Application of the Radial Mode Matching Method to Determine the Resonant Frequency of a Cone Shaped Dielectric Resonator

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ABSTRACT
The paper presents an application of the radial mode matching method to determine the resonant frequency of cone-shaped dielectric resonators. The results of calculations with the use of the developed computer program are presented. The accuracy of the resonator cone projection was investigated taking into account a different number of regions into which the conical resonator was divided. The influence of the number of the considered waveguide modes on the accuracy of the calculation of the resonance frequency was also analyzed. The results of the calculations were compared with the results obtained with the QuickWave electromagnetic simulator. These results are convergent, which proves the correctness of the adopted assumptions and the full usefulness of the radial mode matching method for calculating the resonance frequencies of the cone-shaped dielectric resonator.

INDEX TERMS
Dielectric resonators, Maxwell’s equations, radial mode matching method, cone shaped resonators.

I. INTRODUCTION
Dielectric resonators employed in microwave filters and antennas have variety of shapes. The most popular are, easily produced, cylindrical dielectric resonators but they are not the best ones taking into account the spurious modes. The cylindrical dielectric resonators have inconvenient distribution of spurious resonant modes, which are usually close to the frequency of operation in the selected mode [1], [2]. Much better distribution of spurious resonant modes can be obtained in ring shaped dielectric resonators or other more complicated shapes. The following examples of the shapes of dielectric resonators limited to axisymmetrical structures can be found:
- cylindrical or disc [3], [4],
- disc with curved edge [5],
- wedge disc [5], [6],
- ring [2],
- toroid [5], [6],
- split cylinders (SPDR) and rings [7], [8],
- triple layer cylindrical and ring resonators [9]–[11],
- rings on a rod [6],
- spherical and hemispherical [5], [6], [12]–[14],
- conical [12],
- cup shaped DR [15],
- multi-segment dielectric resonator antennas (MSDRAs) and resonators [16].

Additionally, all of these resonators and antennas can be created with many layers of different dielectrics.

In order to design a filter or an antenna, it is necessary to know the parameters of the components of this device. Therefore, it is necessary to determine the resonant frequency and possibly the Q-factor of the dielectric resonators included in the filter or antenna. In the case of simple resonator structures, e.g. a cylinder or a ring, the calculation of the resonant frequency is not complicated, you can even use analytical relationships, e.g. in the case of a cylinder placed between two infinite metal plates, although the accuracy of analytical formulas is limited. In the case of more complex structures, more advanced numerical methods should be used, e.g. the radial mode matching method or the axial mode matching one, which give the best accuracy of the calculation of the parameters sought.

There are many publications in the literature describing the analysis of resonant structures by the radial modes matching method [17]–[19]. Unfortunately, most often these works concentrate on structures with a small number of regions and
layers, with the exception of the author’s publication [19]. In this work, a detailed description of the radial mode matching method for structures containing up to 20 regions and 20 layers in each one is included.

It is possible to use any electromagnetic simulator, e.g. CST [20], QuickWave [21], HFFS [22], to calculate the resonant frequency of a complex dielectric resonator structure. However, each of these 3D simulators use approximations of partial differential equations, so the accuracy of the calculations is limited. In the 3D simulators the structure is meshed and accuracy of the computations depends on the mesh size. Thus, the accuracy depends on the quantization of space. It can also depend on the quantization of time (in FDTD), dispersion, round-off errors etc. The computation time in some simulators may be much longer than in the mode matching method described in [19]. The price of commercial electromagnetic simulators is also an important factor but it must be added that the 3D electromagnetic simulators can be used to any type of structures, which is their main advantage.

This article presents the possibility of using the radial mode matching method to determine the resonant frequencies of a dielectric resonator in the form of a cone. In order to more accurately represent the lateral surface of the dielectric cone, a previously developed computer program that allows the division of the calculated structure into 20 regions and 20 layers has been extended to 50 regions and 50 layers.

The influence of the accuracy of the dielectric cone mapping in the mode matching method was analyzed, the influence of the quantity of the considered waveguide types was examined and finally the results of the calculations were compared with the results of the calculations given by the QuickWave simulator based on the FDTD method.

II. THE BASICS

Generally, in radial mode matching method the analyzed structure is divided into regions as a function of radius and then into layers within each region across \( z \) axis (assuming cylindrical set of coordinates), Fig. 1. The boundaries of the regions and layers determine the values of a permittivity, i.e. if there is a change in the permittivity value for a given \( r \) value, then for this point we define the region boundary. The layer boundary is determined similarly, but then we move along the \( z \) axis. Finally, the structure composed of cylinders and rings is obtained, within which the permittivity is constant [19].

After such a division of the analyzed structure, the solution of Maxwell’s equations is generated in each layer of a given region, taking into account the properties of the material from which the given fragment of the structure consists. By meeting the boundary conditions along the \( z \) axis, the components of the electromagnetic field in a specific region are determined. Then the longitudinal components of the electromagnetic field in each region are matched to the components of the adjacent region. Finally, the matrix equation (1) is obtained. This equation has a solution if and only if the determinant of the matrix \( W \) is equal to 0.

\[
\text{det} \{ W \} = 0 \tag{1}
\]

The article presents an analysis of the structure consisting of a cone-shaped dielectric resonator placed in a metal cylinder with a height equal to the height of the cone, Fig. 2. The radius of the cylinder is assumed to be much larger than the radius of the base of the cone, so that it can be assumed that the cone is positioned between two endless metal plates. The height and diameter of the base of the cone were 9 mm and the relative permittivity was 36 (a typical value for dielectric resonators).

According to the rules of the radial matching method, the structure was divided into \( N + 1 \) regions, with the \( N + 1 \) region creating the space outside the cone. The radii of the regions were determined assuming that the height of the cone will be divided into \( \text{N parts (h}_N) \) and that each region will have a height equal to the multiple of the value \( h_N \). Hence,
the dependence on the radius of a given region is obtained in the form (2):

$$R_i = R_c - (N - i - 1) \cdot h_N \cdot \frac{R_c - R_1}{L_c - h_N}$$

(2)

where: $R_i$ – is a radius of $i$ region, $N$ – numbers of regions in a cone, $R_c$ – the radius of base of a cone, $R_1$ – the radius of first region, $L_c$ – the height of a cone, $h_N = L_c/N$.

Examples of region radii and layer heights for $N = 9$ are given in Table 1. It was assumed that the first region would be a cylinder with a radius of 0.05 mm. The next eight regions are rings with two layers. The first layer is a dielectric with a relative permittivity of 36, while the second is air. The last, tenth region in the form of a ring consists of only one air layer with a height equal to $L_c$.

As can be seen from Table 1, for such a value of $N$, the mapping of the cone through the cylinder and rings is not very accurate. Therefore, research on the dependence of the mapping on the number of $N$. The results are presented in the next section.

### III. RESULTS OF ANALYSES

This section presents the results of the calculations of the resonant frequency of the dielectric resonator in the form of a cone. The simulations started with determining the resonant frequencies of various modes for a cylindrical resonator placed in a metal cavity with a height equal to the height of the resonator and a radius much larger than the radius of the cylinder. For such a structure, determining the resonance frequency is very simple and does not require advanced methods. The radius of the top base of the cylinder $R_t$ was then reduced to a very small value so that it can be considered a cone (Fig. 3). The results of the calculated resonance frequencies are shown in Figs. 4 to 7. Figs. 4 and 5 show the resonant frequencies of the axisymmetric modes TE (Fig. 4) and TM (Fig. 5). Figs. 6 and 7 show the resonance frequencies for hybrid modes with an index to the $\phi$ axis ($m$) equal to 1 and 2, respectively. The lack of a clear definition of the modes is due to the nature of these modes that can behave, in this case, in a more complicated way than the axisymmetric ones. The calculations were carried out by dividing the cone resonator into 31 regions - 30 inside the resonator and one outside.

| No. of a region | Radius [mm] | Height of first layer [mm] | Eps of first layer | Height of second layer [mm] | Eps of second layer |
|-----------------|-------------|----------------------------|-------------------|---------------------------|-------------------|
| 1               | 0.05        | 9                          | 36                | -                         | -                 |
| 2               | 0.60625     | 8                          | 36                | 1                         | 1                 |
| 3               | 1.1625      | 7                          | 36                | 2                         | 1                 |
| 4               | 1.71875     | 6                          | 36                | 3                         | 1                 |
| 5               | 2.275       | 5                          | 36                | 4                         | 1                 |
| 6               | 2.83125     | 4                          | 36                | 5                         | 1                 |
| 7               | 3.3875      | 3                          | 36                | 6                         | 1                 |
| 8               | 3.94375     | 2                          | 36                | 7                         | 1                 |
| 9               | 4.5         | 1                          | 36                | 8                         | 1                 |
| 10              | 50          | 9                          | 1                 | -                         | -                 |

### FIGURE 3. The analyzed structure.

As would be expected, the resonant frequencies increase as the radius of the top base of the resonator decreases. For the coefficient $m = 2$, two frequencies practically independent of the shape of the resonator can be noticed. These are probably leaky modes.

The essence of the radial mode matching method is the division of the studied structure into regions, as mentioned earlier. The accuracy of the cone projection will depend on the number of these regions. Therefore, calculations were carried out to determine the required number of regions so that the mapping error was as low as possible. The calculation results are presented in Figs. 8 to 12.
The figures show the resonant frequencies of several modes and the relative differences between these frequencies $R_d$. The calculations were presented for the number of regions varying from 11 to 46 (10 and 45 in the resonator area) for several values of the top radius of a truncated cone. The graphs presented in Figs. 8 and 9 refer to the resonator having the shape of an almost cone ($R_t = 0.05$ mm). Fig. 10 shows the results for a resonator, in which the ratio of the radius of the upper base to the lower one is 0.5. On the other hand, Figs. 11 and 12 relate almost to a cylinder.

The relative frequency difference was defined as (3):

$$R_d = \frac{f_N - f_{\infty}}{f_{\infty}} \cdot 100[\%]$$  (3)
where: $R_d$ – the relative difference of frequencies, $f_N$ – the resonant frequency for $N$ regions, $f_\infty$ – the resonant frequency for 46 regions.

By analyzing the results of the calculations presented in the above-mentioned figures, the following conclusions can be drawn:

- the differences in the calculated resonant frequencies decrease with an increase in the number of regions into which the resonator is divided,
- $R_d$ depends on the shape of the resonator,
- the accuracy of the resonant frequency calculation depends on the excited mode,
- the value of $R_d$ does not exceed 6 percent when changing the number of regions from 11 to 46,
- assuming $N$ greater than 40, no frequency calculation error greater than 1 percent will be made.

If the resonator is divided into a larger number of regions, a more accurate representation of the resonator in the form of a cone is obtained. For a resonator with $R_c$ equal to 4.45 mm (compared to 4.5 mm for $R_t$), the accuracy of determining the resonant frequencies decreases, although it seems that the lateral surface of the truncated cone is more accurately reproduced. Here, numerical errors are noticeable, resulting from a small difference in the values of the matched field components at the regions boundaries. For example, when dividing into 31 regions, the subsequent $R_N$ values are: 4.45, 4.451666667, 4.453333333, 4.455, 4.456666667, ..., 4.493333333, 4.495, 4.496666667, 4.498333333, 4.5 mm, so the difference between the consecutive radii does not exceed 0.01 mm, which translates into a small the difference of the calculated field components.

The accuracy of calculations of resonant frequencies for different modes is also due to numerical errors. If field components with small values are matched at the boundaries of the regions, the numerical errors will be greater than in the case of larger ones.

The accuracy of calculating the resonant frequency of the resonator in the radial mode matching method depends on the number of $K$ waveguide modes included in the solution. For simpler resonant structures, 5 to 10 of these modes will suffice. The structure containing the cone-shaped resonator is more complicated, therefore the calculations depending on $K$ were performed. The results are shown in Figs. 13 and 14. Fig. 14 shows the relative frequency changes defined as (4):

$$R_K = \frac{f_K - f_\infty}{f_\infty} \cdot 100[\%] \quad (4)$$
where: $R_R$ – the relative difference of frequencies depended on $K$, $f_K$ – the resonant frequency for $K$ waveguide modes, $f_{\infty}$ – the resonant frequency for 19 waveguide modes.

When analyzing the results presented in Figs. 13 and 14, it should be stated that in the case of a structure with a cone resonator, the number of waveguide modes taken into account should be slightly greater than for simpler structures. It seems that 10 to 15 of these modes should be included in this case.

In order to verify the correctness of the obtained calculation results with other methods, the tested structure was designed in the QuickWave electromagnetic simulator and simulations were performed. The QuickWave simulator is based on the FDTD method [22]. In order to become independent from excitation circuits, in the QuickWave simulator, the tested resonator was stimulated by a point source with a given component of the electromagnetic field. The results of the calculations were compared with the results obtained by the radial modes matching method and are presented in Figs. 15 and 16.

![FIGURE 15. Resonant frequencies of few modes as a function of top radius of cone shaped resonator obtained by means of the radial mode matching method (RMM) and the QuickWave simulator (QW).](image1)

![FIGURE 16. Relative difference of resonant frequencies of few modes as a function of top radius of cone shaped resonator obtained by means of the radial mode matching method and the QuickWave simulator.](image2)

The relative difference of resonant frequencies was defined as (5):

$$R_f = \frac{f_{QW} - f_{RMM}}{f_{RMM}} \cdot 100\% \quad (5)$$

where: $R_f$ – the relative difference of frequencies, $f_{RMM}$ – the resonant frequency obtained by means of the radial mode matching method, $f_{QW}$ – the resonant obtained by means of the QuickWave simulator.

As it results from the presented figures, the resonant frequencies obtained by means on the described method and with the FDTD method in the commercial QuickWave electromagnetic simulator only slightly differ. For TE types, the relative differences do not exceed 1.2 percent, with the greatest differences being for the small upper radius of the truncated cone. The biggest differences are for the type TM$_{011}$. Inconsistencies of the obtained results may result from the non-optimal location of the power source in the simulator.

**IV. CONCLUSION**

The paper presents an analysis of the possibility of using the radial mode matching method to determine the resonant frequencies of a dielectric resonator in the shape of a truncated cone. The results of calculations for different ratios of the radius of the upper to the lower base of the cone are presented. The influence of mapping the shape of the cone resonator in the discussed method through the cylinders and rings (regions) on resonant frequencies and their accuracy were analyzed. On this basis one can state that it was sufficient to divide the cone into 40 regions. The minimum number of waveguide modes considered in the solution was also estimated. It was found that this number should not be lower than 10. The results of the calculations were compared with the results obtained with the use of the commercial electromagnetic simulator QuickWave. The differences in the results were really small. Based on the presented results, it can be concluded that the radial mode of matching method is fully useful for the analysis of a cone-shaped dielectric resonators and can be used to accurately predict resonant frequencies as well as to optimize cone dielectric resonator structures.

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