GBDT of discrete skew-selfadjoint Dirac systems and explicit solutions of the corresponding non-stationary problems

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Dedicated to Rien Kaashoek on the occasion of his 80th anniversary

Abstract

Generalized Bäcklund-Darboux transformations (GBDTs) of discrete skew-selfadjoint Dirac systems have been successfully used for explicit solving of direct and inverse problems of Weyl-Titchmarsh theory. During explicit solving of direct and inverse problems, we considered GBDT of the trivial initial systems. However, GBDT of arbitrary discrete skew-selfadjoint Dirac systems is important as well and we introduce these transformations in the present paper. The obtained results are applied to the construction of explicit solutions of the interesting related non-stationary systems.

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1 Introduction

We consider a discrete skew-selfadjoint Dirac system

\[ y_{k+1}(z) = \left( I_m + \frac{i}{z} C_k \right) y_k(z), \quad C_k = U_k^* j U_k \quad (k \in \mathcal{I}), \quad (1.1) \]
where $I_m$ is the $m \times m$ identity matrix, $U_k$ are $m \times m$ unitary matrices,

$$
   j = \begin{bmatrix}
   I_{m_1} & 0 \\
   0 & -I_{m_2}
   \end{bmatrix} \quad (m_1, m_2 \in \mathbb{N}, \ m_1 + m_2 = m),
$$

(1.2)

and $\mathcal{I}$ is either $\mathbb{N}_0$ or the set $\{k \in \mathbb{N}_0 : 0 \leq k < N < \infty\}$ ($N \in \mathbb{N}$). Here, as usual, $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{N}_0 = 0 \cup \mathbb{N}$.

The relations

$$
   C_k = C_k^*, \quad C_k^2 = I_m
$$

(1.3)

are immediate from the second equality in (1.1).

This paper is a certain prolongation of the papers [3, 10], where direct and inverse problems for Dirac systems (1.1) have been solved explicitly and explicit solutions of the isotropic Heisenberg magnet model have been constructed. We would like to mention also the earlier papers on the cases of the continuous Dirac systems (see, e.g., [8, 9]).

The GBDT version of the Bäcklund-Darboux transformations have been used in [3, 8–10]. Bäcklund-Darboux transformations and related commutation methods (see, e.g., [12, 14, 17] and numerous references therein) are well-known tools in the spectral theory and in the construction of explicit solutions. In particular, the generalized Bäcklund-Darboux transformations (i.e., the GBDT version of the Bäcklund-Darboux transformations) were introduced in [12] and developed further in a series of papers (see [14] for details).

Whereas GBDTs of the trivial initial systems have been used in [3, 8–10] (in particular, initial systems [11] where $C_k \equiv j$ have been considered in [3, 10]), the case of an arbitrary initial discrete Dirac system (1.1) is considered here. In Section 2 we introduce GBDT, construct the so called Darboux matrix and give representation of the fundamental solution of the transformed system (see Theorem 2.2). Note that an explicit representation of the fundamental solutions of the transformed systems in terms of the solutions of the initial systems is one of the main features and advantages of the Darboux transformations.

One of the recent developments of the GBDT theory is connected with its application to the construction of explicit solutions of important dynamical
systems (see, e.g., [5, 13]). In Section 3 of this article, we use the same approach in order to construct explicit solutions of the non-stationary systems corresponding to the systems (1.1).

In the paper, \( \mathbb{C} \) stands for the complex plane and \( \mathbb{C}_+ \) stands for the open upper halfplane. The notation \( \sigma(\alpha) \) stands for the spectrum of the matrix \( \alpha \) and the notation \( \text{diag}\{d_1, d_2, \ldots\} \) stands for the block diagonal matrix with the blocks \( d_1, d_2, \ldots \) on the main diagonal.

2 GBDT of discrete skew-selfadjoint Dirac systems

Each GBDT of the initial system (1.1) is determined by some triple \( \{\alpha, S_0, \Lambda_0\} \) of the \( n \times n \) matrices \( \alpha \) and \( S_0 = S_0^\ast \) and the \( n \times m \) matrix \( \Lambda_0 \) (\( n \in \mathbb{N} \)) such that

\[
\alpha S_0 - S_0 \alpha^* = i\Lambda_0 \Lambda_0^*.
\]

(2.1)

The initial skew-selfadjoint Dirac system has the form (1.1) and the transformed (i.e., GBDT-transformed) system has the form

\[
\tilde{y}_{k+1}(z) = \left(I_m + \frac{i}{z} \tilde{C}_k\right)\tilde{y}_k(z) \quad (k \in \mathcal{I}),
\]

(2.2)

where the potential \( \{\tilde{C}_k\} \) \((k \in \mathcal{I})\) is given by the relations

\[
\Lambda_{k+1} = \Lambda_k + i\alpha^{-1}\Lambda_k C_k,
\]

(2.3)

\[
S_{k+1} = S_k + \alpha^{-1}S_k(\alpha^*)^{-1} + \alpha^{-1}\Lambda_k C_k \Lambda_k^*(\alpha^*)^{-1},
\]

(2.4)

\[
\tilde{C}_k = C_k + \Lambda_k^* S_k^{-1} \Lambda_k - \Lambda_k^* S_{k+1}^{-1} \Lambda_{k+1}, \quad k \in \mathcal{I}.
\]

(2.5)

Here and further in the text we assume that

\[
\det \alpha \neq 0,
\]

(2.6)

and suppose additionally in (2.5) that

\[
\det S_k \neq 0
\]

(2.7)
for \( k \in \mathbb{N}_0 \) or for \( 0 \leq k \leq N \) depending on the choice of the interval \( \mathcal{I} \), on which the Dirac system is considered.

Similar to the proof of [3, (3.7)], using the equality \( C_k^2 = I_m \) from (1.3) and relations (2.1)–(2.4) one easily proves by induction that

\[
\alpha S_k - S_k \alpha^* = i \Lambda_k \Lambda_k^*.
\] (2.8)

**Remark 2.1** Clearly, \( S_k = S_k^* \) and \( \tilde{C}_k = \tilde{C}_k^* \). Further in the text, in Proposition 2.3 we show that \( \tilde{C}_k^2 = I_m \). In Theorem 2.5, we show that under conditions \( S_0 > 0 \) and \( 0, i \notin \sigma(\alpha) \) (\( \sigma(\alpha) \) is the spectrum of \( \alpha \)) we have \( S_k > 0 \) and

\[
\tilde{C}_k = \tilde{U}_k^* j \tilde{U}_k,
\] (2.9)

where \( j \) is given in (1.2) and the matrices \( \tilde{U}_k \) are unitary. The equality (2.9) means that the transformed system (2.2) is again a skew-selfadjoint Dirac system in the sense of the definition (1.1). Before Theorem 2.5 we consider the GBDT-transformed system (2.2) without the requirement (2.9).

The fundamental solutions of (1.1) and (2.2) are denoted by \( w(k, z) \) and \( \tilde{w}(k, z) \), respectively, and are normalized by the conditions

\[
\tilde{w}(0, z) = w(0, z) = I_m.
\] (2.10)

In other words, \( y_k = w(k, z) \) and \( \tilde{y}_k = \tilde{w}(k, z) \) are \( m \times m \) matrix solutions of the initial and transformed systems, respectively, which satisfy the initial conditions (2.10). The so called Darboux matrix corresponding to the transformation of the system (1.1) into (2.2) is given by the transfer matrix function \( w_\alpha \) in Lev Sakhnovich form:

\[
w_\alpha(k, z) = I_m - i \Lambda_k^* S_k^{-1} (\alpha - z I_n)^{-1} \Lambda_k.
\] (2.11)

See [15, 16] as well as [14] and further references therein for the notion and properties of this transfer matrix function. The statement that the Darboux matrix has the form (2.11) may be formulated as the following theorem.
Theorem 2.2 Let the initial Dirac system (1.1) and a triple \( \{\alpha, S_0, \Lambda_0\} \), which satisfies the relations (2.1), (2.6), (2.7) and \( S_0 = S_0^* \), be given. Then, the fundamental solution \( w \) of the initial system and fundamental solution \( \tilde{w} \) of the transformed system (2.2) (determined by the triple \( \{\alpha, S_0, \Lambda_0\} \) via relations (2.3) - (2.5)) satisfy the equality
\[
\tilde{w}(k, z) = w_\alpha(k, -z)w(k, z)w_\alpha(0, -z)^{-1} \quad (k \geq 0),
\] (2.12)
where \( w_\alpha \) has the form (2.11).

Proof. The following equality is crucial for our proof
\[
w_\alpha(k + 1, z) \left( I_m - \frac{i}{z} C_k \right) = \left( I_m - \frac{i}{z} \tilde{C}_k \right) w_\alpha(k, z).
\] (2.13)
(It is easy to see that the important formula [3, (3.16)] is a particular case of (2.13).) In order to prove (2.13), note that according to (2.11) formula (2.13) is equivalent to the formula
\[
\frac{1}{z} \left( \tilde{C}_k - C_k \right) = \left( I_m - \frac{i}{z} \tilde{C}_k \right) \Lambda_k^* S_k^{-1} (zI_n - \alpha)^{-1} \Lambda_k
\] 
\[
- \Lambda_{k+1}^* S_{k+1}^{-1} (zI_n - \alpha)^{-1} \Lambda_{k+1} \left( I_m - \frac{i}{z} C_k \right).
\] (2.14)
Using the Taylor expansion of \((zI_n - \alpha)^{-1}\) at infinity we see that (2.14) is in turn equivalent to the set of equalities:
\[
\tilde{C}_k - C_k = \Lambda_k^* S_k^{-1} \Lambda_k - \Lambda_{k+1}^* S_{k+1}^{-1} \Lambda_{k+1},
\] (2.15)
\[
\Lambda_k^* S_{k+1}^{-1} \alpha^p \Lambda_{k+1} - i \Lambda_{k+1}^* S_{k+1}^{-1} \alpha^{p-1} \Lambda_{k+1} C_k
\] 
\[
= \Lambda_k^* S_k^{-1} \alpha^p \Lambda_k - i \tilde{C}_k \Lambda_k^* S_k^{-1} \alpha^{p-1} \Lambda_k \quad (p > 0).
\] (2.16)
Equality (2.15) is equivalent to (2.5) and it remains to prove (2.16). From (2.3), taking into account \( C_k^2 = I_m \) we have
\[
\alpha \Lambda_{k+1} - i \Lambda_{k+1} C_k = \alpha \Lambda_{k+1} - i \Lambda_k C_k + \alpha^{-1} \Lambda_k = \alpha \Lambda_k + \alpha^{-1} \Lambda_k.
\] (2.17)
Substituting (2.17) into the left hand side of (2.16) and using simple transformations, we rewrite (2.16) in the form
\[
Z_k \alpha^{p-2} \Lambda_k = 0, \quad Z_k := \Lambda_k^* S_{k+1}^{-1} (\alpha^2 + I_n) - \Lambda_k^* S_k^{-1} \alpha^2 + i \tilde{C}_k \Lambda_k^* S_k^{-1} \alpha.
\]
Therefore, in order to prove (2.16) (and so to prove (2.13)) it suffices to show that

\[ \Lambda_k^* S_k^{-1} (\alpha^2 + I_n) = \Lambda_k^* S_k^{-1} \alpha^2 - iC_k \Lambda_k^* S_k^{-1} \alpha, \]  

(2.18)

that is, \( Z_k = 0 \). Relation (2.18) is of interest in itself, since it is an analogue of (2.3) (more precisely of the relation adjoint to (2.3)) when \( \Lambda_k^* S_k^{-1} \) is taken instead of \( \Lambda_k^* \). Such analogues are useful in continuous and discrete GBDT as well as in the construction of explicit solutions of dynamical systems (see, e.g. [5, 13, 14] and references therein).

Taking into account (2.5) and (2.3), we rewrite (2.18) in the form

\[ \Lambda_k^* S_k^{-1} \]

\[ = \Lambda_k^* S_k^{-1} \alpha^2 + i (C_k + \Lambda_k^* S_k^{-1} \Lambda_k) \Lambda_k^* S_k^{-1} \alpha \]

\[ - \Lambda_k^* S_k^{-1} \alpha^2 + i (C_k + \Lambda_k^* S_k^{-1} \Lambda_k) \Lambda_k^* S_k^{-1} \alpha = 0. \]  

(2.19)

Since (2.8) yields

\[ i \Lambda_k \Lambda_k^* S_k^{-1} = \alpha - S_k \alpha^* S_k^{-1}, \]

(2.20)

we rewrite the third line in (2.19) and see that (2.19) (i.e., also (2.18)) is equivalent to

\[ \Lambda_k^* S_k^{-1} \]

\[ = \Lambda_k^* S_k^{-1} (\alpha^2 + I_n) - \alpha - \Lambda_k \Lambda_k^* S_k^{-1} \alpha \]

\[ - \Lambda_k^* S_k^{-1} \alpha^2 + i (C_k + \Lambda_k^* S_k^{-1} \Lambda_k) \Lambda_k^* S_k^{-1} \alpha = 0. \]  

(2.21)

Formula (2.4) implies that

\[ (I_n + \alpha^{-1} \Lambda_k C_k \Lambda_k^* S_k^{-1} \alpha + S_k \alpha^* S_k^{-1} \alpha) = S_{k+1} \alpha^* S_k^{-1} \alpha. \]

Hence, (2.21) is equivalent to

\[ \Lambda_{k+1}^* \alpha^* S_k^{-1} \alpha - \Lambda_k^* S_k^{-1} \alpha^2 + i (C_k + \Lambda_k^* S_k^{-1} \Lambda_k) \Lambda_k^* S_k^{-1} \alpha = 0. \]  

(2.22)

Using again (2.20), we rewrite (2.22) as

\[ \left( \Lambda_{k+1}^* - \Lambda_k^* + i C_k \Lambda_k^* (\alpha^*)^{-1} \right) \alpha^* S_k^{-1} \alpha = 0. \]  

(2.23)
The equality (2.23) is immediate from (2.3), and so (2.18) is also proved. Thus, (2.13) is proved as well.

Next, (2.12) is proved by induction. Clearly (2.10) yields (2.12) for \( k = 0 \). If (2.12) holds for \( k = r \), using (2.12) for \( k = r \) and relations (1.1), (2.2) and (2.13) we write

\[
\tilde{w}(r + 1, z) = \left( I_m + \frac{i}{z} \tilde{C}_r \right) \tilde{w}(r, z) = \left( I_m + \frac{i}{z} \tilde{C}_r \right) w_\alpha(r, -z) w(r, z) w_\alpha(0, -z)^{-1}
\]

\[
= w_\alpha(r + 1, -z) \left( I_m + \frac{i}{z} C_r \right) w(r, z) w_\alpha(0, -z)^{-1}
\]

\[
= w_\alpha(r + 1, -z) w(r + 1, z) w_\alpha(0, -z)^{-1}. \quad (2.24)
\]

Thus, (2.12) holds for \( k = r + 1 \), and so (2.12) is proved. ■

Using (2.13) we prove the next proposition.

**Proposition 2.3** Assume that the matrices \( C_k \) satisfy the second equality in (1.1) and the triple \( \{ \alpha, S_0, \Lambda_0 \} \) satisfies the relations (2.1), (2.6), (2.7), and \( S_0 = S_0^* \). Then, the transformed matrices \( \tilde{C}_k \) given by (2.3)–(2.5) have the following property:

\[
\tilde{C}_2^2 = I_m. \quad (2.25)
\]

**Proof.** It easily follows from (2.8) and (2.11) (see, e.g. [15] or [14, Corollary 1.13]) that

\[
w_\alpha(r, z) w_\alpha(r, \overline{z})^* \equiv I_m. \quad (2.26)
\]

Since \( C_k^2 = I_m \), we have

\[
\left( I_m - \frac{i}{z} C_k \right) \left( I_m + \frac{i}{z} C_k \right) = \left( 1 + \frac{1}{z^2} \right) I_m. \quad (2.27)
\]

In view of (2.26) and (2.27) we derive

\[
w_\alpha(k + 1, z) \left( I_m - \frac{i}{z} C_k \right) \left( I_m + \frac{i}{z} C_k \right) w_\alpha(k + 1, \overline{z})^* = \left( 1 + \frac{1}{z^2} \right) I_m. \quad (2.28)
\]
On the other hand (2.26) yields
\[
\left( I_m - \frac{i}{z} \tilde{C}_k \right) w_{\alpha}(k, z)w_{\alpha}(k, z)^* \left( I_m + \frac{i}{z} \tilde{C}_k \right) = \left( I_m - \frac{i}{z} \tilde{C}_k \right) \left( I_m + \frac{i}{z} \tilde{C}_k \right)
\]
\[
= I_m + \frac{1}{z^2} \tilde{C}_k^2. \tag{2.29}
\]

According to (2.13), the left hand sides of (2.28) and (2.29) are equal, and so we derive
\[
I_m + \frac{1}{z^2} \tilde{C}_k^2 = (1 + \frac{1}{z^2}) I_m, \quad \text{that is}, \quad (2.25) \text{ holds.} \quad \blacksquare
\]

Now, we introduce the notion of an admissible triple \( \{\alpha, S_0, \Lambda_0\} \) and show afterwards that the admissible triples determine \( S_k > 0 \) (\( k \in \mathbb{N}_0 \)). The definition of an admissible triple differs somewhat from the corresponding definition in [3], and the proof that \( S_k > 0 \) uses an idea from [4].

**Definition 2.4** The triple \( \{\alpha, S_0, \Lambda_0\} \) is called admissible if \( 0, i \not\in \sigma(\alpha), S_0 > 0 \) and the matrix identity (2.1) is valid.

**Theorem 2.5** Let an initial Dirac system (1.1) and an admissible triple \( \{\alpha, S_0, \Lambda_0\} \) be given. Then, the conditions of Theorem 2.2 are satisfied. Moreover, we have
\[
S_k > 0 \quad (k \in \mathbb{N}_0), \tag{2.30}
\]
and the transformed system (2.2) is skew-selfadjoint Dirac, that is, (2.9) is valid.

**Proof.** In order to prove (2.30) consider the difference
\[
S_{k+1} - (I_n - i\alpha^{-1})S_k(I_n + i(\alpha^{-1})^*) = S_{k+1} - S_k - \alpha^{-1}S_k(\alpha^{-1})^* + i(\alpha^{-1}S_k - S_k(\alpha^{-1})^*). \tag{2.31}
\]
Using (2.4), (2.8) and the second equality in (1.1), we rewrite (2.31) and derive a useful inequality:
\[
S_{k+1} - (I_n - i\alpha^{-1})S_k(I_n + i(\alpha^{-1})^*) = \alpha^{-1}\Lambda_k C_k\Lambda_k^* (\alpha^{-1})^*
\]
\[
+ \alpha^{-1}\Lambda_k\Lambda_k^* (\alpha^{-1})^* \geq 0. \tag{2.32}
\]

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Since \( 0, i \not\in \sigma(\alpha) \), the sequence \( (I_n - i\alpha^{-1})^{-k}S_k(I_n + i(\alpha^{-1})^*)^{-k} \) \((k \in \mathbb{N}_0)\) is well-defined. In view of (2.32), this sequence is nondecreasing. Hence, taking into account \( S_0 > 0 \) we have \( (I_n - i\alpha^{-1})^{-k}S_k(I_n + i(\alpha^{-1})^*)^{-k} > 0 \), and so (2.30) holds.

Similar to [3, Lemma A.1] one can show that

\[
\sigma(\alpha) \subset \mathbb{C}_+.
\]

That is, one rewrites (2.1) in the form

\[
\left( S_0^{-1/2} \alpha S_0^{-1/2} \right)^* = iS_0^{-1/2} \Lambda_0 \Lambda_0^* S_0^{-1/2},
\]

and from \( S_0^{-1/2} \Lambda_0 \Lambda_0^* S_0^{-1/2} \geq 0 \), the relation \( \sigma(\alpha) = \sigma(S_0^{-1/2} \alpha S_0^{1/2}) \subset \mathbb{C}_+ \) follows. Clearly, (2.33) yields \(-i \not\in \sigma(\alpha)\). Therefore, we may set \( z = -i \) in (2.13) and (taking into account the second equality in (1.1) and formula (2.26)) we obtain

\[
I_m + \tilde{C}_k = w_\alpha(k + 1, -i)(I_m + C_k)w_\alpha(k, i)^* = 2w_\alpha(k + 1, -i)U_k \begin{bmatrix} I_{m_1} & 0 \\ 0 & I_{m_2} \end{bmatrix} U_k^* w_\alpha(k, i)^*.
\]

(2.34)

In the same way, setting in (2.13) \( z = i \) we obtain

\[
I_m - \tilde{C}_k = 2w_\alpha(k + 1, i)U_k^* \begin{bmatrix} 0 & I_{m_2} \\ I_{m_1} & 0 \end{bmatrix} U_k w_\alpha(k, -i)^*.
\]

(2.35)

According to (2.34) and (2.35) the dimension of the subspace of \( \tilde{C}_k \) corresponding to the eigenvalue \( \lambda = -1 \) is more or equal to \( m_2 \) and the dimension of the subspace of \( \tilde{C}_k \) corresponding to the eigenvalue \( \lambda = 1 \) is more or equal to \( m_1 \). Thus, the representation (2.9) is immediate. ■

**Remark 2.6** It follows from (2.9) that \( I_m + \tilde{C}_k \geq 0 \) and that \( I_m + \tilde{C}_k \) has rank \( m_1 \). Hence, (2.31) yields

\[
\begin{bmatrix} I_{m_1} & 0 \\ 0 & I_{m_2} \end{bmatrix} U_k w_\alpha(k + 1, -i)^* = \tilde{q}_k \begin{bmatrix} I_{m_1} & 0 \\ 0 & I_{m_2} \end{bmatrix} U_k w_\alpha(k, i)^*, \quad \tilde{q}_k > 0
\]

(2.36)

for some matrix \( \tilde{q}_k \). In the same way, formulas (2.9) and (2.35) imply that

\[
\begin{bmatrix} 0 & I_{m_2} \end{bmatrix} U_k w_\alpha(k + 1, i)^* = \check{q}_k \begin{bmatrix} 0 & I_{m_2} \end{bmatrix} U_k w_\alpha(k, -i)^*, \quad \check{q}_k > 0.
\]

(2.37)
Now, setting
\[ \tilde{U}_k = W_k := \text{diag}\{\tilde{q}_k^{1/2}, \tilde{q}_k^{1/2}\} \begin{bmatrix} \begin{bmatrix} I_{m_1} & 0 \end{bmatrix} U_k w_\alpha(k, i)^* \\ \begin{bmatrix} 0 & I_{m_2} \end{bmatrix} U_k w_\alpha(k, -i)^* \end{bmatrix} \] (2.38)

we provide expressions for some suitable unitary matrices \(\tilde{U}_k\) in the representations (2.9) of the matrices \(\tilde{C}_k\).

Indeed, according to the definition of \(W_k\) in (2.38) and to the relations (2.34) – (2.37) we have
\[ \tilde{C}_k = \frac{1}{2} \left( (I_m + \tilde{C}_k) - (I_m - \tilde{C}_k) \right) = W_k^* j W_k, \] (2.39)
and it remains to show that \(W_k\) is unitary. In view of (2.26) and (2.38), it is easy to see that \(W_k W_k^*\) is a block diagonal matrix:
\[ W_k W_k^* = \text{diag}\{\hat{\rho}_k, \hat{\rho}_k\} > 0. \] (2.40)

Hence, for \(R_k > 0\) from the polar decomposition \(W_k = R_k V_k (V_k V_k^* = I_m)\) we have \(R_k^2 = \text{diag}\{\hat{\rho}_k, \hat{\rho}_k\}\) (and, in particular, \(R_k\) is block diagonal). Therefore, (2.39) may be rewritten in the form
\[ \tilde{C}_k = V_k^* \text{diag}\{\hat{\rho}_k, -\hat{\rho}_k\} V_k \quad (\hat{\rho}_k > 0, \ \hat{\rho}_k > 0). \]

Comparing (2.9) with the formula above, we see that all the eigenvalues of \(\hat{\rho}_k\) and \(\hat{\rho}_k\) equal 1, that is, \(R_k^2 = R_k = I_m\), and so \(W_k = V_k\). In other words, \(W_k\) is unitary.

### 3  Explicit solutions of the corresponding non-stationary systems

Recall the equalities (2.18):
\[ \Lambda_{k+1}^* S_{k+1}^{-1}(\alpha^2 + I_n) = \Lambda_k^* S_k^{-1} \alpha^2 - i\tilde{C}_k \Lambda_k^* S_k^{-1} \alpha, \quad k \in \mathbb{N}_0, \] (3.1)
which are basic for the construction of explicit solutions of non-stationary systems. Introduce the semi-infinite shift block matrix $\mathbf{S}$ and diagonal block matrix $\tilde{\mathbf{C}}$:

$$
\mathbf{S} := \begin{bmatrix}
0 & I_m & 0 & 0 & \ldots \\
0 & 0 & I_m & 0 & \ldots \\
0 & 0 & 0 & I_m & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots
\end{bmatrix}, \quad \tilde{\mathbf{C}} := \text{diag}\{\tilde{C}_0, \tilde{C}_1, \tilde{C}_2, \ldots\}. \quad (3.2)
$$

The semi-infinite block column $\Psi(t)$ is given by the formula

$$
\Psi(t) = Ye^{ita}, \quad Y = \{Y_k\}_{k=0}^{\infty}, \quad Y_k = \Lambda_k^* S_k^{-1}. \quad (3.3)
$$

It is easy to see that equalities (3.1) and Theorem 2.5 yield the following result.

**Theorem 3.1** Let the initial Dirac system (1.1) and an admissible triple $\{\alpha, S_0, \Lambda_0\}$ be given. Then, the matrices $\Lambda_k, S_k$ and $\tilde{C}_k$ are well-defined via (2.3) – (2.5) for $k \in \mathbb{N}_0$. Moreover, the block vector function $\Psi(t)$ constructed in (3.3) satisfies the non-stationary system

$$
(I - S)\Psi'' + \tilde{C}\Psi' + S\Psi = 0, \quad (3.4)
$$

where $I$ is the identity operator and $\Psi' = \frac{d}{dt}\Psi$.

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