REMARKS ON TOPOLOGICAL ALGEBRAS

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To Victor Ginzburg on his 50th birthday

The note complements “topological” aspects of the chiral algebras story from [BD]. In its first section (which has a whiff of [G] in it) we show that the basic chiral algebra format (chiral operations, etc.) has a precise analog in the setting of topological linear algebra. This provides, in particular, a natural explanation of the passage from chiral to topological algebras from [BD] 3.6. The second section is a brief discussion, in the spirit of [BD] 3.9, of topological algebras similar to rings of chiral differential operators. We also correct some errors from [BD].

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1. Topological tensor products and topological algebras.

1.1. The tensor products. For us, as in [BD] 3.6.1, “topological vector space” is a $k$-vector space equipped with a linear topology assumed (unless stated explicitly otherwise) to be complete and separated. The category of topological vector spaces is denoted by $\text{Top}$. This is an additive Karoubian $k$-category.

The category $\text{Top}$ is quasi-abelian in the sense of [S]. In particular, it is naturally an exact category: the admissible monomorphisms are closed embeddings, the admissible epimorphisms are open surjections.

Remark. If the topology of $B \in \text{Top}$ admits a countable base, then every short exact sequence $0 \to A \to B \to C \to 0$ is split exact, i.e., $B$ is isomorphic to $A \oplus C$.

Let $\{V_i\}_{i \in I}$ be a finite non-empty collection of topological vector spaces. Consider the tensor product $\otimes V_i$. This is an abstract vector space; it carries several natural linear topologies. Namely:

(a) The $\ast$ topology is formed by all subspaces $Q$ of $\otimes V_i$ which satisfy the following property: for every subset $J \subset I$ and vector $v \in \bigotimes_{i \in J} V_i$ there exist open subspaces $P_j \subset V_j$, $j \in J$, such that $Q \supset (\otimes P_j) \otimes v$. Let $\otimes^\ast V_i$ be the corresponding completion. Then for any topological space $F$ a continuous morphism $\otimes^\ast V_i \to F$ is the same as a continuous polylinear map $\times V_i \to F$.

(b) The $!$ topology has base formed by vector subspaces $\sum_{i \in I} P_i \otimes (\bigotimes_{i' \in I \setminus \{i\}} V_{i'})$ where $P_i \subset V_i$ are any open subspaces. The corresponding completion is denoted by $\otimes^! V_i$. Thus $\otimes^! V_i = \varinjlim \otimes (V_i/P_i)$.

Notice that [BD] (in particular, in [BD] 3.6.1) used the notation $\hat\otimes V_i$; the reason for the change of notation will become clear later.

(c) Suppose we have a linear order $\tau$ on $I$, i.e., an identification $\{1, \ldots, n\} \sim I$. It defines on $\otimes V_i$ the $\tau$ topology formed by all subspaces $Q$ which satisfy the following
property: for every $a \in \{1, \ldots, n\}$ and vector $v \in V_{\tau(a+1)} \otimes \ldots \otimes V_{\tau(n)}$ there exists an open subspace $P_a \subset V_{\tau(a)}$ such that $Q \supset V_{\tau(1)} \otimes \ldots \otimes V_{\tau(a-1)} \otimes P_a \otimes v$. The corresponding completion is denoted by $\vec{\otimes}^1 V_i = V_{\tau(1)} \vec{\otimes} \ldots \vec{\otimes} V_{\tau(n)}$.

We refer to morphisms $\otimes V_i \rightarrow F$ which are continuous with respect to the ! topology, i.e., morphisms $\otimes^1 V_i \rightarrow F$, as $\otimes^1$-continuous polylinear maps. Same for the other tensor products.\(^1\)

**Remarks.** (i) Let $U, V$ be topological vector spaces. Suppose first that $V$ is discrete. Then $U \otimes^* V = U \vec{\otimes} V$ is equal to $U \otimes V$ equipped with the ind-topology. Precisely, write $V = \lim V_\alpha$ where $V_\alpha$ runs the directed set of finite-dimensional subspaces of $V$; then $U \otimes V = \lim U \otimes V_\alpha$, and each $U \otimes V_\alpha$ carries an evident topology (as the product of finitely many copies of $U$), which defines the said inductive limit topology on $U \otimes V$.

If $V$ is arbitrary, then $U \vec{\otimes} V = \lim U \vec{\otimes}(V/P)$, the projective limit is taken along the set of all open subspaces $P \subset V$.

(ii) Suppose we have $A \in \text{Top}$ and an associative bilinear product $\cdot : A \otimes A \rightarrow A$. Then $\cdot$ is $\otimes^*$-continuous if and only if the product map $A \times A \rightarrow A$ is continuous; it is $\vec{\otimes}$-continuous if and only if it is continuous and the open left ideals form a base of the topology of $A$;\(^2\) and it is $\otimes^1$-continuous if and only if the open two-sided ideals form a base of the topology.

The tensor products $\otimes^*$ and $\otimes^1$ are commutative and associative; they define tensor structures on $\text{Top}$ which we denote by $\text{Top}^1$ and $\text{Top}^*$. The tensor product $\vec{\otimes}$ is associative but not commutative; it defines a monoidal structure on $\text{Top}$.

**Exercise.** The tensor products are exact.

The * topology on $\otimes V_i$ is stronger than each of the $\tau$ topologies, which in turn are stronger than the ! topology, so the identity map for $\otimes V_i$ gives rise to the natural continuous morphisms

\[(1.1.1) \quad \otimes^* V_i \rightarrow \vec{\otimes}^1 V_i \rightarrow \otimes^1 V_i.\]

**Lemma.** The * topology is equal to the supremum of the $\tau$ topologies for all linear orders $\tau$ on $I$. The ! topology is the infimum of the $\tau$ topologies. I.e., the arrows in (1.1.1) are, respectively, admissible mono- and epimorphism. \(\square\)

**Corollary.** For any pair $U, V$ of topological vector spaces the short sequence

\[(1.1.2) \quad 0 \rightarrow U \otimes^* V \rightarrow U \vec{\otimes} V \oplus V \vec{\otimes} U \rightarrow U \otimes^1 V \rightarrow 0,\]

where the left arrow comes from the diagonal map and the right one from the difference of the projections, is exact. \(\square\)

**1.2. Example.** Suppose $F, G$ are Tate vector spaces (see e.g. [BD] 2.7.7, [D] 3.1). Let $F^*$ be the dual Tate space to $F$, so $F^* \otimes G$ is the vector space of all continuous maps $F^* \rightarrow G$.

\(^1\)So $\otimes^*$-continuous polylinear maps are the same as continuous polylinear maps.

\(^2\)Or, equivalently, the open left ideals form a base of the topology and for every $r \in A$ either of endomorphisms $a \mapsto ar$ or $a \mapsto ar - ra$ of $A$ is continuous.
maps $F \to G$ with finite-dimensional image. Then $F^* \otimes^! G$ identifies naturally with the space $\text{Hom}(F, G)$ of all continuous linear maps $F \to G$, $F^* \otimes^* G$ with the space $\text{Hom}_d(F, G)$ of maps having open kernel, and $F^* \otimes^* G$ with $\text{Hom}_f(F, G) := \text{Hom}_*(F, G) \cap \text{Hom}_d(F, G)$. So the vector spaces $\text{Hom}(F, G)$, $\text{Hom}_c(F, G)$, $\text{Hom}_d(F, G)$, $\text{Hom}_f(F, G)$ carry natural topologies, and (1.1.2) yields a short exact sequence in $\text{Top}$

\[(1.2.1) \quad 0 \to \text{Hom}_f(F, G) \to \text{Hom}_c(F, G) \oplus \text{Hom}_d(F, G) \to \text{Hom}(F, G) \to 0.\]

Let us describe these topological vector spaces explicitly. Choose decompositions $F = F_c \oplus F_d, G = G_c \oplus G_d$ where $F_c, G_c$ are c-lattices, $F_d, G_d$ are d-lattices. Consider the corresponding decomposition of the Hom spaces into the sum of the four subspaces. One has:

- $\text{Hom}(F_c, G_d) = \text{Hom}_c(F_c, G_d) = \text{Hom}_d(F_c, G_d) = \text{Hom}_f(F_c, G_d)$: this is a discrete vector space;
- $\text{Hom}(F_d, G_c) = \text{Hom}_c(F_d, G_c) = \text{Hom}_d(F_d, G_c) = \text{Hom}_f(F_d, G_c)$: this is a compact vector space;
- $\text{Hom}(F_c, G_c) = \text{Hom}_c(F_c, G_c)$ and $\text{Hom}_d(F_c, G_c) = \text{Hom}_f(F_c, G_c)$; the latter topological vector space equals $G_c \otimes F^*_c$ equipped with the ind-topology;
- $\text{Hom}(F_d, G_d) = \text{Hom}_d(F_d, G_d)$ and $\text{Hom}_c(F_d, G_d) = \text{Hom}_f(F_d, G_d)$; the latter topological vector space equals $F^*_d \otimes G_d$ equipped with the ind-topology.

A correction to [BD] 2.7.7. In loc. cit., certain topologies on the Hom spaces were considered. The topology on $\text{Hom}(F, G)$ coincides with the above one, but those on $\text{Hom}_c, \text{Hom}_d$ and $\text{Hom}_f$ are stronger than the above topologies, and they are less natural. Precisely, these topologies differ at terms $\text{Hom}_d(F_c, G_c) = \text{Hom}_f(F_c, G_c)$ and $\text{Hom}_c(F_d, G_d) = \text{Hom}_f(F_d, G_d)$ which are considered in [BD] 2.7.7 as discrete vector spaces. Exact sequence (1.2) coincides with [BD] (2.7.7.1) (up to signs); it is strongly compatible with either of the topologies. These topologies played an auxiliary role: they were used to define (in [BD] 2.7.8) the topology on the Tate extension $\mathfrak{gl}(F)^\flat$ as the quotient topology for the canonical surjective map $\text{End}_c(F) \oplus \text{End}_d(F) \to \mathfrak{gl}(F)^\flat$. Replacing them by the present topologies does not change the quotient topology on $\mathfrak{gl}(F)^\flat$. We suggest to discard the topologies on $\text{Hom}_c, \text{Hom}_d, \text{Hom}_f$ from [BD] 2.7.7, and replace them by those defined above.

1.3. The chiral operations. As in [BD] 1.1.3, the $!$ and $*$ tensor structures on $\text{Top}$ can be seen as pseudo-tensor structures with the spaces of operations $P^!(\{V_i\}, F) := \text{Hom}(\otimes^! V_i, F)$ and $P^*(\{V_i\}, F) := \text{Hom}(\otimes^* V_i, F)$.

The tensor product $\otimes$ is non-commutative, so the definition of the corresponding pseudo-tensor structure requires an appropriate induction:

A digression: Let $\Sigma$ be the linear orders operad (see [BD] 1.1.4). Suppose we have a $k$-category $T$ equipped with an associative tensor product (i.e., a monoidal structure) $\otimes$. For a finite non-empty collection of objects $\{V_i\}_{i \in I}$ and a linear order $\tau \in \Sigma_I$, i.e., $\tau : \{1, \ldots, n\} \to I$, we set $\otimes^\tau V_i := V_{\tau(1)} \otimes \cdots \otimes V_{\tau(n)}$. We define the induced pseudo-tensor structure on $T$ by formula $P^\otimes_I(\{V_i\}, F) := \bigoplus_{\tau \in \Sigma_I} P^!(\{V_i\}, F)^\tau$.
where $P_I^\circ({\{V_i}\}, F)^\tau := \text{Hom}(\otimes^\tau V_i, F)$. The composition of operations is defined in the evident way so that the $\Sigma$-grading is compatible with the composition.

If $\otimes$ is actually commutative, then $P_I^\circ({\{V_i}\}, F) = \text{Hom}(\otimes V_i, F) \otimes \text{Ass}_I$ where $\text{Ass} = k[\Sigma]$ is the associative algebras operad (see [BD] 1.1.7). The construction is functorial with respect to natural morphisms between $\otimes$’s.

Applying this to $\text{Top}$ and $\vec{\otimes}$, we get the chiral operations $P^{\text{ch}} := P^{\vec{\otimes}}$, those from the $\tau$-component of $P^{\text{ch}}$ are called chiral $\tau$ operations. They define the chiral pseudo-tensor structure $\text{Top}^{\text{ch}}$ on $\text{Top}$. This pseudo-tensor structure is representable (see [BD] 1.1.3) with the pseudo-tensor product $\otimes^{ch} V_i = \bigoplus_{\tau \in \Sigma} \otimes^\tau V_i$.

We see that (1.1.1) gives rise to natural transformations

\begin{equation}
(1.3.1) \quad P_I^!(\{V_i\}, F) \otimes \text{Ass}_I \to P_I^{ch}(\{V_i\}, F) \to P_I^*(\{V_i\}, F) \otimes \text{Ass}_I
\end{equation}

compatible with the composition of operations, i.e., the identity functor of $\text{Top}$ lifts naturally to pseudo-tensor functors $\text{Top}^! \otimes \text{Ass} \to \text{Top}^{\text{ch}} \to \text{Top}^* \otimes \text{Ass}$.

Remarks. (i) The arrows in (1.3.1) are injective. If all the $V_i$ are discrete, then they are isomorphisms.

(ii) The endofunctor of $\text{Top}$ which assigns to a topological vector space $V$ same $V$ considered as a discrete space lifts in the obvious manner to a faithful pseudo-tensor endofunctor for all the above pseudo-tensor structures.

Composing (1.3.1) from the left and right with the standard embedding $\text{Lie} \hookrightarrow \text{Ass}$ and projection $\text{Ass} \to \text{Com}$, we get natural morphisms (cf. [BD] 3.2.1, 3.2.2)

\begin{equation}
(1.3.2) \quad P_I^!(\{V_i\}, F) \otimes \text{Lie}_I \to P_I^{ch}(\{V_i\}, F) \to P_I^*(\{V_i\}, F).
\end{equation}

By (1.1.2), in case of two arguments we get an exact sequence

\begin{equation}
(1.3.3) \quad 0 \to P_I^!(\{U, V\}, F) \otimes \text{Lie}_2 \to P_I^{ch}(\{U, V\}, F) \to P_I^*(\{U, V\}, F)
\end{equation}

1.4. Chiral algebras in the topological setting. Let $A$ be a topological vector space. A $\text{Lie}^{ch}$ algebra (or non-unital chiral algebra) structure on $A$ is a Lie bracket $\mu_A : A \otimes^{ch} A \to A$ for the chiral pseudo-tensor structure; we call such $\mu_A$ a chiral product on $A$.

Consider the map $P_2^{ch}({\{A, A\}, A}) \to \text{Hom}(A \vec{\otimes} A, A)$ which assigns to a binary chiral operation its first component.

**Proposition.** This map establishes a bijection between the set of chiral products on $A$ and the set of associative products $A \vec{\otimes} A \to A$.

**Proof.** Our map yields a bijection between the set of skew-symmetric binary chiral operations and $\text{Hom}(A \vec{\otimes} A, A)$. It remains to check that a skew-symmetric

\footnote{Here $\text{Lie}$ is the Lie algebras operad and $\text{Com}$ is the commutative algebras operad, i.e, the unit $k$-operad.}
chiral operation $\mu$ satisfies the Jacobi identity if and only if the corresponding product $A \otimes A \rightarrow A$ is associative.

By Remark (ii) from 1.3 it suffices to consider the case of discrete $A$, so now $A$ is a plain vector space. Let $P(A)$ be the operad of polylinear endooperations on $A$, i.e., $P(A)_I := \text{Hom}(A^I, A)$. By Remark (i) from 1.3 the operad $P^{ch}(A)$ of chiral endooperations of $A$ equals $P(A) \otimes \text{Ass}$.

Let $\cdot_{as} \in \text{Ass}_2$ be the standard associative binary operation. We want to prove that a binary operation $\cdot_A \in P(A)_2$ is associative if and only if the binary operation $\cdot_A \otimes \cdot_{as} - \cdot_A^t \otimes \cdot_{as}^t$ in $P(A) \otimes \text{Ass}$ is a Lie bracket (here $^t$ is the transposition of arguments). Indeed, if $\cdot_A$ is associative, then such is $\cdot_A \otimes \cdot_{as}$, hence the commutator $\cdot_A \otimes \cdot_{as} - \cdot_A^t \otimes \cdot_{as}^t$ is a Lie bracket. The converse statement is a simple computation left to the reader. □

**Remark.** Here is a reformulation of the proposition in a non-commutative algebraic geometry style. We forget about the topologies. Let $A$ be any vector space. Suppose we have an associative algebra structure on $A$. Then for any test associative algebra $R$ the tensor product $A \otimes R$ is naturally an associative algebra, hence a Lie algebra. In other words, let $\phi_A$ be the functor $R \mapsto A \otimes R$ from the category of associative algebras to that of vector spaces; then an associative algebra structure on $A$ yields a Lie algebra structure on $\phi_A$ (i.e., a lifting of $\phi_A$ to the category of Lie algebras). The claim is that this establishes a bijection between the set of associative algebra structures on $A$ and that of Lie algebra structures on $\phi_A$.

For $A \in \text{Top}$ a topological associative algebra structure on $A$ is a continuous associative bilinear product $A \times A \rightarrow A$, i.e., an associative product $A \otimes^* A \rightarrow A$. One can sum up the above proposition as follows (the equivalence of (ii), (iii), and (iv) is Remark (ii) in 1.1):

**Claim.** For $A \in \text{Top}$ the following structures on $A$ are equivalent:

(i) A non-unital chiral algebra structure $\mu_A$;

(ii) An associative product $\cdot_A : A \otimes A \rightarrow A$;

(iii) A topological associative algebra structure such that the open left ideals form a base of the topology of $A$.

(iv) An associative algebra structure such that the open left ideals form a base of the topology of $A$ and the corresponding Lie bracket is continuous. □

A correction to [BD] 3.6.1: In loc. cit. this was recklessly called “topological associative algebra” structure; we suggest to rescind this confusing terminology.

A linguistic comment. The term “chiral” refers to the breaking of symmetry between the right and left movers in physics and is rather awkward in the “purely holomorphic” setting of [BD] (where only one type of movers is present). In the topological setting it looks more suitable for we consider associative products whose left-right asymmetry is enforced by the topology (though now it has to do rather with the time ordering of physicists).

One often constructs topological chiral algebras using the next corollary:
**Corollary.** Let $T$ be an associative algebra equipped with a (not necessary complete) linear topology. Suppose that

- the open left ideals form a base of the topology;
- for every $r$ from a set of associative algebra generators the endomorphisms $t \mapsto [r, t] := rt - tr$ of $T$ are continuous.

Then the completion of $T$ is a chiral algebra.

**Proof.** The endomorphisms $t \mapsto at$ of $T$ are continuous for any $a \in T$ by the first property, so the second property amounts to continuity of the endomorphisms $t \mapsto tr$ for $r$ from our system of generators, hence they are continuous for every $r \in T$. Thus the completion of $T$ carries the structure from (ii) of the claim. □

Suppose we have a topological Lie algebra $L$, i.e., $L$ is a topological vector space equipped with a continuous Lie bracket (i.e., a Lie bracket with respect to $\otimes^s$). We say that $L$ is a topological Lie$^*$ algebra if it satisfies the following technical condition: the open Lie subalgebras form a base of the topology of $L$. Notice that this condition holds automatically if $L$ is a Tate vector space.$^4$

Now the second arrow in (1.3.2) transforms any chiral product $\mu_A$ into a continuous Lie bracket $[\ ]_A$. Equivalently, $[\ ]_A$ is the commutator for the associative product $\cdot_A$. By (iii) of the claim, $[\ ]_A$ is a Lie$^*$ algebra structure on $A$.

Our $\mu_A$ is said to be commutative if $[\ ]_A = 0$. By by Lemma from 1.1 (and the proposition above) a commutative chiral product amounts to a commutative$^*$ algebra structure, i.e., a commutative and associative product $\cdot_A : A \otimes^s A \to A$.

We say that a Lie$^{ch}$ algebra structure is unital if such is the corresponding associative algebra structure. Such algebras are referred to simply as topological chiral algebras.

For a topological chiral algebra $A$ a discrete $A$-module is a left unital $A$-module $M$ (we consider $A$ as a mere associative algebra now) such that the action $A \times M \to M$ is continuous (we consider $M$ as a discrete vector space). Equivalently, this is a discrete unital left $A$-module with respect to $\otimes$ monoidal structure. Denote by $A_{\text{mod}}$ the category of discrete $A$-modules.

Let $\phi : A_{\text{mod}} \to \text{Vec}$ be the forgetful functor (which assigns to a discrete $A$-module its underlying vector space). Then $A$ recovers from $(A_{\text{mod}}, \phi)$:

**Lemma.** $A$ equals the topological associative algebra of endomorphisms of $\phi$. □

1.5. We denote by $\text{Ass}(\text{Top}^*)$ the category of topological associative unital algebras, by $\mathcal{CA}$(\text{Top}) the category of topological chiral algebras, and by $\text{Ass}(\text{Top}^!)$ that of associative unital algebras with respect to $\otimes^!$.

As we have seen in 1.4, the above structures on $A \in \text{Top}$ are the same as an associative product $A \otimes A \to A$ that satisfies stronger and stronger continuity

$^4$Proof: if $P \subset L$ is a c-lattice, then its normalizer $Q$ is an open Lie subalgebra of $L$, hence $P \cap Q$ is also an open Lie subalgebra.
conditions. Thus we have fully faithful embeddings

(1.5.1) \[ \text{Ass}(\text{Top}^1) \hookrightarrow \text{CA}(\text{Top}) \hookrightarrow \text{Ass}(\text{Top}^*). \]

These embeddings admit left adjoint functors

(1.5.2) \[ \text{Ass}(\text{Top}^1) \leftarrow \text{CA}(\text{Top}) \leftarrow \text{Ass}(\text{Top}^*). \]

Namely, for \( A \in \text{Ass}(\text{Top}^*) \) the corresponding chiral algebra \( A^{ch} \) is the completion of \( A \) with respect to the topology whose base is formed by open left ideals, and for \( B \in \text{CA}(\text{Top}) \) the corresponding associative\(^1\) algebra is the completion of \( B \) with respect to the topology whose base is formed by open two-sided ideals (see Remark (ii) in 1.1).

**Remark.** For \( B \in \text{Ass}(\text{Top}^*) \) the category of left unital discrete \( A \)-modules coincides with \( B^{ch}\text{mod}\(^5\).

The forgetful functor from either of the categories of topological algebras above to \( \text{Top} \) which sends a topological algebra to the underlying topological vector space also admits left adjoint. For \( V \in \text{Top} \) the corresponding algebras are denoted by \( T^*V, T^{ch}V, \) and \( T^!V \). These are completions of the “abstract” tensor algebra \( TV := \bigoplus_{n \geq 0} V^{\otimes n} \) with respect to the topology whose base is formed by all subspaces of type \( f^{-1}(U) \), where \( f : TV \rightarrow A \) is a morphism of associative algebras such that \( f|_V : V \rightarrow A \) is continuous, \( A \) is an algebra of our class, and \( U \subset A \) is an open subspace.

**Remarks.** (i) Consider \( T := \bigoplus_{n \geq 0} V^{\otimes^n} \). This is a topological vector space (with the inductive limit topology) and an associative algebra, but the product need not be continuous. If \( V \) is a Tate space, then the product is continuous if \( V \) is either discrete or compact, and not continuous otherwise.\(^5\)

(ii) One obtains \( T^{ch}V \) and \( T^!V \) by applying to \( T^*V \) the functors from (1.5.2). Certainly, \( T^!V = \varprojlim P \) where \( P \) runs the set of an open subspaces of \( V \) and \( T(V/P) \) is its plain tensor algebra (which is a discrete vector space).

A correction to [BD] 3.6.1: In loc. cit. there is a wrong claim that \( T^{ch}V \) equals the direct sum \( \bar{T}V := k \oplus V \oplus V^{\otimes 2} \oplus \ldots \) (equipped with the direct limit topology).

**1.6. A topological \( \mathcal{D} \)-module setting.** Let \( X \) be our curve and \( x \in X \) be a point. We will consider pairs \( (M, \Xi_M) \) where \( M \) is a \( \mathcal{D} \)-modules on \( X \) and \( \Xi_M \) is a topology on \( M \) at \( x \) (see [BD] 2.1.13); such pairs form a \( k \)-category \( \mathcal{M}(X, \text{Top}_x) \). Denote by \( \hat{h}_x(M, \Xi_M) \) the completion of the de Rham cohomology stalk \( h(M)_x \) with respect to our topology (see loc. cit.); we get a functor \( \hat{h}_x : \mathcal{M}(X, \text{Top}_x) \rightarrow \text{Top} \).

We will extend \( \hat{h}_x \) to a pseudo-tensor functor with respect to the \(!, \ast\) and chiral polylinear structures. In order to do this, one needs to explain which operations between \( \mathcal{D} \)-modules are continuous with respect to our topologies.

(a) \( \ast \) operations (cf. [BD] 2.2.20). Let \( (M_i, \Xi_{M_i}), i \in I, \) be a finite non-empty collection of objects of \( \mathcal{M}(X, \text{Top}_x) \).

\(^5\)To check the latter assertion, use the next fact (applied to \( P := V^{\otimes 2}, T_i := V^{\otimes i} \)): if \( P, T_0, T_1, \ldots \in \text{Top} \) are non-discrete and \( P \) is not Tate, then \( \oplus (P \otimes^* T_i) \neq P \otimes^* (\oplus T_i) \).
Lemma. The $\mathcal{D}_{X^i}$-module $\boxtimes M_i$ carries a natural topology $\boxtimes^* \Xi_{M_i}$ at $\Delta^{(I)}(x) = (x, \ldots, x) \in X^I$ such that the corresponding completion of the de Rham cohomology stalk $h(\boxtimes M_i)_{(x, \ldots, x)}$ equals $\otimes^* \hat{h}_x(M_i, \Xi_{M_i})$.

Proof. We want to assign in a natural way to every discrete quotient $T$ of $\otimes^* \hat{h}_x(M_i, \Xi_{M_i})$ a certain quotient of $\boxtimes M_i$ equal to $i_{(x, \ldots, x)}^* T$. Since $\hat{h}_x$ commutes with inductive limits, we can assume that each $M_i$ is a finitely generated $\mathcal{D}_{X^i}$-module. Then $\hat{h}_x(M_i, \Xi_{M_i})$ are all compact, so we can assume that $T = \otimes T_i$ where $T_i$ are discrete (finite-dimensional) quotients of $\hat{h}_x(M_i, \Xi_{M_i})$. So $i_{x*} T_i$ is a quotient of $M_i$, hence $i_{(x, \ldots, x)}^* T = \boxtimes i_{x*} T_i$ is a quotient of $\boxtimes M_i$, and we are done. \]

Let $(N, \Xi_N)$ be another object of $\mathcal{M}(X, \text{Top}_x)$. A * operation $\varphi \in P_i^1(\{M_i\}, N)$ is said to be continuous with respect to our topologies if $\varphi : \boxtimes M_i \to \Delta^{(I)} N$ is continuous with respect to the topologies $\boxtimes^* \Xi_{M_i}$ and $\Delta^{(I)} \Xi_N$. The composition of continuous operations is continuous, so they form a pseudo-tensor structure $\mathcal{M}(X, \text{Top}_x)^*$ on $\mathcal{M}(X, \text{Top}_x)$. By construction, $\hat{h}_x$ lifts to a pseudo-tensor functor

$$\hat{h}_x : \mathcal{M}(X, \text{Top}_x)^* \to \text{Top}^*.$$  

(b) Chiral operations. From now on we will consider a full subcategory $\mathcal{M}(U_x, \text{Top}_x)$ of $\mathcal{M}(X, \text{Top}_x)$ formed by those pairs $(M, \Xi_M)$ that $M = j_{x*} j^*_x M$ where $j_x$ is the embedding $U_x := X \setminus \{x\} \hookrightarrow X$. Suppose that our $(M_i, \Xi_{M_i})$ lie in this subcategory.

Lemma. The $\mathcal{D}_{X^i}$-module $\boxtimes^! M_i$ carries a natural topology $\boxtimes^! \Xi_{M_i}$ at $x$ such that the corresponding completion of $h(\boxtimes^! M_i)_{x}$ is equal to $\boxtimes^! \hat{h}_x(M_i, \Xi_{M_i})$.

Proof. $\boxtimes^* \Xi_{M_i}$ is the topology with base formed by submodules $\boxtimes^! P_i$ of $\boxtimes^! M_i$ where $P_i \subset M_i$ are open submodules for $\Xi_{M_i}$.

Set $\boxtimes^! (M_i, \Xi_{M_i}) := (\boxtimes^! M_i, \boxtimes^! \Xi_{M_i})$. This tensor product makes $\mathcal{M}(U_x, \text{Top}_x)$ a tensor category which we denote by $\mathcal{M}(U_x, \text{Top}_x)^!$. Its unit object is $j_{x*} \omega_{U_x}$ equipped with the topology formed by the open submodule $\omega_X \subset j_{x*} \omega_{U_x}$. The functor $\hat{h}_x$ lifts naturally to a tensor functor

$$\hat{h}_x : \mathcal{M}(U_x, \text{Top}_x)^! \to \text{Top}^!.$$  

(c) Chiral operations. Let $(M_i, \Xi_{M_i})$ be, as above, some objects of $\mathcal{M}(U_x, \text{Top}_x)$, and $j^{(I)} : U^{(I)} \hookrightarrow X^I$ be the complement to the diagonal divisor.

Lemma. The $\mathcal{D}_{X^i}$-module $j^{(I)}_* (j^{(I)})^* \boxtimes M_i$ carries a natural topology $\boxtimes^\chi \Xi_{M_i}$ at $(x, \ldots, x) \in X^I$ such that the corresponding completion of the de Rham cohomology stalk $h(j^{(I)}_* (j^{(I)})^* \boxtimes M_i)_{(x, \ldots, x)}$ is equal to $\boxtimes^\chi \hat{h}_x(M_i, \Xi_{M_i})$.

Proof. We proceed by induction by $|I|$. Let us choose for each $i \in I$ a $\Xi_{M_i}$-open submodule $P_i \subset M_i = j_{x}^* j_x^* M_i$; set $T_i := h(M_i/P_i)_{x}$, so $M_i/P_i = i_{x*} T_i$. Set $I_i := I \setminus \{i\}$. The sequence $0 \to \boxtimes P_i \to \boxtimes M_i \to \bigoplus_{i \in I} (\boxtimes M_{i'}) \boxtimes i_{x*} T_i \to 0$ is short exact over $U^{(I)}$. So we have a short exact sequence

$$0 \to j^{(I)}_* (j^{(I)})^* \boxtimes P_i \to j^{(I)}_* (j^{(I)})^* \boxtimes M_i \to \bigoplus_{i \in I} (j^{(I)}_* (j^{(I)})^* \boxtimes M_{i'}) \boxtimes i_{x*} T_i \to 0.$$
By the induction assumption, for each $i \in I$ the $\mathcal{D}_{X^i}$-module $j_*^{(I)} j_!^{(I)} * \bigotimes_{i' \in I_i} M_{i'}$ carries the topology $\bigotimes_{i' \in I_i} h_{x_i}(M_{i'}, \Xi_{M_{i'}})$ at $(x, \ldots, x)$ with the completed de Rham cohomology stalk $\bigotimes_{i' \in I_i} \hat{h}_{x_i}(M_{i'}, \Xi_{M_{i'}})$. Let us equip the $\mathcal{D}_{X^i}$-module $(j_*^{(I)} j_!^{(I)} * \bigotimes_{i' \in I_i} M_{i'}) \bigotimes_{i' \in I_i} i_{x_i} T_i = i^{X_{x_i} \times X_{x_i}} (j_*^{(I)} j_!^{(I)} * \bigotimes_{i' \in I_i} M_{i'}) \otimes T_i$ with the ind-topology (cf. Remark in 1.1), so the completion of its de Rham cohomology stalk equals $\bigotimes_{i' \in I_i} \hat{h}_{x_i}(M_{i'}, \Xi_{M_{i'}}) \hat{\otimes} T_i$. By (1.6.3), the product of these topologies can be seen as a topology on the quotient module $j_*^{(I)} j_!^{(I)} * \bigotimes_{i' \in I_i} M_{i'}/j_*^{(I)} j_!^{(I)} * \bigotimes_{i' \in I_i} P_i$ which we denote by $\Xi_{(P)}$.

Now our topology $\bigotimes_{i' \in I_i} \Xi_{M_{i'}}$ is formed by all $\mathcal{D}_{X^i}$-submodules $P \subset j_*^{(I)} j_!^{(I)} * \bigotimes_{i' \in I_i} M_{i'}$ such that $P$ contains the submodule $j_*^{(I)} j_!^{(I)} * \bigotimes_{i' \in I_i} P_i$ for some choice of $\Xi_{M_{i'}}$-open $P_i \subset M_i$ and the image of $P$ in $j_*^{(I)} j_!^{(I)} * \bigotimes_{i' \in I_i} M_{i'}/j_*^{(I)} j_!^{(I)} * \bigotimes_{i' \in I_i} P_i$ is $\Xi_{(P)}$-open. $\square$

Let $(N, \Xi_N)$ be another object of $\mathcal{M}(U_x, \text{Top}_x)$ and $\varphi \in P_i^{ch}(\{M_i\}, N)$ be a chiral operation. We say that $\varphi$ is continuous with respect to our topologies if $\varphi : j_*^{(I)} j_!^{(I)} * \bigotimes_{i' \in I_i} M_{i'} \to \Delta_*^{(I)} N$ is continuous with respect to the topologies $\bigotimes_{i' \in I_i} \Xi_{M_{i'}}$ and $\Delta_*^{(I)} \Xi_N$. The composition of continuous operations is continuous, so they form a pseudo-tensor structure $\mathcal{M}(X, \text{Top}_x)^{ch}$ on $\mathcal{M}(U_x, \text{Top}_x)$. By construction, $\hat{h}_x$ lifts to a pseudo-tensor functor

$$\tag{1.6.4} \hat{h}_x : \mathcal{M}(U_x, \text{Top}_x)^{ch} \to \text{Top}^{ch}.$$  

1.7. Example. Let $A$ be a chiral algebra on $U_x$. As in [BD] 3.6.4, we denote by $\Xi_x^{as}$ the topology on $j_{x*} A$ at $x$ whose base is formed by all chiral subalgebras of $j_{x*} A$ that coincide with $A$ on $U_x$. Set $A_x^{as} := \hat{h}_x(j_{x*} A, \Xi_x^{as})$.

Lemma. The chiral product $\mu_A$ is $\Xi_x^{as}$-continuous. $\square$

Therefore, according to (1.6.4), $A_x^{as}$ is a topological chiral algebra. We leave it to the reader to check that the associiative product on $A_x^{as}$ coincides with the one defined in [BD] 3.6.6.

2. Topological cdo.

2.1. The classical limit: from chiral to coisson algebras. Let $A$ be any topological vector space. Below a filtration on $A$ always means an increasing filtration $A_0 \subset A_1 \subset \ldots$ by closed vector subspaces of $A$ such that $A_\infty := \bigcup A_i$ is dense in $A$. We have a graded topological vector space $\text{gr} A$ with components $\text{gr}_i A := A_i/A_{i-1}$.

The vector space $\bigoplus \text{gr}_i A$ carries a natural topology whose base is formed by subspaces $\bigoplus(P \cap A_i)/(P \cap A_{i-1})$ where $P$ is an open subspace of $A$; we denote by $\text{gr} A$ the completion.

Suppose now that $A$ is a topological chiral algebra and $A, \ldots, \text{as}$ above is a ring filtration, i.e., $A_i \cdot A_j \subset A_{i+j}$, $1 \in A_0$; we call such $A$ a chiral algebra filtration. Then $\text{gr} A$ is naturally a topological chiral algebra.
Our $A_\infty$ is a subring of $A$. Consider a topology on $A_\infty$ formed by all left ideals $I$ in $A_\infty$ such that $I \cap A_i$ is open in $A_i$ for every $i$. For any $r \in A_\infty$ the right multiplication endomorphism $a \mapsto ar$ of $A_\infty$ is continuous, so (by Corollary in 1.4) the completion $A_\infty$ of $A_\infty$ with respect to this topology is a topological chiral algebra. The embedding $A_\infty \subset A$ extends by continuity to a morphism of topological chiral algebras $A_\infty \to A$.

**Definition.** A chiral algebra filtration $A$ is **admissible** if $A_\infty \sim A$, i.e., a closed left ideal $I \subset A$ is open if (and only if) each $I \cap A_n$ is open in $A_n$.

**Example.** Consider the algebra $k[[t]]$ equipped with the usual topology. Its chiral algebra filtration $k[[t]]_n := k + kt + \ldots + kt^n$ is not admissible.

Below **topological coisson algebra** means a topological vector space $R$ equipped with a Poisson algebra structure such that the Lie bracket is $(\otimes^*)$-continuous and the product is $\otimes$-continuous. We also demand $R$ to be a topological Lie$^*$ algebra (see 1.4); equivalently, this means that open ideals of the commutative algebra $R$ which are Lie subalgebras form a base of the topology.

Let $A$ be any topological chiral algebra. A chiral algebra filtration on $A$ is said to be **commutative** if $\text{gr}A$ is a commutative algebra. Then $\text{gr}^A$ is a commutative topological chiral algebra, i.e., a commutative$^1$ algebra. The usual Poisson bracket on $\text{gr}A$ extends by continuity to a continuous Lie bracket on $\text{gr}^A$, which makes $\text{gr}A$ a topological coisson algebra.

### 2.2. Topological Lie$^*$ algebroids

Let $R$ be any (unital) topological commutative$^1$ algebra. We denote by $\text{Rmod}^1$ the category of unital $R$-modules in the tensor category $\text{Top}^1$. This is a tensor $k$-category with tensor product $\otimes^1_R$, and an exact category: a short sequence of $\text{Rmod}^1$-modules is exact if it is exact as a sequence in $\text{Top}^1$ (see 1.1). For $M \in \text{Rmod}^1$ and an open ideal $I \subset R$ we write $M_{R/I} := (R/I) \otimes^1_R M = M/\overline{IM}$ (here $\overline{IM}$ is the closure of $IM$); this is a topological $R/I$-module.

For any $M \in \text{Rmod}^1$ we denote by $\text{Sym}^1_RM$ the universal topological commutative$^1$ $R$-algebra generated by $M$, i.e., $\text{Sym}^1_RM = \varprojlim \text{Sym}_{R/J}(M/P)$ where the projective limit is taken with respect all pairs $(J, P)$ where $J \subset R$ is an open ideal and $P \subset M$ an open $R$-submodule such that $J\mathcal{L} \subset P$. Our $\text{Sym}^1_RM$ carries an evident chiral algebra filtration $(\text{Sym}^1_RM)_a = R \oplus M \oplus \ldots \oplus \text{Sym}^1_RM$; here $\text{Sym}^1_RM$ is the symmetric power of $M$ in the tensor category $\text{Rmod}^1$. The filtration is admissible (due to the universality property of $\text{Sym}^1_RM$), and $\text{gr}^1_RM = \text{Sym}^1_RM$ is the universal graded topological commutative$^1$ $R$-algebra generated by $M$ in degree 1.

A topological Lie$^*$ $R$-algebroid is a topological vector space $\mathcal{L}$ equipped with a Lie $R$-algebroid structure such that the Lie bracket is $(\otimes^*)$-continuous and the $R$-action on $\mathcal{L}$ is $\otimes$-continuous. We demand that $\mathcal{L}$ is a topological Lie$^*$ algebra in the sense of 1.4, or, equivalently, that open Lie $R$-subalgebroids of $\mathcal{L}$ form a base of the topology of $\mathcal{L}$. If $\mathcal{L}$ is a topological Lie$^*$ $R$-algebroid, then $\text{Sym}^1_R\mathcal{L}$ is naturally a topological coisson algebra.

**Examples.** (i) Let $L$ be a topological Lie$^*$ algebra that acts continuously on $R$. Then $L_R := R \otimes^1 L$ is naturally a topological Lie$^*$ $R$-algebroid.

(ii) Let $\Omega_R := \varinjlim \Omega_{R/I}$ be the topological $R$-module of continuous differentials
of $R$. Suppose that $R$ is reasonable, formally smooth, and the topology of $R$ admits a countable base; then $Ω_R$ is a Tate $R$-module (see [D] Th. 6.2(iii)). Let $Θ_R$ be the dual Tate $R$-module. Explicitly, $Θ_R = \varprojlim Θ_{R,R/I}$ where for an open ideal $I \subset R$ the topological $R/I$-module $Θ_{R,R/I} = R/I \otimes_R Θ_R$ consists of all continuous derivations $\theta : R \to R/I$; the topology of $Θ_{R,R/I}$ has base formed by $R/I$-submodules that consist of $\theta$ that kill given open ideal $J \subset I$ and finite subset of $R/J$. Our $Θ_R$ is naturally a topological Lie* $R$-algebroid called the tangent algebroid of $R$.

2.3. PBW filtrations. Let $A$ be a topological chiral algebra equipped with a commutative chiral algebra filtration $A$. Set $R := A_0$, $L := \text{gr}_1 A = A_1/A_0$. Then $R$ is a topological commutative$^1$ algebra and $L$ is a topological Lie* $R$-algebroid. By the universality property we have an evident morphism of commutative$^1$ $R$-algebras, which is automatically a morphism of topological coisson algebras,

\[
(2.3.1) \quad \text{Sym}_R^n L \to \widehat{\text{gr}} A,
\]

called the Poincaré-Birkhoff-Witt map.

Lemma. Suppose that the filtration $A$ is admissible and each map $\text{Sym}_R^n L \to \text{gr}_n A$ is an admissible epimorphism (i.e., an open surjection). Then for every open subspace $P \subset A_1$ the closure $\overline{AP} = \overline{A_\infty P}$ of the left ideal generated by $P$ is open. Such ideals form a base of the topology of $A$.

Proof. The second assertion is immediate, once we check the first one. By admissibility, it suffices to check that $\overline{A_{n-1}P}$ is open in $A_n$ for each $n$. By induction, we know that $\overline{A_{n-1}P} \cap A_{n-1}$ is open in $A_{n-1}$, i.e., we have an open $V \subset A_n$ such that $V \cap A_{n-1} \subset \overline{A_{n-1}P} \cap A_{n-1}$. The image $\text{gr}_n \overline{A_{n-1}P}$ of $\overline{A_{n-1}P}$ in $\text{gr}_n A$ is open; indeed, if an open subspace $T$ of $P$ tends to zero, then $A_{n-1}T$ tends to zero, hence $\text{gr}_n \overline{A_{n-1}P}$ contains the image of $\varprojlim (\text{gr}_{n-1} A \cdot \text{gr}_1 P)/(\text{gr}_{n-1} A \cdot \text{gr}_1 T)$ which is an open subspace in $\text{gr}_n A$. Choose $T$ as above such that $A_{n-1}T \subset V$; replacing $V$ by its intersection with the preimage of the open subspace $\text{gr}_n \overline{A_{n-1}T}$ of $\text{gr}_n A$, we can assume that $V = (V \cap A_{n-1}) + (A_{n-1}T)$, hence $V \subset \overline{A_{n-1}P}$, q.e.d.

Definition. We say that $A$ satisfies the weak PBW property if $\text{Sym}_R^n L \sim \text{gr}_n A$ for each $n$; the strong PBW property means that the filtration is admissible and (2.3.1) is an isomorphism. Such a filtration is referred to as weak, resp. strong, PBW filtration.

Remarks. (i) The strong PBW property asserts the existence of large discrete quotients of $A$. Indeed, it amounts to the following two conditions: (a) The filtration is generated by $A_1$, i.e., each $A_n$, $n \geq 1$, equals the closure of $(A_1)^n$; (b) Let $P \subset A_1$ be an open subspace such that $\text{gr}_0 P = P \cap R$ is an ideal in $R$ and $\text{gr}_1 P$ is an $R$-submodule of $\text{gr}_1 A$. Then one can find an open ideal $I \subset A$ such that $I \cap A_1 \subset P$ and the projection $\text{gr}_1 A/\text{gr}_1 I \to \text{gr}_1 A/\text{gr}_1 P$ lifts to a morphism of algebras $\text{gr}_A/\text{gr}_I \to \text{Sym}_R^n/(\text{gr}_1 A/\text{gr}_1 P)$.

(ii) I do not know if every admissible weak PBW filtration automatically satisfies the strong PBW property.

2.4. The chiral envelope of a Lie* algebra. The forgetful functor from the category of topological chiral algebras to that of Lie* algebras $(A, μ_A) \mapsto (A, [ ]_A)$
(see 1.4) admits (as follows easily from 1.5) a left adjoint functor. For a Lie\textsuperscript{*} algebra \( L \) we denote by \( U^\text{ch}(L) \) the corresponding \textit{chiral enveloping} algebra. Explicitly, \( U^\text{ch}(L) \) is the completion of the plain enveloping algebra \( U(L) \) with respect to a topology formed by all the left ideals \( U(L)P \) where \( P \subseteq L \) is an open vector subspace (it satisfies the conditions of Corollary in 1.4).

Our \( U^\text{ch}(L) \) carries a standard filtration \( U^\text{ch}(L) \). defined as the completion of the standard (Poincaré-Birkhoff-Witt) filtration on \( U(L) \). It is admissible and commutative, so the morphism of Lie\textsuperscript{*} algebras \( L \to \text{gr}_1U^\text{ch}(L) \) yields a morphism of topological coisson algebras

\[
\text{Sym}^1L \to \text{gr}U^\text{ch}(L) .
\]

**Lemma.** \((2.4.1)\) is an isomorphism, i.e., \( U^\text{ch}(L) \). is a strong PBW filtration.

**Proof.** By the usual PBW theorem, for every open Lie subalgebra \( P \subseteq L \) one has \( \text{gr}(U(L)/U(L)P) = \text{Sym}^1(L/P) \) (here the quotient \( U(L)/U(L)P \) is equipped with the image of the standard filtration). Such \( P \) form a base of the topology of \( L \) (by the definition of Lie\textsuperscript{*} algebra, see 1.4), and we are done. \( \square \)

**2.5. Chiral extensions of a Lie\textsuperscript{*} algebroid.** We want to prove a similar result for a topological Lie\textsuperscript{*} algebroid. First we need to define its enveloping algebra. This requires (just as in the \( D \)-module setting of [BD] 3.9) an extra structure of chiral extension that we are going to define.

So let \( R \) be a topological commutative\textsuperscript{1} algebra, \( \mathcal{L} \) a topological Lie\textsuperscript{*} \( R \)-algebroid.

Consider for a moment \( R \) as a commutative algebra and \( \mathcal{L} \) as a Lie \( R \)-algebroid in the tensor category \( \text{Top}^\ast \). Let \( \mathcal{L}^\circ \) be a Lie \( R \)-algebroid extension of \( \mathcal{L} \) by \( R \) in \( \text{Top}^\ast \); below we call such \( \mathcal{L}^\circ \) simply a \textit{topological \( R \)-extension} of \( \mathcal{L} \). Explicitly, our \( \mathcal{L}^\circ \) is an extension of topological vector spaces

\[
0 \to R \xrightarrow{i} \mathcal{L}^\circ \xrightarrow{\pi} \mathcal{L} \to 0
\]

(2.5.1)

together with a Lie \( R \)-algebroid structure on \( \mathcal{L}^\circ \) such that \( \pi \) is a morphism of Lie \( R \)-algebroids, \( i \) is a morphism of \( R \)-modules, \( i^\circ := i(1) \) is a central element of \( \mathcal{L}^\circ \); we also demand that \( \mathcal{L}^\circ \) is a Lie\textsuperscript{*} algebra and the \( R \)-action on \( \mathcal{L}^\circ \) is \((\otimes^\ast-)\) continuous.

**Exercise.** \( \mathcal{L}^\circ \) is automatically a topological Lie\textsuperscript{*} algebra (see 1.4).

Our \( \mathcal{L}^\circ \) is automatically an \( R \)-bimodule where the right \( R \)-action is defined by formula \( \ell^\circ r = r\ell^\circ + \ell(r) \), where \( \ell^\circ \in \mathcal{L}^\circ, r \in R, \ell := \pi(\ell^\circ) \), and \( \ell(r) \in R \subseteq \mathcal{L}^\circ \); the right \( R \)-action is continuous as well.

**Definition.** (a) \( \mathcal{L}^\circ \) is called a \textit{classical \( R \)-extension} of \( \mathcal{L} \) if \( \mathcal{L}^\circ \) is a topological \( R \)-algebroid, i.e., the (left) \( R \)-action on \( \mathcal{L} \) is \( \otimes \)-continuous, so we have \( R \otimes^\ast \mathcal{L}^\circ \to \mathcal{L}^\circ \).

(b) \( \mathcal{L}^\circ \) is called a \textit{chiral \( R \)-extension} of \( \mathcal{L} \) if the left and right \( R \)-actions on \( \mathcal{L}^\circ \) are \( \otimes \)-continuous, i.e., we have \( R \otimes^\ast \mathcal{L}^\circ \to \mathcal{L}^\circ \) and \( \mathcal{L}^\circ \otimes^\ast R \to \mathcal{L}^\circ \).

**Example.** As in 2.3, let \( A \) be a chiral algebra equipped with a commutative filtration \( A_\ast \), so \( R := A_0 \) is a commutative\textsuperscript{1} algebra and \( \mathcal{L} := \text{gr}_1A_\ast \) is Lie\textsuperscript{*} \( A_0 \)-algebroid. Set \( \mathcal{L}^\circ := A_1 \); this is an \( R \)-extension of \( \mathcal{L} \). The Lie bracket on \( A_1 \), the
left $A_0$-action on $A_1$, and the adjoint action of $A_1$ on $A_0$ make $L^0$ a topological $R$-extension of the Lie* $R$-algebroid $L$. Since the right $R$-action on $L^0$ equals the right $A_0$-action on $A_1$ that comes from the algebra structure on $A$, it is $\mathcal{O}$-continuous. Therefore $L^0$ is a chiral $R$-extension of $L$.

The topological $R$-extensions of $L$ form naturally a Picard groupoid $\mathcal{P}(L)$; the operation is the Baer sum. More precisely, $\mathcal{P}(L)$ is a $k$-vector space in groupoids: For $L^0_1, L^0_2 \in \mathcal{P}(L)$ and $a_1, a_2 \in k$ the topological $R$-extension $L^{a_1 b_1 + a_2 b_2}$ is defined as the the push-out of $0 \to R \times R \to L^{a_1} \times_L L^{b_2} \to L \to 0$ by the map $R \times R \to R$, $(r_1, r_2) \mapsto a_1 r_1 + a_2 r_2$; the Lie $R$-algebroid structure on it is defined by the condition that the canonical map $L^{a_1} \times_L L^{b_2} \to L^{a_1 b_1 + a_2 b_2}$ is a morphism of Lie $R$-algebroids.

Let $\mathcal{P}_{cl}(L)$, $\mathcal{P}_{ch}(L) \subset \mathcal{P}(L)$ be the subgroupoids of classical and chiral $R$-extensions.

**Lemma.** $\mathcal{P}_{cl}(L)$ is a Picard subgroupoid (actually, a $k$-vector subspace) of $\mathcal{P}(L)$. If $\mathcal{P}_{ch}(L)$ is non-empty, then it is a $\mathcal{P}_{cl}(L)$-torsor.

**Proof.** Let $L^0$ be a topological $R$-extension of $L$; fix $\lambda \in k$. For $\theta \in L^0$, $r \in R$ set $\theta \cdot \lambda \ r = r \theta + \lambda(\theta)$. The operation $\cdot \lambda$ is a right $R$-module structure on $L^0$ (which commutes with the left $R$-module structure). Notice that $\cdot 1$ is the old right $R$-action on $L^0$, and $\cdot 0$ is the left $R$-action.

Suppose we have $L^{b_i} \in \mathcal{P}(L)$ and $\lambda_i, a_i \in k$; here $i = 1, 2$. The right $R$-actions $\cdot \lambda_i$ on $L^{b_i}$ yield a right $R$-action on $L^{a_1 b_1 + a_2 b_2}$, hence a right $R$-action on $L^{a_1 b_1 + a_2 b_2}$ such that the canonical map $L^{a_1} \times_L L^{b_2} \to L^{a_1 b_1 + a_2 b_2}$ is a morphism of right $R$-modules. The latter right $R$-action on $L^{a_1 b_1 + a_2 b_2}$ clearly equals $\cdot a_1 \lambda_1 + a_2 \lambda_2$.

If the right $R$-actions $\cdot \lambda_i$ on $L^{b_i}$ are $\mathcal{O}$-continuous, then the right $R$-actions on $L^{b_1} \times_L L^{b_2}$ and $L^{a_1 b_1 + a_2 b_2}$ are $\mathcal{O}$-continuous as well. Therefore the right $R$-action $\cdot a_1 \lambda_1 + a_2 \lambda_2$ on $L^{a_1 b_1 + a_2 b_2}$ is $\mathcal{O}$-continuous.

Let us call $L^0$ a $\lambda$-chiral $R$-extension if the left $R$-action $R \otimes L^0 \to L^0$ and the right action $\cdot \lambda : L^0 \otimes R \to L^0$ are $\mathcal{O}$-continuous. E.g., 0-chiral extension is the same as classical extension, and 1-chiral extension is the same as chiral extension. We have checked that if $L^{b_i}$ are $\lambda_i$-chiral $R$-extensions, then $L^{a_1 b_1 + a_2 b_2}$ is an $a_1 \lambda_1 + a_2 \lambda_2$-chiral extension. In particular: If $L^{b_i}$ are classical extensions, then $L^{a_1 b_1 + a_2 b_2}$ is a classical extension for every $a_i \in k$. If $L^{b_1}$ is a classical extension, $L^{b_2}$ is a chiral one, then $L^{b_1 + b_2}$ is a chiral extension. If $L^{b_1}$ are chiral extensions, then $L^{b_1 - b_2}$ is a classical extension. We are done.

**Exercises.** (i) A classical extension $L^0$ of $L$ is a chiral extension if and only if for every open ideal $I \subset R$ there is an open ideal $J \subset R$ such that $L(J) \subset I$. (ii) For a topological $R$-extension $L^0$ let $L^{0\tau}$ be the “inverse” $R$-extension: so we have an identification of topological vector spaces $L^{0\tau} \cong L^0$ which commutes with $i$’s and anticommutes with $\pi$’s in (2.5.1), interchanges the left and right $R$-module structures, and identifies the Lie bracket with on $L^{0\tau}$ with minus Lie bracket on $L^0$. Show that if $L^0$ is a classical extension, then $L^{0\tau}$ is a chiral extension if and only if for every open ideal $I \subset R$ there is an open $P \subset L$ such that $P(R) \subset I$.

### 2.6. The enveloping algebra of a chiral Lie algebroid

Let $L^0$ be a chiral extension of a Lie* $R$-algebroid $R$; we call a triple $(R, L, L^0)$ a topological chiral Lie algebroid. These objects form naturally a category $\mathcal{CL}(\text{Top})$. 
Let $\mathcal{CA}^f_c(Top)$ be the category of topological chiral algebras equipped with a commutative filtration. By Example in 2.5, we have a functor $\mathcal{CA}^f_c(Top) \to \mathcal{CL}(Top)$, $(A, A.) \mapsto (A_0, \text{gr}_1 A, A_1)$.

**Proposition.** This functor admits a left adjoint $\mathcal{CL}(Top) \to \mathcal{CA}^f_c(Top)$.

We denote this adjoint functor by $(R, \mathcal{L}, \mathcal{L}^\flat) \mapsto U^c_L(\mathcal{L})^\flat$ and call $U^c_L(\mathcal{L})^\flat$ the *chiral enveloping algebra* of $\mathcal{L}^\flat$. The commutative filtration $U^c_L(\mathcal{L})^\flat$ on $U^c_L(\mathcal{L})^\flat$ is referred to as the *standard* filtration.

**Proof.** Let us define $U^c_L(\mathcal{L})^\flat$ as a universal chiral algebra equipped with a continuous map $\varphi^\flat : \mathcal{L}^\flat \to U^c_L(\mathcal{L})^\flat$ such that $\varphi^\flat$ is a morphism of Lie algebras, its restriction to $R \subset \mathcal{L}^\flat$ is a morphism of chiral (or associative) algebras, and $\varphi^\flat$ is a morphism of $R$-bimodules (with respect to $\varphi^\flat|_R$). To construct it explicitly, consider the “abstract” enveloping algebra $U_R(\mathcal{L})^\flat$ of $\mathcal{L}^\flat$, i.e., take copies of $R$, $\mathcal{L}$, and $\mathcal{L}^\flat$ equipped with discrete topologies; then $U_R(\mathcal{L})^\flat$ is the corresponding chiral enveloping algebra (its topology is discrete). Now $U_R(\mathcal{L})^\flat$ carries a linear topology whose base is formed by all left ideals $U_R(\mathcal{L})^\flat(P + I)$ where $P \subset L$ and $I \subset R$ are open subspaces. Shrinking $P$, $I$ if necessary, one can assume that $P$ is an open Lie subalgebra of $\mathcal{L}^\flat$ and $I$ an open $P$-stable ideal of $R$. This topology satisfies the conditions of Corollary from 1.4, so the corresponding completion is a chiral algebra which equals $U^c_L(\mathcal{L})^\flat$ due to the universality property.

The standard filtration is defined in the usual manner: $U^c_L(\mathcal{L})^\flat_n$ is the closure of the image of $\mathcal{L}^\flat$ for $n = 0$, and is the closure of the image of $n$th power of the image of $\mathcal{L}^\flat$ if $n \geq 1$. It is clearly admissible and commutative. As an object of $\mathcal{CA}^f_c(Top)$, our $U^c_L(\mathcal{L})^\flat$ evidently satisfies the universality property of the statement, and we are done. $\square$

**Remark.** The above explicit construction of $U^c_L(\mathcal{L})^\flat$ implies that a discrete $U^c_L(\mathcal{L})^\flat$-module is the same as a vector space $M$ equipped with a continuous Lie algebra action of $\mathcal{L}^\flat$ such that the action of $R \subset \mathcal{L}^\flat$ is a unital $R$-module structure on $M$, and for $r \in R$, $\ell^\flat \in \mathcal{L}^\flat$, $m \in M$ one has $(r\ell^\flat)m = r(\ell^\flat m)$.

**2.7. Rigidified chiral extensions.** Let $R$ be a topological commutative$^\flat$ algebra, $L$ a topological Lie$^*$ algebra that acts on $R$ in a continuous way. Then $L_R := R \otimes^L L$ is naturally a topological Lie$^*$ $R$-algebroid.

**Lemma.** There is a unique, up to a unique isomorphism, chiral extension $L^\flat_R$ equipped with a Lie$^*$ algebra morphism $L \to L^\flat_R$ which lifts the embedding $L \hookrightarrow L_R$.

**Proof.** We construct $\mathcal{L}^\flat$ explicitly; the uniqueness is clear from the construction.

Forgetting for a moment about the topologies, consider the $L$-rigidified Lie $R$-algebroid $L^\delta_R = R \otimes L$ and its trivialized $R$-extension $L^\sharp_R = L^\delta_R \oplus R$. Our $L^\sharp_R$ contains $L$ as a Lie subalgebra; as in 2.5, it is naturally an $R$-bimodule. For open subspaces $P \subset L$ and $I \subset R$ let $(P, I) \subset L^\sharp_R$ be the vector subspace formed by linear combinations of all vectors $rp, \ell i, i' \in L^\sharp_R$ where $p \in P$, $i, i' \in I$ and $\ell, r$ are

---

$^\flat$It suffices to demand that $\varphi^\flat$ is a morphism of either left or right $R$-modules.
arbitrary elements of \( L, R \). These subspaces form a topology on \( L_R^2 \); we define \( L_R^b \) as the corresponding completion.

Notice that the subspaces \( \langle P, I \rangle \) with \( P \subset L \) an open Lie subalgebra and \( I \subset R \) an open ideal preserved by the \( P \)-action form a base of the topology on \( L_R^1 \). For such \( P, I \) one has \( \langle P, I \rangle = RP + L_R^2 I \). One also has \( \langle P, I \rangle \cap R = I \), so \( L_R^1 \) is an extension of \( R \otimes^L \) by \( R \). The evident morphism \( L \to L_R^1 \) is continuous.

For \( P, I \) as above the subspace \( \langle P, I \rangle \) is a Lie subalgebra and a left \( R \)-submodule (i.e., a Lie \( R \)-subalgebroid) of \( L_R^1 \). The Lie bracket and the left and right actions of \( R \) on \( L_R^2 \) are continuous with respect to our topology, hence \( L_R^b \) is a topological \( R \)-extension of \( L_R \) in the sense of 2.5. Clearly it is a chiral \( R \)-extension, q.e.d.

**Remark.** Here is another description of \( L_R^b \) as a mere topological extension. The map \( L \otimes R \otimes R \otimes L \to L_R^2, \ell \otimes r + r' \otimes \ell' \mapsto \ell r + r' \ell' \), extends by continuity to a continuous morphism \( L \otimes R \otimes R \otimes L \to L_R^b \). It identifies \( L_R^b \) with the quotient of \( L \otimes R \otimes R \otimes L \otimes R \) modulo the closed subspace generated by vectors \( \ell \otimes r - r \otimes \ell - \ell(r) \).

Equivalently, consider the extension \( 0 \to L \otimes^R R \overset{(+, -)}{\to} L \otimes L \otimes^R L \overset{(+, +)}{\to} R \otimes L \to 0 \) (see (1.1.2)); then \( 0 \to R \to L_R^1 \to L_R \to 0 \) is its push-out by the \( L \)-action. The de Rham-Chevalley differential preserves the subalgebra \( C \) and is continuous on it, so \( C \) is a topological commutative DG algebra. □

**2.8. The de Rham-Chevalley chiral \( L^* \)-algebroid.** Let \( R \) be a reasonable topological algebra, \( \mathcal{L} \) a Tate \( R \)-module, \( \mathcal{L}^* \) the dual Tate \( R \)-module (see [D]).

**Exercise.** The \( R \)-module of all continuous \( R \)-n-linear maps \( \mathcal{L} \times \ldots \times \mathcal{L} \to R \) identifies naturally with the \( n \)th tensor power of \( \mathcal{L}^* \) in \( R \text{mod}^1 \).

Suppose we have a \( L^* \)-algebroid structure on \( \mathcal{L} \). Forget for a moment about the topologies and consider \( \mathcal{L} \) as a mere Lie \( R \)-algebroid. We have the corresponding de Rham-Chevalley complex \( C_R(\mathcal{L}) \): this is a commutative DGA whose \( n \)th term equals \( \text{Hom}_R(\Lambda^n_R \mathcal{L}, R) \) and the differential is given by the usual formula. By the exercise, the continuous maps form a graded subalgebra \( C = C_R(\mathcal{L}) \) of \( C_R(\mathcal{L}) \) which equals \( \text{Sym}^R(\mathcal{L}^*[-1]) \).

**Lemma.** The de Rham-Chevalley differential preserves the subalgebra \( C \) and is continuous on it, so \( C \) is a topological commutative DG algebra. □

**Exercise.** A \( L^* \)-algebroid structure on \( \mathcal{L} \) amounts to a differential on the topological graded algebra \( \text{Sym}^R(\mathcal{L}^*[-1]) \).

Our \( C \) carries a natural topological DG \( L^* \)-algebroid \( \mathcal{L}_C \) (cf. [BD] 3.9.16). To construct it, consider \( \mathcal{L} \) as a mere Lie\( L^* \) algebra. It acts on \( \mathcal{C} \) by transport of structure, and this action extends naturally to an action of the contractible Lie\( L^* \) DG algebra \( \mathcal{L}_1 := \mathcal{C} \circ \text{Hom}(\text{id} : \mathcal{L} \to \mathcal{L}) \): namely, the component \( \mathcal{C}[1] \) acts by the evident \( R \)-linear derivations of \( \text{Sym}^R(\mathcal{L}^*[-1]) \). Thus we have the corresponding DG \( L^* \)-algebroid \( \mathcal{L}_C \). By construction, \( \mathcal{L}_C^{\leq -1} = 0 \) and \( \mathcal{L}_C^{-1} = R \otimes \mathcal{L} \). Let \( K \) be the closed DG \( \mathcal{C} \)-submodule of \( \mathcal{L}_C \) generated by \( K^{-1} \subset \mathcal{L}_C^{-1} \) defined as the kernel of the product map \( R \otimes \mathcal{L} \to \mathcal{L} \). Since \( K^{-1} \) acts trivially on \( \mathcal{C} \) and is normalized by the adjoint action of \( \mathcal{L}_+ \), we see that \( K \) is a DG ideal in the Lie\( \mathcal{C} \)-algebroid \( \mathcal{L}_C \). The promised \( \mathcal{L}_C \) is the quotient \( \mathcal{L}_C/K \).
Set \( \mathcal{L}_{1C} := \text{Sym}^1(\mathcal{L}^*[1]) \otimes_R \mathcal{L}[1] \); we consider it as the graded Lie \( \mathcal{C} \)-algebroid generated by the action of the Lie \( \mathcal{C} \)-algebra \( \mathcal{L}[1] \) on \( \mathcal{C} \). The embedding into \( \mathcal{L}_{1C} \) identifies it with \( \mathcal{C} \cdot \mathcal{L}^{-1}[1] \), which is a graded subalgebra of \( \mathcal{L}_{1C} \) not preserved by the differential: in fact, the Kodaira-Spencer map \( \mathcal{L}_{1C} \to \mathcal{L}_{1C}/\mathcal{L}_{1C}^+, \ell \mapsto d(\ell)\text{mod}\mathcal{L}_{1C}^+ \), is an isomorphism. Similarly, consider \( \mathcal{K}_+ := \mathcal{K} \cap \mathcal{L}_{1C}^+ \); then \( \mathcal{K}_+ = \text{Sym}^1(\mathcal{L}^*[1]) \otimes_R \mathcal{K}^{-1} \) and the Kodaira-Spencer map \( \mathcal{K}_+ \to \mathcal{K}/\mathcal{K}_+ \) is an isomorphism. Therefore \( \tilde{\mathcal{K}}_{1C} := \text{the closed } C\text{-submodule of } \mathcal{K}_+ \) generated by \( \tilde{\mathcal{K}}_+ \), equals \( \text{Sym}^1_R(\mathcal{L}^*[1]) \otimes_R \mathcal{L}^{-1}[1] = \text{Sym}^1_R(\mathcal{L}^*[1]) \otimes_R \mathcal{L}[1] \), and the Kodaira-Spencer map \( \mathcal{L}_{1C}^+ \to \mathcal{K}/\mathcal{K}_{1C}^+ \) is an isomorphism.

Remark. We see that the Lie \( \ast \)-algebroid \( \mathcal{L}_{1C}^0 \) is an extension of \( \mathcal{L} \) by \( \mathcal{L}_{1C}^0 = \mathcal{L}^* \otimes_R \mathcal{L} \). It acts naturally on the Tate \( \mathcal{R} \)-module \( \mathcal{L}_{1C}^{-1} = \mathcal{L} \) by the adjoint action. The restriction of this action to the Lie \( \ast \)-algebra \( \mathcal{L}^* \otimes_R \mathcal{L} \) identifies \( \mathcal{L}^* \otimes_R \mathcal{L} \) with the Lie algebra \( \mathfrak{gl}_R(\mathcal{L}) \) of continuous \( \mathcal{R} \)-linear endomorphisms of \( \mathcal{L} \) as in 1.2.

**Proposition.** The topological DG Lie \( \ast \)-algebroid \( \mathcal{L}_{1C} \) admits a unique topological chiral DG extension \( \mathcal{L}_{1C}^0 \).

Proof (cf. [BD] 3.9.17). **Existence.** Consider the rigidified DG chiral extension \( \mathcal{L}_{1C}^\ast \) of the Lie \( \ast \)-algebroid \( \mathcal{L}_{1C} \). Let \( \tilde{\mathcal{K}} \subset \mathcal{L}_{1C}^\ast \) be the closed DG left \( \mathcal{C} \)-subalgebra of \( \mathcal{L}_{1C}^\ast \) generated by \( \tilde{\mathcal{K}}^- \). It remains to prove that the closure of \( \mathcal{C} \cdot \tilde{\mathcal{K}}^- \subset \mathcal{L}_{1C}^\ast \) does not intersect \( \mathcal{R} \). The map \( \mathcal{C}^* \otimes \sigma \tilde{\mathcal{K}}^- \otimes \sigma \mathcal{C}^1 \to \mathcal{L}_{1C}^\ast, c \otimes k \otimes k' \otimes \mathcal{C}^1 \to c \mathcal{C} \otimes k \mathcal{C}^1 \), vanishes on the subspace \( \mathcal{C}^1 \otimes \sigma \tilde{\mathcal{K}}^- \) (see (1.1.2)) since \( \mathcal{K}^- \) acts trivially on \( \mathcal{C} \). So, by (1.1.2), the product map \( \mathcal{C}^1 \otimes \sigma \tilde{\mathcal{K}}^- \to \mathcal{L}_{1C}^\ast \) extends by continuity to \( \mathcal{C}^1 \otimes \tilde{\mathcal{K}}^- \to \mathcal{L}_{1C}^\ast \), and, by \( \mathcal{R} \)-bilinearity, to \( \mu : \mathcal{C}^1 \otimes_R \tilde{\mathcal{K}}^- \to \mathcal{L}_{1C}^\ast \). We know that the composition \( \mathcal{C}^1 \otimes_R \tilde{\mathcal{K}}^- \to \mathcal{L}_{1C}^\ast \) is a closed embedding, so \( \mu \) is a closed embedding whose image does not intersect \( \mathcal{R} \), q.e.d.

**Uniqueness.** By the proposition in 2.5, it suffices to show that every classical DG extension \( \mathcal{L}_{1C}^0 \) of \( \mathcal{L}_{1C} \) admits a unique splitting. The uniqueness of the splitting is clear since \( \mathcal{L}_{1C}^0 \cong \mathcal{L}_{1C}^\ast \) and \( \mathcal{L}_{1C} \) is generated by \( \mathcal{L}_{1C}^\ast \) as a DG Lie \( \ast \)-algebroid. To construct one, consider the embedding \( \mathcal{L}[1] \hookrightarrow \mathcal{L}_{1C}^\ast \). It extends to a morphism of complexes \( \mathcal{L} = \text{Cone}(\text{id}_\mathcal{C}) \to \mathcal{L}_{1C}^\ast \) which is a morphism of Lie \( \ast \)-algebras (as an immediate computation shows). By universality, we get a morphism of DG Lie \( \ast \)-algebroids \( \mathcal{L}_{1C} \to \mathcal{L}_{1C}^\ast \) which vanishes on \( \mathcal{K}^- \). Thus it vanishes on \( \mathcal{K} \), hence we get a promised splitting \( \mathcal{L}_{1C} \to \mathcal{L}_{1C}^\ast \). \( \square \)
Remark. By the R-extension property, the Lie* bracket between $\mathcal{L}_c^{-1}$ and $\mathcal{C}^{1} \subset \mathcal{L}_c^{0}$ equals the standard pairing $\mathcal{L} \times \mathcal{L}^* \to R$. Therefore the subalgebra of $U_c^\theta(L_c)$ generated by these submodules equals the Clifford algebra $\text{Cl} = \text{Cl}_R(\mathcal{L}^* \oplus \mathcal{L})$ of the Tate $R$-module $\mathcal{L}^* \oplus \mathcal{L}$ equipped with the usual hyperbolic quadratic form. It contains the Lie* $R$-subalgebra $L_c^{0}$ which is the Clifford $R$-extension $\mathfrak{gl}_R(\mathcal{L})^{CI}$ of $\mathfrak{gl}_R(\mathcal{L})$.

2.9. $\mathcal{D}$-modules. The most interesting special case of the construction from 2.8 is that of $\mathcal{L} = \Theta_R$ for $R$ as in Example (ii) in 2.2. Then $\mathcal{C}$ is the de Rham DG algebra of $R$.

Remark. In the vertex (or chiral) algebra setting the enveloping algebra $U_c^\text{ch}(\Theta_R)^0$ was first considered in [MSV] under the name of the chiral de Rham complex; for other, essentially equivalent, constructions (in slightly different settings) see [BD] and [KV].

Consider for a moment $U_c^\text{ch}(\Theta_R)^0$ as a non-graded topological chiral algebra (which requires an evident completion). The category of discrete $U_c^\text{ch}(\Theta_R)^0$-modules plays the role of the category of $\mathcal{D}$-modules on the ind-scheme Spec$R$ (this construction, mentioned in [D] 6.3.9, generalizes the constructions from [D] and [KV]).

Exercises. (i) Suppose that $R$ is discrete. Show that for a discrete $U_c^\text{ch}(\Theta_R)^0$-module $M$ the subspace $M^\ell \subset M$ of elements killed by $\Theta_R[1] = L_c^\text{ch}$ is naturally a left $\mathcal{D}$-module on Spec $R$, i.e., an $R$-module equipped with a flat connection. The functor $M \mapsto M^\ell$ is an equivalence between the category of discrete $U_c^\text{ch}(\Theta_R)^0$-modules and that of left $\mathcal{D}$-modules.

(ii) Suppose that $R$ is projective limit of $k$-algebras of finite type, so one has a natural notion of right $\mathcal{D}$-module. Show that for a discrete $U_c^\text{ch}(\Theta_R)^0$-module $M$ the subspace $M^r \subset M$ of elements killed by $\mathcal{C}^{2,1}$ is naturally a right $\mathcal{D}$-module, and the functor $M \mapsto M^r$ is an equivalence between the category of discrete $U_c^\text{ch}(\Theta_R)^0$-modules and that of right $\mathcal{D}$-modules.

More generally, suppose $R$ is arbitrary, and we have a fermion module $V$ in the sense of [D] 5.4.1 over the Clifford algebra $\text{Cl}$ from the second remark in 2.8. Then $V$ yields a chiral extension $\mathcal{L}^V$ of $\mathcal{L}$ defined as follows (cf. [BD] 3.9.20). The Lie* subalgebra $\mathcal{L}_c^{0V}$ normalizes $\text{Cl} \subset U_c^\text{ch}(\mathcal{L}_c)$, and its adjoint action on $\text{Cl}$ factors through $\mathcal{L}_c^0$. Let $\mathcal{L}_c^{0V}$ be the set of pairs $(\tau, \tau_V)$ where $\tau \in \mathcal{L}_c^0$ and $\tau_V$ is a lifting of $\tau$ to $V$, i.e., a continuous endomorphism of $V$ such that $\tau_V(cv) = \tau(c)v + c\tau_V(v)$ for $c \in Cl, v \in V$. One shows easily that $\mathcal{L}_c^{0V}$ is naturally a Lie* $R$-algebroid which is an $R$-extension of $\mathcal{L}_c^0$. Notice that the action on $V$ of $\mathfrak{gl}_R(\mathcal{L})^{CI} \subset \text{Cl}$ identifies the restriction of $\mathcal{L}_c^{0V}$ to $\mathfrak{gl}_R(\mathcal{L})$ with $\mathfrak{gl}_R(\mathcal{L})^{CI}$.

Exercise. For a discrete $U_c^\text{ch}(\Theta_R)^0$-module $M$ the natural action of $\mathcal{L}_c^{0V}$ on $M^V := \text{Hom}_{\text{Cl}}(V, M)$ factors through $\mathcal{L}^V$, and the functor $M \mapsto M^V$ is an equivalence between the category of discrete $U_c^\text{ch}(\Theta_R)^0$-modules and that of discrete modules over the enveloping chiral algebra $U_R^\text{ch}(\mathcal{L})^V$ of $\mathcal{L}^V$.

\footnote{In fact, by Remark in 2.8, $\mathcal{L}_c^0$ identifies with the Atiyah Lie* $R$-algebroid of infinitesimal symmetries of the Tate $R$-modules $\Theta_R$ or $\Omega_R$.}
Remark. The key ingredient of $\mathcal{O}$- and $\mathcal{D}$-module theory in the usual finite-dimensional setting is functoriality of the derived categories with respect to morphisms of varieties. The Clifford module picture permits to recover the pull-back functoriality for appropriately twisted derived categories (the $\mathbb{Z}$-grading of complexes should be labeled by the dimension $\mathbb{Z}$-torsor, and the derived category itself should be understood in an appropriate way). This construction is necessary in order to define the notion of $\mathcal{O}$- or $\mathcal{D}$-module on orbit spaces such as the moduli space of (de Rham) local systems on the formal punctured disc. I hope to return to this subject in a joint work with Gaitsgory.

2.10. The weak PBW theorem for topological chiral algebroids. Let $(R, \mathcal{L}, \mathcal{L}^0)$ be a topological chiral Lie algebroid where $R$ is reasonable. The morphisms $R \to U^{ch}(\mathcal{L})_0$, $\mathcal{L} \to \gr_1 U^{ch}(\mathcal{L})^b$ yield then a morphism of topological coisson algebras (see (2.3.1))

\begin{equation}
\Sym^1_R \mathcal{L} \to \widehat{\gr} U^{ch}_R(\mathcal{L})^b.
\end{equation}

**Theorem.** If $\mathcal{L}$ is a flat $R$-module with respect to $\otimes^1$, then $\Sym^1_R \mathcal{L} \cong \gr U^{ch}_R(\mathcal{L})^b$. Thus the standard filtration satisfies the weak PBW property.

**Proof.** We follow [BD] 3.9.13. Set $U^{ch} := U^{ch}_R(\mathcal{L})^b$ and $U := U_R(\mathcal{L})^b$.

(a) Suppose our chiral extension admits a rigidification, so we have a Lie$^*$ algebra $L$ acting on $R$, $\mathcal{L} = L_R = R \otimes^1 L$, and $\mathcal{L}^b = L^b_R$ is the $L$-rigidified chiral extension.

Then $U$ is the enveloping algebra of $(L, R)$, i.e., the quotient of the free associative algebra generated by $L \oplus R$ modulo the relations saying that the map $L \to U$ is a morphism of Lie algebras and the map $R \to U$ is a morphism of associative algebras which commutes with the $L$-action (where $L$ acts on $U$ via $L \to U$ and the adjoint action). Our $U^{ch}$ is the corresponding chiral enveloping algebra which can be constructed as follows. Consider the topology on $U$ whose base is formed by all left ideals $U(P + I)$ where $P \subset L$ and $I \subset R$ are open subspaces. Shrinking $P, I$ if necessary, one can assume that $P$ is an open Lie subalgebra of $L$ and $I$ an open $P$-stable ideal of $R$. This topology satisfies the conditions from Corollary in 1.4, hence $U^{ch}$ is the completion of $U$ with respect to this topology.

For $(P, I)$ as above the quotient $U/U(P+I)$ coincides with the $L$-module induced from the $P$-module $R/I$. Thus $\gr(U/U(P+I)) = \Sym(L/P) \otimes R/I$ by the usual PBW theorem. Passing to the projective limit with respect to $(P, I)$, we see that the standard filtration satisfies the PBW property (actually, the strong one).

(b) Now let $\mathcal{L}$ be an arbitrary Lie$^*$ $R$-algebroid and $\mathcal{L}^b$ is chiral extension. Let $L$ be a copy of $\mathcal{L}^b$ considered as a mere Lie$^*$ algebra acting on $R$. Consider the corresponding Lie$^*$ $R$-algebroid $L_R = R \otimes^1 L$ and its $L$-rigidified chiral extension $L^b_R$. We have an evident morphism of Lie$^*$ $R$-algebroids $L_R \to \mathcal{L}$ and its lifting to the chiral extensions $L^b_R \to \mathcal{L}^b$. The projection $L^b_R \to L_R$ identifies the kernels of those morphisms. Our $K$ is an $R$-module (in the $\otimes^1$ sense, as a submodule of $L_R$) equipped with a continuous $L$-action (the adjoint one).

Set $\tilde{L}_R := \text{Cone}(K \to L_R)$, $\tilde{L}^b_R := \text{Cone}(K \to L^b_R)$. Then $L_R$ is naturally a DG Lie$^*$ $R$-algebroid, and $\tilde{L}_R$ is its chiral extension. These structures are uniquely
defined by the condition that the embedding \( L_R \to \tilde{L}_R \) is a morphism of Lie\(^*\) \( R\)-algebroids, and that \( L^b_R \to \tilde{L}^b_R \) is a morphism of chiral extensions. Therefore we have a DG chiral algebra \( U_{\tilde{R}}^ch(\tilde{L}_R)^b \).

Set \( \tilde{R} := \text{Sym}^1_R(K[1]) \); this is a commutative\(^*\) graded topological \( R\)-algebra whose component in degree \(-1\) equals \( \Lambda^1_R K \). It carries a natural continuous \( L\)-action. So we have the \((\mathbb{Z}-\text{graded})\) \( \tilde{R}\)-algebroid \( L_{\tilde{R}} \), its \( L\)-rigidified chiral extension \( L^b_{\tilde{R}} \), and the corresponding chiral enveloping algebra \( U^ch_{\tilde{R}}(L_{\tilde{R}})^b \).

**Lemma.** There is a unique isomorphism of \( \mathbb{Z}\)-graded topological chiral algebras

\[
U^ch_{\tilde{R}}(L_{\tilde{R}})^b \cong U^ch_{\tilde{R}}(L_{\tilde{R}})^b
\]

which identifies copies of \( R \) and \( L \) in the degree 0 components, and identifies \( K \subset \tilde{L}^b_{\tilde{R}} \) with \( K \subset \tilde{R} \) in the components of degree \(-1\).

**Proof.** Both algebras are generated by \( R, K, \) and \( L \) with same relations. \( \Box \)

Isomorphism (2.10.2) is identifies the standard filtrations up to a shift by the grading: one has \( U^ch_{\tilde{R}}(L_{\tilde{R}})^{a+n}_n \cong U^ch_{\tilde{R}}(L_{\tilde{R}})_{a+n} \). By (a), \( U^ch_{\tilde{R}}(L_{\tilde{R}})^b \) satisfies the PBW property. So we have an isomorphism of graded DG chiral algebras

\[
\text{Sym}^1_{\tilde{R}} \tilde{L}_R \cong \text{gr} U^ch_{\tilde{R}}(\tilde{L}_R)^b.
\]

(c) Suppose that \( \mathcal{L} \) is \( R\)-flat with respect to \( \otimes^1 \). Then the projection \( \text{Sym}^1_{\tilde{R}} \tilde{L}_R \to \text{Sym}_{\tilde{R}}^1 \mathcal{L} \) is a quasi-isomorphism in the exact category \( R\text{-mod}^1 \). Therefore the differential \( d \) on \( U^ch_{\tilde{R}}(\tilde{L}_R)^b \) is strictly compatible with the standard filtration. Let \( I_n \subset U^ch_{\tilde{R}}(\tilde{L}_R)^b_n \) be the closure of \( K U^ch_{\tilde{R}}(\tilde{L}_R)^b_{n-1} \); then \( I_n = I_m \cap U^ch_{\tilde{R}}(\tilde{L}_R)^b_n \) for any \( m \geq n \), hence \( \text{gr} U^ch_{\tilde{R}}(\mathcal{L})^b \cong \text{gr} U^ch_{\tilde{R}}(\tilde{L}_R)^b / \text{gr} I \cong \text{Sym}^1_{\tilde{R}} \mathcal{L} \), q.e.d. \( \Box \)

**Exercise.** Suppose the topology of \( R \) has a base formed by open reasonable ideals \( I \) that satisfy the next property: The open \( \text{Lie}^* \) \( R\)-subalgebrods \( M \subset \mathcal{L} \) such that \( M(I) \subset I, M \supset I \mathcal{L} \), and \( \mathcal{L}/M \) is a flat \( R/I\)-module, form a base of the topology of \( \mathcal{L}/\mathcal{T}\mathcal{L} \). Then the standard filtration on \( U^ch_{\tilde{R}}(\mathcal{L})^b \) is a strong PBW filtration.

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