The $KH$-Isomorphism Conjecture and Algebraic $KK$-theory

Paul D. Mitchener
University of Sheffield
e-mail: P.Mitchener@sheffield.ac.uk
Web-site: http://www.mitchener.staff.shef.ac.uk
January 15, 2009

Abstract
In this article we prove that the $KH$-assembly map, as defined by Bartels and Lück, can be described in terms of the algebraic $KK$-theory of Cortinas and Thom. The $KK$-theory description of the $KH$-assembly map is similar to that of the Baum-Connes assembly map. In some elementary cases, methods used to prove the Baum-Connes conjecture also apply to the $KH$-isomorphism conjecture.

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Various assembly maps, such as the Baum-Connes assembly map (see [2, 26]) and the Farrell-Jones assembly map (see [7]) can be described using an abstract homotopy-theoretic framework, developed by Davis and Lück in [6]; the machinery is based upon spectra and equivariant homology theories.

Recently, in [1], Bartels and Lück introduced the $KH$-assembly map. The $KH$-assembly map is constructed using the abstract machinery, and resembles the Farrell-Jones assembly map in algebraic $K$-theory. However, the $KH$-assembly map uses homotopy algebraic $K$-theory, as introduced in [27], rather than ordinary algebraic $K$-theory.

Kasparov’s bivariant $K$-theory, or $KK$-theory (see for example [3, 12, 24, 26] for accounts) is in essence a way of organizing certain maps between the $K$-theory groups of $C^*$-algebras. The Baum-Connes assembly map is such a map, and is usually described using $KK$-theory; some work is needed (see [10, 22]) to show that the $KK$-theory assembly map can be constructed using the Davis-Lück machinery.

Recently, in [4], Cortinas and Thom developed a version of bivariant algebraic $K$-theory. This version of $KK$-theory induces maps between the homotopy algebraic $K$-theory groups of discrete algebras. The construction is broadly speaking similar to the construction of $KK$-theory for locally convex algebras in [5], but replaces smooth homotopies with algebraic homotopies.

The purpose of this article is essentially an algebraic version of the converse of [22]; we show that the abstract $KH$-assembly map can be described in terms of algebraic $KK$-theory in terms similar to that of the Baum-Connes conjecture. We also look at some elementary consequences of this description, namely some elementary cases where one can easily show that the $KH$-assembly map is an isomorphism.

It is fairly easy to write down an algebraic $KK$-theory version of the Baum-Connes assembly map. However, to associate this map to the Davis-Lück machinery, we need to write this map at the level of spectra, and moreover to generalize algebraic $KK$-theory from algebras to algebroids, and to equivariant algebras based on groupoids.

Thus, before we look at assembly maps, we describe $KK$-theory spectra for algebroids and groupoid algebras, before looking at some basic properties. These definitions and properties are mainly, but not entirely, easy generalisations of those in [4].
2 Algebroids

The following definition is a slight generalization of the notion of an algebroid in [20].

**Definition 2.1** Let \( R \) be a commutative unital ring. A \( R \)-algebroid, \( \mathcal{A} \), consists of a set of objects, \( \text{Ob}(\mathcal{A}) \), along with a left \( R \)-module, \( \text{Hom}(a,b)_{\mathcal{A}} \) for each pair of objects \( a, b \in \text{Ob}(\mathcal{A}) \), such that:

- We have an associative \( R \)-bilinear composition law
  \[
  \text{Hom}(b,c)_{\mathcal{A}} \times \text{Hom}(a,b)_{\mathcal{A}} \rightarrow \text{Hom}(a,c)_{\mathcal{A}}
  \]

- Given an element \( r \in R \) and morphisms \( x \in \text{Hom}(a,b)_{\mathcal{A}}, \ y \in \text{Hom}(b,c)_{\mathcal{A}} \), the equation \( r(xy) = (rx)y = x(ry) \) holds.

We call an \( R \)-algebroid \( \mathcal{A} \) unital if it is a category. Thus, in a unital \( R \)-algebroid we have an identity element \( 1_a \in \text{Hom}(a,a)_{\mathcal{A}} \) for each object \( a \).

An \( R \)-algebra can be considered to be the same thing as an \( R \)-algebroid with one object.

**Definition 2.2** Let \( \mathcal{A} \) and \( \mathcal{B} \) be \( R \)-algebroids. Then a homomorphism \( \alpha: \mathcal{A} \rightarrow \mathcal{B} \) consists of maps \( \alpha: \text{Ob}(\mathcal{A}) \rightarrow \text{Ob}(\mathcal{B}) \) and \( R \)-linear maps \( \alpha: \text{Hom}(a,b)_{\mathcal{A}} \rightarrow \text{Hom}(\alpha(a),\alpha(b))_{\mathcal{B}} \) that are compatible with the composition law.

Given \( R \)-algebroids \( \mathcal{A} \) and \( \mathcal{B} \), we write \( \text{Hom}(\mathcal{A},\mathcal{B}) \) to denote the set of all homomorphisms.

**Definition 2.3** Let \( \mathcal{A} \) be an \( R \)-algebroid. Then we define the additive completion, \( \mathcal{A}_{\oplus} \), to be the \( R \)-algebroid in which the objects are formal sequences of the form
\[
a_1 \oplus \cdots \oplus a_n \quad a_i \in \text{Ob}(\mathcal{A})
\]
where \( n \in \mathbb{N} \). Repetitions are allowed in such formal sequences. The empty sequence is also allowed, and labelled 0.

The \( R \)-module \( \text{Hom}(a_1 \oplus \cdots \oplus a_n, b_1 \oplus \cdots \oplus b_n)_{\mathcal{A}_{\oplus}} \) is defined to be the set of matrices of the form
\[
\begin{pmatrix}
x_{11} & \cdots & x_{1m} \\
\vdots & \ddots & \vdots \\
x_{n1} & \cdots & x_{nm}
\end{pmatrix}
\]
x_{ij} \in \text{Hom}(a_j,b_i)_{\mathcal{A}}

with element-wise addition and multiplication by elements of the ring \( R \).

The composition law is defined by matrix multiplication.
Given an algebroid homomorphism \( \alpha : A \rightarrow B \), there is an induced homomorphism \( \alpha : A \oplus \rightarrow B \oplus \) defined by writing

\[
\alpha(a_1 \oplus \cdots \oplus a_n) = \alpha(a_1) \oplus \cdots \oplus \alpha(a_n) \quad a_i \in Ob(A)
\]

and

\[
\alpha\left(\begin{array}{ccc}
  x_{1,1} & \cdots & x_{1,m} \\
  \vdots & \ddots & \vdots \\
  x_{n,1} & \cdots & x_{n,m}
\end{array}\right) = \left(\begin{array}{ccc}
  \alpha(x_{1,1}) & \cdots & \alpha(x_{1,m}) \\
  \vdots & \ddots & \vdots \\
  \alpha(x_{n,1}) & \cdots & \alpha(x_{n,m})
\end{array}\right) \quad x_{i,j} \in Hom(a_j, b_i)
\]

With such induced homomorphisms, the process of additive completion defines a functor from the category of \( R \)-algebroids and homomorphisms to itself.

We define the direct sum of two objects \( a = a_1 \oplus \cdots \oplus a_m \) and \( b = b_1 \oplus \cdots \oplus b_n \) in the additive completion \( A \oplus \) by writing

\[
a \oplus b = a_1 \oplus \cdots \oplus a_m \oplus b_1 \oplus \cdots \oplus b_n
\]

In the unital case, the additive completion of a unital \( R \)-algebroid is an additive category; direct sums of objects can be defined as above. The induced functor, \( \alpha \), is an additive functor. Analogous properties hold in the non-unital case.

Finally, given an \( R \)-algebroid \( A \), we would like to define an associated \( R \)-algebra that carries the same information as the additive completion \( A \oplus \). The naive way to do this is to simply form the direct limit

\[
\bigcup_{a_i \in Ob(A)} \text{Hom}(a_1 \oplus \cdots \oplus a_n, a_1 \oplus \cdots \oplus a_n)_{A \oplus}
\]

with respect to the inclusions

\[
\text{Hom}(a \oplus c, a \oplus c)_{A \oplus} \rightarrow \text{Hom}(a \oplus b \oplus c, a \oplus b \oplus c)_{A \oplus}
\]

\[
\left(\begin{array}{cc}
w & x \\
y & z
\end{array}\right) \mapsto \left(\begin{array}{ccc}
w & 0 & x \\
0 & 0 & 0 \\
y & 0 & z
\end{array}\right)
\]

Unfortunately, the above construction is not functorial. We can, however, replace it by an equivalent functorial construction. This construction is essentially the same as that used to define the \( K \)-theory of \( C^* \)-categories in [14] or [21].

Let \( A \) be an \( R \)-algebroid. Then we define \( O_A \) be the category in which the set of objects consists of all compositions of inclusions \( \text{Hom}(a \oplus c, a \oplus c)_{A \oplus} \rightarrow \text{Hom}(a \oplus b \oplus c, a \oplus b \oplus c)_{A \oplus} \) of the form

\[
\left(\begin{array}{cc}
w & x \\
y & z
\end{array}\right) \mapsto \left(\begin{array}{ccc}
w & 0 & x \\
0 & 0 & 0 \\
y & 0 & z
\end{array}\right)
\]

See [18] or [28] for relevant definitions.
A morphism set between two inclusions has precisely one element if the inclusions are composable; otherwise, it is empty.

We can define a functor, $H_A$, from the category $O_A$ to the category of $R$-algebras by associating the $R$-algebra $\text{Hom}(a \oplus c, a \oplus c)_A$ to the inclusion $\text{Hom}(a \oplus c, a \oplus c)_A \to \text{Hom}(a \oplus b \oplus c, a \oplus b \oplus c)_A$. If $i$ and $j$ are composable inclusions, then the one morphism in the set $\text{Hom}(i, j)_{O_A}$ is mapped to the inclusion $i$ itself.

**Definition 2.4** Let $A$ be an $R$-algebroid. Then we define the $R$-algebra $A_H$ to be the colimit of the functor $H_A$.

The following result is obvious from our constructions.

**Proposition 2.5** The assignment $A \oplus \mapsto A_H$ is a covariant functor, and we have a natural transformation $J: A \oplus \to A_H$. The natural transformation $F$ is surjective on each morphism set.

Further, given a homomorphism $\alpha: A \to B$, we have a factorisation

$$A \overset{\alpha'}{\to} B \overset{J}{\to} B_H.$$

\[\square\]

### 3 Tensor Products

**Definition 3.1** Let $A$ and $B$ be $R$-algebroids. Then we define the tensor product $A \otimes_R B$ to be the $R$-algebroid with the set of objects $\text{Ob}(A) \times \text{Ob}(B)$. We write the object $(a, b)$ in the form $a \otimes b$. The $R$-module $\text{Hom}(a \otimes b, a' \otimes b')_A \otimes_R B$ is the tensor product of $R$-modules $\text{Hom}(a, a')_A \otimes_R \text{Hom}(b, b')_B$.

The composition law is defined in the obvious way.

If we view an $R$-algebra as an $R$-algebroid with just one object, we can form the tensor product, $A \otimes_R B$, of an $R$-algebroid $A$ with an $R$-algebra $B$. The objects of the tensor product $A \otimes_R B$ are identified with the objects of the algebroid $A$.

**Definition 3.2** Let $R$ be a commutative ring with an identity element. An $R$-moduloid, $\mathcal{E}$, consists of a collection of objects $\text{Ob}(\mathcal{E})$, along with a left $R$-module, $\text{Hom}(a, b)_\mathcal{E}$ defined for each pair of objects $a, b \in \text{Ob}(\mathcal{E})$.

A homomorphism, $\phi: \mathcal{E} \to \mathcal{F}$, between $R$-moduloids consists of a map $\phi: \text{Ob}(\mathcal{E}) \to \text{Ob}(\mathcal{F})$ and a collection of $R$-linear maps $\phi: \text{Hom}(a, b)_\mathcal{E} \to \text{Hom}(\phi(a), \phi(b))_\mathcal{F}$.

The difference between an $R$-moduloid and an $R$-algebroid is that there is no composition law between the various $R$-bimodules $\text{Hom}(a, b)_\mathcal{E}$. There is of course a forgetful functor, $F$, from the category of $R$-algebroids and homomorphisms to the category of $R$-moduloids and homomorphisms.
Definition 3.3 Let \( A \) be an \( R \)-moduloid. Given objects \( a, b \in \text{Ob}(A) \), let us define

\[
\text{Hom}(a, b)^{(k+1)}_A = \bigoplus_{c_i \in \text{Ob}(A)} \text{Hom}(a, c_1) \otimes_R \text{Hom}(c_1, c_2) \otimes_R \cdots \otimes \text{Hom}(c_k, b)
\]

The tensor algebroid, \( TA \), is the \( R \)-algebroid with the same set of objects as the \( R \)-moduloid \( A \) where the morphism set \( \text{Hom}(a, b)_{TA} \) is the direct sum

\[
\bigoplus_{k=1}^{\infty} \text{Hom}(a, b)^{(k)}_A
\]

Here the the \( R \)-module \( \text{Hom}(a, b)^{(1)}_A \) is simply the morphism set \( \text{Hom}(a, b)_A \).

Composition of morphisms in the tensor category is defined by concatenation of tensors.

Formation of the tensor category defines a functor, \( T \), from the category of \( R \)-moduloids to the category of \( R \)-algebroids. We will abuse notation slightly, and also write \( TA \) to denote the tensor algebroid when \( R \) is already an \( R \)-algebroid. This definition of course ignores the multiplicative structure.

Proposition 3.4 The functor \( T \) is naturally left-adjoint to the forgetful functor \( F \).

Proof: We need a natural \( R \)-linear bijection between the morphism sets \( \text{Hom}(TA, B) \) and \( \text{Hom}(A, FB) \) when \( A \) is an \( R \)-moduloid and \( B \) is an \( R \)-algebroid.

Let \( \alpha: TA \to B \) be a homomorphism of \( R \)-algebroids. The morphism set \( \text{Hom}(a, b)_{TA} \) is the sum

\[
\bigoplus_{k=1}^{\infty} \text{Hom}(a, c_1) \otimes_R \text{Hom}(c_1, c_2) \otimes_R \cdots \otimes \text{Hom}(c_k, b)
\]

We have an induced homomorphism of \( R \)-moduloids, \( G(\alpha): A \to FB \), defined to be \( \alpha \) on the set of objects, and by the restriction \( G(\alpha) = \alpha|_{\text{Hom}(a, b)_A} \) on morphism sets.

Conversely, given an \( R \)-moduloid homomorphism \( \beta: A \to B \), we have an \( R \)-algebroid homomorphism \( H(\beta): TA \to B \), defined to be \( \beta \) on the set of objects, and by the formula

\[
H(\beta)(x_1 \otimes \cdots \otimes x_k) = \beta(x_1) \cdots \beta(x_k)
\]

for each morphism of the form \( x_1 \otimes \cdots \otimes x_k \) in the tensor category.

It is easy to check that the maps \( G \) and \( H \) are \( R \)-linear, natural, and mutually inverse. \( \square \)

Now, let \( A \) be an \( R \)-algebroid. Then there is a canonical homomorphism of \( R \)-moduloids \( \sigma: A \to TA \) defined by mapping each morphism set of the category \( A \) onto the first summand.
Proposition 3.5 Let $\alpha : A \to B$ be a homomorphism of $R$-algebroids. Then there is a unique $R$-algebroid homomorphism $\varphi : TA \to B$ such that $\alpha = \varphi \circ \sigma$.

Proof: We can define the required homomorphism $\varphi : TA \to B$ by writing $\varphi(a) = \alpha(a)$ for each object $a \in Ob(A)$, and

$$\varphi(x_1 \otimes \cdots \otimes x_n) = \alpha(x_n) \cdots \alpha(x_1)$$

for morphisms $x_i \in \text{Hom}(c_i, c_{i+1})_A$. It is easy to see that $\varphi$ is the unique $R$-algebroid homomorphism with the property that $\alpha = \varphi \circ \sigma$. \hfill $\square$

We have a natural homomorphism $\pi : TA \to A$ defined to be the identity on the set of objects, and by the formula

$$\varphi(x_1 \otimes \cdots \otimes x_n) = x_n \cdots x_1$$

for morphisms $x_i \in \text{Hom}(c_i, c_{i+1})$. It follows that there is an $R$-algebroid $JA$ with the same objects as the category $A$, and morphism sets

$$\text{Hom}(a, b)_{JA} = \ker(\pi : \text{Hom}(a, b)_{TA} \to \text{Hom}(a, b)_A)$$

Definition 3.6 A sequence of $R$-algebroids and homomorphisms

$$0 \to I \xrightarrow{j} E \xrightarrow{\pi} B \to 0$$

is called a short exact sequence if the $R$-algebroids $I$, $E$, and $B$ all have the same sets of objects, and for all such object $a$ and $b$ our sequence restricts to a short exact sequence of abelian groups:

$$0 \to \text{Hom}(a, b)_I \to \text{Hom}(a, b)_E \to \text{Hom}(a, b)_B \to 0$$

We call the above short exact sequence $F$-split if there is a homomorphism of $R$-moduloids (but not necessarily $R$-algebroids), $s : B \to E$ such that $j \circ s = 1_C$. We call the homomorphism $s$ an $F$-splitting of the short exact sequence.

Our definitions make it clear that, for an $R$-algebroid $A$, the tensor algebroid fits into a natural short exact sequence

$$0 \to JA \twoheadrightarrow TA \xrightarrow{\pi} A \to 0$$

with natural $F$-splitting $\sigma : A \to TA$.

Definition 3.7 The above short exact sequence is called the universal extension of $A$.

The following result is easy to check.

Proposition 3.8 Let $A$ and $C$ be $R$-algebroids. Then we have a natural $F$-split short exact sequence

$$0 \to (JA) \otimes_R C \to (TA) \otimes_R C \to A \otimes_R C \to 0$$

\hfill $\square$
Theorem 3.9 Let
\[ 0 \rightarrow I \xrightarrow{i} E \xrightarrow{j} A \rightarrow 0 \]
be an \( F \)-split short exact sequence. Then we have a natural homomorphism \( \gamma : JA \rightarrow I \) fitting into a commutative diagram
\[
\begin{array}{ccc}
JA & \rightarrow & TA \\
\downarrow & & \downarrow \\
I & \rightarrow & E \\
\end{array}
\]
\[
\begin{array}{ccc}
& & \rightarrow A \\
\parallel & & \\
I & \rightarrow & E \\
\end{array}
\]
If our \( F \)-splitting is a homomorphism of \( R \)-algebroids, then the homomorphism \( \gamma \) is the zero map.

Proof: Since the exact sequence we are looking at is \( F \)-split, there is an \( R \)-moduloid homomorphism \( s : A \rightarrow E \) such that \( j \circ s = 1_A \). By proposition 3.4, there is a natural \( R \)-algebroid homomorphism \( H(s) : TA \rightarrow E \) fitting into the above diagram.

The homomorphism \( \gamma \) is defined by restriction of the homomorphism \( H(s) \). Now, suppose that the above \( R \)-moduloid homomorphism \( s \) is actually an algebroid homomorphism. Then, by exactness, we have a commutative diagram
\[
\begin{array}{ccc}
JA & \rightarrow & TA \\
\downarrow & & \downarrow \\
I & \rightarrow & E \\
\end{array}
\]
\[
\begin{array}{ccc}
& & \rightarrow A \\
\parallel & & \\
I & \rightarrow & E \\
\end{array}
\]
where the vertical homomorphism on the left is zero, and central vertical homomorphism is the composition \( s \circ \pi \).

Definition 3.10 We call the above homomorphism \( \gamma \) the classifying map of the diagram

4 Algebraic Homotopy and Simplicial Enrichment

For each natural number, \( n \in \mathbb{N} \), write
\[
\mathbb{Z}^\Delta^n = \mathbb{Z}[t_0, \ldots, t_n] / (1 - \sum_{i=0}^n t_i)
\]
In particular, we have \( \mathbb{Z}^\Delta^0 = \mathbb{Z} \) and \( \mathbb{Z}^\Delta^1 = \mathbb{Z}[t] \). The sequence of rings \((\mathbb{Z}^\Delta^n)\) combines with the obvious face and degeneracy maps to form a simplicial ring \( \mathbb{Z}^\Delta \). We refer to [4] for details.
Given an $R$-algebroid $A$, we can also consider $A$ to be a $\mathbb{Z}$-algebroid, and so form tensor products $A \otimes_{\mathbb{Z}} \mathbb{Z}^{\Delta^n}$, and the simplicial $R$-algebroid $A^{\Delta^n}$.

In the case $n = 1$, there are obvious homomorphisms $e_i : A^{\Delta^1} \to A$ defined by the evaluation of a polynomial at a point $i \in \mathbb{Z}$.

**Definition 4.1** Let $f_0, f_1 : A \to B$ be homomorphisms of $R$-algebroids. An elementary homotopy between $f_0$ and $f_1$ is a homomorphism $h : A \to B^{\Delta^1}$ such that $e_0 \circ h = f_0$ and $e_1 \circ h = f_1$.

We call $f_0$ and $f_1$ algebraically homotopic if they are linked by a chain of elementary homotopies. We write $[A, B]$ to denote the set of algebraic homotopy classes of homomorphisms from $A$ to $B$.

Let us view the geometric $n$-simplex $\Delta^n$ as a simplicial set. Given a simplicial set $X$, recall (see for example [1]) that a simplex of $X$ is a simplicial map $f : \Delta^n \to X$. We define the simplex category of $X$, $\Delta \downarrow X$, to be the category in which the objects are the simplices of $X$, and the morphisms are commutative diagrams

$$
\begin{array}{ccc}
\Delta^n & \to & X \\
\downarrow & & \downarrow \\
\Delta^m & \to & X
\end{array}
$$

The simplex category has the feature that the simplicial set $X$ is naturally isomorphic to the direct limit

$$
\lim_{\Delta^n \to X} \Delta^n
$$
taken over the simplex category.

**Definition 4.2** Let $X$ be a simplicial set. Let $A$ be an $R$-algebroid. Then we define $A^X$ to be the direct limit

$$
A^X = \lim_{\Delta^n \to X} A^{\Delta^n}
$$
taken over the simplex category.

The assignment $A^- : X \mapsto A^X$ is a contravariant functor. Let $X$ be a simplicial set with a basepoint, + (in the sense of simplicial sets; see again [1]). Then there is a functorially induced map $A^X \to A = A^+$ arising from the inclusion $+ \hookrightarrow X$. We define

$$
A^{X,+} = \ker(A^X \to A)
$$

Given pointed simplicial sets $X$ and $Y$, it is easy to check that the $R$-algebroids $(A^X)^Y$ and $A^{X \wedge Y}$ are naturally isomorphic. The product $X \wedge Y$ is of course the smash product of pointed simplicial sets. In particular, if $S^m$ and $S^n$ are simplicial spheres, then the $R$-algebroids $(A^{S^m})^{S^n}$ and $A^{S^{m+n}}$ are naturally isomorphic.

Recall from proposition 2.5 that there is a functor $A \mapsto A_H$ from the category of $R$-algebroids to the category of $R$-algebras. Naturality of the relevant constructions immediately gives us the following result.

9
**Proposition 4.3** Let $\mathcal{A}$ be an $R$-algebroid, and let $X$ be a simplicial set. Then the $R$-algebras $(\mathcal{A}_H)^X$ and $(\mathcal{A}^X)_H$ are naturally isomorphic. $\square$

Because of the above proposition, we do not distinguish between the $R$-algebras $(\mathcal{A}_H)^X$ and $(\mathcal{A}^X)_H$, and simplify our notation by writing $\mathcal{A}_H^X$ in both cases.

Now, let $\text{Simp}$ denote the category of simplicial sets. Let $\text{Alg}_R$ denote the category of $R$-algebroids. Then the contravariant functor $\mathcal{A}^-$ can be written as a covariant functor $\mathcal{A}^- : \text{Simp}^{\text{op}} \to \text{Alg}_R$.

Let us call a directed object in the category of $\mathcal{R}$-algebroids and homomorphisms a **directed $\mathcal{R}$-algebroid**. If we write $\text{Alg}_R^{\text{ind}}$ to denote the category of directed $\mathcal{R}$-algebroids, then the above functor has an extension

$$\mathcal{A}^- : \text{Simp}^{\text{op}} \to \text{Alg}_R^{\text{ind}}$$

Given a simplicial set $X$, we can form its subdivision $\text{sd}(X)$; there is a natural simplicial map $h : \text{sd}(X) \to X$. The process of repeated subdivision yields a sequence of simplicial sets, $\text{sd}^n X$:

$$\text{sd}^0 X \xrightarrow{h_X} \text{sd}^1 X \xrightarrow{h_{\text{sd}(X)}} \text{sd}^2 X \leftarrow \cdots$$

**Definition 4.4** Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{R}$-algebroids. Then we define $\text{HOM}(\mathcal{A}, \mathcal{B})$ to be the simplicial set defined by writing

$$\text{HOM}(\mathcal{A}, \mathcal{B})[n] = \lim_{\kappa} \text{Hom}(\mathcal{A}, \mathcal{B}^{\text{sd}^k \Delta^n})$$

The face and degeneracy maps are those inherited from the simplicial $\mathcal{R}$-algebroid $\mathcal{B}^\Delta$.

The following result will by used later on in this article to formulate algebraic $K$-theory in terms of spectra. It is proven in exactly the same way as theorem 3.3.2 in [4].

**Theorem 4.5** Let $\mathcal{A}$ be a simplicial $\mathcal{R}$-algebroid, and let $\mathcal{B}$ be a directed $\mathcal{R}$-algebroid. Then

$$[\mathcal{A}, \mathcal{B}^{S^n,+}] = \pi_n \text{HOM}(\mathcal{A}, \mathcal{B})$$

$\square$

Of course, in the above theorem, we use $S^n$ to denote the simplicial $n$-sphere, and we are using simplicial homotopy groups; see for instance [8] or some other standard reference on simplicial homotopy theory for further details.
5 Path Extensions

Let \( \mathcal{A} \) be an \( R \)-algebroid. Let \( \mathcal{A} \oplus \mathcal{A} \) be the \( R \)-algebroid with the same objects as \( \mathcal{A} \), with morphism sets

\[
\text{Hom}(a, b)_{\mathcal{A} \oplus \mathcal{A}} = \text{Hom}(a, b)_\mathcal{A} \oplus \text{Hom}(a, b)_\mathcal{A}
\]

Write \( \Omega \mathcal{A} = \mathcal{A}^S_{1:+} \). Then we obtain a short exact sequence

\[
0 \rightarrow \Omega \mathcal{A} \rightarrow \mathcal{A}^\Delta^1 (\mathcal{A} \oplus \mathcal{A}) \rightarrow 0
\]

with an \( F \)-splitting \( s: \mathcal{A} \oplus \mathcal{A} \rightarrow \mathcal{A}^\Delta^1 \) given by the formula

\[
s(x, y) = (1 - t)x + ty
\]

If we write \( P\mathcal{A} = \mathcal{A}^\Delta^1_{:+} \), we have a commutative diagram

\[
\begin{array}{ccc}
\Omega \mathcal{A} & \rightarrow & P\mathcal{A} \\
\downarrow & & \downarrow \rho \\
\Omega \mathcal{A} & \rightarrow & \mathcal{A}^\Delta^1 (\mathcal{A} \oplus \mathcal{A}) \\
\downarrow & & \downarrow \\
\mathcal{A} & = & \mathcal{A}
\end{array}
\]

where the homomorphism \( \mathcal{A} \rightarrow \mathcal{A} \oplus \mathcal{A} \) is the inclusion of the second factor, the homomorphism \( \mathcal{A} \oplus \mathcal{A} \rightarrow \mathcal{A} \) is projection onto the first factor, and the map \( \mathcal{A}^\Delta^1 \rightarrow \mathcal{A} \) is the evaluation map \( e_0 \). Further, the complete rows and columns are short exact sequences.

The column on the right has a natural splitting \((1, 0): \mathcal{A} \rightarrow \mathcal{A} \oplus \mathcal{A} \). The second row is the path extension, which has a natural \( F \)-splitting as we observed above. By a diagram chase, it follows that the top row and middle column have natural \( F \)-splittings.

Hence, by theorem 3.9, there is a natural map \( \rho: J\mathcal{A} \rightarrow \Omega \mathcal{A} \).

**Definition 5.1** Let \( f: \mathcal{A} \rightarrow \mathcal{B} \) be a homomorphism of \( R \)-algebroids. Then the **path algebroid** of \( f \) is the unique \( R \)-algebra \( P\mathcal{B} \oplus_{\mathcal{B}} \mathcal{A} \) fitting into a commutative diagram

\[
\begin{array}{ccc}
\Omega \mathcal{B} & \rightarrow & P\mathcal{B} \oplus_{\mathcal{B}} \mathcal{A} \\
\downarrow & & \downarrow \\
\Omega \mathcal{B} & \rightarrow & \mathcal{B}
\end{array}
\]

where the bottom row is the path extension, and the upper row is a short exact sequence.

It is straightforward to check that the path algebroid is well-defined, and that the upper row in the above diagram has a natural \( F \)-splitting. There is therefore a natural map \( \eta(f): J\mathcal{A} \rightarrow \Omega \mathcal{B} \).

We can compose with the map \( i: \Omega \mathcal{A} \rightarrow \mathcal{A}^S_{1} \) to obtain a map \( \eta(f): J\mathcal{A} \rightarrow \mathcal{B}^S_{1} \).

**Definition 5.2** We call the above homomorphism \( \eta(f) \) the **classifying map** of the homomorphism \( f \).
6 The $KK$-theory spectrum

Consider a homomorphism $\alpha: \mathcal{A} \to \mathcal{B}_H$. Then we have a classifying map $\eta(\alpha): J\mathcal{A} \to \mathcal{B}_H^{S^1}$. Given a natural number $m \in \mathbb{N}$, we define the $R$-algebroid $J^m\mathcal{A}$ iteratively, by writing

$$J^0\mathcal{A} = \mathcal{A} \quad J^{k+1}\mathcal{A} = J(J^k\mathcal{A})$$

Given a homomorphism $\alpha: J^{2n}\mathcal{A} \to (\mathcal{B}_H^{S^n})^ {sd \Delta^n}$, we have a structure map

$$\eta(\alpha): J^{2n+1} \to (\mathcal{B}_H^{S^{n+1}})^ {sd \Delta^n}$$

Applying the classifying map construction again to the above, we see that we have a simplicial map $\mathcal{A}$

$$\epsilon: \text{HOM}(J^{2n}\mathcal{A}, \mathcal{B}_H^{S^n}) \to \text{HOM}(J^{2n+2}\mathcal{A}, \mathcal{B}_H^{S^{n+2}}) \cong \Omega \text{HOM}(J^{2n+2}\mathcal{A}, \mathcal{B}_H^{S^{n+1}})$$

**Definition 6.1** We define $\mathbb{K}K(\mathcal{A}, \mathcal{B})$ to be the spectrum with sequence of spaces $\text{HOM}(J^{2n}\mathcal{A}, \mathcal{B}_H^{S^n})$, with structure maps defined as above.

Note that, by proposition 2.3, elements of the space $\text{HOM}(J^{2n}\mathcal{A}, \mathcal{B}_H^{S^n})$ are defined by homomorphisms $\alpha: J^{2n}\mathcal{A} \to (\mathcal{B}_H^{S^n})^ {sd \Delta^n}$.

Recall from [13] that a symmetric spectrum is a spectrum, $E$, equipped with actions of symmetric groups $\Sigma_n \times E_n \to E_n$ that commute with the relevant structure maps. The extra structure means that the smash product, $E \wedge F$ of symmetric spectra $E$ and $F$ can be defined.

**Proposition 6.2** The spectrum $\mathbb{K}K(\mathcal{A}, \mathcal{B})$ is a symmetric spectrum.

**Proof:** There is a canonical action of the permutation group $\Sigma_n$ on the simplicial sphere $S^n \cong S^1 \wedge \cdots \wedge S^1$ defined by permuting the order of the smash product of simplicial circles, and therefore on the space $\text{HOM}(J^{2n}\mathcal{A}, \mathcal{B}_H^{S^n})$.

By construction, the iterated structure map $\epsilon^k: \text{HOM}(J^{2n}\mathcal{A}, \mathcal{B}_H^{S^n}) \to \Omega \text{HOM}(J^{2n+2k}\mathcal{A}, \mathcal{B}_H^{S^{n+k}})$ is $\Sigma_n \times \Sigma_k$-equivariant, and so we have a symmetric spectrum as required. \qed

**Proposition 6.3** Let $\mathcal{B}$ be an $R$-algebroid, and let $k$ and $l$ be natural numbers. Then there is a natural homomorphism $s: J^k(\mathcal{B}^{S^1}) \to (J^l\mathcal{A})^{S^k}$.

**Proof:** The classifying map of the $F$-split exact sequence

$$0 \to (J\mathcal{A})^{S^1} \to (T\mathcal{A})^{S^1} \to \mathcal{A}^{S^1} \to 0$$

is a natural homomorphism $J(\mathcal{A}^{S^1}) \to (J\mathcal{A})^{S^1}$. The homomorphism $s$ is defined by iterating the above construction. \qed
Definition 6.4 Let $A$, $B$, and $C$ be $R$-algebroids. Consider homomorphisms $\alpha: J^{2m}A \rightarrow BS^m_B$ and $\beta: J^{2n}B \rightarrow CS^n_C$. Then we define the product $\alpha \cdot \beta$ to be the composition

$$J^{2m+2n}A \xrightarrow{\alpha} J^{2n}J^{2m}(BS^m_B) \xrightarrow{s} (J^{2n}B)^{S^m_B} \xrightarrow{\beta} CS^{m+n}_C$$

We can of course also define the above in the case of directed $R$-algebroids. This extension of the above definition is needed in the following theorem.

Theorem 6.5 Let $A$, $B$, and $C$ be $R$-algebroids. Then there is a natural map of spectra

$$\mathbb{K}(A, B) \wedge \mathbb{K}(B, C) \rightarrow \mathbb{K}(A, C)$$

defined by the formula

$$\alpha: J^{2m}A \rightarrow (BS^m_B)^{sd} \Delta^k \quad \alpha: J^{2n}B \rightarrow (CS^n_C)^{sd} \Delta^l$$

Further, the above product is associative in the obvious sense. Given homomorphisms $\alpha: A \rightarrow B$ and $\beta: B \rightarrow C$, we have the formula $\alpha \cdot \beta = \beta \circ \alpha$.

Proof: For convenience, we will simply consider homomorphisms of the form $\alpha: J^{2m}A \rightarrow BS^m_B$ and $\beta: J^{2n}B \rightarrow CS^n_C$.

Our construction gives us a natural continuous $S_m \times S_n$-equivariant map $\text{HOM}(J^mA, BS^m_B) \wedge \text{HOM}(J^nB, CS^n_C) \rightarrow \text{HOM}(J^{m+n}A, CS^{m+n}_C)$. Compatibility with the structure maps follows since naturality of the classifying map construction gives us a commutative diagram

$$J^{2m+2n+2}A \xrightarrow{\alpha} J^{2n+2}(BS^m_B) \xrightarrow{s} (J^{2n+2}B)^{S^n_B} \xrightarrow{\eta^s(\beta_B)^{S^n_B}} \Sigma^{m+n+2}CS^{m+n+2}_C$$

We now need to check the statement concerning associativity. Consider homomorphisms

$$\alpha: J^{2m}A \rightarrow BS^m_B \quad \beta: J^{2n}B \rightarrow CS^n_C \quad \gamma: J^{2p}C \rightarrow DS^p_D$$

Then we have a commutative diagram

$$J^{2m+2n+2p}A = J^{2m+2n+2p}A$$

$$\downarrow \quad \downarrow$$

$$J^{2n+2p}(BS^m_B) = J^{2n+2p}(BS^m_B)$$

$$\downarrow \quad \downarrow$$

$$J^{2p}(J^{2n}BS^m_B) \xrightarrow{s} (J^{2n+2p}B)^{S^m_B} \xrightarrow{\eta^s(\beta_B)^{S^m_B}} (J^{2n+2p}B)^{S^m_B}$$

$$\downarrow \quad \downarrow$$

$$J^{2p}(CS^{m+n}_C) \xrightarrow{\eta^s(\beta_B)^{S^n_B}} J^{2p}(CS^{m+n}_C)$$

$$\downarrow \quad \downarrow$$

$$(J^{2p}C)^{S^{m+n}}_C = J^{2p}(CS^{m+n}_C)$$

$$\downarrow \quad \downarrow$$

$$DS^{m+n+p}_D = \Sigma^{m+n+p}DS^{m+n+p}_D$$
But the column on the left is the product \((\alpha \sharp \beta) \sharp \gamma\) and the column on the right is the product \(\alpha \sharp (\beta \sharp \gamma)\) so associativity of the product follows. \(\square\)

**Proposition 6.6** Let \(A, B,\) and \(C\) be \(R\)-algebroids. Then there is a map \(\Delta: \mathbb{K}K(A, B) \to \mathbb{K}K(A \otimes_R C, B \otimes_R C)\). This map is compatible with the product in the sense that we have a commutative diagram

\[
\begin{array}{ccc}
\mathbb{K}K(A, B) \otimes \mathbb{K}K(B, C) & \to & \mathbb{K}K(A, C) \\
\mathbb{K}K(A \otimes_R D, B \otimes_R D) \otimes \mathbb{K}K(B \otimes_R D, C \otimes_R D) & \to & \mathbb{K}K(A \otimes_R D, C \otimes_R D)
\end{array}
\]

where the horizontal maps are defined by the product, and the vertical maps are copies of the map \(\Delta\).

**Proof:** Let \(\alpha: J^{2n}A \to B^{S^n}\) be a homomorphism. Then we have a naturally induced homomorphism \(\alpha \otimes 1: (J^{2n}A) \otimes_R C \to B^{S^n} \otimes_R C\).

There is an obvious natural homomorphism \(\beta: B^{S^n} \otimes_R C \to (B \otimes_R C)^{S^n}\). By proposition 3.8 we have an \(F\)-split short exact sequence

\[
0 \to (JA) \otimes_R C \to (TA) \otimes_R C \to A \otimes_R C \to 0
\]

We thus obtain a natural homomorphism \(\gamma: J(A \otimes_R C) \to (J(A) \otimes_R C)\) as the classifying map of the diagram

\[
\begin{array}{ccc}
A \otimes C & \to & (JA) \otimes_R C \\
\downarrow & & \downarrow \\
0 & \to & (TA) \otimes_R C
\end{array}
\]

We now define the map \(\Delta\) by writing \(\Delta(\alpha) = \beta \circ (\alpha \otimes 1) \circ \gamma^n\). The relevant naturality properties are easy to check. \(\square\)

The following result follows directly from the relevant definitions in the category of symmetric spectra (see [13]) along with the above proposition and theorem.

**Theorem 6.7** Let \(A\) be an \(R\)-algebroid. Then the spectrum \(\mathbb{K}K(A, A)\) is a symmetric ring spectrum.

Let \(B\) be another \(R\)-algebroid. Then the spectrum \(\mathbb{K}K(A, B)\) is a symmetric \(\mathbb{K}K(R, R)\)-module spectrum. \(\square\)

### 7 \(KK\)-theory groups

Note that, by theorem 4.5

\[
\pi_0 \mathbb{K}K(A, B) = \lim_{\to n} [J^{2n}A, B^{S^{2n}}_H]
\]

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and when $B$ is an $R$-algebra, we have a natural isomorphism $B_H \cong B \otimes_R \text{M}_\infty(R)$, where $\text{M}_\infty(R)$ denotes the $R$-algebra of infinite matrices, indexed by $\mathbb{N}$, with entries in the ring $R$.

Thus, if we define

$$KK(A, B) = \pi_0 K A B$$

then in the case where $A$ and $B$ are $R$-algebras, we recover the definition of bivariant algebraic $KK$-theory in [4].

**Definition 7.1** Let $A$ and $B$ be $R$-algebroids. Then we can define the $KK$-theory groups

$$KK_p(A, B) = \pi_p K A B$$

By theorem [4,5] we have natural isomorphisms

$$[A, B^{S^n,+}] \cong \pi_n \text{HOM}(A, B)$$

Hence, by definition of the $KK$-theory spectrum, we know that

$$KK_p(A, B) \cong \lim \pi_n [J^n A, B^{S^n+p}]$$

In section [5] that we defined an $R$-algebroid $\Omega \mathcal{B} = B^{S^n,+}$. Iterating this construction, we see that $\Omega^n \mathcal{B} = B^{S^n,+}$. Hence, when $p \geq 0$ we can write

$$KK_p(A, B) = KK_0(A, \Omega^p \mathcal{B})$$

Similarly, we can write

$$KK_{-p}(A, B) = KK_0(J^p A, B)$$

By definition, elements of the group $KK_0(A, B)$ arise from homomorphisms $\alpha: J^{2n} A \to B^{S^n}_{S^n}$. Given two such homomorphisms $\alpha, \beta: J^{2n} A \to B^{S^n}_{S^n}$, we can define the sum, $\alpha \oplus \beta: J^{2n} A \to B^{S^n}_{S^n}$, by writing

$$(\alpha \oplus \beta)(a) = \alpha(a) \oplus \beta(a)$$

for each object $a \in \text{Ob}(J^n A)$, and

$$(\alpha \oplus \beta)(x) = \begin{pmatrix} \alpha(x) & 0 \\ 0 & \beta(x) \end{pmatrix}$$

for each morphism $x$.

The following result is obvious from the construction of $KK$-theory.

**Proposition 7.2** Let $\alpha: J^{2n} A \to B^{S^n}_{S^n}$ be a homomorphism. Let $[\alpha]$ be the equivalence class defined by the following conditions.

- Let $\alpha, \beta: J^{2n} A \to B^{S^n}_{S^n}$ be algebraically homotopic. Then $[\alpha] = [\beta]$. 

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• Let $\alpha: J^{2n}A \to B^S_{\oplus}$ be a homomorphism, and let $0: A \to B^S_{\oplus}$ be the homomorphism that is zero for each morphism in the category $J^nA$. Then $[\alpha] = [\alpha \oplus 0]$.

• Let $\alpha: J^{2n}A \to B^S_{\oplus}$, and let $\eta(\alpha): J^{2n+2}A \to B^S_{\oplus}$ be the corresponding classifying map. Then $[\eta(\alpha)] = [\alpha]$.

Then the group $KK_0(A, B)$ is the set of equivalence classes of homomorphisms $\alpha: J^nA \to B^S_{\oplus}$. The group operation is defined by the formula $[\alpha \oplus \beta] = [\alpha] + [\beta]$.

At the level of groups, the product is an associative map

$$KK_p(A, B) \otimes KK_q(B, C) \to KK_{p+q}(B, C)$$

Given homomorphisms $\alpha: A \to B$ and $\beta: B \to C$, the product $[\alpha] \cdot [\beta]$ is the equivalence class of the composition, $[\beta \circ \alpha]$.

An essentially abstract argument, as described in section 6.34 of [4] for $R$-algebras rather than $R$-algebroids, yields the following result.

**Theorem 7.3** Let $0 \to A \to B \to C \to 0$ be an $F$-split short exact sequence of $R$-algebroids. Let $D$ be an $R$-algebroid. Then we have natural maps $\partial: KK_p(A, D) \to KK_{p+1}(C, D)$ inducing a long exact sequence of $KK$-theory groups

$$\to KK_p(C, D) \to KK_p(B, D) \to KK_p(A, D) \overset{\beta}{\to} KK_{p+1}(C, D) \to \ldots$$

$\square$

A similar result holds in the other variable; we do not need it here.

**Definition 7.4** We call a homomorphism $\alpha: J^{2n}A \to B^S_{\oplus}$ a $KK$-equivalence if there is an element $[\alpha]^{-1} \in KK_0(B, A)$ such that $[\alpha]^{-1} \cdot [\alpha] = [1_A]$ and $[\alpha] \cdot [\alpha]^{-1} = [1_B]$.

We call two $R$-algebroids $A$ and $B$ $KK$-equivalent if there is a $KK$-equivalence $\alpha: J^{2n}A \to B^S_{\oplus}$.

**Proposition 7.5** Let $\alpha: J^{2n}A \to B^S_{\oplus}$ be a $KK$-equivalence. Then the product with $\alpha$ induces equivalences of spectra

$$KK(B, C) \to KK(A, C) \quad KK(C, A) \to KK(C, B)$$

for every $R$-algebroid $C$. 

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Proof: Let us just look at the map
\[ \alpha^\#: \text{KK}(B, C) \to \text{KK}(A, C) \]
since the other case is almost identical. The map \( \alpha^\# \) induces an isomorphism
\[ \pi_0 \text{KK}(B, C) = KK_0(B, C) \to KK_0(A, C) = \pi_0 \text{KK}(A, C) \]
by definition of the KK-theory group and the term KK-equivalence.

Let \( p \geq 0 \). Replacing the \( R \)-algebroid \( C \) by the \( R \)-algebroid \( \Omega^p C \), we see that the map \( \alpha^\# \) also induces an isomorphism
\[ \pi_p \text{KK}(A, C) = KK_p(B, C) \to KK_p(A, C) = \pi_p \text{KK}(A, C) \]
Thus the map \( \alpha^\# \) is an equivalence of spectra as desired, and we are done.

\[ \square \]

Observe that the category-theoretic concept of natural isomorphism makes sense when we are talking about homomorphisms of unital \( R \)-algebroids.

Lemma 7.6 Let \( \alpha, \beta : A \to B \) be naturally isomorphic homomorphisms of unital \( R \)-algebroids. Then the maps \( \alpha \) and \( \beta \) are simplicially homotopic at the level of KK-theory spectra.

Proof: By definition of the KK-theory spectrum, the space \( \text{KK}(A, B)_0 \) is the simplicial set \( \text{HOM}(A, B_H) \). In this space, the homomorphisms \( \alpha \) and \( \beta \) are the same as the homomorphisms \( \alpha' : A \to B_\oplus \) and \( \beta' : A \to B_\oplus \) defined by writing
\[ \alpha'(a) = \alpha(a) \oplus \beta(a), \quad \beta'(a) = \alpha(a) \oplus \beta(a) \quad a \in \text{Ob}(A) \]
and
\[ \alpha'(x) = \begin{pmatrix} \alpha(x) & 0 \\ 0 & 1 \end{pmatrix}, \quad \beta'(x) = \begin{pmatrix} 1 & 0 \\ 0 & \beta(x) \end{pmatrix} \]
where \( x \in \text{Hom}(a, b)_A \).

Since the homomorphisms \( \alpha \) and \( \beta \) are naturally equivalent, we can find invertible morphisms \( g_a \in \text{Hom}(\alpha(a), \beta(a))_B \) for each object \( a \in \text{Ob}(A) \) such that \( \beta(x)g_a = g_a^{-1}\alpha(x) \) for all \( x \in \text{Hom}(a, b)_A \).

Let
\[ W = \begin{pmatrix} 1 - t^2 & (t^3 - 2t)g_a^{-1} \\ t g_a & 1 - t^2 \end{pmatrix} \]
then the matrix \( W \) is an invertible morphism in the \( R \)-algebroid \( (A \otimes \mathbb{Z}[t])_\oplus \); the inverse is the matrix
\[ W^{-1} = \begin{pmatrix} 1 - t^2 & (2t - t^3)g_a^{-1} \\ -tg_a & 1 - t^2 \end{pmatrix} \]
Further,
\[ e_0(W) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = e_0(W^{-1}) \quad e_1(W) = \begin{pmatrix} 0 & -g_a^{-1} \\ g_a & 0 \end{pmatrix} = -e_1(W^{-1}) \]
Hence
\[ e_0(W \begin{pmatrix} \alpha(x) & 0 \\ 0 & 1 \end{pmatrix} W^{-1}) = \alpha'(x) \]
and
\[ e_1(W \begin{pmatrix} \alpha(x) & 0 \\ 0 & 1 \end{pmatrix} W^{-1}) = \beta'(x) \]

Therefore the homomorphisms \( \alpha' \) and \( \beta' \) are algebraically homotopic. The result now follows by theorem 4.5.

The following result is immediate from the above lemma and proposition 7.5.

**Theorem 7.7** Let \( \mathcal{A} \) and \( \mathcal{A}' \) be equivalent unital \( R \)-algebroids. Let \( \mathcal{B} \) be another \( R \)-algebroid. Then the spectra \( \mathbb{K}K(\mathcal{A}, \mathcal{B}) \) and \( \mathbb{K}K(\mathcal{A}', \mathcal{B}) \) are stably equivalent, and the spectra \( \mathbb{K}K(\mathcal{B}, \mathcal{A}) \) and \( \mathbb{K}K(\mathcal{B}, \mathcal{A}') \) are homotopy-equivalent.

The following result is proved similarly.

**Proposition 7.8** The symmetric spectra \( \mathbb{K}K(\mathcal{A}, \mathcal{B}) \) and \( \mathbb{K}K(\mathcal{A}, \mathcal{B}_0) \) are stably equivalent.

Let \( R \) be a ring where every \( R \)-algebra is central. It is shown in [4] that for an \( R \)-algebra \( A \) there is a natural isomorphism \( KK_n(R, A) \cong KH_n(A) \). The proof depends on certain universal properties of \( KK \)-theory and homotopy \( K \)-theory, and a universal characterisation of algebraic \( KK \)-theory of the type first considered for \( C^* \)-algebras in [11, 25]. The characterisation involves triangulated categories; see [19, 23].

The following result is now immediate.

**Corollary 7.9** Let \( R \) be a ring where every \( R \)-algebra is central. Let \( \mathcal{A} \) be an \( R \)-algebroid that is equivalent to an \( R \)-algebra. Then we have a natural isomorphism
\[ KK_n(R, \mathcal{A}) \cong KH_n(\mathcal{A}) \]

**Definition 7.10** Let \( \mathcal{A} \) be an \( R \)-algebroid. Then we define the homotopy algebraic \( K \)-theory spectrum
\[ \mathbb{KH}(\mathcal{A}) = KK(R, \mathcal{A}) \]

### 8 Modules over Algebroids

The modules we consider in this section were introduced for algebras in [15].
**Definition 8.1** Let $\mathcal{A}$ be an $R$-algebroid. Then a right $\mathcal{A}$-module is an $R$-linear contravariant functor from $\mathcal{A}$ to the category of $R$-modules.

A natural transformation, $T: \mathcal{E} \rightarrow \mathcal{F}$, between two $\mathcal{A}$-modules is called a homomorphism.

We write $\mathcal{L}(\mathcal{A})$ to denote the category of all right $\mathcal{A}$-modules and homomorphisms. The category $\mathcal{L}(\mathcal{A})$ is clearly a unital $R$-algebroid.

**Definition 8.2** Given an object $c \in \text{Ob}(\mathcal{A})$, we define $\text{Hom}(-, c)_\mathcal{A}$ to be the right $\mathcal{A}$-module with spaces $\text{Hom}(-, c)_\mathcal{A}(a) = \text{Hom}(a, c)_\mathcal{A}$; the action of the $R$-algebroid $\mathcal{A}$ is defined by multiplication.

**Definition 8.3** Let $\mathcal{E}$ and $\mathcal{F}$ be right $\mathcal{A}$-modules. Then we define the direct sum, $\mathcal{E} \oplus \mathcal{F}$, to be the right $\mathcal{A}$-module with spaces

$$(\mathcal{E} \oplus \mathcal{F})(a) = \mathcal{E}(a) \oplus \mathcal{F}(a) \quad a \in \text{Ob}(\mathcal{A})$$

We call a right $\mathcal{A}$-module $\mathcal{E}$ finitely generated and projective if there is a right $\mathcal{A}$-module $\mathcal{F}$ such that the direct sum $\mathcal{E} \oplus \mathcal{F}$ is isomorphic to a direct sum of the form

$$\text{Hom}(-, c_1)_\mathcal{A} \oplus \cdots \oplus \text{Hom}(-, c_n)_\mathcal{A}$$

Let us write $\mathcal{L}(\mathcal{A}_{\text{fgp}})$ to denote the category of all finitely-generated projective $\mathcal{A}$-modules and homomorphisms. The following result follows directly from theorem 7.7 and proposition 7.8.

**Proposition 8.4** Let $\mathcal{A}$ and $\mathcal{B}$ be $R$-algebroids. Then there is a natural stable equivalence of spectra

$$\text{KK}(\mathcal{A}, \mathcal{B}) \rightarrow \text{KK}(\mathcal{A}, \mathcal{L}(\mathcal{B}_{\text{fgp}}))$$

**Definition 8.5** Let $\mathcal{A}$ and $\mathcal{B}$ be $R$-algebroids. Then an $(\mathcal{A}, \mathcal{B})$-bimodule, $\mathcal{F}$, is an $R$-algebroid homomorphism $\mathcal{F}: \mathcal{A} \rightarrow \mathcal{L}(\mathcal{B})$. An $(\mathcal{A}, \mathcal{B})$-bimodule $\mathcal{F}$ is termed finitely generated and projective if it is a homomorphism $\mathcal{F}: \mathcal{A} \rightarrow \mathcal{L}(\mathcal{B}_{\text{fgp}})$.

\[\text{Assuming we take all of our right } \mathcal{A}\text{-modules in a given universe, so that the category } \mathcal{L}(\mathcal{A}) \text{ is small.}\]
Let $\mathcal{F}$ be an $(\mathcal{A}, \mathcal{B})$-bimodule. For each object $a \in \text{Ob}(\mathcal{A})$, let us write $\mathcal{F}(-, a)$ to denote the associated right $\mathcal{B}$-module. Then for each object $b \in \text{Ob}(\mathcal{B})$ we have an $R$-module $\mathcal{F}(b, a)$. For each morphism $x \in \text{Hom}(a, a')_{\mathcal{A}}$, we have an induced homomorphism $x: \mathcal{F}(-, a) \rightarrow \mathcal{F}(-, a')$.

Given an $R$-algebroid homomorphism $f: \mathcal{A} \rightarrow \mathcal{B}$, there is an associated $(\mathcal{A}, \mathcal{B})$-bimodule which we will simply label $\mathcal{B}$. The right $\mathcal{B}$-module associated to the object $a \in \text{Ob}(\mathcal{A})$ is $\text{Hom}(-, f(a))_{\mathcal{B}}$. The homomorphisms associated to morphisms in the category $\mathcal{A}$ are defined in the obvious way through the functor $f$.

**Definition 8.6** Let $\mathcal{E}$ be a $\mathcal{A}$-module, and let $\mathcal{F}$ be a $(\mathcal{A}, \mathcal{B})$-bimodule. Let $b \in \text{Ob}(\mathcal{B})$. Then we define the algebraic tensor product $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{F}(b, -)$ to be the $R$-module consisting of all formal linear combinations

$$\lambda_1(\eta_1, \xi_1) + \cdots + \lambda_n(\eta_n, \xi_n)$$

where $\lambda_i \in R$, $\eta_i \in \mathcal{E}(a, \xi_i \in \mathcal{F}(b, a)$, and $a \in \text{Ob}(\mathcal{A})$, modulo the equivalence relation defined by writing

- $(\eta_1 + \eta_2, \xi) \sim (\eta_1, \xi) + (\eta_2, \xi)$
- $(\eta, \xi_1 + \xi_2) \sim (\eta, \xi_1) + (\eta, \xi_2)$
- $(\eta, x\xi) \sim (\eta x, \xi)$ for each morphism $x \in \text{Hom}(a, a')_{\mathcal{A}}$.

Let us write $\eta \otimes \xi$ to denote the equivalence class of the pair $(\eta, \xi)$.

**Definition 8.7** We write $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{F}$ to denote the right $\mathcal{B}$-module defined by associating the $R$-module $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{F}(b, -)$ to an object $b \in \text{Ob}(\mathcal{B})$.

Given a homomorphism of right $\mathcal{A}$-modules, $T: \mathcal{E} \rightarrow \mathcal{E}'$, there is an induced homomorphism $T \otimes 1: \mathcal{E} \otimes_{\mathcal{A}} \mathcal{F} \rightarrow \mathcal{E}' \otimes_{\mathcal{A}} \mathcal{F}$, defined in the obvious way.

Recall that a homomorphism $f: \mathcal{A} \rightarrow \mathcal{B}$ turns the $R$-algebroid $\mathcal{B}$ into an $(\mathcal{A}, \mathcal{B})$-bimodule. Thus, given an $\mathcal{A}$-module $\mathcal{E}$, we can form the tensor product $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{B}$.

We can write $f_* \mathcal{E} = \mathcal{E} \otimes_{\mathcal{A}} \mathcal{B}$. A homomorphism of $\mathcal{A}$-modules, $T: \mathcal{E} \rightarrow \mathcal{E}'$ yields a functorially induced map $f_* T = T \otimes 1: \mathcal{E} \otimes_{\mathcal{A}} \mathcal{B} \rightarrow \mathcal{E}' \otimes_{\mathcal{A}} \mathcal{B}$.

**Theorem 8.8** Let $\mathcal{E}$ be a finitely generated projective right $\mathcal{A}$-module. Then we have a canonical morphism of spectra

$$\mathcal{E} \wedge: \mathbb{K}K(\mathcal{A}, \mathcal{B}) \rightarrow \mathbb{K}K(R, \mathcal{B})$$

This morphism is natural in the variable $\mathcal{B}$ in the obvious sense. Given an $R$-algebroid homomorphism $f: \mathcal{A} \rightarrow \mathcal{A}'$, we have a commutative diagram

$$\begin{array}{ccc}
\mathcal{E} \wedge: & \mathbb{K}K(\mathcal{A}, \mathcal{B}) & \rightarrow \mathbb{K}K(R, \mathcal{B}) \\
\uparrow & & \uparrow \\
\mathcal{E} \otimes_{\mathcal{A}} \mathcal{A}' \wedge: & \mathbb{K}K(\mathcal{A}', \mathcal{B}) & \rightarrow \mathbb{K}K(R, \mathcal{B})
\end{array}$$
Proof: The right $A$-module $E$ defines a homomorphism $R \to \mathcal{L}(A_{\text{fgp}})$ by mapping the one object of $R$, considered as an $R$-algebroid, to the Hilbert module $E$, and the unit 1 to the identity homomorphism on $E$. Thus the Hilbert module $E$ defines an element in the 0-th space of the spectrum $\mathbb{K}(R, \mathcal{L}(A_{\text{fgp}})).$

By proposition [8.4] we have a natural equivalence of symmetric spectra

$$\mathbb{K}(R, \mathcal{L}(A_{\text{fgp}})) \to \mathbb{K}(R, A)$$

By composition of the formal inverse of the above equivalence with the product, we obtain a canonical map

$$E \wedge: \mathbb{K}(A, B) \to \mathbb{K}(R, B)$$

Naturality in the variable $B$ follows by associativity of the product. To prove that the stated diagram is commutative, we need to prove that the diagram

$$E \wedge: \mathbb{K}(\mathcal{L}(A_{\text{fgp}}), B) \to \mathbb{K}(R, B)$$

$$E \otimes_A A' \wedge: \mathbb{K}(\mathcal{L}(A'_{\text{fgp}}), B) \to \mathbb{K}(R, B)$$

is commutative.

The homomorphism $f_*: \mathcal{L}(A_{\text{fgp}}) \to \mathcal{L}(A'_{\text{fgp}})$ is defined by mapping the right $A$-module $E$ to the right $A$-module $E \otimes_A A'$, and the homomorphism $T: E \to \mathcal{F}$ to the homomorphism $T \otimes 1: E \otimes_A A' \to \mathcal{F} \otimes_A A'$.

Now, the map at the bottom of the diagram is defined by the product with the homomorphism $R \to \mathcal{L}(A'_{\text{fgp}})$ defined by the right $A'$-module $E \otimes_A A'$. The composition of the vertical map on the left and the map at the top of the diagram is defined by the product with the composition of the homomorphism $f_*$ and the homomorphism $C \to \mathcal{L}(A)$ defined by the right $A'$-module $E$.

By construction, these two homomorphisms are the same, and we are done. □

We are actually going to use a slight generalisation of the above theorem; the proof is essentially the same as the above.

Definition 8.9 Let $A$ be an $R$-algebroid. Let $a \in Ob(A)$. Then we define the path-component of $a$, $A_{|\text{Ob}(a)}$, to be the full subcategory of $A$ containing all objects $b \in Ob(A)$ such that $\text{Hom}(a, b)_A \neq 0$.

We call a right $A$-module $E$ almost finitely generated and projective if the restriction to each path-component is finitely generated and projective.

Theorem 8.10 Let $E$ be an almost finitely generated projective $A$-module. Then we have a canonical morphism of spectra

$$E \wedge: \mathbb{K}(A, B) \to \mathbb{K}(R, B)$$
This morphism is natural in the variable $B$ in the obvious sense. Given an $R$-algebroid homomorphism $f : A \to A'$, we have a commutative diagram
\[
\begin{array}{c}
\mathcal{E}^\wedge : \KK(A, B) \to \KK(R, B) \\
\uparrow \Downarrow \\
\mathcal{E} \otimes_A A' \wedge : \KK(A', B) \to \KK(R, B)
\end{array}
\]

\[ \square \]

9 Equivariant $KK$-theory

Let $\mathcal{G}$ be a discrete groupoid, and let $R$ be a ring. We can regard $\mathcal{G}$ as a small category in which every morphism is invertible.

**Definition 9.1** A $\mathcal{G}$-algebra over $R$ is a functor from the category $\mathcal{G}$ to the category of $R$-algebras and homomorphisms.

Thus, if $A$ is a $\mathcal{G}$-algebra, then for each object $a \in \text{Ob}(\mathcal{G})$ we have an algebra $A(a)$. A morphism $g \in \text{Hom}(a, b)_\mathcal{G}$ induces a homomorphism $g : A(a) \to A(b)$.

We can regard an ordinary algebra $C$ as a $\mathcal{G}$-algebra by writing $C(a) = C$ for each object $a \in \text{Ob}(\mathcal{G})$ and saying that each morphism in the groupoid $\mathcal{G}$ acts as the identity map.

**Definition 9.2** A $\mathcal{G}$-module over $R$ is a functor from the groupoid $\mathcal{G}$ to the category of $R$-bimodules and $R$-linear maps.

The notation used for $\mathcal{G}$-modules is the same as that used for $\mathcal{G}$-algebras. There is a forgetful functor, $F$, from the category of $\mathcal{G}$-algebras to the category of $\mathcal{G}$-modules.

An equivariant map between $\mathcal{G}$-algebras or $\mathcal{G}$-modules is the same thing as a natural transformation.

**Definition 9.3** Let $A$ and $B$ be $\mathcal{G}$-algebras. Then we define the tensor product $A \otimes_R B$ to be the $\mathcal{G}$-algebra where $(A \otimes_R B)(a) = A(a) \otimes_R B(a)$ for each object $a \in \text{Ob}(\mathcal{G})$, and the $\mathcal{G}$-action is defined by writing $g(x \otimes y) = g(x) \otimes g(y)$ whenever $g \in \text{Hom}(a, b)_\mathcal{G}$, $x \in A(a)$, and $y \in B(a)$.

We define the direct sum $A \oplus B$ to be the $\mathcal{G}$-algebra where $(A \oplus B)(a)$ is the direct sum $A(a) \oplus B(a)$ for each object $a \in \text{Ob}(\mathcal{G})$ and the $\mathcal{G}$-action is defined by writing $g(x \oplus y) = g(x) \oplus g(y)$ whenever $g \in \text{Hom}(a, b)_\mathcal{G}$, $x \in A(a)$, and $y \in B(a)$.

We similarly define tensor products and direct sums of $\mathcal{G}$-modules.

Recall from section 4 that we have a sequence of rings, $(\mathbb{Z}^{\Delta^n})$; these rings combine to form a simplicial ring $\mathbb{Z}^{\Delta}$. If we equip each such ring with the trivial $\mathcal{G}$-action, and $A$ is a $\mathcal{G}$-algebra, we can form the tensor product $A \otimes_\mathbb{Z} \Delta^{\alpha}$.

Just as in section 4, given a $\mathcal{G}$-algebra $A$ and a simplicial set $X$, we can form the $\mathcal{G}$-algebra $A^X$. If the simplicial set $X$ has a basepoint $+$, we also form the $\mathcal{G}$-algebra $A(X, +) = \ker(A^X \to A)$.
**Definition 9.4** Let \( f_0, f_1 : A \to B \) be equivariant maps of \( G \)-algebras. An equivariant map \( h : A \to B \) such that \( e_0 \circ h = f_0 \) and \( e_1 \circ h = f_1 \) is called an *elementary homotopy* between the maps \( f_0 \) and \( f_1 \).

We call \( f_0 \) and \( f_1 \) *algebraically homotopic* if they can be linked by a chain of elementary homotopies. We write \([A, B]_G\) to denote the set of algebraic homotopy classes.

Let us call a directed object in the category of \( G \)-algebras and equivariant maps a *directed \( G \)-algebra*. As in the non-equivariant case, simplicial subdivision gives us a directed \( G \)-algebra \( A^\text{sd}^n \).

**Definition 9.5** Let \( A \) and \( B \) be \( G \)-algebras. Then we define \( \text{HOM}_G(A, B) \) to be the simplicial set defined by writing

\[
\text{HOM}_G(A, B)[n] = \lim_k \text{Hom}(A, B^\text{sd}k)\Delta^n
\]

The face and degeneracy maps are those inherited from the simplicial \( G \)-algebra \( B\Delta \).

The following result is proved in the same way as theorem 3.3.2 in [4].

**Theorem 9.6** Let \( A \) be a \( G \)-algebra, and let \( B \) be a directed \( G \)-algebra. Then

\[
[A, B^S]_G = \pi_n \text{HOM}_G(A, B)
\]

\( \square \)

A *short exact sequence* of \( G \)-algebras is a sequence of \( G \)-algebras and equivariant maps

\[
0 \to A \overset{i}{\to} B \overset{j}{\to} C \to 0
\]

such that the sequence

\[
0 \to A(a) \overset{i}{\to} B(a) \overset{j}{\to} C(a) \to 0
\]

is exact for each object \( a \in \text{Ob}(G) \). A splitting of a short exact sequence is defined in the obvious way.

We call a short exact sequence

\[
0 \to A \to B \to C \to 0
\]

an *\( F \)-split* if there is an equivariant map of \( G \)-modules \( s : C \to B \) such that \( j \circ s = 1_C \). Such a map \( s \) is called an \( F \)-splitting.

**Definition 9.7** Let \( A \) be a \( G \)-module. Then we define the *tensor \( G \)-algebra* \( A^{\otimes k} \) to be the tensor product of \( A \) with itself \( k \) times. We define the *equivariant tensor algebra*, \( TA \), to be the iterated direct sum

\[
TA = \bigoplus_{k=1}^\infty A^{\otimes k}
\]

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Composition of elements in the tensor $\mathcal{G}$-algebra is defined by concatenation of tensors. Formation of the tensor $\mathcal{G}$-algebra defines a functor, $T$ from the category of $\mathcal{G}$-modules to the category of $\mathcal{G}$-algebras. The following result is proved in the same way as proposition 3.4.

**Proposition 9.8** The functor $T$ is naturally adjoint to the forgetful functor $F$.

Given a $\mathcal{G}$-algebra $A$, there is a canonical equivariant map $\sigma: A \rightarrow TA$ of $\mathcal{G}$-modules defined by mapping each morphism set of the $\mathcal{G}$-algebra $A$ onto the first summand. As we should by now expect, there is an associated universal property.

**Proposition 9.9** Let $\alpha: A \rightarrow B$ be an equivariant map between $\mathcal{G}$-algebras. Then there is a unique homomorphism $\varphi: TA \rightarrow B$ such that $\alpha = \varphi \circ \sigma$.

We can thus define a $\mathcal{G}$-algebra $JA$ by writing

$$JA(a) = \ker \pi: TA(a) \rightarrow A(a)$$

for each object $a \in Ob(\mathcal{G})$. The $\mathcal{G}$-action is inherited from the tensor $\mathcal{G}$-algebra. There is a natural short exact sequence

$$0 \rightarrow JA \rightarrow TA \rightarrow A \rightarrow 0$$

with $F$-splitting $\sigma: A \rightarrow TA$. The following result is proved in the same way as theorem 3.9.

**Theorem 9.10** Let

$$0 \rightarrow I \rightarrow E \rightarrow A \rightarrow 0$$

be an $F$-split short exact sequence. Then we have a natural homomorphism $\gamma: JA \rightarrow I$ fitting into a commutative diagram

$$\begin{array}{ccc}
0 & \rightarrow & JA \\
\downarrow & & \downarrow \\
0 & \rightarrow & I
\end{array} \quad \begin{array}{ccc}
& \rightarrow & TA \\
& & \| \\
& \rightarrow & E
\end{array} \quad \begin{array}{ccc}
& & A \\
& & \| \\
& & 0
\end{array}$$

As before, the homomorphism $\gamma$ is called the *classifying map* of the short exact sequence.

The last construction we need to define an equivariant $KK$-theory spectrum is an equivariant version of the path extension. Analogously to the algebroid case, we can write $\Omega A = A^{S^1,+}$ and obtain an $F$-split short exact sequence

$$0 \rightarrow \Omega A \rightarrow A^{\Delta^1} \rightarrow A \oplus A \rightarrow 0$$

As before, a diagram chase enables us to construct a natural classifying map $\rho: JA \rightarrow \Omega A$. 

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Definition 9.11 Let $f: A \rightarrow B$ be an equivariant map of $G$-algebras. Then the path algebra of $f$ is the unique $G$-algebra $PB \oplus_B A$ fitting into a commutative diagram

$$\begin{array}{ccc}
\Omega B & \rightarrow & PB \oplus_B A \\
\| & & \downarrow \\
\Omega B & \rightarrow & PB \\
\end{array}$$

where the bottom row is the path extension, and the upper row is a short exact sequence.

It is straightforward to check that the path algebra is well-defined, and that the upper row in the above diagram has a natural $F$-splitting. There is therefore a natural map $\eta(f): JA \rightarrow \Omega B$.

We can compose with the map $i: \Omega A \rightarrow A S^1$ to obtain a map $\eta(f): JA \rightarrow B S^1$.

Definition 9.12 We call the above equivariant map $\eta(f)$ the classifying map of the equivariant map $f$.

Given a $G$-algebra $A$, let us write $M_\infty(A)$ to denote the direct limit of the $G$-algebras $M_n(A) = A \otimes_R M_n(R)$ under the inclusions

$$x \mapsto \left( \begin{array}{c} x \\ 0 \\ 0 \end{array} \right)$$

Consider an equivariant map $\alpha: A \rightarrow M_\infty(B^{S^n})$, where $A$ and $B$ are $G$-algebras. Then by the above construction, there is an induced classifying map $\eta(\alpha): JA \rightarrow M_\infty(B^{S^{n+1}})$.

Definition 9.13 We define $K_\infty(G)(A, B)$ to be the symmetric spectrum with sequence of spaces $\text{HOM}_G(J^2_n A, M_\infty(B^{S^n}))$. The structure map $\epsilon: \text{HOM}_G(J^{2n} A, M_\infty(B^{S^n})) \rightarrow \text{HOM}_G(J^{2n+2} A, M_\infty(B^{S^{n+1}})) \cong \text{HOM}_G(J^{2n+2} A, M_\infty(B^{S^{n+2}}))$ is defined by applying the above classifying map construction twice, that is to say writing $\epsilon(\alpha) = \eta(\eta(\alpha))$ whenever $\alpha \in \text{HOM}(J^{2n} A, M_\infty(B^{S^{n+1}}))$.

The action of the permutation group $S_n$ is induced by its canonical action on the simplicial sphere $S^n$.

The product is constructed as for $R$-algebroids, in the non-equivariant case.

Definition 9.14 Let $A, B,$ and $C$ be $G$-algebras. Let $\alpha \in \text{HOM}_G(J^{2m} A, M_\infty(B^{S^m}))$ and $\beta \in \text{HOM}_G(J^{2n} B, M_\infty(C^{S^n}))$. Then we define the product $\alpha \sharp \beta$ to be the composition

$$J^{2m+2n} A \overset{f^{2m} \alpha}{\rightarrow} J^{2n} (M_\infty(B^{S^m})) \rightarrow (J^{2n} M_\infty(B))^{S^m} \overset{\beta}{\rightarrow} M_\infty(C)^{S^{m+n}}$$

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Theorem 9.15 Let $A$, $B$, and $C$ be $G$-algebras. Then there is a natural map of spectra
\[ KK_G(A, B) \wedge KK_G(B, C) \to KK_G(A, C) \]
defined by the formula
\[ \alpha \wedge \beta \mapsto \alpha \sharp \beta \]
where $\alpha \in HOM_G(J^m \cdot A, M_\infty(B^{S^n}))$, $\beta \in HOM_G(J^n \cdot B, M_\infty(C^{S^n}))$.

Further, the above product is associative in the usual sense. Given equivariant maps $\alpha: A \to B$ and $\beta: B \to C$, we have the formula $\alpha \sharp \beta = \beta \circ \alpha$.

Corollary 9.16 Let $A$ be a $G$-algebra. Then the spectrum $KK_G(A, A)$ is a symmetric ring spectrum.

Let $B$ be another $G$-algebra. Then the spectrum $KK_G(A, B)$ is a symmetric $KK_G(R, R)$-module spectrum.

Let $\theta: G \to \mathcal{H}$ be a functor between groupoids, and let $A$ be an $\mathcal{H}$-algebra. Abusing notation, we can also regard $A$ as a $G$-algebra; we write $A(a) = A(\theta(a))$ for each object $a \in Ob(G)$, and define a homomorphism $g = \theta(g): A(\theta(a)) \to A(\theta(b))$ for each morphism $g \in Hom(a, b)_G$.

There is an induced map $\theta^*: KK_H(A, B) \to KK_G(A, B)$ defined by the observation that any $\mathcal{H}$-equivariant map is also $G$-equivariant.

Definition 9.17 We call the map $\theta^*: KK_H(A, B) \to KK_G(A, B)$ the restriction map.

Proposition 9.18 Let $\theta: G \to \mathcal{H}$ be a functor between groupoids, and let $A$ and $B$ be $\mathcal{H}$-algebras. Then the restriction map $\theta^*: KK_H(A, B) \to KK_G(A, B)$ is compatible with the product in the sense that we have a commutative diagram
\[
\begin{array}{ccc}
KK_H(A, B) \wedge KK_H(B, C) & \to & KK_H(A, C) \\
\downarrow & & \downarrow \\
KK_G(A, B) \wedge KK_G(B, C) & \to & KK_G(A, C)
\end{array}
\]
where the horizontal map is defined by the product and the vertical maps are restriction maps.

Proof: Abusing notation, we can also regard the $\mathcal{H}$-algebra $A$ as a $G$-algebra; we write $A(a) = A(\theta(a))$ for each object $a \in Ob(G)$, and define a homomorphism $g = \theta(g): A(\theta(a)) \to A(\theta(b))$ for each morphism $g \in Hom(a, b)_G$.

We similarly regard the $\mathcal{H}$-algebra $B$ as a $G$-algebra. An $\mathcal{H}$-equivariant map $\alpha: J^n A \to M_\infty(B^{S^n})$ is also $G$-equivariant. We use this construction to define our map $\theta^*: KK_H(A, B) \to KK_G(A, B)$.

The result is now straightforward to check. \qed
Given a \(G\)-algebra \(A\), we define the \(\text{convolution algebroid}, \ A_G\), to be the algebroid with the same set of objects as the groupoid \(G\), and morphism sets

\[
\text{Hom}(a,b)_{A_G} = \{ \sum_{i=1}^{m} x_i g_i \mid x_i \in A(b), g_i \in \text{Hom}(a,b)_G, \ m \in \mathbb{N} \}
\]

Composition of morphisms is defined by the formula

\[
\left( \sum_{i=1}^{m} x_i g_i \right) \left( \sum_{j=1}^{n} y_j h_j \right) = \sum_{i,j=1}^{m,n} x_i g_i (y_j) g_i h_j
\]

**Theorem 9.19** Let \(G\) be a groupoid, and let A and B be \(G\)-algebras. There is a map

\[
D : \mathbb{K} \mathbb{K}_G(A, B) \to \mathbb{K} \mathbb{K}(A_G, B_G)
\]

which is compatible with the product in the sense that we have a commutative diagram

\[
\begin{array}{ccc}
\mathbb{K} \mathbb{K}_G(A, B) \otimes \mathbb{K} \mathbb{K}_G(B, C) & \to & \mathbb{K} \mathbb{K}(A, C) \\
\| & & \| \\
\mathbb{K} \mathbb{K}(A_G, B_G) \otimes \mathbb{K} \mathbb{K}(B_G, C_G) & \to & \mathbb{K} \mathbb{K}(A_G, C_G)
\end{array}
\]

where the horizontal maps are defined by the product.

**Proof:** Let \(\alpha : J^{2n}A \to M_\infty(B^{S^n})\) be an equivariant map. Then we have a functorially induced homomorphism \(\alpha_* : (J^{2n}A)_G \to M_\infty(B^{S^n})_G\) defined by writing

\[
\alpha_* \left( \sum_{i=1}^{n} x_i g_i \right) = \sum_{i=1}^{n} \alpha(x_i) g_i
\]

We have a natural homomorphism \(\gamma : J(A_G) \to (JA)_G\) defined as the classifying map of the diagram

\[
\begin{array}{cccc}
A_G & & & A_G \\
0 \to (JA)_G \to (TA)_G \to A_G \to 0
\end{array}
\]

For any \(G\)-algebra \(C\), we can regard the \(G\)-algebra \(M_\infty(C)\) as the tensor product \(C \otimes_R M_\infty(R)\). It follows that there is an obvious homomorphism \(\beta : M_\infty(B^{S^n})_G \to M_\infty(B_G)^{S^n}\).

We thus have a map \(D : \mathbb{K} \mathbb{K}_G(A, B) \to \mathbb{K} \mathbb{K}(A_G, B_G)\) defined by writing \(D(\alpha) = \beta \circ \alpha_* \circ \gamma^{2n}\). The relevant naturality properties are easy to check. \(\square\)

**Corollary 9.20** Let \(G\) be a discrete groupoid, and let A and B be \(G\)-algebras. Then the spectrum \(\mathbb{K} \mathbb{K}_G(A_G, B_G)\) is a symmetric \(\mathbb{K} \mathbb{K}_G(R, R)\)-module spectrum. \(\square\)
Let $\mathcal{G}$ be a discrete groupoid, and let $A$ and $B$ be $\mathcal{G}$-algebras. Then we can define groups

$$KK_p^\mathcal{G}(A, B) := \pi_pKK^\mathcal{G}(A, B)$$

The following result is proved in the same way as theorem 7.3.

**Theorem 9.21** Let

$$0 \to A \to B \to C \to 0$$

be an $F$-split short exact sequence of $\mathcal{G}$-algebras. Let $D$ be an $R$-algebroid. Then we have natural maps $\partial : KK_p^\mathcal{G}(A, D) \to KK_{p+1}^\mathcal{G}(C, D)$ inducing a long exact sequence of $KK$-theory groups

$$\cdots \to KK_p^\mathcal{G}(C, D) \to KK_p^\mathcal{G}(B, D) \to KK_p^\mathcal{G}(A, D) \to$$

2

**Theorem 9.22** Let $\theta : \mathcal{G} \to \mathcal{H}$ be an equivalence of discrete groupoids. Let $A$ and $B$ be $\mathcal{H}$-algebras. Then the restriction map $\theta^* : KK^\mathcal{H}(A, B) \to KK^\mathcal{G}(A, B)$ is an isomorphism of spectra.

**Proof:** Since the functor $\theta$ is an equivalence, there is a functor $\phi : \mathcal{H} \to \mathcal{G}$ along with natural isomorphisms $G : \phi \circ \theta \to 1_\mathcal{G}$ and $H : \theta \circ \phi \to 1_\mathcal{H}$.

Thus, for each object $a \in Ob(\mathcal{G})$, there is an isomorphism $H_a \in Hom(\phi \theta(a), a)_\mathcal{G}$. Let $\alpha : J^{2n}A \to M_\infty(B^{S^n})$ be an $\mathcal{H}$-equivariant map. Then the map $\alpha$ can be defined in terms of the restriction $\phi^* \theta^* \alpha : J^{2n}A \to M_\infty(B^{S^n})$ by the formula

$$\alpha(x) = \phi^* \theta^* \alpha(H_a^{-1}xH_a) \quad x \in A(a)$$

Thus the equivariant map $\alpha$ is determined by the restriction $\phi^* \theta^* \alpha$. The natural isomorphism $H$ therefore induces a isomorphism of spectra $H_* : KK^\mathcal{H}(A, B) \to KK^\mathcal{G}(A, B)$ such that $H_* \circ \phi^* \circ \theta^* = 1_{KK^\mathcal{H}(A, B)}$. There is similarly a isomorphism $G_* : KK^\mathcal{G}(A, B) \to KK^\mathcal{G}(A, B)$ such that $G_* \circ \theta^* \circ \phi^* = 1_{KK^\mathcal{G}(A, B)}$.

It follows that the map $\theta^*$ is a isomorphism, and we are done. □

10 **Assembly**

Given a functor, $E$, from the category of $G$-CW-complexes to the category of spectra, we call $E$ $G$-homotopy-invariant if it takes $G$-homotopy-equivalent equivariant maps of $G$-spaces to maps of spectra that induce the same maps between stable homotopy groups.

We call the functor $E$ $G$-excisive if it is $G$-homotopy-invariant, and the collection of functors $X \mapsto \pi_*(E(X)$ forms a $G$-equivariant homology theory.

---

3See for example chapter 20 of [16] for the relevant definitions.
Definition 10.1 Let $G$ be a discrete group. We define the **classifying space for proper actions**, $EG$, to be the $G$-CW-complex with the following properties:

- For each point $x \in X$ the isotropy group $$ G_x = \{ g \in G \mid xg = x \} $$ is finite.
- For a given subgroup $H \leq G$ the fixed point set $EG^H$ is equivariantly contractible if $H$ is finite, and empty otherwise.

The classifying space $EG$ always exists, and is unique up to $G$-homotopy-equivalence; see [2, 6].

The following two results from [6] are the main abstract results on assembly maps we need in this article.

Theorem 10.2 Let $G$ be a discrete group, and let $\mathbb{E}$ be a $G$-homotopy invariant functor from the category of $G$-CW-complexes to the category of spectra. Then there is a $G$-excisive functor $\mathbb{E}'$ and a natural transformation $\alpha: \mathbb{E}' \to \mathbb{E}$ such that the map $$ \alpha: \mathbb{E}'(G/H) \to \mathbb{E}(G/H) $$ is a stable equivalence whenever $H$ is a finite subgroup of $G$.

Further, the pair $(\mathbb{E}', \alpha)$ is unique up to stable equivalence.

Definition 10.3 Let $G$ be a discrete group. Then we define the **orbit category**, $Or(G)$, to be the category in which the objects are the $G$-spaces $G/H$, where $H$ is a subgroup of $G$, and the morphisms are $G$-equivariant maps.

An $Or(G)$-**spectrum** is a functor from the category $Or(G)$ to the category of symmetric spectra.

Theorem 10.4 Let $\mathbb{F}$ be an $Or(G)$-spectrum. Then there is a $G$-excisive functor, $\mathbb{F}'$, from the category of $G$-CW-complexes to the category of spectra such that $\mathbb{F}'(G/H) = \mathbb{F}(G/H)$ whenever $H$ is a subgroup of $G$.

Further, given a functor $\mathbb{F}$ from the category of $G$-CW-complexes to the category of spectra, there is a natural transformation $$ \beta: (\mathbb{F}|_{Or(G)})' \to \mathbb{F} $$ such that the map $$ \beta: (\mathbb{F}|_{Or(G)})'(G/H) \to \mathbb{F}(G/H) $$ is a stable equivalence whenever $H$ is a subgroup of the group $G$.

The constant map $c: EG \to +$ induces a map $c_*: \mathbb{E}'(EG) \to \mathbb{E}(+)$. This map is called the **assembly map**. The corresponding **isomorphism conjecture** is the assertion that this assembly map is a stable equivalence.

---

4As a convention, if we mention a $G$-space, we assume that the group $G$ acts on the right.
**Definition 10.5** Consider a group $G$, and a $G$-space $X$. Then we define the transport groupoid, $X$, to be the groupoid in which the set of objects is the space $X$, considered as a discrete set, and we have morphism sets

$$\text{Hom}(x, y)_X = \{g \in G \mid xg = y\}$$

Composition of morphisms in the transport groupoid is defined by the group operation. There is a faithful functor $i: X \to G$ defined by the inclusion of each morphism set in the group.

Given a ring $R$, a group $G$, and a $G$-algebra $A$ over $R$, let $X$ be a $G$-CW-complex. Then (through the functor $i$), the $G$-algebra $A$ can also be considered an $X$-algebra, and there is a homotopy-invariant functor, $E$, to the category of spectra, defined by writing

$$E(X) = \text{KH}(AX)$$

By theorem 10.2, there is an associated $G$-excisive functor $E'$, and an assembly map

$$\alpha: E'(X) \to \text{KH}(AX)$$

such that the map

$$\alpha: E'(G/H) \to E(G/H)$$

is a stable equivalence whenever $H$ is finite.

**Definition 10.6** The composition of the above map $\beta$ with the map $i_*: \text{KH}(AX) \to \text{KH}(AG)$ induced by the faithful functor $i: X \to G$ is called the $KH$-assembly map for the group $G$ over the ring $R$ with coefficients in the $G$-algebra $A$.

We say that the group $G$ satisfies the $KH$-isomorphism conjecture over $R$ with coefficients in the $G$-algebra $A$ if the assembly map

$$\beta: E(EG) \to K(AG)$$

is a stable equivalence.

The $KH$-assembly map is a variant of the Farrell-Jones assembly map. It was first defined and examined in [1], where in particular its relationship to the Farrell-Jones assembly map in algebraic $K$-theory is analysed.

The following result can be deduced directly from the above definition.

**Theorem 10.7** Consider the $Or(G)$-spectrum

$$E(G/H) = \text{KH}(G/H) = \text{KH}(AG/H)$$

---

5Actually, the Farrell-Jones conjecture and the corresponding $KH$-isomorphism conjecture are usually formulated in terms of virtually cyclic groups rather than finite groups. But by remark 7.4 in [1], the above formulation of the $KH$-isomorphism conjecture is equivalent to the original.
and let \( E' \) be the associated excisive functor.

Let \( X \) be a path-connected space, and let \( c : X \to + \) be the constant map. Then up to stable equivalence the induced map

\[
c_* : E'(X) \to E'(+) \]

is the \( KH \)-assembly map.

11 Algebraic \( KK \)-theory and homology

Let \( G \) be a discrete group, and let \( X \) be a right \( G \)-simplicial complex (or just a \( G \)-complex for short). Let us term \( X \) \( G \)-compact if the quotient \( X/G \) is a finite simplicial complex. Any \( G \)-complex is a direct limit of its \( G \)-compact subcomplexes.

Given a ring \( R \), the right action of \( G \) on the space \( X \) induces a left-action of \( G \) on the simplicial algebroid \( R_X \).

**Definition 11.1** Let \( A \) be a \( G \)-algebra, and let \( X \) be a \( G \)-complex. Then we define the equivariant algebraic \( K \)-homology spectrum of \( X \) with coefficients in \( A \) to be the direct limit

\[
K^G_{\text{hom}}(X; A) = \lim_{\substack{\longrightarrow \\ K \subseteq X \\ G \text{-compact}}} KK_G(R^K, A)
\]

The associated equivariant algebraic \( K \)-homology groups are defined by the formula

\[
K^G_n(X; A) = \pi_n K^G_{\text{hom}}(X; A)
\]

**Theorem 11.2** The functor \( X \mapsto K^G_{\text{hom}}(X; A) \) is \( G \)-homotopy-invariant and excisive. For the one-point space, \( + \), we have \( K^G_{\text{hom}}(+; A) = KK_G(R, A) \).

**Proof:** Suppose that \( X \) and \( Y \) are \( G \)-compact simplicial \( G \)-complexes. Let \( f, g : X \to Y \) be equivariant simplicial maps. Suppose that there is an elementary equivariant simplicial homotopy, \( F : X \times \Delta^1 \to Y \), between \( f \) and \( g \). Then we have an induced equivariant homomorphism \( F^*: R^Y \to R^X \otimes_R R^\Delta^1 = R^X \otimes_Z \mathbb{Z}[t] \) such that \( e_0 F^* = f^* \) and \( e_1 F^* = g^* \).

By construction of equivariant \( KK \)-theory, algebraically \( G \)-homotopic maps \( \alpha, \beta : B \to B' \) induce homotopic maps \( \alpha^*, \beta^* : \mathbb{KK}_G(B', A) \to \mathbb{KK}_G(B, A) \). It follows that the induced maps \( f_*, g_* : \mathbb{KK}_G(X; A) \to \mathbb{KK}_G(Y; A) \) are homotopic.

More generally, two \( G \)-homotopic equivariant simplicial maps \( f, g : X \to Y \) can be linked by a finite chain of elementary homotopies, and the above argument again shows that the induced maps \( f_*, g_* : \mathbb{KK}_G(X; A) \to \mathbb{KK}_G(Y; A) \) are homotopic. Generalising to non-compact \( G \)-complexes and taking direct limits, it follows that the functor \( X \mapsto K^G_{\text{hom}}(X; A) \) is \( G \)-homotopy-invariant.

Let \( X \) and \( C \) be \( G \)-compact simplicial \( G \)-complexes, and suppose we have a subcomplex \( B \subseteq X \) and an equivariant map \( f : B \to C \). Let \( i : B \hookrightarrow X \) be the inclusion map, and let \( F : X \to X \cup_B C \) and \( I : C \to X \cup_B C \) be the maps...
associated to the push-out $X \cup_B C$. Then by functoriality, we have an induced pullback diagram

\[
\begin{array}{ccc}
R^C & \rightarrow & R^B \\
\uparrow & & \uparrow \\
R^{X \cup_B C} & \rightarrow & R^X
\end{array}
\]

It is easy to check that we have a short exact sequence

\[0 \rightarrow R^{X \cup_B C}(F^* I^*) \rightarrow R^X \oplus R^C \rightarrow R^B \rightarrow 0\]

The map $i : B \rightarrow X$ is an inclusion of a subcomplex. It follows that we have an induced inclusion $i_* : R^B \rightarrow R^X$. This induced inclusion is an equivariant map, but not an equivariant homomorphism. Hence the above short exact sequence has an $F$-splitting $(i_*, 0)$.

Therefore, by theorem 9.21 we have natural maps $\partial : K^G_n(X \cup_B C; A) \rightarrow K^G_{n-1}(B; A)$ such that we have a long exact sequence

\[\rightarrow K^G_n(B; A) \xrightarrow{(i_* - f_*)} K^G_n(X; A) \oplus K^G_n(C; A) \xrightarrow{F_* + I_*} K^G_n(X \cup_B C; A) \xrightarrow{\partial} K^G_{n-1}(A) \rightarrow \]

Similar exact sequences can be seen to exist in the non-compact case by taking direct limits.

To check the final axiom required of a $G$-homology theory, let $\{X_i | i \in I\}$ be a family of $G$-compact simplicial $G$-complexes. Let $j_i : X_i \rightarrow \bigoplus_{i \in I} X_i$ be the canonical inclusion. Then by definition of equivariant $K$-homology as a direct limit, the map

\[\bigoplus_{i \in I} (j_i)_* : \bigoplus_{i \in I} K^G_n(X_i; A) \rightarrow K^G_n((\bigoplus_{i \in I} X_i); A)\]

is an isomorphism for all $n \in \mathbb{Z}$.

By definition, for the one-point space, $+$, we have $K^G_{\text{hom}}(+; A) = KK_G(R, A)$.

\[\square\]

12 The Index Map

In [22], the author used natural constructions of analytic $KK$-theory spectra to prove that the Baum-Connes assembly map fits into the general assembly machinery. The $KH$-assembly map is defined using the general machinery; the constructions in this section prove that the map can be described using algebraic $KK$-theory. The methods are somewhat similar to those of [22].

Let $G$ be a discrete group, and let $R$ be a ring. Consider a $G$-algebra $A$, over $R$, and a $G$-compact complex $K$. According to theorem 9.19 we have a natural map of spectra

\[D : KK_G(R^K, A) \rightarrow KK(R^K G, AG)\]

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The algebra $R^K G$ is itself a finitely generated projective $R^K G$-module; let us label this module $E_K$. Then by theorem [X] we have an induced morphism

$$E_K\wedge: \mathbb{K}K(R^K G, AG) \rightarrow \mathbb{K}H(AG)$$

Let $X = E_G$. Then $X$ can be viewed as a $G$-simplicial complex. Combining the above two maps and taking the direct limit over $G$-compact subcomplexes, we obtain a map

$$\beta: \mathbb{K}K_{hom}(X; A) \rightarrow \mathbb{K}H(AG)$$

We call this map the index map, by analogy with the map appearing in the Baum-Connes conjecture (see [2]).

Observe that by definition of the space $E_G$, we can take the space $E_G$ to be a single point if the group $G$ is finite. In this case, the above map is simply the composition

$$\mathbb{K}K G(R, A) \rightarrow \mathbb{K}K(RG, AG) \rightarrow \mathbb{K}K(R, AG) \approx \mathbb{K}H(AG)$$

Now, for a ring $R$, let $M_\infty(R)$ be the $R$-algebra of all infinite matrices over $R$, indexed by $\mathbb{N}$, where almost all entries are zero. Given an $R$-algebra $A$, let $\kappa: A \rightarrow A \otimes_R M_\infty(R)$ be the homomorphism defined by the formula $\kappa(a) = a \otimes P$, where $P \in M_\infty(R)$ is the infinite matrix with 1 at the top left corner, and every other entry zero. By construction of algebraic $KK$-theory, given $R$-algebras $A$ and $B$ the induced map

$$\kappa_+: KK_p(A, B) \rightarrow KK_p(A, B \otimes_R M_\infty(R))$$

is an isomorphism.

Let $G$ be a finite group. Let $M_G(R)$ be the $R$-algebra of matrices with entries in $R$ indexed by the group $G$. Then there is certainly a natural isomorphism

$$M_\infty(R) \cong M_\infty(R) \otimes_R M_G(R)$$

It follows that, given $G$-algebras $A$ and $B$, we have a natural map $\kappa: B \rightarrow B \otimes_R M_G(R)$ inducing isomorphisms

$$\kappa_+: KK_p^G(A, B) \rightarrow KK_p^G(A, B \otimes_R M_G(R))$$

at the level of $KK$-theory groups.

**Lemma 12.1** Let $G$ be a finite group, and let $A$ be a $G$-algebra over a ring $R$. Then there is a natural injective homomorphism $\sigma: AG \rightarrow A \otimes_R M_G(R)$, where the image is the set of all elements of the $R$-algebra $A \otimes_R M_G(R)$ that are fixed by the group $G$.

**Proof:** Let $V_G$ be the $R$-module consisting of all maps from $G$ to the ring $R$. Then $End(V_G) = M_G(R)$. Note that elements of the tensor product $V_G \otimes_R A$ can be viewed as maps $s: G \rightarrow A$. Define homomorphism

$$\rho: G \rightarrow End(V_G \otimes_R A) \quad \pi: A \rightarrow End(V_G \otimes_R A)$$
by the formulae
\[ \rho(g)s(g_1) = s(gg_1) \quad \pi(a)s(g_1) = \pi(g_1(a))s(g_1) \]
respectively, where \( s: G \rightarrow A \) is an element of the module \( V_G \otimes_R A \), \( g, g_1 \in G \), and \( a \in A \).

Then we have an associated injective homomorphism \( \sigma: AG \rightarrow \text{End}(V_G \otimes_R A) \) defined by the formula
\[ \sigma \left( \sum_{i=1}^{m} a_i g_i \right) = \sum_{i=1}^{m} \pi(a_i) \rho(g_i) \]

Now \( \text{End}(V_G \otimes_R A) = A \otimes_R M_G(R) \). It is easy to check that the image is the fixed point set. \( \square \)

The proof of the following result is now adapted from the corresponding result on the Baum-Connes conjecture, where it is sometimes called the Green-Julg theorem. See for example [9] for an elementary account.

**Theorem 12.2** The index map is a stable equivalence for finite groups.

**Proof:** Let \( G \) be a finite group, and let \( A \) be a \( G \)-algebra over the ring \( R \). As we remarked above, we have a natural isomorphism \( \kappa_*: KK^G_p(R, A) \rightarrow KK^G_p(R, A \otimes_R M_G(R)) \). Let \( \sigma_*: \mathbb{K}^G_p(R, AG) \rightarrow KK^G_p(R, (A \otimes_R M_G(R)) \) be induced by the isomorphism in the above lemma.

We have an obvious canonical map
\[ KH_p(AG) = KK^G_p(R, AG) \rightarrow KK^G_p(R, AG) \]
and we can define a homomorphism \( \gamma: KH_p(AG) \rightarrow KK^G_p(R, A) \) fitting into a commutative diagram
\[
\begin{array}{ccc}
KK^G_p(R, AG) & \xrightarrow{\sigma_*} & KK^G_p(R, A \otimes_R M_G(R)) \\
\uparrow & & \uparrow \\
KH_p(AG) & \xrightarrow{\gamma} & KK^G_p(R, A)
\end{array}
\]

We claim that \( \gamma \) is an inverse to the map \( \beta: \mathbb{K}^G_p(R, A) \rightarrow \mathbb{K}(R, AG) \) at the level of groups, thus proving the result. Looking at suspensions, it suffices to prove the case when \( p = 0 \).

Note that, by construction of the algebraic \( KK \)-theory groups, an arbitrary \( KK \)-theory class \( [\alpha] \in KK^0_p(B, C) \) is represented by a homomorphism \( \alpha: J^{2n}B \rightarrow C^{S^n} \otimes_R M_\infty(R) \).

Hence, consider a homomorphism \( \alpha: J^{2n}R \rightarrow A^{S^n} \otimes_R M_\infty(R) \). Then we have a commutative diagram
\[
\begin{array}{ccc}
J^{2n}R & \rightarrow & J^{2n}RG \\
\downarrow & & \downarrow \\
J^{2n}R \otimes_R M_G(R) & \xrightarrow{\alpha} & A^{S^n} \otimes_R M_\infty(R)
\end{array}
\]
where the vertical maps are versions of the map \( \sigma \) in the above lemma.

Now, at the level of \( KK \)-theory groups, the class \( \beta \circ \gamma [\alpha] \) is the composition of the top maps in the above diagram and the vertical map on the right.

On the other hand, the composition of the first map on the top and the first map on the left is simply the stabilisation map \( \kappa \). Hence \([\alpha] = [\alpha \otimes 1] = \beta \circ \gamma [\alpha]\).

We have shown that \( \beta \circ \gamma = 1_{KH(AG)} \).

Conversely, consider an equivariant homomorphism \( \alpha : J^2n R \to (AG)^{S^n} \otimes_R M_\infty(R) \). Then the composition with the map \( \sigma \) defines a class in the \( KK \)-theory group \( KK_G(R, B) \).

The image \( \beta[\alpha] \) is defined by the composition

\[
J^{2n} R \to J^{2n}(RG) \overset{\alpha}{\to} ((AR)^{S^n} G)G
\]

We can form the diagram

\[
\begin{array}{ccc}
J^{2n} RG & \overset{\alpha}{\to} & ((AR)^{S^n} G)G \\
\uparrow & & \uparrow \\
J^{2n} R & \overset{\alpha}{\to} & (AR)^{S^n} G = (AR)^{S^n} G
\end{array}
\]

The left square of the diagram commutes. The right square of the diagram does not commute. The issue is that the middle vertical map gives us a copy of the group \( G \) on the right of the relevant expression, whereas the vertical map on the right gives us a copy of the ring \( M_R(G) \) in the centre of the expression.

If we apply the homomorphism \( s : (A \otimes_R M_R(G))G \to A \otimes_R M_R(G)^{S^n} G \), we see that the square on the right commutes modulo the isomorphism \( s : M_G(R) \otimes_R M_G(R) \to M_G(R) \otimes_R M_G(R) \) defined by writing \( s(x \otimes y) = y \otimes x \).

The map \( s \) is clearly naturally isomorphic to the identity map. It follows by lemma 7.6 that the right square in the above diagram commutes at the level of \( KK \)-theory groups.

Now the composite of the map on the left and the maps on the top row of our diagram are the composition give us the class \( \gamma \circ \beta [\alpha] \). Thus \( \gamma \circ \beta [\alpha] = [\alpha] \).

We see that the homomorphism \( \gamma \) is the inverse of the homomorphism \( \beta \), and we are done.

Given an equivariant map of \( G \)-complexes \( f : X \to Y \), there is an induced functor \( f_* : \mathcal{X} \to \mathcal{Y} \) between the transport groupoids. There is an obvious faithful functor \( i : \mathcal{X} \to G \). If \( A \) is a \( G \)-algebra, it can therefore also be regarded as an \( \mathcal{X} \)-algebra.

Now, let \( K \) be a \( G \)-compact subcomplex. Then we have an induced restriction map

\[
i^* : KK_G(R^K, A) \to KK_{\mathcal{X}}(R^K, A)
\]

By theorem 9.19 there is a natural map

\[
D : KK_{\mathcal{X}}(R^K, A) \to KK(R^K \mathcal{X}, A\mathcal{X})
\]

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Definition 12.3 Let \( x \in X \). Then we write \( \mathcal{E}_K(x) \) to denote the set of collections
\[
\{ \eta_y \in \text{Hom}(x, y)_{R^KX} \mid y \in X \}
\]
such that the formula
\[
\eta_y g = \eta_z
\]
holds for all elements \( g \in G \) such that \( yg = z \).

The assignment \( x \mapsto \mathcal{E}_K(x) \) is an \( R^KX \)-module, which we again label \( \mathcal{E}_K \). The \( R^KX \)-action is defined by composition of morphisms.

Since the action of the group \( G \) on the space \( X \) is not necessarily transitive, the module \( R^KX \) is not necessarily finitely generated and projective. However, the restriction, \( \mathcal{E}_{\text{Or}(z)} \), to the path-component \( (R^KX)_{\text{Or}(z)} \) is isomorphic to the module \( \text{Hom}(-, x)_{R^KX|_{\text{Or}(z)}} \). Thus, the module \( \mathcal{E}_K \) is almost finitely-generated and projective, and we have a canonical morphism
\[
\mathcal{E}_K^\land : \mathbb{K}K(R^KX, AX) \to \mathbb{K}H(AX)
\]

We can compose with the map
\[
D : \mathbb{K}K(R^K, A) \to \mathbb{K}K(R^KX, AX)
\]
to form the composite
\[
\gamma_K : \mathbb{K}K(R^K, A) \to \mathbb{K}(AX)
\]
and take direct limits to obtain a map
\[
\gamma : \mathbb{K}^G_{\text{hom}}(X; A) \to \mathbb{K}(AX)
\]

Proposition 12.4 Let \( f : (X, K) \to (Y, L) \) be a map of pairs of \( G \)-simplicial complexes, where the subcomplexes \( K \) and \( L \) are \( G \)-compact. Let
\[
f^* : R^LX \to R^KX \quad f_* : R^KX \to R^LY
\]
be the obvious induced maps. Then there is an almost finitely generated projective \( R^KX \)-module, \( \theta \), such that \( f_* \theta = \mathcal{E}_L \) and \( f^* \theta = \mathcal{E}_K \).

Proof: Let \( x \in X \). Let \( \theta(x) \) denote the set of collections
\[
\{ \eta_y \in \text{Hom}(x, y)_{R^KX} \mid y \in X \}
\]
such that the formula
\[
\eta_y g = \eta_z
\]
is satisfied for all elements \( g \in G \) such that \( yg = z \).

The assignment \( x \mapsto \theta(x) \) forms an \( R^KX \)-module, \( \theta \). The \( C_0(K)X \)-action is defined by composition of morphisms.

The equations \( f_*(\theta) = \mathcal{E}_L \) and \( f^* \theta = \mathcal{E}_K \) are easily checked. \( \square \)
Lemma 12.5 The map $\gamma: K^G_{\text{hom}}(X; A) \to \mathbb{K}(A \overline{X})$ is natural for proper $G$-complexes.

Proof: Let $f: (X, K) \to (Y, L)$ be a map of pairs of $G$-simplicial complexes, where the subcomplexes $K$ and $L$ are $G$-compact. Then by the above proposition, and naturality of the product and descent map in algebraic $KK$-theory, we have a commutative diagram.

\[
\begin{array}{ccc}
\mathbb{K}(R^K, A) & \to & \mathbb{K}(R^K X, A \overline{X}) \\
\downarrow & & \downarrow \\
\mathbb{K}G(R^K, A) & \to & \mathbb{K}(R^K X, A \overline{X}) \\
\downarrow & & \downarrow \\
\mathbb{K}(R^L, A) & \to & \mathbb{K}(R^L X, A \overline{X}) \\
\downarrow & & \downarrow \\
\mathbb{K}G(R^L, A) & \to & \mathbb{K}(R^L X, A \overline{Y}) \\
\downarrow & & \downarrow \\
\mathbb{K}(R^L Y, A \overline{Y}) & \to & \mathbb{K}(R^L Y, A \overline{Y}) \\
\end{array}
\]

Taking direct limits, the desired result follows. \qed

Lemma 12.6 The composite map $i_\ast \gamma: K^G_{\text{hom}}(X; A) \to \mathbb{K}(A G)$ is the index map.

Proof: Let $K$ be a $G$-compact subcomplex of $X$. The naturality properties of the various descent maps and products give us a commutative diagram

\[
\begin{array}{ccc}
\mathbb{K}G(R^K, A) & \to & \mathbb{K}(R^K G, A G) \\
\downarrow & & \downarrow \\
\mathbb{K}(R^K X, A G) & \to & \mathbb{K}(R^K X, A \overline{X}) \\
\end{array}
\]

Here the top row is the index map, and the composite of the bottom row and the vertical map on the left is the map $\gamma$. Taking direct limits, the desired result follows. \qed

Lemma 12.7 Let $H$ be a finite subgroup of $G$. Then the map $\gamma: K^G_{\text{hom}}(G/H; A) \to \mathbb{K}(A G/H)$ is a stable equivalence of spectra.
Proof: Let $i: H \hookrightarrow G$ be the inclusion isomorphism. Then the group $H$ and groupoid $G/H$ are equivalent. By theorem 9.22 and the naturality of restriction maps, the map $\gamma$ is equivalent to the composition

$$\mathbb{K}K_G(R^{G/H}, A) \xrightarrow{i^*} \mathbb{K}K_H(R^{G/H}, A) \xrightarrow{D} \mathbb{K}K(R^{G/H}H, AH) \xrightarrow{E_G/H} \mathbb{K}K(H)$$

Let $j: R \to R^{G/H}$ be induced by the constant map $G/H \to +$. Then we have a commutative diagram

$$\begin{array}{ccc}
\mathbb{K}K_G(R^{G/H}, A) & \xrightarrow{i^*} & \mathbb{K}K_H(R^{G/H}, A) \\
\downarrow & & \downarrow \\
\mathbb{K}K_H(R^{G/H}, A) & \xrightarrow{j^*} & \mathbb{K}K(R, A) \\
\downarrow & & \downarrow \\
\mathbb{K}K_H(R^{G/H}H, AH) & \xrightarrow{j^*} & \mathbb{K}K_H(RH, AH) \\
\downarrow & & \downarrow \\
\mathbb{K}(AH) & = & \mathbb{K}(AH)
\end{array}$$

A straightforward calculation tells us that the composite $j^*i^*: \mathbb{K}K_G(R^{G/H}, A) \to \mathbb{K}K_H(R, A)$ is a stable equivalence of spectra. The composite map on the right, $\beta: \mathbb{K}K_H(R, A) \to \mathbb{K}(AH)$ is the index map. Since the group $H$ is finite, the space $+$ is a model for the classifying space $E(H, E\mathbb{G})$.

But the index map is a stable equivalence for finite groups by theorem 12.2, so we are done. □

By theorem 11.2 the equivariant $K$-homology functor $\mathbb{K}K^G_{\hom}(\cdot; A)$ is $G$-excisive. We can therefore use the above three lemmas to apply theorem 10.2 to the study of the index map; we immediately obtain the following result.

**Theorem 12.8** Let $\mathbb{E}'$ be a $G$-excisive functor from the category of proper $G$-CW-complexes to the category of symmetric spectra. Suppose we have a natural transformation $\alpha: \mathbb{E}'(X) \to \mathbb{K}(A\overline{X})$ such that the map

$$\alpha: \mathbb{E}'(G/H) \to \mathbb{K}(\overline{AG/H})$$

is a stable equivalence for every finite subgroup, $H$, of the group $G$.

Then, up to stable equivalence, the composite $\alpha i_*: \mathbb{E}'(X) \to \mathbb{K}(AG)$ is the map $\beta$. □

By definition of the $KH$-assembly map, the following therefore holds.

**Corollary 12.9** The index map is the $KH$-assembly map. □

Now, the $KH$-isomorphism conjecture holds for finite groups with any coefficients. It follows from more general results in [11] that the conjecture also holds for the integers.
**Theorem 12.10** Let $G$ be a finitely generated abelian group. Then the $KH$-isomorphism conjecture holds for $G$ with any coefficients.

**Proof:** By the fundamental theorem of abelian groups, we have an isomorphism

$$ G \cong \mathbb{Z}^q \oplus \mathbb{Z}/p_1 \oplus \cdots \oplus \mathbb{Z}/p_k $$

where the $p_i$ are prime numbers. We have classifying spaces for proper actions

$$ E\mathbb{Z} = \mathbb{R} \quad E\mathbb{Z}/p_i = +. $$

Thus the group $G$ has classifying space $\mathbb{R}^q$. The copies of the group $\mathbb{Z}$ act by translation, and the copies of finite groups act trivially.

It follows that we have a commutative diagram

$$
\begin{array}{ccc}
K_n^G(EG; A) & \cong & K_n^\mathbb{Z}(\mathbb{R}; A)^q \oplus K_n^\mathbb{Z}(+; A) \oplus \cdots \oplus K_n^\mathbb{Z}(+; A) \\
\downarrow & & \downarrow \\
KH_n^G(AG) & \cong & KH_n(\mathbb{Z})^q \oplus KH_n(\mathbb{Z}/p_1) \oplus \cdots \oplus KH_n(\mathbb{Z}/p_k)
\end{array}
$$

where the vertical arrows are copy of the assembly map at the level of groups.

We know that the $KH$-isomorphism conjecture holds for finite groups and for the integers. The direct sum of assembly maps on the right is thus an isomorphism.

It follows that the map on the left is also an isomorphism, and we are done.

$\square$

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