Giant Magnon on Deformed $AdS_3 \times S^3$

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Abstract: We study giant magnon solutions for strings moving on a deformed $AdS_3 \times S^3$ background. We impose a conformal gauge on the Polyakov action and proceed with solving the Virasoro constraints. The expressions of the conserved charge $J$ and the energy of a single magnon excitation are then computed. Then we determine the dispersion relation of a giant magnon in the infinite $J$ limit configuration and we find that for $\kappa = 0$ it reduces to celebrated Hofman-Maldacena dispersion relation.

Keywords: AdS/CFT correspondence, Bosonic Strings.
1. Introduction

String theory in suitable space-time backgrounds can have a holographic description in terms of gauge field theories. Type IIB String theory in $AdS_5 \times S^5$ background has been conjectured to be dual to N=4 super Yang-Mills theory in 4-dimensions [1]. The conjectured duality has passed through various non-trivial tests in the past by analyzing the spectrum of quantum string states on $AdS_5 \times S^5$ background and the spectrum of the anomalous dimensions of the N=4 gauge theory operators in the planar limit. Especially in the semi-classical approximation, the theory becomes integrable on both sides of the duality$^1$. In the plane wave background, string theory is integrable and its quantization is simple. All string states are generated by creation oscillators that can be applied to the vacuum state to construct all the modes. For each of these string states, there exist particular dual trace operators in the gauge theory [32, 14]. This is one of the AdS/CFT surprising aspects.

It is also well known that we can construct new integrable models from the given integrable model by applying T-duality transformations [2, 3, 4, 5]. On the other hand, a new class of one parameter integrable deformation of $AdS_5 \times S^5$ supercoset model was proposed in [6], following [7, 8, 9]. This analysis was then extended in [10] where the coordinate form of the bosonic part of the string action was determined and the corresponding string metric and the NS-NS two form were found. The perturbative world-sheet scattering matrix of bosonic particles of the model was also determined there and it was shown that it agrees with the large string tension limit of the q-deformed S-matrix. Finally, the target space interpretation of a given deformation was found recently in [11]$^2$. Further, the deformation of $AdS_3 \times S^3$ and $AdS_2 \times S^2$ were introduced here and it was shown that the corresponding metrics are direct sums of deformed $AdS_n$ and $S^n$ metrics and are given by truncations of the corresponding parts of the deformed ten dimensional metric. Clearly, the integrability of the given models follows from the integrability of the ten dimensional model.

$^1$For reviews of this problems from different point of views, see [32, 33, 34, 35, 36, 37, 38].

$^2$For recent interesting work, see [13].
Many attempts have been undertaken to find out the exact spectrum of string theory in Anti-de-Sitter background. Though our knowledge of the string spectrum in curved backgrounds is still limited, it has been observed that certain sectors of the theory are more tractable. One of them is the sector in which the states carry large angular momentum $J$ or large R-charge in the CFT point of view \cite{14}. In this region, semi-classical string energies yield information on the quantum spectrum of the string \cite{13}. The operator/string correspondence then allows to associate dual long trace operators in the gauge theory side to each string state. The energy eigenvalue $E$ of these string states with respect to time in global coordinates is conjectured to be equal to the scaling dimension $\Delta$ of the dual gauge theory operators. This belief builds on a remarkable proposal \cite{16} relating the Hamiltonian of a Heisenberg’s spin chain system with that of the dilatation operator in N=4 supersymmetric Yang-Mills theory. Said another way, we have uncovered a very promising interplay between string theory, gauge theory and the spin-chain system. This could indeed pave the way for understanding new aspects of this field of research.

One of the most interesting properties of these spin chain systems is the so called magnon excitation. This was initiated in the work of Hofman and Maldacena \cite{17} who showed that these magnon states correspond to the specific configuration of semi-classical string states on $R \times S^2$ \cite{15}. In particular, the giant magnon solution corresponds to operators where one of the $SO(6)$ charge, $J$, is taken to infinity, keeping the $E - J$ fixed\footnote{The Hofman-Maldacena limit: $J \rightarrow \infty$, $\lambda = \text{fixed}$, $p = \text{fixed}$, $E - J = \text{fixed}$}. These excitations satisfy a dispersion relation of the type (in the large ’t Hooft limit ($\lambda$))

$$E - J = \frac{\sqrt{\lambda}}{\pi} \left| \sin \frac{p}{2} \right|,$$  \hspace{1cm} (1.1)

where $p$ is the magnon momentum. Hence after, a lot of work has been devoted to study and generalize to various other magnon states with two and three non vanishing momentum and so on \cite{18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30}.

As was stressed in \cite{11}, it is crucial to find the gauge theory dual to string theory in the deformed geometry. One of the modest steps would be to find the magnon-like dispersion relation for strings moving in a given geometry and this is exactly the goal of this paper. More explicitly, we analyze the motion of the Polyakov string in the deformed geometry $AdS_3 \times S^3$ whose dynamics have a similar form as the dynamics of magnon-like string on $R \times S^2$ \cite{17}. We analyze the equations of motion that follow from the given action and the corresponding Virasoro constraints. We consider infinite $J$ magnon solution when both $E$ and $J$ diverge. Nevertheless, we construct a finite difference out of these two divergent quantities where this result has the form of the magnon-like dispersion relation. The latter has a complicated form which, however, reduces to the standard relation (1.1) in the limit $\kappa \rightarrow 0$. This fact certainly justifies our lacking of the corresponding dual gauge theory interpretation.

This paper is organized as follows. In section (3), we introduce the 3d truncated metric along with the Polyakov string action in this $AdS_3 \times S^3$ background. We derive the corresponding equations of motion and solve them with the specific magnon-like ansatz. In section (3), we consider the limit of infinite angular momentum. A detailed manipulation is

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then presented leading to the giant magnon dispersion relation. Finally, in the conclusion [4], we outline our results and suggest possible extension of present work.

2. Polyakov string in deformed $AdS_3 \times S^3$ background

In this section, we study the Polyakov string in the deformed $AdS_3 \times S^3$ background that was found recently in [11]. The line element has the following form

$$ds^2 = ds^2_{A_3} + ds^2_{S^3},$$

(2.1)

where

$$ds^2_{A_3} = -h(\rho)dt^2 + f(\rho)d\rho^2 + \rho^2 d\psi^2,$$

$$ds^2_{S^3} = \tilde{h}(r)d\varphi^2 + \tilde{f}(r)dr^2 + r^2 d\phi^2,$$

(2.2)

where

$$h = \frac{1 + \rho^2}{1 - \kappa^2 \rho^2}, \quad f = \frac{1}{(1 + \rho^2)(1 - \kappa^2 \rho^2)},$$

$$\tilde{h} = \frac{1 - r^2}{1 + \kappa^2 r^2}, \quad \tilde{f} = \frac{1}{(1 - r^2)(1 + \kappa^2 r^2)}.$$  

(2.3)

where the NS-NS two form vanishes.

We are interested in the geometry (2.2) for finding out the giant magnon solution that is analogue of the giant magnon solution in $AdS_5 \times S^5$ found in [17]. Our starting point is the Polyakov form of the string action in the deformed background

$$S = -\frac{1}{2} \hat{T} \int_{-\pi}^{\pi} d\sigma d\tau \sqrt{-\gamma} \gamma^{\alpha\beta} g_{MN} \partial_\alpha x^M \partial_\beta x^N,$$

(2.4)

where the effective string tension [11] has the form

$$\hat{T} = \frac{\sqrt{\lambda}}{2\pi} \sqrt{1 + \kappa^2},$$

(2.5)

where the undeformed string tension in $AdS$ background has the famous form $T_0 = \frac{\sqrt{\lambda}}{2\pi}$. Further, $\gamma^{\alpha\beta}$ is the world-sheet metric and the modes $x^M, M = 0, \ldots, 9$ parameterize the embedding of the string in the general background.

The variation of the action (2.4) with respect to $x^M$ implies the following equations of motion

$$-\frac{1}{2} \sqrt{-\gamma} \gamma^{\alpha\beta} \partial_K g_{MN} \partial_\alpha x^M \partial_\beta x^N + \partial_\alpha [\sqrt{-\gamma} \gamma^{\alpha\beta} g_{KM} \partial_\beta x^M] = 0.$$  

(2.6)
Further, the variation of the action with respect to the metric implies the constraints

\[ T_{\alpha\beta} = -\frac{2}{\sqrt{-g}} \delta S = \hat{T} \left[ g_{MN} \partial_\alpha x^M \partial_\beta x^N - \frac{1}{2} \gamma_{\alpha\beta\gamma} \gamma^\delta \partial_\gamma x^M \partial_\delta x^N g_{MN} \right] = 0 . \]

(2.7)

Now, looking at the form of the background (2.2), we observe that the action (2.4) is invariant under the following transformations of fields

\[ t'(\tau, \sigma) = t(\sigma, \tau) + \epsilon_t, \]
\[ \psi'(\tau, \sigma) = \psi(\tau, \sigma) + \epsilon_\psi, \]
\[ \phi'(\tau, \sigma) = \phi(\tau, \sigma) + \epsilon_\phi, \]
\[ \varphi'(\tau, \sigma) = \varphi(\tau, \sigma) + \epsilon_\varphi, \]

(2.8)

where \( \epsilon_t, \epsilon_\psi, \epsilon_\phi \) and \( \epsilon_\varphi \) are constants. Then, it is a simple task to determine the corresponding conserved charges

\[ P_t = \hat{T} \int_{-\pi}^{\pi} d\sigma \sqrt{-\gamma} \gamma^{\tau\alpha} g_{tt} \partial_\alpha t , \]
\[ P_\phi = \hat{T} \int_{-\pi}^{\pi} d\sigma \sqrt{-\gamma} \gamma^{\tau\alpha} g_{\phi\phi} \partial_\alpha \phi , \]
\[ P_\psi = \hat{T} \int_{-\pi}^{\pi} d\sigma \sqrt{-\gamma} \gamma^{\tau\alpha} g_{\psi\psi} \partial_\alpha \psi , \]
\[ P_\varphi = \hat{T} \int_{-\pi}^{\pi} d\sigma \sqrt{-\gamma} \gamma^{\tau\alpha} g_{\varphi\varphi} \partial_\alpha \varphi . \]

(2.9)

Note that \( P_t \) is related to the energy as \( P_t = -E \).

Now we are going to find the solution of the equations of motion given above which could be interpreted as a giant magnon. We closely follow the very nice analysis presented in [30].

Let us now consider the following ansatz for obtaining the giant magnon solution

\[ t = -\frac{E}{2\pi\hat{T}} \tau , \quad r = r(\sigma, \tau) , \quad \rho = \rho(\tau) , \]
\[ \varphi = \varphi(\sigma, \tau) , \quad \phi = \text{const.} , \psi = \Psi \tau . \]

(2.10)

We will solve the equations of motions in the conformal gauge where \( \gamma^{\tau \tau} = -1 , \gamma^{\sigma \sigma} = \)
Note that in the given gauge, the constraints (2.7) take the form
\[
T_{\sigma\sigma} = \hat{T}_2 (g_{MN} \partial_{\sigma} x^M \partial_{\sigma} x^N + g_{MN} \partial_{\tau} x^M \partial_{\tau} x^N),
\]
\[
T_{\tau\tau} = \hat{T}_2 (g_{MN} \partial_{\sigma} x^M \partial_{\sigma} x^N + g_{MN} \partial_{\tau} x^M \partial_{\tau} x^N),
\]
\[
T_{\tau\sigma} = \hat{T} g_{MN} \partial_{\sigma} x^M \partial_{\tau} x^N.
\]
(2.11)

Let us now consider the equation of motion for \(X^0 \equiv t\)
\[
\partial_{\alpha} [\sqrt{-g} \eta^{\alpha\beta} g_{tt} \partial_{\beta} t] = 0
\]
(2.12)
that for the ansatz (2.10) implies that \(\rho\) should be constant \(\rho_c\). On the other hand, the equation of motion with respect to \(\rho\) has the form
\[
\frac{\rho(1 + \kappa^2)}{(1 - \kappa^2 \rho^2)^2} - \rho \Psi^2 = 0
\]
(2.13)
that has solution \(\rho_c = 0\) and hence we find \(g_{tt}(\rho_c = 0) = 1\). As a check, note that when we insert (2.10) into the first expression in (2.9) and use \(g_{tt} = 1\) we find
\[
P_t = -\hat{T} \int_{-\pi}^{\pi} d\sigma \frac{E}{2\pi \hat{T}} = -E.
\]
(2.14)

Now, we return to the equation of motion for \(\phi\)
\[
\partial_{\alpha} [g_{\phi\phi} \eta^{\alpha\beta} \partial_{\beta} \phi] = 0
\]
(2.15)
and we see that it is solved form \(\phi = \text{const}\). Finally, the equation of motion for \(\psi\) is obeyed with (2.10) since \(\rho_c = 0\). Then we also have \(P_\psi = 0\).

Now, we proceed to the analysis of the dynamics of \(r\) and \(\varphi\). It is convenient to solve the constraints (2.11) that can be interpreted as the first integrals of the theory instead of solving the equations of motion for \(r\) and \(\varphi\) respectively. Inserting the ansatz (2.10) to the constraints (2.11), we obtain two equations
\[
T_{\sigma\sigma} = T_{\tau\tau} = \hat{T} \left[ -\frac{1}{(2\pi T)^2} E^2 + \tilde{f}(r)(\partial_{\sigma} r)^2 + \tilde{f}(r)(\partial_{\tau} r)^2 + \tilde{h}(r)(\partial_{\sigma} \varphi)^2 + \tilde{h}(r)(\partial_{\tau} \varphi)^2 \right] = 0,
\]
\[
T_{\tau\sigma} = \hat{T} \left[ \tilde{f}(r) \partial_{\tau} \partial_{\sigma} r + \tilde{h}(r) \partial_{\tau} \varphi \partial_{\sigma} \varphi \right] = 0.
\]
(2.16)

Following [17, 30], we search for a solution with the boundary conditions
\[
r(\pi, \tau) - r(-\pi, \tau) = 0, \quad \Delta \varphi = \varphi(\pi, \tau) - \varphi(-\pi, \tau) = p,
\]
(2.17)
where \( p \) is the momentum of the ‘single magnon’ excitation. Since the field \( \varphi \) does not satisfy periodic boundary conditions, this solution corresponds to the open string.

As the next step we introduce the light-cone coordinate \( \varphi \) through the formula

\[
\varphi = \tilde{\varphi} + \omega \tau
\]

and presume that \( \tilde{\varphi} \) and \( r \) depend on \( \tau \) and \( \sigma \) in the following way

\[
r = r(\sigma - \nu \omega \tau) , \quad \tilde{\varphi} = \tilde{\varphi}(\sigma - \nu \omega \tau) .
\]

Inserting these forms into (2.16), we obtain

\[
- \nu \omega \bar{f}(r) r'^2 + \tilde{h}(r)(\omega - \nu \omega \tilde{\varphi}') \tilde{\varphi}' = 0 ,
\]

\[
- C + \bar{f}(r) r'^2 (1 + v^2 \omega^2) + \tilde{h}(r)(\omega - \nu \omega \tilde{\varphi}')^2 + \tilde{h}(r) r'^2 = 0 ,
\]

where

\[
C = \frac{1}{(2\pi \hat{T})^2} E^2 ,
\]

and where now \( r' , \tilde{\varphi}' \) mean derivatives with respect to \( \xi \equiv \sigma - \nu \omega \tau \). If we now combine these equations we obtain

\[
\tilde{\varphi}' = \frac{v}{\tilde{h}(r)} (C - \tilde{h}(r) \omega^2) ,
\]

\[
r'^2 = \frac{\omega^2}{\bar{f}(r) \tilde{h}(r)(1 + v^2 \omega^2)^2} \left( \frac{C}{\omega^2} - \tilde{h}(r) \right) \left( \tilde{h}(r) - v^2 C \right) .
\]

As a check, note that for \( \kappa = 0 \) we have \( \tilde{h}(r) = 1 - r^2 , \bar{f}(r) = \frac{1}{1 - \tau} \). Then, we introduce \( \theta \) as \( r = \cos \theta \) and we obtain

\[
\tilde{\varphi}' = \frac{v}{(1 - v^2 \omega^2) \sin^2 \theta} (C - \omega^2 \sin^2 \theta)
\]

and also

\[
\theta'^2 = \frac{1}{\sin^2 \theta} r'^2 = \frac{\omega^2}{(1 - v^2 \omega^2) \sin^2 \theta} \left( \frac{C}{\omega^2} - \sin^2 \theta \right) (\sin^2 \theta - v^2 C)
\]

that has the same form as the equation that determines giant magnon profile that was derived in [30]. We also see, in agreement with [30], that for this solution the derivative \( r' \) is finite everywhere and vanishes at both points \( r_{\text{min}} \) and \( r_{\text{max}} \) defined as

\[
r_{\text{min}} = \sqrt{\frac{1 - C}{1 + \frac{C}{\omega^2} \kappa^2}} , \quad r_{\text{max}} = \sqrt{\frac{1 - v^2 C}{1 + v^2 \kappa^2 C}} ,
\]

These conditions follow from the requirement that \( \frac{C}{\omega^2} > \tilde{h}(r) \) and \( \tilde{h}(r) - v^2 C > 0 \) in order to have \( r'^2 > 0 \). The opposite condition \( \frac{C}{\omega^2} < \tilde{h}(r) \) and \( \tilde{h}(r) - v^2 C < 0 \) cannot be obeyed when we presume \( v < \frac{1}{\omega} \).
where we presume \( v < \frac{1}{2} \).

As the next step, we insert (2.22) into definition of the conserved charge \( P_\varphi \) and we obtain

\[
P_\varphi = J = 2\hat{T} \int_{r_{\text{min}}}^{r_{\text{max}}} \frac{\omega}{|r'|} \tilde{h}(r)(v\varphi' - 1) = -2\hat{T} \frac{\omega}{|\omega|} \left( \frac{1 + v^2 C^2}{(\frac{C}{\omega} \kappa^2 + 1)} \right) \times
\]

\[
\times \int_{r_{\text{min}}}^{r_{\text{max}}} \frac{dr}{(1 + \kappa^2 r^2) \sqrt{(r^2 - r_{\text{min}}^2)(r_{\text{max}}^2 - r^2)}} (r_{\text{max}}^2 - r^2),
\]

(2.26)

Note that we also have

\[
2\pi = \int_{-\pi}^{\pi} d\sigma = 2 \int_{r_{\text{min}}}^{r_{\text{max}}} \frac{dr}{|r'|} = 2 \frac{(1 - v^2 \omega^2)}{\omega \sqrt{(1 + \frac{C}{\omega} \kappa^2)(1 + v^2 \kappa^2 C)}} \times
\]

\[
\times \int_{r_{\text{min}}}^{r_{\text{max}}} \frac{dr}{\sqrt{(r_{\text{max}}^2 - r^2)(r^2 - r_{\text{min}}^2)}},
\]

(2.27)

Finally we also find

\[
p = \Delta \varphi = 2 \int_{r_{\text{min}}}^{r_{\text{max}}} \frac{dr}{|r'|} \varphi' =
\]

\[
= 2v \frac{\omega^2}{|\omega|} \sqrt{\frac{1 + \frac{C}{\omega} \kappa^2}{1 + v^2 C^2}} \int_{r_{\text{min}}}^{r_{\text{max}}} \frac{dr}{1 - r^2 \sqrt{(r^2 - r_{\text{min}}^2)(r_{\text{max}}^2 - r^2)}}.
\]

(2.28)

In principle we could express these quantities in terms of elliptical integrals. However, in order to derive more transparent results, we restrict ourselves to the case of the infinite giant magnon, where \( J \to \infty \).

3. Infinite \( J \) Giant Magnon

In this section, we discuss the infinite \( J \) giant magnon. This solution corresponds to the situation when \( r_{\text{min}} \to 0 \). From (2.23), we find that it occurs when

\[
\omega^2 = C, \quad r_{\text{max}} = \sqrt{\frac{1 - v^2 \omega^2}{1 + v^2 \omega^2 \kappa^2}}.
\]

(3.1)

Let us calculate \( P_\varphi \) for \( r_{\text{min}} = 0 \)

\[
P_\varphi = 2\hat{T} \frac{\omega r_{\text{max}}}{|\omega|} \sqrt{\frac{1 + v^2 C^2}{1 + \frac{C}{\omega} \kappa^2}} \left[ \sqrt{1 + \kappa^2 r_{\text{max}}^2} \ln \frac{1 + \sqrt{1 + \kappa^2 r_{\text{max}}^2}}{1 - \sqrt{1 + \kappa^2 r_{\text{max}}^2}} + \right]
\]

\[
+ \ln \left[ \frac{1 + \sqrt{1 - (r/r_{\text{max}})^2}}{1 - \sqrt{1 - (r/r_{\text{max}})^2}} \right]_{r_{\text{max}}}
\]

(3.2)
and we see that the second expression diverges as anticipated. However, the same divergence occurs when we calculate the integral

$$2\pi = \int_{-\pi}^{\pi} d\sigma = 2 \int_{r_{\min}}^{r_{\max}} \frac{dr}{|r'|} = -2 \frac{(1 - v^2\omega^2)}{|\omega| \sqrt{(1 + \frac{C}{\omega^2}\kappa^2)(1 + v^2\omega^2)}} \ln \left[ \frac{1 + \sqrt{1 - (r/r_{\max})^2}}{1 - \sqrt{1 - (r/r_{\max})^2}} \right]_{r_{\min}}^{r_{\max}}$$

so that it is natural to regularize the divergence in $P_\varphi$ using (3.3) in order to find dispersion relation that would be analogue of (1.1). Explicitly, let us consider the following combination $2\pi K - P_\varphi$ where we choose the constant $K$ to make this difference finite. It is easy to see that this divergence is canceled out when $K$ is equal to

$$K = -\hat{T}\omega .$$

(3.4)

To proceed further, we choose $\omega = -\sqrt{C} = -\frac{E}{2\pi \hat{T}}$ in order to have positive charge $J$. Then, we finally find

$$K = \frac{E}{2\pi}$$

(3.5)

and hence we have the following dispersion relation

$$E - J = \frac{2\hat{T}}{\kappa} \ln \left[ \frac{1 + \frac{v^2C\kappa^2}{\omega^2}}{1 + \frac{C\kappa^2}{\omega^2}} \sqrt{1 + \kappa^2 r_{\max}^2} \ln \left( \frac{1 + \frac{v^2C\kappa^2}{\omega^2}}{1 + \frac{C\kappa^2}{\omega^2}} \right) \right].$$

(3.6)

Finally, we calculate the momentum defined in (2.28) for $r_{\min} = 0$

$$p = \frac{2v\omega^2}{|\omega|} \sqrt{(1 + \frac{C\kappa^2}{\omega^2})} \int_{0}^{r_{\max}} dr \frac{r}{(1 - r^2)\sqrt{r_{\max}^2 - r^2}} = 2\sin^{-1} r_{\max}$$

(3.7)

so that

$$r_{\max} = \sin \frac{p}{2} .$$

(3.8)

Note that using this relation we can express $v^2C$ as function of $p$

$$v^2C = \frac{1 - \sin^2 \frac{p}{2}}{1 + \kappa^2 \sin^2 \frac{p}{2}} .$$

(3.9)

Plugging this back into (3.6) we find the final form of the giant magnon dispersion relation

$$E - J = \frac{2\hat{T}}{\kappa} \ln \left[ \frac{1 + \frac{\kappa |\sin \frac{p}{2}|}{\sqrt{1 + \kappa^2 \sin^2 \frac{p}{2}}}}{1 - \frac{\kappa |\sin \frac{p}{2}|}{\sqrt{1 + \kappa^2 \sin^2 \frac{p}{2}}}} \right] = \frac{2\hat{T}}{\kappa} \tanh^{-1} \left( \frac{\kappa |\sin \frac{p}{2}|}{\sqrt{1 + \kappa^2 \sin^2 \frac{p}{2}}} \right) .$$

(3.10)

Obviously, we see that it has a much more complicated form than in the case of $\kappa = 0$. At present it is not clear how this dispersion relation could be realized in the dual gauge
theory description at least in some limit. On the other hand, it is instructive to study the
limit \( \kappa \to 0 \) in (3.10). In fact, using

\[
\lim_{\kappa \to 0} \frac{1}{\kappa} \ln \frac{1 + \kappa |\sin \frac{p}{2}|}{\sqrt[2]{1 + \kappa^2 \sin^2 \frac{\pi}{2}}} = \left| \sin \frac{p}{2} \right| \quad (3.11)
\]

we find that (3.10) takes the form

\[
E - J = 2T(\kappa = 0) \sin \frac{p}{2} = \frac{\sqrt{\lambda}}{\pi} \left| \sin \frac{p}{2} \right| \quad (3.12)
\]

in exact agreement with [17]. Further, using the fact that \( \kappa \) is related to the deformations
parameters introduced in [10]

\[
\kappa = \frac{2\eta}{1 - \eta^2}, \quad q = e^{-\frac{\pi}{\nu}} \quad \nu = \frac{2\eta}{1 + \eta^2} = \frac{\kappa}{\sqrt{1 + \kappa^2}} \quad (3.13)
\]

we find that (3.10) can be written as

\[
E - J = \frac{2T_0}{\nu} \sinh^{-1} \left( \frac{\nu}{\sqrt{1 - \nu^2}} \left| \sin \frac{p}{2} \right| \right) \quad (3.14)
\]

and we see that the right side of the equation above precisely agree with the energy of the
magnon excitation calculated in [12]. This fact can be considered as further support of our
analysis.

4. Conclusion

In this paper, we have studied the giant magnon solution in deformed \( \text{AdS}_3 \times S^3 \) background
that was proposed recently in [11]. In the infinite \( J \) limit we were able to compute the
one magnon dispersion relation that has slightly more complicated form than the ordinary
dispersion relation in case of giant magnon in \( \text{AdS}_5 \times S^5 \). However, we have further argued
that this dispersion relation coincides with the latter in the limit \( \kappa \to 0 \).

The present analysis can be extended in various directions. In one event, one can ask
what role these results play on the gauge theory side and what interpretation they provide.
The deformed string metric (2.1) has a curvature singularity at \( \rho = \frac{1}{\kappa} \). For larger values,
the radial coordinate \( \rho \) becomes time-like that suggests that string is confined in the region
\( 0 \leq \rho \leq \frac{1}{\kappa} \). Then it is not completely clear how to proceed with our results on the dual
gauge side. In other words, it would be extremely interesting to understand in detail the
holographic duality between string theory in the deformed geometry and the dual gauge
theory. Clearly, this is an interesting topic for further work. There are more other lines
of investigation worth pursuing. For example, it would be very interesting and challenging
to include the 2-form \( B \) field into our calculations when we consider the giant magnon
solution on the deformed \( \text{AdS}_5 \times S^5 \) background. It would be also interesting to analyze
corresponding spike solutions in given background. These problems are currently under
investigations.
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