Super-zeta functions and regularized determinants associated to cofinite Fuchsian groups with finite-dimensional unitary representations

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Abstract

Let \( M \) be a finite volume, non-compact hyperbolic Riemann surface, possibly with elliptic fixed points, and let \( \chi \) denote a finite dimensional unitary representation of the fundamental group of \( M \). Let \( \Delta \) denote the hyperbolic Laplacian which acts on smooth sections of the flat bundle over \( M \) associated to \( \chi \). From the spectral theory of \( \Delta \), there are three distinct sequences of numbers: The first coming from the eigenvalues of \( L^2 \) eigenfunctions, the second coming from resonances associated to the continuous spectrum, and the third being the set of negative integers. Using these sequences of spectral data, we employ the super-zeta approach to regularization and introduce two super-zeta functions, \( Z_{-}(s, z) \) and \( Z_{+}(s, z) \) that encode the spectrum of \( \Delta \) in such a way that they can be used to define the regularized determinant of \( \Delta - z(1 - z)I \). The resulting formula for the regularized determinant of \( \Delta - z(1 - z)I \) in terms of the Selberg zeta function, see Theorem 5.3, encodes the symmetry \( z \leftrightarrow 1 - z \), which could not be seen in previous works, due to a different definition of the regularized determinant.

1 Introduction

In this article we will develop the super-zeta regularization approach to the spectral data associated to Laplacians which act on smooth sections of flat vector bundles on a finite volume Riemann surface \( M \). In brief, we will use the super-zeta methodology to define two functions, \( Z_{-}(s, z) \) and \( Z_{+}(s, z) \), carrying the information about the spectrum of the Laplacian in such a way that it makes natural to define regularized determinant of \( \Delta - z(1 - z)I \) by formula (1.1) below. We obtain expressions for the Selberg zeta function and the scattering determinant associated to \( M \) in terms of \( Z_{-}(s, z) \) and \( Z_{+}(s, z) \). In some sense, our main results complete by different means, the problem studied in [5] where the author employed a trace formula approach to zeta regularization.

If a given sequence \( \Lambda = \{\lambda_j\} \) is such that \( \lambda_j \) approaches one sufficiently fast, then it is an elementary exercise to define the product of elements of \( \Lambda \); no regularization is needed. However, regularization provides a means by which one can define a product and study its properties when \( \Lambda \) is unbounded. In its most naive interpretation, the regularized product of a sequence, when defined, allows one to write mathematical expressions of the form “\( \infty! = \sqrt{2\pi} \)”, which is often viewed as amusing when first encountered. However, upon further reflection, one views the regularized product of a sequence \( \Lambda = \{\lambda_j\} \) of numbers to be a special value of the zeta function \( \zeta_{\Lambda}(s) = \sum \lambda_j^{-s} \), namely the value \( \exp(-\zeta'_{\Lambda}(0)) \). In this form, one can view the definition of a regularized product as yielding an area of investigation which includes complex analysis,

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meaning the study of the function $\zeta_\Lambda(s)$ of a complex variable $s \in \mathbb{C}$, as well as a type of analytic number theory, as it pertains to special values of meromorphic functions. From this point of view, there exists in the literature various references which develop the elementary study of regularized products and go so far as to include some interesting examples; see, for example, [14].

A somewhat commonplace example for a sequence $\Lambda$ is the set $\{z - \lambda_j\}$ where $z$ is a complex variable and $\{\lambda_j\}$ is the sequence of eigenvalues of a self-adjoint operator, such as the Laplacian which acts on smooth functions on a compact hyperbolic Riemann surface. In this particular setting, it was shown in [21] that the zeta regularized product of $\Lambda$ is closely related to the Selberg zeta function. Further examples are discussed in the well-cited article [11]. As Hawking discusses on page 141 of [11], it is common to develop a means of regularization by starting with the trace of a heat kernel. Unfortunately, there are many instances when such heat kernels are not of trace class, such as when the hyperbolic Riemann surfaces has finite volume yet is not compact. In recognition of this problem the author of [6] writes the following on page 7:

Notice that the generalization to the case of a continuous spectrum is quite simple (the multi-series being just substituted by a multiple integral).

Respectfully, we disagree with this assertion that zeta regularization is simple in the presence of continuous spectrum. Indeed, in [5] the author studied in the problem of expressing the Selberg zeta function as a regularized product, following the approach of [21], which employed the Selberg trace formula, and the concluding result from [5] was not entirely successful. From different studies, various authors have succeeded in defining regularized traces of heat kernels in the setting on finite volume hyperbolic Riemann surfaces, see for example [15], [19] and [18], and from these results one is able to proceed further in developing zeta regularized products. However, in doing so, one does not see very clearly the underlying sequence $\Lambda$ since all analytic consideration has been swept into the study of regularizing the traces of operators.

In the present article we revisit the problem of defining and studying regularized product of the Laplace operator which acts on the space of smooth sections of flat vector bundles which lie over a finite volume hyperbolic Riemann surface. Specifically, let $\Gamma$ be a Fuchsian group of the first kind with $c$ cusps, and assume $c > 0$. Let $M = \Gamma \backslash \mathbb{H}$ be the finite volume, non-compact orbifold quotient space. Let $\chi$ be finite-dimensional unitary representation of $\Gamma$. The Laplacian $\Delta$ on $M$, besides the discrete spectrum, possesses the continuous spectrum with finite multiplicity, which is described through the resonances, meaning the poles of the scattering matrix. We recall that the Phillips-Sarnak philosophy [20] asserts that for a generic surface there might be no non-trivial $L^2$ eigenfunctions; therefore, any general definition of a determinant of the Laplacian must not solely use eigenvalues of $L^2$ eigenfunctions, as is developed in [16] for the special case when $\Gamma$ is arithmetic.

The main purpose of this paper is to undertake a different point of view for regularization which is based on the super-zeta regularization approach as developed by Voros in [28]. Specifically, we define the square of the (super-zeta regularized) determinant of $\Delta - z(1-z)I$ in such a way that it includes both discrete eigenvalues and resonances. Furthermore, our expression for $\det^2(\Delta - z(1-z)I)$ encodes the symmetry $z \leftrightarrow 1-z$, which seems necessary based on the notation but in fact is not true when using a regularized heat trace or Selberg trace formula approach; see [5].

We define two completed zeta functions $Z_+$ and $Z_-$. The set of zeros $N(Z_+)$ of $Z_+$ contains exactly the non-trivial zeros of the Selberg zeta function on $M$ stemming from the discrete eigenvalues of the Laplacian and the resonances, while the set of zeros of $Z_-$ is $1 - N(Z_+)$. We then define two super-zeta functions, $Z_+(s, z)$ and $Z_-(s, z)$ associated to $Z_+$ and $Z_-$, show that $Z_+(s, z)$ and $Z_-(s, z)$ each possess a meromorphic continuation to the whole $s$-plane, taking into account only certain admissible values of $z$. In addition, the continuations of $Z_+(s, z)$ and $Z_-(s, z)$ each are regular at $s = 0$, so then we can define

\[
\det^2(\Delta - z(1-z)I) = \exp \left( - \frac{d}{ds} \left( Z_+(s, z) + Z_-(s, z) \right) \right)_{s=0}.
\]
The above definition is further explained in §3. With this, we prove that
\[
\det^2(\Delta - z(1 - z)I) = \exp(Bz + C) \phi(z) \cdot \left( \frac{Z(z)}{G_1(z)(1 - z/2)^k} \right)^2,
\]
where \(Z(z)\) is the Selberg zeta function associated to \(M\) and \(\chi; \phi(z)\) is the scattering determinant, \(B\) and \(C\) are certain explicitly computable constants, and \(G_1(z)\) is a function given in terms of the Barnes double gamma function and the gamma functions. See §3 and Definition [A.3] for further details.

As a byproduct of our investigation we deduce the expression of the scattering determinant as a regularized determinant, see Corollary 5.5 below. Note also that the above expression yields the symmetric functional equation for the renormalized Selberg zeta function, see also [8].

One of our secondary goals is to allow for ramification points in our Riemann surfaces. Recently, Lee-Peng Teo applied the regularized determinant, in this case, to study the Ruelle zeta-function.

The paper is organized as follows. In Section 2 we define all notation and state necessary background material from the literature. It was our aim to make the article self-contained yet not overly lengthy. In Section 3 we defined and studied the completed zeta functions which formed the basis of our study. Since we chose to study finite volume hyperbolic Riemann surfaces with elliptic points, it was necessary to undertake additional considerations which occur, and those computations constitute a considerable portion of section 3. In section 4 we obtain the meromorphic continuation of the super-zeta functions \(Z_+(s, z)\) and \(Z_-(s, z)\) from which we prove the main results of the article, which are stated in Section 5.

2 Background Information

2.1 Basic notation

Let \(\Gamma \subseteq \text{PSL}(2, \mathbb{R})\) be a Fuchsian group of the first kind acting by fractional linear transformations on the upper half-plane \(\mathbb{H} := \{z \in \mathbb{C} \mid z = x + iy, y > 0\}\). Let \(M\) be the quotient space \(\Gamma \backslash \mathbb{H}\) and \(g\) the genus of \(M\). Denote by \(c\) the number of inequivalent cusps of \(M\) and by \(\{R\}_\Gamma\) the set of inequivalent elliptic classes of elements of \(\Gamma\). For a fixed elliptic representative \(R\), we denote by \(d_R\) the order of element \(R\) and by \(e\) the cardinality of the finite set \(\{R\}_\Gamma\) of inequivalent elliptic classes in \(\Gamma\).

Recall that the hyperbolic volume \(\text{vol}(M)\) of \(M\) is given by the Gauss-Bonnet formula
\[
\text{vol}(M) = 2\pi \left( 2g - 2 + c + \sum_{\{R\}_\Gamma} \left( 1 - \frac{1}{d_R} \right) \right) = 2\pi \left( 2g - 2 + c + e - \sum_{\{R\}_\Gamma} \left( \frac{1}{d_R} \right) \right) . \tag{2.1}
\]

Let \(V\) be an \(h\)-dimensional complex inner-product space and let \(\chi : \Gamma \to \text{GL}(V)\) be a finite-dimensional unitary representation of \(\Gamma\).

Let \(\{S_1, S_2, \ldots, S_c\}\) be parabolic representatives for the cusps of \(\Gamma\). For each \(j = 1 \ldots c\), set
\[V_j = \{v \in V \mid \chi(S_j)v = v\} .\]

Let \(k_j = \text{dim}(V_j)\), and define the degree of singularity, \(k = \sum_j k_j\). If \(k = 0\), we say that \(\chi\) is regular, see [8] Section 1.5 p.28.

For each \(S_j \in \{S_1, S_2, \ldots, S_c\}\), let \(\lambda_{j1}, \ldots, \lambda_{jh}\) be eigenvalues of \(\chi(S_j)\) counted with multiplicity. We can write
\[\lambda_{jp} = e^{2\pi i \beta_{jp}},\]
where \(\beta_{jp} = 0\) for \(1 \leq p \leq k_j\), and \(\beta_{jp} \in (0, 1)\) for \(k_j < p \leq h\), see [8] Section 1.5 p.30.
For \( j \in \{1, \ldots, c\} \) let
\[
\beta_j = \sum_{p = k_j + 1}^{h} \beta_{jp}.
\]

Finally we define
\[
a(\chi) = \left( 2^{h+c} \prod_{j=1}^{c} \prod_{p=k_j+1}^{h} \sin(\pi \beta_{jp}) \right)^{-1}
\tag{2.2}
\]
to be the expression associated to the character \( \chi \) and which appears in the right hand side of the second equation from the top of page 71 \[^{[8]}\] Section 2.4 p.71]. Note that in our paper, \( k = 0 \) since we do not consider higher weight forms.

Let \( \mathcal{H}(\Gamma, \chi) \) be the associated Hilbert space of square-integrable automorphic functions, and let \( \Delta \) be the (non-negative) self-adjoint extension of the Laplacian (see \[^{[23]}\] p. 15-16).

Given a meromorphic function \( f(s) \), we define the null set \( N(f) \) to be \( N(f) = \{ s \in \mathbb{C} | f(s) = 0 \} \) counted with multiplicity. Similarly, \( P(f) \) denotes the polar set, the set of points where \( f \) has a pole.

Our notation is from the well-known sources \[^{[12]}\], \[^{[13]}\] and \[^{[23]}\].

### 2.2 The Gamma function

Let \( \Gamma(s) \) denote the Gamma function. Its poles are all simple and located at each point of \( -\mathbb{N} \), where \( -\mathbb{N} = \{0, -1, -2, \ldots\} \). For \( |\arg s| \leq \pi - \delta \) and \( \delta > 0 \), the asymptotic expansion \[^{[2]}\] p. 20] of \( \log \Gamma(s) \) is given by
\[
\log \Gamma(s) = \frac{1}{2} \log 2\pi + \left( s - \frac{1}{2} \right) \log s - s + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n-1)!} \frac{1}{s^{2n-1}} + g_m(s). \tag{2.3}
\]

Here \( B_n \) are the Bernoulli numbers. Also, for each \( m \), \( g_m(s) \) is a holomorphic function in the right half plane \( \text{Re}(s) > 0 \) such that \( g_m^{(j)}(s) = O(s^{-2n+1-j}) \) as \( \text{Re}(s) \to \infty \) for all integers \( j \geq 0 \), and where the implied constant depends on \( j \) and \( m \).

### 2.3 The double Gamma function

The Barnes double Gamma function is an entire order two function defined by
\[
G(s+1) = (2\pi)^{s/2} \exp \left[ -\frac{1}{2} \left[ (1 + \gamma) s^2 + s \right] \right] \prod_{n=1}^{\infty} \left( 1 + \frac{s}{n} \right)^n \exp \left[ -s + \frac{s^2}{2n} \right], \tag{2.4}
\]
where \( \gamma \) is the Euler constant. Therefore, \( G(s+1) \) has a zero of multiplicity \( n \), at each point \( -n \in \{ -1, -2, \ldots \} \). For \( \text{Re}(s) > 0 \) and as \( s \to \infty \), the asymptotic expansion of \( \log G(s+1) \) is given in \[^{[7]}\] or \[^{[1]}\] Lemma 5.1] by
\[
\log G(s+1) = \frac{s^2}{2} \left( \log s - \frac{3}{2} \right) - \frac{\log s}{12} + \frac{s}{2} \log(2\pi) + \zeta'(1) - \sum_{k=1}^{n} \frac{B_{2k+2}}{k(k+1)s^{2k}} + h_{n+1}(s). \tag{2.5}
\]

Here, \( \zeta(s) \) is the Riemann zeta-function and
\[
h_{n+1}(s) = \frac{(-1)^{n+1}}{s^{2n+2}} \int_{0}^{\infty} \frac{t}{\exp(2\pi t) - 1} \int_{0}^{t^2} \frac{y^{n+1}}{y + s^2} dy \, dt.
\]

By a close inspection of the proof of \[^{[1]}\] Lemma 5.1], it follows that \( h_{n+1}(s) \) is a holomorphic function in the right half plane \( \text{Re}(s) > 0 \) which satisfies the asymptotic relation \( h_{n+1}^{(j)}(s) = O(s^{-2n-2-j}) \) as \( \text{Re}(s) \to \infty \) for all integers \( j \geq 0 \), and where the implied constant depends upon \( j \) and \( n \).
2.4 Automorphic scattering determinant

Let \( \phi(s) \) denote the determinant of the automorphic scattering matrix \( \Phi(s) \) \([23]\) § 2.3 and p. 59]. Note that in \([23]\) they denote \( \phi(s) \) by \( \Delta(s) \).

The function \( \phi(s) \) is meromorphic of order at most two. Furthermore, \( \phi(s) \) is holomorphic for \( \text{Re}(s) > \frac{1}{2} \), except for a finite number of poles, and it satisfies the functional equation

\[
\phi(s)\phi(1 - s) = 1. \tag{2.6}
\]

**Theorem 2.1.** (\([23] \ Thm. 3.5 p. 59\)) For \( \text{Re}(s) > 1 \) we have that

\[
\phi(s) = \left( \frac{\sqrt{\pi} \Gamma(s - \frac{1}{2})}{\Gamma(s)} \right) \sum_{n=1}^{\infty} \frac{d(n)}{g_n^2 s^n} \tag{2.7}
\]

where \( 0 < g_1 < g_2 < \ldots \) and \( d(n) \in \mathbb{C} \) with \( d(1) \neq 0 \).

We will rewrite (2.7) in a slightly different form. Let \( c_1 = -2\log g_1 \neq 0 \), \( c_2 = \log d(1) \), and let \( u_n = g_n / g_1 > 1 \). Then for \( \text{Re}(s) > 1 \) we can write \( \phi(s) = L(s)H(s) \) where

\[
L(s) = \left( \frac{\sqrt{\pi} \Gamma(s - \frac{1}{2})}{\Gamma(s)} \right) e^{c_1 s + c_2} \tag{2.8}
\]

and

\[
H(s) = 1 + \sum_{n=2}^{\infty} \frac{a(n)}{n^{2s}}, \tag{2.9}
\]

where \( a(n) \in \mathbb{C} \). The series (2.9) converges absolutely for \( \text{Re}(s) > 1 \). From the generalized Dirichlet series representation (2.9) of \( H(s) \), it follows that

\[
\frac{d^k}{ds^k} \log H(s) = O(\beta_k^{-\text{Re}(s)}) \quad \text{when} \quad \text{Re}(s) \to +\infty, \tag{2.10}
\]

for some \( \beta_k > 1 \) where the implied constant depends on \( k \in \mathbb{N} \).

For \( 0 \leq \sigma_i \leq 1 \), define \( q(\sigma_i) = \text{[The multiplicity of the pole of } \phi(s) \text{ at } s = \sigma_i] \). The divisor of \( \phi(s) \) consists of the following sets of points \([23] \ pp. 59–60]:

1. Finitely many real zeros of the form \( 1 - \sigma_i \in [0, 1/2) \) for \( i = 1 \ldots T \), each with multiplicity \( q(\sigma_i) \);
2. Finitely many real zeros of the form \( \rho_i > 1/2, i = 1 \ldots N \), where \( N \) is defined to be the sum of the multiplicities;
3. Finitely many real poles of the form \( 1 - \rho_i < 1/2, i = 1 \ldots N \);
4. Finitely many poles \( \sigma_i \in (1/2, 1] \). Each pole has multiplicity \( q(\sigma_i) \).
5. Poles of the form \( 1 - \rho \) and \( 1 - \overline{\rho} \) with \( \text{Re}(\rho) > 1/2 \) and \( \text{Im}(\rho) > 0 \);
6. Zeros of the form \( \rho \) and \( \overline{\rho} \) with \( \text{Re}(\rho) > 1/2 \) and \( \text{Im}(\rho) > 0 \).

Let \( \lambda_i \) be an eigenvalue for the positive, self-adjoint extension \( \Delta \) of the hyperbolic Laplacian. Denote by \( A(\lambda_i) \) the \( \Delta \)-eigenspace corresponding to the eigenvalue \( \lambda_i \). Set \( A_1(\lambda_i) \) to be the subspace of \( A(\lambda_i) \) that is spanned by the incomplete theta series. For each pole \( \sigma_i \in (1/2, 1], i = 1, \ldots, T \) the space \( A_1(\sigma_i(1 - \sigma_i)) \) is non-trivial. In fact from \([12] \ Eq. 3.33 \ p.299\) we have that

\[
q(\sigma_i) = \text{[The multiplicity of the pole of } \phi(s) \text{ at } s = \sigma_i] \leq \dim A_1(\sigma_i(1 - \sigma_i)) \leq k.
\]

The eigenvalue \( \lambda_i = \sigma_i(1 - \sigma_i) \) is called a residual eigenvalue.

\[\text{\footnotesize{1The scattering determinant } }\phi(s)\text{\footnotesize{ is actually real valued on } }\mathbb{R}\text{\footnotesize{ and it follows that } }d(n) \in \mathbb{R}.\]
2.5 Selberg zeta-function

The Selberg zeta function associated to the quotient space $M = \Gamma \backslash \mathbb{H}$, and unitary representation $\chi : \Gamma \to \text{GL}(V)$ is defined for $\text{Re}(s) > 1$ by the absolutely convergent Euler product

$$Z(s) = \prod_{\{P_0\} \in P(\Gamma)} \prod_{n=0}^\infty \det \left( 1 - \chi(P_0) N(P_0)^{-s+n} \right),$$

where $P(\Gamma)$ denotes the set of all primitive hyperbolic conjugacy classes in $\Gamma$, and $N(P_0)$ denotes the norm of $P_0 \in \Gamma$. From the product representation given above, we have for $\text{Re}(s) > 1$ that

$$\log Z(s) = \sum_{\{P_0\} \in P(\Gamma)} \sum_{n=0}^\infty \text{tr} \left( - \sum_{l=1}^\infty \frac{\chi(P_0)^l}{l} N(P_0)^{-s+l} \right) = \sum_{P \in H(\Gamma)} \text{tr}(\chi(P)) \frac{\Lambda(P)}{N(P)^s \log N(P)},$$

where $H(\Gamma)$ denotes the set of all hyperbolic conjugacy classes in $\Gamma$, and $\Lambda(P) = \frac{\log N(P_0)}{1 - N(P_0)^{-s}}$, for the primitive element $P_0$ in the conjugacy class containing $P$ (see [23, p. 83]).

Let $P_{00}$ be the primitive hyperbolic conjugacy class in all of $P(\Gamma)$ with the smallest norm. Setting $\alpha = N(P_{00})^{1/2}$, for $\text{Re}(s) > 2$ and $k \in \mathbb{N}$ we have the following asymptotic formula

$$\frac{d^k}{ds^k} \log Z(s) = O(\alpha^{-\text{Re}(s)}) \quad \text{when} \quad \text{Re}(s) \to +\infty, \quad (2.11)$$

with an implied constant which depends on $k \in \mathbb{N}$. If $\lambda_j$ is an eigenvalue in the discrete spectrum of $\Delta$, let $m(\lambda_j)$ denote its multiplicity. We now state the divisor of the $Z(s)$ (see [23] p. 49, [12] p. 499):

1. Zeros at the points $s_j$ on the line $\text{Re}(s) = \frac{1}{2}$ symmetric relative to the real axis and in $(1/2, 1]$, where each zero $s_j$ has multiplicity $m(s_j) = m(\lambda_j)$ where $s_j (1 - s_j) = \lambda_j$ is an eigenvalue in the discrete spectrum of $\Delta$;

2. Zeros at the points $s_j \in [0, 1/2)$ where $s_j (1 - s_j) = \lambda_j \in [0, 1/4)$ is an eigenvalue in the discrete spectrum of $\Delta$ and the multiplicity $\tilde{m}(s_j)$ is given by $\tilde{m}(s_j) = m(\lambda_j) - q(1 - s_j) \geq 0$; we denote by $K$ the number of eigenvalues $\lambda_j \in [0, 1/4)$ and put $m_j = m(\lambda_j), j = 1, K$ ($q(\cdot)$ was defined in [2.4]).

Note that, in the case when $\lambda_j$ is not the residual eigenvalue, we take $q(1 - s_j) = 0$, i.e. $\tilde{m}(s_j) = m(\lambda_j)$.

3. The point $s = \frac{1}{2}$ can be a zero or a pole, and the order of the point as a divisor is

$$a = 2d_{1/4} - \frac{1}{2} (k - \text{tr} \Phi(\frac{1}{2}))$$

where $d_{1/4}$ denotes the multiplicity of the possible eigenvalue $\lambda = \frac{1}{4}$ of $\Delta$;

4. Poles at $s = -n - \frac{1}{2}$, where $n = 1, 2, \ldots$, each with multiplicity $k$;

5. Finitely many real zeros $1 - \rho_i < 1/2$, where $i = 1 \ldots N$;

6. Zeros at each $s = 1 - \rho, 1 - \overline{\rho}$ where $\rho$ is a zero of $\phi(s)$ with $\text{Re}(\rho) > \frac{1}{2}$ and $\text{Im}(\rho) > 0$;

7. Zeros at points $s = -n \in -\mathbb{N} = \{0, -1, -2, \ldots\}$, with multiplicities

$$m_n = h \frac{\text{vol}(M)}{2\pi} (2n + 1) - \sum_{\{R\} \in \mathbb{R}} \sum_{k=1}^{d_R-1} \frac{\text{tr}(\chi(R))}{d_R} \frac{\sin \left( \frac{k\pi(2n+1)}{d_R} \right)}{\sin \left( \frac{k\pi}{d_R} \right)} . \quad (2.12)$$

The last set of zeros are called trivial zeros. For the purposes of our paper it will be crucial that we give another representation of $m_n$ in such a way that it is clear that they are non-negative integers. Ultimately, we will construct a double gamma based function whose divisor is exactly $m_n$. Finally, note that we will see that, actually $m_0 = 0$. 
3 Construction of the complete zeta functions

3.1 The trivial zeros stemming from the identity motion and elliptic elements

Recall \([2,12]\). We construct an entire function on \( \mathbb{C} \) with zeros at the points \(-n \in \mathbb{N}\) with multiplicities \( m_n \), following ideas of Fisher \([8]\).

**Lemma 3.1.** Suppose that \( \omega \) is a \( d \)-th root of unity, and define the integer \( q \) by \( \omega = \exp \left( \frac{2\pi i q}{d} \right) \), with \( 0 \leq q \leq d - 1 \). For \( n \in \mathbb{N} \), let \( q(n) \in \{0, \ldots, d-1\} \) be the residue of \( n + q \) modulo \( d \) and let \( \tilde{q}(n) \in \{0, \ldots, d-1\} \) be the residue of \( n - q \) modulo \( d \). Then we have

\[
\sum_{k=1}^{d-1} \omega^k \frac{\sin \left( \frac{k\pi(2n+1)}{d} \right)}{\sin \left( \frac{k\pi}{d} \right)} = d - 1 - (q(n) + \tilde{q}(n)).
\]

**Proof.** We follow \([8]\) pages 66–67. \( \square \)

\[
\sum_{k=1}^{d-1} \omega^k \frac{ie^{-i \frac{\pi k}{d} (2n+1)}}{2 \sin \left( \frac{k\pi}{d} \right)} = \sum_{k=1}^{d-1} \frac{e^{-i \frac{\pi k}{d} (q+n)}}{1 - e^{i \frac{\pi k}{d}}} = \lim_{t \to 1} \sum_{m=0}^{\infty} \sum_{r=0}^{d-1} t^{r+md} \sum_{k=1}^{d-1} e^{-i \frac{\pi k}{d} (q+n-r-md)}
\]

\[
= \lim_{t \to 1} \frac{1}{1 - t^d} \left( t^{q(n)d} - \sum_{r=0}^{d-1} t^r \right) = \frac{1}{2} (d-1) - q(n), \quad (3.1)
\]

where the last equality follows from

\[
\sum_{k=1}^{d-1} e^{-i \frac{\pi k}{d} (q+n-r-md)} = \begin{cases} 
  d - 1, & \text{if } r = n + q \mod d \\
  -1, & \text{else}.
\end{cases}
\]

A similar computation yields

\[
\sum_{k=1}^{d-1} \omega^k \frac{ie^{i \frac{\pi k}{d} (2n+1)}}{2 \sin \left( \frac{k\pi}{d} \right)} = - \frac{1}{2} (d-1) + \tilde{q}(n). \quad (3.2)
\]

The lemma now follows by subtracting (3.2) from (3.1). \( \square \)

Assume \( R \) is an order \( d_R \) elliptic element of \( \Gamma \). Recall that \( \chi(R) \) is unitary, acting on the \( h \)-dimensional space \( V \). We will need to simplify expressions of the form \( \text{tr}(\chi^k(R)) \). Since \( \chi(R) \) is can be diagonalized, it follows that

\[
\text{tr}(\chi^k(R)) = \sum_{j=1}^{h} \omega(R)^k_j, \quad (3.3)
\]

where \( \omega(R)_j \), for \( j = 1 \ldots h \) are the eigenvalues (and \( d_R \)-th roots of unity) of \( \chi(R) \).

For each class \( \{R\} \) and \( j = 1 \ldots h \), define integer \( q(R)_j, 0 \leq q(R)_j \leq d_R - 1 \) by

\[
\omega(R)_j = \exp \left( \frac{2\pi i q(R)_j}{d_R} \right). \quad (3.4)
\]

**Definition 3.2.** For an elliptic representative \( R \) in \( \{R\}_\Gamma \), and \( m \in \mathbb{N} \), define \( q_j(R, m), \tilde{q}_j(R, m) \), and \( k_j(R, m), \tilde{k}_j(R, m) \in \mathbb{Z} \) by

\[
q_j(R, m) := m + q(R)_j + d_R k_j(R, m), \quad \tilde{q}_j(R, m) \in \{0, \ldots, d_R - 1\},
\]

\(^2\)Note that our notation is very different from his. Note that his \( k \) represents the weight of his forms, which we take as zero in our paper.
\[ \tau_j(R, m) := m - q(R)_j + d_R \bar{k}_j(R, m) \in \{0, \ldots, d_R - 1\}, \]

define
\[ \alpha(R, m) := \sum_{j=1}^{h} (\tau_j(R, m) + q_j(R, m)), \quad (3.5) \]
\[ k(R, m, j) := k_j(R, m) + \bar{k}_j(R, m), \]
\[ \beta(R, m) := \sum_{j=1}^{h} k(R, m, j). \]

**Lemma 3.3.** With the notation above,
\[ \alpha(R, m) = 2mh + d_R \sum_{j=1}^{h} k(R, m, j) = 2mh + \beta(R, m)d_R, \quad (3.6) \]
\[ k(R, m, j) = \begin{cases} 1, & \text{if } m < q(R)_j \text{ and } m + q(R)_j < d_R \\ -1, & \text{if } m \geq q(R)_j \text{ and } m + q(R)_j \geq d_R \\ 0, & \text{otherwise}. \end{cases} \quad (3.7) \]

**Proof.** Equation (3.6) follows immediately. For Equation (3.7) note that \( k_j(R, m)d_R, \bar{k}_j(R, m)d_R \) are the multiples of \( d_R \) that translate \( m + q(R)_j, m - q(R)_j \) back to the range \( \{0, \ldots, d_R - 1\} \). Noting that \( 0 \leq q(R)_j \leq d_R - 1 \), the derivation of Equation (3.7) is straightforward. \( \Box \)

**Remark 3.4.** In case when \( \chi \) is trivial, \( \beta(R, m) = 0 \) for all \( m \) and all expressions above are significantly simplified.

We give an equivalent expression for \( m_n \). Applying Lemma 3.1,
\[ \sum_{k=1}^{d_R - 1} \text{tr}(\chi^k(R)) \frac{\sin \left( \frac{k\pi(2n+1)}{d_R} \right)}{\sin \left( \frac{k\pi}{d_R} \right)} = h(d_R - 1) - \sum_{j=1}^{h} (\tau_j(R, n) + q_j(R, n)), \]
and also using (2.1), we rewrite (2.12) as
\[ m_n = h \left( 2g - 2 + c + e - \sum_{(R)_r} \frac{1}{d_R} \right) (2n+1) - h \left( 2 \right) \left( 1 \right) (2n+1) - \sum_{(R)_r} \frac{1}{d_R} \left( h(d_R - 1) - \sum_{j=1}^{h} (\tau_j(R, n) + q_j(R, n)) \right) \]
\[ = h \left( 2g - 2 + c + e \right) (2n+1) - \sum_{(R)_r} \frac{1}{d_R} \left( h(2n + d_R) - \sum_{j=1}^{h} (\tau_j(R, n) + q_j(R, n)) \right) \]
\[ = h \left( 2g - 2 + c + e \right) (2n+1) - \sum_{j=1}^{h} \sum_{(R)_r} \frac{1}{d_R} (2n + d_R - (\tau_j(R, n) + q_j(R, n))) \quad (3.8) \]

Consider the function
\[ f_R(s) := \left( \frac{(\Gamma(s))^{d_R}}{(G(s + 1))^2} \right) \prod_{m=0}^{d_R-1} \Gamma \left( s + m \frac{d_R}{d_R} \right)^{-(\tau_j(R, n) + q_j(R, n))} \]

**Lemma 3.5.** The principal branch of \( f_R(s)^{1/d_R} \) is an meromorphic function on all of \( \mathbb{C} \) with zeros (or poles) of order \( -\frac{1}{d_R} (2n + d_R - (\tau_j(R, n) + q_j(R, n))) \naf s = -n, for n \in \mathbb{N}. \)
Proof. If follows from the definitions that the order of \( f_R(s) \) at \( s = -n \) is
\[
-(2n + d_R - (\tilde{q}_j(R, n) + q_j(R, n))).
\]
It follows from the definitions of \( \tilde{q}_j(R, n), q_j(R, n) \) that
\[
d_R | (2n - (\tilde{q}_j(R, n) + q_j(R, n))).
\]
Since \( f_R(s) \) is meromorphic on \( \mathbb{C} \) it can be written as a quotient of two entire functions \( g_R(s)/h_R(s) \) whose zero sets are disjoint. Hence, the order of each zero of both \( g_R, h_R \) must be divisible by \( d_R \). Finally, using the Weierstrass factorization theorem one could construct an entire \( d_R \)-th root of \( g_R, h_R \), and \( f_R \).

From the right hand side of (3.8) we immediately obtain

Lemma 3.6. The following meromorphic function on \( \mathbb{C} \)
\[
G_0(s) := \left( \frac{(G(s+1))^2}{\Gamma(s)} \right)^{\frac{h}{2}} \prod_{j=1}^{h} \prod_{\{R\}_F} \left( \frac{(\Gamma(s))^{d_R}}{(G(s+1))^{d_R}} \prod_{m=0}^{d_R-1} \Gamma \left( \frac{s + m}{d_R} - (\tilde{q}_j(R,m) + q_j(R,m)) \right) \right)^{\frac{1}{d_R}}
\]
has zeros or order \( m_n \) at \( s = -n \), for \( n \in \mathbb{N} \).

For representative \( R \) in \( \{R\}_F \) and \( m \in \mathbb{N} \), Using the Gauss-Bonnet formula (2.1), we can rewrite \( G_0(s) \) as
\[
G_0(s) = \left( \frac{(G(s+1))^2}{\Gamma(s)} \right)^{\frac{h}{2}} \prod_{\{R\}_F} \left( \frac{(\Gamma(s))^{d_R}}{(G(s+1))^{d_R}} \prod_{m=0}^{d_R-1} \Gamma \left( \frac{s + m}{d_R} \right) \right)^{-\alpha(R,m)/d_R}.
\]
Note that the fractional powers of \( G(s+1) \) and \( \Gamma(s) \) are defined via the principal branch of \( \log(z) \).

Finally, set
\[
G_1(s) := \left( \frac{2\pi}{\Gamma(s)} \right)^{\frac{h}{2}} \prod_{\{R\}_F} \left( \frac{(\Gamma(s))^{d_R}}{(\Gamma(s))^{d_R}} \prod_{m=0}^{d_R-1} \Gamma \left( \frac{s + m}{d_R} \right)^{-\alpha(R,m)/d_R} \right)
\]
Note that \( G_1 \) is an meromorphic function on \( \mathbb{C} \) of order two with zeros (or poles) at points \( -n \in -\mathbb{N} \) and corresponding multiplicities \( m_n \). Also note that we added an exponential normalization factor that will later simplify some computations.

3.2 Asymptotic expansion of \( G_1 \)

In order to derive our main results, we need the asymptotic expansion of function \( \log G_1(s) \), as \( s \to \infty \) which is given in the following lemma.

Lemma 3.7. As \( s \to \infty \) we have the following asymptotic expansion of \( \log G_1(s) \):
\[
\log G_1(s) = h \frac{\text{vol}(M)}{2\pi} s \left( \log(s) - \frac{3}{2} \right) + a_1 s \log(s) + b_0 \log(s) + b_0 + O(s^{-1}),
\]
where
\[
a_1 = -h \frac{\text{vol}(M)}{2\pi} \sum_{\{R\}_F} \sum_{m=0}^{d_R-1} \frac{\beta(R,m)}{d_R},
\]
and contants \( a_0 \) and \( b_0 \) will be explicitly given in the proof, equations (3.18) and (3.19), respectively.
Proof. Using (3.6) we obtain

\[
\log \left( \prod_{\{R\}_r} \prod_{m=0}^{d_R-1} \Gamma \left( \frac{s + m}{d_R} \right)^{-\alpha(R,m)/d_R} \right) = - \sum_{\{R\}_r} \sum_{m=0}^{d_R-1} \frac{2mH \log \Gamma \left( \frac{s}{d_R} + \frac{m}{d_R} \right)}{d_R} \log \Gamma \left( \frac{s}{d_R} + \frac{m}{d_R} \right) + \sum_{\{R\}_r} \sum_{m=0}^{d_R-1} \beta(R,m) \log \Gamma \left( \frac{s}{d_R} + \frac{m}{d_R} \right)
\]

(3.13)

For any real numbers \(d > 1\) and \(a \geq 0\), equation (2.23) yields the following expansion as \(s \to \infty\)

\[
\log \Gamma \left( \frac{s}{d} + \frac{a}{d} \right) = \frac{1}{d} s \log s - 1 - \frac{\log d}{d} s + \left( \frac{a}{d} - \frac{1}{2} \right) \log s + \frac{1}{2} \log(2\pi d) - \frac{a}{d} \log d + O(s^{-1}).
\]

Therefore

\[
\sum_{\{R\}_r} \sum_{m=0}^{d_R-1} \frac{2mH \log \Gamma \left( \frac{s}{d_R} + \frac{m}{d_R} \right)}{d_R} = \sum_{\{R\}_r} h \left( 1 - \frac{1}{d_R} \right) [s \log s - 1 - s \log d_R] + (3.14)
\]

\[
+ \sum_{\{R\}_r} h \left( \frac{(d_R - 1)(d_R - 2)}{6d_R} \right) \log s + \sum_{\{R\}_r} h(d_R - 1) \left( \frac{1}{2} \log(2\pi d_R) - \frac{2d_R - 1}{3d_R} \log d_R \right) + O(s^{-1})
\]

and

\[
\sum_{\{R\}_r} \sum_{m=0}^{d_R-1} \beta(R,m) \log \Gamma \left( \frac{s}{d_R} + \frac{m}{d_R} \right) = \sum_{\{R\}_r} \sum_{m=0}^{d_R-1} \beta(R,m) \frac{s \log s - 1 - s \log d_R}{d_R} + \sum_{\{R\}_r} \sum_{m=0}^{d_R-1} \beta(R,m) \log d_R
\]

\[
+ \sum_{\{R\}_r} \sum_{m=0}^{d_R-1} \beta(R,m) \left( \frac{m}{d_R} - \frac{1}{2} \right) \log s + \sum_{\{R\}_r} \sum_{m=0}^{d_R-1} \beta(R,m) \left( \frac{1}{2} \log(2\pi d_R) - \frac{m}{d_R} \log d_R \right) + O(s^{-1})
\]

(3.15)

Next, using (2.5) and (2.3) we obtain

\[
\log \left( \frac{(2\pi)^{-s}(G(s + 1))^2}{\Gamma(s)} \right)^{\frac{\text{vol}(M)}{2\pi}} = \frac{\text{vol}(M)}{2\pi} \cdot \left( s^2 \left( \log(s) - \frac{3}{2} \right) - s \log(s) - 1 \right) + \frac{1}{3} \log(s) + 2\zeta'(-1) - \frac{1}{2} \log(2\pi) + O(s^{-1})
\]

(3.16)

and

\[
\log \left( \prod_{\{R\}_r} d_R^{h(1-1/d_R)s} \left( \Gamma(s) \right)^{\frac{h(1-1/d_R)}{2\pi}} \right)
\]

\[
= \sum_{\{R\}_r} \left[ h(1 - 1/d_R) \left( \frac{1}{2} \log 2\pi + \frac{s}{2} \log(s) - s \right) - h(1 - 1/d_R) s \log(d_R) \right] + O(s^{-1}).
\]

(3.17)

Finally, combining (3.13)–(3.17) with (3.9) we arrive at the expansion

\[
\log G_1(s) = \frac{\text{vol}(M)}{2\pi} \cdot \left( s^2 \left( \log(s) - \frac{3}{2} \right) - s \log(s) - 1 \right) + \frac{1}{3} \log(s) + 2\zeta'(-1) - \frac{1}{2} \log(2\pi)
\]

\[
- \sum_{\{R\}_r} \sum_{m=0}^{d_R-1} \frac{\beta(R,m)}{d_R} s \log s - 1 - s \sum_{\{R\}_r} \sum_{m=0}^{d_R-1} \frac{\beta(R,m)}{d_R} \log d_R.
\]
\[ + \left( h \sum_{\{R\}} \frac{d_R-1}{d_R} \left( \frac{1}{2} - \frac{d_R-2}{6} \right) - \sum_{\{R\}} \sum_{m=0}^{d_R-1} \frac{\beta(R, m)}{d_R} \left( \frac{m}{d_R} - \frac{1}{2} \right) \right) \log s \]
\[ + h \sum_{\{R\}} (d_R - 1) \left( \frac{1}{2d_R} \log(2\pi) - \frac{1}{2} \log(2\pi d_R) + \frac{2d_R - 1}{3d_R} \log d_R \right) \]
\[ - \sum_{\{R\}} \sum_{m=0}^{d_R-1} \beta(R, m) \left( \frac{m}{d_R} \log d_R \right) + O(s^{-1}), \]

hence

\[ \log G_1(s) = h \frac{\text{vol}(M)}{2\pi} s^2 (\log(s) - \frac{3}{2}) + \tilde{a}_1 s (\log(s) - 1) + b_1 s + \tilde{a}_0 \log(s) + b_0 + O(s^{-1}), \]

and it is left to give expressions for \( \tilde{a}_0 \) and \( b_0 \):

\[ \tilde{a}_0 = h \sum_{\{R\}} \frac{d_R-1}{d_R} \left( \frac{1}{2} - \frac{d_R-2}{6} \right) - \sum_{\{R\}} \sum_{m=0}^{d_R-1} \frac{\beta(R, m)}{d_R} \left( \frac{m}{d_R} - \frac{1}{2} \right) + h \frac{\text{vol}(M)}{6\pi}, \] (3.18)

\[ b_0 = h \sum_{\{R\}} (d_R - 1) \left( - \frac{1}{2d_R} \log(2\pi) - \frac{1}{2} \log(2\pi d_R) + \frac{2d_R - 1}{3d_R} \log d_R \right) \]
\[ + h \frac{\text{vol}(M)}{2\pi} \left( 2\zeta'(-1) - \frac{1}{2} \log(2\pi) \right) - \sum_{\{R\}} \sum_{m=0}^{d_R-1} \beta(R, m) \left( \frac{1}{2} \log(2\pi d_R) - \frac{m}{d_R} \log d_R \right). \] (3.19)

3.3 Complete zeta functions

Definition 3.8. We define completed zeta functions \( Z_+ \) and \( Z_- \) as

\[ Z_+(s) = \frac{Z(s)}{G_1(s)(\Gamma(s - 1/2))^k}, \]
\[ Z_-(s) = Z_+(s)\phi(s), \]

where \( G_1(s) \) is defined in (3.9) and \( \phi(s) \) is the scattering determinant.

Note that we have canceled out the trivial zeros and poles of the Selberg zeta-function \( Z(s) \). Hence the zero set \( N(Z_+) \) of \( Z_+ \) consists of the following points:

1. At \( s = \frac{1}{2} \) with multiplicity \( a \) where
   \[ a = 2d_{1/4} + k - \frac{1}{2} \left( k - \text{tr} \Phi\left( \frac{1}{2} \right) \right) = 2d_{1/4} + \frac{1}{2} \left( k + \text{tr} \Phi\left( \frac{1}{2} \right) \right) \geq 0; \]
2. At the points \( s_j \in [0, 1/2] \) where \( s_j(1 - s_j) = \lambda_j \) is an eigenvalue in the discrete spectrum of \( \Delta \) each with multiplicity \( m(\lambda_j) - q(1 - s_j) \geq 0; \)
3. At the points \( s_j \) on the line \( \text{Re}(s) = \frac{1}{2} \) symmetric relative to the real axis and in \( [1/2, 1] \) where each zero \( s_j \) has multiplicity \( m(s_j) = m(\lambda_j) \) where \( s_j(1 - s_j) = \lambda_j \) is an eigenvalue in the discrete spectrum of \( \Delta; \)
4. At each point \( s = 1 - \rho, 1 - \overline{\rho} \) where \( \rho \) is a zero of \( \phi(s) \) with \( \text{Re}(\rho) > \frac{1}{2}, \) and \( \text{Im}(\rho) > 0. \)

From the definition of \( Z_- \), it immediately follows that \( N(Z_-) = 1 - N(Z_+) \). In other words, \( s \) is a zero of \( Z_+ \) if and only if \( 1 - s \) is a zero, necessarily with the same multiplicity, of \( Z_- \).
4 Superzeta functions associated to complete zeta functions \( Z_+ \) and \( Z_- \)

In this section we define superzeta functions associated to completed zeta functions \( Z_+ \) and \( Z_- \) and show that they possess a meromorphic continuation to the whole complex plane, regular at zero.

4.1 Regularized products using superzeta functions

Let \( \mathbb{R}^- = (-\infty, 0] \) be the non-positive real numbers. Let \( \{y_k\}_{k \in \mathbb{N}} \) be the sequence of zeros of an entire function \( f \) of order at most two, repeated with their multiplicities. Let

\[
X_f = \{z \in \mathbb{C} \mid (z - y_k) \notin \mathbb{R}^- \text{ for all } y_k\}.
\]

For \( z \in X_f \), and \( s \in \mathbb{C} \) consider the series

\[
\zeta_f(s, z) = \sum_{k=1}^{\infty} (z - y_k)^{-s},
\]

where the complex exponent is defined using the principal branch of the logarithm with \( \arg z \in (-\pi, \pi) \) in the cut plane \( \mathbb{C} \setminus \mathbb{R}^- \). Since \( f \) is of order at most two, the series \( \zeta_f(s, z) \) converges absolutely for \( \text{Re}(s) > 2 \). Following [25], the series \( \zeta_f(s, z) \) is called the superzeta function associated to the zeros of \( f \), or the simply the superzeta function of \( f \).

If \( \zeta_f(s, z) \) has a meromorphic continuation which is regular at \( s = 0 \), we define the superzeta regularized product associated to \( f \) as

\[
D_f(z) = \exp \left(-\frac{d}{ds} \left. \zeta_f(s, z) \right|_{s=0} \right).
\]

Hadamard’s product formula allows us to write

\[
f(z) = \Delta_f(z) = e^{g(z)} z^r \prod_{k=1}^{\infty} \left( 1 - \frac{z}{y_k} \right) \exp \left[ \frac{z}{y_k} + \frac{z^2}{2y_k^2} \right],
\]

where \( g(z) \) is a polynomial of degree 2 or less, \( r \geq 0 \) is the order of eventual zero of \( f \) at \( z = 0 \), and the other zeros \( y_k \) are listed with multiplicity.

The following proposition, originally due to Voros ([25], [27], [28]) is proven in [9, Prop. 4.1] (see also [10] for a related, more general result).

**Proposition 4.1.** Let \( f \) be an entire function of order two, and for \( k \in \mathbb{N} \), let \( y_k \) be the sequence of zeros of \( f \). Let \( \Delta_f(z) \) denote the Hadamard product representation of \( f \). Assume that for \( n > 2 \) we have the following asymptotic expansion:

\[
\log \Delta_f(z) = \tilde{a}_2 z^2 (\log z - \frac{3}{2}) + b_2 z^2 + \tilde{a}_1 z (\log z - 1) + b_1 z + \tilde{a}_0 \log z + b_0 + \sum_{k=1}^{n-1} a_k z^{\mu_k} + h_n(z),
\]

where \( 1 > \mu_1 > \ldots > \mu_n \to -\infty, \mu_k \neq 0 \), and \( h_n(z) \) is a sequence of holomorphic functions in the sector \( |\arg z| < \theta < \pi, \quad (\theta > 0) \) such that \( \left. h_n^{(j)}(z) \right|_{z=0} = O(|z|^{|\mu_n|-j}) \), as \( |z| \to \infty \) in the above sector, for all integers \( j \geq 0 \).

Then, for all \( z \in X_f \), the superzeta function \( \zeta_f(s, z) \) has a meromorphic continuation to the half-plane \( \text{Re}(s) < 2 \) which is regular at \( s = 0 \). Furthermore, the superzeta regularized product \( D_f(z) \) associated to \( f(s) \) is related to \( \Delta_f(z) \) through the formula

\[
\exp \left(-\frac{d}{ds} \left. \zeta_f(s, z) \right|_{s=0} \right) = D_f(z) = e^{-(b_2 z^2 + b_1 z + b_0)} \Delta_f(z).
\]
4.2 Meromorphic continuation of $Z_+$ and $Z_-$

Let $X_\pm = X_{Z_\pm}$, and for $z \in X_\pm$, denote by $\mathcal{Z}_{\pm}(s, z) := \mathcal{Z}_{Z_{\pm}}(s, z)$ the superzeta functions of $Z_\pm$.

**Proposition 4.2.** For $z \in X_\pm$, the superzeta functions $\mathcal{Z}_{\pm}(s, z)$ have meromorphic continuation to all of $s \in \mathbb{C}$, regular at $s = 0$. Moreover,

$$
\mathcal{Z}_+(0, z) = -h \frac{\text{vol}(M)}{2\pi} z^2 - (\bar{a}_1 + k) z + k - \bar{a}_0 \quad \text{and} \quad \mathcal{Z}_-(0, z) = -h \frac{\text{vol}(M)}{2\pi} z^2 - (\bar{a}_1 + k) z - \bar{a}_0 + \frac{k}{2}
$$

where $\bar{a}_1$ and $\bar{a}_0$ are defined by equations (3.11) and (3.18).

**Proof.** We claim that $Z_+(z)$ and $Z_-(z)$ both are entire, order two functions which satisfy the hypothesis of Proposition 4.1. Indeed, the function $G_1(z)$ is a product of rescaled Barnes double gamma functions, so by using the asymptotic expansion (2.3) of the gamma function, the expansion (2.5) of the Barnes double gamma function, the bound (2.11) for logarithm of the Selberg zeta function, and the asymptotic expansion of the logarithm of the automorphic scattering matrix $\phi(z) = L(z)H(z)$, deduced from (2.8) and (2.10), we can obtain an asymptotic expansion of the form (4.3) for both $Z_+$ and $Z_-$. We refer to the proof of [9, Thm. 6.2] where similar computations are worked out in complete detail. Thus, by Proposition 4.1 both $\mathcal{Z}_{\pm}(s, z)$ have meromorphic continuation to all of $s \in \mathbb{C}$ which are regular at $s = 0$.

To prove the equation (4.5) we need to deduce asymptotic expansion of $\log Z_+(z)$. From the definition (3.8) we have

$$
\log Z_+(z) = \log Z(z) - \log G_1(z) - k \log \Gamma(z - 1/2)
$$

hence, combining (3.10) with

$$
\log \Gamma(z - \frac{1}{2}) = z(\log(z) - 1) - \log(z) + \frac{1}{2} \log(2\pi) + O(z^{-1})
$$

and with (2.11), which yields $\log Z(z) = O(z^{-1})$ we get, as $\text{Re}(z) \to \infty$:

$$
\log Z_+(z) = -h \frac{\text{vol}(M)}{2\pi} z^2 (\log z - \frac{3}{2}) - (\bar{a}_1 + k) z (\log z - 1)
$$

$$
- b_1 z + (k - \bar{a}_0) \log z - b_0 - \frac{k}{2} \log(2\pi) + \sum_{k=1}^{n-1} a_k z^k + b_n(z),
$$

where $\bar{a}_1, b_1, \bar{a}_0$ and $b_0$ are given by formulas (3.11), (3.12), (3.18) and (3.19) respectively.

From the proof of [9, Proposition 4.1] (see specifically [9, Equation 4.9]), and the asymptotic expansion (4.7) we see that for $-1 < \text{Re}(s) < 3$ and $z \in X_+$ we have

$$
\mathcal{Z}_+(s, z) = -2h \frac{\text{vol}(M)}{(s-1)(s-2)} z^{2-s} + \frac{\bar{a}_1 + k}{(s-1)} z^{1-s} + (k - \bar{a}_0) z^{-s} + \frac{1}{\Gamma(s)} f(s, z),
$$

where $f(s, z)$ is a holomorphic function for $\text{Re}(s) > -1/2$. Therefore

$$
\mathcal{Z}_+(0, z) = -h \frac{\text{vol}(M)}{2\pi} z^2 - (\bar{a}_1 + k) z + k - \bar{a}_0
$$

which proves the first formula in (4.5).

Recall that $Z_-(z) = \phi(z)Z_+(z)$. Hence from the asymptotic properties of $\log \phi(z)$, given in (2.8), it follows that

$$
\log Z_-(z) = \log Z_+(z) + \frac{k}{2} \log \pi - \frac{k}{2} \log z + c_1 z + c_2 + o(1) \quad \text{as} \quad z \to \infty.
$$

(4.8)
Here \(c_1\) and \(c_2\) are from \(2.8\), and we used the asymptotic expansions \(4.9\) of \(\log \Gamma(z - \frac{1}{2})\) and asymptotic

\[
\log \Gamma(z) = z \log(z) - 1 - \frac{1}{2} \log(z) + \frac{1}{2} \log(2\pi) + O(z^{-1}),
\]

of \(\log \Gamma(z)\). This, together with \(4.7\) gives

\[
\log Z_-(z) = -\hbar \frac{\text{vol}(M)}{2\pi} z^2 \left(\log z - \frac{3}{2}\right) - (\bar{a}_1 + k) z (\log z - 1)
\]

\[
+ (c_1 - b_1)z + \left(k/2 - \bar{a}_0\right) \log z - b_0 - k/2 \log 2 + c_2 + \log z \sum_{k=1}^{n-1} a_k^2 z^{2k} + g_n(z).
\]

Reasoning analogously as above, using the asymptotic expansion \(4.10\), we deduce the second formula in \(4.5\).

\[
\]

5 Main results

Recall the definition of \(N(Z_\pm)\), the null set of \(Z_\pm(s)\), which is specified in \(3.3\) and the set \(N(Z_-)\), which is symmetric to \(N(Z_+)\) with respect to the mapping \(s \mapsto 1 - s\). One observes that it contains essentially the non-trivial spectral information of the Laplacian: the eigenvalues and the resonances.

To motivate our key definition we perform a purely formal computation. First recall \(4.1\) the definition of the superzeta functions \(Z_{\pm}(s, z) := Z_{Z_{\pm}}(s, z)\). Next, recalling that \(s_j(1 - s_j) = \lambda_j\), we have formally

\[
\exp \left(-\frac{d}{ds}\right) (Z_+(s, z) + Z_-(s, z))|_{s=0} = \prod_j \left((z - s_j)(z - (1 - s_j))\right)^2 \left(z - \rho_j(z - (1 - \rho_j))\right)
\]

\[
= \prod_j (\lambda_j - z)(1 - z) \prod_j (\rho_j(z - (1 - \rho_j)) - z(1 - z)),
\]

where the first product is taken over all discrete eigenvalues of the Laplacian and the second product is taken over all resonances. Therefore, the sum \(Z_+(s, z) + Z_-(s, z)\) satisfies the same formal identity as the zeta function given by formula \(1.3\) in \([5]\). This motivates our definition of the regularized determinant of the Laplacian.

**Definition 5.1.** Let \(Z_{\pm}(s, z) := Z_{Z_{\pm}}(s, z)\) be the superzeta functions of \(Z_{\pm}\). The square of the regularized determinant of \(\Delta - z(1 - z)I\) is defined to be

\[
\det^2(\Delta - z(1 - z)I) = \exp \left(-\frac{d}{ds}\right) (Z_+(s, z) + Z_-(s, z))|_{s=0} = D_{Z_+}(z) D_{Z_-}(z).
\]

Our main result is the following explicit evaluation of \(\det^2(\Delta - z(1 - z)I)\).

**Theorem 5.2.** For \(z \in X_+ \cap X_-\), the regularized determinant of the square of \(\Delta - z(1 - z)I\) is given by the formula

\[
\det^2(\Delta - z(1 - z)I) = \exp \left((2b_1 - c_1)z + 2b_0 + \frac{g}{2}\log(4\pi) - c_2\right) \phi(z) \cdot \left(\frac{Z(z)}{G_1(z)}(\Gamma(z - 1/2))^k\right)^2.
\]

Here \(b_1\) is given by \(3.12\), \(b_0\) is defined in \(3.19\) and \(c_1, c_2\) are from \(2.8\).
Proof. From the expansion (4.7) of \( \log Z_+(z) \), applying Proposition 4.1 we deduce
\[
D_+(z) = \exp(b_1 z + b_0 + \frac{h}{2} \log(2\pi))Z_+(z).
\]
Analogously, the expansion (4.10) together with Proposition 4.1 yields
\[
D_-(z) = \exp((b_1 - c_1)z + b_0 + \frac{h}{2} \log 2 - c_2)Z_-(z).
\]
Combining the expressions for \( D_+ \) and \( D_- \) with definitions of \( Z_+ \) and \( Z_- \) completes the proof.

Definition (5.2) of the square of the regularized determinant is justified by the following functional relation between \( D_+(1-z)D_-(1-z) \) and \( D_+(z)D_-(z) \).

**Theorem 5.3.** For constants \( a(\chi) \) defined by (2.22), \( b_1 \) defined by (3.2), and \( c_1 \) from (2.8), we have the following symmetric functional equation
\[
\exp(-(2b_1 - c_1 + 2 \log a(\chi))Z_+(z)D_-(z)) = \exp(-(2b_1 - c_1 + 2 \log a(\chi))(1-z))D_+(1-z)D_-(1-z).
\]

Proof. First, we will write the function \( \Xi(z) \) introduced in the definition 2.1.4. on p. 115 of (8) (with weight \( k = 0 \)) in our notation:
\[
\Xi(z) = \exp(h \frac{\text{vol}(M)}{2\pi} z(1-z)) \left( \frac{(2\pi)^2 \Gamma(z)}{\Gamma(z+1)^2} \right)^{\frac{h \text{vol}(M)}{2\pi}} \cdot Z(z) \cdot \prod_{(R_\chi)} \frac{\alpha(1-1/dR)}{dR} \prod_{m=0}^{\chi} \frac{\Gamma(1+m)}{\Gamma(1+m)} \cdot \frac{a(\chi)^{-1} g_1^{-1} z(2-2z) - \frac{h}{2} \frac{1}{\rho} \frac{1}{\rho - 1/2}}{2 \left( \frac{z-1/2}{\rho - 1/2} \right)^2} \cdot \prod_{m=1}^{\chi} \left( 1 + \frac{z-\frac{1}{2} - \frac{1}{\sigma m - 2}}{2 \left( \frac{z-1/2}{\sigma m - 1/2} \right)^2} \right)^{-1} \prod_{m=1}^{\chi} \left( 1 + \frac{z-\frac{1}{2} - \frac{1}{\sigma m - 2}}{2 \left( \frac{z-1/2}{\sigma m - 1/2} \right)^2} \right)^{-1} \exp \left( -\frac{1}{2} \frac{z-1/2}{\rho - 1/2} - \frac{1}{2} \frac{z-1/2}{\sigma m - 1/2} \right)
\]
The zeros \( 1 - \sigma_m \) for \( m = 1, \ldots, T \) of the scattering determinant \( \phi \) are defined in §2.4. \( M = \sum_{i=1}^{T} q(\sigma_i) \) is the number of zeros \( \sigma_m \), counted with multiplicities and \( \mathcal{P}(z) \) is a function depending on the resonances of \( \phi \) which is given in (8) Definition 3.2.2, p. 118. In our notation (assuming \( k \neq 0 \), i.e. assuming \( \chi \) is singular) we have
\[
\mathcal{P}(z) = \prod_{n=1}^{N} \left( 1 + \frac{z - \frac{1}{2}}{\rho_n - 1/2} \right)^{-1} \exp \left( -\frac{1}{2} \left( \frac{z - 1/2}{\rho_n - 1/2} \right)^2 \right) \cdot \prod_{\rho} \left( 1 + \frac{z - \frac{1}{2}}{\rho - 1/2} \right)^{-1} \exp \left( -\frac{1}{2} \left( \frac{z - 1/2}{\rho - 1/2} \right)^2 \right)
\]
where the zeros \( \rho_i \), \( i = 1, \ldots, N \) and zeros \( \rho \) are are defined in §2.4.

Using (8) Formula 3.2.1] we see that for \( k \neq 0 \), \( \mathcal{P}(z) \) satisfies the functional relation
\[
\mathcal{P}(1-z) = g_1^{-1} \phi^{-1}(1-z) \prod_{m=1}^{\chi} \frac{\sigma_m - z}{\sigma_m + z - 1} \mathcal{P}(z) \phi(z),
\]
where \( g_1 \) is from (2.7).

Comparing the definition (5.4) of \( \Xi(z) \) with the definition (3.9) of \( G_1(z) \), we arrive at the following relation:
\[
Z(z)G_1(z)^{-1} \Gamma(z - 1/2)^{-k} = \Xi(z) \exp(-h \frac{\text{vol}(M)}{2\pi} z(1-z)) a(\chi)^{1/2} \cdot (z - 1/2)^{1/2} (\frac{1}{2} \frac{1}{\rho} \frac{1}{\rho - 1/2}) g_1 \prod_{m=1}^{\chi} \left( 1 + \frac{z - \frac{1}{2} - \frac{1}{\sigma m - 2}}{2 \left( \frac{z-1/2}{\sigma m - 1/2} \right)^2} \right)^{-1} \mathcal{P}(z),
\]
Since $\Xi(z) = \Xi(1 - z)$ and $\phi^2(\frac{1}{z}) = 1$, taking the square of the expression (5.6), inserting $1 - z$ instead of $z$ and applying the functional relation (5.5), we deduce the following:

$$\left(\frac{Z(1 - z)}{G_1(1 - z)(\Gamma(1/2 - z))^k}\right)^2 = a(\chi)^{2(1 - 2z)} \left(\frac{Z(z)}{G_1(z)(\Gamma(z - 1/2))^k}\right)^2 \phi^2(z),$$

which is equivalent to equation

$$\phi(1 - z) \left(\frac{Z(1 - z)}{G_1(1 - z)(\Gamma(1/2 - z))^k}\right)^2 = a(\chi)^{2(1 - 2z)} \phi(z) \left(\frac{Z(z)}{G_1(z)(\Gamma(z - 1/2))^k}\right)^2.$$

The above equation, together with (5.5) yields that

$$D_+(1 - z)D_-(1 - z) = \exp\left((2b_1 - c_1)(1 - z) + 2b_0 + \frac{3}{2}\log(4\pi) - c_2\right) a(\chi)^{2(1 - 2z)} \phi(z) \cdot \left(\frac{Z(z)}{G_1(z)(\Gamma(z - 1/2))^k}\right)^2,$$

which gives the relation

$$D_+(1 - z)D_-(1 - z) = \exp((2b_1 - c_1)(1 - 2z))a(\chi)^{2(1 - 2z)}D_+(z)D_-(z).$$

**Remark 5.4.** The expression (5.3) can be compared with the results of [5] in the scalar setting ($h = 1$) and the surface is torsion-free. Using the relation

$$\Gamma_2(z) = (2\pi)^z/2\Gamma(z)(G(z + 1))^{-1}$$

between the double gamma function $\Gamma_2(z)$ and the Barnes double gamma function $G(z + 1),$ we can rewrite (5.3) in the notation of [3] (where $h_1$ denotes the degree of singularity):

$$\det^2(\Delta - z(1 - z)I) = (2\pi)^z(2z - 1)^3 \exp\left(-\tilde{B}(2z - 1) + \tilde{C}\right) \phi(z) \cdot (Z(z)Z_\infty(z))^2 (\Gamma(z + 1/2))^{-2h_1},$$

for certain, explicitly computable constants $\tilde{A}, \tilde{B}$ and $\tilde{C}.$

This expression differs from the corrected formula for the square of the determinant of the Laplacian of [3] in constants $\tilde{A}, \tilde{B}$ and $\tilde{C}$ and, more importantly, in the factor $\phi(z).$ (Curiously, the above expression differs from the erroneous formula [5] Equation 1.7, page 445] only in constants $\tilde{A}, \tilde{B}$ and $\tilde{C}.$) This is due to a different definition of the regularized determinant of the Laplacian.

The fact that we have introduced the definition of the determinant of the Laplacian which encodes the natural symmetry $z \leftrightarrow 1 - z$ shows that the approach to zeta regularization through the superzeta functions carrying the spectral information is better suited in in the presence of the continuous spectrum, than the trace formula approach employed in [5], [21] and many other more recent papers.

Our approach immediately yields the following corollary which proves that the scattering determinant can be expressed as the regularized determinant of the superzeta function $Z_-(s, z) - Z_+(s, z),$ modulo the factor $\exp(c_1z + c_2 + \frac{k}{2}\log \pi).$

**Corollary 5.5.** For $z \in X_+ \cap X_-$

$$\phi(z) = (\pi)^{\frac{k}{2}} \exp\left(\frac{-d}{ds}(Z_-(s, z) - Z_+(s, z))|_{s=0}\right).$$ (5.7)

**Proof.** Utilizing the equation (4.8) and applying (4.4) to both $Z_-(s), Z_+(s)$ we see that

$$\frac{D_-(z)}{D_+(z)} = \exp\left(-\frac{k}{2}\log \pi - c_1z - c_2\right) \frac{Z_-(z)}{Z_+(z)},$$
hence
\[
\phi(z) = \frac{Z_-(z)}{Z_+(z)} = \exp \left( \frac{k}{2} \log \pi + c_1 z + c_2 \right) \frac{\exp \left( - \frac{d}{ds} Z_-(s, z) |_{s=0} \right)}{\exp \left( - \frac{d}{ds} Z_+(s, z) |_{s=0} \right)}. \]

This proves (5.7).

A Alternate expressions

The multiplicities \( m_n \) of the trivial zeros of the Selberg zeta function carry important information related to the surface \( M \) and the character \( \chi \). For this reason, we will deduce a different expression for \( m_n \) (see equation (A.6) below) from which it will be obvious that \( m_n \) are non-negative integers. Moreover, we will construct a different order-two entire function \( \tilde{G}_1(s) \), given in terms of the gamma and the Barnes double gamma function (see equation (A.7) below) and such that its null set coincides with the set of negative integers \( -n \) with multiplicities \( m_n \).

A.1 An alternate expression for the multiplicities \( m_n \)

We simplify (2.12) by combining it with (2.1) multiplied by \( h \).

\[
m_n = h (2g - 2 + c + e) (2n+1) - \sum_{j=1}^{\frac{h}{2}} \sum_{\{R\} \cap h} \frac{1}{d_R} \left( 2n + 1 + \sum_{k=1}^{d_R-1} (\omega(R)_{2n+1}^k) \frac{\sin \left( \frac{k\pi(2n+1)}{d_R} \right)}{\sin \left( \frac{k\pi}{d_R} \right)} \right) \tag{A.1}
\]

By (A.1) we can focus on the case of unitary characters rather than unitary representations.

Lemma A.1. Let \( \omega \) be a unitary character of the finite cyclic group \( \{R\} \), where \( R \) is elliptic of order \( d \). Further let \( \omega(R) = \exp(2\pi i q/d) \) for some integer \( q \), with \( 0 \leq q \leq d - 1 \). Let \( n \in \mathbb{N} = \{0, 1, 2, \ldots\} \), then

\[
\frac{1}{d} \left( 2n + 1 + \sum_{k=1}^{d-1} \omega^k(R) \sin \left( \frac{k\pi(2n+1)}{d} \right) \right) = |\{t \in \mathbb{Z} \mid td \in \{-n + q, \ldots, n + q\}\}|, \tag{A.2}
\]

where \( |A| \) denotes the cardinality of the finite set \( A \).

Proof. We apply the identity

\[
\frac{\sin(k\pi(2n+1)/d)}{\sin(k\pi/d)} = \sum_{j=-n}^{n} \exp(2\pi ik j / d)
\]

and obtain

\[
\frac{1}{d} \left( 2n + 1 + \sum_{k=1}^{d-1} \omega^k(R) \sin \left( \frac{k\pi(2n+1)}{d} \right) \right) = \frac{1}{d} (2n+1) + \sum_{k=1}^{d-1} \frac{1}{d} \exp(2\pi ik q / d) \sum_{j=-n}^{n} \exp(2\pi ik j / d) \tag{A.3}
\]

The inner sum on the right is equal to \( d \) if \( d \mid (q + j) \). As \( j \) runs from \(-n, -n+1, \ldots, n\), the number of such \( j \) is equal to the number of multiples of \( d \) in the integer range \(-n + q, \ldots, n + q\). Recalling the factor \( \frac{1}{d} \) out in front proves the lemma.
Lemma A.2. Let $d, q, n$ be integers with $2 \leq d, 0 \leq q \leq d - 1$, and $0 \leq n$.

Then

$$|\{t \in \mathbb{Z} \mid td \in \{-n + q, \ldots n + q\}\}| = \left\lfloor \frac{n + q}{d} \right\rfloor + \left\lfloor \frac{n + d - q}{d} \right\rfloor$$

Here $\lfloor x \rfloor$ denotes the floor function.

Proof. For $a \leq b$, both integers, define $f(a, b, d) = |\{t \in \mathbb{Z} \mid td \in \{a, \ldots b\}\}|$

Now, let $a, b$ be integers with $a < 0$ and $b > 0$. Then

$$f(a, b, d) = f(-a, 0, d) + f(0, b, d) - 1 = f(0, -a, d) + f(0, b, d) + 1 = \left\lfloor \frac{d - a}{d} \right\rfloor + \left\lfloor \frac{b}{d} \right\rfloor \tag{A.4}$$

To prove the lemma, we consider the following cases. The first case, $a = -n + q < 0$, and $b = n + q > 0$ follows from (A.4).

In the case when $n = 0$, the lemma is trivial.

Next, consider the case where $0 < -n + q$. Since $q < d$, and $n \geq 0$, it follows that $0 < -n + q < d$. If $n + q < d$, then it follows that

$$f(-n + q, n + q, d) = 0 = \left\lfloor \frac{n + q}{d} \right\rfloor + \left\lfloor \frac{d + (n - q)}{d} \right\rfloor = 0 + 0,$$

and the lemma is verified in this case.

Finally we are left with the case $0 < -n + q < d$, and $d < n + q$. We shift the integer interval $\{-n + q, \ldots n + q\}$ to the left by $d$, and obtain

$$|\{t \in \mathbb{Z} \mid td \in \{-n + q, \ldots n + q\}\}| = |\{t \in \mathbb{Z} \mid td \in \{-n + q - d, \ldots n + q - d\}\}|.$$

We can apply (A.4) to the shifted interval and we obtain

$$f(-n + q - d, n + q - d, d) = \left\lfloor \frac{d - (-n + q - d)}{d} \right\rfloor + \left\lfloor \frac{n + q - d}{d} \right\rfloor$$

$$= \left\lfloor 1 + \frac{n + d - q}{d} \right\rfloor + \left\lfloor 1 + \frac{n + q}{d} \right\rfloor = \left\lfloor \frac{n + d - q}{d} \right\rfloor + \left\lfloor \frac{n + q}{d} \right\rfloor \tag{A.5}$$

where the last equality follows because both $n + q$ and $n + d - q$ are positive. \hfill \Box

One should note that the above Lemma A.2 is false for arbitrary $n, q, d$.

Combining A.1, Lemma A.1 and Lemma A.2 we arrive at the following alternate expression for $m_n$:

$$m_n = h (2g - 2 + c + e) (2n + 1) - \sum_{\{R\} \in J} \sum_{j=1}^{h} \left( \left\lfloor \frac{n + q(R)_j}{d_R} \right\rfloor + \left\lfloor \frac{n + d_R - q(R)_j}{d_R} \right\rfloor \right) \tag{A.6}$$

A.2 Double gamma function representation of the trivial zeros of the Selberg-zeta function

Recall that the Barnes double Gamma function, which is an entire order two function defined by (A.4) has a zero of multiplicity $n$, at each point $-n \in \{-1, -2, \ldots\}$. We start with the following lemma.

Lemma A.3. Let $d, q, n$ be integers with $2 \leq d, 0 \leq q \leq d - 1$, and $0 \leq n$. Set

$$g(n, q, d) = \left\lfloor \frac{n + q}{d} \right\rfloor + \left\lfloor \frac{n + d - q}{d} \right\rfloor.$$
For \( s \in \mathbb{C} \), define

\[
G_{q,d}(s) = \prod_{m=0}^{d-1} G \left( \frac{s - q + m}{d} \right) G \left( \frac{s - (d - q) + m}{d} \right).
\]

Then \( G_{q,d}(s) \) is an entire function with the set of zeros being the set of negative integers \( \{-n : n = 1, 2, \ldots\} \) and each zero \( s = -n, n = 1, 2, \ldots \) has multiplicity \( g(n, q, d) \).

**Proof.** We study \( G_{q,d}(s) \) at \( s = -n \). Since \( m \in \{0, \ldots, d-1\} \), the number \( \frac{-n-q+m}{d} \) is a negative integer for exactly one value, say \( m = m_1 \), in which case

\[
\left\lfloor \frac{-n-q+m_1}{d} \right\rfloor = -\left\lfloor \frac{n+q}{d} \right\rfloor.
\]

Therefore, \( \prod_{m=0}^{d-1} G \left( \frac{s - q + m}{d} \right) \) has a zero of order \( \left\lfloor \frac{n+q}{d} \right\rfloor \) at \( s = -n \) for any positive integer \( n \). Similarly, \( \prod_{m=0}^{d-1} G \left( \frac{s - (d-q) + m}{d} \right) \) has a zero of order \( \left\lfloor \frac{n+q-d}{d} \right\rfloor \) at \( s = -n \).

Moreover, \( \frac{-n-q+m}{d} = -k \) for some \( k = 1, 2, 3, \ldots \), if \( s = -kd + q - m \) is a negative integer. This shows that there are no zeros of \( \prod_{m=0}^{d-1} G \left( \frac{s - q + m}{d} \right) \) different from negative integers. A similar conclusion holds for \( \prod_{m=0}^{d-1} G \left( \frac{s - (d-q) + m}{d} \right) \) and the proof is complete.

Recall that \( \{R\}_\Gamma \) are the classes of elliptic elements of \( \Gamma \) and that there are \( e \) of them. Further, recall the notation \( e, h \). Recall that \( \omega(R) \), for \( j = 1 \ldots h \) are the eigenvalues (and \( d \)-eigenroots of unity) of \( \chi(R) \), and that \( q(R)_j \), with \( 0 \leq q(R)_j \leq d - 1 \) is defined by (3.4).

**Definition A.4.** With the notation above we define

\[
G_E(s) = \prod_{\{R\}_\Gamma} \prod_{j=1}^{h} \prod_{m=0}^{d_R-1} G \left( \frac{s - q(R)_j + m}{d_R} \right) G \left( \frac{s - (d_R - q(R)_j) + m}{d_R} \right).
\]

and

\[
\tilde{G}_1(s) = \left( G_E(s) \right)^{-1} \left( \frac{(2\pi)^{-s}(G(s+1))^2}{\Gamma(s)} \right)^{h(2g-2+c+e)}
\]

(A.7)

**Lemma A.5.** The function \( \tilde{G}_1(s) \) is a entire of order two with zeros at points \( -n \in -\mathbb{N} \) and corresponding multiplicities \( m_n \).

**Proof.** The function \( \left( \frac{(2\pi)^{-s}(G(s+1))^2}{\Gamma(s)} \right)^{h(2g-2+c+e)} \) possesses zeros at points \( s = -n \) with multiplicity \( h(2g-2+c+e)(2n+1) \), hence the statement follows by combining equation (A.1) with Lemmas 3.1, 3.2 and 3.3.

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