OPTIMAL SPARSE OUTPUT FEEDBACK FOR NETWORKED SYSTEMS WITH PARAMETRIC UNCERTAINTIES

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ABSTRACT. This paper investigates the design of block row/column-sparse distributed static output $H_2$ feedback control for interconnected systems with polytopic uncertainties. The proposed approach is applicable to the networked systems with publisher/subscriber communication topology. We added two additional terms into the optimisation index function to penalise the number of publishers and subscribers. To optimally select a subset of available publishers and/or subscribers in the network, we introduced both an explicit scheme and an iterative process to handle this problem. We demonstrated the effectiveness by using a numerical example. The example showed that the simultaneous identification of favourable networks topologies and design of controller strategy can be achieved by using the proposed method.

1. Introduction. The modern complex systems, e.g., water distribution system, transportation systems, and power distribution networks, are often treated as interconnected large-scale systems, for which the decentralized and distributed control schemes have been presented. The major constraint of the decentralised control is only the local state information is available to stabilize and regulate the subsystems. Thus there is no control network for decentralized control configuration. This can be effective only when the interconnections between the subsystems are not strong [14, 20, 15]. When the interconnections cannot be overlooked, the distributed control framework is often adopted. In this case, each individual subsystem can exploit local states and some of the other neighbor subsystems’ states. Comparing with the decentralised configuration, the distributed solutions can guarantee the stability.
of large-scale network systems when interconnections are strong [19]. Meanwhile, compared to the centralised scheme, it is also less complex and has reduced computational load.

The configuration of the distributed controller network for the interconnected systems is often constrained because of several facts such as implementation-related issues and communication expenses. This limitation in distributed systems is called as information pattern. Different with traditional distributed control structure where all the related sub-controllers utilize the same information, the sub-controllers could acquire different information [16]. Because the fully distributed controller configuration is not always realistic, the design of distributed control systems with imposing a priori limitations on communication network configuration can be a possible solution. Otherwise, designing a control network with the minimum number of communication channels while satisfying a global performance requirements [12, 1] should be considered as another choice for the distributed control systems. Actually, a trade off between the sparsity of the feedback gain matrices and the control performance should be explored [10, 4, 21, 8, 6].

Addressing network sparsification problem, in worst case, one has to test all possible structures, which means a comprehensive examination for numerous structures. This may result an exponential number of growth of the communication links, and it is essentially problematic and impractical to perform. In [18], it is clarified to avoid an exhaustive searching, a compromise choice is either in the searching strategy or in the selection of index function. Another option is to construct a multi-objective optimization of both controller structure and control law design by including secondary index functions into the main cost function, which represents a performance specification of the closed-loop system [10, 17]. This secondary index functions can be constructed as the convex approximations of the original non-convex optimization of ℓ₀-quasi-norms and can be applied to sparsify the distributed controller. The reweighted ℓ₁ (REL1) norm algorithms [10] can be further developed to find the optimal sparse feedback gain. In the REL1 algorithms, the entries of the weighting matrix are calculated at each individual step, which are inversely proportional to the strength of the elements of the gain in the preceding step. This approach has been frequently utilized to sparsify a key matrix at the element-wise level to minimise the number of communication channels in distributed control networks exploiting the so-called bilateral communication scheme [16]. Still, the current REL1 algorithms have disadvantages when the sparsity is defined at a group (e.g. block column or row) level, where the strength of sets of variables (block elements of feedback gain) should be considered. To be more specific, block column/row sparsified feedback gain has an application in the diffusion based networks [16], where a published information in the communication network is available to subscribers, the sub-controllers. Then, the objective is to minimise the number of subscribers/publishers in the system rather than that of bilateral communication links. This is obviously equivalent to seeking for feedback gains with maximum number of block columns/rows with zero off-diagonal blocks. This paper presents a multi-objective optimisation problem by integrating two secondary index functions into the main index function to penalise the number of block columns/rows with non-zero off-diagonal blocks. Then, to cope with the underlying problem, an iterative process is developed, using the relaxed block column/row sparsity promoting penalty functions, to simultaneously penalise the number of subscribers and publishers in the control network. A similar framework has been used to develop
an approach for optimal actuator/sensor selection in over-actuated/sensed systems [2, 3]. The control scheme presented in this manuscript is static output feedback (SOF). Different with most of the approaches in the existing literature, it does not need the information of all the system states. Instead, it only utilizes the available sensor outputs. In addition, the overall networked system considered in this paper involving parametric uncertainties. Thus, this paper proposes a novel scheme for the design of $H_2$-based sparse block column/row-wise SOF for systems with polytope uncertain. An obvious benefit of the proposed SOF is that it is capability to deal with the networked systems whose output matrix has parametric uncertainty; cf. [5].

In this paper, denote a (block) matrix with (block) elements $\mathcal{G}_{ij}$, $i = 1, \ldots, r$, $j = 1, \ldots, r$, as $[\mathcal{G}_{ij}]_{r \times r}$. Moreover, $\text{diag}[\mathcal{G}_{ij}]_{r \times r}$ is utilized to denote a (block) diagonal matrix with (block) elements $\mathcal{G}_{ii}$, $i = 1, \ldots, r$. Furthermore, $\text{col}(w_i(t))_{r = 1}^r$ stands for a (block) vector with (block) entries $w_i(t)$, $i = 1, \ldots, r$.

2. Problem Formulation.

2.1. Problem statement. The following model describes a linear-time-invariant (LTI) large-scale networked system with $h$ subsystems:

\[
\begin{cases}
\dot{\xi}_i(t) = A_i \xi_i(t) + \sum_{j=1, j \neq i}^h A_{ij} \xi_j(t) + B_{2,i} u_i(t) + B_{1,i} w_i(\xi_i), \\
z_i(t) = C_{z,i} \xi_i(t) + \sum_{j=1, j \neq i}^h C_{z,ij} \xi_j(t) + D_{z,i} u_i(t), \\
y_i(t) = C_i \xi_i(t),
\end{cases}
\]  

where $\xi_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$, $z_i \in \mathbb{R}^{n_i}$, and $y_i \in \mathbb{R}^{p_i}$ are the system state, control input, performance output, and output of the $i$-th subsystem, respectively. The matrices in (1) are all constant with appropriate dimensions. $A_{ij}$, $i, j = 1, \ldots, h$, $j \neq i$ is the interactions between the subsystems ($A_{ii}, i = 1, \ldots, h$), i.e., $A_{ij} = 0$ if the subsystem $j$ does not directly influence the subsystem $i$. It is also assumed, without loss of generality, that $m_i \leq q_i \leq n_i$, and rank($B_{2,i}$) = $m_i$. $w_i(t) \in \mathbb{R}^{m_w}$ denotes the external disturbance. Define

\[
\begin{align*}
\xi(t) &= \text{col}(\xi_i(t))_{i=1}^h \in \mathbb{R}^n, \\
u(t) &= \text{col}(u_i(t))_{i=1}^h \in \mathbb{R}^m, \\
y(t) &= \text{col}(y_i(t))_{i=1}^h \in \mathbb{R}^p, \\
z(t) &= \text{col}(z_i(t))_{i=1}^h \in \mathbb{R}^q,
\end{align*}
\]

and

\[
\begin{align*}
A &= \text{diag}[A_i]_{i=1}^h \in [A_{ij}]_{h \times h}, \\
B_2 &= \text{diag}[B_{2,i}]_{i=1}^h, \\
B_1 &= \text{diag}[B_{1,i}]_{i=1}^h, C &= \text{diag}[C_{z,i}]_{i=1}^h, \\
C_z &= \text{diag}[C_{z,ij}]_{i=1}^h \in [C_{z,ij}]_{h \times h}, D_z &= \text{diag}[D_{z,i}]_{i=1}^h, 
\end{align*}
\]

where $A_{ii} = 0$ and $C_{z,ii} = 0$. Based on (1), (2) and (3), the total system can be described as the following state space equations:

\[
\begin{cases}
\dot{\xi}(t) = A\xi(t) + B_2 u(t) + B_1 w(\xi), \\
z(t) = C_z \xi(t) + D_z u(t), \\
y(t) = C \xi(t).
\end{cases}
\]
Assume the system matrices in (4) are all within the polytope
\[
\Delta = \{(A(\lambda), B_2(\lambda), B_1(\lambda), C(\lambda))|(A, B_2, B_1, C) = \sum_{l=1}^{N} \lambda_l (A_l, B_{2,l}, B_{1,l}, C_l), \lambda_l \geq 0, \sum_{l=1}^{N} \lambda_l = 1\},
\]
where $N$ is the number of vertices. Suppose that there exists a static gain $F_y$ such that $A_l + B_{2,l}F_yC_l$ is stable.

In the existing literatures, the frameworks of distributed control design for networked systems are built based on bilateral communication scheme, where the subsystems have two-direction communications. In this case, searching for a sparse feedback gain is equivalent to using less communication channels in the control. However, these frameworks have nothing to do with the so-called publisher/subscriber communication scheme; see Fig. 1. See [16], the diffusion based networks (e.g., the factory instrumentation protocol, such as EN 50170 and IEC 61158/IEC 61784 standards) is considerably different from the bilateral one. In this communication strategy, a published information is available to the sub-controllers that are subscriber in the communication network. In this study, the main target is the optimization of the number of publishers and/or subscribers in the system. This is comparable to investigating for feedback gains with maximum number of block columns/rows with zero off-diagonal blocks. The major aim of this study is highlighted by the following problems:

**Problem 1.** Given the state space representation in Equation (4) of a networked system, choose a subset of available publishers/subscribers and simultaneously find a distributed controller. This controller employs only the available sparse information while minimising the degradation of an optimisation index (e.g., $H_2$ norm of the closed-loop transfer function from $w$ to $z$) in contrast to the circumstance where all the system information are exploited.

To solve Problem 1, firstly build a structure for the design of a controller utilizing a priori specified subset of system information. This structure can be applied to deal with different control network configurations.
Problem 2. Consider a networked system depicted as in Equation (4); build a
distributed controller employing a priori specified subset of system information to
minimise an optimisation metric, e.g., $H_2$ norm of the closed-loop transfer function
from $w$ to $z$.

Because the systems in (1) or (4) have the parametric uncertainties, it is required
to develop a strategy for the design of sparsely distributed controllers for networked
systems. This strategy employs publisher/subscriber communication topology to
cope with the parametric uncertainties. To this end, we investigate the structured
static output feedback synthesis with a $H_2$ performance requirement. We present
some useful definitions as follows.

Definition 2.1. We call a matrix whose elements are either 0 or 1 as a structure
matrix. Let $\Delta = [\Delta_{ij}]_{m \times n}$ be a block matrix with $\Delta_{ij} \in \mathbb{R}^{a_i \times b_j}$, then the structure
matrix of $\Delta$ is obtained as $S(\Delta) = [\gamma_{ij}]_{m \times n}$ with
$$
\gamma_{ij} = \begin{cases} 
0 & \text{if } \Delta_{ij} = 0 \\
1 & \text{otherwise.}
\end{cases}
$$

Definition 2.2. Two matrices $\Delta_1$ and $\Delta_2$ are structurally the same if $S(\Delta_1) = S(\Delta_2)$.

Definition 2.3. The matrix $\Delta_1$ with $S(\Delta_1) = [\gamma_{ij}^1]_{m \times n}$ is structurally subset of $\Delta_2$
with $S(\Delta_2) = [\gamma_{ij}^2]_{m \times n}$ while $\gamma_{ij}^2 - \gamma_{ij}^1 \geq 0$. Denote this as $S(\Delta_1) \subseteq S(\Delta_2)$.

Definition 2.4. A block matrix $\tilde{\Delta}$ is said to be sparse block row-wise/column-wise
if its structure matrix $\tilde{\Gamma}$, i.e. $S(\tilde{\Delta}) = \tilde{\Gamma}$, includes (at least) one row/column of all
zeros.

2.2. $H_2$ State Feedback Control.

Lemma 2.5. (Projection lemma) [1] Given that two matrices $U$ and $V$ are of
column dimension $m$ and a matrix $Z_{m \times m}$ is symmetric, there exists an unstructured
matrix $X$ that satisfies
$$
U^T X V + V^T X^T U + Z < 0,
$$
iff
$$
\begin{cases}
N_U^T Z N_U < 0, \\
N_V^T Z N_V < 0,
\end{cases}
$$
where $N_U$ and $N_V$ are matrices whose columns form a basis of the null spaces of
matrices $U$ and $V$.

Lemma 2.6. [2] Assume a closed loop system is described by $A_{cl}$, $B_{cl}$, $C_{cl}$. Then,
the closed loop system is stable and $||C_{cl}(sI - A_{cl})^{-1}B_{cl}||_2 < \gamma$, if and only if, there
exist matrices $P > 0$ and $Z > 0$ such that
$$
\begin{bmatrix}
A_{cl}P + PA_{cl}^T & * \\
C_{cl}P & -\gamma I
\end{bmatrix} < 0,
$$
$$
\begin{bmatrix}
-Z & * \\
B_{cl} & -P
\end{bmatrix} < 0,
$$
$$
\text{trace}(Z) < 1.
$$
Lemma 2.7. Assume a closed loop system is described by $A_{cl}, B_{cl}, C_{cl}$. Then, the closed loop system is stable and $\|C_{cl}(sI - A_{cl})^{-1}B_{cl}\|_2^2 < \gamma$, if and only if, there exist a scale $\eta > 0$, matrices $X > 0$, $Z > 0$, and $V$, such that

\[
\begin{bmatrix}
-V - V^T & \ast & \ast \\
A_{cl}V + \eta V + X & -2\eta X & \ast \\
C_{cl}V & 0 & -\gamma I
\end{bmatrix} < 0,
\]

(11)

\[
\begin{bmatrix}
-Z & \ast \\
B_{cl} & -X
\end{bmatrix} < 0,
\]

(12)

\[\text{trace}(Z) < 1.\]

Proof. We define the matrices $X$, $Z$, $U$, $N_U$, $V$, and $N_V$ of Lemma 2.5 as follows:

$$X = V,$$

$$Z = \begin{bmatrix}
0 & X & 0 \\
X & -2\eta X & 0 \\
0 & 0 & -\gamma I
\end{bmatrix},$$

(14)

$$U = \begin{bmatrix}
-I & A_{cl}^T + \eta I & C_{cl}^T \\
\end{bmatrix},$$

(15)

$$N_U^T = \begin{bmatrix}
A_{cl} + \eta I & I & 0 \\
C_{cl} & 0 & I
\end{bmatrix},$$

(16)

$$V = \begin{bmatrix}
I & 0 & 0
\end{bmatrix},$$

(17)

$$N_V^T = \begin{bmatrix}
0 & I & 0 \\
0 & 0 & I
\end{bmatrix}. $$

(18)

It can easily be checked that $U^T X V + V^T X^T U + Z < 0$ is equivalent to (11), by letting $X = P$, $N_U^T Z N_U < 0$ in (7) is equivalent to (8), and $N_V^T Z N_V < 0$ in (7) is equivalent to the following trivial inequality

\[
\begin{bmatrix}
-2\eta X & 0 & 0 \\
0 & 0 & -\gamma
\end{bmatrix} < 0.
\]

Then, based on Lemma 2.5 and Lemma 2.6, this lemma can be proved.

Lemma 2.8. The following three statements, involving $X > 0$, $Z > 0$, a general matrix variable $V$ are equivalent.

i) $\exists F$ such that $A + B_2 F$ is stable and $\| (C_z + D_z F)(sI - A - B_2 F)^{-1}B_1 \|_2 < \gamma$.

ii) $\exists X > 0$, $Y$, $Z > 0$ and $V$ such that

\[
\begin{bmatrix}
-(V + V^T) & \ast & \ast \\
A V + B_2 Y + X + V & -2X & \ast \\
C_z V + D_z Y & 0 & -\gamma I
\end{bmatrix} < 0,
\]

(19)

\[
\begin{bmatrix}
-Z & \ast \\
B_1 & -X
\end{bmatrix} < 0,
\]

(20)

\[\text{trace}(Z) < 1,\]

(21)

where $Y = F V$.

Proof. In Lemma 2.7, if let $\eta = 1$, $A_{cl} = A + B_2 F$, $B_{cl} = B_1$, $C_{cl} = C_z + D_z F$, based on Lemmas 2.7 and 2.6 and define $Y = F X$, this lemma can be proved.
It should be emphasized here; as $V + V^T > 0$, $V$ is nonsingular and thus if the LMI (19) is feasible, the state feedback would be derived as $\mathcal{F} = YV^{-1}$. It is also important to note that as post-multiplication retains the row-sparse structure, if $Y$ is row-sparse, the corresponding feedback gain $\mathcal{F} = YV^{-1}$ will trivially be row-sparse. Moreover, the specific LMI characterisation in (19) enables us to use different Lyapunov matrices for each of the related LMI constraints in the problem. The reason is the product terms between the matrix $A$ and the Lyapunov matrix in the LMI (19) disappeared. In such a situation, the feedback gain can be achieved independent of the Lyapunov matrix. This feature has an important implication in the controller design of systems with parametric uncertainties.

The $H_2$ problem by supposing the controller as $u(t) = \mathcal{F}x(t)$ for the system (4) with parametric uncertainty (5), can be embedded in the following optimisation:

$$\begin{align*}
\text{minimise} & \quad \gamma \\
\text{subject to} & \quad (19), (20) \text{ and } (21).
\end{align*}$$

(22)

Also from the item $iii$ of Lemma 2.8, for each vertex $l$, the following inequalities can used:

$$\begin{align*}
&\begin{bmatrix}
-(V_l + V_l^T) & * & * \\
A V_l + B_2 Y_l + X_l + V_l & -2X_l & * \\
C_l V_l + D_2 Y_l & 0 & -\gamma I
\end{bmatrix} < 0, \\
&\begin{bmatrix}
-Z_l & * \\
B_l & -X_l
\end{bmatrix} < 0, \\
&\text{trace}(Z_l) < 1,
\end{align*}$$

(23)\hspace{1cm}(24)\hspace{1cm}(25)

where $X_l > 0$, $Z_l > 0$, $Y_l$ and $V_l$ are variables. However, addressing the optimisation problem in (22) with the above inequalities (for $l = 1, \cdots, N$) constraints by exploiting different $Y_l$ and $V_l$ cannot get a unique state feedback control $\mathcal{F}$. These inequalities in the next subsection will be used to design a sparse row/column-wise SOF for networked systems with parametric uncertain by using the proposed two specific matrix variable transformations.

### 2.3. Sparse row/column-wise $H_2$ SOF

Based on the discussions given previously, we specify the requirements of Problem 2 in the following problem.

**Problem 3.** Given a networked system with the state space representation in Equation (4) involving the parametric uncertainties, design a sparse row/column-wise SOF such that it ensures the $H_2$ performances, i.e. $\|T_{wz}\|_2^2 < \gamma$, while $\mathcal{S}(\mathcal{F}_y) \subseteq \Gamma$, where $\Gamma$ is a priori specified sparse row/column wise structure matrix and $\mathcal{F}_y$ is the SOF.

The SOF problem can be considered as a constrained state feedback problem; i.e. a state feedback (say $\mathcal{F}$) which satisfies the additional constraint $\mathcal{F} = \mathcal{F}_y \mathcal{C}$ [13]. Effective schemes to address a similar non-convex optimisation problem for the design of an $H_\infty$ SOF and mixed $H_2/H_\infty$ SOF are proposed in [13, 5]. In this paper, unlike [13, 5], the output matrix $\mathcal{C}$ belongs to the polytope (5). We now introduce specific LMI decision variables transformations as

$$\begin{align*}
V_l &= \Theta_l V_l \Theta_l^T + \Omega_l V_l \Omega_l^T, \\
Y_l &= Y_l \Omega_l^T,
\end{align*}$$

(26)
and only if $V$ can be set as the following optimisation problem:

$$\text{rank}_p \mathbf{A} \in \mathbb{R}^{n \times p} \text{ and } Y_{\Omega} \in \mathbb{R}^{n \times p}.$$ 

Besides, $\Theta_l = \text{null}(C_l) \in \mathbb{R}^{n \times (n-p)}$ and $\Omega_l \in \mathbb{R}^{n \times n}$ is any matrix that satisfies $C_l \Omega_l = I$. In general form, $\Omega_l$ can be considered as $\Omega_l = C_l^T \Theta_l \Phi_l$, where $\Phi_l \in \mathbb{R}^{(n-p) \times p}$ is a given matrix and $C_l^T (C_l C_l^T)^{-1}$. As seen, the matrix transformation proposed in (26) is essentially different from the ones proposed in [13, 5]. Now by letting the variables $V_l$ and $Y_l$ be (26), the SOF gain can be obtained through the following lemma.

**Lemma 2.9.** Let $V_l = \Theta_l V_{\Omega} \Theta_l^T + \Omega_l V_l \Omega_l^T$ and $Y_l = Y_{\Omega} \Omega_l^T$, then $V_l$ is invertible if and only if $V_{\Omega l}$ is invertible. Also, in such a case, $Y_l V_l^{-1} = F_y C_l$ with $F_y = Y_{\Omega} V_{\Omega}^{-1}$.

**Proof.** Left-multiply $V_l = \Theta_l V_{\Omega} \Theta_l^T + \Omega_l V_l \Omega_l^T$ by $C_l$ to find that $C_l V_l = V_l \Omega_l^T$. As it is guaranteed that $V_l$ is invertible, $C_l V_l$ is of rank $p$, and thereby $V_{\Omega l}$ should have rank $p$. Then, as $V_{\Omega l}$ is a full rank square matrix, it is invertible as well. As a result, $\Omega_l^T = V_{\Omega l}^{-1} C_l V_l$, and thus from (26), we have $Y_l = Y_{\Omega} V_{\Omega}^{-1} C_l V_l$. Now, it can be seen that

$$Y_l V_l^{-1} = Y_{\Omega} V_{\Omega}^{-1} C_l = F_y C_l.$$

\[ \square \]

Now the sparse row/column-wise $H_2$ SOF problem, by exploiting LMI approach, can be set as the following optimisation problem:

$$\text{minimise } \gamma$$

subject to (23), (24), (25), for $l = 1, \cdots, N$,

$$S(Y_{\Omega}) \subseteq \Gamma, S(V_{\Omega}) = I, \text{ and (26)}.$$

**Remark 1.** Matrix $\Phi_l$ plays an important role in the proposed method for SOF design. A trivial choice is $\Phi_l = 0$. However, other choices for $\Phi_l$ can be considered, such as $\Phi_l = (\Theta_l^T \Theta_l)^{-1} \Theta_l^T V_l C_l^T (C_l C_l^T)^{-1}$. However, this choice requires solving the optimisation problem in (22) subject to LMIs in (23)-(25) in advance to find $V_l$ associated with each vertex $l$. Clearly, if no solution can be attained by solving (22), the SOF design problem would not be feasible and no further action is required to be taken for the output feedback problem.

3. **Identifying Favourable Sparse Row/Column-Wise Structures.** The previous section developed a framework for the design of an $H_2$-based SOF while constraining the structure of the feedback gain. In this section, we aim to address the objective mentioned in Problem 1; i.e., seeking for an optimal subset of available publishers/subscribers in the networked system while the $H_2$-norm degradation of the closed-loop system is minimised relative to the fully distributed topology, or equivalently, finding favourable sparse block row/column-wise SOF gains. This problem can also be seen as searching for the redundant publishers and subscribers of the networked system. We indeed aim to construct a multi-objective optimisation program, where the block row/column sparsity of the SOF gain is directly incorporated into the index function. This is encapsulated in the following problem.

**Problem 4.** Given a system with the state space representation in Equation (4), find $F_y = Y_{\Omega} V_{\Omega}^{-1}$ and $\gamma > 0$ in the following optimisation program:

$$\text{minimise } \gamma + \| \Psi_{s} Y_{\Omega} \|_{\text{off-row}} + \| Y_{\Omega} \Psi_{p} \|_{\text{off-col}}$$

subject to the constraints in (27),
where \( Y_\Omega \) is a full decision matrix; i.e. \( S(Y_\Omega) = \Gamma = 1_{m \times p} \), the off-row-0 (off-col-0) is a quasi-norm that counts the number of non-zero off-diagonal block rows (columns) of \( Y_\Omega \), and \( \Psi_s = \text{diag}[\psi_s,i I_m]_{i=1}^m \) (\( \Psi_p = \text{diag}[\psi_p,j I_p]_{j=1}^p \)), with \( \psi_{s,i} \geq 0 \) (\( \psi_{p,j} \geq 0 \)), is a weighting matrix that implies the emphasis on the off-diagonal block row-sparsity (column-sparsity) of \( Y_\Omega \), and thus the SOF \( F_y \). For example, a larger \( \psi_{s,i} \) (\( \psi_{p,j} \)) will lead to not employing i-th subscriber (j-th publisher) in the control system.

Obviously, an intractable combinatorial search is required to address the optimisation problem above, hence the computation time would grow faster than polynomial, as the order of the networked system system grows [17]. A number of convex approximation of quasi-zero-norms are proposed yet, such as \( \ell_1 \)-norm or weighted \( \ell_1 \)-norm [7]. In addition, the paper [7] proposes the reweighted \( \ell_1 \) (REL1) minimisation method which is nothing but an iterative program that solves a sequence of weighted minimisation problems, in which at each iteration the weights are updated based on the previous iteration’s solution. The REL1 algorithm has recently been used by a number of researchers (e.g., see [9], [21]) for the design of sparse controllers for the distributed systems. Nevertheless, the developed REL1 schemes in these references do not promote row/column-sparsity of the feedback gain which is required in (28). Here, we need to develop a novel method in which the variable selection should amount to the selection of the important groups of variables (block rows and/or columns), rather than important individual variables (elements in the feedback gain).

Remark 2. In the existing literature, a scalar is used to weight the sparsity of the feedback gain; cf. [10], in an extended objective function, with the value of this scalar determining the emphasise on the sparsity of the feedback gain. However, in real cases, there may be prior information available about the control network. For example, some communication links can be infeasible or unattractive due to the high implementation costs. In this case, to assist the optimisation-based program proposed for the sparsity pattern recognition, it would be advisable to incorporate this a priori knowledge into the optimisation problem by using different scalars for weighting different off-diagonal block rows (columns). This can be implemented simply by specification of diagonal matrices (\( \Psi_s \) and \( \Psi_p \)) of separate weights corresponding to individual off-diagonal block rows (columns).

Let us now recast the objective function of the optimisation problem (28) as follows

**Problem 5.** Given a system with the state space representation in Equation (1), find \( F_y = Y_\Omega V_\Omega^{-1} \) and \( \gamma > 0 \) in the following optimisation program:

\[
\begin{align*}
\text{minimise} & \quad \gamma + f_s(\Psi_s Y_\Omega) + f_p(Y_\Omega \Psi_p), \\
\text{subject to} & \quad \text{the constraints in (27),}
\end{align*}
\]

excluding the structural constraint on \( Y_\Omega \). Here, \( f_s(\cdot) \) (\( f_p(\cdot) \)) denotes the relaxed off-diagonal block-row-sparsity (block-column-sparsity) promoting function.

The following subsection proposes candidates for \( f_s(\cdot) \) and \( f_p(\cdot) \).
3.1. **REL1 for row sparsity promoting penalty function.** A convex alternative for the non-convex off-row-0 quasi-norm, can be the following function

\[
    f_s(\Psi Y_\Omega) = \sum_{i,j, i \neq j} \mathcal{W}_{s,i} \psi_{s,i} \|Y_\Omega,ij\|_F, \tag{30}
\]

where \(\|\cdot\|_F\) denotes the Frobenius norm, and similarly

\[
    f_p(Y_\Omega \Psi_p) = \sum_{i,j, i \neq j} \|Y_\Omega,ij\|_F \psi_{p,j} W_{p,j}, \tag{31}
\]

is a convex approximation for the non-convex off-col-0 quasi-norm in (29). Moreover, the update rule for \(W_{s,i}\) can be considered as

\[
    W_{s,i}^k = \frac{1}{\sum_{j \neq i} \|F_{(k-1)}^{x,y}ij\|_F + \epsilon}, \tag{32}
\]

where \(k\) denotes the current iteration and we use \(0 < \epsilon \ll 1\) to provide stability and to ensure that a zero valued off-diagonal block row in \(Y_\Omega\) does not strictly prevent a non-zero value at the next step. The weighting matrix will be formed as \(\mathcal{W}_s = [w_{s,ij}]_{h \times h}\), where

\[
    w_{s,ij} = \begin{cases} 
    0_{m_i \times p_i} & \text{if } i = j \\
    W_{s,i}^1 & \text{otherwise.} 
    \end{cases} \tag{33}
\]

As seen, the weights are updated without considering the Frobenius norm of the block diagonal entries of the SOF gain, because local measurements should not raise the communication cost. It is also worth noting that the obtained relaxed sparsity promoting function do not promote sparsity within the blocks, but at the level of block rows. Similarly, \(\mathcal{W}_{p,j}\) can be updated as

\[
    W_{p,j}^k = \frac{1}{\sum_{i \neq j} \|F_{(k-1)}^{x,y}ij\|_F + \epsilon}. \tag{34}
\]

Additionally, we form the weighting matrix as \(\mathcal{W}_p = [w_{p,ij}]_{h \times h}\), where

\[
    w_{p,ij} = \begin{cases} 
    0_{m_i \times p_i} & \text{if } i = j \\
    W_{p,j}^1 & \text{otherwise.} 
    \end{cases} \tag{35}
\]

**Remark 3.** It is worth mentioning that minimising the number of subscribers and publishers or either of them have the possibility of promoting a completely decentralised control structure which is equal to a control network with no subscriber and publisher.

3.2. **An algorithm for solving Problem 5.** Now, define the matrix \(\bar{\tau} = [\tau_{ij}]_{h \times h}\) with

\[
    \tau_{ij} = \begin{cases} 
    0_{m_i \times p_i} & \text{if } i = j \\
    1_{m_i \times p_j} & \text{otherwise.} 
    \end{cases} 
\]

The optimisation problem in (29), by letting \(f_s(Y_\Omega)\) and \(f_p(Y_\Omega)\) as (30) and (31), respectively, is equivalent to

\[
    \text{minimise } \gamma + \text{trace}(\overline{\tau}^T \Psi_s \bar{\Upsilon}_1) + \text{trace}(\mathcal{Y}_2 \Psi_p \overline{\tau}^T) \tag{36}
\]

subject to the constraints in (28),

\[
    -\bar{\Upsilon}_1 \leq \mathcal{W}_s \circ Y_\Omega \leq \bar{\Upsilon}_1, \\
    -\mathcal{Y}_2 \leq \mathcal{W}_p \circ Y_\Omega \leq \mathcal{Y}_2.
\]
where $\circ$ denotes the Hadamard product (entry-wise product). For addressing the above convex problem and identifying a sparse row/column-wise SOF, the following algorithm is proposed.

**Algorithm 1.**

1. Given $\epsilon > 0$, $\alpha > 0$, $\Psi_s > 0$ and $\Psi_p > 0$, initialise $\mathcal{W}_s = [w_{s,ij}]_{h \times h}$, $\mathcal{W}_p = [w_{p,ij}]_{h \times h}$, with $w_{s,ij}$ and $w_{p,ij}$ as in (33) and (35), respectively, by letting $\mathcal{W}_{s,i} = 1$ and $\mathcal{W}_{p,j} = 1$, $k = 1$ and $F_k y = 0$.

2. Solve the minimisation problem (36) to obtain $F_y^\star = Y_\Omega V_\Omega^{-1}$.

3. Update $\mathcal{W}_{s,i}^k$ and $\mathcal{W}_{p,j}^k$ using the update rules in (32) and (34), respectively, and form $\mathcal{W}_s = [w_{s,ij}]_{h \times h}$ and $\mathcal{W}_p = [w_{p,ij}]_{h \times h}$ as in (33) and (35), respectively.

4. If $\|F_y^\star - F_y^k\| \leq \alpha$ go to Step 6, else $F_k y = F_y^k$, $k = k + 1$ and return to Step 2.

5. Let the unnecessary block rows and columns of $F_y^\star$ be zero and return $\Gamma^\star = \mathcal{S}(F_y^\star)$.

Eventually, in order to find the $H_2$ structured SOF with the identified $\Gamma^\star$, we turn to the minimisation problem in (27).

4. **Numerical examples.** Consider a decentralised interconnected system, presented in [16], that consists of four subsystems:

$$A = \begin{bmatrix} -1 & 2 & 1 & -1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ 0.5 & -0.5 & -2 & 0 & 1 & 1 \\ 1 & -1 & -1 & -2 & 0.4 & 0 \\ 0 & 0 & 0 & 1 & -3 & 1 \\ 2 & 0 & -1 & -2 & 0 & -4 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$B_1 = I_6, \quad C_z = \begin{bmatrix} I_6 \\ 0_{5 \times 6} \end{bmatrix}, \quad D_z = \begin{bmatrix} 0_{5 \times 5} \\ I_5 \end{bmatrix}.$$ 

We additionally assume that there exists an uncertainty in the entry (4, 6) of the output matrix $C$ by up to $\pm 50\%$ of its nominal value. As seen this interconnected system is fully coupled and open-loop unstable. Solving the convex problem in (27), by letting $\Gamma = I_{m \times p}$ and assuming a block diagonal structure for $V_l$, $l = 1, 2$ as $V_l = \Theta_l V_\Theta \Theta_l^T + \Omega_l V_\Omega \Omega_l^T$ where $V_\Theta \in \mathbb{R}^{2 \times 2}$ and $V_\Omega = \text{diag}(V_{\Omega,l})_{l=1}^{4 \times 4}$, with $V_{\Omega,l} \in \mathbb{R}$, are symmetric matrices, and $\Phi_l = 0$, results in a true $H_2$-norm of 3.7689 (with nominal $C$). We now exploit Algorithm 1 with $\alpha = 0.01$ and $\epsilon = 0.001$. By increasing $\Psi_s$ and $\Psi_p$, from zero, the number of block rows and columns with non-zero off-diagonal blocks in the SOF gain decreases; see Fig. 2. Once the sparsity structures of controllers are identified for different $\Psi_s$ and $\Psi_p$, the resulting patterns are used to solve (36), by identified $\Gamma$, in order to obtain the $H_2$ block row/column-sparse structured controllers.
5. Conclusions. In this study, the design of optimal sparse block row/column-wise feedback control is investigated for dynamical systems with polytopic uncertainty. Firstly, a LMI-based approach for the design of $H_2$ SOF gain for systems is proposed; it can incorporate extra structural constraints on the feedback gain matrix. Then, we proposed an iterative optimization procedure for the identification of favourable sparse block row/column-wise SOF feedback gains. Thereafter, to obtain the optimal structured gain, the proposed $H_2$-based SOF problem with the identified pattern constraint has been addressed. This approach can be immediately applied in publisher/subscriber networked systems for finding a subset of available publishers/subscribers, which targets the reduction of the communication costs etc. The scheme can also broaden the scope of application of the static output feedback design method for parameter uncertain system. However, it is worth noting that increasing the number of matrix variables in an optimization-based controller design problem may increase computation time. However, it is obvious that there is actually no need to wait for the termination criteria in the REL1 algorithm, and most of them can identify the favorable mode after several iterations. A numerical example clearly demonstrates the effectiveness of the proposed method.

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