Robust Normality Test in the Presence of Outliers

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Abstract. In classical statistics, detecting the normality of the data is one of the essential assumptions. However, if the selected random samples have some outliers, this assumption is violated. It is now evident that if the Jarque-Bera (JB) test is one of the most powerful tests of normality. The study shows that in the presence of outliers, the JB test does not perform well in many situations. Thus, they proposed a robust Jarque-Bera (RJB) test as an alternative. In this article, we incorporate the idea of Gel and Gastwirth and proposed also a modified Jarque-Bera (MJB) test which has more power than the RJB. The results of the real-life example and simulation study shows that the power of MJB is higher in detecting normality of data compared to the JB and RJB test.

1. Introduction

A normality test is used to check the violation of the normality assumption of data. The entire inferential procedure and interpretation of results may be unreliable due to the violation of this assumption. So, it is essential to check the normality assumption before doing any relevant statistical analysis. There are two common ways to check the normality assumption. The easiest way is by using graphical methods. The most popular used graphical test of normality is quantile-quantile (QQ-plot) and histogram, box-plot, and stem-and-leaf plot are the other graphical methods of normality. Another way of checking normality is numerical methods. There are several numerical methods in the literature. Yap and Sim [11] studied and compared the power of Shapiro–Wilk (SW) test, Kolmogorov–Smirnov (KS) test, the Lilliefors (LL) test, Cramer–von Mises (CVM) test, Anderson–Darling (AD) test, the D’Agostino–Pearson test, the Jarque–Bera (JB) test and chi-squared (CSQ) test and found that Jarque–Bera (JB) test is the most powerful test for checking normality of data.

It is now evident that the presence of outliers is very common in real data [9, 2, 7, 6]. Normality assumption is often violated when a dataset contains outliers [8, 4]. Gel and Gastwirth [4] proposed a Robust Jarque–Bera (RJB) test utilizing a robust measure of variance which is less influenced by the outliers. Their study shows that the Robust Jarque- Bera (RJB) test is more powerful than the JB test in detecting moderately heavy-tailed departures from normality. In this article, we proposed another modified version of the Jarque–Bera test, the Modified Jarque–Bera (MJB) test, which gives better power than the Robust Jarque- Bera test proposed by Gel and Gastwirth [4].

The proposed modified Jarque-Bera (MJB) Test is discussed in Section 2. The performance of the proposed MJB test is evaluated through a simulation study in Section 3. A real data example is given in Section 4. Finally, a conclusion is drawn in Section 5.
2. Propose Modified Jarque-Bera (MJB) Test

Bowman and Shenton [1] proposed the Jarque-Bera test which was subsequently derived by Jarque and Bera [5]. According to Jarque and Bera [5] the JB test has optimum asymptotic power properties and good finite sample performance. Based on the sample skewness \( \sqrt{b_1} \) and kurtosis \( b_2 \) the JB statistic is given by

\[
JB = n \left( \frac{\sqrt{b_1}^2}{6} + \frac{(b_2 - 3)^2}{24} \right)
\]

(1)

Under the normality, the JB test statistic follows a chi-squared distribution with two degrees of freedom. A significantly large value of JB leads to the rejection of the normality assumption.

Following the idea of Gel and Gastwirth [4], the robust version of the second central moment is used in the proposed modified Jarque-Bera test. In this study, Normalised Median Absolute Deviation (MAD), which is define as

\[
\phi^2 = \text{MAD}^2 = 1.4826 \left[ \text{median} \left[ \left| X_i - \text{median}(X_i) \right| \right] \right]^2
\]

The JB statistic is based on sample skewness \( \frac{\hat{\mu}_3}{\hat{\mu}_2^{3/2}} \) and kurtosis \( \frac{\hat{\mu}_4}{\hat{\mu}_2^{2}} \). However, the modified sample estimates of skewness and kurtosis are \( \hat{\mu}_3 / \phi^3 \) and \( \hat{\mu}_4 / \phi^4 \) respectively, which leads to the Modified Jarque-Bera (MJB) test statistic as given below.

\[
MJB = n \left( \frac{\hat{\mu}_3}{6 \phi^3} \right)^2 + \frac{n}{216} \left( \frac{\hat{\mu}_4}{\phi^4} - 3 \right)^2
\]

\[
= n \left[ \left( \frac{\hat{\mu}_3}{\phi^3} \right)^2 + \frac{1}{36} \left( \frac{\hat{\mu}_4}{\phi^4} - 3 \right)^2 \right]
\]

2.1 Proposition

Let \( x_1, x_2, ..., x_n \) be an independent random sample, the distribution of modified skewness and kurtosis follows normal distribution with mean zero and variance 6 and 216 respectively. i.e.,

\[
\sqrt{n} \left( \frac{\hat{\mu}_3}{\phi^3} \right) \sim N \left[ 0, \left( \frac{6}{216} \right) \right]
\]

This proposition has been proven by a large number of simulation studies under the assumption of normality. The following figure shows that the proposed Modified Jarque–Bera (MJB) fit well for the chi-square distribution with two degrees of freedom.
3. Simulation Study
In this section, we carry out a simulation experiment to compare the performance of the proposed MJB test with the Jarque-Bera and the Robust Jarque-Bera (RJB) tests. Moreover, other existing tests, Shapiro–Wilk (SW) test, Kolmogorov–Smirnov (KS) test, the Lilliefors (LF) test, and Anderson–Darling (AD) test, are also compared. The estimated powers of these tests under various distributions for different sample sizes are shown in Table 1. The power is calculated followed by Gel and Gastwirth [4] which is in fact no. of times detected non-normality over 10,000 simulations by different normality tests. We consider six different non-normal distributions, the t distribution with 3 and 5 degrees of freedom, the logistic distribution, the double exponential distribution, the contaminated normal distribution, and the exponential distribution. For contaminating normal distributions, 95% observations are generated from the standard normal distribution and the remaining 5% observations are generated from the normal distribution with shifted variance, $\sigma^2=3$. All our results are given based on 5% level of significance.

4. Real Data Examples
A real data example is considered to see the performance of the proposed normality test. However, to represent the results simply, only three normality tests, JB RJB and MJB, are considered. Because it has been confirmed from the simulation study that overall these three tests perform well through the simulation.

4.1 The Water Salinity Data
The water salinity data describes salt concentration on water. The dataset has 28 points and is taken from Rupert and Carrol [10]. We deliberately change one data point to get an outlier (modified data point is shown in the parenthesis) in the original data set. The original and modified data are shown in Table 2. Table 3 shows that when there are no outliers the JB, RJB, and MJB do not reject the normality. However, the scenario has changed for modified data. It shows that for the modified data (with outliers) the JB and the RJB test do not reject the normality at 5% level of significance. However, the proposed MJB test rejects the normality. It is expected that the outlier is the cause of non-normality in a data set. Thus, we may consider that the proposed MJB test has shown a good ability to detect non-normality in the presence of outliers.
### Table 1. Simulated Power of the different tests for Normality

| n  | Test | $t_3$ | $t_5$ | Logistic | Double Exponential | $C/N$ | Exponential |
|----|------|-------|-------|----------|---------------------|-------|-------------|
| 30 | JB   | 0.4650| 0.2542| 0.1483   | 0.3315              | 0.3721| 0.7248      |
|    | SW   | 0.4606| 0.2470| 0.1484   | 0.3529              | 0.3508| 0.9709      |
|    | KS   | 0.0534| 0.0068| 0.0000   | 0.0110              | 0.0087| 0.1128      |
|    | LF   | 0.3366| 0.1540| 0.0952   | 0.2863              | 0.1937| 0.7792      |
|    | AD   | 0.4360| 0.2158| 0.1264   | 0.3675              | 0.2833| 0.9380      |
|    | RJB  | 0.6110| 0.3969| 0.2714   | 0.5578              | 0.4615| 0.8311      |
|    | MJB  | 0.7001| 0.4946| 0.3888   | 0.7020              | 0.5188| 0.8625      |
| 50 | JB   | 0.6730| 0.3901| 0.2245   | 0.5023              | 0.5104| 0.9992      |
|    | SW   | 0.6488| 0.3559| 0.1952   | 0.5105              | 0.4668| 0.9996      |
|    | KS   | 0.1129| 0.0126| 0.0000   | 0.0255              | 0.0134| 0.3681      |
|    | LF   | 0.4909| 0.2151| 0.1141   | 0.4291              | 0.2236| 0.9617      |
|    | AD   | 0.6149| 0.3046| 0.1577   | 0.5365              | 0.3432| 0.9973      |
|    | RJB  | 0.7854| 0.5242| 0.3437   | 0.7209              | 0.5731| 0.9700      |
|    | MJB  | 0.8359| 0.5971| 0.4402   | 0.8199              | 0.6006| 0.9655      |
| 100| JB   | 0.8962| 0.6274| 0.3714   | 0.7831              | 0.6830| 1.0000      |
|    | SW   | 0.8786| 0.5594| 0.3039   | 0.7978              | 0.6227| 1.0000      |
|    | KS   | 0.2355| 0.0266| 0.0022   | 0.0822              | 0.0163| 0.9166      |
|    | LF   | 0.7344| 0.332 | 0.1532   | 0.7023              | 0.2805| 1.0000      |
|    | AD   | 0.8527| 0.4806| 0.2341   | 0.8256              | 0.4413| 1.0000      |
|    | RJB  | 0.9448| 0.7359| 0.5004   | 0.9197              | 0.7232| 1.0000      |
|    | MJB  | 0.9571| 0.7688| 0.5649   | 0.9500              | 0.7236| 0.9992      |
| 200| JB   | 0.9917| 0.8608| 0.5770   | 0.9653              | 0.8873| 1.0000      |
|    | SW   | 0.9892| 0.8154| 0.4857   | 0.9747              | 0.8430| 1.0000      |
|    | KS   | 0.5461| 0.0664| 0.0039   | 0.3529              | 0.0334| 0.9999      |
|    | LF   | 0.9429| 0.5478| 0.2416   | 0.9447              | 0.4175| 1.0000      |
|    | AD   | 0.9850| 0.7372| 0.3887   | 0.9825              | 0.6436| 1.0000      |
|    | RJB  | 0.9971| 0.9159| 0.6991   | 0.9941              | 0.9029| 1.0000      |
|    | MJB  | 0.9973| 0.9182| 0.7231   | 0.9963              | 0.8795| 1.0000      |

### Table 2. The Water Salinity Data (Original and Modified)

| Water Salinity | Water Salinity | Water Salinity | Water Salinity |
|----------------|----------------|----------------|----------------|
| 7.6            | 8.2            | 10.4           | 14.1           |
| 7.7            | 13.2           | 10.5           | 13.5           |
| 4.3            | 12.6           | 7.7            | 11.5           |
| 5.9            | 10.4           | 9.5            | 12.0           |
| 5.0            | 10.8           | 12.0           | 13.0           |
| 6.5            | 13.1           | 12.6           | 14.1           |
| 8.3            | 12.3           | 13.6           | 15.1(22.0)     |


| Test name | Value of test statistics | p-value | Comment (5%) |
|-----------|--------------------------|---------|--------------|
| Original data | JB | 2.0542 | 0.358 | Accepted |
| | RJB | 1.580 | 0.227 | Accepted |
| | MJB | 0.973 | 0.307 | Accepted |
| Modified data | JB | 4.744 | 0.0931 | Accepted |
| | RJB | 4.236 | 0.0601 | Accepted |
| | MJB | 4.6189 | 0.0497 | Rejected |

5. Conclusion
In this article, we focus on the topic of the normality assumption. In particular, the normality assumption in the presence of outliers has been taken great attention. In this respect, a modified Jarque-Bera test has been proposed as a better alternative than the classical Jarque-Bera test and robust Jarque-Bera test proposed by Gel and Gastwirth [4]. The power of the MJB test is also compared with the other existing normality tests. The simulation study and real data example show clear evidence that the proposed statistic has better power than its two competitors. Thus, the proposed modified Jarque-Bera test is recommended to test the normality of the data.

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References
[1] Bowman KO and Shenton LR 1975 Omnibus test contours for departures from normality based on b1 and b2 Biometrika 62 243-250
[2] Cousineau D and Chartier S 2010 Outliers detection and treatment: a review int j psychol res 3 (1) 59-68
[3] Das KR and Imon AHMR 2016 A Brief Review of Tests for Normality Am. J Theo App Stat 5(1) 5-12
[4] Gel YR and Gastwirth JL 2008 A robust modification of the Jarque-Bera test of normality Econ Lett 99 30-32.
[5] Jarque CM and Bera AK 1987 A test for normality of observations and regression residuals Int Stat Rev 55 163–172.
[6] Khalil A, Salahuddin, Mashwani WK, Shafiq M, Hassan S and Kumam W 2020 New advanced outliers detection tests Communications in Statistics - Theory and Methods DOI: 10.1080/03610926.2020.1741630
[7] Kwak, SK and Kim JH 2017 Statistical data preparation: management of missing values and outliers Kore J anesthe 70(4) 407–411
[8] Rana S, Midi H and Imon AHMR 2009 A robust rescaled moment test for normality in regression. J. Math Stat 5 54-62
[9] Roussseeuw PJ and Leroy AM 2003 Robust Regression and Outlier Detection Wiley: New York
[10] Rupert D and Carrol RJ 1980 Trimmed least squares estimation in the linear model J Am Stat Assoc 75 828 – 838
[11] Yap BW and Sim CH 2011 Comparisons of various types of normality tests J Stat Comput Simul 81 2141-2155