Abstract

We develop a theory of T-duality for transitive Courant algebroids. We show that T-duality between transitive Courant algebroids $E \to M$ and $\tilde{E} \to \tilde{M}$ induces a map between the spaces of sections of the corresponding canonical weighted spinor bundles $S_E$ and $S_{\tilde{E}}$ intertwining the canonical Dirac generating operators. The map is shown to induce an isomorphism between the spaces of invariant spinors, compatible with an isomorphism between the spaces of invariant sections of the Courant algebroids. The notion of invariance is defined after lifting the vertical parallelisms of the underlying torus bundles $M \to B$ and $\tilde{M} \to B$ to the Courant algebroids and their spinor bundles. We prove a general existence result for T-duals under assumptions generalizing the cohomological integrality conditions for T-duality in the exact case. Specializing our construction, we find that the T-dual of an exact or a heterotic Courant algebroid is again exact or heterotic, respectively.

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1 Introduction

The concept of $T$-duality appeared first in theoretical physics as a duality between a pair of physical theories related by compactification of a common (possibly hidden) theory along circles of reciprocal radii. Examples include the famous duality between type IIA and type IIB string theories. More generally, it refers to an isomorphism between certain type of structures on a pair of torus bundles over the same manifold [8]. Already in the case of
circle bundles the topology of the bundle typically changes under T-duality [6, 7].

Precise formulations of T-duality are available in the framework of generalized geometry (in the sense of Hitchin) [9, 2]. Recall that the basic idea of generalized geometry is to replace the tangent bundle $TM$ of a manifold $M$ by a Courant algebroid $E$. The first examples of Courant algebroids considered in the literature were the exact Courant algebroids. They are obtained from the generalized tangent bundle $TM := T^*M \oplus TM$ by twisting the canonical Dorfman bracket with a closed 3-form. A more general class of transitive Courant algebroids is the class of heterotic Courant algebroids considered in [2]. These were introduced by Baraglia and Hekmati [2] motivated by T-duality in heterotic string theory. Cavalcanti and Gualtieri [9] developed a theory of $T$-duality for exact Courant algebroids and Baraglia and Hekmati [2] extended it to heterotic Courant algebroids. The approach of [2] is based on reduction of exact Courant algebroids and uses T-duality for the latter algebroids.

In this article we develop T-duality for transitive Courant algebroids. Our theory applies to general Courant algebroids, which might not arise from reduction of an exact Courant algebroid. In fact, it does not use reduction. Our main focus is the systematic study of the interplay of T-duality with Dirac generating operators.

Let $M$ and $\tilde{M}$ be principal $k$-torus bundles over a manifold $B$. We call two transitive Courant algebroids $E$ and $\tilde{E}$ over $M$ and $\tilde{M}$, respectively, $T$-dual if there exists a certain type of isomorphism between the pullbacks of $E$ and $\tilde{E}$ to the fiber product $N = M \times_B \tilde{M}$ (see Definition 49 for details). We show that $T$-duality gives rise to a map between the spaces of sections of the corresponding canonical weighted spinor bundles $S_E$ and $S_{\tilde{E}}$ intertwining the canonical Dirac generating operators, see Theorem 56. More specifically, we obtain compatible isomorphisms between the spaces of (appropriately defined) invariant sections of $E$ and $\tilde{E}$ as well as between the spaces of invariant sections of $S_E$ and $S_{\tilde{E}}$. This implies, in particular, that any invariant geometric structure on the Courant algebroid $E$ gives rise to a corresponding invariant ‘$T$-dual’ geometric structure on $\tilde{E}$. A structure solving a system of partial differential equations defined in terms of the Courant algebroid structure on $E$ will be mapped to a solution of the corresponding system on $\tilde{E}$. Examples include integrability equations as considered in [11] and equations of motion of physical theories such as supergravity. For instance, it was shown in [12, Section 7] that the Hull-Strominger system is invariant under T-duality. We plan to investigate these type of applications in the future.

In Theorem 62 we prove the existence of a $T$-dual $\tilde{E}$ for a class of tran-
sitive Courant algebroids $E$ over a principal torus bundle $M \to B$ under the assumption that certain cohomology classes in $H^2(B, \mathbb{R})$ are integral. The result generalizes a theorem of Bouwknegt, Hannabuss, and Mathai [8] in the exact case, see Section 6.4.1. In the heterotic case we show that the ‘T-dual’ Courant algebroids obtained from our construction are again heterotic, see Proposition 65.

In this paper we only consider Courant algebroids with scalar product of neutral signature. It would be interesting to develop $T$-duality for other classes of Courant algebroids including, in particular, the ‘odd exact’ Courant algebroids studied in [17]. A first step in this direction would be to develop a theory of Dirac generating operators for such Courant algebroids.

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2 Preliminary material

To keep the text reasonably self-contained, we recall, following [10, 11], basic facts we need on transitive Courant algebroids and their canonical Dirac generating operator. We assume that the reader is familiar with the definition of Courant algebroids, Dirac generating operators, generalized connections and $E$-connections. Basic facts on these notions can be found e.g. in [11], the approach and notation of which we preserve along the paper. In this paper we always assume that the Courant algebroids have scalar product of neutral signature. For the definition of densities we keep the conventions from our previous work [11] which coincide with those from [3]. Namely, if $V$ is a vector space of dimension $n$ and $s \in \mathbb{R}$, then the one-dimensional oriented vector space $|\det V|^s$ of $s$-densities on $V$ consists of all maps $\Psi : \Lambda^n V \setminus \{0\} \to \mathbb{R}$ (called $s$-densities) which satisfy $\omega(\lambda \vec{v}) = |\lambda|^s \omega(\vec{v})$, for any $\vec{v} \in \Lambda^n \setminus \{0\}$ and $\lambda \in \mathbb{R} \setminus \{0\}$. Note that, when $s$ is an integer, $|\det V|^s$ is canonically isomorphic to $|\det V|^s \otimes s$ and $|\det V|^s \otimes 2s$ to $(\det V)^{2s}$. Any form $\omega \in \Lambda^n V^s$ defines an $s$-density $|\omega|^s(\vec{v}) = |\omega(v_1, \ldots, v_n)|^s$, where $\vec{v} := v_1 \wedge \cdots \wedge v_n$. If $V$ is oriented then we will identify $\Lambda^n V^s$ and $|\det V|^s$ by the isomorphism which assigns to a positively oriented volume form $\omega \in \Lambda^n V^s$ the density $|\omega|$; the $s$-density $|\omega|^s$ will be denoted by $\omega^s$ when $\omega$ is positively oriented and $|\det V|^s$ by $(\det V)^s$ when $V$ is oriented. The same notation will be used
when $V$ is replaced by a vector bundle.

2.1 The canonical Dirac generating operator

Let $(E, \pi, [\cdot, \cdot], \langle \cdot, \cdot \rangle)$ be a regular Courant algebroid over a manifold $M$, with anchor $\pi : E \to TM$, Dorfman bracket $[\cdot, \cdot]$ and scalar product (of neutral signature) $\langle \cdot, \cdot \rangle$. Let $S$ be an irreducible $\text{Cl}(E)$-bundle (sometimes called a spinor bundle of $E$). We denote by $E \ni v \mapsto \gamma v$ the Clifford action of $E$ on $S$. We assume that $S$ is $\mathbb{Z}_2$-graded and that the gradation is compatible with the Clifford multiplication (this always holds when $E$ is oriented). Let $|\det S^*|^{1/r}$ be the line bundle of $1/r$-densities on $S$, where $r := \text{rk} S$. An $E$-connection $D$ on $S$ induces an $E$-connection on $|\det S^*|^{1/r}$: if $\text{vol}_S \in \Gamma(\Lambda^r S^*)$ is a local volume form on $S$ and $D_v \text{vol}_S = \omega(e) \text{vol}_S$ then $D_e |\text{vol}_S|^{1/r} = \frac{1}{r} \omega(e) |\text{vol}_S|^{1/r}$, for any $e \in E$.

The canonical Dirac generating operator $\mathfrak{d}$ of $E$ acts on sections of the canonical weighted spinor bundle of $E$ determined by $S$. The latter is defined by

$$
\mathfrak{T} := S \otimes |\det S^*|^{1/r} \otimes |\det T^*M|^{1/2} = S \otimes L,
$$

where $S := S \otimes |\det S^*|^{1/r}$ is the canonical spinor bundle of $S$ and $L := |\det T^*M|^{1/2}$. The operator $\mathfrak{d} : \Gamma(S) \to \Gamma(S)$ is given by

$$
\mathfrak{d} = \mathfrak{d} + \frac{1}{4} \gamma_{TD},
$$

where $\mathfrak{d} := \frac{1}{2} \sum_i \gamma_{\tilde{e}_i} D_{\tilde{e}_i}^S$ is the Dirac operator computed with $D^S := D^S \otimes D^L$, $D^S$ is the $E$-connection on $S$ induced by an arbitrary $E$-connection $D^S$ on $S$ compatible with a given generalized connection $D$ on $E$, $D^L$ is the $E$-connection on $L$ defined by $D$ by the rule

$$
D^L_v(\mu) = \mathcal{L}_{\pi(v)} \mu - \frac{1}{2} \text{div}_D(v) \mu, \ v \in E, \ \mu \in \Gamma(L),
$$

where $\text{div}_D(v) := \text{tr}(Dv)$, $\langle \tilde{e}_i, \tilde{e}_j \rangle = \delta_{ij}$ and $TD^D \in \Gamma(\Lambda^3 E^*)$ is the torsion of $D$, viewed as a section of the Clifford bundle $\text{Cl}(E)$ and acting by Clifford multiplication on $S$. The definition of $\mathfrak{d}$ is independent of the choice of generalized connection $D$ and $D$-compatible $E$-connection $D^S$.

2.2 Transitive Courant algebroids

2.2.1 Basic properties

Recall that a scalar product on a Lie algebra is called invariant, if the adjoint representation acts by skew-symmetric endomorphisms. A Lie algebra
endowed with an invariant scalar product is called a quadratic Lie algebra.

Similarly, a vector bundle $G \to M$ endowed with a tensor field $\langle \cdot, \cdot \rangle \in \Gamma(\wedge^2 G^* \otimes G)$ satisfying the Jacobi identity is called a Lie algebra bundle if in a neighborhood of every point $p \in N$ the tensor field has constant coefficients with respect to some local frame. A bundle of quadratic Lie algebras is a Lie algebra bundle $(G, [\cdot, \cdot])$ endowed with an invariant metric $\langle \cdot, \cdot \rangle \in \Gamma(\text{Sym}^2 G^*)$, which we assume of neutral signature.

Let $(G, [\cdot, \cdot], \langle \cdot, \cdot \rangle)$ be a bundle of quadratic Lie algebras over a manifold $M$ and $E := T^*M \oplus G \oplus TM$. Let $pr_G$ and $pr_{TM}$ be the natural projections from $E$ to $G$ and $TM$ respectively. As proved in Theorem 2.3 of [10], any Courant algebroid with underlying bundle $E$, anchor $pr_{TM}$, scalar product

$$\langle \xi + r_1 + X, \eta + r_2 + Y \rangle = \frac{1}{2}(\eta(Y) + \xi(X)) + \langle r_1, r_2 \rangle_G, \xi, \eta \in T^*M, r_1, r_2 \in G,$$

and whose Dorfman bracket satisfies

$$pr_G[r_1, r_2] = [r_1, r_2]_G$$

is defined by data $(\nabla, R, H)$ where $\nabla$ is a connection on the vector bundle $G$, $R \in \Omega^2(M, G)$ and $H \in \Omega^3(M)$ such that $\nabla$ preserves $\langle \cdot, \cdot \rangle_G$ and $[\cdot, \cdot]_G$, the curvature $R^\nabla$ of $\nabla$ is given by

$$R^\nabla(X, Y)r = [R(X, Y), r]_G, \ X, Y \in \mathfrak{X}(M), \ r \in \Gamma(G), \quad (4)$$

and the following relations hold:

$$d^\nabla R = 0, \quad (5)$$

$$dH = \langle R \wedge R \rangle_G. \quad (6)$$

We recall that

$$(d^\nabla R)(X, Y, Z) := \sum_{\sigma \in \mathfrak{S}(X, Y, Z)} (\nabla_X(R(Y, Z)) - R(\mathcal{L}_X Y, Z))$$

and

$$\langle R \wedge R \rangle_G(X, Y, Z, W) := 2 \sum_{\sigma \in \mathfrak{S}(X, Y, Z)} \langle R(X, Y), R(Z, W) \rangle_G,$$

where $X, Y, Z, W \in \mathfrak{X}(M)$ and $\mathfrak{S}(X, Y, Z)$ denotes cyclic permutations over $X, Y, Z$. The Dorfman bracket of $E$ is uniquely determined by the relations

$$[X, Y] = \mathcal{L}_X Y + R(X, Y) + i_Y i_X H$$

$$[X, r] = \nabla_X r - 2\langle i_X R, r \rangle_G$$

$$[r_1, r_2] = [r_1, r_2]_G + 2\langle \nabla r_1, r_2 \rangle_G$$

$$[X, \eta] = \mathcal{L}_X \eta, \ [\eta_1, \eta_2] = [\eta, \eta] = 0, \quad (7)$$
for any \( X, Y \in \mathfrak{X}(M) \), \( \eta_1, \eta_2, \eta \in \Omega^1(M) \) and \( r, r_1, r_2 \in \Gamma(\mathcal{G}) \), together with the condition
\[
[u, v] + [v, u] = 2d\langle u, v \rangle, \quad u, v \in \Gamma(E).
\] (8)

Such a Courant algebroid is called a **standard Courant algebroid**. As proved in [10], any transitive Courant algebroid (i.e. a Courant algebroid with surjective anchor) is isomorphic to a standard Courant algebroid. A **dissection** of a transitive Courant algebroid \( E \) is an isomorphism from \( E \) to a standard Courant algebroid. The quadratic Lie algebra bundle \((\mathcal{G}, [\cdot, \cdot]_\mathcal{G}, \langle \cdot, \cdot \rangle_\mathcal{G})\) which is a summand of a dissection of \( E \) is isomorphic to \( \text{Ker } \pi / (\text{Ker } \pi)^\perp \) (with scalar product and Lie bracket induced from \( E \)), where \( \pi : E \rightarrow TM \) is the anchor of \( E \). The following simple lemma holds.

**Lemma 1.** Let \( E \) be a transitive Courant algebroid with anchor \( \pi : E \rightarrow TM \). Let \((\mathcal{G}_0, [\cdot, \cdot]_0, \langle \cdot, \cdot \rangle_0)\) be a quadratic Lie algebra bundle, isomorphic to \( \text{Ker } \pi / (\text{Ker } \pi)^\perp \). Then \( E \) admits a dissection \( I_0 : E \rightarrow T^*M \oplus \mathcal{G}_0 \oplus TM \).

**Proof.** Start with an arbitrary dissection \( I : E \rightarrow T^*M \oplus \mathcal{G} \oplus TM \), where the target is defined by data \((\nabla, R, H)\) and a quadratic Lie algebra bundle \((\mathcal{G}, [\cdot, \cdot]_\mathcal{G}, \langle \cdot, \cdot \rangle_\mathcal{G})\) The new data
\[
\tilde{\nabla}_X := K \nabla_X K^{-1}, \quad \tilde{R}(X,Y) := KR(X,Y), \quad \tilde{H}(X,Y,Z) := H(X,Y,Z),
\]
where \( K : \mathcal{G} \rightarrow \mathcal{G}_0 \) is an isomorphism of quadratic Lie algebra bundles, together with \((\mathcal{G}_0, [\cdot, \cdot]_0, \langle \cdot, \cdot \rangle_0)\), define a standard Courant algebroid isomorphic to \( T^*M \oplus \mathcal{G} \oplus TM \) (use relations (10) below with \( \Phi := 0 \) and \( \beta := 0 \)). By composing this isomorphism with \( I \) we obtain the required dissection of \( E \). \( \square \)

Let \( E_i := T^*M \oplus \mathcal{G}_i \oplus TM \) \((i = 1, 2)\) be two standard Courant algebroids over a manifold \( M \), defined by quadratic Lie algebra bundles \((\mathcal{G}_i, [\cdot, \cdot]_{\mathcal{G}_i}, \langle \cdot, \cdot \rangle_{\mathcal{G}_i})\) and data \( (\nabla^{(i)}, R_i, H_i) \). As proved in Proposition 2.7 of [10], any fiber preserving Courant algebroid isomorphism \( I_E : E_1 \rightarrow E_2 \) is of the form
\[
I_E(\eta) = \eta, \quad I_E(r) = -2\Phi^*K(r) + K(r), \quad I_E(X) = i_X \beta - \Phi^*\Phi(X) + \Phi(X) + X,
\] (9)
for any \( X \in \mathfrak{X}(M) \), \( r \in \Gamma(\mathcal{G}_1) \) and \( \eta \in \Omega^1(M) \). Above \( \beta \in \Omega^2(M) \), \( K \in \text{Isom}(\mathcal{G}_1, \mathcal{G}_2) \) is an isomorphism of quadratic Lie algebra bundles, \( \Phi \in \Omega^1(M, \mathcal{G}_2) \),
\[
\Phi^*\Phi : T^*M \rightarrow T^*M, \quad (\Phi^*\Phi)(X)(Y) := \langle \Phi(X), \Phi(Y) \rangle_{\mathcal{G}_2},
\]
\[
\Phi^*K : \mathcal{G}_1 \rightarrow T^*M, \quad (\Phi^*K)(r)(X) := \langle K(r), \Phi(X) \rangle_{\mathcal{G}_2}.
\]
for any $X, Y \in \mathcal{X}(M)$ and $r \in \Gamma(\mathcal{G}_1)$, and the next relations are satisfied:

\[
\nabla_X^{(2)} r = K \nabla_X^{(1)} (K^{-1} r) + [r, \Phi(X)]_{\mathcal{G}_2},
\]

\[
KR_1(X, Y) - R_2(X, Y) = (d^{\nabla^{(2)}} \Phi)(X, Y) + [\Phi(X), \Phi(Y)]_{\mathcal{G}_2},
\]

\[
H_1 - H_2 = d\beta + ((KR_1 + R_2) \wedge \Phi)_{\mathcal{G}_2} - c_3,
\]

where $c_3(X, Y, Z) := \langle \Phi(X), [\Phi(Y), \Phi(Z)]_{\mathcal{G}_2} \rangle_{\mathcal{G}_2}$, for any $X, Y, Z \in \mathcal{X}(M)$.

The second and third relations (10) are equivalent with relations (46) and (47) of [10] (easy check) but are written in a simpler form. (We decomposed $\text{pr}_{T*M}(I_E |_{TM})$, which in the notation of [10] is denoted by $\beta$, into its symmetric part $-\langle \Phi(\cdot), \Phi(\cdot) \rangle_{\mathcal{G}_2}$ and skew-symmetric part $\beta$, see relation (44) of [10]).

Under an additional condition on the Courant algebroids $\mathcal{E}_i$ relations (10) can be further simplified, see Lemma 2 below. In the following we denote for simplicity by $\text{Der} \mathcal{G}$ the bundle of skew-symmetric derivations of a bundle of quadratic Lie algebras $(\mathcal{G}, [\cdot, \cdot], \langle \cdot, \cdot \rangle_{\mathcal{G}})$.

**Lemma 2.** Assume that the adjoint actions $\text{ad}_{\mathcal{G}_i} : \mathcal{G}_i \to \text{Der}(\mathcal{G}_i)$ of the Lie algebra bundles $(\mathcal{G}_i, [\cdot, \cdot]_{\mathcal{G}_i})$ of the standard Courant algebroids $\mathcal{E}_i$ are isomorphisms. Then the second relation (10) follows from the first.

**Proof.** From the injectivity of $\text{ad}_{\mathcal{G}_2}$, the second relation (10) holds if and only if

\[
[KR_1(X, Y) - R_2(X, Y), r]_{\mathcal{G}_2} = [(d^{\nabla^{(2)}} \Phi)(X, Y) + [\Phi(X), \Phi(Y)]_{\mathcal{G}_2}, r]_{\mathcal{G}_2}
\]

(11)

for any $r \in \Gamma(\mathcal{G}_2)$. Taking the covariant derivative of the first relation (10) we obtain

\[
[\nabla_Y^{(2)} (\Phi(X)), r]_{\mathcal{G}_2} = [R_2(X, Y), r]_{\mathcal{G}_2} + \nabla_Y^{(2)} r + \nabla_Y^{(2)} (K \nabla_X^{(1)} (K^{-1} r)) - K \nabla_X^{(1)} (K^{-1} \nabla_Y^{(2)} r).
\]

(12)

Now, a straightforward computation which uses the first relation (10), relation (12), and

\[
(d^{\nabla^{(2)}} \Phi)(X, Y) = \nabla_X^{(2)} (\Phi(Y)) - \nabla_Y^{(2)} (\Phi(X)) - \Phi(\mathcal{L} XY)
\]

shows that

\[
[(d^{\nabla^{(2)}} \Phi)(X, Y), r]_{\mathcal{G}_2} = [KR_1(X, Y) - R_2(X, Y), r]_{\mathcal{G}_2} + ((\nabla_X K)(\nabla_Y K^{-1}) - (\nabla_Y K)(\nabla_X K^{-1})) (r),
\]

(13)
where $\nabla$ denotes the connection on $\text{End}(G_1, G_2)$ induced by $\nabla^{(1)}$ and $\nabla^{(2)}$. On the other hand, using the Jacobi identity for $[\cdot, \cdot]_{G_2}$ and the first relation (10), we can compute

$$[[\Phi(X), \Phi(Y)]_{G_2}, r]_{G_2} = ((\nabla_Y K)(\nabla_X K^{-1}) - (\nabla_X K)(\nabla_Y K^{-1})) (r). \quad (14)$$

Relations (13) and (14) imply (11).

We say that two dissections $I_i : E \to T^*M \oplus G_i \oplus TM$ of a transitive Courant algebroid $E$ are related by $(\beta, K, \Phi)$, where $\beta \in \Omega^2(M)$, $K \in \text{Isom}(G_1, G_2)$ and $\Phi : \Omega^1(M, G_2)$, if the isomorphism $I_2 \circ I_1^{-1}$ is given by (9).

The proof of the following proposition is straightforward.

**Proposition 3.** If $I_1 : E_1 \to E_2$ and $I_2 : E_2 \to E_3$ are isomorphisms between standard Courant algebroids $E_i = T^*M \oplus G_i \oplus TM$, defined, according to (9), by $(\beta_1, K_1, \Phi_1)$ and $(\beta_2, K_2, \Phi_2)$ respectively, then $I_2 \circ I_1 : E_1 \to E_3$ is defined by $(\beta_3, K_3, \Phi_3)$ where

$$K_3 := K_2 K_1, \quad \Phi_3 := \Phi_2 + K_2 \Phi_1 \quad (15)$$

and, for any $X, Y \in TM$,

$$\beta_3(X, Y) := (\beta_1 + \beta_2)(X, Y) + \langle \Phi_2(X), K_2 \Phi_1(Y) \rangle_{G_2} - \langle \Phi_2(Y), K_2 \Phi_1(X) \rangle_{G_2}. \quad (16)$$

In particular,

$$(\beta_3 - \Phi_3^* \Phi_3)(X, Y) = (\beta_1 - \Phi_1^* \Phi_1)(X, Y) + (\beta_2 - \Phi_2^* \Phi_2)(X, Y) - 2\langle K_2 \Phi_1(X), \Phi_2(Y) \rangle_{G_3}. \quad (17)$$

### 2.2.2 The canonical Dirac generating operator of a standard Courant algebroid

Let $E = T^*M \oplus G \oplus TM$ be a standard Courant algebroid as above and $S_G$ an irreducible $\text{Cl}(G)$-bundle, with canonical spinor bundle $S_G$. Then $S := \Lambda(T^*M) \hat{\otimes} S_G$ is an irreducible spinor bundle of $E$, with Clifford action

$$\gamma_{\xi + r + X}(\omega \otimes s) = (i_X \omega + \xi \wedge \omega) \otimes s + (-1)^{\omega|} \omega \otimes (r \cdot s), \quad (18)$$

for any $\xi \in T^*M$, $r \in G$, $X \in TM$, $\omega \in \Lambda(T^*M)$ and $s \in S_G$. The canonical weighted spinor bundle of $E$ determined by $S$, as defined in (1), is canonically isomorphic to

$$S = \Lambda(T^*M) \hat{\otimes} S_G \quad (19)$$
owing to the canonical isomorphism

$$\left| \det (\Lambda(TM) \otimes S_G^*) \right|^{1/r} \otimes |\det T^* M|^{1/2} \cong |\det S_G^*|^{1/r} \tag{20}$$

given by

$$|(Z_1 \otimes s_1^*) \wedge \cdots \wedge (Z_N \otimes s_r^*)|^{1/r} \otimes |\alpha_1 \wedge \cdots \wedge \alpha_m|^{1/2} \mapsto |s_1^* \wedge \cdots \wedge s_r^*|^{1/r}, \tag{21}$$

where $N := \text{rk} \Lambda(TM), r := \text{rk} S_G, (s_i^*)$ is a local frame of $S_G^*, (\alpha_i)$ is a local frame of $T^* M,$ and $(Z_i)$ is the local frame of $\Lambda(TM)$ determined by the dual frame $(X_i)$ of $(\alpha_i)$.

As shown in Theorem 67 of [11], the canonical Dirac generating operator $d : \Gamma(S) \rightarrow \Gamma(S)$ takes the form

$$d(\omega \otimes s) = (d\omega - H \wedge \omega) \otimes s + \nabla^{S_G^*}(s) \wedge \omega$$

$$+ \frac{1}{4} (-1)^{\|\omega\|+1} \omega \otimes (C_G s) + (-1)^{\|\omega\|+1} \bar{R}^E(\omega \otimes s), \tag{22}$$

where $\omega \in \Omega(M)$ and $s \in \Gamma(S_G).$ Above $C_G \in \Gamma(\Lambda^3 G^*) \subset \Gamma(\text{Cl}(\mathcal{G}))$ is the Cartan form $C_G(u, v, w) := \langle [u, v]_G, w \rangle_G$ which acts on $s$ by Clifford multiplication, $\nabla^{S_G^*}$ is a connection on $S_G^*$ induced by (any) connection $\nabla^{S_G^*}$ on $S_G$ compatible with $\nabla,$

$$\nabla^{S_G^*}(s) \wedge \omega = \sum_i \alpha_i \wedge \omega \otimes (\nabla_{X_i}^{S_G^*} s)$$

and

$$\bar{R}^E(\omega \otimes s) = \frac{1}{2} \sum_{i,j,k} \langle R(X_i, X_j), r_k \rangle_G (\alpha_i \wedge \alpha_j \wedge \omega) \otimes (\tilde{r}_k s),$$

where $(r_k)$ is a local frame of $\mathcal{G}, (\tilde{r}_k)$ the metrically dual frame (i.e. $\langle r_i, \tilde{r}_j \rangle_G = \delta_{ij}$ for any $i, j$) and $\tilde{r}_k s$ is the Clifford action of $\tilde{r}_k$ on $s.$ Sometimes it will be convenient to write the canonical Dirac generating operator in the form

$$d(\omega \otimes s) = (d\omega \otimes s - H \cdot (\omega \otimes s)) + \sum_i \alpha_i \cdot (\omega \otimes \nabla_{X_i}^{S_G^*} s)$$

$$- \frac{1}{4} C_G \cdot (\omega \otimes s) - \frac{1}{2} \sum_{i,j,k} \langle R(X_i, X_j), r_k \rangle_G \tilde{r}_k \cdot \alpha_i \cdot \alpha_j \cdot (\omega \otimes s), \tag{23}$$

where the dots denote the Clifford action of $\text{Cl}(E) \cong \Lambda E$ on $S.$
3 The bilinear pairing on spinors

In general one cannot define the determinant of a pairing on a vector space. However, when the vector space is of the form $V := V \otimes |det V^*|^{1/n}$, where $V$ is a vector space of dimension $n$, the determinant of a pairing $\langle \cdot, \cdot \rangle$ on $V$ is defined as follows. Consider $\langle \cdot, \cdot \rangle$ as a map $V \rightarrow V^*$, $v \mapsto \langle v, \cdot \rangle$. The determinant $\det \langle \cdot, \cdot \rangle$ as the induced map $\det V \rightarrow \det V^*$, seen as a vector from the tensor product $(\det V^*)^\otimes 2$. Since $V = V \otimes |\det V^*|^{1/n}$, $\det V = \det V \otimes \det V^*$ and $(\det V)^2 \cong (\det V^2) \otimes |\det V^*|^2 \cong (\det V^*)^2$ is canonically isomorphic to $\mathbb{R}$. This means that $\det \langle \cdot, \cdot \rangle$ is a real number. It can be computed as follows: let $(v_i)$ be a basis of $V$ and $I := |v_1 \wedge \cdots \wedge v_n|^{-1/n}$.

The determinant $\det \langle \cdot, \cdot \rangle$ coincides with the determinant of the matrix $A = (a_{ij})$ where $a_{ij} := \langle v_i \otimes l, v_j \otimes l \rangle$. Note that $\det \lambda \langle \cdot, \cdot \rangle = \lambda^n \det \langle \cdot, \cdot \rangle$, for any $\lambda \in \mathbb{R}^*$. The above considerations can be extended to pairings on vector bundles in the obvious way.

Let $(E, \pi, [\cdot, \cdot], \langle \cdot, \cdot \rangle_E)$ be a rank $2n \geq 2$ regular Courant algebroid over a manifold $M$, $S$ an irreducible spinor bundle of $E$ of rank $r$ and $\mathcal{S} = S \otimes \sqrt{|\det S^*|^{1/r}}$ the canonical spinor bundle of $S$.

**Proposition 4.** i) For any $U \subset M$ open and sufficiently small, there is a pairing

$$\langle \cdot, \cdot \rangle_{\mathcal{S}|U} : \Gamma(S|U) \times \Gamma(S|U) \rightarrow C^\infty(U) \quad (24)$$

which is $C^\infty(U)$-linear, satisfies

$$\langle u \cdot s, u \cdot \tilde{s} \rangle_{\mathcal{S}|U} = \langle u, u \rangle_E \langle s, \tilde{s} \rangle_{\mathcal{S}|U}, \quad u \in \Gamma(E|U), s, \tilde{s} \in \Gamma(S|U) \quad (25)$$

and has determinant one if $n > 1$ and $-1$ if $n = 1$. Any two such pairings differ by multiplication by $\pm 1$.

ii) If $n$ is even then the even and odd parts $S^0|U$ and $S^1|U$ of $\mathcal{S}|U$ are orthogonal with respect to $\langle \cdot, \cdot \rangle_{\mathcal{S}|U}$. If $n$ is odd then $S^0|U$ and $S^1|U$ are isotropic with respect to $\langle \cdot, \cdot \rangle_{\mathcal{S}|U}$.

iii) The pairing is symmetric if $n \equiv 0, 1 \pmod{4}$ and skew-symmetric if $n \equiv 2, 3 \pmod{4}$.

iv) Let $D$ be a generalized connection on $E$. The pairing $\langle \cdot, \cdot \rangle_{\mathcal{S}|U}$ is preserved by the $E$-connection $D^S$ induced by (any) $E$-connection $D^S$ on $S$, compatible with $D$.

The remaining part of this section is devoted to the proof of Proposition 4 and to various corollaries. Let $V$ be an $n$-dimensional vector space. We begin by considering the irreducible $\text{Cl}(V \oplus V^*)$-module $\Lambda V^*$ where $V \oplus V^*$ is endowed with its standard metric of neutral signature $\langle X + \xi, Y + \eta \rangle = \langle X, Y \rangle - \langle \xi, \eta \rangle$.
\( \frac{1}{2}(\xi(Y) + \eta(X)) \) and the Clifford action is given by

\[
(X + \xi)\omega := i_X\omega + \xi \land \omega, \quad X \in V, \ \xi \in V^*, \ \omega \in \Lambda V^*.
\]

It is well-known that the vector valued bilinear pairing

\[
\langle \cdot, \cdot \rangle : \Lambda V^* \otimes \Lambda V^* \to \Lambda^n V^*, \quad \langle \omega, \tilde{\omega} \rangle := (\omega^t \land \tilde{\omega})_{\text{top}},
\]

where \( t : \Lambda V^* \to \Lambda V^* \) is defined on decomposable forms by \((\alpha_1 \land \cdots \land \alpha_k)^t := \alpha_k \land \cdots \land \alpha_1 \) and, for a form \( \omega \in \Lambda V^* \), \( \omega_{\text{top}} \in \Lambda^n V^* \) denotes its component of maximal degree, satisfies (25) (see e.g. [14]). Since the metric of \( V \oplus V^* \) has neutral signature, we obtain that (26) is determined (up to multiplication by a non-zero constant), by this property. Note that \( \Lambda^{\text{even}} V^* \) and \( \Lambda^{\text{odd}} V^* \) are orthogonal with respect to the pairing (26) when \( n \) is even and are isotropic when \( n \) is odd. Also it is easy to check that the above pairing is non-degenerate, symmetric if \( n \equiv 0, 1 \) (mod 4) and skew-symmetric if \( n \equiv 2, 3 \) (mod 4). By choosing a volume form on \( V \), we obtain an \( \mathbb{R} \)-valued pairing with the same properties. A pairing with such properties can be constructed on any irreducible Clifford module in neutral signature.

**Lemma 5.** Let \( W \) be an irreducible \( \text{Cl}^{n,n} \)-module. There is a canonical (determined up to multiplication by \( \pm 1 \)) \( \mathbb{R} \)-valued pairing \( \langle \cdot, \cdot \rangle_W \) on \( W := W \otimes |\det W^*|^{1/r} \) (where \( r := \text{rank} W \)) which satisfies (25) and \( \det \langle \cdot, \cdot \rangle_W = 1 \) if \( n > 1 \), respectively \( \det \langle \cdot, \cdot \rangle_W = -1 \) if \( n = 1 \). The pairing is symmetric if \( n \equiv 0, 1 \) (mod 4) and skew-symmetric if \( n \equiv 2, 3 \) (mod 4). Moreover, the even and odd parts \( W^0 \) and \( W^1 \) are orthogonal with respect to \( \langle \cdot, \cdot \rangle_W \) when \( n \) is even and are isotropic when \( n \) is odd.

**Proof.** It remains to prove that we can rescale \( \langle \cdot, \cdot \rangle_W \) such that \( \det \langle \cdot, \cdot \rangle_W = 1 \) or \(-1\). Assume that \( n > 1 \). Using \( \det(\lambda \langle \cdot, \cdot \rangle_W) = \lambda^r \det \langle \cdot, \cdot \rangle_W \) (when \( r := \dim W = 2^n \)) this reduces to showing that \( \det \langle \cdot, \cdot \rangle_W > 0 \) for any bilinear pairing \( \langle \cdot, \cdot \rangle_W \) which satisfies (25). For this, it is sufficient to compute \( \det \langle \cdot, \cdot \rangle_W \) using a basis of \( W \) of the form \( (w_1, \cdots, w_{r/2}, vw_1, \cdots, vw_{r/2}) \) where \( v \in \mathbb{R}^{n \times n} \) satisfies \( \langle v, v \rangle = 1 \). We obtain \( \det \langle \cdot, \cdot \rangle_W = (\det A)^2 \), where \( A = (A_{ij}) \in M_{r/2 \times r/2}(\mathbb{R}) \) with \( A_{ij} = \langle w_i \otimes l, w_j \otimes l \rangle_W \) when \( n \) is even, \( A_{ij} = \langle w_i \otimes l, vw_j \otimes l \rangle_W \) when \( n > 1 \) is odd and \( l := |w_1 \land \cdots \land w_{r/2} \land vw_1 \land \cdots \land vw_{r/2}|^{-1/r} \). In both cases. For \( n = 1 \) we obtain instead \( \det \langle \cdot, \cdot \rangle_W = - (\det A)^2 \).

**Remark 6.** A canonical pairing \( \langle \cdot, \cdot \rangle_W \) can be obtained by starting with any (non-trivial) pairing \( \langle \cdot, \cdot \rangle_W \) on \( W \), which satisfies (25): take a basis \( (w_i) \) of \( W \) and let \( l := |w_1 \land \cdots \land w_r|^{-1/r} \). Then

\[
\langle s \otimes l, \tilde{s} \otimes l \rangle_W = |\det C|^{-1/r} \langle s, \tilde{s} \rangle_W, \quad C := (\langle w_i, w_j \rangle_W)_{i,j}.
\]

(27)
The next lemma concludes the proof of Proposition 4.

**Lemma 7.** Let $U$ be a sufficiently small open subset of $M$. The section of $(\mathcal{S}^* \otimes \mathcal{S}^*)/\pm 1$ defined by the pairings $\langle \cdot, \cdot \rangle_{\mathcal{S}_\mathcal{S}}$ from Lemma 5 lifts to a smooth section $\langle \cdot, \cdot \rangle_{\mathcal{S}|U}$ of $\mathcal{S}^*|U \otimes \mathcal{S}^*|U$, which is preserved by the $E$-connection $D^S$ on $\mathcal{S}$ induced by any generalized connection $D$ on $E$.

**Proof.** Assume that $E|U$ admits a local frame $(e_i)$ with $\langle e_i, e_j \rangle_E = \epsilon_i \delta_{ij}$, where $\epsilon_i = 1$ for $i \leq n$ and $-1$ for $i \geq n+1$. On $\mathbb{R}^{2n}$ we consider the standard basis $(v_i)$ and metric $\langle \cdot, \cdot \rangle_{\mathbb{R}^{2n}}$ defined by $\langle v_i, v_j \rangle_{\mathbb{R}^{2n}} = \epsilon_i \delta_{ij}$. Let $V$ be an irreducible $\text{Cl}(\mathbb{R}^{2n}, \langle \cdot, \cdot \rangle_{\mathbb{R}^{2n}})$-module and $\Sigma := U \times V$, which is an irreducible $\text{Cl}(E|U)$-bundle with Clifford action $\gamma_{\epsilon_i}(p, w) := (p, v_i \cdot w)$, for any $(p, w) \in U \times V$. Since $E$ has neutral signature, $|S|_U = \Sigma \otimes L$ where $L$ is a line bundle and

$$S|_U = \Sigma \otimes |\det \Sigma^*|^{1/r} \otimes L \otimes |L^*|,$$

where $r := \dim V$. The bilinear pairing $\langle \cdot, \cdot \rangle_{\mathcal{S}|U}$ we are looking for is given by

$$\langle s \otimes \tilde{s}, \vartheta \otimes \tilde{\vartheta} \rangle_{|S|_U} = \langle s, \tilde{s} \rangle_{\Sigma \otimes |\det \Sigma^*|^{1/r}} \cdot \|l\|_2^2, \quad s, \tilde{s} \in \Sigma \otimes |\det \Sigma^*|^{1/r}, \quad l \in L \otimes |L^*|,$$

where $\langle s, \tilde{s} \rangle_{\Sigma \otimes |\det \Sigma^*|^{1/r}}$ is the constant pairing on $\Sigma$ induced by a canonical $\mathbb{R}$-valued pairing on $V \otimes |\det V^*|^{1/r}$ (according to Lemma 5) and $l^2 \in C^\infty(U)$ under the canonical isomorphism $(L \otimes |L^*|)^2 = U \times \mathbb{R}$. If

$$D_u(e_k) = 2 \sum_{j < p} \omega_{pj}(u)(e_p \wedge e_j)(e_k), \quad \forall u \in \Gamma(E|U),$$

where $\omega_{pj} \in \Gamma(E^*|U)$, then the $E$-connection $D^\Sigma$ on $\Sigma$ defined by

$$D^\Sigma_u(\sigma_\alpha) := \frac{1}{2} \sum_{i<j} \omega_{ji}(u)e_j e_i \cdot \sigma_\alpha, \quad 1 \leq \alpha \leq r,$$

where $(\sigma_\alpha)$ is a constant frame of $\Sigma$, is compatible with $D$ (see e.g. [11]). Since $\text{trace}(e_i e_j \cdot \cdot) = 0$, $D^\Sigma(\sigma_1 \wedge \cdots \wedge \sigma_r) = 0$ and the $E$-connection induced by $D^\Sigma$ on $\Sigma \otimes |\det \Sigma^*|^{1/r}$, also denoted by $D^\Sigma$, satisfies

$$D^\Sigma_u(\sigma_\alpha \otimes l_\Sigma) = \frac{1}{2} \sum_{i<j} \omega_{ji}(u)(e_j e_i \cdot \sigma_\alpha) \otimes l_\Sigma,$$

where $l_\Sigma := |\sigma_1 \wedge \cdots \wedge \sigma_r|^{-1/r}$. Since $\langle \cdot, \cdot \rangle_{\Sigma \otimes |\det \Sigma^*|^{1/r}}$ is constant in the frame $(\sigma_\alpha \otimes l_\Sigma)$ and the Clifford action of $e_i e_j$ is skew-symmetric with respect to $\langle \cdot, \cdot \rangle_{\Sigma \otimes |\det \Sigma^*|^{1/r}}$ (from the property (25) of $\langle \cdot, \cdot \rangle_{\Sigma \otimes |\det \Sigma^*|^{1/r}}$), we obtain that $D^\Sigma$ preserves $\langle \cdot, \cdot \rangle_{\Sigma \otimes |\det \Sigma^*|^{1/r}}$. Let $D^L$ be an $E$-connection on $L$. Then $D^\Sigma \otimes D^L$ is an $E$-connection on $|S|_U$, compatible with $D$, with the property that the induced connection on $|S|_U$ preserves $\langle \cdot, \cdot \rangle_{\mathcal{S}}$ (easy check). The latter coincides with $D^S|_U$. \qed
As a consequence of Proposition 4 we obtain, for any $U \subset M$ open and sufficiently small, a canonical (unique modulo $\pm 1$) $C^\infty(U)$-bilinear pairing
\[
\langle \cdot, \cdot \rangle_{\hat{S}|U} : \Gamma(\hat{S}|U) \times \Gamma(\hat{S}|U) \to |\det T^*U|, \quad \langle s \otimes l, \tilde{s} \otimes \tilde{l} \rangle_{\hat{S}|U} := \langle s, \tilde{s} \rangle_{\hat{S}|U} l^2, \tag{28}
\]
where $s, \tilde{s} \in \Gamma(\hat{S}|U)$ and $l \in |\det T^*U|^{1/2}$. It satisfies
\[
\langle u \cdot (s \otimes l), u \cdot (\tilde{s} \otimes \tilde{l}) \rangle_{\hat{S}|U} = \langle u, u \rangle_E \langle s \otimes l, \tilde{s} \otimes \tilde{l} \rangle_{\hat{S}|U}, \quad u \in \Gamma(E|U), \quad s \otimes l, \tilde{s} \otimes \tilde{l} \in \Gamma(\hat{S}|U). \tag{29}
\]
When $M$ is oriented, $\langle \cdot, \cdot \rangle_{\hat{S}|U}$ takes values in the bundle $\det T^*U$ of forms of top degree on $U$. A pairing with similar properties (but with values in $(\det T^*U) \otimes \mathbb{C}$) was constructed in Proposition 3.14 of [15].

Given a standard Courant algebroid $T^*M \oplus \hat{G} \oplus TM$, we will often consider, as in the next lemma, an irreducible $\text{Cl}(\hat{G})$-bundle $\hat{S}_G$ of $\hat{G}$. This will always be assumed to be $\mathbb{Z}_2$-graded, with gradation compatible with the Clifford action. In the next lemma by a canonical bilinear pairing of the weighted spinor bundle $\hat{S}_G|U$ of $\hat{S}_G|U$ we mean a smooth $C^\infty(U)$-bilinear pairing
\[
\langle \cdot, \cdot \rangle_{\hat{S}_G|U} : \Gamma(\hat{S}_G|U) \times \Gamma(\hat{S}_G|U) \to C^\infty(U) \tag{30}
\]
of normalized determinant, which satisfies
\[
\langle u \cdot s, u \cdot \tilde{s} \rangle_{\hat{S}_G|U} = \langle u, u \rangle_{\hat{G}} \langle s, \tilde{s} \rangle_{\hat{S}_G|U}, \tag{31}
\]
for any $s, \tilde{s} \in \Gamma(\hat{S}_G|U)$ and $u \in \Gamma(\hat{G}|U)$. With the same argument as in Proposition 4, such a pairing exists when $U \subset M$ is a sufficiently small open set and is unique up to multiplication by $\pm 1$. It is preserved by the connection $\nabla^{\hat{S}_G}$ induced by any connection $\nabla^{\hat{S}_G}$ on $\hat{S}_G$ compatible with $\nabla$.

**Lemma 8.** Assume that $E = T^*M \oplus \hat{G} \oplus TM$ is a standard Courant algebroid over an oriented manifold $M$, defined by a quadratic Lie algebra bundle $(\hat{G}, [-,-]_G, \langle \cdot, \cdot \rangle_G)$ and data $(\nabla, R, H)$. Let $S_G$ be an irreducible $\text{Cl}(\hat{G})$-bundle, $S_G = S_G \otimes |\det S^*|^{1/2}$ the canonical spinor bundle of $S_G$ and $S = \Lambda(T^*M) \otimes S_G$ the corresponding canonical weighted spinor bundle of $E$. For any $U \subset M$ open and sufficiently small, the canonical bilinear pairing $\langle \cdot, \cdot \rangle_{\hat{S}_G|U}$ is given (up to multiplication by $\pm 1$) by
\[
\langle \omega \otimes s, \tilde{\omega} \otimes \tilde{s} \rangle_{\hat{S}_G|U} = (-1)^{|s|(|\omega| + |\tilde{\omega}|)}(\omega^t \wedge \tilde{\omega})_{\text{top}}(s, \tilde{s})_{\hat{S}_G|U}, \tag{32}
\]
where $\langle \cdot, \cdot \rangle_{\hat{S}_G|U}$ is the canonical bilinear pairing of $\hat{S}_G|U$.

**Proof.** The claim is a consequence of the following general statement: if $(V_i, \langle \cdot, \cdot \rangle_i)$ are Euclidian vector spaces with metrics of neutral signature and $S_i$ are irreducible $\text{Cl}(V_i)$-modules of ranks $r_i$, with canonical bilinear pairings
with the canonical Dirac generating operators (see Section 4.3).

The next corollary will be used to show that the pushforward commutes with the canonical Dirac generating operators (see Section 4.3).

**Corollary 9.** In the setting of Lemma 8, let \( \nabla^{S_G} \) be the connection on \( S_G \) induced by an arbitrary connection \( \nabla^{S_{\tilde{G}}} \) on \( S_{\tilde{G}} \), compatible with \( \nabla \). Define \( E \in \text{End} \Gamma(S|U) \) by

\[
E(\omega \otimes s) := (d\omega) \otimes s + \sum_i (\alpha_i \wedge \omega) \otimes \nabla^{S_{\tilde{G}}} X_i s, \quad \omega \in \Omega(U), \ s \in \Gamma(S|U) \tag{35}
\]

where \( (X_i) \) is a local frame of \( TU \), with dual frame \( (\alpha_i) \). Then, for any \( U \subset M \) open and sufficiently small and products \( \omega \otimes s, \tilde{\omega} \otimes \tilde{s} \in \Gamma(S|U) \) of homogeneous elements,

\[
\langle E(\omega \otimes s), \tilde{\omega} \otimes \tilde{s} \rangle_{|U} + \langle \omega \otimes s, E(\tilde{\omega} \otimes \tilde{s}) \rangle_{|U} = (-1)^{|\omega||s|+|\tilde{\omega}|+1+|\omega|+|s|} d \left( \langle s, \tilde{s} \rangle_{S_{\tilde{G}}|U} (\omega^t \wedge \tilde{\omega})_{m-1} \right). \tag{36}
\]

Here \( m \) is the dimension of \( M \) and \( \omega_{m-1} \) denotes the degree \((m-1)\)-component of a form \( \omega \in \Omega(U) \).

**Proof.** We use the expression (32) of the canonical bilinear pairing \( \langle \cdot, \cdot \rangle_{|U} \) of \( S_{|U} = \Lambda(T^*U) \otimes S_{\tilde{G}}|U \). Since \( d(\omega^t) = (-1)^{|\omega|}(d\omega)^t \) we obtain

\[
\langle (d\omega) \otimes s, \tilde{\omega} \otimes \tilde{s} \rangle_{|U} = (-1)^{|s||\omega|+|\tilde{\omega}|+1+|s|} \langle s, \tilde{s} \rangle_{S_{\tilde{G}}|U} \left( d((\omega^t \wedge \tilde{\omega})_{m-1}) + (-1)^{|\omega|+1}(\omega^t \wedge d\tilde{\omega})_{top} \right). \tag{37}
\]
Similarly, since \((\alpha \wedge \omega)^t = \omega^t \wedge \alpha_i\) and using that \(\nabla^S_\varphi\) preserves \(\langle \cdot, \cdot \rangle_{S_\varphi|U}\) we obtain
\[
((\alpha_i \wedge \omega) \otimes \nabla^S_\varphi s, \tilde{\omega} \otimes \tilde{s})_{S_\varphi|U} = (-1)^{s(|\omega|+|\tilde{\omega}|+1)}(\omega^t \wedge \alpha_i \wedge \tilde{\omega})_{[\omega]} X_i \langle s, \tilde{s} \rangle_{S_\varphi|U} - \langle s, \nabla^S_\varphi \tilde{s} \rangle_{S_\varphi|U}. \tag{38}
\]
From (37) and (38) we obtain (36).

4 Dirac generating operator and operations on spinors

4.1 Behaviour of canonical Dirac generating operators under isomorphisms

The next lemma and proposition are stated for transitive Courant algebroids but the same arguments hold in the larger setting of regular Courant algebroids.

Lemma 10. Let \(I_E : E_1 \rightarrow E_2\) be an isomorphism of transitive Courant algebroids over a manifold \(M\) and \(S_i\) irreducible \(\text{Cl}(E_i)\)-bundles \((i = 1, 2)\). Then, for any \(U \subset M\) open and sufficiently small, there is a unique (up to multiplication by a smooth non-vanishing function) isomorphism \(I_{S_1|U} : S_1|U \rightarrow S_2|U\) such that
\[
I_{S_1|U} \circ \gamma_u = \gamma_{I_E(u)} \circ I_{S_1|U}, \quad \forall u \in E_1|U. \tag{39}
\]
The map \(I_{S_1|U}\) is homogeneous \((i.e.\ even\ or\ odd)\). If \(\mathcal{D}_1 \in \text{End} \Gamma(S_1|U)\) is a Dirac generating operator of \(E_1|U\) then \(\mathcal{D}_2 := I_{S_1|U} \circ \mathcal{D}_1 \circ I_{S_1|U}^{-1} \in \text{End} \Gamma(S_2|U)\) is a Dirac generating operator of \(E_2|U\).

Proof. Assume that \(E_1|U\) admits an orthonormal frame \((e_i)\) and let \((\tilde{e}_i) := (I_E(e_i))\) be the corresponding orthonormal frame of \(E_2|U\). Like in the proof of Lemma 7, \(S_i|U = \Sigma_i \otimes L_i\) where \(\Sigma_i := U \times V\) are \(\text{Cl}(E_i)\)-bundles, constructed using an irreducible \(\text{Cl}(\mathbb{R}^{2n}, \langle \cdot, \cdot \rangle_{\mathbb{R}^{2n}})\)-module \(V\) and the orthonormal frames \((e_i)\) and \((\tilde{e}_i)\) respectively, and \(L_i\) are line bundles over \(U\). Restricting \(U\) if necessary, we may assume that \(L_i\) are isomorphic. Let \(I_L : L_1 \rightarrow L_2\) be an isomorphism. Then \(I_{S_1|U} : S_1|U \rightarrow S_2|U\) defined by \(I_{S_1|U}(\sigma \otimes l) := \sigma \otimes I_L(l)\) satisfies (39). The even and odd parts of \(S_1\) are given by \(S_1^0 = \frac{1}{2}(1+\epsilon\gamma_\omega)S\) and \(S_1^1 = \frac{1}{2}(1-\epsilon\gamma_\omega)S\), where \(\epsilon \in \{\pm 1\}\) and \(\omega = e_1 \cdots e_{2n}\), and similarly for the even and odd parts of \(S_2\) (using \(\tilde{\omega} = \tilde{e}_1 \cdots \tilde{e}_n\)). Therefore the statement that
induces an isomorphism \( I_{S|U} \) is homogeneous follows from (39), which implies that \( I_{S|U} \circ \gamma \omega = \gamma \omega \circ I_{S|U} \).
Since \( I_{S|U} \) is homogeneous and \( \partial \gamma \) is odd, we obtain that also \( \partial \gamma \) is odd. The statement that \( \partial \gamma \) satisfies the remaining conditions from the definition of a Dirac generating operator can be checked using (39), which implies
\[
[\partial \gamma, \gamma_{I_E(u)}] = I_{S|U} \circ [\partial \gamma, \gamma_{I_E^{-1}}].
\]

**Remark 11.** i) In general, the isomorphisms \( I_{S|U} \) do not glue together to give an isomorphism \( I_{S} : S_1 \to S_2 \) compatible with \( I_{E} \). However, assume that \( E_1 = E_2 = E \) and let \( S_1 = S_2 = S \) be an irreducible spinor bundle over \( Cl(E) \). If \( I_E \in \text{Aut}(E) \) is of the form \( I_E(u) = \alpha \cdot u \cdot \alpha^{-1} \), where \( \alpha \in \Gamma(\text{Pin}(E)) \), then \( I_S \in \text{Aut}(S) \) defined by \( I_S(s) := \alpha \cdot s, s \in \Gamma(S) \), satisfies (39).

ii) For example, if \( E = T^*M \oplus G \oplus TM \) is in the standard form and \( \beta \in \Omega^2(M) \), then
\[
I_E(\xi + r + X) = \xi + r + X + i_X \beta
\]
can be written as \( I_E(u) = \alpha \cdot u \cdot \alpha^{-1} \) for \( \alpha := e^{-\beta} \) and the induced action on the spinor bundle \( S := \Lambda(T^*M) \otimes S^*_G \) is given by \( I_S(\omega \otimes r) := (e^{-\beta} \wedge \omega) \otimes r \) and is globally defined. If, moreover, \( d\beta = 0 \) then \( I_E \) is a Courant algebroid automorphism (see relations (10)). Similarly, any automorphism \( K \in \text{Aut}(G) \) of the bundle of quadratic Lie algebras \( G \), which belongs to the connected component \( \text{Aut}(G, \langle \cdot, \cdot \rangle_G)_0 \) and is parallel with respect to the connection \( \nabla \) from the data which defines \( E \), defines a Courant algebroid automorphism of \( E \) whose action on spinors is globally defined. In the case of exact odd Courant algebroids [17] one has an additional class of such automorphisms, defined by 1-forms (and called in [17] \( A \)-fields). However, the class of exact odd Courant algebroids is a very special class of Courant algebroids, with \( G \) the trivial rank one vector bundle and \( \nabla \) the standard flat connection.

The isomorphism \( I_{S|U} : S_1|U \to S_2|U \) from Lemma 10 induces an isomorphism \( I_{S|U} : S_1|U \to S_2|U \) between the canonical spinor bundles of \( S_1 \) and \( S_2 \), given by
\[
I_S(s \otimes s_1 \wedge \cdots \wedge s_r) := (I_Ss) \otimes [I_Ss_1 \wedge \cdots \wedge I_Ss_r]^{-1/r}, \tag{40}
\]
where \( s_1 \wedge \cdots \wedge s_r \in \Gamma(\Lambda^r(S_1|U)) \) is non-vanishing. Since \( I_{S|U} \) is unique up to a multiplicative factor, \( I_{S|U} \) is independent of the choice of \( I_{S|U} \), modulo multiplication by \( \pm 1 \) (see also Remark 55 of [11]).

**Lemma 12.** For any \( U \subset M \) open and sufficiently small, the isomorphism \( I_{S|U} \) preserves the canonical bilinear pairings \( \langle \cdot, \cdot \rangle_{S|U} \) of \( S|U \), i.e.
\[
\langle I_Ss, I_S\tilde{s} \rangle_{S|U} = \epsilon\langle s, \tilde{s} \rangle_{S|U}, \text{ for all } s, \tilde{s} \in \Gamma(S_1|U), \text{ where } \epsilon \in \{ \pm 1 \} \text{ is independent of } s, \tilde{s}.
\]
Proof. From relation (39) and the fact that $I_E$ is an isometry, we obtain that bilinear pairing $⟨s, ˜s⟩_{S_{1|U}} := ⟨I_S(s), I_S(˜s)⟩_{S_{2|U}}$ on $S_1$ satisfies (25). Also, $\det ⟨·, ·⟩_{S_{1|U}} = \det ⟨·, ·⟩_{S_{2|U}} = 1$, if $\text{rk} E_1 > 2$ (and $= -1$ if $\text{rk} E_1 = 2$).

From the above lemma, $I_{S|U} := I_{S|U} \otimes \text{Id}_{|\det T^* U|^{1/2}} : S_{1|U} \to S_{2|U}$ satisfies

$$⟨I_{S|U}(s), I_{S|U}(˜s)⟩_{S_{2|U}} = \epsilon⟨s, ˜s⟩_{S_{1|U}}, \ s, ˜s \in \Gamma(S_{1|U})$$

(41)

where $⟨·, ·⟩_{S_{1|U}}$ are the canonical $|\det T^* U|$-valued bilinear pairings of the canonical weighted spinor bundles $S_{i|U}$ of $E_i|U$ determined by $S_i|U$ and $\epsilon \in \{±1\}$ is independent on $s$ and $˜s$.

Notation 13. The isomorphisms $I_{S_i|U}$ and $I_{S_{i|U}}$ are determined only up to multiplication by $±1$. In our computations we will often choose (without repeating it each time) one $I_{S_i|U}$, $I_{S_i|U}$ or $I_{S_{i|U}}$, and refer to it as the isomorphism induced by $I$ (or the isomorphism compatible with $I$) on the spinor bundle, canonical spinor bundle and canonical weighted spinor bundle on $U$, respectively. The arguments will be independent on this choice. A similar convention will be used for the various canonical bilinear pairings like $⟨·, ·⟩_{S_i|U}$ or $⟨·, ·⟩_{S_{i|U}}$ and for the pullback and pushforward on spinors (which will also be uniquely defined only up to multiplication by $±1$, see the next sections).

Remark 14. In the setting of Lemma 10, assume that $E_i = T^* M \oplus G_i \oplus TM$ ($i = 1, 2$) are standard Courant algebroids and that $I_S : S_1 \to S_2$ is defined globally. Let $S_{G_i}$ be irreducible $\text{Cl}(G_i)$-bundles of rank $r$ and $S_i := \Lambda(T^* M) \hat{\otimes} S_{G_i}$. Using (19) and (20), one can show that the isomorphism

$$I_S : S_1 = \Lambda(T^* M) \hat{\otimes} S_{G_1} \to S_2 = \Lambda(T^* M) \hat{\otimes} S_{G_2}$$

induced by $I : E_1 \to E_2$ on the canonical weighted spinor bundles determined by $S_i$ is given in terms of $I_S : S_1 \to S_2$ by

$$I_S(ω \otimes s \otimes |s_i^* \wedge \cdots \wedge s_i^*|^\frac{1}{N}) = |\det (I_S)|^{-\frac{1}{N}} I_S(ω \otimes s) \otimes |\tilde{s}_i^* \wedge \cdots \wedge \tilde{s}_i^*|^\frac{1}{N},$$

(42)

where $ω \otimes s \in \Gamma(S_1)$, $(s_i^*)$ and $(\tilde{s}_i^*)$ are local frames of $S_{G_1}$ and $S_{G_2}$ respectively, $N := \text{rk} Λ(TM)$, $r := \text{rk} S_{G_i}$, and $\det (I_S)$ is the determinant of the representation matrix of $I_S$ in the local frames $(Ω_i \otimes s_j)$ and $(Ω_i \otimes \tilde{s}_j)$ respectively, where $(Ω_i)$ is the local frame of $Λ(T^* M)$ induced by a local frame of $TM$ and $(s_i)$, $(\tilde{s}_i)$ are the frames dual to $(s_i^*)$ and $(\tilde{s}_i^*)$ respectively. (We shall refer to $\det (I_S)$ as the determinant of $I_S$ with respect to the local frames $(s_i)$ and $(\tilde{s}_i)$).
Proposition 15. In the setting of Lemma 10, if $\hat{\phi}_1$ is the canonical Dirac generating operator of $E_1|_U$ then

$$\hat{\phi}_2 = I_{S|U} \circ \hat{\phi}_1 \circ I_{S|U}^{-1}$$

is the canonical Dirac generating operator of $E_2|_U$.

Proof. Let $\nabla^{(1)}$ be a metric connection on $E_1|_U$ and $\nabla^{S_1}$ a compatible connection on $S_1|_U$. Let $D^{(1)}$ and $D^{S_1}$ be the generalized connection on $E_1|_U$ and the $E_1|_U$-connection on $S_1|_U$ defined by $\nabla^{(1)}$ and $\nabla^{S_1}$ respectively. Let $\nabla^{(2)} := I_E \circ \nabla^{(1)} \circ I_E^{-1}$ and $\nabla^{S_2} := I_{S|U} \circ \nabla^{S_1} \circ (I_{S|U})^{-1}$. Then $\nabla^{(2)}$ is a metric connection on $E_2|_U$ and $\nabla^{S_2}$ is compatible with $\nabla^{(2)}$. Let $D^{(2)}$ and $D^{S_2}$ be the generalized connection on $E_2|_U$ and the (compatible) $E_2|_U$-connection on $S_2|_U$, defined by $\nabla^{(2)}$ and $\nabla^{S_2}$. As formula (2) for the canonical Dirac generating operator is independent of the choice of generalized connection (and compatible $E$-connection), we can (and will) choose to compute $\hat{\phi}_1$ and $\hat{\phi}_2$ using $(D^{(1)}, D^{S_1})$ and $(D^{(2)}, D^{S_2})$ respectively.

A straightforward computation using (3) shows that

$$(D^{(2)})^{L}_{I_E(u)} = (D^{(1)})^{L}_{I_E(u)}, \forall u \in E_1|_U, \mu \in \Gamma(L|_U)$$

and

$$(D^{S_2} \otimes (D^{(2)})^{L})_{u} = I_{S|U} \circ (D^{S_1} \otimes (D^{(1)})^{L})_{I_E^{-1}(u)} \circ (I_{S|U})^{-1}.$$ (45)

Relation (45) implies that the Dirac operators $D^{(2)}$ on $S_2|_U$ and $D^{(1)}$ on $S_1|_U$ computed with $D^{S_2} \otimes (D^{(2)})^{L}$ and $D^{S_1} \otimes (D^{(1)})^{L}$ respectively, are related by

$$D^{(2)} = I_{S|U} \circ D^{(1)} \circ (I_{S|U})^{-1}.$$ (46)

On the other hand, it is easy to see that

$$T^{D^{(2)}}(u, v, w) = T^{D^{(1)}}(I_E^{-1}u, I_E^{-1}v, I_E^{-1}w), \forall u, v, w \in E_2|_U$$

which implies that

$$T^{D^{(2)}} = I_E(T^{D^{(1)}}).$$ (47)

where $T^{D^{(i)}} \in \Gamma(\Lambda^2 E_i|_U) \subset \Gamma Cl(E_i|_U)$ and $I_E$ denotes the action induced by the isometry $I_E$ on Clifford algebras. Relations (39) and (47) imply that

$$\gamma_{T^{D^{(2)}}} = I_{S|U} \circ \gamma_{T^{D^{(1)}}} \circ (I_{S|U})^{-1}$$

which, together with (2) and (46), implies our claim.

$\square$
4.2 Pullback of spinors

Let \( f : M \to N \) be a submersion and \( E \) a transitive Courant algebroid over \( N \). Following [16], we recall the definition of the pullback Courant algebroid \( f^* E \). Let \( TM := T^* M \oplus TM \) be the generalized tangent bundle with its standard Courant algebroid structure, given by the Dorfmann bracket

\[
[(\xi + X, \eta + Y)] := \mathcal{L}_X(Y + \eta) - i_Y d\xi,
\]

for any \( X, Y \in \mathfrak{X}(M), \xi, \eta \in \Omega^1(M) \), scalar product \( \langle \xi + X, \eta + Y \rangle := \frac{1}{2}(\xi(Y) + \eta(X)) \) and anchor the natural projection from \( TM \) to \( TM \). Consider the direct product Courant algebroid \( E \times TM \) and let \( a : E \times TM \to T(N \times M) \) be its anchor. Then \( C := a^{-1}(TM_f) \) is a coisotropic subbundle of \( E \times TM \) over the graph \( M_f \subset N \times M \) of \( f \), which we identify with \( M \). Its fiber over \( p \in M \) is given by

\[
C_p := \{(u, \eta + X) \in E_{f(p)} \times TM \mid \pi(u) = (d_p f)(X)\}
\]
and

\[
C^\perp_p := \{(\frac{1}{2}\pi^* \gamma, -(d_p f)^* \gamma) \mid \gamma \in T^*_{f(p)} N \} \subset C_p,
\]

where \( \pi^* : T^* N \to E \) is the dual of the anchor \( \pi : E \to TN \) composed with the natural identification \( E^* \cong E \) induced by the scalar product \( \langle \cdot, \cdot \rangle \) of \( E \). The quotient \( C/C^\perp \) is a Courant algebroid over \( M \cong M_f \) with anchor, scalar product and Courant bracket induced from \( E \times TM \). The Courant algebroid \( C/C^\perp \) was called in [16] the pullback of \( E \) by the map \( f \).

**Lemma 16.** i) Let \( E = T^* N \oplus \mathcal{G} \oplus TN \) be a standard Courant algebroid, defined by a bundle of quadratic Lie algebras \( (\mathcal{G}, [\cdot, \cdot]_\mathcal{G}, \langle \cdot, \cdot \rangle_\mathcal{G}) \) and data \( (\nabla, R, H) \). Then \( f^* E \) is isomorphic to the standard Courant algebroid defined by the bundle of quadratic Lie algebras

\[
(f^* \mathcal{G}, [\cdot, \cdot]_{f^* \mathcal{G}}, \langle \cdot, \cdot \rangle_{f^* \mathcal{G}} := f^* \langle \cdot, \cdot \rangle_\mathcal{G})
\]

together with \((f^* \nabla, f^* R, f^* H)\).

ii) Let \( I : E_1 \to E_2 \) be an isomorphism between two transitive Courant algebroids over \( N \) and \( a_i : E_i \times TM \to T(N \times M) \) the anchors of the direct product Courant algebroids \( E_i \times TM \) \((i = 1, 2)\). Then \( I \) induces a Courant algebroid isomorphism \( I^f : f^* E_1 \to f^* E_2 \) defined by

\[
I^f[(u, \eta + X)] := [(I(u), \eta + X)], \quad \forall (u, \eta + X) \in (C_1)_p,
\]

where \( C_i = (a_i)^{-1}(TM_f) \) and \([I(u), \eta + X] \) denotes the class of \((I(u), \eta + X) \) \((u, \eta + X) \in (C_2)_p \) modulo \((C_2)_p^\perp \).
iii) Let $E$ be a transitive Courant algebroid over $N$. Any dissection of $E$ induces a dissection of $f^1E$. Moreover, if $I_1 : E \to T^*N \oplus \mathcal{G}_* \oplus TN$ are two dissections of $E$, related by $(\beta, K, \Phi)$, then the induced dissections of $f^1E$ are related by $(f^*\beta, f^*K, f^*\Phi)$.

**Proof.** i) We claim that the quadratic Lie algebra bundle $(f^*\mathcal{G}, [\cdot, \cdot]_{f^*\mathcal{G}}, \langle \cdot, \cdot \rangle_{f^*\mathcal{G}})$ together with $(f^*\nabla, f^*R, f^*H)$ define a standard Courant algebroid. The proof reduces to the verification of the conditions stated in Section 2.2.1. The form $f^*R$ is defined by $(f^*R)(X, Y) = R(dfX, dfY) \in \mathcal{G}_{f(p)} = (f^*\mathcal{G})_p$, for any $X, Y \in T_pM$, $p \in M$. To show, for instance, that

$$[d^f\nabla(f^*R)](X, Y, Z) = 0 \quad \text{for all} \quad X, Y, Z \in \mathfrak{X}(M),$$

cf. equation (5), we notice that it holds for any projectable vector fields $X, Y, Z \in \mathfrak{X}(M)$, since

$$(f^*\nabla)_X[(f^*R)(Y, Z)] = f^*[\nabla_{f_*X}R(f_*Y, f_*Z)],$$

$$(f^*R)(\mathcal{L}_X Y, Z) = f^*[R(\mathcal{L}_{f_*X}f_*Y, f_*Z)]$$

and that it is $C^\infty(M)$-linear in all arguments $X, Y, Z$. Here $f_*X$ denotes the vector field on $N$ obtained by projection of a projectable vector field $X \in \mathfrak{X}(M)$. Recall that for projectable vector fields we have $dfX = (f_*X) \circ f$. Relation (52) follows. In a similar way we prove that $(f^*\mathcal{G}, [\cdot, \cdot]_{f^*\mathcal{G}}, \langle \cdot, \cdot \rangle_{f^*\mathcal{G}})$ together with $(f^*\nabla, f^*R, f^*H)$ satisfy the remaining conditions for standard Courant algebroids.

One can show that the map

$$F : T^*M \oplus f^*\mathcal{G} \oplus TM \to f^1E, \quad F(\eta + r + X) := [(r + df(X), \eta + X)]$$

where $\eta \in T^*_pM$, $r \in \mathcal{G}_{f(p)}$, $X \in T_pM$ and $p \in M$ is arbitrary, is a Courant algebroid isomorphism between the standard Courant algebroid defined by the quadratic Lie algebra bundle $(f^*\mathcal{G}, [\cdot, \cdot]_{f^*\mathcal{G}}, \langle \cdot, \cdot \rangle_{f^*\mathcal{G}})$ together with $(f^*\nabla, f^*R, f^*H)$, and $f^1E$.

ii), iii) Claim ii) is an easy check and claim iii) follows by combining claims ii) and iii) and using (9). \qed

Our next aim is to define a pullback from spinors of $E$ to spinors of $f^1E$. At first, we assume that $E$ is a standard Courant algebroid.

**Remark 17.** i) Assume that $E = T^*N \oplus \mathcal{G} \oplus TN$ is a standard Courant algebroid, defined by a bundle of quadratic Lie algebras $(\mathcal{G}, [\cdot, \cdot]_\mathcal{G}, \langle \cdot, \cdot \rangle_\mathcal{G})$ and data $(\nabla, R, H)$. Using the isomorphism (53), we often identify (without repeating it each time) $f^1E$ with the standard Courant algebroid $T^*M \oplus$
\( f^*G \oplus TM \) defined by the quadratic Lie algebra bundle \((f^*G, f^*\langle \cdot, \cdot \rangle_G, f^*\langle \cdot, \cdot \rangle_G)\) and data \((f^*\nabla, f^*R, f^*H)\). We fix an irreducible \(\text{Cl}(G)\)-bundle \(S_G\). Then \(S_{f^*G} := f^*S_G\) is an irreducible \(\text{Cl}(f^*G)\)-bundle and \(f^*S_G = S_{f^*G}\). The natural map

\[
f^* : \Gamma(S_N) = \Omega(N, S_G) \to \Gamma(S_M) = \Omega(M, f^*S_G), \quad \omega \otimes s \to f^*(\omega) \otimes f^*(s) \tag{54}
\]
preserves the \(\mathbb{Z}_2\)-degrees of \(S_N\) and \(S_M\). It is called the \textbf{pullback on spinors}.

ii) Assume in addition that \(f : M \to N\) is endowed with a horizontal distribution. For any \(X \in \mathfrak{x}(N)\), we denote by \(\tilde{X} \in \mathfrak{x}(M)\) the horizontal lift of \(X\) and we define a map

\[
f^* : \Gamma(E) \to \Gamma(f^!E), \quad f^*(\xi + r + X) := f^*(\xi + r) + \tilde{X}. \tag{55}
\]

Let \(\langle \cdot, \cdot \rangle_E\) and \(\langle \cdot, \cdot \rangle_{f^!E}\) be the scalar products of \(E\) and \(f^!E\). As

\[
\langle f^*(u), f^*(v) \rangle_{f^!E} = \langle u, v \rangle_E \circ f, \ u, v \in \Gamma(E),
\]
we obtain an induced map \(f^* : \Gamma \text{Cl}(E) \to \Gamma \text{Cl}(f^!E)\), which satisfies

\[
f^*(u \cdot v) = f^*(u) \cdot f^*(v), \ \forall u, v \in \Gamma \text{Cl}(E) \tag{56}
\]

and

\[
f^*(u \cdot s) = f^*(u) \cdot f^*(s), \ u \in \Gamma \text{Cl}(E), \ s \in \Gamma(S_N). \tag{57}
\]

Assume now that \(E\) is a transitive, but not necessarily standard, Courant algebroid and let \(S_E\) and \(S_{f^!E}\) be canonical weighted spinor bundles of \(E\) and \(f^!E\), determined by irreducible spinor bundles \(S_E\) and \(S_{f^!E}\) respectively. In order to be able to construct a pullback map from \(\Gamma(S_E)\) to \(\Gamma(S_{f^!E})\), we assume that the following condition is satisfied: there is a dissection \(I : E \to E_N = T^*N \oplus G \oplus TN\) of \(E\) and an irreducible \(\text{Cl}(G)\)-bundle \(S_G\), such that \(I\) and the dissection \(I^! : f^!E \to E_M = T^*M \oplus f^*G \oplus TM\) of \(f^!E\) induce \textbf{global isomorphisms} \(I_S : S_E \to \Lambda(T^*N) \otimes S_G\) and \(I^{\dagger}_S : S_{f^!E} \to \Lambda(T^*M) \otimes f^*S_G\) between spinor bundles. We shall often refer to \((I, S_G)\) as an \textbf{admissible pair} for \(S_E\) and \(S_{f^!E}\). Let

\[
I_S : S_E \to S_N = \Lambda(T^*N) \otimes S_G \tag{58}
\]

be the induced (global) isomorphisms between the canonical weighted spinor bundles determined by \(S_E, \Lambda(T^*N) \otimes S_G\), \(S_{f^!E}\) and \(\Lambda(T^*M) \otimes f^*S_G\).
Lemma 18. The map

\[ f^! : \Gamma(S_E) \to \Gamma(S_{f^! E}), \quad f^! := (I_S^f)^{-1} \circ f^* \circ I_S \]

is well defined, up to multiplication by ±1. It is called the pullback on spinors.

Proof. Let

\[ I : E_1 = T^*N \oplus \mathcal{G}_1 \oplus TN \to E_2 = T^*N \oplus \mathcal{G}_2 \oplus TN \]

be an isomorphism between standard Courant algebroids and

\[ I_f^! : f^! E_1 = T^*M \oplus f^* \mathcal{G}_1 \oplus TM \to f^! E_2 = T^*M \oplus f^* \mathcal{G}_2 \oplus T^*M \]

the induced isomorphism between their pullbacks. Let \( S_{\mathcal{G}_i} \) be irreducible \( \text{Cl}(\mathcal{G}_i) \)-bundles, such that \( I \) and \( I_f^! \) induce global isomorphisms

\[ I^S_N : \Lambda(TN) \hat{\otimes} S_{\mathcal{G}_1} \to \Lambda(TN) \hat{\otimes} S_{\mathcal{G}_2}, \quad I_f^S_N : \Lambda(TM) \hat{\otimes} f^* S_{\mathcal{G}_1} \to \Lambda(TM) \hat{\otimes} f^* S_{\mathcal{G}_2}. \]

By considering two admissible pairs for \( S_E \) and \( S_{f^! E} \) the claim reduces to showing that

\[ I_f^S_M \circ f^* = \epsilon f^* \circ I^S_N \]  

(61)

where \( \epsilon \in \{ \pm 1 \} \),

\[ I^S_N : S^1_N \to S^2_N, \quad I_f^S_M : S^1_M \to S^2_M \]

(62)

are the isomorphisms induced by \( I \) and \( I_f^! \) on the canonical weighted spinor bundles

\[ S^i_N = \Lambda(T^*N) \hat{\otimes} S_{\mathcal{G}_i}, \quad S^i_M = \Lambda(T^*M) \hat{\otimes} f^* S_{\mathcal{G}_i} \]

determined by the spinor bundles

\[ S^i_N := \Lambda(T^*N) \hat{\otimes} S_{\mathcal{G}_i}, \quad S^i_M := \Lambda(T^*M) \hat{\otimes} f^* S_{\mathcal{G}_i} \]

where \( S_{\mathcal{G}_i} = S_{\mathcal{G}_i} \otimes |\text{det} S_{\mathcal{G}_i}|^{1/r} \), and \( f^* : S^i_N \to S^i_M \) are defined by (54). In order to prove (61) we fix a distribution \( \mathcal{D} \subset TM \) complementary to Ker \( df \) and we decompose orthogonally \( f^! E_i = V^+_i \oplus V^- \), where \( V^+_i \) and \( V^- \) are given by

\[ (V^+_i)_p = D^*_p \oplus (\mathcal{G}_i)_{f(p)} \oplus D_p \]
\[ (V^-)_p = (\text{Ker } d_pf)^* \oplus \text{Ker } d_pf, \]

for any \( p \in M \). Assume that \( I \) is defined by \( (\beta, K, \Phi) \) as in Section 2.2.1. Then, from Lemma 16 iii), \( I_f^! \) is defined by \( (f^* \beta, f^* K, f^* \Phi) \) and acts as the identity on \( V^- \) while its restriction \( I_f^+ : V^+_i \to V^+_i \) satisfies

\[ (I_f^+)_p(f^* u) = f^*(I_{(f^! p)(u)}), \quad \forall u \in (E_1)_{f(p)}, \quad p \in N, \]

(63)
where \( f^* : (E_i)_{f(p)} \rightarrow (V_i^+) \) are given by (55), constructed using the distribution \( \mathcal{D} \). Consider the spinor bundles

\[
S_i^+ := \Lambda \mathcal{D}^* \otimes f^* S_{G_i}, \quad S_i^- = \Lambda (\text{Ker} \, df)^*
\]

(64)
of \( V_i^+ \) and \( V^- \). Then \( \tilde{S}_M^{i+} := S_i^+ \otimes S_i^- \) is a spinor bundle of \( f^* E_i \), isomorphic to the spinor bundle \( S_M^i \) via the \( \text{Cl}(f^* E_i) \)-bundle isomorphism

\[
T_i : \tilde{S}_M^{i+} \rightarrow S_M^i, \quad T((\omega \otimes s) \otimes \eta) = (-1)^{|\eta|} (\omega \wedge \eta) \otimes s
\]

(65)
where \( \omega \in \Lambda \mathcal{D}^* \) and \( s \in f^* S_{G_i}, \eta \in S^- \) are homogeneous. Let

\[
I^f_{S^+} := f^* \circ I_{S_N} \circ (f^*)^{-1} : S_i^+ \rightarrow S_2^+, \quad (66)
\]

for any \( s \in S_i^+ \) and \( \eta \in S^- \) homogeneous, where \(|\eta|\) and \(|I^f_{S^+}|\) denote the degrees of \( \eta \) and \( I^f_{S^+} \), is compatible with \( I^f \). The isomorphism \( I^f_{S^+} \) induces, via the isomorphisms (65), an isomorphism \( I^f_{S_M} : S_M^1 \rightarrow S_M^2 \) compatible with \( I^f \), which maps \( S_i^+ \subset S_M^1 \) onto \( S_2^+ \subset S_M^2 \) and whose restriction to \( S_1^+ \) coincides with \( I^f_{S^+} \). As we already know, any isomorphism compatible with \( I^f \) and acting between \( S_M^1 \) is uniquely determined up to a multiplicative factor and the isomorphism it induces on the canonical weighted spinor bundles \( S_M^i \) is independent of this factor, up to multiplication by \( \pm 1 \). It remains to show that the isomorphisms \( I^f_{S_M} : S_M^1 \rightarrow S_M^2 \) and \( I_{S_N} : S_N^1 \rightarrow S_N^2 \) induced by \( I^f_{S_M} \) (defined as above) and \( I_{S_N} \) are related by (61). For this, we use Remark 14. Let \( (s_i), (\tilde{s}_i) \) be local frames of \( S_{G_i}, S_{G_2} \) and \( (s^+_i), (\tilde{s}^+_i) \) the dual frames. From Remark (14),

\[
I^f_{S_M}((\omega \otimes s) \otimes |f^* s^+_1 \wedge \cdots \wedge f^* s^+_r|^{1/2}) = I^f_{S_M}((\omega \otimes s) \otimes |f^* \tilde{s}^+_1 \wedge \cdots \wedge f^* \tilde{s}^+_r|^{1/2}) |\det(I_{S_M}^f)|^{-1/n_c} (68)
\]

where \( \omega \in \Lambda (T^*M), s \in f^* S_{G_i}, N_h := \text{rk} (\Lambda \mathcal{D}), N_v := \text{rk} (\Lambda \text{Ker} df), r := \text{rk} S_G, \text{det}(I^f_{S_M}) \) denotes the determinant of \( I^f_{S_M} \) with respect to the local frames \( (f^* s_i) \) and \( (f^* \tilde{s}_i) \). Similarly,

\[
I_{S_N}((\omega \otimes s) \otimes |s^+_1 \wedge \cdots \wedge s^+_r|^{1/2}) = I_{S_N}((\omega \otimes s) \otimes |\tilde{s}^+_1 \wedge \cdots \wedge \tilde{s}^+_r|^{1/2}) |\det(I_{S_N})|^{-1/n_c} (69)
\]
where $\omega \in \Lambda^N(T^*N)$, $s \in S_{G}$ and $\det(I_{S_N})$ is the determinant of $I_{S_N}$ with respect to the local frames $(s_i)$ and $(\tilde{s}_i)$. Using (68), (69) together with
\[
\det(I_{S_M}^f) = \det(I_{S_N}^f)^N = \det(I_{S_N})^N \circ f
\] (70)
we obtain (61). (In the first relation (70) we used the definition of $I_{S_M}^f$ and (67) while in the second relation (70) we used the definition (66) of $I_{S_N}^f$).

**Proposition 19.** Let $f: M \to N$ be a submersion, $E$ a transitive Courant algebroid over $N$ and $S_E$, $S_{f^!E}$ canonical weighted spinor bundles of $E$ and $f^!E$ such that the pullback $f^!: \Gamma(S_E) \to \Gamma(S_{f^!E})$ is defined. Let $\mathfrak{d}_E \in \text{End} \Gamma(S_E)$ and $\mathfrak{d}_{f^!E} \in \text{End} \Gamma(S_{f^!E})$ be the canonical Dirac generating operators of $E$ and $f^!E$. Then
\[
f^! \circ \mathfrak{d}_E = \mathfrak{d}_{f^!E} \circ f^! \tag{71}
\]

Proof. From the invariance of the canonical Dirac generating operators under isomorphisms (see Proposition 15) and the definition of $f^!$ (see (59)) it is sufficient to prove (71) when $E = T^*N \oplus G \oplus TN$ is a standard Courant algebroid, as in Remark 17. With the notation of that remark, we need to show that
\[
f^* \circ \mathfrak{d}_N = \mathfrak{d}_M \circ f^* : \Gamma(S_N) \to \Gamma(S_M) \tag{72}
\]
where $\mathfrak{d}_N$ and $\mathfrak{d}_M$ are the canonical Dirac generating operators of $E$ and $f^!E = T^*M \oplus f^*G \oplus TM$, which can be computed using (22). Let $m$ and $n$ be the dimensions of $M$ and $N$ respectively. Let $(X_i)_{1 \leq i \leq m}$ be a local frame of $TM$ such that $(X_i)_{1 \leq i \leq n}$ are projectable and their projections $(f_*X_i)_{1 \leq i \leq n}$ form a local frame of $TN$ and $(X_i)_{n+1 \leq i \leq m}$ are vertical. Let $(\alpha_i)_{1 \leq i \leq n}$ be the dual frame of $(f_*X_i)_{1 \leq i \leq n}$. Then, using $f_*X_i = 0$ for any $i \geq n + 1$,
\[
\begin{align*}
\hat{R} & = \frac{1}{2} \sum_{i,j \leq n} \langle f^*(R(f_*X_i, f_*X_j)), f^*(\alpha_i \wedge \alpha_j \wedge \omega) \rangle \otimes (f^*(r_k)f^*(s)) \\
\end{align*}
\]
that is,
\[
(\hat{R} \circ f^*)(\omega \otimes s) = (f^* \circ \hat{R})(\omega \otimes s). \tag{73}
\]
On the other hand, if $\nabla^{S_{f^!E}}$ is compatible with $\nabla$ then $\nabla^{S_{f^*G}} := f^* \nabla^{S_{f^!E}}$ is compatible with $f^* \nabla$ and
\[
\nabla^{S_{f^*G}} = f^* \nabla^{S_{f^!E}}. \tag{74}
\]
Relations (73), (74), $C_{f^*G} = f^*C_G$ and the expression of the canonical Dirac generating operator (22) imply (72).

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4.3 Pushforward on spinors

Let $f : M \to N$ be a fiber bundle with compact fibers and $M$, $N$ oriented. Let $E$ a transitive Courant algebroid over $N$. In this section we define a pushforward from spinors of $f^*E$ to spinors of $E$. As for the pullback, we assume first that $E = T^*N \oplus G \oplus TN$ is a standard Courant algebroid, as in Remark 17. We choose an irreducible $Cl(G)$-bundle $S_G$, with canonical spinor bundle $S_G$. Consider an open cover $\mathcal{U} = \{U_i\}$ of $N$ and, for any $U_i \in \mathcal{U}$, a canonical bilinear pairing $\langle \cdot, \cdot \rangle_{S_G|U_i}$ on $\Gamma(S_G|U_i)$. We define $\langle \cdot, \cdot \rangle_{f^*S_G|f^{-1}(U_i)} := f^*\langle \cdot, \cdot \rangle_{S_G|U_i}$, which is a canonical bilinear pairing on $\Gamma(S_{f^*G}|f^{-1}(U_i))$, where $S_{f^*G} = f^*S_G$ is the canonical spinor bundle of $f^*S_G$. We denote by $\langle \cdot, \cdot \rangle_{S_G}$ and $\langle \cdot, \cdot \rangle_{S_{f^*G}}$ the corresponding $\det(T^*U_i)$ and $\det(T^*f^{-1}(U_i))$-valued canonical bilinear pairings on $\Gamma(S_N|U_i)$ and $\Gamma(S_M|f^{-1}(U_i))$, where $S_N = \Lambda(T^*N) \otimes S_G$ and $S_M = \Lambda(T^*M) \otimes f^*S_G$, see relation (32). For any $U_i \in \mathcal{U}$, let

$$f_{*U_i} : \Gamma(S_M|f^{-1}(U_i)) = \Omega(f^{-1}(U_i), f^*S_G) \to \Gamma(S_N|U_i) = \Omega(U_i, S_G)$$

be defined by

$$\int_{U_i} \langle f_{*U_i} s_1, s_2 \rangle_{S_{f^{-1}(U_i)}} = \int_{f^{-1}(U_i)} \langle s_1, f^*s_2 \rangle_{S_{f^{-1}(U_i)}},$$

for all $s_1 \in \Gamma(S_M|f^{-1}(U_i))$ and $s_2 \in \Gamma_c(S_N|U_i)$, where $\Gamma_c(V)$ denotes the space of compactly supported sections of a vector bundle $V$. Using the maps $f_{*U_i}$ and a partition of unity $\{\lambda_i\}$ of $\mathcal{U}$ we obtain a map

$$f_* : \Gamma(S_M) = \Omega(M, f^*S_G) \to \Gamma(S_N) = \Omega(N, S_G)$$

defined by

$$(f_*s) = \sum_i \lambda_i f_{*U_i}(s|f^{-1}(U_i)), \forall s \in \Gamma(S_M).$$

The map (77) is called the pushforward on spinors.

**Remark 20.** Recall that the pushforward on forms $f_* : \Omega(M) \to \Omega(N)$ has the properties

$$f_* \circ d = d \circ f_*, \quad f_*(f^*\alpha \wedge \beta) = \alpha \wedge f_*\beta, \quad \int_M (f^*\alpha) \wedge \beta = \int_N \alpha \wedge f_*\beta.$$

Let $U$ be a local chart over which the fiber bundle $f : M \to N$ is trivial. Then we can identify $f^{-1}(U)$ with $U \times F$, where $F$ is the compact fiber. The decomposition $U \times F$ induces a bigrading on $\Lambda T^*_p M = \Lambda \Lambda^*_x U \otimes \Lambda^*_t F = \bigoplus_{k,t} \Lambda^k \Lambda^*_x U \otimes \Lambda^t \Lambda^*_t F$ for all $p = (x,t) \in U \times F$. Then $f_*\omega = 0$ for every
differential form $\omega$ on $U \times F$ of type $(k, \ell)$, $\ell \neq r = \dim F$. Choosing a positively oriented volume form $\text{vol}_F$ on the fiber $F$, we can write every differential form of type $(k, r)$ as $\omega = h \omega_U \wedge \text{vol}_F$, where $h$ is a function on $U \times F$ and $\omega_U$ is a differential form on $U$. Then

$$f_* \omega = \omega_U \int_F h(x, t) \text{vol}_F(t). \quad (80)$$

So $f_*$ is simply integration over the fibers.

The next lemma provides a concrete formulation for the pushforward on spinors in terms of the pushforward on forms.

**Lemma 21.** For any $\omega \otimes f^s \in \Gamma(S_M)$ such that $s$ is homogenous,

$$f_*(\omega \otimes f^s) = (-1)^{|s| + nr + \frac{r(r-1)}{2}} (f_* \omega) \otimes s, \quad (81)$$

where $n$ and $r$ are the dimensions of $N$ and the fibers of $f$, respectively. In particular, the pushforward is well-defined (i.e. independent on the choice of $U$, partition of unity $\{\lambda_i\}$ and canonical bilinear pairings $\langle \cdot, \cdot \rangle_{S_{f^1(U_i)}}$).

**Proof.** We show that for any $\omega \otimes f^s \in \Gamma(S_M|_{f^{-1}(U_i)})$ with $s$ homogenous and $\tilde{\omega} \otimes \tilde{s} \in \Gamma_c(S_N|_{U_i})$,

$$\int_{U_i} \langle (f_* \omega) \otimes s, \tilde{\omega} \otimes \tilde{s} \rangle_{S_{f^1(U_i)}} = (-1)^{|s| + nr + \frac{r(r-1)}{2}} \int_{f^{-1}(U_i)} \langle \omega \otimes f^s, f^*(\tilde{\omega} \otimes \tilde{s}) \rangle_{S_{f^{-1}(U_i)}}, \quad (82)$$

In order to prove (82), we assume, without loss of generality, that $\omega$, $\tilde{\omega}$ and $\tilde{s}$ are also homogenous. If $|\omega| + |\tilde{\omega}| = m$ (where $m := n + r$) both terms in (82) vanish. Assume now that $|\omega| + |\tilde{\omega}| = m$. Then

$$\int_{f^{-1}(U_i)} \langle \omega \otimes f^s, f^*(\tilde{\omega} \otimes \tilde{s}) \rangle_{S_{f^{-1}(U_i)}} = (-1)^{|s| + m + |\tilde{\omega}|} \int_{f^{-1}(U_i)} f^*(\langle s, \tilde{s} \rangle_{S_{f^0}} \tilde{\omega}) \wedge \omega^t = (-1)^{|s| + m + |\tilde{\omega}|} \int_{U_i} \langle s, \tilde{s} \rangle_{S_{f^0}} \tilde{\omega} \wedge f_*(\omega^t) = (-1)^{r(m - \frac{r+1}{2} - |s|)} \int_{U_i} \langle (f_* \omega) \otimes s, \tilde{\omega} \otimes \tilde{s} \rangle_{S_{U_i}},$$

where we used $f_*(\omega^t) = (f_* \omega)^t (-1)^{\frac{r(r-1)}{2} + r(|\omega| - r)}$, which can be checked using (80) and the third property (79) of the pushforward on forms. Relation (82) is proved and implies (81).

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Remark 22. In the above setting, assume that $f$ is endowed with an horizontal distribution, like in Remark 17 ii). Then

$$f_*(f^*(u) \cdot s) = u \cdot f_* s, \quad \forall u \in \Gamma \text{Cl}(E), \ s \in \Gamma(S_{f^*E}).$$

(83)

where $f^* : \Gamma \text{Cl}(E) \to \Gamma \text{Cl}(f^*E)$ is the map (55).

Assume now that $E$ is a transitive, but not necessarily standard, Courant algebroid. Then we can define the pushforward $f_1 : \Gamma(S_{f^*E}) \to \Gamma(S_E)$ for any canonical weighted spinor bundles $S_E$ and $S_{f^*E}$, for which the pullback $f^1 : \Gamma(S_E) \to \Gamma(S_{f^*E})$ is defined. Namely, we consider an admissible pair $(I : E \to T^*N \oplus G \oplus TN, S_g)$ for $S_E$ and $S_{f^*E}$ and we define the pushforward on spinors

$$f_1 : \Gamma(S_{f^*E}) \to \Gamma(S_E), \ f_1 := (I_S)^{-1} \circ f_* \circ I_{S_{f^*E}}^f$$

(84)

where $f_* : \Gamma(S_M) \to \Gamma(S_N)$ is the map (76). Remark that if $\langle \cdot, \cdot \rangle_{S_E|\nu_i} := (I_S)\ast \langle \cdot, \cdot \rangle_{S_M}$ and $\langle \cdot, \cdot \rangle_{S_{f^*E}|f^{-1}(\nu_i)} := (I_{S_{f^*E}})^{-1} \circ f^* \circ I_{S_{f^*E}}$, then

$$\int_{U_i} \langle f_1 s_1, s_2 \rangle_{S_E|\nu_i} = \int_{f^{-1}(U_i)} \langle s_1, f^1 s_2 \rangle_{S_{f^*E}|f^{-1}(\nu_i)},$$

(85)

for any $s_1 \in \Gamma(S_{f^*E}|f^{-1}(\nu_i))$ and $s_2 \in \Gamma_c(S_E|\nu_i)$, where $f^1 = (I_{S_{f^*E}})^{-1} \circ f^* \circ I_S$, cf. Lemma 18. In particular, (84) is well defined, up to multiplication by $\pm 1$.

Proposition 23. The pushforward $f_1 : \Gamma(f^1E) \to \Gamma(E)$ commutes with the canonical Dirac generating operators, i.e. $f_1 \circ \mathcal{D}_{f^*E} = \mathcal{D}_E \circ f_1$.

Proof. Like in the proof of Proposition 19, it is sufficient to show that

$$f_* \circ \mathcal{D}_M = \mathcal{D}_N \circ f_*,$$

(86)

where we preserve the notation from the proof of that proposition. From (83) we know

$$f_*(f^*(u) \cdot s) = u \cdot f_* s, \quad \forall u \in \Gamma(T^*N \oplus G), \ s \in \Gamma(S_M).$$

(87)

Iterating relation (87) and using (56), we see that (87) holds for any $u \in \Gamma \Lambda(T^*N \oplus G)$. Using the expression (23) of the canonical Dirac generating operator and property (87) of $f_*$ (with $u := H, \alpha_i, C_G$), we obtain that

$$\mathcal{D}_N f_* (\tilde{\omega} \otimes f^* s) = f_* \mathcal{D}_M (\tilde{\omega} \otimes f^* s), \quad \forall \tilde{\omega} \in \Omega(M), \ s \in \Gamma(S_G)$$

(88)

reduces to

$$f_* \mathcal{E}_M (\tilde{\omega} \otimes f^* s) = \mathcal{E}_N f_* (\tilde{\omega} \otimes s),$$

(89)
where
\[ E_N(\omega \otimes s) := (d\omega) \otimes s + \sum_i (\alpha_i \wedge \omega) \otimes \nabla_{X_i} s \]
\[ E_M(\tilde{\omega} \otimes f^* s) := (d\tilde{\omega}) \otimes f^* s + \sum_i ((f^* \alpha_i) \wedge \tilde{\omega}) \otimes f^*(\nabla_{X_i} s), \]
for any \( \omega \in \Omega(N) \), \( \tilde{\omega} \in \Omega(M) \) and \( s \in \Gamma(S_G) \). In order to show (89) it is sufficient to show that for any \( U \subset N \) open and sufficiently small and \( \beta \in \Gamma_c(\mathcal{S}_N|U) \),
\[ \int_U \langle E_N f_*(\tilde{\omega} \otimes f^* s), \beta \rangle_{\mathcal{S}_U} = \int_{f^{-1}(U)} \langle E_M(\tilde{\omega} \otimes f^* s), f^* \beta \rangle_{\mathcal{S}_{f^{-1}(U)}}. \quad (90) \]
From Corollary 9 and \( f^* E_N = E_M f^* \) we have
\[ \int_U \langle E_N f_*(\tilde{\omega} \otimes f^* s), \beta \rangle_{\mathcal{S}_U} = -\int_U \langle f_*(\tilde{\omega} \otimes f^* s), E_N \beta \rangle_{\mathcal{S}_U} \]
\[ = -\int_{f^{-1}(U)} \langle \tilde{\omega} \otimes f^* s, f^* E_N \beta \rangle_{\mathcal{S}_{f^{-1}(U)}} = -\int_{f^{-1}(U)} \langle \tilde{\omega} \otimes f^* s, E_M f^* \beta \rangle_{\mathcal{S}_{f^{-1}(U)}} \]
\[ = \int_{f^{-1}(U)} \langle E_M(\tilde{\omega} \otimes f^* s), f^* \beta \rangle_{\mathcal{S}_{f^{-1}(U)}}, \]
which proves (90).

5 Actions on transitive Courant algebroids

5.1 Basic properties

In this section we consider a class of actions on a transitive Courant algebroid which generalizes torus actions on exact and, more generally, on heterotic Courant algebroids. For the latter types of Courant algebroids, a notion of \( T \)-duality has been developed in [9] and [2] respectively.

Let \( E \) be a transitive Courant algebroid over a manifold \( M \), with anchor \( \pi : E \to TM \), Dorfmann bracket \([\cdot, \cdot]\) and scalar product \( \langle \cdot, \cdot \rangle \). Recall that the automorphism group \( \text{Aut}(E) \) of \( E \) is the group of orthogonal automorphisms \( F : E \to E \) which cover a diffeomorphism \( f : M \to M \), such that
\[ \pi(F(u)) = (d_pf)\pi(u), \ \forall u \in E_p, \ p \in M \]
and the natural map induced by \( F \) on the space of sections of \( E \) preserves the Dorfmann bracket. Its Lie algebra is the Lie algebra \( \text{Der}(E) \) of derivations
of $E$. This is the subalgebra of $\text{End} \Gamma(E)$ of first order linear differential operators $D : \Gamma(E) \to \Gamma(E)$ which satisfy, for any $s, s_1, s_2 \in \Gamma(E)$,

$$
D[s_1, s_2] = [Ds_1, s_2] + [s_1, Ds_2]
\quad
X(s_1, s_2) = \langle Ds_1, s_2 \rangle + \langle s_1, Ds_2 \rangle
\quad
\pi \circ D(s) = L_X \pi(s),
$$

where $X \in \mathfrak{X}(M)$ is a vector field on $M$, uniquely determined by $D$ (from the second relation (91)) and usually denoted by $\pi(D)$.

Let $\mathfrak{g}$ be a Lie algebra acting on $M$ by an infinitesimal action

$$
\psi : \mathfrak{g} \to \mathfrak{X}(M), \ a \mapsto \psi(a) = X_a.
$$

We will always assume (without repeating it each time) that all the infinitesimal actions considered are free, which means that the fundamental vector fields $X_a$ are non-vanishing, for all $a \in \mathfrak{g} \setminus \{0\}$.

**Definition 24.** i) An (infinitesimal) action of $\mathfrak{g}$ on $E$ which lifts $\psi$ is an algebra homomorphism $\Psi : \mathfrak{g} \to \text{Der}(E)$ which satisfies $\pi \Psi(a) = X_a$ for any $a \in \mathfrak{g}$.

ii) Let $\Psi : \mathfrak{g} \to \text{Der}(E)$ be an action of $E$ which lifts $\psi$. An invariant dissection of $E$ is a dissection $I : E \to T^*M \oplus \mathcal{G} \oplus TM$ for which the action

$$
\mathfrak{g} \ni a \to I \circ \Psi(a) \circ I^{-1} \in \text{Der}(T^*M \oplus \mathcal{G} \oplus TM)
$$

preserves the summands $T^*M$, $\mathcal{G}$ and $TM$.

We will only consider (without repeating it each time) infinitesimal actions on Courant algebroids for which there is an invariant dissection. The next proposition shows that this is automatically the case if the infinitesimal action is induced from an action of a compact group.

**Proposition 25.** Let $\Psi : G \to \text{Aut}(E)$ be an action of a compact group $G$ by automorphisms of a Courant algebroid $E$ over $M$, hence covering a group action $\psi : G \to \text{Diff}(M)$. Then $E$ admits a dissection invariant under $\Psi$.

**Proof.** By compactness of $G$ there exists a $G$-invariant positive definite metric $h$ in $E$. Using the auxiliary metric $h$ we can define a $G$-invariant splitting $\sigma_0 : TM \to E$ of the anchor map $\pi : E \to TM$, where $\sigma_0(TM)$ is $h$-orthogonal complement of $\text{Ker} \pi$. The section $\sigma_0$ of $\pi$ can be canonically modified to a $G$-invariant totally isotropic section $\sigma$ defined by

$$
\langle \sigma(X), v \rangle = \langle \sigma_0(X), v - \frac{1}{2} \sigma_0(\pi(v)) \rangle
$$

for all $X \in T_pM$, $v \in E_p$, $p \in M$. If we define $\mathcal{G}$ as the $\langle \cdot, \cdot \rangle$-orthogonal complement of $\pi^*T^*M \oplus \sigma(TM)$, then $E = \pi^*T^*M \oplus \mathcal{G} \oplus \sigma(TM)$ is a $G$-invariant dissection.
In the remaining part of this section we assume that

\[ E = T^*M \oplus \mathcal{G} \oplus TM \]

(92)
is a standard Courant algebroid, defined by a quadratic Lie algebra bundle \((\mathcal{G}, [\cdot, \cdot]_G, \langle \cdot, \cdot \rangle_G)\) and data \((\nabla, R, H)\) as in Section 2.2.1 and we consider in detail the class of actions \(\Psi : \mathfrak{g} \to \text{Der}(E)\) which lift \(\psi\) and preserve the factors \(T^*M, \mathcal{G}\) and \(TM\) of \(E\). From the third condition (91), the restriction of \(\Psi\) to \(TM\) is given by

\[ \Psi(a)(X) = \mathcal{L}_{X_a}X, \quad \forall a \in \mathfrak{g}, \ X \in \mathfrak{X}(M). \quad (93) \]

Since \(X_a\) (with \(a \in \mathfrak{g} \setminus \{0\}\)) are nowhere vanishing we can define

\[ \nabla^\Psi_{X_a}(p) := (\Psi(a)(r))(p), \quad \forall a \in \mathfrak{g}, \ r \in \Gamma(\mathcal{G}), \ p \in M, \]

which is a partial connection on \(\mathcal{G}\).

**Lemma 26.** There is a one to one correspondence between actions \(\Psi : \mathfrak{g} \to \text{Der}(E)\) which lift \(\psi\) and preserve the factors \(T^*M, \mathcal{G}, TM\) of \(E\) and partial connections \(\nabla^\Psi\) on \(\mathcal{G}\) such that the following conditions are satisfied:

i) \(\nabla^\Psi\) is flat and preserves \([\cdot, \cdot]_G\) and \(\langle \cdot, \cdot \rangle_G\):

ii) \(H\) and \(R\) are invariant, i.e. for any \(a \in \mathfrak{g}\),

\[ \mathcal{L}_{X_a}H = 0, \ \mathcal{L}_{\Psi(a)}R = 0 \quad (94) \]

where

\[ (\mathcal{L}_{\Psi(a)}R)(X, Y) := \nabla^\Psi_{X_a}(R(X, Y)) - R(\mathcal{L}_{X_a}X, Y) - R(X, \mathcal{L}_{X_a}Y) \quad (95) \]

for any \(X, Y \in \mathfrak{X}(M)\);

iii) for any \(a \in \mathfrak{g}\), the endomorphism \(A_a := \nabla^\Psi_{X_a} - \nabla_{X_a}\) satisfies

\[ (\nabla X A_a)(r) = [R(X_a, X), r]_G, \quad \forall r \in \Gamma(\mathcal{G}). \quad (96) \]

If i), ii) and iii) are satisfied, then the corresponding action \(\Psi\) acts naturally (by Lie derivative) on the subbundle \(T^*M \oplus TM\) of \(E\), i.e.

\[ \Psi(a)(\xi + X) = \mathcal{L}_{X_a}(\xi + X), \ X \in \mathfrak{X}(M), \ \xi \in \Omega^1(M), \quad (97) \]

and on \(\mathcal{G}\) by

\[ \Psi(a)(r) = \nabla^\Psi_{X_a}r, \quad r \in \Gamma(\mathcal{G}). \quad (98) \]

Moreover, for any \(a \in \mathfrak{g}\), the endomorphism \(A_a\) is a skew-symmetric derivation of \(\mathcal{G}\).
Proof. Let $\Psi$ be an action as in the statement of the lemma. From (93),

$$X_a(X, \eta) = \langle \mathcal{L}_{X_a}X, \eta \rangle + \langle X, \Psi(a)(\eta) \rangle, \quad \forall a \in \mathfrak{g}, \ X \in \mathfrak{X}(M), \ \eta \in \Omega^1(M),$$

and from the fact that $\Psi(a)$ preserves $\Omega^1(M) \subset \Gamma(E)$, we obtain that $\Psi(a)(\eta) = \mathcal{L}_{X_a}\eta$. Relation (97) follows. From our comments above, $\nabla^\Psi$ defined by (98) is a partial connection on $G$. Using (7) we obtain that the relations (91) satisfied by $\Psi$ are equivalent to the following conditions: $R$ and $H$ are invariant, $\nabla^\Psi$ is flat, preserves $\mathfrak{g}$ and $\langle \cdot, \cdot \rangle_G$ and $\langle \cdot, \cdot \rangle_G$, and

$$\begin{align*}
\nabla^\Psi_{X_a}Xr - \nabla_X \nabla^\Psi_{X_a}r - \nabla_{\mathcal{L}_{X_a}X}r &= 0 \\
\mathcal{L}_{X_a}(i_XR, r)_G &= \langle i_X\mathcal{L}_{X_a}X, r \rangle_G + \langle i_Xr, \nabla^\Psi_{X_a}r \rangle_G \\
\mathcal{L}_{X_a}(\nabla_r, \dot{r})_G &= \langle \nabla(\nabla^\Psi_r), \dot{r} \rangle_G + \langle \nabla r, \nabla^\Psi_{\dot{r}} \rangle_G,
\end{align*}$$

(99)

for any $a \in \mathfrak{g}$, $X \in \mathfrak{X}(M)$ and $r, \dot{r} \in \Gamma(G)$. The first relation (99) is equivalent to (96). Since both $\nabla$ and $\nabla^\Psi$ preserve $\langle \cdot, \cdot \rangle_G$ and $\langle \cdot, \cdot \rangle_G$, the endomorphism $A_a$ is a skew-symmetric derivation. The second relation (99) is equivalent to

$$\langle A_a R(X, Y), r \rangle_G + \langle R(X, Y), A_a r \rangle_G = 0$$

(100)

and follows from the skew-symmetry of the endomorphism $A_a$. The third relation (99) follows from the fact that $\nabla$ preserves $\langle \cdot, \cdot \rangle_G$, by writing $\nabla^\Psi_{X_a} = \nabla_{X_a} + A_a$ and using relation (96) together with $R^\mathcal{V}(X_a, X)(r) = [R(X_a, X), r]_G$.

\[ \square \]

Corollary 27. The skew-symmetric derivations $A_a$ from Lemma 26 satisfy

$$\nabla^\Psi_{X_a}(A_a) = A_{[b,a]}, \quad \forall a, b \in \mathfrak{g}.$$  

(101)

Proof. From $\nabla^\Psi_{X_a} = \nabla_{X_a} + A_a$, the flatness of $\nabla^\Psi$ and the expression (4) of $R^\mathcal{V}$, we obtain, for any $r \in \Gamma(G)$,

$$[R(X_a, X_b), r]_G + (d^\mathcal{V}A)(X_a, X_b)(r) + [A_a, A_b](r) = 0, \quad \forall a, b \in \mathfrak{g},$$

(102)

where $[A_a, A_b] := A_aA_b - A_bA_a$ is the commutator of $A_a$ and $A_b$. But

$$(d^\mathcal{V}A)(X_a, X_b)(r) = \left( \nabla_{X_a}A_b - \nabla_{X_b}A_a - A_{[a,b]} \right) r = 2[R(X_b, X_a), r]_G - A_{[a,b]}(r)$$

where we used relation (96) and $[X_a, X_b] = X_{[a,b]}$. We obtain

$$[R(X_b, X_a), r]_G - A_{[a,b]} r + [A_a, A_b](r) = 0, \quad \forall a, b \in \mathfrak{g}, \ r \in G.$$  

(103)
On the other hand,
\[
\nabla^\Psi_{X_b}(A_a)(r) = \nabla^\Psi_{X_b}(A_a(r)) - A_a(\nabla^\Psi_{X_b}r) \\
= \nabla_{X_b}(A_a)(r) + [A_b,A_a](r) \\
= [R(X_a,X_b),r]_G - [A_a,A_b](r) \tag{104}
\]
where in the second equality we used \(\nabla^\Psi_{X_b} = \nabla_{X_b} + A_b\) and in the third equality we used again relation (96). Combining (103) with (104) we obtain (101).

**Remark 28.**

i) The first relation (99) implies that \(\nabla\) is invariant, i.e.
\[
(L_{\Psi(a)}\nabla)_X r := \Psi(a)(\nabla_X r) - \nabla_{L_{X,a}X} r - \nabla_X(\Psi(a)r) = 0,
\]
for any \(a \in \mathfrak{g}, X \in \mathfrak{X}(M)\) and \(r \in \Gamma(G)\).

ii) Like for \(R\), we can define the Lie derivative
\[
(L_{\Psi(a)}\alpha)(X_1, \ldots, X_k) := \Psi(a)(\alpha(X_1, \ldots, X_k)) \\
- \alpha(L_{X_a}X_1, \ldots, X_k) - \cdots - \alpha(X_1, \ldots, L_{X_a}X_k),
\]
for any form \(\alpha \in \Omega^k(M,G)\). The Lie derivative so defined can be extended in the usual way to forms with values in the tensor bundle \(\mathcal{T}(G)\) of \(G\). In particular, for \(\alpha \in \Omega^k(M)\) we simply define \(L_{\Psi(a)}\alpha := L_{X,a}\alpha\). A \(\mathcal{T}(G)\)-valued form \(\alpha\) is called invariant if \(L_{\Psi(a)}\alpha = 0\) for any \(a \in \mathfrak{g}\).

iii) Relation (101) can be written in the equivalent way
\[
L_{\Psi(b)}(A_a) = A_{[b,a]}, \quad \forall a, b \in \mathfrak{g}. \tag{105}
\]
In particular, the endomorphisms \(A_a\) are invariant when \(\mathfrak{g}\) is abelian.

Let \(E_i\) \((i = 1,2)\) be two transitive Courant algebroids over \(M\) and \(\Psi_i : \mathfrak{g} \to \text{Der}(E_i)\) actions which lift \(\psi\). A fiber preserving Courant algebroid isomorphism \(F : E_1 \to E_2\) is called invariant if
\[
\Psi_2(a)(F(u)) = F\Psi_1(a)(u), \quad \forall a \in \mathfrak{g}, u \in \Gamma(E_1). \tag{106}
\]

**Lemma 29.** Let \(E_i\) \((i = 1,2)\) be standard Courant algebroids over \(M\) defined by quadratic Lie algebra bundles \((\mathcal{G}_i, [\cdot, \cdot]_{\mathcal{G}_i}, \langle \cdot, \cdot \rangle_{\mathcal{G}_i})\) and the data \((\nabla^{(i)}, R_i, H_i)\). Assume that \(E_i\) are endowed with actions \(\Psi_i : \mathfrak{g} \to \text{Der}(E_i)\) which lift \(\psi : \mathfrak{g} \to \mathfrak{X}(M)\) and preserve the factors \(T^*M, \mathcal{G}_i\) and \(TM\) of \(E_i\), and that a fiber preserving Courant algebroid isomorphism \(F : E_1 \to E_2\), defined by \((\beta, \Phi, K)\), where \(\beta \in \Omega^2(M), \Phi \in \Omega^1(M, \mathcal{G}_2)\) and \(K \in \text{Isom}(\mathcal{G}_1, \mathcal{G}_2)\), is given, as in (9). Let \(\nabla^{\Psi_i} := (\nabla^{(i)})^{\Psi_i}\) \((i = 1,2)\) be the partial connections associated with \(\nabla^{(i)}\) and \(\Psi_i\). Then \(F\) is invariant if and only if \(K\) maps \(\nabla^{\Psi_1}\) to \(\nabla^{\Psi_2}\) and the forms \(\beta\) and \(\Phi\) are invariant.

**Proof.** The proof uses the expression (7) for the Dorfman bracket. \hfill \(\Box\)
5.1.1 A class of $T^k$-actions

Let $(E = T^*M \oplus G \oplus TM, \langle \cdot, \cdot \rangle, [\cdot, \cdot])$ be a standard Courant algebroid over the total space of a principal $T^k$-bundle $\pi : M \to B$, where $T^k = \mathbb{R}^k/\mathbb{Z}^k$ denotes the $k$-dimensional torus. We assume that $E$ is defined by a bundle of quadratic Lie algebras $(\mathcal{G}, \langle \cdot, \cdot \rangle_\mathcal{G}, [\cdot, \cdot]_\mathcal{G})$ and data $(\nabla, R, H)$, where $\nabla$ is a connection on the vector bundle $\mathcal{G}$ compatible with the tensor fields $\langle \cdot, \cdot \rangle_\mathcal{G}$ and $[\cdot, \cdot]_\mathcal{G}$, $R \in \Omega^2(M, \mathcal{G})$ and $H \in \Omega^1(M)$. Recall that these data satisfy the compatibility equations

$$dH = \langle R \wedge R \rangle, \quad d\nabla R = 0, \quad R^\nabla = \text{ad}_R,$$

where $\langle R \wedge R \rangle_\mathcal{G}$ is abbreviated as $\langle R \wedge R \rangle$ in harmony with the fact that $\langle \cdot, \cdot \rangle_\mathcal{G} = \langle \cdot, \cdot \rangle|_{\mathcal{G} \times \mathcal{G}}$. The Dorfmann bracket, scalar product and anchor of $E$ are then expressed by the usual formulas in terms of the above data.

We assume that the vertical parallelism of $\pi$ is lifted to an action of $t^k$ on $E$,

$$\Psi : t^k = \mathbb{R}^k \to \text{Der}(E), \quad a \mapsto \Psi(a) = (\xi + r + X, L_{X_a}X + \nabla^\Psi_{X_a}r + L_{X_a}X),$$

where $X_a$ is the fundamental vector field of $\pi$ determined by $a \in t^k$ and $\nabla^\Psi$ is a partial flat connection on $\mathcal{G}$. We recall (see Lemma 26) that $\nabla^\Psi$ preserves $[\cdot, \cdot]_\mathcal{G}$ and $\langle \cdot, \cdot \rangle_\mathcal{G}$ and that

$$L_{X_a}R = 0, \quad L_{X_a}H = 0, \quad \langle \nabla_X A_a \rangle(r) = [R(X_a, X), r]|_\mathcal{H},$$

where $A_a := \nabla^\Psi_{X_a} - \nabla_{X_a} \in \text{End}(\mathcal{G})$ is a skew-symmetric derivation, which is invariant since $t^k$ is abelian (see Remark 28). Recall also that $L_{X_a}\nabla = 0$.

We consider

$$\Omega^s_t(M, \mathcal{G}) := \{ \alpha \in \Omega^s(M, \mathcal{G}) \mid L_{\Psi(a)}\alpha = 0, i_{X_a}\alpha = 0, \forall a \in t^k \},$$

the space of basic $\mathcal{G}$-valued $s$-forms on $M$. The space $\Omega^s_t(M)$ of basic scalar valued $s$-forms on $M$ can be defined similarly and coincides with $\pi^*\Omega^s(B) \cong \Omega^s(B)$. The analogous fact for $\Omega^s_t(M, \mathcal{G})$ is stated in the next proposition.

**Proposition 30.** $\Omega^s_t(M, \mathcal{G}) \cong \pi^*\Omega^s(B) \otimes \Gamma_t \mathcal{G}$, where $\Gamma_t \mathcal{G}$ denotes the space of $t^k$-invariant (i.e. $\nabla^\Psi$-parallel) sections.

**Proof.** Let $U \subset B$ be an open set such that $\Lambda^s T^*B|_U$ is trivial. Then any horizontal form $\alpha \in \Omega^s(\pi^{-1}(U), \mathcal{G})$ (i.e. $i_{X_a}\alpha = 0$ for any $a \in t^k$) can be written as $\alpha = \sum_{i}(\pi^*\beta_i) \otimes s_i$ where $(\beta_i)$ is a basis of $\Lambda^s T^*B|_U$ and $s_i \in \Gamma(\mathcal{G}|_{\pi^{-1}(U)})$. Then $L_{\Psi(a)}\alpha = \sum_{i}(\pi^*\beta_i) \otimes L_{\Psi(a)}s_i$ from where we deduce that $\Omega^s_t(\pi^{-1}(U), \mathcal{G}) = \pi^*\Omega^s(\mathcal{G}|_{\pi^{-1}(U)})$. Using a partition of unity in $B$ one can deduce that the same holds globally for $U = B$. \qed
Lemma 31. For any invariant section where for any form \( \omega \) is \( \text{ad} \) 

From now on we will assume that the partial connection \( \nabla \) on the vector bundle \( G \) which allows to freely write decomposable elements as \( \omega \otimes r \) or as \( r \otimes \omega \). Let \((e_i)\) be a basis of \( t^k \), \( X_i := X_{e_i} \) the associated fundamental vector fields and \( A_i := A_{e_i} = \nabla^\Psi_{X_i} - \nabla_{X_i} \in \text{End}(G) \). We choose a connection \( \mathcal{H} \) on the principal bundle \( \pi : M \to B \), with connection form \( \theta = \sum_{i=1}^k \theta_i e_i \). We introduce the connection \( \nabla^\theta := \nabla + \sum_{i=1}^k \theta_i \otimes A_i \) (109) on the vector bundle \( G \). The curvature \( R^\nabla \) of \( \nabla \) and \( R^\theta \) of \( \nabla^\theta \) are related by 

\[
R^\nabla = R^\theta - \sum_{i=1}^k (d\theta_i) \otimes A_i + \sum_{i=1}^k \theta_i \wedge \text{ad}_{R(X_i, \cdot)} - \frac{1}{2} \sum_{i,j} (\theta_i \wedge \theta_j) \otimes [A_i, A_j],
\]

where for any form \( \omega \in \Omega^s(M, G) \) (in particular \( \omega := R(X_i, \cdot) \)) we define \( \text{ad}_\omega \in \Omega^s(M, \text{End} G) \) by 

\[
(\text{ad}_\omega)(Y_1, \cdots, Y_s)(r) := [\omega(Y_1, \cdots, Y_s), r]|_G, \ \forall Y_i \in \mathfrak{X}(M).
\]

Lemma 31. For any invariant section \( r \in \Gamma\psi(G) \), the \( G \)-valued 1-form \( \nabla^\theta r \) is basic.

Proof. The form \( \nabla^\theta r \) is horizontal, since 

\[
\nabla^\theta_{X_i} r = \nabla^\Psi_{X_i} r = \mathcal{L}_{\psi(e_i)} r = 0, \ \forall 1 \leq i \leq k.
\]

On the other hand, as \( \nabla, \theta \) and, from Remark 28 iii), \( A_i \) are \( t^k \)-invariant, so is \( \nabla^\theta \) and 

\[
\mathcal{L}_{\psi(a)}(\nabla^\theta r) = \mathcal{L}_{\psi(a)}(\nabla^\theta) r + \nabla^\theta \mathcal{L}_{\psi(a)} r = 0, \quad (111)
\]

which implies that \( \nabla^\theta r \) is \( t^k \)-invariant. \( \square \)

Assumption 32. From now on we will assume that the partial connection \( \nabla^\Psi \) has trivial holonomy. Then we can define a bundle \( G_B \to B \) whose fiber over \( p \in B \) is 

\[
G_B|p := \Gamma\psi(G|\pi^{-1}(p)),
\]

the vector space of \( \nabla^\Psi \)-parallel sections of \( G \) over the torus \( \pi^{-1}(p) \). We identify \( G = \pi^*G_B, \Gamma\psi(G) = \pi^*\Gamma(G_B) \) and 

\[
\Omega^s_B(M, G) = \pi^*\Omega^s(B) \otimes \pi^*\Gamma(G_B) \cong \Omega^s(B, G_B).
\]

For a basic form \( \alpha \in \Omega^s_B(M, G) \) we shall denote by \( \alpha^B \in \Omega^s(B, G_B) \) the corresponding form in the above identification. (Note that by working locally in a flow box for the vertical foliation of \( M \to B \), we can always assume that \( \nabla^\Psi \) has trivial holonomy. We recall that a flow box is a domain \( V \subset M \) such that for all \( p \in \pi(V) \subset B \) the manifolds \( \pi^{-1}(p) \cap V \) are diffeomorphic to \( \mathbb{R}^k \).)
Lemma 33. i) The bundle $G_B$ inherits a bracket $[,]_{G_B}$ and a scalar product $(\cdot,\cdot)_{G_B}$ which make $(G_B,[,]_{G_B},(\cdot,\cdot)_{G_B})$ into a quadratic Lie algebra bundle.

ii) The connection $\nabla^\theta$ induces a connection $\nabla^{\theta,B}$ on $G_B$, which preserves $[,]_{G_B}$ and $(\cdot,\cdot)_{G_B}$. The curvature $R^\theta$ of $\nabla^\theta$ is the pullback of the curvature $R^\theta(B)$ of $\nabla^\theta_B$, i.e. $(R^\theta)^B = R^\theta_B$.

Proof. i) The claim follows from Lemma 26 i) and the definition of $G_B$.

ii) From Lemma 31, $\nabla^\theta$ induces a connection $\nabla^{\theta,B}$ on $G_B$, defined by

$$\nabla^{\theta,B} r^B = (\nabla^\theta \pi^* r^B)^B, \forall r^B \in \Gamma(G_B).$$

Since $A_i$ are skew-symmetric derivations, and $\nabla$ preserves $[,]_G$ and $(\cdot,\cdot)_G$, we obtain that $\nabla^\theta$ preserves these tensor fields as well. We deduce that $\nabla^{\theta,B}$ preserves $[,]_{G_B}$ and $(\cdot,\cdot)_{G_B}$. The last statement is trivial.

In order to describe the Courant algebroid $E$ together with the action $\Psi : t^k \to \Gamma(E)$ in terms of structures on the base manifold $B$ of the torus bundle, we need to interpret equations (107) and (108) on $B$. As $H$ and $R$ are invariant, they are of the form

$$H = H^{(3)} + \theta_i \wedge H^{i(2)} + \theta_i \wedge \theta_j \wedge H^{ij(1)} + H^{ijk(0)} \theta_i \wedge \theta_j \wedge \theta_k,$$

$$R = R^{(2)} + \theta_i \wedge R^{i(1)} + R^{ij(0)} \theta_i \wedge \theta_j$$

(112)

where $H^{(3)}, H^{i(2)}, H^{ij(1)}, R^{(2)}, R^{i(1)}, R^{ij(0)}$ are basic and for simplicity of notation we omit the summation signs. In the next lemma we denote by $d^{\theta,B}$ the exterior covariant derivative defined by $\nabla^{\theta,B}$. Let $A^B_a$ be the section of $\text{End}(G_B)$ defined by the invariant section $A_a \in \text{End}(G)$, where $a \in t^k$.

Lemma 34. i) The compatibility equations listed in (107) are satisfied if and
only if the following conditions hold:

\[ dH^B_{(3)} + H^i^B \wedge (d\theta_i)^B = \langle R^B_{(2)} \wedge R^B_{(2)} \rangle_{G_B} \] (113)

\[ dH^{p,B}_{(2)} + 2H^{p,B}_{(1)} \wedge (d\theta_i)^B = -2\langle R^B_{(2)} \wedge R^{p,B}_{(1)} \rangle_{G_B} \] (114)

\[ dH^{pq,B}_{(1)} + 3H^{pq,B}_{(0)} (d\theta_i)^B = 2\langle R^{pq,B}_{(0)} , R^B_{(2)} \rangle_{G_B} - \langle R^{pq,B}_{(1)} , R^B_{(1)} \rangle_{G_B} \] (115)

\[ 3dH^{pq,B}_{(0)} + 2(\langle R^{pq,B}_{(0)} , R^B_{(1)} \rangle_{G_B} + \langle R^{pq,B}_{(0)} , R^{p,B}_{(1)} \rangle_{G_B} + \langle R^{qs,B}_{(0)} , R^{q,B}_{(1)} \rangle_{G_B}) = 0 \] (116)

\[ \langle R^{ij,B}_{(0)} , R^{q,B}_{(1)} \rangle_{B} \theta_i \wedge \theta_j \wedge \theta_p \wedge \theta_q = 0 \] (117)

\[ d\theta_i^B + R^{i,B}_{(1)} \wedge (d\theta_i)^B = 0 \] (118)

\[ d\theta_i^B R^{p,B}_{(2)} + R^{i,B}_{(1)} \wedge (d\theta_i)^B = 0 \] (119)

\[ A_p^B \wedge R^{q,B}_{(1)} - A_q^B \wedge R^{p,B}_{(1)} = 2\nabla^B_{\theta_i} R^{pq,B}_{(0)} \] (120)

\[ A^B_p R^{p,q,B}_{(0)} + A^B_q R^{q,p,B}_{(0)} + A^B_s R^{qs,B}_{(0)} = 0 \] (121)

\[ R^{\theta,B} = (d\theta_i)^B \otimes A_i^B + \text{ad}_{R^{(2)}} \] (122)

\[ \text{ad}_{R^{ij,B}_{(0)}} = \frac{1}{2}[A_i^B , A_j^B], \] (123)

where

\[ \text{ad} : G_B \rightarrow \text{Der}(G_B), \quad \text{ad}_u(v) = [u,v]_{G_B} \] (124)

is the adjoint representation in the Lie algebra bundle \((G_B, [\cdot, \cdot]_{G_B})\) and \(1 \leq p, q, s \leq k\) are arbitrary.

ii) If the compatibility relations (107) are satisfied, then the third relation (108) is satisfied as well if and only if

\[ \nabla^B_{\theta_i} A_i^B = [R^{i,B}_{(1)}(X) , r]_{G_B} , \forall X \in \mathfrak{X}(M). \] (125)

Proof. i) The equations (113)-(117) are obtained from the first relation (107), by comparing

\[ dH = dH_{(3)} + (d\theta_i) \wedge H^i_{(2)} - \theta_i \wedge dH^i_{(2)} + 2(d\theta_i) \wedge \theta_j \wedge H^j_{(1)} \]

\[ + \theta_i \wedge \theta_j \wedge dH^j_{(1)} + 3H^j_{(0)} (d\theta_i) \wedge \theta_j \wedge \theta_s + (dH^j_{(0)}) \wedge \theta_i \wedge \theta_j \wedge \theta_s \]

with

\[ \langle R \wedge R \rangle = \langle R_{(2)} \wedge R_{(2)} \rangle + 2\theta_i \wedge \langle R_{(2)} \wedge R^i_{(1)} \rangle + 2\theta_i \wedge \theta_j \wedge \langle R_{(2)} , R^i_{(0)} \rangle \]

\[ - \theta_i \wedge \theta_j \wedge \langle R^i_{(1)} \wedge R^j_{(1)} \rangle + 2\theta_i \wedge \theta_j \wedge \theta_s \wedge \langle R^j_{(0)} , R^i_{(1)} \rangle \]

\[ + \langle R^j_{(0)} , R^i_{(0)} \rangle \theta_i \wedge \theta_j \wedge \theta_p \wedge \theta_q. \]

using \(d\theta_i \in \Omega^2(B)\), that the exterior derivative maps basic forms to basic forms, that the operation \((\alpha, \beta) \mapsto \langle \alpha \wedge \beta \rangle\) maps a pair of \(G\)-valued basic forms
to a basic scalar valued form and then interpreting the resulting relations on $B$. The equations (118)-(121) are obtained from the second relation (107), by computing

$$0 = d^\nabla R = d^\nabla R_{(2)} + (d\theta_i) \land R^i_{(1)} - \theta_i \land d^\nabla R^i_{(1)} + (\nabla R^i_{(0)}) \land \theta_i \land \theta_j + 2R^i_{(0)} \otimes (d\theta_i) \land \theta_j$$

$$= d^\theta R_{(2)} - (\theta_i \otimes A_j) \land R_{(2)} + R^i_{(1)} \land d\theta_i - \theta_j \land \left( d^\theta R^i_{(1)} - (\theta_i \otimes A_j) \land R^i_{(1)} \right)$$

$$+ (\nabla \theta R^i_{(0)}) \land \theta_i \land \theta_j - A_i (R^i_{(0)}) \theta_i \land \theta_j \land \theta_k + 2R^i_{(0)} \otimes (d\theta_i) \land \theta_j,$$  (126)

identifying the horizontal and vertical parts in the last expression of (126) and interpreting the result on $B$. The remaining equations (122) and (123) are obtained by writing $R^\nabla$ in terms of $R^\theta$ as in (110) and identifying the horizontal and vertical parts in $R^\nabla = ad_R$.

ii) The third relation (108) is equivalent to relation (125), together with relation (123). \qed

Since $\nabla^{\theta,B}$ preserves $[,]_{\mathfrak{g}_B}$, the endomorphism $R^{\theta,B}(X,Y)$ of $\mathcal{G}_B$ is a derivation, for any $X,Y \in \mathfrak{X}(B)$. Recall that $A^B_i \in \text{End}(\mathcal{G}_B)$ is also a derivation. The conditions from Lemma 34 simplify considerably when the adjoint representation (124) of the Lie algebra bundle $(\mathcal{G}_B, [,]_{\mathfrak{g}_B})$ is an isomorphism. Then

$$A^B_i = \text{ad}_{r^B}, \quad R^{\theta,B}(X,Y) = \text{ad}_{r^{\theta,B}(X,Y)}$$  (127)

for $r^B \in \Gamma(\mathcal{G}_B)$ and $r^{\theta,B} \in \Omega^2(B, \mathcal{G}_B)$. From the Bianchi identity we obtain that $r^{\theta,B}$ is $d^\theta$-closed.

**Remark 35.** i) Consider the class $\mathcal{C}$ of quadratic Lie algebras $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ for which the adjoint representation is an isomorphism onto the Lie algebra of skew-symmetric derivations of $\mathfrak{g}$. Every semi-simple Lie algebra endowed with its Killing form (or any other invariant scalar product) belongs to this class. Since the center of a quadratic Lie algebra coincides with $[\mathfrak{g}, \mathfrak{g}]^\perp$, there is no non-zero solvable Lie algebra in $\mathcal{C}$. Nevertheless, the class $\mathcal{C}$ is strictly larger than the class of semi-simple quadratic Lie algebras. For instance, the affine Lie algebra $\mathfrak{so}(3) \ltimes \mathfrak{so}(3)^* \cong \mathfrak{so}(3) \ltimes \mathbb{R}^3$ can be endowed with the invariant scalar product of neutral signature defined by duality. The adjoint representation is faithful and one can easily check that all skew-symmetric derivations are inner. There exist solvable Lie algebras with faithful adjoint representation for which all derivations are inner [18]. However these do not admit any invariant scalar product as we have already remarked.

ii) The adjoint representation of the Lie algebra bundle $(\mathcal{G}_B, [,]_{\mathfrak{g}_B})$ is an isomorphism if and only if the same is true for the Lie algebra bundle
More precisely, relations \((\text{adjoint representation of the Lie algebra bundle } (G, \mathcal{G}, [\cdot, \cdot]_G))\) of the Courant algebroid \(E\). Courant algebroids with this property will be described in Proposition 67.

**Corollary 36.** Let \(\pi : M \to B\) be a principal \(T^k\)-bundle and \(\mathcal{H}\) a principal connection on \(\pi\), with connection form \(\theta = \sum_{i=1}^{k} \theta_i e_i \in \Omega^1(M, \mathbb{T}^k)\), where \((e_i)\) is a basis of \(\mathbb{T}^k\). There is a one to one correspondence between

1. standard Courant algebroids \(E = T^*M \oplus \mathcal{G} \oplus TM\) for which the adjoint action of the Lie algebra bundle \((G, \mathcal{G}, [\cdot, \cdot]_G)\) is an isomorphism, together with an action \(\Psi : \mathfrak{t}^k \to \text{Der}(E)\) which lifts the vertical parallelism of \(\pi\), preserves the factors \(T^*M, \mathcal{G}\) and \(TM\) of \(E\), and for which the flat partial connection \(\nabla^\Psi\) has trivial holonomy

and

2. quadratic Lie algebra bundles \((\mathcal{G}_B, \langle \cdot, \cdot \rangle_{\mathcal{G}_B}, [\cdot, \cdot]_{\mathcal{G}_B})\) over \(B\), whose adjoint action is an isomorphism, together with a connection \(\nabla^B\) on the vector bundle \(\mathcal{G}_B\) which preserves \(\langle \cdot, \cdot \rangle_{\mathcal{G}_B}\) and \([\cdot, \cdot]_{\mathcal{G}_B}\), sections \(r^B \in \Gamma(\mathcal{G}_B)\) \((1 \leq i \leq k)\), a 3-form \(H^B_{(3)} \in \Omega^3(B)\), 2-forms \(H^B_{(2)} \in \Omega^2(B)\), 1-forms \(H^B_{(1)} \in \Omega^1(B)\) and constants \(c_{ijp} \in \mathbb{R}\) \((1 \leq i, j, p \leq k)\) such that

\[
dH^B_{(3)} = \langle \mathfrak{r}^B \wedge \mathfrak{r}^B \rangle_{\mathcal{G}_B} - (H^B_{(2)}) + 2 \langle \mathfrak{r}^B, r^B_{p} \rangle_{\mathcal{G}_B} - \langle r^B, r^B_{j} \rangle_{\mathcal{G}_B} (d\theta_j)^B \wedge (d\theta_i)^B,
\]

\[
dH^B_{(2)} = 2 \langle \langle \nabla^B r^B_{p}, r^B_{j} \rangle_{\mathcal{G}_B} - H^B_{(1)} \rangle_{\mathcal{G}_B} \wedge (d\theta_i)^B - 2 \langle \mathfrak{r}^B \wedge \nabla^B r^B_{p} \rangle_{\mathcal{G}_B} - \langle \nabla^B r^B_{p} \wedge \nabla^B r^B_{j} \rangle_{\mathcal{G}_B},
\]

\[
dH^B_{(1)} = -3c_{ijp} (d\theta_i)^B + \langle \mathfrak{r}^B, [r^B_{p}, r^B_{j}]_{\mathcal{G}_B} \rangle_{\mathcal{G}_B} - \langle \nabla^B r^B_{p} \wedge \nabla^B r^B_{j} \rangle_{\mathcal{G}_B},
\]

where \(\mathfrak{r}^B \in \Omega^2(B, \mathcal{G}_B)\) is related to the curvature \(\mathfrak{r}^B\) of the connection \(\nabla^B\) by \(\mathfrak{r}^B(X, Y) = \text{ad}_{\mathfrak{r}^B(X, Y)}\) for any \(X, Y \in \mathfrak{X}(B)\).

**Proof.** The claim follows from Lemma 34, by letting \(\nabla^B := \nabla^\theta B\) and simplifying the relations from this lemma, using in an essential way that the adjoint representation of the Lie algebra bundle \((G, \mathcal{G}, [\cdot, \cdot]_G)\) is an isomorphism. More precisely, relations \((123), (125)\) and \((122)\) determine \(R^B_{(0)}\), \(R^B_{(1)}\) and \(R^B_{(2)}\) respectively by

\[
R^B_{(0)} = \frac{1}{2} \langle [r^B_{i}, r^B_{j}]_{\mathcal{G}_B}, \mathfrak{r}^B \rangle_{\mathcal{G}_B}, \quad R^B_{(1)} = \nabla^B r^B_{i}, \quad R^B_{(2)} = \mathfrak{r}^B - (d\theta_i)^B \wedge r^B_{i}.
\]

Relation \((116)\) with \(R^B_{(0)}\) and \(R^B_{(1)}\) given by \((129)\) implies that

\[
H^B_{pq,s} = \frac{1}{3} \langle [r^B_{p}, r^B_{q}]_{\mathcal{G}_B}, r^B_{s} \rangle_{\mathcal{G}_B} + c_{pq,s}
\]
for some constants $c_{pqrs}$. Written in terms of $r^B$ rather than $R^{B}_{(2)}$, relations (113), (114), (115) become relations (128). The remaining relations from Lemma 34, with $R^{ij,B}_{(0)}, R^{i,B}_{(1)}, R^B_{(2)}$ and $H^{pqrs,B}_{(0)}$ as above and $A_i = \text{ad}_{r^B_i}$ are satisfied.

**Example 37.** Under the assumptions of Corollary 36, let $(\mathcal{G}_B, \langle \cdot, \cdot \rangle_{\mathcal{G}_B}, [\cdot, \cdot]_{\mathcal{G}_B})$ be a quadratic Lie algebra bundle over $B$, whose adjoint action is an isomorphism, together with a connection $\nabla^B$ on the vector bundle $\mathcal{G}_B$ which preserves $\langle \cdot, \cdot \rangle_{\mathcal{G}_B}$ and $[\cdot, \cdot]_{\mathcal{G}_B}$. Choose arbitrary sections $r^B_i \in \Gamma(\mathcal{G}_B)$ ($1 \leq i \leq k$) and define, for any $i, j, s$,

\[
H^{ij,s,B}_{(0)} := -\frac{1}{3} \langle [r^B_i, r^B_j]_{\mathcal{G}_B}, r^B_s \rangle_{\mathcal{G}_B} \tag{131}
\]

and

\[
H^{ij,B}_{(1)} := \frac{1}{2} \left( \langle \nabla^B r^B_i, r^B_j \rangle_{\mathcal{G}_B} - \langle \nabla^B r^B_j, r^B_i \rangle_{\mathcal{G}_B} \right). \tag{132}
\]

With these choices, the third relation (128) is satisfied. For any forms $H^{B}_{(3)}$ and $H^{ij,B}_{(2)}$, such that

\[
K_i := H^{ij,B}_{(2)} + 2\langle r^B_i, r^B_j \rangle_{\mathcal{G}_B} - \langle r^B_i, r^B_j \rangle_{\mathcal{G}_B} (d\theta_j)^B \tag{133}
\]

is closed and

\[
dH^{B}_{(3)} = \langle r^B \wedge r^B \rangle_{\mathcal{G}_B} - K_i \wedge (d\theta_i)^B \tag{134}
\]

the relations (128) are satisfied and we thus obtain a standard Courant algebroid together with an action $\Psi : \mathfrak{k} \to \text{Der}(E)$ lifting the vertical parallelism of the principal torus bundle $\pi : M \to B$. Note that 2-forms $H^{ij,B}_{(2)}$ as required in the above construction do always exist and are unique up to addition of closed forms whereas $H^{B}_{(3)}$ exists if and only if the closed form $\langle r^B \wedge r^B \rangle_{\mathcal{G}_B} - K_i \wedge (d\theta_i)^B$ is exact. It is also unique up to addition of a closed form.

### 5.2 Invariant spinors

Let $E$ be a transitive Courant algebroid over an oriented manifold $M$ and $\Psi : \mathfrak{g} \to \text{Der}(E)$ an action on $E$, which lifts an action $\psi : \mathfrak{g} \to X(M), a \mapsto X_a$ of $\mathfrak{g}$ on $M$. Let $S$ be a canonical weighted spinor bundle of $E$. Our aim in this section is to define an action of $\mathfrak{g}$ on $\Gamma(S)$. In order to find a proper definition we assume that $\Psi$ integrates to a Lie group action

\[
G \to \text{Aut}(E), \ g \mapsto I^g_E : E \to E
\]
such that $I_E^g$ induces a globally defined isomorphism $I_S^g : S \to S$, for any $g \in g$. Recall that

$$I_S^g \circ \gamma_u = \gamma_{I_E^g(u)} \circ I_S^g, \ \forall g \in G, \ u \in E.$$ 

Consider a curve $g = g(t)$ of $G$ with $g(0) = e$ and $g'(0) = a$. We choose $I_S^{g(t)}$ depending smoothly on $t$ and such that $I_S^{g(0)} = \text{Ids}$. Replacing in the above relation $g$ by $g(t)$ and taking the derivative at $t = 0$ we obtain that

$$\Psi^S(a) := \frac{d}{dt} \big|_{t=0} I_S^{g(t)} \in \text{End } \Gamma(S)$$

satisfies

$$\Psi^S(a) \gamma_s = \gamma_{\Psi^S(a)s} + \gamma_a \Psi^S(a)s, \ \forall u \in \Gamma(E), \ s \in \Gamma(S), \ a \in g.$$ (135)

In the following we do not assume that $\Psi$ integrates to an action of $G$.

**Proposition 38.** i) There is a unique linear map

$$\Psi^S : g \to \text{End } \Gamma(S)$$

which satisfies relation (135), the Leibniz rule

$$\Psi^S(a)(fs) = f\Psi^S(a)(s) + X_a(f)s, \ \forall f \in C^\infty(M), \ s \in \Gamma(S), \ a \in g.$$ (136)

and, for any $U \subset M$ open and sufficiently small, preserves the canonical $\det(T^*U)$-valued bilinear pairing $\langle \cdot, \cdot \rangle_{S|U}$ of $\Gamma(S|U)$, i.e.

$$\mathcal{L}_{X_a} \langle s, \tilde{s} \rangle_{S|U} = \langle \Psi^S(a)s, \tilde{s} \rangle_{S|U} + \langle s, \Psi^S(a)\tilde{s} \rangle_{S|U}, \ \forall s, \tilde{s} \in \Gamma(S|U), \ a \in g.$$ (137)

ii) The map $\Psi^S : g \to \text{End } \Gamma(S)$ satisfies

$$[\Psi^S(a), \Psi^S(b)] = \Psi^S[a, b], \ \forall a, b \in g.$$ (138)

It is called the action on spinors induced by $\Psi$.

The remaining part of this section is devoted to the proof of Proposition 38. For uniqueness, let $\Psi^S$ and $\tilde{\Psi}^S$ be two maps which satisfy the required conditions. Then $F(a) := \Psi^S(a) - \tilde{\Psi}^S(a)$ is $C^\infty(M)$-linear and commutes with the Clifford action. Hence $F(a) = \lambda(a) \text{Id}_g$, for $\lambda(a) \in C^\infty(M)$. Since $F(a)$ is skew-symmetric with respect to $\langle \cdot, \cdot \rangle_{S|U}$, we obtain $\lambda(a) = 0$. The uniqueness follows. For existence we need the following two lemmas.

**Lemma 39.** Let $E_i$ $(i = 1, 2)$ be two transitive Courant algebroids over $M$ with actions $\Psi_i : g \to \text{Der}(E_i)$, which lift $\psi$. Let $I_E : E_1 \to E_2$ be an invariant isomorphism and, for any $U \subset M$ open and sufficiently small, $I_{S|U} : S_1|U \to S_2|U$ the induced isomorphism between canonical weighted spinor bundles of $E_i|U$. Let $\Psi_{S_i} : g \to \text{End } \Gamma(S_i|U)$ $(i = 1, 2)$ be two maps related by

$$\Psi^{S_2}(a) = I_{S|U} \circ \Psi^{S_1}(a) \circ (I_{S|U})^{-1}, \ \forall a \in g.$$ (139)

Then $\Psi^{S_i}$ satisfies the conditions from Proposition 38 if and only if $\Psi^{S_2}$ does.
Proof. Let $\Psi^S_1 : g \rightarrow \text{End} \Gamma(S_1|_U)$ be a map which satisfies the conditions from Proposition 38 and $\Psi^S_2 : g \rightarrow \text{End} (S_2|_U)$ be defined by (139). The map $\Psi^S_2$ obviously satisfies (136) and (138) and, from (41), it satisfies (137) as well. Using $I_{S|U} \circ \gamma_u = \gamma_{I_E(u)} \circ I_{S|U}$ and relation (135) satisfied by $\Psi^S_1$, we obtain

$$\Psi^S_2(a) \gamma_u(s) = \gamma_{I_E(a)} I_{E^{-1}}(u) s + \gamma_u \Psi^S_2(a) s, \quad \forall a \in g, \quad u \in \Gamma(E_1|_U), \quad s \in \Gamma(S_2|_U).$$

(140)

Since $I_E$ is invariant, $\Psi^S_2(a) = I_E \circ \Psi^S_1(a) \circ I_{E^{-1}}$ and we obtain that $\Psi^S_2$ satisfies (135).

Let $E_M = T^*M \oplus G \oplus TM$ be a standard Courant algebroid defined by a quadratic Lie algebra bundle $(G, [\cdot, \cdot]_G, \langle \cdot, \cdot \rangle_G)$ and data $(\nabla, R, H)$, with action

$$\Psi : g \mapsto \text{Der}(E_M), \quad \Psi(a)(\xi + r + X) := \mathcal{L}_{X_a} \xi + \nabla^\Psi_{X_a} r + \mathcal{L}_{X_a} X$$

(141)

which lifts an action

$$\psi : g \mapsto \mathfrak{X}(M), \quad a \mapsto X_a$$

of $g$ on $M$. Let $S_G$ be an irreducible $\text{Cl}(G)$-bundle, $S_G = S_G \otimes \vert \det S^* \vert^{1/r}$ the canonical spinor bundle and $S_M := \Lambda(T^*M) \otimes S_G$ the canonical weighted spinor bundle of $E$ determined by $S_G$.

**Lemma 40.** The map

$$\Psi^{SM} : g \rightarrow \text{End} (S_M), \quad \Psi^{SM}(a)(\omega \otimes s) := (\mathcal{L}_{X_a} \omega) \otimes s + \omega \otimes \nabla^\Psi_{X_a} s$$

(142)

for any $a \in g$, $\omega \in \Omega(M)$ and $s \in \Gamma(S_G)$, satisfies the conditions from Proposition 38. Above $\nabla^\Psi_{S_G}$ is the partial connection on $S_G$ induced by any partial connection $\nabla^\Psi_{S_M}$ on $S_G$, compatible with the partial connection $\nabla^\Psi$.

**Proof.** Relation (136) is obviously satisfied. To prove relation (137) we recall that $\langle \cdot, \cdot \rangle_{S|U}$ is given by (32), where $\langle \cdot, \cdot \rangle_{S_G|U}$ is a canonical bilinear pairing of $\Gamma(S_G|_U)$. Relation (137) follows from a computation which uses (32),

$$X_a(s, \tilde{s})_{S_G|U} = \langle \nabla^\Psi_{X_a} S_G, s \rangle_{S_G|U} + \langle s, \nabla^\Psi_{X_a} \tilde{s} \rangle_{S_G|U}$$

(143)

and the fact that $\nabla^\Psi_{S_G}$ preserves the degree of sections of $S_G$. (Relation (143) follows from the fact that $\nabla^\Psi$ preserves $\langle \cdot, \cdot \rangle_G$, which is of neutral signature, and $\nabla^\Psi_{S_G}$ is induced by any partial connection $\nabla^\Psi_{S_M}$ on $S_G$ compatible with $\nabla^\Psi$. The argument is similar to the one used in the proof of Lemma 7). In order to prove (135), decompose $u = \xi + r + X$. Then

$$\Psi(a)(u) = \mathcal{L}_{X_a} (\xi + X) + \nabla^\Psi_{X_a} r,$$
from where we deduce that
\[
\gamma_{\Psi(a)(u)}(\omega \otimes s) = \gamma_{\mathcal{L}_{\mathcal{X}_a}(\xi + \mathcal{X})}(\omega \otimes s) + \gamma_{\nabla^\Psi_{X_a}}(\omega \otimes s)
\]
\[
= (i_{\mathcal{L}_{\mathcal{X}_a}}\mathcal{X}_a + (\mathcal{L}_{\mathcal{X}_a}\xi) \wedge \omega) \otimes s + (-1)^{|\omega|} \omega \otimes (\nabla^\Psi_{X_a} r)s. \tag{144}
\]

Similar computations show that
\[
\Psi^{\mathcal{S}M}(a)\gamma_u(\omega \otimes s) = \mathcal{L}_{\mathcal{X}_a}(i_X\omega + \xi \wedge \omega) \otimes s + (i_X\omega + \xi \wedge \omega) \otimes \nabla^\Psi_{X_a} s \tag{145}
\]
\[
+ (-1)^{|\omega|}(\mathcal{L}_{\mathcal{X}_a}\omega \otimes (rs) + \omega \otimes \nabla^\Psi_{X_a}s(r))
\]
\[
\gamma_u\Psi^{\mathcal{S}M}(a)(\omega \otimes s) = (i_X\mathcal{L}_{\mathcal{X}_a}\omega + \xi \wedge \mathcal{L}_{\mathcal{X}_a}\omega) \otimes s + (-1)^{|\omega|}(\mathcal{L}_{\mathcal{X}_a}\omega) \otimes (rs) \tag{146}
\]
\[
+ (i_X\omega + \xi \wedge \omega) \otimes \nabla^\Psi_{X_a}s + (-1)^{|\omega|} \omega \otimes (r\nabla^\Psi_{X_a}s).
\]

Combining relations (144), (145) and (146) and using that \(\nabla^\Psi_{\mathcal{S}}\) is compatible with \(\nabla^\Psi\), we obtain (135). Relation (138) follows from the definition of the map \(\Psi^{\mathcal{S}M}\) and the flatness of \(\nabla^\Psi_{\mathcal{S}G}\) (which is a consequence of the flatness of \(\nabla^\Psi\)).

We conclude the proof of Proposition 38 by choosing an invariant dissection \(I : E \to E_M\) and isomorphisms \(I_{\mathcal{S}M}\mid_{U_i} : \mathcal{S}\mid_{U_i} \to \mathcal{S}^{\mathcal{S}M}\mid_{U_i}\) compatible with \(I\mid_{U_i}\), where \(U = \{U_i\}\) is a cover of \(M\) with sufficiently small open subsets. Using Lemmas 39 and 40, we obtain that the map \(\Psi^{\mathcal{S}} : g \to \text{End}(\mathcal{S})\) defined by
\[
\Psi^{\mathcal{S}}(a)(s)|_{U_i} := (I_{\mathcal{S}M}\mid_{U_i})^{-1} \circ \Psi^{\mathcal{S}M}(a) \circ I_{\mathcal{S}M}\mid_{U_i}(s)|_{U_i}, \quad \forall s \in \Gamma(S)
\]
satisfies the conditions from Proposition 38.

**Definition 41.** A section of the canonical weighted spinor bundle \(\mathcal{S}\) is an invariant spinor if it is annihilated by the operators \(\Psi^{\mathcal{S}}(a)\), for all \(a \in g\).

**Notation 42.** Given an action \(\Psi : g \to \text{Der}(E)\) on a transitive Courant algebroid \(E\), we shall denote by \(\Gamma_g(\mathcal{S})\) the vector space of invariant spinors. Similarly, \(\Gamma_g(E)\) will denote the vector space of invariant sections of \(E\).

**Lemma 43.** In the setting of Proposition 38,
\[
\hat{\mathcal{A}} \circ \Psi^{\mathcal{S}}(a) = \Psi^{\mathcal{S}}(a) \circ \hat{\mathcal{A}}, \quad \forall a \in g, \tag{147}
\]
where \(\hat{\mathcal{A}}\) is the canonical Dirac generating operator of \(E\).

**Proof.** From Proposition 15 and Lemma 39, it is sufficient to prove the statement for the Courant algebroid \(E_M\) considered in Lemma 40 with action
\[ \Psi^S_M \text{ defined by (142)}. \] We need to show that for any \( a \in \mathfrak{g}, \omega \in \Omega(M) \) and \( s \in \Gamma(S_G^\omega) \)

\[ \mathfrak{d}_M \Psi^S_M(a)(\omega \otimes s) = \Psi^S_M(a) \mathfrak{d}_M(\omega \otimes s) \tag{148} \]

where \( \mathfrak{d}_M \in \text{End}(S_M) \) is the Dirac generating operator of \( E_M \). We consider an invariant local frame \((X_i)\) of \( TM \). Since \( \nabla^\Psi \) is flat we may (and will) take the local frame \((r_k)\) of \( \mathcal{G} \) to be \( \nabla^\Psi \)-parallel. Since \( \nabla^\Psi \) preserves the scalar product \( \langle \cdot, \cdot \rangle_G \), the \( \langle \cdot, \cdot \rangle_G \)-dual frame \((\tilde{r}_k)\) is also \( \nabla^\Psi \)-parallel. Since \( \nabla^\Psi \) preserves the Lie bracket \([\cdot, \cdot]_G \), the Cartan form \( C_G \) is \( \nabla^\Psi \)-parallel.

Since \( R, X_i \) and \( r_k \) are invariant,

\[ \mathcal{L}_{X_i}(R(X_i, X_j), r_k)_G = 0, \quad \forall a \in \mathfrak{g}. \tag{149} \]

From (149), \( \nabla^\Psi C_G = 0 \), the fact that \( \nabla^\Psi, S_G^\omega \) is compatible with \( \nabla^\Psi \) and the expressions (22), (142) for \( \mathfrak{d} \) and \( \Psi^S \), we see that relation (147) reduces to

\[ \nabla^\Psi S_G^\omega \nabla^S_G s = \nabla^S_G \nabla^\Psi S_G^\omega s, \quad \forall a \in \mathfrak{g}, s \in \Gamma(S_G), \tag{150} \]

where, we recall, \( \nabla^S_G \) is the connection on \( S_G \) induced by any connection on \( S_G \) compatible with \( \nabla^\Psi \) and similarly for the partial connections \( \nabla^\Psi, S_G^\omega \) and \( \nabla^\Psi \). For any \( a \in \mathfrak{g} \), let \( A_a := \nabla^\Psi_{X_a} - \nabla_{X_a} \). Then

\[ \nabla^\Psi S_G^\omega s = \nabla^S_G s - \frac{1}{2} A_a \cdot s \tag{151} \]

where \( A_a \cdot s \) denotes the Clifford action of \( A_a \in \Gamma(\Lambda^2 \mathcal{G}) \subset \Gamma \text{Cl}(\mathcal{G}) \) on \( s \in \Gamma(S_G) \) (see e.g. Proposition 53 of [11] for more details). From (151), (96) and \( \mathcal{L}_{X_a} X_i = 0 \) we deduce that (150) is equivalent to

\[ R^{S_G^\omega}(X_a, X_i)s + \frac{1}{2} (\text{ad}_{R(X_a, X_i)})s = 0, \tag{152} \]

where \( (\text{ad}_{R(X_a, X_i)})s \) means the Clifford action of \( \text{ad}_{R(X_a, X_i)} := [\text{ad}_{R(X_a, X_i)}, \cdot]_G \in \Gamma(\Lambda^2 \mathcal{G}) \subset \Gamma \text{Cl}(\mathcal{G}) \) on \( s \). In order to prove (152) we remark first that both endomorphisms \( R^{S_G^\omega}(X_a, X_i) \) and \( (\text{ad}_{R(X_a, X_i)}) \) of \( S_G \) are trace free (the statement for \( R^{S_G^\omega}(X_a, X_i) \) is a consequence of the fact that \( \nabla^S_G \) is induced by a connection \( \nabla^S_G \) on \( S_G \)). On the other hand, since \( \nabla^S_G \) is compatible with \( \nabla \), we obtain that \( T := R^{S_G^\omega}(X_a, X_i) \in \text{End}(S_G) \) satisfies

\[ T(rs) = (R^\nabla(X_a, X_i)r)s + rT(s), \quad \forall r \in \mathcal{G}, s \in S_G. \tag{153} \]

The same relation is satisfied by \( T := -\frac{1}{2} \text{ad}_{R(X_a, X_i)} \) acting by Clifford multiplication (here we use that \( R^\nabla(X_a, X_i)(r) = \text{ad}_{R(X_a, X_i)}(r) \) and relation \( \omega(r) = -\frac{1}{2}[\omega, r]_\text{Cl} \), for any \( \omega \in \Lambda^2 \mathcal{G} \subset \text{Cl}(\mathcal{G}) \), where \( [\omega, r]_\text{Cl} = \omega r - r\omega \) denotes the commutator of \( \omega \) and \( r \) in the Clifford algebra and \( \omega(r) \) the action of \( \omega \in \Lambda^2 \mathcal{G} \cong \mathfrak{so}(\mathcal{G}) \) on \( r \in \mathcal{G} \).

To summarize: both \( R^{S_G^\omega}(X_a, X_i) \) and \( -\frac{1}{2} (\text{ad}_{R(X_a, X_i)}) \) are trace-free and satisfy (153). Since \( \langle \cdot, \cdot \rangle_G \) has neutral signature, they coincide. \( \Box \)
5.3 Pullback actions and spinors

Let \( f : M \to N \) be a submersion and

\[
\psi^M : \mathfrak{g} \to \mathfrak{X}(M), \ a \mapsto X^M_a \\
\psi^N : \mathfrak{g} \to \mathfrak{X}(N), \ a \mapsto X^N_a 
\]

be \( f \)-related infinitesimal actions, i.e. \( X^N_a \circ f = df X^M_a \) for all \( a \in \mathfrak{g} \). Let \( E \) be a transitive Courant algebroid over \( N \) with anchor \( \pi : E \to TN \) and

\[
\mathfrak{g} \ni a \mapsto \Psi(a) \in \text{Der}(E)
\]

be an action on \( E \) which lifts \( \psi^N \). Recall that the pullback Courant algebroid \( f^! E \) is the quotient bundle \( C/C^\perp \) over \( M \) (identified with the graph \( M_f \) of \( f \)), where, for any \( p \in M \),

\[
C_p := \{(u, \mu + X) \in E_{f(p)} \times T_p M : \pi(u) = (d_p f)(X)\}
\]

\[
C^+ \subset C_p := \{(\frac{1}{2} \pi^*(\gamma), -f^* \gamma) \in T_f \pi M : \gamma \in \mathfrak{X}(f^{-1}(U))\}
\]

with the Courant algebroid structure defined at the beginning of Section 4.2. For \( U \subset N \) open, a section of \( C|_{f^{-1}(U)} \) (and the induced section of \( (f^! E)|_{f^{-1}(U)} \)) of the form \( (f^* u, \mu + X) \) where \( u \in \Gamma(E|_U) \), \( X \in \mathfrak{X}(f^{-1}(U)) \) is \( f \)-projectable with \( f_* X = \pi(u) \) and \( \mu \in \Omega^1(f^{-1}(U)) \), will be called distinguished. Let \( U = \{U_i\} \) be an open cover of \( N \), with sufficiently small sets \( U_i \). Any section of \( C|_{f^{-1}(U_i)} \) is a \( C^\infty(f^{-1}(U_i)) \)-linear combination of distinguished sections. For each \( U_i \) we define

\[
\hat{\Psi}^{U_i} : \mathfrak{g} \to \text{End}(C|_{f^{-1}(U_i)}), \tag{154}
\]

such that it satisfies the Leibniz rule

\[
\hat{\Psi}^{U_i}(a)(fs) = X^M_a(f)s + f\hat{\Psi}^{U_i}(a)(s), \tag{155}
\]

for any \( a \in \mathfrak{g} \), \( f \in C^\infty(f^{-1}(U_i)) \), \( s \in \Gamma(C|_{f^{-1}(U_i)}) \), and on distinguished sections is given by

\[
\hat{\Psi}^{U_i}(a)(f^* u, \mu + X) := (f^*(\Psi(a)u), \mathcal{L}_{X^M_a}(\mu + X)). \tag{156}
\]

Lemma 44. The map \( \Psi : \mathfrak{g} \to \text{End}(C) \) given by

\[
\hat{\Psi}(a)(s)|_{f^{-1}(U_i)} = \hat{\Psi}^{U_i}(a)(s|_{f^{-1}(U_i)}) \tag{157}
\]

is well defined, preserves \( \Gamma(C^\perp) \), and induces an action

\[
f^! \Psi : \mathfrak{g} \to \text{Der}(f^! E) \tag{158}
\]

which lifts \( \psi^M \). It is called the pullback action of \( \Psi \).
Proof. The statement that $\hat{\Psi}$ is well defined reduces to showing that for any $U_k, U_p \in \mathcal{U}$, if
\[
\sum_i \lambda_i (f^*u_i, \mu_i + X_i) = 0
\]
where $\lambda_i \in C^\infty(f^{-1}(U_k \cap U_p))$ and $(f^*u_i, \mu_i + X_i)$ are distinguished sections on $f^{-1}(U_k \cap U_p)$, then
\[
\sum_i \left( X^M_{\alpha} (\lambda_i) f^*u_i + \lambda_i f^*\Psi(a)(u_i) \right) = 0. \tag{159}
\]
This follows by writing $u_i \in \Gamma(E|_{U_k \cap U_p})$ in terms of a frame of $E|_{U_k \cap U_p}$ and using the Leibniz rule for $\Psi(a)$ and that $X^M_{\alpha}$ projects to $X^N_{\alpha}$. The map $\hat{\Psi}(a)$ takes values in $\Gamma(C)$ since for any distinguished section $(f^*u, \mu + X)$, we have
\[
\pi \Psi(a)(u) = L_{X^N_{\alpha}} \pi(u) = f_* L_{X^M_{\alpha}} X.
\]
It preserves $\Gamma(C^1)$ since
\[
\Psi(a) \pi^*(\gamma) = \pi^*(L_{X^N_{\alpha}} \gamma), \forall \gamma \in \Omega^1(N).
\]
Since $\Psi$ satisfies the relations (91), also $f^! \Psi$ does (easy check). \hfill \Box

When $E = E_N := T^*N \oplus \mathcal{G} \oplus TN$ is a standard Courant algebroid, and $\Psi = \Psi^N : g \to \text{Der}(E_N)$ preserves the factors $T^*N, \mathcal{G}$ and $TN$ of $E_N$, the pullback action $f^! \Psi^N$ has a concrete formulation, as follows. Assume that $E_N$ is defined by a bundle of quadratic Lie algebras $(\mathcal{G}, [\cdot, \cdot]_{\mathcal{G}}, \langle \cdot, \cdot \rangle_{\mathcal{G}})$ and data $(\nabla, R, H)$. Recall that we identify $f^! E$ with the standard Courant algebroid $E_M := T^*M \oplus f^* \mathcal{G} \oplus TM$ defined by the bundle of quadratic Lie algebras $(f^* \mathcal{G}, f^*[\cdot, \cdot]_{\mathcal{G}}, f^* \langle \cdot, \cdot \rangle_{\mathcal{G}})$ and data $(f^* \nabla, f^* R, f^* H)$, using the canonical isomorphism $F$ defined by (53). Using this identification, we obtain an action
\[
\Psi^M : g \to \text{Der}(E_M), \quad \Psi^M(a) := F^{-1} \circ (f^! \Psi^N)(a) \circ F
\]
of $g$ on $E_M$.

**Lemma 45.** In the above setting, assume that
\[
\Psi^N(a)(\xi + r + X) := L_{X^N_{\alpha}} \xi + \nabla^\Psi_{X^N_{\alpha}} r + L_{X^N_{\alpha}} X, \tag{160}
\]
where $\xi \in \Omega^1(N), r \in \Gamma(\mathcal{G})$ and $X \in \mathfrak{X}(N)$. Then
\[
\Psi^M(a)(\xi + r + X) := L_{X^M_{\alpha}} \xi + (f^* \nabla)^\Psi_{X^M_{\alpha}} r + L_{X^M_{\alpha}} X, \tag{161}
\]
where $\xi \in \Omega^1(M), r \in \Gamma(f^* \mathcal{G})$ and $X \in \mathfrak{X}(M)$. 46
Proof. The isomorphism $F$ given by (53) induces an isomorphism $F : \Gamma(E_M) \to \Gamma(f^1E_N)$ which satisfies
\[
F(\xi + f^*r + X) = [(f^*(r + f_*X), \xi + X)]
\]
where $r \in \Gamma(\mathcal{G})$, $X \in \mathfrak{X}(M)$ is $f$-projectable and $\xi \in \Omega^1(M)$. (In the right hand side of (162) $r + f_*X \in \Gamma(\mathcal{G} \oplus TN) \subset \Gamma(E_N)$). Then
\[
(f^i\Psi^S)(a) \circ F(\xi + f^*r + X) = [(f^*(\nabla^\Psi_{X^g}r + \mathcal{L}_{X^g}f_*X), \mathcal{L}_{X^g}(\xi + X))],
\]
and, applying $F^{-1}$, we obtain (161).

The next proposition states several compatibilities between pullback actions, isomorphisms, pullback and pushforward on spinors.

**Proposition 46.** i) Let $(E_i, \Psi_i)$ $(i = 1, 2)$ be transitive Courant algebroids over $N$ with actions $\Psi_i : \mathfrak{g} \to \text{Der}(E_i)$ which lift $\psi^N$. If $I : (E_1, \Psi_1) \to (E_2, \Psi_2)$ is invariant with respect to $\Psi_i$, then $I^f : (f^1E_1, f^i\Psi_1) \to (f^1E_2, f^i\Psi_2)$ is invariant with respect to $f^i\Psi_i$.

ii) In the setting of Lemma 44, assume that $M$ and $N$ are oriented and let $\Psi^S : \mathfrak{g} \to \text{End}(S_E)$ and $(f^i\Psi)^S : \mathfrak{g} \to \text{End}(S_{f^1E})$ be the actions on canonical weighted spinor bundles, induced by the actions $\Psi : \mathfrak{g} \to \text{Der}(E)$ and $f^i\Psi : \mathfrak{g} \to \text{Der}(f^iE)$. Assume that the pullback $f^i : \Gamma(S_E) \to \Gamma(S_{f^1E})$ is defined and there is an admissible pair $(I : E \to T^*N \oplus \mathcal{G} \oplus TN, S_G)$ for $S_E$ and $S_{f^1E}$ such that $I$ is invariant, cf. Section 4.2. Then
\[
f^i \circ \Psi^S(a) = (f^i\Psi)^S(a) \circ f^i, \ \forall a \in \mathfrak{g}.
\]
If, in addition, $f : M \to N$ has compact fibers then also the pushforward $f_1 : \Gamma(S_{f^1E}) \to \Gamma(S_E)$ is defined and
\[
f_1 \circ (f^i\Psi)^S(a) = (-1)^{m|\xi|+nr+r(r-1)/2} \Psi^S(a) \circ f_1, \ \forall a \in \mathfrak{g},
\]
where $m, n$ and $r$ are the dimension of $M$, $N$ and the fibers of $f$.

**Proof.** i) We need to check that
\[
I^f \circ f^i\Psi_1(a)[(f^*u, \eta + X)] = f^i\Psi_2(a) \circ I^f[(f^*u, \eta + X)]
\]
for any distinguished section $[(f^*u, \eta + X)]$ of $f^1E_1$, which follows by applying the definitions of $I^f$ and $f^i\Psi_1$ and using that $I$ is invariant.

ii) Using the admissible pair $(I, S_G)$ and Lemma 39, we can assume, without loss of generality, that
\[
E = E_N = T^*N \oplus \mathcal{G} \oplus TN, \ \ f^1E = E_M = T^*M \oplus f^*\mathcal{G} \oplus TM
\]
47
and \( \Psi = \Psi^N \), \( f^!\Psi = \Psi^M \) are given by (160) and (161) respectively. Let \( S_N := \Lambda(T^*N) \otimes S_G \) and \( S_M := \Lambda(T^*M) \otimes f^*S_G \) be the canonical weighted spinor bundles of \( E_N \) and \( E_M \), determined by \( S_G \) and its pullback \( S_{f^*G} := f^*S_G \) respectively. From the definition of \( f^! \), we need to show that

\[
f^*\Psi^N(a)(\omega \otimes s) = \Psi^M(a)(f^*\omega \otimes f^*s)
\]

(165)

for any \( \omega \otimes s \in \Gamma(S_N) \), where \( \Psi^N : g \to \text{End} \Gamma(S_N) \) and \( \Psi^M : g \to \text{End} \Gamma(S_M) \) are the induced actions on spinors (given by Lemma 40) and \( f^* \) is the map (54). Relation (165) follows from (142), \( f^*\mathcal{L}_{X_N^\omega} = \mathcal{L}_{X_M^M(f^*\omega)} \) and \( f^*(\nabla^{\Psi,S_G}) = (\nabla^{\Psi,f^*S_G})_{X_M}(f^*s) \) for any \( s \in \Gamma(S_G) \) (the latter being a consequence of \( \nabla^{f^*\Psi} = f^*\nabla^\Psi \)). The statement for the pushforward can be proved by a similar argument, which uses that \( f_*\mathcal{L}_{X_M^M \omega} = \mathcal{L}_{X_M^M f_* \omega} \) for any form \( \omega \) on \( M \) and Lemma (21).

6 T-duality

6.1 Definition of T-duality

Let \( \pi : M \to B \) and \( \tilde{\pi} : \tilde{M} \to B \) be principal bundles over the same manifold \( B \) with structure group the \( k \)-dimensional torus \( T^k \). For notational convenience, we will denote the structure group of \( \tilde{\pi} \) by \( \tilde{T}^k \) and its Lie algebra by \( \tilde{t}^k \). We assume that \( M \), \( \tilde{M} \) and \( B \) are oriented. Let

\[
\text{Lie}(T^k) = t^k \ni a \mapsto \psi^M(a) := X^M_a, \quad \tilde{t}^k \ni a \mapsto \psi^{\tilde{M}}(a) := X^{\tilde{M}}_a,
\]

be the vertical parallelism of \( \pi \) and \( \tilde{\pi} \). We denote by

\[
N := M \times_B \tilde{M} := \{(m, \tilde{m}) \in M \times \tilde{M} \mid \pi(m) = \tilde{\pi}(\tilde{m})\}
\]

the fiber product of \( M \) and \( \tilde{M} \) and by \( \pi_N : N \to M \) and \( \tilde{\pi}_N : N \to \tilde{M} \) the natural projections. The actions of \( T^k \) on \( M \) and \( \tilde{T}^k \) on \( \tilde{M} \) induce naturally an action of \( T^{2k} = T^k \times \tilde{T}^k \) on \( N \), with infinitesimal action

\[
t^{2k} \ni a \to \psi^N(a) = X^N_a,
\]

where, for any \( a \in t^k := t^k \oplus 0 \subset t^{2k} \),

\[
(\pi_N)_*X^N_a = X^M_a, \quad (\tilde{\pi}_N)_*X^N_a = 0,
\]

and for any \( a \in \tilde{t}^k := 0 \oplus t^k \subset t^{2k} \),

\[
(\pi_N)_*X^N_a = 0, \quad (\tilde{\pi}_N)_*X^N_a = X^{\tilde{M}}_a.
\]
Let $E$ and $\hat{E}$ be transitive Courant algebroids over $M$ and $\hat{M}$, and assume they come with actions

\[ \Psi : t^k \to \text{Der}(E), \quad \hat{\Psi} : \hat{t}^k \to \text{Der}(\hat{E}), \]

which lift $\psi^M$ and $\psi^\hat{M}$, such that there are invariant dissections $I : E \to T^*M \oplus G \oplus TM$ and $\hat{I} : \hat{E} \to T^*\hat{M} \oplus \hat{G} \oplus T\hat{M}$ with the property that the partial connections $\nabla^\Psi$ and $\nabla^{\hat{\Psi}}$ on $G$ and $\hat{G}$, induced by the actions, have trivial holonomy (it is easy to see that this condition is independent of the choice of invariant dissections). The pullback Courant algebroids $\pi^1_N E$ and $\hat{\pi}^1_N \hat{E}$ inherit the pullback actions (see Lemma 44)

\[ \pi^1_N \Psi : t^k \to \text{Der}(\pi^1_N E), \quad \hat{\pi}^1_N \hat{\Psi} : \hat{t}^k \to \text{Der}(\hat{\pi}^1_N \hat{E}) \quad (166) \]

which lift the infinitesimal actions $t^k \ni a \to X^N_a$ and $\hat{t}^k \ni a \to X^N_a$ respectively. The situation is summarized in the following commutative diagram, in which the arrows pointing down are quotient maps with respect to principal $T^k$-actions: $B = M/T^k = \hat{M}/\hat{T}^k = N/T^{2k}$, $M = N/\hat{T}^k$, $\hat{M} = N/T^k$ ($T^{2k} = T^k \times \hat{T}^k$).

\[
\begin{array}{ccc}
t^k \circ \pi^1_N E & \longrightarrow & N \\
\pi_N & \downarrow & \hat{\pi}_N \\
t^k \circ E & \longrightarrow & M \\
\end{array}
\]

\[
\begin{array}{ccc}
& \pi^1_N E \nearrow \hat{t}^k & \\
\hat{M} \世代 \hat{E} \nearrow \hat{t}^k & \leftarrow & \hat{t}^k \circ \hat{E}
\end{array}
\]

The next lemma extends the action $\pi^1_N \Psi$ to a $t^{2k}$-action and states some of the properties of this $t^{2k}$-action.

**Lemma 47.** i) The map

\[ \Psi_{\pi^1_N E} : t^{2k} \to \text{Der}(\pi^1_N E) \]

which on $t^k$ coincides with $\pi^1_N \Psi$ and the evaluation of which on any $b \in \hat{t}^k$ satisfies the Leibniz rule

\[ \Psi_{\pi^1_N E}(b)(f s) = X^N_b(f)s + f \Psi_{\pi^1_N E}(b)(s), \forall f \in C^\infty(N), \ s \in \Gamma(\pi^1_N E) \]

and on distinguished sections $[(\pi^1_N(u), \xi + X)]$ of $\pi^1_N E$ is given by

\[ \Psi_{\pi^1_N E}(b)[(\pi^1_N(u), \xi + X)] = [(0, \mathcal{L}_{X^N_b}(\xi + X))] \quad (167) \]
is a well defined action on $\pi^1_N E$.

ii) Let $(E_1, \Psi_1)$ be another transitive Courant algebroid over $M$ with an action $\Psi_1 : t^k \to \text{Der}(E_1)$ which lifts $\psi^M$. If $I : E \to E_1$ is an isomorphism invariant with respect to $\Psi$ and $\Psi_1$, then the pullback isomorphism $I^\pi_N : \pi^1_N E \to \pi^1_N E_1$ is invariant with respect to $\Psi^E$ and $\Psi^E_{E_1}$ (the latter defined as $\Psi^E_{E_1}$, using $\Psi_1$ instead of $\Psi$).

**Proof.** Claim i) follows from arguments similar to the proof of Lemma 44. To prove claim ii), we need to show that

$$I^\pi_N \circ \Psi^E_{E_1}(a)(s) = \Psi^E_{E_1}(a) \circ I^\pi_N(s), \quad \forall a \in t^k, \ s \in \Gamma(\pi^1_N E).$$  \hspace{1cm} (168)

Relation (168) with $a \in t^k$, follows from Proposition 46 i). Relation (168) with $a \in t^k$ follows by assuming that is a distinguished section and using (167) together with the definition of $I^\pi_N$, cf. equation (51). \hfill \Box

**Lemma 48.** i) If $E = T^* M \oplus G \oplus T M$ is a standard Courant algebroid and $\Psi$ preserves the summands $T^* M$, $G$, $TM$ of $E$, then $\Psi^E_{E_1}$ preserves the summands $T^* N$, $\pi^*_N G$, $TN$ of $\pi^1_N E = T^* N \oplus \pi^*_N G \oplus TN$. The partial connection $\nabla^{\Psi^E_{E_1}}$ on $\pi^*_N G$ associated to $\Psi^E_{E_1}$ is the pullback of the partial connection $\nabla^\Psi$ on $G$ associated to $\Psi$:

$$\nabla^{\Psi^E_{E_1}}(\pi^*_N r) = \pi^*_N \nabla^\Psi r, \quad \nabla^{\Psi^E_{E_1}}(\pi^*_N r) = 0, \quad \forall r \in \Gamma(G), \ a \in t^k, \ b \in \tilde{t}^k.$$  \hspace{1cm} (169)

ii) A section of $\pi^*_N G$ is $\nabla^{\Psi^E_{E_1}}$-parallel if and only if it is the pullback by $\pi_N$ of a $\nabla^\Psi$-parallel section of $G$ (or the pullback by $\Pi := \pi \circ \pi_N$ of a section of $G_B$).

**Proof.** Claim i) follows from an argument similar to the proof of Lemma 45. Claim ii) follows immediately from claim i) (recall the definition of the bundle $G_B \to B$ from Section 5.1.1). \hfill \Box

When $E$ is a standard Courant algebroid like in Lemma 47 ii), the partial connection $\nabla^{\Psi^E_{E_1}}$ will be denoted by $\nabla^{\Psi, \pi_N E}$. Let $S_G$ be an irreducible $\text{Cl}(G)$-bundle, with canonical spinor bundle $S_G$. Then $S_{\pi^*_N G} = \pi^*_N S_G$ is the canonical spinor bundle of the irreducible $\text{Cl}(\pi^*_N G)$-bundle $S_{\pi^*_N G} := \pi^*_N S_G$ and the partial connection $\nabla^{\Psi^E_{E_1}, \pi_N S_G}$ on $S_{\pi^*_N G}$ induced by any partial connection on $S_{\pi^*_N G}$ compatible with $\nabla^{\Psi, \pi_N E}$ is the pullback of the partial connection $\nabla^{\Psi, S_G}$ on $S_G$ induced by any partial connection on $S_G$ compatible with $\nabla^\Psi$, that is,

$$\nabla^{\Psi, \pi_N S_G}(\pi^*_N \nabla^{\Psi, S_G})_{X^a_N} = (\pi^*_N \nabla^{\Psi, S_G})_{X^a_N}, \quad \forall a \in t^k.$$  \hspace{1cm} (170)
In a similar way, we construct an action \( \tilde{\Psi} \tilde{\pi}_N^1 \tilde{E} : \mathfrak{t}^2 \rightarrow \text{Der}(\tilde{\pi}_N^1 \tilde{E}) \) which extends \( \tilde{\pi}_N^1 \tilde{\Psi} \). When \( \tilde{E} = T^*M \oplus \tilde{G} \oplus TM \) is a standard Courant algebroid and \( \tilde{\Psi} \) preserves the factors \( T^*M, \tilde{G} \) and \( TM \) of \( \tilde{E} \), we use the notation \( \nabla_{\tilde{\Psi}, \tilde{\pi}_N^1 \tilde{E}} \) for the partial connection \( \nabla_{\tilde{\Psi}, \tilde{\pi}_N^1 \tilde{E}}^\pi_N \tilde{G} \). It is related to \( \nabla_{\tilde{\Psi}} \) by relations analogous to (170).

From now on the Courant algebroids \( \pi_N^1 E \) and \( \tilde{\pi}_N^1 \tilde{E} \) will be considered with the \( t^2 \)-actions \( \Psi \pi_N^1 E \) and \( \tilde{\Psi} \tilde{\pi}_N^1 \tilde{E} \).

**Definition 49.** The Courant algebroids \( E \) and \( \tilde{E} \) are called \( T \)-dual if there is an invariant fiber preserving Courant algebroid isomorphism

\[
F : \pi_N^1 E \rightarrow \tilde{\pi}_N^1 \tilde{E}
\]

such that the following non-degeneracy condition is satisfied. Let

\[
I : E \rightarrow T^*M \oplus G \oplus TM, \quad \tilde{I} : \tilde{E} \rightarrow T^*\tilde{M} \oplus \tilde{G} \oplus \tilde{TM},
\]

be dissections of \( E \) and \( \tilde{E} \), and

\[
I^{\pi_N} : \pi_N^1 E \rightarrow T^*N \oplus \pi_N^*G \oplus TN, \quad \tilde{I}^{\tilde{\pi}_N} : \tilde{\pi}_N^1 \tilde{E} \rightarrow T^*N \oplus \tilde{\pi}_N^*\tilde{G} \oplus TN
\]

the induced dissections of \( \pi_N^1 E \) and \( \tilde{\pi}_N^1 \tilde{E} \). Let \( (\beta, \Phi, K) \), where \( \beta \in \Omega^2(N), \Phi \in \Omega^1(N, \tilde{\pi}_N^*\tilde{G}) \) and \( K \in \text{Isom}(\pi_N^*G, \tilde{\pi}_N^*\tilde{G}) \), be the data which defines the isomorphism

\[
\tilde{I}^{\tilde{\pi}_N} \circ F \circ (I^{\pi_N})^{-1} : T^*N \oplus \pi_N^*G \oplus TN \rightarrow T^*N \oplus \tilde{\pi}_N^*\tilde{G} \oplus TN
\]

(according to relation (9) from Section 2.2.1). Then

\[
\beta - \Phi^*\Phi : \text{Ker} (d\pi_N) \times \text{Ker} (d\tilde{\pi}_N) \rightarrow \mathbb{R}
\]

(171) is non-degenerate.

**Definition 50.** The above definition is independent of the choice of dissections.

**Proof.** Let \( I_i : E \rightarrow T^*M \oplus G_i \oplus TM \) (\( i = 1, 2 \)) be two dissections of \( E \). Then

\[
\tilde{F}_i := \tilde{I}^{\tilde{\pi}_N} \circ F \circ (I_i^{\pi_N})^{-1} : T^*N \oplus \pi_N^*G_i \oplus TN \rightarrow T^*N \oplus \tilde{\pi}_N^*\tilde{G} \oplus TN
\]

satisfy

\[
\tilde{F}_2 = \tilde{F}_1 \circ (I_1 \circ I_2^{-1})^{\pi_N}.
\]

(172)
Assume that the dissections $I_1$ and $I_2$ are related by $(\beta, K, \Phi)$. Then from Lemma 16 iii) the induced dissections of $\pi^*_N E$ are related by $(\pi^*_N \beta, \pi^*_N K, \pi^*_N \Phi)$, i.e. the Courant algebroid isomorphism

$$(I_1 \circ I_2^{-1})^\pi_N : T^* N \oplus \pi^*_N G_2 \oplus TN \rightarrow T^* N \oplus \pi^*_N G_1 \oplus TN$$

is given by (9), with $(\beta, K, \Phi)$ replaced by $(\pi^*_N \beta, \pi^*_N K, \pi^*_N \Phi)$. The independence of the non-degeneracy condition (171) on the dissection of $E$ follows from (17). The independence on the dissection of $\tilde{E}$ can be proved similarly.

Remark 51. Unlike the $T$-duality for exact Courant algebroids, the definition of $T$-dual transitive Courant algebroids $E$ and $\tilde{E}$ is not symmetric with respect to $E$ and $\tilde{E}$, in general. This follows from the lack of symmetry in the non-degeneracy condition from Definition 49.

Lemma 52. Let $(E_1, \Psi_1)$ and $(\tilde{E}_1, \tilde{\Psi}_1)$ be transitive Courant algebroids over $M$ and $\tilde{M}$, together with actions

$$\Psi_1 : t^k \rightarrow \text{Der}(E_1), \quad \tilde{\Psi}_1 : \tilde{t}^k \rightarrow \text{Der}(\tilde{E}_1)$$

which lift $\psi^M$ and $\psi^{\tilde{M}}$ respectively. Assume that

$$G : E_1 \rightarrow E, \quad \tilde{G} : \tilde{E}_1 \rightarrow \tilde{E}$$

(173)

are invariant, fiber preserving Courant algebroid isomorphisms. If $E$ and $\tilde{E}$ are $T$-dual, then also $E_1$ and $\tilde{E}_1$ are $T$-dual.

Proof. Let $F : \pi^*_N E \rightarrow \tilde{\pi}^*_N \tilde{E}$ be an isomorphism which satisfies the conditions from Definition 49. Then the isomorphism

$$F_1 := (\tilde{G}^{\pi N})^{-1} \circ F \circ G^{\pi N} : \pi^*_N E_1 \rightarrow \tilde{\pi}^*_N \tilde{E}_1.$$  

(174)

satisfies the same conditions. (For the invariance of $\tilde{G}^{\pi N}$ and $G^{\pi N}$ we use Lemma 47 ii)).

The next lemma states the conditions that two standard Courant algebroids are $T$-dual. Let $E = T^* M \oplus \mathcal{G} \oplus TM$ and $\tilde{E} = T^* \tilde{M} \oplus \tilde{\mathcal{G}} \oplus T\tilde{M}$ be standard Courant algebroids over $M$ and $\tilde{M}$, defined by a quadratic Lie algebra bundle $(\mathcal{G}, [\cdot, \cdot]_\mathcal{G}, \langle \cdot, \cdot \rangle_\mathcal{G})$ and data $(\nabla^E, R, H)$ and, respectively, a quadratic Lie algebra bundle $(\tilde{\mathcal{G}}, [\cdot, \cdot]_{\tilde{\mathcal{G}}}, \langle \cdot, \cdot \rangle_{\tilde{\mathcal{G}}})$ and data $(\nabla^{\tilde{E}}, \tilde{R}, \tilde{H})$ (as there are various connections involved, we choose to use the notation $\nabla^{\tilde{E}}$ rather than $\nabla$ for the connection which is part of the data which defines $\tilde{E}$; a similar convention is used for $E$). Let $\Psi : t^k \rightarrow \text{Der}(E)$ and $\tilde{\Psi} : \tilde{t}^k \rightarrow \text{Der}(\tilde{E})$ be actions which lift $\psi^M$ and $\psi^{\tilde{M}}$ and preserve the factors of $E$ and $\tilde{E}$.
Lemma 53. The standard Courant algebroids $E$ and $\tilde{E}$ are $T$-dual if and only if there are invariant forms $\beta \in \Omega^1(N)$ and $\Phi \in \Omega^1(N, \tilde{\pi}_N^*\mathcal{G})$ and a quadratic Lie algebra bundle isomorphism $K \in \text{Isom}(\pi_N^*\mathcal{G}, \tilde{\pi}_N^*\mathcal{G})$ which maps $\nabla^{\Psi, \pi_N^*E}$ to $\nabla^{\Psi, \tilde{\pi}_N^*\tilde{E}}$ such that the non-degeneracy condition (171) and the following relations hold:

\[
(\tilde{\pi}_N^*\nabla^\tilde{E})_X r = K(\pi_N^*\nabla^E)_X (K^{-1} r) + [r, \Phi(X)]_{\tilde{\pi}_N^*\tilde{G}}, \quad (175)
\]

\[
K\pi_N^*R - \tilde{\pi}_N^*\tilde{R} = d\tilde{\pi}_N^*\nabla^E \Phi + c_2, \quad (176)
\]

\[
\pi_N^*H - \tilde{\pi}_N^*\tilde{H} = d\beta + \langle (K\pi_N^*R + \tilde{\pi}_N^*\tilde{R}) \wedge \Phi \rangle_{\tilde{\pi}_N^*\tilde{G}} - c_3, \quad (177)
\]

where $c_2 \in \Omega^2(N, \tilde{\pi}_N^*\tilde{G})$ and $c_3 \in \Omega^3(N)$ are defined by

\[
c_2(X, Y) := [\Phi(X), \Phi(Y)]_{\tilde{\pi}_N^*\tilde{G}},
\]

\[
c_3(X, Y, Z) := \langle [\Phi(X), [\Phi(Y), \Phi(Z)]]_{\tilde{\pi}_N^*\tilde{G}} \rangle_{\tilde{\pi}_N^*\tilde{G}},
\]

for any $X, Y, Z \in \mathfrak{x}(N)$.

Proof. Since $E$ and $\tilde{E}$ are standard Courant algebroids, so are $\pi_N^*E$ and $\tilde{\pi}_N^*\tilde{E}$ and an invariant isomorphism $F : \pi_N^*E \to \tilde{\pi}_N^*\tilde{E}$ is defined by invariant data $(\beta, K, \Phi)$, where $\beta \in \Omega^2(N)$, $K \in \text{Isom}(\pi_N^*\mathcal{G}, \tilde{\pi}_N^*\tilde{G})$ and $\Phi \in \Omega^1(N, \tilde{\pi}_N^*\tilde{G})$. The above relations coincide with relations (10), applied to $F$ and $X, Y, Z \in \mathfrak{x}(N)$. □

We end this section with a simple lemma on the existence of preferred dissections of $T$-dual transitive Courant algebroids.

Lemma 54. Let $E$ and $\tilde{E}$ be $T$-dual transitive Courant algebroids (not necessarily in the standard form). Then $E$ and $\tilde{E}$ admit invariant dissections of the form

\[
I : E \to T^*M \oplus \pi^*\mathcal{G}_B \oplus TM, \quad \tilde{I} : \tilde{E} \to T^*\tilde{M} \oplus \tilde{\pi}^*\tilde{\mathcal{G}}_B \oplus T\tilde{M}, \quad (178)
\]

where $(\mathcal{G}_B, \langle \cdot, \cdot \rangle_{\mathcal{G}_B}, [\cdot, \cdot]_{\mathcal{G}_B})$ is a quadratic Lie algebra bundle on $B$.

Proof. Let $I : E \to E_M = T^*M \oplus \mathcal{G} \oplus TM$ and $\tilde{I} : \tilde{E} \to \tilde{E}_M = T^*\tilde{M} \oplus \tilde{\mathcal{G}} \oplus T\tilde{M}$ be invariant dissections of $E$ and $\tilde{E}$. From Lemma 52, $E_M$ and $\tilde{E}_M$ are $T$-dual. Let $F : \pi_N^*E_M \to \pi_N^*E_{\tilde{M}}$ be an invariant isomorphism, and assume it is defined by data $(\beta, K, \Phi)$. Since $F$ is invariant, the quadratic Lie algebra bundle isomorphism $K : \pi_N^*\mathcal{G} \to \pi_N^*\tilde{\mathcal{G}}$ maps $\nabla^{\Psi, \pi_N^*E}$ to $\nabla^{\Psi, \pi_N^*\tilde{E}}$. From Lemma 48 ii), $K = \Pi^*K_B$ where $K_B : \mathcal{G}_B \to \tilde{\mathcal{G}}_B$ is a quadratic Lie algebra bundle isomorphism and $\mathcal{G} = \pi^*\mathcal{G}_B$, $\tilde{\mathcal{G}} = \tilde{\pi}^*\tilde{\mathcal{G}}_B$ (see Section 5.1.1). Using Lemma 1, we change the dissection $I$ to obtain a new invariant dissection of $\tilde{E}$ with $\tilde{\mathcal{G}} = \tilde{\pi}^*\tilde{\mathcal{G}}_B \cong \tilde{\pi}^*\mathcal{G}_B$ replaced by $\tilde{\pi}^*\mathcal{G}_B$. □
6.2 $T$-duality and spinors

Assume that $E$ and $\tilde{E}$ are $T$-dual transitive Courant algebroids and let

$$F : \pi^1_N E \to \tilde{\pi}^1_N \tilde{E}$$

be an invariant isomorphism as in Definition 49. Let $S_E, S_{\tilde{E}}, S_{\pi^1_N E}$ and $S_{\tilde{\pi}^1_N \tilde{E}}$ be canonical weighted spinor bundles of $E$, $\tilde{E}$, $\pi^1_N E$ and $\tilde{\pi}^1_N \tilde{E}$ respectively, such that the pullbacks $\pi^1_N : \Gamma(S_E) \to \Gamma(S_{\pi^1_N E})$ and $\tilde{\pi}^1_N : \Gamma(S_{\tilde{E}}) \to \Gamma(S_{\tilde{\pi}^1_N \tilde{E}})$ are defined. We consider an admissible pair $(I, S_{\tilde{g}})$ for $S_E$ and $S_{\pi^1_N E}$, and an admissible pair $(\tilde{I}, \tilde{S}_{\tilde{g}})$ for $S_{\tilde{E}}$ and $S_{\tilde{\pi}^1_N \tilde{E}}$, with invariant dissections

$$I : E \to E_M = T^* M \oplus \pi^* G_B \oplus TM, \quad \tilde{I} : \tilde{E} \to \tilde{E}_M = T^* \tilde{M} \oplus \tilde{\pi}^* \tilde{G}_B \oplus T \tilde{M}$$

such that $S_{\tilde{g}} = \pi^* S_B$ and $\tilde{S}_{\tilde{g}} = \tilde{\pi}^* \tilde{S}_B$, where $S_B$ is an irreducible $\text{Cl}(G_B)$-bundle and $\tilde{S}_B$ is an irreducible $\text{Cl}(\tilde{G}_B)$-bundle. We assume that the isomorphism $F_S : S_{\pi^1_N E} \to S_{\tilde{\pi}^1_N \tilde{E}}$ induced by $F$ is globally defined. This is equivalent to assuming that the isomorphism $(F_1)_S : S_N \to \tilde{S}_N$ compatible with $F_1 := \tilde{I}^{\pi_N} \circ F \circ (I^{\pi_N})^{-1} : \pi^1_N E_M \to \tilde{\pi}^1_N \tilde{E}_M$ is globally defined, where $S_N = \Lambda(T^* N) \otimes \Pi^* S_B$ and $\tilde{S}_N = \Lambda(T^* \tilde{N}) \otimes \Pi^* \tilde{S}_B$ are spinor bundles of $\pi^1_N E_M = T^* N \oplus \Pi^* G_B \oplus TN$ and $\tilde{\pi}^1_N \tilde{E}_M = T^* \tilde{N} \oplus \Pi^* \tilde{G}_B \oplus T \tilde{N}$ respectively (and $\Pi = \pi \circ \pi_N = \tilde{\pi} \circ \tilde{\pi}_N$).

**Remark 55.** When the dissections are chosen such that $G_B = \tilde{G}_B$ as quadratic Lie algebra bundles (see Lemma 54) and $S_B = \tilde{S}_B$ as $\text{Cl}(G_B)$-bundles, $F_1$ is an automorphism of the vector bundle

$$\pi^1_N E_M = \tilde{\pi}^1_N \tilde{E}_M = T^* N \oplus \Pi^* G_B \oplus TN$$

with scalar product

$$\langle \xi + \Pi^* (r_1) + X, \eta + \Pi^* (r_2) + Y \rangle = \frac{1}{2} (\xi(Y) + \eta(X)) + \Pi^* (r_1, r_2)_{G_B} \quad (179)$$

and $(F_1)_S$ is an automorphism of the spinor bundle $S_N = \Lambda(T^* N) \otimes \Pi^* S_B$ of $T^* N \oplus \Pi^* G_B \oplus TN$. If $F_1$ belongs to the image of the exponential map $exp : \mathfrak{so}(T^* N \oplus \Pi^* G_B \oplus TN) \to SO_0(T^* N \oplus \Pi^* G_B \oplus TN)$, then $(F_1)_S$ is automatically globally defined (cf. Remark 11). In fact, in that case $F_1$ can be lifted to $(F_1)_S$ using the exponential map for $\text{spin}(T^* N \oplus \Pi^* G_B \oplus TN)$.

**Theorem 56.** i) The map

$$\tau := (\tilde{\pi}_N) : F_S \circ \pi^1_N : \Gamma(S_E) \to \Gamma(S_{\tilde{E}}) \quad (180)$$

54
intertwines the canonical Dirac generating operators of \( E \) and \( \tilde{E} \) and maps invariant spinors to invariant spinors. In particular, there is the following commutative diagram

\[
\begin{array}{ccc}
\Gamma_{\pi_N}^t(S_E) & \xrightarrow{\rho} & \Gamma_{\pi_{\tilde{N}}}^t(S_{\tilde{E}}) \\
\downarrow & & \downarrow \\
\Gamma_{\rho^*}^t(S_E) & \xrightarrow{\tau} & \Gamma_{\rho^*}^t(S_{\tilde{E}})
\end{array}
\]

ii) There is an isomorphism \( \rho : \Gamma_{\rho^*}^t(E) \to \Gamma_{\rho^*}^t(\tilde{E}) \) of \( C^\infty(B) \)-modules which preserves Courant brackets, scalar products and is compatible with \( \tau \), i.e.

\[
\tau(\gamma_u s) = \gamma_{\rho(u)} \tau(s), \quad [\rho(u), \rho(v)]_E = \rho[u, v]_E, \quad \langle \rho(u), \rho(v) \rangle_E = \langle u, v \rangle_E,
\]

for any \( u, v \in \Gamma_{\rho^*}^t(E) \) and \( s \in \Gamma_{\rho^*}^t(S_E) \).

The claim from Theorem 56 i) concerning the canonical Dirac generating operators follows from Propositions 15, 19 and 23. The remaining claims from Theorem 56 will be proved in the next two lemmas.

**Lemma 57.** In the above setting, \( E_M \) and \( \tilde{E}_M \) are T-dual. The assertions of Theorem 56 hold for the pair \( (E, \tilde{E}) \) and canonical weighted spinor bundles \( S_E \) and \( S_{\tilde{E}} \) if and only if they hold for the pair \( (E_M, \tilde{E}_M) \) and canonical weighted spinor bundles \( S_M := \Lambda(T^*M) \otimes \pi^*S_B \) and \( \tilde{S}_M := \Lambda(T^*\tilde{M}) \otimes \tilde{\pi}^*\tilde{S}_B \) of \( E_M \) and \( \tilde{E}_M \) respectively.

**Proof.** The fact that \( E_M \) and \( \tilde{E}_M \) are T-dual follows from Lemma 52. We will assume that the assertions of Theorem 56 hold for \( (E, \tilde{E}) \) and show that they hold for \( (E_M, \tilde{E}_M) \) as well. The same arguments prove also the converse statement. Using relation (174), we obtain that the map \( \tau_1 \) defined for the pair \( (E_M, \tilde{E}_M) \) as is defined \( \tau \) for the pair \( (E, \tilde{E}) \), that is,

\[
\tau_1 := (\tilde{\pi}_N)_* \circ (F_1)_S \circ \pi^*_N : \Gamma(S_M) \to \Gamma(S_{\tilde{M}}),
\]

(182)

where

\[
\pi^*_N : \Gamma(S_M) \to \Gamma(S_N) = \Omega(N, \Pi^*S_B)
\]

\[
(\tilde{\pi}_N)_* : \Gamma(S_N) = \Omega(N, \Pi^*\tilde{S}_B) \to \Gamma(S_{\tilde{M}})
\]

are the pullback and pushforward maps (54) and (77), is related to \( \tau \) by \( \tau_1 = \epsilon I_\Sigma \circ \tau \circ (I_\Sigma)^{-1} \). Here \( \epsilon \in \{\pm 1\} \) and \( I_\Sigma : S_E \to S_M, \tilde{I}_\Sigma : S_{\tilde{E}} \to S_{\tilde{M}} \) are induced by \( I, \tilde{I} \). As \( I \) and \( \tilde{I} \) are invariant, \( I_\Sigma \) and \( \tilde{I}_\Sigma \) map invariant spinors to
invariant spinors. This is true also for \( \tau \). We obtain that \( \tau_1 \) maps invariant spinors to invariant spinors. Define

\[
\rho_1 : \Gamma^\psi(E_M) \to \Gamma^\psi(\tilde{E}_M), \quad \rho_1 := \tilde{I} \circ \rho \circ I^{-1}.
\]

Since \((\tau, \rho)\) satisfy (181), so do \((\tau_1, \rho_1)\).

**Lemma 58.** The statements from Theorem 56 hold for the pair \((E_M, \tilde{E}_M)\) and canonical weighted spinor bundles \(\mathbb{S}_M\) and \(\mathbb{S}_{\tilde{M}}\).

**Proof.** i) We prove that the map \(\tau_1\) defined by (182) maps invariant spinors to invariant spinors. We denote by \(\nabla^\Psi\) and \(\tilde{\nabla}^\Psi\) the partial connections defined by the actions on the standard Courant algebroids \(E_M\) and \(\tilde{E}_M\). For the various other partial connections (like \(\nabla^\psi, \pi_S E_M, \nabla^\psi, \pi_S^t S, \nabla^\psi, \Pi^t S_B\) etc) we use the definition explained after Lemma 47 (keeping in mind that \(S_G = \pi^t S_B\) and \(\pi^t S_G = \Pi^t S_B\)). We prove that if \(\omega \otimes s \in \Gamma(\mathbb{S}_M)\) is \(t^k\)-invariant then \(\pi^*_N(\omega \otimes s) \in \Gamma(\mathbb{S}_N)\) is \(t^k\)-invariant. The \(t^k\)-invariance of \(\pi^*_N(\omega \otimes s)\) follows from Proposition 46 ii). In order to prove that \(\pi^*_N(\omega \otimes s)\) is \(\tilde{t}^k\)-invariant, we apply the formula

\[
\Psi_N^s(a)\pi^*_N(\omega \otimes s) = L_{X_N^a}(\pi^*_N(\omega) \otimes \pi^*_N s + (\pi^*_N s) \otimes \nabla^\psi, \Pi^t S_B(\pi^*_N s)).
\]

If \(a \in \tilde{t}^k\) then

\[
L_{X_N^a}(\pi^*_N(\omega)) = \pi^*_N L_{(\pi_N^*(\omega)), X_N^a}\omega = 0,
\]

since \((\pi_N), X_N^a = 0\). On the other hand, from (170), we deduce that

\[
\nabla^\psi, \Pi^t S_B(\pi^*_N s) = \pi^*_N(\nabla^\psi, \pi_S^t S_B(\pi^*_N s)) = 0
\]

by using again \((\pi_N), X_N^a = 0\). It follows that \(\pi^*_N(\omega \otimes s)\) is \(\tilde{t}^k\)-invariant. We have proven that \(\pi^*_N(\omega \otimes s)\) is \(t^k\)-invariant. From Lemma 39, \((F_1)_*\pi^*_N(\omega \otimes s)\) is \(t^k\)-invariant (in particular, \(\tilde{t}^k\)-invariant) and from Proposition 46 ii) we obtain that \(\tau_1(\omega \otimes s)\) is \(\tilde{t}^k\)-invariant, as needed.

ii) Let \(u = \xi + r + X \in \Gamma^\psi(E_M)\). Then \(X\) and \(\xi\) are invariant with respect to the standard (by Lie derivatives) action of \(t^k\) on \(M\) and \(r\) is \(\nabla^\psi\)-parallel. We obtain

\[
L_{X_N^a}(\pi^*_N(\xi)) = 0, \quad \forall a \in t^{2k}, \quad \nabla^\psi, \pi_S^t E_M(\pi^*_N r) = 0,
\]

where in the second relation (185) we used (169). We claim that there is a unique \(t^{2k}\)-invariant lift \(\tilde{X}_0 \in \mathbb{X}_{t^{2k}}(N)\) of \(X\) with the property that

\[
\text{pr}_{T^\ast N} F_1(\pi^*_N(\xi + r) + \tilde{X}_0) = \tilde{\pi}_N^*(\tilde{\xi})
\]

for an (invariant) 1-form \(\tilde{\xi} \in \Omega^1(\tilde{M})\). To prove the claim we assume that the isomorphism \(F_1\) is defined by data \((\beta, K, \Phi)\) as in Section 2.2.1, where
\( \beta \in \Omega^2(N), \ K \in \text{Isom}(\Pi^*G_B, \Pi^*\tilde{G}_B) \) and \( \Phi \in \Omega^1(N, \Pi^*\tilde{G}_B) \). Let \( \tilde{X} \) be an arbitrary \( t^{2k} \)-invariant lift of \( X \). Then

\[
\text{pr}_{T^N}F_1(\pi_N^*(\xi + r) + \tilde{X}) = \tilde{X}, \ \text{pr}_{\tilde{G}_B}F_1(\pi_N^*(\xi + r) + \tilde{X}) = K(\pi_N^*r) + \Phi(\tilde{X})
\]

\[
\text{pr}_{T^N}F_1(\pi_N^*(\xi + r) + \tilde{X}) = \pi_N^*(\xi) + 2\Phi^*K(\pi_N^*r) + i_{\tilde{X}}\beta - \Phi^*\Phi(\tilde{X}). \quad (187)
\]

From (185) \( \pi_N^*(\xi + r) \) is \( t^{2k} \)-invariant and we obtain that \( F_1(\pi_N^*(\xi + r) + \tilde{X}) \) is also \( t^{2k} \)-invariant. The non-degeneracy of (171) together with the \( t^{2k} \)-invariance of \( \text{pr}_{T^N}F_1(\pi_N^*(\xi + r) + \tilde{X}) \) and the last formula (187) imply that there is a unique \( t^{2k} \)-invariant lift \( \tilde{X}_0 \) of \( X \) such that \( \text{pr}_{T^N}F_1(\pi_N^*(\xi + r) + \tilde{X}_0) \) is horizontal with respect to \( \tilde{\pi}_N \) and invariant, hence basic. For this lift relation (186) holds.

On the other hand, since \( F_1(\pi_N^*(\xi + r) + \tilde{X}_0) \) is \( t^{2k} \)-invariant, its projection to \( \Pi^*\tilde{G}_B \) is \( \nabla^{\tilde{\psi}} \)-invariant and parallel and is therefore the pullback of a \( \nabla^{\tilde{\psi}} \)-parallel section \( \tilde{r} \) of \( \tilde{\pi}^*\tilde{G}_B \). To summarize,

\[
F_1(\pi_N^*(\xi + r) + \tilde{X}_0) = \pi_N^*(\tilde{\xi} + \tilde{r}) + \tilde{X}_0 \quad (188)
\]

where \( \tilde{\xi} \in \Omega^1(\tilde{M}) \) is \( \tilde{t}^k \)-invariant and \( \nabla^{\tilde{\psi}}(\tilde{r}) = 0 \). We define

\[
\rho_1(u) := \tilde{\xi} + \tilde{r} + (\tilde{\pi}_N)_*\tilde{X}_0. \quad (189)
\]

Obviously, \( \rho_1(u) \) is \( \tilde{t}^k \)-invariant and the resulting map \( \rho_1 : \Gamma_{\tilde{t}^k}(E_M) \to \Gamma_{\tilde{t}^k}(\tilde{E}_M) \) is \( C^\infty(B) \)-linear. It remains to prove that \( (\rho_1, \tau_1) \) satisfy relations (181). In order to prove the first relation (181), let \( u := \xi + r + X \in \Gamma_{t^k}(E_M) \), \( \rho_1(u) = \tilde{\xi} + \tilde{r} + (\tilde{\pi}_N)_*\tilde{X}_0 \) constructed as above, \( s \in \Gamma_{\tilde{t}^k}(S_M) \) and \( \sigma \in \Gamma(S_N) \).

Using (57) and Remark 22,

\[
\pi_N^*\gamma_u(s) = \gamma_{\pi_N^*(\xi + r) + \tilde{X}_0}^*\pi_N^*s, \quad (\tilde{\pi}_N)_*\gamma_{\tilde{\pi}_N^*(\xi + r) + \tilde{X}_0}^*\sigma = \gamma_{\rho_1(u)}(\tilde{\pi}_N)_*\sigma \quad (190)
\]

and we write

\[
\tau_1\gamma_u(s) = (\tilde{\pi}_N)_*(F_1)_B(\pi_N)^*\gamma_u(s) = (\tilde{\pi}_N)_*(F_1)_B\gamma_{\pi_N^*(\xi + r) + \tilde{X}_0}^*\pi_N^*s
\]

\[
= (\tilde{\pi}_N)_*\gamma_{\tilde{\pi}_N^*(\xi + r) + \tilde{X}_0}^*(F_1)_B\pi_N^*(s) = \gamma_{\rho_1(u)}(\tilde{\pi}_N)_*{(F_1)}_B(\pi_N)^*(s)
\]

\[
= \gamma_{\rho_1(u)}(\tau_1(s)),
\]

where in the third equality we used Lemma 10 and relation (188). The first relation of (181) is proved. The second relation of (181) follows from the next computation, which uses the first relation of (181) together with \( \tau_1 \circ \phi_M = \phi_M \circ \tau_1 \) proved in part i) of Theorem 56 (where \( \phi_M \) and \( \phi_M \) are
the Dirac generating operators of $E_M$ and $\tilde{E}_M$ acting on $\Gamma(S_M)$ and $\Gamma(S_{\tilde{M}})$ respectively). For any $u, v \in \Gamma_{\psi}(E_M)$ and $s \in \Gamma_{\psi}(S_M)$, we have
\[
\gamma_{\rho_1[u,v]E_M} \tau_1(s) = \tau_1 \gamma_{\langle u,v \rangle E_M}(s) = \tau_1 [[d_M, \gamma_u], \gamma_v](s) = \tau_1 [\gamma_{\rho_1(u)}, \gamma_{\rho_1(v)}] \tau_1(s) = \gamma_{\rho_1(u), \rho_1(v)} \tau_1(s).
\]
In order to prove the third relation of (181), we remark that for any $u \in \Gamma_{\psi}(E_M)$, $\langle u, u \rangle_{E_M}$ is $\psi$-invariant and hence is the pullback of a function $g \in C^\infty(B)$. Similarly, $\langle \rho_1(u), \rho_1(u) \rangle_{\tilde{E}_{\tilde{M}}}$ is the pullback of a function $\tilde{g} \in C^\infty(B)$. We need to show that $g = \tilde{g}$. This follows from the next computation which uses the first relation of (181):
\[
\tilde{\pi}^*(g) \tau_1(s) = \tau_1 (\pi^*(g)) = \tau_1 (\langle u, u \rangle_{E_M} s) = \tau_1 \gamma_u^2(s) = \gamma_{\rho_1(u)} \tau_1(s) = \langle \rho_1(u), \rho_1(u) \rangle_{\tilde{E}_{\tilde{M}}} \tau_1(s) = \tilde{\pi}^*(\tilde{g}) \tau_1(s).
\]
From the third relation of (181) we obtain that $\rho_1$ is an isomorphism (of vector spaces and even of $C^\infty(B)$-modules). The proof of the theorem is completed.

**Corollary 59.** The map
\[
\tau := (\tilde{\pi}_N)_! \circ F_3 \circ \pi_1^! : \Gamma_{\psi}(S_E) \to \Gamma_{\psi}(S_{\tilde{E}})
\]
(191)
is an isomorphism of $C^\infty(B)$-modules.

**Proof.** This follows from the irreducibility of the spinor bundles together with the fact that $\tau$ is $C^\infty(B)$-linear, is not the zero map and intertwines the Clifford multiplications in the commutative diagram
\[
\begin{array}{ccc}
\Gamma_{\psi}(E) \times \Gamma_{\psi}(S_E) & \longrightarrow & \Gamma_{\psi}(S_{\tilde{E}}) \\
\rho \times \tau & \downarrow & \tau \\
\Gamma_{\psi}(\tilde{E}) \times \Gamma_{\psi}(S_{\tilde{E}}) & \longrightarrow & \Gamma_{\psi}(S_{\tilde{E}}).
\end{array}
\]

**Remark 60.** As in the $T$-duality for exact or heterotic Courant algebroids, the map $\rho$ constructed in Theorem 56 can be interpreted as an isomorphism between Courant algebroids over $B$ (see [2] and [9]).

In the next remark we discuss Theorem 56 without the assumption that $F_3$ is globally defined.
Remark 61. i) We claim that the isomorphism \((F_1)_S\) introduced before Remark 55 (hence also \(F_\tilde{S}\)) is always defined on subsets of \(N\) of the form \(\Pi^{-1}(V)\), where \(V \subset B\) is open and sufficiently small. Indeed, \((F_1)_S|_U \in Isom(S_N|_U, \tilde{S}_N|_U)\) is defined, whenever \(U \subset N\) is open and sufficiently small (see Lemma 10). Letting \(V := \Pi(U)\), we can find (reducing \(V\) is necessary) invariant frames \((s_i)\) and \((\tilde{s}_i)\) of \(S_N\) and \(\tilde{S}_N\) on \(\Pi^{-1}(V)\). With respect to these frames, \((F_1)_S|_U\) is given by

\[
(F_1)_S|_U(s_i) = \sum_j C_{ji} \tilde{s}_j
\]  

(192)

for some functions \(C_{ji} \in C^\infty(U)\). From the invariance of \((F_1)_S|_U\), we deduce that \(\mathcal{L}_{X^a} C_{ji} = 0\) for any \(a \in t^{2k}\), i.e. \(C_{ji} = \Pi^*_0(e_{ji})\) where \(e_{ji} \in C^\infty(V)\) and \(\Pi_0 : U \to V\) is the restriction of \(\Pi\). Using that all \(\pi_N E_M, \tilde{\pi}_N \tilde{E}_M, S_N\) and \(\tilde{S}_N\) admit invariant frames on \(\Pi^{-1}(V)\), we deduce from the compatibility of \((F_1)_S|_U\) with \(F_1|_U\) that

\[
(F_1)_S|_{\Pi^{-1}(V)}(s_i) := \sum_j \Pi^*(e_{ji}) \tilde{s}_j,
\]  

(193)

defined on \(\Pi^{-1}(V)\), is compatible with \(F_1|_{\Pi^{-1}(V)}\).

ii) From the above, the map

\[
\tau_V := (\tilde{\pi}_N)_* \circ F_\tilde{S} \circ \pi_N^1 : \Gamma(S_E|_{\pi^{-1}(V)}) \to \Gamma(S_E|_{\tilde{\pi}^{-1}(V)})
\]

is defined. Theorem 56 still holds, the only difference being that \(\tau\) is replaced by the locally defined maps \(\tau_V\), for any \(V \subset B\) open and sufficiently small. (The isomorphism \(\rho\) remains defined globally.)

6.3 Existence of a \(T\)-dual

Let \(\pi : M \to B\) be a principal \(T^k\)-bundle and \(\mathcal{H}\) a principal connection on \(\pi\), with connection form \(\theta = \sum_{i=1}^k \theta_i e_i \in \Omega^1(M, t^k)\), where \((e_i)\) is a basis of \(t^k\) such that \(T^k = t^k/\text{span}_\mathbb{Z}\{e_i\}\). Let \((E, \Psi)\) be a standard Courant algebroid with an action \(\Psi : t^k \to \text{Der}(E)\) which lifts the vertical parallelism of \(\pi\), defined by a quadratic Lie algebra bundle \((\mathcal{G}_B, [\cdot, \cdot]_{\mathcal{G}_B}, \langle \cdot, \cdot \rangle_{\mathcal{G}_B})\) whose adjoint representation is an isomorphism, a connection \(\nabla^B\) on \(\mathcal{G}_B\) which preserves \([\cdot, \cdot]_{\mathcal{G}_B}\) and \(\langle \cdot, \cdot \rangle_{\mathcal{G}_B}\), a 3-form \(H^B_{(3)}\), 2-forms \(H^B_{(2)}\) and sections \(r^B_i \in \Gamma(\mathcal{G}_B)\) \((1 \leq i \leq k)\) as in Example 37 (see also Corollary 36). We denote by \((e^i)\) the dual basis of \((e_i)\) and by \(\tilde{T}^k = (t^k)^*/\text{span}_\mathbb{Z}\{e^i\}\) the dual torus.

Theorem 62. Assume that the closed forms \(K_i\) defined by (133) represent integral cohomology classes in \(H^2(B, \mathbb{R})\) and let \(\tilde{\pi} : \tilde{M} \to B\) be a principal
A standard Courant algebroid \( \tilde{\theta} = \sum_{i=1}^{k} \tilde{\theta}_i e^i \), such that \((d\tilde{\theta}_i)^B = K_i\) for any \(i\). Then \(E\) admits a \(T\)-dual \(\tilde{E}\), defined on \(\tilde{M}\) and

\[
\left[ \sum_{i=1}^{k} (d\tilde{\theta}_i)^B \wedge (d\tilde{\theta}_i)^B \right] = \langle (\tau^B \wedge \tau^B)_{\mathfrak{g}_B} \rangle \in H^4(B, \mathbb{R}).
\]

**Proof.** From the expression (133) of \(K_i = (d\tilde{\theta}_i)^B\), we have

\[
(d\tilde{\theta}_i)^B = H^i_{(2)} + 2 \langle \tau^B, \tilde{\tau}^B \rangle_{\mathfrak{g}_B} - \langle \tilde{\tau}^B, \tilde{\tau}^B \rangle_{\mathfrak{g}_B}(d\tilde{\theta}_j)^B.
\]

We consider the data formed by the quadratic Lie algebra bundle

\[
(\tilde{\mathcal{G}}_B, [\cdot, \cdot]_{\mathfrak{g}_B}, \langle \cdot, \cdot \rangle_{\mathfrak{g}_B}) := (\mathcal{G}_B, [\cdot, \cdot]_{\mathfrak{g}_B}, \langle \cdot, \cdot \rangle_{\mathfrak{g}_B}),
\]

connection \(\tilde{\nabla}^B := \nabla^B\), sections \(\tilde{\tau}^B \in \Gamma(\mathcal{G}_B)\) (arbitrarily chosen), 3-form \(\tilde{H}^B_{(3)} := H^B_{(3)}\) and 2-forms

\[
\tilde{H}^i_{(2)} := (d\tilde{\theta}_i)^B - 2 \langle \tau^B, \tilde{\tau}^B \rangle_{\mathfrak{g}_B} + \langle \tilde{\tau}^B, \tilde{\tau}^B \rangle_{\mathfrak{g}_B}(d\tilde{\theta}_j)^B.
\]

From (196), the 2-form

\[
\tilde{\mathcal{K}}_i := \tilde{H}^i_{(2)} + 2 \langle \tau^B, \tilde{\tau}^B \rangle_{\mathfrak{g}_B} - \langle \tilde{\tau}^B, \tilde{\tau}^B \rangle_{\mathfrak{g}_B}(d\tilde{\theta}_j)^B = (d\tilde{\theta}_i)^B
\]

is closed. Since

\[
d\tilde{H}^i_{(3)} = dH^i_{(3)} = \langle \tau^B \wedge \tau^B \rangle_{\mathfrak{g}_B} - K_i \wedge (d\tilde{\theta}_i)^B = \langle \tau^B \wedge \tau^B \rangle_{\mathfrak{g}_B} - \tilde{\mathcal{K}}_i \wedge (d\tilde{\theta}_i)^B,
\]

we obtain from Example 37 a standard Courant algebroid \((\tilde{E}, \tilde{\Psi})\) together with an action which lifts the vertical parallelism of \(\tilde{\pi}\), such that

\[
\tilde{H}^{\text{pair},B}_{(0)} := -\frac{1}{3} \langle [\tilde{\tau}^B, \tilde{\tau}^B]_{\mathfrak{g}_B} - \tilde{\tau}^B, \tilde{\tau}^B \rangle_{\mathfrak{g}_B},
\]

\[
\tilde{H}^{ij,B}_{(1)} := \frac{1}{2} \langle \langle \nabla^B \tilde{\tau}^B_{ij}, \tilde{\tau}^B \rangle_{\mathfrak{g}_B} - \langle \nabla^B \tilde{\tau}^B_{ij}, \tilde{\tau}^B \rangle_{\mathfrak{g}_B} \rangle.
\]

and the \(\mathcal{G}_B\)-valued forms \(\tilde{\Phi}^{ij,B}_{(0)}, \tilde{\Phi}^{i,B}_{(1)}\) and \(\tilde{\Phi}^{i,B}_{(2)}\) given by the tilde analogue of (129). The quadratic Lie algebra bundles of \(E\) and \(\tilde{E}\) are the pullbacks of \((\mathcal{G}_B, [\cdot, \cdot]_{\mathfrak{g}_B}, \langle \cdot, \cdot \rangle_{\mathfrak{g}_B})\) and, as vector bundles with scalar products,

\[
\pi_N^* E = \tilde{\pi}_N^* \tilde{E} = T^* N \oplus \Pi^* \mathcal{G}_B \oplus T N,
\]

where the scalar products are given by (179) and \(\Pi = \pi \circ \pi_N = \tilde{\pi} \circ \tilde{\pi}_N\).

We claim that \(E\) and \(\tilde{E}\) are \(T\)-dual, i.e. not only that \(\pi_N^* E\) and \(\tilde{\pi}_N^* \tilde{E}\) are isomorphic as Courant algebroids but that one can choose the isomorphism
to be invariant and such that the non-degeneracy condition (171) is satisfied. Such an isomorphism $F : \pi^1_N E \rightarrow \tilde{\pi}^1_N \tilde{E}$ (if it exists) is given by a triple $(\beta, K, \Phi)$, where $\beta \in \Omega^2(N)$, $\Phi \in \Omega^1(N, \Pi^* \mathcal{G}_B)$ and $K = \Pi^* K_B$ where $K_B \in \text{Aut}(\mathcal{G}_B)$ is a quadratic Lie algebra bundle automorphism (see the proof of Lemma 54). Let

$$\nabla^\theta := \nabla^E + \theta_i \otimes \text{ad}_{r_i} = \pi^* \nabla^B$$

$$\nabla^{\tilde{\theta}} := \nabla^\tilde{E} + \tilde{\theta}_i \otimes \text{ad}_{\tilde{r}_i} = \tilde{\pi}^* \nabla^B$$

be the connections on $E$ and $\tilde{E}$ defined before Lemma 31, where $r_i := \pi^*(r_i^B)$, $\tilde{r}_i := \tilde{\pi}^*(\tilde{r}_i^B)$ and to simplify notation we continue to omit the summation sign and we denote by the same symbol ‘ad’ the adjoint action in the Lie algebra bundles $\mathcal{G}_B$, $\tilde{\mathcal{G}}$, $\mathcal{G}$ or their pullbacks to $N$. Then

$$\tilde{\pi}^*_N \nabla^{\tilde{E}} = \Pi^* \nabla^B - (\tilde{\pi}^*_N \tilde{\theta}_i) \otimes \Pi^* (\text{ad}_{\tilde{r}_i^B})$$

$$\pi^*_N \nabla^E = \Pi^* \nabla^B - (\pi^*_N \theta_i) \otimes \Pi^* (\text{ad}_{r_i^B}).$$

(197)

With these preliminary remarks, we now consider separately the relations from Lemma 53 and we look for $(\beta, K = \Pi^* K_B, \Phi)$ such that these relations are satisfied. Relation (175) can be written in the equivalent way

$$\Pi^* \left( K_B (\nabla^B) K_B^{-1} - \nabla^B \right) = (\pi^*_N \theta_i) \otimes \Pi^* (\text{ad}_{K_B(r_i^B)}) - (\tilde{\pi}^*_N \tilde{\theta}_i) \otimes (\Pi^* (\text{ad}_{r_i^B})) + \text{ad} \circ \Phi.$$  

(198)

Letting

$$K_B := \text{Id}_{\mathcal{G}_B}, \quad \Phi := (\tilde{\pi}^*_N \tilde{\theta}_i) \otimes \Pi^* (\tilde{r}_i^B) - (\pi^*_N \theta_i) \otimes \Pi^* (r_i^B),$$

(199)

relation (198) is obviously satisfied. Relation (176) is automatically satisfied from Lemma 2 and our hypothesis that the adjoint representation of $(\mathcal{G}_B, [\cdot, \cdot]_{\mathcal{G}_B})$ is an isomorphism. It remains to find an invariant 2-form $\beta \in \Omega^2(N)$ such that relation (177) is satisfied. Now, a straightforward computation which uses the definition of $\Phi$ shows that the 3-form

$$c_3(X, Y, Z) := \langle \Phi(X), [\Phi(Y), \Phi(Z)]_{\Pi^* \mathcal{G}_B} \Pi^* \mathcal{G}_B, \forall X, Y, Z \in \mathfrak{X}(N)$$

is given by

$$c_3 = \frac{1}{6} \left( \langle [\tilde{r}_i^B, \tilde{r}_j^B]_{\mathcal{G}_B}, \tilde{r}_k^B \tilde{\theta}_s \wedge \tilde{\theta}_i \wedge \tilde{\theta}_j - \langle [r_i^B, r_j^B]_{\mathcal{G}_B}, r_k^B \theta_s \wedge \theta_i \wedge \theta_j \rangle \right)$$

$$+ \frac{1}{2} \left( \langle [\tilde{r}_i^B, r_j^B]_{\mathcal{G}_B}, \tilde{r}_k^B \theta_s \wedge \theta_i \wedge \theta_j - \langle [\tilde{r}_j^B, \tilde{r}_k^B]_{\mathcal{G}_B}, r_i^B \theta_s \wedge \tilde{\theta}_i \wedge \tilde{\theta}_j \rangle, \right),$$

(200)

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where we identify forms on \( M, \tilde{M} \) or \( B \) with their pullback to \( N \) (we omit the pullback signs) and we denote \( \langle \cdot, \cdot \rangle_{\mathcal{G}_B} \) by \( \langle \cdot, \cdot \rangle \) for simplicity. On the other hand,

\[
\pi^*_N R = R^{B}_{(2)} + \theta_i \wedge R^{ij}_B + R^{ij}_{(0)} \otimes (\theta_i \wedge \theta_j),
\]

\[
\tilde{\pi}^*_N \tilde{R} = \tilde{R}^{B}_{(2)} + \tilde{\theta}_i \wedge \tilde{R}^{ij}_B + \tilde{R}^{ij}_{(0)} \otimes (\tilde{\theta}_i \wedge \tilde{\theta}_j),
\]

where we recall that

\[
R^{ij}_{(0)} = \frac{1}{2}[r^i_{(1)}, r^j_{(1)}]_{\mathcal{G}_B}, \quad R^{i,j}_B = \nabla^B r^i_B, \quad R^{B}_{(2)} = r^B - d\theta_i \otimes r^i_B
\]

and similarly for \( \tilde{R}^{ij}_{(0)}, \tilde{R}^{i,j}_B \) and \( \tilde{R}^{B}_{(2)} \), with \( r^i_B \) replaced by \( \tilde{r}^i_B \) and \( \theta_i \) replaced by \( \tilde{\theta}_i \). From (201) and (202) we obtain that

\[
\langle (\pi^*_N R + \tilde{\pi}^*_N \tilde{R}) \wedge \Phi \rangle_{\Pi^* \mathcal{G}_B} = -\theta_i \wedge \tilde{\theta}_j \wedge d(r^i_B, \tilde{r}^j_B) + \langle (2\pi^*_N r^i_B - \theta_i \wedge \tilde{r}^j_B) \wedge \tilde{\theta}_j \rangle + \langle (2\pi^*_N r^i_B - (d\theta_i) \otimes r^i_B) \wedge (d\tilde{\theta}_i) \otimes \tilde{r}^j_B \rangle \wedge \theta_j + \frac{1}{2}\langle \langle [r^i_B, r^j_B], r^i_B \rangle \theta_i \wedge \theta_j \wedge \tilde{\theta}_p + \langle [r^i_B, r^j_B], r^p_B \rangle \tilde{\theta}_i \wedge \tilde{\theta}_j \wedge \theta_p \rangle \rangle
\]

\[
+ \langle \langle [\tilde{r}^i_B, \tilde{r}^j_B], \tilde{r}^i_B \rangle \tilde{\theta}_i \wedge \tilde{\theta}_j \wedge \tilde{\theta}_p + \langle [\tilde{r}^i_B, \tilde{r}^j_B], \tilde{r}^p_B \rangle \theta_i \wedge \theta_j \wedge \theta_p \rangle \rangle
\]

\[
+ \langle \nabla^B r^i_B, r^j_B \rangle \wedge \theta_i \wedge \theta_j - \langle \nabla^B \tilde{r}^i_B, \tilde{r}^j_B \rangle \wedge \tilde{\theta}_i \wedge \tilde{\theta}_j \rangle.
\]

We write the 2-form \( \beta \) as

\[
\beta = \beta_{(2)} + \theta_i \wedge \beta^i_{(1)} + \tilde{\theta}_i \wedge \tilde{\beta}^i_{(1)} + f_{ij} \theta_i \wedge \tilde{\theta}_j
\]

where \( \beta_{(2)}, \beta^i_{(1)}, \tilde{\beta}^i_{(1)} \) and \( f_{ij} \) are defined on \( B \), so that

\[
d\beta = d\beta_{(2)} + d\theta_i \wedge \beta^i_{(1)} + d\tilde{\theta}_i \wedge \tilde{\beta}^i_{(1)}
\]

\[
- \theta_i \wedge (d\beta^i_{(1)} + f_{ij} d\tilde{\theta}_j) + \tilde{\theta}_i \wedge (-d\tilde{\beta}^i_{(1)} + f_{ij} d\theta_j)
\]

\[
- df_{ij} \wedge \theta_j \wedge \theta_i.
\]

Finally, we write, as in Section 5.1.1,

\[
\pi^*_N H = H^B_{(3)} + \theta_i \wedge H^i_{(2)} + \theta_i \wedge \theta_j \wedge H^{ij}_{(1)} + H^{ij}_{(0)} \theta_i \wedge \theta_j \wedge \theta_s
\]

\[
\tilde{\pi}^*_N \tilde{H} = \tilde{H}^B_{(3)} + \tilde{\theta}_i \wedge \tilde{H}^i_{(2)} + \tilde{\theta}_i \wedge \tilde{\theta}_j \wedge \tilde{H}^{ij}_{(1)} + \tilde{H}^{ij}_{(0)} \tilde{\theta}_i \wedge \tilde{\theta}_j \wedge \tilde{\theta}_s.
\]
Using the expressions of $H_{(0)}^{ij,s,B}$ and $\tilde{H}_{(0)}^{ij,s,B}$, and (200), (203), (204) and (205), we obtain that relation (177) reduces to the following relations:

$$H^B_{(3)} - \tilde{H}^B_{(3)} - d\beta(2) - (d\tilde{\theta}_i)^B \wedge \beta_1^i - (d\tilde{\theta}_j)^B \wedge \tilde{\beta}_1^j = 0$$
$$d\beta_1^i + f_{ij}(d\tilde{\theta}_j)^B = -H^i_{(2)} + \langle \tilde{r}_i^B, \tilde{r}_j^B \rangle (d\tilde{\theta}_j)^B - 2\langle \tilde{r}_i^B, r_j^B \rangle (d\theta_j)^B$$
$$d\tilde{\beta}_1^i - f_{ji}(d\theta_j)^B = \tilde{H}^i_{(2)} - \langle \tilde{r}_i^B, \tilde{r}_j^B \rangle (d\theta_j)^B + 2\langle \tilde{r}_i^B, \tilde{r}_j^B \rangle - \langle \tilde{r}_i^B, r_j^B \rangle (d\theta_j)^B$$
$$df_{ij} = d\langle \tilde{r}_i^B, r_j^B \rangle. \tag{206}$$

Recall now that $H^B_{(3)} = \tilde{H}^B_{(3)}$ and

$$(d\tilde{\theta}_i)^B = H^i_{(2)} + 2\langle \tilde{r}_i^B, \tilde{r}_j^B \rangle (d\tilde{\theta}_j)^B$$
$$(d\theta_i)^B = \tilde{H}^i_{(2)} + 2\langle \tilde{r}_i^B, \tilde{r}_j^B \rangle - \langle \tilde{r}_i^B, r_j^B \rangle (d\theta_j)^B.$$

It follows that $\beta(2) := 0$, $\beta_1^i := 0$, $\tilde{\beta}_1^i := 0$ and $f_{ij} := \langle \tilde{r}_i^B, \tilde{r}_j^B \rangle - \delta_{ij}$ satisfy relations (206). We obtain that the 2-form

$$\beta := ((\tilde{r}_i^B, \tilde{r}_j^B) - \delta_{ij})\theta_i \wedge \tilde{\theta}_j$$

satisfies (177). The existence of $F$ is proved. It is clear that it is invariant. The non-degeneracy condition (171) is satisfied, since

$$\beta(X_a, X_b) = -\langle \tilde{r}_a^B, \tilde{r}_b^B \rangle + \delta_{ab}, \quad (\Phi^*\Phi)(X_a, X_b) = -\langle \tilde{r}_a, r_b \rangle. \quad \square$$

### 6.4 Examples of T-duality

In this section we apply Theorem 62 to various classes of transitive Courant algebroids. In particular, we recover, in our setting, the T-duality for exact Courant algebroids [9] and the T-duality for heterotic Courant algebroids [2].

#### 6.4.1 T-duality for exact Courant algebroids

Let $E = T^*M \oplus TM$ be an exact Courant algebroid over the total space of a principal $T^k$-bundle $\pi : M \to B$, with Dorfmann bracket $[\cdot, \cdot]_H$ twisted by an invariant, closed, 3-form $H \in \Omega^3(M)$, that is,

$$[\xi + X, \eta + Y]_H := \mathcal{L}_X(Y + \eta) - i_Y d\xi + i_{Y} i_{X} H, \tag{207}$$

for any $X, Y \in \mathfrak{X}(M)$, $\xi, \eta \in \Omega^1(M)$, scalar product $\langle \xi + X, \eta + Y \rangle := \frac{1}{2}(\xi(Y) + \eta(X))$ and anchor the natural projection from $E$ to $TM$. The action of $T^k$ on $M$ lifts naturally to an action on $E$. Assuming that $H|_{\Lambda^2(\text{Ker} \pi)} = 0$, we obtain a Courant algebroid of the type described in Example 37. Choose
a connection $\mathcal{H}$ on $\pi$, with connection form $\theta = \sum_{i=1}^{k} \theta_i e_i$ (where $(e_i)$ is a basis of $t^k$) and write

$$H = H_{(3)} + \sum_{i=1}^{k} \theta_i \wedge H_{(2)}^i,$$

where $H_{(3)}$ and $H_{(2)}^i$ are basic. If $[H] \in H^3(M, \mathbb{R})$ is an integral cohomology class, then so is $[H_{(2)}^i] \in H^2(B, \mathbb{R})$ (for any $i$) and Theorem 62 can be applied.

We recover the existence of a $T$-dual for exact Courant algebroids, which was proved in [8] (see also Proposition 2.1 of [9]).

### 6.4.2 Heterotic $T$-duality

Let $G$ be a compact semi-simple Lie group, with a fixed invariant scalar product of neutral signature $\langle \cdot, \cdot \rangle_G$ on $g = \text{Lie}(G)$. Let $\sigma : P \to M$ be a principal $G$-bundle and $\mathcal{H}$ a connection on $\sigma$. By definition, the heterotic Courant algebroid defined by the principal $G$-bundle $\sigma : P \to M$, connection $H$ and a 3-form $H \in \Omega^3(M)$ is the standard Courant algebroid $E = T^*M \oplus G \oplus TM$ with the following properties:

1. $(\mathcal{G}, [\cdot, \cdot]_{\mathcal{G}}, \langle \cdot, \cdot \rangle_{\mathcal{G}})$, as a quadratic Lie algebra bundle, is given by the adjoint bundle $g_P := P \times_{\text{Ad}} g$. Recall that sections $r \in \Gamma(g_P)$ are invariant vertical vector fields on $P$ and can be identified with functions $f : P \to g$ which satisfy the equivariance condition $f(pg) = \text{Ad}_{g^{-1}}f(p)$ for any $p \in P$ and $g \in G$. We shall use the notation $r \equiv f$ to denote this identification. Since the Lie bracket $[\cdot, \cdot]_g$ and scalar product $\langle \cdot, \cdot \rangle_g$ of $g$ are $\text{Ad}$-invariant, they induce a Lie bracket and a scalar product on $g_P$, which make $g_P$, a quadratic Lie algebra bundle. The Lie bracket of $g_P$ so defined coincides with the usual Lie bracket of invariant, vertical vector fields on $P$.

2. The connection $\nabla$ which is part of the data $(\nabla, R, H)$ which defines the standard Courant algebroid $E$ is induced by $\mathcal{H}$; $R = R^H \in \Omega^2(M, g_P)$ is the curvature of $\mathcal{H}$ and $dH = \langle R^H \wedge R^H \rangle_g$.

The following proposition describes all invariant scalar products on compact semi-simple Lie algebras. A similar description can be given for arbitrary reductive Lie algebras. Recall that a semi-simple Lie algebra is called compact if it is the Lie algebra of a compact group.

**Proposition 63.** Let $g = \bigoplus_{i=1}^{\ell} k_i g_i$ be the decomposition of a real semi-simple Lie algebra into its simple ideals $g_i$, of multiplicity $k_i \geq 1$. Assume
that \( \mathfrak{g} \) is compact (or, more generally, that none of the \( \mathfrak{g}_i \) has an invariant complex structure). Then every invariant scalar product on \( \mathfrak{g} \) is of the form

\[
\sum B_i \otimes b_i, \tag{208}
\]

where \( B_i \) is the Killing form of \( \mathfrak{g}_i \) and \( b_i \) is a scalar product on \( \mathbb{R}^{k_i} \). The scalar product (208) is of neutral signature if and only if \( \sum (\dim \mathfrak{g}_i) p_i = \sum (\dim \mathfrak{g}_i) q_i \), where \( (p_i, q_i) \) is the signature of \( b_i \).

**Proof.** We compute the space of invariant symmetric bilinear forms on \( \mathfrak{g} \) as

\[
(\text{Sym}^2 \mathfrak{g}^*)^p = \bigoplus \text{Sym}^2 (\mathfrak{g}_i)^{p_i} \otimes \text{Sym}^2 (\mathfrak{R}^{k_i})^* + \bigoplus (\Lambda^2 \mathfrak{g}_i)^{q_i} \otimes \Lambda^2 (\mathfrak{R}^{k_i})^*.
\]

Since \( \mathfrak{g}_i \) is simple and not complex, every invariant bilinear form on \( \mathfrak{g}_i \) is a multiple of \( B_i \) and the right-hand side reduces to \( \bigoplus B_i \otimes \text{Sym}^2 (\mathfrak{R}^{k_i})^* \). This implies the first claim. The second claim follows from observing that the signature \( (p, q) \) of (208) is given by \( p = -\sum (\dim \mathfrak{g}_i) p_i, \ q = -\sum (\dim \mathfrak{g}_i) q_i \). \( \square \)

Assume that \( M \) is the total space of a principal \( T^k \)-bundle \( \pi : M \to B \) and that \( \sigma : P \to M \) is the pullback of a principal \( G \)-bundle \( \sigma_0 : P_0 \to B \). Then, for any \( m \in M \) and \( g \in T^k \) there is a natural identification between the fibers \( P_m := \sigma^{-1}(m), \ P_{mg} := \sigma^{-1}(mg) \) and \( (P_0)_{\pi(m)} := \sigma_0^{-1}(\pi(m)) \), and the \( T^k \)-action on \( M \) lifts naturally to an action on \( P \) (such that \( g \) acts as the identity map between \( P_m \) and \( P_{mg} \) in the above identification). We deduce that on any heterotic Courant algebroid \( E = T^* M \oplus \mathfrak{g}_P \oplus TM \) there is an induced action

\[
\Psi : t^k \to \text{Der}(E), \ \Psi(a)(\xi + r + X) := \mathcal{L}_{X^M_\sigma} \xi + \mathcal{L}_{X^P_\sigma} r + \mathcal{L}_{X^M_\sigma} X \quad \tag{209}
\]

where \( X^M_\sigma \) and \( X^P_\sigma \) denote the fundamental vector fields of the \( T^k \)-action on \( M \) and \( P \) defined by \( a \in t^k \) and \( r \in \Gamma(\mathfrak{g}_P) \) is viewed as an invariant vertical vector field on \( P \). If \( r \equiv f \), then \( \mathcal{L}_{X^P_\sigma} \equiv X^P_\sigma(f) \).

Following [2], we shall be interested in heterotic Courant algebroids defined by \( \sigma \) and a particular class of connections \( \mathcal{H} := \mathcal{H}^\sigma \) on \( \sigma \). More precisely, we consider a connection \( \mathcal{H}^\sigma \) on the principal \( T^k \)-bundle \( \pi : M \to B \), with connection form \( \theta = \sum_{i=1}^k \theta_i e_i \in \Omega^1(M, t^k) \) (where \( (e_i) \) is a basis of \( t^k \)), a connection \( \mathcal{H}^\sigma_0 \) on the principal \( G \)-bundle \( \sigma_0 : P_0 \to B \), with connection form \( A_0 \in \Omega^1(P_0, \mathfrak{g}) \) and a \( G \times T^k \)-equivariant function \( \hat{\nu} : P \to (t^k)^* \otimes \mathfrak{g} \). They define a connection \( \mathcal{H}^\sigma \) on \( \sigma \), with connection form

\[
A := \pi_0^* A_0 - \langle \sigma^* \theta, \hat{\nu} \rangle = \pi_0^* A_0 - \sum_{i=1}^k \sigma^* \theta_i \otimes \hat{\nu}_i, \quad \tag{210}
\]
where \( \pi_0 : P \to P_0 \) is the natural projection, \( \langle \cdot, \cdot \rangle \) denotes the natural contraction between \( t^k \) and \( (t^k)^* \), and \( \hat{v}_i = \langle \hat{v}, e_i \rangle : P \to g \). From the equivariance of \( \hat{v} \), the functions \( \hat{v}_i \) define sections of \( g_{P_0} = \pi^* g_{P_0} \) which are pullback of sections of \( g_{P_0} \), i.e., \( \hat{v}_i = \pi^* \hat{v}_i^B \) for \( \hat{v}_i^B \in \Gamma(g_{P_0}) \) (we use the same notation for the functions \( \hat{v}_i \), \( \hat{v}_i^B \) and the corresponding sections of \( g_P \) and \( g_{P_0} \) respectively). We shall denote by \( \hat{X}^\sigma_0, \hat{Y}^p_0, \hat{Y}^A_0 \), the horizontal lifts of \( X \in \mathfrak{X}(B) \) and \( Y \in \mathfrak{X}(M) \) with respect to \( \mathcal{H}^{\sigma_0}, \pi_0^* \mathcal{H}^{\sigma_0} \) and \( \mathcal{H}^\tau \) respectively. Here \( \pi_0^* \mathcal{H}^{\sigma_0} \subset TP \) denotes the \( G \)-invariant horizontal distribution in \( \sigma : P \to M \) defined by \( (\pi_0^* \mathcal{H}^{\sigma_0})_p = (d_p \pi_0)^{-1} \mathcal{H}^{\sigma_0}_{\pi_0(p)} \), \( p \in P \). It coincides with the kernel of the connection form \( \pi_0^* A_0 \).

**Lemma 64.** Let \( (E = T^* M \oplus g_P \oplus TM, \Psi) \) be the heterotic Courant algebroid defined by \( \sigma : P \to M \), the connection \( \mathcal{H}^\tau \) with connection form \( (210) \) and a 3-form \( H \in \Omega^3(M) \) such that \( dH = (R^H \wedge R^H)_g \), together with the \( t^k \)-action \( (209) \). Then the connection \( \nabla^g_0 \) on \( g_P \) defined in \( (109) \) is the pullback of the connection \( \nabla^\sigma_0 \) on \( g_{P_0} \) induced by \( \mathcal{H}^{\sigma_0} \) (in the notation of Corollary 36, \( G_B = g_{P_0} \) and \( \nabla^{g_B} = \nabla^{\sigma_0} \)).

Proof. We claim that the horizontal lift \( \hat{X}_a^\sigma_0 A_0 \in \mathfrak{X}(P) \) of the \( \pi \)-vertical vector field \( X_a^M \) determined by \( a \in t^k \) coincides with the fundamental vector field \( X_a^P \) of the \( T^k \)-action on \( P \), i.e.

\[
\hat{X}_a^\sigma_0 A_0 = X_a^P, \forall a \in t^k. \tag{211}
\]

In order to prove \( (211) \), let \( U \subset B \) be open and sufficiently small such that, over \( U \), \( \sigma_0 \) is the trivial \( G \)-bundle and

\[
\pi_0 : P|_{\pi^{-1}(U)} = \pi^{-1}(U) \times G \to P_0|_U = U \times G, \quad \pi_0(p, g) = (\pi(p), g).
\]

For any \( X \in \mathfrak{X}(\pi^{-1}(U)) \),

\[
\hat{X}_a^\sigma_0 A_0 = X - \langle (\pi_0^* A_0)(X), f_i^* \rangle X_i^P, \tag{212}
\]

where \( (f_i) \) is a basis of \( g \) with dual basis \( (f_i^*) \) and for \( f \in g \), \( X_f^P \) is the left invariant vector field on \( G \) determined by \( f \) (viewed as a vector field on \( \pi^{-1}(U) \times G \)). On the other hand, \( X_a^M \in \mathfrak{X}(\pi^{-1}(U)) \), viewed as a vector field on \( P = \pi^{-1}(U) \times G \), satisfies \( (\pi_0)_* X_a^M = 0 \), since \( \pi_* X_a^M = 0 \) and \( \pi = \pi \times \text{Id} \) in our trivializations. Applying \( (212) \) to \( X := X_a^M \) and using \( (\pi_0)_* X_a^M = 0 \) we obtain \( \hat{X}_a^\sigma_0 A_0 = X_a^M \). On the other hand, the action of \( T^k \) on \( P = \pi^{-1}(U) \times G \) is given by \( R_g(m, \tilde{g}) = (mg, \tilde{g}) \) which implies that \( X_a^P = X_a^M \). Relation \( (211) \) follows.

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Let $\nabla$ be the connection on $g_P$ induced by $H^\sigma$. Its covariant derivative is given by
\[
\nabla_X r \equiv \tilde{X}^A = \tilde{X}^\pi \sigma A_0(f) - \theta_i(X) \text{ad}_{\hat{v}_i} \circ f, \quad X \in \mathfrak{X}(M),
\]
where $r \in \Gamma(g_P)$ and $r \equiv f$. Here we have used that
\[
\tilde{X}^A = \tilde{X}^\pi \sigma A_0 + \theta_i(X) X^P \hat{v}_i
\]
and the $G$-equivariance of $f$, which implies $X^P_{\hat{v}}(f) = -\text{ad}_{v} \circ f$ for all $v \in g$. Applying relation (213) to $X := X^M_a$ and using (211) we obtain
\[
\nabla_{X^M_a} r \equiv X^P_a(f) - \langle a, e_{i*} \rangle \text{ad}_{\hat{v}_i} \circ f, \quad \forall a \in t^k, \tag{214}
\]
which implies that the skew-symmetric derivation $A_a$ of $g_P$, from Lemma 26, is given by
\[
A_a(r) = (\mathcal{L}_{X^P_a} - \nabla_{X^M_a})r \equiv \text{ad}_{(\hat{v}_a)} \circ f, \quad \forall a \in t^k. \tag{215}
\]
From its definition (109) and relations (213), (215), $\nabla^\theta$ is given by
\[
\nabla^\theta_X r = \nabla_X r + \sum_{i=1}^{k} \theta_i(X) A_i(r) \equiv \tilde{X}^\pi \sigma A_0(f), \tag{216}
\]
which implies that $\nabla^\theta = \pi^* \nabla^A_0$ as needed. \qed

Since $\nabla^{A_0}$ preserves the Lie bracket and scalar product of $g_{P_0}$, its curvature takes values in the bundle of skew-symmetric derivations of $g_{P_0}$ and is of the form $\text{ad}_{r^A_0}$ where $r^A_0 \in \Omega^2(B, g_{P_0})$ (since $g$ is semi-simple). Like in (112), we decompose $H \in \Omega^3(M)$ using the connection $\theta$.

**Proposition 65.** In the above setting, assume that
\[
H_{ij,B}^{(0)} := -\frac{1}{3} \langle [\hat{v}_i^B, \hat{v}_j^B]_{\theta}, \hat{v}_a^B \rangle \theta
\]
and that the (closed) forms
\[
H_{ij,B}^{(1)} := \frac{1}{2} \left( (\nabla^{A_0} \hat{v}_i^B, \hat{v}_j^B)_{\theta} - (\nabla^{A_0} \hat{v}_j^B, \hat{v}_i^B)_{\theta} \right), \tag{217}
\]
and that the (closed) forms
\[
H_{ij,B}^{(2)} + 2 \langle r^{A_0}, \hat{v}_i^B \rangle_{\theta} - \langle \hat{v}_i^B, \hat{v}_j^B \rangle_{\theta} (d\theta)_{ij}B. \tag{218}
\]
represent integral cohomology classes. Then $(E, \Psi)$ admits a $T$-dual which is a heterotic Courant algebroid.
Proof. The conditions (217) mean that \((E, \Psi)\) belongs to the class of standard Courant algebroids with \(t^k\)-action described in Example 37 (in the notation of that example, \(r_i = \hat{v}_i\) and \(r^B_i = \hat{v}^B_i\)). Let \((\tilde{E}, \tilde{\Psi})\) be a \(T\)-dual of \((E, \Psi)\), provided by Theorem 62. Then \((\tilde{E}, \tilde{\Psi})\) is defined on the total space of a principal \(T^k\)-bundle \(\tilde{\pi} : \tilde{M} \to B\), with connection form \(\tilde{\theta} = \sum_{i=1}^k \tilde{\theta}_i e^i\), in terms of arbitrarily chosen sections \(\tilde{r}^B_i \in \Gamma(g_{P_0})\). We define the pullback bundle \(\tilde{\sigma} : \tilde{P} \to \tilde{M}\) of \(\sigma_0 : P_0 \to B\) by the map \(\tilde{\pi}\). The arguments from Theorem 62 and the above lemma show that \(\tilde{E}\) is the heterotic Courant algebroid defined by the principal \(G\)-bundle \(\tilde{\sigma}\), connection \(\tilde{H}^\sigma\) with connection form

\[
\tilde{A} = \tilde{\pi}_0^* A_0 - \sum_{i=1}^k \tilde{\sigma}^* \tilde{\theta}_i \otimes \tilde{r}_i
\]

where \(\tilde{\pi}_0 : \tilde{P} \to P_0\) is the natural projection, \(\tilde{r}_i = \tilde{\pi}^* (\tilde{r}^B_i) \in \Gamma(g_{\tilde{P}})\) and 3-form \(\tilde{H}\) is constructed as in Theorem 62 (in particular, \(\tilde{H}_{ij, B}^{(0)}\) and \(\tilde{H}_{ij, B}^{(1)}\) are given by (217) with \(\hat{v}^B_i\) replaced by \(\tilde{r}^B_i\)).

\[\blacksquare\]

Remark 66. The above treatment provides an alternative viewpoint for the heterotic \(T\)-duality developed in [2]. Heterotic Courant algebroids can be obtained from exact Courant algebroids by a reduction procedure described in [2] and the heterotic \(T\)-duality from [2] was obtained as a reduction of the \(T\)-duality for exact Courant algebroids [9]. Our approach is more direct and makes no reference to exact Courant algebroids.

In our setting it is natural to relax the definition of a heterotic Courant algebroid [2] by allowing as structure groups of the principal bundle not only compact semi-simple Lie groups but any connected Lie group \(G\) such that

\[P_1\) \, Ad : G \to Aut(\langle \cdot, \cdot \rangle_{g})_0\) is a covering for some invariant scalar product \(\langle \cdot, \cdot \rangle_{g}\) on \(g = Lie\, G\). (Equivalently, \(\text{ad} : g \to Der(g, \langle \cdot, \cdot \rangle_{g})\) is an isomorphism onto the Lie algebra of skew-symmetric derivations, cf. Remark 35.) As before, we restrict to scalar products of neutral signature.

The resulting Courant algebroids are transitive and the corresponding bundles of quadratic Lie algebras \(G\) have the property

\[P_2\) \, ad : G \to Der(G)\) is an isomorphism.

Proposition 67. The class of transitive Courant algebroids \(E \to M\) over simply connected manifolds for which the bundle of quadratic Lie algebras \(G\) has the property \(P_2\) coincides with the above (relaxed) class of heterotic Courant algebroids.
Proof. We first remark that the fibers $(\mathcal{G}, [\cdot, \cdot]_\mathcal{G}, \langle \cdot, \cdot \rangle_\mathcal{G})|_p, p \in M$, are all isomorphic to a fixed quadratic Lie algebra $(\mathfrak{g}, \langle \cdot, \cdot \rangle_\mathfrak{g})$. In fact, for a transitive Courant algebroid $E$, any two fibers of $\mathcal{G}$ are related by parallel transport, which preserves the tensor fields $[\cdot, \cdot]_\mathcal{G}$ and $\langle \cdot, \cdot \rangle_\mathcal{G}$. Note that $\mathcal{G}$ satisfies $P_2$ if and only if $\mathfrak{g}$ satisfies $P_1$.

Let us fix a basis a basis in $\mathfrak{g}$. The connection $\nabla$ in the bundle $\mathcal{G}$ induces a connection in the bundle $P$ of standard frames of $\mathcal{G}$. A frame is called standard if its structure constants and the Gram matrix of the scalar product coincide with those of the underlying quadratic Lie algebra $(\mathfrak{g}, \langle \cdot, \cdot \rangle_\mathfrak{g})$ with respect to the fixed basis in $\mathfrak{g}$. The structure group of $P$ is $\text{Aut}(\mathfrak{g}, \langle \cdot, \cdot \rangle_\mathfrak{g})$ and can be always reduced to the connected group $\text{Aut}(\mathfrak{g}, \langle \cdot, \cdot \rangle_\mathfrak{g})_0$ by holonomy reduction if $M$ is simply connected. The property $P_2$ implies $\text{Lie Aut}(\mathfrak{g}, \langle \cdot, \cdot \rangle_\mathfrak{g})_0 \cong \mathfrak{g}$ and then $G := \text{Aut}(\mathfrak{g}, \langle \cdot, \cdot \rangle_\mathfrak{g})_0$ satisfies $P_1$. In that case we can rewrite the bundle $\mathcal{G}$ as the adjoint bundle with connection induced from the connection in the principal $G$-bundle $P$. This shows that $E$ belongs to the (relaxed) class of heterotic Courant algebroids.

References

[1] A. Alekseev, P. Xu, Derived brackets and courant algebroids, unpublished, available at http://www.math.psu.edu/ping/anton-final.pdf (2001).

[2] D. Baraglia, P. Hekmati: Transitive Courant algebroids, String structures and T-duality, Adv. Theor. Math. Phys. 19 (3) (2015), p. 613-672.

[3] N. Berline, E. Getzler, M. Vergne: Heat Kernels and Dirac Operators, Springer-Verlag 1992.

[4] H. Bursztyn, G. R. Cavalcanti, M. Gualtieri: Reduction of Courant algebroids and generalized complex structures, Adv. Math. 211 (2007), p. 726-765.

[5] H. Bursztyn, G. R. Cavalcanti, M. Gualtieri: Generalized Kähler and hyper-Kähler quotients, Contemporary Mathematics, Poisson Geometry in Mathematics and Physics, vol. 450, p. 61-77.

[6] P. Bouwknegt, J. Evslin, V. Mathai: T-Duality: Topology Change from H-flux, Comm. Math. Phys. 249 (2004), p. 283-415.

[7] P. Bouwknegt, J. Evslin, V. Mathai: On the Topology and Flux of T-Dual Manifolds, Phys. Rev. Lett. 92(2004)018.
[8] P. Bouwknegt, K. Hannabuss, V. Mathai: *T-duality for principal torus bundles*, JHEP03(2004)018.

[9] G. R. Cavalcanti, M. Gualtieri: *Generalized complex geometry and T-duality*. A Celebration of the Mathematical Legacy of Raoul Bott, CRM Proceedings and Lecture Notes, American Mathematical Society (2010), p. 341-366.

[10] Z. Chen, M. Stiényon, P. Xu: *On regular Courant algebroids*, J Symplectic Geom. **11** (1) (2013), p. 1-24.

[11] V. Cortés, L. David: *Generalized connections, spinors and integrability of generalized structures on Courant algebroids*, Moscow Math. J. 21 (4) (2021), p. 695-736.

[12] M. García-Fernández: *Ricci flow, Killing spinors and T-duality in generalized geometry*, Adv. Math. **350** (2019), p. 1059-1108.

[13] M. Gualtieri: *Generalized complex geometry*, Ann. Math. **174** (1) (2011), p. 75-123.

[14] M. Gualtieri: *Generalized complex geometry*, Ph.D thesis, University of Oxford, 2004.

[15] M. Grützmann, J.-P. Michel, P. Xu: *Weyl quantisation of degree 2 symplectic graded manifolds*, arxiv:1410.3346.

[16] D. Li-Bland, E. Meinrenken: *Courant algebroids and Poisson geometry*, Int. Math. Res. Notices **11** (2009), p. 2106-2145.

[17] R. Rubio: *Generalized geometry of type B_n*, Ph.D Thesis, University of Oxford, 2014 (134 pages).

[18] E. L. Stitzinger: *On Lie algebras with only inner derivations*, J. Algebra **105** (1987), 341-343.

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