Stochastic Optimal Control and BSDEs with Logarithmic Growth

KHALED BAHLALI † and BRAHIM EL ASRI ‡

Abstract In this paper, we study the existence of an optimal strategy for the stochastic control of diffusion in general case and a saddle-point for zero-sum stochastic differential games. The problem is formulated as an extended BSDE with logarithmic growth in the z-variable and terminal value in some $L^p$ space. We also show the existence and uniqueness of solution of this BSDE.

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1 Introduction

In this paper we study BSDE with the applications to stochastic control and stochastic zero-sum differential games.

We consider a backward stochastic differential equation (BSDE) with generator $\varphi$ and terminal condition $\xi$

$$Y_t = \xi + \int_t^T \varphi(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s, \quad t \in [0, T]$$

(1.1)

where $(B_t)_{t \geq 0}$ is a standard Brownian motion. Such equations have been extensively studied since the first paper of E. Pardoux and S. Peng [15]. We will consider the case when $\varphi$ is allowed to have logarithmic growth $(|z| \ln^2(|z|))$ in the z-variable. Moreover, we will allow $\xi$ to be unbounded.

1) [10] showed the existence of an optimal stochastic control in the stochastic control of diffusions, in the case where the drift term of equation $f$ which defines the controlled system is bounded. In the same bounded case the existence of a saddle-point for a zero-sum stochastic differential game can be proved in a similar way.

2) [11] established the existence of an optimal stochastic control in the stochastic control of diffusions, in the case where the running reward function $h$ is bounded. In the same bounded case the existence of a saddle-point for a zero-sum stochastic differential game can be proved in a similar way.

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†IMATH, UFR Sciences, USVT, B.P. 132, 83957 La Garde Cedex, France. e-mail: bahlali@univ-tln.fr

‡Institut für Stochastik Friedrich-Schiller-Universität Jena Ernst-Abbe-Platz 2 07743 Jena, Germany; e-mail: brahim.el-asri@uni-jena.de
Our aim in this work is to relax the boundedness assumption on drift term of equation $f$ functionals and the running reward function $h$. Therefore the main objective of our work, and this is the novelty of the paper, is to show the existence of an optimal strategy for the stochastic control of diffusion. The main idea consists to showed the existence and uniqueness of the solution of BSDE and characterize the value function as a solution of BSDE.

This paper is organized as follows: In Section 2, we present the assumptions and we formulate the problem. In Section 3, we give the the main result on existence and uniqueness of the solution of BSDE. In Section 4, we state some estimates of the solutions from which we derive some integrability properties of the solution. In Section 5, we give estimate between two solutions and the proof of Theorem 3.1 and 3.2. In Section 6, we introduce the optimal stochastic control problem and we give the connection between optimal stochastic control problem and the zero-sum stochastic differential games and the BSDE. We show the value function as a solution of BSDE.

2 Assumptions and formulation of the problem

Let $(\Omega, \mathcal{F}, P)$ be a fixed probability space on which is defined a standard $d$-dimensional Brownian motion $B = (B_t)_{0 \leq t \leq T}$ whose natural filtration is $(\mathcal{F}_t^0 := \sigma\{B_s, s \leq t\})_{0 \leq t \leq T}$. Let $\mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ be the completed filtration of $(\mathcal{F}_t^0)_{0 \leq t \leq T}$ with the $P$-null sets of $\mathcal{F}$. We consider the following assumptions,

(H.1) \[ E[|\xi|^{(\ln(CT)+2)+2}] < +\infty. \]

(H.2) (i) Assume $\varphi$ is continuous in $(y,z)$ for almost all $(t,w)$;  
   (ii) There exist a constant positive $c_0$ and a process $\eta_t$ satisfying  
   \[ E \left[ \int_0^T \eta_s^{\ln(Cs)+2} ds \right] < +\infty. \]
   and such that for every $t, \omega, y, z$ :  
   \[ |\varphi(t,\omega,y,z)| \leq \eta_t + c_0|z|\sqrt{\ln(|z|)}. \]

(H.3) There exist $v \in L^{q'}(\Omega \times [0,T]; \mathbb{R}_+)$ (for some $q' > 0$) and a real valued sequence $(A_N)_{N \geq 1}$ and constants $M_2 \in \mathbb{R}_+$, $r > 0$ such that: 
   i) $\forall N > 1, 1 < A_N \leq N^r$.  
   ii) $\lim_{N \to \infty} A_N = \infty$.  
   iii) For every $N \in \mathbb{N}$, and every $y, y', z, z'$ such that $|y|, |y'|, |z|, |z'| \leq N$, we have  
   \[
   (y - y') \left( \varphi(t,\omega,y,z) - \varphi(t,\omega,y',z') \right) 1_{\{v_t(\omega) \leq N\}} \leq M_2 |y - y'|^2 \log A_N \\
   + M_2 |y - y'| |z - z'| \sqrt{\log A_N} \\
   + M_2 \frac{\log A_N}{A_N}.
   \]

3 The main results

The main objective of this paper is to focus on the existence and uniqueness of the solution of equation under the previous assumptions.
We denote by $\mathbb{E}$ the set of $\mathbb{R} \times \mathbb{R}^d$-valued processes $(Y, Z)$ defined on $\mathbb{R}_+ \times \Omega$ which are $\mathcal{F}_t$-adapted and such that: $||(Y, Z)||^2 = \mathbb{E}(\sup_{0 \leq t \leq T} |Y_t|^2 + \int_0^T |Z_s|^2 ds) < +\infty$. The couple $(\mathbb{E}, ||.||)$ is then a Banach space.

For $N \in \mathbb{N}^*$, we define

$$\rho_N(\varphi) = E \int_0^T \sup_{|y|,|z| \leq N} |\varphi(s, y, z)| ds. \quad (3.1)$$

**Definition 3.1.** A solution of equation (1.1) is a couple $(Y, Z)$ which belongs to the space $(\mathbb{E}, ||.||)$ and satisfies equation (1.1).

The main result of this section are the following two theorems.

**Theorem 3.1.** Assume that (H.1), (H.2) and (H.3) are satisfied. Then, equation (1.1) has a unique solution.

In the following, we give a stability result for the solution with respect to the data $(\varphi, \xi)$. Roughly speaking, if $\varphi_n$ converges to $\varphi$ in the metric defined by the family of semi-norms $(\rho_N)$ and $\xi_n$ converges to $\xi$ in $L^2(\Omega)$ then $(Y^n, Z^n)$ converges to $(Y, Z)$ in some reflexive Banach space which we will precise below. Let $(\varphi_n)$ be a sequence of functions which are measurable for each $n$. Let $(\xi_n)$ be a sequence of random variables which are $\mathcal{F}_T$-measurable for each $n$ and such that $\sup_n \mathbb{E}(|\xi_n|^\ln(CT+2)+2) < +\infty$. We will assume that for each $n$, the BSDE corresponding to the data $(\varphi_n, \xi_n)$ has a (not necessarily unique) solution. Each solution of the BSDE $(\varphi_n, \xi_n)$ will be denoted by $(Y^n, Z^n)$. We consider the following assumptions,

(H.5) For every $N$, $\rho_N(\varphi_n - \varphi) \rightarrow 0$ as $n \rightarrow \infty$.

(H.6) $E(|\xi_n - \xi|^\ln(CT+2)+2) \rightarrow 0$ as $n \rightarrow \infty$.

(H.7) there exist a constant positive $c_0$ and $\eta$ satisfying

$$\mathbb{E} \left[ \int_0^T \eta^\ln(Cs+2) + 2 ds \right] < +\infty,$$

and such that:

$$\sup_n |\varphi_n(t, \omega, y, z)| \leq \eta_t + c_0|z|\sqrt{\ln(|z|)}$$

**Theorem 3.2.** Let $\varphi$ and $\xi$ be as in Theorem 3.1. Assume that (H.5), (H.6), and (H.7) are satisfied. Then, for all $q < 2$ we have

$$\lim_{n \rightarrow +\infty} \left( \mathbb{E} \sup_{0 \leq t \leq T} |Y^n_t - Y_t|^q + \mathbb{E} \int_0^T |Z^n_s - Z_s|^q ds \right) = 0.$$

**Remark 3.1.** The conclusions of the previous theorems remain valid if, instead of hypothesis (H2)-(ii), we assume the following more general condition :

(H2)-(iii) There exist a constants positive $c_0$, $0 < \alpha' < 2$ and a process $\eta$ satisfying

$$\mathbb{E} \left[ \int_0^T \eta^\ln(Cs+2) + 2 ds \right] < +\infty,$$

and such that for every $t, \omega, y, z$:

$$|\varphi(t, w, y, z)| \leq \eta_t + |y|^\alpha' + c_0|z|\sqrt{\ln(|z|)}$$

3
4 Proofs

To prove Theorem 3.1 and Theorem 3.2, we need the following lemmas.

Lemma 4.1. Let $(Y, Z)$ be a solution of the above BSDE, where $(\xi, \varphi)$ satisfies the assumptions $(H1)$ and $(H2)$. Then there exists a constant $C_T$, such that:

$$
\mathbb{E} \sup_{t \in [0, T]} |Y_t|^{\ln(Ct+2)+2} \leq C_T \mathbb{E}\left[ |\xi|^{\ln(CT+2)+2} + \int_0^T \eta_s^{\ln(Cs+2)+2} ds \right].
$$

Proof. For some constant $C$ large, let us consider the function from $[0, T] \times \mathbb{R}$ into $\mathbb{R}^+$ defined by.

$$
u(t, x) = |x|^{\ln(Ct+2)+2}.
$$

Then

$$
u_t = \frac{C}{Ct+2} \ln(|x|) |x|^{\ln(Ct+2)+2}, \quad \nu_x = (\ln(Ct+2)+2) |x|^{\ln(Ct+2)+1} \text{sgn}(x)
$$

and $\nu_{xx} = (\ln(Ct+2)+2)(\ln(Ct+2)+1) |x|^{\ln(Ct+2)}$, with the notation $\text{sgn}(x) = -1_{x \leq 0} + 1_{x > 0}$. For $k \geq 0$, let $\tau_k$ be the stopping time defined as follows:

$$
\tau_k = \inf\{t \geq 0, \int_0^T (\ln(Cs+2)+2)^2 |Y_s|^{2\ln(Cs+2)+2} |Z_s|^2 ds \geq k\} \wedge T.
$$

Next using Itô’s formula yields:

$$
\left| Y_{t \wedge \tau_k} \right|^{\ln(Ct+2)+2} - \left| Y_{\tau_k} \right|^{\ln(Ct+2)+2} - \int_{t \wedge \tau_k}^{\tau_k} \frac{C}{Cs+2} \ln(|Y_s|) |Y_s|^{\ln(Cs+2)+2} ds
$$

$$
- \frac{1}{2} \int_{t \wedge \tau_k}^{\tau_k} |Z_s|^2 (\ln(Cs+2)+2)(\ln(Cs+2)+1) |Y_s|^{\ln(Cs+2)} ds
$$

$$
+ \int_{t \wedge \tau_k}^{\tau_k} (\ln(Cs+2)+2) |Y_s|^{\ln(Cs+2)+1} \text{sgn}(Y_s) f(s, Y_s, Z_s) ds
$$

$$
- \int_{t \wedge \tau_k}^{\tau_k} (\ln(Cs+2)+2) |Y_s|^{\ln(Cs+2)+1} \text{sgn}(Y_s) Z_s dB_s,
$$

$$
\leq \left| Y_{\tau_k} \right|^{\ln(Ct+2)+2} - \int_{t \wedge \tau_k}^{\tau_k} \frac{C}{Cs+2} \ln(|Y_s|) |Y_s|^{\ln(Cs+2)+2} ds
$$

$$
- \frac{1}{2} \int_{t \wedge \tau_k}^{\tau_k} |Z_s|^2 (\ln(Cs+2)+2)(\ln(Cs+2)+1) |Y_s|^{\ln(Cs+2)} ds
$$

$$
+ \int_{t \wedge \tau_k}^{\tau_k} (\ln(Cs+2)+2) |Y_s|^{\ln(Cs+2)+1} (\eta_s + c_0 |Z_s| \sqrt{\ln(|Z_s|)}) ds
$$

$$
- \int_{t \wedge \tau_k}^{\tau_k} (\ln(Cs+2)+2) |Y_s|^{\ln(Cs+2)+1} \text{sgn}(Y_s) Z_s dB_s.
$$

By Young’s inequality it hold true that:

$$
(\ln(Cs+2)+2) |Y_s|^{\ln(Cs+2)+1} \eta_s \leq |Y_s|^{\ln(Cs+2)+2} + (\ln(Cs+2)+2) |Y_s|^{\ln(Cs+2)+1} \eta_s^{\ln(Cs+2)+2}.
$$
For \( |y| \) large enough and the last inequality there exists \( C_1 \) such that:

\[
| Y_{t\wedge \tau_k} |^{\text{ln}(C't+2)+2} = | Y_{\tau_k} |^{\text{ln}(C't+2)+2} - \int_{t\wedge \tau_k}^{\tau_k} C_1 \ln(| Y_s |) | Y_s |^{\text{ln}(C's+2)+2} \, ds
\]

\[
- \frac{1}{2} \int_{t\wedge \tau_k}^{\tau_k} |Z_s|^2 (\text{ln}(C's + 2) + 2)(\text{ln}(C's + 2) + 1) | Y_s |^{\text{ln}(C's+2)} \, ds
\]

\[
+ \int_{t\wedge \tau_k}^{\tau_k} (\text{ln}(C's + 2) + 2) | Y_s |^{\text{ln}(C's+2)+1} c_0 |Z_s| \sqrt{\ln(|Z_s|)} ds
\]

\[
+ \int_{t\wedge \tau_k}^{\tau_k} (\text{ln}(C's + 2) + 2)^{\text{ln}(C's+2)+1} | Y_s |^{\text{ln}(C's+2)+2} \, ds
\]

\[
- \int_{t\wedge \tau_k}^{\tau_k} (\text{ln}(C's + 2) + 2) | Y_s |^{\text{ln}(C's+2)+1} \, ds
\]

\[
\leq | Y_{\tau_k} |^{\text{ln}(C't+2)+2} - \int_{t\wedge \tau_k}^{\tau_k} (\text{ln}(C's + 2) + 2)(\text{ln}(C's + 2) + 1) | Y_s |^{\text{ln}(C's+2)} \left[ \frac{C_1 \ln(| Y_s |) | Y_s |^2}{(\text{ln}(C's + 2) + 2)(\text{ln}(C's + 2) + 1)} + \frac{|Z_s|^2}{2} - \frac{\text{ln}(C's + 2) + 2}{(\text{ln}(C's + 2) + 2)(\text{ln}(C's + 2) + 1)} | Y_s | c_0 |Z_s| \sqrt{\ln(|Z_s|)} \right] \, ds
\]

\[
+ \int_{t\wedge \tau_k}^{\tau_k} (\text{ln}(C's + 2) + 2)^{\text{ln}(C's+2)+1} | Y_s |^{\text{ln}(C's+2)+2} \, ds
\]

\[
- \int_{t\wedge \tau_k}^{\tau_k} (\text{ln}(C's + 2) + 2) | Y_s |^{\text{ln}(C's+2)+1} \, ds
\]

There exist a constants \( C_2 \) and \( C_3 \) (\( C_2 > 2C_0^2 \))

\[
| Y_{t\wedge \tau_k} |^{\text{ln}(C't+2)+2} \leq \quad (4.1)
\]

\[
| Y_{\tau_k} |^{\text{ln}(C't+2)+2} - \int_{t\wedge \tau_k}^{\tau_k} (\text{ln}(C's + 2) + 2)(\text{ln}(C's + 2) + 1)
\]

\[
| Y_s |^{\text{ln}(C's+2)} \left[ C_2 \ln(| Y_s |) | Y_s |^2 + \frac{|Z_s|^2}{2} - C_3 | Y_s | |Z_s| \sqrt{\ln(|Z_s|)} \right] \, ds
\]

\[
+ \int_{t\wedge \tau_k}^{\tau_k} (\text{ln}(C's + 2) + 2)^{\text{ln}(C's+2)+1} | Y_s |^{\text{ln}(C's+2)+2} \, ds
\]

\[
- \int_{t\wedge \tau_k}^{\tau_k} (\text{ln}(C's + 2) + 2) | Y_s |^{\text{ln}(C's+2)+1} \, ds
\]

Now we show that

\[
C_3 | Y_s | |Z_s| \sqrt{\ln(|Y_s|)} \leq \frac{|Z_s|^2}{2} + C_2 \ln(| Y_s |) | Y_s |^2 \quad (4.2)
\]

if \( |Z_s| \leq | Y_s | \). \( (4.2) \) is obviously true. Assume \( |Z_s| > | Y_s | \). Denote \( a_s = \frac{|Z_s|}{|Y_s|} \). Then

\[
C_3 | Y_s | |Z_s| \sqrt{\ln(|Y_s|)} \leq C_3 a_s Y_s^2 \sqrt{\ln(|Y_s|)} + \sqrt{\ln(|Y_s|)}
\]

\[
\frac{|Z_s|^2}{2} + C_2 \ln(| Y_s |) | Y_s |^2 \leq \left[ \frac{a_s^2}{2} + C_2 \ln(| Y_s |) \right] | Y_s |^2
\]

Obviously

\[
C_3 a_s \sqrt{\ln(a_s)} \leq \frac{a_s^2}{2} + 2C_0^2 \ln(| Y_s |)
\]

Assume \( r \) is the constant such that \( C_3 \sqrt{\ln(r)} = \frac{r}{4} \). If \( a_s \geq r \),

\[
C_3 a_s \sqrt{\ln(a_s)} \leq \frac{a_s^2}{4}
\]
If $a_s \leq r$, and $|y|$ large enough then

$$C_3 a_s \sqrt{\ln(a_s)} \leq C_3 r \sqrt{\ln(r)} \leq \frac{C_2}{2} \ln(|Y_s|).$$

Then (4.2) holds. Finally taking the limit in both sides as $k \to +\infty$ and the lemma is proved.

Lemma 4.2. Let $(Y, Z)$ be a solution of the above BSDE. Then there exists a real constant $C_p$ depending only on $p$ such that:

$$\mathbb{E} \left[ \left( \int_0^T |Z_s|^2 ds \right)^{p/2} \right] \leq C_p \mathbb{E} \left[ |\xi|^p + \sup_{t \in [0,T]} |Y_t|^{p+2\ln(2)} + \left( \int_0^T |\eta_s|^2 ds \right)^{\frac{p}{2}} \right].$$

Proof. Applying Itô’s formula to the process $Y_t$ and the function $y \mapsto y^2$ yields:

$$|Y_0|^2 + \int_0^T |Z_s|^2 ds = |\xi|^2 + 2 \int_0^T Y_s \varphi(s, Y_s, Z_s) ds - 2 \int_0^T Y_s Z_s dB_s$$

$$\leq |\xi|^2 + 2 \int_0^T |Y_s| |\eta_s| + c_0 |Z_s| \sqrt{\ln(|Z_s|)} ds - 2 \int_0^T Y_s Z_s dB_s.$$

As we have

$$2 |Y_s| |\eta_s| \leq |Y_s|^2 + |\eta_s|^2,$$

and for any $\varepsilon > 0$ we have:

$$\sqrt{2\varepsilon \ln(|z|)} = \sqrt{\ln(|z|^{2\varepsilon})} \leq |z|^{\varepsilon}.$$

Then plug the two last inequalities in the previous one to obtain:

$$|Y_0|^2 + \int_0^T |Z_s|^2 ds$$

$$\leq |\xi|^2 + \sup_{s \leq T} |Y_s|^2 + \int_0^T |\eta_s|^2 ds + \frac{2}{\sqrt{2\varepsilon}} \int_0^T |Y_s| |Z_s|^{1+\varepsilon} ds - 2 \int_0^T Y_s Z_s dB_s.$$

We now choose $0 < \varepsilon < 1$ and by young’s inequality it holds true that:

$$2 \frac{|Y_s|}{\sqrt{2\varepsilon}} |Z_s|^{1+\varepsilon} \leq \frac{1 - \varepsilon}{2} \left( \frac{2}{\sqrt{2\varepsilon}} \right)^2 |Y_s|^2 + \frac{1 + \varepsilon}{2} |Z_s|^2.$$

Then, there exists a positive constant $c_\varepsilon$

$$|Y_0|^2 + \int_0^T |Z_s|^2 ds \leq |\xi|^2 + \sup_{s \leq T} |Y_s|^2 + \int_0^T |\eta_s|^2 ds$$

$$+ c_\varepsilon \sup_{s \leq T} |Y_s|^{2-\varepsilon} + \frac{1 + \varepsilon}{2} \int_0^T |Z_s|^2 ds - 2 \int_0^T Y_s Z_s dB_s.$$

For $|y|$ large enough and $\varepsilon \leq \frac{\ln(2)}{2+\ln(2)}$ then
\[|Y_0|^2 + \int_0^T |Z_s|^2 ds \leq |\xi|^2 + c_\varepsilon \sup_{s \leq T} |Y_s|^{2+\ln(2)} + \int_0^T |\eta_s|^2 ds + \frac{1 + \varepsilon}{2} \int_0^T |Z_s|^2 ds - 2 \int_0^T Y_s Z_s dB_s.\]

Then we obtain:

\[
\begin{align*}
\mathbb{E} \left( \int_0^T |Z_s|^2 ds \right)^{p/2} &\leq C_p \mathbb{E} \left[ \left( \int_0^T |Y_s|^2 |Z_s|^2 ds \right)^{p/4} \right] \\
&\leq \tilde{C}_p \left( \mathbb{E} \left[ \left( \sup_{t \in [0,T]} |Y_t|^p \right)^{p/2} \left( \int_0^T |Z_s|^2 ds \right)^{p/4} + \varepsilon \frac{p}{2} \left( \int_0^T |Z_s|^2 ds \right)^{p/2} \right] \right) \\
&\leq \frac{C_2^2}{\beta} \left( \mathbb{E} \left[ \left( \sup_{t \in [0,T]} |Y_t|^p \right) \right] + \mathbb{E} \left[ \left( \int_0^T |Z_s|^2 ds \right)^{p/2} \right] \right).
\end{align*}
\]

Next thanks to BDG’s inequality and for any \(\beta > 0\) we have:

\[
\begin{align*}
\mathbb{E} \left[ \left( \int_0^T Y_s Z_s dB_s \right)^{p/2} \right] &\leq C_p \mathbb{E} \left[ \left( \int_0^T |Y_s|^2 |Z_s|^2 ds \right)^{p/4} \right] \\
&\leq \tilde{C}_p \left( \mathbb{E} \left[ \left( \sup_{t \in [0,T]} |Y_t|^p \right)^{p/2} \left( \int_0^T |Z_s|^2 ds \right)^{p/4} \right] \right) \\
&\leq \frac{C_2^2}{\beta} \left( \mathbb{E} \left[ \left( \sup_{t \in [0,T]} |Y_t|^p \right) \right] + \mathbb{E} \left[ \left( \int_0^T |Z_s|^2 ds \right)^{p/2} \right] \right).
\end{align*}
\]

Choosing \(\beta\) and \(\varepsilon\) small enough to obtain the desired result.

**Lemma 4.3.** If \((H.2)\) holds then,

\[
\mathbb{E} \int_0^T |\varphi(s, Y_s, Z_s)|^{\overline{\alpha}} ds \leq K \left[ 1 + \mathbb{E} \int_0^T \eta_s^2 ds + \mathbb{E} \int_0^T |Z_s|^2 ds \right]
\]

where \(\overline{\alpha} = \min(2, \frac{\overline{\alpha}}{\alpha})\) and \(K\) is a positive constant which depends on \(c_0\) and \(T\).

**Proof.** Observe that assumption \((H.2)\) implies that there exist \(c_1 > 0\) and \(0 \leq \alpha < 2\) such that:

\[
|\varphi(t, \omega, y, z)| \leq \eta_t + c_1 |z|^\alpha.
\]

We successively use Assumption \((H.3)\) and inequality (4.3) to show that

\[
\begin{align*}
\mathbb{E} \int_0^T |f(s, Y_s, Z_s)|^{\overline{\alpha}} ds &\leq \mathbb{E} \int_0^T (\eta_s + c_0 |z| \sqrt{\ln(|z|)})^{\overline{\alpha}} ds \\
&\leq \mathbb{E} \int_0^T (\eta_s + c_1 |Z_s|^{\overline{\alpha}} ds \\
&\leq (1 + c_{\overline{\alpha}}) \mathbb{E} \int_0^T (\eta_s^{\overline{\alpha}} + (|Z_s|)^{\overline{\alpha}}) ds \\
&\leq (1 + c_{\overline{\alpha}}) \mathbb{E} \int_0^T ((1 + \eta_s)^{\overline{\alpha}} + (1 + |Z_s|)^{\overline{\alpha}}) ds \\
&\leq (1 + c_{\overline{\alpha}}) \mathbb{E} \int_0^T ((1 + \eta_s)^2 + (1 + |Z_s|)^2) ds \\
&\leq (1 + c_{\overline{\alpha}}) (4T + \mathbb{E} \int_0^T (\eta_s^2 + |Z_s|^2) ds)
\end{align*}
\]
Lemma 4.4 is proved.

**Lemma 4.4.** There exists a sequence of functions \((\varphi_n)\) such that,
(a) For each \(n\), \(\varphi_n\) is bounded and globally Lipschitz in \((y,z)\) a.e. \(t\) and \(P\)-a.s.\(\omega\).
(b) \(\sup_n |\varphi_n(t,\omega,y,z)| \leq \eta_0 + c_0|z|\sqrt{\ln(|z|)}\), \(P\)-a.s., a.e. \(t \in [0,T]\).
(c) For every \(N\), \(\rho_N(\varphi_n - \varphi) \to 0\) as \(n \to \infty\).

**Proof** Let \(\varepsilon_n : \mathbb{R}^2 \to \mathbb{R}_+\) be a sequence of smooth functions with compact support which approximate the Dirac measure at 0 and which satisfy \(\int \varepsilon_n(u)du = 1\). Let \(\psi_n\) from \(\mathbb{R}^2\) to \(\mathbb{R}_+\) be a sequence of smooth functions such that \(0 \leq |\psi_n| \leq 1\), \(\psi_n(u) = 1\) for \(|u| \leq n\) and \(\psi_n(u) = 0\) for \(|u| \geq n + 1\). We put, \(\varepsilon_{q,n}(t, y, z) = \int \varphi(t, (y, z) - u)\alpha_q(u)d\psi_n(y, z)\). For \(n \in \mathbb{N}^*\), let \(q(n)\) be an integer such that \(q(n) \geq n + n^0\). It is not difficult to see that the sequence \(\varphi_n := \varepsilon_{q(n), n}\) satisfies all the assertions (a)-(c).

Using Lemma 4.1, Lemma 4.2, Lemma 4.3, Lemma 4.4 and standard arguments of BSDEs, one can prove the following estimates.

**Lemma 4.5.** Let \(\varphi\) and \(\xi\) be as in Theorem 3.1. Let \((\varphi_n)\) be the sequence of functions associated to \(\varphi\) by Lemma 4.4. Denote by \((Y^{\varphi_n}, Z^{\varphi_n})\) the solution of equation \((E^{\varphi_n})\). Then, there exist constants \(K_1, K_2, K_3\) and a universal constant \(\ell\) such that

\[
\begin{align*}
&\text{a) } \sup_n \mathbb{E} \int_0^T |Z_s^{\varphi_n}|^2 ds \leq K_1 \\
&\text{b) } \sup_n \mathbb{E} \sup_{0 \leq t \leq T} (|Y_t^{\varphi_n}|^2) \leq \ell K_1 := K_2 \\
&\text{c) } \sup_n \mathbb{E} \int_0^T |\varphi_n(s, Y_s^{\varphi_n}, Z_s^{\varphi_n})|^\overline{\alpha} ds \leq K_3
\end{align*}
\]

where \(\overline{\alpha} = \min(2, \frac{2}{\alpha})\).

After extracting a subsequence, if necessary, we have

**Corollary 4.1.** There are \(Y \in \mathbb{L}^2(\Omega, L^\infty[0,T])\), \(Z \in \mathbb{L}^2(\Omega \times [0,T])\), \(\Gamma \in \mathbb{L}^\infty(\Omega \times [0,T])\) such that

\[
\begin{align*}
Y^{\varphi_n} &\rightharpoonup Y, \text{ weakly star in } \mathbb{L}^2(\Omega, L^\infty[0,T]) \\
Z^{\varphi_n} &\rightharpoonup Z, \text{ weakly in } \mathbb{L}^2(\Omega \times [0,T]) \\
\varphi_n(., Y^{\varphi_n}, Z^{\varphi_n}) &\rightharpoonup \Gamma, \text{ weakly in } \mathbb{L}^{\overline{\alpha}}(\Omega \times [0,T]),
\end{align*}
\]

and moreover

\[
Y_t = \xi + \int_t^T \Gamma_s ds - \int_t^T Z_s dW_s, \forall t \in [0,T].
\]

The following lemma, were established in [31], is a direct consequence of Hölder’s and Schwarz’s inequalities and the fact that \(ab \leq \frac{\alpha^2}{2}a^2 + \frac{1}{2\alpha^2}b^2\) for each \(\alpha > 0\) and each real numbers \(a, b\).

**Lemma 4.6.** For every \(\beta \in [1,2]\), \(A > 0\), \((y)_{i=1..d} \subset \mathbb{R}\), \((z)_{i=1..d, j=1..r} \subset \mathbb{R}\) we have,

\[
A|y||z| - \frac{1}{2}|z|^2 + \frac{2 - \beta}{2}|y|^{-2}|yz|^2 \leq \frac{1}{\beta - 1}A^2|y|^2 - \frac{\beta - 1}{4}|z|^2.
\]

This lemma remains valid in multidimensional case.
5 Estimate between two solutions

The key estimate is given by,

**Lemma 5.1.** For every $R \in \mathbb{N}$, $\beta \in ]1, \min (3 - \frac{2}{\nu}, 2) [\), $\delta' < (\beta - 1) \min (\frac{1}{4M^2}, \frac{3 - \frac{2}{\nu} - \beta}{2rM^2})$ and $\varepsilon > 0$, there exists $N_0 > R$ such that for all $N > N_0$ and $T' \leq T$:

$$
\limsup_{n,m \to +\infty} E \sup_{(T' - \delta')^+ \leq t \leq T'} |Y^\varphi_n - Y^\varphi_m|^\beta + E \int_{(T' - \delta')^+}^T |Z^\varphi_n - Z^\varphi_m|^2 \frac{2 - \delta'}{2r} ds \\
\leq \varepsilon + \frac{\ell}{\beta - 1} e^{C_N \delta'} \limsup_{n,m \to +\infty} E |Y^\varphi_n - Y^\varphi_m|^\beta.
$$

where $\nu_R = \sup \{ (A_N)^{-1}, N \geq R \}$, $C_N = \frac{2M^2}{(1 - 1)^2}$ and $\ell$ is a universal positive constant.

**Proof.** To simplify the computations, we assume (without loss of generality) that assumption (H3)-C-iii) holds without the multiplicative term $1 1$. For every $e > 0$ and $\beta < \min \{ (3 - \frac{2}{\nu}, 2) \}$. Itô’s formula shows that,

$$
e^{Ct} \Delta_t^\varphi + C \int_t^T e^{Cs} \Delta_s^\varphi ds
\leq e^{C'T} \Delta_{T'}^\varphi + \beta \int_t^{T'} e^{Cs} \Delta_s^\varphi \left( Y^\varphi_n - Y^\varphi_m \right) \left( \varphi_n(s, Y^s_n, Z^s_n) - \varphi_m(s, Y^s_m, Z^s_m) \right) ds
\leq \frac{\beta}{2} \int_t^{T'} e^{Cs} \Delta_s^{\frac{\varphi}{s} - 1} \left( Y^\varphi_n - Y^\varphi_m \right) \left( Z^\varphi_n - Z^\varphi_m \right) ds
\leq \frac{\beta}{2} \left( Y^\varphi_n - Y^\varphi_m \right) \left( Z^\varphi_n - Z^\varphi_m \right) ds.
$$

Put $\Phi(s) = |Y^\varphi_n| + |Y^\varphi_m| + |Z^\varphi_n| + |Z^\varphi_m|$. Then

$$
e^{Ct} \Delta_t^\varphi + C \int_t^T e^{Cs} \Delta_s^\varphi ds
\leq e^{C'T} \Delta_{T'}^\varphi - \beta \int_t^{T'} e^{Cs} \Delta_s^{\frac{\varphi}{s} - 1} \left( Y^\varphi_n - Y^\varphi_m \right) \left( Z^\varphi_n - Z^\varphi_m \right) ds
\leq \frac{\beta}{2} \int_t^{T'} e^{Cs} \Delta_s^{\frac{\varphi}{s} - 1} \left( Y^\varphi_n - Y^\varphi_m \right) \left( Z^\varphi_n - Z^\varphi_m \right) ds
\leq \frac{\beta}{2} \left( Y^\varphi_n - Y^\varphi_m \right) \left( Z^\varphi_n - Z^\varphi_m \right) ds
\leq J_1 + J_2 + J_3 + J_4,
$$

where $J_1$ is a universal positive constant.
where
\[
J_1 := \beta \int_t^{T'} e^{Cs} \Delta_s^\frac{\beta}{\beta - 1} (Y_s^\varphi_n - Y_s^\varphi_m) (\varphi_n(s, Y_s^\varphi_n, Z_s^\varphi_n) - \varphi_m(s, Y_s^\varphi_m, Z_s^\varphi_m)) \mathbb{I}_{\{\Phi(s) > N\}} ds.
\]
\[
J_2 := \beta \int_t^{T'} e^{Cs} \Delta_s^\frac{\beta}{\beta - 1} (Y_s^\varphi_n - Y_s^\varphi_m) (\varphi_n(s, Y_s^\varphi_n, Z_s^\varphi_n) - \varphi(s, Y_s^\varphi_n, Z_s^\varphi_n)) \mathbb{I}_{\{\Phi(s) \leq N\}} ds.
\]
\[
J_3 := \beta \int_t^{T'} e^{Cs} \Delta_s^\frac{\beta}{\beta - 1} (\varphi(s, Y_s^\varphi_n, Z_s^\varphi_n) - \varphi(s, Y_s^\varphi_m, Z_s^\varphi_m)) \mathbb{I}_{\{\Phi(s) \leq N\}} ds.
\]
\[
J_4 := \beta \int_t^{T'} e^{Cs} \Delta_s^\frac{\beta}{\beta - 1} (\varphi(s, Y_s^\varphi_m, Z_s^\varphi_m) - \varphi_m(s, Y_s^\varphi_m, Z_s^\varphi_m)) \mathbb{I}_{\{\Phi(s) \leq N\}} ds.
\]

We shall estimate $J_1$, $J_2$, $J_3$, $J_4$. Let $\kappa = 3 - \frac{2}{\beta} - \beta$. Since $\frac{\beta - 1}{2} + \frac{\beta}{2} + \frac{1}{\kappa} = 1$, we use Hölder inequality to obtain
\[
J_1 \leq \beta e^{CT'} \frac{1}{N^\kappa} \int_t^{T'} \Delta_s^{\frac{\beta - 1}{\beta - 2}} \Phi(s) |\varphi_n(s, Y_s^\varphi_n, Z_s^\varphi_n) - \varphi_m(s, Y_s^\varphi_m, Z_s^\varphi_m)| ds
\]
\[
\leq \beta e^{CT'} \frac{1}{N^\kappa} \left[ \int_t^{T'} \Delta_s ds \right]^{\frac{\beta - 1}{\beta - 2}} \left[ \int_t^{T'} \Phi(s)^2 ds \right]^{\frac{1}{\beta - 2}} \times \left[ \int_t^{T'} |\varphi_n(s, Y_s^\varphi_n, Z_s^\varphi_n) - \varphi_m(s, Y_s^\varphi_m, Z_s^\varphi_m)| ds \right]^{\frac{1}{\beta - 2}}.
\]

Since $|Y_s^\varphi_n - Y_s^\varphi_m| \leq \Delta_s^{\frac{1}{2}}$, it easy to see that
\[
J_2 + J_4 \leq 2 \beta e^{CT'} [2N^2 + \nu_1] \frac{\beta - 1}{\beta - 2} \left[ \int_t^{T'} \sup_{|y|, |z| \leq N} |\varphi_n(s, y, z) - \varphi(s, y, z)| ds \right.
\]
\[
+ \left. \int_t^{T'} \sup_{|y|, |z| \leq N} |\varphi_m(s, y, z) - \varphi(s, y, z)| ds \right].
\]

Using assumption (H3), we get
\[
J_3 \leq \beta M_2 \int_t^{T'} e^{CS} \Delta_s^{\frac{\beta}{\beta - 1}} \left[ |Y_s^\varphi_n - Y_s^\varphi_m|^2 \log A_N \right.
\]
\[
+ \frac{\log A_N}{A_N} = 1 + |Y_s^\varphi_n - Y_s^\varphi_m| \left| Z_s^\varphi_n - Z_s^\varphi_m \right| \sqrt{\log A_N} \right] \mathbb{I}_{\{\Phi(s) < N\}} ds
\]
\[
\leq \beta M_2 \int_t^{T'} e^{CS} \Delta_s^{\frac{\beta}{\beta - 1}} \left[ \Delta_s \log A_N + |Y_s^\varphi_n - Y_s^\varphi_m| \left| Z_s^\varphi_n - Z_s^\varphi_m \right| \sqrt{\log A_N} \right] \mathbb{I}_{\{\Phi(s) \leq N\}} ds.
\]
We choose \( C = C_N = \frac{2M^2 \beta}{\beta - 1} \log A_N \), then we use Lemma 4.6 to show that

\[
e^{C_N t} \frac{\Delta^2}{t} + \frac{\beta(\beta - 1)}{4} \int_0^t e^{C_N s} \Delta^\frac{\beta}{s} \frac{1}{s} |Z^\varphi_s - Z^\varphi_m|^2 ds
\leq e^{C_N T} \frac{\Delta^2}{T} - \beta \int_0^T e^{C_N s} \Delta^\frac{\beta}{s} \left( Y^\varphi_s - Y^\varphi_m, (Z^\varphi_s - Z^\varphi_m) dW_s \right) + \beta e^{C_N T} \frac{1}{N^\kappa} \left[ \mathbb{E} \int_0^T \Delta s ds \right] \delta^\frac{1}{2} \times \left[ \mathbb{E} \int_0^T \Phi(s)^2 ds \right]^\frac{1}{2}
\times \left[ \mathbb{E} \int_0^T \sup_{|y|,|z| \leq N} |\varphi_n(s, y, z) - \varphi_n(s, y, z)| ds \right]
\leq \beta [2N^2 + \nu_1] \frac{\beta - 1}{2} \mathbb{E} \left[ \int_0^T \sup_{|y|,|z| \leq N} |\varphi_n(s, y, z) - \varphi(s, y, z)| ds \right]
\leq \beta [2N^2 + \nu_1] \frac{\beta - 1}{2} \mathbb{E} \left[ \int_0^T \sup_{|y|,|z| \leq N} |\varphi_n(s, y, z) - \varphi(s, y, z)| ds \right]
\]

Burkholder’s inequality and Hölder’s inequality (since \( \frac{(\beta - 1)}{2} + \frac{\kappa}{2} + \frac{1}{\beta} = 1 \)) allow us to show that there exists a universal constant \( \ell > 0 \) such that \( \forall \delta' > 0 \),

\[
\mathbb{E} \sup_{T' - \delta' \leq t \leq T'} \left[ e^{C_N t} \Delta^2 + \mathbb{E} \int_0^T \Delta s ds \right] \leq \frac{\ell}{\beta - 1} e^{C_N T} \mathbb{E} \left[ \int_0^T \Delta s ds \right] \delta^\frac{1}{2} \times \left( \mathbb{E} \int_0^T \Phi(s)^2 ds \right)^\frac{1}{2}
\times \left[ \mathbb{E} \int_0^T \sup_{|y|,|z| \leq N} |\varphi_n(s, y, z) - \varphi(s, y, z)| ds \right]
\]

We use Lemma 4.4 and Lemma 4.5 to obtain, \( \forall \delta > R \),

\[
\mathbb{E} \sup_{T' - \delta \leq t \leq T'} |Y^\varphi_t - Y^\varphi_m| \leq \mathbb{E} \int_0^T \Delta s ds \leq \beta - 1 \mathbb{E} \left[ \int_0^T \Delta s ds \right] \delta^\frac{1}{2} \times \left( \mathbb{E} \int_0^T \Phi(s)^2 ds \right)^\frac{1}{2}
\times \left[ \mathbb{E} \int_0^T \sup_{|y|,|z| \leq N} |\varphi_n(s, y, z) - \varphi(s, y, z)| ds \right]
\]

\[
\leq \frac{\ell}{\beta - 1} e^{C_N T} \mathbb{E} \left[ \int_0^T \Delta s ds \right] \delta^\frac{1}{2} \times \left( \mathbb{E} \int_0^T \Phi(s)^2 ds \right)^\frac{1}{2}
\times \left[ \mathbb{E} \int_0^T \sup_{|y|,|z| \leq N} |\varphi_n(s, y, z) - \varphi(s, y, z)| ds \right]
\]
Hence for $\delta' < (\beta - 1) \min\left(\frac{1}{2M^2}, \frac{\kappa}{2r^2M^2} \beta\right)$ we derive

\[
\frac{2M^2\delta' \beta}{(A_N)^{\beta-1}} \rightarrow_{N \to \infty} 0
\]

and

\[
\frac{2M^2\delta' \beta}{(A_N)^{\beta-1}} \rightarrow_{N \to \infty} 0.
\]

Passing to the limits first on $n$ and next on $N$, and using assertion (c) of lemma 4.4

**Remark 5.1.** To deal with the case which take account of the process $v_1$ appearing in assumption (H3), it suffices to take $\Phi(s) := |Y_s^1| + |Y_s^2| + |Z_s^1| + |Z_s^2| + v_s$ in the proof of Lemma 5.1

**Proof of Theorem 3.1.** Taking successively $T' = T$, $T' = (T - \delta')^+$, $T' = (T - 2\delta')^+$ ... in Lemma 5.1 we obtain, for every $\beta \in ]1, \min\left(3 - \frac{2}{\alpha}, 2\right)$]

\[
\lim_{n,m \to +\infty} \left(\mathbb{E} \sup_{0 \leq t \leq T} |Y_t^{\phi_n} - Y_t^{\phi_m}|^\beta + \mathbb{E} \int_0^T \frac{|Z_s^{\phi_n} - Z_s^{\phi_m}|^2}{(|Y_s^{\phi_n} - Y_s^{\phi_m}|^2 + \nu_R)^{\frac{2-\beta}{2}}} ds \right) = 0.
\]

But by Schwarz inequality we have

\[
\mathbb{E} \int_0^T |Z_s^{\phi_n} - Z_s^{\phi_m}| ds \leq \left(\mathbb{E} \int_0^T \frac{|Z_s^{\phi_n} - Z_s^{\phi_m}|^2}{(|Y_s^{\phi_n} - Y_s^{\phi_m}|^2 + \nu_R)^{\frac{2-\beta}{2}}} ds \right)^{\frac{1}{2}} \left(\mathbb{E} \int_0^T (|Y_s^{\phi_n} - Y_s^{\phi_m}|^2 + \nu_R)^{\frac{2-\beta}{2}} ds \right)^{\frac{1}{2}}
\]

Since $\beta > 1$, Lemma 4.5 allows us to show that

\[
\lim_{n \to +\infty} \left(\mathbb{E} \sup_{0 \leq t \leq T} |Y_t^{\phi_n} - Y_t|^\beta + \mathbb{E} \int_0^T |Z_s^{\phi_n} - Z_s| ds \right) = 0.
\]

In particular, there exists a subsequence, which we still denote $(Y^{\phi_n}, Z^{\phi_n})$, such that

\[
\lim_{n \to +\infty} (|Y_t^{\phi_n} - Y_t| + |Z_t^{\phi_n} - Z_t|) = 0 \quad a.e. \ (t, \omega).
\]

On the other hand

\[
\mathbb{E} \int_0^T |\varphi_n(s, Y_s^{\phi_n}, Z_s^{\phi_n}) - \varphi(s, Y_s^{\phi_n}, Z_s^{\phi_n})| ds
\]

\[
\leq \mathbb{E} \int_0^T |\varphi_n(s, Y_s^{\phi_n}, Z_s^{\phi_n}) - f(s, Y_s^{\phi_n}, Z_s^{\phi_n})| \mathbb{I}_{\{|Y_s^{\phi_n}| + |Z_s^{\phi_n}| \leq N\}} ds
\]

\[
+ \mathbb{E} \int_0^T |\varphi_n(s, Y_s^{\phi_n}, Z_s^{\phi_n}) - f(s, Y_s^{\phi_n}, Z_s^{\phi_n})| \frac{(|Y_s^{\phi_n}| + |Z_s^{\phi_n}|)^{2-\frac{\beta}{2}}}{N^{2-\frac{\beta}{2}}} \mathbb{I}_{\{|Y_s^{\phi_n}| + |Z_s^{\phi_n}| \geq N\}} ds
\]

\[
\leq \rho_N (\varphi_n - \varphi) + \frac{2K_3^\frac{1}{2} [TK_2 + K_1]^{1-\frac{1}{2}}}{N^{2-\frac{\beta}{2}}}.
\]

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Passing to the limit first on $n$ and next on $N$ we obtain

$$
\lim_{n} E \int_{0}^{T} |\varphi_n(s, Y_s^{\varphi_n}, Z_s^{\varphi_n}) - \varphi(s, Y_s^{\varphi_n}, Z_s^{\varphi_n})|ds = 0.
$$

Finally, we use (H.1), Lemma 4.3 and Lemma 4.5 to show that,

$$
\lim_{n} E \int_{0}^{T} |\varphi_n(s, Y_s^{\varphi_n}, Z_s^{\varphi_n}) - \varphi(s, Y_s, Z_s)|ds = 0.
$$

The existence is proved.

**Uniqueness.** Let $(Y, Z)$ and $(Y', Z')$ be two solutions of equation $(E^f)$. Arguing as previously one can show that: for every $R > 2$, $\beta \in ]1, \min \left(3 - \frac{2}{\alpha}, 2 \right)$, $\delta' < (\beta - 1) \min \left(\frac{1}{4M_2^2}, \frac{3-2\alpha-\beta}{2rM_2^2\beta}\right)$ and $\varepsilon > 0$ there exists $N_0 > R$ such that for all $N > N_0$, $\forall T' \leq T$,

$$
\mathbb{E} \sup_{(T'-\delta')^+ \leq t \leq T'} |Y_t - Y'_t|^\beta + \mathbb{E} \int_{(T'-\delta')^+}^{T'} \frac{|Z_s - Z'_s|^2}{rM_2^2} ds 
\leq \varepsilon + \frac{\ell}{\beta - 1} \mathbb{E} |Y_{T'} - Y'_{T'}|^\beta.
$$

Again, taking successively $T' = T$, $T' = (T - \delta')^+$, $T' = (T - 2\delta')^+\ldots$, we establish the uniqueness of solution. Theorem 3.1 is proved.

**Proof of Theorem 3.2.** Also as in the proof of Theorem 3.1 we show that, for every $R > 2$, $\beta \in ]1, \min \left(3 - \frac{2}{\alpha}, 2 \right)$, $\delta' < (\beta - 1) \min \left(\frac{1}{4M_2^2}, \frac{3-2\alpha-\beta}{2rM_2^2\beta}\right)$ and $\varepsilon > 0$, there exists $N_0 > R$ such that for all $N > N_0$, for all $T' \leq T$:

$$
\limsup_{n \to +\infty} \mathbb{E} \sup_{(T'-\delta')^+ \leq t \leq T'} |Y^n_t - Y_t|^\beta + \mathbb{E} \int_{(T'-\delta')^+}^{T'} \frac{|Z^n_s - Z_s|^2}{rM_2^2} ds 
\leq \varepsilon + \frac{\ell}{\beta - 1} \mathbb{E} |Y^n_{T'} - Y_{T'}|^\beta.
$$

Again as in the proof of Theorem 3.1 taking successively $T' = T$, $T' = (T - \delta')^+$, $T' = (T - 2\delta')^+\ldots$, we establish the convergence in the whole interval $[0, T]$. In particular, we have for every $q < 2$, $\lim_{n \to +\infty} (|Y^n - Y|^q) = 0$ and $\lim_{n \to +\infty} (|Z^n - Z|^q) = 0$ in measure $P \times dt$. Since $(Y^n)$ and $(Z^n)$ are square integrable, the proof is finished by using an uniform integrability argument. Theorem 3.2 is proved.

6 Application to stochastic and control

In all the following $\Omega = \mathcal{C}([0, T], \mathbb{R}^m)$ is the space of continuous functions from $[0, T]$ to $\mathbb{R}^m$.

Let us consider a mapping $\sigma : (t, w) \in [0, T] \times \Omega \to \sigma(t, w) \in \mathbb{R}^m \otimes \mathbb{R}^m$ satisfying the following:

(1.1) $\sigma$ is $P$-measurable.
There exists a constant $C$ such that $|\sigma(t, w) - \sigma(t, w')| \leq C||w - w'||_t$ and $|\sigma(t, w)| \leq C(1 + ||w||_t)$, where for any $w, w' \in \Omega^2$ and $t \leq T$, $||w||_t = \sup_{s \leq t} |w_s|$.

For any $(t, w) \in [0, T] \times \Omega$, the matrix $\sigma(t, w)$ is invertible and $|\sigma^{-1}(t, w)| \leq C$ for some constants $C$.

Let $x_0 \in \mathbb{R}^m$ and $x = (x_t)_{t \leq T}$ be the solution of the following standard functional differential equation:

$$x_t = x_0 + \int_0^t \sigma(s, x)dB_s, \quad t \leq T; \quad (6.1)$$

the process $(x_t)_{t \leq T}$ exists, since $\sigma$ satisfies (1.1) - (1.3) (see, e.g., [16] page 375. Moreover,

$$\mathbb{E}[(||x||_T)^n] < +\infty, \quad \forall n \in [1, +\infty], \quad (6.2)$$

### 6.1 Stochastic control of diffusions

Let $A$ be a compact metric space and $U$ be the space of $P$-measurable processes $u := (u_t)_{t \leq T}$ with value in $A$. Let $f : [0, T] \times \Omega \times A \to \mathbb{R}^m$ be such that:

(1.4) For each $a \in A$, the function $(t, w) \to f(t, w, a)$ is predictable.

(1.5) For each $(t, w)$, the mapping $a \to f(t, w, a)$ is continuous.

(1.6) There exists a real constant $K > 0$ such that

$$|f(t, w, a)| \leq K(1 + ||w||_t), \quad \forall 0 \leq t \leq T, w \in \Omega, a \in A. \quad (6.3)$$

For any given admissible control strategy $u \in U$, the exponential process

$$\Lambda^u_t = \exp\{\int_0^T \sigma^{-1}(s, x)f(s, x, u_s)dB_s - \frac{1}{2} \int_0^T |\sigma^{-1}(s, x)f(s, x, u_s)|^2ds\}$$

$0 \leq t \leq T$, is a martingale under all these assumptions; namely, $\mathbb{E}[\Lambda^u_T] = 1$ (see Karatzas and Shreve (1991), pages 191 and 200 for this result). Then the Girsanov theorem guarantees that the process

$$B^u_t = B_t - \int_0^t \sigma^{-1}(s, x)f(s, x, u_s)ds, \quad 0 \leq t \leq T, \quad (6.4)$$

is a Brownian motion with respect to the filtration $\mathcal{F}_t$, under the new probability measure

$$P^u(B) = \mathbb{E}[\Lambda^u_T.1_B], \quad B \in \mathcal{F}_T,$$

which is equivalent to $P$. It is now clear from the equations (6.1) and (6.15) that

$$x_t = x_0 + \int_0^t f(s, x, u_s)ds + \int_0^t \sigma(s, x)dB^u_s, \quad 0 \leq t \leq T, \quad (6.5)$$

holds almost surely. This will be our model for a controlled stochastic functional differential equation, with the control appearing only in the drift term.

In order to specify the objective of our stochastic game of control and stopping. Let us now consider the followings:
(1.6) \( h : [0, T] \times \Omega \times A \rightarrow \mathbb{R} \) is measurable and for each \((t, w)\) the mapping \( a \rightarrow h(t, w, a) \) is continuous. In addition there exists a real constant \( K > 0 \) such that
\[
|h(t, w, a)| \leq K(1 + ||w||_t), \quad \forall 0 \leq t \leq T, w \in \Omega, a \in A. \tag{6.6}
\]

(1.7) \( g_1 : [0, T] \times \Omega \rightarrow \mathbb{R} \) and is continuous function and there exists a real positive constant \( C \) such that:
\[
|g_1(t, w)| \leq C(1 + ||w||_t), \quad \forall (t, w) \in [0, T] \times \Omega. \tag{6.7}
\]

We shall study a stochastic control with one player. The controller, who chooses an admissible control strategy \( u \in U \) to minimize this amount
\[
\int_0^T h(s, x, u_s)ds + g_1(T, x_T). \tag{6.8}
\]

It is thus in the best interest of the controller to make the amount (6.8) as small as possible, at least on the average. We are thus led to a stochastic control, with
\[
J(u) = \mathbb{E}^u \left[ \int_0^T h(s, x, u_s)ds + g_1(T, x_T) \right]. \tag{6.9}
\]

The problem we are interested in is finding an intervention strategies \( u^* \), for controller such that for any \( u \in U \), we have
\[
J(u^*) \leq J(u).
\]

Then \( u^* \) is called an optimal control for the problem. Now let us set
\[
H(t, x, z, u_t) = z\sigma^{-1}(t, x)f(t, x, u_t) + h(t, x, u_t) \quad \forall (t, x, z, u_t) \in [0, T] \times \Omega \times \mathbb{R}^m \times A. \tag{6.10}
\]

The function \( H \) is called the Hamiltonian associated with stochastic control such that:
\[
(2.1) \forall z \in \mathbb{R}^m, \text{ the process } (H(t, x, z, u_t))_{t \leq T} \text{ is } \mathcal{P}-\text{measurable.}
\]

Lemma 6.1. *The Hamiltonian \( H \) satisfies (H.2) and (H.3).*

Proof For (H.2), it is not difficult to show that for every \((t, x, z, u_t) \in [0, T] \times \Omega \times \mathbb{R}^m \times A \) and \(|z| \) large enough, there exist a constants \( C \) and \( c_0 \) such that:
\[
|H(t, x, z, u_t)| \leq Cexp(||x||_t) + c_0|z| \ln^2(|z|). \tag{6.11}
\]

To prove that \( H \) satisfies assumption (H.3), it is enough to take \( v_t := \exp |f(t, x, u_t)| \). Indeed, we have
\[
(y - y')(H(t, x, z, u_t) - H(t, x, z', u_t)) \mathbb{I}_{\{|f(t, x, u_t)|^2 \leq N\}} \leq |y - y'| \cdot |z - z'| f(t, x, u_t) \mathbb{I}_{\{|f(t, x, u_t)|^2 \leq \log N\}} \leq |y - y'| \cdot |z - z'| \sqrt{\log A_N}
\]

To complete the proof, we shall show that \( \exp |f(t, x, u_t)|^2 \) belongs to \( L^q(\Omega \times [0, T]; \mathbb{R}_+) \) for some \( q > 0 \). We have,
\[
\mathbb{E} \int_0^T \exp(qf(s, x, u_s)^2)ds \leq \mathbb{E} \int_0^T \exp(2qK^2(1 + \sup_{s \leq T} |x_s|^2))ds \\
\leq \exp(2qK^2T)\mathbb{E} \int_0^T \exp(2qK^2 \sup_{s \leq T} |x_s|^2)ds \\
\leq T \exp(2qK^2T)\mathbb{E} \exp(2qK^2 \sup_{s \leq T} |x_s|^2)
\]

And, since \(\sigma\) is with linear growth, it is well known that \(E \exp(2qK^2 \sup_{s \leq T} |x_s|^2) < \infty\) for \(q\) small enough.

To begin with let us define the notion of solution of the reflected BSDE associated with the triple \((H, g_2, g_1)\) which we consider throughout this paper.

In order to construct a stochastic control, we need to do is find an admissible control strategy \(u^*() \in U\) for our stochastic control.

The Hamiltonian function defined in (6.10) attains its infimum over the set \(A\) at some \(u^* \equiv u^*(t, x, p) \in A\), for any given \((t, x, p) \in [0, T] \times \Omega \times \mathbb{R}^m\), namely,

\[
\inf_{u \in A} H(t, x, u, p) = H(t, x, u^*(t, x, p), p).
\]

(This is the case, for instance, if the set \(A\) is compact and the mapping \(u \rightarrow H(t, x, u, p)\) continuous.) Then it can be shown (see Lemma 1 in Benes (1970), that the mapping \(u^* : ([0; T] \times \Omega \times \mathbb{R}^m \rightarrow A\) can be selected to be \(\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^m)\)-measurable.

Now let \(H^*(t, x, z) = \inf_{u \in A} H(t, x, u, z)\) where \(x\) is the solution of (6.1). Let \((Y_t)_{t \leq T}\) be the process constructed as in Theorem 3.1 with \((H^*, g_1)\). Using once again Theorem 3.1 there exists a unique pair \((Y_t, Z_t)_{t \leq T}\) such that

\[
\begin{cases}
(Y, Z) \in (\mathbb{E}, ||.||); \\
Y_t = g_1(T, x_T) + \int_t^T H^*(s, x, Z_s)\,ds - \int_t^T Z_sdB_s, \; t \in [0, T].
\end{cases}
\]

We are ready to give the main result of this section.

**Theorem 6.1.** The admissible control \(u^*\) is optimal for the stochastic control; i.e., it satisfies

\[
J(u^*) = Y_0 \leq J(u) \; \forall u \in U.
\]

Additionally, \(Y_0\) is the value of the stochastic control, i.e.,

\[
Y_0 = \inf_{u \in U} J(u).
\]

**Proof :** Let us show that \(Y_0 = J(u^*)\). It follows that

\[
Y_0 = g_1(T, x_T) + \int_0^T H^*(s, x, Z_s)\,ds - \int_0^T Z_sdB_s \\
= g_1(T, x_T) + \int_0^T h(s, x, u^*(s, x, Z_s))\,ds - \int_0^T Z_sdB_s^u.
\]
As \( \left( \int_0^t Z_s dB_s \right)_{t \leq T} \) is an \((\mathbb{F}_t, P^u)\)-martingale, taking expectation we get
\[
Y_0 = \mathbb{E}^u[Y_0] = \mathbb{E}^u\left[ g_1(T, x_T) + \int_0^T h(s, x, u^*(s, x, Z_s)) ds \right],
\]
because \( Y_0 \) is \( \mathbb{F}_0 \)-measurable, and hence deterministic. Now \( P \)-a.s., and also \( P^u \)-a.s. (since they are equivalent probabilities). Then
\[
Y_0 = J(u^*).
\]

Next let \( u \in \mathcal{U} \). Let us show that
\[
Y_0 \leq J(u).
\]
\[
Y_0 = g_1(T, x_T) + \int_0^T H(s, x, u, Z_s) ds - \int_0^T Z_s dB_s
\]
\[
\leq g_1(T, x_T) + \int_0^T H(s, x, u, Z_s) ds - \int_0^T Z_s dB_s
\]
\[
= g_1(T, x_T) + \int_0^T h(s, x, u(s, x, Z_s)) ds - \int_0^T Z_s dB_s^u,
\]

Once more \( \left( \int_0^t Z_s dB_s \right)_{t \leq T} \) is an \((\mathbb{F}_t, P^u)\)-martingale; then taking the expectation with respect to \( P^u \) and taking into account the fact that \( Y_0 \) is deterministic, we obtain
\[
Y_0 = \mathbb{E}^u[Y_0] \leq \mathbb{E}^u\left[ g_1(T, x_T) + \int_0^T h(s, x, u(s, x, Z_s)) ds \right],
\]
then \( Y_0 \leq J(u) \). The proof is now complete.

### 6.2 Stochastic zero-sum differential games

Let \( A \) (resp. \( B \)) be a compact metric space and \( \mathcal{U} \) (resp. \( \mathcal{V} \)) be the space of \( \mathcal{P} \)-measurable processes \( u = (u_t)_{t \leq T} \) (resp. \( v = (v_t)_{t \leq T} \)) with value in \( A \) (resp. \( B \)). Let \( f : [0, T] \times \Omega \times A \times B \to \mathbb{R}^m \) be such that:

(1.4) For each \( a \in A \) and \( b \in B \), the function \((t, x) \to f(t, w, a, b)\) is predictable.

(1.5) For each \((t, w)\), the mapping \((a, b) \to f(t, w, a, b)\) is continuous.

(1.6) There exists a real constant \( K > 0 \) such that
\[
|f(t, w, a, b)| \leq K(1 + ||w||_t), \quad \forall 0 \leq t \leq T, w \in \Omega, a \in A, b \in B.
\]

For any given admissible control strategy \((u, v) \in \mathcal{U} \times \mathcal{V}\), the exponential process
\[
\Lambda^{(u,v)} = \exp \{ \int_0^T \sigma^{-1}(s, x)f(s, x, u_s, v_s) dB_s - \frac{1}{2} \int_0^T |\sigma^{-1}(s, x)f(s, x, u_s, v_s)|^2 ds \}
\]
\( 0 \leq t \leq T \), is a martingale under all these assumptions; namely, \( \mathbb{E}[\Lambda_T^{(u,v)}] = 1 \) (see Karatzas and Shreve (1991), pages 191 and 200 for this result). Then the Girsanov theorem guarantees that the process
\[
B^{(u_t, v_t)} = B_t - \int_0^t \sigma^{-1}(s, x)f(s, x, u_s, v_s) ds, \quad 0 \leq t \leq T,
\]
\[17\]
is a Brownian motion with respect to the filtration $\mathcal{F}_t$, under the new probability measure

$$P^{(u,v)}(B) = \mathbb{E}[\Lambda_T^{(u,v)} \cdot 1_B], \quad B \in \mathcal{F}_T,$$

which is equivalent to $P$. It is now clear from the equations (6.1) and (6.15) that

$$x_t = x_0 + \int_0^t f(s, x, u_s, v_s)ds + \int_0^t \sigma(s, x)dB^{(u_s, v_s)}, \quad 0 \leq t \leq T, \tag{6.16}$$

holds almost surely.

Let us now consider the followings:

(1.6) $h : [0, T] \times \Omega \times A \times B \to \mathbb{R}$ is measurable and for each $(t, w)$ the mapping $(a, b) \to h(t, w, a, b)$ is continuous. In addition there exists a real constant $K > 0$ such that

$$|h(t, w, a, b)| \leq K(1 + ||w||_t), \quad \forall 0 \leq t \leq T, \ w \in \Omega, \ a \in A, \ b \in B. \tag{6.17}$$

(1.7) $g_1 : [0, T] \times \mathbb{R}^m \to \mathbb{R}$ and is continuous function and there exists a real positive constant $C$ such that:

$$|g_1(t, w)| \leq C(1 + ||w||_t), \quad \forall (t, w) \in [0, T] \times \mathbb{R}^m. \tag{6.18}$$

We shall study a stochastic zero-sum differential games. Then the payoff corresponding to $u \in \mathcal{U}$ and $v \in \mathcal{V}$ is

$$J(u, v) = \mathbb{E}^{u,v}[\int_0^T h(s, x, u_s, v_s)ds + g_1(T, x_T)]. \tag{6.19}$$

where $u \in \mathcal{U}$ (resp. $v \in \mathcal{U}$) is the strategy of the first (resp. second) player. The first player looks for minimize $J(u,v)$, when the second looks for maximize the same $J(u,v)$. We are concerned by the problem of the existence of a saddle-point for this game, i.e the existence of an admissible contol $(u^*, v^*)$ which satisfies:

$$J(u^*, v) \leq J(u^*, v^*) \leq J(u, v^*), \quad (u, v) \in \mathcal{U} \times \mathcal{V}$$

We introduce the hamiltonian function defined by:

$$H(t, x, z, u_t, v_t) = z\sigma^{-1}(t, x)f(t, x, u_t, v_t) + h(t, x, u_t, v_t) \quad \forall (t, x, z, u_t, v_t) \in [0, T] \times \mathbb{R} \times A \times B,$$

and we suppose that the Isaacs’ condition is satisfied:

\[ (H) \quad \sup_{v \in \mathcal{V}} \inf_{u \in \mathcal{U}} H(t, x, p, u, v) = \inf_{u \in \mathcal{U}} \sup_{v \in \mathcal{V}} H(t, x, p, u, v) \quad \forall (t, x, p) \in [0, T] \times \Omega \times \mathbb{R}^m \tag{6.20} \]

(2.1) $\forall z \in \mathbb{R}^m$, the process $(H(t, x, z, u_t, v_t))_{t \leq T}$ is $\mathcal{P}$-measurable.

Then using a selection theorem we get easily the:

**Lemma 6.2.** $(H)$ is equivalent to the following assumption:

There exists $u^*(t, x, p), v^*(t, x, p) \in \mathcal{P} \otimes \mathcal{B}(\mathbb{R}^m)$- measurable valued respectively in $\mathcal{U}$ and $\mathcal{V}$ such that:

$$H(t, x, p, u^*(t, x, p), v(t, x, p)) \leq H(t, x, p, u^*(t, x, p), v^*(t, x, p)) \leq H(t, x, p, u(t, x, p), v^*(t, x, p)) \quad \forall u, v, t, x, p.$$  

Moreover $u^*$ and $v^*$ satisfy

$$H(t, x, p, u^*(t, x, p), v^*(t, x, p)) = \sup_{v \in \mathcal{V}} \inf_{u \in \mathcal{U}} H(t, x, p, u, v) = \inf_{u \in \mathcal{U}} \sup_{v \in \mathcal{V}} H(t, x, p, u, v).$$
Lemma 6.3. The Hamiltonian $H$ satisfies (H.2) and (H.3).

The proof of the following results are similar to the proof of lemma (6.1).

Proposition 6.1. 1) For all $(u, v) \in U \times V$, let $(Y^{u,v}, Z^{u,v})$ be the solution of the BSDE with the generator $(H(t, x, p, u, v), g_1(T, x_T))$ then $J(u, v) = Y_0^{u,v}$. 2) Similarly, let $(Y^*, Z^*)$ be the solution of BSDE with generator $(H(t, x, p, u^*(t, x), v^*(t, x, p)), g_1(T, x_T))$ and define $(\tilde{u}, \tilde{v}) \in U \times V$ by $(\tilde{u}, \tilde{v}) = (u^*(t, x, Z^*_t), v^*(t, x, Z^*_t))_{t \leq T}$, then $J(\tilde{u}, \tilde{v}) = Y_0^*$.

Theorem 6.2. The strategy $(\tilde{u}, \tilde{v})$ is a saddle-point for the game.

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