DELTA SHOCKS AND VACUUM STATES IN VANISHING PRESSURE LIMITS OF SOLUTIONS TO THE RELATIVISTIC EULER EQUATIONS FOR GENERALIZED CHAPLYGIN GAS

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Abstract. The Riemann solutions for the relativistic Euler equations for generalized Chaplygin gas are considered. It is rigorously proved that, as the pressure vanishes, they tend to the two kinds of Riemann solutions to the zero-pressure relativistic Euler equations, which include a delta shock formed by a weighted $\delta$-measure and a vacuum state.

1. Introduction. The Euler system of conservation laws of energy and momentum in special relativity reads [8, 17, 20]

\[
\begin{align*}
\left\{ \frac{(p + \rho c^2)}{c^2(c^2 - v^2)} v^2 + \rho \right\}_t + \left\{ \frac{(p + \rho c^2)}{c^2 - v^2} \right\}_x &= 0, \\
\left\{ \frac{(p + \rho c^2)}{c^2 - v^2} v \right\}_t + \left\{ \frac{(p + \rho c^2)}{c^2 - v^2} v^2 + p \right\}_x &= 0,
\end{align*}
\]

where $p$, $\rho$ and $v$ represent the proper energy density, pressure and particle speed, respectively, and the constant $c$ is the speed of light. The system (1.1) with the equation of state

\[
p(\rho) = -A\rho^{-\alpha}, \quad 0 < \alpha < 1, \quad A > 0
\]

is called the generalized Chaplygin gas dynamics. System (1.1) models the dynamics of plane waves in special relativistic fluids (see [31, 32, 33, 36]) in a two dimensional Minkowski time-space $(x^0, x^1)$:

\[
\text{div} \, T = 0,
\]

where $T$ is the stress-energy tensor for the fluid:

\[
T^{ij} = (p + \rho c^2)u^iu^j + p\eta^{ij},
\]

with all indices running from 0 to 1 with $x^0 = ct$. In (1.4),

\[
\eta^{ij} = \delta_{ij} = \text{diag}(-1, 1)
\]

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denotes the flat Minkowski metric, $u$ the 2-velocity of the fluid particle (the velocity of the frame of isotropy of the perfect fluid), and $\rho$ the mass-energy density of the fluid as measured in units of mass in a reference frame moving with the fluid particle.

In recent years, astrophysicists have growing interests in the Chaplygin gas dynamics, which replaces the polytropic equation of state $p(\rho) = \rho^\gamma \ (\gamma > 1)$ (e.g., see [8, 9, 15, 30]) with $p(\rho) = -\rho^{-1}$, which was introduced by Chaplygin [7], Tsien [34] and von Karman [18] as a suitable mathematical approximation for calculating the lifting force on a wing of an airplane in aerodynamics. A Chaplygin gas owns a negative pressure and occurs in certain theories of cosmology. Such a gas has been advertised as a possible model for dark energy [2, 3, 13, 23]. The typical feature of the Chaplygin gas dynamics is that the $\delta$-shocks appear in non-zero pressure cases. The Riemann problem was solved for the nonrelativistic case by Brenier [4], relativistic case by Cheng and Yang [11], followed by its vanishing pressure limit problem by Yin and Song [40].

The Chaplygin gas dynamics was generalized in which the pressure is expressed by $p(\rho) = -A\rho^{-\alpha}, \ \alpha \in (0, 1), \ A > 0$, which has been advertised as a suitable model for dark energy [1, 12, 24, 41]. Also in this system, $\delta$-shocks appear in non-zero pressure cases. Probably the first mathematical results on this system were given by Wang [35] for the nonrelativistic case, where the Riemann problem was solved and the interactions of elementary waves were discussed. Recently, its vanishing pressure limit problem was studied by Sheng, Wang and Yin [28]. For the relativistic case, the Riemann problem was solved by Huang and Shao [16]. The main contribution of the present paper is that in the vanishing pressure limit, the solutions converge in the distributional sense to those of the pressureless relativistic Euler equations.

For the isentropic Chaplygin gas Euler equations, Brenier [4] firstly studied the 1-D Riemann problem and obtained solutions with concentration when initial data belong to a certain domain in the phase plane. Furthermore, Guo, Sheng and Zhang [14] abandoned this constraining and constructively obtained the global solutions to the 1-D Riemann problem, in which the $\delta$-shock developed. Moreover, they also systematically studied the 2-D Riemann problem for isentropic Chaplygin gas equations. For the 2-D case, we can also refer to Serre [22], in which he studied the interaction of the pressure waves for the 2-D isentropic irrotational Chaplygin gas and constructively proved the existence of transonic solutions for two cases, saddle and vortex of 2-D Riemann problem. Recently, Sheng, Wang and Yin [28] and Wang [35] studied the Riemann problem for the generalized Chaplygin gas and obtained the solutions to the Riemann problem and the interactions of elementary waves. The Riemann solutions to the transport equations in zero-pressure flow in gas dynamics were presented by Sheng and Zhang [29], in which delta shocks and vacuum states appeared.

The idea of vanishing pressure limits dates back to the paper by Li [19], in which he proved that for the the isentropic Euler equations with $p = T\rho$, when temperature drops to zero, the Riemann solution containing two shock waves converges to the delta shock solution to the transport equations and the solution containing two rarefaction waves converges to the solution involving vacuum to the transport equations. Chen and Liu [9] studied the formation of $\delta$-shocks and vacuum states and the limits of the Riemann solutions to the isentropic Euler equations for polytropic gas as $\varepsilon \to 0$, in which they took the equation of state as $P = \varepsilon p$ for $p = \rho^{\gamma}/\gamma \ (\gamma > 1)$. Further, they also obtained the same results for the Euler equations for nonisentropic fluids in [10]. See Yin and Sheng [38] the result of the relativistic
Euler equations for polytropic gas, Yin and Song [39, 40] for Chaplygin gas, Sheng, Wang and Yin [28] for the isentropic Euler equations for the generalized Chaplygin gas, Yang and Wang [37] for modified Chaplygin gas, Shen [26] for the isentropic magnetogasdynamics equations, Shen and Sun [27] for the perturbed Aw-Rascle model, Mitrovic and Nedeljkov [21] for the generalized pressureless gas dynamics model with a scaled pressure term, etc.

For the Chaplygin gas [40], both the characteristic fields are linearly degenerate, thus the classical elementary waves only involves contact discontinuities. The rarefaction wave curves and the shock wave curves are actually coincided to the so-called contact discontinuities, and the δ-shocks appear in non-zero pressure cases. Compared with the Chaplygin gas case, the characteristic fields for the generalized Chaplygin gas here are both genuinely nonlinear, while nonclassical solutions, namely, delta shock wave type solutions, also appear, besides classical solutions (the rarefaction waves and shock waves), which is an interesting point. The mechanism is different during the formation of singular solutions from that of the pressureless fluids [9, 29]. Unlike the polytropic gas case, considering the unbounded singularity in Riemann solutions, the limits of Riemann solutions here are expected to behave some difference and more interesting points during the process of vanishing pressure limit.

The mathematical difficulties caused by this fact as compared to the Chaplygin gas case by Yin and Song [40] or their later work [39] are that the shock rarefaction limit. The mathematical difficulties caused by this fact as compared to the Chaplygin gas case by Yin and Song [40] or their later work [39] are that the shock wave curves are given by

\[
\frac{v - v_-}{c^2 - v_-^2} = -\frac{\sqrt{\Theta(\rho, \rho_-)}}{1 - v_- \sqrt{\Theta(\rho, \rho_-)}}, \quad v < v_-, \quad \rho > \rho_-
\]

with \(\rho > \rho_-\) for a 1-shock curve \(S_1(\rho_-, v_-)\), and \(\rho < \rho_-\) for a 2-shock curve \(S_2(\rho_-, v_-)\), where

\[
\Theta(\rho, \rho_-) = \frac{A(-\rho^{-\alpha} + \rho_-^{-\alpha})(\rho - \rho_-)}{(-A\rho^{-\alpha} + \rho_-c^2)(-A\rho^{-\alpha} + \rho_+c^2)},
\]

the rarefaction wave curves are given by

\[
R_1(\rho_-, v_-) : \begin{cases}
\xi = \frac{c^2}{c^2 - v_- \sqrt{A\rho^{-\frac{1+\alpha}{2}}}} \\
\quad \left(\frac{c+v_-}{c-v_-}\right)^{\frac{2\alpha}{\alpha+1}} = \left(\frac{c-v_-}{c+v_-}\right)^{\frac{2\alpha}{\alpha+1}}, \quad \rho < \rho_-, \quad \rho > \rho_-.
\end{cases}
\]

and

\[
R_2(\rho_-, v_-) : \begin{cases}
\xi = \frac{c^2}{c^2 + v_- \sqrt{A\rho^{-\frac{1+\alpha}{2}}}} \\
\quad \left(\frac{c+v_-}{c-v_-}\right)^{\frac{2\alpha}{\alpha+1}} = \left(\frac{c-v_-}{c+v_-}\right)^{\frac{2\alpha}{\alpha+1}}, \quad \rho > \rho_-.
\end{cases}
\]

However, for the Chaplygin gas case (\(\alpha = 1\) in (1.2)) by Yin and Song [40], the shock wave curves are given by the following explicit formulas

\[
S_1(\rho_-, v_-) : \sigma_1 = \frac{c^2(\rho v - \sqrt{A})}{\rho c^2 - \sqrt{A}v} = \frac{c^2(\rho_- v_-- \sqrt{A})}{\rho_- c^2 - \sqrt{A}v_--}, \quad \rho > \rho_-,
\]
and
\[
S_2(\rho, v_+) : \sigma_2 = \frac{c^2(\rho v + \sqrt{A})}{\rho c^2 + \sqrt{A}v} = \frac{c^2(\rho_+ v_+ + \sqrt{A})}{\rho_+ c^2 + \sqrt{A}v_+}, \quad \rho_+ \leq \rho,
\]
the rarefaction wave curves are given by
\[
R_1(\rho, v_+) : \xi = \frac{c^2(\rho v - \sqrt{A})}{\rho c^2 - \sqrt{A}v} = \frac{c^2(\rho_+ v_+ - \sqrt{A})}{\rho_+ c^2 - \sqrt{A}v_+}, \quad \rho_+ \leq \rho,
\]
and
\[
R_2(\rho, v_+) : \xi = \frac{c^2(\rho v + \sqrt{A})}{\rho c^2 + \sqrt{A}v} = \frac{c^2(\rho_+ v_+ + \sqrt{A})}{\rho_+ c^2 + \sqrt{A}v_+}, \quad \rho_+ \leq \rho.
\]
The rarefaction wave curves and the shock wave curves are coincident in the phase plane, which actually correspond to contact discontinuities. Similarly, for the non-relativistic case by Wang [35], Sheng, Wang and Yin [28], the shock wave curves are given by
\[
S_1(\rho, v_+) : \begin{cases} 
\sigma_1 = v_+ - \left( \frac{\rho |p|}{\rho_+ |p|} \right)^{1/2}, \\
v = v_+ - \left( \frac{1}{\rho_+ |\sigma|} \right) (\rho - \rho_+), 
\end{cases} \quad \rho_+ \leq \rho,
\]
and
\[
S_2(\rho, v_+) : \begin{cases} 
\sigma_2 = v_+ + \left( \frac{\rho_+ |p|}{\rho |p|} \right)^{1/2}, \\
v = v_+ + \left( \frac{1}{\rho |\sigma|} \right) (\rho - \rho_+), 
\end{cases} \quad \rho_+ \leq \rho.
\]
The rarefaction wave curves are given by
\[
R_1(\rho, v_+) : \begin{cases} 
\xi = v - \sqrt{A}o \rho^{-\frac{1+\alpha}{2}}, \\
v - 2\sqrt{A}o \rho^{-\frac{1+\alpha}{2}} = v_+ - 2\sqrt{A}o \rho_+^{-\frac{1+\alpha}{2}}, 
\end{cases} \quad \rho_+ \leq \rho,
\]
and
\[
R_2(\rho, v_+) : \begin{cases} 
\xi = v + \sqrt{A}o \rho^{-\frac{1+\alpha}{2}}, \\
v + 2\sqrt{A}o \rho^{-\frac{1+\alpha}{2}} = v_+ + 2\sqrt{A}o \rho_+^{-\frac{1+\alpha}{2}}, 
\end{cases} \quad \rho_+ \leq \rho.
\]
To overcome these difficulties, we carefully analyze the global behavior of the shock wave curves and rarefaction wave curves in the phase plane, geometric properties of shock wave curves and rarefaction wave curves in the physical space, and the behavior of Riemann solutions of (1.1)-(1.2). For example, in Sections 3, in Case \( v_- = v_+ \), in order to prove
\[
\lim_{A \to 0} \sigma_2 = \frac{\left( -\frac{A}{\rho_+^2} + \rho_+ c^2 \right) v_- - \frac{v_+}{\rho_+ - \rho_+} \sqrt{c^2 - v_+^2} \sqrt{c^2 - v_-^2} + \frac{v_+}{c^2 - v_+^2} \left( \frac{\rho_+ - \rho_+}{\rho_+ - \rho_+} + c^2 \right)}{\left( -\frac{A}{\rho_+^2} + \rho_+ c^2 \right) v_- - \frac{v_+}{\rho_+ - \rho_+} \sqrt{c^2 - v_+^2} \sqrt{c^2 - v_-^2} + \frac{v_+}{c^2 - v_+^2} \left( \frac{\rho_+ - \rho_+}{\rho_+ - \rho_+} + c^2 \right) + 1}\]
we first establish \( \lim_{A \to 0} \frac{v_- - v_+}{\rho_+ - \rho_+} = 0 \).

The organization of this paper is as follows. In Sections 2 and 3, we display some results on the Riemann solutions to system (1.1)-(1.2) and the zero-pressure
relativistic Euler equations. In Section 4, we investigate the limits of the Riemann solutions and the formation of delta shocks and vacuum states as pressure vanishes.

2. Riemann problem for system (1.1)-(1.2). In this section, we discuss the Riemann solutions of (1.1) and (1.2) with initial data

$$(\rho, v)(0, x) = (\rho_{\pm}, v_{\pm}), \quad \pm x > 0,$$

where $\rho_{\pm} > 0$ and $v_{\pm}$ are arbitrary constants. Here the physically relevant region for solutions is

$$V = \left\{ (\rho, v) : \rho^{\alpha+1} \geq \frac{A\alpha}{c^2}, |v| < c \right\},$$

that is, the sonic speed $\sqrt{p(\rho)}$ should be not more than the light speed $c$. Obviously, the physically relevant region (2.2) is different from that for polytropic gas. The detailed study of the Riemann solutions of (1.1)-(1.2) can be found in [16, 25]. For the general knowledge about the Riemann problem for hyperbolic conservation laws, see [6, 5] for example.

The eigenvalues of system (1.1) and (1.2) are

$$\lambda_1 = \frac{c^2(v - \sqrt{A\alpha\rho^{\alpha+1}})}{c^2 - v\sqrt{A\alpha\rho^{\alpha+1}}}, \quad \lambda_2 = \frac{c^2(v + \sqrt{A\alpha\rho^{\alpha+1}})}{c^2 + v\sqrt{A\alpha\rho^{\alpha+1}}},$$

and the corresponding right eigenvectors

$$\vec{r}_1 = \left( -\frac{1}{c^2 - v^2}, \frac{\sqrt{A\alpha\rho^{\alpha+1}}}{\rho c^2 - A\rho^{-\alpha}} \right)^T, \quad \vec{r}_2 = \left( 1, \frac{\sqrt{A\alpha\rho^{\alpha+1}}}{\rho c^2 - A\rho^{-\alpha}} \right)^T.$$

So system (1.1) with (1.2) is strictly hyperbolic. The first and second characteristic fields are genuinely nonlinear with $\nabla \lambda_1 \cdot \vec{r}_1 \neq 0$ and $\nabla \lambda_2 \cdot \vec{r}_2 \neq 0$ for $0 < \alpha < 1$, in which $\nabla$ denotes the gradient with respect to $(\rho, v)$. Therefore, in classical sense, the associated waves are rarefaction waves or shocks.

Since system (1.1), (1.2) and the Riemann data (2.1) are invariant under stretching of coordinates: $(t, x) \rightarrow (at, \alpha x)$ $(\alpha$ is a constant), we seek the self-similar solution

$$(\rho, v)(t, x) = (\rho, v)(\xi), \quad \xi = \frac{x}{t}.$$

Then, the Riemann problem (1.1), (1.2) and (2.1) is reduced to the following boundary value problem of ordinary differential equations:

$$\begin{cases}
-\xi \left( (p + \rho c^2)\frac{v^2}{c^2 - v^2} + \rho \right) \frac{\partial}{\partial \xi} + \left( (p + \rho c^2)\frac{v^2}{c^2 - v^2} \right) \frac{\partial}{\partial \xi} = 0, \\
-\xi \left( (p + \rho c^2)\frac{v^2}{c^2 - v^2} \right) + \left( (p + \rho c^2)\frac{v^2}{c^2 - v^2} + p \right) = 0,
\end{cases}$$

(2.3)

with $(\rho, v)(\pm \infty) = (\rho_{\pm}, v_{\pm})$.

For any smooth solution, system (2.3) can be rewritten as

$$\begin{pmatrix}
\frac{(c^2 + p'(\rho))c^2v - (c^4 + p'(\rho)v^2)\xi}{c^2(v^2 + p'(\rho))} & \frac{(p(\rho) + \rho c^2)(c^2 + v^2 - 2v\xi)}{c^2 - v^2} \\
\frac{(c^2 + p'(\rho))c^2v - (c^4 + p'(\rho)v^2)\xi}{c^2(v^2 + p'(\rho))} & \frac{(p(\rho) + \rho c^2)(2c^2v - \xi c^2 - \xi v^2)}{c^2 - v^2}
\end{pmatrix}
\begin{pmatrix}
\rho \xi \\
v \xi
\end{pmatrix} = 0.$$

(2.4)

It provides either general solutions (constant states)

$$(\rho, v)(\xi) = \text{constant (} \rho > \left( \frac{A\alpha}{c^2} \right)^{\frac{1}{\alpha+1}} \text{)}.$$
holds that the sets of states that can be connected to the state \((\rho, v)\) on the right by a centered rarefaction wave in the 1-family are as follows:

\[
R_1(\rho_-, v_-) : \begin{cases}
\xi = \frac{c^2(v - \sqrt{A\alpha\rho - \frac{1 + \alpha}{2}})}{c^2 - v\sqrt{A\alpha\rho - \frac{1 + \alpha}{2}}}, \\
\frac{c + v}{c - v} \left(\frac{c + \sqrt{\rho - \frac{1 + \alpha}{2}}}{c - \sqrt{\rho - \frac{1 + \alpha}{2}}} \right) \sqrt{\frac{\rho}{\rho - \frac{1 + \alpha}{2}}}
\end{cases}
\]

Similarly, for a given left state \((\rho_-, v_-)\), the rarefaction wave curves which are the sets of states that can be connected to \((\rho_-, v_-)\) on the right in the 2-family are as follows:

\[
R_2(\rho_-, v_-) : \begin{cases}
\xi = \frac{c^2(v + \sqrt{A\alpha\rho - \frac{1 + \alpha}{2}})}{c^2 + v\sqrt{A\alpha\rho - \frac{1 + \alpha}{2}}}, \\
\frac{c + v}{c - v} \left(\frac{c + \sqrt{\rho - \frac{1 + \alpha}{2}}}{c - \sqrt{\rho - \frac{1 + \alpha}{2}}} \right) \sqrt{\frac{\rho}{\rho - \frac{1 + \alpha}{2}}}
\end{cases}
\]

Similarly, for a given state \((\rho_+, v_+)\), one can obtain \(R_1(\rho_+, v_+)\) or \(R_2(\rho_+, v_+)\).

For a bounded discontinuous solution at \(\xi = \sigma\), the Rankine-Hugoniot condition holds

\[
\begin{align*}
-\sigma \left[ \left( -\frac{A}{\rho^\alpha} + \rho c^2 \right) \frac{v^2}{c^2(c^2 - v^2)} + \rho \right] + \left[ \left( -\frac{A}{\rho^\alpha} + \rho c^2 \right) \frac{v}{c^2 - v^2} \right] &= 0, \\
-\sigma \left[ \left( -\frac{A}{\rho^\alpha} + \rho c^2 \right) \frac{v^2}{c^2} + \left[ \left( -\frac{A}{\rho^\alpha} + \rho c^2 \right) \frac{v^2}{c^2 - v^2} - \frac{A}{\rho^\alpha} \right] \right] &= 0,
\end{align*}
\]

where \(\sigma = \rho - \rho_-\) and \(\sigma\) is the velocity of the discontinuity. The Lax entropy conditions imply that

\[
\rho > \rho_- \quad (1\text{-shock}), \quad \rho < \rho_- \quad (2\text{-shock}).
\]

For a given state \((\rho_-, v_-)\), the possible states \((\rho, v)\) that can be connected to \((\rho_-, v_-)\) on the right by a 1-(or 2-)shock wave are as follows:

\[
\frac{v - v_-}{c^2 - v_-^2} = -\frac{\sqrt{\Theta(\rho_+, \rho_-)}}{1 - v_- \sqrt{\Theta(\rho_+, \rho_-)}}, \quad v < v_-.
\]
with \( \rho > \rho_- \) for a 1-shock curve \( S_1(\rho_-, \rho_-) \), and \( \rho < \rho_- \) for a 2-shock curve \( S_2(\rho_-, \rho_-) \), where

\[
\Theta(\rho, \rho_-) = \frac{A(-\rho_-^{\alpha} + \rho_-^{\alpha})(\rho - \rho_-)}{(-A\rho_-^{\alpha} + \rho_-c^2)(-A\rho^{\alpha} + \rho c^2)}.
\]

Similarly, for a given right state \((\rho_+, v_+)\), two shock curves consist of those states \((\rho, v)\) that can be connected to \((\rho_+, v_+)\) on the left by a 1-(or 2-)shock wave as follows:

\[
\frac{v - v_+}{c^2 - v_+^2} = \frac{\sqrt{\Theta(\rho_+, \rho)}}{1 + v_+^2\sqrt{\Theta(\rho_+, \rho)}}, \quad v > v_+,
\]

with \( \rho < \rho_+ \) for a 1-shock curve \( S_1(\rho_+, v_+) \), and \( \rho > \rho_+ \) for a 2-shock curve \( S_2(\rho_+, v_+) \).

**Lemma 2.1.** [16] The 1-shock wave curve \( S_1(\rho_-, v_-) \) has a straight line \( v = v^- \) as its asymptote, and the 2-shock wave curve \( S_2(\rho_+, v_+) \) has a straight line \( v = v^+ \) as its asymptote, where

\[
v^- = v_- - (c^2 - v_-^2)\frac{\sqrt{A\rho_-^{\frac{1+\alpha}{2}}}}{c^2 - v_-\sqrt{A\rho_-^{\frac{1+\alpha}{2}}}}, \quad v^+ = v_+ + (c^2 - v_+^2)\frac{\sqrt{A\rho_+^{\frac{1+\alpha}{2}}}}{c^2 + v_+\sqrt{A\rho_+^{\frac{1+\alpha}{2}}}.
\]

And the curves \( S_2(\rho_-, v_-) \) and \( R_1(\rho_-, v_-) \) have singularity points \( ((\sqrt{\rho/c})^{\frac{1+\alpha}{2}}, -c) \) and \( ((\sqrt{\rho/c})^{\frac{1+\alpha}{2}}, c) \), respectively.

In the phase plane \( \rho > (\frac{A\rho}{c^2})^{\frac{1+\alpha}{2}}, -c < v < c \), through point \((\rho_-, v_-)\), we draw the elementary wave curves \( R_1, R_2, S_1 \) and \( S_2 \), respectively. In addition, we draw an \( S_3 \) curve, which is determined as follows (see [16])

\[
v + (c^2 - v^2)\frac{\sqrt{A\rho_-^{\frac{1+\alpha}{2}}}}{c^2 + v\sqrt{A\rho_-^{\frac{1+\alpha}{2}}}} = v^- = v_- - (c^2 - v_-^2)\frac{\sqrt{A\rho_-^{\frac{1+\alpha}{2}}}}{c^2 - v_-\sqrt{A\rho_-^{\frac{1+\alpha}{2}}}.
\]

Then the phase plane is divided into five regions \( I, II, III, IV \), and \( V(\rho_-, v_-) \) (see Fig. 1).

By the analysis method in phase plane, for any given state \((\rho_+, v_+)\), one can construct the Riemann solutions as follows:

1. \((\rho_+, v_+) \in I(\rho_-, v_-) : R_1 + R_2;
2. \((\rho_+, v_+) \in II(\rho_-, v_-) : R_1 + S_2;
3. \((\rho_+, v_+) \in III(\rho_-, v_-) : S_1 + R_2;
4. \((\rho_+, v_+) \in IV(\rho_-, v_-) : S_1 + S_2;
5. \((\rho_+, v_+) \in V(\rho_-, v_-) : Delta shock wave.
Lemma 2.2. [16] For arbitrary state \((\rho_+, v_+) \in V(\rho_-, v_-)\), 1-shock curve \(S_1(\rho_-, v_-)\) does not intersect 2-shock curve \(S_2(\rho_+, v_+)\).

Now let us consider the delta shock wave solution. When \((\rho_+, v_+) \in V(\rho_-, v_-)\), we have

\[
\lambda_1^+ = \frac{c^2(v_+ - \sqrt{A\alpha \rho_+^{1+\alpha}})}{c^2 - v_+ \sqrt{A\alpha \rho_+^{1+\alpha}}} < \lambda_2^+ = \frac{c^2(v_+ + \sqrt{A\alpha \rho_+^{1+\alpha}})}{c^2 + v_+ \sqrt{A\alpha \rho_+^{1+\alpha}}},
\]

\[
< \lambda_1^- = \frac{c^2(v_- - \sqrt{A\alpha \rho_-^{1+\alpha}})}{c^2 - v_- \sqrt{A\alpha \rho_-^{1+\alpha}}} < \lambda_2^- = \frac{c^2(v_- + \sqrt{A\alpha \rho_-^{1+\alpha}})}{c^2 + v_- \sqrt{A\alpha \rho_-^{1+\alpha}}},
\]

which means that the characteristic lines from initial data will overlap in a domain \(\Omega\) as shown in Fig. 2. So, singularity must happen in \(\Omega\). It is easy to know that the singularity is impossible to be a jump with finite amplitude because the Rankine-Hugoniot condition is not satisfied on the bounded jump. In other words, there is
no solution which is piecewise smooth and bounded. Motivated by [29], we seek solutions with delta distribution at the jump.

For system (1.1) and (1.2), the definition of solutions in the sense of distributions can be given as follows.

**Definition 2.3.** A pair \((\rho, v)\) constitutes a solution of (1.1) and (1.2) in the sense of distributions if it satisfies

\[
\begin{align*}
\int_0^\infty \int_-\infty^+ \left( \left( \frac{A}{\rho^2} + \rho \phi^2 \right) \frac{v^2}{c^2 - v^2} + \rho \right) \phi \tau + \left( \frac{A}{\rho^2} + \rho \phi^2 \right) \frac{v^2}{c^2 - v^2} \phi_x \right) dx dt &= 0, \\
\int_0^\infty \int_-\infty^+ \left( \left( \frac{A}{\rho^2} + \rho \phi^2 \right) \frac{v}{c^2 - v^2} \right) \phi \tau + \left( \frac{A}{\rho^2} + \rho \phi^2 \right) \frac{v^2}{c^2 - v^2} - \frac{A}{\rho^2} \phi_x \right) dx dt &= 0,
\end{align*}
\]

(2.14)

for all test functions \(\varphi \in C_0^\infty(R^+ \times R^1)\).

Moreover, we define a two-dimensional weighted delta function in the following way.

**Definition 2.4.** A two-dimensional weighted delta function \(w(s)\delta_L\) supported on a smooth curve \(L = \{(t(s), x(s)) : a < s < b\}\) is defined by

\[
\langle w(s)\delta_L, \varphi \rangle = \int_a^b w(s)\varphi(t(s), x(s)) ds,
\]

(2.15)

for all test functions \(\varphi \in C_0^\infty(R^+ \times R^1)\).

Let us consider a solution of (1.1) and (1.2) of the form

\[
(\rho, v)(t, x) = \begin{cases}
(\rho_-, v_-), & x < \sigma t, \\
(\rho(t)\delta(x - \sigma t), \sigma), & x = \sigma t, \\
(\rho_+, v_+), & x > \sigma t,
\end{cases}
\]

(2.16)

where \(\sigma\) is a constant, \(w(t) \in C^1[0, +\infty)\), and \(\delta(\cdot)\) is the standard Dirac measure. \(x(t), w(t)\) and \(\sigma\) are the location, weight and velocity of the delta shock respectively. Similar to [4, 11, 14], we define \(\frac{1}{\rho^\alpha}\) as follows:

\[
\frac{1}{\rho^\alpha} = \begin{cases}
\frac{1}{\rho_+^\alpha}, & x < \sigma t, \\
0, & x = \sigma t, \\
\frac{1}{\rho_-^\alpha}, & x > \sigma t.
\end{cases}
\]

Then the following generalized Rankine-Hugoniot condition holds (see [16]):

\[
\begin{cases}
\frac{dx}{dt}(t) = \sigma, \\
\frac{d}{dt} \left( w(t) \frac{c^2}{c^2 - \sigma^2} \right) = \sigma \left[ \frac{-A}{\rho^2} + \rho \phi^2 \right] \frac{v^2}{c^2(c^2 - v^2)} + \rho - \left[ \frac{-A}{\rho^2} + \rho \phi^2 \right] \frac{v}{c^2 - v^2}, \\
\frac{d}{dt} \left( w(t) \frac{c^2}{c^2 - \sigma^2} \right) = \sigma \left[ \frac{-A}{\rho^2} + \rho \phi^2 \right] \frac{v}{c^2(c^2 - v^2)} - \left[ \frac{-A}{\rho^2} + \rho \phi^2 \right] \frac{v^2}{c^2 - v^2} - \frac{A}{\rho^2},
\end{cases}
\]

(2.17)

where \([\rho] = \rho_+ - \rho_-\), with initial data

\[
(x, w)(0) = (0, 0).
\]

(2.18)
In addition, to guarantee uniqueness, the delta shock wave should satisfy the entropy condition
\[
\lambda^+ = \frac{c^2(v_+ + \sqrt{\lambda_0 c^2_+})}{c^2 + v_+ \sqrt{\lambda_0 c^2_+}} < \sigma < \lambda^- = \frac{c^2(v_- - \sqrt{\lambda_0 c^2_-})}{c^2 - v_- \sqrt{\lambda_0 c^2_-}},
\] (2.19)
which means that all the characteristic lines on both sides of the delta shock wave are incoming.

Integrating (2.17) from 0 to \(t\) with initial data (2.18), we have
\[
\begin{align*}
  &\begin{cases}
    x = \sigma t, \\
    wc^2 = (\sigma E - F)t, \\
    wc^2\sigma = (\sigma F - G)t,
  \end{cases} \\
\end{align*}
\] (2.20)
where
\[
E = \left[\left(-\frac{A}{\rho^{\alpha}} + \rho c^2\right) \frac{v^2}{c^2(v^2 - \rho^2)} + \rho\right], \\
F = \left[\left(-\frac{A}{\rho^{\alpha}} + \rho c^2\right) \frac{v}{c^2(v^2 - \rho^2)}\right], \\
G = \left[\left(-\frac{A}{\rho^{\alpha}} + \rho c^2\right) \frac{v^2}{c^2(v^2 - \rho^2)} - \frac{A}{\rho^{\alpha}}\right].
\] (2.21)

Solving (2.20) with the entropy condition (2.19), we have, when \(E = 0\),
\[
\begin{align*}
x(t) &= \frac{G}{2F} t, \quad \sigma = \frac{G}{2F}, \\
w(t) &= -F \left(1 - \left(\frac{G}{2Fc}\right)^2\right) t;
\end{align*}
\] (2.22)
when \(E \neq 0\),
\[
\begin{align*}
x(t) &= \frac{F + \sqrt{F^2 - EG}}{E} t, \quad \sigma = \frac{F + \sqrt{F^2 - EG}}{E}, \\
w(t) &= \sqrt{F^2 - EG} \left(1 - \left(\frac{F + \sqrt{F^2 - EG}}{cE}\right)^2\right) t.
\end{align*}
\] (2.23)

3. Riemann problem for the zero-pressure relativistic Euler equations.
The Riemann solutions to the zero-pressure relativistic Euler equations were presented by Sheng and Yin [38]. The Riemann problem to the zero-pressure relativistic Euler equations is
\[
\begin{align*}
  &\begin{cases}
    \left(\frac{\rho}{c^2(v^2 - \rho^2)}\right)_t + \left(\frac{\rho v}{c^2(v^2 - \rho^2)}\right)_x = 0, \\
    \left(\frac{\rho v}{c^2(v^2 - \rho^2)}\right)_t + \left(\frac{\rho v^2}{c^2(v^2 - \rho^2)}\right)_x = 0,
  \end{cases}
\end{align*}
\] (3.1)
with initial data
\[
(\rho, v)(0, x) = (\rho_{\pm}, v_{\pm}), \quad \pm x > 0.
\] (3.2)

The system (3.1) has a double eigenvalue \(\lambda = v\) and only one right eigenvector \(\vec{\tau} = (1, 0)^T\). The system is obviously linearly degenerate by \(\nabla \lambda \cdot \vec{\tau} \equiv 0\). As usual, we seek the self-similar solution
\[
(\rho, v)(t, x) = (\rho, v)(\xi), \quad \xi = \frac{x}{t}.
\]
Then, Riemann problem (3.1) and (3.2) is reduced to the following boundary value problem of ordinary differential equations:
\[
\begin{align*}
  &\begin{cases}
    -\xi\left(\frac{\rho}{c^2(v^2 - \rho^2)}\right)_\xi + \left(\frac{\rho v}{c^2(v^2 - \rho^2)}\right)_\xi = 0, \\
    -\xi\left(\frac{\rho v}{c^2(v^2 - \rho^2)}\right)_\xi + \left(\frac{\rho v^2}{c^2(v^2 - \rho^2)}\right)_\xi = 0,
  \end{cases}
\end{align*}
\] (3.3)
with \((\rho, v)(\pm \infty) = (\rho_{\pm}, v_{\pm})\).

For any smooth solution, system (3.3) can be rewritten as

\[
\begin{pmatrix}
\frac{v - \xi}{c^2 - v^2} \\
\frac{\rho(c^2 + v^2 - 2v\xi)}{(c^2 - v^2)^2} \\
\frac{\rho(2v^2 - c^2\xi - v^2\xi)}{(c^2 - v^2)^2}
\end{pmatrix}
\begin{pmatrix}
\rho \\
v \\
v\xi
\end{pmatrix}
= 0.
\tag{3.4}
\]

It provides either the general solution (constant state)

\[(\rho, v)(\xi) = \text{constant} \quad (\rho > 0)\]

or the singular solution

\[
\begin{cases}
\rho = 0, \\
v = \xi,
\end{cases}
\tag{3.5}
\]

which is called the vacuum state (see [29]), where \(v(\xi)\) is an arbitrary smooth function.

For a bounded discontinuity at \(\xi = \sigma\), the Rankine-Hugoniot condition holds:

\[
\begin{cases}
-\sigma\left[\frac{\rho}{c^2 - v^2}\right] + \left[\frac{\rho v}{c^2 - v^2}\right] = 0, \\
-\sigma\left[\frac{\rho v}{c^2 - v^2}\right] + \left[\frac{\rho v^2}{c^2 - v^2}\right] = 0,
\end{cases}
\tag{3.6}
\]

where \([q] = q_+ - q_-\) denotes the jump of \(q\) across the discontinuity. By solving (3.6), we obtain

\[J : \xi = \sigma = v_- (= \lambda_-) = v_+ (= \lambda_+),\]
\tag{3.7}

which is a contact discontinuity. It is a slip line and just the characteristic of solutions on both its sides in \((t, x)\)-plane. Then two states \((\rho_-, v_-)\) and \((\rho_+, v_+)\) can be connected by \(J\), if and only if \(v_- = v_+\).

The Riemann problem (3.1) and (3.2) can be solved by contact discontinuities, vacuum or delta shock wave connecting two constant states \((\rho_{\pm}, v_{\pm})\).

For the case \(v_- < v_+\), there is no characteristic passing through the region \(v_-t < x < v_+t\) and the vacuum appears in this region. The solution can be expressed as

\[
(\rho, v)(\xi) =
\begin{cases}
(\rho_-, v_-), & -\infty < \xi < v_-, \\
(0, \xi), & v_- \leq \xi \leq v_+, \\
(\rho_+, v_+), & v_+ < \xi < +\infty,
\end{cases}
\tag{3.8}
\]

For the case \(v_- = v_+\), it is easy to see that the constant states \((\rho_{\pm}, v_{\pm})\) can be connected by a contact discontinuity.
For the case $v_- > v_+$, the characteristic lines from initial data will overlap in a domain $\Omega$, as shown in Fig. 3. Motivated by [29], we seek solutions with delta distribution at the jump.

To do so, a two-dimensional weighted delta function $w(s)\delta_L$ supported on a smooth curve $L = \{ (t(s), x(s)) : a < s < b \}$ is defined by

$$\langle w(s)\delta_L, \varphi \rangle = \int_a^b w(s)\varphi(t(s), x(s))ds, \quad (3.9)$$

for any $\varphi \in C^\infty_0(R^+ \times R^1)$.

Let us consider a solution of (3.1)-(3.2) of the form

$$(\rho, v)(t, x) = \begin{cases} (\rho_-, v_-), & x < x(t), \\ (w(t)\delta(x - x(t)), v_\delta), & x = x(t), \\ (\rho_+, v_+), & x > x(t), \end{cases} \quad (3.10)$$

where $v_\delta$ is a constant, $x(t), w(t) \in C^1[0, +\infty)$, and $\delta(\cdot)$ is the standard Dirac measure. $x(t), w(t)$ and $v_\delta$ are the location, weight and velocity of the delta shock, respectively. Then the following generalized Rankine-Hugoniot condition holds:

$$\begin{align*}
\frac{dx(t)}{dt} &= v_\delta, \\
\frac{dw(t)}{dt} &\left( \frac{c^2 - v_\delta^2}{c^2 - v^2} \right) = v_\delta \left[ \frac{\rho}{c^2 - v^2} \right] - \left[ \frac{\rho v}{c^2 - v^2} \right], \\
\frac{dv(t)}{dt} &\left( \frac{c^2 - v_\delta^2}{c^2 - v^2} \right) = v_\delta \left[ \frac{\rho v}{c^2 - v^2} \right] - \left[ \frac{\rho v^2}{c^2 - v^2} \right],
\end{align*} \quad (3.11)$$

where $\rho = \rho_+ - \rho_-$, with initial data

$$(x, w)(0) = (0, 0). \quad (3.12)$$

In addition, to guarantee uniqueness, the delta shock wave should satisfy the entropy condition

$$v_+ < v_\delta < v_-, \quad (3.12)$$
which means that all the characteristic lines on either side of a delta shock run into the line of the delta shock in the \((t,x)\)-plane, this implies that a delta shock is an overcompressive shock.

Solving (3.11) with initial data (3.12), we have, when \([\frac{\rho}{c^2-v_+^2}] = 0\),

\[
v_6 = \frac{1}{2} (v_- + v_+), \quad x(t) = \frac{1}{2} (v_- + v_+) t, \]

\[
w(t) = \left( c^2 - \frac{(v_- + v_+)^2}{4} \right) \left( \frac{\rho - v_-}{c^2 - v_-^2} - \frac{\rho + v_+}{c^2 - v_+^2} \right) t;
\]

when \([\frac{\rho}{c^2-v_+^2}] \neq 0\),

\[
v_6 = \frac{\sqrt{\frac{\rho}{c^2-v_+^2}} + v_+}{\sqrt{\frac{\rho}{c^2-v_+^2}} + \sqrt{\frac{\rho}{c^2-v_-^2}}}, \quad x(t) = v_6 t, \]

\[
w(t) = \sqrt{\frac{\rho}{c^2-v_+^2}} \frac{\rho + v_+}{c^2 - v_+^2} (v_- - v_+) (c^2 - v_6^2) t. \quad (3.13)
\]

4. The limits of Riemann solutions of (1.1)-(1.2) as \(A \to 0\). In this section, we study the limits of the Riemann solutions of (1.1)-(1.2) as \(A \to 0\). According to the values of \(v_-\) and \(v_+\), we divide our discussion into the following three cases.

4.1. Case \(v_- > v_+\).

**Lemma 4.1.** If \(v_- > v_+\), then there exist \(A_0 > A_1 > 0\), such that \((\rho_+, v_+) \in IV \cup V(\rho_-, v_-)\) as \(A < A_0\), and \((\rho_+, v_+) \in V(\rho_-, v_-)\) as \(0 < A < A_1\).

**Proof.** It can be derived from (2.10) that all possible states \((\rho, v)\) that can be connected to the left state \((\rho_-, v_-)\) on the right by a 1-shock wave \(S_1\) or a 2-shock wave \(S_2\) should satisfy

\[
\frac{v - v_-}{c^2 - v_-^2} = -\frac{\sqrt{\Theta(\rho, \rho_-)}}{1 - v_- \sqrt{\Theta(\rho, \rho_-)}}, \quad v < v_-; \quad \text{(4.1)}
\]

with \(\rho > \rho_-\) for \(S_1\), and \(\rho < \rho_-\) for \(S_2\), where

\[
\Theta(\rho, \rho_-) = \frac{A(-\rho_-^{\alpha} + \rho_+^{\alpha})(\rho - \rho_-)}{(-A\rho_-^{\alpha} + \rho_- c^2)(-A\rho_+^{\alpha} + \rho_+ c^2)}.
\]

The states \((\rho, v)\) that can be connected to the left state \((\rho_-, v_-)\) on the right directly by a \(\delta\)-shock wave \(S_3\) should satisfy

\[
S_3: \quad \frac{c^2(v + \sqrt{A \rho^{\alpha} - v_- c^2})}{c^2 + v \sqrt{A \rho^{\alpha} - v_- c^2}} = \frac{c^2(v_- - \sqrt{A \rho_-^{\alpha} - v_- c^2})}{c^2 - v_- \sqrt{A \rho_-^{\alpha} - v_- c^2}}. \quad (4.2)
\]

If \(\rho_+ \neq \rho_-\) and \((\rho_+, v_+) \in IV \cup V(\rho_-, v_-)\), by (4.1) we have

\[
\frac{v_+ - v_-}{c^2 - v_-^2} < -\frac{\sqrt{\Theta(\rho_+, \rho_-)}}{1 - v_- \sqrt{\Theta(\rho_+, \rho_-)}}, \quad \text{(4.3)}
\]

which implies that

\[
\rho_+^{\alpha} \rho_-^{\alpha} A^2 - c^2(\rho_+^{1-\alpha} + \rho_-^{1-\alpha}) + \left(\frac{c^2 - v_+ v_-}{v_+ - v_-}\right)^2 (-\rho_+^{\alpha} + \rho_-^{\alpha})(\rho_+ - \rho_-) A c^4 \rho_+ \rho_- > 0. \quad (4.4)
\]
Thus we can take
\[ A_0 = \frac{c^2(\rho_1^{1-\alpha} + \rho_0^{1-\alpha}) + \left(\frac{c^2 - v_+ v_-}{v_+ - v_-}\right)^2(-\rho_1^{1-\alpha} + \rho_0^{1-\alpha})(\rho_- - \rho_0) - \sqrt{\Delta}}{2\rho_1^{1-\alpha} \rho_0^{1-\alpha}}, \]
where
\[ \Delta = \left(c^2(\rho_1^{1-\alpha} + \rho_0^{1-\alpha}) + \left(\frac{c^2 - v_+ v_-}{v_+ - v_-}\right)^2(-\rho_1^{1-\alpha} + \rho_0^{1-\alpha})(\rho_- - \rho_0)\right)^2 - 4c^4 \rho_1^{1-\alpha} \rho_0^{1-\alpha} \]
\[ = c^4(\rho_1^{1-\alpha} - \rho_0^{1-\alpha})^2 + 2c^2(\rho_1^{1-\alpha} + \rho_0^{1-\alpha})\left(\frac{c^2 - v_+ v_-}{v_+ - v_-}\right)^2(-\rho_1^{1-\alpha} + \rho_0^{1-\alpha})(\rho_- - \rho_0) \]
\[ + \left(\frac{c^2 - v_+ v_-}{v_- - v_+}\right)^2(-\rho_1^{1-\alpha} + \rho_0^{1-\alpha})(\rho_- - \rho_0)\right)^2 > 0. \]

Then we have \((\rho_+, v_+) \in IV \cup V(\rho_-, v_-)\) when \(A < A_0\).

Next, if \(\rho_+ \neq \rho_0\) and \((\rho_+, v_+) \in V(\rho_-, v_-)\), by (4.2) we have
\[ \frac{c^2(v_+ + \sqrt{\rho_+} \rho_+^{\frac{1+\alpha}{2}})}{c^2 + v_+ \sqrt{\rho_+} \rho_+^{\frac{1+\alpha}{2}}} < \frac{c^2(v_- - \sqrt{\rho_-} \rho_-^{\frac{1+\alpha}{2}})}{c^2 - v_- \sqrt{\rho_-} \rho_-^{\frac{1+\alpha}{2}}}, \] which is equivalent to
\[ (v_+ - v_-)A + (\rho_+^{1+\alpha} + \rho_-^{1+\alpha})(v_+ v_- - c^2)\sqrt{A} + c^2 \rho_+^{1+\alpha} \rho_-^{1+\alpha} (v_+ - v_-) > 0. \]

Taking
\[ \sqrt{A_1} = \left(\rho_+^{1+\alpha} + \rho_-^{1+\alpha}\right)\left(c^2 - v_+ v_-\right) - \sqrt{\left(\rho_+^{1+\alpha} + \rho_-^{1+\alpha}\right)^2(c^2 - v_+ v_-)^2 - 4c^4 \rho_+^{1+\alpha} \rho_-^{1+\alpha} (v_+ - v_-)^2}, \]
we have \((\rho_+, v_+) \in V(\rho_-, v_-)\) when \(A < A_1\). It is not difficult to show that \(A_1 < A_0\).

If \(\rho_+ = \rho_0\), then \((\rho_+, v_+) \in IV \cup V(\rho_-, v_-)\) for any \(A > 0\) and moreover \((\rho_+, v_+) \in V(\rho_-, v_-)\), if
\[ (v_+ - v_-)A + 2\rho_+^{1+\alpha} (v_+ v_- - c^2)\sqrt{A} + c^2 \rho_+^{1+\alpha} (v_+ - v_-) > 0. \]

Thus we can take
\[ \sqrt{A_1} = \frac{c^2 - v_+ v_- - \sqrt{(c^2 - v_+^2)(c^2 - v_-^2)}}{v_+ - v_-} \rho_+^{\frac{1+\alpha}{2}}. \]

Obviously we have \((\rho_+, v_+) \in V(\rho_-, v_-)\) when \(0 < A < A_1\). The proof is completed.

\[ \Box \]

**Remark 1.** From Lemma 4.1, it can be seen that the shock wave curves \(S_1\) and \(S_2\) become steeper when \(A\) decreases, and \((\rho_+, v_+) \in IV(\rho_-, v_-)\) when \(A_1 < A < A_0\).

There is no delta shock wave in the Riemann solution of (1.1)-(1.2) for a fluid with strong pressure. As pressure decreases, the delta shock wave occurs in the Riemann solution.

When \(A_1 < A < A_0\), namely \((\rho_+, v_+) \in IV(\rho_-, v_-)\), suppose that \((\rho_*, v_*)\) is the intermediate state connected with \((\rho_-, v_-)\) by a 1-shock \(S_1\) with speed \(\sigma_1\) and \((\rho_+, v_+)\) by a 2-shock \(S_2\) with speed \(\sigma_2\). Then \((\rho_*, v_*)\) is determined by
\[ \frac{v_+ - v_-}{c^2 - v_-^2} = -\frac{\sqrt{\Theta(\rho_*, \rho_-)}}{1 - v_\sqrt{\Theta(\rho_*, \rho_-)}}, \quad \rho_* > \rho_-, \] (4.6)
and
\[
\frac{v_* - v_+}{c^2 - v^2_+} = \frac{\sqrt{\Theta(p_+, \rho_*)}}{1 + v_+ \sqrt{\Theta(p_+, \rho_*)}}, \quad \rho_* > \rho_+.
\] (4.7)

Then, combining (4.6) and (4.7), we have
\[
v_- - v_+ = \left(\frac{c^2 - v^2_+}{1 - v_- \sqrt{\Theta(p_+, \rho_-)}}\right) + \left(\frac{c^2 - v^2_+}{1 + v_+ \sqrt{\Theta(p_+, \rho_*)}}\right), \quad \rho_* > \rho_+.
\] (4.8)

where
\[
\Theta(p_+, \rho_-) = \frac{A(-\rho_*^\alpha + \rho_-^\alpha)(\rho_* - \rho_-)}{(-A\rho_*^\alpha + \rho_- c^2)(-A\rho_*^\alpha + \rho_* c^2)}.
\]

\[
\Theta(p_+, \rho_*) = \frac{A(-\rho_*^\alpha + \rho_*^\alpha)(\rho_+ - \rho_*)}{(-A\rho_*^\alpha + \rho_+ c^2)(-A\rho_*^\alpha + \rho_* c^2)}.
\]

Letting \(A \to A_1\) in (4.8) yields
\[
\left(\frac{c^2 - v^2_+}{1 - v_- \sqrt{\Theta(p_+, \rho_-)}}\right) + \left(\frac{c^2 - v^2_+}{1 + v_+ \sqrt{\Theta(p_+, \rho_*)}}\right) = v_- - v_+.
\] (4.9)

Hence, from the fact that \(v_- - v_+ = \left(\frac{c^2 - v^2_+}{1 - \sqrt{A_1 \rho_-}}\right) + \left(\frac{c^2 - v^2_+}{1 + \sqrt{A_1 \rho_+}}\right)\), we deduce that
\[
\lim_{A \to A_1} \rho_* = +\infty.
\] (4.10)

Furthermore, we have the following result.

**Lemma 4.2.** Set \(\sigma = \frac{c^2(v_- - \sqrt{A_1 \rho_-})}{c^2 - v^2_+} = \frac{c^2(v_+ + \sqrt{A_1 \rho_+})}{c^2 + v^2_+}\). Then, we have
\[
\lim_{A \to A_1} v_* = \lim_{A \to A_1} \sigma_1 = \lim_{A \to A_1} \sigma_2 = \sigma,
\] (4.11)

\[
\lim_{A \to A_1} \int_{\sigma_1}^{\sigma_2} \rho_* c^2 dx = \left[\sigma \left(-\frac{A_1}{\rho^2} + \rho c^2\right) \frac{v^2}{c^2 (c^2 - v^2) + \rho} - \left(-\frac{A_1}{\rho^2} + \rho c^2\right) \frac{v^2}{c^2 (c^2 - v^2)}\right]t.
\] (4.12)

**Proof.** Letting \(A \to A_1\) in (4.6) and noting (4.10), we have
\[
\lim_{A \to A_1} v_* = v_- - \left(\frac{c^2 - v^2_+}{1 - v_- \sqrt{\Theta(p_+, \rho_-)}}\right) = v_- - \left(\frac{c^2 - v^2_+}{1 - \sqrt{A_1 \rho_-}}\right) = \sigma.
\] (4.13)

Based on the Rankine-Hugoniot condition (2.8), we have
\[
\sigma_1 = \left(-\frac{A}{\rho^2} + \rho c^2\right) \frac{v_+}{c^2 - v^2_+} - \left(-\frac{A}{\rho^2} + \rho c^2\right) \frac{v_-}{c^2 - v^2_+},
\] (4.14)

\[
\sigma_2 = \left(-\frac{A}{\rho^2} + \rho c^2\right) \frac{v_+}{c^2 - v^2_+} - \left(-\frac{A}{\rho^2} + \rho c^2\right) \frac{v_-}{c^2 - v^2_+},
\] (4.15)
which leads to
\[
\lim_{A \to A_1} \sigma_1 = \lim_{A \to A_1} \sigma_2 = \frac{\varepsilon^2 \sigma}{c^2 - \sigma^2} + 1 = \sigma.
\] (4.16)

Thus it can be seen from (4.16) that the two shocks \( S_1 \) and \( S_2 \) will coalesce when \( A \) arrives at \( A_1 \).

From (4.14) and (4.15), it follows that
\[
(\sigma_1 - \sigma_2)((-\frac{A}{\rho^*} + \rho_+ c^2) \frac{v^2}{c^2(c^2 - v^2)} + \rho_+) = \left[ (-\frac{A}{\rho^*} + \rho c^2) \frac{v}{c^2 - v^2} \right]
\]
\[
+ \sigma_1((-\frac{A}{\rho^*} + \rho_+ c^2) \frac{v^2}{c^2(c^2 - v^2)} - \rho_-) - \sigma_2((-\frac{A}{\rho^*} + \rho_+ c^2) \frac{v^2}{c^2(c^2 - v^2)} + \rho_+),
\] (4.17)

which implies that
\[
\lim_{A \to A_1} (\sigma_1 - \sigma_2)((-\frac{A}{\rho^*} + \rho_+ c^2) \frac{v^2}{c^2(c^2 - v^2)} + \rho_+) = \lim_{A \to A_1} (\sigma_1 - \sigma_2)((\frac{\rho_+ v^2}{c^2 - v^2} + \rho_+)
\]
\[
= \left[ (-\frac{A_1}{\rho^*} + \rho c^2) \frac{v}{c^2 - v^2} \right] - \sigma \left[ (-\frac{A_1}{\rho^*} + \rho c^2) \frac{v^2}{c^2(c^2 - v^2)} + \rho \right].
\] (4.18)

Hence, we have
\[
\lim_{A \to A_1} \int_{\sigma_1 t}^{\sigma_2 t} \frac{\rho_+ c^2}{c^2 - v^2} dx = \lim_{A \to A_1} \left( \sigma_2 - \sigma_1 \right) \frac{\rho_+ c^2}{c^2 - v^2} t
\]
\[
= \left[ \sigma \left[ (-\frac{A_1}{\rho^*} + \rho c^2) \frac{v^2}{c^2(c^2 - v^2)} + \rho \right] - \left[ (-\frac{A_1}{\rho^*} + \rho c^2) \frac{v}{c^2 - v^2} \right] \right] t.
\] (4.19)

The proof is completed.

It can be concluded from Lemma 4.2 that the two shocks \( S_1 \) and \( S_2 \) will coincide when \( A \) tends to \( A_1 \). Now, we give the following results which give a very nice depiction of the limit in the case \( v_- > v_+ \).

**Theorem 4.3.** Let \( v_- > v_+ \). For each fixed \( A \in (A_1, A_0) \), assuming that \((\rho, v)\) is a solution containing two shocks \( S_1 \) and \( S_2 \) of (1.1)-(1.2) with Riemann initial data (2.1), constructed in Section 2, it is obtained that as \( A \to A_1 \), \((\rho, v)\) converges in the sense of distributions, and the limit functions \( \frac{\rho c^2}{c^2 - v^2} \) and \( \frac{\rho_+ c^2}{c^2 - v^2} \) are the sums of a step function and a \( \delta \)-measure with weights
\[
\left( \sigma \left[ (-\frac{A_1}{\rho^*} + \rho c^2) \frac{v^2}{c^2(c^2 - v^2)} + \rho \right] - \left[ (-\frac{A_1}{\rho^*} + \rho c^2) \frac{v}{c^2 - v^2} \right] \right) t
\]
and
\[
\left( \sigma \left[ (-\frac{A_1}{\rho^*} + \rho c^2) \frac{v}{c^2 - v^2} \right] - \left[ (-\frac{A_1}{\rho^*} + \rho c^2) \frac{v^2}{c^2(c^2 - v^2)} - \frac{A_1}{\rho^*} \right] \right) t,
\]
respectively, which form a delta shock wave solution of (1.1)-(1.2) when \( A = A_1 \).

**Proof.** Let \( \xi = x/t \). Then for each fixed \( A \in (A_1, A_0) \), the Riemann solution can be written as
\[
(\rho, v)(\xi) = \begin{cases} 
(\rho_-, v_-), & \xi < \sigma_1, \\
(\rho_+, v_+), & \xi > \sigma_2, \\
(\rho_+, v_+), & \sigma_1 < \xi < \sigma_2,
\end{cases}
\] (4.20)

which satisfies the following weak formulations:
\[
\int_{-\infty}^{+\infty} \left( (-\frac{A}{\rho^*} + \rho c^2) \frac{v^2}{c^2(c^2 - v^2)} + \rho \right)(\xi) \xi v'(\xi) d\xi
\]
\[
+ \int_{-\infty}^{+\infty} \left( -\frac{A}{\rho^2} \cdot \frac{v^2}{c^2(c^2 - v^2)} \right)(\xi)\psi(\xi)d\xi + \int_{-\infty}^{+\infty} \left( \frac{\rho c^2}{c^2 - v^2} \right)(\xi)\psi(\xi)d\xi
- \int_{-\infty}^{+\infty} \left( (-\frac{A}{\rho^2} + \rho c^2) \frac{v}{c^2 - v^2} \right)(\xi)\psi(\xi)d\xi = 0,
\]
and
\[
+ \int_{-\infty}^{+\infty} \left( (-\frac{A}{\rho^2} + \rho c^2) \frac{v^2}{c^2 - v^2} \right)(\xi)\psi(\xi)d\xi + \int_{-\infty}^{+\infty} \left( \frac{\rho v c^2}{c^2 - v^2} \right)(\xi)\psi(\xi)d\xi
- \int_{-\infty}^{+\infty} \left( (-\frac{A}{\rho^2} + \rho c^2) \frac{v^2}{c^2 - v^2} - \frac{A}{\rho^2} \right)(\xi)\psi(\xi)d\xi = 0,
\]
for any test function \( \psi \in C_0^\infty(-\infty, +\infty) \).

The first integral on the left-hand side of (4.21) can be decomposed into
\[
\left\{ \int_{-\infty}^{\sigma_1} + \int_{\sigma_1}^{\sigma_2} + \int_{\sigma_2}^{+\infty} \right\} \left( (-\frac{A}{\rho^2} + \rho c^2) \frac{v^2}{c^2(c^2 - v^2)} + \rho \right)(\xi)\psi(\xi)d\xi,
\]
which equals
\[
\left( -\frac{A}{\rho^2} + \rho c^2 \right) \frac{v^2}{c^2(c^2 - v^2)} \int_{-\infty}^{\sigma_1} \psi(\xi)d\xi
- \left( -\frac{A}{\rho^2} + \rho c^2 \right) \frac{v^2}{c^2(c^2 - v^2)} \int_{-\infty}^{\sigma_1} \psi(\xi)d\xi
- \left( -\frac{A}{\rho^2} + \rho c^2 \right) \frac{v^2}{c^2(c^2 - v^2)} \int_{\sigma_1}^{\sigma_2} \psi(\xi)d\xi
- \left( -\frac{A}{\rho^2} + \rho c^2 \right) \frac{v^2}{c^2(c^2 - v^2)} \int_{\sigma_1}^{\sigma_2} \psi(\xi)d\xi
+ \left( -\frac{A}{\rho^2} + \rho c^2 \right) \frac{v^2}{c^2(c^2 - v^2)} \int_{\sigma_2}^{+\infty} \psi(\xi)d\xi
- \left( -\frac{A}{\rho^2} + \rho c^2 \right) \frac{v^2}{c^2(c^2 - v^2)} \int_{\sigma_2}^{+\infty} \psi(\xi)d\xi.
\]
The last integral on the left-hand side of (4.21) can be calculated by
\[
- \left\{ \int_{-\infty}^{\sigma_1} + \int_{\sigma_1}^{\sigma_2} + \int_{\sigma_2}^{+\infty} \right\} \left( -\frac{A}{\rho^2} + \rho c^2 \right) \frac{v}{c^2 - v^2} \psi(\xi)d\xi
= -\left( -\frac{A}{\rho^2} + \rho c^2 \right) \frac{v}{c^2 - v^2} \psi(\sigma_1)
- \left( -\frac{A}{\rho^2} + \rho c^2 \right) \frac{v}{c^2 - v^2} \psi(\sigma_2)
- \left( -\frac{A}{\rho^2} + \rho c^2 \right) \frac{v}{c^2 - v^2} \psi(\sigma_1).
\]
Combining (4.21), (4.24) and (4.25) together, we obtain
\[
\int_{-\infty}^{+\infty} \left( \frac{\rho c^2}{c^2 - v^2} \right)(\xi)\psi(\xi)d\xi
- \left( -\frac{A}{\rho^2} + \rho c^2 \right) \frac{v^2}{c^2(c^2 - v^2)} \int_{-\infty}^{\sigma_1} \psi(\xi)d\xi
+ \left( -\frac{A}{\rho^2} + \rho c^2 \right) \frac{v^2}{c^2(c^2 - v^2)} \int_{\sigma_1}^{\sigma_2} \psi(\xi)d\xi
+ \left( -\frac{A}{\rho^2} + \rho c^2 \right) \frac{v^2}{c^2(c^2 - v^2)} \int_{\sigma_2}^{+\infty} \psi(\xi)d\xi
- \left( -\frac{A}{\rho^2} + \rho c^2 \right) \frac{v^2}{c^2(c^2 - v^2)} \int_{-\infty}^{\sigma_1} \psi(\xi)d\xi
+ \left( -\frac{A}{\rho^2} + \rho c^2 \right) \frac{v^2}{c^2(c^2 - v^2)} \int_{\sigma_1}^{\sigma_2} \psi(\xi)d\xi
+ \left( -\frac{A}{\rho^2} + \rho c^2 \right) \frac{v^2}{c^2(c^2 - v^2)} \int_{\sigma_2}^{+\infty} \psi(\xi)d\xi
- \left( -\frac{A}{\rho^2} + \rho c^2 \right) \frac{v^2}{c^2(c^2 - v^2)} \int_{-\infty}^{\sigma_1} \psi(\xi)d\xi
+ \left( -\frac{A}{\rho^2} + \rho c^2 \right) \frac{v^2}{c^2(c^2 - v^2)} \int_{\sigma_1}^{\sigma_2} \psi(\xi)d\xi
+ \left( -\frac{A}{\rho^2} + \rho c^2 \right) \frac{v^2}{c^2(c^2 - v^2)} \int_{\sigma_2}^{+\infty} \psi(\xi)d\xi
- \left( -\frac{A}{\rho^2} + \rho c^2 \right) \frac{v^2}{c^2(c^2 - v^2)} \int_{-\infty}^{\sigma_1} \psi(\xi)d\xi
+ \left( -\frac{A}{\rho^2} + \rho c^2 \right) \frac{v^2}{c^2(c^2 - v^2)} \int_{\sigma_1}^{\sigma_2} \psi(\xi)d\xi
+ \left( -\frac{A}{\rho^2} + \rho c^2 \right) \frac{v^2}{c^2(c^2 - v^2)} \int_{\sigma_2}^{+\infty} \psi(\xi)d\xi
- \left( -\frac{A}{\rho^2} + \rho c^2 \right) \frac{v^2}{c^2(c^2 - v^2)} \int_{-\infty}^{\sigma_1} \psi(\xi)d\xi
+ \left( -\frac{A}{\rho^2} + \rho c^2 \right) \frac{v^2}{c^2(c^2 - v^2)} \int_{\sigma_1}^{\sigma_2} \psi(\xi)d\xi
+ \left( -\frac{A}{\rho^2} + \rho c^2 \right) \frac{v^2}{c^2(c^2 - v^2)} \int_{\sigma_2}^{+\infty} \psi(\xi)d\xi.
\]
\[-\left(\frac{A}{\rho_\sigma^2} + \rho_\sigma c^2\right) \frac{v_-^2}{c^2(c^2 - v_-^2)} + \rho_\sigma)(\sigma_2 - \sigma_1) \cdot \frac{\sigma_2\psi(\sigma_2) - \sigma_1\psi(\sigma_1)}{\sigma_2 - \sigma_1} \]
\[+ \left(\frac{\rho_\sigma c^2}{c^2 - v_+^2}\right)(\sigma_2 - \sigma_1) \cdot \frac{1}{\sigma_2 - \sigma_1} \int_{\sigma_1}^{\sigma_2} \psi(\xi)d\xi + \left(\frac{A}{\rho_\sigma^2} + \rho_- c^2\right) \frac{v_-}{c^2 - v_-^2}\psi(\sigma_1) \]
\[-\left(\frac{A}{\rho_\sigma^2} + \rho_+ c^2\right) \frac{v_+}{c^2 - v_+^2}\psi(\sigma_2) + \left(\frac{A}{\rho_\sigma^2} + \rho_+ c^2\right)(\sigma_2 - \sigma_1)v_+ \cdot \frac{\psi(\sigma_2) - \psi(\sigma_1)}{\sigma_2 - \sigma_1}. \tag{4.26} \]

Taking the limit \( A \to A_1 \) in (4.26), noting (4.18) and the fact that both \( \psi \in C^0[\infty, +\infty) \) and \( \lim_{A \to A_1} v_\sigma = \lim_{A \to A_1} \sigma_1 = \lim_{A \to A_1} \sigma_2 = \sigma \), we deduce that

\[
\lim_{A \to A_1} \int_{-\infty}^{+\infty} \left( \frac{\rho_\sigma^2}{c^2 - v_2^2}(\xi - \rho_\sigma c^2(\xi - \sigma))\right)\psi(\xi)d\xi
\]
\[
= \left(\sigma \left[\left(\frac{\rho_\sigma}{\rho\sigma} + \rho c^2\right) \frac{v^2}{c^2(c^2 - v^2)} + \rho\right] - \left[\left(\frac{A_1}{\rho\sigma} + \rho c^2\right) \frac{v^2}{c^2(c^2 - v^2)}\right]\right)\psi(\sigma) \tag{4.27} \]

where \( a_0(\xi) = a_- + [a]H(\xi) \) and \( H \) is the Heaviside function.

Similarly, the first integral on the left-hand side of (4.22) can be decomposed into three parts as

\[
\left\{ \int_{-\infty}^{\sigma_1} + \int_{\sigma_2}^{+\infty} \right\} \left(\frac{A}{\rho\sigma} + \rho c^2\right) \frac{v^2}{c^2 - v^2}\psi(\xi)\psi'(\xi)d\xi, \tag{4.28} \]

which equals

\[
\left(\frac{A}{\rho_\sigma^2} + \rho_- c^2\right) \frac{v_-}{c^2 - v_-^2}\sigma_1\psi(\sigma_1) - \left(\frac{A}{\rho_\sigma^2} + \rho_- c^2\right) \frac{v_-}{c^2 - v_-^2}\int_{-\infty}^{\sigma_1} \psi(\xi)d\xi
\]
\[-\left(\frac{A}{\rho_\sigma^2} + \rho_+ c^2\right) \frac{v_+}{c^2 - v_+^2}\sigma_2\psi(\sigma_2) + \left(\frac{A}{\rho_\sigma^2} + \rho_+ c^2\right) \frac{v_+}{c^2 - v_+^2}\int_{\sigma_2}^{+\infty} \psi(\xi)d\xi
\]+\left(\frac{A}{\rho_\sigma^2} + \rho_+ c^2\right) \frac{v_+}{c^2 - v_+^2}\psi(\sigma_2) - \sigma_1\psi(\sigma_1) - \left(\frac{A}{\rho_\sigma^2} + \rho_+ c^2\right) \frac{v_+}{c^2 - v_+^2}\int_{\sigma_1}^{\sigma_2} \psi(\xi)d\xi. \tag{4.29} \]

The last integral on the left-hand side of (4.22) can be calculated by

\[
-\left\{ \int_{-\infty}^{\sigma_1} + \int_{\sigma_2}^{+\infty} \right\} \left(\frac{A}{\rho\sigma} + \rho c^2\right) \frac{v^2}{c^2 - v^2} - \frac{A}{\rho\sigma}\psi(\sigma_1)
\]
\[= -\left(\frac{A}{\rho_\sigma^2} + \rho_- c^2\right) \frac{v_-}{c^2 - v_-^2} - \frac{A}{\rho_\sigma^2}\psi(\sigma_1)
\]
\[+ \left(\frac{A}{\rho_\sigma^2} + \rho_+ c^2\right) \frac{v_+}{c^2 - v_+^2} - \frac{A}{\rho_\sigma^2}\psi(\sigma_2)
\]-\left(\frac{A}{\rho_\sigma^2} + \rho_+ c^2\right) \frac{v_+}{c^2 - v_+^2} - \frac{A}{\rho_\sigma^2}\psi(\sigma_1). \tag{4.30} \]

Summarizing (4.22), (4.29) and (4.30) leads to

\[
\int_{-\infty}^{+\infty} \left(\frac{\rho_\sigma v^2}{c^2 - v^2}\right)\psi(\xi)d\xi = \]
As done previously, one can deduce easily from (4.31) that

\[
\lim_{A \to A_1} \int_{-\infty}^{+\infty} \left( \frac{\rho v c^2}{c^2 - v^2} \sigma \right) (\xi) - \rho_0 v_0 c^2 (\xi - \sigma) \psi(\xi) d\xi = \left( \sigma \left( -\frac{A_1}{\rho^2} + \rho c^2 \frac{v}{c^2 - v^2} \right) - \left( -\frac{A_1}{\rho^2} + \rho c^2 \frac{v}{c^2 - v^2} - \frac{A_1}{\rho^2} \right) \right) \psi(\sigma),
\]

which is true for any \( \psi \in C^\infty_0(-\infty, +\infty) \).

Finally, we study the limits of \( \frac{\rho v c^2}{c^2 - v^2} \) and \( \frac{\rho_0 v_0 c^2}{c^2 - v^2} \) as \( A \to A_1 \), by tracing the time-dependence of weights of the \( \delta \)-measure. Let \( \phi(x,t) \in C^\infty_0((-\infty, +\infty) \times [0, +\infty)), \) then we have

\[
\lim_{A \to A_1} \int_{0}^{+\infty} \int_{-\infty}^{+\infty} \frac{\rho v c^2}{c^2 - v^2} (x) \phi(x,t) dx dt = \lim_{A \to A_1} \int_{0}^{+\infty} \int_{-\infty}^{+\infty} \frac{\rho_0 v_0 c^2}{c^2 - v^2} (x) \phi(x,t) dx dt.
\]

Regarding \( t \) as a parameter and applying (4.27), one can easily see that

\[
\lim_{A \to A_1} \int_{-\infty}^{+\infty} \frac{\rho v c^2}{c^2 - v^2} (\xi) \phi(\xi, t) d\xi = \int_{-\infty}^{+\infty} \frac{\rho_0 v_0 c^2}{c^2 - v^2} (\xi - \sigma) \phi(\xi, t) d\xi
\]

\[+ \left( \sigma \left( -\frac{A_1}{\rho^2} + \rho c^2 \frac{v^2}{c^2 - v^2} + \rho \right) - \left( -\frac{A_1}{\rho^2} + \rho c^2 \frac{v^2}{c^2 - v^2} \right) \right) \phi(\sigma, t)
\]

\[= \frac{1}{t} \int_{-\infty}^{+\infty} \frac{\rho_0 v_0 c^2}{c^2 - v^2} (x - \sigma t) \phi(x, t) dx
\]

\[+ \left( \sigma \left( -\frac{A_1}{\rho^2} + \rho c^2 \frac{v^2}{c^2 - v^2} + \rho \right) - \left( -\frac{A_1}{\rho^2} + \rho c^2 \frac{v^2}{c^2 - v^2} \right) \right) \phi(\sigma, t).
\]

Substituting (4.34) into (4.33), we have

\[
\lim_{A \to A_1} \int_{0}^{+\infty} \int_{-\infty}^{+\infty} \frac{\rho v c^2}{c^2 - v^2} (x) \phi(x,t) dx dt = \int_{0}^{+\infty} \int_{-\infty}^{+\infty} \frac{\rho_0 v_0 c^2}{c^2 - v^2} (x - \sigma t) \phi(x, t) dx dt
\]

\[+ \int_{0}^{+\infty} t \left( \sigma \left( -\frac{A_1}{\rho^2} + \rho c^2 \frac{v^2}{c^2 - v^2} + \rho \right) - \left( -\frac{A_1}{\rho^2} + \rho c^2 \frac{v^2}{c^2 - v^2} \right) \right) \phi(\sigma, t) dt.
\]
By the definition, the last term on the right-hand side of (4.35) equals to
\[ w_1(t) = \left( \sigma \left[ \left( -\frac{A_1}{\rho^3} + \rho c^2 \right) \frac{v^2}{c^2 - v^2} + \rho \right] - \left[ \left( -\frac{A_1}{\rho^3} + \rho c^2 \right) \frac{v}{c^2 - v^2} \right] \right) t. \]

With the same reason as before, we arrive at
\[
\lim_{A \to A_1} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\rho \rho^2}{c^2 - v^2} \phi(x,t)dxdt = \int_{0}^{+\infty} \int_{-\infty}^{+\infty} \frac{\rho_0 v_0 c^2}{c^2 - v^2} (x - \sigma t) \phi(x,t)dxdt 
+ \int_{0}^{+\infty} t \left[ \left( -\frac{A_1}{\rho^3} + \rho c^2 \right) \frac{v}{c^2 - v^2} \right] - \left[ \left( -\frac{A_1}{\rho^3} + \rho c^2 \right) \frac{v^2}{c^2 - v^2} + \frac{A_1}{\rho^3} \right] \phi(\sigma t, t)dt. \quad (4.36)
\]

The last term on the right-hand side of (4.36) equals to \( \langle w_2(t) \delta_S, \phi(\cdot, \cdot) \rangle \), where
\[ w_2(t) = \left( \sigma \left[ \left( -\frac{A_1}{\rho^3} + \rho c^2 \right) \frac{v}{c^2 - v^2} \right] - \left[ \left( -\frac{A_1}{\rho^3} + \rho c^2 \right) \frac{v^2}{c^2 - v^2} + \frac{A_1}{\rho^3} \right] t. \]

The proof is completed. \( \square \)

Now, we study the limit behavior of the Riemann solution of (1.1) and (1.2) in the case \( (\rho_+, v_+) \in V(\rho_-, v_-) \) with \( v_- > v_+ \) as \( A \to 0 \).

**Theorem 4.4.** If \( (\rho_+, v_+) \in V(\rho_-, v_-) \) with \( v_- > v_+ \), as \( A \to 0 \), the limit of the Riemann solution of (1.1) and (1.2) with initial data (2.1) is just the Riemann solution of the zero-pressure relativistic Euler equations (3.1) with the same initial data, which contains a delta shock wave besides two constant states.

**Proof.** If \( (\rho_+, v_+) \in V(\rho_-, v_-) \) with \( v_- > v_+ \), the Riemann solution of (1.1)-(1.2) contains a delta shock wave besides two constant states \((\rho_\pm, v_\pm)\), and the strength and propagation speed of which can be determined by (2.23) for \( E \neq 0 \). Rewrite \( \sigma \) in (2.23) as
\[
\sigma = \frac{\left( \rho_+ c^2 - \frac{A}{\rho^3} \right) \frac{v_+}{c^2 - v_+} - \left( \rho_- c^2 - \frac{A}{\rho^3} \right) \frac{v_-}{c^2 - v_-} + \sqrt{\Delta}}{\left( c^2 - \frac{A^2}{\rho^3} \right) \frac{\rho_+}{c^2 - v_+} - \left( c^2 - \frac{A^2}{\rho_-} \right) \frac{\rho_-}{c^2 - v_-}}, \quad (4.37)
\]
where
\[
\Delta = \left( c^2 - \frac{A}{\rho_+} \right) \frac{\rho_+ v_+}{c^2 - v_+} - \left( c^2 - \frac{A}{\rho_-} \right) \frac{\rho_- v_-}{c^2 - v_-} \right)^2 \\
- \left( c^2 - \frac{A^2}{\rho_+ + 1} \right) \frac{\rho_+}{c^2 - v_+} - \left( c^2 - \frac{A^2}{\rho_- + 1} \right) \frac{\rho_-}{c^2 - v_-} \right) \\
\times \left( \left( -\frac{A}{\rho_+ + 1} + c^2 \right) \frac{\rho_+ v_+^2}{c^2 - v_+^2} - \frac{A}{\rho_+} - \left( -\frac{A}{\rho_- + 1} + c^2 \right) \frac{\rho_- v_-^2}{c^2 - v_-^2} + \frac{A}{\rho_-} \right).
\]
Taking the limit \( A \to 0 \) in (4.37) leads to
\[
\lim_{A \to 0} \sigma = v_- \sqrt{\frac{\rho_-}{c^2 - v_-^2}} + v_+ \sqrt{\frac{\rho_+}{c^2 - v_+^2}} = v_\delta. \quad (4.38)
\]
It can be derived from (2.23) and (4.38) that
\[
\lim_{A \to 0} w(t) = \sqrt{\frac{\rho_-}{c^2 - v_-^2}} \frac{\rho_+}{c^2 - v_+^2} (v_- - v_+) (c^2 - v_\delta^2) t.
\]
Thus, the limit values of \( \sigma \) and \( w(t) \) are identical with (3.13). For the special case \( E = 0 \), the same result can be derived from (2.22), as proposed for the delta shock wave.
wave in the Riemann solution of the zero-pressure relativistic Euler equations (3.1) with the same initial data. The proof is completed.

4.2. Case $v_- < v_+$.

Lemma 4.5. If $v_- < v_+$, then there exists $A_0 > 0$ such that $(\rho_+, v_+) \in I(\rho_-, v_-)$ when $0 < A < A_0$.

Proof. If $\rho_0 = \rho_+$, then $(\rho_+, v_+) \in I(\rho_-, v_-)$ for any $A > 0$. Thus, we only need to consider the case $\rho_0 \neq \rho_+$.

It can be derived from (2.6) and (2.7) that all possible states $(\rho, v)$ that can be connected to the left state $(\rho_-, v_-)$ on the right by a 1-rarefaction wave $R_1$ or a 2-rarefaction wave $R_2$ should satisfy

$$ R_1: \left(\frac{c+v}{c-v}\right) \left(\frac{c-\sqrt{A} \rho_+^{\frac{1}{2}}}{c+\sqrt{A} \rho_-^{\frac{1}{2}}}\right)^{\frac{2c+\alpha}{c+\alpha}} = \left(\frac{c+v_-}{c-v_-}\right) \left(\frac{c-\sqrt{A} \rho_-^{\frac{1}{2}}}{c+\sqrt{A} \rho_-^{\frac{1}{2}}}\right)^{\frac{2c+\alpha}{c+\alpha}}, \quad v > v_- \quad \rho < \rho_-, $$

$$ R_2: \left(\frac{c+v}{c-v}\right) \left(\frac{c+\sqrt{A} \rho_+^{\frac{1}{2}}}{c-\sqrt{A} \rho_-^{\frac{1}{2}}}\right)^{\frac{2c+\alpha}{c+\alpha}} = \left(\frac{c+v_-}{c-v_-}\right) \left(\frac{c+\sqrt{A} \rho_-^{\frac{1}{2}}}{c-\sqrt{A} \rho_-^{\frac{1}{2}}}\right)^{\frac{2c+\alpha}{c+\alpha}}, \quad v > v_- \quad \rho > \rho_. $$

If $(\rho_+, v_+) \in I(\rho_-, v_-)$, then we can see from (4.39)-(4.40) that

$$ \left(\frac{c+v}{c-v}\right) \left(\frac{c-\sqrt{A} \rho_+^{\frac{1}{2}}}{c+\sqrt{A} \rho_-^{\frac{1}{2}}}\right)^{\frac{2c+\alpha}{c+\alpha}} > \left(\frac{c+v_-}{c-v_-}\right) \left(\frac{c-\sqrt{A} \rho_-^{\frac{1}{2}}}{c+\sqrt{A} \rho_-^{\frac{1}{2}}}\right)^{\frac{2c+\alpha}{c+\alpha}}, \quad \text{if } \rho_+ < \rho_-, $$(4.41)

and

$$ \left(\frac{c+v}{c-v}\right) \left(\frac{c+\sqrt{A} \rho_+^{\frac{1}{2}}}{c-\sqrt{A} \rho_-^{\frac{1}{2}}}\right)^{\frac{2c+\alpha}{c+\alpha}} > \left(\frac{c+v_-}{c-v_-}\right) \left(\frac{c+\sqrt{A} \rho_-^{\frac{1}{2}}}{c-\sqrt{A} \rho_-^{\frac{1}{2}}}\right)^{\frac{2c+\alpha}{c+\alpha}}, \quad \text{if } \rho_+ > \rho_-.$$ (4.42)

This implies that

$$ (a-b) \rho_+^{\frac{1}{2}} (\rho_+-\rho_+^{\frac{1}{4}}) A + |\rho_+^{\frac{1}{2}} - \rho_-^{\frac{1}{4}}| \cdot c (a+b) \sqrt{A} + (b-a)c^2 < 0, $$

where

$$ a = \left(\frac{c+v}{c-v_-}\right) \frac{a+b}{c-v_-} \quad \text{and} \quad b = \left(\frac{c+v_-}{c-v_-}\right) \frac{a+b}{c-v_-}. $$

By $v_+ > v_-$, it is easy to see that $a > b$. Thus we can take

$$ \sqrt{A_0} = -\frac{|\rho_+^{\frac{1}{2}} - \rho_-^{\frac{1}{4}}| \cdot c(a+b) + \sqrt{\Delta}}{2(a-b)\rho_+^{\frac{1}{2}}}, $$

where

$$ \Delta = c^2(a+b)^2(\rho_+^{\frac{1}{2}} - \rho_-^{\frac{1}{4}})^2 + 4c^2(a-b)^2\rho_+^{\frac{1}{2}} \rho_-^{\frac{1}{4}} > 0. $$

Obviously we have $\rho_+ \in I(\rho_-, v_-)$ when $0 < A < A_0$. The proof is completed.

Next, we study the formation of vacuum states in the Riemann solutions of (1.1) and (1.2) in the case $(\rho_+, v_+) \in I(\rho_-, v_-)$ with $v_- < v_+$ as $A \to 0$. When $v_- < v_+$,
by Lemma 4.5, for any given $A \in (0, A_0)$, the Riemann solution consists of a 1-rarefaction wave $R_1$, a 2-rarefaction wave $R_2$ and an intermediate state $(\rho_*, v_*)$ besides two constant states $(\rho_\pm, v_\pm)$, which are as follows

$$
R_1:\left\{
\begin{array}{l}
\xi = \lambda_1 = \frac{c^2(v-\sqrt{A\rho_+^{\frac{1+\alpha}{2}}})}{c^2-v\sqrt{A\rho_+^{\frac{1+\alpha}{2}}}}, \\
\frac{(c+v)(c-\sqrt{A\rho_+^{\frac{1+\alpha}{2}}})}{c-v}\left(\frac{c-\sqrt{A\rho_+^{\frac{1+\alpha}{2}}}}{c}\right) = \frac{(c+v_\pm)(c-\sqrt{A\rho_+^{\frac{1+\alpha}{2}}})}{c-v_\pm}\left(\frac{c-\sqrt{A\rho_+^{\frac{1+\alpha}{2}}}}{c}\right), \quad \rho_* \leq \rho \leq \rho_-.
\end{array}
\right.
$$

(4.44)

$$
R_2:\left\{
\begin{array}{l}
\xi = \lambda_2 = \frac{c^2(v+\sqrt{A\rho_-^{\frac{1+\alpha}{2}}})}{c^2+v\sqrt{A\rho_-^{\frac{1+\alpha}{2}}}}, \\
\frac{(c+v)(c+\sqrt{A\rho_-^{\frac{1+\alpha}{2}}})}{c-v}\left(\frac{c-\sqrt{A\rho_-^{\frac{1+\alpha}{2}}}}{c}\right) = \frac{(c+v_\pm)(c+\sqrt{A\rho_-^{\frac{1+\alpha}{2}}})}{c-v_\pm}\left(\frac{c-\sqrt{A\rho_-^{\frac{1+\alpha}{2}}}}{c}\right), \quad \rho_* \leq \rho \leq \rho_+.
\end{array}
\right.
$$

(4.45)

From the second equations of (4.44) and (4.45), we can derive

$$
\left(\frac{c-\sqrt{A\rho_+^{\frac{1+\alpha}{2}}}}{c+\sqrt{A\rho_+^{\frac{1+\alpha}{2}}}}\right)^2 = \left(\frac{(c+v_\pm)(c-v_\pm)}{(c-v_\pm)(c+v_\pm)}\right) = \left(\frac{(c-\sqrt{A\rho_-^{\frac{1+\alpha}{2}}})(c-\sqrt{A\rho_+^{\frac{1+\alpha}{2}}})}{(c+\sqrt{A\rho_-^{\frac{1+\alpha}{2}}})(c+\sqrt{A\rho_+^{\frac{1+\alpha}{2}}})}\right),
$$

(4.46)

for any given $\rho_\pm > 0$, if $\lim_{A \to 0} \rho_* = K \in (0, \min(\rho_-, \rho_+)]$, then (4.46) leads to $v_- = v_+$, which contradicts with $v_- < v_+$. Therefore, $\lim_{A \to 0} \rho_* = 0$, which implies that a vacuum occurs. We can employ (4.46) and calculate to obtain

$$
\frac{A}{\rho_*^{1+\alpha}} = c^2\left(\frac{1-\phi(A)}{1+\phi(A)}\right),
$$

(4.47)

where

$$
\phi(A) = \sqrt{\left(\frac{(c+v_\pm)(c-v_\pm)}{(c-v_\pm)(c+v_\pm)}\right)^{\frac{1+\alpha}{2}} - \left(\frac{(c-\sqrt{A\rho_-^{\frac{1+\alpha}{2}}})(c-\sqrt{A\rho_+^{\frac{1+\alpha}{2}}})}{(c+\sqrt{A\rho_-^{\frac{1+\alpha}{2}}})(c+\sqrt{A\rho_+^{\frac{1+\alpha}{2}}})}\right)}.
$$

By $v_+ > v_-$, it is easy to see that

$$
\lim_{A \to 0} \phi(A) = \left(\frac{(c+v_\pm)(c-v_\pm)}{(c-v_\pm)(c+v_\pm)}\right)^{\frac{1+\alpha}{2+\alpha}} = : \bar{\phi} < 1,
$$

which implies that

$$
\lim_{A \to 0} \frac{A}{\rho_*^{1+\alpha}} = c^2\left(\frac{1-\bar{\phi}}{1+\bar{\phi}}\right)^2.
$$

Then, by $p_* = -A\rho_*^{-\alpha}$, we have

$$
\lim_{A \to 0} p_* = -\lim_{A \to 0} \frac{A}{\rho_*^{1+\alpha}} = 0.
$$

It can be derived from (4.44) that

$$
\lim_{A \to 0} \frac{(c+v)(c-\sqrt{A\rho_-^{\frac{1+\alpha}{2}}}^{\frac{1+\alpha}{2}})}{(c+\sqrt{A\rho_-^{\frac{1+\alpha}{2}}}^{\frac{1+\alpha}{2}})} = \frac{c+v_\pm}{c-v_\pm}.
$$

(4.48)
The uniform boundedness of $\rho(\xi)$ with respect to $A$ leads to $\lim_{A \to 0} v = v_-$ on $R_1$. Then, from the first equation of (4.44), we get that $\lim_{A \to 0} \lambda_1 = v_-$. Similarly, we can obtain that

$$\lim_{A \to 0} \lambda_2 = v_+ \quad \text{and} \quad \lim_{A \to 0} v(\xi) = \xi, \quad \text{for} \quad \xi \in (v_-, v_+). \quad (4.49)$$

Then from above we have proved the following results.

**Theorem 4.6.** In the case $(\rho_+, v_+) \in I(\rho_-, v_-)$ with $v_- < v_+$, when $A \to 0$, $p_*$ and $p_*$ vanish, and two rarefaction waves $R_1$ and $R_2$ become two contact discontinuities $v = v_-$ and $v = v_+$, respectively, connecting the constant states $(\rho_+, v_+)$ with the vacuum $(\rho = 0)$.

**Theorem 4.7.** In the case $(\rho_+, v_+) \in I(\rho_-, v_-)$ with $v_- < v_+$, when $A \to 0$, the limit of the Riemann solution of (1.1) and (1.2) with initial data (2.1) is just the Riemann solution of the zero-pressure relativistic Euler equations (3.1) with the same initial data, which contains two contact discontinuities $\xi = x/t = v_\pm$ and a vacuum state besides two constant states.

4.3. Case $v_- = v_+$. In this case, $(\rho_+, v_+) \in II \cup III(\rho_-, v_-)$, and the Riemann solution contains a 1-rarefaction wave $R_1$ and a 2-shock wave $S_2$ for $\rho_+ < \rho_-$ or a 1-shock wave $S_1$ and a 2-rarefaction wave $R_2$ for $\rho_+ > \rho_-$. Particularly, if $\rho_+ = \rho_-$, the Riemann solution is a constant state $(\rho_-, v_-)$.

Fig. 4. Riemann solution when $(\rho_+, v_+) \in II(\rho_-, v_-)$ in the phase plane.

For the case $\rho_+ < \rho_-$, we have $(\rho_+, v_+) \in II(\rho_-, v_-)$ obviously, as shown in Fig. 4, the Riemann solution of (1.1)-(1.2) can be expressed as

$$(\rho, v)(\xi) = \begin{cases} 
(\rho_-, v_-), & -\infty < \xi \leq \lambda_1(\rho_-, v_-), \\
R_1, & \lambda_1(\rho_-, v_-) \leq \xi \leq \lambda_1(\rho_+, v_+), \\
(\rho_+, v_+), & \lambda_1(\rho_+, v_+) \leq \xi < \sigma_2, \\
(\rho_*, v_*), & \sigma_2 < \xi < +\infty,
\end{cases} \quad (3.50)$$

where $\sigma_2$ is the propagation speed of $S_2$ and $R_1$ consists of the states $(\rho, v)$ satisfying (2.6).
By (2.6) and (2.11), the intermediate state \((\rho_*, v_*)\) between \(R_1\) and \(S_2\) satisfies
\[
\frac{c + v_*}{c - v_*} \left( \frac{c - \sqrt{A\rho_*}}{c + \sqrt{A\rho_*}} \right)^2 = \frac{c + v_0}{c - v_0} \left( \frac{c - \sqrt{A\rho_0}}{c + \sqrt{A\rho_0}} \right)^2, \tag{3.51}
\]
and
\[
v_* - v_+ = \frac{\sqrt{\Theta(\rho_+, \rho_*)}}{c^2 - v_+^2}
\tag{3.52}
\]
where
\[
\Theta(\rho_+, \rho_*) = \frac{A(-\rho_+^{\alpha} + \rho_+^{\alpha})(\rho_+ - \rho_*)}{(-A\rho_+^{\alpha} + \rho_*c^2)(-A\rho_+^{\alpha} + \rho_+ c^2)},
\]
with \(\rho_+ < \rho_* < \rho_-.\)

Letting \(A \to 0\) in (3.51), one can immediately get \(\lim v_* = v_0\), noticing the fact that \(\rho_*\) is bounded. Furthermore, taking the limit \(A \to 0\) in (3.52) leads to
\[
\lim_{A \to 0} \frac{v_* - v_+}{\sqrt{\frac{1}{\rho_*^2} - \frac{1}{\rho_+^2}}(\rho_* - \rho_+)} = \lim_{A \to 0} \frac{(c^2 - v_+^2)}{1 + v_+\sqrt{\Theta(\rho_+, \rho_*)}} = 0,
\]
which implies that
\[
\lim_{A \to 0} \frac{v_* - v_+}{\sqrt{\frac{1}{\rho_*^2} - \frac{1}{\rho_+^2}}(\rho_* - \rho_+)} = \lim_{A \to 0} \frac{v_* - v_+}{\sqrt{\frac{1}{\rho_*^2} - \frac{1}{\rho_+^2}}(\rho_* - \rho_+)} = 0,
\]
due to the fact that \(\rho_+ < \rho_* < \rho_-.\) Thus, it can be derived from the Rankine-Hugoniot condition (2.8) that
\[
\lim_{A \to 0} \sigma_2 = \lim_{A \to 0} \frac{(-A\rho_*^{\alpha} + \rho_+ c^2) \frac{v_* - v_+}{\sqrt{c^2 - v_*^2}} - (-A\rho_*^{\alpha} + \rho_* c^2) \frac{v_* - v_+}{\sqrt{c^2 - v_*^2}}}{(-A\rho_*^{\alpha} + \rho_+ c^2) \frac{v_* - v_+}{\sqrt{c^2 - v_*^2}} + \rho_* - (-A\rho_*^{\alpha} + \rho_* c^2) \frac{v_* - v_+}{\sqrt{c^2 - v_*^2}}}, \tag{3.55}
\]
which leads to
\[
\lim_{A \to 0} \sigma_2 = \lim_{A \to 0} \frac{(-A\rho_*^{\alpha} + \rho_+ c^2) \frac{v_* - v_+}{\sqrt{c^2 - v_*^2}} + \rho_* - (-A\rho_*^{\alpha} + \rho_+ c^2) \frac{v_* - v_+}{\sqrt{c^2 - v_*^2}}}{(-A\rho_*^{\alpha} + \rho_+ c^2) \frac{v_* - v_+}{\sqrt{c^2 - v_*^2}} + \rho_* - (-A\rho_*^{\alpha} + \rho_+ c^2) \frac{v_* - v_+}{\sqrt{c^2 - v_*^2}}} \frac{c^2 - v_*^2}{c^2 - v_*^2} + 1
= \frac{c^2 - v_*^2}{c^2 - v_*^2} + 1 = v_-. \quad \tag{3.56}
\]
Meanwhile, we have
\[
\lim_{A \to 0} \lambda_1(\rho_*, v_*) = \lim_{A \to 0} \frac{c^2(v_* - \sqrt{A\rho_+^{\alpha+1}})}{c^2 - v_* \sqrt{A\rho_+^{\alpha+1}}} = v_-, \tag{3.57}
\]
and
\[
\lim_{A \to 0} \lambda_1(\rho_-, v_-) = \lim_{A \to 0} \frac{c^2(v_- - \sqrt{A\rho_-^{\alpha+1}})}{c^2 - v_- \sqrt{A\rho_-^{\alpha+1}}} = v_. \tag{3.58}
\]
Thus it can be seen from (3.56)-(3.58) that as \( A \to 0 \), the rarefaction wave \( R_1 \) and shock wave \( S_2 \) converge to one contact discontinuity with the propagation speed \( v_- \), which connects the constant states \((\rho_\pm, v_\pm)\), see Fig. 5.

\[ J : x = v_- t \]

\[ S_2 : x = \sigma_2 t \]

\( R_1 \)

\( S_2 \): \( x = \sigma_2 t \)

\( (\rho_-, v_-) \)

\( S_1 \), \( R_1 \), \( R_2 \)

\( (\rho_+, v_+) \)

\( (\rho_*, v_*) \)

\( (\rho_+, v_+) \)

Fig. 5. Riemann solution when \((\rho_+, v_+) \in II(\rho_-, v_-)\) in \((t, x)\)-plane.

\[ \rho \]

\[ v \]

\( S_2 \)

\( S_1 \)

\( R_1 \)

\( R_2 \)

\( V \)

\( IV \)

\( III \)

\( II \)

\( I \)

\( (\rho_*, v_*) \)

\( (\rho_-, v_-) \)

\( (\rho_+, v_+) \)

\( R_2 \)

For the case \( \rho_+ > \rho_- \), \((\rho_+, v_+) \in III(\rho_-, v_-)\), as shown in Fig. 6, the Riemann solution of (1.1)-(1.2) can be given by

\[
(\rho, v)(\xi) = \begin{cases} 
(\rho_-, v_-), & -\infty < \xi < \sigma_1, \\
(\rho_*, v_*), & \sigma_1 < \xi < \lambda_2(\rho_*, v_*), \\
R_2, & \lambda_2(\rho_*, v_*) \leq \xi \leq \lambda_2(\rho_+, v_+), \\
(\rho_+, v_+), & \lambda_2(\rho_+, v_+) \leq \xi < +\infty, 
\end{cases}
\]  

(3.59)

where \( \sigma_1 \) is the propagation speed of \( S_1 \) and \( R_2 \) consists of the states \((\rho, v)\) satisfying (2.7).
By (2.7) and (2.11), the intermediate state \((\rho^*, v^*)\) between \(R_1\) and \(S_2\) satisfies
\[
\left(\frac{c + v_+}{c - v_+}\right)\left(\frac{c + \sqrt{A\rho_+^{\frac{14-a}{2}}}}{c - \sqrt{A\rho_+^{\frac{14-a}{2}}}}\right)^\frac{2\sqrt{A}}{3+a} = \left(\frac{c + v_*}{c - v_*}\right)\left(\frac{c + \sqrt{A\rho_*^{\frac{14-a}{2}}}}{c - \sqrt{A\rho_*^{\frac{14-a}{2}}}}\right)^\frac{2\sqrt{A}}{3+a},
\]
and
\[
v_* - v_- = \frac{-\sqrt{\Theta(\rho_*, \rho_-)}}{1 - v_- \sqrt{\Theta(\rho_*, \rho_-)}}, \quad v_* < v_-,
\]
with \(\rho_- < \rho_* < \rho_+\), where
\[
\Theta(\rho_*, \rho_-) = \frac{A(-\rho_*^{-\alpha} + \rho_-^{-\alpha})(\rho_* - \rho_-)}{(-A\rho_*^{-\alpha} + \rho_- c^2)(-A\rho_-^{-\alpha} + \rho_* c^2)}.
\]
Similarly, we have
\[
\lim_{A \to 0} \lambda_2(\rho, v_*) = \lim_{A \to 0} \lambda_2(\rho_+, v_+) = \lim_{A \to 0} \sigma_2 = v_-.
\]
Thus as \(A \to 0\), the rarefaction wave \(R_2\) and shock wave \(S_1\) converge to a contact discontinuity with the propagation speed \(v_-\), which connects the constant states \((\rho_\pm, v_\pm)\), see Fig. 7.

![Fig. 7. Riemann solution when \((\rho_+, v_+)\) ∈ III(\(\rho_-, v_-\)) in \((t, x)\)-plane.](image)

From the above discussion, we can summarize our results as follows.

**Theorem 4.8.** In the case \(v_- = v_+\), as \(A \to 0\), the Riemann solution of (1.1) and (1.2) with initial data (2.1) converges to a contact discontinuity connecting the constant states \((\rho_\pm, v_\pm)\), which is just the Riemann solution of the zero-pressure relativistic Euler equations (3.1) with the same initial data.

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