OPERATOR INEQUALITIES OF JENSEN TYPE

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Abstract. We present some generalized Jensen type operator inequalities involving sequences of self-adjoint operators. Among other things, we prove that if \( f : [0, \infty) \rightarrow \mathbb{R} \) is a continuous convex function with \( f(0) \leq 0 \), then

\[
\sum_{i=1}^{n} f(C_i) \leq f \left( \sum_{i=1}^{n} C_i \right) - \delta_f \sum_{i=1}^{n} C_i \leq f \left( \sum_{i=1}^{n} C_i \right) - \frac{1}{2} \sum_{i=1}^{n} C_i (M - \frac{1}{2}) \]

for all operators \( C_i \) such that \( 0 \leq C_i \leq M \leq \sum_{i=1}^{n} C_i \) (\( i = 1, \ldots, n \)) for some scalar \( M \geq 0 \), where \( \widetilde{C}_i = \frac{1}{2} - \left| \frac{C_i}{M} - \frac{1}{2} \right| \) and \( \delta_f = f(0) + f(M) - 2f \left( \frac{M}{2} \right) \).

1. Introduction and Preliminaries

Let \( \mathcal{B}(\mathcal{H}) \) be the \( C^* \)-algebra of all bounded linear operators on a complex Hilbert space \( \mathcal{H} \) and \( I \) denote the identity operator. If \( \dim \mathcal{H} = n \), then we identify \( \mathcal{B}(\mathcal{H}) \) with the \( C^* \)-algebra \( M_n(\mathbb{C}) \) of all \( n \times n \) matrices with complex entries. Let us endow the real space \( \mathbb{B}(\mathcal{H}) \) of all self-adjoint operators in \( \mathcal{B}(\mathcal{H}) \) with the usual operator order \( \leq \) defined by the cone of positive operators of \( \mathbb{B}(\mathcal{H}) \).

If \( T \in \mathbb{B}(\mathcal{H}) \), then \( m = \inf \{ \langle Tx, x \rangle : \|x\| = 1 \} \) and \( M = \sup \{ \langle Tx, x \rangle : \|x\| = 1 \} \) are called the bounds of \( T \). We denote by \( \sigma(J) \) the set of all self-adjoint operators on \( \mathcal{H} \) with spectra contained in \( J \). All real-valued functions are assumed to be continuous in this paper. A real valued function \( f \) defined on an interval \( J \) is said to be operator convex if \( f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda)f(B) \) for all \( A, B \in \sigma(J) \) and all \( \lambda \in [0,1] \). If the function \( f \) is operator convex, then the so-called Jensen operator inequality \( f(\Phi(A)) \leq \Phi(f(A)) \) holds for any unital positive linear map \( \Phi \) on \( \mathbb{B}(\mathcal{H}) \) and any \( A \in \sigma(J) \). The reader is referred to \([3, 4, 8]\) for more information about operator convex functions and other versions of the Jensen operator inequality. It should be remarked that if \( f \) is a real convex function, but not operator convex, then the Jensen operator inequality may not hold. To see this, consider the convex (but not operator convex) function \( f(t) = t^4 \) defined on \([0, \infty)\) and the positive mapping \( \Phi : \mathcal{M}_3(\mathbb{C}) \rightarrow \mathcal{M}_2(\mathbb{C}) \) defined by

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\[ \Phi((a_{ij})_{1 \leq i,j \leq 3}) = (a_{ij})_{1 \leq i,j \leq 2} \] for any \( A = (a_{ij})_{1 \leq i,j \leq 3} \in \mathcal{M}_3(\mathbb{C}) \). If
\[ A = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}, \]
then there is no relationship between
\[ f(\Phi(A)) = \begin{pmatrix} 36 & 46 \\ 46 & 59 \end{pmatrix} \quad \text{and} \quad \Phi(f(A)) = \begin{pmatrix} 36 & 48 \\ 48 & 68 \end{pmatrix} \]
in the usual operator order.

Recently, in [6] a version of the Jensen operator inequality was given without operator convexity as follows:

**Theorem A.** [6, Theorem 1] Let \( (A_1, \ldots, A_n) \) be an \( n \)-tuple of operators \( A_i \in \mathbb{B}(\mathcal{H}) \) with bounds \( m_i \) and \( M_i \), \( m_i \leq M_i \), and let \( (\Phi_1, \ldots, \Phi_n) \) be an \( n \)-tuple of positive linear mappings \( \Phi_i \) on \( \mathbb{B}(\mathcal{H}) \) such that \( \sum_{i=1}^n \Phi_i(I) = I \). If
\[ (m_C, M_C) \cap [m_i, M_i] = \emptyset \quad (1.1) \]
for all \( 1 \leq i \leq n \), where \( m_C \) and \( M_C \) with \( m_C \leq M_C \) are bounds of the self-adjoint operator \( C = \sum_{i=1}^n \Phi_i(A_i) \), then
\[ f \left( \sum_{i=1}^n \Phi_i(A_i) \right) \leq \sum_{i=1}^n \Phi_i(f(A_i)) \quad (1.2) \]
holds for every convex function \( f : J \to \mathbb{R} \) provided that the interval \( J \) contains all \( m_i, M_i \); see also [7].

Another variant of the Jensen operator inequality is the so-called Jensen–Mercer operator inequality [5] asserting that if \( f \) is a real convex function on an interval \([m, M]\), then
\[ f \left( M + m - \sum_{i=1}^n \Phi_i(A_i) \right) \leq f(M) + f(m) - \sum_{i=1}^n \Phi_i(f(A_i)), \]
where \( \Phi_1, \ldots, \Phi_n \) are positive linear maps on \( \mathbb{B}(\mathcal{H}) \) with \( \sum_{i=1}^n \Phi_i(I) = I \) and \( A_1, \ldots, A_n \in \sigma([m, M]) \).

Recently, in [9] an extension of the Jensen–Mercer operator inequality was presented as follows:

**Theorem B.** [9, Corollary 2.3] Let \( f \) be a convex function on an interval \( J \). Let \( A_i, B_i, C_i, D_i \in \sigma(J) \) \( (i = 1, \ldots, n) \) such that \( A_i + D_i = B_i + C_i \) and \( A_i \leq m \leq B_i, C_i \leq M \leq D_i \). Let \( \Phi_1, \ldots, \Phi_n \) be positive linear maps on \( \mathbb{B}(\mathcal{H}) \) with
\[ \sum_{i=1}^{n} \Phi_i(I) = I. \]

Then
\[ f \left( \sum_{i=1}^{n} \Phi_i(B_i) \right) + f \left( \sum_{i=1}^{n} \Phi_i(C_i) \right) \leq \sum_{i=1}^{n} \Phi_i(f(A_i)) + \sum_{i=1}^{n} \Phi_i(f(D_i)). \quad (1.3) \]

The authors of [9] used inequality (1.3) to obtain some operator inequalities. In particular, they gave a generalization of the Petrović operator inequality as follows:

**Theorem C.** [9, Corollary 2.5] Let \( A, D, B_i \in \sigma(J) \) \( (i = 1, \cdots, n) \) such that \( A + D = \sum_{i=1}^{n} B_i \) and \( A \leq m \leq B_i \leq M \leq D \) \( (i = 1, \cdots, n) \) for two real numbers \( m < M \). If \( f \) is convex on \( J \), then
\[ \sum_{i=1}^{n} f(B_i) \leq (n - 1) f \left( \frac{1}{n-1} A \right) + f(D). \]

If \( f : [0, \infty) \to \mathbb{R} \) is a convex function such that \( f(0) = 0 \), then
\[ f(a) + f(b) \leq f(a + b) \quad (1.4) \]
for all scalars \( a, b \geq 0 \). However, if the scalars \( a, b \) are replaced by two positive operators, this inequality may not hold. For example if \( f(t) = t^2 \) and \( A, B \) are the following two positive matrices
\[
A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix},
\]
then a straightforward computation reveals that there is no relationship between \( A^2 + B^2 \) and \( (A + B)^2 \) under the operator order. Many authors tried to obtain some operator extensions of (1.4). In [10], it was shown that
\[ f(A + B) \leq f(A) + f(B) \]
for all non-negative operator monotone functions \( f : [0, \infty) \to [0, \infty) \) if and only if \( AB + BA \) is positive.

Another operator extension of (1.4) was established in [9]

**Theorem D.** [9, Corollary 2.9] If \( f : [0, \infty) \to [0, \infty) \) is a convex function with \( f(0) \leq 0 \), then \( f(A) + f(B) \leq f(A + B) \) for all invertible positive operators \( A, B \) such that \( A \leq MI \leq A + B \) and \( B \leq MI \leq A + B \) for some scalar \( M \geq 0 \).

Some other operator extensions of (1.4) can be found in [1, 2, 11]. In this paper, as a continuation of [9], we extend inequality (1.3), refine (1.3) and improve some
of our results in [9]. Some applications such as further refinements of the Petrović operator inequality and the Jensen–Mercer operator inequality are presented as well.

2. Results

To presenting our results, we introduce the abbreviation:

\[
\delta_f = f(m) + f(M) - 2f \left( \frac{m + M}{2} \right)
\]

for \( f : [m, M] \to \mathbb{R}, m < M \).

We need the following lemma may be found in [7, Lemma 2]. We give a proof for the sake of completeness.

Lemma 2.1. Let \( A \in \sigma([m, M]) \), for some scalars \( m < M \). Then

\[
f(A) \leq \frac{M - A}{M - m} f(m) + \frac{A - m}{M - m} f(M) - \delta_f \tilde{A} \tag{2.1}
\]

holds for every convex function \( f : [m, M] \to \mathbb{R} \), where

\[
\tilde{A} = \frac{1}{2} - \frac{1}{M - m} \left| A - \frac{m + M}{2} \right|.
\]

If \( f \) is concave on \([m, M]\), then inequality (2.1) is reversed.

Proof. First assume that \( a, b \in [m, M] \) and \( \lambda \in [0, 1/2] \) so that \( \lambda \leq 1 - \lambda \). Then

\[
f(\lambda a + (1 - \lambda)b) = f \left( 2\lambda \frac{a + b}{2} + (1 - 2\lambda)b \right)
\leq 2\lambda f \left( \frac{a + b}{2} \right) + (1 - 2\lambda)f(b)
= \lambda f(a) + (1 - \lambda)f(b) - \lambda \left( f(a) + f(b) - 2f \left( \frac{a + b}{2} \right) \right).
\]

It follows that

\[
f(\lambda a + (1 - \lambda)b)
\leq \lambda f(a) + (1 - \lambda)f(b) - \min\{\lambda, 1 - \lambda\} \left( f(a) + f(b) - 2f \left( \frac{a + b}{2} \right) \right) \tag{2.2}
\]
for all \(a, b \in [m, M]\) and all \(\lambda \in [0, 1]\). If \(t \in [m, M]\), then by using (2.2) with \(\lambda = M-t/M-m\), \(a = m\) and \(b = M\) we obtain

\[
f(t) = f\left(\frac{M-t}{M-m}m + \frac{t-m}{M-m}M\right) \leq \frac{M-t}{M-m}f(m) + \frac{t-m}{M-m}f(M)
- \min\left\{\frac{M-t}{M-m}, \frac{t-m}{M-m}\right\} \left(f(m) + f(M) - 2f\left(\frac{m+M}{2}\right)\right)
\]

(2.3)

for any \(t \in [m, M]\). Since \(\min\{\frac{M-t}{M-m}, \frac{t-m}{M-m}\} = \frac{1}{2} - \frac{1}{M-m}|t - \frac{m+M}{2}|\), we have from (2.3) that

\[
f(t) \leq \frac{M-t}{M-m}f(m) + \frac{t-m}{M-m}f(M)
- \left(\frac{1}{2} - \frac{1}{M-m}\right)\left(t - \frac{m+M}{2}\right) \left(f(m) + f(M) - 2f\left(\frac{m+M}{2}\right)\right),
\]

(2.4)

for all \(t \in [m, M]\). Now if \(A \in \sigma([m, M])\), then by utilizing the functional calculus to (2.4) we obtain (2.1). \(\square\)

In the next theorem we present a generalization of [9, Theorem 2.1].

**Theorem 2.2.** Let \(\Phi_i, \overline{\Phi}_i, \Psi_i, \overline{\Psi}_i\) be positive linear mappings on \(\mathcal{B}(\mathcal{H})\) such that \(\sum_{i=1}^{n_1} \Phi_i(I) = \alpha I, \sum_{i=1}^{n_2} \overline{\Phi}_i(I) = \beta I, \sum_{i=1}^{n_3} \Psi_i(I) = \gamma I, \sum_{i=1}^{n_4} \overline{\Psi}_i(I) = \delta I\) for some real numbers \(\alpha, \beta, \gamma, \delta > 0\). Let \(A_i (i = 1, \ldots, n_1), D_i (i = 1, \ldots, n_2), C_i (i = 1, \ldots, n_3)\) and \(B_i (i = 1, \ldots, n_4)\) be operators in \(\sigma(J)\) such that \(A_i \leq m \leq B_i, C_i \leq M \leq D_i\) for two real numbers \(m < M\). If

\[
\frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(A_i) + \frac{1}{\delta} \sum_{i=1}^{n_2} \overline{\Phi}_i(D_i) = \frac{1}{\gamma} \sum_{i=1}^{n_3} \Psi_i(C_i) + \frac{1}{\beta} \sum_{i=1}^{n_4} \overline{\Psi}_i(B_i),
\]

(2.5)

then

\[
f\left(\frac{1}{\gamma} \sum_{i=1}^{n_3} \Psi_i(C_i)\right) + f\left(\frac{1}{\beta} \sum_{i=1}^{n_4} \overline{\Psi}_i(B_i)\right) \leq \frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(f(A_i)) + \frac{1}{\delta} \sum_{i=1}^{n_2} \overline{\Phi}_i(f(D_i)) - \delta f\tilde{X}
\]

\[
\leq \frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(f(A_i)) + \frac{1}{\delta} \sum_{i=1}^{n_2} \overline{\Phi}_i(f(D_i))
\]

(2.6)

holds for every convex function \(f : J \to \mathbb{R}\), where

\[
\tilde{X} = 1 - \frac{1}{M-m}\left(\frac{1}{\beta} \sum_{i=1}^{n_4} \overline{\Psi}_i(B_i) - \frac{m+M}{2}\right) + \frac{1}{\gamma} \sum_{i=1}^{n_3} \Psi_i(C_i) - \frac{m+M}{2}\right).
\]

If \(f\) is concave, then the reverse inequalities are valid in (2.6).
Proof. We prove only the case when \( f \) is convex. Let \([m, M] \subseteq J\). It follows from the convexity of \( f \) on \( J \) that
\[
f(t) \geq \frac{M-t}{M-m} f(m) + \frac{t-m}{M-m} f(M)
\]
for all \( t \in J \setminus [m, M] \). Hence, by \( A_i \leq m \) and \( D_i \geq M \) we have
\[
f(A_i) \geq \frac{M - A_i}{M-m} f(m) + \frac{A_i - m}{M-m} f(M) \quad (i = 1, \ldots, n_1)
\]
and similarly
\[
f(D_i) \geq \frac{M - D_i}{M-m} f(m) + \frac{D_i - m}{M-m} f(M) \quad (i = 1, \ldots, n_2).
\]
Applying the positive linear mappings \( \Phi_i \) and \( \Psi_i \), respectively, to both sides of (2.8) and (2.9) and summing we get
\[
\frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(f(A_i)) \geq \frac{M - \frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(A_i)}{M-m} f(m) + \frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(A_i) - m \quad (2.10)
\]
and
\[
\frac{1}{\beta} \sum_{i=1}^{n_2} \Phi_i(f(D_i)) \geq \frac{M - \frac{1}{\beta} \sum_{i=1}^{n_2} \Phi_i(D_i)}{M-m} f(m) + \frac{1}{\beta} \sum_{i=1}^{n_2} \Phi_i(D_i) - m \quad (2.11)
\]
On the other hand, taking into account that \( m \leq \frac{1}{\beta} \sum_{i=1}^{n_1} \Psi_i(B_i), \frac{1}{\gamma} \sum_{i=1}^{n_3} \Psi_i(C_i) \leq M \) and using Lemma 2.1 we obtain
\[
f \left( \frac{1}{\beta} \sum_{i=1}^{n_2} \Psi_i(B_i) \right) \leq \frac{M - \frac{1}{\beta} \sum_{i=1}^{n_1} \Psi_i(B_i)}{M-m} f(m) + \frac{1}{\beta} \sum_{i=1}^{n_1} \Psi_i(B_i) - m \quad (2.12)
\]
and
\[
f \left( \frac{1}{\gamma} \sum_{i=1}^{n_2} \Psi_i(C_i) \right) \leq \frac{M - \frac{1}{\gamma} \sum_{i=1}^{n_3} \Psi_i(C_i)}{M-m} f(m) + \frac{1}{\gamma} \sum_{i=1}^{n_3} \Psi_i(C_i) - m \quad (2.13)
\]
where \( \overline{B} = \frac{1}{2} - \frac{1}{M-m} \left| \frac{1}{\beta} \sum_{i=1}^{n_1} \Psi_i(B_i) - \frac{m + M}{2} \right| \) and \( \overline{C} = \frac{1}{2} - \frac{1}{M-m} \left| \frac{1}{\gamma} \sum_{i=1}^{n_3} \Psi_i(C_i) - \frac{m + M}{2} \right| \).

Adding two inequalities (2.12) and (2.13) and putting
\[
\overline{X} = 1 - \frac{1}{M-m} \left( \frac{1}{\beta} \sum_{i=1}^{n_2} \Psi_i(B_i) - \frac{m + M}{2} \right) + \frac{1}{\gamma} \sum_{i=1}^{n_3} \Psi_i(C_i) - \frac{m + M}{2}
\]
we obtain

\[
\begin{align*}
&f\left(\frac{1}{\beta} \sum_{i=1}^{n_1} \Psi_i(B_i)\right) + f\left(\frac{1}{\gamma} \sum_{i=1}^{n_2} \Psi_i(C_i)\right) \\
&\leq \frac{2M - \frac{1}{\beta} \sum_{i=1}^{n_1} \Psi_i(B_i) - \frac{1}{\gamma} \sum_{i=1}^{n_2} \Psi_i(C_i)}{M - m} f(m) \\
&+ \frac{\frac{1}{\gamma} \sum_{i=1}^{n_2} \Psi_i(D_i) - 2m}{M - m} (\bar{f} - \bar{X}) \\
&= \frac{2M - \frac{1}{\beta} \sum_{i=1}^{n_1} \Phi_i(A_i) - \frac{1}{\gamma} \sum_{i=1}^{n_2} \Phi_i(D_i)}{M - m} f(m) \\
&+ \frac{\frac{1}{\gamma} \sum_{i=1}^{n_2} \Phi_i(D_i) - 2m}{M - m} (\bar{f} - \bar{X}) \quad \text{(by (2.5))}
\end{align*}
\]

which is the first inequality in (2.6).

Furthermore, \( m \leq \frac{1}{\beta} \sum_{i=1}^{n_1} \Psi_i(B_i), \frac{1}{\gamma} \sum_{i=1}^{n_2} \Psi_i(C_i) \leq M \). The numerical inequality

\[
\left| t - \frac{m + M}{2} \right| \leq \frac{M - m}{2} \quad (m \leq t \leq M)
\]

yields that

\[
\left| \frac{1}{\beta} \sum_{i=1}^{n_1} \Psi_i(B_i) - \frac{m + M}{2} \right| + \left| \frac{1}{\gamma} \sum_{i=1}^{n_2} \Psi_i(C_i) - \frac{m + M}{2} \right| \leq M - m.
\]

Therefore \( \bar{X} \geq 0 \). Moreover, \( f \) is convex on \([m, M]\). Hence \( \delta f \geq 0 \). So the second inequality in (2.6) holds. \( \square \)

Remark 2.3. We can conclude some other versions of inequality (2.6). In fact, under the assumptions in Theorem 2.2 the following inequalities hold true:

1. \( \frac{1}{\gamma} \sum_{i=1}^{n_1} \Psi_i(f(C_i)) + \frac{1}{\beta} \sum_{i=1}^{n_4} \Psi_i(f(B_i)) \leq f\left(\frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(A_i)\right) + f\left(\frac{1}{\delta} \sum_{i=1}^{n_2} \Phi_i(D_i)\right) - \delta f \bar{X}_2 \)

\[
\leq f\left(\frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(A_i)\right) + f\left(\frac{1}{\delta} \sum_{i=1}^{n_2} \Phi_i(D_i)\right);
\]

2. \( f\left(\frac{1}{\gamma} \sum_{i=1}^{n_1} \Psi_i(C_i)\right) + \frac{1}{\beta} \sum_{i=1}^{n_4} \Psi_i(f(B_i)) \leq f\left(\frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(A_i)\right) + \frac{1}{\delta} \sum_{i=1}^{n_2} \Phi_i(f(D_i)) - \delta f \bar{X}_3 \)

\[
\leq f\left(\frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(A_i)\right) + \frac{1}{\delta} \sum_{i=1}^{n_2} \Phi_i(f(D_i)),
\]

in which

\[
\bar{X}_2 = 1 - \frac{1}{M - m} \left[ \frac{1}{\gamma} \sum_{i=1}^{n_1} \Psi_i \left( \left| C_i - \frac{M + m}{2} \right| \right) + \frac{1}{\beta} \sum_{i=1}^{n_4} \Psi_i \left( \left| B_i - \frac{M + m}{2} \right| \right) \right],
\]

\[
\bar{X}_3 = 1 - \frac{1}{M - m} \left[ \frac{1}{\gamma} \sum_{i=1}^{n_1} \Psi_i(C_i) - \frac{M + m}{2} \right] + \frac{1}{\beta} \sum_{i=1}^{n_4} \Psi_i \left( \left| B_i - \frac{M + m}{2} \right| \right).
\]
Before giving an example, we present some special cases of Theorem 2.2 which are useful in our applications. The next corollary provides a refinement of [9, Theorem 2.1].

**Corollary 2.4.** Let $f$ be a convex function on an interval $J$. Let $A, B, C, D \in \sigma(J)$ such that $A + D = B + C$ and $A \leq m \leq B, C \leq M$ for two real numbers $m < M$. If $\Phi$ is a unital positive linear map on $\mathbb{B}(\mathcal{H})$, then

$$f(\Phi(B)) + f(\Phi(C)) \leq \Phi(f(A)) + \Phi(f(D)) - \delta_f \bar{X},$$

where

$$\bar{X} = 1 - \frac{1}{M - m} \left( \Phi(B) - \frac{m + M}{2} \right) + \left( \Phi(C) - \frac{m + M}{2} \right).$$

In particular,

$$f(B) + f(C) \leq f(A) + f(D) - \delta_f \bar{X} \leq f(A) + f(D). \quad (2.15)$$

If $f$ is concave on $J$, then inequalities (2.14) and (2.15) are reversed.

Another special case of Theorem 2.2 leads to a refinement of [9, Corollary 2.3].

**Corollary 2.5.** Let $f$ be a convex function on an interval $J$. Let $A_i, B_i, C_i, D_i \in \sigma(J)$ $(i = 1, \cdots, n)$ such that $A_i + D_i = B_i + C_i$ and $A_i \leq m \leq B_i, C_i \leq M \leq D_i$ $(i = 1, \cdots, n)$. Let $\Phi_1, \cdots, \Phi_n$ be positive linear mappings on $\mathbb{B}(\mathcal{H})$ with $\sum_{i=1}^{n} \Phi_i(I) = I$. Then

1. $f \left( \sum_{i=1}^{n} \Phi_i(B_i) \right) + f \left( \sum_{i=1}^{n} \Phi_i(C_i) \right) \leq \sum_{i=1}^{n} \Phi_i(f(A_i)) + \sum_{i=1}^{n} \Phi_i(f(D_i)) - \delta_f \bar{X}_1$

$$\leq \sum_{i=1}^{n} \Phi_i(f(A_i)) + \sum_{i=1}^{n} \Phi_i(f(D_i));$$

2. $\sum_{i=1}^{n} \Phi_i(f(B_i)) + \sum_{i=1}^{n} \Phi_i(f(C_i)) \leq f \left( \sum_{i=1}^{n} \Phi_i(A_i) \right) + f \left( \sum_{i=1}^{n} \Phi_i(D_i) \right) - \delta_f \bar{X}_2$

$$\leq f \left( \sum_{i=1}^{n} \Phi_i(A_i) \right) + f \left( \sum_{i=1}^{n} \Phi_i(D_i) \right);$$

3. $\sum_{i=1}^{n} \Phi_i(f(B_i)) + f \left( \sum_{i=1}^{n} \Phi_i(C_i) \right) \leq f \left( \sum_{i=1}^{n} \Phi_i(D_i) \right) + \sum_{i=1}^{n} \Phi_i(f(A_i)) - \delta_f \bar{X}_3$

$$\leq f \left( \sum_{i=1}^{n} \Phi_i(D_i) \right) + \sum_{i=1}^{n} \Phi_i(f(A_i));$$
where

\[ \tilde{X}_1 = 1 - \frac{1}{M - m} \left[ \sum_{i=1}^{n} \Phi_i(B_i) - \frac{m + M}{2} \right] + \left[ \sum_{i=1}^{n} \Phi_i(C_i) - \frac{m + M}{2} \right], \]

\[ \tilde{X}_2 = 1 - \frac{1}{M - m} \left[ \sum_{i=1}^{n} \Phi_i \left( \left| B_i - \frac{m + M}{2} \right| \right) + \sum_{i=1}^{n} \Phi_i \left( \left| C_i - \frac{m + M}{2} \right| \right) \right], \]

\[ \tilde{X}_3 = 1 - \frac{1}{M - m} \left[ \sum_{i=1}^{n} \Phi_i \left( \left| B_i - \frac{m + M}{2} \right| \right) + \sum_{i=1}^{n} \Phi_i(C_i) - \frac{m + M}{2} \right]. \]

Now we give an example to show that how Theorem 2.2 works.

**Example 2.6.** Let \( n_i = 1 \) for \( i = 1, 2, 3, 4 \) and let \( f(t) = t^4 \). The function \( f \) is convex but not operator convex [3]. Let \( \Phi, \Psi, \Psi = \Phi \) in which

\[ \Phi : \mathcal{M}_3(\mathbb{C}) \to \mathcal{M}_2(\mathbb{C}), \quad \Phi((a_{ij})_{1 \leq i, j \leq 3}) = (a_{ij})_{1 \leq i, j \leq 2}. \]

If

\[ A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 2 \\ 1 & 2 & -5 \end{pmatrix}, \quad D = \begin{pmatrix} 9 & 1 & 1 \\ 1 & 10 & 2 \\ 1 & 2 & 15 \end{pmatrix}, \quad C = \begin{pmatrix} 4 & 1 & 2 \\ 1 & 4 & 1 \\ 2 & 1 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 6 & -1 & 1 \\ -1 & 7 & 1 \\ 1 & 1 & 5 \end{pmatrix}, \]

then \( \Phi(A) + \Phi(D) = \Phi(C) + \Phi(B) \) and \( A \leq 2.2I \leq C, B \leq 8I \leq D \). Also \( \delta_f = 2766.4 \) and \( \tilde{X} = \begin{pmatrix} 0.655 & 0.345 \\ 0.345 & 0.655 \end{pmatrix} \), whence

\[ (\Phi(C))^4 + (\Phi(B))^4 = \begin{pmatrix} 1891 & -859 \\ -859 & 3022 \end{pmatrix}. \]

This shows that inequalities in (2.6) can be strict.
Moreover,
\[ (\Phi(A)^4 + \Phi(B^4) - \Phi(A^4) - (\Phi(B))^4) = \begin{pmatrix} 884 & 1735 \\ 1735 & 3396 \end{pmatrix} \not= 0 \]
\[ (\Phi(A)^4 + \Phi(B^4) - \Phi(A^4) - (\Phi(B))^4) = \begin{pmatrix} 921 & 1797 \\ 1797 & 3505 \end{pmatrix} \not= 0. \]

Hence there is no relationship between the right hand sides of inequalities in Corollary 2.5.

**Corollary 2.7.** Let \( f \) be a convex function on an interval \( J \). Let \( A_i, B_i, C_i, D_i, i = 1, \ldots, n \), be operators in \( \sigma(J) \). If \( A_i \leq m \leq C_i, B_i \leq M \leq D_i, i = 1, \ldots, n \), for two real numbers \( m < M \) and
\[
\sum_{i=1}^{n} (A_i + D_i) = \sum_{i=1}^{n} (C_i + B_i),
\]
then
\[
f \left( \sum_{i=1}^{n} C_i \right) + f \left( \sum_{i=1}^{n} B_i \right) \leq f \left( \sum_{i=1}^{n} A_i \right) + f \left( \sum_{i=1}^{n} D_i \right) - \delta_{f,n} \tilde{X}_n,
\]
\[
\leq f \left( \sum_{i=1}^{n} A_i \right) + f \left( \sum_{i=1}^{n} D_i \right)
\]
(2.17)

and
\[
\sum_{i=1}^{n} f(C_i) + \sum_{i=1}^{n} f(B_i) \leq \sum_{i=1}^{n} f(A_i) + \sum_{i=1}^{n} f(D_i) - \delta f \left( \sum_{i=1}^{n} (\tilde{C}_i + \tilde{B}_i) \right)
\]
\[
\leq \sum_{i=1}^{n} f(A_i) + \sum_{i=1}^{n} f(D_i)
\]
(2.18)
in which \( \delta_{f,n} = f(nm) + f(nM) - 2 f \left( \frac{nm+nM}{2} \right) \) and
\[
\tilde{X}_n = 1 - \frac{1}{nM-nm} \left[ \left| \sum_{i=1}^{n} C_i - \frac{nM+nm}{2} \right| + \left| \sum_{i=1}^{n} B_i - \frac{nM+nm}{2} \right| \right].
\]

If \( f \) is concave, then inequalities (2.17) and (2.18) are reversed.

**Proof.** We prove only inequality (2.17) in the convex case. It follows from \( A_i \leq m \leq C_i, B_i \leq M \leq D_i, (i = 1, \ldots, n) \) that
\[
\sum_{i=1}^{n} A_i \leq mnI \leq \sum_{i=1}^{n} C_i, \sum_{i=1}^{n} B_i \leq MnI \leq \sum_{i=1}^{n} D_i.
\]
Using the same reasoning as in the proof of Theorem 2.2 we get

\[ f \left( \sum_{i=1}^{n} C_i \right) + f \left( \sum_{i=1}^{n} B_i \right) \leq 2Mn - \sum_{i=1}^{n} (C_i + B_i) \frac{f(mn)}{Mn - mn} f(Mn) - \delta_{f,n,\tilde{X}_n} \]

which give the first inequality in (2.17). It is easy to see that \( \delta_{f,n,\tilde{X}_n} \geq 0 \), whence the second inequality derived. \( \square \)

3. Applications

Using the results in Section 2, we provide some applications which are refinements of some well-known operator inequalities. As the first, we give a refinement of the operator Jensen–Mercer inequality.

**Corollary 3.1.** Let \( \Phi_1, \ldots, \Phi_n \) be positive linear maps on \( \mathfrak{B}(\mathcal{H}) \) with \( \sum_{i=1}^{n} \Phi_i(I) = I \) and \( B_1, \ldots, B_n \in \sigma([m, M]) \) for two scalars \( m < M \). If \( f \) is a convex function on \( [m, M] \), then

\[ f \left( m + M - \sum_{i=1}^{n} \Phi_i(B_i) \right) \leq f(m) + f(M) - \sum_{i=1}^{n} \Phi_i(f(B_i)) - \delta_{f,\tilde{B}} \]

\[ \leq f(m) + f(M) - \sum_{i=1}^{n} \Phi_i(f(B_i)), \]

where \( \tilde{B} = 1 - \frac{1}{M - m} \left[ \sum_{i=1}^{n} \Phi_i \left( \left| B_i - \frac{m + M}{2} \right| \right) + \sum_{i=1}^{n} \Phi_i(B_i) - \frac{m + M}{2} \right] \).

**Proof.** Clearly \( m \leq B_i \leq M \) \( (i = 1, \ldots, n) \). Set \( C_i = M + m - B_i \) \( (i = 1, \ldots, n) \). Then \( m \leq C_i \leq M \) and \( B_i + C_i = m + M \) \( (i = 1, \ldots, n) \). Applying inequality (3) of Corollary 2.5 when \( A_i = mI \) and \( D_i = MI \) we obtain the desired inequalities. \( \square \)

The next result provides a refinement of the Petrović inequality for operators.
Corollary 3.2. If \( f : [0, \infty) \rightarrow \mathbb{R} \) is a convex function and \( B_1, \ldots, B_n \) are positive operators such that \( \sum_{i=1}^{n} B_i = MI \) for some scalar \( M > 0 \), then

\[
\sum_{i=1}^{n} f(B_i) \leq f\left(\sum_{i=1}^{n} B_i\right) + (n-1)f(0) - \delta_f \tilde{B} \leq f\left(\sum_{i=1}^{n} B_i\right) + (n-1)f(0),
\]

where \( \tilde{B} = \frac{n}{2} - \sum_{i=1}^{n} \frac{|B_i|}{M} - \frac{1}{2} \).

Proof. It follows from \( 0 \leq B_i \leq M \) that

\[
f(B_i) \leq \frac{M - B_i}{M - 0} f(0) + \frac{B_i - 0}{M - 0} f(M) - \delta_f \tilde{B}_i \quad (i = 1, \ldots, n).
\]

Summing above inequalities over \( i \) we get

\[
\sum_{i=1}^{n} f(B_i) \leq \frac{nM - \sum_{i=1}^{n} B_i}{M} f(0) + \frac{\sum_{i=1}^{n} B_i}{M} f(M) - \delta_f \sum_{i=1}^{n} \tilde{B}_i
\]

\[
= (n - 1)f(0) + f\left(\sum_{i=1}^{n} B_i\right) - \delta_f \tilde{B} \quad \text{(by } \sum_{i=1}^{n} B_i = M)\]

\[
\leq (n - 1)f(0) + f\left(\sum_{i=1}^{n} B_i\right) \quad \text{(by } \delta_f \tilde{B} \geq 0)\]

where \( \tilde{B} = \frac{n}{2} - \sum_{i=1}^{n} \frac{|B_i|}{M} - \frac{1}{2} \). \( \square \)

As another consequence of Theorem 2.2, we present a refinement of the Jensen operator inequality for real convex functions. The authors of [9] introduce a subset \( \Omega \) of \( \mathbb{B}_h(\mathcal{H}) \times \mathbb{B}_h(\mathcal{H}) \) defined by

\[
\Omega = \left\{ (A, B) \mid A \leq m \leq \frac{A + B}{2} \leq M \leq B, \text{ for some } m, M \in \mathbb{R} \right\}.
\]

We have the following result.

Corollary 3.3. Let \( f \) be a convex function on an interval \( J \) containing \( m, M \). Let \( \Phi_i, i = 1, \ldots, n, \) be positive linear mappings on \( \mathbb{B}(\mathcal{H}) \) with \( \sum_{i=1}^{n} \Phi_i(I) = I \).

If \( (A_i, D_i) \in \Omega, \ i = 1, \ldots, n, \) then

\[
f\left(\sum_{i=1}^{n} \Phi_i\left(\frac{A_i + D_i}{2}\right)\right) \leq \sum_{i=1}^{n} \Phi_i\left(\frac{f(A_i) + f(D_i)}{2}\right) - \delta_f \bar{X}
\]

\[
\leq \sum_{i=1}^{n} \Phi_i\left(\frac{f(A_i) + f(D_i)}{2}\right), \quad (3.1)
\]
where
\[
\tilde{X} = \frac{1}{2} - \frac{1}{M - m} \left| \sum_{i=1}^{n} \Phi_i \left( \frac{A_i + D_i}{2} \right) - \frac{m + M}{2} \right|.
\]

If \( f \) is concave, then inequalities in (3.1) are reversed.

Proof. Putting \( B_i = C_i = \frac{A_i + D_i}{2} \) and using inequality (1) of Corollary 2.5, we conclude the desired result. \( \square \)

Note that utilizing Corollary 2.5, we even be able to obtain a converse of the Jensen operator inequality. For this end, under the assumptions in the Corollary 3.3 we have
\[
\sum_{i=1}^{n} \Phi_i \left( f \left( \frac{A_i + D_i}{2} \right) \right) \leq \frac{1}{2} \left[ f \left( \sum_{i=1}^{n} \Phi_i(A_i) \right) + f \left( \sum_{i=1}^{n} \Phi_i(D_i) \right) \right] - \delta f \tilde{X}
\]
\[
\leq \frac{1}{2} \left[ f \left( \sum_{i=1}^{n} \Phi_i(A_i) \right) + f \left( \sum_{i=1}^{n} \Phi_i(D_i) \right) \right],
\]
where
\[
\tilde{X} = \frac{1}{2} - \frac{1}{M - m} \sum_{i=1}^{n} \Phi_i \left( \left| \frac{A_i + D_i}{2} - \frac{m + M}{2} \right| \right).
\]

Note that the function \( f \) need not to be operator convex. Let us give an example to illustrate these inequalities.

Example 3.4. Let \( n = 1 \) and the unital positive linear map \( \Phi : \mathcal{M}_3(\mathbb{C}) \to \mathcal{M}_2(\mathbb{C}) \) be defined by
\[
\Phi((a_{ij})_{1 \leq i,j \leq 3}) = (a_{ij})_{1 \leq i,j \leq 2}
\]
for each \( A = (a_{ij})_{1 \leq i,j \leq 3} \in \mathcal{M}_3(\mathbb{C}) \). Consider the convex function \( f(t) = e^t \) on \([0, \infty)\). If
\[
A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 7 & -1 & 0 \\ -1 & 6 & 0 \\ 0 & 0 & 7 \end{pmatrix},
\]
then \( 0 \leq A \leq 2I \leq \frac{A + D}{2} \leq 5I \leq D \), i.e., \((A, D) \in \Omega\). Hence it follows from (3.1) that
\[
f \left( \Phi \left( \frac{A + D}{2} \right) \right) \leq \Phi \left( \frac{f(A) + f(D)}{2} \right) - \delta f \tilde{X}
\]
\[
\leq \Phi \left( \frac{f(A) + f(D)}{2} \right),
\]

where
\[
\tilde{X} = \frac{1}{2} - \frac{1}{M - m} \sum_{i=1}^{n} \Phi_i \left( \left| \frac{A_i + D_i}{2} - \frac{m + M}{2} \right| \right).
\]
in which $\delta_f = 89.6$ and $\bar{X} = \begin{pmatrix} 0.1 & 0.1 \\ 0.1 & 0.2 \end{pmatrix}$.

It should be mentioned that in the case when $f$ is operator convex, under the assumptions in Corollary 3.3 we have even more:

\[
f \left( \sum_{i=1}^{n} \Phi_i \left( \frac{A_i + D_i}{2} \right) \right) \leq \sum_{i=1}^{n} \Phi_i \left( f \left( \frac{A_i + D_i}{2} \right) \right) \quad \text{(by the Jensen inequality)}
\]

\[
\leq \frac{1}{2} \left[ f \left( \sum_{i=1}^{n} \Phi_i(A_i) \right) + f \left( \sum_{i=1}^{n} \Phi_i(D_i) \right) \right] - \delta_f \bar{X} \quad \text{(by (3.2))}
\]

\[
\leq \frac{1}{2} \left[ \sum_{i=1}^{n} \Phi_i(f(A_i) + f(D_i)) \right] - \delta_f \bar{X} \quad \text{(by the Jensen inequality)}
\]

\[
\leq \sum_{i=1}^{n} \Phi_i \left( \frac{f(A_i) + f(D_i)}{2} \right) \quad \text{(since $\delta_f \bar{X} \geq 0$)}
\]

**Corollary 3.5.** If $f$ is a convex function on an interval $J$ containing $m, M$, then

\[
f(\lambda A + (1-\lambda)D) \leq \lambda f(A) + (1-\lambda)f(D) - \delta_f \bar{X} \leq \lambda f(A) + (1-\lambda)f(D)
\]

(3.3)

for all $(A, D) \in \Omega$ and all $\lambda \in [0, 1]$, where $\bar{X} = \frac{1}{2} - \frac{1}{M-m} \left| \frac{A+D-M-m}{2} \right|$. If $f$ is concave, then inequality (3.3) is reversed.

**Proof.** Put $n = 1$ and let $\Phi$ be the identity map in Corollary 3.3 to get

\[
f \left( \frac{A + D}{2} \right) \leq \frac{f(A) + f(D)}{2} - \delta_f \bar{X} \leq \frac{f(A) + f(D)}{2}
\]

for any $(A, D) \in \Omega$, which implies (3.3) by the continuity of $f$. \qed

Regarding to obtain an operator version of (3.4), it is shown in [9] that if $f : [0, \infty) \rightarrow [0, \infty)$ is a convex function with $f(0) \leq 0$, then

\[
f(A) + f(B) \leq f(A + B)
\]

(3.4)

for all strictly positive operators $A, B$ for which $A \leq M \leq A + B$ and $B \leq M \leq A + B$ for some scalar $M$. We give a refined extension of this result as follows.

**Theorem 3.6.** If $f : [0, \infty) \rightarrow \mathbb{R}$ is a convex function with $f(0) \leq 0$ then

\[
\sum_{i=1}^{n} f(C_i) \leq f \left( \sum_{i=1}^{n} C_i \right) - \delta_f \sum_{i=1}^{n} \bar{C}_i \leq f \left( \sum_{i=1}^{n} C_i \right)
\]

(3.5)
for all positive operators $C_i$ such that $C_i \leq M \leq \sum_{i=1}^{n} C_i$ ($i = 1, \ldots, n$) for some scalar $M \geq 0$. If $f$ is concave, then the reverse inequality is valid in (3.5).

In particular, if $f$ is convex, then

$$f(A) + f(B) \leq f(A + B) - \delta f \tilde{X} \leq f(A + B)$$

for all positive operators $A, B$ such that $A \leq MI \leq A + B$ and $B \leq MI \leq A + B$ for some scalar $M \geq 0$, where $\tilde{X} = 1 - \frac{A}{M} - \frac{1}{2} - \frac{B}{M} - \frac{1}{2}$.

Proof. Without loss of generality let $M > 0$. Lemma 2.1 implies that

$$f(C_i) \leq \frac{MI - C_i}{M - 0} f(0) + \frac{C_i}{M - 0} f(M) - \delta f \tilde{C}_i = \frac{C_i}{M} f(M) - \delta f \tilde{C}_i \quad (i = 1, \ldots, n)$$

since $f(0) \leq 0$. Summing the above inequalities over $i$ we get

$$\sum_{i=1}^{n} f(C_i) \leq \frac{\sum_{i=1}^{n} C_i}{M} f(M) - \delta f \sum_{i=1}^{n} \tilde{C}_i. \quad (3.6)$$

Since the spectrum of $\sum_{i=1}^{n} C_i$ is contained in $[M, \infty) \subset [0, \infty) \setminus [0, M)$, we have

$$f \left( \sum_{i=1}^{n} C_i \right) \geq \frac{MI - \sum_{i=1}^{n} C_i}{M - 0} f(0) + \sum_{i=1}^{n} C_i \geq \frac{\sum_{i=1}^{n} C_i}{M} f(M) \quad (\text{since } MI \leq \sum_{i=1}^{n} C_i \text{ and } f(0) \leq 0). \quad (3.7)$$

Combining two inequalities (3.6) and (3.7), we reach to the desired inequality (3.5).

\[ \Box \]

**Theorem 3.7.** Let $A, B, C, D \in \sigma(J)$ such that $A \leq m \leq B, C \leq M \leq D$ for two real numbers $m < M$. If $f$ is a convex function on $J$ and any one of the following conditions

(i) $B + C \leq A + D$ and $f(m) \leq f(M)$

(ii) $A + D \leq B + C$ and $f(M) \leq f(m)$

is satisfied, then

$$f(B) + f(C) \leq f(A) + f(D) - \delta f \tilde{X} \leq f(A) + f(D), \quad (3.8)$$

where $\tilde{X} = 1 - \frac{1}{M - m} \left( \frac{B - M + m}{2} \right) + \frac{C - M + m}{2}$.

If $f$ is concave and any one of the following conditions

(iii) $B + C \leq A + D$ and $f(M) \leq f(m)$

(iv) $A + D \leq B + C$ and $f(M) \leq f(m)$

is satisfied, then inequality (3.8) is reversed.
Proof. Let \( f \) be convex and (i) is valid. It follows from Lemma 2.1 that
\[
\frac{f(M) - f(m)}{M - m} B + \frac{f(m)M - f(M)m}{M - m} - \delta_f \left( \frac{1}{2} - \frac{1}{M - m} \left| B - \frac{M + m}{2} \right| \right)
\]
and
\[
\frac{f(M) - f(m)}{M - m} C + \frac{f(m)M - f(M)m}{M - m} - \delta_f \left( \frac{1}{2} - \frac{1}{M - m} \left| C - \frac{M + m}{2} \right| \right).
\]
Summing above inequalities we get
\[
f(B) + f(C) \leq \frac{f(M) - f(m)}{M - m} (B + C) + 2 \frac{f(m)M - f(M)m}{M - m} - \delta_f \tilde{X}
\]
\[
\leq \frac{f(M) - f(m)}{M - m} (A + D) + 2 \frac{f(m)M - f(M)m}{M - m} - \delta_f \tilde{X} \quad \text{(by (i))}
\]
\[
= \frac{f(M) - f(m)}{M - m} A + \frac{f(m)M - f(M)m}{M - m} + \frac{f(M) - f(m)}{M - m} D + \frac{f(m)M - f(M)m}{M - m} - \delta_f \tilde{X}
\]
\[
\leq f(A) + f(D) - \delta_f \tilde{X} \quad \text{(by (2.8) and (2.9))}
\]
\[
\leq f(A) + f(D) \quad \text{(by } \delta_f \tilde{X} \geq 0 )
\]
The other cases can be verified similarly. \( \square \)

Applying the above theorem to the power functions we get

**Corollary 3.8.** Let \( A, B, C, D \in \mathbb{B}_h(\mathcal{H}) \) be such that \( I \leq A \leq m \leq B, C \leq M \leq D \) for two real numbers \( m < M \). If one of the following conditions

(i) \( B + C \leq A + D \) and \( p \geq 1 \)

(ii) \( A + D \leq B + C \) and \( p \leq 0 \)

is satisfied, then
\[
B^p + C^p \leq A^q + D^q - \delta_p \tilde{X} \leq A^q + D^q
\]
for each \( q \geq p \), where
\[
\delta_p = m^p + M^p - 2 \left( \frac{m + M}{2} \right)^p, \quad \tilde{X} = 1 - \frac{1}{M - m} \left( \left| B - \frac{M + m}{2} \right| + \left| C - \frac{M + m}{2} \right| \right).
\]

Proof. Let (i) be valid. Applying Theorem 3.7 for \( f(t) = t^p \), it follows
\[
B^p + C^p \leq A^p + D^p - \delta_p \tilde{X}
\]
\[
\leq A^q + D^q - \delta_p \tilde{X} \quad \text{(by } q \geq p )
\]
\[
\leq A^q + D^q \quad \text{(by } \delta_p \tilde{X} \geq 0 )
\]
The other cases may be verified similarly. \( \square \)
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References

1. T. Ando and X. Zhan, *Norm inequalities related to operator monotone functions*, Math. Ann. 315 (1999), 771–780.
2. K. M.R. Audenaert and J.S. Aujla *On norm sub-additivity and super-additivity inequalities for concave and convex functions*, arXiv:1012.2254v2.
3. T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond–Pečarić Method in Operator Inequalities*, Zagreb, Element, 2005.
4. M. Kian and M.S. Moslehian, *Operator inequalities related to Q-class functions*, Math. Slovaca, (to appear).
5. A. Matković, J. Pečarić and I. Perić, *A variant of Jensen’s inequality of Mercer’s type for operators with applications*, Linear Algebra Appl. 418 (2006), 551–564.
6. J. Mićić, Z. Pavić and J. Pečarić, *Jensen’s inequality for operators without operator convexity*, Linear Algebra Appl. 434 (2011), 1228–1237.
7. J. Mićić, J. Pečarić and J. Perić, *Refined Jensen’s operator inequality with condition on spectra*, Oper. Matrices 7 (2013), 293–308.
8. M.S. Moslehian, *Operator extensions of Hua’s inequality*, Linear Algebra Appl. 430 (2009), no. 4, 1131–1139.
9. M.S. Moslehian, J. Mićić and M. Kian, *An operator inequality and its consequences*, Linear Algebra Appl. (2012), .
10. M.S. Moslehian and H. Najafi. *Around operator monotone functions*, Integral Equations Operator Theory 71 (2011), 575–582.
11. M. Uchiyama, *Subadditivity of eigenvalue sums*, Proc. Amer. Math. Soc. 134 (2006), 1405–1412.

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