Steady-state structure of relativistic collisionless shocks

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We explore analytically the structure of relativistic shock and solitary wave solutions in collisionless plasmas. In the wave frame of reference, a cold plasma is flowing from one end and impacting on a low velocity plasma. First we show that under astrophysical conditions, a cold electron-positron plasma is unstable with respect to a two-stream instability in the interface between these regions. The instability heats the inflowing cold plasma rapidly, on a timescale comparable to the inverse of its plasma frequency. We then derive time-independent equations to describe the resulting hot state of the pair plasma, and describe the conditions under which the spatially uniform solution is the unique stable solution for the post shock conditions. We also examine plasmas composed of cold protons and hot electrons, and show that the spatially uniform solution is the unique stable solution there as well. We state the shock jump conditions which connect a cold, electron-proton plasma to a hot electron-proton plasma. The generic feature evident in all of these models is that the plasma’s initial, directed kinetic energy gets almost completely converted into heat. The magnetic field plays the role of catalyst which can induce the plasma instability, but our solutions indicate that the macroscopic field only gets amplified by a factor of $\sim 3$ in the frame of the shock.

52.35.Tc, 52.35.Qz, 52.35.Sb, 52.60.+h

I. INTRODUCTION

Collisionless shock waves are common in astrophysical environments. They are a generic product of cosmic explosions, such as supernovae \cite{1,2} (which result in non-relativistic shocks), and gamma-ray bursts \cite{3,4} (which yield highly relativistic shocks). Understanding their structure and dynamics is crucial for modeling pulsar magnetospheres \cite{5}, active galactic nuclei \cite{6}, structure formation in the intergalactic medium \cite{7}, and a local phenomenon such as the Earth’s bow shock in the solar wind \cite{8}. Despite the wide range of applications related to collisionless shocks, their theoretical modeling is still at a primitive state, and so astrophysicists often resort to collisional fluid equations in modeling extra-solar systems with no proper justification.

Because of the disparate physics used to describe collisional and collisionless plasmas, it is important to have an independent model of collisionless shocks. In a collisional plasma, the inter-collision mean free path sets the scale for the shock transition layer. Frequent collisions generate a Maxwellian distribution of post-shock particle energies, and all particle species eventually come to share a common temperature \cite{9,10}. However, shocks also occur in plasmas for which binary collisions are completely negligible \cite{11,12,13}. In this case, plasma instabilities act through macroscopic electromagnetic fields to bring about the shock wave, and possibly establish thermal equilibrium \cite{8}.

Several aspects of collisionless shocks have been explored in the literature. Steady, soliton solutions have been found \cite{14} for cold, magnetically dominated pair plasmas. But when the plasma kinetic energy is comparable to the Poynting energy, particles are magnetically reflected and execute looping motions. Simulations have suggested \cite{11,12,15} that these looping motions are unstable, and that a shock front forms in the region of magnetic reflection. Downstream of the shock, the plasma kinetic energy was seen to be mainly converted into thermal energy. Since the simulations allowed particle motion in only two dimensions, it is unclear whether or not the resulting momentum distribution is isotropic. These simulations have been complemented by analytic studies \cite{16,17,18} of the two-stream instability, which is operative whenever the fluid is composed of counter-moving streams. Since the aforementioned looping orbits present such a case, these analytical arguments suggest that the two-stream instability would dissipate the particles’ ordered motion on time scales comparable to the inverse plasma frequency. As the instability is saturated, the particles’ momentum distribution isotropizes, and the magnetic field receives significant amplification.

In several astrophysical situations, and in particular the generation of gamma-ray burst afterglows, it is important to understand what happens when a relativistic shock wave impinges on a cold plasma carrying a small amount of energy in its magnetic field, compared to the kinetic energy of the shock. In §II we will investigate the stability of a multi-stream configuration that typically results at such an interface. We will derive a simple dispersion relation, valid when the plasma frequency is much greater than the cyclotron frequency, which indicates that this configuration is unstable. This instability acts on a time scale comparable to that observed in simulations, and provides a promising driver for heating the gas and creating the necessary jump conditions of collisionless, relativistic shocks. The instability eventually saturates and converts the ordered bulk velocity of the particles into a three-dimensional
velocity distribution (thus, playing a role similar to collisions in a normal fluid). In §III, we will develop a stationary fluid model to describe the the post-shock plasma. After developing the appropriate equations, we will examine two cases in particular. First, we will investigate hot pair plasmas (§IV), and show that no stationary, continuous shock solutions exist. We will further derive the conditions under which oscillatory solutions might exist, and show that they are similar to those obtained earlier for cold plasmas [1]. In §V, we will investigate the case of an electron-proton plasma. Here, the two species have very different dynamical length scales, and so we will construct a simple model in which the electrons are hot, but the protons are cold. As a first approximation, we will average over the behaviour of the electrons, and calculate the resulting fields and the motion of the protons. Finally, we will consider a plasma in which both the electrons and the protons are hot. We will derived the jump conditions, and briefly discuss the possibilities for spatial structure. §VI will summarize the main conclusions of this work.

II. THE INSTABILITY

Previous studies have revealed that a generic feature of an initially cold plasma carrying relatively little energy in its magnetic field is the development of multiple streams [11,12]. The condition that there be little energy in the magnetic field can be stated in terms of the dimensionless ratio \(\sigma = B^2/8\pi\gamma^2 n mc^2 \ll 1\), where \(B\) is the plasma’s self-magnetic field, \(\gamma\) is the Lorentz factor corresponding to the mean fluid velocity, \(n\) is the proper number density, \(m\) is the mass per particle, and \(c\) is the speed of light. For simplicity, we suppose that the direction of the magnetic field is initially perpendicular to the mean fluid velocity. Then the definition of \(\sigma\) indicates that it is invariant with respect to Lorentz boosts in the direction of the bulk flow. This fact allows for a simple calculation of \(\sigma\) in the plasma’s rest frame: for electrons in the interstellar medium, \(\sigma \approx 5 \times 10^{-9} (B/\mu G)^2 (n/cm^3)^{-1}\). This small value suggests that multiple streams play an important role in the dynamics of shock and solitary waves in the interstellar medium.

Simulations [11,12] have suggested that multi-stream situations are unstable, and thus would require a time-dependent treatment. We take an analytical approach to the question of stability by applying the collisionless Boltzmann equation,

\[
\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + e \left( \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) \cdot \frac{\partial f}{\partial \mathbf{p}} = 0, \tag{1}
\]

to both species comprising a pair plasma. Since \(\sigma \ll 1\), the initial fields have negligible impact on length scales of the order of \(c/\omega_p\), where \(\omega_p\) is the plasma frequency. Then we could analyze the stability locally (over distances \(\leq c/\omega_p\)) with the fields completely neglected. Their primary role is to serve as catalysts which induce multiple streams. With this, Eq. (1) becomes

\[
\frac{\partial f_0}{\partial t} + \mathbf{v} \cdot \frac{\partial f_0}{\partial \mathbf{x}} = 0. \tag{2}
\]

The initial two-stream distribution function satisfying Eq. (2) is locally of the form

\[
f_0(x_0, \mathbf{p}) = \frac{n}{2} \left[ \delta (p_x - p_{x0}) + \delta (p_x + p_{x0}) \right] \delta (p_y) \delta (p_z). \tag{3}
\]

The two species have values of \(p_{y0}\) that are equal in magnitude and opposite in sign because the magnetic field bends particles of opposite charge in opposite directions. As a consequence, it is impossible to Lorentz boost to a frame where both species have vanishing \(p_{y0}\). It is this nonzero value of \(p_{y0}\) that distinguishes our model from the classical two-stream instability.

We now calculate the effects of a perturbation in the fields and the distribution functions with the spacetime dependence \(\exp(i(\mathbf{k} \cdot \mathbf{x} - \omega t))\). Faraday’s Law implies that the perturbed magnetic field is related to the perturbed electric field by \(\mathbf{k} \times \mathbf{E} = \omega \mathbf{B}\). Using this in conjunction with Eq. (1) while keeping only first-order terms in the perturbed quantities yields

\[
(-i\omega + i\mathbf{k} \cdot \mathbf{v}) \delta f = -e \frac{\partial f_0}{\partial \mathbf{p}} \cdot \left[ \mathbf{E} + \omega^{-1} \mathbf{v} \times (\mathbf{k} \times \mathbf{E}) \right]. \tag{4}
\]

As suggested by [18], we specialize to the case in which \(\mathbf{E}\) is along the anisotropy axis, \(\hat{x}\), and \(\mathbf{k}\) is along the \(y\)-direction. Expanding the triple vector product, we find that

\[
\delta f = \frac{-ieE}{\omega - kv_y} \left[ \left( 1 - \frac{kv_y}{\omega} \right) \frac{\partial f_0}{\partial p_z} + \frac{kv_x}{\omega} \frac{\partial f_0}{\partial p_y} \right]. \tag{5}
\]
For weak fields, the relation between \( \mathbf{E} \) and the displacement vector \( \mathbf{D} \) is linear, \( D_\alpha = \epsilon_{\alpha\beta} E_\beta \). From this, the definition of \( \mathbf{D} \) in terms of the polarization vector, \( \mathbf{P} = \frac{1}{\epsilon} (\mathbf{D} - \mathbf{E}) \), and the relation between the polarization and the current, \( \partial \mathbf{P} / \partial t = \mathbf{J} \), we obtain the following relation between the current and the electric field:

\[
\sum_{\text{species}} \left[ e \int d^3 \mathbf{p} v_\alpha \delta f \right] = \frac{-i \omega}{4\pi} \left( \epsilon_{\alpha\beta} - \delta_{\alpha\beta} \right) E_\beta. \tag{6}
\]

A scalar equation is obtained by contracting with \( E_\alpha \). Since the initial fields are negligible by assumption, the permittivity tensor may be decomposed into longitudinal and transverse parts. The transverse piece is given by

\[
\left( \epsilon_{\alpha\beta} - \delta_{\alpha\beta} \right) E_\beta = (\epsilon_t - 1) E_\alpha. \tag{7}
\]

And so we obtain an equation for \( \epsilon_t \):

\[
\frac{m \omega}{4\pi e^2} (\epsilon_t - 1) = \int d^3 \mathbf{p} \frac{p_x}{\gamma \omega - kp_y/m} \left[ \left( 1 - \frac{k p_y}{\gamma m \omega} \right) \frac{\partial}{\partial p_x} + \frac{k p_x}{\gamma m \omega} \frac{\partial}{\partial p_y} \right] \sum_{\text{species}} f_0. \tag{8}
\]

The delta functions make the integrations straightforward. We integrate by parts to obtain

\[
\frac{m \omega}{4\pi e^2} (\epsilon_t - 1) = \frac{-2n}{\gamma_0 \omega} \left( 1 - v_{x0}^2 \right) - \frac{2n k^2 v_{x0}^2}{\gamma_0 \omega} \left( \frac{\omega^2 + k^2 v_{y0}^2 - 3 \omega^2 v_{y0}^2 + k^2 v_{i0}^4}{(\omega^2 - k^2 v_{y0}^2)^2} \right), \tag{9}
\]

where the velocities have been normalized by the speed of light. To obtain a dispersion relation for transverse, electromagnetic waves, we set the permittivity tensor equal to \( k \) where the velocities have been normalized by the speed of light. To obtain a dispersion relation for transverse, electromagnetic waves, we set the permittivity tensor equal to \( k \). For simplicity we normalize the frequency in terms of the relativistic plasma frequency, \( \omega_p \), and the wave number by \( \omega_p / c \). Solving for \( \omega \) leads to a cubic equation in \( \omega^2 \),

\[
0 = \omega^6 + (2v_{x0}^2 - 2 - k^2 v_{y0}^2 - k^2) \omega^4 + (k^4 v_{y0}^2 + 2 k^2 v_{y0}^2 + 4 k^2 v_{y0}^2 - k^2 v_{x0}^2 + 2 k^2 v_{x0}^2 v_{y0}^2) \omega^2 - k^6 v_{y0}^4 - 2 k^4 v_{y0}^2 (v_{x0}^2 + v_{y0}^2). \tag{10}
\]

This cubic depends on two parameters, viz., the two components of the velocity.

The exact solution to this equation is quite complicated and we will not pause to write it in its entirety. Instead, we mention a few special cases. First note that when \( p_{x0} = 0 \), the frequency is strictly real [See Eq. (9)]. Thus any instability must arise from having overlapping streams of the same particle species. The solution \( \omega(k) \) is plotted for two cases in Fig. 1. As expected, when \( p_{y0} = 0 \), the frequency is either strictly real or strictly imaginary. This is the classic two-stream instability. As \( p_{y0} \) becomes comparable to \( p_{x0} \), we see that the plasma is still unstable; but in contrast to the previous case, both real and imaginary parts of the frequency are nonzero. The growth rate increases monotonically with \( k \) and asymptotically approaches a number of the order of the plasma frequency, while the real part of the frequency grows approximately linearly with \( k \) when \( c k \gg \omega_p \). The instability is most powerful for large \( k \), or alternatively, over small distances. This result is self-consistent with our initial restriction of studying phenomena over distance scales shorter than \( c/\omega_p \).
FIG. 1. The solid line depicts the growth rate of the standard two stream instability as a function of wave number. Each stream has a Lorentz factor of 100. The dashed and dotted lines show the real and imaginary parts, respectively, of the frequency for the modified two-stream instability discussed in §II. Here also the overall Lorentz factor is 100, but the $x$- and $y$-components of the velocity are equal in magnitude. Only when the $y$-component is nonzero are both the real and imaginary parts of the frequency nonzero.

III. BASIC FLUID EQUATIONS

In the rest frame of a planar shock front, cold material is flowing at a highly supersonic speed from $x = -\infty$. The results derived in the previous section imply that a cold, low-$\sigma$ plasma inevitably develops multiple streams and becomes unstable around the shock front. Simulations have shown that the instability tends to heat the plasma and eventually saturate [11,12]. After saturation, we assume for simplicity that the collisionless plasma approaches a steady state. In this section we develop the time-independent equations necessary for the fluid description of this state.

We consider the following boundary conditions at $x = -\infty$. In the plasma's rest frame, we allow for a magnetic field in $+\hat{z}$-direction. A planar shock wave travels along the $\hat{x}$-direction with a constant Lorentz factor, $\gamma_{sh}$. We next boost to the rest frame of the shock and assume that the plasma has reached a quiescent state, so that all physical quantities are independent of time in this frame. Moreover, we take the shock front to be infinite in the $\hat{y}$- and $\hat{z}$-directions; therefore, all physical quantities can only depend on the $x$-coordinate, which represents the perpendicular distance to the shock front. The Lorentz boost to the shock frame generates an electric field in the $\hat{y}$-direction. Other components of the fields and velocities are assumed to be initially zero.

For the sake of simplicity, we work strictly with dimensionless quantities. Electromagnetic fields are normalized by a reference value, $B_0$; spatial components of the four-velocity are normalized by a reference value $\gamma_0\beta_0c$, while the time component are normalized by $\gamma_0$. Distances are normalized by $\gamma_0 m_r c^2 / |e| B_0$ where $m_r$ is a reference mass to be determined by the particle species of interest. Finally, we define two dimensionless variables $\sigma = B_0^2 / 8\pi \gamma_0^2 m_r n_0 c^2$ and $T = P_0 / n_0 m_r c^2$. Note that these reference values are not, in general, equal to the values given by the boundary conditions at $x = -\infty$.

The equations of continuity can be obtained by integrating Eq. (1) over momentum. The resulting equations can be immediately integrated yielding

$$nu_x = \text{constant},$$

(11)
where $u_x$ is the $x$-component of the four-velocity. Faraday’s law shows that the $y$- and $z$-components of the electric field are constant, and can therefore be evaluated from the boundary conditions. The condition, $\nabla \cdot \mathbf{B} = 0$, ensures that the $x$-component of the magnetic field remains zero. Poisson’s equation determines how the longitudinal electric field develops,

$$ \frac{dE_x}{dx} = \frac{1}{2\sigma} \left( \frac{\gamma_i}{u_{xi}} - \frac{\gamma_e}{u_{xe}} \right). \quad (12) $$

The subscript $e$ denotes quantities pertaining to electrons, and the subscript $i$ denotes quantities pertaining to the ions (either positrons or protons). The $x$-component of Ampere’s Law is satisfied identically; the $y$ and $z$ components are

$$ \frac{dB_y}{dx} = \frac{\beta_0}{2\sigma} \left( \frac{u_{zi}}{u_{xi}} - \frac{u_{ze}}{u_{xe}} \right), \quad (13) $$

$$ \frac{dB_z}{dx} = \frac{\beta_0}{2\sigma} \left( \frac{u_{ye}}{u_{xe}} - \frac{u_{pe}}{u_{xe}} \right). \quad (14) $$

We assume that each species of particles has a stress-energy tensor of the form $T^{\mu\nu} = \text{diag}(\rho, P, P, P)$. Though $P$ is not strictly the pressure in a collisionless fluid, we will refer to it as such throughout this paper for the sake of brevity. To these we add the stress-energy tensor of the electromagnetic field, and set the divergence of the sum equal to zero to obtain the Euler equations [see Eqs. (13)-19]. Only three of these equations are independent; therefore, we will only concern ourselves with the spatial components. To complete our set of equations, we need two equations of state for the electron energy density and pressure, and two equations of state for the corresponding ionic quantities. We assume that an adiabatic description is valid, with adiabatic index $\Gamma$ between one and two [12,13,19]. Then we close our set of equations with

$$ P = P_0 \left( \frac{n}{n_0} \right)^\Gamma \quad (15) $$

$$ P = (\Gamma - 1) \left( \rho - m_ne^2 \right) c^2 \quad (16) $$

Eqs. (11),(13), and (16) allow us to eliminate $n$, $P$, and $\rho$, and so express the remaining equations completely in terms of the four-velocities and electromagnetic fields. As described above, each species has three equations to describe its four-velocity, which are given by Eqs. (17)-19. Because of our boundary conditions on $B_y$ and $u_z$, it is now clear that they will remain zero.

$$ \left[ \frac{\Gamma T (2 - \Gamma)}{(\Gamma - 1) u_x^{-1}} + \frac{m}{m_r} - \frac{\Gamma T}{\gamma_i^2 \beta_0^2 u_x^{1+1}} \right] \frac{du_x}{dx} = \frac{\text{sgn}(e)}{\beta_0} \left( \frac{\gamma E_x}{u_x} + B_z \beta_0 u_y - B_y \beta_0 u_z \right) \quad (17) $$

$$ \left[ \frac{\Gamma T}{(\Gamma - 1) u_x^{1-1}} + \frac{m}{m_r} \right] \frac{du_y}{dx} = \Gamma T \frac{u_y}{u_x} \frac{du_x}{dx} + \frac{\text{sgn}(e)}{\beta_0} \left( \frac{\gamma}{u_x} - B_z \right) \quad (18) $$

$$ \left[ \frac{\Gamma T}{(\Gamma - 1) u_x^{1-1}} + \frac{m}{m_r} \right] \frac{du_z}{dx} = \Gamma T \frac{u_z}{u_x} \frac{du_x}{dx} + \frac{\text{sgn}(e)}{\beta_0} B_y \quad (19) $$

Our problem is to solve the remaining six, coupled, non-linear differential equations for $B_z$, (or hereafter simply $B$), $E_x$, and the $x$ and $y$ components of the electron and ion four-velocities. This problem is simplified by the fact that three integrations can be immediately performed. These represent the conservation of the $x$ and $y$ components of momentum flux and the conservation of energy flux. The equation corresponding to the conservation of the $z$-component of momentum flux is satisfied identically. If $T^{\mu\nu}$ represents the combined energy-momentum tensor of all species of particles and the fields, then these integrals may be obtained most readily by realizing that $T^{\mu1} = constant$. The resulting equations are given below.

$$ 2\sigma (B - 1) = \sum_{\text{species}} \left[ \frac{m}{m_r} (1 - \gamma) \frac{m}{m_r} \frac{\Gamma T}{\Gamma - 1} (1 - \gamma u_x^{-1}) \right] \quad (20) $$
\[
\sigma \left( B^2 - 1 + E_x^2 \right) = \sum_{\text{species}} \left[ \frac{m}{m_r} \beta_0^2 (1 - u_x) + \frac{\Gamma T}{\Gamma - 1} \beta_0^2 (1 - u_x^{2-\gamma}) + \frac{T}{\gamma_0} (1 - u_x^{\gamma}) \right]
\]  \hspace{1cm} (21)

\[
2\sigma E_x = \sum_{\text{species}} \left[ \frac{m}{m_r} \beta_0 u_y + \frac{\Gamma T}{\Gamma - 1} \beta_0 u_y u_x^{1-\gamma} \right]
\]  \hspace{1cm} (22)

IV. PAIR PLASMAS

In this section we apply our model to pair plasmas. We choose the reference mass in the definition of \( \sigma \) and \( T \) to be the electron mass. Because both species have the same mass, the \( x \)-components of their four-momentum will be equal, and the \( y \)-components will have equal magnitude and opposite sign. From Eq. (12), we see that no longitudinal electric field will develop. Thus we need to determine spatial structure of the three quantities \( u_x, u_y, \) and \( B \), which are linked by Eqs. (20) and (21). The simplest possible solution is one that is constant in space. This solution can be used to assemble a simple picture of a low-\( \sigma \) collisionless shock. A plasma, with zero temperature at \( x = -\infty \), at some point encounters a shock wave generated by the modified two-stream instability of Sec. II. We expect the shock width to be of order \( c/\omega_p \) (\( \ll |e| B/\gamma mc \)), and so we approximate it as a discontinuity. Then Eqs. (20)-(22) may be used to obtain a constant solution for the post-shock plasma [19]. In situations where \( \sigma \ll 1 \), the jump conditions approach the classical hydrodynamic limit, given by [20], as might have been expected on the basis of our equations of state.

Next we examine under what conditions the constant solution is unique. Since the jump conditions of [19] were unique, the equations of a hot plasma cannot admit continuous shock wave solutions. Solitary wave solutions for a cold pair plasma were explored by [14]. Here we extend their results to hot plasmas. From Eqs. (20) and (21), it is clear that all quantities must be bounded. From Eq. (22), we can consider \( B^2 \) to be a function of \( u_x \). As \( u_x \) goes to zero through positive values, \( B^2 \to -\infty \). As \( u_x \) increases, \( B^2 \) increases monotonically, eventually becomes positive, reaches a maximum, and then decreases monotonically. Therefore the reality of \( B \) implies the existence of a maximum and minimum value of \( u_x \), and also an upper limit on the magnitude of the magnetic field. We do not consider the case where \( u_x \) is less than zero so as to avoid instabilities.

Eqs. (17) and (18) show that the derivatives of the four-velocity diverge at a certain value of \( u_x \). This occurs because of the presence of the pressure terms, and is absent in the zero-temperature case. This divergence might be remedied by more exact equations of state. The presence of this divergence, however, complicates the case under consideration. This divergence occurs at a value of \( u_x \) such that \( B \) is real; in fact, it occurs at value of \( u_x \) such that the right hand side of Eq. (22) is maximized. So we must now only look for a bounded solution, but a bounded solution for which both turning points of \( u_x \) are on the same side of this singularity.

On each side of the singularity, there is exactly one point where \( B = 0 \). Oscillatory solutions require at least two extrema of \( u_x \), so it is clear that we must find at least one extremum of \( u_x \) where \( u_y = 0 \). But in fact we must find at least two. For suppose that there exists an oscillatory solution on the right of the singularity. Then there must exist a minimum of \( u_x \) such that \( u_y = 0 \) in between the singularity and the point where \( B = 0 \). Examination of the derivative of Eq. (17) indicates that \( B > 0 \) for a minimum of \( u_x \) at such a point. Since the only place where \( B \) could go through zero is at a maximum of \( u_x \), we see that if \( B \) ever becomes less than zero there no longer exists the possibility of minimum, and so the ‘solution’ will inevitably become singular. Applying similar reasoning to the left side of the singularity, we see that at any maximum of \( u_x \), \( B > 0 \). We conclude as follows: any possible oscillatory solutions must have \( B > 0 \) everywhere. As a corollary, there must exist at least two turning points of \( u_x \) (and also \( B \)) where \( u_y = 0 \).

| \( T \) | \( \gamma_0 = 10 \) | \( \gamma_0 = 100 \) | \( \gamma_0 = 1000 \) |
|---|---|---|---|
| 0 | 4.6 | 50 | 5.1 \times 10^4 |
| 0.01 | 6.5 | 75 | 7.7 \times 10^2 |
| 0.01 | 39 | 2.9 \times 10^2 | 1.6 \times 10^5 |
| 100 | 3.0 \times 10^3 | 2.1 \times 10^4 | 1.1 \times 10^5 |
| 10000 | 3.0 \times 10^4 | 2.1 \times 10^5 | 1.1 \times 10^6 |

TABLE I. Minimum values of \( \sigma \) required for oscillating, steady-state solutions in a collisionless pair plasma. A value of 4/3 is assumed for the adiabatic index.
V. ELECTRON-PROTON PLASMAS

A. Cold protons, hot electrons

Next we investigate the steady-state properties of a plasma consisting of cold protons and hot electrons. In this section, we choose the reference mass in the definition of $\sigma$ and $T$ to be the proton mass. As derived in Sec. II, there are only three independent quantities once Eqs. (23)–(25) are accounted for. As with pair plasmas, a trivial steady-state solution exists in which all quantities remain constant throughout space. We now consider perturbations about this solution.

Because the electron-to-proton mass ratio is very small, quantities pertaining to the electrons vary considerably over a much shorter distance than quantities pertaining to the protons. In order to reduce the number of dependent variables and make analytical progress, we begin by investigating the spatial structure of the plasma on length scales of order the proton Larmor radius. We assume that the electrons undergo oscillatory motion about the spatially uniform solution, and then average the governing equations over several electron orbits. Because the motion of the electrons is bounded by assumption, we see that their averaged velocity derivatives vanish. Eqs. (17) and (18) for the protons go through essentially unchanged. The electrons’ contribution to the right hand side of Eq. (14) vanishes, and their contribution to the right hand side of Eq. (12) is just a constant number. In Eqs. (20)–(22) we average over powers of the electron bulk velocity. So as not to complicate the resulting equations, we take $< u_x^2 > = < u_x >^2$. Our justification for doing this rests in the facts that $p \approx 1$ in almost all of the terms, and in that we are looking for solutions that stay close to the constant solution.

With the above averaging, the problem is again reduced to solving for a single dependent variable given two fixed parameters, $\sigma$ and $\gamma_0$. Here $\gamma$ is defined in terms of the proton mass. Our averaging procedure evidently has the added advantage that there is no explicit dependence on the electrons’ equation of state. To obtain a single equation, we use Eq. (22) to eliminate the electric field and Eq. (21) to eliminate $u_x$. Eq. (20) can then be used to obtain a quadratic equation for $u_y^2$, with coefficients depending on the magnetic field:

$$0 = u_y^4 + 8\sigma \left( 2\gamma - 1 - \frac{\sigma}{\beta_0^2} + \frac{\gamma B^2}{\beta_0^2} \right) u_y^2$$

$$+ 16\sigma^3 \left[ \frac{\gamma}{\beta_0^2} - 2 - 4\gamma + 4(1 + 2\sigma) B - \left( 4\sigma + 2\frac{\gamma}{\beta_0^2} \right) B^2 + \frac{\gamma}{\beta_0^2} B^4 \right].$$  (23)
To eliminate $u_y$, we define a new independent variable $d\tau = dx/u_x$. Then, (the averaged) Eq. (14) illustrates that $u_y$ is simply proportional to the derivative of $B$. To obtain extremal values of $B$, we set $u_y = 0$ in Eq. (23). The result is a quartic polynomial which can be immediately deflated since $B = 1$ is a double root. The resulting quadratic can be solved to obtain the two additional roots

$$B = -1 \pm \beta_0 \sqrt{4 + 2/\sigma}. \quad (24)$$

We now investigate what bounded solutions are possible. Eq. (24) can easily be solved with the quadratic formula for $(dB/d\tau)^2$ yielding

$$u_y^2 = -8\sigma^2 + 4\sigma + \frac{4\sigma^2 B^2}{\beta_0^2} \pm \frac{8\sigma^2}{\beta_0} \sqrt{\frac{2B^2}{(2 + \frac{1}{\sigma}) B + \beta_0^2 \left(1 - \frac{1}{2\sigma}\right)^2 + \frac{1}{\sigma}}} \quad (25)$$

A sign ambiguity results; however, once the sign of the radical is chosen, it will remain unchanged provided that $\beta_0 > 1/\sqrt{2}$. For lesser values of $\beta_0$, the radicand may go through zero; but since the zeros of the radicand are in general different from the extrema of $B$, this would violate the condition that $u_y$ be real and there would be no solution. Thus, once the sign of the radical it will remain fixed. Additional intuitive insight may be gained by considering $B$ to be a generalized coordinate [8]. Then the differential equation has the form $\frac{1}{2}B^2 + \Phi (B) = 0$, i.e. Kinetic Energy + Potential Energy = 0. From this we can see that ‘energy’ is conserved; thus, there will be no shock wave solutions.

We now investigate the possibility of oscillatory solutions for $\sigma \ll 1$. There are two possible extremal values of $B$ (given by Eq. (24)) which may be turning points of a physical solution. First note that if $\sigma \ll 1$, then the root $B = 1$ occurs when we choose the minus sign in Eq. (25). To determine the sign which would make the other two roots extrema, we substitute the values from Eq. (24) into Eq. (25). For $\beta_0 \approx 1$ and $\sigma \ll 1$, the terms outside of the radical of Eq. (25) are negative, indicating that the plus sign is the appropriate choice. We can set these terms to zero to see what choice of parameters results. Solving for $\sigma$, we see that if $\beta_0 > 1/\sqrt{2}$, it is impossible to have $B$ given by Eq. (24) be roots of Eq. (25) with the minus sign. Therefore, if a oscillating solution exists in the relativistic case, the extrema of $B$ are given by Eq. (24) and we must choose the plus sign in Eq. (25).

However, the requirement that $u_x > 0$ precludes the possibility of an oscillating solution. For we have

$$u_x = 1 - \frac{\sigma}{\beta_0^2} (B^2 - 1) - \frac{u_y^2}{4\sigma}. \quad (26)$$

Provided $\beta_0 > 1/\sqrt{2}$, the point $B = 1$ is in between the two extremal values of $B$. Therefore, we evaluate Eq. (26) for $B = 1$. This gives $u_x = -1 + 4\sigma$, which is certainly less than zero in many astrophysical situations where $\sigma \sim 10^{-9}$ (See §II). Thus, there are no oscillating solutions, and the constant solution is unique.

**B. Hot Protons, Hot Electrons**

Finally, we consider the case where both the electrons and the protons are hot, i.e. they both satisfy the equations of state (13) and (14). Again, we define $\sigma$ and $T$ in terms of the proton mass. At $x = -\infty$ we take both species to be cold, but at some later stage we suppose that they have undergone a shock, in which both fluids were heated. We can then find the uniform post-shock solution as follows. Eqs. (12) and (14) show that both species have equal velocities. Eq. (17) shows that $u_y = 0$ for both species, and therefore Eq. (22) indicates that the longitudinal electric field vanishes. Eqs. (24) and (21) may be used in conjunction with Eq. (19) to determine $B$, $u_x$, and the total pressure. It is impossible from these considerations to determine how the total pressure is split between the electrons and the protons. Table I shows the downstream values of these quantities for several choices of $\gamma_0$. We have found the jump conditions to be independent of $\sigma$ provided that $\sigma$ is less than approximately 0.1.

| Lorentz Factor | $B_{\Delta s}/B_{us}$ | $u_{x,\Delta s}/u_{x,us}$ | $T_i + T_e = P_{0i}/n_0m_p + P_{0e}/n_0m_p$ |
|----------------|-----------------------|--------------------------|------------------------------------------|
| $10^2$         | 3.04                  | $3.5 \times 10^{-5}$     | 3.54                                     |
| $10^3$         | 3.00                  | $3.5 \times 10^{-4}$     | 16.7                                     |
| $10^4$         | 3.00                  | $3.5 \times 10^{-5}$     | 77.4                                     |

**TABLE II.** The shock jump conditions for a low-$\sigma$, electron-proton plasma for various shock Lorentz factors. Upstream of the shock, the plasma is assumed to be cold. But far downstream of the shock, the temperature has significantly increased. The jump conditions are essentially independent of $\sigma$, provided $\sigma \ll 1$. The adiabatic index $\Gamma$ has been taken to be 4/3.
Analytical study of non-constant solutions of these differential equations is much more complicated than in the preceding cases. Here again, one must deal with coupled differential equations. One can try averaging over the electrons’ orbits, but solving for $u_{y,\text{protons}}$ is no longer straightforward, since it is no longer easy to eliminate $u_{x,\text{protons}}$. We have, however, numerically integrated these equations with $\sigma \ll 1$ and $\gamma_0 \gg 1$, and have failed to find satisfactory solutions.

VI. CONCLUSIONS

We have examined the equations of a hot pair plasma, a plasma of hot electrons and cold protons, and finally a plasma in which both electrons and protons are hot. In none of these cases were continuous shock wave solutions found. However, we have seen that a cold plasma with a small, embedded magnetic field is subject to a version of the two-stream instability, which is a likely mechanism for the generation of collisionless shocks. The standard two-stream instability can only be saturated by nonlinear effects because of its aperiodic nature, indicating that the magnetic field may be significantly amplified [8]. But with this modified two-stream instability, both the real and imaginary parts of the frequency are nonzero, and so kinetic effects such as collisionless damping and resonance broadening may play an important role in the eventual saturation. The final level of magnetic field amplification is unclear. Since the growth rate of the instability is much higher than the cyclotron frequency, we expect this instability to be a dominant influence in the formation of a collisionless shock.

We have also investigated the possible existence of soliton solutions in a hot, pair plasma. As with cold plasmas, we have found that such solutions exist only for $\sigma \gg 1$. Thus, the only time-independent solution is one that is independent of the spatial coordinate. This simple structure has been observed in numerical simulations [8], which do suggest spatially uniform solutions, with physical quantities equal to the values predicted by the jump conditions.

We have also considered a model consisting of hot electrons and cold protons. Since we have averaged over the electrons’ orbits, this model is essentially independent of the electrons’ equations of state. In this case again we have found that the unique solution for a relativistic, low-$\sigma$ plasma is the spatially uniform one. Finally, we obtained the jump conditions for the state of a hot plasma of electrons and protons that was initially cold and had undergone a collisionless shock. As in the other cases, we have found that almost all of the plasma energy is converted into pressure. The magnetic field gets amplified by a factor of $\sim 3$ in the shock frame, and the post-shock value of $\sigma$ is only amplified by a factor of $\sim \gamma_{sh}$. In many astrophysical situation, this leaves the fraction of energy contained in the magnetic field still far below the equipartition limit. Numerical simulations of relativistic collisionless shocks would be very useful in understanding whether these jump conditions are appropriate, and whether or not the adopted model of a quiescent, low-$\sigma$ plasma is adequate. In particular, such simulations would also provide information on the relative thermal Lorentz factors for the electrons and protons, which is of fundamental importance for models of the relativistic blast wave in gamma-ray burst afterglows [8].

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