Quantum optical technologies
for metrology, sensing and imaging

Jonathan P. Dowling, Fellow, OSA, and Kaushik P. Seshadreesan, Student Member, OSA,
Hearne Institute for Theoretical Physics and Department of Physics and Astronomy, Louisiana State University,
Baton Rouge, LA 70803, USA

Abstract—Over the past 20 years, bright sources of entangled photons have led to a renaissance in quantum optical interferometry. Optical interferometry has been used to test the foundations of quantum mechanics and implement some of the novel ideas associated with quantum entanglement such as quantum teleportation, quantum cryptography, quantum lithography, quantum computing logic gates, and quantum metrology. In this paper, we focus on the new ways that have been developed to exploit quantum optical entanglement in quantum metrology to beat the shot-noise limit, which can be used, e.g., in fiber optical gyroscopes and in sensors for biological or chemical targets. We also discuss how this entanglement can be used to beat the Rayleigh diffraction limit in imaging systems such as in LIDAR and optical lithography.

Keywords—Quantum entanglement, Quantum sensors, Quantum metrology, Heisenberg limit

I. INTRODUCTION

Sensors are an integral part of many modern technologies that touch our day to day lives, e.g., automotive technologies, the global positioning system (GPS) and mobile telecommunications, to name a few. They also get widely used in industrial applications, e.g., in manufacturing and machinery, in petroleum-well mapping, oil refineries, chemical processes and medicine. It is desired for a sensor to capture the faintest of signals. The capability of a sensor to do so is largely dictated by its noise or precision characteristics. Hence, metrology—the study of precision measurements—plays a fundamental role in the design of sensors.

Quantum mechanics, being a fundamental theory of nature, has a bearing on the performance of technologies that are based on information processing, e.g., computation, communication and cryptography. It is thus imperative to consider quantum mechanics in order to determine the ultimate limits of these technologies. To this end, the classical theories of computation, communication and cryptography—which revolutionized the technological world in the past half a century or so—have been revisited to study the effects of quantum mechanics. This has lead to exciting new possibilities in these areas, such as quantum algorithms for fast integer factorization, fast database search, quantum teleportation, superdense coding and quantum key distribution [1]–[5]. Likewise, metrology, which is also a science based on information acquisition, processing, and estimation, has been revisited to include the effects of quantum mechanics too. Quantum metrology [6] has been found to enable measurements with precisions that surpass the classical limit, and has grown into an exciting new area of research with potential applications, e.g., in gravitational wave detection [7], quantum positioning and clock synchronization [8], quantum frequency standards [9], quantum sensing [10], [11], quantum radar and LIDAR [12], [13], quantum imaging [14], [15] and quantum lithography [16]–[18].

Fig. 1. A schematic of a typical quantum parameter estimation setup. Probes prepared in suitable quantum states are made to evolve through a unitary process $U(\varphi)$, which is an optical interferometer in our case. The process imparts information about the unknown parameter of interest on to the probes. The probes are then detected at the output, and the measurement outcomes used to estimate the unknown value of the parameter.

Quantum metrology offers a theoretical framework that can be used to analyze the precision performance of measurement devices that employ quantum-mechanical probes containing nonclassical effects such as entanglement or squeezing. It relies on the theory of quantum parameter estimation [19]–[21]. Consider the typical scenario of parameter estimation described in Fig. 1, where we want to estimate an unknown parameter associated with the unitary dynamics generated by a known physical process. We prepare probes in suitable quantum states, evolve them through the process, and measure the probes at the output using a suitable detection strategy. We then compare the input and output probe states, which allows us to estimate the unknown parameter of the physical process. Let us suppose that the generating Hamiltonian is linear in the number of probes. When $N$ classical probes (probes with no quantum effects) are used, the precision is limited by a scaling given by $1/\sqrt{N}$; known as the shot-noise limit. This scaling arises from the central limit theorem of statistics. On the other hand, probes with quantum entanglement can reach below the shot-noise limit and determine the unknown parameter with a precision that can scale as $1/N$; known as the Heisenberg limit [22].

Optical metrology uses light interferometry as a tool to
perform precision measurements. The most basic optical interferometer is a two-mode device with an unknown relative phase (between the two modes). This unknown phase can be engineered to carry information about different quantities of interest in different contexts, e.g., it is related to the strength of a magnetic field in an optical magnetometer, a gravitational wave at LIGO (light interferometer gravitational wave observation), etc. Fig. 2 shows a conventional optical interferometer in the Mach Zehnder configuration. The input to the classical interferometer is a coherent laser source, and the detection is based on intensity difference measurement. When a coherent light of average photon number $n$ is used, the precision of phase estimation is limited by the shot-noise of $1/\sqrt{n}$ associated with the intensity fluctuations at the output, which have their origin in the vacuum fluctuations of the quantized electromagnetic field that enter the device through the unused input port $b_0$.

However, quantum optical metrology enables sub-shot-noise phase estimation. In a seminal work in the field, Caves [23] showed that when the nonclassical squeezed vacuum state is mixed instead of the vacuum state in the unused port of the same interferometer, sub-shot-noise precision that scales as $1/\sqrt{N}$ can be attained. Subsequently, two-mode squeezed states were shown to enable phase estimation at a precision of $1/\sqrt{N}$ [24]. With the advancement in single-photon technology, finite photon number states containing quantum entanglement were also proposed and studied in quantum optical metrology. This includes the $N00N$ states [25], which are Schrödinger cat-like, maximally mode-entangled states of two modes, where the $N$ photons are in superposition of all $N$ photons being in one mode or the other; the Holland-Burnett states [22] and the Berry-Wiseman states [26], to name a few. All these states were found to be capable of attaining the Heisenberg limit $1/N$. The above theoretical results have led to many experimental demonstrations of sub-shot-noise metrology with finite photon number states [27]–[32].

Along with the different quantum states of light, a plethora of detection strategies have also been investigated. This includes homodyne and heterodyne detection [33], the canonical phase measurement [34] (which can be mimicked by an adaptive measurement [35]), photon number counting [36], [37], and photon number parity [38]. These measurement schemes have been shown to be capable of attaining the optimal precisions of different quantum states of light.

More recently, numerous studies have been devoted to investigating the effects of photon loss, dephasing noise and other decoherence phenomena, on the precision of phase estimation in quantum optical metrology. Useful lower bounds on precision, and optimal quantum states that attain those bounds, have been identified in some such scenarios both numerically and analytically [39]–[44].

In this paper, we focus on interferometry with entangled states of finite photon number, in particular, those based on the $N00N$ states. We focus on some recent experiments that demonstrate $N00N$ state interferometry for modest photon numbers $N$, with potential applications in quantum technology [10], [11], [45]–[47]. The $N00N$ states not only attain the Heisenberg limit of $1/N$ in phase precision—known as super-sensitivity, but are also capable of phase resolution below the Rayleigh diffraction limit—known as super-resolution [25].

The paper is organized as follows. In section II, we discuss some basic concepts of quantum optical metrology. This includes the different available representations to study two-mode quantum interferometry, and the methods of quantum parameter estimation theory that are used to analyze the interferometric output statistics to estimate the unknown phase. In section III, we discuss quantum optical metrology. We begin by introducing quantum entanglement, which is the driving-force behind the quantum enhancement, and describe the different available methods to generate entanglement for optical metrology. We describe an interferometric scheme that is known to achieve the optimal Heisenberg limit in phase precision. Section IV is devoted to quantum technologies for metrology, sensing and imaging. In this section, we present results from a few recent experiments that have demonstrated the benefits of quantum optical interferometry with the $N00N$ states for such technological applications. In Section V, we conclude with a brief discussion on some recent considerations in quantum optical interferometry and a summary.

II. BASIC CONCEPTS

We now discuss the basic concepts that underlie quantum optical metrology, namely, those of quantum optical interferometry and quantum parameter estimation.

A. Classical optical interferometry

Before we describe quantum interferometry, let us briefly examine the conventional coherent laser light interferometer described in Fig. 2 in purely classical terms. The input laser beam is split into two beams of equal intensities by the first 50:50 beamsplitter. These beams then gather an unknown relative phase as they pass through the device. They are then recombined on the final beamsplitter, and the average intensity difference between the two output beams is measured. A simple classical optics calculation tells us that the intensities at the output ports may be written in terms of the input intensity $I_{a_0}$ and the relative phase $\varphi$ as

$$I_{a_2} = I_{a_0} \sin^2(\varphi/2),$$

$$I_{b_2} = I_{a_0} \cos^2(\varphi/2).$$

(1)
This implies the intensity difference between the two output ports is \( M(\varphi) = I_b - I_a = I_{a_0} \cos \varphi \)—sinusoidal fringes that can be observed when the relative phase is varied.

The precision with which one can estimate an unknown relative phase based on the measurement of \( M \), in terms of the phase error, or the minimum detectable phase, \( \Delta \varphi \), may be determined to a good approximation using the following linear error-propagation formula:

\[
\Delta \varphi = \frac{\Delta M}{|dM/d\varphi|} = \frac{\Delta M}{I_{a_0} \sin \varphi}.
\]  

(2)

The above equation suggests that at a local value of phase \( \varphi = \pi/2 \), the precision of phase estimation can be made arbitrarily small by measuring the intensity difference \( M \) with infinite precision, and further by making the input intensity \( I_a \) arbitrarily large. However, quantum mechanics rules out the possibility of measuring intensities with infinite precision, i.e., with \( \Delta M = 0 \). This is because photon detection is intrinsically a quantum phenomenon, where what is actually measured is not a continuously varying intensity signal, but rather the discrete number of quanta of energy, or photons, that are absorbed by the detector. This absorption process is inherently stochastic due to the vacuum fluctuations of the quantized electromagnetic field, and in the case of the coherent laser light the photon numbers detected obey a Poisson distribution. This limits the precision of phase estimation in classical interferometry to \( \Delta \varphi \sim 1/\sqrt{\pi} \), where \( \pi \) is the intensity of the input laser beam.

B. Quantum optical interferometry

In order to give a fully quantum treatment of two-mode optical interferometry, we now introduce a quantized mode of the electromagnetic field and describe its various states. We then discuss the most relevant linear optical transformations for a pair of independent photonic modes in interferometry, namely the beam splitter and phase shifter transformations. This is followed by considering Hermitian operators as measurement observables at the output of the interferometer, and the calculation of precision of phase estimation based on the error propagation formula.

1) Quantum states of a single mode of the electromagnetic field: A quantized mode of the electromagnetic field is completely described by its creation and annihilation operators, \( \hat{a} \) and \( \hat{a}^\dagger \), which satisfy the commutation relation \([\hat{a}, \hat{a}^\dagger] = 1\). They are defined by their action on the number states of the mode, \(|n\rangle\)—also called Fock states, given by:

\[
\hat{a}|n\rangle = \sqrt{n}|n-1\rangle, \quad \hat{a}^\dagger |n\rangle = \sqrt{n+1}|n+1\rangle.
\]  

(3)

Pure states of the single-mode field (vectors in Hilbert space) can be expressed in terms of the action of a suitable function of the mode creation and annihilation operators on the vacuum state \(|0\rangle\). For example, a Fock state \(|n\rangle\) can be expressed as \( \frac{\hat{a}^\dagger^n|0\rangle}{\sqrt{n!}} \), where \(|0\rangle\) is the vacuum state. The coherent state can be written as:

\[
|\alpha\rangle = \hat{D}(\alpha)|0\rangle,
\]  

(4)

where \( \hat{D}(\alpha) = \exp(\alpha \hat{a}^\dagger - \alpha^* \hat{a}) \) is called the displacement operator, and \( \alpha \) is a complex number that denotes the coherent amplitude of the state. Both the Fock states and the Coherent states form complete bases (the coherent states in fact form an over-complete basis). Therefore, any pure state of the quantum single-mode field can be expressed in terms of these states. More generally, any state of the single-mode field, including mixed states, which are ensembles of pure states, can be written in terms of these states in the form of a density operator. Density operators are positive semi-definite trace one operators. The positive semi-definiteness condition enforces that the eigenvalues of the state are non-negative real numbers, so that they can be interpreted as valid probabilities. The trace one condition further ensures that these probabilities sum up to one, thereby making the state a normalized state. For example, the most general state of a single-mode field can be written in the Fock basis as the following density operator:

\[
\hat{\rho} = \sum_{n,n'} p_{n,n'}|n\rangle\langle n'|, \quad \text{Tr}(\rho) = 1, \quad \rho \geq 0.
\]  

(5)

Alternatively, a quantized mode can be described in terms of quasi-probability distributions in the phase space of eigenvalues \( x \) and \( p \) of the quadrature operators of the mode \( \hat{x} \) and \( \hat{p} \). These operators are defined in terms of the creation and annihilation operators of the mode as \( \hat{x} = \hat{a}^\dagger + \hat{a} \) and \( \hat{p} = i(\hat{a}^\dagger - \hat{a}) \), respectively. The Wigner distribution of a single-mode state can be obtained from its density operator of the form in (5) as:

\[
W(\alpha) = \frac{1}{2\pi^2} \int d^2\alpha \text{Tr} \left\{ \hat{\rho} \hat{D}(\alpha) \right\} e^{-\alpha \alpha^* - \alpha^* \alpha},
\]  

(6)

where \( \hat{\alpha} = \hat{x} + i\hat{p} \) and \( \alpha = x + ip \).

2) Quantum states and dynamics in the Mach-Zehnder interferometer: In the quantum description of the Mach-Zehnder interferometer (MZI), we associate mode creation and annihilation operators with each of the two modes. Here, we call them \( \hat{a}_i, \hat{a}_i^\dagger \) and \( \hat{b}_i, \hat{b}_i^\dagger \), \( i \in \{0, 1, 2\} \), where the different values of \( i \) refer to the modes at the input, inside, and output of the interferometer. The two modes of an MZI could be spatial modes or polarization modes.

Consider the propagation of the input quantum states of the two modes through the different linear optical elements present in the MZI. In the so-called Heisenberg picture, the propagation can be viewed as a transformation of the mode operators via a scattering matrix \( M_i \):

\[
M_i = \begin{bmatrix} 0_a & 0_b \\ b_0 & 0_a \end{bmatrix} = M_i^{-1} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}.
\]  

(7)

The scattering matrices corresponding to a 50:50 beam splitter and a phase shifter are given by

\[
\hat{M}_{BS} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}, \quad \hat{M}_\varphi = \begin{bmatrix} 1 \quad e^{-i\varphi} \\ 0 \quad 0 \end{bmatrix},
\]  

(8)

respectively. (Note that this form for \( \hat{M}_{BS} \) holds for beamsplitters that are constructed as a single dielectric layer, in which case the reflected and the transmitted beams gather a relative
The relation between the complex variables is similarly given by
\[ M_{\text{MZI}} = M_{\text{BS}} M_{\varphi} M_{\text{BS}} \]
and is found to be:
\[ \hat{M}_{\text{MZI}} = i e^{-i \frac{\varphi}{2}} \left[ \begin{array}{cc} \sin \left( \frac{\varphi}{2} \right) & \cos \left( \frac{\varphi}{2} \right) \\ \cos \left( \frac{\varphi}{2} \right) & -\sin \left( \frac{\varphi}{2} \right) \end{array} \right] \tag{9} \]
(Note that the overall scattering matrix has been suitably renormalized.)

In terms of phase space quasi-probability distributions such as the Wigner distribution function, the propagation through the Mach-Zehnder interferometer can be similarly described by relating the initial complex variables in the Wigner function to their final expressions as:
\[ W_{\text{out}}(\alpha, \beta) = W_{\text{in}}[\alpha_0(\alpha, \beta), \beta_0(\alpha, \beta)] \tag{10} \]
The relation between the complex variables is similarly given in terms of the two-by-two scattering matrices \( M \):
\[ \left[ \begin{array}{c} \alpha_0 \\ \beta_0 \end{array} \right] = \hat{M}^{-1} \left[ \begin{array}{c} \alpha_1 \\ \beta_1 \end{array} \right] \tag{11} \]
\( \alpha_0, \beta_0, \alpha_1, \) and \( \beta_1 \) being the complex amplitudes of the field in the modes \( a_0, b_0, a_1, \) and \( b_1 \), respectively. The approach based on phase space probability distributions is particularly convenient and powerful when dealing with Gaussian states, namely, states that have a Gaussian Wigner distribution, and Gaussian operators \[ \{45\}. \] Examples include the coherent state, the squeezed vacuum state and the thermal state \[ \{49\}. \] This is due to the fact that a Gaussian distribution is completely described by its first and second moments, and there exist tools based on the algebra of the symplectic group that can be used to propagate the mean and covariances of Gaussian states of any number of independent photonics modes.

3) **The Schwinger model:** The Schwinger model presents an alternative way to describe quantum states and their dynamics in a MZI \[ \{50\}. \] The model is based on an interesting relationship between the algebra of the mode operators of two independent photonic modes and the algebra of angular momentum.

Consider the following functions of the mode operators of a pair of independent photonic modes, say, \( a_1, a_1^\dagger, b_1, \) and \( b_1^\dagger \):
\[ \hat{J}_x = \frac{1}{2}(\hat{a}_1^\dagger \hat{b}_1 + \hat{b}_1^\dagger \hat{a}_1), \quad \hat{J}_y = \frac{1}{2i}(\hat{a}_1^\dagger \hat{b}_1 - \hat{b}_1^\dagger \hat{a}_1), \quad \hat{J}_z = \frac{1}{2}(\hat{a}_1^\dagger \hat{a}_1 - \hat{b}_1^\dagger \hat{b}_1) \tag{12} \]
(Note that we have chosen the mode index “1”, which in the MZI of Fig. 2 corresponds to the modes past the first beamsplitter. This is relevant when we discuss operational usefulness of the \( \hat{J}_q \) operators shortly.) They can be shown to obey the SU(2) algebra of angular momentum operators, namely, \[ \{\hat{J}_q, \hat{J}_r\} = i \hbar \epsilon_{q,r,s} \hat{J}_s, \] where \( \epsilon \) is the antisymmetric tensor and where \( q, r, s \in \{x, y, z\}. \) Based on this relation, a two-mode \( N \)-photon pure state gets uniquely mapped on to a pure state in the spin-\( N/2 \) subspace of the angular momentum Hilbert space, i.e.,
\[ |n_a, n_b\rangle \rightarrow |j = \frac{n_a + n_b}{2}, m = \frac{n_a - n_b}{2}\rangle. \tag{13} \]

The propagation of the quantized single-mode field can be viewed in terms of the Schwinger representation as a SU(2) group transformation generated by the angular momentum operators \( \hat{J}_x, \hat{J}_y \) and \( \hat{J}_z \). For example, the beamsplitter transformation of \[ \{8\} \] can be written as:
\[ \left[ \begin{array}{c} \hat{a}_0 \\ \hat{b}_0 \end{array} \right] = U_{\text{BS}} \left[ \begin{array}{c} \hat{a}_1 \\ \hat{b}_1 \end{array} \right] U_{\text{BS}}^\dagger, \tag{14} \]
where \( U_{\text{BS}} = \exp(i \varphi \hat{J}_y) \), and likewise, the transformation due to the phase shifter inside the interferometer can be described as
\[ \left[ \begin{array}{c} \hat{a}_1 \\ \hat{b}_1 \end{array} \right] \rightarrow U_{\varphi} \left[ \begin{array}{c} \hat{a}_1 \\ \hat{b}_1 \end{array} \right] U_{\varphi}^\dagger, \tag{15} \]

Using the SU(2) algebra of the angular momentum operators and the Baker-Hausdorff lemma \[ \{51\}, \] the overall unitary transformation corresponding to the MZI can be expressed as \( \hat{U}_{\text{MZI}} = \exp(-i \varphi \hat{J}_y) \). Operationally, for any given two-mode state, the operator \( \hat{J}_z \) tracks the photon number difference between the two modes inside the interferometer (which is \( \propto \hat{a}_1^\dagger \hat{a}_1 - \hat{b}_1^\dagger \hat{b}_1 \)). Similarly, it can be shown using the SU(2) commutation relations that the operators \( \hat{J}_x \) and \( \hat{J}_y \) track the photon number differences at the input (which is \( \propto \hat{a}_0^\dagger \hat{a}_0 - \hat{b}_0^\dagger \hat{b}_0 \)) and the output (which is \( \propto \hat{a}_2^\dagger \hat{a}_2 - \hat{b}_2^\dagger \hat{b}_2 \)), respectively.

4) **Measurement and phase estimation:** After propagating the two-mode quantum state through the MZI, we measure the output state (most generally a density operator \( \hat{\rho} \)) using a suitable Hermitian operator \( \hat{O} \) as the measurement observable. For example, the measurement observable corresponding to the intensity difference detection of the conventional MZI described in Fig. 2 is the photon number difference operator \( \hat{O} = \hat{b}_2^\dagger \hat{b}_2 - \hat{a}_2^\dagger \hat{a}_2 \). Another interesting detection scheme that has been found to be optimal for many input states is the photon number parity operator \[ \{19\}, \{22\} - \{24\} \] of one of the two output modes, e.g., the parity operator of mode \( \hat{a}_2 \) is given by \( \hat{P} = (-1)^{\hat{a}_2^\dagger \hat{a}_2} \). The measured signal corresponding to any observable \( \hat{O} \) is given by \( \langle \hat{O} \rangle = \text{Tr}(\hat{O} \hat{\rho}) \). Further, the precision with which the unknown phase \( \varphi \) can be estimated using the chosen detection scheme, to a good approximation, is given using the error propagation formula as
\[ \Delta \varphi = \frac{\Delta \hat{O}}{d \langle \hat{O} \rangle / d \varphi}, \tag{16} \]
where \( \Delta \hat{O} = \sqrt{\langle \hat{O}^2 \rangle - \langle \hat{O} \rangle^2} \).

As an example, consider the coherent light interferometer of Fig. 2. The output state is determined using the scattering matrix of \( \{9\} \) as
\[ |\alpha\rangle |0\rangle \rightarrow |i \alpha \sin(\varphi/2)e^{-i\varphi/2}\rangle |i \alpha \cos(\varphi/2)e^{-i\varphi/2}\rangle. \tag{17} \]
The output signal for the measurement operator $\hat{O} = \hat{b}_1^\dagger \hat{b}_2 - \hat{a}_1^\dagger \hat{a}_2$ corresponding to intensity difference detection is

$$\langle \hat{O} \rangle = |\alpha|^2(\cos^2(\varphi/2) - \sin^2(\varphi/2)) = |\alpha|^2 \cos \varphi, \quad (18)$$

which matches the classical result. The second moment $\langle \hat{O}^2 \rangle$ for the output state is $|\alpha|^4 \cos^2 \varphi + |\alpha|^2$, with which we can then ascertain the precision of phase estimation possible with the coherent light interferometer and intensity difference measurement to be

$$\Delta \varphi = \sqrt{\frac{|\alpha|^4 \cos^2 \varphi + |\alpha|^2 - |\alpha|^4 \cos^2 \varphi}{|\alpha|^2 \sin \varphi}} = \frac{1}{\sqrt{\pi} \sin \varphi}, \quad (19)$$

where $\pi$ is the average photon number of the coherent state. Say the unknown phase $\varphi$ is such that $\varphi - \theta$ is a small real number, where $\theta$ is a control phase. Then, the precision is optimal when $\theta$ is an odd multiple of $\pi/2$, attaining $\Delta \varphi = 1/\sqrt{\pi}$, which is the quantum shot-noise limit.

In the above, it is possible to get rid of the dependence on the actual value of phase by considering the fringe visibility observable $\nu$. The visibility observable accomplishes this by keeping track of not only the photon number difference, but also the total photon number observed.

### C. Quantum parameter estimation

Although the linear error propagation formula described in [16] provides a good approximation for the precision of estimation of the unknown phase $\varphi$ of an optical interferometer, in order to determine the absolute lower bound on precision that is possible in a given interferometric scheme, one has to resort to the theory of parameter estimation. We now briefly review the quantum theory of parameter estimation. There exist two main paradigms in parameter estimation, (i) where the unknown parameter is assumed to hold a deterministic value, (ii) where the unknown parameter is assumed to be intrinsically random. Here, we focus on the first one.

Consider $N$ identical copies of a quantum state that has interacted with the unknown parameter of interest and holds information about it. Since the state carries the information about the parameter of interest, say $\varphi$, let us denote it as $\hat{\varphi}_x$. Now, consider a set of data points $x = \{x_1, x_2, \ldots, x_n\}$ that are obtained from the $N$ copies of $\hat{\varphi}_x$ as outcomes of a generalized quantum measurement. The generalized measurement is a positive operator-valued measure (POVM), which is a collection of positive operators $\Lambda_\mu$, with the index $\mu \in \{1, 2, \ldots, M\}$ denoting the outcome of the measurement, whose probability of occurrence for a state $\rho$, is given by $p(\mu) = \text{Tr}\{\rho \Lambda_\mu\}$. The elements of a POVM add up to the identity $\sum_\mu \Lambda_\mu = I$, which ensures that $p(\mu)$ is a valid probability distribution. Since the data points are obtained by measuring identical copies of the quantum state, the $x_i$s are realizations of independent and identically distributed random variables $X_i$, $i \in \{1, 2, \ldots, \nu\}$ that are distributed according to some probability distribution function $p_\varphi(X)$. The goal is to apply a suitable estimation rule $\hat{\varphi}_x$ to the data points, to obtain a good estimate for the unknown parameter $\varphi$.

When estimation rule $\hat{\varphi}_x$ is applied to a set of data points $x$, a good measure of precision for the resulting estimate $\hat{\varphi}_x(x)$ is its mean-square error, given by:

$$\Delta^2 \hat{\varphi}_x = E[(\hat{\varphi}_x(x) - \varphi)^2], \quad (20)$$

where $E$ denotes expectation value. For any estimation rule $\hat{\varphi}_x$, which is unbiased, i.e.,

$$E[\hat{\varphi}_x(x)] = \varphi, \quad (21)$$

the Cramer-Rao theorem of classical estimation theory lower bounds the mean-square error as

$$\Delta^2 \hat{\varphi}_x \geq \frac{1}{\nu F(p_\varphi)}, \quad (22)$$

where $F(p_\varphi)$ is known as the Fisher information of the probability distribution given by

$$F(p_\varphi) = F_{Cl}(\hat{\varphi}_\varphi; \Lambda_\mu) = \mathbb{E}\left[\frac{d^2}{d\varphi^2} \log p_\varphi\right]. \quad (23)$$

The above lower bound is called the classical Cramer-Rao bound. It gives the optimal precision of estimation that is possible when both the parameter-dependent quantum state and the measurement scheme are specified. Estimation rules that attain the classical Cramer-Rao bound are called efficient estimators. The maximum-likelihood estimator is an example of an efficient estimator that attains the lower bound in the asymptotic limit.

The quantum theory of parameter estimation further provides an ultimate lower bound on precision of estimation when the quantum state alone is specified. It goes by the name of quantum Cramer-Rao bound, and is given by

$$\Delta^2 \hat{\varphi}_x \geq \frac{1}{\nu F_Q(\hat{\varphi}_\varphi)}, \quad (24)$$

where $F_Q(\hat{\varphi}_\varphi)$ is known as the quantum Fisher information, which is defined as the optimum of the classical Fisher information over all possible generalized measurements:

$$F_Q(\hat{\varphi}_\varphi) = \max_{\Lambda_\mu} F_{Cl}(\hat{\varphi}_\varphi; \Lambda_\mu). \quad (25)$$

A measurement scheme that attains this lower bound is called an optimal measurement scheme. The symmetric logarithmic derivation operation is one such measurement, which is known to be optimal for all quantum states $|\varphi\rangle$.

In the case of entangled pure states, the quantum Fisher information takes the simplified expression given by

$$F_Q = 4\Delta^2 H,$$

where $\hat{H}$ is the generator of parameter evolution. This gives rise to a generalized uncertainty relation between the generating Hamiltonian of parameter evolution and the estimator that is used for estimating the unknown value of the parameter, given by

$$\Delta^2 \hat{\varphi}_x \Delta^2 H \geq \frac{1}{4\nu} \Delta^2 \hat{\varphi}_x, \quad (26)$$
for a generating Hamiltonian $\hat{H}$, and where $\nu$ is the number of data points gathered from measuring identical copies of the state.

III. QUANTUM OPTICAL METROLOGY

Having discussed the necessary tools, we now describe quantum optical metrology. We begin the section with a brief account of the nonclassical effects that form the source of the quantum advantage in optical metrology, namely entanglement and squeezing.

A. Entanglement

Quantum mechanics allows for correlations between physical systems beyond those allowed in classical physics. Entanglement [56] is the most prominent manifestation of such quantum correlations. Entanglement is considered by many as the hallmark feature of quantum mechanics, and is widely believed to be the source of the quantum advantage over classical techniques in quantum information processing technologies such as quantum computing, communication and cryptography. A quantum state is said to be entangled if it is anything but a separable state. For example, in the bipartite case (i.e. when there are two subsystems, say $A$ and $B$), separable states are of the form

$$\hat{\rho}_{AB} = \sum_x p(x) \hat{\rho}_A^x \otimes \hat{\rho}_B^x, \quad p(x) \geq 0, \sum_x p(x) = 1, \quad (27)$$

where $\hat{\rho}_A^x$ and $\hat{\rho}_B^x$ are density operators. Entangled states can be made to violate a Bell’s inequality, the latter being bounds on correlations possible in classical, local hidden-variable theories [57].

Entanglement is also thought to be the driving force behind the enhancements possible in quantum metrology over classical approaches. The quantum Fisher information of $N$ independent probes in a separable state, i.e., without quantum entanglement, cannot exceed $N$. Since this value of the quantum Fisher information corresponds to precision at the shot-noise limit based on (25), thus, separable states cannot beat the shot-noise limit. On the other hand, the quantum Fisher information of entangled states can exceed this bound. In fact it has been shown that the Fisher information of a $N$-particle state being greater than $N$ is a sufficient condition for multipartite entanglement [6]. Entangled states are therefore capable of achieving sub-shot-noise precision. However, it is important to note that the presence of entanglement is a necessary, but not sufficient condition for achieving sub-shot-noise precisions.

In other words, not all entangled states offer a quantum enhancement to precision metrology [59]. When the generator of parameter evolution $\hat{H}$ is linear in the number of probes, according to (26), the quantum Fisher information of a state containing $N$ probes can at best attain a value of $N^2$, which corresponds to the Heisenberg limit in the precision of parameter estimation.

In two-mode optical interferometry, e.g., of the type in Fig. (3), the relevant type of entanglement to consider is entanglement between the two modes past the first beamsplitter, namely $a_1$ and $b_1$. The most well-known mode entangled states are the $N00N$ states \([N :: 0]_{a_1,b_1} = \frac{1}{\sqrt{2}}(|N\rangle_{a_1}|0\rangle_{b_1} + |0\rangle_{a_1}|N\rangle_{b_1})\), \((28)\) where $a_1$ and $b_1$ denote the two modes past the first beamsplitter. The $N00N$ has a quantum Fisher information of $N^2$ and hence is capable of achieving the Heisenberg limit in phase estimation. It is known that both the photon number difference operator and the photon number parity operator are optimal for Heisenberg-limited phase estimation with the $N00N$ states \([25], [60]\). Another example of finite photon number states that are known to be capable of Heisenberg-limited precision are the Holland-Burnett states $|N\rangle_{a_1}|N\rangle_{b_1}$, which result in a mode-entangled state inside the interferometer.

B. Squeezed light

In the indefinite photon number (continuous variable) regime, entanglement is intimately connected to another nonclassical effect—squeezing. Squeezed light [49] refers to minimum uncertainty states of light whose fluctuations with respect to one of any two orthogonal quadratures in phase space has been reduced at the expense of increased fluctuations in the other. They are described mathematically using the squeezing operator. The single-mode squeezing operator acting on a mode $\hat{a}_0$ is given by:

$$\hat{S}(\xi) = \exp \left( \frac{1}{2} (\xi \hat{a}_0^2 - \xi^* \hat{a}_0^* \hat{a}_0) \right), \quad (29)$$

where $\xi = re^{i\theta}$, $r$ and $\theta$ being the squeezing parameter and squeezing angle, respectively. The squeezed vacuum state, which is the state corresponding to the action of the squeezing operator in (29) on the vacuum state is given by

$$|\xi\rangle = \hat{S}(\xi)|0\rangle = \sum_{m=0}^{\infty} \frac{\left(2m\right)!}{2^{2m} \left(m!\right)^2} \tanh^m r \cosh^2 r |2m\rangle. \quad (30)$$

It has a mean photon number of $\pi = \sinh^2 r$. There are numerous ways to generate squeezed light. The most common method is based on degenerate parametric down conversion using nonlinear crystals that contain second order ($\chi^{(2)}$) susceptibility. When a $\chi^{(2)}$ nonlinear crystal is pumped with photons of frequency $\omega_p$, some of these pump photons get converted into a pair of photons—of frequencies $\omega_p/2$, which are in the single-mode squeezed vacuum state of (30).

The connection between squeezing and entanglement is unveiled when two single-mode squeezed vacuum beams are mixed on a beam splitter of the type described in (3). The state that results past the beamsplitter is given by the two-mode squeezed vacuum state

$$|\xi\rangle = \hat{S}_2(\xi)|0\rangle_{a_1}|0\rangle_{b_1} (\xi = re^{i(\theta+\pi/2)})$$

$$= \frac{1}{\cosh r} \sum_{n=0}^{\infty} (-1)^n e^{in(\theta+\pi/2)} (\tanh r)^n |n\rangle_{a_1} |n\rangle_{b_1}, \quad (31)$$
where $\tilde{S}_2(\xi) = \exp \left( \xi \hat{a}_1^{\dagger} \hat{b}_1 - \xi^* \hat{a}_1 \hat{b}_1 \right)$ is the two-mode squeezing operator. This state is mode-entangled as the state of the two modes cannot be written in a separable form. The two-mode squeezing operator can itself also be implemented using a non-degenerate parametric conversion process, where once again a strong pump emitting photons at frequency $\omega_p$ interacts with a nonlinear crystal containing a second order nonlinearity, generating pairs of photons at frequencies $\omega_{a_1}$ and $\omega_{b_1}$, such that $\omega_{a_1} + \omega_{b_1} = \omega_p$.

The scheme of mixing coherent light with squeezed vacuum light, originally considered by Caves, also generates mode entanglement past the first beamsplitter. This provides an alternative explanation for the sub-shot-noise phase precision capabilities of the scheme. We now describe the interferometry with coherent light mixed with squeezed vacuum light in a bit more detail, and show its stronger connection to maximal mode-entanglement of the type present in the $N00N$ states.

C. Coherent-mixed with squeezed vacuum light interferometry

The interferometry based on mixing coherent light and squeezed vacuum light, as mentioned before, was where the possibility of sub-shot-noise phase estimation was originally unearthed. This scheme has been revisited recently. Hofmann and Ono [62] showed that when these inputs are mixed in equal intensities, namely such that $\sinh^2 r = |\alpha|^2 = \pi/2$ (for any value of average photon number $\bar{n}$), then the state that results past the mixing splitter is such that each $N$-photon component in the state has a fidelity higher than 90% with the corresponding $N00N$ state. Therefore, this scheme has been widely used to generate $N00N$ states in experiments. Afek et al. [29], [63] used this scheme to thereby generate $N00N$ states of up to $N = 5$; the state of the art in the generation of such states. Fig. 3 shows a schematic of this interferometry, photon number distribution of an $N$-photon component in the generated state, its $N00N$ fidelity, and the observed $N$-fold coincidence fringes.

Further, Pezze and Smerzi [64] calculated the classical Cramer-Rao bound for the interferometer with coherent light and squeezed vacuum light along with photon number detection at the output, and found it to be:

$$F_{C1} = |\alpha|^2 e^{2r} + \sinh^2 r.$$  \hspace{1cm} (32)

When the average photon numbers of the two inputs are about the same, i.e., $\sinh^2 r = |\alpha|^2 = \pi/2$, the classical Fisher information is approximately $\bar{n}^2 + \bar{n}/2$, which results in Heisenberg scaling for the phase precision, namely $\Delta \phi = 1/(\sqrt{\bar{n}})$, where $\bar{n}$ is the number of data points gathered from measurement identical copies of the state. It has been shown that photon number parity also attains the same phase precision at the Heisenberg scaling in this interferometry [53].

IV. QUANTUM TECHNOLOGIES WITH ENTANGLED PHOTONS

In this section, we review some recent experiments that have demonstrated the enhanced sensing and imaging capabilities of entangled photons in interferometry with the $N00N$ states.

In particular, these experiments focus on the small photon-number regime, which is relevant for sensing and imaging delicate material systems such as biological specimen, single molecules, cold quantum gases and atomic ensembles. We also discuss a recent experiment based on the $N00N$ state for enhanced spatial resolution for applications in quantum lithography.

A. Quantum metrology and sensing

Several experiments based on the $N00N$ states have demonstrated phase estimation beyond the shot-noise limit, and achieving the Heisenberg limit. Here, we briefly mention about two experiments, which have used $N00N$ states to measure useful quantities mapped on to the optical phase under realistic conditions of photon loss and other decoherence. The first one is by Crespi et al. [11] (see Fig. 4), where $N = 2 N00N$ states.
were used to measure the concentration of a blood protein in an aqueous buffer solution. The experiment used an opto-fluidic device, which consists of a waveguide interferometer whose one arm passes through a microfluidic channel containing the solution. The concentration-dependent refractive index of the solution causes a relative phase shift between the two arms of the interferometer, which is then detected using coincidence photon number detection. The $N = 2 N_{00}N$ states were generated using Hong-Ou-Mandel interference with entangled photon pairs from a parametric down conversion source. At the output, an array of telecommunication optical fibers were used to collect the photons, which were then detected with coincidence detection using four single-photon avalanche photo-diodes. The photons were detected with a fringe visibility of about 87% in the case where the microchannel had a transmissivity of only about 61% due to photon loss. The experiment achieved a sensitivity below the shot-noise limit.

In another experiment, Wolfgramm et al. [10] used $N = 2$ polarization $N_{00}N$ states and Faraday rotation to probe a Rubidium atomic spin ensemble in a non-destructive manner. Atomic spins ensembles find application in optical quantum memory, quantum-enhanced atom interferometry, etc. Such atomic spin ensembles, when interacted with via optical measurements, e.g., to store or readout quantum information in a quantum memory or to produce spin-squeezing in atom interferometry, inherently suffer from scattering induced depolarization noise. Also, there is photon loss due to the scattering of the optical probes off the ensemble. In order to minimize loss, the experiment generated narrowband $N_{00}N$ states of about 90% fidelity and purity, at a frequency detuned four Doppler widths from the nearest Rb-85 resonance containing matter-resonant indistinguishable photons. The photons at the output were detected using condense photon number detection with a fringe visibility of > 90%. The experiment achieved a sensitivity that was five standard deviations better than the shot-noise limit.

### B. Quantum imaging

Another important application of optical phase measurement is that of microscopy and imaging. In biology, the technique of differential interference contrast microscopy is widely used to image biological samples. The depth resolution of the images produced by this technique is related to the signal-to-noise ratio (SNR) of the measurement. In the case of classical laser-light-based imaging, for a given light intensity, this is limited by the shot-noise limit in phase precision. While one way to enhance the SNR is to raise the illumination power, this might have undesirable effects on delicate, photosensitive samples such as biological tissues, ice crystals, etc. Quantum metrology, however, can provide an enhancement to the SNR without having to increase the illumination power, and therefore could be of significant help in this scenario. In one of the first works on the use of quantum metrology for phase imaging, Brida et al. [65] showed that entangled photon pairs can provide sub-shot-noise imaging of absorbing samples. Later, Taylor et al. [66] showed that squeezed light could be used to achieve sub-shot-noise sensitivities in micro-particle tracking, with applications in tracking diffusive biological specimen in realtime.

We describe two recent experiments that have used entangled photons for phase super-sensitive imaging. The first one is by Ono et al. [47], where $N = 2 N_{00}N$ states were used in a laser confocal microscope in conjunction with a differential
interference contrast microscope (LCM-DIM) to demonstrate quantum-enhanced microscopy. Figures 5 and 6 show their experimental setup and results, respectively. The LCM-DIM works based on polarization interferometry, where the $H$ and $V$ modes are separated using a polarization beamsplitter or a Nomarski prism, and made to pass through different spatial parts of the sample. These modes, depending on the local refractive index and the structure of the sample, experience different phase shifts, whose difference is then measured at the output. The experiment used $N = 2$ polarization $N00N$ states generated via Hong-Ou-Mandel interference with about 98% fidelity, and the photons were detected at the output using photon number parity measurement with a fringe visibility of about 95%. The microscope attained an SNR 1.35-times better than the shot noise limit.

In another experiment, Israel et al. [45] used $N = 2$ and $N = 3$ $N00N$ states in quantum polarization light microscopy (QPLM) to image a quartz crystal. In QPLM, a birefringent sample causes the $H$ and $V$ modes to experience a differential phase shift, which is then measured at the output to image the sample. The $N00N$ states for the experiment were generated from the mixing of coherent light and squeezed vacuum light in equal intensities discussed in Section III-C, and the photons were detected using an array of single photon counting modules. The experiment achieved quantum-enhanced imaging with sensitivities close to the Heisenberg limit.

C. Quantum lithography

Lithography relies on the creation and detection of spatial interference fringes to etch ultra fine features on a chip. While classical light lithography is limited by the Rayleigh diffraction limit, as mentioned before, the $N00N$ states can beat this limit—a result known as super-resolution [16]–[18]. A few independent experiments with $N = 2$ $N00N$ states had earlier demonstrated this result, however, it was realized that the $N00N$ state lithography suffers from the problem that the efficiency of detection $N$ photons in the same spatial location decreases exponentially in $N$.

In a new theoretical development, a counter-measure was suggested based on the optical centroid measurement [67], which does not require all the photons to arrive at the same spatial point. The optical centroid measurement is based on an array of detectors that keep track of every $N$ photon detection event irrespective of which detectors fired. Then the average position of the photons is obtained via post-processing. In a recent experiment, using $N = 2, 3, 4$ $N00N$ states and the optical centroid measurement, Rozema et al. [46] for the
first time demonstrated a scalable implementation of quantum super-resolving interferometry with a visibility of interference fringes nearly independent of \( N \).

V. DISCUSSION

In this paper, we presented a brief overview of quantum optical metrology, with an emphasis on quantum technologies that have been demonstrated with the \( N^00N \) states. We introduced some basic concepts of quantum interferometry and quantum parameter estimation, and the notions of entanglement and squeezing. We then presented an interferometric scheme based on the mixing of coherent and squeezed vacuum light on a beam splitter, which generates an output whose \( N \)-photon components are approximately \( N^00N \) states. We then presented some state-of-the-art experiments that use \( N^00N \) states generated using this technique (or other techniques in some cases) for technological applications such as quantum-enhanced biosensing, imaging, and spatial resolution lithography.

This review is by no means representative of everything there is to quantum optical metrology. For a comprehensive review of the field, please refer [68], [69]. We did not discuss the approach to quantum metrology via the Bayesian method [70], where the unknown parameter is assumed to be inherently random, and thus distributed according to an unknown probability distribution. Optimal states in this paradigm have also been identified, and adaptive protocols have been designed, which implement the optimal measurements for such states based on measurement settings that continually changed based on the results previously obtained [26], [28], [30], [35]. Also, we did not go into the details of how the effects of photon loss and decoherence such as collective dephasing noise due to the thermal motion of optical components or laser noise, etc, are handled in optical metrology. A large body of work in the recent literature has dealt with identifying useful lower bounds on phase precision in the presence of such decoherence [40], [42]. Further, optimal quantum states of light that attain these bounds in the presence of decoherence have been identified in the asymptotic limit of a large number of photons [59].

It must be mentioned here that the \( N^00N \) states are highly susceptible to photon loss, or other types of decoherence in the limit of a large photon number \( N \). Nevertheless, the use of the \( N^00N \) state in the experiments discussed here is justified, since the \( N^00N \) states still remain optimal for relatively small photon numbers. In fact, in noisy, decoherence-ridden interferometry, given a large finite photon number constraint, it has been recently shown that the best strategy for phase estimation is to divide the total number of photons into smaller independent packets or “clusters”, where each cluster is prepared in a \( N^00N \) state [59]. These clusters are then to be sent through the interferometer one at a time. (The optimal size of the clusters will depend on the decoherence strength inside the interferometer.) In addition, alternatives to the \( N^00N \) states, of the form \(|N - M\rangle\langle N - M| + |N - M\rangle\langle M|/2\), where \( M \leq N \) (also known as the \( m^N \) states) have been proposed for interferometry in the presence of photon loss [71], [72]. Such states, when suitably chosen, offer the same benefits as the \( N^00N \) states, while being more robust against photon loss than the latter. Both the \( N^00N \) states and the \( m^N \) states of a moderate number of photons, have been shown to perform optimally in the presence of collective dephasing noise [73]. Therefore, the \( m^N \) states may provide a way to perform quantum metrology in the presence of both photon loss and collective dephasing noise in suitable regimes of photon numbers and the decoherence parameters.

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