ON $C^0$-CONTINUITY OF THE SPECTRAL NORM ON NON-SYMPLECTICALLY ASPHERICAL MANIFOLDS

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Abstract. Our first main result states that on rational symplectic manifolds the spectral norm $\gamma$ of a Hamiltonian is close to $\lambda_0 Z$ if the Hamiltonian generates a time-1 map that is $C^0$-close to $Id$ where $\lambda_0$ denotes the rationality constant. As a corollary, we prove the $C^0$-continuity of spectral norms on complex projective spaces which provides an alternative method to the result of Shelukhin.

Our second main result states that the spectral norm $\gamma$ on $\text{Ham}(M, \omega)$ is $C^0$-continuous when $(M, \omega)$ is negative monotone.

These extend the results on the $C^0$-continuity of spectral norms proven for $\mathbb{R}^{2n}$ (Viterbo), closed surfaces (Seyfaddini), symplectically aspherical manifolds (Buhovsky-Humilière-Seyfaddini) and complex projective spaces (Shelukhin).

We also discuss some applications of the $C^0$-continuity of spectral norms including the Arnold conjecture in the context of $C^0$-symplectic topology to describe the rigidity of Hamiltonian homeomorphisms.

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1. Introduction and main results

This article addresses the question of $C^0$-continuity of spectral norms, which are extracted from the action spectrum of a Hamiltonian via Floer theory. The concept of spectral invariants/norms was first introduced by Viterbo in [Vit] where he also proved their $C^0$-continuity on $\mathbb{R}^{2n}$. Spectral invariants/norms were later generalized by Schwarz [Sch] and Oh [Oh] and further progress on their $C^0$-continuity were made by Seyfaddini on closed surfaces [Sey1], Buhovsky-Humilière-Seyfaddini on aspherical symplectic manifolds and Shelukhin on complex projective spaces [Sh].

In this article, we prove some results concerning the spectral norm of Hamiltonians that generates a time-1 map that is $C^0$-close to $Id$ on rational symplectic manifolds, which directly imply the $C^0$-continuity of spectral norms on a certain class of monotone symplectic manifolds including $\mathbb{C}P^n$. This provides an alternative approach to the result of Shelukhin. We also prove the $C^0$-continuity of spectral norms on negative monotone manifolds.

1.1. Background on spectral norms. Let $(M, \omega)$ be a symplectic manifold.

$(M, \omega)$ is said to be rational if $\langle \omega, \pi_2(M) \rangle = \lambda_0 \mathbb{Z}$ for some constant $\lambda_0 > 0$. We refer to the constant $\lambda_0$ as the rationality constant. $(M, \omega)$ is said to be monotone (resp. negative monotone) if $\omega|_{\pi_2(M)} = \lambda \cdot c_1|_{\pi_2(M)}$ for some positive (resp. negative) constant $\lambda$ where $c_1$ denotes the first Chern class of $TM$. We refer to the constant $\lambda$ as the monotonicity constant. Of course, (negative) monotone symplectic manifolds are rational. The positive generator of $\langle c_1, \pi_2(M) \rangle$ is called the minimal Chern number i.e. $\langle c_1, \pi_2(M) \rangle = N\mathbb{Z}$. When $\omega|_{\pi_2(M)} = c_1|_{\pi_2(M)} = 0$, the symplectic manifold is called (symplectically) aspherical.

Example.

(1) An important example of a monotone symplectic manifold is the complex projective space equipped with the standard Fubini-Study form $(\mathbb{C}P^n, \omega_{FS})$. The minimal Chern number $N$ of $(\mathbb{C}P^n, \omega_{FS})$ is $N = n + 1$.

(2) An important class of negative monotone symplectic manifolds is the following submanifolds of $\mathbb{C}P^n$:

$$\{(z_0 : z_1 : \cdots : z_n) \in \mathbb{C}P^n : z_0^k + z_1^k + z_2^k + \cdots + z_n^k = 0\}$$

for $k > n + 1$. The minimal Chern number of such symplectic manifolds is $N = k - (n + 1)$. 

Let $\text{Ham}(M, \omega), \text{Ham}^{C^0}(M, \omega)$ denote respectively the group of Hamiltonian diffeomorphisms and homeomorphisms: by a Hamiltonian homeomorphism, we mean a homeomorphism which is a $C^0$-limit of Hamiltonian diffeomorphisms.

For a Hamiltonian $H \in C^\infty(S^1 \times M, \mathbb{R})$ on $(M, \omega)$, one can define a symplectic invariant called the spectral invariant $\rho(H, a)$ for each non-zero homology class $a \in H_*(M) \setminus \{0\}$. Roughly speaking, they are action values at which the homology class $a$ appears in the filtered Hamiltonian Floer homology of $H$ where the filtration is with respect to the action of orbits of $H$: see Section 2 for a detailed definition. The spectral norm

$$\gamma : C^\infty(S^1 \times M, \mathbb{R}) \to \mathbb{R}$$

is defined as

$$\gamma(H) := \rho(H, [M]) + \rho(\overline{H}, [M])$$

where $\overline{H}$ is the Hamiltonian that generates $(\phi^t_H)^{-1}$.

Since $\gamma$ is invariant under homotopy (i.e. if $\phi^t_H \sim \phi^t_G$ rel. endpoints, then $\gamma(H) = \gamma(G)$), it can be seen as a map defined on the universal cover of $\text{Ham}(M, \omega)$, namely

$$\gamma : \widetilde{\text{Ham}}(M, \omega) \to \mathbb{R}.$$  

When $(M, \omega)$ is symplectically aspherical, we have the following: if $\phi_H = \phi_G$, then

$$\gamma(H) = \gamma(G).$$

In other words, $\gamma$ descends to a map on $\text{Ham}(M, \omega)$ i.e.

$$\gamma : \text{Ham}(M, \omega) \to \mathbb{R},$$

$$\gamma(\phi) := \gamma(H)$$

for any $H$ such that $\phi = \phi_H$. See [Sch] for details.

However, this is no longer true for non-aspherical symplectic manifolds. Precisely, it can happen that $\gamma(H) \neq \gamma(G)$ even if $\phi_H = \phi_G$ i.e. $\gamma$ does not descend to a map on $\text{Ham}(M, \omega)$. In this case, in order to define spectral norms on $\text{Ham}(M, \omega)$, we define as follows:

$$\gamma : \text{Ham}(M, \omega) \to \mathbb{R},$$

$$\gamma(\phi) := \inf_{\phi = \phi_H} \gamma(H).$$

1.2. The case of rational symplectic manifolds. Our first result concerns the value of a "modified spectral norm" of Hamiltonians that generate a time-1 map which is $C^0$-close to $Id$ where the symplectic manifold is rational. If the symplectic manifold is monotone, then the "modified spectral norm" is simply the difference of two spectral invariants.

**Definition 1.** Let $(M, \omega)$ be any closed symplectic manifold and $a, b \in H_*(M) \setminus \{0\}$. We define the following:

$$\gamma_{a,b} : C^\infty(S^1 \times M, \mathbb{R}) \to \mathbb{R},$$
\[ \gamma_{a,b}(H) := \rho(H, a) + \rho(\Pi, b). \]

**Remark 2.** Of course, \( \gamma_{[M], [M]} = \gamma \) where \( \gamma \) is the usual spectral norm.

**Theorem 3.** Let \((M, \omega)\) be a rational symplectic manifold and \(a, b \in H_*(M) \setminus 0\).
For any \( \varepsilon > 0 \), there exists \( \delta > 0 \) satisfying the following:
If \( d_{C^0}(id, \phi_H) < \delta \), then
\[
|\gamma_{a,b}(H) - l \cdot \lambda_0| < \varepsilon
\]
for some integer \( l \in \mathbb{Z} \) depending on \( a, b \in H_*(M) \setminus 0 \) and \( H \).

**Remark 4.** For strongly semi-positive symplectic manifolds, by Lemma 5.1. in [Ost] (See also Lemma 2.2 in [EP2]), we have the following alternative expression of Theorem 3:
Let \( a, b \in H_*(M) \setminus 0 \). For any \( \varepsilon > 0 \), there exists \( \delta > 0 \) satisfying the following:
If \( d_{C^0}(id, \phi_H) < \delta \), then
\[
|\rho(H, a) - \rho(H, b) - l \cdot \lambda_0| < \varepsilon
\]
for some integer \( l \in \mathbb{Z} \) depending on \( a, b \in H_*(M) \setminus 0 \) and \( H \).

This implies that if the spectral norm is bounded around \( Id \) by a real number which is strictly smaller than the rationality constant, then it is \( C^0 \)-continuous.

**Corollary 5.** Let \((M, \omega)\) be a rational symplectic manifold.
Assume that there exist constants \( 0 < \kappa < 1 \) and \( \delta' > 0 \) such that if \( \phi \in Ham(M, \omega) \), \( d_{C^0}(id, \phi) \leq \delta' \), then \( \gamma(\phi) \leq \kappa \cdot \lambda_0 \).

Then, the spectral norm is \( C^0 \)-continuous i.e.
\[ \gamma : (Ham(M, \omega), d_{C^0}) \to \mathbb{R} \]
is continuous.

**Proof.** (Corollary 5)
It is enough to prove the continuity at \( Id \) since \( |\gamma(\phi) - \gamma(\psi)| \leq \gamma(\psi^{-1} \phi) \).
For a given \( \varepsilon \in (0, \frac{1}{2}(1 - \kappa) \lambda_0) \), take \( \delta > 0 \) as in Theorem 3.
Let
\[ \phi \in Ham(M, \omega), \ d_{C^0}(Id, \phi) < \min\{\delta, \delta'\}. \]
There exists a Hamiltonian \( H \) such that \( \phi_H = \phi \) and
\[
\gamma(H) < \gamma(\phi) + \varepsilon < \kappa \cdot \lambda_0 + \frac{1}{2} (1 - \kappa) \lambda_0
\]
\[
= \frac{1}{2} (1 + \kappa) \lambda_0 < \lambda_0 - \varepsilon.
\]
Thus, by Theorem 3
\[ \gamma(H) < \varepsilon. \]
Thus,
\[ \gamma(\phi) \leq \gamma(H) < \varepsilon. \]
This implies the continuity of \( \gamma \) at \( Id \) and hence completes the proof of Corollary 5. \( \square \)
1.3. The case of monotone symplectic manifolds. We see that a certain class of monotone symplectic manifolds satisfies the assumptions in Corollary 5. Stimulated by Corollary 5, we investigate bounds of spectral norms.

Our first result in this direction is the following.

**Theorem 6.** Let \((M^{2n}, \omega)\) be a monotone symplectic manifold.

1. For any \(\varepsilon > 0\), there exists \(\delta > 0\) such that if \(d_{C^0}(\text{id}, \phi_H) < \delta\), then \(\gamma(H) < \frac{4n}{N} \cdot \lambda_0 + \varepsilon\).

2. If \(N > 4n\), then the spectral norm is \(C^0\)-continuous i.e. \(\gamma : (\text{Ham}(M, \omega), d_{C^0}) \to \mathbb{R}\) is continuous. Moreover, \(\gamma\) extends continuously to \(\text{Ham}^{C^0}(M, \omega)\).

The author does not know any example of a symplectic manifold meeting the assumptions in the second property.

Our next result in this direction is the following. Before stating the theorem, we would like to point out that the first property of Theorem 7 was proven by Kislev-Shelukhin [KS] prior to the author under the setting of Lagrangian Floer homology. Nevertheless, we state the version of Hamiltonian Floer homology.

**Theorem 7.** Let \((M^{2n}, \omega)\) be a monotone symplectic manifold having a minimal Chern number \(N > n\).

Assume that there exist \(\psi \in \pi_1(\text{Ham}(M, \omega))\) and a section class \(\sigma\) of the Hamiltonian fibration \(\tilde{M}_\psi \to S^2\), such that its Seidel element \(S_{\psi, \sigma} \in QH_*(M)\) satisfies the following:

- \((S_{\psi, \sigma})^k = [pt]\) for some \(k \in \mathbb{N}\) where \([pt]\) denotes the generator of \(H_0(M)\).
- \((S_{\psi, \sigma})^{k' \cdot l'} = [M] \cdot q^{-l'}\) for some \(k', l' \in \mathbb{N}\) where \([M]\) denotes the fundamental class and \(q\) denotes the generator of the Novikov ring of \((M, \omega)\).

Then the spectral norm satisfies the following.

1. \(\forall \phi \in \text{Ham}(M, \omega)\),
   \[\gamma(\phi) \leq \frac{n}{N} \cdot \lambda_0\]

2. The spectral norm is \(C^0\)-continuous i.e. \(\gamma : (\text{Ham}(M, \omega), d_{C^0}) \to \mathbb{R}\) is continuous. Moreover, \(\gamma\) extends continuously to \(\text{Ham}^{C^0}(M, \omega)\).

To the best of the authors knowledge, the only example meeting the assumptions in Theorem 7 is \((\mathbb{C}P^n, \omega_{FS})\) which implies the following properties of \((\mathbb{C}P^n, \omega_{FS})\). Note that these properties of \((\mathbb{C}P^n, \omega_{FS})\) were proven prior to the author: the first property by Kislev-Shelukhin [KS] and the second by Shelukhin [Sh]. It is interesting to point out that Shelukhin obtained the
result in the context of a conjecture of Viterbo while we obtain it as a direct
consequence of the first property.

**Corollary 8.** Let \( (\mathbb{C}P^n, \omega_{FS}) \) be the complex projective space equipped with
the Fubini-Study form.

\[ \forall \phi \in \text{Ham}(\mathbb{C}P^n, \omega_{FS}), \]
\[ \gamma(\phi) \leq \frac{n}{n+1} \cdot \lambda_0 \]
where \( \lambda_0 \) denotes the rationality constant.

(1) The spectral norm is \( C^0 \)-continuous i.e.
\[ \gamma: (\text{Ham}(\mathbb{C}P^n, \omega_{FS}), d_{C^0}) \rightarrow \mathbb{R} \]
is continuous. Moreover, \( \gamma \) extends continuously to \( \tilde{\text{Ham}}(\mathbb{C}P^n, \omega_{FS}) \).

1.4. The case of negative monotone symplectic manifolds. By employing a similar method to the proof of Theorem 3, we obtain the following
result where the symplectic manifold is negative monotone.

**Theorem 9.** Let \( (M, \omega) \) be a negative monotone symplectic manifold.

For any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that if \( H \in C^\infty(S^1 \times M, \mathbb{R}) \) and \( d_{C^0}(id, \phi_H) < \delta \), then \( \gamma(H) < \varepsilon \).

This implies that spectral norms descend to \( \text{Ham}(M, \omega) \) from \( \tilde{\text{Ham}}(M, \omega) \).

**Corollary 10.** Let \( (M, \omega) \) be a negative monotone symplectic manifold.

If \( \phi_H = \phi_G \) where \( H, G \in C^\infty(S^1 \times M, \mathbb{R}) \), then \( \gamma(H) = \gamma(G) \)
i.e. \( \gamma: \tilde{\text{Ham}}(M, \omega) \rightarrow \mathbb{R} \) descends to \( \gamma: \text{Ham}(M, \omega) \rightarrow \mathbb{R} \).

It also implies the \( C^0 \)-continuity of the spectral norm.

**Corollary 11.** Let \( (M, \omega) \) be a negative monotone symplectic manifold.

The spectral norm is \( C^0 \)-continuous i.e.
\[ \gamma: (\text{Ham}(M, \omega), d_{C^0}) \rightarrow \mathbb{R} \]
is continuous. Moreover, \( \gamma \) extends continuously to \( \tilde{\text{Ham}}^{C^0}(M, \omega) \).

**Proof.** (Corollary 10 and 11)

(1) If \( \phi_H = \phi_G \), then \( d_{C^0}(id, \phi_H^{-1} \circ \phi_G) = 0 \). Thus for any \( \varepsilon > 0 \),
\[ \gamma(G) \leq \gamma(H) + \gamma(\#G) \leq \gamma(H) + \varepsilon. \]
\( \varepsilon \) is arbitrary so \( \gamma(G) \leq \gamma(H) \). By changing the role of \( H \) and \( G \), we obtain the opposite inequality and thus \( \gamma(H) = \gamma(G) \).

(2) Once we know that spectral norms are well-defined on \( \text{Ham}(M, \omega) \),
the \( C^0 \)-continuity at \( Id \) follows directly from Theorem 9.

At \( \phi \in \text{Ham}(M, \omega) \), for any \( \varepsilon > 0 \), if we take \( d_{C^0}(\phi, \psi) \) small
enough so that \( d_{C^0}(id, \phi^{-1} \circ \psi), d_{C^0}(id, \psi^{-1} \circ \phi) < \delta \)
(\( \delta \) is taken as in Theorem 9), then
\[ \gamma(\psi) \leq \gamma(\phi) + \gamma(\phi^{-1} \circ \psi) < \gamma(\phi) + \varepsilon. \]

This implies the \( C^0 \)-continuity of \( \gamma \) at an arbitrary point \( \phi \in \text{Ham}(M, \omega) \). \( \square \)

1.5. Application 1: \( C^0 \)-continuity of barcodes. In [PS], Polterovich-Shelukhin defined barcodes of Hamiltonian diffeomorphisms on symplectic manifolds that are symplectically aspherical. Developping the idea in [PS], Polterovich-Shelukhin-Stojisavljević considered barcodes on monotone symplectic manifolds in [PSS] by fixing a degree to achieve "finiteness". Later, Le Roux-Seyfaddini-Viterbo extended the notion of barcodes to allow them to have infinitely many intervals in order to define barcodes of Hamiltonian diffeomorphisms on spheres in [LSV]. Their method extends directly to the case of (negative) monotone symplectic manifolds and thus we can define barcodes of Hamiltonian diffeomorphisms on (negative) monotone symplectic manifolds. We will denote this map by

\[ B : \text{Ham}(M, \omega) \rightarrow \text{Barcodes} \]

and call it the barcode map, provided that \( (M, \omega) \) is (negative) monotone. See Section 2.4 for the precise definition of the barcode map and \( \text{Barcodes} \).

\[ d_{\text{bot}}(B(\phi), B(\psi)) \leq \frac{1}{2} \gamma(\psi^{-1} \phi). \]

Thus, the \( C^0 \)-continuity of the spectral norm implies the \( C^0 \)-continuity of the barcode map \( B \). This allows us to define a barcode of Hamiltonian homeomorphisms i.e. homeomorphisms that are \( C^0 \)-limits of Hamiltonian diffeomorphisms.

Corollary 12. Let \( (M, \omega) \) be a negative monotone symplectic manifold. The barcode map is \( C^0 \)-continuous i.e.

\[ B : (\text{Ham}(M, \omega), d_{C^0}) \rightarrow (\text{Barcodes}, d_{\text{bot}}) \]

is continuous. Moreover, \( B \) extends continuously to \( \text{Ham}^{C^0}(M, \omega) \).

Remark 13. Of course, (2) of Theorem 8 directly implies the \( C^0 \)-continuity of barcodes in the case of \((\mathbb{C}P^n, \omega_{FS})\). This is a result of Shelukhin: see Corollary 6 in [Sh].

1.6. Application 2: The \( C^0 \)-Arnold conjecture. The (homological) Arnold conjecture states the following.

For \( \phi \in \text{Ham}(M, \omega) \) where \( (M^{2n}, \omega) \) denotes a closed symplectic manifold,

\[ \text{Fix}(\phi) \geq \text{cl}(M) \]
where
\[ \text{cl}(M) := \# \max \{k + 1 : \exists a_1, a_2, \cdots, a_k \in H_{<2n}(M), a_1 \cap a_2 \cap \cdots \cap a_k \neq 0 \} . \]

Here, \( \cap \) denotes the intersection product.

Buhovsky-Humilière-Seyfaddini discovered that Hamiltonian homeomorphisms do not satisfy the original version of the Arnold conjecture:

**Theorem 14.** ([BHS1]) Let \((M, \omega)\) be any closed symplectic manifold of dimension \( \geq 4 \). There exists a Hamiltonian homeomorphism having only one fixed point.

However, in [BHS2], the authors point out that on aspherical symplectic manifolds, if one counts the total number of spectral invariants in stead of the fixed points, Hamiltonian homeomorphism do satisfy a rigidity property similar to the Arnold conjecture.

Following their work, in this article we prove a rigidity result of Hamiltonian homeomorphisms on negative monotone symplectic manifolds and \( \mathbb{C}P^n \) which could be considered as a \( C^0 \)-version of the homological Arnold conjecture. To state the theorem, we define some concepts.

**Definition 15.** Let \((M^{2n}, \omega)\) be a symplectic manifold. For a Hamiltonian \( H \) and homology classes \( a, b \in H_*(M) \setminus 0 \), define
\[ \sigma_{a,b}(H) := \rho(H, a) - \rho(H, b) \]
where \( \rho(H, \cdot) \) denotes the spectral invariants of \( H \).

In the case of negative monotone symplectic manifolds and \( (\mathbb{C}P^n, \omega_{FS}) \), we define \( \sigma_{a,b} \) for Hamiltonian diffeomorphisms by
\[ \sigma_{a,b} : \text{Ham}(M, \omega) \to \mathbb{R}, \]
\[ \sigma_{a,b}(\phi) := \inf_{\phi \in \text{Ham}} \sigma_{a,b}(H) \]
and in fact, \( \sigma_{a,b} \) is \( C^0 \)-continuous i.e.
\[ \sigma_{a,b} : (\text{Ham}(M, \omega), d_{C^0}) \to \mathbb{R} \]
is continuous. The \( C^0 \)-continuity of \( \sigma_{a,b} \) allows us to define \( \sigma_{a,b} \) for Hamiltonian homeomorphisms. See Section 4.2 for details.

Now we are ready to state a \( C^0 \)-version of the homological Arnold conjecture. Recall that, a subset \( A \subset M \) is homologically non-trivial if for every open neighborhood \( U \) of \( A \) the map \( i_* : H_j(U) \to H_j(M) \), induced by the inclusion \( i : U \to M, \) is non-trivial for some \( j > 0 \). Clearly, homologically non-trivial sets are infinite.

**Theorem 16.** Let \((M^{2n}, \omega)\) be either a negative monotone symplectic manifold with a minimal Chern number \( N \geq n \) or \((\mathbb{C}P^n, \omega_{FS})\).

Let \( \phi \in \overline{\text{Ham}}^{C^0}(M, \omega) \). If there exist homology classes \( a, b \in H_{<2n}(M) \setminus 0 \) such that \( \sigma_{a,a \cap b}(\phi) = 0 \), then \( \text{Fix}(\phi) \) is homologically non-trivial, hence is infinite.
1.7. **Application 3: The displaced disks problem.** A topological group $G$ is a Rokhlin group if it possesses a dense conjugacy class i.e. $\exists \phi \in G$ such that $C(\phi) := \{ \psi^{-1}\phi \psi : \psi \in G \}$ is dense. F. Béguin, S. Crovisier, and F. Le Roux formulated the following "displaced disks problem" in order to answer if the group of area-preserving homeomorphisms on a sphere is a Rokhlin group or not.

**Question.** For $r > 0$, define

$$G_r := \{ \phi \in \overline{\text{Ham}}_{C^0}(M, \omega) : \phi(f(B_r)) \cap f(B_r) = \emptyset \}$$

where $f : B_r \to (M, \omega)$ is a symplectic embedding. Does the $C^0$-closure of $G_r$ contains $Id$ for some $r > 0$?

This original version which was for $(M, \omega) = (S^2, \omega_{\text{area}})$ was solved by Seyfaddini in [Sey2] as a consequence of his earlier result on the $C^0$-continuity of spectral norms on closed surfaces [Sey1]. The same question on other symplectic manifolds were considered also in the context of the $C^0$-continuity of spectral norms: see [BHS2] for the case of aspherical symplectic manifolds and [Sh] for the case of $\mathbb{CP}^n$. Here we add the case of negative monotone symplectic manifolds.

**Theorem 17.** Let $(M, \omega)$ be a negative monotone symplectic manifold.

For every $r > 0$, there exists $\delta > 0$ such that if $\phi \in \overline{\text{Ham}}_{C^0}(M, \omega)$ displaces a symplectically embedded ball of radius $r$, then $d_{C^0}(Id, \phi) > \delta$.

The following follows immediately from this.

**Corollary 18.** Let $(M, \omega)$ be a negative monotone symplectic manifold.

$\overline{\text{Ham}}_{C^0}(M, \omega)$ seen as a topological group with respect to the $C^0$-topology is not a Rokhlin group.

**Remark 19.** As results in this section follows from the $C^0$-continuity of $\gamma$, (2) of Theorem 8 will allow us to obtain similar results in the case of $(\mathbb{CP}^n, \omega_{FS})$. This was considered by Shelukhin in [Sh].

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Let \((M, \omega)\) be a symplectic manifold. A Hamiltonian \(H\) on \(M\) is a smooth time dependent function \(H : S^1 \times M \to \mathbb{R}\). We define its Hamiltonian vector field \(X_H\) by \(-dH_t = \omega(X_H, \cdot)\). The Hamiltonian flow of \(H\), denoted by \(\phi^t_H\), is by definition the flow of \(X_H\). A Hamiltonian diffeomorphism is a diffeomorphism which arises as the time-one map of a Hamiltonian flow. The set of all Hamiltonian diffeomorphisms is denoted by \(\text{Ham}(M, \omega)\).

Denote the set of smooth contractible loops in \(M\) by \(\mathcal{L}M\) and consider its universal cover. Two elements in the universal cover, say \([z_1, w_1]\) and \([z_2, w_2]\), are called equivalent if their boundary sum \(w_1 \# w_2\) satisfies \(\omega(w_1 \# w_2) = 0\) and \(c_1(w_1 \# w_2) = 0\). We denote by \(\tilde{\mathcal{L}}M\) the space of equivalence classes.

For a Hamiltonian \(H\), define the action functional \(A_H : \tilde{\mathcal{L}}M \to \mathbb{R}\) by

\[
A_H([z, w]) := \int_0^1 H(t, z(t))dt - \int_{D^2} w^*\omega
\]

where \(w : D^2 \to M\) is a capping of \(z : S^1 \to M\). Note that in general, the action functional depends on the capping and not only on the loop.

Critical points of this functional, which is denoted by \(\text{Crit}(A_H)\), are precisely the capped 1-periodic Hamiltonian flows of \(H\). The set of critical values of \(A_H\) is called the action spectrum and is denoted by \(\text{Spec}(H)\).

We briefly explain some notions of indices used later to construct Floer homology. The Maslov index

\[
\mu : \pi_1(Sp(2n)) \to \mathbb{Z}
\]

maps a loop of symplectic matrices to an integer. For a capped periodic orbit of a loop of Hamiltonian diffeomorphisms \(\psi \in \pi_1(\text{Ham}(M, \omega))\) denoted by \([\psi^t(x), w]\), we define its Maslov index \(\mu([\psi^t(x), w])\) via the trivialization of \(w^*TM\) and the loop of symplectic matrices \(d\psi^t(x) : T_xM \to T_{\psi^t(x)}M\).

The definition of Maslov indices cannot be directly applied to periodic orbits of a Hamiltonian \(H\) since given a periodic orbit \([\phi^t_H(x), w]\), \(d\phi^t_H(x) : T_xM \to T_{\phi^t(x)}M\) might not define a loop. To overcome this difficulty, Conley-Zehnder modified the definition of the Maslov index and introduced the Conley-Zehnder index

\[
\mu_{\text{CZ}} : \{A : [0, 1] \to Sp(2n) | A(1) : \text{non-degenerate}\} \to \mathbb{Z}
\]

which maps paths of symplectic matrices to integers. Thus, as in the case of Maslov indices, we define the Conley-Zehnder index of a non-degenerate periodic orbit of a Hamiltonian \(H [\phi^t_H(x), w]\) (i.e. \(d\phi_H(x) : T_xM \to T_{\phi(x)}M\) is non-degenerate), denoted by \(\mu_{\text{CZ}}([\phi^t_H(x), w])\), via the trivialization of \(w^*TM\) and the path of symplectic matrices \(d\phi^t_H(x) : T_xM \to T_{\phi^t(x)}M\).

The following elementary properties are often used to calculate the action.

**Proposition 20.** Let \((M, \omega)\) be a symplectic manifold. Assume the Hamiltonian paths generated by \(H\) and \(G\) are homotopic rel. end points

i.e. \(\exists u : [0, 1] \times [0, 1] \to \text{Ham}(M, \omega)\) such that
Proposition 22. Let $x \in \text{Fix}(\phi_H) = \text{Fix}(\phi_G)$ and $[\phi_H^t(x), w]$ be a capped orbit of $H$. Then the action of the capped orbit $[\phi_G^t(x), w']$ where $w' := w \# u$ coincides with the action of $[\phi_H^t(x), w]$:

$$\mathcal{A}_H([\phi_H^t(x), w]) = \mathcal{A}_G([\phi_G^t(x), w'])$$

Proposition 21. Let $(M, \omega)$ be a symplectic manifold.

1. For any Hamiltonian $H \in C^\infty(S^1 \times M, \mathbb{R})$,

$$\overline{\Pi}(t, x) := -H(t, \phi_H^t(x))$$

generates a time-1 map $\phi_H^{-1}$.

2. For any Hamiltonian $H \in C^\infty(S^1 \times M, \mathbb{R})$,

$$\overline{H}(t, x) := -H(-t, x)$$

equivalently generates a time-1 map $\phi_H^{-1}$.

3. $\overline{H}$ and $\overline{H}$ generates Hamiltonian paths that are homotopic rel. end points.

Proposition 22. Let $(M, \omega)$ be a symplectic manifold.

1. For any Hamiltonians $H, G \in C^\infty(S^1 \times M, \mathbb{R})$,

$$H \# G(t, x) := H(t, x) + G(t, (\phi_H^t)^{-1}(x))$$

generates a time-1 map $\phi_H \circ \phi_G$.

2. For any Hamiltonians $H, G \in C^\infty(S^1 \times M, \mathbb{R})$,

$$H \wedge G(t, x) := \begin{cases} G(2t, x) & (0 \leq t \leq 1/2) \\ H(2t - 1, \phi_G(x)) & (1/2 \leq t \leq 1) \end{cases}$$

also generates a time-1 map $\phi_H \circ \phi_G$.

3. $H \# G$ and $H \wedge G$ generates Hamiltonian paths that are homotopic rel. end points.

The following two propositions will be used in Section 3.1.

Proposition 23. Let $(M, \omega)$ be a symplectic manifold, $U$ a simply connected non-empty open set and $H$ a Hamiltonian such that $\phi_H(p) = p$ for all $p \in U$. Take any $x_0 \in U$ and a capping $w_0 : D^2 \to M$ of the orbit $\phi_H^1(x_0)$ and fix them.

For any $x \in U$, define a capping $w_x : D^2 \to M$ of the orbit $\phi_H^t(x)$ by

$$w_x(s e^{2\pi i t}) := \phi_H^t(c(s)) \# w_0$$

where $c : [0, 1] \to M$ is a smooth path from $x_0$ to $x$ and $\phi_H^t(c(s)) \# w_0$ denotes the gluing of $\phi_H^t(c(s))$ and $w_0$ along $\phi_H^t(x_0)$. Then we have the following:

1. $\mathcal{A}_H([\phi_H^t(x), w_x]) = \mathcal{A}_H([\phi_H^t(x_0), w_0])$.

2. $\mu([\phi_H^t(x), w_x]) = \mu([\phi_H^t(x_0), w_0])$. 
Proof. (1) It follows from \( \frac{d}{dt} A_H([\phi^t_H(c(s)), \phi^t_H(c(s)) \# w_0]) = 0 \).

(2) We can extend smoothly the trivialization of \( w_0^*TM \) to \( w_*^*TM \) and this extension is unique up to homotopy since \( U \) is simply connected.

\( \square \)

Proposition 24. Let \((M, \omega)\) be a symplectic manifold, \(H\) a Hamiltonian and \([\phi^t_H(x), w]\) any capped 1-periodic orbit of \(H\). Then

(1) \( \overline{w} : D^2 \to M, \overline{w}(se^{2\pi i t}) := w(se^{2\pi i(-t)}) \) is a capping of the orbit \( \phi^t_H(x) \)

(2) \( \mu([\phi^t_H(x), w]) = -\mu([\phi^t_H(x), \overline{w}]) \)

(3) \( A_H([\phi^t_H(x), w]) = -A_H([\phi^t_H(x), \overline{w}]) \) where \( \tilde{H}(t, x) := -H(-t, x) \).

Proof. (Proposition 24)

(1) We change the coordinate according to the direction of the orbit.

(2) The disks \( w \) and \( \overline{w} \) are geometrically equivalent with opposite orientation \((\overline{w}(s, t) = w(s, -t)).\) Thus, if \( \{Z_1, Z_2, \cdots, Z_{2n}\} \) gives a symplectic basis of \( w^*TM \), then a symplectic basis of \( \overline{w}^*TM \) is given by \( \{\overline{Z}_1, \overline{Z}_2, \cdots, \overline{Z}_{2n}\} \) where \( \overline{Z}_i(s, t) := Z_i(s, -t). \) Since \( \phi^t_H = \phi^{\overline{t}}_H \), the Maslov index of symplectic paths of the orbits \( \phi^t_H(x) \) and \( \phi^{\overline{t}}_H(x) \) have opposite signs.

(3) \( A_H([\tilde{H}(\phi^t_H(x), \overline{w})]) = \int_0^1 \tilde{H}(t, \tilde{H}(\phi^t_H(x)) dt - \int_{D^2} \overline{w}^* \omega \\
= -\int_0^1 H(t, \phi^t_H(x)) ds - (-\int_{D^2} w^* \omega) = -A_H([\phi^t_H(x), w]). \)

\( \square \)

2.1. Hamiltonian Floer theory. We fix a ground field \( \mathbb{F} \) throughout this section. We say that a Hamiltonian \( H \) is non-degenerate if \( \Delta \cap \Gamma_\phi \) where

\[ \Gamma_\phi := \{(x, \phi(x)) \in M \times M\} \]

and \( \Delta \) denotes the diagonal set. The Floer chain complex of (non-degenerate) \( H, CF_*(H) \), is the vector space spanned by \( \text{Crit}(A_H) \) over the ground field \( \mathbb{F} \) and graded by the Conley-Zehnder index \( \mu_{CZ} \). The boundary map counts certain solutions of a perturbed Cauchy-Riemann equation for a chosen \( \omega \)-compatible almost complex structure \( J \) on \( TM \), which can be viewed as isolated negative gradient flow lines of \( A_H \). This gives us a chain complex \((CF_*(H), \partial)\) called the Floer chain complex. Its homology is called the Floer homology of \((H, J)\) and is denoted by \( HF_*(H, J) \). Often it is abbreviated to \( HE_*(H) \) as Floer homology does not depend on the choice of an almost complex structure.

Recapping of a capped orbit changes the action and the Conley-Zehnder index as follows:

- \( A_H([z, w \# A]) = A_H([z, w]) - \omega(A) \).
- \( \mu_{CZ}([z, w \# A]) = \mu_{CZ}([z, w]) - 2c_1(A) \).
We define the filtered Floer complex of $H$ by
\[ CF^\tau_*(H) := \{ \sum a_z z : a_z \in CF_*(H) : A_H(z) < \tau \}. \]
Since the Floer boundary map decreases the action, $(CF^\tau_*(H), \partial)$ forms a chain complex. The filtered Floer homology of $H$ which is denoted by $HF^\tau_*(H)$ is the homology defined by the chain complex $(CF^\tau_*(H), \partial)$.

It is useful to clarify our convention of the Conley-Zehnder index since conventions change according to literature. We fix our convention as follows: let $f$ denote a $C^2$-small Morse function. For every critical point $x$ of $f$, we require that
\[ \mu_{CZ}([x, w_x]) = i(x) \]
where $i$ denotes the Morse index and $w_x$ is the trivial capping.

2.2. Quantum homology and Seidel representation. We sketch some basic definitions and properties concerning the quantum homology. Throughout this section, we fix a ground field $\mathbb{F}$.

Let $(M, \omega)$ be a closed symplectic manifold. Define
\[ \Gamma := \pi_2(M) / (\ker(\omega) \cap \ker(c_1)). \]
The Novikov ring $\Lambda_\omega$ is defined by
\[ \Lambda_\omega := \{ \sum a_A \otimes e^A : a_A \in \mathbb{F}, \forall \tau \in \mathbb{R}, \#\{a_A \neq 0, \omega(A) < \tau\} < \infty \}. \]

The quantum homology of $(M, \omega)$ is defined by
\[ QH_*(M) := H_*(M) \otimes_{\mathbb{F}} \Lambda_\omega. \]
The quantum homology has a ring structure with respect to the quantum product denoted by $\ast$. It is defined as follows:
\[ \forall a, b, c \in H_*(M), \ (a \ast b) \circ c := \sum_{A \in \Gamma} GW_{3,A}(a, b, c) \otimes e^A \]
where $\circ$ denotes the intersection product and $GW_{3,A}$ denotes the 3-pointed Gromov-Witten invariant in the class $A$. See [MS2] for details.

Remark 25. Assume $(M, \omega)$ either monotone, negative monotone or rational and $c_1|_{\pi_2(M)} = 0$.

In these cases, we have $\Gamma \simeq \mathbb{Z}$ with a generator $A$ such that
\[ \omega(A) = \lambda_0, \ \langle \omega, \pi_2(M) \rangle = \lambda_0 \mathbb{Z}, \]
\[ c_1(A) = \pm N, \ \langle c_1, \pi_2(M) \rangle = N \mathbb{Z} \ (if \ N < \infty). \]

Furthermore, the Novikov ring is the ring of formal Laurent series $\mathbb{F}[[q]]$ where $q := e^{-A}$. Thus any element $a \in QH_*(M)$ can be written in the following form:
\[ a = \sum_{k \in \mathbb{Z}} a_k \cdot q^k, \ a_k \in H_*(M) \]
where $a_k = 0$ for sufficiently large $k$. 
The quantum product can also be expressed in a simple manner.

\[ \forall a, b \in H_\ast(M), \quad a \ast b = a \cap b + \sum_{k>0} (a \ast b)_k \cdot q^{-k}. \]

The series on the right hand side runs over only non-positive powers since the elements of \( \Gamma \) appearing in the sum represents pseudo-holomorphic spheres and pseudo-holomorphic spheres has non-negative \( \omega \)-area (remember that \( \omega(q) = -\lambda_0 \)).

One should be careful that depending on whether \( (M, \omega) \) is monotone or negative monotone, the effect of \( q \) to the degree changes since \( q \) represents a pseudo-holomorphic sphere such that \( \omega(q) = -\lambda_0 \). Precisely, when \( (M, \omega) \) is monotone we have \( c_1(q) = -N \) and when \( (M, \omega) \) is negative monotone we have \( c_1(q) = +N \).

**Example.** The quantum homology group of \( (\mathbb{C}P^n, \omega_{FS}) \) is expressed as follows:

\[
QH_\ast(\mathbb{C}P^n) = \mathbb{F}[q][u] / \langle u^{(n+1)} = [\mathbb{C}P^n \cdot q^{-1}] \rangle
\]

where \( u \in H_{2n-2}(\mathbb{C}P^n) \) denotes the projective hyperplane class, \( q \) denotes the generator of the Novikov ring (see the remark above) and \( u^{(n+1)} := u \ast u \ast \cdots \ast u \ (n+1\text{-times}) \).

There is a canonical isomorphism called the PSS-isomorphism between Floer homology and quantum homology which will be denoted by \( \Phi \):

\[
\Phi : QH_\ast(M) \sim \rightarrow HF_\ast(H).
\]

PSS-isomorphism preserves the ring structure:

\[
\Phi(a) \ast_{pp} \Phi(b) = \Phi(a \ast b), \quad a, b \in QH_\ast(M)
\]

where \( \ast_{pp} \) denotes the pair-of-pants product.

Next, we briefly explain the Seidel representation \( S \) where the idea goes back to Seidel [Sei]. Let \( H \) be a non-degenerate Hamiltonian. Given a loop of Hamiltonian diffeomorphism \( \psi \in \pi_1(Ham(M, \omega)) \), we have the following bijection between loops in \( M \):

\[
z(t) \mapsto (\psi^t)^{-1}(z(t)).
\]

Recall that the generators of the Floer chain complex \( CF_\ast(H) \) were capped orbits of \( H \) and that given a loop \( \psi \in \pi_1(Ham(M, \omega)) \), one can construct a Hamiltonian fibration over \( S^2 \) with fiber \( (M, \omega) : \tilde{M}_\psi \rightarrow S^2 \).

In fact, with an arbitrary choice of a (pseudo-holomorphic) section \( \sigma \) of the Hamiltonian fiber bundle \( \tilde{M}_\psi \rightarrow S^2 \), one can lift the bijection to an isomorphism of capped orbits:

\[
S_{\psi, \sigma} : CF_\ast(H) \rightarrow CF_\ast(\psi^*H)
\]

\[
[z, w] \mapsto (\psi, \sigma)^*[z, w].
\]

where \( \psi^*H \) is the Hamiltonian generating \( (\psi^t)^{-1} \circ \phi_H^t(x) \). See [MS2] Section 12.5. for details.
This induces an isomorphism between Floer homologies:

\[ S_{\psi, \sigma} : HF_*(H) \to HF_*(\psi^*H) \]

and one can consider it as an isomorphism between quantum homologies via PSS-isomorphism:

\[ S_{\psi, \sigma} : QH_*(M) \to QH_*(M). \]

This isomorphism has a following simple expression:

For \( S_{\psi, \sigma} \), there exists a quantum homology class \( a \) such that

\[ \forall b \in QH_*(M), \quad S_{\psi, \sigma}(b) = a \ast b. \]

Thus, we often identify \( S_{\psi, \sigma} \) and this quantum homology class \( a \) i.e. we see \( S_{\psi, \sigma} \) as a quantum homology class. In this text, we persist on this identification.

Seidel representation satisfies the following property:

For \( \psi_1, \psi_2 \in \pi_1(\text{Ham}(M, \omega)) \) and sections \( \sigma_1, \sigma_2 \),

\[ S_{\psi_2, \sigma_2} \ast S_{\psi_1, \sigma_1} = S_{(\psi_2, \sigma_2)\#(\psi_1, \sigma_1)}. \]

2.3. **Spectral invariants.** Let \( i^\tau : CF^*_\tau(H) \to CF_\tau(H) \) be the natural inclusion map and denote by \( i^*_\tau : HF^*_\tau(H) \to HF_\tau(H) \) the induced map on homology.

For a quantum homology class \( a \in QH_*(M) \), define the spectral invariant by

\[ \rho(H, a) := \inf \{ \tau \in \mathbb{R} : \Phi(a) \in \text{Im}(i^\tau_\tau) \}. \]

The concept of spectral invariants was first introduced by Viterbo for \( \mathbb{R}^{2n} \) [Vit] and later by Schwarz for aspherical symplectic manifolds [Sch] and Oh for closed symplectic manifolds [Oh].

We list some basic properties of spectral invariants.

**Proposition 26.** Spectral invariants satisfy the following properties where \( H, G \) are Hamiltonians:

1. For any \( a \in QH_*(M) \),

\[ |\rho(H, a) - \rho(G, a)| \leq E(H \# G) \]

where

\[ E(H) := \int_{t \in [0, 1]} \{ \sup_x H_t(x) - \inf_x H_t(x) \} dt. \]

2. For any \( a \in QH_*(M) \),

\[ \rho(H, a) \in \text{Spec}(H). \]

3. For any \( a, b \in QH_*(M) \),

\[ \rho(H \# G, a \ast b) \leq \rho(H, a) + \rho(G, b). \]

4. Let \( U \) be a non-empty subset of \( M \).

\[ \rho(H, [M]) \leq e(\text{Supp}(H)) := \inf \{ E(G) : \phi_G(\text{Supp}(H)) \cap \text{Supp}(H) = \emptyset \}. \]
Let $f : M \to \mathbb{R}$ be an autonomous Hamiltonian. For a sufficiently small $\varepsilon > 0$, we have

$$\rho(\varepsilon f, a) = \rho_{LS}(\varepsilon f, a) = \varepsilon \cdot \rho_{LS}(f, a)$$

where $\rho_{LS}(f, a)$ is the topological quantity defined by

$$\rho_{LS}(f, a) := \inf \{ \tau : a \in \text{Im}(H_\ast\{ f \leq \tau \}) \to H_\ast(M) \}.$$ 

For $\psi \in \pi_1(\text{Ham}(M, \omega))$, a section $\sigma$ of the Hamiltonian fibration $\tilde{M}_\psi \to S^2$, and $a \in QH_\ast(M)$ we have

$$\rho(\psi^*H, a) = \rho(H, S_{\psi, \sigma} * a) + c(\sigma)$$

where

$$(\psi^*H)_t := (H_t - K_t) \circ \psi^t, \quad \psi^t := \phi^t_K, \quad \phi^t_{\psi^*H} = (\psi^t)^{-1} \circ \phi^t_H$$

and $c(\sigma)$ denotes a constant depending only on $\sigma$.

**Remark 27.** For a set $A$, $e(A) := \inf \{ E(G) : \phi_G(A) \cap A = \emptyset \}$ is called the displacement energy.

The spectral norm of $H$ is defined by

$$\gamma(H) := \rho(H, [M]) + \rho(\overline{M}, [M])$$

where $[M]$ denotes the fundamental class.

When the symplectic manifold is symplectically aspherical or monotone, we have the equality $\rho(\overline{M}, [M]) = -\rho(H, [pt])$ (see Lemma 2.2 in [EP1]), thus in these cases we can also describe spectral norms as follows:

$$\gamma(H) := \rho(H, [M]) - \rho(H, [pt]).$$

We also define a spectral norm of a Hamiltonian diffeomorphism $\phi$ by

$$\gamma(\phi) := \inf_{\phi_H = \phi} \gamma(H).$$

We define the $C^0$-distance of Hamiltonian diffeomorphisms by

$$d_{C^0}(\phi, \psi) := \max_{x \in M} d(\phi(x), \psi(x))$$

where $d$ denotes the distance on $M$ induced by the Riemannian metric on $M$. Note that the topology induced by the $C^0$-distance is independent of the choice of the Riemannian metric.

Whether if spectral norms are continuous with respect to the $C^0$-topology on $\text{Ham}(M, \omega)$ caught interest of a lot of symplectic geometers. First progress in this question was made by Viterbo [Vit] where he proved their $C^0$-continuity in the case of $\mathbb{R}^{2n}$. Later, Seyfaddini proved the case of closed surfaces [Sey1]. Recently Buhovsky-Humilière-Seyfaddini gave an affirmative answer to the case of aspherical symplectic manifolds [BHS2] and Shelukhin for $\mathbb{C}P^n$ [Sh].
### 2.4. Barcodes

A finite barcode is a finite set of intervals $B = \{I_j\}_{1 \leq j \leq N}$. The bottleneck distance between two finite barcodes $B$ and $B'$. Let us briefly recall the definition.

Two finite barcodes $B, B'$ are said to be $\delta$-matched if, after deleting some intervals of length less than $2\delta$, there exists a bijective matching between the intervals of $B$ and $B'$ such that the endpoints of the matched intervals are less than $\delta$ of each other. Then the define the bottleneck distance between two finite barcodes as follows.

$$d_{\text{bot}}(B, B') := \inf \{\delta > 0 : \exists \text{ matching between } B \text{ and } B'\}.$$  

Since the space of all finite barcodes equipped with the bottleneck distance is not a complete metric space, we will need to allow certain non-finite barcodes to achieve its completeness. We define a barcode $B = \{I_j\}_{j \in \mathbb{N}}$ to be a collection of intervals such that for any $\delta > 0$ only finitely many of the intervals $I_j$ are of length greater than $\delta$. We will denote this class of barcodes by $\text{Barcodes}$. Observe that the bottleneck distance extends to $\text{Barcodes}$.

The space $(\text{Barcodes}, d_{\text{bot}})$ is indeed the completion of the space of finite barcodes.

Given a barcode $B = \{I_j\}_{j \in \mathbb{N}}$ and $c \in \mathbb{R}$, define $B + c = \{I_j + c\}_{j \in \mathbb{N}}$, where $I_j + c$ is the interval obtained by adding $c$ to the endpoints of $I_j$. Define an equivalence relation $\sim$ by $B \sim B'$ if $B' = B + c$ for some $c \in \mathbb{R}$. We will denote the quotient space of $\text{Barcodes}$ with the relation $\sim$ by $\hat{\text{Barcodes}}$.

Barcodes of Hamiltonian diffeomorphisms were first defined in [PS] on symplectic manifolds that are symplectically aspherical. [PSS] considered barcodes on monotone symplectic manifolds by fixing a degree to achieve "finiteness". Later, Le Roux-Seyfaddini-Viterbo extended the notion of barcodes to allow them to have infinitely many intervals in order to define barcodes of Hamiltonian diffeomorphisms on spheres in [LSV] and their method easily extends to (negative) monotone symplectic manifolds. We explain briefly how to map a Hamiltonian diffeomorphism on (negative) monotone symplectic manifolds to a barcode following [LSV]: the readers are invited to [LSV] for a more detailed explanation. See also [PSS].

Given a Hamiltonian $H$ and an integer $k \in \mathbb{Z}$, its filtered $k$-th Floer homology groups $\{HF^\tau_k(H)\}_{\tau \in \mathbb{R}}$ forms a persistence module. Note that, if we do not fix a degree, the filtered Floer homology group will not be a persistence module since the existence of pseudo-holomorphic spheres will prohibit us to have the "finiteness condition". One can correspond a persistence module to a barcode in a canonical way and we denote the barcode obtained by the filtered $k$-th Floer homology groups by $B_k(H)$ and we define

$$B(H) := \sqcup_k B_k(H).$$

Note that $B(H)$ is not exactly a standard barcode since it contains infinitely many bars. Nevertheless, the bottleneck distance extends to this class of barcodes.
For two Hamiltonians $H, G$ such that $\phi_H = \phi_G$, their Floer homologies coincide up to shifts of indices and action filtrations i.e. $HF^\tau_+(H) \cong HF^{r+\tau_0}_+(G)$ for some $k_0 \in \mathbb{Z}, \tau_0 \in \mathbb{R})$. Thus their barcodes might be different i.e. $B(H) \neq B(G) \in \text{Barcodes}$ but their quotient barcodes coincide: $B(H) = B(G) \in \text{Barcodes}$. Therefore, we define the barcode of $\phi$ as follows.

$$B(\phi) := B(H) \in \text{Barcodes}$$

for any $H$ such that $\phi_H = \phi$.

Hence, we obtain a map

$$B : \text{Ham}(M, \omega) \to \hat{\text{Barcodes}}.$$ 

Kislev-Shelukhin [KS] proved the following inequality to estimate the bottleneck distance between barcodes of $\phi, \psi \in \text{Ham}(M, \omega)$:

$$d_{\text{bot}}(B(\phi), B(\psi)) \leq \frac{1}{2} \gamma(\phi^{-1} \psi).$$

This implies that once we obtain the $C^0$-continuity of $\gamma$, the map

$$B : (\text{Ham}(M, \omega), d_{C^0}) \to (\hat{\text{Barcodes}}, d_{\text{bot}})$$

is continuous.

### 3. Proof of the main results

In this section, we prove the results claimed in the introduction. We start from the case of negative monotone symplectic manifolds since the proof is based on a similar idea to the case of rational symplectic manifolds but it is simpler.

#### 3.1. The case of negative monotone symplectic manifolds

We prove Theorem [9]. It is achieved by combing the following two propositions.

The first proposition considers Hamiltonian diffeomorphisms that does not move any point on a given open set.

**Proposition 28.** Let $(M, \omega)$ be a negative monotone symplectic manifold and $U$ an arbitrary simply connected non-empty open set in $M$.

For any $\varepsilon > 0$, there exists $\delta > 0$ such that if $H \in C^\infty(S^1 \times M, \mathbb{R})$, $d_{C^0}(id, \phi_H) < \delta$ and $\phi_H(x) = x$ for all $x \in U$, then $\gamma(H) < \varepsilon$.

The second proposition, proven by Buhovsky-Humilière-Seyfaddini [BHS2], claims that given a Hamiltonian diffeomorphism $\phi$, one can always deform the Hamiltonian diffeomorphism $\phi \times \phi^{-1}$ to a Hamiltonian diffeomorphism that does not move any point on a certain open set by composing with a both $C^0$- and $\gamma$-small Hamiltonian diffeomorphism.

**Proposition 29.** ([BHS2]) Let $(M, \omega)$ be any closed symplectic manifold. For any $\varepsilon > 0$, there exists a non-empty open ball $B$ satisfying the following properties:

For any $\varepsilon' > 0$, there exists $\delta' > 0$ such that if $\phi \in \text{Ham}(M, \omega), d_{C^0}(id_M, \phi) < \delta'$, then there exist $G \in C^\infty(S^1 \times M \times M, \mathbb{R})$ such that
\( (1) \gamma(G) < \varepsilon \)
\( (2) d_{C^0}(id_{M \times M}, \phi_G) < \varepsilon' \)
\( (3) (|\phi \times \phi^{-1}| \circ \phi_G|_{B \times B} = id_{B \times B} \)

Let us postpone the proof of these propositions and see how Theorem 9 is obtained.

Proof. (Theorem 9)

Given an \( \varepsilon > 0 \), we can take a ball \( B \) in \( M \times M \) as in Proposition 29. For this ball \( B \), we apply Proposition 28 and obtain \( \delta > 0 \) as in the proposition. Note that if \( (M, \omega) \) is negative monotone, then so is \( (M \times M, \omega \oplus \omega) \). Now, by Proposition 29, there exists \( \delta' > 0 \) such that if \( d_{C^0}(id_{M}, \phi) < \delta' \), then there exist \( G \in C^\infty(S^1 \times M \times M, \mathbb{R}) \) such that

\[
\begin{align*}
(1) & \gamma(G) < \varepsilon \\
(2) & d_{C^0}(id_{M \times M}, \phi_G) < \delta \\
(3) & (|\phi \times \phi^{-1}| \circ \phi_G|_{B \times B} = id_{B \times B} )
\end{align*}
\]

Take any Hamiltonian \( H \) generating \( \phi \): \( \phi_H = \phi \). \((H \oplus \overline{H}) \#G \) generates \( (\phi \times \phi^{-1}) \circ \phi_G \) so by Proposition 28 we have

\[
\gamma((H \oplus \overline{H}) \#G) < \varepsilon.
\]

Therefore,

\[
\gamma(H \oplus \overline{H}) \leq \gamma((H \oplus \overline{H}) \#G) + \gamma(G) = \gamma(H \oplus \overline{H}) \#G + \gamma(G) < \varepsilon + \varepsilon = 2\varepsilon.
\]

\( \gamma(H \oplus \overline{H}) = 2\gamma(H) \) (by Theorem 5.1. in [EP2]), so \( \gamma(H) < \varepsilon \). This proves the theorem. \( \square \)

We now prove Proposition 28. We invite the readers to [BHS2] for the proof of Proposition 29.

Proof. (Proposition 28)

Take a Morse function \( f : M \to \mathbb{R} \) whose critical points are located in \( U \). We assume that \( f \) is \( C^2 \)-small enough so that its Hamiltonian flow does not admit any non-trivial periodic points and that \( osc(f) \) := \( \max f - \min f < \varepsilon \).

Since \( \phi_f \) has no fixed points in \( M \setminus U \), there exists \( \delta > 0 \) such that

\[ \forall x \in M \setminus U, \ d(x, \phi_f(x)) > \delta. \]

We then take \( \phi_H, C^0 \)-close enough to the Id so that

\[ \text{Crit}(f) = \text{Fix}(\phi_f \circ \phi_H). \]

In fact, \( \text{Crit}(f) \subset \text{Fix}(\phi_H \circ \phi_f) \) is obvious. We see that if \( \phi_H, C^0 \)-close enough to \( Id \), then \( \text{Crit}(f) = \text{Fix}(\phi_f \circ \phi_H) \). Let \( x \in \text{Fix}(\phi_f \circ \phi_H) \).

(1) Assume \( x \in U \). Then, \( \phi_f(x) = \phi_f \circ \phi_H(x) = x \) and since \( \phi_f \) has only trivial fixed points, \( x \in \text{Crit}(f) \).
(2) Assume \( x \in M \setminus U \). Then,
\[
d_{C^0}(x, \phi_f \circ \phi_H(x)) \geq d_{C^0}(\phi_H(x), \phi_f \circ \phi_H(x)) - d_{C^0}(\phi_H(x), \phi_f \circ \phi_H(x), x) = \delta - d_{C^0}(id, \phi_H).
\]

We take \( \phi_H \in C^0 \)-close enough to \( Id \) to make the last equation positive.

Then, \( x \notin \operatorname{Fix}(\phi_f \circ \phi_H) \). Thus \( x \in \operatorname{Fix}(\phi_f \circ \phi_H) \) implies \( x \in U \) and \( x = \phi_f \circ \phi_H(x) = \phi_f(x) \). Thus \( x \in \operatorname{Crit}(f) \).

Thus, for any \( x \in \operatorname{Crit}(f) = \operatorname{Fix}(\phi_f \circ \phi_H) \), its orbit is \( \phi_H^t(x) = \phi_H^t(x) \) and thus,
\[
\operatorname{Spec}(H \# f) = \operatorname{Spec}(H) + \{ f(x) : x \in \operatorname{Crit}(f) \}
\]
\[
\{ f(x) + A_H([\phi^t_H(x), w]) : x \in \operatorname{Crit}(f), [\phi^t_H(x), w] \in \operatorname{Crit}(A_H) \}.
\]

Take any \( x_0 \in \operatorname{Crit}(f) \) and a capping \( w_0 : D^2 \to M \) of the orbit \( \phi^t_H(x_0) \) i.e. \( w_0(e^{2\pi it}) = \phi^t_H(x_0) \). We fix this capped orbit \( [\phi^t_H(x_0), w_0] \) in the sequel.

For any \( x \in \operatorname{Crit}(f) \), define a capping \( w_x : D^2 \to M \) of the orbit \( \phi^t_H(x) \) by
\[
w_x(se^{2\pi it}) := \phi^t_H(c(s))\#w_0
\]
where \( c : [0, 1] \to U \) is a smooth path from \( x_0 \) to \( x \) and \( \phi^t_H(c(s))\#w_0 \) denotes the gluing of \( \phi^t_H(c(s)) \) and \( w_0 \) along \( \phi^t_H(x_0) \).

Recall that \( \gamma(H) = \rho(H, [M]) + \rho(\overline{H}, [M]) \) and we will estimate \( \rho(H, [M]) \) and \( \rho(\overline{H}, [M]) \) separately.

By the triangle inequality,
\[
\rho(H, [M]) \leq \rho(H \# f, [M]) + \rho(\overline{f}, [M]).
\]
For the first term we know that
\[
\rho(\overline{f}, [M]) = \rho(-f, [M]) \leq \epsilon
\]
since \( \text{osc}(f) < \epsilon \).

For the second term,
\[
\rho(H \# f, \cdot) \in \operatorname{Spec}(H \# f)
\]
so there exists a point \( x \in \operatorname{Crit}(f) \) and a sphere \( A : S^2 \to M \) such that
\begin{itemize}
  \item \( A_{H \# f}([\phi^t_H(x), w_x \# A]) = \rho(H \# f, [M]) \).
  \item \( \mu_{CZ}([\phi^t_H(x), w_x \# A]) = \deg([M]) = 2n. \)
\end{itemize}
Be careful that the sphere \( A \) plays the role to correct the capping of the capped orbit \( [\phi^t_H(x), w_x] \) to achieve the appropriate capped orbit which realizes the spectral invariant \( \rho(H \# f, [M]) \).

The action and the index can be rewritten in the following way where \( i \) denotes the Morse index:
\begin{itemize}
  \item \( A_{H \# f}([\phi^t_H(x), w_x \# A]) = f(x) + A_H([\phi^t_H(x), w_x]) - \omega(A). \)
  \item \( \mu_{CZ}([\phi^t_H(x), w_x \# A]) = i(x) + 2\mu([\phi^t_H(x), w_x]) - 2c_1(A). \)
\end{itemize}
Thus we get the following two equations.
\begin{itemize}
  \item \( \rho(H \# f, [M]) = f(x) + A_H([\phi^t_H(x), w_x]) - \omega(A). \)
  \item \( 2n = i(x) + 2\mu([\phi^t_H(x), w_x]) - 2c_1(A). \)
\end{itemize}
In the same way, there exist a point \( y \in \text{Crit}(f) \) and a sphere \( B : S^2 \to M \) such that

- \( \rho(\partial f, [M]) = f(y) + \mathcal{A}([\varphi_H(x), \overline{w_x}]) - \omega(B) \).
- \( 2n = i(y) + 2\mu([\varphi_H(y), \overline{w_y}]) - 2c_1(B) \).

Here, the capping \( \overline{w_y} \) is \( \overline{w_y}(se^{2\pi it}) := w_y(se^{2\pi i(-t)}) \).

Thus, by adding the two equations of the action, we obtain the following.

\[
\gamma(H) = 2\rho(-f, [M]) + \rho(H f, [M]) + \rho(\overline{H} \# f, [M]) = 2\rho(-f, [M]) + f(y) + \mathcal{A}(\varphi_H(x), w_x) + \mathcal{A}(\varphi_H(y), \overline{w_y}) - \omega(A + B).
\]

Also, by adding the two equalities of the index, we obtain the following.

\[
4n = i(x) + i(y) + 2\mu([\varphi_H^t(x), w_x]) + 2\mu([\varphi_H^t(y), \overline{w_y}]) - 2c_1(A + B).
\]

By Proposition 23 and 24 stated earlier, the two terms of action (resp. index) cancel each other in the first (resp. second) equality. Precisely, the first equality becomes

\[
\gamma(H) \leq 4\epsilon - \omega(A + B),
\]

and the second equality becomes

\[
4n = i(x) + i(y) - 2c_1(A + B).
\]

Now, since \( i(x), i(y) \) are Morse indices, we have

\[
0 \leq i(x), i(y) \leq 2n = \dim(M)
\]

and thus,

\[
0 \leq 4n + 2c_1(A + B) \leq 4n.
\]

Thus,

\[
-2n \leq c_1(A + B) \leq 0.
\]

In particular, by the negative monotonicity of \((M, \omega)\), we have

\[
\omega(A + B) = \lambda \cdot c_1(A + B) \geq 0.
\]

Therefore,

\[
\gamma(H) \leq 4\epsilon.
\]

This completes the proof. \(\square\)
3.2. The case of rational symplectic manifolds. We prove Theorem 3.

The idea of the proof is similar to the case of negative monotone symplectic manifolds. The main technical differences to prove Theorem 3 are

1. Here we compose an appropriate symplectomorphism to a Hamiltonian (i.e. take a conjugation of a Hamiltonian diffeomorphism) so that the "extra term" vanishes.
2. Here we estimate $\rho(\mathcal{H} \oplus H, \cdot)$ while in the negative monotone case we estimated $\gamma(\mathcal{H} \oplus H, \cdot)$.

We will need the following proposition to find an appropriate symplectomorphism to compose to the Hamiltonian as explained above.

**Proposition 30.** Let $(M, \omega)$ be a closed symplectic manifold. Fix an arbitrary point $x_0 \in M$. There exists a constant $C > 0$ satisfying the following property:

For any point $x \in M$, there exists a symplectomorphism $\psi$ such that

1. $\psi(x) = x_0$
2. $\|d\psi\| \leq C$

**Proof.** (Proposition 30)

First of all, take a Darboux chart $U_0$ (an open subset of $M$) centered at $x_0$:

$$f_0 : U_0 \cong B(2r) \subset \mathbb{R}^{2n}, f_0(x_0) = 0.$$

For any $p \in M$, there exists a Hamiltonian diffeomorphism $\phi_p$ such that $\phi_p(x_0) = p$. Since $M = \bigcup_{p \in M} \phi_p(U'_0)$ where $U'_0 := f_0^{-1}(B(r))$, by the compactness of $M$, there exists a finite subcovering: $\exists \{p_j \in M\}_{1 \leq j \leq m}$ such that $M = \bigcup_{1 \leq j \leq m} \phi_{p_j}(U'_0)$. Denote $U_j := \phi_{p_j}(U_0)$, $U'_j := \phi_{p_j}(U'_0)$ and $\phi_j := \phi_{p_j}$.

On $B(2r) \subset \mathbb{R}^{2n}$ for any point $p \in B(r)$, we can take a Hamiltonian $A_p$ having its support in $B(2r)$ such that $\phi_{A_p}(p) = 0$. (Consider a Hamiltonian that generates a Hamiltonian diffeomorphism that is a linear translation on $B(r)$.) Furthermore, it is possible to choose them so that $\|dA_p\|$ is bounded by a constant $C' > 0$ depending only on the radius $r$ and independent of the point $p \in B(r)$.

Now, for any point $x \in M$, there exists $U'_j$ such that $x \in U'_j$. For $\phi_j^{-1}(x) \in U'_0 \subset U_0$, consider the Hamiltonian $A := A_x$ as explained above by identifying $U_0$ and $B(2r)$ and let

$$\psi := \phi_{A \circ f_0} \circ \phi_j^{-1}.$$

Note that $A$ was a smooth function defined on $B(2r)$ with a compact support. Thus, $A \circ f_0$ is supported in $U_0$ so we can see $A \circ f_0$ as a smooth function defined on $M$ by setting $(A \circ f_0)|_{M \setminus U_0} = 0$. Hence, $\phi_{A \circ f_0}$ is a Hamiltonian diffeomorphism.

$\psi$ satisfies

$$\psi(x) = x_0,$$

$$\|d\psi\| \leq \|dA\| \cdot \max_{1 \leq j \leq m} \|d\phi_j\|^{-1}.$$
Now we are ready to prove Theorem 3.

Proof. (Theorem 3)

The proof is similar to the proof of Theorem 2. For a given $\varepsilon > 0$, we take a ball $B$ (inside a Darboux chart) as in Proposition 29, namely a ball that has a displacement energy less than $\varepsilon$ appearing above is taken with respect to this small $B$. For the open set $B \times B$, consider a Morse function $F : M \to \mathbb{R}$ as in the proof of Proposition 28, namely, a Morse function whose critical points are all in $B \times B$ and is $C^2$-small enough so that its Hamiltonian flow does not admit any non-trivial periodic points and that $\text{osc}(F) := \max F - \min F < \varepsilon$. This also implies that $\phi_F$ has no fixed points in $M \setminus B \times B$. Thus, there exists $\delta > 0$ such that for all $x \in M \setminus B \times B$, we have $d(x, \phi_F(x)) > \delta$.

For any $\varepsilon' > 0$, we can take $\delta' > 0$ as in Proposition 29. Let $C > 0$ be the constant as in Proposition 30, taken with respect to the point $x_0$. For $\phi_H$ such that $d_{C^0}(\text{Id}, \phi_H) < \delta'G$ take any of its fixed points and denote it $x_*$. Denote by $\psi$ the symplectomorphism as in Proposition 30 with respect to $x_*$, i.e., $\psi(x_*) = x_0$ and $\|d\psi\| \leq C$ where the constant $C$ is independent of $\phi_H$.

Let $H' := H \circ \psi$. We have

$$d_{C^0}(\text{Id}, \phi_{H'}) = d_{C^0}(\text{Id}, \psi^{-1}\phi_H\psi) = d_{C^0}(\psi, \psi^{-1}\phi_H) \leq \|d\psi\| d_{C^0}(\text{Id}, \phi_H) \leq \delta'G \cdot C = \delta'.$$

Here, we take $\varepsilon' > 0$ small enough so that

$$d_{C^0}(\text{Id}, (\phi_{H'}^{-1} \times \phi_{H'}) \circ \phi_G) \leq \delta$$

$$(d_{C^0}(\text{Id}, (\phi_{H'}^{-1} \times \phi_{H'}) \circ \phi_G) \leq d_{C^0}(\text{Id}, \phi_G) + d_{C^0}(\phi_G, (\phi_{H'}^{-1} \times \phi_{H'}) \circ \phi_G) = d_{C^0}(\text{Id}, \phi_G) + d_{C^0}(\text{Id}, \phi_{H'}^{-1} \times \phi_{H'}) \leq \varepsilon' + \delta')$$

where $G \in C^\infty(S^1 \times M \times M)$ is the Hamiltonian as in Proposition 29. $\delta' > 0$ appearing above is taken with respect to this small $\varepsilon'$.

Therefore $\text{Fix}((\phi_{H'}^{-1} \times \phi_{H'}) \circ \phi_G \circ \phi_F) = \text{Crit}(F)$ so as in the proof of Proposition 28 the spectral invariant $\rho((\mathbb{H} \oplus H')\#G\#F, a \otimes b)$ has the following form.

$$\rho((\mathbb{H} \oplus H')\#G\#F, a \otimes b) = F(x, y) + A((\mathbb{H} \oplus H')\#G)(\phi_{(\mathbb{H} \oplus H')\#G}^t((x, y))) + (\omega \oplus \omega)(A_1)$$

for some $(x, y) \in \text{Crit}(F) \subset B \times B$ and $A_1 \in \pi_2(M \times M)$. Here, $\phi_{(\mathbb{H} \oplus H')\#G}^t((x, y))$ denotes an arbitrary chosen capped orbit of the orbit $\phi_{(\mathbb{H} \oplus H')\#G}^t((x, y))$ and the sphere $A_1$ plays the role to correct the capping of the orbit. From now on, we fix this arbitrary chosen capped orbit $\phi_{(\mathbb{H} \oplus H')\#G}^t((x, y))$. The right hand side is independent of the choice of $x \in M$ so we complete the proof. □
By Proposition 23 (2), we obtain
\[ A_{(\mathcal{H} \oplus H')\#G}(\phi^t_{\mathcal{H} \oplus H'}\#G)((x, y))) = A_{(\mathcal{H} \oplus H')\#G}(\phi^t_{\mathcal{H} \oplus H'}\#G)((x_0, x_0)). \]

Recall that \( G = (0 \oplus \mathcal{H}) \# \overline{Q} \# (0 \oplus H') \# Q \) where \( Q \) is an autonomous Hamiltonian which generates a time-1 map that switches the coordinate \( i.e. \ (p, q) \mapsto (q, p) \) in \( B \times B \). Since it is a rotation around the origin, it fixes the origin \( (x_0, x_0) \) for all time \( t \): \( \phi_Q^t((x_0, x_0)) \equiv (x_0, x_0) \).

Thus,
\[ A_{(\mathcal{H} \oplus H')\#G}(\phi^t_{\mathcal{H} \oplus H'}\#G)((x_0, x_0))) \]
\[ = \int Q(\phi^t_Q(x_0, x_0)) dt + \int (0 \oplus H')(t, x_0, \phi^t_{H'}(x_0)) dt - \omega(\phi^t_H(x_0)) + \int \mathcal{Q}(\phi^t_Q(x_0, x_0)) dt + \int (0 \oplus \mathcal{H}')(t, x_0, \phi^t_{\mathcal{H}'}(x_0)) dt - \omega(\phi^t_{\mathcal{H}'}(x_0)) + (\omega + \omega)(A_2) \]
where \( A_2 \) denotes the sphere to achieve an appropriate capping of the orbit.
Thus, by employing Proposition 23 (3) for \( \int H'_t(\phi^t_H(x_0)) dt \) and \( \int \mathcal{H}'_t(\phi^t_{\mathcal{H}'}(x_0)) dt \), we obtain,
\[ A_{(\mathcal{H} \oplus H')\#G}(\phi^t_{\mathcal{H} \oplus H'}\#G)((x_0, x_0))) = (\omega + \omega)(A_2). \]

Therefore,
\[ |\rho((\overline{\mathcal{H}} \oplus H')\#G \# \mathcal{F}, a \otimes b) - \rho(\mathcal{H} \oplus H', a \otimes b)| \]
\[ \leq \rho(G \# \mathcal{F}, [M \times M]) \]
\[ \leq \rho(G, [M \times M]) + \rho(F, [M \times M]) < 3\varepsilon. \]

Note that the final line uses,
\[ \rho(F, [M \times M]) \leq \max(f) < \varepsilon \]
and
\[ \rho(G, [M \times M]) = \rho((0 \oplus \overline{\mathcal{H}}) \# \overline{Q} \# (0 \oplus H') \# Q, [M \times M]) \]
\[ \leq \rho((0 \oplus \overline{\mathcal{H}}) \# \overline{Q} \# (0 \oplus H'), [M \times M]) + \rho(Q, [M \times M]) \]
\[ = \rho(\mathcal{Q}, [M \times M]) + \rho(Q, [M \times M]) \]
\[ \leq \varepsilon(Supp(\mathcal{Q})) + \varepsilon(Supp(Q)) \ (Proposition \ 26) \]
\[ \leq 2\varepsilon(B \times B) < 2\varepsilon \]
by our choice of the ball \( B \).

Here,
\[ \rho((\overline{\mathcal{H}} \oplus H')\#G \# \mathcal{F}, a \otimes b) \]
\[ = F(x, y) + A_{(\overline{\mathcal{H}} \oplus H')\#G}(\phi^t_{(\overline{\mathcal{H}} \oplus H')\#G}((x, y))) + (\omega + \omega)(A_1) \]
\[ = F(x, y) + (\omega + \omega)(A_2) + (\omega + \omega)(A_1) \]
\[ = F(x, y) + l \cdot \lambda_0 \]
for some integer \( l \in \mathbb{Z} \) such that \( (\omega + \omega)(A_1 + A_2) = l \cdot \lambda_0 \) and
\[ \rho(\overline{\mathcal{H}} \oplus H', a \otimes b) = \gamma_{a, b}(H') = \gamma_{a, b}(H \circ \psi) = \gamma_{a, b}(H). \]
Putting all together,
\[ |\gamma_{a,b}(H) - l' \cdot \lambda_0| \leq 4\varepsilon. \]
Hence we complete the proof. \(\square\)

3.3. The case of monotone symplectic manifolds. First we prove Theorem 6.

Proof. (Theorem 6)

It is almost identical to the proof of Theorem 9. We consider \((M \times M, \omega \oplus \omega)\) and first apply Proposition 29 to \(\phi_H \times \phi_H^{-1}\) and reduce it to the situation where we can work under the setting as in Proposition 28. Denote by \(G\) the Hamiltonian as in Proposition 29.

Notice that in the proof of Proposition 28, we do not use the negative monotonicity of the symplectic manifold up to the point where we obtain equations

- \(\gamma(H) \leq 4\varepsilon - \omega(A + B)\),
- \(-4n = \dim(M \times M) \leq c_1(A + B) \leq 0\).

(In fact, in the proof of Proposition 28, we only use the negative monotonicity precisely at the point in the proof where we write "In particular, by the negative monotonicity of \((M, \omega), \cdots\")

Thus these equations holds in the case of monotone symplectic manifolds too. By the monotonicity, we have

\[ \omega(A + B) = \lambda \cdot c_1(A + B) := l \cdot \lambda N = l\lambda_0 \]

for some \(l \in \mathbb{Z}\) that satisfies, by the second equation,

\[ -\frac{4n}{N} \leq l \leq 0. \]

Thus,

\[ \gamma(H) \leq 4\varepsilon - \omega(A + B) \leq 4\varepsilon + \frac{4n}{N} \cdot \lambda_0. \]

\(\square\)

Next we prove Theorem 7.

Proof. (Theorem 7)

Let \(\phi \in Ham(M, \omega)\) and take any Hamiltonian \(H\) such that \(\phi_H = \phi\). Let \(\tilde{\psi} \in \tilde{\pi}_1(Ham(M, \omega))\) be the loop of Hamiltonian diffeomorphisms as in the statement. Recall that the generator of the Novikov ring \(q\) satisfies \(\omega(q) = -\lambda_0\) and \(c_1(q) = -N\) since \((M, \omega)\) is monotone.

Denote \(a := S_{\psi, \sigma} \in QH_\ast(M)\) and \(a^k := a \ast a \ast \cdots \ast a\ (k\text{-times})\).

First of all, notice that

- \(\deg(a^k) = \deg([pt]) = 0\),
- \(\deg(a^k) = \deg([M] \cdot q^{-l'}) = 2n + 2Nl'\),
- For any \(m \in \mathbb{N}\), \(\deg(a^m) = m \cdot \deg(a) - (m - 1) \cdot 2n\).
These equations will give us the following:

\[
\frac{k'}{k} = \frac{Nl'}{n}
\]

and our assumption \(N > n\) implies \(k' > k\).

By Proposition 26, we get the following.

- \(\gamma(H) = \rho(H, [M]) - \rho(H, [pt]) = \rho(H, [M]) - \rho(H, a^k)\),
- \(\gamma(\psi H) = \rho(H, S_{\psi, \sigma} [M]) - \rho(H, S_{\psi, \sigma} a^k) = \rho(H, a) - \rho(H, a^{(k+1)})\),
- \(\gamma((\psi^2) H) = \rho(H, a^{2}) - \rho(H, a^{(k+2)})\).

\[\ldots\]

- \(\gamma((\psi^{k'-k}) H) = \rho(H, a^{(k'-k)}) - \rho(H, a^{k'})\)
  \[= \rho(H, a^{(k'-k)}) - \rho(H, [M]) + l' \lambda_0.\]
- \(\gamma((\psi^{k'-k+1}) H) = \rho(H, a^{(k'-k+1)}) - \rho(H, a^{(k+1)}) + l' \lambda_0.\)

\[\ldots\]

We used that for \(k' - k \leq j \leq k' - 1\),
\[\rho(H, a^{(j+k)}) = \rho(H, a^j) + l' \lambda_0.\]

Adding up these \(k'\)-equations will give us the following.
\[\sum_{0 \leq j \leq k'-1} \gamma((\psi^j) H) = kl' \cdot \lambda_0.\]

The spectral norm of \(\phi\) is not greater than any of \(\gamma((\psi^j) H)\) so,
\[k' \cdot \gamma(\phi) \leq kl' \cdot \lambda_0.\]

Recall the equation \(\frac{k'}{k} = \frac{Nl'}{n}\) we deduced from the degree calculation earlier in the proof and this will give us the following.
\[\gamma(\phi) \leq \frac{n}{N} \cdot \lambda_0.\]

\[\square\]

Proof. (Corollary 8)

We explain briefly that \(\mathbb{C}P^n\) meets the assumptions in Theorem 7. Consider a loop of Hamiltonian diffeomorphism of \(\mathbb{C}P^n\) defined by
\[\psi^t([z_0 : z_1 : \cdots : z_{n-1} : z_n]) := [z_0 : e^{2\pi it} z_1 : e^{2\pi it} z_2 : \cdots : e^{2\pi it} z_{n-1} : e^{2\pi it} z_n].\]

It is known that there exists a section class \(\sigma\) such that \(S_{\psi, \sigma} = [CP^{n-1}]\) where \([CP^{n-1}]\) denotes the generator of \(H_{2n-2}([CP^n])\). See Example 9.6.1 and
Proposition 9.6.4 in [MS2]. This shows that \( \mathbb{CP}^n \) satisfies the assumptions in Theorem [7].

4. Proof of the applications

4.1. The displaced disks problem. We prove Theorem 17. It is a consequence of the energy-capacity inequality proven by Usher in [Ush]:

Let \( B \) a symplectically embedded ball of radius \( r \). If \( \phi(B) \cap B = \emptyset \), then

\[ \pi r^2 \leq \gamma(\phi). \]

Proof. (Theorem 17)

Let \( \phi \in \text{Ham}^0(M, \omega) \) such that \( \phi \neq Id \). Assume \( \phi(U) \cap U = \emptyset \) where \( U \) denotes a symplectically embedded ball of radius \( r \). The energy-capacity inequality and the \( C^0 \)-continuity of \( \gamma \) gives us \( \gamma(\phi) \geq \pi r^2 \). This completes the proof and solves the displaced disk problem negatively.

4.2. The \( C^0 \)-Arnold conjecture. We prove a rigidity property of Hamiltonian homeomorphisms of \( \mathbb{CP}^n \) and negative monotone symplectic manifolds with a minimal Chern number \( N \geq n \). As written in the introduction, spectral invariants seem to be the right tool to illuminate rigidity of Hamiltonian homeomorphisms. We start with looking at properties of \( \sigma_{a,b} \) defined earlier in Section 1.6.

**Proposition 31.** Let \( (M^{2n}, \omega) \) be a symplectic manifold. For Hamiltonians \( H, G \), we have the following triangle inequality:

\[ \forall a, b \in H_*(M), \ |\sigma_{a,b}(H) - \sigma_{a,b}(G)| \leq \gamma(H \# G). \]

Proof.

\[ \sigma_{a,b}(H) - \sigma_{a,b}(G) = \rho(H, a) - \rho(H, b) - (\rho(G, a) - \rho(G, b)) \]

\[ \leq \rho(G \# H, [M]) + \rho(H \# G, [M]) = \gamma(H \# G). \]

By changing the role of \( H \) and \( G \), we get \( \sigma_{a,b}(G) - \sigma_{a,b}(H) \leq \gamma(H \# G) \) too. This completes the proof.

Proposition 31 allows us to define the following:

Let \( (M^{2n}, \omega) \) be a negative monotone symplectic manifold and \( a, b \in H_*(M) \).

\[ \forall \phi \in \text{Ham}(M, \omega), \ \sigma_{a,b}(\phi) := \sigma_{a,b}(H) \text{ for any } H \text{ such that } \phi_H = \phi. \]

Note that the well-definedness is due to Corollary 10.

Similarly, we define \( \sigma_{a,b} \) for Hamiltonian diffeomorphisms on \( \mathbb{CP}^n \) as follows:

\[ \sigma_{[\mathbb{CP}^k], [\mathbb{CP}^l]}(\phi) := \begin{cases} \inf_{\phi_H = \phi} \sigma_{[\mathbb{CP}^k], [\mathbb{CP}^l]}(H) & (\text{if } k \geq l) \\ \sup_{\phi_H = \phi} \sigma_{[\mathbb{CP}^k], [\mathbb{CP}^l]}(H) & (\text{if } k < l) \end{cases}. \]

**Remark 32.**

1. We will abbreviate \( \sigma_{[\mathbb{CP}^k], [\mathbb{CP}^l]} \) by \( \sigma_{2k,2l} \).
(2) \[ \sigma_{2k,2l} \geq 0 \quad (\text{if } k \geq l) \]
\[ \sigma_{2k,2l} \leq 0 \quad (\text{if } k \leq l) \]

since if \( k \geq l \), then \( \rho(H, [\mathbb{C}P^k]) \geq \rho(H[C\mathbb{P}^l]) \)
for every Hamiltonian \( H \).

(3) Thus in particular for every \( k, l \in \mathbb{Z} \), \( \sigma_{2k,2l} \) is finite.

Corollary 33. Let \((M^{2n}, \omega)\) be either a negative monotone symplectic manifold or \((\mathbb{C}P^n, \omega_{FS})\). We have the following triangle inequality:
\[
\forall \phi, \psi \in \text{Ham}(M, \omega), \quad |\sigma_{a,b}(\phi) - \sigma_{a,b}(\psi)| \leq \gamma(\phi^{-1}\psi).
\]

Proof. We only explain the case of \((\mathbb{C}P^n, \omega_{FS})\) since the other is simpler. Without loss of generality, we can assume \( k \leq l \). By the triangle inequality,
\[
\sigma_{2k,2l}(H \# G) \leq \sigma_{2k,2l}(H) + \gamma(G).
\]
Take an infimum on both sides as in the definition.
\[
\sigma_{2k,2l}(\phi \psi) \leq \inf_{H=\phi, G=\psi} \sigma_{2k,2l}(H \# G) \leq \sigma_{2k,2l}(\phi) + \gamma(\psi).
\]
Since \( \sigma_{2k,2l} \) are finite,
\[
\sigma_{2k,2l}(\phi \psi) - \sigma_{2k,2l}(\phi) \leq \gamma(\psi).
\]
This implies the triangle inequality
\[
\forall \phi, \psi \in \text{Ham}(M, \omega), \quad |\sigma_{2k,2l}(\phi) - \sigma_{2k,2l}(\psi)| \leq \gamma(\phi^{-1}\psi).
\]

This corollary and the \( C^0 \)-continuity of \( \gamma \) implies the \( C^0 \)-continuity of \( \sigma_{a,b} \). This allows us to define \( \sigma_{a,b} \) for Hamiltonian homeomorphisms i.e. for a Hamiltonian homeomorphism \( \phi \), define \( \sigma_{a,b}(\phi) := \lim_{n \to \infty} \sigma_{a,b}(\phi_n) \) where \( \phi_n \in \text{Ham}(M, \omega) \), \( \phi_n \overset{C^0}{\to} \phi \).

We are now ready to prove Theorem 16.

Proof. (Theorem 16)

First of all, notice that in either cases,
\[
a, b \in H_{*<2n}(M), \quad a \ast b = a \cap b
\]
and thus
\[
\sigma_{a,a \cap b} = \sigma_{a,a \ast b}
\]
provided that \( a, b, a \cap b \) are all non-trivial homology classes. Here, \( \ast \) and \( \cap \) denotes respectively the quantum product and the intersection product. This is because,

(1) (Negative monotone case)
\[
a \ast b = a \cap b + \sum_{k>0} (a \ast b)_k \cdot q^{-k}.
\]

Be careful that the sum in the quantum product takes only positive powers and that since \((M, \omega)\) is negative monotone, \( q \) represents a
pseudo-holomorphic sphere such that $\omega(q) = -\lambda_0$ and $c_1(q) = +N$. (See Remark 25)

\[
\forall k \geq 0, \deg((a \ast b)_k \cdot q^{-k}) = \deg(a) + \deg(b) - 2n < 2n + 2n - 2n = 2n.
\]

However,

\[
\forall k > 0, \deg((a \ast b)_k \cdot q^{-k}) = \deg((a \ast b)_k) - 2N \cdot (-k) \geq 0 + 2N \geq 2n
\]
since for each $k > 0$, $(a \ast b)_k$ represents a homology class. Thus by the previous inequality we have

\[
\forall k > 0, (a \ast b)_k = 0.
\]

(2) (The case of $\mathbb{C}P^n$) Once again we have

\[
a \ast b = a \cap b + \sum_{k > 0} (a \ast b)_k \cdot q^{-k}
\]

but this time, since $\mathbb{C}P^n$ is monotone, $q$ represents a pseudo-holomorphic sphere such that $\omega(q) = -\lambda_0$ and $c_1(q) = -N = -(n + 1)$. (See Remark 25) If $(a \ast b)_k$ is a non-trivial homology class for $k > 0$, then

\[
\forall k > 0, \deg(a \cap b) = \deg((a \ast b)_k \cdot q^{-k}) = \deg((a \ast b)_k) - 2(-N) \cdot (-k) \leq 2n - 2(n + 1) = -2.
\]

Since $a \cap b$ represents a non-trivial homology class, this is absurd and

\[
\forall k > 0, (a \ast b)_k = 0.
\]

It is enough to prove that an arbitrary open neighborhood $U$ of Fix$(\phi)$ is homologically non-trivial. Let $f : M \to \mathbb{R}$ be a sufficiently $C^2$-small smooth function such that $f < 0$ on $M \setminus U$ and $f|_U = 0$. Since the negative monotone case is simpler than the case of $(\mathbb{C}P^n, \omega_{FS})$, we prove the latter.

First of all, take a sequence $\phi_j \in Ham(M, \omega)$, $j \in \mathbb{N}$ such that $d_{C^0}(\phi, \phi_j) \leq 1/j$. Take a subsequence $\{j_k\}_{k \in \mathbb{N}}$ so that for each $k$,

\[
\gamma(\phi^{-1}_j \phi_{j_k}) < 1/k.
\]

Next, for each $k$, take a Hamiltonian $H_k$ such that

\[
\sigma_{a,a \ast b}(H_k) \leq \sigma_{a,a \ast b}(\phi_{j_k}) + 1/k.
\]

As in [BHS2], there exist $k_0 \in \mathbb{N}$ such that if $k \geq k_0$, then

\[
\rho(H_k \# f, a) = \rho(H_k, a)
\]

for all $a \in H_s(M)$. For $k \geq k_0$,

\[
\rho(H_k, a \ast b) = \rho(H_k \# f, a \ast b) \leq \rho(H_k, a) + \rho(f, b)
\]

and thus,

\[
-\sigma_{a,a \ast b}(H_k) \leq \rho(f, b).
\]
By our choices of $\phi_{j,k}$ and $H_k$, we have the following.
\[ \sigma_{a,a^*b}(H_k) \leq \sigma_{a,a^*b}(\phi_{j,k}) + 1/k \leq \sigma_{a,a^*b}(\phi) + \gamma(\phi^{-1}\phi_{j,k}) + 1/k \leq \sigma_{a,a^*b}(\phi) + 2/k = 2/k. \]
Thus,
\[ -2/k \leq -\sigma_{a,a^*b}(H_k) \leq \rho(f,b). \]
By taking a limit $k \to +\infty$, we obtain
\[ 0 \leq \rho(f,b). \]
Thus,
\[ 0 \leq \rho(f,b) \leq \rho(f,\lfloor M \rfloor) \leq 0. \]
The last inequality follows from $f \leq 0$. This implies that $U$ is homologically non-trivial.

\[\square\]

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