauschildt, P. H. 2003, A&A, 402, 701
Barker, J., & Kolb, U. 2003, MNRAS, 340, 623
Burrows, A., & Liebert, J. 1993, Rev. Mod. Phys., 65, 301
Chandrasekhar, S. 1933, MNRAS, 93, 390
