Affine Connection Induced from The Horizontal lift $^H\nabla$ on a Cross-section

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February 6, 2014

Abstract

The main purpose of present paper is to study the affine connection induced from the horizontal lift $\nabla$ on the cross-section $\beta_\vartheta (M_n)$ determined by a vector field $\vartheta$ in $M_n$ with respect to the adapted frame of $\beta_\vartheta (M_n)$.

Keywords: Horizontal lift, Affine connection, Cross-section, Lie derivative.

2010 AMS Classification:53C05, 53B05, 53C07

1. Introduction

Let $M_n$ be an $n$-dimensional differentiable manifold of class $C^\infty$ an $T_p (M_n)$ the tangent space at a point $P$ of $M_n$, that is, the set of all tangent vectors of $M_n$ at $P$. Then the set

$$T (M_n) = \bigcup_{P \in M_n} T_P (M_n),$$

is by definition, tangent bundle over the manifold $M_n$ [1].

Let $M_n$ be a Riemannian manifold with metric $g$ whose components in a coordinate neighborhood $U$ are $g_{ij}$, and denote by $\Gamma^k_{ij}$ the Christoffel symbols formed with $g_{ij}$. If $U$ being a neighborhood of $M_n$, then the horizontal lift $^Hg$ of $g$ has components

$$^Hg = \begin{pmatrix} \Gamma^m_i g_{mj} + \Gamma^m_j g_{im} & g_{ij} \\ g_{ij} & 0 \end{pmatrix}$$

with respect to the induced coordinates $(x^h, y^h)$ in $\pi^{-1} (U) \subset T (M_n)$, where $\Gamma^m_i = y^l \Gamma^m_{lj}$, $\Gamma^m_i$ being the components of the affine connection in $M_n$.

Now we shall define the horizontal lift $\nabla$ of the affine connection $\nabla$ in $M_n$ to $T (M_n)$ by the conditions

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\[ \nabla_{VX} VY = 0, \quad \nabla_{VX} H Y = 0 \]
\[ \nabla_{UX} VY = U (\nabla_X Y), \quad \nabla_{UX} H Y = U (\nabla_X Y), \quad (1) \]

for \( X, Y \in \mathfrak{g}_1(M) \). From (1), the horizontal lift of \( \nabla \) has components \( \Gamma_{JI}^K \) such that

\[
\begin{align*}
\Gamma_{ji}^k &= \Gamma_{ji}^k, & \Gamma_{ji}^k &= \Gamma_{ji}^k = \Gamma_{ji}^k = 0, \\
\Gamma_{ji}^k &= y^s \partial_s \Gamma_{ji}^k - y^s R_{sjit}, & \Gamma_{ji}^k &= \Gamma_{ji}^k = \Gamma_{ji}^k = \Gamma_{ji}^k = 0, & (2)
\end{align*}
\]

with respect to the induced coordinates in \( T(M) \), where \( \Gamma_{ji}^k \) are the components of \( \nabla \) in \( M \) [6].

Let a vector-field in a manifold \( M \), then the vector field defines a cross-section in the tangent bundle \( T(M) \). Tensor fields and connections on a cross-section in the tangent bundle was studied by Houh and Ishihara [1], Tani [3], Yano [4]. Affine connections induced from \( \nabla \) on the cross-section \( \beta_\vartheta(M) \) was studied by Yano and Ishihara [6].

We suppose that there is given a vector field \( \vartheta \) in an \( n \)-dimensional manifold \( M \). Then the correspondence \( p \rightarrow \vartheta_p, \vartheta_p \) being the value of \( \vartheta \) at \( p \in M \), determines a mapping \( \beta_\vartheta : M \rightarrow T(M) \) and the \( n \)-dimensional submanifold \( \beta_\vartheta(M) \) of \( T(M) \) is called the cross-section determined by \( \vartheta \). If the vector field \( \vartheta \) has local components \( \vartheta^k(x) \) in \( M \). Then the cross-section \( \beta_\vartheta(M) \) is locally expressed by

\[
\begin{align*}
x^h &= x^h, & y^h &= \vartheta^h(x) \quad (3)
\end{align*}
\]

with respect to the induced coordinates \( (x^A) = (x^h, y^h) \) in \( T(M) \). Differentiating (3), we see that \( n \) tangent vectors \( B_{(j)} \) to \( \beta_\vartheta(M) \) have components

\[
B^A_j = \frac{\partial x^A}{\partial x^j}
\]

i.e.,

\[
B_{(j)} : (B^A_j) = \left( \begin{array}{c} \delta^h_j \\ \partial_j \vartheta^h \end{array} \right) \quad (4)
\]

with respect to the induced coordinates \( T(M) \).

On the other hand, since a fibre is locally expressed by \( x^h = const., y^h = y^h \), \( y^h \) being considered as parameters.

\[
C_{(j)} : (C^A_j) = \left( \begin{array}{c} 0 \\ \delta^h_j \end{array} \right) \quad (5)
\]

are tangent to the fibre.

We now consider in \( \pi^{-1}(U) \), \( U \) being coordinate neighborhood of \( M \), \( 2n \) local vector fields \( B_{(j)} \) and \( C_{(j)} \) along \( \beta_\vartheta(M) \), represented respectively by
\[ B_{(j)} = B \frac{\partial}{\partial x^j}, \quad C_{(j)} = C \frac{\partial}{\partial x^j}. \]

They form a local family of frames \( \{ B_{(j)}, C_{(j)} \} \) along \( \beta_\vartheta(M_n) \), which is called the adapted frame of \( \beta_\vartheta(M_n) \) in \( \pi^{-1}(U) \) [6].

2. Affine Connection Induced from \( \nabla \) on a Cross-Section

We suppose that \( M_n \) is a manifold with affine connection \( \nabla \). Thus the tangent bundle \( T(M_n) \) of \( M_n \) is a manifold with affine connection \( \nabla \) which is the horizontal lift of \( \nabla \). We now study the affine connection induced from \( \nabla \) on the cross-section \( \beta_\vartheta(M_n) \) determined by a vector field \( \vartheta \) in \( M_n \) with respect to the adapted frame of \( \beta_\vartheta(M_n) \).

The linear connection \( \nabla \) on the cross-section \( \beta_\vartheta(M_n) \) induced from \( \nabla \) is defined by connection components \( {'}^\Gamma_{ji} \) given by [6]

\[ {'}^\Gamma_{ji} = \left( \partial_j B^A_i + {^\Gamma^A}_{MN} B^M_j B^N_i - {'}^\Gamma_{ji}^A h \right) B^h_A, \quad (6) \]

where \( {^\Gamma^A}_{MN} \) are the connection components of \( \nabla \) with respect to the induced coordinates in \( T(M_n) \) and \( B^h_A \) are defined by

\[ (B^h_A, C^h_A) = \left( B^A_j, C^A_j \right)^{-1} \]

and hence

\[ (B^h_A) = \left( \delta^h_j, 0 \right), \quad (C^h_B) = \left( -\partial_j \vartheta^h, \delta^h_j \right). \quad (7) \]

Substituting (2) for \( {^\Gamma^A}_{MN} \), (4), (5) and (7) in (6), we find

\[ {'}^\Gamma_{ji} = \Gamma_{ji}, \quad (8) \]

where \( \Gamma_{ji} \) are components of \( \nabla \) in \( M_n \).

From (6) we see that

\[ \partial_j B^A_i + {^\Gamma^A}_{MN} B^M_j B^N_i - {'}^\Gamma_{ji}^A h = H^k_{ji} C^A_k, \quad (9) \]

i.e., that the left hand side is a linear combinations of \( C^A_k \), where the coefficients \( H^k_{ji} \) will be found in the sequel. To find the coefficients \( H^k_{ji} \), we put \( A = h \) in (9) and hence obtain

\[ H^k_{ji} = \partial_j \partial_h \vartheta^k + \vartheta^t \partial_t \Gamma_{ji}^h + \vartheta^t R^h_{tji} + \Gamma^h_{mi} \partial_j \vartheta^m + \Gamma^h_{jn} \partial_i \vartheta^n - \Gamma^h_{ji} \partial_h \vartheta^h \quad (10) \]

which are components \( L_0 \Gamma_{ji}^k \) of the Lie derivative of the affine connection \( \nabla \) with respect to \( \vartheta [5] \). Thus, representing the left-hand side of (9) by \( ^\nabla_j B^A_i \), we have from (10)

\[ ^\nabla_j B^A_i = (L_0 \Gamma_{ji}^h + \vartheta^t R^h_{tji}) C^A_h. \quad (11) \]

Thus we have
Proposition 1 If $\vartheta^t R^h_{tji} = 0$, then $\nabla_j B^i = \left( L_\vartheta \Gamma^h_{ji} + \vartheta^t R^h_{tji} \right) C^A_h$ is the equation of Gauss for the cross-section $\beta_\vartheta(M_n)$ determined by a vector field $\vartheta$ in $M_n$ to $T(M_n)$.

Proposition 2 In order that the cross-section in $T(M_n)$ determined by a vector field $\vartheta$ in $M_n$ with affine connection $\nabla$ be totally geodesic with respect to $\nabla$ it is necessary and sufficient that respectively $\vartheta$ is an infinitesimal affine transformation in $M_n$, i.e., that $L_\vartheta \nabla = 0$ and $\vartheta^t R^h_{tji} = 0$, where $R^h_{tji}$ is components of the curvature tensor $R$ of $\nabla$.

By means of (9), the equation (11) reduces to

$$\nabla B^i (j) = \Gamma^h_{ji} B^i (h) + H^h_{ji} C^i (h). \quad (12)$$

We now have

$$\overline{R} (B(k), B(j)) B(i) = \nabla B(k) \nabla B(j) B(i) - \nabla B(i), \quad (13)$$

$\overline{R}$ being the curvature tensor of $\nabla$ because $[B(j), B(i)] = 0$. Thus, denoting by $R^h_{kji}, B(h)$ the components of the curvature tensor $R$ of $\nabla$, we have from (13)

$$\overline{R} (B(k), B(j)) B(i) = R^h_{kji} B(h) + \left\{ \nabla_k \left( L_\vartheta \Gamma^h_{ji} \right) - \nabla_j \left( L_\vartheta \Gamma^h_{ki} \right) \right\} C^i (h)$$

$$\nabla_k \left( \vartheta^t R^h_{tji} \right) - \nabla_j \left( \vartheta^t R^h_{tki} \right) \quad (14)$$

which reduces to

$$\overline{R} (B(k), B(j)) B(i) = R^h_{kji} B(h) + \left\{ \nabla_k \left( L_\vartheta \Gamma^h_{ji} \right) - \nabla_j \left( \vartheta^t R^h_{tki} \right) \right\} C^i (h)$$

$$\nabla_k \left( \vartheta^t R^h_{tji} \right) - \nabla_j \left( \vartheta^t R^h_{tki} \right) \quad (15)$$

where the well know formula [5]

$$\nabla_k \left( L_\vartheta \Gamma^h_{ji} \right) - \nabla_j \left( L_\vartheta \Gamma^h_{ki} \right) = L_\vartheta R^h_{tji}$$

from (15), we have

Proposition 3 In order that $\overline{R} (X, Y) Z$ evaluated for vector fields $X, Y$ and $Z$ tangent to the cross-section determined by a vector field $\vartheta$ in $M_n$, $R$ being curvature tensor of an affine connection $\nabla$, be always tangent to the cross-section, it is necessary and sufficient that respectively the Lie derivative $L_\vartheta R$ of $R$ with respect to $\vartheta$ in $M_n$ vanishes, i.e., $L_\vartheta R = 0$ and $\nabla_k \left( \vartheta^t R^h_{tji} \right) - \nabla_j \left( \vartheta^t R^h_{tki} \right) = 0$.
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