Semigroup identities of supertropical matrices

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Abstract
We prove that, for any $n$, the monoid of all $n \times n$ supertropical matrices extending tropical matrices satisfies nontrivial semigroup identities; in particular, such supertropical triangular matrices admit exactly the same identities satisfied by tropical triangular matrices. These semigroup identities are carried over to labeled-weighted digraphs with double arcs.

Keywords
Tropical (max-plus) matrices · Supertropical matrices · Idempotent semirings · Semigroup identities · Semigroup representations · Weighted digraphs

1 Introduction
The tropical (max-plus) semiring is the set $\mathbb{T} := \mathbb{R} \cup \{-\infty\}$ equipped with the operations of maximum and summation

$$a \vee b := \max\{a, b\}, \quad a + b := \text{sum}\{a, b\},$$

serving respectively as addition and multiplication. It extends to the supertropical semiring $\mathbb{ST}$, defined over the disjoint union $\mathbb{ST} := \mathbb{R} \cup \{-\infty\} \cup \mathbb{R}^\nu$ of two copies of $\mathbb{R}$, together with $\emptyset := -\infty$, see [8, 14, 22]. The members of $\mathbb{R}^\nu$ are denoted by $a^\nu := \nu(a)$, for $a \in \mathbb{R}$, where $\nu$ is the projection $\nu : \mathbb{ST} \rightarrow \mathbb{R}^\nu \cup \{\emptyset\}$, i.e., $\nu(a^\nu) = a^\nu$ for all every $a^\nu \in \mathbb{R}^\nu \cup \{\emptyset\}$. The semiring $\mathbb{ST}$ has the total ordering

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\[ 0 < a < a'^\nu < b < b'^\nu \] for any \( a < b \) in \( \mathbb{R} \), which determines the addition

\[
x \lor y := \begin{cases} 
\max\{x, y\}, & \text{if } \nu(x) \neq \nu(y), \\
\nu(x), & \text{if } \nu(x) = \nu(y), 
\end{cases}
\]

where \( \max\{x, y\} \) is taken with respect to the ordering of \( \mathbb{ST} \) and \( \emptyset \) is the zero element. The multiplication of \( \mathbb{ST} \) is given by

\[
x \cdot y := \begin{cases} 
s\{x, y\}, & \text{if } x, y \in \mathbb{R}, \\
\sum\{\nu(x), \nu(y)\}, & \text{otherwise},
\end{cases}
\]

where the identity element is \( 1 := 0 \). In this extension \( x \lor x = \nu(x) \) for every \( x \in \mathbb{ST} \), and thus \( \mathbb{ST} \) is not additively idempotent like \( \mathbb{T} \), i.e., \( x \lor x = x \) for every \( x \). With this arithmetic \( \mathbb{T}^\nu := \mathbb{R}^\nu \cup \{0\} \) is a semiring ideal, isomorphic to \( \mathbb{T} \). Elements \( x, y \in \mathbb{ST} \) are \( \nu \)-equivalent, written \( x \cong_\nu y \), iff \( \nu(x) = \nu(y) \).

All \( n \times n \) matrices over any semiring \( R \), and in particular over \( \mathbb{T} \) and \( \mathbb{ST} \), form a multiplicative monoid \( \mathcal{M}_n(R) \), whose multiplication is induced from the operations of \( R \) in the familiar way. The (upper) triangular matrices form a submonoid of \( \mathcal{M}_n(R) \), denoted \( \mathcal{U}_n(R) \).

In [10, 11], and independently in [29], it was proved that \( \mathcal{U}_n(\mathbb{T}) \) admits nontrivial semigroup identities, which have been also studied in [2, 3, 27]. Later, following several partial results [4, 13, 16, 31], in [17, Theorem 3.7] we proved that the matrix monoid \( \mathcal{M}_n(\mathbb{T}) \) also admits nontrivial semigroup identities. In this note we extend these results to supertropical matrices in \( \mathcal{M}_n(\mathbb{ST}) \).

**Corollary 4.4.** The monoid \( \mathcal{U}_n(\mathbb{ST}) \) satisfies the same semigroup identities as \( \mathcal{U}_n(\mathbb{T}) \), for every \( n \in \mathbb{N} \).

**Corollary 4.7.** The monoid \( \mathcal{M}_n(\mathbb{ST}) \) satisfies a nontrivial semigroup identity, for every \( n \in \mathbb{N} \).

An explicit inductive construction of semigroup identities admitted by \( \mathcal{M}_n(\mathbb{T}) \) has been provided in [17], essentially relying on identities of nonsingular matrices [10]. Theorem 4.6 proves that semigroup identities for \( \mathcal{M}_n(\mathbb{ST}) \) can be composed from semigroup identities of \( \mathcal{M}_n(\mathbb{T}) \).

Obviously, since \( \mathcal{M}_n(\mathbb{ST}) \) contains a copy of \( \mathcal{M}_n(\mathbb{T}) \), all the identities satisfied by \( \mathcal{M}_n(\mathbb{ST}) \) are also satisfied by \( \mathcal{M}_n(\mathbb{T}) \). Computations with SageMath have shown that, for \( \mathcal{M}_2(\mathbb{T}) \) shorter nontrivial identities than those in [16] are not identities for \( \mathcal{M}_2(\mathbb{ST}) \).

The results in this note are a further step in the study of supertropical matrices [8, 18, 22–26] and have immediate consequences in representations of semigroups and topological-combinatorial objects. These matrices pave the way for a new type of linear representations of semigroups, enhancing former representations [12], and establish a tool to extract semigroup identities.

**Corollary 4.8.** Any semigroup that is faithfully represented in \( \mathcal{M}_n(\mathbb{ST}) \) satisfies a nontrivial semigroup identity.
While tropical matrices correspond uniquely to weighted digraphs (see e.g. [1, 7]), which play a central role in many pure and applied subjects of study, supertropical matrices correspond to weighted digraphs with possible double arcs (e.g. quivers). By this correspondence, we conclude that the semigroup identities of $M_n(\mathbb{ST})$ are carried over to walks on labeled-weighted digraphs with double arcs (Corollary 4.9). Besides digraphs, supertropical matrices provide a framework for representations of additional, more complicated combinatorial objects, such as matroids, lattices, and simplicial complexes [19, 20].

2 Preliminaries

We begin with a brief relevant background on semigroup identities, tropical matrices, and tropical polynomials.

2.1 Semigroup identities

The free monoid $\mathcal{A}^*$ of finite words is the monoid generated by a finite alphabet $\mathcal{A}$—a finite set of letters—whose identity element is the empty word. The length $\ell(w)$ of a word $w$ is the number of its letters. Excluding the empty word from $\mathcal{A}^*$, the free semigroup $\mathcal{A}^+$ is obtained.

A (nontrivial) semigroup identity is a formal equality $u = v$, written as a pair $\langle u, v \rangle$, of words $u \neq v$ in $\mathcal{A}^+$, cf. [30]. Allowing $u$ and $v$ to be the empty word as well, i.e., $u, v \in \mathcal{A}^*$, a monoid identity is received. An identity $\langle u, v \rangle$ is an $n$-letter identity, if $u$ and $v$ together involve at most $n$ different letters from $\mathcal{A}$. The length of $\langle u, v \rangle$ is defined to be $\max\{\ell(u), \ell(v)\}$.

A semigroup $\mathcal{S} := (\mathcal{S}, \cdot)$ satisfies a semigroup identity $\langle u, v \rangle$, if

$$\phi(u) = \phi(v) \text{ for every semigroup homomorphism } \phi : \mathcal{A}^+ \longrightarrow \mathcal{S}.$$  \hspace{0.5cm} (2.1)

Id($\mathcal{S}$) denotes the set of all semigroup identities satisfied by $\mathcal{S}$.

Theorem 2.1 ([10, Theorem 3.10]) A semigroup that satisfies an $n$-letter identity, where $n \geq 2$, also satisfies a 2-letter identity of the same length.

Therefore, in this view, regarding the existence of semigroup identities, we assume that $\mathcal{A} = \{a, b\}$.

Notation 2.2 ([17, Notation 1.2]) Given a word $w \in \mathcal{A}^+$ and elements $s, t \in \mathcal{S}$, we write $w[s, t]$ for the evaluation of $w$ in $\mathcal{S}$, obtained by substituting $a \mapsto s$, $b \mapsto t$, where $s, t \in \mathcal{S}$. Similarly, we write $\langle u, v \rangle[s, t]$ for the pair of evaluations $u[s, t]$ and $v[s, t]$ in $\mathcal{S}$ of the words $u$ and $v$.

In the certain case that $\mathcal{S} = \mathcal{A}^+$, to indicate that for $u, v \in \mathcal{A}^+$ the evaluation $w[u, v]$ is again a word in $\mathcal{A}^+$, we use the particular notation $w[u, v]$. Similarly, we write $\langle u, v \rangle[s, t]$ for $\langle u, v \rangle[s, t]$. 

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With these notations, condition (2.1) can be rephrased as

\[
\langle u, v \rangle \in \text{Id}(\mathcal{S}) \text{ iff } u[s, t] = v[w_1, w_2] \text{ for every } s, t \in \mathcal{S}.
\]

Clearly, \(\langle u, v \rangle \in \text{Id}(\mathcal{S})\) implies \(\langle u, v \rangle [w_1, w_2] \in \text{Id}(\mathcal{S})\) for any \(w_1, w_2 \in \mathcal{S}^+\).

For the monoid \(\mathcal{S} = M_n(T)\) of all \(n \times n\) tropical matrices, utilizing the machinery of weak CSR expansions [28], we have proved:

**Theorem 2.3** ([17, Theorem 3.7]) The monoid \(\mathcal{M}_n(\mathbb{T})\) satisfies a nontrivial semigroup identity for every \(n \in \mathbb{N}\).

In addition, [17] provides an inductive construction of semigroup identities satisfied by \(\mathcal{M}_n(\mathbb{T})\) which relies on identities of nonsingular tropical matrices [10].

### 2.2 Matrix semigroups

To deal with triangular matrices and general matrices at the same time, we use the formalism of [3]. Let \((\Gamma, \varrho)\) be a finite set \(\Gamma\) equipped with a reflexive-transitive binary relation \(\varrho\). Let \(S \subseteq R\) be a subset of a semiring \(R\), and let \(S = S \cup \{0_R\}\). (In this note \(R\) is taken to be \(T\) or \(ST\), where \(0_R = -\infty\).) Write \(\Gamma(S)\) for the set of all functions \(F : \Gamma \times \Gamma \rightarrow S\), viewed as matrices whose rows and columns are indexed by \(\Gamma\), and \(F_{i,j}\) for the image of \((i, j)\). Define

\[
\Gamma(S) := \begin{cases} 
\{A \in S^{\Gamma \times \Gamma} \mid A_{i,j} \neq 0_R \Rightarrow i \varrho j\}, & \text{if } 0_R \in S, \\
\{A \in S^{\Gamma \times \Gamma} \mid A_{i,j} \neq 0_R \Leftrightarrow i \varrho j\}, & \text{if } 0_R \notin S.
\end{cases}
\]

When \(S\) is a semiring, similar to the case of matrices, a multiplication on \(\Gamma(S)\) is defined by:

\[
(F \cdot G)_{i,j} = \bigvee_{k \in \Gamma} F_{i,k} G_{k,j}.
\]

The transitivity of \(\varrho\) ensures that appropriate entries of the product are \(0_R\). This operation is clearly associative, and thus endows \(\Gamma(S)\) with a structure of semigroup. If \(\Gamma = \{1, \ldots, n\}\) and \(\varrho\) is the complete binary relation (resp. the standard linear order), then \(\Gamma(S) = \mathcal{M}_n(S)\) (resp. \(\Gamma(S) = \mathcal{U}_n(S)\)). In the sequel, \(n\) denotes the size of \(\Gamma\), whose elements are labeled by \(1, \ldots, n\).

### 2.3 Tropical polynomials

The semiring \(\mathbb{T}[\lambda_1, \ldots, \lambda_m]\) consists of polynomials in \(m\) variables \(\lambda_1, \ldots, \lambda_m\) of the form

\[
f = \bigvee_{I \in \Omega} \alpha_I \lambda_1^{i_1} \cdots \lambda_m^{i_m},
\]
where $\Omega \subset \mathbb{N}^m$ is a finite nonempty set of multi-indices $i = (i_1, \ldots, i_m)$ for which $\alpha_i \neq 0$. A polynomial $f$ is say to be flat, if all $\alpha_i = \beta_i$ for some fixed $\beta_i \in \mathbb{T}$. By substitution, each polynomial $f \in \mathbb{T}[\lambda_1, \ldots, \lambda_m]$ determines a function

$$\tilde{f} : T^m \to \mathbb{T}, \quad (a_1, \ldots, a_m) \mapsto \tilde{f}(a_1, \ldots, a_m) := \bigvee_{i \in \Omega} a^i_1 \cdots a^i_m.$$  

It is well known that the map $f \mapsto \tilde{f}$ is not injective, and that $f$ can be reduced to have only those monomials needed to describe $\tilde{f}$.

Writing a polynomial $f = \bigvee_{i \in \Omega} f_i$ as a sum of monomials $f_i$, we say that a monomial $f_j$ is inessential, if $\tilde{f_j}(a) \leq \tilde{f}(a)$ for every $a = (a_1, \ldots, a_m) \in \mathbb{R}^m$, cf. [8]. A monomial $f_j$ is essential, if $\tilde{f_j}(a) > \bigvee_{i \neq j} \tilde{f_i}(a)$ for some $a \in \mathbb{R}^m$, and is quasi-essential, if it is inessential but $\tilde{f_j}(a) = \bigvee_{i \neq j} \tilde{f_i}(a)$ for some $a \in \mathbb{R}^m$. The essential part $f^{es}$ of $f$ consists of all those monomials that are essential for $f$. The polynomial $f^{es}$ is unique [16, Lemma 1.6], with $\tilde{f} = \tilde{f}^{es}$ [16, Proposition 1.5]. Therefore, for any $f, g \in \mathbb{T}[\lambda_1, \ldots, \lambda_m]$, $\tilde{f} = \tilde{g}$ iff $f^{es} = g^{es}$.

For each monomial $f_i$ of a polynomial $f$ of the form (2.2) there is the degree map

$$\delta : f_i \mapsto \mathbb{N}^m, \quad \alpha_i \lambda_1^{i_1} \cdots \lambda_m^{i_m} \mapsto (i_1, \ldots, i_m).$$

(2.3)

The Newton polytope $\Delta(f)$ of a polynomial $f = \bigvee_{i \in \Omega} a^i_1 \cdots a^i_m$ is the convex hull in $\mathbb{R}^m$ of the set

$$\{\delta(f_i) \mid f_i \text{ is a monomial of } f\},$$

i.e., the convex hull of the set of multi-indices $\Omega \subset \mathbb{N}^m$. This lattice polytope $\Delta(f)$ has a subdivision

$$\Delta_{div}(f) : \Delta(f) = \Delta_1 \cup \cdots \cup \Delta_\ell$$  

(2.4)

into disjoint lattice polytopes $\Delta_1, \ldots, \Delta_\ell$, determined by projecting the upper part of the convex hull of the points $(i_1, \ldots, i_m, \alpha_i) \in \mathbb{R}^{m+1}$ onto $\Delta(f) \subset \mathbb{R}^m$.

The subdivision of $\Delta(f)$ yields a duality between faces of $\Delta_{div}(f)$ and faces of the tropical hypersurface determined by $\tilde{f}$ (cf. [5]), and consequently a one-to-one correspondence between essential monomials of $f$ and vertices of $\Delta_{div}(f)$, i.e., vertices of the $\Delta_i$’s in (2.4). This gives the condition that a monomial $f_i$ is essential for $f$ iff $\delta(f_i)$ is a vertex of $\Delta_{div}(f)$, and furthermore $\tilde{f} = \tilde{g}$ iff $\Delta_{div}(f) = \Delta_{div}(g)$, for any $f, g \in \mathbb{T}[\lambda_1, \ldots, \lambda_m]$. 

A quasi-essential monomial $f_i$ of $f$ corresponds to a lattice point of $\Delta_{div}(f)$ which is not a vertex, implying that $f$ has at least two essential monomials $f_j$ and $f_k$ for which $\tilde{f_j}(a) = \tilde{f_k}(a) = \tilde{f}(a)$ for some $a \in \mathbb{R}^m$. When $f$ is a flat polynomial, $\Delta(f) = \Delta_{div}(f)$ and $\Delta_{div}(f)$ has no internal vertices. Therefore, in this case, the essential monomials of $f$ correspond only to vertices of $\Delta(f)$.

Let $\lambda = (\Lambda_{i,j})$ and $\Sigma = (\Sigma_{i,j})$ be two (square) matrices of the same size as those in $\Gamma(\mathbb{T})$ whose entries are variables $\Lambda_{i,j}$ and $\Sigma_{i,j}$, if $i \neq j$, and $-\infty$ otherwise. Let
\((u, v) \in \text{Id}(\Gamma(\mathbb{T}))\) be an identity of \(\Gamma(\mathbb{T})\), i.e., \(u[A, B] = v[A, B]\) for any \(A, B \in \Gamma(\mathbb{T})\). This means that \(u[\Lambda, \Sigma]\) and \(v[\Lambda, \Sigma]\) define the same function \(\Gamma(\mathbb{T})^2 \rightarrow \Gamma(\mathbb{T})\), which restricts to at most \(n^2\) entry-functions \(\Gamma(\mathbb{T})^2 \rightarrow \mathbb{T}\). Namely, substituting \(\Lambda\) and \(\Sigma\) into \(u\) and \(v\), each entry \(u[\Lambda, \Sigma]_{i,j}\) and \(v[\Lambda, \Sigma]_{i,j}\) is a polynomial in at most \(2n^2\) variables for which \(u[\Lambda, \Sigma]_{i,j} = v[\Lambda, \Sigma]_{i,j}\), and thus

\[
\Delta_{\text{div}}(u[\Lambda, \Sigma]_{i,j}) = \Delta_{\text{div}}(v[\Lambda, \Sigma]_{i,j}). \tag{2.5}
\]

These polynomials, which we denote by

\[
f^u_{i,j} := u[\Lambda, \Sigma]_{i,j}, \quad f^v_{i,j} := v[\Lambda, \Sigma]_{i,j},
\]

are flat polynomials of the form

\[
\bigvee_{i \in \Omega} \Lambda_{i_1}^{\ell_{i_1}} \cdots \Lambda_{i_n}^{\ell_{i_n}} \Sigma_{1_1}^{m_{1_1}} \cdots \Sigma_{n_n}^{m_{n_n}}, \quad i = (\ell_{i_1}, \ldots, \ell_{i_n}, m_{1_1}, \ldots, m_{n_n}) \in \mathbb{N}^{2n^2}. \tag{2.6}
\]

Therefore, (2.5) reads as

\[
\Delta(f^u_{i,j}) = \Delta(f^v_{i,j}).
\]

Note that \(f^u_{i,j}\) and \(f^v_{i,j}\) may have quasi-essential monomials. We write \(\widetilde{f}^u_{i,j}(A, B)\) for the evaluation of the function \(f^u_{i,j} : \Gamma(\mathbb{T})^2 \rightarrow \mathbb{T}\) at \(A = (A_{k,i}), B = (B_{k,i})\).

### 3 Tropical matrices versus digraphs

An \(n \times n\) matrix \(A = (A_{i,j})\) over any semiring \(R\) is associated uniquely to the **weighted digraph** \(G(A) := (\mathcal{Y}, \varepsilon^\circ)\) over the **node set** \(\mathcal{Y} := \{1, \ldots, n\}\) with a **directed arc** \(\varepsilon_{i,j} := (i, j) \in \varepsilon^\circ\) of **weight** \(A_{i,j}\) from \(i\) to \(j\) for every \(A_{i,j} \neq 0_R\). \(G(A)\) is called the digraph of the matrix \(A\), and conversely \(B \in \mathcal{M}_n(R)\) is said to be the matrix of the weighted digraph \(G'\), if \(G' = G(B)\). An arc \(\varepsilon_{i,j} \in \varepsilon^\circ\) is called a **loop**.

A **walk** \(\gamma\) on \(G(A)\) is a sequence of arcs \(\varepsilon_{i_1,j_1}, \ldots, \varepsilon_{i_m,j_m}\), with \(j_k = i_{k+1}\) for every \(k = 1, \ldots, m - 1\). We write \(\gamma := \gamma_{i,j}\) to indicate that \(\gamma\) is a walk from \(i = i_1\) to \(j = j_m\). The **length** \(\ell(\gamma)\) of a walk \(\gamma\) is the number of its arcs. The **weight** \(\omega(\gamma)\) of \(\gamma\) is the product of the weights of its arcs, counting repeated arcs, taken with respect to the multiplicative operation of the semiring \(R\).

**Definition 3.1** ([17, Definition 1.7]) The **labeled-weighted digraph** \(G(A, B)\), written **lw-digraph**, of matrices \(A, B \in \mathcal{M}_n(R)\) is the digraph over the nodes \(\mathcal{Y} := \{1, \ldots, n\}\) with a directed arc \(\varepsilon_{i,j}\) from \(i\) to \(j\) labeled a of weight \(A_{i,j}\) for every \(A_{i,j} \neq 0_R\) and a directed arc from \(i\) to \(j\) labeled \(b\) of weight \(B_{i,j}\) for every \(B_{i,j} \neq 0_R\). A walk \(\gamma = \varepsilon_{i_1,j_1}, \ldots, \varepsilon_{i_m,j_m}\) on \(G(A, B)\) is labeled by the sequence of arcs’ labels along \(\gamma\), from \(\varepsilon_{i_1,j_1}\) to \(\varepsilon_{i_m,j_m}\), which is a word \(w\) in \(\{a, b\}^\ast\). A walk labeled \(w\) is often denoted by \(\gamma_w\). (In particular, every walk \(\gamma_w\) has length \(\ell(w)\).) The weight \(\omega(\gamma)\) of \(\gamma\) is the product of its arcs’ weights with respect to the multiplicative operation of the semiring \(R\).
The lw-digraph $G(A, B)$ may have parallel arcs, but with different labels. Note that, when $A = B$, each arc $\varepsilon_{i, j}$ of $G(A, A)$ is doubled with the same label $a$ and the same weight $A_{i, j}$, so, when considering labeled-weighted walks in $G(A, A)$, we often identify $G(A, A)$ with the digraph $G(A)$. With this definition, for tropical matrices, we have the following proposition.

**Proposition 3.2** Given a word $w \in \{a, b\}^+$ and matrices $A, B \in M_n(\mathbb{T})$, the $(i, j)$-entry of the matrix $w\llbracket A, B \rrbracket$ is the maximum over the weights of all walks $\gamma_{i, j} = \gamma_w$ on $G(A, B)$ from $i$ to $j$ labeled by $w$.

Let $\gamma$ and $\gamma'$ be walks on an lw-digraph $G(A, B)$ of $A, B \in M_n(R)$. We define $\mathcal{E}(\gamma) = (\mathfrak{A}^\gamma, \mathfrak{B}^\gamma)$ to be the pair of matrices $\mathfrak{A}^\gamma = (\mathfrak{A}^\gamma_{i, j})$ and $\mathfrak{B}^\gamma = (\mathfrak{B}^\gamma_{i, j})$ in $M_n(\mathbb{N})$ whose $(i, j)$-entry is respectively the number of occurrences, called **multiplicity**, of the arc $\varepsilon_{i, j}$ labeled $a$ and $b$ in $\gamma$. The multiplicity of an arc is 0, if it is not included in $\gamma$, or it does not exist in $G(A, B)$. We call $\mathcal{E}(\gamma)$ the **configuration** of $\gamma$, and write $\gamma \equiv_{\mathcal{E}} \gamma'$, if $\mathcal{E}(\gamma') = \mathcal{E}(\gamma)$. Namely, if $\gamma \equiv_{\mathcal{E}} \gamma'$, then $\gamma$ and $\gamma'$ consist exactly of the same arcs with the same multiplicities, but not necessarily with the same ordering. Then, the equivalence $\gamma \equiv_{\mathcal{E}} \gamma'$ implies that $\omega(\gamma) = \omega(\gamma')$ and $\ell(\gamma) = \ell(\gamma')$. To simplify notations, we sometimes identify $\mathcal{E}(\gamma)$ with the point

$$\mathcal{E}(\gamma) := (\mathfrak{A}^\gamma_{1, 1}, \ldots, \mathfrak{A}^\gamma_{n, n}, \mathfrak{B}^\gamma_{1, 1}, \ldots, \mathfrak{B}^\gamma_{n, n}) \in \mathbb{N}^{2n^2}.$$  

We next elaborate on the result of Theorem 2.3, especially enhancing the interplay between matrix identities, lw-diagraphs, and polynomials.

**Remark 3.3** Let $A, B \in M_n(\mathbb{T})$. Given a word $w \in \{a, b\}^+$, let $\gamma_w$ be a walk on $G(A, B)$ from $i$ to $j$ labeled by $w$. Let $f := f_{i, j}^w$ be the polynomial $w\llbracket A, \Sigma \rrbracket_{i, j}$ of the form (2.6). The point $\mathcal{E}(\gamma_w) \in \mathbb{N}^{2n^2}$ is the image $\delta(f)$ of a monomial $f_i$ of $f$ for which $\omega(\gamma_w) = f_i(A, B)$. Thereby a correspondence between labeled walks $\gamma_w$ on $G(A, B)$ from $i$ to $j$ and monomials $f_i$ of $f$ is obtained. (Different walks may correspond to a same monomial.) We denote by $\chi(\gamma_w)$ the monomial corresponding to a walk $\gamma_w$, and thus have $\delta(\chi(\gamma_w)) = \mathcal{E}(\gamma_w)$, cf. (2.3). Hence, if $\chi(\gamma_w') \neq \chi(\gamma_w'')$, then $\mathcal{E}(\gamma_w') \neq \mathcal{E}(\gamma_w'')$.

If $\gamma_w$ is a walk of highest weight, then the corresponding monomial $f_i = \chi(\gamma_w)$ is either essential or quasi-essential. The latter case implies that there are at least two other walks $\gamma_w', \gamma_w''$ of weight $\omega(\gamma_w)$ from $i$ to $j$ labeled $w$, since $f_i$ corresponds to a lattice point of $\Delta(f)$ which is not a vertex, cf. §2.3.

Informally, fixing a word $w$, we have a correspondence $\chi$ between walks and monomials, given by associating an arc $\varepsilon_{i, j}$ labeled $a$ to the variable $A_{i, j}$ and an arc $\varepsilon_{i, j}$ labeled $b$ to the variable $B_{i, j}$. The powers of these variables in a monomial are the arcs’ multiplicities, encoded by the configuration of a corresponding walk.

Combining the above graph view, applied to tropical matrices, and the polynomial view from §2.3, we have the following result.

**Lemma 3.4** Let $(u, v) \in \text{Id}(\Gamma(\mathbb{T}))$ be an identity for $\Gamma(\mathbb{T})$, and let $G(A, B)$ be the lw-digraph of $A, B \in \Gamma(\mathbb{T})$. Fix $i$ and $j$, and let $\gamma_u$ be a walk labeled $u$ of highest weight from $i$ to $j$ on $G(A, B)$. Let $f := f_{i, j}^u = u\llbracket A, \Sigma \rrbracket_{i, j}$ and $g := f_{i, j}^v = v\llbracket A, \Sigma \rrbracket_{i, j}$.

\[ \square \] Springer
(i) If \( \delta(\chi(\gamma_u)) \) is a vertex of \( \Delta(f) \), then there exists a walk \( \gamma_v \cong \gamma_u \) from \( i \) to \( j \) labeled \( v \) of weight \( \omega(\gamma_u) \) on \( G(A, B) \).

(ii) If \( \delta(\chi(\gamma_u)) \) is not a vertex of \( \Delta(f) \), then there are at least two walks \( \gamma'_v, \gamma''_v \not\cong \gamma_u \) from \( i \) to \( j \) labeled \( v \) of weight \( \omega(\gamma_u) \).

(iii) If there exists another walk \( \gamma'_v \not\cong \gamma_u \) labeled \( u \) with \( \omega(\gamma_u) = \omega(\gamma_u') \), then there are walks \( \gamma'_v \not\cong \gamma_v \) labeled \( v \) such that \( \omega(\gamma_v) = \omega(\gamma'_v) = \omega(\gamma_u) \).

**Proof** Clearly, since \( (u[A, B], i, j) = (v[A, B], i, j) \), there exists a walk \( \gamma_v \) labeled \( v \) from \( i \) to \( j \) for which \( \omega(\gamma_u) = \omega(\gamma_v) \). Since \( (u, v) \in \text{Id}(\Gamma(T)) \), \( f = u[A, \Sigma]i, j \), and \( g = v[A, \Sigma]i, j \), we have \( \tilde{f} = \tilde{g} \) for the functions \( \tilde{f}, \tilde{g} \); in particular, \( \tilde{f}(A, B) = \tilde{g}(A, B) \), and thus \( \Delta(f) = \Delta(g) \), cf. §2.3. Since \( \gamma_u \) is of highest weight, and therefore \( \gamma_v \) is also of highest weight, there exist quasi-essential monomials \( f_i, g_i \) of \( f, g \) such that
\[
\tilde{f}_i(A, B) = (\chi(\gamma_u))(A, B) = \omega(\gamma_u) = (\chi(\gamma_v))(A, B) = \tilde{g}_i(A, B).
\]

Let \( p = \delta(f_i) \) and \( q = \delta(g_i) \) be the lattice points corresponding to these monomials in \( \Delta(f) \) and \( \Delta(g) \). That is \( p = \mathcal{C}(\gamma_u) = \delta(\chi(\gamma_u)) \) and \( q = \mathcal{C}(\gamma_v) = \delta(\chi(\gamma_v)) \). Recall that \( \Delta(f) = \Delta(g) \).

(i): If \( p \) is a vertex of \( \Delta(f) \), then \( p \) is also a vertex of \( \Delta(g) \). So, there is a walk \( \gamma_v \) for which \( p = q \), and thus \( \mathcal{C}(\gamma_u) = \mathcal{C}(\gamma_v) \). Hence \( \gamma_v \cong \gamma_u \).

(ii): If \( p \) is not a vertex of \( \Delta(f) \), then \( f_i \) is a quasi-essential monomial of \( f \), cf. §2.3, and \( f \) has at least two essential monomials \( f_j \) and \( f_k \) corresponding to vertices of \( \Delta(f) \) for which \( \tilde{f}(A, B) = \tilde{f}_j(A, B) = \tilde{f}_k(A, B) = f_j(A, B) \). Each of these monomials is associated to a walk labeled \( u \) of weight \( \omega(\gamma_u) \). Thus, by part (i), there are two walks \( \gamma'_v, \gamma''_v \) labeled \( v \) with \( \omega(\gamma'_v) = \omega(\gamma''_v) = \omega(\gamma_u) \).

(iii): If there are two walks \( \gamma_u \not\cong \gamma'_v \) labeled \( u \) of highest weight, then each walk is associated to a different monomial of \( f \). These monomials correspond to different lattice points \( p \) and \( p' \) in \( \Delta(f) \). If both \( p \) and \( p' \) are vertices of \( \Delta(f) \), then we are done by part (i). Otherwise, one of these lattice points is not a vertex of \( \Delta(f) \), and the proof follows from part (ii).

In the case that the binary relation \( \varrho \) is a partial order, we have the following property.

**Lemma 3.5** If \( \varrho \) is a partial order, for any given configuration there is at most one walk having a finite weight.

**Proof** A walk \( \gamma_w \) on \( G(A, B) \) with a finite weight follows the ordering \( \varrho \). Each non-loop arc in \( \gamma_w \) is unique and, therefore, \( \gamma_w \) is completely determined by the positions of its non-loop arcs and the multiplicities of its loops. These are uniquely determined by the configuration \( \mathcal{C}(\gamma_w) \). \(\square\)

## 4 Semigroup identities of supertropical matrices

Matrices over the semiring \( \text{ST} \) are matrices with entries in \( \text{ST} \); \( \mathbb{M}_n(\text{ST}) \) denotes the monoid of all \( n \times n \) matrices over \( \text{ST} \). The map \( \nu : \text{ST} \rightarrow \mathbb{T}^n \) extends entry-wise...
Given an identity \( \nu(A) = (v(A_{i,j})) \), inducing the \( \nu \)-equivalence \( \equiv_\nu \) on matrices in \( \mathcal{M}_n(\mathbb{T}) \), i.e., \( A = (A_{i,j}) \equiv_\nu (B_{i,j}) = B \) if \( A_{i,j} \equiv_\nu B_{i,j} \) for every \( (i, j) \). The \( \nu \)-equivalence relation is a congruence on \( \mathcal{M}_n(\mathbb{T}) \), i.e., if \( A \equiv_\nu B \), then \( AC \equiv_\nu BC \) and \( CA \equiv_\nu CB \) for any \( C \in \mathcal{M}_n(\mathbb{T}) \). By the structure of \( \mathbb{T} \), for any matrix \( A \in \mathcal{M}_n(\mathbb{T}) \) there exists a unique matrix \( \hat{A} \) with entries in \( \mathbb{T} = \mathbb{R} \cup \{0\} \) such that \( \hat{A} \equiv_\nu A \).

Similar to matrices in \( \mathcal{M}_n(\mathbb{T}) \), each matrix \( A \in \mathcal{M}_n(\mathbb{ST}) \) is associated with a unique weighted digraph \( G(A) \), cf. §3, while now arcs have their weights from \( \mathbb{ST} \). The lw-digraph \( G(A, B) \) of matrices \( A, B \in \mathcal{M}_n(\mathbb{ST}) \) is as defined in Definition 3.1. A walk \( \gamma_w \) on \( G(A, B) \) has weight \( \omega(\gamma_w) \) in \( \mathbb{R}^v \), if one of its arcs has a weight in \( \mathbb{R}^v \). Otherwise, \( \gamma_w \) has weight in \( \mathbb{R} \). A walk \( \gamma_w \) from \( i \) to \( j \) is said to have highest \( \nu \)-weight, if \( \nu(\omega(\gamma_w)) \) is maximal among the weights of all walks labeled \( w \) from \( i \) to \( j \).

Each arc \( e_{i,j} \) of \( G(A, B) \) having weight in \( \mathbb{R}^v \) can be interpreted as a double arc, i.e., two parallel arcs from \( i \) to \( j \) with the same label and the same weight in \( \mathbb{R} \). \( G(A, B) \) denotes the lw-digraph taken with this interpretation.

**Lemma 4.1** Given an identity \( \langle u, v \rangle \in \text{Id}(\Gamma(\mathbb{T})) \), then \( u[uA, B] \equiv_\nu v[A, B] \) for any \( A, B \in \Gamma(\mathbb{ST}) \).

**Proof** The monoid \( \Gamma(\mathbb{T}^v) \) of all matrices over \( \mathbb{T}^v \) is isomorphic to \( \Gamma(\mathbb{T}) \), and thus \( \nu(u[uA, B]) = \nu(v[A, B]) \), which by definition means \( u[uA, B] \equiv_\nu v[A, B] \). \( \square \)

**Remark 4.2** Lemma 3.4 remains valid for matrices in \( \Gamma(\mathbb{ST}) \). Indeed, mark each arc whether its weight is in \( \mathbb{R} \) or \( \mathbb{R}^v \), apply the lemma for the images \( \nu(A) \) and \( \nu(B) \) of the matrices \( A \) and \( B \) to find the walks, then recover the walks’ weight (which could be either \( \mathbb{R} \) or in \( \mathbb{R}^v \)) from the marking of their arcs.

We are now ready to prove our first result.

**Theorem 4.3** If \( (\Gamma, \varrho) \) is a partial ordered set, then \( \Gamma(\mathbb{T}) \) and \( \Gamma(\mathbb{ST}) \) satisfy the same semigroup identities.

**Proof** Since \( \Gamma(\mathbb{T}) \) embeds in \( \Gamma(\mathbb{ST}) \), all identities satisfied by \( \Gamma(\mathbb{ST}) \) are also satisfied by \( \Gamma(\mathbb{T}) \). Conversely, we verify that every identity \( \langle u, v \rangle \in \text{Id}(\Gamma(\mathbb{T})) \) is satisfied by \( \Gamma(\mathbb{ST}) \). Let \( A, B \in \Gamma(\mathbb{ST}), U := u[uA, B], \) and \( V := v[A, B] \). By Lemma 4.1, it is enough to show that, for every \( (i, j) \), \( U_{i,j} \in \mathbb{R}^v \) implies \( V_{i,j} \in \mathbb{R}^v \). Assume that \( U_{i,j} \in \mathbb{R}^v \) for some fixed \( (i, j) \) and set \( f := f_{i,j}^u = u[A, \Sigma]_{i,j} \).

If there are two walks from \( i \) to \( j \) labeled by \( u \) of highest \( \nu \)-weight with different configurations, Lemma 3.4 (iii) and Remark 4.2 imply that there are two walks from \( i \) to \( j \) labeled by \( v \) of highest \( \nu \)-weight with different configurations; therefore \( V_{i,j} \in \mathbb{R}^v \).

Otherwise, all maximally \( \nu \)-weighted walks from \( i \) to \( j \) labeled by \( u \) have the same configuration, but Lemma 3.5 says that such a walk \( \gamma_u \) is unique, and has weight in \( \mathbb{R}^v \), since \( U_{i,j} \in \mathbb{R}^v \). Since a maximal weight is achieved on at least one vertex of \( \Delta(f) \), then \( \delta(\gamma_u) \) is a vertex of \( \Delta(f) \). From Lemma 3.4 (i) and Remark 4.2 it follows that there exists a walk \( \gamma_v \) labeled by \( v \) with the same weight, whence \( V_{i,j} \in \mathbb{R}^v \). \( \square \)

Theorem 4.3 readily implies the following.

**Corollary 4.4** The monoid \( \mathcal{U}_n(\mathbb{ST}) \) satisfies the same semigroup identities as \( \mathcal{U}_n(\mathbb{T}) \), for every \( n \in \mathbb{N} \).
The existence of identities for general matrices in \( \mathcal{M}_n(\mathbb{T}) \) follows from the next lemma.

**Lemma 4.5** Given an identity \( \langle u, v \rangle \in \text{Id}(\mathcal{M}_n(\mathbb{T})) \), then \( u[\mathbb{A}, B] = v[\mathbb{A}, B] \) for any \( A \cong_v B \) in \( \mathcal{M}_n(\mathbb{T}) \).

**Proof** Let \( U := u[\mathbb{A}, B] \) and \( V := v[\mathbb{A}, B] \). Since \( A \cong v B \), then \( U \cong v V \). Moreover, \( \hat{A} = \hat{B} \), and thus

\[
u[\hat{A}, \hat{B}] = u[\hat{A}, \hat{A}] = Z = v[\hat{A}, \hat{A}] = v[\hat{A}, \hat{B}]. (\ast)
\]

Hence, \( Z \) is a matrix in \( \mathcal{M}_n(S^n) \), obtained as the \( l(u) \) power of the matrix \( \hat{A} \) in \( \mathcal{M}_n(\mathbb{T}) \). Accordingly, an entry \( Z_{i,j} \) of \( Z \) corresponds to a walk of highest weight from \( i \) to \( j \) of length \( l(u) \) on the diagraph \( G(\hat{A}) \).

Let \( U' := u[\hat{A}, \hat{B}] \) and \( V' := v[\hat{A}, \hat{B}] \), which are equal by \((\ast)\). Fixing \((i, j)\), the entry \( U_{i,j} \) (resp. \( V_{i,j} \)) corresponds to walks \( \Phi_u \) (resp. \( \Phi_v \)) from \( i \) to \( j \) labeled \( u \) (resp. \( v \)) of highest \( v \)-weight on \( G(A, B) \). In particular, \( \omega(\gamma_u) \cong_v \omega(\gamma_v') \) for all \( \gamma_u, \gamma_v' \in \Phi_u \). The same holds for \( \Phi_v \). Note that, \( U_{i,j} \in \mathbb{T}^n \) (resp. \( V_{i,j} \in \mathbb{T}^n \)) implies \( U_{i,j} \in \mathbb{T}^n \) (resp. \( V_{i,j} \in \mathbb{T}^n \)).

We need to prove that, if \( U_{i,j} \in \mathbb{T}^n \), then also \( V_{i,j} \in \mathbb{T}^n \). If \( U_{i,j} = 0 \), then \( V_{i,j} = 0 \) by \((\ast)\), and hence \( V_{i,j} = 0 \). If both \( U_{i,j} \) and \( V_{i,j} \) are in \( \mathbb{R} \), then \( U_{i,j} = U_{i,j} = V_{i,j} = V_{i,j} \) by \((\ast)\). Otherwise, say \( U_{i,j} \) has a value in \( \mathbb{R}^n \), for which we have two cases:

(a) \( \Phi_u \) includes a walk \( \gamma_u \) having weight in \( \mathbb{R}^n \).

(b) \( \Phi_u \) includes only walks \( \gamma_u \) having weight in \( \mathbb{R} \).

Let \( f_{i,j}^u = u[\mathbb{A}, \mathbb{S}]_{i,j} \) and \( f_{i,j}^v = v[\mathbb{A}, \mathbb{S}]_{i,j} \), cf. §2.3.

Case (a): \( \omega(\gamma_u) \in \mathbb{R}^n \) implies that one of the arcs composing \( \gamma_u \), say \( \varepsilon_{s,t} \), has weight in \( \mathbb{R}^n \). If \( C(\gamma_u) \) is a vertex of \( \Delta(f_{i,j}^u) \), then by Lemma 3.4(i), in the view of Remark 4.2, \( \Phi_u \) contains a walk \( \gamma_v \) with \( C(\gamma_v) = C(\gamma_u) \). This means that \( \gamma_v \) consists of the same arcs as \( \gamma_u \), in particular \( \varepsilon_{s,t} \) is an arc of \( \gamma_v \) with \( \omega(\varepsilon_{s,t}) \in \mathbb{R}^n \), implying that \( \omega(\gamma_v) \in \mathbb{R}^n \). Thus, \( V_{i,j} \in \mathbb{R}^n \), and hence \( V_{i,j} = U_{i,j} \).

If \( C(\gamma_u) \) is not a vertex of \( \Delta(f_{i,j}^u) \), then by Lemma 3.4(ii) and Remark 4.2, \( \Phi_u \) contains at least two walks \( \gamma_v', \gamma_v'' \) with \( \omega(\gamma_v') = \omega(\gamma_v'') = \omega(\gamma_u) \). Thus \( V_{i,j} = \omega(\gamma_v') + \omega(\gamma_v'') = \omega(\gamma_v) \in \mathbb{R}^n \), and hence \( V_{i,j} = U_{i,j} \).

Case (b): Since \( U_{i,j} \in \mathbb{R}^n \), \( \Phi_u \) contains at least two walks \( \gamma_u', \gamma_u'' \) with weight in \( \mathbb{R} \), and thus \( \gamma_u' \) and \( \gamma_u'' \) are also walks on \( G(\hat{A}) \) – the lw-digraph of \( v[\hat{A}, \hat{B}] \) with \( \hat{A} = \hat{B} \), cf. \((\ast)\). Therefore, \( \omega(\gamma_u') + \omega(\gamma_u'') = \omega(\gamma_u) \in \mathbb{R}^n \), implying that \( V_{i,j} \in \mathbb{R}^n \), and hence \( U_{i,j} = V_{i,j} \).

\[\square\]

**Theorem 4.6** Let \( \langle u, v \rangle \) and \( \langle u', v' \rangle \) be identities for \( \mathcal{M}_n(\mathbb{T}) \). The monoid \( \mathcal{M}_n(S^n) \) admits the semigroup identity \( \langle u[\mathbb{A}, \mathbb{B}], v[\mathbb{A}, \mathbb{B}] \rangle = \langle u'[\mathbb{A}, \mathbb{B}], v'[\mathbb{A}, \mathbb{B}] \rangle \).

**Proof** \( U' := u'[\mathbb{A}, \mathbb{B}] \cong_v v'[\mathbb{A}, \mathbb{B}] := V' \) by Lemma 4.1, then \( u[U', V'] = v[U', V'] \) by Lemma 4.5.

\[\square\]

**Corollary 4.7** The monoid \( \mathcal{M}_n(S^n) \) admits nontrivial semigroup identities.
Given a semiring $R$, a (linear) representation of a semigroup $\mathcal{S}$ is a semigroup homomorphism
\[
\rho : \mathcal{S} \rightarrow \mathcal{M}_n(R),
\]
i.e., $\rho(st) = \rho(s)\rho(t)$ for any $s, t \in \mathcal{S}$. $\rho$ is said to be faithful, if it is injective.

**Corollary 4.8** Any semigroup which is faithfully represented in $\mathcal{M}_n(\mathbb{ST})$ satisfies a nontrivial semigroup identity.

Considering the lw-digraphs $G(A, B)$ with double arcs associated to matrices $A, B \in \mathcal{M}_n(\mathbb{ST})$, Theorem 4.6 receives the following meaning.

**Corollary 4.9** For any labeled-weighted digraph $G$ with arcs labeled by $\{a, b\}$, possibly with double arcs, there are two different words $u, v \in \{a, b\}^+$, such that, for any pair $(i, j)$ of nodes of $G$, the highest weight of walks from $i$ to $j$ is the same for walks labeled $u$ and for walks labeled $v$. In addition, if there is a unique walk of highest weight labeled $u$, then there is a unique walk of highest weight labeled $v$. Moreover, there is a pair of words that works for all digraphs having a given number of nodes.

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