An Infinite-Dimensional Variational Inequality Formulation and Existence Result for Dynamic User Equilibrium with Elastic Demands

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Abstract

This paper is concerned with dynamic user equilibrium (DUE) with elastic travel demand (E-DUE). We present and prove a variational inequality (VI) formulation of E-DUE using measure-theoretic argument. Moreover, existence of the E-DUE is formally established with a version of Brouwer’s fixed point theorem in a properly defined Hilbert space. The existence proof requires the effective delay operator to be continuous, a regularity condition also needed to ensure the existence of DUE with fixed demand (Han et al., 2013c). Our proof does not invoke the \textit{a priori} upper bound of the departure rates (path flows).

**Keywords:** dynamic user equilibrium; elastic demand; variational inequalities; differential variational inequalities; existence

1 Introduction

This paper is concerned with an elastic demand extension of the fixed-demand dynamic traffic assignment model originally presented in Friesz et al. (1993) and discussed subsequently by Friesz and Mookherjee (2006); Friesz et al. (2011, 2013); Friesz and Meimand (2013). As such, it is concerned with a specific type of dynamic traffic assignment known as dynamic user equilibrium (DUE) for which travel cost, including delay as well as early and late arrival penalties, are equilibrated and demand is determined endogenously.

1.1 Introductory remarks on dynamic user equilibrium

In the past two decades there have been many efforts to develop a theoretically sound formulation of dynamic user equilibrium especially in continuous time. As is pointed out by Friesz et al. (2011), DUE models tend to be comprised of five essential components:

1. a model of path delay;

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2. flow dynamics;

3. flow propagation constraints;

4. a route and departure time choice model; and

5. a model of demand growth (evolution).

Dynamic user equilibrium models from the early 1990s forward have been largely concerned with the sub-models 1 through 4 above, concentrating on the within-day time scale for which travelers make routing and departure time decisions. Items 1 through 3 above form a sub-model known as the *dynamic network loading* (DNL), which determines arc- and/or path-specific volumes and flows as well as experienced path delays when departure rates (path flows) are known for each path. Item 4 aims at expressing mathematically the notion of Nash-like equilibrium conditions. Item 5, demand evolution, occurs on the day-to-day time scale and allows travel demands to be updated.

There are multiple means of expressing the Nash-like notion of a dynamic user equilibrium, including the following:

(i) an infinite-dimensional variational inequality \cite{Friesz1993, Smith1994, Smith1995}

(ii) a nonlinear complementarity problem \cite{Wie2002, Han2011}

(iii) a differential variational inequality \cite{Friesz2001, Friesz2013, Friesz2011, Friesz2006}

(iv) a differential complementarity system \cite{Pang2011}

The variational inequality representation is presently the primary mathematical form employed for dynamic user equilibrium. One of the advantages of such variational inequality formulation is that it subsumes almost all simultaneous route-and-departure time choice DUE models regardless of the dynamic network loading procedure employed. In fact, the arc dynamics and flow propagation constraints can be naturally embedded in the *effective delay operator* which is viewed as a mapping between two Hilbert spaces. However, it would be a mistake to think that somehow the VI formulation is an “easier” formulation since the effective delay operators are generally not knowable in closed form; in fact, the delay operators may be non-analytic and may need to be derived from an embedded delay model, data combined with response surface methodology, or data combined with inverse modeling. Analytical results of the effective delay operator is crucial for understanding qualitative properties of dynamic user equilibrium such as existence, uniqueness and convergence of certain computational schemes. The existence of DUE requires that the delay operator is continuous, which is a consequence of a generalization of Brouwer’s fixed point theorem \cite{Browder1968}. In addition, the uniqueness of a DUE solution further requires that the delay operator is strongly monotone \cite{Nagurney1993}. Friesz and Mookherjee \cite{Friesz2006} propose and test a fixed point algorithm implemented in continuous time to solve the differential variational inequality (DVI) formulation of DUE \cite{Friesz2001}; that algorithm requires monotonic effective delay operators to assure convergence.
1.2 Some history of DUE with elastic demand

In this section we review some of the few prior efforts to model DUE with elastic travel demand. This review is based in part on Friesz and Meimand (2013). Most of the studies of DUE reported in the DTA literature are about dynamic user equilibrium with constant travel demand for each origin-destination pair. We denote such a dynamic user equilibrium with fixed demand by F-DUE. It is, of course, not generally true that travel demand is fixed, even for short time horizons. Arnott et al. (1993) and Yang and Huang (1997) directly consider elastic travel demand; their work possesses a limited relationship to the analysis presented in this paper, for their work is concerned with a simple bottleneck instead of a nontrivial network, which is our focus.

Yang and Meng (1998) extend a simple bottleneck model to a general queuing network with known elastic demand functions for each origin-destination (OD) pair. Wie et al. (2002) study a version of dynamic user equilibrium with elastic demand, using a complementarity formulation that requires path delays to be expressible in closed form. Szeto and Lo (2004) study dynamic user equilibrium with elastic travel demand when network loading is based on the cell transmission model (CTM); their formulation is based on discrete time and is expressed as a finite-dimensional variational inequality (VI). Han et al. (2011) study dynamic user equilibrium with elastic travel demand for a network whose traffic flows are also described by CTM.

Although Friesz et al. (2011) show that analysis and computation of dynamic user equilibrium with constant travel demand is tremendously simplified by stating it as a differential variational inequality (DVI), they do not discuss how elastic demand may be accommodated within a DVI framework. Friesz and Meimand (2013) extend the DVI formulation to an elastic demand setting. Such a DVI formulation of elastic demand DUE is not straightforward. In particular, the DVI presented therein has both infinite-dimensional and finite-dimensional terms; moreover, for any given origin-destination pair, inverse travel demand corresponding to a dynamic user equilibrium depends on the terminal value of a state variable representing cumulative departures. The DVI formulation achieved in that paper is significant because it allows the still emerging theory of differential variational inequalities to be employed for the analysis and computation of solutions of the elastic-demand DUE problem when simultaneous departure time and route choice are within the purview of users, all of which constitutes a foundation problem within the field of dynamic traffic assignment.

A good review of recent insights into abstract differential variational inequality theory, including computational methods for solving such problems, is provided by Pang and Stewart (2008). Also, differential variational inequalities involving the kind of explicit, agent-specific control variables employed herein are presented in Friesz (2010).

1.3 Main results

This paper further advances the knowledge of continuous-time E-DUE based on Friesz and Meimand (2013) in terms of formulation and qualitative properties. In particular, we state and prove an equivalent variational inequality (VI) formulation of the E-DUE problem. The proof uses a measure-theoretic argument, and does not invoke the optimal control framework which is the primary methodology employed by Friesz and Meimand (2013) to establish the DVI formulation. Such a result is presented in Theorem 3.1.

As an immediate application of the proposed variational inequality formulation, we will analyze and establish existence result for the E-DUE problem. As commented in Han et al. (2013c), the most obvious approach to establishing existence is to convert the problem to an
equivalent variational inequality problem and then apply a version of Brouwer’s fixed point existence theorem. Nearly all proofs of DUE existence employ Brouwer’s fixed point theorem, either implicitly or explicitly. One statement of Brouwer’s theorem appears as Theorem 2 of Browder (1968). Approaches based on Brouwer’s theorem require the set of feasible path flows under consideration to be compact and convex in a topological vector space, and typically involve the \textit{a priori} bound on path flows. For instance, using the link delay model introduced by Friesz et al. (1993), Zhu and Marcotte (2000) show that a route choice (RC) dynamic user equilibrium exists under certain regularity conditions. In their modeling framework, the departure rate at each origin is given as \textit{a priori} and assumed to be bounded from above. Thus one is assured that all path flows are automatically uniformly bounded. In Wie et al. (2002), the existence of an arc-based user equilibrium is established under the assumption that the path flows are \textit{a priori} bounded.

Difficulties arise in the proof of a general existence theorem from two aspects: (i) in a continuous-time setting, the set of feasible path flows is not compact; and (ii) the assumption of \textit{a priori} boundedness of path flows, which is usually required for a topological argument, does not arise from any behavioral argument or theory. The existence proof provided by this paper manages to overcome these two major difficulties. Regarding item (i) above, we employ successive finite-dimensional approximations of the feasible path flows set, which allows Brouwer’s fixed point theorem to be applied. Regarding item (ii), we propose an in-depth analysis and computation of the path flows under minor assumptions on the travelers’ disutility functions.

Existence result for the elastic demand case is further complicated by the fact that the corresponding VI formulation has both infinite-dimensional and finite-dimensional terms (see Theorem 3.1 below). In order to apply Browder’s theorem (Browder, 1968), one needs to work in an extended Hilbert space that is a product of an infinite-dimensional space and a finite-dimensional space, and define appropriate inner product that allows compactness and weak topology to be properly defined. It is significant that our existence result for E-DUE, stated and proved in Theorem 5.6 does not rely on the \textit{a priori} upper bound of path flows and can be established for any dynamic network loading sub-model with reasonable and weak regularity conditions.

1.4 Organization

The rest of this paper is organized as follows. Section 2 recaps several key notations and concepts that are essential for the subsequent discussion. In Section 3 we present one of the main results of this paper: the variational inequality (VI) formulation of the simultaneous route-and-departure choice dynamic user equilibrium with elastic travel demand (E-DUE). Section 4 briefly reviews the differential variational inequality formulation proposed by Friesz and Meimand (2013). Section 5 establishes the existence result for E-DUE based on the VI formulation proposed in Section 3. The proof overcomes several difficulties known to other researchers, including non-compactness of the feasible set and the \textit{a priori} boundedness of path flows.

2 Notation and essential background

The time interval of analysis is a single commuting period or “day” expressed as

\[ [t_0, t_f] \subset \mathbb{R}^1 \]
where $t_f > t_0$, and both $t_0$ and $t_f$ are fixed. Here, as in all DUE modeling, the single most crucial ingredient is the path delay operator, which provides the delay on any path $p$ per unit of flow departing from the origin of that path; it is denoted by

$$D_p(t, h) \quad \forall p \in \mathcal{P}$$

where $\mathcal{P}$ is the set of all paths employed by travelers, $t$ denotes departure time, and $h$ is a vector of departure rates. From these, we construct effective unit path delay operators $\Psi_p(t, h)$ by adding the so-called schedule delay $f(t + D_p(t, h) - T_A)$; that is

$$\Psi_p(t, h) = D_p(t, h) + f(t + D_p(t, h) - T_A) \quad \forall p \in \mathcal{P} \quad (2.1)$$

where $T_A$ is the desired arrival time and $T_A < t_f$. The function $f(\cdot)$ assesses a penalty whenever

$$t + D_p(t, h) \neq T_A \quad (2.2)$$

since $t + D_p(t, h)$ is the clock time at which departing traffic arrives at the destination of path $p \in \mathcal{P}$. We stipulate that each $\Psi_p(\cdot, h) : [t_0, t_f] \rightarrow \mathbb{R}_+^1 \quad \forall p \in \mathcal{P}$

is measurable and strictly positive. We employ the obvious notation

$$\left(\Psi_p(\cdot, h) : p \in \mathcal{P}\right) \in \mathbb{R}^{||P||}$$

to express the complete vector of effective delay operators.

It is now well known that path delay operators may be obtained from an embedded delay model, data combined with response surface methodology, or data combined with inverse modeling. Unfortunately, regardless of how derived, realistic path delay operators do not possess the desirable property of monotonicity; they may also be non-differentiable.

### 2.1 Review on dynamic user equilibrium with fixed demand (F-DUE)

For the completeness of our presentation, in this section we recap the definition of DUE with fixed travel demand, originally articulated by [Friesz et al. (1993)](http://example.com).

Let us introduce the fixed trip matrix $(Q_{ij} : (i, j) \in \mathcal{W})$, where each $Q_{ij} \in \mathbb{R}_+^1$ is the fixed travel demand between origin-destination pair $(i, j) \in \mathcal{W}$, where $\mathcal{W}$ is the set of origin-destination pairs. Note that $Q_{ij}$ represents traffic volume, not flow. Finally we let $\mathcal{P}_{ij} \subset \mathcal{P}$ be the set of paths connecting origin-destination pair $(i, j) \in \mathcal{W}$. As mentioned earlier, $h$ is the vector of path flows $h = \{h_p : p \in \mathcal{P}\}$. We denote the space of square integrable functions on the real interval $[t_0, t_f]$ by $L^2[t_0, t_f]$. We stipulate that each path flow is square integrable, that is

$$h \in \left(L^2_+[t_0, t_f]\right)^{||P||}$$

where $\left(L^2_+[t_0, t_f]\right)^{||P||}$ is the positive cone of the $||P||$-fold product of the Hilbert space $L^2[t_0, t_f]$. Each element $h = (h_p : p \in \mathcal{P}) \in \left(L^2_+[t_0, t_f]\right)^{||P||}$ is interpreted as a vector of departure-time densities, or more simply path flows, measured at the entrance of the first arc of the relevant path. It will be seen that these departure time densities are defined only up to a set of measure zero. With this in mind, let $\nu$ denote a Lebesgue measure on $[t_0, t_f]$, and for each measurable set $S \subseteq [t_0, t_f]$, let $\forall_\nu(t \in S)$ denote the phrase for $\nu$-almost all $t \in S$. If $S = [t_0, t_f]$, then we may at times simply write $\forall_\nu(t)$. 

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We write the flow conservation constraints as

$$\sum_{p \in P} \int_{t_0}^{t_f} h_p(t) \, dt = Q_{ij} \quad \forall (i, j) \in W \quad (2.3)$$

where (2.3) consists of Lebesgue integrals. Using the notation and concepts we have mentioned, the feasible region for DUE when effective delay operators are known is

$$\Lambda_0 = \left\{ h \geq 0 : \sum_{p \in P} \int_{t_0}^{t_f} h_p(t) \, dt = Q_{ij} \quad \forall (i, j) \in W \right\} \subseteq \left( L^2_+ [t_0, t_f] \right)^{|P|} \quad (2.4)$$

In order to define an appropriate concept of minimum travel costs in the present context, we require the measure-theoretic analog of the infimum of a set of numbers. In particular, for any measurable set \( S \subseteq [t_0, t_f] \) with \( \nu(S) > 0 \), and any measurable function \( f : S \to \mathbb{R} \), the essential infimum of \( f(\cdot) \) on \( S \) is given by

$$\text{essinf}\{f(s) : s \in S\} = \sup\{x \in \mathbb{R} : \nu\{s \in S : f(s) < x\} = 0\} \quad (2.5)$$

Note that for each \( x > \text{essinf}\{f(s) : s \in S\} \) it must be true by definition that \( \nu\{s \in S : f(s) < x\} > 0 \).

Let us define the essential infimum of effective travel delays \( v_p = v_p(h) = \text{essinf}\{\Psi_p(t, h) : t \in [t_0, t_f]\} \geq 0 \quad \forall p \in P \)

\( v_{ij} = v_{ij}(h) = \min\{v_p(h) : p \in P_{ij}\} \quad \forall (i, j) \in W \)

The following definition of dynamic user equilibrium was first articulated by [Friesz et al. (1993)]:

**Definition 2.1. (Dynamic user equilibrium)** A vector of departure rates (path flows) \( h^* \in \Lambda_0 \) is a dynamic user equilibrium if

$$h^*_p(t) > 0, \ p \in P_{ij} \implies \Psi_p(t, h^*) = v_{ij}(h^*) \quad \forall \nu(t) \in [t_0, t_f], \ \forall (i, j) \in W \quad (2.6)$$

$$\Psi_p(t, h^*) \geq v_{ij}(h^*) \quad \forall \nu(t) \in [t_0, t_f], \ \forall (i, j) \in W \quad (2.7)$$

We denote this equilibrium by \( \text{DUE} (\Psi, \Lambda_0, [t_0, t_f]) \).

Using measure theoretic arguments, [Friesz et al. (1993)] establish that a dynamic user equilibrium is equivalent to the following variational inequality under suitable regularity conditions:

$$\text{find } h^* \in \Lambda_0 \text{ such that }$$

$$\sum_{p \in P} \int_{t_0}^{t_f} \Psi_p(t, h^*)(h_p - h^*_p) \, dt \geq 0 \quad \forall h \in \Lambda_0$$

$$VI(\Psi, \Lambda_0, [t_0, t_f]) \quad (2.8)$$

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3The demand satisfaction expressed via Lebesgue integrals is not enough to assure that the path flows are bounded. This has been the major hurdle to proving existence without the a priori invocation of bounds on path flows.
2.2 Dynamic user equilibrium with elastic demand (E-DUE)

The general setup of DUE with elastic demand is similar to that of the fixed demand case, with the exception that the total travel demand \( Q_{ij} \) between an origin-destination pair \((i, j) \in W\) is no longer a prescribed constant. Rather, transportation demand is assumed to be expressed as the following invertible function

\[
Q_{ij} = F_{ij}[v]
\]

for each origin-destination pair \((i, j) \in W\), where \( W \) is the set of all origin-destination pairs and \( v \) is a concatenation of origin-destination minimum travel cost \( v_{ij} \) associated with \((i, j) \in W\). That is, we have that

\[
v_{ij} \in \mathbb{R}^+ \quad \text{and} \quad v = (v_{ij} : (i, j) \in W) \in \mathbb{R}^{\vert W\vert}_+
\]

Note that to say \( v_{ij} \) is a minimum travel cost means it is the minimum cost for all departure time choices and all route choices pertinent to origin-destination pair \((i, j) \in W\). Further note that \( Q_{ij} \) is the unknown cumulative travel demand between \((i, j) \in W\) that must ultimately arrive by time \( t_f \).

We will also find it convenient to form the complete vector of travel demands by concatenating the origin-specific travel demands to obtain

\[
Q = (Q_{ij} : (i, j) \in W) \in \mathbb{R}^{\vert W\vert}_+
\]

The inverse demand function for every \((i, j) \in W\) is

\[
v_{ij} = \Theta_{ij}[Q]
\]

and we naturally define

\[
\Theta[Q] = (\Theta_{ij}[Q] : (i, j) \in W) \in \mathbb{R}^{\vert W\vert}_+
\]

As a consequence, we employ the following set of feasible departure flows when the travel demand between each origin-destination pair is unknown.

\[
\tilde{\Lambda} = \left\{ (h, Q) : h \geq 0, \sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} h_p(t) \, dt = Q_{ij}, \forall (i, j) \in W \right\} \subset (L^2[t_0, t_f])^{|\mathcal{P}|} \times \mathbb{R}^{\vert W\vert}_+
\]

where \((L^2[t_0, t_f])^{|\mathcal{P}|} \times \mathbb{R}^{\vert W\vert}_+\) is the direct product of the \(|\mathcal{P}|\)-fold product of Hilbert spaces consisting of square-integrable path flows, and the \(|W|\)-dimensional Euclidean space consisting of vectors of elastic travel demands.

With preceding preparation, we are in a place where the simultaneous route-and-departure-choice dynamic user equilibrium with elastic demand can be rigorously defined, as follows.

**Definition 2.2. (Dynamic user equilibrium with elastic demand)** A pair \((h^*, Q^*) \in \tilde{\Lambda}\), where \( h^* \) is a vector of departure rates (path flows) and \( Q^* \) is the associated vector of travel demands, is said to be a dynamic user equilibrium with elastic demand if for all \((i, j) \in W\),

\[
h^*_p(t) > 0, \quad p \in \mathcal{P}_{ij} \implies \Psi_p(t, h^*) = \Theta_{ij}[Q^*] \quad \forall \nu(t) \in [t_0, t_f] \tag{2.10}
\]

\[
\Psi_p(t, h^*) \geq \Theta_{ij}[Q^*] \quad \forall \nu(t) \in [t_0, t_f], \quad \forall p \in \mathcal{P}_{ij} \tag{2.11}
\]
3 The variational inequality formulation of E-DUE

Experience with differential games suggests that the DUE problem with elastic demand can be expressed as a variational inequality, as shown in the theorem below.

Theorem 3.1. (E-DUE equivalent to a variational inequality) Assume \( \Psi_p(\cdot, h) : [t_0, t_f] \rightarrow \mathbb{R}_{++} \) is measurable and strictly positive for all \( p \in \mathcal{P} \) and all \( h \) such that \( (h, Q) \in \tilde{\Lambda} \). Also assume that the elastic travel demand function is invertible with inverse \( \Theta_{ij}[\cdot] \) for all \((i, j) \in \mathcal{W} \). Then a pair, \((h^*, Q^*) \in \tilde{\Lambda} \), is a DUE with elastic demand as in Definition 2.2 if and only it solves the following variational inequality:

\[
\begin{align*}
\text{find } (h^*, Q^*) \in \tilde{\Lambda} \text{ such that } & \\
\sum_{p \in \mathcal{P}} \int_{t_0}^{t_f} \Psi_p(t, h^*)(h_p - h^*_p)dt - \sum_{(i, j) \in \mathcal{W}} \Theta_{ij}[Q^*] [Q_{ij} - Q^*_{ij}] & \geq 0 \\
\forall (h, Q) \in \Lambda 
\end{align*}
\]

(3.12)

Proof. \((i)[\text{Necessity}]\) Given a DUE solution with elastic demand \((h^*, Q^*) \in \tilde{\Lambda} \), we easily deduce from (2.10) and (2.11) that for any \((h, Q) \in \tilde{\Lambda} \),

\[
\sum_{p \in \mathcal{P}} \int_{t_0}^{t_f} \Psi_p(t, h^*)(h_p - h^*_p)dt - \sum_{(i, j) \in \mathcal{W}} \Theta_{ij}[Q^*] [Q_{ij} - Q^*_{ij}]
\]

\[
= \sum_{(i, j) \in \mathcal{W}} \left( \sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} \Psi_p(t, h^*)h_p(t)dt - \Theta_{ij}[Q^*] \cdot Q_{ij} \right)
\]

\[
- \sum_{(i, j) \in \mathcal{W}} \left( \sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} \Psi_p(t, h^*)h^*_p(t)dt - \Theta_{ij}[Q^*] \cdot Q^*_{ij} \right)
\]

\[
= \sum_{(i, j) \in \mathcal{W}} \left( \sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} \Psi_p(t, h^*)h_p(t)dt - \Theta_{ij}[Q^*] \cdot Q_{ij} \right)
\]

\[
- \sum_{(i, j) \in \mathcal{W}} \left( \sum_{p \in \mathcal{P}_{ij}} \Theta_{ij}[Q^*] \cdot \int_{t_0}^{t_f} h^*_p(t)dt - \Theta_{ij}[Q^*] \cdot Q^*_{ij} \right)
\]

\[
= \sum_{(i, j) \in \mathcal{W}} \left( \sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} \Psi_p(t, h^*)h_p(t)dt - \Theta_{ij}[Q^*] \cdot Q_{ij} \right)
\]

\[
- \sum_{(i, j) \in \mathcal{W}} \Theta_{ij}[Q^*] \cdot \left( \sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} h^*_p(t)dt - Q^*_{ij} \right)
\]

\[
= \sum_{(i, j) \in \mathcal{W}} \left( \sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} \Psi_p(t, h^*)h_p(t)dt - \Theta_{ij}[Q^*] \cdot Q_{ij} \right) \quad (3.13)
\]
Observe that in (3.13), given any \((i, j) \in \mathcal{W}\),
\[
\sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} \Psi_p(t, h^*) h_p(t) dt - \Theta_{ij}[Q^*] \cdot Q_{ij}
\geq v_{ij}^* \sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} h_p(t) dt - \Theta_{ij}[Q^*] \cdot Q_{ij} = v_{ij}^* \left( \sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} h_p(t) dt - Q_{ij} \right) = 0 \quad (3.14)
\]
where \(v_{ij}^*\) is the essential infimum of \(\Psi_p(\cdot, h^*)\) on \([t_0, t_f]\) for all \(p \in \mathcal{P}_{ij}\), and is equal to \(\Theta_{ij}[Q^*]\) according to (2.10).

As an immediate consequence of (3.13) and (3.14), the following inequality holds for all \((h, Q) \in \tilde{\Lambda}\).
\[
\sum_{p \in \mathcal{P}} \int_{t_0}^{t_f} \Psi_p(t, h^*) (h_p - h_p^*) dt - \sum_{(i, j) \in \mathcal{W}} \Theta_{ij} [Q^*] (Q_{ij} - Q_{ij}^*) \geq 0 \quad (3.15)
\]

(ii)[Sufficiency] Assume that (3.15) holds for any \((h, Q) \in \tilde{\Lambda}\), then \(h^*\) is a solution to the DUE problem with fixed demand given by \(Q^*\), since in this case the second term in the left hand side of (3.15) vanishes and we recover the well-known VI for the fixed demand case, see (2.8). Therefore by Definition 2.1 for any \((i, j) \in \mathcal{W}\),
\[
\begin{cases}
  h_p^*(t) > 0, p \in \mathcal{P}_{ij} \implies \Psi_p[t, h^*(t)] = v_{ij}^* \\
  \Psi_p(t, h^*) > v_{ij}^* \implies h_p^*(t) = 0
\end{cases} \quad \forall \nu(t) \in [t_0, t_f]
\]
In order to show that \((h^*, Q^*)\) is a DUE with elastic demand using definition (2.10) and (2.11), it suffices to establish that \(v_{ij}^* = \Theta_{ij}[Q^*]\), \(\forall (i, j) \in \mathcal{W}\). We proceed as follows.

Fix arbitrary \((k, l) \in \mathcal{W}\) such that \(Q_{kl}^* \neq 0\). We define the pair \((\hat{h}, \hat{Q}) \in \tilde{\Lambda}\) as
\[
\hat{h}_p(t) = \begin{cases}
  a h_p^*(t) & p \in \mathcal{P}_{kl} \\
  h_p^*(t) & p \in \mathcal{P} \setminus \mathcal{P}_{kl}
\end{cases} \quad \forall t \in [t_0, t_f]
\]
\[
\hat{Q}_{ij} = \begin{cases}
  a Q_{kl}^* & (i, j) = (k, l) \\
  Q_{ij}^* & (i, j) \in \mathcal{W} \setminus (k, l)
\end{cases}
\]
where \(a \in \mathbb{R}_{++}\) is an arbitrary positive parameter. Substituting \((h, Q)\) for \((\hat{h}, \hat{Q})\), the left hand side of (3.15) becomes
\[
\begin{align*}
\sum_{p \in \mathcal{P}} \int_{t_0}^{t_f} \Psi_p(t, h^*) (\hat{h}_p - h_p^*) dt & - \sum_{(i, j) \in \mathcal{W}} \Theta_{ij} [Q^*] \left( \hat{Q}_{ij} - Q_{ij}^* \right) \\
= \sum_{p \in \mathcal{P}_{kl}} \int_{t_0}^{t_f} \Psi_p(t, h^*) (a h_p^* - h_p^*) dt & + \sum_{p \in \mathcal{P} \setminus \mathcal{P}_{kl}} \int_{t_0}^{t_f} \Psi_p(t, h^*) (h_p^* - h_p^*) dt \\
- \Theta_{kl}[Q^*] (a Q_{kl}^* - Q_{kl}^*) & - \sum_{(i, j) \in \mathcal{W} \setminus (k, l)} \Theta_{ij}[Q^*] (Q_{ij}^* - Q_{ij}^*) \\
= (a - 1) \sum_{p \in \mathcal{P}_{kl}} \int_{t_0}^{t_f} \Psi_p(t, h^*) h_p^*(t) dt & - (a - 1) \Theta_{kl}[Q^*] Q_{kl}^* \\
= (a - 1) v_{kl}^* Q_{kl}^* & - (a - 1) \Theta_{kl}[Q^*] Q_{kl}^*
\end{align*}
\]
\footnote{For any origin-destination pair \((i, j)\) such that \(Q_{ij}^* = 0\), the logical statements (2.10) and (2.11) automatically hold, since in this case \(h_p^*(\cdot)\) vanishes almost everywhere for all \(p \in \mathcal{P}_{ij}\).}
We conclude from (3.15) that
\[(a - 1) (v^k_l - \Theta kl [Q^*]) Q^k_l \geq 0\]
Since \(a \in \mathbb{R}^{++}\) is arbitrary, there must hold \(v^k_l = \Theta kl [Q^*]\), for any \((k, l) \in \mathcal{W}\). The proof is complete.

4 The differential variational inequality formulation of E-DUE

The notion of dynamic user equilibrium can be alternatively illustrated and analyzed using the mathematical paradigm of differential variational inequality (DVI) \cite{Friesz et al., 2001, 2011}. In particular, the demand satisfaction can be easily rewritten as a two-point boundary value problem
\[
\frac{dy_{ij}}{dt} = \sum_{p \in P_{ij}} h_p(t), \quad y_{ij}(t_0) = 0, \quad y_{ij}(t_f) = Q_{ij} \quad \forall (i, j) \in \mathcal{W} \tag{4.16}
\]
where \(y_{ij}(\cdot)\) is interpreted as the cumulative departure curve. In the context of the elastic demand case, we assume there are unknown terminal states \(y_{ij}(t_f)\), for all \((i, j) \in \mathcal{W}\), which are the realized DUE travel demands. Moreover, for each origin-destination pair \((i, j) \in \mathcal{W}\), inverse travel demand is expressed as
\[
v_{ij} = \Theta_{ij} [y(t_f)] \tag{4.17}
\]

The differential variational inequality formulation of E-DUE is first stated and proved by Friesz and Meimand (2013) using optimal control theory.

**Theorem 4.1. (E-DUE equivalent to a differential variational inequality)** Assume \(\Psi_p(\cdot, h) : [t_0, t_f] \rightarrow \mathbb{R}^{++}_{\times}\) is measurable and strictly positive for all \(p \in \mathcal{P}\) and all \(h \in \Lambda_1\).
Also assume that the elastic travel demand function is invertible, with inverse \( \Theta_{ij} (Q) \) for all \((i, j) \in W\). A vector of departure rates (path flows) \( h^* \in \Lambda_1 \) is a dynamic user equilibrium with associated demand \( Q^* (t_f) \) if and only if \( h^* \) solves the following DVI(\( \Psi, \Theta, t_0, t_f \)).

\[
\sum_{p \in \mathcal{P}} \int_{t_0}^{t_f} \Psi_p(t, h^*)(h_p - h^{*}_p)dt - \sum_{(i, j) \in W} \Theta_{ij} [y^*(t_f)] \left[ y_{ij}(t_f) - y^*_{ij} (t_f) \right] \geq 0 \quad \forall h \in \Lambda_1
\]

\[
DV\overline{I}(\Psi, \Theta, t_0, t_f)
\]

(4.20)

Proof. See Friesz and Meimand (2013).

\[ \square \]

5 Existence of dynamic user equilibrium with elastic demand

In this section, we will establish existence result for \( VI(\Psi, \Theta, t_0, t_f) \), an equivalent formulation of E-DUE. Our proposed approach is meant to incorporate the most general dynamic network loading sub-model with minimum regularity requirements, and to yield existence of E-DUE without invoking the a priori upper bound on path flows. In subsequent analysis, we will rewrite \( VI(\Psi, \Theta, t_0, t_f) \) as a variational inequality problem in an extended Hilbert space and then employ a version of the Brouwer’s Fixed Point Theorem (Browder, 1968). To this end, we introduce the product space \( E \doteq (L^2[t_0, t_f])^{\mathcal{P}} \times \mathbb{R}^{\mathcal{W}} \), which is a Hilbert space with the induced inner product defined as follows

\[
\langle X, Y \rangle_E \doteq \sum_{i=1}^{\mathcal{P}} \int_{t_0}^{t_f} \xi_i(t) \cdot \eta_i(t) dt + \sum_{j=1}^{\mathcal{W}} u_j v_j \quad (5.21)
\]

\[
X = (\xi_1(\cdot), \ldots, \xi_{|\mathcal{P}|}(\cdot), u_1, \ldots, u_{|\mathcal{W}|}) \in E \quad (5.22)
\]

\[
Y = (\eta_1(\cdot), \ldots, \eta_{|\mathcal{P}|}(\cdot), v_1, \ldots, v_{|\mathcal{W}|}) \in E \quad (5.23)
\]

Let us recall the VI formulation presented in Section 3, the set \( \bar{\Lambda} \) of admissible pair \((h, Q)\) can now be embedded in the extended space \( E \):

\[
\bar{\Lambda} = \left\{ (h, Q) \in (L^2_+[t_0, t_f])^{\mathcal{P}} \times \mathbb{R}^{\mathcal{W}} : \sum_{p \in \mathcal{P}, i} \int_{t_0}^{t_f} h_p(t) dt = Q_{ij} \quad \forall (i, j) \in \mathcal{W} \right\} \subset E
\]

In view of the inverse demand function \( \Theta = (\Theta_{ij} : (i, j) \in \mathcal{W}) \), we introduce notation

\[
\Theta^- \doteq (-\Theta_{ij} : (i, j) \in \mathcal{W}) : \mathbb{R}^{\mathcal{W}}_+ \rightarrow \mathbb{R}^{\mathcal{W}}_-
\]

We next define the mapping

\[
\mathcal{F} : \bar{\Lambda} \rightarrow E, \quad (h, Q) \mapsto (\Psi(\cdot, h), \Theta^- (Q)) \quad (5.24)
\]

where \((h, Q) \in \bar{\Lambda}, \Psi(\cdot, h) \in (L^2_+[t_0, t_f])^{\mathcal{W}}, \Theta^- (Q) \in \mathbb{R}^{\mathcal{W}}_+ \). Such a mapping is clearly well-defined. With the preceding discussion, the VI formulation of the DUE problem with elastic demand is readily rewritten as the following infinite-dimensional variational inequality in the extended Hilbert space.
find $X^* \in \tilde{\Lambda}$ such that

$$
\langle F(X^*), X - X^* \rangle_E \geq 0 \\
\forall X \in \tilde{\Lambda}
$$

where $X = (h, Q)$ and $X^* = (h^*, Q^*)$. Problem (5.25) is written in the generic form of variational inequality, which allows analysis regarding solution existence to be carried out in a framework provided by Browder (1968).

### 5.1 Existence result for $VI(F, \Theta, \tilde{\Lambda}, t_0, t_f)$

Our qualitative analysis regarding solution existence for the variational inequality (5.25) is based on the following extension of Brouwer’s fixed point theorem to topological vector spaces.

**Theorem 5.1.** (Browder, 1968) Let $K$ be a compact convex subset of the locally convex topological vector space $V$, $T$ a continuous (single-valued) mapping of $K$ into $V^*$, where $V^*$ is the dual space of $V$. Then there exists $u_0$ in $K$ such that

$$
\langle T(u_0), u_0 - u \rangle \geq 0 \\
\text{for all } u \in K.
$$

**Proof.** See Browder (1968).

In preparation for our existence proof, we recap several key results from functional analysis that facilitate our presentation. In particular, we note the following facts. The reader is referred to Royden and Fitzpatrick (1988) for more detailed discussion on these subjects.

**Proposition 5.2.** The space of square-integrable real-valued functions on a compact interval $[t_0, t_f]$, denoted by $L^2[t_0, t_f]$, is a locally convex topological vector space. In addition, the $|P|$-fold product of these spaces, denoted by $(L^2[t_0, t_f])^{|P|}$, is also a locally convex topological vector space.

**Proposition 5.3.** The dual space of $(L^2[t_0, t_f])^{|P|}$ has a natural isomorphism with $(L^2[t_0, t_f])^{|P|}$. The dual space of the Euclidean space $\mathbb{R}^{|W|}$ consisting of columns of $|W|$ real numbers is interpreted as the space consisting of rows of $|W|$ real numbers. As a consequence, the dual space of $(L^2[t_0, t_f])^{|P|} \times \mathbb{R}^{|W|}$ is again $(L^2[t_0, t_f])^{|P|} \times \mathbb{R}^{|W|}$.

**Proposition 5.4.** In a metric space (therefore topological vector space), the notion of compactness is equivalent to the notion of sequential compactness, that is, every infinite sequence has a convergent subsequence.

Theorem 5.1 is immediately applicable for showing that $VI(F, \Theta, \tilde{\Lambda}, t_0, t_f)$ has a solution if (1) $F$ is continuous; and (2) $\tilde{\Lambda} \subset E$ is compact. Unfortunately, such compactness does not generally hold for the problem we study herein. To overcome such an obstacle, we proceed in a similar way as in Han et al. (2013) by considering finite-dimensional approximations of the underlying infinite-dimensional Hilbert space. Another major hurdle that stymied many researchers is the a priori upper bound of path flows. Such bound is important for a topological argument that we will rely on in the proof, but does not arise from any physical
or behavioral perspective of traffic modeling. In fact, as observed by Bressan and Han (2011), the equilibrium path flows could very well become unbounded or even produce dirac-delta, if no additional assumptions are made regarding exogenous parameters of the Nash-like game, such as travelers’ disutility functions.

The following assumptions are key to our analysis of path flows. The first assumption, (A1), poses hypothesis on drivers’ perceived arrival costs; the second assumption, (A2), is concerned with the model of link/path dynamics and can be easily satisfied by existing models such as the Vickrey model (Vickrey, 1969; Han et al., 2013a,b), the LWR-Lax model (Friesz et al., 2013), and the Lighthill-Whitham-Richards model (Daganzo, 1994; Lighthill and Whitham, 1955; Richards, 1956).

(A1). The function \( f(\cdot) \) appearing in (2.1) is continuous on \([t_0, t_f]\) and satisfies

\[
 f(t_2) - f(t_1) \geq \Delta(t_2 - t_1) \quad \forall t_0 \leq t_1 < t_2 \leq t_f
\]

for some \( \Delta > -1 \)

(A2). The first-in-first-out (FIFO) rule is obeyed on a path level. In addition, each link \( a \in A \) in the network has a finite exit flow capacity \( M_a < \infty \).

Remark 5.5. Assumption (A1) is employed in an in-depth analysis of the network flow later in the proof of Theorem 5.6. The reader is referred to Han et al. (2013c) for the motivation and generality of such an assumption. Notice that if \( f(\cdot) \) is continuously differentiable, then (A1) is equivalent to requiring that \( f'(t) > -1, t \in [t_0, t_f] \).

With the preceding preparation, we are now ready to state and prove the main result of this section.

Theorem 5.6. Assume that the effective delay operator \( \Psi : \Lambda_0 \rightarrow (L^2[t_0, t_f])^{|P|} \) is continuous. In addition, let assumptions (A1) and (A2) hold. If the inverse demand function \( \Theta : \mathbb{R}_{++}^{\mid W\mid} \rightarrow \mathbb{R}_{++}^{\mid W\mid} \) is continuous, then the variational inequality \( VI(F, \Theta, \tilde{\Lambda}, t_0, t_f) \) has a solution.

Proof. Given that \( \Psi \) and \( \Theta \) are both continuous mappings, it is straightforward to verify by definition (5.24) that the mapping \( F : \tilde{\Lambda} \rightarrow E \) is also continuous.

Since Theorem 5.1 cannot be directly applied to obtain a solution of the infinite-dimensional VI, let us instead employ the technique from Han et al. (2013d) by considering finite-dimensional approximations of \( \tilde{\Lambda} \). More specifically, consider for each \( n \geq 1 \) the uniform partition of \([t_0, t_f]\) by \( n \) sub-intervals \( I^1, \ldots, I^n \). Define the finite-dimensional subset of \( \tilde{\Lambda} \):

\[
\tilde{\Lambda}_n = \left\{ (h_1(\cdot), \ldots, h_{|P|}(\cdot), Q_1, \ldots, Q_{|W|}) \in \tilde{\Lambda} : h_i(\cdot) \text{ is constant on each } I^j \quad \forall 1 \leq j \leq n, \quad \forall 1 \leq i \leq |P| \right\}
\]

Moreover, it is not restrictive to assume that there is an upper bound on the elastic demand for each origin-destination pair. That is, there exists a vector \( U = (U_{ij}) \in \mathbb{R}_{++}^{\mid W\mid} \) such that

\[
0 \leq Q_{ij} \leq U_{ij} \quad \forall (i, j) \in W, \quad \forall (h, Q) \in \tilde{\Lambda}
\]
We can then show that $\tilde{\Lambda}_n$ defined as such is convex and compact in $\tilde{\Lambda}$ for each $n \geq 1$. The proof is postponed until Lemma 7.1 of the Appendix for the clarity and concision of our presentation.

We are now in a position where Theorem 5.1 applies to each $\tilde{\Lambda}_n$. In other words, fix $n \geq 1$, there exists some $X^{n,*} = (h^{n,*}, Q^{n,*}) \in \tilde{\Lambda}_n$ such that

$$\langle F(X^{n,*}), X^n - X^{n,*} \rangle_E \geq 0 \quad \forall X^n \in \tilde{\Lambda}_n$$

(5.28)

As a consequence of (5.28), we have

$$\sum_{p \in P} \int_{t_0}^{t_f} \Psi_p(t, h^{n,*}) h^{n,*}(t) dt - \sum_{(i, j) \in W} \Theta_{ij}[Q^{n,*}]Q^{n,*}_{ij}$$

$$\leq \sum_{p \in P} \int_{t_0}^{t_f} \Psi_p(t, h^{n,*}) h^n(t) dt - \sum_{(i, j) \in W} \Theta_{ij}[Q^{n,*}]Q^n_{ij}$$

(5.29)

for all $(h^n, Q^n) \in \tilde{\Lambda}_n$. In particular, for all $h^n \in (L^2[t_0, t_f])^{|P|}$ such that

$$\sum_{p \in P} \int_{t_0}^{t_f} h^n_p(t) dt = Q^n_{ij} \quad \forall (i, j) \in W$$

inequality (5.29) becomes

$$\sum_{p \in P} \int_{t_0}^{t_f} \Psi_p(t, h^{n,*}) h^{n,*}(t) dt \leq \sum_{p \in P} \int_{t_0}^{t_f} \Psi_p(t, h^{n,*}) h^n(t) dt$$

(5.30)

Recall that $h^{n,*}(\cdot)$ is piecewise constant, thus (5.30) implies the following:

$$h^n_p(t) > 0, \quad t \in I^k \implies \int_{I^k} \Psi_p(t, h^{n,*}) dt = \min_{1 \leq j \leq n} \int_{I^j} \Psi_p(t, h^{n,*}) dt$$

(5.31)

for all $p \in P_{ij}$, $(i, j) \in W$. We invoke Lemma 7.2 from the Appendix and find a constant $M \in \mathbb{R}^{++}$ such that

$$h^n_p(t) \leq M \quad \forall t \in [t_0, t_f], \quad \forall p \in P, \quad \forall n \geq 1$$

With the uniform upper bound $M$ on the path flows $h^{n,*}$ and upper bound $U_{ij}$ for each $Q_{ij}$, $(i, j) \in W$, one can extract a subsequence $\{X^{n_k,*}\}$ out of $\{X^{n,*}\}$ such that $X^{n_k,*}$ converges weakly to some $X^* \in \tilde{\Lambda}$ as $k \to \infty$, where the weak topology on $E$ is defined via the inner product $\langle \cdot, \cdot \rangle_E$.

In view of (5.28), by strong continuity of $F$ and the weak convergence $X^{n,*} \to X^*$, we conclude that

$$\langle F(X^*), X - X^* \rangle_E \geq 0 \quad \forall X \in \tilde{\Lambda}$$

Remark 5.7. The existence of an E-DUE requires that the embedded network loading procedure yields a delay operator that is continuous. Such regularity condition coincides with that of the fixed demand case, see Han et al. (2013a). Theorem 5.6 subsumes all notions of simultaneous-route-and-departure choice user equilibrium regardless of the arc dynamic, flow propagation and delay model employed.
6 Conclusion

We have shown that dynamic network user equilibrium based on simultaneous departure time and route choice in the presence of elastic travel demand may be formulated as a variational inequality (VI). Such a result is a nontrivial extension of the VI formulation established by Friesz et al. (1993) for dynamic user equilibrium with fixed travel demand. The proof relies on a measure-theoretic argument and provides unique insights on the qualitative properties of E-DUE such as existence, which is also established in this paper.

Similar to the fixed demand case, existence of the elastic demand case is easily analyzed using the framework of Brouwer’s fixed point theorem for infinite-dimensional variational inequalities proposed in Browder (1968). Nevertheless, the variational inequality for the elastic demand case has both infinite- and finite-dimensional terms. The proof of existence requires construction of an extended space that subsumes both parts, as well as an appropriate choice of inner product that allows compactness and weak topology to be defined. It is significant that our proof does not rely on the \textit{a priori} upper bound on path flows.

We are conducting further investigation of the E-DUE problem in terms of computation, based on the DVI formulation. In particular, the optimal control framework inherent in the DVI formulation allows us to device an iterative scheme based on a fixed-point-problem reformulation. Convergence of such a scheme requires some sort of monotonicity to be articulated and proved, such as those mentioned in Friesz et al. (1993), which is the focus of ongoing research.

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7 Appendix

Lemma 7.1. Assume that there exists a vector \( U = (U_{ij}) \in \mathbb{R}^{\mid W \mid} \) such that

\[
0 \leq Q_{ij} \leq U_{ij} \quad \forall (i, j) \in W, \quad \forall (h, Q) \in \tilde{\Lambda}
\]

Then for each \( n \geq 1 \), the subset

\[
\tilde{\Lambda}_n \doteq \left\{ (h_1(\cdot), \ldots, h_{|P|}(\cdot), Q_1, \ldots, Q_{|W|}) \in \tilde{\Lambda} : h_i(\cdot) \text{ is constant on each } I^j, \quad \forall 1 \leq j \leq n, \quad \forall 1 \leq i \leq |P| \right\}
\]

is compact and convex in \( \tilde{\Lambda} \).

Proof. We begin with verifying convexity. Let \( \tilde{X} = (\tilde{h}, \tilde{Q}) \) and \( \tilde{X} = (\tilde{h}, \tilde{Q}) \) be any two elements of \( \tilde{\Lambda}_n \). Given any \( \alpha \in (0, 1) \), we have that

\[
\sum_{p \in P} \int_{t_0}^{t_f} \left( \alpha \tilde{h}_p(t) dt + (1 - \alpha) \tilde{Q}_p(t) \right) dt = \alpha \tilde{Q}_{ij} + (1 - \alpha) \tilde{Q}_{ij} \quad \forall (i, j) \in W
\]

Moreover, \( \alpha \tilde{h}_p(\cdot) + (1 - \alpha) \tilde{h}_p(\cdot) \) clearly remains constant on each sub-interval \( I^j, j = 1, \ldots, n \), for all \( p \in P \). We thus conclude that \( \alpha \tilde{X} + (1 - \alpha) \tilde{X} = \tilde{\Lambda}_n \).

Next, let us investigate compactness. From now on let us fix \( n \geq 1 \). In light of Proposition 5.4 it suffices to establish sequential compactness for \( \tilde{\Lambda}_n \). We consider an arbitrary infinite sequence \( \{X^k\}_{k \geq 1} \subset \tilde{\Lambda}_n \) where \( X^k = (h^k, Q^k) \). For each \( k \geq 1 \) and \( p \in P \), let \( \mu^k_{p,j} = (\mu^k_{p,j}) \in \mathbb{R}^n_+ \) be such that

\[
\mu^k_{p,j} = h^k_{p,j}(t) \quad t \in I^j, \quad \forall j = 1, \ldots, n
\]

We then define \( \mu^k \in \mathbb{R}^{|P|}_+ \) to be the concatenation of all vectors \( \mu^k_{p,j}, p \in P \). We also notice that the vectors \( \mu^k, k \geq 1 \) are uniformly bounded by the constant

\[
\max_{(i,j) \in W} U_{ij} \cdot \frac{n}{t_f - t_0}
\]
Thus by the Bolzano-Weierstrass theorem, there exists a convergent subsequence \( \{ \mu^{k'} \}_{k' \geq 1} \). It is immediately verifiable that the corresponding subsequence \( h^{k'} \) converges uniformly on \( [t_0, t_f] \) and also in the \( L^2 \) norm.

Moreover, by virtue of the uniform bounds \( U_{ij} \), \( (i, j) \in W \), there exists a further subsequence \( \{ k'' \} \subset \{ k' \} \) such that \( Q^{k''} \) is a convergent subsequence according to the Bolzano-Weierstrass theorem. Thus, the subsequence \( \{ X^{k''} \}_{k'' \geq 1} \) converges with respect to the norm induced by inner product \((7.33)\).

\[ \]

\textbf{Lemma 7.2.} Assume that \((A1)\) and \((A2)\) hold. Then there exists \( M > 0 \) such that for all \((h^{n,*}, Q^{n,*})\) satisfying \((5.28)\), there hold

\[ h^{n,*}_p(t) \leq M \quad \forall t \in [t_0, t_f], \quad \forall p \in P, \quad \forall n \geq 1 \quad (7.33) \]

\textbf{Proof.} In view of \((A2)\), we are prompted to define the following constant

\[ M^{\text{max}} := \max_{a \in A} M_a < + \infty \]

where \( A \) is the set of links of the network. Recalling the constant \( \Delta \) from \((A1)\), we choose constant \( M \) such that

\[ M > \frac{3M^{\text{max}}}{\Delta + 1} \]

We claim that \((7.33)\) holds for such \( M \). Otherwise, if \((7.33)\) fails, there must exist some \( m \geq 1, q \in P \) and \( 1 \leq j \leq m \) such that

\[ h^{m,*}_q(t) \equiv \lambda > M \quad t \in I^j \]

Without losing generality, we assume that \( j > 1 \) and consider the interval \( I^{j-1} \). By possibly modifying the value of the function \( \Psi_q(\cdot, h^{m,*}) \) at one point without changing the measure-theoretic nature of the problem, we obtain \( t^* \in I^{j-1} \) such that

\[ \Psi_q(t^*, h^{m,*}) = \sup_{t \in I^{j-1}} \Psi_q(t, h^{m,*}) \]

We denote by \( \tau_q(t, h^{m,*}) \) the time of arrival at destination of driver who departs at time \( t \) along path \( q \). According to the first-in-first-out (FIFO) principle, we deduce that \( \forall t \in I^j \),

\[ (t-t^j) \lambda \leq \int_{t^j}^{t} h^{m,*}_q(t) \, dt \leq M^{\text{max}} (\tau_q(t, h^{m,*}) - \tau_q(t^*, h^{m,*})) \]

where \( t^j \) is the left boundary of the interval \( I^j \). We then have the following estimation:

\[ \Psi_q(t, h^{m,*}) - \Psi_q(t^*, h^{m,*}) = D_q(t, h^{m,*}) + f(\tau_q(t, h^{m,*}) - T_A) - D_q(t^*, h^{m,*}) - f(\tau_q(t^*, h^{m,*}) - T_A) \]

\[ \geq \tau_q(t, h^{m,*}) - \tau_q(t^*, h^{m,*}) - (t - t^*) + \Delta (\tau_q(t, h^{m,*}) - \tau_q(t^*, h^{m,*})) \]

\[ = (\Delta + 1)(\tau_q(t, h^{m,*}) - \tau_q(t^*, h^{m,*})) - (t - t^*) \]

\[ \geq (\Delta + 1) \frac{\lambda}{M^{\text{max}}} (t - t^j) - (t - t^*) \quad \forall t \in I^j \quad (7.34) \]
Integrating (7.35) with respect to $t$ over interval $I^j$ shows the following:

$$\int_{I^j} \Psi_q(t, h_{m,*}) \, dt - (t^{j+1} - t^j) \Psi_q(t^*, h_{m,*}) \geq \frac{(t^{j+1} - t^j)^2}{2} \cdot \frac{(\Delta + 1)\lambda}{M_{\text{max}}} + (t^{j+1} - t^j) \cdot \left( t^* - \frac{t^j + t^{j+1}}{2} \right) \tag{7.36}$$

where $t^j$, $t^{j+1}$ are respectively the left and right boundary of $I^j$. Since $t^* \in I^{j-1}$, we have that

$$t^* - \frac{t^j + t^{j+1}}{2} \geq -\frac{3}{2}(t^{j+1} - t^j)$$

With this observation, (7.36) becomes

$$\int_{I^j} \Psi_q(t, h_{m,*}) \, dt - (t^{j+1} - t^j) \Psi_q(t^*, h_{m,*}) \geq \frac{(t^{j+1} - t^j)^2}{2} \cdot \frac{(\Delta + 1)\lambda}{M_{\text{max}}} - \frac{3}{2}(t^{j+1} - t^j)^2$$

$$= \frac{(t^{j+1} - t^j)^2}{2} \left( \frac{(\Delta + 1)\lambda}{M_{\text{max}}} - 3 \right) > 0$$

This implies

$$\int_{I^j} \Psi_q(t, h_{m,*}) \, dt > \int_{I^{j-1}} \Psi(t, h_{m,*}) \, dt$$

which yields contradiction to (5.31). This finishes the proof. \hfill \Box