The Logarithmic Spiral Conjecture

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Abstract. When searching for a planar line, if given no further information, one should adopt a logarithmic spiral strategy (although unproven).

This brief paper is concerned entirely with geometry in the plane and continues a thought in [1]. If a line intersects a circle in one or two points, we say that the line strikes the circle. If a line intersects a circle in exactly one point (that is, if the line is tangent to the circle), we say that the line touches the circle.

Let $f$ be a nonnegative, continuously differentiable function on $\mathbb{R}$ satisfying
\[
\lim_{\theta \to -\infty} f(\theta) = 0, \quad \lim_{\theta \to \infty} f(\theta) = \infty.
\]
The polar curve $r = f(\theta)$ intersects every line in the plane, that is, $f$ is a spiral. (Reason: for each $R > 0$, there exists $\Theta$ so large that $\theta > \Theta$ implies $f(\theta) > R$. Any line striking the circle $r = R$ must therefore intersect the curve $r = f(\theta)$. Since $R$ was arbitrary, the statement follows.) Existence of intersection points is only the beginning of our study.

Consider the set $\Sigma$ of all lines that strike the circle $r = R$. The spiral $r = f(\theta)$ possesses a first intersection point $\theta$ with each line in $\Sigma$; let $\theta_1$ denote the supremum of all such $\theta$. Loosely put, $\theta_1$ constitutes the worst case scenario when seeking all members of $\Sigma$ via the search strategy $r = f(\theta)$. Clearly $\theta_1$ depends on $R$ and $\theta_1 = -\infty$ when $R = 0$.

The cost of finding all lines in $\Sigma$, starting from the origin, can be quantified by the arclength
\[
\Lambda(f) = \int_{-\infty}^{\theta_1} \sqrt{f(\theta)^2 + f'(\theta)^2} \, d\theta.
\]
We naturally wish to minimize $\Lambda(f)$ as a function of $f$, for fixed $R$. Our focus is on the following asymptotic inequality.

Conjecture 1.
\[
\lim_{R \to \infty} \frac{\Lambda(f)}{R} \geq 13.8111351795\ldots
\]

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with equality if and only if \( f(\theta) \sim C e^{\kappa \theta} \) as \( \theta \to \infty \), where \( \kappa = 0.2124695594 \ldots \) and \( C > 0 \) is arbitrary.

The two numerical constants appear precisely in [2], along with detailed treatment of the special case of a logarithmic spiral \( f(\theta) = e^{\kappa \theta} \). Difficulties arise in the general case, owing to the vast variety of spirals permitted.

A sketch of a geometric proof of Conjecture 1 was published in [3, 4, 5]. The first part claimed that an optimal spiral must be similar with respect to both rotations and dilations about the origin; the second part claimed that such a highly symmetric spiral must necessarily be a logarithmic spiral. The second part, in fact, is true via the solution of a well-known functional equation [6]. We doubt, however, that any purely geometric proof of the first part can be rigorously correct (although appealing). A more careful analysis, based on the calculus of variations, is perhaps mandatory.

0.1. Examples. We repeat certain steps employed in [2], suitably generalized.

**Lemma 2.** The distance between the line \( Ax + By + C = 0 \) and the origin is \( \frac{|C|}{\sqrt{A^2 + B^2}} \).

**Lemma 3.** The equation of a line tangent to the spiral \( r = f(\theta) \) is \( y - f(\theta) \sin(\theta) = m(x - f(\theta) \cos(\theta)) \), where \( \theta \) corresponds to the point of tangency and the slope is given by

\[
 m = \frac{f'(\theta) \sin(\theta) + f(\theta) \cos(\theta)}{f'(\theta) \cos(\theta) - f(\theta) \sin(\theta)}.
\]

**Proof of Lemma 3.** Clearly

\[
 \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(f(\theta) \sin(\theta))'}{(f(\theta) \cos(\theta))'} = \frac{f'(\theta) \sin(\theta) + f(\theta) \cos(\theta)}{f'(\theta) \cos(\theta) - f(\theta) \sin(\theta)}.
\]

**Theorem 4.** Let \( L \) denote the first line that is both tangent to the spiral \( r = f(\theta) \) and tangent to the circle \( r = R \). The tangency point \( \theta_0 \) of \( L \) with the spiral satisfies the equation

\[
 R^2(f(\theta)^2 + f'(\theta)^2) = f(\theta)^4.
\]

**Proof of Theorem 4.** Apply Lemma 2 with \( A = m, B = -1 \) and \( C = f(\theta)(\sin(\theta) - m \cos(\theta)) \) to obtain \((1 + m^2)R^2 = f(\theta)^2(\sin(\theta) - m \cos(\theta))^2 \). Substituting the expression for \( m \) from Lemma 3 gives the desired equation.

We emphasize that, on the one hand, \( \theta_0 \) is where the spiral first intersects a line that touches the circle \( r = R \) (the touching occurs elsewhere). On the other hand, \( \theta_1 \) is just above where the spiral last intersects a new line that strikes the circle \( r = R \) (the striking, again, occurs elsewhere). If the function \( f \) is strictly increasing, then
in the interval \( \theta_0 < \theta < \theta_1 \), the spiral intersects all other lines that touch \( r = R \); at \( \theta = \theta_1 \), repetition begins so we stop there. Suppose that we are given a spiral \( r = f(\theta) \) for which \( f(\theta) \neq C e^{\kappa \theta} \) for any \( \kappa > 0, C > 0 \). Clearly

\[
\frac{\Lambda(f)}{R} \geq \frac{1}{R} \int_{-\infty}^{\theta_0} \sqrt{f(\theta)^2 + f'(\theta)^2} \, d\theta,
\]

and thus if we demonstrate that the right hand side \( \to \infty \) or is at least > 13.82, then this is consistent with Conjecture 1.

As a first example, consider Archimedes’ spiral

\[
f(\theta) = \begin{cases} 
\kappa \theta & \text{if } \theta \geq 0, \\
0 & \text{if } \theta < 0.
\end{cases}
\]

From Theorem 4, it follows that \( R^2 (1 + \theta^2) = \kappa^2 \theta^4 \) and hence

\[
\theta_0 = \frac{R}{\kappa} \sqrt{\frac{1}{2} \left(1 + \sqrt{1 + \frac{4\kappa^2}{R^2}}\right)} \geq \frac{R}{\kappa} \sqrt{\frac{1}{2} (1 + 1)} \geq \frac{R}{\kappa}.
\]

Consequently, the normalized arclength is bounded from below by

\[
\frac{\kappa}{R} \int_0^{\theta_0} \sqrt{1 + \theta^2} \, d\theta \geq \frac{\kappa}{R} \int_0^{\theta_0} \theta \, d\theta = \frac{\kappa}{2R} \theta_0^2 \geq \frac{R^2}{2\kappa} \to \infty
\]
as \( R \to \infty \). Alternatively, we can avoid solving for \( \theta_0 \) altogether: From \( R^2 (1 + \theta^2) = \kappa^2 \theta^4 \), deduce that

\[
R = \frac{\kappa \theta^2}{\sqrt{1 + \theta^2}} \leq \kappa \theta^2
\]
and hence that \( \theta \to \infty \) as \( R \to \infty \). Here we obtain

\[
\frac{\kappa}{R} \int_0^{\theta_0} \theta \, d\theta = \frac{\kappa}{2R} \theta_0^2 = \frac{\kappa}{2} \frac{1 + \theta_0^2}{2} \theta_0^2 = \frac{1}{2} \frac{1 + \theta_0^2}{\kappa \theta_0^2} \geq \frac{\theta_0}{2} \to \infty
\]
as \( \theta_0 \to \infty \) (and thus as \( R \to \infty \)). This latter device will be useful in the following examples. See Figure 1 for an illustration.

Consider next the spiral

\[
f(\theta) = \begin{cases} 
e^{\theta^a} & \text{if } \theta \geq 0, \\
-ne^{-|\theta|^a} & \text{if } \theta < 0.
\end{cases}
\]
Figure 1: The first contact point that the spiral $r = \theta$ has with a line tangent to the circle $R = 6$ is at $\theta_0 = 348.4^\circ$. The second contact point with the line is at $\theta_1 = 641.5^\circ$. Incidentally, the line is tangent to $R = 6$ at $339.1^\circ < \theta_0$. 


for a fixed exponent $a > 0$. From Theorem 4, it follows that $R^2(1 + a^2 \theta^{2a-2}) = e^{2\theta^a}$, that is,

$$R = \frac{e^{\theta^a}}{\sqrt{1 + a^2 \theta^{2a-2}}} \leq e^{\theta^a}.$$  

Hence $\theta \to \infty$ as $R \to \infty$. If $a > 1$, the normalized arclength is bounded from below by

$$\frac{1}{R} \left( \int_{-\infty}^{0} \sqrt{1 + a^2 |\theta|^{2a-2}} e^{-\theta^a} d\theta + \int_{0}^{\theta_0} \sqrt{1 + a^2 \theta^{2a-2}} e^{\theta^a} d\theta \right)$$

$$\geq \frac{1}{R} \left( \int_{-\infty}^{0} a |\theta|^{a-1} e^{-\theta^a} d\theta + \int_{0}^{\theta_0} a \theta^{a-1} e^{\theta^a} d\theta \right)$$

$$= \frac{1}{R} \left( 1 + e^{\theta_0^a} - 1 \right) = \sqrt{1 + a^2 \theta_0^{2a-2}} \geq a \theta_0^{a-1} \to \infty$$

as $\theta_0 \to \infty$ (and thus as $R \to \infty$). If $0 < a < 1$, the normalized arclength is bounded by

$$\frac{1}{R} \left( \int_{-\infty}^{0} e^{-|\theta|^a} d\theta + \int_{0}^{\theta_0} e^{\theta^a} d\theta \right)$$

$$\geq \frac{1}{R} \left( \int_{-\infty}^{0} e^{-|\theta|^a} d\theta + \int_{0}^{\theta_0} e^{\theta^a} d\theta \right) \geq \frac{1}{R} \left( 0 + \int_{0}^{\theta_0} e^{\theta^a} d\theta \right)$$

and we have asymptotics

$$\frac{1}{R} \int_{0}^{\theta_0} e^{\theta^a} d\theta \sim \frac{1}{R} \left( \frac{1}{a} \theta_0^{1-a} e^{\theta_0^a} \right) = \frac{1}{a} \sqrt{1 + a^2 \theta_0^{2a-2}} \theta_0^{1-a} \to \infty$$

as $\theta_0 \to \infty$ (and thus as $R \to \infty$). Only the case $a = 1$ remains, which is covered in [2]. This is compelling (but not completely convincing) evidence that the Logarithmic Spiral Conjecture is valid.

Consider finally the spiral

$$f(\theta) = \begin{cases} \theta^b e^\theta & \text{if } \theta \geq 0, \\ 0 & \text{if } \theta < 0 \end{cases}$$

for a fixed exponent $b > 0$. From Theorem 4, we obtain

$$R = \frac{\theta^{b+1} e^\theta}{\sqrt{\theta^2 + (b+\theta)^2}} \leq \frac{1}{\sqrt{2}} \theta^b e^\theta,$$
hence \( \theta \to \infty \) as \( R \to \infty \). Clearly \( \theta_1 \geq \theta_0 + \pi \) on geometric grounds. Therefore the normalized arclength is bounded from below by

\[
\frac{1}{R} \int_{0}^{\theta_0 + \pi} \sqrt{\theta^2 + (b + \theta)^2} \theta^{b-1} e^{\theta} d\theta \geq \frac{\sqrt{2}}{R} \int_{0}^{\theta_0 + \pi} \theta^{b} e^{\theta} d\theta
\]

and we have asymptotics

\[
\frac{\sqrt{2}}{R} \int_{0}^{\theta_0 + \pi} \theta^{b} e^{\theta} d\theta \sim \frac{\sqrt{2}}{R} \left( (\theta_0 + \pi)^{b} e^{\theta_0 + \pi} \right)
\]

\[= \sqrt{2} \frac{(\theta_0 + \pi)^{b} e^{\theta_0 + \pi}}{\theta_0^{b+1} e^{\theta_0}} \sqrt{\theta_0^2 + (b + \theta_0)^2}
\]

\[\to 2e^\pi > 13.82
\]

as \( \theta_0 \to \infty \).

The logarithmic spiral appears with regard to another planar search problem \[7\], but the techniques of Gal & Chazan do not seem to apply here. A min-mean analog of our Conjecture 1 could also be formulated, starting with \[2\].

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