On the analyticity of the trajectories of the particles in the patch problem for 2D Euler and aggregation equations.

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August 1, 2019

Abstract

We give a proof of the analyticity in time for the particle trajectories associated with the solutions of some transport equations when the initial datum is a patch. These results are obtained from a precise study of the Beurling transform, which provides estimates for the solutions of some new equations satisfied by the lagrangian flow.

1 Introduction

Let us consider some classical transport equations in the plane. One of these is the 2-D Euler equation, in vorticity form, for an incompressible inviscid fluid:

\[
\begin{cases}
\partial_t \omega + v \cdot \nabla \omega = 0, \\
\omega(0) = \omega_0,
\end{cases}
\]

(E)

where \( \omega \) is a scalar function representing this vorticity, and \( v \) is the velocity of the fluid. In this case the vorticity and the velocity of the fluid are related by the Biot-Savard law

\[ v(t) = \omega(t) * \left( \frac{x}{|x|^2} \right), \]

where \( x \in \mathbb{R}^2 \).

Another equation, related with biological systems, is the aggregation equation

\[
\begin{cases}
\partial_t \rho + v \cdot \nabla \rho = 0, \\
\rho(0) = \rho_0,
\end{cases}
\]

(\(\tilde{A}\))

\( \rho \) representing the density of mass of an irrotational inviscid and compressible fluid, where

\[ v(t) = \rho(t) * \left( \frac{x}{|x|^2} \right), \]

Equation (\(\tilde{A}\)) is closely related to the continuity equation

\[
\begin{cases}
\partial_t \rho + \text{div}(\rho v) = 0, \\
\rho(0) = \rho_0,\end{cases}
\]

(A)
See [BLL12] for the relationship between equations \( A \) and \( \tilde{A} \) and problems in Science and Technology.

A classical theorem due to Yudovich asserts that for any initial datum in \( (L^\infty \cap L^p)(\mathbb{R}^2) \), where \( 1 < p < \infty \), there is a unique weak solution for the equation \( E \). Using similar arguments an analogous result is obtained for the equation \( A \), and in both cases the solution depends continuously on the time variable and is as regular in the space variables as the datum (see [Jud63], [Che98], [MB02], [BLL12]).

The aim of this paper is to prove real analyticity of the particle trajectories of these fluids. These trajectories are globally described by a flow associated to the velocity field \( v \). More precisely we will prove its analyticity in time. This regularity in time has largely already been studied (see [Che92], [Her18], [Shn12], [FZ14], [Sue11], [Ser95], [HB14]).

We will use complex notation, owing to the presence of some intrinsic objects of complex analysis and also because it simplifies both the arguments and notation.

### 1.1 Basic notation

In \( \mathbb{C} \) we consider the standard coordinate \( z = x + iy \). Then
\[
\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)
\]
and we have, identifying \( v = (v_1, v_2) \) with \( v = v_1 + iv_2 \), that
\[
\text{curl } v = 2 \Im \left( \frac{\partial v}{\partial z} \right) = \frac{1}{i} \left( \frac{\partial v}{\partial \zeta} - \frac{\partial \bar{v}}{\partial \bar{\zeta}} \right)
\]
and
\[
\text{div } v = 2 \Re \left( \frac{\partial v}{\partial z} \right) = \frac{\partial v}{\partial \zeta} + \frac{\partial \bar{v}}{\partial \bar{\zeta}}.
\]

We will denote by \( m \) the Lebesgue measure in \( \mathbb{R}^2 \), associated to the standard volume form in \( \mathbb{C}, \frac{1}{i} \, dz \wedge d\bar{z} \).

The conjugate Cauchy transform, inverting the operator \( \frac{\partial}{\partial z} \), will be denoted by
\[
\tilde{C}[\varphi](z) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\varphi(\zeta)}{\zeta - z} \, d\bar{z} \wedge dz = \frac{1}{\pi} \int_{\mathbb{C}} \frac{\varphi(\zeta)}{\zeta - z} \, dm,
\]
defined for suitable functions \( \varphi \). Then
\[
\frac{\partial}{\partial z} \tilde{C}[\varphi] = \varphi
\]
and the derivative
\[
\tilde{B}[\varphi](z) = \left( \frac{\partial}{\partial \bar{z}} \tilde{C}[\varphi] \right)(z) = \text{p. v.} \, \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\varphi(\zeta)}{(\zeta - z)^2} \, d\bar{z} \wedge dz = \text{p. v.} \, \frac{-1}{\pi} \int_{\mathbb{C}} \frac{\varphi(\zeta)}{(\zeta - z)^2} \, dm,
\]
the conjugate Beurling transform.

Let $\phi$ a test function supported in the unit ball, whose integral is equal to 1 and such that $\phi(0) = 1$, we consider, for a distribution $T$ the limit

$$
\lim_{\epsilon \to 0} \langle T, \phi_{x_0, \epsilon} \rangle,
$$

(1)

where $\phi_{x_0, \epsilon}(x) = \frac{1}{\epsilon^2} \phi\left(\frac{x-x_0}{\epsilon}\right)$.

If this limit exists at some point $x_0$ and is independent of the choice of $\phi$ we call it density function of $T$ at $x_0$ and denote it by $\Theta(T, x_0)$.

In fact, we have

**Lemma 1.** If $\Omega \Subset \mathbb{C}$ a domain such that $\partial \Omega \in C^1$, then, if $T_{\chi \Omega}$ is the distribution given by $\chi_{\Omega}$, then we have

$$
\Theta(T_{\chi \Omega}, z_0) = \begin{cases} 
1 & \text{if } z_0 \in \Omega, \\
\frac{1}{2} & \text{if } z_0 \in \partial \Omega, \\
0 & \text{if } z_0 \in \Omega^c.
\end{cases}
$$

**Proof.** Let $\rho$ be a conveniently chosen defining function for $\Omega$ (see [Bur06]). $x_0 \in \partial \Omega$ and then consider

$$
\kappa(x; x_0) = (\nabla \rho(x_0), x - x_0)
$$

and

$$
\omega(x; x_0) = \rho(x) - \kappa(x; x_0).
$$

Then, taking the function $\phi_{x_0, \epsilon}$ used in the definition of (1), we have

$$
\int_{B_r(z_0) \cap \Omega} \phi_{z_0, \epsilon}(z) \, dm(z) = \int_{B_r(z_0) \cap \{\rho < 0\}} \phi_{z_0, \epsilon}(z) \, dm(z) = (*),
$$

and since

$$
\{\rho < 0\} = (\{\rho < 0\} \cap \{\kappa < 0\}) \cup (\{\rho < 0\} \cap \{\kappa \geq 0\}),
$$

then

$$
(* = \int_{B_r(z_0) \cap \{\kappa < 0\}} \phi_{z_0, \epsilon}(z) \, dm(z)
+ \int_{B_r(z_0) \cap \{\rho < 0\} \cap \{\kappa \geq 0\}} \phi_{z_0, \epsilon}(z) \, dm(z)
- \int_{B_r(z_0) \cap \{\rho \geq 0\} \cap \{\kappa < 0\}} \phi_{z_0, \epsilon}(z) \, dm(z).
$$

So, after a rotation

$$
\lim_{\epsilon \to 0} \int_{B_r(z_0) \cap \{\kappa < 0\}} \phi_{z_0, \epsilon}(z) \, dm(z) = \lim_{\epsilon \to 0} \int_{B_r(z_0) \cap \mathbb{R}^2_+} \phi_{z_0, \epsilon}(z) \, dm(z) = \frac{1}{2}.
$$
and
\[
\left| \int_{B_\epsilon(z_0) \cap \{ \rho < 0 \} \cap \{ \kappa \geq 0 \}} \phi_{z_0, \epsilon}(z) \, dm(z) - \int_{B_\epsilon(z_0) \cap \{ \rho \geq 0 \} \cap \{ \kappa < 0 \}} \phi_{z_0, \epsilon}(z) \, dm(z) \right|
\leq \frac{1}{\epsilon^2} \{ m(B_\epsilon(z_0) \cap \{ 0 \leq \kappa < -\omega \}) + m(B_\epsilon(z_0) \cap \{ -\omega \leq \kappa < 0 \}) \}.
\]

Now, the set \( B_\epsilon(z_0) \cap \{ 0 \leq \kappa < -\omega \} \) is contained in a cylinder that has the direction of \( \nabla \rho(z_0) \parallel \nabla \rho(z_0) \) and tall \( \epsilon \) and the radius of the basis equal to \( \epsilon \), so its measure vanishes as \( \epsilon \to 0 \).

The other term is similar.

We will use the space \( C^{k, \gamma}(U) \), were \( U \) is an open subset of the plane, \( k \) is a non-negative integer and \( 0 < \gamma < 1 \). It is the space of functions with continuous derivatives up to the order \( k \) such that each derivative of order \( k \) extends to a \( \gamma \)-Hölder function in the closure of \( U \).

In this paper we will mainly use the spaces \( C^{k, \gamma}(U) \) for \( k = 0, 1 \), equipped with the norms
\[
\|f\|_\gamma = \|f\|_\infty + \sup_{z \neq w, z, w \in U} \frac{|f(z) - f(w)|}{|z - w|^\gamma}
\]
and
\[
\|f\|_{1, \gamma} = \|f\|_\infty + \|\nabla f\|_\gamma = \|f\|_\infty + \left\| \frac{\partial f}{\partial \bar{z}} \right\|_\gamma.
\]

We let \( C^\omega \) stand for a space of real analytic functions and \( D(A) \) for the compactly supported \( C^\infty \) functions whose support is constrained in a set \( A \subset \mathbb{R}^n \).

### 1.2 Statement of results

Let \( \Omega \subset \mathbb{C} \) be a bounded domain such that \( \partial \Omega \in C^{1, \gamma} \) for \( \gamma \in (0, 1) \).

Let \( f \in C^\infty(\mathbb{C}) \) and consider the distribution \( T = f T_{\chi_\Omega} \). Then it is an immediate fact that \( T \) has a density function
\[
\Theta(T, z) = \begin{cases} f(z) & \text{if } z \in \Omega, \\ \frac{f(z)}{2} & \text{if } z \in \partial \Omega, \\ 0 & \text{if } z \notin \Omega. \end{cases}
\]

(2)

We will consider
\[
\varpi_0(z) = \Theta(T, z)
\]
as the initial datum of a Cauchy problem (i.e. vorticity for the Euler equation and density for aggregation and transport equations) and then we define
\[
v_0 = C[\varpi_0]
\]
as the initial velocity. We have (cf. [AIM09, section 4.3.2]) that \( v_0 \in \text{Lip}(1, \mathbb{C}) \cap C^\infty(\mathbb{C} \setminus \partial \Omega) \).

We will establish the analyticity in time for the flow associated to the problems \((E), (A)\) and \(\tilde{A}\) in the case in which the initial datum of patches, namely the initial datum \( \rho_0 \) or \( \omega_0 \) is the characteristic function of \( \Omega \). In the current literature, this is the patch problem.

For doing so, we first study the local analyticity of solutions of the general problem \((3)\) below. In fact, we have

**Theorem 1.** Given \( \varpi_0 \) as in \((2)\), if \( \beta_0 : \mathbb{C} \to (0, \infty) \) is a function, define

\[
\Upsilon = \{(z, t) : z \in \mathbb{C}; \quad t \in (-\beta_0(z), \beta_0(z))\},
\]

and

\[
\Upsilon_{\partial \Omega} = \{(z, t) : z \in \mathbb{C} \setminus \partial \Omega; \quad t \in (-\beta_0(z), \beta_0(z))\}.
\]

Let \( a : \Upsilon \to \mathbb{C} \) a function in \( C^\infty(\Upsilon_{\partial \Omega}) \) and such that \( t \to a(z, t) \) is analytic in \((-\beta_0(z), \beta_0(z))\) for every \( z \in \mathbb{C} \).

Then the problem

\[
\begin{cases}
\frac{\partial^2 \psi}{\partial t \partial z} - \frac{\partial^2 \psi}{\partial t \partial \bar{z}}(z,t) = \varpi_0(z) \left(1 + a(z, t) t\right), \\
\psi(z,0) = z,
\end{cases}
\tag{3}
\]

has a solution \( \psi \) in \( \Upsilon \), such that \( t \to \psi(z, t) \) is in \( C^\omega((-\beta_0(z), \beta_0(z))\) for every \( z \in \mathbb{C} \).

The main consequence of this theorem concerns the solution for the patch problem for Euler and aggregation equations. The solution \( \omega \) or \( \rho \) of these problems give rise to a (velocity) field \( v \) that can be written in complex form.

For the case \( \omega(t) \ast \left(\frac{x}{|x|^2}\right) \), this is

\[
v(t) = \omega(t) \ast \left(\frac{y - ix}{|z|^2}\right) = (i \omega(t)) \ast \left(\frac{1}{z}\right) = \pi \hat{C}[i \omega(t)],
\]

and for \( \rho(t) \ast \left(\frac{x + iy}{|z|^2}\right) \) is

\[
v(t) = \rho(t) \ast \left(\frac{x + iy}{|z|^2}\right) = \rho(t) \ast \left(\frac{1}{z}\right) = \pi \hat{C}[ho(t)].
\]

So \( \frac{\partial v}{\partial t} = \omega \) or \( \rho \), where \( \omega \) is the vorticity in Euler’s equation \((\text{div} \, v = 0)\) or the density of mass in the aggregation equation \((\text{curl} \, v = 0)\).

In both cases there exist an associated flow,

\[
\psi(z, t) = z + \int_0^t v(\psi(z, \tau), \tau) \, d\tau,
\]

in \( \mathbb{C} \times [-T, T] \). Moreover
Theorem 2. Let $\Omega \subset \mathbb{C}$ a bounded domain such that $\partial\Omega \in C^{1,\gamma}$ for $\gamma \in (0,1)$. If $\psi$ is the flow corresponding to the solution of the equations (E), (A) or (A) with initial condition $\chi_{\Omega}$, then the function

$$t \to \psi(z, t), \quad z \in \mathbb{C}$$

is in $C^\omega(I)$, where $I$ is the interval of existence of the flow.

1.3 Plan of the paper

The remaining of the paper is devoted to the proof of the two theorems above in three sections.

In section 2 we prove Theorem 2 showing that, since for both Euler and aggregation the vorticity or the density are transported by the corresponding flow, then Theorem 1 applies locally providing an analytic solution near $t = 0$.

The persistence of the regularity, established in [Che93] and [BC93] in the Euler case and in [BGLV16] in the aggregation case allows the extension of this solution at all values of $t$. Then the uniqueness of the solution, established in [Jud63] (see also [Che1] or [BC93] in the Euler case and in [BLL12] in the aggregation case) shows that the flows are analytic.

In section 3 we prove Theorem 1 following a standard a priori method based in power series developments for $\psi$ in the equation

$$\frac{\partial^2 \psi}{\partial t \partial z} \frac{\partial \psi}{\partial z} - \frac{\partial^2 \psi}{\partial t \partial \bar{z}} \frac{\partial \psi}{\partial \bar{z}} = \varpi_0(z) e^{2 \Re[\varpi_0(z)]} t,$$

where $\varpi_0$ is essentially $i\omega_0$ or $\rho_0$, that changes the PDE problem to a system of functional equations. Our procedure is inspired in the paper [FZ14], showing analyticity of flows in a different context.

Section 4 is devoted to some very technical facts. The proof of Theorem 1 heavily relies on some formulas and precise bounds for the (conjugate) Beurling transform on domains with regular and bounded boundary. This technical result is the statement of Theorem 3 in section 4.

2 Proof of Theorem 2

First of all we show that we can reduce Euler and aggregation equations, to Theorem 1 at least locally.

Lemma 2. Let $V \subset \mathbb{C}$ an open subset, $\alpha: V \to (0, \infty)$. Consider in

$$U = \{(z, t) : z \in V, t \in (-\alpha(z), \alpha(z))\}$$

a complex valued function $a$, such that there exist functions $v$ and $\psi$ defined in $U$ and regular enough such that

$$\frac{\partial v}{\partial z}(z, t) = a(z, t)$$
\[
\frac{\partial \psi}{\partial t} (z, t) = v(\psi(z, t), t),
\]

then
\[
\left( \frac{\partial^2 \psi}{\partial t \partial z} - \frac{\partial^2 \psi}{\partial t \partial \bar{z}} \frac{\partial \psi}{\partial \bar{z}} \right) (z, t) = a(\psi(z, t), t) \left( \left| \frac{\partial \psi}{\partial z} \right|^2 - \left| \frac{\partial \psi}{\partial \bar{z}} \right|^2 \right) (z, 0) e^{2 \int_0^t \Re \{a(\psi(z, \tau), \tau)\} \, d\tau}.
\]

**Proof.** Taking derivatives with respect to \( z \) in (4) we have
\[
\frac{\partial^2 \psi}{\partial t \partial z} (z, t) = \frac{\partial v}{\partial \zeta}(\psi(z, t), t) \frac{\partial \psi}{\partial z} (z, t) + \frac{\partial v}{\partial \bar{\zeta}}(z, t) \frac{\partial \bar{\psi}}{\partial \bar{z}} (z, t) = a(\psi(z, t), t) \frac{\partial \psi}{\partial z} (z, t) + \frac{\partial v}{\partial \zeta}(z, t) \frac{\partial \bar{\psi}}{\partial \bar{z}} (z, t)
\]
and multiplying by \( \frac{\partial \psi}{\partial z} \) we have
\[
\frac{\partial^2 \psi}{\partial t \partial z} (z, t) \frac{\partial \psi}{\partial z} (z, t) = a(\psi(z, t), t) \left| \frac{\partial \psi}{\partial z} \right|^2 (z, t) + \frac{\partial v}{\partial \zeta}(z, t) \frac{\partial \bar{\psi}}{\partial \bar{z}} (z, t) \frac{\partial \psi}{\partial z} (z, t).
\]

Also taking derivatives with respect to \( \bar{z} \) in (4), we have
\[
\frac{\partial^2 \psi}{\partial t \partial \bar{z}} (z, t) = \frac{\partial v}{\partial \zeta}(\psi(z, t), t) \frac{\partial \psi}{\partial \bar{z}} (z, t) + \frac{\partial v}{\partial \bar{\zeta}}(z, t) \frac{\partial \bar{\psi}}{\partial \bar{z}} (z, t) = a(\psi(z, t), t) \frac{\partial \bar{\psi}}{\partial \bar{z}} (z, t) + \frac{\partial v}{\partial \zeta}(z, t) \frac{\partial \psi}{\partial z} (z, t)
\]
and multiplying by \( \frac{\partial \bar{\psi}}{\partial \bar{z}} \) we have
\[
\frac{\partial^2 \psi}{\partial t \partial \bar{z}} (z, t) \frac{\partial \bar{\psi}}{\partial \bar{z}} (z, t) = a(\psi(z, t), t) \left| \frac{\partial \bar{\psi}}{\partial \bar{z}} \right|^2 (z, t) + \frac{\partial v}{\partial \zeta}(z, t) \frac{\partial \psi}{\partial z} (z, t) \frac{\partial \bar{\psi}}{\partial \bar{z}} (z, t).
\]

Subtracting (6) from (5), and taking advantage of a cancellation we conclude that
\[
\left( \frac{\partial^2 \psi}{\partial t \partial z} - \frac{\partial^2 \psi}{\partial t \partial \bar{z}} \frac{\partial \psi}{\partial \bar{z}} \right) (z, t) = a(\psi(z, t), t) \left( \left| \frac{\partial \psi}{\partial z} \right|^2 - \left| \frac{\partial \psi}{\partial \bar{z}} \right|^2 \right).
\]

Moreover, we have
\[
\frac{\partial}{\partial t} \left( \left| \frac{\partial \psi}{\partial z} \right|^2 - \left| \frac{\partial \psi}{\partial \bar{z}} \right|^2 \right) (z, t) = 2 \Re \left\{ \left( \frac{\partial^2 \psi}{\partial t \partial z} - \frac{\partial^2 \psi}{\partial t \partial \bar{z}} \frac{\partial \psi}{\partial \bar{z}} \right) (z, t) \right\} = 2 \Re \{a(\psi(z, t), t)\} \left( \left| \frac{\partial \psi}{\partial z} \right|^2 - \left| \frac{\partial \psi}{\partial \bar{z}} \right|^2 \right) (z, t),
\]

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and if \(|\frac{\partial \psi}{\partial z}|^2 - |\frac{\partial \psi}{\partial \bar{z}}|^2| (z, t)\) never vanishes on \(U\), then

\[
\left(\left|\frac{\partial \psi}{\partial z}\right|^2 - \left|\frac{\partial \psi}{\partial \bar{z}}\right|^2\right)(z, t) = \left(\left|\frac{\partial \psi}{\partial z}\right|^2 - \left|\frac{\partial \psi}{\partial \bar{z}}\right|^2\right)(z, 0) e^{2 \int_0^t \Re\{a(\psi(z, \tau), \tau)\} d\tau}.
\]

Remark. The jacobian of \(\psi\) as considered a map from \(\mathbb{R}^2\) into itself is, in complex coordinates,

\[
J_z \psi(z, t) = \left(\frac{\left|\frac{\partial \psi}{\partial z}\right|^2}{\left|\frac{\partial \psi}{\partial \bar{z}}\right|^2}\right)(z, t).
\]

Now

1. In the case of the \textit{vortex patch problem}, we start by considering, for the case of the \textit{Euler equation}, the \textbf{purely imaginary valued} function defined on \(\mathbb{C}\) by

\[
\varpi_0 = ic\left\{\chi \Omega + \frac{1}{2} \chi \partial \Omega\right\},
\]

where \(c \in \mathbb{R}\).

Then

\[
\nu_0(z) = \hat{C}[\varpi_0](z)
\]

and by Yudovich’s theorem there exist functions \(\varpi\) and \(v\) defined in \(\mathbb{C} \times \mathbb{R}\) such that \(\varpi \in L^\infty(\mathbb{C} \times \mathbb{R}) \cap L^\infty_{\text{loc}}(\mathbb{R}; L^p(\mathbb{C}))\) for \(1 < p < \infty\) and takes its values in \(i \mathbb{R}\), and \(v\) satisfies

\[
\frac{\partial v}{\partial z}(z, t) = \frac{1}{2} \varpi(z, t)
\]

in the distributions sense, and the couple \((\varpi, v)\) satisfy the Euler equation \((E)\) in the standard weak sense.

Moreover, there exists a unique function \(\psi \in \mathcal{C}(\mathbb{C} \times \mathbb{R}; \mathbb{C})\) such that

\[
\psi(z, t) = z + \int_0^t v(\psi(z, \tau), \tau) d\tau,
\]

and there is a constant \(C > 0\) such that for any \(t \in \mathbb{R}\),

\[
\psi(z, t) = I \in \mathcal{C}^{-c_{\psi(=0)(L^p;L^\infty)}}(\mathbb{C}).
\]

Then, from the particular shape of \(\varpi_0\), we have that if \(U \subset \Omega\) or \(U \subset \mathbb{C} \setminus \bar{\Omega}\), then \(\nu_0 \in \mathcal{C}^\infty(U)\) and then, using for instance Proposition 8.3 in [MB02], we can conclude that for any \(t \in \mathbb{R}\),

\[
w(z, t) \in \mathcal{C}^\infty(\psi(U, t))
\]

and

\[
v(z, t) \in \mathcal{C}^\infty(\psi(U, t)).
\]

Moreover, both the velocity and the flow given by the function \(\psi\) in the theorem are globally defined with respect to the time variable and in general are regular beyond the continuity in the \(z\) variable.
The flow $\psi$ also inherits the local space regularity (after derivation under the integral sign). Moreover, using Theorem 1.3.1 of Chapter 1 in [H97], we have that $\psi \in C^1([0, \infty) \times U)$ and $\frac{\partial \psi}{\partial t}(\cdot, t) \in C^1(U)$.

Then the incompressibility of the fluid can be written as

$$\left| \frac{\partial \psi}{\partial z} \right|^2 - \left| \frac{\partial \psi}{\partial \bar{z}} \right|^2 \equiv 1,$$

and the vorticity is constant along the flow lines.

From these facts and Lemma 2, we get the relationship between the flow and the vorticity

$$\left( \frac{\partial^2 \psi}{\partial t \partial z} \frac{\partial \psi}{\partial z} - \frac{\partial^2 \psi}{\partial t \partial \bar{z}} \frac{\partial \psi}{\partial \bar{z}} \right)(z, t) = \frac{1}{2} \varpi_0(z),$$

in the region of $C \times \mathbb{R}$ where it makes sense.

The formula above falls in the so-called lagrangian approach. This approach was introduced by A. Cauchy in [Can16] and used by several authors since then (e.g., [FZ14] or [Shn12]).

2. For the case of the aggregation equation, we consider the real valued function defined on $C$

$$\varpi_0 = c \left\{ \chi_{\Omega} + \frac{1}{2} \chi_{\partial \Omega} \right\},$$

where $c \in \mathbb{R}$.

Again

$$v_0(z) = \mathcal{C}[\varpi_0](z)$$

and, as proven in [BLL12 Theorems 2.3, 2.4 and 3.1], there exists a constant $T = T(c)$ and functions $\varpi$ and $v$ defined in $C \times [0, T)$ such that $\varpi \in L^\infty(C \times [0, T)) \cap C([0, T); L^1(C))$, also $\varpi(\cdot, t)$ has bounded support for each $t \in [0, T)$, and the function

$$\frac{\partial v}{\partial z}(z, t) = \frac{1}{2} \varpi(z, t)$$

in a weak sense. The functions $\varpi$ and $v$ are unique, solving the equation

$$\begin{cases} 
\frac{\partial \varpi}{\partial t} + 2 \Re(\frac{\partial (\varpi v)}{\partial z}) = 0, \\
\varpi(\cdot, 0) = \varpi_0.
\end{cases} \quad (A)$$

Moreover, $\varpi(\cdot, t)$ has compact support for each $t \in [0, T)$.

All this implies that $v_0 \in \text{Lip}(1, C) \cap C^\infty(C \setminus \partial \Omega)$, and since

$$v(z, t) = \mathcal{C}[\varpi(\cdot, t)](z)$$

also (cf. [AIM09])

$$v \in L^\infty(C \times [0, T)) \cap C([0, T); \text{Lip}(1, C)).$$
Then (cf. [Che98, Theorem 5.2.1]), there exists a unique function \( \psi \in \mathcal{C}(\mathbb{C} \times [0, T]; \mathbb{C}) \) such that

\[
\psi(z, t) = z + \int_0^t v(\psi(z, \tau), \tau) \, d\tau,
\]

and \((\varpi, v)\) and there is a constant \( C > 0 \) such that for any \( t \in \mathbb{R} \),

\[
\psi(\cdot, t) - I \in \mathcal{C}^{c-\mathcal{C}(\mathbb{L}^p \cap \mathbb{L}^\infty)(\mathbb{C})} \mathbb{C}^c.(\mathbb{C}).
\]

It is then clear that

\[
\left( \left| \frac{\partial \psi}{\partial z} \right|^2 - \left| \frac{\partial \psi}{\partial \bar{z}} \right|^2 \right)(z, 0) = 1.
\]

As in the case of the Euler equation, for each \( t \in [0, T) \) and \( U \in \mathbb{C} \setminus \partial \Omega \), we have that \( \psi(U, t) \in \mathbb{C} \setminus \partial \Omega \). Then, from the particular shape of \( \varpi_0 \), we have that if \( U \subset \Omega \) or \( U \in \mathbb{C} \setminus \bar{\Omega} \), then \( \psi(U) \in \mathbb{C}^{c}(U) \) and then, using for instance Proposition 8.3 in [MB92], we can conclude that for any \( t \in \mathbb{R} \),

\[
w(\cdot, t) \in \mathcal{C}^{c}(\psi(U, t))
\]

and

\[
v(\cdot, t) \in \mathcal{C}^{c}(\psi(U, t)).
\]

Moreover, both the velocity and the flow given by the function \( \psi \) in the theorem are globally defined with respect to the time variable and in general are regular beyond the continuity in the \( z \) variable. This is because the local regularity is preserved by the Cauchy transform, and so the regularity of initial conditions is propagated by solutions of ordinary differential equations.

The flow \( \psi \) also inherits the local space regularity (after derivation under the integral sign). Moreover, using Theorem 1.3.1 of Chapter 1 in [H97], we have that \( \psi \in \mathcal{C}^1(U \times [0, T)) \) and \( \frac{\partial \psi}{\partial t} (\cdot, t) \in \mathcal{C}^1(U) \).

Then we have that (cf. [BLL12]) the transport of the density by the flow satisfies

\[
\varpi(\psi(z, t), t) = \frac{\varpi_0(z)}{1 - \varpi_0(z)} t
\]

in \( \mathbb{C} \setminus \partial \Omega \).

These facts allow us to formulate the relationship between the flow and the density, in the region of \( \mathbb{C} \times [0, T) \) where it makes sense, as

\[
\left( \frac{\partial^2 \psi}{\partial t \partial z} \frac{\partial \psi}{\partial z} - \frac{\partial^2 \psi}{\partial t \partial \bar{z}} \frac{\partial \psi}{\partial \bar{z}} \right)(z, t) = \frac{1}{2} \frac{\varpi_0(z)}{1 - \varpi_0(z)} t^c \int_0^t \frac{\varpi_0(\tau)}{1 - \varpi_0(\tau)} \, d\tau
\]

\[
= \frac{c}{2(1 - ct)^2} \chi^\Omega.
\]

Applying Theorem 1 to both cases, we have:
(a) In the Euler’s case there exists a number \( T_1 > 0 \) and a function \( \psi: \mathbb{C} \to \mathbb{C} \), such that \( t \to \psi(z, t) \) is real analytic in \((-T_1, T_1)\).

If \( T_1 = +\infty \) there is nothing else to say. Otherwise, at time \( \pm T_1 \), from the theorem of persistence of regularity ([Che98], [BC93]), we have that \( \partial \Omega_{\pm T_1} \in \mathcal{C}^{1,\gamma} \) and we can iterate the procedure and use the uniqueness of the solution to obtain \( t \)-analyticity in \((-T_2, T_2)\) for \( T_2 > T_1 \). This implies that the set where the of analyticity is open and closed in \( \mathbb{R} \), so it is \( \mathbb{R} \).

(b) For the aggregation equation, the procedure is similar. We only have to take in account that now \( t > 0 \), and if \( c < 0 \) the analyticity will occur for \( t \in [0, \infty) \) and for \( c > 0 \), for \( t \in [0, \frac{1}{c}) \). The only necessary ingredient in the proof is the persistence of the regularity in this case.

The point is the persistence of the regularity in the case of the problem \( (\tilde{A}) \). The rescaling

\[
s(t) = \ln \left( \frac{1}{1 - ct} \right)
\]

and

\[
\tilde{\wp}(z, s) = \frac{1 - ct(s)}{c} \wp(z, t(s))
\]

transforms the problem

\[
\begin{cases}
\frac{\partial \wp}{\partial t} + 2\Re(\frac{\partial (\wp v)}{\partial z}) = 0, \\
v(z, t) = -C[\wp(\cdot, t)](z), \\
\wp(\cdot, 0) = \wp_0,
\end{cases} \tag{A}
\]

in the problem

\[
\begin{cases}
\frac{\partial \tilde{\wp}}{\partial s} + \Re(\tilde{\nu} \frac{\partial \tilde{\wp}}{\partial z}) = 0, \\
\tilde{v}(z, s) = -C[\tilde{\wp}(\cdot, s)](z), \\
\tilde{\wp}(\cdot, 0) = \chi_\Omega,
\end{cases} \tag{\tilde{A}}
\]

which is a transport equation with initial datum the indicator function of \( \Omega \), a region with \( \mathcal{C}^{1,\gamma} \) boundary.

For the problem \( (\tilde{A}) \) it is proven in [BGLV16] that for every \( s \in [0, \infty) \), if \( \tilde{\psi} \) is the corresponding flow, then \( \psi(\partial \Omega, s) \) is a \( \mathcal{C}^{1,\gamma} \) embedded submanifold of \( \mathbb{C} \) of real dimension 1. Since the rescalings above do not affect the \( z \) variable, the same regularity is true in the case of \( (A) \), for \( t \in [0, \frac{1}{c}) \).

From Theorem\( \|I \| \) there exists a number \( T_1 > 0 \) and a function \( \psi: \mathbb{C} \to \mathbb{C} \), such that \( t \to \psi(z, t) \) is real analytic in \((-T_1, T_1)\).

If \( T_1 = +\infty \) there is nothing else to say. Otherwise, after a time \( T_1 \), we have

\[
\wp(z, T_1) = \frac{c}{1 - c T_1} \wp_0(z),
\]

that must be used as initial density to iterate the procedure, because the boundary \( \partial \Omega_{T_1} \) is \( \mathcal{C}^{1,\gamma} \), getting a new \( T_2 > T_1 \) and analyticity in \((-T_2, T_2)\).

Again an argument of connectivity and the uniqueness conclude that the flow is analytic in \([0, \frac{1}{c})\).
3 Proof of Theorem 1

There are functions

\[ a^{(s)} : \mathbb{C} \to \mathbb{C}, \]

for each \( s \in \mathbb{N} \setminus \{0\} \) such that

\[ \varpi_0(z)(1 + a(z, t)t) = \varpi_0(z) + \sum_{s=1}^{\infty} a^{(s)}(z) t^s, \]

for \( t \in (-\beta_0(z), \beta_0(z)) \).

It is worth to remark that \( a^{(s)}(z) = 0 \) for every \( z \in \mathbb{C} \setminus \bar{\Omega} \).

Assume that there exists a family of functions

\[ \xi^{(s)} : \mathbb{C} \to \mathbb{C}, \quad s \in \mathbb{N}, \]

having first order derivatives with respect to \( z \) and \( \bar{z} \) at each point of \( \mathbb{C} \setminus \partial \Omega \) and such that

\[ \psi(z, t) = \sum_{s=0}^{\infty} \xi^{(s)}(z) t^s, \]

then

\[ \left( \frac{\partial^2 \psi}{\partial t \partial \bar{z}} \frac{\partial \psi}{\partial z} - \frac{\partial^2 \psi}{\partial \bar{t} \partial z} \frac{\partial \psi}{\partial \bar{z}} \right)(z, t) = \sum_{s=0}^{\infty} \left\{ \sum_{k=0}^{s} (k + 1) \left( \frac{\partial \xi^{(k+1)}}{\partial z} (z) \frac{\partial \xi^{(s-k)}}{\partial \bar{z}} (z) \right. \right. \]

\[ \left. \left. - \frac{\partial \xi^{(k+1)}}{\partial \bar{z}} (z) \frac{\partial \xi^{(s-k)}}{\partial z} (z) \right) \right\} t^s. \]

If \( \psi \) is a solution analytic in \( t \) in a neighborhood of \( \mathbb{C} \times \{0\} \) of the problem

\[ (3), \]

then

\[ \xi^{(0)}(z) \equiv z \]

and if \( z \in \mathbb{C} \setminus \partial \Omega \), then

\[ \begin{cases} \frac{\partial \xi^{(1)}}{\partial z} (z) = \varpi_0(z) \overset{\text{defn}}{=} a^{(0)}(z), \\ \frac{\partial \xi^{(s+1)}}{\partial z} (z) = \frac{a^{(s)}(z)}{s+1} - \frac{1}{s+1} \sum_{k=0}^{s-1} (k+1) (\frac{\partial a^{(s-k)}}{\partial z} (z) - \frac{\partial a^{(s-k)}}{\partial \bar{z}} (z)), \quad s \in \mathbb{N} \setminus \{0\}. \end{cases} \]

The system \( (7) \) provides for each \( \in \mathbb{N} \setminus \{0\} \) the \( z \)-derivative of \( \xi^{(s)} \) in terms of the first order derivatives of the functions \( \xi^{(l)} \), where \( 1 \leq l < s \), if \( s > 1 \), and in terms of \( \varpi_0 \) in the case \( s = 1 \).

This means that, in the distributions sense

\[ \xi^{(1)}(z) = \mathcal{C}[\varpi_0](z), \]

and we have
Proposition 1. The function $\bar{\mathcal{C}}[\varpi_0](z)$ is in $C^\infty(C \backslash \Omega) \cap \text{Lip} (1, \mathbb{C})$ and has a decay at the infinity of the type
\[
\frac{1}{\max \{R_0, d(z, \partial \Omega)\}}.
\]
Moreover, if $T^{(1)}$ is the distribution corresponding to $\frac{\partial \mathcal{C}[\varpi_0]}{\partial z}$, then
\[
\Theta(T^{(1)}, z) = \bar{\varpi}_0(z)
\]
for all $z \in \mathbb{C}$.

Proof. Of the last part
\[
\langle \frac{\partial T_{\mathcal{C}[\varpi_0]}}{\partial z}, \phi_{z_0, \epsilon} \rangle = -\langle T_{\mathcal{C}[\varpi_0]}, \frac{\partial \phi_{z_0, \epsilon}}{\partial z} \rangle = -\langle T_{\mathcal{C}[\varpi_0]}, \frac{\partial \phi_{z_0, \epsilon}}{\partial z} \rangle = -\int_{\mathbb{C}} \frac{\partial \phi_{z_0, \epsilon}}{\partial z}(z) \text{dm}_2(z) = \int_{\mathbb{C}} \phi_{z_0, \epsilon}(z) \text{dm}_2(z).
\]

In the remaining we use the following theorem on the boundedness of the conjugate Beurling transform

Theorem 3. Let $\Omega$ be a domain such that $\partial \Omega \in C^{1, \gamma}$, where $\gamma \in (0, 1)$ and $g$ a function defined at every $z \in \mathbb{C}$, $g \in C^\infty(C \backslash \partial \Omega)$ and $\chi_{\Omega} g$ extends to a (unique) function $g_- \in \text{Lip}(\gamma, \Omega)$, $\chi_{\Omega} g$ extends to a (unique) function $g_+ \in \text{Lip}(\gamma, \mathbb{C} \backslash \Omega)$ and for $z \in \partial \Omega$, $g(z) = \frac{1}{2} (g_+(z) + g_-(z))$.

We also assume that there is a constant $R_0 > 0$ depending only on $\Omega$ such that if
\[
U_{R_0} = \{ z \in \mathbb{C} : d(z, \Omega) < R_0 \},
\]
then for $g$ there is a constant $C(g) > 0$ such that for $z \in \mathbb{C} \backslash U_{R_0}$,
\[
|g(z)| \leq \frac{C(g)}{\max \{R_0, d(z, \partial \Omega)^2\}}.
\]

Then $\mathcal{B}[g](z)$ is well defined for each $z \in \mathbb{C}$ and
\[
\chi_{\Omega} \mathcal{B}[g] \in C^\infty(\Omega) \cap \text{Lip}(\gamma, \Omega),
\]
\[
\chi_{\Omega \backslash \Omega} B[g] \in C^\infty(C \backslash \Omega) \cap \text{Lip}(\gamma, \mathbb{C} \backslash \Omega),
\]
and for any $z \in \partial \Omega$ we have
\[
\mathcal{B}[g](z) = \frac{1}{2} \left\{ \lim_{w \to z; w \in \Omega} \chi_{\Omega} \mathcal{B}[g](w) + \lim_{w \to z; w \in \mathbb{C} \backslash \Omega} \chi_{\Omega \backslash \Omega} \mathcal{B}[g](w) \right\}. \tag{8}
\]
Moreover, there exists a constant $K = K(\gamma, \Omega, R_0) > 0$ such that
\[
\|\bar{B}[g]\|_{L^\infty(C)} \leq K \|g\|_\gamma,
\]
\[
\|\bar{B}[g]\|_{\gamma, \Omega} \leq K \|g\|_\gamma,
\]
and
\[
\|\bar{B}[g]\|_{\gamma, C \setminus \Omega} \leq K \|g\|_\gamma.
\]
Moreover, for any $z \in (U_{R_0} \cup \Omega)^c$,
\[
|g(z)| \leq K C(g) \frac{1 + \ln d(z, \Omega)}{\max\{R_0^2, d(z, \partial \Omega)^2\}} \|g\|_\gamma.
\tag{9}
\]

Using Theorem 3 we have that

**Proposition 2.** If $\tilde{T}^{(1)}$ is the distribution corresponding to $\frac{\partial \psi_0^{(1)}}{\partial \bar{z}}$, then
\[
\Theta(\tilde{T}^{(1)}, z) = \bar{B}[\psi_0](z)
\]
for all $z \in \mathbb{C}$.

So $\tilde{T}^{(1)}$ is given by a function whose decay at $\infty$ is of type
\[
\frac{1 + \ln d(z, \Omega)}{\max\{R_0^2, d(z, \partial \Omega)^2\}}.
\tag{10}
\]

**Proof.** The density part.
\[
\left\langle \frac{\partial T_{C[\psi_0]}}{\partial \bar{z}}, \psi_{2\epsilon, \epsilon} \right\rangle = -\left\langle T_{C[\psi_0]}, \frac{\partial \psi_{2\epsilon, \epsilon}}{\partial \bar{z}} \right\rangle
\]
\[
= \frac{i}{2\pi} \int_{\mathbb{C}} \left\{ \int_{\mathbb{C}} \frac{\psi_0(\zeta)}{\zeta - \bar{z}} dm_2(\zeta) \right\} \frac{\partial \psi_{2\epsilon, \epsilon}(z)}{\partial \bar{z}} dm_2(z)
\]
\[
= \frac{i}{2\pi} \int_{\mathbb{C}} \left\{ \int_{\mathbb{C}} \frac{\partial \psi_{2\epsilon, \epsilon}(z)}{\zeta - \bar{z}} dm_2(z) \right\} \psi_0(\zeta) dm_2(\zeta) = (*)
\]
and using the Stokes formula and Theorem 3, we have
\[
\int_{\mathbb{C}} \frac{\partial \psi_{2\epsilon, \epsilon}(z)}{\zeta - \bar{z}} dm_2(z) = \lim_{\epsilon \to 0} \int_{\mathbb{C} \setminus B_\epsilon(\zeta)} \frac{\partial \psi_{2\epsilon, \epsilon}(z)}{\zeta - \bar{z}} dm_2(z) = (**)\]
and
\[
\int_{\mathbb{C} \setminus B_\epsilon(\zeta)} \frac{\partial \psi_{2\epsilon, \epsilon}(z)}{\zeta - \bar{z}} dm_2(z) = \frac{1}{2i} \int_{\mathbb{C} \setminus B_\epsilon(\zeta)} \frac{\partial \psi_{2\epsilon, \epsilon}(z)}{\zeta - \bar{z}} d\bar{z} \wedge dz
\]
\[
= \frac{1}{2i} \int_{\partial B_\epsilon(\zeta)} \frac{\phi_{2\epsilon, \epsilon}(z)}{\zeta - \bar{z}} dz - \frac{1}{2i} \int_{\mathbb{C} \setminus B_\epsilon(\zeta)} \frac{\phi_{2\epsilon, \epsilon}(z)}{(\zeta - \bar{z})^2} d\bar{z} \wedge dz,
\]
so
\[
(**) = \frac{1}{2i} \lim_{\epsilon \to 0} \int_{\mathbb{C} \setminus B_\epsilon(\zeta)} \frac{\phi_{2\epsilon, \epsilon}(z)}{(\zeta - \bar{z})^2} d\bar{z} \wedge dz.
\]
and then
\[
(*) = \int_{\mathbb{C}} \phi_{z_0, \varepsilon}(\zeta) \left( \frac{1}{\pi} \, p. \, v. \int_{\mathbb{C}} \frac{\overline{\phi}(\zeta)}{(\zeta - z)^2} \, dm_2(\zeta) \right) \, dm(z).
\]

The decay estimate \((10)\) is a consequence of \(9\) in Theorem \(3\). \(\square\)

Now, for \(s > 1\) we will define \(T^{(s)}\) and \(\tilde{T}^{(s)}\), the distributions corresponding to \(\frac{\partial T^{(s)}}{\partial z}\) and \(\frac{\partial \tilde{T}^{(s)}}{\partial z}\) respectively, and then \(\Theta(T^{(s)}, z) = \theta^{(s)}(z)\) and \(\Theta(\tilde{T}^{(s)}, z) = \eta^{(s)}(z)\). All these objects exist, by a direct application of the previous proposition to the cases \(s' < s\).

So, taking densities, the system \((7)\) becomes
\[
\begin{aligned}
\theta^{(1)}(z) &= \varpi_0(z), \\
\eta^{(1)}(z) &= B[\varpi_0(z)], \\
\theta^{(s+1)}(z) &= \frac{\varphi^{(s)}(z)}{s+1} - \frac{1}{s+1} \sum_{k=0}^{s-1} (k+1) \\
&\quad \times \{ \theta^{(k+1)}\theta^{(s-k)} - \eta^{(k+1)}\eta^{(s-k)} \}(z), \quad \text{for } s \in \mathbb{N} \setminus \{0\}.
\end{aligned}
\tag{11}
\]

Then we consider, in \(\mathbb{C}\), the decomposition
\[
\theta^{(j)} = \chi_\Omega \theta^{(j)} + \chi_{\partial \Omega} \theta^{(j)} + \chi_{\partial_0 \Omega} \theta^{(j)} = \phi^{(j)} + \psi^{(j)} + \Phi^{(j)}.
\tag{12}
\]

So
\[
(s + 1) \left( \phi^{(s+1)} + \psi^{(s+1)} + \Phi^{(s+1)} \right)(z)
= \frac{\varphi^{(s)}(z)}{s+1} - \sum_{k=0}^{s-1} (k+1) \left\{ \left( \phi^{(k+1)} + \psi^{(k+1)} + \Phi^{(k+1)} \right) \left( \phi^{(s-k)} + \psi^{(s-k)} + \Phi^{(s-k)} \right) \right\}(z)
- B\left[ \phi^{(k+1)} + \psi^{(k+1)} + \Phi^{(k+1)} \right] B\left[ \phi^{(s-k)} + \psi^{(s-k)} + \Phi^{(s-k)} \right](z)
= \frac{\varphi^{(s)}(z)}{s+1} - \sum_{k=0}^{s-1} (k+1) \left\{ \left( \phi^{(k+1)} + \psi^{(k+1)} + \Phi^{(k+1)} \right) \left( \phi^{(s-k)} + \psi^{(s-k)} + \Phi^{(s-k)} \right) \right\}(z)
- B\left[ \phi^{(k+1)} + \psi^{(k+1)} + \Phi^{(k+1)} \right] B\left[ \phi^{(s-k)} + \psi^{(s-k)} \right](z),
\]

consequently, if we define
\[
\Phi[ ] = \chi_\Omega B[ ], \quad \Psi[ ] = \chi_{\partial \Omega} B[ ], \quad \text{and} \quad \Gamma[ ] = \chi_{\partial_0 \Omega} B[ ],
\]
we have
\[
(s + 1) \phi^{(s+1)} = \frac{\varphi^{(s)}(z)}{s+1} - \sum_{k=0}^{s-1} (k+1) \left\{ \phi^{(k+1)} \phi^{(s-k)} - \Phi[ \phi^{(k+1)} ] \Phi[ \phi^{(s-k)} ] \right\}
+ \sum_{k=0}^{s-1} (k+1) \left\{ \Phi[ \psi^{(k+1)} ] \Phi[ \phi^{(s-k)} ] + \Psi[ \psi^{(k+1)} ] \Phi[ \phi^{(s-k)} ] + \Phi[ \psi^{(k+1)} ] \Phi[ \psi^{(s-k)} ] \right\},
\]

15
\[(s + 1) \psi^{(s+1)} = - \sum_{k=0}^{s-1} (k + 1) \left\{ \psi^{(k+1)} \overline{\psi^{(s-k)}} - \Psi[\psi^{(k+1)}] \overline{\psi^{(s-k)}} \right\} + \sum_{k=0}^{s-1} (k + 1) \left\{ \Psi[\phi^{(k+1)}] \overline{\phi^{(s-k)}} + \Psi[\psi^{(k+1)}] \overline{\phi^{(s-k)}} \right\} + \Psi[\phi^{(k+1)}] \overline{\phi^{(s-k)}} \]

and

\[(s + 1) \beta^{(s+1)} = \frac{a^{(s)}(z)}{s+1} - \sum_{k=0}^{s-1} (k + 1) \left\{ \beta^{(k+1)} \overline{\beta^{(s-k)}} - \Gamma[\phi^{(k+1)}] \overline{\Gamma^{(s-k)}} \right\} + \sum_{k=0}^{s-1} (k + 1) \left\{ \Gamma[\phi^{(k+1)}] \overline{\Gamma^{(s-k)}} + \Gamma[\psi^{(k+1)}] \overline{\Gamma^{(s-k)}} \right\} + \Gamma[\psi^{(k+1)}] \overline{\Gamma^{(s-k)}} \],

in \( \Omega \),

\[(s + 1) \psi^{(s+1)} = \sum_{k=0}^{s-1} (k + 1) \left\{ \psi^{(k+1)} \overline{\psi^{(s-k)}} - \Psi[\psi^{(k+1)}] \overline{\psi^{(s-k)}} \right\} + \sum_{k=0}^{s-1} (k + 1) \left\{ \Psi[\phi^{(k+1)}] \overline{\psi^{(s-k)}} + \Psi[\psi^{(k+1)}] \overline{\phi^{(s-k)}} \right\} + \Psi[\phi^{(k+1)}] \overline{\phi^{(s-k)}} \]

and

\[(s + 1) \beta^{(s+1)} = \frac{a^{(s)}(z)}{s+1} - \sum_{k=0}^{s-1} (k + 1) \left\{ \beta^{(k+1)} \overline{\beta^{(s-k)}} - \Gamma[\phi^{(k+1)}] \overline{\Gamma^{(s-k)}} \right\} + \sum_{k=0}^{s-1} (k + 1) \left\{ \Gamma[\phi^{(k+1)}] \overline{\Gamma^{(s-k)}} + \Gamma[\psi^{(k+1)}] \overline{\Gamma^{(s-k)}} \right\} + \Gamma[\psi^{(k+1)}] \overline{\Gamma^{(s-k)}} \],

in \( \partial \Omega \).

And now we can determine inductively the functions \( \phi, \psi \) and \( \beta \).

**Proposition 3.** If \( \phi^{(1)} = \chi_{\Omega} \varpi_0 \), \( \psi^{(1)} = 0 \) and \( \beta^{(1)} = \chi_{\partial \Omega} \varpi_0 \), then for any \( s \in \mathbb{N} \setminus \{0, 1\} \), we have that \( \phi^{(s)} \in \text{Lip}(\gamma, \bar{\Omega}) \), \( \psi^{(s)} \in \text{Lip}(\gamma, \Omega^c) \) and \( \beta^{(s)} \in \text{Lip}(\gamma, \partial \Omega) \).

**Proof.** The formulas above allow, for any \( s \geq 2 \), the control of \( \phi^{(s+1)} \) and \( \psi^{(s+1)} \) in terms of \( \phi^{(l)} \) and \( \psi^{(r)} \), for all \( 1 \leq r, l \leq s \).

In the case of \( s = 1 \), Theorem 3 gives the desired estimate.

For \( s > 1 \), we need the existence and estimates of terms of type \( B[f^{(l)}] B[f^{(r)}] \) for \( 1 \leq r, l \leq s \), where \( f^{(s)} \) states for any of the functions in the statement.

The estimates in \( C \setminus \Omega \) provided by (9) in Theorem 3 imply that

\[ B[f^{(l)}] B[f^{(r)}] = O \left( \left( \frac{\ln d(z, \partial \Omega)}{d(z, \partial \Omega)} \right)^{\frac{1}{4}} \right) \]

and then the product is integrable and satisfies the hypotheses Theorem 3 with the corresponding estimates.

So we can perform the iteration.

Let us control now the Hölder size of \( \phi, \psi \) and \( \beta \).

If \( \| \| \) denotes \( \| \|_{\gamma, \bar{\Omega}} \) or \( \| \|_{\gamma, \Omega^c} \), we have

\[ \| \phi^{(1)} \| = \| \varpi_0 \| \overset{\text{def}}{=} \frac{1}{2} \alpha \]
and
\[ \|\psi^{(1)}\| = 0. \]

The remaining terms have the control
\[
\|\phi^{(s+1)}\| \leq \frac{1}{s+1} \|\varpi_0\| \left( \frac{2\|\varpi_0\|}{s!} \right)^s
\]
\[
+ \frac{1}{s+1} \sum_{k=0}^{s-1} (k+1) \left\{ \|\phi^{(k+1)}\| \|\phi^{(s-k)}\| + \|\Phi[\phi^{(k+1)}]\| \|\Phi[\phi^{(s-k)}]\| \right\}
\]
\[
+ \frac{1}{s+1} \sum_{k=0}^{s-1} (k+1) \left\{ \|\Phi[\phi^{(k+1)}]\| \|\Phi[\psi^{(s-k)}]\|
\]
\[
+ \|\Phi[\psi^{(k+1)}]\| \|\Phi[\phi^{(s-k)}]\| \right\}. \]

and
\[
\|\psi^{(s+1)}\| \leq \frac{1}{s+1} \sum_{k=0}^{s-1} (k+1) \left\{ \|\psi^{(k+1)}\| \|\psi^{(s-k)}\| + \|\Psi[\psi^{(k+1)}]\| \|\Psi[\psi^{(s-k)}]\| \right\}
\]
\[
+ \frac{1}{s+1} \sum_{k=0}^{s-1} (k+1) \left\{ \|\Psi[\phi^{(k+1)}]\| \|\Psi[\psi^{(s-k)}]\|
\]
\[
+ \|\Psi[\psi^{(k+1)}]\| \|\Psi[\phi^{(s-k)}]\| + \|\Psi[\phi^{(k+1)}]\| \|\Psi[\phi^{(s-k)}]\| \right\}. \]

If $f = \phi^{(s)}$ or $\psi^{(s)}$, using Theorem 3 we have that $\|\Phi[f]\|$ and $\|\Psi[f]\| \leq K \|f\|$, and then, using the notation $\alpha_p = \|\phi^{(p)}\|$ and $\beta_p = \|\psi^{(p)}\|$,.

\[
\alpha_{s+1} \leq \frac{1}{s+1} \frac{2\alpha}{s!} \left( 1 + K^2 \right) \frac{1}{s+1} \sum_{k=0}^{s-1} (k+1) \alpha_{k+1} \alpha_{s-k}
\]
\[
+ K^2 \frac{1}{s+1} \sum_{k=0}^{s-1} (k+1) \left\{ \alpha_{k+1} \beta_{s-k} + \beta_{k+1} \alpha_{s-k} + \beta_{k+1} \beta_{s-k} \right\}, \]

and
\[
\beta_{s+1} \leq \frac{1}{s+1} \frac{2\alpha}{s!} \left( 1 + K^2 \right) \frac{1}{s+1} \sum_{k=0}^{s-1} (k+1) \beta_{k+1} \beta_{s-k}
\]
\[
+ K^2 \frac{1}{s+1} \sum_{k=0}^{s-1} (k+1) \left\{ \alpha_{k+1} \beta_{s-k} + \beta_{k+1} \alpha_{s-k} + \alpha_{k+1} \alpha_{s-k} \right\}. \]

So
Proposition 4. If

\[ A_s = \sum_{k=0}^{s-1} (k + 1) \alpha_{k+1} \alpha_{s-k}, \]

\[ B_s = \sum_{k=0}^{s-1} (k + 1) \beta_{k+1} \beta_{s-k}, \]

and

\[ C_s = \sum_{k=0}^{s-1} (k + 1) \{ \alpha_{k+1} \beta_{s-k} + \beta_{k+1} \alpha_{s-k} \}, \]

then we have

\[ (s + 1) \alpha_{s+1} \leq 2 \alpha^s + (1 + K^2) A_s + K^2 (C_s + B_s) \]

and

\[ (s + 1) \beta_{s+1} \leq (1 + K^2) B_s + K^2 (C_s + A_s). \]

The proof is just a combinatorial computation.

Let us consider, next, the polynomic functions

\[ f_1(\xi) = \frac{\alpha}{2} \xi, \]
\[ g_1(\xi) = 0, \]
\[ f_N(\xi) = \sum_{p=0}^{N} \alpha_p \xi^p, \]

and

\[ g_N(\xi) = \sum_{q=0}^{N} \beta_q \xi^q, \]

where \( \alpha_0 = \beta_0 = 0. \)

Then the previous proposition implies that for \( \xi \in [0, \infty), \)

\[ f'_{N+1}(\xi) = \sum_{s=0}^{N} (s + 1) \alpha_{s+1} \xi^s \]

\[ \leq \frac{\alpha}{2} + \sum_{s=0}^{N} 2 \frac{\alpha^s}{s!} + (1 + K^2) \sum_{s=0}^{N} A_s \xi^s + K^2 \sum_{s=0}^{N} (C_s + B_s) \xi^s \]

and

\[ g'_{N+1}(\xi) = \sum_{s=0}^{N} (s + 1) \beta_{s+1} \xi^s \leq (1 + K^2) \sum_{s=0}^{N} B_s \xi^s + K^2 \sum_{s=0}^{N} (C_s + A_s) \xi^s. \]
Then a simple development and direct estimates imply that
\[ \sum_{s=0}^{N} A_s \xi^s \leq f'_N f_N, \]
\[ \sum_{s=0}^{N} B_s \xi^s \leq g'_N g_N, \]
and
\[ \sum_{s=0}^{N} C_s \xi^s \leq f'_N g_N + f_N g'_N, \]
so
\[ f'_{N+1}(\xi) \leq \frac{a}{2} + 2 \sum_{s=0}^{N} \frac{\alpha_{s+1}}{s!} (1 + K^2) \left( \frac{1}{2} f_N^2 \right)' + K^2 \left( \left( \frac{1}{2} g_N^2 \right)' + (f_N g_N)' \right) \]
\leq 3 e^{a} + (1 + K^2) \left( \frac{1}{2} f_N^2 \right)' + K^2 \left( \left( \frac{1}{2} g_N^2 \right)' + (f_N g_N)' \right) \]
and
\[ g'_{N+1}(\xi) \leq (1 + K^2) \left( \frac{1}{2} g_N^2 \right)' + K^2 \left( \left( \frac{1}{2} f_N^2 \right)' + (f_N g_N)' \right) \]
and integrating
\[ f_{N+1}(\xi) \leq c + 3 e^{a} \xi + \frac{1 + K^2}{2} f_N^2 + K^2 \left( \frac{1}{2} g_N^2 + f_N g_N \right) \]
and
\[ g_{N+1}(\xi) \leq d + \frac{1 + K^2}{2} g_N^2 + K^2 \left( \frac{1}{2} f_N^2 + f_N g_N \right) \]

Adding these two inequalities and taking and considering that \( f_N(0) = g_N(0) = 0 \), we have that
\[ (f_{N+1} + g_{N+1}) \leq 3 e^{a} \xi + \frac{1 + K^2}{2} (f_N + g_N)^2. \]
Then, if \( h_N = f_N + g_N \), we have
\[ h_{N+1} \leq 3 e^{a} \xi + \frac{1 + K^2}{2} h_N^2, \tag{13} \]
and then

**Proposition 5.** For any \( N \in \mathbb{N} \setminus \{0\} \), let \( h_N \) a sequence of functions satisfying \( (13) \) we have
\[ h_N(\xi) \leq \frac{2}{1 + K^2} \]
for \( \xi \in (0, \frac{1}{12 e^{a} (1 + K^2)}) \).
Proof. We want to prove that there exists $L > 0$ such that if
\[ h_1(\xi) = 3e^\alpha \xi \leq L \]
and for any $N \in \mathbb{N} \setminus \{0\}$, if $h_N \leq L$, then $h_{N+1} \leq L$. By the previous inequality, we have that it is enough that $L$ satisfies
\[ 3e^\alpha \xi \leq L \]
and
\[ 3e^\alpha \xi + \frac{1 + K^2}{2} L^2 - L \leq 0. \]

Using the notation $c = 3e^\alpha \xi$ and $R = \frac{1 + K^2}{2}$, the polynomial
\[ Rx^2 - x + c = R \left( x - \frac{1 - \sqrt{1 - 4 Rc}}{2R} \right) \left( x - \frac{1 + \sqrt{1 - 4 Rc}}{2R} \right) \]
has two real positive different roots whenever $0 < c \leq \frac{1}{4R}$, and, independently of $c \in (0, \frac{1}{4R})$, the value $\frac{1}{R}$ is the mean value of the roots.

In this case $L$ must be in the interval determined by the roots, so if $c = \frac{\mu}{4R}$ and $0 < \mu < 1$, the smallest root is $\frac{1 - \sqrt{1 - 4 Rc}}{2R}$ an it is larger than $c$. so any number in the interval
\[ \left( \frac{1 - \sqrt{1 - 4 Rc}}{2R}, \frac{1 + \sqrt{1 - 4 Rc}}{2R} \right) \]
can be chosen as $L$, and in particular $h_N(\xi) \leq \frac{1}{R}$ for any $N$ and any $c \in (0, \frac{1}{4R})$.

Back to the decomposition (12) and the fact that $\alpha = 2 \|\varpi_0\|$, the previous proposition implies that
\[ \sum_{s=0}^{N} \left\| \frac{\partial \xi(s)}{\partial z} \right\| \tau^s \leq h_N(\tau) \leq \frac{2}{1 + K^2} \]
for every $N$ and $\tau \in [0, \frac{1}{12e^\alpha \|\varpi_0\| \left( 1 + K^2 \right)}].$

Since the functions $\frac{\partial \xi(s)}{\partial z}$ are in $C^{1, \gamma}$ and, as the recurrence shows, they decay at $\infty$ of order $\frac{1}{12e^\alpha \|\varpi_0\| \left( 1 + K^2 \right)}$, then
\[ \xi(s) = \mathcal{C} \left[ \frac{\partial \xi(s)}{\partial z} \right] \]
is in $C^{1, \gamma}$ (cf. Theorems 4.3.11 and 4.3.12 in [AIM09]). Moreover
\[ \|\xi(s)\|_{L^\infty} \leq K_0 \left\| \frac{\partial \xi(s)}{\partial z} \right\|_\gamma. \]

Consequently, for any $N$,
\[ \left| \sum_{s=0}^{N} \xi(s)(z)t^s \right| \leq \sum_{s=0}^{N} \|\xi(s)\|_{L^\infty} |t|^s \leq K_0 \sum_{s=0}^{N} \left\| \frac{\partial \xi(s)}{\partial z} \right\|_\gamma |t|^s \]
and the last term is uniformly bounded in $N$, for $|t| \leq \frac{1}{12e^\alpha \|\varpi_0\| \left( 1 + K^2 \right)}$.

This implies the existence of an analytic solution of the equation (3), providing the proof of Theorem 1.
4 Proof of Theorem 3

This is the most technical section of the paper. We divide it in several subsections, where we exhibit some structural and geometric facts about the Beurling transform, necessary for our purposes, and then we perform the uniform and Lipschitz estimates, necessary for the proof of Theorem 1.

Let us consider a function $f$ defined on a domain $W \subset \mathbb{C}$ with bounded $C^{1,\gamma}$-regular boundary.

In general, we will also denote by $f$ the extension of $f$ by 0 to $\overline{W}$, provided it exists.

Whenever it be necessary, we will specialize $f = \phi$ and then $W = \text{supp}(\phi)$, or $f = \psi$ and $W = \text{supp}(\psi)$, respectively.

4.1 The geometric lemma

The metric and geometric properties of $\partial \Omega$ play a crucial role in the behavior of the Beurling transform. The next lemma is a synthesis of these properties.

To describe the geometry of the boundary

**Lemma 3.** Let $\Omega \subset \mathbb{C}$ be a bounded domain such that $\partial \Omega \in C^{1,\gamma}$.

There exists $0 < R_0 < 1$ such that if $U_{R_0}(\partial \Omega) := \bigcup_{z \in \partial \Omega} B_{R_0}(z)$, then there exists $R_1$ such that for any $z_0 \in U_{R_0}$ the level set $\{\rho = \rho(z_0)\} \cap B_{R_1}(z_0)$ coincides with the graph of a function.

Moreover $\varphi_{z_0}$ is $C^{1,\gamma}$ and $\varphi_{z_0}(0) = \varphi'(z_0)(0) = 0$.

**Remark.** The function $\varphi_{z_0}$ is defined on a segment of the tangent line to the level set of $\rho$ across $z_0$.

Then

$$\rho(z_0 + s \eta(z_0) + \varphi_{z_0}(s) \eta(z_0)) = \rho(z_0).$$

**Proof.** Let $\rho \in C^{1,\gamma}$ be the defining function for $\Omega$, as in [Bur06]. Since $\partial \Omega$ is a compact set, let

$$m = \min\{||\nabla \rho(z)|| : z \in \partial \Omega\}.$$

Since $\nabla \rho \in \text{Lip}(\gamma)$, then for

$$R = \left(\frac{m}{2\sqrt{2} ||\rho||_\gamma}\right)^\frac{1}{\gamma},$$

we define $U_R = \bigcup_{\zeta \in \partial \Omega} B_R(\zeta)$. It is an open neighborhood of $\partial \Omega$, and for every $z \in U$,

$$||\nabla \rho(z)|| \geq \frac{m}{2}.$$
Fix $z_0 \in U_{R_2}$. The vectors $\eta(z_0) = \frac{\nabla \rho(z_0)}{\|\nabla \rho(z_0)\|}$ and $\tau(z_0) = \eta(z_0)^\perp$ form an orthonormal basis of $T_{z_0}(\mathbb{R}^2)$, and the map

$$\Xi : \mathbb{R}^2 \to \mathbb{C}$$

given by

$$\Xi(s,t) = z_0 + s \tau(z_0) + t \eta(z_0)$$

is a rigid movement.

The function $r(s,t) = \rho \circ \Xi(s,t)$ satisfies

$$\nabla r(0,0) = \|\nabla \rho(z_0)\| e_2.$$ 

Let us consider the map

$$\Psi(s,t) = \left(s, r(s,t) - r(0,0) \right)\|\nabla \rho(z_0)\|.$$ 

We have that $\Psi(0,0) = (0,0)$ and

$$\det J\Psi(0,0) = \frac{\partial \rho}{\partial y}(z_0) \|\nabla \rho(z_0)\|.$$ 

Next, $\Psi(s,t) = \Psi(s',t')$ iff $s = s'$ and $r(s,t) = r(s',t')$, so, for some $\xi$ in the open interval delimited by $t$ and $t'$, we have

$$0 = r(s,t) - r(s,t') = \frac{\partial r}{\partial t}(s,\xi)(t-t')$$

$$= \frac{\partial r}{\partial t}(0,0)(t-t') + \left( \frac{\partial r}{\partial t}(s,\xi) - \frac{\partial r}{\partial t}(0,0) \right)(t-t'),$$

so

$$0 \geq \left| \frac{\partial r}{\partial t}(0,0) \right| |t-t'| - \left| \frac{\partial r}{\partial t} \right|_{\text{Lip}(\gamma)} |t-t'|^{1+\gamma}$$

$$= \left\{ \|\nabla \rho(z_0)\| - \left| \frac{\partial r}{\partial t} \right|_{\text{Lip}(\gamma)} \right\} |t-t'|,$$

and if we choose $R_1$ in such a way that if $z_0 \in \overline{U}_{R_1}$, then $B_{R_1}(z) \subset U_{R_2}$ and

$$R_1 \leq \left( \frac{\|\nabla \rho(z_0)\|}{\frac{1}{4} \left| \frac{\partial r}{\partial t} \right|_{\text{Lip}(\gamma)}} \right)\gamma,$$

then

$$0 \geq \frac{1}{2} \|\nabla \rho(z_0)\| |t-t'|.$$

So $\Psi$ is one-to-one in $B_{R_1}(0,0)$. 

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Choose, now, $0 < R_2 < R_1$. Then, if $s^2 + t^2 = R_2^2$, we have

$$\|\Psi(s, t)\|^2 = s^2 + \frac{(r(s, t) - r(0, 0))^2}{\|\nabla \rho(z_0)\|^2}\]

$$= s^2 + \left(t + \frac{(\nabla r(s', t') - \nabla r(0, 0), (s, t))}{\|\nabla \rho(z_0)\|}\right)^2$$

$$\geq R_2^2 \left\{1 + \left(\frac{(\nabla r(s', t') - \nabla r(0, 0), u)}{\|\nabla \rho(z_0)\|}\right)^2 - 2 |(u, e_2)| \frac{(\nabla r(s', t') - \nabla r(0, 0), u)}{\|\nabla \rho(z_0)\|}\right\} = (*)$$

where $u$ is a unitary vector. Moreover

$$(*) \geq R_2^2 \left\{1 - \frac{\|\nabla r\|_{\text{Lip} (\gamma)} R_2^2}{\|\nabla \rho(z_0)\|}\right\},$$

and if

$$R_2 \leq \left(\frac{\|\nabla \rho(z_0)\|}{4\|\nabla r\|_{\text{Lip} (\gamma)}}\right)^\frac{1}{2},$$

then

$$d(\Psi(\partial B_{R_2}(0, 0), (0, 0)) \geq \frac{R_2}{\sqrt{2}}.$$  

This implies that for $(p, q) \in B_{\frac{R_2}{\sqrt{2}}}(0, 0)$, the function $g(s, t) = \|\Psi(s, t) - (p, q)\|^2$ has a local minimum at some point $(s_0, t_0) \in B_{R_2}(0, 0)$, and then

$$\begin{cases}
2(s_0 - p) + 2\left(\frac{(\nabla r(s_0, t_0) - r(0, 0), \nabla \rho(z_0))}{\|\nabla \rho(z_0)\|}\right) = 0, \\
2\left(\frac{(\nabla r(s_0, t_0) - r(0, 0), \nabla \rho(z_0))}{\|\nabla \rho(z_0)\|}\right) = 0.
\end{cases}$$

And we can change $R_2$ by $R_3$ so that $\frac{\partial \Phi}{\partial t}$ do not vanish at $B_{R_3}(0, 0)$ and the other properties still hold.

So, there is a new choice of the region $U$ such that $\|\nabla \rho\|$ is uniformly bounded from below on $U$, and then there are choices of $R', R'' > 0$ such that for every $z_0 \in U$ the corresponding map $\Psi$ composed with the rigid movement describe above map $C^{1, \gamma}$-diffeomorphically $B_{R'}(z_0)$ in $B_{R''}(z_0)$.

If $\Phi = \Psi^{-1}$, then $\Phi$ is a $C^{1, \gamma}$-diffeomorphism such that $\Phi(0, 0) = (0, 0)$. Moreover $\Psi_1(s, t) = s$ and

$$\Psi_2(\{(s, t) : r(s, t) = r(0, 0)\}) = 0,$$

so the level set $\{r(s, t) = r(0, 0)\} \cap B_{R'}(0, 0)$ coincides wit the set $\{t = \Phi_2(s, 0)\}$.

Taking a convenient rectangle inside the balls, we have the function $\varphi$ in the statement of the theorem.

Finally, since essentially $\varphi(s) = \Phi_2(s, 0)$, we have that

$$\varphi'(0) = \frac{\partial \Phi_2}{\partial p}(0, 0),$$

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and since
\[ J\Phi(0, 0) = (J\Psi(0, 0))^{-1} \]
we have \( \varphi'(0) = 0 \).

Using the rigid movement above, we can take the construction backward to \( \mathbb{C} \) and get the theorem. \( \square \)

### 4.2 Decomposition of the singularities

Set, for any \( z \in \mathbb{C} \),
\[ d(z) = d(z, \partial \Omega) \quad \text{and} \quad \delta(z) = \max\{d(z), \frac{R_0}{2}\}. \]

#### Proposition 6.

Let \( \Omega \subset \mathbb{C} \) be a bounded domain with boundary of class \( C^{1, \gamma} \).

If \( f \in C^\gamma(\overline{\Omega}) \cap C^\gamma(\mathbb{C} \setminus \Omega) \cap L^p(\mathbb{C}) \) with \( p > 1 \), then there exists
\[ B[f](z) = \lim_{\epsilon \to 0} B_\epsilon[f](z) \]
for every \( z \in \mathbb{C} \).

Moreover
\[ B[f](z) = Q[f](z) + L[f](z) + f(z) \Theta_{\Omega}^{\frac{R_0}{2}}(z), \]
where
\[ Q[f](z) = \int_{\mathbb{C} \setminus B_{\delta(z)}(z)} f(\zeta) \frac{1}{|\zeta - z|^2} \, dm(\zeta), \]
\[ L[f](z) = \int_{B_{\delta(z)}(z)} \frac{f(\zeta) - f(z)}{|\zeta - z|^\gamma} \frac{|\zeta - z|^\gamma}{|\zeta - z|^2} \, dm(\zeta), \]
and
\[ \Theta_{\Omega}^{\frac{R_0}{2}}(z) = \begin{cases} 0 & \text{if } d(z) > 0, \\ \int_{B_{\frac{R_0}{2}}(z) \cap \partial \Omega \cap \{\kappa z > 0\}} - \int_{B_{\frac{R_0}{2}}(z) \cap \partial \Omega \cap \{\kappa z < 0\}} \frac{dm(\zeta)}{|\zeta - z|^2} & \text{if } d(z) = 0. \end{cases} \]

**Remark.** The term \( \Theta_{\Omega}^{\frac{R_0}{2}}(z) \) is an intrinsic geometric object. Sometimes we will also use it in the form
\[ \int_{B_{\frac{R_0}{2}}(z) \cap W} \frac{dm(\zeta)}{|\zeta - z|^2}. \]

**Remark.** In order to avoid notation, from now on we will use in the proofs of results the notation \( b_z(\zeta) \) instead of \( \frac{dm(\zeta)}{|\zeta - z|^2} \).

**Remark.** Since the Beurling transform is a classical Calderon-Zygmund operator, the existence of the principal value is well known for functions in many different classes (See [Duo01], Corollary 5.8), nevertheless, we need here the existence of the principal value at each point in the plane, for functions in the aforementioned class.

**Proof.** Let \( z \in \mathbb{C} \).
• If \( d(z) > 0 \), then for \( 0 < \epsilon < d(z) \) we have

\[
B_{\epsilon}[f](z) = \int_{C \setminus B_{\epsilon}(z)} f b_z = \int_{C \setminus B_{\epsilon}(z)} f b_z + \int_{B_{\epsilon}(z) \setminus B_{\epsilon}(z)} f b_z.
\]

In the case of \( z \notin \Omega \), the second term is 0 and we have

\[
B[f](z) = \lim_{\epsilon \to 0} B_{\epsilon}[f](z) = \int_{C \setminus B_{\epsilon}(z)} f b_z.
\]

• In the case of \( z \in \Omega \), we have

\[
B_{\epsilon}[f](z) = \int_{C \setminus B_{\epsilon}(z)} f b_z + \int_{B_{\epsilon}(z) \setminus B_{\epsilon}(z)} (f - f(z)) b_z
\]

\[
= \int_{C \setminus B_{\epsilon}(z)} f b_z + \int_{B_{\epsilon}(z) \setminus B_{\epsilon}(z)} \frac{f(\zeta) - f(z)}{\zeta - z} \frac{(\zeta - z)^\gamma}{|\zeta - z|^2} dm(\zeta),
\]

and in the second term has a weakly singular kernel acts against an integrable function, so the limit exists

\[
B[f](z) = \lim_{\epsilon \to 0} B_{\epsilon}[f](z)
\]

\[
= \int_{C \setminus B_{\epsilon}(z)} f b_z + \int_{B_{\epsilon}(z) \setminus B_{\epsilon}(z)} \frac{f(\zeta) - f(z)}{\zeta - z} \frac{|\zeta - z|\gamma}{|\zeta - z|^2} dm(\zeta).
\]

If \( d(z) \leq \frac{R_0}{2} \), then

\[
\int_{C \setminus B_{\epsilon}(z)} f b_z = \int_{C \setminus B_{\frac{R_0}{2}}(z)} f b_z + \int_{B_{\frac{R_0}{2}}(z) \setminus B_{\epsilon}(z)} f b_z,
\]

and applying again the cancellation lemma (below), we have that the second integral is

\[
\int_{C_{\frac{R_0}{2}}(z)} \frac{f(\zeta) - f(z)}{\zeta - z} \frac{|\zeta - z|\gamma}{|\zeta - z|^2} dm(\zeta),
\]

where \( C_{\frac{R_0}{2}} = B_{\frac{R_0}{2}}(z) \setminus B(z) \).

Implementing these facts in the formula for \( B[f](z) \) above, we have the result.

• If \( d(z) = 0 \), then for any \( \epsilon < \frac{R_0}{2} \) we have

\[
B_{\epsilon}[f](z) = \int_{C \setminus B_{\frac{R_0}{2}}(z)} f b_z + \int_{C_{\frac{R_0}{2}}(z)} f b_z.
\]

The second term, is

\[
\int_{C_{\frac{R_0}{2}}(z) \cap \Omega} f b_z = \int_{C_{\frac{R_0}{2}}(z) \cap \Omega} (f - f(z)) b_z + f(z) \int_{C_{\frac{R_0}{2}}(z) \cap \Omega} b_z
\]

\[
= \int_{C_{\frac{R_0}{2}}(z) \cap \Omega} \frac{f(\zeta) - f(z)}{\zeta - z} \frac{|\zeta - z|\gamma}{|\zeta - z|^2} dm(\zeta)
\]

\[\]
From these two terms, the first one is weakly singular integral and has a limit
\[
\int_{B_{R_0}(z) \cap \Omega} \frac{f(\zeta) - f(z)}{|\zeta - z|^\gamma} \frac{1}{(\zeta - \bar{z})^2} \, dm(\zeta).
\]

The second term, according to the Beurling geometric lemma (below) has a limit as \( \epsilon \to 0 \), that is
\[
f(z) \left\{ \int_{B_{R_0}(z) \cap \Omega \cap \{\kappa z > 0\}} - \int_{B_{R_0}(z) \cap \Omega' \cap \{\kappa z < 0\}} \right\} b_z.
\]

\[\Box\]

**Lemma 4 (Beurling-Geometric lemma).** If \( \Omega \subset \mathbb{C} \) and \( \partial \Omega \) is compact and \( C^{1,\gamma} \), then there exists
\[
\lim_{\epsilon \to 0} \int_{C_{\epsilon R}^2(\zeta \cap \Omega)} \frac{dm(\zeta)}{(\zeta - \bar{z})^2} = \left\{ \int_{B_{R_0}(z) \cap \Omega \cap \{\kappa z > 0\}} - \int_{B_{R_0}(z) \cap \Omega' \cap \{\kappa z < 0\}} \right\} \frac{dm(\zeta)}{(\zeta - \bar{z})^2}.
\]

**Proof.** First of all, we have
\[
\int_{C_{\epsilon R}^2(\zeta \cap \Omega)} b_z = \int_{C_{\epsilon R}^2(\zeta \cap \{\kappa z \leq 0\})} b_z + \left\{ \int_{B_{R_0}(z) \cap \Omega \cap \{\kappa z > 0\}} - \int_{B_{R_0}(z) \cap \Omega' \cap \{\kappa z < 0\}} \right\} b_z,
\]

and
\[
\int_{C_{\epsilon R}^2(\zeta \cap \{\kappa z \leq 0\})} b_z = \int_{\epsilon}^{R_0} \frac{dr}{r} \int_0^{\pi} e^{2i\theta} \, d\theta = 0.
\]

From the geometrical lemma, we have that
\[
\int_{C_{\epsilon R}^2(\zeta \cap \{\kappa z \leq 0\})} b_z + \left\{ \int_{B_{R_0}(z) \cap \Omega \cap \{\kappa z > 0\}} - \int_{B_{R_0}(z) \cap \Omega' \cap \{\kappa z < 0\}} \right\} b_z
\]
\[
= \int_{\epsilon}^{R_0} \frac{dr}{r} \int_{I(r)} e^{2i\theta} \, d\theta = \int_{\epsilon}^{R_0} \vartheta_{\Omega}(z; r) \frac{dr}{r^{1-\gamma}},
\]

where
\[
\vartheta_{\Omega}(z; r) = \frac{\cos \alpha(r) - 1 + i \sin \alpha(r) - 1 - \cos \beta(r) - i \sin \beta(r)}{r^\gamma}.
\]

By the geometric lemma\[\Box\] we have
\[
|\vartheta(r)| \leq 2M
\]
and then the limit exists and is equal to
\[
\int_{\epsilon}^{R_0} \vartheta_{\Omega}(z; r) \frac{dr}{r^{1-\gamma}}.
\]

\[\Box\]
The next fact is proved in Lemma 3 in [MOV09] in a more general situation. For completeness we give a proof adapted to this case.

**Lemma 5** (Cancellation lemma). If \( z \in B_{R_0}(z) \) and \( \epsilon < R_0 - |z - w| \), then

\[
\int_{B_{R_0}(w) \setminus B_{\epsilon}(z)} \frac{dm(\zeta)}{(\zeta - \bar{z})^2} = 0.
\]

**Proof.** In complex coordinates, the integral is

\[
\frac{1}{2i} \int_{B_{R_0}(w) \setminus B_{\epsilon}(z)} \frac{d\zeta \wedge d\bar{\zeta}}{(\zeta - \bar{z})^2} = \frac{i}{2} \left( \int_{\partial B_{R_0}(w)} - \int_{\partial B_{\epsilon}(z)} \right) \frac{d\zeta}{\zeta - \bar{z}},
\]

and

\[
\int_{\partial B_{\epsilon}(z)} \frac{d\zeta}{\zeta - \bar{z}} = \int_{0}^{2\pi} i e^{2i\theta} d\theta = 0,
\]

and

\[
\int_{\partial B_{R_0}(w)} \frac{d\zeta}{\zeta - \bar{z}} = \int_{0}^{2\pi} \frac{iR_0 e^{i\theta} d\theta}{R_0 e^{-i\theta} + w - z} = R_0 \int_{0}^{2\pi} \frac{e^{i\theta} i e^{i\theta} d\theta}{R_0 + (w - z) e^{i\theta}} = R_0 \int_{\beta B_{\epsilon}(0)} \frac{\tau d\tau}{R_0 + (w - z) \tau} = 0,
\]

because \( \frac{\tau}{R_0 + (w - z) \tau} \) is a function holomorphic in \( \tau \), as \( \frac{R_0}{|w - z|} > 1 \).


\[
4.3 \quad \text{The jump formula for } \bar{\mathcal{B}}
\]

Now we prove the jump formula

\[
\bar{\mathcal{B}}[g](z) = \frac{1}{2} \left\{ \lim_{w \to z; w \in \Omega} \chi_{\Omega} \bar{\mathcal{B}}[g](w) + \lim_{w \to z; w \in \mathbb{C} \setminus \Omega} \chi_{\mathbb{C} \setminus \Omega} \bar{\mathcal{B}}[g](w) \right\}
\]

that is identity (8) in Theorem 3.

**Remark.** Jump formulas of this type, for Calderón–Zygmund operators in potential theory appear in [HMT10]. For the special case of the conjugate Beurling transform we give a simple proof to keep the exposition more self-contained.

As we will see in section 4.4 and 4.5 we have that \( \chi_{\Omega} \bar{\mathcal{B}}[g] \in \text{Lip}_{\gamma}(\bar{\Omega}) \) and \( \chi_{\mathbb{C} \setminus \Omega} \bar{\mathcal{B}}[g] \in \text{Lip}_{\gamma}(\mathbb{C} \setminus \Omega) \), and then the limits \( \lim_{w \to z; w \in \Omega} \chi_{\Omega} \bar{\mathcal{B}}[g](w) \) and \( \lim_{w \to z; w \in \mathbb{C} \setminus \Omega} \chi_{\mathbb{C} \setminus \Omega} \bar{\mathcal{B}}[g](w) \) both exist and we can choose \( w = z \pm \lambda \eta(z) \), where \( \eta(z) \) is the unit vector, normal exterior to \( \partial \Omega \) at \( z \).

Also, if \( g_{\pm} \) are the Lipschitz extensions of \( g \) to \( \Omega^c \) and \( \bar{\Omega} \), respectively, we have, for \( w \in B_{\frac{R_0}{2}}(z) \), the following facts
• If \( w \in \Omega \), then

\[
\bar{B}[g_-](w) = \int_{\Omega} (g_- - g_-(w)) b_w + g_-(w) \int_{\Omega} b_w
\]

\[
= \int_{\Omega \cap B_{|z-w|}(z)} (g_- - g_-(w)) b_w
\]

\[
+ \int_{\Omega \cap B_{|z-w|}(z)} (g_- - g_-(w)) b_w + g_-(w) \int_{\Omega} b_w
\]

\[
= (I)(w) + (II)(w) + g_-(w) (III)(w).
\]

For the integral \((I)(w)\), we have immediately that for any fixed \( \zeta \),

\[
\chi_{\Omega \cap B_{|z-w|}(z)}(\zeta) (g_- (\zeta) - g_-(w)) b_w(\zeta) \to w \to z \chi_{\Omega} (\zeta) (g_- (\zeta) - g_-(z)) b_2(\zeta),
\]

and also

\[
|\chi_{\Omega \cap B_{|z-w|}(z)}(\zeta) (g_- (\zeta) - g_-(w)) b_w(\zeta)| \leq \|g_-\|_{\text{Lip}(\gamma, \bar{\Omega})} \frac{2}{|\zeta - z|^{2-\gamma}},
\]

and by the dominated convergence theorem,

\[
(I)(w) \to w \to z \int_{\Omega} (g_- - g_-(z)) b_z.
\]

The term \((II)(w)\) is controlled by

\[
\|g_-\|_{\text{Lip}(\gamma, \bar{\Omega})} \int_{\Omega \cap B_{|z-w|}(z)} \frac{1}{|\zeta - w|^{2-\gamma}} dm(\zeta)
\]

and the last integral is bounded by

\[
\int_{\Omega \cap B_{|z-w|}(w)} \frac{1}{|\zeta - w|^{2-\gamma}} dm(\zeta) \leq 2\pi \int_{0}^{3|z-w|} \frac{1}{r^{1-\gamma}} dr = 2\pi 3^\gamma |z - w|^\gamma,
\]

and then

\[
(II)(w) \to w \to z 0.
\]

Also, if \( \phi \in \mathcal{D}(C) \) such that \( \phi \equiv 1 \) in a ball containing \( \bar{\Omega} \), then

\[
\bar{B}[g_+](w) = \bar{B}[(1 - \phi) g_+](w) + \bar{B}[\phi g_+](w)
\]

\[
= (IV)(w) + \int_{\mathbb{C} \cap (\Omega \cup B_{|z-w|}(z))} \phi (g_+ - g_+(z)) b_w
\]

\[
+ \int_{\Omega \cap B_{|z-w|}(z)} \phi (g_+ - g_+(z)) b_w + g_+(z) \int_{\Omega} \phi b_w
\]

\[
= (IV)(w) + (V)(w) + (VI)(w) + g_+(z) (VII)(w).
\]
It is immediate that
\[
\lim_{w \to z} (IV)(w) = B[(1 - g) g_+](z).
\]
By arguments similar to those used for (I)(w), we have that
\[
\lim_{w \to z} (V)(w) = \int_{\Omega^c} \varrho (g_+ - g_+ (z)) b_z,
\]
and, analogously to (II)(w), we have that
\[
\lim_{w \to z} (VI)(w) = 0.
\]

![Figure 1](image)

So, as all limits exist, we have
\[
\lim_{w \to z; w \in \Omega} \chi_\Omega B[g](w)
= \int_{\Omega} (g_- - g_-(z)) b_z + B[(1 - g) g_+](z)
+ \int_{\Omega^c} \varrho (g_+ - g_+ (z)) b_z + g_-(z) \lim_{w \to z} \int_{\Omega} b_w + g_+(z) \lim_{w \to z} \int_{\Omega^c} \varrho b_w
= B[g_-](z) + B[g_+](z) + g_-(z) \left\{ \lim_{w \to z} \int_{\Omega} b_w - \int_{\Omega^c} \right\}
+ g_+(z) \left\{ \lim_{w \to z} \int_{\Omega^c} \varrho b_w - \int_{\Omega} \right\}
= B[g](z) + g_-(z) \left\{ \lim_{w \to z} \int_{\Omega^c} b_w - \int_{\Omega} \right\}
+ g_+(z) \left\{ \lim_{w \to z} \int_{\Omega^c} \varrho b_w - \int_{\Omega} \right\}.
\]
Lemma 6. Let $W \subset \mathbb{C}$ a domain with compact $C^{1,\gamma}$ boundary. Let $z \in \partial W$ and $\eta = \eta(z)$ be the normal exterior vector at $z$. Consider the points $w = z \pm \lambda \eta$.

For $h \in \mathcal{D}(\mathbb{C})$ we have

$$\frac{1}{2} \left\{ \lim_{\lambda \to 0; \ w = z \pm \lambda \eta} \int_W h b_w + \lim_{\lambda \to 0; \ w = z \pm \lambda \eta} \int_W h b_w \right\} = \int_W h b_z.$$

Proof. First of all, for $w \notin \partial W$, using Stokes theorem we have

$$\int_W h b_w = \frac{1}{2i} \int_W h(\zeta) \frac{d\zeta \wedge d\bar{\zeta}}{(\zeta - \bar{w})^2} = \frac{1}{2i} \left\{ \int_W \frac{\partial h}{\partial \zeta}(\zeta) \frac{d\zeta \wedge d\bar{\zeta}}{\zeta - \bar{w}} - \int_{\partial W} h(\zeta) \frac{d\zeta}{\zeta - \bar{w}} \right\} = (*) .$$

The integral

$$\int_{\partial W} h(\zeta) \frac{d\zeta}{\zeta - w} = \int_{\partial W \setminus B_{1/2} - w} h(\zeta) \frac{d\zeta}{\zeta - w} + \int_{\partial W \cap B_{1/2} - w} h(\zeta) \frac{d\zeta}{\zeta - w}.$$
and
\[
\int_{\partial W \cap B_2 \mid z-w (z)} \frac{d\zeta}{\zeta - \overline{w}} = \int_{\{\kappa_z = 0\} \cap B_2 \mid z-w (z)} \frac{d\zeta}{\zeta - \overline{w}} + \left\{ \int_{\partial W \cap B_2 \mid z-w (z)} \frac{d\zeta}{\zeta - \overline{w}} - \int_{\{\kappa_z = 0\} \cap B_2 \mid z-w (z)} \frac{d\zeta}{\zeta - \overline{w}} \right\}
\]
\[
= (I) + (II),
\]
and the second term is
\[
(II) = -\left\{ \int_{\{\kappa_z < 0\} \cap W \cap \partial B_2 \mid z-w (z)} \frac{d\zeta}{\zeta - \overline{w}} + \int_{\{\kappa_z > 0\} \cap W \cap \partial B_2 \mid z-w (z)} \frac{d\zeta}{\zeta - \overline{w}} \right\}
\]
All the integrals are well defined for \( w = z \pm \lambda \eta \), and the geometric cancellation lemma and the facts that both \( \{\kappa_z > 0\} \cap W \cap \partial B_2 \mid z-w (z) \) and \( \{\kappa_z < 0\} \cap W \cap \partial B_2 \mid z-w (z) \) tend to 0 as \( r \) tends to 0, imply that
\[
\lim_{w \to z} (II) = 0.
\]
After a translation and a rotation,
\[
(I) = \int_{-2 \mid z-w \mid}^{2 \mid z-w \mid} \frac{dx}{x - i\lambda} = \int_{-2 \mid z-w \mid}^{2 \mid z-w \mid} \frac{x}{x^2 + \lambda^2} dx + i \lambda \int_{-2 \mid z-w \mid}^{2 \mid z-w \mid} \frac{1}{x^2 + \lambda^2} dx
\]
\[
= \ln(x^2 + \lambda^2)^{\mid z-w \mid} + i \operatorname{arctan}\left(\frac{x}{\lambda}\right)^{\mid z-w \mid}
\]
\[
= 2i \operatorname{arctan}\left(\frac{2 \mid z-w \mid}{\lambda}\right),
\]
and then
\[
\lim_{w \to z} (I) = i \pi.
\]
Then,
\[
\frac{1}{2} \left\{ \lim_{\lambda \to 0} \left[ \lim_{w \to z-\lambda \eta} \int_W h b_w + \lim_{\lambda \to 0} \left[ \lim_{w \to z+\lambda \eta} \int_W h b_w \right] \right] \right\}
\]
\[
= \frac{1}{2} \left\{ \int_W \frac{\partial h}{\partial \zeta}(\zeta) \frac{d\zeta}{\zeta - \overline{w}} - \frac{1}{2} \left( \lim_{\lambda \to 0} \int_{\partial W \backslash B_2 \mid z \lambda (z)} h(\zeta) \frac{d\zeta}{\zeta - z + i \lambda} \right) \right\}
\]
because in the second term
\[
\lim_{w \to z} (I) = -i \pi.
\]
Then
\[
(**) = \frac{1}{2i} \left\{ \int_{W} \frac{\partial h}{\partial \zeta}(\zeta) \frac{d\zeta \wedge d\bar{\zeta}}{\zeta - \bar{w}} \right.
- \left. \frac{1}{2} \lim_{\lambda \to 0} \int_{\partial W \setminus B_{\lambda}(z)} h(\zeta) \left( \frac{1}{\zeta - z - i\lambda} + \frac{1}{\zeta - z + i\lambda} \right) d\zeta \right\}
= \frac{1}{2i} \left\{ \int_{W} \frac{\partial h}{\partial \zeta}(\zeta) \frac{d\zeta \wedge d\bar{\zeta}}{\zeta - \bar{w}} \right. - \left. \frac{1}{2} \lim_{\lambda \to 0} \int_{\partial W \setminus B_{\lambda}(z)} h(\zeta) \left( \frac{2(\bar{\zeta} - \bar{z})}{(\bar{\zeta} - \bar{z})^2 + \lambda^2} \right) d\zeta \right\}.
\]

On the other hand,
\[
\int_{W} h_{b_2} = \lim_{\lambda \to 0} \int_{W \setminus B_{\lambda}(z)} \frac{h(\zeta)}{(\bar{\zeta} - \bar{z})^2} dm(z) = \lim_{\lambda \to 0} I_{\lambda},
\]
and by Cauchy–Green’s formula (cf. [H"90])
\[
I_{\lambda} = \frac{1}{2i} \left\{ - \int_{\partial W \setminus B_{\lambda}(z)} h(\zeta) \frac{d\zeta \wedge d\bar{\zeta}}{\zeta - \bar{\zeta}} + \int_{\partial B_{\lambda}(z) \cap W} h(\zeta) \frac{d\zeta}{\zeta - \bar{\zeta}} \right. \left. + \int_{W \setminus B_{\lambda}(z)} \frac{\partial h}{\partial \zeta}(\zeta) \frac{d\zeta \wedge d\bar{\zeta}}{\zeta - \bar{\zeta}} \right\},
\]
and as in the previous developments
\[
I_{\lambda} \to_{\lambda \to 0} \frac{1}{2i} \left\{ - \text{p. v.} \int_{\partial W} h(\zeta) \frac{d\zeta \wedge d\bar{\zeta}}{\zeta - \bar{\zeta}} + \int_{W} \frac{\partial h}{\partial \zeta}(\zeta) \frac{d\zeta \wedge d\bar{\zeta}}{\zeta - \bar{\zeta}} \right\},
\]
where the fact that
\[
\int_{0}^{\pi} e^{2i\theta} d\theta = 0
\]
is used.

### 4.4 Uniform estimates

Let us assume WLOG that $0 \in \Omega$, and $d(0, \partial \Omega) = \max\{d(z, \partial \Omega) : z \in \Omega\}$ and also that $R_0$ in Lemma 3 is not larger than 1.

The parameter
\[
\Delta(z) = \frac{1}{\max\{|z|^2, d(z)^2\}},
\]
for $z \in (\Omega \cup U_{R_0})^c$ plays an important role in the estimates.

We have

**Proposition 7.** Let $\phi \in C^\infty(\Omega) \cap \text{Lip}(\gamma, \bar{\Omega})$. Let $\psi \in C^\infty(C \setminus \bar{\Omega}) \cap \text{Lip}(\gamma, C \setminus \Omega)$, satisfying that for a fixed constant $C(\psi)$,
\[
|\psi(z)| \leq \frac{C(\psi)}{\delta(z)^2},
\]
for $z \in (\Omega \cup U_{R_0})^c$.

Then there exists a constant $K = K(\gamma, \Omega, R_0)$ such that
1. For \( f = \phi \) or \( \psi \), we have
\[
\| \chi_\Omega B[f] \|_\infty \leq K \| f \|_\gamma.
\]

2. For \( f \) as above and \( z \in \Omega^c \cap U_{R_0} \), we have
\[
\| \chi_{C\setminus \Omega} B[f](z) \| \leq K \| f \|_\gamma.
\]

3. For \( z \in C \setminus (\Omega \cup U_{R_0})^c \), we have
\[
\| (\chi_{C\setminus \Omega} B[\phi])(z) \| \leq K \| \phi \|_\gamma \delta(z)^2.
\]

4. For \( z \in C \setminus (\Omega \cup U_{R_0})^c \), we have
\[
\| (\chi_{C\setminus \Omega} B[\psi])(z) \| \leq K (1 + \| \psi \|_\gamma) C(\psi) \left\{ \frac{1}{\delta(z)^2} + \frac{1}{\Delta(z)^2} \left( 1 + \ln(\Delta(z)) \right) \right\}.
\]

**Remark.** In particular, if \( f = \chi_\Omega \) then
\[
\| \chi_\Omega B[\chi_\Omega] \|_\infty \leq K
\]
and
\[
\| (\chi_{C\setminus \Omega} B[\chi_\Omega])(z) \| \leq K \left( \chi_{U_{R_0}}(z) + \frac{\chi_{C\setminus (U_{R_0} \cup \Omega)}(z)}{\delta(z)^2} \right).
\]

### 4.4.1 Proof of Proposition 7

We have to estimate \( Q[f](z) \) and \( L[f](z) \) in different situations depending on the support of \( f \) and the position of the point \( z \).

For this, we have

1) **Estimates for** \( L \):

**Proposition 8.** If \( z \in \Omega \setminus U_{R_0} \), there exits a constant \( C_1 = C_1(\gamma, \Omega) > 0 \) such that
\[
|L[\phi](z)| \leq C_1 \| \phi \|_\gamma
\]
and
\[
L[\psi](z) = 0.
\]

**Proof.** We have now that \( z \in \Omega, \delta(z) = d(z) > R_0 \) and \( r \leq d(z) \).

For the first part,
\[
|L[\phi](z)| \leq \| \phi \|_\gamma 2\pi \int_0^{d(z)} r^{\gamma-1} dr = \| \phi \|_\gamma 2\pi \frac{d(z)^\gamma}{\gamma} \leq \| \phi \|_\gamma 2\pi \frac{\text{diam} \Omega}{\gamma}.
\]

For the second part, we have \( L[\psi](z) = 0. \)

---

\( ^4C_1 = 2\pi \frac{\text{diam} \Omega}{\gamma}. \)
Proposition 9. If \( z \in \overline{U_{R_0}} \), there exists a constant \( \bar{C}_2 = C_2(\gamma, \Omega, R_0) > 0 \) such that both for \( f = \phi, \psi \), we have

\[
|L[f](z)| \leq \bar{C}_2 \|f\|_\gamma.
\]

Proof. For \( z \in \overline{U_{R_0}} \), let \( \tau \in \partial \Omega \) such that \( d(z) = |z - \tau| \) and \( \kappa_\tau \) defining the half space determined by the tangent line to \( \partial \Omega \) across \( \tau \) and containing the inward normal vector to \( \partial \Omega \) in \( \tau \). Then

\[
L[f](z) = \left( \int_{B_t(z)(\kappa_\tau \leq 0)} + \int_{B_t(z)(\kappa_\tau > 0)} \right) \frac{f(\zeta) - f(z)}{|\zeta - z|^\gamma} \frac{|\zeta - z|^\gamma}{(\zeta - \bar{z})^2} d\mu(\zeta).
\]

If \( f = \phi \), in the case of \( z \in \Omega \), we have

\[
L[\phi](z) = \int_{B_t(z)(\kappa_\tau \leq 0)} \frac{\phi(\zeta) - \phi(z)}{|\zeta - z|^\gamma} \frac{|\zeta - z|^\gamma}{(\zeta - \bar{z})^2} d\mu(\zeta)
- \int_{B_t(z)(\kappa_\tau > 0)} \frac{\phi(\zeta) - \phi(z)}{|\zeta - z|^\gamma} \frac{|\zeta - z|^\gamma}{(\zeta - \bar{z})^2} d\mu(\zeta)
- \int_{\Omega} \frac{\phi(\zeta) - \phi(z)}{|\zeta - z|^\gamma} \frac{|\zeta - z|^\gamma}{(\zeta - \bar{z})^2} d\mu(\zeta) = (I) + (II) + (III) + (IV).
\]

The integral

\[
|(I)| \leq \|\phi\|_\gamma \frac{2\pi}{\gamma} d(z)^\gamma
\]

as in the previous proposition.

Concerning the terms (II) and (III), the integrals there are extended to the subsets of \( B_{t}(z) \) located between the tangent line to \( \partial \Omega \) at \( \tau \) and \( \partial \Omega \) itself. Also \( z \) is not in the domain of integration. Then, using the lemma "(Aux. 1)"; we have

\[
|(II)| \leq \|\phi\|_\infty \frac{4 \delta(z)^\gamma}{\gamma},
\]

and the integral

\[
|(III)| \leq \|\phi\|_\gamma \frac{4 \delta(z)^\gamma}{\gamma}.
\]

\footnote{\(C_2 \) estimates \( \|f\|_\gamma \left( \frac{1}{\gamma} \left( \frac{B_0}{R_0} \right)^\gamma + \frac{1}{\gamma} \left( \frac{R_0}{B_0} \right)^\gamma \right) + \|f\|_\infty \left( \frac{3}{\gamma} + \frac{4 \delta(z)^\gamma}{\gamma} \right). \)}
Finally, the integral in \((IV)\) can be written as
\[
\left( \int_{B_{\delta(z)} \cap \{ \kappa \tau \geq 0 \}} - \int_{B_{\delta(z)} \cap \{ \kappa \tau \geq 0 \} \cap \{ \rho \leq 0 \}} \right) b_z,
\]
and using Lemmas 7 and 8 we have the estimate
\[
|\(IV)\| \leq \|\phi\|_\infty \left( \frac{\pi}{2} + \frac{4 \delta(z) \gamma}{\gamma} \right).
\]
Finally, un this case \(\frac{R_0}{2} \leq \delta(z) \leq R_0\), and this concludes. \(\square\)

The case of \(f = \psi\) is completely similar.

Lemma 7. Under the conditions and notation of Proposition 9, the integrals
\[
\int_{B_{\delta(z)} \cap \{ \kappa \tau \leq 0 \} \cap \{ \rho > 0 \}} b_z
\]
and
\[
\int_{B_{\delta(z)} \cap \{ \kappa \tau \geq 0 \} \cap \{ \rho \leq 0 \}} b_z
\]
are bounded by
\[
\frac{4 R_0^\gamma}{\gamma}.
\]

Proof. Concerning the terms \((II)\) and \((III)\), the integrals are extended to the subsets of \(B_{\delta(z)}(z)\) located between the tangent line to \(\partial \Omega\) at \(w_0\) and \(\partial \Omega\) itself.

Then, we can take coordinates centered at \(w_0\) given by the frame \(\eta(w_0) = \frac{2 \delta \rho(w_0)}{\|\nabla \rho(w_0)\|}, \tau(w_0) = i \eta(w_0)\), so any point \(z \in \mathbb{C}\) is of the form
\[
z = w_0 + \lambda \eta(w_0) + \mu \tau(w_0)
\]
and
\[
\kappa_{w_0}(z) = \lambda \|\nabla \rho(w_0)\|
\]
and
\[
\rho(z) = \kappa_{w_0}(z) + \omega(\sqrt{\lambda^2 + \mu^2}) \sqrt{\lambda^2 + \mu^2},
\]
where \(\omega\) is the modulus of continuity of the derivatives of \(\rho\) at \(w_0\).

Then the isometric map
\[
P: \mathbb{C} \to \mathbb{C}
\]
given by
\[
\phi(\mu + i \lambda) = w_0 + \lambda \eta(w_0) + \mu \tau(w_0)
\]
transforms 0 in \(w_0\), \(-i d(z_0)\) in \(z_0\), the real and the imaginary axis in the lines across \(w_0\) directed by \(\eta(w_0)\) and \(\tau(w_0)\) respectively. Also
\[
P(B_{\delta(z)}(-i d(z_0))) = B_{\delta(z)}(z).
\]
\[\text{We change the notation and use } w_0 \text{ instead of } \tau.\]
Then, if

\[ A_1 = \{ \mu + i \lambda \in \mathbb{C} : |\mu| \leq \sqrt{R_0^2 - d(z)^2}, \lambda \|\nabla \rho(w_0)\| + \omega(\sqrt{\lambda^2 + \mu^2}) \sqrt{\lambda^2 + \mu^2} \leq 0 \} \]

and

\[ A_2 = \{ \mu + i \lambda \in \mathbb{C} : |\mu| \leq \sqrt{R_0^2 - d(z)^2}, \lambda \|\nabla \rho(w_0)\| + \omega(\sqrt{\lambda^2 + \mu^2}) \sqrt{\lambda^2 + \mu^2} > 0 \}, \]

we have that

\[
(II) = \int_{B_{\delta(z)}(\lambda > 0) \cap A_1} \frac{1}{(\mu - i \lambda + i d(z))^2} \, dm(\mu, \lambda)
\]

\[
= \int_{B_{\delta(z)}(\lambda > 0) \cap A_1} \frac{1}{(\mu - i \lambda + i d(z))^2} \, dm(\mu, \lambda)
\]

\[
= \int_{-\sqrt{R_0^2 - d(z)^2}}^{\sqrt{R_0^2 - d(z)^2}} d\mu \int_{0}^{\varphi_+(\mu)} \frac{1}{(\mu - i \lambda + i d(z))^2} \, d\lambda
\]

\[
= \frac{1}{i} \int_{-\sqrt{R_0^2 - d(z)^2}}^{\sqrt{R_0^2 - d(z)^2}} \frac{\varphi_+(\mu)}{\mu - i \varphi_+(\mu) + i d(z)} \, d\mu.
\]

Then

\[
|\langle II \rangle| \leq \int_{-\sqrt{R_0^2 - d(z)^2}}^{\sqrt{R_0^2 - d(z)^2}} \frac{|\varphi_+(\mu)|}{|\mu - i (\varphi_+(\mu) + d(z))|} \, d\mu
\]

\[
\leq C \int_{-\sqrt{R_0^2 - d(z)^2}}^{\sqrt{R_0^2 - d(z)^2}} \frac{|\mu|^\gamma}{\sqrt{\mu^2 + (\varphi_+(\mu) + d(z))^2}} \, d\mu
\]

\[
\leq 2C \int_{0}^{\sqrt{R_0^2 - d(z)^2}} \frac{1}{\mu^{1-\gamma}} \, d\mu = \frac{2C}{\gamma} \left( \sqrt{R_0^2 - d(z)^2} \right)^{\gamma}.
\]

\[ \square \]

**Lemma 8.**

\[
\left| \int_{B_{\delta(z)}(\kappa, \gamma \geq 0)} b_z \right| \leq \frac{\pi}{2}
\]

**Proof.** It is clear that if \( \delta(z) \geq \frac{R_0}{2} \) or if \( z = \tau \), then

\[
\int_{B_{\delta(z)}(\kappa, \gamma \geq 0)} b_z = 0
\]

by the cancellation property. So we assume that \( 0 < \delta(z) < \frac{R_0}{2} \) and then our integral is

\[
\int_{B_{\frac{R_0}{2}}(z)(\kappa, \gamma \geq 0)} b_z.
\]

\[ ^4 \text{Here } \mu \text{ is the name for a coordinate. Also } R_0 \text{ must be } \delta(z) \text{ in this proof.} \]
Also by the cancellation property, and after a rigid movement, our integral is

\[
\int_{B(\frac{\alpha}{R}) \cap (\Im(\zeta) \geq \alpha)} B_z = \int_{B(\frac{\alpha}{R})} \frac{2r}{R} \int_{\alpha}^{\arcsin(\frac{\alpha}{R})} e^{2i\theta} d\theta
\]

\[
= \int_{\alpha}^{\arcsin(\frac{\alpha}{R})} \frac{1}{2i} \left( e^{-2i \arcsin(\frac{\alpha}{R})} - e^{2i \arcsin(\frac{\alpha}{R})} \right)
\]

\[
= \int_{\alpha}^{\arcsin(\frac{\alpha}{R})} \frac{1}{2i} \left( e^{-i \arcsin(\frac{\alpha}{R})} - e^{i \arcsin(\frac{\alpha}{R})} \right)
\]

\[
= -2 \int_{\alpha}^{\arcsin(\frac{\alpha}{R})} \frac{1}{r} \frac{\alpha}{R} \sqrt{1 - \left(\frac{\alpha}{R}\right)^2} = \ast,
\]

so we have, for \( t = \frac{\alpha}{R} \), that

\[
\ast = -2 \int_{\frac{\alpha}{R}}^{1} \sqrt{1 - t^2} dt = -(\arcsin(t) + t \sqrt{1 - t^2})|_{\frac{\alpha}{R}}^{1}
\]

\[
= \arcsin\left(\frac{2\alpha}{R_0}\right) + \frac{2\alpha}{R_0} \sqrt{1 - \left(\frac{2\alpha}{R_0}\right)^2} - \arcsin 1 \leq \frac{\pi}{2}.
\]

**Proposition 10.** If \( z \in \mathbb{C} \setminus \overline{U_{R_0} \cup \Omega} \), then

\[
L[\phi](z) = 0,
\]

and there exists a constant \( C_3 = C_3(\gamma, \Omega, R_0) \) such that

\[
|L[\psi](z)| \leq C_3 \left(1 + \|\psi\|_\gamma\right) C(\psi) \left\{ \frac{1}{\max\{R_0^2, d(z)^2\}} + \frac{1}{\max\{d(z), |z|\}^2} \left\{ 1 + \ln(\max\{d(z), |z|\}) \right\} \right\}.
\]

Here we are assuming that \( \|\psi\|_\gamma > 0 \) and also that \( 0 \in \Omega \setminus U_{R_0} \).

**Proof.** The case of \( \phi \) is immediate.

In the case of \( \psi \), for any choice of a positive \( \alpha(z) < \delta(z) \), we have

\[
L[\psi](z) = \int_{B(\frac{\alpha(z)}{R})} \frac{\psi(\zeta) - \psi(z)}{|\zeta - z|^\gamma} \frac{|\zeta - z|^\gamma}{(\zeta - z)^2} dm(\zeta)
\]

\[
= \left( \int_{B(\frac{\alpha(z)}{R})} + \int_{B(\frac{\alpha(z)}{R}) \setminus B(\frac{\alpha(z)}{R})} \right) \frac{\psi(\zeta) - \psi(z)}{|\zeta - z|^\gamma} \frac{|\zeta - z|^\gamma}{(\zeta - z)^2} dm(\zeta)
\]

\[
= (I)_{\alpha(z)} + (II)_{\alpha(z)}.
\]
Since
\[ |\psi(\zeta)| \leq |\psi(z)| + \|\psi\|_\gamma |\zeta - z|^\gamma \]
then for \( \zeta \in B_{\alpha(z)}(z) \) we have
\[ |\psi(\zeta)| \leq |\psi(z)| + \|\psi\|_\gamma \alpha(z) \gamma \]
and if \( \alpha(z) \leq (\max\{\frac{|\psi(z)|}{\|\psi\|_\gamma}, \frac{1}{\max\{R_0^2, d(z)^2\}}\})^{\frac{1}{\gamma}} \), we have
\[ |\psi(\zeta)| \leq 2 |\psi(z)| \]
and then
\[ |(I)_{\alpha(z)}| \leq \|\psi\|_\gamma \int_{B_{\alpha(z)}(z)} \frac{dm(\zeta)}{|\zeta - z|^{2-\gamma}} \]
\[ = 2\pi \|\psi\|_\gamma \int_{\epsilon}^{\alpha(z)} \frac{dr}{r^{1-\gamma}} \]
\[ \leq 2\pi \|\psi\|_\gamma \frac{\alpha(z)^\gamma}{\gamma} = \frac{2\pi}{\gamma} |\psi(z)|. \]

The term
\[ (II)_{\alpha(z)} = \left( \int_{B_{\alpha(z)}(z) \cap U_{R_0}} + \int_{B_{\alpha(z)}(z) \cap (B_{\alpha(z)}(z)) \setminus U_{R_0}} \right) \frac{\psi(\zeta) - \psi(z)}{|\zeta - z|^\gamma} \frac{|\zeta - z|^\gamma}{(\zeta - z)^2} dm(\zeta) \]
\[ = (III) + (IV)_{\alpha(z)}. \]

The term
\[ (III) = \left( \int_{B_{\alpha(z)}(z) \cap U_{R_0}} + \int_{B_{\alpha(z)}(z) \cap (U_{R_0} \setminus U_{\frac{R_0}{2}})} \right) \frac{\psi(\zeta) - \psi(z)}{|\zeta - z|^\gamma} \frac{|\zeta - z|^\gamma}{(\zeta - z)^2} dm(\zeta) \]
\[ = (V) + (VI). \]

Since for \( \zeta \in U_{R_0} \) we have \( |\zeta - z| \geq \frac{d(z)}{2} \), then\(^1\)
\[ |(V)| \leq \|\psi\|_\infty \frac{4 m(\bar{U}_{R_0})}{\max\{R_0^2, d(z)^2\}}. \]

The integral of the term \((VI)\), extends to \( \zeta \in U_{R_0} \setminus U_{\frac{R_0}{2}} \). Then, if \( d(z) \geq \frac{3R_0}{2} \) we have
\[ |\zeta - z| \geq ||\zeta - \varsigma| - |\varsigma - z|| \geq d(z) - R_0 \geq \max\{\frac{d(z)}{3}, \frac{3R_0}{2}\} \]
so
\[ |(VI)| \leq \|\psi\|_\infty \frac{36 m(\bar{U}_{R_0})}{\max\{R_0^2, d(z)^2\}}. \]

\(^1\)In all this part, the terms are also bounded by \( \frac{1}{R_0^2} \) with the same constants.
If \( R_0 \leq d(z) < \frac{3R_0}{2} \), then \( d(z) - R_0 \leq |\zeta - z| \leq d(z) \) and then

\[
| (VI) | \leq \| \psi \|_\gamma 2\pi \int_{d(z)-R_0}^{d(z)} \frac{dr}{r^{1+\gamma}} \leq \| \psi \|_\gamma \frac{2\pi}{\gamma} d(z)^{\gamma} = \| \psi \|_\gamma \frac{\Theta}{\Theta} \frac{R_0^{2+\gamma}}{\max\{R_0^2, d(z)^2\}}.
\]

The term

\[
(IV)_{\alpha(z)} = \left( \int_{C_{\alpha(z)}^{d(z)}} \phi(z) + \int_{C_{\alpha(z)}^{d(z)}} \psi(z) b_{\zeta} \right) \psi(z) = (VII)_{\alpha(z)} + (VIII),
\]

and we can use the estimate

\[
| \psi(z) | \leq \frac{C(\psi)}{\max\{R_0^2, d(z)^2\}}.
\]

Since\(^8\) for \( \zeta \in B_{d(z)}(z) \), we have \( d(\zeta) > \frac{d(z)}{2} \), we have

\[
| (VII)_{\alpha(z)} | \leq C(\psi) \int_{C_{\alpha(z)}^{d(z)}} \frac{dm(\zeta)}{d(\zeta)^2} \leq C(\psi) \int_{C_{\alpha(z)}^{d(z)}} \frac{dm(\zeta)}{|\zeta - z|^2} = C(\psi) \int_{C_{\alpha(z)}^{d(z)}} \frac{dm(\zeta)}{2 \alpha(z)} \int \frac{d|x|}{r}
\]

and if \( \frac{|\psi(z)|}{\| \psi \|_\gamma} \geq \frac{1}{\max\{R_0^2, d(z)^2\}} \), then

\[
(\ast) \leq C(\psi) \frac{8\pi}{d(z)^2} \left\{ \ln(d(z)) \| \psi \|_\gamma \right\} + \ln(2 | \psi(z) |^{\frac{\gamma}{2}}),
\]

and in the opposite case we have

\[
(\ast) \leq C(\psi) \frac{8\pi}{d(z)^2} \left\{ \ln(d(z)) + \ln(2 \max\{R_0^2, d(z)^2\}) \right\}.
\]

For the term \((VIII)\), we consider

\[
C_{\alpha(z)}^{d(z)} \setminus U_{R_0} = E_1(z) \cup E_2(z),
\]

where \( E_1(z) \) and \( E_2(z) \) are the intersection of the domain of integration with the set

\[
\{ \zeta \in \mathbb{C} : d(\zeta) \geq |\zeta - z| \}
\]

or its complement. Then

\[
(VIII) = \left( \int_{E_1(z)} + \int_{E_2(z)} \right) \psi b_{\zeta} = (IX) + (X),
\]

\(^8\)In the case of \( \frac{d(z)}{2} > \alpha(z) \). Otherwise the first term is equal to 0.
and the term
\[ |(IX)| \leq C(\psi) \int_{C \setminus B_{\frac{d(z)}{2}}(z)} \frac{dm(\zeta)}{|\zeta - z|^4} \leq C(\psi) 2\pi \int_{d(z)}^{\infty} \frac{dr}{r^3} \leq \frac{8\pi}{d(z)^2}. \]

Let
\[ M = \max\{|w| : w \in (U \cup \Omega)\}. \]

The term
\[ (X) = \left( \int_{E_2(z) \cap B_{2M}(0)} + \int_{E_2(z) \setminus B_{2M}(0)} \right) \psi b_2 = (XI) + (XII) \]
since in general \( d(\zeta) \geq |\zeta| - M \), we have that if \( |\zeta| \geq 2M \), then \( d(\zeta) \geq \frac{|\zeta|}{2} \) and then
\[ |(XII)| \leq C(\psi) 4 \int_{C_{\frac{d(z)}{2}}(z) \setminus B_{2M}(0)} \frac{dm(\zeta)}{|\zeta|^2 |\zeta - z|^2}. \]

If \( |z| \geq 5M \), then \( d(z) \geq 3M \) and for \( r < 3M \), the balls \( B_{2M}(0) \) and \( B_{\frac{d(z)}{2}}(z) \) are mutually disjoint and we consider then the decomposition in disjoint sets
\[ C_{\frac{d(z)}{2}}(z) \setminus B_{2M}(0) = A_1 \cup A_2 \cup A_3, \]
where
\[ A_1 = C_{\frac{d(z)}{2}}(z) \setminus B_{2|z|}(0), \]
\[ A_2 = \{ \zeta \in B_{2|z|}(0) \setminus B_{2M}(0) : |\zeta| \leq |\zeta - z| \}, \]
and
\[ A_3 = \{ \zeta \in B_{2|z|}(0) \setminus B_{\frac{d(z)}{2}}(z) : |\zeta| \geq |\zeta - z| \}. \]

If \( |\zeta| > 2|z| \), then \( \frac{1}{2} |\zeta| \leq |\zeta - z| \leq \frac{1}{2} |\zeta| \), so the integral over \( A_1 \) is bounded by
\[ \int_{C_{\frac{d(z)}{2}}(z) \setminus B_{2|z|}(0)} \frac{4dm(\zeta)}{|\zeta|^4} = \frac{4\pi}{|z|^2}. \]

Since for \( \zeta \in A_2, |\zeta - z| \geq \frac{|z|}{2} \), the integral over \( A_2 \) is bounded by
\[ \frac{4}{|z|^2} \int_{C_{\frac{d(z)}{2}}(0)} \frac{dm(\zeta)}{|\zeta|^2 |\zeta - z|^2} \leq \frac{8\pi}{|z|^2} \int_{2M}^{\infty} \frac{dr}{r} = \frac{8\pi}{|z|^2} \ln \left( \frac{|z|}{M} \right). \]

Finally, for \( \zeta \in A_3 \) we have that \( |\zeta| \geq \frac{|z|}{2} \) and the integral in \( A_3 \) is bounded by
\[ \frac{4}{|z|^2} \int_{B_{2|z|}(0) \setminus B_{\frac{d(z)}{2}}(z)} \frac{dm(\zeta)}{|\zeta - z|^2} \leq \frac{4}{|z|^2} \int_{B_{4|z|}(0) \setminus B_{\frac{d(z)}{2}}(z)} \frac{dm(\zeta)}{|\zeta - z|^2} \]
\[ = \frac{8\pi}{|z|^2} \ln \left( \frac{8|z|}{d(z)} \right). \]
If $|z| \leq 5M$, then the set

$$C_{\frac{d(z)}{2}}(z) \setminus B_{2M}(0)$$

decomposes in a disjoint union of the resulting intersection with the set

$$\{ |\zeta| > |\zeta - z| \}$$

and its complement, namely $G_1$ and $G_2$. Then

$$\int_{G_1} \frac{dm(\zeta)}{|\zeta|^2 |\zeta - z|^2} \leq \int_{\Omega \setminus B_{\frac{d(z)}{2}}(z)} \frac{dm(\zeta)}{|\zeta - z|^4} = \frac{16\pi}{d(z)^2},$$

and

$$\int_{G_2} \frac{dm(\zeta)}{|\zeta|^2 |\zeta - z|^2} \leq \int_{\Omega \setminus B_{2M}(0)} \frac{dm(\zeta)}{|\zeta|^4} = \frac{16\pi}{4M^2} \leq \frac{16\pi}{|z|^2}. $$

The integral

$$|XI| \leq \int_{E_{2z} \cap \Omega \setminus B_{2M}(0)} |\zeta|^2 |\zeta - z|^2 \leq \int_{B_{2M}(0)} dm(\zeta) \leq C(\psi) \frac{64\pi M^2}{R^2 \cdot 4} \frac{4}{d(z)^2}. $$

2) Estimates for $Q$:

**Proposition 11.** There exists a constant $C_4 = C_4(\Omega, \frac{1}{R_0})$ such that if $z \in \Omega \cup U_{R_0}$, we have

$$|Q[\phi](z)| \leq C_4 \|\phi\|_{\infty}$$

and

$$|Q[\psi](z)| \leq C_4 (\|\psi\|_{\infty} + C(\psi)).$$

**Proof.** A) In the case of $z \in \Omega \setminus U_{R_0}$, we have $\delta(z) = d(z) \geq R_0$ and

$$Q[f](z) = \int_{\Omega \setminus B_{d(z)}(z)} f b_z.$$ 

Then

$$Q[\phi](z) = \int_{\Omega \setminus B_{d(z)}(z)} \phi b_z$$

and

$$|Q[\phi](z)| \leq \|\phi\|_{\infty} \frac{m(\Omega \cup U_{R_0})}{R^2}.$$

Also

$$Q[\psi](z) = \int_{\Omega \setminus B_{d(z)}(z)} \psi b_z = \int_{\Omega \setminus U_{R_0}} \psi b_z$$

$$= \int_{\Omega \setminus U_{R_0}} \psi b_z + \int_{\Omega \setminus U_{R_0}} \psi b_z = (I) + (II),$$

and (I) has the same control as $Q[\phi](z)$.  

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The integral
\[ |(II)| \leq C(\psi) \int_{C \setminus (\Omega \cup U_{R_0})} \frac{dm(\zeta)}{\max\{R_0^2, d(\zeta)^2\} |\zeta - z|^2}, \]
and the set
\[ C \setminus (\Omega \cup U_{R_0}) = E_1 \cup E_2, \]
where \( E_1 \) is the intersection of \( C \setminus (\Omega \cup U_{R_0}) \) with
\[ \{d(\zeta) \geq |\zeta - z|\} \]
and \( E_2 \) with the complement, we have that the integral is
\[ \int_{E_1} \frac{dm(\zeta)}{d(\zeta)^2 |\zeta - z|^2} + \int_{E_2} \frac{dm(\zeta)}{d(\zeta)^2 |\zeta - z|^2} = (III) + (IV), \]
and
\[ (III) \leq \int_{E_1} \frac{dm(\zeta)}{|\zeta - z|^4} \leq \int_{C \setminus B_{R_0}(z)} \frac{dm(\zeta)}{|\zeta - z|^4} \leq \frac{2\pi}{3R^3}. \]

For the integral \((IV)\), we have that in our domain, \( d(\zeta) \geq R_0 \) and
\[ |\zeta - z| \leq |\zeta - \tau| + |\tau - z| \leq d(\zeta) + \text{diam}(\Omega), \]
where \( |\zeta - \tau| = d(z) \). This implies that \( E_2 \subset B_{R_0 + 3 \text{diam}(\Omega)}(z_0) \) for \( z_0 \in \Omega \) a chosen point (in does not matter which one) and so
\[ |(IV)| \leq \frac{m(B_{R_0 + 3 \text{diam}(\Omega)} + d(0))}{R_0^4}. \]

B) In the case of \( z \in U_{R_0} \), we have
\[ Q[f](z) = \int_{C \setminus B_{R_0}(z)} f b_z = \left( \int_{C \setminus (B_{R_0}(z) \cup U_{R_0} \cup \Omega)} + \int_{(U_{R_0} \cup \Omega) \setminus B_{R_0}(z)} \right) f b_z = (I) + (II). \]
In the case \( f = \phi \), the integral
\[ (I) = 0. \]

The integral
\[ |(II)| \leq \|\phi\|_{\infty} \frac{m(\Omega \cup U_{R_0})}{R_0^4} \]
in all cases.

In the case of \( f = \psi \),
\[ |(I)| \leq C(\psi) \int_{C \setminus (\Omega \cup U_{R_0})} \frac{dm(\zeta)}{\max\{R_0^2, d(\zeta)^2\} |\zeta - z|^2} \]
and it has the same control of \((II)\).
Proposition 12. There exists a constant $C_5 = C_5(\Omega, \frac{1}{R_0})$ such that if $z \in \mathbb{C} \setminus (\Omega \cup U_{R_0})$, then
\[
|Q[\phi](z)| \leq C_5 \|\phi\|_{\infty} \frac{1}{\max\{R_0^2, d(z)^2\}}
\]
and
\[
|Q[\psi](z)| \leq C_5 \left\{ \|\psi\|_{\infty} \frac{1}{\delta(z)^2} + C(\psi) \frac{1}{\max\{|z|^2, d(z)^2\}} \right\} \left( 1 + \ln \max\{|z|, d(z)\} \right).
\]

Proof. If $z \in \mathbb{C} \setminus (\Omega \cup U_{R_0})$, then
\[
Q[\phi](z) = \int_{\Omega \setminus B_{R_0}(z)} \phi b_z
\]
and is bounded by
\[
\|\phi\|_{\infty} \frac{m(\Omega \cup U_{R_0})}{\delta(z)^2},
\]
leading to the statement.

Also
\[
Q[\psi](z) = \int_{\mathbb{C} \setminus (\Omega \cup B_{R_0}(z))} \psi b_z
\]
and is bounded by
\[
\|\psi\|_{\infty} \frac{m(U_{R_0})}{\delta(z)^2}.
\]

The integral
\[
|(I)| \leq \|\psi\|_{\infty} \frac{m(U_{R_0})}{\delta(z)^2}.
\]

The term
\[
|(I)| \leq C(\psi) \int_{\mathbb{C} \setminus (\Omega \cup U_{R_0} \cup B_{R_0}(z))} \frac{d\mu(\zeta)}{\max\{R_0^2, d(\zeta)^2\} |\zeta - z|^2},
\]
and we consider
\[
\mathbb{C} \setminus (\Omega \cup B_{R_0}(z)) = E_1(z) \cup E_2(z),
\]
where $E_1(z)$ and $E_2(z)$ are the intersection of the domain of integration with the set
\[
\{ \zeta \in \mathbb{C} : d(\zeta) \geq |\zeta - z| \}
\]
or its complement. Then the integral in $(I)$ can be decomposed as
\[
\left( \int_{E_1(z)} + \int_{E_2(z)} \right) \varphi b_z = (III) + (IV),
\]
and the term
\[
|(III)| \leq \int_{\mathbb{C} \setminus B_{R_0}(z)} \frac{d\mu(\zeta)}{|\zeta - z|^2} \leq 2\pi \int_{d(z)}^{\infty} \frac{dr}{r^3} \leq \frac{\pi}{d(z)^2}.
\]
The term

$$ (IV) = \left( \int_{E_2(z) \cap B_{2M}(0)} + \int_{E_2(z) \setminus B_{2M}(0)} \right) \varphi_b \varepsilon = (V) + (VI) $$

since in general $d(\zeta) \geq ||\zeta| - M|$, we have that if $|\zeta| \geq 2M$, then $d(\zeta) \geq \frac{|\zeta|}{2}$ and then

$$ ||(VI)|| \leq 4 \int_{C \setminus (B_{2M}(0) \cup B_{d(z)}(\zeta))} \frac{dm(\zeta)}{|\zeta|^2 |\zeta - z|^2}. $$

If $|z| \geq 5M$, then $d(z) \geq 3M$ and for $r < 3M$, the balls $B_{2M}(0)$ and $B_{d(z)}(\zeta)$ are mutually disjoint and we consider the decomposition in disjoint sets

$$ C \setminus (B_{2M}(0) \cup B_{d(z)}(\zeta)) = A_1 \cup A_2 \cup A_3, $$

where

$$ A_1 = C \setminus B_{||z||}(0), $$

$$ A_2 = \{ \zeta \in B_{||z||}(0) \setminus B_{2M}(0) : |\zeta| \leq |\zeta - z| \}, $$

and

$$ A_3 = \{ \zeta \in B_{||z||}(0) \setminus B_{d(z)}(\zeta) : |\zeta| \geq |\zeta - z| \}. $$

If $|\zeta| > 2|z|$, then $\frac{1}{2} |\zeta| \leq |\zeta - z| \leq \frac{1}{2} |\zeta|$, so the integral over $A_1$ is bounded by

$$ \int_{C \setminus (B_{||z||}(0))} \frac{4 dm(\zeta)}{|\zeta|^4} = \frac{4\pi}{|z|^2}. $$

Since for $\zeta \in A_2$, $|\zeta - z| \geq \frac{|z|}{2}$, the integral over $A_2$ is bounded by

$$ \frac{4}{|z|^2} \int_{C_{2M}(z)} \frac{dm(\zeta)}{|\zeta|^2 |\zeta - z|^2} \leq \frac{8\pi}{|z|^2} \int_{2M}^{|z|} \frac{dr}{r} = \frac{8\pi}{|z|^2} \ln \left( \frac{|z|}{M} \right). $$

Finally, for $\zeta \in A_3$ we have that $|\zeta| \geq \frac{|z|}{2}$ and the integral in $A_3$ is bounded by

$$ \frac{4}{|z|^2} \int_{B_{||z||}(0) \setminus B_{d(z)}(\zeta)} \frac{dm(\zeta)}{|\zeta - z|^2} \leq \frac{4}{|z|^2} \int_{B_{d(z)}(0) \setminus B_{d(z)}(\zeta)} \frac{dm(\zeta)}{|\zeta - z|^2} \leq \frac{8\pi}{|z|^2} \ln \left( \frac{8|z|}{d(z)} \right). $$

If $|z| \leq 5M$, then the set

$$ C \setminus (B_{2M}(0) \cup B_{d(z)}(\zeta)) $$

7In fact the ball of radius $d(z)$ should be subtracted, but the inequality is maintained with this domain of integration.
decomposes in a disjoint union of the resulting intersection with the set
\[ \{|\zeta| > |\zeta - z|\} \]
and its complement, namely \( G_1 \) and \( G_2 \). Then
\[
\int_{G_1} \frac{dm(\zeta)}{|\zeta|^2 |\zeta - z|^2} \leq \int_{C \setminus B_{2 \gamma}(z)} \frac{dm(\zeta)}{|\zeta - z|^4} = \frac{16\pi}{d(z)^2},
\]
and
\[
\int_{G_2} \frac{dm(\zeta)}{|\zeta|^2 |\zeta - z|^2} \leq \int_{C \setminus B_{2 \gamma}(0)} \frac{dm(\zeta)}{|\zeta|^4} = \frac{16\pi}{4M^2} \leq \frac{16\pi}{25|z|^2}.
\]
The integral
\[
|\langle V \rangle| \leq \int_{E_2(z) \cap B_{2 \gamma}(0)} |\phi b_z| \leq \int_{B_{2 \gamma}(0)} \frac{dm(\zeta)}{R_0^2 |\zeta - z|^2} \leq \frac{64\pi M^2}{R_0^2} \frac{4}{d(z)^2}.
\]

4.5 Lipschitz estimates

We assume the same hypotheses, definitions and notation of the previous sections and subsections.

**Proposition 13.** There exists a constant \( K = K(\gamma, \Omega, R_0) \) such that for \( \phi \) and \( \psi \) satisfying the conditions of Proposition 7 we have, for \( f = \phi \) or \( \psi \), that, if both \( z, w \in \Omega \), or both \( z, w \in \mathbb{C} \setminus \bar{\Omega} \), then
\[
\frac{|B[f](z) - B[f](w)|}{|z - w|^{\gamma}} \leq K_0 \|f\|_{\gamma}.
\]

**Remark.** The theorem implies that \( B[f] \) has a Lipschitz extension to \( \bar{\Omega} \) and also to \( \mathbb{C} \setminus \bar{\Omega} \) but these extensions differ in a jump along \( \partial\Omega \).

4.5.1 Proof of Proposition 13

We will estimate the quotients in the left hand side of the inequalities in the statement above, only in the case of \( |z - w| \leq \frac{R_0}{4} \).

Also, most of the proof goes along for \( \phi \) or \( \psi \) with no distinction, so unless it be necessary, we will use \( f \) for \( \phi \) or \( \psi \).

The goal is to estimate \( \frac{|B[f](z) - B[f](w)|}{|z - w|^{\gamma}} \) for \( z, w \in \Omega \) and \( z, w \in \mathbb{C} \setminus \bar{\Omega} \).

Then we have in general that
\[
\frac{|B[f](z) - B[f](w)|}{|z - w|^{\gamma}} \leq \frac{|Q[f](z) - Q[f](w)|}{|z - w|^{\gamma}} + \frac{|L[f](z) - L[f](w)|}{|z - w|^{\gamma}} + \frac{|f(z) \Theta_{\Omega}^{\frac{R_0}{\gamma}}(z) - f(w) \Theta_{\Omega}^{\frac{R_0}{\gamma}}(w)|}{|z - w|^{\gamma}} = (I) + (II) + (III).
\]

Since \( \Theta_{\Omega}^{\frac{R_0}{\gamma}} \) is a bounded function supported in \( \partial\Omega \), then \( (III) = 0 \).
The term

\[ Q[f](z) - Q[f](w) = \int_{C \setminus B_R(z)} f(b_z) - \int_{C \setminus B_R(w)} f(b_w) = \int_{C \setminus (B_R(z) \cup B_R(w))} f(b_z - b_w) + \int_{B_R(w) \setminus B_R(z)} f(b_z) - \int_{B_R(z) \setminus B_R(w)} f(b_w) = (1) + (2). \]

And we have

**Lemma 9.**

\[ b_z(\xi) - b_w(\xi) = - (\bar{z} - \bar{w}) (\bar{z} + \bar{w} - 2 \bar{\xi}) b_z(\xi) b_w(\xi). \]

**Proof.** It is a direct computation. \(\square\)

Then

\[ (1) = - (\bar{z} - \bar{w}) \int_{C \setminus (B_R(z) \cup B_R(w))} f(\xi) (\bar{z} + \bar{w} - 2 \bar{\xi}) b_z(\xi) b_w(\xi). \]

Also

\[ L[f](z) - L[f](w) = \int_{B_R(z)} \frac{f(\xi) - f(z)}{|\xi - z|^\gamma} \left( \frac{|\xi - z|^\gamma}{(\xi - z)^2} \right) dm(\xi) \]

\[ - \int_{B_R(w)} \frac{f(\xi) - f(w)}{|\xi - w|^\gamma} \left( \frac{|\xi - w|^\gamma}{(\xi - w)^2} \right) dm(\xi) \]

\[ = \int_{B_R(w) \setminus B_R(z)} \left[ \frac{f(\xi) - f(z)}{|\xi - z|^\gamma} \left( \frac{|\xi - z|^\gamma}{(\xi - z)^2} \right) - \frac{f(\xi) - f(w)}{|\xi - w|^\gamma} \left( \frac{|\xi - w|^\gamma}{(\xi - w)^2} \right) \right] dm(\xi) \]

\[ + \int_{B_R(z) \setminus B_R(w)} \frac{f(\xi) - f(z)}{|\xi - z|^\gamma} \left( \frac{|\xi - z|^\gamma}{(\xi - z)^2} \right) \]

\[ - \int_{B_R(w) \setminus B_R(z)} \frac{f(\xi) - f(w)}{|\xi - w|^\gamma} \left( \frac{|\xi - w|^\gamma}{(\xi - w)^2} \right) dm(\xi) \]

\[ = (3) + (4). \]

1. In general, the integral

\[ (1) = - (\bar{z} - \bar{w}) \int_{C \setminus (B_R(z) \cup B_R(w))} f(\xi) \frac{\bar{w} - \bar{\xi} + \bar{z} - \bar{\xi} - 2(\bar{\xi} - \bar{z})}{(\bar{\xi} + \bar{\xi} - \bar{z})^2(\bar{z} - \bar{\xi} + \bar{w} - \bar{\xi})^2} dm(\xi), \]

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and since $\tau - z = \frac{z + w}{2} - z = \frac{w - z}{2} \overset{\text{def}}{=} a$ and $\tau - w = -\frac{w - z}{2} = -a$, we have

$$ (1) = 2(\bar{z} - \bar{w}) \int_{C \setminus (B_{\delta(z)}(\zeta) \cup B_{\delta(w)}(w))} f(\zeta) \frac{\zeta - \bar{\tau}}{(\zeta - \bar{\tau} + a)(\zeta - \bar{\tau} - a)} \, dm(\zeta) \overset{\text{def}}{=} 2(\bar{z} - \bar{w}) K[f](z, w). $$

Now, we have that, if

$$ \lambda = \sqrt{\frac{\delta(z)^2 - |a|^2 + \delta(w)^2 - |a|^2}{2}}, $$

then $B_\lambda(\tau) \subset B_{\delta(z)}(\zeta) \cup B_{\delta(w)}(w)$, and, since $f$ is a bounded function, then

$$ |K[f](z, w)| \leq \int_{B_\lambda(\tau)} \left| \frac{f(\zeta)}{(\zeta - \bar{\tau} + a)(\zeta - \bar{\tau} - a)} \right| \, d\zeta \leq \|f\|_\infty \int_0^{2\pi} \int_0^\infty \frac{r^2}{|e^{-i\theta} + \bar{a}|} \left| e^{i\theta} - \bar{a} \right|^2 \, dr \, d\theta = 2\pi \|f\|_\infty \int_0^{2\pi} \frac{r^2}{|a|^4 + r^2} \, dr = \frac{2\pi \|f\|_\infty}{\|f\|_\infty} \int_0^{2\pi} \frac{r^2}{|a|^4 + r^2} \, dr \leq \frac{2\pi \sqrt{2} \|f\|_\infty}{\lambda}, $$

by the lemma below, and the choice of $v_0$.

**Lemma 10.** If $v_0 \leq \frac{R_0}{\sqrt{2}}$, then

$$ \int_0^{2\pi} \frac{1}{|e^{-i\theta} + \bar{a}|^2} \left| e^{i\theta} - \bar{a} \right|^2 \, d\theta = \frac{2\pi}{|a|^4 + r^2}. $$

**Proof.** Since

$$ |e^{-i\theta} + \bar{a}|^2 \left| e^{i\theta} - \bar{a} \right|^2 = |r^2 e^{-2i\theta} - \bar{a}\bar{\bar{a}}|^2 = |r^2 e^{-2i\theta} - \bar{a}\bar{\bar{a}}|^2 = (r^2 e^{-2i\theta} - \bar{a}\bar{\bar{a}})(r^2 e^{2i\theta} - \bar{a}\bar{\bar{a}}) = r^4 + |a|^4 - (a^2 r^2 e^{-2i\theta} + \bar{a}\bar{\bar{a}} r^2 e^{2i\theta}) = -a^2 r^2 \zeta^4 + (r^4 + |a|^4) \zeta^2 - a^2 r^2, $$

for $\zeta = e^{i\theta}$.

This is a polynomial in $\zeta$, and the roots are $\pm \frac{r}{a}$ and $\pm i \frac{a}{r}$, so if $r \neq |a|$, then

$$ \int_0^{2\pi} \frac{1}{|r^2 e^{-2i\theta} - \bar{a}\bar{\bar{a}}|^2} \, d\theta = \frac{1}{i} \int_{\Xi} \frac{\zeta}{-a^2 r^2 \zeta^4 + (r^4 + |a|^4) \zeta^2 - a^2 r^2} \, d\zeta = \frac{-1}{a^2 r^2 i} \int_{\Xi} \frac{\zeta}{(\zeta - \frac{r}{a})(\zeta + \frac{a}{r})(\zeta - i \frac{a}{r})(\zeta + i \frac{a}{r})} \, d\zeta = (\ast). $$

---

*See note 3.*

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If $v_0 \leq \frac{R_0}{\sqrt{2}}$, then $\lambda \geq \sqrt{\frac{R_0^2}{4} - |a|^2} \geq |a|$. Then, by the residue theorem, we have

$$\lambda \geq \frac{-2\pi}{\bar{a}^2 r^2 |a|^4 + r^4} = \frac{2\pi}{|a|^4 + r^4}.$$ 

\(\square\)

For the remaining terms, both in the decomposition of $Q$ and $L$, the absolute and mutual positions of $z$ and $w$, specially related to $\partial \Omega$, will play an important role.

For this purpose, we will consider the following (four) situations:

1) If $B_{\bar{\delta}(w)}(w) \cap B_{\bar{\delta}(z)}(z) = \emptyset$, equivalent to the fact that $\bar{\delta}(w) + \delta(z) \leq |z - w|$ can never happen because then $\frac{R_0}{2} + \frac{R_0}{2} \leq \frac{R_0}{\sqrt{2}}$.

2) The cases of $B_{\bar{\delta}(w)}(w) \subset B_{\bar{\delta}(z)}(z)$, or $B_{\delta(z)}(z) \subset B_{\delta(w)}(w)$, coresponding respectively to the facts that $\delta(w) + |z - w| \leq \delta(z)$ or $\delta(z) + |z - w| \leq \delta(w)$, so is $\frac{|z - w|}{\delta(z) + \delta(w)} \leq 1 - \frac{2\delta(w)}{\delta(z) + \delta(w)}$, or

$$\frac{|z - w|}{\delta(z) + \delta(w)} \leq 1 - \frac{2\delta(z)}{\delta(z) + \delta(w)},$$

respectively.

The situations are completely symmetric and in each case only one term in (2) survives.

3) The case of the conditions

$$\begin{cases} B_{\bar{\delta}(w)}(w) \cap B_{\bar{\delta}(z)}(z) \neq \emptyset, \\ B_{\delta(w)}(w) \cap B_{\delta(z)}(z) \neq \emptyset, \end{cases}$$

or reciprocally, are both satisfied.

In this case, we have

$$\begin{cases} \delta(w) + \delta(z) \geq |z - w|, \\ \delta(z) \leq |z - w| + \delta(w), \end{cases}$$

or

$$1 - \frac{2\delta(w)}{\delta(z) + \delta(w)} \leq \frac{|z - w|}{\delta(z) + \delta(w)} \leq 1,$$

and, simultaneously,

$$1 - \frac{2\delta(z)}{\delta(z) + \delta(w)} \leq \frac{|z - w|}{\delta(z) + \delta(w)} \leq 1.$$

2. For the term (2), let us consider first the case 2). WLOG we assume that we are in the first situation of this case. Then

$$\int_{B_{\delta(w)}(w) \setminus B_{\delta(z)}(z)} f b_z.$$
If \( z \notin U_{R_0 \gamma} \), we have \( \delta(z) = d(z) \), and since \( \delta(w) \geq |z - w| + \delta(z) \), then also \( \delta(w) = d(w) \), and

\[
(2) = \int_{B_d(w) \setminus B_{d(z)}(z)} f b_z
\]

\[
= \int_{B_d(w) \setminus B_{d(z)}(z)} \frac{f(\zeta) - f(z)}{|\zeta - z|^\gamma} \frac{|\zeta - z|^\gamma}{(\zeta - z)^2} \, dm(\zeta),
\]

by the cancellation Lemma 5 and since \( B_{\delta(w)}(w) \subset \Omega \) or \( B_{\delta(w)}(w) \subset \Omega^c \), we have clearly that

\[
|(2)| \leq \|f\|_{\gamma} 2\pi \int_{d(z)}^{d(w)} \frac{dr}{r^{1-\gamma}} = \|f\|_{\gamma} \frac{2\pi}{\gamma} \{d(w) - d(z)\gamma\} \leq \|f\|_{\gamma} \frac{2\pi}{R_0^1-\gamma} |w - z|.
\]

If \( z \in U_{R_0 \gamma} \), we have that \( \delta(z) = \frac{R_0}{2} \), and

\[
\delta(w) \geq |z - w| + \delta(z) = |z - w| + \frac{R_0}{2} \geq \frac{R_0}{2},
\]

then

\[
(2) = \int_{B_d(w) \setminus B_{\frac{R_0}{2}}(z)} \frac{f(\zeta) - f(z)}{|\zeta - z|^\gamma} \frac{|\zeta - z|^\gamma}{(\zeta - z)^2} \, dm(\zeta),
\]

so

\[
|(2)| \leq \|f\|_{\gamma} 2\pi \int_{\frac{R_0}{2}}^{d(w) + |z - w|} \frac{dr}{r^{1-\gamma}}
\]

\[
= \|f\|_{\gamma} \frac{2\pi}{\gamma} \left\{ (d(w) + |z - w|)^{\gamma} - \left(\frac{R_0}{2}\right)^\gamma \right\} \leq \|f\|_{\gamma} \frac{2\pi}{R_0^1-\gamma} \left( (d(w) - \frac{R_0}{2} + |z - w|) \right)
\]

\[
\leq \|f\|_{\gamma} \frac{2\pi}{R_0^1-\gamma} (d(w) - d(z) + |z - w|),
\]

having the same estimate as above.

In the situation of the case 3),

\[
(2) = \int_{B_{\delta(w)}(w) \setminus B_{\delta(z)}(z)} f b_z - \int_{B_{\delta(z)}(z) \setminus B_{\delta(w)}(w)} f b_w.
\]

If \( z, w \notin U_{\frac{R_0}{2}} \), since \( |z - w| \leq \frac{R_0}{4} \), we have that \( z, w \in \Omega \) or \( z, w \in \Omega^c \) are the only possibilities, and also that

\[
(2) = \int_{B_d(w) \setminus B_{d(z)}(z)} f b_z - \int_{B_d(z) \setminus B_d(w)} f b_w.
\]
Since $B_{d(w)}(w) \subset B_{d(z)+|z-w|}(z)$, then the first term

$$\left| \int_{B_{d(w)}(w) \setminus B_{d(z)}(z)} f b_z \right|$$

$$\leq \|f\|_{L^\infty} \frac{1}{d(z)^2} m(C_{d(z)}^{d(z)+|z-w|}(z))$$

$$= \pi \|f\|_{L^\infty} \frac{d(z) + |z - w| + d(w)}{d(z)^2} (d(z) + |z - w| - d(w))$$

$$\leq \frac{4 \pi \|f\|_{L^\infty}}{R_0} \frac{|z - w|}{d(z)} \frac{d(z) + |z - w| + d(w)}{d(z)} \leq \frac{16 \pi \|f\|_{L^\infty}}{R_0} |z - w|,$$

because $d(w) \leq d(z) + |z - w|$. The other term is similar.

- If $w \notin U_{R_0}$, but $z \in U_{R_0}$, we have that

$$2) = \int_{B_{d(w)}(w) \setminus B_{R_0}(z)} f b_z - \int_{B_{R_0}(z) \setminus B_{d(w)}(w)} f b_w,$$

and we consider several subcases, considering in each one $f = \phi$ and $f = \psi$ separately.

  - If $z, w \in \Omega$, then (2) is

$$\int_{B_{d(w)}(w) \setminus B_{R_0}(z)} \phi b_z - \int_{B_{R_0}(z) \setminus B_{d(w)}(w)} \phi b_w$$

$$= (21)_{\phi} = (211)_{\phi} - (212)_{\phi},$$

or

$$- \int_{B_{R_0}(z)} \psi b_w = (21)_{\phi}.$$

Now,

$$| (211)_{\phi} | \leq \|\phi\|_{L^\infty} \int_{B_{d(w)}(w) \setminus B_{R_0}(z)} \frac{1}{|z - \zeta|^2} \, dm$$

$$\leq \frac{4 \|\phi\|_{L^\infty}}{R_0^2} m(B_{d(w)}(w) \setminus B_{R_0}(z))$$

$$= \frac{4 \pi \|\phi\|_{L^\infty}}{R_0^2} ((d(w) + |z - w|)^2 - (R_0)^2)$$

$$\leq \frac{16 \pi \|\phi\|_{L^\infty} (\text{diam}(\Omega) + R_0)}{R_0^2} |w - z|.$$
Also
\[
| (212) | \phi \leq \frac{4 \| \phi \|_\infty}{R_0^2} m(B_{R_0}(z) \setminus B_{d(w)}(w))
\]
\[
\leq \frac{4 \| \phi \|_\infty}{R_0^2} m(C_{d(w)}^{R_0 + |z-w|}(w))
\]
\[
\leq \frac{4 \pi \| \phi \|_\infty}{R_0^2} \left( \frac{R_0}{2} + |z-w| - d(w) \right) \left( \frac{R_0}{2} + |z-w| + d(w) \right)
\]
\[
\leq \frac{16 \pi \| \phi \|_\infty}{R_0^2} (\text{diam}(\Omega) + R_0) |w - z|.
\]

The term
\[
| (21) \psi | \leq \| \psi \|_\infty \int_{B_{R_0}(z) \cap \Omega^c} |b_w| \leq \| \psi \|_\infty \frac{4}{R_0^2} m(C_{d(z)}^{R_0}(z))
\]
\[
\leq \| \psi \|_\infty \frac{2 \pi}{R_0^2} \left( \frac{R_0}{2} \right)^2 - d(z)^2
\]
\[
\leq \| \psi \|_\infty \frac{2 \pi}{R_0^2} (\text{diam}(\Omega) + R_0) \left( d(w) - d(z) \right),
\]
because \( d(w) \geq \frac{R_0}{2} \).

- If \( z, w \in \Omega^c \), then (2) is
\[
- \int_{B_{R_0}(z) \cap \Omega^c} \phi b_w = (22)_\phi,
\]
or
\[
\int_{B_{d(w)}(w) \setminus B_{R_0}(z)} \psi b_z - \int_{B_{R_0}(z) \setminus B_{d(w)}(w)} \psi b_w = (22)_\psi.
\]
The term \((22)_\phi\) is analogous to \((21)_\psi\), so
\[
| (22)_\phi | \leq \| \phi \|_\infty \frac{2 \pi}{R_0^2} (\text{diam}(\Omega) + R_0) (d(w) - d(z)).
\]

Also the term
\[
(22)_\psi = (221)_\psi - (222)_\psi
\]
is analogous to \((21)_\phi\) and
\[
| (221)_\psi | \leq \frac{4 \pi \| \phi \|_\infty}{R_0^2} \left( d(w)^2 - \left( \frac{R_0}{2} \right)^2 \right)
\]
\[
\leq \frac{4 \pi \| \phi \|_\infty}{R_0^2} \left( d(w) - \frac{R_0}{2} \right) \left( d(w) + \frac{R_0}{2} \right)
\]
\[
\leq \frac{8 \pi (\text{diam}(\Omega) + R_0) \| \phi \|_\infty}{R_0^2} (d(w) - d(z)).
\]
The term \((222)_\psi\) is similar to the previous one.
• If \( z, w \in U_{\frac{\delta}{4}} \), we have that

\[
(2) = \int_{B_{\frac{\delta}{4}}(w) \setminus B_{\frac{\delta}{4}}(z)} f b_z - \int_{B_{\frac{\delta}{4}}(z) \setminus B_{\frac{\delta}{4}}(w)} f b_w,
\]

and then

\[
|(2)| \leq 2 \frac{\|f\|_{L^\infty}}{R_0} m(B_{\frac{\delta}{4}}(w) \setminus B_{\frac{\delta}{4}}(z))
\]

and the previous procedure applies.

3. Now, the term

\[
(3) = \int_{B_{\frac{\delta}{4}}(w) \cap B_{\delta(z)}(z)} \left[ \frac{f(\zeta) - f(z)}{|\zeta - z|^\gamma} - \frac{f(\zeta) - f(w)}{|\zeta - w|^\gamma} \right] \frac{|\zeta - z|^\gamma}{(\zeta - z)^2} \left( \frac{f(\zeta) - f(w)}{|\zeta - w|^\gamma} - \frac{f(\zeta) - f(z)}{|\zeta - z|^\gamma} \right) dm(\zeta).
\]

If \(|z - w| < \frac{\delta}{4}\), then \( z, w \in B_{\frac{\delta}{4}}(w) \cap B_{\delta(z)}(z) \subset B_{\delta(z)}(w) \cap B_{\delta(z)}(z) \), and we can reduce the study to this situation.

In such case, \( d(z, \partial B_{\delta(z)}(w)) = \delta(w) - |z - w| \) and \( d(w, \partial B_{\delta(z)}(z)) = \delta(z) - |z - w| \), so the maximum radius of a ball centered at \( z \) and contained in \( B_{\delta(z)}(z) \cap B_{\delta(z)}(w) \) is equal to \( \min\{\delta(z), \delta(w) - |z - w|\} \). For \( w \) we have an analogous expression. And since \( \delta(z) \geq \frac{\delta}{4} \geq 2|z - w| \), and analogously for \( w \), we can consider, defining

\[
k^2(\zeta) = |\zeta - z|^\gamma,
\]

\[
\Delta^\gamma(\zeta) = \frac{f(\zeta) - f(z)}{k^2(\zeta)},
\]

the decomposition

\[
(3) = \int_{B_{\delta(z)}(w) \cap B_{\delta(z)}(z)} \left[ \Delta^\gamma_k k^2 b_z - \Delta^\gamma_{k_0} k^2 b_w \right]
\]

\[
= \int_{B_{\frac{|z - w|}{2}}(z)} \Delta^\gamma_k k^2 b_z - \int_{B_{\frac{|z - w|}{2}}(w)} \Delta^\gamma_{k_0} k^2 b_w
\]

\[
+ \int_{(B_{\frac{|z - w|}{2}}(w) \cap B_{\delta(z)}(z)) \setminus B_{\frac{|z - w|}{2}}(z)} \Delta^\gamma_k k^2 b_z
\]

\[
- \int_{(B_{\delta(z)}(w) \cap B_{\delta(z)}(z)) \setminus B_{\frac{|z - w|}{2}}(w)} \Delta^\gamma_{k_0} k^2 b_w
\]

\[
= (31) + (32).
\]

Now,

\[
|(31)| \leq 2 \|f\|_\gamma 2\pi \int_0^{\frac{|z - w|}{2}} \frac{d\gamma}{\gamma^{1-\gamma}} = \frac{4\pi \|f\|_\gamma}{\gamma^{2\gamma}} |z - w|^\gamma.
\]
Since for $\tau = \frac{z + w}{2}$ we have $B_{|z - w|}(z) \cup B_{|z - w|}(w) \subset B_{|z - w|}(\tau)$, we have that

$$\int_{B_{|z - w|}(\tau) \setminus B_{|z - w|}(z)} \Delta^\gamma_k b_z - \int_{B_{|z - w|}(\tau) \setminus B_{|z - w|}(w)} \Delta^\gamma_w b_w$$

$$+ \int_{(B_{|z - w|}(w) \cap B_{|z - w|}(z)) \setminus B_{|z - w|}(\tau)} [\Delta^\gamma_k b_z - \Delta^\gamma_w b_w]$$

$$(32) = (321) + (322),$$

and the term

$$|(321)| \leq 2 \|f\|_\gamma \frac{4\pi}{\gamma} \left( 3 \frac{3}{2} - \left( \frac{1}{2} \right)^\gamma \right) |z - w|^{\gamma}.$$  

For the term (322), we have that

$$\Delta^\gamma_k b_z - \Delta^\gamma_w b_w = (f(\zeta) - f(z)) b_z(\zeta) - (f(\zeta) - f(w)) b_w(\zeta) = (\ast).$$

- If $z \notin \Omega_0$, then $|z - w| < \frac{R_0}{4}$, we have that $w, \tau = \frac{z + w}{2} \in B_{R_0}(z)$, so all are in $\Omega$ or all are in $\Omega^C$, and then, using the decomposition

$$(\ast) = (f(\zeta) - f(\tau)) (b_z - b_w)(\zeta) + (f(w) - f(\tau)) b_w(\zeta) - (f(z) - f(\tau)) b_z(\zeta),$$

we have

$$\int_{(B_{|z - w|}(w) \cap B_{|z - w|}(z)) \setminus B_{|z - w|}(\tau)} \Delta^\gamma_k (b_z - b_w)$$

$$+ (f(w) - f(\tau)) \int_{(B_{|z - w|}(w) \cap B_{|z - w|}(z)) \setminus B_{|z - w|}(\tau)} b_w$$

$$- (f(z) - f(\tau)) \int_{(B_{|z - w|}(w) \cap B_{|z - w|}(z)) \setminus B_{|z - w|}(\tau)} b_z$$

$$= (3221) + (3222).$$

For the integral (3321) we use the change of variables

$$\zeta \rightarrow s = \frac{\zeta - \tau}{a} = \phi(\zeta),$$

where $a = \frac{z + w}{2}$, then $\phi(\tau) = 0$ and $J\phi = |a|^{-2}$.

We have

$$k^\gamma(\zeta)(b_z - b_w)(\zeta) = |\zeta - \tau|^\gamma \left\{ \frac{1}{(\zeta - z)^2} - \frac{1}{(\zeta - w)^2} \right\}$$

$$= |a|^\gamma \left\{ \frac{1}{(\bar{a} (\bar{s} + 1))^2} - \frac{1}{(\bar{a} (\bar{s} - 1))^2} \right\}$$

$$= \frac{|a|^\gamma}{\bar{a}^2} \frac{-|s|^2 4s}{(\bar{s} + 1)^2 (\bar{s} - 1)^2}.$$
And then, since $\phi((B_{\delta(w)}(w)) = B_{\delta(w)}(-1)$, $\phi((B_{\delta(z)}(z)) = B_{\delta(z)}(1)$ and $\phi((B_{|z-w|}(\tau)) = B_2(0)$, we have that

$$\int \frac{\gamma}{(s+1)^2} \frac{|s|^{1+\gamma}}{(s+1)^2 (s-1)^2} dm(s),$$

so

$$|\int \frac{\gamma}{(s+1)^2} \frac{|s|^{1+\gamma}}{(s+1)^2 (s-1)^2} dm(s)| \leq 16\pi.$$

**Proof.** Using Stokes formula,

$$\int_{(B_{\delta(w)}(w) \cap B_{\delta(z)}(z)) \backslash B_{|z-w|}(\tau)} b_w$$

$$= \frac{1}{2i} \left\{ \int_{\partial B_{\delta(w)}(w)} + \int_{B_{\delta(w)}(w) \cap \partial(B_{\delta(z)}(z))} - \int_{\partial B_{|z-w|}(\tau)} \right\} \frac{d\zeta}{\zeta - \bar z}$$

$$= (I) + (II) - (III).$$

Then using the change $\zeta = \tau + 2 |a| e^{i\theta}$, we have

$$(III) = \int_{\partial B_{\frac{|a|}{\delta(z)}}(\tau)} \frac{d\zeta}{\zeta - \bar a} = \int_0^{2\pi} \frac{2i |a| e^{i\theta} d\theta}{2 |a| e^{i\theta} - \bar a}$$

$$= 2i |a| \int_0^{2\pi} \frac{e^{2i\theta} d\theta}{2 |a| - e^{i\theta}} = 2 |a| \int \frac{s ds}{2 |a| - s} = 0,$$

by the Cauchy formula.

The integral

$$|(II)| \leq \frac{1}{\delta(z)} \int_{\partial B_{\delta(z)}(z)} |d\zeta| = 2\pi.$$

In the same way, in our situation,

$$|(I)| \leq \frac{1}{\delta(z) - |z-w|} \int_{\partial B_{\delta(w)}(w)} |d\zeta| = 2\pi \frac{\delta(w)}{\delta(w) - |z-w|}.$$

If $d(w) \leq \frac{R_0}{2}$, then

$$|(I)| \leq 2\pi \frac{\frac{R_0}{2} - \frac{R_0}{4}}{\frac{R_0}{2} - \frac{R_0}{4}} = 4\pi.$$
If \( d(w) > \frac{R_0}{2} \), then

\[
|I| \leq 2\pi \frac{d(w)}{d(w) - \frac{R_0}{2}} \leq 2\pi \frac{d(w)}{\frac{R_0}{2}} = 4\pi.
\]

This implies that

\[
|I| \leq \frac{32\pi}{2^\gamma} |z - w|^{n+1} \|f\|_{\gamma}.
\]

- The situation is completely equivalent to the case of \( w \notin U_{\frac{R_0}{4}} \).

- If \( z, w \in U_{\frac{R_0}{4}} \), then \( \delta(z) = \delta(w) = \frac{R_0}{2} \), and

\[
(322) = \int_{(B_{R_0}(w) \cap B_{\frac{R_0}{2}}(z)) \setminus B_{\frac{R_0}{4}}(w)} \left[ (f(\zeta) - f(z))b_z(\zeta) - (f(\zeta) - f(w))b_w(\zeta) \right] d\zeta
\]

\[
= \int_{(B_{R_0}(w) \cap B_{\frac{R_0}{2}}(z)) \setminus B_{\frac{R_0}{4}}(w)} (f(\zeta) - f(\tau')) [b_z(\zeta) - b_w(\zeta)]
\]

\[
+ (f(\tau') - f(z)) \int_{(B_{R_0}(w) \cap B_{\frac{R_0}{2}}(z)) \setminus B_{\frac{R_0}{4}}(w)} b_z(\zeta)
\]

\[
- (f(\tau') - f(w)) \int_{(B_{R_0}(w) \cap B_{\frac{R_0}{2}}(z)) \setminus B_{\frac{R_0}{4}}(w)} b_z(\zeta)
\]

\[
= (3223) + (3224),
\]

where \( \tau' \) is a point in \( \partial\Omega \) at minimum distance of \( \tau \).

Now, using the Lemma 9, we have

\[
(3223) = -\left( \bar{z} - \bar{w} \right) \int_{(B_{R_0}(w) \cap B_{\frac{R_0}{2}}(z)) \setminus B_{\frac{R_0}{4}}(w)} \frac{f(\zeta) - f(\tau')}{|\zeta - \tau'|^\gamma}
\]

\[
\times k_2(\zeta) (\bar{z} + \bar{\zeta} - 2 \bar{\zeta}) b_z(\zeta) b_w(\zeta).
\]

If \( \tau' \in W \) satisfies that \( d(\tau, W) = |\tau - \tau'| \), then, since

\[
|\tau' - \tau| \leq |\tau - z| = \frac{|z - w|}{2},
\]

then

\[
|z - \tau'| \leq |z - \tau| + |\tau - \tau'| \leq 2|a|
\]

and

\[
|\zeta - \tau'| \leq |\zeta - \tau| + |\tau - \tau'| \leq |\zeta - \tau| + |a|,
\]

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so

$$|\zeta - \tau'| \leq (|\zeta - \tau| + |a|)^\gamma \leq \frac{\gamma |\zeta - \tau|}{|a|^{1-\gamma}},$$

and then

$$|(3223)| \leq \left| \frac{|z - w|}{|a|^{1-\gamma}} \|f\| \gamma \right| \times \int_{(B_{\frac{\delta(w)}{4}}(w) \cap B_{\delta(w)}(z)) \setminus B_{\frac{\delta(w)}{2}}(\tau)} \frac{|\zeta - \tau| \gamma (|\zeta - \tau| + |\zeta - w|)}{|\zeta - z|^2 |\zeta - w|^2} \, dm(\zeta)$$

and the estimate is identical to the case of (322).

Also, a repetition of the arguments shows that the term (3224) has the same estimate as the term (3222).

4. For the term

$$(4) = \int_{B_{\delta(w)}(z) \setminus B_{\delta(w)}(w)} f(\zeta) - f(z) \frac{|\zeta - z|^{\gamma}}{|\zeta - z|^\gamma (\zeta - z)^2} - \int_{B_{\delta(w)}(w) \setminus B_{\delta(w)}(z)} f(\zeta) - f(w) \frac{|\zeta - w|^{\gamma}}{|\zeta - w|^\gamma (\zeta - w)^2} \, dm(\zeta) = (41) + (42).$$

The term

$$|(41)| \leq \|f\| \gamma \int_{B_{\delta(w)}(z) \setminus B_{\delta(w)}(w)} \frac{dm(\zeta)}{|\zeta - z|^{1-\gamma}},$$

and since $|z - w| \leq \frac{\delta(w)}{4} < \delta(w), \delta(z)$, then we have that $z \in B_{\delta(w)}(w)$. Moreover, if $|\zeta - z| < \delta(w) - |z - w|$, then $\delta(w) > |\zeta - z| + |z - w| \geq |\zeta - w|$. This implies that

$$B_{\delta(w) - |z - w|}(z) \subset B_{\delta(w)}(w)$$

and then, since $\delta(w) - |z - w| < \delta(z)$, otherwise the domain of integration is empty, the previous integral is bounded by

$$\int_{C_{\delta(w) - |z - w|}(z)} \frac{dm(\zeta)}{|\zeta - z|^{1-\gamma}}$$

$$= 2\pi \int_{\delta(w) - |z - w|} dr \frac{dr}{r^{1-\gamma}} = 2\pi \frac{\delta(z)^\gamma - (\delta(w) - |z - w|)^\gamma}{\gamma}$$

$$= 2\pi \frac{\delta(z) - (\delta(w) - |z - w|)}{\delta(w) - |z - w| + (\delta(w) - |z - w|)^{1-\gamma}}$$

$$\leq 2\pi \frac{\delta(z) - \delta(w) + |z - w|}{(\delta(w) - |z - w|)^{1-\gamma}} \leq 2\pi \frac{\delta(z) - \delta(w) + |z - w|}{(\frac{\delta(w)}{4})^{1-\gamma}}.$$

The term (42) is symmetric and analogous to (41).
With the previous arguments we have the conjugate Beurling transforms of \( \phi \) and \( \psi \) are Hölder at \( \Omega \) or the interior of \( \Omega^c \). The following lemma completes the case of \( \partial \Omega \).

**Lemma 12.** If \( V = \Omega \) or \( V = (\bar{\Omega})^c \) and \( f \in \text{Lip} (\gamma, V) \) and \( \|f\|_{\gamma} < +\infty \), then \( f \) extends to a Lipschitz function on \( \bar{V} \), with the same Lipschitz norm.

**Proof.** Since \( \partial W \) is compact, then \( f \) is uniformly continuous in \( \bar{U}_{R_0} \).

If \( z, w \in \partial \Omega \) and \( |z - w| < \frac{R_0}{4} \), for \( U_{\frac{R_0}{4}} \cap W \) we have

\[
f(z) - f(w) = f(z) - f(z') + f(z') - f(w') + f(w') - f(w) = (1) + (2) + (3).
\]

If \( |z' - z|, |w' - w| < \min\{\delta, \frac{|z - w|}{3}\} \), then

\[
(1), (3) \leq M |z - w|^{\gamma}.
\]

Also

\[
|z' - w'| \leq \frac{5}{3} |z - w|,
\]

so

\[
(2) \leq M |z - w|^{\gamma}.
\]

\[\square\]

Finally, we can apply this theorem to the last term and we have

**Corollary 1.** There exists a constant \( C_1 \), depending only on \( R_0, \nu_0, \text{diam} (\Omega), \gamma \), such that

\[
|f(z) \Theta_{\Omega}^{\frac{R_0}{4}} (z) - f(w) \Theta_{\Omega}^{\frac{R_0}{4}} (w)| \leq |z - w|^{\gamma} \|f\|_{\gamma} C_1.
\]

**Acknowledgements**: The authors are grateful to Joan Verdera for some useful conversations and comments on the results of the paper. Both authors are partially supported by grants 2017-SGR-0395 (Generalitat de Catalunya) and MDM-2014-044 (MICINN, Spain). The second named author is also partially supported by MTM-2016-75390 (MINECO, Spain)

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