Suzuki–Ree groups and Tits mixed groups over rings

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ABSTRACT
It is shown that Suzuki–Ree groups can be easily defined by means of comparing two fundamental representations of the ambient Chevalley group in characteristic 2 or 3. This eliminates the distinction between the Suzuki–Ree groups over perfect and imperfect fields and gives a natural definition for the analogs of such groups over commutative rings. As an application of the same idea, we explicitly construct a pair of polynomial maps between the groups of types $B_n$ and $C_n$ in characteristic 2 that compose to the Frobenius endomorphism. This, in turn, provides a simple definition for the Tits mixed groups over rings.

1. Introduction

The classical definition of Suzuki–Ree groups over a perfect field [14] uses the existence of the exceptional root length changing automorphism of the Chevalley group of type $C_2$, $G_2$ or $F_4$ and the endomorphism $\tau$ of the base field that squares to Frobenius. An attempt to carry this definition to the groups over rings faces two obstacles:

- In case of an imperfect base field $F$ the Suzuki–Ree group, $G(F)$ is defined as the subgroup of $G(F')$, where $F'$ is the perfect closure of $F$ [15, II.10]. However, a ring of characteristic $p$ can be embedded into a perfect ring only if it contains no nilpotents of degree $p$ [8].

- The special isogenies of types $B_2/C_2$, $F_4$ and $G_2$ are defined on the elementary generators, and it is not easy to compute the image of an element $g \in G(R) \setminus E(R)$ under this exceptional automorphism. At the same time, almost every actual calculation with the elements of Suzuki–Ree groups, from the very simplest [13] to the complicated [5], starts with the Bruhat decomposition. The latter is sometimes even taken as the definition of the group in question, while being unavailable over rings.

The aim of the present note is to construct these special isogenies explicitly and thus show how to define Suzuki–Ree groups by specific difference-algebraic equations, making them more amenable to calculations. The construction is based on the consideration of two fundamental representations in defining characteristic and thus ties together (a) the Tits’ polarity between points and lines in projective space, and (b) the construction of diagram automorphisms for $G(A_n)$, $G(D_n)$ and $G(E_6)$ via the invariance of the weights lattice and the induced automorphisms of the Hopf algebra $\mathbb{Z}[G]$. The same idea gives an explicit description of the pair of polynomial maps between $G(B_n)$ and $G(C_n)$ in characteristic 2 that compose to the Frobenius endomorphism and, as an application, a simple definition for ‘Tits’ mixed groups over rings.
Let $\Phi$ be a root system of type $C_2, G_2$ or $F_4$, and let $\sigma$ be a mapping induced by the symmetry of its Dynkin diagram. Consider the fundamental weights $\lambda$ and $\mu$ corresponding to the terminal nodes of the Dynkin diagram, the latter are permuted by $\sigma$. Assume that $\lambda$ is the highest weights of the minimal representation of the simply connected Chevalley–Demazure group scheme $G(\Phi)$, then $\mu$ is the highest weight of the adjoint representation in case $\Phi = G_2, F_4$ and of the unique short-roots representation in case $\Phi = C_2$ (Table 1). It so happens that in characteristic 2 (for $C_2$ and $F_4$) or 3 (for $G_2$) the dimension of the irreducible highest weight representation $V(\mu)$ drops and becomes equal $\dim V(\lambda)$.

In what follows a subscript $p$ will denote the restriction to characteristic $p$. Thus $G_p$ means the affine group scheme over $\mathbb{F}_p$ obtained in a natural way from a $\mathbb{Z}$-scheme $G$, and $V(\lambda)_p$ denotes the $G_p$-module obtained from $V(\lambda)$ over $G$. In each case the admissible base for $V(\lambda)_p$ is chosen to satisfy the relations of Matsumoto lemma [9, Lemma 2.3], that is, the base vectors $e_\nu$ (of non-zero weight $\nu$), and $e_0^\alpha$ (of zero weight, one for each of the fundamental roots $\alpha$ that are weights) are such that the elementary root unipotents $x_\alpha(\zeta)$ act by

$$
x_\alpha(\zeta)e_\nu = e_{\nu + \alpha}, \quad \text{if } \nu + \alpha \text{ is not a weight},
$$

$$
x_\alpha(\zeta)e_\nu = e_\nu \pm \zeta e_{\nu + \alpha}, \quad \text{if } \nu + \alpha \neq 0 \text{ is a weight},
$$

$$
x_\alpha(\zeta)v = v, \quad \text{if } \alpha \text{ is not a weight and } v \text{ is a zero–weight vector},
$$

$$
x_\alpha(\zeta)e_\beta^0 = e_\beta^0 \pm \zeta e_\alpha, \quad \text{if } \alpha \text{ is a weight and } \beta \text{ is in the support of } \alpha,
$$

$$
x_\alpha(\zeta)e_{-\beta} = e_{-\beta} \pm 2\zeta e_\alpha, \quad \text{if } \alpha \text{ is a weight}.
$$

Here $v^\alpha(\alpha)$ is a zero weight vector that is a linear combination of $e_\beta^0$ over all $\beta$ in the support of $\alpha$, and all the coefficients are $\pm 1$. In case $\Phi = C_2, F_4$ the particular choice of signs does not matter for our considerations, and for $G_2$ the signs will be fixed in 3.

For the adjoint representation $V(\mu)_p$ of $G(F_4)$ and $G(G_2)$ we take the standard Chevalley generators $e_\nu, \alpha \in \Phi$, and $h_i, i = 1, ..., \text{rk}(\Phi)$ as a basis.

The module $V(\nu)_p$ might not be irreducible, so we will denote by $L(\nu)$ the irreducible $G_p$-module with the highest weight $\nu$.

To see that $\dim L(\lambda) = \dim L(\mu) < \dim V(\mu)_p$, let us construct $L(\mu)$ explicitly as a quotient of $V(\mu)_p$. This amounts to finding the unique maximal proper $G_p$-invariant submodule of $V(\mu)_p$.

If $\Phi = C_2, F_4$, the short-roots Chevalley generators $\{e_\nu \mid \alpha \in \Phi^+\}$ together with $\{h_i \mid \alpha_i \in \Pi^\subset\}$ generate an ideal $s$ of $g(\Phi)_p$, which is the subrepresentation of $V(\mu)_p$. Note that $\dim s = \dim V(\mu)_p - \dim V(\lambda)_p$.

If $\Phi = C_2$, we consider the natural 4-dimensional representation $V = V(\sigma_1)_p$ of $G(C_2) = Sp_4$ and its basis $e_1, e_2, e_-, e_-$, then the 5-dimensional short-roots representation $V(\sigma_2)_p$ can be realized as a submodule of $\wedge^2 V$ spanned by $e\{1,2\}, e\{1,-2\}, e\{-1,1\}, e\{-1,-1\}$ and $e\{1,-2\} - e\{2,-2\}$ (here $e\{i,j\} = e_i \alpha e_j$). The required $G(C_2)_2$-invariant submodule of $V(\sigma_2)_p$ is $\langle e\{1,-2\} + e\{2,-2\} \rangle$.

Alternatively, one can start with the adjoint representation $V(\sigma_{\max})_p$ of $G(C_2)$, but in this case the quotient of $V(\sigma_{\max})_p$ by the ideal $s = \langle e_\alpha, h_1 \mid \alpha \in \Phi^+ \rangle$ is not irreducible, while the quotient by $\langle s, h_2 \rangle$ is. The resulting action of the elementary root unipotents is the same. Also, the realization of $L(\sigma_2)$ as a quotient of $\wedge^2(L(\sigma_1))$ makes the Tits’ polarity between points and lines in projective space $[15, \S 2.3]$ transparent.

Let now $R$ be a commutative ring of characteristic $p$ and let $\tau$ be a Tits endomorphism on $R$, that is, $\tau^2 = \varphi$, the Frobenius endomorphism. $\tau$ induces a mapping on every free $R$-module, in
particular, on $L(\lambda)_R$ and $L(\mu)_R$. Here $L(\lambda)_R = L(\lambda) \otimes R$. The sets of weights of these two representations are isomorphic as posets, see Figure 1, so we can identify the underlying $R$-modules of $L(\lambda)_R$ and $L(\mu)_R$ by identifying the chosen weight vectors according to the ordering of weights.

For an element $g \in G(\Phi, R)$ we will denote by $g/C$ the corresponding operator on $V/C$. The action of the elementary root unipotents $x_a(n)$ on $L(l)$ can be read off the weight diagram of $V(l)$. Namely, assume that $a = a_i + a_{i+1} + \cdots + a_k$, where $a_j$ are some fundamental roots, possibly with repetitions, and assume that there is a chain of weights $\lambda_0 = \lambda_m + \lambda_{m-1} + \alpha$ for each $m = 1, \ldots, n$ and $\lambda_0 - \alpha$ is not a weight. On the diagram this means that there is an increasing path of length $k$ from $\lambda_{m-1}$ to $\lambda_m$ with edges labeled by $i_j, j = 1, \ldots, k$ in some order (see Figure 2 for the case $a = a_1$).

$$x_a(\xi)e_{\lambda_0} = \sum_{k=0}^n \pm \xi^k e_{\lambda_k}$$

For the adjoint and short-roots representations this follows from Serre relations for the associated Lie algebra, since such chains correspond to root strings. Indeed, if $e_x \in g_x$ and $e_y \in g_y$, then $[e_x, e_y] = \pm (r + 1)e_{x + y}$, where $\beta - r\alpha, \ldots, \beta + q\alpha$ is the $\alpha$-string through $\beta$. Since by assumption $\lambda_0 - \alpha$ is not a root, the repeated application of $\text{ad}(e_x)$ maps $e_{\lambda_0}$ to the root subspaces of $\lambda_0 + \alpha, \lambda_0 + 2\alpha$, etc., and the coefficient $r$ grows by 1 on each step. Thus $\text{ad}(e_x)^j \cdot e_{\lambda_0} = \pm j! \cdot e_{\lambda_0}$ and $x_a(\xi) = \exp(\xi \cdot \text{ad}(e_x))$ acts on $e_{\lambda_0}$ as prescribed by the formula above.

In particular, since all the middle vertices in every such chain are quotiented out, as is seen on Figure 1, then $x_{(\xi)}_{\gamma} = x_{\delta}(\xi^p)_\gamma$ for some $\delta$, if $\gamma$ is a short root. Comparing Figure 1 with the weight diagrams reveals the reversed orders of labels, thus $\delta = \sigma(\gamma)$.
We define the Suzuki–Ree groups as
\[ \sigma G(\Phi, R, \tau) = \{ g \in G(\Phi, R) \mid \tau g = g \mu \tau \}. \]

This set is obviously closed under multiplication. To show that it is closed under taking inverses consider the equality \( \tau = g^{-1}_\mu g_\mu \), rewrite the right-hand side as \( g^{-1}_\mu \tau g_\mu \) and multiply both sides by \( g^{-1}_\mu \) from the right. Define
\[
U = \sigma U(\Phi, R, \tau) = U(\Phi, R) \cap \sigma G(\Phi, R, \tau), \\
H = \sigma H(\Phi, R, \tau) = H(\Phi, R) \cap \sigma G(\Phi, R, \tau), \\
B = \sigma B(\Phi, R, \tau) = B(\Phi, R) \cap \sigma G(\Phi, R, \tau), \\
N = \sigma N(\Phi, R, \tau) = N(\Phi, R) \cap \sigma G(\Phi, R, \tau).
\]

**Lemma 1.** If \( F \) is a field, Suzuki–Ree group admits Bruhat decomposition
\[ \sigma G(\Phi, F, \tau) = U N U. \]

**Proof.** For an element \( g \in \sigma G(\Phi, F, \tau) \) let \( g = u h w v \) be its restricted Bruhat decomposition in \( G(\Phi, F) \), that is, \( u \in U(\Phi, F), h \in H(\Phi, F), w \in N(\Phi, F), v \in U(\Phi, F) \cap U^- (\Phi, F) \). Then \( \tau u h w v = u_0 \mu h_0 w_0 v_0 \). Rewrite this as
\[ \tau u h w v = u_\mu h_\mu w_\mu v_\mu \tau^{-1} \]
and note that for any (uni-)triangular matrix \( u \) there exists a (uni-)triangular matrix \( u' \) such that \( \tau u = u' \tau \) (namely, \( u' \) is obtained from \( u \) by the element-wise application of \( \tau \)).
\[ u' h' w = u_\mu h_\mu w_\mu v_\mu \tau, \quad \text{and hence} \quad \tau w = h' u' u_\mu h_\mu w_\mu v_\mu v' \tau. \]

Everything is \( F_\mu \)-linear, the operator in the left-hand side acts naturally on \( V(\lambda, F_\mu) \), an \( F_\mu \)-submodule of \( V(\lambda) \), and so the same applies to the one in the right-hand side. Moreover, in this action \( \tau \) can be discarded, for \( \tau |_{F_\mu} = \text{id} \). Thus both operators \( w_{\lambda} = h'' u'' u_\mu h_\mu w_\mu v_\mu v' \) are written in terms of their restricted Bruhat decompositions in \( G(\Phi, F) \), which is unique, and thus \( w_{\lambda} = w_\mu, u'' u_\mu = v_\mu v' = h'' h_\mu = 1 \). Restoring \( u_\lambda, h_\lambda \) and \( v_\lambda \) from \( u'', h'' \) and \( v' \), one sees that \( g = u h w v \) is the Bruhat decomposition in \( G(\Phi, F, \tau) \).

2. Suzuki group, the case \( \Phi = C_2 \)

It is easy to check that
\[ x_\gamma(\xi) = \begin{cases} 
  x_{\sigma(\gamma)}(\xi)_{\lambda}, & \text{if } \gamma \text{ is long}, \\
  x_{\sigma(\gamma)}(\xi^2), & \text{if } \gamma \text{ is short}.
\end{cases} \]

One can now describe the structure of \( U \) by comparing for \( u = \prod_{\gamma \in C_2^+} x_\gamma(\xi) \in U \) the action of \( \tau u_\lambda \) and \( u_\mu \tau \) on \( e_{\nu} \), where \( \nu \) is the lowest weight, and solving for \( \xi_{\gamma} \). Namely
\[
\sigma U(C_2, R, \tau) \Omega u = x_+(a, b) = x_+(a) x_\beta(a^\gamma) x_{\lambda + \beta}(b) x_{2\lambda + \beta}(a^{\gamma + 2} + b^\gamma),
\]
\[ x_+(a, b)_{\lambda} = \begin{pmatrix} 1 & a & b + a^{\gamma + 1} & ab + b^\gamma + a^{\gamma + 2} \\
 & 1 & a^\gamma & b \\
 & & 1 & a \\
 & & & 1 \end{pmatrix} , \]

Figure 2. The action of \( x_\lambda(\xi) \) on \( v_0 \).
Recall that an element of the torus $H(C_2, R)$ is of the form $h = h_3(\varepsilon_1)h_p(\varepsilon_2)$, where

$$h_j(\varepsilon) = w_j(\varepsilon)w_j(-1), \quad w_j(\varepsilon) = x_j(\varepsilon)x_{-j}(-\varepsilon^{-1})x_j(\varepsilon).$$

Thus $h_1 = \text{diag}(\varepsilon_1, \varepsilon_2/\varepsilon_1, 1/\varepsilon_1, 1/\varepsilon_1)$ and $h_p = \text{diag}(\varepsilon_2, \varepsilon_1^2/\varepsilon_2, \varepsilon_2/\varepsilon_2^2, 1/\varepsilon_2)$, and so $h \in H$ if $\varepsilon_2 = \varepsilon_1^2$ and $h$ is of the form $h(\varepsilon) = \text{diag}(\varepsilon, \varepsilon^{1-1}, \varepsilon^{1-1}, \varepsilon^{-1})$.

Note the following relations:

$$x_+(a, b)x_+(c, d) = x_+(a + c, b + d + a^2c),$$

$$h(\varepsilon)x_+(a, b) = x_+(\varepsilon^{1-2}a, \varepsilon^{1}b),$$

$$[h(\varepsilon), x_+(0, b)] = x_+(0, b + \varepsilon^{1}b),$$

$$[h(\varepsilon), x_+(a, 0)] = x_+(a + \varepsilon^{1-2}, a^{1+1}(\varepsilon^{3-2} + 1)).$$

The torus normalizer $N$ equals $H \cup \widehat{w}_0H$, where $\widehat{w}_0 = \text{antidiag}(1, 1, 1, 1)$. Note that

$$\widehat{w}_0h(\varepsilon) = x_-(0, \varepsilon)^{x_+(\varepsilon^{1-1}, 0)},$$

and so over a field $F$ it follows from Bruhat decomposition that $^\sigma G(C_2, F, \tau) = \langle U, U^- \rangle$. Here

$$U^- = \widehat{w}_0U = \{x_-(a, b) \mid a, b \in R\}, \quad x_-(a, b) = \widehat{w}_0x_+(a, b).$$

If $F \neq \mathbb{F}_2$, the Suzuki group $^\sigma G(C_2, F, \tau)$ is perfect. Indeed, for $\varepsilon \in F \setminus \{0, 1\}$ one has

$$[h(\varepsilon), x_+(0, b/(\varepsilon^7 + 1))] = x_+(0, b),$$

$$[h(\varepsilon), x_+(a/(\varepsilon^{2-1} + 1), 0)] = x_+(a, a^{1+1}/(\varepsilon^{3-2} + 1)).$$

### 3. Small Ree group, the case $\Phi = G_2$

In this case one has to fix the signs of the structure constants appropriately in order to obtain a nice description for $^\sigma U(G_2, R, \tau)$. We construct the Lie algebra and Chevalley group of type $G_2$ in their 7-dimensional representation as simultaneously preserving certain symmetric bilinear and alternating trilinear forms, defined as follows.

Fix a basis $e_i, i = 1, 2, 3, 0, -3, -2, -1$ of $V$, then let $B$ be a bilinear form with Gram matrix $B_e = \text{antidiag}(1, 1, 1, 2, 1, 1, 1)$. Define $T$ to be the unique alternating trilinear form such that $T(e_i, e_j, e_k) = 1$ if $(i, j, k)$ is one of $(0, 1, -1), (0, -2, 2), (0, -3, 3), (1, -2, -3)$ or $(-1, 3, 2)$ and 0 if $(i, j, k)$ is not the permutation of one of the triples above. Note that this description differs from the classical and more symmetric formula for the Dickson form [1], [2, section 5], but suits better for our purposes.

Now we define the $G_2$ Lie algebra and Chevalley group as

$$g(G_2) = \left\{ g \in g_{L_7} \bigg| \forall u, v, w \in V \quad B(gu, v) + B(u, gw) = 0, \right. \left. T(gu, v, w) + T(u, gw, v) + T(u, v, gw) = 0 \right\},$$

$$G(G_2) = \left\{ g \in GL_7 \bigg| \forall u, v, w \in V \quad B(gu, gw) = B(u, v), \right. \left. T(gu, gw, w) = T(u, v, w) \right\}.$$
The elements of the torus in $G$ are of the form $h = h_x(e_1)h_y(e_2)$, where
\[
\begin{align*}
 h_x(e_1) &= \text{diag}(1, 1, e^2, e, 1, 1, e), \\
 h_y(e_2) &= \text{diag}(1, e, 1, e, e, 1, e^2, 1).
\end{align*}
\]

From this one deduces that $h \in H$ if $e_2 = e_1^2$ and $h$ is of the form
\[
 h(e) = \text{diag}(e, e^{r-1}, e^{2-r}, 1, e^{r-2}, e^{1-r}, e^{-1}).
\]

The relations between these elements are
\[
\begin{align*}
 x_+(a, b, c)x_+(a_2, b_2, c_2) &= x_+(a_1 + a_2, b_1 + b_2 + a_1a_2, c_2 + c_1 + a_1b_1 + a_1a_2^2 - a_1^2a_2^2), \\
 h(e)x_+(a, b, c)^{-1} &= x_+(-a, -b + a^{r+1}, -c + ab + a^{r+2}), \\
 h(e)x_+(a, b, c) &= x_+(e^{r-1}a, e^{r-1}b, ec), \\
 [h(e), x_+(0, 0, c)] &= x_+(0, 0, c(e-1)), \\
 [h(e), x_+(0, b, 0)] &= x_+(0, b(e^{-1} - 1), 0), \\
 [h(e), x_+(a, 0, 0)] &= x_+(a(e^{2-r} - 1), a^{r+1}(1 - e^{2-r}), a^{r+2}(e^{2-r} - 1)^2).
\end{align*}
\]

The torus normalizer $N$ equals $H \cup \tilde{w}_0H$, where $\tilde{w}_0 = \text{antidiag}(-1, ..., -1)$. We also define
\[
 U^- = \tilde{w}_0U = \{x_-(a, b, c) \mid a, b, c \in R\}, \quad x_-(a, b, c) = \tilde{w}_0x_+(a, b, c).
\]

It is not known whether $^aG(G_2, F, \tau)$ is perfect when $F$ is an infinite field, but it is always quasi-perfect, with the sole exception $F = \mathbb{F}_3$. Denote by $E$ the elementary subgroup of $G = ^aG(G_2, F, \tau)$, that is, $E = \langle U, U^- \rangle$. 

Lemma 2. If $F \neq \mathbb{F}_3$, then $[G,G] = E$ and is perfect.

Proof. Since $G = \langle U, N \rangle$, one has $[G,G] = \langle [f,g] | f, g \in U \cup N \rangle^G$. Using the relations listed above, it is easy to show that $E \leq [G,G]$. Namely,

$$x_+(0,0,c) = [h(-1),x_+(0,0,c)],$$
$$x_+(0,b,0) = [h(c),x_+(0,b/(c^{-1}-1),0)] \quad \text{for } c \notin \mathbb{F}_3,$$
$$x_+(a,0,0) = [h(-1),x_+(a,0,0)] \cdot x_+(0, *, *).$$

On the other hand, all the commutators $[f,g]$ with $f, g \in U \cup N$ lie in $E$. The inclusion $[N,U] \leq E$ follows from the expression for $h(c)x_+(a,b,c)$ and the definition of $U^-$. It only remains to consider commutators of the form $[N,N]$. But $H$ is abelian, while

$$[h(\varepsilon),w_0h(\eta)] = h(\varepsilon^2) \quad \text{and} \quad [w_0h(\varepsilon),w_0h(\eta)] = h(\eta^2/\varepsilon^3),$$

so $[N,N] = \{h(\varepsilon^2) | \varepsilon \in R^*\} = H^{(2)}$. The coset $\overline{w_0}H^{(2)}$ is contained in $E$ because

$$x_+(0,1/\eta,0)x_+(0,\eta,0)x_+(0,1/\eta,0) = \overline{w_0}h(\eta^{t+1}),$$

and substituting $\eta = \varepsilon^{-t}$ gives $\eta^{t+1} = \varepsilon^2$. Now $H^{(2)} = (\overline{w_0}H^{(2)})^2$.

It follows from the Bruhat decomposition and the inclusion $NU \geq E$ that $E$ is a normal subgroup of $G$. Since it contains the normal generators of $[G,G]$, one concludes that $E \leq [G,G]$.

To show that $E$ is perfect, note that in the commutator expressions for $x_+(a,b,c)$ all factors can be taken from $E$. Indeed,

$$x_+(-\eta^{-1},\eta^{-1-t},\eta^{-2-t})x_+(\eta,0,0)x_+(\eta^{-1},-\eta^{-1-t},\eta^{-2-t}) = \overline{w_0}h(-\eta^{t+2t})$$

and substituting $\eta = \varepsilon^{2-t}$ gives $-\eta^{t+2t} = -\varepsilon^2$, in particular, $h(-1) \in E$.

Remark 1. If the field $F$ is finite, $G = E$.

Proof. If $F$ is finite and $\text{char}(F) = 3$, then $F^* = \{\pm \varepsilon^2 | \varepsilon \in F^*\}$, so $H \leq E$.

4. Simplicity

Lemma 3. If $F \neq \mathbb{F}_p$ is a field, then the commutator subgroup $G = G(\Phi, F, \tau)'$ is simple.

Proof. Since $\tilde{B} = (H \cap G)U$ and $N \cap G$ form a BN-pair for $G$, we will prove the simplicity by appealing to a general result of Tits (see [7, Theorem 11.1.1]). Namely, it suffices to show that $\tilde{B}$ is solvable and core-free, the ambient group $G$ is perfect and that the simple reflections cannot be divided into two commuting subsets. The solvability of $\tilde{B}$ follows from the solvability of the group of upper triangular matrices, the perfectness of $G$ has been proved above, and the Weyl group property is easy to check. It remains to show that the intersection of all conjugates of $\tilde{B}$ is trivial. Note that $\overline{w_0}Bw_0^{-1} \cap B = H$. Now consider $\nu B$ for some $v \in U^-$, $\nu \neq 1$. Its elements are of the form $vh(\varepsilon)uv^{-1}$ for some $t \in F^*$ and $u \in U$. If this element lies in $H$, then $vh(\varepsilon)u = h(\eta)v = v'h(\eta')$ for some $\eta \in F^*$ and $\nu' \in U^-$. By the uniqueness of the Bruhat decomposition $u = 1$ and $\varepsilon = \eta$. But $\nu \neq \nu'$ unless $\varepsilon = 1$, and so $B \cap \overline{w_0}B \cap \nu'B = 1$.

5. Polynomial mappings between $B_n$ and $C_n$

As an application of the same idea, we explicitly construct a pair of polynomial maps between the simply connected Chevalley groups of type $C_n$ and $B_n$ in characteristic 2 that compose to the
Frobenius endomorphism. The existence of such maps allowed Nuzhin and Stepanov [11] to carry the result of Bak and Stepanov about the subring subgroups of symplectic groups [3] to the subgroups of the Chevalley group $G(\mathcal{B}_n, A)$ that contain $E(\mathcal{B}_n, R)$, where $R$ is a subring of a commutative ring $A$.

In order to distinguish the representations of two groups in question, we will denote the representation of the simply connected group scheme $G(\Phi)$ with the highest weight $\lambda$ by $(\Phi, \lambda)$ (by abuse of language, both the representation and the underlying module).

The map $\rho : G(\mathcal{B}_n) \to G(\mathcal{C}_n)$ is constructed by restriction of the natural action of $G(\mathcal{B}_n)$ on $(\mathcal{B}_n, \sigma_1) = \langle e_1, \ldots, e_n, e_n, e_{n-1}, \ldots, e_1 \rangle$ onto $\langle e_1, \ldots, e_n, e_{n-1}, \ldots, e_1 \rangle$ (here $e_i$ are the weight vectors satisfying the relations of [9, Lemma 2.3] and $\langle e_0 \rangle$ is the zero-weight subspace), see the end of the proof of [7, Theorem 11.3.2(ii)]. This restriction results in mapping

$$x_a(t)_{(\mathcal{B}_n, \sigma_1)} \mapsto x_{a^2}(t)_{(\mathcal{C}_n, \sigma_1)},$$

if $a \in \mathcal{B}_n$ is long,

$$x_a(t)_{(\mathcal{B}_n, \sigma_1)} \mapsto x_{a^2}(t^2)_{(\mathcal{C}_n, \sigma_1)},$$

if $a \in \mathcal{B}_n$ is short.

The map $\theta : G(\mathcal{C}_n) \to G(\mathcal{B}_n)$ is constructed in a similar manner, but by considering the representations $(\mathcal{C}_n, \sigma_2)$ and $(\mathcal{B}_n, \sigma_2)$.

The fundamental representation $(\mathcal{C}_n, \sigma_2)$ is a subrepresentation of $\wedge^n V$, where $V$ is the natural $2n$-dimensional module for $Sp_{2n}$. We fix the basis $e_1, \ldots, e_n, e_{n-1}, \ldots, e_1$ of $V$, such that the elementary generators act as

$$T_{ij}(\xi) = e + \xi e_{ij} \pm \xi e_{j-i}, \quad i \neq \pm j,$$

$$T_{i-j}(\xi) = e + \xi e_{i-j}.$$

Here $\xi \in R$ and $i, j \in \{1, \ldots, -1\}$. Denote

$$e_A = \bigwedge_{a \in A} e_a \quad \text{for} \quad A \subseteq \{1, \ldots, -1\}.$$

Here we do not care about the order of factors for we are only interested in the case of characteristic 2. The vectors $e_A$ with $|A| = n$ form a basis of $\wedge^n V$. For $A \subseteq \{1, \ldots, -1\}$ denote $A^\circ = A \cap \{\pm i, \pm j\}$ and $S(A) = A \cap -A$. Define a linear operator

$$X_+ : \wedge^n V \to \wedge^{n-2} V \quad \text{by} \quad X_+ e_A = \sum_{a \in S(A), a > 0} e_{A \cup \{\pm a\}}.$$

Then $V(\sigma_n)_0 = \ker X_+$ (see [6, Ch. VIII, §13.3.4]). Now let us express the action of the elementary generators on $\wedge^n V$.

$$T_{i-j}(\xi) e_A = \bigwedge_{a \in A \cup \{-i\}} e_a \wedge (e_{-i} \pm \xi e_i) = \begin{cases} e_A \pm \xi e_{A \cup \{i\}} & \text{if } A^\circ \in \{\{j\}, \{i, j\}, \{-i, j\}\}, \\
\pm \xi e_{A \cup \{-j\}} & \text{if } A^\circ \in \{\{i\}, \{i, j\}\}, \\
\pm \xi e_{A \cup \{-j\}} & \text{if } A^\circ \in \{-i, \{-i, j\}\}, \\
\pm \xi^2 e_{A \cup \{-i, j\}} & \text{if } A^\circ = \{-i, j\}, \\
e_A & \text{otherwise.} \end{cases}$$

We will now show that $U = \langle e_A, S(A) \neq 0 \rangle \cap \ker X_+$ is invariant in characteristic 2. For an element $u = \sum_k a_k e_{A_k}$ and $B \subseteq \{1, \ldots, -1\}$, $|B| = n-2$ denote
Y(u, B) = \sum_k x_k, \quad \text{where the sum is over such } k \text{ that } A_k = B \cup \{ \pm a \} \text{ for some } a. \]

An element \( u = \sum_i x_i e_{A_i} \) lies in \( \ker X_+ \) if for every \( B \subset \{ 1, \ldots, -1 \}, |B| = n-2 \) the sum \( Y(u, B) \) vanishes. Consider \( u = \sum_k x_k e_{A_k} \), where \( S(A_k) \neq \emptyset \) for every \( k \). Then
\[
T_{i,-i}(\xi)u = u + \xi \sum_{k \in A_k, i \in E_k} x_k e_{A_k \setminus \{ i \}} = u + \xi u',
\]
and all the summands in \( u' \) are of the form \( x e_A \) with \( S(A) \neq \emptyset \). Hence \( T_{i,i}(\xi)u \in U \).

\[
T_{ij}(\xi)u = u + \xi \sum_{k \in A_k, i \in E_k} x_k e_{A_k \setminus \{ i \}} + \xi \sum_{k \in A_k, j \in E_k} x_k e_{A_k \setminus \{ j \}} \equiv \xi \sum_{k : A_k^j = \{ i \}, i \in E_k} x_k e_{A_k \setminus \{ i \}} + \xi \sum_{k : A_k^j = \{ j \}, j \in E_k} x_k e_{A_k \setminus \{ j \}} \quad \text{(mod } U \text{)}
\]

Since \( T_{ij}(\xi)u \in \ker X_+ \), to each summand of \( u \) of the form \( x_k e_{A_k} \) with \( A_k = B \cup \{ \pm j \} \), where \( S(B) = \emptyset \), there corresponds another summand \( x_m e_{A_m}, A_m = B \cup \{ \pm i \} \) with the same \( B \), otherwise \( Y(u, B) \neq 0 \). But then \( A_k \setminus \{ j \} \cup \{ i \} = B \cup \{ i, j \} \neq A_m \setminus \{ i \} \cup \{ j \} \), so the basis elements involved in each of the two sums above coincide. Now note that since \( U \) is defined by linear equations on \( e_A \) with coefficients in \( \mathbb{F}_2 \), one can find a basis of \( U \) that consists of \( \mathbb{F}_2 \)-linear combinations of \( e_A \), thus one can assume that all \( x_k \) in the above sums are actually 1, and so this sum becomes zero in characteristic 2.

Consider the quotient space \( \ker X_+ / U \). This is an \( Sp_{2n} \)-module of dimension \( 2^n \) with the basis \( e_A + U, S(A) = \emptyset \). The action of the induced root element operators \( \tilde{T} \) is described as
\[
\tilde{T}_{i,-i}(\xi)e_A = \begin{cases} e_A + \xi e_{A \setminus \{ i \}} + U, & \text{if } -i \in A, \\ e_A + U, & \text{otherwise.} \end{cases}
\]
\[
\tilde{T}_{ij}(\xi)e_A = \begin{cases} e_A + \xi^2 e_{A \setminus \{ i, j \}} + U, & \text{if } -i, j \in A, \\ e_A + U, & \text{otherwise.} \end{cases}
\]

These are the formulas for the action of the elementary root unipotents in the spin representation of \( B_n \), where the long root unipotents act by squares.

To see that this mapping does indeed send an arbitrary element \( g \) of \( Sp(2n, R) \) to an element of \( \text{Spin}(2n + 1, R) \), one has to check that the operator \( \tilde{g} = \sqrt{g} \), acting on \( ker X_+ / U \), is an element of \( \text{Spin}(2n + 1, R) \) in its \( 2^n \)-dimensional spin representation. The action of \( \tilde{g} \) is given by the formula
\[
\tilde{g}(e_A + U) = \bigwedge_{a \in A} g e_a + U.
\]

Denote \( M = (e_1, \ldots, e_n) \) (so that \( V = H(M) \) in the natural way), then one can identify \( ker X_+ / U \) and \( \wedge M \) by \( e_A + U \mapsto e_A \cap \{1, \ldots, n\} \) and use the canonical isomorphism \( C(2n + 1) = C(H(M)) \cong \text{End}(\wedge M) \) (see [4, Theorem 2.4]). The latter isomorphism maps \( e_i \in M \) to \((y \mapsto e_i \wedge y)\), which is, in turn, identified with the mapping
\[
s_i : e_A + U \mapsto \begin{cases} e_A \setminus \{ i \} + U, & \text{if } -i \in A, \\ U, & \text{if } i \in A. \end{cases}
\]
which is an element of $\text{End}(\ker X_+/U)$. An element $f \in M^* = \langle e_{-n}, ..., e_{-1} \rangle$ is mapped to the unique (anti)derivation $d_f$ of degree $-1$ extending $f$, in particular,

$$d_{e_{-1}}(e_B) = \sum_{c \in B} e_{B \setminus \{i\}} d_{e_{-1}}(e_c) = \begin{cases} e_{B \setminus \{i\}}, & \text{if } i \in B, \\ 0, & \text{if } i \notin B. \end{cases}$$

This operator acts on $\ker X_+/U$ as

$$s_{-i} : e_A + U \mapsto \begin{cases} e_{A \setminus \{i\} \cup \{-i\}} + U, & \text{if } i \in A, \\ U, & \text{if } -i \in A. \end{cases}$$

By definition [4, § 3.1], the special Clifford group of a quadratic module $P$ is

$$\text{SCliff}(P) = \{ u \in C^+(P)^* \mid \pi(u)P \subseteq P \}.$$  

Here $\pi(u)a = \pm uau^{-1}$. Thus to show that $\tilde{g} \in \text{SCliff}(2n + 1)$, one has to check that conjugation by $\tilde{g}$ preserves the linear span $\langle s_1, ..., s_n, id, s_{-n}, ..., s_{-1} \rangle \subseteq \text{End}(\ker X_+/U)$. The $\text{Sp}(2n, R)$-invariance of this subspace is checked by a straightforward computation. Namely, denote $q = \overline{T_{i, i}}(\xi)$ and $r = \overline{T_{g}(\xi)}$, then

$$q_{sj} = \begin{cases} s_{-i} + \xi \text{id} + \xi^2 s_j, & \text{if } j = -i, \\ s_j, & \text{otherwise}, \end{cases} \quad r_{sk} = \begin{cases} s_{-i} + \xi^2 s_{-j}, & \text{if } k = -i, \\ s_j + \xi^2 s_i, & \text{if } k = j, \\ s_k, & \text{otherwise}. \end{cases}$$

The Spin group is defined [4, § 3.2] as the kernel of the norm map $N : \text{SCliff}(P) \to R^\times$. Here $N(x) = xx$, and $x \mapsto \hat{x}$ is the antiautomorphism of $C(P)$ that extends the identity map on $P$. On $C^+(2n + 1) \cong \text{End}(\ker X_+/U)$ it can be defined by $N(x) = JxJ$, where $J_{A,B} = \delta_{A,B} - \delta_{A,-B}$ (with respect to the basis $e_A + U, S(A) = \emptyset$). Consider $K \in \text{End} \wedge^n V$, defined (with respect to $e_A$) by $K_{A,B} = \delta_{A,B}, B = \{1, ..., -1\} \setminus B$. Then if follows from the Laplace expansion in multiple rows that

$$\wedge^n g \cdot K \cdot (\wedge^n g)^t \cdot K \equiv \det(g) \cdot \text{id}, \quad \text{mod } 2,$$

and hence $N(\tilde{g}) = \hat{g}J\hat{g}^\top = \det(\tilde{g}) = 1$, so $\theta : g \mapsto \tilde{g}$ maps $\text{Sp}(2n, R)$ to $\text{Spin}(2n + 1, R)$.

6. Tits mixed groups

The maps $\rho$ and $\theta$, constructed in the previous section, allow one to give a simple definition of the Tits mixed groups over commutative rings. Usually one considers a pair of fields $E \leq F$ of characteristic $p$, the part of an infinite chain of fields

$$\ldots \leq F^2 \leq F^p \leq F^p \leq E \leq E^{1/p} \leq F^{1/p} \leq \ldots$$

The mixed group is then defined [16, § 10.3.2] (see also [10]) as

$$G(\Phi, E, F) = \left\{ \alpha(\xi) : \xi \in \Phi, \xi \in E \text{ if } \alpha \text{ is long, } \xi \in F \text{ if } \alpha \text{ is short} \right\}.$$ 

Over a ring this defines the elementary subgroup, but not the ambient group. Note that $\theta(G(C_n, F)) = G(B_n, F^2, F)$ and that $G(B_n, F^2, E) = G(B_n, E) \cap G(B_n, F^2, F)$. Now since $G(B_n, E, F) \cong G(C_n, F^2, E)$, also $G(B_n, F^2, E) \cong G(C_n, F^2, F^2) = \phi(G(C_n, E, F))$. This definitions of mixed group are easily extended to the groups over rings.

For the mixed groups of types $G_2$ or $F_4$ everything is even easier. In case $\Phi = G_2$, the reduction map $\vartheta : G(G_2)_3 \to \text{End}(L(\mu))$ sends an element of $G(G_2)_3 = G_{ad}(G_2)$ to an element of $G(G_2)_3$, since the preserved forms $B$ and $T$ on $V$ are simply the reductions modulo 3 of the Killing form $\kappa(u, v) = \text{tr}(adu \cdot adv)$ and of the unique $G_{ad}(G_2)$-invariant (alternating)
trilinear form \((u, v, w) \mapsto \kappa([u, v], w)\) on the adjoint module. Then one defines \(G(G_2, F^3, E) = G(G_2, E) \cap \vartheta(G(G_2, F))\).

**Funding**

The research was supported by Russian Science Foundation (RSF) (project No. 17-11-01261).

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