ON THE WAVE EQUATION WITH QUADRATIC NONLINEARITIES IN THREE SPACE DIMENSIONS

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Abstract. The Cauchy problem for the nonlinear wave equation
\[ \Box u = (\partial^2_t - \Delta)u = B_k(u, u), \quad u(0) = u_0, \quad u_t(0) = u_1 \]
in three space dimensions is considered. The data \((u_0, u_1)\) are assumed to belong to \(\dot{H}^s_1(\mathbb{R}^3) \times \dot{H}^s_{-1}(\mathbb{R}^3)\), where \(\dot{H}^s_1\) is defined by the norm
\[ \|f\|_{\dot{H}^s_1} := \|\langle \xi \rangle^s f\|_{L^r_x \cap L^{r'}_x}, \quad \langle \xi \rangle = (1 + |\xi|^2)^{1/2}, \quad \frac{1}{r} + \frac{1}{r'} = 1. \]
Local well-posedness is shown in the parameter range \(2 \geq r > 1, s > 1 + \frac{2}{r}\).
For \(r = 2\) this coincides with the result of Ponce and Sideris, which is optimal on the \(H^s\)-scale by Lindblad’s counterexamples, but nonetheless leaves a gap of \(\frac{1}{2}\) derivative to the scaling prediction. This gap is closed here except for the endpoint case. Corresponding results for \(\Box u = \partial u^2\) are obtained, too.

1. Introduction and Main Results

In this note we consider the Cauchy problem
\[ (1) \quad \Box u = (\partial_t^2 - \Delta)u = B_k(u, u), \quad u(0) = u_0, \quad u_t(0) = u_1 \]
for the nonlinear wave equation in \(\mathbb{R}^3\), where the right hand side is given by
\[ B_1(u, v) = \partial(uv) \quad \text{or} \quad B_2(u, v) = \partial u \partial v \]
with \(\partial \in \{\partial_t, \partial_{x_1}, \partial_{x_2}, \partial_{x_3}\}\), and no special structure of the bilinear forms \(B_k\), \(k \in \{1, 2\}\), such as a null-structure is assumed. Concerning the local well-posedness (LWP) of this problem with data \((u_0, u_1) \in H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3)\) the following is known. For \(s > k + \frac{1}{r}\) energy estimates can be applied to obtain an affirmative result. Ponce and Sideris showed in [15] how to improve this down to \(s > k\) by using Strichartz inequalities. Further progress is possible, if the nonlinearity satisfies a null-condition such as
\[ \tilde{B}_2(u, v) = \langle \nabla_x u, \nabla_x v \rangle - \partial_t u \partial_t v, \]
see the work of Klainerman and Machedon [8], [9], [10], who used wave Sobolev spaces to exploit the null-structure of the bilinear terms, thus reaching LWP for \(s > s_c(k) = k - \frac{1}{2}\), which is here the critical Sobolev regularity by scaling considerations. If no such structure is present in the quadratic term, one has in fact ill-posedness of the Cauchy problem \((1)\) for \(s \leq k\), as the sharp counterexamples of Lindblad show, see [12], [13], [14]. So in general there is a gap of half a derivative between the optimal LWP result on the \(H^s\)-scale and the scaling prediction.

For several important nonlinear dispersive equations in one space dimension - such as cubic NLS and DNLS, KdV, mKdV and its higher order generalizations -

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1
there is a similar gap between the best possible LWP result in $H^s$ and the critical regularity. In the case of cubic nonlinearities this can be closed almost completely by considering data in the spaces $\tilde{H}^r$, defined by the norm
$$\|f\|_{\tilde{H}^r} := \|\xi f\|_{L_t^r}, \quad \langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{r}}, \quad \frac{1}{r} + \frac{1}{r'} = 1,$$
see \cite{3, 4, 7, 5}; for an application in the periodic setting cf. \cite{6}. The purpose of this note is to show that the methods developed in the one-dimensional framework also apply to the nonlinear wave equation \cite{1, 2} in three space dimensions and give LWP for data $(u_0, u_1) \in \tilde{H}^s(\mathbb{R}^3) \times \tilde{H}^{s-1}(\mathbb{R}^3)$, provided $1 < r \leq 2$ and $s > s_k(r) := k + 1 - \frac{3}{2r}$. In the limit $r \to 1$ we can almost reach the space $\tilde{H}^s_{k+1}(\mathbb{R}^3) \times \tilde{H}^s_k(\mathbb{R}^3)$, which is critical by scaling. To prove this result we use an appropriate variant of Bourgain’s Fourier restriction norm method, see \cite{3} Section 2], and estimates for products of two free solutions of half-wave equations. The latter are very much in the spirit of the work of Foschi and Klainerman \cite{11} and can be seen as bilinear substitutes and refinements of the Strichartz inequalities for the three-dimensional wave equation.

2. General arguments, function spaces, and precise statement of results

Following \cite{2} Section 2] we first rewrite \cite{1} as the first order system
\begin{equation}
(i \partial_t + J_x) u_{\pm} = \mp \frac{1}{4} J_x^{-1} B_\delta (u_+ + u_-) \mp \frac{1}{2} J_x^{-1} (u_+ + u_-),
\end{equation}
where $J_x = (1 - \Delta_x)^{\frac{1}{2}}$ is the Bessel potential operator of order $-1$ in the space variable $x$, and $u_{\pm} = u \pm i J_x^{-1} \partial_t u$ so that the initial conditions become
\begin{equation}
u_{\pm} (0) = u_0 \pm i J_x^{-1} u_1 := f_{\pm} \in \tilde{H}^s(\mathbb{R}^3).
\end{equation}
To treat the system \cite{3} with data given by \cite{1} we will use the function spaces $X^r_{s,b}$ defined by their norm
$$\|u\|_{X^r_{s,b}} := \left( \int d\xi d\tau \langle \xi \rangle^{sr} \langle \tau \pm \xi \rangle^{br} |\mathcal{F}u(\xi, \tau)|^r \right)^{\frac{1}{r}}.$$
For $s = b = 0$ we write $\tilde{L}^r_{xt} := X^{r,+}_{0,0} = X^{r,-}_{0,0}$, correspondingly we set $\tilde{H}^r_\delta := \tilde{H}^r_0$. Local solutions are obtained by the contraction mapping principle in the time restriction space
$$X^{r,\pm}_{s,b}(\delta) := \{ u = \tilde{u} |_{[-\delta, \delta] \times \mathbb{R}^3} : \tilde{u} \in X^{r,\pm}_{s,b} \}$$
endowed with the norm
$$\|u\|_{X^{r,\pm}_{s,b}(\delta)} := \inf \{ \|\tilde{u}\|_{X^{r,\pm}_{s,b}} : \tilde{u} |_{[-\delta, \delta] \times \mathbb{R}^3} = u \}.$$
We will always have $b > \frac{1}{2}$, hence $X^{r,\pm}_{s,b} \subset C(\mathbb{R}, \tilde{H}^r_{\delta})$ and $X^{r,\pm}_{s,b}(\delta) \subset C([-\delta, \delta], \tilde{H}^r_{\delta})$, which gives the persistence property of our solutions. To deal with $B_2$ - especially if time derivatives are involved - we also need the norms
$$\|u\|_{X^{r,\pm}_{s,b}} := \|u\|_{X^{r,\pm}_{s,b}} + \|\partial_t u\|_{X^{r,\pm}_{s-1,b}};$$
the corresponding restriction spaces are defined precisely as above. Now our result concerning $B_1$ reads as follows.

**Theorem 1.** Let $1 < r \leq 2$, $s > \frac{3}{r}$, $\frac{1}{r} < b < 1$ and $f_{\pm} \in \tilde{H}^r_{\delta}$. Then there exist $\delta = \delta(||f_+\|_{\tilde{H}^r_{\delta}}, ||f_-\|_{\tilde{H}^r_{\delta}}) > 0$ and a unique solution $(u_+, u_-) \in X^{r,\pm}_{s,b}(\delta) \times X^{r,\pm}_{s,b}(\delta)$ of
Theorem 2. Let $i\partial_x \pm \sigma_b$ be locally Lipschitz continuous. The solution is persistent and the flow map

$$(f_+, f_-) \mapsto (u_+, u_-), \quad \tilde{H}^r_s \times \tilde{H}^r_s \to X^{r,\pm}_{s,b}(\delta) \times X^{r,\mp}_{s,b}(\delta)$$

is locally Lipschitz continuous.

Similarly we can show the following for $B_2$.

Theorem 2. Let $1 < r \leq 2$, $s > \frac{2}{r} + 1$, $\frac{1}{2} < b < 1$ and $f_{\pm} \in \tilde{H}^r_s$. Then there exist $\delta = \delta(\|f_+\|_{\tilde{H}^r_s}, \|f_-\|_{\tilde{H}^r_s}) > 0$ and a unique solution $(u_+, u_-) \in Z^{r,\pm}_{s,b}(\delta) \times Z^{r,\mp}_{s,b}(\delta)$ of (3) with $k = 2$ satisfying the initial condition (4). The solution is persistent and the flow map

$$(f_+, f_-) \mapsto (u_+, u_-), \quad \tilde{H}^r_s \times \tilde{H}^r_s \to Z^{r,\pm}_{s,b}(\delta) \times Z^{r,\mp}_{s,b}(\delta)$$

is locally Lipschitz continuous.

The general LWP theorem (3) Theorem 2.3] reduces the proofs of Theorem 1 and 2 to that of bilinear estimates in $X^{r,\pm}_{s,b}$-norms. The next section is devoted to the proof of the following key estimate.

Theorem 3. Let $1 < r \leq 2$, $b > \frac{1}{2}$, and $\sigma > \frac{r}{2}$. Then

$$(5) \quad \|J_x^\sigma (uv)\|_{\tilde{L}^r_t} + \|J_x^{\sigma-1} \partial_t (uv)\|_{\tilde{L}^r_t} \lesssim \|u\|_{X^{r,\pm}_{s,b}} \|v\|_{X^{r,\mp}_{s,b}},$$

where $[\pm]$ denotes independent signs.

Assume (3) already to be proven. Concerning $B_1$ we then have that for all $b, r$ and $s = \sigma$ according to the assumptions of Theorem (3) and $b' \leq 0$

$$\|J_x^{\sigma-1} \partial_t (uv)\|_{X^{r,\pm}_{s,b'}} \leq \|J_x^{\sigma-1} \partial_t (uv)\|_{\tilde{L}^r_t}$$

$$\lesssim \|J_x^\sigma (uv)\|_{\tilde{L}^r_t} + \|J_x^{\sigma-1} \partial_t (uv)\|_{\tilde{L}^r_t} \lesssim \|u\|_{X^{r,\pm}_{s,b}} \|v\|_{X^{r,\mp}_{s,b}},$$

which combined with (3) Theorem 2.3] leads to Theorem 1 since the linear term on the right of (3) can be trivially taken care of. Similarly for $B_2$ we have with $s = \sigma + 1 > 1 + \frac{1}{2}$ and $r, b, b'$ as before

$$\|J_x^{\sigma-1} (\partial u \partial v)\|_{Z^{r,\pm}_{s,b'}} \leq \|J_x^\sigma (\partial u \partial v)\|_{\tilde{L}^r_t} + \|J_x^{\sigma-1} \partial_t (\partial u \partial v)\|_{\tilde{L}^r_t}$$

$$\lesssim \|\partial u\|_{X^{r,\pm}_{s,b}} \|\partial v\|_{X^{r,\pm}_{s,b}} \lesssim \|u\|_{Z^{r,\pm}_{s,b}} \|v\|_{Z^{r,\pm}_{s,b}},$$

which is sufficient for Theorem 2.

3. Proof of the Key Estimate

Theorem (3) will be a consequence of several bilinear estimates for free solutions of the half-wave equations $(i\partial_t \pm D_x)u = 0$, subject to the initial condition $u(0) = u_0$. So for the remaining part of the paper let $u_{\pm}(t) = e^{\pm itD_x}u_0 = f_x^{-1}e^{\pm it|\xi|}f_x u_0$ and $v_{\pm}(t) = e^{\pm itD_x}v_0$. By the transfer principle - see e. g. [11] Proposition 3.5 or [3] Lemma 2.1] - the proof of (5) essentially reduces to showing that

$$(6) \quad \|J_x^{\sigma-1} \partial_x (u_{\pm} v_{\mp}|_{\pm})\|_{\tilde{L}^r_t} + \|J_x^{\sigma-1} \partial_t (u_{\pm} v_{\mp}|_{\pm})\|_{\tilde{L}^r_t} \lesssim \|u_0\|_{\tilde{H}^r_s} \|v_0\|_{\tilde{H}^r_s}.$$
To prove (3) we make substantial use of the calculations in [1]. By symmetry it suffices to consider the (++)- and (+-)-cases. For both we calculate the space-time Fourier transform of the product. Defining $P_{\pm}(\eta) := \frac{1}{2} - \eta \pm \frac{1}{2} \eta$ with $\nabla P_{\pm}(\eta) = \frac{x}{|x|^2} \pm \frac{x}{|x|^2}$ and using the properties of the $\delta$-distribution we obtain

$$\mathcal{F} u_+ v_+ (\xi, \tau) = c \int_{P_+(\eta) = \tau} \frac{dS_\eta}{\nabla P_+(\eta)} \hat{u}_0(\frac{\xi}{2} - \eta) \hat{v}_0(\frac{\xi}{2} + \eta),$$

for more details see [1] Sections 3 and 4. Observe that the set $\{P_+(\eta) = \tau\}$ ($\{P_-(\eta) = \tau\}$) is an ellipsoid (hyperboloid) of rotation, so the (++)-case ((+-)-case) is henceforth referred to as elliptic (hyperbolic).

3.1. The elliptic case. We choose $0 < s_1, s_2 < \frac{2}{p}$ with $s_1 + s_2 = \frac{2}{p}$ and use Hölder’s inequality to get

$$|\mathcal{F} u_+ v_+ (\xi, \tau)| \lesssim \left( \int_{P_+(\eta) = \tau} \frac{dS_\eta}{\nabla P_+(\eta)} |\frac{\xi}{2} - \eta|^{-s_1} |\frac{\xi}{2} + \eta|^{-s_2} \right)^{\frac{1}{2}} \times \left( \int_{P_+(\eta) = \tau} \frac{dS_\eta}{\nabla P_+(\eta)} |J^\tau_1 u_0(\frac{\xi}{2} - \eta) J^\tau_2 v_0(\frac{\xi}{2} + \eta)|^{r'} \right)^{\frac{1}{r'}}.$$

For the first factor we apply [1] Lemma 4.1 to see that

$$\int_{P_+(\eta) = \tau} \frac{dS_\eta}{\nabla P_+(\eta)} |\frac{\xi}{2} - \eta|^{-s_1} |\frac{\xi}{2} + \eta|^{-s_2} \tau$$

$$= c \int_{-1}^{1} |\tau + |\xi|^{1-s_1} \tau + |\xi|^{1-s_2} \tau|\tau + |\xi|^{1-s_2} \tau| \tau + |\xi|^{1-s_2} \tau|dx$$

$$= c \int_{-1}^{1} \left| \frac{\xi}{|\xi|} + |\xi|^{1-s_1} \frac{\xi}{|\xi|} + |\xi|^{1-s_2} \frac{\xi}{|\xi|} \right| \leq c_{s_1, s_2}.$$

Taking the $L^{r'}_{\xi, \tau}$-norm of the second factor and using the coarea formula we arrive at

$$\|u_+ v_+\|_{L^{r'}_{\xi, \tau}} \lesssim \|J^\tau_1 u_0\|_{L^r_{\xi}} \|J^\tau_2 v_0\|_{L^r_{\xi}}.$$

Unfortunately this argument breaks down, if $s_1 = 0$ or $s_2 = 0$, cf. the necessary condition (9) in [1]. To overcome this difficulty we split $u_+ v_+ = P_\geq (u_+ v_+) + P_\leq (u_+ v_+)$, where

$$\mathcal{F} x P_\geq (f, g)(\xi) := \int_{\xi_1 + \xi_2 = \xi} \hat{f}(\xi_1) \hat{g}(\xi_2) \chi_{\{|\xi_1| \gtrsim |\xi_2|\}} d\xi_1.$$

By the proceeding we have

$$(7) \quad \| P_\geq (u_+ v_+) \|_{L^{r'}_{\xi, \tau}} \lesssim \| J^\tau_3 u_0 \|_{L^r_{\xi}} \| J^\tau_4 v_0 \|_{L^r_{\xi}}.$$ 

To estimate $P_\leq (u_+ v_+)$ we decompose $u_0$ dyadically into $u_0 = \sum_{k \geq 0} P_{\Delta k} u_0$ with $P_{\Delta 0} = F_x^{-1} \chi_{\{|\xi| \leq 1\}} F_x$ and, for $k \geq 1$, $P_{\Delta k} = F_x^{-1} \chi_{\{|\xi| \sim 2^k \}} F_x$, so that

$$\| P_\leq (u_+ v_+) \|_{L^{r'}_{\xi, \tau}} \lesssim \sum_{k \geq 0} \| P_\leq (P_{\Delta k} u_+, v_+) \|_{L^{r'}_{\xi, \tau}}.$$ 

Now by [1] Lemma 12.2 we have

$$(8) \quad \int_{P_+(\eta) = \tau} \frac{dS_\eta}{\nabla P_+(\eta)} |\chi_{\{|\xi| \sim 2^k \}} \eta| \lesssim 2^{2k},$$

hence a Hölder application as above gives

$$\| P_\leq (P_{\Delta k} u_+, v_+) \|_{L^{r'}_{\xi, \tau}} \lesssim 2^{2k} \| P_{\Delta k} u_0 \|_{L^r_{\xi}} \| v_0 \|_{L^r_{\xi}}.$$
Summing up the dyadic pieces and combining the result with (7) we obtain for \( \sigma > \frac{2}{r} \)

\[
\| u_+ v_+ \|_{L^2_x} \lesssim \| J^s_x u_0 \|_{L^2_x} \| v_0 \|_{L^2_x}.
\]

The convolution constraint \( \xi = \xi_1 + \xi_2 = (\frac{\xi}{2} - \eta) + (\frac{\xi}{2} + \eta) \) implies \( \langle \xi \rangle^\sigma \lesssim \langle \xi_1 \rangle^\sigma + \langle \xi_2 \rangle^\sigma = (\frac{\xi}{2} - \eta)^\sigma + (\frac{\xi}{2} + \eta)^\sigma \), and we may exchange \( u_0 \) and \( v_0 \) in (9).

This gives

\[
\| J^s_x (u_+ v_+) \|_{L^2_x} \lesssim \| J^s_x u_0 \|_{L^2_x} \| J^s_x v_0 \|_{L^2_x},
\]

provided \( \sigma > \frac{2}{r} \). In (10) we may clearly replace \( J^s_x (u_+ v_+) \) by \( J^s_x \partial_t (u_+ v_+) \), since in the support of \( F(u_+ v_+) \) we have \( \tau = |\frac{\xi}{2} - \eta| + |\frac{\xi}{2} + \eta| \) and hence \( \langle \xi \rangle^\sigma |\tau| \leq (\frac{\xi}{2} - \eta)^\sigma + (\frac{\xi}{2} + \eta)^\sigma \). Thus we have shown:

**Lemma 1.** Let \( 1 \leq r \leq 2 \) and \( \sigma > \frac{2}{r} \). Then

\[
\| J^s_x (u_+ v_+) \|_{L^2_x} + \| J^s_x \partial_t (u_+ v_+) \|_{L^2_x} \lesssim \| u_0 \|_{\dot{H}^2_x} \| v_0 \|_{\dot{H}^2_x}.
\]

### 3.2. The hyperbolic case.

The estimation in this case goes along similar lines as in section 3.1, as long as

\[
|\frac{\xi}{2} - \eta| + |\frac{\xi}{2} + \eta| \leq c_1 |\xi|.
\]

If (11) is fulfilled, we choose again \( s_1, s_2 \in (0, \frac{2}{r}) \) with \( s_1 + s_2 = \frac{2}{r} \) and obtain from [1] Lemma 4.4] that

\[
\int_{P_{\gamma}(\eta) = \tau} \frac{dS_\gamma}{|\nabla P_{\gamma}(\eta)|} |\frac{\xi}{2} - \eta|^{-s_1 \tau} |\frac{\xi}{2} + \eta|^{-s_2 \tau} \leq c_{s_1, s_2},
\]

which gives

\[
\| u_+ v_- \|_{L^2_x} \lesssim \| J^s_x u_0 \|_{L^2_x} \| J^s_x v_0 \|_{L^2_x},
\]

and hence

\[
\| P_{\gamma}(u_+, v_-) \|_{L^2_x} \lesssim \| J^s_x u_0 \|_{L^2_x} \| v_0 \|_{L^2_x}.
\]

A dyadic decomposition together with (8) shows that

\[
\| P_{\leq}(\Delta_k u_+ v_-) \|_{L^2_x} \lesssim 2^k \| P_{\leq} u_0 \|_{L^2_x} \| v_0 \|_{L^2_x},
\]

and combining (12) and (13) after summation in \( k \) we arrive at

\[
\| u_+ v_- \|_{L^2_x} \lesssim \| J^s_x u_0 \|_{L^2_x} \| v_0 \|_{L^2_x},
\]

provided \( 1 \leq r \leq 2 \), \( \sigma > \frac{2}{r} \), and \( u_+ v_- \) fulfills assumption (11). To fix a partial result concerning the hyperbolic case, let \( P(u, v) \) denote the projection on the domain in Fourier space, where (11) holds. Then, taking into account the arguments at the end of Section 3.1, we have the following estimate.

**Lemma 2.** Let \( 1 \leq r \leq 2 \) and \( \sigma > \frac{2}{r} \). Then

\[
\| J^s_x P(u_+, v_-) \|_{L^2_x} + \| J^s_x \partial_t P(u_+, v_-) \|_{L^2_x} \lesssim \| u_0 \|_{\dot{H}^2_x} \| v_0 \|_{\dot{H}^2_x}.
\]

We turn to the region, where

\[
c_1 |\xi| \leq |\frac{\xi}{2} - \eta| + |\frac{\xi}{2} + \eta|.
\]
Lemma 3. Let $D_r$ for all $s_1, s_2 \geq 0$ and $s_1 + s_2 = 3/2 + \varepsilon$. This gives

$$\int_{P_-(\eta)=\tau} \frac{dS_\eta}{|\nabla P_-(\eta)|} |\eta|^{-s_1 r} |\xi|^{-s_2 r} \chi(\mathbb{E}) = c_1 \int_0^\infty |\tau| + |\xi| |x|^{-s_1 r} |\tau - |\xi| |x|^{-s_2 r} dx = c(|\xi|^{2-(s_1 + s_2)r} \int_0^\infty |\tau| + |x|^{-s_1 r} |\tau - |x|^{-s_2 r} dx \lesssim |\xi|^{2-(s_1 + s_2)r},$$

which in turn implies

$$(15) \quad \|D_x^{s_1 + s_2 - \frac{5}{2}} Q(u_+, v_-)\|_\mathcal{E}_t \lesssim \|J_x^{s_1} u_0\|_\mathcal{E}_t \|J_x^{s_2} v_0\|_\mathcal{E}_t.$$  

Bilinear interpolation of (15) with $r = 1$ and

$$\|u_+ v_-\|_{L_x^r} \lesssim \|J_x^{s_1} u_0\|_{L_x^r} \|J_x^{s_2} v_0\|_{L_x^r}, \quad (s_{1,2} \geq 0, s_1 + s_2 > 1),$$

which follows from Strichartz estimate, gives the sharpened version

$$\|D_x^s Q(u_+, v_-)\|_\mathcal{E}_t \lesssim \|J_x^{s_1} u_0\|_{L_x^r} \|J_x^{s_2} v_0\|_{L_x^r},$$

where $1 \leq r \leq 2$, $s = (1 - \frac{5}{4})(1 + \varepsilon)$, $s_{1,2} \geq 0$ with $s_1 + s_2 = 3 - \frac{5}{4} + \varepsilon$ and $\varepsilon > 0$. If in addition $r > 1$ and $\varepsilon$ is sufficiently small, so that $s \leq 1$, we may replace the $D_x^s$ by $J_x^{s-1} \partial_x$ and hence by $J_x^{s-1} \partial_x$. This gives

$$\|J_x^s \partial_x Q(u_+, v_-)\|_\mathcal{E}_t \lesssim \|J_x^{s_1} u_0\|_{L_x^r} \|J_x^{s_2} v_0\|_{L_x^r}$$

for all $r \in (1, 2]$ and $s_{1,2} \geq 0$ with $s_1 + s_2 > 3 - \frac{s}{2}$. Using once more $|\xi| \leq (\frac{5}{4} - \eta) + (\frac{5}{4} + \eta)$ we conclude for $\sigma > \frac{s}{2}$ that

$$\|J_x^{s-1} \partial_x Q(u_+, v_-)\|_\mathcal{E}_t \lesssim \|u_0\|_{\mathcal{H}_x^\sigma} \|v_0\|_{\mathcal{H}_x^\sigma},$$

which also holds true with $\partial_t$ instead of $\partial_x$, since we are in the hyperbolic case, where $|\tau| \leq |\xi|$. Summarizing we have:

**Lemma 3.** Let $1 < r \leq 2$ and $\sigma > \frac{s}{2}$. Then

$$\|J_x^{s-1} \partial_x Q(u_+, v_-)\|_\mathcal{E}_t + \|J_x^{s-1} \partial_t Q(u_+, v_-)\|_\mathcal{E}_t \lesssim \|u_0\|_{\mathcal{H}_x^\sigma} \|v_0\|_{\mathcal{H}_x^\sigma}.$$  

Now the crucial estimate (8) follows from the Lemmas 1, 2 and 3.

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