UPPER BOUNDS ON THE FIRST EIGENVALUE FOR A DIFFUSION OPERATOR VIA BAKRY-ÉMERY RICCI CURVATURE II

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Abstract. Let $L = \Delta - \nabla \varphi \cdot \nabla$ be a symmetric diffusion operator with an invariant measure $d\mu = e^{-\varphi}dx$ on a complete Riemannian manifold. In this paper we prove Li-Yau gradient estimates for weighted elliptic equations on the complete manifold with $|\nabla \varphi| \leq \theta$ and infinite-dimensional Bakry-Émery Ricci curvature bounded below by some negative constant. Based on this, we give an upper bound on the first eigenvalue of the diffusion operator $L$ on this kind manifold, and thereby generalize a Cheng’s result on the Laplacian case (Math. Z., 143 (1975) 289-297).

1. Introduction and main result

Given $(M, g)$ be an $n$-dimensional complete Riemannian manifold with the Ricci curvature satisfying $\text{Ric}(g) \geq -(n-1)$, Cheng (Theorem 4.2 in [9]) proved an upper bound of the first nontrivial eigenvalue of the Laplacian

$$\lambda_1 \leq \frac{(n-1)^2}{4}.\]

Later, the author [30] extended this result to the case of the diffusion operator with the $m$-dimensional Bakry-Émery Ricci curvature bounded below.

**Theorem A.** (Wu [30]) Let $(M^n, g)$ be an $n$-dimensional ($n \geq 2$) complete Riemannian manifold and $\varphi \in C^2(M^n)$ be a function. Assume that the $m$-dimensional Bakry-Émery Ricci curvature satisfies

$$\text{Ric}_{m,n}(L) \geq -(n-1).$$

Then we have an upper bound estimate on the first nontrivial eigenvalue for the diffusion operator $L = \Delta - \nabla \varphi \cdot \nabla$

$$\lambda_1 \leq \frac{(m-1)(n-1)}{4}.$$

In Theorem A, the $m$-dimensional Bakry-Émery Ricci curvature (see [12, 13, 18]) is defined by

$$\text{Ric}_{m,n}(L) := \text{Ric} + \text{Hess}(\varphi) - \frac{\nabla \varphi \otimes \nabla \varphi}{m-n}.$$
where $Ric$ and $Hess$ denote the Ricci curvature and the Hessian of the metric $g$, respectively, and where $m := \dim_{BE}(L) \geq n$ is called the Bakry-Émery dimension of $L$, which is a constant and $m = n$ if and only if $\varphi$ is constant. When $\varphi$ is constant, $Ric_{m,n}(L) = Ric$ and Theorem A reduces to the Cheng’s theorem by taking $m = n$. The essential tool in proving Theorem A is the Li-Yau’s gradient estimate trick [17], which are originated by Yau [33] (see also Cheng and Yau [10]).

The $m$-dimensional Bakry-Émery Ricci curvature, mentioned in Theorem A, has a natural extension to non-smooth metric measure spaces [20, 25, 26]. We refer to the survey article [28] for details on this tensor. A remarkable feature for $m$-dimensional Bakry-Émery Ricci curvature is that Laplacian comparison theorems hold for $Ric_{m,n}(L)$ that look like the case of Ricci tensor in a $m$-dimensional manifold [29] (see also [11, 18, 30]). This is why many geometric and topological results for manifolds with a lower bound on the Ricci curvature can be easily extended to smooth metric measure spaces with only a $m$-dimensional Bakry-Émery Ricci curvature bounded below. For discussions of this use, see for example Theorem 1.3 in [11], Theorem 5.1 in [18] and Theorem 1.1 in [30].

For the $m$-dimensional Bakry-Émery Ricci curvature, we can allow $m$ to be infinite. In this case this becomes

$$Ric(L) := \lim_{m \to +\infty} Ric_{m,n}(L) = Ric + Hess(\varphi),$$

which is called the $\infty$-dimensional Bakry-Émery Ricci curvature. Bakry and Émery [1] extensively studied this tensor and its relationship to diffusion processes. The $\infty$-dimensional Bakry-Émery Ricci tensor also occurs naturally in many different subjects, see e.g. [11, 12, 19, 23, 32]. Recently, the $\infty$-dimensional Bakry-Émery Ricci curvature has become an important object of study in Riemannian geometry, in large part due to the gradient Ricci soliton equation:

$$Ric(L) = \rho g$$

for some constant $\rho$, which plays an important role in the Ricci flow. For the recent progress on Ricci solitons the reader may refer to Cao’s survey papers [6, 7].

As mentioned in Remark 1.2 (2) of [30], in the above Theorem A, if we let $m = \infty$, then $\frac{(m-1)(n-1)}{4} = \infty$. At this time, we do not obtain any useful upper bound on the first eigenvalue when $Ric(L) \geq -(n-1)$ from Theorem A. Naturally, we ask if we can extend Cheng’s theorem to the case of the diffusion operator $L$ with only the $\infty$-dimensional Bakry-Émery Ricci curvature bounded below. This is not an easy question to answer. One main difficulty is that we can not get a proper local gradient estimate on Riemannian manifolds with only a lower $\infty$-dimensional Bakry-Émery Ricci curvature bound, since local gradient estimates are closely related to Laplacian comparison theorems. However, classical Laplacian comparison theorems can not be directly extended to the case for only a lower bound on the $\infty$-dimensional Bakry-Émery Ricci tensor. This is a main difference compared with the $m$-dimensional Bakry-Émery tensor.

As many recent authors said, when dealing with the $\infty$-dimensional Bakry-Émery Ricci curvature, many geometric and topological results remain true under some assumptions of the potential functions $\varphi$ (see e.g. [11, 28, 29]). Following this idea, in this paper if we assume additionally that the functions $|\nabla \varphi|$ are bounded, we can derive the desired gradient estimates under the $\infty$-dimensional Bakry-Émery
Ricci curvature bounded below, analogous to [11, 29, 30]. Furthermore, we can derive the following useful upper bound on the first eigenvalue of \( L \).

**Theorem 1.1.** Let \((M^n, g)\) be an \( n \)-dimensional \((n \geq 2)\) complete Riemannian manifold and \( \varphi \in C^2(M^n) \) be a function satisfying \( |\nabla \varphi| \leq \theta \) for some constant \( \theta \geq 0 \). Assume that

\[
Ric(L) \geq -(n-1).
\]

Then we have an upper bound estimate on the first nontrivial eigenvalue for the diffusion operator \( L = \Delta - \nabla \varphi \cdot \nabla \)

\[
(1.3) \quad \lambda_1 \leq \frac{1}{4}(n-1+\theta)^2.
\]

When \( \varphi \) is constant, we can take \( \theta = 0 \), and our theorem reduces to Cheng’s theorem [9]. The assumption \( |\nabla \varphi| \leq \theta \) guarantees that the weighted comparison theorem [29] holds when \( Ric(L) \) is bounded below. Su-Zhang [27] provided a simple example to explain that the assumption \( |\nabla \varphi| \) is necessary. Recently, Munteanu-Wang [22] and Su-Zhang [27] also independently obtained this eigenvalue estimate by means of the weighted volume comparison theorem and the eigenvalue variational principle. We emphasize that our proof relies on Li-Yau’s gradient estimates (see Theorem 2.1 in Sect. 2).

**Remark 1.2.** We remark that there exists an example which shows that estimate (1.3) is sharp. Consider \( M = \mathbb{R} \times N^{n-1} \) endowed with the warped product metric

\[
d s_M^2 = dt^2 + \exp(-2t)ds_N^2.
\]

If \( \{e_\alpha\} \) for \( \alpha = 2, \ldots, n \) form an orthonormal basis of the tangent space of \( N \), then \( e_1 = \frac{\partial}{\partial t} \) together with \( \{e_\alpha = \exp(-t)e_\alpha\} \) form an orthonormal basis for the tangent space of \( M \). If \( \varphi(t, x) = \theta t \) for \( (t, x) \in \mathbb{R} \times N^{n-1} \), by the standard computation, then the \( \infty \)-dimensional Bakry-Émery Ricci curvature of \( M \) is

\[
Ric(L)_{1j} = Ric_{M,1j} + \varphi_{1j} = -(n-1)\delta_{1j}
\]

and

\[
Ric(L)_{\alpha\beta} = \exp(2t)Ric_{N,\alpha\beta} - (n-1)\delta_{\alpha\beta}.
\]

Clearly, \( Ric_N \geq 0 \) if and only if \( Ric(L) \geq -(n-1) \). Moreover, we claim that \( \lambda_1 = \frac{(n-1+\theta)^2}{4} \). Indeed, we choose function \( f = \exp(\frac{n-1+\theta}{2} t) \), and have that

\[
\Delta_\varphi f = \frac{d^2 f}{dt^2} - (n-1)\frac{df}{dt} - \frac{d\varphi}{dt} \cdot \frac{df}{dt} = -(n-1+\theta)^2 \frac{f}{4},
\]

since \( \Delta = \frac{\partial^2}{\partial t^2} - (n-1)\frac{\partial}{\partial t} + \exp(2t)\Delta_N \). From this we conclude that \( \lambda_1 = \frac{(n-1+\theta)^2}{4} \) as claimed.

**Remark 1.3.** If \( Ric(L) \equiv -(n-1) \) with \( |\nabla \varphi| \leq \theta \), then \( M \) must be Einstein by [24]. If \( M \) is the gradient expanding Ricci soliton: \( Ric + \nabla \nabla \varphi = \rho g \), \( \rho < 0 \) (see [14] or [3]), we can derive that \( |\nabla \varphi|^2 = -R + 2\rho \varphi \). If we assume \( \varphi \geq c \) for some constant \( c \leq n/2 \), then

\[
|\nabla \varphi|^2 \leq -n\rho + 2\rho \varphi \leq -n\rho + 2c\rho,
\]

where we used \( R \geq n\rho \) [8]. Hence by [24], we conclude that \( M \) is Einstein.
Remark 1.4. In an earlier version of this result, the author proved that for any constant $m_0 \geq n$, the first nontrivial eigenvalue satisfies

$$\lambda_1 \leq \frac{(m_0 - 1)(n - 1) + \frac{m_0 - 1}{m_0 - n} \theta^2}{4}.$$  

This estimate can be optimized by estimate (1.3), pointed out by the referee. Since $m_0$ is arbitrary, letting $m_0 = n + k\theta$, where $k > 0$ is a free parameter, then

$$(m_0 - 1)(n - 1) + \frac{m_0 - 1}{m_0 - n} \theta^2 = (n - 1 + k\theta)(n - 1 + k^{-1}\theta) \geq (n - 1 + \theta)^2,$$

where $m_0 = n$ if and only if $\theta = 0$.

The structure of this paper is organized as follows. In Sect. 2, we derive Li-Yau gradient estimates, i.e., Theorem 2.1. The proof makes use of the author’s previous result in [30], combining the concept of Bakry-Émery tensor used in [18]. In Sect. 3, we apply Theorem 2.1 to the setting of gradient Ricci solitons. Finally in appendix, we give a detailed proof of Theorem 2.1 though the proof method nearly follows from that of Theorem 2.1 in [30]. We include it because we feel it might be useful in other applications.

2. SOME BASIC GRADIENT ESTIMATES

In this section, we will prove an important gradient estimate (see (2.2) below), which also implies the proof of Theorem 1.1.

**Theorem 2.1.** Let $(M^n, g)$ be an $n$-dimensional $(n \geq 2)$ complete Riemannian manifold and $\varphi \in C^2(M^n)$ be a function satisfying $|\nabla \varphi| \leq \theta$ for some constant $\theta \geq 0$. Assume that

$$\text{Ric}(L) \geq -(n - 1)K$$

for some constant $K \geq 0$. Let $f$ be a positive function defined on the geodesic ball $B_p(2R) \subset M^n$ satisfying

$$Lf = -\lambda f$$

for some constant $\lambda \geq 0$. Then for any constant $m_0 \geq n$, there exists a constant $C$ depending on $m_0$ and $n$ such that

$$(2.1) \quad \frac{|\nabla f(x)|^2}{f^2(x)} \leq \frac{2(m_0 - 1)(n - 1) + \epsilon K}{(2 - \epsilon)} + \frac{2(m_0 - 1)\theta^2}{(m_0 - n)(2 - \epsilon)} + C \left( \frac{1 + \epsilon^{-1}}{R^2} + \lambda \right)$$

for all $x \in B_p(R)$ and for any $0 < \epsilon < 2$. Furthermore, if the positive function $f$ is defined on $(M^n, g)$, then for any constant $m_0 \geq n$, we have

$$(2.2) \quad |\nabla \ln f|^2 \leq \frac{(m_0 - 1)(n - 1)K}{2} - \lambda + \frac{m_0 - 1}{2(m_0 - n)} \theta^2$$

$$+ \sqrt{\left( \frac{(m_0 - 1)(n - 1)K + \frac{m_0 - 1}{m_0 - n} \theta^2}{4} \right)^2 - \left( \frac{(m_0 - 1)(n - 1)K + \frac{m_0 - 1}{m_0 - n} \theta^2}{4} \right)^2} \lambda$$

and

$$(2.3) \quad \lambda \leq \frac{(m_0 - 1)(n - 1)K + \frac{m_0 - 1}{m_0 - n} \theta^2}{4}.$$
Remark 2.2. (1) Similar to Remark 1.4, letting $m_0 = n + k\theta$, where $k > 0$ is a free parameter, then
\[(m_0 - 1)(n - 1)K + \frac{m_0 - 1}{m_0 - n}\theta^2 = \left[(n - 1)\sqrt{K} + k\sqrt{K}\theta\right] \left[(n - 1)\sqrt{K} + \frac{\theta}{k\sqrt{K}}\right] \geq \left[(n - 1)\sqrt{K} + \theta\right]^2,
\]
where $m_0 = n$ if and only if $\theta = 0$. Therefore estimates (2.2) and (2.3) can be respectively optimized by
\[
|\nabla \ln f|^2 \leq \frac{\left[(n - 1)\sqrt{K} + \theta\right]^2}{2} - \lambda + \sqrt{\frac{\left[(n - 1)\sqrt{K} + \theta\right]^4}{4} - \left[(n - 1)\sqrt{K} + \theta\right]^2} \lambda
\]
and
\[
\lambda \leq \frac{\left[(n - 1)\sqrt{K} + \theta\right]^2}{4}.
\]

Remark 2.3. If $\text{Ric}(L) \geq 0$, by letting $K = 0$ and then letting $m_0 \to \infty$ in (2.3), we immediately obtain $\lambda_1 \leq \frac{\theta^2}{4}$, which has been proved by Munteanu and Wang in [21] under some weak assumption.

Remark 2.4. Our Li-Yau gradient estimates can be used to prove splitting type theorems for complete manifolds with $\infty$-dimensional Bakry-Émery Ricci curvature. This was done by the author in a separated paper [31]. Recently, Munteanu and Wang [22] obtained similar gradient estimates for weighted harmonic functions ($\lambda = 0$ in Theorem 2.1) under some oscillation of function $\varphi$. Their proof is a mixture of the Bochner identity and the DeGiorgi-Nash-Moser theory.

Proof of Theorem 2.1. We moved our original proof using the Li-Yau gradient estimate method to the appendix because we feel it seems to be tedious. Here we use a simple and direct proof, which was pointed by the referee. We are very grateful to the referee for his valuable comment which leads us to give this quick proof. According to Section 1.5 in X.-D. Li’s paper [18], the conditions $\text{Ric}(L) \geq -(n - 1)K$ and $|\nabla \varphi| \leq \theta$ imply that the $m$-dimensional Bakry-Émery Ricci curvature is bounded from below by a new constant for any $m_0 > n$, i.e.,
\[
\text{Ric}_{m,n}(L) \geq -(n - 1) \left[K + \frac{\theta^2}{(m_0 - n)(n - 1)}\right] := -(n - 1)\tilde{K}.
\]
Using the gradient estimate trick developed in [18], from Theorem 2.1 in [30], we know that there exists a constant $C$ depending on $m_0$ and $n$ such that
\[
\frac{|\nabla f(x)|^2}{f^2(x)} \leq \frac{2(m_0 - 1)(n - 1) + \epsilon \tilde{K}}{2 - \epsilon} + C \cdot \left(1 + \epsilon^{-1} R^2 + \lambda\right)
\]
for all $x \in B_0(R)$ and for any $0 < \epsilon < 2$. If the positive function $f$ is defined on $(M^n, g)$, then
\[
|\nabla \ln f|^2 \leq \frac{(m - 1)(n - 1)\tilde{K}}{2} - \lambda + \sqrt{\frac{(m_0 - 1)^2(n - 1)^2\tilde{K}^2}{4} - (m_0 - 1)(n - 1)\tilde{K}\lambda}
\]
and
\[
\lambda \leq \frac{(m_0 - 1)(n - 1)\tilde{K}}{4}.
\]
Substituting \( \tilde{K} = K + \frac{\theta^2}{(m_0-n)(n-1)} \) into the above inequalities yields (2.1), (2.2) and (2.3).

Below, Theorem 1.1 can be easily obtained by Theorem 2.1.

**Proof of Theorem 1.1.** Let \((M^n, g)\) be an \(n\)-dimensional \((n \geq 2)\) complete Riemannian manifold and \(\varphi \in C^2(M^n)\) be a function satisfying \(|\nabla \varphi| \leq \theta\). Assume that the \(\infty\)-dimensional Bakry-Émery Ricci curvature satisfies \(Ric(L) \geq -(n-1)\).

Let \(\lambda_1\) be the first nontrivial eigenvalue of the diffusion operator \(L = \Delta - \nabla \varphi \cdot \nabla\). Hence \(\lambda_1\) satisfies the equation

\[
Lf = -\lambda_1 f,
\]

where \(f\) is the eigenfunction. Now let \(\lambda = \lambda_1\) and \(K = 1\) in Theorem 2.1. Then estimate (2.3) and Remark 2.2 give the complete proof of Theorem 1.1.

3. Applications of Theorem 2.1 to Ricci solitons

In this section, we discuss the first eigenvalues of the diffusion operator \(L\) on gradient Ricci solitons by Theorem 2.1. Recall that a complete smooth Riemannian manifold \((M^n, g)\) is called a gradient Ricci soliton if equation (1.1) holds. The gradient Ricci solitons are called shrinking, steady and expanding accordingly when \(\rho > 0, \rho = 0\) and \(\rho < 0\).

On the gradient steady Ricci solitons \(Ric + \nabla \nabla \varphi = 0\), there exists a positive constant \(a > 0\) (see [14] or [5]) such that

\[
R + |\nabla \varphi|^2 = a^2.
\]

**Theorem 3.1.** On gradient steady Ricci solitons \(Ric + \nabla \nabla \varphi = 0\), normalized as in (3.1), the first eigenvalue of diffusion operator \(L\) satisfies \(\lambda_1 \leq a^2/4\).

**Remark 3.2.** A stronger result for gradient steady Ricci solitons has been proved by Munteanu and Wang (Proposition 2.4 in [21]) using a different approach.

**Proof.** Since \(R \geq 0\) for any complete gradient steady Ricci soliton [8], by (3.1) we have \(|\nabla \varphi| \leq a\). Hence by Theorem 2.1 in (2.3), letting \(K = 0\) and \(m_0 \to \infty\), we have the desired result.

On the gradient shrinking Ricci solitons \(Ric + \nabla \nabla \varphi = \rho g, \rho > 0\), there exists a constant \(C\) (see [14] or [5]) such that

\[
R + |\nabla \varphi|^2 = 2R + 2\rho \varphi = C.
\]

We normalize \(\varphi\) (by adding a constant) so that \(C = 0\). Hence

\[
|\nabla \varphi|^2 = -R + 2\rho \varphi.
\]

If we let \(\varphi \leq b\) for some positive constant \(b\), then we have

**Theorem 3.3.** On gradient shrinking Ricci solitons \(Ric + \nabla \nabla \varphi = \rho g, \rho > 0\), normalized as in (3.2), if \(\varphi \leq b\) for some positive constant \(b\), then the first eigenvalue of diffusion operator \(L\) satisfies \(\lambda_1 \leq \frac{ab}{2}\).

**Proof.** Since \(R \geq 0\) for any gradient shrinking Ricci soliton [8], by (3.2) we have \(|\nabla \varphi|^2 \leq 2\rho \varphi\). Since we assume \(\varphi \leq b, |\nabla \varphi| \leq \sqrt{2\rho b}\). Hence by Theorem 2.1 in (2.3), letting \(K = 0\) and \(m_0 \to \infty\), we have \(\lambda_1 \leq \frac{ab}{2}\).
Remark 3.4. As we all know, for complete noncompact gradient shrinking Ricci solitons, function $\varphi$ is quadratic growth of the distance function unless it is trivial [7]. Hence our theorem is only used to the compact case.

Remark 3.5. Recently, A. Futaki, H.-Z. Li and X.-D. Li [13] proved a lower bound estimate for the first non-zero eigenvalue of the diffusion operator on compact Riemannian manifolds. As an application, they derived a lower bound estimate for the diameter of compact gradient shrinking Ricci solitons.

If we assume $\varphi > n/2$ on compact shrinking Ricci solitons, we also have an upper bound of the first eigenvalue. Indeed, tracing shrinking Ricci solitons yields

$$R + \Delta \varphi = n\rho.$$ Combining this with (3.2) we have

$$\Delta \varphi = n\rho - 2\rho \varphi.$$ Letting $\hat{\varphi} = \varphi - n/2 > 0$, then

$$\Delta \varphi \hat{\varphi} = -2\rho \hat{\varphi}.$$ Hence by the definition of $\lambda_1$, we conclude that $\lambda_1 \leq 2\rho$.

4. APPENDIX

In this section we shall give a detailed proof of Theorem 2.1. The proof nearly follows from that of Theorem 2.1 in [30]. We include it because we feel it might be useful in other applications. For example, the course of proving gradient estimate is an important step for proving splitting type theorems with $\infty$-dimensional Bakry-Émery Ricci curvature (see [31]).

**Detailed proof of Theorem 2.1.** Step 1. The proof spirit is the same as the arguments used in the proofs of Lemma 2.1, Theorem 2.2 and Theorem 1.3 in [18]. Here our proof exactly follows from that of Theorem 2.1 in [30] (see also [15]) with little modification. We firstly prove the following inequality (4.1). Define $h := \ln f$. Then $Lh = -|\nabla h|^2 - \lambda$. Direct calculation shows that

$$L|\nabla h|^2 = 2h^2_{ij} + 2(R_{ij} + \nabla^2 \varphi) h_i h_j - 2(\nabla h, \nabla |\nabla h|^2).$$ Choose a local orthonormal frame $\{e_1, e_2, ..., e_n\}$ near any such given point so that at the given point $\nabla h = |\nabla h|e_1$. Then we can write

$$|\nabla |\nabla h|^2| = 4 \sum_{j=1}^{n} \left( \sum_{i=1}^{n} h_i h_{ij} \right)^2 = 4h_{11}^2 \cdot \sum_{i=1}^{n} h_{1i}^2 = 4|\nabla h|^2 \cdot \sum_{i=1}^{n} h_{1i}^2.$$
On the other hand, we carefully estimate the term \( h_{ij}^2 \) in (4.1). Notice

\[
\begin{align*}
    h_{ij}^2 &\geq h_{11}^2 + 2 \sum_{\alpha=2}^{n} h_{1\alpha}^2 + \sum_{\alpha=2}^{n} h_{\alpha\alpha}^2 \\
    &\geq h_{11}^2 + 2 \sum_{\alpha=2}^{n} h_{1\alpha}^2 + \frac{1}{n-1} \left( \sum_{\alpha=2}^{n} h_{\alpha\alpha} \right)^2 \\
    &= h_{11}^2 + 2 \sum_{\alpha=2}^{n} h_{1\alpha}^2 + \frac{1}{n-1} (\Delta h - h_{11})^2 \\
    &= h_{11}^2 + 2 \sum_{\alpha=2}^{n} h_{1\alpha}^2 + \frac{1}{n-1} \left( \nabla h^2 + \lambda + h_{11} - \varphi_i h_i \right)^2 \\
    &\geq h_{11}^2 + 2 \sum_{\alpha=2}^{n} h_{1\alpha}^2 + \frac{1}{n-1} \left[ \left( \nabla h^2 + \lambda + h_{11} \right)^2 - \left( \varphi_i h_i \right)^2 \right] \\
\end{align*}
\]

for any constant \( m_0 (\geq n) \). Since \(|\nabla \varphi| \leq \theta\), we have

\[
(4.3) \quad h_{ij}^2 \geq \frac{m_0}{m_0 - 1} \sum_{i=1}^{n} h_{ii}^2 + \frac{2(m_0 - 1)}{m_0 - 1} \frac{\left( \nabla h^2 + \lambda \right)^2}{m_0 - 1} + \frac{2h_{11}(\nabla h^2 + \lambda)}{m_0 - 1} + \frac{\theta^2 |\nabla h|^2}{m_0 - n}.
\]

Note that \( 2h_{11} = \langle \nabla |\nabla h|^2, \nabla h \rangle - |\nabla h|^2 \). Substituting this into (4.3) yields

\[
(4.4) \quad h_{ij}^2 \geq \frac{m_0}{m_0 - 1} \sum_{i=1}^{n} h_{ii}^2 + \frac{2(m_0 - 1)}{m_0 - 1} \frac{\left( \nabla h^2 + \lambda \right)^2}{m_0 - 1} + \frac{\langle \nabla |\nabla h|^2, \nabla h \rangle}{|\nabla h|^2} - \frac{\theta^2 |\nabla h|^2}{m_0 - n}.
\]

Putting (4.4), (4.2) and (4.3) together, we deduce

\[
L |\nabla h|^2 \geq \frac{2m_0}{m_0 - 1} \sum_{i=1}^{n} h_{ii}^2 + \frac{2(m_0 - 1)}{m_0 - 1} \frac{\left( \nabla h^2 + \lambda \right)^2}{m_0 - 1} + \frac{\langle \nabla |\nabla h|^2, \nabla h \rangle}{|\nabla h|^2} - \frac{2\theta^2 |\nabla h|^2}{m_0 - n} + 2(R_{ij} + \nabla_i \nabla_j \varphi) h_i h_j - 2 \langle \nabla h, \nabla |\nabla h|^2 \rangle
\]

\[
= \frac{m_0}{2(m_0 - 1)} \frac{|\nabla |\nabla h|^2|^2}{|\nabla h|^2} + \frac{2(m_0 - 1)}{m_0 - 1} \frac{\left( \nabla h^2 + \lambda \right)^2}{m_0 - 1} + 2Ric(L)(\nabla h, \nabla h)
\]

\[
- \frac{2\theta^2 |\nabla h|^2}{m_0 - n} + \left[ \frac{2\lambda}{(m_0 - 1)|\nabla h|^2} - \frac{2m_0 - 4}{m_0 - 1} \right] \cdot \langle \nabla |\nabla h|^2, \nabla h \rangle.
\]

Using \( Ric(L) \geq -(n-1)K \), the function \( h := \ln f \) satisfies

\[
(4.5) \quad L |\nabla h|^2 \geq \frac{m_0}{2(m_0 - 1)} \frac{|\nabla |\nabla h|^2|^2}{|\nabla h|^2} + \frac{2(m_0 - 1)}{m_0 - 1} \frac{\left( \nabla h^2 + \lambda \right)^2}{m_0 - 1} - 2 \left( (n-1)K + \frac{\theta^2}{m_0 - n} \right) |\nabla h|^2
\]

\[
+ \left[ \frac{2\lambda}{(m_0 - 1)|\nabla h|^2} - \frac{2m_0 - 4}{m_0 - 1} \right] \cdot \langle \nabla |\nabla h|^2, \nabla h \rangle
\]

for all \( x \in B_p(R) \).

**Step 2.** To obtain estimates (2.1) and (2.2), we apply the diffusion operator \( L \) to a suitable function, and then use the maximum principle argument.

(i) Now we introduce a cut-off function. Let

\[
\phi(x) := \eta \left( \frac{\rho(x)}{R} \right),
\]
where \( \eta(t) \) is a non-negative cut-off function such that \( \eta(t) = 1 \) for \( 0 \leq t \leq 1 \), \( \eta(t) = 0 \) for \( t \geq 2 \) and \( 0 \leq \eta(t) \leq 1 \) for \( 1 < t < 2 \). Furthermore, take the derivatives of \( \eta \) to satisfy

\[
-C \eta^{1/2}(r) \leq \eta' \leq 0 \quad \text{and} \quad \eta'' \geq -C,
\]

where \( 0 < C < \infty \) is a universal constant. Here \( \rho(x) \) denotes the distance from some fixed \( p \in M^n \). Using an argument of Calabi [4] (see also [10] or [16]), we can assume without loss of generality that \( \phi(x) \in C^2(M^n) \) with support in \( B_p(2R) \).

Since \( \text{Ric}(L) \geq -(n-1)K \) and \( \nabla \phi \geq -\theta \), by the weighted Laplacian comparison theorem (Theorem 1.1 (a) in [29]),

\[
L \rho \leq (n-1)\sqrt{K} \coth(\sqrt{K} \rho) + \theta.
\]

Note that

\[
L \phi = \frac{\eta' L \rho}{R} + \frac{\eta'' |\nabla \rho|^2}{R^2}.
\]

According to the definition of \( \eta \), the function \( \phi \) satisfies

\[
L \phi \geq -C_1 \left[ (\sqrt{K} + \theta)R^{-1} + R^{-2} \right],
\]

where \( C_1 \) is a constant, depending only on \( n \) and \( C \), and

\[
\frac{|\nabla \phi|^2}{\phi} \leq C_2 R^2,
\]

where \( C_2 \) is also a constant, depending only on \( C \).

(ii) Let \( G := \phi \cdot |\nabla h|^2 \). Using inequality (4.5), we obtain

\[
L G = (L \phi) \cdot |\nabla h|^2 + 2 \langle \nabla \phi, \nabla |\nabla h|^2 \rangle + \phi \cdot L |\nabla h|^2
\]

\[
\geq L \phi \cdot G + 2 \left( \frac{\nabla \phi}{\phi} \cdot \nabla G \right) - 2 \left( \frac{\nabla \phi}{\phi} \cdot \nabla \phi \right) \frac{m_0}{m_0 - 1} \cdot \frac{|\nabla |\nabla h|^2|^2}{|\nabla h|^2}
\]

\[
- 2 \left[ (n-1)K + \frac{\theta^2}{m_0 - n} \right] G - \frac{2m_0 - 4}{m_0 - 1} \cdot \langle \nabla h, |\nabla G|^2 \rangle + \frac{2m_0 - 4}{m_0 - 1} \cdot \frac{2 \lambda G + \phi \lambda^2}{\phi} G
\]

\[
+ \frac{2 \lambda G}{(m_0 - 1)|\nabla h|^2} + \frac{2 \lambda \cdot (\nabla h, \nabla \phi)}{m_0 - 1} + \frac{2 \lambda \cdot (\nabla h, \nabla \phi)}{m_0 - 1} \left( \phi^{-1} G^2 + 2 \lambda G + \phi \lambda^2 \right).
\]

In the following, we will estimate ‘bad’ terms on the right hand side (or RHS for short) of (4.8). On one hand,

\[
|\nabla G|^2 = |\nabla (\phi \cdot |\nabla h|^2)|^2
\]

\[
= |\nabla \phi|^2 \cdot |\nabla h|^4 + 2 \phi |\nabla h|^2 \cdot \langle \nabla \phi, \nabla |\nabla h|^2 \rangle + \phi^2 |\nabla |\nabla h|^2|^2
\]

\[
= \frac{|\nabla \phi|^2}{\phi^2} \cdot G^2 + 2 \left( \frac{\nabla \phi}{\phi} \cdot \nabla G \right) \cdot G + \phi^2 |\nabla |\nabla h|^2|^2
\]

This implies

\[
\phi \cdot \frac{|\nabla |\nabla h|^2|^2}{|\nabla h|^2} = \frac{|\nabla G|^2}{G} + \frac{|\nabla \phi|^2}{\phi^2} \cdot G - \frac{2 \langle \nabla \phi, \nabla G \rangle}{\phi} G.
\]

On the other hand, we can easily have

\[
2 \left( \frac{\nabla h, \nabla \phi}{\phi} \right) G \geq -2 |\nabla \phi| \phi^{-\frac{3}{2}} G^\frac{3}{2} \quad \text{and} \quad -2 \langle \nabla h, \nabla \phi \rangle \geq -2 |\nabla \phi| \phi^{-\frac{3}{2}} G^\frac{3}{2}.
\]
Substituting (4.10) and (4.11) into the RHS of (4.8) gives
\[
LG \leq \frac{L \phi}{\phi} \cdot G + \frac{m_0 - 2}{m_0 - 1} \cdot \frac{\langle \nabla \phi, \nabla G \rangle}{\phi} - \frac{3m_0 - 4}{2(m_0 - 1)} \cdot \frac{|\nabla \phi|^2}{\phi^2} \cdot G \\
+ \frac{m_0}{2(m_0 - 1)} \cdot \frac{|\nabla G|^2}{G} + \left[ \frac{4 \lambda m_0}{m_0 - 1} - 2(n - 1)K - \frac{2 \theta^2}{m_0 - n} \right] \cdot G \\
- \frac{2m_0 - 4}{m_0 - 1} \cdot \langle \nabla h, \nabla G \rangle - \frac{2m_0 - 4}{m_0 - 1} \cdot |\nabla \phi| \phi^{-\frac{1}{2}} G \frac{\lambda}{2} \\
- \frac{2 \lambda}{m_0 - 1} \cdot |\nabla \phi| \phi^{-\frac{1}{2}} G \frac{\lambda}{2} + \frac{2 \phi^{-1} G^2}{m_0 - 1} + \frac{2 \lambda^2 \phi}{m_0 - 1}.
\]

Let \( x_0 \in B_\rho(2R) \subset M^n \) be a point where \( G \) achieves a maximum. By the maximum principle, we have
\[
(4.12) \quad LG(x_0) \leq 0, \quad \nabla G(x_0) = 0.
\]

All further calculations in this proof will be at \( x_0 \). Multiplying both sides of (4.11) by \( (m_0 - 1) \phi \) and using (4.12), then (4.11) reduces to
\[
0 \geq LG \geq (m_0 - 1)L \phi \cdot G - \frac{3m_0 - 4}{2} \cdot \frac{|\nabla \phi|^2}{\phi} \cdot G \\
+ \left[ \frac{4 \lambda (m_0 - 1)(n - 1)K - \frac{2(1 - \theta)^2}{m_0 - n} \right] \cdot G - (3m_0 - 4) \cdot |\nabla \phi| \phi^{-\frac{1}{2}} G \frac{\lambda}{2} \\
- \frac{2 \lambda}{m_0 - 1} \cdot |\nabla \phi| \phi^{-\frac{1}{2}} G \frac{\lambda}{2} + 2G^2 + 2\lambda^2 \phi^2.
\]

Combining this with the above estimates of \( \phi \) (4.6) and (4.7), we get
\[
(4.13) \quad 0 \geq - \left( C_3 R^{-1}(\sqrt{K} + \theta) + C_4 R^{-2} \right) G + \left[ \frac{4 \lambda \phi - 2(m_0 - 1)(n - 1) \phi K - \frac{2(1 - \theta)^2}{m_0 - n} \right] \cdot G \\
- \frac{C_5 R^{-1}}{2} \frac{G^2}{C_6 \lambda R^{-1} G} + 2G^2 + 2\lambda^2 \phi^2,
\]

where \( C_3 \) is some constant depending on \( m_0 \), \( n \) and \( C \); \( C_4 \) and \( C_5 \) are fixed constants depending on \( m_0 \) and \( C \); and \( C_6 \) is also a constant, depending only on \( C \).

Since \( x_0 \) is the maximum point of the function \( G \) and \( \phi = 1 \) on \( B_\rho(R) \), hence
\[
\phi(x_0)|\nabla h|^2(x_0) \geq \sup_{B_\rho(R)} |\nabla h|^2(x).
\]

On the other hand, using the fact that
\[
\phi(x_0)|\nabla h|^2(x_0) \leq \phi(x_0) \sup_{B_\rho(2R)} |\nabla h|^2(x),
\]
we conclude that \( \sigma(R) \leq \phi(x_0) \leq 1 \), where \( \sigma(R) \) is defined by
\[
\sigma(R) := \frac{\sup_{B_\rho(R)} |\nabla h|^2(x)}{\sup_{B_\rho(2R)} |\nabla h|^2(x)}.
\]

Applying this to (4.13) yields
\[
(4.14) \quad 0 \geq - \left[ C_3 R^{-1}(\sqrt{K} + \theta) + C_4 R^{-2} - 4 \lambda \sigma(R) + 2(m_0 - 1)(n - 1)K + \frac{2(1 - \theta)^2}{m_0 - n} \right] \cdot G \\
- C_5 R^{-1} G^2 + C_6 \lambda R^{-1} G + 2G^2 + 2\lambda^2 \sigma^2(R).
\]
Below we want to estimate some ‘bad’ terms of the RHS of (4.14). Using the Schwarz inequality, we have the following three inequalities:

\[-C_5 R^{-1} G^2 \geq -\epsilon G^2 - \frac{C_4^2}{4} \epsilon^{-1} R^{-2} G,\]

\[-C_6 R^{-1} \lambda^2 \geq -\epsilon \lambda^2 - \frac{C_4^2}{4} \epsilon^{-1} R^{-2} G,\]

and

\[-C_7 R^{-1} \sqrt{K} \geq -\epsilon K - \frac{C_4^2}{4} \epsilon^{-1} R^{-2}\]

for all \(\epsilon > 0\). Hence at the point \(x_0\), (4.14) can be rewritten as (4.15)

\[
0 \geq - \left[ C_7 (1 + \epsilon^{-1}) R^{-2} - 4 \lambda \sigma(R) + (2(m_0 - 1)(n - 1) + \epsilon) K + \frac{2(m_0 - 1)\theta^2}{m_0 - n} \right] G + (2 - \epsilon) G^2 + 2 \lambda^2 \sigma^2(R) - \epsilon \lambda^2,
\]

where \(C_7\) is some constant, depending only on \(m_0, n\) and \(C\). Now we have a quadratic inequality in \(G\). If \(\epsilon < 2\), then by (4.15) we get

\[
A^2 - 4(2 - \epsilon) \lambda^2 (2\sigma^2(R) - \epsilon) \geq 0,
\]

and an upper bound

\[
(4.17) \quad G(x_0) \leq \frac{A + \sqrt{A^2 - 4(2 - \epsilon) \lambda^2 (2\sigma^2(R) - \epsilon)}}{2(2 - \epsilon)},
\]

where \(A := C_7 (1 + \epsilon^{-1}) R^{-2} - 4 \lambda \sigma(R) + (2(m_0 - 1)(n - 1) + \epsilon) K + \frac{2(m_0 - 1)\theta^2}{m_0 - n}\).

We will see that (4.10) and (4.17) imply estimates (2.1), (2.2) and (2.3) in Theorem 2.1. In fact for any \(x \in B_p(R)\),

\[|\nabla h|^2(x) = \phi(x)|\nabla h|^2(x) \leq G(x_0).\]

Combining this with (4.17) and noticing that \(0 \leq \sigma(R) \leq 1\), we have that

\[
(4.18) \quad \frac{|\nabla f|^2}{f^2}(x) \leq \frac{2A + \sqrt{4(2 - \epsilon) \lambda^2 (2\sigma^2(R) - \epsilon)}}{2(2 - \epsilon)} \leq \frac{2(m_0 - 1)(n - 1) + \epsilon K}{(2 - \epsilon)} + \frac{2(m_0 - 1)\theta^2}{(m_0 - n)(2 - \epsilon)} + \tilde{C} [(1 + \epsilon^{-1}) R^{-2} + \lambda]
\]

for all \(x \in B_p(R)\) and for any \(0 < \epsilon \leq 2\sigma^2(R) < 2\), where \(\tilde{C}\) is a constant, depending only on \(m_0, n\) and \(C\). Hence the proof of the estimate (2.1) is finished.

In the following we assume that \(f\) is defined on \(M^n\). For any \(0 < \epsilon < 2\), if we take \(R \to \infty\) in (4.17), then \(\sigma(R) \to 1\) and (4.17) becomes

\[
|\nabla h|^2(x) \leq \frac{-4\lambda + (2(m_0 - 1)(n - 1) + \epsilon) K}{2(2 - \epsilon)} + \frac{(m_0 - 1)\theta^2}{(m_0 - n)(2 - \epsilon)} + \sqrt{\frac{-4\lambda + (2(m_0 - 1)(n - 1) + \epsilon) K + \frac{2(m_0 - 1)\theta^2}{m_0 - n}}{2(2 - \epsilon)} - 4(2 - \epsilon) \lambda^2}.
\]

Letting \(\epsilon \to 0+\), we obtain (2.2).
At last, since $f$ is defined on $M^n$, $\sigma(R) \to 1$ in (4.16) as $R \to \infty$. Then taking $\epsilon \to 0$, the inequality (4.16) becomes
\[
\left[ (m_0 - 1)(n-1)K + \frac{m_0 - 1}{m_0 - n} \theta^2 \right]^2 \leq \left[ (m_0 - 1)(n-1)K + \frac{m_0 - 1}{m_0 - n} \theta^2 \right] \lambda \geq 0.
\]
Thus
\[
\lambda \leq \frac{(m_0 - 1)(n-1)K + \frac{m_0 - 1}{m_0 - n} \theta^2}{4}.
\]
This finishes the proof of Theorem 2.1. □

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