E$_8$ BUNDLES AND RIGIDITY

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Abstract. In this paper, we establish rigidity and vanishing theorems for Dirac operators twisted by E$_8$ bundles.

INTRODUCTION

Let $X$ be a closed smooth connected manifold which admits a nontrivial $S^1$ action. Let $P$ be an elliptic differential operator on $X$ commuting with the $S^1$ action. Then the kernel and cokernel of $P$ are finite dimensional representations of $S^1$. The equivariant index of $P$ is the virtual character of $S^1$ defined by

$$
\text{Ind}(g, P) = \text{tr}|_g \ker P - \text{tr}|_g \text{coker} P,
$$

for $g \in S^1$. We call that $P$ is rigid with respect to this circle action if $\text{Ind}(g, P)$ is independent of $g$.

It is well known that classical operators: the signature operator for oriented manifolds, the Dolbeault operator for almost complex manifolds and the Dirac operator for spin manifolds are rigid [2]. In [30], Witten considered the indices of Dirac-like operators on the free loop space $LX$. The Landweber-Stong-Ochanine elliptic genus ([20], [28]) is just the index of one of these operators. Witten conjectured that these elliptic operators should be rigid. See [19] for a brief early history of the subject. Witten’s conjecture were first proved by Taubes [29] and Bott-Taubes [4]. Hirzebruch [13] and Krichever [15] proved Witten’s conjecture for almost complex manifold case. Various aspects of mathematics are involved in these proofs. Taubes used analysis of Fredholm operators, Krichever used cobordism, Bott-Taubes and Hirzebruch used Lefschetz fixed point formula. In [22, 23], using modularity, Liu gives simple and unified proof as well as various generalizations of the Witten conjecture. Several new vanishing theorems are also found in [22, 23]. Liu-Ma [24, 25] and Liu-Ma-Zhang [26, 27] established family versions of rigidity and vanishing theorems.

In this paper, we study rigidity and vanishing properties for Dirac operators twisted by E$_8$ bundles. Let $X$ be an even dimensional closed spin manifold and $D$ the Dirac operator on $X$. Let $P$ be an (compact-)E$_8$ principal bundle over $X$. Let $W$ be the vector bundle over $X$ associated to the complex adjoint representation $\rho$ of E$_8$. The twisted Dirac operator $D^W$ plays a prominent role in string theory and M theory. In [31], the index of such twisted operator is discovered as part of the phase of the M-theory
action. In [8], the partition function in M-theory, involving the index theory of an $E_8$ bundle, is compared with the partition function in type IIA string theory described by K-theory to test M-theory/Type IIA duality. In this paper, we are interested in the equivariant index of the operator $D^W$ and establish rigidity and vanishing theorems for this operator.

More precisely, let $X$ be a $2k$-dimensional closed spin manifold, which admits a nontrivial $S^1$ action. Let $P$ be an (compact-) $E_8$ principal bundle over $X$ such that the $S^1$ action on $X$ can be lifted to $P$ as a left action which commutes with the free action of $E_8$ on $P$. Let $W$ be the complex vector bundle associated to the complex adjoint representation of $E_8$ mentioned above. Then the $S^1$ action on $P$ naturally induces an action on $W$ by $g \cdot [s, v] = [g \cdot s, v]$, where $[s, v]$ with $s \in P, v \in \mathbb{C}^{248}$, is the equivalent classes defining the elements in $W$ by the equivalent relations $(s, v) \sim (s \cdot h, \rho(h^{-1}) \cdot v)$ for $h \in E_8$. Let $X^{S^1}$ be the fixed point manifold and $\pi$ be the projection from $X^{S^1}$ to a point $pt$. Let $u$ be a fixed generator of $H^2(BS^1, \mathbb{Z})$. We have the following theorem:

**Theorem 0.1.** Assume the action only has isolated fixed points and the restriction of the equivariant characteristic class $\frac{1}{36}c_2(W)_{S^1} - p_1(TX)_{S^1}$ to $X^{S^1}$ is equal to $n \cdot \pi^* u^2$ for some integer $n$.

(i) If $n < 0$, then $\text{Ind}(g, D^W)$ is independent of $g$ and equal to $-\text{Ind}(D^{T_{C^X}})$, minus the index of the Rarita-Schwinger operator. In particular, one has $\text{Ind}D^W = -\text{Ind}D^{T_{C^X}}$ and when $k$ is odd, i.e. $\dim X \equiv 2 \pmod{4}$, one has $\text{Ind}(g, D^W) \equiv 0$.

(ii) If $n = 0$, then $\text{Ind}(g, D^W)$ is independent of $g$. Moreover, when $k$ is odd, one has $\text{Ind}(g, D^W) \equiv 0$.

(iii) If $n = 2$ and $k$ is odd, then $\text{Ind}(g, D^W) \equiv 0$.

Actually we have established rigidity and vanishing results in more general settings concerning the twisted spin$^c$ Dirac operators. See Theorem 2.1 and Theorem 2.2 for details. The above theorem is a corollary of Theorem 2.1. We prove our theorems by studying the modularity of Lefschetz numbers of certain elliptic operators involving the basic representation of the affine Kac-Moody algebra of $E_8$. In the rest of the paper, we will first briefly review the Jacobi theta functions and the basic representation for the affine $E_8$ by following [16] (see also [17]) as the preliminary knowledge in Section 1 and then state our theorems as well as give their proofs in Section 2.

### 1. Preliminaries

#### 1.1. Jacobi theta functions.

The four Jacobi theta-functions are defined as follows (cf. [5]),

\[
\theta(z, \tau) = 2q^{1/8} \sin(\pi z) \prod_{j=1}^{\infty} [(1 - q^j)(1 - e^{2\pi^2 \sqrt{-1} j} q^j)(1 - e^{-2\pi^2 \sqrt{-1} j} q^j)],
\]
\( (1.2) \quad \theta_1(z, \tau) = 2q^{1/8} \cos(\pi z) \prod_{j=1}^{\infty} [(1 - q^j)(1 + e^{2\pi \sqrt{-1}z} q^j)(1 + e^{-2\pi \sqrt{-1}z} q^j)], \)

\( (1.3) \quad \theta_2(z, \tau) = \prod_{j=1}^{\infty} [(1 - q^j)(1 - e^{2\pi \sqrt{-1}z} q^{j-1/2})(1 - e^{-2\pi \sqrt{-1}z} q^{j-1/2})], \)

\( (1.4) \quad \theta_3(z, \tau) = \prod_{j=1}^{\infty} [(1 - q^j)(1 + e^{2\pi \sqrt{-1}z} q^{j-1/2})(1 + e^{-2\pi \sqrt{-1}z} q^{j-1/2})], \)

where \( q = e^{2\pi \sqrt{-1} \tau}, \ \tau \in \mathbb{H}, \) the upper half plane.

They are all holomorphic functions for \((z, \tau) \in \mathbb{C} \times \mathbb{H},\) where \( \mathbb{C} \) is the complex plane.

Let \( \theta'(0, \tau) = \frac{\partial }{\partial z} \theta(z, \tau)|_{z=0}. \) One has the following Jacobi identity (c.f. [5]),

\( (1.5) \quad \theta'(0, \tau) = \pi \theta_1(0, \tau) \theta_2(0, \tau) \theta_3(0, \tau). \)

Let

\[ SL(2, \mathbb{Z}) := \left\{ \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \bigg| a_1, a_2, a_3, a_4 \in \mathbb{Z}, \ a_1a_4 - a_2a_3 = 1 \right\} \]

be the modular group. Let \( S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) be the two generators of \( SL(2, \mathbb{Z}). \) Their actions on \( \mathbb{H} \) are given by

\[ S : \tau \mapsto -\frac{1}{\tau}, \ \ T : \tau \mapsto \tau + 1. \]

The actions on theta-functions by \( S \) and \( T \) are given by the following transformation formulas (c.f. [5]),

\( (1.6) \quad \theta(z+1, \tau) = e^{\frac{\pi i \tau}{4}} \theta(z, \tau), \ \ \theta(z, -1/\tau) = \frac{1}{\sqrt{-1}} \left( \frac{\tau}{\sqrt{-1}} \right)^{1/2} e^{\pi \sqrt{-1} \tau z^2} \theta(\tau z, \tau); \)

\( (1.7) \quad \theta_1(z+1, \tau) = e^{\frac{\pi i \tau}{4}} \theta_1(z, \tau), \ \ \theta_1(z, -1/\tau) = \left( \frac{\tau}{\sqrt{-1}} \right)^{1/2} e^{\pi \sqrt{-1} \tau z^2} \theta_2(\tau z, \tau); \)

\( (1.8) \quad \theta_2(z+1) = \theta_3(z, \tau), \ \ \theta_2(z, -1/\tau) = \left( \frac{\tau}{\sqrt{-1}} \right)^{1/2} e^{\pi \sqrt{-1} \tau z^2} \theta_1(\tau z, \tau); \)

\( (1.9) \quad \theta_3(z+1) = \theta_2(z, \tau), \ \ \theta_3(z, -1/\tau) = \left( \frac{\tau}{\sqrt{-1}} \right)^{1/2} e^{\pi \sqrt{-1} \tau z^2} \theta_3(\tau z, \tau). \)

One also has the following formulas about how the theta functions vary along the lattice \( \Gamma = \{ a + b\tau | a,b \in \mathbb{Z} \} \) (c.f. [5]),

\( (1.10) \quad \theta(z + a, \tau) = (-1)^a \theta(z, \tau), \ \ \theta(z + b\tau, \tau) = (-1)^b e^{2\pi \sqrt{-1}\tau z - \pi \sqrt{-1}b^2 \tau} \theta(z, \tau); \)
The basic representation for the affine $E_8$. In this subsection we briefly review the basic representation for the affine $E_8$ following [16] (see also [17]).

Let $\mathfrak{g}$ be the (complex) Lie algebra of $E_8$. Let $\langle , \rangle$ be the Killing form on $\mathfrak{g}$. Let $\tilde{\mathfrak{g}}$ be the affine Lie algebra corresponding to $\mathfrak{g}$ defined by

$$\tilde{\mathfrak{g}} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g} \oplus \mathbb{C}c,$$

with bracket

$$[P(t) \otimes x + \lambda c, Q(t) \otimes y + \mu c] = P(t)Q(t) \otimes [x, y] + \langle x, y \rangle \text{ Res}_{t=0} \left( \frac{dP(t)}{dt}Q(t) \right) c.$$

Let $\mathfrak{g}$ be the affine Kac-Moody algebra obtained from $\tilde{\mathfrak{g}}$ by adding a derivation $t \frac{d}{dt}$ which operates on $\mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}$ in an obvious way and sends $c$ to 0.

The basic representation $V(\Lambda_0)$ is the $\mathfrak{g}$-module defined by the property that there is a nonzero vector $v_0$ (highest weight vector) in $V(\Lambda_0)$ such that $cv_0 = v_0$, $(\mathbb{C}[t] \otimes \mathfrak{g} \oplus \mathbb{C}t \frac{d}{dt}) v_0 = 0$. Setting $V_i := \{ v \in V(\Lambda_0) | t \frac{d}{dt} v = -iv \}$ gives a $\mathbb{Z}_4$-gradation by finite dimensional subspaces. Since $[\mathfrak{g}, t \frac{d}{dt}] = 0$, each $V_i$ is a representation of $\mathfrak{g}$. Moreover, $V_1$ is the adjoint representation of $E_8$.

Fix a basis $\{Z_i\}_{i=1}^8$ for the Cartan subalgebra. The character of the basic representation is given by

$$\text{ch}(z_1, z_2, \cdots, z_8, \tau) := \sum_{i=0}^{\infty} (\text{ch}V_i)(z_1, z_2, \cdots, z_8)q^i = \varphi(\tau)^{-8} \Theta_{\mathfrak{g}}(z_1, z_2, \cdots, z_8, \tau),$$

where $\varphi(\tau) = \prod_{n=1}^{\infty} (1 - q^n)$ so that $\eta(\tau) = q^{1/24} \varphi(\tau)$ is the Dedekind $\eta$ function; $\Theta_{\mathfrak{g}}(z_1, z_2, \cdots, z_8, \tau)$ is the theta function defined on the root lattice $Q$ by

$$\Theta_{\mathfrak{g}}(z_1, z_2, \cdots, z_8, \tau) = \sum_{\gamma \in Q} q^{||\gamma||^2/2e^{2\pi \sqrt{-1}\tau}(\sum_{i=1}^{8} z_i \gamma_i)}.$$

It is proved in [10] (cf. [11]) that there is a basis for the $E_8$ root lattice such that

$$\Theta_{\mathfrak{g}}(z_1, \cdots, z_8, \tau) = \frac{1}{2} \left( \prod_{l=1}^{8} \theta_8(z_l, \tau) + \prod_{l=1}^{8} \theta_1(z_l, \tau) + \prod_{l=1}^{8} \theta_2(z_l, \tau) + \prod_{l=1}^{8} \theta_3(z_l, \tau) \right).$$
2. $E_8$ Bundles and Rigidity

In this section we prove two rigidity and vanishing theorems for spin$^c$ manifolds with $E_8$ principal bundles. Theorem 0.1 is deduced from the first one (Theorem 2.1).

Let $X$ be a $2k$ dimensional closed spin$^c$ manifold, which admits a non-trivial $S^1$ action that preserves the spin$^c$ structure. Let $L$ be the complex line bundle associated with the spin$^c$ structure of $X$. It’s the associated line bundle of the $U(1)$-bundle $Q/spin(2k) \to Q/spin^c(2k) \cong X$, where $Q$ is the spin$^c(2k)$ principal bundle over $X$ determined by the spin$^c$ structure. We denote the first equivariant Chern class of $L$ by $c_1(X)_{S^1}$. Then $P$ be an $E_8$ principal bundle over $X$ such that the $S^1$ action on $X$ can be lifted to $P$ as a left action which commutes with the free action of $E_8$ on $P$. Let $W$ be the vector bundle associated to the complex adjoint representation of $E_8$ mentioned above. Then the $S^1$ action on $P$ naturally induces an action on $W$ as described in the introduction.

Let $g^{TX}$ be a Riemannian metric on $X$. Let $\nabla^{TX}$ be the Levi-Civita connection associated to $g^{TX}$. Denote the complexification of $TX$ by $T_CX$. Let $g^{TX}$ and $\nabla^{TX}$ be the induced Hermitian metric and Hermitian connection on $T_CX$. Let $h^L$ be a Hermitian metric on $L$ and $\nabla^L$ be a Hermitian connection. Let $\overline{L}$ be the complex conjugate of $L$ with the induced Hermitian metric and connection. Assume that the $S^1$ action on $X$ preserves the metrics and connections involved. Let $S_c(TX) = S_{c,+}(TX) \oplus S_{c,-}(TX)$ denote the bundle of spinors associated to the spin$^c$ structure, $(TX, g^{TX})$ and $(L, h^L)$. Then $S_c(TX)$ carries induced Hermitian metric and connection preserving the above $\mathbb{Z}_2$-grading. Let $D_{c,\pm} : \Gamma(S_{c,\pm}(TX)) \to \Gamma(S_{c,\mp}(TX))$ denote the induced spin$^c$ Dirac operators (cf. [21]). If $V$ is an equivariant complex vector bundle over $X$ with equivariant Hermitian metric $h^V$ and Hermitian connection $\nabla^V$, let $D_{c,\pm}^V : \Gamma(S_{c,\pm}(TX) \otimes V) \to \Gamma(S_{c,\mp}(TX) \otimes V)$ denote the induced twisted spin$^c$ Dirac operators.

Theorem 2.1. Assume the action only has isolated fixed points and the restriction of the equivariant characteristic class

$$\frac{1}{30} c_2(W)_{S^1} + 3 c_1(X)_{S^1}^2 - p_1(TX)_{S^1}$$

to $X_{S^1}$ is equal to $n \cdot \pi^* u^2$ for some integer $n$.

(i) If $n < 0$, then

$$\text{Ind}(g, D_{c,+}^{(1+T)} \otimes W) + \text{Ind}(g, D_{c,+}^{(1+\overline{T})} \otimes (T_CX - (L^2 + \overline{L}^2) + (L + \overline{L}))) \equiv 0.$$

In particular,

$$\text{Ind}D_{c,+}^{(1+T)} \otimes W + \text{Ind}D_{c,+}^{(1+\overline{T})} \otimes (T_CX - (L^2 + \overline{L}^2) + (L + \overline{L})) = 0.$$

(ii) If $n = 0$, then

$$\text{Ind}(g, D_{c,+}^{(1+\overline{T})} \otimes W) + \text{Ind}(g, D_{c,+}^{(1+T)} \otimes (T_CX - (L^2 + \overline{L}^2) + (L + \overline{L}))) = 0.$$
is independent of \(g\). Moreover, when \(k\) is odd, one has
\[
\text{Ind}(g, D_{c,+}^{(1+\overline{T})\otimes W}) + \text{Ind}(g, D_{c,+}^{(1+\overline{T})\otimes (\nabla \text{c}X-(L^2+\overline{T}^2)+(L+\overline{T}))}) \equiv 0.
\]

(iii) If \(n = 2\) and \(k\) is odd, then
\[
\text{Ind}(g, D_{c,+}^{(1+\overline{T})\otimes W}) + \text{Ind}(g, D_{c,+}^{(1+\overline{T})\otimes (\nabla \text{c}X-(L^2+\overline{T}^2)+(L+\overline{T}))}) \equiv 0.
\]

**Proof.** Let \(g = e^{2\pi \sqrt{-1}t} \in S^1\) be the generator of the action group. Let \(X_{S^1} = \{p\}\) be the set of fixed points. Let \(TX|p = E_1 \oplus \cdots \oplus E_k\) be the decomposition of the tangent bundle into the \(S^1\)-invariant 2-planes. Assume that \(g\) acts on \(E_j\) by \(e^{2\pi \sqrt{-1}t_{\alpha_j}}\), \(\alpha_j \in \mathbb{Z}\). Assume \(g\) acts on \(L|p\) by \(e^{2\pi \sqrt{-1}ct}\), \(c \in \mathbb{Z}\). Clearly,
\[
p_1(TM| p)_{S^1} = (2\pi \sqrt{-1})^2 \sum_{j=1}^{k} \alpha_j^2 t^2, \quad c_1(L| p)_{S^1} = 2\pi \sqrt{-1}ct.
\]

Denote \(L \oplus \overline{L}\) by \(L_C\). If \(E\) is a complex vector bundle over \(X\), set \(\tilde{E} = E - \text{c rk}(E) \in K(X)\).

Let \(\Theta(X, L, \tau)\) be the virtual complex vector bundle over \(X\) defined by
\[
\Theta(X, L, \tau) := \left(\bigotimes_{m=1}^{\infty} S_{q^m}(\tilde{\text{c}}X) \right) \otimes \left(\bigotimes_{u=1}^{\infty} \Lambda_{q^u}(\overline{L_C}) \right)
\]
\[
\otimes \left(\bigotimes_{v=1}^{\infty} \Lambda_{-q^{v-1/2}}(\overline{L_C}) \right) \otimes \left(\bigotimes_{w=1}^{\infty} \Lambda_{q^{w-1/2}}(\overline{L_C}) \right),
\]

Let \(W_i (i = 0, 1, \cdots)\) be the associated bundles \(P \times_{\rho_i} V_i\), where \(V_i\)'s are the representations of \(E_8\) as in §1.2. Then \(W = W_1\).

Consider the twisted operator
\[
D_{c,+}^{(1+\overline{T})\otimes \Theta(X, L, \tau) \otimes (\varphi^8(\tau) \sum_{i=0}^{\infty} W_i q^i)}.
\]

Expanding \(q\)-series, we have
\[
\Theta(X, L, \tau) \otimes (\varphi^8(\tau) \sum_{i=0}^{\infty} W_i q^i)
\]
\[
=(1 + (T_{\text{c}}X - 2k)q + O(q^2)) \otimes (1 + \tilde{L}_{\text{c}}q + O(q^2))
\]
\[
\otimes (1 - \tilde{L}_{\text{c}}q^{1/2} - 2\tilde{L}_{\text{c}}q + O(q^{3/2})) \otimes (1 + \tilde{L}_{\text{c}}q^{1/2} - 2\tilde{L}_{\text{c}}q + O(q^{3/2}))
\]
\[
\otimes (1 - 8q + O(q^2)) \otimes (1 + Wq + O(q^2))
\]
\[
=1 + (W - 8 + T_{\text{c}}X - 2k - 3\tilde{L}_{\text{c}} - \tilde{L}_{\text{c}} \otimes \tilde{L}_{\text{c}})q + O(q^2).
\]

It’s not hard to see that \(\tilde{L}_{\text{c}} \otimes \tilde{L}_{\text{c}} = L^2 + \overline{L}^2 - 4(L + \overline{L}) + 6\). So
\[
D_{c,+}^{(1+\overline{T})\otimes \Theta(M, L, \tau) \otimes (\varphi^8(\tau) \sum_{i=0}^{\infty} W_i q^i)}
\]
\[
=D_{c,+}^{(1+\overline{T})} + D_{c,+}^{(1+\overline{T})\otimes (W + T_{\text{c}}X - (L^2+\overline{L}^2)+(L+\overline{L})-8-2k)}q + O(q^2).
\]
By the Atiyah-Bott-Segal-Singer Letchselz fixed point formula, for the twisted operator $D^{(1+L)\otimes\Theta(X,L,\tau)\otimes(e^8(\tau)\sum_{i=0}^{\infty} W_i q^i)_c}$, the equivariant index

$$I(t, \tau) = \sum_{p} \left\{ \frac{1}{2(2\pi - 1)^k} \prod_{j=1}^{k} \frac{\theta'(0, \tau) \theta_1(\alpha_j t, \tau) \theta_3(0, \tau)}{\theta(\alpha_j t, \tau) \theta_1(0, \tau) \theta_3(0, \tau)} \right\}

(2.5)

\cdot \varphi^8(\tau) \cdot \left( \sum_{i=0}^{\infty} \text{ch}(W_i|_p) S^1 q^i \right).$$

On the fixed point $p$, fixing an element $s \in P|_p$, one can define a map $f_s : S^1 \to E_8$ by $g \cdot s = s \cdot f_s(g)$. It’s not hard to check that $f_s$ is a group homomorphism. Moreover, for $h \in E_8$, we have

$$g \cdot (s \cdot h) = (g \cdot s) \cdot h = s \cdot f_s(g) \cdot h = (s \cdot h) \cdot (h^{-1} f_s(g) h).$$

As all the maximal tori in $E_8$ are conjugate, then one may choose $s \in P|_p$ such that $f_s$ maps $S^1$ into the maximal torus $t$ that corresponds to the Cartan subalgebra such that the theta function $\Theta_3(z_1, \cdots, z_8, \tau)$ appears as in (1.16). For any unitary representation $\rho : E_8 \to U(N)$, let $\mathfrak{F}$ be a maximal torus of $U(N)$ that contains $\rho(t)$. Let

$$\hat{\mathfrak{F}} \overset{\hat{\rho}}{\longrightarrow} \overset{\hat{f}_s}{\longrightarrow} \overset{\hat{\mathfrak{S}}}{\longrightarrow} S^1$$

be the induced maps on the character groups. Assume $\hat{f}_s(z_i) = \beta_i t$. Let $\{x_i\}$ are basis for $\hat{\mathfrak{F}}$. By definition,

$$(\text{ch}\rho)(z_1, z_2, \cdots, z_8) = \sum_{i=1}^{N} e^{\hat{\rho}(x_i)},$$

and therefore

$$(\text{ch}\rho)(\beta_1 t, \beta_2 t, \cdots, \beta_8 t)$$

$= \hat{f}_s((\text{ch}\rho)(z_1, z_2, \cdots, z_8))$

$= \sum_{i=1}^{N} e^{(\hat{f}_s \circ \hat{\rho})(x_i)}$

$= \text{ch}((P \times_\rho \mathbb{C}^N)|_p)_{S^1}.$

So for each $i$, we have $\text{ch}(W_i|_p)_{S^1} = (\text{ch}V_i)(\beta_1 t, \beta_2 t, \cdots, \beta_8 t)$. Then by (1.14) and (1.16), we have

$$\varphi^8(\tau) \cdot \left( \sum_{i=0}^{\infty} \text{ch}(W_i|_p)_{S^1} q^i \right)$$

(2.6)

$$= \frac{1}{2} \left( \prod_{l_1=1}^{8} \theta(\beta_1 t, \tau) + \prod_{l_1=1}^{8} \theta_1(\beta_1 t, \tau) + \prod_{l_1=1}^{8} \theta_2(\beta_1 t, \tau) + \prod_{l_1=1}^{8} \theta_3(\beta_1 t, \tau) \right).$$
Comparing both sides of (2.6), we can see by direct computation that
\begin{equation}
30 \cdot (2\pi \sqrt{-1})^2 \sum_{l=1}^{8} \beta_l^2 t^2 = c_2(W|_p)_{S^1}.
\end{equation}

By (2.5) and (2.6), we have
\begin{equation}
I(t, \tau) = \sum_p \left\{ \frac{1}{(2\pi \sqrt{-1})^k} \prod_{j=1}^{k} \theta(0, \tau) \theta(\alpha_j t, \tau) \frac{\theta_1(c t, \tau)}{\theta_1(0, \tau)} \frac{\theta_2(c t, \tau)}{\theta_2(0, \tau)} \frac{\theta_3(c t, \tau)}{\theta_3(0, \tau)} \right\}.
\end{equation}

From the transformation laws of theta functions (1.10)-(1.13), for $a, b \in 2\mathbb{Z}$, it’s not hard to see that
\[ I(t + a \tau + b, \tau) = e^{-\pi \sqrt{-1} n (b^2 \tau + 2br)} I(t, \tau). \]

Since when restricted to fixed points, \( \frac{1}{30} c_2(W)_{S^1} + 3c_1(L)_{S^1} - p_1(TX)_{S^1} \) is equal to $n \cdot \pi^* u^2$, then for each fixed point, from (2.1) and (2.7) we have
\[ \sum_{l=1}^{8} \beta_l^2 + 3c^2 - \sum_{j=1}^{k} \alpha_j^2 = n \]
and therefore
\begin{equation}
I(t + a \tau + b, \tau) = e^{-\pi \sqrt{-1} n (b^2 \tau + 2br)} I(t, \tau).
\end{equation}

It’s easy to deduce from (1.6) that
\[ \theta'(0, \tau + 1) = e^{\pi \sqrt{-1} \tau} \theta'(0, \tau), \quad \theta'(0, -1/\tau) = \frac{1}{\sqrt{-1}} \left( \frac{\tau}{\sqrt{-1}} \right)^{1/2} \tau \theta'(0, \tau). \]

Using the above two formulas and the transformation laws of theta functions (1.6)-(1.9), we have
\begin{equation}
I(t, \tau + 1) = I(t, \tau)
\end{equation}
and
\begin{equation}
I\left( \frac{t}{\tau}, -\frac{1}{\tau} \right) = \tau^{k+4} e^{\pi \sqrt{-1} \frac{(\sum_{j=1}^{k} \beta_j^2 + 3c^2 - \sum_{j=1}^{k} \alpha_j^2)}{a^2}} I(t, \tau) = \tau^{k+4} e^{\pi \sqrt{-1} \frac{a^2}{n}} I(t, \tau).
\end{equation}

(2.9)-(2.11) tell us that $I(t, \tau)$ obeys the transformation laws that a Jacobi form (see [9]) should satisfy.

Next we shall prove that $I(t, \tau)$ is holomorphic for $(t, \tau) \in \mathbb{C} \times \mathbb{H}$. First, we have the following lemma:

**Lemma 2.1.** $I(t, \tau)$ is holomorphic for $(t, \tau) \in \mathbb{R} \times \mathbb{H}$. 

The proof of this lemma is almost verbatimly same as the proof of Lemma 1.3 in [22]. We shall prove that \(I(t, \tau)\) is actually holomorphic on \(\mathbb{C} \times \mathbb{H}\). The possible polar divisor of \(I(t, \tau)\) can be written in the form \(t = \frac{m(c\tau+d)}{l}\) for integers \(m, l, c, d\) with \((c, d) = 1\). Assume \(\frac{m(c\tau+d)}{l}\) is a pole for \(I(t, \tau)\). Find integers \(a, b\) such that \(ad - bc = 1\). Consider the function \(I\left(\frac{t}{c\tau+a}, \frac{d\tau-b}{c\tau+a}\right)\).

By (2.10) and (2.11), it’s easy to see that

\[
I\left(\frac{t}{c\tau+a}, \frac{d\tau-b}{c\tau+a}\right) = f(t, \tau) \cdot I(t, \tau),
\]

where \(f(t, \tau)\) is an entire function of \(t\) for every \(\tau \in \mathbb{H}\). If \(\tau' = \frac{a\tau+b}{c\tau+d}\), then \(\tau = \frac{d\tau'-b}{c\tau'+a}\) and \(\frac{m(c\tau'+a)+d}{l}\) is a pole for the function \(I\left(t, \frac{d\tau'-b}{c\tau'+a}\right)\).

However by (2.12), we have

\[
I\left(\frac{m(c\tau'+a)+d}{l}, \frac{d\tau'-b}{c\tau'+a}\right) = I\left(\frac{m}{l}, \tau'\right) \cdot I\left(\frac{m}{l}, \tau'\right).
\]

As \(\frac{m}{l}\) is real, by Lemma 2.1, we get a contradiction. Therefore \(I(t, \tau)\) is holomorphic for \((t, \tau) \in \mathbb{C} \times \mathbb{H}\).

Combining the transformation formulas (2.9)-(2.11) and the holomorphicity of \(I(t, \tau)\) on \(\mathbb{C} \times \mathbb{H}\), we see that \(I(t, \tau)\) is a weak Jacobi form of index \(\frac{n}{2}\) and weight \(k + 4\) over \((2\mathbb{Z})^2 \times SL(2, \mathbb{Z})\). Here by weak Jacobi form, we don’t require the regularity condition at the cusp but only require that at the cusp \(g\) appears with nonnegative powers only. We refer to [9] for the precise definition of the Jacobi forms.

If \(n = 0\), by (2.9), we see that \(I(t, \tau)\) is holomorphic on the torus \(\mathbb{C}/2\mathbb{Z} + 2\mathbb{Z}\tau\) and therefore must be independent of \(t\). So, by (2.4), we see that

\[
\text{Ind}(g, D_{c,+}^{(1+L)}) + \text{Ind}(g, D_{c,+}^{(1+L)\otimes(W+TcX-(L^2+L^2)+(L+L)-8-2k)})
\]

are both independent of \(g\). So

\[
\text{Ind}(g, D_{c,+}^{(1+L)\otimes W}) + \text{Ind}(g, D_{c,+}^{(1+L)\otimes(TcX-(L^2+L^2)+(L+L))})
\]

must be independent of \(g\). The index density of the operator

\[
D_{c,+}^{(1+L)\otimes W} + D_{c,+}^{(1+L)\otimes(TcX-(L^2+L^2)+(L+L))}
\]
involves the characteristic forms

\[ \hat{A}(TM), e^{c_1(L)/2}(1 + e^{-c_1(L)}), \text{ch}(W), \text{ch}(T_C M), \text{ch}(L + L'), \text{ch}(L^2 + L'^2), \]

which are all of degree 4 (noting that \( W \) is the complexification of the real adjoint representation of compact \( E_8 \)). Therefore by the Atiyah-Singer index theorem, \( \text{Ind} D(g, D(1+L)c, + \text{Ind} D(g, D(1+L)c, + (T_C X - (L^2 + L'^2) + (L + L'))) \equiv 0 \).

This finishes the proof of part (ii).

If \( n \neq 0 \), i.e in the case of nonzero anomaly, we need the following two lemmas.

**Lemma 2.2** (Theorem 1.2 in [9]). Let \( I \) be a weak Jacobi form of index \( m \) and weight \( h \). Then for fixed \( \tau \), if not identically 0, \( I \) has exactly \( 2m \) zeros in any fundamental domain for the action of the lattice on \( \mathbb{C} \).

**Lemma 2.3** (Theorem 2.2 in [9]). Let \( I \) be a weak Jacobi form of index \( m \) and weight \( h \). If \( m = 1 \) and \( h \) is odd, then \( I \) is identically 0.

We would like to point that Lemma 2.2 and Lemma 2.3 are stated in [9] for Jacobi forms. However, as in the proofs of them no regularity condition at the cusp are used, we state them here for weak Jacobi forms. See [9] for details.

If \( n < 0 \), then by Lemma 2.2, \( I(t, \tau) \equiv 0 \), therefore

\[ \text{Ind}(g, D_{c,+}^{(1+L)c, + (T_C X - (L^2 + L'^2) + (L + L'))}) \equiv 0. \]

So part (i) follows.

If \( n = 2 \), as the the weight of \( I(t, \tau) \) is \( k + 4 \), so part (iii) similarly follows clearly from Lemma 2.3.

\[ \square \]

Theorem 0.1 can be easily deduced from Theorem 2.1 as follows.

**Proof of Theorem 0.1:** When \( X \) is a spin manifold, \( L \) is trivial and \( D_{c,+} = D \).

By the Atiyah-Hirzebruch vanishing theorem ([2]), we have \( \text{Ind}(g, D) \equiv 0 \). Moreover by the Witten rigidity theorem ([29, 4, 22], the operator \( D_{T_C X} \) is rigid. i.e. \( \text{Ind}(g, D_{T_C X}) \equiv \text{Ind} D_{T_C X} \). Also note that \( \text{Ind} D_{T_C X} \) equals to 0 when \( k \) is odd. Then the three parts in Theorem 0.1 easily follow from the corresponding three parts in Theorem 2.1. \[ \square \]

For Spin\(^c\) manifolds, we have rigidity and vanishing theorem for another type of twisted operators.
Theorem 2.2. Assume the action only has isolated fixed points and the restriction of the equivariant characteristic class

\[ \frac{1}{30} c_2(W)_{S^1} + c_1(X)_{S^1}^2 - p_1(TX)_{S^1} \]

to \(X_{S^1}\) is equal to \(n \cdot \pi^* u^2\) for some integer \(n\).

(i) If \(n < 0\), then

\[ \text{Ind}(g, D_{c,+}^{(1-L) \otimes W}) + \text{Ind}(g, D_{c,+}^{(1-L) \otimes (T_C X - (L + \bar{L}))}) \equiv 0. \]

In particular,

\[ \text{Ind}D_{c,+}^{(1-L) \otimes W} + \text{Ind}D_{c,+}^{(1-L) \otimes (T_C X - (L + \bar{L}))} = 0. \]

(ii) If \(n = 0\), then

\[ \text{Ind}(g, D_{c,+}^{(1-L) \otimes W}) + \text{Ind}(g, D_{c,+}^{(1-L) \otimes (T_C X - (L + \bar{L}))}) \]

is independent of \(g\). Moreover, when \(k\) is even, one has

\[ \text{Ind}(g, D_{c,+}^{(1-L) \otimes W}) + \text{Ind}(g, D_{c,+}^{(1-L) \otimes (T_C X - (L + \bar{L}))}) \equiv 0. \]

(iii) If \(n = 2\) and \(k\) is even, then

\[ \text{Ind}(g, D_{c,+}^{(1-L) \otimes W}) + \text{Ind}(g, D_{c,+}^{(1-L) \otimes (T_C X - (L + \bar{L}))}) \equiv 0. \]

Proof. We will use same notations as in the proof of Theorem 2.1.

Let \(\Theta^*(X, L, \tau)\) be the virtual complex vector bundles over \(X\) defined by

\[ \Theta^*(X, L, \tau) := \left( \bigotimes_{m=1}^{\infty} S_{q^m}(T_C X) \right) \otimes \left( \bigotimes_{u=1}^{\infty} \Lambda_{-q^u} (\wedge_C) \right). \]

Consider the twisted operator

\[ D_{c,+}^{(1-L) \otimes \Theta^*(X, L, \tau) \otimes (\varphi^8(\tau) \sum_{i=0}^\infty W_i q^i)}. \]

Expanding \(q\)-series, we have

\[ \Theta^*(X, L, \tau) \otimes (\varphi^8(\tau) \sum_{i=0}^\infty W_i q^i) \]

\[ = (1 + (T_C X - 2k)q) \otimes (1 - L_C q + O(q^2)) \]

\[ \otimes (1 - 8q + O(q^2)) \otimes (1 + Wq + O(q^2)) \]

\[ = 1 + (W + T_C X - (L + \bar{L}) - 2k - 6)q + O(q^2). \]

So

\[ D_{c,+}^{(1-L) \otimes \Theta^*(X, L, \tau) \otimes (\varphi^8(\tau) \sum_{i=0}^\infty W_i q^i)} \]

\[ = D_{c,+}^{(1-L)} + D_{c,+}^{(1-L) \otimes (W + T_C X - (L + \bar{L}) - 2k - 6)q + O(q^2)}. \]
By the Atiyah-Bott-Segal-Singer Letschetz fixed point formula, for this twisted operator $D_{c,+}(1-L)\otimes\Theta^*(X,L,\tau)\otimes(\phi^8(\tau)\sum_{i=0}^{\infty}W_iq^i)$, the equivariant index (2.16)
\[
\begin{align*}
J(t,\tau) &= 2\sum_p \frac{1}{(2\pi\sqrt{-1})^k} \prod_{j=1}^{k} \frac{\theta'(0,\tau)}{\theta(\alpha_j t,\tau)} \frac{\theta(ct,\tau)}{\theta_1(0,\tau)\theta_2(0,\tau)\theta_3(0,\tau)} \\
&\quad \cdot \phi^8(\tau) \cdot \left( \sum_{i=0}^{\infty} \text{ch}(W_i|p)q^i \right) \\
&= \sum_p \frac{1}{(2\pi\sqrt{-1})^k} \prod_{j=1}^{k} \frac{\theta'(0,\tau)}{\theta(\alpha_j t,\tau)} \frac{\theta(ct,\tau)}{\theta_1(0,\tau)\theta_2(0,\tau)\theta_3(0,\tau)} \\
&\quad \cdot \left( \prod_{l=1}^{8} \theta(\beta_l t,\tau) + \prod_{l=1}^{8} \theta_1(\beta_l t,\tau) + \prod_{l=1}^{8} \theta_2(\beta_l t,\tau) + \prod_{l=1}^{8} \theta_3(\beta_l t,\tau) \right).
\end{align*}
\]

As when restricted to fixed points, $\frac{1}{8}c_2(W)_{S^1} + c_1(L)_{S^1}^2 - p_1(TX)_{S^1}$ is equal to $n \cdot \pi^*u^2$, then for each fixed point, we have
\[
\sum_{l=1}^{8} \beta_l^2 + c^2 - \sum_{j=1}^{k} \alpha_j^2 = n.
\]

Therefore, similar to (2.9), one can show that for $a, b \in 2\mathbb{Z}$ (2.17)
\[
J(t + a\tau + b, \tau) = e^{-\pi\sqrt{-1}n(b^2\tau + 2br)}J(t, \tau).
\]

One can also show that (2.18)
\[
J(t, \tau + 1) = J(t, \tau)
\]
and
\[
J \left( \frac{t}{\tau}, -\frac{1}{\tau} \right) = \tau^{k+3} e^{\frac{\pi\sqrt{-1}nt^2}{\tau}} J(t, \tau).
\]

So similar to $I(t, \tau)$ in the proof of Theorem 2.1, combing Lemma 2.1 and the above transformation laws, we can prove that $J(t, \tau)$ is a weak Jacobi form of index $\frac{n}{2}$ and weight $k + 3$ over $(2\mathbb{Z})^2 \ltimes SL(2, \mathbb{Z})$.

Then one can prove the three parts of Theorem 2.2 almost the same as those in Theorem 2.1. The only difference one needs to notice is that by the Atiyah-Singer index theorem, $\text{Ind}D_{c,+}(1-L)\otimes\Theta^*(X,L,\tau)\otimes(\phi^8(\tau)\sum_{i=0}^{\infty}W_iq^i)$ must be 0 when the dimension of the manifold is divisible by 4 as the index density of the operator $D_{c,+}(1-L)\otimes\Theta^*(X,L,\tau)\otimes(\phi^8(\tau)\sum_{i=0}^{\infty}W_iq^i)$ is a differential form of degree $4l + 2$. \qed
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