New group iterative schemes for solving the two-dimensional anomalous fractional sub-diffusion equation

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Abstract

In this paper, new group iterative schemes are developed for the numerical solution of two-dimensional anomalous fractional sub-diffusion equation subject to specific initial and Dirichlet boundary conditions. The new group relaxation iterative schemes are derived from the combination of standard and rotated (skewed) five-point modified implicit finite difference approximations. The results derived from the conducted numerical experiments show that fractional explicit de-coupled group (FEDG) iterative method has a significantly less computational cost in terms of CPU-timings as compared to the other iterative schemes, without threatening compromising accuracies.

Keywords: Riemann-Liouville fractional derivative, fractional implicit standard point, fractional implicit rotated point, anomalous fractional sub-diffusion equation.

2020 MSC: 65N06, 65M06, 65M12.

1. Introduction

In recent years, there has been a growing interest in the field of fractional differential equations (FDEs). FDEs have fascinated growing significance because they have several applications in different fields of science and engineering \cite{19, 27}. Many types of FDEs cannot be solved analytically, therefore a number of mathematical methods, techniques, and approximations have been proposed for solving partial differential equation of fractional order due to their importance in representing the various physical problems \cite{1, 2, 7, 8, 20, 21}. However, the study of exact analytic solution of some types FDEs are also present in literature \cite{9, 18, 24, 25}. Fractional anomalous diffusion equations are a special type of fractional sub-diffusion equation and are a class of important FDEs. Several mathematicians have solved such type of research problems and their modified forms by different techniques. Li and Wang \cite{22} suggested an improved efficient difference method for modified anomalous sub-diffusion equation with a nonlinear source term. They utilized shifted Grunwald-Letnikov fractional derivative instead of Riemann-Liouville derivative and second-order interpolation formula for nonlinear source term. Zhuang et al. \cite{28} proposed

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doi: 10.22436/jmcs.022.02.03
Received: 2020-03-31 Revised: 2020-04-23 Accepted: 2020-05-24
new solution and analytic techniques for anomalous sub-diffusion equation by using implicit difference scheme and they proved the stability and convergence by energy method. Sokolov and Klafter [26] proposed a model that differentiate procedure to become less anomalous as time growth by the addition of a secondary time fractional derivative acting on a diffusion operator. In this paper, we assume such type of model and will applying our grouping strategies. Consider the following two-dimensional time-fractional anomalous fractional sub-diffusion equation together with source term,
\[
\frac{\partial u(x,y,t)}{\partial t} = \frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}} \left[ \frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2} \right] + f(x,y,t),
\] (1.1)
where, \(0 < \alpha < 1\) and \(\frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}}\) is the the Riemann-Liouville fractional derivatives of fractional order \(1 - \alpha\) with the following initial and boundary conditions,
\[
u(x,y,0) = g(x,y), \quad u(0,y,t) = g_1(y,t), \quad u(L,y,t) = g_2(y,t), \quad u(x,0,t) = g_3(x,t), \quad u(x,L,t) = g_4(x,t),
\]
in the region \(\Omega = \{(x,y,t)/ \quad 0 < x,y < L, 0 \leq t \leq T\} \).

The Riemann-Liouville fractional derivatives are defined as follows,
\[
\frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}}u(x,y,t) = \frac{1}{\Gamma(\alpha)} \frac{\partial}{\partial t} \int_0^t \frac{u(x,y,\xi)}{(t-\xi)^{1-\alpha}} d\xi = \frac{\partial}{\partial t} I_{0+}^\alpha u(x,y,t).
\]
Here
\[
I_{0+}^\alpha u(x,y,t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(x,y,\xi)}{(t-\xi)^{1-\alpha}} d\xi
\]
is the Riemann-Liouville integral of fractional order \(\alpha\). In this paper, we use the following two Lemmas.

**Lemma 1.1.** If \(u(t) \in C^1[0,T]\), then the Riemann-Liouville fractional integral of order \(0 < \gamma < 1\) is defined as follows,
\[
I_{0+}^\gamma u(t_k) = \frac{\tau^\gamma}{\Gamma(\gamma+1)} \sum_{s=0}^{k-1} b_s^\gamma u(t_{k-s}) + R_{k}^\gamma,
\]
where, \(b_s^\gamma = (s+1)^\gamma - s^\gamma, \quad s = 0,1,2 \cdots k, \quad |R_{k}^\gamma| \leq C \tau b_k^\gamma.

**Lemma 1.2.** The coefficients \(b_k^\gamma\) in Lemma 1, satisfy the following properties
(a) \(b_k^\gamma = 1, b_k^\gamma > 0, \quad \forall \ k = 0,1,2,\cdots\);
(b) \(b_{k-1}^\gamma > b_k^\gamma, \quad \forall \ k = 0,1,2,\cdots\);
(c) For any positive constant \(C > 0\) such that \(\tau \leq C b_k^\gamma \tau^\gamma, \quad \forall \ k = 0,1,2,\cdots\);
(d) \(\sum_{s=0}^{k} b_s^\gamma \tau^\gamma = (k+1)^\gamma \leq T^\gamma, \quad \forall \ 0 < \gamma < 1\).

The discretization of our proposed research problem in terms of partial differential equation by utilizing the finite difference approximations, generates a sparse system of simultaneous linear equations. For the solution of this linear system, iterative methods are very suitable. In the literature, many iterative methods have been suggested by various authors for solving the linear system. Moreover, several studies on iterative methods have been proposed to speed up the convergence rate in solving sparse linear system. One of them is the development of group iterative methods that possess the capabilities to reduce the convergence timings of the iterative process.

Since 1980s, the group iterative schemes in combination with the finite difference have been utilized for the numerical solution of several types of two-dimensional PDEs in literature. Evans (1985) [14] utilized Explicit Group (EG) iterative method by designing 4-point block iterative scheme for 2D Poisson problem. Based on this ideology, a series of EG iterative schemes have extensively been investigated by Evans and Yousif (1986) [17], Evans and Sahimi (1988) [16], Evans and Hasan (2003) [15] and Kew and Ali (2015) [23] for the numerical solution of various types of PDEs. Since these grouping strategies on integer derivative
for PDEs work well, therefore the efforts are now being made to apply these techniques on FDEs. In this case, the preliminary work has been done by Balasim and Ali (2016-17) [10–12] on two-dimensional time-fractional diffusion and advection-diffusion PDEs, by utilizing Caputo’s fractional derivative, where the time-fractional derivative of order \(\alpha\) lies between 0 and 1. In addition, the most recent work by Ali and Ali (2018-19) [3–5] on two-dimensional time-fractional wave, damped wave and hyperbolic telegraph PDEs, by using the same definition of Caputo’s fractional derivative, where the order of time-fractional derivative \(\alpha\in (1,2)\). In this paper, we step forward towards the numerical solution of more complicated 2D time-fractional anomalous sub-diffusion equation by applying by Riemann-Liouville fractional derivative and grouping methods, and results were found according to our expectation.

The origination of the paper will be as follows. We will discretize our proposed problem by applying the implicit finite difference approximation and derive FISP iterative scheme in Section 2. In Section 3, FEG iterative scheme will be derived by applying a group of 4-points in the solution domain. FIRP iterative scheme will be constructed by rotating the \(x-y\) axis clockwise through an angle of 45° to the standard mesh in Section 4. In Section 5, FEDG iterative scheme will be established based on FIRP iterative scheme. The numerical experiments and conclusion will be present in Sections 6 and 7, respectively.

2. Fractional implicit standard point (FISP) iterative scheme

Discretize the solution domain by defining the same step sizes \(h\) in both \(x\) and \(y\) directions, i.e., \(\Delta x = \Delta y = h\) as \(h = \frac{1}{n}\) and \(\tau = \frac{1}{l}\), for the positive integers \(n\) and \(l\), for the space and time respectively. The grid points for the space interval \([0,1]\) are designated as \(x_i = ih\) and \(y_j = jh\), \(i, j = 0, 1, 2, \ldots, n\), and grid points for time are chosen as \(t_k = k\tau, k = 0,1,2,\ldots, l\).

We utilize Riemann-Liouville derivative for time-fractional case and implicit difference approximation for second-order space derivatives on the right side, while forward difference approximation with respect to time on left side of Eq. (1.1). Riemann-Liouville integral operator defined in Lemma (1.1) will automatically be utilized when we use Riemann-Liouville fractional derivative for solving the 2D anomalous fractional sub-diffusion equation.

In this way, the following fractional implicit standard point (FISP) iterative scheme at the point \((x_i, y_j, t_{k+1})\) is obtained,

\[
\begin{align*}
    u_{i,j}^{k+1} - u_{i,j}^k &= \frac{1}{r_1} \sum_{s=0}^{k-1} b_s^{(\alpha)} \delta_x^2 (u_{i,j}^{k-s+1} - u_{i,j}^{k-s}) + \frac{1}{r_2} \sum_{s=0}^{k-1} b_s^{(\alpha)} \delta_y^2 (u_{i,j}^{k-s+1} - u_{i,j}^{k-s}) + \tau f_{i,j}^{k+1} + R_{i,j,k}^x
\end{align*}
\]

for \(i, j = 1, 2, 3, \ldots, n-1, k = 0,1,2,\ldots, l\).

Here, the operators \(\delta_x^2\) and \(\delta_y^2\) are defined as,

\[
\begin{align*}
    \delta_x^2 u_{i,j}^{k+1} &= u_{i-1,j}^{k+1} - 2u_{i,j}^{k+1} + u_{i+1,j}^{k+1} \\
    \delta_y^2 u_{i,j}^{k+1} &= u_{i,j-1}^{k+1} - 2u_{i,j}^{k+1} + u_{i,j+1}^{k+1}
\end{align*}
\]

Moreover, \(r_1 = \frac{\tau^\alpha k_1}{(\Delta x)^{2(1+\alpha)}}\), \(r_2 = \frac{\tau^\alpha k_2}{(\Delta y)^{2(1+\alpha)}}\),

\[
|R_{i,j,k}^x| \leq C b_k^{(\alpha)} \tau^\alpha (\tau + \tau (\Delta x)^2).
\]

An alternative form of the above equation is

\[
\begin{align*}
    u_{i,j}^{k+1} &= \frac{1}{(1 - r_1 \delta_x^2 - r_2 \delta_y^2)} [(1 + r_1 (b_1^{(\alpha)} - 1) \delta_x^2 + r_2 (b_1^{(\alpha)} - 1) \delta_y^2) u_{i,j}^k \\
    &\quad - (b_k^{(\alpha)} (r_1 \delta_x^2 + r_2 \delta_y^2)) u_{i,j}^1 + \sum_{s=1}^{k-1} (b_s^{(\alpha)} - b_{s-1}^{(\alpha)}) (r_1 \delta_x^2 + r_2 \delta_y^2) u_{i,j}^{k-s} + \tau f_{i,j}^{k+1}].
\end{align*}
\]
Applying the operators $\delta_x^2$ and $\delta_y^2$ on $u_{i,j}^{k+1}$ and $u_{i,j}^k$, we get the following expression,

$$
(1 + 2r_1 + 2r_2)u_{i,j}^{k+1} = (r_1)(u_{i-1,j}^{k} + u_{i+1,j}^{k}) + (r_2)(u_{i,j-1}^{k} + u_{i,j+1}^{k+1})
$$

$$
+ (r_1(b_1^{(\alpha)} - 1))(u_{i-1,j}^{k} + u_{i+1,j}^{k}) + (r_2(b_1^{(\alpha)} - 1))(u_{i,j-1}^{k} + u_{i,j+1}^{k+1})
$$

$$
+ (1 - 2r_1(b_1^{(\alpha)} - 1) - 2r_2(b_1^{(\alpha)} - 1))u_{i,j}^{k} - (b_2^{(\alpha)}(r_1\delta_x^2 + \delta_y^2))u_{i,j}^{1}
$$

$$
\sum_{s=1}^{k-1} (b_s^{(\alpha)} - b_1^{(\alpha)})(r_1\delta_x^2 + \delta_y^2)u_{i,j}^{k-s} + \tau r_{i,j}^{k+1},
$$

for all $i, j = 1, 2, 3, \ldots, n - 1$, $k = 0, 1, 2, 3, \ldots, l$ with initial and boundary conditions,

$$
u_{i,j}^0 = g(x_i, y_j), \quad u_{i,j}^k = g_1(y_j, t_k), \quad u_{i,j}^k = g_2(y_j, t_k), \quad u_{i,0}^k = g_3(x_i, t_k), \quad u_{i,L}^k = g_4(x_i, t_k),
$$

$0 < x, y < L, 0 \leq t \leq T$.

3. Fractional explicit group (FEG) iterative scheme

The fractional explicit group (FEG) iterative scheme can be formulated by applying Eq. (2.1) on a group of four points of the solution domain. This will result in a $4 \times 4$ system of equations as follows:

$$
\begin{pmatrix}
  k_1 & k_2 & 0 & k_3 \\
  k_2 & k_1 & k_3 & 0 \\
  0 & k_1 & k_2 & 0 \\
  0 & 0 & k_2 & k_1
\end{pmatrix}
\begin{pmatrix}
  u_{i-1,j}^{k} \\
  u_{i,j}^{k} \\
  u_{i,j+1}^{k} \\
  u_{i,j-1}^{k+1}
\end{pmatrix}
= \begin{pmatrix}
  (r_1)(u_{i-1,j}^{k} + u_{i+1,j}^{k}) + (r_2)(u_{i,j-1}^{k} + u_{i,j+1}^{k+1})
  + (r_1(b_1^{(\alpha)} - 1))(u_{i-1,j}^{k} + u_{i+1,j}^{k}) + (r_2(b_1^{(\alpha)} - 1))(u_{i,j-1}^{k} + u_{i,j+1}^{k+1})
  + (1 - 2r_1(b_1^{(\alpha)} - 1) - 2r_2(b_1^{(\alpha)} - 1))u_{i,j}^{k} - (b_2^{(\alpha)}(r_1\delta_x^2 + \delta_y^2))u_{i,j}^{1}
  \sum_{s=1}^{k-1} (b_s^{(\alpha)} - b_1^{(\alpha)})(r_1\delta_x^2 + \delta_y^2)u_{i,j}^{k-s} + \tau r_{i,j}^{k+1},
\end{pmatrix}
$$

where, $k_1 = 1 + 2r_1 + 2r_2$, $k_2 = -r_1$, $k_3 = -r_2$.
Rewrite matrix Eq. (3.1) as,

\[
\begin{pmatrix}
  u_{i,j}^{k+1} \\
  u_{i+1,j}^{k+1} \\
  u_{i,j+1}^{k+1} \\
  u_{i+1,j+1}^{k+1}
\end{pmatrix} = \frac{1}{M} \begin{pmatrix}
  p_1 & p_2 & p_3 & p_4 \\
  p_2 & p_1 & p_4 & p_3 \\
  p_3 & p_4 & p_1 & p_2 \\
  p_4 & p_3 & p_2 & p_1
\end{pmatrix} \begin{pmatrix}
  (r_1)u_{i-1,j}^{k+1} + (r_2)u_{i,j}^{k+1} + T_1 \\
  (r_1)u_{i+1,j}^{k+1} + (r_2)u_{i,j+1}^{k+1} + T_2 \\
  (r_1)u_{i,j+1}^{k+1} + (r_2)u_{i+1,j+1}^{k+1} + T_3 \\
  (r_1)u_{i,j}^{k+1} + (r_2)u_{i+1,j}^{k+1} + T_4
\end{pmatrix},
\]

where,

\[ M = (1 + 2r_1 + 2r_2)^4 - 2(1 + 2r_1 + 2r_2)^2[(r_1)^2 + (r_2)^2] + [(r_1)^2 - (r_2)^2]^2, \]

\[ p_1 = (1 + 2r_1 + 2r_2)((1 + 2r_1 + 2r_2)^2 - (r_1)^2 - (r_2)^2), \]

\[ p_2 = (r_1)((1 + 2r_1 + 2r_2)^2 - (r_1)^2 + (r_2)^2), \]

\[ p_3 = 2(1 + 2r_1 + 2r_2)(r_1)/r_2, \]

\[ p_4 = (r_2)^2(1 + 2r_1 + 2r_2^2) + (r_1)^2 - (r_2)^2]. \]

The FEG iterative method decreases the computational cost by reducing the total number of arithmetic operations used in the iterative procedure. It earlier stored all the arithmetic operations of each 4-point calculations in the form of a group and afterward this group treated as a single element in the entire iterative process generated by Eq. (3.2) by reducing CPU-timings.

4. Fractional implicit rotated point (FIRP) iterative scheme

Another implicit difference scheme can be constructed by rotating Eq. (2.1) an angle 45° to the standard mesh [6]. We have the following fractional implicit rotated point (FIRP) iterative scheme,

\[
(1 + r_1 + r_2)u_{i,j}^{k+1} = \frac{1}{2}(r_1)(u_{i-1,j}^{k+1} + u_{i-1,j-1}^{k+1} + u_{i+1,j+1}^{k+1}) + \frac{1}{2}(r_2)(u_{i+1,j+1}^{k+1} + u_{i,j-1}^{k+1}) \\
+ \frac{1}{2}(r_1)(b_1^{(\alpha)} - 1)(u_{i+1,j}^{k+1} + u_{i,j}^{k+1}) + \frac{1}{2}(r_2)(b_1^{(\alpha)} - 1)(u_{i,j}^{k+1} + u_{i-1,j}^{k+1}) \\
+ \frac{1}{2}(r_1)(b_1^{(\alpha)} - 1)(u_{i,j}^{k+1} - u_{i-1,j}^{k+1}) + \frac{1}{2}(r_2)(b_1^{(\alpha)} - 1)(u_{i,j}^{k+1} - u_{i+1,j}^{k+1})
\]

\[ + \sum_{s=1}^{k-1}(b_1^{(\alpha)} - b_1^{(\alpha)})(r_1\delta_x^2 + r_2\delta_y^2)u_{i,j}^{k-s} + \tau f_{i,j}^{k+1}, \]

for all \(i, j = 1, 2, 3, \ldots, n - 1, k = 0, 1, 2, 3, \ldots, l\) with initial and boundary conditions,

\[ u_{0,j}^0 = g(x, y_j), \quad u_{0,j}^k = g_1(y_j, t_k), \quad u_{k,j}^k = g_2(y_j, t_k), \quad u_{k,0}^k = g_3(x_k, t_k), \quad u_{i,l}^k = g_4(x_i, t_k), \]

\[ 0 < x, y < L, 0 \leq t \leq T. \]

5. Fractional explicit de-coupled group (FEDG) iterative scheme

Similar to FEG iterative scheme, the fractional explicit de-coupled group (FEDG) iterative scheme can be constructed by applying Eq. (4.1) on a group of four points of the solution domain. Since FEDG iterative scheme has the rotation factor due the involvement of Eq. (4.1), therefore, at the end, we got two independent set of matrix equation. In this way, we have a 4 x 4 system of equations of the following form:

\[
\begin{pmatrix}
  l_1 & l_3 & 0 & 0 \\
  l_3 & l_1 & 0 & 0 \\
  0 & l_1 & l_2 & 0 \\
  0 & 0 & l_2 & l_1
\end{pmatrix} \begin{pmatrix}
  u_{i,j}^{k+1} \\
  u_{i+1,j+1}^{k+1} \\
  u_{i,j+1}^{k+1} \\
  u_{i,j+1}^{k+1}
\end{pmatrix} = \frac{1}{M} \begin{pmatrix}
  (r_1)u_{i-1,j}^{k+1} + (r_2)u_{i,j}^{k+1} + T_1^* \\
  (r_1)u_{i+1,j}^{k+1} + (r_2)u_{i,j+1}^{k+1} + T_2^* \\
  (r_1)u_{i,j+1}^{k+1} + (r_2)u_{i+1,j+1}^{k+1} + T_3^* \\
  (r_1)u_{i,j}^{k+1} + (r_2)u_{i+1,j}^{k+1} + T_4^*
\end{pmatrix},
\]
where, $l_1 = 1 + r_1 + r_2$, $l_2 = -\frac{1}{2}(r_1)$, $l_3 = -\frac{1}{2}(r_2)$,

\[ T_1^* = \frac{1}{2}(r_1(b_1^{(\alpha)} - 1))(u_{k-1,i,j+1}^k + u_{k+1,i,j-1}^k) + \frac{1}{2}(r_2(b_1^{(\alpha)} - 1))(u_{k+1,i,j+1}^k + u_{k-1,i,j-1}^k) \\
+ \{1 - r_1(b_1^{(\alpha)} - 1) - r_2(b_1^{(\alpha)} - 1)\}u_{k,i,j}^k - b_k^{(\alpha)}(r_1\delta_x^2 + r_2\delta_y^2)u_{k+1,i,j}^k \\
+ \sum_{s=1}^{k-1}(b_{s-1}^{(\alpha)} - b_s^{(\alpha)})(r_1\delta_x^2 + r_2\delta_y^2)u_{k-s,i,j}^k \tau r_{i+1,j}^{k+1}, \]

\[ T_2^* = \frac{1}{2}(r_1(b_1^{(\alpha)} - 1))(u_{k+1,i,j+2}^k + u_{k-1,i+2,j}^k) + \frac{1}{2}(r_2(b_1^{(\alpha)} - 1))(u_{k+1,i+2,j+2}^k + u_{k-1,i+2,j}^k) \\
+ \{1 - r_1(b_1^{(\alpha)} - 1) - r_2(b_1^{(\alpha)} - 1)\}u_{k,i,j+1}^k - b_k^{(\alpha)}(r_1\delta_x^2 + r_2\delta_y^2)u_{k+1,i,j+1}^k \\
+ \sum_{s=1}^{k-1}(b_{s-1}^{(\alpha)} - b_s^{(\alpha)})(r_1\delta_x^2 + r_2\delta_y^2)u_{k-s,i,j+1}^k \tau r_{i+j+1,i}^{k+1}, \]

\[ T_3^* = \frac{1}{2}(r_1(b_1^{(\alpha)} - 1))(u_{k-1,i,j+2}^k + u_{k+1,i,j-1}^k) + \frac{1}{2}(r_2(b_1^{(\alpha)} - 1))(u_{k+1,i+2,j}^k + u_{k-1,i+2,j}^k) \\
+ \{1 - r_1(b_1^{(\alpha)} - 1) - r_2(b_1^{(\alpha)} - 1)\}u_{k,i+1,j}^k - b_k^{(\alpha)}(r_1\delta_x^2 + r_2\delta_y^2)u_{k+1,i+1,j}^k \\
+ \sum_{s=1}^{k-1}(b_{s-1}^{(\alpha)} - b_s^{(\alpha)})(r_1\delta_x^2 + r_2\delta_y^2)u_{k-s,i+1,j}^k \tau r_{i+1,i}^{k+1}, \]

\[ T_4^* = \frac{1}{2}(r_1(b_1^{(\alpha)} - 1))(u_{k-1,i+2,j}^k + u_{k+1,i,j}^k) + \frac{1}{2}(r_2(b_1^{(\alpha)} - 1))(u_{k+1,i,j+2}^k + u_{k-1,i,j+2}^k) \\
+ \{1 - r_1(b_1^{(\alpha)} - 1) - r_2(b_1^{(\alpha)} - 1)\}u_{k,i+j}^k - b_k^{(\alpha)}(r_1\delta_x^2 + r_2\delta_y^2)u_{k+1,i+j}^k \\
+ \sum_{s=1}^{k-1}(b_{s-1}^{(\alpha)} - b_s^{(\alpha)})(r_1\delta_x^2 + r_2\delta_y^2)u_{k-s,i+j}^k \tau r_{i+j,j}^{k+1}. \]

The matrix Eq. (5.1) can be written as pair of matrix equations,

\[
\begin{pmatrix}
{u_{k+1,i,j}^k} \\
{u_{k+1,i,j+1}^k}
\end{pmatrix} = \frac{1}{S_1} \begin{pmatrix}
{s_1} & s_2 & s_1 \\
{s_2} & s_1 & s_1
\end{pmatrix} \begin{pmatrix}
\frac{1}{2}(r_1)(u_{k+1,i,j+1}^k + u_{k+1,i,j-1}^k) + \frac{1}{2}(r_2)(u_{k+1,i,j+1}^k + u_{k+1,i,j-1}^k) + T_1^* \\
\frac{1}{2}(r_1)(u_{k+1,i,j+1}^k + u_{k+1,i,j-1}^k) + \frac{1}{2}(r_2)(u_{k+1,i,j+1}^k + u_{k+1,i,j-1}^k) + T_2^*
\end{pmatrix}, \tag{5.2}
\]

and

\[
\begin{pmatrix}
{u_{k+1,i,j}^k} \\
{u_{k+1,i,j+1}^k}
\end{pmatrix} = \frac{1}{S_2} \begin{pmatrix}
{s_1} & s_2 & s_1 \\
{s_2} & s_1 & s_1
\end{pmatrix} \begin{pmatrix}
\frac{1}{2}(r_1)(u_{k+1,i,j+1}^k + u_{k+1,i,j-1}^k) + \frac{1}{2}(r_2)(u_{k+1,i,j+1}^k + u_{k+1,i,j-1}^k) + T_1^* \\
\frac{1}{2}(r_1)(u_{k+1,i,j+1}^k + u_{k+1,i,j-1}^k) + \frac{1}{2}(r_2)(u_{k+1,i,j+1}^k + u_{k+1,i,j-1}^k) + T_2^*
\end{pmatrix}, \tag{5.3}
\]

where,

\[ S_1 = (1 + r_1 + r_2)^2 - 1/4(r_2)^2, \]
\[ S_2 = (1 + r_1 + r_2)^2 - 1/4(r_1)^2, \]
\[ s_1^* = s_1 = 1 + r_1 + r_2, \quad s_2 = 1/2(r_2), \quad s_2^* = 1/2(r_1). \]

The iterative process can be started by either matrix Eqs. (5.2) or (5.3). Once the convergence is attained, the values on the remaining points of the solution domain can be evaluated using FISP formula as described in Section 2.

6. Numerical experiments and results

We perform numerical experiments in Mathematica 11 software on a PC of 2GB-RAM to test the capability of the derived schemes FISP, FIRP, FEG, and FEDG in Sections 2, 3, 4, and 5, respectively. The
The exact analytical solution is $E_\infty = \max_{0 \leq k \leq l} \left( \max_{0 \leq j \leq n} (|U_{ij}^k - u_{ij}^k|) \right)$, where, $E_\infty$ denotes the maximum error.

**Example 6.1.** Consider the following time-fractional two dimensional anomalous fractional sub-diffusion equation [13],

$$\frac{\partial u(x, y, t)}{\partial t} = \frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}} \left( \frac{\partial^2 u(x, y, t)}{\partial x^2} + \frac{\partial^2 u(x, y, t)}{\partial y^2} \right) + f(x, y, t), \quad 0 \leq t \leq 1,$$

where,

$$f(x, y, t) = e^{x+y}[(1 + \alpha)t^{\alpha} - \frac{2\Gamma(2 + \alpha)}{\Gamma(1 + 2\alpha)}].$$

The initial and boundary conditions are given by

$$u(x, y, 0) = g(x, y) = 0, \quad 0 < x, y < 1,$$

$$u(0, y, t) = g_1(y, t) = t^{1+\alpha}e^y, \quad u(1, y, t) = g_2(y, t) = t^{1+\alpha}e^{1+y},$$

$$u(x, 0, t) = g_3(x, t) = t^{1+\alpha}e^x, \quad u(x, 1, t) = g_4(x, t) = t^{1+\alpha}e^{1+x}.$$

The exact analytical solution is $u(x, y, t) = t^{1+\alpha}e^{x+y}$.

| Table 1: Numerical results of FISP, FIRP, FEG, and FEDG methods for different value of $\alpha$ of Example 6.1. |
|-----------------|-----------------|-----------------|-----------------|
| $h^{-1}$ | Method | Time(s) | Ave. Err | Max. Err | Time(s) | Ave. Err | Max. Err | Time(s) | Ave. Err | Max. Err |
| 25 | FISP | 119.32 | 69 | 8.8454e-02 | 9.5767e-02 | 98.654 | 56 | 6.3380e-02 | 7.4052e-02 | 77.939 | 46 | 4.7322e-02 | 5.2935e-02 |
| | FIRP | 77.875 | 42 | 8.4231e-02 | 9.5667e-02 | 48.672 | 29 | 6.6743e-02 | 7.3952e-02 | 38.874 | 22 | 4.7432e-02 | 5.2817e-02 |
| | FEG | 87.534 | 43 | 8.3836e-02 | 9.5532e-02 | 56.765 | 30 | 6.8043e-02 | 7.3872e-02 | 44.725 | 23 | 4.7432e-02 | 5.2737e-02 |
| | FEDG | 44.987 | 23 | 8.3706e-02 | 9.5405e-02 | 32.193 | 21 | 6.8373e-02 | 7.3752e-02 | 31.953 | 13 | 4.7542e-02 | 5.2637e-02 |
| 50 | FISP | 166.97 | 111 | 8.7546e-02 | 9.5755e-02 | 143.92 | 88 | 6.8633e-02 | 7.3652e-02 | 133.98 | 76 | 4.7322e-02 | 5.2547e-02 |
| | FIRP | 84.785 | 67 | 8.7886e-02 | 9.5255e-02 | 74.723 | 47 | 6.8443e-02 | 7.3562e-02 | 62.730 | 39 | 4.7052e-02 | 5.2447e-02 |
| | FEG | 94.098 | 66 | 8.7868e-02 | 9.5135e-02 | 84.097 | 47 | 6.8353e-02 | 7.3412e-02 | 66.973 | 41 | 4.7432e-02 | 5.2317e-02 |
| | FEDG | 55.765 | 31 | 8.7306e-02 | 9.8055e-02 | 67.923 | 26 | 6.9535e-02 | 7.3002e-02 | 51.849 | 21 | 4.7922e-02 | 5.2257e-02 |
| 75 | FISP | 256.56 | 213 | 7.7056e-02 | 8.9395e-02 | 217.76 | 101 | 5.8433e-02 | 6.9852e-02 | 196.82 | 91 | 3.7742e-02 | 4.3812e-02 |
| | FIRP | 133.93 | 97 | 7.7636e-02 | 8.5855e-03 | 104.93 | 53 | 5.8933e-03 | 6.9732e-03 | 79.492 | 47 | 3.7632e-03 | 4.3787e-03 |
| | FEG | 145.77 | 98 | 7.7996e-02 | 8.5735e-03 | 117.67 | 54 | 5.8073e-03 | 6.9692e-03 | 79.309 | 48 | 3.7842e-03 | 4.3647e-03 |
| | FEDG | 105.23 | 56 | 7.7426e-02 | 8.5685e-03 | 78.395 | 33 | 5.8713e-03 | 6.9502e-03 | 62.874 | 27 | 3.7732e-03 | 4.3567e-03 |
| 100 | FISP | 398.32 | 342 | 7.7006e-02 | 8.5588e-03 | 345.84 | 303 | 5.8883e-03 | 6.3342e-03 | 284.942 | 227 | 3.7042e-03 | 4.3477e-03 |
| | FIRP | 207.36 | 166 | 7.7846e-03 | 8.5475e-03 | 174.73 | 145 | 5.8633e-03 | 6.9492e-03 | 147.986 | 142 | 3.7742e-03 | 4.3337e-03 |
| | FEG | 218.76 | 166 | 7.7256e-03 | 8.5375e-03 | 185.37 | 145 | 5.8673e-03 | 6.9492e-03 | 156.307 | 143 | 3.7732e-03 | 4.3257e-03 |
| | FEDG | 145.450 | 78 | 7.7457e-03 | 8.5225e-03 | 96.621 | 56 | 5.5333e-03 | 6.9242e-03 | 126.953 | 64 | 3.7652e-03 | 4.3177e-03 |

The numerical results are calculated in terms of elapsed time (in second), number of iterations (Ite.), average absolute error (Ave. Error) and maximum absolute error (Max. Error) by FISP, FIRP, FEG and FEDG iterative methods and these are shown in Table 1. In general, results show that the execution timings of FEDG method is only about $(15 - 24\%)$ of FISP, $(50 - 75\%)$ of FIRP and $(25 - 49\%)$ of FEG method. Figure 1 shows the graph of elapsed time of fractional explicit and group iterative methods against the mesh sizes at $\alpha = 0.25$, while Figures 2 and 3 illustrate the graph of numerical approximate solution by FEDG method at $\alpha = 0.75$. Moreover, since $\tau$ is fixed and as we increases the step size h, the maximum error $E_\infty$ become significantly small as shown in Table 1. Moreover, $E_\infty$ approaches towards zero as $\alpha$ varies from 0 to 1.
7. Conclusion

In this work, we have successfully derived the point and group iterative schemes in the numerical solution of two-dimensional time-fractional anomalous sub-diffusion equations. The numerical experiment is performed to test the authenticity of the derived iterative schemes. In our findings, we observed that the numerical results obtained from FEDG method are significantly efficient in terms of number of iterations and execution of timings than any other method in this paper. In future, we have several open research problems, specially 2D higher-order compact FDEs and 3D PDEs, on which grouping techniques can be applied.

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