Vector boson in constant electromagnetic field

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Abstract

The propagator and complete sets of in- and out-solutions of wave equation, together with Bogoliubov coefficients, relating these solutions, are obtained for vector $W$-boson (with gyromagnetic ratio $g = 2$) in a constant electromagnetic field. When only electric field is present the Bogoliubov coefficients are independent of boson polarization and are the same as for scalar boson. When both electric and magnetic fields are present and collinear, the Bogoliubov coefficients for states with boson spin perpendicular to the field are again the same as in scalar case. For $W^-$ spin parallel (antiparallel) to the magnetic field the Bogoliubov coefficients and contributions to the imaginary part of the Lagrange function in one loop approximation are obtained from corresponding expressions for scalar case by substitution $m^2 \rightarrow m^2 + 2eH$ ($m^2 \rightarrow m^2 - 2eH$). For gyromagnetic ratio $g = 2$ the vector boson interaction with constant electromagnetic field is described by the functions, which can be expected by comparing wave functions for scalar and Dirac particle in constant electromagnetic field.

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1 Introduction

Vector bosons occupy an intermediate place between low spin particles (with spin 0 and 1/2) and higher spin particles. As such they can share some of the problems encountered in considering higher spin particle interaction with strong electromagnetic field. The most conspicuous feature of vector boson interaction in case of $g = 2$ is the appearance of tachionic modes in overcritical magnetic field. The ways to deal with this problem in the framework of non-abelian theories are analyzed in [1]. But are there any others? According to [2] just in case $g = 2$ there are problems in treating pair production by electric field, using the method of diagonalization of Hamiltonian. This is surprising in view of successful calculation of the Lagrange function of constant field in one loop approximation [3]. We calculate pair production by constant field, using Bogoliubov coefficients (which contain all the information about this process) and obtained, as expected, the results in agreement with [3] and [4].

2 Vector boson in constant electric field

We assume $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ and denote $e = |e|$. The wave function of $W^-$ boson ($g = 2$) in a source-free space (where $\partial_{\mu} F^{\mu\nu} = 0$) satisfies the equation [1, 5]

$$(-D_\sigma D^\sigma + m^2)\psi_\mu - 2ieF_{\mu\nu}\psi^\nu = 0$$

(1)

and constraint

$$D_\mu \psi^\mu = 0, \quad D_\mu = \partial_\mu + ieA_\mu.$$  

(2)

Taking the vector potential in the form $A_3 = -Et, A_1 = A_2 = A_0 = 0$, we find from (1) that the equations for $\psi^1$ and $\psi^2$ are the same as in scalar case

$$(-D^2 + m^2)\psi^{1,2} = 0.$$

(3)

For $\psi^3$ and $\psi^0$ we get from (1)

$$(-D^2 + m^2)\psi^3 - 2ieE\psi^0 = 0, \quad (-D^2 + m^2)\psi^0 - 2ieE\psi^3 = 0.$$  

(4)

Introducing $\psi^\pm = \psi^0 \pm \psi^3$ we rewrite (4) as follows

$$(-D^2 + m^2 \mp 2ieE)\psi^\pm = 0,$$

i.e. it is obtainable from (3) by substitution $m^2 \to m^2 \mp 2ieE$. We see that the wave function for vector boson can be obtained from corresponding wave function for scalar boson by simple rules.

Now we shall do this. The positive-frequency in-solution for (negatively charged) scalar boson we denote as $^+\psi^\bar{p}$. Here $\bar{p} = (p_1, p_2, p_3)$; this index will be dropped in what follows. Then [6]

$$^+\psi \propto D_\nu(\tau) \exp i\bar{p}\bar{x}.$$  

(6)

Here $D_\nu(\tau)$ is parabolic cylinder function [7] and

$$\nu = \frac{i\lambda}{2} - \frac{1}{2}, \quad \tau = -\sqrt{2eE} e^{-i\pi} (t - \frac{p_3}{eE}), \quad \lambda = \frac{m^2 + p_1^2 + p_2^2}{eE}.$$  

(7)
For vector boson we get

\[ +\psi = \begin{bmatrix} \psi^0 \\ \psi^1 \\ \psi^2 \\ \psi^3 \end{bmatrix} = \begin{bmatrix} c_+ D_{\nu+1}(\tau) + c_- D_{\nu-1}(\tau) \\ c_1 D_{\nu}(\tau) \\ c_2 D_{\nu}(\tau) \\ c_+ D_{\nu+1}(\tau) - c_- D_{\nu-1}(\tau) \end{bmatrix} e^{i\vec{p} \cdot \vec{x}}. \] (8)

Here \( \psi^1 = +\psi^1, \psi^2 = +\psi^2 \) and

\[ \psi^0 \equiv +\psi^0 = \frac{1}{2}(+\psi^+ + +\psi^-), \quad \psi^3 \equiv +\psi^3 = \frac{1}{2}(+\psi^+ - +\psi^-), \] (9)

\[ +\psi^\pm = 2c_{\pm} D_{\nu\pm1} e^{i\vec{p} \cdot \vec{x}}. \]

\( D_{\nu\pm1}(\tau) \) is obtained from \( D_{\nu}(\tau) \) in (6-7) by substitution \( m^2 \to m^2 \mp 2iE \). The arbitrary coefficients \( c_1, c_2, c_\pm \equiv +c_\pm \) determine polarization of vector boson. They are not independent due to the constraint (2):

\[ c_1 p_1 + c_2 p_2 + \sqrt{2eE}e^{i\vec{p} \cdot \vec{x}} [(1 + \nu)_+ c_+ - c_-] = 0. \] (10)

For negative-frequency in-solution (for scalar boson) instead of (6) we have

\[ -\psi \propto [D_{\nu}(\tau)]^* e^{i\vec{p} \cdot \vec{x}}. \] (11)

Star means complex conjugation. Similarly to (8) the parabolic cylinder functions entering in \( -\psi^\pm \) are obtained from \( [D_{\nu}(\tau)]^* \) in (11) by substitutions \( m^2 \to m^2 \mp 2iE \). So

\[ -\psi = \begin{bmatrix} c_+ D_{\nu\pm1}(\tau^*) + c_- D_{\nu\pm1}(\tau^*) \\ c_1 D_{\nu\pm1}(\tau^*) \\ c_2 D_{\nu\pm1}(\tau^*) \\ c_+ D_{\nu\pm1}(\tau^*) - c_- D_{\nu\pm1}(\tau^*) \end{bmatrix} e^{i\vec{p} \cdot \vec{x}}. \] (12)

(In (12) \( c_\pm = -c_\pm \) and similarly in other cases) The constraint takes the form

\[ c_1 p_1 + c_2 p_2 + \sqrt{2eE}e^{-i\vec{p} \cdot \vec{x}} [-c_+ + \nu c_-] = 0. \] (13)

Nothing prevents us to assume that \( c_1, c_2 \) in (12) are the same as in (8).

The negative-frequency out-solution is obtained from positive frequency in-solution by changing sign of \( \tau \) in parabolic cylinder functions in (8):

\[ -\psi = \begin{bmatrix} c_+ D_{\nu+1}(-\tau) + c_- D_{\nu-1}(-\tau) \\ c_1 D_{\nu}(-\tau) \\ c_2 D_{\nu}(-\tau) \\ c_+ D_{\nu+1}(-\tau) - c_- D_{\nu-1}(-\tau) \end{bmatrix} e^{i\vec{p} \cdot \vec{x}}, \quad -c_\pm = +c_\pm, \] (14)

see (112a). The constraint has the form

\[ c_1 p_1 + c_2 p_2 + \sqrt{2eE}e^{i\vec{p} \cdot \vec{x}} [-c_- (1 + \nu)c_+] = 0. \] (15)
Similarly we find the positive frequency out-solution from $\psi$ in (12) by changing sign of $\tau^*$

$$+\psi = \begin{bmatrix} c_1 D_{\nu^* - 1}(-\tau^*) + c_- D_{\nu^* + 1}(-\tau^*) \\ c_1 D_{\nu^*}(-\tau^*) \\ c_2 D_{\nu^*}(-\tau^*) \\ c_1 D_{\nu^* - 1}(-\tau^*) - c_- D_{\nu^* + 1}(-\tau^*) \end{bmatrix} e^{i\vec{p} \cdot \vec{x}}. \quad (16)$$

The corresponding constraint is

$$c_1 \psi_1 + c_2 \psi_2 - \sqrt{2eE} e^{-i\vec{p} \cdot \vec{x}} [\nu^+ c_- + c_+] = 0. \quad (17)$$

Now for scalar boson the in- and out-solutions are related by [6]

$$+\psi_n = c_{1n}^+ \psi_n + c_{2n}^- \psi_n, \quad (18)$$
$$-\psi_n = c_{2n}^+ \psi_n + c_{1n}^- \psi_n,$$

$$c_{1n} = \frac{\sqrt{2\pi}}{\Gamma((1 - i\lambda)/2)} \exp[-\frac{\pi}{4} (\lambda - i)], \quad c_{2n} = \exp[-\frac{\pi}{2} (\lambda + i)],$$

$$|c_{1n}|^2 - |c_{2n}|^2 = 1.$$

Subscript $n$ means a set of quantum numbers. Here $n = \vec{p}$. By straightforward calculation similar to scalar case we find that (18) hold also for vector boson and that

$$+c_- = \frac{i}{\nu} c_- = -c_-, \quad (19)$$
$$-c_+ = -i(1 + \nu) c_+ = -c_+ = i(1 + \nu) c_+.$$

These relations guarantee that all wave functions $\pm \psi, \pm \psi$ are normalized in the same manner and that any constraint can be obtained from any other using (19).

As seen from (18) in constant electric field the Bogoliubov coefficients $c_{1n}, c_{2n}$ does not depend on boson polarization. Thus the imaginary part of the Lagrange function is simply $3 \text{Im} L_{\text{spin0}}$ in agreement with [3,4].

## 3 Vector boson in constant electromagnetic field

Now we add to constant electric field the collinear constant magnetic field. Assuming $A_2 = H x_1$ we have for $\psi_1$ and $\psi_2$ from (1)

$$(-D^2 + m^2)\psi_1 - 2ieH \psi_2 = 0, \quad (-D^2 + m^2)\psi_2 + 2ieH \psi_1 = 0. \quad (20)$$

Introducing

$$\tilde{\psi}_1 = \psi_1 - i\psi_2, \quad \tilde{\psi}_2 = \psi_1 + i\psi_2, \quad \psi_1 = \frac{1}{2} (\tilde{\psi}_1 + \tilde{\psi}_2), \quad \psi_2 = \frac{i}{2} (\tilde{\psi}_1 - \tilde{\psi}_2), \quad (21)$$

we rewrite (20) as

$$(-D^2 + m^2 + 2eH)\tilde{\psi}_1 = 0, \quad (-D^2 + m^2 - 2eH)\tilde{\psi}_2 = 0, \quad (22)$$
i.e. $\tilde{\psi}_{1,2}$ are obtained from scalar boson wave function by substitutions $m^2 \to m^2 \pm 2eH$. So we may write

$$\tilde{\psi}_1 \propto 2c_1 D_{n-1}(\zeta), \quad \tilde{\psi}_2 \propto 2c_2 D_{n+1}(\zeta), \quad \zeta = \sqrt{2eH(x_1 + \frac{p_2}{eH})}. \tag{23}$$

Thus we have instead of (8)

$$+\psi_{p_2p_3n} = \left[ [c_+ D_{\nu+1}(\tau) + c_- D_{\nu-1}(\tau)] D_n(\zeta) \right]$$

$$+ i[c_1 D_{n-1}(\zeta) + c_2 D_{n+1}(\zeta)] D_{\nu}(\tau)$$

$$+ [c_+ D_{\nu+1}(\tau) - c_- D_{\nu-1}(\tau)] D_n(\zeta) \right] e^{i(p_2 x_2 + p_3 x_3)}, \tag{24}$$

and similarly for other $\psi$. Here

$$\nu = \frac{i\lambda}{2} - \frac{1}{2}, \quad \lambda = \frac{m^2 + eH(2n + 1)}{eE}, \tag{25}$$

The constraints can be obtained from previous ones by substitution

$$c_1 p_1 + c_2 p_2 \to -i\sqrt{2eH}(1 + n)c_2 - c_1. \tag{26}$$

We note here that $D_{\mu}\psi^\mu$ is proportional to scalar wave function

$$D_n(\zeta) D_{\nu}(\tau) \exp[i(p_2 x_2 + p_3 x_3)]$$

(which is dropped in the expressions like (10) with modification (26), or in (116)). The equations (67) and (98) were used to obtain the constraints. It follows from derivation that the presence of $c_1$ in the r.h.s. of (26) is due to the assumption that $D_{n-1}(\zeta)$ in (24) is not zero, i.e. $n \geq 1$.

Using (24) and (26), we can build three polarization states $\psi(i, x), i = 1, 2, 3$, see Sec. 7. For these states the minimal values of $n$ in (25) are $-1, 0, 1$ correspondingly. Thus the Bogoliubov coefficients depend on all four quantum numbers $(n = p_2, p_3, n, i)$ through minimal $n$.

Taking into account that $2\text{Im}L = \sum_n \ln(1 + |c_{2n}|^2)$ it is easy to show that in agreement with [4]

$$\text{Im}2L_{\text{spin1}} = 2 \times 3\text{Im}L_{\text{spin0}} + \{\ln[1 + e^{-\pi m^2 - eH}] - \ln[1 + e^{-\pi m^2 + eH}]\} \frac{\alpha}{\pi} EHVT. \tag{27}$$

The factors outside braces give the statistical weight of "correcting" states, see eqs. (3.6), (3.7) in [6].

The Bogoliubov coefficients permit to find the transition probability from any initial to any final state (with any occupation numbers) [6]. For example, if the initial state is vacuum, we have for the cell with set of quantum numbers $n = p_2, p_3, n, i$

$$|c_{1n}|^{-2} \{1 + w_n + w_n^2 + w_n^3 + \cdots\} = 1, \quad w_n = \frac{|c_{2n}|^2}{|c_{1n}|^2}. \tag{28}$$

Term $|c_{1n}|^{-2}w_n^k$ gives the probability for production of $k$ pairs, $k = 0, 1, 2, \cdots$. The events in cells with different quantum numbers are independent.
4 Propagator of free vector boson

We may take the wave functions of a free vector boson with momentum \( p^\mu = (p^0, 0, 0, p^3) \) in the form

\[
\psi^\mu(i, x) = \frac{u^\mu(i)}{\sqrt{2|p^0|}} e^{ip \cdot x}, \quad \eta^{\mu\nu} = \text{diag}(-1, 1, 1, 1), \quad \mu = 0, 1, 2, 3, \tag{29}
\]

\[
u(1) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad u(2) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad u(3) = \frac{1}{m} \begin{bmatrix} p_3 \\ 0 \\ 0 \\ p^0 \end{bmatrix}.
\]

These solutions satisfy the wave equation (1) and constraint (2) with switched off external field. Summing \( \psi^\mu(i, x)\psi^{\nu*}(i, x') \) over polarization, we find

\[
\sum_{i=1}^{3} \psi^\mu(i, x)\psi^{\nu*}(i, x') = \frac{1}{2|p^0|} \begin{bmatrix} \frac{p^2}{m^2} & 0 & 0 & \frac{p^3 p^0}{m^2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{p^3 p^0}{m^2} & 0 & 0 & \frac{(p^0)^2}{m^2} \end{bmatrix} e^{i(p, x-x')}. \tag{30}
\]

If instead of linear polarization states (29) we use helicity states (cf. §16 in [8]), we get the same result (30). In general we have to replace the matrix in the r.h.s. of (30) by \( \eta^{\mu\nu} + p^\mu p^\nu/m^2 \). Only the presence of this matrix in the expression for propagator (similar to (51)) differs this case from scalar particle case. For this reason the vector boson propagator can be obtained from scalar one

\[
G^{\text{spin}0}(x, x') = \frac{1}{(2\pi)^4} \int d^4p \frac{e^{i(p, x-x')}}{p^2 + m^2 - i\varepsilon} = \frac{1}{(4\pi)^2} \int_0^\infty \frac{ds}{s^2} e^{-ism^2 + i(x-x')^2}, \tag{31}
\]

considered as a unit matrix over discrete indices, by acting on it with the differential operator:

\[
G^{\mu\nu}(x, x') = \left( \eta^{\mu\nu} - \frac{1}{m^2} \frac{\partial^2}{\partial x_\mu \partial x_\nu} \right) G^{\text{spin}0}(x, x'). \tag{32}
\]

As the scalar boson propagator satisfies the equation

\[
(-\partial_\mu \partial^\mu + m^2)G^{\text{spin}0}(x, x') = \delta^4(x - x'), \tag{33}
\]

we have for vector boson

\[
(-\partial_\sigma \partial^\sigma + m^2)G^{\mu\nu}(x, x') = (\eta^{\mu\nu} - \frac{1}{m^2} \frac{\partial^2}{\partial x_\mu \partial x_\nu})\delta^4(x - x'), \tag{34}
\]

i.e. on the r.h.s. stands not simply \( \delta^4(x - x') \). The complication is due to the existence of constraint. This circumstance prevent us from using the well-known methods of constructing propagators of scalar and spinor particles in an external field [9, 10]. An elegant way to circumvent this difficulty is given by Vanyashin and Terentyev [3].
5 Vector boson propagator in a constant magnetic field

To write down the propagator we need the complete set of orthonormalized solutions. The orthonormalization is performed with the help of expression for vector current [5]

\[ J_\mu = -i\{\psi^{\mu*}(D_\mu \psi - D_\nu \psi_\nu) - (D_\mu^* \psi^*_\nu - D_\nu^* \psi^*_\mu)\psi^\nu\}, \quad D_\mu = \partial_\mu + ieA_\mu. \] (35)

By the way our expression for \( D_\mu \) in (35) coincide with that in [5]; although our \( \eta_{\mu\nu} \) differs in sign, we also substitute \( e \to -e \), using \( e = |e| \) and assuming by analogy with electron that \( W^- \) is particle.

In space without field the expression for \( J_\mu \) up to divergence terms can be written in form similar to scalar case, see §15 in [8]. It is remarkable that with constraint (2) the same is true in the presence of field. Indeed,

\[-\psi^{\mu*} D_\nu \psi_\mu = -\partial_\nu(\psi^{\mu*} \psi_\mu) + \psi_\mu D_\nu^* \psi^{\mu*}. \] (36)

But the last term on the r.h.s. is zero due to (2) for boson with \( g = 2 \). Similarly

\[(D_\nu^* \psi^*_\mu)\psi^\nu = \partial_\nu(\psi^{\mu*} \psi_\mu) - \psi_\mu D_\nu^* \psi^\nu = \partial_\nu(\psi^{\mu*} \psi_\mu). \] (37)

So

\[ J_\mu = -i\{\psi^{\mu*} D_\nu \psi_\nu - (D_\mu^* \psi^*_\nu)\psi^\nu - \partial_\nu[\psi^{\mu*} \psi_\mu - \psi^{\nu*} \psi^*_\nu]\}. \] (38)

To normalize wave functions we need only \( J_0 \). Straightforward calculations show that the divergence terms does not contribute to \( J_0 \) for considered fields. Then

\[ J^0 = -J_0 = i\{\psi^{\mu*} D_0 \psi_\nu - (D_0^* \psi^*_\nu)\psi^\nu\}. \] (39)

Correspondingly we use for orthonormalization the expression

\[ J^0(\psi', \psi) = i\{\psi'^{\mu*} D_0 \psi_\nu - (D_0^* \psi'^*_\nu)\psi^\nu\}. \] (40)

For our vector-potentials \( A_0(x) = 0 \). Then \( D_0 = \frac{\partial}{\partial t} \) and

\[ J^0(\psi', \psi) = i\{\psi'^*_k \frac{\partial}{\partial t} \psi_k - \psi'^{0*} \frac{\partial}{\partial t} \psi^0\}. \] (41)

The sum over \( k \) runs from 1 to 3.

The positive-frequency solution of the wave equation (1) with \( A_\mu(x) = \delta_{\mu 2} H x_1 \) has the form

\[ \psi^\mu_{\nu,p_2,p_3,n} = \begin{bmatrix} e^{iD_n(\zeta)} \\ c_1 D_{n-1}(\zeta) + c_2 D_{n+1}(\zeta) \\ i[c_1 D_{n-1}(\zeta) - c_2 D_{n+1}(\zeta)] \\ c_3 D_n(\zeta) \end{bmatrix} e^{i(p_{2x_2} + p_{3x_3} - p^0 t)}. \] (42)

The elements of this column correspond to \( \mu = 0, 1, 2, 3 \),

\[ \zeta = \sqrt{2eH(x_1 + \frac{p_2}{eH})}, \quad p^0 = \sqrt{m^2 + p_3^2 + eH(2n + 1)}. \]

The coefficients \( c \), determining the boson polarization, satisfy the constraint

\[-ip^0 c^0 + ip_3 c_3 + \sqrt{2eH}(n + 1)c_2 - c_1 = 0. \] (43)
For states with polarizations $c'$ and $c$ we get from (41) and (42)
\[ J^0(\psi', \psi) = 2p_0 \{ 2c'_1 c_1 D^2_{n-1}(\zeta) + 2c'_2 c_2 D^2_{n+1}(\zeta) + (c'_3 c_3 - c''0 c^0) D^2_n(\zeta) \}, \tag{44} \]
Integrating over $x_1$, we find
\[ \int_{-\infty}^{\infty} dx_1 J^0(\psi', \psi) = 2p_0 n! \sqrt{\frac{\pi}{eH}} \left\{ \frac{2}{n} c'_1 c_1 + 2(n + 1) c'_2 c_2 + c'_3 c_3 - c''0 c^0 \right\}, \tag{45} \]
Using orthonormalization conditions
\[ \int dx_1 J^0(\psi(i, x), \pm \psi(j, x)) = \pm \delta_{ij}, \quad i, j = 1, 2, 3, \tag{46} \]
and constraint (43), we find the following positive-frequency polarization states
\[ \psi^\mu(1, x) = N(1) \begin{bmatrix} \sqrt{2} e^{i(p_{2x} + p_{3x} - p^0 t)} \sqrt{2} e^{i(p_{2x} + p_{3x} - p^0 t)} \end{bmatrix}_{\mu} D_n(\zeta) \tag{47} \]
Here
\[ \mu = 0, 1, 2, 3, \quad m^2 = m^2 + eH(2n + 1), \quad p^0 = \sqrt{m^2 + p^2 + eH(2n + 1)}, \]
\[ N(1) = n_1 N_0, \quad N_0 = \left( \frac{eH}{\pi} \right)^{\frac{1}{4}} \frac{1}{\sqrt{2|p^0| n!}}, \quad n_1 = \frac{1}{\sqrt{2(n + 1)m^2(m^2 + eHn)}} \tag{48} \]
\[ \psi(2, x) = n_2 N_0 \begin{bmatrix} p_3 D_n(\zeta) \\ 0 \\ 0 \\ p^0 D_n(\zeta) \end{bmatrix} e^{i(p_{2x} + p_{3x} - p^0 t)}, \quad n_2 = \frac{1}{\sqrt{m^2}}, \tag{49} \]
\[ \psi(3, x) = n_3 N_0 \begin{bmatrix} \sqrt{2} e^{i(p_{2x} + p_{3x} - p^0 t)} \sqrt{2} e^{i(p_{2x} + p_{3x} - p^0 t)} \end{bmatrix}_{\mu} D_n(\zeta) \tag{47} \]n_3 = \sqrt{n} \sqrt{2m^2(m^2 + eHn)} \tag{50} \]
We detach from normalization factors $N(i)$ the normalization factor $N_0$ of scalar wave function, because we shall concentrate our attention on the differences from scalar case. We note also that $N(3) \propto \Gamma^{-\frac{1}{2}}(n)$ and is zero for $n = 0$. So for state $\psi(3, x)$ only $n = 1, 2, 3 \cdots$ are possible. The same follows from the fact that the constraint (43) cannot be satisfied because $c_1$ is absent in it for $n = 0$. 

Now we are in a position to build up vector boson propagator. We start from the expression (which is a special case of more general result derived in Sec. 6, see (80-81))

\[
G^{\mu\nu}(x, x') = i \int \frac{dp_3}{2\pi} \int \frac{dp_2}{2\pi} \sum_{n=-1}^{\infty} \sum_{i=1}^{3} \left\{ _{+}\psi^{\mu}(i, x) + _{+}\psi^{\nu}(i, x') , \quad t > t' \right. \\
\left. - _{-}\psi^{\mu}(i, x) - _{-}\psi^{\nu}(i, x') , \quad t < t' \right\}
\] (51)

We denote in the following the previous quantity \( p^0 \) as \( E_n \) and use the relations

\[
-\frac{1}{2\pi i} \int_{-\infty}^{\infty} dp^0 \frac{e^{-ip^0(t-t')}}{(p^0 - E_n + i\epsilon)(p^0 + E_n - i\epsilon)} = \frac{1}{2E_n} \left\{ e^{-iE_n(t-t')} , \quad t > t' \right.
\]
\[
\left. e^{iE_n(t-t')} , \quad t < t' \right\};
\] (52)

\[
\frac{1}{i(E_n^2 - (p^0)^2)} = \int_{0}^{\infty} ds e^{-is(E_n^2 - p^0)}
\]

to rewrite (51) in the form \( (p^0 = -p_0) \)

\[
G^{\mu\nu}(x, x') = i \sqrt{\frac{eH}{\pi}} \sum_{n=-1}^{\infty} \int \frac{dp_2}{2\pi} \int \frac{dp_3}{2\pi} \int \frac{dp^0}{2\pi} \int_{0}^{\infty} ds a^{\mu\nu}(x, x') \frac{1}{n!} \times
\]
\[
e^{-is(m_2^2 + p_2^2 - p_0^2) + ip_2(x_2 - x'_2) + p_3(x_3 - x'_3) - p_0(t - t')} , \quad m_2^2 = m^2 + eH(2n + 1).
\] (53)

We note that the lower line on r.h.s. of (52) is obtained from the upper line by substitution \( t \leftrightarrow t' \), which does not change anything, as the r.h.s. may be written in the form \( (2E_n)^{-1} \exp[-iE_n|t - t'|] \). The explicitly symmetric in \( t, t' \) form of the l.h.s. is

\[
\int_{0}^{\infty} ds e^{-isE_n^2} \int_{-\infty}^{\infty} \frac{dp^0}{2\pi} e^{ip_0^2 - ip^0(t-t')} = \frac{e^+}{2\sqrt{\pi}} \int_{0}^{\infty} \frac{ds}{\sqrt{s}} \exp[-isE_n^2 - i(t - t')^2/4s].
\] (54)

We first get the propagator for scalar particle in proper time representation [10]. We substitute in (53) \( a^{\mu\nu}(x, x') \) by \( D_n(\zeta)D_n(\zeta') \). Then using the formula

\[
D_n(\zeta) = \sqrt{\frac{2}{\pi}} e^{\frac{\zeta^2}{2}} \int_{0}^{\infty} dy y^n e^{-\frac{y^2}{2}} \cos(\zeta y - \frac{n\pi}{2}),
\] (55)

we find

\[
\sum_{n=0}^{\infty} \frac{D_n(\zeta)D_n(\zeta')}{n!} e^{-ir(2n+1)} = (2 \sin 2\tau)^{-\frac{3}{4}} \exp[-i\frac{\pi}{4} + i\frac{(\zeta - \zeta')^2}{8 \tan \tau} - i\frac{(\zeta + \zeta')^2}{8 \cot \tau}],
\]
\[
\tau = eHs; \quad \zeta' = \sqrt{2eH(x'_1 + \frac{p_2}{eH})}.
\] (56)

Subsequent integration over \( p_2 \) gives

\[
\int_{-\infty}^{\infty} \frac{dp_2}{2\pi} \sum_{n=0}^{\infty} \frac{D_n(\zeta)D_n(\zeta')}{n!} e^{-ir(2n+1) + ip_2z_2} =
\]

\[ -i \sqrt{\frac{eH}{\pi}} (4 \sin \tau)^{-1} \exp[-i \frac{eHz_2(x_1 + x'_1)}{2} + i \frac{eH(z_1^2 + z_2^2)}{4 \tan \tau}], \quad z_\mu = x_\mu - x'_\mu. \] (57)

Using also

\[ \int_{-\infty}^{\infty} \frac{dp_3}{2\pi} \int_{-\infty}^{\infty} \frac{dp^0}{2\pi} \exp[i(p_3z_3 - p^0z^0) - is(p_3^2 - p^0_3)] = \frac{1}{4\pi s} \exp[i \frac{z_3^2 - z_0^2}{4s}], \] (58)

we find \[ 9, 10, 6, 11].

\[ G_{\text{spin0}}(x, x') = \frac{eH}{(4\pi)^2} \int_0^\infty \frac{ds}{s \sin eHs} \times \]

\[ \exp[-i \frac{eHz_2(x_1 + x'_1)}{2}] \exp[-ism^2 + i \frac{z_3^2 - z_0^2}{4s} + i \frac{(z_1^2 + z_2^2)eH}{4 \tan eHs}]. \] (59)

We shall show now how to obtain \[ \mu\nu(x, x') \] in (53) and how to turn it into differential matrix, which, when inserted in the integrand in (59), will give the propagator of vector boson. As a preliminary we write down two formulae directly related to (56):

\[ \sum_{n=-1}^{\infty} \frac{D_{n+1}(\zeta)D_{n+1}(\zeta')}{(n+1)!} e^{-i\pi(2n+1)} = e^{2i\pi} \sum_{n=0}^{\infty} \frac{D_n(\zeta)D_n(\zeta')}{n!} e^{-i\pi(2n+1)}, \] (60)

\[ \sum_{n=1}^{\infty} \frac{D_{n-1}(\zeta)D_{n-1}(\zeta')}{(n-1)!} e^{-i\pi(2n+1)} = e^{-2i\pi} \sum_{n=0}^{\infty} \frac{D_n(\zeta)D_n(\zeta')}{n!} e^{-i\pi(2n+1)}. \] (61)

We see that the expressions in (60) and (61) only by factors \( e^{2i\tau} \) and \( e^{-2i\tau} \) differ from scalar case.

Now we go back to \( \mu\nu(x, x') \). As seen from (51) and (53)

\[ a^{\mu\nu}(x, x') \propto \sum_{i=1}^{3} \psi^\mu(i, x)\psi^{\nu*}(i, x'). \] (62)

For example, we consider \( a^{11}(x, x') \). We see from (49) that \( \psi^1(2, x) = 0 \), i.e. the term with \( i = 2 \) does not contribute to \( a^{11}(x, x') \). According to (47) the contribution from term with \( i = 1 \) is

\[ n_1^2m_1^4D_{n+1}(\zeta)D_{n+1}(\zeta'), \quad n_1^2 = \frac{1}{2(n+1)m_1^2(m^2 + eHn)}. \] (63)

Term with \( i = 3 \) gives

\[ n_3^2[-(m^2 + eHn)D_{n-1}(\zeta) + eHD_{n-1}(\zeta)][-(m^2 + eHn)D_{n-1}(\zeta') + eHD_{n+1}(\zeta')], \]

\[ n_3^2 = \frac{n}{2m^2(m^2 + eHn)}. \] (64)

Now we have \( a^{11}(x, x') \) as a sum of (63) and (64):

\[ a^{11}(x, x') = \frac{1}{2(m^2 + eHn)} \left( \frac{m_1^2}{n+1} + \frac{(eH)^2n}{m^2} \right) D_{n+1}(\zeta)D_{n+1}(\zeta') + \]

\[ \frac{1}{2(m^2 + eHn)} \left( \frac{m_1^2}{n+1} + \frac{(eH)^2n}{m^2} \right) D_{n+1}(\zeta)D_{n+1}(\zeta') + \]
\begin{equation}
\frac{n(m^2 + eHn)}{2m^2} D_{n-1}(\zeta) D_{n-1}(\zeta') - \frac{eHn}{2m^2} [D_{n+1}(\zeta) D_{n-1}(\zeta') + D_{n-1}(\zeta) D_{n+1}(\zeta')].
\end{equation}

Next we note that
\begin{equation}
\frac{1}{m^2 + eHn} \left( \frac{m^2}{n+1} + \frac{(eH)^2n}{m^2} \right) = \frac{1}{n+1} + \frac{eH}{m^2},
\end{equation}
i.e. the undesirable factor \( \frac{1}{m^2 + eHn} \), contained in \( n_1^2 \) and \( n_2^3 \) in (63) and (64), disappears in the sum (65).

Further we use the relations, see eqs.(8.2.15-16) in [7]
\begin{equation}
\left( \frac{d}{d\zeta} + \frac{\zeta}{2} \right) D_n(\zeta) = nD_{n-1}(\zeta), \quad \left( \frac{d}{d\zeta} - \frac{\zeta}{2} \right) D_n(\zeta) = -D_{n+1}(\zeta).
\end{equation}

We write down also the sum and difference of these expressions:
\begin{equation}
2 \frac{d}{d\zeta} D_n(\zeta) = nD_{n-1}(\zeta) - D_{n+1}(\zeta), \quad \zeta D_n(\zeta) = nD_{n-1}(\zeta) + D_{n+1}(\zeta).
\end{equation}

Now it is easy to verify that
\begin{equation}
a^{11}_1(x, x') = \frac{D_{n+1}(\zeta) D_{n+1}(\zeta')}{2(n+1)} + \frac{n}{2} D_{n-1}(\zeta) D_{n-1}(\zeta') + \frac{2eH}{m^2} \frac{\partial^2}{\partial\zeta \partial\zeta'} D_n(\zeta) D_n(\zeta').
\end{equation}

The first term on the r.h.s. of (69) works in eq. (60), the second term is used in (61); the necessary factor \( n! \) comes from \( N_0 \), see (48). The third term can be written as
\begin{equation}
\frac{1}{m^2} \frac{\partial^2}{\partial x^1 \partial x_1'} D_n(\zeta) D_n(\zeta').
\end{equation}

In a similar manner we find the other \( a^{\mu\nu}(x, x') = a^{\mu\nu*}(x', x) \). It is easy to verify that the differential operator \( A^{\mu\nu}(x, x') \), corresponding to \( a^{\mu\nu}(x, x') \), has the form
\begin{equation}
A^{\mu\nu} = B^{\mu\nu} + C^{\mu\nu}, \quad C^{\mu\nu} = \frac{1}{m^2} \Pi^\mu(x) \Pi^{\nu*}(x'),
\end{equation}

\begin{equation}
\Pi^\mu(x) = -i \frac{\partial}{\partial x^\mu} + eA^\mu(x), \quad \Pi^{\nu*}(x') = i \frac{\partial}{\partial x'^\nu} + eA^\nu(x').
\end{equation}

In our case
\begin{equation}
A^\mu(x) = \delta_{\mu2} Hx_1, \quad \Pi^0(x) = i \frac{\partial}{\partial t}, \quad \Pi^{0*}(x') = -i \frac{\partial}{\partial t'}.
\end{equation}

The nonzero \( B^{\mu\nu} \) are
\begin{equation}
B^{11} = B^{22} = \cos \tau, \quad B^{21} = -B^{12} = \sin \tau, \quad B^{33} = -B^{00} = 1.
\end{equation}

The difference of \( B^{\mu\nu} \) from \( \eta^{\mu\nu} \) is due to the interaction of boson magnetic moment with magnetic field. We may say that \( B^{\mu\nu} \) with \( \mu, \nu = 1, 2 \) are ”dressed” by magnetic field.

Thus
\begin{equation}
G^{\mu\nu}(x, x') = \frac{eH}{(4\pi)^2} \int_0^\infty \frac{ds}{s \sin esm^2} e^{-ism^2} A^{\mu\nu} \times
\end{equation}
\[
\exp[-\frac{ieHz_2(x_1 + x'_1)}{2}]\exp[i\frac{z_2^2 - z_0^2}{4s} + \frac{i}{4}(z_1^2 + z_2^2)eH \cot eHs], \quad z_\mu = x_\mu - x'_\mu. \quad (74)
\]

It is somewhat surprising that this representation does not coincide with Vanyashin-Terentyev representation [3] with switched off electric field. Possibly these are two different representations of one and the same propagator and it would be interesting to verify this supposition.

## 6 Propagator of vector boson in constant electric field

First, we shall give the generalization of (51) for the case, when external field can create pairs [12, 6]. To this end we write

\[
G(x, x')_{abs} = i < 0_{out}|T(\Psi(x)\Psi^\dagger(x'))|0_{in} >= < 0_{out}|0_{in} > G(x, x'), \quad (75)
\]

where \(T\) is chronological ordering operator,

\[
\Psi(x) = \sum_n [a_{n, out}^+ \psi_n(x) + b_{n, out}^+ \psi_n(x)], \quad \Psi^\dagger(x) = \sum_n [a_{n, in}^+ \psi_n^*(x) + b_{n, in}^+ \psi_n^*(x)] \quad (66)
\]

As usual, \(a_n\) and \(b_n\) are operators of destruction of particle and antiparticle in a state with quantum numbers \(n:\)

\[
\Psi^\dagger(x')|0_{in} >= \sum_k \psi_k^+(x') a_{k, in}^\dagger |0_{in} >, < 0_{out}|\Psi(x) = < 0_{out}|\sum_n a_{n, out}^+ \psi_n(x). \quad (77)
\]

For \(t > t'\) from (75), (77) we have

\[
G(x, x')_{abs} = i \sum_{n, k}^+ \psi_n(x) + \psi_k^*(x') < 0_{out}|a_{n, out}^+ a_{k, in}^\dagger |0_{in} >, \quad t > t'. \quad (78)
\]

The Bogoliubov transformations in our case have the form (cf. eq. (18))[6]

\[
a_{n, out}^\dagger = c_{1, n}^* a_{n, in}^\dagger + c_{2, n} b_{n, in}, \quad b_{n, out} = c_{2, n}^* a_{n, in}^\dagger + c_{1, n} b_{n, in}. \quad (79)
\]

From the first eq. in (79) it follows \(a_{k, out}^\dagger |0_{in} >= c_{1, k} a_{k, in}^\dagger |0_{in} >.\) We substitute \(a_{k, in}^\dagger |0_{in} >\) from here into (78) and use the commutation relation \([a_{k, out}, a_{n, out}^\dagger] = \delta_{kn}\). Then we obtain

\[
G(x, x')_{abs} = < 0_{out}|0_{in} i \sum_n^+ \psi_n(x) + \psi_n^*(x') \frac{1}{c_{1, n}}, \quad t > t'. \quad (80)
\]

Similarly for \(t < t'\) we find

\[
G(x, x')_{abs} = < 0_{out}|0_{in} i \sum_n^- \psi_n(x) - \psi_n^*(x') \frac{1}{c_{1, n}}, \quad t < t'. \quad (81)
\]

If an external field does not create pairs, the obtained expressions go into (51).
The transition current \((41)\) in terms of states \(\psi, \psi\) in \((8)\) takes the form
\[
J^0(\psi, \psi) = \sqrt{2eE} e^{i\frac{\pi\lambda}{4}}[c_1' + c_2' + 2i(c_2' - c_1 + c_3' + c_4')].
\] (82)

Here we have used eq. (8.2.11) in [7] (and its complex conjugate):
\[
D_{\nu+1}(\tau^*) \frac{d}{dt} D_{\nu-1}(\tau) = \sqrt{2eE} \exp\left[\frac{\pi\lambda}{4}\right] =
\]
\[
-D_{\nu-1}(\tau^*) \frac{d}{dt} D_{\nu+1}(\tau) = D_{\nu}^∗(\tau^*)i \frac{d}{dt} D_{\nu}(\tau).
\] (83)

The constraint is given in \((10)\). Using \((82)\), \((8)\) and \((10)\) we find the following \(\psi\) polarization states:
\[
+\psi(1, x) = N(1) \begin{bmatrix} p_2 \sqrt{\frac{eE}{2}} e^{i\frac{i\pi\lambda}{4}} [D_{\nu+1}(\tau) - \nu D_{\nu-1}(\tau)] \\ 0 \\ m_2^2 D_{\nu}(\tau) \\ p_2 \sqrt{\frac{eE}{2}} e^{i\frac{i\pi\lambda}{4}} [D_{\nu+1}(\tau) + \nu D_{\nu-1}(\tau)] \end{bmatrix} e^{ip\cdot x}, \quad N(i) = n_1 N_0,
\]
\[
N(1) = n_1 N_0, \quad n_1 = \sqrt{\frac{1}{m_2^2 (m^2 + p_1^2)}}, \quad N_0 = \frac{1}{2} e^{-\frac{i\pi\lambda}{8}},
\] (85)
\[
+\psi(2, x) = N(2) \begin{bmatrix} D_{\nu+1}(\tau) + (1 + \nu) D_{\nu-1}(\tau) \\ 0 \\ 0 \\ D_{\nu+1}(\tau) - (1 + \nu) D_{\nu-1}(\tau) \end{bmatrix} e^{ip\cdot x},
\]
\[
n_2 = \sqrt{\frac{eE}{2m_2}}, \quad m_2^2 = m^2 + p_1^2 + p_2^2,
\] (86)
\[
+\psi(3, x) = N(3) \begin{bmatrix} p_1 \sqrt{\frac{eE}{2}} e^{i\frac{i\pi\lambda}{4}} [D_{\nu+1}(\tau) - \nu D_{\nu-1}(\tau)] \\ (m^2 + p_1^2) D_{\nu}(\tau) \\ p_1 p_2 D_{\nu}(\tau) \\ p_1 \sqrt{\frac{eE}{2}} e^{i\frac{i\pi\lambda}{4}} [D_{\nu+1}(\tau) + \nu D_{\nu-1}(\tau)] \end{bmatrix} e^{ip\cdot x},
\]
\[
n_3 = \frac{1}{\sqrt{m_2^2 (m^2 + p_1^2)}},
\] (87)

The \(\psi\) polarization states can be obtained from these ones with the help of eqs. \((19)\) (see also \((16)\)):
\[
+\psi(1, x) = N(1) \begin{bmatrix} p_2 \sqrt{\frac{eE}{2}} e^{-\frac{i\pi\lambda}{4}} [(1 + \nu) D_{\nu^*-1}(-\tau^*) + D_{\nu^*+1}(-\tau^*)] \\ 0 \\ m_2^2 D_{\nu}(\tau^*) \\ p_2 \sqrt{\frac{eE}{2}} e^{-\frac{i\pi\lambda}{4}} [(1 + \nu) D_{\nu^*-1}(-\tau^*) - D_{\nu^*+1}(-\tau^*)] \end{bmatrix} e^{ip\cdot x},
\] (88)
Here

It can be derived similarly to the magnetic case, but the role of eq. (52) plays the relation

\[ \psi \]

In eqs. (85-90) the states \( \psi(i, x) \) are characterized by \( p_1, p_2, p_3, i; \nu \) and \( \lambda \) are given in (7).

We note here that the transition current \( J^0(\psi, +\psi) \) in terms of \( +c \) has the same form as \( J^0(\psi', +\psi) \) in terms of \( +c \), see (82). Similar statement is true for negative-frequency states. Taking into account that \( \nu + 1 = -\nu^* \), see (7), we have from (19)

\[ +c^*_+ + c_+ = +c^*_+ + c_+ = -c^*_- c_+ = -c^*_- c_+ . \]  

So

\[ J^0(\psi(i, x), +\psi(j, x)) = J^0(\psi(i, x), +\psi(j, x)) \propto \delta_{ij}, \]  

and

\[ J^0(\psi(i, x), -\psi(j, x)) = J^0(\psi(i, x), -\psi(j, x)) = -J^0(\psi(i, x), +\psi(j, x)). \]

As earlier we shall focus our attention on differences from scalar case in expression similar to (53). The proper-time representation of propagator for scalar particle is [12]:

\[ G(x, x')_{\text{spin}0} = \frac{eE}{(4\pi)^2} \exp\left[\frac{i}{2} eE(t + t') z_3 \right] \times \int_0^\infty \frac{ds}{s \sinh eEs} \exp[-ism^2 + \frac{i}{4s}(z_1^2 + z_2^2) + \frac{i}{4}eE(z_3^2 - z_0^2) \coth eEs] . \]  

It can be derived similarly to the magnetic case, but the role of eq. (52) plays the relation [12, 6]

\[ \sqrt{2} \int_0^\infty \frac{d\theta}{\sqrt{\sinh 2\theta}} \exp\left\{ -i2\pi\theta - \frac{i}{8} \left[ \frac{(T + T')^2}{\coth \theta} + \frac{(T - T')^2}{\tanh \theta} \right] \right\} = \]

\[ = \Gamma \left( i\kappa + \frac{1}{2} \right) \left\{ \begin{array}{ll}
D_{-i\kappa-\frac{1}{2}}(\chi)D_{-i\kappa-\frac{1}{2}}(-\chi'), & T > T' \\
D_{-i\kappa-\frac{1}{2}}(-\chi)D_{-i\kappa-\frac{1}{2}}(\chi'), & T < T'.
\end{array} \right. \]  

Here

\[ \theta = eEs, \quad T = \sqrt{2eE(t - \frac{P_3}{eE})}, \quad T' = \sqrt{2eE(t' - \frac{P_3}{eE})}, \]

\[ \chi = -\tau^* = e^{\frac{i}{4}T}, \quad \chi' = e^{\frac{i}{4}T'}, \quad \kappa = \frac{\lambda}{2} = \frac{m^2}{2eE}. \]
The lower line on the r.h.s. of (93) can be obtained from the upper line by substitution $T \leftrightarrow T'$. As seen from the l.h.s. of (93) this does not change the value of (93), cf. with remark after eq. (53).

By analogy with magnetic case we expect the appearance of factors $e^{\pm 2\theta}$ in the integrand of (93), cf. with eqs. (73), (60-61). To make the insertion possible, we have to rotate clockwise the integration contour by some angle. This is in line with Vanyashin-Terentyev approach [3]. In this way by substitution $\kappa \rightarrow \kappa + i$ we get from (93)

$$
\sqrt{2} \int_C \frac{d\theta}{\sqrt{\sinh 2\theta}} \exp \left\{ -i2\tau + 2\theta - \frac{i}{8} \left[ \frac{(T + T')^2}{\coth \theta} + \frac{(T - T')^2}{\tanh \theta} \right] \right\} = 
$$

$$
= \Gamma \left( i\tau - \frac{1}{2} \right) \left\{ \begin{array}{l} D_{-i\tau + \frac{1}{2}}(\chi)D_{-i\tau + \frac{1}{2}}(-\chi'), \quad T > T' \\ D_{-i\tau + \frac{1}{2}}(-\chi)D_{-i\tau + \frac{1}{2}}(\chi'), \quad T < T' \end{array} \right. 
$$

(95)

Similarly, substituting $\tau \rightarrow \tau - i$ in (93) we get

$$
\sqrt{2} \int_0^\infty \frac{d\theta}{\sqrt{\sinh 2\theta}} \exp \left\{ -i2\tau - 2\theta - \frac{i}{8} \left[ \frac{(T + T')^2}{\coth \theta} + \frac{(T - T')^2}{\tanh \theta} \right] \right\} = 
$$

$$
= \Gamma \left( i\tau + \frac{3}{2} \right) \left\{ \begin{array}{l} D_{-i\tau - \frac{1}{2}}(\chi)D_{-i\tau - \frac{1}{2}}(-\chi'), \quad T > T' \\ D_{-i\tau - \frac{1}{2}}(-\chi)D_{-i\tau - \frac{1}{2}}(\chi'), \quad T < T' \end{array} \right. 
$$

(96)

Integration over $p_3$, contained in the sum over $n$ in (80-81) gives $(T, T' \text{ are function of } p_3$, see (94))

$$
\int_{-\infty}^\infty \frac{dp_3}{2\pi} \exp[ip_3 z_3 - \frac{i}{8}(T + T')^2 \tanh \theta] = 
$$

$$
\frac{1}{2} e^{-\frac{i\pi}{4}} \sqrt{\frac{eE \coth \theta}{\pi}} \exp \left\{ \frac{i z_3^2 e E}{4 \tanh \theta} + \frac{ie E z_3 (t + t')}{2} \right\}, \quad z_3 = x_3 - x'_3.
$$

Further calculations leading to (92) are similar to magnetic case.

Now we look for differences from scalar case. First, we rewrite relations (67-68) between parabolic cylinder functions for present case:

$$
\frac{d}{d\tau^{**}} \tau^{**} D_{\nu^{**}}(\tau^{**}) = \nu^{*} D_{\nu^{**}-1}(\tau^{**}),
$$

$$
\frac{d}{d\tau^{**}} \tau^{**} D_{\nu^{-}}(\tau^{**}) = -D_{\nu^{**}+1}(\tau^{**}),
$$

$$
2 \frac{d}{d\tau^{**}} D_{\nu^{**}}(\tau^{**}) = \nu^{*} D_{\nu^{**}-1}(\tau^{**}) - D_{\nu^{**}+1}(\tau^{**}),
$$

$$
\tau^{**} D_{\nu^{**}}(\tau^{**}) = \nu^{*} D_{\nu^{**}-1}(\tau^{**}) + D_{\nu^{**}+1}(\tau^{**}).
$$
Other necessary relations are obtained from these by substitution $\tau'' \to -\tau^*$.

Now taking into account that

$$c_{1n} = \frac{\sqrt{2\pi}}{\Gamma(-i\kappa + \frac{1}{2})} \exp[-\frac{\pi\kappa}{2} + \frac{i\pi}{4}], \quad i\frac{\pi}{c_{1n} N_0^2} = \frac{\exp[\frac{3i\pi}{4}]}{2\sqrt{\pi eE}} \Gamma(i\kappa + \frac{1}{2}),$$

(100)

we can write the propagator in the form

$$G^{\mu\nu}(x, x') = \frac{\exp[\frac{3i\pi}{4}]}{2\sqrt{\pi eE}} \int \frac{d^3p}{(2\pi)^3} a^{\mu\nu}(x, x') e^{i(\vec{p}, \vec{x} - \vec{x}')},$$

(101)

The scalar particle propagator can be obtained from the r.h.s. of (101), if we substitute $a^{\mu\nu}(x, x')$ by the expression (93). As an example we calculate now $a^{33}(x, x')$. For $t > t'$ we have

$$a^{33}(x, x') \propto \sum_{i=1}^{3} +\psi^3(i, x) + \psi^{3*}(i, x').$$

(102)

The first term in the sum is

$$+\psi^3(1, x) + \psi^{3*}(1, x') \propto -\frac{i eE}{2}\tau^* D_{\nu^*}(-\tau^*)\tau'' D_{\nu^*}(\tau'') \frac{p_2^2}{m_\perp (m^2 + p_1^2)}.$$

(103)

We have used here the second equation in (99) and the one obtained from it by substitution $\tau'' \to -\tau^*$. Similarly,

$$+\psi^3(3, x) + \psi^{3*}(3, x') \propto -\frac{i eE}{2}\tau^* D_{\nu^*}(-\tau^*)\tau'' D_{\nu^*}(\tau'') \frac{p_1^2}{m^2 (m^2 + p_1^2)}.$$

(104)

Summing (103) and (104), we get

$$-\frac{i eE}{2}\tau^* D_{\nu^*}(-\tau^*)\tau'' D_{\nu^*}(\tau'')\left[\frac{p_2^2}{m_\perp (m^2 + p_1^2)} + \frac{p_1^2}{m^2 (m^2 + p_1^2)}\right].$$

(105)

The expression in square brackets can be simplified:

$$\frac{1}{m^2 + p_1^2} \left(\frac{p_2^2}{m_\perp} + \frac{p_1^2}{m^2}\right) = \frac{1}{m^2} - \frac{1}{m_\perp}.$$

(106)

The undesirable factor $(m^2 + p_1^2)^{-1}$, present in (103) and (104), disappears in the sum (105).

The first term on the r.h.s. of (106) gives the following contribution to (105):

$$-\frac{i eE}{2m^2}\tau^* D_{\nu^*}(-\tau^*)\tau'' D_{\nu^*}(\tau'') = \frac{1}{m^2} (p_3 - eEt)(p_3 - eEt') D_{\nu^*}(-\tau^*) D_{\nu^*}(\tau'').$$

(107)

This is already the desired form. Now we rewrite the contribution from second term on the r.h.s. of (106) to (105) in the initial form (i.e. before using second equation in (99)):

$$\frac{i eE}{2m_\perp} \left[(-1 + \nu)^2 D_{\nu^* - 1}(-\tau^*) D_{\nu^* - 1}(\tau'') + (1 + \nu) D_{\nu^* + 1}(-\tau^*) D_{\nu^* - 1}(\tau'') + (1 + \nu) D_{\nu^* - 1}(-\tau^*) D_{\nu^* + 1}(\tau'') - D_{\nu^* + 1}(-\tau^*) D_{\nu^* - 1}(\tau'')\right].$$

(108)
This expression still contains undesirable factor \( \frac{1}{m_\perp^2} \). But we must take into account the contribution from term with \( i = 2 \) in (102):

\[
\psi^3(2, x) + \psi^{3*}(2, x') \propto \frac{ieE}{2m_\perp^2}(1 + \nu)[-D_{\nu-1}(-\tau^*)D_{\nu+1}(\tau'')] - \frac{1}{\nu}D_{\nu+1}(-\tau^*)D_{\nu-1}(\tau'') - \nu D_{\nu-1}(-\tau^*)D_{\nu-1}(\tau'') - D_{\nu+1}(-\tau^*)D_{\nu-1}(\tau'').
\]  

(109)

It is easy to see that in the sum of (108) and (109) undesirable terms are cancelled and unpleasant denominator \( m_\perp^2 = -ieE(1 + 2\nu) \) disappears:

\[
(108) + (109) = \frac{1}{2}((1 + \nu)D_{\nu-1}(-\tau^*)D_{\nu-1}(\tau'') + \frac{1}{\nu}D_{\nu+1}(-\tau^*)D_{\nu+1}(\tau'')).
\]  

(110)

Thus \( a_{33}(x, x') \) is given by the sum of expressions (107) and (110). The first term on the r.h.s. of (110) is used in (96) and the second term in (95). In the same manner we find all other \( a_{\mu\nu}(x, x') \). Similarly to the magnetic case we have

\[
G^{\mu\nu}(x, x') = \frac{eE}{(4\pi)^2} \int_C \frac{ds}{s \sinh eEs} A^{\mu\nu} \exp[i\frac{eE}{2}z_3(t + t')] \times
\]

\[
\exp[-ism^2 + \frac{i}{4s}(z_1^2 + z_2^2) + \frac{i}{4s}(z_3 - z_0^2)eE \coth eEs].
\]  

(111)

Here \( A^{\mu\nu} \) has the form (71), but the vector-potential is \( A_\mu(x) = -\delta_{\mu3} Et \). The nonzero \( B^{\mu\nu} \) are

\[
B^{11} = B^{22} = 1, \quad B^{33} = -B^{00} = \cosh 2eEs, \quad B^{30} = -B^{03} = \sinh 2eEs.
\]  

(112)

We see that electric field dresses \( B^{\mu\nu} \) with \( \mu, \nu = 3, 0 \).

Moving on to the case \( t < t' \), we note that according to (19)

\[
-c_\pm = -c_\pm, \quad -c_\pm = -c_\pm.
\]  

(112a)

It follows from here that \( -\psi(-\psi) \) is obtained from \( +\psi(+\psi) \) by changing sign of arguments of parabolic cylinder functions and sign of \( \psi^0 \) and \( \psi^3 \). The overall change of sign of \( \psi(2, x) \) does not tell on corresponding term in (102). In \( \psi(1, x) \) and \( \psi(3, x) \) changing sign of \( \psi^0 \), \( \psi^3 \) and arguments \( \tau^*, \tau'^* \) is equivalent to changing sign of only arguments \( \tau^*, \tau'^* \), when \( \psi^0 \) and \( \psi^3 \) are expressed through the left hand sides of (99). Now, as expected, it follows from (93-96) that \( G^{\mu\nu}(x, x') \) retains the same form (111) for \( t < t' \).

### 7 Propagator of vector boson in constant electromagnetic field

After we have considered separately the magnetic and electric fields, the building up of the propagator of vector boson in both fields meets with no new problems. We take vector-potential in the form

\[
A_\mu(x) = \delta_{\mu2} H x_1 - \delta_{\mu3} Et.
\]  

(113)
The transition current between states \( +\psi' \) and \( +\psi \) is

\[
J^0(\psi', \psi) = 2\left( [\sqrt{e^2EH} e^{\frac{\pi}{4}} [D_{\nu+1}(\tau) - \nu D_{\nu-1}(\tau)] D_n(\zeta) \right) i \frac{d}{dt} D_{\nu}(\tau) - \frac{d}{dt} D_{\nu-1}(\tau) D_n(\zeta) \]  

Taking into account (84) and integrating over \( x_1 \) we get

\[
\int_{-\infty}^{\infty} dx_1 J^0(\psi', \psi) = n! \sqrt{2\pi E H} 2e^{\frac{\pi}{4}} \left[ \frac{1}{n} c' c_1 + (1 + n) c_2 c_2 + i(c_- c_+ - c'_- c'_+) \right] \]  

The constraint has the form

\[
\sqrt{2eH} [(1 + n)c_2 - c_1] + \sqrt{2eE} e^{-\frac{i\pi}{4}} [ + c_- - (1 + \nu) + c_+] = 0. \]  

Using (115) and (116) we find the \( +\psi \) polarization states (in the following the factor \( e^{i(p_2x_2 + p_3x_3)} \) is dropped for brevity):

\[
+\psi(1, x) = N(1) \left[ (1 + n) \sqrt{e^2EH} e^{\frac{\pi}{4}} [D_{\nu+1}(\tau) - \nu D_{\nu-1}(\tau)] D_n(\zeta) \right] i \frac{d}{dt} D_{\nu}(\tau) D_{\nu+1}(\tau) D_n(\zeta) \]  

\[
N(i) = n_i N_0, \quad N_0 = \left( \frac{H}{2\pi E} \right)^\frac{n}{4} \exp \left[ -\frac{\pi}{4} \right], \]  

\[
n_1 = \frac{1}{\sqrt{2m_1^2(m^2 + eHn)(1 + n)}}, \]  

\[
+\psi(2, x) = N(2) \left[ [D_{\nu+1}(\tau) - (1 + \nu) D_{\nu-1}(\tau)] D_n(\zeta) \right] i \frac{d}{dt} D_{\nu}(\tau) \]  

\[
n_2 = \sqrt{\frac{eE}{2m_2}}, \]  

\[
+\psi(3, x) = N(3) \left[ \sqrt{\frac{e^2EH}{2m^2}} [D_{\nu+1}(\tau) - \nu D_{\nu-1}(\tau)] D_n(\zeta) \right] e^{\frac{i\pi}{4}} [D_{\nu}(\tau) - (m^2 + eHn) D_{\nu-1}(\zeta) + eHD_{\nu+1}(\zeta) ] \]  

\[
n_3 = \sqrt{\frac{n}{2m^2(m^2 + eHn)}}, \]  

To obtain polarization states of \( +\psi \) (or \( -\psi, -\psi \) for that matter) we use again (19) (cf. with (88-90), (47-50)). Then we get

\[
+\psi(1, x) = \]
The first and the fourth lines on the r.h.s. of (122) and (124) can be written in more compact form with the aid of relations obtainable from (99) by substitution \( \tau^* \rightarrow -\tau^* \).

Further calculations are quite similar to those in Sections (5) and (6). The result is evident, of course, beforehand: now \( A^{\mu\nu} \) is given by (71) with vector-potential (113) and all nonzero \( B^{\mu\nu} \) are "dressed", see (73) and (112). The propagator of scalar particle has the form

\[
G_{\text{spin}0}(x, x') = \frac{e^2EH}{(4\pi)^2} \int_0^\infty \frac{ds}{\sinh eEs \sin eHs} \exp\{-ism^2 + (125)
\]

\[
\frac{i}{4}[z_1^2 + z_2^2]eH \cot eHs + (z_3^2 - z_0^2)eE \coth eEs] + \frac{i}{2}[eEz_3(t + t') - eHz_2(x_1 + x_1')],
\]

\( z_\mu = x_\mu - x'_\mu \).

This expression is in agreement with Ritus calculations [10-11]. The presence of phase factor \( e^{i\pi} \) in his formulas is due to difference in definition of propagator. We note also that (125) is symmetric in \( t, t' \) and \( G_{\text{spin}0}(x, x', e) = G_{\text{spin}0}(x', x, -e) \). Thus

\[
G^{\mu\nu}(x, x') = \frac{e^2EH}{(4\pi)^2} \int_0^\infty \frac{ds}{\sinh eEs \sin eHs} A^{\mu\nu} \exp\{-ism^2 + (126)
\]

\[
\frac{i}{4}[z_1^2 + z_2^2]eH \cot eHs + (z_3^2 - z_0^2)eE \coth eEs] + \frac{i}{2}[eEz_3(t + t') - eHz_2(x_1 + x_1')].
\]

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