Unified formalism for higher-order variational problems and its applications in optimal control

LEONARDO COLOMBO*
Instituto de Ciencias Matemáticas (CSIC-UAM-UC3M-UCM).
C/ Nicolás Cabrera 15. 28049 Madrid. Spain

PEDRO DANIEL PRIETO-MARTÍNEZ†
Departamento de Matemática Aplicada IV. Edificio C-3, Campus Norte UPC
C/ Jordi Girona 1. 08034 Barcelona. Spain

May 20, 2014

Abstract

In this paper we consider an intrinsic point of view to describe the equations of motion for higher-order variational problems with constraints on higher-order trivial principal bundles. Our techniques are an adaptation of the classical Skinner-Rusk approach for the case of Lagrangian dynamics with higher-order constraints. We study a regular case where it is possible to establish a symplectic framework and, as a consequence, to obtain a unique vector field determining the dynamics. As an interesting application we deduce the equations of motion for optimal control of underactuated mechanical systems defined on principal bundles.

Key words: Higher-order systems, Lagrangian and Hamiltonian mechanics, Underactuated mechanical system, Constrained variational calculus, Optimal control, Skinner-Rusk formalism

AMS s. c. (2000): 70H50, 22E70, 49J15, 53C80.

*e-mail: leo.colombo@icmat.es
†e-mail: peredaniel@ma4.upc.edu
Contents

1 Introduction 3

2 Mathematical background 4
   2.1 The Lagrangian-Hamiltonian unified formalism 4
   2.2 The constraint algorithm 5
   2.3 Higher-order tangent bundles 6
   2.4 Higher-order Hamilton equations in $M \times G$ 8

3 Geometric formalism for higher-order variational problems 9
   3.1 Unconstrained problem 9
      3.1.1 Geometrical setting 9
      3.1.2 Dynamical equation 12
      3.1.3 A theoretical example 15
   3.2 Constrained problem 17
      3.2.1 Geometrical setting 17
      3.2.2 Dynamical equation 18

4 Application to optimal control of underactuated mechanical systems 21
   4.1 Optimal control of an underactuated vehicle 22

5 Conclusions and further research 25
1 Introduction

The aim of this work is to describe, in an intrinsic way, higher-order Euler-Lagrange equations on trivial principal bundles. The main motivation is the analysis of a class of optimal control problems that are important in a wide range of contexts, such as the search of less cost devices in industrial processes, aerospace and defense, marine and automotive systems, electro-mechanical systems and robotic, among others. The optimal control problem consists in finding a trajectory of the state variables and control inputs, solution of the controlled Euler-Lagrange equations given initial and final conditions, and minimizing a cost function.

The study of higher-order tangent bundles has been developed in the last decades for different reasons, as a training field to understand field theory, theoretical physics, relativistic mechanics, classification of higher-order symmetries, among others [9, 10, 19, 21, 22, 23, 31, 39, 42]. In the last decade, higher-order variational problems had an extraordinary impact in the design and planning of trajectories, interpolation problems in Riemannian manifolds, optimization problems, optimal control applications, higher-order jet groups and particle methods, image registration methods for computational anatomy, etc. [7, 14, 25, 26, 32, 33, 35, 38].

In 1983, R. Skinner and R. Rusk introduced a formulation for the dynamics of an autonomous mechanical system which combines the Lagrangian and Hamiltonian features [44]. The aim of this formulation is to obtain a common framework for both regular and singular dynamics, obtaining simultaneously the Lagrangian and Hamiltonian formulations of the dynamics. Over the years, the Skinner-Rusk framework, or unified framework, has been extended in many directions: explicit time-dependent systems using a jet bundle language [3, 17], vakonomic mechanics and the comparison between the solutions of vakonomic and nonholonomic mechanics [16], higher-order dynamical systems [40, 41], first-order and higher-order classical field theories [8, 24, 45], and optimal control applications [2, 4, 13].

There is an interesting class of mechanical control systems, underactuated mechanical systems, which are characterized by the fact that there are more degrees of freedom than actuators. This type of systems is quite different from a mathematical and engineering perspective than fully actuated control systems (that is, where all the degrees of freedom are actuated). Underactuated systems include spacecraft, underwater vehicles, mobile robots, helicopters, wheeled vehicles, mobile robots, underactuated manipulators... (see [1, 5, 6] and references therein).

Our main objective in this paper is to characterize geometrically the equations of motion for a higher-order autonomous system with constraints using an extension of the Skinner-Rusk formalism for higher-order trivial principal bundles, and apply this to the optimal control problems of an underactuated mechanical system. The main results of this work can be found in Section 3, where we give a general method to deal with explicit and implicit, constrained and unconstrained mechanical systems.

The organization of the paper is as follows. In Section 2 we introduce some geometric constructions which are used along the paper. In particular, the Skinner-Rusk formalism for first-order mechanical systems, the Gotay-Nester-Hinds algorithm, some geometric aspects of higher-order tangent bundles, and Hamilton equations for a Hamiltonian system defined on the cotangent bundle of a higher-order trivial principal bundle. In Section 3 we introduce the Pontryagin bundle $T^*((T^{k-1}M) \times G \times kG \times kG^*)$, where we introduce the dynamics using a presymplectic Hamiltonian formalism, and we deduce the $k$th-order Euler-Lagrange equations in this context. Since the system is presymplectic, it is necessary to analyze the consistency of the dynamics using a constraint algorithm. We show that our techniques are easily adapted to the case of constrained dynamics. As an illustration of the applicability of our formulation, we analyze in Section 4 the case of underactuated control of mechanical systems and, as a particular
example, the optimal control problem of an underactuated vehicle in $SE(2) \times S^1$.

All the manifolds are real, second countable and $C^\infty$. The maps and the structures are assumed to be $C^\infty$. Sum over repeated indices is understood.

2 Mathematical background

In this section we give the notation used along this work, and the basic mathematical background needed about higher-order tangent bundles. There is also a sketch of the Gotay-Nester-Hinds algorithm and the Skinner-Rusk formalism for first-order systems. Finally we will derive Hamilton equations for higher-order trivial principal bundles.

2.1 The Lagrangian-Hamiltonian unified formalism

(See [44] for details.)

Let $Q$ be a $n$-dimensional smooth manifold modeling the configuration space of a first-order dynamical system with $n$ degrees of freedom, and $L \in C^\infty(TQ)$ a Lagrangian function describing the dynamics of the system.

Let us consider the bundle $W = TQ \times_Q T^*Q$. This bundle is endowed with canonical projections over each factor, namely $\text{pr}_1 : W \to TQ$ and $\text{pr}_2 : W \to T^*Q$. Using these projections, and the canonical projections of the tangent and cotangent bundle of $Q$, we introduce the following diagram

\[
\begin{array}{ccc}
TQ \times_Q T^*Q & \xrightarrow{\text{pr}_1} & TQ \\
\downarrow{\tau_Q} & & \downarrow{\pi_Q} \\
Q & \xrightarrow{\text{pr}_2} & T^*Q
\end{array}
\]

Local coordinates in $W$ are constructed as follows: if $(U, \varphi)$ is a local chart of $Q$ with $\varphi = (q^A)$, $1 \leq A \leq n$, then the induced local charts in $TQ$ and $T^*Q$ are $(\tau_Q^{-1}(U), (q^A, v^A))$ and $(\pi_Q^{-1}(U), (q^A, p_A))$, respectively. Therefore, the natural coordinates in $W$ are $(q^A, v^A, p_A)$. Observe that $\dim W = 3n$. Using these coordinates, the above projections have the following local expressions

\[
\text{pr}_1(q^A, v^A, p_A) = (q^A, v^A) ; \quad \text{pr}_2(q^A, v^A, p_A) = (q^A, p_A).
\]

The bundle $W$ is endowed with some canonical geometric structures. First, let $\theta \in \Omega^1(T^*Q)$ be the Liouville 1-form on the cotangent bundle and $\omega = -d\theta \in \Omega^2(T^*Q)$ the canonical symplectic form on $T^*Q$. From this we can define a 2-form $\Omega$ in $W$ as

\[
\Omega := \text{pr}_2^* \omega \in \Omega^2(W).
\]

It is clear that $\Omega$ is a closed 2-form, since

\[
\Omega = \text{pr}_2^*(-d\theta) = -d\text{pr}_2^* \theta.
\]
Nevertheless, this form is degenerate, and therefore is a presymplectic form. This is easy to check in coordinates. Bearing in mind the local expression of the canonical symplectic form of the cotangent bundle, which is \( \omega = dq^A \wedge dp_A \), and the local expression of the projection \( pr_2 \) given above, we have

\[
\Omega = pr_2^*(dq^A \wedge dp_A) = pr_2^*(dq^A) \wedge pr_2^*(dp_A) = d pr_2^*(q^A) \wedge d pr_2^*(p_A) = dq^A \wedge dp_A .
\]

From this local expression, it is clear that the kernel of \( \Omega \) is given locally by,

\[
\ker \Omega = \left\langle \frac{\partial}{\partial v^A} \right\rangle = X^V_{pr_2}(W),
\]

where \( X^V_{pr_2}(W) \) denotes the module of vector fields of \( W \) which are vertical with respect to the projection \( pr_2 \) (that is, \( X^V_{pr_2}(W) = \ker(T pr_2) \)). Therefore the 2-form \( \Omega \) is degenerate.

**Definition 1** Let \( p \in Q \) be a point, \( v_p \in T_p Q \) a tangent vector at \( p \), and \( \alpha_p \in T^*_p Q \) a covector on \( p \). Then we define the coupling function \( C \in C^\infty(W) \) as

\[
C : W \rightarrow \mathbb{R}
\]

\[
(p, v_p, \alpha_p) \mapsto \langle \alpha_p, v_p \rangle
\]

where \( \langle \alpha_p, v_p \rangle \equiv \alpha_p(v_p) \) is the canonical pairing between elements of \( T_p Q \) and \( T^*_p Q \).

If we consider a local chart on \( p \in Q \) such that \( \alpha_p = p_A dq^A \big|_p \), \( v_p = v^A \frac{\partial}{\partial q^A} \big|_p \), then the local expression of \( C \) is

\[
C(p, v_p, \alpha_p) = \langle \alpha_p, v_p \rangle = \left\langle p_A dq^A \big|_p , v^A \frac{\partial}{\partial q^A} \big|_p \right\rangle = p_A v^A \big|_p .
\]

Finally, we define the Hamiltonian function \( H \in C^\infty(W) \) by

\[
H = C - pr_1^* \mathcal{L},
\]

whose local expression is

\[
H(q^A, v^A, p_A) = p_A v^A - \mathcal{L}(q^A, v^A).
\]

Hence, we have constructed a presymplectic Hamiltonian system \( (W, \Omega, H) \). The dynamics for this systems is given by equation

\[
i(X)\Omega = dH ,
\]

where \( X \in \mathfrak{x}(W) \) is the Hamiltonian vector field of the system.

**2.2 The constraint algorithm**

In this subsection we briefly review the constraint algorithm for presymplectic systems. (See [28, 29, 30] for details).

By definition, if \( (M_1, \Omega) \) is a symplectic manifold then the equation

\[
i(X)\Omega = \alpha
\]

(1)
has a unique solution \( X \in \mathcal{X}(M_1) \) for every \( \alpha \in \Omega^1(M_1) \) that we consider. Nevertheless, if \( \Omega \) is closed and degenerate (that is, presymplectic), then the above equation may not have a solution defined on the whole manifold \( M_1 \), but only in some points of \( M_1 \). The tuple \((M_1, \Omega, \alpha)\) is said to be a presymplectic system. The aim of the Gotay-Nester-Hinds algorithm, or constraint algorithm, is to find a final submanifold \( M_f \hookrightarrow M_1 \) such that the equation (1) has solutions in \( M_f \) (if such submanifold exists). More precisely, the constraint algorithm returns the maximal submanifold \( M_f \) of \( M_1 \) such that there exists a vector field \( X \in \mathcal{X}(M_f) \) satisfying equation (1) with support on \( M_f \).

The algorithm proceeds as follows. Since \( \Omega \) is degenerate, then equation (1) has no solution in general, or the solutions are not defined everywhere. In the most favorable case, equation (1) admits a global (but not necessarily unique) solution \( X \in \mathcal{X}(M_1) \). Otherwise, we select the subset of points of \( M_1 \), where such a solution exists, that is,

\[
M_2 := \{ p \in M_1 : \text{there exists } X_p \in T_pM_1 \text{ satisfying } i(X_p)\Omega_p = \alpha_p \} = \{ p \in M_1 : (i(Y)\alpha)(p) = 0 \text{ for every } Y \in \ker \Omega \},
\]

and we assume that it is a submanifold of \( M_1 \). Then, equation (1) admits a solution \( X \) defined at all points of \( M_2 \), but \( X \) is not necessarily tangent to \( M_2 \), and thus it does not necessarily induce a dynamics on \( M_2 \). So we impose a tangency condition along \( M_2 \), and we obtain a new submanifold

\[
M_3 := \{ p \in M_2 : \text{there exists } X_p \in T_pM_2 \text{ satisfying } i(X_p)\Omega_p = \alpha_p \}.
\]

A solution \( X \) to equation (1) does exist in \( M_3 \) but, again, such an \( X \) is not necessarily tangent to \( M_3 \), and this condition must be required. Following this process, we obtain a sequence of submanifolds

\[
\cdots M_l \hookrightarrow \cdots \hookrightarrow M_2 \hookrightarrow M_1
\]

where the general description of \( M_{l+1} \) is

\[
M_{l+1} := \{ p \in M_l : \text{there exists } X_p \in T_pM_l \text{ satisfying } i(X_p)\Omega_p = \alpha_p \}.
\]

If the algorithm terminates at a nonempty set, in the sense that at some \( s \geq 1 \) we have \( M_{l+1} = M_l \) for every \( l \geq s \), then we say that \( M_s \) is the final constraint submanifold which is denoted by \( M_f \). It may still happen that \( \dim M_f = 0 \), that is, \( M_f \) is a discrete set of points, and in this case the system does not admit a proper dynamics. But in the case when \( \dim M_f > 0 \), by construction, there exists a well-defined solution \( X \) of equation (1) along \( M_f \).

### 2.3 Higher-order tangent bundles

In this subsection we recall some basic facts of the higher-order tangent bundle theory. We particularize our construction to the case when the configuration space is a Lie group \( G \). (See [9, 23, 42] for details.)

Let \( Q \) be a \( n \)-dimensional smooth manifold, and \( k \in \mathbb{N} \). The \( k \)-th order tangent bundle of \( Q \), denoted by \( T^kQ \), is the \( (k + 1)n \)-dimensional smooth manifold made of the \( k \)-jets of curves \( \phi : \mathbb{R} \rightarrow Q \) with source at \( 0 \in \mathbb{R} \); that is, \( T^kQ = J^k_0(\mathbb{R}, Q) \). It is a submanifold of \( J^k(\mathbb{R}, Q) \). A point in \( T^kQ \) is denoted \( j^k_0\phi \), where \( \phi \) is a representative of the equivalence class.

We have the following natural projections: if \( r \leq k \),

\[
\rho^k_r : T^kQ \longrightarrow T^rQ ; \quad \beta^k : T^kQ \longrightarrow Q \quad ; \quad j^k_0\phi \mapsto \rho^k_r(j^k_0\phi) \quad ; \quad j^k_0\phi \mapsto \beta^k(j^k_0\phi) \mapsto \phi(0) .
\]
Observe that $\rho^k_0 = \beta^k$, where $T^0Q$ is canonically identified with $Q$, $\rho^s_r \circ \rho^k_s = \rho^k_r$ for every $r \leq s \leq k$, and $\rho^k_k = \text{Id}_{T^kQ}$.

From a local chart $(U, \varphi)$ of $Q$, where $\varphi = (\varphi^A)$, $1 \leq A \leq n$, the induced local coordinates in $T^kQ$ are constructed as follows: let $\phi: \mathbb{R} \to Q$ be a curve such that $\phi(0) \in U$. Then, denoting $\varphi^A = \varphi^A \circ \phi$, the point $j^k_0 \phi$ is given in $(\beta^k)^{-1}(U)$ as $(q^A_0, \ldots, q^A_k) = (q^A_{\xi_i})$, $0 \leq i \leq k$, where

$$q^A_0 = \varphi^A(0), \quad q^A_i = \frac{d^i \varphi^A}{dt^i} \bigg|_{t=0}.$$  

When there is no risk of confusion, we use the standard conventions, $q^A_0 = q^A$, $q^A_1 = \dot{q}^A$ and $q^A_2 = \ddot{q}^A$. Using these coordinates, the local expression of the canonical projections are

$$\rho^k_r(q^A_0, \ldots, q^A_k) = (q^A_0, \ldots, q^A_r) \quad \beta^k_r(q^A_0, \ldots, q^A_k) = (q^A_r).$$

Now, assume that $Q = G$ is a finite dimensional Lie group, and let us consider the left-multiplication on itself

$$G \times G \to G \quad (g, h) \mapsto gh.$$  

If we denote $\mathcal{L}_g(h) = gh$ for every $g, h \in G$ the left-translation, it is obvious that $\mathcal{L}_g: G \to G$ is a diffeomorphism for every $g \in G$.

**Remark:** The same is valid for the right-translation, but in the sequel we only work with the left-translation, for the sake of simplicity.

The left-translation enables us to trivialize the tangent and cotangent bundles of $G$ as follows

$$TG \to G \times \mathfrak{g} \quad (g, \dot{g}) \mapsto (g, \xi) = (g, g^{-1}\dot{g}) = (g, T_g \mathcal{L}_{g^{-1}}\dot{g}) \quad \xrightarrow{\text{linear}} \quad (g, \alpha_g) \mapsto (g, \alpha) = (g, T_e \mathcal{L}_g(\alpha_g)),$$

where $\mathfrak{g} = T_eG$ is the Lie algebra of $G$ and $e \in G$ is the neutral element of the group.

For higher-order tangent bundles, we can also use the left-translation to identify the $k$th-order tangent bundle of $G$, $T^kG$, with $G \times k\mathfrak{g}$ as follows: if $g: I \subseteq \mathbb{R} \to G$ is a curve, we define

$$\Upsilon^k: T^kG \to G \times k\mathfrak{g} \quad j^k_0g \mapsto (g(0), g^{-1}(0)\dot{g}(0), \frac{d}{dt}\bigg|_{t=0}(g^{-1}(t)\dot{g}(t)), \ldots, \frac{d^{k-1}}{dt^{k-1}}\bigg|_{t=0}(g^{-1}(t)\dot{g}(t))).$$

It is clear that $\Upsilon^k$ is a diffeomorphism. If we denote by $\xi(t) = g^{-1}(t)\dot{g}(t)$, we can rewrite the above expression as

$$\Upsilon^k(j^k_0g) = (g, \xi^0, \xi^1, \ldots, \xi^{k-1}),$$

where

$$g = g(0) \quad \xi^i = \frac{d^i}{dt^i} \bigg|_{t=0}(g^{-1}(t)\dot{g}(t)), \quad 0 \leq i \leq k - 1.$$  

We will indistinctly use the notation $\xi^0 = \xi, \xi^1 = \dot{\xi}$, where there is no danger of confusion.

In this case, the canonical projections $\rho^k_r$ and $\beta^k$ are denoted by

$$\tau^k_r: T^kG \to T^rG \quad j^k_0g \mapsto j^r_0g \quad ; \quad \tau^k_r: T^kG \to G \quad j^k_0g \mapsto g(0).$$

Using the previous identifications, we have

$$\tau^k_r(g, \xi^0, \ldots, \xi^{k-1}) = (g, \xi^0, \ldots, \xi^{r-1}) \quad \tau^k_r(g, \xi^0, \ldots, \xi^{k-1}) = g.$$  

As before, $\tau^k_r \circ \tau^k_s = \tau^k_r$, $\tau^k_0 = \tau^k_r$, and $\tau^k_k = \text{Id}_G$. 
2.4 Higher-order Hamilton equations in \( M \times G \)

Let us consider the manifold \( Q = M \times G \), where \( M \) is a \( m \)-dimensional smooth manifold and \( G \) is a finite dimensional Lie group. Using the results of Section 2.3, we have

\[
T^*(T^{k-1}Q) = T^*(T^{k-1}(M \times G)) \simeq T^*(T^{k-1}M) \times G \times (k-1)\mathfrak{g} \times k\mathfrak{g}^*.
\]

In order to geometrically derive Hamilton equations for higher-order variational problems we need to equip the previous space with a symplectic structure. Thus, we construct a Liouville 1-form \( \theta \) and a canonical symplectic 2-form \( \omega \) by pull-backing the canonical Liouville forms in \( T^*(T^{k-1}M) \) and \( G \times (k-1)\mathfrak{g} \times k\mathfrak{g}^* \). Let \( (q_i^A, p_i^A) \) be the natural coordinates in \( T^*(T^{k-1}M) \), and denote by \( \xi \in (k-1)\mathfrak{g} \) and \( \alpha \in k\mathfrak{g}^* \) with components \( \xi = (\xi_0, \ldots, \xi_{k-2}) \) and \( \alpha = (\alpha_0, \ldots, \alpha_{k-1}) \). Then, after a straightforward computation, we deduce that

\[
\theta_{(q_i^A, p_i^A, g, \xi, \alpha)}(\bar{F}^A_i, G^A_i, \xi_1, \nu^1) = p_i^A \bar{F}^A_i + \left< \alpha, \xi_1 \right>,
\]

\[
\omega_{(q_i^A, p_i^A, g, \xi, \alpha)}((\bar{F}^A_i, G^A_i, \xi_1, \nu^1), (\bar{F}^A_i, G^A_i, \xi_2, \nu^2)) = \sum_{i=0}^{k-1} (\bar{F}^A_i G^A_i - \bar{F}^A_i \tilde{G}^A_i) + \left< \nu^2, \xi_1 \right> - \left< \nu^1, \xi_2 \right> + \left< \alpha_0, [\xi_0^0, \xi_0^1] \right>
\]

where \( \xi_a \in \mathfrak{g} \) and \( \nu^a \in \mathfrak{g}^* \), \( a = 1, 2 \) with components \( \xi_a = (\xi_a^0) \) and \( \nu^a = (\nu_a^0), \) \( 0 \leq i \leq k-1 \), where each component \( \xi_a^i \in \mathfrak{g} \) and \( \nu_i^a \in \mathfrak{g}^* \). Observe that \( \alpha_0 \) comes from the identification \( T^*G \simeq G \times \mathfrak{g}^* \).

Given a Hamiltonian function \( H \in C^\infty(T^*(T^{k-1}(M \times G))) \), we compute

\[
dH_{(q_i^A, p_i^A, g, \xi, \alpha)}(\bar{F}^A_i, G^A_i, \xi_2, \nu^2) = \bar{F}^A_i \frac{\partial H}{\partial q_i^A}(q_i^A, p_i^A, g, \xi, \alpha) + G^A_i \frac{\partial H}{\partial p_i^A}(q_i^A, p_i^A, g, \xi, \alpha) + \left< \xi_2, \frac{\partial H}{\partial \xi_0} \right>
\]

\[
+ \sum_{i=0}^{k-2} \frac{\partial H}{\partial \xi_i}(q_i^A, p_i^A, g, \xi, \alpha) \xi_i^{i+1} + \sum_{i=0}^{k-2} \left< \nu_i^2, \frac{\partial H}{\partial \xi_i}(q_i^A, p_i^A, g, \xi, \alpha) \right>.
\]

Now we can derive Hamilton equations for a higher-order dynamical system in a trivial principal bundle. First, let us compute the Hamiltonian vector field \( X_H \in \mathfrak{X}(T^*(T^{k-1}M) \times G \times (k-1)\mathfrak{g} \times k\mathfrak{g}^*) \) satisfying the geometric equation \( i(X_H)\omega = dH \). If \( X_H \) is locally given by \( X_H(q_i^A, p_i^A, g, \xi, \alpha) = (\bar{F}^A_i, G^A_i, \xi_1, \nu^1) \), then the previous equation gives the following system of equations

\[
\bar{F}^A_i = \frac{\partial H}{\partial p_i^A}(q_i^A, p_i^A, g, \xi, \alpha),
\]

\[
G^A_i = -\frac{\partial H}{\partial q_i^A}(q_i^A, p_i^A, g, \xi, \alpha),
\]

\[
\xi_1 = \frac{\partial H}{\partial \alpha}(q_i^A, p_i^A, g, \xi, \alpha),
\]

\[
\nu_i^1 = -\mathcal{L}_g \frac{\partial H}{\partial g}(q_i^A, p_i^A, g, \xi, \alpha) + \lambda \xi_i^0 \alpha_0,
\]

\[
\nu_{i+1}^1 = -\frac{\partial H}{\partial \xi_i}(q_i^A, p_i^A, g, \xi, \alpha), \quad 0 \leq i \leq k-2.
\]
Finally, if \( \gamma(t) = (q^A_i(t), p^A_i(t), g(t), \xi(t), \alpha(t)) \) is an integral curve of \( X_H \), then from the condition \( X_H \circ \gamma = \dot{\gamma} \) we obtain the higher-order Hamilton equations

\[
\begin{align*}
\dot{q}^A_i &= \frac{\partial H}{\partial p^A_i}(q^A_i, p^A_i, g, \xi, \alpha), & 0 \leq i \leq k - 1, \\
\dot{p}^A_i &= -\frac{\partial H}{\partial q^A_i}(q^A_i, p^A_i, g, \xi, \alpha), & 0 \leq i \leq k - 1, \\
\dot{g} &= g \frac{\partial H}{\partial \alpha_0}(q^A_i, p^A_i, g, \xi, \alpha), \\
\dot{\xi}^i &= \frac{\partial H}{\partial \alpha_i}(q^A_i, p^A_i, g, \xi, \alpha), & 1 \leq i \leq k - 1, \\
\dot{\alpha}_0 &= -L_g \frac{\partial H}{\partial g}(q^A_i, p^A_i, g, \xi, \alpha) + ad^*_{\partial H/\partial \alpha_0} \alpha_0, \\
\dot{\alpha}_{i+1} &= -\frac{\partial H}{\partial \xi^i}(q^A_i, p^A_i, g, \xi, \alpha), & 0 \leq i \leq k - 2.
\end{align*}
\]

3 Geometric formalism for higher-order variational problems

In this section, we describe the main results of the paper. First, we intrinsically derive the equations of motion for Lagrangian systems defined on higher-order trivial principal bundles and, finally, we extend the results to the case of variationally constrained problems.

3.1 Unconstrained problem

3.1.1 Geometrical setting

Let \( Q \) be a finite dimensional smooth manifold modeling the configuration space of a \( k \)th-order dynamical system, and let \( L \in C^\infty(T^kQ) \) be a Lagrangian function describing the dynamics of the system. Consider the Pontryagin bundle \( W = T^kQ \times_{T^{k-1}Q} T^*(T^{k-1}Q) \) in a similar way as in [40]. Now, if we take \( Q = M \times G \), where \( M \) is a \( m \)-dimensional smooth manifold and \( G \) is a finite dimensional Lie group, we have

\[
\begin{align*}
\mathcal{W} &= T^k(M \times G) \times_{T^{k-1}(M \times G)} T^*(T^{k-1}(M \times G)) \\
&\simeq (T^kM \times_{T^{k-1}M} T^*(T^{k-1}M)) \times (T^kG \times_{T^{k-1}G} T^*(T^{k-1}G)) = \mathcal{W}_M \times \mathcal{W}_G
\end{align*}
\]

where we denote \( \mathcal{W}_M := T^kM \times_{T^{k-1}M} T^*(T^{k-1}M) \) and \( \mathcal{W}_G := T^kG \times_{T^{k-1}G} T^*(T^{k-1}G) \). Using left-trivialization and the results in Section [2,3] we have the following identifications

\[
T^kG \simeq G \times k \mathfrak{g}; \quad T^*(T^{k-1}G) \simeq G \times (k-1)\mathfrak{g} \times k \mathfrak{g}^*,
\]

and therefore the manifold \( \mathcal{W}_G \) admits the identification

\[
\mathcal{W}_G \simeq G \times k \mathfrak{g} \times k \mathfrak{g}^*.
\]
Taking into account all the previous comments, we can consider the diagram illustrating the situation

\[
\begin{array}{c c c c}
\rho_{k-1} & T^{k-1}M \\
\downarrow & \downarrow & \downarrow \\
\pi_{k-1} & T^k(M) \quad & W_{M} & \quad & \pi_{k-1} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\pi_{k-1} & W_{G} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\bar{\pi}_{k-1} & G \times k\mathfrak{g} & \quad & \bar{\pi}_{k-1} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\bar{\pi}_{k-1} & G \times (k-1)\mathfrak{g} \times k\mathfrak{g}^* & \quad & \bar{\pi}_{k-1} \\
\end{array}
\]

where all the maps are the canonical projections.

Let \((q^A_i, q_k^A, p_A^j), \) where \(0 \leq i \leq k-1\) and \(1 \leq A \leq m,\) be a set of local coordinates in \(\mathcal{W}_M\) (see [40] for details), and \((g, \xi, \xi^{k-1}, \alpha),\) where \(\xi = (\xi_0^0, \ldots, \xi_{k-2}^0) \in (k-1)\mathfrak{g}\) and \(\alpha = (\alpha_0, \ldots, \alpha_{k-1}) \in k\mathfrak{g}^*,\) a set of local coordinates in \(\mathcal{W}_G\) (see [13] for details). Then, the induced natural coordinates in \(\mathcal{W} = \mathcal{W}_M \times \mathcal{W}_G\) are \((q^A_i, q_k^A, p_A^j, g, \xi, \xi^{k-1}, \alpha).\) Using these coordinates, the above projections have the following local expressions

\[
\begin{align*}
\text{pr}_1(q^A_i, q_k^A, p_A^j, g, \xi, \xi^{k-1}, \alpha) &= (q^A_i, q_k^A, p_A^j) \quad ; \quad \text{pr}_2(q^A_i, q_k^A, p_A^j, g, \xi, \xi^{k-1}, \alpha) = (g, \xi, \xi^{k-1}) \\
\bar{\text{pr}}_1(q^A_i, q_k^A, p_A^j) &= (q^A_i, q_k^A) \quad ; \quad \bar{\text{pr}}_2(q^A_i, q_k^A, p_A^j) = (q^A_i, p_A^j) \\
\overline{\text{pr}}_1(g, \xi, \xi^{k-1}, \alpha) &= (g, \xi, \xi^{k-1}) \quad ; \quad \overline{\text{pr}}_2(g, \xi, \xi^{k-1}, \alpha) = (g, \xi, \alpha)
\end{align*}
\]

The bundle \(\mathcal{W}\) is endowed with some canonical geometric structures. First, let \(\omega_{k-1} \in \Omega^2(T^*(T^{k-1}M))\) and \(\omega_{G \times (k-1)\mathfrak{g}} \in \Omega^2(G \times (k-1)\mathfrak{g} \times k\mathfrak{g}^*)\) be the canonical symplectic forms in \(T^*(T^{k-1}M)\) and \(T^*(T^{k-1}G) \simeq G \times (k-1)\mathfrak{g} \times k\mathfrak{g}^*,\) respectively. Then, we can consider the presymplectic forms \(\Omega_M = \bar{\text{pr}}_2^*\omega_{k-1} \in \Omega^2(\mathcal{W}_M)\) and \(\Omega_G = \overline{\text{pr}}_2^*\omega_{G \times (k-1)\mathfrak{g}} \in \Omega^2(\mathcal{W}_G).\) Then we define the following presymplectic form in \(\mathcal{W}\)

\[\Omega = \text{pr}_1^*\Omega_M + \text{pr}_2^*\Omega_G \in \Omega^2(\mathcal{W}).\]

Observe that since \(\ker \Omega_M = \mathfrak{X}^V(\bar{\text{pr}}_2)(\mathcal{W}_M)\) and \(\ker \Omega_G = \mathfrak{X}^V(\overline{\text{pr}}_2)(\mathcal{W}_G),\) we have

\[\ker \Omega = \mathfrak{X}^V(\bar{\text{pr}}_2 \circ \text{pr}_1)(\mathcal{W}) \cap \mathfrak{X}^V(\text{pr}_2)(\mathcal{W}) + \mathfrak{X}^V(\overline{\text{pr}}_2 \circ \text{pr}_2)(\mathcal{W}) \cap \mathfrak{X}^V(\text{pr}_1)(\mathcal{W}).\]

In natural coordinates, recall that the forms \(\omega_{k-1}\) and \(\omega_{G \times (k-1)\mathfrak{g}}\) are locally given by

\[
\begin{align*}
(\omega_{k-1})(q^A_i, p_A^j) &= \sum_{i=0}^{k-1} \left( F_{A}^i A^i - F_{A}^i A^i \right) \\
(\omega_{G \times (k-1)\mathfrak{g}})(g, \xi, \nu) &= \langle \nu, \xi \rangle - \langle \nu_1, \xi_1 \rangle + \langle \alpha, \xi_0 \rangle \\
&= \sum_{i=0}^{k-1} \left( \langle \nu_i, \xi_i \rangle - \langle \nu_1, \xi_1 \rangle \right) + \langle \alpha, \xi_0 \rangle
\end{align*}
\]

where \(\xi = (\xi_0^0, \ldots, \xi_{k-1}^0) \in k\mathfrak{g},\) \(\nu = (\nu_0^0, \ldots, \nu_{k-1}^0) \in k\mathfrak{g}^*\) (\(a = 1, 2,\)) \((q^A_i, p_A^j) \in T^*(T^{k-1}M)\) and \((F_{A}^i A^i), (F_{A}^i A^i) \in T_{(q^A_i, p_A^j)}T^*(T^{k-1}M).\) Therefore, the presymplectic form \(\Omega \in \Omega^2(\mathcal{W})\) is locally given by

\[\Omega(q^A_i, q_k^A, p_A^j, g, \xi^{k-1}, \alpha)(X_1, X_2) = \sum_{i=0}^{k-1} \left( F_{A}^i A^i - F_{A}^i A^i + \langle \nu_i, \xi_i \rangle - \langle \nu_1, \xi_1 \rangle \right) + \langle \alpha, \xi_0 \rangle + \langle \xi_0, \xi_0 \rangle,\]

(3)
where $X_1 = (\tilde{F}^A, \tilde{F}^k, \tilde{G}^i_A, \xi_1, \xi_k^1, \nu^1)$, $X_2 = (\tilde{F}^A, \tilde{F}^k, \tilde{G}^i_A, \xi_2, \xi_k^2, \nu^2) \in \mathcal{X}(W)$. Moreover, a local basis for $\ker \Omega$ is

$$\ker \Omega = \left( \frac{\partial}{\partial q^A_k}, \frac{\partial}{\partial \xi^{k-1}} \right).$$

(4)

The second relevant canonical structure in $W$ is the coupling function $\mathcal{C} \in C^\infty(W)$. First, since $T^kM$ is canonically embedded into $T(T^k(M))$, we can define a canonical pairing between the elements of $T^*(T^{k-1}M)$ and the elements of $T^kM$ as a function in $C^\infty(W_M)$. Indeed, let $p \in T^kM$ be a point, $q = p^*_k(p)$ its projection to $T^{k-1}M$ and $\alpha_q \in T^*_q(T^{k-1}M)$ a covector. Then, the function $\mathcal{C}_M \in C^\infty(W_M)$ is defined as

$$\mathcal{C}_M : T^kM \times_{T^{k-1}M} T^*(T^{k-1}M) \rightarrow \mathbb{R} \quad (p, \alpha_q) \mapsto \langle \alpha_q, j_k(p)q \rangle_{k-1},$$

where $j_k : T^kM \hookrightarrow T(T^k(M))$ is the canonical embedding, $j_k(p)_q \in T_q(T^{k-1}M)$ the corresponding tangent vector to $T^{k-1}M$ in $q$, and $\langle \cdot, \cdot \rangle_{k-1} : T(T^k(M)) \times T^*(T^{k-1}M) \rightarrow \mathbb{R}$ the canonical pairing. In natural coordinates, if $p = (q^A_0, \ldots, q^A_k)$, then $q = p^*_k(p) = (q^A_0, \ldots, q^A_{k-1})$, and the canonical embedding is locally given by $j_k(p) = (q^A_0, \ldots, q^A_{k-1}, q^A_k)$. Hence, if $j_k(p)_q = \frac{\partial}{\partial q^A_k} \bigg|_q \in T_q(T^{k-1}M)$ and $\alpha_q = \left. p^*_A dq^A_i \right|_q \in T^*_q(T^{k-1}M)$, then the local expression of the function $\mathcal{C}_M$ is

$$\mathcal{C}_M(q^A_i, q^A_k, p^A_j) = \left. p^*_A dq^A_i \right|_q \cdot$$

On the other hand, we can define a canonical pairing in $W_G \simeq G \times k\mathfrak{g} \times k\mathfrak{g}^*$ as a function $\mathcal{C}_G \in C^\infty(W_G)$ as follows

$$\mathcal{C}_G : G \times k\mathfrak{g} \times k\mathfrak{g}^* \rightarrow \mathbb{R} \quad (g, \xi, \xi^{k-1}, \alpha) \mapsto \langle \alpha_i, \xi^i \rangle,$$

Bearing in mind the above constructions, we can give the following definition.

**Definition 2** The coupling function $\mathcal{C} \in C^\infty(W)$ is defined as

$$\mathcal{C} = \text{pr}_1^* \mathcal{C}_M + \text{pr}_2^* \mathcal{C}_G.$$  

(5)

In the induced natural coordinates of $W$, bearing in mind the local expressions of both $\mathcal{C}_M$ and $\mathcal{C}_G$, and the coordinate expressions of the projections $\text{pr}_1$ and $\text{pr}_2$, we have that the coupling function $\mathcal{C} \in C^\infty(W)$ is locally given by

$$\mathcal{C}(q^A_i, q^A_k, p^A_j, g, \xi, \xi^{k-1}, \alpha) = \sum_{i=0}^{k-1} \left( p^*_A dq^A_i + \langle \alpha_i, \xi^i \rangle \right).$$

Now, given a Lagrangian function $\mathcal{L} \in C^\infty(T^k(M \times G))$, we can define the Hamiltonian function $H \in C^\infty(W)$ as

$$H = \mathcal{C} - \pi^* \mathcal{L},$$

(6)

where $\pi : W \rightarrow T^k(M \times G)$ is the natural projection, and whose local expression is

$$H(q^A_i, q^A_k, p^A_j, g, \xi, \xi^{k-1}, \alpha) = \sum_{i=0}^{k-1} \left( p^*_A dq^A_i + \langle \alpha_i, \xi^i \rangle \right) - \mathcal{L}(q^A_i, q^A_k, g, \xi, \xi^{k-1}).$$
3.1.2 Dynamical equation

The dynamical equation for a presymplectic Hamiltonian system \((\mathcal{W}, \Omega, H)\) is geometrically written as

\[
i(X)\Omega = dH, \quad \text{for } X \in \mathfrak{X}(\mathcal{W}).
\]  

(7)

Then, following [30], we have

Proposition 1 A solution \(X \in \mathfrak{X}(\mathcal{W})\) to equation (7) exists only on the points of the submanifold \(W_c \hookrightarrow \mathcal{W}\) defined by

\[
W_c = \{p \in \mathcal{W}: (i(Y)dH)(p) = 0, \quad \forall Y \in \ker \Omega\}.
\]

In natural coordinates, since \(dH \in \Omega^1(\mathcal{W})\) is locally given by

\[
dH_{(q^A, p^A, q^i, p^i, \xi, \xi)}(Y) = -\tilde{F}_0^A \frac{\partial L}{\partial q^0_A} + \tilde{F}_i^{A+1} \left( p^i_A - \frac{\partial L}{\partial q^i_{A+1}} \right) + \tilde{G}^A_i q^{A+1}_i,
\]

(8)

where \(Y = (\tilde{F}_i^{A}, \tilde{F}_i^k, \tilde{G}^i_A, \xi_2, \xi_2^k, \nu^2) \in \mathfrak{X}(\mathcal{W})\), and \(\ker \Omega\) has local basis \([41]\), we have

\[
i(Y)dH = \begin{cases} 
p^{k-1}_A - \frac{\partial L}{\partial q^k_A}, & \text{if } Y = \frac{\partial}{\partial q^k_A}, \\
\alpha_{k-1} - \frac{\partial L}{\partial \xi^{k-1}}, & \text{if } Y = \frac{\partial}{\partial \xi^{k-1}}.
\end{cases}
\]

Therefore, \(W_c \hookrightarrow \mathcal{W}\) is locally defined by the constraints

\[
p^{k-1}_A - \frac{\partial L}{\partial q^k_A} = 0 ; \quad \alpha_{k-1} - \frac{\partial L}{\partial \xi^{k-1}} = 0.
\]

Now, let us compute the local expression of equation (7). Let \(X \in \mathfrak{X}(\mathcal{W})\) be a generic vector field locally given by

\[
X = F_i^A \frac{\partial}{\partial q^A_i} + F_k^A \frac{\partial}{\partial q^A_k} + G^i_A \frac{\partial}{\partial p^A_i} + \xi_0 \frac{\partial}{\partial g} + \xi_i^{A+1} \frac{\partial}{\partial \xi^i} + \nu_0^i \frac{\partial}{\partial \alpha_i} = (F_i^A, F_k^A, G_A^i, \xi_1, \xi_1^k, \nu^1).
\]

(9)

Then, using (3) and (8), we have the following system of equations

\[
F_i^A = q^i_{A+1},
\]

(10)

\[
G^0_A = \frac{\partial L}{\partial q^A_0}, \quad G^i_A = \frac{\partial L}{\partial q^A_i} - p^{i-1}_A,
\]

(11)

\[
p^{k-1}_A - \frac{\partial L}{\partial q^k_A} = 0,
\]

(12)

\[
\xi_1^i = \xi^i,
\]

(13)

\[
\nu_0^i = L_g^A \frac{\partial L}{\partial g} + ad_{\xi_0^i} \alpha_0, \quad \nu_1^{i+1} = \frac{\partial L}{\partial \xi^i} - \alpha_i,
\]

(14)

\[
\alpha_{k-1} - \frac{\partial L}{\partial \xi^{k-1}} = 0.
\]

(15)
Therefore, the vector field $X$ solution to equation (7) is locally given by

$$X = q_{i+1}^A \frac{\partial}{\partial q_i^A} + F_k^A \frac{\partial}{\partial q_k^A} + \frac{\partial L}{\partial q_i^A} \frac{\partial}{\partial p_{i}^A} + \left( \frac{\partial L}{\partial q_i^A} - p_{j}^{j-1} \right) \frac{\partial}{\partial p_{j}^A}$$

$$+ \xi^0 \frac{\partial}{\partial g} + \xi^{i+1} \frac{\partial}{\partial \xi^i} + \xi^k \frac{\partial}{\partial \xi^{k-1}} + \left( L_g^* \frac{\partial L}{\partial g} + ad^*_\xi \alpha_0 \right) \frac{\partial}{\partial g} + \left( \frac{\partial L}{\partial \xi^i} - \alpha_i \right) \frac{\partial}{\partial \alpha_{i+1}}.$$

Observe that equations (12) and (15) are compatibility conditions that say that the vector field $X$ exists with support on a submanifold defined locally by these equations. Hence, we recover in coordinates the result stated in Proposition 1.

The coefficients $F_k^A$ and $\xi^k$ are yet to be determined. Nevertheless, recall that $X$ is a vector field in $W$ that exists at support on $W_c$. Hence, we must study the tangency of $X$ along the submanifold $W_c$; that is, we must require $L(X)\xi|_{W_c} \equiv X(\xi)|_{W_c} = 0$ for every constraint function $\xi$ defining $W_c$. Thus, taking into account that $W_c$ is locally defined by equations (12) and (15), the tangency condition for $X$ along $W_c$ gives the following equations

$$\frac{\partial L}{\partial q_{k-1}^A} - p_{k-2}^A = q_{i+1}^B \frac{\partial^2 L}{\partial q_i^A \partial q_k^A} + F_k^B \frac{\partial^2 L}{\partial q_k^A \partial q_k^A}$$

$$+ \xi^0 L_g^* \frac{\partial^2 L}{\partial g \partial q_k^A} + \xi^{i+1} \frac{\partial^2 L}{\partial q^i \partial q_k^A} + \xi^k \frac{\partial^2 L}{\partial q_{k-1} \partial q_k^A},$$

$$\frac{\partial L}{\partial \xi_{k-2}^k} - \alpha_{k-2} = q_{i+1}^B \frac{\partial^2 L}{\partial q_i^A \partial \xi_{k-1}^A} + F_k^B \frac{\partial^2 L}{\partial q_k^A \partial \xi_{k-1}^A}$$

$$+ \xi^0 L_g^* \frac{\partial^2 L}{\partial g \partial \xi_{k-1}^A} + \xi^{i+1} \frac{\partial^2 L}{\partial q^i \partial \xi_{k-1}^A} + \xi^k \frac{\partial^2 L}{\partial \xi_{k-1} \partial \xi_{k-1}^A}.$$  

(16)

These equations enable us to determine the remaining coefficients $F_k^A$ and $\xi^k$ of the vector field $X$. Observe that if the Hessian matrix of $L$ with respect to the highest-order “velocities”, $q^A_k$ and $\xi^{k-1}$, is invertible, that is,

$$\det \left( \begin{array}{cc} \frac{\partial^2 L}{\partial q_k^A \partial q_k^A} & \frac{\partial^2 L}{\partial q_k^A \partial \xi_{k-1}^A} \\ \frac{\partial^2 L}{\partial \xi_{k-1}^A \partial q_k^A} & \frac{\partial^2 L}{\partial \xi_{k-1}^A \partial \xi_{k-1}^A} \end{array} \right) (p) \neq 0, \text{ for every } p \in T^k M \times G \times k_g,$$

then the previous system of equations has a unique solution for $F_k^A$ and $\xi^k$, thus obtaining a unique vector field $X \in \mathfrak{X}(W)$ solution to the equation (7). In particular, the constraint algorithm finishes at the first step. Otherwise, new constraints may arise from equations (16), and the algorithm continues if necessary.

**Remark:** In the particular case when the Hessian matrix of the Lagrangian function is a block diagonal matrix, that is,

$$\frac{\partial^2 L}{\partial q_i^A \partial q_j^A} = 0 \text{ and } \frac{\partial^2 L}{\partial q_i^A \partial \xi_j^k} = 0, \text{ for every } 1 \leq A \leq m, 0 \leq i \leq k, 0 \leq j \leq k-1,$$

then equations (16) become

$$\frac{\partial L}{\partial q_{k-1}^A} - p_{k-2}^A = q_{i+1}^B \frac{\partial^2 L}{\partial q_i^A \partial q_k^A} + F_k^B \frac{\partial^2 L}{\partial q_k^A \partial q_k^A},$$

$$\frac{\partial L}{\partial \xi_{k-2}^k} - \alpha_{k-2} = \xi^0 L_g^* \frac{\partial^2 L}{\partial g \partial q_{k-1}^A} + \xi^{i+1} \frac{\partial^2 L}{\partial q^i \partial q_{k-1}^A} + \xi^k \frac{\partial^2 L}{\partial \xi_{k-1} \partial q_{k-1}^A}.$$
In this case, to solve the equation (17) in \( W \) is equivalent to solve separately the corresponding equations in \( W_M \) and \( W_G \) following the patterns in [10] and [13], respectively, and then take \( X = X_M + X_G \) as a solution of equation (17), where \( X_M \in \ker pr_2 \) is a vector field \( pr_1 \)-related with the vector field solution to the equation in \( W_M \) and \( X_G \in \ker pr_1 \) is a vector field \( pr_2 \)-related with the solution of the equation in \( W_G \).

Now, let \( \gamma: \mathbb{R} \to W \) be an integral curve of \( X \) locally given by
\[
\gamma(t) = (q^A_1(t), q^A_k(t), p^j_A(t), g(t), \xi^i(t), \alpha_i(t)).
\] (17)

From the condition \( X \circ \gamma = \dot{\gamma} \) we obtain the following system of differential equations for the component functions of \( \gamma \)
\[
p^0_A = \frac{\partial L}{\partial q^A_1} \ , \quad p^j_A = \frac{\partial L}{\partial q^A_k} - p^{j-1}_A,
\] (18)
\[
\dot{q}^A_i = q^{A+1}_i ,
\] (19)
\[
\dot{g} = g\xi^0 \ , \quad \dot{\xi}^i = \xi^i \ ,
\] (20)
\[
\dot{\alpha}_0 = L_g^{*} \frac{\partial L}{\partial g} + ad_{\xi^0}^{*}\alpha_0 \ , \quad \dot{\alpha}_{i+1} = \frac{\partial L}{\partial \xi^i} - \alpha_i \ ,
\] (21)
in addition to equations (12) and (13). Now, using equations (12) in combination with equations (19) we obtain the \( k \)th-order Euler-Lagrange equations
\[
\sum_{i=0}^{k} (-1)^i \frac{d}{dt} \left. \frac{\partial L}{\partial q^A_i} \right|_{\gamma} = 0 \ .
\] (22)

On the other hand, using equations (15) in combination with equations (21) we obtain the \( k \)th-order trivialized Euler-Lagrange equations
\[
\left( \frac{d}{dt} - ad_{\xi^0}^{*} \right) \sum_{i=0}^{k-1} (-1)^i \left. \frac{d}{dt} \frac{\partial L}{\partial \xi^i} \right|_{\gamma} = L_g^{*} \left. \frac{\partial L}{\partial g} \right|_{\gamma} \ .
\] (23)

Therefore, a dynamical trajectory \( \gamma: \mathbb{R} \to W \) of the system must satisfy the following local equations
\[
\sum_{i=0}^{k} (-1)^i \left. \frac{d}{dt} \frac{\partial L}{\partial q^A_i} \right|_{\gamma} = 0 \ , \quad \left( \frac{d}{dt} - ad_{\xi^0}^{*} \right) \sum_{i=0}^{k-1} (-1)^i \left. \frac{d}{dt} \frac{\partial L}{\partial \xi^i} \right|_{\gamma} = L_g^{*} \left. \frac{\partial L}{\partial g} \right|_{\gamma} \ .
\]

Remark: The above equations may be compatible or not. If not, a constraint algorithm must be used in order to find a final submanifold where the above equations have a solution (if such a submanifold exists).

Finally, if the Lagrangian function \( L \in C^\infty(T^k M \times G \times k\mathfrak{g}) \) is left-invariant, that is,
\[
L(q^A_i, q^A_k, g, \xi^i) = L(q^A_i, q^A_k, h, \xi^i) ,
\]
for all \( g, h \in G \), then we can define the reduced Lagrangian \( \ell \in C^\infty(T^k M \times k\mathfrak{g}) \) by
\[
\ell(q^A_i, q^A_k, \xi^i) = L(q^A_i, q^A_k, e, \xi^i) ,
\]
and therefore equations (23) become the \( k \)th order Euler-Poincaré equations
\[
\left( \frac{d}{dt} - ad_{\xi^0}^{*} \right) \sum_{i=0}^{k-1} (-1)^i \left. \frac{d}{dt} \frac{\partial \ell}{\partial \xi^i} \right|_{\gamma} = 0 \ .
\] (24)
Observe that equations (22) remain the same with the reduced Lagrangian function, just replacing $\mathcal{L}$ by $\ell$.

Equations (22) and (23) are exactly the same that one of the authors derived in [11] using variational techniques. Our derivation allows us to identify straightforwardly the geometric preservation of the system, for instance, preservation of the Hamiltonian or (pre)symplecticity of the flow.

### 3.1.3 A theoretical example

Now, we give a theoretical example inspired by the applications in Clebsch variational principle and continuum mechanics studied in [27, 34].

Let us consider the particular case when the manifold $M$ is the dual of a real vector space, that is, $M = V^*$, where $V$ is a finite dimensional real vector space. In this case we have the following identifications

$$T^k V^* \simeq (k + 1)V^* \quad , \quad T^* V^* \simeq V^* \times V$$

$$T^*(T^k V^*) \simeq k(V^* \times V)$$

Using these identifications, we have

$$\mathcal{W}_{V^*} = T^k V^* \times_{\mathcal{T} V^*} T^* (T^k V^*) \simeq (k + 1)V^* \times kV$$

Since $V$ and $V^*$ have global charts of coordinates defined by any basis, we will denote an element of $(k + 1)V^*$ by $(\mu, \mu_k) \equiv (\mu_0, \mu_k) \equiv (\mu_0, \ldots, \mu_k)$, where $\mu_j \in V^*$ for every $0 \leq j \leq k$, and an element of $kV$ will be denoted by $(v) \equiv (v^0, \ldots, v^{k-1})$, where $v^j \in V$ for every $0 \leq j \leq k - 1$. Then, an element of $\mathcal{W}_{V^*}$ will be denoted $(\mu, \mu_k, v) \equiv (\mu_0, \ldots, \mu_{k-1}, \mu_k, v^0, \ldots, v^{k-1})$.

The canonical projections $\tilde{pr}_1: (k + 1)V^* \times kV \to (k + 1)V^*$ and $\tilde{pr}_2: (k + 1)V^* \times kV \to kV^* \times kV$ are given by

$$\tilde{pr}_1(\mu, \mu_k, v) = (\mu, \mu_k) \quad ; \quad \tilde{pr}_2(\mu, \mu_k, v) = (\mu, v).$$

Let $\omega_{kV^*} \in \Omega^2(k(V^* \times V))$ be the canonical symplectic form in $k(V^* \times V)$, which is given by

$$(\omega_{kV^*})_{(\mu, \nu)}((\beta^1, u_1^i), (\beta^2, u_2^j)) = \sum_{i=0}^{k-1} (\langle u_2^j, \beta_1^i \rangle_{V^*} - \langle u_1^i, \beta_2^j \rangle_{V^*}) = \sum_{i=0}^{k-1} (\beta_1^i u_2^j - \beta_2^j u_1^i).$$

where $\langle \cdot, \cdot \rangle_{V^*}$ is the canonical pairing between the elements of $V^*$ and its dual $V^{**} \simeq V$. We define the presymplectic form in $\mathcal{W}_{V^*}$ as $\Omega_{V^*} = \tilde{pr}_2^* \omega_{kV^*} \in \Omega^2(\mathcal{W}_{V^*})$. This 2-form is given locally by

$$(\Omega_{V^*})_{(\mu, \mu_k, \nu)}((\beta^1, \mu_k^i, u_1^i), (\beta^2, \mu_k, u_2^j)) = \sum_{i=0}^{k-1} (\beta_1^i u_2^j - \beta_2^j u_1^i).$$

Now, we define the canonical pairing in $\mathcal{W}_{V^*} \simeq (k + 1)V^* \times kV$ as a function $C_{V^*} \in C^\infty(\mathcal{W}_{V^*})$ as follows:

$$C_{V^*}: (k + 1)V^* \times kV \longrightarrow \mathbb{R} \quad (\mu, \mu_k, v) \longmapsto \langle v^i, \mu_{i+1} \rangle_{V^*} = \mu_{i+1}(v^i).$$

With these elements, we can follow the patterns in Section 3.1. Let us consider the manifold $Q = V^* \times G$. Then we consider the Pontryagin bundle

$$\mathcal{W} \simeq (k + 1)V^* \times kV \times G \times k\mathfrak{g} \times k\mathfrak{g}^*.$$
Then, the diagram in Section 3.1 becomes

\[
\begin{array}{c}
(k + 1)V^* \times kV \\
\downarrow \text{pr}_1 \\
(k + 1)V^* \\
\downarrow \mu_{k-1} \\
kV^* \\
\downarrow \tau_{k-1, V^*} \\
kV^* \times kV \\
\downarrow \tau_k \\
W \\
\downarrow \text{pr}_1 \\
(k + 1)V^* \times kV \\
\downarrow \text{pr}_2 \\
G \times k\mathbb{g} \times k\mathbb{g}^* \\
\downarrow \tau_{k-1, \mathbb{g}} \\
G \times k\mathbb{g} \\
\downarrow \tau_k \\
G \times (k - 1)\mathbb{g} \times k\mathbb{g}^* \\
\downarrow \text{pr}_2
\end{array}
\]

Let \((\mu, \mu_k, v, g, \xi, \xi^k, \alpha)\) be local coordinates in \(W_{V^*}\) and \((g, \xi, \xi^k, \alpha)\) are local coordinates in \(W_G\). Then, the induced local coordinates in \(W\) are \((\mu, \mu_k, v, g, \xi, \xi^k, \alpha)\). The canonical projections \(\text{pr}_1: (k + 1)V^* \times kV \times G \times k\mathbb{g} \times k\mathbb{g}^* \to (k + 1)V^* \times kV\) and \(\text{pr}_2: (k + 1)V^* \times kV \times G \times k\mathbb{g} \times k\mathbb{g}^* \to G \times k\mathbb{g} \times k\mathbb{g}^*\) are given in these coordinates by

\[\text{pr}_1(\mu, \mu_k, v, g, \xi, \xi^k, \alpha) = (\mu, \mu_k, v) \quad \text{and} \quad \text{pr}_2(\mu, \mu_k, v, g, \xi, \xi^k, \alpha) = (g, \xi, \xi^k, \alpha)\]

The presymplectic form \(\Omega \in \Omega^2(W)\) defined in (2) is now given by

\[
\Omega(\mu, \mu_k, v, g, \xi, \xi^k, \alpha)(X_1, X_2) = \sum_{i=0}^{k-1} \left( \beta_1^i (u_2^i) - \beta_2^i (u_1^i) + \langle v_1^i, \xi_1^i \rangle - \langle v_2^i, \xi_2^i \rangle \right) + \langle \alpha_0, [\xi_1^0, \xi_2^0] \rangle,
\]

where \(X_1 = (\beta_1^1, \beta_1^2, u_1^1, \xi_1^1, \xi_2^1, \nu^1), X_2 = (\beta_2^1, \beta_2^2, u_2^1, \xi_2^1, \xi_2^1, \nu^2) \in \mathcal{X}(W)\). The coupling function \(C \in C^\infty(W)\) is

\[
C(\mu, \mu_k, v, g, \xi, \xi^k, \alpha) = \sum_{i=0}^{k-1} (\mu_{i+1}(v^i) + \langle \alpha_i, \xi_i^i \rangle).
\]

Let \(\mathcal{L} \in C^\infty((k + 1)V^* \times G \times k\mathbb{g})\) be a \(k\)-th order Lagrangian function. We define the Hamiltonian function \(H \in C^\infty(W)\) by

\[
H(\mu, \mu_k, v, g, \xi, \xi^k, \alpha) = \sum_{i=0}^{k-1} (\mu_{i+1}(v^i) + \langle \alpha_i, \xi_i^i \rangle) - \mathcal{L}(\mu, \mu_k, g, \xi, \xi^k, \alpha).
\]

Now, if \(X \in \mathcal{X}(W)\) is a vector field, locally given by

\[
X = \beta_i \frac{\partial}{\partial \mu_i} + \beta_k \frac{\partial}{\partial \mu_k} + u^i \frac{\partial}{\partial v^i} + \xi_1 \frac{\partial}{\partial g} + \xi_1 \frac{\partial}{\partial \xi_1} + \nu_1 \frac{\partial}{\partial \alpha} = (\beta, \beta_k, u^i, \xi_1 \xi_1, \nu_1),
\]

then the dynamical equation

\[i(X)\Omega = dH,\]
Therefore, the vector field $X$ solution to the dynamical equation is locally given by

$$X = \mu_{i+1} \frac{\partial}{\partial \mu_i} + \beta_k \frac{\partial}{\partial \mu_k} + \frac{\partial L}{\partial v_i} \frac{\partial}{\partial v^0} + \left( \frac{\partial L}{\partial \mu_i} - v^{i-1} \right) \frac{\partial}{\partial v^1} + \xi^i \frac{\partial}{\partial g} + \xi^i \frac{\partial}{\partial \xi^1} + \left( L^*_g \frac{\partial L}{\partial g} + \alpha_0 \right) \frac{\partial}{\partial \alpha_0} + \left( \frac{\partial L}{\partial \xi^i} - \alpha_i \right) \frac{\partial}{\partial \alpha_{i+1}}$$

Finally, the tangency condition along the submanifold $\mathcal{W}_c$ defined locally by the constraints

$$v^{k-1} - \frac{\partial L}{\partial \mu_k} = 0 \quad ; \quad \alpha_{k-1} - \frac{\partial L}{\partial \xi^k} = 0 \quad (25)$$

gives the following system of equations for the remaining coefficients $\beta_k$ and $\xi^k$

$$\begin{align*}
\frac{\partial L}{\partial \mu_{k-1}} - v^{k-2} &= \mu_{i+1} \frac{\partial^2 L}{\partial \mu_i \partial \mu_k} + \beta_k \frac{\partial^2 L}{\partial \mu_k \partial \mu_k} + \xi^0 \frac{\partial^2 L}{\partial g \partial \mu_k} + \xi^{i+1} \frac{\partial^2 L}{\partial \xi^i \partial \mu_k} + \xi^k \frac{\partial^2 L}{\partial \xi^k \partial \mu_k}, \\
\frac{\partial L}{\partial \xi^{k-2} - \alpha_{k-2}} &= \mu_{i+1} \frac{\partial^2 L}{\partial \mu_i \partial \xi^{k-1}} + \mu_k \frac{\partial^2 L}{\partial \mu_k \partial \xi^{k-1}} + \xi^0 \frac{\partial^2 L}{\partial g \partial \xi^{k-1}} + \xi^{i+1} \frac{\partial^2 L}{\partial \xi^i \partial \xi^{k-1}} + \xi^k \frac{\partial^2 L}{\partial \xi^k \partial \xi^{k-1}}.
\end{align*}$$

If the matrix

$$\begin{pmatrix}
\frac{\partial^2 L}{\partial \mu_k \partial \mu_k} & \frac{\partial^2 L}{\partial \mu_k \partial \xi^{k-1}} \\
\frac{\partial^2 L}{\partial \mu_k \partial \mu_k} & \frac{\partial^2 L}{\partial \mu_k \partial \xi^{k-1}} \\
\frac{\partial^2 L}{\partial \xi^{k-1} \partial \mu_k} & \frac{\partial^2 L}{\partial \xi^{k-1} \partial \xi^{k-1}}
\end{pmatrix},$$

is regular for every point $p \in (k+1)V^* \times G \times k_g$, then by a direct computation the previous equations have a unique solution for $\mu_{k-1}$ and $\xi^k$.

### 3.2 Constrained problem

#### 3.2.1 Geometrical setting

As in Section 3.1 let $Q$ be a finite dimensional smooth manifold modeling the configuration space of a $k$th-order dynamical system. Now we assume that the dynamics of the system are constrained. Geometrically, the Lagrangian function containing the dynamical information of the system is defined at support on a submanifold of $T^kQ$. Let $j_N: N \hookrightarrow T^kQ$ be the constraint submanifold, with codim $\mathcal{N} = n$, and $\mathcal{L}_N \in C^\infty(\mathcal{N})$ the Lagrangian function describing the dynamics of the constrained dynamical system.
Let us consider the submanifold $W = N \times_{T^k-1} T^*(T^k-1Q)$ of $T^kQ \times_{T^k-1} T^*(T^k-1Q)$ with canonical embedding $i_W: W \hookrightarrow T^kQ \times_{T^k-1} T^*(T^k-1Q)$ and natural projection $\text{pr}_N: W \rightarrow N$. If we take $Q = M \times G$, where $M$ is a $m$-dimensional smooth manifold and $G$ a finite dimensional Lie group, then we have $W = N \times_{T^k-1} M \times_{T^k-1} G^*$. 

Now, using the results given in Section 3.1, we can define a closed 2-form in $W$ as $\Omega = i^*_W \Omega \in \Omega^2(W)$, where $\Omega \in \Omega^2(N)$ is the presymplectic form defined in (2), and a Hamiltonian function $H = i^*_W C - \text{pr}_N^* L_N \in C^\infty(W)$, where $C \in C^\infty(N)$ is the coupling function defined in (5). With these elements we can state the dynamical equation for the constrained problem, which is

$$i(X)\Omega = dH. \quad (26)$$

Since $N \hookrightarrow T^k(M \times G)$ is an arbitrary $n$-codimensional submanifold, we do not have a natural set of coordinates in $W$, and therefore the local study of the equation can not be done in a general setting. For this reason, we adopt an “extrinsic point of view”, that is, we will work in the bundle $W$, and then require the solutions to lie in the submanifold $W = N \times_{T^k-1} M \times_{T^k-1} G^*$.

In order to do this, we must construct a Hamiltonian function $H \in C^\infty(W)$ using the Lagrangian function $L_N \in C^\infty(N)$ containing the dynamical information of the system. Hence, let $L \in C^\infty(T^k(M \times G))$ be an arbitrary extension of $L_N$, and let $H$ be the Hamiltonian function defined in (6) using this arbitrary extension of the Lagrangian function $L_N$.

### 3.2.2 Dynamical equation

The extrinsic dynamical equation for a constrained dynamical system is

$$i(X)\Omega - dH \in \text{ann}(TW), \quad \text{for } X \in \mathfrak{x}(W) \text{ tangent to } W. \quad (27)$$

where $\text{ann}(D)$ denotes the annihilator of a distribution $D \subset TW$. Observe that this equation is clearly equivalent to (26).

Then, following [30] we have

**Proposition 2** A solution to the equation (27) exists only on the points of the submanifold $W_c \hookrightarrow W$ defined by

$$W_c = \{ p \in W: (i(Y)dH)(p) \in \text{ann}(T_pW), \forall Y \in \ker \Omega \}.$$

In natural coordinates, let $\Phi^a \in C^\infty(T^k(M \times G))$, $1 \leq a \leq n$, be local functions defining the submanifold $N \hookrightarrow T^k(M \times G)$, that is,

$$N = \{ p \in T^k(M \times G): \Phi^a(p) = 0, 1 \leq a \leq n \}.$$

In an abuse of notation, we also denote by $\Phi^a$ the pull-back of the constraint functions to $W$. Then, the annihilator of $TW$ is locally given by

$$\text{ann}(TW) = (d\Phi^a).$$

Therefore, the equation defining the submanifold $W_c$ may be written locally as

$$i(Y)dH = \lambda_a d\Phi^a, \forall Y \in \ker \Omega,$$
where $\lambda_a, 1 \leq a \leq n$ are the Lagrange multipliers. Then, bearing in mind the local expression (8) of $dH$ and (4) of $\ker \Omega$, the equations defining locally the submanifold $W_c$ are

$$p^i_A - \frac{\partial L}{\partial q_k^A} + \lambda_a \frac{\partial \Phi^a}{\partial q_k^A} = 0 \quad ; \quad \alpha_{k-1} - \frac{\partial L}{\partial \xi_{k-1}} + \lambda_a \frac{\partial \Phi^a}{\partial \xi_{k-1}} = 0 \quad ; \quad \Phi^a = 0.$$ 

Now, let us compute the local expression of equation (27). If we assume that $N$ is determined by the vanishing of the $n$ functions $\Phi^a$, then equation (27) may be rewritten as

$$i(X) \Omega - dH = \lambda_a d\Phi^a,$$

where $\lambda_a$ are Lagrange multipliers to be determined. Then, bearing in mind the local expression (8) of $dH$ and (3) of $\Omega$, taking a generic vector field locally given by (9) we obtain the following system of equations

$$C^0_A = \frac{\partial L}{\partial q_0^A} - \lambda_a \frac{\partial \Phi^a}{\partial q_0^A}, \quad G^i_A = \frac{\partial L}{\partial q_i^A} - \lambda_a \frac{\partial \Phi^a}{\partial q_i^A} - p^{i-1}_A, \quad F^i_A = q_i^A, \quad (28)$$

$$p^{k-1}_A - \frac{\partial L}{\partial q_k^A} + \lambda_a \frac{\partial \Phi^a}{\partial q_k^A} = 0, \quad (30)$$

$$\nu_0^1 = L_g^*(\frac{\partial L}{\partial g} - \lambda_a \frac{\partial \Phi^a}{\partial g}) + ad^*_{\xi_0} \alpha_0, \quad \nu_{i+1}^1 = \frac{\partial L}{\partial \xi_i} - \lambda_a \frac{\partial \Phi^a}{\partial \xi_i} - \alpha_i, \quad (32)$$

$$\alpha_{k-1} - \frac{\partial L}{\partial \xi_{k-1}} + \lambda_a \frac{\partial \Phi^a}{\partial \xi_{k-1}} = 0, \quad (33)$$

$$\Phi^a(q_i^A, q_k^A, g, \xi, \xi_{k-1}) = 0. \quad (34)$$

Therefore, the vector field $X$ solution to equation (27) is locally given by

$$X = q_{i+1}^A \frac{\partial}{\partial q_i^A} + F_k^A \frac{\partial}{\partial q_k^A} + \left( \frac{\partial L}{\partial q_0^A} - \lambda_a \frac{\partial \Phi^a}{\partial q_0^A} \right) \frac{\partial}{\partial p^0_A} + \left( \frac{\partial L}{\partial q_i^A} - \lambda_a \frac{\partial \Phi^a}{\partial q_i^A} - p^{i-1}_A \right) \frac{\partial}{\partial p^i_A}$$

$$+ \xi_0 \frac{\partial}{\partial \xi_0} + \xi_{i+1} \frac{\partial}{\partial \xi_{i+1}} + \xi_k \frac{\partial}{\partial \xi_k} + \left( L_g^* \left( \frac{\partial L}{\partial g} - \lambda_a \frac{\partial \Phi^a}{\partial g} \right) + ad^*_{\xi_0} \alpha_0 \right) \frac{\partial}{\partial \alpha_0}$$

$$+ \left( \frac{\partial L}{\partial \xi_i} - \lambda_a \frac{\partial \Phi^a}{\partial \xi_i} - \alpha_i \right) \frac{\partial}{\partial \alpha_{i+1}}.$$ 

Observe that equations (30), (33) and (34) do not involve coefficient functions of the vector field $X$: they are pointwise algebraic relations, stating that the vector field $X$ exists with support on a submanifold defined locally by these equations. Hence, we recover locally the result stated in Proposition 2.

The coefficients $F^i_k$ and $\xi^k_i$ of the vector field and the Lagrange multipliers $\lambda_a$ remain undetermined. Nevertheless, from Proposition 2 we know that the vector field $X$ exists only at support on the submanifold $W_c$. Hence, we must require the vector field $X$ to be tangent to $W_c$, that is, we must impose $L(X) \zeta \big|_{W_c} = 0$ for every constraint function $\zeta$ defining $W_c$. Then, taking into account that $W_c$ is locally defined by equations (30), (33) and (34), the tangency
condition for \( X \) along \( \mathcal{W}_c \) gives the following equations

\[
\frac{\partial L}{\partial q_k^A} - p_k^{k-2} = q_{i+1}^B \left( \frac{\partial^2 L}{\partial q_k^B \partial q_k^A} - \lambda_a \frac{\partial^2 \Phi^a}{\partial q^2_k} \right) + F_k^B \left( \frac{\partial^2 L}{\partial q_k^A \partial q_k^A} - \lambda_a \frac{\partial^2 \Phi^a}{\partial q_k^2} \right) + \xi^0 \xi_q^g \left( \frac{\partial^2 L}{\partial g \partial q_k^A} - \lambda_a \frac{\partial^2 \Phi^a}{\partial g \partial q_k^A} \right) + \xi^{i+1} \left( \frac{\partial^2 L}{\partial \xi \partial q_k^A} - \lambda_a \frac{\partial^2 \Phi^a}{\partial \xi \partial q_k^A} \right) + \xi^k \frac{\partial \Phi^a}{\partial q_k^A} + F_k^A \frac{\partial \Phi^a}{\partial q_k^A} + \xi^0 \xi_q^g \frac{\partial \Phi^a}{\partial g} + \xi^{i+1} \frac{\partial \Phi^a}{\partial \xi^i} + \xi^k \frac{\partial \Phi^a}{\partial \xi^k} = 0.
\]

If we denote by \( \Omega_{\mathcal{W}_c} \) the pullback of the presymplectic 2-form \( \Omega \) to \( \mathcal{W}_c \), then we deduce the following theorem.

**Theorem 1 (\( \mathcal{W}_c, \Omega_{\mathcal{W}_c} \)) is a symplectic manifold if and only if**

\[
\frac{\partial^2 L}{\partial q_k^B \partial q_k^A} - \lambda_a \frac{\partial^2 \Phi^a}{\partial q^2_k} \begin{pmatrix}
\frac{\partial \Phi^a}{\partial q_k^A} \\
\frac{\partial \Phi^a}{\partial q_k^A} \\
\frac{\partial \Phi^a}{\partial q_k^A} \\
\frac{\partial \Phi^a}{\partial q_k^A} \\
... \\
\frac{\partial \Phi^a}{\partial q_k^A} \\
\frac{\partial \Phi^a}{\partial q_k^A}
\end{pmatrix}
= 0
\]

is nondegenerate along \( \mathcal{W}_c \).

(Proof) The proof of this theorem is a straightforward computation using Theorem 4.1 in [14] and Theorem 3.3 in [13].

Now, let \( \gamma: \mathbb{R} \to \mathcal{W} \) be an integral curve of \( X \) locally given by (17). Then the condition \( X \circ \gamma = \dot{\gamma} \) gives the following system of differential equations for the component functions of \( \gamma \)

\[
\dot{q}^A_i = q_{i+1}^A, \quad \dot{p}^0_A = \frac{\partial L}{\partial q^A_0} - \lambda_a \frac{\partial \Phi^a}{\partial q^A_0}, \quad \dot{p}^i_A = \frac{\partial L}{\partial q^A_i} - \lambda_a \frac{\partial \Phi^a}{\partial q^A_i} - p_{i-1}^A, \quad \dot{g} = g^{\xi^0}, \quad \dot{\xi}^{i+1} = \xi^i,
\]

\[
\dot{\alpha}_0 = L^g \left( \frac{\partial L}{\partial g} - \lambda_a \frac{\partial \Phi^a}{\partial g} \right) + a d^{\alpha}_\xi \alpha_0, \quad \dot{\alpha}_i = \frac{\partial L}{\partial \xi^i} - \lambda_a \frac{\partial \Phi^a}{\partial \xi^i} - \alpha_i,
\]

in addition to equations (30), (33) and (34). Now, using equations (30) in combination with (37) we obtain the \( k \)th order constrained Euler-Lagrange equations

\[
\sum_{i=0}^{k} (-1)^i \frac{d^i}{dt^i} \left( \frac{\partial L}{\partial q^A_i} - \lambda_a \frac{\partial \Phi^a}{\partial q^A_i} \right) \bigg|_{\gamma} = 0.
\]
On the other hand, using equations \textbf{(33)} in combination with \textbf{(37)} we obtain the \textit{k}th order trivialized constrained Euler-Lagrange equation

\[
\left( \frac{d}{dt} - ad_{t_0}^* \right) \sum_{i=0}^{k-1} (-1)^i \frac{d^i}{dt^i} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}^A} - \lambda_a \frac{\partial \Phi^a}{\partial \dot{q}^i} \right) \gamma = \mathcal{L}^*_g \left( \frac{\partial \mathcal{L}}{\partial g} - \lambda_a \frac{\partial \Phi^a}{\partial g} \right) \gamma. \tag{41}
\]

Therefore, a dynamical trajectory \( \gamma : \mathbb{R} \to \mathcal{W} \) of the system must satisfy the equations \textbf{(40)} and \textbf{(41)}, in addition to \( \Phi^a(q^A_i(t), q^A_k(t), g(t), \xi^i(t)) = 0 \).

Finally, if both the extended Lagrangian function \( \mathcal{L} \in C^\infty(T^k M \times G \times k g) \) and the constraint functions \( \Phi^a \in C^\infty(T^k M \times G \times k g) \) are left-invariant, then we can define the reduced Lagrangian function \( \ell \in C^\infty(T^k M \times k g) \) and the reduced constraint functions \( \phi^a \in C^\infty(T^k M \times k g) \) as

\[
\ell(q^A_i, q^A_k, \xi^i) = \mathcal{L}(q^A_i, q^A_k, e, \xi^i), \quad \phi^a(q^A_i, q^A_k, \xi^i) = \Phi^a(q^A_i, q^A_k, e, \xi^i),
\]

and then equations \textbf{(41)} become

\[
\left( \frac{d}{dt} - ad_{t_0}^* \right) \sum_{i=0}^{k-1} (-1)^i \frac{d^i}{dt^i} \left( \frac{\partial \ell}{\partial \dot{q}^i} - \lambda_a \frac{\partial \phi^a}{\partial \dot{q}^i} \right) \gamma = 0.
\]

Note that equations \textbf{(40)} remain the same, just replacing \( \mathcal{L} \) by \( \ell \) and \( \Phi^a \) by \( \phi^a \).

### 4 Application to optimal control of underactuated mechanical systems

In this section we study optimal control problems for underactuated mechanical systems (or superarticulated mechanical systems following the terminology in \textbf{[1]}). The presence of underactuated mechanical systems is ubiquitous in engineering applications as a result, for instance, of design choices motivated by the search of less cost devices or as a result of a failure regime in fully actuated mechanical systems. The underactuated systems include spacecraft, underwater vehicles, mobile robots, helicopters, wheeled vehicles, mobile robots, underactuated manipulators, etc.

Let \( U \subseteq \mathbb{R}^r \) be the control manifold where \( u(t) \in U \) is the control parameter. We assume that all the control systems are controllable, that is, for any two points \( q_0 \) and \( q_T \) in the configuration space \( Q \), there exists an admissible control \( u(t) \) defined on some interval \([0, T]\) such that the system with initial condition \( q_0 \) reaches the point \( q_T \) in time \( T \) (see \textbf{[6]} for details).

Let us consider that the configuration space \( Q \) of the system is a trivial principal bundle, that is, \( Q = M \times G \), where \( M \) is an \( m \)-dimensional smooth manifold and \( G \) a finite dimensional Lie group. Let \( \mathcal{L} \in C^\infty(TM \times g) \) be a left-trivialized Lagrangian function, where \( g \) is the Lie algebra of \( G \).

The Euler-Lagrange equations with controls are

\[
\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial q^A} \right) - \frac{\partial \mathcal{L}}{\partial q^A} = u_a \mu^a(q), \quad (1 \leq A \leq m) \tag{42}
\]

\[
\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \xi} \right) - ad_{t_0}^* \left( \frac{\partial \mathcal{L}}{\partial \xi} \right) = u_a \eta^a(q),
\]

where \( \mathcal{B}^a = \{ (\mu^a, \eta^a) \} \), \( \mu^a(q) \in T^*_q M \), \( \eta^a(q) \in g^* \), \( a = 1, \ldots, r \), is a set of independent sections of the bundle \( \pi : T^* M \times g^* \to M \), and \( u_a \) are admissible controls.
We complete $\mathcal{B}^{a}$ to a basis $\{\mathcal{B}^{a}, \mathcal{B}^{\alpha}\}$ of $\Gamma(\tau)$, and let us consider its dual basis $\{\mathcal{B}_{a}, \mathcal{B}_{\alpha}\}$, that is, a basis of $\Gamma(\tau)$, where $\tau: TM \times g \to M$. Observe that $\Gamma(\tau) = \mathcal{X}(M) \times C^{\infty}(M, g)$ (see [20] for details). This basis induces coordinates $(q^{A}, \dot{q}^{A}, \xi^{i}, \dot{\xi}^{i})$ on $TM \times g$.

If we denote $\mathcal{B}_{a} = \{\{X_{a}, \Xi_{a}\}\} \subset \Gamma(\tau)$ and $\mathcal{B}_{\alpha} = \{X_{\alpha}, \Xi_{\alpha}\} \subset \Gamma(\tau)$, where $X_{a}, \dot{X}_{a} \in \mathcal{X}(M)$ are locally given by $X_{a}(q) = X_{a}^{A}(q) \frac{\partial}{\partial q^{A}}|q$, $\dot{X}_{a}(q) = X_{a}^{A}(q) \frac{\partial}{\partial q^{A}}|q$, and $\Xi_{a}(q), \Xi_{\alpha}(q) \in g$, with $q \in M$, then equations (42) can be rewritten as

$$\left(\frac{d}{dt} \left(\frac{\partial L}{\partial q^{A}}\right) - \frac{\partial L}{\partial \dot{q}^{A}}\right) X_{a}^{A}(q) + \left(\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^{A}}\right) - ad_{\dot{q}^{A}} \frac{\partial L}{\partial q^{A}}\right) \Xi_{a}(q) = u_{a},$$

$$\left(\frac{d}{dt} \left(\frac{\partial L}{\partial q^{A}}\right) - \frac{\partial L}{\partial \dot{q}^{A}}\right) X_{\alpha}^{A}(q) + \left(\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^{A}}\right) - ad_{\dot{q}^{A}} \frac{\partial L}{\partial q^{A}}\right) \Xi_{\alpha}(q) = 0. \tag{43}$$

**Optimal control problem.** The optimal control problem consists in finding a trajectory $(q(t), \dot{q}(t), \xi(t), u(t))$ of the state variables and control inputs solving equation (43) given initial and final conditions $(q(0), \dot{q}(0), \xi(0))$ and $(q(T), \dot{q}(T), \xi(T))$, respectively, and minimizing the following functional

$$A(q, \dot{q}, \xi, u) = \int_{0}^{T} C(q(t), \dot{q}(t), \xi(t), u(t)) dt,$$

where $C: (TM \times g) \times U \to \mathbb{R}$ is a cost function.

Following [5], to solve this optimal control problem is equivalent to solve the following second-order variational problem with second-order constraints

$$\min \bar{L}(q^{A}, \dot{q}^{A}, \ddot{q}^{A}, \xi^{i}, \ddot{\xi}^{i})$$

subject to $\Phi^{\alpha}(q^{A}, \dot{q}^{A}, \ddot{q}^{A}, \xi^{i}, \ddot{\xi}^{i})$, $\alpha = 1, \ldots, m$

where $\bar{L}, \Phi^{\alpha} \in C^{\infty}(T^{2}M \times 2g)$ are given by

$$\bar{L}(q^{A}, \dot{q}^{A}, \ddot{q}^{A}, \xi^{i}, \ddot{\xi}^{i}) = C \left(q^{A}, \dot{q}^{A}, \xi^{i}, \ddot{\xi}^{i}, F_{a}(q^{A}, \dot{q}^{A}, \ddot{q}^{A}, \xi^{i}, \ddot{\xi}^{i})\right),$$

where $C$ is the cost function and

$$F_{a}(q^{A}, \dot{q}^{A}, \ddot{q}^{A}, \xi^{i}, \ddot{\xi}^{i}) = \left(\frac{d}{dt} \left(\frac{\partial L}{\partial q^{A}}\right) - \frac{\partial L}{\partial \dot{q}^{A}}\right) X_{a}^{A}(q) + \left(\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^{A}}\right) - ad_{\ddot{q}^{A}} \frac{\partial L}{\partial q^{A}}\right) \Xi_{a}(q).$$

The Lagrangian $\bar{L}$ is subjected to the second-order constraints:

$$\Phi^{\alpha}(q^{A}, \dot{q}^{A}, \ddot{q}^{A}, \xi^{i}, \ddot{\xi}^{i}) = \left(\frac{d}{dt} \left(\frac{\partial L}{\partial q^{A}}\right) - \frac{\partial L}{\partial \dot{q}^{A}}\right) X_{\alpha}^{A}(q) + \left(\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^{A}}\right) - ad_{\ddot{q}^{A}} \frac{\partial L}{\partial q^{A}}\right) \Xi_{\alpha}(q).$$

4.1 Optimal control of an underactuated vehicle

Consider a rigid body moving in special Euclidean group of the plane $SE(2)$ with a thruster to adjust its pose. The configuration of this system is determined by a tuple $(x, y, \theta, \gamma)$, where $(x, y)$ is the position of the center of mass, $\theta$ is the orientation of the blimp with respect to a fixed basis, and $\gamma$ the orientation of the thrust with respect to a body basis. Therefore, the configuration manifold is $Q = SE(2) \times S^{1}$ (see [6] and references therein), where $(x, y, \theta)$ are the local coordinates of $SE(2)$ and $\gamma$ is the local coordinate of $S^{1}$.
The Lagrangian of this system is given by its kinetic energy
\[
\mathcal{L}(x, y, \theta, \gamma, \dot{x}, \dot{y}, \dot{\theta}, \dot{\gamma}) = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} J_1 \dot{\theta}^2 + \frac{1}{2} J_2 (\dot{\theta} + \dot{\gamma})^2,
\]
and the input forces are
\[
F^1 = \cos(\theta + \gamma) \, dx + \sin(\theta + \gamma) \, dy - p \sin \gamma d\theta \quad ; \quad F^2 = d\gamma,
\]
where the control forces that we consider are applied to a point on the body with distance \(p > 0\) from the center of mass (\(m\) is the mass of the rigid body), along the body \(x\)-axis. Note this system is an example of underactuated mechanical system when the configuration space is a trivial principal bundle.

The system is invariant under the left multiplication of the Lie group \(G = SE(2)\):
\[
\Phi : SE(2) \times SE(2) \times S^1 \longrightarrow SE(2) \times S^1 \quad ((a, b, \alpha), (x, y, \theta, \gamma)) \longmapsto (x \cos \alpha - y \sin \alpha + a, x \sin \alpha + y \cos \alpha + b, \theta + \alpha, \gamma).
\]
A basis of the Lie algebra \(\mathfrak{se}(2) \simeq \mathbb{R}^3\) of \(SE(2)\) is given by
\[
e_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},
\]
from we have that
\[
[e_1, e_2] = e_3, \quad [e_1, e_3] = -e_2, \quad [e_2, e_3] = 0.
\]
Thus, we can write down the structure constants as
\[
C_{31}^2 = C_{23}^1 = -1, C_{13}^2 = C_{32}^1 = 1,
\]
and all others vanish. An element \(\xi \in \mathfrak{se}(2)\) is of the form \(\xi = \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3\); therefore the reduced Lagrangian \(\ell : TS^1 \times \mathfrak{se}(2) \rightarrow \mathbb{R}\) is given by
\[
\ell(\gamma, \tilde{\gamma}, \xi) = \frac{1}{2} m (\dot{\xi}_1^2 + \dot{\xi}_2^2) + \frac{J_1 + J_2}{2} \dot{\xi}_3^2 + J_2 \xi_3 \ddot{\gamma} + \frac{J_2}{2} \dot{\gamma}^2.
\]
Then the reduced Euler-Lagrange equations with controls are given by (see, for example, [11] and [12])
\[
\begin{align*}
m \dot{\xi}_1 &= u_1 \cos \gamma, \\
m \dot{\xi}_2 + (J_1 + J_2) \xi_1 \xi_3 + J_2 \xi_1 \ddot{\gamma} - m \xi_1 \ddot{\xi}_3 &= u_1 \sin \gamma, \\
(J_1 + J_2) \dot{\xi}_3 + J_2 \ddot{\gamma} - m \xi_2 (\dot{\xi}_1 + \xi_3) &= -u_1 p \sin \gamma, \\
J_2 (\dot{\xi}_3 + \ddot{\gamma}) &= u_2.
\end{align*}
\]
On the other hand, choosing the adapted basis \(\{\mathcal{B}_a, \mathcal{B}_\alpha\}\) the modified equations of motion [13] read in this case as
\[
\begin{align*}
m (\cos \gamma \ddot{\xi}_1 + \sin \gamma (\dot{\xi}_2 - \xi_1 \xi_3)) + (J_1 + J_2) \xi_1 \xi_3 \sin \gamma + J_2 \ddot{\xi}_1 \sin \gamma &= u_1, \\
m (\cos \gamma (\dot{\xi}_2 - \xi_1 \dot{\xi}_3) - \sin \gamma \xi_1) + \dot{\xi}_1 \xi_3 (J_1 + J_2) \cos \gamma + J_2 \xi_1 \dot{\gamma} \cos \gamma &= 0, \\
\frac{J_1 + J_2}{p} (\dot{\xi}_3 + p \xi_1 \dot{\xi}_3) + \frac{J_2}{p} (\dot{\gamma} + p \xi_1 \dot{\gamma}) + m \left( \dot{\xi}_2 - \dot{\xi}_1 \xi_3 - \frac{\xi_2 \xi_1 + \xi_3 \xi_2}{p} \right) &= 0, \\
J_2 (\dot{\xi}_3 + \ddot{\gamma}) &= u_2.
\end{align*}
\]
Now, we can study the optimal control problem that consists, as mentioned before, on finding a trajectory of state variables and control inputs satisfying the previous equations from given initial and final conditions \((\gamma(0), \dot{\gamma}(0), \xi(0)), (\gamma(T), \dot{\gamma}(T), \xi(T))\) respectively, and extremizing the cost functional
\[
\int_0^T (\rho_1 u_1^2 + \rho_2 u_2^2) \, dt,
\]
where \(\rho_1\) and \(\rho_2\) are non-zero constants.

The related optimal control problem is equivalent to the second-order Lagrangian problem with second-order constraints defined as follows. Extremize
\[
\tilde{A} = \int_0^T \tilde{L}(\xi, \dot{\xi}, \gamma, \dot{\gamma}) \, dt,
\]
subject to second-order constraints given by the functions
\[
\begin{align*}
\Phi^1 &= m(\cos \gamma (\dot{\xi}_2 - \xi_1 \xi_3) - \sin \gamma \dot{\xi}_1) + \xi_1 \xi_3 (J_1 + J_2) \cos \gamma + J_2 \xi_1 \dot{\gamma} \cos \gamma, \\
\Phi^2 &= \frac{J_1 + J_2}{p} (\dot{\xi}_3 + p \xi_1 \dot{\xi}_3) + \frac{J_2}{p} (\dot{\gamma} + p \xi_1 \dot{\gamma}) + m \left( \dot{\xi}_2 - \xi_1 \xi_3 - \frac{\xi_2 \xi_1 + \xi_3 \xi_2}{p} \right).
\end{align*}
\]
Here, \(\tilde{L} : T^2S^1 \times 2SE(2) \to \mathbb{R}\) is defined by
\[
\tilde{L}(\gamma, \dot{\gamma}, \ddot{\gamma}, \xi, \dot{\xi}) = \rho_1 \left( m(\cos \gamma \dot{\xi}_1 + \sin \gamma (\dot{\xi}_2 - \xi_1 \xi_3)) + (J_1 + J_2) \xi_1 \xi_3 \sin \gamma + J_2 \xi_1 \dot{\gamma} \sin \gamma \right)^2 + \rho_2 J_2^2 (\dot{\xi}_3 + \dot{\gamma})^2,
\]
which basically is the cost function \(C = \rho_1 u_1^2 + \rho_2 u_2^2\) in terms of the new variables.

The submanifold \(M\) of \(T^2S^1 \times SE(2) \times 2SE(2)\) is given by the constraints
\[
\begin{align*}
\ddot{\gamma} &= -\frac{mp}{J_2} \left( \dot{\xi}_2 - \xi_1 \xi_3 - \frac{\xi_1 + \xi_3}{p} \right) - \frac{J_1 + J_2}{J_2} (\dot{\xi}_3 + p \xi_1 \dot{\xi}_3) - p \xi_1 \dot{\gamma}, \\
\dot{\xi}_1 &= \frac{1}{\tan \gamma} \left( \frac{J_1 + J_2}{m} \xi_1 \xi_3 + \frac{J_2}{m} \xi_1 \dot{\gamma} + \dot{\xi}_2 - \xi_1 \xi_3 \right).
\end{align*}
\]

We consider the submanifold \(\overline{M} = M \times T^*(TS^1) \times 2SE(2)^*\) with induced coordinates
\[
(\gamma, \dot{\gamma}, \dot{\gamma}, g, \xi_1, \xi_2, \xi_3, \dot{\xi}_1, \dot{\xi}_2, \dot{\xi}_3, \eta_1, \eta_2, p_1, p_2, p_3, \tilde{p}_1, \tilde{p}_2, \tilde{p}_3),
\]
and the restriction \(\overline{L}_M\) of \(\tilde{L}\) given by
\[
\overline{L}_M = \rho_1 \left[ m \cos \gamma \tan \gamma \left( \frac{J_1 + J_2}{m} \xi_1 \xi_3 + \frac{J_2}{m} \xi_1 \dot{\gamma} + \dot{\xi}_2 - \xi_1 \xi_3 \right) + (J_1 + J_2) \xi_1 \xi_3 \sin \gamma + J_2 \xi_1 \dot{\gamma} \sin \gamma \\
+ \sin \gamma (\dot{\xi}_2 - \xi_1 \xi_3) \right]^2 + \rho_2 J_2^2 \left[ \dot{\xi}_3 - \frac{mp}{J_2} \left( \dot{\xi}_2 - \xi_1 \xi_3 - \frac{\xi_1 + \xi_3}{p} \right) - \frac{J_1 + J_2}{J_2} (\dot{\xi}_3 + p \xi_1 \dot{\xi}_3) - p \xi_1 \dot{\gamma} \right]^2.
\]

Observe that we use the intrinsic formulation in the submanifold \(M\) because the constraints enable us to write the variables \(\ddot{\gamma}\) and \(\dot{\xi}_1\) in terms of the others, and thus it is easy to determine a subset of intrinsic coordinates.

Then, we can write the equations of motion of the optimal control problem for this underactuated system. For simplicity, we consider the particular case \(J_1 = J_2 = 1\) and \(m = p = 1\) then
the equations of motion of the optimal control system are:

\[
\begin{align*}
\dot{\xi}_i &= \frac{d}{dt}\xi_i, \quad \dot{\gamma} = \frac{d}{dt}\gamma, \quad \dot{\tilde{\gamma}} = \frac{d}{dt}\tilde{\gamma}, \quad i = 1, 2, 3, \\
\dot{\eta}_1 &= 2\rho_1 \left( \left( \frac{\cos \gamma}{\tan \gamma} + \sin \gamma \right) \cdot \mathbf{A} \right) \left( \cos \gamma - \frac{\sin \gamma}{\tan \gamma} - \frac{1}{\cos^2 \gamma + \tan^2 \gamma} \right) + \lambda_1 \sin \gamma \mathbf{A} \left( 1 + \frac{1}{\tan \gamma} \right), \\
\dot{\eta}_2 &= 2\xi_1 \rho_1 \left( \left( \frac{\cos \gamma}{\tan \gamma} + \sin \gamma \right)^2 \cdot \mathbf{A} \right) - \xi_1 (\lambda_1 \cos \gamma + \lambda_2 - 2B\rho_2) - \eta_1, \\
\dot{p}_1 &= 2\rho_1 \left( \left( \frac{\cos \gamma}{\tan \gamma} + \sin \gamma \right)^2 \cdot \mathbf{A} \right) \left( 2B - \lambda_2 \right)(\xi_3 + \dot{\gamma} - \xi_2) - \lambda_1 \cos \gamma(\dot{\gamma} + \xi_3), \\
\dot{p}_2 &= (\lambda_2 - 2B\rho_2)(\xi_1 + \xi_3), \\
\dot{p}_3 &= 2\xi_1 \rho_1 \left( \left( \frac{\cos \gamma}{\tan \gamma} + \sin \gamma \right)^2 \cdot \mathbf{A} \right) - \lambda_1 \cos \gamma \xi_1 - (\xi_1 - \xi_2)(\lambda_2 + 2\rho_2B) \\
\dot{p}_i &= \text{ad}^*_\xi p_i, \quad i = 1, 2, 3.
\end{align*}
\]

where

\[
\xi = (\xi_1, \xi_2, \xi_3) \quad ; \quad \mathbf{A} = \xi_1 \xi_3 + \xi_1 \dot{\gamma} + \dot{\xi}_2 \quad ; \quad \mathbf{B} = \xi_3 + \xi_1 \xi_3 + \dot{\xi}_2 - \xi_2 \xi_1 - \xi_2 \xi_3 + \xi_1 \dot{\gamma}
\]

and the coadjoint operator is just the cross product, \(\text{ad}^*_\xi p = \xi \times p\) using the identification of \(\mathfrak{se}(2)\) with \(\mathbb{R}^3\). One can check that, using different techniques, these equations are the equations of control in the classical literature as \([6]\). Also one can compare these equations with the equations obtained using variational tools in \([11]\) and \([12]\).

In all cases we additionally have the reconstruction equation

\[
g(t) = g(t)(\xi_1(t)e_1 + \xi_2(t)e_2 + \xi_3(t)e_3)
\]

with boundary conditions \(g(t_0)\) and \(g(t_f)\), where \(g(t) = (x(t), y(t), \theta(t))\).

Finally, the regularity condition is given by the matrix

\[
\mathbf{A} = \begin{pmatrix}
2\rho_2 & 0 & 2\rho_2 & 0 & 1 \\
0 & 2\rho_1 \cos^2 \gamma & 2\rho_1 \sin \gamma \cos \gamma & 0 & -\sin \gamma & 0 \\
0 & 2\rho_1 \sin \gamma \cos \gamma & 2\rho_1 \sin^2 \gamma & 0 & \cos \gamma & 1 \\
2\rho_2 & 0 & 0 & 2\rho_2 & 0 & 2 \\
0 & -\sin \gamma & \cos \gamma & 0 & 0 & 0 \\
1 & 0 & 1 & 2 & 0 & 0
\end{pmatrix},
\]

whose determinant is

\[
\det \mathbf{A} = 4\rho_1 \rho_2 \sin^4 \gamma + 4\rho_1 \rho_2 \cos^4 \gamma + 8\rho_1 \rho_2 \sin^2 \gamma \cos^2 \gamma = 4\rho_1 \rho_2 (\sin^2 \gamma + \cos^2 \gamma)^2 = 4\rho_1 \rho_2 \neq 0.
\]

Therefore the algorithm stabilizes at the first constraint submanifold \(\mathcal{W}_c\). Moreover, there exists a unique solution of the dynamics, the vector field \(X \in \mathfrak{X}(\mathcal{W}_c)\) which satisfies \(i(X)\Omega_{\mathcal{W}_c} = dH_{\mathcal{W}_c}\). In consequence, we have a unique control input which extremizes (minimizes) the objective function \(\mathcal{A}\). If we take the flow \(F_t : \mathcal{W}_c \rightarrow \mathcal{W}_c\) of the solution vector field \(X\) then we have that \(F_t^* \Omega_{\mathcal{W}_c} = \Omega_{\mathcal{W}_c}\).

5 Conclusions and further research

We have defined, following an intrinsic point of view, the equations of motion for constrained variational higher-order Lagrangian problems on higher-order trivial principal bundles. As a
particular case, we obtain the higher-order Lagrange-Poincaré equations (see [11, 25]). As an interesting application we deduce the equations of motion for optimal control of underactuated mechanical systems defined on principal bundles. These systems appear in numerous engineering and scientific fields. In this sense we study the optimal control of an underactuated vehicle.

Moreover, in a future paper we will generalize the presented construction of higher-order Euler-Lagrange equations to the case of non-trivial principal bundles and in the context of Lie algebroids. This last approach will be interesting because we may derive the equations of motion for different cases as, for instance, higher-order Euler-Poincaré equations, Lagrange-Poincaré equations and the reduction by morphisms in a unified way (see [20, 37, 43]).

The case of optimal control problems for mechanical systems with nonholonomic constraints will be also studied using some of the ideas exposed through the paper (see [15, 18, 36] for more details). Finally, we would like to point out that a slight modification of the techniques presented in this work would allow to approach the Clebsch-Pontryagin optimal control problem (see [27, 34]).

Acknowledgments

We wish to thank Prof. D. Martín de Diego for fruitful discussions and comments. We acknowledge the financial support of the Ministerio de Ciencia e Innovación (Spain), projects MTM 2010-21186-C02-01, MTM2011-22585 and MTM2011-15725-E; AGAUR, project 2009 SGR:1338.; IRSES-project “Geomech-246981”; and ICMAT Severo Ochoa project SEV-2011-0087. P.D. Prieto-Martínez wants to thank the UPC for a Ph.D grant, and L. Colombo wants to thank CSIC for a JAE-Pre grant.

References

[1] J. Baillieul, The geometry of controlled mechanical systems, Mathematical control theory Springer New York 1999 pp. 322–354.

[2] M. Barbero Liñán, A. Echeverría-Enríquez, D. Martín de Diego, M.C. Muñoz-Lecanda, and N. Román Roy, “Skinner-rusk unified formalism for optimal control systems and applications”, J. Math. Phys. A: Math. Theor. 40(40) (2007) 12071–12093.

[3] M. Barbero Liñán, A. Echeverría-Enríquez, D. Martín de Diego, M.C. Muñoz-Lecanda, and N. Román-Roy, “Unified formalism for non-autonomous mechanical systems”, J. Math. Phys. 49(6) (2008) 062902.

[4] R. Benito and D. Martín de Diego, Hidden symplecticity in Hamilton’s principle algorithms, Differential geometry and its applications Matfyzpress, Prague 2005 pp. 411–419.

[5] A.M. Bloch, Nonholonomic Mechanics and Control, Interdisciplinary Applied Mathematics Series, vol. 24, Springer-Verlag, New York 2003.

[6] F. Bullo and A. Lewis, Geometric Control of Mechanical Systems: Modeling, Analysis, and Design for Simple Mechanical Control Systems, Texts in Applied Mathematics, vol. 49, Springer-Verlag, New York 2005.

[7] M. Camarinha, F. Silva Leite, and P. Crouch, “On the geometry of Riemannian cubic polynomials”, Differential Geom. Appl. 15(2) (2001) 107–135.

[8] C.M. Campos, M. de León, D. Martín de Diego, and J. Vankerschaver, “Unambiguous formalism for higher-order lagrangian field theories”, J. Phys A: Math Theor. 42 (2009) 475207.

[9] F. Cantrijn, M. Crampin, and W. Sarlet, “Higher-order differential equations and higher-order Lagrangian mechanics”, Math. Proc. Cambridge Philos. Soc. 99(3) (1986) 565–587.
[10] J.F. Cariñena and C. López, “The time-evolution operator for higher-order singular Lagrangians”, *Internat. J. Modern Phys. A* **7**(11) (1992) 2447–2468.

[11] L. Colombo, F. Jiménez, and D. Martín de Diego, “Optimal Control and higher-order mechanics for systems with symmetries”, arXiv:1209.6315 [math-ph], 2012.

[12] L. Colombo, F. Jiménez, and D. Martín de Diego, “Second-order euler-poincaré equations for trivial principal bundles”, *AIP Conference Proceedings* **1460** (2012) 185–191.

[13] L. Colombo and D. Martín de Diego, “On the geometry of higher-order variational problems on Lie groups”, arXiv:1104.3221 [math-ph], 2011.

[14] L. Colombo, D. Martín de Diego, and M. Zuccalli, “Optimal control of underactuated mechanical systems: a geometric approach”, *J. Math. Phys.* **51**(8) (2010) 083519.

[15] J. Cortés, M. de León, J.C. Marrero, D. Martín de Diego, and E. Martínez, “A survey of Lagrangian mechanics and control on Lie algebroids and groupoids”, *Int. J. Geom. Methods Mod. Phys.* **3**(3) (2006) 509–558.

[16] J. Cortés, M. de León, D. Martín de Diego, and S. Martínez, “Geometric description of vakonomic and nonholonomic dynamics. Comparison of solutions”, *SIAM J. Control Optim.* **41**(5) (2002) 1389–1412.

[17] J. Cortés, S. Martínez, and F. Cantrijn, “Skinner-Rusk approach to time-dependent mechanics”, *Physics Letters A* **300** (2002) 250–258.

[18] M. de León, F. Jiménez, and D. Martín de Diego, “Hamiltonian dynamics and constrained variational calculus: continuous and discrete settings”, *J. Phys. A* **45**(20) (2012) 205204, 29. MR 2924849

[19] M. de León and E.A. Lacomba, “Lagrangian submanifolds and higher-order mechanical systems”, *J. Phys. A* **22**(18) (1989) 3809–3820.

[20] M. de León, J.C. Marrero, and E. Martín, “Lagrangian submanifolds and dynamics on Lie algebroids”, *J. Phys. A* **38**(24) (2005) R241–R308.

[21] M. de León and D. Martín de Diego, “Classification of symmetries for higher order Lagrangian systems”, *Extracta Math.* **9**(1) (1994) 32–36.

[22] M. de León, P. Pitanga, and P.R. Rodrigues, “Symplectic reduction of higher order Lagrangian systems with symmetry”, *J. Math. Phys.* **35**(12) (1994) 6546–6556.

[23] M. de León and P.R. Rodrigues, *Generalized classical mechanics and field theory*, North-Holland Math. Studies, vol. 112, Elsevier Science Publishers B.V., Amsterdam 1985.

[24] A. Echeverría-Enríquez, C. López, J. Marín-Solano, M.C. Muñoz-Lecanda, and N. Román-Roy, “Lagrangian-Hamiltonian unified formalism for field theory”, *J. Math. Phys.* **45**(1) (2004) 360–380.

[25] F. Gay-Balmaz, D.D. Holm, D.M. Meier, T.S. Ratiu, and F.-X. Vialard, “Invariant higher-order variational problems”, *Comm. Math. Phys.* **309**(2) (2012) 413–458.

[26] F. Gay-Balmaz, D.D. Holm, D.M. Meier, T.S. Ratiu, and F.-X. Vialard, “Invariant higher-order variational problems II”, *J. Nonlinear Sci.* **22**(4) (2012) 553–597.

[27] F. Gay-Balmaz, D.D. Holm, and T.S. Ratiu, “Geometric dynamics of optimization”, *Commun. Math. Sci.* **11**(1) (2013) 163–231.

[28] M.J. Gotay and J.M. Nester, “Presymplectic Lagrangian systems. I. The constraint algorithm and the equivalence theorem.”, *Ann. Inst. H. Poincaré Sect. A* **30**(2) (1979) 129–142.

[29] M.J. Gotay and J.M. Nester, “Presymplectic Lagrangian systems. II. The second-order equation problem.”, *Ann. Inst. H. Poincaré Sect. A* **32**(1) (1980) 1–13.

[30] M.J. Gotay, J.M. Nester, and G. Hinds, “Presymplectic manifolds and the Dirac-Bergmann theory of constraints”, *J. Math. Phys.* **19**(11) (1978) 2388–2399.

[31] X. Gracia, J.M. Pons, and N. Román-Roy, “Higher-order Lagrangian systems: Geometric structures, dynamics and constraints”, *J. Math. Phys.* **32**(10) (1991) 2744–2763.

[32] S. Grillo and M. Zuccalli, “Variational reduction of Lagrangian systems with general constraints”, *J. Geom. Mech.* **4**(1) (2012) 49–88.
[33] J. Hinkle, P. Muralidharan, and P.T. Fletcher, “Polynomial regression on Riemannian manifolds”, Computer Vision ECCV 2012. Lectures notes in Computer Science 7574 (2012) 1–14.

[34] D.D. Holm, Geometric mechanics. Part II, , Imperial College Press, London 2008 Rotating, translating and rolling.

[35] I.I. Hussein and A.M. Bloch, “Dynamic coverage optimal control for multiple spacecraft interferometric imaging”, J. Dyn. Control Syst. 13(1) (2007) 69–93.

[36] I.I. Hussein and A.M. Bloch, “Optimal control of underactuated nonholonomic mechanical systems”, IEEE Trans. Automat. Control 53(3) (2008) 668–682.

[37] D. Iglesias, J.C. Marrero, D. Martín de Diego, and D. Sosa, “Singular Lagrangian systems and variational constrained mechanics on Lie algebroids”, Dyn. Syst. 23(3) (2008) 351–397.

[38] H.O. Jacobs, T.S. Ratiu, and M. Desbrun, “On the coupling between an ideal fluid and immersed particles”, arXiv:1208.6561 [math-ph], 2012.

[39] O. Krupkova, “Higher-order mechanical systems with constraints”, J. Math. Phys. 41(8) (2000) 5304.5324.

[40] P.D. Prieto-Martínez and N. Román-Roy, “Lagrangian-Hamiltonian unified formalism for autonomous higher-order dynamical systems”, J. Phys. A: Math. Teor. 44(38) (2011) 385203.

[41] P.D. Prieto-Martínez and N. Román-Roy, “Unified formalism for higher-order non-autonomous dynamical systems”, J. Math. Phys. 53(3) (2012) 032901.

[42] D.J. Saunders, The geometry of jet bundles, London Mathematical Society, Lecture notes series, vol. 142, Cambridge University Press, Cambridge, New York 1989.

[43] D.J. Saunders, “Prolongations of Lie groupoids and Lie algebroids”, Houston J. Math. 30(3) (2004) 637–655 (electronic).

[44] R. Skinner and R. Rusk, “Generalized Hamiltonian dynamics. I. Formulation on $T^*Q \oplus TQ$”, J. Math. Phys. 24(11) (1983) 2589–2594.

[45] L. Vitagliano, “The Lagrangian-Hamiltonian Formalism for Higher Order Field Theories”, J. Geom. Phys. 60 (2010) 857–873.