Abstract. In this paper we obtain 32 canonical forms for 3D piecewise smooth vector fields presenting the so called cusp-fold singularity. All these canonical forms are topologically distinct and collect the main topological aspects of the singularities described as kind of the tangencies involved and positions of the sliding, escaping and crossing regions. Also, one-parameter bifurcations of these canonical forms are presented and the topologically equivalent piecewise smooth vector fields are obtained.

1. Introduction

It is well known that Ordinary Differential Equations (ODEs for short) can be used to model a large range of real world phenomena (see [14]). However, in some of these phenomena it occurs a sudden change in the law that governs the model. When these sudden changes occurs, a non smooth ODE is a better choice to model the system and a piecewise smooth vector field (PSVF for short) must to be considered. The pioneering works concerning this subject are the works [1, 11]. After it, the literature was widely improved and some books about the subject are [10, 13, 15], among others.

Examples of such applications are HIV or cancer treatment where the patient receives drugs in a period and does not receives for another period in order to provide a better recovery of the organism or a protocol of containment due to the COVID (see [4, 5, 6, 12, 16], among others). We stress that the previously quoted papers [4, 6] are applied models where the 3D cusp-fold singularity (see the formal definition below) appears in a natural way. Moreover, a better description of the dynamics of such applied models were not stated with more details in the literature due to the hardness of understanding the behavior around the cusp-fold singularity. Here we follow a Thom-Smale program and consider the smallest codimension 3D cusp-fold singularities and its unfoldings. Of course, higher codimension 3D cusp-fold singularities must be analyzed in future works.

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Along this paper we will prove the existence of 32 canonical forms (see Theorem 1) that cover all topological types of cusp-fold singularities when we are interested in the kind of tangential contact occurring along $S_{\pm}$, the position of the distinguished regions sliding, escaping and crossing and the way by means the trajectory of $Z_{\pm}$ is arriving or departing from the cusp singularity. Also, we prove the equivalence among a specific canonical form and all other PSVF presenting a cusp-fold singularity presenting the same set of aspects that we are interested (see Theorem 2). Moreover, codimension one bifurcations are considered, the bifurcation diagrams are exhibited and topological equivalences are provided (see Theorem 3).

The paper is organized as follows: In Section 2 we establish some terminology and the main results of the paper. In Section 3 we prove Theorem 1. In Section 4 we prove Theorem 2. In Section 5 we prove Theorem 3. We conclude the paper with Section 6 where we collect some aspects proved along the paper.

2. Setting the problem

2.1. Basic theory concerning PSVFs

In this paper we study 3D PSVFs in two zones, i.e., the space is partitioned in two zones by means of a codimension one switching manifold $\Sigma = \{x \in \mathbb{R}^3 \mid f(x) = 0\}$ for some smooth function $f : \mathbb{R}^3 \to \mathbb{R}$ having $0 \in \mathbb{R}$ as a regular value. Then, two smooth vector fields $Z_+$ and $Z_-$ are considered in each one of these zones and we define $\Omega$ being the space of PSVFs $Z : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$Z(q) = \begin{cases} Z_+(q), & \text{for } q \in \Sigma^+, \\ Z_-(q), & \text{for } q \in \Sigma^- . \end{cases}$$

We endow $\Omega$ with the product topology. The trajectories of $Z$ are solutions of $\dot{q} = Z(q)$ and we will accept it to be multi-valued in points of $\Sigma$.

These smooth vector fields can present trajectories tangent to $\Sigma$. In 3D, these tangencies will occurs, generically, in curves $S_+$ and $S_-$, respectively, embedded in $\Sigma$.

For practical purposes, the contact between the smooth vector field $Z_+$ and the switching manifold $\Sigma$ is characterized by the Lie derivative given by the expression $Z_+ f(p) = \langle \nabla f(p), Z_+(p) \rangle$ where $\langle \ldots \rangle$ is the usual inner product in $\mathbb{R}^2$. Moreover, $Z_+^i f(p) = \langle \nabla Z_+^{i-1} f(p), Z_+(p) \rangle$ for $i \geq 2$. The same for $Z_-$.

When $Z_\pm$ has a quadratic contact with $\Sigma$ at $p$ we say that $p$ is a fold singularity. When this contact is cubic, we say that $p$ is a cusp singularity. The fold singularities can be partitioned in two groups: the visible fold singularities with are those ones where the tangent trajectory remains at the
same zone where its smooth vector field is defined and the invisible fold singularities with are those ones where the tangent trajectory is placed at the zones where its vector field is not defined.

Generically, a cusp singularity $p$ is isolated over $S_\pm$ and in one side of $p$ it appear invisible fold singularities over $S_\pm$ and at the other side it appear visible fold singularities. We get a fold-fold singularity (respectively, cusp-fold singularity) if such a point is a fold singularity for both $Z_+$ and $Z_-$ (respectively, is a fold singularity for one of them and a cusp singularity for the other one).

The 3D PSVFs presenting cusp-fold singularities can have very complicated dynamics. In fact, it is an infinite codimension singularity since it has (as we will see next in the paper) a fold-fold singularity of infinite codimension at its bifurcation set (the proof of this fact can be seen in [2, 8]). Moreover, $Z_\pm$ can be nonlinear and with very complicated expression. For some previous references concerning 3D cusp-fold singularities see [7, 9] and for 2D cusp-fold singularities see [3].

Despite of the tangencies, the smooth vector fields $Z_\pm$ are transversal to $\Sigma$. The region $\Sigma^a$ (respectively, $\Sigma^e$) of $\Sigma$ where both $Z_\pm$ are arriving (respectively, departing from) $\Sigma$ is called the sliding region (respectively, escaping region). The region $\Sigma^c$ of $\Sigma$ where one of the vector fields $Z_\pm$ is arriving and the other one is departing transversely from $\Sigma$ is called a crossing region.

2.2. The main results.

**Theorem 1.** The next 32 linear PSVFs defined in $\mathbb{R}^3$ have cusp-fold singularities topologically distinct at the origin. They are given by all possible choices on the parameters $\alpha, \beta, \gamma, \mu, \theta \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$ in

$$Z(x, y, z) = \begin{cases} Z_+(x, y, z) = \begin{pmatrix} \text{Sgn}(\alpha)y \\ \text{Sgn}(\beta) \\ \text{Sgn}(\gamma)x \\ 0 \end{pmatrix}, & z \geq 0 \\
Z_-(x, y, z) = \begin{pmatrix} \text{Sgn}(\mu) \\ \text{Sgn}(\theta)y \end{pmatrix}, & z \leq 0. \end{cases}$$

(2)

**Remark 1** (Notation). We will say that a cusp-fold singularity is of type $(+++,-+)$ if in Equation (2) we get all the 3 parameters $(\alpha, \beta, \gamma)$ of $Z_+$ and the 2 parameters $(\mu, \theta)$ of $Z_-$ positive. In an analogous way, we define all the 32 types of cusp-fold singularities given by $(\pm \pm \pm, \pm \pm)$. 
\textbf{Definition 1.} We will say that two PSVFs $Z = (Z_+, Z_-)$ and $\tilde{Z} = (\tilde{Z}_+, \tilde{Z}_-)$ are \textbf{weak-equivalent} if they have the same kind of tangential contact points occurring along $S_\pm$ and $\tilde{S}_\pm$, including the orientation of the trajectories passing through it, and the position of the distinguished regions sliding, escaping and crossing is the same in relation to the boundaries $S_\pm \subset \Sigma$ and $\tilde{S}_\pm \subset \tilde{\Sigma}$.

\textbf{Theorem 2.} Let $\tilde{Z} = (\tilde{Z}_+, \tilde{Z}_-)$ be a PSVF defined in a small neighborhood $\tilde{U} \subset \mathbb{R}^3$ of a point $p \in \Sigma$. Suppose that $p$ is an isolated cusp singularity of $\tilde{Z}_+$ and a fold singularity of $\tilde{Z}_-$. If the curves the $\tilde{S}_+$ and $\tilde{S}_-$ are transversal at $p$ then $\tilde{Z}$ is weak-equivalent to one of the previous cusp-folds presented in Theorem 1.

Also, we will consider a one-parameter bifurcation of the PSVFs presenting a cusp-fold singularity in Theorem 1.

\textbf{Theorem 3.} Consider the PSVFs given by the following perturbation of Equation (2):

\begin{equation}
Z_\lambda(x, y, z) = \begin{cases}
Z_+(x, y, z) = \begin{pmatrix}
\text{Sgn}(\alpha)y \\
\text{Sgn}(\beta) \\
\text{Sgn}(\gamma)x
\end{pmatrix}, & z \geq 0 \\
0 & \text{otherwise}
\end{cases}
\end{equation}

\begin{equation}
Z_\lambda^-(x, y, z) = \begin{pmatrix}
\text{Sgn}(\mu) \\
\text{Sgn}(\theta)y - \lambda
\end{pmatrix}, & z \leq 0
\end{equation}

where $\lambda \in (-\varepsilon, \varepsilon)$ with $\varepsilon$ being a small positive real number. The variation of the parameters $\alpha, \beta, \gamma, \mu, \theta \in \mathbb{R}^*$ produce one-parameter unfoldings of the 32 cusp-fold singularities presented in Theorem 1. Moreover, from Theorem 2, the configuration on $\Sigma$ (kinds of tangency points and position of sliding, crossing and escaping regions) obtained in these unfoldings are the same of an unfolding of a general PSVF presenting a cusp-fold with the hypothesis of Theorem 2 and, as consequence, there exists a weak-equivalence between the unfoldings of the perturbations of the canonical forms (3) and the unfolding of the perturbations of general cusp-fold singularities with the hypotheses of Theorem 2.
3. Proof of Theorem 1

Consider the vector fields $Z_\pm$ defined by Equation (2). A direct inspection shows that the Lie derivatives with respect to $Z_+$ are given by:

$$
Z_+ f(x, y, z) = \text{Sgn}(\gamma)x, \\
Z_2^+ f(x, y, z) = \text{Sgn}(\alpha\gamma)y, \\
Z_3^+ f(x, y, z) = \text{Sgn}(\alpha\beta\gamma).
$$

This shows that $S_+$ is the $y$–axis and the origin is a cusp point of the vector field $Z_+$. Note that $Z_+ f, Z_2^+ f$ and $Z_3^+ f$ depend directly on the sign of the parameters $\alpha, \beta$ and $\gamma$. This generates eight topologically distinct configurations (see Figure 1).

![Figure 1](image_url)

**Figure 1.** The eight configurations for $Z_+$. 

Such settings can be explained by combining the following cases: if $Z_3^+ f > 0$ (respectively $Z_3^+ f < 0$) we have $\alpha\beta\gamma > 0$ (respectively $\alpha\beta\gamma < 0$) and the integral curve of $Z_+$ passing through the origin, whose parameterization is
given by \( \xi(t) = (\frac{1}{2}t^2\alpha\beta, t\beta, \frac{1}{6}t^3\alpha\beta\gamma) \), evolves to \( \Sigma^+ \) (receptively, \( \Sigma^- \)); if \( \alpha\gamma > 0 \) (respectively \( \alpha\gamma < 0 \)) we have \( Z^+_xf > 0 \) (respectively \( Z^+_xf < 0 \)) when \( y > 0 \) and \( Z^+_xf < 0 \) (respectively \( Z^+_xf > 0 \)) when \( y < 0 \), which ensures that the positive part of the \( y \)-axis is filled by visible (respectively invisible) fold points of \( Z^+ \) while the negative part is filled by invisible (respectively visible) fold points of \( Z^+ \); if \( \alpha\gamma > 0 \) (respectively \( \alpha\gamma < 0 \)) we have \( Z^+_xf > 0 \) (respectively \( Z^+_xf < 0 \)) when \( x > 0 \) and \( Z^+_xf < 0 \) (respectively \( Z^+_xf > 0 \)) when \( x < 0 \) which means that the angle between \( Z^+_f(x, y, z) \) and \( \nabla f(x, y, z) \) belongs to \([0, \pi/2)\) (respectively, \((\pi/2, \pi)\)) when \( x > 0 \) while the angle between \( Z^-f(x, y, z) \) and \( \nabla f(x, y, z) \) belongs to \((\pi/2, \pi)\) (respectively, \([0, \pi/2)\)) when \( x < 0 \).

In the same way, we obtain that the Lie derivatives with respect to \( Z^-f \) are given by:

\[
Z^-f(x, y, z) = \text{sgn}(\theta)y, \\
Z^2_f(x, y, z) = \text{sgn}(\mu\theta).
\]

This shows that \( S^- \) is the \( x \)-axis and the sign of the parameters \( \mu \) and \( \theta \) directly influence \( Z^-f \) and \( Z^2_f \). This generates four configurations that are also topologically distinct (see Figure 2). These configurations are obtained by combining the following cases: if \( \mu\theta > 0 \) (respectively \( \mu\theta < 0 \)) we have \( Z^2_- > 0 \) (respectively \( Z^2_- < 0 \)), which guarantees that the \( x \)-axis is formed by invisible (respectively visible) fold points of \( Z^- \); if \( \theta > 0 \) (respectively \( \theta < 0 \)) we have \( Z^-f > 0 \) (respectively \( Z^-f < 0 \)) when \( y > 0 \) and \( Z^-f < 0 \) (respectively \( Z^-f > 0 \)) when \( y < 0 \) which means that the angle between \( Z^-(x, y, z) \) and \( \nabla f(x, y, z) \) belongs to \([0, \pi/2)\) (respectively, \((\pi/2, \pi)\)) when \( y > 0 \) while the angle between \( Z^-f(x, y, z) \) and \( \nabla f(x, y, z) \) belongs to \((\pi/2, \pi)\) (respectively, \([0, \pi/2)\)) when \( y < 0 \).

Finally, combining the eight possibilities illustrated in Figure 1 with the four possibilities represented in Figure 2 we obtain the 32 cusp-fold singularities topologically distinct at the origin and Theorem 1 is proved.

Denote \( \Sigma^{c+} \) (respectively, \( \Sigma^{c-} \)) the set of points \( q \in \Sigma^c \) such that the angle between, both \( Z_+(q) \) and \( Z_-(q) \), and \( \nabla f(x, y, z) \) belongs to \([0, \pi/2)\) (respectively, \((\pi/2, \pi)\)). Since \( Z_+(f(x, y, x)Z^-f(x, y, x) = \text{sgn}(\gamma\theta)xy \) we get four configurations for the sets \( \Sigma^{c+}, \Sigma^c \) and \( \Sigma^c \) (see Figure 3) whose analysis is similar to what we did earlier.

4. PROOF OF THEOREM 2

In this section we will prove Theorem 2. Let \( Z(x, y, z) \) be given by (2). Before to continue, let us distinguish \( \Sigma^{c+} \subset \Sigma^c \) (respectively, \( \Sigma^{c-} \subset \Sigma^c \)) if the angle between \( Z_+ \) and \( \nabla f \) belongs to \([0, \pi/2)\) (respectively, \((\pi/2, \pi)\)).
Assume the hypothesis $\alpha, \beta, \gamma > 0, \mu, \theta > 0$ and that $Z$ is defined in a small neighborhood $U \subset \mathbb{R}^3$ of the origin. The configuration is given by item (A) of Figure 1 and item (A) of Figure 2. For the other configurations of $Z$, the construction of the homomorphism will be analogous.

Let $\tilde{Z}(x, y, z) = (\tilde{Z}_+, \tilde{Z}_-)$ be a PSVF defined in a small neighborhood $\tilde{U} \subset \mathbb{R}^3$ of a point $p \in \tilde{\Sigma}$, such that $p$ is an isolated cusp singularity of $\tilde{Z}_+$ and a fold singularity of $\tilde{Z}_-$. Furthermore, assume that the tangency curves $\tilde{S}_\pm$ are transversal at $p$, the $\tilde{Z}_+$-trajectory passing though $p$ is arriving $\tilde{\Sigma}_+$, on the right of $p$ the curve $\tilde{S}_1 \subset \tilde{S}_+$ is fulfilled of visible fold singularities, at the left
of $p$ the curve $\tilde{S}_2 \subset \tilde{S}_+$ is fulfilled of invisible fold singularities, the fold points of $\tilde{Z}_-$ are invisible.

Now let us construct the homomorphism $h$. The first step is to define $h(0) = p$. Let $\gamma_0$ be the cusp orbit of $Z_+$ of length $|A(0)|$, and $\gamma_p$ be the cusp orbit of $\tilde{Z}_+$ of length $|A(p)|$, where $A(0)$ and $A(p)$ denotes the arc of the trajectory through 0 and $p$, bounded by $U$ and $\tilde{U}$, respectively. The second step is to define $h(\gamma_0) = \gamma_p$ by arc length parametrization. In fact, let $u$ be a point on the arc $A(0)$, see Figure 5. We define $h(u)$ as the image of this point on the arc $A(p)$, such that the length $|\hat{0}u|$ of arc $\hat{0}u$ divides the arc length $A(0)$ in the same proportion such that $|ph(u)|$ divides the arc length $A(p)$, i.e.,

$$\frac{|\hat{0}u|}{|A(0)|} = \frac{|ph(u)|}{|A(p)|}.$$

Now, let $\tilde{L}_1$ be the length of $\tilde{S}_1 \cap \tilde{U}$, where the vector field $\tilde{Z}_+$ presents visible folds. We consider $L_1$ being the length of $S_1 \cap U$, where the vector field $(+++, ++)$ presents visible folds. We identify $S_1$ and $\tilde{S}_1$ by arc length parametrization. Analogously, we make the same for $\tilde{S}_2$ and $S_2$ (respectively, $\tilde{S}_-$ and $S_-$). So $h(S_+) = S_+$ and $h(S_-) = S_-.$

By $q \in S_+ \setminus \{0\}$ (respectively, $\tilde{q} \in \tilde{S}_+ \setminus \{p\}$), we mark a line segment $R_q$ (resp. $R_{\tilde{q}}$), orthogonal to $\Sigma$ in such a way that this segment reaches an orbit of $Z_+$ (resp. $\tilde{Z}_+$) in $q_1$ (resp. $\tilde{q}_1$), see Figure 5. Since $S_+$ and $\tilde{S}_+$ are identified, we identify each $R_\alpha$ with $R_{\tilde{q}}$ by arc length parametrization.

Let $W$ (resp. $\tilde{W}$) be the foliation generated by the arcs $R_\alpha$ (resp. $R_{\tilde{q}}$). We divide $W = W_1 \cup W_2$ where $W_1$ contains points of $S_1$ and $W_2$ contains points of $S_2$. Analogously for $\tilde{W}$. See Figure 4.

For every point $q_1 \in W_2$ (resp. $\tilde{q}_1 \in \tilde{W}_2$), by Implicit Function Theorem (IFT for short), there exists a smallest time $t_0 < 0$ (resp. $\tilde{t}_0 < 0$) depending on $q_1$ (resp. $\tilde{q}_1$), such that $\Phi_{Z_+}(q_1, t_0) = q_0 \in \Sigma$ (resp. $\Phi_{\tilde{Z}_+}(\tilde{q}_1, \tilde{t}_0) = \tilde{q}_0 \in \tilde{\Sigma}$), where $\Phi_T$ denotes the flow of the vector field $T$.

Identify the orbit arcs $\gamma_{q_0}^q(Z_+)$ and $\gamma_{\tilde{q}_0}^{\tilde{q}}(\tilde{Z}_+)$ by arc length parametrization. Again by IFT, there exists a smallest time $t_2 > 0$ (resp. $\tilde{t}_2 > 0$) depending on $q_1$ (resp. $\tilde{q}_1$), such that $\Phi_{Z_+}(q_1, t_2) = q_2 \in \Sigma$ (resp. $\Phi_{\tilde{Z}_+}(\tilde{q}_1, \tilde{t}_2) = \tilde{q}_2 \in \tilde{\Sigma}$). Identify the orbit arcs $\gamma_{q_1}^{q_2}(Z_+)$ and $\gamma_{\tilde{q}_1}^{\tilde{q}_2}(\tilde{Z}_+)$ by arc length parametrization.

Since $S_-$ and $\tilde{S}_-$ are invisible folds of $Z_-$ and $\tilde{Z}_-$, respectively, we repeat the previous approach.
Let \( w_1 \in \Sigma^+ \), such that there exists a smallest time \( |t_1| \) such that \( \varphi_{Z_+}(t_1, w_1) = s_1 \in W_1 \), see Figure 6. We have that \( h(s_1) = \tilde{s}_1 \), since \( W_1 \) and \( \tilde{W}_1 \) are identified. So we identify the arcs of trajectories \( A(s_1) \) and \( A(\tilde{s}_1) \), limited by \( U \) and \( \tilde{U} \), respectively. In more details, we define \( h(w_1) \) such that,

\[
\frac{|\overset{\sim}{W}_1 s_1|}{|A(s_1)|} = \frac{|h(w_1)\tilde{s}_1|}{|A(\tilde{s}_1)|}.
\]

Now let us identify \( \Sigma^{c+} \cup \Sigma^s \) and \( \tilde{\Sigma}^{c+} \cup \tilde{\Sigma}^s \). In \( \tilde{\Sigma}^{c+} \cup \tilde{\Sigma}^s \) let us construct a kind of system of coordinates in the following way: since \( \tilde{S}_1 \) and \( \tilde{S}_- \) are transversal, consider curves on \( \tilde{\Sigma}^{c+} \cup \tilde{\Sigma}^s \) that are parallel to \( \tilde{S}_1 \) and \( \tilde{S}_- \). So, a point \( \tilde{p} \in \tilde{\Sigma}^{c+} \) belongs to one, and just one, intersection of these curves. If the curve parallel to \( \tilde{S}_1 \) is placed at a distance \( x \) of \( \tilde{S}_1 \) and if the curve parallel to \( \tilde{S}_- \) is placed at a distance \( y \) of \( \tilde{S}_- \) we say that the coordinate of \( \tilde{p} \) is \( "(x, y)" \). So, we identify an arbitrary point \( p = (x, y) \in \Sigma^{c+} \) with the point \( \tilde{p} = "(x, y)" \) and \( h(\Sigma^{c+} \cup \Sigma^s) = \tilde{\Sigma}^{c+} \cup \tilde{\Sigma}^s \).

Let \( r_1 \in \Sigma^+ \), such that there exists a smallest time \( |t_2| \) such that \( \Phi_{Z_+}(t_2, r_1) = s_2 \in \Sigma^{c+} \cup \Sigma^s \). We have that \( h(s_2) = \tilde{s}_2 \in \tilde{\Sigma}^{c+} \cup \tilde{\Sigma}^s \). So, we identify the arcs \( A(s_2) \) and \( A(\tilde{s}_2) \) by length arc.
So, there exists an homeomorphism $h : U \to \tilde{U}$ taking trajectories of $Z$ in trajectories of $\tilde{Z}$ and preserving the kind of tangencies and the position of the distinguished regions on the switching manifolds. As a consequence, Theorem 2 is proved.

5. Proof of Theorem 3 – The bifurcation scenario

Consider $Z_\lambda^\pm$ defined in Equation 3 whose Lie derivatives are given by:

\begin{align*}
Z_\lambda^+ f(x, y, z) &= \text{Sgn}(\theta)y - \lambda, \\
Z_\lambda^- f(x, y, z) &= \text{Sgn}(\mu \theta).
\end{align*}

Note that $Z_\lambda^\pm f(x, y, z) = 0$ when $y = \text{Sgn}(\theta)\lambda$. This shows that the set $S_{Z_\lambda^\pm}$ can be seen as a translation of the set $S_-$ (see Figure 7).

Also, since $Z_\lambda^- f = \mu \theta = Z_\lambda^+ f$ we still have that $S_{Z_\lambda^-}$ is composed by invisible (respectively visible) fold points of $Z_\lambda^+$ when $\mu \theta > 0$ (respectively $\mu \theta < 0$).
As we saw in section 3, the positive part of the $y$–axis is filled by visible (respectively invisible) fold points of $Z_+$ if $\alpha \gamma > 0$ (respectively $\alpha \gamma < 0$) while the negative part is filled by the invisible (respectively visible) fold points of $Z_+$.

Therefore, the point $p = (0, Sgn(\theta) \lambda, 0)$ is a fold-fold singularity for $Z$ which assumes the following possibilities:

- visible - visible: if $\alpha \gamma < 0, \mu \theta < 0, \lambda \theta < 0$ or $\alpha \gamma > 0, \mu \theta < 0, \lambda \theta > 0$,
- visible - invisible: if $\alpha \gamma < 0, \mu \theta > 0, \lambda \theta < 0$ or $\alpha \gamma > 0, \mu \theta > 0, \lambda \theta > 0$,
- invisible - visible: if $\alpha \gamma > 0, \mu \theta < 0, \lambda \theta < 0$ or $\alpha \gamma < 0, \mu \theta < 0, \lambda \theta > 0$,
- invisible - invisible: if $\alpha \gamma > 0, \mu \theta > 0, \lambda \theta < 0$ or $\alpha \gamma < 0, \mu \theta < 0, \lambda \theta > 0$.

The distribution of regions $\Sigma^{c\pm}, \Sigma^e$ and $\Sigma^s$ are similar to those shown in Figure 3 whose borders are delimited by the sets $S_{Z_-}$ and $S_+ (y – axis)$ as the example illustrated in Figure 8.

The equivalence between the unfoldings follow directly from the proof of Theorem 2. In fact, the difference is that it occurs a translation on $S_-$ but the approach used in the proof of Theorem 2 can be repeated with the same steps.

Figure 8. Distribution of regions $\Sigma^{c\pm}, \Sigma^e$ and $\Sigma^s$ considering $\gamma, \theta > 0$.

6. Concluding remarks

Along this paper we present 32 topologically distinct canonical forms of the 3D cusp-fold singularity for PSVFs and a 1-parameter unfolding of each of them. A very important final observation is that it proves that a general cusp-fold singularity has infinite codimension if we consider the notion of codimension given in [2, 8]. In fact, in [2, 8] it is proved that the 3D invisible-invisible fold-fold singularity has infinite codimension. Since in the bifurcation scenario of the cusp-fold singularities of types $(\pm, \pm, \pm, ++)$ and $(\pm, \pm, \pm, --)$ it appears an invisible-invisible fold-fold singularity at the unfolding, and in a general situation, this invisible-invisible fold-fold singularity can have infinite codimension, we conclude that this is also the codimension of
a general 3D cusp-fold singularity. As consequence, the dynamics of the cusp-fold singularity is still more sophisticated then the obtained at the so called invisible-invisible fold-fold singularity (or Teixeira-singularity, T-singularity).

**Data Availability**

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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