TOPOLOGY OF T-VARIETIES

ANTONIO LAFACE, ALVARO LIENDO, AND JOAQUIN MORAGA

ABSTRACT. A T-variety is a variety X endowed with an effective action of an algebraic torus T. In [2,3], Altmann, Hausen and Süß gave a combinatorial description of T-varieties by means of divisorial fans. In this paper we study the topology of T-varieties in terms of this description. We compute the fundamental group of a T-variety with log-terminal singularities, the cohomology groups of an affine complexity-one T-variety and the cohomology ring and Chow ring of a shellable, complete, Q-factorial, complexity-one T-variety.

CONTENTS

Introduction 1
1. Combinatorial description of T-varieties 2
2. Fundamental group of T-varieties 4
3. The cohomology ring of a complexity one T-variety 10
4. Topology of complexity-one T-varieties 17
References 23

INTRODUCTION

In this paper, we study normal algebraic varieties endowed with an effective action of an algebraic torus $T = (\mathbb{C}^*)^n$, the so-called T-varieties. Given a T-variety $X$ we define its complexity as $\dim X - \dim T$. Such T-varieties admit a combinatorial description starting with the well-known case of complexity-zero T-varieties, i.e., toric varieties (see e.g. [8] and [13, Chapter 1]). Then, the complexity-one case was systematically studied in [13, Chapter 2 and 4], [17] and [9]. Finally, in [2,3] a combinatorial description is provided for arbitrary T-varieties.

The topology of toric varieties has been well studied, for instance in the books [7, Chapter 12] and [10, Chapter 5]. In particular, the fundamental group, the cohomology groups, the cohomology ring and the Chow ring of toric varieties are known in different degrees of generality. In this paper we generalize those results to the case of T-varieties. In the toric setting, these objects depends on the combinatorial and geometric structure of its defining fan, while in higher complexity...
they also depend on the topology of a normal semi-projective variety \( Y \) of dimension \( \text{dim} \ X - \text{dim} \ T \), which is a kind of quotient (chow quotient) of \( X \) by the torus action.

In the following, we describe the main results in this paper. Given a divisorial fan \( S \) on \( (Y, N_\mathbb{Q}) \) (see Section 1), in Section 2 we construct a finitely generated abelian group \( N(S) \), which is a quotient of \( N_\mathbb{Q} \), and a open subset of \( Y \) denoted by Loc(S). The fundamental group of \( X(S) \) is given by the following.

**Theorem 1.** Let \( X(S) \) be a T-Variety with log-terminal singularities. Then
\[
\pi_1(X(S)) \simeq N(S) \times \pi_1(\text{Loc}(S)).
\]

Our next main result concerns the rational Chow ring \( A^*(\bar{X}(S)_\mathbb{Q}) \) of the complexity-one \( T \)-variety \( \bar{X}(S) \), which appears in the construction of \( X(S) \) given in Section 1, for a divisorial fan \( S \) on \( Y = \mathbb{P}^1 \). The theorem holds for a class of complete and simplicial divisorial fans called shellable, introduced in Definition 3.9. This notion generalizes the usual shellable condition for fans. Given a divisorial fan \( S \) on \( (\mathbb{P}^1, N_\mathbb{Q}) \), in Section 3 we construct the polynomial ring \( \mathbb{Q}[D : D \text{ invariant divisor}] \) and the ideal \( I \) generated by all the linear relations between the invariant divisors, plus the monomials corresponding to subsets of invariants divisors with empty intersection (see Notation 3.10).

**Theorem 2.** Let \( S \) be a complete, simplicial and shellable divisorial fan on \( \mathbb{P}^1 \). Then we have an isomorphism
\[
\mathbb{Q}[D : D \text{ invariant divisor}] / I \rightarrow A^*(\bar{X}(S)_\mathbb{Q}), \quad D + I \mapsto [D].
\]

Finally, we study the canonical map from rational Chow groups to rational Borel-Moore homology groups of complexity-one \( T \)-varieties coming from divisorial fans satisfying the conditions of Definition 3.9 and prove that is an isomorphism.

**Theorem 3.** Let \( S \) be a complete, simplicial and shellable divisorial fan on \( \mathbb{P}^1 \). Then the canonical map from Chow groups to Borel-Moore homology
\[
A_k(\bar{X}(S)_\mathbb{Q}) \rightarrow H_{2k}(\bar{X}(S); \mathbb{Q}),
\]
is an isomorphism.

The paper is organized as follows: In Section 1 we introduce the combinatorial description of \( T \)-varieties and introduce some notation. In Section 2 we study the fundamental group of a \( T \)-variety with log-terminal singularities. In Section 3 we compute the Chow ring and cohomology ring, with rational coefficients, of complexity-one \( T \)-varieties under some conditions (see Definition 3.9). And finally Section 4 deals with the cohomology groups of affine complexity-one \( T \)-varieties, see also [1] for a computation of the rational intersection cohomology of rational contraction-free complexity-one \( T \)-varieties.

1. **Combinatorial description of \( T \)-varieties**

Let \( N \) be a finitely generated abelian group. We denote by \( N_\mathbb{Q} := N \otimes \mathbb{Z}_\mathbb{Q} \) the associated rational vector space and by \( M := \text{Hom}(N, \mathbb{Z}) \) the dual of \( N \). Let \( \sigma \) be a pointed cone in \( N_\mathbb{Q} \). Given a polyhedron \( \Delta \subseteq N_\mathbb{Q} \), its tail cone is \( \sigma(\Delta) = \{ v \in N_\mathbb{Q} : v + \Delta \subseteq \Delta \} \), we say that \( \Delta \) is a \( \sigma \)-polyhedron if \( \sigma(\Delta) = \sigma \). We denote by \( \text{Pol}_\mathbb{Q}(N, \sigma) \) the set of \( \sigma \)-polyhedra in \( N_\mathbb{Q} \), observe that this set endowed with the
Minkowski sum is a semigroup. Given a variety $Y$ we denote by $\text{CaDiv}_{\geq 0}(Y)$ the semigroup of effective Cartier divisors. A polyhedral divisor on $(Y, N)$ is a formal sum
\[ D := \sum_{D} \Delta_D \otimes D \in \text{Pol}_{\geq 0}(N, \sigma) \otimes_{\mathbb{Z}_{\geq 0}} \text{CaDiv}_{\geq 0}(Y), \]
where $\Delta_D$ are convex $\sigma$-polyhedra in $N$, we call $\sigma$ the tailcone of $D$. We will allow the coefficients to be the empty set $\emptyset \in \text{Pol}_{\geq 0}(N, \sigma)$ satisfying the condition $\emptyset + \Delta := \emptyset$. The open loci of $D$ is
\[ \text{Loc}(D) := Y \setminus \bigcup_{\Delta_D = \emptyset} D, \]
the support of $D$ is $\text{supp}(D) := \text{Loc}(D) \cap \bigcup_{\Delta_D \neq \sigma} D$ and the trivial locus of $D$ is $V_D := \text{Loc}(D) \setminus \text{supp}(D)$.

Given an open subset $V \subseteq Y$, we define the restriction of the polyhedral divisor $D$ to $V$ as the polyhedral divisor $D|_V$ on $(V, N)$ given by:
\[ D|_V := \begin{cases} \sum_{\Delta_D \cap V \neq \emptyset} \Delta_D \otimes D|_V & \text{if } V \subseteq V_D, \\ \sigma(D) \otimes D & \text{for some } D \in \text{CaDiv}_{\geq 0}(V), V \subseteq V_D. \end{cases} \]

**Definition 1.1.** An intersection set of $D$ is a non-empty intersection of divisors of $\text{supp}(D)$. The intersection set is called maximal intersection set if it does not properly contain an intersection set. Given a maximal intersection set $M$ the polyhedron $\sum_{M \subseteq D} \Delta_D$ is called a maximal polyhedron of $D$. Given $y \in Y$, we denote by $D_y$ the polyhedron $\sum_{y \in D} \Delta_D$.

Given a polyhedral divisor $D$ on $(Y, N)$ with tail cone $\sigma$, we have an evaluation map
\[ \sigma^v \to \text{Div}(Y), \quad u \mapsto D(u) := \sum_{v \in \Delta_D} \min_{v \in \Delta_D} (u, v) \cdot D. \]

**Definition 1.2.** If $D$ is a polyhedral divisor on $(Y, N)$ with tail cone $\sigma$ such that $D(u)$ is a semiample divisor for every $u \in \sigma^v$ and $D(u)$ is big for $u \in \text{relint}(\sigma^v)$, then we say that $D$ is a p-divisor.

We denote by $X(\Sigma)$ the toric variety associated to the fan $\Sigma$. Now we generalize this notation to affine $\mathbb{T}$-Varieties. Given a p-divisor $D$ on $(Y, N)$ with tail cone $\sigma$, we denote by $\tilde{X}(D)$ the relative spectrum of the coherent sheaf of algebras $\mathcal{A}(D) := \bigoplus_{u \in M \cap \sigma^v} \mathcal{D}(u)$ on $\text{Loc}(D)$ and recall that there is a good quotient
\[ \pi: \tilde{X}(D) \to \text{Loc}(D), \]
induced by the inclusion of sheaves $\mathcal{O}_Y \to \mathcal{A}(D)$. This gives rise to an affine $\mathbb{T}$-variety $X(D) := \text{Spec}(\text{Loc}(D), \mathcal{A}(D))$, with a proper birational morphism $\tilde{X}(D) \to X(D)$.

Given two p-divisors $D, D'$ on $(Y, N)$, we write $D' \subseteq D$ if $\Delta_{D'} \subseteq \Delta_D$ for each $D \in \text{CaDiv}_{\geq 0}(Y)$. Observe that if $D' \subseteq D$, then we have an induced map $X(D') \to X(D)$. We say that $D'$ is a face if such map is an open embedding. The intersection $D \cap D'$ of $D'$ and $D$ is the polyhedral divisor $\sum_{D}(\Delta_D \cap \Delta'_{D}) \otimes D$.

**Definition 1.3.** A divisorial fan $\mathcal{S}$ is a finite set of p-divisors on $(Y, N)$ such that the intersection of any two $\mathbb{T}$-divisors of $\mathcal{S}$ is a face of both and $\mathcal{S}$ is closed under taking intersections.
We denote by \( X(S) \) the \( T \)-Variety obtained by gluing the affine \( T \)-Varieties \( X(D^i) \) along the open subvarieties \( X(D^i \cap D^j) \) for each \( D^i, D^j \in S \). The set \( \{ \sigma(D) \mid D \in S \} \) form the so-called tailfan \( \Sigma(S) \) of \( S \). The locus of \( S \) is the open set

\[
\text{Loc}(S) := \bigcup_{D \in S} \text{Loc}(D),
\]

the support of \( S \) is \( \text{supp}(S) := \bigcup_{D \in S} \text{supp}(D) \) and the trivial locus of \( S \) is the open set \( V_S := \bigcap_{D \in S} V_D \). For any divisorial fan \( S \) and open subset \( V \subseteq Y \) we define the restriction of the divisorial fan \( S \) to \( V \) as \( S|_V := \{ D|_V \mid D \in S \} \).

### 2. Fundamental Group of \( T \)-Varieties

Given a \( p \)-divisor \( D \) on a variety \( Y \), we denote by \((N_D)_\mathbb{Q}\) the subspace of \( N_\mathbb{Q} \) generated by the subset

\[
\{ v_1 - v_2 \mid y \in Y, v_1, v_2 \in D_y \}.
\]

We denote by \( N_D := N \cap (N_D)_\mathbb{Q} \) the sublattice of \( N \) of the integer points of \((N_D)_\mathbb{Q}\) and by \( N(D) \) the quotient \( N/N_D \). Given a divisorial fan \( S \) on a variety \( Y \), we denote by \( N_S \) the sublattice of \( N \) generated by \( \{N_D \mid D \in S\} \) and by \( N(S) \) the following quotient

\[
N(S) := N/N_S.
\]

This section is devoted to prove Theorem 1.

**Remark 2.1.** The log-terminal assumption of Theorem 1 is essential. Indeed, let \( D \) be any \( p \)-divisor on a projective curve \( Y \) of positive genus, then by [14, Corollary 5.4] the \( T \)-variety \( X := X(D) \) is not log-terminal. Moreover, since \( Y = \text{Loc}(D) \) is projective, \( X \) has an attractive fixed point and thus \( \pi_1(X) \) is trivial, while \( \pi_1(\text{Loc}(D)) \) is non-trivial.

**Definition 2.2 (See [12]).** We say that a divisorial fan \( S \) on \((Y, N_\mathbb{Q})\) is contraction-free if for all \( D \in S \) the locus of \( D \) is affine.

Observe that \( S \) is contraction-free if for every \( D \in S \) the contraction morphism \( \bar{X}(D) \to X(D) \) is an isomorphism. Recall that for any divisorial fan \( S \) we have a rational quotient map \( \pi: X(S) \to \text{Loc}(S) \) and if \( S \) is contraction-free observe that this rational quotient map is indeed a morphism \( \pi: X(S) \to \text{Loc}(S) \). Given a contraction-free divisorial fan \( S \) on \((Y, N_\mathbb{Q})\) and an open subset \( V \subseteq Y \), we will adopt the following notation

\[
U := \pi^{-1}(V) \quad V_i := V \cap V_S \quad U_i := \pi^{-1}(V_i).
\]

In order to prove Theorem 1, we will prove Proposition 2.3 via relaxing the hypothesis successively in Lemmas 2.4, 2.5, 2.7 and 2.8. In what follows we will assume that \( Y \) is a smooth variety and that all the divisors of the following four lemmas have tail cone \( \sigma(D) = \sigma \). In order to abbreviate the notation we will assume that \( V \) is contained in \( \text{Loc}(D) \) in the following lemmas.

**Proposition 2.3.** Let \( S \) be a contraction-free divisorial fan on \((Y, N_\mathbb{Q})\) such that \( X(S) \) is smooth and \( V \subseteq Y \) an open subset. Then the trivialization \( t: U_i \simeq T_N \times V_i \)
induces the following commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & K \\
\downarrow & \cong & \downarrow \cong \\
0 & \rightarrow & K_S \times K_V \rightarrow N(\Sigma(S)) \times \pi_1(V_i) \rightarrow \alpha \times \beta N(S|_V) \times \pi_1(V) \rightarrow 0
\end{array}
\]

where \(\alpha: N(\Sigma(S)) \rightarrow N(S|_V)\) is induced by the inclusion \(N(S|_V) \times \pi_1(V)\) and \(\beta: \pi_1(V_i) \rightarrow \pi_1(V)\) is induced by the inclusion \(V_i \rightarrow V\).

Recall that given a \(p\)-divisor \(D\) on a variety \(Y\) we have a good quotient \(\pi: \bar{X}(D) \rightarrow Y\), that for any open subset \(V \subseteq Y\) we have an isomorphism \(U_i \cong X(\sigma(D)) \times V_i\) and given a point \(y \in V_i\) we have a commutative diagram

\[
\begin{array}{ccc}
T_N & \xrightarrow{\iota \rightarrow i(v)} & T_N \times V_i \\
\downarrow & & \downarrow \\
X(\sigma(D)) & \rightarrow & \bar{X} \rightarrow Y
\end{array}
\]

where the vertical arrows are open embeddings and \(X(\sigma(D)) \rightarrow \bar{X}\) is the inclusion of the fiber \(\pi^{-1}(y) = X(\sigma(D))\) over \(y\). Passing to the fundamental group and using [7, Theorem 12.1.10] we have the following commutative diagram

\[
\begin{array}{ccc}
1 & \rightarrow & N \\
\downarrow & \rightarrow & \downarrow \\
N(\sigma(D)) & \rightarrow & \pi_1(\bar{X}) \rightarrow \pi_1(Y) \rightarrow 1
\end{array}
\]

If \(\bar{X}\) and \(Y\) are smooth, by [15, Lemma 1.5.C] the rows are exact and by [7, Theorem 12.1.5] the vertical arrows are surjective.

**Lemma 2.4.** Let \(D\) be a \(p\)-divisor on \((Y, N_D)\) such that \(X\) is smooth. If there is only one maximal intersection set, and \(N(D)\) is trivial, then Proposition 2.3 holds for \(V = Y\) affine.

**Proof.** \(Y\) is affine so we have \(\bar{X} = X\). Since there is only one maximal intersecting set there exists a point \(y \in Y\) in the intersection of all divisors of \(\text{supp}(D)\). Moreover, \(N(\bar{D})\) is trivial, the polyhedron \(D_y\) is full dimensional in \(N_D\). By [2, Proposition 7.6] we have that the fiber \(\pi^{-1}(y)\) of the good quotient \(\pi: \bar{X} \rightarrow Y\), contains a fixed point \(p\), with respect to the torus action. Let \(p_0 \in \bar{X}\) be a point on an irreducible fiber \(\pi^{-1}(y_0)\) of \(\pi\). By [7, Theorem 12.1.5] the inclusion \(T_N \rightarrow \pi^{-1}(y_0)\) induces a surjective homomorphism \(\iota^*: \pi_1(T_N) \rightarrow \pi_1(\pi^{-1}(y_0))\), so that any loop \(\eta\) in the fiber is homotopically equivalent to a loop in \(T_N\). In particular, we can assume without loss of generality that \(\eta(t) = \alpha(t) \cdot q\), where \(\alpha\) is a loop in \(T_N\) with base point the identity. Let \(\gamma\) be a path from \(q_0\) to \(p\). The homotopy \(H: I^2 \rightarrow \bar{X}\), defined by \((t, s) \mapsto \alpha(t) \cdot \gamma(s)\), contracts the loop \(\alpha\) to the constant loop at \(p\). Thus the homomorphism \(N \rightarrow \pi_1(\bar{X})\) is trivial and the first statement of Proposition 2.3 holds. The second statement follows from the commutative diagram (2.1). □
Lemma 2.5. Let $D$ be a $p$-divisor on $(Y, N_\mathbb{Q})$ such that $X$ is smooth. If there is only one maximal intersection set, any divisor of $\text{supp}(D)$ is principal and $D$ have at least one polyhedral coefficient of positive dimension, then Proposition 2.3 holds for $V = Y$ affine.

Proof. Write $D = \sum_D \Delta_D \otimes D$. For each $D$ let $v_D \in N$ such that $\Delta_D - v_D \subseteq (N_D)_\mathbb{Q}$ and let $D' := \sum_D (\Delta_D - v_D) \otimes D$. By hypothesis the divisor $D - D'$ is principal and thus the sheaves of algebras $A(D)$ and $A(D')$ are isomorphic. Being $D'$ contained in a proper subspace $(N_D)_\mathbb{Q}$ of $N_\mathbb{Q}$ we conclude that $X = \text{Spec}_{\text{Loc}(D)} A(D) \simeq \text{Spec}_{\text{Loc}(D')} A(D')$ is isomorphic to the cartesian product $T_{N(D)} \times \text{Spec}_{\text{Loc}(D)} A(D_0')$, where $D_0'$ is the polyhedral divisor $D'$ whose coefficients lie in $(N_D)_\mathbb{Q}$. Then, the first statement follows by Lemma 2.4. The second statement follows by the commutative diagram (2.1). \[□\]

Remark 2.6. Suppose that all the polyhedral coefficients of $D$ are points and write $D = \sum_D q_D \otimes D$. Let $L$ be the line spaned by two polyhedral coefficients of $D$. For any point $q_D$ choose $v_D \in N$ such that $q_D - v_D \in L$. Let $l \in L \cap N$ and denote by $D' := \sum_D (q_D - v_D - l) \otimes D$. As in the previous paragraph we can write $\bar{X} \simeq T_{N(l)} \times \text{Spec}_{\text{Loc}(D)} A(D_0')$, where $D_0'$ is the polyhedral divisor $D'$ whose coefficients lies in $N_l \simeq \mathbb{Q}$.

Lemma 2.7. Let $D$ be a $p$-divisor on $(Y, N_\mathbb{Q})$ such that $X$ is smooth. If there is only one maximal intersection set and any divisor of $\text{supp}(D)$ is principal, then Proposition 2.3 holds for $V = Y$ affine.

Proof. By Lemma 2.5 we reduce to the case when all the polyhedral coefficients of $D$ are points and using remark 2.6 we can reduce to the case $N \simeq \mathbb{Z}$. Observe that in this case the morphism $X = \bar{X} \to Y$ is a topological $\mathbb{C}^*$-fibration. Let $m \in \mathbb{Z}_{>0}$ such that all the points of $mD$ are integrals, then we have that $X(mD) \simeq \mathbb{C}^* \times Y$. Observe that the morphism $X(D) \to X(mD)$ is the quotient by a cyclic group of order $m$. Both $p$-divisors have the same trivial open subset and we have the following commutative diagram

$$
\begin{array}{c}
\mathbb{C}^* \times V_D \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \\
\mathbb{C}^* \times Y \\
\downarrow \\
X(D)
\end{array}
$$

where the vertical arrows are open embeddings. Passing to the fundamental group we have the following commutative diagram

$$
\begin{array}{c}
\mathbb{Z} \times \pi_1(V_D) \\
\downarrow \quad \quad \downarrow \\
\mathbb{Z} \times \pi_1(Y) \\
\downarrow \\
\pi_1(X(D))
\end{array}
$$

The vertical arrows are surjective morphisms by [7, Theorem 12.1.5]. By the commutativity of the diagram we conclude that $\pi_1(X(D)) \simeq \mathbb{Z} \times (\pi_1(V_D)/K)$, where $K$ is a normal subgroup of $\pi_1(V_D)$. On the other hand, we have the long exact
sequence of the fibration $X(\mathcal{D}) \to Y$:

$$
\begin{array}{c}
\pi_1(\mathbb{C}^*) \longrightarrow \pi_1(X(\mathcal{D})) \longrightarrow \pi_1(Y) \\
\approx \hspace{1cm} \approx \hspace{1cm} \\
\mathbb{Z} \longrightarrow \mathbb{Z} \times (\pi_1(\mathcal{V}_D)/K) \longrightarrow \pi_1(Y) \longrightarrow 0.
\end{array}
$$

Choosing $\mathbb{C}^*$ to be a fiber of a point in the trivial open subset of $Y$ we can see that the homomorphism $\mathbb{Z} \to \mathbb{Z} \times (\pi_1(\mathcal{V}_D)/K)$ is the inclusion on the first component. Thus $\pi_1(X(\mathcal{D})) \simeq \mathbb{Z} \times \pi_1(Y)$ and the result follows. □

**Lemma 2.8.** Let $\mathcal{D}$ be a $p$-divisor on $(Y, N_{pq})$ such that $X$ is smooth. Then Proposition 2.3 holds.

**Proof.** We split the proof in four steps.

(1) Observe that if Proposition 2.3 holds for the open sets $V^1 \subseteq V^2 \subseteq Y$ then we have a commutative diagram

$$
\begin{array}{ccc}
N(\sigma(\mathcal{D})) \times \pi_1(V^1) & \longrightarrow & N(\sigma(\mathcal{D})) \times \pi_1(V^2) \\
\downarrow^{\alpha^1 \times \beta^1} & & \downarrow^{\alpha^2 \times \beta^2} \\
N(\mathcal{D}|_{V^1}) \times \pi_1(V^1) & \longrightarrow & N(\mathcal{D}|_{V^2}) \times \pi_1(V^2)
\end{array}
$$

where $i_{12}: V^1 \to V^2$ is the inclusion. We conclude that the bottom horizontal map is of the form $s_{12} \times i_{12}^*$, where $s_{12}$ is the surjection induced by $N_{\mathcal{D}|_{V^2}} \to N_{\mathcal{D}|_{V^1}}$.

(2) If Proposition 2.3 holds for the open sets $V^1, V^2 \subseteq Y$ and $V^1 \cap V^2$ then by Seifert Van-Kampen Theorem [11, Theorem 1.20] we have two push-out diagrams

$$
\begin{array}{ccc}
N(\mathcal{D}|_{V^1 \cap V^2}) & \longrightarrow & N(\mathcal{D}|_{V^1}) \\
\downarrow & & \downarrow \\
N(\mathcal{D}|_{V^2}) & \longrightarrow & N(\mathcal{D}|_{V^1 \cup V^2}) \\
\downarrow & & \downarrow \\
\pi_1(V^1 \cap V^2) & \longrightarrow & \pi_1(V^1 \cap V^2)
\end{array}
$$

$$
\begin{array}{ccc}
\pi_1(V^1 \cap V^2) & \longrightarrow & \pi_1(V^1) \\
\downarrow & & \downarrow \\
\pi_1(V^2) & \longrightarrow & \pi_1(V^1 \cup V^2)
\end{array}
$$

Taking the free product of the two diagrams and using Step (1) we get the following push-out diagram

$$
\begin{array}{ccc}
\pi_1(U^1 \cap U^2) & \longrightarrow & \pi_1(U^1) \\
\downarrow^{i_1} & & \downarrow^{i_2} \\
\pi_1(U^2) & \longrightarrow & N(\mathcal{D}|_{V^1 \cup V^2}) \times \pi_1(V^1 \cup V^2)
\end{array}
$$

where the maps $i_1$ and $i_2$ are induced by the inclusion of the corresponding open subsets. Then $\pi_1(U_1 \cup U_2)$ is isomorphic to $N(\mathcal{D}|_{V^1 \cup V^2}) \times \pi_1(V^1 \cup V^2)$.
Moreover, observe that we have a commutative diagram

\[
\begin{array}{ccc}
\pi_1(U^1_t \cap U^2_t) & \to & \pi_1(U^1_t) \\
\downarrow & & \downarrow \\
\pi_1(U^2_t) & \to & \pi_1(U^1_t \cup U^2_t) \\
\downarrow & & \downarrow \\
\pi_1(U^1_t \cup U^2_t) & \to & \pi_1(U^1 \cup U^2) \\
\end{array}
\]

where \( p_i \) is the composition \( \pi_1(U^1_t) \to \pi_1(U^i) \to \pi_1(U^1 \cup U^2) \) induced by the two inclusion maps and the square diagram is a push-out. Then by universal property of the push-out diagram the unique homomorphism \( p \) is induced by the inclusion \( U^1_t \cup U^2_t \subseteq U^1 \cup U^2 \). Moreover we have a commutative diagram

\[
\begin{array}{ccc}
\pi_1(U^1_t \cup U^2_t) & \xrightarrow{p} & \pi_1(U^1 \cup U^2) \\
\downarrow & & \downarrow \\
N(\sigma(\mathcal{D})) \times \pi_1(V^1_t \cup V^2_t) & \xrightarrow{p'} & N(\mathcal{D})|_{V^1 \cup V^2} \times \pi_1(V^1 \cup V^2) \\
\end{array}
\]

where \( p' \) is induced by the inclusions \( N_{\sigma(\mathcal{D})} \subseteq N_{\mathcal{D}|_{V^1 \cup V^2}} \) and \( V^1_t \cup V^2_t \subseteq V^1 \cup V^2 \). Thus, Proposition 2.3 holds for \( V^1 \cup V^2 \).

(3) Suppose that Proposition 2.3 holds for the open subsets \( V_1, \ldots, V_k \) of \( Y \) and for every intersection of this sets. We conclude by Step (2) that Proposition 2.3 also holds for \( \bigcup_{i=1}^{k} V_i \). Indeed, suppose by induction that it holds for \( \bigcup_{i=1}^{k-1} V_i \) and \( V_k \cap (\bigcup_{i=1}^{k-1} V_i) = \bigcup_{i=1}^{k-1} (V_k \cap V_i) \), then by Step (2) it also holds for \( \bigcup_{i=1}^{k} V_i \).

(4) Let \( V \subseteq Y \) be any open subset. Consider a finite affine open cover \( V_i \) of \( V \) such that every divisor of \( \text{supp}(\mathcal{D}) \) is principal at any \( V_i \), and \( \mathcal{D}|_{V_i} \) has only one maximal intersection set. Let \( V' \) be any finite intersection of the \( V_i \)'s, then \( V' \) is an open affine set, so \( \mathcal{D}|_{V'} \) is a \( p \)-divisor on \( V' \) with only one maximal intersection set. By Lemma 2.7 we conclude that Proposition 2.3 holds for \( V' \). Thus, we are in situation of Step (3) and Proposition 2.3 also holds for \( V \).

\[ \square \]

**Proof of Proposition 2.3.** First we prove the Theorem in the case \( V = Y \). We proceed by induction on the number \( n \) of \( p \)-divisors of \( S \). If \( n = 1 \) then \( X(S) = X(\mathcal{D}) \) and results follows from Lemma 2.8. Suppose that \( S \) is the set of \( p \)-divisors \( \{\mathcal{D}^1, \ldots, \mathcal{D}^n\} \) and the result is true for \( n - 1 \). Assume that \( \mathcal{D}^n \) is maximal with respect to the inclusion. Using the induction hypothesis on the divisorial fans

\[
S_1 := \{\mathcal{D}^n\}, \quad S_2 := \{\mathcal{D}^1, \ldots, \mathcal{D}^{n-1}\}, \quad S' := \{\mathcal{D}^1 \cap \mathcal{D}^n, \ldots, \mathcal{D}^{n-1} \cap \mathcal{D}^n\}.
\]
By a similar argument as in the proof of Lemma 2.8 we have a commutative diagram

\[
\begin{array}{ccc}
N(\sigma(S')) \times \pi_1(V_{S'}) & \longrightarrow & N(\sigma(S_i)) \times \pi_1(V_{S_i}) \\
\downarrow & & \downarrow \\
N(S') \times \pi_1(Loc(S')) & \overset{\alpha \times \beta_i}{\longrightarrow} & N(S_i) \times \pi_1(Loc(S_i)),
\end{array}
\]

for each \(i \in \{1, 2\}\). Thus the homomorphisms induced by the inclusion \(X(S') \to X(S_i)\) is \(\alpha_i \times \beta_i\) where on each factor the corresponding homomorphism is induced by the inclusions \(Loc(S) \to Loc(S_i)\) and \(N_{S_i} \to N_S\), for each \(i\). We have the following push-out diagram

\[
\begin{array}{ccc}
N(S') \times \pi_1(Loc(S')) & \overset{\alpha_1 \times \beta_1}{\longrightarrow} & N(S_i) \times \pi_1(Loc(S_i)) \\
\downarrow & & \downarrow \\
N(S_2) \times \pi_1(Loc(S_2)) & \longrightarrow & N(S) \times \pi_1(Loc(S_1) \cup Loc(S_2)),
\end{array}
\]

then the isomorphism \(\pi_1(X(S)) \to N(S) \times \pi_1(Loc(S))\) follows by the Seifert Van-Kampen Theorem. We now look at the homomorphism \(\pi_1(U_S) \to \pi_1(X(S))\). We have a commutative diagram

\[
\begin{array}{ccc}
N(\sigma(S')) \times \pi_1(V_{S'}) & \longrightarrow & N(\sigma(S_i)) \times \pi_1(V_{S_i}) \\
\downarrow & & \downarrow \pi_1(X(S)) \quad \pi_1(X(S)) \\
N(\sigma(S_2)) \times \pi_1(V_{S_2}) & \longrightarrow & N(\sigma(S)) \times \pi_1(V_{S})
\end{array}
\]

where all the arrows are induced by inclusions and \(p_i\) is the composition

\[N(\sigma(S_i)) \times \pi_1(V_{S}) \to N(S_i) \times \pi_1(V_{S_i}) \to N(S) \times \pi_1(Loc(S))\]

for each \(i\). Then by the universal property of the push-out diagram we have a commutative diagram

\[
\begin{array}{ccc}
\pi_1(U_S) & \longrightarrow & \pi_1(X(S)) \\
\downarrow \cong & & \downarrow \cong \\
N(\sigma(S)) \times \pi_1(V_{S}) & \overset{p'}{\longrightarrow} & N(S) \times \pi_1(Loc(S))
\end{array}
\]

where \(p'\) is induced by the inclusions \(N_{\sigma(S)} \subseteq N_S\) and \(V_S \subseteq Loc(S)\). The statement follows. \(\square\)

**Proof of Theorem 1.** Reasoning as in the proof of Proposition 2.3, by gluing affine \(T\)-varieties, it is enough to prove the affine case.

Let \(D\) be a \(p\)-divisor on \((Y, N_0)\) such that \(X(D)\) have log-terminal singularities. Let \(r_S: X(S) \to \bar{X}(D)\) be a toroidal resolution of singularities of \(\bar{X}(D)\) such that \(S\) is contraction-free and \(D\) has the same locus and support than \(S\), see [14]. Thus,
we have a commutative diagram

\[
\begin{array}{ccc}
X(\sigma(D)) \times V_D & \xrightarrow{S_D} & X(D) \\
\downarrow r & \nearrow i_D & \downarrow r_S \\
\tilde{X}(D) & \xleftarrow{\acute{\text{r}}_S} & X(S)
\end{array}
\]

where \(i_D\) and \(i_S\) are inclusions, \(r\) and \(r_S\) are birational contractions and \(S_D\) is defined as the composition \(r \circ i_D\). Passing to the fundamental groups, \((r \circ r_S)_*\) is an isomorphism by [16, Theorem 1.1]. Being \(X(S)\) smooth and \(N(D) = N(S)\) we have the isomorphisms \(\pi_1(X(D)) \cong \pi_1(X(S)) \cong N(S) \times \pi_1(\text{Loc}(S)) = N(D) \times \pi_1(\text{Loc}(D))\). Being \(i_D\) and \(i_S\) inclusions of open subsets we have that \((i_D)_*\) and \((i_S)_*\) are surjective. We deduce that the kernel of the three homomorphisms \((s_D)_*, (i_D)_*\) and \((i_S)_*\) are equal. Thus \((s_D)_*: N(\sigma(D)) \times \pi_1(V_D) \to N(D) \times \pi_1(\text{Loc}(D))\) is \(\alpha \times \beta\), where \(\alpha\) is the surjection induced by the inclusion \(N_{\sigma(D)} \subseteq N_D\) and \(\beta\) is the surjection induced by the inclusion \(V_D \subseteq \text{Loc}(D)\).

3. The Cohomology Ring of a Complexity One \(T\)-Variety

From now, we will consider divisorial fans \(S\) on \((\mathbb{P}^1, N_\mathbb{Q})\), such that \(\tilde{X}(S)\) is complete and \(\mathbb{Q}\)-factorial and describe the Cohomology ring and the Chow Ring under certain conditions (see Definition 3.9). Recall that using [14, Example 2.5] we can see in \(S\) when \(\tilde{X}(S)\) is a \(\mathbb{Q}\)-factorial variety. The results of this sections are generalizations of well-known Theorems, see [10, Chapter 5].

**Definition 3.1.** Given a \(\sigma\)-polyhedron \(\triangle \subseteq N_\mathbb{Q}\), we denote by \(V(\triangle)\) its set of vertices and by \(N(\triangle)\), or simply \(N\), its normal fan consisting of the regions where the function \(\sigma^\vee \to \mathbb{Q}, u \mapsto \min_{v \in \triangle} \langle u, v \rangle\), is lineal. The cones of \(N\) are in one-to-one dimension-reversing correspondence with the faces \(F \leq \triangle\) via the bijection

\[
F \mapsto \lambda(F) := \{ u \in M_\mathbb{Q} \mid \langle F, u \rangle = \min(\triangle, u) \}.
\]

Given a \(\sigma\)-polyhedron \(\triangle\) we define the affine toric bouquet \(X(\triangle) := \text{Spec}(\mathbb{C}[N])\), where \(\mathbb{C}[N] := \bigoplus_{u \in \sigma^\vee \cap M} \mathbb{C}x^u\) as a \(\mathbb{C}\)-vector space and the multiplication is given by

\[
\chi^u \cdot \chi^{u'} := \begin{cases} 
\chi^{u+u'} & \text{if } u, u' \text{ belong to a common cone of } N, \\
0 & \text{otherwise.}
\end{cases}
\]

The ring \(\mathbb{C}[N]\) is not an integral domain, hence \(X(\triangle)\) is not a variety, but is a scheme, nevertheless since \(\mathbb{C}[N]\) is reduced, \(X(\triangle)\) have a decomposition into a finite union of irreducible toric varieties

\[
X(\triangle) = \bigcup_{v \in V(\triangle)} X(\mathbb{Q}_{\geq 0} \cdot (\triangle - v)).
\]

These irreducible toric varieties are intersecting along \(T\)-invariant divisors, hence the big torus is still acting on \(X(\triangle)\). Observe that the orbits of \(X(\triangle)\) are in one-to-one dimension-reversing correspondence with the faces \(F \leq \triangle\). We denote by \(\triangle(k)\) the set of faces of codimension \(k\) and given a face \(F \in \triangle(k)\) we denote by \(O_F\) its corresponding torus orbit of dimension \(k\). Given \(\triangle\) a polyhedral complex consisting of a finite number of \(\sigma_i\)-polyhedra \(\triangle_i\), we can glue the affine toric bouquets \(X(\triangle_i)\) and \(X(\triangle_j)\) along \(X(\triangle_i \cap \triangle_j)\) to obtain a toric bouquet \(X(\triangle)\). We denote by \(\triangle(k)\) the set of faces of codimension \(k\) of \(\triangle\).
Proposition 3.2. The Chow group $A_k(X)$ of an arbitrary toric bouquet $X = X(\Delta)$ is generated by the classes of the orbit closures $[\overline{O_F}]$ for $F \in \Delta(k)$.

Proof. We denote by $X_i$ the union of all orbits corresponding to faces of codimension at most $i$. This gives a filtration $X = X_n \supseteq X_{n-1} \supseteq \cdots \supseteq X_1 = \emptyset$ by closed subschemes. The complement of $X_{i-1}$ in $X_i$ is a disjoint union of $|\Delta(i)|$ torus orbits of dimension $i$. We conclude by the exact sequence \cite[Proposition 1.8]{10}

$$A_k(X_{i-1}) \to A_k(X_i) \to A_k(X_i - X_{i-1}) \to 0,$$

and by induction on $i$. □

Definition 3.3. In $\overline{X}(S)$ we have two kind of $T$-invariant cycles. Given a point $y \in Y$ and a face $F$ of $S_y$ we have a $T$-invariant cycle $[\overline{O_F}] \in A_*(\overline{X}(S))$. We call such cycle 	extit{vertical} if $y$ is a closed point of $Y$ and 	extit{horizontal} if $y$ is the generic point of $Y$. If $y \in Y$ is the generic point and $\rho \in S_y = \Sigma(S)$ is a ray, we denote the corresponding 	extit{horizontal divisor} by $D_\rho$ and if $y \in Y$ is a closed point and $v \in V(S_y)$ is a vertex, we denote the corresponding 	extit{vertical divisor} by $D_{(v,y)}$.

Proposition 3.4. Given a divisorial fan $S$ on a curve $Y$ the Chow group $A_k(\overline{X}(S))$ is generated by the classes of the horizontal and vertical cycles of dimension $k$.

Proof. Let $p_1, \ldots, p_r$ be the support of the divisorial fan $S$. Recall that each $\pi^{-1}(p_i)$ is a toric bouquet and that we have an isomorphism

$$\pi^{-1}(Y - \{p_1, \ldots, p_r\}) \simeq X(\Sigma(S)) \times (Y - \{p_1, \ldots, p_r\}).$$

Recall that by Proposition 3.2 the Chow group $A_k(\bigcup_{i=1}^r \pi^{-1}(p_i))$ is generated by vertical cycles of dimension $k$, while using the above isomorphism we see that the Chow group $A_k(\pi^{-1}(Y - \{p_1, \ldots, p_r\}))$ is generated by vertical and horizontal cycles of dimension $k$. Using the exact sequence of Chow groups relating the closed subscheme $\bigcup_{i=1}^r \pi^{-1}(p_i)$ and its complement the result follows. □

Definition 3.5. Given a polyhedral complex $\Delta$ on $N_{\mathbb{Q}}$, we say that $\Delta$ is 	extit{shellable} if we can order the maximal polyhedra $\Delta_1, \ldots, \Delta_k$ of $\Delta$ such that for each $i$ the following set ordered by inclusion

$$\{ F \leq \Delta_i \mid F \text{ is not contained in } \bigcup_{j<i} \Delta_j \},$$

has a unique minimal element denoted by $G_i$.

For example the fan of a projective and simplicial toric variety is always shellable by \cite[Section 5.2, Lemma]{10}. Observe that for each face $F$ of a shellable polyhedral complex $\Delta$ there is a unique $i$ such that $G_i \subseteq F \subseteq \Delta_i$.

Definition 3.6. We say that a polyhedral complex $\Delta$ is 	extit{simplicial} if for each vertex $v \in \Delta$ and maximal polyhedron $v \in \Delta \in \Delta$ the cone $Q \cdot (v - \Delta)$ is simplicial.

Let $\Delta$ be a shellable polyhedral complex and $\Delta_1, \ldots, \Delta_k$, the order induced in its maximal polyhedra. for $1 \leq i \leq k$ we define the subvarieties of $X(\Delta)$

$$Y_i := \bigcup_{G_i \subseteq F \subseteq \Delta_i} O_F, \quad Z_i := Y_i \cup Y_{i+1} \cup \cdots \cup Y_k.$$
Lemma 3.7. Each $Z_i$ is closed, $Z_1 = X(\Delta)$ and $Z_i - Z_{i+1} = Y_i$. Moreover, if $\Delta$ is simplicial then each $Y_i$ is the quotient of an affine space by a finite group.

Proof. Recall that for each face $F$ of $S_\eta$ there is a unique $i$ such that $G_i \subseteq F \subseteq \Delta_i$, then $\pi^{-1}(y)$ is the disjoint union of the sets $Y_i$. The closure of $O_F$ is the union of all the orbits $O_{F'}$ with $F' \supseteq F$, then we conclude that each $Z_i$ is closed. The last assertion follows from the fact that each irreducible toric component of $\pi^{-1}(y)$ is a simplicial toric variety. \hfill $\square$

Proposition 3.8. Let $\Delta$ be a shellable and simplicial polyhedral complex with complete support. Then the classes $[O_{G_i}]$ form a basis for $A_\ast(X(\Delta))_\mathbb{Q} \cong H_{2\ast}(X(\Delta); \mathbb{Q})$ and $H_q(X(\Delta)) = 0$ for $q$ odd.

Proof. In the proof all homologies and Chow groups are over $\mathbb{Q}$. Recall that $X(\Delta) = Z_1$. We prove, by descending induction on $i$, that the canonical map $A_\ast(Z_i) \to H_\ast(Z_i)$ is an isomorphism, that the classes $[O_{G_i}]$, for any $j \geq i$, form a basis of $A_\ast(Z_i)$ and that $H_q(Z_i) = 0$ for $q$ odd. Following Fulton [10, Pag. 103] we have a commutative diagram of Chow groups and Borel-Moore homology with rational coefficients

$$
\begin{array}{cccc}
A_p(Z_{i+1}) & \longrightarrow & A_p(Z_i) & \longrightarrow & A_p(Y_i) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & H_{2p+1}(Y_i) & \longrightarrow & H_{2p}(Z_i) & \longrightarrow & H_{2p}(Y_i) & \longrightarrow & H_{2p-1}(Z_{i+1}) & \longrightarrow & \cdots
\end{array}
$$

Since $Y_i$ is the quotient of an affine space by a finite group we conclude that $A_k(Y_i) \cong H_{2k}(Y_i) \cong \mathbb{Q}$ is generated by the class of $Y_i$ for $k = \dim Y_i$ and otherwise both spaces are trivial. By induction hypothesis $H_q(Z_{i+1}) = 0$ for $q$ odd. The statement follows. \hfill $\square$

Consider a divisorsal fan $\mathcal{S}$ on $\mathbb{P}^1$, with support $\{p_1, \ldots, p_r\}$, such that $X(\mathcal{S})$ is complete and $\mathbb{Q}$-factorial. We denote by $S_i$ the polyhedral complex corresponding to the toric bouquet $\pi^{-1}(p_i)$, for $i \in \{1, \ldots, r\}$ and by $S_0$ the polyhedral complex corresponding to the toric variety $\pi^{-1}(p_0)$, where $p_0$ is a general point. Assume that $S_i$ is shellable with ordered maximal polyhedra $\triangle_1^i, \ldots, \triangle_{k_i}^i$ and minimal elements $G_1^i, \ldots, G_{k_i}^i$, for any $i$. We denote by

$$B_i := \{[O_{G_j^i}] : j \in \{1, \ldots, k_i\}\}
$$

the $\mathbb{Q}$-vector space basis of $A_k(X(S_i))$. For each face $F$ of $S_0$ we denote by $S(F)_i$ the set of faces of $S_i$ which have the same tailcone and dimension of $F$. For each $F' \in S(F)_i$ we denote by $v(F')$ its unique vertex. We define the 0-graded linear map of $\mathbb{Q}$-vector spaces

$$j_i : A_\ast(X(S_0)) \to A_\ast(X(S_i)), \quad [O_{G_j^i}] \mapsto \sum_{F' \in S(G_j^i)_i} \mu(v(F'))[O_{F'}],$$

for $j \in \{1, \ldots, i_0\}$. Observe that $j_i$ is well defined, since $B_0$ is a $\mathbb{Q}$-vector space basis of $A_\ast(X(S_0))$ by Proposition 3.8.

Definition 3.9. We say that $\mathcal{S}$ is complete if $X(\mathcal{S})$ is a complete variety. We say that $\mathcal{S}$ is simplicial if $X(\mathcal{S})$ is a $\mathbb{Q}$-factorial variety. We say that $\mathcal{S}$ is shellable if $j_{i,k}$ is injective for each $k \in \mathbb{N}$ and $i \in \{1, \ldots, r\}$.
For each face $F$ of $S_i$, we denote by $i(F)$ the minimum positive integer such that $F \subseteq \Delta_{i(F)}$. Observe that a sufficient condition for $j_{i;k}$ to be injective is that for each $1 \leq n < m \leq k_0$ we have that
\[
\min\{i(F) \mid F \in S(G_{m;i})\} < \min\{i(F) \mid F \in S(G_{n;i})\}.
\]
Indeed in this case the representative matrix of $j_i$ with respect to the basis $B_0$ of $A_*(X(S_0))$ and $B_i$ of $A_*(X(S_i))$ contains a triangular submatrix of full rank.

**Notation 3.10.** Given a complete, simplicial and shellable divisorial fan $S$ on $\mathbb{P}^1$, we denote by $\mathbb{Q}[D_\rho, D_{(p,v)}]$ the polynomial ring over $\mathbb{Q}$ generated by elements $D_\rho$ and $D_{(p,v)}$ for $\rho \in \Sigma(S)(1)$ and $v \in V(S_p)$, with $p \in \mathbb{P}^1$.

For each face $F$ of a slice $S_p$ with $p \in \mathbb{P}^1$, we have a corresponding monomial in $\mathbb{Q}[D_\rho, D_{(p,v)}]$, which is the product of all the elements $D_\rho$ and $D_{(p,v)}$ such that $v \in F$ and $\rho \in \sigma(F)$. We denote such element by $p(F)$. For each cone $\sigma$ of the fan $\Sigma(S)$, we have a corresponding monomial in $\mathbb{Q}[D_\rho, D_{(p,v)}]$, which is the product of all the elements $D_\rho$ with $\rho \in \sigma$. We denote such element by $p(\sigma)$.

We denote by $I$ the ideal generated by
\[
(3.1) \quad \text{div}(f^u) := \sum_{\rho \in \Sigma(S)(1)} \langle p, u \rangle D_\rho + \sum_{p \in \mathbb{P}^1, v \in V(S_p)} \mu(v)\langle v, u \rangle + \text{ord}_p f \cdot D_{(p,v)},
\]
for each rational function $f \in \mathbb{C}(\mathbb{P}^1)$ and character lattice $\chi^u \in M$ and by all the monomials which are not of the form $p(F)$ nor $p(\sigma)$.

**Proof of Theorem 2.** In the proof we omit the rational coefficients in order to abbreviate notation. We start by proving the first statement. Let $U = X(S) - \cup_{i=1}^r \pi^{-1}(p_i)$. We have a commutative diagram of Chow groups and Borel-Moore homology with rational coefficients
\[
\begin{array}{cccc}
A_k(U^c) & \xrightarrow{i} & A_k(X(S)) & \xrightarrow{=} & A_k(U) & \xrightarrow{i} & 0 \\
H_{2k+1}(U) & \xrightarrow{i} & H_{2k}(X(S)) & \xrightarrow{=} & H_{2k}(U) & \xrightarrow{i} & H_{2k-1}(U^c).
\end{array}
\]

The first vertical arrow is an isomorphism by Proposition 3.8. Using that $A_k(\mathbb{P}^1 - \{p_1, \ldots, p_r\}) \to H_{2k}(\mathbb{P}^1 - \{p_1, \ldots, p_r\})$ is an isomorphism for each $k$, K"unneth formula for Borel-Moore homology [6, Section 2.6.19] and Proposition 3.8, we see that the third vertical arrow is an isomorphism. Moreover $H_{2k-1}(U^c) = 0$ by Proposition 3.8. We define an injective homomorphism
\[
A_*(X(S_0)) \to A_*(U^c)
\]
in the following way. Given the orbit closure $V$ of $X(S_0)$ we map its class $[V] \in A_*(X(S_0))$ to the class of $i_*^*([V \times \mathbb{P}^1 - \{p_1, \ldots, p_r\}])$, where $i_* : X(S_i) \to X(S)$ is the inclusion of the reduced fiber. This map extends to an injective homomorphism
\[
i : \mathbb{Q}^{-1} \otimes \mathbb{Q} A_*(X(S_0)) \to A_*(U^c) \cong \bigoplus_{i=1}^r A_*(X(S_i))
\]
defined by
\[(m_1, \ldots, m_{r-1}) \otimes [V] \mapsto (m_1j_1([V]), \ldots, m_{r-1}j_{r-1}([V]), -m_r([V])),\]
where \( m = m_1 + \cdots + m_{r-1} \). The image of \( i \) is in the kernel of \( A_* (U^c) \rightarrow A_* (\overline{X}(S)) \). Indeed, \((m_1, \ldots, m_{r-1}) \otimes \mathbb{Q}[V] \) is mapped to the divisor of a rational function \( f \chi^0 \), where \( f \) has order \( m_i \) at \( p_i \), for \( i \in \{1, \ldots, r-1\} \) and order \(- \sum_{i=1}^{r-1} m_i \) at \( p_r \). Then we have the following commutative diagram

\[
\begin{array}{c}
\mathbb{Q}^{r-1} \otimes_{\mathbb{Q}} A_k(X(S_0)) \\
\downarrow i \quad \downarrow j \\
K \quad H_k(X(S_0)) \\
\downarrow \quad \downarrow \quad \downarrow \Rightarrow \quad \downarrow \quad \downarrow \quad \downarrow \Rightarrow \\
A_k(U^c) \quad A_k(\overline{X}(S)) \quad A_k(U) \quad 0 \\
\uparrow \quad \uparrow \Rightarrow \quad \uparrow \Rightarrow \quad \uparrow \\
H_{2k}(U^c) \quad H_{2k}(\overline{X}(S)) \quad H_{2k}(U) \quad 0,
\end{array}
\]

where \( K \) is the kernel of \( A_k(U^c) \rightarrow A_k(\overline{X}(S)) \), \( i \) is the injection induced by the above homomorphism, and \( j \) is the unique arrow making the diagram commutative. By the four lemma the third vertical arrow is surjective, then \( j \) is injective. Since

\[
\dim(A_k(X(S_0)))_\mathbb{Q} = \dim(H_k(X(S_0))_\mathbb{Q})
\]

for each \( k \) by Proposition 3.8, all the vertical arrows are isomorphisms.  

Now we prove three Lemmas concerning the structure of the ring \( \mathbb{Q}[D_{p_i}, D_{(p,v)}] \), in order to prove Theorem 3. Recall that given a complete, simplicial and shellable divisorial fan \( S \) on \( \mathbb{P}^1 \) with support \( \{p_1, \ldots, p_r\} \) we denote by \( \Delta_1, \ldots, \Delta_k \), the ordered maximal polyhedra of \( S_i \) and we denote by \( G^i_1, \ldots, G^i_k \), the minimal elements for any \( i \). Moreover, we will denote by \( \Delta_1, \ldots, \Delta_k, \) the ordered maximal cones of \( \Sigma(S) \) and we denote by \( G_1, \ldots, G_k \) the minimal elements. We assume that the shellability induced in \( S_0 \) and \( \Sigma(S) \) are the same.

**Lemma 3.11.** Let \( S \) be a divisorial fan on \( \mathbb{P}^1 \) as above. Let \( \emptyset \neq G \subseteq H \subseteq F \) be faces in \( S_i \), with \( i \in \{0, \ldots, r\} \). Then there exists an element on \( I \) of the form

\[
p(H) - \sum_j m_j p(H_j),
\]

where \( H_j \) are faces of \( S_i \), with \( G \subset H_j \) and \( H_j \subset F \), and \( m_j \in \mathbb{Q} \).

**Proof.** Let \( V(F) = \{v_1, \ldots, v_k\} \), and \( \sigma(F) = \{p_1, \ldots, p_{\nu}\} \). We prove the case where \( v_1 \in V(H) \) is not in \( V(G) \) (the case where exists \( \rho \in \sigma(H) \), not in \( \sigma(G) \) is analogous). Recall that the set

\[
S := \{(\rho_1, 0), \ldots, (\rho_{\nu}, 0), (v_1, 1), \ldots, (v_k, 1)\},
\]

is linearly independent in \( \mathbb{N}_0 \otimes \mathbb{Q} \). Consider an element \( u' \in M \otimes \mathbb{Z} = \text{Hom}(\mathbb{N} \otimes \mathbb{Z}, \mathbb{Z}) \) which vanishes at all the elements of \( S - \{(v_1, 1)\} \) and \( u'(v_1, 1) = 1 \). Let \( u \) be the restriction of \( u' \) to \( M \) and consider a rational function \( f \in \mathbb{C}(\mathbb{P}^1) \) with \( \text{ord}_{p_i}(f) = u'(0, 1) \) and \( \text{ord}_{p_{\nu}}(f) = 0 \) for \( k \neq i \). Recall that \( \text{div}(f \chi^u) \) is in \( I \) by definition. Moreover, by (3.1) we have

\[
\text{div}(f \chi^u) := \sum_{\rho \in \sigma(F)} \langle \rho, u \rangle D_\rho + \sum_{v \in V(F)} \mu(v) (\langle v, u \rangle + \text{ord}_p f) \cdot D_{(p, v)}
\]

\[
+ D_{(p_i, v_1)} + \sum_{p \in \mathbb{P}^1 - \{p_i\}, v \in V(S_p)} \mu(v) (\langle v, u \rangle + \text{ord}_p f) \cdot D_{(p, v)},
\]
where all but the summand $D_{(p,v_1)}$ correspond to vertices or rays which are not in $\mathcal{V}(F)\cup \sigma(F)$. Multiplying $\text{div}(f\chi^v)$ by $p(H)/D_{(p,v_1)}$, subtracting all the monomials which are not of the form $p(F)$, with $F$ a face of $\mathcal{S}_i$, and observing that $p(G)$ divides $p(H)/D_{(p,v_1)}$, we get the result.

\begin{lemma}
Let $\mathcal{S}$ be a divisorial fan as above and let $\mathcal{S}_i$ be a slice with $i \in \{0, \ldots, r\}$. The ideal $I_i := (D_{(p,v)} + I \mid v \in \mathcal{S}_i)$ of $\mathbb{Q}[D_p, D_{(p,v)}]/I$ is generated as a $\mathbb{Q}$-vectorial space by the set $\{p(G_j^i) + I \mid j \in \{1, \ldots, k_i\}\}$. 
\end{lemma}

\begin{proof}
First we prove that $I_i$ is generated by square-free monomials. To this aim it is enough to show that for any face $F$ of $\mathcal{S}_i$ and $v \in \mathcal{V}(F)$ the element $D_{(p,v)}p(F)$ and $D_rp(F)$ are equivalent to sum of square-free monomials modulo $I_i$. Following the proof of Lemma 3.11 we can find an element $\text{div}(f\chi^v) \in I$ which contains $D_{(p,v)}p(F)$ as a summand and the other summands corresponds to vertices and rays which are not in $F$. Multiplying this element by $p(F) + I$ we conclude that $D_{(p,v)}p(F)$ is equal to a linear combination of square-free monomials. A similar argument applies to $D_rp(F)$.

By descending induction on $k$, we prove that if $G_k^i \subseteq H \subseteq F_k$ then $p(H) + I$ is in the submodule generated by $p(G_k^i) + I$ with $j \geq k$. If $H = G_k^i$ we are done, otherwise we can apply Lemma 3.11 to $\emptyset \neq G_k^i \subseteq H \subseteq F_k$ and conclude by the induction hypothesis.

In the following proof, given a face $F \in \mathcal{S}_p$, with $i \in \{1, \ldots, r\}$, we will denote by $D_F$ the $p$-divisor with support $\{p_1, \ldots, p_r\}$ and slices $D_{F,p} = F$ and $D_{F,p_j} = \emptyset$, if $j \neq i$. Recall that we have a open embedding $X(D_F) \to \tilde{X}(\mathcal{S})$.

\begin{lemma}
Let $\mathcal{S}$ be a complete, simplicial and shellable divisorial fan on $\mathbb{P}^1$. Then every $T$-invariant prime subvariety of $\tilde{X}(\mathcal{S})$ is the complete intersection of $T$-invariant divisors.
\end{lemma}

\begin{proof}
Recall that, according to Definition 3.3, any $T$-invariant prime subvariety of $\tilde{X}(\mathcal{S})$ is of the form $\overline{O_F}$, where $F$ is a face of the slice $\mathcal{S}_y$ and $y \in \mathbb{P}^1$ is a point. If $y$ is a closed point, then $\overline{O_F}$, with $F \in \mathcal{S}_y$, equals $\bigcup_{F' \supseteq F} O_{F'}$. Observe that $F' \supseteq F$ if and only if $\sigma(F') \supseteq \sigma(F)$ and $\mathcal{V}(F') \supseteq \mathcal{V}(F)$, so that the following equality holds:

\[
\overline{O_F} = \bigcap_{v \in \mathcal{V}(F)} D_{(y,v)} \cap \bigcap_{\rho \in \sigma(F)} D_{\rho}.
\]

If $y$ is the generic point and $p \in \mathbb{P}^1$ is a closed point, localizing to a toric neighborhood of $p$ we have the equality

\[
\overline{O_F} \cap \pi^{-1}(p) = \bigcup_{F' \supseteq \sigma(F)} O_{F'},
\]

where the $\subseteq$ inclusion is clear, while the $\supseteq$ inclusion is due to the fact that if $F' \in \mathcal{S}_p$ and $\sigma(F') \not\supset \sigma(F)$, then $X(D_{F'})$ is an open subset of $\tilde{X}(\mathcal{S})$ which contains $O_{F'}$ and is disjoint from $\overline{O_F}$. We conclude that

\[
\overline{O_F} = \bigcup_{F' \supseteq \sigma(F)} O_{F'} = \bigcap_{\rho \in \sigma(F)} \overline{O_F} = \bigcap_{\rho \in \sigma(F)} D_{\rho}.
\]
Proof of Theorem 3. We have a canonical homomorphism
\[ \mathbb{Q}[D_{\rho}, D_{(p,v)}] \rightarrow A^*(\tilde{X}(S)), \quad D \mapsto [D] \]
which maps every \(D_{\rho}\) (resp. \(D_{(p,v)}\)) to the class of the corresponding divisor. In \(\tilde{X}(S)\) every \(T\)-invariant cycle is the intersection of \(T\)-invariant divisors by Lemma 3.13, then the above homomorphism is surjective. Recall by \([4, \text{Theorem 26}]\) that given a rational function \(f\) of \(\mathbb{P}^1\) and a character \(\chi^u\) of the torus acting on \(\tilde{X}(S)\), the element \(\text{div}(f\chi^u) \in \mathbb{Q}[D_{\rho}, D_{(p,v)}]\) is in the kernel of the homomorphism. Consider a monomial \(m\) of the form
\[ D_{p_1} \ldots D_{p_k}, D_{(p_{k+1},v_1)} \ldots D_{(p_{k+4},v_k)} \in \mathbb{Q}[D_{\rho}, D_{(p,v)}]. \]
If there exist \(1 \leq i < j \leq k\) such that \(p_i \neq p_j\) then the image of \(m\) in \(A^*(\tilde{X}(S))\) is zero because we are intersecting two divisors which are in different fibers of the quotient morphism \(\tilde{X}(S) \rightarrow \mathbb{P}^1\). If all the \(p_i\)'s are equal, we can localize to the toric neighborhood \(X(\Sigma_i)\) of \(\pi^{-1}(p_i)\) and see that there exists no face \(F\) in \(S_{p_i}\) such that \(\sigma(F) = (p_1, \ldots, p_{k'})\) and \(\mathcal{V}(F) = \{v_1, \ldots, v_k\}\) if and only if there exists no cone in \(\Sigma_i\) generated by \(\{(p_1,0), \ldots, (p_{k'},0),(v_1,1), \ldots, (v_k,1)\}\). The last condition is equivalent to ask for the divisors in \(m\) to have empty intersection. If this is the case then \(m\) is in the kernel of the homomorphism. If there is no \(D_{(p_{k+1},v_1)}\) in the monomial, and there is no cone \(\sigma \in \Sigma(S)\) such that \(p_1, \ldots, p_{k'}\) generates \(\sigma\), then the horizontal divisors in \(m\) have empty intersection by \([10, \text{Section 5.2}]\). Thus we have a well-defined surjective homomorphism
\[ \phi: \mathbb{Q}[D_{\rho}, D_{(p,v)}]/I \rightarrow A^*(\tilde{X}(S)). \]
In order to conclude we prove that there is a subset of \(\mathbb{Q}[D_{\rho}, D_{(p,v)}]/I\) which generates it as a \(\mathbb{Q}\)-vector space and whose image is a \(\mathbb{Q}\)-basis of the Chow ring.

Observe that the subring of \(A^*(\tilde{X}(S))\) generated by \(\{D_{\rho} \mid \rho \in \Sigma(S)(1)\}\) is isomorphic to \(A^*(X(\Sigma(S))\) \(\mathbb{Q}\), and by Proposition \([10, \text{Chapter 5}]\) it is generated as \(\mathbb{Q}\)-vectorial space by the monomials \(p(G_j)\), with \(j \in \{1, \ldots, k\}\).

From now, we use the notation of the proof of Theorem 3. The restriction of \(\phi\) to \(I_i\) induces a surjective homomorphism \(\alpha_i: I_i \rightarrow A^*(X(S_i))\) of \(\mathbb{Q}\)-vector spaces which maps \(\{p(G_j) + I \mid j \in \{1, \ldots, k_i\}\}\) into a \(\mathbb{Q}\)-basis of the codomain. Then \(\alpha_i\) is an isomorphism by 3.12. Consider a rational function \(f \in \mathbb{C}(\mathbb{P}^1)\) with \(\text{ord}_{p_i}(f) = 1\) and \(\text{ord}_{p_{k'}}(f) = -1\). Expanding the product \(p(G_j) \text{div}(f\chi^u) \in I\) we deduce the following
\[ p(G_j^0) + I = \sum_{v \in \mathcal{V}(S_i)} \mu(v)p(G_j)D_{(p_i,v)} + I. \]
For each \(i\), we define the \(\mathbb{Q}\)-linear map
\[ I_0 \rightarrow I_i, \quad p(F) + I \mapsto \sum_{v \in \mathcal{V}(S_i)} \mu(v)p(F)D_{(p_i,v)} + I. \]
which makes commute the following diagram
\[
\begin{array}{ccc}
A_*(X(S_0)) & \xrightarrow{\jmath_i} & A_*(X(S_i)) \\
\approx & & \approx \\
I_0 & \xleftarrow{\cong} & I_i \\
\end{array}
\]
\[
\begin{array}{ccc}
Q[D_{\rho}, D_{(p,v)}]/I & \xrightarrow{\phi} & A_*(\tilde{X}(S)).
\end{array}
\]
In particular, for each $i$ we can choose a set of elements $B_i$ of $I_i$, which are linear combination of the monomials $p(G_j^i) + I$, such that the image of $B_i$ via $\phi$ is a basis of $\text{coker}(j_i)$. Observe that $\{p(G_j^i) + I \mid j \in \{1, \ldots, k_0\}\} \cup B_i$ generates $I_i$ as a $\mathbb{Q}$-vector space for each $i$. Being $\tilde{X}(S)$ a $\mathbb{Q}$-factorial variety and using Theorem 3 we have that

$$A^r(\tilde{X}(S)) \simeq A_*(\tilde{X}(S)) \simeq A_{r-2}(X(\Sigma(S))) \oplus A_r(X(S_0)) \bigoplus_{i=1}^{r} \text{coker}(j_i).$$

We conclude that the set

$$\{p(G_j) + I \mid j \in \{1, \ldots, k\}\} \cup \{p(G_j^0) + I \mid j \in \{1, \ldots, k_0\}\} \bigcup_{i=1}^{r} B_i,$$

is a subset of $\mathbb{Q}[D_{p_i}, D_{p,p_i}] / I$ which generates its as a $\mathbb{Q}$-vector space and whose image is a $\mathbb{Q}$-basis of the Chow ring. \qed

The following example gives the dimensions of cohomology groups with rational coefficients of a complete $\mathbb{Q}$-factorial threefold with shellable divisorial fan.

**Example 3.14.** Let $S$ be shellable divisorial fan on $\mathbb{P}^1$ such that $\tilde{X}(S)$ is a complete $\mathbb{Q}$-factorial threefold. Denote by $p_1, \ldots, p_r$ the support of $S$ and by $S_i$ the slice at the point $p_i$. We denote by $s_i$ the number of maximal polyhedra in $S_i$. Using Theorem 3 we can compute the dimension of its cohomology groups with rational coefficients

$$\dim_\mathbb{Q}(H^0(\tilde{X}(S))) = 1, \quad \dim_\mathbb{Q}(H^2(\tilde{X}(S))) = \sum_{i=1}^{r} |V(S_i)| + |\Sigma(S)| - r - 2,$$

$$\dim_\mathbb{Q}(H^4(\tilde{X}(S))) = \sum_{i=1}^{r} (s_i - |V(S_i)|) - r|\Sigma(S)| + r + 1, \quad \dim_\mathbb{Q}(H^6(\tilde{X}(S))) = 1.$$

**Remark 3.15.** Observe that the conclusion of Lemma 3.13 is no longer true if we substitute $\tilde{X}(S)$ with $X(S)$. As an example consider the quadric $Q = V(x_1x_2 + x_3x_4 + x_5x_6)$ of $\mathbb{P}^5$. It admits an effective action of $(\mathbb{K}^*)^3$ and thus is a $\mathbb{T}$-variety of complexity one, so that $Q = X(\Sigma(S))$ for some divisorial fan $S$ on $\mathbb{P}^1$. On the other hand $Q$ is isomorphic to the Pl"{u}cker embedding of the Grassmannian $G(2, 4)$ and its Chow ring of $Q$ is well known: $A^1(X) \cong \mathbb{Z}$ is generated by the class of a hyperplane section and $A^2(X) \cong \mathbb{Z}^2$. Hence $A^*(X)$ cannot be generated by classes of invariant divisors.

### 4. Topology of complexity-one $\mathbb{T}$-varieties

In this section $Y$ denotes a smooth curve. Let $D = \sum_{i=1}^{r} \Delta_{p_i} \otimes p_i$ denote a $p$-divisor on a curve $Y$ with tailcone $\sigma$. We define the degree of $D$ as

$$\text{deg}(D) := \sum_{i=1}^{r} \Delta_{p_i} \subseteq N_{\mathbb{Q}}.$$

For a $p$-divisor $D$ on a curve $Y$ we have that $\text{deg}(D) \neq 0 \iff \text{Loc}(D) = Y$. If $D$ is a polyhedral divisor on $Y$ then $D$ is a $p$-divisor if and only if $\text{deg}(D) \subseteq \sigma$ and $D(u)$ has a principal multiple for all $u \in \sigma^\vee$ with $u^\perp \cap (\text{deg}(D)) \neq 0$.

Now we turn to compute the higher homology and cohomology of contraction-free complexity-one $\mathbb{T}$-varieties. Given $D = \sum_{i=1}^{r} \Delta_{p_i} \oplus p_1$ be a $p$-divisor on a
curves $Y$, we consider the open set $V := \prod_{i=1}^{r} V_i := \mathbb{C}$ which is an analytic neighborhood of the support of $D$. We will write $U := \pi^{-1}(V)$ and $U_i := \pi^{-1}(V_i)$ for each $i$. We recall that we have an equality $U_i := X(\sigma_i)$ for some cone $\sigma_i$ in $\Delta_i$. For each polyhedral $\Delta$, we denote by $N_\Delta := N/N_{\Delta_\sigma}$, where $N_{\Delta_\sigma} := \mathbb{Z} \cap \{v_1 - v_2 \mid v_1, v_2 \in \Delta\}$. For each $i$ we have that $N'(\sigma_i) \simeq N(\Delta_i)$ and we will denote by $N_{\Delta_i}$ the kernel of the surjective homomorphism $N(\sigma) \to N(\Delta_i)$, then we can identify $N(\Delta_i)$ with a subspace of $N(\sigma)$ via the isomorphism $N(\sigma) \simeq N(\Delta_i) \oplus N_{\Delta_i}$ for each $i$. We will use the following notation

$$
\begin{align*}
N^P_i(1) & := N_{\Delta_i} \cap \langle N_{\Delta_{i+1}}, \ldots, N_{\Delta_r} \rangle, \\
N^P_i(2) & := N_{\Delta_i} \cap N(\Delta_{i+1}) \cap \cdots \cap N(\Delta_r), \\
N^P_i(3) & := N(\Delta_i) \cap \langle N_{\Delta_{i+1}}, \ldots, N_{\Delta_r} \rangle, \\
N^P_i(4) & := N(\Delta_i) \cap N(\Delta_{i+1}) \cap \cdots \cap N(\Delta_r).
\end{align*}
$$

Observe that for each $i$ the subspaces $N^P_i(1), N^P_i(2), N^P_i(3)$ and $N^P_i(4)$ are pairwise disjoint and generate $N(\sigma)$. To abbreviate notation, given natural numbers $a_1, a_2, a_3, a_4$ we will write

$$\wedge^{a_1, a_2, a_3, a_4} N^P_i := \wedge^{a_1} N^P_i(1) \otimes_{\mathbb{Z}} \wedge^{a_2} N^P_i(2) \otimes_{\mathbb{Z}} \wedge^{a_3} N^P_i(3) \otimes_{\mathbb{Z}} \wedge^{a_4} N^P_i(4),$$

and

$$H_k(N^P) := \bigoplus_{i=1}^{r-1} \left( \bigoplus_{(a_1, a_2, a_3, a_4) \in \mathcal{I}_k} \wedge^{a_1, a_2, a_3, a_4} N^P_i \right),$$

where $\mathcal{I}_k := \{(a_1, a_2, a_3, a_4) \in \mathbb{N}^4 \mid a_1 + a_2 + a_3 + a_4 = k, (a_1 + a_2)(a_1 + a_3) \neq 0\}$.

**Proposition 4.1.** Let $D = \Delta_p \oplus p$ be a $p$-divisor on a curve $Y$. Then the following statements hold.

1. If $D$ have affine locus, then $H_k(\tilde{X}(D))$ is isomorphic to

   $$\wedge^k N^P_i(4) \oplus \left( \wedge^{k-1} N(\sigma) \otimes_{\mathbb{Z}} H_1(Y) \right) \oplus H_k(N^P).$$

2. If $D$ have complete locus, then $H_k(\tilde{X}(D))$ is isomorphic to

   $$H_k(X(\sigma) \times Y).$$

**Lemma 4.2.** Let $N, N', N''$ be free finitely generated $\mathbb{Z}$-modules. Then the following statements hold.

1. Let $f \colon N' \to N$ be a surjective homomorphism with kernel $K$. Then the kernel of $\wedge^k f \colon \wedge^k N' \to \wedge^k N$ is isomorphic to

   $$\bigoplus_{i+j=k \atop i,j \neq 0} \wedge^i K \otimes_{\mathbb{Z}} \wedge^j N.$$

2. Let $K_1, \ldots, K_r$ be submodules of $N'$ and let $K := \langle K_{i+1}, \ldots, K_r \rangle$. Then the kernel of the sum homomorphism $K_1 \oplus \cdots \oplus K_r \to M$ is isomorphic to

   $$\bigoplus_{i=1}^{r-1} K_i \cap K_i.$$
Let \( f : N' \to N \) be a surjective homomorphism with kernel \( K \) and \( i : N' \to N'' \) be a homomorphism. Then the cokernel of the induced homomorphism \( (f, i) : N' \to N \oplus N'' \) is isomorphic to 
\[ N''/i(K). \]

Proof of Proposition 4.1. We prove the first assertion. Recall that \( N'(\sigma) \cong N(\sigma) \oplus \mathbb{Z} \), then for each \( k \) we have
\[ \wedge^k N'(\sigma) \cong \wedge^k N(\sigma) \oplus \wedge^{k-1} N(\sigma). \]

Using \[ \text{Lemma 18} \] we have that
\[ H_k(U \cap U_D) \cong \bigoplus_{i=1}^r \wedge^k N'(\sigma) \cong \wedge^k N(\sigma) \oplus \wedge^{k-1} N(\sigma) \oplus \mathbb{Z} Z' \]
\[ H_k(U_D) \cong \wedge^k N(\sigma) \oplus \wedge^{k-1} N(\sigma) \oplus \mathbb{Z} (H_1(Y) \oplus Z'). \]

In order to compute \( H_k(X(D)) \) we will study the homomorphism
\[ i_k : H_k(U \cap U_D) \to H_k(U) \oplus H_k(U_D). \]

We denote by \( i_{k,1} \) and \( i_{k,2} \) the coordinates of \( i_k \). Then we can write
\[ i_{k,2} : \wedge^k N(\sigma) \oplus \mathbb{Z} Z' \oplus \wedge^{k-1} N(\sigma) \oplus \mathbb{Z} Z' \to \wedge^k N(\sigma) \oplus \wedge^{k-1} N(\sigma) \oplus \mathbb{Z} (H_1(Y) \oplus Z'), \]
\[ i_{k,1} : \mathbb{Z} (\Delta_i) \to \bigoplus_{i=1}^r \wedge^k N(\Delta_i), \]
where \( i_{k,2} \) is induced by the sum homomorphisms \( Z' \to \mathbb{Z} \) and the injection \( Z' \to H_1(Y) \oplus Z' \) and \( i_{k,1} \) corresponds on each factor to the \( k \)-th wedge homomorphism of the projection \( N(\sigma) \to N(\Delta_i) \) and the zero homomorphism on \( \wedge^{k-1} N(\sigma) \).

Then, the kernel of \( i_k \) corresponds to the elements of \( \mathbb{Z} (\Delta_i) \) which are in the kernel of the surjection \( \bigoplus_{i=1}^r \mathbb{Z} (\Delta_i) \to \bigoplus_{i=1}^r \wedge^k N(\Delta_i) \) and in the kernel of the sum homomorphism \( \bigoplus_{i=1}^r \wedge^k N(\sigma) \to \wedge^k N(\sigma) \). Using part (1) and (2) of Lemma 4.2 we conclude that the kernel of \( i_k \) is isomorphic to
\[ \bigoplus_{i=1}^{r-1} \left( \bigoplus_{a_1, a_2, a_3, a_4 \in I_k} \wedge^{a_1,a_2,a_3,a_4} N_i^D \right), \]
Observe that this is a free finitely generated abelian group.

On the other hand, \( i_{k,1} \) is a surjection, then using part (1) and (3) of Lemma 4.2 we conclude that the cokernel of \( i_k \) is isomorphic to
\[ \wedge^k N_i^D(4) \oplus (\wedge^{k-1} N(\sigma) \otimes \mathbb{Z} H_1(Y)) \]
Then the first assertion follows.

For the second assertion observe that as \( D \) have complete locus, then by [4, Lemma 18] we have that \( N(\Delta_i) = N(\sigma) \) for each \( i \). Then we have that \( N_i^D(1) = N_i^D(2) = N_i^D(3) = 0 \) and \( N_i^D(4) = N(\sigma) \) for each \( i \). Then the result follows from the first part.

Now we turn to compute the cohomology of affine complexity-one \( \mathbb{T} \)-varieties. Given a cone \( \sigma \subseteq N \) we denote by \( M(\sigma) := \sigma \cap M \). We recall from \[ \text{Proposition 12.3.1} \] that \( H^k(X(\sigma)) \cong \wedge^k M(\sigma) \) for each \( k \). Then, given a \( p \)-divisor \( D := \sum_{i=1}^r \Delta_i \oplus p_i \) on a semi-projective curve \( Y \). Observe that for each \( i \) we have isomorphisms \( M(\sigma) := M(\Delta_i) \oplus M_{\Delta_i} \) and \( M'(\sigma_i) \cong M(\Delta_i) \), where \( M_{\Delta_i} := (N_{\Delta_i})^\perp \) for each \( i \).
We will use the following notation
\[ M_i^D(1) := M_{\Delta_{i+1}} \cap \langle M(\Delta_1), \ldots, M(\Delta_i) \rangle, \]
\[ M_i^D(2) := M_{\Delta_{i+1}} \cap M(\Delta_1) \cap \cdots \cap M(\Delta_i), \]
\[ M_i^D(3) := M(\Delta_{i+1}) \cap \langle M(\Delta_1), \ldots, M(\Delta_i) \rangle, \]
\[ M_i^D(4) := M(\Delta_1) \cap \cdots \cap M(\Delta_{i+1}). \]
Observe that the subspaces \( M_i^D(1) \), \( M_i^D(2) \), \( M_i^D(3) \) and \( M_i^D(4) \) are pairwise disjoint and generate \( M(\sigma) \). To abbreviate notation, given natural numbers \( a_1, a_2, a_3, a_4 \) we will write
\[ \wedge^{a_1, a_2, a_3, a_4} H_k(M^D) := \wedge^{a_1} M_i^D(1) \otimes_{\mathbb{Z}} \wedge^{a_2} M_i^D(2) \otimes_{\mathbb{Z}} \wedge^{a_3} M_i^D(3) \otimes_{\mathbb{Z}} \wedge^{a_4} M_i^D(4), \]
and
\[ H^k(M^D) := \bigoplus_{i=1}^{r-1} \left( \bigoplus_{(a_1, a_2, a_3, a_4) \in I_{k-1}} \wedge^{a_1, a_2, a_3, a_4} M_i^D \right). \]

**Proposition 4.3.** Let \( \mathcal{D} := \sum_{i=1}^{r} \Delta_{p_i} \otimes_{\mathbb{Z}} 1_{p_i} \) be a \( p \)-divisor on a curve \( Y \). Then the following statements hold.

- If \( \mathcal{D} \) have affine locus then \( H^k(\bar{X}(\mathcal{D})) \) is isomorphic to \( \bigwedge^{k,M} M^D(4) \oplus \left( \bigwedge^{k-1} M(\sigma) \otimes_{\mathbb{Z}} H^1(Y) \right) \oplus H^k(M^D) \).
- If \( \mathcal{D} \) have complete locus then \( H^k(\bar{X}(\mathcal{D})) \) is isomorphic to \( H^k(X(\sigma) \times Y) \oplus H^k(X(\sigma)) \).

**Lemma 4.4.** Let \( M \) be a free finitely generated \( \mathbb{Z} \)-module, \( K_1, \ldots, K_r \) be submodules and \( K' := K_1 \cap \cdots \cap K_r \). Then the cokernel of the homomorphism
\[ \bigoplus_{i=1}^{r} K_i \oplus M \to M', \]
given by \((k_1, \ldots, k_r, m) \mapsto (m - k_1, \ldots, m - k_r)\) is isomorphic to \( \bigoplus_{i=1}^{r-1} M/\langle K_i^i, K_{i+1} \rangle \).

**Proof of Proposition 4.3.** We proof the first assertion. Recall that \( M'(\sigma) \simeq M(\sigma) \oplus \mathbb{Z} \), then for each \( k \) we have
\[ \bigwedge^{k,M'} M(\sigma) \simeq \bigwedge^{k,M} M(\sigma) \oplus \bigwedge^{k-1,M} M(\sigma). \]
Using [7, Proposition 12.3.1] we see that
\[ H^k(U \cap U_D) \simeq \bigoplus_{i=1}^{r} \bigwedge^{k,M'} M(\sigma) \simeq \bigwedge^{k,M} M(\sigma) \oplus \bigwedge^{k-1,M} M(\sigma) \oplus \mathbb{Z}^r \oplus \bigwedge^{k-1,M(\Delta_i)} \bigwedge^{k,M} M(\Delta_i), \]
\[ H^k(U) \simeq \bigoplus_{i=1}^{r} \bigwedge^{k,M} M(\Delta_i), \]
\[ H^k(U_D) \simeq \bigwedge^{k,M(\sigma)} \oplus \bigwedge^{k-1,M(\sigma)} \oplus \mathbb{Z}(H^1(Y) \oplus \mathbb{Z}^r). \]
In order to compute \( H^k(X(\mathcal{D})) \) we will study the homomorphism
\[ i^k: H^k(U) \oplus H^k(U_D) \to H^k(U \cap U_D). \]
We denote by $i^{k,1}$ and $i^{k,2}$ the homomorphism $i^k$ restricted to $H^k(U)$ and $H^k(U_D)$ respectively. Then we can write

$$i^{k,1}: \bigoplus_{i=1}^r \wedge^k M(\Delta_i) \to \bigoplus_{i=1}^r (\wedge^k M(\sigma) \oplus \wedge^{k-1} M(\sigma)),$$

$$i^{k,2}: \wedge^k M(\sigma) \oplus \wedge^{k-1} M(\sigma) \otimes \mathbb{Z}(H^1(Y) \oplus \mathbb{Z}^r) \to \wedge^k M(\sigma) \otimes \mathbb{Z}^r \oplus \wedge^{k-1} M(\sigma) \otimes \mathbb{Z}^r,$$

where $i^{k,1}$ is the $k$-th wedge product of the inclusion $M(\Delta_i) \to M(\sigma)$ on each component and $i^{k,2}$ is the homomorphism induced by $\mathbb{Z} \to \mathbb{Z}^r, z \mapsto (\ldots, z)$ and the projection $H^1(Y) \oplus \mathbb{Z}^r \to \mathbb{Z}^r$.

Then the kernel of $i^k$ is isomorphic to $\wedge^{k-1} M(\sigma) \otimes \mathbb{Z} H^1(Y)$ direct sum with the kernel of the homomorphism

$$\bigoplus_{i=1}^r \wedge^k M(\sigma) \oplus \wedge^k M(\sigma) \to (\wedge^k M(\sigma))^r,$$

which is isomorphic to $(\wedge^k(M(\Delta_1) \cap \cdots \cap M(\Delta_r)))^{r-1}$. Observe that the kernel of $i^k$ is a free finitely generated abelian group. On the other hand, the cokernel of $i^k$ is the cokernel of homomorphism 4.1. Then, using Lemma 4.4 we see that the cokernel of $i^k$ is isomorphic to

$$\bigoplus_{i=1}^{r-1} \left( \bigoplus_{(a_1,a_2,a_3,a_4) \in I_{k-1}} \wedge^{a_1,a_2,a_3,a_4} M^J_i \right).$$

Then the first assertion follows.

For the second assertion observe that as $D$ have complete locus, then by [4, Lemma 18] we have that $M(\Delta_i) = M(\sigma)$ for each $i$. Then we have that $M^P_k(1) = M^P_k(2) = M^P_k(3) = 0$ and $M^P_k(4) = M(\sigma)$ for each $i$. Then the result follows from the first part.

**Remark 4.5.** Proposition 4.3 can be obtained from Proposition 4.1 and the Universal coefficient theorem for cohomology. In this case using the Universal coefficient theorem correspond to take duality $N(\sigma) \to M(\sigma)$.

Let $S$ be a divisorial fan on a curve $Y$, we will denote by $I$ an ordered set of indexes $i$ such that $D^i \in S$ is a maximal $p$-divisor with respect to the inclusion. We denote by $I^k$ the set of increasing $k$-sequences of elmenets of $I$.

**Proposition 4.6.** Let $S$ be a divisorial fan on a curve $Y$ such that for each $i \in I$, $D^i$ has affine locus and full-dimensional tailcone. Then

$$H^2(\bar{X}(S)) \simeq \ker \left( \bigoplus_{(i,j) \in I^2} M^{D^i \cap D^j}(4) \to \bigoplus_{(i,j,k) \in I^3} M^{D^i \cap D^j \cap D^k}(4) \right) \oplus H^2(\text{Loc}(S)),$$

where the homomorphism $M^{D^i \cap D^j}(4) \to M^{D^i \cap D^j \cap D^k}(4)$, is induced by the inclusion of $p$-divisors $D^i \cap D^j \cap D^k \subseteq D^i \cap D^j$.

**Proof.** Consider the affine open cover of $X(S)$ given by $U := \{ X(D^i) \mid i \in I \}$. We have an spectral sequence of the covering $U$

$$E_1^{p,q} = \bigoplus_{\gamma=(i_0,\ldots,i_p) \in I^{p+1}} H^q(X(D^{i_0}) \cap \cdots \cap X(D^{i_p})) \Rightarrow H^{p+q}(X(S)).$$
Observe that
\[ E^{p,0}_1 = \bigoplus_{\gamma = (i_0, \ldots, i_p) \in I^{p+1}} \mathbb{Z}. \]
Thus, \( E^{p,0}_1 = 0 \) for \( p > 0 \) and \( E^{0,0}_2 = \mathbb{Z} \). The maximal cones of \( \Sigma(S) \) are full-dimensional, then by Proposition 4.3 we have that \( E^{1,q}_1 = \bigoplus_{i \in I} H^q(X(D^i)) = 0 \), for all \( q > 1 \) and \( E^{0,1}_1 = \bigoplus_{i \in I} H^1(\text{Loc}(D^i)). \) It follows that \( E^{0,q}_2 = 0 \) for all \( q > 1 \). Then \( E^{2,0}_2 \) and \( E^{0,2}_1 \) are zero for all \( r \geq 2 \). Moreover, the differentials into and out of \( E^{1,1}_1 \) are zero for all \( r \geq 2 \). Thus, \( E^{2,1}_1 = E^{1,1}_1 \simeq H^2(X(S)). \) Then, we can compute \( H^2(X(S)) \) from the complex
\[ E^{0,1}_1 \rightarrow E^{1,1}_1 \rightarrow E^{2,1}_1. \]
By Proposition 4.1 we have that \( H^2(X(S)) \) is the homology of the following complex
\[ \bigoplus_{i \in I} H^1(\text{Loc}(D^i)) \rightarrow \bigoplus_{(i,j) \in I^2} M^{D^i \cap D^j}(4) \oplus H^1(\text{Loc}(D^i \cap D^j)) \]
\[ \rightarrow \bigoplus_{(i,j,k) \in I^3} M^{D^i \cap D^j \cap D^k}(4) \oplus H^1(\text{Loc}(D^i \cap D^j \cap D^k)). \]
Observe that this group is isomorphic to
\[ \ker \left( \bigoplus_{(i,j) \in I^2} M^{D^i \cap D^j}(4) \rightarrow \bigoplus_{(i,j,k) \in I^3} M^{D^i \cap D^j \cap D^k}(4) \right), \]
direct sum with the homology of the complex
\[ \bigoplus_{i \in I} H^1(\text{Loc}(D^i)) \rightarrow \bigoplus_{(i,j) \in I^2} H^1(\text{Loc}(D^i \cap D^j)) \rightarrow \bigoplus_{(i,j,k) \in I^3} H^1(\text{Loc}(D^i \cap D^j \cap D^k)), \]
which is isomorphic to \( H^2(\text{Loc}(S)) \). \( \Box \)

In the following example we use Theorem 1 and Proposition 4.1 to compute the cohomology groups of a particular affine \( \mathbb{T} \)-variety.

**Example 4.7.** Consider a divisorial fan \( S \) on \((Y, \mathbb{Z}^2)\) with \( Y \) a curve. Let \( \{y_1, y_2\} \subseteq Y \) be the support of \( S \). Assume that the fan of the generic fiber is generated by \( e_1 \) and the slices \( S_1 \) and \( S_2 \) over \( y_1 \) and \( y_2 \) respectively corresponds to the polyhedra
\[ \Delta_1 := \langle 0, 0, 0, 0, 1, 0 \rangle + e_1, \quad \Delta_2 := \langle 0, 0, 0, 0, 0, 1 \rangle + e_1. \]
Using Theorem 1 we observe that \( \pi_1(\tilde{X}(S)) \simeq \pi_1(Y) \), then \( H_1(\tilde{X}(S)) \simeq H_1(Y) \). Moreover, using Proposition 4.1 we can compute
\[ N^P_1(1) = \{0\}, \quad N^P_1(2) = \langle e_2 \rangle, \quad N^P_1(3) = \langle e_3 \rangle, \quad N^P_1(4) = 0, \]
\[ H_3(N^P) \simeq \mathbb{Z}, \quad \text{and 0 otherwise}. \]
Then, we conclude that
\[ H_0(\tilde{X}(S)) \simeq \mathbb{Z}, \quad H_1(\tilde{X}(S)) \simeq H_1(Y), \quad H_2(\tilde{X}(S)) \simeq H_1(Y) \otimes \mathbb{Z} \mathbb{Z}^2, \]
\[ H_3(\tilde{X}(S)) \simeq H_1(Y) \oplus \mathbb{Z}, \quad H_k(\tilde{X}(S)) \simeq 0, \quad \text{for } k \geq 4. \]
References

[1] Marta Agustin Vicente and Kevin Langlois, *Intersection cohomology for rational projective contraction-free T-varieties of complexity one*, 2014. arXiv:1412.7634v2 [math.AG], 5p.

[2] Klaus Altmann and Jürgen Hausen, *Polyhedral divisors and algebraic torus actions*, Math. Ann. 334 (2006), no. 3, 557–607. ↑1, 5

[3] Klaus Altmann, Jürgen Hausen, and Hendrik Süß, *Gluing affine torus actions via divisorial fans*, Transform. Groups 13 (2008), no. 2, 215–242. ↑1

[4] Klaus Altmann, Nathan Owen Itoen, Lars Petersen, Hendrik Süß, and Robert Vollmert, *The geometry of T-varieties*, Contributions to algebraic geometry, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2012, pp. 17–69. ↑16, 19, 21

[5] Anders Björner, Michel Las Vergnas, Bernd Sturmfels, Neil White, and Günter M. Ziegler, *Oriented matroids*, 2nd ed., Encyclopedia of Mathematics and its Applications, vol. 46, Cambridge University Press, Cambridge, 1999. ↑11

[6] Neil Chriss and Victor Ginzburg, *Representation theory and complex geometry*, Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, MA, 2010. Reprint of the 1997 edition. ↑13

[7] John B. Little and Henry K. Schenck David A. Cox, *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011. ↑1, 5, 6, 19, 20

[8] Michel Demazure, *Sous-groupes algébriques de rang maximum du groupe de Cremona*, Ann. Sci. École Norm. Sup. (4) 3 (1970), 507–588 (French). ↑1

[9] Hubert Flenner and Mikhail Zaidenberg, *Normal affine surfaces with C∗-actions*, Osaka J. Math. 40 (2003), no. 4, 981–1009. ↑1

[10] William Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993. The William H. Roever Lectures in Geometry. ↑1, 10, 11, 12, 16

[11] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002. ↑7

[12] Nathan Owen and Vollmert Robert Ilten, *Upgrading and downgrading torus actions*, J. Pure Appl. Algebra 217 (2013), no. 9, 1583–1604. ↑4

[13] G. Kempf, Finn Faye Knudsen, D. Mumford, and B. Saint-Donat, *Toroidal embeddings. I*, Lecture Notes in Mathematics, Vol. 339, Springer-Verlag, Berlin-New York, 1973. ↑1

[14] Alvaro Liendo and Hendrik Süß, *Normal singularities with torus actions*, Tohoku Math. J. (2) 65 (2013), no. 1, 105–130. ↑4, 9, 10

[15] Madhav V. Nori, *Zariski’s conjecture and related problems*, Ann. Sci. École Norm. Sup. (4) 16 (1983), no. 2, 305–344. ↑5

[16] Shigeharu Takayama, *Local simple connectedness of resolutions of log-terminal singularities*, Internat. J. Math. 14 (2003), no. 8, 825–836. ↑10

[17] Dmitri A. Timashev, *Torus actions of complexity one*, Torus topology, 2008, pp. 349–364. ↑1

Departamento de Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile
E-mail address: alface@udec.cl

Instituto de Matemática y Física, Universidad de Talca, Casilla 721, Talca, Chile
E-mail address: aliendo@inst-mat.utalca.cl

Departamento de Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile
E-mail address: joamoraga@udec.cl