1. Introduction

In the pioneering work [7], Bénilan and Crandall showed that for the class of homogeneous operators \( A \) of order \( \alpha > 0 \) with \( \alpha \neq 1 \), defined on a normed space \( (X, \| \cdot \|_X) \), every solution of the differential inclusion

\[
\frac{du}{dt} + A(u(t)) \ni 0
\]

satisfies the global regularity estimate

\[
\limsup_{h \to 0^+} \frac{\|u(t+h) - u(t)\|_X}{h} \leq 2L \frac{\|u_0\|_X}{|\alpha - 1|} \frac{1}{t} \quad \text{for every } t > 0.
\]
Here, \( A \subseteq X \times X \) might be multi-valued and is called \textit{homogeneous of order } \( \alpha \) if
\[
A(\lambda u) = \lambda^\alpha A u \quad \text{for all } \lambda \geq 0 \text{ and } u \in D(A).
\]
Moreover, to obtain (1.2), it is assumed that there is a family \( \{T_t\}_{t \geq 0} \) associated with \( A \) of Lipschitz continuous mappings \( T_t \) on \( X \) of constant \( L \) such that
\[
u(t) = T_t u_0 \quad \text{for every } t \geq 0,
\]
is (in some given sense) a solution of (1.1) for some initial value \( u_0 \in X \). We refer to Definition 3.2 and Definition 3.5 for the different notions of solutions.

Further, if \( X \) is equipped with a partial ordering \( \preceq \) such that \( (X, \preceq) \) defines an ordered vector space, and if for this ordering, the family \( \{T_t\}_{t \geq 0} \) is order-preserving (that is, (2.17) below holds), then every positive\(^1\) solution \( u \) of (1.1) satisfies the point-wise estimate
\[
(\alpha - 1) \frac{du}{dt}(t) \geq - \frac{u}{f} \quad \text{in } D' \text{ for every } t > 0.
\]

Estimates of the form (1.2) describe an instantaneous and global regularizing effect of solutions \( u \) of (1.1), since they imply that the solution \( u \) of (1.1) is locally Lipschitz continuous in \( t \in (0, +\infty) \). Further (1.5) provides a rate of dissipativity involved in the differential inclusion (1.1).

It is the aim of this paper to extend the theory developed in [7] to the important case \( \alpha = 0 \); in other words, for the class of homogeneous operators \( A \) of order zero (see Definition 2.1 below). Important examples of this class of operators include the (negative) total variational flow operator \( Au = -\Delta_1 u := -\text{div} \left( \frac{Du}{|Du|} \right) \), also known as (negative) 1-Laplacian, or the 1-fractional Laplacian
\[
Au = (-\Delta)^s u(x) := \text{PV} \int_X \frac{u(y) - u(x)}{|u(y) - u(x)|} \frac{dy}{|x - y|^{d+s}}, \quad s \in (0,1).
\]
In our first main result (Theorem 2.3), we establish the global regularity estimate (1.2) for order \( \alpha = 0 \) and for solutions \( u \) of differential inclusions with a forcing term:
\[
\frac{du}{dt} + A(u(t)) \supseteq f(t) \quad \text{on } (0,T),
\]
where \( f : [0,T] \to X \) is an integrable function, and \( T > 0 \). In Corollary 2.4 and Corollary 2.6, we provide the resulting inequality when \( f \equiv 0 \) and the right hand-side derivative \( \frac{du}{dt}(t) \) of \( u \) exists at \( t > 0 \).

In many applications (cf Section 5), \( X \) is given by the classical Lebesgue space \( (L^r, \| \cdot \|_r) \), \( (1 \leq r \leq \infty) \). If \( \{T_t\}_{t \geq 0} \) is a semigroup satisfying an \( L^q\)-\( L^r\)-regularity estimate
\[
\|T_t u_0\|_r \leq C e^{\omega t} \|u_0\|^q_q t^{-q} \quad \text{for all } t > 0, \text{ and } u_0 \in L^q,
\]
for \( \omega \in \mathbb{R}, \gamma = \gamma(q,r,d), \delta = \delta(q,r,d) > 0 \), and some (or for all) \( 1 \leq q < r \), then we show in Corollary 2.6 that combining (1.2) with (1.7) yields
\[
\limsup_{h \to 0+} \frac{\|u(t + h) - u(t)\|_r}{h} \leq C L^2 \delta^{q+1} \omega^{q+1} \|u_0\|^\gamma_q^{-q}.
\]
\(^1\)Here, we call a measurable function \( u \) \textit{positive} if \( u \geq 0 \) for the given partial ordering \( \preceq \).
Regularity estimates similar to (1.7) have been studied recently by many authors (see, for example, [13, 19, 14] covering the linear theory, and [12] the nonlinear one and there references therein).

In Theorem 2.7, Corollary 2.9 and Corollary 2.10 we generalize the pointwise estimate (1.5) to the homogeneous order $\alpha = 0$.

We emphasize that the regularizing effect of solutions $u$ of (1.1) remains true with a slightly different inequality (see Corollary 2.12) if the homogeneous operator $A$ is perturbed by a Lipschitz mapping $F$. This is quite surprising since $F$ might not be homogenous and hence, the operator $A + F$ is also not homogeneous.

In Section 3, we consider the class of quasi accretive operators $A$ (see Definition 3.1) and outline how the property that $A$ is homogeneous of order zero is passed on to the semigroup $\{T_t\}_{t \geq 0}$ generated by $-A$ (see the paragraph after Definition 3.2). In particular, we discuss when solutions $u$ of (1.1) are differentiable a.e. in $t > 0$.

The fact that every Lipschitz continuous mappings $u : [0, T] \to X$ is differentiable almost everywhere on $(0, T)$ depends on the underlying geometry of the given Banach space $X$; this property is well-known as the Radon-Nikodým property of a Banach space. The Lebesgue space $L^1$ has not this property, but alone from the physical point of view, $L^1$ is for many models not avoidable. In [8], Bénilan and Crandall developed the celebrated theory of completely accretive operators $A$ (in $L^1$). For this class of operators, it is known that for each solutions $u$ of (1.1) in $L^1$, the derivative $\frac{du}{dt}$ exists in $L^1$. These results have been extended recently to the notion of quasi completely accretive operators in [12]. In Section 4, we study regularity estimates of the form (1.2) for $\alpha = 0$ satisfied by solutions $u$ of (1.1), where $A$ is a quasi completely accretive operator of homogeneous order zero. In fact, the two operators $-\Delta_1$ and $(-\Delta_1)^s$ mentioned above, belong exactly to this class of operators. Thus, our two main examples of differential inclusions discussed in Section 5 are

\begin{align}
\frac{du}{dt} - \text{div} \left( \frac{Du}{|Du|} \right) + f(\cdot, u) &\geq 0 \quad (1.9) \\
\frac{du}{dt} + PV \int_{\Sigma} \frac{(u(y) - u(x))}{|u(y) - u(x)|} \frac{dy}{|x-y|^{d+1}} + f(\cdot, u) &\geq 0, \quad (1.10)
\end{align}

respectively equipped with some boundary conditions on a domain $\Sigma$ in $\mathbb{R}^d$, $d \geq 1$. In (1.9) and (1.10), the function $f$ is a Carathéodory function, which is Lipschitz continuous in the second variable with constant $\omega > 0$ uniformly with respect to the first variable (see Section 5 for more details).

Note, if the right hand-side derivative $\frac{du}{dt}(t)$ of a solution $u$ of (1.1) exists at every $t \in (0, 1]$, then (1.2) for $\alpha = 0$ becomes

\begin{equation}
\|Au(t)\|_X \leq 2L \left\| \frac{\|u_0\|_X}{t} \right\| \quad \text{for every } t > 0.
\end{equation}

Here, it is worth mentioning that if the operator $A$ in (1.1) is linear (that is, $\alpha = 1$), then inequality (1.11) means that $-A$ generates an analytic semigroup $\{T_t\}_{t \geq 0}$ (cf [4, 17]). Thus, it is interesting to see that a similar regularity inequality such as (1.11), in particular, holds for certain classes of nonlinear operators.
In addition, if $\|\cdot\|_X$ is the induced norm by an inner product $(\cdot, \cdot)_X$ of a Hilbert space $X$ and $A$ is a sub-differential operators $\partial \varphi$ on $X$, then inequality (1.11) is also satisfied by solutions of (1.1) (cf [11]). In [2], inequality (1.11) was shown to hold for solutions of (1.9) with $f \equiv 0$ and equipped with Neumann boundary conditions.

2. Main results

Suppose $X$ is a linear vector space and $\|\cdot\|_X$ a semi-norm on $X$. Then, the main object of this paper is the following class of operators.

**Definition 2.1.** An operator $A$ on $X$ is said to be **homogeneous of order zero** if for every $u \in D(A)$ and $\lambda \geq 0$, one has that $\lambda u \in D(A)$, and $A$ satisfies (1.3) for $\alpha = 0$.

**Remark 2.2.** It follows necessarily from (1.3) that for every homogeneous operator $A$ of order $\alpha > 0$, one has that $0 \in A0$. But for homogeneous operators $A$ of order zero, the property $0 \in A0$ does not need to hold.

Now, assume that for the operator $A$ on $X$ and for given $f : [0, T] \to X$ and $u_0 \in X$, the function $u \in C^1([0, T]; X)$ is a classical solution of the differential inclusion (1.6) with forcing term $f$ satisfying initial value $u(0) = u_0$. If $A$ is homogeneous of order zero, then for $\lambda > 0$, the function

$$v(t) = \lambda^{-1}u(\lambda t), \quad (t \in [0, T]),$$

satisfies

$$\frac{dv}{dt}(t) = \frac{du}{dt}(\lambda t) \in -A(u(\lambda t)) + f(\lambda t) = -A(v(t)) + f(\lambda t)$$

for every $t \in (0, T)$ with initial value $v(0) = \lambda^{-1}u(0) = \lambda^{-1}u_0$. Thus, if for every $t \in [0, T]$, we denote

$$T_t(u_0, f) := u(t) \quad \text{for every } u_0 \text{ and } f,$$

where $u$ is the unique classical solution $u$ of (1.6) with initial value $u(0) = u_0$, then the above reasoning shows that the homogeneity of $A$ is reflected in

$$\lambda^{-1}T_{\lambda t}(u_0, f) = T_t(\lambda^{-1}u_0, f(\lambda \cdot)) \quad \text{for every } \lambda > 0,$$

and all $t \in [0, T]$. Identity (2.2) together with standard growth estimates of the form

$$e^{-\omega t} \|T_t(u_0, f) - T_t(a, \hat{f})\|_X$$

for every $0 \leq s \leq t \leq T$, (for some $\omega \in \mathbb{R}$ and $L \geq 1$) are the main ingredients to obtain global regularity estimates of the form (1.2). This leads to our first main result.

**Theorem 2.3.** For a subset $C \subseteq X$, let $\{T_t\}_{t=0}^T$ be a family of mappings $T_t : C \times L^1(0, T; X) \to C$ satisfying (2.3), (2.2), and $T_t(0, 0) \equiv 0$ for all $t \geq 0$. Then for every
$u_0 \in C, f \in L^1(0,T;X)$, and $t \in (0,T], h > 0$, one has that
\[
\|T_{t+h}(u_0,f) - T_t(u_0,f)\|_X
\leq \frac{|h|}{t} L e^{\omega t} \left[ 2\|u_0\|_X + (1 + \frac{h}{t}) \int_0^t e^{-\omega s} \left\| \frac{f(s + \frac{h}{t}s) - f(s)}{\frac{h}{t}} \right\|_X ds + \int_0^t e^{-\omega s} \|f(s)\|_X ds \right].
\]
(2.4)

In particular, if
\[
V(f,t) := \limsup_{\xi \to 0} \int_0^t e^{-\omega s} \left\| \frac{f(s + \xi s) - f(s)}{\xi} \right\|_X ds,
\]
then the family $\{T_t\}_{t \geq 0}$ satisfies
\[
\limsup_{h \to 0^+} \left\| \frac{T_{t+h}(u_0,f) - T_t(u_0,f)}{h} \right\|_X
\leq \frac{L e^{\omega t}}{t} \left[ 2\|u_0\|_X + V(f,t) + \int_0^t e^{-\omega s} \|f(s)\|_X ds \right].
\]
(2.5)

for every $t > 0, u_0 \in C, f \in L^1(0,T;X)$, and if $f$ is locally absolutely continuous and differentiable a.e. on $(0,T)$, then
\[
\limsup_{h \to 0^+} \left\| \frac{T_{t+h}(u_0,f) - T_t(u_0,f)}{h} \right\|_X
\leq \frac{L e^{\omega t}}{t} \left[ 2\|u_0\|_X + \int_0^t e^{-\omega s} \|f'(s)\|_X ds + \int_0^t e^{-\omega s} \|f(s)\|_X ds \right].
\]
(2.6)

Moreover, if the right hand-side derivative $\frac{d}{dt} T_t(u_0,f)$ exists (in $X$) at $t > 0$, then
\[
\left\| \frac{d}{dt} T_t(u_0,f) \right\|_X \leq \frac{L e^{\omega t}}{t} \left[ 2\|u_0\|_X + V(t,f) + \int_0^t e^{-\omega s} \|f(s)\|_X ds \right].
\]
(2.7)

\textbf{Proof.} Let $u_0 \in C, f \in L^1(0,T;X)$, and for $t > 0$, let $h \neq 0$ satisfying $1 + \frac{h}{t} \geq 0$. Then, choosing $\lambda = 1 + \frac{h}{t}$ in (2.2) gives
\[
T_{t+h}(u_0,f) - T_t(u_0,f) = T_{\lambda t}(u_0,f) - T_t(u_0,f)
= \left( 1 + \frac{h}{t} \right) T_t \left[ \left( 1 + \frac{h}{t} \right)^{-1} u_0, f(\cdot + \frac{h}{t}) \right] - T_t(u_0,f)
\]
(2.8)

and so,
\[
T_{t+h}(u_0,f) - T_t(u_0,f)
= \left( 1 + \frac{h}{t} \right) \left[ T_t \left[ \left( 1 + \frac{h}{t} \right)^{-1} u_0, f(\cdot + \frac{h}{t}) \right] - T_t(u_0,f(\cdot + \frac{h}{t})) \right]
+ \left( 1 + \frac{h}{t} \right) \left[ T_t \left[ u_0, f(\cdot + \frac{h}{t}) \right] - T_t(u_0,f) \right]
+ \left[ \left( 1 + \frac{h}{t} \right) - 1 \right] T_t(u_0,f).
\]
(2.9)
Thus, by applying (2.3) and since $T_t(0,0) \equiv 0$, one sees that 
\[
\|T_{t+h}(u_0, f) - T_t(u_0, f)\|_X 
\leq (1 + \frac{h}{T}) \left\| T_t \left[ \left(1 + \frac{h}{T}\right)^{-1} u_0, f(\cdot + \frac{h}{T}) \right] - T_t(u_0, f(\cdot + \frac{h}{T})) \right\|_X 
+ (1 + \frac{h}{T}) \left\| T_t(u_0, f(\cdot + \frac{h}{T})) - T_t(u_0, f) \right\|_X 
+ \left[ \left(1 + \frac{h}{T}\right)^{-1} - 1 \right] \|T_t(u_0, f)\|_X 
\leq (1 + \frac{h}{T}) L e^{\omega t} \left\| \left(1 + \frac{h}{T}\right)^{-1} u_0 - u_0 \right\|_X 
+ (1 + \frac{h}{T}) L \int_0^t e^{\omega(t-s)} \| f(s + \frac{h}{T}s) - f(s) \|_X \ ds 
+ L e^{\omega t} \left( \left(1 + \frac{h}{T}\right)^{-1} - 1 \right) \left( \|u_0\|_X + \int_0^t e^{-\omega s} \| f(s) \|_X \ ds \right).
\]
From this is clear that (2.4)-(2.7) follows.

In the case $f \equiv 0$, then the mapping $T_t$ given by (2.1) only depends on the initial value $u_0$, that is,
\[
T_t u_0 = T_t(u_0,0) \quad \text{for every } u_0 \text{ and } t \geq 0.
\]
In this case, the estimates in Theorem 2.3 reduce to the following one.

**Corollary 2.4.** Let $\{T_t\}_{t \geq 0}$ be a family of mappings $T_t : C \to C$ defined on a subset $C \subseteq X$ satisfying
\[
\|T_t u_0 - T_t \hat{u}_0\|_X \leq L e^{\omega t} \| u_0 - \hat{u}_0 \|_X \quad \text{for all } t \geq 0, u, \hat{u} \in C,
\]
\[
\lambda^{-1} T_{\lambda t} u_0 = T_t [\lambda^{-1} u_0] \quad \text{for all } \lambda > 0, t \geq 0 \text{ and } u_0 \in C,
\]
and $T_t 0 \equiv 0$ for all $t \geq 0$. Then, for every $u_0 \in C$ and $t, h > 0$, one has that
\[
\|T_{t+h} u_0 - T_t u_0\|_X \leq 2 \frac{h}{T} L e^{\omega t} \| u_0 \|_X.
\]
In particular, the family $\{T_t\}_{t \geq 0}$ satisfies
\[
\limsup_{h \to 0^+} \frac{\|T_{t+h} u_0 - T_t u_0\|_X}{h} \leq 2L e^{\omega t} \|u_0\|_X \quad \text{for every } t > 0, u_0 \in C.
\]
Moreover, if the right hand-side derivative $\frac{d}{dt} T_t u_0$ exists (in $X$) at $t > 0$, then
\[
\left\| \frac{d}{dt} T_t u_0 \right\|_X \leq 2L e^{\omega t} \|u_0\|_X \frac{\gamma}{t^\delta}.
\]

For our next corollary, we recall the following well-known definition.

**Definition 2.5.** Let $C$ be a subset of $X$. Then, a family $\{T_t\}_{t \geq 0}$ of mappings $T_t : C \to C$ is called a semigroup if $T_{t+s} u = T_t \circ T_s u$ for every $t, s \geq 0, u \in C$.

**Corollary 2.6.** Let $\{T_t\}_{t \geq 0}$ be a semigroup of mappings $T_t : C \to C$ defined on a subset $C \subseteq X$ and suppose, there is a second vector space $Y$ with semi-norm $\| \cdot \|_Y$ such that $\{T_t\}_{t \geq 0}$ satisfies the following $Y$-$X$-regularity estimate
\[
\|T_t u_0\|_X \leq M e^{\omega t} \frac{\|u_0\|_Y}{t^\delta} \quad \text{for every } t > 0 \text{ and } u_0 \in C.
\]
for some $M, \gamma, \delta > 0$ and $\hat{\omega} \in \mathbb{R}$. If for $u_0 \in C$, $\{T_t\}_{t \geq 0}$ satisfies \eqref{eq:2.14}, then

$$\limsup_{h \to 0^+} \frac{\|T_{t+h}u_0 - T_tu_0\|_X}{h} \leq 2^{\delta+2}LM \varepsilon^{\frac{1}{2}(\omega+\hat{\omega})t} \frac{\|u_0\|_Y}{t^{\delta+1}}.$$  

Moreover, if the right hand-side derivative $\frac{dT_tu_0}{dt}$ exists (in $X$) at $t > 0$, then

$$\left\| \frac{dT_tu_0}{dt} \right\|_X \leq 2^{\delta+2}LM \varepsilon^{\frac{1}{2}(\omega+\hat{\omega})t} \frac{\|u_0\|_Y}{t^{\delta+1}}.$$  

Proof. Since $\{T_t\}_{t \geq 0}$ is a semigroup, one sees by \eqref{eq:2.14} and \eqref{eq:2.16} that

$$\limsup_{h \to 0^+} \frac{\|T_{t+h}u_0 - T_tu_0\|_X}{h} = \limsup_{h \to 0^+} \frac{\|T_{t+\frac{h}{2}}(T_{\frac{h}{2}}u_0) - T_{t+\frac{h}{2}}(T_{\frac{h}{2}}u_0)\|_X}{h} \leq 4L e^{\omega\frac{h}{2}} \frac{\|T_{t+2u_0}\|_X}{t} \leq 2^{\delta+2}LM e^{\frac{1}{2}(\omega+\hat{\omega})t} \frac{\|u_0\|_Y}{t^{\delta+1}}.$$  

Next, suppose that there is a partial ordering “$\leq$” on $X$ such that $(X, \leq)$ is an ordered vector space. Then, we can state the following theorem.

**Theorem 2.7.** Let $(X, \leq)$ be an ordered vector space, $C$ be a subset of $X$, and $\{T_t\}_{t \geq 0}$ be a family of mappings $T_t : C \to C$ defined on a subset $C \subseteq X$ satisfying

\begin{equation}
\tag{2.17}
\text{for every } u_0, \hat{u}_0 \in C \text{ satisfying } u_0 \leq \hat{u}_0, \text{ one has } T_tu_0 \leq T_t\hat{u}_0 \text{ for all } t \geq 0.
\end{equation}

and

\begin{equation}
\tag{2.18}
\lambda^{-1}T_{t\lambda}u_0 = T_t[\lambda^{-1}u_0] \quad \text{for all } \lambda > 0, t \geq 0 \text{ and } u_0 \in C.
\end{equation}

Then for every $u_0 \in C$ satisfying $u_0 \geq 0$, one has

\begin{equation}
\tag{2.19}
\frac{T_{t+h}u_0 - T_tu_0}{h} \leq \frac{1}{t}T_tu_0 \quad \text{for every } t, h > 0.
\end{equation}

Before giving the proof of Theorem 2.7, we state the following definition.

**Definition 2.8.** If $(X, \leq)$ is an order vector space then a family $\{T_t\}_{t \geq 0}$ of mappings $T_t : C \to C$ defined on a subset $C \subseteq X$ is called order preserving if $\{T_t\}_{t \geq 0}$ satisfies \eqref{eq:2.17}.

Proof of Theorem 2.7. Since $\left(1 + \frac{h}{T_t}\right)^{-1} < 1$, one has that $\left(1 + \frac{h}{T_t}\right)^{-1}u_0 \leq u_0$. Then, by \eqref{eq:2.8} for $f \equiv 0$ and \eqref{eq:2.17}, one finds

$$T_{t+h}u_0 - T_tu_0 = \left(1 + \frac{h}{T_t}\right) T_t \left(1 + \frac{h}{T_t}\right)^{-1}u_0 - T_tu_0 \leq \frac{h}{T_t}T_t \left(1 + \frac{h}{T_t}\right)^{-1}u_0 \leq \frac{h}{T_t}T_tu_0,$$

from where one sees that \eqref{eq:2.19} holds. \qed
By Theorem 2.7, if the derivative \( \frac{dT_t u_0}{dt} \) exists (in \( X \)) at \( t > 0 \), then we can state the following.

**Corollary 2.9.** Under the hypotheses of Theorem 2.7, suppose that for \( u_0 \in C \) satisfying \( u_0 \geq 0 \), the right hand-side derivative \( \frac{dT_t u_0}{dt} \) exists (in \( X \)) at \( t > 0 \), then

\[
\frac{dT_t u_0}{dt} \leq \frac{1}{t} T_t u_0.
\]

Further, we can conclude from Theorem 2.7 the following result.

**Corollary 2.10.** In addition to the hypotheses of Theorem 2.7, suppose that there is a linear functional \( \Lambda : X \to \mathbb{R} \) satisfying

\[
(2.20) \quad \Delta x \geq 0 \quad \text{for every } x \in X \text{ satisfying } x \geq 0
\]

and

\[
(2.21) \quad \Delta T_t u_0 = \Lambda u_0 \quad \text{for every } t \geq 0 \text{ and } u_0 \in X \text{ satisfying } u_0 \geq 0.
\]

Then, the following estimate holds for each \( \nu \in \{+,-\} \),

\[
(2.22) \quad \Lambda [T_{t+h} u_0 - T_t u_0]^{\nu} \leq \frac{h}{t} \Lambda x \quad \text{for all } t, h > 0, u_0 \in C \text{ with } u_0 \geq 0.
\]

**Example 2.11.** If \( X = L^q(\Sigma, \mu) \) for some \( \Sigma \)-measure space \( (\Sigma, \mu) \) and \( 1 \leq q \leq \infty \), then an example for \( \Delta \) satisfying (2.20) and (2.21) is given by

\[
\Delta x = \int_{\Sigma} x \mu \quad \text{for every } x \in X.
\]

**Proof of Corollary 2.10.** Let \( u_0 \in C \) with \( u_0 \geq 0 \), and \( t, h > 0 \). Then we note first that by (2.21),

\[
0 = \Delta T_{t+h} u_0 - \Delta T_t u_0 = \Lambda (T_{t+h} u_0 - T_t u_0)
\]

and since

\[
\Lambda (T_{t+h} u_0 - T_t u_0) = \Lambda [T_{t+h} u_0 - T_t u_0]^+ - \Lambda [T_{t+h} u_0 - T_t u_0]^-,\n\]

one has that

\[
(2.23) \quad \Lambda [T_{t+h} u_0 - T_t u_0]^+ = \Lambda [T_{t+h} u_0 - T_t u_0]^-.\n\]

Further, by Theorem 2.7 and since \( T_t u_0 \geq 0 \), it follows from the definition of \( [x]^+ = \max \{x, 0\} \), \( x \in \mathbb{R} \), that

\[
(2.24) \quad [T_{t+h} u_0 - T_t u_0]^+ \leq \frac{h}{t} T_t u_0.
\]

By the linearity of \( \Lambda \) and by (2.20), one has that \( x \leq y \) yields \( \Lambda x \leq \Lambda y \). Thus applying \( \Lambda \) to (2.24) leads to (2.22) for \( \nu = "+" \). Moreover, by (2.23), inequality (2.22) also holds for \( \nu = "-" \). This completes the proof of this corollary. \( \square \)

For the last result of this section, we consider the following differential inclusion

\[
(2.25) \quad \frac{du}{dt} + A(u(t)) + F(u(t)) \geq 0 \quad \text{on } (0, +\infty),
\]

for some operator \( A \subseteq X \times X \) and a Lipschitz-continuous mapping \( F : X \to X \) with Lipschitz constant \( \omega \geq 0 \) and satisfying \( F(0) = 0 \). As for the differential inclusion (1.6) and the case \( f \equiv 0 \), suppose, there is a subset \( C \subseteq X \) and a family \( \{T_t\}_{t \geq 0} \) of mappings \( T_t : C \to C \) associated with \( A \) through the relation that
for every given $u_0 \in C$, the function $u$ defined by (1.4) is the unique solution of (2.25) with initial value $u(0) = u_0$. On the other hand, setting

$$f(t) := -F(u(t)), \quad (t \geq 0),$$

one has that

$$T_t(u_0, f) = u(t) = T_tu_0 \quad \text{for every } t \geq 0, u_0 \in C.$$ 

Thus, by Theorem 2.3 for $T = +\infty$ we have the following estimates.

**Corollary 2.12.** Let $F : X \to X$ be a Lipschitz continuous mapping with Lipschitz-constant $\omega > 0$ and satisfying $F(0) = 0$. Suppose, there is a subset $C \subseteq X$, and a family $\{T_t\}_{t \geq 0}$ of mappings $T_t : C \to C$ satisfying

$$\|T_tu_0\|_X \leq e^{\omega t} \|u_0\|_X \quad \text{for all } t \geq 0, u_0 \in C,$$

and in relation with (2.27), suppose that $\{T_t\}_{t \geq 0}$ satisfies (2.2) and (2.3) for $f$ given by (2.26). Then for every $u_0 \in C$, and $t, h > 0$ such that $|h|/t < 1$, one has that

$$\left\|\frac{T_{t+h}u_0 - T_tu_0}{h}\right\|_X \leq \left[2e^{\omega t} \int_0^t e^{-\omega s} ds + \omega \int_0^t e^{\omega s} \int_s^t e^{-\omega r} dr ds \right] e^{\omega t} L \|u_0\|_X.$$

Moreover, if the derivative $\frac{d}{dt}T_tu_0$ exists (in $X$) for a.e. $t > 0$, then

$$\left\|\frac{d}{dt}T_tu_0\right\|_X \leq e^{\omega t} L \left[2e^{\omega t} \int_0^t e^{-\omega s} ds + \omega \int_0^t e^{\omega s} \int_s^t e^{-\omega r} dr ds \right] \frac{\|u_0\|_X}{t}$$

for a.e. $t > 0$.

For the proof of this corollary, we will employ the following version of Gronwall’s lemma.

**Lemma 2.13.** Let $a \in L^1(0, T)$, $B : [0, T] \to \mathbb{R}$ be an absolutely continuous function, and $v \in L^\infty(0, T)$ satisfy

$$v(t) \leq \int_0^t a(s)v(s) \, ds + B(t) \quad \text{for a.e. } t \in (0, T).$$

Then,

$$v(t) \leq B(0) e^{\int_0^t a(s) ds} + \int_0^t e^{\int_0^s a(r) \, dr} B'(s) \, ds \quad \text{for a.e. } t \in (0, T).$$

We now give the proof of Corollary 2.12.

**Proof.** Let $u_0 \in C$, and $t, h > 0$ such that $|h|/t < 1$. Then, by the hypotheses of this corollary, we are in the position to apply Theorem 2.3 to $T_t(u_0, f)$ for $f$ given by (2.26). Then by (2.4), one finds

$$\left\|\frac{T_{t+h}u_0 - T_tu_0}{h}\right\|_X \leq L e^{\omega t} \left[2\|u_0\|_X + \int_0^t e^{-\omega s} \|F(T_tu_0)\|_X ds + \right.$$  

$$+ \left(1 + \frac{h}{T}\right) \int_0^t e^{-\omega s} \left\|\frac{F(T_tu_0)}{T_tu_0} - F(T_su_0)\right\|_X ds \right].$$
Since $F$ is globally Lipschitz continuous with constant $\omega > 0$, $F(0) = 0$ and by (2.28), it follows that
\[
\|T_{t+h}u_0 - T_tu_0\|_X \leq L e^{\omega t} \left( (2 + \omega t)\|u_0\|_X + \frac{(1 + \frac{h}{t})}{t} \omega \int_0^t e^{-\omega s} \left\| \frac{T_{s+h}u_0 - T_su_0}{h} \right\|_X ds \right).
\]
(2.31)

Since $|h|/t < 1$,
\[
e^{-\omega t} \|T_{t+h}u_0 - T_tu_0\|_X \leq L \left( (2 + \omega t)\|u_0\|_X + 2\omega \int_0^t e^{-\omega s} \left\| \frac{T_{s+h}u_0 - T_su_0}{h} \right\|_X ds \right).
\]

Due to (2.31), we can apply Gronwall’s lemma to
\[B(t) = L(2 + \omega t)\|u_0\|_X \quad \text{and} \quad a(t) = L2\omega e^{-\omega t}.
\]
Then, one sees that (2.29) holds. Now, suppose that the derivative $\frac{d}{dt} T_tu_0$ exists (in $X$) for a.e. $t > 0$, then by (2.7), the Lipschitz continuity of $F$ and by (2.28), one has that
\[e^{-\omega t} \left\| \frac{d}{dt} T_tu_0 \right\|_X \leq 2\|u_0\|_X + \omega \int_0^t e^{-\omega s} \left\| \frac{d}{ds} T_su_0 \right\|_X ds + \omega t \|u_0\|_X
\]
for a.e. $t > 0$. Now, applying Gronwall’s lemma to
\[B(t) = L(2 + \omega t)\|u_0\|_X \quad \text{and} \quad a(t) = L\omega e^{-\omega t},
\]
leads to (2.30). This completes the proof of this corollary.

3. ACCRETIVE OPERATORS OF HOMOGENEOUS ORDER ZERO

Suppose $X$ is Banach space with norm $\|\cdot\|_X$. Then, we begin this section with the following definition.

**Definition 3.1.** For $\omega \in \mathbb{R}$, an operator $A$ on $X$ is called $\omega$-quasi $m$-accretive operator on $X$ if $A$ is accretive, that is, for every $(u,v)$, $(\tilde{u},\tilde{v}) \in A$ and every $\lambda \geq 0$,
\[\|u - \tilde{u}\|_X \leq \|u - \tilde{u} + \lambda (\omega(u - \tilde{u}) + v - \tilde{v})\|_X.
\]
and if for $A$ the range condition
\[Rg(I + \lambda A) = X \quad \text{for some (or equivalently, for all) } \lambda > 0, \lambda \omega < 1,
\]
holds.

If $A$ is $\omega$-quasi $m$-accretive operator, then the classical existence theorem [9, Theorem 6.5] (cf [6, Corollary 4.2]), for every $u_0 \in D(A)^X$ and $f \in L^1(0,T;X)$, there is a unique mild solution $u \in C([0,T];X)$ of (1.6).
Definition 3.2. For given $u_0 \in \overline{D(A)}^\times$ and $f \in L^1(0, T ; X)$, a function $u \in C([0, T]; X)$ is called a mild solution of the inhomogeneous differential inclusion (1.6) with initial value $u_0$ if $u(0) = u_0$ and for every $\varepsilon > 0$, there is a partition $\tau_\varepsilon = 0 = t_0 < t_1 < \cdots < t_N = T$ and a step function

$$u_{\varepsilon, N}(t) = u_0 \mathbb{1}_{\{t=0\}}(t) + \sum_{i=1}^{N} u_i \mathbb{1}_{(t_i, t_{i+1}]}(t) \quad \text{for every } t \in [0, T]$$

satisfying

$$t_i - t_{i-1} < \varepsilon \quad \text{for all } i = 1, \ldots, N,$$

$$\sum_{N=1}^{N} \int_{t_{i-1}}^{t_i} \|f(t) - \bar{f}_i\| \, dt < \varepsilon \quad \text{where } \bar{f}_i := \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} f(t) \, dt,$$

$$\frac{u_i - u_{i-1}}{t_i - t_{i-1}} + Au_i \supseteq \bar{f}_i \quad \text{for all } i = 1, \ldots, N,$$

and

$$\sup_{t \in [0, T]} \|u(t) - u_{\varepsilon, N}(t)\|_X < \varepsilon.$$

In particular, if $A$ is $\omega$-quasi $m$-accretive, and if for given $u_0 \in \overline{D(A)}^\times$, $f \in L^1(0, T ; X)$, the function $u : [0, T] \to X$ is the unique mild solution of (1.6) with initial value $u(0) = u_0$, then by (2.1) the family $\{T_t\}_{t=0}^{T}$ defines a semigroup of $\omega$-quasi contractions $T_t : \overline{D(A)}^\times \times L^1(0, T ; X) \to \overline{D(A)}^\times$ for $C = \overline{D(A)}^\times$; that is, $\{T_t\}_{t=0}^{T}$ satisfies

- (semigroup property) $T_{t+s} = T_t \circ T_s$ for every $t, s \in [0, T]$;
- (strong continuity) for every $(u_0, f) \in \overline{D(A)}^\times \times L^1(0, T ; X)$, $t \mapsto T_t(u_0, f)$ belongs to $C([0, T]; X)$;
- (\(\omega\)-quasi contractivity) $T_t$ satisfies (2.3)

Furthermore, keeping $f \equiv 0$ and only varying $u_0 \in \overline{D(A)}^\times$, shows that by

$$T_t u_0 = T_t(u_0, 0) \quad \text{for every } t \geq 0.$$

defines a strongly continuous semigroup $\{T_t\}_{t \geq 0}$ of $\omega$-quasi contractions $T_t : \overline{D(A)}^\times \to \overline{D(A)}^\times$. For the family $\{T_t\}_{t \geq 0}$ on $\overline{D(A)}^\times$, the operator

$$A_0 := \left\{ (u_0, v) \in X \times X \mid \lim_{h \downarrow 0} \frac{T_h(u_0, 0) - u_0}{h} = v \text{ in } X \right\}$$

is an $\omega$-quasi accretive well-defined mapping $A_0 : D(A_0) \to X$ and called the infinitesimal generator of $\{T_t\}_{t \geq 0}$. Under additional conditions on the geometry of the Banach space $X$ (see Definition 3.7), one has that $A_0 \subseteq A$. Thus, we say (ignoring the abuse of details) that both families $\{T_t\}_{t \geq 0}^T$ on $\overline{D(A)}^\times \times L^1(0, T ; X)$ and $\{T_t\}_{t \geq 0}^{T}$ on $\overline{D(A)}^\times$ are generated by $-A$.

In application, usually $X$ is given by the Lebesgue space $L^\infty(\Sigma, \mu)$ (or $L^r(\Sigma, \mu)$ for $1 \leq r < \infty$) and $Y$ is given by $L^1(\Sigma, \mu)$ (or $L^q(\Sigma, \mu)$ for some $1 \leq q < r$) for some $\sigma$-finite measure space $(\Sigma, \mu)$. Then, $L^1$-$L^\infty$-decay estimates are intimately connected with abstract Sobolev inequalities satisfied by the infinitesimal generator $-A$ of the semigroup $\{T_t\}_{t \geq 0}$. For more details to the linear semigroup...
theory we refer to the monograph [19] and to [12] for the nonlinear semigroup theory.

Moreover (cf [9, Chapter 4.3]), for given $u_0 \in \overline{D(A)^\infty}$ and any step function $f = \sum_{i=1}^N f_i \mathbf{1}_{(t_{i-1}, t_i)} \in L^1(0, T; X)$, let $u : [0, T] \to X$ given by
\begin{equation}
(3.2) \quad u(t) = u_0 \mathbf{1}_{\{t=0\}}(t) + \sum_{i=1}^N u_i(t) \mathbf{1}_{(t_{i-1}, t_i]}(t)
\end{equation}
is the unique mild solution of (1.6), where $u_i$ is the unique mild solution of
\begin{equation}
(3.3) \quad \frac{du_i}{dt} + A(u_i(t)) \ni f_i \quad \text{on} \quad (t_{i-1}, t_i), \quad \text{and} \quad u_i(t_{i-1}) = u_{i-1}(t_{i-1}).
\end{equation}
Then for every $i = 1, \ldots, N$, the semigroup $\{T_i\}_{i=0}^T$ is obtained by the exponential formula
\begin{equation}
(3.4) \quad T_i(u(t_{i-1}), f_i) = u_i(t) = \lim_{n \to \infty} \left[ \int_{t_{i-1}}^{t_i} \right]_0^1 u(t_{i-1}) \quad \text{in} \quad C([t_{i-1}, t_i]; X)
\end{equation}
for every $i = 1, \ldots, N$, where for $\mu > 0$, $J^{A_i}_\mu = (I + \mu A_i)^{-1}$ is the resolvent operator of the operator $A_i$ and $A_i := \{(x, y - f_i) : (x, y) \in A\}$.

As for classical solutions, the fact that $A$ is homogeneous of order zero, is also reflected in the notion of mild solution and so in $\{T_i\}_{i=0}^T$. This is shown in our next lemma.

**Lemma 3.3.** Let $A$ be a $\omega$-quasi m-accrative and $\{T_i\}_{t \geq 0}$ be the semigroup on $\overline{D(A)^\infty} \times L^1_{\text{loc}}([0, +\infty); X)$ generated by $-A$. If $A$ is homogeneous of order zero, then $\{T_i\}_{t \geq 0}$ satisfies \((2.2)\) for every $(u_0, f) \in \overline{D(A)^\infty} \times L^1(0, T; X)$.

**Proof.** For every $\mu > 0, v \in X$, and $\lambda > 0$, one has that
\begin{equation}
J^{A_i}_\mu \left[ \lambda^{-1} v \right] = u \quad \text{if and only if} \quad u + \mu A_i u \ni \lambda^{-1} v,
\end{equation}
which if $A$ is homogeneous of order zero, is equivalent to
\begin{equation}
\lambda u + \lambda \mu A_i (\lambda u) \ni v \quad \text{or} \quad J^{A_i}_\mu v = \lambda u.
\end{equation}
Therefore,
\begin{equation}
(3.5) \quad \lambda^{-1} J^{A_i}_\mu v = J^{A_i}_{\lambda \mu} \left[ \lambda^{-1} v \right] \quad \text{for all} \quad \lambda, \mu > 0, v \in X.
\end{equation}
Now, for $u_0 \in \overline{D(A)^\infty}$ and a partition
\begin{equation}
\pi : 0 = t_0 < t_1 < \cdots < t_N = T \quad \text{of} \quad [0, T]
\end{equation}
let $f = \sum_{i=1}^N f_i \mathbf{1}_{(t_{i-1}, t_i]} \in L^1(0, T; X)$ be a step function and $u$ be the unique mild solution of (1.6) for $f$. Then $u$ is given by (3.2), were on each subinterval $(t_{i-1}, t_i), u_i$ is the unique mild solution of (3.3). For $t > n, n \in \mathbb{N}$, and $\lambda \in (0, 1]$, apply (3.5) to
\begin{equation}
\mu = \frac{t}{n} \quad \text{and} \quad v = J^{A_i}_{\lambda \mu} \left[ \lambda^{-1} u_0 \right].
\end{equation}
Then,
\begin{equation}
\left[ J^{A_i}_{\lambda} \right]^2 \left[ \lambda^{-1} u_0 \right] = J^{A_i}_{\lambda} \left[ \lambda^{-1} J^{A_i}_{\lambda \mu} u_0 \right] = \lambda^{-1} \left[ J^{A_i}_{\lambda \mu} \right]^2 u_0.
\end{equation}
Iterating this equation $n$-times, one finds that
\[ \lambda^{-1} \left[ f_{A_1}^n \right] u_0 = \left[ \lambda^{-1} u_0 \right] \]
and so, by (3.4) sending $n \to +\infty$ in the latter equation, yields on the one site
\[ \lim_{n \to +\infty} \lambda^{-1} \left[ f_{A_1}^n \right] u_0 = \lambda^{-1} u_1(\lambda t) = \lambda^{-1} u(\lambda t) \]
for every $t \in [0, \frac{t_1}{\lambda}]$, and on the other side
\[ \lim_{n \to +\infty} \left[ f_{A_1}^n \right] u_0 = v(t) \]
for every $t \in [0, \frac{t_1}{\lambda}]$, where $v$ is the unique mild solution of (3.3) for $i = 1$ on
$(0, \frac{t_1}{\lambda})$ with initial value $v(0) = \lambda^{-1} u_0$. By uniqueness of the two limits, we
have thereby shown that
\[ \lambda^{-1} T_{\lambda t}(u_0, f_1) = T_t(\lambda^{-1} u_0, f_1) \mathbb{1}_{(0, \frac{t_1}{\lambda})} \quad \text{for every } t \in \left[ 0, \frac{t_1}{\lambda} \right]. \]
Similarly, for every $i = 2, 3, \ldots, N$, replacing in (3.6) $u_0$ by $u(t_{i-1})$ (where
\( u(t_{i-1}) = u(\lambda \frac{t_{i-1}}{\lambda}) = v(\frac{t_{i-1}}{\lambda}) \), $A_1$ by $A_i$, and $\frac{t_{i-1}}{\lambda}$ by $\frac{t_i}{\lambda}$ gives
\[ \lambda^{-1} \left[ f_{A_i}^n \right] u(t_{i-1}) = \left[ \lambda^{-1} v(\frac{t_{i-1}}{\lambda}) \right] \]
and by sending $n \to +\infty$, limit (3.4) leads one one side to
\[ \lim_{n \to +\infty} \lambda^{-1} \left[ f_{A_i}^n \right] u(t_{i-1}) = \lambda^{-1} u(\lambda t) \]
and on the other side,
\[ \lim_{n \to +\infty} \left[ f_{A_i}^n \right] \lambda^{-1} v(\frac{t_{i-1}}{\lambda}) = v(t) \]
for every $t \in \left[ \frac{t_{i-1}}{\lambda}, \frac{t_i}{\lambda} \right]$, where $v$ is the unique mild solution of (3.3) for $i$ on
\( (\frac{t_{i-1}}{\lambda}, \frac{t_i}{\lambda}) \) with initial value $v(\frac{t_{i-1}}{\lambda}) = \lambda^{-1} v(\frac{t_{i-1}}{\lambda}) = \lambda^{-1} u(t_{i-1})$. Therefore, and
since $u$ is given by (3.2), we have shown that
\[ \lambda^{-1} T_{\lambda t}(u(t_{i-1}), f_i) = T_t(\lambda^{-1} u(t_{i-1}), f_i) \mathbb{1}_{(\frac{t_{i-1}}{\lambda}, \frac{t_i}{\lambda})} \quad \text{for } t \in \left[ \frac{t_{i-1}}{\lambda}, \frac{t_i}{\lambda} \right]. \]
Since for every step function $f$ on a partition $\pi$ of $[0, T]$, $u$ is given by (3.2), we
have thereby shown that (2.2) holds if $f$ is a step function. Now, by (2.3), an
approximation argument shows that if $A$ is homogeneous of order zero, then
the semigroup $\{ T_t \}_{t \geq 0}$ on $D(A)^\infty \times L^1(0, T; X)$ generated by $-A$ satisfies (2.2).
□

By the above Lemma and Theorem 2.3, we can now state the following.
Corollary 3.4. For \( \omega \in \mathbb{R} \), suppose \( A \) is an \( \omega \)-quasi \( m \)-accretive operator on a Banach space \( X \), and \( A \) is homogeneous of order zero satisfying \( 0 \in A0 \). Then, for every \((u_0, f) \in D(A) \times L^1(0, T; X)\), the semigroup \( \{T_t\}_{t \geq 0} \) of mapping \( T_t : D(A) \times L^1(0, T; X) \rightarrow D(A) \) generated by \(-A\) satisfies (2.4)-(2.7).

For having that regularity estimate (2.7) (respectively, (2.15)) is satisfied by the semigroup \( \{T_t\}_{t \geq 0} \), one requires that each mild solution \( u \) of (1.6) (respectively, of (1.1)) is differentiable and a stronger notion of solutions of (1.6). The next definition is taken from [9, Definition 1.2] (cf [6, Chapter 4]).

Definition 3.5. A locally absolutely continuous function \( u[0, T] : \rightarrow X \) is called a strong solution of differential inclusion (1.1) if \( u \) is differentiable a.e. on \((0, T)\), and for a.e. \( t \in (0, T) \), \( u(t) \in D(A) \) and \( f(t) - \frac{du}{dt}(t) \in A(u(t)) \).

The next characterization of strong solutions of (1.1) highlights the important point of a.e. differentiability.

Proposition 3.6 ([9, Theorem 7.1]). Let \( X \) be a Banach space, \( f \in L^1(0, T; X) \) and for \( \omega \in \mathbb{R} \), \( A \) be \( \omega \)-quasi \( m \)-accretive in \( X \). Then \( u \) is a strong solution of the differential inclusion (1.6) on \([0, T]\) if and only if \( u \) is a mild solution on \([0, T]\) and \( u \) is "absolutely continuous" on \([0, T]\) and differentiable a.e. on \((0, T)\).

Of course, every strong solution \( u \) of (1.6) is a mild solution of (1.6), absolutely continuous and differentiable a.e. on \([0, T]\). Moreover, the differential inclusion (1.6) admits mild and Lipschitz continuous solutions if \( A \) is \( \omega \)-quasi \( m \)-accretive in \( X \) (cf [9, Lemma 7.8]). But absolutely continuous vector-valued functions \( u : [0, T] \rightarrow X \) are not, in general, differentiable a.e. on \((0, T)\). However, if one assumes additional geometric properties on \( X \), then the latter implication holds true. Our next definition is taken from [9, Definition 7.6] (cf [5, Chapter 1]).

Definition 3.7. A Banach space \( X \) is said to have the Radon-Nikodým property if every absolutely continuous function \( F : [a, b] \rightarrow X \), \( (a, b \in \mathbb{R}, a < b) \), is differentiable almost everywhere on \((a, b)\).

Known examples of Banach spaces \( X \) admitting the Radon-Nikodým property are:

- **(Dunford-Pettis)** if \( X = Y^* \) is separable, where \( Y^* \) is the dual space of a Banach space \( Y \);
- if \( X \) is reflexive.

We emphasize that \( X_1 = L^1(\Sigma, \mu), X_2 = L^\infty(\Sigma, \mu) \), or \( X_3 = C(M) \) for a \( \sigma \)-finite measure space \((\Sigma, \mu)\), or respectively, for a compact metric space \((M, d)\) don’t have, in general, the Radon-Nikodým property (cf [5]). Thus, it is quite surprising that there is a class of operators \( A \) (namely, the class of completely accretive operators, see Section 4 below), for which the differential inclusion (1.6) nevertheless admits strong solutions (with values in \( L^1(\Sigma, \mu) \) or \( L^\infty(\Sigma, \mu) \)).

Now, by Corollary 3.4 and Proposition 3.6, we can conclude the following results. We emphasize that one crucial point in the statement of Corollary 3.8 below is that due to the uniform estimate (2.7), one has that for all initial values \( u_0 \in D(A)^\times \), the unique strong solution \( u \) of (1.6) satisfying \( u(0) = u_0 \) is a strong solution, and not only for \( u_0 \in D(A) \).
Corollary 3.8. For $\omega \in \mathbb{R}$, suppose $A$ is an $\omega$-quasi $m$-accretive operator on a Banach space $X$ admitting the Radon-Nikodým property, and $\{T_t\}_{t \geq 0}$ is the semigroup on $\overline{D(A)} \times L^1(0,T;X)$ generated by $-A$. If $A$ is homogeneous of order zero satisfying $0 \in A0$, then for every $u_0 \in \overline{D(A)}$ and $f \in BV(0,T;X)$, the unique mild solution $u$ of (1.6) satisfying $u(0) = u_0$ is a strong solution and satisfies (2.7) for every $t > 0$.

Now by Corollary 2.12 and Proposition 3.6, we obtain the following result when $A$ is perturbed by a Lipschitz mapping.

Corollary 3.9. Suppose $X$ is a Banach space with the Radon-Nikodým property, $F : X \to X$ be a Lipschitz continuous mapping with Lipschitz-constant $\omega > 0$ satisfying $F(0) = 0$, $A$ an $m$-accretive operator on $X$, and $\{T_t\}_{t \geq 0}$ is the semigroup on $\overline{D(A)}$, generated by $-(A + F)$. If $A$ is homogeneous of order zero satisfying $0 \in A0$, then (2.30) holds for every $u_0 \in \overline{D(A)}$ and a.e. $t > 0$.

If the Banach space $X$ and its dual space $X^*$ are uniformly convex, then (cf \[6, Theorem 4.6\]) for every $u_0 \in D(A)$, $f \in W^{1,1}(0,T;X)$, the mild solution $u(t) = T_t(u_0,f)$, $(t \geq 0)$, of (1.6) is a strong solution of (1.6), $u$ is everywhere differentiable from the right, $\frac{du}{dt}$ is right continuous, and

$$\frac{du}{dt}(t) + (A - f(t))u(t) = 0$$

for every $t \geq 0$,

where for every $t \in [0,T]$, $(A - f(t))u$ denotes the minimal selection of $A - f(t)$ defined by

$$(A - f(t))^\circ := \left\{ (u,v) \in A - f(t) \mid \|v\|_X = \inf_{\hat{v} \in Au - f(t)} \|\hat{v}\|_X \right\}.$$ 

Thus, under those assumptions on $X$ and by Proposition 3.6, we can state the following three corollaries. We begin by stating the inhomogeneous case.

Corollary 3.10. Suppose $X$ and its dual space $X^*$ are uniformly convex, for $\omega \in \mathbb{R}$, $A$ is an $\omega$-quasi $m$-accretive operator on $X$, and $\{T_t\}_{t \geq 0}$ is the semigroup on $\overline{D(A)} \times L^1(0,T;X)$ generated by $-A$. If $A$ is homogeneous of order zero satisfying $0 \in A0$, then for every $u_0 \in \overline{D(A)}$ and $f \in W^{1,1}(0,T;X)$,

$$\| (A - f(t))^\circ T_t(u_0,f) \|_X \leq \frac{e^{\omega t}}{t} \left[ 2\|u_0\|_X + \int_0^t e^{-\omega s} \|f'(s)\|_X \, ds + \int_0^t e^{-\omega s} \|f(s)\|_X \, ds \right]$$

for every $t > 0$.

The following corollary states the homogeneous case.

Corollary 3.11. Suppose $X$ and its dual space $X^*$ are uniformly convex, for $\omega \in \mathbb{R}$, $A$ is an $\omega$-quasi $m$-accretive operator on $X$, and $\{T_t\}_{t \geq 0}$ is the semigroup on $\overline{D(A)}$ generated by $-A$. If $A$ is homogeneous of order zero satisfying $0 \in A0$, then

$$\| A^\circ T_t u_0 \|_X \leq 2e^{\omega t} \frac{\|u_0\|_X}{t}$$

for every $t > 0$ and $u_0 \in \overline{D(A)}$.

The last corollary states the case when $A$ is perturbed by a Lipschitz mapping. This follows from [6, Theorem 4.6] and Corollary 2.12.
Corollary 3.12. Suppose \( X \) and its dual space \( X^* \) are uniformly convex, \( F : X \to X \) be a Lipschitz continuous mapping with Lipschitz-constant \( \omega > 0 \) satisfying \( F(0) = 0 \), \( A \) an \( m \)-accretive operator on \( X \), and \( \{T_t\}_{t \geq 0} \) is the semigroup on \( D(A)^m \) generated by \( -(A + F) \). If \( A \) is homogeneous of order zero satisfying \( 0 \in A0 \), then for every \( u_0 \in D(A)^m \),
\[
\left\| \frac{dT_t u_0}{dt} \right\|_X \leq e^{\omega t} \left[ 2e^{\omega t} \int_0^t e^{-\omega s} \, ds + \omega \int_0^t e^{\omega s} \, ds \right] \frac{\|u_0\|_X}{t}
\]
for every \( t > 0 \).

4. Completely Accretive Operators of Homogeneous Order Zero

In [8], Bénilan and Crandall introduced the celebrated class of completely accretive operators \( A \) and showed that there spaces without the Radon-Nikodym property, but if \( A \) is homogeneous of order \( \alpha > 0 \) with \( \alpha \neq 1 \), then the mild solutions of differential inclusion (1.1) involving \( A \) are strong solutions. In this section we will see that this also happen for completely accretive of homogeneous order zero.

4.1. General framework. We begin by outlining our framework and then provide a brief introduction to the class of completely accretive operators.

For the rest of this paper, suppose \( (\Sigma, B, \mu) \) is a \( \sigma \)-finite measure space, and \( M(\Sigma, \mu) \) the space of \( \mu \)-a.e. equivalent classes of measurable functions \( u : \Sigma \to \mathbb{R} \). For \( u \in M(\Sigma, \mu) \), we write \( [u]^+ \) to denote \( \max\{u, 0\} \) and \( [u]^− = −\min\{u, 0\} \). We denote by \( L^q(\Sigma, \mu) \), \( 1 \leq q \leq \infty \), the corresponding standard Lebesgue space with norm
\[
\|u\|_q = \left\{ \left( \int_{\Sigma} |u|^q \, d\mu \right)^{1/q} \right\}^{1/q}
\]
if \( 1 \leq q < \infty \),
\[
\inf \left\{ k \in [0, +\infty] \mid |u| \leq k \ \mu\text{-a.e. on } \Sigma \right\}
\]
if \( q = \infty \).

For \( 1 \leq q < \infty \), we identify the dual space \( (L^q(\Sigma, \mu))^\prime \) with \( L^{q\prime}(\Sigma, \mu) \), where \( q\prime \) is the conjugate exponent of \( q \) given by \( 1 = \frac{1}{q} + \frac{1}{q\prime} \).

Next, we first briefly recall the notion of Orlicz spaces (cf [18, Chapter 3]). A continuous function \( \psi : [0, +\infty) \to [0, +\infty) \) is an \( N \)-function if it is convex, \( \psi(s) = 0 \) if and only if \( s = 0 \), \( \lim_{s \to 0^+} \psi(s)/s = 0 \), and \( \lim_{s \to \infty} \psi(s)/s = \infty \). Given an \( N \)-function \( \psi \), the Orlicz space
\[
L^\psi(\Sigma, \mu) := \left\{ u \in M(\Sigma, \mu) \mid \int_{\Sigma} \psi \left( \frac{|u|}{\alpha} \right) \, d\mu < \infty \text{ for some } \alpha > 0 \right\}
\]
and equipped with the Orlicz-Minkowski norm
\[
\|u\|_\psi := \inf \left\{ \alpha > 0 \mid \int_{\Sigma} \psi \left( \frac{|u|}{\alpha} \right) \, d\mu \leq 1 \right\}.
\]

With those preliminaries in mind, we are now in the position to recall the notation of completely accretive operators introduced in [8] and further developed to the \( \omega \)-quasi case in [12].
Let $J_0$ be the set given by
\[ J_0 = \{ j : \mathbb{R} \to [0, +\infty) \mid j \text{ is convex, lower semicontinuous, } j(0) = 0 \}. \]

Then, for every $u, v \in M(\Sigma, \mu)$, we write
\[ u \ll v \quad \text{if and only if} \quad \int_M [j(u)] \, d\mu \leq \int_M [j(v)] \, d\mu \quad \text{for all } j \in J_0. \]

**Remark 4.1.** Due to the interpolation result [8, Proposition 1.2], for given $u, v \in M(\Sigma, \mu)$, the relation $u \ll v$ is equivalent to the two conditions
\[
\begin{cases}
\int_M (u - k)^+ \, d\mu \leq \int_M (v - k)^+ \, d\mu & \text{for all } k > 0 \\
\int_M (u + k)^- \, d\mu \leq \int_M (v + k)^- \, d\mu & \text{for all } k > 0.
\end{cases}
\]

Thus, the relation $\ll$ is closely related to the theory of rearrangement-invariant function spaces (cf [10]). Another, useful characterization of relation " $u \ll v$ " is the following (cf [8, Remark 1.5]): for $u, v \in M(\Sigma, \mu)$, $u \ll v$ if and only if $u^+ \ll v^+$ and $u^- \ll v^-$. Further, the relation $\ll$ on $M(\Sigma, \mu)$ has the following properties. We omit the easy proof of this proposition.

**Proposition 4.2.** For every $u, v, w \in M(\Sigma, \mu)$, one has that
1. $u^+ \ll u, u^- \ll -u$;
2. $u \ll v$ if and only if $u^+ \ll v^+$ and $u^- \ll v^-$;
3. (positive homogeneity) if $u \ll v$ then $au \ll av$ for all $a > 0$;
4. (transitivity) if $u \ll v$ and $v \ll w$ then $u \ll w$;
5. If $u \ll v$ then $|u| \ll |v|$;
6. (convexity) for every $u \in M(\Sigma, \mu)$, the set $\{ w \mid w \ll u \}$ is convex.

With these preliminaries in mind, we can now state the following definitions.

**Definition 4.3.** A mapping $S : D(S) \to M(\Sigma, \mu)$ with domain $D(S) \subseteq M(\Sigma, \mu)$ is called a complete contraction if
\[ Su - S\tilde{u} \ll u - \tilde{u} \quad \text{for every } u, \tilde{u} \in D(S). \]

More generally, for $L > 0$, we call $S$ to be an $L$-complete contraction if
\[ L^{-1}Su - L^{-1}S\tilde{u} \ll u - \tilde{u} \quad \text{for every } u, \tilde{u} \in D(S), \]
or for $L = e^{\omega t}$ with $\omega \in \mathbb{R}$ and $t \geq 0$, $S$ is then also called an $\omega$-quasi complete contraction.

**Remark 4.4.** Choosing $j_Q(\cdot) = ||\cdot||_q \in J_0$ if $1 \leq q < \infty$ and $j_\infty(\cdot) = ([\cdot]^+ - k]^+ \in J_0$ for $k \geq 0$ large enough if $q = \infty$, or $j_{\psi, \alpha}(\cdot) = \psi([\cdot]^+ / \alpha)$ for any $N$-function $\psi$ and $\alpha > 0$ shows that for each $L$-complete contraction $S : D(S) \to M(\Sigma, \mu)$ with domain $D(S) \subseteq M(\Sigma, \mu)$, the mapping $L^{-1}S$ is order-preserving and contractive respectively with respect to the $L^q$-norm for all $1 \leq q \leq \infty$, and the $L^\infty$-norm for any $N$-function $\psi$.

Now, we can state the definition of completely accretive operators.
Definition 4.5. An operator $A$ on $M(\Sigma, \mu)$ is called completely accretive if for every $\lambda > 0$, the resolvent operator $I_\lambda$ of $A$ is a complete contraction, or equivalently, if for every $(u_1, v_1), (u_2, v_2) \in A$ and $\lambda > 0$, one has that

$$u_1 - u_2 \preccurlyeq u_1 - u_2 + \lambda(v_1 - v_2).$$

If $X$ is a linear subspace of $M(\Sigma, \mu)$ and $A$ an operator on $X$, then $A$ is $m$-completely accretive on $X$ if $A$ is completely accretive and satisfies the range condition (3.1). Further, for $\omega \in \mathbb{R}$, an operator $A$ on a linear subspace $X \subseteq M(\Sigma, \mu)$ is called $\omega$-quasi $(m)$-completely accretive in $X$ if $A + \omega I$ is $(m)$-completely accretive in $X$.

Before stating a useful characterization of completely accretive operators, we first need to introducing the following function spaces. Let

$$L^{1+\infty}(\Sigma, \mu) := L^1(\Sigma, \mu) + L^\infty(\Sigma, \mu) \quad \text{and} \quad L^{1\cap \infty}(\Sigma, \mu) := L^1(\Sigma, \mu) \cap L^\infty(\Sigma, \mu)$$

be the sum and the intersection space of $L^1(\Sigma, \mu)$ and $L^\infty(\Sigma, \mu)$, which respectively equipped with the norms

$$||u||_{1+\infty} := \inf \left\{ \left(\int \frac{1}{|t|} \right)^{1/2} df(t) \mid u = u_1 + u_2, \quad u_1 \in L^1(\Sigma, \mu), u_2 \in L^\infty(\Sigma, \mu) \right\},$$

$$||u||_{1\cap \infty} := \max \left\{ ||u||_1, ||u||_\infty \right\}$$

are Banach spaces. In fact, $L^{1+\infty}(\Sigma, \mu)$ and $L^{1\cap \infty}(\Sigma, \mu)$ are respectively the largest and the smallest of the rearrangement-invariant Banach function spaces (cf [10], Chapter 3.1). If $\mu(\Sigma)$ is finite, then $L^{1+\infty}(\Sigma, \mu) = L^1(\Sigma, \mu)$ with equivalent norms, but if $\mu(\Sigma) = \infty$ then $L^{1+\infty}(\Sigma, \mu)$ contains $\bigcup_{1 \leq q \leq \infty} L^q(\Sigma, \mu)$. Further, we will employ the space

$$L_0(\Sigma, \mu) := \left\{ u \in M(\Sigma, \mu) \mid \int \frac{1}{|t|} \right\} \left\{ \left(\int \frac{1}{|t|} \right)^{1/2} df(t) \mid u = u_1 + u_2, \quad u_1 \in L^1(\Sigma, \mu), u_2 \in L^\infty(\Sigma, \mu) \right\},$$

which equipped with the $L^{1+\infty}$-norm is a closed subspace of $L^{1+\infty}(\Sigma, \mu)$. In fact, one has (cf [8]) that $L_0(\Sigma, \mu) = L^1(\Sigma, \mu) \cap L^\infty(\Sigma, \mu)_{1+\infty}$. Since for every $k \geq 0$, $T_k(s) := [s - k]^+$ is a Lipschitz mapping $T_k : \mathbb{R} \to \mathbb{R}$ and by Chebyshev’s inequality, one see that $L^q(\Sigma, \mu) \to L_0(\Sigma, \mu)$ for every $1 \leq q < \infty$ (and $q = \infty$ if $\mu(\Sigma) < \infty$), and $L^q(\Sigma, \mu) \to L_0(\Sigma, \mu)$ for every $N$-function $\psi$.

Proposition 4.6 ([12]). Let $P_0$ denote the set of all functions $T \in C^\infty(\mathbb{R})$ satisfying $0 \leq T' \leq 1$, $T'$ is compactly supported, and $x = 0$ is not contained in the support supp$(T)$ of $T$. Then for $\omega \in \mathbb{R}$, an operator $A \subseteq L^0(\Sigma, \mu) \times L_0(\Sigma, \mu)$ is $\omega$-quasi completely accretive if and only if

$$\int \frac{1}{|t|} (u - \tilde{u})(v - \tilde{v}) \, d\mu + \omega \int \frac{1}{|t|} (u - \tilde{u})(v - \tilde{v}) \, d\mu \geq 0$$

for every $T \in P_0$ and every $(u, v), (\tilde{u}, \tilde{v}) \in A$.

Remark 4.7. For convenience, we dot the unique extension of $\{T_t\}_{t \geq 0}$ on $L^p(\Sigma, \mu)$ or $L^1(\Sigma, \mu)$ again by $\{T_t\}_{t \geq 0}$.

Definition 4.8. A Banach space $X \subseteq M(\Sigma, \mu)$ with norm $||\cdot||_X$ is called normal if the norm $||\cdot||_X$ has the following property:

$$\left\{ \begin{array}{l}
\text{for every } u \in X, v \in M(\Sigma, \mu) \text{ satisfying } v \ll u, \\
\text{one has that } v \in X \quad \text{and} \quad ||v||_X \leq ||u||_X.
\end{array} \right.$$
Remark 4.9. It is important to point out that if $X$ is a normal Banach space, then for every $u \in X$, one always have that $u^+, u^-$ and $|u| \in X$. To see this, recall that by (1) Proposition 4.2, if $u \in X$, then $u^+ \ll u$ and $u^- \ll -u$. Thus, $u^+$ and $u^- \in X$ and since $|u| = u^+ + u^-$, one also has that $|u| \in X$.

The dual space $(L_0(\Sigma, \mu))^*$ of $L_0(\Sigma, \mu)$ is isometrically isomorphic to $L^{1\cap\infty}(\Sigma, \mu)$. Thus, a sequence $(u_n)_{n \geq 1}$ in $L_0(\Sigma, \mu)$ is said to be weakly convergent to $u$ in $L_0(\Sigma, \mu)$ if

$$\langle v, u_n \rangle := \int_\Sigma v u_n \, d\mu \to \int_\Sigma v u \, d\mu \quad \text{for every } v \in L^{1\cap\infty}(\Sigma, \mu).$$

For the rest of this paper, we write $\sigma(L_0, L^{1\cap\infty})$ to denote the weak topology on $L_0(\Sigma, \mu)$. For this weak topology, we have the following compactness result.

Proposition 4.10 ([8, Proposition 2.11]). Let $u \in L_0(\Sigma, \mu)$. Then, the following statements hold.

(1) The set $\left\{ v \in M(\Sigma, \mu) \mid v \ll u \right\}$ is $\sigma(L_0, L^{1\cap\infty})$-sequentially compact in $L_0(\Sigma, \mu)$;

(2) Let $X \subseteq M(\Sigma, \mu)$ be a normal Banach space satisfying $X \subseteq L_0(\Sigma, \mu)$ and

$$\text{(4.1)} \quad \text{for every } u \in X, \ (u_n)_{n \geq 1} \subseteq M(\Sigma, \mu) \text{ with } u_n \ll u \text{ for all } n \geq 1$$

$$\text{and } \lim_{n \to +\infty} u_n(x) = u(x) \mu\text{-a.e. on } \Sigma, \text{ yields } \lim_{n \to +\infty} u_n = u \text{ in } X.$$

Then for every $u \in X$ and sequence $(u_n)_{n \geq 1} \subseteq M(\Sigma, \mu)$ satisfying

$$u_n \ll u \text{ for all } n \geq 1 \quad \text{and} \quad \lim_{n \to +\infty} u_n = u \sigma(L_0, L^{1\cap\infty})\text{-weakly in } X,$$

one has that

$$\lim_{n \to +\infty} u_n = u \quad \text{in } X.$$

Note, examples of normal Banach spaces $X \subseteq L_0(\Sigma, \mu)$ satisfying (4.1) are $X = L^p(\Sigma, \mu)$ for $1 \leq p < \infty$ and $L_0(\Sigma, \mu)$.

To complete this section we state the following Proposition summarizing statements from [12, Proposition 2.9 & Proposition 2.10], which we will need in the sequel (cf [8] for the case $\omega = 0$).

Proposition 4.11. For $\omega \in \mathbb{R}$, let $A$ be $\omega$-quasi completely accretive in $L_0(\Sigma, \mu)$.

(1.) If there is a $\lambda_0 > 0$ such that $\text{Rg}(I + \lambda A)$ is dense in $L_0(\Sigma, \mu)$, then for the closure $\overline{A}$ of $A$ in $L_0(\Sigma, \mu)$ and every normal Banach space with $X \subseteq L_0(\Sigma, \mu)$, the restriction $\overline{A}_X := \overline{A} \cap (X \times X)$ of $A$ on $X$ is the unique $\omega$-quasi $m$-completely accretive extension of the part $A_X = A \cap (X \times X)$ of $A$ in $X$.

(2.) For a given normal Banach space $X \subseteq L_0(\Sigma, \mu)$, and $\omega \in \mathbb{R}$, suppose $A$ is $\omega$-quasi $m$-completely accretive in $X$, and $\{T_t\}_{t \geq 0}$ be the semigroup generated by $-A$ on $D(A)^\times$. Further, let $\{S_t\}_{t \geq 0}$ be the semigroup generated by $-\overline{A}$, where $\overline{A}$ denotes the closure of $A$ in $X^{1\cap\infty}$. Then, the following statements hold.


(a) The semigroup \( \{S_t\}_{t \geq 0} \) is \( \omega \)-quasi completely contractive on \( \overline{D(A)}^{1+\infty} \), \( T_t \) is the restriction of \( S_t \) on \( \overline{D(A)}^{\infty} \), \( S_t \) is the closure of \( T_t \) in \( L^{1+\infty}(\Sigma, \mu) \), and

\[
S_t u_0 = L^{1+\infty}(\Sigma, \mu) - \lim_{n \to +\infty} \left( I + \frac{t}{n} A \right)^{-n} u_0 \quad \text{for all } u_0 \in \overline{D(A)}^{1+\infty} \cap X;
\]

(b) If there exists \( u \in L^{1+\infty}(\Sigma, \mu) \) such that the orbit \( \{T_t u \mid t \geq 0\} \) is locally bounded on \( \mathbb{R}_+ \) with values in \( L^{1+\infty}(\Sigma, \mu) \), then, for every N-function \( \psi \), the semigroup \( \{T_t\}_{t \geq 0} \) can be extrapolated to a strongly continuous, order-preserving semigroup of \( \omega \)-quasi contractions on \( \overline{D(A)}^{\infty} \cap L^{1+\infty}(\Sigma, \mu) \) (respectively, on \( \overline{D(A)}^{\infty} \cap L^{1+\infty}(\Sigma, \mu) \)), and to an order-preserving semigroup of \( \omega \)-quasi contractions on \( \overline{D(A)}^{\infty} \cap L^{1+\infty}(\Sigma, \mu) \). We denote each extension of \( T_t \) on those spaces again by \( T_t \).

(c) The restriction \( A_X := \overline{A} \cap (X \times X) \) of \( A \) on \( X \) is the unique \( \omega \)-quasi m-complete extension of \( A \) in \( X \); that is, \( A = A_X \).

(d) The operator \( A \) is sequentially closed in \( X \times X \) equipped with the relative \( \sigma(L_0(\Sigma, \mu), L^{1+\infty}) \)-topology.

(e) The domain of \( A \) is characterized by

\[
D(A) = \left\{ u \in \overline{D(A)}^{1+\infty} \cap X \mid \exists v \in X \text{ s.t. } e^{-\omega t} \frac{S_t u - u}{t} \ll v \text{ for small } t > 0 \right\};
\]

(f) For every \( u \in D(A) \), one has that

\[
\lim_{t \to 0^+} \frac{S_t u - u}{t} = -A^2 u \quad \text{strongly in } L_0(\Sigma, \mu).
\]

4.2. The subclass of homogeneous operators of order zero. As mentioned in Section 3, the Banach spaces \( X_1 = L^1(\Sigma, \mu) \) and \( X_2 = L^\infty(\Sigma, \mu) \) don’t have, the Radon-Nikodým property. But for the class of quasi m-completely accretive operators \( A \) defined on a normal Banach space \( X \subseteq M(\Sigma, \mu) \), for semigroup \( \{T_t\}_{t \geq 0} \) generated by \( -A \), the time-derivative \( \frac{d}{dt} T_{tu} \) exists in \( X \) at every \( t > 0 \) for every \( u_0 \in \overline{D(A)}^{\infty} \). This fact follows from the following compactness result. Here, the partial ordering “\( \leq \)” is the standard one defined by \( u \leq v \) for \( u, v \in M(\Sigma, \mu) \) if \( u(x) \leq v(x) \) for \( \mu \)-a.e. \( x \in \Sigma \), and we use the symbol \( \to \) for indicating continuous embeddings.

Lemma 4.12. Let \( X \subseteq L_0(\Sigma, \mu) \) be a normal Banach space satisfying (4.1). For \( \omega \in \mathbb{R} \), let \( \{T_t\}_{t \geq 0} \) be a family of mappings \( T_t : C \to C \) defined on a subset \( C \subseteq X \) of \( \omega \)-quasi complete contractions satisfying (2.18) and \( T_0 = 0 \) for all \( t \geq 0 \). Then, for every \( u_0 \in C \) and \( t > 0 \), the set

\[
\left\{ \frac{T_{t+h} u_0 - T_t u_0}{h} \mid h \neq 0, t + h > 0 \right\}
\]

is \( \sigma(L_0, L^{1+\infty}) \)-weakly sequentially compact in \( L_0(\Sigma, \mu) \).

Proof. Let \( u_0 \in C \), \( t > 0 \), and \( h \neq 0 \) such that \( t + h > 0 \). Then by taking \( \lambda = 1 + \frac{h}{t} \) in (2.18), one sees that

\[
|T_{t+h} u_0 - T_t u_0| = |\lambda T_t \left[ \lambda^{-1} u_0 \right] - T_t u_0|
\]
\[
\leq \lambda \left| T_t \left[ \lambda^{-1} u_0 \right] - T_t u_0 \right| + |1 - \lambda| |T_t u_0|.
\]
Since \(T_t\) is an \(\omega\)-quasi complete contraction, by (3) of Proposition 4.2, and since \(T_t 0 = 0, (t \geq 1)\), one has that
\[
\lambda e^{-\omega t} \left| T_t \left[ \lambda^{-1} u_0 \right] - T_t u_0 \right| \ll |1 - \lambda| |u_0|
\]
and
\[
|1 - \lambda| e^{-\omega t} |T_t u_0| \ll |1 - \lambda| |u_0|.
\]
Since the set \(\{ w \mid w \ll |1 - \lambda| |u_0| \}\) is convex (see (6) of Proposition 4.2), we can conclude that
\[
\frac{1}{2} e^{-\omega t} |T_{t+h} u_0 - T_t u_0| \ll |1 - \lambda| |u_0| = \frac{|t|}{t} |u_0|
\]
and hence, by (3) of Proposition 4.2,
\[
\frac{|T_{t+h} u_0 - T_t u_0|}{|h|} \ll 2 e^{\omega t} \frac{|u_0|}{t}.
\]
Since for every \(u \in M(\Sigma, \mu)\), one always has that \(u^+ \ll |u|\), the transitivity of “\(\ll\)" (see (4) of Proposition 4.2) implies for
\[
f_h := \frac{T_{t+h} u_0 - T_t u_0}{|h|}, \quad \text{one has that} \quad f_h^+ \ll 2 e^{\omega t} \frac{|u_0|}{t}.
\]
Therefore, by (1) of Proposition 4.10, the two sets \(\{ f_h^+ \mid h \neq 0, t + h > 0 \}\) and \(\{|f_h| \mid h \neq 0, t + h > 0 \}\) are \(\sigma(L_0, L^{1,\infty})\)-weakly sequently compact in \(L_0(\Sigma, \mu)\), and since \(f_h^+ = |f_h| - f_h^+\) and \(f_h = f_h^+ - f_h^-\), we have thereby shown that the claim of this lemma holds.

With these preliminaries in mind, we can now state the regularization effect of the semigroup \(\{T_t\}_{t \geq 0}\) generated by a \(\omega\)-quasi \(m\)-completely accretive operator of homogeneous order zero.

**Theorem 4.13.** Let \(X \subseteq L_0(\Sigma, \mu)\) be a normal Banach space satisfying (4.1). For \(\omega \in \mathbb{R}\), let \(A\) be a \(\omega\)-quasi \(m\)-completely accretive in \(X\), and \(\{T_t\}_{t \geq 0}\) be the semigroup generated by \(-A\) on \(\overline{D(A)^\times}\). If \((0, 0) \in A\) and \(A\) is homogeneous of order zero, then for every \(u_0 \in \overline{D(A)^\times}\) and \(t > 0\), \(\frac{dT_t u_0}{dt}\) exists in \(X\) and
\[
|A^\circ T_t u_0| \leq 2 e^{\omega t} \frac{|u_0|}{t} \quad \text{\(\mu\)-a.e. on} \ \Sigma.
\]
In particular,
\[
\left\| \frac{dT_t u_0}{dt} \right\| \leq 2 e^{\omega t} \frac{|u_0|}{t} \quad \text{for every} \ t > 0,
\]
and
\[
\frac{dT_t u_0}{dt} \leq T_t u_0 \quad \text{\(\mu\)-a.e. on} \ \Sigma \text{for every} \ t > 0 \text{if} \ u_0 \geq 0,
\]
for every \(u_0 \in \overline{D(A)^\times}\) (then \(\|\cdot\|\) denotes the norm on \(X\)), respectively, for every \(u_0 \in \overline{D(A)^\times} \cap L^{1,\infty}(\Sigma, \mu)^\times\) (then \(\|\cdot\|\) is the \(L^p\)-norm) for every \(N\)-function \(\psi\) or for every \(1 \leq \psi \equiv p < \infty\), and for every \(u_0 \in \overline{D(A)^\times} \cap L^{\infty}(\Sigma, \mu)\) (where then \(\|\cdot\|\) is the \(L^{\infty}\)-norm).
Proof. Let \( u_0 \in \overline{D(A)}^\times, t > 0, \) and \( (h_n)_{n \geq 1} \subseteq \mathbb{R} \) be a zero sequence such that \( t + h_n > 0 \) for all \( n \geq 1 \). Due to Lemma 3.3, we can apply Lemma 4.12. Thus, there is a \( z \in L_0(\Sigma, \mu) \) and a subsequence \( (h_{k_n})_{n \geq 1} \) of \( (h_n)_{n \geq 1} \) such that

\[
\lim_{n \to +\infty} \frac{T_{t+h_{k_n}}u_0 - T_tu_0}{h_{k_n}} = z \quad \text{weakly in } L_0(\Sigma, \mu).
\]

Moreover, by (2e) of Proposition 4.11, one has that \( (T_tu_0, -z) \in A \). Thus (2f) of the same proposition 4.11 yields that \( z = -A^\circ T_tu_0 \) and

\[
\lim_{n \to 0} \frac{T_{t+h_{k_n}}u_0 - T_tu_0}{h_{k_n}} = -A^\circ u_0 \quad \text{strongly in } L_0(\Sigma, \mu).
\]

After possibly passing to another subsequence, we have that limit (4.10) holds also \( \mu \text{-a.e. on } \Sigma \). Since \( 2e^{-\omega t} \frac{|u_0|}{t} \in X \) and \( X \subseteq L_0(\Sigma, \mu), (2) \) of Proposition 4.10 implies that

\[
\lim_{h \to 0} \frac{T_{t+h}u_0 - T_tu_0}{h} \in X \quad \text{for every } \mu \text{-a.e. on } \Sigma.
\]

Thus and since by (4.5),

\[
\frac{|T_{t+h}u_0 - T_tu_0|}{|h_{k_n}|} \leq 2e^{-\omega t} \frac{|u_0|}{t} \quad \text{for all } n \geq 1,
\]

sending \( n \to +\infty \) in the last inequality, gives (4.6). In particular, by Corollary 2.4, one has that (4.7) holds for the norm \( \| \cdot \|_X \) on \( X \) and by Theorem 2.7 that (4.8) holds. Moreover, we have that \( -A^\circ T_tu_0 = \frac{dT_tu_0}{dt}, \mu \text{-a.e. on } \Sigma \) for every \( t > 0 \). Thus, sending \( h \to 0+ \) in (4.5) shows that (4.6) holds. Further, by the \( \mu \text{-a.e.-limit (4.11)}, \) applying Fatou’s lemma to (4.5) yields that (4.7) holds for the \( L^p \)-norm for every \( N \)-function \( \psi \) and the \( L^p \)-norm \( 1 \leq p < \infty \). Since (4.7) holds for all \( p < \infty \), sending \( 1 \leq p \to +\infty \) completes the proof of this theorem. \( \square \)

Remark 4.14 (Open problem). We emphasize that the crucial point in the previous proof is that due to the zero-order homogeneity of \( A \), the set (4.4) is \( \sigma(L_0, L^{1(\infty)}) \)-weakly sequently compact in \( L_0(\Sigma, \mu) \) and hence, for every \( t > 0, T_tu_0 \in D(\overline{A}) \) and

\[
\frac{dT_tu_0}{dt} = \lim_{h \to 0+} \frac{T_{t+h}u_0 - T_tu_0}{h} = -A^\circ T_tu_0 \quad \text{exists in } X.
\]

We believe that this remains true if the infinitesimal generator of the semigroup \( \{T_t\}_{t \geq 0} \) is of the form \( A + F \) where \( A \) is homogeneous of order zero and \( F \) is Lipschitz-continuous. But so far, we are not able to show this result.

As a final result of this section, we state the following decay estimates for semigroups generated by the perturbed operator \( A + F \). Here, we write \( L_{\infty}^{\infty} \) for the intersection space \( L_{\infty}^{\infty}(\Sigma, \mu) \rightarrow L^\infty(\Sigma, \mu) \).

**Theorem 4.15.** Let \( F : M(\Sigma, \mu) \rightarrow M(\Sigma, \mu) \) be a mapping such that for every \( N \)-function \( \psi \) and for \( \psi \equiv 1 \) and \( \psi \equiv +\infty \), the restriction \( F_{\psi} : L^\psi(\Sigma, \mu) \rightarrow L^\psi(\Sigma, \mu) \) is Lipschitz continuous with constant Lipschitz \( \omega > 0 \) and \( F(0) = 0 \). Let \( A \) be an \( m \)-completely accretive operator on normal Banach space \( X \subseteq L_0(\Sigma, \mu) \), and \( \{T_t\}_{t \geq 0} \) the
semigroup generated by \(- (A + F)\) on \(\overline{D(A)}^X\). If \((0, 0) \in A\) and \(A\) is homogeneous of order zero, then

\[
\left\| \frac{dT_t u_0}{dt} \right\|_\psi \leq \left[ 2 e^{\omega t} \int_0^t e^{-\omega s} ds + \omega \int_0^t e^{\omega t} \int_s^t e^{-\omega r} dr ds \right] \frac{e^{\omega t} \| u_0 \|_\psi}{t}
\]

for every \(t > 0\), and every \(u_0 \in \overline{D(A)}^X \cap L^{1,\infty}(\Sigma, \mu)^{\psi}\) for every \(N\)-function \(\psi\), and every \(1 \leq \psi \equiv p < \infty\), and for every \(u_0 \in \overline{D(A)}^X \cap L^{q,\infty}(\Sigma, \mu)\) for \(q = \infty\) (where then \(\| \cdot \|_\psi\) is the \(L^\infty\)-norm).

**Proof.** Since \((0, 0) \in (A + F), T_t 0 = 0\) for all \(t \geq 0\). Thus, \(u \equiv 0 \in L^{1,\infty}(\Sigma, \mu)\) such that \(\{ T_t u | t \geq 0 \}\) is locally bounded in \(\text{IR}_+\). Thus, by Proposition 4.11, for every \(N\)-function \(\psi\) (respectively, for \(\psi \equiv 1\) and \(\psi \equiv \infty\)) each \(T_t\) admits a unique extension (which we denote again by \(T_t\)) of an \(\omega\)-quasi contractions on \(\overline{D(A)}^X \cap L^{1,\infty}(\Sigma, \mu)^{\psi}\) with respect to the \(L^\psi\)-norm. In addition, the family \(\{ T_t \}_{t \geq 0}\) remains a semigroup satisfying (2.28) and in relation with (2.27), \(\{ T_t \}_{t \geq 0}\) satisfies (2.2) and (2.3) for \(f\) given by (2.26). Further, for \(1 \leq \psi \equiv q < \infty\), \(L^q(\Sigma, \mu)\) and its dual space \(L^{q'}(\Sigma, \mu)\) are uniformly convex. Therefore, by Corollary 2.12 and Proposition 3.6, for every \(u_0 \in \overline{D(A)}^X \cap L^{1,\infty}(\Sigma, \mu)^{\psi}\) for every \(t > 0\), \(\frac{dT_t u_0}{dt} \) exists in \(L^q(\Sigma, \mu)\),

\[
\frac{dT_t u_0}{dt} = \lim_{h \to 0^+} \frac{T_{t+h} u_0 - T_t u_0}{h}
\]

and (4.12) holds. Moreover, by Corollary 2.12, one has that (2.29) holds for the \(L^\psi\)-norm and every \(u_0 \in \overline{D(A)}^X \cap L^{1,\infty}(\Sigma, \mu)^{\psi}\) and every \(N\)-function \(\psi\), respectively for the \(L^1\)-norm and every \(u_0 \in \overline{D(A)}^X \cap L^{1,\infty}(\Sigma, \mu)^{\psi}\). Thus and by (4.13), sending \(h \to 0^+\) in (2.29) one obtains that (4.12) holds for all \(N\)-function \(\psi\) and \(q = 1\).

Next, let \(u_0 \in \overline{D(A)}^X \cap L^{q,\infty}(\Sigma, \mu)\) for some \(1 \leq q_0 < +\infty\) and \(t > 0\). We assume \(\| \frac{dT_t u_0}{dt} \|_\infty > 0\) (otherwise, there is nothing to show). Then, for every \(s \in (0, \| \frac{dT_t u_0}{dt} \|_\infty)\) and every \(q_0 \leq q < \infty\), Chebyshev’s inequality yields

\[
\mu \left( \left\{ \left| \frac{dT_t u_0}{dt} \right| \geq s \right\} \right)^{1/q} \leq \frac{\| \frac{dT_t u_0}{dt} \|_q}{s}
\]

and so, by (4.12),

\[
s \mu \left( \left\{ \left| \frac{dT_t u_0}{dt} \right| \geq s \right\} \right)^{1/q} \leq \left[ 2 e^{\omega t} \int_0^t e^{-\omega s} ds + \omega \int_0^t e^{\omega t} \int_s^t e^{-\omega r} dr ds \right] \frac{e^{\omega t} \| u_0 \|_q}{t}.
\]

Thus and since \(\lim_{q \to \infty} \| u_0 \|_q = \| u_0 \|_\infty\), sending \(q \to +\infty\) in the last inequality, yields

\[
s \leq \left[ 2 e^{\omega t} \int_0^t e^{-\omega s} ds + \omega \int_0^t e^{\omega t} \int_s^t e^{-\omega r} dr ds \right] \frac{e^{\omega t} \| u_0 \|_\infty}{t}
\]

and since \(s \in (0, \| \frac{dT_t u_0}{dt} \|_\infty)\) was arbitrary, we have thereby shown that (4.12) also holds for \(q = \infty\). 

\(\square\)
5. Application

Throughout this section, let $\Sigma$ be an open set of $\mathbb{R}^d$ and the Lebesgue space $L^1(\Sigma)$ is equipped with the classical Lebesgue measure. Suppose $f : \Sigma \times \mathbb{R} \to \mathbb{R}$ is a Lipschitz-continuous Carathéodory function, that is, $f$ satisfies the following three properties:

\begin{align}
(5.1) & \quad f(\cdot, u) : \Sigma \to \mathbb{R} \text{ is measurable on } \Sigma \text{ for every } u \in \mathbb{R}, \\
(5.2) & \quad f(x, 0) = 0 \text{ for a.e. } x \in \Sigma, \text{ and} \\
(5.3) & \quad |f(x, u) - f(x, \hat{u})| \leq \omega |u - \hat{u}| \quad \text{for all } u, \hat{u} \in \mathbb{R}, \text{ a.e. } x \in \Sigma.
\end{align}

Then, for every $1 \leq q < \infty$, $F : L^q(\Sigma) \to L^q(\Sigma)$ defined by

$$
F(u)(x) := f(x, u(x)) \quad \text{for every } u \in L^q(\Sigma)
$$

is the associated Nemytskii operator on $L^q(\Sigma)$. Moreover, by (5.3), $F$ is globally Lipschitz continuous on $L^q(\Sigma)$ with constant $\omega > 0$ and $F(0)(x) = 0$ for a.e. $x \in \Sigma$.

5.1. Decay estimates of the total variational flow. In this subsection, we consider the perturbed total variational flow operator (1-Laplace operator) given by

$$
\Delta_1 u := -\Delta u + f(x, u) \quad \text{with } \Delta_1 u = \text{div} \left( \frac{Du}{|Du|} \right),
$$

equipped with either Neumann boundary conditions on a bounded domain $\Sigma$ in $\mathbb{R}^d$, $d \geq 1$.

Here, we use the following notation. A function $u \in L^1(\Sigma)$ is said to be a function of bounded variation in $\Sigma$, if the distributional partial derivatives $D_1 u := \frac{\partial u}{\partial x_1}, \ldots, D_d u := \frac{\partial u}{\partial x_d}$ are finite Radon measures in $\Sigma$, that is, if

$$
\int_{\Omega} u D_i \varphi \, dx = - \int_{\Omega} \varphi \, dD_i u
$$

for all $\varphi \in C^\infty_c(\Omega)$, $i = 1, \ldots, d$. The linear vector space of functions $u \in L^1(\Sigma)$ of bounded variation in $\Sigma$ is denoted by $BV(\Sigma)$. Further, we set $Du = (D_1 u, \ldots, D_d u)$ for the distributional gradient of $u$. Then, $Du$ belongs to the class $M^b(\Omega, \mathbb{R}^d)$ of $\mathbb{R}^d$-valued bounded Radon measure on $\Omega$, and we either write $|Du|(\Sigma)$ or $\int_{\Sigma} |Du|$ to denote the total variation measure of $Du$. The space $BV(\Sigma)$ equipped with the norm $\|u\|_{BV(\Sigma)} := \|u\|_{L^1(\Sigma)} + |Du|(\Sigma)$ forms a Banach space. Further, let

$$
X_1(\Sigma) = \left\{ z \in L^\infty(\Sigma, \mathbb{R}^d) \mid \text{div}(z) := \sum_{i=1}^d D_i z \in L^1(\Sigma) \right\},
$$

$\text{sign}_0(s), (s \in \mathbb{R})$, is the classical sign function with the additional property that $\text{sign}_0(0) = 0$, and for every $k > 0$, $T_k(s) := [k - |k - |s||^\top \text{sign}_0(s), (s \in \mathbb{R})$.

The Neumann total variational flow operator. In [2] (see also [3]), the negative total variational flow operator (1-Laplace operator) $-\Delta_1^N$ in $L^1(\Sigma)$ equipped
with Neumann boundary conditions was introduced by

\[-\Delta^N_1 = \left\{ (u, v) \in L^1(\Sigma) \times L^1(\Sigma) \mid T_k(u) \in BV(\Sigma) \forall k > 0 \& \exists z \in X_1(\Sigma) \right. \]

such that \( \|z\|_\infty \leq 1 \& (5.4) \) holds

where

\[
\begin{align*}
\left\{ v = -\text{div}(z) \quad & \text{in } D'(\Sigma), \quad \text{and} \\
\int_{\Sigma} (\xi - T_k(u)) v \, dx & \leq \int_{\Sigma} z \cdot D\xi \, dx - \int_{\Sigma} |D\xi| \mid
\end{align*}
\]

(5.4)

for every \( \xi \in W^{1,1}(\Sigma) \cap L^\infty \) and all \( k > 0 \). Moreover, the negative Neumann 1-Laplace operator \(-\Delta^N_1\) is \( m \)-completely accretive in \( L^1(\Sigma) \) with dense domain. Therefore, under the hypotheses (5.1)–(5.3), the operator \(-\Delta^N_1 + F\) is \( \omega \)-quasi \( m \)-completely accretive on \( L^1(\Sigma) \) (cf [12]). Now, it is not difficult to see that \(-\Delta^N_1\) is homogeneous of order zero and \( 0 \in -\Delta^N_1 0 \). Thus, by Theorem 4.13 and Theorem 4.15, we can state the following regularity result.

**Corollary 5.1.** For every \( 1 \leq q < \infty \) and \( u_0 \in L^q(\Sigma) \) (respectively \( u_0 \in L^{1,\infty}(\Sigma) \) if \( q = \infty \)), the unique mild solution \( u \) of problem

\[
\begin{align*}
\frac{du}{dt} - \text{div} \left( \frac{Du}{|Du|} \right) + f(x, u) & = 0 \quad \text{on } \Sigma \times (0, +\infty), \\
D_t u & = 0 \quad \text{on } \partial\Sigma \times (0, +\infty), \\
u(0) & = u_0 \quad \text{on } \Sigma \times \{t = 0\},
\end{align*}
\]

(5.5)

is a strong solution satisfying (4.12). Moreover, if \( f \equiv 0 \), then either for every \( 1 \leq \psi \equiv p \leq \infty \) or \( N \)-function \( \psi \) and every \( u_0 \in L^\psi(\Sigma) \), the unique mild solution \( u \) of problem (5.5) satisfies (4.7) and (4.8).

**Remark 5.2 (The Dirichlet boundary case).** In [1] (cf [3]), existence and uniqueness of the the parabolic initial boundary-value problem

\[
\begin{align*}
\frac{du}{dt} - \text{div} \left( \frac{Du}{|Du|} \right) & = 0 \quad \text{on } \Sigma \times (0, +\infty), \\
u(0) & = \varphi \quad \text{on } \partial\Sigma \times (0, +\infty), \\
u(0) & = u_0 \quad \text{on } \Sigma \times \{t = 0\},
\end{align*}
\]

(5.6)

associated with the total variational flow equipped with (inhomogeneous) Dirichlet boundary conditions was established. For every boundary term \( \varphi \in L^1(\Sigma) \), the negative Dirichlet total variational flow operator (1-Laplace operator) \( \Delta^D_1 u := \text{div} \left( \frac{Du}{|Du|} \right) \) is \( m \)-completely accretive in \( L^1(\Sigma) \). But only in the homogeneous case \( \varphi \equiv 0 \), the operator \( \Delta^D_1 \) is homogeneous of order zero. Thus, the same statement as given in Corollary 5.1 holds in the Dirichlet case with \( \varphi \equiv 0 \).

### 5.2. Decay estimates of the nonlocal total variational Flow.

In this very last section, we consider for \( 0 < s < 1 \), the perturbed fractional 1-Laplace operator

\[
Au := \text{PV} \int_{\Sigma} \frac{(u(y) - u(x))}{|u(y) - u(x)|} \frac{dy}{|x - y|^{d+s}} + f(x, u)
\]
equipped with either Dirichlet on a domain $\Sigma$ in $\mathbb{R}^d$ or with or vanishing conditions if $\Sigma = \mathbb{R}^d, d \geq 1$.

For $0 < s < 1$, let $W_0^{s,1}(\Sigma)$ be the Banach space given by

$$W_0^{s,1}(\Sigma) = \{ u \in L^1(\Sigma) \mid [u]_{s,1} < \infty \text{ and } u = 0 \text{ a.e. on } \mathbb{R}^d \setminus \Sigma \}$$

equipped with the norm $\| \cdot \|_{W_0^{s,1}} := \| \cdot \| + [\cdot]_{s,1}$, where

$$[u]_{s,1} := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|}{|x - y|^{d+s}} dy \, dx \quad \text{for every } u \in W_0^{s,1}(\Sigma).$$

Further, let $B_{L^{\infty}}$ denote the closed unit ball of all anti-symmetric $\eta \in L^{\infty}(\mathbb{R}^d \times \mathbb{R}^d)$, that is,

$$\eta(x,y) = -\eta(y,x) \quad \text{for a.e. } (x,y) \in \mathbb{R}^d \times \mathbb{R}^d \quad \text{and } \| \eta \|_{\infty} \leq 1.$$ Then, it was shown in [16, Section 3.] that the fractional Dirichlet 1-Laplace operator $(-\Delta_1^D)^s$ in $L^2(\Sigma)$ can be realized by (the graph)

$$(-\Delta_1^D)^s = \left\{ (u,v) \in L^2(\Sigma) \times L^2(\Sigma) \mid u \in W_0^{s,1}(\Sigma) \& \exists \eta \in B_{L^{\infty}} \text{ s.t. (5.7) holds} \right\}$$

where

$$\eta(x,y) \in \text{sign}(u(x) - u(y)) \quad \text{for a.e. } (x,y) \in \mathbb{R}^d \times \mathbb{R}^d$$

and

$$\frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\eta(x,y)(\zeta(x) - \zeta(y))}{|y - x|^{d+s}} dy \, dx = \int_\Sigma v(x) \zeta(x) \, dx$$

for all $\xi \in W_0^{s,1}(\Sigma) \cap L^2(\Sigma)$, and $(-\Delta_1^D)^s$ is $m$-completely accretive in $L^2(\Sigma)$ with dense domain $D((-\Delta_1^D)^s)$ in $L^2(\Sigma)$. One immediately sees that $(-\Delta_1^D)^s$ is homogeneous of order zero and $0 \in (-\Delta_1^D)^s 0$. Moreover, under the hypotheses (5.1)-(5.3), the operator $(-\Delta_1^D)^s + F$ is $\omega$-quasi $m$-completely accretive on $L^2(\Sigma)$. Thus, by Theorem 4.15, we have the following regularity result.

Corollary 5.3. For every $1 \leq q < \infty$ and $u_0 \in L^q(\Sigma)$ (respectively $u_0 \in L^{1,\infty}(\Sigma)$ if $q = \infty$), the unique mild solution $u$ of problem

$$\begin{cases}
\frac{du}{dt} + PV \int_\Sigma \frac{u(x) - u(y)}{|u(y) - u(x)|} \, dy = 0 & \text{on } \Sigma \times (0, +\infty), \\
[du]{\partial} = f(x,u) = 0 & \text{on } \partial \Sigma \times (0, +\infty), \\
u(0) = u_0 & \text{on } \Sigma \times \{ t = 0 \},
\end{cases}$$

is a strong solution satisfying (4.12). Moreover, if $f \equiv 0$, then either for every $1 \leq q \equiv p < \infty$ or N-function $\psi$ and every $u_0 \in L^q(\Sigma)$, the unique mild solution $u$ of problem (5.5) satisfies (4.7) and (4.8).

Further, since for every $0 < s < 1$ and $d \geq 1$, the following (fractional) Sobolev inequality (cf [15, Theorem 14.29])

$$\| u \|_{s,1} \leq C \| u \|_{s,1} \quad \text{for every } u \in W_0^{s,1}(\Sigma),$$
holds for a constant $C = C(d,s) > 0$, we have that the fractional Dirichlet 1-Laplace operator $A = (-\Delta^D)^s$ satisfies the following abstract Sobolev inequality
\[
\|u\|_{s}^r \leq C \|u\|_{\Sigma_2} \quad \text{for every } (u,v) \in A
\]
with parameters $r = \frac{d}{d+s} > 1$ and $\sigma = 1$, where $\|\cdot\|_{\Sigma_2}$ denote the $L^2$-inner product. Thus, by [12, Theorem 1.2], the semigroup $\{T_t\}_{t \geq 0}$ generated by $\{-(\Delta^D)^s + F\}$ satisfies the following $L^2$-$L^\infty$-regularity estimate
\[
\|T_t u\|_{L^\infty} \leq \tilde{C} t^{-\frac{d-s}{2}} e^{\left(\frac{d}{2(q+1)}+1\right)\omega t} \|u_0\|_{L^\infty} \quad \text{for every } t > 0
\]
and $u_0 \in L^2(\Sigma)$. Furthermore, for every $q > \frac{d-s}{s}$, one has
\[
\|T_t u\|_{L^\infty} \leq \tilde{C} t^{-\frac{d-s}{2}} e^{\left(\frac{d}{2(q+1)}+1\right)\omega t} \|u_0\|_{L^\infty} \quad \text{for every } t > 0
\]
and $u_0 \in L^{\frac{d}{q}}(\Sigma)$. Thus, by Corollary 2.6 and inequality (4.12) for $q = \infty$, we also have the following estimate.

**Corollary 5.4.** Let $d \geq 1$, $0 < s < 1$ and $q > \frac{d-s}{s}$. Then for every $u_0 \in L^{\frac{d}{q}}(\Sigma)$, the unique solution $u$ of (5.8) satisfies
\[
\left\|\frac{du}{dt}\right\|_{L^\infty} \leq \tilde{C} C_\omega(t/2) 2^{\frac{d-s}{2(q+1)}+1} \omega^t \left(\frac{d}{2(q+1)} \frac{d}{q}+1\right) t \|u_0\|_{L^{\frac{d}{q}}} \quad \text{for every } t > 0,
\]
where $C_\omega(t)$ is the constant in (4.12).

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