Universal Algorithms: Beyond the Simplex

Daron Anderson
Department of Computer Science and Statistics
Trinity College Dublin
Ireland

Douglas Leith
Department of Computer Science and Statistics
Trinity College Dublin
Ireland

Editor: Kevin Murphy and Bernhard Schölkopf

Abstract

The bulk of universal algorithms in the online convex optimisation literature are variants of the Hedge (exponential weights) algorithm on the simplex. While these algorithms extend to polytope domains by assigning weights to the vertices, this process is computationally unfeasible for many important classes of polytopes where the number $V$ of vertices depends exponentially on the dimension $d$. In this paper we show the Subgradient algorithm is universal, meaning it has $O(\sqrt{N})$ regret in the antagonistic setting and $O(1)$ pseudo-regret in the i.i.d setting, with two main advantages over Hedge: (1) The update step is more efficient as the action vectors have length only $d$ rather than $V$; and (2) Subgradient gives better performance if the cost vectors satisfy Euclidean rather than sup-norm bounds. This paper extends the authors’ recent results for Subgradient on the simplex. We also prove the same $O(\sqrt{N})$ and $O(1)$ bounds when the domain is the unit ball. To the authors’ knowledge this is the first instance of these bounds on a domain other than a polytope.

Keywords: sequential decision making, regret minimisation, subgradient, online convex optimisation, Birkhoff polytope

1. Introduction

Universal algorithms for online learning are algorithms which simultaneously achieve $O(\sqrt{N})$ regret for adversarial loss sequences and $O(1)$ pseudo-regret for i.i.d loss sequences, where $N$ is the number of time steps for which the algorithm is run. In this paper we show that the lazy, anytime variant Subgradient algorithm is universal when the domain is a polytope. Namely, for antagonistic cost vectors drawn from the unit ball the regret is $O(\sqrt{N})$ and for i.i.d cost vectors in the unit ball the pseudo-regret is $O(1/\Delta)$ where $\Delta$ is the suboptimality gap (defined below). These bounds are dimension independent. To the authors’ knowledge this is the first proof of these two bounds for a domain other than the simplex. We prove similar bounds for the Euclidean unit ball. We get $O(\sqrt{N})$ antagonistic regret and i.i.d pseudo-regret $O(1/\|a\|)$ for $a$ the expected cost vector. Again
the bounds are dimension independent. These results generalise the authors’ earlier work (Anderson and Leith, 2019) establishing the universal nature of Subgradient on the simplex.

These results are significant for a number of reasons. Firstly, because Subgradient can be efficiently run on many polytopes where the number of vertices \( V \) is large relative to the dimension \( d \) (as is frequently the case, for example the cube has \( V = 2^d \) vertices). For comparison the bulk of the literature on universal algorithms focuses on variants of the Hedge algorithm (Kivinen and Warmuth, 1997) where the domain is the simplex, and provides no obvious algorithm to efficiently solve problems for many vertices. Given a problem on a \( d \)-dimensional polytope with \( V \) vertices \( \{v_1, v_2, \ldots, v_V\} \) one naive approach is to lift the problem to the \( V \)-simplex by mapping each vertex of the simplex to a vertex of the polytope. See Section 3.1 for details. The recent result of Mourtada and Gaïffas (2019) says running Hedge on the lifted problem gives regret bounds \( O(L_\infty \sqrt{\log(V)N}) \) and \( O(L_\infty^2 \log(V)/\Delta) \) for \( L_\infty = \max \{|a_n \cdot v_j| : j \leq V\} \). This lifting procedure has two drawbacks compared to Subgradient:

1. High computational cost. To run Hedge we must update action vectors of length \( V \). For example the \( O(L_\infty \sqrt{dN}) \) bound for a \( d \)-dimensional cube has little practical value because the \( O(2^d) \) complexity makes the algorithm unfeasible for even moderate values of \( d \). See Tables 1 and 2 for examples with even worse \( \Omega(d!) \) cost.

2. The procedure ignores Euclidean bounds on the cost vectors. For example given cost vectors in the Euclidean unit ball, there is no guarantee the lifted cost vectors are in the unit ball. Moreover even if the lifted vectors are in the unit ball, some dependence on dimension is inevitable since Hedge is tailored to deal instead with \( \infty \)-norm bounds. Indeed we can only use the result of Mourtada and Gaïffas (2019) by observing \( L_\infty = \|A_i\|_\infty = \max \{|a_i \cdot v_j| : j \leq V\} \).

These problems do not occur for Subgradient because (1) it only update vectors of length \( d \) and (2) is naturally suited to Euclidean bounds. The most expensive part of Subgradient is projecting onto the domain. Thus when this can be done efficiently we can solve problems on polytopes with many vertices. Moreover if the cost vectors satisfy good Euclidean bounds then we can get dimension-independent regret bounds.

A second reason our results are interesting is that the Subgradient and Hedge algorithms are popular and widely used so improved results have immediate broad application, plus earlier lines of research on universal algorithms required the development of complicated algorithms purpose-built to be universal, whereas Subgradient and Hedge are simple and predate this line of research.

Thirdly, our analysis is quite different from those existing in the literature for Hedge-type algorithms on the simplex. The proof strategy is to follow the sequence of unprojected actions and show the projected actions snap to the optimal vertex with high probability. For comparison Hedge-type algorithms can only approach the optimal vertex asymptotically. We also make use of vector concentration results which seem to be new in this context.

1.1 Related Work

There has been much recent interest in so-called universal algorithms that achieve \( O(\sqrt{N}) \) regret in the antagonistic setting but give much better performance for easier data sets. For example see Bubeck and Slivkins (2012); Zimmert and Seldin (2018); Seldin and Slivkins (2014); Wei and Luo
The above algorithms deal with the bandit setting. They only apply to the simplex and are purpose-built to be universal. The cleanest picture of what is possible is provided by Zimmert and Seldin who prove pseudo-regret bounds $\sqrt{dN \log(N)}$ and $\log(N)/\Delta$, and Auer and Chiang who show this is only possible if we deal with adversarial pseudo-regret and not adversarial expected regret.

In the full-information setting Mourtada and Gaïffas (2019) have proved the familiar Hedge (Exponential Weights) algorithm is universal. This is surprising because Hedge algorithm is particularly simple and predates the recent interest in universal algorithms. For example see Kivinen and Warmuth (1997). In the same spirit the authors (Anderson and Leith, 2019) have proved the Subgradient algorithm is universal on the simplex.

Huang et al. (2016) is the only other paper the authors know that deals with universal algorithms on polytopes and the unit ball. For polytopes and i.i.d cost vectors $a_1, a_2, \ldots$ where $a = \mathbb{E}[a_n]$ has a unique minimiser, they show Follow-the-leader gives expected regret $O(L^2 d/r^2)$. Here all $\|a_n\|_\infty \leq L_\infty$ and $r$ is the largest distance we can move the expected cost without changing the minimiser. To get a universal algorithm they use the Prod($\mathcal{A}, \mathcal{B}$) algorithm of Sani et al. (2014) to combine FTL with Subgradient. This preserves their $O(1)$ bound for the i.i.d setting but gives only a $O(\sqrt{N \log N})$ bound for the antagonistic setting.

Their analysis is quite different from our own since it focuses on the number of times the optimal vertex in hindsight changes. The fact that $a$ has a unique minimiser is needed to make this happen on expectation only finitely many times. The analysis of Prod($\mathcal{A}, \mathcal{B}$) is also unlike that used for our main result.

The largest part of Huang et al. (2016) however concerns curved domains. In particular they show Subgradient gives bounds $O(\log N)$ and $O(\sqrt{N})$ on the Euclidean ball. This is a corollary to their much more general results about FTL on strongly curved domains. We manage to improve their first bound to $O(1)$ by working from first principles.

For many special polytopes there are purpose-built efficient alternatives to the lifting procedure mentioned in the Introduction. For example Helmbold and Warmuth (2009) efficiently learn permutations; and Warmuth and Kuzmin (2008) efficiently learn $k$-element subsets of some $\{1, 2, \ldots, n\}$.

See Kalai and Vempala (2016) and the references therein for methods to efficiently learn paths on a graph. It is not obvious whether any of these methods adapt to give $O(1)$ regret in the i.i.d case.

1.2 Results and Contribution

The main novelty of this paper is an analysis of a universal algorithm that is both computationally feasible and valid on a domain other than the simplex. Our analysis has a different flavour to that for Hedge-type algorithms in the existing literature. This is due to how Subgradient can snap to the correct vertex in finite time, unlike Hedge which can only approach the correct vertex asymptotically.

Theorem 2 in Section 1 says that running Subgradient on the unit ball with cost vectors $\|a_n\| \leq L$ gives i.i.d pseudo-regret $O(L^2/\|a\|)$ for $a = \mathbb{E}[a_n]$ the expected cost vector. The bound is independent of the dimension. This is better than the $O(\log N)$ bound of Huang et al. (2016). On the other hand their bound holds for any strongly convex domain with a smooth boundary.
Theorem 8 in Section 3 says that running Subgradient on a polytope \( P \) gives i.i.d pseudo-regret \( O(D^2L^2/\Delta) \) independent of dimension. Here \( D = \max\{\|x - y\| : x, y \in P\} \) is the diameter of the polytope and the suboptimality gap \( \Delta \) is defined as follows: Let \( V \) be the vertex set and \( V^* = \arg\min\{a \cdot v : v \in V\} \) the optimisers. Choose \( v_1 \in V^* \) and \( v_2 \in \arg\min\{a \cdot v : v \in V - V^*\} \) and define \( \Delta = a \cdot (v_2 - v_1) \).

In Section 3.1 we specialise our \( O(D^2L^2/\Delta) \) bound to some particularly well-studied classed of polytopes. In Table 1 we compare our bounds for Subgradient to those for Hedge. Most examples have \( D \) depending on dimension. Under Euclidean bounds on cost vectors we find for all examples Subgradient scales better with dimension than Hedge. In particular for the (signed) permutahedron the Hedge bounds have an extra factor of \( d \) and \( d^2 \) in the antagonistic and i.i.d cases.

In Section 4 we reformulate Theorem 8 to replace the Euclidean bounds on the cost vectors with the intrinsic bounds \( \max\{|a_n \cdot (x - y)| : x, y \in P\} \leq L_\infty \). This generalises the standard \( \infty \)-norm bound for problems on the simplex. Theorem 24 says that running Subgradient on \( P \) gives i.i.d pseudo-regret of order \( O\left(D^2L^2_\infty W_2^2/\Delta\right) \). Here the width \( W \) (see Definition 20) is the number obtained by first discarding all directions perpendicular to the affine hull of \( P \) and then taking the smallest number \( w \) such that \( P \) is contained between two hyperplanes of distance \( w \) apart.

In Section 4.1 we consider the examples from Section 3.1 under the intrinsic bounds rather than Euclidean. In Table 2 we see the Simplex and Birkhoff Polytope scale better with dimension for Hedge while the permutahedrons scale slightly better for Subgradient. However for all examples other than the simplex Hedge quickly becomes unfeasible.

In Section 5 we discuss the computational cost of running Subgradient and some open problems and possible improvements.

To the authors’ knowledge the widths of the polytopes in Tables 1 and 2 do not appear elsewhere in the literature. Computing the widths is nontrivial, and we use a probabilistic counting trick famously attributed to Paul Erdős (Alon and Spencer, 2004) and suggested by David E Speyer (2019). In fact we could not find a modern treatment of the width of the simplex. See Appendix A.

**Terminology and Notation**

Throughout \( d \) is the dimension of the online optimisation problem. The cost vectors \( a_1, a_2, \ldots \in \mathbb{R}^d \) are realisations of a sequence of i.i.d random variables with each \( E[a_i] = a \). When we write \( b_1, b_2, \ldots \) for the cost vectors we make no assumptions on whether they are i.i.d or otherwise. Unless otherwise specified we assume bounds of the form \( \|a_i - a\| \leq R \) and \( \|a_i\| \leq L \) for \( \|\cdot\| \) the Euclidean norm.

In the problem setup we are given a compact convex set \( \mathcal{X} \subset \mathbb{R}^d \) called the domain or action set. On turn \( n \) we know \( b_1, b_2, \ldots, b_{n-1} \) and must select an action \( x_n \in \mathcal{X} \). In the antagonistic setting our goal is to minimise the regret \( \sum_{i=1}^N a_i \cdot (x_i - x^*) \) for the best fixed action \( x^* \in \arg\min \left\{ \sum_{i=1}^N a_i \cdot x : x \in \mathcal{X} \right\} \) in hindsight. In the i.i.d setting our goal is to minimise the pseudo-regret \( \sum_{i=1}^N a \cdot (x_i - x^*) \) for \( x^* \in \arg\min \{a \cdot x : x \in \mathcal{X}\} \).
We write $S_d$ for the $d$-simplex $\{x \in \mathbb{R}^d : x(j) \geq 0 \text{ and } x(1) + \ldots + x(d) = 1\}$. The diameter of $X$ is $\max\{\|x - y\| : x, y \in P\}$. Throughout $P \subset \mathbb{R}^d$ is a polytope. Namely the convex hull of some finite set $V = \{v_1, v_2, \ldots, v_V\}$ of vertices. See Section 4 of Gallier (2008) for the equivalent characterisation of a polytope as the solution to a set of affine inequalities.

By an affine subspace of $\mathbb{R}^d$ we mean a translation of a vector subspace. The affine hull of $A \subset \mathbb{R}^d$ is the smallest affine subspace containing $A$. The dimension of an affine subspace is the dimension of the corresponding vector subspace. The dimension of a polytope is the dimension of its affine hull. By a face of $P$ we mean the intersection of $P$ with any tangent plane. By a facet of $P$ we mean a face whose affine hull has dimension 1 less than that of the polytope.

For any function $f : X \to \mathbb{R}$ we write $\arg\min\{f(x) : x \in X\}$ for the set of minimisers. Each linear function on a polytope is minimised on some vertex. For each $a \in \mathbb{R}^d$ it follows $\arg\min\{a \cdot x : x \in P\}$ is the convex hull of $\arg\min\{a \cdot x : x \in V\}$. For any $y \in \mathbb{R}^d$ we write $P_X(y) = \arg\min\{\|x - y\| : x \in X\}$ for the Euclidean projection onto $X$. The normal cone to $X$ at $v \in X$ is the set $N_X(v) = \{u \in \mathbb{R}^d : u \cdot x \leq u \cdot v \text{ for all } x \in X\}$. Note for $u \in X$ in the interior the normal cone is empty.

### Algorithm 1: Lazy, Anytime Subgradient Algorithm

**Data:** Action set $X \subset \mathbb{R}^d$. Base point $y_1 \in \mathbb{R}^d$. Parameter $\eta > 0$.

1. select action $x_1 = P_X(y_1)$
2. pay cost $a_1 \cdot x_1$
3. for $n = 2, 3, \ldots$ do
   4.  receive $a_{n-1}$
   5.  $y_n = y_1 - \eta \left(\frac{a_1 + \ldots + a_{n-1}}{\sqrt{n} - 1}\right)$
   6.  select action $x_n = P_X(y_n)$
   7.  pay cost $a_n \cdot x_n$

Pseudo-code for the lazy, anytime variant of Subgradient is shown as Algorithm 1. It is lazy in the sense that the quantity $y_{n+1}$ that is projected is proportional to the sum $a_1 + \ldots + a_n$ of the loss vectors whereas in greedy variants $y_{n+1}$ is replaced with the object $x_n - a_n/\sqrt{n}$ that depends only on the previous action and newest cost vector. The lazy aspect of the algorithm is important since greedy variants of Subgradient are known not to be universal, see Anderson and Leith (2019) Section 4.1.

### 2. The Unit Ball

The Subgradient algorithm is among the simplest and most familiar algorithms for online linear optimisation. Subgradient has antagonistic regret $O(\sqrt{N})$. For the original proof see Zinkevich (2003). For a modern exposition see Chapter 2 of Shalev-Shwartz (2012). For a self-contained proof of the anytime case see Anderson and Leith (2019) Appendix A.
Theorem 1 Given cost vectors $b_1, b_2, \ldots, b_N$ with all $\|b_i\| \leq L$ Algorithm 1 with parameter $\eta$ has regret satisfying

$$\sum_{i=1}^{N} b_i \cdot (x_i - x^*) \leq LD + \left( \frac{1}{2\eta} \|X\|^2 + 2\eta L^2 \right) \sqrt{N}$$

for $\|X\| = \max \{ \|x - y_1\| : x \in X \}$ and $D = \max \{ \|x - y\| : x, y \in X \}$ the diameter of $X$. In particular for $y_1 \in X$ and $\eta = \|X\|/2L$ we have

$$\sum_{i=1}^{N} b_i \cdot (x_i - x^*) \leq LD + 2L\|X\|\sqrt{N} \leq 3LD\sqrt{N}.$$  

Our first main theorem says Subgradient algorithm on the Euclidean unit ball has $O(1)$ pseudo-regret in the i.i.d setting. For comparison Huang et al. (2016) get a $O(\log N)$ bound as a corollary to their more general results about strongly curved domains. Our proof proceeds from first principles hence is longer with a stronger conclusion.

Theorem 2 Suppose the cost vectors $a_1, a_2, \ldots$ are i.i.d with all $\|a_i\| \leq L$ and $\|a_i - a\| \leq R$ and $E[a_i] = a$. Suppose we run Algorithm 1 on the unit ball with base point $y_1 = 0$ and parameter $\eta > 0$. The pseudo-regret satisfies

$$E \left[ \sum_{i=1}^{\infty} a \cdot (x_i - x^*) \right] \leq 2L + \left( \frac{1}{2\eta} + 2\eta L^2 + \sqrt{2\pi R} \right) \left( 1 + \frac{2\sqrt{2R}}{\|a\|} + \frac{2\|a\|}{\|a\|} \right) + 10R^2 \|a\|$$

In particular for $\eta = 1/2L$ we have

$$E \left[ \sum_{i=1}^{\infty} a \cdot (x_i - x^*) \right] \leq (4L + \sqrt{2\pi R}) + \frac{(4L + 2\sqrt{2R})^2 + 10R^2}{\|a\|}$$

The rest of the section is the proof of Theorem 2. We consider separately an initial and final segment of the turns. First we look at the final segment that starts on turn $M = \left\lceil \frac{4\eta R^2}{b^2} + 8R^2 \right\rceil$. By performing a rotation we can assume $a = (b, 0, \ldots, 0)$ for some $b \geq 0$. Then clearly $x^* = (-1, 0, \ldots, 0)$. We keep this assumption for the remainder of the section. The first lemma shows how the error terms $\sum_{i=1}^{n} (a - a_i)$ being small leads to small regret.

Lemma 3 Under the hypotheses of Theorem 2 suppose $n \geq M$ and the error terms have $\frac{1}{\sqrt{b}} \|\sum_{i=1}^{n} (a - a_i)\| < \frac{\sqrt{\pi}}{2} b$. Then we have

$$a \cdot (x_{n+1} - x^*) \leq b \left( 1 - \sqrt{1 - \frac{4\|\varepsilon\|}{b^2} n} \right).$$

Proof Write $\varepsilon = \frac{1}{\sqrt{b}} \sum_{i=1}^{n} (a - a_i)$. First we show the point $y_{n+1}$ from Algorithm 1 has first coordinate at most $-1$ and so $x_{n+1}$ is in the lower half $\{x \in \mathbb{R}^d : \|x\|^2 = 1$ and $x_1 \leq 0\}$ of the unit
sphere. To prove this write
\[ y_{n+1} = -\frac{\eta}{\sqrt{n}} \sum_{i=1}^{n} a_i = -\eta\sqrt{n}a - \frac{\eta}{\sqrt{n}} \sum_{i=1}^{n} (a_i - a) = -\eta\sqrt{n}a + \eta \varepsilon. \]

Hence the first coordinate is 
\[-\eta\sqrt{n}b + \eta \varepsilon \leq -\eta\sqrt{n}b + \eta \| \varepsilon \| \leq -\eta\sqrt{n}b + \eta \| \varepsilon \| \leq -\eta \| \varepsilon \| \] which is less than \(-1\) since \(n \geq \frac{4}{\eta^2 \varepsilon^2}\). Since \(y_{n+1}\) is outside the ball its Euclidean projection onto the ball is
\[ \frac{y_{n+1}}{\| y_{n+1} \|} \] and so,\(x_{n+1} = -\frac{\eta}{\sqrt{n}} \sum_{i=1}^{n} a_i = -\frac{\eta}{\sqrt{n}} (\sqrt{n}a + \varepsilon) = -\sqrt{n}a + \varepsilon - \frac{\sqrt{n}b - \varepsilon_1, -\varepsilon_2, \ldots, -\varepsilon_d}{\sqrt{n}a + \varepsilon}. \]

Since \(x_{n+1} = (z_1, \ldots, z_d)\) is on the lower half of the sphere we have \(z_1 = -\sqrt{1 - z_2^2 - \ldots - z_d^2}\) and the pseudo-regret for that round is \(a \cdot (x_{n+1} - x^*) = b(z_1 + 1) = b\left(1 - \sqrt{1 - z_2^2 - \ldots - z_d^2}\right)\).

To get a bound use (1) to write
\[ z_2^2 + \ldots + z_d^2 = \varepsilon_2^2 + \ldots + \varepsilon_d^2 \leq \frac{\| \varepsilon \|^2}{\| \sqrt{n}a + \varepsilon \|^2} \leq \left(\frac{\| \varepsilon \|}{\| \sqrt{n}a \| - \| \varepsilon \|} \right)^2 = \left(\frac{\| \varepsilon \|}{\sqrt{n}b - \| \varepsilon \|} \right)^2 \]

Since \(\| \varepsilon \| \leq \frac{\eta \varepsilon}{\sqrt{2}}\) the denominator is at least \(\frac{\eta \varepsilon}{\sqrt{2}}\) and the above gives \(z_2^2 + \ldots + z_d^2 \leq \frac{4 \| \varepsilon \|^2}{\eta^2 b^2} n\). Hence \(z_1 \leq -\sqrt{1 - \frac{4 \| \varepsilon \|^2}{\eta^2 b^2} n}\) and plugging this into \(b(1 + z_1)\) we get the result. \(\blacksquare\)

The previous lemma says small error leads to small regret. The next lemma shows the error shrinks fast enough that the regret gives a convergent series.

**Lemma 4** Under the hypotheses of Theorem 2 we have \(E \left[ \sum_{n>M} \infty a \cdot (x_n - x^*) \right] \leq \frac{10 R^2}{b} \)

**Proof** Suppose \(n \geq M\) and let \(\varepsilon\) be defined as in the previous Lemma. For each \(\delta \in [0, 1]\) it is straightforward to check
\[ 1 - \sqrt{1 - \frac{4 \| \varepsilon \|^2}{\eta^2 b^2} n} < \delta \iff \| \varepsilon \| < \sqrt{\frac{2 \delta - \delta^2 b \sqrt{n}}{2}}. \]

Since \(\sqrt{2 \delta - \delta^2} \leq 1\) the right-hand-side implies \(\| \varepsilon \| \leq \frac{\sqrt{2}}{\eta^2} b\). It follows from the previous lemma that if the right-hand-side occurs we have \(a \cdot (x_{n+1} - x^*) \leq b \delta\).

Theorem 29 says \(\| \varepsilon \| < \frac{\sqrt{2 \delta - \delta^2 b \sqrt{n}}}{2}\) occurs with probability at least \(1 - 2 \exp\left(\frac{(2 \delta - \delta^2) b^2}{8 R^2 n}\right)\).

Hence the CDF \(F(\delta) = P\left(\frac{a \cdot (x_{n+1} - x^*)}{b} \leq \delta\right)\) dominates the function
\[ f(\delta) = \begin{cases} 1 - 2 \exp\left(\frac{(2 \delta - \delta^2) b^2}{8 R^2 n}\right) & 0 < \delta \leq 1 \\ 1 - 2 \exp\left(\frac{b^2}{8 R^2 n}\right) & 1 < \delta \leq 2 \\ 1 & 2 < \delta \end{cases} \]
The second line comes from plugging $\delta = 1$ into the first line. The third line comes from how $a \cdot (x_{n+1} - x^*) \leq 2b$. For the random variable $X = \frac{a \cdot (x_{n+1} - x^*)}{b}$ Lemma 16 of (Anderson and Leith, 2019) says

$$
\mathbb{E}[X] \leq \int_0^\infty (1 - F(x)) dx \leq 2 \int_0^1 \exp \left( - \frac{(2x - x^2)b^2}{2R^2n} \right) dx + 2 \exp \left( - \frac{b^2}{2R^2n} \right).
$$

For $x \in [0, 1]$ we have $x \geq x^2$ hence $2x - x^2 \geq x$ and the first integral is at most

$$
\int_0^1 \exp \left( - \frac{nb^2}{2R^2x} \right) dx = \frac{8R^2}{nb^2} \left( \exp \left( - \frac{nb^2}{2R^2} \right) - 1 \right) \leq \frac{8R^2}{nb} \exp \left( - \frac{nb^2}{2R^2} \right).
$$

It follows that

$$
\mathbb{E}[a \cdot (x_{n+1} - x^*)] \leq \frac{16R^2}{nb} \exp \left( - \frac{b^2}{2R^2n} \right) + 2b \exp \left( - \frac{b^2}{2R^2n} \right).
$$

Summing from $M = \left[ \frac{4}{\eta^2b^2} + \frac{8R^2}{b^2} \right]$ to infinity we have

$$
\mathbb{E} \left[ \sum_{n>M}^\infty a \cdot (x_n - x^*) \right] \leq \frac{16R^2}{b} \sum_{n>M}^\infty \frac{1}{n} \exp \left( - \frac{nb^2}{2R^2} \right) + 2b \sum_{n>M}^\infty \exp \left( - \frac{b^2}{2R^2n} \right)
\leq \frac{16R^2}{b} \int_M^\infty \frac{1}{x} \exp \left( - \frac{b^2}{2R^2x} \right) dx + 2b \int_M^\infty \exp \left( - \frac{b^2}{2R^2n} \right) dx
\leq \frac{16R^2}{b} \int_M^\infty \frac{1}{x} \exp \left( - \frac{b^2}{2R^2x} \right) dx + \frac{16R^2}{b} \exp \left( - \frac{b^2}{2R^2M} \right)
\leq \frac{16R^2}{b} \int_M^\infty \frac{1}{x} \exp \left( - \frac{b^2}{2R^2x} \right) dx + \frac{16R^2}{e} \frac{1}{b}
$$

By Lemma 32 in Appendix B the remaining integral is at most $\frac{\log(2)}{e}$. Hence we get

$$
\mathbb{E} \left[ \sum_{n>M}^\infty a \cdot (x_n - x^*) \right] \leq \frac{16R^2}{b} \frac{\log(2)}{e} + \frac{16}{e} \frac{R^2}{b} = \frac{16}{e} \frac{1 + \log(2)}{b} R^2 \leq \frac{10R^2}{b}.
$$

This completes the analysis of the final segment. For the initial segment we will use Theorem 1. Since the theorem refers to the regret and not the pseudo-regret we first need the following.

**Lemma 5** For $M = \left[ \frac{4}{\eta^2\|a\|^2} + \frac{8R^2}{\|a\|^2} \right]$ we have

$$
\mathbb{E} \left[ \sum_{i=1}^M (a - a_i) \cdot (x_i - x^*) \right] \leq \sqrt{2R} \left( 1 + \frac{2}{\eta\|a\|} + \frac{2\sqrt{2R}}{\|a\|} \right)
$$

**Proof** The unit ball has diameter 2. Hence Lemma 32 in Appendix B gives says the right-hand-side is at most $\sqrt{2R}/\sqrt{M}$. To bound $\sqrt{M}$ write

$$
\sqrt{M} \leq \left( 1 + \frac{4}{\eta^2\|a\|^2} + \frac{8R^2}{\|a\|^2} \right)^{1/2} \leq 1 + \sqrt{\frac{4}{\eta^2\|a\|^2} + \frac{8R^2}{\|a\|^2}} = 1 + \frac{2}{\eta\|a\|} + \frac{2\sqrt{2R}}{\|a\|}
$$

(3)
substitute the right-hand-side into $\sqrt{2\pi R\sqrt{M}}$ to get the stated bound.

Lemma 6 For $M = \left\lceil \frac{4}{\eta^2 \|a\|^2} + \frac{8R^2}{\|a\|^2} \right\rceil$ the pseudo-regret up to $M$ satisfies

$$\mathbb{E} \left[ \sum_{i=1}^{M} a \cdot (x_i - x^*) \right] \leq \left( \frac{1}{2\eta} + 2\eta L^2 + \sqrt{2\pi R} \right) \left( 1 + \frac{2}{\eta \|a\|} + \frac{2\sqrt{2}R}{\|a\|} \right).$$

Proof To bound over $n \leq M$ we use Theorem 1 to get

$$\sum_{i=1}^{M} a_i \cdot (x_i - x^*) \leq 2L + \left( \frac{1}{2\eta} + 2\eta L^2 \right) \sqrt{M}.$$ The left-hand-side is the regret and not the pseudo-regret. To get pseudo-regret add $\sum_{i=1}^{M} (a - a_i) \cdot (x_i - x^*)$ to both sides to get

$$\mathbb{E} \left[ \sum_{i=1}^{M} a \cdot (x_i - x^*) \right] \leq 2L + \left( \frac{1}{2\eta} + 2\eta L^2 \right) \sqrt{M} + \mathbb{E} \left[ \sum_{i=1}^{M} (a - a_i) \cdot (x_i - x^*) \right].$$

The left-hand-side is the pseudo-regret. The first two terms on the right are constant. Use the previous lemma to bound the final term and (3) to bound $\sqrt{M}$. We see the above is at most

$$2L + \left( \frac{1}{2\eta} + 2\eta L^2 + \sqrt{2\pi R} \right) \left( 1 + \frac{2}{\eta \|a\|} + \frac{2\sqrt{2}R}{\|a\|} \right).$$

Combining the initial and final bounds from Lemmas 6 and 4 respectively we have proved our main theorem.

Theorem 2 Suppose the cost vectors $a_1, a_2, \ldots$ are i.i.d with all $\|a_i\| \leq L$ and $\|a_i - a\| \leq R$ and $\mathbb{E}[a] = a$. Suppose we run Algorithm 1 on the unit ball with base point $y_1 = 0$ and parameter $\eta > 0$. The pseudo-regret satisfies

$$\mathbb{E} \left[ \sum_{i=1}^{\infty} a \cdot (x_i - x^*) \right] \leq 2L + \left( \frac{1}{2\eta} + 2\eta L^2 + \sqrt{2\pi R} \right) \left( 1 + \frac{2}{\eta \|a\|} + \frac{2\sqrt{2}R}{\|a\|} \right) + \frac{10R^2}{\|a\|}.$$ In particular for $\eta = 1/2L$ we have

$$\mathbb{E} \left[ \sum_{i=1}^{\infty} a \cdot (x_i - x^*) \right] \leq (4L + \sqrt{2\pi R}) + \left( \frac{4L + 2\sqrt{2}R}{\|a\|} + \frac{10R^2}{\|a\|} \right) + \frac{10R^2}{\|a\|}.$$ Taking $R = 2L$ we get the order bound

Corollary 7 Under the hypotheses of Theorem 2 running Algorithm 1 with parameter $\eta = 1/2L$ gives pseudo-regret of order $O(L^2/\|a\|)$ independent of the dimension.
3. Polytopes

Henceforth $\mathcal{P} \subset \mathbb{R}^d$ is a polytope with vertex set $\mathcal{V} \subset \mathcal{P}$. Write $D = \max\{\|x - y\| : x, y \in \mathcal{P}\}$ for the diameter of $\mathcal{P}$ and $\|\mathcal{P}\| = \max\{\|x - y_1\| : x \in \mathcal{P}\}$ for the maximum distance from $\mathcal{P}$ to the basepoint. Write $\mathcal{V}^* = \mathcal{V} \cap \text{argmin}\{a \cdot x : x \in \mathcal{P}\}$ for the set of vertices where $a$ is minimised. For each $v \in \mathcal{V} - \mathcal{V}^*$ and $v^* \in \mathcal{V}^*$ the gap $\Delta_v = a \cdot (v - v^*)$ is positive. Define $\Delta = \min\{\Delta_v : \Delta_v > 0\}$ and $\Delta_d = \max\{\Delta_v : v \in \mathcal{V}\}$.

In this section we prove the Subgradient algorithm on $\mathcal{P}$ has pseudo-regret $O(L^2/\Delta)$ in the i.i.d setting. The bounds are in terms of the Euclidean norm. In Section 3 we give more natural bounds for the polytope in question.

**Theorem 8** Let $\mathcal{P} \subset \mathbb{R}^d$ be a polytope with diameter $D$. Let $\alpha \geq 3$ be arbitrary and define $\beta = \frac{1}{3} - \frac{1}{\alpha}$. Suppose the cost vectors $a_1, a_2, \ldots$ are i.i.d with all $\|a_i\| \leq L$ and $\|a_i - a\| \leq R$ for $E[a_i] = a$. Suppose we run Algorithm 1 with domain $\mathcal{P}$ and parameter $\eta > 0$. The pseudo-regret satisfies

$$
E\left[\sum_{i=1}^{\infty} a \cdot (x_i - x^*)\right] \leq LD + \left(\frac{\|\mathcal{P}\|^2}{2\eta} + 2\eta L^2 + \frac{\sqrt{\pi}}{2} RD\right) \left(\frac{3\alpha D^2}{2\eta\Delta} + \frac{\eta\Delta}{\alpha D^2}\right) + \frac{4R^2 D^2}{\beta^2} \left(\frac{1}{\Delta_d} + \frac{2}{\Delta}\right) \exp\left(-\frac{1}{2} \left(\frac{\alpha\beta \|\mathcal{P}\|}{\eta R}\right)^2\right)
$$

for $\|\mathcal{P}\| = \max\{\|x - y_1\| : x \in \mathcal{P}\}$. In particular for $y_1 \in \mathcal{P}$ and $\eta = D/2L$ we have

$$
E\left[\sum_{i=1}^{\infty} a \cdot (x_i - v_1)\right] \leq \left(2LD + \sqrt{\frac{\pi}{2}} RD\right) \left(\frac{30LD}{\Delta} + \frac{11}{20}\right) + \frac{15R^2 D^2}{\Delta}.
$$

The rest of the section is a proof of Theorem 8. The proof is broken into several lemmas. The first follows from the definition $N_X(z) = \{u \in \mathbb{R}^d : u \cdot x \leq u \cdot z \text{ for all } x \in X\}$ of the normal cone.

**Lemma 9** Let $\mathcal{X} \subset \mathbb{R}^d$ be convex with $x \in \mathcal{X}$ and $-b \in N_X(x)$. For the tangent plane $Q = \{z \in \mathbb{R}^d : b \cdot z = b \cdot x\}$ at $v$ in the $b$-direction we have

$$
Q \cap \mathcal{X} = \{z \in \mathcal{X} : -b \in N_X(z)\} = \text{argmin}\{b \cdot z : z \in \mathcal{X}\}.
$$

The next is Proposition 2.3 (i) of Ziegler (1995).

**Lemma 10** Each face $F$ of $\mathcal{P}$ is the convex hull $F \cap \mathcal{V}$.

The next is proved in Appendix C.

**Lemma 11** The set $\text{argmin}\{a \cdot x : x \in \mathcal{P}\}$ is the convex hull of $\mathcal{V}^*$.

The fourth lemma gives a lower bound on the angle between $-a$ and all the normal directions at a suboptimal vertex. It is also proved in the appendix.
Lemma 12 For each \( v \notin \mathcal{V} - \mathcal{V}^* \) the quantity \( \theta_v = \min \left\{ \frac{a \cdot u}{\|a\|\|u\|} : u \in N_P(v) \right\} \) satisfies

\[
\theta_v \geq \frac{1/2}{1 + D^2 \|a\|^2/\Delta_v^2} = 1
\]

Hence \( \theta_v > -1 \), and the quantities \( \phi_v = \theta_v + 1 \) are positive.

By relabelling the vertices we can assume \( \mathcal{V} = \{v_1, \ldots, v_V\} \) where the gaps \( \Delta_j = \Delta_{v_j} \) have 0 = \( \Delta_1 \leq \Delta_2 \leq \cdots \leq \Delta_V \). For ease of notation assume \( \Delta_1 < \Delta_2 < \cdots < \Delta_V \). Later we remove this limitation.

The proof strategy is in principle the same as Anderson and Leith (2019). Let each \( \mathcal{V}_n \) be the convex hull of \( \{v_1, \ldots, v_n\} \) and each \( C_n = \bigcup \{N_P(x) : x \in \mathcal{V}_n\} \) be the normal cone of \( \mathcal{V}_n \). Observe \( \{v_1\} = \mathcal{V}_1 \subset \mathcal{V}_2 \subset \cdots \mathcal{V}_V = \mathcal{V} \) and \( C_1 \subset C_2 \subset \cdots \subset C_V \). We claim as \( n \) increases the vector \( y_1 - \eta \sqrt{n}a \) moves into \( C_{V-1}, C_{V-2}, \ldots, C_1 \) in turn. Hence the projection moves into \( \mathcal{V}_{V-1}, \mathcal{V}_{V-2}, \ldots, \mathcal{V}_2, \mathcal{V}_1 \) respectively.

The vector \( y_n \) in Algorithm 1 is a noisy version of \( y_1 - \eta \sqrt{n}a \). Thus when \( n \) is sufficiently large and the noise sufficiently small the projection \( x_n \) is in the convex hull of \( \{v_1, \ldots, v_j\} \) and the regret is at most \( \Delta_j \). Moreover as \( n \) increases \( y_1 - \eta \sqrt{n}a \) moves deeper into \( \mathcal{V}_j \) and the probability of the noise pushing it back out drops exponentially. The next lemma makes this precise.

Henceforth define the error terms \( \varepsilon_{n+1} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (a - a_i) \). By definition we have each \( y_{n+1} = y_1 - \eta \sqrt{n}a + \varepsilon_{n+1} \).

Lemma 13 Let \( v \in \mathcal{V} - \mathcal{V}^* \) and \( \alpha \geq 3 \) be arbitrary. Define \( \beta = \frac{1}{3} - \frac{1}{\alpha} \) and suppose

\[
n > \left( \frac{\alpha\|\mathcal{P}\|}{\eta\|a\|} \right)^2 \max \left\{ 1, \frac{1}{2\phi_v} \right\} \quad \|\varepsilon_{n+1}\| < \beta\sqrt{n}\|a\| \min \left\{ 1, \sqrt{2\phi_v} \right\}.
\]

Then \( v \) is not in the tangent plane at \( x_{n+1} \) in the \( y_{n+1} - x_{n+1} \) direction.
Proof: For ease of notation write \( x, y \) and \( \varepsilon \) instead of \( x_{n+1}, y_{n+1} \) and \( \varepsilon_{n+1} \). Recall \( x \) is the projection of \( y = y_1 - \eta \sqrt{\alpha} a + \eta \varepsilon \) onto \( P \). To see \( y \notin P \) write
\[
\|y_1 - y\| = \| \eta \sqrt{\alpha} - \varepsilon \| \geq \eta \sqrt{\alpha} \|a\| - \eta \|\varepsilon\| \geq (1 - \beta) \eta \sqrt{\alpha} \|a\| \\
\geq (1 - \beta) \alpha \|P\| = \left(2 - \sqrt{\alpha^2 + 1}\right) \alpha \|P\| \geq \|P\|
\]

Since \( \|y_1 - y\| \geq \|P\| \) we have by definition \( y \notin P \). Hence \( y - x \in N_P(x) \) and the plane \( Q = \{ z \in \mathbb{R}^d : (y - x) \cdot z = (y - x) \cdot x \} \) is tangent at \( x \).

For a contradiction suppose \( v \in Q \). In the notation of Lemma 9 write \(-b = y - x\). Then \( v \in Q \cap P \) and the lemma says \( v \in \{ z \in P : -b \in N_P(z) \} \) so \( y - x \in N_P(v) \). Then Lemma 12 says
\[
a \cdot (y - x) \|a\| \|y - x\| \geq \min \left\{ a \cdot u : u \in N_P(v) \right\} = \theta_v.
\]

To reach a contradiction it is enough to show \( a \cdot (y - x) \|a\| \|y - x\| \leq \theta_v \). To that end write
\[
a \cdot (y - x) \|a\| \|y - x\| = \frac{1}{2} \left( a \cdot \frac{y - x}{\|y - x\|} \right)^2 - 1
\]

To make the right-hand-side less than \( \theta_v \) it is enough to make \( \left| \frac{a}{\|a\|} + \frac{(y-x)}{\|y-x\|} \right| < \sqrt{2 \theta_v} \). To that end write \( y - x = X - \eta \sqrt{\alpha} a \) for \( X = \eta \varepsilon + (y_1 - x) \). Then we have
\[
a \cdot \frac{y_n - x_n}{\|y_n - x_n\|} = \frac{a}{\|a\|} + \frac{X - \eta \sqrt{\alpha} a}{\|X - \eta \sqrt{\alpha} a\|} = \frac{X}{\|X - \eta \sqrt{\alpha} a\|} + \left(1 - \frac{\eta \sqrt{\alpha} \|a\|}{\|X - \eta \sqrt{\alpha} a\|}\right) \frac{a}{\|a\|}.
\]
Taking norms the triangle inequality gives
\[
\left\| \frac{a}{\|a\|} + \frac{y_n - x_n}{\|y_n - x_n\|} \right\| \leq \left\| \frac{X}{\|X - \eta \sqrt{\alpha} a\|} \right\| + \left| 1 - \frac{\eta \sqrt{\alpha} \|a\|}{\|X - \eta \sqrt{\alpha} a\|} \right|
\]

For the second term write \( A = \eta \sqrt{\alpha} a - X \) and \( B = \eta \sqrt{\alpha} a \) and use the reverse-triangle-inequality to see the numerator is \( \|A\| - \|B\| \leq \|A\| - \|B\| \leq \|A - B\| = \|X\| \). Thus we have
\[
\left\| \frac{a}{\|a\|} + \frac{y_n - x_n}{\|y_n - x_n\|} \right\| \leq \frac{2 \|X\|}{\|X - \eta \sqrt{\alpha} a\|} \tag{4}
\]

Now we bound the denominator of the above. The first assumption implies \( \|P\| < \frac{\eta \sqrt{\alpha} \|a\|}{\alpha} \). Hence the first terms of the max and min give
\[
\|X\| \leq \eta \|\varepsilon\| + \|y_1 - x_n\| < \eta \sqrt{\alpha} \|a\| \beta + \|P\| < (\beta + 1/\alpha) \eta \sqrt{\alpha} \|a\| = \frac{\eta \sqrt{\alpha} \|a\|}{3} \tag{5}
\]
\[
\eta \sqrt{\alpha} \|a\| - \|X\| > \frac{\eta \sqrt{\alpha} \|a\|}{3} = \frac{2 \eta \sqrt{\alpha} \|a\|}{3} \tag{6}
\]
From (6) we see the denominator in (4) has
\[
\|X - \eta\sqrt{n}a\| \geq \|\eta\sqrt{n}a - \|X\|\| > \frac{2\eta\sqrt{n}\|a\|}{3},
\] (7)
and so
\[
\left\| \frac{a}{\|a\|} + \frac{(y_n - x_n)}{\|y_n - x_n\|} \right\| \leq 2\|X\| \left( \frac{3}{2\eta\sqrt{n}\|a\|} \right) = \frac{3\|X\|}{\eta\sqrt{n}\|a\|}.
\] (8)
To bound the numerator write the first assumption to see \(\|P\| < \frac{\eta\sqrt{n}\|a\|}{\alpha} \sqrt{2\phi_v} \). Hence the \(\phi_v\) terms in the assumptions give
\[
\|X\| \leq \eta\|\varepsilon_n\| + \|y_1 - x_n\| \leq \beta\eta\sqrt{n}\|a\|\sqrt{2\phi_v} + \|P\|
\]
\[
< (\beta + 1/\alpha)\eta\sqrt{n}\|a\|\sqrt{2\phi_v} = \frac{\eta\sqrt{n}\|a\|}{3} \sqrt{2\phi_v}
\] (9)
Finally from (8) and the above we conclude
\[
\left\| \frac{a}{\|a\|} + \frac{(y_n - x_n)}{\|y_n - x_n\|} \right\| \leq \frac{3}{\eta\sqrt{n}\|a\|} \frac{\eta\sqrt{n}\|a\|}{3} \sqrt{2\phi_v} = \sqrt{2\phi_v}.
\]
\[\]
Next we relate the previous lemma to the regret.

**Lemma 14** Let \(\alpha, \beta\) be as in the previous lemma. Let \(j \in \{2, 3, \ldots, V\}\) be arbitrary and suppose
\[
\|P\| = \frac{\alpha\|P\|}{\eta\|a\|} + \frac{D^2\|a\|^2}{\Delta_j^2} \leq \frac{1}{\Delta_j^2}
\]
then \(a \cdot (x_{n+1} - v_1) \leq \Delta_{j-1}^2\).

**Proof** For \(v = v_j\) the above implies the bounds from Lemma 13 hold. Thus \(v_j \notin Q\) for \(Q = \{x \in \mathbb{R}^d : (y_{n+1} - x) \cdot x = (y_{n+1} - x_{n+1}) \cdot x_{n+1}\}\) the tangent plane at \(x_{n+1}\) in the \(y_{n+1} - x_{n+1}\) direction. Moreover since \(\Delta_j < \Delta_{j+1} < \ldots < \Delta_V\) the bounds from Lemma 13 hold when \(\Delta_j\) is replaced by any of \(\Delta_{j+1}, \ldots, \Delta_V\). Hence all \(v_j, v_{j+1}, \ldots, v_V\) \(\notin Q\). Lemma 10 implies the face \(Q \cap P\) of \(P\) is contained in the convex hull of \(\{v_1, v_2, \ldots, v_j\}\). Since \(x_{n+1} \in Q\) we have \(a \cdot (x_{n+1} - v_1) \leq \Delta_{j-1}\) ■

Now we apply concentration inequalities to show the error condition in Lemma 14 is increasingly likely to hold as \(n\) grows. Let \(\alpha \geq 3\) be fixed. Define \(\beta = \frac{1}{3} - \frac{1}{\alpha}\) and
\[
N = \left( \frac{\alpha\|P\|}{\eta\|a\|} \right)^2 \left( 1 + \frac{D^2\|a\|^2}{\Delta_j^2} \right) + 1 \quad r_j = \beta\|a\| \left( 1 + \frac{D^2\|a\|^2}{\Delta_j^2} \right)^{-1/2}
\] (10)
Like before we derive separate bounds over an initial and final segment \(\{1, 2, \ldots, N\}\) and \(\{N+1, N+2, \ldots\}\) of the turns. For the final segment Theorem 29 combined with Lemma 14 gives the following bound.
Lemma 15 Let $\alpha, \beta, N$ and $r_j$ be as defined in (10). For $n > N$ and each $j \in \{2, \ldots, V\}$ we have

$$P(a \cdot (x_{n+1} - v_1) > \Delta_j - 1) \leq 2 \exp \left( - \frac{r_j^2}{2R^2} n \right).$$

In particular for $j = 2$ we have

$$P(a \cdot (x_{n+1} - v_1) > 0) \leq 2 \exp \left( - \frac{r_2^2}{2R^2} n \right).$$

We are ready to derive our bound over the final segment.

Lemma 16 Let $\alpha, \beta, N$ be as defined in (10). We have

$$\sum_{n=N}^{\infty} E[a \cdot (x_{n+1} - v_1)] \leq \frac{4R^2 D^2}{\beta^2} \left( \frac{1}{\Delta_d} + \frac{2}{\Delta} \right) \exp \left( - \frac{1}{2} \left( \frac{\alpha \beta \|P\|}{\eta R} \right)^2 \right).$$

Proof Lemma 15 says the complementary CDF $F(t) = P(a \cdot (x_{n+1} - v_1) > t)$ is dominated by the piecewise function

$$f(x) = \begin{cases} 
2 \exp \left( - \frac{r_2^2}{2R^2} n \right) & 0 < x \leq \Delta_2 \\
2 \exp \left( - \frac{r_k^2}{2R^2} n \right) & \Delta_{k-1} < x \leq \Delta_k \text{ with } k \geq 3 \\
0 & \Delta_V < x 
\end{cases}$$

Lemma 16 of Anderson and Leith (2019) says $E[a \cdot (x_{n+1} - v_1)] \leq \int_0^\infty F(t) dt$ and so

$$E[a \cdot (x_{n+1} - v_1)] \leq \int_0^\infty f(t) dt = \int_0^{\Delta_V} f(t) dt = 2\Delta_2 \exp \left( - \frac{r_2^2}{2R^2} n \right) + 2 \sum_{k=3}^{V} (\Delta_k - \Delta_{k-1}) \exp \left( - \frac{r_k^2}{2R^2} n \right).$$

Now sum from $N$ to infinity to see $\sum_{n=N}^{\infty} E[a \cdot (x_{n+1} - v_1)]$ is at most

$$2\Delta_2 \sum_{n=N}^{\infty} \exp \left( - \frac{r_2^2}{2R^2} n \right) + 2 \sum_{n=N}^{\infty} \sum_{k=3}^{V} (\Delta_k - \Delta_{k-1}) \exp \left( - \frac{r_k^2}{2R^2} n \right) \leq 2\Delta_2 \int_{N-1}^{\infty} \exp \left( - \frac{r_2^2}{2R^2} x \right) dx + 2 \int_{N-1}^{\infty} \sum_{k=3}^{V} (\Delta_k - \Delta_{k-1}) \exp \left( - \frac{r_k^2}{2R^2} x \right) dx$$

$$= 4R^2 \frac{\Delta_2}{r_2^2} \exp \left( - \frac{r_2^2}{2R^2} (N - 1) \right) + 4R^2 \sum_{k=3}^{V} \frac{\Delta_k - \Delta_{k-1}}{r_k^2} \exp \left( - \frac{r_k^2}{2R^2} (N - 1) \right)$$

Since $\Delta_2 \leq \Delta_3 \leq \ldots \leq \Delta_V$ we have $r_2 \leq r_3 \leq \ldots \leq r_V$ and the above implies
\[
\sum_{n=N}^{\infty} \mathbb{E}[a \cdot (x_{n+1} - v_1)] \leq 4R^2 \left( \frac{\Delta_2}{r_2^2} + \frac{V}{\sum_{k=3}^N (\Delta_k - \Delta_{k-1})} \right) \exp \left( - \frac{r_2^2}{2R^2} (N - 1) \right)
\]

\[
\leq 4R^2 \left( \frac{\Delta_2}{r_2^2} + \frac{V}{\sum_{k=3}^N (\Delta_k - \Delta_{k-1})} \right) \exp \left( - \frac{r_2^2}{2R^2} \left( \frac{\alpha \|P\|}{\eta |a|} \right)^2 \left( 1 + \frac{D^2 |a|^2}{\Delta^2} \right) \right)
\]

\[
= 4R^2 \left( \frac{\Delta_2}{r_2^2} + \frac{V}{\sum_{k=3}^N (\Delta_k - \Delta_{k-1})} \right) \exp \left( - \frac{1}{2} \left( \frac{\alpha \|P\|}{\eta R} \right)^2 \right)
\]

where the last line follows from expanding the definition (10) of \( r_2 \) and cancelling terms. Now expand each \( r_k \) to see the above is

\[
\frac{4R^2}{\beta^2} \left( \frac{\Delta_2}{2} + \frac{D^2 |a|^2}{\Delta_2^2} \right) + \frac{V}{\sum_{k=3}^N (\Delta_k - \Delta_{k-1})} \exp \left( - \frac{1}{2} \left( \frac{\alpha \|P\|}{\eta R} \right)^2 \right)
\]

\[
= \frac{4R^2}{\beta^2} \left( \frac{\Delta_2}{2} + \frac{D^2}{\Delta_2} + \frac{D^2 V}{\sum_{k=3}^N \Delta_k - \Delta_{k-1}} \right) \exp \left( - \frac{1}{2} \left( \frac{\alpha \|P\|}{\eta R} \right)^2 \right)
\]

\[
= \frac{4R^2}{\beta^2} \left( \frac{\Delta V}{\|a\|^2} + D^2 \left( \frac{1}{\Delta_2} \frac{\Delta_3 - \Delta_2}{\Delta_3^2} + \ldots + \frac{\Delta V - \Delta V - \Delta V}{\Delta V} \right) \right) \exp \left( - \frac{1}{2} \left( \frac{\alpha \|P\|}{\eta R} \right)^2 \right)
\]

Lemma 17 of (Anderson and Leith, 2019) says the sum involving \( \Delta_2, \ldots, \Delta V \) is at most \( \frac{2}{\Delta_2} \). Hence the entire expression is at most

\[
\frac{4R^2}{\beta^2} \left( \frac{\Delta V}{\|a\|^2} + \frac{2D^2}{\Delta} \right) \exp \left( - \frac{1}{2} \left( \frac{\alpha \|P\|}{\eta R} \right)^2 \right).
\]

To see \( \frac{\Delta V}{\|a\|^2} \leq \frac{D^2}{\Delta} \) recall \( \Delta V = a \cdot (v_V - v_1) \leq \|a\| D \) and so \( \frac{1}{\|a\|} \leq \frac{D}{\Delta V} \). Hence the first term is at most \( \frac{\Delta V}{\|a\|^2} \leq \frac{D^2}{\Delta V} = \frac{D^2}{\Delta_2} \).

Next we derive a bound over the initial segment.

**Lemma 17** Let \( \alpha, \beta, N \) as defined in (10). We have

\[
\mathbb{E} \left[ \sum_{i=1}^N a \cdot (x_i - v_1) \right] \leq LD + \left( \frac{\|P\|^2}{2\eta} + 2\eta L^2 + \sqrt{\frac{\pi}{2} RD} \right) \left( \frac{3}{2} \frac{\alpha D^2}{\eta \Delta} + \frac{\eta \Delta}{\alpha D^2} \right).
\]

**Proof** Theorem 1 says

\[
\sum_{i=1}^N a_i \cdot (x_i - v_1) \leq LD + \left( \frac{\|P\|^2}{2\eta} + 2\eta L^2 \right) \sqrt{N}.
\]
By Lemma 32 in Appendix B we have
\[ E \left[ \sum_{i=1}^{N} (a - a_i) \cdot (x_i - v_1) \right] \leq \sqrt{\frac{\pi}{2}} RD \sqrt{N}. \]

Adding the two lines together and taking expectation we get
\[ E \left[ \sum_{i=1}^{N} a \cdot (x_i - v_1) \right] \leq LD + \left( \frac{\|P\|^2}{2\eta} + 2\eta L^2 + \sqrt{\frac{\pi}{2}} RD \right) \sqrt{N}. \quad (11) \]

It remains to bound \( \sqrt{N} \). The definition (10) says
\[ \sqrt{N} \leq \sqrt{\left( \frac{\alpha D}{\eta \|a\|} \right)^2 \left( 1 + \frac{D^2 \|a\|^2}{\Delta^2} \right) + 2}. \]

By concavity we have \( \sqrt{x + 2} \leq \sqrt{x} + \frac{2}{2\sqrt{x}} = \sqrt{x} + \frac{1}{\sqrt{x}} \) and so
\[ \sqrt{N} \leq \frac{\alpha D}{\eta \|a\|} \sqrt{1 + \frac{D^2 \|a\|^2}{\Delta^2} + \frac{\|a\|}{\alpha D} \left( 1 + \frac{D^2 \|a\|^2}{\Delta^2} \right)^{-1/2}} \]
\[ \leq \frac{\alpha D}{\eta \|a\|} \sqrt{1 + \frac{D^2 \|a\|^2}{\Delta^2} + \frac{\|a\|}{\alpha D} \left( \frac{D^2 \|a\|^2}{\Delta^2} \right)^{-1/2}} \]
\[ = \frac{\alpha D}{\eta \|a\|} \sqrt{1 + \frac{D^2 \|a\|^2}{\Delta^2} + \frac{\|a\|}{\alpha D \Delta^2}}. \]

Again by concavity square-root has
\[ \sqrt{1 + \frac{D^2 \|a\|^2}{\Delta^2}} \leq \frac{D \|a\|}{\Delta} + \frac{\Delta}{2D \|a\|}. \]

Hence we have
\[ \sqrt{N} \leq \frac{\alpha D^2}{\eta \Delta} + \frac{\alpha \Delta}{2\eta \|a\|^2} + \frac{\eta \Delta}{\alpha D^2}. \]

To remove the dependence on \( \|a\| \) recall \( \Delta = a \cdot (v_2 - v_1) \leq \|a\| D \) and so \( \frac{1}{\|a\|} \leq \frac{D}{\Delta} \). Hence the middle term is at most \( \frac{\alpha D^2}{2\eta \Delta} \). Going back to (11) we get
\[ E \left[ \sum_{i=1}^{N} a \cdot (x_i - v_1) \right] \leq LD + \left( \frac{\|P\|^2}{2\eta} + 2\eta L^2 + \sqrt{\frac{\pi}{2}} RD \right) \left( \frac{3 \alpha \|P\|^2}{2 \eta \Delta} + \frac{\eta \Delta}{\alpha D^2} \right). \]

In the special case \( \Delta_2 < \Delta_3 < \ldots < \Delta_V \) The main theorem follows from combining the bounds in Lemma 17 and 16 over the initial and final segments. In the more general case the proof is the same, except we must deal with the distinct elements \( \Delta(2) < \Delta(3) < \ldots < \Delta(V') \) of \( \{\Delta_2, \Delta_3, \ldots, \Delta_V\} \) rather than the gaps themselves.
Theorem 8 Let $\mathcal{P} \subset \mathbb{R}^d$ be a polytope with diameter $D$. Let $\alpha \geq 3$ be arbitrary and define $\beta = \frac{1}{3} - \frac{1}{\alpha}$. Suppose the cost vectors $a_1, a_2, \ldots$ are i.i.d with all $\|a_i\| \leq L$ and $\|a_i - a\| \leq R$ for $E[a_i] = a$. Suppose we run Algorithm 1 with domain $\mathcal{P}$ and parameter $\eta > 0$. The pseudo-regret satisfies

$$
\mathbb{E} \left[ \sum_{i=1}^{\infty} a \cdot (x_i - x^*) \right] \leq LD + \left( \frac{\|\mathcal{P}\|^2_{2\eta}}{2\eta} + 2\eta L^2 + \sqrt{\frac{\pi}{2}} RD \right) \left( \frac{3\alpha D^2}{2\eta D} + \frac{\eta D}{\alpha D^2} \right) + \frac{4R^2D^2}{\beta^2} \left( \frac{1}{\Delta_d} + \frac{2}{\Delta} \right) \exp \left( \frac{1}{2} \left( \frac{\alpha \beta \|\mathcal{P}\|}{\eta R} \right)^2 \right)
$$

(12)

for $\|\mathcal{P}\| = \max\{\|x - y_1\| : x \in \mathcal{P}\}$. In particular for $y_1 \in \mathcal{P}$ and $\eta = D/2L$ we have

$$
\mathbb{E} \left[ \sum_{i=1}^{\infty} a \cdot (x_i - v_1) \right] \leq \left( 2LD + \sqrt{\frac{\pi}{2}} RD \right) \left( \frac{30LD}{\Delta} + \frac{1}{3} \right) + \frac{15R^2D^2}{\Delta}.
$$

Proof The first bound comes from combining Lemmas 16 and 17. To get the second bound observe any such $y_1$ gives $\|\mathcal{P}\| = D$. First first round $\frac{1}{\Delta_d}$ up to $\frac{1}{\Delta}$. Then plug in $\|\mathcal{P}\| = D$ and $\eta = D/2L$ and simplify to get

$$
LD + \left( 2LD + \sqrt{\frac{\pi}{2}} RD \right) \left( \frac{3\alpha LD}{\Delta} + \frac{\Delta}{2\alpha DL} \right) + \frac{12R^2D^2}{\beta^2\Delta} \exp \left( -2 \left( \frac{\alpha \beta L}{\eta R} \right)^2 \right).
$$

(13)

Since $\Delta = a \cdot (v_2 - v_1) \leq LD$ we can bound the term $\frac{\Delta}{2\alpha DL} \leq \frac{1}{2\alpha}$. Since we can take $R = 2L$ we can replace the exponent $2 \left( \frac{\alpha \beta L}{\eta R} \right)^2$ with $\frac{\alpha^2 \beta^2}{2}$. It follows (13) is at most

$$
LD + \left( 2LD + \sqrt{\frac{\pi}{2}} RD \right) \left( \frac{3\alpha D}{\Delta} L + \frac{1}{2\alpha} \right) + \frac{12R^2D^2}{\beta^2\Delta} \exp \left( -\frac{\alpha^2 \beta^2}{2} \right).
$$

The bound is difficult to optimise algebraically. One parameter that gives coefficients of the same order is $\alpha = 10$. Then $\beta = \frac{1}{3} - \frac{1}{10} = \frac{7}{30}$ and $\alpha \beta = \frac{7}{3}$ and the above becomes

$$
LD + \left( 2LD + \sqrt{\frac{\pi}{2}} RD \right) \left( \frac{30LD}{\Delta} + \frac{1}{20} \right) + 12 \left( \frac{30}{7} \right)^2 \frac{R^2D^2}{\Delta} \exp \left( -\frac{49}{18} \right) \leq \left( 2LD + \sqrt{\frac{\pi}{2}} RD \right) \left( \frac{30LD}{\Delta} + \frac{11}{20} \right) + \frac{15R^2D^2}{\Delta}
$$

Since Theorem 8 holds for $R = 2L$ we have the order bound.
Corollary 18  Under the hypotheses of the second part of Theorem 8 running Algorithm 1 with base point \( y_1 \in \mathcal{P} \) and parameter \( \eta = D/2L \) gives pseudo-regret of order \( O(L^2 D^2 / \Delta) \) independent of the dimension.

3.1 Examples with Euclidean Bounds

In this section we look at some well-studied classes of polytopes. We compare our regret bounds for Subgradient to those obtained by lifting the problem and running Hedge.

| Poltope | \( D \) | \( V \) | \( L^\infty \) | Subgradient | Hedge |
|---------|--------|--------|--------|-------------|-------|
| \( d \)-Simplex | 2 | \( d \) | \( L \) | \( L \sqrt{N} \) | \( L^2 / \Delta \) |
| | | | | \( L \sqrt{\log(d) N} \) | \( L^2 \log(d) / \Delta \) |
| \( d \)-Cube | \( 4d \) | \( 2^d \) | \( L \sqrt{d} \) | \( L \sqrt{d N} \) | \( L^2 d / \Delta \) |
| | | | | \( Ld \sqrt{N} \) | \( L^2 d^2 / \Delta \) |
| \( \mathcal{B}(n) \) | \( 2n \) | \( n! \) | \( L \sqrt{n} \) | \( L n \sqrt{\log(n) N} \) | \( L^2 n^2 \log(n) / \Delta \) |
| | | | | \( L \) | \( L^2 n^2 \log(n) / \Delta \) |
| \( \mathcal{P}(d) \) | \( d^3 / 3 \) | \( d! \) | \( L d^{3/2} \) | \( L d^{3/2} \sqrt{N} \) | \( L^2 d^3 / \Delta \) |
| | | | | \( L d^{5/2} \sqrt{\log(d) N} \) | \( L^2 d^5 \log(d) / \Delta \) |
| \( \mathcal{P}_\pm(d) \) | \( 4d^3 / 3 \) | \( 2^d d! \) | \( L d^{3/2} \) | \( L d^{3/2} \sqrt{N} \) | \( L^2 d^3 / \Delta \) |
| | | | | \( L d^{5/2} \sqrt{\log(d) N} \) | \( L^2 d^5 \log(d) / \Delta \) |

Table 1: Comparison of order bounds for Subgradient and Hedge under the Euclidean bounds \( \|a_n\| \leq L \) on cost vectors.

First we describe the lifting procedure in detail: Suppose we have a problem on a polytope \( \mathcal{P} \subset \mathbb{R}^d \) with vertices \( \{v_1, v_2, \ldots, v_V\} \). We will define an auxiliary problem on the \( V \)-simplex. Let \( \phi : \mathbb{R}^V \to \mathbb{R}^d \) be the unique linear map with each \( \phi e_j = v_j \). For cost vectors \( a_1, a_2, \ldots \in \mathbb{R}^d \) define the auxiliary cost vectors \( A_1, A_2, \ldots \in \mathbb{R}^V \) by each \( A_i(j) = a_i \cdot v_j \). Run Hedge on the auxiliary problem to get actions \( y_1, y_2, \ldots \) in the \( V \)-simplex.

Since the auxiliary problem space has dimension \( V \) the results of Mourtada and Gaïffas (2019) say \( y_1, y_2, \ldots \) give \( O(L^\infty \sqrt{\log(V) N}) \) regret in the antagonistic case and pseudo-regret \( O(L^2_\infty \log(V) / \Delta) \) in the i.i.d case. Here \( L_\infty = \max\{|A_i(j)| : i \leq N, j \leq V\} \).

To get a bound for the original problem observe the auxiliary regret is \( \sum_{i=1}^N A_i \cdot (y_i - e_j) \). By linearity this equals \( \sum_{i=1}^N a_i \cdot (\phi y_i - v_j) \). We conclude the action sequence \( x_n = \phi y_n \) in the original
problem satisfies the same two regret bounds. The Hedge bounds in Table 1 refer to playing this sequence. To bound \( L_\infty \) in terms of the original problem write \( |A_i(j)| = |a_i \cdot v_j| \leq \|a_i\|\|v_j\| \leq LF \) for \( L = \max\{\|a_i\| : i \leq N\} \) and \( F = \max\{\|v_j\| : j \leq d\} \).

In Table 1 the cost vectors are assumed to satisfy the Euclidean bound \( \|a_n\| \leq L \). The regret bounds for Subgradient come from our Theorems 1 and 8. Note the theorems are dimension-independent but the later examples have the diameter \( D \) depending on dimension. Hence \( d \) appears in those bounds.

In the first column \( D \) is the diameter of the polytope. For the first three examples \( D \) is exact while for \( P(d) \) and \( P_\pm(d) \) it is exact as \( d \to \infty \). In the second column \( V \) is the number of vertices. In the first three examples \( L_\infty \) is exact. For \( P(d) \) and \( P_\pm(d) \) the exact value is \( L_\infty = L \sqrt{\frac{d(d+1)(2d+1)}{6}} \) which is obtained (Weisstein) using the formula \( \sum_{n=1}^{d} n^2 = \frac{d(d+1)(2d+1)}{6} \). The bounds in the last two columns are order-bounds.

In all examples Subgradient has a weaker dependence on dimension than Hedge. This is because \( d \) affects the Hedge bounds twice. First through the explicit \( \log(V) \) factor and second through \( L_\infty = L \max\{\|v_j\| : j \leq V\} \) which is dimension dependent in the later examples. For Subgradient the dimension only contributes once.

4. Intrinsic Bounds on the Cost Vectors

The bound in Theorems 1 and 8 fall short in the following case: Suppose \( P \subseteq \mathbb{R}^3 \) is contained in the subspace \( \mathbb{R}^2 \times \{0\} \) and the cost vectors \( a_n = (0,0,1) \) are perpendicular to the subspace. Then each cost vector is constant over the domain and the regret minimisation problem is trivial. However this is not reflected in the \( O(LD\sqrt{N}) \) and \( O(L^2D^2/\Delta) \) bounds from the theorems since \( L = \max \|a_n\| = 1 \) gives nontrivial bounds.

The solution in this case is to replace each \( a_n \) with the zero vector, and observe the regret and Subgradient actions are unchanged. Thus we can apply our theorems to the modified problem with \( L = 0 \) to get the correct bound.

To deal with the general case we introduce the following bounds

\[
\max\{|a_n \cdot (x - y)| : x, y \in P\} \leq L_\infty \quad \max\{|(a_n - a) \cdot x| : x, y \in P\} \leq R_\infty \quad (14)
\]

In Section 3 we considered Euclidean bounds on the cost vectors. For comparison the bounds (14) generalise the standard \( \infty \)-norm bounds for Hedge. For example the first bound is equivalent to each \( |a_n(k) - a_n(j)| \leq L_\infty \). Since the Hedge actions are unchanged by translating the cost vectors we can replace each \( a_n(k) \) with \( a_n(k) - \frac{\max_j a_n(j) - \min_j a_n(j)}{2} \) to get \( \|a_n\|_\infty \leq L_\infty/2 \).

If the simplex is replaced with a general polytope \( P \) the generalisation of the above is to replace each \( a_n \) with \( a_n + c_n \) where each \( c_n \) is perpendicular to \( P \). The strategy for this section is to convert (14) into Euclidean bounds for \( \|a_n + c_n\| \) and where \( c_n \) is chosen to make the new cost vectors parallel to \( P \). We prove this does not change the regret or actions, and then apply Theorem 8 to the new cost vectors. First we give a geometric interpretation of the bounds (14).
Suppose \( \mathcal{P} = [-1, 1]^2 \) is the unit square. The quantity \( \max\{|a_n \cdot (x-y)| : x, y \in \mathcal{P} \} = 2(\|a_n(1)\|_\infty + \|a_n(2)\|_\infty) \) is achieved by the vertices \( x = (\text{sign } a_n(1), \text{sign } a_n(2)) \) and \( y = -x \). Hence the first bound in (14) becomes the 1-norm bound \( \|a_n\|_1 \leq \frac{L_\infty}{2} \). More generally if \( \mathcal{P} \) is the rectangle \([-\alpha, \alpha] \times [-\beta, \beta]\), the set of allowed cost vectors is the 1-ball rescaled to height \( \frac{L_\infty}{\beta} \) and width \( \frac{L_\infty}{\alpha} \). If \( \mathcal{P} \) is wide in some direction we allow only cost vectors with small components in that direction. Conversely if \( \mathcal{P} \) is narrow in some direction we allow cost vectors with large components in that direction. This suggests the definition of the direction in which \( \mathcal{P} \) is narrowest.

**Definition 19** Let \( \mathcal{P} \subset \mathbb{R}^d \) be a polytope with interior. For each \( \ell \in \mathbb{R}^d \) the set \( \ell \cdot \mathcal{P} = \{ \ell \cdot p : p \in \mathcal{P} \} \) is an interval \([x, y]\) for some \( x, y \in \mathbb{R} \). Define \( W_\ell = |x - y| \) and define the width of \( \mathcal{P} \) as \( W = \min\{W_\ell : \|\ell\| = 1\} \).

The width of \( \mathcal{P} \) is the smallest distance \( W \) such that \( \mathcal{P} \) can be sandwiched between two parallel hyperplanes distance \( W \) apart. To the authors’ knowledge the notion of width does not appear in the existing optimisation literature to describe the shape of an action set. It appears elsewhere, for example in the study of mean widths of simplices (see Litvak (2018) and the references within); discrete geometry (Barvinok, 2017); and variants of Tarski’s plank problem about covering a given convex set with copies of some prescribed shapes (see Bezdek (2013) and the references within).

Since the simplex is contained in the infinitely thin plane \( \sum_{j=1}^{d} x(i) = 1 \), Definition 19 suggests it should have width zero. This definition is not useful here, since as mentioned earlier our goal is to ignore the perpendicular directions, as they have no bearing on the regret. The useful definition is the following.

**Definition 20** Let \( \mathcal{P} \subset \mathbb{R}^d \) be a polytope with affine hull \( U + t \) for some \( t \in \mathbb{R}^d \) and vector subspace \( U \subset \mathbb{R}^d \). Define the width of \( \mathcal{P} \) as \( W = \min\{W_\ell : \ell \in U \text{ and } \|\ell\| = 1\} \).

Definitions 19 and 20 are equivalent if \( U = \mathbb{R}^d \). In particular this holds if \( \mathcal{P} \) has interior. Since \( \mathcal{P} \) always has interior in its affine hull, the meaning of Definition 20 is to translate \( \mathcal{P} \) to contain the origin, discard all directions perpendicular to \( U \), and apply Definition 19 to \( \mathcal{P} \) as a subset of \( U \) rather than \( \mathbb{R}^d \).

Below are examples of widths of familiar polytopes. See Appendix A for proofs.

**Examples**

1. The cube \([-1, 1]^d\) has width 2.

2. The \(d\)-simplex \( \{x \in \mathbb{R}^d : \text{all } x(j) \geq 0 \text{ and } x(1) + \ldots + x(d) = 1\} \) has width \( 2/\sqrt{d} \) for \( d \) even. For \( d \) odd the width is \( 2/\sqrt{d} \) as \( d \to \infty \).

3. The Birkhoff Polytope \( \mathcal{B}(n) \) is the set of nonnegative \( n \times n \) matrices with all row and column sums equal to 1. Equivalently \( \mathcal{B}(n) \) is the convex hull of the \( n! \) permutation matrices. The width is at least \( 2/\sqrt{n-1} \).
The permutahedron $\mathcal{P}(d)$ is the set of vectors $x \in \mathbb{R}^d$ with entries $\{x(1), \ldots, x(d)\} = \{1, 2, \ldots, d\}$. Equivalently $\mathcal{P}(d)$ is the convex hull of $\{(\sigma(1), \ldots, \sigma(d)) : \sigma \in S_d\}$ for $S_d$ the permutation group. The width satisfies

$$W \geq \sqrt{\frac{5d^2 + 8d + 4}{6}}$$

$$\liminf_{d \to \infty} \frac{W}{d} \geq \sqrt{5}/6.$$

The signed permutahedron $\mathcal{P}_\pm$ is the convex hull of the vectors $(\pm \sigma(1), \ldots, \pm \sigma(d))$ for all choices of signs and permutation $\sigma \in S_d$. The width satisfies

$$W \geq 2\sqrt{\frac{2d^2 + 3d + 1}{6}}$$

$$\liminf_{d \to \infty} \frac{W}{d} \geq \sqrt{4}/3.$$

Now we begin to relate the width to Euclidean bounds.

**Lemma 21** Suppose the polytope $\mathcal{P}$ has affine hull $U + t$. Suppose the vector $c \in U$ has $\max\{|c \cdot (x - y)| : x, y \in \mathcal{P}\} \leq L$. Then $\|c\| \leq L/W$.

**Proof** Since $c/\|c\| = \ell$ is a unit vector we have

$$W_\ell = \max\{|\ell \cdot (x - y)| : x, y \in \mathcal{P}\} = \max\left\{\frac{1}{\|c\|} \cdot \frac{1}{\|c\|} \cdot (x - y) : x, y \in \mathcal{P}\right\}$$

$$= \frac{1}{\|c\|} \max\{|c \cdot (x - y)| : x, y \in \mathcal{P}\} \leq \frac{L}{\|c\|}.$$ But by definition $W \leq W_\ell$. Hence $W \leq L/\|c\|$ and $\|c\| \leq L/W$ as required. \qed

We wish to apply Lemma 21 where $c$ is a cost vector. There is no reason to believe the cost vectors are in $U$ as assumed in the lemma. Hence we must show the actions are unchanged if we replace each cost vector with its projection onto $U$. This amounts to showing the projection onto a convex set factors through the projection onto its affine hull.

**Lemma 22** Suppose the convex set $X \subset \mathbb{R}^d$ has affine hull $U + t$. For each $p \in \mathbb{R}^d$ we have $P_X(p) = P_X(P_U(p))$.

**Proof** Since rotations commute with projections we can assume $U = \{x \in \mathbb{R}^d : x_1 = 0\}$. Write $p = (p_1, \ldots, p_d)$ and $t = (t_1, \ldots, t_d)$. Clearly $P_U(p) = (0, p_2, \ldots, p_d)$. Since $X \subset U + t$ we have $x_1 = t_1$ for all $x \in X$. Hence $P_X(p)$ is the unique minimiser over $x \in X$ to

$$\|p - x\|^2 = \sum_{j=1}^{d}(p_j - x_j)^2 = (p_1 - t_1)^2 + \sum_{i=2}^{d}(p_i - x_i)^2.$$ Since $P_U(p) = (q_1, q_2, \ldots, q_d) = (0, p_2, \ldots, p_d)$ we see $P_X(P_U(p))$ is the unique minimiser over $x \in X$ to

$$\|P_U(p) - x\|^2 = \sum_{j=1}^{d}(q_j - x_j)^2 = t_1^2 + \sum_{i=2}^{d}(p_i - x_i)^2.$$
Since \( P_X(p), P_X(P_U(p)) \in U \) they have first coordinate \( t_1 \). The other coordinates are obtained by minimising \( \sum_{i=2}^{d} (p_i - x_i)^2 \) over the projection of \( X \) onto the last \( d - 1 \) coordinates. Hence we have \( P_X(p) = P_X(P_U(p)) \).

**Lemma 23** Suppose the domain \( X \) has affine hull \( U + t \). Let \( c_1, c_2, \ldots \) be the projections of the cost vectors \( b_1, b_2, \ldots \) onto \( U \). The actions chosen by Algorithm 1 given \( c_1, c_2, \ldots \) are the same as those given \( b_1, b_2, \ldots \).

**Proof** Given cost vectors \( b_1, b_2, \ldots \) Algorithm 1 selects actions

\[
x_{n+1} = P_X \left( y_1 - \eta \frac{b_1 + \ldots + b_n}{\sqrt{n}} \right).
\]

Lemma 22 says the right-hand-side is unchanged if we replace the argument with its projection onto \( U \). Since projection onto a vector subspace is a linear function we have

\[
x_{n+1} = P_X \left( P_U(y_1) - \eta \frac{P_U(b_1) + \ldots + P_U(b_n)}{\sqrt{n}} \right) = P_X \left( P_U(y_1) - \eta \frac{c_1 + \ldots + c_n}{\sqrt{n}} \right).
\]

Since all \( c_i \in U \) we have \( c_i = P_U(c_i) \). Hence the above equals

\[
P_X \left( P_U(y_1) - \eta \frac{P_U(c_1) + \ldots + P_U(c_n)}{\sqrt{n}} \right) = P_X \circ P_U \left( y_1 - \eta \frac{c_1 + \ldots + c_n}{\sqrt{n}} \right).
\]

Use Lemma 22 to remove the \( P_U \) from the above and get

\[
x_{n+1} = P_X \left( y_1 - \eta \frac{c_1 + \ldots + c_n}{\sqrt{n}} \right).
\]

These are just the actions given \( c_1, c_2, \ldots \) as required.

Now combine Lemmas 21 and 23 with Theorem 8 to get a pseudo-regret bound in terms of the intrinsic bounds (14) on the cost vectors.

**Theorem 24** Let \( P \subset \mathbb{R}^d \) be a polytope with diameter \( D \) and width \( W \). Suppose the cost vectors \( a_1, a_2, \ldots \) are i.i.d with all \( |a_n \cdot (x - y)| \leq L_\infty \) and \( |(a_n - a) \cdot (x - y)| \leq R_\infty \) for \( x, y \in P \). Then Algorithm 1 with domain \( P \) and \( y_1 \in P \) and \( \eta = DW/2L \) gives pseudo-regret bound

\[
\mathbb{E} \left[ \sum_{i=1}^{\infty} a \cdot (x_i - v_1) \right] \leq \left( \frac{2L_\infty D}{W} + \sqrt{\frac{\pi RD}{2W}} \right) \left( \frac{30L_\infty D}{W \Delta} + \frac{1}{3} \right) + \frac{15R_\infty^2 D^2}{W^2 \Delta}.
\]

**Corollary 25** Under the hypotheses of the second part of Theorem 24 running Algorithm 1 with parameter \( \eta = DW/2L_\infty \) gives pseudo-regret of order \( O \left( \frac{L_\infty^2 D^2}{W^2 \Delta} \right) \) independent of the dimension.
Now Combine Lemmas 21 and 23 with Theorem 1 to get the corresponding bound for regret.

**Theorem 26** Let $\mathcal{P} \subset \mathbb{R}^d$ be a polytope with diameter $D$ and width $W$. Suppose the cost vectors $b_1, b_2, \ldots$ have $|b_n \cdot (x - y)| \leq L_\infty$ for all $x, y \in \mathcal{P}$. Then Algorithm 1 with domain $\mathcal{P}$ and $y_1 \in \mathcal{P}$ and parameter $\eta = DW/2L_\infty$ gives regret bound

$$\sum_{i=1}^{N} b_i \cdot (x_i - x^*) \leq \frac{3L_\infty D}{W} \sqrt{N}.$$ 

### 4.1 Examples with Intrinsic Bounds

Here we examine the polytopes from Section 3.1 under the intrinsic bounds (14) on the cost vectors rather than the Euclidean bounds considered earlier. See Appendix A for the definitions and properties of the Polytopes in Table 2. The Columns $D, W$ and $V$ are the diameter, width and number of vertices. The values for $D$ are exact except for $\mathcal{P}(d)$ and $\mathcal{P}_\pm(d)$ where they are exact as $d \to \infty$. The values for $W$ are exact for the simplex and cube and are lower bounds for $\mathcal{B}(n), \mathcal{P}(d), \mathcal{P}_\pm(d)$.

| Polytope | $D^2$ | $W^2$ | $V$ | Subgradient | $L_\infty D \sqrt{N}$ | $L_\infty^2 D^2 W^2/\Delta$ |
|----------|-------|-------|-----|-------------|-----------------------|--------------------------|
| $d$-Simplex | 2 | $\frac{2}{d - 1}$ | $d$ | Subgradient | $L_\infty \sqrt{d} N$ | $L_\infty^2 d^2/\Delta$ |
| $d$-Cube | $4d^2$ | 4 | $2^d$ | Subgradient | $L_\infty \sqrt{d} N$ | $L_\infty^2 d^2/\Delta$ |
| $B(n)$ | $2n^2$ | $\frac{4}{n - 1}$ | $n!$ | Subgradient: | $L_\infty n \sqrt{N}$ | $L_\infty^2 n^2/\Delta$ |
| $\mathcal{P}(d)$ | $\frac{d^3}{3}$ | $\frac{5d^2}{6}$ | $d!$ | Subgradient | $L_\infty \sqrt{d} N$ | $L_\infty^2 d^2/\Delta$ |
| $\mathcal{P}_\pm(d)$ | $\frac{2d^3}{3}$ | $\frac{4d^2}{3}$ | $2^d d!$ | Subgradient | $L_\infty \sqrt{d} N$ | $L_\infty^2 d^2/\Delta$ |

Table 2: Comparison of order bounds for Subgradient and Hedge under the intrinsic bounds on cost vectors $|a_n \cdot (x - y)| \leq L_\infty$ and $|(a_n - a) \cdot (x - y)| \leq R_\infty$ for all $x, y \in \mathcal{P}$.

The antagonistic and i.i.d bounds for Subgradient come from Theorems 24 and 26 respectively. The bounds for Hedge refer to the lifting procedure detailed in Section 3.1 and come from Mourtada and Gaïffas (2019).
Since Hedge is tailored to the simplex it is no surprise the algorithm performs better than Subgradient on the simplex. The simplex is the only example above where it is feasible to run Hedge for high dimensions.

For the cube the Hedge and Subgradient bounds have the same order, but only Subgradient is feasible for high dimensions. For $B(n)$ the Subgradient bounds are not much worse than the Hedge bounds for small $n$. For example $n = 10$ gives an extra factor of $\sqrt{n/\log(n)} \approx 2.084 \ldots$. On the other hand there are $n! = 3628800$ vertices and running Hedge is computationally unfeasible. For comparison the main cost of Subgradient is projecting onto $B(n)$. This can be done with cost $O(n^2)$ if we are satisfied with an approximately feasible point. See Section 5.1 for discussion.

For the signed permutahedron the Subgradient bounds are slightly better than the Hedge bounds. This suggests the permutahedron is rounder than the Birkhoff polytope. Unfortunately this same roundness means the polytope has $2^n$ facets. This makes the cost of projecting using Lagrange multipliers prohibitively large. Fortunately there exist more sophisticated methods (Lim and Wright, 2016; Negrinho and Martins, 2014) that use the polytope’s structure to project with cost $O(n)$.

5. Open Problems

Here we discuss computational costs of running Subgradient on polytopes. We also mention some problems left open by our research and possible future directions.

5.1 Computational Cost

The advantage of Subgradient over Hedge is it only requires updating vectors of length $d$ rather than $V$. The most expensive part of the algorithm is projecting onto the domain. For polytopes with few facets this can be done using Lagrange multipliers with cost proportional to that of computing the Lagrangian. For a polytope with $F$ faces there are $F$ linear constraints, each with $d$ terms. Hence the cost of computing the Lagrangian is $O(dF)$.

For the Birkhoff polytope (see Section 4 and Appendix A) the constraints are particularly simple. We can write

$$B(n) = \{ x \in \mathbb{R}^{n \times n} : x_{ij} \geq 0 \text{ and } \sum_{k=1}^{d} x_{i,k} \text{ for all } i,j \leq d \}.$$ 

There are $n^2$ constraints with one term and $2n$ with $n$ terms. Hence the cost is only $O(d) = O(n^2)$.

On the other hand the Permutahedron has $2^d$ faces, one for each subset of $\{1,2,\ldots,d\}$. One general trick to optimise over a polytope $\mathcal{P}$ with many faces is to find a so-called extended formulation (Conforti et al., 2010; Kaibel, 2011; Goemans, 2015; Rahmanian et al., 2016). That means a polytope $\mathcal{B}$ with fewer faces and linear map $\phi : \mathcal{B} \to \mathcal{P}$. Then rather than optimise the given function $f$ over $\mathcal{P}$ we can optimise $\phi \circ f$ over $\mathcal{B}$. In particular the permutahedron $\mathcal{P} = \mathcal{P}(d)$ has extended formulation $\mathcal{B} = \mathcal{B}(d)$ and $\phi(x_{ij}) = \sum_{j=1}^{d}(j x_{1j}, \ldots, j x_{dj})$ and we get $O(d^2)$ cost for projecting.
There also exist purpose-built algorithms (Li et al. (2018)) for Birkhoff and sophisticated dual-
problem methods for the permutahedron (Lim and Wright 2016. Negrinho and Martins, 2014) to
project with cost $O(d)$.

Unfortunately none of the methods mentioned above compute the exact projection, but only an
iterative approximation that might lie outside the action set.

One idea to rescue an almost-feasible point is to shrink the domain slightly and project onto
the smaller domain. After sufficiently many iterations the approximation will lie inside the original
domain. Then we attempt to bound the distance between the original and shrunken solutions in
terms of the change of domain.

We would like to know how many iterations are needed to get within a prescribed distance of
the projection. The methods of Nedić and Ozdaglar (2009) show how to get a prescribed constraint
violation. Unfortunately the relationship between constraint violation and distance from the polytope
depends on how sharp the corners are. For polytopes with very sharp corners there are points
with small constraint violation but large distance.

To play a Birkhoff matrix it is in practice necessary to decompose into a linear combination of
permutation matrices. This turns out to be helpful, since methods for finding a decomposition can
be used to rescue an almost feasible point. When applied to an almost Birkhoff matrix the methods
yield a decomposition of a nearby Birkhoff matrix. For discussion see Helmbold and Warmuth (2009)
Section 4.2.

One way to avoid almost-feasible points is to use self-concordant barriers (Nemirovski and Todd,
2008). This has the advantage of giving a feasible sequence converging to the solution. However it
comes at the cost of inverting a $d \times d$ Hessian matrix (or solving the corresponding linear system) at
each iteration. Since the system has $d^2$ coefficients this adds an extra $\Omega(d^2)$ to the computational
cost. This is cheaper than Hedge but still becomes unfeasible faster than Lagrange multipliers as $d$
grows.

5.2 Barrier Functions

Given a polytope $P$ another approach than Subgradient or reduction to Hedge is to use barrier
functions. For simplicity suppose $P \subset \mathbb{R}^n$ has interior and has $F$ facets. We can write $P = \{x \in \mathbb{R}^d : \phi_i(x) \geq 0 \text{ for } i = 1, 2, \ldots, F\}$ for some affine functions $\phi_i : \mathbb{R}^d \rightarrow \mathbb{R}$ corresponding to the facets.
This gives the following algorithm.

The Barrier algorithm is a generalisation of Hedge where facets take prominence, rather than the
reduction to Hedge (see Introduction) where vertices take prominence. For example Hedge is a
special case of Barrier for $\phi_j(x) = x(j)$ and $\eta_j = 1$.

It would be interesting to know if the Barrier algorithm is universal, and how the computational
cost and regret bounds compare to Subgradient and Hedge. We suspect Barrier outperforms the
other two on the cube, and by symmetry we should select $\eta_j$ to be equal. For more general polytopes
we imagine the tuning of the parameters $\eta_j$ is important.
Algorithm 2: Anytime Barrier Function Algorithm

Data: Polytope $P \subset \mathbb{R}^d$. Base Point $x_1 \in P$. Affine functions $\phi_1, \ldots, \phi_F : \mathbb{R}^d \to \mathbb{R}$. Parameters $\eta_1, \ldots, \eta_F > 0$.

1. select action $x_1$
2. pay cost $a_1 \cdot x_1$
3. for $n = 2, 3, \ldots$ do
   4. receive $a_{n-1}$
   5. select action $x_n = \arg\min_{x \in P} \left( \sum_{i=1}^{F} \eta_i \phi_i(x) \log(\phi_i(x)) + \frac{1}{\sqrt{n-1}} \sum_{i=1}^{n-1} a_i \cdot x \right)$
   6. pay cost $a_n \cdot x_n$

5.3 Higher-Order Estimates

One shortcoming of Theorems 8 and 24 is they do not recover the bound from Theorem 2 when the polytope approximates the ball. For finer approximations we have $D, W \to 2$ but the suboptimality gap $\Delta \to 0$. Hence the $O\left( \frac{L^2 D^2}{W^2} \right)$ bound from Theorem 24 goes to infinity rather than the finite $O\left( \frac{L^2}{\|a\|} \right)$ bound from Theorem 2. To fix this one idea is to replace the estimate from Lemma 12

$$\theta_v \geq \frac{1/2}{1 + D^2 \|a\|^2 / \Delta_v^2} - 1$$

with something that contains more information about the polytope. For example a second-order estimate in terms of the dihedral angles and side-lengths of the polytope. For example small sides and large angles means the polytope curves upwards from its lowest point at a faster rate. This should give better estimates than Lemma 12.

On a similar note we predict that by explicitly computing $\theta_v$ for the polytopes in Table 2 one can derive better bounds.

5.4 Smooth Domains

Subgradient can be run on arbitrary domains. But for some domains it fails to be universal. See Example 16 of Anderson and Leith (2019) where we prove the domains

$$\mathcal{Y}_\alpha = \{(x, y) \in [-1, 1] \times [0, 1] : y \geq x^\alpha \} \text{ for } \alpha > 2$$

(15)
can give pseudo-regret $\Omega(N^{1/2-\varepsilon})$ for any $\varepsilon > 0$. Thus the i.i.d pseudo-regret can be almost as bad as the antagonistic regret.

For smooth strongly convex domains Huang et al. (2016) show Follow-the-leader gives $O(\log N)$ expected regret. For example the domains $\mathcal{Y}_\alpha$ with $\alpha \leq 2$. They then use the Prod($A, B$) algorithm of Sani et al. (2014) to get $O(\log N)$ and $O(\sqrt{N \log N})$ bounds.

It seems harder to find a general condition on the boundary that gives an $O(1)$ rather than $O(\log N)$ bound. It seems even harder to find a condition that covers both smooth boundaries and
polytopes which are **infinitely curved** at the vertices. Indeed Huang et al. prove their bounds for smooth domains and polytopes separately.

In the smooth case suppose \( F : \mathbb{R} \to \mathbb{R} \) is convex and minimised at \( F(0) = 0 \). Let the domain be \( \mathcal{X} = \{(x, y) \in \mathbb{R}^2 : y \geq F(x)\} \) and the cost-vectors be i.i.d with \( \mathbb{E}[a_i] = (0, 1) \). By similar methods to Theorem 2, and the standard trick of replacing series with integrals, we can show Subgradient has \( O(1) \) pseudo-regret if

\[
\int_{-\varepsilon}^{\varepsilon} \frac{F(x)F''(x)}{F'(x)^3} < \infty \quad \text{for some } \varepsilon > 0. \tag{16}
\]

Since 0 is the minimiser \( \lim_{x \to 0} F'(x) = 0 \) and the integrand goes to infinity at zero. If this happens slowly enough the integral is finite. Note since the integrand exists away from zero (16) is equivalent to the integral over say \([-1, 1]\) being finite.

For example let \( \alpha \geq 1 \) and \( F(x) = x^\alpha \). The integrand is proportional to \( x^{1-\alpha} \). For \( \alpha > 2 \) the integral is infinite. For \( \alpha = 2 \) the integral is infinite but the integral from \( 1/N \) to 1 has order \( O(\log N) \) and we get the same \( O(\log N) \) bound of Huang et al. (2016) .

For \( \alpha < 2 \) the integral is finite. It is tempting to look for a convex function whose rotated graph looks like \( x^{3/2} \) say and satisfies (16) at every point. No such functions exist, since \( x^{3/2} \) is not second-differentiable at the origin, and Alexandrov’s theorem (Howard, 1998; Aleksandrov, 1939) says every convex function has a second derivative almost everywhere.

The authors are unaware of any physical meaning for (16) or relationship to known geometric properties. It conceivably such a relationship exists, since powers of higher derivatives already appear across geometry. For example the *curvature* of a function \( f : \mathbb{R} \to \mathbb{R} \) at \( x \) is defined by the equally nasty-looking formula

\[
\kappa = \frac{f''(x)}{(1 + f'(x)^2)^{3/2}}
\]

and the definition of a *self-concordant barrier function* in optimisation theory is the inequality

\[
f'''(x)^2 \leq (2f''(x))^3.
\]

**Acknowledgements**

This work was supported by Science Foundation Ireland grant 16/IA/4610.

**Appendix A: Dimensions of Polytopes**

Here we derive bounds for the width and diameter of the polytopes in Tables 1 and 2. To our knowledge the widths of Examples 3-5 do not appear in the literature at all. Thanks to David E Speyer (2019) for suggesting the probabilistic counting trick used in those examples.

Recall Definition 20 of the width: Given a polytope \( \mathcal{P} \subset \mathbb{R}^d \) the affine hull has the form \( U + t \) for some vector subspace \( U \subset \mathbb{R}^d \). For each \( \ell \in \mathbb{R}^d \) define \( W_\ell = \max\{\ell \cdot (x - y) : x, y \in \mathcal{P}\} \). The width of \( \mathcal{P} \) is \( W = \min\{W_\ell : \ell \in U \text{ and } \|\ell\| = 1\} \).
Example 1 For the cube $[-1, 1]^d$ we have $W_\ell = 2$ for $\ell = (1, 0, \ldots, 0)$. We claim the width is $W = 2$. To that end suppose $\|\ell\| = 1$. We must show $\ell \cdot (x - y) \geq 2$ for some $x, y \in [-1, 1]^d$. By permuting the coordinates we can assume $\ell = (a_1, \ldots, a_n, b_1, \ldots, b_m)$ for $a_i \geq 0$ and $b_i \leq 0$. Then $\ell$ is maximised (minimised) over the cube at $\pm p$ where $p = (1, \ldots, 1, -1, \ldots, -1)$ has exactly $n$ positive entries. For $x = p$ and $y = -p$ we have $\ell \cdot (x - y) = 2\ell \cdot p = 2((a_1 + \ldots + a_n) - (b_1 + \ldots + b_m)) = 2(|a_1| + \ldots + |a_n| + |b_1| + \ldots + |b_m|) = 2\|\ell\| \geq 2\|\ell\| = 2$.

Example 1.5 The cube has diameter $2\sqrt{d}$ due to the following lemma.

Lemma 27 Each polytope $\mathcal{P}$ has vertices $u, v \in \mathcal{P}$ with $\|u - v\| = \max\{\|x - y\| : x, y \in \mathcal{P}\}$.

Proof Moreover we claim for each $x \in \mathcal{P}$ that $D_x = \max\{\|x - y\| : y \in \mathcal{P}\}$ is achieved when $y$ is a vertex. To prove that write $y = \sum_{i=1}^{V} \lambda_i v_i$ as a convex combination of the vertices. Then we have

$$\|x - y\| = \|x - \sum_{i=1}^{V} \lambda_i v_i\| = \left\|\sum_{i=1}^{V} \lambda_i x - \sum_{i=1}^{V} \lambda_i v_i\right\| \leq \sum_{i=1}^{V} \lambda_i \|x - v_i\| \leq \max_{i \leq V} \|x - v_i\|.$$

Thus $D_x$ is maximised when $y$ is the vertex $v$ that maximises $\|x - v\|$. Likewise we have $\|x - v\| \leq \|u - v\|$ for some vertex $u$. Since $x, y$ are arbitrary we get the result. \qed

Example 2 The $d$-simplex $\mathcal{S}$ has width asymptotically equal to $2/\sqrt{d}$. The authors were unable to find a modern proof of this fact. The standard proof seems to be Alexander (1977). Unfortunately the author refers to fundamental properties of convex sets that were perhaps more well-known at the time. Therefore we refer to (1.9) of Gritzmann and Klee (1992).

Theorem 28 Suppose the polytope $\mathcal{P} \subset \mathbb{R}^N$ has non-empty interior and width $W$. There exists a unit vector $\ell \in \mathbb{R}^N$ and faces $A, B$ of $\mathcal{P}$ and points $a \in A$ and $b \in B$ such that

\begin{align*}
(1) \{\ell \cdot x : x \in \mathcal{P}\} &\text{ has length } W & (2) \dim(A - B) &\geq N - 1.
\end{align*}

In particular $\dim A + \dim B \geq N - 1$.

Note the given proof has a typographical error. For the proof to work we must use (1.8) of that paper to take the points $q_{\pm}$ of the form $q_{\pm} = \pm q$ for some $q \in \mathbb{P}_{\ell_2}$ of $\mathcal{P}$. Then $\ell \cdot (q_+ - q_-)$ is the width of $\mathbb{P}_{\ell_2}$. As stated in the proof $q_+ = q$ is in the relative interior of some facet $F$ of the symmetric polytope $\mathbb{P}_{\ell_2}$ and so $q_- = -q$ is in the relative interior of the facet $-F$. Theorem 3.1.2 of Weibel (2007) says $\mathcal{P}$ has faces $A, B$ with $F = \mathbb{P}_{\ell_2}/2$. Hence $N - 1 = \dim F = \dim \left(\mathbb{P}_{\ell_2}/2\right) = \dim(A - B) \leq \dim A + \dim B$. Choose $a \in A$ and $b \in B$ with $a/b = q$. It follows from (1.4) and (1.5) of Gritzmann and Klee (1992) that $\mathcal{P}$ and $\mathbb{P}_{\ell_2}$ have the same width. Thus $W = \ell \cdot (q_+ - q_-) = 2 \ell \cdot q = 2 \ell \cdot \left(\frac{a+b}{2}\right) = \ell \cdot (a - b)$ as required.
Example 2.5 The distance between any two vertices of the simplex is \( \sqrt{2} \). Thus the diameter is \( D = \sqrt{2} \).

Example 3 The Birkhoff Polytope \( B \) is the set of nonnegative \( n \times n \) matrices with all row and column sums equal to 1. Equivalently the convex hull of the \( n! \) permutation matrices. Identify each permutation \( \sigma \in S_n \) with the corresponding matrix. We claim \( W \geq 2/\sqrt{n-1} \). To prove this let \( \ell \in \mathbb{R}^{n\times n} \) have \( \|\ell\|^2 = 1 \) and all \( \sum_{i=1}^{n} \ell^i_j = \sum_{j=1}^{n} \ell^i_j = 0 \).

The proof uses a probabilistic counting trick. Let \( \sigma \in S_n \) be a uniformly-chosen counting matrix and consider the random variables \( X = \ell \cdot \sigma \). We can write

\[
\sum_{\sigma \in S_n} (\ell \cdot \sigma)^2 = \sum_{\sigma \in S_n} \left( \sum_{i} \ell^i_{\sigma(i)} \right)^2 = \sum_{\sigma \in S_n} \left( \sum_{i} (\ell^i_{\sigma(i)})^2 + \sum_{i \neq j} \ell^i_{\sigma(i)} \ell^j_{\sigma(j)} \right)
\]

For each pair \( (i, j) \) the term \((\ell^i_j)^2\) appears in the expansion \((n-1)!\) times, since this is the number of permutations with \( \sigma(i) = j \). For each tuple \( (i, j, a, b) \) with \( i \neq j \) and \( a \neq b \) the term \( \ell^i_a \ell^j_b \) appears \((n-2)!\) times since there are \((n-2)!\) permutations with \( \sigma(i) = a \) and \( \sigma(j) = b \). Hence the above equals

\[
(n-1)! \sum_{i,j \leq n} (\ell^i_j)^2 + (n-2)! \sum_{i \neq j, a \neq b} \ell^i_a \ell^j_b = (n-1)! + (n-2)! \sum_{i \neq j, b \neq a} \ell^i_a \ell^j_b
\]

where we have used \( \|\ell\|^2 = 1 \) to simplify the first term. For the second term, since row \( j \) and column \( b \) sum to zero we have
\[
\sum_{i,a} \ell_i^a \sum_{j \neq i} \ell_j^a = \sum_{i,a} \ell_i^a \sum_{j \neq i} (-\ell_j^a) = \sum_{i,a} \ell_i^a \sum_{j \neq i} \ell_j^a = \sum_{i,a} \ell_i^a (-\ell_i^a) = \sum_{i,a} (\ell_i^a)^2 = 1.
\]

We conclude
\[
\sum_{\sigma \in S_n} (\ell \cdot \sigma)^2 = (n-1)! + (n-2)! = n(n-2)!
\]

Hence \( X = \ell \cdot \sigma \) has variance \( n(n-2)!/n! = 1/(n-1) \) and standard deviation \( 1/\sqrt{n-1} \). Popoviciu’s inequality says the standard deviation is at most \( \max X - \min X \). From this we get \( \max X - \min X \geq 2/\sqrt{n-1} \) as required.

**Example 3.5** The diameter is achieved for any pair of permutation matrices with no nonzero entries in common. Thus we have \( D^2 = 2n \).

**Example 4** The permutahedron \( P \) is the set of vectors \( x \in \mathbb{R}^d \) with entries \( \{x_1, \ldots, x_d\} = \{1, 2, \ldots, d\} \). Equivalently \( P \) is the convex hull of \( \{(\sigma(1), \ldots, \sigma(d)) : \sigma \in S_d\} \). Identify the permutation \( \sigma \in S_d \) with the vector \( (\sigma(1), \ldots, \sigma(d)) \in S_d \). We claim
\[
W \geq \sqrt{\frac{5d^2 + 8d + 4}{6}} \quad \liminf_{d \to \infty} \frac{|P|}{d^d} \geq \sqrt{\frac{5}{6}}
\]
We use the same variance trick with \( X = \ell \cdot \sigma \)
\[
\sum_{\sigma \in S_n} (\ell \cdot \sigma)^2 = \sum_{\sigma \in S_n} \left( \sum_i \sigma(i) \ell_i \right)^2 = \sum_{\sigma \in S_n} \left( \sum_i \sigma(i)^2 \ell_i^2 + \sum_{i \neq j} \sigma(i) \sigma(j) \ell_i \ell_j \right)
\]
For each \( \sigma \) we have \( \sigma(1)^2 + \ldots + \sigma(d)^2 = \frac{d(\sigma+1)(2\sigma+1)}{6} \). Thus the sum of coefficients in the first sum is
\[
\sum_{\sigma \in S_n} \sum_i \sigma(i)^2 = \frac{d(\sigma+1)(2\sigma+1)}{6} d! = \frac{d(\sigma+1)(\sigma+1)!}{6}
\]
By symmetry each \( \ell_i^2 \) appears in the expansion with multiplicity \( \frac{(2\sigma+1)(\sigma+1)!}{6} \). The sum of coefficients in the second sum is
\[
\sum_{\sigma \in S_n} \sum_{i \neq j} \sigma(i)\sigma(j) = \frac{1}{2} \sum_{\sigma \in S_n} \left( (\sigma(1) + \ldots + \sigma(d))^2 - \sigma(1)^2 - \ldots - \sigma(d)^2 \right)
\]

\[
= \frac{1}{2} \sum_{\sigma \in S_n} \left( \frac{d(d+1)^2}{2} - \frac{d(d+1)(2d+1)}{6} \right)
\]

\[
= \frac{1}{2} \sum_{\sigma \in S_n} \left( \frac{d^2(d+1)^2}{2} - \frac{d(d+1)(2d+1)}{6} \right)
\]

\[
= \frac{1}{2} \sum_{\sigma \in S_n} \frac{3d^4 + 4d^3 - d}{6} = \frac{(3d^4 + 4d^3 - d)d!}{12}
\]

Since there are \(d(d-1)\) choices for the pair \((i, j)\) with \(i \neq j\) we have by symmetry that each \(\ell_i \ell_j\) appears in the expansion with multiplicity \(\frac{(3d^4 + 4d^3 - d)(d-1)!}{12}\). Thus we have shown

\[
\sum_{\sigma \in S_n} (\ell \cdot \sigma)^2 = \frac{(2d+1)(d+1)!}{6} \sum_{j=1}^{d} \ell_i^2 + \frac{(3d^3 + 4d^2 - 1)(d-1)!}{12} \sum_{i \neq j} \ell_i \ell_j
\]

To simplify the first term recall \(\sum_{j=1}^{d} \ell_i^2 = ||\ell||^2 = 1\). For the second write

\[
\sum_{i \neq j} \ell_i \ell_j = \frac{1}{2} \left( (\ell_1 + \ldots + \ell_d)^2 - \ell_1^2 - \ldots - \ell_d^2 \right) - \frac{||\ell||^2}{2} = -\frac{1}{2}.
\]

Thus we have

\[
\sum_{\sigma \in S_n} (\ell \cdot \sigma)^2 = \frac{(2d+1)(d+1)!}{6} - \frac{(3d^3 + 4d^2 - 1)(d-1)!}{24}
\]

\[
\geq \frac{(2d+1)(d+1)!}{6} - \frac{(3d^2 + 4d)d!}{24} = \frac{4(2d+1)(d+1) - (3d^2 + 4d)d!}{24}
\]

and the variance is

\[
\frac{4(2d+1)(d+1) - (3d^2 + 4d)}{24} = \frac{8d^2 + 12d + 4 - (3d^2 + 4d)}{24} = \frac{5d^2 + 8d + 4}{24}
\]

and standard deviation

\[
\sqrt{\frac{5d^2 + 8d + 4}{24}} = \frac{1}{2} \sqrt{\frac{5d^2 + 8d + 4}{6}}
\]

Like before we see \(\max X - \min X \geq \sqrt{\frac{5d^2 + 8d + 4}{6}}\) as required. For large \(d\) the above is approximately \(\sqrt{\frac{5}{6}d}\).
Example 4.5 We claim the diameter is achieved for the vertices \( v = (1, 2, \ldots, d) \) and \( w = (d, d - 1, \ldots, 1) \). For suppose \( \sigma \) and \( \mu \) are vertices. By symmetry we can assume \( \mu \) is the identity. For some \( m \leq d \) we have \( \sigma(m) = 1 \). Suppose \( m \neq d \). We can write

\[
\|\sigma - \mu\|^2 = (1 - \sigma(1))^2 + \ldots + (m - \sigma(m))^2 + \ldots + (d - \sigma(d))^2
\]

\[
= \sum_{n=1}^{d} n^2 + \sum_{n=1}^{d} \sigma(n)^2 - 2 \sum_{n=1}^{d} n\sigma(n) = 2 \sum_{n=1}^{d} n^2 - 2 \sum_{n=1}^{d} n\sigma(n).
\]

The first term is independent of \( \sigma \). Hence to maximise \( \|\sigma - \mu\|^2 \) we must minimise \( \sum_{n=1}^{d} n\sigma(n) \).

We will prove a more general statement. Suppose \( 0 < x_1 < x_2 < \ldots < x_d \) and \( y_1 > y_2 > \ldots > y_d > 0 \). We claim that \( \sum_{n=1}^{d} x_n y_{\sigma(n)} \) is not a minimiser. It follows \( \|\sigma - \mu\|^2 \) is maximised for \( \sigma = w \) and \( \mu = v \).

For odd \( d = 22 \) is twice the sum of odd squares. To compute this recall the sum of the first \( d \) squares (Weisstein) is \( \frac{d(d+1)(2d+1)}{6} \). Hence the sum of the first \( d/2 \) even squares is

\[
\sum_{n=1}^{d/2} (2n)^2 = 4 \sum_{n=1}^{d/2} n^2 = 4 \sum_{n=1}^{d/2} \frac{n^2}{2} = 4 \frac{(\frac{d}{2} + 1)(\frac{d}{2} + 1)}{6} = \frac{d(d + 2)(d + 1)}{6}.
\]

The sum of odd squares is the sum of all squares minus the sum of even squares and so equals

\[
\frac{d(d + 1)(2d + 1)}{6} - \frac{d(d + 2)(d + 1)}{6} = \frac{d(d + 1)(d - 1)}{6} = \frac{d(d^2 - 1)}{6}.
\]

For odd \( d \) we see \( D^2 \) is twice the sum \( (d - 1)^2 + (d - 3)^2 + \ldots + 2^2 \) of the first \( \frac{d - 1}{2} \) even squares.

By the above it equals \( \frac{(d-1)(d+1)d}{6} \) and so \( D^2 = \frac{d(d^2 - 1)}{3} \) like before.
**Example 5** The signed permutahedron $P_{\pm}$ is the convex hull of the vectors $(\pm \sigma(1), \ldots, \pm \sigma(d))$ for all choices of signs and permutation $\sigma \in S_d$. We claim

$$|P_{\pm}| \geq 2\sqrt{\frac{2d^2 + 3d + 1}{6}}$$

and

$$\liminf_{d \to \infty} \frac{|P_{\pm}|}{d} \geq \sqrt{\frac{8}{6}}.$$ 

For $S^d = \{-1, 1\}^d$ we can write

$$P_{\pm} = \{(s_1\sigma(1), \ldots, s_d\sigma(d)) : \sigma \in S_d, s \in S^d\}$$

Write $s\sigma = (s_1\sigma(1), \ldots, s_d\sigma(d))$ and consider the random variables $X = \ell \cdot s\sigma$.

$$\sum_{s \in S^d} \sum_{\sigma \in S_d} (\ell \cdot s\sigma)^2 = \sum_{s \in S^d} \sum_{\sigma \in S_d} \left( \sum_i \sigma(i)\ell_i \right)^2 = \sum_{s \in S^d} \sum_{\sigma \in S_d} \left( \sum_i s_i^2\sigma(i)^2\ell_i^2 + \sum_{i \neq j} s_is_j\sigma(i)\sigma(j)\ell_i\ell_j \right)$$

By symmetry the second part vanishes leaving

$$\sum_{s \in S^d} \sum_{\sigma \in S_d} \sum_i \sigma(i)^2\ell_i^2 = \sum_{s \in S^d} \left( \frac{(2d + 1)(d + 1)!}{6} \sum_i \ell_i^2 \right) = 2^d(d+1)(d+1)!$$

The first equality uses the argument from the previous example to compute the coefficients. The second equality uses $\|\ell\|^2 = 1$. Since $|S^d \times S_d| = 2^d d!$ the variance is

$$\frac{(2d + 1)(d + 1)}{6} = \frac{2d^2 + 3d + 1}{6}$$

Like before we see

$$\max X - \min X \geq 2\sqrt{\frac{2d^2 + 3d + 1}{6}}$$

as required. For large $d$ the above is approximately $\sqrt{8/6}d$.

**Example 5.5** The diameter is achieved for some pair $v, w$ of vertices. Similar to Example 4.5 we see $\|v - w\|^2$ is maximised for $v = (1, 2, \ldots, d)$ and $w = (-1, -2, \ldots, -d)$. Then

$$D^2 = \sum_{n=1}^d (2n)^2 = 4 \sum_{n=1}^d n^2 = 4 \frac{d(d + 1)(2d + 1)}{6} = \frac{2d(d + 1)(2d + 1)}{3}. $$
Appendix B: Probability

Our main concentration result follows from Theorem 3.5 of (Pinelis, 1994). See (Anderson and Leith, 2019) Appendix C for discussion.

**Theorem 29** Suppose the i.i.d sequence \(a_1, a_2, \ldots\) takes values in \(\mathbb{R}^d\). Suppose for \(E[a_i] = a\) we have \(\|a_i - a\| \leq R\). Then for each \(r \geq 0\) we have

\[
P\left( \left\| \sum_{i=1}^{n} (a_i - a) \right\| \geq \sqrt{nr} \right) \leq 2 \exp\left( -\frac{r^2}{2R^2} \right).
\]

For real-valued martingales there exist one-sided versions of the above without the leading factor of 2. For the definition of a martingale and proof of the following see Gamarnik (2013).

**Theorem 30 (Azuma-Hoeffding)** Suppose \(X_1, X_2, \ldots\) is a real-valued martingale difference sequence with each \(|X_i| \leq R\). Then for each \(r > 0\) we have

\[
P\left( \sum_{i=1}^{n} X_i \geq \sqrt{nr} \right) \leq \exp\left( -\frac{r^2}{2R^2} \right).
\]

The next lemma is used to bound the pseudo-regret in terms of the regret.

**Lemma 31** Let \(a_1, a_2, \ldots\) be an i.i.d sequence of cost vectors and \(x_1, x_2, \ldots\) the actions of Algorithm 1. The random variables \(X_i = (a - a_i) \cdot (x_i - x^*)\) define a martingale difference sequence with respect to the filtration generated by \(a_1, a_2, \ldots\).

**Proof** We must show each \(E[X_i|a_1, a_2, \ldots a_{n-1}] = 0\). That means for each set \(U = a_1^{-1}(U_1) \cap \ldots \cap a_{n-1}^{-1}(U_{n-1})\) in the algebra generated by \(a_1, a_2, \ldots a_{n-1}\) we have \(\int_U X_n dP = 0\). To that end write each \(B(i) = a_i^{-1}(U_i)\) and observe the indicator \(1_{B(i)}\) is a measurable function of \(a_1, \ldots, a_{n-1}\). Now write

\[
\int_U X_n dP = \int_U (a - a_n) \cdot (x_n - x^*) dP = \int_U (a - a_n) \cdot (x_n - x^*) 1_{B(1)} \cdot \ldots 1_{B(n-1)} dP.
\]

Recall \(x(n)\) is a function of \(a_1, \ldots, a_{n-1}\). Since all \(a_i\) are independent we can distribute to get

\[
\int_U (a - a_n) \cdot (x_n - x^*) dP = \int_U (a - a_n) dP \cdot \int_U (x_n - x^*) 1_{B(1)} \cdot \ldots 1_{B(n-1)} dP.
\]

Since \(E[a_n] = a\) the above is zero as required.

Next we apply the previous lemma.

**Lemma 32** Suppose we run Algorithm 1 on the domain \(X\) with diameter \(D\). For each \(M \in \mathbb{N}\) we have

\[
E \left[ \sum_{i=1}^{M} (a - a_i) \cdot (x_i - x^*) \right] \leq \sqrt{\frac{\pi}{2}} DR\sqrt{M}.
\]
Proof Lemma 31 says $X_i = (a - a_i) \cdot (x_i - x^*)$ is a martingale difference sequence with respect to $a_1, a_2, \ldots$. Since $|X_i| = |(a - a_i) \cdot (x_i - x^*)| \leq \|a - a_i\| \|x_i - x^*\| \leq DR$ the Azuma-Hoeffding inequality says
\[
P\left( \sum_{i=1}^{M} (a - a_i) \cdot (x_i - x^*) > t \right) \leq \exp\left( -\frac{t^2}{2D^2R^2M} \right).
\]
By Lemma 16 of (Anderson and Leith 2019) we can bound the expectation
\[
E \left[ \sum_{i=1}^{M} (a - a_i) \cdot (x_i - x^*) \right] \leq \int_0^\infty \exp\left( -\frac{t^2}{2D^2R^2M} \right) dt = \frac{1}{2} \sqrt{2\pi MR^2} = \sqrt{\frac{\pi}{2}} DR\sqrt{M}
\]
where we have used (Nicholas and Yates, 1950) to evaluate the Gaussian integral.

Appendix C: Convex Geometry

Here we prove two of the preliminary lemmas in Section 2.

Lemma 11 The set $\arg\min \{a \cdot x : x \in P\}$ is the convex hull of $V^*$.

Proof Clearly $V^* \cap \arg\min \{a \cdot x : x \in P\} = V^*$. Once we show $\arg\min \{a \cdot x : x \in P\}$ is a face of $P$ we can apply Lemma 10. To that end write $\arg\min \{a \cdot x : x \in P\} = V^*$ for the tangent plane $Q = \{x \in \mathbb{R}^d : a \cdot x = a_0\}$ for $a_0 = \min\{a \cdot x : x \in P\}$. Now apply the previous lemma.

Lemma 12 For each $v \not\in V - V^*$ the quantity $\theta_v = \min \left\{ \frac{a \cdot u}{\|a\|\|u\|} : u \in N_P(v) \right\}$ satisfies
\[
\theta_v \geq \frac{1/2}{1 + D^2\|a\|^2/\Delta_v^2} - 1
\]
Hence $\theta_v > -1$ and the quantities $\phi_v = \theta_v + 1$ are positive.

Proof By performing a rotation we can assume $a = (\|a\|, 0, \ldots, 0)$. Then we have $\Delta_v = a \cdot (v - v^*) = \|a\|(v_1 - v_1^*)$ and so $v_1 - v_1^* = \Delta_v/\|a\|$. For each normal $u \in N_P(v)$ we know $P$ is contained in the half-space $\{x \in \mathbb{R}^d : u \cdot x \leq u \cdot v\}$. Hence for each $v^* \in V^*$ we have $u \cdot v^* \leq u \cdot v$. Expand the inequality to get
\[
u_1(v_1^* - v_1) \leq u_2(v_2 - v_2^*) + \ldots + u_d(v_d - v_d^*) \]
\[-u_1 \frac{\Delta_v}{\|a\|} \leq u_2(v_2 - v_2^*) + \ldots + u_d(v_d - v_d^*) \]
\[\leq \sqrt{u_2^2 + \ldots + u_d^2}D = \sqrt{\|u\|^2 - u_1^2}D \]
where the last line uses Cauchy Schwarz. First assume $u_1 \leq 0$. Since both sides are nonnegative we take squares and simplify to get

$$\frac{u_1^2}{\|u\|^2} \leq \frac{1}{1 + \Delta_\nu^2/D^2\|a\|^2}$$  \hspace{1cm} (17)$$

Now recall $\frac{a}{\|a\|} = (1,0,\ldots,0)$ and write

$$\frac{2a \cdot u}{\|a\|\|u\|} = \left\| \frac{a}{\|a\|} + \frac{u}{\|u\|} \right\|^2 - 2 = \left(1 - \frac{u_1}{\|u\|}\right)^2 + \frac{u_1^2 + \ldots + u_d^2}{\|u\|^2} - 2 \hspace{1cm} (18)$$

Since $u/\|u\|$ is a unit vector we have

$$\frac{u_1^2 + \ldots + u_d^2}{\|u\|^2} \geq 1 - \frac{1}{1 + \Delta_\nu^2/D^2\|a\|^2} = \frac{1}{1 + D^2\|a\|^2/\Delta_\nu^2}$$ \hspace{1cm} (19)

where we have used (17) for the inequality. Combining (18) and (19) we have

$$\frac{a \cdot u}{\|a\|\|u\|} \geq \frac{1/2}{1 + D^2\|a\|^2/\Delta_\nu^2 - 1}.$$

Now assume $u_1 \geq 0$. Then the left-hand-side is nonnegative. Since the right-hand-side is negative the above holds. Hence it holds for all $u \in N_P(v)$ and the result follows.

**Appendix D: An Integral**

Here we bound a non-elementary integral that appears in Section 1.

**Lemma 33** For any $b,R,\eta \geq 0$ and $M = \left\lceil \frac{4\eta b^2 + 8R^2}{b^2} \right\rceil$ we have

$$\int_M^\infty \frac{1}{x} \exp \left(- \frac{b^2}{8R^2} x \right) dx \leq \frac{\log 2}{e}$$

**Proof** For $z > 0$ define the Exponential integral function $E_1$ in keeping with Alzer (1997) as $E_1(z) = \int_z^\infty \frac{e^{-x}}{z} dx$. For $A = b^2/8R^2$ we can substitute $z = Ax$ and write the integral as

$$\int_M^\infty \frac{1}{Ax} \exp \left(- \frac{b^2}{8R^2} x \right) dx = \int_M^\infty \frac{e^{-Ax}}{x} dx = \int_{AM}^\infty \frac{e^{-z}}{z} dz = E_1(AM)$$

By (Gautschi, 1998) formula (5.2) we have the inequality

$$\frac{e^{-z}}{2} \log \left(1 + \frac{2}{z}\right) \leq E_1(z) \leq e^{-z} \log \left(1 + \frac{1}{z}\right). \hspace{1cm} (20)$$

and so $E_1(AM) \leq e^{-AM} \log \left(1 + \frac{1}{AM}\right)$. Since $e^{-z}$ and $\log \left(1 + \frac{1}{z}\right)$ are decreasing so is the right-hand-side of (20). Hence if $AM \geq 1$ the integral is at most $\log(2)/e$. To that end write

$$AM = \frac{b^2}{8R^2} \left\lceil \frac{4\eta^2 b^2 + 8R^2}{b^2} \right\rceil \geq \frac{b^2}{8R^2} \left(\frac{4\eta^2 b^2 + 8R^2}{b^2}\right) = \frac{1}{2\eta^2 R^2} + 1 \geq 1.$$
References

Aleksandr Danilovich Aleksandrov. Almost everywhere existence of the second differential of a convex function and some properties of convex surfaces connected with it. *Leningrad State Univ. Annals [Uchenye Zapiski] Math. Ser.*, (6), 1939.

Ralph Alexander. The width and diameter of a simplex. *Geometriae Dedicata*, 6(1):87–94, 1977.

Noga Alon and Joel H Spencer. *Paul Erdős and the Probabilistic Method*. John Wiley & Sons, 2004. URL http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.636.3738&rep=rep1&type=pdf.

Horst Alzer. On some inequalities for the incomplete gamma function. *Math. Comput.*, 66(218):771–778, April 1997. ISSN 0025-5718. doi: 10.1090/S0025-5718-97-00814-4. URL http://dx.doi.org/10.1090/S0025-5718-97-00814-4.

Daron Anderson and Douglas Leith. Optimality of the subgradient algorithm in the stochastic setting. *Arxiv E-prints*, 2019.

Peter Auer and Chao-Kai Chiang. An algorithm with nearly optimal pseudo-regret for both stochastic and adversarial bandits. *CoRR*, abs/1605.08722, 2016. URL http://arxiv.org/abs/1605.08722.

Alexander Barvinok. 7: Lattice points and lattice polytopes. In *Handbook of discrete and computational geometry*, pages 185–210. Chapman and Hall/CRC, 2017.

Károly Bezdek. Tarskis plank problem revisited. In *Geometryintuitive, discrete, and convex*, pages 45–64. Springer, 2013.

Sébastien Bubeck and Aleksandrs Slivkins. The best of both worlds: stochastic and adversarial bandits. *CoRR*, abs/1202.4473, 2012. URL http://arxiv.org/abs/1202.4473.

Nicolo Cesa-Bianchi, Yishay Mansour, and Gilles Stoltz. Improved second-order bounds for prediction with expert advice. *Machine Learning*, 66(2-3):321–352, 2007. URL https://arxiv.org/pdf/1809.01382.pdf#cite.gaillard2014secondorder

Michele Conforti, Gérard Cornuéjols, and Giacomo Zambelli. Extended formulations in combinatorial optimization. *4OR*, 8(1):1–48, 2010.

Pierre Gaillard, Gilles Stoltz, and Tim Van Erven. A second-order bound with excess losses. In *Conference on Learning Theory*, pages 176–196, 2014. URL https://arxiv.org/pdf/1402.2044.pdf.
Jean Gallier. Notes on Convex Sets, Polytopes, Polyhedra Combinatorial Topology, Voronoi Diagrams and Delaunay Triangulations. *Arxiv E-prints*, 2008. URL https://arxiv.org/pdf/0805.0292.pdf.

David Gamarnik. 15.070J: Advanced Stochastic Processes. MIT OpenCourseWare, 2013. URL https://ocw.mit.edu/courses/sloan-school-of-management/15-070j-advanced-stochastic-processes-fall-2013/.

Walter Gautschi. The incomplete gamma functions since Tricomi. In *Tricomi’s Ideas and Contemporary Applied Mathematics, Atti dei Convegni Lincei, n. 147*, Accademia Nazionale dei Lincei, pages 203–237, 1998. URL https://www.cs.purdue.edu/homes/wxg/selected_works/section_02/155.pdf.

Michel X Goemans. Smallest compact formulation for the permutahedron. *Mathematical Programming*, 153(1):5–11, 2015.

Peter Gritzmann and Victor Klee. Inner and outerj-radii of convex bodies in finite-dimensional normed spaces. *Discrete & Computational Geometry*, 7(3):255–280, 1992.

David P Helmbold and Manfred K Warmuth. Learning permutations with exponential weights. *Journal of Machine Learning Research*, 10(Jul):1705–1736, 2009. URL http://www.jmlr.org/papers/volume10/helmbold09a/helmbold09a.pdf.

Ralph Howard. Alexandrov's theorem on the second derivatives of convex functions via Rademacher's theorem on the first derivative of Lipschitz functions. University of South Carolina, Columbia, 1998. URL http://people.math.sc.edu/howard/Notes/alex.pdf.

Ruitong Huang, Tor Lattimore, András György, and Csaba Szepesvári. Following the leader and fast rates in linear prediction: curved constraint sets and other regularities. In *Advances in Neural Information Processing Systems*, pages 4970–4978, 2016.

Volker Kaibel. Extended formulations in combinatorial optimization. *arXiv preprint arXiv:1104.1023*, 2011.

Adam Tauman Kalai and Santosh Vempala. Efficient algorithms for on-line optimization. *Journal of Computer and System Sciences*, 71, 2016.

Jyrki Kivinen and Manfred Warmuth. Exponentiated Gradient versus Gradient Descent for Linear Predictors. *Information and Computation*, (132):1–63, 1997.

Xudong Li, Defeng Sun, and Kim-Chuan Toh. On the efficient computation of a generalized Jacobian of the projector over the Birkhoff polytope. *Mathematical Programming*, 2018. URL https://arxiv.org/pdf/1702.05934.pdf.
Cong Han Lim and Stephen J. Wright. Efficient bregman projections onto the permutahedron and related polytopes. In Arthur Gretton and Christian C. Robert, editors, Proceedings of the 19th International Conference on Artificial Intelligence and Statistics, volume 51 of Proceedings of Machine Learning Research, pages 1205–1213, Cadiz, Spain, 09–11 May 2016. PMLR. URL http://proceedings.mlr.press/v51/lim16.html.

Alexander E Litvak. Around the simplex mean width conjecture. In Analytic Aspects of Convexity, pages 73–84. Springer, 2018.

Haipeng Luo and Robert E Schapire. Achieving all with no parameters: Adanormalhedge. In Conference on Learning Theory, pages 1286–1304, 2015. URL https://arxiv.org/pdf/1301.0534.pdf.

Jaouad Mourtada and Stéphane Gaïffas. On the optimality of the Hedge algorithm in the stochastic regime. Journal of Machine Learning Research, 20:1–28, 2019.

Angelina Nedić and Asuman Ozdaglar. Approximate primal solutions and rate analysis for dual subgradient methods. SIAM Journal on Optimization, 19(4):1757–1780, 2009.

Renato Negrinho and André F. T. Martins. Orbit regularization. In Proceedings of the 27th International Conference on Neural Information Processing Systems - Volume 2, NIPS’14, pages 3221–3229, Cambridge, MA, USA, 2014. MIT Press. URL https://www.cs.cmu.edu/~negrinho/assets/papers/nips2014_main.pdf.

Arkadi S Nemirovski and Michael J Todd. Interior-point methods for optimization. Acta Numerica, 17:191–234, 2008. URL https://pdfs.semanticscholar.org/3a60/cfc1a4e076276d98a7ecac562c75062022ed.pdf

C. P. Nicholas and R. C. Yates. The probability integral. The American Mathematical Monthly, 57(6):412–413, 1950. ISSN 00029890, 19300972. URL http://www.jstor.org/stable/2307644

Iosif Pinelis. Optimum bounds for the distributions of martingales in Banach spaces. The Annals of Probability, 22(4):1679–1706, 10 1994. doi: 10.1214/aop/1176988477. URL https://doi.org/10.1214/aop/1176988477.

Holakou Rahmanian, David P Helmbold, and SVN Vishwanathan. Online learning of combinatorial objects via extended formulation. arXiv preprint arXiv:1609.05374, 2016.

Amir Sani, Gergely Neu, and Alessandro Lazaric. Exploiting easy data in online optimization. Proceedings of the 27th International Conference on Neural Information Processing Systems - Volume 1, pages 810–818, 2014. URL https://www.researchgate.net/publication/279258445_Exploiting_easy_data_in_online_optimization
Yevgeny Seldin and Gábor Lugosi. An improved parametrization and analysis of the EXP3++ algorithm for stochastic and adversarial bandits. *CoRR*, abs/1702.06103, 2017. URL http://arxiv.org/abs/1702.06103.

Yevgeny Seldin and Aleksandrs Slivkins. One practical algorithm for both stochastic and adversarial bandits. In Eric P. Xing and Tony Jebara, editors, *Proceedings of the 31st International Conference on Machine Learning*, volume 32 of *Proceedings of Machine Learning Research*, pages 1287–1295, Beijing, China, 22–24 Jun 2014. PMLR. URL http://proceedings.mlr.press/v32/seldinb14.html.

Shai Shalev-Shwartz. Online learning and online convex optimization. *Found. Trends Mach. Learn.*, 4(2):107–194, February 2012. ISSN 1935-8237. URL http://dx.doi.org/10.1561/2200000018.

David E. Speyer. How Wide is the Birkhoff Polytope? *Math Overflow*, 2019. URL https://mathoverflow.net/questions/339297/how-wide-is-the-birkhoff-polytope.

Tim Van Erven, Peter Grünwald, Nishant A Mehta, Mark Reid, Robert Williamson, et al. Fast rates in statistical and online learning. 2015.

Manfred K Warmuth and Dima Kuzmin. Randomized online pca algorithms with regret bounds that are logarithmic in the dimension. *Journal of Machine Learning Research*, 9(Oct):2287–2320, 2008.

Chen-Yu Wei and Haipeng Luo. More adaptive algorithms for adversarial bandits. *CoRR*, abs/1801.03265, 2018. URL http://arxiv.org/abs/1801.03265

Christophe Weibel. Minkowski sums of polytopes. Technical report, EPFL, 2007.

Eric Weisstein. Power sum. *MathWorld — A Wolfram Web Resource*. URL http://mathworld.wolfram.com/PowerSum.html.

Günter M Ziegler. Lectures on polytopes. *Springer-Verlag New York*, 1995.

Julian Zimmert and Yevgeny Seldin. An optimal algorithm for stochastic and adversarial bandits. *CoRR*, abs/1807.07623, 2018. URL http://arxiv.org/abs/1807.07623.

Martin Zinkevich. Online convex programming and generalized infinitesimal gradient ascent. *Proceedings of the Twentieth International Conference on International Conference on Machine Learning*, pages 928–935, 2003. URL http://www.cs.cmu.edu/~maz/publications/techconvex.pdf.