Toroidal LNRF-velocity profiles in thick accretion discs orbiting rapidly rotating Kerr black holes

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Abstract. We show that in the equatorial plane of marginally stable thick discs (with uniformly distributed specific angular momentum $\ell(r, \theta) = \text{const}$) the orbital velocity relative to the LNRF has a positive radial gradient in the vicinity of black holes with $a > 0.99979$. The change of sign of the velocity gradient occurs just above the center of the thick toroidal discs, in the region where stable circular geodesics of the Kerr spacetime are allowed. The global character of the phenomenon is given in terms of topology changes of the von Zeipel surfaces (equivalent to equivelocity surfaces in the tori with $\ell(r, \theta) = \text{const}$). Toroidal von Zeipel surfaces exist around the circle corresponding to the minimum of the equatorial LNRF velocity profile, indicating a possibility of development of some vertical instabilities in those parts of marginally stable tori with positive gradient of the LNRF velocity. Eventual oscillatory frequencies connected with the phenomenon are given in a coordinate-independent form.

Keywords: Kerr black holes, orbital velocity, locally non-rotating frames, accretion discs, oscillations

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1. INTRODUCTION

High frequency (kHz) quasi-periodic oscillations (QPOs) with frequency ratios 3:2 (and sometimes 3:1) are observed in microquasars (see, e.g., [1, 2]). The same frequency ratios of QPOs in mHz are observed in the Galactic Center black hole Sgr A* [3, 4]. It is assumed now that the QPOs are related to the parametric or forced resonance [5] of the radial and vertical epicyclic oscillations or one of the epicyclic and orbital oscillations in accretion discs [6, 7, 8].

The oscillations could be related to both the thin Keplerian discs [9, 10] or the thick, toroidal accretion discs [11, 12]. The parametric resonance of the radial and vertical oscillations in the thin discs can explain the QPOs with the $\omega_0/\omega_0 = 3 : 2$ frequency ratio observed in all the microquasars and can put strong limits on the rotational parameter of their central black holes related to the limits on their mass [13].

Aschenbach [4] conjectured relation between the 3:2 and 3:1 resonance orbits by relating their Keplerian orbital velocities at $r_{3:2}$ and $r_{3:1}$ to be $\Omega_K(r_{3:1}; a) = 3\Omega_K(r_{3:2}; a)$, fixing thus the rotational parameter of black holes at the value of $a = a_f = 0.99616$. Further, he proposed that excitation of the oscillations at $r = r_{3:1}$ can be related to two changes of sign of the radial gradient of the Keplerian orbital velocity as measured in the locally non-rotating frame (LNRF) [18] that occurs in vicinity of $r = r_{3:1}$ for black holes with $a > 0.9953$. While the assumption of frequency commensurability of Keplerian orbits at $r_{3:1}$ and $r_{3:2}$ seems to be rather artificial because distant parts of the Keplerian disc have to be related, we consider the positive radial gradient of orbital velocity in LNRF nearby the $r_{3:1}$ orbit around black holes with $a > 0.9953$ to be a physically interesting phenomenon, even if a direct mechanism relating this to triggering of the excitation of radial (and vertical) epicyclic oscillations is unknown. We also show (Table 1) that physically relevant, i.e. coordinate-independent, frequencies, given by maximal positive radial gradient of the LNRF-orbital velocity, take locally values from tens to hundreds of Hz and for stationary observers at infinity tens of Hz (for typical stellar-mass black holes $M \sim 10M_\odot$), which likely disables possibility to identify these frequencies directly with those of high frequency QPOs [13].

Because the accretion-disc regime will vary from thin Keplerian disc to thick toroidal disc with variations of accretion flow, we shall study here, without addressing details of the mechanism, whether the orbital velocity in LNRF can have positive gradient also for matter orbiting black holes in marginally stable thick discs with uniform distribution of the specific angular momentum ($\ell(r, \theta) = \text{const}$), leading to a possibility to excite oscillations in the thick-disc accretion regime. Note that the assumption of uniform distribution of the specific angular momentum can be relevant at least at the inner parts of the thick disc and that matter in the disc follows nearly geodesic circular orbits nearby the centre of the disc and in the vicinity of its inner edge determined by the cusp of its critical equipotential
surface, see [14, 15].

In thick tori, it is necessary to have information about the character of the Aschenbach’s phenomenon also outside the equatorial plane. We shall obtain such information by introducing the notion of von Zeipel radius $\mathcal{R}$, analogous to the radius of gyration $\mathcal{R}^*$ introduced for the case of Kerr spacetimes in the framework of optical geometry by Abramowicz et al. [16], generalizing in one special way the definition used for static spacetimes [17]. The von Zeipel radius is defined in such a way that for the marginally stable tori the von Zeipel surfaces, i.e., the surfaces of constant values of $\mathcal{R}$, coincide with surfaces of constant orbital velocity relative to the LNRF.

2. TOROIDAL MARGINA LLY STABLE ACCRETION DISCS

The perfect fluid stationary and axisymmetric toroidal discs are characterized by 4-velocity field $U^\mu = (U^t, 0, 0, U^\theta)$ with $U^t = U^t(r, \theta)$, $U^\theta = U^\theta(r, \theta)$, and by the distribution of specific angular momentum $\ell = -U_\phi/U_t$. The angular velocity of orbiting matter, $\Omega = U^\theta/U^t$, is then related to $\ell$ by the formula

$$\Omega = -\frac{\ell g_{tt} + g_{t\phi}}{\ell g_{tt} + g_{\phi\phi}}. \quad (1)$$

The marginally stable toroids are characterized by the uniform distribution of specific angular momentum $\ell = \ell(r, \theta) = \text{const}$ and they are fully determined by the spacetime structure through equipotential surfaces of the potential $W = W(r, \theta)$ defined by the relation [14]

$$W - W_{\text{in}} = \ln \left( \frac{U_1}{(U_1)_{\text{in}}} \right), \quad (U_1)^2 = \frac{g_{t\phi}^2 - g_{tt}g_{\phi\phi}}{g_{tt}^2 + 2g_{t\phi}g_{\phi\phi}}; \quad (2)$$

the subscript “in” refers to the inner edge of the disc.

In the Kerr spacetimes with the rotational parameter assumed to be $a > 0$, the relevant metric coefficients in the standard Boyer-Lindquist coordinates read:

$$g_{tt} = -\frac{\Delta - a^2 \sin^2 \theta}{\Sigma}, \quad g_{t\phi} = -2ar \sin^2 \theta \Sigma, \quad g_{\phi\phi} = \frac{A \sin^2 \theta}{\Sigma}, \quad (3)$$

where

$$\Delta = r^2 - 2r + a^2, \quad \Sigma = r^2 + a^2 \cos^2 \theta, \quad A = (r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta. \quad (4)$$

The geometrical units, $c = G = 1$, together with putting the mass of the black hole equal to one, $M = 1$, are used to obtain completely dimensionless formulae hereafter. The relation (1) for the angular velocity of matter orbiting the black hole acquires the form

$$\Omega = \Omega(r, \theta; a, \ell) = \frac{(\Delta - a^2 \sin^2 \theta)\ell + 2ar \sin^2 \theta}{(A - 2lr) \sin^2 \theta} \quad (5)$$

and the potential $W$, defined in Eq. (2), has the explicit form

$$W = W(r, \theta; a, \ell) = \frac{\Sigma \Delta \sin^2 \theta}{(r^2 + a^2 - a\ell)^2 \sin^2 \theta - (\ell - a \sin^2 \theta)^2}. \quad (6)$$

3. THE ORBITAL VELOCITY IN LNRF

The locally non-rotating frames are given by the tetrad of 1-forms [18]

$$e^{(r)} = \left( \frac{\Sigma \Delta}{A} \right)^{1/2} \text{d}r, \quad e^{(r)} = \left( \frac{\Sigma}{\Delta} \right)^{1/2} \text{d}r, \quad e^{(\theta)} = \Sigma^{1/2} \text{d}\theta, \quad e^{(\phi)} = \left( \frac{A}{\Sigma} \right)^{1/2} \sin \theta (\text{d}\phi - \omega \text{d}r), \quad (7)$$

where

$$\omega = -\frac{g_{t\phi}}{g_{\phi\phi}} = \frac{2ar}{A}. \quad (8)$$

Toroidal LNRF-velocity profiles in thick accretion discs orbiting rapidly rotating Kerr black holes...
is the angular velocity of LNRF. The azimuthal component of 3-velocity in LNRF reads

$$\gamma^{(\phi)}_{\text{LNRF}} = \frac{U^{\mu}_{\text{LNRF}}(\phi)}{U^{\nu}_{\text{LNRF}}(\nu)} = \frac{A \sin \theta}{\Sigma \sqrt{\Delta}} (\Omega - \omega).$$  \hspace{1cm} (9)$$

Substituting for the angular velocities $\Omega$ and $\omega$ from the relations (5) and (8), respectively, we arrive at the formula

$$\gamma^{(\phi)}_{\text{LNRF}} = \frac{A(\Delta - a^2 \sin^2 \theta) + 4a^2 r^2 \sin^2 \theta \ell}{\Sigma \sqrt{\Delta} (\Delta - 2ar) \sin \theta} \ell.$$

(10)

We focus our investigation to the motion in the equatorial plane, $\theta = \pi/2$. Formally, this velocity vanishes for $r \to \infty$ and $r \to r_+ = 1 + \sqrt{1 - a^2}$, where the event horizon is located, i.e., there must be a change of its radial gradient for any case of values of the parameters $a$ and $\ell$, contrary to the case of Keplerian orbits characterized by the Keplerian distributions of the angular velocity and the specific angular momentum

$$\Omega = \Omega_K(r; a) \equiv \frac{1}{r^{3/2} + a}, \quad \ell = \ell_K(r; a) \equiv \frac{r^2 - 2ar^{1/2} + a^2}{r^{3/2} - 2r^{1/2} + a},$$

(11)

where the azimuthal component of the 3-velocity in LNRF in the equatorial plane, $\theta = \pi/2$, reads

$$\gamma^{(\phi)}_K(r; a) = \frac{(r^2 + a^2)^2 - a^2 \Delta - 2ar(r^{3/2} + a)}{r^2(r^{3/2} + a) \sqrt{\Delta}}$$

(12)

and formally diverges at $r = r_+$.

Of course, for both thick tori and Keplerian discs we must consider the limit on the disc extension given by the innermost stable orbit. For Keplerian discs this is the marginally stable geodetical orbit, while for the thick tori this is an unstable circular geodesic kept stable by pressure gradients and located between the marginally bound and the marginally stable geodetical orbits, with the radius being determined by the specific angular momentum $\ell = \text{const} \in (\ell_{\text{ms}}, \ell_{\text{mb}})$ through the equation $\ell = \ell_K(r; a); \ell_{\text{ms}} (\ell_{\text{mb}})$ denotes specific angular momentum of the circular marginally stable (marginally bound) geodesic.

The radial gradient of the equatorial orbital velocity of thick discs reads

$$\frac{\partial \gamma^{(\phi)}}{\partial r} = \frac{[\Delta + (r - 1)r[r(r^2 + a^2) - 2a(\ell - a)] - r(3r^2 + a^2)\Delta]}{[r(r^2 + a^2) - 2a(\ell - a)]^2 \sqrt{\Delta}} \ell,$$

(13)

so that it changes its orientation at radii determined for a given $\ell \in (\ell_{\text{ms}}, \ell_{\text{mb}})$ by the condition

$$\ell = \ell_{\text{ex}}(r; a) \equiv a + r^2[(r^2 + a^2)(r - 1) - 2r\Delta]/[2a(\Delta + r(r - 1))].$$

(14)

Detailed discussion [19] shows that two changes of sign of $\partial \gamma^{(\phi)}/\partial r$ can occur for Kerr black holes with the rotational parameter $a > a_c(\text{black}) \approx 0.99979$. The interval of relevant values of the specific angular momentum $\ell \in (\ell_{\text{ms}}(a), \ell_{\text{ex}}(\text{max})(a))$ grows with $a$ growing up to the critical value of $a_c(\text{mb}) \approx 0.99998$. For $a > a_c(\text{mb})$, the interval of relevant values of $\ell \in (\ell_{\text{ms}}(a), \ell_{\text{mb}}(a))$ is narrowing with growing of the rotational parameter up to $a = 1$, which corresponds to a singular case where $\ell_{\text{ms}}(a = 1) = \ell_{\text{mb}}(a = 1) = 2$. Notice that the situation becomes to be singular only in terms of the specific angular momentum; it is shown (see [18]) that for $a = 1$ both the total energy $E$ and the axial angular momentum $L$ differ at $r_{\text{ms}}$ and $r_{\text{mb}}$, respectively, but their combination, $\ell \equiv L/E$, giving the specific angular momentum, coincides at these radii.

4. VON ZEIPERL SURFACES

It is useful to find global characteristics of the phenomenon that is manifested in the equatorial plane as the existence of a small region with positive gradient of the LNRF velocity. A physically reasonable way of defining a global quantity

Toroidal LNRF-velocity profiles in thick accretion discs orbiting rapidly rotating Kerr black holes September 22, 20183
characterizing rotating fluid configurations in terms of the LNRF orbital velocity is to introduce, so-called, von Zeipel radius defined by the relation

\[ R \equiv \frac{\ell}{y_{\text{LNRF}}^{\varphi}(\varphi)} = (1 - \omega \ell) \hat{\rho} \tag{15} \]

which generalizes in another way as compared with [16] the Schwarzschildian definition of the gyration radius \( \hat{\rho} \). (For more details see [19].)

In the case of marginally stable tori with \( \ell(r, \theta) = \text{const} \), the von Zeipel surfaces, i.e., the surfaces of \( R(r, \theta; a, \ell) = \text{const} \), coincide with the equivelocity surfaces \( y_{\text{LNRF}}^{\varphi}(r, \theta; a, \ell) = \text{const} \). For the tori in the Kerr spacetimes, there is

\[ R(r, \theta; a, \ell) = \frac{\Sigma \sqrt{\Delta(A - 2a\ell r) \sin \theta}}{A(\Delta - a^2 \sin^2 \theta) + 4a^2 r^2 \sin^2 \theta}. \tag{16} \]

Toroidal LNRF-velocity profiles in thick accretion discs orbiting rapidly rotating Kerr black holes.
FIGURE 2. Profiles of the equatorial orbital velocity of marginally stable tori in LNRF in terms of the radial Boyer-Lindquist coordinate (left column). For comparison, the profiles of the orbital velocity of Keplerian discs in Kerr spacetimes with the same rotational parameter $a$ are added (right column). For thick discs, values of $\ell = $ const are appropriately chosen; commonly, $\ell = \ell_{\text{ms}}$ is used giving the maximal value of the velocity difference in between the local extrema, and representing the limiting case of marginally stable thick discs.

Topology of the von Zeipel surfaces can be directly determined by the behaviour of the von Zeipel radius (16) in the equatorial plane

$$R(r, \theta = \pi/2; a, \ell) = r(r^2 + a^2 - 2a(\ell - a)) r\sqrt{\Delta}. \quad (17)$$

The local minima of the function (17) determine loci of the cusps of the von Zeipel surfaces, while its local maximum (if it exists) determines a circle around which closed toroidally shaped von Zeipel surfaces are concentrated (see Fig. 1). Notice that the minima (maximum) of $R(r, \theta = \pi/2; a, \ell)$ correspond(s) to the maxima (minimum) of $V^{(\phi)}_{\text{LNRF}}(r, \theta = \pi/2; a, \ell)$, therefore, the inner cusp is always physically irrelevant being located outside of the toroidal configuration of perfect fluid. Behaviour of the von Zeipel surfaces nearby the center and the inner edge of the thick tori orbiting Kerr black holes with $a > a_{c(\text{thick})} \doteq 0.99979$, i.e., the existence of the von Zeipel surface with an outer cusp or the surfaces with toroidal topology, suggests possibility of strong instabilities in both the vertical and radial direction and a tendency for development of some vortices crossing the equatorial plane. We plan studies of these expected phenomena in future.

5. DISCUSSION AND CONCLUSIONS

It is useful to discuss both the qualitative and quantitative aspects of the phenomenon of the positive gradient of LNRF orbital velocity. In the Kerr spacetimes with $a > a_{c(\text{thick})}$, changes of sign of the gradient of $V^{(\phi)}(r, a)$ must occur closely above the center of relevant toroidal discs, at radii corresponding to stable circular geodesics of the spacetime, where the radial and vertical epicyclic frequencies are also well defined.
In two interesting cases, behaviour of \( \gamma^{(\phi)}(r; a, \ell) \) is illustrated in Fig. 2; for comparison, profiles of the Keplerian velocity \( \gamma^{(\phi)}_K(r; a) \) are included. With \( a \) growing in the region of \( a \in (a_c(\text{thick}), 1) \), the difference \( \Delta \gamma^{(\phi)} = \gamma^{(\phi)}_{\max} - \gamma^{(\phi)}_{\min} \) grows as well as the difference of radii, \( \Delta r \equiv r_{\max} - r_{\min} \), where the local extrema of \( \gamma^{(\phi)}(r; a, \ell) \) occur, see Fig. 3.

In terms of the redefined rotational parameter \( (1 - a) \), the changes of sign of gradient of the LNRF orbital velocity of marginally stable thick disc occur for discs orbiting Kerr black holes with \((1 - a) < 1 - a_{c(\text{thick})} = 2.1 \times 10^{-4}\) which is more than one order lower than the value \( 1 - a_{c(\text{thin})} = 4.7 \times 10^{-3}\) found by Aschenbach [4] for the changes of sign of the gradient of the orbital velocity in Keplerian, thin discs. Moreover, in the thick discs, the velocity difference, \( \Delta \gamma^{(\phi)} = \gamma^{(\phi)}_{\max} - \gamma^{(\phi)}_{\min} \), is smaller but comparable with those in the thin discs (see Fig. 3). In fact, we can see that for \( a \to 1 \), the velocity difference in the thick discs \( \Delta \gamma^{(\phi)}_{\text{thick}} \approx 0.02 \), while for the Keplerian discs it goes even up to \( \Delta \gamma^{(\phi)}_{\text{thin}} \approx 0.07 \). These are really huge velocity differences, being expressed in units of \( c \).

Following Aschenbach [4], we can define the typical frequency of the mechanism for excitation of oscillations by the maximum slope of the positive gradient of \( \partial \gamma^{(\phi)}/\partial \rho \) in the region of the changes of its sign by the relation

\[
\Omega_{\text{crit}}^c = 2\pi \left. \frac{\partial \gamma^{(\phi)}}{\partial \rho} \right|_{\max}.
\]  

The “oscillatory” frequency has to be determined numerically. We have done it for both Keplerian discs and the marginally stable discs with \( \ell = \ell_{\text{ms}} = \text{const} \), see Fig. 4 and Table 1. It is interesting that in Keplerian discs with \( a \approx 0.996 \), there is \( \Omega_{\text{crit}}^c \approx \Omega_K \approx \Omega_K/3 \), i.e., \( \Omega_{\text{crit}}^c \) can be related to the resonant phenomena at the radius where the epicyclic frequencies are in the 3:1 ratio [4]. However, it is more correct to consider \( \partial \gamma^{(\phi)}/\partial \tilde{R} \) where \( \tilde{R} \) is the physically relevant (coordinate-independent) proper radial distance, as it is more convenient for estimation of physically realistic characteristic frequencies related to local physics in the disc. Then the critical frequency for possible excitation of oscillations is given by the relation

\[
\Omega_{\text{crit}}^c = 2\pi \left. \frac{\partial \gamma^{(\phi)}}{\partial \tilde{R}} \right|_{\max}, \quad d\tilde{R} = \sqrt{g_{rr}} dr = \sqrt{\Delta} dr.
\]  

Of course, such a locally defined “oscillatory” frequency, confined to the orbiting LNRF-observers, should be further related to distant observers by the relation (taken at B–L coordinate \( r \) corresponding to \( (\partial \gamma^{(\phi)}/\partial r)_{\max} \))

\[
\Omega_{\text{crit}}^c = \sqrt{(-g_{rr} + 2\omega g_{\phi\phi} + \omega^2 g_{\phi\phi}) \Omega_{\text{crit}}^c}.
\]

Similarly, an analogical relation can be written also for the critical frequency \( \Omega_{\text{crit}}^c \), giving the circular frequency \( \Omega_{\text{circ}}^c \). Because the velocity gradient related to the proper distance is suppressed in comparison with those related Toroidal LNRF-velocity profiles in thick accretion discs orbiting rapidly rotating Kerr black holes

September 22, 2018

6
FIGURE 4. Critical “oscillatory” frequency for excitation of epicyclic oscillations, introduced by Aschenbach [4], as a function of the rotational parameter of the black hole in terms of both the B–L coordinate radius ($\Omega^c_{\text{crit}}$) and the proper radial distance ($\Omega^R_{\text{crit}}$).

(a) Keplerian discs. (b) Marginally stable (non-Keplerian) discs with constant specific angular momentum $\ell = \ell_{\text{ms}}$. (c) Comparison of critical frequencies for Keplerian $\Omega^c_{\text{crit}}$ and non-Keplerian $\Omega^m_{\text{crit}}$ discs in terms of the proper radius. (d) Positive parts of the “coordinate” and “proper” radial gradient $\partial \gamma^{(\phi)} / \partial r$ and $\partial \gamma^{(\phi)} / \partial \tilde{R}$ for a given value of the rotational parameter $a$.  

to the coordinate distance, there is $\Omega^R_{\text{crit}} < \Omega^c_{\text{crit}}$. The situation is illustrated in Fig. 4. Moreover, Fig. 4d shows mutual behaviour of the “coordinate” and “proper” radial gradient $\partial \gamma^{(\phi)} / \partial r$ and $\partial \gamma^{(\phi)} / \partial \tilde{R}$ in region between the local minimum and the outer local maximum of the orbital velocity $\gamma^{(\phi)}$ for an appropriately chosen value of the rotational parameter $a$. Characteristic frequencies $f = \Omega / 2\pi$, where $\Omega$ corresponds to the particular circular frequencies $\Omega^c_{\text{crit}}$, $\Omega^c_{\infty}$, $\Omega^R_{\text{crit}}$, $\Omega^R_{\infty}$, respectively, are given in Table 1.

We can conclude that in constant specific angular momentum tori, the effect discovered by Aschenbach is elucidated by topology changes of the von Zeipel surfaces. In addition to one self-crossing von Zeipel surface existing for all values of the rotational parameter $a$, for $a > a_{c(\text{thick})}$, the second self-crossing surface together with toroidal surfaces occur. Toroidal von Zeipel surfaces exist under the newly developing cusp, being centered around the circle corresponding to the minimum of the equatorial LNRF velocity profile. Further, the behaviour of von Zeipel surfaces in marginally stable tori orbiting Kerr black holes with $a > a_{c(\text{thick})}$, strongly suggests a possibility of development of both the vertical and vortical instabilities because of the existence of the critical surface with a cusp, located above the center of the torus and the toroidal von Zeipel surfaces located under the cusp. The effect of “velocity gradient sign changes” can be very important as a trigger instability mechanism for oscillations observed in QPOs. Of course, further studies directed both to the theoretically well founded, detailed physical mechanisms for triggering of oscillations in the equilibrium tori with general specific angular momentum distribution, and the link to observations, are necessary and planned for the future.

Finally, we would like to call attention to the fact that signs’ changes of the radial gradient of orbital velocity in LNRF occur nearby the $r = r_{3:1}$ orbit, while in the vicinity of the $r = r_{3:2}$ orbit, $\partial \gamma^{(\phi)}_{\text{LNRF}} / \partial r < 0$ for all values of $a$ for both the Keplerian discs and the marginally stable toroidal discs with all allowed values of $\ell$. Clearly, the parametric resonance, which is the strongest one for ratios of the epicyclic frequencies $\Omega_V / \Omega_R = 3/2$ works at the $r = r_{3:2}$ orbit,
while its effect is much smaller at the radius \( r = r_{3,1} \) with \( \Omega_\gamma/\Omega_\kappa = 3/1 \) [9]. Therefore, the forced resonance, triggered by the changes of \( \partial \gamma (r) / \partial r \), will be important for the 3:1 resonance. Notice that the forced resonance at \( r = r_{3,1} \) can generally result in observed QPOs frequencies with 3:2 ratio due to the beat frequencies allowed for the forced resonance [20]; however it seems to be irrelevant in the case of microquasars, as all observed frequencies lead to the values of the rotational parameter \( a < a_c(\text{thick}) \) as shown in [13].

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**TABLE 1.** Characteristic frequencies in units \((M/M_\odot)^{-1}\) Hz \((M/M_\odot)\) is the mass of the Kerr black hole in units of mass of the Sun.) related to the circular frequencies \(\Omega = 2\pi f\) defined in the text are given for appropriate values of the black hole spin. Maximal values of the frequencies related to the stationary observer at infinity are bold-faced.

| \(1 - a\) | Keplerian discs | Fluid tori |
|----------|----------------|------------|
|          | \(f^r\) | \(f^r_\infty\) | \(f^R\) | \(f^R_\infty\) | \(f^\Rbar\) | \(f^\Rbar_\infty\) |
| \(4.5 \times 10^{-3}\) | 356 | 86 | 121 | 29 | 3617 | 767 | 1130 | 248 |
| \(4 \times 10^{-3}\) | 1303 | 303 | 432 | 102 | 22982 | 2203 | 4352 | 607 |
| \(3 \times 10^{-3}\) | 3617 | 767 | 1130 | 248 | 26857 | 35617 | 767 | 1130 |
| \(1 \times 10^{-3}\) | 12179 | 1849 | 3061 | 536 | 36593 | 1565 | 4816 | 590 |
| \(5 \times 10^{-4}\) | 17132 | 2126 | 3789 | 592 | 36593 | 1565 | 4816 | 590 |
| \(2 \times 10^{-4}\) | 23301 | 2203 | 4382 | 607 | 26857 | 35617 | 767 | 1130 |
| \(1 \times 10^{-4}\) | 26857 | 2126 | 4579 | 603 | 36593 | 1565 | 4816 | 590 |
| \(1 \times 10^{-5}\) | 36593 | 1565 | 4816 | 590 | 10940 | 657 | 1447 | 135 |
| \(1 \times 10^{-6}\) | 42556 | 1001 | 4841 | 588 | 16271 | 589 | 1718 | 147 |
| \(1 \times 10^{-9}\) | 49250 | 201 | 4844 | 588 | 23277 | 185 | 1807 | 150 |

Toroidal LNRF-velocity profiles in thick accretion discs orbiting rapidly rotating Kerr black holes September 22, 2018.