Classical Gravitation as free Membrane Dynamics

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I. INTRODUCTION

General Relativity is commonly regarded as the correct approach to non-quantum gravitation. Einstein’s theory views gravity as a manifestation of the curvature of the 4-D space-time. Several authors have proposed to consider this physical curved 4-D space-time as a membrane embedded in a flat space-time of higher dimension called the host space. This point of view is computationally convenient and also extremely natural in the context of modern string and brane theory. The aim of the present article is to complement the existing literature on this topic. Our main conclusion is that the embedding approach to GR can be successfully implemented in a large variety of contexts and provides some undeniable computational and conceptual advantages. Here follows a summary of our principal results.

We first introduce two new classes of embeddings (modeled after Nash’s classical free embeddings) and explain why these two classes are particularly natural from the physical point of view. Although they typically require host spaces of higher dimensions than most embeddings proposed by various authors, these new classes of embeddings present the important physical advantage of being deformable, and therefore physically more realistic. In particular, given an arbitrary space-time, any embedding of this space-time which belongs to one of the two new classes can be deformed to obtain an embedding for gravitational waves propagating in this space-time.

We then give explicit examples of embeddings in both classes for the standard Minkowski space-time, the Schwarzschild black hole and gravitational waves propagating in flat space-time. We then propose new variational principles which give back Einstein’s General Relativity by viewing the 4-D space-time as a membrane moving in a flat host space. Some of the variational principles involve new border terms previously not considered by previous authors. Actually, the issue of constructing actions which deliver the equations of standard General Relativity in terms of embedding functions has been often addressed in the literature. Our work is the first to propose a solution to this long standing problem. We finally show that the embedding point of view permits a particularly simple and physically enlightening treatment of the initial value problem in relativistic gravitation.

II. FREE EMBEDDINGS

A. Generalities about embeddings

1. What is an embedding?

We denote the physical 4-D space-time by $\mathcal{M}$ and its Lorentzian, possibly curved metric by $g$. Space-time indices running from 0 (1) to 3 will be indicated by Greek (Latin) letters and the metric signature will be $(+,−,−,−)$. The covariant derivative for tensor fields defined on $\mathcal{M}$ is, as usual, the derivative operator associated with the Levi-Civita connection of the metric $g$. We also consider a ‘host’-space $\mathcal{E}_N$ i.e. an $N$-dimensional Lorentzian flat space with metric $\eta$ and choose a system of $N$ coordinates $Y^A$, $A = 0, \ldots, N−1$, in the host-space $\mathcal{E}_N$.

To view the physical 4-D space-time as embedded in the host-space is tantamount to saying that an arbitrary point $P$ in $\mathcal{M}$ can be considered as a point of $\mathcal{E}_N$ as well. We thus define an embedding by a set of $N$ functions $y^A(P)$, $A = 0,\ldots,N−1$, which represent the $Y^A$
coordinates of the space-time point $P$. Note that these functions are scalars with respect to coordinate changes on the space-time $\mathcal{M}$. Let us now choose a system of four coordinates $x^\mu$, $\mu = 0, 1, 2, 3$ on the physical space-time $\mathcal{M}$. The squared line element $ds^2$ between two infinitesimal points of $\mathcal{M}$ reads, with obvious notations:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu; \quad (1)$$

but the same squared line element can also be evaluated by viewing both points as belonging to the host-space; this leads to

$$ds^2 = \eta_{AB} dy^A dy^B \quad (2)$$
or

$$ds^2 = \eta_{AB} y^A_\mu y^B_{\nu} dx^\mu dx^\nu, \quad (3)$$

where $y^A_\mu$ denotes the partial differentiation of $y^A$ with respect to $x^\mu$. This partial derivative actually coincides with the covariant derivative $y^A_\mu$ of $y^A$ with respect to $x^\mu$ because, as noted earlier, the function $y^A$ is a scalar respect to coordinate changes on $\mathcal{M}$. Equating (1) and (3) delivers the important relation:

$$g_{\mu\nu} = \eta_{AB} y^A_\mu y^B_{\nu}$$

$$\equiv y^\mu_\mu \cdot y^\nu_\nu, \quad (4)$$

which is manifestly covariant with respect to coordinate changes on $\mathcal{M}$.

2. Existence of embeddings

It is a well known result that a given Lorentzian (or Riemannian) metric manifold can be embedded into a flat host space of higher dimension. Constructive and existence theorems in the local [23, 24] as well as in the global sense [17, 25, 26] give conditions on the minimal dimension of the host space, for closed and open manifolds (see also [27], and the references in the review [11]). The minimal dimension of the host-space needed to embed locally a generic 4-dimensional space-time is $N = 10$. Usually less dimensions are needed for vacuum space-times [18, 19].

It has however been argued heuristically by Deser et al. [4] that embeddings cannot a priori be used with profit by physicists. This conclusion essentially rests on an intuition gained from studying the so-called trivial embedding of 4-D Minkovski space-time into itself, which cannot be deformed to accomodate standard gravitational waves. The way out of this possible problem is conceptually extremely simple. It consists in working only with particular embeddings which do admit deformations. This is where the notion of freeness [17, 27] enters the picture.

B. Free embeddings

1. Definitions

Put simply, free, $q$-free and spatially free embeddings are three particular classes of embeddings which share the common property of being by definition deformable to accommodate linear variations of the metric tensor. Let us now present the technicalities which motivate the three definitions we are about to give.

Consider a given embedding of the form (4) and let $\delta y$ be an arbitrary perturbation or deformation of this embedding. We assume that the vectors $y_\mu$, $\mu = 0, \ldots, 3$ of the host space are linearly independent. Note that this condition is necessary for the metric to be invertible, since from eq.(4), the linear dependence of these vectors would imply directly the existence of a nonzero eigenvector of the metric matrix $g_{\mu\nu}$ with zero eigenvalue.

Varying (4), we obtain at first order in $\delta y$:

$$\delta g_{\mu\nu} = 2(y_\mu \cdot \delta y_\nu) - 2 y_{\mu\nu} \cdot \delta y. \quad (5)$$

The embedding variation is made up of two contributions, one tangent to the 4-D space-time and the other one normal to it; we thus write

$$\delta y = \delta W^\mu y_\mu \oplus \delta y_\perp. \quad (6)$$

Equation (5) then becomes:

$$\delta g_{\mu\nu} = 2 \delta W_{(\mu;\nu)} - 2 y_{\mu\nu} \cdot \delta y_\perp. \quad (7)$$

As (5) is manifestly covariant with respect to coordinate changes on $\mathcal{M}$. The vectors $y_{\mu\nu}$, $\mu, \nu = 0, \ldots, 3$ of the host space define the second fundamental form and are normal to the 4-D embedded space-time: $y_{\mu\nu} \cdot y_\alpha = 0$ (see appendix A).

Now, the embedding $y$ will be useful physically if it can be deformed to accommodate an arbitrary perturbation of the metric $\delta g$. This means that the physically useful embeddings are those for which equation (7) can be solved in $\delta W_{\mu\nu}$ and $\delta y_\perp$ for any arbitrarily given $\delta g_{\mu\nu}$. Let us now introduce the definitions of free, $q$-free and spatially $q$-free embeddings.

**Definition 1: Free embeddings.** An embedding (4) is said to be free if, for any metric perturbation $\delta g_{\mu\nu}$ and any choice of tangential variation $\delta W_{\mu\nu}$, the 10 equations (7) can be solved in the normal variations $\delta y_\perp$.

This definition originated with Nash’ work on embeddings of Riemannian manifold and is now standard. Nash actually chose to work with vanishing tangent variations and the normal variations can then be obtained by algebraic methods only (see [27] for a modern discussion). Locally, the dimension of the host space of Nash’ free embeddings must be $N \geq 14$.

The material presented in the following sections (in particular, the various action principles discussed in Section III) makes it natural to introduce two other classes of embeddings, the $q$-free and spatially free embeddings.
These classes have never been considered by earlier authors and we now give their definitions: 

**Definition 2: q-free embeddings.** An embedding is q-free if, for an arbitrary metric perturbation, the 10 equations are equivalent to:

(i) \( q \geq 6 \) linearly independent linear combinations of which can be solved in the normal variations, no matter what the tangential variations are, and

(ii) \( 10 - q \) remaining equations that can be solved in the tangential variations, independently of the solution obtained in (i) for the normal variations.

Notice that \( 10 \geq q \geq 6 \), because there are 4 independent tangential variations and thus the number of equations in (ii) must lie between 0 and 4. Note also that, by definition, any free embedding is automatically a 10-free embedding.

The host-space dimension for a q-free embedding must be \( N \geq q + 4 \) in order to accommodate the linearly independent 4-dimensional tangent space and the q-dimensional subspace of the normal space which solve the equations in (i). Since \( q \geq 6 \), we have \( N \geq 10 \).

**Definition 3: Spatially free embeddings.** An embedding is spatially free if there exist a coordinate system \((x^0, x^1)\) on the 4-D space-time in which the 6 vector fields \( y_{ij}, \ i, j = 1, 2, 3 \), are linearly independent.

Spatially free embeddings are particularly important when working in the so-called 3+1 formalism. They form a subclass of the q-free embeddings. Indeed, let \( y \) be a spatially free embedding and let \((x^0, x^1)\) be the coordinate system in which the 6 vector fields \( y_{ij}, \ i, j = 1, 2, 3 \), are linearly independent. In this coordinate system, the \((i, j)\) components of \( y \) can be solved for \( \delta y \). The remaining components of \( y \) are the \((0, \mu)\) components; taken together, they constitute a system of four inhomogeneous first order differential equations for the fields \( \delta W_{\mu} \), that can be solved by integration along \( x^0 \). Thus, any spatially free embedding is at least 6-free, hence q-free.

The three above definitions are relevant for physics if at least some general relativistic space-times admit embeddings which are either free, q-free or spatially free. We prove in the next section that this is so by constructing explicit examples of free, q-free and spatially free embeddings for several space-times of physical interest, including a Schwarzschild black hole. Whether all general relativistic space-times admit embeddings in at least one of these three classes remains however an open problem.

### C. Examples

We now give explicit examples of embeddings of physically relevant space-times belonging to the classes defined in the last section.

They represent thus the first examples in the literature of embeddings whose deformations can, by construction, be properly mapped to metric deformations, so that, for example, gravitational waves can be described as embedding waves.

Let the space-time coordinates be denoted by \((x^0, x^j)\), where \(\{x^j, j = 1, 2, 3\}\) are ‘spatial’ coordinates.

We have developed a very simple method to construct q-free embeddings for an interesting class of space-times, including the flat and Schwarzschild 4-dimensional space-times. The method consists in splitting the host space as a direct sum of two flat subspaces, (i) the ‘base’ host space, with flat Euclidean metric diag\((-1, \ldots, -1)\), and (ii) the ‘extra’ host space, with flat Lorentzian metric diag\((1, -1, \ldots, -1)\) (one and only one time-like direction). We will describe the contribution of each subspace to the total embedding in two steps, followed by the explicit examples.

**1. First Step: The base embedding**

On the base host space, we define a ‘base embedding’ \(Z(x^j)\), depending only on the spatial coordinates, with the following properties:

- **P1.** The metric induced by \(Z\) is flat: \(Z_{ij} \cdot Z_{jk} = -\delta_{jk}\)

- **P2.** The 9 vectors \(\{Z_{ij}, Z_{ik}\}\) are linearly independent for all \(x^j \in B\), where \(B\) is a given 3-dimensional subset of \(\mathbb{R}^3\).

Among the several types of base embeddings that can be constructed we have chosen the following 11-dimensional one, for its simplicity:

\[
Z(x^j) = \begin{pmatrix}
 f_1(\xi_1 x^1) \\
 f_2(\xi_2 x^2) \\
 f_3(\xi_3 x^3) \\
 \cos(g_1(\xi_1 x^1)) \cos(g_2(\xi_2 x^2)) \cos(g_3(\xi_3 x^3)) \\
 \cos(g_1(\xi_1 x^1)) \cos(g_2(\xi_2 x^2)) \sin(g_3(\xi_3 x^3)) \\
 \cos(g_1(\xi_1 x^1)) \sin(g_2(\xi_2 x^2)) \cos(g_3(\xi_3 x^3)) \\
 \cos(g_1(\xi_1 x^1)) \sin(g_2(\xi_2 x^2)) \sin(g_3(\xi_3 x^3)) \\
 \sin(g_1(\xi_1 x^1)) \cos(g_2(\xi_2 x^2)) \cos(g_3(\xi_3 x^3)) \\
 \sin(g_1(\xi_1 x^1)) \cos(g_2(\xi_2 x^2)) \sin(g_3(\xi_3 x^3)) \\
 \sin(g_1(\xi_1 x^1)) \sin(g_2(\xi_2 x^2)) \cos(g_3(\xi_3 x^3)) \\
 \sin(g_1(\xi_1 x^1)) \sin(g_2(\xi_2 x^2)) \sin(g_3(\xi_3 x^3))
\end{pmatrix},
\]

where the functions \(f_j, g_j\) and the parameters \(\xi_j\) are chosen so that \(Z\) satisfies properties P1 and P2. Property P1 is ensured if there exist real functions \(u_j(s)\) such that:

\[
f_j'(s) + i g_j'(s) = \xi_j^{-1} e^{i u_j(s)}, \quad j = 1, 2, 3,
\]

where \(i^2 = -1\) and prime denotes derivative with respect to \(s\). Property P2 is ensured if the real functions \(u_j(s)\) satisfy:

\[
0 < u_j(s) < \pi, \quad j = 1, 2, 3
\]

\[
\Delta \equiv (v_1 v_2 v_3)^2 + (v_1 v_2 v_3)^2 + (v_1 v_2 v_3)^2 + (v_1 v_2 v_3)^2 > 0
\]
where \( v_j \equiv \xi_j u_j' (\xi_j x^j) \) and \( w_j \equiv \sin^2 u_j (\xi_j x^j) \), for \( j \) fixed. A particular solution of the previous inequalities is
\[
 u_1(s) = \frac{\pi}{4}, \quad u_2(s) = u_3(s) = \frac{1}{4} (\pi + \arctan s).
\]

With this solution, \( Z \) is a base embedding on \( B = \mathbb{R}^3 \setminus \{ x^1 \in \mathbb{R}, x^2 = \pm \infty, x^3 = \pm \infty \} \) and this embedding is not self-intersecting (i.e., for any two sets of 3 spatial coordinates \( p \) and \( q \), \( Z(p) = Z(q) \iff p = q \)).

2. **Second step: the extra embedding functions**

It is easy to show that, by adding extra embedding functions to this base embedding, the general embedding \( y(x^0, x^j) = Y(x^0, x^j) \otimes Z(x^j) \) satisfies automatically the following properties:

P1'. The metric induced by \( y \) decomposes as:
\[
 ds^2 = Y_{\mu \nu} d^\mu dx^\nu - \delta_{jk} dx^j dx^k \quad (10)
\]

P2'. The embedding \( y(x^\mu) \) is spatially free for \( x^j \in B \).

Appropriate choices of these extra embedding functions \( Y \) generate spatially free embeddings of the flat Minkovski space-time and of the Schwarzschild black hole.

3. **First example: Flat 4-D space-time embedded into flat (1+11)-D host space**

Here, to accommodate the extra embedding, only one extra dimension (necessarily time-like) is needed. The extra embedding function is \( Y_{\text{Flat}}(x^\mu) = (x^0) \). The \((1+11)\)-dimensional host space metric is \( \eta_{AB} = \text{diag}(1,-1,\ldots,-1) \), and it follows from eq. (10) that the induced space-time metric is \( g_{\mu \nu} = \text{diag}(1,-1,-1,\ldots) \). This 12-dimensional embedding is 6-free for \( x^j \in B = \mathbb{R}^3 \setminus \{ x^1 \in \mathbb{R}, x^2 = \pm \infty, x^3 = \pm \infty \} \).

4. **Second example: Schwarzschild 4-D black hole (Kerr-Schild coordinates) into (1+14)-D host space**

In this case, to accommodate the extra embedding we need 3 extra dimensions, one time-like and two space-like. The metric components \( g_{\mu \nu}(x^0, x^1, x^2, x^3) \) of this black hole in Kerr-Schild coordinates do not depend explicitly on \( x^0 \), and we use the following extra embedding:
\[
 Y^0_M(x^\mu) = x^0 - M h_1 \left( \frac{r}{M} \right),
 Y^1_M(x^\mu) = \frac{6 \sqrt{\frac{M}{r}}}{\zeta} \frac{2 M}{r} \cos \left( \frac{M}{\sqrt{6 M^2}} \left( x^0 - M h_2 \left( \frac{r}{M} \right) \right) \right),
 Y^2_M(x^\mu) = \frac{6 \sqrt{\frac{M}{r}}}{\zeta} \frac{2 M}{r} \cos \left( \frac{M}{\sqrt{6 M^2}} \left( x^0 - M h_2 \left( \frac{r}{M} \right) \right) \right).
\]

Each spatial coordinate \( x^j \) ranges from \(-\infty \) to \( \infty \). The parameter \( \zeta \) must satisfy \( \zeta^2 \geq 1 \). Finally, the functions \( h_1(s) \) and \( h_2(s) \) must solve the differential equations:
\[
 h_1'(s) = \frac{2(1+h_2'(s))^2}{s}, \quad h_2'(s) = \frac{2 s - \sqrt{s^2 + (54 - 27 s - s^2)^2}}{(s-2)s^2}.
\]

They are well-behaved for all \( s > 0 \), even near the horizon \( s = 2 \). Analytical expressions for the functions \( h_1(s) \) and \( h_2(s) \) can be obtained when \( \zeta = 1 \). Figs. 1 and 2 have been constructed using these expressions.

The extra embedding functions \( Y^0_M, Y^1_M, Y^2_M \) account for the causal structure of the black hole. After rescaling the time \( x^0 \), the radial coordinate \( r \) and the embedding functions in units of \( M \), we plot them from eqs. (11) in Figs. 1 and 2 (interior and exterior region, respectively) as two-dimensional surfaces parameterized by \( \tau = x^0/M, s = r/M \). These are helicoidal surfaces with a pitch \( \Delta \tau = \pi \sqrt{M} \). We remark that the interior and exterior regions are smoothly connected. The black hole metric is equal to the metric induced by the plotted surface plus the (flat) metric \( Z_{\mu \nu} \cdot Z_{\mu \nu} = -\delta_{jk} \) induced by the base embedding (not plotted). The event horizon \( r = 2M \), denoted in both figures, is an upgoing helix subtending an angle of \( \pi/4 \) with respect to the rescaled axis \( Y^0/M \).

The interior region \( 0 < r < 2M \) is represented in Fig. 1. We can see the event horizon \( r = 2M \) as a light-like helix which is close to the vertical axis. The space-like helix \( r = 0.1M \) is bounding the figure. The actual surface extends to infinity as \( r \to 0 \), subtending asymptotically an angle of \( \pi/4 \) with respect to the rescaled axis \( Y^0/M \) as \( r \approx 0 \); all physical trajectories get trapped in the interior region, approaching the helix at infinity \( \tau \to 0 \). This graphical interpretation is possible since the plotted surface gives the main contribution to the black hole metric in the region \( r \approx 0 \). Indeed, the (flat metric) contribution coming from the base embedding (not plotted) is proportional to \( r^2/M^2 \) and is therefore negligible.

The exterior region \( r > 2M \) is represented in Fig. 2 by the spiral surface that approaches the vertical axis as \( r \to \infty \), folding and folding indefinitely, never reaching the axis. Notice that in this region, the above mentioned base embedding’s contribution (not plotted) to the black hole metric, being proportional to \( r^2/M^2 \), is much more important than the spatial contribution coming from the plotted surface, so that this exterior region plot gives only partial information about the causal structure of the black hole.
FIG. 1: Schwarzschild black hole embedding, eq.(11). The helicoidal surface shown represents a piece of the interior zone \( r < 2M \), bounded by the wider, space-like helix \( r = 0.1M \) and by the light-like helix \( r = 2M \) (the event horizon).

5. Third example: Waves around the embedding of flat space-time

Let us check that the 6-free embedding of flat space-time introduced in section II C 3 allows for the representation of gravitational waves as embedding waves by computing explicitly the perturbation of this embedding that corresponds to the well-known \(^2\) plane wave: \( \delta g_{\mu\nu} = L_{\mu\nu} e^{i\mathbf{p} \cdot \mathbf{x}} \), with \( p_\mu p^\mu = 0 \), \( L_{0\mu} = 0 \) and \( L_{jk} p^k = 0 \). We solve eq.(7) for the embedding variations by first taking vanishing tangent variations \( \delta W_\mu = 0 \). We then split the normal variations as the embedding itself: \( \delta y_\perp = 0 \oplus \delta Z_\perp \), where \( \delta Z_\perp \equiv Z_{jk} \delta f^{jk}; \delta f^{12} \) and \( \delta f^{11} \) are explicitly given by:

\[
\begin{align*}
\delta f^{12} &= -\frac{\delta g_{12}}{4v_1 w_2}, \\
(2\Delta) \delta f^{11} &= -\delta g_{11} \left((v_2 v_3)^2 + (v_2 w_3)^2 + (w_2 v_3)^2\right) \\
&+ \delta g_{22} w_1 w_2 v_3^2 + \delta g_{33} w_1 v_2^2 w_3,
\end{align*}
\]

and the other components are obtained by cyclic permutations of \((1, 2, 3)\). The functions \( v_1, w_2 \) and \( \Delta \) were defined in Section II C 1.

The embedding waves are thus simply plane waves, modulated with some smooth functions related to this particular embedding.

III. ACTION PRINCIPLES FOR EMBEDDING THEORY

We have introduced in the previous sections the (purely kinematical) concept of free embedding, which is by definition deformable to accommodate linear variations of the metric tensor, and thus gravitational waves. Our next goal, therefore, is to find a satisfactory action principle for general relativity in terms of embedding variables, in order to obtain what we will call ‘free embedding field theory for gravity’. This theory that requires host space dimension \( N \geq 14 \), turns out to be equivalent to GR, not only at the levels of EOM but also for perturbations (waves around the general solution). We then consider the ‘q-free embedding theory for gravity’, that allows us to reduce the minimal dimension of the host space from 14 to 10.

Finally, we introduce a third, alternative family of action principles that depend on Lagrange multipliers. The metric variations and embedding variations are independent and their relation (eq.(4)) is given as an EOM. These alternative theories are also shown to be equivalent to
usual GR, both at the levels of EOM and of perturbations (waves around the general solution), provided that the embedding is $g$-free.

## A. Hilbert action in terms of embedding variables

Let us denote the 4-D space-time manifold and its 3-D boundary by $\mathcal{M}$ and $\partial \mathcal{M}$ respectively and let $N$ be the dimension of the host space. Consider the usual Einstein action principle but written in terms of embedding variables

$$
S[y] = \frac{1}{8\pi G} \int_{\mathcal{M}} R[g[y]] \sqrt{-|g[y]|} d^4x + \frac{1}{4\pi G} \int_{\partial \mathcal{M}} K[y]
$$

where $g[y]$ means that $g$ is replaced everywhere by its definition, eq. (1). The scalar curvature $R[g[y]]$ is given explicitly by $R = \gamma_{\alpha\alpha} \cdot \gamma_{\beta\beta} - \gamma_{\alpha\beta} \cdot \gamma_{\alpha\beta}$. Note that the scalar curvature is quadratic and of second order only in the embedding functions $y$ (see appendix A). Finally, $K[y]$ is the trace of the extrinsic curvature of the boundary. The $K$-boundary term is here, as usual, to cancel boundary terms that depend on the normal derivatives of $g_{\mu\nu}$, and allows to impose the boundary conditions $\delta g_{\mu\nu} = 0$.

We thus assume from now on the vanishing boundary conditions (v.b.c.) $\delta g_{\mu\nu} = 0$ on $\partial \mathcal{M}$. These boundary conditions explicitly read, using eq. (17),

$$
0 = 2 \delta W_{(\mu;\nu)} - 2 y_{\mu;\nu} \cdot \delta y_\perp.
$$

Let us consider the variation of action $S$, in two steps. First, we obtain in the standard way

$$
\delta S = - \frac{1}{4\pi G} \int_{\partial \mathcal{M}} \delta W_\mu G^{\mu\nu} d\Sigma_\nu + \frac{1}{4\pi G} \int_{\mathcal{M}} G^{\mu\nu} y_{\mu;\nu} \cdot \delta y_\perp \sqrt{-|g|} d^4x,
$$

where $d\Sigma_\nu$ is the normal unitary 3-volume element on the boundary. The above boundary term comes from the differential dependence of the metric in terms of the embedding functions. Note that it has nothing to do with the $K[y]$-boundary term in eq. (13).

## B. Free embedding theory equivalent to General Relativity

We assume in this section the simplest case which is that the embedding is free throughout the manifold and at its boundary. From local existence theorems, this will require the dimension of the host space to be $N \geq 14$. By definition of free embedding (see section II B 1), the 10 $N$-dimensional host-space vectors $\{y_{\mu;\nu}, \mu, \nu = 0, \ldots, 3\}$ are linearly independent.

In this simple case, we can consistently take $\delta W_\mu = 0$ throughout the manifold. The boundary conditions on the embedding functions $\{y_{\mu;\nu}, \mu, \nu = 0, \ldots, 3\}$ now imply, because of the freeness of the embedding, $\delta y_\perp = 0$ on the boundary.

Using eq. (13), under arbitrary normal embedding variations $\delta y_\perp$ in the bulk we obtain the following Euler-Lagrange equations:

$$
G^{\mu\nu}[y] y_{\mu;\nu} = 0
$$

It is worthwhile at this point to warn the reader against a common misinterpretation. In connection with string-like models with arbitrary $N$, the equations for the minimal hypervolume membrane, $g^{\mu\nu} y_{\mu;\nu} = 0$, are correctly understood as constraints on the second fundamental form $y_{\mu;\nu} \parallel [8]$. But the analogy of these equations with eqs. (17), with $G^{\mu\nu}$ replacing $g^{\mu\nu}$ leads to a widespread error: to also understand eqs. (17) as constraints on the second fundamental form.$^4$ For free embeddings this is clearly a wrong interpretation since, as we have shown, when a free embedding solves eq. (17), it follows $G^{\mu\nu} = 0$ in a perfectly consistent way. On the other hand, for the minimal hypervolume membrane this interpretation is correct, since if a free embedding solved the corresponding equation, then the contradictory result $g^{\mu\nu} = 0$ would follow.
C. \textit{q-free embedding theories equivalent to General Relativity}

The earliest work we know on the variational principle for gravitation in terms of embedding variables is the ‘gravitation \textit{à la string}’ due to Regge and Teitelboim [3] (RT). This approach is well-known not to be equivalent to general relativity [3, 4, 6]. The reason for this is as follows. By requiring isometric embedding only, the authors set \( N = 10 \) as the dimension of the flat host space. They use essentially the action \( I_{\text{13}} \) as a starting point, and the equations of motion are again \( I_{\text{17}} \). However, in this case the equations no longer imply the Einstein equations since the embedding cannot be free, for the dimension of the host space is \( N = 10 < 14 \). Indeed, there are only 6 independent EOM.

The free embedding theories introduced in the previous section are equivalent to GR but require a host space of dimension \( N \geq 14 \). The following question therefore naturally arises: ‘are there embedding theories equivalent to GR, based on a host space of dimension 10 \( \leq N < 14 \)?’

1. 6-free \( N = 10 \) case

In this section we consider the most challenging case, \( N = 10 \). In other words, is it possible to modify \( N = 10 \) RT theory and make it equivalent to GR? We now show that the only way to make a \( N = 10 \) RT theory equivalent to GR is to require the embedding to be 6-free together with appropriate boundary conditions on the embedding variations.

In order to be equivalent to GR, a theory should have the same number (ten) of independent EOM. However, as pointed out by Regge and Teitelboim [3], Deser et al. [4], Pavšič [5], and Franke and Tapia [8], it follows from \( y_{\mu
u} = 0 \) that the second fundamental form \( y_{\mu
u} \) has at most 6 independent components (instead of ten). There are therefore at most 6 independent EOM \( I_{\text{17}} \). Note that they are \textit{exactly} 6 independent equations (by definition 2, section \[\text{11B1}\]) when the embedding is 6-free. What happened to the other 4?

The answer comes from the fact that the RT boundary conditions do not allow for an arbitrary variation \( \delta y_{\mu
u} \) in the bulk, which is not consistent with the standard derivation of the Einstein equations from the action principle \( I_{\text{13}} \) (see eq. \( I_{\text{15}} \)). The possibility of having arbitrary \( \delta y_{\mu
u} \) in the bulk depends on the boundary conditions imposed on the embedding functions in the theory. But, as we will see below, the naive RT choice of v.b.c. on the variations of embedding functions and their normal derivatives imposes constraints on the bulk metric variations, independently of the 6-freeness property. However, an arbitrary variation \( \delta y_{\mu
u} \) in the bulk is allowed when the boundary conditions: (a) \( \partial W_{\mu} \) arbitrary and (b) \( \delta y_{\mu
u} \) given by eqs. \( I_{\text{14}} \) on the boundary, are used instead of the RT v.b.c.

To give a specific example, consider variations around a \( N = 10 \), 6-free and spatially free embedding of an open neighborhood within the flat space-time \( \mathbb{R} \times \mathbb{R}^3 \). Here the relevant boundaries for the action principle are the space-like boundaries \( x^0 = 0 \), \( x^0 = 1 \). We assume for simplicity the following property for the embedding functions: \( y_{0\mu} = 0 \) throughout the open neighborhood. Now suppose that both \( \delta y_{\mu
u} \) and \( \delta W_{\mu} \) are zero on the boundaries.

Following the general equations \( I_{\text{7}} \) there are, in the bulk, 4 differential equations we must solve for the tangent embedding variations: \( \delta y_{\mu
u} = 2 \delta W_{(0,\nu)} \), \( \nu = 0, 1, 2, 3 \), and 6 algebraic equations for the normal embedding variations: \( \delta y_{ij} = 2 \delta W_{(i,j)} - 2 y_{ij} \cdot \delta y_{\perp} \), \( i, j = 1, 2, 3 \). The last 6 equations imply \( \delta y_{ij} = 0 \) on the boundaries, with no condition in the bulk. However, the first 4 equations give constraints: \( \nu = 0 \) component implies \( \delta W_0(x^0, x^i) |_{x^0=0} = \frac{1}{2} \int_0^1 \delta y_{00}(z, x^i) dz \), and the \( \nu = j \) components imply \( \delta W_j(x^0, x^i) |_{x^0=0} = \int_0^1 \delta y_{0j}(z, x^i) dz - \frac{\partial}{\partial x^0} \int_0^1 dz \int_0^1 ds \delta y_{00}(s, x^i) \). But these expressions are zero for the v.b.c. on the embedding variations, so we obtain \( \int_0^1 \delta y_{00}(z, x^i) dz = 0 \), \( \int_0^1 \delta y_{0j}(z, x^i) dz = 0 \), \( \int_0^1 \delta y_{00}(s, x^i) ds = 0 \).

Conversely (see eqs. \( I_{\text{7}} \)), the assumption of arbitrary metric variations in the bulk imply that, at space-like portions of the boundary, all 4 embedding tangent variations \( \delta W_{\mu} \) must be arbitrary. Recalling that the action variation \( I_{\text{16}} \) depends on these tangent variations, we are led to 4 extra Euler-Lagrange equations on the boundary, that will turn out to be the usual constraints of GR, see eq. \( I_{\text{13}} \) below.

We now turn to the proof that, in the more general case of curved space-times embedded in a spatially free manner, these extra equations will be enough to show the equivalence of the Einstein equations with the Euler-Lagrange equations obtained from the action \( I_{\text{13}} \) under the new boundary conditions: \( \delta W_{\mu} \) arbitrary and \( \delta y_{\perp} \) given by eqs. \( I_{\text{14}} \), which admit solutions because of spatial freeness. For simplicity, we will assume the following properties for the space-time and the embedding: (i) the space-time is globally hyperbolic, which allows to define a global time coordinate, and (ii) the embedding is 6-free and, with respect to the latter time coordinate, it is spatially free (see Definition 3 in section \[\text{11B1}\]). We define the resulting theory as a 6-free embedding theory of gravity.

Using the general equation \( I_{\text{16}} \) for the variation of the action, we consider arbitrary variations \( \delta W_{\mu} \) on the boundary, and arbitrary variations \( \delta y_{\perp} \) in the bulk. The Euler-Lagrange equations for this theory are thus:

On space-like portions of \( \partial M \): \( n_{\mu} G^{\mu\nu} = 0 \), \( I_{\text{18}} \)

where \( n_{\mu} \) is the unit normal to the boundary, and

On \( M \): \( G^{\mu\nu} y_{\mu\nu} = 0 \). \( I_{\text{19}} \)
Using the appropriate time slicing, which exists in globally hyperbolic space-times, the Euler-Lagrange equations at the space-like boundary are \( G^{0\mu}|_{t=t_0} = 0 \). On the other hand, spatial freeness of the 10-dimensional embedding implies the relations \( y_\partial y_\mu = A^{ij}_\mu y_{ij} \), where \( A^{ij}_\mu \) are tensor fields defined by the embedding. Therefore the bulk eqs. (19) become

\[
G^{ij} + G^{00}A^{ij}_0 + 2G^{0k}A^{ij}_k = 0 .
\]

(20)

Using this equation we can rewrite \( G^{\mu\nu} = 0 \) as a system of four first order, homogeneous partial differential equations for the unknowns \( G^{0\mu} \). Recalling that the initial condition is \( G^{0\mu}|_{t=t_0} = 0 \), we obtain \( G^{0\mu} = 0 \) in the bulk, and then eqs. (20) imply \( G^{\mu\nu} = 0 \) in the bulk. We have thus shown the equivalence between GR and 6-free, spatially free embedding theory.

2. 6-free \( N > 10 \) case

The case \( N > 10 \) corresponds to the addition of extra embedding functions to the theory, and all derivations above apply with minimal obvious modifications.

3. Waves in 6-free embedding theory

We now explicitly demonstrate that the above embedding theory linearized about the solution of the EOM \( G^{\mu\nu} = 0 \) is equivalent to standard linearized GR, and thus contains the standard gravitational waves. To wit, we compute the variations of eqs. (18–19), and evaluate them on the general solution, which satisfies the EOM \( G^{\mu\nu} = 0 \). In the rest of this subsection the covariant derivative corresponds to the metric satisfying the EOM. We obtain:

\[
\begin{align*}
\text{On } \partial M & : \quad n_\mu \delta G^{\mu\nu} = 0 , \\
\text{On } M & : \quad \delta G^{\mu\nu} y_{\mu\nu} = 0 .
\end{align*}
\]

(21)

(22)

The tensor \( \delta G^{\mu\nu} \) is covariantly conserved when the EOM hold. This can be deduced from the variation of the identity \( G^{\mu\nu} = 0 \). We get \( \delta G^{\mu\nu} \equiv 0 \) and, using the EOM \( G^{\mu\nu} = 0 \), \( \delta G^{\mu\nu} = 0 \) follows. Then, the tensor \( \delta G^{\mu\nu} \equiv 0 \) satisfies the same equations as the tensor \( G \). Repeating the arguments already used above we conclude that for spatially free 6-free embeddings the solution is \( \delta G^{\mu\nu} = 0 \), just as expected for the standard perturbation theory of the Einstein equations in GR. However, this time one has to compute the variations with respect to the embedding variables. Though it is possible to write down the resulting equations, in our opinion it gives no further insight to do it. Instead we notice that the perturbations of the embedding functions must propagate in a proper way, because of the proved correspondence that exists, for spatially free 6-free embeddings, between the variations of the metric and those of the embedding, eqs. (7).

We refer the reader to the explicit example of waves around flat space-time in section II C 5.

4. The case of \( q \)-free embeddings with \( 6 < q < 10 \)

This case is intermediate between the \( N \geq 14 \) free (i.e., 10-free) case (section II B 5) and the \( N \geq 10 \) 6-free case (sections II C 1–II C 2).

In this case the EOM (17) are exactly \( q > 6 \) equations (by definition 2, section II B 1). Therefore there will be only \( 10 - q < 4 \) missing equations. These equations will be obtained as boundary equations like eqs. (18), by assuming arbitrary boundary conditions on \( 10 - q \) out of the four \( \delta W_\mu \). Following the same type of arguments as those between eqs. (18) and eqs. (20), we can conclude that if the embedding is spatially free and \( q \)-free with \( N \geq 4 + q \), the EOM are equivalent to GR.

D. Action Principles with independent metric and embedding functions

This section deals with a class of action principles where the metric \( g_{\mu\nu} \) and the embedding functions \( y \) are considered independent. Relation (1) between metric and embedding therefore appears as an EOM.

Consider the following action

\[
S_n[g, y, \lambda] = \frac{1}{8\pi G} \int_M d^4x \sqrt{-g} R[g] + \frac{1}{4\pi G} \int_{\partial M} K[g] + \frac{1}{8\pi G} \int_M d^4x \sqrt{-g} \lambda^{\mu_1\nu_1...\mu_n\nu_n} (g_{\mu_1\nu_1} - y_{\mu_1} \cdot y_{\nu_1}) \cdots (g_{\mu_n\nu_n} - y_{\mu_n} \cdot y_{\nu_n}) ,
\]

(23)

where \( R[g], K[g] \) are the Ricci scalar and the extrinsic curvature of the boundary \( \partial M \), both expressed in terms of the metric, and \( \lambda \) is a Lagrange multiplier.

1. The case \( n = 1 \)

This case has already been presented in the literature \cite{4}, with \( \text{v.b.c.} \) on the embedding variables; these \( \text{v.b.c.} \).
make the theory inequivalent to GR. Let us consider now the action \( (23) \) in the context of free and \( q \)-free embeddings and allow for arbitrary tangent variations \( \delta W_\mu \) of the embedding.

The variation of the \( S_1 \) reads:

\[
(8\pi G) \delta S_1 = \int_M \left[ \delta \lambda^{\mu \nu} (g_{\mu \nu} - y_{,\mu} \cdot y_{,\nu}) + \delta g_{\mu \nu} \left( \lambda^{\mu \nu} + \frac{1}{2} g^{\mu \nu} \lambda^{\alpha \beta} (g_{\alpha \beta} - y_{,\alpha} \cdot y_{,\beta}) - G^{\mu \nu}[g] \right) \right] \sqrt{-g} d^4 x + 2 \int_{\partial M} \delta \bar{W}_{\mu} \lambda_{\mu \nu} d\Sigma_\nu + 2 \int_M \left[ \delta \bar{W}_{\mu} \lambda_{\mu \nu} + \delta y_{,\perp} \cdot y_{,\mu} \lambda^{\mu \nu} \right] \sqrt{-g} d^4 x, \tag{24}
\]

where the Einstein tensor \( G^{\mu \nu}[g] \) is viewed as a functional of the metric. Remember that \( g_{\mu \nu} \) and \( y \) are treated as independent at this stage; their variations are therefore independent too and eq. \( (24) \) thus delivers three Euler-Lagrange equations in the bulk. The Euler-Lagrange equation with respect to the Lagrange multiplier \( \lambda^{\mu \nu} \) gives, as a constraint, the relation between the metric and the embedding functions, eq. \( (4) \); replacing this equation into the Euler-Lagrange equation with respect to the normal embedding variations we get the EOM \( G^{\mu \nu} y_{,\mu \nu} = 0 \) in the bulk, which is formally identical to the equation obtained from the action principle presented in the previous section. For free embeddings this EOM implies the Einstein equations, \( G^{\mu \nu} = 0 \). However, for \( q \)-free embeddings, equivalence with GR can be obtained by supplementing the \( 10 - q \) missing equations by the boundary equations coming from the arbitrary variation of \( \delta W_\mu \) in eq. \( (24) \) (see previous section).

2. The case \( n = 2 \)

Variation of eq. \( (28) \) now reads

\[
(8\pi G) \delta S_2 = \int_M \left[ \delta \lambda^{\mu \nu \rho \sigma} (g_{\rho \sigma} - y_{,\rho} \cdot y_{,\sigma}) (g_{\mu \nu} - y_{,\mu} \cdot y_{,\nu}) + \delta g_{\mu \nu} \left( 2 \lambda^{\mu \nu} + \frac{1}{2} g^{\mu \nu} \lambda^{\alpha \beta} (g_{\alpha \beta} - y_{,\alpha} \cdot y_{,\beta}) - G^{\mu \nu}[g] \right) \right] \sqrt{-g} d^4 x + 4 \int_{\partial M} \delta \bar{W}_{\mu} \lambda_{\mu \nu} d\Sigma_\nu + 4 \int_M \left[ \delta \bar{W}_{\mu} \lambda_{\mu \nu} + \delta y_{,\perp} \cdot y_{,\mu} \lambda^{\mu \nu} \right] \sqrt{-g} d^4 x, \tag{25}
\]

where \( \lambda^{\mu \nu \rho \sigma} \equiv \lambda^{\mu \nu \rho \sigma} (g_{\rho \sigma} - y_{,\rho} \cdot y_{,\sigma}) \).

The Euler-Lagrange equations stemming from the first two variations in eq. \( (25) \) are equivalent to:

\[
g_{\mu \nu} = y_{,\mu} \cdot y_{,\nu}, \quad \bar{G}^{\mu \nu}[g] = 0, \tag{26}
\]

which imply Einstein dynamics for the embedding variables. Equations \( (26) \) also imply that both remaining surface and bulk variations in eq. \( (24) \) vanish. At this stage, \( \lambda^{\mu \nu \rho \sigma} \) remains completely arbitrary.

It thus seems that, at the level of EOM, freeness is not needed to recover Einstein equations. However this is misleading because the existence of propagating gravitational waves equivalent to those of GR for this theory needs, as we are now going to prove, that the embedding be \( q \)-free.

Let us perturb the EOM of the \( S_2 \) embedding theory by varying the embedding variables \( y^A \), the metric \( g_{\mu \nu} \) and the tensor \( \lambda^{\mu \nu \rho \sigma} \). After the perturbation has been performed to first order, we can use the EOM wherever we want. Recall that here \( G^{\mu \nu} \) depends only on \( g_{\mu \nu} \). First note that the variation \( \delta G^{\mu \nu} \) vanishes because of the EOM. The other variations give, in the bulk:

\[
- \delta G^{\mu \nu} + 2 \lambda^{\mu \nu \rho \sigma} \delta (g_{\sigma \rho} - y_{,\sigma} \cdot y_{,\rho}) = 0, \tag{27}
\]

\[
0 = y_{,\mu} \left[ \lambda^{\mu \nu \rho \sigma} \delta (g_{\sigma \rho} - y_{,\sigma} \cdot y_{,\rho}) \right]_{,\nu} + y_{,\mu \nu} \lambda^{\mu \nu \rho \sigma} \delta (g_{\sigma \rho} - y_{,\sigma} \cdot y_{,\rho}). \tag{28}
\]

On the boundary we get:

\[
\text{On } \partial M : \quad n_{\mu} \lambda^{\mu \nu \rho \sigma} \delta (g_{\sigma \rho} - y_{,\sigma} \cdot y_{,\rho}) = 0. \tag{29}
\]

Now, using eq. \( (27) \), eqs. \( (28) \) become:

\[
y_{,\mu} \delta G^{\mu \nu} + y_{,\mu \nu} \delta G^{\mu \nu} = 0, \quad \text{on } M, \tag{30}
\]

\[
n_{\mu} \delta G^{\mu \nu} = 0, \quad \text{on } \partial M. \tag{31}
\]
The first term in eq. (30) vanishes (see the discussion after eq. (22)). Equations (30–31) are thus equivalent to eqs. (21–22) and these are in turn equivalent to Einstein equations when the embedding is $g$-free. Note finally that it is necessary to assume that the tensor $\lambda^{\mu\nu\alpha\beta}$ is invertible in order to deduce from eq. (27) the embedding equation $\delta(g_{\mu\nu} - y_{\mu} \cdot y_{\nu}) = 0$, which permits the identification $g_{\mu\nu} = y_{\mu} \cdot y_{\nu}$ of the metric in terms of the embedding at the EOM and at the perturbative level as well; this is what makes possible a description of gravitational waves in terms of embedding waves.

IV. VACUUM INITIAL VALUE FORMULATION OF GR IN EMBEDDING VARIABLES

In this section we discuss the possibility of treating the Einstein equations as a dynamical system in terms of the embedding functions. We present some preliminary results regarding the initial value formulation of the vacuum Einstein equations in the embedding approach. Note that, in the usual formulation, the numerical integration schemes suffer from instabilities (e.g., pure gauge modes and violation of constraints [24, 31]), which destroy their performance in finite time. In this context, the direct numerical integration of the equations in the embedding variables is certainly worth developing.

Another motivation for future numerical work is that it could provide information on some interesting theoretical questions. For example: is $g$-freeness (or spatially free $g$-freeness) kept as long as one integrates in time the evolution equations in embedding variables?

Let us assume for simplicity that the dimension of the host space is $N = 10$.

A. The coordinate system and the embedding variables

Let us assume that the host space has only one time-like coordinate: $\eta_{AB} = \text{diag}(1, -1, \ldots, -1)$, and that the space-time is globally hyperbolic. One can then introduce a coordinate system $(\tau, \lambda^i)$, with $\tau$ a global time-like coordinate and $\partial^\tau > 0$. Let the initial embedding, defined by the functions $y(\tau = 0, \lambda^i)$, be 6-free and spatially free with respect to this coordinate system.

Let $u \equiv \frac{\tau}{\sqrt{g}}$; this vector is timelike and verifies:

$$
\begin{align*}
    u \cdot u &= 1 \\
    u \cdot y_i &= 0,
\end{align*}
$$

(32)

where $F_i = \frac{\partial y_i}{\partial \tau}$. Let $(u^\tau, u^i)$ be the components of $u$ in the basis \{$y_{\tau}, y_i$\}, i.e., $u = u^\tau y_{\tau} + u^i y_i$. One has $u^\tau = (u^\tau)^{-1} = \sqrt{g^\tau}$ and $u_i = 0$. Defining now the lapse function $N(\tau, \lambda^i)$ and the shift vector $N^i(\tau, \lambda^i)$ by

$$
\begin{align*}
    N &= (u^\tau)^{-1}, \\
    N^i &= -(u^\tau)^{-1} u^i,
\end{align*}
$$

(33)

one gets the following evolution equation for $y$:

$$
y_{\tau} = N u + N^i y_i.
$$

(34)

In terms of the functions $(y, N, N^i)$, the metric components read $g_{ij} = y_{i} \cdot y_{j}$, $g_{\tau i} = -g_{ij} N^j$, $g_{\tau \tau} = N^2 + g_{ij} N^i N^j$. The condition for $y_{\tau}$ to be time-like is $g_{\tau \tau} > 0$.

Our definition of lapse and shift agrees with the standard one (see Wald [2]). It is well known that these functions are not dynamical and that fixing them improperly can cause problems in the numerical integration of Einstein equations [32]. Because of coordinate transformation invariance, 4 components of $y$ can be chosen as arbitrary functions of $(\tau, \lambda^i)$. The corresponding 4 equations in [33] then fix $(N, N^i)$ once $u$ is known.

In this way, the evolution equations (33) allow to propagate in time the embedding functions $y$ provided the vector $u$ is known. As shown in the next section, the choice of $u$ as the relevant variables ‘conjugate to $y$’ is the natural one to deal with Einstein equations.

B. The vacuum Einstein equations

As usual, the vacuum Einstein equations $G^{\tau j} = 0$ and $G^{\tau \tau} = 0$ are 4 constraints on the initial data; in the present case the initial data are the values of $(y, u)$ at $\tau = 0$. The first three equations are linear in the derivatives $u_{ij}$, while the last one is algebraic and quadratic in $u$:

$$
\begin{align*}
    \tilde{g}^{ij} (y_{ij} \cdot u_{k} - y_{jk} \cdot u_{i}) &= 0, & i &= 1, 2, 3, \\
    \tilde{g}^{ij} \tilde{g}^{kl} (y_{jk} \cdot y_{il} - y_{ij} \cdot y_{kl}) &= 0;
\end{align*}
$$

(35)

here, $\tilde{g}^{ij}$ is the 3–inverse of $g_{ij}$ and $y_{jk} = S \cdot y_{jk} - u_i u_j$, where the spatial projector $S$ is defined by $S_{AB} = \eta_{AB} - \tilde{g}^{ij} y_{A,i} y_{B,j}$. The projector and the metric components appearing in eqs. (35) are clearly given by the initial data.

Let us define the ‘acceleration’ $a$ by

$$
\frac{\partial u}{\partial \tau} = Na + N^i u_i.
$$

(36)

Since $(N, N^i)$ have been fixed by a choice of coordinates, the time evolution of $u$ will be completely determined if the acceleration $a$ is known. Its normal components are obtained (when the embedding is spatially free) from the vacuum Einstein equations $R_{ij} = 0$:

$$
y_{ij} \cdot a = u_i \cdot S \cdot u_j + \tilde{g}^{kl} (y_{ik} \cdot y_{jl} - y_{ij} \cdot y_{kl})
$$

(37)

and its tangent components satisfy:

$$
\begin{align*}
    u \cdot a &= 0, \\
    y_{i} \cdot a &= -(\ln N)_i.
\end{align*}
$$

(38)

These expressions are obtained from straightforward manipulation of the differentiated versions of eqs. (22) and eqs. (34).

Equations (30–38) can thus be used to propagate in time the vector $u$. 
V. CONCLUSION

A. Summary

We have revisited the embedding approach on General Relativity which views the 4-D, possibly curved, physical space-time as a membrane floating in a flat host space-time of higher dimension. We have first introduced two new classes of embeddings, both based on Nash’ notion of freeness. All embeddings in these classes are deformable and, therefore, allow for a description of gravitational waves; explicit examples of such embeddings have been given for both Minkowskian space-time and the Schwarzschild black hole. We have also presented new variational principles which deliver General Relativity as a field theory for embedding variables. Einstein’s dynamics thus appears as free membrane dynamics in the host space. We have finally considered the general relativistic initial value formulation in terms of embedding variables and argued that this new point of view sheds new light on this particularly difficult issue.

B. Discussion

This article proposes what is, to the best of our knowledge, the first consistent embedding approach to non quantum gravitation. Previous attempts have been marred by essentially two problems. The first one concerns the possibility of deforming the embedding to accommodate for gravitational waves. Our approach is the first to use the notion of freeness introduced by Nash, and variations thereof, to solve this problem. The second problem is linked with the possibility of constructing an action principle which would deliver Einstein’s theory in terms of the embedding variables. Previous attempts \([3, 4, 7, 8]\) failed in this respect because the action functionals were used with the wrong boundary conditions, and also because the considered embeddings were not free.

We have constructed explicitly free embeddings of several general relativistic space-times of astrophysical importance (for example, a Schwarzschild black hole). The approach developed in this article is thus relevant to physics. Whether all general relativistic space-times can be embedded freely in a flat space-time of higher dimension remains however an open question.

We would like to end this article by mentioning a few assets offered by the embedding point of view on GR and references therein). For example, embedding variables seem ideally suited to the semiclassical study of the spontaneous creation or destruction of universes out of a quantum vacuum \([33]\). More generally, the embedding point of view surely appears as the right tool to study problems involving changes in the topology of space-time. In fact, in our example of flat space-time, the extra parameters and functions (appearing explicitly in the embedding but not in the metric) can be chosen in order to change the topology of space-time from \(\mathbb{R} \times \mathbb{R}^3\) to \(\mathbb{R} \times \mathbb{R}^2 \times S^1\). In this context, the problem of averaging statistically the geometry of (classical) space-time has recently been solved for situations in which the topology of space-time is fixed \([28, 34]\). Does the embedding point of view permit a more general treatment?

Finally, it is certainly interesting to investigate how field quantization on the (flat) host space translates on the embedded space-time. The free embedding theory of gravity introduced in this paper, precisely because it deals with \(q\)-free embeddings, is suitable for perturbative quantum theory, as opposed to the old approaches (see, for example, \([8]\)). To cite just a few interesting questions: does the resulting 4-D quantum theory depend, at fixed space-time coordinates \(x^\mu\), on the choice of the embedding? We refer the reader to discussions of this topic in the context of RT theory \([3, 9]\), and in the context of embedding theory of induced gravity in \([9]\). What new insight does the embedding point of view bring to the Unruh effect \([21]\)? And how does black hole thermodynamics appear in the embedding point of view? (See \([33, 36]\) for an account of these two last topics in the context of GR).

APPENDIX A: CHRISTOFFEL SYMBOLS, COVARIANT DERIVATIVE, SECOND FUNDAMENTAL FORM, NORMAL PROJECTOR, AND CURVATURE TENSOR IN TERMS OF EMBEDDING FUNCTIONS

Following Dirac \([37]\) we define the Christoffel symbols in terms of the embedding functions: \(\Gamma^\alpha_{\mu\nu} = y^\alpha,_{\mu} \cdot y^\mu,_{\nu}\). The covariant derivative is defined as usual in terms of the Christoffel symbols. For example, the second covariant derivative of a scalar function \(\phi(x^\mu)\) is \(\phi_{,\mu\nu} = \phi_{,\mu\nu} - \phi_{,\alpha} \Gamma^\alpha_{\mu\nu}\). It is interesting to see that, when \(\phi\) is replaced by the embedding functions (which are scalars by definition) we get what is known as the second fundamental form: \(y_{,\mu\nu} = y_{,\mu\nu} - y_{,\alpha} \Gamma^\alpha_{\mu\nu} = y_{,\mu\nu} - y_{,\alpha} (y^\alpha \cdot y_{,\nu}) = N \cdot y_{,\mu\nu}\), where \(N^{AB} = \eta^{AB} - (y^A),_{\alpha}(y^B),_{\alpha}\) is the normal projector, whose kernel is by definition the tangent space \(T_p(M)\), generated by the tangent vectors \(y_{,\nu}(x^\nu), \mu = 0, \ldots, 3\). Then, the second fundamental form is a set of vectors in the normal space \((y_{,\mu\nu})_p \subset N_p(M)\), \(\mu, \nu = 0, \ldots, 3\) with \(y_{,\mu\nu} \cdot y_{,\alpha} = 0\).

Finally we compute the Riemann and Einstein tensors in terms of the embedding functions. The Riemann curvature tensor depends on second (partial) derivatives of the metric tensor, which itself depends on first derivatives of the embedding functions. Therefore one could naively expects the curvature tensor to depend
on third order derivatives of the embedding functions. However, all terms containing third order derivatives vanish. The Riemann tensor thus depends on second derivatives only and reads $R^{\alpha\beta\gamma\delta} = y_{\mu\alpha} \cdot y_{\nu\beta} - y_{\mu\nu} \cdot y_{\alpha\beta}$. The Einstein tensor is similarly written $G_{\alpha\beta} = g^{\rho\sigma} \left( y_{\rho\alpha} \cdot y_{\sigma\beta} - y_{\rho\beta} \cdot y_{\sigma\alpha} \right) \left[ \kappa_{\alpha}^{\lambda} \delta_{\beta}^{\nu} - \frac{1}{2} g_{\alpha\beta} y^{\lambda\nu} \right]$. Geometrically, it is natural that only second covariant derivatives of embedding functions appear in these tensors: the curvature radii of the embedded manifold along the principal axes depend essentially on the normal components of the matrix of second derivatives of the embedding functions (the second fundamental form).

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[38] The action for the minimal hypervolume membrane is $S_B[\rho^4] = \int_M \sqrt{\vert g \vert} \rho^4 \vert d^4x \vert$.
[39] It remains an open problem if there exist $q$-free embeddings that solve the membrane equations: a careful study must be made on the membrane action principle and the boundary conditions on the embedding variations.