NON-EXISTENCE OF TORICALLY MAXIMAL HYPERSURFACES

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Abstract. Torically maximal curves (known also as simple Harnack curves) are real algebraic curves in the projective plane such that their logarithmic Gauss map is totally real. In this paper we show that hyperplanes in projective spaces are the only torically maximal hypersurfaces of higher dimensions.

1. Introduction

Torically maximal curves, also known as simple Harnack curves, were introduced and studied in [Mik00]. Since then, they have appeared in several areas of mathematics. Finding their reasonable higher dimensional counterparts is an open and challenging problem (cf. e.g. [AIM06]). In this note we explore a direct generalisation of toric maximality for projective hypersurfaces proposed in [Mik01, Section 3.4]. We show that when \( n \geq 3 \), hyperplanes in projective spaces are the only torically maximal hypersurfaces in this sense.

Let \( X \) be an algebraic hypersurface of \((\mathbb{C}^*)^n\) defined by the equation \( P(z_1, \ldots, z_n) = 0 \). We denote by \( \Delta(X) \) the Newton polytope of the polynomial \( P(z_1, \ldots, z_n) \), and by \( \overline{X} \) the topological closure of \( X \) in the toric variety \( \text{Tor}(X) \) defined by \( \Delta(X) \). Note that \( X \subset (\mathbb{C}^*)^n \) determines \( \Delta(X) \) only up to a translation in \( \mathbb{Z}^n \), however this does not play a role in what follows.

**Definition 1.1.** We say that a hypersurface \( X \subset (\mathbb{C}^*)^n \) is torically non-singular if the polytope \( \Delta(X) \) is \( n \)-dimensional and the intersection of \( \overline{X} \) with each torus orbit of \( \text{Tor}(X) \) is non-singular in this orbit. If \( X \) is torically non-singular then \( \overline{X} \) is transverse to all torus orbits of \( \text{Tor}(X) \).

We say that \( X \subset (\mathbb{C}^*)^n \) is torically projective if \( \text{Tor}(X) = \mathbb{C}P^n \).

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1
Following [Kap91], we define the logarithmic Gauß map of a non-singular hypersurface $X \subset (\mathbb{C}^*)^n$ as

$$\gamma_X : X \rightarrow \mathbb{C}P^{n-1} (z_1, \ldots, z_n) \mapsto [z_1 \frac{\partial P}{\partial z_1}(z_1, \ldots, z_n) : \ldots : z_n \frac{\partial P}{\partial z_n}(z_1, \ldots, z_n)].$$

The map $\gamma_X$ is just the usual Gauß map after the reparameterisation of $X$ with the help of a local branch of the holomorphic logarithm restricted to $X$; clearly the map $\gamma_X$ does not depend on the chosen branch of the logarithm.

It is proved in [Mik00, Section 3.2] that when $X$ is torically non-singular, the map $\gamma_X$ extends to an algebraic map $\gamma_X : X \rightarrow \mathbb{C}P^{n-1}$ such that

$$\deg(\gamma_X) = \text{Vol}_n(\Delta(X)),$$

where $\text{Vol}_n$ denotes the lattice volume of an $n$-dimensional polytope (i.e. $n!$ times the Euclidean volume).

Since $\gamma_X$ is a map between 2 manifolds of the same dimension, the fibre $\gamma_X^{-1}(y)$ is finite for almost all $y$ in $\mathbb{C}P^{n-1}$. Our first result is that $\gamma_X$ is actually finite in the case of torically projective hypersurfaces, that is $\gamma_X^{-1}(y)$ is finite for any $y \in \mathbb{C}P^{n-1}$.

**Theorem 1.2.** If $X \subset (\mathbb{C}^*)^n$ is a torically non-singular projective hypersurface, then the logarithmic Gauß map $\gamma_X : X \rightarrow \mathbb{C}P^{n-1}$ is finite.

We then investigate the existence of torically maximal hypersurfaces. Given a real algebraic subvariety $X$ of a complex toric variety, we denote by $\mathbb{R}X$ the real part of $X$. We say that a real algebraic map $f : X \rightarrow Y$ between 2 real algebraic varieties is almost totally real if $f^{-1}(x) \subset \mathbb{R}X$ for any $x \in \mathbb{R}Y \setminus S$ where $S$ is some subspace of $f(X) \cap \mathbb{R}Y$ of positive codimension. If $S$ is empty, then the map $f$ is said to be totally real.

**Definition 1.3.** A torically non-singular real algebraic hypersurface $X$ of $(\mathbb{C}^*)^n$ is said to be almost torically maximal if the map $\gamma_X$ is almost totally real.

A torically non-singular real algebraic hypersurface $X$ of $(\mathbb{C}^*)^n$ is said to be torically maximal if $X$ is non-singular and the map $\gamma_X$ is totally real.

**Remark 1.4.** Note that the above definition of (almost) torically maximal is axiomatizing [Mik01, Proposition 26] rather than making use of [Mik01, Definition 10]. In particular, we do not require $X$ or $\overline{X}$ to be maximal in the sense the Smith-Thom inequality (see for example [BR90] for the Smith-Thom inequality).

Any almost torically maximal hypersurface is torically maximal if $n \leq 2$. When $n = 1$, the variety $X$ is torically maximal if and only if all roots of $P(z_1)$ in $\mathbb{C}^*$ are simple and real. When $n = 2$, the real curve $X$ is torically maximal if and only if $\mathbb{R}X$ is a simple Harnack curve, see [PR11] Lemma 2.2 and Theorem 3.5. It is proved in [Mik00] that the topological type of the pair $((\mathbb{R}^*)^2, \mathbb{R}X)$ is uniquely determined by $\Delta(X)$ when $X$ is a simple Harnack curve (see also [Bru15] for an alternative proof).

The logarithmic Gauß map of a hyperplane in $\mathbb{C}P^n$ has degree 1, therefore any real hyperplane is almost torically maximal, and hence torically maximal by Theorem 1.2. The next theorem asserts that this is the only possible
example of almost torically projective maximal hypersurfaces as soon as \( n \geq 3 \).

**Theorem 1.5.** Let \( n \geq 3 \) and \( X \subset (\mathbb{C}^*)^n \) be an almost torically maximal projective hypersurface. Then \( \overline{X} \) is a hyperplane.

In the case of torically maximal hypersurfaces, the previous theorem can be extended to any Newton polytope.

**Theorem 1.6.** Let \( n \geq 3 \) and \( X \subset (\mathbb{C}^*)^n \) be a torically maximal hypersurface. Then \( \text{Tor}(X) = \mathbb{C}P^n \) and \( \overline{X} \) is a hyperplane.

**Remark 1.7.** Note that Theorem 1.5 can be deduced as a corollary of Theorem 1.2 and Theorem 1.6. Nevertheless, its direct proof is quite simple, so we prove it independently of Theorem 1.6.

Let us make some comments about Theorems 1.5 and 1.6 and further generalisations of simple Harnack curves. First, we do not know whether there exist almost torically maximal hypersurfaces \( X \subset (\mathbb{C}^*)^n \) which are not torically maximal. However, Section 4 provides an example of a singular hypersurface for which the logarithmic Gauß map is almost totally real but not totally real. Therefore the assumption of smoothness of \( \overline{X} \) (which is a part of the definition of toric maximality) is essential in Theorem 1.6.

Next, Theorems 1.5 and 1.6 may be a hint that the direct generalisation of toric maximality proposed in [Mik01, Section 3.4] in dimension at least 3 can be weakened. For example, relaxing the smoothness assumption on \( X \subset (\mathbb{C}^*)^n \) in Definition 1.3 may produce meaningful objects (see [Lan15] for the case of generalised simple Harnack curves). Additionally, it is worthwhile to consider real subvarieties of higher codimension. There is a natural generalisation of the logarithmic Gauß map where the target is now a Grassmannian, and also a generalisation of (almost) torically maximal real algebraic varieties of any codimension. Products of torically maximal hypersurfaces give examples of toricmaximal subvarieties of codimension \( > 1 \). So far we do not know of other examples.

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2. Properties of the logarithmic Gauss map

If \( X \subset (\mathbb{C}^*)^n \) is a torically non-singular hypersurface, then \( \overline{X} \cap Y \) is by definition also a torically non-singular hypersurface for any torus orbit \( Y \) of \( \text{Tor}(X) \). In particular the logarithmic Gauß map \( \gamma_{\overline{X} \cap Y} \) is well defined. The following lemma is straightforward.

**Lemma 2.1.** Let \( X \subset (\mathbb{C}^*)^n \) be a torically non-singular hypersurface. Then for any torus orbit \( Y \) of \( \text{Tor}(X) \), the logarithmic Gauß map of \( \overline{X} \cap Y \) coincides with the restriction of \( \gamma_X \) to \( \overline{X} \cap Y \). Furthermore, if the face of \( \Delta(X) \) corresponding to \( Y \) is parallel to a linear space \( L \subset \mathbb{R}^n \), then the image of the restriction of \( \gamma_X \) to \( \overline{X} \cap Y \) lies in the projectivisation of \( L \otimes \mathbb{C} \) in \( \mathbb{C}P^{n-1} \).
Proof of Theorem 1.2. Lemma 2.1 implies that for any \( x \in \mathbb{C}P^{n-1} \), the fibre \( \psi^{-1}(x) \) is disjoint from at least 1 toric divisor of \( \mathbb{C}P^n \), which is a hyperplane. Since any positive-dimensional subvariety of \( \mathbb{C}P^n \) intersects any hyperplane, all fibres \( \psi^{-1}(x) \) have to be a finite collection of points. \( \square \)

Remark 2.2. When \( X \) is torically non-singular, but not necessarily projective, the above argument can be used to show that any curve contained in the fibre \( \psi^{-1}(x) \) must be contained in the closure of a subtorus translate of \((\mathbb{C}^*)^n\).

Theorem 1.2 immediately implies the following.

Corollary 2.3. If \( X \subset (\mathbb{C}^*)^n \) is an almost torically maximal projective hypersurface, then \( X \) is torically maximal.

The following theorem about totally real morphisms is used to restrict the topology of \( \mathbb{R}X \). Note that in [KS15] a totally real morphism is called real fibered.

Theorem 2.4. [KS15] Theorem 2.19] Let \( X \) and \( Y \) be non-singular real algebraic varieties of the same dimension, and let \( \phi : X \to Y \) be a totally real morphism. Then \( d_\phi : T_xR X \to T_{\phi(x)}R Y \) is an isomorphism for all \( x \in \mathbb{R}X \).

We outline the proof of the above theorem for completeness, referring the reader to [KS15] for details. Since it is a local statement, we may assume that both \( X \) and \( Y \) are real open neighbourhoods of 0 in \( \mathbb{C}^n \). Firstly, notice that the statement is true when \( X \) and \( Y \) are 1-dimensional: if a real map \( \phi : (\mathbb{C},0) \to (\mathbb{C},0) \) is ramified at 0, then \( \phi \) is locally given by \( z \mapsto z^d \) for \( d \geq 2 \) which is clearly not totally real.

For \( n > 1 \), if \( d_\phi \) is not injective for some \( x \in \mathbb{R}X \), then choose a real line \( L \subset \mathbb{C}^n \) such that \( \phi(x) \in L \) and \( T_{\phi(x)}\mathbb{R}X \cap d_\phi(T_x\mathbb{R}X) = \{0\} \). Consider the real algebraic curve \( C = \phi^{-1}(L) \subset X \), and its normalisation \( \pi : \tilde{C} \to C \). The composition \( \phi \circ \pi : \tilde{C} \to L \) is also a totally real map. Since the theorem is true for maps between curves, this map is unramified over the real locus. However, for any point \( \tilde{x} \in \tilde{C} \) such that \( \pi(\tilde{x}) = x \in \mathbb{R}C \subset \mathbb{R}X \), the differential satisfies \( d_{\tilde{x}}(\phi \circ \pi) = d_\phi \circ d_{\tilde{x}} \). Therefore the image of \( d_{\tilde{x}}(\phi \circ \pi) \) is zero by the assumption that \( T_{\phi(x)}\mathbb{R}X \cap d_\phi(T_x\mathbb{R}X) = \{0\} \). This gives a contradiction and the theorem follows.

It follows from Theorem 2.4 that the logarithmic Gauß map induces a covering map \( \mathbb{R}X \to \mathbb{R}P^{n-1} \) if \( X \subset (\mathbb{C}^*)^n \) is a torically maximal hypersurface. For \( n > 1 \), there are only 2 connected coverings of \( \mathbb{R}P^n \), namely \( \mathbb{R}P^n \to \mathbb{R}P^n \) of degree 1 and \( S^n \to \mathbb{R}P^n \) of degree 2. Hence the degree of the covering map \( \mathbb{R}X \to \mathbb{R}P^{n-1} \) is determined by the topology of \( \mathbb{R}X \) when \( n \geq 3 \), and Formula (1) implies the following.

Corollary 2.5. Let \( X \subset (\mathbb{C}^*)^n \) be a torically maximal hypersurface with \( n \geq 3 \). Then \( \mathbb{R}X \) is a disjoint union of \( k \) connected components homeomorphic to \( S^{n-1} \), and \( l \) connected components homeomorphic to \( \mathbb{R}P^{n-1} \). Furthermore, the integers \( k \) and \( l \) satisfy
\[
\deg(\gamma) = \text{Vol}_n(\Delta(X)) = 2k + l.
\]
3. TORICALLY MAXIMAL HYPERSURFACES

Let \(X \subset (\mathbb{C}^*)^3\) be a torically maximal surface. By Lemma 2.1, for each 2-dimensional torus orbit \(Y\) of \(\text{Tor}(X)\), the curve \(Z = X \cap Y\) is a simple Harnack curve. By [Mik00], the intersections of \(Z\) with the toric boundary divisors of \(\text{Tor}(Z)\) are real and contained in a single component of \(\mathbb{R}Z\). Call this connected component the outer circle of the curve \(Z\) and denote it by \(O(Z)\).

**Lemma 3.1.** Let \(X \subset (\mathbb{C}^*)^3\) be a torically maximal surface. Then there exists a connected component of \(\mathbb{R}X\) containing all outer circles of \(X\).

We call this connected component of \(\mathbb{R}X\) the outer component of \(X\).

**Proof.** Each outer circle is an embedded circle contained in some connected component of \(\mathbb{R}X\). Facets \(F\) and \(F'\) of \(\Delta(X)\) intersect in an edge \(E\) if and only if the corresponding outer circles \(O(Z)\) and \(O(Z')\) intersect transversally in exactly \(\text{Length}(E)\) points, where \(\text{Length}(E)\) is the lattice length of \(E\), i.e.

\[
\text{Length}(E) := |E \cap \mathbb{Z}^3| - 1.
\]

In particular, \(O(Z)\) and \(O(Z')\) are contained in the same connected component of \(\mathbb{R}X\). The facets of \(\Delta(X)\) are connected via the edges, therefore there exists a single connected component \(\mathbb{R}X\) containing the outer circles of all boundary curves of \(X\).

**Proposition 3.2.** Let \(X \subset (\mathbb{C}^*)^3\) be a torically maximal surface. Then the outer component of \(X\) is homeomorphic to \(\mathbb{R}P^2\), and \(\Delta(X)\) is a tetrahedron with all edges of lattice length 1.

**Proof.** Let us denote by \(C\) the outer component of \(X\). By Corollary 2.5, it is homeomorphic to either \(S^2\) or \(\mathbb{R}P^2\). Suppose \(C\) is homeomorphic to \(S^2\). Since any 2 closed curves in \(S^2\) intersecting transversally do so in an even number of points, we deduce that each edge of \(\Delta(X)\) has an even lattice length. A facet \(F\) of \(\Delta(X)\) has at least 3 edges, therefore the lattice perimeter of every facet \(F\) satisfies

\[
\sum_{E \in \mathcal{E}(F)} \text{Length}(E) \geq 6,
\]

where \(\mathcal{E}(F)\) denotes the set of edges of \(F\).

On the other hand, by [Mik00] we have

\[
\deg(\gamma_X|_{O(Z)}) = \sum_{E \in \mathcal{E}(\Delta(Z))} \text{Length}(E) - 2,
\]

for any outer circle \(O(Z)\) of \(\overline{X}\). Since the restriction of the logarithmic Gauss map \(\gamma_X\) to \(C\) has degree 2, Equation \((2)\) gives

\[
\sum_{E \in \mathcal{E}(\Delta(Z))} \text{Length}(E) - 2 \leq 2.
\]

Therefore, \(\sum_{E \in \mathcal{E}(F)} \text{Length}(E) \leq 4\) for any facet \(F\) of \(\Delta(X)\), which yields a contradiction to the lower bound of the lattice perimeter of \(F\) given above.
So $C$ is homeomorphic to $\mathbb{R}P^2$ and the restriction of the logarithmic Gauß map $\gamma_X|c : C \to \mathbb{R}P^2$ is 1-1. Equation (2) gives
\[ \sum_{E \in E(\Delta(Z))} \text{Length}(E) - 2 = 1, \]
which implies that each facet $F$ of $\Delta(X)$ is a lattice triangle, and that $\text{Length}(E) = 1$ for all edges $E$ of $\Delta(X)$. In particular, each outer circle $O(Z)$ intersects some other outer circle $O(Z')$ transversally in a single point. Hence each outer circle realises the non-zero class in $H_1(C, \mathbb{Z}/2\mathbb{Z})$. But then any 2 outer circles intersect, that is to say each pair of faces of $\Delta(X)$ must share an edge. This implies that $\Delta(X)$ is a lattice tetrahedron, and the proposition is proved.

**Proof of Theorem 1.5.** By Corollary 2.3 the hypersurface $X$ is torically maximal. So the case $n = 3$ follows immediately from Proposition 3.2. If $X \subset (\mathbb{C}^*)^n$ is torically maximal and projective for $n > 3$ then by intersecting $X$ with a 3-dimensional torus orbit of Tor$(X)$, we would obtain a torically maximal projective surface, which by above is a plane. Therefore $X$ must be a hyperplane.

Recall that given a lattice polytope $F$ of dimension $k$ in $\mathbb{R}^n$, its *lattice volume* is defined as
\[ \text{Vol}_k(F) = \frac{\text{Vol}^E_k(F)}{\text{Vol}^E_k(\Pi_F)}, \]
where $\text{Vol}^E_k$ denotes any Euclidean volume in the affine span $V_F$ of $F$, and $\Pi_F$ is any lattice simplex whose vertices form an affine basis of $V_F \cap \mathbb{Z}^n$. We say that $F$ is unimodular if $\text{Vol}_k(F) = 1$.

An $n$-dimensional lattice polytope $\Delta \subset \mathbb{R}^n$ is said to be smooth in dimension 1 if for every 1-dimensional face $E$ of $\Delta$, there exist $n - 1$ outward primitive integer normal vectors to the facets adjacent to $E$ that can be completed to a basis of $\mathbb{Z}^n$. If $\Delta$ is smooth in dimension 1, then the corresponding toric variety has singularities only at 0-dimensional torus orbits. If $X \subset (\mathbb{C}^*)^n$ is a torically maximal hypersurface, then its Newton polytope $\Delta(X)$ is smooth in dimension 1 since $X$ is non-singular.

**Lemma 3.3.** Let $n \geq 3$, and let $\Delta \subset \mathbb{R}^n$ be an $n$-dimensional lattice simplex smooth in dimension 1 such that all its facets are unimodular. Then $\Delta$ is unimodular.

**Proof.** Denote by $(e_1, \ldots, e_n)$ the canonical basis of $\mathbb{R}^n$, and choose a facet $F$ of $\Delta$. Since $F$ is unimodular, it can be assumed, up to an integer affine transformation of $\mathbb{R}^n$, that the vertices of $F$ are $0, e_1, \ldots, e_{n-1}$. There is 1 additional vertex $a$ of $\Delta$ with
\[ a = (a_1, \ldots, a_{n-1}, v) \in \mathbb{Z}^n. \]
Note that $\text{Vol}_n(\Delta) = |\text{det}(e_1, \ldots, e_{n-1}, a)| = v$.

By assumption, the facet of $\Delta$ which is the convex hull of all vertices of $\Delta$ except $e_i$ is also unimodular, so there is a vector $c = (c_1, \ldots, c_n) \in \mathbb{Z}^n$ such that
\[ \text{det}(c, e_1, \ldots, e_i, \ldots, e_{n-1}, a) = \pm (ve_i - a_ie_n) = \pm 1. \]
Therefore, the primitive integer normal vectors to this facet are \( \pm (a_i e_n - v e_i) \).

The condition that \( \Delta \) is smooth in dimension 1 implies that at each edge \( E \) of \( \Delta \), the primitive integer outward normal vectors of the facets of \( \Delta \) adjacent to \( E \) form a subset of a basis of \( \mathbb{Z}^n \). Applying this condition at the edge \([0, e_1] \) of \( \Delta \), we deduce that there must exist \( c = (c_1, \ldots, c_n) \in \mathbb{Z}^n \), such that

\[
\det (c, a_2 e_n - v e_2, \ldots, a_{n-1} e_n - v e_{n-1}, e_n) = \pm c_1 \cdot v^{n-2} = \pm 1
\]

Therefore, \( \text{Vol}_n(\Delta) = v = 1 \) and \( \Delta \) is unimodular as stated. \( \square \)

**Remark 3.4.** In dimension 3, there are tetrahedra, with unimodular faces which are not unimodular. For example, the convex hull of

\[
(0, 0, 0), (1, 0, 0), (0, 1, 0), (1, p, q),
\]

for every pair of \( p, q \) with \( \gcd(p, q) = 1 \), has unimodular facets but the volume of the tetrahedron is \( q \). So this polytope is not unimodular for \( q > 1 \). Notice that it fails to be smooth in dimension 1 along the edge joining \((1, 0, 0)\) and \((0, 1, 0)\).

**Proof of Theorem 1.4.** The theorem is proved by induction on \( n \), starting with \( n = 3 \) as the base case. Recall that \( \Delta(X) \) is smooth in dimension 1 if \( X \subset (\mathbb{C})^n \) is a torically maximal hypersurface.

Let \( X \subset (\mathbb{C})^n \) be a torically maximal surface. By Corollary 2.5, the real part \( \mathbb{R}X \) is a disjoint union of \( k \) connected components homeomorphic to \( S^{n-1} \) and \( l \) connected components homeomorphic to \( \mathbb{R}P^{n-1} \), such that

\[
\text{Vol}_n(\Delta(X)) = 2k + l.
\]

By Proposition 3.2, the outer component of \( X \) is homeomorphic to \( \mathbb{R}P^2 \), and \( \Delta(X) \) is a tetrahedron with all edge lengths equal to 1. In particular we have \( \sum_{E \in \mathcal{E}(\Delta)} \text{Length}(E) = 6 \).

Let us denote by \( \beta_* (M; \mathbb{Z}/2\mathbb{Z}) \) the sum of all \( \mathbb{Z}/2\mathbb{Z} \) Betti numbers of a manifold \( M \). The Smith-Thom inequality states that (cf e.g. [BR90])

\[
\beta_* (\mathbb{R}X; \mathbb{Z}/2\mathbb{Z}) \leq \beta_* (X; \mathbb{Z}/2\mathbb{Z}).
\]

(3)

The total sum of Betti numbers for \( \mathbb{R}P^2 \) and \( S^2 \) are

\[
\beta_* (\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z}) = 3 \quad \text{and} \quad \beta_* (S^2; \mathbb{Z}/2\mathbb{Z}) = 2,
\]

so that

\[
\beta_* (\mathbb{R}X; \mathbb{Z}/2\mathbb{Z}) = 3l + 2k = 2l + \text{Vol}_3(\Delta).
\]

Moreover, Khovanskii’s formula [Kho78] for the Euler characteristic of the complex hypersurface \( X \) gives

\[
\beta_* (X; \mathbb{Z}/2\mathbb{Z}) = \text{Vol}_3(\Delta(X)) - \sum_{F \in \mathcal{F}(\Delta(X))} \text{Area}(F) + \sum_{E \in \mathcal{E}(\Delta(X))} \text{Length}(E),
\]

(5)

where \( \mathcal{F}(\Delta(X)) \) denotes the set of facets of \( \Delta(X) \). Combining Equations 4, 5, 3 yields

\[
2l + \text{Vol}_3(\Delta) \leq \text{Vol}_3(\Delta) - \sum_{F \in \mathcal{F}(\Delta)} \text{Area}(F) + \sum_{E \in \mathcal{E}(\Delta)} \text{Length}(E),
\]
which further implies that
\[ \sum_{F \in \mathcal{F}(\Delta)} \text{Area}(F) \leq 4. \]

Therefore, each facet of \( \Delta \) is unimodular. Since \( \Delta \) is also non-singular in dimension 1, Lemma 3.3 implies that \( \Delta \) is itself unimodular. Hence \( \text{Tor}(X) = \mathbb{CP}^3 \) and \( \overline{X} \) is a hyperplane.

Now proceed by induction for \( n > 3 \). Suppose \( X \subset (\mathbb{C}^*)^n \) is a torically maximal hypersurface. By induction, each facet \( F \) of \( \Delta(X) \) is unimodular, the corresponding toric divisor \( T_F \) is \( \mathbb{CP}^n \), and \( \overline{X} \cap T_F \) is a hyperplane. In particular, the intersection \( \mathbb{RP} \cap T_F \) is connected for all facets \( F \).

Therefore, similarly to Lemma 3.1, there is a single connected component \( C \) of \( \mathbb{RP} \) which contains all intersections \( \mathbb{RP} \cap T_F \) when \( F \) runs over all facets of \( \Delta(X) \). Let \( F \) be a facet of \( \Delta(X) \), and let \( A \) be a 2-dimensional face of \( \Delta(X) \) intersecting \( F \) along an edge \( E \). Hence \( \mathbb{RP} \cap T_F \) and \( \mathbb{RP} \cap T_A \) intersect transversally, and their intersection is \( \mathbb{RP} \cap T_F \) which is a single point by the unimodularity of \( F \). Hence the class realised by \( \mathbb{RP} \cap T_F \) in \( H_{n-2}(C; \mathbb{Z}/2\mathbb{Z}) \) is non-trivial. In particular \( H_{n-2}(C; \mathbb{Z}/2\mathbb{Z}) \neq 0 \), and \( C \) is homeomorphic to \( \mathbb{RP}^{n-1} \). Furthermore, for any 2 facets \( F \) and \( F' \) of \( \Delta(X) \), the intersection of \( \mathbb{RP} \cap T_F \) and \( \mathbb{RP} \cap T_{F'} \) realises the non-zero homology class in \( H_{n-3}(C; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \). In particular, the intersection is non-empty and of dimension \( n - 3 \). On the other hand, the intersection of \( \mathbb{RP} \cap T_F \) and \( \mathbb{RP} \cap T_{F'} \) is of codimension 2 if and only if \( F \) and \( F' \) intersect in a face of \( \Delta(X) \) of codimension 2. This implies that every pair of facets of \( \Delta(X) \) must meet in a codimension 2 face. Therefore, the polytope \( \Delta(X) \) has at most \( n + 1 \) facets, all of which are \( n - 1 \) dimensional unimodular lattice simplices. Since \( \Delta(X) \) is also smooth in dimension 1 by assumption, applying Lemma 3.3 completes the proof.

4. A SINGULAR TORICALLY MAXIMAL SURFACE

We end the paper with an example showing that the hypothesis that any singularities of \( \text{Tor}(X) \) are contained in the 0-dimensional torus orbits is essential in Theorem 1.6. Let \( \Delta \subset \mathbb{R}^3 \) be the simplex with vertices
\[(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 2),\]
and let \( X \subset (\mathbb{C}^*)^3 \) be a non-singular real algebraic surface with \( \Delta(X) = \Delta \). Up to a real toric change of coordinates, the surface \( X \) has equation
\[az_3^2 + z_3 + z_2 + z_1 + 1 = 0\]
with \( a \in \mathbb{R}^\times \).

The variety \( \text{Tor}(X) \) is singular along the orbit \( Y \) corresponding to the edge \( e = [(1, 0, 0), (0, 1, 0)] \). Namely, the surface \( \overline{X} \) has an ordinary double point at \( p = \overline{X} \cap Y \). The blow-up of \( \text{Tor}(X) \) along \( Y \) is a non-singular toric variety \( Z \), and the proper transform \( \widetilde{X} \) of \( \overline{X} \) is non-singular. Note that \( \widetilde{X} \) is simply the blow-up of \( \overline{X} \) at the point \( p \). We denote by \( C \) the corresponding \((-2)\)-curve in \( \widetilde{X} \). The logarithmic Gauss map \( \gamma_X : X \to \mathbb{CP}^2 \) extends to a map \( \widetilde{\gamma}_X : \widetilde{X} \to \mathbb{CP}^2 \) that contracts the exceptional curve \( C \) to a point. In
particular, the map $\gamma_X: X \to \mathbb{CP}^2$ is the composition of the blow-down map with the map $\tilde{\gamma}_X$.

**Proposition 4.1.** The map $\gamma_X$ is totally real for $a \in (0, \frac{1}{4})$.

Note that even if $a \in (0, \frac{1}{4})$, the map $\tilde{\gamma}_X$ is almost totally real, but not totally real since it contracts the curve $C$ to a point.

**Proof.** One has

$$\gamma_X(z_1, z_2, z_3) = [z_1 : z_2 : 2az_3^2 + z_3].$$

Given $(\gamma_1 : \gamma_2 : \gamma_3) \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}$, determining the real points in the fibre $\gamma_X^{-1}(\gamma_1 : \gamma_2 : \gamma_3)$ reduces to solve the system

$$(S) \quad \begin{cases} az_3^2 + z_3 + z_2 + z_1 + 1 = 0 \\ z_1 = s\gamma_1 \\ z_2 = s\gamma_2 \\ 2az_3^2 + z_3 = s\gamma_3 \end{cases}$$

in the variables $z_1, z_2, z_3 \in \mathbb{R}$ and $s \in \mathbb{R}^*$. Since the triangle with vertices $(0, 0, 0), (1, 0, 0), (0, 1, 0)$ is unimodular, it follows from Lemma [22] that $\gamma_X^{-1}(\gamma_1 : \gamma_2 : 0)$ contains at least 1 real point. Since it is a degree 2 map, the whole fibre must be contained in $\mathbb{R}X$. Similarly, we have $\gamma_X^{-1}(\gamma_1 : \gamma_2 : \gamma_3) \subset \mathbb{R}X$ if $2\gamma_1 + 2\gamma_2 + \gamma_3 = 0$.

Assume now that $\gamma_3 = 1$ and $2\gamma_1 + 2\gamma_2 + 1 \neq 0$. Then the system $(S)$ reduces to the system

$$\begin{cases} -az_3^2 + s(\gamma_1 + \gamma_2 + 1) + 1 = 0 \\ 2az_3^2 + z_3 = s \end{cases}$$

Substituting $s = 2az_3^2 + z_3$ in the first equation we obtain

$$a(2\gamma_1 + 2\gamma_2 + 1)z_3^2 + (\gamma_1 + \gamma_2 + 1)z_3 + 1 = 0.$$ 

This is a degree 2 equation in the variable $z_3$ whose discriminant is

$$(\gamma_1 + \gamma_2 + 1)^2 - 4a(2\gamma_1 + 2\gamma_2 + 1) = (\gamma_1 + \gamma_2 + 1)^2 - 8a(\gamma_1 + \gamma_2 + 1) + 4a.$$ 

The polynomial $P(x) = x^2 - 8ax + 4a$ has discriminant

$$16a(4a - 1),$$

and so is negative if $a \in (0, \frac{1}{4})$. In this case $P(\gamma_1 + \gamma_2 + 1)$ is positive, and $\gamma_X^{-1}(\gamma_1 : \gamma_2 : 1)$ is composed of 2 points in $\mathbb{R}X$. \qed

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