THREE SOLUTIONS TO A STEKLOV PROBLEM INVOLVING
THE WEIGHTED $p(\cdot)$-LAPLACIAN

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Abstract. This paper is concerned with a nonlinear Steklov boundary-value
problem involving weighted $p(\cdot)$-Laplacian. Using the Ricceri’s variational
principle, we obtain the existence of at least three weak solutions in double
weighted variable exponent Sobolev space.

1. Introduction

The purpose of the present paper is to study the following Steklov problem of
the type

$$
\begin{aligned}
\text{div} \left( a(x) |\nabla u|^{p(x)-2} \nabla u \right) &= b(x) |u|^{p(x)-2} u, \quad x \in \Omega \\
a(x) |\nabla u|^{p(x)-2} \frac{\partial u}{\partial \nu} &= \lambda f(x, u), \quad x \in \partial \Omega,
\end{aligned}
$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a bounded smooth domain, $\frac{\partial u}{\partial \nu}$ is the outer unit normal
derivative on $\partial \Omega$, $\lambda > 0$ is a real number, $p$ is a continuous function on $\Omega$, i.e.
$p \in C(\Omega)$ with $\inf_{y \in \Omega} p(y) > N$, the function $f : \partial \Omega \times \mathbb{R} \to \mathbb{R}$ will be specified later,
a(x) and b(x) are weight functions.

Nonlinear and variational problems involving $p(x)$-Laplacian operator has at-
tracted great attention in recent years. Because several physical problems, such
as electrorheological fluids, elastic mechanics, stationary thermo-rheological vis-
cous flows of non-Newtonian fluids, exponential growth and image processing, can
be modeled by such kind of equations, see [13], [21], [22]. In recent years $p(x)$-
Laplacian equations have attracted great attention, see [6], [10], [11], [12], [18], [22].
Moreover, Steklov problems involving $p(x)$-Laplacian operator have been studied
by many authors, see [2], [3], [4], [7], [8], [9], [14]. In 2008, Deng [9] consid-
ered the eigenvalue of $p(x)$-Laplacian Steklov problem and proved the existence of infinitely
many eigenvalue sequences. Moreover, Mostofa et al. [4] obtained the existence
and multiplicity of solutions of the nonlinear Steklov boundary-value problem using
Ricceri’s result in weighted variable exponent Sobolev spaces.

In this paper, we define double weighted variable exponent Sobolev spaces and
give some basic properties of these spaces. We also obtain more general results
than above mentioned references, especially [4], under some suitable conditions.
Finally, we show that the problem (1.1) has at least three weak solutions due to
the approach of Ricceri [20].

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problem, Ricceri’s variational principle.
2. Notation and Preliminaries

In this section, we give some definitions and basic informations about weighted variable Lebesgue and Sobolev spaces to find out the solution of the problem (1.1). A normed space \((X, \| \cdot \|_X)\) is called a Banach function space (shortly BF-space), if Banach space \((X, \| \cdot \|_X)\) is continuously embedded into \(L^1_{\text{loc}}(\Omega)\), briefly \(X \hookrightarrow L^1_{\text{loc}}(\Omega)\), i.e. for any compact subset \(K \subset \Omega\) there is some constant \(c_K > 0\) such that \(\|f|_K\|_{L^1(\Omega)} \leq c_K \|f\|_X\) for every \(f \in X\). Moreover, a normed space \(X\) is compactly embedded in a normed space \(Y\), briefly \(X \hookrightarrow Y\), if \(X \hookrightarrow Y\) and the identity operator \(I : X \rightarrow Y\) is compact, equivalently, \(I\) maps every bounded sequence \((x_i)_{i \in \mathbb{N}}\) into a sequence \((I(x_i))_{i \in \mathbb{N}}\) that contains a subsequence converging in \(Y\). Suppose that \(X\) and \(Y\) are two Banach spaces and \(X\) is reflexive. Then \(I : X \rightarrow Y\) is a compact operator if and only if \(I\) maps weakly convergent sequences in \(X\) onto convergent sequences in \(Y\). More details can be found in [1]. Suppose that \(\Omega\) is a bounded open domain of \(\mathbb{R}^N\) with a smooth boundary \(\partial \Omega\) and \(p \in C_+ (\overline{\Omega})\), where

\[
C_+ (\overline{\Omega}) = \left\{ p \in C (\overline{\Omega}) : \inf_{x \in \overline{\Omega}} p(x) > 1 \right\}.
\]

For any \(p \in C_+ (\overline{\Omega})\), we denote

\[
1 < p^- = \inf_{x \in \Omega} p(x) \leq p^+ = \sup_{x \in \Omega} p(x) < \infty.
\]

Let \(p \in C_+ (\overline{\Omega})\). The variable exponent Lebesgue space \(L^{p(\cdot)}(\Omega)\) consists of all measurable functions \(u\) such that \(\varrho_{p(\cdot)}(u) < \infty\), equipped with the Luxemburg norm

\[
\|u\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \varrho_{p(\cdot)} \left( \frac{u}{\lambda} \right) \leq 1 \right\},
\]

where

\[
\varrho_{p(\cdot)}(u) = \int_{\Omega} |u(x)|^{p(x)} \, dx.
\]

The space \(L^{p(\cdot)}(\Omega)\) is a Banach space with respect to \(\| \cdot \|_{p(\cdot)}\). If \(p(\cdot) = p\) is a constant function, then the norm \(\| \cdot \|_{p(\cdot)}\) coincides with the usual Lebesgue norm \(\| \cdot \|_p\), see [10]. A measurable and locally integrable function \(a : \Omega \rightarrow (0, \infty)\) is called a weight function. Define the weighted variable exponent Lebesgue space by

\[
L^{p(\cdot)}_a(\Omega) = \left\{ u : \Omega \rightarrow \mathbb{R} \text{ measurable and } \int_{\Omega} |u(x)|^{p(x)} a(x) \, dx < +\infty \right\}
\]

with the Luxemburg norm

\[
\|u\|_{p(\cdot), a} = \inf \left\{ \tau > 0 : \varrho_{p(\cdot), a} \left( \frac{u}{\tau} \right) \leq 1 \right\},
\]

where

\[
\varrho_{p(\cdot), a}(u) = \int_{\Omega} |u(x)|^{p(x)} a(x) \, dx.
\]
The space \( L^p_a(\Omega) \) is a Banach space with respect to \( \| \cdot \|_{p(\cdot),a} \). Moreover, \( u \in L^p_a(\Omega) \) if and only if \( \| u \|_{p(\cdot),a} = \| u^{1/p(\cdot)} \|_{p(\cdot)} < \infty \). It is known that the relationships between \( \theta_{p(\cdot),a} \) and \( \| \cdot \|_{p(\cdot),a} \) as

\[
\min \left\{ \theta_{p(\cdot),a}(u)^{1/p(\cdot)} \right\} \leq \| u \|_{p(\cdot),a} \leq \max \left\{ \theta_{p(\cdot),a}(u)^{1/p(\cdot)} \right\}
\]

and

\[
\min \left\{ \| u \|_{p(\cdot),a}^{-p(\cdot)}, \| u \|_{p(\cdot),a}^{p(\cdot)} \right\} \leq \theta_{p(\cdot),a}(u) \leq \max \left\{ \| u \|_{p(\cdot),a}^{-p(\cdot)}, \| u \|_{p(\cdot),a}^{p(\cdot)} \right\}
\]

are satisfied. Also, if \( 0 < C_1 \leq a(x) \) for all \( x \in \Omega \), then we have \( L^p_a(\Omega) \hookrightarrow L^p(\Omega) \), since one easily sees that

\[
C_1 \int_{\Omega} |u(x)|^{p(x)} \, dx \leq \int_{\Omega} |u(x)|^{p(x)} a(x) \, dx
\]

and \( C_1 \| u \|_{p(\cdot)} \leq \| u \|_{p(\cdot),a} \). Moreover, the dual space of \( L^p_a(\Omega) \) is \( L^q(\Omega) \), where \( \frac{1}{p(\cdot)} + \frac{1}{q(\cdot)} = 1 \) and \( a^∗ = a^{1-q(\cdot)} = a^{-\frac{1}{p(\cdot)-1}} \). Let \( a : \Omega \rightarrow (0, \infty) \). In addition, the space \( L^p_a(\partial \Omega) \) can be defined by

\[
L^p_a(\partial \Omega) = \left\{ u \mid u : \partial \Omega \rightarrow \mathbb{R} \text{ measurable and } \int_{\partial \Omega} |u(x)|^{p(x)} a(x) \, d\sigma < +\infty \right\}
\]

equipped with the Luxemburg norm, where \( d\sigma \) is the measure on the boundary. Then the space \( L^p_a(\partial \Omega) \) is a Banach space with respect to \( \| \cdot \|_{p(\cdot),a} \). If \( a \in L^\infty(\Omega) \), then we get \( L^p_a = L^p(\cdot) \).

**Theorem 1.** (see [4]) If \( a^{-\frac{1}{p(\cdot)-1}} \in L^1_{\text{loc}}(\Omega) \), then \( L^p_a(\Omega) \hookrightarrow L^1_{\text{loc}}(\Omega) \hookrightarrow D'(\Omega) \), that is, every function in \( L^p_a(\Omega) \) has distributional (weak) derivative, where \( D'(\Omega) \) is distribution space.

**Remark 1.** If \( a^{-\frac{1}{p(\cdot)-1}} \notin L^1_{\text{loc}}(\Omega) \), then the embedding \( L^p_a(\Omega) \hookrightarrow L^1_{\text{loc}}(\Omega) \) need not hold.

**Definition 1.** Let \( a^{-\frac{1}{p(\cdot)-1}} \in L^1_{\text{loc}}(\Omega) \). We set the weighted variable exponent Sobolev space \( W^{k,p(\cdot)}_a(\Omega) \) by

\[
W^{k,p(\cdot)}_a(\Omega) = \left\{ u \in L^p_a(\Omega) : D^\alpha u \in L^p_a(\Omega), 0 \leq |\alpha| \leq k \right\}
\]

equipped with the norm

\[
\| u \|_{k,p(\cdot),a} = \sum_{0 \leq |\alpha| \leq k} \| D^\alpha u \|_{p(\cdot),a}
\]

where \( \alpha \in \mathbb{N}^N_0 \) is a multi-index, \( |\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_N \) and \( D^\alpha = \frac{\partial^{(|\alpha|)}}{\partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}} \).

It is known that \( W^{k,p(\cdot)}_a(\Omega) \) is a reflexive Banach space. In particular, the space \( W^{1,p(\cdot)}_a(\Omega) \) is defined by

\[
W^{1,p(\cdot)}_a(\Omega) = \left\{ u \in L^p_a(\Omega) : |\nabla u| \in L^p_a(\Omega) \right\}.
\]
The function \( \varrho_{1,p,(.)} : W_{a}^{1,p,(.)}(\Omega) \to [0, \infty) \) is shown as \( \varrho_{1,p,(.)}(u) = \varrho_{p,(.)}(u) + \varrho_{p,(.)}(\nabla u) \). Also, the norm \( \|u\|_{1,p,(.)} = \|u\|_{p,(.)} + \|\nabla u\|_{p,(.)} \) makes the space \( W_{a}^{1,p,(.)}(\Omega) \) a Banach space. 

Let \( a^{-\frac{1}{p-1}} \in L_{loc}^{1}(\Omega) \) and \( b^{-\frac{1}{p-1}} \in L_{loc}^{1}(\Omega) \). The double weighted variable exponent Sobolev space \( W_{a,b}^{1,p,(.)}(\Omega) \) is defined by 

\[
W_{a,b}^{1,p,(.)}(\Omega) = \left\{ u \in L_{b}^{p,(.)}(\Omega) : \nabla u \in L_{a}^{p,(.)}(\Omega) \right\}
\]

equipped with the norm 

\[
\|u\|_{1,p,(.),a,b} = \|\nabla u\|_{p,(.),a} + \|u\|_{p,(.),b}.
\]

Since \( a^{-\frac{1}{p-1}} \in L_{loc}^{1}(\Omega) \) and \( b^{-\frac{1}{p-1}} \in L_{loc}^{1}(\Omega) \), then it can be seen that \( L_{a}^{p,(.)}(\Omega) \hookrightarrow L_{loc}^{1}(\Omega) \) and \( L_{b}^{p,(.)}(\Omega) \hookrightarrow L_{loc}^{1}(\Omega) \). Therefore, the double weighted variable exponent Sobolev space \( W_{a,b}^{1,p,(.)}(\Omega) \) is well-defined. The dual space of \( W_{a,b}^{1,p,(.)}(\Omega) \) is \( W_{a^{-1},b^{-1}}^{1,q,(.)}(\Omega) \), where \( \frac{1}{a^{-1}} + \frac{1}{b^{-1}} = 1 \) and \( a^{*} = a^{-\frac{1}{p-1}}, b^{*} = b^{-\frac{1}{p-1}} \). Moreover, the space \( W_{a,b}^{1,p,(.)}(\Omega) \) is a separable and reflexive Banach space.

**Proposition 1.** (see [17]) Let \( I(u) = \int_{\Omega} \left( a(x) |\nabla u|^{p(x)} + b(x) |u|^{q(x)} \right) dx \). For all \( u \in W_{a,b}^{1,p,(.)}(\Omega) \),

(i) if \( \|u\|_{1,p,(.),a,b} \geq 1 \), then the inequality \( \|u\|_{p,(.),a,b}^{p^{-}} \leq I(u) \leq \|u\|_{1,p,(.),a,b}^{p^{+}} \) is satisfied.

(ii) if \( \|u\|_{1,p,(.),a,b} \leq 1 \), then the inequality \( \|u\|_{p,(.),a,b}^{p^{-}} \leq I(u) \leq \|u\|_{1,p,(.),a,b}^{p^{+}} \) is satisfied.

The following a compact embedding theorem of \( W_{a,b}^{1,p,(.)}(\Omega) \) into \( C(\overline{\Omega}) \) plays an important role in this paper. For the proof, we use the method in [15, Theorem 2.11].

**Theorem 2.** Let \( a^{-\alpha(\cdot)} \in L^{1}(\Omega) \) with \( \alpha(x) \in \left( \frac{N}{p(x)}, \infty \right) \cap \left[ \frac{1}{p(x)-1}, \infty \right) \). If we define the variable exponent \( p_{\alpha}(x) = \frac{\alpha(x)p(x)}{\alpha(x)+1} \) with \( N < p_{\alpha}^{-} \), then we have the compact embedding \( W_{a,b}^{1,p,(\cdot)}(\Omega) \hookrightarrow C(\overline{\Omega}) \).

**Proof.** First we will show that the continuous embedding \( W_{a,b}^{1,p,(\cdot)}(\Omega) \hookrightarrow W_{a,b}^{1,p_{\alpha}(\cdot)}(\Omega) \) is valid. Let \( u \in W_{a,b}^{1,p,(\cdot)}(\Omega) \). Then we write that \( u \in L_{b}^{p_{\alpha}(\cdot)}(\Omega) \) and \( \nabla u \in L_{a}^{p,(\cdot)}(\Omega) \). Using Hölder’s inequality with \( q(x) = \frac{p(x)}{p_{\alpha}(x)} = \frac{\alpha(x)+1}{\alpha(x)} \) and \( q'(x) = \alpha(x) + 1 \), we have 

\[
\int_{\Omega} |\nabla u(x)|^{p_{\alpha}(x)} \, dx = \int_{\Omega} |\nabla u(x)|^{\frac{\alpha(x)p(x)}{\alpha(x)+1}} \, dx \\
= \int_{\Omega} |\nabla u(x)|^{\frac{\alpha(x)p(x)}{\alpha(x)+1}} a^{-\frac{\alpha(x)}{\alpha(x)+1}(x)} |\nabla u(x)|^{\alpha(x)+1} \, dx \\
\leq 2 \left\| a^{-\frac{\alpha(x)}{\alpha(x)+1}} |\nabla u|^{\frac{\alpha(x)p(x)}{\alpha(x)+1}} \right\|_{\alpha(x)+1} \left\| a^{-\frac{\alpha(x)}{\alpha(x)+1}} \right\|_{\alpha(x)+1}.
\]
It is well known that \( g_{r(.)}(u) < \infty \) if and only if \( \| u \|_{r(.)} < \infty \). Since \( a^{-\alpha(.)} \in L^1(\Omega) \), then \( \left\| a^{-\alpha(.)} \right\|_{\alpha(.)+1} < C_2 < \infty \). Thus, we obtain

\[
(2.1) \quad \int_{\Omega} |\nabla u(x)|^{p_\ast(x)}\, dx \leq C_3 \left\| a^{\frac{\alpha(.)}{\alpha(.)+1}} |\nabla u|^{\frac{\alpha(.)+1}{\alpha(.)+1}} \right\|_{\alpha(.)+1}^{\alpha(.)+1}.
\]

In general, we can suppose that \( \int |\nabla u(x)|^{p_\ast(x)}\, dx > 1 \). Because if \( \int |\nabla u(x)|^{p_\ast(x)}\, dx \leq 1 \), then \( \nabla u \in L^{p_\ast(.)}(\Omega) \) and \( u \in L^{p_\ast(.)}(\Omega) \) due to \( p_\ast(.) < p(.) \). Thus we have \( u \in W^{1,p_\ast(.)}(\Omega) \). If \( \int a(x)|\nabla u|^{p(x)}\, dx \leq 1 \), then by \((2.1)\) and Proposition 1 we have

\[
\| \nabla u \|_{p_\ast} \leq C_3 \left\| a^{\frac{\alpha(.)}{\alpha(.)+1}} |\nabla u|^{\frac{\alpha(.)+1}{\alpha(.)+1}} \right\|_{\alpha(.)+1}^{\alpha(.)+1} \leq C_3 \| \nabla u \|_{p(.),a}^{\alpha(.)+1}.
\]

This follows

\[
(2.2) \quad \| \nabla u \|_{p_\ast} \leq C_4 \| \nabla u \|_{p(.),a},
\]

where \( C_4 = \frac{\alpha(.)+1}{\alpha(.)} > 0 \). On the other hand, if \( \int a(x)|\nabla u|^{p(x)}\, dx > 1 \), then by \((2.1)\) and Proposition 1 we obtain

\[
\| \nabla u \|_{p_\ast} \leq C_3 \left\| a^{\frac{\alpha(.)}{\alpha(.)+1}} |\nabla u|^{\frac{\alpha(.)+1}{\alpha(.)+1}} \right\|_{\alpha(.)+1}^{\alpha(.)+1} \leq C_3 \| \nabla u \|_{p(.),a}^{\alpha(.)+1},
\]

or equivalently

\[
(2.3) \quad \| \nabla u \|_{p_\ast} \leq C_4 \| \nabla u \|_{p(.),a}^\beta,
\]

where \( \beta = \frac{\alpha(.)+1}{\alpha(.)} > \frac{\alpha(.)+1}{\alpha(.)+1} \). If we consider the \((2.2)\) and \((2.3)\), then we have \( \nabla u \in L^{p_\ast(.)}(\Omega) \). Therefore, we have \( u \in W^{1,p_\ast(.)}(\Omega) \). Hence, the inclusion \( W^{1,p_\ast(.)}(\Omega) \subset W^{1,p_\ast(.)}(\Omega) \) is satisfied. Using the Banach closed graph theorem, we get

\[
W^{1,p_\ast(.)}(\Omega) \hookrightarrow W^{1,p_\ast(.)}(\Omega).
\]

Since \( p_\ast > N \) and \( W^{1,p_\ast(.)}(\Omega) \hookrightarrow W^{1,p_\ast(.)}(\Omega) \) it follows that \( W^{1,p_\ast(.)}(\Omega) \hookrightarrow C(\overline{\Omega}) \) and \( W^{1,p_\ast(.)}(\Omega) \hookrightarrow C(\overline{\Omega}) \). This completes the proof. \( \square \)

**Corollary 1.** Since \( W^{1,p_\ast(.)}(\Omega) \hookrightarrow C(\overline{\Omega}) \), then there exists a \( C_5 > 0 \) such that

\[
\| u \|_\infty \leq C_5 \| u \|_{1,p_\ast(.)},\alpha,b
\]

for any \( u \in W^{1,p_\ast(.)}(\Omega) \), where \( \| u \|_\infty = \sup_{x \in \overline{\Omega}} u(x) \) for \( u \in C(\overline{\Omega}) \).
For \( A \subset \Omega \), denote by \( \theta^{-}(A) = \inf_{x \in A} \theta(x) \) and \( \theta^{+}(A) = \sup_{x \in A} \theta(x) \). For any \( x \in \partial \Omega \) and \( r \in C(\partial \Omega, \mathbb{R}) \) with \( r^{-} = \inf_{x \in \partial \Omega} r(x) > 1 \), we define

\[
\theta^{\theta}(x) = (\theta(x))^{\theta} = \begin{cases} 
\frac{(N-1)\theta(x)}{N-N\theta(x)} & \text{if } \theta(x) < N, \\
\frac{1}{\theta(x)} & \text{if } \theta(x) \geq N,
\end{cases}
\]

\[
\theta_{r(x)}^{\theta}(x) = \frac{r(x) - 1}{r(x)} \theta^{\theta}(x).
\]

**Theorem 3.** (see [9]) Assume that the boundary of \( \Omega \) possesses the cone property and \( \theta \in C(\overline{\Omega}) \) with \( \theta^{-} > 1 \). Suppose that \( a \in L^{r(-)}(\partial \Omega) \), \( r \in C(\partial \Omega) \) with \( r(x) > \frac{\theta^{\theta}(x)}{\theta_{r(x)}^{\theta}(x)} \) for all \( x \in \partial \Omega \). If \( q \in C(\partial \Omega) \) and \( 1 \leq q(x) < \theta_{r(x)}^{\theta}(x) \) for all \( x \in \partial \Omega \), then there is a compact embedding from \( W^{1,\theta(-)}(\Omega) \) into \( L^{q(-)}(\partial \Omega) \). In particular, there is a compact embedding from \( W^{1,\theta(-)}(\Omega) \) into \( L^{q(-)}(\partial \Omega) \), where \( 1 \leq q(x) < \theta_{r(x)}^{\theta}(x) \) for all \( x \in \partial \Omega \).

**Corollary 2.** All conditions in Theorem 2 and Theorem 3 are satisfied. If \( q \in C(\partial \Omega) \) and \( 1 \leq q(x) < p_{r(x)}^{\theta}(x) \), for all \( x \in \partial \Omega \), then we have \( W^{1,p_{r(x)}^{\theta}(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial \Omega) \). This yields that \( W^{1,p_{r(x)}^{\theta}(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial \Omega) \). Moreover, we obtain \( W^{1,p_{r(x)}^{\theta}(x)}(\Omega) \hookrightarrow W^{1,p_{r(x)}^{\theta}(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial \Omega) \) for all \( 1 \leq q(x) < p_{r(x)}^{\theta}(x) \) for all \( x \in \partial \Omega \).

**Theorem 4.** (see [20]) Let \( X \) be a separable and reflexive real Banach space; \( \Phi : X \to \mathbb{R} \) a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on \( X^{*} \); \( \Psi : X \to \mathbb{R} \) a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that

(i) \( \lim_{|u| \to \infty} (\Phi(u) + \lambda \Psi(u)) = \infty \) for all \( \lambda > 0 \),
(ii) there are \( r \in \mathbb{R} \) and \( u_{0}, u_{1} \in X \) such that \( \Phi(u_{0}) < r < \Phi(u_{1}) \),
(iii) \( \inf_{u \in \Phi^{-1}((-\infty,r])} \Psi(u) \geq \frac{(\Phi(u_{2}) - r)\Psi(u_{0}) + (r - \Phi(u_{0}))\Psi(u_{2})}{\Phi(u_{2}) - \Phi(u_{0})} \).

Then there exist an open interval \( \Lambda \subset (0, \infty) \) and a positive constant \( \rho > 0 \) such that for any \( \lambda \in \Lambda \) the equation \( \Phi'(u) + \lambda \Psi'(u) = 0 \) has at least three solutions in \( X \) whose norms are less than \( \rho \).

**Proposition 2.** (see [17]) Let us consider the functional \( \Phi(u) = \int_{\Omega} \frac{1}{|x|^{\gamma}} \left( a(x)|\nabla u|^{p(x)} + b(x)|u|^{p(x)} \right) dx \) for all \( u \in W^{1,p_{\theta}(\cdot)}_{a,b}(\Omega) \). Then, we have

(i) \( \Phi : W^{1,p_{\theta}(\cdot)}_{a,b}(\Omega) \to \mathbb{R} \) is sequentially weakly lower semicontinuous and \( \Phi \in C^{1} \left( W^{1,p_{\theta}(\cdot)}_{a,b}(\Omega), \mathbb{R} \right) \). Moreover, the derivative operator \( \Phi' \) of \( \Phi \) define as

\[
(\Phi'(u), v) = \int_{\Omega} \left( a(x)|\nabla u|^{p(x)-2} \nabla u \nabla v + b(x)|u|^{p(x)-2} uv \right) dx
\]

for all \( u, v \in W^{1,p_{\theta}(\cdot)}_{a,b}(\Omega) \).

(ii) \( \Phi' : W^{1,p_{\theta}(\cdot)}_{a,b}(\Omega) \to W^{-1,q_{\theta}(\cdot)}_{a,b}(\Omega) \) is a continuous, bounded and strictly monotone operator.
We call that for any $W^1_\Psi$ exists and continuous. Moreover, $v$ for all $u$.

$(\Psi')$ is a mapping of type $(S_\Psi)$, i.e., if $u_n \to u$ in $W_1^{1,p}(\Omega)$ and $\limsup_{n \to \infty} \langle \Psi'(u_n) - \Psi'(u), u_n - u \rangle \leq 0$, then $u_n \to u$ in $W_1^{1,p}(\Omega)$.

$(\Psi') : W_1^{1,p}(\Omega) \to W_1^{-1,q}(\Omega)$ is a homeomorphism.

### 3. MAIN RESULTS

In this paper, we assume that $a^{-\alpha}(\cdot) \in L^1_{loc}(\Omega)$ and $b^+ \in L^1_{loc}(\Omega)$, $a^{-\alpha}(\cdot) \in L^1(\Omega)$ with $\alpha(x) \in \left(\frac{N}{p(x)}, \infty\right) \cap \left[\frac{1}{p(x)-1}, \infty\right)$ and $N < p^-$. Moreover, the function $f : \partial \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and satisfies the following conditions:

$(F1)$ $|f(x,t)| \leq k(x) + c|t|^{s(x)-1}$ for all $(x,t) \in \partial \Omega \times \mathbb{R}$ where $k(x) \in L^{\frac{s(x)}{s(x)-1}}(\partial \Omega)$, $k(x) \geq 0$ and $s(x) \in C_+(\partial \Omega)$, $1 < s^- = \inf_{x \in \partial \Omega} s(x) \leq s^+ = \sup_{x \in \partial \Omega} s(x) < p^-$ with $s(x) < p^0(x)$, for all $x \in \partial \Omega$.

$(F2)$ $(i)$ $f(x,t) < 0$ for all $(x,t) \in \partial \Omega \times \mathbb{R}$, when $|t| \in (0,1)$,

$(ii)$ $f(x,t) \geq M > 0$, when $|t| \in (t_0, \infty)$, $t_0 > 1$.

**Definition 2.** We call that $f \in W_1^{1,p}(\Omega)$ is a weak solution of the problem $(1.1)$ if

$$\int_{\Omega} a(x) |\nabla u|^{p(x)-2} \nabla u \nabla v dx + \int_{\Omega} b(x) |u|^{p(x)-2} u v dx - \lambda \int_{\partial \Omega} f(x,u) v d\sigma = 0$$

for all $v \in W_1^{1,p}(\Omega)$. The corresponding energy functional of the problem $(1.1)$

$$J(u) = \Phi(u) + \lambda \Psi(u)$$

where the functionals $\Phi, \Psi$ from $W_1^{1,p}(\Omega)$ into $\mathbb{R}$ as $\Phi(u) = \int_{\Omega} \frac{1}{p(x)} \left( a(x) |\nabla u|^{p(x)} + b(x) |u|^{p(x)} \right) dx,$

$$\Psi(u) = - \int_{\partial \Omega} F(x,u) d\sigma + \int_{0}^{\epsilon} f(x,y) dy.$$

By [13] Proposition 3.1] and Proposition 4 we get $\Phi, \Psi \in C^1\left(W_1^{1,p}(\Omega), \mathbb{R}\right)$ with the derivatives given by

$$\langle \Phi'(u), v \rangle = \int_{\Omega} \left( a(x) |\nabla u|^{p(x)-2} \nabla u \nabla v + b(x) |u|^{p(x)-2} u v \right) dx,$$

$$\langle \Psi'(u), v \rangle = - \int_{\partial \Omega} f(x,u) v d\sigma,$$

for any $u, v \in W_1^{1,p}(\Omega)$. Since $\Psi' : W_1^{1,p}(\Omega) \to W_1^{-1,q}(\Omega)$ is a homeomorphism by Proposition 4 it is obvious that $(\Phi')^{-1} : W_1^{-1,q}(\Omega) \to W_1^{1,p}(\Omega)$ exists and continuous. Moreover, $\Psi' : W_1^{1,p}(\Omega) \to W_1^{-1,q}(\Omega)$ is completely continuous due to the assumption $(F1)$ in [4] Theorem 2.9, which implies $\Psi' : W_1^{1,p}(\Omega) \to W_1^{-1,q}(\Omega)$ is compact.

We note that the operator $J$ is a $C^1\left(W_1^{1,p}(\Omega), \mathbb{R}\right)$ functional and the critical points of J are weak solutions of the problem $(1.1)$.

Now, we are ready to give our main result.
Theorem 5. If the conditions (F1) and (F2) are valid, then there exist an open interval \( \Lambda \subset (0, \infty) \) and a positive constant \( \rho > 0 \) such that for any \( \lambda \in \Lambda \), the problem \((P)\) has at least three solutions in \( W_{a,b}^{1,p(\cdot)}(\Omega) \) whose norms are less than \( \rho \).

Proof. To prove this theorem, we first verify the condition (i) of Theorem 4. In fact, by Proposition 2 we have

\[
\Phi(u) \geq \frac{1}{p^*} \int_{\Omega} \left( a(x) |\nabla u|^{p(x)} + b(x) |u|^{p(x)} \right) \, dx
\]

(3.1)

\[
= \frac{1}{p^*} I(u) \geq \frac{1}{p^*} \|u\|_{1,p(\cdot),a,b}^{-p^*}
\]

for any \( u \in W_{a,b}^{1,p(\cdot)}(\Omega) \) with \( \|u\|_{1,p(\cdot),a,b} > 1 \).

On the other hand, by \((F1)\) and the H"older inequality, we get

\[
-\Psi(u) = \int_{\partial \Omega} F(x,u) \, d\sigma = \int_{\partial \Omega} \left( \int_0^{u(x)} f(x,t) \, dt \right) \, d\sigma
\]

\[
\leq \int_{\partial \Omega} \left( k(x) |u(x)| + \frac{c}{s(x)} |u(x)|^{s(x)} \right) \, d\sigma
\]

(3.2)

\[
\leq 2 \|k\|_{\frac{1}{1+s(\cdot)},\partial \Omega} \|u\|_{s(\cdot),\partial \Omega} + \frac{c}{s} \int_{\partial \Omega} |u(x)|^{s(x)} \, d\sigma.
\]

By Corollary 2 it is obtained that \( W_{a,b}^{1,p(\cdot)}(\Omega) \hookrightarrow L^{s(\cdot)}(\partial \Omega) \) and

\[
\int_{\partial \Omega} |u(x)|^{s(x)} \, d\sigma \leq \max \left\{ \|u\|_{s(\cdot),\partial \Omega}^{s-}, \|u\|_{s(\cdot),\partial \Omega}^{s+} \right\} \leq C_6 \|u\|_{1,p(\cdot),a,b}^{s+}.
\]

(3.3)

If we use (3.2) and (3.3), then we get

\[
-\Psi(u) \leq C_7 \|k\|_{\frac{1}{1+s(\cdot)},\partial \Omega} \|u\|_{1,p(\cdot),a,b} + \frac{c}{s} C_6 \|u\|_{1,p(\cdot),a,b}^{s+}.
\]

(3.4)

For any \( \lambda > 0 \) we can obtain

\[
\Phi(u) + \lambda \Psi(u) \geq \frac{1}{p^*} \|u\|_{1,p(\cdot),a,b}^{p^*} - \lambda C_7 \|k\|_{\frac{1}{1+s(\cdot)},\partial \Omega} \|u\|_{1,p(\cdot),a,b} - \frac{c}{s} \lambda C_6 \|u\|_{1,p(\cdot),a,b}^{s^+}
\]

by (3.1) and (3.4). Since \( 1 < s^+ < p^- \), then \( \lim_{\|u\|_{1,p(\cdot),a,b} \to \infty} (\Phi(u) + \lambda \Psi(u)) = \infty \) for all \( \lambda > 0 \) and (i) is verified.

By \( \frac{\partial F(x,t)}{\partial t} = f(x,t) \) and (F2), it is obtained that \( F(x,t) \) is increasing for \( t \in (t_0, \infty) \) and decreasing for \( t \in (0,1) \), uniformly with respect to \( x \in \partial \Omega \), and \( F(x,0) = 0 \). In addition, \( F(x,t) \to \infty \) when \( t \to \infty \) due to \( F(x,t) \geq Mt \) uniformly for \( x \). Then there exists a real number \( \delta > t_0 \) such that

\[
F(x,t) \geq 0 = F(x,0) \geq F(x,\tau), \text{ for all } x \in \partial \Omega, \ t > \delta, \ \tau \in (0,1).
\]

Let \( \beta, \gamma \) be two real numbers such that \( 0 < \beta < \min \{1, C_5\} \), where \( C_5 \) is given in Corollary 1 and \( \gamma > \delta \) (\( \gamma > 1 \)) satisfies \( \gamma p^- \|b\|_1 > 1 \). If we use (3.5), we have
F(x, t) \leq F(x, 0) = 0 \text{ for } t \in [0, \beta], \text{ and}

\begin{equation}
(3.6) \quad \int_{\partial \Omega} \sup_{0 \leq t \leq \beta} F(x, t) d\sigma \leq \int_{\partial \Omega} F(x, 0) d\sigma = 0.
\end{equation}

Moreover, due to \( \gamma > \delta \) and (3.5) we obtain \( \int_{\partial \Omega} F(x, \delta) d\sigma > 0 \) and

\begin{equation}
(3.7) \quad \frac{1}{C_5^p} \frac{\beta^+}{\gamma^p} \int_{\partial \Omega} F(x, \delta) d\sigma > 0.
\end{equation}

If we consider (3.6) and (3.7), then we get

\[ \int_{\partial \Omega} \sup_{0 \leq t \leq \alpha} F(x, t) d\sigma \leq 0 < \frac{1}{p^+} \frac{\beta^+}{\gamma^p} \int_{\partial \Omega} F(x, \delta) d\sigma. \]

Define \( u_0, u_1 \in W_{a, b}^{1,p} (\Omega) \) with \( u_0(x) = 0 \) and \( u_1(x) = \gamma \) for any \( x \in \Omega \). If we take \( r = \frac{1}{p^+} \left( \frac{\beta}{C_5} \right)^{p^+} \), then \( r \in (0, 1) \), \( \Phi(u_0) = \Psi(u_0) = 0 \) and

\[ \Phi(u_1) = \int_{\Omega} \frac{1}{p(x)} b(x) \gamma^{p(x)} dx \geq \frac{\gamma^{p^-}}{p^+} \int_{\Omega} b(x) dx = \frac{1}{p^+} \gamma^{p^-} \| b \|_1 \]

\[ \geq \frac{1}{p^+} > r. \]

Thus we have \( \Phi(u_0) < r < \Phi(u_1) \) and

\[ \Psi(u_1) = -\int_{\partial \Omega} F(x, u_1) d\sigma = -\int_{\partial \Omega} F(x, \gamma) d\sigma < 0. \]

Then (ii) of Theorem 4 is verified.

On the other hand, we have

\[ -\frac{(\Phi(u_1) - r) \Psi(u_0) + (r - \Phi(u_0)) \Psi(u_1)}{\Phi(u_1) - \Phi(u_0)} = -r \frac{\Psi(u_1)}{\Phi(u_1)} \int_{\partial \Omega} F(x, \gamma) d\sigma \]

\[ = r \frac{1}{p(x)} b(x) \gamma^{p(x)} dx > 0. \]

Now, we consider the case \( u \in W_{a, b}^{1,p} (\Omega) \) with \( \Phi(u) \leq r \leq 1 \). Due to \( \frac{1}{p^+} I(u) \leq \Phi(u) \leq r \), we have

\[ I(u) \leq p^+ r = \left( \frac{\beta}{C_5} \right)^{p^+} < 1. \]

By Proposition 1 we get \( \| u \|_{1,p^+} < 1 \) and

\[ \frac{1}{p^+} \| u \|_{1,p^+}^p \leq \frac{1}{p^+} I(u) \leq \Phi(u) \leq r. \]

If we consider Corollary 1 then we get

\[ \| u(x) \| \leq C_5 \| u \|_{1,p^+} \leq C_5 (p^+ r)^{\frac{1}{p^+}} = \beta \]

for all \( u \in W_{a, b}^{1,p} (\Omega) \) and \( x \in \Omega \) with \( \Phi(u) \leq r \).
This follows that
\[
-\inf_{u \in \Phi^{-1}((\infty, r])} \Psi(u) = \sup_{u \in \Phi^{-1}((\infty, r])} -\Psi(u) \leq \int_{\partial\Omega} \sup_{0 \leq t \leq \beta} F(x, t) d\sigma \leq 0.
\]
Then we have
\[
-\inf_{u \in \Phi^{-1}((\infty, r])} \Psi(u) < r \int_{\partial\Omega} F(x, \gamma) d\sigma \int_{\Omega} p(x) b(x) \gamma p(x) dx
\]
and
\[
\inf_{u \in \Phi^{-1}((\infty, r])} \Psi(u) > \frac{(\Phi(u_1) - r) \Psi(u_0) + (r - \Phi(u_0)) \Psi(u_1)}{\Phi(u_1) - \Phi(u_0)}.
\]
Thus condition (iii) of Theorem 4 is obtained. This completes the proof. \(\square\)

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