Fisher Metric, Geometric Entanglement and Spin Networks

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Starting from recent results on the geometric formulation of quantum mechanics, we propose a new information geometric characterization of entanglement for spin network states in the context of quantum gravity. For the simple case of a single-link fixed graph (\textit{Wilson line}), we detail the construction of a Riemannian Fisher metric tensor and a symplectic structure on the graph Hilbert space, showing how these encode the whole information about separability and entanglement. In particular, the Fisher metric defines an entanglement monotone which provides a notion of distance among states in the Hilbert space. In the maximally entangled gauge-invariant case, the entanglement monotone is proportional to a power of the area of the surface dual to the link thus supporting a connection between entanglement and the (simplicial) geometric properties of spin network states.

We further extend such analysis to the study of non-local correlations between two non-adjacent regions of a generic spin network graph characterized by the bipartite unfolding of an Intertwiner state. Our analysis confirms the interpretation of spin network bonds as a result of entanglement and to regard the same spin network graph as an information graph, whose connectivity encodes, both at the local and non-local level, the quantum correlations among its parts. This gives a further connection between entanglement and geometry.

I. INTRODUCTION

Background-independent candidates to a full theory of Quantum Gravity, such as Loop Quantum Gravity [1–5], the modern incarnation of the canonical quantization programme for the gravitational field, together with its covariant counterpart (spin foam models), based on simplicial gravity techniques, and Group Field Theory (GFT) [6–9], a closely related formalism sharing the same type of fundamental degrees of freedom, propose a picture of the microscopic quantum structure of spacetime, where at very small scales continuum space, time and geometry dissolve into non-geometric, combinatorial and algebraic (group-theoretic) entities. These entities can be described in terms of spin networks, graphs coloured by irreducible representations of the local gauge group of gravity (the Lorentz group, then usually gauge-fixed to SU(2)). Quantum spin network states represent elementary excitations of spacetime itself, and geometric observables are operators acting on them. For example, areas and volumes correspond to quantum operators which are diagonalized on spin network states. The heuristic picture for spin networks is therefore that of “grains of space”.

The key issue, then, becomes the reconstruction or “emergence” of continuum spacetime and geometry from such microscopic building blocks, in some approximate regime of the quantum dynamics. This issue is intertwined with, but goes also much beyond, the difficulties with defining a notion of locality in spacetime due to diffeomorphism invariance.

Various strategies for addressing this open issue are being explored. At a more formal level, they all aim at a better control over the regime of the fundamental theory involving a large number of fundamental degrees of freedom, and therefore rest on the renormalization of quantum gravity models. This line of research has witnessed a tremendous progress in the context of renormalization of group field theory models [10], which amounts automatically to a renormalization of the corresponding spin foam amplitudes, as well as in the context of spin foam models understood as generalised lattice gauge theories [11]. At a more physical level, the effective continuum dynamics emerging from quantum gravity models has been studied, for example, in the formalism of group field theory condensate cosmology in [12, 13], so far mostly limited to spatially homogenous and isotropic universes\textsuperscript{1}.

There are many hints that entanglement and tools from quantum information theory should play a crucial role both in the characterization of the intrinsic properties of the quantum texture of spacetime and in the reconstruction of its geometry. For instance, in a different, but still quantum gravity-related context, recent developments in AdS/CFT have shown that the entanglement of spatial regions on the boundary is directly related to the connectivity of the bulk regions thus suggesting that our three-dimensional space is held together by quantum entanglement [16]. Later works based on the so-called Ryu-Takayanagi formula, which relates the entanglement entropy in a conformal field theory to the area of a minimal surface in its holographic dual [17], have also shown that the stress-energy tensor near the boundary of a bulk spacetime region can be reconstructed from the entanglement

\textsuperscript{1} A first investigation of anisotropic GFT condensate configurations can be found in [14] and a first step towards the analysis of inhomogeneities is taken in [15].
Also on the side of fundamental quantum gravity formalisms based on spin networks, there are many proposals to use quantum information to reconstruct geometrical notions such as distance in terms of the entanglement on spin network states [20–22]. The idea is that in a purely relational, background independent context only correlations have a physical meaning and it seems reasonable to regard spin networks themselves as networks of quantum correlations between regions of space and then derive geometrical properties from the intrinsic information content of the theory. In fact, there has been also a lot of activity, recently, in connecting spin network states and tensor networks, which are a crucial tool for controlling the entanglement structure of many-body quantum states [23], and in using the same connection to extract information about the entanglement entropy encoded in spin network states [24, 25]. In particular, in [25], a precise dictionary between tensor networks, spin networks and group field theory states has been established, and used as a basis for a derivation of the Ryu-Takayanagi formula in a quantum gravity context. Other results on entanglement in spin network states can be found in [26, 27]. Moreover, part of the cited work on spin foam renormalization [11] relies as well on tensor network techniques, providing another route for understanding how the entanglement of spin network states affects the continuum limit of quantum gravity models.

This work aims at providing further insights on the transition “from pregeometry to geometry” by introducing the tools of the geometric formulation of quantum mechanics (GQM) in the quantum gravity context. Indeed, the usual Hilbert space of a quantum mechanical system can be equipped with a Kähler manifold structure inheriting both a Riemannian metric tensor and a symplectic structure from the underlying complex projective space of rays. This is the so-called Fubini-Study Hermitian tensor whose real and imaginary parts provide us with a metric and a symplectic structure, respectively. For pure states, this metric is the Fisher-Rao metric, well known in the context of information geometry. Powerful techniques of differential geometry can be thus imported in QM and, in particular, the entanglement properties of a composite system can be characterized in a purely tensorial fashion [28, 29]. Indeed, the pulled-back Hermitian tensor on orbit submanifolds of quantum states, related by unitary transformations, decomposes in block matrices which encode all the information about the separability or entangled nature of the fiducial state of the given orbit. Of particular interest are the off-diagonal blocks of the metric which encode the information on quantum correlations between the subsystems and define an entanglement measure interpreted as a distance with respect to the separable case.

Along the same line, we can use quantum tensors to characterize the entanglement on spin network states. The advantages of this formalism are both computational and conceptual. Indeed, unlike the calculations involving entanglement entropy, it does not require the explicit knowledge of the Schmidt coefficients. Moreover, the key structures of the formalism are built purely from the space of states without introducing additional external structures. We will give further motivations for the use of such geometric techniques in a quantum gravity context, in the coming sections.

The paper is organized as follows. Section II introduces the basics of Geometric Quantum Mechanics focusing on the characterization of entanglement by means of the tensorial structures defined on the manifold of pure quantum states. In Sections III–VI we try then such GQM formalism on the structure of correlations of the spin networks states, for two different simple examples, in the generalised context of GFT abstract graphs. Specifically, we first consider the case of a single-link graph regarded as a bipartite system correlating the spin states at its endpoints and explicitly discuss the two extreme cases of a separable and a maximally entangled state (Sec. IV). Then in Section V we discuss the entanglement resulting from the gluing of two links. Finally, in Sec. VI we extend the analysis to the study of non-local correlations characterizing the bipartite unfolding of an Intertwiner state. Detailed computations and an explicit example are reported in Appendix A and B, respectively. In Section VII we collect our results and discuss future investigations.

To keep the treatment self-contained we added some further appendices at the end of the paper which give a review of the notion of spin networks in QG. We first introduce the geometric, or embedded, definition of states of quantum geometry as given in the canonical framework of LQG (Appendix C), hence we move to a generalised description provided by GFT, where graphs become abstract networks comprised by pre-geometric quanta of space of purely combinatorial and algebraic nature (Appendix D).

II. A GEOMETRIC APPROACH TO QUANTUM MECHANICS

According to the probabilistic interpretation of quantum mechanics, we usually identify (pure) states with equivalence classes (rays) of state vectors $|\psi\rangle$ with respect to multiplication by a non-zero complex number $\lambda$. The space of rays $R(\mathcal{H})$ is a differential manifold identified with the complex projective space $\mathbb{C}P(\mathcal{H})$ associated with $\mathcal{H}$ [59]. The manifold structure of this space requires that we replace all objects, whose definition depends on the linear structure on $\mathcal{H}$, with tensorial geometrical entities which preserve their meaning under general transformations and not just linear ones [56].

In this section we briefly recall how to construct tensorial quantities on the space of states of a quantum system by focusing on those tensors on $\mathcal{H}$ which can be identified with the pull-back of tensorial objects defined on the
underlying complex projective space. We therefore describe the procedure of pull-back on orbit submanifolds of quantum states with respect to the action of unitary representations of Lie groups. Eventually, we shall apply such a procedure to derive a tensorial characterization of quantum entanglement for composite systems [29].

A. Classical tensors on pure states

Let $\mathcal{H} \cong \mathbb{C}^N$ be a finite-dimensional Hilbert space of dimension $N$ and denote by $\{|e_j\rangle\}_{j=1,\ldots,N}$ its (orthonormal) basis. We can introduce complex coordinate functions $c_j$ on $\mathcal{H}$ by setting $c_j(\psi) = e_j(\psi)$, for any $\psi \in \mathcal{H}$. By replacing functions with their exterior differentials we may associate with the Hermitian inner product on quantum state vectors $\langle \cdot | \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ a Hermitian covariant tensor on quantum-state-valued sections of the tangent bundle $T\mathcal{H}$ defined by [54, 55, 57]

$$h = \langle d\psi \otimes d\psi \rangle := \sum_j d\overline{c}_j \otimes dc_j ,$$

such that

$$\langle d\psi \otimes d\psi \rangle (X_\psi, X_{\psi'}) = \langle \psi | \psi' \rangle$$

for any vector field $X_\psi : \phi \mapsto (\phi, \psi), \forall \phi \in \mathcal{H}$. This essentially amounts to identify $\mathcal{H}$ with the tangent space $T_\psi \mathcal{H}$ at each point of the base manifold.

The decomposition of the coordinate functions $c_j$ into real and imaginary part, say $c_j = x_j + iy_j$, that is to replace $\mathcal{H}$ with its realification $\mathcal{H}_\mathbb{R} := \mathbb{R}e(\mathcal{H}) \oplus \mathbb{R}m(\mathcal{H}) \cong \mathbb{R}^{2N}$, allows to identify an Euclidean metric and a symplectic structure on $\mathcal{H}_\mathbb{R}$ respectively with the real and imaginary part of the Hermitian tensor (1), i.e.

$$h = g + i\omega = \delta_{jk}(dx^j \otimes dx^k + dy^j \otimes dy^k) + i\delta_{jk}(dx^j \otimes dy^k - dy^j \otimes dx^k) .$$

These two tensors are related by a $(1,1)$-tensor field $J = \delta_{jk}(dx^j \otimes \frac{\partial}{\partial x^k} - dy^j \otimes \frac{\partial}{\partial y^k})$ playing the role of a complex structure. The real differential manifold $\mathcal{H}_\mathbb{R}$ is thus equipped with a Kähler manifold structure [57].

Coming back to the space of rays, it is well known [58, 59] that the equivalence classes of state vectors identifying points of the complex projective space $\mathbb{CP}(\mathcal{H}) \cong R(\mathcal{H})$ can be represented by rank-one projectors $\rho = \frac{|\psi \rangle \langle \psi|}{\langle \psi | \psi \rangle} \in D^1(\mathcal{H}) \subset u^*(\mathcal{H})$ called pure states which satisfy the properties $\rho^1 = \rho, \rho^2 = \rho, \text{Tr} \rho = 1$. Inheriting the differential calculus from $u^*(\mathcal{H})$, we define an operator-valued $(0,2)$-tensor $d\rho \otimes d\rho$ which may be turned into a covariant tensor by evaluating it on the state $\rho$ itself, i.e.

$$\text{Tr} (d\rho \otimes d\rho) .$$

The pull-back of the tensor (4) from $R(\mathcal{H})$ to $\mathcal{H}_0 \equiv \mathcal{H} - \{0\}$ along the (momentum) map

$$\mu : \mathcal{H}_0 \ni |\psi\rangle \mapsto \rho = \frac{|\psi \rangle \langle \psi|}{\langle \psi | \psi \rangle} \in R(\mathcal{H}) \cong D^1(\mathcal{H}) \subset u^*(\mathcal{H})$$

(5)

gives the so-called Fubini-Study Hermitian tensor [60]

$$h_{FS} = \frac{\langle d\psi \otimes d\psi \rangle}{\langle \psi | \psi \rangle} - \frac{\langle d\psi | d\psi \rangle}{\langle \psi | \psi \rangle} \otimes \frac{\langle d\psi | d\psi \rangle}{\langle \psi | \psi \rangle} ,$$

(6)

whose real and imaginary parts define a metric and a symplectic structure on $\mathcal{H}_0$. Therefore, according to the diagram

$$\begin{array}{ccc}
\mathcal{H}_0 & \xrightarrow{\mu} & u^*(\mathcal{H}) \\
\pi \downarrow & & \downarrow \iota \\
\mathcal{R}(\mathcal{H}) & \cong & D^1(\mathcal{H}) \\
\end{array}$$

(7)

the space of pure quantum states naturally inherits a Kähler structure from $\mathcal{H}_0$.

In particular, one can exploit the above construction to describe specific manifolds of states of the quantum system under consideration. Examples of manifolds of quantum states are provided for instance by coherent states [61] or by the stratified manifold of density states, where each stratus contains density states with fixed rank [56]. Indeed, given a finite-dimensional manifold $M$ and $\iota_M : M \hookrightarrow \mathcal{H}$ the embedding of $M$ into $\mathcal{H}$, the induced pull-back $i^*_M$ of the Hermitian tensor (4) or (6) defines a covariant Riemannian metric tensor and a closed (symplectic in a non-degenerate case) 2-form on $M$.

In this spirit it has been shown in [62] that, if $M$ is the space (of parameters) of probability distributions associated to quantum states, the Fisher-Rao metric tensor used in statistics and information theory [63] can be obtained from the Fubini-Study tensor defined on the space of pure quantum states. In what follows we will then take (4) to be the definition of the quantum Fisher tensor.

A convenient way to identify submanifolds of quantum states, which will turn very useful for characterizing entanglement of composite systems, consists in considering orbits originated from some fiducial state. Specifically, if $M$ admits the structure of a Lie group $G$, the orbits $G/\mathcal{O}$

$$\mathcal{O} \cong G / G_0 = \{ |g\rangle = U(g) |0\rangle | g \in G \} / \sim ,$$

(8)

generated by the action of a unitary representation $U(g)$

\[3\]

Due to the present state-of-the-art of infinite dimensional differential geometry, methods from differential geometry are much more effective when the identified submanifold has finite dimension. Fortunately many situations of great physical interest like those emerging in quantum computation are concerned with finite dimensional manifolds of quantum states.

\[3\]

Here $G_0$ is the isotropy group of the state $|0\rangle$, i.e. the subgroup of elements of $G$ which leave the state $|0\rangle$ unchanged, and $\sim$ is the equivalence relation with respect to such an action.
of $\mathcal{G}$ upon a normalized fiducial state $|0\rangle \in \mathcal{H}_0$ identify submanifolds of quantum states $|g\rangle$ when we consider an embedding map via the group action

$$\phi_0 : \mathcal{G} \ni g \mapsto |g\rangle = U(g)|0\rangle \in \mathcal{H}_0.$$  

(9)

Correspondingly on $\mathcal{R}(\mathcal{H})$ we identify orbit submanifolds of pure quantum states with respect to the co-adjoint action on some fiducial pure state $\rho_0 = \frac{|0\rangle \langle 0|}{|\langle 0|0\rangle|}$ respectively given by $^4$

$$\mathcal{K}_+ = \left\{ \frac{1}{2} \text{Tr}(\rho_0 [R(x_j), R(x_k)]_+) \right\} \theta^j \otimes \theta^k ,$$

$$\mathcal{K}_- = \frac{1}{2} \text{Tr}(\rho_0 [R(x_j), R(x_k)]_-) \theta^j \wedge \theta^k.$$  

(14)

(15)

B. Quantum Fisher tensor for bipartite N-level systems

Such a pull-back procedure can be extended also to the case of a composite system $[28, 29]$. The correlation properties of the fiducial state $\rho_0$ are captured by the tensorial structures induced on the orbits of the action of local unitary groups which define submanifolds of states with fixed amount of entanglement.

Let $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \cong \mathbb{C}^{N_A} \otimes \mathbb{C}^{N_B}$ be the Hilbert space of a composite system consisting of two $N$ level systems $A$ and $B$ with number of levels respectively given by $N_A = \dim \mathcal{H}_A$ and $N_B = \dim \mathcal{H}_B$. For the sake of clarity, in what follows we will denote by $\otimes$ the usual tensor product of spaces and by $\otimes_F$ the product of forms. So let $\rho_0$ be a fiducial pure state in $D^1(\mathcal{H}_A \otimes \mathcal{H}_B)$ and $\mathcal{G}_0$ its isotropy group, we want to compute now the pull-back of the Hermitian quantum Fisher tensor

$$\text{Tr}(\rho d\rho \otimes d\rho)$$  

(16)

on the orbits

$$\mathcal{O} \cong U(N_A) \times U(N_B)/\mathcal{G}_0$$  

(17)

of unitarily related (pure) quantum states $\rho = U \rho_0 U^{-1}$, induced by the co-adjoint action of the unitary group on $\rho_0$ with respect to the product representation

$$U = U_A \otimes U_B = (U_A \otimes 1_B) \cdot (1_A \otimes U_B).$$  

(18)

The corresponding Lie algebra representation $u(\mathcal{H}_A) \oplus u(\mathcal{H}_B)$ is provided by means of the following realization

$$R(X_j) = \left\{ \begin{array}{ll}
\{\sigma_j^{(A)} \otimes 1_B & \text{for } 1 \leq j \leq N_A^2 \\
1_A \otimes \sigma_j^{(B)} & \text{for } N_A^2 + 1 \leq j \leq N_A^2 + N_B^2 
\end{array} \right.$$  

(19)

of the infinitesimal generators of the one-dimensional subgroup of $U(N_A) \times U(N_B)$. In the following, for both

$^4$ $[\cdot, \cdot]_\pm$ respectively denote the anticommutator and the commutator and we use the shorthand notation

$$\theta^a \otimes \theta^b = \frac{1}{2} (\theta^a \otimes \theta^b + \theta^b \otimes \theta^a), \quad \theta^a \wedge \theta^b = \frac{1}{2} (\theta^a \otimes \theta^b - \theta^b \otimes \theta^a) .$$

for the symmetrized and antisymmetrized product of forms.
subsystems we will adopt a short-hand notation of indices $a, b$ without specifying their range of values. Therefore, being $U^{-1}dU$ a left-invariant 1-form, it can be decomposed as $^5$

$$U^{-1}dU = i\sigma^{(A)}_a \theta^a_A \otimes 1_B + 1_A \otimes i\sigma^{(B)}_b \theta^b_B,$$  \hspace{1cm} (20)

where \{\theta_A\} and \{\theta_B\} denote a basis of left-invariant 1-forms on the corresponding Lie group representation acting on the subsystem $A, B$ respectively. The operator-valued 1-form (12) can be thus written as

$$d\rho = U [i\sigma^{(A)}_a \theta^a_A \otimes 1_B, \rho_0]_U^{-1} + U [1_A \otimes i\sigma^{(B)}_b \theta^b_B, \rho_0]_U^{-1},$$  \hspace{1cm} (21)

from which it follows that the pull-back of the Hermitian tensor (16) on the orbit submanifold starting from the fiducial state $\rho_0$ then reads as

$$K = K^{(A)}_{ab} \theta^a_A \otimes \theta^b_A + K^{(AB)}_{ab} \theta^a_A \otimes \theta^b_B + K^{(B)}_{ab} \theta^a_B \otimes \theta^b_A + K^{(AB)}_{ab} \theta^a_B \otimes \theta^b_B,$$  \hspace{1cm} (22)

with

$$\begin{align*}
K^{(A)}_{ab} &= -\text{Tr}(\rho_0 [\sigma^a_A \otimes 1_B, \rho_0]_U [\sigma^b_A \otimes 1_B, \rho_0]_U), \\
K^{(AB)}_{ab} &= -\text{Tr}(\rho_0 [\sigma^a_A \otimes 1_B, \rho_0]_U [\sigma^b_A \otimes 1_B, \rho_0]_U), \\
K^{(B)}_{ab} &= -\text{Tr}(\rho_0 [1_A \otimes \sigma^b_B, \rho_0]_U [1_A \otimes \sigma^b_B, \rho_0]_U), \\
K^{(AB)}_{ab} &= -\text{Tr}(\rho_0 [1_A \otimes \sigma^b_B, \rho_0]_U [1_A \otimes \sigma^b_B, \rho_0]_U).
\end{align*}$$  \hspace{1cm} (23)

Now, being $\rho_0^a = \rho_0^b = \rho_0$ for a pure state $\rho_0$, a direct computation shows that the pulled-back Hermitian tensor $K$ decomposes into a Riemannian metric $K_+$ and a symplectic structure $K_-$

$$K = K_+ + iK_- = \begin{pmatrix} K^{(A)}_{ab} & K^{(AB)}_{ab} \\ K^{(BA)}_{ab} & K^{(B)}_{ab} \end{pmatrix} + i \begin{pmatrix} 0 & 0 \\ 0 & K^{(AB)}_{ab} \end{pmatrix},$$  \hspace{1cm} (24)

with

$$\begin{align*}
K^{(A)}_{ab} &= \frac{1}{2} \text{Tr}(\rho_0 [\sigma^a_A, \sigma^b_A]_+ \otimes 1_B) - \text{Tr}(\rho_0 \sigma^a_A \otimes 1_B) \text{Tr}(\rho_0 \sigma^b_A \otimes 1_B), \\
K^{(B)}_{ab} &= \frac{1}{2} \text{Tr}(\rho_0 1_A \otimes [\sigma^b_B, \sigma^b_B]_+) - \text{Tr}(\rho_0 1_A \otimes \sigma^b_B) \text{Tr}(\rho_0 1_A \otimes \sigma^b_B), \\
K^{(AB)}_{ab} &= \frac{1}{2} \text{Tr}(\rho_0 \sigma^a_A \otimes \sigma^b_B) - \text{Tr}(\rho_0 \sigma^a_A \otimes \sigma^b_B), \\
K^{(BA)}_{ab} &= \frac{1}{2} \text{Tr}(\rho_0 \sigma^a_A \otimes \sigma^b_B) - \text{Tr}(\rho_0 \sigma^a_A \otimes \sigma^b_B).
\end{align*}$$  \hspace{1cm} (25)

$^5$ Here we use the decomposition of the exterior differential operator $d = d_A \otimes 1_B + 1_A \otimes d_B$ acting on a product representation (18).

- when $\rho_0$ is separable the off-diagonal blocks of the metric component vanish and the pulled-back Hermitian tensor $K$ decomposes into a direct sum $K_A \otimes K_B$ of Hermitian tensors associated with the two subsystems;

- when $\rho_0$ is maximally entangled the symplectic component vanishes.

In particular, the information about (quantum) correlations between the two subsystems is encoded in the off-diagonal block-coefficient $N_A^2 \times N_B^2$ and $N_B^2 \times N_A^2$ matrices $K^{(AB)}$ and $K^{(BA)}$ which allow us to define an entanglement monotone given by [64]

$$E = \frac{N^2}{4(N^2 - 1)} \text{Tr}(K^{(AB)} T K^{(AB)}) = \frac{N^2}{4(N^2 - 1)} \text{Tr}(K^{(BA)} T K^{(BA)}),$$  \hspace{1cm} (28)

where $N_\leq = \min(N_A, N_B)$. Such a measure of entanglement is directly related to a geometric definition of distance between entangled and separable states introduced in [65]:

$$\Sigma = \text{Tr}(R^T R) \quad \text{with} \quad R := \rho_0 - \rho_0^{(A)} \otimes \rho_0^{(B)},$$  \hspace{1cm} (29)

and $\rho_0^{(A,B)} = \text{Tr}_{B,A}(\rho_0)$ the reduced states. Indeed, as shown in [65], the geometric distance $\Sigma$ can be computed to be

$$\Sigma = \frac{1}{N_\leq} \text{Tr}(K^{(AB)} T K^{(AB)}),$$  \hspace{1cm} (30)

thus giving a geometrical interpretation of $E$ itself.
III. GEOMETRY OF QUANTUM SPIN NETWORK STATES

In the microscopic description of spacetime provided by the background-independent approaches to quantum gravity, [1–9], quantum states of space geometry are described in terms of spin networks [1–3]. In the language of tensor networks [23], spin networks are symmetric tensor network states given by collections of quantum tensor states characterised by 3D rotation invariance (the Lorentz group in LQG, usually gauge-fixed to SU(2)) linked to each other by group holonomy actions encoding the change of frame from one tensor to the next. This set of frame transformations translates into graphs coloured by irreducible representations of the local gauge group. In the end, this leaves with a graph with group representations associated to its links, with group intertwiners associated to its nodes, which is the usual characterization of spin network states [1–3].

In the context of LQG, spin networks are by construction embedded into continuum 3D manifolds, from which they partly inherit a natural geometric characterisation. In related approaches, like Tensor Models or Group Field Theories, as well as in spin foam models based on simplicity ideas, spin networks associated to emerging random geometries are not embedded, hence they must be interpreted as abstract graphs defined with no reference to any background notions of space, time or geometry [48], and can be at best associated with quantized simplicial (piecewise-flat) geometries. A short review of both the LQG embedded and GFT abstract spin networks description is given in Appendix C.

In absence of a background metric structure, and due to the dynamical nature of any additional discrete ‘quantum geometric’ variable that can be associated to the spin network graph, adjacent regions of a spin network will not necessarily correspond to regions of space that are ‘close’ in a geometric sense. Moreover, 3D spatial geometry will generically be realised as a quantum superposition of abstract non-embedded entities each of which having a different connectivity (i.e., a different graph structure), what is local in one term of the superposition, with respect to the combinatorial and algebraic data characterizing it, will in general not be local in others [45, 46]. Still, a given region of a spin network can be localized in a combinatorial sense, with respect to other parts of the graph.

In this framework, the general algorithmic procedure to construct tensorial geometric structures on the space of states of a given quantum theory becomes a new crucial tool to reconsider notions as ‘close’ and “far” in terms of quantum correlations between subregions of the spin network graph or more generally as relations between different spin network configurations (states) in the Hilbert space. This is the main focus of our work.

Given the highly intricate structure of the spin network Hilbert space, as a preliminary step along this line, we start our analysis by focusing on a set of states which constitute the fundamental building blocks of the spin network description, and more generally of any tensor network representation of lattice gauge theory: the Wilson line states and the intertwiner states, respectively providing the basic structure for links and nodes of the network.

IV. FUBINI–STUDY TENSOR FOR THE SINGLE LINK STATE

The correlation structure of spin network states is encoded in the connectivity of the underlying graphs, which translates into entanglement between the fundamental vertex states connected by links. To a given graph $\Gamma$ (see Appendix C) is associated a Hilbert space $H_\Gamma \cong L^2[SU(2)^L]$, where $L$ indicates the number of links comprising the graph. A basis for $H_\Gamma$ can be naturally derived starting from the Peter-Weyl theorem [43], which gives the unitary equivalence

$$L^2[SU(2)^L] \cong \bigotimes_{\ell=1}^{L} V^{(j_\ell)} \otimes V^{(j_\ell)^*} ,$$

where $V^{(j)}$ denotes the $(2j+1)$-dimensional linear space carrying the irreducible representation of $SU(2)$ and for any $j \in \frac{1}{2} \mathbb{N}$, the system $\{ |j, m\rangle \}_{-j \leq m \leq j}$ is orthonormal, i.e.,

$$V^{(j)} = \text{span} \{ |j, m\rangle \}_{-j \leq m \leq j} ,$$

while $V^{(j)^*}$ is its dual vector space. A function $\psi_\Gamma \in H_\Gamma$ can be decomposed as

$$\psi_\Gamma = \sum_{j_\ell, m_\ell, n_\ell} f_{m_1, ..., m_L, n_1, ..., n_L} D_{m_1 n_1}^{(j_1)}(h_1) ... D_{m_L n_L}^{(j_L)}(h_L) ,$$

where $D_{m n}^{(j)}(h) = \langle j, m | D^{(j)}(h) | j, n \rangle$ are the Wigner D-matrix elements corresponding to the spin-$j$ irreducible representations of the group elements $h_\ell \in SU(2)$ labelling the link. An orthonormal basis for the Hilbert space $H_\Gamma$ is thus provided by

$$\langle \tilde{h} | \Gamma; \tilde{j}, \tilde{m}, \tilde{n} \rangle \equiv \left( \prod_{\ell=1}^{L} \sqrt{2j_\ell + 1} \right) D_{m_1 n_1}^{(j_1)}(h_1) ... D_{m_L n_L}^{(j_L)}(h_L) ,$$

where the compact vectorial notation $\tilde{j}, \tilde{m}, \tilde{n}$ denotes the spin labels of the unitary irreducible representations of $SU(2)$ associated with each link of the graph, and similarly for the corresponding group elements. The simplest

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6 The possibility of defining spin network states in a more abstract, combinatorial way has been considered also within the canonical LQG approach [50, 51].
Any state in $H_\gamma$ can be therefore expanded in the spin basis as in (35) with coefficients given by

$$c^j_{mn} \equiv \langle j, m, n | \psi_\gamma \rangle = \int dh \psi_\gamma [h] \langle j, m, n | h \rangle$$

(39)

where the generic Wilson line state $|j, m, n\rangle$ defines the matrix element of the representation of the holonomy along the link,

$$\langle h | j, m, n \rangle := \sqrt{2j + 1} D^{(j)}_{mn}(h).$$

(36)

The orthogonality relations of the Wigner representation matrices $D^{(j)}_{mn}$ ensure the normalization of the basis states

$$|j', m', n' \rangle = \delta_{jj'} \delta_{m m'} \delta_{n n'},$$

(37)

together with the decomposition of the identity

$$1 = \sum_{jmn} |j, m, n\rangle \langle j, m, n|.$$

(38)

Any state in $H_\gamma$ can be expressed in terms of the algebraic data of the spin-network graph $(j, m$ and $n$ in this specific situation).

Now, by recalling the notation adopted in Section II, for the link state in spin basis we can associate

$$|e_a\rangle \leftrightarrow |j, m, n\rangle \equiv |e^{(j)}_{mn}\rangle$$

$$c_a(\psi) \leftrightarrow c^j_{mn} = \langle j, m, n | \psi \rangle$$

$$|d\psi\rangle = \sum_a d_{ca} |e_a\rangle \leftrightarrow |d\psi_\gamma\rangle = \sum_{jmn} d^j_{mn} |j, m, n\rangle$$

and thereby derive

$$\langle d\psi_\gamma \otimes d\psi_\gamma \rangle = \sum_{jmn} \langle d\psi_\gamma | j, m, n \rangle \langle j, m, n | d\psi_\gamma \rangle$$

$$= \sum_{jmn} \int_{SU(2)} dh \langle d\psi_\gamma | j, m, n \rangle \langle j, m, n | h(A) \rangle \langle h(A) | d\psi_\gamma \rangle$$

$$= \sum_{jmn} \int_{SU(2)} dh \left( \langle d\psi_\gamma | j, m, n \rangle \langle j, m, n | h(A) \rangle \cdot \langle h(A) | j, m', n' \rangle \langle j, m', n' | d\psi_\gamma \rangle \right)$$

$$= (2j + 1) \sum_{jmn} \left( \int_{SU(2)} dh \frac{D^{(j)}_{mn}(h(A)) D^{(j')}_{m'n}(h(A))}{(2j + 1)} \right) d^j_{mn} \otimes d^j_{mn}.$$

(40)

Similarly, we have

$$\langle \psi_\gamma | d\psi_\gamma \rangle = \tau^j_{mn} d^j_{mn} \quad \text{(with sum over } j, m, n \text{).}$$

(41)

Finally, the pull-back to the Hilbert space of the Fubini-Study Hermitian tensor is given by:

$$K_{H_\gamma} = \frac{\langle d\psi_\gamma \otimes d\psi_\gamma \rangle}{\langle \psi_\gamma | d\psi_\gamma \rangle} - \frac{\langle d\psi_\gamma | \psi_\gamma \rangle \otimes \langle \psi_\gamma | d\psi_\gamma \rangle}{\langle \psi_\gamma | \psi_\gamma \rangle^2}$$

$$= \frac{d\tau^j_{mn} \otimes d\tau^j_{mn}}{\sum_{mn} |c^j_{mn}|^2} - \frac{d\tau^j_{mn} c^j_{mn} \otimes d\tau^j_{mn} d\tau^j_{mn}}{\left( \sum_{mn} |c^j_{mn}|^2 \right)^2}. $$

(42)

A. Pull-back on orbit submanifolds of quantum states in $H_\gamma$

To pull-back the Hermitian tensor (42) on orbit submanifolds of quantum states we need to understand what are the objects entering the diagram (11) in the specific case under examination.

Let us therefore choose (35) to be our fiducial Wilson line state, i.e.

$$|0\rangle \equiv |\psi_\gamma\rangle = \sum_{jmn} c^j_{mn} |j, m, n\rangle,$$

(43)

where we recall from Sec. II A that $|0\rangle$ denotes a fiducial fixed state starting from which the orbits under the group action are generated, without any reference to specific properties of the selected state. Since we are considering spin basis states $|j, m, n\rangle$ constructed with the common eigenstates of the operator $J^2$ and one of the $J$'s (say $J_z$), i.e., with a fixed orientation (say the $z$-axis) of the magnetic moments at the endpoints of the link$^7$, the only transformations that we can perform on such states are those generated by the operators $J_1, J_2, J_3$ which have a well-defined action on the basis states. The group $G$ acting on $H_\gamma$ is thus given by the group $SU(2)$. Therefore, the diagram (11) which explains the various levels

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$^7$ We may also consider a more general situation in which we have an additional degree of freedom to take into account a different direction of the magnetic moment. As discussed in [68], in this case the basis states are given by $|j, \hat m, \hat n\rangle$, where $\hat m$ simply denotes the new direction ($\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta$) obtained by rotating the direction $\hat z = (0, 0, 1)$. This kind of states can be used for instance to account a non-completely precise face matching of polyhedra glued along faces dual to the graph edges which will give some torsion thus providing a generalization of Regge geometries as twisted geometries [69, 70].
at which the (co-adjoint) orbit $O$ is embedded in the projective Hilbert space $\mathcal{R}(\mathcal{H}_\gamma)$ now becomes

\[
SU(2) \xrightarrow{\phi_0} S(\mathcal{H}_\gamma)
\]

\[
U(1) \quad \quad \quad \quad U(1)
\]

\[
SU(2)/U(1) \xrightarrow{\pi_0} \mathcal{R}(\mathcal{H}_\gamma)
\]

\[
SU(2)/\mathbb{U}_0 \overset{U(1)}{\cong} O
\]

Being $\mathcal{H}_\gamma$ given by the direct sum of fixed-$j$ Hilbert spaces, i.e.

\[
\mathcal{H}_\gamma \cong \bigoplus_j \mathcal{H}^{(j)} \cong \bigoplus_j \mathcal{V}^{(j)} \otimes \mathcal{V}^{(j)*},
\]

\[ (45) \]

the $SU(2)$ action on the fiducial state $|0\rangle$ is given component-wise. The embedding of the Lie group into $\mathcal{H}_\gamma - \{0\}$ is then realized by means of the action of a spin-$j$-representation on each element of the sum (45), that is

\[
\phi_0 : SU(2) \ni h \rightarrow |h\rangle = U(h)|0\rangle \in \mathcal{H}_\gamma - \{0\},
\]

\[ (46) \]

where the unitary representation $U : SU(2) \rightarrow \text{Aut}(\mathcal{H}_\gamma)$ is given by

\[
U(h)|0\rangle = \sum_{j,m,n} c_{mn}^{j} U^{(j)}(h)|j,m,n\rangle,
\]

\[ (47) \]

and

\[
U^{(j)}(t) = e^{iR^{(j)}(X^{k})t}k
\]

\[ (48) \]

$R^{(j)}(X^{k}) \equiv J_{k}$ denoting the set of Hermitian operators which represent the $SU(2)$ generators. Similarly, the corresponding embedding of $\mathbb{G} \equiv SU(2)$ into the space of rays is given by the co-adjoint action map

\[
\tilde{\phi}_0 : h \rightarrow U^{(j)}(h)\rho_0 U^{(j)^*}(h).
\]

\[ (49) \]

The pull-back of the Hermitian tensor (42) to the co-adjoint orbit starting from a pure fiducial state $\rho_0$ decomposes into a direct sum of the corresponding tensors on each $\mathcal{H}^{(j)}_\gamma$. We will then focus on a fixed-$j$ block which, according to Eq. (13), will be given by

\[
\mathcal{K} = \mathcal{K}_{kl} \theta^k \otimes \theta^l,
\]

\[ (50) \]

with coefficients

\[
\mathcal{K}_{kl} = \text{Tr}(\rho_0 J_k J_l) - \text{Tr}(\rho_0 J_k)\text{Tr}(\rho_0 J_l).
\]

\[ (51) \]

Moreover, in the case of a pure state, by using the explicit expression for the fiducial state

\[
\rho_0 = |0\rangle \langle 0| = \frac{|\psi_\gamma\rangle}{\langle \psi_\gamma |}.
\]

\[ (52) \]

we find the pulled-back tensor on the corresponding orbits in the Hilbert space:

\[
\mathcal{K}_{kl} = \frac{|0\rangle \langle 0| - |0\rangle \langle 0|}{\langle 0|0\rangle^2} = \frac{|\psi_\gamma\rangle \langle \psi_\gamma | - |\psi_\gamma\rangle \langle \psi_\gamma |}{\langle \psi_\gamma |\psi_\gamma \rangle^2} = \langle \psi_\gamma |J_k J_l| \psi_\gamma \rangle - \langle \psi_\gamma |J_k \rangle \langle J_l | \psi_\gamma \rangle.
\]

\[ (53) \]

We then see that the Hermitian tensor on the orbits coincides with the covariance matrix of the $SU(2)$ generators. Indeed, starting from the definition of the covariance matrix whose entry in the $k\text{th}$ row and $l\text{th}$ column is

\[
\text{Cov}(J)_{kl} = \langle (J_k - \langle J_k \rangle)(J_l - \langle J_l \rangle) \rangle,
\]

\[ (54) \]

we have

\[
\langle (J_k - \langle J_k \rangle)(J_l - \langle J_l \rangle) \rangle = \langle (J_k J_l - J_k \langle J_l \rangle - J_l \langle J_k \rangle + \langle J_k \rangle \langle J_l \rangle) \rangle
\]

\[
\equiv \langle J_k J_l \rangle - \langle J_k \rangle \langle J_l \rangle - \langle J_l \rangle \langle J_k \rangle + \langle J_k \rangle \langle J_l \rangle
\]

\[
\equiv \langle J_k J_l \rangle - \langle J_k \rangle \langle J_l \rangle.
\]

The tensor (53) therefore will measure the correlations in the fluctuations of the operators $J$. The non-commutativity of such operators implies that the covariance matrix (54) is not symmetric, but if we remem- ber the decomposition of the Hermitian tensor in its real symmetric and imaginary skew-symmetric part, we find a metric tensor

\[
\mathcal{K}_{kl} = \frac{1}{2} \text{Re} \left( (J_k - \langle J_k \rangle)(J_l - \langle J_l \rangle) \right)
\]

\[ (55) \]

\[
\equiv \text{Re} \left( \langle J_k J_l \rangle - \langle J_k \rangle \langle J_l \rangle \right)
\]

\[ (56) \]

and a symplectic structure

\[
\mathcal{K}_{[kl]} = \text{Im} \left( \frac{1}{2} \langle J_k J_l \rangle - \langle J_k \rangle \langle J_l \rangle \right) = \frac{1}{2} \langle \varepsilon_{klr} J_r \rangle_0.
\]

\[ (57) \]

where we have used the commutation relations $[J_k, J_l]_\gamma = i\varepsilon_{klr} J_r$ of the Lie algebra $su(2)$.

**B. Link as an entangled pair of spherical harmonics**

The single link space at fixed $j$ provides the simplest bipartite spin network system, given by the tensor product Hilbert space

\[
\mathcal{H}^{(j)}_\gamma \cong \mathcal{V}^{(j)} \otimes \mathcal{V}^{(j)*}.
\]

\[ (58) \]
As we choose an orthonormal basis in the two subspaces, as shown in (32), the single-link state (35) will generally read

\[ |\psi_j^{(3)}\rangle = \sum_{mn} c_{mn}^{(j)} |j, n\rangle \otimes |j, m\rangle \]

\[ = \sum_{mn} c_{mn}^{(j)} |j, n\rangle \otimes |j, m\rangle = \sum_{mn} c_{mn}^{(j)} \langle j, n | j, m \rangle_{[j,m,n]} \]

namely, as a composite state of two semi-link states analogue to two spherical harmonics \( Y_a^\pm(h) = |h\rangle[j, m]\). Therefore, we can characterize the entanglement of the bipartite system (58) by means of the quantum Fisher tensor description introduced in II.B. Once again, as in IV.A, we restrict our analysis to the cases of states with fixed \( j \), that is no sum over \( j \) in Eq. (59).

According to the diagram (44), we select a fiducial pure state

\[ \rho_0 \in D^1(\mathcal{V}^{(j)} \otimes \mathcal{V}^{(j)^*}) \cong \mathcal{R}(\mathcal{V}^{(j)} \otimes \mathcal{V}^{(j)^*}) = \mathcal{R}(\mathcal{H}^{(j)}), \]

and then we consider the product representation

\[ \phi_0 : G \equiv SU(2) \times SU(2) \rightarrow Aut(\mathcal{V}^{(j)} \otimes \mathcal{V}^{(j)^*}), \]

providing the following embedding map

\[ G \ni g \mapsto \rho_g = U(g)\rho_0 U^\dagger(g) \in \mathcal{R}(\mathcal{V}^{(j)} \otimes \mathcal{V}^{(j)^*}) \]

with \( U(g) = e^{iR(X_k)g} \). Infinitesimal generators \( R(X_k) \) are realized as the tensor products between the identity of a subsystem and the spin operators \( J_k \) representing the \( su(2) \) algebra in terms of self-adjoint operators on the Hilbert space \( \mathcal{V}^{(j)} \) (cfr. Eq. (19)). Thus, according to Sec. II.B, we find that the pull-back of the Hermitian Fisher tensor \( \text{Tr}(\rho d\rho \otimes d\rho) \) from \( \mathcal{R}(\mathcal{H}^{(j)}) = \mathcal{R}(\mathcal{V}^{(j)} \otimes \mathcal{V}^{(j)^*}) \) to the co-adjoint orbit

\[ \mathcal{O}_{\rho_0} := SU(2) \times SU(2)/G_{\rho_0}, \]

where \( G_{\rho_0} \) is the isotropy group of the fiducial state\(^8\), decomposes into a symmetric Riemannian and a skewsymmetric (pre-)symplectic component

\[ \mathcal{K}_{kl} = \mathcal{K}_{(kl)} + i\mathcal{K}_{[kl]} = \begin{pmatrix} A & C \\ C^\dagger & B \end{pmatrix} + i \begin{pmatrix} D_A & 0 \\ 0 & D_B \end{pmatrix}, \]

with \( 3 \times 3 \) blocks given by

\[ \begin{aligned}
A_{ab} &= \frac{1}{2} \text{Tr}(\rho_0 [J_a, J_b]_+ \otimes \mathbb{1}) - \text{Tr}(\rho_0 J_a \otimes \mathbb{1}) \text{Tr}(\rho_0 J_b \otimes \mathbb{1}) \\
B_{ab} &= \frac{1}{2} \text{Tr}(\rho_0 \mathbb{1} \otimes [J_a, J_b]_+) - \text{Tr}(\rho_0 \mathbb{1} \otimes J_a) \text{Tr}(\rho_0 \mathbb{1} \otimes J_b) \\
C_{ab} &= \text{Tr}(\rho_0 J_a \otimes J_b) - \text{Tr}(\rho_0 J_a \otimes \mathbb{1}) \text{Tr}(\rho_0 \mathbb{1} \otimes J_b) \\
(D_A)_{ab} &= \frac{1}{2} \text{Tr}(\rho_0 J_a \otimes [J_b, J_-] \otimes \mathbb{1}) \\
(D_B)_{ab} &= \frac{1}{2} \text{Tr}(\rho_0 \mathbb{1} \otimes [J_a, J_b]_-) \\
\end{aligned} \]

Therefore, we see that if \( \rho_0 \) is maximally entangled, that is the reduced states are maximally mixed

\[ \rho_0^{(A)} = \rho_0^{(B)} = \frac{1}{\dim \mathcal{V}^{(j)}} \mathbb{1}_{A.B} = \frac{1}{2j+1} \sum_{j,m} \rho_j^{(j)}, \]

then

\[ (D_A)_{ab} = \frac{1}{2} \text{Tr}(\rho_0^{(B)} [J_a, J_b]_-) \times \text{Tr}([J_a, J_b]_-) = 0, \]

and similarly for \( (D_B)_{ab} \). On the other hand, if \( \rho_0 \) is separable, i.e., \( \rho_0 = \rho_0^{(A)} \otimes \rho_0^{(B)} \), then

\[ C_{ab} = \text{Tr}(\rho_0^{(A)} J_a \otimes \rho_0^{(B)} J_b) - \text{Tr}(\rho_0^{(A)} J_a \otimes \rho_0^{(B)} J_b) \text{Tr}(\rho_0^{(A)} \otimes \rho_0^{(B)} J_b) \]

\[ = \text{Tr}(\rho_0^{(A)} J_a) \text{Tr}(\rho_0^{(B)} J_b) - \text{Tr}(\rho_0^{(A)} J_a) \text{Tr}(\rho_0^{(B)} J_b) \text{Tr}(\rho_0^{(A)} \otimes \rho_0^{(B)} J_b) = 0. \]

Thus, as stated in Sec. II.B, information about the separability or entanglement of the fiducial state \( \rho_0 \) is encoded into the different blocks of the pulled-back Hermitian tensor on the orbit of unitarily related states starting from \( \rho_0 \). Indeed, the vanishing of the symplectic tensor for a maximally entangled state \( \rho_0 \) corresponds to a vanishing separability while the off-diagonal blocks of the Riemannian tensor are responsible for the entanglement degree of the state \( \rho_0 \) and allow us to define an associated entanglement monotone \( \text{Tr}(C^T C) \) which identifies an entanglement measure geometrically interpreted as a distance between entangled and separable states. As we will discuss later in this work, since we are regarding the link as resulting from the entanglement of semilinks, such entanglement monotone gives us a measure of the existence of the link itself and so of the graph connectivity.

C. Two limiting cases: maximally entangled and separable states

In order to visualize the considerations of the previous section, let us focus on the two extreme cases respectively given by a maximally entangled and a separable single link state, and compute explicitly the pull-back of the Hermitian tensor on the orbit having that state as fiducial state. To this aim, we start by considering the Schmidt

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\(^8\) The topology of the orbit will thus depend on the isotropy group of the selected fiducial state. We refer to [72, 73] for a general discussion.
decomposition [74] of the normalized state (59):

$$|\psi^{(j)}_\gamma\rangle = \sum_k \lambda_k |j,k\rangle \otimes \langle j,k|.$$  \hspace{1cm} (68)

In the maximally entangled case all Schmidt coefficients are equal and, according to the normalization condition $\langle \psi^{(j)}_\gamma | \psi^{(j)}_\gamma\rangle = 1$, they are given by:

$$\lambda_k = \frac{1}{\sqrt{2j+1}} \quad \forall k \in [-j,+j],$$ \hspace{1cm} (69)

thus yielding a maximally entangled state

$$|\psi^{(j)}_\gamma\rangle = \frac{1}{\sqrt{2j+1}} \sum_k |j,k\rangle \otimes \langle j,k|,$$ \hspace{1cm} (70)

which is nothing but the gauge-invariant loop state $|\psi_L\rangle$. Indeed, such a state corresponds to glue the two endpoints of the link into a bivalent vertex and contract their magnetic moments with an intertwiner provided by the normalized identity in $\nu^{(j)}$, i.e.:

$$|\psi_L\rangle = \sum_{k,k'} \frac{\delta_{k,k'}}{\sqrt{2j+1}} |j,k\rangle \otimes \langle j,k'| = \sum_{k,k'} i_{k,k'} |j,k\rangle \otimes \langle j,k'|.$$ \hspace{1cm} (71)

Therefore, concerning the open single line state regarded as an entangled state of two semilinks, there is a close relationship between maximal entanglement and gauge-invariance. It is actually the gauge-invariance requirement to be responsible for the appearence of entanglement in gluing open spin network states. This is realized by identifying the maximally entangled state (70) with the closed loop state, i.e.

$$\begin{pmatrix}
\frac{1}{3}j(j+1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3}j(j+1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{3}j(j+1) & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{3}j(j+1) & 0 & 0 & \frac{1}{3}j(j+1) & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3}j(j+1) & 0 & 0 & \frac{1}{3}j(j+1) & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{3}j(j+1) & 0 & 0 & \frac{1}{3}j(j+1) & 0 & 0 & 0 \\
\end{pmatrix}$$ \hspace{1cm} (75)

from which, using the decomposition $\mathcal{K}_{k\ell} = \mathcal{K}_{(k\ell)} + i\mathcal{K}_{[k\ell]}$ , we see that the real symmetric part $\mathcal{K}_{(k\ell)}$ decomposes in the block-diagonal matrices $A, B$ and the two equal block-off-diagonal matrices $C$, according to

$$\mathcal{K}_{(k\ell)} = \begin{pmatrix} A & C \\ C & B \end{pmatrix}$$ \hspace{1cm} (76)

$$\mathcal{H}_{\text{max.ent.}} = \mathcal{H}_{\text{loop}} = \text{Inv}_{SU(2)}[\nu^{(j)} \otimes \nu^{(j)*}] \subset \mathcal{H}_\gamma^{(j)}.$$ \hspace{1cm} (72)

However, it should be stressed that the correspondence between gauge invariance and maximal entanglement holds for basis states. Indeed, one can consider gauge invariant superpositions of spin networks, in particular those corresponding to generic cylindrically functions. In this case, the presence of the modes would imply that the states are gauge invariant, but they do not maximize entanglement. Nevertheless, since here we are interested in showing how the GQM machinery introduced in Sec. II explicitly works, this provides a useful simple example to test the tensorial characterization of entanglement of Sec. II B. We will then move to the more interesting case of intertwiner entanglement in Sec. VI.

Hence, taking the maximally entangled loop state (71) as our fiducial state, we are interested in the corresponding pulled-back Hermitian tensor on the orbit starting from it. The pure state density matrix $\rho_0 \in D^1(\nu^{(j)} \otimes \nu^{(j)*})$ associated with it is given by

$$\rho_0 = |\psi_L\rangle \langle \psi_L| = \frac{1}{2j+1} \sum_{k,k'} (|j,k\rangle \langle j,k'|) \otimes (|j,k\rangle \langle j,k|),$$ \hspace{1cm} (73)

such that the reduced states are diagonal with eigenvalues exactly given by the square of the Schmidt coefficients, e.g.

$$\langle \rho_0 \rangle_A = \text{Tr}_B(\rho_0) = \frac{1}{2j+1} \sum_k |j,k\rangle \langle j,k| = \frac{I_j}{\text{dim} V^{(j)}}.$$ \hspace{1cm} (74)

Hence, by using Eqs. (64,65), after lengthy but straightforward calculations, the pull-back of the Hermitian tensor $\mathcal{K}$ on the orbit $O_{\rho_0}$ of Eq. (63) takes the following form (see the appendix of [75] for details)

$$A = B = \begin{pmatrix}
\frac{1}{3}j(j+1) & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3}j(j+1) & 0 & 0 & 0 \\
0 & 0 & \frac{1}{3}j(j+1) & 0 & 0 \\
0 & 0 & 0 & \frac{1}{3}j(j+1) & 0 \\
0 & 0 & 0 & 0 & \frac{1}{3}j(j+1) \\
\end{pmatrix},$$ \hspace{1cm} (77)

$$C = \begin{pmatrix}
\frac{1}{3}j(j+1) & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3}j(j+1) & 0 & 0 & 0 \\
0 & 0 & \frac{1}{3}j(j+1) & 0 & 0 \\
0 & 0 & 0 & \frac{1}{3}j(j+1) & 0 \\
0 & 0 & 0 & 0 & \frac{1}{3}j(j+1) \\
\end{pmatrix},$$ \hspace{1cm} (78)
while the imaginary skewsymmetric part $K_{[k \ell]}$

$$K_{[k \ell]} = \begin{pmatrix} \frac{1}{2}[j_1(j_1 + 1) - k_1^2] & 0 & 0 \\ -\frac{1}{2}k_1 & \frac{1}{2}[j_1(j_1 + 1) - k_1^2] & 0 \\ 0 & 0 & k_1(k_1 - k_2) \end{pmatrix}$$

(87)

gives a vanishing symplectic structure, as expected for the maximally entangled case. Moreover, by using the off-diagonal blocks (77) of the Riemannian symmetric part, we have

$$\text{Tr}(C^TC) = \sum_{a,b=1}^3 C_{ab}^2 = \frac{1}{3}[j(j+1)]^2.$$  

(79)

The associated entanglement monotone (28) is given by

$$\mathcal{E} = \frac{(2j+1)^2}{4[(2j+1)^2 - 1]} \frac{1}{3}[j(j+1)]^2$$

$$= \frac{1}{48}j(j+1)[4j(j+1)+1],$$

(80)

from which we see that, as expected for the separable case, we have vanishing off-diagonal block matrices $C$ while the geometric distance $\Sigma$ defined in (30) reads as

$$\Sigma = \frac{1}{3(2j+1)^2}[j(j+1)]^2.$$  

(81)

Let us notice that the entanglement measure $\mathcal{E}$ depends only on the area eigenvalue $j(j+1)$. Moreover, for large $j$, Eqs. (79) and (80) coincide up to a numerical factor, while in the same limit the geometric distance $\Sigma$ is independent of $j$ as it should.

On the other extreme, if we consider a separable fiducial state, the two spin states do not talk with each other and may have in general different spins, i.e.:

$$|0\rangle = |j_1, k_1\rangle \otimes |j_2, k_2\rangle.$$  

(82)

The corresponding pure state density matrix is given by

$$\rho_0 = \rho_0^{(A)} \otimes \rho_0^{(B)} = (|j_1, k_1\rangle \langle j_1, k_1|) \otimes (|j_2, k_2\rangle \langle j_2, k_2|).$$  

(83)

Hence, the pull-back of the Hermitian tensor $K$ on the orbit $O_{\rho_0}$ will take the following form [75]

$$K = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} \frac{1}{2}[j_2(j_2 + 1) - k_2^2] & 0 & 0 \\ 0 & \frac{1}{2}[j_2(j_2 + 1) - k_2^2] & 0 \\ 0 & 0 & k_2(k_2 - k_1) \end{pmatrix}$$

(85)

and a direct sum

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2}[j_2(j_2 + 1) - k_2^2] & 0 \\ 0 & 0 & k_2(k_2 - k_1) \end{pmatrix},$$

(88)

$$C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and an imaginary skewsymmetric part

$$K_{[k \ell]} = \begin{pmatrix} D_A & 0 \\ 0 & D_B \end{pmatrix}$$

(89)

of two decoupled Hermitian tensors $K_A$ and $K_B$ one for each subsystem. Moreover, a further decomposition of the Hermitian tensor (84) as $K_{k \ell} = K_{(k \ell)} + iK_{[k \ell]}$, gives a symmetric real part

$$K_{(k \ell)} = \begin{pmatrix} A & C \\ C & B \end{pmatrix}$$

(86)

with

$$A = \begin{pmatrix} \frac{1}{2}[j_1(j_1 + 1) - k_1^2] & 0 & 0 \\ 0 & \frac{1}{2}[j_1(j_1 + 1) - k_1^2] & 0 \\ 0 & 0 & k_1(k_1 - k_2) \end{pmatrix}$$

(87)

while the geometric distance $\Sigma$ defined in (30) reads as
with
\[
D_A = \begin{pmatrix}
0 & \frac{1}{2}k_1 & 0 \\
-\frac{1}{2}k_1 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}, \quad D_B = \begin{pmatrix}
0 & \frac{1}{2}k_2 & 0 \\
-\frac{1}{2}k_2 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}.
\]
(90)

Finally, we have
\[
\text{Tr}(C^T C) = 0,
\]
(91)
i.e., coherently with its interpretation as a distance from the separable state, the entanglement measure associated with the block-off-diagonal matrices \(C\) is zero in the unentangled case.

\section{V. Gluing Links by Entanglement}

Let us proceed a little step further with respect to what shown in the previous section, and consider now the description of the entanglement resulting from the gluing of two lines into one. The bipartite Hilbert space is given by two copies of a single link Hilbert space with fixed but different spin labels, i.e.
\[
\mathcal{H} = \mathcal{H}^{(j_1)}_{\gamma_1} \otimes \mathcal{H}^{(j_2)}_{\gamma_2},
\]
(92)
and the fiducial state is chosen to be a single line state coming from the gluing of two other links, thus admitting the following expression
\[
|0\rangle \equiv |\psi_\gamma\rangle = \frac{1}{\sqrt{2^j + 1}} \sum_{m,n,k,\ell} c_{mn} |j, m, k\rangle \otimes |j, \ell, n\rangle \delta_{k,\ell}.
\]
(93)
The local \(SU(2)\) gauge-invariance requirement at the gluing point \(v \equiv \gamma_1(1) = \gamma_2(0)\), implemented by the bivalent intertwiner \(\delta_{k,\ell}/\sqrt{2^j + 1}\) contracting the magnetic numbers of the glued endpoints, forces the two spins to be equal, i.e. \(j_1 = j_2 = j\). In other words, \(|0\rangle\) is a locally \(SU(2)\)-invariant state in \(\mathcal{H}\), that is
\[
|0\rangle \in \mathcal{H}^{(j)}_\gamma \subset \mathcal{H}, \quad \gamma = \gamma_1 \circ \gamma_2.
\]
(94)
However, in order to compute an entanglement measure which can be interpreted as the distance of our fiducial state from the separable one, we need to consider the action \(\phi\) of a Lie group \(\mathbb{G}\) on \(\mathcal{H}\) and not only on the gauge-reduced level \(\tilde{\phi} : \mathbb{G}/SU(2) \to \mathcal{H}^{(j)}_\gamma\). Therefore, the underlying scheme of the construction of the pulled-back Hermitian tensor on the orbit of states with fixed amount of entanglement will be given by the following diagram
\[
\begin{array}{c}
\mathbb{G} \xrightarrow{\phi_0} \mathcal{H}^{(j_1)}_{\gamma_1} \otimes \mathcal{H}^{(j_2)}_{\gamma_2} \\
SU(2) \downarrow \text{“gluing”} \downarrow SU(2) \\
\mathbb{G}/SU(2) \xrightarrow{\pi_0} \mathcal{H}^{(j)}_{\gamma_1 \circ \gamma_2} \\
\mathbb{G}/\mathbb{G}_0 \xrightarrow{\sim} \mathcal{O}
\end{array}
\]
(95)
We recall that the group \(\mathbb{G}\) is a group of local unitary transformations which as such do not modify the degree of entanglement along the orbit starting at the selected fiducial state. In the specific case under consideration, the group \(\mathbb{G}\) is \(SU(2)\) and its action on the bipartite Hilbert space (92) is realized through a product representation
\[
U(\mathcal{H}) = U(\mathcal{H}^{(j_1)}_{\gamma_1}) \otimes U(\mathcal{H}^{(j_2)}_{\gamma_2}),
\]
(96)
whose infinitesimal generators are given by the \(SU(2)\)-generators tensored by the identity of one of the subsystems. Indeed, each subsystem Hilbert space reads as
\[
\mathcal{H}^{(j_i)}_{\gamma_i} \cong \mathcal{V}^{(j_i)} \otimes \mathcal{V}^{(j_i)*} \quad (i = 1, 2),
\]
(97)
and so the bipartite Hilbert space (92) can be regarded as
\[
\mathcal{H} \cong (\mathcal{V}^{(j_1)} \otimes \mathcal{V}^{(j_1)*}) \otimes (\mathcal{V}^{(j_2)} \otimes \mathcal{V}^{(j_2)*}).
\]
(98)
The gluing operation \(\gamma = \gamma_1 \circ \gamma_2\) corresponds to select the subspace
\[
\mathcal{V}^{(j_1)} \otimes \text{Inv}_{SU(2)} \mathcal{V}^{(j_1)} \otimes \mathcal{V}^{(j_2)} \subset \mathcal{H},
\]
(99)
which reduces to
\[
\mathcal{V}^{(j)} \otimes \mathcal{V}^{(j)*} \cong \mathcal{H}^{(j)}_{\gamma}, \quad j = j_1 = j_2
\]
(100)
since, according to the Schur’s lemma [43], when we have only two spin representations the invariant bivalent intertwining operator \(\mathcal{V}^{(j_1)} \to \mathcal{V}^{(j_2)}\) is either proportional to the identity if \(j_1 = j_2\) or zero if \(j_1 \neq j_2\), i.e., the invariant subspace is trivial.

We are thus brought back to the situation of the previous section. The pulled-back Hermitian tensor \(\mathcal{K}\) is again given by the pull-back of (65) with a fiducial state now given by (93) and the spin operators \(J\) act nontrivially only at the free endpoints of the resulting new link. Hence, there is no need to repeat our calculations and we only notice that, coherently with the general considerations of Section II, we have:
For the skew-symmetric part:

\[
(D_A)_{ab} = \langle 0| [J_a, J_b] - \otimes 1 |0 \rangle \\
= \frac{1}{2j + 1} \sum_{m \ell m' \ell'} c_{mn} c_{m'n'} \langle j, m, \ell | [J_a, J_b] - | j, m, \ell \rangle \delta_{\ell', \ell''} \delta_{n', n}
\]

\[
= \frac{1}{2j + 1} \sum_{m m' n} c_{m'n'} c_{mn} \langle j, m', \ell' | [J_a, J_b] - | j, m, \ell \rangle
\]

\[
= \sum_{m m' n} c_{m'n'} c_{mn} \langle j, m' | [J_a, J_b] - | j, m \rangle
\]

being \( \langle j, \ell | j, \ell \rangle = 1 \). We then see that when the fiducial state is maximally entangled, i.e., all the Schmidt coefficients are equal, we end up with the trace of the commutator which is zero.

By similar arguments, when \( |0 \rangle \) is separable, we see that the block-off-diagonal matrices of the symmetric part vanish:

\[
C_{ab} = \langle 0 | J_a \otimes J_b | 0 \rangle - \langle 0 | J_a \otimes 1 | 0 \rangle \langle 0 | 1 \otimes J_b | 0 \rangle
\]

\[
= \langle J_a | J_b \rangle - \langle J_a | J_b \rangle = 0 .
\]

Let us stress again that here for simplicity we content our analysis to fixed spin \( j \) labelling the \( SU(2) \) irreducible representations associated to the links. Nevertheless, we could also consider the action of a product unitary representation on the full Hilbert spaces \( \mathcal{H}_{\gamma_1} \otimes \mathcal{H}_{\gamma_2} \) where, as noticed in section IV A, the action on each subsystem will be realized in terms of the action on each element of the direct sum \( \mathcal{H}_{\gamma_i} \cong \bigoplus_{j_i} \mathcal{H}_{j_i} \), \( i = 1, 2 \). Obviously, the resulting expression of the Fisher tensor will be more complicated in this case but it will acquire the following structure which takes into account different kinds of correlations between the subsystems. The diagonal blocks will refer to different subspaces in the direct sum characterized by a different value of the spin labels. Within each block, the off-diagonal terms will therefore encode correlations between fixed-spin subspaces. On the other hand, the off-diagonal blocks of the full tensor will encode possible correlations between different spin configurations. This would allow to extend the study of correlations also to the gluing of full Wilson line states and, more generally, of open spin network states. We will discuss such cases elsewhere.

VI. UNFOLDING INTEERTWINER INTO A PAIR OF ENTANGLED QDITS

We shall now extend our tensorial approach to a class of quantum spin network states with a more involved nonlocal correlation structure.

In particular we look at the most concise description for a bounded region of quantum space in the framework of LQG spin networks, given by polyhedra dual to a superpositions of \( n \)-valent intertwiner states (see Appendix C for a detailed definition of intertwiner), with support on a single vertex graphs with \( n \) links. In absence of non-trivial internal curvature degrees of freedom [77], one can dually think of such region as a portion of 3d space flatly embedded in 4d. The irreducible representations carried by the open edges are dual to boundary patches comprising the quantum surface of the convex flat polyhedron dual to the intertwiner.

In particular, we can consider an ideal partition of the surface boundary, by dividing \( n \) into two sets \( n_A = \{ \{ \ell_k \} \in \partial A \} \) and \( n_B = \{ \{ \ell_k \} \in \partial B \} \) and think of the intertwiner state as a bipartite quantum mechanical system constrained by an overall \( SU(2) \) gauge invariance.

The full intertwiner Hilbert space, in the notation given (C16), reads

\[
\mathcal{H}_V = \bigoplus_{\{ j_k \}} \text{Inv}_{SU(2)} \left[ \bigotimes_{e \in V} \mathcal{V}_{j_k} \right]
\]

\[
\cong L^2 \left( SU(2)^L / SU(2) \right)
\]

This space has infinite dimension, due to the direct sum structure over the spin-\( J \) representations. Therefore, as for the bipartite link in IVB, we choose to further restrict our analysis to the finite dimensional case of the single intertwiner space at fixed \( \{ j_k \} \),

\[
\mathcal{H}_V^{\{ j_k \}} = \text{Inv}_{SU(2)} \left[ \bigotimes_{e \in V} \mathcal{V}_{j_k} \right]
\]

This is still considerably more involved than a single link. However, we must point out that, from the point of view of both LQG and GFT, it remains a drastic truncation of the set of (kinematical) degrees of freedom, and of the possibly relevant states, particularly from the point of view of a reconstruction of an approximate continuum spacetime and geometry. Indeed, it implies a truncation to a finite set of degrees of freedom loosing the functional aspects of the theory.\(^9\)

A. The bipartite system

We consider then a generic single intertwiner space as a bipartite quantum mechanical system constrained by an overall \( SU(2) \) gauge invariance. To the given separation of the boundary degrees of freedom, we can associate two boundary spaces, respectively defined by the tensor product of the \( SU(2) \) irreps labelling the edges

\[
\mathcal{H}_A \equiv \bigotimes_{e \in \partial A} \mathcal{V}_{j_e} \quad \text{and} \quad \mathcal{H}_B \equiv \bigotimes_{e \in \partial B} \mathcal{V}_{j_e} .
\]
In the intertwiner, the two boundary sets of degrees of freedom are constrained by the gauge invariance at the vertex. Indeed, we can rewrite (104) as

$$\mathcal{H}_{\nu}^{(j_\nu)} = \text{Inv}_{SU(2)}[\mathcal{H}_A \otimes \mathcal{H}_B] \quad (106)$$

Such a constraint is at the root of the non-local quantum correlations among the edges.

Let us then proceed by taking the decomposition of the two tensor product spaces in (105) in direct sums of irreducible representations: we write each subspace $\mathcal{H}_{A,B}$ as $\bigoplus_{k} \mathcal{V}^k \otimes D_k$, where $\mathcal{V}^k$ is a $(2k+1)$-dimensional re-coupled spin-$k$ irrep of $SU(2)$, coming with its degeneracy space $D_k$. In these terms, we write

$$\mathcal{H}_{\nu}^{(j_\nu)} = \text{Inv}_{SU(2)}[\mathcal{H}_A \otimes \mathcal{H}_B]$$

$$= \text{Inv}_{SU(2)} \left[ \bigoplus_{k,y} (\mathcal{V}^k_A \otimes \{j_{e\alpha A}\}D_k) \otimes (\mathcal{V}^y_B \otimes \{j_{e\alpha B}\}D_y) \right]$$

$$= \bigoplus_{k,y} \delta_{k,y} \left[ (\mathcal{V}^k_A \otimes \mathcal{V}^y_B) \otimes \{j_{e\alpha A}\}D_k \otimes \{j_{e\alpha B}\}D_y \right]$$

$$= \bigoplus_k \{j_{e\alpha A}\} \mathcal{H}^{(k)}_A \otimes \{j_{e\alpha B}\} \mathcal{H}^{(k)}_B$$

$$= \bigoplus_k \{j_{e\alpha A}\} \mathcal{H}^{(k)}_A \otimes \{j_{e\alpha B}\} \mathcal{H}^{(k)}_B \quad (107)$$

where, by gauge invariance, the two re-coupled spin irreps appearing in the second line are tensored to form a trivial representation for each $k$. Consistently, the dimension of the single intertwiner space reduces to

$$N_{\nu} \equiv \dim(\mathcal{H}_{\nu}^{(j_\nu)}) = \dim \left[ \bigoplus_k (\mathcal{V}^k_A \otimes \mathcal{V}^k_B) \otimes (D_k^A \otimes D_k^B) \right]$$

$$= \sum_k \dim D_k^A \cdot \dim D_k^B = \sum_k N_A^k N_B^k.$$  

(108)

where we define $N_{A,B}^k \equiv d_k^{(n_A,n_B)} = \dim D_k^{(A,B)}$, the dimensions of the degeneracy spaces.

Starting from the decomposition in (107), a convenient basis in the two subsystems $A$ and $B$ is labeled by three numbers, respectively $|k,m,\alpha_k\rangle$ and $|k,m,\beta_k\rangle$, with $\alpha_k, \beta_k$ giving the number of the different irreducible representations $\mathcal{V}^k_{A,B}$ for given $k$ [76]. A basis for the single intertwiner space is then written as

$$|k,\alpha_k,\beta_k\rangle = \sum_{m=-k}^{k} \frac{(-1)^{k-m}}{\sqrt{2k+1}} |k,-m,\alpha_k\rangle_A \otimes |k,m,\beta_k\rangle_B \quad (109)$$

Given the peculiar tensor structure of the unfolded intertwiner space, each basis state can be represented as a tensor product state on three subspaces [76],

$$|k,\alpha_k,\beta_k\rangle \equiv |k\rangle_{\mathcal{V}^k\otimes \mathcal{V}^k} \otimes |\alpha_k\rangle_{D^A_k} \otimes |\beta_k\rangle_{D^B_k} \quad (110)$$

where the generic $|\zeta_k\rangle$ labels a basis vector of $D_k$, with $\zeta_k$ running from 1 to $N_k = \dim D_k$. Therefore, a generic state vector in $\mathcal{H}_{\nu}^{(j_\nu)}$ is given by a superposition of product basis states of the three subspaces,

$$|\psi_{\nu}\rangle = \sum_{k,\alpha_k,\beta_k} c_{k,\alpha_k,\beta_k}^{(j_\nu)} |k\rangle_{\mathcal{V}^k\otimes \mathcal{V}^k} \otimes |\alpha_k\rangle_{D^A_k} \otimes |\beta_k\rangle_{D^B_k}. \quad (111)$$

In particular, we focus our analysis on a specific class of states, generically written as

$$|\psi_{\nu}k\rangle = \sum_{\alpha_k,\beta_k} c_{\alpha_k,\beta_k}^{(k,j_\nu)} |k\rangle_{\mathcal{V}^k\otimes \mathcal{V}^k} \otimes |\alpha_k\rangle_{D^A_k} \otimes |\beta_k\rangle_{D^B_k}, \quad (112)$$

with no sum over $k$. This means that we discard any quantum correlation among the tensored irreps space and the degeneracy spaces and we look at the entanglement induced by correlations among the degeneracy spaces only. This partially reduces the complexity of the problem.

By fixing the virtual link spin $k$, the unfolded intertwiner space (107) admits the following tensor product structure

$$\mathcal{H}_{\nu}^{(k,j_\nu)} = \{j_{e\alpha A}\} \mathcal{H}^{(k)}_A \otimes \{j_{e\alpha B}\} \mathcal{H}^{(k)}_B$$

$$= (\mathcal{V}^k_A \otimes \{j_{e\alpha A}\}D_k) \otimes (\mathcal{V}^k_B \otimes \{j_{e\alpha B}\}D_k)$$

$$= (\mathcal{V}^k_A \otimes \mathcal{V}^k_B) \otimes (D^A_k \otimes D^B_k).$$

A generic pure state density matrix $\rho_{\nu} = |\psi_{\nu}\rangle \langle \psi_{\nu}| \in D^1(\mathcal{H}^{(k,j_\nu)}_{\nu})$ will then have the following simplified form

$$\rho_{\nu} \equiv |\psi_{\nu}\rangle \langle \psi_{\nu}| = \rho^{(k)} \otimes \rho^{(k)}_{AB}; \quad (113)$$

with

$$\rho^{(k)} = |k\rangle \langle k| \in D^1(\mathcal{V}^k_A \otimes \mathcal{V}^k_B), \quad (114)$$

and

$$\rho^{(k)}_{AB} = \sum_{\alpha_k,\beta_k} c_{\alpha_k,\beta_k}^{(k,j_\nu)} \sigma_{\alpha_k,\beta_k}^{(k,j_\nu)} |\alpha_k\rangle \langle \alpha_k'| \otimes |\beta_k\rangle \langle \beta_k'|, \quad (115)$$

in $D^1(D^A_k \otimes D^B_k)$. With respect to the total space $\mathcal{H}_{\nu}^{(k,j_\nu)}$ at fixed $k$, the pure state $\rho_{\nu}$ is separable and can be factorized into a tensor product between a pure state $\rho^{(k)}$ involving only the spin $k$ irreps and a pure state $\rho^{(k)}_{AB}$ possibly entangling the degeneracy spaces. Since there are no correlations among the tensored irreps space and the degeneracy spaces, we can trace out $\rho^{(k)}$ and focus our attention only on the state $\rho^{(k)}_{AB}$ over the degeneracy spaces.

In these terms, for each $k$, we can effectively treat the intertwiner state as an entangled pure state on the bipartite degeneracy space $D^A_k \otimes D^B_k$. In this sense, we
can describe the degeneracy structure of the unfolded intertwiner state as a couple of entangled quantum $N$-level systems with number of levels provided by the degeneracy factors $N_k^X = \dim D^X_k \equiv g_k^{(nx)}$, $X = A, B$, respectively. Along the lines of the derivation given in Section II, we now proceed in investigating the correlation structure of

\[ \tau \]

tertwire state as a couple of entangled quantum

can describe the degeneracy structure of the unfolded in-

natural form. To this aim, we use the fiducial bipartite state (115).

B. Quantum Fisher Tensor on $D^1(D^A_k \otimes D^B_k)$

Our goal consists now in computing the full quantum Fisher tensor on the orbit submanifolds identified by the fiducial bipartite state (115).

In order to calculate the entanglement monotone in (28) explicitly, we need to put the block-coefficient matrices (25), (27) of the Hermitian tensor $K$ in a more manageable form. To this aim, we use the standard (or natural) basis over complex numbers for the $u(N)$ Lie algebras associated with the two subsystems, that is

\[ \sigma_a^{(A)} \rightarrow \tau_{aa}^{(A)} \equiv |a_k \rangle \langle a_k' |, \quad \sigma_b^{(B)} \rightarrow \tau_{bb}^{(B)} \equiv |b_k \rangle \langle b_k' | \quad (116) \]

\[
\begin{align*}
K_{(aa)}^{(A)} & = \frac{1}{2} \text{Tr} \left( \rho_0 \tau_{aa}^{(A)} \otimes I_B \right) - \text{Tr} \left( \rho_0 \tau_{bb}^{(B)} \otimes I_B \right), \\
K_{(bb)}^{(B)} & = \frac{1}{2} \text{Tr} \left( \rho_0 \tau_{bb}^{(B)} \otimes I_B \right) - \text{Tr} \left( \rho_0 \tau_{aa}^{(A)} \otimes I_B \right), \\
K_{(AA)}^{(B)} & = \text{Tr} \left( \rho_0 \tau_{aa}^{(A)} \otimes I_B \right) - \text{Tr} \left( \rho_0 \tau_{bb}^{(B)} \otimes I_B \right), \\
K_{(BB)}^{(A)} & = \text{Tr} \left( \rho_0 \tau_{bb}^{(B)} \otimes I_B \right) - \text{Tr} \left( \rho_0 \tau_{aa}^{(A)} \otimes I_B \right), \\
K_{(ab)}^{(A)} & = \frac{1}{2} \text{Tr} \left( \rho_0 \tau_{aa}^{(A)} \otimes I_B \right) - \text{Tr} \left( \rho_0 \tau_{bb}^{(B)} \otimes I_B \right), \\
K_{(ba)}^{(B)} & = \frac{1}{2} \text{Tr} \left( \rho_0 \tau_{bb}^{(B)} \otimes I_B \right) - \text{Tr} \left( \rho_0 \tau_{aa}^{(A)} \otimes I_B \right), \\
K_{(ab)}^{(B)} & = \frac{1}{2} \text{Tr} \left( \rho_0 \tau_{aa}^{(A)} \otimes I_B \right) - \text{Tr} \left( \rho_0 \tau_{bb}^{(B)} \otimes I_B \right), \\
K_{(ba)}^{(A)} & = \frac{1}{2} \text{Tr} \left( \rho_0 \tau_{bb}^{(B)} \otimes I_B \right) - \text{Tr} \left( \rho_0 \tau_{aa}^{(A)} \otimes I_B \right).
\end{align*}
\]

(119)

(120)

The fiducial state (115) can be thus written as

\[ \rho_0 = \sum_{\alpha, \beta, \alpha', \beta'} c_{\alpha \beta} \tau_{\alpha' \beta'} \tau_{\alpha \alpha'}^{(A)} \otimes \tau_{\beta \beta'}^{(B)}. \quad (121) \]

where we do not explicitly write the superscripts $(k, \{j_e\})$

\[ \theta_{11} = \theta_\theta + \theta_3, \quad \theta_{12} = \theta_1 - i \theta_2, \quad \theta_{21} = \theta_1 + i \theta_2, \quad \theta_{22} = \theta_\theta - \theta_3. \]

10 For instance, in the $U(2)$ case we have:

\[ \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]

\[ \tau_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \tau_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \tau_{21} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tau_{22} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \]

Therefore

\[ \sigma_0 = \sigma_1 + \tau_{22}, \quad \sigma_1 = \sigma_1 + \tau_{21}, \quad \sigma_2 = \sigma_2 + i(\tau_{21} - \tau_{12}), \quad \sigma_3 = \tau_{11} - \tau_{22} \]

from which, by imposing that $U^{-1}dU = i \theta_\theta \theta_\theta = i \tau_{kk} \theta_{kk'}$, it is easy to see that
Therefore, when this is the case, the matrix elements of the fiducial state (121) can be written as:

\[
K_{(aa')^A}^{(AB)} = \sum_{\alpha, \alpha'} \lambda_{\alpha} \tilde{\lambda}_{\alpha'} c_{\alpha \alpha'}^{(A)} \otimes \sum_{\beta, \beta'} \lambda_{\beta} \tilde{\lambda}_{\beta'} c_{\beta \beta'}^{(B)},
\]

(122)

\[
K_{(bb')^A}^{(AB)} = \frac{1}{2} \left( \delta_{ab} \sum_{\beta} \lambda_{\beta} \tilde{\lambda}_{\beta} + \delta_{a'b'} \sum_{\beta} \lambda_{\beta} \tilde{\lambda}_{\beta} \right) - \\
- \sum_{\beta} \lambda_{\beta} \tilde{\lambda}_{\beta} \sum_{\delta} \delta_{\delta \delta} c_{\beta \delta} ,
\]

(123)

\[
K_{(aa')^A, (bb')^B}^{(B)} = \frac{1}{2} \left( \delta_{ab} \sum_{\beta} \lambda_{\beta} \tilde{\lambda}_{\beta} - \delta_{a'b'} \sum_{\beta} \lambda_{\beta} \tilde{\lambda}_{\beta} \right) .
\]

(124)

Similar results hold for \( K^{(BA)} \) and the \( K^{(B)} \) blocks of the symmetric and antisymmetric part, respectively.

Finally, omitting for the moment the constant factor in front of the trace in Eq. (28), the entanglement monotone \( \mathcal{E} \) is given by (cfr. Eqs. (A6,A7)):

\[
\mathcal{E} = \sum_{aa'cc'} \left( c_{aa'} c_{cc'} - \sum_{\beta} \lambda_{\beta} \tilde{\lambda}_{\beta} \sum_{\gamma} c_{\gamma c'} c_{\gamma' c} \right)^2 .
\]

(125)

Here we see the advantage of choosing the standard basis. Indeed, in this basis, our measure of entanglement \( \mathcal{E} \), as well as all the block-coefficient matrices of the tensor \( K \), are written directly in terms of the coefficients \( c \) of the fiducial state.

C. Some special cases

Let us finally check that the blocks of the tensor \( K \) actually encode the information about separability or entanglement by considering some explicit choice of the fiducial state \( \rho_0 \). In particular we have to check that the off-diagonal blocks, and hence the entanglement monotone \( \mathcal{E} \), vanish when \( \rho_0 \) is separable, while the symplectic part vanishes when \( \rho_0 \) is maximally entangled. So if \( \rho_0 \) is separable, the coefficients \( c_{\alpha \beta} \) factorize as \( \lambda_{\alpha} \lambda_{\beta} \) and the fiducial state (121) can be written as:

\[
\rho_0 = \sum_{\alpha, \alpha'} \lambda_{\alpha} \tilde{\lambda}_{\alpha'} c_{\alpha \alpha'}^{(A)} \otimes \sum_{\beta, \beta'} \lambda_{\beta} \tilde{\lambda}_{\beta'} c_{\beta \beta'}^{(B)} .
\]

(126)

Therefore, when this is the case, the matrix elements of the off-diagonal blocks (122) are given by

\[
K_{(aa')^A}^{(AB)} = \lambda_{a'} \lambda_{\beta} \tilde{\lambda}_{\beta} - \lambda_{a} \tilde{\lambda}_{\alpha} \left( \sum_{\beta} \lambda_{\beta} \tilde{\lambda}_{\beta} \right) \cdot \lambda_{a'} \lambda_{b'} \tilde{\lambda}_{b'} = 0
\]

(127)

where we have used the normalization condition \( \sum_{\beta} |\lambda_{\beta}|^2 = 1 \). Thus, being \( \mathcal{K}^{(AB)} = 0 \) in the separable case, also the entanglement measure \( \mathcal{E} \) defined in (28) obviously vanishes as it can be directly checked from Eq. (125).

On the other hand, if \( \rho_0 \) is maximally entangled, then the coefficients \( c_{\alpha \beta} \) are given by \( \delta_{\alpha \beta} / \sqrt{N_{\leq}} \), with \( N_{\leq} = \min(N_A, N_B) \) and the fiducial state can be written as:

\[
\rho_0 = \frac{1}{N_{\leq}} \sum_{\alpha, \alpha'} c_{\alpha \alpha'}^{(A)} \otimes \tilde{c}_{\alpha \alpha'}^{(B)} .
\]

(128)

From the expression (A5) we then see that

\[
K_{(aa')^A, (bb')^B}^{(B)} = \frac{1}{2N_{\leq}} \left( \delta_{ab} \sum_{\beta} \delta_{\beta \beta} - \delta_{a'b'} \sum_{\beta} \delta_{\beta \beta} \right)
\]

(129)

i.e., as expected, the symplectic part vanishes in the maximally entangled case. Moreover, in this case the entanglement measure (28) is given by

\[
\mathcal{E} = \sum_{aa'cc'} \left( \frac{1}{N_{\leq}} \delta_{a'c'} \delta_{a c} - \frac{1}{N_{\leq}^2} \sum_{\beta} \delta_{\beta \beta} \sum_{\gamma} \delta_{\gamma \gamma} \delta_{\gamma' \gamma} \right)^2
\]

(130)

then explicitly, we have\(^{11}\)

\[
\mathcal{E} = \frac{N_{\leq}^2}{4(N_{\leq}^2 - 1)} \operatorname{Tr} \left( K^{(AB)} T K^{(AB)} \right) = \frac{1}{4} ,
\]

(131)

and the distance with respect to the separable state (29) takes the following value:

\[
\mathcal{E} = \frac{N_{\leq}^2}{4(N_{\leq}^2 - 1)} \operatorname{Tr} \left( K^{(AB)} T K^{(AB)} \right) = \frac{2}{N_{\leq}^2} .
\]

(132)

\(^{11}\) The explicit example of a simple spin network graph with all spins fixed to \( \frac{1}{2} \) is discussed in appendix B.
Finally, let us consider the intermediate case of a generic entangled fiducial state, that is
\[ c_{\alpha \beta} = f(\alpha) \delta_{\alpha \beta}, \quad (134) \]
where \( f(\alpha) \) is a complex functions satisfying the normalization condition \( \sum_{\alpha} |f(\alpha)|^2 = 1 \).

The fiducial state (121) can be thus written as
\[ \rho_0 = \sum_{\alpha, \alpha'} f(\alpha) f(\alpha') \tau^{(A)}_{\alpha \alpha' \otimes} \tau^{(B)}_{\alpha \alpha'}; \quad (135) \]
hence we have
\[ \text{Tr} \left( K^{(AB)} T K^{(AB)} \right) = \sum_{a a' c c'} \left( f(a') \overline{\tau}(a) \delta_{a a'} \delta_{c c} - \sum_{\beta} f(a') \overline{\tau}(a) \delta_{a a} f(\gamma) \overline{\tau}(\gamma) \delta_{c c} \right)^2 \]
\[ = \sum_{a a' c c'} \left( f(a') \overline{\tau}(a) \delta_{a a'} \delta_{c c} - f(a') \overline{\tau}(a) \delta_{a a} f(\gamma) \overline{\tau}(\gamma) \delta_{c c} \right)^2 \]
\[ = \sum_{a} \overline{\tau}(a)^2 \sum_{a'} f(a')^2 + \sum_{a a' c c'} f(a')^2 \overline{\tau}(a) \delta_{a a'} f(\gamma)^2 \overline{\tau}(\gamma) \delta_{c c} - 2 \sum_{a a' c c'} f(a')^2 \overline{\tau}(a) \delta_{a a'} f(\gamma) \overline{\tau}(\gamma) \delta_{c c} \]
\[ = \sum_{a} \overline{\tau}(a)^2 \sum_{a'} f(a')^2 - \sum_{ac} |f(\alpha)|^4 |f(\gamma)|^2 \delta_{ac} = \sum_{a} \overline{\tau}(a)^2 \sum_{a'} f(a')^2 - \sum_{a} |f(\alpha)|^6 . \]

Therefore, if the functions \( f \) are real\(^{12}\), Eq. (136) reduces to:
\[ \text{Tr} \left( K^{(AB)} T K^{(AB)} \right) = 1 - \sum_{a} |f(\alpha)|^6 . \quad (137) \]

In particular, we see that when \( f(\alpha) = 1/\sqrt{N_c} \), \( \forall \alpha = 1, \ldots, N_c \), we recover the result (130) for the maximally entangled case.

To sum up, we collect the above results in the following table:

| \( \rho_0 \)         | \( c_{\alpha \beta} \)                                      | \( \text{Tr} \left( K^{(AB)} T K^{(AB)} \right) \) |
|----------------------|---------------------------------------------------------------|--------------------------------------------------|
| separable            | \( \lambda_\alpha \lambda_\beta \)                         | 0                                               |
| maximally entangled  | \( \delta_{\alpha \beta} / \sqrt{N_c} \)                   | \( 1 - \frac{1}{N_c} \)                        |
| entangled            | \( f(\alpha) \delta_{\alpha \beta}, f(\alpha) \in \mathbb{C} \) | \( \sum_{\alpha} \overline{\tau}(\alpha)^2 \sum_{\alpha'} f(\alpha')^2 - \sum_{\alpha} |f(\alpha)|^6 \) |
|                                    | \( f(\alpha) \delta_{\alpha \beta}, f(\alpha) \in \mathbb{R} \) | \( 1 - \sum_{\alpha} |f(\alpha)|^6 \) |

VII. CONCLUSIONS AND OUTLOOK

Motivated by the idea that, in the background independent framework of a quantum theory of gravity, entanglement is expected to play a key role in the reconstruction of spacetime geometry, this work is a preliminary investigation towards the possibility of using the formalism of Geometric Quantum Mechanics (GQM) to give a fully tensorial characterization of entanglement on spin network states. Given that such states also carry an intrinsic quantum geometric characterization in terms of the algebraic data labeling them, interpreted in the sense of simplicial geometry, an additional issue is to relate any geometric notion encoded in their entanglement proper-

\(^{12}\) This is actually the case of a Schmidt decomposition.
ties with such simplicial geometry. Our analysis focused first on the simple case of a single link graph state for which we define a dictionary to construct a Riemannian metric tensor and a symplectic structure on the space of states. The manifold of (pure) quantum states was then stratified in terms of orbits of equally entangled states showing that the block-coefficient matrices of the corresponding pulled-back tensors fully encode the information about separability and entanglement. In particular, the off-diagonal blocks $C$ define an entanglement monotone $E \propto \text{Tr}(C^TC)$, directly related to the geometric distance with respect to the separable state. Such a construction provides:

1. A formalism which fits well to a purely relational interpretation of the link as an elementary process describing the quantum correlations between its endpoints.

2. A quantitative characterization of graph connectivity by means of the entanglement monotone $E$ which comes to be a measure of the existence of the process/link.

3. A connection between the GQM formalism and the geometric properties of the quantum states through entanglement. In the maximally entangled case, which for the single link corresponds to a gauge-invariant loop, the entanglement monotone is actually proportional to a power of the corresponding expectation value of the area operator.

As a second step, we applied the construction to the case of a spin network intertwiner state, dual to a fundamental volume of space. In this framework, we focussed on the Hilbert space of the single $N$-valent intertwiner and we regarded the whole system as a bipartite one, where each subsystem is $n$-level and the number of levels is determined by the degeneracy of the two virtual intertwiner spaces, resulting from the unfolding of the initial vertex. We then studied the resulting quantum correlations using our GQM formalism. The series of analytic results derived for the entanglement monotone $E$ support its interpretation as a measure of spatial connectivity.

In fact, our interest in considering intertwiner states goes beyond their role of fundamental structural tensors attached to nodes. Indeed, more generally, within a reduction by gauge fixing scheme, intertwiner spaces provides a synthetic coarse grained description for a generic closed region of quantum space with boundary and non-trivial internal degrees of freedom (see Fig. 1). Assuming bulk flatness, i.e. the absence of curvature degrees of freedom, a generic region of a spin network is effectively described by an intertwiner between the $n$ links puncturing the dual surface. From the point of view of the surface, a state of geometry of that region is described by a superposition of the possible $n$-valent intertwiners [21].

It has been proposed in [22] that a notion of distance between two regions of space should be derived in terms of the entanglement between the two regions $A$ and $B$ of the underlying spin network induced by the rest of the network. Our geometric approach along the same line, though still limited to the simplified case of a single $n$-valent intertwiner, however suggests that, in the flat bulk case, the measure of entanglement does not depend on the simplicial bulk distance between the two non adjacent regions, which always trivialized due to the gauge invariance of the state, but only on the representations of the boundary states. Overall, these results may be intended as the starting point of a program whose final goal aims to understand in full generality how the tensorial structures defined on the space of spin network states can be used to characterize their geometric features, alongside with (or as an alternative to) their simplicial geometric interpretation, and, in particular, how they can help in the reconstruction of the continuum (quantum) geometry of spacetime. We have seen that even at the simplest level of a single link we can already grasp some connections between entanglement and geometry. Obviously, the cases considered in this paper being so simple, we need to extend our construction to more general cases. Let us then close by sketching some possible future developments:

- The general setup we have used to compute entanglement properties of the bipartite system associated to two regions of a spin network state should now be extended, and the calculations generalized. This can be done in several directions, corresponding to the progressive removal of the various approximations we have imposed on our system, in this work. One is the introduction of curvature degrees of freedom, or of a proper coarse graining procedure to deal with them. Another is the inclusion of the sum over spin degrees of freedom in the calculations, i.e. allowing for a superposition
of spin network states while keeping the underlying combinatorial structure fixed. More ambitiously, we need to learn to control the entanglement properties of superpositions over graph structures. For the latter goal, the GFT formalism may be the one with the greatest potential. In order to generalize our entanglement calculation to superpositions of spin network graphs, one possibility is to adopt the tensor model techniques already used for the calculation of the entanglement entropy of horizon states built out of spin networks in [27].

• We may focus on coherent states and exploit their interpretation as semiclassical states to study the classical limit of the metric tensor. Let us also notice that in this case we are selecting a particular family of states with their own parameter space. Moreover, since the parameter space of coherent states is isomorphic to the classical phase space of discrete geometries associated to spin network states, establishing a correspondence between the (entanglement) properties measured in terms of information geometry and the same simplicial geometry should be rather direct. This should also enable us to exploit the connection between the Fubini-Study and the Fisher-Rao metrics and the related tools of information geometry (see for example [79] where the quantum metric is derived from relative entropy).

• The analysis of entanglement with classical tensors has been extended also to the case of mixed states [82]. The case of Gibbs states, where expectation values of geometric observables such as area play the role of the parameters of the exponential family of maximally mixed states, would be interesting ion view of its application to the study of black holes. In particular, once we generalize our construction to include curvature degrees of freedom in the bulk of our spatial region, any subsequent coarse-graining of the same curvature degrees of freedom will result in mixed states. In other words, one could associate the mixed nature of quantum states being considered to the curvature degrees of freedom having been traced out.

• A further interesting aspect concerns the very interpretation of these tensors in those cases where the space of states is a tensor product of boundary states spaces of a process. The case of the single link, where the Hermitian tensor can be associated with an amplitude from an initial to a final spin state, may be generalized to a full (spin foam) path integral amplitude, meant as a process generating a region of space-time. In this case, the Fubini-Study metric would provide a metric for the space-time region. This setting has interesting formal analogies with the general boundary formalism [83, 84].

We finally wish to comment about the novelty and relevance of the problem treated in the paper: the new element is the application of geometric quantum mechanics to the study of correlations on spin network states. The idea of using information theoretic tools in the context of quantum gravity is certainly not new, but the use of geometry in the context of quantum information has recently seen a great development (see [79–81] and references therein) and its application to quantum gravity is certainly new. These geometric techniques, in our opinion, will allow to import quantum information tools in quantum gravity in a more efficient and fruitful way, than it has been done so far. Tensorial structures defined on the space of quantum states of whatever dynamical system one is analyzing have the advantage of being basis independent and intrinsic. The results that we have obtained about the relation between spin networks connectivity and entanglement are certainly preliminary and for the moment limited to pure states. However, along with the quantum gravity perspective considered, the most compelling feature of this formalism resides in the possibility of generalizing the treatment to the case of mixed states, something precluded to any analysis built on entanglement entropy, as well as to the case of multipartite entanglement. Both aspects are expected to play a fundamental role once coarse graining and entanglement renormalisation schemes will be considered, for example in the study of the continuum limit leading to the emergence of semiclassical geometry.

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Appendix A: Computation in the standard basis

In this appendix we report the explicit computations of the block-coefficient matrices (119) of the Hermitian quantum Fisher tensor on the orbit generated from the fiducial state

$$\rho_0 = \sum_{\alpha\alpha'\beta\beta'} c_{\alpha\beta} \bar{c}_{\alpha'\beta'} \tau_{\alpha\alpha'} \otimes \tau_{\beta\beta'} \cdot \quad (A1)$$

$$K^{(AB)}_{(aa')(bb')} = \sum_{\alpha\alpha'\beta\beta'} c_{\alpha\beta} \bar{c}_{\alpha'\beta'} \text{Tr} \left( \tau_{\alpha\alpha'} \otimes \tau_{\beta\beta'} \tau_{bb'} \right) - \sum_{\alpha\alpha'\beta\beta'} c_{\alpha\beta} \bar{c}_{\alpha'\beta'} \text{Tr} \left( \tau_{\alpha\alpha'} \otimes \tau_{\beta\beta'} \tau_{bb'} \right) \sum_{\gamma'\delta'} c_{\gamma\delta} \bar{c}_{\gamma'\delta'} \text{Tr} \left( \tau_{\gamma\gamma'} \otimes \tau_{\delta\delta'} \tau_{bb'} \right)$$

$$= \sum_{\alpha\alpha'\beta\beta'} c_{\alpha\beta} \bar{c}_{\alpha'\beta'} \text{Tr}_A \left( \rho_{\alpha\alpha'} \tau_{\alpha\alpha'} \right) \text{Tr}_B \left( \tau_{\beta\beta'} \tau_{bb'} \right) + \sum_{\alpha'\beta'\beta\alpha} c_{\alpha'\beta'} \delta_{\alpha\alpha'} \delta_{\beta\beta'} \delta_{\delta\delta'} - \sum_{\alpha\alpha'\beta\beta'} c_{\alpha\beta} \bar{c}_{\alpha'\beta'} \delta_{\alpha\alpha'} \delta_{\beta\beta'} \sum_{\tau_{\gamma'\delta'}} \delta_{\gamma'\gamma} \delta_{\delta'\delta}$$

$$= c_{ab'} \bar{c}_{ab} - \sum_{\beta} c_{a\beta} \bar{c}_{\alpha\beta} \cdot \sum_{\gamma} c_{\gamma b'} \bar{c}_{b'\gamma}$$

(A2)

where in the third equality we used the relations $\text{Tr}_A \left( \tau_{\alpha\alpha'} \tau_{\alpha\alpha'} \right) = \delta_{\alpha\alpha'} \delta_{\alpha\alpha'}$ and $\text{Tr}_A \left( \tau_{\gamma\gamma'} \right) = \delta_{\gamma\gamma'}$ (and similarly for the $\tau^{(B)}$’s) which can be easily deduced from Eq. (117). A similar result holds for the block $K^{(BA)}$.

As regards the other blocks, by using the commutation and anti-commutation relations

$$[\tau_{aa'}, \tau_{bb'}]_{\pm} = \delta_{ab} \tau_{ab} \pm \delta_{ab} \tau_{ab} \cdot \quad X = A, B \quad (A3)$$

we get

$$K^{(A)}_{(aa')(bb')} = \frac{1}{2} \sum_{\alpha\alpha'\beta\beta'} c_{\alpha\beta} \bar{c}_{\alpha'\beta'} \text{Tr}_A \left( \tau_{\alpha\alpha'} \tau_{\alpha\alpha'} \tau_{bb'} \right) \text{Tr}_B \left( \tau_{\beta\beta'} \tau_{bb'} \right) - \sum_{\alpha\alpha'\beta\beta'} c_{\alpha\beta} \bar{c}_{\alpha'\beta'} \text{Tr}_A \left( \tau_{\alpha\alpha'} \tau_{\alpha\alpha'} \tau_{bb'} \right) \cdot \quad (A4)$$

$$\cdot \sum_{\gamma'\delta'} \text{Tr}_A \left( \tau_{\gamma'\gamma' \tau_{bb'}} \right) \text{Tr}_B \left( \tau_{\delta\delta'} \right)$$

$$= \frac{1}{2} \sum_{\alpha\alpha'\beta\beta'} c_{\alpha\beta} \bar{c}_{\alpha'\beta'} \text{Tr}_A \left( \tau_{\alpha\alpha'} \tau_{\alpha\alpha'} \tau_{bb'} \right) \text{Tr}_B \left( \tau_{\beta\beta'} \right) + \sum_{\alpha\alpha'\beta\beta'} c_{\alpha\beta} \bar{c}_{\alpha'\beta'} \cdot \sum_{\alpha'\beta'\beta\alpha} \delta_{\alpha\alpha'} \delta_{\beta\beta'} \delta_{\delta\delta'}$$

$$= \frac{1}{2} \sum_{\alpha\alpha'\beta\beta'} c_{\alpha\beta} \bar{c}_{\alpha'\beta'} \left( \delta_{\alpha\alpha'} \delta_{\beta\beta'} \delta_{\delta\delta'} + \delta_{\alpha\alpha'} \delta_{\beta\beta'} \delta_{\delta\delta'} \right) - \sum_{\alpha\alpha'\beta\beta'} c_{\alpha\beta} \bar{c}_{\alpha'\beta'} \cdot \sum_{\alpha'\beta'\beta\alpha} \delta_{\alpha\alpha'} \delta_{\beta\beta'} \delta_{\delta\delta'}$$

$$= \frac{1}{2} \left( \delta_{ab} - \sum_{\beta} \delta_{ab} \bar{c}_{b'\beta} \sum_{\beta} c_{b\beta} \bar{c}_{b'\beta} \right) - \frac{1}{2} \left( \delta_{ab} - \sum_{\beta} \delta_{ab} \bar{c}_{b'\beta} \sum_{\beta} c_{b\beta} \bar{c}_{b'\beta} \right) \cdot \quad (A5)$$
and

\[ K_{[aa']}^{(A)} \frac{[bb']}{[bb']} = \frac{1}{2} \sum_{\alpha \alpha' \beta \beta'} c_{\alpha \beta} \tau_{\alpha' \beta'} T_A \left( \tau_{\alpha' \beta'}^A, [\tau_{\alpha' \beta'}^A]^{-1} \right) T_B \left( \tau_{\beta' \beta}^B \right) \]
\[ = \frac{1}{2} \sum_{\alpha \alpha' \beta} c_{\alpha \beta} \tau_{\alpha' \beta} T_A \left( [\tau_{\alpha \alpha'}^A]^T, [\tau_{\alpha' \beta}^A]^{-1} \right) \]
\[ = \frac{1}{2} \sum_{\alpha \alpha' \beta} c_{\alpha \beta} \tau_{\alpha' \beta} \left[ \delta_{\alpha' b} T_A \left( [\tau_{\alpha' \alpha'}^A]^T - \delta_{\alpha' b} [\tau_{\alpha' \alpha'}^A]^{-1} \right) \right] \]
\[ = \frac{1}{2} \sum_{\alpha \alpha' \beta} c_{\alpha \beta} \tau_{\alpha' \beta} \left( \delta_{\alpha' b} \delta_{\alpha' a} - \delta_{\alpha b} \delta_{\alpha' a'} \right) \]
\[ = \frac{1}{2} \left( \delta_{\alpha' b} T_b \left( \delta_{\alpha' b} \sum_{\beta} \tau_{a \beta} - \delta_{\alpha b} \sum_{\beta} \tau_{a \beta} \right) \right). \] (A5)

Similar results hold for the \( K^{(E)} \) blocks of the symmetric and antisymmetric part, respectively.

Finally, let us consider the entanglement measure (28).

First of all, let us compute the product \( K^{(AB)} K^{(AB)} T \) whose matrix elements, according to Eq. (A2), are given by

\[ (K^{(AB)} K^{(AB)} T)_{(aa')}^{(bb')} = \sum_{cc'} K^{(AB)}_{(aa')} K^{(AB)}_{(cc')} (bb') = \sum_{cc'} K^{(AB)}_{(aa')} K^{(AB)}_{(cc')} (bb') \]
\[ \cdot \sum_{cc'} \left[ \left( c_{a c'} \tau_{a c} - \sum_{\beta} c_{a' \beta} \tau_{a \beta} \sum_{\gamma} c_{\gamma c'} \tau_{\gamma c} \right) \cdot \left( c_{b c'} \tau_{b c} - \sum_{\delta} c_{b' \delta} \tau_{b \delta} \sum_{\zeta} c_{\zeta c'} \tau_{\zeta c} \right) \right] \]
\[ \cdot \sum_{cc'} \left( \sum_{\beta} c_{a c'} \tau_{a c} \sum_{\gamma} c_{\gamma c'} \tau_{\gamma c} - \sum_{\delta} c_{b c'} \tau_{b c} \sum_{\zeta} c_{\zeta c'} \tau_{\zeta c} \right). \] (A6)

Now taking the trace of Eq.(A6) amounts to set \( a = b, a' = b' \) and to sum over \( a, a' \). Therefore, omitting for the moment the constant factor in front of the trace in Eq. (28), we have:

\[ E = \sum_{aa'cc'} c_{a c'}^2 \tau_{ac}^2 + \sum_{\gamma} c_{\gamma c'} \tau_{\gamma c} \sum_{\delta} c_{\delta c'} \tau_{\delta c}^2 - 2 c_{a' c'} \tau_{ac} \sum_{\beta} c_{a' \beta} \tau_{a \beta} \sum_{\gamma} c_{\gamma c'} \tau_{\gamma c} \]
\[ = \sum_{aa'cc'} \left( c_{a c'} \tau_{ac} - \sum_{\beta} c_{a' \beta} \tau_{a \beta} \sum_{\gamma} c_{\gamma c'} \tau_{\gamma c} \right)^2. \] (A7)

**Appendix B: Spin-\( \frac{1}{2} \) Graph and Large \( n \) Correlations**

Let us consider the case of a spin network graph \( \Gamma \) whose edges are all labeled by spins fixed at the fundamental representation, i.e., \( j_e = \frac{1}{2} \) \( \forall e \in \partial A \cup \partial B \equiv \partial R \). The specific structure of the starting graph is not relevant for our analysis and the only assumption we make is that the gauge reduction procedure leads to a single intertwiner graph \( \Gamma_R \) between the \( 2n \) \( (n \in \mathbb{N}) \) SU(2)-representations defining the boundary \( \partial R \) with no loops carrying curvature excitations. The number of boundary edges must be necessarily even since there does not exist any intertwiner between an odd number of \( \frac{1}{2} \)-spin representations.

With such a choice of spin labels, we may unfold the single \( 2n \)-valent intertwiner into two \( (n + 1) \)-valent vertices coupling the two boundary sets of \( n \) spin-\( \frac{1}{2} \) edges with a virtual link labeled by a fixed spin \( k \). As long as
On the other hand, when $k \leq n$, the dimension of the corresponding degeneracy spaces $D^A_k$ and $D^B_k$ can be then expressed in terms of binomial coefficients as [77]:

$$N = N_A^k = N_B^k = d^{(n)}_k = \binom{2n}{n+k} - \binom{2n}{n+k+1}$$

$$= \frac{2k+1}{n+k+1} \binom{2n}{n+k}.$$  

Therefore, with this separation of boundary degrees of freedom, the gauge-reduced unfolded intertwiner state can be actually described as a couple of entangled $N$-level systems with the same number of levels given by (B1).

Let us analyze the large $n$ behaviour of the quantum correlations between the two subsystems as a function of $k$. We focus on a fiducial state $|0\rangle \equiv |\psi_{TR}\rangle$ with all coefficients $c^{(k,\frac{1}{2})}_{\gamma(\ell),\gamma'_{\ell}}$ equal to $\frac{1}{\sqrt{N}}$, whose corresponding pure state density matrix is given by

$$\rho_0 = \frac{1}{N} \sum_{\alpha_k,\alpha'_k=1}^N \tau^{(A)}_{\alpha_k\alpha'_k} \otimes \tau^{(B)}_{\alpha_k\alpha'_k}. \quad \text{(B2)}$$

In section V-E we found that, for such a choice of the fiducial state, the entanglement measure constructed with the off-diagonal blocks of the pulled-back metric tensor on the orbit starting at $\rho_0$ is given by

$$\text{Tr} \left( K^{(AB)} T K^{(AB)} \right) = 1 - \frac{1}{N^2}. \quad \text{(B3)}$$

For large $n$, say $n \gg 1$, the degeneracy factors (B1) admit the following asymptotic expression [76]

$$N = d^{(n)}_k \sim \frac{2^{2n+1}}{\sqrt{\pi n}} \frac{x}{(1+x)\sqrt{1-x^2}} e^{-n\varphi(x)}, \quad \text{(B4)}$$

with

$$\varphi(x) = (1+x) \log(1+x) + (1-x) \log(1-x), \quad \text{(B5)}$$

for $x = \frac{k}{n} \in [0,1]$. So now, when $k$ is much smaller than $n$ or equivalently $x \to \varepsilon$ with $\varepsilon \ll 1$, up to terms $o(\varepsilon^2)$ we have

$$N = d^{(n)}_k \sim \frac{2^{2n+1}}{\sqrt{\pi n}} \varepsilon + o(\varepsilon^2). \quad \text{(B6)}$$

On the other hand, when $k$ becomes comparable with $n$ (that is $x \to 1$), $\varphi(x) \sim 2 \log 2$ and $d^{(n)}_k$ goes to infinity. Hence, we have:

$$N = d^{(n)}_k \sim \begin{cases} \frac{2^{2n+1}}{\sqrt{\pi n}} \varepsilon & \text{for} \quad x \to \varepsilon \quad (k \ll n) \\ \infty & \text{for} \quad x \to 1 \quad (k \simeq n) \end{cases}. \quad \text{(B7)}$$

The entanglement monotone (B3) then exhibits the following large $n$ behaviours

$$\text{Tr} \left( K^{(AB)} T K^{(AB)} \right) \sim \begin{cases} 1 - \frac{1}{N^2} & \text{for} \quad k \ll n \\ \frac{1}{1} & \text{for} \quad k \simeq n \end{cases}.$$  

Therefore, being (B6) positive and greater than 1 (for large $n$ the numerator is greater than the denominator), the entanglement measure (B8) reaches its maximum value when $k$ becomes comparable with $n$.

**Appendix C: Spin Networks in the Embedded Canonical Framework**

In Loop Quantum Gravity, Einstein’s theory of General Relativity (GR) is recast into the form of a gauge theory with structure group $Spin(1,3)$, plus the additional gauge symmetries resulting from the space-time diffeomorphism invariance. A partial fixing of the $Spin(1,3)$ invariance leads to a phase space description of the classical theory in terms of connections $A$ of a principal $SU(2)$-bundle over spacelike hypersurfaces $\Sigma$, embedded in a spacetime manifold $M$, and sections $E$ of the associated vector bundle over $\Sigma$, whose pull back are Lie algebra valued pseudo two forms.

The two forms $E$ encode the information about the 3d geometry on $\Sigma$, while the $A$ carries the information about the extrinsic curvature of $\Sigma$ in $M$.

The conjugated variables $(A,E)$, with standard Poisson brackets, define a (Yang–Mills like) phase space, which is then reduced by the imposition of the $SU(2)$ Gauss constraint, the spatial diffeomorphism constraint and the Hamiltonian constraint, respectively implementing the internal local gauge symmetry and the symmetry under diffeomorphisms.

The Dirac quantization procedure, before the imposition of the diffeomorphism constraints, leads to Hilbert spaces $H_\Gamma$ associated to graphs embedded in the canonical manifold.\(^ {13}\)

Let $\Gamma \subset \Sigma$ be a graph, i.e., a finite and ordered collection of smooth oriented paths $\gamma_\ell \in \Sigma$ with $\ell = 1, \ldots, L$ meeting at most at their endpoints (such paths will be called the links or the edges of the graph, while the intersection points will be called nodes or vertices), and let $\psi : SU(2)^L \rightarrow \mathbb{C}$ be a (smooth) (cylindrical) function $\psi_\Gamma(h_1, \ldots, h_L)$ of $L$ group elements. These group elements are interpreted as parallel transports $h_\ell(A) \equiv h_{\gamma_\ell}(A)$ of the connection $A$ along the links $\gamma_\ell$ of the graph $\Gamma$, embedded in the canonical hypersurface. The linear space of such cylindrical functionals w.r.t. a given graph

\(^ {13}\) For a comprehensive introduction to the LQG quantum geometry states we refer to the literature cited in the introduction, as well as to [31–39].
$\Gamma$ can be turned into a Hilbert space by equipping it with the following scalar product

$$\langle \psi(\Gamma) | \psi'(\Gamma) \rangle \equiv \int \prod_{\ell=1}^{L} d\mu_{\ell} \psi(h_{1}, \ldots, h_{L}) \psi'(h_{1}, \ldots, h_{L}) ,$$

(1)

where $d\mu_{\ell}$ are $L$ copies of the (left- and right-invariant) Haar measure of $SU(2)$. The inner product (1) is invariant under $SU(2)$ gauge transformations acting as left or right multiplications on the arguments of the wave functions $\psi$, depending on whether the gauge transformation is associated to the starting or end point of the link to which each argument is referring to. This is a direct consequence of the invariance of the Haar measure. One then needs to construct a Hilbert space out of the space of all cylindrical functions for all graphs $\Gamma \subset \Sigma$:

$$\bigcup_{\Gamma \subset \Sigma} \mathcal{H}_{\Gamma} .$$

(2)

To do this, we need to define a scalar product for cylindrical functions based on different graphs. Such a scalar product can be deduced from that on $\mathcal{H}_{\Gamma}$ as follows. The construction is based on the introduction of the so-called cylindrical equivalence relations which reflect properties of the underlying continuum connection field, and it is therefore directly inspired by the continuum embedding of the graphs $\Gamma$, and thus on the origin of the quantum states $\psi$ as coming from the canonical quantization of a continuum field theory. The details of the construction are not so important for our purposes. Essentially, the (unconstrained) kinematical Hilbert space of the theory can be casted as a direct sum of single graph-based Hilbert spaces

$$\mathcal{H}_{\text{kin}} = \bigcup_{\Gamma \subset \Sigma} \mathcal{H}_{\Gamma} \sim \bigoplus_{\Gamma \subset \Sigma} \mathcal{H}_{\Gamma} ,$$

(3)

where the individual graph-based Hilbert spaces $\mathcal{H}_{\Gamma}$ correspond to $\mathcal{H}_{\Gamma}$ without zero modes, i.e., where the spins $j_{\ell}$ never take the value zero. This space can then be understood as a Hilbert space over “generalized” connections on $\Sigma$ with the so-called Ashtekar-Lewandowski measure $d\mu_{AL}$ [40–42],

$$\mathcal{H}_{\text{kin}} \cong L^{2}(A, d\mu_{AL}) .$$

(4)

In our analysis, we mainly focus on the spaces associated to single graphs $\Gamma$, thus with $\mathcal{H}_{\Gamma}$, and the cylindrical equivalence conditions will not play much of a role.

From such fundamental (unconstrained) kinematical Hilbert space $\mathcal{H}_{\text{kin}}$, the physical space $\mathcal{H}_{\text{phys}}$ is derived from a series of reduction processes under the imposition of the Gauss and diffeomorphism constraints.

1. Gauge-invariant states and spin network basis

In this work we content ourselves with the kinematical structure of LQG, encoded in $\mathcal{H}_{\text{kin}}^{0}$, obtained from $\mathcal{H}_{\text{kin}}$ after the imposition of the Gauss constraint only, which is common to a large extent also to the group field theory formalism, as we will discuss.

The solutions of the quantum Gauss constraint form the Hilbert space $\mathcal{H}_{\text{kin}}^{0}$ of $SU(2)$-gauge invariant states, i.e.:

$$\mathcal{H}_{\text{kin}}^{0} = \text{Inv}_{SU(2)}[\mathcal{H}_{\text{kin}}] ,$$

(5)

which can be defined by the same construction outlined in the previous section, but starting from gauge invariant spaces $\mathcal{H}_{\Gamma}^{0}$ associated to all possible graphs $\Gamma$. From the transformation of continuum parallel transports under $SU(2)$ transformations, it follows that a gauge transformation acts only on the nodes of the graph. Therefore, the gauge-invariance requirement for cylindrical functions translates into the requirement of invariance under the action of the group at the nodes, i.e.:

$$\psi_{\Gamma}^{0}(h_{1}, \ldots, h_{L}) = \psi_{\Gamma}(g(\gamma(0))h_{1}g^{-1}(\gamma(1)), \ldots, g(\gamma_{L}(0))h_{L}g^{-1}(\gamma_{L}(1))) ,$$

(6)

where $\gamma_{i}(0)$ (resp. $\gamma_{i}(1)$) indicates the starting (resp. end) point of the link $i$ of the graph $\Gamma$. The above invariance can be implemented by group averaging

$$\int \prod_{v=1}^{V} dg_{v} f(g(\gamma_{i}^{v})h_{1}g^{-1}(\gamma_{i}^{v}), \ldots, g(\gamma_{L}^{v}(0))h_{L}g^{-1}(\gamma_{L}^{v})) ,$$

(7)

where $V$ is the number of nodes (vertices) of the graph $\Gamma$ and we write $\gamma^{v} \equiv \gamma(0)$ (resp. $\gamma^{v} \equiv \gamma(1)$) for short notation. In the spin representation of each function, this corresponds to inserting on each node $v$ of the graph the following projector,

$$\mathcal{I}_{v} = \int dg \prod_{\ell \in v} D^{(j_{\ell})}(g) .$$

(8)

with

$$\prod_{\ell \in v} D^{(j_{\ell})}_{\text{me}}(g) \in \bigotimes_{v} \mathcal{H}^{(j_{v})} = \bigotimes_{v} (\mathcal{V}^{(j_{v})} \otimes \mathcal{V}^{(j_{v})^{*}})$$

(9)

$\mathcal{V}^{(j_{v})}$ denoting the $SU(2)$ irreducible spin-$j_{v}$ representation spaces. Therefore, by using the decomposition of the tensor product $\bigotimes_{\ell \in v} \mathcal{H}^{(j_{\ell})}$ into irreducible representations

$$\bigotimes_{\ell \in v} \mathcal{H}^{(j_{\ell})} = \bigoplus_{i} \mathcal{H}^{(j_{i})} ,$$

(10)

we find that $\mathcal{I}_{v}$ projects onto the gauge invariant part of
\[ \bigotimes_{\ell \in v} \mathcal{H}^{(j_{\ell})}, \text{namely the singlet space } \mathcal{H}^{(0)}: \]

\[ \mathcal{I}_v : \bigotimes_{\ell \in v} \mathcal{H}^{(j_{\ell})} \rightarrow \mathcal{H}^{(0)}. \tag{C11} \]

Being \( \mathcal{I}_v \) a projector, it can be decomposed in terms of a basis \( \{ \alpha \} \) of \( \mathcal{H}^{(0)} \) and its dual as

\[ \mathcal{I}_v = \sum_{\alpha=1}^{\dim \mathcal{H}^{(0)}} \alpha_{\alpha} \alpha^{*} \in \mathcal{H}^{(0)} \otimes \mathcal{H}^{(0)^*}, \tag{C12} \]

from which, together with the decomposition of \( \bigotimes_{\ell \in v} \mathcal{H}^{(j_{\ell})} = \bigotimes_{\ell \in \text{in}} \mathcal{H}^{(j_{\ell})} \otimes \bigotimes_{\ell \in \text{out}} \mathcal{H}^{(j_{\ell})} \) between ingoing and outgoing links of the vertex \( v \), it follows that \( \mathcal{I}_v \) is the invariant map between the representation spaces associated with the edges joined at the node \( v \), i.e.: \( \mathcal{I}_v : \bigotimes_{\ell \in \text{in}} \mathcal{H}^{(j_{\ell})} \rightarrow \bigotimes_{\ell \in \text{out}} \mathcal{H}^{(j_{\ell})}. \tag{C13} \)

Such invariants are called intertwiners. Hence, if we have an \( p \)-valent node, the intertwiner is an element of the invariant subspace \( \text{Inv}_{SU(2)}[\mathcal{H}^{(j_1)} \otimes \cdots \otimes \mathcal{H}^{(j_p)}] \) of the tensor product space between the \( p \) irreducible representations associated to the links joining that node. However, such a procedure is possible only if some conditions necessary to have an invariant subspace are satisfied. For instance, in the case of a 3-valent node, there exists an intertwiner space only if the spin numbers \( j_1, j_2, j_3 \) labelling the representations associated to the three links satisfy the Clebsch-Gordan condition:

\[ |j_1 - j_2| \leq j_3 \leq j_1 + j_2. \tag{C14} \]

For a \( p \)-valent node (with \( p > 3 \)) the space \( \mathcal{H}^{(0)} \) can have a larger dimension and the construction consists of adding first two irreducible representations, then the third, and so on, thus giving rise to a decomposition in virtual 3-valent nodes in which virtual links are labelled by spins \( k \) satisfying the condition (C14).

Since the projector (C12) acts only on the nodes of the graph that labels the basis of \( \mathcal{H}_{\text{kin}} \), we can write the result of the action of \( \mathcal{I}_v \) on elements of \( \mathcal{H}_{\text{kin}} \) as a linear combination of products of representation matrices \( D_{n_{\text{in}} n_{\text{out}}}(h_{\ell}(A)) \) contracted with intertwiners. This leads us to give the following

**Definition:** A triplet \( (\Gamma, \vec{j}, \vec{i}) \) representing a graph \( \Gamma \) embedded in \( \Sigma \) whose \( L \) links are colored by the spins \( \vec{j} = (j_1, \ldots, j_L) \) and whose \( V \) nodes are labelled by intertwiners \( \vec{i} = (i_1, \ldots, i_V) \) is called a spin network \( S \) embedded in \( \Sigma \) associated with the graph \( \Gamma \). A spin network state \( |S\rangle \equiv |\Gamma; \vec{j}, \vec{i}\rangle \) is the cylindrical function over the spin network \( S \) associated with the graph \( \Gamma \) which can be written as

\[ \langle A|\Gamma; \vec{j}, \vec{i}\rangle = \psi_{\Gamma, \vec{j}, \vec{i}}[A] = \bigotimes_{\ell} D^{(j_{\ell})}(h_{\ell}(A)) \cdot \bigotimes_{v} i_{\ell}, \tag{C15} \]

Figure 2. Heuristic picture of a minimal chunk of quantum space: a tetrahedron dual to a single node of a spin network state. A discrete 3d quantum geometry is realised by a superposition of quantum tetrahedra, glued together by specific adjacency conditions.

where \( D^{(j_{\ell})}(h_{\ell}(A)) \) are the spin irreducible representations of the holonomy along each link and \( \cdot \) denotes the contraction with the intertwiners whose indices (hidden for simplicity) can be reconstructed from the connectivity of the graph.

These states form a complete orthonormal basis for \( \mathcal{H}^{0}_{\Gamma} \) [44], given by

\[ \mathcal{H}^{0}_{\Gamma} = \bigoplus_{j_{\ell}} \bigotimes_{v} \text{Inv}_{SU(2)}[\bigotimes_{\ell \in v} \mathcal{H}^{(j_{\ell})}] \tag{C16} \]

\[ \cong L^2(SU(2)^L/SU(2)^V) \]

for a fixed graph \( \Gamma \) with \( L \) links and \( V \) nodes. Spin network states diagonalize geometric operators such as area and volume. In particular, the face dual to each link \( \ell \) has an area proportional to the spin label \( j_{\ell} \), and each region around a node \( v \) has a volume determined by the intertwiner \( i_{\ell} \). This matches the identification of spin networks with states of quantum polyhedra dual to the nodes of the graph [31] and, more heuristically, traduces into a picture where 3d space-like surfaces are represented by a collection of “chunks” (the polyhedra dual to the nodes, see Fig. (2)) with quantized volume, which share surfaces whose area is determined by the spin of the dual link connecting them. Due to the embedding into a differentiable manifold, the algebraic data set colouring the LQG spin network graph provides a notion of quantum geometry which is at the same time discrete and relational [47].

**Appendix D: Spin Networks as Quantum Many-Body States**

A different approach along the same line, consists in starting from abstract spin network structures, with states defined independently of any embedding into a continuum manifold, hence with no reference to background notions of space, time or geometry [48]. This is the way spin networks appear in the group field theory.
Since in what follows it is important to distinguish different nodes, the possibility of defining spin network states in a more abstract, combinatorial way has been considered also within the canonical LQG approach. 14

Along this line, in particular, spin networks can be reformulated as quantum “many-body” systems, where closed graphs result from a precise gluing prescription among individual single vertex open graphs (tensors) dual to fundamental volumes of space. Let us consider a closed graph $\Gamma$ with $V$ $d$-valent vertices labelled by the index $i = 1, \ldots, V$ and denote the set of its edges by

$$L(\Gamma) = (\{1, \ldots, V\} \times \{1, \ldots, d\})^2 \quad (D1)$$

such that

$$[(ia)(ia)] \notin L(\Gamma) , \quad [(ia)(jb)] \in L(\Gamma) \quad (D2)$$

where the last condition specifies the connectivity of the graph telling us the existence of a directed edge connecting the $a$-th link at the $i$-th node to the $b$-th link at the $j$-th node, with source $i$ and target $j$. A generic cylindrical function based on the graph $\Gamma$ will be a function of the group elements $h^{ab}_{ij} \in G$ ($G \equiv SU(2)$ in LQG) assigned to each link $\ell := [(ia)(jb)] \in L(\Gamma)$.

$$\psi_T(h^{11}_{12}, h^{21}_{31}, \ldots) = \psi_T(h^{ab}_{ij}) \in \mathcal{H}_T \cong L^2(G^L/G^V), \quad (D3)$$

with $h_{ij} = h_{ji}^{-1}$ and we impose gauge invariance at each vertex $i$ of the graph, i.e.:

$$\psi_T(\{h_{ij}\}) = \psi_T(\{g_i h_{ij}^{-1} g_i^{-1}\}) \quad \forall g_i \in G. \quad (D4)$$

Consider now a new Hilbert space given by

$$\mathcal{H}_V \cong L^2(G^d \times V / G^V), \quad (D5)$$

whose generic element will be a function of $d \times V$ group elements

$$\varphi(\{g^a_i\}) = \varphi(g^1_1, \ldots, g^a_d, \ldots, g^V_1, \ldots, g^V_d) \in \mathcal{H}_V \quad (D6)$$

satisfying the gauge invariance at the vertices of the graph, i.e.: $\forall \alpha \in G,$

$$\varphi(\ldots, g^a_i, \ldots, g^b_i, \ldots) = \varphi(\ldots, \alpha_i g^a_i, \ldots, \alpha_j g^b_i, \ldots). \quad (D7)$$

As in LQG, the measure of the Hilbert space is taken to be the Haar measure. The interpretation of such functions is that each $\varphi$ is associated to a $d$-valent graph formed by $V$ disconnected components, each corresponding to a single $d$-valent vertex and $d$ 1-valent vertices, which are called open spin network vertices.

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14 The possibility of defining spin network states in a more abstract, combinatorial way has been considered also within the canonical LQG approach [50, 51].

15 Since in what follows it is important to distinguish different nodes and the links joining them, we admit a certain excessive complexity of notation using more indices to label links $(ab)$ and their source and target nodes $(ij)$. 

![Figure 3. Gluing of open spin network vertices to form a spin network closed graph.](image-url)
• $J_{ij}^{ab}$ label the representations of the group $G$ and
$D_{(j)}^{(i)}$ are the corresponding representation matrices
whose indices refer to the start and end vertex of
the edge $[(ia)(jb)]$ to which the group element $h_{ij}^{ab}$ is
attached;

• $C^{(j)}_{i{\mathcal{I}}} \Gamma$ are the normalized intertwiners for the
group $G$, attached in pairs to the vertices, resulting
from the gauge-invariance requirement, a basis of
which is labelled by additional quantum numbers
$\mathcal{I}$. These intertwiners contract all indices of both
nodes and of the representation functions, leaving
a gauge-invariant function of spin variables only.

By using a similar decomposition for the function $\varphi$, the
group averaging expression of $\psi$ in terms of $\varphi$ can be
written as

$$
\psi_{\mathcal{T}}(\{h_{ij}^{ab} = g_i^a(g_j^b)^{-1}\}) = \int \prod_{[(ia)(jb)]} d\alpha_{ij}^{ab} \sum_{\{\vec{J}_i, \vec{m}_i\}} \varphi_{\vec{J}, \vec{m}_i, \{\mathcal{I}_i\}},
$$

$$
= \sum_{\{J_{ij}^{ab}\}, \{m_j\}, \{\mathcal{I}_i\}} \varphi_{\vec{J}_i, \vec{m}_i, \{\mathcal{I}_i\}} \prod_{[(ia)(jb)]} \delta_{j_i^a j_j^b} \delta_{m_i^a, m_j^b} D_{m_i^a m_j^b}^{(J_{ij})} (g_i^a(g_j^b)^{-1}) C_{\vec{J}_i, \vec{m}_i, \{\mathcal{I}_i\}}^{(j)} (D10)
$$

from which, comparing with (D9), we get the gluing formula in spin representation

$$
\psi_{\mathcal{T}}(\{J_{ij}\}, \{\mathcal{I}_i\}) = \sum_{\{\vec{m}_i\}} \varphi_{\vec{J}_i, \vec{m}_i, \{\mathcal{I}_i\}} \prod_{[(ia)(jb)]} \delta_{j_i^a j_j^b} \delta_{m_i^a, m_j^b} .
$$

This means that LQG states can be regarded as linear
combinations of disconnected open spin network states
with additional conditions enforcing the gluing and en-
coding the connectivity of the graph. Explicitly, Eq.
(D11) shows that such conditions basically correspond
to insert intertwiners given by the identity map at the
bivalent vertices where the open links are pairwise glued.
In order to deal with graphs with an arbitrary number of
vertices, we consider the Hilbert space

$$
\mathcal{H} = \bigoplus_{V=0}^{\infty} \mathcal{H}_V . \quad (D12)
$$

Eq. (D8), or equivalently (D10), shows that there is
a correspondence between LQG states and states in
$\mathcal{H}$. This is actually more than a correspondence at the
level of sets of states since it is possible to prove that
the scalar product in $\mathcal{H}_V$ for the special class of states
corresponding to closed graphs induces the standard
LQG kinematical scalar product for cylindrical functions
$\psi_{\mathcal{T}} \in \mathcal{H}_T$ based on a fixed graph (see [48] for details).
This means that, assuming that the graph $\Gamma$ has $V$
vertices, $\mathcal{H}_T$ can be embedded into $\mathcal{H}_V$ faithfully, i.e.,
preserving the scalar product.

Still, even though they agree exactly for each $\mathcal{H}_T$,
it is important to stress the main differences between
the new Hilbert space $\mathcal{H}$ and $\mathcal{H}_V^{0}$,

1) The Hilbert space $\mathcal{H}$ in (D12) is defined by taking
the direct sum over all the Hilbert spaces $\mathcal{H}_V \supset \mathcal{H}_T$
with fixed number of vertices without introducing
any cylindrical equivalence class. As such, unlike
the LQG case, zero modes are now included in the
Hilbert space.

2) In the new Hilbert space, states associated to dif-
ferent graphs are organized in a different way w.r.t.
the LQG space. Indeed, states associated to graphs
with different number of vertices are orthogonal,
but those associated to different graphs but with the
same number of vertices are not orthogonal.

The functions $\varphi(\vec{g}_1, \ldots \vec{g}_V)$ can be understood as “many-
body” wave functions for $V$ quanta corresponding to the
$V$ open spin network vertices to which the function refers.
Indeed, each state can be decomposed into products of
“single-particle”, “single-vertex” states

$$
|\varphi\rangle = \sum_{\{\vec{x}_i\}, \ldots \vec{x}_V} \varphi_{\vec{x}_1 \cdots \vec{x}_V} |\vec{x}_1\rangle \otimes \cdots \otimes |\vec{x}_V\rangle , \quad (D13)
$$

which in the group representation reads as

$$
\varphi(g) \equiv \langle g | \varphi \rangle = \sum_{\{\vec{x}_i\}} \varphi_{\vec{x}_1 \cdots \vec{x}_V} \langle \vec{g}_1 | \vec{x}_1 \rangle \cdots \langle \vec{g}_V | \vec{x}_V \rangle \quad (D14)
$$

where the complete basis of single-vertex wave functions
is given by wave functions for individual spin network
vertices, i.e. $|\vec{x}\rangle = |J, \vec{m}, \{\mathcal{I}\}\rangle$, with

$$
|\psi_{\vec{x}}(\vec{g})\rangle = \langle \vec{g} | \vec{x}\rangle = \left( \prod_{\ell=1}^{d} D_{m_{\ell} n_{\ell}}^{(J_{\ell})} \right) C_{n_1 \cdots n_d}^{J_1 \cdots J_d} \{\mathcal{I}\} . \quad (D15)
$$
The normalization condition for the $\varphi$ is provided by

$$\int \prod_{v=1}^{V} d\vec{g} \tilde{\varphi}(\vec{g}_1, \ldots, \vec{g}_V) \varphi(\vec{g}_1, \ldots, \vec{g}_V) = \sum_{(x_v)} \tilde{\varphi}(x_v) \varphi(x_v), \quad (D16)$$

where we have used the normalization condition of single-particle wave functions

$$\int d\vec{g} \tilde{\psi}_\vec{v}(\vec{g}) \psi_{\vec{v}}(\vec{g}) = \delta_{\vec{v}, \vec{v}'}. \quad (D17)$$

The functions $\varphi$ are exactly the many-body wave functions for point particles living on the group manifold $\mathbb{G}^d$, whose classical phase space is $(T^* \mathbb{G})^d \cong (\mathbb{G} \times \mathbb{G}^*)^d$ which is also the classical phase space of a single polyhedron dual to a $d$-valent spin network vertex. The resulting picture of the microstructure of spacetime is thus based on glued pre-geometric fundamental building blocks. This is the general picture underlying the GFT formalism. This is even more evident in a 2nd quantized, Fock space reformulation to the same Hilbert space $\mathcal{H}$, where building blocks are created and annihilated and their gluing corresponds to interactions of combinatorial nature [48, 52]. Along with the examples given in Section V, also the gluing of open spin networks can be naturally understood in terms of entanglement of their spin network degrees of freedom, so that the GFT Hilbert space fits very well our pre-geometric approach to quantum spacetime and the idea of reconstructing geometry from entanglement.

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