ON CONNECTION BETWEEN VARIATIONAL SYMMETRIES
AND ALGEBRAIC STRUCTURES

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Abstract. In the work we present a rather general approach for finding connections between the symmetries of $B_u$-potentials, variational symmetries, and algebraic structures, Lie-admissible algebras and Lie algebras. In order to do this, in the space of the generators of the symmetries of the functionals we define such bilinear operations as $(S, T)$-product, $\mathcal{G}$-commutator, commutator. In the first part of the work, to provide a complete description, we recall needed facts on $B_u$-potential operators, invariant functionals and variational symmetries. In the second part we obtain conditions, under which $(S, T)$-product, $\mathcal{G}$-commutator, commutator of symmetry generators of $B_u$-potentials are also their symmetry generators. We prove that under some conditions $(S, T)$-product turns the linear space of the symmetry generators of $B_u$-potentials into a Lie-admissible algebra, while $\mathcal{G}$-commutator and commutator do into a Lie algebra. As a corollary, similar results were obtained for the symmetry generators of potentials, $B_u \equiv I$, where the latter is the identity operator. Apart of this, we find a connection between the symmetries of functionals with Lie algebras, when they have bipotential gradients. Theoretical results are demonstrated by examples.

Keywords: variational symmetry, transformation generator, Lie-admissible algebra, Lie algebra, $(S, T)$-product, $\mathcal{G}$-commutator, commutator.

Mathematics Subject Classification: 47G40, 70S10

1. Introduction

Symmetries and the first integrals play an important role in mathematics, mechanics, physics. After work [1], a high interest to studying symmetry properties and finding the conservation laws is related with fundamental monographs [2], [3]. To find the first integrals by means of the variational symmetries one has to study the question on existence of the action functional, that is, to solve the inverse problem of the calculus of variations including that for the equations with non-potential operators. The construction of direct and indirect variational formulations for various types of equations and systems were studied, for instance, in works [4], [5], [6], [7], [8], [9], [10], [11], [12]. In works [13], [14], there was established a relation between the symmetries of Euler and non-Euler functionals with the first integrals of the corresponding motion equations. The methods for studying symmetry properties of operator equations with the second time derivative developed in [15], [16] allow one to find their first integrals including the case of non-potentiality of the operators of these equations. It was shown in monograph [17] that the symmetries of Euler functionals are also symmetries of the corresponding Euler-Lagrange equations. In works [18], [19], similar results were obtained in the general case for non-Euler functionals, to which equations with quasi-potential operators correspond. A role of algebraic structures associated with motion equations is well-known in the mechanics of finite-dimensional
Moreover, in work [24], there were obtained the conditions under which the (with respect to the commutator. These studies were continued in works [13], [14], [18], [19]. Moreover, in work [24], there were obtained the conditions under which the (8, 7)-product, the S-commutator, the commutator of the symmetry generators for the operator equations are also the symmetry generators and a relation between the symmetries of operator equations and Lie-admissible algebras and Lie algebras was found.

In view of the said above, there naturally arises a problem on establishing a relation between the variational symmetries and algebraic structures, Lie-admissible algebras and Lie algebras. The present work is devoted to this question.

Below we shall make use of the following definitions and theorems.

Below we follow the notations and terminology of works [6], [19], [24].

In view of the said above, there naturally arises a problem on establishing a relation between the variational symmetries and algebraic structures, Lie-admissible algebras and Lie algebras. The present work is devoted to this question.

Below we shall make use of the following definitions and theorems. Let $U$, $V$ be linear normed spaces over the field of real numbers $\mathbb{R}$.

**Definition 2.1** ([6]). An operator $N : D(N) \subset U \to V$ is called $B_u$-potential on the set $D(N)$ relative to a bilinear form $\Phi : V \times V \to \mathbb{R}$ if there exist a linear operator $B_u : D(B_u) \subset V \to V$ and a Gâteaux differentiable functional $F_N : D(F_N) = D(N) \to \mathbb{R}$ such that

$$
\delta F_N[u, h] = \Phi(N(u), B_u h) \quad \text{for all} \quad u \in D(N), \quad h \in D(N_u', B_u),
$$

where $D(N_u', B_u) = D(N'_u) \cap D(B_u)$.

A functional $F_N$ is called a $B_u$-potential of the operator $N$ and $N$ is called a $B_u$-gradient of the functional $F_N$.

**Theorem 2.1** ([6]). Let a Gâteaux differentiable operator $N : D(N) \subset U \to V$ and a bilinear form $\Phi : V \times V \to \mathbb{R}$ be such that for all fixed elements $u \in D(N)$, $g, h \in D(N_u', B_u)$ the function $\varepsilon \to \Phi(N(u + \varepsilon h), B_u g)$ is a continuously differentiable on the segment $[0, 1]$. Then the operator $N$ is $B_u$-potential in a simply connected domain $D(N)$ with respect to the above bilinear form if and only if the condition holds:

$$
\Phi(N'_u h, B_u g) + \Phi(N(u), B'_u(g; h)) = \Phi(N'_u g, B_u h) + \Phi(N(u), B'_u(h; g)) \tag{2.1}
$$

for all $u \in D(N)$, $g, h \in D(N'_u, B_u)$. At that, the $B_u$-potential of $F_N$ is determined by the formula

$$
F_N[u] = \int_0^1 \Phi(N(\hat{u}(\lambda)), B_{\hat{u}(\lambda)}(u - u_0)) \, d\lambda + F_N[u_0], \tag{2.2}
$$

where $\hat{u}(\lambda) = u_0 + \lambda(u - u_0)$ and $u_0$ is a fixed element in $D(N)$.

If $B_u \equiv I$ is an identical operator, then functional (2.2) becomes

$$
F_N[u] = \int_0^1 \Phi(N(\hat{u}(\lambda)), u - u_0) \, d\lambda + F_N[u_0]. \tag{2.3}
$$

On $D(N)$ we consider an infinitely small transformation defined by the formula

$$
\pi = u + \varepsilon S(u). \tag{2.4}
$$

The operator $S$ is called the transformation generator.
**Definition 2.2** ([19]). Functional (2.2) is called invariant with respect to transformation (2.4) if
\[ F_N[u + \varepsilon S(u)] = F_N[u] + r(u, \varepsilon S(u)) \quad \forall u \in D(N), \tag{2.5} \]
and
\[ \lim_{\varepsilon \to 0} \frac{r(u, \varepsilon S(u))}{\varepsilon} = 0. \]

We note that in this case transformation (2.4) is called a symmetry of functional (2.2) and the operator \( S \) is called the symmetry generator. The symmetries of the functionals are also called variational symmetries.

**Definition 2.3** ([19]). An operator \( N : D(N) \subset U \to V \) is called quasi-\( B_u \)-potential on the set \( D(N) \) with respect to a bilinear form \( \Phi : V \times V \to \mathbb{R} \) if there exists a linear operator \( B_u : D(B_u) \subset V \to V \), a Gâteaux differentiable functional \( F : D(F) = D(N) \to \mathbb{R} \) and a density of a non-\( B_u \)-potential force \( \Lambda(u) \) such that
\[ \delta F[u, h] + \Phi(\Lambda(u), B_u h) = \Phi(N(u), B_u h) \quad \text{for all} \quad u \in D(N), \quad h \in D(N'_u, B_u). \]

Let an operator \( N \) in an equation
\[ N(u) = 0 \tag{2.6} \]
is quasi-\( B_u \)-potential on \( D(N) \) with respect to a continuous non-degenerate bilinear form \( \Phi : V \times V \to \mathbb{R} \).

This means that the operator \( \tilde{N} = N - \Lambda \) is \( B_u \)-potential on \( D(N) \) with respect to \( \Phi \).

Then the corresponding functional reads as
\[ F[u] = \int_0^1 \Phi(\tilde{N}(\hat{u}(\lambda)), B_{\hat{u}(\lambda)}(u - u_0)) d\lambda + F[u_0]. \tag{2.7} \]

**Theorem 2.2** ([19]). Transformation (2.4) is a symmetry of functional (2.7) on \( D(N) \) if and only if
\[ \Phi(\tilde{N}(u), B_u S(u)) = 0 \quad \forall u \in D(N). \tag{2.8} \]

Following [6], we denote by \( A(U) \) the linear space of the operators mapping \( U \) into \( U \) with the usual operation of summing operators and the multiplication by a number in the field \( \mathbb{R} \) and we define a \((\mathcal{S}, \mathcal{T})\)-product of two operators as
\[ (S_1, S_2)(u) = S'_1uS_uS'_2uS_2(u) - S'_2u\mathcal{T}_uS_1(u), \tag{2.9} \]
and a \( \mathcal{S} \)-commutator as
\[ [S_1, S_2]\mathcal{S}(u) = S'_1u\mathcal{S}_uS'_2uS_2(u) - S'_2u\mathcal{S}_uS_1(u) \tag{2.10} \]
and a commutator
\[ [S_1, S_2](u) = S'_1uS'_2uS_2(u) - S'_2uS_1(u). \tag{2.11} \]

It was proved in [6] that a linear space \( A(U) \) is an algebra over the field \( \mathbb{R} \) with respect to the \((\mathcal{S}, \mathcal{T})\)-product. This algebra is denoted by \( \langle A(U); (\mathcal{S}, \mathcal{T}) \rangle \).

**Theorem 2.3** ([6]). If linear operators \( S_u : U \to U \) and \( \mathcal{T}_u : U \to U \) are such that the condition holds:
\[ \tilde{S}'_u(v; \tilde{S}_uh) = \tilde{S}'_u(h; \tilde{S}_uv) \quad \text{for all} \quad h, u, v \in U, \]
where \( \tilde{S}_u \equiv S_u + \mathcal{T}_u \), then the algebra \( \langle A(U); (\mathcal{S}, \mathcal{T}) \rangle \) is a Lie-admissible algebra.
3. Variational symmetries and Lie-admissible algebras

**Theorem 3.1.** If $S_1, S_2$ are symmetry generators of functional (2.7) and if there exist operators $S_u, T_u : D(N'_u, B_u) \to D(N'_u, B_u)$ such that for all $u \in D(N)$, $h, v \in D(N'_u, B_u)$ the condition

$$
\Phi(\tilde{N}' h, B_u v) + \Phi(\tilde{N}(v; S_u h)) = \Phi(\tilde{N}' h, B_u h) + \Phi(\tilde{N}(v; T_u v)),
$$

holds, then $(S, T)$-product (2.9) is also the symmetry generator of this functional.

Proof. We have

$$
\Phi(\tilde{N}(u + \varepsilon T_u S_1(u)), B_{u + \varepsilon T_u S_1(u)} S_2(u + \varepsilon T_u S_1(u))) = 0 \quad \text{for all} \quad u \in D(N),
$$
or

$$
\Phi(\tilde{N}' S_u S_1(u), S_2(u) + \Phi(\tilde{N}(u, B'_u S_2(u); T_u S_1(u)))) + \Phi(\tilde{N}(u, B_u S_2 S_1(u); T_u S_1(u))) = 0.
$$

Similarly,

$$
\Phi(\tilde{N}' h, B_u S_1(u)) + \Phi(\tilde{N}(h; S_u S_2(u))) + \Phi(\tilde{N}(h; B_u S_2 S_1(u))) = 0.
$$

Deducting the first identity and taking into consideration condition (3.1), we obtain:

$$
\Phi(\tilde{N}(u), B_u (S_1, S_2)(u)) = 0.
$$

Thus, $(S, T)$-product (2.9) is also a symmetry generator of functional (2.7), see Theorem 2.2. The proof is complete.

**Theorem 3.2.** If $S_1, S_2$ are the symmetry generators of functional (2.7) and if there exist operators $S_u, T_u : D(N'_u, B_u) \to D(N'_u, B_u)$ such that for all $u \in D(N)$, $h, v \in D(N'_u, B_u)$ the conditions

$$
\Phi(\tilde{N}' S_u h, B_u v) + \Phi(\tilde{N}(v; S_u h)) = \Phi(\tilde{N}' h, B_u h) + \Phi(\tilde{N}(v; T_u v)),
$$

$$
\tilde{G}'_u (v; \tilde{G}_u h) = \tilde{G}'_u (h; \tilde{G}_u v),
$$

hold, where $\tilde{G}_u \equiv S_u + T_u$, then the symmetry generators of functional (2.7) form a Lie-admissible algebra with respect to $(S, T)$-product (2.9).

Proof. The statement follows from Theorems 3.1 and 2.3. Let $B_u \equiv I$ be the identity operator. In this case functional (2.7) casts into the form

$$
F[u] = \int_0^1 \Phi(\tilde{N}(\lambda u), u - u_0) \, d\lambda + F[u_0],
$$

and Theorems 3.1 and 3.2 are formulated as follows.

**Theorem 3.3.** If $S_1, S_2$ are symmetry generators of functional (3.4) and if there exist the operators $S_u, T_u : D(N'_u) \to D(N'_u)$ such that for all $u \in D(N)$, $h, v \in D(N'_u)$ the condition

$$
\Phi(\tilde{N}' S_u h, v) = \Phi(\tilde{N}' h, v)
$$

holds, then $(S, T)$-product (2.9) is also the symmetry generator of this functional.

**Theorem 3.4.** If $S_1, S_2$ are symmetry generators of functional (3.4) and if there exist the operators $S_u, T_u : D(N'_u) \to D(N'_u)$ such that for all $u \in D(N)$, $h, v \in D(N'_u)$ the conditions

$$
\Phi(\tilde{N}' S_u h, v) = \Phi(\tilde{N}' h, v),
$$

$$
\tilde{G}'_u (v; \tilde{G}_u h) = \tilde{G}'_u (h; \tilde{G}_u v),
$$

hold, where $\tilde{G}_u \equiv S_u + T_u$, then the symmetry generators of functional (3.4) form a Lie-admissible algebra with respect to $(S, T)$-product (2.9).
4. Variational symmetries and Lie algebras

**Theorem 4.1.** If $S_1$, $S_2$ are the symmetry generators of functional (2.7) and if there exist an operator $S_u : D(N'_u, B_u) \to D(N'_u, B_u)$ such that for all $u \in D(N)$, $h, v \in D(N'_u, B_u)$ the condition

$$
\Phi(\tilde{N}'_u S_u h, B_u v) + \Phi(\tilde{N}(u), B'_u(v; S_u h)) = \Phi(\tilde{N}'_u S_u v, B_u h) + \Phi(\tilde{N}(u), B'_u(h; S_u v))
$$

(4.1) holds, then $S$-commutator (2.10) is also the symmetry generator for this functional.

The proof of this theorem is similar to the proof of Theorem 3.1 as $S_u \equiv S_u = T_u$.

**Theorem 4.2.** If $S_1$, $S_2$ are symmetry generators of functional (2.7) and if there exists an operator $S_u : D(N'_u, B_u) \to D(N'_u, B_u)$ such that for all $u \in D(N)$, $h, v \in D(N'_u, B_u)$ the conditions

$$
\Phi(\tilde{N}'_u S_u h, B_u v) + \Phi(\tilde{N}(u), B'_u(v; S_u h)) = \Phi(\tilde{N}'_u S_u v, B_u h) + \Phi(\tilde{N}(u), B'_u(h; S_u v)),
$$

(4.2)

and

$$
S'_u(v; S_u h) = S'_u(h; S_u v)
$$

(4.3) hold, then the symmetry generators of functional (2.7) form a Lie algebra with respect to $S$-commutator (2.10).

**Proof.** This statement follows from Theorems 4.1 and 2.3.

**Theorem 4.3.** If the operator $N$ in equation (2.6) is quasi-$B_{1u}$-potential ($i = 1, 2$) on $D(N)$ with respect to a continuous non-degenerate bilinear form $\Phi : V \times V \to \mathbb{R}$, that is, the operator $\tilde{N} = N - \Lambda$ is biopotential, $S_1$, $S_2$ are symmetry generators of functional (2.7) as $B_u = B_{1u}$, the inverse operator $B_{1u}^{-1}$ is well-defined and for all $u \in D(N)$, $h, v \in D(N'_u, B_{1u}, B_{2u})$ the condition

$$
\Phi(\tilde{N}(u), B_{1u} S'_u (v; h) - B_{1u} S'_u (h; v)) = 0
$$

(4.4) holds, where $S_u = B_{1u}^{-1} B_{2u}$, then $S$-commutator (2.10) is also a symmetry generator of functional [2.7]. If moreover

$$
S'_u(v; S_u h) = S'_u(h; S_u v)
$$

(4.5) for all $u \in D(N)$, $h, v \in D(N'_u, B_{1u}, B_{2u})$, then the symmetry generators of functional [2.7] form a Lie algebra with respect to $S$-commutator (2.10).

**Proof.** By formula (2.1) we obtain

$$
\Phi(\tilde{N}'_u S_u h, B_{1u} v) + \Phi(\tilde{N}(u), B'_{1u}(v; S_u h)) = \Phi(\tilde{N}'_u v, B_{2u} h) + \Phi(\tilde{N}(u), B'_{1u}(S_u h; v))
$$

$$
= \Phi(\tilde{N}'_u v, B_{2u} h) + \Phi(\tilde{N}(u), B'_{1u}(S_u h; v))
$$

$$
= \Phi(\tilde{N}'_u h, B_{2u} v) - \Phi(\tilde{N}(u), B'_{2u}(h; v))
$$

$$
+ \Phi(\tilde{N}(u), B'_{2u}(v; h)) + \Phi(\tilde{N}(u), B'_{1u}(S_u h; v))
$$

$$
= \Phi(\tilde{N}'_u h, B_{1u} S_u v) - \Phi(\tilde{N}(u), B'_{1u}(S_u h; v))
$$

$$
- \Phi(\tilde{N}(u), B'_{1u}(S_u h; v)) + \Phi(\tilde{N}(u), B'_{1u}(S_u h; v))
$$

$$
+ \Phi(\tilde{N}(u), B'_{1u}(S_u h; v)) + \Phi(\tilde{N}(u), B'_{1u}(S_u h; v))
$$

$$
= \Phi(\tilde{N}'_u S_u v, B_{1u} h) + \Phi(\tilde{N}(u), B'_{1u}(h; S_u v))
$$

$$
- \Phi(\tilde{N}(u), B'_{1u}(S_u v; h)) - \Phi(\tilde{N}(u), B'_{1u}(S_u v; h))
$$

$$
+ \Phi(\tilde{N}(u), B'_{1u}(S_u v; h)) + \Phi(\tilde{N}(u), B'_{1u}(S_u v; h))
$$

$$
= \Phi(\tilde{N}'_u S_u v, B_{1u} h) + \Phi(\tilde{N}(u), B'_{1u}(h; S_u v))
$$

$$
+ \Phi(\tilde{N}(u), B'_{1u}(S_u v; h)) - \Phi(\tilde{N}(u), B'_{1u}(S_u v; h)).
$$
In view of condition (4.4), identity (4.6) becomes
\[ Φ(\tilde{N}_u′S_u h, B_{1u}v) + Φ(\tilde{N}(u), B_{1u}'(v; S_u h)) = Φ(\tilde{N}_u′S_u v, B_{1u}h) + Φ(\tilde{N}(u), B_{1u}'(h; S_u v)). \]
Therefore, condition (4.1) is satisfied and by Theorem 4.1, \( G \)-commutator (2.10) is also a symmetry generator of functional (2.7). If moreover condition (4.5) holds, then by Theorem 4.2, the symmetry generators of functional (2.7) form a Lie algebra with respect to \( G \)-commutator (2.10). The proof is complete.

**Theorem 4.4.** If \( S_1, S_2 \) are the symmetry generators of functional (2.7), then commutator (2.11) is also the symmetry generator of this functional.

**Proof.** The statement follows from Theorem 4.1. We note that \( G \equiv I \), where \( I \) is the identity operator and this is why condition (4.1) is satisfied since in this case it ensures quasi-\( B_u \)-potentiality of the operator \( N \) in equation (2.6). The proof is complete.

**Theorem 4.5.** The symmetry generators of functional (2.7) form a Lie algebra with respect to operation (2.11).

**Proof.** This statement follows from Theorems 4.2 and 4.4. In this case \( S_u' \) is the zero operator and this is why condition (4.3) is satisfied identically. The proof is complete.

Thus, in certain cases, Theorems 3.1, 4.1 and 4.4 can be employed for constructing symmetry generators of functional (2.7) by at least two known symmetry generators.

Let \( B_u \equiv I \) be the identity operator. In this case functional (2.7) becomes (3.4) and Theorems 4.1, 4.2, 4.4, 4.5 are formulated as follows.

**Theorem 4.6.** If \( S_1, S_2 \) are symmetry generators of functional (3.4) and if there exists an operator \( S_u : D(N_u') \to D(N_u') \) such that for all \( u \in D(N), h, v \in D(N_u') \) the condition
\[ Φ(\tilde{N}_u'S_u h, v) = Φ(\tilde{N}_u'S_u v, h) \] (4.7)
holds, then \( G \)-commutator (2.10) is also a symmetry generator of this functional.

**Theorem 4.7.** If \( S_1, S_2 \) are symmetry generators of functional (3.4) and if there exists an operator \( S_u : D(N_u') \to D(N_u') \) such that for all \( u \in D(N), h, v \in D(N_u') \) the conditions
\[ Φ(\tilde{N}_u'S_u h, v) = Φ(\tilde{N}_u'S_u v, h), \] (4.8)
\[ S_u'(v; S_u h) = S_u'(h; S_u v) \] (4.9)
hold, then the symmetry generators of functional (3.4) form a Lie algebra with respect to \( G \)-commutator (2.10).

**Theorem 4.8.** If \( S_1, S_2 \) are symmetry generators of functional (3.4), then their commutator (2.11) is also a symmetry generator for this functional.

**Theorem 4.9.** Symmetry generators of functional (3.4) form a Lie algebra with respect to operation (2.11).

5. **Examples**

1. We consider an equation
\[ N(u) \equiv u_{tt} - u_{xx} + u_x = 0, \quad (x, t) \in \Omega = (a, b) \times (t_0, t_1). \] (5.1)
We let
\[ D(N) = \{ u \in U = C^∞(\Omega) : u|_{t=t_0} = \varphi_1(x), u|_{t=t_1} = \varphi_2(x) (x \in (a, b)), u|_{x=a} = \psi_1(t), u|_{x=b} = \psi_2(t) (t \in (t_0, t_1)) \}. \] (5.2)
where $\varphi_i \in C[a, b], \psi_i \in C[t_0, t_1], i = 1, 2$. We observe that the operator $N$ in (5.1) is quasipotential on set $D(N)$ with respect to a classical bilinear form

$$\Phi(v, g) = \int_0^t \int_a^b v(x, t) g(x, t) \, dx \, dt.$$  

In this case

$$\tilde{N}(u) = u_{tt} - u_{xx}, \quad \Lambda(u) = u_x.$$

The corresponding functional is the form

$$F[u] = -\frac{1}{2} \int_0^t \int_a^b (u_t^2 - u_x^2) \, dx \, dt.$$  

(5.3)

We shall assume that $u_x \in D(N'_u)$ and $u_t \in D(N'_u)$.

The operators $S_1 = D_x$ and $S_2 = D_t$ are symmetry generators of functional (5.3). This is implied by Theorem 2.2 as $B_u \equiv I$ is the identity operator since

$$\Phi(N, S_1(u)) = \int_0^t \int_a^b (u_{tt} - u_{xx}) u_x \, dx \, dt = \int_0^t \int_a^b \left( -u_t u_x - \frac{1}{2} D_x (u_x^2) \right) \, dx \, dt$$

$$= \int_0^t \int_a^b \left( -\frac{1}{2} D_x (u_t^2) - \frac{1}{2} D_x (u_x^2) \right) \, dx \, dt = 0$$

and

$$\Phi(N, S_2(u)) = \int_0^t \int_a^b (u_{tt} - u_{xx}) u_t \, dx \, dt = \int_0^t \int_a^b \left( \frac{1}{2} D_t (u_t^2) + u_x u_{tx} \right) \, dx \, dt$$

$$= \int_0^t \int_a^b \left( \frac{1}{2} D_t (u_t^2) + \frac{1}{2} D_t (u_x^2) \right) \, dx \, dt = 0.$$

Let us also assume that

$$\frac{\partial^{i+1} u}{\partial t^i \partial x^i} \in D(N'_u), \quad i \in \mathbb{N}.$$

Condition (3.5) holds with $S_u \equiv S = D_x$ and $T_u \equiv T = -D_x$. Indeed,

$$\Phi(N'_u S_h, v) = \int_0^t \int_a^b (D_t - D_{xx}) h_x \cdot v \, dx \, dt$$

$$= \int_0^t \int_a^b (h_{tx} - h_{xxx}) v \, dx \, dt = -\int_0^t \int_a^b (v_{tx} - v_{xxx}) h \, dx \, dt$$

$$= -\int_0^t \int_a^b (D_t - D_{xx}) v_x \cdot h \, dx \, dt = \Phi(N'_u T v, h).$$

Then by Theorem 3.3 ($S, T$)-product of generators $S_1$ and $S_2$

$$(S_1, S_2)(u) = S'_1 S S_2(u) - S'_2 S T S_1(u) = D_x D_x u_t - D_t (-D_x) u_x = 2 u_{txx}$$

is also a symmetry generator of functional (5.3).
In this case $\mathcal{S}_u \equiv \mathcal{S} + \mathcal{T} = D_x - D_x = 0$ and this is why condition \((3.7)\) is also satisfied. By Theorem \([3.4]\) symmetry generators of functional \((5.3)\) form a Lie-admissible algebra with respect to \((\mathcal{S}, \mathcal{T})\)-product
\[
(S_1, S_2)(u) = S'_{1u} D_x S_2(u) + S'_{2u} D_x S_1(u).
\]

2. We consider an equation
\[
N(u) \equiv u_t + uu_{xx} + u^2_x = 0, \quad (x, t) \in Q = (a, b) \times (t_0, t_1).
\]
Let
\[
D(N) = \{ u \in U = C^\infty(\overline{Q}) : u|_{t=t_0} = \varphi_1(x), \; u|_{t=t_1} = \varphi_2(x) \; (x \in (a, b)), \; u|x=a = u|x=b = 0 \},
\]
where $\varphi_i \in C[a, b], \; i = 1, 2$.

We observe that operator $N$ of form \((5.4)\) is quasi-$B_{1u}$-potential on set $D(N)$ \((5.5)\) with respect to a classical bilinear form
\[
\Phi(v, g) = \int_{t_0}^{t_1} \int_{a}^{b} v(x, t)g(x, t)dxdt.
\]
In this case
\[
\tilde{N}(u) = uu_{xx} + u^2_x, \quad \Lambda(u) = u_t, \quad B_{1u} \equiv B_1 = D_x^{-1} D_x^{-1},
\]
where
\[
D_x^{-1}v(x, t) = \int_{a}^{x} v(y, t)dy.
\]
The corresponding functional reads as
\[
F[u] = \frac{1}{6} \int_{t_0}^{t_1} \int_{a}^{b} u^3 dxdt.
\]
Operator $\tilde{N}$ of form \((5.7)\) is $B_{2u}$-potential on set $D(N)$ \((5.5)\) with respect to bilinear form \((5.6)\), where $B_{2u} = uI$ and $I$ is the identity operator. Thus, operator $N$ in \((5.4)\) is quasi-$B_{1u}$-potential \((i = 1, 2)\) on $D(N)$ \((5.5)\) with respect to bilinear form \((5.6)\).

The operators $S_1 = D_x$ and $S_2(u) = uu_x$ are the symmetry generators of functional \((5.8)\). This is implied from Theorem \([2.2]\) since
\[
\Phi(\tilde{N}(u), B_1 S_1(u)) = \int_{t_0}^{t_1} \int_{a}^{b} (uu_{xx} + u^2_x) D_x^{-1} D_x^{-1} u_x dxdt = \frac{1}{2} \int_{t_0}^{t_1} \int_{a}^{b} D_{xx} u^2 \cdot D_x^{-1} D_x^{-1} u_x dxdt
\]
\[
= \frac{1}{2} \int_{t_0}^{t_1} \int_{a}^{b} u^2 u_x dxdt = \frac{1}{6} \int_{t_0}^{t_1} \int_{a}^{b} D_x u^3 dxdt = 0
\]
and
\[
\Phi(\tilde{N}(u), B_1 S_2(u)) = \int_{t_0}^{t_1} \int_{a}^{b} (uu_{xx} + u^2_x) D_x^{-1} D_x^{-1} (uu_x) dxdt
\]
\[
= \frac{1}{2} \int_{t_0}^{t_1} \int_{a}^{b} D_{xx} u^2 \cdot D_x^{-1} D_x^{-1} (uu_x) dxdt
\]
\[
\frac{1}{2} \int_{t_0}^{t_1} \int_a^b u^3 u_x dx dt = \frac{1}{8} \int_{t_0}^{t_1} \int_a^b D_x u^4 dx dt = 0.
\]

Condition (4.4) holds since

\[ G_u v = D_{xx} (uv), \quad G'_u (v; h) = D_{xx} (vh) \]

and

\[
B_{1u} G'_u (v; h) - B_{1u} G'_u (h; v) = D^{-1}_x D^{-1}_x (D_{xx} (vh)) - D^{-1}_x D^{-1}_x (D_{xx} (hv)) = vh - hv = 0.
\]

Then by Theorem 4.3 \( \mathcal{S} \)-commutator

\[
[S_1, S_2]_\mathcal{S} (u) = S'_{1u} G_u S_2 (u) - S'_{2u} G_u S_1 (u)
\]

\[
= D_x D_{xx} \left( u^2 u_x \right) - \left( u_x I + u D_x \right) D_{xx} (u u_x)
\]

\[
= 9 u_{xx} u_x^2 + 3 u_{xx} u_{xxx} + 3 u_x^2
\]

is a symmetry generator of functional (5.8).

We note that in this case condition (4.5) is not satisfied since

\[
G'_u (v; S_u h) = D_{xx} (v D_{xx} (uh)), \quad G'_u (h; S_u v) = D_{xx} (h D_{xx} (uv)).
\]

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