The double cover of cubic surfaces branched along their Hessian

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Abstract

We prove the relation between the Hodge structure of the double cover of a nonsingular cubic surface branched along its Hessian and the Hodge structure of the triple cover of \( \mathbb{P}^3 \) branched along the cubic surface. And we introduce a method to study the infinitesimal variations of Hodge structure of the double cover of the cubic surface. Using these results, we compute the Néron-Severi lattices for the double cover of a generic cubic surface and the Fermat cubic surface.

1 Introduction

Let \( X \subset \mathbb{P}^3 \) be a nonsingular cubic surface over the complex numbers \( \mathbb{C} \). It is well-known that \( X \) contains 27 lines in \( \mathbb{P}^3 \). A point \( p \in X \) is called an Eckardt point if there are three lines through \( p \) on \( X \). The classification of nonsingular cubic surfaces by the configuration of their Eckardt points is given in the book [1]. Although the configuration of the Eckardt points varies by a deformation of \( X \), the Néron-Severi lattice for \( X \) is constant. In order to detect the difference of the configuration of the Eckardt points, we consider the Néron-Severi lattice for the double cover of \( X \) branched along its Hessian. Let \( B \subset X \) be the zeros of the Hessian of the defining equation of \( X \). Then \( B \) has at most node as its singularities, and a point \( p \in X \) is a node of \( B \) if and only if \( p \) is an Eckardt point on \( X \). Therefore an Eckardt point on \( X \) corresponds to an ordinary double point on the finite double cover \( Y' \) over \( X \) branched along \( B \). Let \( \phi : Y \to X \) be the composition of the minimal resolution of \( Y' \) and the finite double cover. Then an Eckardt point \( e \) on \( X \) corresponds to the \((-2)\)-curve \( \phi^{-1}(e) \) on \( Y \), and a line \( L \) on \( X \) splits by the pull-back \( \phi^* \) into two \((-3)\)-curves \( L^+ \) and \( L^- \) on \( Y \), where

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we can chose the component $L^+$ of $\phi^*L$ so that the union of 27 rational curves $\bigcup_{L} L^+$ is a disjoint union. We remark that $Y$ is a minimal surface of general type with the geometric genus 4, and the double cover $\phi$ is the canonical morphism of $Y$. In this paper, we regard an Eckardt point $e$ on $X$ as the class $[\phi^{-1}(e)]$ in the Néron-Severi lattice of $Y$, and we compute the Hodge structure on $H^2(Y, \mathbb{Z})$.

There is another way to study cubic surfaces by using the Hodge structure of some associated variety. Let $\rho : V \to \mathbb{P}^3$ be the triple Galois cover branched along a cubic surface $X$. The Hodge structure on $H^3(V, \mathbb{Z})$ with the Galois action was considered by Allcock, Carlson and Toledo [11] to understand the moduli space of cubic surfaces as a ball quotient. In this paper, we investigate the relation between the Hodge structures $H^2(Y, \mathbb{Z})$ and $H^3(V, \mathbb{Z})$, and we prove that there is an isomorphism

$$\left( \bigwedge^2 H^3(V, \mathbb{Q})(1) \right)^{\text{Gal}(\rho)} \cong \frac{H^2(Y, \mathbb{Q}^+)}{\sum_{L} QL^+}$$

(1.1)

of Hodge structures. More precise statement in $\mathbb{Z}$-coefficients is given in Theorem 5.8. We remark that $V$ is a nonsingular cubic 3-fold in $\mathbb{P}^4$, and the Hodge structures of cubic 3-folds were studied by Clemens-Griffiths [3] and Tjurin [12]. Let $S$ be the set of lines on a nonsingular cubic 3-fold $V \subset \mathbb{P}^4$. It is a nonsingular projective surface, which is called the Fano surface of lines on $V$. Then the isomorphisms of Hodge structures $H^3(V, \mathbb{Z})(1) \cong H^1(S, \mathbb{Z})$ and $\bigwedge^2 H^1(S, \mathbb{Q}) \cong H^2(S, \mathbb{Q})$ are proved there. In order to relate the Hodge structure $H^2(Y, \mathbb{Q})$ with $H^2(S, \mathbb{Q})$, we regard the surface $Y$ as a kind of variety of lines. Let $\Lambda(\mathbb{P}^3)$ be the Grassmannian variety of all lines in $\mathbb{P}^3$. We show that $Y$ is isomorphic to the variety

$$Y_3 = \{(p, L) \in \mathbb{P}^3 \times \Lambda(\mathbb{P}^3) \mid L \text{ intersects } X \text{ at } p \text{ with the multiplicity } \geq 3\},$$

and the double cover $\phi : Y \to X$ corresponds to the first projection $Y_3 \to X; (p, L) \mapsto p$. Then the second projection $Y_3 \to \Lambda(\mathbb{P}^3); (p, L) \mapsto L$ is a birational morphism to its image $Z_3 \subset \Lambda(\mathbb{P}^3)$, and the Fano surface $S$ of the triple cover $V$ of $\mathbb{P}^3$ is a triple cover of $Z_3$ by $S \to Z_3; L \mapsto \rho(L)$. By the isomorphism $H^2(S, \mathbb{Q})^{\text{Gal}(\rho)} \cong H^2(Z_3, \mathbb{Q}) \cong \frac{H^2(Y, \mathbb{Q})}{\sum_{L} QL^+}$, we get the isomorphism (1.1).

By using this isomorphism (Theorem 5.8), we compute the Néron-Severi lattice $\text{NS}(Y)$ of $Y$. For a generic cubic surface $X$, we prove the theorem of Noether-Lefschetz type (Theorem 6.1), which says that $\text{NS}(Y)$ is generated by $(-3)$-curves on $Y$ corresponding to lines on $X$ for a generic cubic surface. We use the theory of the infinitesimal variations of Hodge structures [2] to compute that the rank of $\text{NS}(Y)$ is 28 for a generic cubic surface $X$. We introduce a method to compute the Hodge cohomology $H^2(Y, \Omega^1_Y)$ for $Y$, which is a generalization of the classical method by Griffiths [6]. And it enables us to compute the infinitesimal variations of Hodge structure of $Y$. In order to prove that the $(-3)$-curves on $Y$ generate the Néron-Severi group over $\mathbb{Z}$, we need the computation of the determinant of the lattice, for which the identification in Theorem 5.8
is used. For a special cubic surface, the rank of \( \text{NS}(Y) \) is greater than 28. If \( X \) is the Fermat cubic surface, then \( \text{NS}(Y) \) is of rank \( h^1(Y, \Omega_Y^1) = 44 \), and the \( \mathbb{Q} \)-vector space \( \mathbb{Q} \otimes \text{NS}(Y) \) is generated by \((-2)\)-curves corresponding to their Eckardt points and \((-3)\)-curves corresponding to lines on \( X \). More precisely, the generator of \( \text{NS}(Y) \) over \( \mathbb{Z} \) is given in Theorem 6.6. For the proof of Theorem 6.6 we use the computation of the Néron-Severi lattice of the Fano surface \( S \) for the Fermat cubic 3-fold by Roulleau [10].

The contents of this paper is the followings. In Section 2, we introduce the variety \( Y_3 \) for a nonsingular cubic surface \( X \), and compute the numerical invariants for the surface \( Y_3 \). In Section 3, we prove that the first projection \( Y_3 \rightarrow X \) is the double cover branched along the Hessian \( B \). And we compute the intersection number on \( Y = Y_3 \) of the curve \( \phi^{-1}(e) \) corresponding to an Eckardt point \( e \) on \( X \) and the curves \( L^\pm \) corresponding to \( \mathbb{Q} \)-vector space \( \mathbb{Q} \otimes \text{NS}(Y) \) is generated by \((-2)\)-curves corresponding to their Eckardt points and \((-3)\)-curves corresponding to lines on \( X \). More precisely, the generator of \( \text{NS}(Y) \) over \( \mathbb{Z} \) is given in Theorem 6.6. For the proof of Theorem 6.6 we use the computation of the Néron-Severi lattice of the Fano surface \( S \) for the Fermat cubic 3-fold by Roulleau [10].

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2 Varieties of lines

We denote by \( \Lambda(\mathbb{P}^n) \) the Grassmannian variety of all lines in the projective space \( \mathbb{P}^n \) over the complex numbers \( \mathbb{C} \), and by \( \mathcal{O}_{\Lambda(\mathbb{P}^n)}(1) \) the line bundle which gives the Plücker embedding of \( \Lambda(\mathbb{P}^n) \). We denote by \( \Gamma(\mathbb{P}^n) \) be the flag variety of all pairs \( (p,L) \) of a point \( p \in \mathbb{P}^n \) and a line \( L \subset \mathbb{P}^n \) which contains the point \( p \);

\[
\Gamma(\mathbb{P}^n) = \{(p,L) \in \mathbb{P}^n \times \Lambda(\mathbb{P}^n) \mid p \in L\}.
\]

We remark that their canonical bundles are given by \( K_{\Lambda(\mathbb{P}^n)} \simeq \mathcal{O}_{\Lambda(\mathbb{P}^n)}(-n-1) \) and \( K_{\Gamma(\mathbb{P}^n)} \simeq \Phi^* \mathcal{O}_{\mathbb{P}^n}(-2) \otimes \Psi^* \mathcal{O}_{\Lambda(\mathbb{P}^n)}(-n) \), where \( \Phi : \Gamma(\mathbb{P}^n) \rightarrow \mathbb{P}^n \) is the first projection and \( \Psi : \Gamma(\mathbb{P}^n) \rightarrow \Lambda(\mathbb{P}^n) \) is the second projection. Let \( \mathcal{Q}_{\Lambda(\mathbb{P}^n)} = \{H^0(L, \mathcal{O}_{\mathbb{P}^n}(1)|_L)\}_{L \in \Lambda(\mathbb{P}^n)} \) be the tautological bundle on \( \Lambda(\mathbb{P}^n) \), and let \( \mathcal{S} \) be the subbundle of \( \Psi^* \mathcal{Q}_{\Lambda(\mathbb{P}^n)} \) whose fiber at \( (p,L) \in \Gamma(\mathbb{P}^n) \) is

\[
\mathcal{S}(p,L) = \text{Ker}(H^0(L, \mathcal{O}_{\mathbb{P}^n}(1)|_L) \longrightarrow H^0(p, \mathcal{O}_{\mathbb{P}^n}(1)|_p)).
\]

Then the Chow ring of \( \Gamma(\mathbb{P}^n) \) is

\[
\text{CH}(\Gamma(\mathbb{P}^n)) \simeq \mathbb{Z}[s,t]/(t^{n+1}, \sum_{i=0}^{n} s^{n-i}t^i),
\]
where \( s = c_1(S) \) and \( t = c_1(\Phi^*\mathcal{O}_P(1)) \) (cf. \[5\] (14.6)).

Let \( X \subseteq P^3 \) be a nonsingular cubic surface. We define subvarieties of \( \Gamma(P^3) \) by

\[
Y_m = \{(p, L) \in \Gamma(P^3) \mid L \text{ intersects } X \text{ at } p \text{ with the multiplicity } \geq m\}
\]

for \( 1 \leq m \leq 3 \) and

\[
Y_\infty = \{(p, L) \in \Gamma(P^3) \mid L \text{ is contained in } X\}.
\]

By the first projection \( \Phi, Y_1 \) is a \( P^2 \)-bundle over \( X \), and \( Y_2 \) is a \( P^1 \)-bundle over \( X \). By \[5\] Theorem 3.5, \( Y_3 \) is a nonsingular projective irreducible surface, and the first projection \( \Phi|_{Y_3} \) is a generically finite morphism of degree 2 over \( X \). Since \( X \) contains 27 lines in \( P^3 \), \( Y_\infty \) is a disjoint union of 27 rational curves.

Let \( F \in H^0(P^3, \mathcal{O}_{P^3}(3)) \) be a section which define the cubic surface \( X \). The restriction \( F|_L \in H^0(L, \mathcal{O}_{P^3}(3)|_L) \) is contained in the image of the natural injective homomorphism

\[
S(p, L)^{\otimes m} \otimes H^0(L, \mathcal{O}_{P^3}(3 - m)|_L) \longrightarrow H^0(L, \mathcal{O}_{P^3}(3)|_L)
\]

if and only if the pair \((p, L)\) is contained in \( Y_m \). Hence, for \( 1 \leq m \leq 3 \), the subvariety \( Y_m \) is defined as the zeros of a regular section of the vector bundle

\[
\frac{\Psi^* \text{Sym}^3 \mathcal{O}_{\text{A}(P^3)}}{\mathcal{S}^{\otimes m} \otimes \Psi^* \text{Sym}^{3-m} \mathcal{O}_{\text{A}(P^3)}} \cong \Phi^* \mathcal{O}_{P^3}(4 - m) \otimes \Psi^* \text{Sym}^{m-1} \mathcal{O}_{\text{A}(P^3)}
\]
on \( \Gamma(P^n) \), where the isomorphism is given in \[9\] \textsection 2.

**Proposition 2.1.** \( Y_3 \) is a minimal surface of general type with the geometric genus \( p_g(Y_3) = 4 \), the irregularity \( q(Y_3) = 0 \) and the square of the canonical divisor \( K_{Y_3}^2 = 6 \), and the first projection \( \Phi|_{Y_3} \) is the canonical map of the surface \( Y_3 \).

**Proof.** Since

\[
\begin{cases}
\mathcal{O}_{\Gamma(P^3)}(Y_1) \cong \Phi^* \mathcal{O}_{P^3}(3), \\
\mathcal{O}_{Y_1}(Y_2) \cong (\Phi^* \mathcal{O}_{P^3}(2) \otimes \mathcal{S})|_{Y_1} \cong (\Phi^* \mathcal{O}_{P^3}(1) \otimes \Psi^* \mathcal{O}_{\text{A}(P^3)}(1))|_{Y_1}, \\
\mathcal{O}_{Y_2}(Y_3) \cong (\Phi^* \mathcal{O}_{P^3}(1) \otimes \mathcal{S}^\otimes 2)|_{Y_2} \cong (\Phi^* \mathcal{O}_{P^3}(-1) \otimes \Psi^* \mathcal{O}_{\text{A}(P^3)}(2))|_{Y_2}
\end{cases}
\]

and \( K_{\Gamma(P^3)} = \Phi^* \mathcal{O}_{P^3}(-2) \otimes \Psi^* \mathcal{O}_{\text{A}(P^3)}(-3) \), we have

\[
\begin{cases}
K_{Y_1} \cong (\Phi^* \mathcal{O}_{P^3}(1) \otimes \Psi^* \mathcal{O}_{\text{A}(P^3)}(-3))|_{Y_1}, \\
K_{Y_2} \cong (\Phi^* \mathcal{O}_{P^3}(2) \otimes \Psi^* \mathcal{O}_{\text{A}(P^3)}(-2))|_{Y_2}, \\
K_{Y_3} \cong (\Phi^* \mathcal{O}_{P^3}(1))|_{Y_3}.
\end{cases}
\]

Since \( H^i(\Gamma(P^3), \Phi^* \mathcal{O}_{P^3}(-3)) = 0 \) and \( H^i(\Gamma(P^3), \Phi^* \mathcal{O}_{P^3}(-2)) = 0 \) for any \( i \), the restriction induces isomorphisms

\[
\begin{cases}
H^i(\Gamma(P^3), \mathcal{O}_{\Gamma(P^3)}) \cong H^i(Y_1, \mathcal{O}_{Y_1}), \\
H^i(\Gamma(P^3), \Phi^* \mathcal{O}_{P^3}(1)) \cong H^i(Y_1, (\Phi^* \mathcal{O}_{P^3}(1))|_{Y_1})
\end{cases}
\]
for any \(i\). Since \(H^i(\Gamma(\mathbb{P}^3), \Phi^*\mathcal{O}_{\mathbb{P}^3}(j) \otimes \Psi^*\mathcal{O}_{\Lambda(\mathbb{P}^3)}(-1)) = 0\) for any \(i\) and \(j\), we have \(H^i(Y_1, (\Phi^*\mathcal{O}_{\mathbb{P}^3}(j) \otimes \Psi^*\mathcal{O}_{\Lambda(\mathbb{P}^3)}(-1))|_{Y_1}) = 0\) for any \(i\) and \(j\), hence the restriction induces isomorphisms

\[
\begin{cases}
H^i(Y_1, \mathcal{O}_{Y_1}) \simeq H^i(Y_2, \mathcal{O}_{Y_2}), \\
H^i(Y_1, (\Phi^*\mathcal{O}_{\mathbb{P}^3}(1))|_{Y_1}) \simeq H^i(Y_2, (\Phi^*\mathcal{O}_{\mathbb{P}^3}(1))|_{Y_2})
\end{cases}
\]

for any \(i\), and the dimension of these cohomology groups are

\[
h^i(Y_2, \mathcal{O}_{Y_2}) = h^i(\Gamma(\mathbb{P}^3), \mathcal{O}_{\Gamma(\mathbb{P}^3)}) = \begin{cases} 1 & \text{if } i = 0, \\ 0 & \text{if } i \neq 0, \end{cases}
\]

and

\[
h^i(Y_2, (\Phi^*\mathcal{O}_{\mathbb{P}^3}(1))|_{Y_2}) = h^i(\Gamma(\mathbb{P}^3), (\Phi^*\mathcal{O}_{\mathbb{P}^3}(1))) = \begin{cases} 4 & \text{if } i = 0, \\ 0 & \text{if } i \neq 0. \end{cases}
\]

By the exact sequence

\[
0 \longrightarrow K_{Y_2} \longrightarrow (\Phi^*\mathcal{O}_{\mathbb{P}^3}(1))|_{Y_2} \longrightarrow K_{Y_3} \longrightarrow 0
\]

and the duality

\[
H^i(Y_2, K_{Y_2}) \simeq H^{3-i}(Y_2, \mathcal{O}_{Y_2})^\vee,
\]

we have \(p_g(Y_3) = 4\) and \(q(Y_3) = 0\), and \(\Phi|_{Y_3}\) is the canonical map. Since \(K_{Y_3} \simeq (\Phi^*\mathcal{O}_{\mathbb{P}^3}(1))|_{Y_3}\) is nef and the image of the canonical map is the surface \(X\), the surface \(Y_3\) is a minimal surface of general type.

Since \(Y_3\) is defined as the zeros of a regular section of the vector bundle

\[
\frac{\Psi^*\text{Sym}^3 Q_{\Lambda(\mathbb{P}^3)}}{S \otimes 3} \simeq \Phi^*\mathcal{O}_{\mathbb{P}^3}(1) \otimes \Psi^*\text{Sym}^2 Q_{\Lambda(\mathbb{P}^3)},
\]

its class in the Chow ring of \(\Gamma(\mathbb{P}^3)\) is

\[
[Y_3] = c_3(\Phi^*\mathcal{O}_{\mathbb{P}^3}(1) \otimes \Psi^*\text{Sym}^2 Q_{\Lambda(\mathbb{P}^3)}) = 6s^2t + 15st^2 + 6t^3 \in \text{CH}^3(\Gamma(\mathbb{P}^3)),
\]

hence

\[
K_{Y_3}^2 = \deg (c_1(\Phi^*\mathcal{O}_{\mathbb{P}^3}(1))^2 \cdot [Y_3]) = 6.
\]

Remark 2.2. Proposition 2.1 implies that the Hodge number \(h^1(Y_3, \Omega^1_{Y_3}) = 44\). Minimal surfaces with such numerical invariants are classified by Horikawa, and \(Y_3\) is of type Ib in \([7]\). Since \(Y_3\) is simply connected by \([7\) Theorem 12.1], we have \(H_1(Y_3, \mathbb{Z}) = 0\), hence \(H^i(Y_3, \mathbb{Z})\) has no torsion element for any \(i\).
Since the cubic surface $X$ is recovered as the image of the canonical map of $Y_3$, we have the following Torelli type theorem.

**Corollary 2.3.** The isomorphism class of the cubic surface $X$ is uniquely determined by the isomorphism class of $Y_3$.

**Proposition 2.4.** Each component of $Y_\infty$ is a $(-3)$-curve on $Y_3$.

*Proof.* Since $\mathcal{O}_{Y_3}(Y_\infty) \simeq S^{\otimes 3}|_{Y_3}$, the self intersection number of $Y_\infty$ on $Y_3$ is

$$(Y_\infty, Y_\infty) = \deg \left( c_1(S^{\otimes 3})^2 \cdot [Y_3] \right) = -81.$$ 

The self intersection number of a component of $Y_\infty$ is less than $-1$ because $K_{Y_3}$ is nef, and the component is not a $(-2)$-curve because its image by the canonical map is a line in $\mathbb{P}^3$. Since $Y_\infty$ is a disjoint union of 27 rational curves, each component of $Y_\infty$ is $(-3)$-curve on $Y_\infty$. \hfill \square

**Remark 2.5.** The second projection

$$\Psi|_{Y_3} : Y_3 \longrightarrow \Lambda(\mathbb{P}^3); (p, L) \longmapsto L,$$

is birational to its image $Z_3 = \Psi(Y_3)$, which induces an isomorphism $Y_3 \setminus Y_\infty \simeq Z_3 \setminus Z_\infty$, where $Z_\infty = \{ L \in \Lambda(\mathbb{P}^3) \mid L \subset X \}$ is equal to the singular locus of $Z_3$.

## 3 The double cover branched along Hessian

For simplicity, we denote the first projection $\Phi|_{Y_3} : Y_3 \to X$ by $\phi : Y \to X$. Let $R$ be the ramification divisor of $\phi : Y \to X$. Since $R$ is the zeros of the determinant of the differential $d\phi : T_Y \to \phi^*T_X$, its class in $\text{CH}^1(Y)$ is

$$[R] = c_1(K_Y \otimes \phi^*K_X) = c_1((\phi^*\mathcal{O}_{\mathbb{P}^3}(2))|_Y).$$

We denote by $B = \phi_*R$ the branch divisor of $\phi$. Let $F(x_0, \ldots, x_3) \in \mathbb{C}[x_0, \ldots, x_3]$ be a cubic polynomial which defines the nonsingular cubic surface $X$.

**Proposition 3.1.** $B \subset X$ is the zeros of the Hessian

$$\det \left( \frac{\partial^2 F}{\partial x_i \partial x_j} \right)_{0 \leq i, j \leq 3} \in H^0(X, \mathcal{O}_{\mathbb{P}^3}(4)|_X).$$

*Proof.* For $p = [a_0 : a_1 : a_2 : a_3] \in \mathbb{P}^3$, if $a_0 \neq 0$, then there is an isomorphism

$$\mathbb{P}^2 \xrightarrow{\sim} \Phi^{-1}(p) \subset \Gamma(\mathbb{P}^3); \quad q = [b_1 : b_2 : b_3] \longmapsto (p, L_{(p,q)}),$$
where \( L_{(p,q)} \) denotes the line through the points \( p \) and \([0 : b_1 : b_2 : b_3] \) in \( \mathbb{P}^3 \);

\[
L_{(p,q)} = \{(a_0 t_0 : a_1 t_1 : \cdots : a_3 t_0 + b_3 t_1) \in \mathbb{P}^3 \mid [t_0 : t_1] \in \mathbb{P}^1 \}.
\]

For \( 0 \leq i \leq 3 \), we set a polynomial \( F_i(x, z) \) on variables \((x_0, \ldots, x_3, z_1, \ldots, z_3) \) inductively by

\[
F_0(x, z) = F(x_1, \ldots, x_3)
\]

and

\[
F_i(x, z) = \frac{1}{i} \sum_{j=1}^{3} \frac{\partial F_{i-1}}{\partial x_j}(x, z) z_j.
\]  \hspace{1cm} (3.1)

Since

\[
F(a_0 t_0, a_1 t_1, a_2 t_0 + b_2 t_1, a_3 t_0 + b_3 t_1)
= F_0(a, b)t_0^3 + F_1(a, b)t_0^2t_1 + F_2(a, b)t_0t_1^2 + F_3(a, b)t_1^3,
\]

if \( p \in X \), then

\[
\phi^{-1}(p) \simeq \{q = [b_1 : b_2 : b_3] \in \mathbb{P}^2 \mid F_1(a, b) = 0, \ F_2(a, b) = 0\}.
\]

\( p \in X \) is contained in \( B \) if and only if there exists \([b_1 : b_2 : b_3] \in \mathbb{P}^2 \) such that \( F_1(a, b) = F_2(a, b) = 0 \) and the rank of the matrix

\[
\begin{pmatrix}
\frac{\partial F_1}{\partial x_1}(a, b) & \frac{\partial F_1}{\partial x_2}(a, b) & \frac{\partial F_1}{\partial x_3}(a, b) \\
\frac{\partial F_2}{\partial x_1}(a, b) & \frac{\partial F_2}{\partial x_2}(a, b) & \frac{\partial F_2}{\partial x_3}(a, b)
\end{pmatrix}
= \begin{pmatrix}
\sum_{j=1}^{3} \frac{\partial F}{\partial x_j}(a) b_j & \sum_{j=1}^{3} \frac{\partial^2 F}{\partial x_j \partial x_j}(a) b_j & \sum_{j=1}^{3} \frac{\partial^2 F}{\partial x_j \partial x_j}(a) b_j
\end{pmatrix}
\]

is less than 2. Since \( \left( \frac{\partial F}{\partial x_1}(a), \frac{\partial F}{\partial x_2}(a), \frac{\partial F}{\partial x_3}(a) \right) \neq (0, 0, 0) \), the condition on the rank of the matrix is equivalent to the existence of \( b_0 \in \mathbb{C} \) such that

\[
b_0 \left( \frac{\partial F}{\partial x_1}(a) \right) + (b_1, b_2, b_3) \begin{pmatrix}
\frac{\partial^2 F}{\partial x_1 \partial x_1}(a) & \frac{\partial^2 F}{\partial x_1 \partial x_2}(a) & \frac{\partial^2 F}{\partial x_1 \partial x_3}(a) \\
\frac{\partial^2 F}{\partial x_2 \partial x_1}(a) & \frac{\partial^2 F}{\partial x_2 \partial x_2}(a) & \frac{\partial^2 F}{\partial x_2 \partial x_3}(a) \\
\frac{\partial^2 F}{\partial x_3 \partial x_1}(a) & \frac{\partial^2 F}{\partial x_3 \partial x_2}(a) & \frac{\partial^2 F}{\partial x_3 \partial x_3}(a)
\end{pmatrix}
= 0.
\]

Then \( F_1(a, b) = 0 \) implies \( F_2(a, b) = 0 \), because

\[
F_2(a, b) = \frac{1}{2} \begin{pmatrix} b_1 & b_2 & b_3 \end{pmatrix} \begin{pmatrix}
\frac{\partial^2 F}{\partial x_1 \partial x_1}(a) & \frac{\partial^2 F}{\partial x_1 \partial x_2}(a) & \frac{\partial^2 F}{\partial x_1 \partial x_3}(a) \\
\frac{\partial^2 F}{\partial x_2 \partial x_1}(a) & \frac{\partial^2 F}{\partial x_2 \partial x_2}(a) & \frac{\partial^2 F}{\partial x_2 \partial x_3}(a) \\
\frac{\partial^2 F}{\partial x_3 \partial x_1}(a) & \frac{\partial^2 F}{\partial x_3 \partial x_2}(a) & \frac{\partial^2 F}{\partial x_3 \partial x_3}(a)
\end{pmatrix}
\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}
\]

\[
= - \frac{b_0}{2} \begin{pmatrix} \frac{\partial F}{\partial x_1}(a) & \frac{\partial F}{\partial x_2}(a) & \frac{\partial F}{\partial x_3}(a) \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = - \frac{b_0}{2} F_1(a, b).
\]
Hence, \( p \in X \) is contained in \( B \) if and only if there exists \([b_0 : b_1 : b_2 : b_3] \in \mathbb{P}^3\) such that

\[
\begin{pmatrix} b_0 & b_1 & b_2 & b_3 \\ \frac{\partial F}{\partial x_1}(a) & \frac{\partial^2 F}{\partial x_1^2}(a) & \frac{\partial^2 F}{\partial x_1 x_2}(a) & \frac{\partial^2 F}{\partial x_1 x_3}(a) \\ \frac{\partial F}{\partial x_2}(a) & \frac{\partial^2 F}{\partial x_2^2}(a) & \frac{\partial^2 F}{\partial x_2 x_1}(a) & \frac{\partial^2 F}{\partial x_2 x_3}(a) \\ \frac{\partial F}{\partial x_3}(a) & \frac{\partial^2 F}{\partial x_3^2}(a) & \frac{\partial^2 F}{\partial x_3 x_1}(a) & \frac{\partial^2 F}{\partial x_3 x_2}(a) \end{pmatrix} = (0 \ 0 \ 0 \ 0),
\]

and it is equivalent to

\[
0 = \det \begin{pmatrix} \frac{\partial F}{\partial x_1}(a) & \frac{\partial^2 F}{\partial x_1^2}(a) & \frac{\partial^2 F}{\partial x_1 x_2}(a) & \frac{\partial^2 F}{\partial x_1 x_3}(a) \\ \frac{\partial F}{\partial x_2}(a) & \frac{\partial^2 F}{\partial x_2^2}(a) & \frac{\partial^2 F}{\partial x_2 x_1}(a) & \frac{\partial^2 F}{\partial x_2 x_3}(a) \\ \frac{\partial F}{\partial x_3}(a) & \frac{\partial^2 F}{\partial x_3^2}(a) & \frac{\partial^2 F}{\partial x_3 x_1}(a) & \frac{\partial^2 F}{\partial x_3 x_2}(a) \end{pmatrix} = \frac{a_0^2}{4} \cdot \det \begin{pmatrix} \frac{\partial^2 F}{\partial x_1 x_2}(a) & \frac{\partial^2 F}{\partial x_1 x_3}(a) & \frac{\partial^2 F}{\partial x_1 x_3}(a) \\ \frac{\partial^2 F}{\partial x_2 x_3}(a) & \frac{\partial^2 F}{\partial x_2 x_3}(a) & \frac{\partial^2 F}{\partial x_2 x_3}(a) \\ \frac{\partial^2 F}{\partial x_3 x_1}(a) & \frac{\partial^2 F}{\partial x_3 x_1}(a) & \frac{\partial^2 F}{\partial x_3 x_1}(a) \end{pmatrix}.
\]

Hence \( B \) is defined by the Hessian on \( X \setminus \{x_0 \neq 0\} \). In the same way, we can show that \( B \) is defined by the Hessian on \( X \setminus \{x_i \neq 0\} \) for \( 1 \leq i \leq 3 \).

Let \( E \) be the sum of all components of \( R \) which contract to points by \( \phi \), and let \( D \) be the divisor such that \( R = D + E \). For a line \( L \) on \( X \), we denote by \( L^+ \) the corresponding component of \( Y_\infty \);

\[
L^+ = \{(p, L') \in \Gamma(\mathbb{P}^3) \mid L' = L\}.
\]

Let \( L^- \) be the other component of \( \phi^*(L) \) dominating \( L \) by \( \phi \), and let \( Y^-_\infty \) be the sum of \( L^- \) for all lines on \( X \). A point \( p \) on the cubic surface \( X \) is called an Eckardt point if there are three lines through \( p \) on \( X \).

**Theorem 3.2.** The divisor \( D \) is a disjoint union of nonsingular curves, \( E \) is a disjoint union of \((-2)\)-curves on \( Y \), and \( Y^-_\infty \) is a disjoint union of \((-3)\)-curves on \( Y \). The divisors \( R + Y_\infty \), \( R + Y^-_\infty \) and \( E + Y_\infty + Y^-_\infty \) are reduced simple normal crossing divisors. The branch divisor \( B \) has at most nodes as its singularities, and the singular locus of \( B \) is equal to the set of Eckardt points of \( X \). A line \( L \) on \( X \) intersects \( B \) at two points with each multiplicity 2, and

\[
\phi^* L = L^+ + L^- + \sum_{e \in L^- \cap \text{Sing}(B)} \phi^{-1}(e).
\]
Lemma 3.3. Let $F(x) = \sum c_{ijk}x_0^{3-i-j-k}x_1^ix_2^jx_3^k$ be an equation of a nonsingular cubic surface $X$, and let $p$ be a point on $X$.

1. If $\phi^{-1}(p)$ is a set of distinct two points, then $F(x)$ is normalized by a transformation of the homogeneous coordinate to satisfy $p = [1 : 0 : 0 : 0]$, $c_{000} = c_{100} = c_{010} = c_{200} = c_{020} = 0$ and $c_{001} = c_{110} = 1$.

2. If $\phi^{-1}(p)$ is a point, then $F(x)$ is normalized by a transformation of the homogeneous coordinate to satisfy $p = [1 : 0 : 0 : 0]$, $c_{000} = c_{100} = c_{010} = c_{200} = c_{110} = 0$ and $c_{001} = c_{020} = c_{210} = 1$.

3. If $\phi^{-1}(p) \simeq \mathbb{P}^1$, then $F(x)$ is normalized by a transformation of the homogeneous coordinate to satisfy $p = [1 : 0 : 0 : 0]$, $c_{000} = c_{100} = c_{010} = c_{200} = c_{110} = c_{020} = c_{210} = c_{120} = 0$ and $c_{001} = 3c_{300} = 3c_{030} = 1$.

Proof. First, we can chose a homogeneous coordinate $[x_0 : \cdots : x_3]$ as $p = [1 : 0 : 0 : 0]$. Then $p \in X$ implies that $c_{000} = 0$. Since $X$ is nonsingular at $p$, $(c_{100}, c_{010}, c_{001}) \neq (0, 0, 0)$. We may assume that $c_{001} \neq 0$. By the transformation

$$
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{pmatrix} \mapsto \begin{pmatrix}
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  c_{100} & c_{010} & c_{001}
\end{pmatrix} \begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{pmatrix},
$$

we may assume that $(c_{100}, c_{010}, c_{001}) = (0, 0, 1)$.

1. We consider the case where $\phi^{-1}(p)$ is a set of distinct two points. Then the quadratic form $c_{200}x_1^2 + c_{110}x_1x_2 + c_{020}x_2^2$ is factorized into independent linear forms:

$$
c_{200}x_1^2 + c_{110}x_1x_2 + c_{020}x_2^2 = (\alpha_1x_1 + \alpha_2x_2)(\beta_1x_1 + \beta_2x_2).
$$

By the transformation

$$
\begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix} \mapsto \begin{pmatrix}
  \alpha_1 & \alpha_2 \\
  \beta_1 & \beta_2
\end{pmatrix} \begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix},
$$

$F(x)$ is normalized to satisfy $(c_{200}, c_{110}, c_{020}) = (0, 1, 0)$.

2. We consider the case where $\phi^{-1}(p)$ is a point. Then the quadratic form $c_{200}x_1^2 + c_{110}x_1x_2 + c_{020}x_2^2$ is the square of a nonzero linear form:

$$
c_{200}x_1^2 + c_{110}x_1x_2 + c_{020}x_2^2 = (\alpha_1x_1 + \alpha_2x_2)^2.
$$
and we may assume that \( \alpha_2 \neq 0 \). By the transformation

\[
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ \alpha_1 & \alpha_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},
\]

we may assume that \( (c_{200}, c_{110}, c_{020}) = (0, 0, 1) \). If \( c_{210} \neq 0 \), then by the transformation

\[
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} \sqrt{c_{210}} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},
\]

\( F(x) \) is normalized to satisfy \( c_{210} = 1 \). If \( c_{210} = 0 \) and \( c_{300} \neq 0 \), then by the transformation

\[
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -1/3c_{300} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},
\]

\( F(x) \) is normalized to satisfy \( c_{210} = 1 \). If \( (c_{300}, c_{210}) = (0, 0) \), then \( X \) is singular at \([a : 1 : 0 : 0]\), where \( a \) is a root of the quadratic equation

\[
\frac{\partial F}{\partial x_3}(s, 1, 0, 0) = s^2 + c_{101}s + c_{201} = 0.
\]

3. We consider the case where \( \phi^{-1}(p) \simeq \mathbb{P}^1 \). Then we have \( (c_{200}, c_{110}, c_{020}) = (0, 0, 0) \), and the cubic form \( c_{300}x_1^3 + c_{210}x_1^2x_2 + c_{120}x_1x_2^2 + c_{030}x_2^3 \) is factorized into nonzero linear forms;

\[
c_{300}x_1^3 + c_{210}x_1^2x_2 + c_{120}x_1x_2^2 + c_{030}x_2^3 = (\alpha_1x_1 + \alpha_2x_2)(\beta_1x_1 + \beta_2x_2)(\gamma_1x_1 + \gamma_2x_2).
\]

We have \( \alpha_1\beta_2 - \alpha_2\beta_1 \neq 0, \beta_1\gamma_2 - \beta_2\gamma_1 \neq 0, \) and \( \gamma_1\alpha_2 - \gamma_2\alpha_1 \neq 0, \) because for example if \( \alpha_1\beta_2 - \alpha_2\beta_1 = 0 \), then \( X \) is singular at \([a : -\alpha_2 : \alpha_1 : 0]\), where \( a \) is a root of the quadratic equation

\[
\frac{\partial F}{\partial x_3}(s, -\alpha_2, \alpha_1, 0) = s^2 + (c_{011}\alpha_1 - c_{101}\alpha_2)s + (c_{021}\alpha_1^2 - c_{111}\alpha_1\alpha_2 + c_{201}\alpha_2^2) = 0.
\]

Let \( \omega \in \mathbb{C} \) be a primitive 3-nd root of unity. By the transformation

\[
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} \frac{\alpha_1\beta_2\gamma_1 + \alpha_1\beta_1\gamma_2 + \alpha_2\beta_1\gamma_2 - \alpha_2\beta_2\gamma_1\omega - \alpha_1\beta_2\gamma_2\omega}{\sqrt{d}} & -\frac{\alpha_2\beta_1\gamma_2 - \alpha_2\beta_2\gamma_1\omega - \alpha_1\beta_2\gamma_2\omega}{\sqrt{d}} \\ -\frac{\alpha_1\beta_2\gamma_1 + \alpha_1\beta_1\gamma_2 + \alpha_2\beta_1\gamma_2 - \alpha_2\beta_2\gamma_1\omega - \alpha_1\beta_2\gamma_2\omega}{\sqrt{d}} & \frac{\alpha_2\beta_1\gamma_2 - \alpha_2\beta_2\gamma_1\omega - \alpha_1\beta_2\gamma_2\omega}{\sqrt{d}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},
\]

where

\[
d = \det\left(\begin{array}{cc} \alpha_1\beta_2\gamma_1 + \alpha_1\beta_1\gamma_2 + \alpha_2\beta_1\gamma_2 - \alpha_2\beta_2\gamma_1\omega - \alpha_1\beta_2\gamma_2\omega \\ -\alpha_1\beta_2\gamma_1 + \alpha_2\beta_1\gamma_2 - \alpha_2\beta_2\gamma_1\omega - \alpha_1\beta_2\gamma_2\omega \end{array}\right) = (\omega - \omega^2)(\alpha_1\beta_2 - \alpha_2\beta_1)(\beta_1\gamma_2 - \beta_2\gamma_1)(\gamma_1\alpha_2 - \gamma_2\alpha_1),
\]

\( F(x) \) is normalized to satisfy \( (c_{300}, c_{210}, c_{120}, c_{020}) = (\frac{1}{3}, 0, 0, \frac{1}{3}) \).
Proof of Theorem 3.2. For $p \in X$, by Lemma 3.3 we may assume that $p = [1:0:0:0]$, $c_{000} = c_{100} = c_{010} = 0$ and $c_{001} = 1$. Then

$$X \setminus \{x_0 \neq 0\} \approx \{(\xi_1, \xi_2, \xi_3) \in \mathbb{C}^3 \mid F(1, \xi_1, \xi_2, \xi_3) = 0\},$$

and $(\xi_1, \xi_2)$ gives a local coordinate of $X$ at $p$ because $\frac{\partial F}{\partial x_3}(p) = c_{001} \neq 0$. For $[s_1 : s_2] \in \mathbb{P}^1$, we set a line on $\mathbb{P}^3$ by

$$L_{[s_1:s_2]} = \{[x_0 : \cdots : x_3] \in \mathbb{P}^3 \mid s_1x_2 = s_2x_1, \ x_3 = 0\},$$

which intersects $X$ at $p$ with multiplicity $\geq 2$. For $0 \leq i \leq 3$, we set a polynomial by

$$f_i(\xi_1, \xi_2, \xi_3) = F_i(1, \xi_1, \xi_2, \xi_3),$$

where $F_i(x, z)$ is the polynomial defined in (3.1). Then $Y$ is locally defined by these polynomials on a neighborhood of $(p, L_{[1:0]}) \in Y;

$$Y \approx \{(\xi_1, \xi_2, \xi_3) \in \mathbb{C}^3 \mid f_0(\xi, \zeta) = f_1(\xi, \zeta) = f_2(\xi, \zeta) = 0\}.$$

In order to give a local coordinate of $Y$, we divide the case into three types.

1. The case where $\phi^{-1}(p)$ is a set of distinct two points. By Lemma 3.3, we may assume that $c_{000} = c_{100} = c_{010} = c_{200} = c_{202} = 0$ and $c_{001} = c_{110} = 1$. Then we have $\phi^{-1}(p) = \{(p, L_{[1:0]}), (p, L_{[0:1]})\}$. Since

\[
\begin{vmatrix}
\frac{\partial f_0}{\partial \xi_1}(0,0,0,0) & \frac{\partial f_0}{\partial \xi_2}(0,0,0,0) & \frac{\partial f_0}{\partial \xi_3}(0,0,0,0) \\
\frac{\partial f_1}{\partial \xi_1}(0,0,0,0) & \frac{\partial f_1}{\partial \xi_2}(0,0,0,0) & \frac{\partial f_1}{\partial \xi_3}(0,0,0,0) \\
\frac{\partial f_2}{\partial \xi_1}(0,0,0,0) & \frac{\partial f_2}{\partial \xi_2}(0,0,0,0) & \frac{\partial f_2}{\partial \xi_3}(0,0,0,0)
\end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = -1 \neq 0,
\end{vmatrix}
\]

$(\xi_1, \xi_2)$ gives a local coordinate of $Y$ at $(p, L_{[1:0]})$ and $\phi$ is a local isomorphism in a neighborhood of $(p, L_{[1:0]})$. When $L_{[1:0]}$ is contained in $X$, $L_{[1:0]}^+ \subset Y$ is locally isomorphic to $\{(\xi_1, \xi_2) \mid \xi_2 = 0\}$, and when $L_{[0:1]}$ is contained in $X$, $L_{[0:1]}^- \subset Y$ is locally isomorphic to $\{(\xi_1, \xi_2) \mid \xi_1 = 0\}$. Hence, if $(p, L_{[1:0]}) \in L_{[1:0]}^+ \cap L_{[0:1]}^-$, then $L_{[1:0]}^+$ intersects $L_{[0:1]}^-$ transversally at $(p, L_{[1:0]}) \in Y$. In the same way, we can see the picture of a neighborhood of $(p, L_{[0:1]})$.

2. The case where $\phi^{-1}(p)$ is a point. By Lemma 3.3, we may assume that $c_{000} = c_{100} = c_{010} = c_{200} = c_{110} = 0$ and $c_{001} = c_{202} = c_{210} = 1$. Then $\phi^{-1}(p) = \{(p, L_{[1:0]})\}$. Since

\[
\begin{vmatrix}
\frac{\partial f_0}{\partial \xi_1}(0,0,0,0) & \frac{\partial f_0}{\partial \xi_2}(0,0,0,0) & \frac{\partial f_0}{\partial \xi_3}(0,0,0,0) \\
\frac{\partial f_1}{\partial \xi_1}(0,0,0,0) & \frac{\partial f_1}{\partial \xi_2}(0,0,0,0) & \frac{\partial f_1}{\partial \xi_3}(0,0,0,0) \\
\frac{\partial f_2}{\partial \xi_1}(0,0,0,0) & \frac{\partial f_2}{\partial \xi_2}(0,0,0,0) & \frac{\partial f_2}{\partial \xi_3}(0,0,0,0)
\end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ 0 & c_{010} & 1 \\ 1 & c_{202} & c_{101} \end{vmatrix} = 1 \neq 0,
\end{vmatrix}
\]
there are holomorphic functions \( \varphi_2(\xi_1, \zeta_2), \varphi_3(\xi_1, \zeta_2) \) and \( \mu_3(\xi_1, \zeta_2) \) on a neighborhood of \((\xi_1, \zeta_2) = (0, 0)\) such that

\[
\varphi_2(0, 0) = 0, \quad \varphi_3(0, 0) = 0, \quad \mu_3(0, 0) = 0
\]

and

\[
f_i(\xi_1, \varphi_2(\xi_1, \zeta_2), \varphi_3(\xi_1, \zeta_2), \zeta_2, \mu_3(\xi_1, \zeta_2)) = 0
\]

for \(0 \leq i \leq 2\). We remark that

\[
\varphi_2(\xi_1, \zeta_2) \equiv -3c_{300}\xi_1 + (-9c_{300}^2c_{101} + 9c_{300}c_{120}c_{101} + 9c_{300}c_{201} - 3c_{300}c_{101})\xi_1^2
\]

\[
+ (-6c_{300}c_{101} + 6c_{300}c_{120} - 2)\xi_1\zeta_2 - \zeta_2^2 \mod (\xi_1^3, \xi_1^2\zeta_2, \xi_1\zeta_2^2, \zeta_2^3),
\]

\[
\varphi_3(\xi_1, \zeta_2) \equiv -9c_{300}^2\xi_1 \mod (\xi_1^3, \xi_1^2\zeta_2, \xi_1\zeta_2^2, \zeta_2^3),
\]

\[
\mu_3(\xi_1, \zeta_2) \equiv (9c_{300}^2c_{101} - 9c_{300}c_{120} + 3c_{300})\xi_1^2 + 6c_{300}\xi_1\zeta_2 \mod (\xi_1^3, \xi_1^2\zeta_2, \xi_1\zeta_2^2, \zeta_2^3).
\]

Then \((\xi_1, \zeta_2)\) is a local coordinate of \(Y\) at \((p, L_{[1:0]}\)) and \(R = D\) is locally isomorphic to \(\{(\xi_1, \zeta_2) \mid \frac{\partial \varphi_2}{\partial \zeta_2}(\xi_1, \zeta_2) = 0\}\), and it is nonsingular at \((p, L_{[1:0]}\)) because

\[
\frac{\partial^2 \varphi_2}{\partial \xi_1^2}(0, 0) = -2 \neq 0.
\]

There is a holomorphic function \(\sigma(\xi_1)\) on a neighborhood of \(\xi_1 = 0\) such that \(\sigma(0) = 0\) and \(\frac{\partial \varphi_2}{\partial \zeta_2}(\xi_1, \sigma(\xi_1)) = 0\). Then \(B \subset X\) is locally isomorphic to \(\{(\xi_1, \xi_2) \mid \xi_2 = \varphi_2(\xi_1, \sigma(\xi_1))\}\), and it is nonsingular at \(p\). When \(L_{[1:0]}\) is contained in \(X\), we have \(c_{300} = 0\) and there is a holomorphic function \(\eta_2(\xi_1, \zeta_2)\) such that \(\varphi_2(\xi_1, \zeta_2) = \zeta_2\eta_2(\xi_1, \zeta_2)\). Then \(L_{[1:0]}^+ \subset Y\) is locally isomorphic to \(\{(\xi_1, \zeta_2) \mid \zeta_2 = 0\}\), and \(L_{[1:0]}^- \subset Y\) is locally isomorphic to \(\{(\xi_1, \zeta_2) \mid \eta_2(\xi_1, \zeta_2) = 0\}\).

Since

\[
\begin{pmatrix}
\frac{\partial^2 \varphi_2}{\partial \xi_1 \partial \zeta_2}(0, 0) & \frac{\partial \varphi_2}{\partial \xi_1}(0, 0) & \frac{\partial \varphi_2}{\partial \zeta_2}(0, 0) \\
\frac{\partial^2 \varphi_2}{\partial \zeta_2}(0, 0) & \frac{\partial \varphi_2}{\partial \xi_1}(0, 0) & \frac{\partial \varphi_2}{\partial \partial \zeta_2}(0, 0)
\end{pmatrix}
= \begin{pmatrix}
-2 & 0 & -2 \\
-2 & 1 & -1
\end{pmatrix},
\]

\(D\) intersects \(L_{[1:0]}^+\) and \(L_{[1:0]}^-\) transversally, and \(L_{[1:0]}^+\) intersects \(L_{[1:0]}^-\) transversally at \((p, L_{[1:0]}\)) \(Y\). Since \(L\) is locally isomorphic to \(\{(\xi_1, \xi_2) \mid \xi_2 = 0\}\) and

\[
\begin{aligned}
\varphi_2(\xi_1, \sigma(\xi_1))|_{\xi_1=0} &= 0, \\
\frac{d}{d\xi_1}(\varphi_2(\xi_1, \sigma(\xi_1)))|_{\xi_1=0} &= 0, \\
\frac{d^2}{d\xi_1^2}(\varphi_2(\xi_1, \sigma(\xi_1)))|_{\xi_1=0} &= 2 \neq 0,
\end{aligned}
\]

\(L\) intersects \(B\) at \(p\) with multiplicity 2.

3. The case where \(\phi^{-1}(p) \simeq \mathbb{P}^1\). By Lemma 3.3 we may assume that \(c_{000} = c_{100} =\)
there are holomorphic functions \( \varphi_1(\xi_2, \zeta_2), \varphi_3(\xi_2, \zeta_2) \) and \( \mu_3(\xi_2, \zeta_2) \) on a neighborhood of \( \{(\xi_2, \zeta_2) \mid \xi_2 = 0\} \) such that
\[
\varphi_1(0, \zeta_2) = 0, \quad \varphi_3(0, \zeta_2) = 0, \quad \mu_3(0, \zeta_2) = 0
\]
and
\[
f_i(\varphi_1(\xi_2, \zeta_2), \xi_2, \varphi_3(\xi_2, \zeta_2), \zeta_2, \mu_3(\xi_2, \zeta_2)) = 0
\]
for \( 0 \leq i \leq 2 \). We remark that
\[
\begin{align*}
\varphi_1(\xi_2, \zeta_2) &\equiv -\xi_2^2 \xi_2 + (c_{101} + c_{011} \zeta_2 + c_{021} \zeta_2^2) \xi_2 \quad \text{mod} \ (\xi_2^3), \\
\varphi_3(\xi_2, \zeta_2) &\equiv 0 \mod (\xi_2^3), \\
\mu_3(\xi_2, \zeta_2) &\equiv (-\xi_2 - \xi_2^3) \xi_2 \quad \text{mod} \ (\xi_2^3).
\end{align*}
\]
There is a holomorphic function \( \eta_1(\xi_2, \zeta_2) \) such that
\[
\varphi_1(\xi_2, \zeta_2) = \xi_2 \eta_1(\xi_2, \zeta_2).
\]
Since \( R \) is locally isomorphic to \( \{(\xi_2, \zeta_2) \mid \xi_2 \neq 0\} \), \( E \) is locally isomorphic to \( \{(\xi_2, \zeta_2) \mid \xi_2 = 0\} \) and \( D \) is locally isomorphic to \( \{(\xi_2, \zeta_2) \mid \eta = 0\} \). We remark that \( L_{[1,\lambda]} \subset X \) if and only if \( \lambda^3 + 1 = 0 \). Hence \( p \) is an Eckardt point on \( X \).
We assume that \( \lambda^3 + 1 = 0 \). Then \( L_{[1,\lambda]} \) is locally isomorphic to \( \{(\xi_1, \xi_2) \mid \xi_2 = \lambda \xi_1\} \) and \( \phi^* L_{[1,\lambda]} \) is locally isomorphic to \( \{(\xi_2, \zeta_2) \mid \xi_2 \equiv \lambda \varphi(\xi_2, \zeta_2) \} \), hence \( L^{+}_{[1,\lambda]} + L^{-}_{[1,\lambda]} \) is locally isomorphic to \( \{(\xi_2, \zeta_2) \mid 1 = \lambda \eta_1(\xi_2, \zeta_2) \} \). Since \( \eta_1(0, \zeta_2) = -\xi_2^2 \),
\[
(0, \zeta_2) \in L^{+}_{[1,\lambda]} + L^{-}_{[1,\lambda]} \iff 1 = -\lambda \xi_2^2 \iff \xi_2^2 = \lambda^2.
\]
Then \( L^{+}_{[1,\lambda]} \) intersects \( E \) transversally at \( (p, L^{+}_{[1,\lambda]}) \) by
\[
\frac{\partial}{\partial \xi_2}(1 - \lambda \eta_1) \bigg|_{(\xi_2, \zeta_2) = (0, \lambda)} = 2\lambda^2 \neq 0,
\]
and \( L^{-}_{[1,\lambda]} \) intersects \( E \) transversally at \( (p, L^{-}_{[1,\lambda]}) \) by
\[
\frac{\partial}{\partial \xi_2}(1 - \lambda \eta_1) \bigg|_{(\xi_2, \zeta_2) = (0, -\lambda)} = -2\lambda^2 \neq 0.
\]
Since \( \frac{\partial n}{\partial \xi_2}(0, \zeta_2) = -2\zeta_2 \),
\[
(0, \zeta_2) \in D \iff \zeta_2 = 0.
\]
Then \( D \) intersects \( E \) transversally at \((p, L_{[1:0]})\) by
\[
\frac{\partial^2 n_1}{\partial \zeta_2^2}(0,0) = -2 \neq 0.
\]
There is a holomorphic function \( \sigma(\xi_2) \) on a neighborhood of \( \xi_2 = 0 \) such that \( \sigma(0) = 0 \) and \( \frac{\partial n}{\partial \xi_2}(\xi_2, \sigma(\xi_2)) = 0 \). Then the image \( B_1 \) of the local component of \( D \) at \((p, L_{[1:0]})\) by \( \sigma(\xi_2) \) is locally isomorphic to \( \{(\xi_1, \xi_2) \mid \xi_1 = \varphi_1(\xi_2, \sigma(\xi_2))\} \). Since
\[
\frac{\partial}{\partial \xi_2}(\varphi_1(\xi_2, \sigma(\xi_2)))|_{\xi_2=0} = 0,
\]
\( B_1 \) intersects \( L_{[1:1]} \) transversally at \( p \).
In the same way, we can show that \( D \) intersects \( E \) transversally at \((p, L_{[0:1]})\), and there is a holomorphic function \( \tau(\xi_1) \) on a neighborhood of \( \xi_1 = 0 \) such that \( \frac{d\tau}{d\xi_1}(0) = 0 \) and the image \( B_2 \) of the local component of \( D \) at \((p, L_{[0:1]})\) by \( \phi \) is locally isomorphic to \( \{(\xi_1, \xi_2) \mid \xi_2 = \tau(\xi_1)\} \). Then \( B_2 \) intersects \( L_{[1:1]} \) and \( B_1 \) transversally at \( p \). This implies that \( B \) has a node at \( p \), and \( L_{[1:1]} \) intersects \( B \) at \( p \) with multiplicity 2.

By the above observation, we have
\[
\phi^* L = L^+ + L^- + \sum_{e \in L \cap \text{Sing}(B)} \phi^{-1}(e)
\]
for a line \( L \) on \( X \), and \( B \cap L \) is a set of distinct two point because \((B, L) = 4\). Hence we have
\[
(L^-. L^-) = (L^-. \phi^* L - L^+ - \sum_{e \in L \cap \text{Sing}(B)} \phi^{-1}(e))
\]
\[
= (L. L) - (L^-. L^+ + \sum_{e \in L \cap \text{Sing}(B)} \phi^{-1}(e)) = -1 - 2 = -3.
\]
Each component of \( E \) corresponds to an Eckardt point on \( X \), and it is a \((-2)\)-curve on \( Y \), because \( \phi \) is the canonical map of \( Y \) by Proposition 2.1.

Remark 3.4. There are at most two Eckardt points on a line \( L \subset X \), hence there are at most 18 Eckardt points on \( X \). If \( X \) has 18 Eckardt points, then \( X \) is isomorphic to the Fermat cubic surface \([11]\).

Remark 3.5. Let \( \phi' : Y' \to X \) be the finite double cover of \( X \) branched along \( B \). Then \( Y' \) may have ordinary double points, and \( Y \) is the minimal resolution of \( Y' \).

Remark 3.6. By Theorem 3.2 for lines \( L_1, L_2, L \) on \( X \) and Eckardt points \( e_1, e_2, e \) on
$X$, the intersection numbers on $Y$ are computed by

$$(L_1^.L_2^+) = (L_1^.L_2^-) = \begin{cases} 0 & \text{if } L_1 \neq L_2, \\ -3 & \text{if } L_1 = L_2, \end{cases}$$

$$(L_1^.L_2^-) = \begin{cases} 0 & \text{if } L_1 \cap L_2 = \emptyset, \\ 1 & \text{if } L_1 \cap L_2 \text{ is a point which is not an Eckardt point,} \\ 0 & \text{if } L_1 \cap L_2 \text{ is a point which is an Eckardt point,} \\ 0 & \text{if } L_1 = L_2 \text{ and there are two Eckardt points on } L_1 = L_2, \\ 1 & \text{if } L_1 = L_2 \text{ and there is only one Eckardt point on } L_1 = L_2, \\ 2 & \text{if } L_1 = L_2 \text{ and there are no Eckardt points on } L_1 = L_2, \end{cases}$$

$$(\phi^{-1}(e_1).\phi^{-1}(e_2)) = \begin{cases} 0 & \text{if } e_1 \neq e_2, \\ -2 & \text{if } e_1 = e_2, \end{cases}$$

$$(L^+.\phi^{-1}(e)) = (L^-.\phi^{-1}(e)) = \begin{cases} 0 & \text{if } e \notin L, \\ 1 & \text{if } e \in L. \end{cases}$$

**Proposition 3.7.** Any $(-2)$-curve on $Y$ is a component of $E$, and any $(-3)$-curve on $Y$ is a component of $Y_\infty + Y^-_\infty$.

**Proof.** Let $C$ be a $(-2)$-curve on $Y$. Since $(\phi_*C. \mathcal{O}_\mathbb{P}^3(1)|_X) = (C, K_Y)$, the image of $C$ by the morphism $\phi$ is a point on $X$, hence $C$ is a component of $E$. Let $C$ be a $(-3)$-curve on $Y$. Since $(\phi_*C. \mathcal{O}_\mathbb{P}^3(1)|_X) = (C, K_Y)$, the image of $C$ by the morphism $\phi$ is a line on $X$, hence $C$ is a component of $Y_\infty + Y^-_\infty$. \hfill \square

**Remark 3.8.** We can check that the divisor $Y_\infty + Y^-_\infty$ is connected. Hence, if a divisor $W$ on $Y$ is a disjoint union of irreducible components of $Y_\infty + Y^-_\infty$, and $W$ contains a component of $\phi^*L$ for any line $L$ on $X$, then $W = Y_\infty$ or $W = Y^-_\infty$.

Let $\psi = \psi|_Y : Y \to Z = Z_3 \subset \lambda(\mathbb{P}^3)$ be the second projection in Remark 2.5, and let $[\mathcal{O}_Z(1)] \in H^2(Z, \mathbb{Z})$ be the class of a hyperplane section by the Plücker embedding $\Lambda(\mathbb{P}^3) \subset \mathbb{P}^5$. Let $Z_\infty$ be the set of all lines on the cubic surface $X$. For a line $L_0 \in Z_\infty$, we set $Z_\infty(L_0) = \{L \in Z_\infty \mid L_0 \neq L, \ L_0 \cap L \neq \emptyset\}$, which is a set of 10 lines.

**Proposition 3.9.** There are the following relations in the Néron-Severi group $\text{NS}(Y)$:

$$\psi^*[\mathcal{O}_Z(3)] = \phi^*[\mathcal{O}_X(3)] + \sum_{L \in Z_\infty} L^+ \quad (3.2)$$

and

$$\psi^*[\mathcal{O}_Z(1)] = 3\phi^*L_0 - L_0^+ + \sum_{L \in Z_\infty(L_0)} L^+ \quad (3.3)$$

for any line $L_0 \in Z_\infty$.  

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Proof. Since $Y_{\infty} = \bigsqcup_{L \in Z_{\infty}} L^+$, the relation (3.2) is given by

$$\mathcal{O}_Y(Y_{\infty}) \simeq S^{\otimes 3}|_Y \simeq \psi^* \mathcal{O}_Z(3) \otimes \phi^* \mathcal{O}_X(-3).$$

For $L_0 \in \Lambda(\mathbb{P}^3)$,

$$H_{L_0} = \{ L \in \Lambda(\mathbb{P}^3) \mid L_0 \cap L \neq \emptyset \}$$

is a hyperplane section by the Plücker embedding $\Lambda(\mathbb{P}^3) \subset \mathbb{P}^5$. We prove that

$$\psi^* H_{L_0} = 2L_0^+ + 3L_0^- + 3 \sum_{e \in L_0 \cap \text{Sing}(B)} \phi^{-1}(e) + \sum_{L \in Z_{\infty}(L_0)} L^+$$

for $L_0 \in Z_{\infty}$. It gives the relation (3.3) by the relation in Theorem 3.2. For $(p, L) \in \psi^{-1}(H_{L_0}) \subset Y$, if $p \in L_0$ then

$$(p, L) \in \phi^{-1}(L_0) = L_0^+ \cup L_0^- \cup \bigcup_{e \in L_0 \cap \text{Sing}(B)} \phi^{-1}(e),$$

and if $p \notin L_0$ then $L \subset X$. Hence the support of $\psi^* H_{L_0}$ is

$$\psi^{-1}(H_{L_0}) = L_0^+ \cup L_0^- \cup \bigcup_{e \in L_0 \cap \text{Sing}(B)} \phi^{-1}(e) \cup \bigcup_{L \in Z_{\infty}(L_0)} L^+.$$ 

We compute the multiplicity of each component.

1. The case where there are no Eckardt points on the line $L_0$. We set integers $a_+$, $a_-$ and $a_L$ by

$$\psi^*[\mathcal{O}_Z(1)] = \psi^* H_{L_0} = a_+ L_0^+ + a_- L_0^- + \sum_{L \in Z_{\infty}(L_0)} a_L L^+. $$

Since $(\psi^*[\mathcal{O}_Z(1)]. L^+) = 0$ for $L \in Z_{\infty}$,

$$0 = (\psi^* H_{L_0}. L^+) = \begin{cases} -3a_+ + 2a_- & \text{if } L = L_0, \\ a_- - 3a_L & \text{if } L \in Z_{\infty}(L_0). \end{cases}$$

By the relation (3.2),

$$(\psi^*[\mathcal{O}_Z(3)]. L_0^-) = (\phi^*[\mathcal{O}_X(3)]. L_0^-) + (L_0^+. L_0^-) + \sum_{L \in Z_{\infty}(L_0)} (L^+. L_0^-) = 3 + 2 + 10,$$

hence we have

$$5 = (\psi^* H_{L_0}. L_0^-) = 2a_+ - 3a_- + \sum_{L \in Z_{\infty}(L_0)} a_L.$$

These equations imply that $a_+ = 2$, $a_- = 3$ and $a_L = 1$ for $L \in Z_{\infty}(L_0)$. 

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2. The case where there is only one Eckardt point \( e \). We denote by \( Z_\infty(e, L_0) \subset Z_\infty(L_0) \) the set of two lines through the point \( e \). We set integers \( a_+, a_-, b \) and \( a_L \) by

\[
\psi^*[OZ(1)] = \psi^* H_{L_0} = a_+ L_0^+ + a_- L_0^- + b\phi^{-1}(e) + \sum_{L \in Z_\infty(L_0)} a_L L^+.
\]

Since \( (\psi^*[OZ(1)]. L^+) = 0 \) for \( L \in Z_\infty \),

\[
0 = (\psi^* H_{L_0}, L^+) = \begin{cases} 
-3a_+ + a_- + b & \text{if } L = L_0, \\
2a_- - 3a_L & \text{if } L \in Z_\infty(L_0) \setminus Z_\infty(e, L_0), \\
b - 3a_L & \text{if } L \in Z_\infty(e, L_0).
\end{cases}
\]

By the relation (3.2),

\[
(\psi^*[OZ(3)]. L^-) = (\phi^*[OX(3)]. L^-) + (L_0^+. L^-) + \sum_{L \in Z_\infty(L_0)} (L^+. L^-) = 3 + 1 + 8
\]

and

\[
(\psi^*[OZ(3)]. \phi^{-1}(e)) = (\phi^*[OX(3)]. \phi^{-1}(e)) + (L_0^+. \phi^{-1}(e)) + \sum_{L \in Z_\infty(L_0)} (L^+. \phi^{-1}(e))
\]

\[
= 0 + 1 + 2,
\]

hence we have

\[
4 = (\psi^* H_{L_0}, L^-) = a_+ - 3a_- + b + \sum_{L \in Z_\infty(L_0) \setminus Z_\infty(e, L_0)} a_L
\]

and

\[
1 = (\psi^* H_{L_0}, \phi^{-1}(e)) = a_+ + a_- - 2b + \sum_{L \in Z_\infty(e, L_0)} a_L.
\]

These equations imply that \( a_+ = 2, a_- = 3, b = 3 \) and \( a_L = 1 \) for \( L \in Z_\infty(L_0) \).

3. The case where there are two Eckardt points \( e_1, e_2 \) on the line \( L_0 \). We set integers \( a_+, a_-, b_1, b_2 \) and \( a_L \) by

\[
\psi^*[OZ(1)] = \psi^* H_{L_0} = a_+ L_0^+ + a_- L_0^- + b_1\phi^{-1}(e_1) + b_2\phi^{-1}(e_2) + \sum_{L \in Z_\infty(L_0)} a_L L^+.
\]

Since \( (\psi^*[OZ(1)]. L^+) = 0 \) for \( L \in Z_\infty \),

\[
0 = (\psi^* H_{L_0}, L^+)
\]

\[
= \begin{cases} 
-3a_+ + b_1 + b_2 & \text{if } L = L_0, \\
a_- - 3a_L & \text{if } L \in Z_\infty(L_0) \setminus (Z_\infty(e, L_0) \cup Z_\infty(e_2, L_0)), \\
b_1 - 3a_L & \text{if } L \in Z_\infty(e_1, L_0).
\end{cases}
\]
By the relation 3.2,
\[(\psi^*[\mathcal{O}_Z(3)]. L_0^-) = (\phi^*[\mathcal{O}_X(3)]. L_0^-) + \sum_{L \in Z_{\infty}(L_0^\perp)} (L^+. L_0^-) = 3 + 0 + 6 \]
and
\[(\psi^*[\mathcal{O}_Z(3)]. \phi^{-1}(e_i)) = (\phi^*[\mathcal{O}_X(3)]. \phi^{-1}(e_i)) + \sum_{L \in Z_{\infty}(L_0^\perp)} (L^+. \phi^{-1}(e_i)) \]
= 0 + 1 + 2,

hence we have
\[3 = (\psi^* H_{L_0^-}. L_0^-) = -3a_- + b_1 + b_2 + \sum_{L \in Z_{\infty}(L_0) \setminus (Z_{\infty}(e_1, L_0) \cup Z_{\infty}(e_2, L_0))} a_L \]
and
\[1 = (\psi^* H_{L_0^-}. \phi^{-1}(e_i)) = a_+ + a_- - 2b_i + \sum_{L \in Z_{\infty}(e_i, L_0)} a_L. \]

These equations imply that \(a_+ = 2, a_- = 3, b_1 = 3, b_2 = 3\) and \(a_L = 1\) for \(L \in Z_{\infty}(L_0)\).

\[\square\]

4 Periods of cubic 3-folds

We review some works on cubic 3-folds by Clemens-Griffiths \[3\] and Tjurin \[12\]. Let \(V \subset \mathbb{P}^4\) be a nonsingular cubic 3-folds. We define a subvariety \(W\) of \(\mathbb{P}^4 \times \Lambda(\mathbb{P}^4)\) by
\[W = \{(p, L) \in \mathbb{P}^4 \times \Lambda(\mathbb{P}^4) \mid p \in L \subset V\},\]
and we define a subvariety \(S\) of \(\Lambda(\mathbb{P}^4)\) by
\[S = \{L \in \Lambda(\mathbb{P}^4) \mid L \subset V\},\]
which is a nonsingular surface and called the Fano surface of lines on \(V\). The first projection \(\phi : W \to V\) is a generically finite morphism of degree 6, and the second projection \(\psi : W \to S\) is a \(\mathbb{P}^1\)-bundle.

**Theorem 4.1** (Clemens-Griffiths \[3\], Theorem 11.19). The homomorphism
\[\phi_* \circ \psi^* : H^3(S, \mathbb{Z}) \longrightarrow H^3(V, \mathbb{Z})\]
is an isomorphism of Hodge structures.
Let $J$ be the intermediate Jacobian of the Hodge structure $H^3(V, \mathbb{Z})$. Then the complex torus $J$ is a principally polarized abelian variety of dimension 5. We denote by $\theta \in H^2(J, \mathbb{Z})$ the class of the polarization. Let $A$ be the Albanese variety of $S$, and $\iota : S \to A$ the Albanese morphism. By Theorem 4.3, there is a natural isomorphism $A \simeq J$ of abelian varieties. Let us denote by $\theta \in H^2(A, \mathbb{Z})$ the corresponding principal polarization on $A$. The primitive part of $H^2(A, \mathbb{Z})$ is defined as the space

$$H^2_{\text{prim}}(A, \mathbb{Z}) = \ker (\theta^{\otimes 4}: H^2(A, \mathbb{Z}) \to H^{10}(A, \mathbb{Z}); \alpha \mapsto \theta^{\otimes 4} \cup \alpha),$$

and the primitive part of $H^2(S, \mathbb{Z})$ is defined as the space

$$H^2_{\text{prim}}(S, \mathbb{Z}) = \ker ([\mathcal{O}_S(1)]: H^2(S, \mathbb{Z}) \to H^4(S, \mathbb{Z}); \beta \mapsto [\mathcal{O}_S(1)] \cup \beta),$$

where $[\mathcal{O}_S(1)] \in H^2(S, \mathbb{Z})$ is the class of a hyperplane section by the Plücker embedding $\Lambda(P^4) \subset P^9$. We define a symmetric form on $H^2(A, \mathbb{Z})$ by

$$\langle , \rangle_A : H^2(A, \mathbb{Z}) \times H^2(A, \mathbb{Z}) \to \mathbb{Z}; (\alpha_1, \alpha_2) \mapsto \deg \left( \left( \frac{\theta^{\otimes 3}}{3!} \cup \alpha_1 \cup \alpha_2 \right) \cap [A] \right),$$

and a symmetric form on $H^2(S, \mathbb{Z})$ by

$$\langle , \rangle_S : H^2(S, \mathbb{Z}) \times H^2(S, \mathbb{Z}) \to \mathbb{Z}; (\beta_1, \beta_2) \mapsto \deg ((\beta_1 \cup \beta_2) \cap [S]).$$

We remark that these symmetric forms give polarizations of Hodge structures on the primitive part $H^2_{\text{prim}}(A, \mathbb{Z})$ and $H^2_{\text{prim}}(S, \mathbb{Z})$.

**Proposition 4.2.** The homomorphism $\iota^* : H^2(A, \mathbb{Z}) \to H^2(S, \mathbb{Z})$ induces the isomorphism

$$(H^2_{\text{prim}}(A, \mathbb{Z}), \langle , \rangle_A) \simeq (H^2_{\text{prim}}(S, \mathbb{Z}), \langle , \rangle_S)$$

of polarized Hodge structures.

**Proof.** By [3] Lemma 9.13 and (10.14), the homomorphism $\iota^* : H^2(A, \mathbb{Z}) \to H^2(S, \mathbb{Z})$ is injective with a finite cokernel. By [4] (2.3.5), the homology group $H_1(S, \mathbb{Z})$ has no torsion element, and the cokernel of $\iota_* : H_2(S, \mathbb{Z}) \to H_2(A, \mathbb{Z})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Hence $H^2(S, \mathbb{Z})$ has no torsion element, and the cokernel of $\iota^* : H^2(A, \mathbb{Z}) \to H^2(S, \mathbb{Z})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Since $[\iota(S)] = \frac{\theta^{\otimes 3}}{6} \in H^6(A, \mathbb{Z})$ by [3] Proposition 13.1, we have

$$\iota_*((\iota^*\alpha_1 \cup \iota^*\alpha_2) \cap [S]) = (\alpha_1 \cup \alpha_2) \cap \iota_*[S] = (\alpha_1 \cup \alpha_2) \cap \left( \frac{\theta^{\otimes 3}}{6} \cap [A] \right) = \left( \frac{\theta^{\otimes 3}}{6} \cup \alpha_1 \cup \alpha_2 \right) \cap [A]$$

for $\alpha_1, \alpha_2 \in H^2(A, \mathbb{Z})$, hence the homomorphism $\iota^*$ is compatible with the symmetric forms. Let $\tau \in H^2(S, \mathbb{Z})$ be the class of an incidence divisor [3] §2. Since $3\tau = [\mathcal{O}_S(1)]$ by [3] §10, the primitive part $H^2_{\text{prim}}(S, \mathbb{Z})$ is equal to the space orthogonal to $\tau$. Since $2\tau = \iota^*\theta$ by [3] Lemma 11.27, we have

$$\iota_*((2\tau \cup \iota^*\alpha) \cap [S]) = \iota_*((\iota^*\theta \cup \iota^*\alpha) \cap [S]) = \left( \frac{\theta^{\otimes 4}}{6} \cup \alpha \right) \cap [A]$$

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for any $\alpha \in H^2(A,\mathbb{Z})$. Hence we have a commutative diagram of exact sequences

$$
\begin{array}{cccccc}
0 & & 0 & & \\
\downarrow & & \downarrow & & \\
H^2_{\text{prim}}(A,\mathbb{Z}) & \longrightarrow & H^2_{\text{prim}}(S,\mathbb{Z}) & \longrightarrow & \\
\downarrow & & \downarrow & & \\
0 & \longrightarrow & H^2(A,\mathbb{Z}) & \longrightarrow & H^2(S,\mathbb{Z}) & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 \\
\theta^{1/4} & \downarrow & \iota^* & \downarrow & \tau \\
\mathbb{Z} & = & \\
& & \\
\end{array}
$$

Since $\theta$ is a principal polarization, the image of the homomorphism

$$
\frac{\theta^{1/4}}{12} : H^2(A,\mathbb{Z}) \to \mathbb{Z}; \alpha \mapsto \deg \left( \left( \frac{\theta^{1/4}}{12} \cup \alpha \right) \cap [A] \right)
$$

is $2\mathbb{Z}$. And the image of the homomorphism

$$
\tau : H^2(S,\mathbb{Z}) \to \mathbb{Z}; \alpha \mapsto \deg \left( (\tau \cup \alpha) \cap [S] \right)
$$

is not contained in $2\mathbb{Z}$, because $\deg (\tau^{1/2} \cap [S]) = 5 \notin 2\mathbb{Z}$ by [3, (10.8)]. Hence $\tau : H^2(S,\mathbb{Z}) \to \mathbb{Z}$ is surjective, and $\iota^* : H^2_{\text{prim}}(A,\mathbb{Z}) \to H^2_{\text{prim}}(S,\mathbb{Z})$ is an isomorphism.

\section{5 Periods of cubic surfaces}

Let $X \subset \mathbb{P}^3$ be a nonsingular cubic surface defined by $F(x_0,\ldots,x_3) \in \mathbb{C}[x_0,\ldots,x_3]$. Let $V \subset \mathbb{P}^4$ be the cubic 3-fold defined by $F(x_0,\ldots,x_3) + x_4^3 \in \mathbb{C}[x_0,\ldots,x_4]$. Then the projection

$$
\rho : V \to \mathbb{P}^3; \ [x_0 : \cdots : x_3 : x_4] \mapsto [x_0 : \cdots : x_3]
$$

is the triple Galois cover branched along the cubic surface $X$. Let $S$ be the Fano surface of lines on $V$. Then the Galois group $\text{Gal}(\rho) \simeq \mathbb{Z}/3\mathbb{Z}$ of the cover $\rho$ acts on the surface $S$.

\textbf{Lemma 5.1.} Let $L$ be a line in $\mathbb{P}^4$. If $L$ is contained in $V$, then its image $\rho(L) \subset \mathbb{P}^3$ by $\rho$ is a line in $\mathbb{P}^3$, and it is contained in $X$ or intersects $X$ at only one point with multiplicity 3.

\textbf{Proof.} Let $H_4 \subset \mathbb{P}^4$ be the hyperplane defined by the equation $x_4 = 0$. If $L$ is contained in $H_4 \cap V$, then it is clear that $\rho(L)$ is a line contained in $X$. We assume that $L \cap H_4$ is a point $[a_0 : \cdots : a_3 : 0] \in \mathbb{P}^4$. By taking a point $[b_0 : \cdots : b_3 : 1] \in L \setminus H_4$, the line $L$ is written as

$$
L = \{ [a_0 t_0 + b_0 t_1 : \cdots : a_3 t_0 + b_3 t_1 : t_1] \in \mathbb{P}^4 \mid [t_0 : t_1] \in \mathbb{P}^1 \}.
$$
If $L \subset V$, then
\[
F(a_0t_0 + b_0t_1, \ldots, a_3t_0 + b_3t_1) + t_1^3 = 0 \in C[t_0, t_1].
\]
Since $F(b_0, \ldots, b_3) + 1 = 0$ and $F(a_0, \ldots, a_3) = 0$, we have $(b_1, \ldots, b_3) \neq (0, \ldots, 0)$ and $[a_0 : \cdots : a_3] \neq [b_0 : \cdots : b_3]$, hence
\[
\rho(L) = \{[a_0t_0 + b_0t_1 : \cdots : a_3t_0 + b_3t_1] \in P^3 \mid [t_0 : t_1] \in P^1\}.
\]
is a line in $P^3$. Since $F(a_0t_0 + b_0t_1, \ldots, a_3t_0 + b_3t_1) = -t_1^3$, the line $\rho(L)$ intersects $X$ at the point $[a_0 : \cdots : a_3] \in P^3$ with multiplicity 3. \hfill \Box

Let $Z = Z_3$ be the surface in Remark 2.5. By Lemma 5.1, the line $\rho(L)$ represents a point of $Z$ for a line $L$ on $V$. Let us abuse notation by
\[
\rho : S \longrightarrow Z; \ L \longmapsto \rho(L).
\]
We set
\[
S_{\infty} = \{L \in \Lambda(P^4) \mid L \subset V \cap H_4\},
\]
which is a set of 27 points on $S$.

**Lemma 5.2.** $\rho : S \rightarrow Z$ is the quotient morphism by the Gal($\rho$)-action, and $S_{\infty}$ is the set of the fixed point by the Gal($\rho$)-action on $S$.

**Proof.** Let $\omega \in C$ be a primitive 3-rd root of unity. The automorphism
\[
\sigma : V \longrightarrow V; \ [x_0 : \cdots : x_3 : x_4] \longmapsto [x_0 : \cdots : x_3 : \omega x_4]
\]
is a generator of the Galois group Gal($\rho$). For a line $L$ on $V$, we have $\rho(L) = \rho(\sigma(L))$, and if $L = \sigma(L)$, then $L$ is contained in $H_4$. Hence $S_{\infty}$ is the set of fixed points of the Gal($\rho$)-action on $S$.

Let
\[
L' = \{[a_0t_0 + b_0t_1 : \cdots : a_3t_0 + b_3t_1] \in P^3 \mid [t_0 : t_1] \in P^1\}
\]
be a line in $P^3$ which intersects $X$ at $[a_0 : \cdots : a_3]$ with multiplicity $\geq 3$. Then there exists $c \in C$ such that
\[
F(a_0t_0 + b_0t_1, \ldots, a_3t_0 + b_3t_1) = ct_1^3.
\]
If a line
\[
L = \{[a_0t_0 + b_0t_1 : \cdots : a_3t_0 + b_3t_1 : a_4t_0 + b_4t_1] \in P^4 \mid [t_0 : t_1] \in P^1\}
\]
is contained in $V$, then
\[
-(a_4t_0 + b_4t_1)^3 = F(a_0t_0 + b_0t_1, \ldots, a_3t_0 + b_3t_1) = ct_1^3,
\]
hence $a_4 = 0$ and $b_4^3 = -c$. This imply that the morphism $\rho : S \rightarrow Z$ is surjective, and the fiber at $L' \in Z$ is contained in a Gal($\rho$)-orbit. \hfill \Box
Remark 5.3. Each singularity of $Z$ is isomorphic to the quotient of $\mathbb{C}^2$ by the cyclic group generated by the action $(a, b) \mapsto (\omega a, \omega b)$. Hence we have

$$H^i(Z, Z \setminus Z_\infty, Z) \simeq \begin{cases} (Z/3Z)^{\oplus 27} & \text{if } i = 3, \\ Z^{\oplus 27} & \text{if } i = 4, \\ 0 & \text{if } i \neq 3, 4. \end{cases}$$

Let $\phi : Y = Y_3 \to X$ be the double cover branched along its Hessian, and let $Y_\infty$ be the distinguished divisor on $Y$ which is introduced in Section 2. By Remark 5.3, the restriction homomorphism $H^2(Z, \mathbb{Z}) \to H^2(Z \setminus Z_\infty, \mathbb{Z}) \simeq H^2(Y \setminus Y_\infty, \mathbb{Z})$ is injective with a finite cokernel, hence $\psi^* : H^2(Z, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$ is injective. Since $H^2(Y, \mathbb{Z})$ is torsion free, $H^2(Z, \mathbb{Z})$ is also torsion free. The period integral

$$H^0(Y, \Omega^1_Y((\log Y)_\infty)) \to \text{Hom}(H_2(Y \setminus Y_\infty, \mathbb{Z}), \mathbb{C}); \omega \mapsto [\gamma \mapsto \int_\gamma \omega]$$

defines Hodge structures of pure weight 2 on $H^2(Z, \mathbb{Z})$ and $H^2(Z \setminus Z_\infty, \mathbb{Z})$. For $\gamma \in H^2(Z \setminus Z_\infty, \mathbb{Z})$, there is a unique $\bar{\gamma} \in H^2(Z, \mathbb{Q})$ such that the restriction of $\bar{\gamma}$ to $H^2(Z \setminus Z_\infty, \mathbb{Q})$ is equal to the class of $\gamma$ in the rational cohomology group. We define the primitive part of $H^2(Z, \mathbb{Z})$ and $H^2(Z \setminus Z_\infty, \mathbb{Z})$ by

$$H^2_{\text{prim}}(Z, \mathbb{Z}) = \text{Ker} \left( [\mathcal{O}_Z(1)] : H^2(Z, \mathbb{Z}) \to H^4(Z, \mathbb{Z}); \gamma \mapsto [\mathcal{O}_Z(1)] \cup \gamma \right),$$

$$H^2_{\text{prim}}(Z \setminus Z_\infty, \mathbb{Z}) = \text{Ker} \left( [\mathcal{O}_Z(1)] : H^2(Z \setminus Z_\infty, \mathbb{Z}) \to H^4(Z, \mathbb{Q}); \gamma \mapsto [\mathcal{O}_Z(1)] \cup \gamma \right).$$

We define symmetric forms on $H^2(Z, \mathbb{Z})$ and $H^2(Z \setminus Z_\infty, \mathbb{Z})$ by

$$\langle \cdot, \cdot \rangle : H^2(Z, \mathbb{Z}) \times H^2(Z, \mathbb{Z}) \to \mathbb{Z}; (\gamma_1, \gamma_2) \mapsto \text{deg}((\gamma_1 \cup \gamma_2) \cap [Z]),$$

$$\langle \cdot, \cdot \rangle : H^2(Z \setminus Z_\infty, \mathbb{Z}) \times H^2(Z \setminus Z_\infty, \mathbb{Z}) \to \mathbb{Q}; (\gamma_1, \gamma_2) \mapsto \text{deg}((\gamma_1 \cup \gamma_2) \cap [Z]).$$

These symmetric forms give polarizations of Hodge structures on the primitive part $H^2_{\text{prim}}(Z, \mathbb{Z})$ and $H^2_{\text{prim}}(Z \setminus Z_\infty, \mathbb{Z})$.

**Proposition 5.4.** The homomorphism

$$H^2(Z \setminus Z_\infty, \mathbb{Z}) \overset{\rho^*}{\to} H^2(S \setminus S_\infty, \mathbb{Z}) \simeq H^2(S, \mathbb{Z})$$

induces an isomorphism $H^2(Z \setminus Z_\infty, \mathbb{Z})_{\text{free}} \simeq H^2(S, \mathbb{Z})^{\text{Gal}(\rho)}$ of Hodge structures and an isomorphism

$$(H^2_{\text{prim}}(Z \setminus Z_\infty, \mathbb{Z})_{\text{free}}, \langle \cdot, \cdot \rangle) \simeq (H^2_{\text{prim}}(S, \mathbb{Z})^{\text{Gal}(\rho)}, \langle \cdot, \cdot \rangle)$$

of polarized Hodge structures.
Proof. Since $\rho : S \backslash S_\infty \to Z \backslash Z_\infty$ is a finite étale Galois cover, we have the Cartan-Leray spectral sequence

$$E_2^{p,q} = H^p(\text{Gal}(\rho), H^q(S \backslash S_\infty, \mathbb{Z})) \Rightarrow H^{p+q}(Z \backslash Z_\infty, \mathbb{Z}).$$

Since the $\text{Gal}(\rho)$-action on $H^0(S \backslash S_\infty, \mathbb{Z}) \simeq H^0(S, \mathbb{Z}) \simeq \mathbb{Z}$ is trivial, we have

$$H^p(\text{Gal}(\rho), H^0(S \backslash S_\infty, \mathbb{Z})) \simeq \begin{cases} \mathbb{Z} & \text{if } p = 0, \\ 0 & \text{if } p \text{ is odd}, \\ \mathbb{Z}/3\mathbb{Z} & \text{if } p \neq 0 \text{ is even}. \end{cases}$$

Since $H^1(S \backslash S_\infty, \mathbb{Z}) \simeq H^1(S, \mathbb{Z}) \simeq H^2(V, \mathbb{Z})$ is a free $\mathbb{Z}$-module of rank 10 and the $\text{Gal}(\rho)$-action has no invariant part, it is regarded as a free $\mathbb{Z}[\omega]$-module of rank 5, where $\mathbb{Z}[\omega] \simeq \mathbb{Z}[\text{Gal}(\rho)]/(\sum_{\sigma \in \text{Gal}(\rho)} \sigma)$ is the ring of Eisenstein integers $[\mathbb{P} (2.2)]$. Hence we have

$$H^p(\text{Gal}(\rho), H^1(S \backslash S_\infty, \mathbb{Z})) \simeq \begin{cases} (\mathbb{Z}/3\mathbb{Z})^{\oplus 5} & \text{if } p \text{ is odd}, \\ 0 & \text{if } p \text{ is even}. \end{cases}$$

By the spectral sequence, the homomorphism

$$H^2(Z \backslash Z_\infty, \mathbb{Z}) \rightarrow H^0(\text{Gal}(\rho), H^2(S \backslash S_\infty, \mathbb{Z})) \simeq H^2(S, \mathbb{Z})^{\text{Gal}(\rho)}$$

is surjective, and its kernel is of order $3^6$. Since $\rho^*\mathcal{O}_Z(1) = \mathcal{O}_S(1)$, we have

$$\rho_*(([\mathcal{O}_S(1)] \cup \rho^*\bar{\gamma}) \cap [S]) = ([\mathcal{O}_Z(1)] \cup \bar{\gamma}) \cap \rho_*[S] = ([\mathcal{O}_Z(1)] \cup \bar{\gamma}) \cap 3[\mathbb{Z}]$$

for $\gamma \in H^2(Z \backslash Z_\infty, \mathbb{Z})$, hence $\gamma \in H^2_{\text{prim}}(Z \backslash Z_\infty, \mathbb{Z})$ if and only if $\rho^*\bar{\gamma} \in H^2_{\text{prim}}(S, \mathbb{Q})$. And we have

$$\deg ((\rho^*\bar{\gamma}_1 \cup \rho^*\bar{\gamma}_2) \cap [S]) = \deg ((\bar{\gamma}_1 \cup \bar{\gamma}_2) \cap \rho_*[S]) = 3 \deg ((\bar{\gamma}_1 \cup \bar{\gamma}_2) \cap [Z])$$

for $\gamma_1, \gamma_2 \in H^2(Z \backslash Z_\infty, \mathbb{Z})$. \qed

Remark 5.5. In the similar way, we can prove that the coinvariant part of the $\text{Gal}(\rho)$-action on $H^2(S, \mathbb{Z})$ is isomorphic to $H^2(Z \backslash Z_\infty, \mathbb{Z})$. By the duality $H^2(Z \backslash Z_\infty, \mathbb{Z}) \simeq H_2(Z, \mathbb{Z}) \simeq H^2(Z, \mathbb{Z})$, we have a commutative diagram

$$\begin{array}{ccc}
H^2(S \backslash S_\infty, \mathbb{Z})^{\text{Gal}(\rho)} & \xleftarrow{\rho^*} & H^2(Z \backslash Z_\infty, \mathbb{Z})_{\text{free}} \simeq H_2(Z, \mathbb{Z})_{\text{free}} \\
\uparrow{\simeq} & & \uparrow{\simeq} \\
H^2(S, \mathbb{Z})^{\text{Gal}(\rho)} & \xleftarrow{\rho^*} & H^2(Z, \mathbb{Z}) \simeq H_2(Z \backslash Z_\infty, \mathbb{Z}) \xrightarrow{\nu} H^2(S \backslash S_\infty, \mathbb{Z})_{\text{Gal}(\rho)}.
\end{array}$$

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Remark 5.6. The restriction \( H^2(Y, \mathbb{Z}) \to H^2(Y \setminus Y_\infty, \mathbb{Z}) \) induces an isomorphism

\[
\frac{H^2(Y, \mathbb{Z})}{\sum_{L \in \mathbb{Z}_\infty} \mathbb{Z}L^+} \simeq H^2(Y \setminus Y_\infty, \mathbb{Z}) \simeq H^2(Z \setminus Z_\infty, \mathbb{Z}),
\]

and the injection \( \psi^* : H^2(Z, \mathbb{Z}) \to H^2(Y, \mathbb{Z}) \) induces an isomorphism

\[
H^2(Z, \mathbb{Z}) \simeq \left( \sum_{L \in \mathbb{Z}_\infty} \mathbb{Z}L^+ \right)^{\perp} \subset H^2(Y, \mathbb{Z}),
\]

where \( \perp \) means the orthogonal complement in the unimodular lattice

\[
\langle \ , \ \rangle_Y : H^2(Y, \mathbb{Z}) \times H^2(Y, \mathbb{Z}) \to \mathbb{Z}; (\gamma_1, \gamma_2) \mapsto \deg ((\gamma_1 \cup \gamma_2) \cap [Y]).
\]

Proposition 5.7. The homomorphism

\[
H^2(X, \mathbb{Z}) \xrightarrow{\phi^*} H^2(Y \setminus Y_\infty, \mathbb{Z}) \simeq H^2(Z \setminus Z_\infty, \mathbb{Z})
\]

induces an isomorphism

\[
\frac{H^2_{\text{prim}}(X, \mathbb{Z})}{3H^2_{\text{prim}}(X, \mathbb{Z})} \simeq H^2(Z \setminus Z_\infty, \mathbb{Z})_{\text{tor}}
\]

of abelian groups.

Proof. Since \( \psi^*H^2(Z, \mathbb{Z}) = \left( \sum_{L \in \mathbb{Z}_\infty} \mathbb{Z}L^+ \right)^{\perp} \subset H^2(Y, \mathbb{Z}) \), the primitive closure of the sublattice \( \sum_{L \in \mathbb{Z}_\infty} \mathbb{Z}L^+ \) in \( H^2(Y, \mathbb{Z}) \) is \( \left( \psi^*H^2(Z, \mathbb{Z}) \right)^{\perp} \subset H^2(Y, \mathbb{Z}) \), hence the torsion part of \( H^2(Z \setminus Z_\infty, \mathbb{Z}) \) is

\[
H^2(Z \setminus Z_\infty, \mathbb{Z})_{\text{tor}} \simeq \left( \frac{H^2(Y, \mathbb{Z})}{\sum_{L \in \mathbb{Z}_\infty} \mathbb{Z}L^+} \right)_{\text{tor}} \simeq \frac{\left( \psi^*H^2(Z, \mathbb{Z}) \right)^{\perp}}{\sum_{L \in \mathbb{Z}_\infty} \mathbb{Z}L^+}.
\]

By the proof of Proposition 5.4, this is an abelian group of order \( 3^6 \), hence the sublattice \( \left( \psi^*H^2(Z, \mathbb{Z}) \right)^{\perp} \subset H^2(Y, \mathbb{Z}) \) is of rank 27 and

\[
\det \left( \psi^*H^2(Z, \mathbb{Z}) \right)^{\perp} = (3^6)^{-2} \cdot \det \left( \sum_{L \in \mathbb{Z}_\infty} \mathbb{Z}L^+ \right) = -3^{15}.
\]

Since \( H^2_{\text{prim}}(X, \mathbb{Z}) \) is generated by the difference of two lines on \( X \), by Proposition 3.9, we have \( 3\phi^*H^2_{\text{prim}}(X, \mathbb{Z}) \subset \sum_{L \in \mathbb{Z}_\infty} \mathbb{Z}L^+ \) and \( \phi^*H^2_{\text{prim}}(X, \mathbb{Z}) \subset \left( \psi^*H^2(Z, \mathbb{Z}) \right)^{\perp} \). By Remark 3.6, we can directly compute the determinant of the sublattice \( \phi^*H^2_{\text{prim}}(X, \mathbb{Z}) + \sum_{L \in \mathbb{Z}_\infty} \mathbb{Z}L^+ \subset H^2(Y, \mathbb{Z}) \), that is \( \det \left( \phi^*H^2_{\text{prim}}(X, \mathbb{Z}) + \sum_{L \in \mathbb{Z}_\infty} \mathbb{Z}L^+ \right) = -3^{15} \). Hence we have

\[
\phi^*H^2_{\text{prim}}(X, \mathbb{Z}) + \sum_{L \in \mathbb{Z}_\infty} \mathbb{Z}L^+ = \psi^*H^2(Z, \mathbb{Z})^{\perp}.
\]
This implies that the homomorphism
\[(\mathbb{Z}/3\mathbb{Z})^{\oplus 6} \cong \frac{H^2_{\text{prim}}(X, \mathbb{Z})}{3H^2_{\text{prim}}(X, \mathbb{Z})} \rightarrow \frac{(\psi^*H^2(Z, \mathbb{Z}))}{\sum_{L \in \mathbb{Z}_\infty} ZL^+}\]
is surjective. Since the order of these groups are both equal to 3^6, it is an isomorphism.

By proposition \[5.7\] and Remark \[5.6\], we have the isomorphism
\[
\frac{H^2(Y, \mathbb{Z})}{\phi^*H^2_{\text{prim}}(X, \mathbb{Z}) + \sum_{L \in \mathbb{Z}_\infty} ZL^+} \cong H^2(Z \setminus Z_\infty, \mathbb{Z})_{\text{free}}.
\]
We denote by \((\bigwedge^2 H^3(V, \mathbb{Z}))_{\text{0}}\) the subspace of \((\bigwedge^2 H^3(V, \mathbb{Z}))\) orthogonal to \([\psi^*O_Z(1)] \in H^2(Y, \mathbb{Z})\). We denote by \((\bigwedge^2 H^3(V, \mathbb{Z}))_{\text{0}}\) the kernel of the homomorphism
\[
\bigwedge^2 H^3(V, \mathbb{Z}) \rightarrow \mathbb{Z}; \quad \alpha_1 \wedge \alpha_2 \mapsto \deg ((\alpha_1 \cup \alpha_2) \cap [V]),
\]
and denote by \(H^3(V, \mathbb{Z})(1)\) the Hodge structure of weight 1 which is defined from the Hodge structure \(H^3(V, \mathbb{Z})\) by the shift of the weight.

**Theorem 5.8.** There is a natural injective homomorphism
\[
\left(\bigwedge^2 H^3(V, \mathbb{Z})(1)\right)_{\text{Gal}(\rho)} \rightarrow \frac{H^2(Y, \mathbb{Z})}{\phi^*H^2_{\text{prim}}(X, \mathbb{Z}) + \sum_{L \in \mathbb{Z}_\infty} ZL^+}
\]
with the cokernel \(\mathbb{Z}/2\mathbb{Z}\), which induces an isomorphism
\[
\left(\bigwedge^2 H^3(V, \mathbb{Z})(1)\right)_{\text{0}} \cong \left(\frac{H^2(Y, \mathbb{Z})}{\phi^*H^2_{\text{prim}}(X, \mathbb{Z}) + \sum_{L \in \mathbb{Z}_\infty} ZL^+}\right)_{\text{0}}
\]
of Hodge structures.

**Proof.** By Theorem \[4.1\] [Lemma 9.13 and (10.14)], Proposition \[5.4\], Remark \[5.6\] and Proposition \[5.7\], we have the following sequence of homomorphisms of Hodge structures;
\[
\begin{align*}
\left(\bigwedge^2 H^3(V, \mathbb{Z})(1)\right)_{\text{Gal}(\rho)} & \cong \left(\bigwedge^2 H^1(S, \mathbb{Z})\right)_{\text{Gal}(\rho)} & & \cong \left(\bigwedge^2 H^1(A, \mathbb{Z})\right)_{\text{Gal}(\rho)} \\
H^2(S \setminus S_\infty, \mathbb{Z})_{\text{Gal}(\rho)} & \cong H^2(S, \mathbb{Z})_{\text{Gal}(\rho)} & & \supset H^2(A, \mathbb{Z})_{\text{Gal}(\rho)} \\
H^2(Z \setminus Z_\infty, \mathbb{Z})_{\text{free}} & \cong H^2(Y \setminus Y_\infty, \mathbb{Z})_{\text{free}} & & \supset \frac{H^2(Y, \mathbb{Z})}{\phi^*H^2_{\text{prim}}(X, \mathbb{Z}) + \sum_{L \in \mathbb{Z}_\infty} ZL^+}.
\end{align*}
\]
Since \((\wedge^2 H^3(V, \mathbb{Z})(1))_0\) corresponds to \(H^2_{\text{prim}}(A, \mathbb{Z})\), and \((\phi^* H^2_{\text{prim}}(X, \mathbb{Z})+\sum_{L \in \mathbb{Z}_\infty} \mathbb{Z}L^+)\) corresponds to \(H^2_{\text{prim}}(S, \mathbb{Z})^{\text{Gal}(\rho)}\), by Proposition 4.2 we have the isomorphism
\[
\left(\bigwedge^2 H^3(V, \mathbb{Z})(1)\right)_0^{\text{Gal}(\rho)} \cong \left(\phi^* H^2_{\text{prim}}(X, \mathbb{Z})+\sum_{L \in \mathbb{Z}_\infty} \mathbb{Z}L^+\right)_0.
\]

We denote by \(A_i\) the positive definite root lattice of type \(A_i\), and by \(1\) the trivial lattice of rank 1.

**Proposition 5.9.** There are isomorphisms of lattices:
\[
(H^2(Z \setminus Z_\infty, \mathbb{Z})_{\text{free}}, \langle , \rangle_Z) \simeq (\frac{1}{3} \cdot 1) \oplus (-\frac{1}{3} \cdot 1) \oplus (\frac{1}{3} A_2) \oplus (-\frac{1}{3} A_2)^6,
\]
\[
(H^2_{\text{prim}}(Z \setminus Z_\infty, \mathbb{Z})_{\text{free}}, \langle , \rangle_Z) \simeq (-\frac{1}{3} A_4) \oplus (\frac{1}{3} A_2) \oplus (-\frac{1}{3} A_2)^6,
\]
\[
(H^2(Z, \mathbb{Z}), \langle , \rangle_Z) \simeq (3 \cdot 1) \oplus (-3 \cdot 1) \oplus (\frac{1}{3} A_2) \oplus (-A_2)^6,
\]
\[
(H^2_{\text{prim}}(Z, \mathbb{Z}), \langle , \rangle_Z) \simeq (-3 A_4) \oplus A_2^6 \oplus (-A_2)^6.
\]

We define an alternating form on \(H^1(A, \mathbb{Z})\) by
\[
\langle , \rangle_A : H^1(A, \mathbb{Z}) \times H^1(A, \mathbb{Z}) \rightarrow \mathbb{Z}; (\alpha_1, \alpha_2) \mapsto \deg \left( \left( \frac{\theta^4}{4!} \cup \alpha_1 \cup \alpha_2 \right) \cap [A] \right).
\]

**Lemma 5.10** ([1] (2.7)). There is a basis \((v_0, \ldots, v_4)\) of the \(\mathbb{Z}[\omega]\)-module \(H^1(A, \mathbb{Z})\) such that
\[
\left( \langle v_i, v_j \rangle_A \right)_{0 \leq i, j \leq 4} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]
\[
\left( \langle v_i, \omega v_j \rangle_A \right)_{0 \leq i, j \leq 4} = \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

**Proof of Proposition 5.9.** Using the basis in Lemma 5.10, the class of the principal polarization is
\[
\theta = -v_0 \cup \omega v_0 + \sum_{i=1}^4 v_i \cup \omega v_i \in H^2(A, \mathbb{Z}).
\]

We set \(\tau = \frac{1}{2} \theta \in H^2(A, \mathbb{Q})\). Then \(\tau\) corresponds to the class of an incidence divisor on \(S\), and \(H^2(S, \mathbb{Z})\) is identified with the sublattice in \(H^2(A, \mathbb{Q})\) generated by \(\tau\) and \(H^2(A, \mathbb{Z})\). We define sublattices in \(H^2(A, \mathbb{Q})\) by
\[
U_0 = \bigoplus_{i=0}^4 \mathbb{Z}v_i \cup \omega v_i \subset H^2(A, \mathbb{Z}),
\]
\[ \tilde{U}_0 = \mathbb{Z} \tau + U_0 = \mathbb{Z} \tau \bigoplus \bigoplus_{i=1}^{4} \mathbb{Z} v_i \cup \omega v_i \subset H^2(A, \mathbb{Q}), \]

\[ U'_0 = \mathbb{Z} (v_0 \cup \omega v_0 + v_1 \cup \omega v_1) \bigoplus \bigoplus_{i=1}^{3} \mathbb{Z} (v_{i+1} \cup \omega v_{i+1} - v_i \cup \omega v_i) \subset H^2(A, \mathbb{Z}) \]

and

\[ U_{i,j} = \mathbb{Z} v_i \cup v_j \bigoplus \mathbb{Z} v_i \cup \omega v_j \bigoplus \mathbb{Z} \omega v_i \cup v_j \bigoplus \mathbb{Z} \omega v_i \cup \omega v_j \subset H^2(A, \mathbb{Z}) \]

for \(0 \leq i < j \leq 4\). Then we have orthogonal decompositions of lattices

\[ H^2(S, \mathbb{Z}) = \tilde{U}_0 \bigoplus \bigoplus_{0 \leq i < j \leq 4} U_{i,j}, \]

\[ H^2_{\text{prim}}(S, \mathbb{Z}) \simeq H^2_{\text{prim}}(A, \mathbb{Z}) = U'_0 \bigoplus \bigoplus_{0 \leq i < j \leq 4} U_{i,j}, \]

which are compatible with the \( \text{Gal}(\rho) \)-action. The \( \text{Gal}(\rho) \)-action on \( \tilde{U}_0 \simeq 1 \oplus (-1)^{\oplus 4} \) and \( U'_0 \simeq (-A_4) \) are trivial, and the invariant parts of the \( \text{Gal}(\rho) \)-action on \( U_{i,j} \) are

\[ U_{i,j}^\text{Gal}(\rho) = \mathbb{Z} (v_i \cup v_j + \omega v_i \cup \omega v_j) \bigoplus \mathbb{Z} (v_i \cup v_j + \omega v_i \cup \omega v_j + v_0) \simeq A_2 \]

for \(1 \leq j \leq 4\), and

\[ U_{i,j}^\text{Gal}(\rho) = \mathbb{Z} (v_i \cup v_j + \omega v_i \cup \omega v_j + v_i \cup v_j) \bigoplus \mathbb{Z} (v_i \cup v_j + \omega v_i \cup \omega v_j + v_i) \simeq (-A_2) \]

for \(1 \leq i < j \leq 4\). Hence we have

\[ (H^2(S, \mathbb{Z})^\text{Gal}(\rho), \langle , \rangle_S) \simeq 1 \oplus (-1)^{\oplus 4} \oplus A_2^{\oplus 4} \oplus (-A_2)^{\oplus 6} \]

and

\[ (H^2_{\text{prim}}(S, \mathbb{Z})^\text{Gal}(\rho), \langle , \rangle_S) \simeq (-A_4) \oplus A_2^{\oplus 4} \oplus (-A_2)^{\oplus 6}. \]

By Proposition 5.4, we have the results for lattices \( H^2(Z \setminus Z_{\infty}, \mathbb{Z})_{\text{free}} \) and \( H^2_{\text{prim}}(Z \setminus Z_{\infty}, \mathbb{Z})_{\text{free}} \). In the similar way, the statements for lattices \( H^2(Z, \mathbb{Z}) \) and \( H^2_{\text{prim}}(Z, \mathbb{Z}) \) can be proved.

**Proposition 5.11.**

\[ \phi^* H^2_{\text{prim}}(X, \mathbb{Z}) + \sum_{L \in Z_{\infty}} \mathbb{Z} L^+ = (\psi^* H^2(Z, \mathbb{Z}))^\perp, \]

\[ \phi^* H^2(X, \mathbb{Z}) + \sum_{L \in Z_{\infty}} \mathbb{Z} L^+ = (\psi^* H^2_{\text{prim}}(Z, \mathbb{Z}))^\perp. \]
Proof. The first equality has been proved in the proof of Proposition 5.7. Since
\[ \psi^*H^2_{\text{prim}}(Z, Z) = (\phi^*H^2(X, Z) + \sum_{L \in \mathbb{Z}} ZL^+) \perp, \]
we have
\[ \phi^*H^2(X, Z) + \sum_{L \in \mathbb{Z}} ZL^+ \subset (\psi^*H^2_{\text{prim}}(Z, Z)) \perp, \]
which are sublattices of rank 28. We compute the determinant of these lattices. By Proposition 5.9, we have
\[ \det H^2_{\text{prim}}(Z, Z) = 3^{14} \cdot (\det (-A_4) \cdot (\det A_2)^4 \cdot (\det (-A_2))^6) = 3^{14} \cdot 5, \]
hence \(\det (\psi^*H^2_{\text{prim}}(Z, Z)) \perp = -3^{14} \cdot 5.\) On the other hand, by Remark 3.6, we can directly compute the determinant of the sublattice
\[ \phi^*H^2(X, Z) + \sum_{L \in \mathbb{Z}} ZL^+ \subset H^2(Y, Z), \]
that is
\[ \det \left( \phi^*H^2(X, Z) + \sum_{L \in \mathbb{Z}} ZL^+ \right) = -3^{14} \cdot 5, \]
hence we have the second equality.

\[ \square \]

6 Néron-Severi lattice

The Néron-Severi group \(\text{NS}(Y)\) of the surface \(Y\) is the subgroup of \(H^2(Y, \mathbb{Z})\) generated by algebraic cycles. Since \(H^2(X, \mathbb{Z})\) is generated by algebraic cycles,
\[ \text{NS}(Y)_0 = \phi^*H^2(X, \mathbb{Z}) + \sum_{L \in \mathbb{Z}} \mathbb{Z}L^+ \subset H^2(Y, \mathbb{Z}) \]
is contained in \(\text{NS}(Y)\). By the proof of Proposition 5.11, \(\text{NS}_0(Y)\) is a sublattice of rank 28 with the determinant \(-3^{14} \cdot 5.\) If there are no Eckardt points on \(X\), then \(\text{NS}(Y)_0 = \sum_{L \in \mathbb{Z}} (ZL^+ + ZL^-)\).

Theorem 6.1. \(\text{NS}(Y) = \text{NS}(Y)_0\) for a generic cubic surface \(X\).

The idea of the proof is based on the theory of infinitesimal variations of Hodge structure [2, Section 3]. Let \(\mathcal{M} \subset \mathbf{P}(H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))^\vee)\) be the space of smooth cubic surfaces, and let \(\mathcal{Y} \to \mathcal{M}\) be the family of the surface \(Y\). We define a homomorphism by
\[ \epsilon : H^1(Y, \Omega^1_Y) \to \text{Hom}(T_M([F]), H^2(Y, \mathcal{O}_Y)); \omega \mapsto [\xi \mapsto c(\kappa(\xi) \cup \omega)], \]
where \(T_M([F])\) is the tangent space of \(\mathcal{M}\) at \([F] \in \mathcal{M}, Y\) is the fiber of \(\mathcal{Y} \to \mathcal{M}\) at \([F] \in \mathcal{M}, \kappa(\xi) \in H^1(Y, T_Y)\) is the Kodaira-Spencer class of \(\xi \in T_M([F]),\) and
\[ c : H^2(Y, T_Y \otimes \Omega^1_Y) \to H^2(Y, \mathcal{O}_Y) \]
is the contraction homomorphism. We remark that \(\mathbf{C} \otimes \mathbb{Z} \text{NS}(Y)\) is isomorphic to the kernel of \(\epsilon\) for a generic \([F] \in \mathcal{M}.\)
Proposition 6.2. The homomorphism $\epsilon : H^1(Y, \Omega^1_Y) \to \text{Hom} \left( T_M([F]), H^2(Y, \mathcal{O}_Y) \right)$ is of rank 16.

The computation of the infinitesimal variations of Hodge structure for $Y$ is given in Section 7 and Proposition 6.2 will be proved there.

Proof of Theorem 6.1. By Proposition 5.11, $\frac{\text{NS}(Y)}{\text{NS}_0(Y)}$ has no torsion element, and by Proposition 6.2, the rank of $\text{NS}(Y)$ is 28 for a generic cubic surface $X$. Hence we have $\frac{\text{NS}(Y)}{\text{NS}_0(Y)} = 0$ for a generic $X$. \qed

Next we study the surface $Y$ for the Fermat cubic surface $X$. Let $X \subset \mathbb{P}^3$ be the cubic surface defined by $F = x_0^3 + \cdots + x_3^3$. Then the triple Galois cover $V$ of $\mathbb{P}^3$ branched along $X$ is the Fermat cubic 3-fold defined by $\tilde{F} = x_0^3 + \cdots + x_3^3 + x_4^3$. We set a point $e_{i,j}^\alpha$ on $V$ by

$$e_{i,j}^\alpha = \{ [x_0 : \cdots : x_4] \in \mathbb{P}^4 \mid x_i + \alpha x_j = 0, \ x_k = 0 \text{ for } k \in \{0, 1, \ldots, 4\} \setminus \{i, j\} \}$$

for $0 \leq i < j \leq 4$ and $\alpha \in \mathbb{C}$ with $\alpha^3 = 1$. The point $e_{i,j}^\alpha$ corresponds to an elliptic curve $E_{i,j}^\alpha$ on the Fano surface $S$ of lines on $V \subset \mathbb{P}^4$ by

$$E_{i,j}^\alpha = \{ L \in S \mid e_{i,j}^\alpha \in L \}. $$

Theorem 6.3 (Roulleau [10], Theorem 3.13). For the Fermat cubic 3-fold $V$, the Néron-Severi lattice $\text{NS}(S)$ is of rank 25 with the determinant $3^{18}$, and

$$\text{NS}(S) = \mathbb{Z} \tau + \sum_{0 \leq i < j \leq 4} (\mathbb{Z} E_{i,j}^1 + \mathbb{Z} E_{i,j}^\omega + \mathbb{Z} E_{i,j}^{\omega^2}),$$

where $\tau$ is the class of an incidence divisor.

By using Theorem 6.3, we compute the Néron-Severi lattice $\text{NS}(Y)$ for the Fermat cubic surface $X$. The branch divisor $B$ of the double cover $\phi : Y \to X$ is the sum of the elliptic curves

$$B_k = \{ [x_0 : \cdots : x_3] \in X \mid x_k = 0 \}$$

for $0 \leq k \leq 3$, because the Hessian of $F$ is $6^4 x_0 x_1 x_2 x_3$. Let $D_k$ be the irreducible component of the ramification divisor $R$ of $\phi : Y \to X$ which corresponds to $B_k$, and let $E_{i,j}^\alpha$ be the irreducible component of the ramification divisor $R$ which corresponds to the Eckardt point

$$\rho(e_{i,j}^\alpha) = \{ [x_0 : \cdots : x_3] \in \mathbb{P}^3 \mid x_i + \alpha x_j = 0, \ x_k = 0 \text{ for } k \in \{0, 1, 2, 3\} \setminus \{i, j\} \}$$

for $0 \leq i < j \leq 3$ and $\alpha \in \mathbb{C}$ with $\alpha^3 = 1$. We remark that $D_k$ is an elliptic curve, and the irreducible decomposition of the ramification divisor is

$$R = \sum_{k=0}^3 D_k + \sum_{0 \leq i < j \leq 3} (E_{i,j}^1 + E_{i,j}^\omega + E_{i,j}^{\omega^2}).$$
Remark 6.4. For a line \( L \) on the Fermat cubic surface \( X \) and an Eckardt point \( e \) on \( X \), the intersection numbers on \( Y \) are computed by

\[
(D_k, D_l) = \begin{cases} 
0 & \text{if } k \neq l, \\
-3 & \text{if } k = l,
\end{cases}
\]

\[
(D_k, L^+) = (D_k, L^-) = 0,
\]

\[
(D_k, \phi^{-1}(e)) = \begin{cases} 
0 & \text{if } e \notin B_k, \\
1 & \text{if } e \in B_k.
\end{cases}
\]

Lemma 6.5. There is an isomorphism

\[
\chi : \text{NS}(S)^{\text{Gal}(\rho)} \xrightarrow{\sim} \frac{\text{NS}(Y)}{\phi^*H^2_{\text{prim}}(X, \mathbb{Z}) + \sum_{L \in \mathbb{Z}_\infty} \mathbb{Z}L^+}
\]

such that

\[
\begin{align*}
\chi(\tau) &= \pi(\phi^* L) & \text{for a line } L \text{ on } X, \\
\chi(\mathcal{E}^\alpha_{i,j}) &= \pi(E^\alpha_{i,j}) & \text{for } 0 \leq i \leq j \leq 3 \text{ and } \alpha^3 = 1, \\
\chi(\mathcal{E}^1_{k,4} + \mathcal{E}^{\omega}_{k,4} + \mathcal{E}^{\omega^2}_{k,4}) &= \pi(D_k) & \text{for } 0 \leq k \leq 3,
\end{align*}
\]

where \( \pi \) denotes the natural surjective homomorphism

\[
\pi : \text{NS}(Y) \longrightarrow \frac{\text{NS}(Y)}{\phi^*H^2_{\text{prim}}(X, \mathbb{Z}) + \sum_{L \in \mathbb{Z}_\infty} \mathbb{Z}L^+}.
\]

Proof. By Proposition 5.4, Remark 5.6 and Proposition 5.7, we have the isomorphism of Hodge structures

\[
H^2(S, \mathbb{Z})^{\text{Gal}(\rho)} \xrightarrow{\sim} \frac{H^2(Y, \mathbb{Z})}{\phi^*H^2_{\text{prim}}(X, \mathbb{Z}) + \sum_{L \in \mathbb{Z}_\infty} \mathbb{Z}L^+},
\]

and this induces the isomorphism

\[
\chi : \text{NS}(S)^{\text{Gal}(\rho)} \xrightarrow{\sim} \frac{\text{NS}(Y)}{\phi^*H^2_{\text{prim}}(X, \mathbb{Z}) + \sum_{L \in \mathbb{Z}_\infty} \mathbb{Z}L^+}.
\]

Since \( 3\tau = [\mathcal{O}_S(1)] = \rho^*[\mathcal{O}_Z(1)] \) by \cite{8}, §10, and \( \pi(\psi^*[\mathcal{O}_Z(1)]) = \pi(3\phi^* L) \) by Proposition 3.9, we have \( \chi(3\tau) = \pi(3\phi^* L) \). Since \( \frac{\text{NS}(Y)}{\phi^*H^2_{\text{prim}}(X, \mathbb{Z}) + \sum_{L \in \mathbb{Z}_\infty} \mathbb{Z}L^+} \) is torsion free, we have \( \chi(\tau) = \pi(\phi^* L) \). The triple cover \( \rho : S \rightarrow Z \) induces a triple cover \( \mathcal{E}^\alpha_{i,j} \rightarrow \psi(E^\alpha_{i,j}) \) for \( 0 \leq i \leq j \leq 3 \), and an isomorphism \( \mathcal{E}^\alpha_{k,4} \sim \psi(D_k) \) for \( 0 \leq k \leq 3 \). These imply that \( \chi(\mathcal{E}^\alpha_{i,j}) = \pi(E^\alpha_{i,j}) \) and \( \chi(\mathcal{E}^1_{k,4} + \mathcal{E}^{\omega}_{k,4} + \mathcal{E}^{\omega^2}_{k,4}) = \pi(D_k) \). \[\square\]
Theorem 6.6. For the Fermat cubic surface $X$, the Néron-Severi lattice $\text{NS}(Y)$ is of rank 44 with the determinant $-3^{12}$, and

\[
\text{NS}(Y) = \sum_{L \in \mathbb{Z}_\infty} (ZL^+ + ZL^-) + \sum_{0 \leq i < j \leq 3} (ZE_{i,j}^1 + ZE_{i,j}^\omega + ZE_{i,j}^{\omega^2}) + \sum_{0 \leq k \leq 3} ZD_k.
\]

Proof. By Theorem $6.3$, we have

\[
\text{NS}(S)^{\text{Gal}(\rho)} = Z\tau + \sum_{0 \leq i < j \leq 3} (Z\mathcal{E}_{i,j}^1 + Z\mathcal{E}_{i,j}^\omega + Z\mathcal{E}_{i,j}^{\omega^2}) + \sum_{0 \leq k \leq 3} Z(\mathcal{E}_{k,4}^1 + \mathcal{E}_{k,4}^\omega + \mathcal{E}_{k,4}^{\omega^2}).
\]

By Lemma $6.5$, we have

\[
\text{NS}(Y) = \phi^* \text{NS}(X) + \sum_{L \in \mathbb{Z}_\infty} ZL^+ + \sum_{0 \leq i < j \leq 3} (ZE_{i,j}^1 + ZE_{i,j}^\omega + ZE_{i,j}^{\omega^2}) + \sum_{0 \leq k \leq 3} ZD_k,
\]

and by Remark $6.6$ and Remark $6.4$, we can directly compute the determinant of the lattice. \qed

Remark 6.7. The sublattice

\[
\sum_{L \in \mathbb{Z}_\infty} (ZL^+ + ZL^-) + \sum_{0 \leq i < j \leq 3} (ZE_{i,j}^1 + ZE_{i,j}^\omega + ZE_{i,j}^{\omega^2})
\]

is of rank 44 with the determinant $-2^2 \cdot 3^{12}$, hence it is a sublattice of index 2 in $\text{NS}(Y)$.

7 Infinitesimal variations of Hodge structure

In this section, we compute the infinitesimal variations of Hodge structure for the surface $Y \subset \Gamma(\mathbb{P}^3)$, and we prove Proposition $6.2$. The method is introduced in [9] as a theory of Jacobian rings. Let $Y = Y_3 \subset Y_2 \subset Y_1 \subset \Gamma(\mathbb{P}^3)$ be the varieties defined in Section $2$. Let

\[
\begin{array}{c}
\mathcal{Y}_3 \subset \mathcal{M} \times \Gamma(\mathbb{P}^3) \\
\downarrow \searrow \\
\mathcal{M}
\end{array}
\]

be the family of the surface $Y_3$. Let

\[
\kappa : T_{\mathcal{M}}([F]) \longrightarrow H^1(Y_3, T_{Y_3}),
\]

be the Kodaira-Spencer map. By the duality, Proposition $6.2$ is a corollary of the following proposition.
Proposition 7.1. The homomorphism

\[ T_M([F]) \otimes H^0(Y_3, \Omega^2_{Y_3}) \rightarrow H^1(Y_3, \Omega^1_{Y_3}); \; \xi \otimes \omega \rightarrow c(\kappa(\xi) \cup \omega) \]

is of rank 16, where \( c \) is the contraction homomorphism

\[ c : H^1(Y_3, T_{Y_3} \otimes \Omega^2_{Y_3}) \sim H^1(Y_3, \Omega^1_{Y_3}). \]

Let \( S_{p3} \) be the kernel of the homomorphism \( \mathcal{O}_{p3} \otimes V \rightarrow Q_{p3} \simeq \mathcal{O}_{p3}(1) \), where \( V = H^0(P^3, \mathcal{O}_{p3}(1)) \). Let \( S_{\Lambda(p3)} \) be the kernel of the homomorphism \( \mathcal{O}_{\Lambda(p3)} \otimes V \rightarrow Q_{\Lambda(p3)} \).

Then we have the natural exact sequence

\[ 0 \rightarrow \psi^* S_{\Lambda(p3)} \xrightarrow{\sigma} \phi^* S_{p3} \xrightarrow{\lambda} \psi^* Q_{\Lambda(p3)} \xrightarrow{\tau} \phi^* Q_{p3} \rightarrow 0 \]

of vector bundles on \( \Gamma(P^3) \), and we have the exact sequence

\[ 0 \rightarrow \mathcal{O}_{\Gamma(p3)} \xrightarrow{\lambda} \phi^* S_{p3} \otimes \psi^* Q_{\Lambda(p3)} \xrightarrow{\tau \times \sigma} \phi^* (S_{p3} \otimes Q_{p3}) \oplus \psi^* (S_{\Lambda(p3)} \otimes Q_{\Lambda(p3)}) \xrightarrow{\sigma \oplus (-\tau)} \psi^* S_{\Lambda(p3)} \otimes \phi^* Q_{p3} \rightarrow 0. \]

Since the homomorphism

\[ T_{p3 \times \Lambda(p3)|\Gamma(p3)} \simeq \phi^* (S_{p3} \otimes Q_{p3}) \oplus \psi^* (S_{\Lambda(p3)} \otimes Q_{\Lambda(p3)}) \xrightarrow{\sigma \oplus (-\tau)} \psi^* S_{\Lambda(p3)} \otimes \phi^* Q_{p3} \]

is identified with the natural homomorphism to the normal bundle \( T_{p3 \times \Lambda(p3)|\Gamma(p3)} \rightarrow N_{\Gamma(p3)/p3 \times \Lambda(p3)} \), we have the exact sequence

\[ 0 \rightarrow \mathcal{O}_{\Gamma(p3)} \xrightarrow{\lambda} \phi^* S_{p3} \otimes \psi^* Q_{\Lambda(p3)} \rightarrow T_{\Gamma(p3)} \rightarrow 0. \tag{7.1} \]

Let \( (x_0, \ldots, x_3) \) be a basis of the vector space \( V = H^0(P^3, \mathcal{O}_{p3}(1)) \), and let \( (x_0^\vee, \ldots, x_3^\vee) \) be the dual basis of \( (x_0, \ldots, x_3) \).

Lemma 7.2.

\[ H^0(Y_2, T_{\Gamma(p3)}|Y_2) \simeq \frac{V^\vee \otimes V}{C \cdot \sum_{i=0}^3 x_i^\vee \otimes x_i} \]

Proof. The natural homomorphism \( \mathcal{O}_{\Gamma(p3)} \otimes V^\vee \otimes V \rightarrow \phi^* S_{p3}^\vee \otimes \psi^* Q_{\Lambda(p3)} \) induces the isomorphism \( V^\vee \otimes V \simeq H^0(\Gamma(P^3), \phi^* S_{p3}^\vee \otimes \psi^* Q_{\Lambda(p3)}) \). By the exact sequence (7.1), we have

\[ H^0(\Gamma(P^3), T_{\Gamma(p3)}) \simeq \frac{V^\vee \otimes V}{C \cdot \sum_{i=0}^3 x_i^\vee \otimes x_i}, \]

and we can prove that \( H^0(\Gamma(P^3), T_{\Gamma(p3)}) \simeq H^0(Y_2, T_{\Gamma(p3)}|Y_2) \) by the restriction. \( \square \)
We define filtration on $\Psi^* \text{Sym}^3 \mathcal{Q}_{\Lambda(\mathbb{P}^3)}$ by

$$\text{Fil}^i = \text{Fil}^i \Psi^* \text{Sym}^3 \mathcal{Q}_{\Lambda(\mathbb{P}^3)} = S^i \otimes \Psi^* \text{Sym}^{3-i} \mathcal{Q}_{\Lambda(\mathbb{P}^3)} \subset \Psi^* \text{Sym}^3 \mathcal{Q}_{\Lambda(\mathbb{P}^3)},$$

where $S$ denotes the line bundle defined as the kernel of the homomorphism $\Psi^* \mathcal{Q}_{\Lambda(\mathbb{P}^3)} \xrightarrow{\tau} \Phi^* \mathcal{Q}_{\mathbb{P}^3}$. For $G \in \text{Sym}^3 V$, we denote by $[G]_i$ the image of $G$ by the natural homomorphism $\text{Sym}^3 V \rightarrow H^0(\Lambda(\mathbb{P}^3), \text{Fil}^0, \text{Fil}^i)$. We define the sheaf of $\mathcal{O}_{\mathbb{P}^2}$-modules $\mathcal{N}$ as the cokernel of the homomorphism

$$\mathcal{O}_{\mathbb{P}^2} \xrightarrow{[F]_3, \mathbb{P}^2} \mathcal{N},$$

and denote by $[G]_{i,Y_j}$ its restriction to $H^0(Y_j, \text{Fil}^0, \text{Fil}^i|_{Y_j})$. We remark that $Y_j$ is the zeros of the regular section $[F]_j$, and if $i \geq j$ then $[F]_{i,Y_j}$ is contained in $H^0(Y_j, \text{Fil}^0, \text{Fil}^i|_{Y_j})$. We define the sheaf of exact sequence

$$0 \rightarrow T_{Y_2}(-\log Y_3) \rightarrow T_{\Gamma(\mathbb{P}^3)|Y_2} \rightarrow \mathcal{N} \rightarrow 0.$$

**Proof.** By the definition of $\mathcal{N}$, we have the exact sequence

$$0 \rightarrow \mathcal{N}_{Y_3/Y_2} \rightarrow \mathcal{N} \rightarrow \mathcal{N}_{Y_2/\Gamma(\mathbb{P}^3)} \rightarrow 0,$$

where we remark that $\mathcal{N}_{Y_2/\Gamma(\mathbb{P}^3)} \simeq \text{Fil}^0, \text{Fil}^2|_{Y_2}$, and $\mathcal{N}_{Y_3/Y_2}$ is the cokernel of the homomorphism

$$\mathcal{O}_{\mathbb{P}^2} \xrightarrow{[F]_3, \mathbb{P}^2} \mathcal{N}. $$

Since the kernel of the composition $T_{\Gamma(\mathbb{P}^3)|Y_2} \rightarrow \mathcal{N} \rightarrow \mathcal{N}_{Y_3/Y_2}$ is identified with $T_{Y_2}$, we have the homomorphism $T_{Y_2} \rightarrow \mathcal{N}_{Y_3/Y_2}$ and its kernel is identified with $T_{Y_2}(-\log Y_3)$. \qed

**Lemma 7.4.**

$$H^0(Y_2, \mathcal{N}) \simeq \frac{V \otimes \text{Sym}^2 V}{C \cdot \sum_{i=0}^{3} x_i \otimes \frac{\partial F}{\partial x_i}},
Proof. By the homomorphism
\[ \Psi^* \text{Sym}^2 Q_{\Lambda(P^3)} \to \Phi^* Q_{P^2} \otimes \Psi^* \text{Sym}^2 Q_{\Lambda(P^3)}; \ abc \mapsto \tau(a) \otimes bc + \tau(b) \otimes ca + \tau(c) \otimes ab, \]
we have the isomorphism
\[ \frac{\text{Fil}^0}{\text{Fil}^3} \simeq \Phi^* Q_{P^2} \otimes \Psi^* \text{Sym}^2 Q_{\Lambda(P^3)}. \]
The natural homomorphism \( O_{\Gamma(P^3)} \otimes V \otimes \text{Sym}^2 V \to \Phi^* Q_{P^2} \otimes \Psi^* \text{Sym}^2 Q_{\Lambda(P^3)} \) induces the isomorphism
\[ V \otimes \text{Sym}^2 V \simeq H^0(\Gamma(P^3), \Phi^* Q_{P^2} \otimes \Psi^* \text{Sym}^2 Q_{\Lambda(P^3)}), \]
and we can prove that
\[ H^0(\Gamma(P^3), \frac{\text{Fil}^0}{\text{Fil}^3}) \simeq H^0(Y_2, \frac{\text{Fil}^0}{\text{Fil}^3}|_{Y_2}). \]

By the exact sequence in Lemma 7.3, we have the exact sequence
\[ 0 \to O_{Y_2} \xrightarrow{[F]_{3, Y_2}} \frac{\text{Fil}^0}{\text{Fil}^3}|_{Y_2} \to \mathcal{N} \to 0, \]
we have
\[ H^0(Y_2, \mathcal{N}) \simeq \frac{V \otimes \text{Sym}^2 V}{C \cdot \sum_{i=0}^3 x_i \otimes \partial F \partial x_i}. \]

Lemma 7.5. The kernel of the homomorphism
\[ H^1(Y_2, T_{Y_2}(- \log Y_3)) \to H^1(Y_2, T_{\Gamma(P^3)}|_{Y_2}) \]
is identified with the cokernel of the injective homomorphism
\[ \delta \circ \nu : V^\vee \otimes V \to V \otimes \text{Sym}^2 V; \ x_j^\vee \otimes A \mapsto A \otimes \frac{\partial F}{\partial x_j} + \sum_{i=0}^3 x_i \otimes A \frac{\partial^2 F}{\partial x_i \partial x_j}. \]

Proof. By the exact sequence in Lemma 7.3, we have the exact sequence
\[ H^0(Y_2, T_{\Gamma(P^3)}|_{Y_2}) \to H^0(Y_2, \mathcal{N}) \to H^1(Y_2, T_{Y_2}(- \log Y_3)) \to H^1(Y_2, T_{\Gamma(P^3)}|_{Y_2}). \]
By Lemma 7.2 and Lemma 7.4, we can check that \( H^0(Y_2, T_{\Gamma(P^3)}|_{Y_2}) \to H^0(Y_2, \mathcal{N}) \) is induced by the homomorphism
\[ \frac{V^\vee \otimes V}{C \cdot \sum_{i=0}^3 x_i^\vee \otimes x_i} \to \frac{V \otimes \text{Sym}^2 V}{C \cdot \sum_{i=0}^3 x_i \otimes \frac{\partial F}{\partial x_i}}; \ x_j^\vee \otimes A \mapsto A \otimes \frac{\partial F}{\partial x_j} + \sum_{i=0}^3 x_i \otimes A \frac{\partial^2 F}{\partial x_i \partial x_j}. \]
We remark that the homomorphism $\delta \circ \nu$ is the composition of injective homomorphisms $\nu : V^\vee \otimes V \to \text{Sym}^3 V$ and

$$\delta : \text{Sym}^3 V \longrightarrow V \otimes \text{Sym}^2 V; \; G \longmapsto \sum_{i=0}^{3} x_i \otimes \frac{\partial G}{\partial x_i}.$$ 

$\square$

Remark 7.6. Since $H^1(Y_2, T_{Y_2}(-Y_3)) = 0$, the homomorphism

$$H^1(Y_2, T_{Y_2}(-\log Y_3)) \longrightarrow H^1(Y_3, T_{Y_3})$$

is injective.

Lemma 7.7. The Kodaira-Spencer map $\kappa : T_M([F]) \to H^1(Y_3, T_{Y_3})$ is computed by the homomorphism

$$\kappa : T_M([F]) \simeq \frac{\text{Sym}^3 V}{C \cdot F} \longrightarrow \frac{V \otimes \text{Sym}^2 V}{(\delta \circ \nu)(V^\vee \otimes V)} \subset H^1(Y_3, T_{Y_3}); \; G \longmapsto \sum_{i=0}^{3} x_i \otimes \frac{\partial G}{\partial x_i},$$

and its image $\kappa(T_M([F]))$ is identified with the cokernel of the injective homomorphism $\nu : V^\vee \otimes V \to \text{Sym}^3 V$.

Proof. Let $(F, G_1, \ldots, G_{19})$ be a basis of $\text{Sym}^3 V \simeq H^0(P^3, \mathcal{O}_{P^3}(3))$. We have a local coordinate of $M$ at $[F] \in M \subset P(H^0(P^3, \mathcal{O}_{P^3}(3))^\vee)$ by

$$(\mu_1, \ldots, \mu_{19}) \longmapsto F - \sum_{i=1}^{19} \mu_i G_i,$$

and the tangent space of $M$ at $[F]$ is identified with $\frac{\text{Sym}^3 V}{C \cdot F}$ by

$$T_M([F]) \simeq \frac{\text{Sym}^3 V}{C \cdot F}; \; \frac{\partial}{\partial \mu_j} \longmapsto G_j.$$ 

We have the commutative diagram of exact sequences

$$0 \longrightarrow T_{Y_3} \longrightarrow T_{Y_3}|_{Y_3} \longrightarrow \mathcal{O}_{Y_3} \otimes T_M([F]) \longrightarrow 0$$

$$0 \longrightarrow T_{Y_3} \longrightarrow T_{\Gamma(P^3)}|_{Y_3} \longrightarrow \mathcal{N}_{Y_3/\Gamma(P^3)} \longrightarrow 0,$$

where $T_{Y_3}|_{Y_3} \to T_{\Gamma(P^3)}|_{Y_3}$ is induced by the natural projection and $\tilde{\kappa}$ is defined by

$$\tilde{\kappa} : T_M([F]) \longrightarrow H^0(Y_3, \frac{\text{Fil}^0}{\text{Fil}^3}_{Y_3}) \simeq H^0(Y_3, \mathcal{N}_{Y_3/\Gamma(P^3)}); \; \frac{\partial}{\partial \mu_j} \longmapsto [G_j]_{3,Y_3}.$$
We can compute the homomorphism $\tilde{\kappa}$ by

$$\tilde{\kappa} : T_M([F]) \longrightarrow \frac{V \otimes \text{Sym}^2 V}{C \cdot F} \approx H^0(Y_2, N \cap Y_3) \subset H^0(Y_3, \mathcal{N}_{Y_3/\Gamma(P^3)});$$

and $\tilde{\kappa}$ induces the homomorphism

$$\kappa : T_M([F]) \approx \text{Sym}^3 V \longrightarrow V \otimes \text{Sym}^2 V \subset H^1(Y_2, T_{Y_2}(-\log Y_3)) \subset H^1(Y_3, T_{Y_3}).$$

\[\square\]

**Lemma 7.8.** $H^0(Y_2, (\Phi^* Q_{P^3} \otimes T_{\Gamma(P^3)})|_{Y_2})$ is naturally identified with the cokernel of the injective homomorphism

$$\alpha : V \oplus V \longrightarrow V \otimes V^\vee \otimes V; \quad A \oplus B \longmapsto \sum_{i=0}^3 (x_i \otimes x_i^\vee \otimes A + B \otimes x_i^\vee \otimes x_i)$$

**Proof.** By the exact sequence

$$0 \longrightarrow \Psi^* Q_{\Lambda(P^3)} \longrightarrow \Phi^* Q_{P^3} \otimes V^\vee \otimes \Psi^* Q_{\Lambda(P^3)} \longrightarrow \Phi^* (Q_{P^3} \otimes S_{P^3}^\vee) \otimes \Psi^* Q_{\Lambda(P^3)} \longrightarrow 0,$$

$H^0(Y_2, (\Phi^* (Q_{P^3} \otimes S_{P^3}^\vee) \otimes \Psi^* Q_{\Lambda(P^3)})|_{Y_2})$ is identified with the cokernel of the injective homomorphism

$$\lambda_0 : V \longrightarrow V \otimes V^\vee \otimes V; \quad A \longmapsto \sum_{i=0}^3 x_i \otimes x_i^\vee \otimes A.$$

By the exact sequence \((7.1)\), we have the exact sequence

$$0 \longrightarrow \Phi^* Q_{P^3} \longrightarrow \Phi^* (Q_{P^3} \otimes S_{P^3}^\vee) \otimes \Psi^* Q_{\Lambda(P^3)} \longrightarrow \Phi^* Q_{P^3} \otimes T_{\Gamma(P^3)} \longrightarrow 0,$$

and $H^0(Y_2, (\Phi^* Q_{P^3} \otimes T_{\Gamma(P^3)})|_{Y_2})$ is identified with the cokernel of the injective homomorphism

$$V \longrightarrow \frac{V \otimes V^\vee \otimes V}{\lambda_0(V)}; \quad B \longmapsto \sum_{i=0}^3 B \otimes x_i^\vee \otimes x_i.$$

\[\square\]

**Lemma 7.9.** $H^0(Y_2, (\Phi^* Q_{P^1})|_{Y_2} \otimes \mathcal{N})$ is naturally identified with the cokernel of the injective homomorphism

$$\beta : V \oplus V \longrightarrow \text{Sym}^2 V \otimes \text{Sym}^2 V; \quad A \oplus B \longmapsto \sum_{i=0}^3 \left( \frac{\partial F}{\partial x_i} \otimes A x_i + B x_i \otimes \frac{\partial F}{\partial x_i} \right)$$

\[36\]
Lemma 7.10. \( H^1(Y_2, \Omega^2_{Y_2}(\log Y_3)) \) is naturally identified with the cokernel of the injective homomorphism

\[ \nu_1 : V \otimes V^\vee \otimes V \to \Sym^2 V \otimes \Sym^2 V; \quad A \otimes x_j^\vee \otimes B \mapsto AB \otimes \frac{\partial F}{\partial x_j} + \sum_{i=0}^3 A_{x_i} \otimes B \frac{\partial^2 F}{\partial x_i \partial x_j}. \]

Proof. Since

\[ \Omega^2_{Y_2}(\log Y_3) \simeq \Omega^3_{Y_2}(Y_3) \otimes T_{Y_2}(\log Y_3) \simeq (\Phi^* \mathcal{Q}_{\mathcal{P}^3})|_{Y_2} \otimes T_{Y_2}(\log Y_3), \]

we have the exact sequence

\[ 0 \to \Omega^2_{Y_2}(\log Y_3) \to (\Phi^* \mathcal{Q}_{\mathcal{P}^3} \otimes T_{\Gamma(\mathcal{P}^3)})|_{Y_2} \to (\Phi^* \mathcal{Q}_{\mathcal{P}^3})|_{Y_2} \otimes \mathcal{N} \to 0 \]

by Lemma 7.3 and we can check that \( H^1(Y_2, (\Phi^* \mathcal{Q}_{\mathcal{P}^3} \otimes T_{\Gamma(\mathcal{P}^3)})|_{Y_2}) = 0 \). By Lemma 7.8 and Lemma 7.9 \( H^1(Y_2, \Omega^2_{Y_2}(\log Y_3)) \) is identified with the cokernel of the homomorphism

\[ \frac{V \otimes V^\vee \otimes V}{\alpha(V \oplus V)} \to \frac{\Sym^2 V \otimes \Sym^2 V}{\beta(V \oplus V)}; \quad A \otimes x_j^\vee \otimes B \mapsto AB \otimes \frac{\partial F}{\partial x_j} + \sum_{i=0}^3 A_{x_i} \otimes B \frac{\partial^2 F}{\partial x_i \partial x_j}, \]

and it is injective because \( H^0(Y_2, \Omega^2_{Y_2}(\log Y_3)) = 0 \). Since the homomorphism \( \nu_1 \) induces an isomorphism \( \alpha(V \oplus V) \simeq \beta(V \oplus V) \), the homomorphism \( \nu_1 \) is injective. \( \square \)
Proof of Proposition 7.1. By Lemma 7.7 and Lemma 7.10 we have a commutative diagram of exact sequences

\[
\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
V \otimes V^\vee \otimes V & = & V \otimes V^\vee \otimes V \\
1 \otimes \nu & \downarrow & \downarrow \nu_1 \\
V \otimes \text{Sym}^3 V & \xrightarrow{\delta_1} & \text{Sym}^2 V \otimes \text{Sym}^2 V \\
\downarrow & & \downarrow \\
H^0(Y_3, \Omega^2_{Y_3}) \otimes \kappa(T_M([F])) & \to & H^1(Y_3, \Omega^1_{Y_3}) \\
0, & & \\
\end{array}
\]

where we remark that

\[
V \simeq H^0(Y_2, (\Phi^*Q_{P_3})|_{Y_2}) \simeq H^0(Y_2, \Omega^2_{Y_2}(Y_3)) \simeq H^0(Y_3, \Omega^2_{Y_3}),
\]

\[
\frac{\text{Sym}^2 V \otimes \text{Sym}^2 V}{\nu_1(V \otimes V^\vee \otimes V)} \simeq H^1(Y_2, \Omega^2_{Y_2}(\log Y_3)) \subset H^1(Y_3, \Omega^1_{Y_3})
\]

and the homomorphism \(\delta_1\) is defined by

\[
\delta_1 : V \otimes \text{Sym}^3 V \longrightarrow \text{Sym}^2 V \otimes \text{Sym}^2 V; \quad A \otimes B \longmapsto \sum_{i=0}^3 A x_i \otimes \frac{\partial B}{\partial x_i}.
\]

Since \(\delta_1\) is injective, the homomorphism \(H^0(Y_3, \Omega^2_{Y_3}) \otimes \kappa(T_M([F])) \to H^1(Y_3, \Omega^1_{Y_3})\) is also injective, hence the dimension of its image is equal to 16. \(\square\)

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