Locally optimal measurement-based quantum feedback with application to multi-qubit entanglement generation

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(Dated: July 6, 2018)

We present a general approach to measurement-based quantum feedback that employs independent variation of quantum state-based and proportional (PaQS) feedback components to obtain locally optimal feedback protocols. To demonstrate the power of the method, we apply it to generation of high-fidelity multipartite entanglement with an emphasis on remote entanglement, requiring spatially local feedback Hamiltonians. The symmetry of both measurement and feedback operators is found to be essential for construction of effective protocols. We show that entangled states can be reached with fidelity approaching unity under non-Markovian feedback control protocols, while Markovian protocols resulting from optimizing the feedback unitaries on ensemble averaged states still yield fidelities above 94%. Application of the PaQS approach to generation of N-qubit W, Dicke and GHZ states shows that such entangled states can be efficiently generated with high fidelity, for up to N = 100 in some cases.

Entanglement [1] is a crucial resource for quantum information science, with applications in secure cryptography [2], long-range quantum state transfer [3], quantum computation [4], quantum-enhanced sensors [5–7] and quantum simulation [8]. Many applications, particularly large-scale quantum information processing, will require modular quantum devices that are able to talk to each other [9, 10]. However, the objects that we would like to entangle often have little or no direct interaction to enable entanglement generation, either because they are separated by significant distances, or because of intrinsically weak physical interactions. The former is the case in most quantum networking applications, as well as in nitrogen vacancy centers, for which large inter qubit spacing is necessary to maintain coherence properties [11]. The latter is the case for dilute neutral atoms, which have received considerable interest for generation of spin-squeezed states involving large numbers of entangled atoms [12, 13].

When no direct interaction is available, joint measurement on all parties offers a practical method to project the total system into a desired entangled state. This has been demonstrated by interfering spontaneously emitted optical photons emitted from (artificial) atomic systems [14–17], and also by performing cavity-QED-based joint measurements on superconducting qubits [18] or cold neutral atoms [19].

However, the stochastic nature of a quantum measurement prevents such an approach from succeeding with unit probability target [20]. In order to improve the probability of reaching an entangled state, or to obtain it deterministically, one can then add control via feedback unitary operations. Several experimental and theoretical works have shown that this approach can yield deterministic remote entanglement generation between two qubits [21, 23]. For multi-partite entanglement generation, Ref. [13] experimentally demonstrated a feedback strategy for deterministically preparing an ensemble of atoms near the maximally spin-squeezed subspace. Previously, Refs. [24], [12], and [25] gave state-based i.e., non-Markovian protocols for deterministic preparation of Dicke states. Generation of multi-spin states has also been discussed in the theoretical control literature in the context of state stabilization [26–28]. In the conclusion of Ref. [12], the authors emphasize both the experimental difficulty of implementing non-Markovian protocols and the importance of constructive methods to derive feedback control laws, but leave these issues as open problems for future research.

In the current work, we address both of these technical issues by deriving a general method for constructing locally optimal feedback protocols. Our approach is based on introduction of time-dependent feedback unitaries composed of two independently varying components, one of which depends linearly on the measurement outcome with a state-dependent coefficient, while the other depends only on the quantum state. We term this approach “quantum state and proportional feedback” (PaQS feedback), to emphasize the increased flexibility offered by these two independent feedback components. We also show how this construction can be modified to guarantee Markovianity, meaning that each feedback operation only depends on the immediately proceeding measurement outcome, which greatly simplifies experimental implementation. Although the result is applicable to any system in which measurement-based feedback is possible, we demonstrate this formalism by giving a systematic treatment of entanglement generation in three-qubit systems and beyond, emphasizing the crucial role of symmetry in achieving high fidelity. We shall consider here generation of both the N-qubit GHZ states and the full range of N-qubit symmetric Dicke states, from W states to half-filled states, which are maximally spin
squeezed. In most cases, our protocols are characterized by a high degree of symmetry, and derived using general analytic techniques that allow us study up to 100 qubits. These advantages offer remarkably simple experimental implementations of the feedback controller.

Our protocols rely on the fact that realistic measurements acquire only partial information over a finite time interval. This so-called continuous measurement formalism \cite{29} may in principle be applied to any system in which the measurement signal is collected with high efficiency, but has proven particularly useful for superconducting qubits \cite{18,23,32,34}. Feedback consists of applying additional control unitary operations conditional on the outcome of the continuous measurement. To maintain compatibility with the desired application of remote entanglement, we restrict the control unitaries to local rotations on each qubit.

The remainder of the paper is organized as follows. In section I we summarize the evolution of a general quantum state under continuous measurement and feedback as described by the stochastic master equations. Next, we develop a general formalism for efficiently computing locally optimal protocols in the context of any measurement-based feedback system. Section II and III then develop locally optimal protocols for generation of Dicke and GHZ states respectively, based on this formalism. An important result of section II is the construction of a general method for deriving locally optimal Markovian feedback protocols. In Section IV we provide a summary of the results and an outlook for future work. Key calculational details and supplementary materials are presented in the Appendices, along with an additional technique for deriving feedback protocols based on entanglement measures.

I. CONTINUOUS MEASUREMENT WITH LOCALLY OPTIMAL FEEDBACK

In this section, We first introduce the theoretical framework of continuous measurements with feedback. Then we will find the locally optimal feedback control which will be our main protocol to generate Dicke state and GHZ state in later sections.

A. Continuous measurement and feedback

We consider a joint measurement that is realized by an indirect simultaneous measurement on multiple qubits. Such measurements are routinely made for superconducting qubits in the dispersive regime, using homodyne detection of cavity transmission \cite{18,23,32,34}. For a perfect measurement, the readout voltage of the signal is given by

$$dV = \langle X \rangle (t) dt + \frac{dW}{\sqrt{8k}} \quad (1)$$

where $X$ is the measurement operator which will be specified in detail below, $\langle \cdot \rangle$ denotes the trace average of this, $dW$ is a Wiener increment satisfying $dW(t) dW(t') = \delta(t - t') dt$ that represents the quantum uncertainty in the homodyne detection (represented by white Gaussian noise in the continuous limit \cite{25}, \cite{36}) and $k$ is the measurement strength.

The evolution of the quantum state conditioned on this measurement signal is given by a stochastic master equation (SME), given by

$$d\rho = -\frac{i}{\hbar} [H_S, \rho] dt + 2k D[X] \rho(t) dt + \sqrt{2k H[X]} \rho dW \quad (2)$$

Here $H_S$ is the system Hamiltonian which is not relevant to this analysis and will henceforth be set to zero, $D[X] \rho = X \rho X^\dagger - \frac{1}{2} (X^\dagger X \rho + \rho X^\dagger X)$, and $H[X] \rho = X \rho + \rho X^\dagger - \langle X + X^\dagger \rangle \rho$. The second, deterministic, term of order $dt$ describes the dephasing effect of the measurement and is just the usual dissipator term in the Markovian master equation. It describes the unconditioned dynamics of an open system coupled to an external bath, i.e., the dynamics after averaging over all possible measurement records. The third, stochastic, term of order $dW$ continuously updates $\rho$ based on knowledge gained from the measurement.

To incorporate feedback into such a continuous measurement process, we select an infinitesimally small time interval, say from $t$ to $t + dt$. Evolution under the SME over a short period of time may be generally expressed via operators as

$$\rho_{t+dt}^M = \frac{\Omega_{dV} \rho_t \Omega_{dV}^\dagger}{Tr(\Omega_{dV} \rho_t \Omega_{dV}^\dagger)} \quad (3)$$

where the superscript $M$ denotes the state resulting from the measurement, and $\Omega_{dV}$ is a set of measure-
moment operators defining a positive operator valued measure (POVM) \( \sum_{n_1}^{N} \Omega_{AV} \) with every subset \([v_2, v_1]\) of the range of \( dV \). The complete set of POVM operators is only constrained by the normalization condition \( I = \int dV \Omega_{AV} \). One may recover the SME Eq. (2) from Eq. (3) by taking

\[
\Omega_{AV} = \left( \frac{4k}{\pi dt} \right)^{\frac{1}{4}} \exp \left[ -2kdt \left( \frac{dV}{dt} - X \right)^2 \right],
\]

(4)

where the prefactor is determined by the normalization.

As the qubits are assumed to be too remote to allow direct interactions, we construct a unitary feedback operator in the form of local rotations on each of the \( N \) qubits,

\[
U_F(\theta_1, \hat{n}_1; \theta_2, \hat{n}_2; \ldots; \theta_N, \hat{n}_N) = \bigotimes_{i=1}^{N} U(\theta_i, \hat{n}_i).
\]

(5)

Here \((\theta_i, \hat{n}_i)\) are the rotation angle parameters for the \( i \)th qubit and \( U(\theta, \hat{n}) = e^{-i\hat{n} \cdot \vec{\sigma}} \) is the rotation operator on a single qubit. \( \vec{\sigma} = (X, Y, Z) \) denotes the usual Pauli operators. These rotation angles will be determined based on the measurement result. For simplicity, we assume that these rotations are realized instantaneously.

After adding the feedback, the complete evolution of the qubits over a single infinitesimal cycle of measurement and feedback (see Fig. 1) is described by the following:

\[
\rho_{i+dt} = U_F \rho_i^{\text{M}} U_F^\dagger = U_F \frac{\Omega_{AV} \rho_i \Omega_{AV}^\dagger}{\text{Tr}(\Omega_{AV} \rho_i \Omega_{AV}^\dagger)} U_F^\dagger,
\]

(6)

where the superscript \( \text{c} \) indicates this state is obtained after the feedback control.

### B. Locally optimal control protocol

Now we are ready to state the locally optimal control problem in our case and find the solution. Suppose we have a state \( \rho_t \) at time \( t \), and that we ultimately wish to reach some target state \( |\psi_T\rangle \). In this work we shall focus primarily on the fidelity with respect to the target state as the cost function, whose optimization determines the parameter of the feedback operator. If we choose the feedback operator to take the generic form \( U_F(\theta) = e^{-i\theta H_F/\hbar} \), then the fidelity after feedback is given by

\[
\mathcal{F}_{t+dt}(\theta) = \langle \psi_T | U_F(\theta) (\rho_t + d\rho) U_F^\dagger(\theta) | \psi_T \rangle.
\]

(7)

For convenience, we shall set \( \hbar = 1 \) from now on. Our local optimality condition is given by

\[
\mathcal{G} \equiv \frac{\partial \mathcal{F}_{t+dt}(\theta)}{\partial \theta} = \langle \psi_T | \left[ U_F'(\theta) (\rho_t + d\rho) U_F^\dagger(\theta) + h.c. \right] | \psi_T \rangle
\]

(8)

where \( \rho^c = U_F(\theta)(\rho_t + d\rho)U_F^\dagger(\theta) \). To find an analytical form of the solution \( \theta^* \), we observe that \( d\rho \) is infinitesimal in the limit of \( dt \to 0 \). If as we shall assume \( \rho_t \) is already optimized from the previous time step, then typically \( \theta^* \) will be \( \mathcal{O}(d\rho) \) as well (we deal with possible exceptions below). We can therefore parameterize the rotation angle as

\[
\theta^* = A_1(t) dt + A_2(t) dt.
\]

(9)

Expanding \( U_F \) to second order in \( dW \) and making use of Ito’s lemma [36] yields

\[
U \equiv I - iA_1 H_F dW - (iA_2 H_F + A_2^2 H_F^2) dt.
\]

(10)

Substituting this in Eq. (8), together with the stochastic master equation Eq. (2) for \( d\rho_t \), leads to the following explicit form for \( \mathcal{G} \):

\[
\mathcal{G} = -i \langle \psi_T | [H_F, \rho_t] | \psi_T \rangle - i \langle \psi_T | [H_F, D[Y] \rho_t] dt + \mathcal{H}[Y] \rho_t dW | \psi_T \rangle - A_1 \langle \psi_T | [H_F, [H_F, \rho_t]] | \psi_T \rangle dW
\]

\[
- i A_1^2 \langle \psi_T | [H_F, D[H_F] \rho_t] | \psi_T \rangle dt - \langle \psi_T | [H_F, [H_F, A_1 \mathcal{H}[Y] \rho_t + A_2 \rho_t]] | \psi_T \rangle dt,
\]

(11)

where we have defined \( Y = \sqrt{2k} X \) to suppress \( k \) in the result. We now solve Eq. (11) order by order in \( dW \). Despite the large number of terms, it is nevertheless possible to solve for \( A_1 \) and \( A_2 \) in complete generality. The only assumption we make is that the optimal rotation was applied at the immediately preceding time step, so that \( \mathcal{F}_t(\theta) \) is maximized at \( \theta = 0 \). This is required for consistency with the assumption that the input state \( \rho_t \) is already optimized and \( \theta^* \) is infinitesimal. It implies that \( \partial \mathcal{F}_t(\theta)/\partial \theta\big|_{\theta=0} = i \langle \psi_T | [H_F, \rho_t] | \psi_T \rangle = 0 \), so that the first term in Eq. (11) may be dropped. Terms proportional to \( dW \) yield a linear equation in \( A_1 \), which is easily solved. Once \( A_1 \) is known, terms proportional to \( dW^2 = dt \) are gathered to yield another linear equation, this time for \( A_2 \). The final result in full form is
Using Eq. (1), the locally optimal feedback rotation can also be written as
\[
\theta^* = \sqrt{8k}A_1dV + (A_2 - 2A_1\langle Y \rangle)dt, \tag{13}
\]
from which we see that \( A_1 \) can be identified with a proportional feedback term, while the second term - dependent on \( A_1, A_2 \), and the state at time \( t \) - can be identified with an additional time dependent effective Hamiltonian drive.

Inserting the form of \( \theta^* \) into equation (3) and expanding it to second order in \( dW \) yields the following SME for evolution under the locally optimal control:
\[
d\rho = D[Y]\rho dt + H[Y]\rho dW - i(A_1dW + A_2dt)[H_F, \rho] + A_2^2D[H_F]\rho dt - iA_1[H_F, Y\rho + \rho Y^\dagger]dt. \tag{14}
\]

So far, we have assumed that the optimal angle is infinitesimal. However, Eq. (3) only guarantees that the solution \( \theta^* \) is a local extremum and does not guarantee that it is necessarily a maximum. A sufficient condition for it to be a local maximum is that the second derivative of the fidelity function evaluated at \( \theta^* \) be negative:
\[
\frac{\partial^2 F_t}{\partial \theta^2} \bigg|_{\theta=\theta^*} = -\langle \psi_T| [H_F, [H_F, \rho^*]]|\psi_T \rangle < 0. \tag{15}
\]
A detailed discussion of this second derivative test is given in Appendix A. Failure of this test indicates presence of a local minimum from the infinitesimal solution. Then we will need a large (i.e., non-infinitesimal) rotation to maximize the fidelity by searching for a local maximum over the entire angle range. As we demonstrate in subsequent sections, when combined with symmetry requirements, infinitesimal rotations work well for the preparation of W states, and also for Dicke states more generally. However, the GHZ state requires large rotation angles, illustrating an important exception to the infinitesimal optimal rotation equation.

Note that the functions \( A_1 \) and \( A_2 \) are dependent on both the initial and target states, as well as on the state at time \( t \). Dependence on the current state implies implicit dependence on the full measurement record, yielding a potentially non-Markovian feedback protocol. In the next section, we describe how it may be modified to define a Markovian feedback protocol. It should also be noted that \( A_1 \) and \( A_2 \) can in principle become singular if \( H_F \) commutes with \( \rho \). However, when \( [H_F, \rho] = 0 \), feedback will have no effect on the state, implying that \( F_t \) is constant. In this situation one may simply set \( \theta^* = 0 \) and proceed to the next iteration, so that the singular denominator is never encountered. Other singular cases may be treated with a global search over all possible values of \( \theta \). At the initial step, we assume that the controller chooses the rotation angle \( \theta^* \) that ensures a global maximum of \( F_{t=0} \). During evolution of the state, the above protocol typically continues to pick \( \theta^* \) as the maximum of \( F_t \) and thus maintains the system on a locally (time-)optimal trajectory. However even if Eq. (15) remains negative, it is quite possible that the nearest local maximum of \( F_t \) can fail to be the global maximum. The only way to catch such instances is to occasionally undertake a brute-force maximization of \( F_t \) and to thereby check whether the local maximum identified by Eq. (12) is also a global maximum. Developing efficient routines for finding such 'shortcuts' remains an interesting direction for future work.

Although the general optimization formalism above has been illustrated using fidelity with respect to a desired target state as the cost function, it could straightforwardly be applied to other linear cost functions. Going beyond linear protocols, appendix A shows how entanglement measures can be used as cost functions, with a protocol to generate the 3-qubit GHZ state by optimizing the 3-tangle. This necessitates significant differences in the protocol, since the cost function is not only nonlinear but also invariant under local unitaries.

## II. GENERATION OF W AND DICKE STATES

We now apply the locally optimal protocol developed in Section I to the generation of both W states and general Dicke states. We first consider W states. The W state represents one type of three qubit entanglement and is given by
\[
|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle), \tag{16}
\]
that possesses zero three-way entanglement but maximally retains bipartite entanglement on loss of a qubit [37]. To prepare the W state, we first choose a measurement observable
\[
X_W = Z_1 + Z_2 + Z_3, \tag{17}
\]
where \( Z_i \) is the Pauli operator along the \( \hat{z}_i \) axis for qubit \( i \). The W state is an eigenstate of \( X_W \), with eigenvalue 1. This implies that for any initial state \( \rho_0 \), the long time
limit of evolution under continuous measurement of $X_W$ in the absence of feedback will result in projection onto the W state with probability $p_W = (W|\rho_0|W)$. To increase this success probability, we apply the fidelity-optimized protocol of the previous, choosing the target state as $|\psi_T\rangle = |W\rangle$. To choose a proper feedback rotation operator, we generalize the locally optimal two-qubit feedback unitary used to generate the 2-qubit W state $[|10\rangle + |01\rangle]/\sqrt{2}$ in Ref. [21], which was shown to provide a globally optimal protocol for both maximal fidelity and concurrence [22]. As in that work, we restrict the evolution of each qubit to lie in the $xz$ plane ($\phi_i = 0, i = \{1, 2, 3\}$), with local rotations around the $y$ axis having equal angles for each of the three qubits, i.e., $\theta_1 = \theta_2 = \theta_3 = \theta$. The latter condition is consistent with the fact that both the target W state and the observable $X_W$ are symmetric with respect to any permutation of the three qubits. For the W state, the rotation operator in Eq. (6) then takes the following form

$$U_W^+(\theta) = e^{-i\frac{\pi}{4}(Y_1 + Y_2 + Y_3)},$$

where $Y_i$ are the Pauli operators along the $\hat{y}_i$ axis for qubit $i$. If our initial state is also symmetric to permutation of the qubits, then this condition guarantees that the state will only evolve in the symmetric subspace $H^S_W$ spanned by

$$|\phi^1_W\rangle = |000\rangle,$$

$$|\phi^2_W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle),$$

$$|\phi^3_W\rangle = \frac{1}{\sqrt{3}}(|011\rangle + |101\rangle + |110\rangle),$$

$$|\phi^4_W\rangle = |111\rangle,$$

in which all states are invariant to particle exchange. To see this, consider an arbitrary operator $P$ from the permutation group $S_3$. This clearly commutes with both $X_W$ and $U_W^+(\theta)$ in Eqs. (17) and (18), so that we may use Eq. (6) to obtain

$$P \rho_{i+dt}^t \rho_{i+dt}^t = U_W^t P \rho_{i+dt}^t U_W^+ U_W^+ U_W^+ = U_W^t \frac{\Omega_{dV} P \rho_{i}^t \Omega_{dV}^t}{Tr(\Omega_{dV} \rho_{i} \Omega_{dV}^t)} U_W^+ (19)$$

$$= U_W^t \frac{\Omega_{dV} \rho_{i}^t \Omega_{dV}^t}{Tr(\Omega_{dV} \rho_{i} \Omega_{dV}^t)} U_W^+ = \rho_{i+dt}^t \frac{\Omega_{dV} P \rho_{i}^t \Omega_{dV}^t}{Tr(\Omega_{dV} \rho_{i} \Omega_{dV}^t)}$$

Experimentally it is generally easy to prepare qubits in a product state, so we shall assume without loss of generality that the initial state is $|000\rangle$. We emphasize that the choice of symmetrized equal rotation angles in Eq. (18) is not just a simplification but is actually essential to generate the W state with high fidelity at long times. This can be seen by the following analysis. The eigenspace of the measurement observable $X_W$ to which the W state belongs is triply degenerate and spanned by $|001\rangle, |010\rangle, |100\rangle$. Without the symmetry constraint on the rotation angles, a general measurement observable $X'$ will not be able to single out the W state from other states in this subspace. After we impose this symmetry, however, the W state becomes a non-degenerate eigenstate of $X_W$, and it becomes then possible to select out the W state with a suitably designed sequence of feedback operations. We will discuss this role of symmetry further when developing feedback protocols for the GHZ state, where it places important physical constraints on the measurement observable.

To numerically test on our protocol proposed in previous section (Eqs. (9) and (12)), we start from the product state $|000\rangle$. Before evolving this state, we first locally rotate it with $U_W^t (\theta)$ to a state with maximum fidelity. Following this initial rotation, we evolve under the local optimization protocol, computing the optimal feedback coefficient at every time step using Eq. (12).

Fig. 7 shows the resulting trajectory ensemble average (TEA) fidelity (red solid line) with respect to the W state, obtained by averaging over 1000 trajectories. The TEA results are seen to saturate at fidelity $\sim 1$ at a time of $\sim 1.5 \mu$s. We note that in order to ensure that the fidelity saturates at 1, it is necessary to occasionally apply non-infinitesimal rotations when the
second derivative test fails (Eq. (15)). Detailed analysis of the TEA ensemble showed that approximately 100 out of $10^4$ trajectories, i.e., $\sim 1\%$ required at least one non-infinitesimal rotation angle (for a total of $\sim 10^5$ time steps). Without feedback, the fidelity remains constant out of 10 of the TEA ensemble showed that approximately 100

This TEA-based protocol implicitly assumes knowledge of the entire measurement record for $\rho$, since the optimal angle for the feedback unitary at any time $t$ depends on the input state state $\rho_t$. By induction it implies knowledge of the state at all times. Thus it is intrinsically a non-Markovian protocol. For experimental implementation, this requires ability to do real-time state estimation, which is both challenging and time-consuming. An attractive alternative is the average state locally optimal (ASLO) protocol developed in Ref. [21]. In the ASLO protocol, instead of feeding back based on the controlled state in Eq. (3), we use an unconditioned state that is obtained by averaging over the entire measurement record prior to the current time step

$$\rho_{t+dt} = \int_{-\infty}^{\infty} dV \, Tr(\Omega_{dV} \rho_{t} \Omega_{dV}^\dagger) $$

$$= \int_{-\infty}^{\infty} dV \, U_F \rho_{t} \Omega_{dV}^\dagger U_F^\dagger. $$

The state $\rho_t$ along the evolution is then understood as an averaged state with deterministic dynamics. This is now a Markovian protocol, since the dependence on the previous measurement history has been removed by the averaging. The feedback angle $\theta^*$ (Eqs. (12) and (13)) at time $t$ is determined from the unconditioned average state $\bar{\rho}_t$ and the current measurement record without regard to the earlier history. An SME for $\bar{\rho}$ may be simply obtained by averaging over (i.e., dropping) the terms proportional to $dW$ in Eq. (14), resulting in an efficient simulation of an arbitrary ASLO feedback protocol.

Simulating the Markovian protocol of Eq. (24) requires only a single time evolution for the averaged state. The averaging procedure in Eq. (24) is a mathematical step that corresponds to precisely the averaging over trajectories with different measurement records and hence over quantum noise histories that is done in an experiment. The feedback unitary characterized by $A_1(t)$ and $A_2(t)$ for the averaged state can be applied to any individual realization of the state at each instant. This provides significant advantages for experimental implementation, since $A_1(t)$ and $A_2(t)$ can be pre-calculated with the same procedure as described for the TEA approach in Section 1B and the resulting feedback operation applied to each experimental trajectory without the need for real-time state estimation.

Fig. 3 shows the evolution of the optimal feedback angle coefficients $A_1(t)$ and $A_2(t)$ evaluated by this ASLO protocol. The coefficient $A_2(t)$ determines the average value of $\theta^*$, while $A_1(t)$ determines its variance. It is evident that the average value is considerably smaller than the variance at all except the earliest times in the evolution.

Using the optimized ASLO feedback angles, Eq. (13), with $\rho_t$ replaced by the unconditioned state $\bar{\rho}_t$ to control the averaged state dynamics gives rise to a fidelity that is shown in Fig. 7 by the dashed red line. The same initial condition is used here as for the TEA protocol, i.e., the 3-qubit product state $|000\rangle$. It is apparent that while the W state is reached with high fidelity within a comparable time of approximately 600 ns, the ASLO protocol nevertheless saturates slightly below unity, at $\sim 0.98$. The origin of this difference may lie in the different sampling of the density matrix that is enabled by the TEA and ASLO approaches. Note that while the procedure for generating the coefficients $A_1(t)$ and $A_2(t)$ is identical, Eq. (24), the input density matrices are different, with the TEA approach sampling these from many trajectories while the ASLO approach takes just one averaged trajectory. However, these two approaches can be identical in the situation which feedback cancels measurement noise exactly, as is the case for two-qubit optimal control [22]. It should be noted that although the ASLO protocol does not achieve unit fidelity, one can nevertheless still produce unit-fidelity states by adding a final projective measurement. As the symmetry reduction has removed all degeneracy from the measurement operator $X$, the measurement outcome now uniquely determines the state. Although the success probability under the ASLO protocol is less than one, it has been significantly enhanced by feedback. This argument applies to all protocols in which the symmetry reduction is made, so that one may thereby interpret the final ASLO fidelity as a success probability for perfect state preparation.

We can also easily extend the above approach to gen-
eral Dicke states, which are defined as
\[ |N, k\rangle = \frac{1}{\sqrt{\binom{N}{k}}} \sum_{P \in S_N} P(|0\rangle^\otimes (N-k) \otimes |1\rangle^\otimes k) \] (25)
where \( P \) is an operator belonging to the permutation group \( S_N \) on \( N \) qubits. When \( k = 1 \), we have the usual \( W \) state. When \( k = \frac{N}{2} \) (\( N \) even or \( k = \frac{N+1}{2} \) when \( N \) is odd), we have the half-filled Dicke state. We shall measure the symmetric observable
\[ X_D^N = Z_1 + Z_2 + \cdots + Z_N. \] (26)
Imposing the permutation group symmetry on \( N \) qubits allows the Dicke states to be represented within the symmetric subspace of dimension \( N+1 \). Each state in this subspace is just a Dicke state with \( k \) excitations, which is also associated with a non-degenerate eigenspace of the observable \( X_D^N \). In particular, the \( W \) state for \( N \) qubits, defined as
\[ |W\rangle = \frac{1}{\sqrt{N}} (|10\cdots0\rangle_N + |01\cdots0\rangle_N + \cdots + |00\cdots1\rangle_N) \] (27)
belongs to the eigenspace of \( X_D \) with eigenvalue \( (N-1) - 1 = N - 2 \). This implies that the computational resources required to compute the feedback protocol scale only polynomially with the number of qubits, which is a huge improvement compared to the exponential scaling of the full Hilbert space dimension.

The results of ASLO calculations with the locally optimal protocol of Section VIB (see Eq. (4)) for Dicke states with variable excitation number \( k \) for up to \( N = 48 \) qubits, are shown in Fig.4. The insets show the fidelity reached for variable excitation \( k \) of \( N = 48 \) qubits (upper) and for single excitation \( (k = 1, \ W \text{ state}) \) of up to \( N = 100 \) qubits (lower). While we do not plot \( A_1 \) and \( A_2 \), we note that \( A_2 = 0 \) for half-filled Dicke states of odd \( N \). It is seen that for all target states shown here, the final fidelity remains well above 0.9, showing that a simple Markovian-type feedback strategy exists to prepare any of these maximally spin-squeezed states deterministically.

III. GENERATION OF GHZ STATES

The GHZ state
\[ |GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \] (28)
is the three-qubit state with both maximal multi-particle entanglement and maximal fragility of three-way entanglement \[ 27,39 \]. To prepare this state we use the following symmetric measurement observable
\[ X_G^S = Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1. \] (29)
Together, these two symmetries induce a symmetric subspace \( \mathcal{H}_G^S \) of the full Hilbert space. The basis vectors of \( \mathcal{H}_G^S \) are expressed in the computational basis as
\[ |\phi_{G1}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \] (30)
\[ |\phi_{G2}\rangle = \frac{1}{\sqrt{6}}(|011\rangle + |101\rangle + |110\rangle + |100\rangle + |010\rangle + |001\rangle). \] (31)
\( |\phi_{G1}\rangle, i = 1, 2 \) are eigenstates of \( X_G^S \) with eigenvalues \( e_1 = +3 \) and \( e_2 = -1 \). Within this subspace, the GHZ state \( |\phi_{G1}\rangle \) is then a non-degenerate eigenstate of \( X_G^S \). Thus, if we are able to restrict the dynamics governed by Eq. (6) to this subspace, when combined together with a suitably designed feedback protocol, a continuous measurement of \( X_G^S \) should be able to extract the GHZ state. Just as was done for the \( W \) state above, this can be achieved by imposing the symmetry of \( X_G^S \) on the feedback rotation operator \( U_F \). We define the latter here by a single rotation around the \( x \) axis, which is consistent with the bit-flip symmetry, and set the rotation angles to be equal for all three qubits, to be consistent with the permutation symmetry. This yields the rotation operator
\[ U_F^G(\theta) = e^{-i\frac{\theta}{2}(X_1+X_2+X_3)}. \] (32)
If the initial state is within \( \mathcal{H}_G^S \), then the action of \( U_F^G(\theta) \) will ensure that the subsequent evolution remains in \( \mathcal{H}_G^S \).

Similar to the analysis above for the \( W \) state, we then determine the locally optimal angle \( \theta^* \) by maximizing the fidelity at each time step
\[ \mathcal{F}_G(t + dt) = \langle GHZ | \rho(t + dt^*) | GHZ \rangle, \] (33)
where \( \rho(t + dt) \) is again the output state given by Eq. (9). Full details of the procedure to determine the optimal
angle for the GHZ target state are given in Appendix \[B\]. Unlike the optimization for the W state where the second derivative test failed in a small number of instances, the GHZ state protocol fails this test very often (Eq. \[(15)\]). Indeed, as shown in Appendix \[B\] the optimal angle is found to be always equal to either 0 or $\frac{\pi}{2}$. Consequently, the SME Eq. \[(14)\], which assumes infinitesimal rotations, cannot be used to simulate the dynamics and instead we must use the full POVM equation Eq. \[(6)\].

The GHZ state allows an interesting alternative optimization approach, deriving from the fact that after the dynamics are constrained to the symmetric subspace, the dimension of the Hilbert space is reduced from eight to two. Consequently, under these constrained dynamics the three-qubit problem can be mapped to a single qubit problem. This mapping is described explicitly in Appendix \[B\] where it is shown that this allows an alternative approach to determination of the optimal angle that also results in an optimal angle of either 0 or $\frac{\pi}{2}$.

Fig. 5 shows the time dependence of the fidelity of formation of the GHZ state obtained using the locally optimal protocol within the ASLO approach, with the symmetrized measurement $X_G^S$, and using as initial condition the complete superposition state

$$\frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle).$$ \[(34)\]

We see that in this situation the fidelity under the ASLO protocol asymptotically approaches unity.

It is instructive to compare the performance of this protocol with its symmetrized measurement operator, to that obtained from feedback control based on measurements of an observable not respecting this permutation symmetry. The one-body observable

$$X_G = 2Z_1 - Z_2 - Z_3$$ \[(35)\]

presents such an observable. This measurement operator is permutation symmetric only with respect to exchange of the last two qubits, and the target state is then contained in a degenerate eigenspace with eigenvalue zero that is spanned by $\frac{1}{\sqrt{2}}(|000\rangle \pm |111\rangle)$. While we can still impose the symmetry condition on the feedback rotation operator, the measurement is now unable to distinguish the target GHZ state from another state $\frac{1}{\sqrt{2}}(|000\rangle - |111\rangle)$ within the degenerate eigenspace. We therefore expect that any feedback protocol based on this measurement will be unlikely to reach unit fidelity. In fact, the results achieved with this protocol, shown as the blue line in Fig. 5, are of very low quality.

We can also generalize our protocol to the GHZ state of $N$ qubits, which is defined as

$$|GHZ\rangle_N = \frac{1}{\sqrt{2}}(|00\cdots0\rangle_N + |11\cdots1\rangle_N).$$ \[(36)\]

The observable we use is still the two-body symmetrized observable

$$X_G^S = \sum_{i<j} Z_i Z_j.$$ \[(37)\]

After imposing both the permutation and spin-flipping symmetry, the corresponding symmetric subspace $\mathcal{H}_G^S$ now has dimension $\frac{N+1}{2}$ when $N$ is odd, and dimension $\frac{N}{2} + 1$ when $N$ is even. Explicitly, the symmetrized basis
\[ |\phi_G^1\rangle = \frac{1}{\sqrt{2}} (|00\ldots0\rangle_N + |11\ldots1\rangle_N) \]  
\[ |\phi_G^2\rangle = \frac{1}{\sqrt{2N}} \sum_{P\in S_N} P(1 + \Pi) |10\ldots0\rangle_N \]  
\[ \vdots \]  
\[ |\phi_G^n\rangle = \frac{1}{\sqrt{2N}} \sum_{P\in S_N} P(1 + \Pi) \underbrace{|11\ldots10\ldots0\rangle}_m \]  
where \( \Pi = X_1 X_2 \ldots X_N \) is the spin-flipping operator. The index \( m \) indicates how many spins are pointing downward and runs from 0 to either \( \frac{N-1}{2} \) (\( N \) odd) or \( \frac{N}{2} \) (\( N \) even). Note that every basis vector \( |\phi_G^m\rangle \) is a non-degenerate eigenstate of the observable \( X_G^m \). In particular, the GHZ state is just \( |\phi_G^2\rangle \), with eigenvalue \( (N/2) \). Following the same procedure as for \( n = 3 \), we choose the feedback rotation operator to be of the form  
\[ U_F(\theta) = \bigotimes_{i=1}^N e^{-i\frac{\theta}{2} X_i} \]  
and start from the symmetric initial state \( |\psi(0)\rangle = \bigotimes_{i=1}^N \frac{|0\rangle + |1\rangle}{\sqrt{2}} \). The locally optimal rotation angles are calculated using the same local expansion approach employed for \( n = 3 \) qubits (see Appendix B 1).

Results obtained with the trajectory ensemble approach for generation of the \( n = 4 \) GHZ state using this locally optimal GHZ protocol are shown in Fig. 6. Undertaking these non-Markovian locally optimal calculations for the GHZ state is significantly more expensive than the corresponding calculations for the W state in Section III. However, our results for \( n = 3 \) and \( n = 4 \) suggest that the locally optimal approach may be similarly be scaled beyond \( n = 4 \) to generation of many-body GHZ states, as was done for the Dicke states in the previous section.

As noted in Section IIIA above, it is also possible to construct feedback protocols based on cost functions providing a direct measure of entanglement, rather using as cost function the fidelity with a specific target state. While, as seen above, the latter choice simplifies many calculations, it does however require that a single specific target state is singled out. This ignores the possibility that one might be able to do better by targeting a different target state that is nevertheless locally equivalent to the desired target state. In Appendix C, we present a method for deriving locally optimal protocols based on entanglement measures, and apply it to directly optimize the three-tangle, a measure of entanglement for three qubits that achieves its maximal unit value for a GHZ state and all locally equivalent states.

### IV. SUMMARY AND OUTLOOK

We have presented a general, analytic construction of locally optimal measurement-based feedback protocols in which a time-dependent feedback unitary is designed with two independent components, one of which depends linearly on the measurement outcome with a state-dependent coefficient, while the other depends only on the quantum state. We term this approach “quantum state and proportional feedback”, or PaQS feedback, to emphasize the increased flexibility offered by these two independent feedback components. We showed that the resulting protocol can be modified to generate an average state locally optimal (ALSO), or Markovian protocol that can be efficiently implemented.

We demonstrated the effectiveness of this PaQS feedback by applying it to the generation of W, Dicke and GHZ states for \( n = 3 - 100 \) qubits, highlighting the utility of symmetry constraints. The non-Markovian control protocols were found to saturate at unit fidelity with the target state in all cases, while the ALSO approach was found to saturate at slightly lower than unit fidelity for the W and Dicke states. It is remarkable that the significant constraints imposed by Markovianity yield only a slight reduction in performance.

Several interesting challenges derive from systematization of this PaQS feedback approach. One question is how to choose the functional form of the feedback operators. A second general question is which classes of quantum states may be generated given specific choices of the measurement and feedback operators. A third issue is the rate of approach to the target state. Recent work has shown that global exponential stabilization of two-level systems is possible by feedback based on a combination of quantum state and measurement proportional control similar to that introduced here [40]. Whether global and or exponential stabilization is possible for higher dimensional quantum systems is an important question for further work.

Another interesting question for further study is the relative benefits of different cost functions. This work focused primarily on the use of a maximal quantum state fidelity cost function. However for the \( n = 3 \) GHZ state we also derived a feedback protocol based on optimizing an entanglement measure, the three-qubit tangle, which is invariant to single qubit rotations. This required a modified feedback protocol in which the state is controlled before the measurement. The tangle is one of several invariants for three-qubit states. Investigation of feedback protocols based on optimization of the corresponding invariants for higher numbers of qubits is a challenging topic for future work.

### ACKNOWLEDGMENTS

This work was supported by Laboratory Directed Research and Development (LDRD) funding from Lawrence Berkeley National Laboratory, provided by the U.S. Department of Energy, Office of Science under Contract No. DE-AC02-05CH11231.
Appendix A: Second Derivative Test for Local Optimal Control

In this Appendix we discuss the situation when the infinitesimal solution Eqn. [9] with Eqn. [12] fails to pass the second derivative condition, Eq. [15]. Using the same expansion as in Section I B, we have

\[
\langle \psi_T \rangle^2 [H, [H, \rho]] \psi_T \rangle + \langle \psi_T \rangle [H, [H, Y \rho + \rho Y^T]] - iA_1[H, [H, [H, \rho]]] \psi_T \rangle dW
\]

\[
+ \langle \psi_T \rangle [H, [H, D[Y] \rho]] + A_2^2[H, [H, D[H] \rho]] - i[H, [H, A_1(Y \rho + \rho Y^T) + A_2\rho]]] \psi_T \rangle dt > 0
\]

There will always be some non-zero probability that this condition is violated. Specifically, when

\[
\theta^* = 0 \text{ or } \theta^* = \pi/2, \text{ depending on the sign of the state-dependent term in parentheses.}
\]

2. Mapping of \( N = 3 \) GHZ state to a single qubit state

Since the symmetric subspace obtained by imposing the \( S_3 \) permutation symmetry on three qubits, has dimension two, we can map the three-qubit problem into an effective single-qubit representation. The geometric meaning of the resulting control protocol will become clear below.

In order to ensure that the state stays in the symmetric subspace, we have to apply a symmetric rotation around the qubit axes:

\[
U_F = e^{-i \frac{\pi}{4} (X_1 + X_2 + X_3)}.
\]

Then the symmetric subspace within which the dynamics takes place is spanned by

\[
|\tilde{0}\rangle = \frac{1}{\sqrt{2}} \left( |000\rangle + |111\rangle \right)
\]

\[
|\tilde{1}\rangle = \frac{1}{\sqrt{6}} \left( |100\rangle + |010\rangle + |001\rangle + |011\rangle + |101\rangle + |110\rangle \right)
\]

The \( |\tilde{0}\rangle \) basis is just the GHZ state we are trying to generate. Let us see how encoded operations are realized in this subspace. First, the rotation axis \( \Sigma \equiv X_1 + X_2 + X_3 \)
where $\hat{\Sigma}$ is equivalent to
\[
\hat{\Sigma} = \begin{pmatrix} 0 & \sqrt{3} \\ \sqrt{3} & 2 \end{pmatrix} = \hat{I} + \sqrt{3}\hat{X} - \hat{Z}
\]
We can drop the identity here since it only gives a global phase. Then we have
\[
\hat{\Sigma} = \sqrt{3}\hat{X} - \hat{Z} = 2\hat{n} \cdot \hat{\sigma},
\]
where $\hat{n} = (\frac{\sqrt{3}}{2}, 0, \frac{1}{2})$ is a unit vector lying in the $x - z$ plane which gives us the rotation axis, and $\hat{\sigma}$ is just the vector of the effective Pauli matrices. So in the effective single qubit picture, the rotation operator $\hat{\sigma}$ and the corresponding rotation angle $\theta$ are given by
\[
\hat{\sigma} = \frac{\sqrt{3}}{2} \hat{X} - \frac{1}{2} \hat{Z} \tag{B7}
\]
\[
\theta = 2\theta \tag{B8}
\]
Note that the rotation angle in the effective single qubit space is equal to twice that in the original space.

Now let us look at the measurement process, which is given by the encoded operator
\[
\hat{X}_{\text{obs}} = \begin{pmatrix} \langle 0|X_{\text{obs}} |0 \rangle & \langle 0|X_{\text{obs}} |1 \rangle \\ \langle 1|X_{\text{obs}} |0 \rangle & \langle 1|X_{\text{obs}} |1 \rangle \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} = \hat{I} + 2\hat{Z} \tag{B9}
\]
The evolution equation is given by Eq. (2) of the main text. In our case, the Hamiltonian is zero, and for the terms deriving from the measurement, we have
\[
D[\hat{I} + 2\hat{Z}] = D[2\hat{Z}] \tag{B10}
\]
\[
\mathcal{M}[\hat{I} + 2\hat{Z}] = \mathcal{M}[2\hat{Z}] \tag{B11}
\]
Dropping the identity in the measured observable, we find that the measurement in the effective single qubit subspace is a measurement along the encoded $z$ axis, with a four-fold increase in the measurement strength, i.e.,
\[
\hat{X}_{\text{obs}} = Z \tag{B12}
\]
where $\tilde{k}$ is the effective measurement strength in the single qubit space.

It is easy to show that the initial state $|\psi\rangle_0 = (\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle))^{\otimes 3}$ becomes
\[
|\tilde{\psi}\rangle_0 = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle \tag{B14}
\]
in the effective single qubit space, or in Bloch vector form,
\[
\tilde{\rho}_0 = |\tilde{\psi}\rangle_0\langle\tilde{\psi}|_0 \sim (\frac{\sqrt{3}}{2}, 0, -\frac{1}{2}) \tag{B15}
\]
The fidelity with respect to the GHZ state is then given in terms of this Bloch vector by
\[
f = \langle 0|\rho|0 \rangle = 1 + \tilde{z} \tag{B16}
\]
where $\tilde{z}$ is the $z$ component of the Bloch vector in the effective single qubit space. Clearly the rotation angle has to be equal to either 0 or $\frac{\pi}{2}$, in order to ensure that this fidelity is optimal at each time step.

### Appendix C: Tangle-based protocols

Here we present an alternative locally optimal protocol for generation of a three-qubit GHZ state that is based on optimization of the three-tangle $\tau$, which provides a measure for tripartite entanglement [41, 42]. This protocol has the advantage that it not only differentiates between the two distinct types of tripartite entanglement of three-qubit states, but is also invariant under local rotations of the state. Under this measure the GHZ state reaches the upper bound, with value $\tau = 1$, while all two-particle entanglements are zero [43, 44]. In the following...
we shall maximize the tangle under measurements of the symmetrized two-body operator $X_{SG}$, Eq. (29).

We note first that starting from a pure state, the conditioned state (Eq. (3)) after the weak measurement will still be pure. So if we avoid averaging the state along the evolution over measurement outcomes, the state will remain pure at all times. This allows us to use the pure state definition of the three-tangle \[ \tau = \tau(ABC) = \tau(A|BC) - \tau(A|B) - \tau(B|C), \] \[(C1)\]
where the quantities on the right hand side (referred to as “two-tangles”) are given by squares of the relevant concurrences \[42\]. This considerably simplifies the determination of the feedback angles, since computing the tangle for a mixed state can be very difficult, involving determination of a convex roof extension \[45\].

Since the tangle is invariant under our feedback unitary operations, the maximization procedure used for the fidelity cost function in the main text of the paper does not work here. Instead, we determine the optimal angle by maximizing the expected increase in tangle after measurement. The feedback angle at each infinitesimal time step is now computed as follows. At time $t$, the (pure) input state $|\psi\rangle_t$ is rotated using the feedback control operator $U_G^{\tau}(t)$ (Eq. (32))

\[ |\psi\rangle_c(t) = U_G^{\tau}(t)|\psi\rangle_t, \]
\[(C2)\]
with the rotation angle parameter determined as described below. We then make a weak measurement on
the controlled state:

\[ |\psi|^c_{t+dt} = \frac{\Omega_{dt}|\psi|^c_t}{\|\Omega_{dt}|\psi|^c_t\|} \]

(C3)

\[ = \frac{\Omega_{dt} U^c_F(t)|\psi|^c_t}{\|\Omega_{dt} U^c_F(t)|\psi|^c_t\|} \]  

(C4)

Note that we are now controlling the state before the measurement instead of after measurement: we choose to do this because a local rotation on a pure state will not change the value of the tangle for the state, so the tangle is not affected by the control.

Now the choice of rotation should not be determined by a particular measurement outcome that occurs after imposition of the control. Therefore in order to obtain the optimal rotation angle while avoiding issues of causality, we may simply average the tangle over all possible measurement outcomes and choose the control rotation as the value maximizing this average, i.e.,

\[ U^c_F(t) = \arg\max_{U^c_F(t)} \int dV \tau(|\psi|^c_{t+dt}). \]  

(C5)

This requires sampling values of rotation angle and evaluating the average over measurement outcomes for each case. The state is then evolved forward by acting with the measurement after the optimal rotation, yielding the evolution described by Eq. (C3) with \( U^c_F(t) \) replaced by \( U^c_F(t) \).

Fig. 9a shows that when this tangle-based protocol is implemented using the two-body measurement observable \( X^c_G \), both the value of the three-tangle \( \tau \) and the corresponding fidelity \( F_G \) appear to asymptotically reach a value of one, although on a slower timescale than the corresponding fidelity under the fidelity based approach (compare with Fig. 3). In contrast, when the tangle-based optimization is used with the non-symmetrized one-body observable \( X_G \) for measurement, a significantly lower value of the tangle is obtained, with an asymptotic value of approximately 0.7 being reached. It is thus evident again that a protocol based on symmetrized two-body observable measurements significantly outperforms a protocol based on measurement with a non-fully symmetrized observable.

The three-tangle \( \tau \) is one of five non-trivial polynomial invariants that characterize normalized three-qubit states [43, 44]. Our work suggests that optimization of multiple invariants might be useful for construction of feedback protocols to systematically generate arbitrarily entangled three-qubit states. For three-qubit states, \( \tau \) achieves its maximal value for the GHZ state and all other invariants automatically reach the boundary value. In this case, optimizing the tangle alone then guarantees that the other invariants reach the correct values for the state. This is not the case for other states in general. One alternative choice of cost function in more general situations is to use the sum of the squared differences between the invariants of the current state and the target state, which can act as a measure of the distance between the two states.

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