Quantitative evaluation of an active Chemotaxis model in Discrete time.

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Abstract
A system of $N$ particles in a chemical medium in $\mathbb{R}^d$ is studied in a discrete time setting. Underlying interacting particle system in continuous time can be expressed as

$$dX_i(t) = [-(I-A)X_i(t) + \nabla h(t, X_i(t))]dt + dW_i(t), \quad X_i(0) = x_i \in \mathbb{R}^d \forall i = 1, \ldots, N$$

$$\frac{\partial}{\partial t} h(t,x) = -\alpha h(t,x) + D \Delta h(t,x) + \frac{\beta}{n} \sum_{i=1}^{N} g(X_i(t), x), \quad h(0, \cdot) = h(\cdot).$$

(0.1)

where $X_i(t)$ is the location of the $i$th particle at time $t$ and $h(t, x)$ is the function measuring the concentration of the medium at location $x$ with $h(0, x) = h(x)$. In this article we describe a general discrete time non-linear formulation of the model (0.1) and a strongly coupled particle system approximating it. Similar models have been studied before (Budhiraja et al.(2010)) under a restrictive compactness assumption on the domain of particles. In current work the particles take values in $\mathbb{R}^d$ and consequently the stability analysis is particularly challenging. We provide sufficient conditions for the existence of a unique fixed point for the dynamical system governing the large $N$ asymptotics of the particle empirical measure. We also provide uniform in time convergence rates for the particle empirical measure to the corresponding limit measure under suitable conditions on the model.

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1 Introduction

There have been a surge of significant research activities aimed towards understanding the dynamics of collective behavior of a multi-agent system in the time limit. Motivations for such problems come from various examples of self organizing systems such as consensus formation in opinion dynamics [11], active chemotaxis [3], self organized networks [13], large communication systems [12], multi target tracking [6], swarm robotics [14] (additional applications can be found in [15]) etc. One of the basic challenges is to understand how a large group of autonomous agents with decentralized local interactions that gives rise to a coherent behavior.

In this paper we consider a reduced model motivated by both [3],[5] for a system of interacting agents in a stochastic diffusing environment, variations of which have been proposed (see [3],[14] and references

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\[ dX_i(t) = \left[ -(I - A)X_i(t) + \nabla h(t, X_i(t)) + \frac{1}{N} \sum_{j=1, j \neq i}^N K(X_i(t), X_j(t)) \right] dt + dW_i(t), \quad (1.1) \]

\[ \frac{\partial}{\partial t} h(t, x) = -\alpha h(t, x) + D \triangle h(t, x) + \frac{\beta}{N} \sum_{i=1}^N g(X_i(t), x), \quad h(0, \cdot) = h(\cdot). \]

Here \( W_i, i = 1, ..., N \) are independent Brownian motions that drive the state process \( X_i \) of the interacting particles. The interaction between the particles arises directly from the evolution equation (1.1) and indirectly through the underlying potential field \( h \) which changes continuously according to a diffusion equation and through the aggregated input of the \( N \) particles. One example of such an interaction is in Chemotaxis where cells preferentially move towards a higher chemical concentration and themselves release chemicals into the medium, in response to the local information on the environment, thus modifying the potential field dynamically over time. In this context, \( h(t, x) \) represents the concentration of a chemical at time \( t \) and location \( x \). Diffusion of the chemical in the medium is captured by the Laplacian in (1.1) and the constant \( \alpha > 0 \) models the rate of decay or dissipation of the chemical. The first equation in (1.1) describes the motion of a particle in terms of diffusion process with a drift consisting of three terms. The first term models a restoring force towards the origin where origin represents the natural rest state of the particles. The second term is the gradient of the chemical concentration and captures the fact that particles tend to move particularly towards regions of higher chemical concentration. Finally the third term captures the interaction (e.g. attraction or repulsion) between the particles. Contribution of the agents to the chemical concentration field is given through the last term in the second equation. The function \( g \) captures the agent response rules and can be used to model a wide range of phenomenon [15].

In [3] the authors considered a discrete time model which captures some of the key features of the dynamics in (1.1) and studied several long time properties of the system. One aspect that greatly simplified the analysis of [3] is that the state space of the particles is taken to be a compact set in \( \mathbb{R}^d \). However this requirement is restrictive and may be unnatural for the time scales at which the particle evolution is being modeled. In [14] authors had considered a number of variations of (1.1). The theoretical properties obtained in this work on the long time behavior of the particle system can be also applied for such systems with some minor modifications.

We now give a general description of the \( N \)-particle system that gives a discrete time approximation of the mechanism outlined above. The space of real valued bounded measurable functions on \( S \) is denoted as \( BM(S) \). Borel \( \sigma \) field on a metric space will be denoted as \( B(S) \). \( C_b(S) \) denotes the space of all bounded and continuous functions \( f : S \rightarrow \mathbb{R} \). For a measurable space \( S \), \( \mathcal{P}(S) \) denotes the space of all probability measures on \( S \). For \( k \in \mathbb{N} \), let \( \mathcal{P}_k(\mathbb{R}^d) \) be the space of \( \mu \in \mathcal{P}(\mathbb{R}^d) \) such that

\[ \| \mu \|_k := \left( \int |x|^k d\mu(x) \right) \frac{1}{k} \leq \infty. \]

Consider a system of \( N \) interacting particles that evolve in \( \mathbb{R}^d \) governed by a random dynamic chemical field according to the following discrete time stochastic evolution equation given on some probability space \((\Omega, \mathcal{F}, P)\). Suppose that the chemical field at time instant \( n \) is given by a nonnegative \( C^1 \) (i.e. continuously differentiable) real function on \( \mathbb{R}^d \) satisfying \( \int_{\mathbb{R}^d} \eta(x) dx = 1 \). Then, given that particle state at time instant \( n \) is \( x \) and the empirical measure of the particle states at time \( n \) is \( \mu \), the particle state \( X^+ \) at time \((n + 1)\) is given as

\[ X^+ = Ax + \delta f(\nabla \eta(x), \mu, x, \epsilon) + B(\epsilon), \quad (1.2) \]

where \( A \) is a \( d \times d \) matrix, \( \delta \) is a small parameter, \( \epsilon \) is a \( \mathbb{R}^m \) valued random variable with probability law \( \theta \) and \( f : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \times \mathbb{R}^d \times \mathbb{R}^m \rightarrow \mathbb{R}^d \) is a measurable function. Here we consider a somewhat more general form of dependence of the particle evolution on the concentration profile than the additive form that appears in (1.1). Additional assumptions on \( A, \theta, f \) will be introduced shortly. Nonlinearity (modeled by \( f \) and \( B \)) of the system can be very general and as described below. Denote by \( X_i^+ \equiv X_i^{1,N} \) (a \( \mathbb{R}^d \) valued random variable) the state of the \( i \)-th particle \((i = 1, ..., N)\) and by \( \eta_n^N \) the chemical concentration field
at time instant \( n \). Let \( \mu_n^N := \frac{1}{N} \sum_{i=1}^N \delta_{X_i^n} \) be the empirical measure of the particle values at time instant \( n \). The stochastic evaluation equation for the \( N \)-particle system is given as

\[
X_{n+1}^i = AX_n^i + \delta f(\nabla \eta^N_n(X_n^i), \mu_n^N, X_n^i, \epsilon_{n+1}) + B(\epsilon_{n+1}), \quad i = 1, \ldots, N, \quad n \in \mathbb{N}_0. \tag{1.3}
\]

In (1.3) \( \{\epsilon_n^i, i = 1, \ldots, N, \quad n \geq 1 \} \) is an i.i.d. array of \( \mathbb{R}^m \) valued random variables with common probability law \( \theta \). Here \( \{X_0^i, i = 1, \ldots, N\} \) are assumed to be exchangeable with common distribution \( \mu_0 \) where \( \mu_0 \in \mathcal{P}(\mathbb{R}^d) \). Note that in the notation we have suppressed the dependence of the sequence \( \{X_n^i\} \) on \( N \).

We now describe the evolution of the chemical field approximating the second equation in (1.1) and its interaction with the particle system. A transition probability kernel on \( S \) is a map \( P : S \times \mathcal{B}(S) \to [0, 1] \) such that \( P(x, \cdot) \in \mathcal{P}(S) \ \forall x \in S \) and for each \( A \in \mathcal{B}(S), \ P(\cdot, A) \in BM(S) \). Given the concentration profile at time \( n \) is a \( C^1 \) probability density \( \eta \) on \( \mathbb{R}^d \) and the empirical measure of the state of \( N \)-particles at time instant \( n \) is \( \mu \), the concentration probability density \( \eta^+ \) at time \( (n + 1) \) is given by the relation

\[
\eta^+(y) = \int_{\mathbb{R}^d} \eta(x) R_\mu^N(x, y) l(dx)
\]

where \( l \) denotes the Lebesgue measure on \( \mathbb{R}^d \), and \( R_\mu^N(x, y) \) is the Radon-Nikodym derivative of the transition probability kernel with respect to the Lebesgue measure \( l(dy) \) on \( \mathbb{R}^d \). The kernel \( R_\mu^N \) is given as follows. We considered the same model as introduced in [3]. Let \( P \) and \( P' \) be two transition probability kernels on \( \mathbb{R}^d \). For \( \mu \in \mathcal{P}(\mathbb{R}^d) \) and \( \alpha \in (0, 1) \) define the transition probability kernel \( R^\alpha_\mu \) on \( \mathbb{R}^d \) as

\[
R^\alpha_\mu(x, C) := (1 - \alpha)P(x, C) + \alpha \mu P'(C), \quad x \in \mathbb{R}^d, C \in \mathcal{B}(\mathbb{R}^d).
\]

Here \( P \) represents the background diffusion of the chemical concentration while \( \delta_\mu P' \) captures the contribution to the field by a particle with location \( x \). So the kernel \( P' \) gives a spike at origin which can be approximated by a smooth density function as \( P(x, dy) = \frac{1}{\sqrt{2\pi\lambda}} e^{-\frac{(x-y)^2}{2\lambda}} \) with very small \( \lambda > 0 \). The parameter \( \alpha \) given a convenient way for combining the contribution from the background diffusion and the individual particles. For each \( x \in \mathbb{R}^d \), both \( P(x, \cdot) \) and \( P'(x, \cdot) \) are assumed to be absolutely continuous with respect to Lebesgue measure and throughout this article we will denote the corresponding Radon-Nikodym derivatives with the same notations \( P(x, \cdot) \) and \( P'(x, \cdot) \) respectively. Additional properties of \( P \) and \( P' \) will be specified shortly. The evolution equation for the chemical field is then given as

\[
\eta^N_{n+1}(y) = \int_{\mathbb{R}^d} \eta^N_n(x) R^\alpha_\mu(x, y) l(dx).
\]

In contrast to the model studied in [5], the situation here is somewhat more involved. Note that \( \{X_n(N)\}_{n \geq 0} := (X_0^N, X_2^N, \ldots, X_n^N)_{n \geq 0} \) is not a Markov process and in order to get a Markovian state descriptor one needs to consider \( \{X_n(N), \eta^N_n\}_{n \geq 0} \) which is a discrete time Markov chain with values in \( (\mathbb{R}^d)^N \times \mathcal{P}(\mathbb{R}^d) \).

We will show that as \( N \to \infty \), \( (\mu^N_n, \eta^N_n)_{n \in \mathbb{N}_0} \) converges to a deterministic nonlinear dynamical system \( (\mu_n, \eta_n)_{n \in \mathbb{N}_0} \) with methods followed in [3]. We established further sharp quantitative bounds (with techniques used in [10] and [5]) for weakly interacting particle system jointly with the stochastic field potential to the nonlinear system of interest. For both polynomial and exponential concentration bound it requires further constraints on the tail of the transition kernels \( P, P' \) used in modeling the diffusive environment. One major motivation of the current article is giving a sharp uniform in time quantitative estimate for the particle system \( (\mu^N_n, \eta^N_n) \) to the non-linear system of interest \( (\mu_n, \eta_n) \) so that any functional of the form \( \langle \phi_1, \mu_n \rangle + \langle \phi_2, \eta_n \rangle \) can be approximated by \( \frac{1}{N} \sum_{i=1}^N \phi_1(X_i^n) + \langle \phi_2, \eta^N_n \rangle \) with desired precision. Previous work on concentration bounds for similar particle system in discrete time includes [8] but that involves a Dobrushin type stability condition which is not very effective if the particles are assumed to come from a non-compact domain. A very recent work [4] addresses several quantitative bounds for Chemotaxis model motivated by Patlak-keller-segel type non-linear equations.

The following notations will be used in this article. \( \mathbb{R}^d \) will denote the \( d \)-dimensional Euclidean space with the usual Euclidean norm \(|\cdot|\). The set of natural numbers (resp. whole numbers) is denoted by \( \mathbb{N} \) (resp. \( \mathbb{N}_0 \)). Cardinality of a finite set \( S \) is denoted by \(|S|\). For \( x \in \mathbb{R}^d \), \( \delta_x \) is the Dirac delta measure on \( \mathbb{R}^d \) that puts a
unit mass at location $x$. The supremum norm of a function $f : S \to \mathbb{R}$ is $\|f\|_{\infty} = \sup_{x \in S} |f(x)|$. When $S$ is a metric space, the Lipschitz seminorm of $f$ is defined by $\|f\|_1 = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)}$, where $d$ is the metric on the space $S$. For a bounded Lipschitz function $f$ on $S$ we define $\|f\|_{BL} := \|f\|_1 + \|f\|_{\infty}$. Lip$_1(S)$ (resp. BL$_1(S)$) denotes the class of Lipschitz (resp. bounded Lipschitz) functions $f : S \to \mathbb{R}$ with $\|f\|_1$ (resp. $\|f\|_{BL}$) bounded by 1. Occasionally we will suppress $S$ from the notation and write Lip$_1$ and BL$_1$ when clear from the context. For a Polish space $S$, $\mathcal{P}(S)$ is equipped with the topology of weak convergence. A convenient metric metrizing this topology on $\mathcal{P}(S)$ is given as $\beta(\mu, \gamma) = \sup \left\{ \int f \, d\mu - \int f \, d\gamma : \|f\|_{BL} \leq 1 \right\}$ for $\mu, \gamma \in \mathcal{P}(S)$. For a signed measure $\mu$ on $\mathbb{R}^d$, we define $\langle f, \gamma \rangle := \int f \, d\gamma$ whenever the integral makes sense. The space $\mathcal{P}_1(\mathbb{R}^d)$ will be equipped with the Wasserstein-1 distance that is defined as follows:

$$W_1(\mu_0, \gamma_0) := \inf_{\pi \in \mathcal{P}(X \times Y)} E[|X - Y|], \quad \mu_0, \gamma_0 \in \mathcal{P}_1(\mathbb{R}^d),$$

where the infimum is taken over all $\mathbb{R}^d$ valued random variables $X, Y$ defined on a common probability space and where the marginals of $X, Y$ are $\mu_0$ and $\gamma_0$ respectively. From Kantorovich-Rubenstein duality (cf. [17]) one sees the Wasserstein-1 distance has the following characterization

$$W_1(\mu_0, \gamma_0) = \sup_{f \in \text{Lip}_1(\mathbb{R}^d)} |\langle f, \mu_0 - \gamma_0 \rangle|, \quad \mu_0, \nu_0 \in \mathcal{P}_1(\mathbb{R}^d). \tag{1.6}$$

For a signed measure $\mu$ on $(S, \mathcal{B}(S))$, the total variation norm of $\mu$ is defined as $|\mu|_{TV} := \sup \|\mu\|_{\leq} |\langle f, \mu \rangle|$. Probability distribution of a $S$ valued random variable $X$ will be denoted as $\mathcal{L}(X)$. Convergence in distribution of a $S$ valued sequence $\{X_n\}_{n \geq 1}$ to a $S$ valued random variable $X$ will be written as $X_n \Rightarrow X$.

A finite collection $\{Y_1, Y_2, \ldots, Y_N\}$ of $S$ valued random variables is called exchangeable if

$$\mathcal{L}(Y_1, Y_2, \ldots, Y_N) = \mathcal{L}(Y_{\pi(1)}, Y_{\pi(2)}, \ldots, Y_{\pi(N)})$$

for every permutation $\pi$ on the $N$ symbols $\{1, 2, \ldots, N\}$. Let $\{Y_i^N, i = 1, \ldots, N\}_{N \geq 1}$ be a collection of $S$ valued random variables, such that for every $N$, $\{Y_1^N, Y_2^N, \ldots, Y_N^N\}$ is exchangeable. Let $\nu_N = \mathcal{L}(Y_1^N, Y_2^N, \ldots, Y_N^N)$. The sequence $\{\nu_N\}_{N \geq 1}$ is called $\nu$-chaotic (cf. [16]) for a $\nu \in \mathcal{P}(\mathcal{S})$, if for any $k \geq 1$, $f_1, f_2, \ldots, f_k \in C_b(\mathcal{S})$, one has

$$\lim_{N \to \infty} \langle f_1 \otimes f_2 \otimes \ldots \otimes f_k \otimes 1 \ldots \otimes 1, \nu_N \rangle = \prod_{i=1}^k \langle f_i, \nu \rangle. \tag{1.7}$$

Denoting the marginal distribution on first $k$ coordinates of $\nu_N$ by $\nu_N^k$, equation (1.7) says that, for every $k \geq 1$, $\nu_N^k \to \nu^{\otimes k}$. The gradient of a real differentiable function $f$ on $\mathbb{R}^d$ denoted by $\nabla f$ is defined as the $d$ dimensional vector field $\nabla f := \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_d} \right)^t$. For a function $f : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$

$$\nabla_x f(x, y) := \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_d} \right)^t.$$ 

The function $\nabla_y f(x, y)$ is defined similarly. Absolute continuity of a measure $\mu$ with respect to a measure $\nu$ will be denoted by $\mu \ll \nu$. We will denote the Radon-Nikodym derivative of $\mu$ with respect to $\nu$ by $\frac{d\mu}{d\nu}$. For $f \in BM(\mathcal{S})$ and a transition probability kernel $P$ on $S$, define $PF \in BM(\mathcal{S})$ as $PF(\cdot) = \int_S f(y)P(\cdot, dy)$. For any closed subset $B \subseteq S$, and $\mu \in \mathcal{P}(B)$, define $\mu P \in \mathcal{P}(S)$ as $\mu P(A) = \int_A P(x, A)\mu(dx)$. For a matrix $B$ the usual operator norm is denoted by $\|B\|$. 

## 2 Description of the nonlinear system:

We now describe the nonlinear dynamical system obtained on taking the limit $N \to \infty$ of $(\mu_N, \eta_N^0)$. Given a $C^1$ density function $\rho$ on $\mathbb{R}^d$ and $\mu \in \mathcal{P}(\mathbb{R}^d)$, define a transition probability kernel $Q^{\mu, \rho}$ on $\mathbb{R}^d$ as

$$Q^{\rho, \mu}(x, C) = \int_{\mathbb{R}^m} 1_{\{Ax + \delta f(\nabla \rho(z), \mu, x, z) + B(z) \in C\}} \theta(dz), \quad (x, C) \in \mathbb{R}^d \times \mathcal{B}(\mathbb{R}^d).$$
With an abuse of notation we will also denote by $Q^{\rho, \mu}$ the map from $BM(\mathbb{R}^d)$ to itself, defined as

$$Q^{\rho, \mu} \phi(x) = \int_{\mathbb{R}^d} \phi(y) Q^{\rho, \mu}(x, dy), \quad \phi \in BM(\mathbb{R}^d), x \in \mathbb{R}^d.$$  

For $\mu, \mu_1 \in \mathcal{P}(\mathbb{R}^d)$, let $\mu Q^{\rho, \mu_1} \in \mathcal{P}(\mathbb{R}^d)$ be defined as

$$\mu Q^{\rho, \mu_1}(C) = \int_{\mathbb{R}^d} Q^{\rho, \mu_1}(x, C) \mu(dx), \quad C \in \mathcal{B}(\mathbb{R}^d).$$  

(2.1)

Note that $\mu Q^{\rho, \mu_1} = \mathcal{L}(AX + \delta f(\nabla \rho(X), \mu_1, X, \epsilon) + B(\epsilon))$ where $\mathcal{L}(X, \epsilon) = \mu \otimes \theta$.

Define $\mathcal{P}_1(\mathbb{R}^d) := \{\mu \in \mathcal{P}_1(\mathbb{R}^d) : \mu \ll d \mathcal{H}^d$ is continuously differentiable and $||\nabla \frac{d\mu}{d\mathcal{H}^d}||_1 < \infty\}$. For notational simplicity we will identify an element in $\mathcal{P}_1(\mathbb{R}^d)$ with its density and denote both by the same symbol. Define the map $\Psi : \mathcal{P}(\mathbb{R}^d) \times \mathcal{P}_1(\mathbb{R}^d) \to \mathcal{P}(\mathbb{R}^d) \times \mathcal{P}(\mathbb{R}^d)$ as

$$\Psi(\mu, \eta) = (\mu Q^{\rho, \mu}, \eta R^\alpha), \quad (\mu, \eta) \in \mathcal{P}(\mathbb{R}^d) \times \mathcal{P}_1(\mathbb{R}^d).$$  

(2.2)

Under suitable assumptions (which will be introduced in Section 3) it will follow that for every $(\mu, \eta) \in \mathcal{P}_1(\mathbb{R}^d) \times \mathcal{P}_1(\mathbb{R}^d)$, $\eta^+$ defined by (1.4) is in $\mathcal{P}_1(\mathbb{R}^d)$ and $\mu Q^{\rho, \mu}$ defined by (2.1) is in $\mathcal{P}_1(\mathbb{R}^d)$. Thus (under those assumptions) $\Psi$ is a map from $\mathcal{P}_1(\mathbb{R}^d) \times \mathcal{P}_1(\mathbb{R}^d)$ to itself. Using the above notation we see that $\{(X^1_n, ..., X^n_N, \mu_N, \eta_N)\}_{n \geq 0}$ is a $(\mathbb{R}^d)^N \times \mathcal{P}_1(\mathbb{R}^d) \times \mathcal{P}_1(\mathbb{R}^d)$ valued discrete time Markov chain defined recursively as follows. Let $X_k(N) \equiv (X^1_k, X^2_k, ..., X^N_k)$, and $\eta_N^k$ be the initial chemical field which is a random element of $\mathcal{P}_1(\mathbb{R}^d)$. Let $\mathcal{F}_0 = \sigma\{X_0(N), \eta_0^N\}$. Then, for $k \geq 1$

$$\begin{aligned}
\begin{cases}
P(X_k(N) \in C | \mathcal{F}_{k-1}^N) = \bigotimes_{i=1}^N \delta_{X^i_k} Q^{\eta^i_{k-1}, \mu^i_{k-1}}(C) & \forall C \in \mathcal{B}(\mathbb{R}^d)^N, \\
\mu^i_k = \frac{1}{N} \sum_{i=1}^N \delta_{X^i_k}, \\
\eta^i_k = \eta^i_{k-1} R^{\alpha}_{\mu^i_{k-1}} \\
\mathcal{F}_{k} = \sigma\{\eta^N_{k-1}, X_k(N)\} \cup \mathcal{F}_{k-1}^N.
\end{cases}
\end{aligned}$$  

(2.3)

We will call this particle system as $\mathbb{IPS}_1$. We next describe a nonlinear dynamical system which is the formal Vlasov-McKean limit of the above system, as $N \to \infty$. Given $(\mu_0, \eta_0) \in \mathcal{P}_1(\mathbb{R}^d) \times \mathcal{P}_1(\mathbb{R}^d)$ define a sequence $\{(\mu_n, \eta_n)\}_{n \geq 0}$ in $\mathcal{P}_1(\mathbb{R}^d) \times \mathcal{P}_1(\mathbb{R}^d)$ as

$$\begin{aligned}
\mu_{n+1} = \mu_n Q^{\eta_n, \mu_n}, \\
\eta_{n+1} = \eta_n R^{\alpha}_{\mu_n}, \quad n \geq 0.
\end{aligned}$$  

(2.4)

Using (2.2) the above evolution can be represented as

$$\begin{aligned}
(\mu_{n+1}, \eta_{n+1}) = \Psi(\mu_n, \eta_n), \quad n \in \mathbb{N}_0.
\end{aligned}$$  

(2.5)

As in [5], the starting point of our investigation on long time asymptotics of the above interacting particle system will be to study the stability properties of (2.4). We identify $\eta, \eta' \in \mathcal{P}(\mathbb{R}^d)$ that are equal a.e under the Lebesgue measure on $\mathbb{R}^d$. From a computational point of view we are approximating $(\mu_n, \eta_n)$ by $(\mu_n^N, \eta^N_n)$ uniformly in time parameter $n$, with explicit uniform concentration bounds. Such results are particularly important for developing sampling methods for approximating the steady state distribution of the mean field models such as in (2.4).

The third equation in (2.3) makes the simulation of $\mathbb{IPS}_1$ numerically challenging. In section 3 we will mention another particle system (based on the second particle system in [3]) referred to as $\mathbb{IPS}_2$ which also gives an asymptotically consistent approximation of (2.4) and is computationally more tractable. We show in Theorem 3.2 that under conditions that include a Lipschitz property of $f$ (Assumptions 1 and 2), smoothness assumptions on the transition kernels of the background diffusion of the chemical medium (Assumption 4) the Wasserstein-1($\mathcal{W}_1$) distance between the occupation measure of the particles along with the chemical medium $(\mu_n^N, \eta_n^N)$ and $(\mu_n, \eta_n)$ converges to 0, for every time instant $n$. Under an additional condition on the contractivity of $A$ and $\delta$, $\alpha$ being sufficiently small we show that the nonlinear system (2.5) has a unique fixed point and starting from an arbitrary initial condition, convergence to the fixed point occurs at a geometric rate. Using these results we next argue in Theorem 1 that under some integrability conditions (Assumption 7-8), as $N \to \infty$, the empirical occupation measure of the $N$-particles
and density of the chemical medium at time instant $n$, namely $(\mu_n^N, \eta_n^N)$ converges to $(\mu_n, \eta_n)$ in the $W_1$ distance, in $L^1$, uniformly in $n$. This result in particular shows that the $W_1$ distance between $(\mu_n^N, \eta_n^N)$ and the unique fixed point $(\mu_\infty, \eta_\infty)$ of (2.5) converges to zero as $n \to \infty$ and $N \to \infty$ in any order. We next show that for each $N$, there is unique invariant measure $\Theta_N^\infty$ of the N-particle dynamics with integrable first moment and this sequence of measures is $\mu_\infty$-chaotic, namely as $N \to \infty$, the projection of $\Theta_N^\infty$ on the first $k$-coordinates converges to $\mu_\infty^k$ for every $k \geq 1$. This propagation of chaos property all the way to $n = \infty$ crucially relies on the uniform in time convergence of $(\mu_n^N, \eta_n^N)$ to $(\mu_\infty, \eta_\infty)$. Such a result is important since it says that the steady state of a $N$-dimensional fully coupled Markovian system has a simple approximate description in terms of a product measure when $N$ is large. This result is key in developing particle based numerical schemes for approximating the fixed point of the evolution equation (2.5). Next we present some uniform in time concentration bounds of $W_1(\mu_n^N, \mu_n) + W_1(\eta_n^N, \eta_n)$. Proof is very similar to that of Theorem 3.8 of [5] so we only provide a sketch after showing necessary conditions.

3 Main Results:

We now introduce our main assumptions on the problem data. Recall that $\{X_i^0, i = 1, \ldots, N\}$ is assumed to be exchangeable with common distribution $\mu_0$. We assume further $(\mu_0, \eta_0) \in P_1(\mathbb{R}^d) \times P_1^*(\mathbb{R}^d)$. For a $d \times d$ matrix $B$ we denote its norm by $\|B\|$, i.e. $\|B\| = \sup_{x \in \mathbb{R}^d \setminus \{0\}} \frac{|Bx|}{|x|}$.

**Assumption 1** The error distribution $\theta$ is such that $\int_A(z) \theta(dz) := \sigma \in (0, \infty)$ where

$$A_1(\epsilon) := \sup_{\{x_1, x_2, y_1, y_2 \in \mathbb{R}^d, \mu_1, \mu_2 \in P_1(\mathbb{R}^d), \mu_1 \neq \mu_2, x_1 \neq x_2, y_1 \neq y_2\}} \frac{|f(y_1, \mu_1, x_1, \epsilon) - f(y_2, \mu_2, x_2, \epsilon)|}{|x_1 - x_2| + |y_1 - y_2| + W_1(\mu_1, \mu_2)}.$$  (3.1)

It follows that $\forall x, y \in \mathbb{R}^d, \mu \in P_1(\mathbb{R}^d)$,

$$|f(y, \mu, x, \epsilon)| \leq (|y| + \|\mu\|_1 + |x|)A_1(\epsilon) + A_2(\epsilon)$$  (3.2)

where $A_2(\epsilon) := f(0, 0, \epsilon)$.

Recall the function $B : \mathbb{R}^m \to \mathbb{R}^d$ introduced in (1.2).

**Assumption 2** The error distribution $\theta$ is such that

$$\int_{\mathbb{R}^m} \left(A_2(z) + |B(z)|\right) \theta(dz) < \infty.$$

**Assumption 3** $\eta_0^N$ (the density function) is a Lipschitz function on $\mathbb{R}^d$ and $\eta_0^N \in P_1^*(\mathbb{R}^d)$.

Assumptions 4 and 5 on the kernels $P$ and $P'$ hold quite generally. In particular, they are satisfied for Gaussian kernels.

**Assumption 4** There exist $t_P^S \in (0, 1]$ and $t_{P'}^S \in (0, \infty)$ such that for all $x, y, x', y' \in \mathbb{R}^d$

$$|\nabla_y P(x, y) - \nabla_y P(x', y')| \leq t_P^S (|y - y'| + |x - x'|)$$  (3.3)

$$|\nabla_y P'(x, y) - \nabla_y P'(x', y')| \leq t_{P'}^S (|y - y'| + |x - x'|).$$  (3.4)

Furthermore

$$\sup_{x \in \mathbb{R}^d} \{|\nabla_y P(x, 0)| \vee |\nabla_y P'(x, 0)|\} < \infty.$$  (3.5)

Using the Lipschitz property in (3.3) and the growth condition (3.6) one has the linear growth property for some $M_P^\infty \in (0, \infty)$

$$\sup_{x \in \mathbb{R}^d} |\nabla_y P(x, y)| \leq M_P^\infty (1 + |y|).$$  (3.6)

A similar inequality holds for $P'$ from (3.4) with $M_{P'}^\infty \in (0, \infty)$. 


Denote \((1 - \alpha)l_P^\alpha + \alpha l_P^\gamma\) by \(l_{P,P'}^\alpha\).

**Assumption 5** For every \(f \in \text{Lip}_1(\mathbb{R}^d)\), \(Pf\) and \(P'f\) are also Lipschitz and

\[
\sup_{f \in \text{Lip}_1(\mathbb{R}^d)} \sup_{x \neq y \in \mathbb{R}^d} \frac{|Pf(x) - Pf(y)|}{|x - y|} := l(P) < \infty
\]

Also \(l(P')\) defined as above for \(P'\) is finite.

**Assumption 6** Both \(P(x, \cdot)\) and \(P'(x, \cdot)\) are such that for any compact set \(K \subset \mathbb{R}^d\), the families of probability measures \(\{P(x, \cdot) : x \in K\}\) and \(\{P'(x, \cdot) : x \in K\}\) are both uniformly integrable.

Let \(\max\{l(P), l(P')\} = l_{P,P'}\).

**Remark 3.1** Assumption 5 is satisfied if \(P, P'\) are given as follows. For any \(f \in C_b(\mathbb{R}^d)\), let

\[
Pf(\cdot) := Ef(g_1(\cdot, \varepsilon_1)), \quad P'f(\cdot) := Ef(g_2(\cdot, \varepsilon_2))
\]

(3.7)

where \(\varepsilon_1, \varepsilon_2\) are \(\mathbb{R}^m\) valued random variables and \(g_1, g_2 : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}\) are maps with following properties:

\[
E(G_1(\varepsilon_1)) \leq l(P) \quad \text{and} \quad E(G_2(\varepsilon_2)) \leq l(P'),
\]

(3.8)

where

\[
G_1(y) := \sup_{x_1 \neq x_2} \frac{g_1(x_1, y) - g_1(x_2, y)}{|x_1 - x_2|} \quad \text{and} \quad G_2(y) := \sup_{x_1 \neq x_2} \frac{g_2(x_1, y) - g_2(x_2, y)}{|x_1 - x_2|}.
\]

Simulation of the system is numerically intractable due to the step that involves the updating of \(\eta^N_{n-1}\) to \(\eta^N_n\). This requires computing the integral in (1.4) which, since \(R^\alpha_k\) is a mixture of two transition kernels, over time leads to an explosion of terms in the mixture that need to be updated. An approach (proposed in [3]) that addresses this difficulty is, without directly updating \(\eta^N_{n-1}\), to use the empirical distribution of the observations drawn independently from \(\eta^N_{n-1}\).

Denote \(\bar{X}_0(N)\) by \((\bar{X}^{1,N}_0, \ldots, \bar{X}^{N,N}_0)\) a sample of size \(N\) from \(\mu_0\). Let \(M \in \mathbb{N}\). The new particle scheme will be described as a family \((\bar{X}_k(N), \bar{\mu}^N_k, \bar{\eta}^M_{k-1})_{k \in \mathbb{N}}\) of \((\mathbb{R}^d)^N \times \mathcal{P}(\mathbb{R}^d) \times \mathcal{P}^*(\mathbb{R}^d)\) valued random elements on some probability space defined recursively as follows. Set \(\bar{X}_0(N) = (\bar{X}_0^{1,N}, \ldots, \bar{X}_0^{N,N}), \bar{\eta}^M_0 = \eta_0, \bar{\mathcal{F}}^M_0 = \sigma(\bar{X}^N(0))\). For \(k \geq 1\)

\[
\begin{align*}
\bar{\mu}^N_k &= \frac{1}{N} \sum_{i=1}^N \delta_{\bar{X}^{i,N}_k}, \\
P(\bar{X}_k(N) \in C|\bar{\mathcal{F}}^{M,N}_{k-1}) &= \bigotimes_{i=1}^N (\delta_{\bar{X}^{i,N}_{k-1}} \cdot \mathcal{Q}^{\bar{\mu}^{M,N}_{k-1}})(C) \quad \forall C \in \mathcal{B}(\mathbb{R}^d)^N, \\
\bar{\eta}^{M}_{k-1} &= (1 - \alpha)(S^M(\bar{\eta}^{M}_{k-1})P) + \alpha \bar{\mu}^{N}_{k-1}P', \\
\bar{\mathcal{F}}^{M,N}_{k} &= \sigma(\bar{\eta}^{M}_{k}, \bar{X}_k(N)) \lor \bar{\mathcal{F}}^{M,N}_{k-1}
\end{align*}
\]

(3.10)

where \(S^M(\bar{\eta}^{M}_{k-1})\) is the random measure defined as \(\frac{1}{M} \sum_{i=1}^M \delta_{\bar{Y}^{i,M}_{k-1}}\) where \(\{\bar{Y}^{i,M}_{k-1}\}_{i=1,...,M}\) conditionally on \(\bar{\mathcal{F}}^{M,N}_{k-1}\), are \(M\) i.i.d distributed according to \(\bar{\eta}^{M}_{k-1}\). We will call this particle system as \(\mathbb{P}\mathbb{S}\_2\). We remark that our notation is not accurate since both the quantities \(\bar{\mu}^N_k, \bar{\eta}^M_k\) depend on \(M, N\). The superscripts only describe the number of particles/samples used in the procedure to combine them. Note that like \(\mathbb{P}\mathbb{S}\_1\), here \((\bar{X}_k(N), \bar{\eta}^{M}_k)_{k \geq 0}\) is not a Markov chain on \((\mathbb{R}^d)^N \times \mathcal{P}^*(\mathbb{R}^d)\) anymore. Rather \((\bar{X}^{N}(k), \bar{\eta}^{M}_k, S^M(\bar{\eta}^{M}_k))_{k \geq 0}\) is a discrete time Markov chain on \((\mathbb{R}^d)^N \times \mathcal{P}^*(\mathbb{R}^d) \times \mathcal{P}_1(\mathbb{R}^d)\).

For any random variable \(Z\) we denote \(E[Z|\bar{\mathcal{F}}^{M,N}_{k}]\) by \(E^{M,N}_{k}[Z]\). The following result shows that the particle systems in (2.3) and (3.10) approximate the dynamical system in (2.4) as \(N\) (respectively \(\min\{M, N\}\) for \(\mathbb{P}\mathbb{S}\_2\)) becomes large for a fixed time instant.

**Proposition 3.2** Suppose Assumptions 1, 2, 4 and 5 hold.
(a) Consider the particle system \( \mathbb{IPS}_1 \) in (1.3.1.5). Suppose the sampling of the exchangeable datapoints \( X_0(N) \equiv (X^1_0, X^2_0, \ldots, X^N_0) \) is exchangeable and \( \{\mathcal{L}(X_0(N))\}_{N \in \mathbb{N}} \) is \( \mu_0 \)-chaotic. Suppose \( E\mathcal{W}_1(\eta_0^n, \eta_0) \to 0 \) as \( N \to \infty \). Then, as \( N \to \infty \)
\[
E \left[ \mathcal{W}_1(\mu_n^N, \mu_n) + \mathcal{W}_1(\eta_n^n, \eta_n) \right] \to 0 \tag{3.11}
\]
for all \( n \geq 0 \) where \( \mu_n, \eta_n \) are as in (2.4).

(b) Consider the second particle system \( \mathbb{IPS}_2 \). Suppose that in addition Assumption 6 holds. Suppose the sampling of the exchangeable datapoints \( X_0(N) \equiv (X^1_0, X^2_0, \ldots, X^N_0) \) is exchangeable and \( \{\mathcal{L}(X_0(N))\}_{N \in \mathbb{N}} \) is \( \mu_0 \)-chaotic. Then as \( \min\{N, M\} \to \infty \)
\[
E \left[ \mathcal{W}_1(\bar{\mu}_n^N, \mu_n) + \mathcal{W}_1(\bar{\eta}_n^M, \eta_n) \right] \to 0 \tag{3.12}
\]
for all \( n \geq 0 \).

As a consequence of Proposition 3.2, we have a finite time propagation of chaos result of the following form. Let \( \nu_n^N = \mathcal{L}(X^1_n, X^2_n, \ldots, X^n_n) \).

**Corollary 3.3** Under Assumptions as in Proposition 3.2 the family \( \{\nu_n^N\}_{n \geq 1} \) is \( \mu_n \) chaotic for every \( n \geq 1 \).

As noted in introduction, the primary goal is studying long time properties of (1.3) and the non-linear dynamical system (2.4). Following proposition identifies the range of values of the modeling parameters that leads to stability of the system.

**Proposition 3.4** Suppose Assumptions (1)-(5) hold. Then there exist \( \omega_0, \alpha_0, \delta_0 \in (0, 1) \) such that for all \( \|A\| < \omega_0, \alpha \in (0, \alpha_0), \) and \( \delta \in (0, \delta_0) \). The map \( \Psi \) defined in (2.2) has a unique fixed point \( (\mu_\infty, \eta_\infty) \) in \( \mathcal{P}_1(\mathbb{R}^d) \times \mathcal{P}_1^\alpha(\mathbb{R}^d) \).

Now we will give more stringent conditions under which a non-asymptotic bound on convergence rates of the particle system to the deterministic nonlinear dynamics and their consequences for the steady state behavior can be established.

**Assumption 7** For some \( \tau > 0 \),
\[
\mu_0 \in \mathcal{P}_{1+\tau}(\mathbb{R}^d), \quad \int A_1(z)^{1+\tau} \theta(dz) := \sigma_1(\tau) < \infty \quad \int \left( A_2(z) + |B(z)| \right)^{1+\tau} \theta(dz) := \sigma_2(\tau) < \infty.
\]

We need to impose the following condition on \( P, P' \) for uniform in time convergence.

**Assumption 8** For some \( \langle |x|^{1+\tau}, \eta_0 \rangle < \infty \). There exist \( m_\tau(P) \) and \( M_\tau(P') \) in \( \mathbb{R}^+ \) such that following holds for all \( x \in \mathbb{R}^d \)
\[
\int_{\mathbb{R}^d} |y|^{1+\tau} P(x, dy) \leq m_\tau(P) (1 + |x|^{1+\tau}), \quad \text{and} \quad \int_{\mathbb{R}^d} |y|^{1+\tau} P'(x, dy) \leq m_\tau(P') (1 + |x|^{1+\tau}).
\]

Now we state a generalization of the Proposition 3.2, which gives the convergence rate of
\[
E \left\{ \mathcal{W}_1(\mu_n^N, \mu_n) + \mathcal{W}_1(\eta_n^M, \eta_n) \right\} \to 0
\]
uniformly over all \( n \geq 0 \) in a nonasymptotic manner.

Recall \( l_\nu^\alpha, l_\nu^\alpha \) introduced in Assumption 3. For \( \alpha \in (0, 1) \), let \( l_\nu^\alpha = (1 - \alpha)l_\nu + \alpha l_\nu^\alpha \). With the notations of Assumption 1 we define
\[
a_0 := \frac{1 - \|A\|}{\sigma(2 + l_\nu^\alpha)}.
\]

For \( (\mu_n, \eta_n), (\mu'_n, \eta'_n) \in \mathcal{P}_1(\mathbb{R}^d) \times \mathcal{P}_1^\alpha(\mathbb{R}^d) \) define the following distance on \( \mathcal{P}_1(\mathbb{R}^d) \times \mathcal{P}_1^\alpha(\mathbb{R}^d) \)
\[
\mathcal{W}_1((\mu_n, \eta_n), (\mu'_n, \eta'_n)) := \mathcal{W}_1(\mu_n, \mu'_n) + \mathcal{W}_1(\eta_n, \eta'_n).
\]
Theorem 1 Consider the particle system $\mathbb{IPS}_2$. Suppose Assumptions (1)-(5) and Assumptions (7),(8) hold for some $\tau > 0$. Let $N_1 := \min\{M, N\}$. Also assume $\delta \in (0, a_0)$, $(1 - \alpha)c_\ast(P) < 1$ and

$$\max \left\{ (\|A\| + \delta \sigma(2 + l^\alpha_{P,P'})) + \alpha l(P) \right\} + \delta \sigma \max \left\{ \alpha l^\gamma_{P}, (1 - \alpha)l^\gamma_{P} \right\} < 1,$$

Then there exists $\theta < 1$, and $a \in (0, \infty)$ such that for each $n \geq 0$, the upperbound $b(N_1, \tau, d)$ of

$$EW_1\left((\mu^n_N, \eta^n_M, (\mu_n, \eta_n)\right) - a^n EW_1\left((\mu^0_N, \eta^0_M, (\mu_0, \eta_0)\right)$$

can be expressed as

$$b(N_1, \tau, d) = C \left\{ \begin{array}{ll}
N_1^{-\max\{\frac{1}{d}, \frac{1}{\tau}, \frac{1}{\delta}\}} & \text{if } d = 1, \tau \neq 1, \\
N_1^{\frac{1}{d}} \log N_1 & \text{if } d = 1, \tau = 1, \\
N_1^{-\frac{1}{d}} \log N_1 + N_1^{-\frac{1}{\tau}} & \text{if } d = 2, \tau \neq 1, \\
N_1^{-\frac{1}{d}} (\log N_1)^2 & \text{if } d = 2, \tau = 1, \\
N_1^{-\max\{\frac{1}{d}, \frac{1}{\tau}, \frac{1}{\delta}\}} & \text{if } d > 2, \tau \neq \frac{1}{d-1}, \\
N_1^{-\frac{1}{d}} \log N_1 & \text{if } d > 2, \tau = \frac{1}{d-1}, \\
\end{array} \right. \quad (3.13)$$

where the value of the constant $C$ will vary for each of the cases.

Remark 3.5 For the first particle system (1.3.1-5) similar results hold where the explicit bounds are given in terms of number of particles $N$ instead of $N_1$. For $\mathbb{IPS}_2$ if the initial sampling scheme of $X_0(N) \equiv (X_0^1, X_0^2, ..., X_0^N)$ is $\mu_0$-chaotic then using the fact $EW_1(\mu^0_N, \mu_0) \to 0$ as $N \to \infty$, it follows from the conclusion of the Theorem 1

$$\sup_{n \geq 0} EW_1\left((\mu^n_N, \eta^n_M, (\mu_n, \eta_n)\right) \to 0$$

as $\min\{N, M\} \to \infty$. For the first particle system in (1.3.1-5), if $EW_1(\eta^N_0, \eta_0) \to 0$ as $N \to \infty$, and $X_0(N) \equiv (X_0^1, X_0^2, ..., X_0^N)$ is $\mu_0$-chaotic then following

$$\sup_{n \geq 0} EW_1\left((\mu^n_n, \eta^n_N, (\mu_n, \eta_n)\right) \to 0$$

holds for $N \to \infty$.

One consequence of above theorem and Proposition 3.4 will be the following interchange of limit results which is analogous to Corollary 3.5 from [5].

Corollary 3.6 Under conditions of Theorem 1

$$\limsup_{n \to \infty} \limsup_{n \to \infty} \sup_{n \geq 0} EW_1((\mu^n_N, \eta^n_M, (\mu_n, \eta_n)) = \limsup_{n \to \infty} \limsup_{n \to \infty} EW_1((\mu^n_N, \eta^n_M, (\mu_n, \eta_n)) = 0.$$ （3.14）

Suppose Assumptions of Theorem 1 hold and let $(\mu_\infty, \eta_\infty)$ be the fixed point of the map $\Psi$ of (2.5). We are interested in establishing a propagation of chaos result for $n = \infty$. Recall for $\mathbb{IPS}_2$, $S^M(\eta^n_M)$ is the random measure defined as $\frac{1}{M} \sum_{i=1}^{M} \delta_{Y^n_i,M}$ where $\{Y^n_i\}_{i=1,...,M}$ conditionally on $F_n$, are $M$ i.i.d distributed $\mathbb{R}_d$ valued random variables according to $\eta^n_M$. Denote $Y_n(M) := (Y^n_1^M, ..., Y^n_M^M)$.

Theorem 2 Consider the second particle system $\mathbb{IPS}_2$. Suppose Assumptions 1,2,4,5 hold with conditions

$$\delta \in (0, a_0), \quad \sum_{i=0}^{\infty} (1 - \alpha)^i \int_{\mathbb{R}_d} |y| P^i P^i(0, dy) < \infty.$$

Then for every $N, M \geq 1$, the Markov process $(\tilde{X}^N(n), \eta^n_M, S^M(\eta^n_M))_{n \geq 0}$ on $(\mathbb{R}^d)^N \times \mathcal{P}_C(\mathbb{R}^d) \times \mathcal{P}^C(\mathbb{R}^d)$ has a unique invariant measure $\Theta_{\infty}^{N, M}$ if following holds

$$\max \left\{ (\|A\| + \delta \sigma(2 + l^\alpha_{P,P'})) + \alpha l(P) \right\} + \delta \sigma \max \left\{ \alpha l^\gamma_{P}, (1 - \alpha)l^\gamma_{P} \right\} < 1.$$
Let $\Theta^1_{\infty,M}$ be the marginal distribution on $(\mathbb{R}^d)^N$ of the first co-ordinate of $\Theta^1_{\infty,M}$. Suppose additionally Assumption 4.3 and Assumption 7.8 hold with further condition for some $\tau > 0$

$$(1-\alpha)l_\tau(P) < 1.$$ 

Then $\Theta^1_{\infty,M}$ is $\mu_\infty$- chaotic, where $\mu_\infty$ is defined in Proposition 3.4.

**Remark 3.7** For first particle system $(\mathbb{IP}_3)$ similar steady state result holds for the discrete time Markov chain $(X^N(n),\eta^N_n)_{n \geq 0}$ on $(\mathbb{R}^d)^N \times P^1_{\infty}(\mathbb{R}^d)$.

### 3.1 Concentration Bounds:

In order to obtain uniform in time concentration bounds of $W_1((\mu^N_n, \eta^N_n), (\mu_n, \eta_n))$ we proceed according to those in Theorem 3.7 and Theorem 3.8 of [5] respectively. Here we establish two different types of concentration bounds. The first one is with initial non iid (i.e initial samples are $\mu_0$ chaotic) assumption and the second one is without that.

**Assumption 9** (i) For some $K \in (1, \infty)$, $A_1(x) \leq K$ for $\theta$ a.e. $x \in \mathbb{R}^m$.

(ii) There exists $\alpha \in (0, \infty)$ such that $\int e^{\alpha|x|} \mu_0(dx) < \infty$ and there exists $\alpha(\delta) \in (0, \alpha)$ such that

$$\int_{\mathbb{R}^m} e^{\alpha(\delta)(A_2(z)+\frac{|h(z)|}{\delta})} \theta(dz) < \infty.$$ 

**Assumption 10** Suppose there exists functions $h_1(\cdot), h_2(\cdot), h_3(\cdot), h'_1(\cdot), h'_2(\cdot), h'_3(\cdot)$ such that $h_2(0) = 0$, $h'_2(0) = 0$; and constants $l_{h_1} \in (0,1], l_{h'_1} \in (0,\infty)$ such that $h_1(x)$, and $h'_1(x)$ are respectively $h_1$ and $h'_1$ Lipschitz. There exists $\alpha \in (0, \infty)$ such that following hold for all $\alpha \in (0, \alpha)$

$$\int e^{\alpha |x|} P(x, dy) \leq e^{h_2(\alpha)} (e^{\alpha h_1(x)} + e^{h_3(\alpha)}), \quad \int e^{\alpha |y|} P'(x, dy) \leq e^{h'_2(\alpha)} (e^{\alpha h'_1(x)} + e^{h'_3(\alpha)}). \quad (3.15)$$

**Remark 3.8** (a) For Gaussian transition kernel $P(x, dy) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-y)^2}{2\lambda^2}} dy$, one has

$$\int e^{\alpha |y|} P(x, dy) = e^{\frac{\alpha^2}{2\lambda^2}} \left[ e^{-\alpha^2 \Phi(-\frac{x}{\lambda})} + e^{\alpha^2 \Phi(\alpha \lambda + \frac{x}{\lambda})} \right],$$

where $\Phi(\cdot)$ is the cumulative distribution function of Normal distribution. So (3.15) holds with $h_1(x) = x$, $h_3(x) = 0$, $h_2(\alpha) = \frac{\lambda^2 \alpha^2}{2\lambda^2}$.

(b) For Bi-exponential kernel $P(x, dy) = \frac{1}{2\pi} e^{-\frac{|x-y|}{\lambda}} dy$ one has

$$\int e^{\alpha |y|} P(x, dy) = e^{\alpha x} \left[ \frac{1}{1-\alpha^2 \lambda^2} \right].$$

So (3.15) holds under condition $\alpha < \frac{1}{\lambda}$. Note that any kernel with tail lighter than exponential (like Gaussian) will satisfy (3.15) for all $\alpha$, where for kernels with exponential like tail will have a specific restriction on $\alpha$.

(c) We worked here only for $l_{h_1} = 1$ as the upper bound. It only influences in the choice of $\alpha$ for which

$$\sup_{n \geq 0} \sup_{M \geq 1} \mathbb{E} \left[ e^{\alpha |x|}, \tilde{\eta}^M_n \right] < \infty. \quad (3.16)$$

For $l_{h_1} = 1$ one has a definite upper bound of $\alpha_1$. More precisely denoting $\alpha h_1(0) \sum_{j=0}^i l_{h_1} + \sum_{j=0}^i h_2(\alpha l_{h_1})$ by $g(i)$ if $g(i)$ is linear in $i$ (happens only for $l_{h_1} = 1$) then there exists $\alpha^* > 0$ such that (3.16) holds for $\alpha_1 < \alpha^*$. On the other hand if $g(i)$ is bounded, then $\sup_{n \geq 0} \sup_{M \geq 1} \mathbb{E} \left[ e^{\alpha_1 |x|}, \tilde{\eta}^M_n \right]$ will remain finite for all $\alpha_1 > 0$. If $g(i)$ is exponential in $i$ (when $l_{h_1} > 1$) then the upper bound of $\sup_{n \geq 0} \sup_{M \geq 1} \mathbb{E} \left[ e^{\alpha_1 |x|}, \tilde{\eta}^M_n \right]$ will diverge.
With $\tau, \sigma_1(\tau)$ defined above in Assumption 7 let
\[
a(\tau) := \frac{4^{-\tau} - ||A||^{1+\tau}}{\sigma_1(\tau)[1 + (1 + l_{PP})^{1+\tau}]}.
\]

**Theorem 3** (a) (Polynomial Concentration) Let $N_1 = \min\{M, N\}$. Suppose Assumptions (1-5) and Assumptions (7), (8) hold for some $\tau > 0$. Suppose that $\delta \in (0, a(\tau)], (1 - \alpha)l(P) < 1$ and
\[
\max\left\{\left(\|A\| + \delta\sigma(2 + l_{PP}^\alpha) + \alpha l(P'), (1 - \alpha)l(P)\right)\right\} + \delta\sigma \max\{\alpha l(P), (1 - \alpha)l(P)\} < 1.
\]
Then there exists $\nu > 1, \gamma \in (0, 1)$, $N_0 \in \mathbb{N}_0$ and $C_1 \in (0, \infty)$ such that for all $\varepsilon > 0$, and for all $n \geq 0$,
\[
P(W_1((\mu_n^N, \eta_n^M), (\mu_n, \eta_n)) > \varepsilon) \leq P(W_1((\mu_0^N, \eta_0^M), (\mu_0, \eta_0)) > \gamma \nu^n \varepsilon) + C_1 \varepsilon^{-(1 + \alpha)} N_1^{\frac{4^{\alpha - 1}}{\sigma_1(\tau)}}
\]
for all $N_1 > N_0$ ($\max\{1, \log^+ \varepsilon\}$). (3.18)

(b) (Exponential Concentration) Let $N_1 = \min\{M, N\}$. Suppose that Assumptions 9 and 10 hold with (3.18). Suppose $\delta \in \left[0, \frac{1 - \|A\|}{(2 + l_{PP}^\alpha)\varepsilon}\right]$ and $\alpha_1 \in \left[0, \min\{\alpha^*, \frac{\alpha(\delta \varepsilon)}{8}\}\right]$ where
\[
\alpha^* |h_1(0)| + h_2(\alpha^*) = -\log(1 - \alpha).
\]
Then there exists $N_0 \in \mathbb{N}, \nu > 1, \gamma \in (0, 1)$ and $C_2 \in (0, \infty)$ such that for all $\varepsilon > 0$
\[
P(W_1((\mu_n^N, \eta_n^M), (\mu_n, \eta_n)) > \varepsilon) \leq P(W_1((\mu_0^N, \eta_0^M), (\mu_0, \eta_0)) > \gamma \nu^n \varepsilon) + e^{-C_1 \varepsilon N_1^{1/d+2}},
\]
for all $n \geq 0, N_1 \geq N_0 \max\{(\frac{1}{\varepsilon} \log^+ \frac{1}{\varepsilon})^{d+2}, (\varepsilon^{(d+2)/(d-1)})^d\}$, if $d > 1$; and
\[
P(W_1((\mu_n^N, \eta_n^M), (\mu_n, \eta_n)) > \varepsilon) \leq P(W_1((\mu_0^N, \eta_0^M), (\mu_0, \eta_0)) > \gamma \nu^n \varepsilon) + e^{-C_1 \varepsilon \alpha^* N_1^{1/d+2}},
\]
for all $n \geq 0, N_1 \geq N_0 \max\{(\frac{1}{\varepsilon} \log^+ \frac{1}{\varepsilon})^{d+2}, 1\}$, if $d = 1$.

**Remark 3.9** (a) Similar concentration bounds hold for the first particle system $\mathbb{IPS}_1$.

(b) Here the nonlinearity in the kernel of the nonlinear Markov process has a linear structure (linear combination of $P$ and $\mu P'$) which is handled through $W_1$ distance. It can be further generalized for any nonlinear Markov process where the nonlinearity in the kernel depends on the higher order moments (of $\delta$th order) of the law of the chain, then working with $W_\delta$ distance would yield similar results.

Note that the bounds in Theorems 3 are not dimensions independent while the initial sampling assumptions are not restrictive. It will be interesting to see if one can get sharper bounds under stronger conditions than above theorems. The following result shows that such bounds can be obtained in cases where initial locations of $N$ particles are i.i.d. and under a more stringent condition on other parameters.

**Theorem 4** Consider the first particle system $\mathbb{IPS}_1$ with initial condition $\eta_0^N \equiv \eta_0$. Suppose that $\{X_0^{i,N}\}_{i=1,\ldots,N}$ are i.i.d. with common distribution $\mu_0$ for each $N$. Let
\[
C_1 := \delta K \max\{1, (1 - \alpha)(1 + l_{PP}^\alpha)\max\left\{\|A\| + \delta K(1 + l_{PP}^\alpha), \alpha l(P')\right\} \left[\|A\| + \delta K(1 + l_{PP}^\alpha) - \max\{\alpha l(P'), (1 - \alpha)l(P)\}\right] \right\}
\]
\[
\chi_1 := \delta K \max\left\{\|A\| + \delta K(1 + l_{PP}^\alpha), \alpha l(P'), (1 - \alpha)l(P)\right\} + C_1
\]
Suppose that Assumptions 1, 4, 5 and 9 hold with conditions $\chi_1 \in (0, 1), \delta \in \left[0, \frac{1 - ||A\|}{(2 + l_{PP}^\alpha)\varepsilon}\right]$ and $\alpha_1 < \frac{\delta\varepsilon}{8}$. Then there exist $a_1, a_2, a_1', a_2', a_1''$ in $(0, \infty)$ and $N_0, N_1, N_2$ for all $\varepsilon > 0$
\[
\sup_{n \geq 0} P(W_1(\mu_n^N, \mu_n) > \varepsilon) \leq \left\{\begin{array}{ll}
a_1 e^{-N_2 a_2(\varepsilon^2 / \varepsilon^2)} \mathbb{1}_{\{d=1\}} & \text{if } N \geq N_1 \max\left\{\frac{1}{\varepsilon}, \frac{1}{\varepsilon^2}\right\}, \\
a_1' e^{-N_2 a_2'(\varepsilon^2 / \varepsilon^2)} \mathbb{1}_{\{d=2\}} & \text{if } N \geq N_2 \max\left\{\frac{1}{\varepsilon}, \left(\frac{\log(2 + \varepsilon)}{\varepsilon}\right)^2\right\}, \\
a_1'' e^{-N_3 a_2''(\varepsilon^2 / \varepsilon^2)} \mathbb{1}_{\{d>2\}} & \text{if } N \geq N_3 \max\left\{\frac{1}{\varepsilon}, \frac{1}{\varepsilon^2}\right\}.
\end{array}\right.
\]
Remark 3.10 (a) If Assumption 9 is strengthened to \( \int e^{\alpha(\delta)\left(A^2(z) + \frac{1}{B(z)}\right)} \theta(dz) < \infty \) for some \( \alpha(\delta) > 0 \) then one can strengthen the conclusion of Theorem 4 as follows: For \( \delta, \alpha \) sufficiently small there exist \( N_0, a_1, a_2 \in (0, \infty) \) and a nonincreasing function \( \varsigma_2 : (0, \infty) \to (0, \infty) \) such that \( \varsigma_2(t) \downarrow 0 \) as \( t \uparrow \infty \) and for all \( \varepsilon > 0 \) and \( N \geq N_0 \varsigma_2(\varepsilon) \)

\[
\sup_{n \geq 0} P[\mathcal{W}_1(\mu_n^N, \mu_n) > \varepsilon] \leq a_1 e^{-N \alpha a_2 \varepsilon^2}.
\]

(b) Here stability condition (3.18) which is a crucial assumption for Lemma 5.4 is not used. Such is the power of the coupling that we used in Theorem 4.

4 Discussion and Conclusion

This article describes a modified version of discrete time particle approximation scheme described in [3] which incorporates the evolution of particles in a non-compact domain. A similar form of stability condition is obtained under which the nonlinear system has a unique fixed point. Our contribution is computing the quantitative nonasymptotic bounds on these approximation schemes and how these relate to the conditions on the tail and smoothness of the transition kernels \( P, P' \) that were used to model the diffusive environment. As an additional result we obtained the propagation of chaos result of the particle scheme at time \( n = \infty \).

There are few questions and remarks that should be addressed in future.

(a) Theorem 4 is developed exclusively for \( \Pi PS_1 \). For \( \Pi PS_2 \) we would have an extra term \( \mathcal{W}_1 \left(S^M(\bar{\mu}_{n-1}^N), \bar{\mu}_{n-1}^M \right) \) in the expression of \( \mathcal{W}_1(\mu_n^N, \mu_n) \). Now the problem will arise in computing sharper (than (5.109)) bound of

\[
P[\mathcal{W}_1 \left(S^M(\bar{\mu}_{n-1}^N), \bar{\mu}_{n-1}^M \right) > \varepsilon] = EP[\mathcal{W}_1 \left(S^M(\bar{\mu}_{n-1}^N), \bar{\mu}_{n-1}^M \right) > \varepsilon | \mathcal{F}_n^{M,N}] .
\]

Concentration bound of the conditional probability can be given in terms of random \( \langle e^{\alpha_1|z|}, \bar{\mu}_{n-1}^M \rangle \) and getting an explicit relationship of the bound with the conditional exponential moment is unavailable. After taking expectation it is impossible conclude whether the inequality of upper bound still holds or not. Illustratively if the conditional concentration bound of \( P[\mathcal{W}_1 \left(S^M(\bar{\mu}_{n-1}^N), \bar{\mu}_{n-1}^M \right) > \varepsilon | \mathcal{F}_n^{M,N}] \) is a concave function of \( \langle e^{\alpha_1|z|}, \bar{\mu}_{n-1}^M \rangle \) then by Jensen’s inequality reasonable conclusion would hold but to our knowledge such explicit relationship is not present in literature.

(b) The concentration bounds established in [10] for \( \mathcal{W}_1 \) distance of empirical distribution of i.i.d observations to the true distribution is sharp however their method can be applied here only for \( \Pi PS_1 \) as done in Theorem 4 using the well known coupling construction that works for all Vlasov McKean type systems. Without using that coupling, we attempted to use the grid based methods of [10] in order to find sharper bounds for \( P[\mathcal{W}_1 \left(\bar{\mu}_{n-1}^N, \bar{\mu}_{n-1}^M \right), \Psi(\bar{\mu}_{n-1}^N, \bar{\mu}_{n-1}^M) > \varepsilon | \mathcal{F}_n^{M,N}] \) along the line of Theorem 3. We faced similar problem as in the previous remark. Since one can derive a bound for \( P[\mathcal{W}_1 \left(\bar{\mu}_{n-1}^N, \bar{\mu}_{n-1}^M \right), \Psi(\bar{\mu}_{n-1}^N, \bar{\mu}_{n-1}^M) > \varepsilon | \mathcal{F}_n^{M,N}] \) keeping \( \langle e^{\alpha_1|z|}, \bar{\mu}_{n-1}^M \rangle, \langle e^{\alpha_1|z|}, \bar{\mu}_{n-1}^N \rangle \) as constants but we do not know explicit structure how these bounds are functionally depending on \( \langle e^{\alpha_1|z|}, \bar{\mu}_{n-1}^M \rangle, \langle e^{\alpha_1|z|}, \bar{\mu}_{n-1}^N \rangle \), so that unconditionally we can conclude something useful. These issues will be addressed in future.

5 Proofs

The following two elementary lemmas give a basic moment bound that will be used in the proofs. We denote the function \( f(\cdot, \cdot, x) + \frac{B(x)}{\delta} \) by \( f_\delta(\cdot, \cdot, x) \).

Lemma 5.1 For an interacting particle system illustrated in (1.3) and (1.5),

(a) Suppose Assumptions 1, 2 and 4 hold. Then, for every \( n \geq 1 \), \( M_n = \sup_{N \geq 1} E|X_n^i| < \infty \). Moreover if Assumption 1 holds, then under \( \delta \in (0, a_0) \) then \( \sup_{n \geq 1} M_n < \infty \).
(b) With the assumptions in part (a) suppose additionally Assumption 7 holds for some \( \tau > 0 \) and suppose \( \delta \in (0, a(\tau)^{\frac{1}{1+\tau}}) \). Then

\[
\sup_{N \geq 1} \sup_{n \geq 1} E|X_{n}^{i}|^{1+\tau} < \infty,
\]

where in limit \( a(\tau)^{\frac{1}{1+\tau}} \to a_0 \) as \( \tau \to 0^+ \).

**Remark 5.1** Note that the same bound for \( \sup_{n} \sup_{M \geq 1} E|\bar{X}_{n+1}^{i}| \) and \( \sup_{n} \sup_{M \geq 1} E|\bar{X}_{n+1}^{i}|^{1+\tau} \) also hold for IPS\(_{2}\) under same condition on \( \delta \).

### 5.0.1 Proof of Lemma 5.1

(a) We prove the second statement. Proof of the first statement is similar. For each \( n \geq 1 \) and \( i = 1, \ldots, N \), applying Assumption 1 on particle system in (1.3) with definitions of \( A_1(\cdot) \) and \( A_2(\cdot) \)

\[
|X_{n+1}^{i}| \leq \|A\| |X_{n}^{i}| + \delta A_{1}(\epsilon_{n+1}^{i}) \|\nabla \eta_{n}^{N}(X_{n}^{i})\| + \|\mu_{n}^{N}\|_{1} + |X_{n}^{i}| + \delta A_{2}(\epsilon_{n+1}^{i}) + |B(\epsilon_{n+1}^{i})|.
\]

(5.1)

Now by Assumption 4 using DCT one has

\[
\nabla \eta_{n+1}(y) = \int_{\mathbb{R}^{d}} \eta_{n}(x)[\nabla_{y} R_{\mu_{n}}^{\alpha}(x, y)]dx
\]

(5.2)

for every \( y \) since from Assumption 4 \( \sup_{x \in \mathbb{R}^{d}} |\nabla_{y} R_{\mu_{n}}^{\alpha}(x, y)| \leq l_{P_{P'}}^{\alpha, \sigma} |y| + \sup_{x \in \mathbb{R}^{d}} (1 - \alpha) |\nabla_{y} P(x, 0)| + \alpha |\nabla_{y} P(x, 0)| \). Applying the same condition followed by the inequality \( |\nabla \eta_{n+1}(y)| \leq \int_{\mathbb{R}^{d}} \eta_{n}(x)[\nabla_{y} R_{\mu_{n}}^{\alpha}(x, y)]dx \), one has

\[
|\nabla \eta_{n}(y)| \leq l_{P_{P'}}^{\alpha, \sigma} |y| + c_{P_{P'}}^{\alpha, \sigma}.
\]

(5.3)

Also note by exchangeability \( E\|\mu_{n}^{N}\|_{1} = E\int |x|\mu_{n}^{N}(dx) = E|X_{n}^{i}| \). Taking expectation in (5.1) and using (5.3) and independence between \( \epsilon_{n+1}^{i} \) and \( \{X_{n}^{i}\}_{i=1}^{N} \), one has

\[
E|X_{n+1}^{i}| \leq (\|A\| + \delta \sigma \left( 2 + l_{P_{P'}}^{\alpha, \sigma} \right) E|X_{n}^{i}| + \delta \sigma c_{P_{P'}}^{\alpha, \sigma} + \sigma_{2}(\delta)).
\]

(5.4)

The assumption on \( \delta \) implies that \( \gamma := \|A\| + \delta \sigma \left( 2 + l_{P_{P'}}^{\alpha, \sigma} \right) \in (0, 1) \). A recursion on (5.4) will give \( M_{n} \leq \gamma^{n} E|X_{0}^{i}| + \frac{\delta \sigma c_{P_{P'}}^{\alpha, \sigma} + \sigma_{2}(\delta)}{1 - \gamma} \), from which the result follows.

(b) By Holder’s inequality for any three nonnegative real numbers \( a, b, c, d \)

\[
(a + b + c + d)^{1+\tau} \leq 4\tau(a^{1+\tau} + b^{1+\tau} + c^{1+\tau} + d^{1+\tau}).
\]

(5.5)

Starting with (5.1), applying (5.5), and Assumption 1, on (5.1) we have

\[
|X_{n+1}^{i}|^{1+\tau} \leq 4^{\tau} \left[ \|A\|^{(1+\tau)}|X_{n}^{i}|^{1+\tau} + \left( \delta A_{1}(\epsilon_{n+1}^{i}) \left( 1 + l_{P_{P'}}^{\alpha, \sigma} \right) |X_{n}^{i}| \right)^{1+\tau} + \delta A_{2}(\epsilon_{n+1}^{i}) \|\mu_{n}^{N}\|_{1} \right]^{1+\tau} + \delta^{1+\tau} \left[ A_{1}(\epsilon_{n+1}^{i})c_{P_{P'}}^{\alpha, \sigma} + A_{2}(\epsilon_{n+1}^{i}) + \frac{|B(\epsilon_{n+1}^{i})|}{\delta} \right]^{1+\tau}.
\]

For any convex function \( \phi(\cdot) \), applying Jensen’s inequality one gets \( \phi(\|\mu_{n}^{N}\|_{1}) \leq \int \phi(x)|\mu_{n}^{N}(dx) = \frac{1}{N} \sum_{i=1}^{N} \phi(X_{n}^{i}) \). Using \( \phi(x) = x^{1+\tau} \), after taking expectation one gets following recursive equation for \( E|X_{n+1}^{i}|^{1+\tau} \),

\[
E|X_{n+1}^{i}|^{1+\tau} \leq 4^{\tau} \left[ \|A\|^{(1+\tau)} + \delta^{1+\tau} \sigma_{1}(\tau) \left( 1 + l_{P_{P'}}^{\alpha, \sigma} \right)^{1+\tau} + 1 \right] E|X_{n}^{i}|^{1+\tau} + \delta^{1+\tau} 8^{\tau} \left[ \sigma_{1}(\tau)c_{P_{P'}}^{\alpha, \sigma} + \sigma_{2}(\delta, \tau) \right].
\]

Note that for our condition on \( \delta \), \( \kappa_{1} := 4^{\tau} \left[ \|A\|^{(1+\tau)} + \delta^{1+\tau} \sigma_{1}(\tau) \left( 1 + l_{P_{P'}}^{\alpha, \sigma} \right)^{1+\tau} + 1 \right] < 1 \). Thus

\[
\sup_{n \geq 1} E|X_{n}^{i}|^{1+\tau} \leq \kappa_{n}^{1} E|X_{0}^{i}|^{1+\tau} + \frac{\delta^{1+\tau} 8^{\tau} \left[ \sigma_{1}(\tau)c_{P_{P'}}^{\alpha, \sigma} + \sigma_{2}(\delta, \tau) \right]}{1 - \kappa_{1}}.
\]

(5.6)
Lemma 5.2 Suppose Assumptions 1, 2, 4 and 5 hold.

(a) Consider the interacting particle system described in (1.3) and (1.5). Then, for every \( n \geq 1 \),
\[
\langle |x|, \eta_n \rangle < \infty, \quad \sup_{N \geq 1} E \langle |x|^N, \eta_n^N \rangle < \infty. \tag{5.7}
\]
Moreover if Assumption 1 holds, then under conditions
\[
\delta \in (0, a_0), \quad \text{and} \quad \sum_{i=0}^{\infty} (1 - \alpha)^i \int_{\mathbb{R}^d} |y|^i P^i P^i(0, dy) < \infty, \tag{5.8}
\]
one has \( \sup_{n \geq 1} \langle |x|, \eta_n \rangle < \infty \).
Additionally assuming \( \sup_{N \geq 1} E \langle |x|^N, \eta_0^N \rangle < \infty \) one gets
\[
\sup_{n \geq 1} \sup_{N \geq 1} E \langle |x|, \eta_n^N \rangle < \infty.
\]

(b) With the assumptions in part (a) suppose additionally Assumption 7, 8 hold for some \( \tau > 0 \) and suppose \( \delta \in (0, a(\tau)^{\frac{1}{\tau}}) \). Then with condition \( (1 - \alpha) a_n(P) < 1 \) one has \( \sup_{n \geq 1} \langle |x|^{1+\tau}, \eta_n \rangle < \infty \). Additionally assuming \( \sup_{N \geq 1} E \langle |x|^{1+\tau}, \eta_0^N \rangle < \infty \) one gets \( \sup_{n \geq 1} \sup_{N \geq 1} E \langle |x|^{1+\tau}, \eta_n^N \rangle < \infty \), where in limit \( a(\tau)^{\frac{1}{\tau}} \to a_0 \) as \( \tau \to 0^+ \).

Remark 5.2 The second condition in (5.8) is very general. It doesn't impose any condition on \( \alpha \in (0, 1) \). The condition holds for all transition kernels \( P(x, \cdot), P'(x, \cdot) \) with finite first moment. Only thing one needs to check
\[
\int_{\mathbb{R}^d} |y|^P P^i(0, dy) = g(i)
\]
where \( g(i) \) is some polynomial in \( i \) (For Gaussian it's linear). If \( g(\cdot) \) is an exponential function then it will impose a further lower bound condition on \( \alpha \).

Corollary 5.3 For \( \mathbb{IPS}_2 \) same conclusion about \( \bar{\eta}_n^M \) holds as \( \eta_n^N \) in first particle system specified in Lemma 5.2 under same set of conditions on \( \delta, \alpha \). Note that \( \bar{\eta}_0^M = \eta_0 \), so we don’t need to assume anything about the initial sampling scheme like \( \sup_{M \geq 1} \langle |x|, \bar{\eta}_0^M \rangle < \infty \) (or \( \sup_{M \geq 1} E \langle |x|^{1+\tau}, \eta_0^M \rangle < \infty \)) since they automatically hold for \( \eta_0 \in \mathcal{P}_1(\mathbb{R}^d) \) (or \( \eta_0 \in \mathcal{P}_1^L(\mathbb{R}^d) \)) respectively.

5.0.2 Proof of Lemma 5.2

We will start with the second part of part (a) of the lemma. First part will follow similarly. We will show if \( \eta_0 \in \mathcal{P}_1^L(\mathbb{R}^d) \) then \( \eta_n \in \mathcal{P}_1(\mathbb{R}^d) \) for all \( n \geq 1 \). Note that
\[
\eta_{k+1} = \sum_{i=0}^{k} \alpha(1 - \alpha)^i \mu_{k-i} P^i P^i + (1 - \alpha)^k \eta_0 P^{k+1}. \tag{5.9}
\]
From Assumption 5, it is obvious that \( P^i P^i f \) is \( l(P') l(P)^i \) Lipschitz if \( f \) is a 1-Lipschitz function. It implies \( |P^i P^i f(x) - P^i P^i f(0)| \leq l(P') l(P)^i |x| \) for any \( f \) \( \in \mathcal{L}_1(\mathbb{R}^d) \). Since \( |x| \) is 1-Lipschitz, one has
\[
P^i P^i |x| \leq l(P') l(P)^i |x| + \int_{\mathbb{R}^d} |y| P^i P^i(0, dy).
\]
Using this inequality one has from (5.9)
\[
\langle |x|, \eta_{k+1} \rangle = \sum_{i=0}^{k} \alpha (1 - \alpha)^i \langle |x|, \mu_{k-i}P^i \rangle + (1 - \alpha)^{k+1} \langle |x|, \eta_0 P^{k+1} \rangle
\]
\[
\leq \sum_{i=0}^{k} [\alpha (1 - \alpha)^i \langle l(P)l(P)^i|x, \mu_{k-i}\rangle] + \alpha \sum_{i=0}^{\infty} (1 - \alpha)^i \int_{\mathbb{R}^d} |y|P^i\delta(0, dy) + [(1 - \alpha)l(P)]^{k+1} \langle |x|, \eta_0 \rangle
\]
\[
\leq \alpha l(P) \left( \sup_{n \in \mathbb{N}} \langle |x|, \mu_n \rangle \right) \sum_{i=0}^{k} [(1 - \alpha)l(P)]^i + \alpha \sum_{i=0}^{\infty} (1 - \alpha)^i \int_{\mathbb{R}^d} |y|P^i\delta(0, dy)
\]
\[
+ [(1 - \alpha)l(P)]^{k+1} \langle |x|, \eta_0 \rangle.
\]
(5.10)
By Assumption 5, \(l(P) \leq 1\), implies \((1 - \alpha)l(P) < 1\). From similar derivation done in Lemma 5.1, one has \(\sup_{k \in \mathbb{N}} \langle |x|, \eta_k \rangle < \infty\) if \(\delta \in (0, a_0)\). The result follows using all the conditions
\[
\sup_{k \in \mathbb{N}} \langle |x|, \eta_k \rangle < \infty.
\]
For \(E \langle |x|, \eta_k^N \rangle\) note that for any function \(f\),
\[
\langle f, \eta_k^N \rangle = \sum_{i=0}^{k} \alpha (1 - \alpha)^i \langle f, \mu_{k-i}P^i \rangle + (1 - \alpha)^{k+1} \langle f, \eta_0 P^{k+1} \rangle.
\]
(5.11)
From Lemma 5.1 \(\sup_{n \geq 0} \sup_{N \geq 1} E \langle |x|, \mu_n^N \rangle < \infty\) for \(\delta \in (0, a_0)\). Putting \(f(x) = |x|\), then expanding \(\langle |x|, \eta_n^N \rangle\) similarly like (5.10) after taking expectation one gets a similar bound and finiteness of \(\sup_n \sup_{N \geq 1} E \langle |x|, \eta_n^N \rangle\) follows from that.

\[\square\]

**Proof of Lemma 5.2(b):** From (5.9),
\[
\langle \eta_{k+1}, |x|^{1+\tau} \rangle = \sum_{i=0}^{k} \alpha (1 - \alpha)^i \langle \mu_{k-i}P^i, |x|^{1+\tau} \rangle + (1 - \alpha)^{k+1} \langle \eta_0 P^{k+1}, |x|^{1+\tau} \rangle.
\]
(5.12)
From Assumption 8 we get the following recursion for \(a_i := \langle \mu P^i, |x|^{1+\tau} \rangle\) for any measure \(\mu \in \mathcal{P}_{1+\tau}(\mathbb{R}^d)\)
\[
a_i = \langle \mu P^i, |x|^{1+\tau} \rangle \leq m_\tau(P)(1 + a_{i-1})
\]
(5.13)
since \(P|x|^{1+\tau} \leq m_\tau(P)(1 + |x|^{1+\tau})\) from Assumption 8. Using the fact \(a_0 := \langle \mu, P|x|^{1+\tau} \rangle \leq m_\tau(P)(1 + \langle \mu, |x|^{1+\tau} \rangle)\), we finally have
\[
\langle \eta_{k+1}, |x|^{1+\tau} \rangle \leq \alpha \sum_{i=0}^{k} (1 - \alpha)^i \left[ m_\tau(P) \frac{l^i(P) - 1}{m_\tau(P) - 1} + m_\tau(P) l^i(P) \left[ 1 + \langle |x|^{1+\tau}, \mu_{k-i} \rangle \right] \right]
\]
\[
+ (1 - \alpha)^{k+1} \left[ m_\tau(P) \frac{l^{k+1}(P) - 1}{m_\tau(P) - 1} + l^{k+1}(P) \langle \eta_0, |x|^{1+\tau} \rangle \right].
\]
(5.14)
Under condition \(\delta \in (0, a(\tau^{1+\tau}))\) and \((1 - \alpha)m_\tau(P) < 1\) one gets \(\sup_n \langle \eta_n, |x|^{1+\tau} \rangle < \infty\). Similarly the same bound can be derived for \(\sup_n \sup_{N \geq 1} E \langle |x|^{1+\tau}, \eta_n^N \rangle\) under the same set of conditions.

\[\square\]

**5.0.3 Proof of Corollary 5.3**

To prove the Corollary about \(\tilde{\eta}_n^M\), define the random operator \(S^M \circ P\) acting on the probability measure \(\mu\) on \(\mathbb{R}^d: \mu(S^M \circ P) = (S^M(\mu))P\). Note the following recursive form of \(\tilde{\eta}_n^M:\)
\[
\tilde{\eta}_{k+1}^M = \sum_{i=0}^{k} [\alpha (1 - \alpha)^i \tilde{\eta}_{k-i}^M P^i(S^M \circ P)^i] + (1 - \alpha)^{k+1}\eta_0(S^M \circ P)^{k+1}.
\]
(5.15)
Note that for any function $f$ one has
\[ E \langle \mu(S^M \circ P), f \rangle = E \langle S^M(\mu), Pf \rangle = \langle \mu, Pf \rangle = \langle \mu P, f \rangle. \]

Now by expanding $\mu(S^M \circ P)^k$ one gets,
\[ \mu(S^M \circ P)^k = [\mu(S^M \circ P)^{k-1}] (S^M \circ P) = S^M(\mu(S^M \circ P)^{k-1})P. \]

Taking expectation one has
\[ E \langle \mu(S^M \circ P)^k, f \rangle = E \langle S^M(\mu(S^M \circ P)^{k-1})P, f \rangle = E \langle \mu(S^M \circ P)^{k-1}, Pf \rangle = E \langle \mu(S^M \circ P)^{k-1}P, f \rangle. \]

Continuing this calculation $k-1$ times one has $E \langle \mu(S^M \circ P)^k, f \rangle = \langle \mu P^k, f \rangle$ which leads to the following expression
\[ E \langle \bar{\mu}_{k-1}^N P'(S^M \circ P)^i, f \rangle = EE \left[ \langle \bar{\mu}_{k-1}^N P'(S^M \circ P)^i, f \rangle \bigg| \mathcal{F}_{k-i}^M \right] \]
\[ = E \left[ \langle \mu_{k-1}^N P^k, f \rangle \right] = E \left[ \langle \mu_{k-1}^N, P^k f \rangle \right]. \tag{5.16} \]

The corollary is proved by observing (5.16). The same bound holds for both $E \langle \bar{\eta}_{n}^M, f \rangle$, $E \langle \eta_{n}^M, f \rangle$ because of the similarity of bounds of $E \langle f, \mu_n^N \rangle$, and $E \langle f, \bar{\mu}_n^N \rangle$ for $f(x) = |x|, |x|^2, |x|^3, e^{|x|}$ which follows from Remark 5.1.

\[ \square \]

### 5.1 Proof of Proposition 3.2

We will prove part (b) of the theorem. Part (a) will follow similarly. We will start with the following lemma.

**Lemma 5.3** (a) *Under Assumptions 1,2,4, for every $\epsilon > 0$ and $n \geq 1$, there exists a compact set $K_{\epsilon,n} \in \mathcal{B}(\mathbb{R}^d)$ such that*

\[ \sup_{M,N \geq 1} E \left\{ \int_{K_{\epsilon,n}} |x| \left( \mu_n^N(dx) + \mu_{n-1}^N \eta_{n-1}^M(dx) \right) \right\} < \epsilon. \]

(b) *Suppose Assumptions 1,2,4,5,6 hold. Then for every $\epsilon > 0$ and $k \geq 1$, there exists a compact set $K_{\epsilon,k} \in \mathcal{B}(\mathbb{R}^d)$ such that*

\[ \sup_{M,N \geq 1} E \langle |x|1_{K_{\epsilon,k}}, S^M(\bar{\eta}_k^M) + \eta_k^M \rangle < \epsilon. \]

*This part of the lemma is exclusively for part (b) of the Proposition 3.2.*

**Proof:** Note that for any non-negative $\phi : \mathbb{R}^d \to \mathbb{R}$,

\[ E \int \phi(x) \mu_n^N(dx) = \frac{1}{N} \sum_{k=1}^{N} E \phi(X_n^k) = E \phi(X_n^1), \tag{5.17} \]

\[ E \int \phi(x) \mu_{n-1}^N \eta_{n-1}^M(dx) = \frac{1}{N} \sum_{i=1}^{N} E \left( E(\langle \phi, \delta_{X_n^i} \eta_{n-1}^M \rangle | \mathcal{F}_n) \right) \]
\[ = \frac{1}{N} \sum_{i=1}^{N} E \phi \left( AX_n^i + \delta_{f_{\delta}}(X_n^i, \mu_n^N, \nabla \eta_{n-1}^M(X_n^i, \epsilon_{n+1})) \right) \]
\[ = \frac{1}{N} \sum_{i=1}^{N} E \phi(X_n^{i+1}) = E \phi(X_n^{1+1}). \tag{5.18} \]
To get the desired result from above equalities it suffices to show that
the family \( \{X_n^{i,N}, i = 1, \ldots, N; M, N \geq 1 \} \) is uniformly integrable for every \( n \geq 0 \). \hfill (5.19)

We will prove (5.19) by induction on \( n \). Once more we suppress \( N \) from the super-script. Clearly by our assumptions \( \{X_n^i, i = 1, \ldots, N; N \geq 1 \} \) is uniformly integrable. Now suppose that the Statement (5.19) holds for some \( n \). Note that from (5.1) and (5.3)

\[
|X_{n+1}^i| \leq ||A|||X_n^i| + A_1(e_{n+1}^i)|[\nabla \eta_n^N(X_n^i)] + ||\mu_n^N||_1 + |X_n^i| + \delta A_2(e_{n+1}^i) + |B(e_{n+1}^i)|.
\]

\[
\leq ||A|||X_n^i| + A_1(e_{n+1}^i)|[\nabla \eta_n^N(X_n^i)] + ||\mu_n^N||_1 + (1 + \|\nabla \eta_n^N\|_1)|X_n^i| + \delta A_2(e_{n+1}^i) + |B(e_{n+1}^i)| + \delta c_{PP} \cdot A_1(e_{n+1}^i).
\]

From Assumptions 1 and 2 the families \( \{A_1(e_{i,n}^i); i \geq 1\}, \{A_2(e_{i,n}^i); i \geq 1\} \{B_2(e_{i,n}^i) \} \) are uniformly integrable. Now by exchangeability, \( \frac{1}{N} \sum_{i=1}^{N} |X_n^i| = E\left[|X_n^i| \mid \sigma\left( \frac{1}{N} \sum_{i=1}^{N} \delta X_n^i \right) \right] \). If \( \{X_n: \alpha \in \Gamma_1\} \) is uniformly integrable, and \( \{\sigma_{\beta}, \beta \in \Gamma_2\} \) is a collection of \( \sigma \)-fields where \( \Gamma_1, \Gamma_2 \) are arbitrary index sets, then \( E(X_n \mid \sigma_{\beta}) \) is also a uniformly integrable family. It follows that \( \frac{1}{N} \sum_{i=1}^{N} |X_n^i|, N \geq 1 \) is a uniformly integrable family from induction hypothesis. Using (5.19) again along with independence between \( \epsilon_{i,n}^1, i = 1, \ldots, N \) and \( \{X_n^i: i = 1, \ldots, N; N \geq 1\} \) yield that the family \( \{X_{n+1}^i: i = 1, \ldots, N; N \geq 1\} \) is uniformly integrable. The result follows. \( \square \)

**Proof of Lemma 5.3(b):** Note that \( S^M(\bar{\eta}_n^M) = \frac{1}{M} \sum_{i=1}^{M} \delta_{Y_k^i, \eta} \) where \( \{Y_k^i, M\}_{i=1}^{M} \) are i.i.d from \( \bar{\eta}_n^M \).

So for any non-negative function \( \phi \) we have

\[
E(\phi, S^M(\bar{\eta}_n^M)) = E \left[ \frac{1}{M} \sum_{i=1}^{M} \phi(Y_k^i, M) \mid \mathcal{F}_{\eta}^M \right] = \frac{1}{M} \sum_{i=1}^{M} \phi(Y_k^i, M) = E \left[ \phi(Y_k^i, M) \mid \mathcal{F}_{\eta}^M \right].
\]

We will prove the result if we can show the family

\( \{Y_k^i, M, i = 1, \ldots, M; M, N \geq 1\} \) is uniformly integrable for every \( k \geq 0 \). \hfill (5.21)

We will prove (5.21) through induction on \( k \). For \( k = 0 \), the result follows trivially since \( \{Y_0^i, M, i = 1, \ldots, M; M, N \geq 1\} \) are i.i.d from \( \eta_0 \). Suppose it holds for \( k = n \). We will show that both,

\( S^M(\bar{\eta}_n^M \mid P: M, N \geq 1 \} \) and \( \{\mu_n^N \mid P: N \geq 1 \} \) are uniformly integrable families of probability measures. \hfill (5.22)

Then from the structure \( \bar{\eta}_n^M = (1 - \alpha)S^M(\bar{\eta}_n^M) \alpha^{M - 1} \mu_n^N \) it is evident that \( \{\bar{\eta}_n^M: M, N \geq 1\} \) is uniformly integrable which equivalently implies \( \{Y_k^i, M: i = 1, \ldots, M; M, N \geq 1\} \) is UI too. On proving the first assertion in (5.22), note that due to the exchangeability of \( \{Y_k^i, M: i = 1, \ldots, M\} \), one has

\[
S^M(\bar{\eta}_n^M) = E \left[ \delta_{Y_k^i, M} \mid \sigma \left( \frac{1}{M} \sum_{i=1}^{M} \delta_{Y_k^i, M} \right) \right].
\]

We know that if \( \{Z_{\alpha}, \alpha \in \Gamma_1\} \) is a uniformly integrable family and \( \{\mathcal{H}_{\beta}, \beta \in \Gamma_2\} \) is a collection of \( \sigma \)-fields where \( \Gamma_1, \Gamma_2 \) are arbitrary index sets, then \( E(Z_\alpha \mid \mathcal{H}_{\beta}) \) is a uniformly integrable family. So from (5.23) it suffices to prove that \( \{\delta_{Y_k^i, M} \mid P: i = 1, \ldots, M; M, N \geq 1\} \) is uniformly integrable. Define a function \( f_k(\cdot) \) such that, \( f_k(x) = 0 \) if \( |x| \in [0, \frac{k}{2}] \) and \( f_k(x) = |x| \) if \( |x| \geq k \) and linear in between range. Then by construction \( f_k(\cdot) \) is Lipschitz with coefficient 2 and \( x.E.1_{|x| > k} \leq f_k(x) \) for all \( x \in \mathbb{R}^d \).

By Assumption 6 we have that \( P(z, \cdot): z \in K \) is uniformly integrable. So taking the compact set \( K = \{ |x| \leq k \} \) assuming \( Y_k^i, M \) has unconditional law \( m_n^i \) for all \( i = 1, \ldots, M, \) the quantity

\[
\int_{|z| > L} y_{1, (K^c)} P(z, dy) m_n^i(dz) \leq \int_{|z| > L} \left[ f_k(y) P(z, dy) \right] m_n^i(dz)
\]

\[
\leq \int_{|z| > L} \left[ |P f_k(0)| + 2L |P(z)| \right] m_n^i(dz)
\]

\[
\leq P f_k(0) \int_{|z| > L} m_n^i(dz) + 2L \int_{|z| > L} |z| m_n^i(dz).
\]
The display in (5.24) follows from Assumption 5 and using Lipschitz property of \( f_k \). After taking supremum in the set \( \{i = 1, \ldots, M; M, N \geq 1\} \) in both sides of (5.25), second part of R.H.S goes to 0, as \( L \to \infty \) by induction hypothesis. About the first part \( P_{f_k}(0) \) goes to 0 as \( k \to \infty \) by D.C.T since (\( \int |y|P(0, dy) < \infty \)) and also \( \int_{|z|>L} m_k^N \langle dz \rangle \) converges to 0 (as \( L \) goes to \( \infty \)) due to the tightness of \( \{m_k^N : i = 1, \ldots, M; M, N \geq 1\} \) which also follows from induction hypothesis. The second assertion that \( \{\tilde{\mu}_N^n P' : N \geq 1\} \) is uniformly integrable follows similarly through induction.

□

We will proceed to the main proof via induction on \( n \in \mathbb{N} \) for the quantity \( E \left[ \mathcal{W}_1(\tilde{\mu}_N^n, \mu_n) + \mathcal{W}_1(\tilde{\eta}_N^n, \eta_n) \right] \). For \( n = 0 \), we will first show that \( EW_1(\tilde{\mu}_0^N, \mu_0) \to 0 \) as \( N \to \infty \). From [16] we have

\[
(\tilde{X}_0^N, \tilde{X}_1^N, \ldots, \tilde{X}_N^N) \text{ is } \mu_0\text{-chaotic} \iff \tilde{\mu}_0^N \text{ converges weakly to } \mu_0 \text{ in probability } \iff \beta(\tilde{\mu}_0^N, \mu_0) \to 0.
\]

From Lemma 5.3 one can construct \( K_{0,0} \), compact ball containing 0, so that \( E \langle |x|1_{K_{0,0}}, \tilde{\mu}_0^N \rangle < \frac{\beta}{2} \) and \( \langle |x|1_{K_{0,0}}, \mu_0 \rangle < \frac{\beta}{2} \) hold. So using the fact for any \( f \in \text{Lip}_1(\mathbb{R}^d) \) with \( f(0) = 0 \), one has \( \sup_{x \in \mathbb{R}^d} |f(x)| \leq |x| \).

\[
EW_1(\tilde{\mu}_0^N, \mu_0) = E \sup_{f \in \text{Lip}_1(\mathbb{R}^d)} |\langle f, \tilde{\mu}_0^N - \mu_0 \rangle| = E \sup_{f \in \text{Lip}_1(\mathbb{R}^d), f(0)=0} |\langle f, \tilde{\mu}_0^N - \mu_0 \rangle| \\
\leq E \sup_{f \in \text{Lip}_1(\mathbb{R}^d), f(0)=0} |\langle f1_{K_{0,0}}, \tilde{\mu}_0^N - \mu_0 \rangle| + E \langle |x|1_{K_{0,0}}, \tilde{\mu}_0^N \rangle + \langle |x|1_{K_{0,0}}, \mu_0 \rangle \\
\leq \text{diam}(K_{0,0})E|\beta(\tilde{\mu}_0^N, \mu_0) + \epsilon \quad (5.26)
\]

In the last display we used the fact that \( \sup_{x \in K_{0,0}} |f(x)| \leq \text{diam}(K_{0,0}) \). Note that \( \beta(\tilde{\mu}_0^N, \mu_0) \) is bounded by 2 (so Uniformly Integrable) and \( \beta(\tilde{\mu}_0^N, \mu_0) \to 0 \) implies \( E|\beta(\tilde{\mu}_0^N, \mu_0) \to 0 \) as \( N \to \infty \) proving the assertion (3.12) for \( n = 0 \). Suppose it holds for \( n \leq k \). We start with the following triangular inequality

\[
W_1(\mu_{k+1}^N, \mu_{k+1}) \leq W_1(\mu_k^N, \mu_{k+1}) + W_1(\tilde{\mu}_k^N Q^n_{\mu_k^N}, \tilde{\mu}_{k+1}^N) \\
+W_1(\tilde{\mu}_k^N Q^n_{\mu_k^N}, \mu_{k+1}). \quad (5.27)
\]

Consider the third term of (5.27). From the general calculations followed by (5.45)-(5.47), we have the following estimate,

\[
W_1(\tilde{\mu}_k^N Q^n_{\mu_k^N}, \mu_{k+1}) \leq \left( \|A\| + \delta \sigma(2 + I_{p,p^*}) \right) W_1(\tilde{\mu}_k^N, \mu_k). \quad (5.28)
\]

Now we consider the first term of the right hand side of (5.27). We will use Lemma 5.3(a). Fix \( \epsilon > 0 \) and let \( K_\epsilon \) be a compact set in \( \mathbb{R}^d \) such that

\[
\sup_{N \geq 1} E \left\{ \int_{K_\epsilon^c} |x|(|\tilde{\mu}_{k+1}^N(dx) + \tilde{\mu}_k^N Q^n_{\tilde{\mu}_k^N}(dx)) \right\} < \epsilon.
\]

Let \( \text{Lip}_1^0(\mathbb{R}^d) := \{f \in \text{Lip}_1(\mathbb{R}^d) : f(0) = 0\} \). Then,

\[
E \sup_{\phi \in \text{Lip}_1^0(\mathbb{R}^d)} |\langle \phi, \tilde{\mu}_{k+1}^N - \tilde{\mu}_k^N Q^n_{\tilde{\mu}_k^N} \rangle| = E \sup_{\phi \in \text{Lip}_1^0(\mathbb{R}^d)} |\langle \phi, \tilde{\mu}_{k+1}^N - \tilde{\mu}_k^N Q^n_{\tilde{\mu}_k^N} \rangle| \\
\leq E \sup_{\phi \in \text{Lip}_1^0(\mathbb{R}^d)} |\langle \phi1_{K_\epsilon}, \tilde{\mu}_{k+1}^N - \tilde{\mu}_k^N Q^n_{\tilde{\mu}_k^N} \rangle| + \epsilon. \quad (5.29)
\]

We will now apply Lemma A.1 in the Appendix. Note that for any \( \phi \in \text{Lip}_1^0(\mathbb{R}^d) \), \( \sup_{x \in K_\epsilon} |\phi(x)| \leq \text{diam}(K_\epsilon) := m_\epsilon \).

Thus with notation as in Lemma A.1

\[
\sup_{\phi \in \text{Lip}_1^0(\mathbb{R}^d)} |\langle \phi1_{K_\epsilon}, \tilde{\mu}_{k+1}^N - \tilde{\mu}_k^N Q^n_{\tilde{\mu}_k^N} \rangle| \leq \max_{\phi \in \text{Lip}_1^0(\mathbb{R}^d)} |\langle \phi, \tilde{\mu}_{k+1}^N - \tilde{\mu}_k^N Q^n_{\tilde{\mu}_k^N} \rangle| + 2\epsilon. \quad (5.30)
\]
where we have denoted the restrictions of $\bar{\mu}_{k+1}^N$ and $\bar{\beta}_k^N Q_{\bar{y}_k}^N$ to $K_\epsilon$ by the same symbols. Using the above inequality in (5.29), we obtain
\[
EW_1(\bar{\mu}_{k+1}^N, \bar{\beta}_k^N Q_{\bar{y}_k}^N) \leq \sum_{\phi \in \mathcal{F}^N_{m_1,1}(K_\epsilon)} E\{\phi, \bar{\mu}_{k+1}^N - \bar{\mu}_k^N Q_{\bar{y}_k}^N\} + 3\epsilon.
\] (5.31)

Using Lemma A.2 we see that the first term on the right hand side can be bounded by $\frac{2m_t|\mathcal{F}_{m_t}^N(K_\epsilon)|}{\sqrt{N}}$.

Consider the second term of R.H.S of (5.27). From Assumption 4 applying DCT one has
\[
\nabla \bar{\eta}_k^N(y) = (1 - \alpha) \int S^M(\bar{\eta}_{k-1}^N)(dx)\nabla_y P(x,y) + \alpha \int \bar{\mu}_k^N(dx)\nabla_y P'(x,y),
\]
(5.32)
\[
\nabla \eta_k(y) = (1 - \alpha) \int \eta_{k-1}(dx)\nabla + \alpha \int \mu_k(dx)\nabla_y P'(x,y).
\] (5.33)

Suppose $\bar{X}_k$ is a random variable conditioned on $\mathcal{F}_{M,N}^N$ is distributed with law $\bar{\mu}_k^N$. Then almost surely $\mathcal{W}_1(\bar{\mu}_k^N Q_{\bar{y}_k}^N, \bar{\beta}_k^N Q_{\bar{y}_k}^N)$ is
\[
\leq \sup_{g \in \text{Lip}_1(\mathbb{R}^d)} \mathcal{E}^{M,N}_k \left[ g(A\bar{X}_k + \delta f_0(\nabla \bar{\eta}_k^N(\bar{X}_k), \bar{\mu}_k^N, \bar{X}_k, \epsilon)) \right.
\]
\[
- g(A\bar{X}_k + \delta f_0(\nabla \eta_k(\bar{X}_k), \bar{\mu}_k^N, \bar{X}_k, \epsilon)) \left. \right] \leq \delta \sigma \mathcal{E}^{M,N}_k \left[ |\nabla \bar{\eta}_k^N(\bar{X}_k) - \nabla \eta_k(\bar{X}_k)| \right]
\]
\[
\leq \delta \sigma (1 - \alpha) \int \left| \int \{ S^M(\bar{\eta}_k^M) - \eta_k \} (dx)\nabla_y P(x,y) \right| \bar{\mu}_k^N(dy)
\]
\[
+ \delta \sigma \alpha \int \left| \int \{ \bar{\mu}_k^N - \mu_k \} (dx)\nabla_y P'(x,y) \right| \bar{\mu}_k^N(dy)
\]
\[
\leq \delta \sigma (1 - \alpha) \int \mathcal{W}_1(S^M(\bar{\eta}_k^M), \eta_k) + \delta \sigma \alpha \eta_{\mathcal{P}} \mathcal{W}_1(\bar{\mu}_k^N, \mu_k).
\] (5.34)

(5.34) follows by using Assumption 4. About the first term in (5.34) note that from triangular inequality,
\[
EW_1(S^M(\bar{\eta}_k^M), \eta_k) \leq EW_1(S^M(\bar{\eta}_k^M), \bar{\eta}_k^M) + EW_1(\bar{\eta}_k^M, \eta_k).
\] (5.35)

The first term in (5.35) can be written as
\[
EW_1(S^M(\bar{\eta}_k^M), \eta_k) \leq E \sup_{f \in \text{Lip}_1(\mathbb{R}^{d+1})} |\langle f, 1_{K_{\epsilon,1}}, S^M(\bar{\eta}_k^M) - \eta_k^M \rangle| + E \langle |x|, 1_{K_{\epsilon,1}}, S^M(\bar{\eta}_k^M) \rangle
\]
\[
+ E \langle |x|, 1_{K_{\epsilon,1}}, \eta_k^M \rangle.
\] (5.36)

By Lemma 5.3(b), for a specified $\epsilon > 0$, one can construct a compact set $K_{\epsilon,1}$ containing 0 such that,
\[
\sup_{M,N \geq 1} E \langle |x|, 1_{K_{\epsilon,1}}, S^M(\bar{\eta}_k^M) + \eta_k^M \rangle < \epsilon.
\]

Denote $m_{k,\epsilon} = \text{diam}(K_{k,\epsilon})$. Using Lemma A.1 we have the L.H.S of (5.36)
\[
E \mathcal{E}^{M,N}_k \left[ \sup_{f \in \text{Lip}_1(\mathbb{R}^{d+1})} |\langle f, 1_{K_{\epsilon,1}}, S^M(\bar{\eta}_k^M) - \eta_k^M \rangle| \right] + \epsilon \leq E \mathcal{E}^{M,N}_k \left[ \sup_{f \in \mathcal{F}_{m_{k,\epsilon,1}}^\nu(K_{k,\epsilon})} \max_{\phi \in \mathcal{F}_{m_{k,\epsilon,1}}^\nu(K_{k,\epsilon})} |\langle \phi, S^M(\bar{\eta}_k^M) - \eta_k^M \rangle| \right] + 2\epsilon
\]
where (5.36) follows from similar arguments used in (5.31). Note that the Lemma 5.3 also suggests the compact set $K_{k,\epsilon}$ is non-random, which only depends on $k$ and $\epsilon$ only. So from the display above we have
\[
E \mathcal{E}^{M,N}_k \left[ \sum_{\phi \in \mathcal{F}_{m_{k,\epsilon,1}}^\nu(K_{k,\epsilon})} |\langle \phi, S^M(\bar{\eta}_k^M) - \eta_k^M \rangle| \right] + 2\epsilon \leq \sum_{\phi \in \mathcal{F}_{m_{k,\epsilon,1}}^\nu(K_{k,\epsilon})} E |\langle \phi, S^M(\bar{\eta}_k^M) - \eta_k^M \rangle| + 2\epsilon
\] (5.37)

Using Lemma A.2 we get the final bound of the first term in R.H.S of (5.37) as $\frac{2m_t|\mathcal{F}_{m_t}^N(K_{k,\epsilon})|}{\sqrt{M}}$. Combining this estimate with (5.28),(5.31) and (5.34) we now have
\[
EW_1(\bar{\mu}_{k+1}^N, \mu_{k+1}) \leq (|A| + \delta \sigma (2 + \bar{\eta}_{\mathcal{P}}) + \delta \sigma \alpha \bar{\eta}_{\mathcal{P}}) EW_1(\bar{\mu}_k^N, \mu_k) + \delta \sigma (1 - \alpha) \bar{\eta}_{\mathcal{P}} EW_1(\bar{\eta}_k^M, \eta_k)
\]
\[
+ \frac{2\delta \sigma (1 - \alpha) \bar{\eta}_{\mathcal{P}} m_{k,\epsilon} \mathcal{F}_{m_{k,\epsilon,1}}^\nu(K_{k,\epsilon})}{\sqrt{M}} + \frac{2m_t|\mathcal{F}_{m_t}^N(K_{k,\epsilon})|}{\sqrt{N}} + (3 + 2\delta \sigma (1 - \alpha) \bar{\eta}_{\mathcal{P}}) \epsilon.
\] (5.38)
For the term $EW_1(\tilde{\eta}^M_{k+1}, \eta_{k+1})$, we start with the following recursive form

$$\tilde{\eta}^M_{k+1} - \eta_{k+1} = (1 - \alpha) \left[ S^M(\tilde{\eta}^M_k) - \tilde{\eta}^M_k \right] P + (1 - \alpha) \left[ \tilde{\eta}^M_k - \eta_k \right] P + \alpha \left[ \tilde{\mu}^N_k - \mu_k \right] P'$$

which leads to the following inequality

$$W_1(\tilde{\eta}^M_{k+1}, \eta_{k+1}) \leq (1 - \alpha) l(P) W_1(S^M(\tilde{\eta}^M_k), \tilde{\eta}^M_k) + (1 - \alpha) l(P) W_1(\tilde{\eta}^M_k, \eta_k) + \alpha l(P') W_1(\tilde{\mu}^N_k, \mu_k). \quad (5.40)$$

Using earlier estimates one has the final estimate for

$$EW_1(\tilde{\eta}^M_{k+1}, \eta_{k+1}) \leq 2(1 - \alpha) l(P) \frac{m_{k,c} |F^c_{m_{k,c}}(K_{k,c})|}{\sqrt{M}} + (1 - \alpha) l(P) W_1(\tilde{\eta}^M_k, \eta_k) + \alpha l(P') W_1(\tilde{\mu}^N_k, \mu_k) + 2(1 - \alpha) l(P) \epsilon. \quad (5.41)$$

Adding (5.38) and (5.41), using induction hypothesis and sending $M, N \to \infty$ we have

$$EW_1(\tilde{\mu}^N_{k+1}, \mu_{k+1}) + EW_1(\tilde{\eta}^M_{k+1}, \eta_{k+1}) \leq (3 + 2\delta \sigma (1 - \alpha) l^P + 2(1 - \alpha) l(P)) \epsilon.$$

Since $\epsilon > 0$ arbitrary, the result follows.

Part (a) can be proved similarly. The change will come from the structural difference of $\tilde{\eta}^N_k$ and $\tilde{\eta}^M_k$ because of the change in the updating kernel. So the term coming from the quantity $S^M(\tilde{\eta}^M_k) - \tilde{\eta}^M_k$ won’t appear here. Hence we get the following final estimate

$$E \left[ W_1(\mu^N_{k+1}, \mu_{k+1}) + W_1(\eta^M_{k+1}, \eta_{k+1}) \right] \leq \left[ \|A\| + \delta \sigma (2 + 1 l^P_{P'}) + \delta \sigma \alpha l^P_{P'} + \alpha l(P') \right] EW_1(\mu^N_k, \mu_k) + \left[ \delta \sigma (1 - \alpha) l^P + (1 - \alpha) l(P) \right] EW_1(\eta^M_k, \eta_k) + 3\epsilon + \frac{2m_{\epsilon} |F^c_{m_{\epsilon}}(K_{\epsilon})|}{\sqrt{N}}$$

from which the result follows by induction.

\[\square\]

### 5.2 Proof of Proposition 3.4

The techniques that we used is very similar with the contraction based method that was used in [3]. We will start with the following lemma and then prove the Proposition 3.4 using it. Define the following distance on $\mathcal{P}_1(\mathbb{R}^d) \times \mathcal{P}_1^1(\mathbb{R}^d)$ for $(\mu_n, \eta_n), (\mu'_n, \eta'_n) \in \mathcal{P}_1(\mathbb{R}^d) \times \mathcal{P}_1^1(\mathbb{R}^d)$

$$W_1((\mu_n, \eta_n), (\mu'_n, \eta'_n)) := W_1(\mu_n, \mu'_n) + W_1(\eta_n, \eta'_n).$$

Note that it is a complete separable metric of the space $\mathcal{P}_1(\mathbb{R}^d) \times \mathcal{P}_1^1(\mathbb{R}^d)$.

**Lemma 5.4** Let $\mu_0, \mu'_0 \in \mathcal{P}_1(\mathbb{R}^d)$ and $\eta_0, \eta'_0 \in \mathcal{P}_1^1(\mathbb{R}^d)$. Suppose Assumptions 1, 2, 4 and 5 hold. Then the transformation $\Psi : \mathcal{P}_1(\mathbb{R}^d) \times \mathcal{P}_1^1(\mathbb{R}^d) \to \mathcal{P}_1(\mathbb{R}^d) \times \mathcal{P}_1^1(\mathbb{R}^d)$ is well defined if following hold

$$\delta < \alpha_0 \quad \text{and} \quad \sum_{i=0}^{\infty} (1 - \alpha)^i \int_{\mathbb{R}^d} |y|^P P^i(0, dy) < \infty. \quad (5.42)$$

Moreover if Assumptions 4, 3 and 5 hold with the following condition:

$$\max \left\{ \left( \|A\| + \delta \sigma (2 + 1 l^P_{P'}) + \alpha l(P') \right), (1 - \alpha) l(P) \right\} + \delta \sigma \max \{ \alpha l^P_{P'} + (1 - \alpha) l^P_{P'} \} < 1. \quad (5.43)$$

Then there exist a $\theta \in (0, 1)$ and a constant $a_1 \in (0, \infty)$ such that for any $n \in \mathbb{N}$,

$$W_1(\Psi^n(\mu_0, \eta_0), \Psi^n(\mu'_0, \eta'_0)) \leq a_1 \theta^n.$$

**Remark 5.4** The condition (5.43) implies the first condition of (5.42) while the second one is very general.
5.2.1 Proof of Lemma 5.4

For fixed \( \mu_0, \mu'_0 \in \mathcal{P}_1(\mathbb{R}^d) \) and \( \eta_0, \eta'_0 \in \mathcal{P}^*_1(\mathbb{R}^d) \) define the following quantities for \( n \geq 1 \)

\[
(\mu_n, \eta_n) = \Psi^n(\mu_0, \eta_0), \quad (\mu_n', \eta_n') = \Psi^n(\mu'_0, \eta'_0) \quad \text{and} \quad \Psi^0 = I.
\]

First we will show that under transformation \( \Psi \) the \( (\mu_n, \nu_n) \in \mathcal{P}_1(\mathbb{R}^d) \times \mathcal{P}^*_1(\mathbb{R}^d) \) for \( (\mu_0, \nu_0) \in \mathcal{P}_1(\mathbb{R}^d) \times \mathcal{P}^*_1(\mathbb{R}^d) \), so that the quantity \( W_1(\mu_n, \mu_n', \nu_n, \nu_n') \) is well defined. Note that, if \( \delta \in (0, a_0) \), then \( \gamma = \|A\| + \delta \sigma \left( 2 + I_{P_P'P'}^\alpha \right) \in (0, 1) \), implying

\[
\langle |x|, \mu_n \rangle \leq \gamma^n \langle |x|, \mu_0 \rangle + \frac{\delta[\sigma c_{PP'}^\alpha + \sigma_2]}{1 - \gamma},
\]

which follows similarly from the proof of Lemma 5.1(a). It means if \( \delta \in (0, a_0) \) and \( \langle |x|, \mu_0 \rangle < \infty \) hold, then \( \mu_n \in \mathcal{P}_1(\mathbb{R}^d) \) for all \( n \geq 1 \). Under conditions in (5.42) one also has \( \sup_{n \geq 0} \langle |x|, \eta_n \rangle < \infty \) for all \( n \in \mathbb{N} \). One has \( \nabla \eta_{n+1}(y) = \int_{\mathbb{R}^d} \eta_n(x) [\nabla_y R_{\mu_n}^\alpha(x, y)] dx \) by Assumption 4 using DCT. From that condition it follows that for any \( n \geq 1 \), \( \|\nabla \eta_n(\cdot)\|_1 < (1 - \alpha) \|I_{P_P'} + \alpha I_{P_P'}^\alpha\| < \infty \) showing \( \eta_n \in \mathcal{P}^*_1(\mathbb{R}^d) \) for all \( n > 0 \) if \( \eta_0 \in \mathcal{P}^*_1(\mathbb{R}^d) \).

Now we will go back to the proof of the second part of the lemma regarding the contraction part. Assume \( n \geq 2 \). The first term of \( W_1((\mu_n, \eta_n), (\mu_n', \eta_n')) \) can be expressed as

\[
W_1(\mu_n, \mu_n') = W_1(\mu_n Q^{n-1, \mu_n-1}, \mu_n' Q^{n-1, \mu_n'-1}) \leq W_1(\mu_{n-1} Q^{n-1, \mu_n-1}, \mu_{n-1}' Q^{n-1, \mu_n'-1}) + W_1(\mu_n' Q^{n-1, \mu_n'-1}, \mu_{n-1}' Q^{n-1, \mu_n'-1}) = T_1 + T_2.
\]

The last inequality (5.45) follows from Assumption 1. As a consequence of Assumption 4 from (5.2) it follows that

\[
|\nabla \eta_{n+1}(X) - \nabla \eta_{n+1}(Y)| \leq \int_{\mathbb{R}^d} \eta_n(x) |\nabla_y R_{\mu_n}^\alpha(x, X) - \nabla_y R_{\mu_n}^\alpha(x, X)| dx \leq (1 - \alpha) \int_{\mathbb{R}^d} \eta_n(x) |\nabla_y P(x, X) - \nabla_y P(x, Y)| dx + \alpha \int_{\mathbb{R}^d} \eta_n(x) |\nabla_y \mu_n P'(X) - \nabla_y \mu_n P'(Y)| \leq \|I_{P_P'}^\alpha\|_1 |X - Y|.
\]

With that estimate, taking infimum at R.H.S of (5.45) with all possible couplings of \((X, Y)\) with marginals respectively \( \mu_{n-1} \) and \( \mu_{n-1}' \), one gets

\[
T_1 = W_1(\mu_{n-1} Q^{n-1, \mu_n-1}, \mu_{n-1}' Q^{n-1, \mu_n'-1}) \leq (\|A\| + \delta \sigma (2 + I_{P_P'}^\alpha)) W_1(\mu_{n-1}, \mu_{n-1}').
\]
Let \( X \) be a \( \mathbb{R}^d \) valued random variable with law \( \mu'_{n-1} \). Now about the term \( T_2 \),

\[
T_2 = \mathcal{W}_1(\mu'_{n-1}Q^{\eta'_{n-1},\mu'_{n-1}}, \mu'_{n-1}Q^{\eta'_{n-1},\mu'_{n-1}})
\]

\[
\leq \sup_{g \in \text{Lip}_1(\mathbb{R}^d)} E\left[ g(AX + \delta f_\delta(\nabla \eta_{n-1}(X), \mu'_{n-1}, X, \epsilon)) - g(AX + \delta f_\delta(\nabla \eta'_n(X), \mu'_{n-1}, X, \epsilon)) \right]
\]

\[
\leq \delta \sigma E\left| \nabla \eta_{n-1}(X) - \nabla \eta'_n(X) \right|
\]

\[
\leq \delta \sigma E\left\| \int_{\mathbb{R}^d} \eta_{n-2}(x)(\nabla_y R^\alpha_{\mu'_{n-2}}(x, X))dx - \int_{\mathbb{R}^d} \eta'_n(x)(\nabla_y R^\alpha_{\mu'_{n-2}}(x, X))dx \right\|
\]

\[
\leq \alpha \delta \sigma \int_{\mathbb{R}^d} \eta_{n-2}(x) E\left| \nabla_y \mu_{n-2}P'(X) - \nabla_y \mu'_{n-2}P'(X) \right| dx
\]

\[
+(1 - \alpha) \delta \sigma E\left\| \int_{\mathbb{R}^d} \nabla_y P(x, X)(\eta_{n-2}(x) - \eta'_n(x))dx \right\|
\]

\[
= T_2^{(1)} + T_2^{(2)}
\]

(5.48)

Note that

\[
T_2^{(1)} := \alpha \delta \sigma \int_{\mathbb{R}^d} \eta_{n-2}(x) \int_{\mathbb{R}^d} \mu'_{n-1}(dz) \int_{\mathbb{R}^d} \left( \mu_{n-2}(dy) \nabla_y P'(y, z) - \mu_{n-2}(dy) \nabla_y P'(y, z) \right) dx
\]

(5.49)

Since from Assumption 4 \( \nabla_y P'(\bar{x}, \cdot) \) is a Lipschitz function with coefficient \( l_\sigma \), the first integrand in (5.49) will be bounded by \( l_\sigma \mathcal{W}_1(\mu_{n-2}, \mu'_{n-2}) \) which gives

\[
T_2^{(1)} \leq \alpha \delta \sigma l_\sigma \mathcal{W}_1(\mu_{n-2}, \mu'_{n-2}).
\]

(5.50)

Now using Assumption 3 the second term \( T_2^{(2)} \) gives similarly

\[
T_2^{(2)} = (1 - \alpha) \delta \sigma E\left\| \int_{\mathbb{R}^d} \nabla_y P(x, X)(\eta_{n-2}(x) - \eta'_n(x))dx \right\|
\]

\[
\leq (1 - \alpha) \delta \sigma \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \nabla_y P(x, y) \{ \eta_{n-2}(x) - \eta'_n(x) \} \mu'_{n-1}(dy)
\]

\[
\leq (1 - \alpha) \delta \sigma l_\sigma \mathcal{W}_1(\eta_{n-2}, \eta'_n).
\]

(5.51)

Using the Assumption 5 we have

\[
\mathcal{W}_1(\eta_n, \eta'_n) \leq (1 - \alpha) l(P)\mathcal{W}_1(\eta_{n-1}, \eta'_{n-1}) + \alpha l(P)\mathcal{W}_1(\mu_{n-1}, \mu'_{n-1})
\]

(5.52)

Combining (5.50), (5.51) and (5.52) we have the following recursion for \( n \geq 2 \),

\[
\mathcal{W}_1(\mu_n, \mu'_n) + \mathcal{W}_1(\eta_n, \eta'_n) \leq \left( \|A\| + \delta \sigma(2 + l_\sigma \sigma) \right) \mathcal{W}_1(\mu_{n-1}, \mu'_{n-1}) + \alpha l(P)\mathcal{W}_1(\mu_{n-1}, \mu'_{n-1}) + (1 - \alpha) \delta \sigma l_\sigma \mathcal{W}_1(\eta_{n-2}, \eta'_n) + (1 - \alpha) l(P)\mathcal{W}_1(\eta_{n-1}, \eta'_{n-1}).
\]

(5.53)

Define a sequence \( a_n := \mathcal{W}_1(\mu_n, \mu'_n) + \mathcal{W}_1(\eta_n, \eta'_n) \), for \( n \geq 2 \) and and first two terms we set them to be

\[
a_0 := \mathcal{W}_1(\mu_0, \mu'_0) + \mathcal{W}_1(\eta_0, \eta'_0), \quad a_1 := \mathcal{W}_1(\mu_1, \mu'_1) + \mathcal{W}_1(\eta_1, \eta'_1)
\]

which are well defined for \( \mu_0, \mu'_0 \in \mathcal{P}_f(\mathbb{R}^d) \) and \( \eta_0, \eta'_0 \in \mathcal{P}_f^1(\mathbb{R}^d) \). Then from (5.53) and denoting \( c_1 := \max \left\{ \left( \|A\| + \delta \sigma(2 + l_\sigma \sigma) \right), (1 - \alpha) l(P) \right\}, c_2 := \delta \sigma \max \{ \alpha l(P), (1 - \alpha) l_\sigma \} \) following holds

\[
a_n \leq c_1 a_{n-1} + c_2 a_{n-2}
\]

(5.54)

for \( n \geq 2 \). Given \((\omega, \delta, \alpha)\) if there exists a \( \theta \in (0, 1) \) for which the following inequality holds

\[
\frac{c_1}{\theta} + \frac{c_2}{\theta^2} \leq 1,
\]

(5.55)

then denoting \( \lambda = \frac{c_1}{\theta} \), we have

\[
a_n \leq \left[ \theta \left( 1 - \frac{\lambda}{\theta} \right) \right] a_{n-1} + \theta \lambda a_{n-2} \quad \Leftrightarrow \quad a_n + \lambda a_{n-1} \leq \theta (a_{n-1} + \lambda a_{n-2}).
\]

(5.56)
Existence of a solution $\theta \in (0,1)$ satisfying (5.55) is valid under $c_1 + c_2 < 1$ which is equivalent to the condition

$$\max \left\{ \left( \|A\| + \delta \sigma (2 + l_{P'}^\alpha) \right), (1 - \alpha) l(P) \right\} + \delta \sigma \max \{ a l_{P'}^\alpha, (1 - \alpha) l_{P'}^\alpha \} < 1$$  \hspace{1cm} (5.57)$$

in (5.43) satisfied by $(\delta, \alpha, \|A\|)$. From (5.57) it follows

$$a_n \leq a_n + \lambda a_{n-1} \leq \theta^{n-1}[a_1 + \lambda a_0]$$

for $n \geq 2$. Since

$$W_1(\eta_1, \eta'_1) = W_1(\eta_0 R_{\mu_0}^\alpha, \eta'_0 R_{\mu'_0}^\alpha) \leq (1 - \alpha) l(P) W_1(\eta_0, \eta'_0) + \alpha l(P') W_1(\mu_0, \mu'_0),$$

$$W_1(\mu_1, \mu'_1) = W_1(\mu_0 Q^{\nu_0, \mu_0}, \mu'_0 Q^{\nu'_0, \mu'_0}) \leq W_1(\mu_0Q^{\nu_0, \mu_0}, \mu'_0Q^{\nu'_0, \mu'_0}) + W_1(\mu'_0 Q^{\nu'_0, \mu'_0}, \mu'_0 Q^{\nu'_0, \mu'_0}) \leq \left( \|A\| + \delta \sigma (2 + l_{P'}^\alpha) \right) W_1(\mu_0, \mu'_0) + \delta \sigma E |\nabla \eta_0(X) - \nabla \eta'_0(X)|$$

where $X \sim \mu'_0$. Final estimate for $a_n$ is

$$a_n \leq \theta^{n-1}\left[ \max \left\{ \|A\| + \delta \sigma (2 + l_{P'}^\alpha) + \alpha l(P'), (1 - \alpha) l(P) \right\} + \lambda \right] a_0 + \delta \sigma E |\nabla \eta_0(X) - \nabla \eta'_0(X)| .$$

Since $X \sim \mu'_0 \in P_1(\mathbb{R}^d)$ and $\nabla \eta_0, \nabla \eta'_0$ have linear growth (since $\eta_0, \eta'_0 \in P_1^*(\mathbb{R}^d)$), the second term inside the bracket is finite. A general formula can be observed for $a_n$

$$W_1(\Psi^n(\mu_0, \eta_0), \Psi^n(\mu'_0, \eta'_0)) \leq \theta^n \left[ a W_1((\mu_0, \eta_0), (\mu'_0, \eta'_0)) + b W_1(\mu_0 Q^{\nu_0, \mu_0}, \mu'_0 Q^{\nu'_0, \mu'_0}) \right]$$

where

$$a = \max \left\{ \|A\| + \delta \sigma (2 + l_{P'}^\alpha) + \alpha l(P'), (1 - \alpha) l(P) \right\} + \lambda, \quad b = \frac{1}{\theta} .$$

Observe that the quantity inside the bracket of RHS of (5.58) is finite for $\mu_0, \mu'_0 \in P_1(\mathbb{R}^d)$ and $\eta_0, \eta'_0 \in P_1^*(\mathbb{R}^d)$. Hence proved the lemma.

We now complete the proof of the theorem. Given $l(P') < 1$ from Assumption (5), one can always find $(\omega_0, \alpha_0, \delta_0) \in (0,1) \times (0,1) \times (0,1)$ for which (5.57) holds under

$$\|A\| < \omega_0, \quad \alpha < \alpha_0, \quad \delta < \delta_0 .$$

For existence we need to show that under $W_1((\cdot, \cdot), (\cdot, \cdot))$ distance $P_1(\mathbb{R}^d) \times P_1^*(\mathbb{R}^d)$ is complete. From Lemma 5.4 one can choose $(\omega, \alpha, \delta)$ such that (5.43) holds. It follows that using the $\theta$ from that lemma the sequence $\{ \Psi^n(\mu_0, \eta_0) \} \in P_1^*(\mathbb{R}^d)$ is a cauchy sequence in $P_1(\mathbb{R}^d) \times P_1^*(\mathbb{R}^d)$ which is a complete metric space under $W_1((\cdot, \cdot), (\cdot, \cdot))$. So there exists a $(\mu_\infty, \eta_\infty) \in P_1(\mathbb{R}^d) \times P_1^*(\mathbb{R}^d)$ such that $\Psi^n(\mu_0, \eta_0) \rightarrow (\mu_\infty, \eta_\infty)$ as $n \rightarrow \infty$. Our assertion for existence will be proved if we prove $\eta_\infty \in P_1^*(\mathbb{R}^d)$. Given the initial condition $\|\nabla \eta_0(x)\|_1 < \infty$, we will always have from (5.2) $\|\nabla \eta_k(x)\|_1 < \infty \quad \forall \quad k > 1$. Note that for $\eta_0 \in P_1^*(\mathbb{R}^d)$, one has $\eta_k \in P_1^*(\mathbb{R}^d)$ for all $k$. This implies $\eta_\infty \in P_1^*(\mathbb{R}^d)$. So

$$(\mu_\infty, \eta_\infty) \in P_1(\mathbb{R}^d) \times P_1^*(\mathbb{R}^d) .$$

Observe further for $\theta \in (0,1)$ in (5.58) of Lemma 5.4

$$W_1(\Psi^n(\mu_0, \eta_0), (\mu_\infty, \eta_\infty)) = W_1(\Psi^n(\mu_0, \eta_0), \Psi^n(\mu_\infty, \eta_\infty)) \leq \theta^n [ a W_1((\mu_0, \eta_0), (\mu_\infty, \eta_\infty)) + b W_1(\mu_\infty Q^{\nu_\infty, \mu_\infty}, \mu_\infty Q^{\nu_\infty, \mu_\infty}) ] .$$

Uniqueness of fixed points follows immediately from (5.59).

$\square$
5.3 Proof of Theorem 1

We will prove part (b) of the theorem. Part (a) will follow similarly. We need to prove the following Lemma first.

Lemma 5.5 Consider the second particle system $\mathbb{IP} \mathbb{S}_2$. Suppose that Assumptions 7, 8 hold. Denote $N_1 = \min \{ N, M \}$. Then there exist a constant $C \in (0, \infty)$ such that the upper-bound $b(\tau, d)$ of the quantity $\sup_{k \geq 1} E W_1((\bar{\mu}^N_k, \bar{\eta}^M_k), (\bar{\mu}^N_{k-1}, \bar{\eta}^M_{k-1}))$ can be given as $b(N_1, \tau, d)$ as defined in Theorem 1. The constant $C$ will vary for different cases.

5.3.1 Proof of Lemma 5.5

We start with the fact that

$$EW_1((\bar{\mu}^N_k, \bar{\eta}^M_k), (\bar{\mu}^N_{k-1}, \bar{\eta}^M_{k-1})) = EW_1((\bar{\mu}^N_k, \bar{\mu}^N_{k-1}Q^{\bar{\eta}^N_{k-1}, \bar{\mu}^N_{k-1}}) + EW_1((\bar{\eta}^M_k, \bar{\eta}^M_{k-1}R^{\bar{\eta}^M_{k-1}}))$$

$$\leq EW_1((\bar{\mu}^N_k, \bar{\mu}^N_{k-1}Q^{\bar{\eta}^N_{k-1}, \bar{\mu}^N_{k-1}}) + (1 - \alpha)EW_1(S^M(\bar{\eta}^M_{k-1}), \bar{\eta}^M_{k-1})$$

$$= EW_1((\bar{\mu}^N_k, \bar{\mu}^N_{k-1}Q^{\bar{\eta}^N_{k-1}, \bar{\mu}^N_{k-1}}) + (1 - \alpha)E \left[ EW_1(S^M(\bar{\eta}^M_{k-1}), \bar{\eta}^M_{k-1})|F_{k-1} \right].$$

(5.60)

In order to bound both terms in (5.60) we borrow the following formulation from [10] about the convergence rate of empirical distribution of iid random variables to its common distribution, where the key idea of bounding Wasserstein distance came from the constructive quantization context [9]. A similar idea was also developed in [1]. We will maintain the same notation used in [10]. Let $\Pi$ be the natural partition of $(-1, 1]^d$ into $2^d$ translations of $(-2^{-l}, 2^{-l}]d$. Define a sequence of sets $\{ B_n \}_{n \geq 0}$ such that $B_0 := (-1, 1]^d$ and, for $n \geq 1$, $B_n := (-2^n, 2^n]^d \setminus (-2^{n-1}, 2^{n-1})^d$. For a set $F \subset \mathbb{R}^d$ denote the set $2^n F$ as $\{2^n \cdot x : x \in F \}$. For any two probability measures $\mu$ and $\nu$, combining Lemma 5 and 6 of [10] one has the following inequality for the Wasserstein-1 distance,

$$W_1(\mu, \nu) \leq 3C \left(2^{1/2} \sum_{n \geq 0} 2^n \sum_{l \geq 0} \sum_{F \in \Pi} \left[ \mu(2^n F \cap B_n) - \nu(2^n F \cap B_n) \right] \right),$$

(5.61)

where $C$ is a constant depends only on $d$. We denote $a_{k,M,N}^{i} := \delta_{\vec{X}_{k-1}^{i}} - \delta_{\vec{X}_{k-1}^{i}}Q^{\bar{\eta}^N_{k-1}, \bar{\mu}^N_{k-1}}$. It follows that $\bar{\mu}^N_{k-1}Q^{\bar{\eta}^N_{k-1}, \bar{\mu}^N_{k-1}} = \frac{1}{N} \sum_{i=1}^{N} a_{k,M,N}^{i}$. Note that on conditioned upon $F_{k-1}$, the family of signed measures $\{a_{k,M,N}^{i}\}_{i=1,\ldots,M}$ is an independent class of measures while unconditionally they are just identical. Using the fact that for any set $A \in \mathcal{B}(\mathbb{R}^d)$, $\delta_{\vec{X}_{k-1}^{i}}(A) \left| F_{k-1} \right. \sim \text{Bernoulli}(\delta_{\vec{X}_{k-1}^{i}}Q^{\bar{\eta}^N_{k-1}, \bar{\mu}^N_{k-1}}(A))$, we have

$$E \left[ \left( a_{k,M,N}^{i}(A) \right)^2 \mid F_{k-1} \right] = \delta_{\vec{X}_{k-1}^{i}}Q^{\bar{\eta}^N_{k-1}, \bar{\mu}^N_{k-1}}(A) \left[ 1 - \delta_{\vec{X}_{k-1}^{i}}Q^{\bar{\eta}^N_{k-1}, \bar{\mu}^N_{k-1}}(A) \right] \leq \delta_{\vec{X}_{k-1}^{i}}Q^{\bar{\eta}^N_{k-1}, \bar{\mu}^N_{k-1}}(A)$$

(5.62)

which implies the unconditional expectation $E \left[ \left( a_{k,M,N}^{i}(A) \right)^2 \right] \leq P \left[ \vec{X}_{k-1}^{i} + \beta \delta_{\vec{X}_{k-1}^{i}}Q^{\bar{\eta}^N_{k-1}, \bar{\mu}^N_{k-1}}(A) \right] \in A$. Using all these we have

$$E \left[ \bar{\mu}^N_k(A) - \bar{\mu}^N_{k-1}Q^{\bar{\eta}^N_{k-1}, \bar{\mu}^N_{k-1}}(A) \right]^2 = E \left[ \frac{1}{N} \sum_{i=1}^{N} a_{k,M,N}^{i}(A) \right]^2 \leq \frac{E \left[ a_{k,M,N}^{i}(A) \right]^2}{N} \leq \frac{E \left[ \left( a_{k,M,N}^{i}(A) \right)^2 \mid F_{k-1} \right]}{N} \left[ 1 - \delta_{\vec{X}_{k-1}^{i}}Q^{\bar{\eta}^N_{k-1}, \bar{\mu}^N_{k-1}}(A) \right].$$

Using these with Cauchy-Schwarz inequality one gets following bound

$$E \left[ \bar{\mu}^N_k(A) - \bar{\mu}^N_{k-1}Q^{\bar{\eta}^N_{k-1}, \bar{\mu}^N_{k-1}}(A) \right] \leq \min \left\{ \sqrt{E \left[ \bar{\mu}^N_k(A) - \bar{\mu}^N_{k-1}Q^{\bar{\eta}^N_{k-1}, \bar{\mu}^N_{k-1}}(A) \right]} \frac{\left[ 1 - \delta_{\vec{X}_{k-1}^{i}}Q^{\bar{\eta}^N_{k-1}, \bar{\mu}^N_{k-1}}(A) \right]}{N}, \frac{2E \left[ \delta_{\vec{X}_{k-1}^{i}}Q^{\bar{\eta}^N_{k-1}, \bar{\mu}^N_{k-1}}(A) \right]}{N} \right\}.$$

(5.63)
where second term inside the bracket of RHS of (5.63) follows trivially. Denoting the whole constant in R.H.S of (5.61) as \( C_d \), we have

\[
EW_1(\hat{\mu}_k^N, \hat{\mu}_{k-1}^N Q_{\hat{\mu}_{k-1}^N} - \hat{\mu}_{k-1}^N) \leq C_d \sum_{n \geq 0} \sum_{l \geq 0} 2^n 2^{-l} E \left[ \hat{\mu}_k^N (2^n F \cap B_n) - \hat{\mu}_{k-1}^N Q_{\hat{\mu}_{k-1}^N} (2^n F \cap B_n) \right]
\] (5.64)

Note that \( \# p_1 = 2^d \). Using Cauchy-Schwarz inequality with (5.63) and Jensen’s inequality \( \sqrt{X} \leq \sqrt{EX} \) for non-negative random variable \( X \), the last sum \( E \sum_{F \in p_1} [\hat{\mu}_k^N (2^n F \cap B_n) - \hat{\mu}_{k-1}^N Q_{\hat{\mu}_{k-1}^N} (2^n F \cap B_n)] \) in the R.H.S of (5.64) can be bounded by

\[
\leq \min \left\{ 2^d \left[ E \left[ \frac{1}{N} \delta_{X_{k-1}} (Q_{\hat{\mu}_{k-1}^N} (B_n)) \right] \right]^\frac{1}{2}, 2E \left[ \delta_{X_{k-1}} (Q_{\hat{\mu}_{k-1}^N} (B_n)) \right] \right\}.
\] (5.65)

Now using Remark 5.1 along with Lemma 5.1, if \( \delta \in (0, a(\tau)) \) the quantity \( \sup_{n \geq 0} \sup_{M,N \geq 1} E|X_i^{\tau + \tau} := b(\tau) < \infty \), one has by Chebyshev inequality for \( n \geq 1 \),

\[
\sup_{k \geq 1} E \left[ \delta_{X_{k-1}} (Q_{\hat{\mu}_{k-1}^N} (B_n)) \right] \leq \sup_{k \geq 1} P[|X_k| > 2^{n-1}] \leq \frac{b(\tau)}{2^{(1+\tau)(n-1)}} = b(\tau) 2^{-(1+\tau)(n-1)}.
\]

Note that \( a(\tau) \rightarrow a_0 \) as \( \tau \rightarrow 0 \) and \( \delta \in (0, a_0) \), we can find \( \tau_0 \in (0, a(\tau)) \) such that \( \delta \in (0, a(\tau_0) \rightarrow a_0) \). So the bound in (5.65) can be restated as

\[
\sup_{k \geq 1} EW_1(\hat{\mu}_k^N, \hat{\mu}_{k-1}^N Q_{\hat{\mu}_{k-1}^N}) \leq C_d \sum_{n \geq 0} \sum_{l \geq 0} 2^n 2^{-l} \min \left\{ 2^d \left[ \frac{b(\tau) 2^{-(1+\tau)(n-1)}}{N}, 2b(\tau) 2^{-(1+\tau)(n-1)} \right] \right\}
\] (5.66)

where \( b(\tau) \) is just a constant and the last display is obtained by accumulating upper bounds of all the constants to \( C_d \). Now proceeding exactly like step 1 to step 4 of the proof of Theorem 1 (for \( p = 1, q = 1+\tau \) in [10] one gets the following bounds

\[
\sup_{k \geq 1} EW_1(\hat{\mu}_k^N, \hat{\mu}_{k-1}^N Q_{\hat{\mu}_{k-1}^N}) = C \begin{cases} N^{-\max \left\{ \frac{1}{2}, \frac{1}{1+\tau} \right\}} & \text{if } d = 1, \tau \neq 1, \\ N^{-\frac{1}{2} \log (1 + N) + \frac{1}{1+\tau}} & \text{if } d = 2, \tau \neq 1, \\ N^{-\max \left\{ \frac{1}{2}, \frac{1}{1+\tau} \right\}} & \text{if } d > 2, \tau \neq \frac{1}{d-1}. \end{cases}
\]

Now we will fill the gaps for each of the three special cases \( \tau = 1, \tau = 1 \) and \( \tau = \frac{1}{d-1} \) of three regimes respectively \( d = 1, d = 2 \) and \( d > 2 \). We note that one can generalize the choice of \( l_{N,\varepsilon} \) done in step 1 of Theorem 1 of [10] where \( l_{N,\varepsilon} \) could be taken as \( \frac{1}{d \log N} \vee 0 \) instead of \( \frac{1}{d \log 2} \vee 0 \) though it doesn’t change the conclusion of the main theorem. After step 1 with \( p = 1, q = 1+\tau, \varepsilon = 2^{-(1+\tau)n} \) one will get

\[
\sum_{l \geq 0} 2^{-l} \min \left\{ 2^d \sqrt{\frac{\varepsilon}{N}}, \varepsilon \right\} = C \begin{cases} \min \left\{ \varepsilon, \left( \frac{\varepsilon}{N} \right)^{\frac{1}{2}} \right\} & \text{if } d = 1, \\ \min \left\{ \varepsilon, \left( \frac{\varepsilon}{N} \right)^{\frac{1}{2}} \log (\varepsilon N) \right\} & \text{if } d = 2, \\ \min \left\{ \varepsilon, \varepsilon (\varepsilon N)^{-\frac{1}{2}} \right\} & \text{if } d > 2, \end{cases}
\]

where the constant C will vary from case to cases. Suppose \( d = 1 \). From (5.66) for general \( \tau > 0 \) one has

\[
\sup_{k \geq 1} EW_1(\hat{\mu}_k^N, \hat{\mu}_{k-1}^N Q_{\hat{\mu}_{k-1}^N}) \leq C_d' \sum_{n \geq 0} 2^n \min \left\{ 2^{-(1+\tau)n}, \left( \frac{2^{(1+\tau)n}}{N} \right)^{\frac{1}{2}} \right\}.
\] (5.67)

Note that for \( n \geq n_{N,\tau} := \frac{\log N}{(1+\tau) \log 2} \), one has \( 2^{-(1+\tau)n} \leq \left( \frac{2-(1+\tau)n}{N} \right)^{\frac{1}{2}} \). So for \( \tau = 1 \),

\[
\sum_{n \geq 0} 2^n \min \left\{ 2^{-(2n)}, \left( \frac{2^{2n}}{N} \right)^{\frac{1}{2}} \right\} \leq \sum_{n < n_{N,1}} 2^n \left( \frac{2^{2n}}{N} \right)^{\frac{1}{2}} + \sum_{n \geq n_{N,1}} 2^{-n} = n_{N,1} N^{-\frac{1}{2}} + C 2^{-n_{N,1}} = N^{-\frac{1}{2}} \frac{\log N}{2 \log 2} + C N^{-\frac{1}{2}}.
\] (5.68)
Finally using Jensen inequality
\[ E \leq C_d \sum_{n \geq 0} 2^n \min \left\{ 2^{-2n}, \left( \frac{2^{-2n}}{N} \right)^{\frac{1}{d}} \left[ \log \left( 2^{-2n} N \right) \lor 0 \right] \right\}. \]

For \( \tau = 1, \varepsilon = 2^{-2n} \). Note that if \( n < n_N^{(2)} := \log N - \log_2 (\log N) \), then one has
\[ \varepsilon = 2^{-2n} > \left( \frac{2^{-2n}}{N} \right)^{\frac{1}{d}} \left[ \log \left( 2^{-2n} N \right) \lor 0 \right] \]
\[ \leq \sum_{n < n_N^{(2)}} 2^n \left( \frac{2^{-2n}}{N} \right)^{\frac{1}{d}} \left[ \log \left( 2^{-2n} N \right) \lor 0 \right] + \sum_{n \geq n_N^{(2)}} 2^{-n} \leq n_N^{(2)} \frac{\log N}{N^{\frac{1}{d}}} + C 2^{-n_N^{(2)}} \]
\[ \leq C_1 N^{-\frac{1}{d}} \left[ (\log N)^2 - \log N \log_2 (\log N) \right] + C_2 \frac{\log N}{\sqrt{N}}. \quad (5.69) \]

By proceeding similarly, for all non regular cases we will end up getting the following results (the constant \( C \) will vary from case to case):
\[ \sup_{k \geq 1} EW_1(\tilde{\mu}_k^N, \tilde{\mu}_{k-1}^N Q_{\tilde{b}k-1, \tilde{\mu}_{k-1}}^N) = C \begin{cases} N^{-\frac{1}{d}} \log N + N^{-\frac{1}{d}} & \text{if } d = 1, \tau = 1, \\ N^{\frac{1}{d}} \left[ (\log N)^2 - \log N \log_2 (\log N) \right] + \frac{\log N}{N^{\frac{1}{d}}} & \text{if } d = 2, \tau = 1, \\ \log_2 N + N^{\frac{1}{d}} & \text{if } d > 2, \tau = \frac{1}{d-1}. \end{cases} \]

Now about the second term of (5.60) using (5.61), the upperbound of \( EW_1(S^M(\tilde{\eta}_k^{M-1})(\tilde{\eta}_k^{M-1})) \) is
\[ 3C_2^{(1+\frac{1}{d})} \sum_{n \geq 0} 2^n \sum_{l \geq 0} 2^{-l} E \sum_{F \in F_l} \left[ S^M(\tilde{\eta}_k^{M-1})(2^n F \cap B_n) - \tilde{\eta}_k^{M-1}(2^n F \cap B_n) \right] \cdot \quad (5.70) \]

By Cauchy Schwarz inequality and using Jensen inequality \( E \sqrt{X} \leq \sqrt{EX} \) for a nonnegative random variable \( X \), one gets the upperbound of
\[ E \left[ \sum_{F \in F_{k-1}} \left[ S^M(\tilde{\eta}_k^{M-1})(2^n F \cap B_n) - \tilde{\eta}_k^{M-1}(2^n F \cap B_n) \right] \right] \leq 2^{\frac{d}{2}} \left[ \sum_{F \in F_{k-1}} \frac{1}{M} \sum_{i=1}^{M} \delta_{Y_{\eta_k}}(2^n F \cap B_n) \right] \cdot \quad (5.71) \]

Using similar argument used in (5.62) the R.H.S of (5.71) will be less than
\[ 2^{\frac{d}{2}} \left[ \frac{\sum_{F \in F_{k-1}} \tilde{\eta}_k^{M-1}(2^n F \cap B_n) \left( 1 - \tilde{\eta}_k^{M-1}(2^n F \cap B_n) \right)}{M} \right] \leq 2^{\frac{d}{2}} \left[ \frac{\tilde{\eta}_k^{M-1}(B_n)}{M} \right] \cdot \quad (5.72) \]

Finally using Jensen inequality \( E \sqrt{X} \leq \sqrt{EX} \), and from Corollary 5.3 followed by Lemma 5.2(b) denoting \( c(\tau) := \sup_{k \geq 1} \sup_{M \geq 1} E \left( \langle x \rangle^{1+\tau}, \tilde{\eta}_k^{M-1} \right) \) one gets
\[ \sup_{k \geq 1} E \sum_{F \in F_{k-1}} \left[ S^M(\tilde{\eta}_k^{M-1})(2^n F \cap B_n) - \tilde{\eta}_k^{M-1}(2^n F \cap B_n) \right] \leq 2^{\frac{d}{2}} \sup_{k \geq 1} E \left[ \frac{\langle x \rangle^{1+\tau}, \tilde{\eta}_k^{M-1}}{M} \right] \cdot \quad (5.73) \]
Hence the conclusion about the upper bound of $\text{EW}_1(S^M(\tilde{\eta}^M_{k-1}), \tilde{\eta}^M_{k-1})$ will be similar to the first term of (5.60). It will be a function of the sample size of the concentration gradient $M$ in place of $N$ in the bound of $\text{EW}_1(\tilde{\mu}^N_k, \tilde{\mu}^N_{k-1}Q^{\tilde{\eta}^M_{k-1}, \tilde{\eta}^M_{k-1}})$. Combining this with the conclusion about the first term of (5.60) we can state the bound in terms of $N_1 = \min\{M, N\}$ and the result of Lemma 5.5 will follow.

\[\square\]

Now we will complete the theorem. Observe the following identity

\[
(\tilde{\mu}^N_n, \tilde{\eta}^M_n) - (\mu_n, \eta_n) = \sum_{i=1}^n \left[ \Psi^{(n-i)}(\tilde{\mu}^N_i, \tilde{\eta}^M_i) - \Psi^{(n-i)}(\mu_n, \eta_n) \right] + \left[ \Psi^n(\tilde{\mu}^N_0, \tilde{\eta}^M_0) - \Psi^n(\mu_0, \eta_0) \right].
\]

Using Triangular inequality and Lemma 5.4 following holds

\[
\begin{align*}
\mathcal{W}_1\left( (\tilde{\mu}^N_n, \tilde{\eta}^M_n), (\mu_n, \eta_n) \right) &\leq \sum_{i=1}^n \mathcal{W}_1\left( \Psi^{(n-i)}(\tilde{\mu}^N_i, \tilde{\eta}^M_i), \Psi^{(n-i)}(\mu_n, \eta_n) \right) + \mathcal{W}_1\left( \Psi^n(\tilde{\mu}^N_0, \tilde{\eta}^M_0), \Psi^n(\mu_0, \eta_0) \right) \\
&\leq \sum_{i=1}^n \theta^{n-i} \left[ a\mathcal{W}_1\left( (\tilde{\mu}^N_i, \tilde{\eta}^M_i), \Psi(\tilde{\mu}^N_{i-1}, \tilde{\eta}^M_{i-1}) \right) + b\mathcal{W}_1\left( \tilde{\mu}_{M,N}^{(i-1)}Q_{\tilde{\eta}^M_{i-1}}, \tilde{\eta}^M_{i-1}Q_{\mu_{M,N}}^{(i-1)} \right) \right] \\
&\quad + \theta^n \left[ a\mathcal{W}_1\left( (\tilde{\mu}^N_0, \tilde{\eta}^M_0), (\mu_0, \eta_0) \right) + b\mathcal{W}_1(\mu_0Q^{\tilde{\eta}^M_0}, \mu_0Q^{\eta_0}) \right]
\end{align*}
\]

(5.74)

where (5.74) follows from (5.58) with specified constants $a$ and $b$ and $\tilde{\mu}_{M,N}^{(i-1)} := \tilde{\mu}^N_{i-1}Q^{\tilde{\eta}^M_{i-1}}\tilde{\mu}^N_{i-1}$. Let $X_i^{M,N}$ be a random variable, conditioned on $\mathcal{F}_i^{M,N}$, sampled from $\tilde{\mu}^{(i-1)}_{M,N}$. We have

\[
\begin{align*}
\mathcal{W}_1\left( \tilde{\mu}_{M,N}^{(i-1)}Q^{\tilde{\eta}^M_{i-1}}, \tilde{\eta}^M_{i-1}Q_{\mu_{M,N}}^{(i-1)} \right) &\leq \sup_{g \in \text{Lip}_1([0,1])} \left\| g(AX_i^{M,N} + \delta \sigma(\nabla \tilde{\eta}_{i-1}^{M,N}, \tilde{\mu}_{M,N}^{(i-1)}, X_i^{M,N}, \epsilon)) - g(AX_i^{M,N}) \right\| \\
&\quad + \delta \sigma \left( \| \nabla \tilde{\eta}_{i-1}^{M,N}(X_i^{M,N}) - \nabla \tilde{\eta}_{i-1}^{M,N}(X_i^{M,N}) \|_{\mathcal{F}_i^{M,N}} \right) \\
&= (1 - \alpha) \int \left[ \int \left( S^M(\tilde{\eta}_{i-1}^{M,N}) - \tilde{\eta}_{i-1}^{M,N} \right) (dx) \nabla_y P(x, y) \right] (\tilde{\mu}_{i-1}^{N}Q^{\tilde{\eta}^M_{i-1}})(dy) \\
&\leq \mathcal{I}_k^N(1 - \alpha)\mathcal{W}_1\left( S^M(\tilde{\eta}_{i-1}^{M,N}), \tilde{\eta}^M_{i-1} \right).
\end{align*}
\]

(5.75)

Last display follows from Assumption 4. Since $\tilde{\eta}_0^M = \eta_0$, one has

\[
\mathcal{W}_1(\mu_0Q^{\tilde{\eta}^M_0}, \mu_0Q^{\eta_0}) = 0.
\]

(5.76)

Combining the results (5.75),(5.76), with (5.74) we get for each $n$,

\[
\begin{align*}
\mathcal{W}_1\left( (\tilde{\mu}^N_n, \tilde{\eta}^M_n), (\mu_n, \eta_n) \right) &\leq \frac{a}{1 - \theta} \sup_{k \geq 1} \mathcal{W}_1\left( (\tilde{\mu}_k^N, \tilde{\eta}_k^M), \Psi(\tilde{\mu}_{k-1}^N, \tilde{\eta}_{k-1}^M) \right) \\
&\quad + \frac{b}{1 - \theta} \sup_{k \geq 1} \mathcal{W}_1\left( S^M(\tilde{\eta}_{k-1}^{M,N}), \tilde{\eta}^M_{k-1} \right) + a\theta^n \mathcal{W}_1\left( (\tilde{\mu}^N_0, \tilde{\eta}^M_0), (\mu_0, \eta_0) \right).
\end{align*}
\]

(5.77)

Using Lemma 5.5 the result follows.

\[\square\]

5.4 Proof of Corollary 3.6:

Using triangular inequality and from (5.58) one gets

\[
\begin{align*}
\text{EW}_1\left( (\tilde{\mu}^N_n, \tilde{\eta}^M_n), (\mu_\infty, \eta_\infty) \right) &\leq \mathcal{W}_1\left( (\mu_n, \eta_n), (\mu_\infty, \eta_\infty) \right) + \text{EW}_1\left( (\tilde{\mu}^N_n, \tilde{\eta}^M_n), (\mu_n, \eta_n) \right) \\
&\leq \theta^n \left[ a\mathcal{W}_1(\mu_0, \eta_0), (\mu_\infty, \eta_\infty) \right] + b\mathcal{W}_1(\mu_0Q^{\eta_\infty}, \mu_0Q^{\eta_\infty}) + \text{EW}_1\left( (\tilde{\mu}^N_n, \tilde{\eta}^M_n), (\mu_n, \eta_n) \right).
\end{align*}
\]

(5.78)
Combining this with (5.77) we get

\[ EW_1 (\tilde{\mu}_n^N, \tilde{\eta}_n^M, (\mu_\infty, \eta_\infty)) \leq \theta^n \left[ a W_1 ((\mu_0, \eta_0), (\mu_\infty, \eta_\infty)) + b W_1 (\mu_0 Q^n, \mu_0 Q^n) \right] + \frac{\alpha}{1 - \theta} \sup_{k \geq 1} EW_1 (\tilde{\mu}_k^N, \tilde{\eta}_k^M, (\mu_\infty, \eta_\infty)) + \frac{b \delta(1 - \alpha)}{1 - \theta} \sup_{k \geq 1} EW_1 (\tilde{\mu}_k^M, \tilde{\eta}_k^M, (\mu_\infty, \eta_\infty)). \]

The result is obvious after using Lemma 5.5.

\[ \square \]

### 5.5 Proof of Theorem 2:

Fix \( N \) and \( M \). Define \( \Theta_n^{N,M} \in \mathbb{P}((\mathbb{R}^N)^N \times \mathbb{P}_1(\mathbb{R}^d) \times \mathbb{P}(\mathbb{R}^d)) \) as

\[ \langle \phi, \Theta_n^{N,M} \rangle = \frac{1}{n} \sum_{j=1}^n E \phi (\tilde{X}_j(N), \eta_j^M, S^M(\eta_j^M)), \quad \phi \in BM ((\mathbb{R}^N)^N \times \mathbb{P}_1(\mathbb{R}^d) \times \mathbb{P}(\mathbb{R}^d)) \]  

(5.79)

for \( N \geq 1, M \geq 1 \) and \( n \in \mathbb{N}_0 \) where \( \{ (\tilde{X}_j(N), \eta_j^M, S^M(\eta_j^M)), j \in \mathbb{N}_0, i = 1, \ldots, N \} \) are as defined in the context of IPS\(_2\). Note that \((\mathbb{R}^N)^N \times \mathbb{P}_1(\mathbb{R}^d) \times \mathbb{P}(\mathbb{R}^d)\) is a complete separable metric space with metric

\[ d((x, \mu_1, \mu_2), (y, \mu_3, \mu_4)) := \| x - y \| + \frac{1}{2} \| \mu_1 - \mu_2 \| + \frac{1}{2} \| \mu_3 - \mu_4 \| \]

where \( \| \cdot \| \) is the metric on \((\mathbb{R}^N)^N \times \mathbb{P}_1(\mathbb{R}^d) \times \mathbb{P}(\mathbb{R}^d)\). From Lemma 5.1 and 5.2 it follows that, for each \( N, M \geq 1 \), the sequence \( \{ \Theta_n^{N,M}, n \geq 1 \} \) is relatively compact (By Prohorov’s Theorem) and using Assumption 1 it is easy to see that any limit point \( \Theta^N,M_N \) of \( \Theta_n^{N,M} \) (as \( n \to \infty \)) is an invariant measure of the Markov chain \( \{ X_n(N), \tilde{\eta}_n^M, S^M(\tilde{\eta}_n^M) \}_{n \geq 0} \) and from Lemma 5.1 it satisfies \( \int_{(\mathbb{R}^N)^N \times \mathbb{P}_1(\mathbb{R}^d) \times \mathbb{P}(\mathbb{R}^d)} | x | \Theta_N,M_N (dx) < \infty \) (Taking the norm of the product space as \( | (x, y, z) | = \| x \| + \frac{1}{2} \| y \| + \frac{1}{2} \| z \| \)) where \( (x, y, z) \in (\mathbb{R}^N)^N \times \mathbb{P}_1(\mathbb{R}^d) \times \mathbb{P}(\mathbb{R}^d) \). Uniqueness of invariant measure can be proved by the following simple coupling argument (see for example [5]): Suppose \( \Theta_M.M_N, \tilde{\Theta}_M^{N,M} \) are two invariant measures that satisfy \( \int_{(\mathbb{R}^N)^N \times \mathbb{P}_1(\mathbb{R}^d) \times \mathbb{P}(\mathbb{R}^d)} | x | \Theta_N,M_N (dx) < \infty, \int_{(\mathbb{R}^N)^N \times \mathbb{P}_1(\mathbb{R}^d) \times \mathbb{P}(\mathbb{R}^d)} | \tilde{x} | \tilde{\Theta}_M^{N,M} (dx) < \infty \).

Let \( (X_0(N), \tilde{\eta}_0^M, S^M(\tilde{\eta}_0^M)) \) and \( (\tilde{X}_0(N), \tilde{\eta}_0^M, S^M(\tilde{\eta}_0^M)) \) with probability laws \( \Theta_n^N,M \) and \( \tilde{\Theta}_n^{M,N} \) respectively be given on a common probability space under same noise sequence (i.e in which an i.i.d. array of \( M \) valued random variables \( \{ \epsilon_i^n, i = 1, \ldots, N, n \geq 1 \} \) are defined that is independent of \( (X_0(N), \tilde{\eta}_0^M, \tilde{X}_0(N), \tilde{\eta}_0^M) \) with common probability law \( \theta \) and the evolution equations are following.

\[
X_{n+1}^i = A X_n^i + \delta f_{\epsilon}(X_n^i, \nabla \eta_n^M(X_n^i), \tilde{\mu}_n^i, \epsilon_n^i, \mu_n^M) = \frac{1}{N} \sum_{i=1}^N \delta X_n^i, \quad \tilde{X}_{n+1}^i = A \tilde{X}_n^i + \delta f_{\epsilon}(\tilde{X}_n^i, \nabla \tilde{\eta}_n^M(\tilde{X}_n^i), \tilde{\mu}_n^i, \epsilon_n^i, \tilde{\mu}_n^M) = \frac{1}{N} \sum_{i=1}^N \delta \tilde{X}_n^i,
\]

where recall \( f_{\epsilon}(\cdot, \cdot, x) = f(\cdot, \cdot, x) + \frac{\beta(x)}{\delta} \). Note that

\[ W_1 (\frac{1}{N} \sum_{i=1}^N \delta X_n^i, \frac{1}{N} \sum_{i=1}^N \delta \tilde{X}_n^i) \leq \frac{1}{N} \sum_{i=1}^N | X_i - \tilde{X}_i | \]

(5.80)

for any two arrays \( \{ X_i \}_{i=1}^N \) and \( \{ \tilde{X}_i \}_{i=1}^N \). Using the independence of the noise sequence along with (5.80) and Assumption 1 we have

\[
E | X_{n+1}^i - \tilde{X}_{n+1}^i | \leq E | X_{n}^i - \tilde{X}_n^i | + \delta \frac{1}{N} \sum_{j=1}^N E | X_j^i - \tilde{X}_j^i | + \delta E | \nabla \eta_n^M(X_n^i) - \nabla \tilde{\eta}_n^M(\tilde{X}_n^i) |.
\]

(5.81)

Now applying Assumption 4 (doing similar calculations as in (5.48),(5.50),(5.51)) following inequality holds

\[
E | \nabla \eta_n^M(X_n^i) - \nabla \tilde{\eta}_n^M(\tilde{X}_n^i) | \leq E | \nabla \eta_n^M(X_n^i) - \nabla \tilde{\eta}_n^M(\tilde{X}_n^i) | + E | \nabla \tilde{\eta}_n^M(\tilde{X}_n^i) - \nabla \tilde{\eta}_n^M(\tilde{X}_n^i) | \leq l_{pp}^N E | X_n^i - \tilde{X}_n^i | + \alpha l_{pp}^N E W_1 (\mu_{n-1}^N, \tilde{\mu}_{n-1}^N) + (1 - \alpha) l_{pp}^N E W_1 (S^M(\eta_{n-1}^N), S^M(\tilde{\eta}_{n-1}^N)).
\]

(5.82)
Note that (5.80) implies
\[ E[\mathcal{W}_1(S^M(\eta_{k-1}^M), S^M(\tilde{\eta}_{k-1}^M)) | F_{k-1}^M] \leq \mathcal{W}_1(\eta_{k-1}^M, \tilde{\eta}_{k-1}^M) \]  
from which following holds from (5.82)
\[ E[\nabla \eta_{n}^M (X_n^i) - \nabla \tilde{\eta}_{n}^M (\tilde{X}_n^i)] \leq \frac{l_{pp}}{l_{pp}} E[X_n^i - \tilde{X}_n^i] + \alpha l_{pp} E[X_{n-1}^i - \tilde{X}_{n-1}^i] + (1 - \alpha) l_{pp} E W_1(\eta_{n-1}^M, \tilde{\eta}_{n-1}^M). \]  
We also have
\[ W_1(\eta_{n+1}^M, \tilde{\eta}_{n+1}^M) \leq (1 - \alpha) l(P) W_1(S^M(\eta_n^M), S^M(\tilde{\eta}_n^M)) + \alpha l(P') W_1(\mu_n^M, \tilde{\mu}_n^M) \] 
and after taking expectation
\[ E W_1(\eta_{n+1}^M, \tilde{\eta}_{n+1}^M) \leq (1 - \alpha) l(P) E W_1(\eta_n^M, \tilde{\eta}_n^M) + \alpha l(P') E|X_n^i - \tilde{X}_n^i|. \] 
Letting \( A^{(M,N)}_{n+1} := \frac{1}{N} \sum_{i=1}^{N} |X_n^i - \tilde{X}_n^i| + W_1(\eta_{n+1}^M, \tilde{\eta}_{n+1}^M) \), we have the following recursion relation combining (5.81),(5.84) and (5.86)
\[ E A^{(M,N)}_{n+1} \leq \max \left\{ \left\{ \| A \| + \delta(2 + l_{pp}^\alpha) + \alpha l(P') \right\}, (1 - \alpha) l(P) \right\} E A^{(M,N)}_{n+1} + \delta \max \left\{ (1 - \alpha) l_{pp}^\alpha, \alpha l_{pp} \right\} E A^{(M,N)}_{n+1} \] 
which is the same recursion as in (5.54). Now for the chosen \( \delta, \alpha \) satisfying (5.57) there exists a \( \theta \in (0,1) \) such that
\[ E A^{(M,N)}_{n+1} \leq \theta^{n-1}[E A^{(M,N)}_{0} + E A^{(M,N)}_{1}]. \] 
Also, since \( \Theta^{N,M}_\infty \) and \( \tilde{\Theta}^{N,M}_\infty \) are invariant distributions, for every \( n \in \mathbb{N}_0 \), \( (X_{n+1}(N), \eta_{n+1}^M, S^M(\eta_{n+1}^M)) \) is distributed as \( \Theta^{N,M}_\infty \) and \( (\tilde{X}_{n+1}(N), \tilde{\eta}_{n+1}^M, S^M(\tilde{\eta}_{n+1}^M)) \) is distributed as \( \tilde{\Theta}^{N,M}_\infty \). Thus \( (X_{n+1}(N), \eta_{n+1}^M, S^M(\eta_{n+1}^M)) \) and \( (\tilde{X}_{n+1}(N), \tilde{\eta}_{n+1}^M, S^M(\tilde{\eta}_{n+1}^M)) \) define a coupling of random variables with laws \( \Theta^{N,M}_\infty \) and \( \tilde{\Theta}^{N,M}_\infty \) respectively. From (5.88) we then have
\[ W_1(\Theta^{N,M}_\infty, \tilde{\Theta}^{N,M}_\infty) \leq Ed((X_N(N), \eta_n^M, S^M(\eta_n^M)), (\tilde{X}_N(N), \tilde{\eta}_n^M, S^M(\tilde{\eta}_n^M))) \leq E A^{M,N}_n \to 0, \] 
as \( n \to \infty \). So there exists a unique invariant measure \( \Theta^{N,M}_\infty \in \mathcal{P}_1((\mathbb{R}^d)^N \times \mathcal{P}_1(\mathbb{R}^d) \times \mathcal{P}(\mathbb{R}^d)) \) for this Markov chain and, as \( n \to \infty \),
\[ \Theta^{N,M}_n \to \Theta^{N,M}_\infty. \] 
This proves the first part of the theorem. Denote \( \Theta^{N,M}_\infty (\cdot, \mathcal{P}_1(\mathbb{R}^d), \mathcal{P}(\mathbb{R}^d)) \) by \( \Theta^{N,M}_\infty \) and \( \Theta^{N,M}_\infty (\cdot, \mathcal{P}_1(\mathbb{R}^d), \mathcal{P}(\mathbb{R}^d)) \) by \( \Theta^{N,M}_{\infty}(\cdot) \).
Define \( r_N : (\mathbb{R}^d)^N \to \mathcal{P}(\mathbb{R}^d) \) as
\[ r_N(x_1, \ldots, x_N) = \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i}, \quad (x_1, \ldots, x_N) \in (\mathbb{R}^d)^N. \] 
Let \( \nu_{\infty}^{N,M} = \Theta^{1,N,M}_\infty \circ r_{N}^{-1} \) and \( \nu_{\infty}^{N,M} = \Theta^{1,N,M}_\infty \circ r_{N}^{-1} \). In order to prove that \( \Theta^{1,N,M}_\infty \) is \( \mu_{\infty} \)-chaotic, it suffices to argue that (cf. [16])
\[ \nu_{\infty}^{N,M} \to \delta_{\mu_{\infty}} \text{ in } \mathcal{P}(\mathcal{P}(\mathbb{R}^d)), \text{ as } N, M \to \infty. \] 
We first argue that as \( n \to \infty \)
\[ \nu_{n}^{N,M} \to \nu_{\infty}^{N,M} \text{ in } \mathcal{P}(\mathcal{P}(\mathbb{R}^d)). \] 
It suffices to show that \( \langle F, \nu_{n}^{N,M} \rangle \to \langle F, \nu_{\infty}^{N,M} \rangle \) for any continuous and bounded function \( F : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R} \). But this is immediate on observing that
\[ \langle F, \nu_{n}^{N,M} \rangle = \langle F \circ r_{N}, \Theta^{1,N,M}_n \rangle, \quad \langle F, \nu_{\infty}^{N,M} \rangle = \langle F \circ r_{N}, \Theta^{1,N,M}_\infty \rangle, \] 

\[ 29 \]
the continuity of the map $r_N$ and the weak convergence of $\Theta_{n,M}$ to $\Theta_{\infty,M}$. Next, for any $f \in BL_1(P(\mathbb{R}))$

$$\left| \langle f, \nu_{\infty,M}^N \rangle - \langle f, \delta_{\infty} \rangle \right| = \left| \frac{1}{n} \sum_{j=1}^{n} Ef(\hat{\mu}_j^N) - f(\mu_\infty) \right| \leq \frac{1}{n} \sum_{j=1}^{n} EW_1(\hat{\mu}_j^N, \mu_\infty).$$

Fix $\epsilon > 0$. For every $N, M \in \mathbb{N}$ there exists $n_0(N, M) \in \mathbb{N}$ such that for all $n \geq n_0(N, M)$

$$EW_1(\hat{\mu}_N^N, \mu_\infty) \leq \limsup_{n \to \infty} EW_1(\hat{\mu}_n^N, \mu_\infty) + \epsilon.$$ 

Thus for all $n, N, M \in \mathbb{N}$

$$\left| \langle f, \nu_{\infty,M}^N \rangle - \langle f, \delta_{\infty} \rangle \right| \leq \frac{n_0(N, M)}{n} \max_{1 \leq j \leq n_0(N, M)} EW_1(\hat{\mu}_j^N, \mu_\infty) + \limsup_{n \to \infty} EW_1(\hat{\mu}_n^N, \mu_\infty) + \epsilon. \quad (5.92)$$

Finally

$$\limsup_{N,M \to \infty} |\langle f, \nu_{\infty,M}^N \rangle - \langle f, \delta_{\infty} \rangle| = \limsup_{\min \{N,M\} \to \infty} \limsup_{n \to \infty} |\langle f, \nu_{\infty,M}^N \rangle - \langle f, \delta_{\infty} \rangle| \leq \limsup_{\min \{N,M\} \to \infty} \limsup_{n \to \infty} EW_1(\hat{\mu}_n^N, \mu_\infty) + \epsilon \leq \epsilon,$$

where the first equality is from (5.91), the second uses (5.92) and the third is a consequence of Corollary 3.6. Since $\epsilon > 0$ is arbitrary, we have (5.90) and the result follows.

□

5.6 Proof of Concentration bounds:

5.6.1 Proof of Theorem 3 (a):

We start with the following lemma where we establish a concentration bound for $W_1\left(\left(\hat{\mu}_n^N, \eta_n^M, \Psi(\hat{\mu}_n^N - 1, \eta_n^M)\right)\right)$ for each fixed time $n \in \mathbb{N}$ and then combine it with the estimate in (5.74) in order to get the desired result.

Lemma 5.6 Let $N_1 = \min\{M, N\}$. Assumptions (1-4) and Assumptions (7),(8) hold for some $\tau > 0$. Suppose that $\delta \in (0, a(\tau) P^{-1})$, and $(1 - \alpha)\mu(P) < 1$. Then there exist

$$a_1, a_2, a_3, a_1', a_2', a_3' \in (0, \infty)$$

such that for all $\epsilon, R > 0, n \in \mathbb{N}$, and $N_1 \geq \max\{1, a_1(\frac{R}{d})^{d+2}\}$

$$P(W_1\left(\left(\hat{\mu}_n^N, \eta_n^M, \Psi(\hat{\mu}_n^N - 1, \eta_n^M)\right)\right) > \epsilon) \leq a_3 \left( e^{-a_2 \frac{N_1^2}{R^2}} + \frac{R^{-\tau}}{\epsilon} \right), \quad (5.93)$$

$$P(W_1\left(S^M(\eta_n^M - 1, \eta_n^M)\right) > \epsilon) \leq a_3' \left( e^{-a_2' \frac{N_1^2}{R^2}} + \frac{R^{-\tau}}{\epsilon} \right). \quad (5.94)$$

5.6.2 Proof of Lemma 5.6

Second concentration bound will follow by proceeding as Lemma 4.5 of [5]. The proof relies on an idea of restricting measures to a compact set and estimates on metric entropy [2] (see also [17]). The basic idea is to first obtain a concentration bound for the $W_1$ distance between the truncated law and its corresponding empirical law in a compact ball of radius $R$ and getting a tail estimate from Lemma 5.2 and Corollary 3.5 after conditioning by $\mathcal{F}_{n+1}^{N,M}$. With the notations (for example $\mu_R$ is the truncated measure of $\mu$ restricted on a ball $B_R(0)$ of $R$ radius) introduced in Lemma 4.5 of [5] we sketch the proof of the second bound.

With that notation the truncated version of $\eta_{n+1}^M$ is denoted by $\eta_{n+1}^M$. Suppose $\{Y_{n+1}^{i,M} : i = 1, \ldots, M\}$ are iid from $\eta_{n+1}^M$ conditioned on $\mathcal{F}_{n+1}^{N,M}$, where $\{Z_i^{M,R} : i = 1, \ldots, M\}$ are iid from $\eta_{n+1}^M$ conditioned under $\mathcal{F}_{n+1}^{N,M}$. Define

$$X_{n+1}^{i,M} = \begin{cases} Y_{n+1}^{i,M} & \text{when } |Y_{n+1}^{i,M}| \leq R, \\ Z_{n+1}^{i,M} & \text{otherwise}. \end{cases}$$
Note that \( P(X_{n-1}^i \in A \mid \mathcal{F}_{n-1}^M) = P(Z_{n-1}^i \in A \mid \mathcal{F}_{n-1}^M) \). Denote \( S^M(\tilde{\eta}_{n-1}^M, R) := \frac{M}{n} \sum_{i=1}^{M} \delta_{X_{n-1}^i} \). Now denoting \( a(1+\tau) := \sup_{n \geq 0} \sup_{M,N} E \langle |x|^1+\tau, \tilde{\eta}_{n-1}^M \rangle \), from (5.80) we have

\[
P[ W_1( S^M(\tilde{\eta}_{n-1}^M, R), S^M(\tilde{\eta}_{n-1}^M)) > \frac{\varepsilon}{3} ] \leq 3 \frac{E[ W_1( S^M(\tilde{\eta}_{n-1}^M, R), S^M(\tilde{\eta}_{n-1}^M)) ]}{\varepsilon} \leq 3 \frac{E[ W_1( S^M(\tilde{\eta}_{n-1}^M, R), S^M(\tilde{\eta}_{n-1}^M)) ]}{\varepsilon} \leq 6 a(1+\tau) \frac{R^{-\tau}}{\varepsilon}.
\]

(5.95)

Now using Azuma Hoeffding inequality as done in display (4.35) of Lemma 4.5 in [5] one has

\[
P[ W_1( S^M(\tilde{\eta}_{n-1}^M, R), \tilde{\eta}_{n-1}^M) > \frac{\varepsilon}{3} ] \leq \max \{ 2, \frac{16 R}{\varepsilon} (2\sqrt{d} + 1) 3^{\frac{d}{2}} (\sqrt{d} + 1)^d \} e^{-\frac{M^2 \varepsilon^2}{288 d^2}}.
\]

(5.96)

From the definition of \( \tilde{\eta}_{n-1}^M, R \)

\[
P[ W_1( \tilde{\eta}_{n-1}^M, \tilde{\eta}_{n-1}^M) > \frac{\varepsilon}{3} ] \leq 6 E[ |Y_{n-1}^i| 1_{|Y_{n-1}^i| R} ] \leq 3 a(1+\tau) \frac{R^{-\tau}}{\varepsilon}.
\]

(5.97)

Using triangular inequality

\[
W_1( S^M(\tilde{\eta}_{n-1}^M, R), \tilde{\eta}_{n-1}^M) \leq W_1( S^M(\tilde{\eta}_{n-1}^M, R), S^M(\tilde{\eta}_{n-1}^M)) + W_1( S^M(\tilde{\eta}_{n-1}^M, R), \tilde{\eta}_{n-1}^M) + W_1( \tilde{\eta}_{n-1}^M, \tilde{\eta}_{n-1}^M)
\]

combining (5.95),(5.96) and (5.97) the result (5.109) will follow.

The first one (5.108) follows by noting that

\[
P[ W_1( \tilde{\eta}_{n-1}^M, \tilde{\eta}_{n-1}^M) > \frac{\varepsilon}{2} ] \leq P[ W_1( \tilde{\eta}_{n-1}^M, \tilde{\eta}_{n-1}^M) > \frac{\varepsilon}{2} ] + P[ W_1( S^M(\tilde{\eta}_{n-1}^M, \tilde{\eta}_{n-1}^M) ) > \frac{\varepsilon}{2} ]
\]

(5.98)

Proceeding like Lemma 4.5 of [5] the bound for the first term in RHS of (5.98) can be established.

\( \Box \)

### 5.6.3 Proof of Theorem 3(a)

Combining (5.74),(5.75) and (5.76) it follows that

\[
W_1 \left( (\tilde{\eta}_{n-1}^M), (\mu_n) \right) \leq \sum_{i=1}^{n} \theta_{n-1}^i \left[ a W_1 \left( (\tilde{\eta}_{n-1}^M) \right) + b\tilde{\eta}_{n-1}^M \right] + a\theta_{n-1}^M \left[ a W_1 \left( (\tilde{\eta}_{n-1}^M) \right) + b\tilde{\eta}_{n-1}^M \right] + a\theta_{n-1}^M \left[ a W_1 \left( (\tilde{\eta}_{n-1}^M) \right) + b\tilde{\eta}_{n-1}^M \right]
\]

(5.99)

Denoting \( c_1 := \max \left\{ \left\langle \left( \|A\| + \delta \sigma (2 + \tilde{T}_{P^2}^{\alpha}) \right) + \delta l(P), (1 - \alpha)l(P) \right\rangle \right\}, \) \( c_2 := \delta \sigma \max \left\{ \alpha l(P), (1 - \alpha)l(P) \right\} \)

\[
W_1 \left( (\tilde{\eta}_{n-1}^M) \right) \leq \sum_{i=1}^{n} \theta_{n-1}^i \left[ a W_1 \left( (\tilde{\eta}_{n-1}^M) \right) + b\tilde{\eta}_{n-1}^M \right] + a\theta_{n-1}^M \left[ a W_1 \left( (\tilde{\eta}_{n-1}^M) \right) + b\tilde{\eta}_{n-1}^M \right] + a\theta_{n-1}^M \left[ a W_1 \left( (\tilde{\eta}_{n-1}^M) \right) + b\tilde{\eta}_{n-1}^M \right]
\]

(5.99)

Denote \( c_1 := \max \left\{ \left\langle \left( \|A\| + \delta \sigma (2 + \tilde{T}_{P^2}^{\alpha}) \right) + \delta l(P), (1 - \alpha)l(P) \right\rangle \right\}, \) \( c_2 := \delta \sigma \max \left\{ \alpha l(P), (1 - \alpha)l(P) \right\} \)

\[
c_1 := \max \left\{ \left\langle \left( \|A\| + \delta \sigma (2 + \tilde{T}_{P^2}^{\alpha}) \right) + \delta l(P), (1 - \alpha)l(P) \right\rangle \right\}, \) \( c_2 := \delta \sigma \max \left\{ \alpha l(P), (1 - \alpha)l(P) \right\} \)

\[
W_1 \left( (\tilde{\eta}_{n-1}^M) \right) \leq \sum_{i=1}^{n} \theta_{n-1}^i \left[ a W_1 \left( (\tilde{\eta}_{n-1}^M) \right) + b\tilde{\eta}_{n-1}^M \right] + a\theta_{n-1}^M \left[ a W_1 \left( (\tilde{\eta}_{n-1}^M) \right) + b\tilde{\eta}_{n-1}^M \right] + a\theta_{n-1}^M \left[ a W_1 \left( (\tilde{\eta}_{n-1}^M) \right) + b\tilde{\eta}_{n-1}^M \right]
\]

(5.99)

\[
W_1 \left( (\tilde{\eta}_{n-1}^M) \right) \leq \sum_{i=1}^{n} \theta_{n-1}^i \left[ a W_1 \left( (\tilde{\eta}_{n-1}^M) \right) + b\tilde{\eta}_{n-1}^M \right] + a\theta_{n-1}^M \left[ a W_1 \left( (\tilde{\eta}_{n-1}^M) \right) + b\tilde{\eta}_{n-1}^M \right] + a\theta_{n-1}^M \left[ a W_1 \left( (\tilde{\eta}_{n-1}^M) \right) + b\tilde{\eta}_{n-1}^M \right]
\]

(5.99)

\[
W_1 \left( (\tilde{\eta}_{n-1}^M) \right) \leq \sum_{i=1}^{n} \theta_{n-1}^i \left[ a W_1 \left( (\tilde{\eta}_{n-1}^M) \right) + b\tilde{\eta}_{n-1}^M \right] + a\theta_{n-1}^M \left[ a W_1 \left( (\tilde{\eta}_{n-1}^M) \right) + b\tilde{\eta}_{n-1}^M \right] + a\theta_{n-1}^M \left[ a W_1 \left( (\tilde{\eta}_{n-1}^M) \right) + b\tilde{\eta}_{n-1}^M \right]
\]

(5.99)

Since \( g_0(0) = c_2 + c_1 - 1 < 0 \) (from the assumption), \( g_0(0) = c_2 + c_1 - 1 < 0 \) and \( g(\cdot) \) is continuous. So there exists a \( \gamma > 0 \) such that \( g_0(\gamma) < 0 \) or equivalently

\[
\frac{c_1}{1 - \gamma} + \frac{c_2}{(1 - \gamma)^2} < 1.
\]
So there exists a \( \theta \in (0, 1 - \gamma) \) such that statement of Lemma 5.4 holds. Now using that \( \gamma \) from (5.99) one has

\[
P\left[ W_1 \left( (\bar{\mu}_n^N, \bar{\eta}_n^M), (\mu_n, \eta_n) \right) > \varepsilon \right] \leq P \left[ \bigcup_{i=1}^{n} \{ a \theta^{n-i} W_1 \left( (\bar{\mu}_i^N, \bar{\eta}_i^M), \Psi(\bar{\mu}_{i-1}^N, \bar{\eta}_{i-1}^M) \right) > \frac{\gamma}{2} (1 - \gamma)^{n-i} \varepsilon \} \right] + \\
\sum_{i=1}^{n} \left\{ b l_n^W (1 - \alpha) \theta^{n-i} W_1 \left( S^M (\bar{\eta}_{i-1}^M), \bar{\eta}_{i-1}^M \right) > \frac{\gamma}{2} (1 - \gamma)^{n-i} \varepsilon \right\} \bigcup_{i=1}^{n} \left\{ \theta^n W_1 \left( (\bar{\mu}_0^N, \bar{\eta}_0^M), (\mu_0, \eta_0) \right) > \gamma (1 - \gamma)^n \varepsilon \right\}
\]

Let \( \beta_1 = \frac{\gamma \varepsilon}{2a}, \beta_2 = \frac{\gamma \varepsilon}{2bl_n^W (1 - \alpha)} \beta_3 = \gamma \varepsilon \). Note that \( \nu := \left( \frac{1 - \gamma}{\theta} \right) > 1 \), from our choice of \( \gamma \). Therefore denoting \( \beta := \min\{ \beta_1, \beta_2 \} \), \( N_1 \geq a_1 \left( \frac{R}{\beta \nu} \right)^{d+2} \) \( \forall 1 \) implies \( N_1 \geq a_1 \left( \frac{R}{\beta \nu} \right)^{d+2} \) \( \forall 1 \) for all \( n \in \mathbb{N}_0 \) and a consequence of Lemma 5.6 gives

\[
P\left[ W_1 \left( (\bar{\mu}_n^N, \bar{\eta}_n^M), (\mu_n, \eta_n) \right) > \varepsilon \right] \leq \sum_{i=1}^{n} P \left[ W_1 \left( (\bar{\mu}_i^N, \bar{\eta}_i^M), \Psi(\bar{\mu}_{i-1}^N, \bar{\eta}_{i-1}^M) \right) > \beta_1 \nu^{n-i} \right] + \\
\sum_{i=1}^{n} P \left[ W_1 \left( S^M (\bar{\eta}_{i-1}^M), \bar{\eta}_{i-1}^M \right) > \beta_2 \nu^{n-i} \right] + P \left[ W_1 \left( (\bar{\mu}_0^N, \bar{\eta}_0^M), (\mu_0, \eta_0) \right) > \beta_3 \nu^n \right]
\]

(5.100)

Now proceeding similarly like the proof of Theorem 3.7 of [5] through optimizing the value of \( R \) the conclusion will follow.

5.6.4 Proof of Theorem 3(b)

Second part regarding the exponential concentration bound will follow similarly (like Theorem 3.8 of [5]) under the following lemmas on uniform exponential integrability.

**Lemma 5.7** Suppose Assumptions 9 and 10 hold. Suppose there exists \( \alpha^* > 0 \) such that

\[
\alpha^* |h_1(0)| + h_2(\alpha^*) = - \log(1 - \alpha).
\]

Then for all \( \alpha_1 \in [0, \min \{ \alpha^*, \frac{\alpha(\delta)}{\delta} \} \) and \( \delta \in \left[ 0, \frac{1 - ||A||}{2(1 + ||A||)} \right) \),

\[
\sup_{n \geq 0} \sup_{M,N \geq 1} \mathbb{E} e^{\alpha_1 |X_{n+1}^N|} < \infty, \quad \sup_{n \geq 0} \sup_{M,N \geq 1} \mathbb{E} \left( e^{\alpha_1 |x|}, \bar{\eta}_n^M \right) < \infty.
\]

(5.101)

**Proof.** We will start by proving the second inequality. Note that from Corollary 5.3 the conditions for \( \sup_{n \geq 0} \sup_{M,N \geq 1} \mathbb{E} e^{\alpha_1 |x|}, \bar{\eta}_n^M \) < \( \infty \) are same as the conditions for \( \sup_{n \geq 0} \sup_{M,N \geq 1} \mathbb{E} e^{\alpha_1 |x|}, \eta_n^N \) < \( \infty \) in \( \mathbb{E} \) and from Lemma 5.2 they are again same as the conditions for finiteness of \( \sup_{n \geq 0} \mathbb{E} e^{\alpha_1 |x|}, \eta_n \). Note that

\[
\left( \eta_{k+1}, e^{\alpha_1 |x|} \right) = \sum_{i=0}^{k} \left[ \alpha (1 - \alpha)^i \left( \mu_{k-i}, P^i P^*, e^{\alpha_1 |x|} \right) \right] + (1 - \alpha)^{k+1} \left( \eta_0 P^{k+1}, e^{\alpha_1 |x|} \right).
\]

(5.102)
Now from Assumption 10, using lipchitz property $|h_n(x)| \leq l_{h_n}|x| + |h_1(0)|$ one has $\langle \mu_k \mathcal{P}^n, e^{\alpha_1|x|} \rangle \leq e^{\alpha_1|\mu_k(0)|} + e^{\alpha_1|h_2(0)|} \langle \mu_k \mathcal{P}^{n-1}, e^{\alpha_1h_1(|x|)} \rangle + e^{\alpha_1(\alpha_1)} + h_3(\alpha_1) + h_2(\alpha_1)$. So we have an upperbound of $\langle \mu \mathcal{P}^n, e^{\alpha_1|x|} \rangle$ that is

\[
\begin{align*}
&\leq \sum_{k=0}^{i-1} \langle \mu_k \mathcal{P}^n, e^{\alpha_1h_1(|x|)} \rangle + \sum_{k=0}^{i-1} \left( \sum_{k=1}^{\infty} \left( h_1(\alpha_1h_1(|x|)) + h_2(\alpha_1h_1(|x|)) + e^{\alpha_1(\alpha_1) + h_3(\alpha_1) + \alpha_1h_1(|x|)} \right) \right) \\
&\leq \langle \mu \mathcal{P}^n, e^{\alpha_1|x|} \rangle e^{\alpha_1h_1(|x|)} + \frac{e^{\alpha_1h_1(|x|)}}{1 - e^{\alpha_1h_1(|x|)}} - 1.
\end{align*}
\]

Last inequality follows since $h_2(\cdot), h_3(\cdot)$ are non-decreasing and $h_1 \leq 1$. Using (5.102) under the condition $\sup_{n \geq 0} \langle e^{\alpha_1|x|}, \mu_n \rangle < \infty$ (which we prove shortly) we conclude that $\sup_{k \geq 0} \langle \eta_{k+1}, e^{\alpha_1|x|} \rangle < \infty$ or equivalently $\sum_{k=0}^{\infty} (1 - \alpha)^k e^{\alpha_1h_1(|x|)} < \infty$ if there exists an $\alpha_1$ such that $\alpha_1|\mu_k(0)| + h_2(\alpha_1) + \log(1 - \alpha) < 0$. Since $g(\alpha_1) := \alpha_1|\mu_k(0)| + h_2(\alpha_1)$, is an increasing function of $\alpha_1$ and $g(0) = 0$. From the definition of $\alpha^*$ we can always find $0 < \alpha < \alpha^*$ such that $\sup_{n \geq 0} \langle e^{\alpha_1|x|}, \eta_n \rangle < \infty$.

Now we prove $\sup_{n \geq 0} \langle e^{\alpha_1|x|}, \mu_n \rangle < \infty$ or equivalently the first term in (5.101). Note that from (5.1) for $n \geq 1$

\[
|X_{n+1,i}^i| \leq \|A\|X_i^i + \delta A_i(e_n(\alpha_{n+1})[\nabla \eta_i^N(X_i)]) + \|\mu_n\| \|X_i^i\| + \delta A_2(e_n(\alpha_{n+1})) + |B(e_n(\alpha_{n+1}))| \\
\leq X_i^i \left( \|A\| + \delta K \left( 1 + t_{P,P}^{\alpha_i} \right) \right) + \delta K \|\mu_n\| + \delta A_2(e_n(\alpha_{n+1}) + Kc_{P,P}^{\alpha_i} + B(e_n(\alpha_{n+1})) + \frac{\delta}{\delta} \right).
\]

Now from the choice $\alpha_1 \leq \frac{\alpha(\delta)}{\delta}$, taking expectation after having exponentialual

\[
Ee^{\alpha_1|X_{n+1,i}|} \leq e^{\alpha_1|X_{n,i}|[\|A\| + \delta K \left( 1 + t_{P,P}^{\alpha_i} \right)]} + \alpha_1 \delta K \|\mu_n\| \mathcal{E}_1(\alpha_1)
\]

where $\mathcal{E}_1(\alpha_1) = e^{\alpha_1 \delta K \mathcal{E}_{P,P}^{\alpha_i}} \int e^{\alpha_1 \delta \left( A_2(z) + |B(z)| \right)} \theta(dz)$. We note that from Assumption 10 there always exist $\alpha^{**} < \frac{\alpha(\delta)}{\delta}$, $c_3$ such that for all $\alpha_1 \in (0, \alpha^{**})$

\[
\mathcal{E}_1(\alpha_1) \leq e^{c_3 \alpha_1}.
\]

Using conditioning argument we have

\[
E e^{\alpha_1|X_{n,i}|[\|A\| + \delta K \left( 1 + t_{P,P}^{\alpha_i} \right)]} + \alpha_1 \delta K \|\mu_n\| \leq \mathcal{E}_1(\alpha_1)
\]

where (5.105) follows from exchangeability of $\{X_{n,i}^i\}_{i=1,...,N}$. Observing $\|\mu_n\| \leq \int |x| \mu_n(dx)$ and using Jensen’s inequality applied to the function $x \mapsto e^{\alpha_1x^2}$, we have after taking expectation

\[
E \left[ e^{\alpha_1 \delta K \mathcal{E}_1(\alpha_1)} \left( \frac{1}{N} \sum_{i=1}^{N} e^{\alpha_1 |X_{n,i}^i|} \left( \|A\| + \delta K \left( 1 + t_{P,P}^{\alpha_i} \right) \right) \right) \right] \leq \mathcal{E}_1(\alpha_1)
\]

Since $f_1(x) := e^{\alpha_1 \delta K x}$ and $f_2(x) := e^{\alpha_1 \delta K \left( 1 + t_{P,P}^{\alpha_i} \right)}$ are both non-decreasing, so putting $\mu = \mu_n^N$ almost surely in the following inequality $\int f_1(x) f_2(x) \mu(dx) \geq \int f_1(x) \mu(dx) \int f_2(y) \mu(dy)$ and taking expectation we have

\[
E \left[ \frac{1}{N} \sum_{i=1}^{N} e^{\alpha_1 \delta K |X_{n,i}^i|} \left( \|A\| + \delta K \left( 1 + t_{P,P}^{\alpha_i} \right) \right) \right] \leq \mathcal{E}_1(\alpha_1)
\]

Since $f_1(x) := e^{\alpha_1 \delta K x}$ and $f_2(x) := e^{\alpha_1 \delta K \left( 1 + t_{P,P}^{\alpha_i} \right)}$ are both non-decreasing, so putting $\mu = \mu_n^N$ almost surely in the following inequality $\int f_1(x) f_2(x) \mu(dx) \geq \int f_1(x) \mu(dx) \int f_2(y) \mu(dy)$ and taking expectation we have

\[
E \left[ \frac{1}{N} \sum_{i=1}^{N} e^{\alpha_1 \delta K |X_{n,i}^i|} \left( \|A\| + \delta K \left( 2 + t_{P,P}^{\alpha_i} \right) \right) \right] \leq \mathcal{E}_1(\alpha_1)
\]
From our choice of $\delta, \kappa := \|A\| + \delta K (2 + l_P^n) \in (0, 1)$. Denoting $F_{n+1}(\alpha_1) := Ee^{\alpha_1|X_{n+1}^i|}$ from (5.103) we have the following recursive inequality:

$$F_{n+1}(\alpha_1) \leq F_n(\alpha_1\kappa_1)\mathcal{E}_1(\alpha_1).$$

(5.106)

Iterating the above inequality we have for all $n \geq 1$

$$F_n(\alpha_1) \leq F_0(\alpha_1) \prod_{j=0}^{n-1} \mathcal{E}_1(\alpha_1\kappa_1^j) \leq F_0(\alpha_1)e^{c_3\alpha_1\sum_{j=0}^{n-1}\kappa_1^j} \leq F_0(\alpha_1)e^{c_3\alpha_1/(1-\kappa_1)}$$

where the second inequality is a consequence of (5.104).

Note further for the system in (2.4) let $\{X_n\}_{n \in \mathbb{N}_0}$ be defined as the random variables with laws $\mathcal{L}(X_n) := \mu_n$ for $n \in \mathbb{N}_0$. Then starting similarly from

$$|X_{n+1}| \leq |X_n|[\|A\| + \delta K (1 + l_P^n\alpha)] + \delta K\|\mu_n\|_1 + \delta(A_2(\epsilon_{n+1}) + K\epsilon_{n+1} + \frac{B(\epsilon_{n+1})}{\delta})$$

using the inequality $\int f_1(x)f_2(x)\mu(dx) \geq \int f_1(x)\mu(dx) \int f_2(y)\mu(dy)$ (similar to Lemma 4.11 of [5]) one can prove

$$\sup_{n \geq 0} \left\langle e^{\alpha_1|x|}, \mu_n \right\rangle \leq \left\langle e^{\alpha_1|x|}, \mu_0 \right\rangle e^{\frac{\alpha_1}{\kappa_1}}.$$  

(5.107)

under same conditions on $\delta, \alpha_1$. This is needed for proving $\sup_{n \geq 0} \left\langle e^{\alpha_1|x|}, \eta_n \right\rangle < \infty$. The result follows.

Lemma 5.8 Then there exist $a_1, a_2, a_3, a'_1, a'_2, a'_3 \in (0, \infty)$ such that for all $\epsilon, R > 0$ and $n \in \mathbb{N}$, and $N_1 \geq \max \{1, \tilde{a}_1(R)^{d+2}\}$

$$P[|W_1 ((\tilde{\mu}_n^N, \tilde{\eta}_n^N), \Psi(\tilde{\mu}_n^{N-1}, \tilde{\eta}_n^{N-1})) > \epsilon]\leq a_3 \left(e^{-a_2 \frac{N_1^2}{\epsilon^2}} + \tilde{B}_1(\alpha_1)\frac{e^{-\alpha_1 R}}{\epsilon}\right),$$

(5.108)

and

$$P[|W_1 (S^M(\tilde{\eta}_n^{N-1}, \tilde{\eta}_n^{N-1})) > \epsilon] \leq a'_3 \left(e^{-a'_2 \frac{N_1^2}{\epsilon^2}} + \tilde{B}_2(\alpha_1)\frac{e^{-\alpha_1 R}}{\epsilon}\right).$$

(5.109)

5.6.5 Proof of Lemma 5.8:

Follows from similar decompositions given in Lemma 5.6 and Lemma 4.7 of [5].

5.6.6 Proof of Theorem 3(b):

Starting from (5.99), the conclusion will follow by applying Lemma 5.8 in (5.100).

5.7 Proof of Theorem 4

We will start by introducing a coupling. Consider a system of $\mathbb{R}^d$ valued auxiliary random variables $\{Y^{i,N}_n, i = 1, \ldots, N\}_{n \geq 0}$ defined as follows.

$$Y^{i,N}_n = A^{i,N}_n + \delta f(\nabla \eta_n(Y^{i,N}_n, \mu_n, Y^{i,N}_n, \epsilon_{n+1}) + B(\epsilon_{n+1}), i = 1, \ldots, N, n \in \mathbb{N}_0.$$  

(5.110)
Now for each \( n \in \mathbb{N} \), \( \{ Y_{i,i}^{i,N}, i = 1, \ldots, N \} \) is a set of \( \mathbb{R}^d \) valued iid random variables under initial assumption \( \mathcal{L}((X_{0}^{i,N})_{i=1,\ldots,N}) = \mu_{0}^{\otimes N} \). Suppose \( \zeta_{n}^{N} := \frac{1}{n} \sum_{i=1}^{N} \delta_{Y_{i,i}^{i,N}} \). The following Lemma will make a connection between \( \zeta_{n}^{N} \) and \( \mu_{n}^{N} \).

**Lemma 5.9** (Coupling with the auxiliary system) Suppose Assumptions 1, 4, 5 and 9 hold. Then for every \( n \geq 0 \) and \( N \geq 1 \), with the \( C_{1} \), and \( C_{2} \) defined in (3.19), (3.20)

\[
W_{1}(\mu_{n+1}^{N}, \mu_{n+1}) \leq W_{1}(\zeta_{n+1}^{N}, \mu_{n+1}) + C_{1} \sum_{k=0}^{n} \chi_{1}^{n-k} W_{1}(\zeta_{k}^{N}, \mu_{k}).
\]  

(5.111)

**Proof.** Since by Assumption 1 and \( A_{1}(\epsilon) \leq K \), we have for each \( j = 1, \ldots, N \)

\[
|X_{n+1}^{j} - Y_{n+1}^{j,N}| \leq \|A\|\|X_{n}^{j} - Y_{n}^{j,N}|| + \delta K \left\{ |\nabla \eta_{n}^{N}(X_{n}^{j,N}) - \nabla \eta_{n}(Y_{n}^{j,N})| + |X_{n}^{j,N} - Y_{n}^{j,N}|| + W_{1}(\mu_{n}^{N}, \mu_{n}) \right\}
\]

Using the calculations in (5.46), (5.48), (5.49) and (5.51)

\[
|\nabla \eta_{n}^{N}(X_{n}^{j,N}) - \nabla \eta_{n}(Y_{n}^{j,N})| \leq |\nabla \eta_{n}^{N}(X_{n}^{j,N}) - \nabla \eta_{n}(Y_{n}^{j,N})| + |\nabla \eta_{n}(Y_{n}^{j,N}) - \nabla \eta_{n}(Y_{n}^{j,N})|
\]

\[
\leq l_{P,\mu}^{\alpha} |X_{n}^{j} - Y_{n}^{j,N}| + (1 - \alpha) l_{P}^{\alpha} W_{1}(\eta_{n-1}^{N}, \eta_{n-1}) + \alpha l_{P}^{\alpha} W_{1}(\mu_{n-1}^{N}, \mu_{n-1})
\]

Thus

\[
|X_{n+1}^{j,N} - Y_{n+1}^{j,N}| \leq \left[ \|A\| + \delta K (1 + l_{P,\mu}^{\alpha}) \right] |X_{n}^{j} - Y_{n}^{j,N}| + \delta K \left[ W_{1}(\mu_{n}^{N}, \mu_{n})
\right.
\]

\[
+ (1 - \alpha) l_{P}^{\alpha} W_{1}(\eta_{n-1}^{N}, \eta_{n-1}) + \alpha l_{P}^{\alpha} W_{1}(\mu_{n-1}^{N}, \mu_{n-1})
\]

(5.112)

Using (5.112) as the recursion on \( a_{n+1}^{j} := |X_{n+1}^{j} - Y_{n+1}^{j,N}| \) with \( a_{0}^{j} = 0 \), we get

\[
a_{n+1}^{j} \leq \delta K \sum_{k=1}^{n} \left[ \|A\| + \delta K (1 + l_{P,\mu}^{\alpha}) \right]^{n-k} \left[ W_{1}(\mu_{k}^{N}, \mu_{k}) + (1 - \alpha) l_{P}^{\alpha} W_{1}(\eta_{k-1}^{N}, \eta_{k-1})
\right.
\]

\[
+ \alpha l_{P}^{\alpha} W_{1}(\mu_{k-1}^{N}, \mu_{k-1}) \].
\]  

(5.113)

Denote \( \|A\| + \delta K (1 + l_{P,\mu}^{\alpha}) \) by \( \chi \). Observe that

\[
W_{1}(\eta_{n-1}^{N}, \eta_{n-1}) = (1 - \alpha) l(P) W_{1}(\eta_{n-2}^{N}, \eta_{n-2}) + \alpha l(P) W_{1}(\mu_{n-2}^{N}, \mu_{n-2}).
\]

Denote the quantity in the third bracket of RHS of (5.113) by \( b_{k} \). Using (5.114) and \( \eta_{0}^{N} = \eta_{0} \) we have

\[
b_{k} = W_{1}(\mu_{k}^{N}, \mu_{k}) + (1 - \alpha) l_{P}^{\alpha} W_{1}(\eta_{k-1}^{N}, \eta_{k-1}) + \alpha l_{P}^{\alpha} W_{1}(\mu_{k-1}^{N}, \mu_{k-1})
\]

\[
= W_{1}(\mu_{k}^{N}, \mu_{k}) + (1 - \alpha) l_{P}^{\alpha} \alpha l(P) \left[ (1 - \alpha) l(P) \right]^{k-2} W_{1}(\mu_{k}^{N}, \mu_{k}) + \alpha l_{P}^{\alpha} W_{1}(\mu_{k-1}^{N}, \mu_{k-1})
\]

\[
\leq c_{4} \sum_{i=0}^{k} c_{5}^{k-i} W_{1}(\mu_{i}^{N}, \mu_{i}).
\]

(5.115)

where \( c_{4} := \max\{1, (1 - \alpha) l_{P}^{\alpha} \} \) and \( c_{5} := \max\{\alpha l_{P}^{\alpha}, (1 - \alpha) l(P)\} \). Thus from (5.113) we have

\[
a_{n+1}^{j} \leq \delta K c_{4} \sum_{k=0}^{n} \chi^{n-k} \sum_{i=0}^{k} c_{5}^{k-i} W_{1}(\mu_{i}^{N}, \mu_{i}).
\]

(5.116)

Now applying Lemma A.3 we have

\[
a_{n+1}^{j} \leq \delta K c_{4} \sum_{i=0}^{n} W_{1}(\mu_{i}^{N}, \mu_{i}) \left[ \frac{\chi^{n+1-i} - c_{5}^{n+1-i}}{\chi - c_{5}} \right]
\]

\[
\leq \delta K c_{7} \sum_{i=0}^{n} \chi^{n+1-i} W_{1}(\mu_{i}^{N}, \mu_{i})
\]

(5.117)
where \( \chi_2 := \max\{\chi, c_3\} \) and \( c_7 := \frac{c_4}{|\chi - c_5|} \). Note that from (5.80) we have for all \( n \geq 0, \)
\[
W_1(\zeta_n^N, \mu_n^N) \leq \frac{1}{N} \sum_{j=1}^{N} a_j^n.
\]
Combining the result above and using triangle inequality in (5.117)
\[
W_1(\zeta_n^N, \mu_n^N) \leq \delta K c_7 \sum_{k=0}^{n} \chi_2^{n+1-k} W_1(\zeta_k^N, \mu_k^N) + \delta K c_7 \sum_{k=0}^{n} \chi_2^{n+1-k} W_1(\zeta_k^N, \mu_k).
\]

Applying Lemma A.3 with
\[
a_n = \chi_2^{-n} W_1(\zeta_n^N, \mu_n^N), \quad b_n = \delta K c_7 \sum_{k=0}^{n-1} \chi_2^{-k} W_1(\eta_k^N, \mu_k), \quad p_n = \delta K c_7, \quad n \geq 0.
\]
We have
\[
\chi_2^{-(n+1)} W_1(\zeta_n^N, \mu_n^N) \leq b_{n+1} + \sum_{k=0}^{n} (\delta K c_7)^2 \sum_{i=0}^{k-1} \chi_2^{-i} W_1(\zeta_i^N, \mu_i) (1 + \delta K c_7)^{-k-1}
\]
\[
= b_{n+1} + \sum_{i=0}^{n-1} (\delta K c_7)^2 \chi_2^{-i} W_1(\zeta_i^N, \mu_i) \sum_{m=0}^{n-i-1} (1 + \delta K c_7)^m
\]
\[
= b_{n+1} + (\delta K c_7) \chi_2^{-i} W_1(\zeta_i^N, \mu_i) [(1 + \delta K c_7)^{n-i} - 1]. \tag{5.118}
\]

Simplifying (5.118) one gets
\[
W_1(\zeta_n^N, \mu_n^N) \leq \delta K c_7 \sum_{k=0}^{n} \chi_2^{n+1-k} W_1(\zeta_k^N, \mu_k) + \sum_{k=0}^{n} (\delta K c_7) \chi_2^{n+1-k} W_1(\zeta_k^N, \mu_k) [(1 + \delta K c_7)^{n-k} - 1]
\]
\[
= \sum_{k=0}^{n} (\delta K c_7) \chi_2^{n+1-k} W_1(\zeta_k^N, \mu_k) (1 + \delta K c_7)^{n-k}.
\]
\[
= \delta K c_7 \chi_2 \sum_{k=0}^{n} (\chi_2 + \delta K c_7 \chi_2)^{n-k} W_1(\zeta_k^N, \mu_k).
\]
Note that \( \delta K c_7 \chi_2 = C_1 \) and \( \chi_2 + C_2 = \chi_1 \) as defined in (3.19) (3.20) respectively. Thus we have
\[
W_1(\zeta_n^N, \mu_n^N) \leq C_1 \sum_{k=0}^{n} \chi_1^{n-k} W_1(\zeta_k^N, \mu_k).
\]
The result now follows by an application of triangle inequality. \( \Box \)

5.7.1 Proof of Theorem 4

Since \( \chi_1 < 1 \). So we can find \( \gamma > 0 \) such that \( \chi_1 < 1 - \gamma \). Taking that \( \gamma \), we have \( \nu_1 := \frac{1-\gamma}{\chi_1} > 1 \). For any \( \epsilon > 0 \), From Lemma 4
\[
P[W_1(\mu_n^N, \mu_n) > \epsilon] \leq P[W_1(\zeta_n^N, \mu_n) > \epsilon \gamma] + \sum_{i=0}^{n} P[C_1 \chi_1^{n-1-i} W_1(\zeta_i^N, \mu_i) \geq \epsilon (1 - \gamma)^{n-i}]
\]
\[
= P[W_1(\zeta_n^N, \mu_n) > \epsilon \gamma] + \sum_{i=1}^{n} P[W_1(\zeta_{n-i}^N, \mu_{n-i}) \geq \frac{\gamma \epsilon \chi_1}{C_1} \nu^i]. \tag{5.119}
\]
\[
= P[W_1(\zeta_n^N, \mu_n) > \epsilon \gamma] + \sum_{i=1}^{n} P[W_1(\zeta_{n-i}^N, \mu_{n-i}) \geq \frac{\gamma \epsilon \chi_1}{C_1} \nu^i] + \sum_{i=i_0+1}^{n} P[W_1(\zeta_{n-i}^N, \mu_{n-i}) \geq \frac{\gamma \epsilon \chi_1}{C_1} \nu^i].
\]
where \( i_\varepsilon := \max\{i \geq 0 : \frac{\gamma_1 \varepsilon}{C_1} \nu_i < 1\} \). Note that for \( \delta \in \left[0, \frac{1-\|A\|}{(2+1/p_\nu^*)K}\right] \), and \( \alpha_1 \in (0, \frac{a_0}{\delta}) \) from (5.107) we have \( \sup_{n \geq 0} (e^{a_1|x|}, \mu_n) < \infty \). That implies from the statement of Theorem 2 of [10] that for all \( N > 0 \),

\[
P[\mathcal{W}_1(\zeta_n^N, \mu_n) \geq \varepsilon] \leq a(N, \varepsilon)1_{\{\varepsilon \leq 1\}} + b(N, \varepsilon).
\]

where \( a(N, \varepsilon) = e^{-c_1N\varepsilon^2}1_{\{d=1\}} + e^{-c_1N(\log(2+\frac{\varepsilon}{\delta}))^2}1_{\{d=2\}} + e^{-c_1N\varepsilon^4}1_{\{d>2\}} \) and \( b(N, \varepsilon) = e^{-c_1N\varepsilon} \). In order to prove (3.21) we will prove only for one case \( d > 2 \). Rest will follow similarly. There exists \( C_1', C_2', C_3' \)

\[
\sum_{i=1}^{n} P[\mathcal{W}_1(\zeta_n^N, \mu_n-i) \geq \frac{\gamma_1 \varepsilon}{C_1} \nu_i] \leq \sum_{i=1}^{n} e^{C_i \varepsilon N \nu_i}
\]

(5.120)

\[
\sum_{i=1}^{n} P[\mathcal{W}_1(\zeta_n^N, \mu_n-i) \geq \frac{\gamma_1 \varepsilon}{C_1} \nu_i] \leq \sum_{i=1}^{n} a(N, \frac{\gamma_1 \varepsilon}{C_1} \nu_i) \leq \sum_{i=1}^{n} e^{-C_1' \varepsilon N \nu_i} \leq \sum_{i=1}^{n} e^{-C_2' \varepsilon N \nu_i} \leq \sum_{i=1}^{n} e^{-C_3' \varepsilon N \nu_i}
\]

(5.121)

\[
P[\mathcal{W}_1(\zeta_n^N, \mu_n) > \varepsilon] \leq e^{-C_3' \varepsilon N \nu_i}
\]

(5.122)

Suppose \( k_0 \) such that \( \nu_i \geq k_0 \) for all \( i \geq 1 \). Combining (5.120),(5.121),(5.122) we have for all \( N > 1 \) and \( a_2'' = k_0 \min\{C_1', C_2', C_3'\} \).

\[
\sup_n P[\mathcal{W}_1(\mu_n^N, \mu_n) > \varepsilon] \leq \sum_{i=0}^{\infty} e^{-a_2'' N \varepsilon N \nu_i} \leq \frac{e^{-a_2'' N \varepsilon N \nu_i}}{1 - e^{-a_2'' N \varepsilon N \nu_i}}
\]

(5.123)

Now there exists \( N_3 := -\frac{1}{a_2''} \log(1 - \frac{1}{a_2''}) \) such that \( N \geq N_3 \max\{\frac{1}{a_2''}, \frac{1}{a_2' \varepsilon}\} \) we have

\[
\sup_n P[\mathcal{W}_1(\mu_n^N, \mu_n) > \varepsilon] \leq a_2'' e^{-a_2'' N \varepsilon N \nu_i}.
\]

\[ \square \]

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Appendix

The first part of the following lemma is an immediate consequence of Ascoli-Arzela theorem where as the second follows from Lemma 5 in [7].

Lemma A.1 (a) For a compact set \( K \) in \( \mathbb{R}^d \) let \( \mathcal{F}_{a,b}(K) \) be the space of functions \( f : K \rightarrow \mathbb{R} \) such that \( \sup_{x \in K} |f(x)| \leq a \) and \( |f(x) - f(y)| \leq b|x - y| \) for all \( x, y \in K \). Then for any \( \varepsilon > 0 \) there is a finite subset \( \mathcal{F}_{a,b}^c(K) \) of \( \mathcal{F}_{a,b}(K) \) such that for any signed measure \( \mu \)

\[
\sup_{f \in \mathcal{F}_{a,b}(K)} |\langle f, \mu \rangle| \leq \max_{g \in \mathcal{F}_{a,b}^c(K)} |\langle g, \mu \rangle| + \varepsilon |\mu|_{TV}.
\]

The next lemma is straightforward.

Lemma A.2 Let \( P : \mathbb{R}^d \times \mathcal{B}(\mathbb{R}^d) \rightarrow [0, 1] \) be a transition probability kernel. Fix \( N \geq 1 \) and let \( y_1, y_2, \ldots, y_N \in \mathbb{R}^d \). Let \( X_1, X_2, \ldots, X_N \) be independent random variables such that \( \mathcal{L}(X_i) = \delta_{y_i}P \). Let \( f \in BM(\mathbb{R}^d) \) and let \( m^N_0 = \frac{1}{N} \sum_{i=1}^{N} \delta_{y_i}, m^N_1 = \frac{1}{N} \sum_{i=1}^{N} \delta_{X_i} \). Then

\[
E|\langle f, m^N_1 - m^N_0 P \rangle| \leq \frac{2\|f\|_{\infty}}{\sqrt{N}}.
\]
The following is a discrete version of Gronwall’s lemma.

Lemma A.3 (a) Let \( \{a_i\}_{i=0}^{\infty}, \{b_i\}_{i=0}^{\infty}, \{p_i\}_{i=0}^{\infty} \) be non-negative sequences. Suppose that

\[
a_n \leq b_n + \sum_{k=0}^{n-1} p_k a_k \quad \text{for all } n \geq 0.
\]

Then

\[
a_n \leq b_n + \sum_{k=0}^{n-1} \left[ p_k b_k \left( \prod_{j=k+1}^{n-1} (1 + p_j) \right) \right] \quad \text{for all } n \geq 0.
\]

(b) For any \( a, b > 0 \) and \( \{C_i\}_{i \geq 0} \) be a non-negative sequence of elements, then for all \( n \geq 0 \)

\[
\sum_{k=0}^{n} a^{n-k} \sum_{i=0}^{k} b^{k-i} C_i = \sum_{i=0}^{n} C_i \left[ \frac{a^{n+1-i} - b^{n+1-i}}{a - b} \right].
\]

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