Some Applications of Augmentation Quotients

Deepak Gumber
School of Mathematics and Computer Applications
Thapar University, Patiala - 147 004, India
E-mail: dkgumber@yahoo.com

Abstract
We give some applications of augmentation quotients of free group rings in group theory.

2000 Mathematics Subject Classification: 16S34, 20C07.

Keywords: integral group ring, augmentation quotient, subgroups determined by ideals.

1 Introduction

Let $ZG$ denote the integral group ring of a group $G$ and $\Delta(G)$ its augmentation ideal. Let $\{\gamma_n(G)\}_{n \geq 1}$ be the lower central series of $G$. We also write $G'$ for $\gamma_2(G) = [G, G]$, the derived group of $G$. When $G$ is free, then integral group ring is known as free group ring. Let $\Delta_n(G)$ denote the $n$-th associative power of $\Delta(G)$ with $\Delta_0(G) = ZG$. The additive abelian group $\Delta_n(G)/\Delta_{n+1}(G)$ is known as the $n$-th augmentation quotient and has been intensively studied during the last forty years. Vermani[7] has given a notable survey article about work done on augmentation quotients. In this short note we are interested in the applications of augmentation quotients in group theory. Henceforth, unless or otherwise stated, $F$ is a free group and $R$ is a normal subgroup of $F$. Hurley and Sehgal[4] identified the subgroup $F \cap (1 + \Delta^2(F)\Delta^n(R))$ for all $n \geq 1$ and then using the fact that $\Delta(F)\Delta^m(R)/\Delta^2(F)\Delta^n(R)$ is free abelian for all $n \geq 1$ [1], they showed that the group $\gamma_{n+1}(R)/\gamma_{n+2}(R)\gamma_{n+1}(R \cap F')$ is a free abelian group for all $n \geq 1$. Gruenberg [1, Lemma III.5] proved that $\Delta^n(F)\Delta^m(R)/\Delta^{n+1}(F)\Delta^m(R)$ is a free abelian group for all $m, n \geq 1$. When $R$ is an arbitrary subgroup of $F$, Karan and Kumar [5] proved that the groups $\Delta^n(F)\Delta^m(R)/\Delta^{n+1}(F)\Delta^m(R)$, $\Delta^n(F)\Delta^m(R)/\Delta^{n-1}(F)\Delta^{n+1}(R)$ and $\Delta^n(F)\Delta^m(R)/\Delta^n(F)\Delta^{m+1}(R)$ are free abelian for all $m, n \geq 1$. They gave the complete description of all these groups and explicit bases of first two groups. As a consequence of their results they proved that $R'/[R', R \cap F']$ is a free abelian group. Gumber et. al. [2] proved that $\Delta^p(R)\Delta^q(R)/\Delta^p(R)\Delta^q(R)\Delta^{n+1}(F)\Delta^q(R)$ is free abelian for all $p, q, n \geq 1$ and as a consequence showed that $\gamma_3(R)/\gamma_4(R)[R \cap F', R \cap F', R]$ is a free abelian.
Proof of Theorem A

Theorem 2.3 is free abelian. Then containing \( \Delta \) of the transversal \( S \)

Lemma 2.2 \[ \begin{align*}
\phi & \text{ maps } \Delta^1(U) \text{ onto } \Delta^1(\Gamma(H)) \text{ for all } \Gamma \in G. \end{align*} \]

Theorem 2.1 For \( m \geq 2 \), let \( L^{(m)} = \sum_{n \geq m} L_n. \)

\[ \Delta^{n-1}(G)\Delta(H) + \Delta^{n-2}(G)\Delta(H) + \cdots + \Delta(G)\Delta(\gamma_{n-1}(G)) + \Delta(\gamma_n(G)) \oplus L^{(n)}. \]

Let \( U \) be a group and \( W \) be a left transversal of a subgroup \( V \) of \( U \) with \( 1 \in W \). Then every element of \( U \) can be uniquely written as \( wv, w \in W, v \in V \).

Let \( \phi : ZU \to ZV \) be the onto homomorphism of right \( ZV \)-modules which on the elements of \( U \) is given by \( \phi(wv) = v, w \in W, v \in V \). The homomorphism \( \phi \) maps \( \Delta(U)J \) onto \( \Delta(V)J \) for every ideal \( J \) of \( ZV \). In particular, by the choice of the transversal \( S \) of \( H \) in \( G \), we have \( \phi \mid L^{(n)} = 0 \). The homomorphism \( \phi \) is usually called the filtration map.

We shall also need the following results:

Lemma 2.2 \[ \begin{align*}
\text{Let } G \text{ be a group, } K \text{ a subgroup of } G, \text{ and } J \text{ an ideal of } ZG \text{ containing } \Delta^2(K). \text{ Then } G \cap (1 + J + \Delta(K)) = (G \cap (1 + J))K. \end{align*} \]

Theorem 2.3 \[ \begin{align*}
\text{Let } G \text{ be a group with a normal subgroup } H \text{ such that } G/H \text{ is free abelian. Then } G \cap (1 + \Delta^n(G) + \Delta(G)\Delta(H)) = \gamma_n(G)H' \text{ for all } n \geq 1. \end{align*} \]

3 Proof of Theorem A

To avoid repeated and prolonged expressions, we write

\[ \begin{align*}
A &= \Delta^4(R) + \Delta^2(R)\Delta(R \cap F') + \Delta(R)\Delta([R, R \cap F']) \\
B &= \Delta^3(R) + \Delta^3(R)\Delta(R \cap F') + \Delta^2(R)\Delta([R, R \cap F']).
\end{align*} \]
Proposition 3.1 \( R \cap (1 + \Delta^3(R) + \Delta(R)\Delta(R \cap F') + \Delta([R, R \cap F'])) = [R, R \cap F']. \)

**Proof.** Proof is easy and follows by Lemma 2.2 and Theorem 2.3.

Proposition 3.2 \( R \cap (1 + A) = \gamma_4(R)[R, R \cap F', R \cap F']. \)

**Proof.** Since \( \gamma_4(R) - 1 \subset \Delta^3(R) \) and \( [R, R \cap F', R \cap F'] - 1 \subset \Delta^2(R)\Delta(R \cap F') + \Delta(R)\Delta([R, R \cap F']) \), it follows that \( \gamma_4([R, R \cap F', R \cap F'] \subset R \cap (1 + A) \). For the reverse inequality, we let \( w \in R \) such that \( w - 1 \in A \) and proceed to show that \( w \equiv 1 \pmod{\gamma_4(R)[R, R \cap F', R \cap F']} \). Since \( R/R \cap F' \) is free-abelian, using Theorem 2.1 repeatedly we have

\[
A = \Delta(\gamma_4(R)) + L^{(4)} + \Delta(R)\Delta^2(R \cap F') + \Delta(R')\Delta(R \cap F') + L^{(2)}\Delta(R \cap F') + \Delta(R)\Delta([R, R \cap F']).
\]

Now since \( R \cap (1 + A) \subset R \cap F' \), using the filtration map \( \phi : ZR \to Z(R \cap F') \), it follows that

\[
R \cap (1 + A) \subset (R \cap F') \cap (1 + \Delta^3(R \cap F') + \Delta(R')\Delta(R \cap F') + \Delta([R, R \cap F']) + \Delta(\gamma_4(R)))
\]

\[
\subset (R \cap F') \cap (1 + \Delta^3(R \cap F') + \Delta(R')\Delta(R \cap F') + \Delta([R, R \cap F']) + \Delta(\gamma_4(R)))
\]

\[
\subset [R, R \cap F', R \cap F'] \gamma_4(R)
\]

where last equality follows by Theorem 2.4 and second last equality follows by Lemma 2.3.

Proposition 3.3 \( R \cap (1 + B) = \gamma_5(R)[R, R \cap F', R \cap F', R \cap F'][[R, R \cap F'], [R, R \cap F']]. \)

**Proof.** As in the above proposition, it is sufficient to prove that if \( w \in R \) is such that \( w - 1 \in B \), then

\[
w \equiv 1 \pmod{\gamma_5(R)[R, R \cap F', R \cap F', R \cap F'][[R, R \cap F'], [R, R \cap F']].}
\]

Using Theorem 2.1 repeatedly, we have

\[
\Delta^5(R) + \Delta^3(R)\Delta(R \cap F') + \Delta^2(R)\Delta([R, R \cap F'])
\]

\[
= \Delta(R)\Delta(\gamma_4(R)) + \Delta(\gamma_5(R)) + L^{(5)} + \Delta(R)\Delta^3(R \cap F') + L^{(2)}\Delta^2(R \cap F') + \Delta(R)\Delta(R' \cap F') + \Delta(\gamma_3(R))\Delta(R \cap F') + L^{(3)}\Delta(R \cap F') + \Delta(R)\Delta(R \cap F') + \Delta([R, R \cap F']) + \Delta(R')\Delta([R, R \cap F']) + L^{(2)}\Delta([R, R \cap F']).
\]

3
Applying filtration map \( \phi : ZR \to Z(R \cap F') \), we have

\[
R \cap (1 + \Delta^3(R) + \Delta^2(R) \Delta(R \cap F') + \Delta^2(R) \Delta([R, R \cap F'])) = (R \cap F') \cap (1 + \Delta^4(R \cap F') + \Delta(R') \Delta^2(R \cap F') + \Delta(\gamma_3(R)) \Delta(R \cap F') + \Delta^2(R \cap F') \Delta([R, R \cap F'])) + \Delta(R') \Delta([R, R \cap F']) \gamma_5(R)
\]

Now since \( R \cap F' / R' \) is free-abelian, a use of similar arguments with left replaced by right and the left \( ZR' \)-homomorphism \( \phi : Z(R \cap F') \to ZR' \) implies that

\[
(R \cap F') \cap (1 + \Delta^4(R \cap F') + \Delta(R') \Delta^2(R \cap F') + \Delta(\gamma_3(R)) \Delta(R \cap F')) + \Delta(R') \Delta([R, R \cap F']) \gamma_5(R) = \gamma_5(R)[R, R \cap F', R \cap F', R \cap F']([R, R \cap F'], [R, R \cap F']).
\]

**Proof of Theorem A:** From [6], it follows that

\[
\Delta^3(F) \cap \Delta^2(R) = \Delta^3(R) + \Delta(R) \Delta(R \cap F') + \Delta(R) \Delta([R, R \cap F']),
\]

and since \( \Delta(R)ZF \) is a free right \( ZF \)-module [3, Proposition I.1.12], we have

\[
\Delta^m(R) \Delta^3(F) \cap \Delta^{m+2}(R) = \Delta^{m+3}(R) + \Delta^{m+1}(R) \Delta(R \cap F') + \Delta^m(R) \Delta([R, R \cap F'])
\]

for all \( m \geq 0 \). The natural homomorphism

\[
\eta : \Delta^{m+2}(R) \to \Delta^m(R) \Delta^2(F) / \Delta^m(R) \Delta^3(F)
\]

has \( ker \phi = \Delta^{m+3}(R) + \Delta^{m+1}(R) \Delta(R \cap F') + \Delta^m(R) \Delta([R, R \cap F']) \) in view of the above intersection. Thus \( \Delta^{m+2}(R) / \Delta^{m+3}(R) + \Delta^{m+1}(R) \Delta(R \cap F') + \Delta^m(R) \Delta([R, R \cap F']) \) is free-abelian. Again, the homomorphism

\[
\theta : \gamma_{m+2}(R) \to \Delta^{m+2}(R) / \Delta^{m+3}(R) + \Delta^{m+1}(R) \Delta(R \cap F') + \Delta^m(R) \Delta([R, R \cap F'])
\]

defined as \( x \to (x - 1) \), \( x \in \gamma_{m+2}(R) \) has \( ker \theta \) equal to

\[
\gamma_{m+2}(R) \cap (1 + \Delta^{m+3}(R) + \Delta^{m+1}(R) \Delta(R \cap F') + \Delta^m(R) \Delta([R, R \cap F'])).
\]

Therefore

\[
\gamma_{m+2}(R) \cap (1 + \Delta^{m+3}(R) + \Delta^{m+1}(R) \Delta(R \cap F') + \Delta^m(R) \Delta([R, R \cap F'])),
\]

is free-abelian for all \( m \geq 0 \). The proof now follows by putting \( m = 0, 1, 2 \) in the above group and using Propositions 3.1, 3.2, and 3.3 respectively.
References

[1] Gruenberg K.W., Cohomological Topics in Group Theory, Lecture Notes in Mathematics, Springer, Berlin, 143 (1970).

[2] D.K. Gumber, R. Karan and I. Pal, Some augmentation quotients of integral group rings, Proc. Indian Acad. Sci. (Math. Sci.) 118 (2010), 537-546.

[3] N. Gupta, Free group rings, Contemporary Math., Amer. Math. Soc. 66 (1987).

[4] T. Hurley and S. Sehgal, Groups related to fox subgroups, Comm. Algebra 28 (2000) 1051-1059.

[5] R. Karan and D. Kumar, Augmentation quotients of free group rings, Algebra Colloq. 12 (2005) 597-606.

[6] R. Karan, D. Kumar and L.R. Vermani, Some intersection theorems and subgroups determined by certain ideals in integral group rings-II, Algebra Colloq. 9 (2002), 135-142.

[7] L.R. Vermani, Augmentation quotients of integral group rings, Groups-Koria’94 (Pusan), de Gruyter, Berlin (1995) 303-15.

[8] L.R. Vermani, A. Razdan and R. Karan, Some remarks on subgroups determined by certain ideals in integral group rings, Proc. Indian Acad. Sci. (Math. Sci.) 103 (1993), 249-256.

[9] K.I. Tahara, L.R. Vermani and Atul Razdan, On generalized third dimension subgroups, Canad. Math. Bull. 41 (1998), 109-117.