Relaxation of the order-parameter statistics and dynamical confinement

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Abstract – We study the relaxation of the local ferromagnetic order in the quantum Ising chain in a slant field with both longitudinal and transverse components. After preparing the system in a fully polarised state, we analyse the time evolution of the entire probability distribution function (PDF) of the magnetisation within a block of ℓ spins. We first analyse the effect of confinement on the gaussification of the PDF for large ℓ, showing that the melting of initial order is suppressed when the longitudinal field is aligned to initial magnetisation while it is sped up when it is in the opposite direction. Then we study the thermalisation dynamics. In the paramagnetic region, the PDF quickly shows thermal features. Conversely, in the ferromagnetic phase, when confinement takes place, the relaxation suffers a typical slowing down which depends on the interplay between the strength of the longitudinal field, the density of excitations, and the direction of the initial polarisation. Even when the initial magnetisation is aligned oppositely to the longitudinal field, confinement prevents thermalisation in the accessible timescale, as it is neatly bared by the PDF.

Introduction. – Isolated many-body quantum systems are fundamental theoretical and experimental laboratories to benchmark our understanding of elementary processes in nature. When brought out-of-equilibrium, they may show diametrically opposed behaviour depending on various aspects that come into play. Low-dimensionality, quantum integrability, and disorder are just a few physical features that induce specific responses of the system, and therefore affect the relaxation dynamics toward the local steady state (see the reviews [1–5]).

Recently, confinement of quantum excitations was brought to the fore as one mechanism to explain a number of unexpected non-equilibrium phenomena in condensed-matter physics [6–24]. For example, when a quantum Ising chain in a transverse field is brought out of equilibrium, excitations propagate ballistically and correlations spread accordingly, leading to the renowned light-cone effect [25]. However, with an additional longitudinal field in the ordered phase, the system experiences a completely different behaviour because excitations are confined into mesons [6].

It is worth to investigate the effects of the confinement within the genuinely quantum mechanical framework of the Probability Distribution Function (PDF), which gives a complete characterisation of the associated observable, thus going much beyond the mere averages. PDFs have been investigated in many contexts, but in spite of the large literature regarding equilibrium setups [26–42], results on non-equilibrium states are still scarce [43–49]. Actually, by analysing the PDF dynamics, we address the following questions altogether: Is there a remnant local order enhanced by the confinement of excitations? Does it depend on the explicit breaking of the $\mathbb{Z}_2$ symmetry? Can confinement affect thermalisation and ferromagnetic order in the thermodynamic limit?

The Letter is organised as follows: First we introduce the model and briefly review some of its features; we then define the observables of interest and outline the dynamical protocol. After few details on the numerical technique, we focus on the results of the PDF dynamics. Finally we draw our conclusions.

Model. – We consider the quantum Ising spin chain with transverse ($h_z$) and longitudinal ($h_x$) fields, described by the Hamiltonian

$$H = - \sum_j \sigma^z_j \sigma^z_j + h_z \sum_j \sigma^z_j - h_x \sum_j \sigma^x_j,$$  \hspace{1cm} (1)

where $\sigma^\alpha_j$ are Pauli matrices acting on the site $j$. When $h_x = 0$ the model can be mapped to non-interacting spin-
less fermions, and it is characterised by: (i) a symmetry broken phase (for \( |h_z| < 1 \)) with non-zero order parameter \( \langle \sigma_j^z \rangle \); (ii) a paramagnetic phase (for \( |h_z| > 1 \)) without magnetic order; (iii) a non-equilibrium dynamics driven by stable excitations which ballistically propagates throughout the system. Conversely, when \( h_z \neq 0 \), the model is no more integrable, and the quasi-particles may experience a different fate depending on the presence of confinement. Such effect takes place within the ferromagnetic region of the unperturbed Ising model. To understand the mechanism, let us focus on the quasi-classical regime where \( |h_z| \ll 1 \) and the magnetic fluctuations are small: here the excitations are dilute, and correspond to freely moving kinks connecting regions with opposite magnetisation \( \langle \sigma_j^z \rangle \neq 0 \). By adding a finite \( h_z \), the many-body spectrum is non-perturbatively modified: the true ground state is the one polarised along the magnetic field and the energy of a domain in the opposite direction increases linearly with its size. This originates an effective strong interaction between consecutive kinks which therefore get confined in a finite spatial region, similarly to mesons in quantum chromodynamics.

**Full counting statistics.** We are interested in the probability distribution function, or Full Counting Statistics (FCS), \( P_\ell(m) = \langle \delta(M_\ell - m) \rangle \), of the order-parameter in a subsystem of \( \ell \) sites

\[
M_\ell = \frac{1}{2} \sum_{j=1}^{\ell} \sigma_j^z. \tag{2}
\]

Several past studies in a variety of models considered the PDFs of either observables within one (or few) sites or global ones, but for our aims it is fundamental to focus on the FCS within a block. Indeed, \( M_\ell \) is the right observable to understand relaxation to a statistical ensemble since on the one hand it generically relaxes (differently from global observables) and, on the other, it can have a thermodynamic behaviour like those expected in the post-quench evolution in a confined phase.

Any deviation from gaussianity is a signature of anomalous behaviour like those expected in the post-quench evolution in a confined phase.

Another important aspect concerns thermalisation. For a generic chaotic model, thermalisation should occur in the long time limit and therefore the PDF should approach the characteristic thermal distribution \( P_\ell(m) = \text{Tr} [\delta(M_\ell - m) g] \) where \( g = e^{-\beta H}/\text{Tr}(e^{-\beta H}) \) with the inverse temperature \( \beta \) fixed by the energy of the initial state. It has been proposed that confinement can prevent thermalisation, at least for numerical accessible times.

In the following we will first discuss how gaussification depends on the quench parameters and how confinement makes it more difficult. Only after we move to thermalisation and to the effects of confinement. The two phenomena have some connections, but they are different: gaussification is expected at any time for large \( \ell \), thermalisation takes place at large time for arbitrary \( \ell \).

**Quench protocol and numerical technique.** We consider the following quench protocol: (i) at time \( t = 0 \), the system is prepared in the initial state \( |\Psi_0\rangle \) which is fully polarised, namely \( |\uparrow\rangle \equiv |\uparrow\uparrow\uparrow\ldots\rangle \) where \( \sigma_j^z|\uparrow\rangle = |\uparrow\rangle \) (ii) thereafter the system evolves unitarily \( |\Psi(t)\rangle = \exp(-iHt)|\Psi_0\rangle \) with the Hamiltonian (1).

At \( t = 0 \) the PDF is \( P_\ell(m) = \delta(m,\ell/2) \); the subsequent time evolution strongly depends on the Hamiltonian parameters in particular on the sign of \( h_z \); while for \( h_z > 0 \), \( |\Psi_0\rangle \) is a low-energy state of the post-quench Hamiltonian, for \( h_z < 0 \), \( |\Psi_0\rangle \) falls in middle of the spectrum.

We numerically investigate the real-time dynamics in a finite system with \( L = 80 \) lattice sites and open boundary conditions. At each instant of time the state is described by a normalised Matrix Product State (MPS) \( |\Psi(t)\rangle = \sum_{\sigma_1,\ldots,\sigma_L} v_1 \cdot \prod_{j=1}^{\ell} \mathbf{A}^{(j)}(t) \cdot v \langle \sigma_1,\ldots,\sigma_L | \), where \( v_\alpha = \delta_{\alpha,1} \) is the boundary vector, and \( \mathbf{A}^{(j)} \) are matrices whose (auxiliary) dimension is at most \( \chi_{\max} = 500 \). The time evolution is obtained by employing the Time-Evolving Block-Decimation (TEBD) algorithm, with second-order Suzuki-Trotter decomposition of the evolution operator with time step \( dt = 0.01 \). Doing so, depending on the quench parameters, we can reach a maximum time \( t_{\max} \simeq 10 \), within reasonable truncation errors. The MPS representation is very suitable to evaluate the generating function \( F_\ell(\lambda) \), where \( \lambda \) is moved in \([ -\pi, \pi ]\) with step \( d\lambda = 0.01 \), and the subsystem with \( \ell \) sites is in the middle of the chain to minimise boundary effects. \( P_\ell(m) \) is eventually obtained by taking the Fourier transform of the numerically interpolated \( F_\ell(\lambda) \).

**Gaussification and memory of the initial order.** Since the quench pumps an extensive energy into the system, a finite correlation length should build up during the time evolution (a 1D system cannot have critical points at finite energy density). How fast and how large depend on the post-quench parameters \( h_z \) and \( h_x \). In general, when the subsystem size is sufficiently larger than the
time-dependent correlation length, the PDF should fairly match the Gaussian approximation (4), where the only parameters are the first two cumulants. In Fig. 1 we show the results for some representative quenches: For quenches to the paramagnetic phase, the time-dependent correlation length is very small and, for subsystem with $\ell = 20$, we found a good agreement with the Gaussian (4) at any time; moreover we do not find any noticeable qualitative difference depending on the sign of $h_x$ or its absolute value. Indeed, here tuning the Hamiltonian parameters has the only effect of changing the characteristic relaxation time.

The PDF dynamics after quenching to the ferromagnetic phase is completely different. We first discuss what happens for small $|h_x|$ showing in Fig. 1 the data for $h_x = \pm 0.1$ and up to time $t = 8$. For finite but large $\ell$ (20 in Fig. 1) the PDF is still not Gaussian and shows a very mild dependence on the sign of $h_x$. For $h_x = 0$, in Refs. [47, 48], it has been argued that the deviations from a gaussian are related to the memory of the initial local order. Our conclusion is that, at least for relative short times, the presence of a small longitudinal field does not qualitatively alter this effect.

As $|h_x|$ increases, the values of $\ell \gg 1$ for which there is gaussianification depends strongly on the sign of $h_x$. In Fig. 2 we analyse the PDF for $|h_x| = 0.1, 0.4$ at fixed time $t = 8$ with varying the subsystem size $\ell$. For $h_x < 0$ (i.e. initial polarisation discord to the magnetic field), the PDF is approximately gaussian for relatively small $\ell$ for $h_x = -0.4$. This is not the case for $h_x > 0$ when (for $h_x = 0.4$) we still observe large deviations from Gaussian at $\ell = 40$. In other words, the memory of the initial order is enhanced by a strong longitudinal field in the direction of the initial polarisation; instead the local order is easily melted when the longitudinal field is in the opposite direction. Microscopically, this behaviour follows from the fact that, by increasing the absolute value of $h_x$, while for $h_x < 0$ the initial state has larger (and almost uniform) overlaps with eigenstates of the post-quench Hamiltonian in the middle of the spectrum, for $h_x > 0$ the larger contributions are from the low-energy states, with small dependence on $h_x$.

**Paramagnetic thermalisation and ferromagnetic confinement.** – When quenching to the paramagnetic (and critical) phase, it has been firmly established that the system thermalises in a standard way [24], independently of the sign of $h_x$. The analysis of the PDF in Fig. 3 (for quenches to $h_z = 1$ and $1.25$ with $h_x = \pm 0.1$) confirms this scenario. The PDF clearly does what we would expect: The average moves toward the thermal average and the distribution broadens approaching its thermal expectation in a uniform manner. The main reason why we report these expected results here is to contrast them with what happens in the ferromagnetic phase.

When quenching to the ferromagnetic region, the confinement strongly affects the relaxation [6–11, 13]. A representative set of our results for the dynamics of the or-

Fig. 1: PDF of the local order parameter with $\ell = 20$ and different times, for quenches toward the paramagnetic (right) and ferromagnetic (left) phase with $h_x > 0$ (top row) and $h_x < 0$ (bottom). TEBD exact results (symbol) are compared with the Gaussian approximation in Eq. (4) (full lines).

Fig. 2: Order parameter PDF at time $t = 8$ for different subsystem sizes $\ell$, and quenches toward the ferromagnetic phase. TEBD exact results (symbol) are compared with the Gaussian approximation in Eq. (4) (full lines).
der parameter statistics are in Fig. 4 for $h_x = \pm 0.1$ and $h_z = 0.5$ and $h_z = 0.75$. These parameters have been chosen to not be deeply in the ferromagnetic/confining phase, but still we observe many non-thermal features. While the precise value of $h_z$ only quantitatively changes the dynamics, the sign of $h_z$ provides very different features which are very nicely captured by the PDF. This difference stems from the fact that for $h_z > 0$ the initial state has energy close to the ground state, while for $h_z < 0$ it is in the middle of the many-body spectrum (and naively and erroneously one could expect to thermalise quicker).

For $h_z > 0$, the PDF remains localised around the initial value $\ell/2$ which is not far from the asymptotic mean value. Anyhow, neither the shift toward the mean value nor the broadening is uniform and the PDF has more than one maximum sometimes. The magnetisation shows the known oscillations related to the masses of the characteristic bound states [6,56]. The same oscillations are present in all cumulants, explaining the non-uniform relaxation.

However, the strongest qualitative evidence for the lack of thermalisation appears when quenching to $h_z < 0$. In this case, the initial magnetisation is $\ell/2 > 0$, while the thermal distribution is centred around a negative value. For small $h_z$, there are few dilute and heavy excitations (mesons): they cannot scatter and so the PDF is stuck on the initial wrong side of $m$, likely for an exponentially long time [11]. This is visible in Fig. 4 for $h_z = -0.1$ and $h_z = 0.5$. When the transverse field is increased, mesons become lighter (so larger typical size), their density larger and they may be produced in pairs of opposite momenta. Consequently they can scatter and drift toward equilibrium. (In other words, some mesons have size larger than the localisation length $\xi_{loc} \sim h_z \sqrt{|h_x|}$ [11].) Indeed a slow drift of the PDF toward the right side of $m$ may be seen in Fig. 4 for $h_z = 0.75$ and $h_z = -0.1$ (still, at the larger accessible time, it is very far from thermal PDF). However, also at these values of $h_z$ there exist heavier dilute mesons that can have a considerable overlap with the initial state; they will remain localised for an exponential time. In conclusion, the interplay between the density of mesons and their masses (i.e. between the strength of the quench and the confinement) manifests in a two-step relaxation of the PDF of $M_t$: (i) the lightest excitations may be abundantly produced (even in pairs) and cause part of the PDF to relax with a time $\propto t$; (ii) the dilute heaviest excitations can only melt after an exponential time causing the freezing of another part of the PDF.

The phenomenology we just described for $h_z < 0$ is just a dynamical manifestation of the Schwinger effect [57], i.e. the decay of the false vacuum, as already described for other observables [11,18]. Such decay takes place via tunnelling effect for the production of pairs of mesons which is exponentially suppressed exactly as outlined above.

**Conclusions.** We studied the relaxation dynamics of the order parameter statistics in the quantum Ising
chain under magnetic confinement. We inspected different regimes, by varying the Hamiltonian couplings. We found that a finite value of a longitudinal field may affect the gaussification of the order-parameter distribution function, depending whether it is tuned in the same direction of the initial polarisation or not. This phenomenon is somehow related to the memory of the original local order.

We showed that the PDF is an ideal quantity to qualitatively show the lack of thermalisation (in numerically accessible time windows) in the presence of confinement, despite the non-integrability of the model. In particular, we argued that an eventual relaxation can happen in a two-step process with the lightest mesons relaxing quickly and the heaviest ones remaining frozen (likely for an exponentially long time). It is natural to wonder whether similar PDFs could shed some light also for other models that tend to avoid thermalisation, like those with quantum scars [58].

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