POLARISED LIE GROUPS CONTACTOMORPHIC TO STRATIFIED GROUPS

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Abstract. For a stratified group $G$, we construct a class of polarised Lie groups, which we call modifications of $G$, that are locally contactomorphic to it. Vice versa, we show that if a polarised group is locally contactomorphic to a stratified group $G$, whose Lie algebra has finite Tanaka prolongation, then it must be a modification of $G$.

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1. Introduction

In this article, we consider the following question: given a stratified group $G$, we wish to characterise those polarised Lie groups that are locally contactomorphic to $G$. Here a polarisation on a Lie group is the choice of a left-invariant and bracket-generating subbundle of the tangent bundle, and a contactomorphism between polarised Lie groups is a diffeomorphism that preserves the polarisation. Stratified groups carry a canonical polarisation. Given a stratified group, we will construct a class of polarised Lie groups that are contactomorphic to $G$, which we
will call modifications of $G$. The key tool for our construction will be Tanaka prolongation theory.

Before diving into the technical details of our main results, we first provide some framework. The problem under study is relevant in different areas, such as Tanaka prolongation theory, CR geometry, sub-Riemannian geometry, and control theory. In the setting of Tanaka’s theory, it is known that the infinitesimal automorphisms of the polarisation associated to a stratified Lie algebra are encoded by its full Tanaka prolongation (see, e.g., [13, 15, 17]). If the Lie algebra is not stratified, however, all we can conclude is that every infinitesimal automorphism induces an infinitesimal automorphism on its stratified symbol. In this paper, we construct classes of polarised Lie algebras that are not stratified, but that have the same space of infinitesimal automorphisms as their stratified symbol.

Our study has potential applications to geometric control theory. Given a non-holonomic control system, the motion planning problem consists in finding a curve tangent to the polarisation that connects two given points in the ambient space. Nilpotent Lie groups are the widest class of nonholonomic systems for which an exact solution to the motion planning problem is known, see [6]. Contactomorphisms are equivalences of motion planning problems. Thus, our method detects classes of non-nilpotent nonholonomic systems that are equivalent to nilpotent ones.

Furthermore, our findings have consequences in metric geometry. On a polarised Lie group, one may define a left-invariant sub-Riemannian distance. In metric geometry, it is natural to study the equivalence of metric spaces up to isometries, bi-Lipschitz mappings, conformal and quasiconformal mappings. For example, if two stratified groups are (locally) quasiconformal, then their Lie algebras are isomorphic ([14]). If two nilpotent Lie groups are isometric, then they are isomorphic ([7, 9]). It is an open question to determine whether two nilpotent Lie groups that are globally bi-Lipschitz to one another are indeed isomorphic. In [3], the authors study the Lie groups that can be made isometric to a given nilpotent Lie group, endowed with a left-invariant distance. (See also [5] for the Riemannian case.) In this sense, our work follows [3], because contactomorphisms are locally bi-Lipschitz.

In sub-Riemannian geometry, one of the major open problems is to determine whether the conclusions of Sard Theorem hold for the endpoint map, which is a canonical map from an infinite dimensional path space to the underlying finite dimensional manifold. The set of critical values for the endpoint map is also known as abnormal set, being the set of endpoints of abnormal extremals leaving the base point. In the context of Lie groups, perhaps the most general positive results have been proved in [11]. Here the authors prove that the abnormal set has measure zero in the case of 2-step stratified groups and several other examples. This property for the abnormal set is preserved by contactomorphisms between sub-Riemannian manifolds. It then comes out from our results that every modification of a stratified group satisfies the Sard Theorem, if the stratified group does.

Now we will present our main results in detail. Recall that a Tanaka prolongation of a stratified Lie algebra $\mathfrak{g}$ through a Lie subalgebra $\mathfrak{g}_0$ of the Lie algebra of derivations of $\mathfrak{g}$ that preserve the stratification is the maximal graded Lie algebra that contains $\mathfrak{g} + \mathfrak{g}_0$. When $\mathfrak{g}_0$ is chosen as the whole set of strata preserving derivations, we obtain the full Tanaka prolongation. When the prolongation algebra is finite dimensional, we obtain a graded Lie algebra $\mathfrak{p} = \mathfrak{g} \oplus \mathfrak{q}$. Modifications of $\mathfrak{g}$ are then defined to be subalgebras $\mathfrak{s}$ of $\mathfrak{p}$ of the same dimension of $\mathfrak{g}$ that are transversal to $\mathfrak{q}$. It turns out that there is a linear map $\sigma : \mathfrak{g} \to \mathfrak{q}$ of which the
modification is the graph. If \( V \) is the first layer of \( \mathfrak{g} \), then the set \( \{ v + \sigma(v) : v \in V \} \) defines a polarisation on a Lie group whose Lie algebra is the modification \( \mathfrak{s} \).

Our first main result is that every modification of a stratified Lie group \( G \) is locally contactomorphic to \( G \) (Theorem 4.1). Our Theorem 4.2 shows that this construction is natural, in the sense that if a polarised Lie group \( S \) is contactomorphic to a stratified group \( G \), and the Lie algebra of \( G \) has finite full Tanaka prolongation, then \( S \) is a modification.

A key tool in the study of local contactomorphisms is the quotient manifold \( M = P/Q \), where \( P \) and \( Q < P \) are the Lie groups with Lie algebras \( \mathfrak{p} \) and \( \mathfrak{q} \) respectively. The polarisation of \( G \) induces a polarisation on \( M \) and \( G \) embeds in \( M \) as an open subset, see Proposition 2.10. Moreover, if \( S \) is a modification of \( G \), then an open neighborhood of the identity in \( S \) can be also embedded into \( M \), see Proposition 3.1. The composition of such embeddings induce a local contactomorphism between \( G \) and \( S \). If the group \( G \) is rigid, i.e., its full tanaka prolongation is finite dimensional, then all contactomorphisms between \( G \) and \( S \) arise in this way, see Theorem 4.2.

It remains open whether this statement holds true without asking that the full Tanaka prolongation is finite. While we cannot prove the theorem in this generality, examples suggest that it may be true. More precisely, in Subsection 5.1, we show that all three dimensional sub-Riemannian structures are modifications of the Heisenberg group with respect to a suitable finite dimensional Tanaka prolongation, even though the full prolongation of the Heisenberg Lie algebra is infinite dimensional. This justifies the following conjecture.

**Conjecture.** Suppose that \( G \) is a stratified Lie groups and that \( S \) is a polarised Lie group that is locally contactomorphic to \( G \). Then there is a finite Tanaka prolongation of \( \text{Lie}(G) \) in which \( \text{Lie}(S) \) is a modification of \( \text{Lie}(G) \).

In Subsection 5.2 we explicitly compute some modifications of the free nilpotent Lie group with two generators and step four, \( F_{24} \). It comes out that one may construct examples of non-nilpotent Lie groups that are contactomorphic to \( F_{24} \). We also find a nilpotent, non-stratified, polarised Lie group that is globally contactomorphic to \( F_{24} \). Finally, in Subsection 5.3, we study all the modifications of an ultra-rigid stratified group, that is, a stratified group whose only derivation is the dilation. It turns out that such modifications are all solvable and the only nilpotent one is the stratified group itself.

The paper is organized as follows. In Section 2, we fix the notation and establish the framework in which we will be working. We consider stratified algebras and their Tanaka prolongations, we define the corresponding Lie groups and fix a polarisation on them. In Section 3, we define the modifications of a stratified algebra and those of a stratified group. Therefore, we prove the results that will be the building blocks for the proof of our main claims, which are contained in Section 4. Finally, we apply our modification technic to a number of examples in Section 5.

## 2. Notation and preliminaries

### 2.1. Polarizations and Tanaka prolongations

Given a connected, smooth manifold \( M \), a polarisation of \( M \) is the choice of a subbundle \( \Delta_M \) of the tangent bundle \( TM \) that is bracket generating, i.e., with the property that the sections of \( \Delta_M \) bracket generate all the sections of \( TM \). Given two polarised manifolds \((M, \Delta_M)\) and \((N, \Delta_N)\), a contactomorphism between \( M \) and \( N \) is a diffeomorphism \( f : M \to N \) such that \( f_*(\Delta_M) = \Delta_N \). We denote by \( \Gamma(TM) \) the space of vector fields on \( M \). A vector field \( V \in \Gamma(TM) \) on a polarised manifold \((M, \Delta_M)\) is a contact vector field if its flow is made of contactomorphisms. For a Lie group \( S \), we shall always consider left-invariant polarisations \( \Delta_S \). The pair \((S, \Delta_S)\) is
called a polarised group. The identity element will be denoted by \( e_S \), or simply \( e \) if no confusion arises. We denote by \( G \) a stratified group, that is, a connected and simply connected Lie group whose Lie algebra decomposes as \( \mathfrak{g} = \bigoplus_{i=-s}^{i} \mathfrak{g}_i \), with \( [\mathfrak{g}_{-1}, \mathfrak{g}_j] = \mathfrak{g}_{j-1} \) for every \(-s + 1 \leq j \leq -1\). On a stratified group we will always consider the left-invariant polarisation \( \Delta_G \) for which \( (\Delta_G)_{e_G} = \mathfrak{g}_{-1} \). In a stratified group \( G \) we consider the strata preserving derivations

\[
\text{Der}(\mathfrak{g}) := \{ u \in \text{End}(\mathfrak{g}) : u(\mathfrak{g}_{-1}) \subset \mathfrak{g}_{-1}, \text{ and } u[X,Y] = [u(X),Y] + [X,u(Y)] \forall X,Y \in \mathfrak{g} \}.
\]

Given a subalgebra \( \mathfrak{g}_0 \) of \( \text{Der}(\mathfrak{g}) \), we define the Tanaka prolongation of \( \mathfrak{g} \) through \( \mathfrak{g}_0 \) as the (possibly infinite) maximal graded Lie algebra \( \text{Prol}(\mathfrak{g}, \mathfrak{g}_0) = \bigoplus_{k \geq -s} \mathfrak{g}_k \) which contains \( \mathfrak{g} \oplus \mathfrak{g}_0 \). When \( \mathfrak{g}_0 = \text{Der}(\mathfrak{g}) \), we call \( \text{Prol}(\mathfrak{g}, \mathfrak{g}_0) \) the full Tanaka prolongation of \( \mathfrak{g} \). It is not difficult to see that the latter contains all prolongations. We say that \( G \) is rigid if the full Tanaka prolongation has finite dimension. When it is clear from the context and the prolongation under consideration is finite dimensional, we shall denote \( \text{Prol}(\mathfrak{g}, \mathfrak{g}_0) \) by \( \mathfrak{p} \), the nonnegative part \( \bigoplus_{k \geq 0} \mathfrak{g}_k \) by \( \mathfrak{q} \), and the positive part \( \bigoplus_{k \geq 0} \mathfrak{g}_k \) by \( \mathfrak{p}_+ \). See [13, 15, 17] for further details on Tanaka prolongation.

2.2. The groups \( P \) and \( Q \) and their quotient \( M \). In the following, we establish a number of properties of the Lie groups that correspond to the Lie algebras introduced above. Let \( \bar{P} \) be the connected and simply connected Lie group whose Lie algebra is a finite dimensional Tanaka prolongation \( \mathfrak{p} \) of a stratified Lie algebra \( \mathfrak{g} \). Let \( \bar{Q} \) be the connected subgroup of \( \bar{P} \) whose Lie algebra is \( \mathfrak{q} \).

The set \{ \( \delta_\lambda : \lambda > 0 \) \} of mappings on \( \mathfrak{p} \) defined by \( \delta_\lambda(X) = \lambda^i X \) for \( X \in \mathfrak{g}_i \), is a one parameter family of automorphisms of \( \mathfrak{p} \). By abuse of notation, we write \( \delta_\lambda \) for the corresponding automorphisms of the group \( \bar{P} \). Such maps exist because \( \bar{P} \) is simply connected.

**Lemma 2.1.** Denote by \( \text{exp}_P : \mathfrak{p} \rightarrow \bar{P} \) the exponential map of \( \bar{P} \). Then \( \text{exp}_P \) is injective on \( \mathfrak{g} \) and on \( \bigoplus_{k \geq 1} \mathfrak{g}_k \).

**Proof.** Let \( v, w \in \mathfrak{g} \) such that \( \text{exp}_P(v) = \text{exp}_P(w) \). Since \( v, w \in \mathfrak{g} \), then \( \lim_{\lambda \rightarrow \infty} \delta_\lambda v = \lim_{\lambda \rightarrow \infty} \delta_\lambda w = 0 \). Let \( \lambda \geq 1 \) be such that both \( \delta_{\lambda v} \) and \( \delta_{\lambda w} \) belong to a neighborhood \( U \) of 0 in \( \mathfrak{p} \) on which the exponential map \( \text{exp}_P \) is injective. Then \( \text{exp}_P(\delta_{\lambda v}) = \delta_{\lambda}(\text{exp}_P(v)) = \delta_{\lambda}(\text{exp}_P(w)) = \text{exp}_P(\delta_{\lambda} w) \). By the injectivity of \( \text{exp}_P \) on \( U \), we have \( \delta_{\lambda v} = \delta_{\lambda} w \). Since \( \delta_{\lambda} \) is a linear isomorphism, we conclude that \( v = w \). A similar argument proves that \( \text{exp}_P \) is injective on \( \bigoplus_{k \geq 1} \mathfrak{g}_k \).

By Lemma 2.1, the canonical immersion \( G \hookrightarrow \bar{P} \) induced by \( \mathfrak{g} \hookrightarrow \mathfrak{p} \) is injective.

We are going to show that \( G \) is closed in \( \bar{P} \). We prove two lemmas first.

**Lemma 2.2.** The intersection of \( G \) with \( \bar{Q} \) is trivial.

**Proof.** Since \( \delta_1(\mathfrak{g}) = \mathfrak{g} \) and \( \delta_1(\mathfrak{q}) = \mathfrak{q} \), then \( \delta_1(G) = G \) and \( \delta_1(\bar{Q}) = \bar{Q} \), for all \( \lambda > 0 \). Since \( \mathfrak{g} \) is nilpotent, \( G = \text{exp}_P(\mathfrak{g}) \).

Let \( x \in G \cap \bar{Q} \), then \( x = \text{exp}_P(v) \) for some \( v \in \mathfrak{g} \) and \( \lim_{\lambda \rightarrow \infty} \delta_\lambda(x) = \text{exp}_P(\lim_{\lambda \rightarrow \infty} \delta_\lambda v) = \text{exp}_P \). It follows that the curve \( \gamma : [0,1] \rightarrow \bar{P} \), \( \gamma(t) = \delta_{(1-t)x} \), extends to a continuous path \( [0,1] \rightarrow \bar{P} \), and \( \gamma(0) = e_P \), \( \gamma(1) = x \), lying in \( G \). Since \( \delta_\lambda(x) \in \bar{Q} \) for all \( \lambda > 0 \), then \( \gamma \) lies in \( \bar{Q} \) as well.

Since \( \mathfrak{g} \oplus \mathfrak{q} = \mathfrak{p} \), there are open neighborhoods \( U \subset \mathfrak{g} \) and \( V \subset \mathfrak{q} \) of 0 such that \( \Omega = \text{exp}_P(U) \cap \text{exp}_P(V) \) is an open neighborhood of \( e_P \) in \( \bar{P} \) and the following holds: The connected component of \( \Omega \cap G \) containing \( e_P \) is \( \text{exp}_P(U) \), the connected component of \( \Omega \cap \bar{Q} \) containing \( e_P \) is \( \text{exp}_P(V) \), and \( \text{exp}_P(U) \cap \text{exp}_P(V) = \{e_P\} \).

Since \( \gamma \) joins \( x \) to \( e_P \) continuously, then \( \gamma([0,1]) \cap \Omega \) lies in both the connected components of \( \Omega \cap G \) and \( \Omega \cap \bar{Q} \) containing \( e_P \), i.e., \( \gamma([0,1]) \cap \Omega \subset \text{exp}_P(U) \cap \text{exp}_P(V) = \{e_P\} \). This implies that \( x = e_P \).
Lemma 2.3 (Lemma on Lie groups). Let $G$ be a Lie subgroup of a Lie group $P$ and let $\iota : G \rightarrow P$ the inclusion. The image $\iota(G)$ is not closed in $P$ if and only if there is a sequence $\{g_n\}_{n \in \mathbb{N}} \subset G$ such that $\lim_{n \rightarrow \infty} g_n = e$ and $\lim_{n \rightarrow \infty} \iota(g_n) = eP$.

Proof. Recall that $G$ is closed in $P$ if and only if $\iota$ is an embedding. So, if such a sequence exists then $\iota(G)$ is not closed in $P$. We need to prove the converse implication.

Let $\rho$ be any left-invariant Riemannian distance on $G$. Then $\rho$ is complete and in particular closed balls are compact. Let $\{g_n\}_{n \in \mathbb{N}} \subset G$ be a sequence such that $\lim_{n \rightarrow \infty} \iota(g_n) = p \in P$. If there is $R > 0$ such that $\rho(e_G, g_n) \leq R$ for all $n$, then there is a subsequence $g_{n_k}$ converging to some $g_\infty \in G$. Since the immersion $\iota : G \rightarrow P$ is continuous, we obtain $\iota(g_\infty) = p$, hence $p \in \iota(G)$.

So, if $\iota(G)$ is not closed, then there is a sequence $\{g_n\}_{n \in \mathbb{N}} \subset G$ such that $\lim_{n \rightarrow \infty} \iota(g_n) = p \in P$ but $g_n \rightarrow \infty$ in $G$. Let $\{g_{n_k}\}_k$ be a subsequence such that $\rho(g_{n_k}, g_{n_k+1}) > k$ for $k \in \mathbb{N}$ and define $h_k = g_{n_k} g_{n_k+1}$. Then $h_k \rightarrow \infty$ in $G$, because $\rho(e_G, h_k) = \rho(e_G, g_{n_k}^2 g_{n_k+1}) = \rho(g_{n_k}, g_{n_k+1}) > k$ for all $k$. However, $\iota(h_k) = \iota(g_{n_k}^2) \iota(g_{n_k+1}) \rightarrow p$ in $P$ as $k \rightarrow \infty$. □

Lemma 2.4. The immersed group $G$ is closed in $\bar{P}$.

Proof. We prove that, if $\{v_n\}_{n \in \mathbb{N}} \subset \mathfrak{g}$ is a sequence so that $\lim_{n \rightarrow \infty} \exp_P(v_n) = e_P$, then $\lim_{n \rightarrow \infty} v_n = 0$. By Lemma 2.3 and $\exp_P(\mathfrak{g}) = G$, this claim implies that $G$ is closed in $\bar{P}$.

Let $\{v_n\}_{n \in \mathbb{N}} \subset \mathfrak{g}$ be a sequence with $\lim_{n \rightarrow \infty} \exp_P(v_n) = e_P$. Let $U \subset \mathfrak{g}$ and $W \subset \mathfrak{q}$ be open neighborhoods of 0 such that the map $U \times W \rightarrow \exp_P(u) \exp_P(w)$ is a diffeomorphism onto its image. Then, for $n$ large enough, there are $u_n \in U$ and $w_n \in W$ so that $\exp_P(u_n) \exp_P(w_n) = \exp_P(v_n)$. Therefore, $\exp_P(u_n)^{-1} \exp_P(v_n) = \exp_P(w_n) \in Q \cap G$. By Lemma 2.2, we have $\exp_P(u_n) = \exp_P(v_n)$. By Lemma 2.1, we have $u_n = v_n$. Since $\exp_P(u_n) \rightarrow e_P$, then $v_n = u_n \rightarrow 0$. □

Corollary 2.5. The immersed group $\bar{Q}$ is closed in $\bar{P}$.

Proof. This is a consequence of Lemma 2.4 and part (iii) of Lemma 2.15 in [3] □

Since $\bar{Q}$ is closed, we may consider the homogeneous manifold $M := \bar{P}/\bar{Q}$ with quotient projection $\pi : \bar{P} \rightarrow M$. The action of $\bar{P}$ may have a non-trivial kernel

$$K := \{p \in \bar{P} : p.x = x \forall x \in M\} = \bigcap_{p \in \bar{P}} pQp^{-1}.$$ 

Lemma 2.6. The kernel $K$ of the action of $\bar{P}$ on $M$ is discrete and contained in $\bar{Q}$. Moreover, if $p \in K$, then $\delta_{\lambda p} = p$ for all $\lambda > 0$.

Proof. Clearly $K$ is a normal and closed subgroup of $\bar{P}$ and it is contained in $\bar{Q}$. Let $v \in \text{Lie}(K)$, the Lie algebra of $K$. Then for some positive integer $\ell$, we may write $v = v_0 + \cdots + v_\ell$, with $v_i \in \mathfrak{g}_i$ for every $i = 0, \ldots, \ell$. Since $\text{Lie}(K)$ is an ideal in $\mathfrak{p}$ contained in $\mathfrak{q}$, it follows in particular that for all $i = 0, \ldots, \ell$,

$$[[\ldots[[v, y_1], y_2], \ldots], y_{\ell+1}] \in \mathfrak{g}_{-\ell-1} \cap \mathfrak{q} = \{0\},$$

for every $y_1, \ldots, y_{\ell+1} \in \mathfrak{g}_{-1}$. By definition of Tanaka prolongation, this implies that $v = 0$. Therefore, the Lie algebra of $K$ is trivial and so $K$ is discrete.

Since $K = \bigcap_{p \in \bar{P}} xQx^{-1}$, it is clear that $\delta_{\lambda K} \subset K$ for all $\lambda > 0$. However, since $\lambda \mapsto \delta_{\lambda p}$ is a continuous curve passing through $p$, we must have $\delta_{\lambda p} = p$ when $p \in K$. □
From Lemma 2.6 it follows that $P := \hat{P}/K$ and $Q := \hat{Q}/K$ are Lie groups, that $Q$ is closed in $P$ and $M = P/Q$. Moreover, the maps $\delta_\lambda$ are automorphisms of $P$ as well, for all $\lambda > 0$. Since $G \cap K = \{e\}$, the group $G$ is embedded in $P$ with $G \cap Q = \{e\}$.

Remark 2.7. If we are given $G$ and $Q$ inside $P$, for instance as matrix groups, we may want to visualise the action of $P$ on $M$ as a local action of $P$ on $G$. In other words, if $p \in P$, then there may be open subsets $U_p, V_p \subset G$ and a contactomorphism $f_p : U_p \rightarrow V_p$ that corresponds to the action of $p$ on $M$, i.e., $f_p(g_1)$ is the only $g_2 \in G$, if it exists, such that $\{g_1Q\} \cap G = \{g_2\}$. In general, such construction is not possible for all $p \in P$, but if $p$ is near enough to $e_P$, then $U_p$, $V_p$ and $f_p$ do exist. The fact that such $f_p$ is a contactomorphism will be proved in Proposition 2.8.

2.3. Polarizations on $G$, $P$ and $M$. We denote by $\pi : P \rightarrow M$ the quotient map, with $M = P/Q$. If $p \in P$ and $m \in M$, we use the notation $p.m$ or $p(m)$ for the action of $p$ on $m$. In such contexts, we will identify elements $p \in P$ with smooth diffeomorphisms $p : M \rightarrow M$.

Recall that on $G$ we have the polarisation $\Delta_G$ with $(\Delta_G)_e = g_{-1}$. We define on $P$ the polarisation $\Delta_P$ such that $(\Delta_P)_e = g_{-1} \oplus \mathfrak{q}$. Notice that $\Delta_G = \Delta_P \cap TG$. Define $\Delta_M := d\pi(\Delta_P)$ which is a subset of $TM$. We shall prove that $\Delta_M$ is a $P$-invariant polarisation on $M$.

Proposition 2.8. The set $\Delta_M \subset TM$ is a $P$-invariant, bracket generating subbundle of $M$. In particular, $(M, \Delta_M)$ is a polarised manifold and the diffeomorphisms $p : M \rightarrow M$ are contactomorphisms.

Proof. Notice that $\Delta_M$ is a $P$-invariant subset of $TM$. In order to show that $\Delta_M$ is a subbundle, we need to prove that, if $p_1, p_2 \in P$ are such that $\pi(p_1) = \pi(p_2)$, then

\begin{equation}
(1) \quad d\pi((\Delta_P)_{p_1}) = d\pi((\Delta_P)_{p_2}).
\end{equation}

Since $p \circ \pi = \pi \circ L_p$ for all $p \in P$, then (1) is equivalent to $d(p_2^{-1} \circ \pi \circ L_{p_1})([\Delta_P]_c) = d\pi([\Delta_P]_c)$. Let $p = p_1$ and $q \in Q$ such that $p_2 = p_1q$. Then $p_2^{-1} \circ \pi \circ L_{p_1} = \pi \circ L_{q^{-1}}$ and thus (1) is also equivalent to

\begin{equation}
(2) \quad \text{Ad}_q([\Delta_P]_c) \mod \mathfrak{q} = ([\Delta_P]_c) \mod \mathfrak{q}.
\end{equation}

Since $\text{Ad}$ is a homomorphism and every $q \in Q$ is the finite product of exponential elements, it’s enough that we show (2) for $q = \exp y$, $y \in \mathfrak{q}$. Denote by $y_0$ the projection of $y$ on $\mathfrak{g}_0$. Let $w \in \mathfrak{g}_{-1} \oplus \mathfrak{q}$ and denote by $w_{-1}$ its projection on $\mathfrak{g}_{-1}$. Then

\begin{equation*}
\text{Ad}_q w \mod \mathfrak{q} = e^{\text{ad}(y_0)} w \mod \mathfrak{q} = e^{\text{ad}(y_0)} w_{-1} \mod \mathfrak{q}.
\end{equation*}

Since $e^{\text{ad}(y_0)} : \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_{-1}$ is a bijection, we conclude that $\text{Ad}_q([\Delta_P]_c) \mod \mathfrak{q} = \mathfrak{g}_{-1} \mod \mathfrak{q}$. This proves (2) and therefore (1).

Finally, we need to show that $\Delta_M$ is bracket generating. Remind that, for an analytic subbundle of an analytic manifold, being bracket generating is equivalent to being connected by curves tangent to the subbundle, and that quotients of Lie groups and invariant subbundles are all analytic. Thus, let $m_0 = \pi(p_0)$ and $m_1 = \pi(p_1)$ in $M$. Then there is a $C^1$-curve $\gamma : [0, 1] \rightarrow P$ such that $\gamma(0) = p_0$, $\gamma(1) = p_1$, and $\gamma'(t) \in \Delta_P$ for all $t \in [0, 1]$. Hence, the curve $\pi \circ \gamma : [0, 1] \rightarrow M$ goes from $m_0$ to $m_1$ and is clearly tangent to $\Delta_M$. \qed

Proposition 2.9. The restriction $\pi|_G : (G, \Delta_G) \rightarrow (M, \Delta_M)$ is a contactomorphism onto its image, which is an open subset of $M$.\[\square\]
2.4. Orbits of $G$ in $M$. If $f : M \to M$ is a contactomorphism, it can be restricted to $\Omega := \pi(G) \cap f^{-1}(\pi(G))$ so that $f : \Omega \to f(\Omega)$ is a contactomorphism on $G$, where we identify $G$ with $\pi(G)$. How large is $\Omega$ depends on how large is the complement of $\pi(G)$ in $M$, because $G \setminus \Omega = f^{-1}(M \setminus \pi(G)) \cap \pi(G)$. For instance, if $\pi(G) = M$, then clearly $\Omega = M$, while if $\pi(G)$ is dense in $M$, then $\Omega$ is also dense in $G$.

For this reason we are interested in the orbits of $G$ in $M$. Indeed, the following results imply in particular that the boundary of $\Omega$ is locally the zero set of a analytic map and that $\Omega = G$ if $q = p_0$.

Proposition 2.11. Each orbit of $G$ in $M$ is either open or nowhere dense. In particular, there are (at most) countably many open orbits of $G$ in $M$ and the complement of their union is locally the zero set of an analytic map.

Proof. For $m \in M$, define $\phi_m : G \to M$, $\phi_m(x) = x.m$. Notice that, for all $m \in M$ and $g, x \in G$, $\phi_m(x) = x.m = g.(g^{-1}x).m = g \circ \phi_m \circ L_{g^{-1}}(x)$. Differentiating at $g$ we get,

$$d\phi_m|_g = dg|_m \circ d\phi_m|_e \circ dL_{g^{-1}}|_g.$$  \hfill (3)

Therefore, the rank of $d\phi_m|_g$ does not depend on $g$ and, for all $m \in M$, $\phi_m$ is an embedding $G \to M$ if and only if $d\phi_m|_e : g \to T_mM$ is bijective. Since each orbit of $G$ in $M$ is of the form $\phi_m(G)$ for some $m \in M$, we obtain that it is either open or nowhere dense.

Notice that the map $m \mapsto d\phi_m|_e$ is analytic (when written in analytic coordinates). Indeed, if $\Phi : G \times M \to M$ is the action $\Phi(g, m) = g.m$, which is analytic, then $d\phi_m|_e = d\Phi|_{(e, m)}|_q \times \{0\}$ is analytic in $m$. In particular, the function $m \mapsto \det(d\phi_m|_e)$, which is well defined only in coordinates, is analytic and its zeros correspond to points $m \in M$ whose orbit $G.m$ is nowhere dense.

Proposition 2.12. If $q = p_0$, then $G$ is normal in $P$ and therefore $P = G \times Q$. In particular, for all $p \in P$ there are $g \in G$ and $q \in Q$ such that $p = gq$, i.e., $\pi(p) = \pi(g)$. So, $\pi(G) = M$.

Proof. If $q = p_0$, then $G$ is normal in $P$ and therefore $P = G \times Q$. In particular, for all $p \in P$ there are $g \in G$ and $q \in Q$ such that $p = gq$, i.e., $\pi(p) = \pi(g)$. So, $\pi(G) = M$. □
2.5. Contactomorphisms of $M$ when $G$ is rigid. This section contains a number of lemmas that are preparatory for the proofs of Theorems 4.4 and 4.6. Relative to a vector $X \in T_P P$, we denote by $\tilde{X}$ the left-invariant vector field $\tilde{X}(p) = dL_p|_{\mathfrak{e}}[X]$, and by $X^\dagger$ the right-invariant vector field $X^\dagger(p) = dR_p|_{\mathfrak{e}}[X]$. Similarly, we denote by $\mathfrak{p}$ the Lie algebra of left-invariant vector fields and by $\mathfrak{p}^\dagger$ the Lie algebra of right-invariant vector fields on $P$.

Lemma 2.13. Let $\ell : \mathfrak{p} \to \mathfrak{g}$ be a Lie algebra automorphism with $\ell(q) = q$ and $\ell(\mathfrak{g}_{-1} \oplus \mathfrak{q}) = \mathfrak{g}_{-1} \oplus \mathfrak{q}$. Then there are a unique contactomorphic Lie group automorphism $L : P \to P$ with $L^\pi \circ \pi = \pi \circ L$.

Proof. If $\ell : \mathfrak{p} \to \mathfrak{g}$ is a Lie algebra automorphism with $\ell(q) = q$, then the induced Lie group automorphism $L : P \to P$ is such that $L(K) = K$, where $K$ is the kernel of the action of $P$ on $M$. It follows that there is a Lie group automorphism $L : P \to P$ such that $L_* = \ell$.

If $L : P \to P$ is a Lie group automorphism with $L(Q) = Q$, then it is well known that there is a unique diffeomorphism $L^\pi : M \to M$ such that $L^\pi \circ \pi = \pi \circ L$.

Now, suppose that $\ell(\mathfrak{g}_{-1} \oplus \mathfrak{q}) = \mathfrak{g}_{-1} \oplus \mathfrak{q}$, i.e., $L_*(\Delta_P)_e = (\Delta_P)_e$. Since $\Delta_P$ is left-invariant, then $L$ is a contactomorphism of $(P, \Delta_P)$. Finally, we prove that $L^\pi$ is a contactomorphism. Let $x \in \Delta_P$ and $\pi \in P$. Then
\[
d(l^\pi)_{|x}|[\pi(x)] = d(\pi \circ L)|_{\pi(x)} = d\pi|_{\pi(x)} = \pi(L^{\pi}|_{\pi(x)}).
\]

For the following two claims, see [15, 17]

Theorem 2.14. If $\mathfrak{p}$ is the full Tanaka prolongation of $\mathfrak{g}$ and it is finite dimensional, then $\pi_* \mathfrak{p}^\dagger \subset \Gamma(TM)$ is the set of all germs of contact vector fields on $M$. More precisely, on the one hand $\pi_* \mathfrak{p}^\dagger$ are contact vector fields of $(M, \Delta_M)$; On the other hand, if $U \subset M$ is open and $V \subset \Gamma(TU)$ is a contact vector field, then there is a unique $X \in \mathfrak{p}$ such that $V = \pi_* X^\dagger|_U$.

Lemma 2.15. Suppose that $\mathfrak{g}$ is rigid and that $\mathfrak{p}$ is the full Tanaka prolongation. Let $U \subset M$ open and $f : U \to M$ be a map such that $f(0) = 0$ and $f : U \to f(U)$ is a diffeomorphism with $df(\Delta_M) \subset \Delta_M$. Then there exists a unique Lie group automorphism $L : P \to P$ such that $f = L^\pi|_U$ as in Lemma 2.13. Such $L$ satisfies also $L(Q) = Q$ and $L_* (\Delta_P) = \Delta_P$.

Remark 2.16. In the Lemma 2.15 above, one can assume $f$ to be only smooth of class $C^2$. The upgrade of the regularity works like in [13].

Lemma 2.17. Suppose that $\mathfrak{g}$ is rigid and that $\mathfrak{p}$ is the full Tanaka prolongation. Let $D : \mathfrak{p} \to \mathfrak{g}$ be a derivation such that $D(q) \subset \mathfrak{q}$ and $D(\Delta_P) \subset \Delta_P$. Then there is $X \in \mathfrak{q}$ such that $D = \text{ad}_X$.

Proof. The one-parameter group of Lie algebra automorphisms $\ell_t := e^{tD}$ are such that $\ell_t(q) = q$ and $\ell_t(\Delta_P) = (\Delta_P)$. By Lemma 2.13, they induce a one-parameter group of Lie group automorphism $L_t$ on $P$ and a one-parameter group of contactomorphism $L^\pi_t : M \to M$.

Since $L^\pi_t$ is a one-parameter group of contactomorphisms on $M$ and by Theorem 2.14, there is $V \in \pi_* \mathfrak{p}^\dagger$ such that $L^\pi_t$ is its flow. Let $X \in \mathfrak{p}$ be such that $\pi_*(X^\dagger) = V$. Since $L^\pi_t(0) = 0$ and thus $V(0) = 0$, we have $X \in \mathfrak{q}$.

Notice that $L^\pi_t(m) = \exp(tX)m$ for all $m \in M$. Therefore, if $p \in P$ and $\pi(p)$, then
\[
\pi(L_t(p)) = L^\pi_t(\pi(p)) = \exp(tX).\pi(p) = \pi(\exp(tX)p) = \pi(\exp(tX)p \exp(-tX)) = \pi(C_{\exp(tX)}p),
\]
where \( C_{aP} = apa^{-1} \) is the conjugation by \( a \in P \). Since by Lemma 2.15 the lift of a contactomorphism from \( M \) to \( P \) is unique, we conclude that \( C_{\exp(tX)} = Lt \).

Finally, for all \( t \in \mathbb{R} \) we have

\[
ed^{tadX} = \text{Ad}_{\exp(tX)} = dC_{\exp(tX)}|_e = dL_t|_e = e^{tD}
\]

and thus \( D = ad_X \).

\[\square\]

3. Modifications of stratified groups

A polarised Lie algebra is a pair \((\mathfrak{s}, \mathfrak{s}_{-1})\) where \( \mathfrak{s} \) is a Lie algebra and \( \mathfrak{s}_{-1} \) is a bracket-generating subspace. We say that two polarised Lie algebras \((\mathfrak{s}, \mathfrak{s}_{-1})\) and \((\mathfrak{s}', \mathfrak{s}'_{-1})\) are isomorphic if there is a Lie algebra isomorphism \( \phi : \mathfrak{s} \to \mathfrak{s}' \) such that \( \phi(\mathfrak{s}_{-1}) = \mathfrak{s}'_{-1} \).

Given a stratified Lie algebra \( \mathfrak{g} \) and a finite dimensional Tanaka prolongation \( \mathfrak{p} = \text{Prol}(\mathfrak{g}, \mathfrak{g}_0) \), a modification of \( \mathfrak{g} \) in \( \mathfrak{p} \) is a polarized Lie algebra \((\mathfrak{s}, \mathfrak{s}_{-1})\) where \( \mathfrak{s} \subset \mathfrak{p} \) is a subalgebra such that \( \mathfrak{p} = \mathfrak{s} \oplus \mathfrak{q} \) and \( \mathfrak{s}_{-1} = (\mathfrak{g}_{-1} \oplus \mathfrak{q}) \cap \mathfrak{s} \). In other words, a modification of \( \mathfrak{g} \) in \( \mathfrak{p} \) is a subalgebra \( \mathfrak{s} \subset \mathfrak{p} \) of the form

\[
\mathfrak{s} := \{ X + \sigma(X) : X \in \mathfrak{g} \},
\]

for some \( \sigma : \mathfrak{g} \to \mathfrak{q} \) linear, endowed with the polarization

\[
\mathfrak{s}_{-1} := \{ X + \sigma(X) : X \in \mathfrak{g}_{-1} \}.
\]

Notice that \( \mathfrak{s}_{-1} \) bracket generates \( \mathfrak{s} \). Indeed, on the one hand, \( \mathfrak{s} \) has the same dimension as \( \mathfrak{g} \). On the other hand, one can easily check that, for iterated brackets of length \( k \geq 0 \), we have

\[
\left( [g_{-1}, \ldots, [g_{-1}, g_{-1}] \ldots] \mod \bigoplus_{j \geq -k} \mathfrak{g}_j \right) = \left( [g_{-1}, \ldots, [g_{-1}, g_{-1}] \ldots] \mod \bigoplus_{j \geq -k} \mathfrak{g}_j \right).
\]

If \( S \) is the connected Lie subgroup of \( P \) with \( T_eS = \mathfrak{s} \), we call the pair \((S, \Delta_S)\) modification of \( G \) in \( P \), where \( \Delta_S|_e = \mathfrak{s}_{-1} \). If \((\mathfrak{s}', \mathfrak{s}'_{-1})\) is a polarized Lie algebra that is isomorphic to a modification of \( \mathfrak{g} \) in \( \mathfrak{p} \), then we just say that \( \mathfrak{s} \) is a modification of \( \mathfrak{g} \). Similarly, a modification of \( G \) is any polarized group \((S, \Delta_S)\) whose Lie algebra is a modification of \( \mathfrak{g} \).

**Proposition 3.1.** Let \( S \) be a modification of \( G \) in \( P \). The restriction \( \pi|_S : S \to M \) is a contactomorphism when restricted from a neighbourhood of \( e_S \) to one of \( \pi(e_S) \).

**Proof.** We denote by \( o \) the base point \( \pi(e_S) \) in \( M \). Observe that \( d(\pi|_S)_e : T_eS = \mathfrak{s} \to T_oM \) is the restriction to \( \mathfrak{s} \) of \( d\pi|_\mathfrak{p} : \mathfrak{p} \to T_oM \). Since the kernel of \( d\pi|_\mathfrak{p} \) is \( \mathfrak{q} \), and since \( \mathfrak{q} \cap \mathfrak{s} = \{0\} \), \( d(\pi|_S)_e \) is injective. Moreover, \( \dim \mathfrak{s} = \dim \mathfrak{g} = \dim M \), so that \( d(\pi|_S)_e \) is a linear isomorphism. In particular, \( \pi|_S \) is a diffeomorphism between two open neighbourhoods of \( e_S \) and \( o \), respectively. Finally, on the one hand

\[
d(\pi|_S)((\Delta_S)_e) = d\pi((\Delta_P \cap TS)_e) \subseteq \Delta_M,
\]

while on the other hand \( \dim((\Delta_S)_e) = \dim(\mathfrak{g}_{-1}) = \dim((\Delta_M)_{\pi(s)}) \) for all \( s \in S \). So, at all points \( s \) where \( d(\pi|_S)_e \) is injective we have \( d(\pi|_S)_s(\mathfrak{s}_{-1}) = (\Delta_M)_{\pi(s)} \).

By Proposition 3.1, both maps

\[
\psi^G_S = \pi|_{\mathfrak{s}_-1}^{-1} \circ \pi|_G : U_G \to U_S
\]

are contactomorphism between a neighborhood \( U_G \) of \( e_G \) in \( G \) and a neighborhood \( U_S \) of \( e_S \) in \( S \). They are one the inverse of the other. In different situations we may be more interested in one direction than the other.

**Remark 3.2.** If we are given \( G \) and \( S \) in \( P \) (for instance as matrix groups), then for any \( s \in S \) the image \( \psi^G_S(s) \) is the only element \( g \) of \( G \), if it exists, such that \((sQ) \cap G = \{g\} \). Such an element is unique because \( \pi : G \to P/Q \) is injective. ♥
Lemma 3.3. The differential $d\psi^G_S|_{e_S} : \mathfrak{g} \to \mathfrak{s}$ is the map $X \mapsto X + \sigma(X)$.

Proof. The thesis follows from the identities $d\psi^G_S|_{e_S} = (d\pi|_{\mathfrak{g}}|_s)^{-1} \circ (d\pi|_{\mathfrak{g}}|_|\mathfrak{q})$ and $d\pi|_e(X + \sigma X) = d\pi|_e(X)$.

4. Main results

In this section we will present all main results of the paper. Let's repeat the notation we are using. We denote by $G$ a stratified Lie group whose Lie algebra is $\mathfrak{g} = \bigoplus_{j=-s}^{1} \mathfrak{g}_j$. The Lie algebra $\mathfrak{p} = \bigoplus_{j=-s}^{\infty} \mathfrak{p}_j$ is a finite dimensional Tanaka prolongation of $\mathfrak{g}$ and $\mathfrak{q} = \bigoplus_{j=0}^{\infty} \mathfrak{p}_j$, while $P$ and $Q < P$ are the corresponding Lie groups. So we also have $G < P$. We denote by $\pi$ the quotient $P/Q$ and by $\psi$ the quotient projection $P \to M$. The vector bundles $\Delta_G$, $\Delta_P$ and $\Delta_M$ are the left-invariant vector bundle generated by $\mathfrak{g}_{-1}$, $\mathfrak{g}_{-1} \oplus \mathfrak{q}$ and $d\pi|_e(\mathfrak{g}_{-1})$, respectively.

A stratified Lie algebra is rigid if its full Tanaka prolongation is finite-dimensional. A stratified group is rigid if its Lie algebra is rigid.

Theorem 4.1. A modification of a stratified Lie group $G$ is locally contactomorphic to $G$.

Proof. Let $\mathfrak{g}$ be the Lie algebra of $G$ and $\mathfrak{p} = \mathfrak{g} \oplus \mathfrak{q}$ a finite Tanaka prolongation of $\mathfrak{g}$. Let $\mathfrak{s} \subset \mathfrak{p}$ be a modification of $\mathfrak{g}$, i.e., a Lie subalgebra such that $\mathfrak{p} = \mathfrak{s} \oplus \mathfrak{q}$. Let $P, Q < P$ and $S < P$ be the corresponding Lie groups. On $P$ we have the polarisation $\Delta_P$ induced by $\mathfrak{g}_{-1} \oplus \mathfrak{q}$, which is so that $\Delta_G = \Delta_P \cap TG$ and $\Delta_S = \Delta_P \cap TS$. Let $M = P/Q$ endowed with the polarisation $\Delta_M$ given by the push forward of $\Delta_P$ via the quotient map $\pi : P \to M$. From Proposition 3.1 that the restrictions $\pi|_G : (G, \Delta_G) \to (M, \Delta_M)$ and $\pi|_S : (S, \Delta_S) \to (M, \Delta_M)$ are contactomorphisms near to the identity element. It follows that

$$\psi^G_S = \pi|_S^{-1} \circ \pi|_G : U_G \to U_S$$

is a contactomorphism from a neighborhood $U_G$ of $e_G$ in $G$ to a neighborhood $U_S$ of $e_S$ in $S$.

Theorem 4.2. Suppose that $G$ is a rigid stratified group and that $(S, \Delta_S)$ is a polarized Lie group that is locally contactomorphic to $G$. Then $(S, \Delta_S)$ is a modification of $G$.

Proof. Let $\mathfrak{p}$ be the full Tanaka prolongation of $\mathfrak{g}$. Let $\psi : U_S \to U_G$ be a contactomorphism from an open subset $U_S \subset S$ to $U_G \subset G$. Up to composing $\psi$ with left translations on $S$ and on $G$, we may assume $\psi(e_S) = e_G$.

Let $\mathfrak{s}^1 \subset \Gamma(TS)$ be the Lie algebra of right-invariant vector fields on $S$. Since $\mathfrak{s}^1$ is made of contact vector fields on $S$ and since the Tanaka prolongation of $\mathfrak{g}$ coincides canonically with the Lie algebra of germs of contact vector fields on $G$, $\psi_* : \Gamma(TU_S) \to \Gamma(TU_G)$ gives an injective Lie algebra morphism $\psi_* : \mathfrak{s}^1 \hookrightarrow \mathfrak{p}$.

Notice that if $X \in \mathfrak{s}^1$ is such that $\psi_*X(e_G) = 0$, then $X = 0$. Therefore, $\psi_*(\mathfrak{s}^1) \cap \mathfrak{q} = \{0\}$. Since $S$ and $G$ have the same dimension, we obtain that

$$\psi_*(\mathfrak{s}^1) = \{X + \sigma X : X \in \mathfrak{g}\}$$

for some linear map $\sigma : \mathfrak{g} \to \mathfrak{q}$.

Finally, since $d\psi((\Delta_S)_{e_S}) = (\Delta_G)_{e_G}$, we obtain that

$$\psi_*X \in (\mathfrak{s}^1) : X(e_G) \in (\Delta_S)_{e_S} = \{X + \sigma X : X \in \mathfrak{g}_{-1}\}.$$

We conclude that $(\psi_*(\mathfrak{s}^1), d\psi((\Delta_S)_{e_S}))$ is a modification of $\mathfrak{g}$ in $\mathfrak{p}$.

Remark 4.3. In the case $G$ is not rigid, i.e., the full Tanaka prolongation of $\mathfrak{g}$ is infinite dimensional, then the argument in the proof of Theorem 4.2 does not work. However, the example of the Heisenberg group, which is not rigid, shows
that it may still be possible to obtain as modifications all Lie groups that are locally contactomorphism to $G$.

Let $\text{Aut}(\mathfrak{p}, g)$ be the group of Lie algebra automorphisms of $\mathfrak{p}$ that induce contactomorphism on $M$. By Lemma 2.13, we have

$$\text{Aut}(\mathfrak{p}, g) = \{ \phi \in \text{Aut}(\mathfrak{p}) : \phi(q) = q, \phi(\mathfrak{g}_{-1} \oplus q) = \mathfrak{g}_{-1} \oplus q \}.$$ 

This group plays a crucial role in the classification of modifications of a stratified Lie algebra, as we shall show in Theorem 4.6.

**Theorem 4.4.** Suppose $g$ is rigid and let $\mathfrak{p}$ be its full Tanaka prolongation. Denote by $o$ the point $\pi(e) \in M$. Let $U \subset M$ open and $f : U \to f(U) \subset M$ be a $\Delta$-function such that $f(o) = o$ and $df(\Delta_M) \subset \Delta_M$. Then there is a Lie group automorphism $L : P \to P$ such that $L_\ast \in \text{Aut}(\mathfrak{p}, g)$ and $\pi(L(x)) = f(\pi(x))$ for all $x \in \pi^{-1}(U)$. In particular, $f$ extends uniquely to a contactomorphism $M \to M$.

**Proof.** This is a direct consequence of Lemma 2.15.

**Theorem 4.5.** Suppose $g$ is rigid and let $\mathfrak{p}$ be its full Tanaka prolongation. The Lie algebra of $\text{Aut}(\mathfrak{p}, g)$ is $\{ \text{ad}_X : X \in \mathfrak{q} \}$. In particular, the connected component of the identity in $\text{Aut}(\mathfrak{p}, g)$ is $\{ \text{Ad}_x : x \in Q \}$.

**Proof.** This is a direct consequence of Lemma 2.17.

**Theorem 4.6.** Suppose $g$ is rigid. If $s, s'$ are two modifications of $\mathfrak{g}$ in $\mathfrak{p}$, and if there is an isomorphism $\phi : s \to s'$ such that $\phi(s \cap (\mathfrak{g}_{-1} \oplus \mathfrak{q})) = s' \cap (\mathfrak{g}_{-1} \oplus \mathfrak{q})$, then there is a unique $\ell \in \text{Aut}(\mathfrak{p}, g)$ such that $\phi = \ell \vert_s$.

**Proof.** Let $S, S' \subset P$ be the subgroups of $P$ whose Lie algebra are $s$ and $s'$ respectively, endowed with the polarizations induced by $P$, e.g., $\Delta_S = \Delta_P \cap TS$. The map $\phi$ defines a local contactomorphism $\Phi : \Omega \to \Phi(\Omega), \Omega \subset S$ open with $e \in \Omega$. We may assume that $\pi| \Omega : \Omega \to \pi(\Omega) \subset M$ and $\pi| \Phi(\Omega) : \Phi(\Omega) \to \pi(\Phi(\Omega)) \subset M$ are contactomorphisms, see Proposition 3.1. Define $U := \pi(\Omega)$ and $f := \pi \circ \Phi \circ \pi| \Omega^{-1} : U \to f(U) = \pi(\Phi(\Omega))$. The map $f$ is then a contactomorphism. By Theorem 4.4, there is a Lie group automorphism $L : P \to P$ such that $f(\pi(p)) = \pi(L(p))$ for all $p \in \pi^{-1}(U)$. Now, we claim that the map $L_\ast : \mathfrak{p} \to \mathfrak{p}$ restricted to $s$ is equal to $\phi$. Indeed, if $X \in s$, then

$$L_\ast[X] = L_\ast[X^\dagger] = \pi^{-1} \circ \pi_\ast \circ L_\ast \circ \pi^{-1} \circ \pi_\ast[X^\dagger] = \pi^{-1} \circ f_\ast \circ \pi_\ast[X^\dagger] = \pi^{-1} \circ (\pi \circ \Phi \circ \pi| \Omega^{-1})_\ast \circ \pi_\ast[X^\dagger] = \Phi_\ast[X^\dagger] = \phi(X).$$

5. Examples

In this section we consider a few applications of our main results. First, we observe that every left-invariant three dimensional contact structure in $\mathbb{R}^3$ is a modification of the Heisenberg group, and consequently is locally contactomorphic to it. Although this is of course a consequence of the more general Darboux Theorem, we believe it is a good example for presenting our techniques. Second, we study some modifications of the free nilpotent Lie algebra $f_{24}$. In this case we are able to find a nilpotent modification $(N, \Delta_N)$ of the stratified group $F_{24}$ corresponding to $f_{24}$ that is globally contactomorphic to $F_{24}$, but not isomorphic. In particular, if we endow $N$ and $F_{24}$ with left-invariant sub-Riemannian distances, our example shows two nilpotent Lie groups that are bi-Lipschitz on every compact set but not isomorphic.
5.1. Modifications of the Heisenberg group.

We study the consequences of the results of the previous section in the case where \( g \) is the three-dimensional Heisenberg algebra. We will show that all three dimensional left-invariant contact structures are modifications of the Heisenberg group in a finite-dimensional prolongation. Our study is based on a classification of three-dimensional Lie algebras due to several authors. We summarise the results we need in the following theorem.

**Theorem 5.1.** Let \((s, s_{-1})\) be a three-dimensional polarised Lie algebra such that \(\dim(s_{-1}) = 2\). Then there is a basis \((f_1, f_2, f_3)\) of \(s\) with \(s_{-1} = \text{span}\{f_1, f_2\}\) such that exactly one of the following cases occurs:

(A) \[ f_1, f_2 = f_3, [f_1, f_3] = \alpha f_2 + \beta f_3 \text{ and } [f_2, f_3] = 0, \text{ for some } \alpha \in \mathbb{R} \text{ and } \beta \in \{0, 1\}. \]

In this case \(s\) is solvable and the non-isomorphic cases are exactly the following four:

(A.1) \[ f_1, f_2 = f_3, [f_1, f_3] = 0 \text{ and } [f_2, f_3] = 0; \]

(A.2) \[ f_1, f_2 = f_3, [f_1, f_3] = f_2 \text{ and } [f_2, f_3] = 0; \]

(A.3) \[ f_1, f_2 = f_3, [f_1, f_3] = -f_2 \text{ and } [f_2, f_3] = 0; \]

(A.4) \[ f_1, f_2 = f_3, [f_1, f_3] = \alpha f_2 + f_3 \text{ and } [f_2, f_3] = 0. \]

(B) \[ f_1, f_2 = f_3, [f_1, f_3] = -f_2, [f_2, f_3] = f_1. \]

In this case \(s = \mathfrak{su}(2)\) is simple.

(C) \[ f_1, f_2 = f_3, [f_1, f_3] = -f_1, [f_2, f_3] = f_2. \]

In this case \(s = \mathfrak{sl}(2, \mathbb{R})\) is simple.

(D) \[ f_1, f_2 = f_3, [f_1, f_3] = f_2, [f_2, f_3] = -f_1. \]

In this case \(s = \mathfrak{sl}(2, \mathbb{R})\) is simple.

Part of the proof of Theorem 5.1 is based on the following result, see [2].

**Lemma 5.2.** Let \(S\) be a three-dimensional solvable Lie group endowed with a left-invariant sub-Riemannian structure \((\Delta_s, g)\). There exist vectors \(e_1, e_2, e_3\) linearly independent in \(s\), \(\alpha \in \mathbb{R}\) and \(\beta \geq 0\) such that \(e_1, e_2\) is an orthonormal basis of \((\Delta_s)_e\) and

\[
\begin{align*}
[e_1, e_2] &= e_3, \quad [e_1, e_3] = \alpha e_2 + \beta e_3, \quad [e_2, e_3] = 0.
\end{align*}
\]

**Proof of Theorem 5.1.** If \(s\) is a three dimensional Lie algebra, then it is either solvable or simple. Indeed, the claim follows from the Levi decomposition and the fact that there are no simple Lie groups of dimension 1 or 2. If \(s\) is simple, then the \((s, s_{-1})\) falls into one the cases (B), (C) or (D), see [1]. Notice that the cases (C) and (D) are not isomorphic as polarised Lie algebras because \(\text{ad}_{f_1}\) is a reflection of \(s_{-1}\) in case (C), while in case (D) it is a rotation.

If \(s\) is solvable, then we apply Lemma 5.2 and obtain case (A). However, since the classification in Lemma 5.2 is up to isometry, we have to further discriminate to obtain non-isomorphic subcases. So, if \(f_1, f_2, f_3\) is a basis of \(s\) with \(s_{-1} = \text{span}\{f_1, f_2\}, [f_1, f_2] = f_3\) and \([f_2, f_3] = 0\), then we must have

\[
\begin{align*}
&f_1 = a_1^1 e_1 + a_1^2 e_2 \\
f_2 = a_2^2 e_2 \\
f_3 = a_3^1 a_3^2 e_3,
\end{align*}
\]

for some real coefficients. The third bracket relation is

\[ [f_1, f_3] = \alpha(a_1^1)^2 f_2 + \beta a_1^1 f_3. \]
Since $\alpha \in \mathbb{R}$ and $\beta \geq 0$, in each case we can choose $a_i^j$ in the following way:

\begin{align*}
\alpha = \beta = 0 & \quad a_1^1 = 1, \quad a_1^2 = 0, \quad a_2^2 = 1 : \quad [f_1, f_3] = 0 \\
\beta > 0, \quad \alpha \in \mathbb{R} & \quad a_1^1 = \frac{1}{\beta}, \quad a_1^2 = 0, \quad a_2^2 = 1 : \quad [f_1, f_3] = \frac{\alpha}{\beta^2} f_2 + f_3 \\
\beta = 0, \quad \alpha > 0 & \quad a_1^1 = \frac{1}{\sqrt{\alpha}}, \quad a_1^2 = 0, \quad a_2^2 = 1 : \quad [f_1, f_3] = f_2 \\
\beta = 0, \quad \alpha < 0 & \quad a_1^1 = \frac{1}{\sqrt{|\alpha|}}, \quad a_1^2 = 0, \quad a_2^2 = 1 : \quad [f_1, f_3] = -f_2
\end{align*}

Now, we want to show that cases (A.1), (A.2), (A.3) and (A.4) are not isomorphic to each other. Notice that $\ell := \text{span}\{f_3\} = [s_{-1}, s_{-1}]$ and $s^{(2)} := [s, s]$ are invariant under isomorphisms of polarised Lie algebras.

First, case (A.1) is not isomorphic to the others because in case (A.1) we have $s^{(2)} = \text{span}\{f_3\}$ while in all other cases we have $s^{(2)} = \text{span}\{f_2, f_3\}$.

Second, case (A.4) is not isomorphic to the others because in case (A.4) we have $[\ell, s_{-1}] \not\subseteq s_{-1}$ while in all other cases we have $[\ell, s_{-1}] \subseteq s_{-1}$.

Third, for different choices of $\alpha \in \mathbb{R}$ in case (A.4) we get non-isomorphic polarised Lie algebras: To prove this, we shall show that the parameter $\alpha$ is independent of the choice of the basis. So, suppose that $g_1, g_2, g_3 \in s$ form another basis with $s_{-1} = \text{span}\{g_1, g_2\}$. Then one easily shows that $g_1 = x f_1 + y f_2$, $g_2 = \mu f_2$ and $g_3 = \lambda f_3$, for some $x, y, \lambda, \mu \in \mathbb{R}$ with $\frac{y}{\mu} = 1$. Moreover, $[g_1, g_3] = \frac{\lambda y}{\mu^2} g_2 + x g_3$, which implies $x = 1$ and $\lambda = \alpha'$.

Finally, cases (A.2) and (A.3) are not isomorphic to each other, because in case (A.2) it holds $\text{ad}_{f_1^{(2)}} = Id|_{s^{(2)}}$, while in case (A.3) it holds $\text{ad}_{f_1^{(2)}} = -Id|_{s^{(2)}}$.

It is well known that the full Tanaka prolongation of the Heisenberg Lie algebra $g$ is infinite. However, there is a number a different choices of subalgebras $g_0 \subset \text{Der}(g)$ that generate finite dimensional prolongations. We shall show that these finite prolongations are enough to recover all polarized Lie groups that are locally contactomorphic with the Heisenberg group, i.e., all three dimensional Lie groups with a non-trivial polarization.

**Theorem 5.3.** Let $(s, s_{-1})$ be a three dimensional polarised Lie algebra such that $\dim(s_{-1}) = 2$. Then there is a finite-dimensional prolongation $\mathfrak{p}$ of the Heisenberg Lie algebra $\mathfrak{h}$ so that $(s, s_{-1})$ is isomorphic to a modification in $\mathfrak{p}$.

**Proof.** Let us fix the notation for the Heisenberg Lie algebra. Fix a basis $e_1, e_2, e_3$ so that $[e_1, e_2] = e_3$, and choose $s_{-1} = \text{span}\{e_1, e_2\}$. The space $\text{Der}(g)$ of the strata preserving derivations of $g$ may be identified with $\mathfrak{gl}(2, \mathbb{R})$.

First, we consider

$g_0 := \{D \in \text{Der}(g) : D(e_1) \subseteq \mathbb{R} e_1 \text{ and } D(e_2) \subseteq \mathbb{R} e_2\}$.

In this case, $\text{Pro}(g, g_0) = \mathfrak{sl}(3, \mathbb{R}) = g \oplus q$ (see, e.g., [4]), where $g$ is identified with the Lie algebra generated by

\begin{equation}
e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\end{equation}

and $q$ is the set of matrices in $\mathfrak{sl}(3, \mathbb{R})$ of the form

\begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix}
The modifications of $\mathfrak{g}$ in $\mathfrak{sl}(3, \mathbb{R})$ are the subalgebras of $\mathfrak{sl}(3, \mathbb{R})$ of the form $\{X + \sigma(X) : X \in \mathfrak{g}\}$, for some linear map $\sigma : \mathfrak{g} \to \mathfrak{q}$. We show that all three dimensional Lie algebras with a bracket generating plane are graphs of such a $\sigma$:

Case (A): If $\mathfrak{s}$ is solvable, then define $\sigma$ by the assignments:

$$\sigma(e_1) = \begin{pmatrix} 2\beta & 0 & 0 \\ \alpha & -\beta & 0 \\ 0 & 0 & -\beta \end{pmatrix}, \quad \sigma(e_2) = \sigma(e_3) = 0.$$ 

It is easy to check that vectors $f_i := e_i + \sigma(e_i)$, $i = 1, 2, 3$, satisfy the bracket relations of case (A) in Theorem 5.1.

Case (B): For this case, we choose

$$\sigma(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \sigma(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad \sigma(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$ 

Case (C): we obtain the brackets in (C) by choosing

$$\sigma(e_1) = 0, \quad \sigma(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \sigma(e_3) = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$ 

Case (D): In this case we use the finite prolongation $\mathfrak{su}(2,1)$ of the Heisenberg algebra, as in [8, p313]. Let

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$ 

The Lie algebra $\mathfrak{su}(2,1)$ is given by $3 \times 3$ complex matrices $A$ with zero trace and such that $A^*J + JA = 0$, where $A^*$ is the hermitian transpose of $A$. Define the Lie algebra automorphism $\theta : \mathfrak{su}(2,1) \to \mathfrak{su}(2,1)$, $\theta A := JA$. Define

$$X = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & -i \\ 0 & -i & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 2i & 0 & 2i \\ 0 & 0 & 0 \\ -2i & 0 & -2i \end{pmatrix},$$

$$H = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} i & 0 & 0 \\ 0 & -2i & 0 \\ 0 & 0 & i \end{pmatrix},$$

$$\theta X = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & -i & 0 \end{pmatrix}, \quad \theta Y = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \theta Z = \begin{pmatrix} 2i & 0 & -2i \\ 0 & 0 & 0 \\ 2i & 0 & -2i \end{pmatrix}.$$ 

The grading of $\mathfrak{su}(2,1)$ is

$$\mathfrak{g}_{-2}(\mathfrak{g}) = \text{span}[Z]$$
$$\mathfrak{g}_{-1}(\mathfrak{g}) = \text{span}[X,Y]$$
$$\mathfrak{g}_{0}(\mathfrak{g}) = \text{span}[H,U]$$
$$\mathfrak{g}_{1}(\mathfrak{g}) = \text{span}[\theta X, \theta Y]$$
$$\mathfrak{g}_{2}(\mathfrak{g}) = \text{span}[\theta Z],$$

where $\mathfrak{g}_{-2}(\mathfrak{g}) \oplus \mathfrak{g}_{-1}(\mathfrak{g}) \oplus \mathfrak{g}_{0}(\mathfrak{g}) \oplus \mathfrak{g}_{1}(\mathfrak{g}) \oplus \mathfrak{g}_{2}(\mathfrak{g}) = \mathfrak{g}$ is the Heisenberg Lie algebra: notice that $[X,Y] = Z$ while $[X,Z] = [Y,Z] = 0$. So, $\mathfrak{q} = \text{span}[H,U,\theta X,\theta Y,\theta Z]$. Define $\sigma : \mathfrak{g} \to \mathfrak{q}$ by
setting

\[ \sigma X := -\frac{1}{16} \theta X + i \frac{9}{16} \theta Y = \begin{pmatrix} 0 & -i \frac{1}{2} & 0 \\ -i \frac{5}{8} & 0 & i \frac{1}{8} \\ 0 & -i \frac{1}{2} & 0 \end{pmatrix}, \]

\[ \sigma Y := -i \frac{9}{16} \theta X - \frac{1}{16} \theta Y = \begin{pmatrix} 0 & -i \frac{1}{2} & 0 \\ i \frac{5}{8} & 0 & -i \frac{1}{8} \\ 0 & -i \frac{1}{2} & 0 \end{pmatrix}, \]

\[ \sigma Z := -i \frac{9}{4} H + \frac{1}{4} U - \frac{5}{16} \theta Z = \begin{pmatrix} -i \frac{13}{2} & 0 & -i \frac{123}{8} \\ 0 & -i \frac{1}{2} & 0 \\ -i \frac{13}{2} & 0 & i \frac{1}{8} \end{pmatrix}. \]

One can easily check that \( f_1 = X + \sigma X \), \( f_2 = Y + \sigma Y \) and \( f_3 = Z + \sigma Z \) form a basis of a Lie subalgebra of \( \mathfrak{su}(2,1) \) satisfying the relations of Case (D).

\[ \square \]

Remark 5.4. The map \( \sigma \) above can easily be found using the software Maple and it is not the unique.

We conclude this section discussing more in detail the case of the group of rigid motions of the plane as a modification of the Heisenberg group. At a group level, we may represent points in the Heisenberg group \( H \) as matrices in \( SL(3,\mathbb{R}) \) by

\[ H(x_1, x_2, x_3) := \begin{pmatrix} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix}, \]

for \( x_1, x_2, x_3 \in \mathbb{R} \).

The Lie algebra of the the group of rigid motions of the plane \( E(2) \) corresponds to the case (A) with \( \alpha = -1 \) and \( \beta = 0 \). The corresponding representation in \( \mathfrak{sl}(3,\mathbb{R}) \) given in the previous theorem is the span of the vectors

\[ f_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad f_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]

At the group level, the points of \( E(2) \) inside \( SL(3,\mathbb{R}) \) are parametrized by

\[ R(y_1, y_2, y_3) := \begin{pmatrix} \cos y_1 & \sin y_1 & y_3 \\ -\sin y_1 & \cos y_1 & y_2 \\ 0 & 0 & 1 \end{pmatrix}, \]

where \( y_1 \in \mathbb{R}/(2\pi \mathbb{Z}) \) and \( y_2, y_3 \in \mathbb{R} \).

With the procedure described in Remark 3.2, we find the mapping \( E(2) \to H \):

\[ R(y_1, y_2, y_3) \mapsto H(\tan y_1, y_2, y_3), \]

which is defined on the domain \( (-\pi/2, \pi/2) \times \mathbb{R}^2 \).

5.2. Modifications of the free nilpotent Lie group \( F_{24} \). We consider with the free nilpotent Lie algebra \( f_{24} = \text{span}\{e_i : i = 1, \ldots, 8\} \). The Lie brackets are

\[ [e_2, e_1] = e_3, \quad [e_3, e_1] = e_4, \quad [e_3, e_2] = e_5, \]

\[ [e_4, e_1] = e_6, \quad [e_5, e_1] = e_7, \quad [x_4, e_2] = e_7, \quad [e_5, e_2] = e_8. \]

It is known that the full Tanaka prolongation of \( f_{24} \) is \( p = f_{24} \oplus \text{Der}(g) \), with \( \text{Der}(g) \simeq \mathfrak{gl}(2, \mathbb{R}) \) (see [16]). Therefore, the modifications of \( f_{24} \) are subalgebras of \( p \) that are graphs of some linear map \( \sigma : f_{24} \to \mathfrak{gl}(2, \mathbb{R}) \). Here we only consider \( \sigma \) that on the basis of \( f_{24} \) is zero except for \( \sigma(e_1) \). Imposing that the graph is a Lie algebra, a direct computation shows that

\[ \sigma(e_1) = \begin{pmatrix} a & 0 \\ c & b \end{pmatrix}, \]
where $a, b, c \in \mathbb{R}$. We obtain a three parameter family $\mathfrak{s}(a, b, c)$ of Lie algebras with basis $f_1, \ldots, f_8$ and brackets

$$
[f_2, f_1] = f_3 - bf_2, \quad [f_3, f_1] = (a + b)f_3 - f_4, \quad [f_3, f_2] = f_5,
$$

$$
[f_4, f_1] = cf_5 + (2a + b)f_4 - f_6, \quad [f_4, f_2] = f_7, \quad [f_5, f_1] = (a + 2b)f_5 - f_7,
$$

$$
[f_1, f_0] = 2c f_2 + (3a + b)f_6, \quad [f_1, f_1] = cf_8 + (2a + 2b)f_7,
$$

$$
[f_1, f_8] = (a + 3b)f_8, \quad [f_5, f_2] = f_8.
$$

In particular, setting $a = b = 0$ gives a one parameter family of nilpotent Lie algebras $\mathfrak{s}(c)$. We now find the contact mapping $\Psi$ from $S(c)$ to $F_{24}$ when $c = 1$, as in Remark 3.2. In this case we have $f_1 = e_1 + \sigma(e_1)$ and $f_i = e_i$ for $i = 2, \ldots, 8$. Then every point in $S(1)$ is of the form $\exp_p(\sum x_i f_i)$. Following [12], $\exp_p(\sum x_i f_i) = (\mathcal{E}_{F_{24}}(x_1 \sigma(e_1); \sum x_i e_i), \exp_{GL}(x_1 \sigma(e_1))) \in F_{24} \times GL(2, \mathbb{R})$, where $\mathcal{E}_{F_{24}}(x_1 e_1; \sum x_i e_i) = \gamma(1)$ and $\gamma : [0, 1] \to F_{24}$ is the solution of

$$
\begin{cases}
\gamma'(t) & = dL_{\gamma(t)} \exp_{GL}(t x_1 \sigma(e_1))(\sum x_i e_i) \\
\gamma(0) & = e_{F_{24}}.
\end{cases}
$$

The image of this point via $\Psi$ is going to be that element $p \in F_{24}$ such that $gQ = \exp\big(\sum x_i f_i\big)Q$, i.e.,

$$
\Psi \left( \exp\big(\sum x_i f_i\big) \right) = \mathcal{E}_{F_{24}}(x_1 e_1; \sum x_i e_i).
$$

To compute this, first,

$$
v := \exp_{GL}(t x_1 \sigma(e_1)) \left( \sum x_i e_i \right)
= (x_1, x_1^2 t + x_2, x_3, x_4, x_5 + tx_1 x_4, x_6, x_7 + 2tx_1 x_6, x_8 + tx_1 x_7 + t^2 x_1^2 x_6).
$$

Second, we need to compute $dL_{\gamma} v$ using the Baker—Campbell—Hausdorff formula:

$$
dL_{\gamma} v = \frac{d}{dh} \bigg|_{h=0} \exp^{-1}(\exp(\gamma) \exp(hv)) = v + \frac{1}{2}[\gamma, v] + \frac{1}{12}[\gamma, [\gamma, v]].
$$

The system of differential equations $\dot{\gamma} = dL_{\gamma} v$ that we obtain is

$$
\begin{align*}
\dot{\gamma}_1 & = x_1 \\
\dot{\gamma}_2 & = tx_1^2 + x_2 \\
\dot{\gamma}_3 & = -\frac{1}{2}tx_1^2 \gamma_1 - \frac{1}{2}x_2 \gamma_1 + \frac{1}{2}x_1 \gamma_2 + x_3 \\
\dot{\gamma}_4 & = \frac{1}{12}tx_1^2 \gamma_1 + \frac{1}{12}x_2 \gamma_1^2 - \frac{1}{12}(\gamma_1 \gamma_2 - 6 \gamma_3) x_1 - \frac{1}{2}x_3 \gamma_1 + x_4 \\
\dot{\gamma}_5 & = \frac{1}{12}x_2 \gamma_1 \gamma_2 - \frac{1}{12} \gamma_1 \gamma_2^2 + \frac{1}{12}(x_2^2 \gamma_1 \gamma_2 + 6x_2^2 \gamma_3 + 12x_1 x_4) t - \frac{1}{2}x_3 \gamma_2 \\
& \quad + \frac{1}{2}x_2 \gamma_3 + x_5 \\
\dot{\gamma}_6 & = \frac{1}{12}x_3 \gamma_1^2 - \frac{1}{12}(\gamma_1 \gamma_3 - 6 \gamma_4) x_1 - \frac{1}{2}x_4 \gamma_1 + x_6 \\
& \quad \quad - \frac{1}{12}(\gamma_2 \gamma_3 - 6 \gamma_5)x_1 - \frac{1}{2}x_5 \gamma_1 - \frac{1}{2}x_4 \gamma_2 + \frac{1}{2}x_2 \gamma_4 + x_7 \\
\dot{\gamma}_7 & = \frac{1}{6}x_3 \gamma_1 \gamma_2 - \frac{1}{12}x_2 \gamma_1 \gamma_3 - \frac{1}{12}(x_2^2 \gamma_1 \gamma_3 + 6x_1 x_4 \gamma_1 - 6x_1^2 \gamma_4 - 24x_1 x_6) t \\
& \quad - \frac{1}{12}(\gamma_2 \gamma_3 - 6 \gamma_5)x_1 - \frac{1}{2}x_5 \gamma_1 - \frac{1}{2}x_4 \gamma_2 + \frac{1}{2}x_2 \gamma_4 + x_7 \\
\dot{\gamma}_8 & = t^2 x_1^2 x_6 + \frac{1}{12}x_3 \gamma_2^2 - \frac{1}{12}x_2 \gamma_2 \gamma_3 - \frac{1}{12}(x_1^2 \gamma_2 \gamma_3 + 6x_1 x_4 \gamma_2 - 6x_1^2 \gamma_5 - 12x_1 x_7) t \\
& \quad - \frac{1}{2}x_5 \gamma_2 + \frac{1}{2}x_2 \gamma_5 + x_8.
\end{align*}
$$
Third, we need to integrate this system of ODEs with initial conditions $\gamma_i(0) = 0$ for every $i = 1, \ldots, 8$. The solution is

$$
\begin{align*}
\gamma_1(t) &= tx_1 \\
\gamma_2(t) &= \frac{1}{2} t^2 x_1^2 + tx_2 \\
\gamma_3(t) &= -\frac{1}{12} t^3 x_1^3 + tx_3 \\
\gamma_4(t) &= tx_4 \\
\gamma_5(t) &= -\frac{1}{240} t^5 x_1^5 + \frac{1}{12} t^3 x_1^3 x_3 + \frac{1}{2} t^2 x_1 x_4 + tx_5 \\
\gamma_6(t) &= \frac{1}{720} t^5 x_1^5 + tx_6 \\
\gamma_7(t) &= \frac{1}{720} t^5 x_1^5 + \frac{1}{360} t^4 x_1^2 x_2 + t^2 x_1 x_6 + tx_7 \\
\gamma_8(t) &= \frac{1}{5040} t^7 x_1^7 + \frac{1}{720} t^6 x_1^2 x_2^2 + \frac{1}{720} (x_1^3 x_2^2 + 3 x_1^4 x_3) t^5 \\
&\quad - \frac{1}{12} (x_1 x_2 x_4 - x_1^2 x_5 - 4 x_1^2 x_6) t^3 + \frac{1}{2} t^2 x_1 x_7 + tx_8.
\end{align*}
$$

Therefore, the mapping from $S(1)$ to $G$ is $\Psi : (x_1, \ldots, x_8) \mapsto \gamma(1)$, which is a global, surjective smooth contactomorphism and in particular biLipschitz on every compact set. Notice, however, that this is not a global quasiconformal mapping.

5.3. Modifications of ultra-rigid stratified groups. A stratified Lie algebra $\mathfrak{g}$ is called ultra-rigid if the only automorphisms of $\mathfrak{g}$ preserving the stratifications are dilations, see [10]. In particular, the full Tanaka prolongation of such $\mathfrak{g}$ is $\mathfrak{p} = \mathfrak{g} \rtimes \mathbb{R}$, as semi-direct product of Lie algebras. In this section we describe all modifications in $\mathfrak{g} \rtimes \mathbb{R}$ and their equivalence relation. Many results do not need the assumption of $\mathfrak{g}$ being ultra-rigid, so we assume this hypothesis only when needed.

Let $\mathfrak{g} = \bigoplus_{j=-\infty}^{\infty} \mathfrak{g}_j$ be a stratified Lie algebra. Let $D : \mathfrak{g} \to \mathfrak{g}$ be the linear map with $Dv = jv$ for $v \in \mathfrak{g}_{-j}$. Notice that $D$ is a derivation of $\mathfrak{g}$ that preserves the layers and that $\delta_v = e^{tD} : \mathfrak{g} \to \mathfrak{g}$ are the dilations.

The semi-direct product $\mathfrak{p} := \mathfrak{g} \rtimes \mathbb{R}$ is the Lie algebra whose Lie brackets are $[(0, a), (Y, 0)] = (aDY, 0)$ hence $[(X, a), (Y, b)] = ([X, Y] + aDY - bDX, 0)$.

**Proposition 5.5.** Let $\sigma : \mathfrak{g} \to \mathfrak{g}$ be a linear map and set $\mathfrak{s} := \{(X, \sigma X) : X \in \mathfrak{g}\}$. The vector space $\mathfrak{s}$ is a Lie subalgebra of $\mathfrak{g} \rtimes \mathbb{R}$ if and only if $\bigoplus_{j=-\infty}^{\infty} \mathfrak{g}_j \subset \ker \sigma$.

**Proof.** First, we note that $\mathfrak{s}$ is a Lie algebra if and only if, for all $X, Y \in \mathfrak{g}$,

$$
\sigma([X, Y]) + (\sigma X)(\sigma Y) - (\sigma Y)(\sigma DX) = 0. \tag{6}
$$

\[\square\] Suppose $\mathfrak{s}$ is a Lie algebra, i.e., (6) holds for all $X, Y \in \mathfrak{g}$. We prove $\bigoplus_{j=-\infty}^{\infty} \mathfrak{g}_j \subset \ker \sigma$ by induction on $j$. If $X, Y \in \mathfrak{g}_{-1}$, then $DX = X$ and $DY = Y$, thus (6) implies $\sigma([X, Y]) = 0$. Since $\mathfrak{g}_{-2} = [\mathfrak{g}_{-1}, \mathfrak{g}_{-1}]$, it follows that $\mathfrak{g}_{-2} \subset \ker \sigma$. Now, suppose that $\mathfrak{g}_{-k} \subset \ker \sigma$ for $k \geq 2$. If $X \in \mathfrak{g}_{-1}$ and $Y \in \mathfrak{g}_{-k}$, then (6) implies that $\sigma([X, Y]) = 0$. Since $\mathfrak{g}_{-k-1} = [\mathfrak{g}_{-1}, \mathfrak{g}_{-k}]$, it follows that $\mathfrak{g}_{-k-1} \subset \ker \sigma$. We conclude that $\bigoplus_{j=-\infty}^{\infty} \mathfrak{g}_j \subset \ker \sigma$.

\[\square\] Suppose $\bigoplus_{j=-\infty}^{\infty} \mathfrak{g}_j \subset \ker \sigma$. By the bilinearity of the expression, we need to show that (6) holds only when $X \in \mathfrak{g}_i$ and $Y \in \mathfrak{g}_j$ for some $i$ and $j$. Since $\sigma$ is non-zero only on the first layer, the only non-trivial instance of (6) is for $X, Y \in \mathfrak{g}_{-1}$. In this case, $\sigma([X, Y]) = 0$, and $(\sigma X)(\sigma Y) - (\sigma Y)(\sigma DX) = (\sigma X)(\sigma Y) - (\sigma Y)(\sigma X) = 0$. Therefore, (6) is satisfied and $\mathfrak{s}$ is a Lie algebra. \[\square\]
Lemma 5.6. The Lie algebra automorphisms $\phi : p \to p$ such that $\phi(\{0\} \times \mathbb{R}) = \{0\} \times \mathbb{R}$ and $\phi(g_{-1} \times \mathbb{R}) = g_{-1} \times \mathbb{R}$ are exactly those of the form $\phi(X, a) = (\phi_1 X, a)$ for some Lie algebra automorphism $\phi_1 : g \to g$ that preserves the layers.

Proof. On the one hand, if $\phi_1 : g \to g$ is a Lie algebra automorphism that preserves the layers, then $\phi(X, a) = (\phi_1 X, a)$ is clearly a Lie algebra automorphism $\phi : p \to p$ with $\phi(\{0\} \times \mathbb{R}) = \{0\} \times \mathbb{R}$ and $\phi(g_{-1} \times \mathbb{R}) = g_{-1} \times \mathbb{R}$, because $\phi_1 D = D\phi_1$.

On the other hand, if $\phi : p \to p$ is a Lie algebra automorphism, then $\phi(g \times \{0\}) = g \times \{0\}$ because $g \times \{0\} = [p, p]$. Suppose also that $\phi(\{0\} \times \mathbb{R}) = \{0\} \times \mathbb{R}$ and $\phi(g_{-1} \times \mathbb{R}) = g_{-1} \times \mathbb{R}$. Then $\phi(X, a) = \phi(X, 0) + \phi(0, a) = (\phi_1(X), 0) + (0, \phi_2(a))$ and $\phi_1(g_{-1}) = g_{-1}$. This implies that $\phi_1(\delta_j) = \delta_j$ for all $j$, as one can prove by induction on $j$. Notice that, for all $X \in g$ and all $a \in \mathbb{R}$,

$$
\phi_2(a)D\phi_1 X = [(0, \phi_2(a)), (\phi_1 X, 0)] = \phi(\phi([0, a]), (X, 0)) = \phi(aDX, 0) = a\phi_1 DX.
$$

For every $X \in g_{-1}$, $DX = X$ and $D\phi_1 X = \phi_1 X$, hence $\phi_2(a)\phi_1 X = a\phi_1 X$, i.e., $\phi_2(a) = a$. □

Proposition 5.7. Suppose that $g$ is ultra-rigid, i.e., $p = g \times \mathbb{R}$ is its full Tanaka prolongation. The set of all non-isomorphic modifications of $g$ is parametrized by $g_{-1}^*/\mathbb{R}_{>0}$. Moreover, all modifications of $g$ in $p$ are solvable and the only nilpotent one is $g$ itself.

Proof. The set of all modifications of $g$ in $p$ can be identified with $g_{-1}^*$ by Proposition 5.5. where $\sigma \in g_{-1}^*$ is identified with $\sigma(\sum_j v_j) = \sigma(v_{-1})$ for $\sum_j v_j \in g$ and the modification $a_\sigma := \{(X, \sigma X) : X \in g\} \subset p$. Since $g$ is rigid, by Theorem 4.6 two modifications $\sigma, \tau \in g_{-1}^*$ are isomorphic if and only if there is a Lie algebra automorphism $\phi : p \to p$ with $\phi(\{0\} \times \mathbb{R}) = \{0\} \times \mathbb{R}$ and $\phi(g_{-1} \times \mathbb{R}) = g_{-1} \times \mathbb{R}$ such that $\phi(a_\sigma) = a_\tau$. Therefore, by Lemma 5.6, two modifications $\sigma, \tau \in g_{-1}^*$ are isomorphic if and only if there is a Lie algebra automorphism $\phi_1 : g \to g$ such that, for all $X \in g$,

$$(\phi_1 X, \sigma X) = (\phi_1 X, \tau \phi_1 X),$$

i.e., $\sigma X = \tau \phi_1 X$ for all $X \in g_{-1}$. Now, since $g$ is ultra-rigid, $\phi_1 = \delta_\lambda$ for some $\lambda > 0$. Therefore, two modifications $\sigma, \tau \in g_{-1}^*$ are isomorphic if and only if there is $\lambda > 0$ such that $\sigma = \lambda \tau$.

Finally, notice that all modifications of $g$ in $p$ are solvable, because $p$ itself is solvable. Moreover, the only nilpotent modification is $g$ itself. Indeed, if $s \not= g$, then there is $X \in g_{-1}$ with $\sigma X \not= 0$, so that, if $Y \in g_s$ is nonzero, then $[(X, \sigma X), (Y, 0)] = \sigma s X (Y, 0)$, where $s$ is the step of $g$. Therefore, we obtain that $(Y, 0) \in [\sigma, [\ldots, [\sigma, s \ldots]]]$ for any order of brackets, that is, $s$ is not nilpotent. □

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