Evaluation of Error Probability of Classification
Based on the Analysis of the Bayes Code
Shota Saito and Toshiyasu Matsushima

Abstract

Suppose that we have two training sequences generated by parametrized distributions \( P_{\theta^*_1} \) and \( P_{\theta^*_2} \), where \( \theta^*_1 \) and \( \theta^*_2 \) are unknown. Given training sequences, we study the problem of classifying whether a test sequence was generated according to \( P_{\theta^*_1} \) or \( P_{\theta^*_2} \). This problem can be thought of as a hypothesis testing problem and the weighted sum of type I error and type II error is analyzed. To prove the result, we utilize the analysis of the codeword lengths of the Bayes code. It is shown that the bound of the probability of error is characterized by the terms involving Rényi divergence, the dimension of a parameter space, and the ratio of the length between the training sequences and the test sequence.

Index Terms
Bayes code, Chernoff information, classification, hypothesis testing, Rényi divergence

I. INTRODUCTION

In this paper, we consider the following problem. Let \( y_1 = y_{1,1}, \ldots, y_{1,N} \in A^N \) be a sequence called the 1st training sequence and \( y_2 = y_{2,1}, \ldots, y_{2,N} \in A^N \) be a sequence called the 2nd training sequence. For simplicity, we assume that \( A \) is a finite set in Sections 2 and 3, but our result holds in the case where \( A \) is an infinite set (see the discussion in Section 4). Suppose that \( y_{1,1}, \ldots, y_{1,N} \) are drawn i.i.d. from a probability mass function \( P(\cdot|\theta^*_1) \) and \( y_{2,1}, \ldots, y_{2,N} \) are drawn i.i.d. from a probability mass function \( P(\cdot|\theta^*_2) \).\(^1\) We assume that parametric models \( \{P(\cdot|\theta_1) : \theta_1 \in \Theta \subset \mathbb{R}^d\} \) and \( \{P(\cdot|\theta_2) : \theta_2 \in \Theta \subset \mathbb{R}^d\} \) are known, but the true parameters \( \theta^*_1 \in \Theta \) and \( \theta^*_2 \in \Theta \) are not known, where \( \Theta \subset \mathbb{R}^d \) is a \( d \)-dimensional parameter space. We denote by \( D := \{y_1, y_2\} \) and \( \theta := (\theta_1, \theta_2) \).

Let \( x = x_1, \ldots, x_n \in A^n \) be a sequence called the test sequence. Regarding the length of training sequences \( N \) and the length of a test sequence \( n \), we assume that \( N = \alpha n \) for some \( \alpha > 0 \). Suppose that \( x_1, \ldots, x_n \) are drawn i.i.d. from either \( P(\cdot|\theta^*_1) \) or \( P(\cdot|\theta^*_2) \), but we do not know whether it is from \( P(\cdot|\theta^*_1) \) or \( P(\cdot|\theta^*_2) \). Then, the problem is described as follows:

Given a training data set \( D \) and a test sequence \( x \), we attempt to classify a test sequence \( x \) whether it is generated according to \( P(\cdot|\theta^*_1) \) or \( P(\cdot|\theta^*_2) \).

This work was supported in part by JSPS KAKENHI Grant Numbers JP17K00316, JP17K06446, JP18K11585, JP19K04914, and JP19K14989. Shota Saito and Toshiyasu Matsushima are with the Department of Applied Mathematics, Waseda University, Tokyo 169-8555, Japan (e-mail: shota@aoni.waseda.jp; toshimat@waseda.jp).

\(^1\)In this paper, we sometimes use the notation \( P_\theta(\cdot) \) instead of \( P(\cdot|\theta) \).
As we show in Section 2, this problem can be thought of as a hypothesis testing problem and several previous studies have investigated the probability of error of the hypothesis testing (e.g., [4], [5], [7], [9], [10], [11], [12]). Among these works, the most related work is [7] in which prior distributions of both hypotheses and prior distributions of parameters $w(\theta_1)$ and $w(\theta_2)$ were assumed and the weighted sum of type I error and type II error was investigated.

In this paper, we investigate the probability of error which was analyzed in [7] in a different way from [7]. Merhav and Ziv [7] used the method of types to derive their results. On the other hand, we notice the close relationship between the probability of error and the codeword lengths of the Bayes code (e.g., [1], [3], [6]), and use the analysis of its codeword lengths to derive the result. It is shown that the bound of the probability of error is characterized by the terms involving Rényi divergence [8], the dimension of a parameter space $d$, and the ratio $\alpha$ of the length between the training sequences and the test sequence.

The rest of this paper is organized as follows. In Section II, we formulate the above problem as a hypothesis testing problem and define a decision rule and the probability of error which we investigate in this paper. In Section III, utilizing the analysis of codeword lengths of the Bayes code, we analyze the probability of error. First, Section III-A describes the Bayes code and the analysis of its codeword lengths. Next, Section III-B defines the Rényi divergence, which is an important quantity in producing our result. Finally, Section III-C shows our main result. In Section IV, we discuss our result and compare it with the previous result [7]. In Section V, we conclude this paper.

II. FORMULATION OF HYPOTHESIS TESTING

The problem which we described in Section I can be thought of as a hypothesis testing problem with the following two hypotheses:

- $H_1$: the 1st training sequence $y_1$ and the test sequence $x$ are generated according to the same distribution.
- $H_2$: the 2nd training sequence $y_2$ and the test sequence $x$ are generated according to the same distribution.

A decision rule of this hypothesis testing problem is defined as follows:

**Definition 1:** A decision rule $\Lambda(D)$, derived from the training data set $D$, is a partition of the space $A^n$ of all possible test sequences into two disjoint regions $\Lambda_1(D)$ and $\Lambda_2(D)$ whose union equals $A^n$, i.e., $\Lambda_1(D) \cap \Lambda_2(D) = \emptyset$, $\Lambda_1(D) \cup \Lambda_2(D) = A^n$. If $x \in \Lambda_1(D)$, $H_1$ is accepted.

We assign prior probabilities to two hypotheses and define the probability of error as follows:

**Definition 2:** The conditional error probability $P_{\Lambda}(e|D)$ associated with a decision rule $\Lambda = \Lambda(D)$ is defined as

$$P_{\Lambda}(e|D) := \sum_{i=1}^{2} P(H_i) \sum_{x \in \Lambda_i(D)} P(x|D, H_i),$$

where $P(H_i)$ is a prior probability of hypothesis $H_i$, $\Lambda_i(D)$ is the complement set of $\Lambda_i(D)$, and $P(x|D, H_i)$ is a conditional probability mass function of $x$ given $D$ and the fact that the hypothesis $H_i$ is true.
When we assume
\[ w(\theta) = \prod_{j=1}^{2} w(\theta_j), \]
\[ P(D|\theta) = \prod_{j=1}^{2} P(y_j|\theta_j), \]
\[ P(H_i) = \frac{1}{2} \quad (i = 1, 2), \]
\[ P(x|D, \theta, H_i) = P(x|\theta_i), \]
and use the Bayes theorem, we have
\[ P(\Lambda(e|D) = \frac{1}{2} \sum_{i=1}^{2} \sum_{x \in \Lambda_i(D)} P(x|y_i, H_i), \] (1)
where \( P(x|y_i, H_i) \) is calculated as
\[ P(x|y_i, H_i) = \frac{\int_{\Theta} P(x|\theta_i)P(y_i|\theta_i)w(\theta_i)d\theta_i}{\int_{\Theta} P(y_i|\theta_i)w(\theta_i)d\theta_i}. \] (2)

The decision rule \( \Lambda^* = \Lambda^*(D) = \{\Lambda_1^*(D), \Lambda_2^*(D)\} \) which minimizes (1) is given by
\[ \Lambda_1^*(D) = \{x \in \mathcal{A}^n : P(x|y_1, H_1) \geq P(x|y_2, H_2)\}, \]
\[ \Lambda_2^*(D) = \{x \in \mathcal{A}^n : P(x|y_2, H_2) \geq P(x|y_1, H_1)\}, \]
where ties are broken arbitrarily. The conditional error probability
\[ P_{\Lambda^*}(e|D) = \frac{1}{2} \sum_{i=1}^{2} \sum_{x \in \Lambda_i^*(D)} P(x|y_i, H_i) \] (3)
is the quantity which we shall investigate in Section III.

III. EVALUATION OF \( P_{\Lambda^*}(e|D) \)

A. Bayes Code

Consider a source sequence \( s^k = s_1 \ldots s_k \), where \( s_1 \ldots s_k \) are drawn i.i.d. from a probability mass function \( P(\cdot|\xi^*) \). Suppose that a class of parametrized distribution of a source \( \{P(\cdot|\xi) : \xi \in \Xi \subset \mathbb{R}^d\} \) is known, but the true parameter \( \xi^* \in \Xi \) is unknown. For a lossless compression in this situation, the Bayes code is one of the major universal codes (see, e.g., [1], [3], [6]). For a source sequence \( s^k \), the coding probability \( P_C(s^k) \) of the Bayes code is defined so that it minimizes the Bayes risk function defined as
\[ \int_{\Xi} w(\xi) \sum_{s^k} P(s^k|\xi) \ln \frac{P(s^k|\xi)}{P_C(s^k)} d\xi, \] (4)
where \( w(\xi) \) is a prior probability density function of \( \xi \). In other words, the Bayes code is the optimal code in the sense that it minimizes the mean codeword length averaged with the prior probability density function \( w(\xi) \). The coding probability \( P_C(s^k) \) which minimizes (4) is given by \( \int_{\Xi} P(s^k|\xi)w(\xi)d\xi \) (see, e.g., [6]), and the codeword length of the Bayes code \( \ell_{\text{Bayes}}(s^k) \) is
\[ \ell_{\text{Bayes}}(s^k) = -\ln \int_{\Xi} P(s^k|\xi)w(\xi)d\xi. \]
The mean codeword length of the Bayes code was analyzed up to constant terms (see, e.g., [1], [3]). Furthermore, the codeword length of the Bayes code for an individual source sequence $s^k$ was evaluated (see, e.g., [1], [3]). In view of the result in [1], we have the following lemma.

**Lemma 1 ([1]):** Under a stationary memoryless source and Conditions 1–3 in [1], we have

$$\ell_{\text{Bayes}}(s^k) = \ln \frac{1}{P(s^k|x^*)} + \frac{d}{2} \ln \frac{k}{2\pi} + O(\ln \ln k) \quad \text{a.s.}$$

(5)

**B. Rényi Divergence**

The Rényi divergence is defined as follows:

**Definition 3 ([8]):** For two probability mass functions $p(z), q(z)$ and $\lambda \in (0, 1)$, the Rényi divergence of order $\lambda$ is defined as

$$D_\lambda(p||q) := \frac{1}{\lambda - 1} \ln \sum_z p(z)^\lambda q(z)^{1-\lambda}. $$

The quantity $\sup_{\lambda \in (0, 1)} (1-\lambda) D_\lambda(p||q)$ is known as the Chernoff information. Also, the Rényi divergence contains the relative entropy as its special case, i.e., when $\lambda \to 1$, we have $\lim_{\lambda \to 1} D_\lambda(p||q) = D(p||q)$, where $D(p||q)$ denotes the relative entropy.

**C. Main Result: Bound on $P_{\Lambda^*}(e|D)$**

The next theorem shows our main result.

**Theorem 1:** Under Conditions 1–3 in [1], we have

$$-\frac{1}{n} \ln P_{\Lambda^*}(e|D) \geq \sup_{\lambda \in (0, 1)} (1-\lambda) D_\lambda(P_{\theta_1^*}||P_{\theta_2^*}) + \frac{d}{2n} \ln \left(1 + \frac{1}{\alpha}\right) + O\left(\frac{\ln \ln n}{n}\right) \quad \text{a.s.}$$

(6)

**Proof:** From (3), we have

$$P_{\Lambda^*}(e|D) = \frac{1}{2} \sum_{x \in \mathcal{A}^n} \min \{P(x|y_1, H_1), P(x|y_2, H_2)\}$$

$$\leq \sum_{x \in \mathcal{A}^n} P(x|y_1, H_1)^\lambda P(x|y_2, H_2)^{1-\lambda},$$

(7)

for all $\lambda \in (0, 1)$, where the inequality is due to $1/2 \leq 1$ and $\min\{a, b\} \leq a^\lambda b^{1-\lambda}$ for all $a, b \in \mathbb{R}_+$ and all $\lambda \in (0, 1)$.

Next, we evaluate $P(x|y_i, H_i)$ ($i = 1, 2$) in (7). From Lemma 1, for $i = 1, 2$, we have

$$\int_{\Theta} P(y_i|\theta_i) w(\theta_i) d\theta_i = P(y_i|\theta_i^*) \left(\frac{N}{2\pi}\right)^{-\frac{d}{2}} e^{O(\ln \ln N)} \quad \text{a.s.}$$

(8)

$$\int_{\Theta} P(x|\theta_i) P(y_i|\theta_i) w(\theta_i) d\theta_i = P(x|\theta_i^*) P(y_i|\theta_i^*) \left(\frac{n+N}{2\pi}\right)^{-\frac{d}{2}} e^{O(\ln \ln (n+N))} \quad \text{a.s.}$$

(9)

Substituting (8) and (9) for (2) and recalling that $N = \alpha n$, we obtain

$$P(x|y_i, H_i) = P(x|\theta_i^*) \left(\frac{1+\alpha}{\alpha}\right)^{-\frac{d}{2}} e^{O(\ln \ln n)} \quad \text{a.s.}$$

$$= \prod_{j=1}^{n} P(x_j|\theta_i^*) \left(\frac{1+\alpha}{\alpha}\right)^{-\frac{d}{2}} e^{O(\ln \ln n)} \quad \text{a.s.}$$

(10)
for \( i = 1, 2 \).

Finally, by the combination of (7) and (10) and some calculation, we complete the proof. \( \square \)

IV. DISCUSSION

As \( n \to \infty \), the right-hand side of (6) approaches the Chernoff information \( \sup_{\lambda \in (0, 1)} (1 - \lambda) D_{\lambda}(P_{\theta_1} \| P_{\theta_2}) \), which is the best asymptotic achievable exponent of the weighted sum of type I error and type II error when two probability distributions are known (see, e.g., [2]).

Next, we compare our result with the previous result [7]. Merhav and Ziv [7] considered the case where training sequences and a test sequence are generated from a Markov source, but their results can be applied for i.i.d. setup as we have described in this paper. From Theorem 2 of [7], it holds that

\[
\lim_{n \to \infty} -\frac{1}{n} \ln P_{\Lambda^*}(e|D) = \min \left\{ \inf_{\theta \in U_{\theta_1}} \rho(\theta_1, \theta), \inf_{\theta \in U_{\theta_2}} \rho(\theta_2, \theta) \right\} \text{ a.s.,} \tag{11}
\]

where

\[
\rho(\theta, \theta) := (1 + \alpha) \left\{ H \left( \frac{\alpha}{1 + \alpha} P_{\theta_1} + \frac{1}{1 + \alpha} P_{\theta} \right) - \frac{\alpha}{1 + \alpha} H(P_{\theta_1}) - \frac{1}{1 + \alpha} H(P_{\theta}) \right\},
\]

and \( U_{\theta_i} := \{ \theta \in \Theta : \rho(\theta_i, \theta) \geq \rho(\theta_j, \theta) \} \). The previous result (11) was asymptotic analysis in the sense that the exponent of the probability of error was evaluated in the case of \( n \to \infty \). On the other hand, our result (6) is non-asymptotic analysis and the exponent of the probability of error is evaluated up to \( O(\ln \ln n / n) \). Further, it should be noted that the assumption that a source alphabet \( \mathcal{A} \) is a finite set is crucial in [7] because their analysis relies on the method of types. On the other hand, our result holds for both a finite set \( \mathcal{A} \) and an infinite set \( \mathcal{A} \) because (5) holds for a continuous random variable (see, e.g., [1]).

V. CONCLUSION

We have evaluated the exponent of the error probability \( P_{\Lambda^*}(e|D) \) up to \( O(\ln \ln n / n) \) and shown that the bound of \( P_{\Lambda^*}(e|D) \) is characterized by the Chernoff information \( \sup_{\lambda \in (0, 1)} (1 - \lambda) D_{\lambda}(P_{\theta_1} \| P_{\theta_2}) \) and the term involving \( d \) (the dimension of a parameter space) and \( \alpha \) (the ratio of the length between the training sequences and the test sequence).

REFERENCES

[1] B. S. Clarke and A. R. Barron, "Information-theoretic asymptotics of Bayes methods," IEEE Trans. Inf. Theory, vol.36, no.3, pp.453–471, May 1990.

[2] T. M. Cover and J. A. Thomas, Elements of Information Theory, 2nd ed. New York, USA: John Wiley & Sons, 2006.

[3] M. Gotoh, T. Matsushima, and S. Hirasawa, “A generalization of B. S. Clarke and A. R. Barron’s asymptotics of Bayes codes for FSMX sources,” IEICE Trans. Fundamentals, vol.E81-A, no.10, pp.2123–2132, Oct. 1998.

[4] M. Gutman, “Asymptotically optimal classification for multiple tests with empirically observed statistics,” IEEE Trans. Inf. Theory, vol. 35, no. 2, pp.401–408, Mar. 1989.

The notation \( H(\cdot) \) denotes the entropy.
[5] B. G. Kelly, A. B. Wagner, T. Tularak, and P. Viswanath, “Classification of homogeneous data with large alphabets,” IEEE Trans. Inf. Theory, vol. 59, no. 2, pp. 782–795, Feb. 2013.

[6] T. Matsushima, H. Inazumi, and S. Hirasawa, “A class of distortionless codes designed by Bayes decision theory,” IEEE Trans. Inf. Theory, vol. 37, no. 5, pp. 1288–1293, Sept. 1991.

[7] N. Merhav and J. Ziv, “A Bayesian approach for classification of Markov sources,” IEEE Trans. Inf. Theory, vol. 37, no. 4, pp. 1067–1071, July 1991.

[8] A. Rényi, “On measures of entropy and information,” in Proc. 4th Berkley Symposium on Mathematics, Statistics and Probability, pp. 547–561, 1961.

[9] J. Unnikrishnan, “Asymptotically optimal matching of multiple sequences to source distributions and training sequences,” IEEE Trans. Inf. Theory, vol. 61, no. 1, pp. 452–468, Jan. 2015.

[10] J. Unnikrishnan and D. Huang, “Weak convergence analysis of asymptotically optimal hypothesis tests,” IEEE Trans. Inf. Theory, vol. 62, no. 7, pp. 4285–4299, July 2016.

[11] L. Zhou, V. Y. F. Tan, and M. Motani, “Second-order asymptotically optimal statistical classification,” in Proc. IEEE Int. Symp. Inf. Theory (ISIT), Paris, France, pp. 231–235, July 2019.

[12] J. Ziv, “On classification with empirically observed statistics and universal data compression,” IEEE Trans. Inf. Theory, vol. 34, no. 2, pp. 278–286, Mar. 1988.