The Star Height Hierarchy Vs. The Variable Hierarchy

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Abstract

The star height hierarchy (resp. the variable hierarchy) results in classifying µ-terms into classes according to the nested depth of fixed point operators (resp. to the number of bound variables). We prove, under some assumptions, that the variable hierarchy is a proper refinement of the star height hierarchy. We mean that the non collapse of the variable hierarchy implies the non collapse of the star height hierarchy. The proof relies on the combinatorial characterization of the two hierarchies.

Keywords. µ-calculi, hierarchies, games, strategies, combinatorial problems.

1 Introduction

Roughly speaking, a µ-calculus in the abstract sense of Arnold and Niwiński [1, §2] is a set of syntactic entities and a set of formal operations. The latter consists in the fixed point operators µ and ν and the substitution operation. These syntactic entities come with and intended interpretation over a class of complete lattices. Each entity $t$ is interpreted as a monotonic mapping from $t^{ar(t)}$ to $L$, where $ar(t)$ is the arity of $t$, that corresponds to the free variables of $t$, and $L$ is a complete lattice. The entities $\mu x.t$ and $\nu x.t$ of a µ-calculus are interpreted respectively as the least and greatest parametrized fixed point of the interpretation of $t$. The substitution is interpreted by means of the functional composition.

Hierarchies and logical expressiveness are at the core of fixed point theory. The most known and well studied measure of the complexity of the µ-calculi is the alternation depth of its µ-terms [12 8 2], that is, the number of alternations between µ and ν. As a consequence, it is possible to construct a hierarchy of µ-terms according to the alternation depth measure. The variable hierarchy [5 4] results in classifying µ-terms into classes according to the number of bound variables i.e. fixed point variables. By introducing the variable hierarchy for the propositional modal µ-calculus [11] and showing that it does not collapse [6], the authors managed to separate Parikh’s Game Logic [13] from the modal
μ-calculus and solve a long standing open problem.

The third hierarchy is the star height hierarchy \[10]\[9]: the μ-terms are classified into levels of a hierarchy according to the nested depth of the application of fixed point operators (or the iteration operator). The star height problem was first asked in formal language theory and consists in answering whether all regular languages can be expressed using regular expressions of bounded star height. This question have been answered by Eggan in \[10\], where he gave examples of regular languages of star height \(n\) for every \(n \in \mathbb{N}\). The star height problem was asked later for regular trees \[9\], the latter are finite or infinite trees with only finite many distinct subtrees, up to isomorphism. Regular trees form the free iteration theory and they might be written by means of iterative theory expressions. These expressions use an iteration operator, denoted \(\dagger\), which is interdefinable with the Kleene’s \(*\) operator in the case of matrix iteration theories \[7\] \[\S\]9. The star height problem can be asked in a general way for iteration theories \[7\], where the dagger \(\dagger\) operator is considered.

The alternation depth hierarchy and the variable hierarchy are orthogonal, however the variable hierarchy and the star height hierarchy are intuitively close. In this paper we show that the variable hierarchy is a proper refinement of the star height hierarchy. That is, the non collapse of the former implies the non collapse of the latter. The key observation is that the combinatorial measure which characterizes the variable hierarchy (i.e. the entanglement) is lower than the combinatorial measure which characterizes the star height hierarchy (i.e. the rank).

**Preliminaries and Notations**

A digraph \(G = (V_G, E_G)\) is a set of vertices \(V_G\) and a binary relation \(E_G \subseteq V_G \times V_G\). \(G\) is strongly connected if for each two vertices \(v_1, v_2 \in V_G\) there exists a path in \(G\) from \(v_1\) to \(v_2\). A strongly connected component of \(G\) is a maximal strongly connected subgraph of \(G\). We shall write \(scc(v)\) for the strongly connected component of \(G\) that contains the vertex \(v\). A strongly connected component is trivial if it reduces to a single vertex without loops. We shall write \(SCC(G)\) for the set of the non trivial strongly connected components of \(G\). We define a transitive relation \(\prec_G\) on \(SCC(G)\) as follows: \(G_1 \prec_G G_2\) if and only if \(G_1 \neq G_2\) and there is a path in \(G\) from a vertex of \(G_1\) to a vertex of \(G_2\). If \(G_1 \in SCC(G)\) then let \(G_1^+ = \{ G_2 \in SCC(G) \mid G_1 \prec_G G_2 \}\). If no confusion will arise then we shall write \(\prec\) instead of \(\prec_G\).
The $\mu$-calculi: syntax and semantics

We recall the definition of a $\mu$-calculus as given in [1, §2]. Let $E$ be a set of objects or entities and let $\text{Var}$ be a fixed countable set of variables. The variables in $\text{Var}$ will be denoted by $x, y, z, \ldots$. A mapping $\rho : \text{Var} \rightarrow E$ is called a substitution. If $\rho$ is a substitution into some set $E$, $x$ a variable, and $e$ and element of $E$, we denote by $\rho\{ e/x \}$ the substitution $\rho'$ defined by $\rho'(x) = e$ and $\rho'(y) = \rho(y)$ if $y \neq x$. More generally, if $x_1, \ldots, x_n$ are distinct variables and if $e_1, \ldots, e_n$ are elements of $E$, then $\rho\{ e_1/x_1, \ldots, e_n/x_n \}$ is the substitution $\rho'$ defined by

$$
\rho'(y) = \begin{cases} 
eq e_i & \text{if } y \in \{ x_1, \ldots, x_n \}, \\
\rho(y) & \text{if } y \notin \{ x_1, \ldots, x_n \}
\end{cases}
$$

Definition 2.1. A $\mu$-calculus is a tuple $(T, \text{id}, \text{ar}, \text{comp}, \mu, \nu)$, where

- $T$ is an arbitrary set, its elements are the $\mu$-terms of the $\mu$-calculus.
- $\text{id}$ is a mapping from $\text{Var}$ to $T$. We denote by $\hat{x}$ the element $\text{id}(x)$ in $T$.
- $\text{ar}$ is a mapping associating to each $t \in T$ a subset of $\text{Var}$ called the arity of $t$. If $x \in \text{ar}(t)$, we say that $x$ occurs free in $t$, and the elements of $\text{ar}(t)$ are called the free variables of $t$.
- $\text{comp}$ is a mapping associating a $\mu$-term $\text{comp}(t, \rho)$ with any $\mu$-term $t$ and any substitution $\rho$; we shall write also $t[\rho]$.
- $\mu$ and $\nu$ are two mappings from $\text{Var} \times T$ to $T$, the value of the mapping $\theta$ on $x$ and $t$ is written $\theta x. t$, for $\theta = \mu, \nu$.

Moreover, a $\mu$-calculus should satisfy further axioms, see [1, §2].

Semantics Let $(T, \text{id}, \text{ar}, \text{comp}, \mu, \nu)$ be a $\mu$-calculus. A $\mu$-interpretation of $T$ is a pair $(L, I)$ where $L$ is a complete lattice and $I$ is a function that associates to each $\mu$-term $t$ a monotonic mapping $t : L^{\text{ar}(t)} \rightarrow L$ such that the substitution is interpreted as the functional composition and $\mu x. t$ (resp. $\nu x. t$) is interpreted as the least (resp. greatest) parametrized fixed point of the interpretation of $t$.

Many syntactic entities can be structured to have a shape of a $\mu$-calculus. For instance, this happened to infinite words [1, §5], automata [1, §7], and parity games [14]. The interpretation of an automaton, viewed as an entity of the $\mu$-calculus, is the language which it accepts.
3 The star height hierarchy

Definition 3.1. Let $t$ be a $\mu$-term and $\text{Comp}_0$ be the set of $\mu$-terms without application of fixed point operators $\mu$ and $\nu$. The star height of $t$ is defined as follows:

$$h(t) = \begin{cases} 0 & \text{if } t \in \text{Comp}_0 \\ \max\{ h(t'), h(\rho(x_1)), \ldots, h(\rho(x_n)) \} & \text{if } t = \text{comp}(t', \rho) \text{ where } x_i \in \text{ar}(t') \\ 1 + h(t') & \text{if } t = \theta x.t' \text{ where } \theta = \mu, \nu \end{cases}$$

The rank: a digraph measure for the star height

In [10] Eggan defined a complexity measure of digraphs, called the feedback number, that captures the minimal star height of regular languages. The minimal star height of a regular language is exactly the feedback number of the minimal digraph of the expressions defining this language. This measure has been formulated in a more natural way by Courcelle et al. in [9], and they rename it the rank. There, they solved the star height problem for regular trees, and showed that the minimal star height of a regular tree is exactly the rank of the minimal digraph of the tree.

Definition 3.2. The rank of a digraph $G$ is defined as follows:

- if $\text{SCC}(G) = \emptyset$, then $r(G) = 0$,
- if $\text{SCC}(G) = \{ G \}$, then $r(G) = 1 + \min\{ r(G \setminus v) \mid v \in V_G \}$,
- otherwise, $r(G) = \max\{ r(G') \mid G' \in \text{SCC}(G) \}$.

Note that $r(G) = 0$ if $G$ is acyclic. If $G$ is strongly connected and $r(G) = 1$ then $G$ contains a vertex whose removal makes the digraph acyclic. It is not hard to argue that the rank of an undirected path on $n$ vertices is $\lceil \log(n) \rceil$.

Thief and Cops games for the rank

To establish the relation between the rank and the entanglement, in a first step, we rephrase the definition of the rank in terms of games and strategies in the most direct way.

Definition 3.3. The rank game $R(G, k), k \geq 0$ played alternatively between a Thief and Cops on the digraph $G$ is defined as follows.

- Its positions are of the form $(G', P, n)$ where $0 \leq n \leq k$, $G'$ is a subgraph of $G$, and $P \in \{ \text{Thief, Cops} \}$ such that
  - the starting position is $(G, \text{Thief}, k)$,
  - if $\text{SCC}(G) = \emptyset$ or $n = 0$ then the play halts.
If SCC(G) = \{G_1, \ldots, G_l\}, (possibly l = 1) then Thief chooses some G_i and moves from (G, Thief, n) to (G_i, Cops, n).

- If the position is (G, Cops, n) then Cops choose v ∈ V_G and move to (G \ v, Thief, n − 1).

Thief wins a play if and only if its final position (G', P, n) is such that SCC(G) ≠ ∅ and n = 0.

We define \( \mathcal{R}(G) \) to be the minimum k such that Cops have a winning strategy in the rank game \( \mathcal{R}(G, k) \).

**Proposition 3.4.** Let G be a digraph, then \( \mathcal{R}(G) \) equals r(G).

**Proof.** We have just rephrased the definition of the rank by means of games and strategies following the game theoretic tradition. That is, Cops play the role of the minimizer and Thief plays the role of the maximizer.

Now, in order to compare the rank with the entanglement in an easy way, we shall give a useful variant of the rank games. The idea is that, whenever SCC(G) = \{G_1, \ldots, G_l\} and Thief moves from (G, Thief, n) to (G_i, Cops, n), for some i ∈ \{1, \ldots, l\}, then he is allowed later, and at any moment, to come back and move to (G_j, Cops, n) where G_i, G_j ∈ SCC(G) and G_i ≺ G_j.

**Definition 3.5.** Let G be a digraph and k ≥ 0. We define the rank game with come back \( \mathcal{R}^B(G, k) \) between Thief and Cops on the digraph G as follows:

Its positions are of the form (G', P, L, n) where 0 ≤ n ≤ k, G' is a subgraph of G, P ∈ \{Thief, Cops\}, and L is a set of quadruplet of the form (G, P, L, n) such that

- the starting position is (G, Thief, ∅, k),
- if (SCC(G) = ∅ or n = 0) and L = ∅ then the play halts.

If SCC(G) = \{G_1, \ldots, G_l\}, (possibly l = 1) then Thief has two kinds of moves:

- he chooses some G_i ∈ SCC(G) and moves from

\[
(G, \text{Thief}, L, n) \rightarrow (G_i, \text{Cops}, G_i^\succ \times (\text{Cops}, L, n) \cup L, n)
\]

(forward move)

where \{G_1, \ldots, G_l\} \times (\text{Cops}, L, n) =_{def} \{ (G_1, \text{Cops}, L, n), \ldots, (G_l, \text{Cops}, L, n) \}.

- or he moves from

\[
(G', \text{Thief}, L, n) \rightarrow B \text{ where } B \in L
\]

(come back move)

\(^1\)Observe that in this case G is strongly connected.

\(^2\)Observe that there is no infinite play.
If the position is \((G, Cops, n)\) then

Cops choose \(v \in V_G\) and move to \((G \setminus v, Thief, n - 1)\).

Thief wins a play if and only if its final position \((G', P, n)\) is such that \(SCC(G) \neq \emptyset\) and \(n = 0\).

We define \(R^B(G)\) to be the minimum \(k\) such that Cops have a winning strategy in the rank game \(R^B(G, k)\).

**Fact 3.6.** There is no infinite play in the game \(R^B(G, k)\).

**Lemma 3.7.** Let \(G\) be a digraph, then Thief has a winning strategy in \(R^B(G, k)\) if and only if he has a winning strategy in \(R(G, k)\). Therefore \(r(G) = \mathcal{R}(G) = \mathcal{R}^B(G)\).

**Proof.** First, if Thief has a winning strategy in \(R(G, k)\) then he also has a winning strategy in \(R^B(G, k)\) i.e. the latter being without using come back moves.

Second, if Thief has a winning strategy in \(R^B(G, k)\) which uses a come back move of the form \((G', Thief, L, n) \rightarrow B\) then he was able to move early to \(B\). \(\square\)

## 4 The variable hierarchy

In order to compute the minimum number of bound (i.e. fixed point) variables needed in a \(\mu\)-term up to \(\alpha\)-conversion, a digraph measure is required, that is the entanglement. The entanglement of a finite digraph \(G\), denoted \(\mathcal{E}(G)\), was defined in [5] by means of some games \(\mathcal{E}(G, k), k = 0, \ldots, |V_G|\). The game \(\mathcal{E}(G, k)\) is played on \(G\) by Thief against Cops, a team of \(k\) cops as follows. Initially all the cops are placed outside the digraph, Thief selects and occupies an initial vertex of \(G\). After Thief’s move, Cops may do nothing, may place a cop from outside the digraph onto the vertex currently occupied by Thief, may move a cop already on the graph to the current vertex. In turn Thief must choose an edge outgoing from the current vertex whose target is not already occupied by some cop and move there. If no such edge exists, then Thief is caught and Cops win. Thief wins if he is never caught. The entanglement of \(G\) is the least \(k \in \mathbb{N}\) such that \(k\) cops have a strategy to catch the thief on \(G\). It will be useful to formalize these notions.

**Definition 4.1.** The entanglement game \(\mathcal{E}(G, k)\) of a digraph \(G\) is defined by:

- Its positions are of the form \((v, C, P)\), where \(v \in V_G, C \subseteq V_G\) and \(|C| \leq k\), \(P \in \{Cops, Thief\}\).

- Initially Thief chooses \(v_0 \in V_G\) and moves to \((v_0, \emptyset, Cops)\).

- Cops can move from \((v, C, Cops)\) to \((v, C', Thief)\) where \(C'\) can be

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\(^3\)Observe that in this case \(G\) is strongly connected.
\[ C : \text{Cops skip,} \]
\[ C \cup \{ v \} : \text{Cops add a new Cop on the current position,} \]
\[ (C \setminus \{ x \}) \cup \{ v \} : \text{Cops move a placed Cop to the current position.} \]

- **Thief can move from** \((v, C, \text{Thief})\) to \((v', C, \text{Cops})\) if \((v, v') \in E_G\) and \(v' \notin C\).

*Every finite play is a win for Cops, and every infinite play is a win for Thief.*

\[ \mathcal{E}(G), \text{the entanglement of } G, \text{ is the minimum } k \in \{0, \ldots, |V_G|\} \text{ such that Cops have a winning strategy in } \mathcal{E}(G, k). \]

Observe that an undirected path has entanglement at most 2. The following Proposition, see \([3, \S 3.4]\), provides a useful variant of entanglement games.

**Proposition 4.2.** Let \(\tilde{\mathcal{E}}(G, k)\) be the game played as the game \(\mathcal{E}(G, k)\) apart that Cops are allowed to retire a number of cops placed on the digraph. That is, Cops moves are of the form

- \((g, C, \text{Cops}) \rightarrow (g, C', \text{Thief})\) (generalized skip move),
- \((g, C, \text{Cops}) \rightarrow (g, C' \cup \{ g \}, \text{Thief})\) (generalized replace move),

where in both cases \(C' \subseteq C\). Then Cops have a winning strategy in \(\mathcal{E}(G, k)\) if and only if they have a winning strategy in \(\tilde{\mathcal{E}}(G, k)\).

### 4.1 An ad hoc variant of entanglement games

The game theoretic definition of the entanglement, Definition 4.1 and even the variant given in Proposition 4.2 refers to some rules which are not very close to the rules of the rank games with come back which characterize the rank, Definition 3.5. And hence we can not establish the relation between the rank and the entanglement in an easy way. Therefore we shall give an equivalent variant of entanglement games, denoted \(\mathcal{E}^V(G, k)\), with the property that its rules are close to those of the rank games.

First we explain informally the new features of this game w.r.t the games for entanglement. In the game \(\mathcal{E}^V(G, k)\) Cops are allowed to skip, add a cop, replace a cop, retire a number of cops, and moreover we would like that they can put a cop on a vertex situated anywhere in the digraph. However, the latter move is not allowed by entanglement rules. In order to make it possible, Cops should keep in reserve a set \(Vir\) of virtual cops for this purpose: whenever Cops decide to put a cop on an arbitrary vertex \(w\) then they should reserve a cop for this purpose and this cop can not be used until Thief visits vertex \(w\). And at this moment, the virtual cop must be placed on \(w\).
Definition 4.3. The game $\mathcal{E}^V(G,k)$ is defined as the entanglement game $\mathcal{E}(G,k)$ apart that its positions are of the form $(v,C,Vir,P)$ where $Vir \subseteq V_G$ and $|C \cup Vir| \leq k$. Besides the old Cops’ moves, the latter act on the set $C$, Cops can move from $(v,C,Vir,P)$ to $(v,C',Vir',Thief)$ such that:

- if $v \in Vir$ then Cops must update $C' = C \cup \{v\}$ and $Vir' = Vir \setminus \{v\}$,
- if $v \notin Vir$ then Cops may update $Vir' = (Vir \setminus A) \cup \{w\}$ where $w \in V_G$ and $A \subseteq Vir$.

Lemma 4.4. Let $G$ be a digraph. Cops have a winning strategy in $\mathcal{E}^V(G,k)$ if and only if they have a winning strategy in $\mathcal{E}(G,k)$.

Proof. First, a winning strategy for Cops in $\mathcal{E}(G,k)$ is still winning for them in $\mathcal{E}^V(G,k)$, the latter does not refer to virtual cops. In This case every position $(v,C,Vir,P)$ in $\mathcal{E}(G,k)$ is matched with the position $(v,C,P)$ in $\mathcal{E}(G,k)$ where $Vir = \emptyset$.

Second, a Cops’ winning strategy in $\mathcal{E}^V(G,k)$ is mapped to a Cops’ winning strategy in $\mathcal{E}(G,k)$ as follows. Every position $(v,C,P)$ of $\mathcal{E}(G,k)$ is matched with the position $(v,C,Vir,P)$ of $\mathcal{E}^V(G,k)$.

A Thief’s move from $v$ to $v'$ in $\mathcal{E}(G,k)$ is simulated by the same move from $v$ to $v'$ in $\mathcal{E}^V(G,k)$. Indeed this simulation is possible because Thief is allowed in $\mathcal{E}^V(G,k)$ to cross a vertex which is occupied by a virtual cop.

Assume that the position $(v,C,Cops)$ is matched with the position $(v,C,Vir,Cops)$ and consider a Cop’s move $M = (v,C,Vir,Cops) \rightarrow (v,C',Vir',Thief)$ in $\mathcal{E}^V(G,k)$. The move $M$ is simulated in $\mathcal{E}(G,k)$ by the move $(v,C,Cops) \rightarrow (v,C',Thief)$.

5 The star height hierarchy vs. the variable hierarchy

We are ready to state the main result of this paper.

Theorem 5.1. Let $G$ be a digraph. The entanglement of $G$ is lower or equal to the rank of $G$.

Proof. To prove that $\mathcal{E}(G) \leq r(G)$ it is enough to prove $\mathcal{E}^V(G) \leq \mathcal{R}^B(G)$. Because Lemma 4.4 shows that $\mathcal{E}(G) = \mathcal{E}^V(G)$ and Lemma 5.7 shows that $r(G) = \mathcal{R}^B(G)$. Let $k = \mathcal{E}^V(G)$, we shall construct a winning strategy for Cops in the game $\mathcal{E}^V(G,k)$ out of a Cops’ winning strategy in $\mathcal{R}^B(G,k)$.

Every position of the form $(v,C,Vir,P)$ in $\mathcal{E}^V(G,k)$ is matched with a position of the form $(G',P,L,n)$ in $\mathcal{R}^B(G,k)$ such that if the strongly connected component of the subgraph $G \setminus (C \cup Vir)$ which contains $v$, denoted by $scc(v)$, is not trivial then $scc(v) = G'$.

\footnote{Which are the skip, the add and the generalized replace.}
Let us consider a Thief’s move in $E^V(G, k)$ of the form $M = (v, C, Vir, Thief) \rightarrow (w, C, Vir, Cops)$. If $scc(w)$ is trivial, then Cops just skip in $R^B(G, k)$. Observe that either Thief will reach a vertex without successors (where he loses), or he enters a non trivial strongly connected component. Otherwise i.e. $scc(w)$ is not trivial, the move $M$ is simulated in $R^B(G, k)$ according to $w$:

- if $w \in V_{G'}$ then $M$ is simulated by the move $(G', Thief, L, n) \rightarrow (G'', Cops, L, n)$ where $G''$ is the strongly connected component of $G'$ containing $w$,

- if $w \notin V_{G'}$ then the move $M$ is simulated by the come back move $(G', Thief, L, n) \rightarrow (G^-, Cops, L', m)$ where $(G^-, Cops, L', m) \in L$ and $w \in V_{G^-}$.

A Cops’ move of the form $N = (G', Cops, L, n) \rightarrow (G' \setminus w, Thief, L, n - 1)$ in $R^B(G, k)$ is simulated in $E^V(G, k)$ according to the nature of the position $(G', Cops, L, n)$.

- if the position $(G'', Thief, L'', n)$ that precedes $(G', Cops, L, n)$ was in the same strongly connected component i.e. $G' \subset G''$ (note that $L = L''$), then the move $N$ is simulated either by $(v, C, Vir, Cops) \rightarrow (v, C \cup \{v\}, Vir, Thief)$ if $v = w$, or by $(v, C, Vir, Cops) \rightarrow (v, C, Vir \cup \{w\}, Thief)$ otherwise.

- if the position $(G', Cops, L, n)$ comes from a come back move $(G'', Thief, L'', m) \rightarrow (G', Cops, L, n)$ then $N$ is simulated by $(v, C, Vir, Cops) \rightarrow (v, C \setminus (C \cap V_{G^-}), Vir \setminus (Vir \cap V_{G^-}), Thief)$.

Let us define $G^-$. There exists just one position $\gamma^-$ such that (i) $\gamma^-$ has the same predecessor of the position $(G', Cops, L, n)$ in the game $R^B(G, k)$ and (ii) the position $(G'', Thief, L'', m)$ has been reached from the position $\gamma^-$. We define $G^-$ to be the digraph associated to $\gamma^-$, i.e. $\gamma^-$ is of the form $(G^-, Cops, L^-, n^-)$.

Finally, besides Theorem 5.1 we give further conditions under which the non collapse of the variable hierarchy implies the non collapse of the star height hierarchy of a given $\mu$-calculus $\mathbb{L}_\mu$.

The proof schema of the strictness of the variable hierarchy consists essentially in the construction of hard $\mu$-terms of arbitrary entanglement $[6, 4]$. A $\mu$-term $t$ is said to be hard (w.r.t entanglement) if for every $\mu$-term $t'$ which is equivalent to $t$ we have that the entanglement of $t$ (viewed as a digraph) is lower or equal to the entanglement of $t'$ up to a constant. To argue that the star height of $\mathbb{L}_\mu$ is also infinite, it suffices to construct hard $\mu$-terms of arbitrary entanglement such that the entanglement of each $\mu$-term equals its rank. It follows from Theorem 5.1 that such $\mu$-terms are also hard w.r.t the star height.

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