Two spin measurements in exchange interaction quantum computers

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We propose and analyse a method for single shot measurement of the total spin of a two electron system in a coupled quantum dot or donor impurity structure, which can be used for readout in a quantum computer. The spin can be inferred by observing spin dependent fluctuations of charge between the two sites, via a nearby electrometer. Realistic experimental parameters indicate that the fidelity of the measurement can be larger than 0.9999 with existing or near-future technology.

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Semiconductor technology is rapidly reaching the level where quantum systems comprising of one or two electron spins can be confined and coherently manipulated within a single nanostructure \cite{1, 2}. These experiments are important from a fundamental point of view, and will also pave the way to new applications in quantum information processing (QIP) \cite{3, 4, 5}. A key requirement for these QIP schemes is the realization of a single shot readout technique, whereby the spin state of a one- or two-electron system can be determined in a single measurement run. Such measurements are important both for the ongoing experimental development of spin qubit systems (e.g. for characterizing qubit parameters, and studying the physics of decoherence processes) and, ultimately, for developing scalable QIP architectures.

Recently, elegant experiments have demonstrated single shot readout of single electron spins, via optical techniques in a defect center in diamond \cite{6}, and using an all-electrical technique involving spin to charge conversion in a quantum dot, together with charge detection with a nearby quantum point contact (QPC) \cite{7}. A number of other methods based on spin-to-charge conversion have also been proposed \cite{8, 9, 10, 11, 12, 13}. Of particular interest are readout schemes in donor impurity implementations of QIP \cite{10, 11, 12, 13, 14, 15}, in which an electric field is applied to induce spin-dependent polarization of a two-electron double-donor system. It is believed that these proposals may fail due to the short lifetime of the quasi-bound two electron system under the large $E$ field required to observe a significant polarization \cite{10, 11}. In this paper we discuss an alternative scheme, whereby the total spin of a two electron system can be inferred by observing spin dependent fluctuations of charge between two tunnel coupled quantum dots (CQD) \cite{22}, even in the absence of an external electric field. Our scheme therefore avoids the lifetime issue in donor impurity systems, and furthermore may be easily integrated into proposed QIP architectures, as it does not require magnetic tunnel barriers, tunnel coupling to electron reservoirs, or external R.F. fields. The scheme can be used for read out of a single spin qubit (together with an ancilla spin), read out of an encoded qubit \cite{11}, or, together with appropriate single spin operations, as a two-qubit readout device.

Spin dependent charge fluctuations can be observed by continuously measuring the CQD system with an electrometer adjacent to one of the dots (see Fig. 1). For the purposes of this work, we model the electrometer as a QPC, although it might equally be a single electron transistor. Note that the QPC has a dual role in this scheme: it acts both as a noise source which can induce inelastic transitions in the CQD system, and as a detector to observe these transitions.

In what follows, we first describe a model for the coupled CQD-QPC system, and subsequently derive master equations for both the unconditional and conditional dynamics of the CQD system. We use these results to simulate individual runs of the measurement scheme, and to characterize the detector output for the different measurement outcomes. We show that observing the detector output leads to a quantum measurement in the singlet-triplet basis: for a singlet state (total spin $S = 0$) one observes fluctuations in the output current, while for a triplet state ($S = 1$), these fluctuations are energetically suppressed, owing to the Pauli principle. Finally we determine the single-shot singlet-triplet measurement time and fidelity for realistic experimental parameters.

\textit{Model.} A number of authors have considered the problem of a continuously observed single charge in a coupled dot system \cite{12, 13, 14, 15, 16}. Here, we adopt the qua-
H = \hbar\Delta(t)\sum_{\sigma=\uparrow,\downarrow}(a_{i\sigma}^\dagger a_{i\sigma} + a_{i\sigma}^\dagger a_{i\sigma}) + uU \sum_{i=1,2} n_{i\uparrow} n_{i\downarrow}, \tag{1}

where \(a_{i\sigma}^{(t)}\) is the fermionic annihilation (creation) operator for an electron on site \(i\) with spin \(\sigma\), \(n_{i\sigma} \equiv a_{i\sigma}^\dagger a_{i\sigma}\), \(\Delta(t)\) is the tunnelling amplitude between sites, and \(U\) is the on-site Coulomb energy. We allow \(\Delta(t)\) to be time dependent, since this can lead to shorter measurement times (see below), although in what follows, we frequently suppress this time dependence for clarity. \(H\) is spanned by the four singly occupied states \(|\sigma_1\sigma_2\rangle\) (where \(\sigma_1 = \uparrow, \downarrow\) denotes the spin on dot \(i\)) and two doubly occupied states \(|d_i\rangle = a_{i\uparrow}^\dagger a_{i\downarrow}^\dagger|0\rangle\) \((i = 1, 2)\). The eigenstates of \(H\) are as follows. The singly occupied triplet states \(|\uparrow\downarrow\rangle\) \(|\downarrow\uparrow\rangle\) and \(|\uparrow\uparrow\rangle\) \(|\downarrow\downarrow\rangle\) form a degenerate subspace with eigenvalue 0. The remaining states (with total spin \(S = 0\)) are \(|s_0\rangle = 2^{-\frac{1}{2}}\cos(\theta/2)(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) + 2^{-\frac{1}{2}}\sin(\theta/2)(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle), |s_1\rangle = 2^{-\frac{1}{2}}|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle, \) \(|s_2\rangle = 2^{-\frac{1}{2}}|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\), where \(\theta = \tan^{-1}(4\Delta(U)/E)\). These states have eigenvalues \(-J, U + J, \) \(U\) and \(-J\), respectively, where \(J(t) = (\sqrt{U^2 + 16\Delta^2(t)} - U)/2 \approx 4\Delta^2(t)/U\) is the exchange splitting between the lowest energy singlet and triplet states. \(J(t)\) can be varied over many orders of magnitude by varying a surface gate voltage \([\underline{3}, 10]\).

The entire system (i.e. CQD coupled to a QPC) has a Hamiltonian \(H_{\text{tot}} = H + H_{\text{tun}} + H_{\text{heads}}\) where

\[ H_{\text{tun}} = \hbar \sum_{k,q,\sigma} (T_{kq} + \chi_{kq} n_{1}) a_{k\sigma}^\dagger a_{Rq\sigma} + \text{h.c.,} \tag{2} \]

\[ H_{\text{heads}} = \hbar \sum_{k,\sigma} \omega_{k\sigma} a_{k\sigma}^\dagger a_{k\sigma} + \hbar \sum_{q,\sigma} \omega_{Rq\sigma} a_{Rq\sigma}^\dagger a_{Rq\sigma}. \tag{3} \]

Here, \(a_{k\sigma} (a_{Rq\sigma}^\dagger)\) are fermionic annihilation operators for electrons in the \(k\)th \((q\)th) mode on the left \((right)\) side of the barrier with spin \(\sigma\) and angular frequency \(\omega_{k\sigma} (\omega_{Rq\sigma})\). Tunnelling between the left and right baths is described by the factor \(T_{kq} + \chi_{kq} n_{1}\), where \(n_{1} = n_{1\uparrow} + n_{1\downarrow}\) is the total occupancy of dot ‘1’. \(T_{kq} \approx T_{av}\) and \(\chi_{kq} \approx \chi_{av}\) are assumed to vary slowly over the energy range where tunnelling is allowed.

\textbf{Unconditional master equation.} We now derive an unconditional master equation (UME) for the evolution of the reduced density matrix of the CQD system, \(\rho(t)\). The UME is obtained by first transforming to an interaction picture in which the dynamics are governed by \(H_{I}(t) = U^\dagger(t)H_{\text{tun}}U(t)\), with \(U(t) = e^{-i[H^{+}H_{\text{heads}}]t}\). In the Born and Markov approximations \([17]\), the resulting dynamics for the CQD system is given by

\[ \dot{\rho}_{I} = \text{tr}_{\text{heads}}\left\{ -\frac{1}{\hbar^2} \int_{-\infty}^{t} dt' \{ H_{I}(t), [H_{I}(t'), \rho_{I}(t')\otimes\rho_{L}\otimes\rho_{R}]\}\right\}. \tag{4} \]

Here, \(\rho_{I}(t) = U^\dagger(t)\rho(t)U(t)\), and \(\rho_{L}\) and \(\rho_{R}\) are the lead density matrices, given by Fermi-Dirac distributions with chemical potentials \(\mu_{L}\) and \(\mu_{R}\). The interaction picture tunnelling Hamiltonian may be written explicitly as \(H_{I}(t) = \sum_{k,q,\sigma}[T_{kq} + \chi_{kq} n_{1}(t)]e^{i(\omega_{k\sigma} - \omega_{Rq\sigma})t}a_{k\sigma}^\dagger a_{Rq\sigma} + \text{h.c.,}\) where the time dependence of \(n_{1}(t)\) is given by \(n_{1}(t) = \frac{1}{2} - e^{-i(J+U)t} \sin(\theta/2)|s_{0}\rangle<s_{2}| - e^{iJt}\cos(\theta/2)|s_{1}\rangle<s_{2}| + \text{h.c..}\) Note that Eq. \(4\) corresponds exactly to the first non-vanishing term in a perturbative expansion of \(\rho_{I}(t)\) in terms of \(H_{I}(t)\) \([17]\).

We now make two further controlled approximations. First, we make a rotating wave approximation, setting very rapidly oscillating factors \(e^{i\alpha_{\sigma}t} \to 0\), where \(\alpha_{\sigma} = (\mu_{L} - \mu_{R})/\hbar\). This is valid for \(\alpha_{\sigma} \gg \nu^{2}V\), where \(\nu = \sqrt{4\pi g_{L} g_{R} R_{av}}\) is a dimensionless coupling constant, \(g_{i}\) is the density of states in lead \(i\), and \(V = (\mu_{L} - \mu_{R})/\hbar\). Second, we neglect small terms of order \(\nu^{2}\sqrt{4V}\), which is valid when \(J \ll V\). At low temperatures \(k_{B} T \ll \hbar V\), and taking \(V \geq J + U\), the UME is

\[ \dot{\rho}(t) = -\frac{i}{\hbar}[H, \rho(t)] + \sum_{\sigma=\uparrow,\downarrow} D_{\sigma}[c_{\sigma}]\rho(t) = \mathcal{L}\rho(t), \tag{5} \]

where \(D_{\sigma}[c] = \mathcal{J}[\sigma]\rho - \mathcal{A}[\sigma][c]\rho, \mathcal{J}[\sigma][c]\rho = c_{\sigma}\rho_{\sigma}, \mathcal{A}[\sigma][c]\rho = \langle c \mid c \rangle\rho\). The Lindblad operators are given by

\[ c_{1} = \sqrt{2V} - (J + U) \sin\frac{\theta}{2}|s_{2}\rangle<s_{0}|, \tag{6} \]

\[ c_{2} = \sqrt{2V} + (J + U) \sin\frac{\theta}{2}|s_{0}\rangle<s_{2}|, \tag{7} \]

\[ c_{3} = \sqrt{2V} \left[ \mathcal{T} + \nu - \nu \cos\frac{\theta}{2}|s_{1}\rangle<s_{2}| + \text{h.c.}\right]. \tag{8} \]

where \(\mathcal{T} = \sqrt{4\pi g_{L} g_{R} R_{av}}\). The \(c_{i}\)’s can be associated with different types of tunnelling process in the QPC. \(c_{1,2}\) correspond to inelastic transitions, in which electrons tunnelling through the QPC exchange energy with the CQD system, and are accompanied by transitions between the low-energy, singly occupied state, and the high energy, doubly occupied state. \(c_{3}\) corresponds to a quasi-elastic transition, in which electrons tunnel through the QPC without changing energy.

\textbf{Conditional dynamics.} Individual measurement runs can be simulated using a conditional master equation (CME), which describes the evolution of the CQD system conditioned by the observed detector output, and also permits calculation of the current power spectrum. To derive the CME, we use an explicit model of the measurement process in terms of projective measurements of the number of electrons that have tunnelled through the...
V = 5 which means that these equations are valid for timescales Eqs. (5) and (9), we have used a course graining in time, |↓↑⟩⟨↓↑| coherent oscillations between the singly occupied states is not a significant restriction. This behaviour is reflected in the QPC output [Fig. 2(c)]: when both electrons are localized on dot ‘1’ (‘2’), the current is smaller (larger) than when the system is in a singly occupied state. This is reminiscent of a random telegraph switching (RTS) signal. At the first jump, the system becomes localized in the singlet subspace [Fig. 2(b)]. This is because the doubly occupied state |d1⟩⟨d1| is a spin-singlet state. Doubly occupied triplet states are energetically forbidden, and therefore when a transition into the doubly occupied subspace is observed, one can infer that the total spin of the system is a spin-singlet. In Fig. 3 no jump into the doubly occupied subspace is observed, and the detector output current remains constant (up to shot noise) This leads to a gradual increase in the observer’s confidence that the QCD system is in a spin-triplet state [Fig. 3(b)]. In both cases, the final state of the system is localized in either the singlet or triplet subspace, and a strong measurement of the total spin of the system in the singlet-triplet basis is obtained.

Detector power spectra. The possible detector outputs can be characterized by the power spectrum of the current, I(t), through the QPC. The power spectrum is given by $S(\omega) = 2 \int_0^\infty dr G(\tau)e^{-i\omega \tau}$, where $G(\tau) = E[I(t+\tau)I(t)] - E[I(t+\tau)]E[I(t)]$, and $E[\ldots]$ denotes the classical expectation. Following Refs. 13, 14, 15, we have $G(\tau) = e^2[\sum_n \mathcal{J}[c_n]e^{i\omega \tau} \mathcal{J}[c_n]_{\rho_{\infty}}^2 + \sum_n \mathcal{J}[c_n]_{\rho_{\infty}}^2] + \int_0^\infty dr G(\tau)e^{-i\omega \tau}$, where $\rho_{\infty}$ is a steady state solution of the UME, Spectra corresponding to the different measurement outcomes can be found by evaluating $G(\tau)$ for different $\rho_{\infty}$, corresponding to steady states localized in the singlet or triplet subspaces.

For a singlet state outcome, the steady state is $\rho_{S,\infty} = [(V + U + J)|s_0⟩⟨s_0| + (V - U - J)|s_1⟩⟨s_1| + (V - U - J)|s_2⟩⟨s_2|]/(3V - U - J)$. In the parameter regime of interest (\(\omega, J, \nu^2V \ll J + U \leq V\)), the power spectrum is well approximated by

$$S_S(\omega) = 2e^2 \mathcal{I}^2, 2J, \nu^2V \left(\frac{V - U - J}{\omega^2 - J^2} + \frac{J}{\omega^2(\nu^2V)^2}\right),$$

where $\mathcal{I} = e^2(T + \nu)^2V$ is the average current through
the detector, and \( \delta t = \varepsilon^2 (2T \nu) V \). In the limit \( J \ll \nu^2 V \), Eq. (10) corresponds to a RTS process with switching rate \( t_{\text{RTS}}^{-1} = J^2/(4\nu^2 V) \) which is the rate at which the system jumps between the states \( |d_1\rangle \) and \( |d_2\rangle \). For a triplet state outcome, the steady state is an arbitrary state in the (singly occupied) triplet subspace. The power spectrum only contains a shot noise component, and is given by \( S_T(\omega) = 2\nu I \). Power spectra for both singlet and triplet outcomes are illustrated in Fig. 4.

**Measurement time and fidelity.** The total time required to perform a single run of the singlet-triplet measurement scheme, \( t_{\text{meas}} \), has two components, \( t_{\text{meas}} = t_{\text{rel}} + t_{\text{det}} \), where \( t_{\text{rel}} \) is the time taken for a singlet state to relax into the doubly occupied subspace, and \( t_{\text{det}} = t_{\text{RTS}} \approx 4\nu^2 V/J^2 \) is the time required to determine the presence or absence of the RTS signal. By analysing Eq. (8), we find \( t_{\text{rel}}^{-1} \approx \nu^2 J(V - U)/U \). \( t_{\text{meas}} \) can be minimized by varying \( J(t) \) over the course of the measurement, such that different values of \( J(t) \) are obtained during the ‘relaxation’ and ‘detection’ phases of the measurement. In order to minimize the \( t_{\text{rel}} \), \( J(t) \) should be as large as possible, but to obtain a strong RTS signal, we require \( J_{\text{det}} \lesssim \nu^2 V \). We now evaluate \( t_{\text{rel}} \) and \( t_{\text{det}} \) for two possible implementations [4, 5]. For P donor qubits in Si [4], we take \( U = 43.8 \text{ meV, } V = 45.4 \text{ meV} \), \( \nu = 6 \times 10^{-5} \) (corresponding to a change in conductance of 5 \( \% \) for each electron added to dot ‘1’, assuming \( T = 0.3 \). \( J_{\text{rel}} = 1.0 \text{ meV and } J_{\text{det}} = 0.8 \nu^2 V = 2.2 \mu \text{eV} \)), which gives \( t_{\text{rel}} = 300 \text{ ns and } t_{\text{det}} = 1.51 \text{ ns} \). For quantum dot qubits in GaAs [5], we take \( U = 1 \text{ meV, } V = 2 \text{ meV, } \nu = 6 \times 10^{-5} \), \( J_{\text{rel}} = 0.1 \text{ meV and } J_{\text{det}} = 0.8 \nu^2 V = 0.11 \mu \text{eV} \), giving \( t_{\text{rel}} = 110 \text{ ns and } t_{\text{det}} = 34.4 \text{ ns} \).

The fidelity of the measurement is approximately \( F \sim 1 - \frac{t_{\text{meas}}}{t_{\text{mix}}} \), where \( t_{\text{mix}} \) is the mixing time for unwanted transitions between the singlet and triplet subspaces. Owing to the form of the detector-dot interaction in Eq. (8), the detector back-action does not induce such mixing transitions, and therefore the dominant contribution to \( t_{\text{mix}} \) is due to interactions with the environment. Although a lower bound of 200 \( \mu \text{s} \) for the singlet-triplet relaxation time in a single quantum dot has been measured [1], presently no data exists for the two-electron-two dot case. Therefore we take \( t_{\text{mix}} \sim \min\{T_1, T_2\} \), where \( T_1 \) and \( T_2 \) are the single spin relaxation and dephasing times, respectively (both processes can lead to transitions between the singlet and triplet subspaces). For P donors in Si, \( T_1 \) can be hours [18], and \( T_2 = 50 \text{ ms} \) was recently measured [19], giving \( F \sim 1 - 5 \times 10^{-6} \). For GaAs qubits, \( T_1 = 0.8 \text{ ms} \) has been measured [20], and it is believed that \( T_2 \approx T_1 \) [20], thus \( F \sim 1 - 10^{-5} \). These results are particularly promising for the implementation of single shot singlet-triplet measurements with the fidelity required for fault tolerant quantum computation [21].

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[22] Note that our scheme is applicable to both quantum dot or donor impurity based implementations, although hereafter we will refer to the electron sites as ‘dots’.

[23] In order to prevent direct ionization of the donors by the detector noise, we require $V \leq E_{\text{bind}}$, where $E_{\text{bind}} = 45.5$ meV is the single electron binding energy for phosphorous donors in silicon.