REGULAR PARTITIONS OF GENTLE GRAPHS

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Dedicated to Endre Szemerédi on the occasion of his eightieth birthday.

Abstract. Szemerédi’s Regularity Lemma is a very useful tool of extremal combinatorics. Recently, several refinements of this seminal result were obtained for special, more structured classes of graphs. We survey these results in their rich combinatorial context. In particular, we stress the link to the theory of (structural) sparsity, which leads to alternative proofs, refinements and solutions of open problems. It is interesting to note that many of these classes present challenging problems. Nevertheless, from the point of view of regularity lemma type statements, they appear as “gentle” classes.

Contents

Introduction 2

Part 1. Preliminaries and survey of some regularity lemmas 4
1. Model theory background 4
2. Graph theoretic background 5
3. Random-free regularity lemmas 9

Part 2. Regularity for gentle graphs 15
4. Set-defined classes: both stable and semi-algebraic regularity 15
5. Order-defined classes 18
6. Inherited regularity of 2-covered classes 19
7. Regularity for classes 2-covered by embedded m-partite cographs 22
8. Regularity and non-regularity for nowhere dense classes 26

Conclusion 29
Acknowledgments 30
References 30

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Szemerédi’s Regularity Lemma. Szemerédi’s Regularity Lemma [77] is a very useful tool in extremal graph theory. Informally, the lemma states that the vertices of every sufficiently large graph can be partitioned into a bounded number of parts so that the edges between almost all pairs of different parts behave in a sense like random graphs. Let us give the formal definitions.

Definition 1. Let $G$ be a graph and let $A, B \subseteq V(G)$ be two disjoint non-empty subsets of vertices. We write $E(A, B)$ for the set of edges with one end in $A$ and the other end in $B$. We define the density of the pair $(A, B)$ as

$$\text{dens}(A, B) := \frac{|E(A, B)|}{|A||B|}.$$ 

Definition 2. Let $\varepsilon > 0$, let $G$ be a graph and let $A, B \subseteq V(G)$ be two disjoint non-empty subsets of vertices. We call the pair $(A, B)$ $\varepsilon$-regular if, for all subsets $A' \subseteq A$ and $B' \subseteq B$ with $|A'| \geq \varepsilon |A|$ and $|B'| \geq \varepsilon |B|$, we have

$$|\text{dens}(A', B') - \text{dens}(A, B)| \leq \varepsilon.$$ 

This uniform distribution of edges is typical in random bipartite graphs.

Definition 3. A partition $V = V_1 \cup V_2 \cup \ldots \cup V_k$ of a set into disjoint parts is called an equipartition if $||V_i| - |V_j|| \leq 1$ for $1 \leq i < j \leq k$.

Theorem 4 (Szemerédi’s Regularity Lemma [77]). For every real $\varepsilon > 0$ and integer $m \geq 1$ there exists two integers $M$ and $N$ with the following property. For every graph $G$ with $n \geq N$ vertices there exists an equipartition of the vertex set into $k$ classes $V_1, \ldots, V_k$, $m \leq k \leq M$, such that all but at most $\varepsilon k^2$ of the pairs $(V_i, V_j)$ are $\varepsilon$-regular.

Szemerédi’s Regularity Lemma is a high level approximation scheme for large graphs and has many applications. It may be seen as an essential theoretical justification for the introduction and the study of the so-called stochastic block model in statistics [40]. In this sense, the presence of densities brings this result closer to a random graph model than to a graph approximation.

Szemerédi’s Regularity Lemma is not only a fundamental result in graph theory, it also led to extensions and new proofs in several other mathematical areas such as analysis [51], information theory [78], number theory [38], hypergraphs [69, 37, 19] and relational structures [8], algebra [79] and algebraic geometry [22]. There are countless applications of the regularity lemma. One of the key uses of the lemma is to transfer results from random graphs, which are much easier to handle, to the class of all graphs of a given edge density. We refer to the papers [4, 45] for extensive background on the applications of the regularity lemma.

As shown by Conlon and Fox in [16], the exceptional pairs cannot be avoided in the statement of Szemerédi’s Regularity Lemma, as witnessed by the example of half-graphs. A half-graph is a bipartite graph with vertices $a_1, \ldots, a_n, b_1, \ldots, b_n$ for some integer $n \geq 1$ and edges $\{a_i, b_i\}$ for $1 \leq i \leq j \leq n$. The bound $M$ for the number of parts in the partition of the graph is very large: it has to grow as a tower of $2$’s of height $\Omega(\varepsilon^{-2})$, as proved by Fox and Lovász in [23], extending earlier results of Gowers [36].
Consequently, it is natural to ask if restrictions on the graph being partitioned might result in a stronger form of regularity. Such restricted versions of the regularity lemma were established for example in [1, 6, 12, 13, 26, 54, 66, 75, 76]. These results establish for example a polynomial number of parts, stronger forms of regularity, the absence of exceptional pairs, etc., in restricted graph classes. Results of this kind will be simply called “regularity lemmas” in this paper.

In the first part of this work we survey these results in their rich combinatorial context. After this survey part, the stage is set for our further study of regularity properties of low complexity graph classes.

**Sparsity and low complexity classes.** Szemerédi’s regularity lemma is useful only for dense graphs. For sparse graphs, i.e. graphs with a sub-quadratic number of edges, it becomes trivial, as every balanced partition into a suitable constant number of pieces is $\varepsilon$-regular. Nevertheless, various regularity lemmas for sparse graphs exist, see e.g. [43, 44, 68, 33, 70]. In this paper, sparse graphs are considered in the context of combinatorially defined classes of graphs, which recently formed a very active area. This is referred to briefly as “sparsity”, with key notions, such as bounded expansion, nowhere denseness, quasi-wideness, etc. (see e.g. [58]), as well as notions from geometric and structural graph theory. All of this will be reviewed below in Section 2.

We will focus on numerous questions that arise when we consider dense graphs that are constructed from sparse graphs, e.g. map graphs, which are induced subgraphs of squares of planar graphs. The operations of taking a graph power and taking induced subgraphs are special cases of logical transduction and of logical interpretations, which are studied in model theory. As a second example, graphs of bounded cliquewidth, or equivalently, bounded rankwidth, are first-order transductions of tree-orders. In this sense, model theory offers a very convenient way to construct graphs from other well behaved structures via interpretations and transductions. Also, some of the stronger forms of the regularity lemma are based on model theoretic notions, e.g. for graphs defined in distal theories [12, 75] or graphs with a stable edge relation [54].

One of the essential tools in the study of sparse classes are the so-called low treedepth decompositions [56, 57, 58, 59], referred to as $p$-covers by classes with bounded treedepth in this paper (see Section 6). This tool has been extended to structurally bounded expansion classes, that is, to graph classes obtained as transductions of classes with bounded expansion, which turn out to be characterized by the existence of $p$-covers by classes with bounded shrubdepth [29]. This type of decomposition has been extended to graphs with linear rankwidth (which are 2-covered by classes with bounded embedded shrubdepth [61]) and to classes $p$-covered by classes with bounded rankwidth [49]. We will see in Section 6 that there is a nice interplay between the notions of graph regularity and the existence of 2-covers by classes of graphs with small complexity. For the first time, we consider base classes consisting of embedded $m$-partite cographs, which generalize bounded treedepth, bounded shrubdepth, and bounded embedded shrubdepth. Then we consider the more general case of a base class with bounded rankwidth.
Our results. In Section 4 we introduce set-defined classes, which are semi-algebraic and have bounded order-dimension. As a consequence, they enjoy both stable regularity (Theorem 23) and semi-algebraic regularity (Theorem 26). We give important examples of set-defined classes and prove that set-defined classes are a dense analog of degenerate classes (Theorem 35).

In Section 6 we consider 2-covers of a class by another class and how these 2-covers transport properties like being distal-defined, semi-algebraic, set-defined, having bounded VC-dimension or bounded order-dimension, as well as the existence of polynomial \( \varepsilon \)-nice partitions (just as in the distal regularity lemma Theorem 28).

In Section 7, we show that classes 2-covered by a class of embedded \( m \)-partite cographs are semi-algebraic, and hence satisfy the semi-algebraic regularity lemma. Moreover, we give an explicit construction for the construction of an \( \varepsilon \)-nice partition with explicit bound for the number of parts (Corollary 6) in the style of the distal regularity lemma.

In Section 8, we study regularity properties of nowhere dense classes, and characterize nowhere dense classes in terms of the regularity properties of the \( d \)-powers of the graphs in the class (Theorem 48). On a negative side, we prove that there exists a nowhere dense class that not only is not distal-defined, but also does not allow \( \varepsilon \)-nice partitions for any \( \varepsilon < 1 \) (Corollary 8).

Summarizing we provide many new regularity lemmas for the sparse classes defined by the means of combinatorial and model theoretical tools.

Part 1. Preliminaries and survey of some regularity lemmas

1. Model theory background

1.1. Structures. A language \( L \) is a set of function symbols, relation symbols and constant symbols. To each function symbol \( f \in L \) and each relation symbol \( R \in L \) we associate an arity. Let \( L \) be a language. An \( L \)-structure \( M \) consists of a nonempty set \( M \), called the universe or domain of \( M \), a function \( f_M : M^k \to M \) for each \( k \)-ary function symbol \( f \in L \), a relation \( R_M \subseteq M^k \) for each \( k \)-ary relation symbol \( R \in L \) and an element \( c_M \in M \) for each constant symbol \( c \in L \). The functions \( f_M, R_M \) and \( c_M \) are called the interpretations of \( f, R \) and \( c \) in \( M \). If no confusion can arise we do not distinguish between a symbol and its interpretation.

Example 5. The standard language for real closed fields is \( L_{RCF} = \{+,\cdot,<,0,1\} \), where \( + \) and \( \cdot \) are binary function symbols, \( < \) is a binary relation symbol and \( 0, 1 \) are constant symbols. The field of real numbers \( (\mathbb{R},+,\cdot,<,0,1) \), where \(+,\cdot,<,0,1\) are interpreted as usual, is a prototypical model of the theory of real closed fields.

A graph can be seen as a structure over the language \( L_{graph} = \{E\} \), where \( E \) is a binary relation symbol. For this, we identify a symmetric and irreflexive binary relation with a set of undirected edges.

1.2. First-order logic, interpretations and transductions. We use standard first-order logic and refer to [39] for more background. When the language \( L \) is clear from context, we shall use the term formula for an \( L \)-formula, that is a
first-order formula in the language $L$. We use the standard abbreviations $\varphi \rightarrow \psi$ for $\neg \varphi \lor \psi$, $\varphi \leftrightarrow \psi$ for $(\varphi \rightarrow \psi) \land (\psi \rightarrow \varphi)$, and $\varphi \oplus \psi$ for $(\varphi \land \neg \psi) \lor (\neg \varphi \land \psi)$. A theory is a set of sentences (formulas without free variables). A model of a theory $T$ is a structure $\mathcal{M}$ that satisfies all the sentences in $T$. We do not distinguish between a theory and the class of all its models.

We write $\varphi(x_1, \ldots, x_k)$ to indicate that the free variables of $\varphi$ are among $x_1, \ldots, x_k$. We usually write $\bar{x}$ for the tuple $(x_1, \ldots, x_k)$ and leave it to the context to determine the length $k$ of the tuple. Let $\mathcal{M}$ be an $L$-structure. Every $L$-formula $\varphi(\bar{x})$ defines a relation $\varphi(\mathcal{M}) = \{ \bar{a} \in M^{|\bar{x}|} : \mathcal{M} \models \varphi(\bar{a}) \}$. A relation $R$ on $\mathcal{M}$ is called definable (without parameters) if there is a formula $\varphi(\bar{x})$ such that $R = \varphi(\mathcal{M})$. A graph $G$ is definable in $\mathcal{M}$ if, for some integer $k$, we have $V(G) = M^k$ and $E(G) \subseteq M^k \times M^k$ is definable on $\mathcal{M}$. A class $\mathcal{C}$ of graphs is definable in a class $\mathcal{D}$ of structures if there is an integer $k$ and a formula $\varphi(\bar{x})$ with $2k$ free variables such that each $G \in \mathcal{C}$ is defined by $\varphi$ in some $\mathcal{M} \in \mathcal{D}$.

Interpretations and transductions provide a very useful formalism to encode (classes of) structures inside other (classes of) structures and to lift results from one structure to the other. For our purpose it will be sufficient to define interpretations of graphs in structures. A simple interpretation $I$ of graphs in $L$-structures consists of two $L$-formulas $\nu(x)$ and $\eta(x, y)$, where $\eta$ is symmetric (i.e. $\eta(x, y) \leftrightarrow \eta(y, x)$) and irreflexive (i.e. $\neg \eta(x, x)$). If $\mathcal{M}$ is an $L$-structure, then $I(\mathcal{M})$ is the graph with vertex set $\nu(\mathcal{M})$ and edge set $\eta(\mathcal{M}) \cap \nu(\mathcal{M})^2$. For sake of simplicity, we will also allow to define simple interpretations by means of a non-symmetric and/or reflexive formula $\psi(x, y)$ by implicitly considering the formula $\neg (x = y) \land (\psi(x, y) \lor \psi(y, x))$.

Transductions allow an additional coloring of the elements of the structures, which gives additional encoding power. This is formalized as follows. For languages $L, L^+$, if $L \subseteq L^+$, then $L^+$ is called an expansion of $L$ and $L$ is called a reduct of $L^+$. If $\mathcal{M}^+$ is an $L^+$-structure, then the structure $\mathcal{M}$ obtained from $\mathcal{M}^+$ by “forgetting” the relations in $L^+ \setminus L$ is called the $L$-reduct of $\mathcal{M}^+$, and $\mathcal{M}^+$ is called an $L^+$-expansion of $\mathcal{M}$. If $L^+$ is an expansion of $L$ by a set of unary relation symbols, and $\mathcal{M}$ is an $L$-structure, then we call any $L^+$-expansion $\mathcal{M}^+$ of $\mathcal{M}$ a monadic lift of $\mathcal{M}$. A transduction $T$ is the composition of a monadic lift followed by a simple interpretation $I$. Let $\mathcal{C}$ and $\mathcal{D}$ be classes of $L$-structures and graphs, respectively. We say that $\mathcal{D}$ is a transduction of $\mathcal{C}$ if there exists a simple interpretation $I$ of graphs in $L^+$-structures, where $L^+$ is a monadic expansion of $L$, such that for every $G \in \mathcal{D}$ there exists a lift $\mathcal{M}^+$ of some structure $\mathcal{M} \in \mathcal{C}$ such that $G = I(\mathcal{M}^+)$.

2. Graph theoretic background

We consider finite, simple and undirected graphs. For a graph $G$ we write $V(G)$ for its vertex set and $E(G)$ for its edge set. For $A, B \subseteq V(G)$ we write $E(A, B)$ for the set of edges with one end in $A$ and one end in $B$. A partition of $V(G)$ is a family of pairwise disjoint subsets $V_1, \ldots, V_k \subseteq V(G)$ whose union is $V(G)$. A bipartite graph is a graph with a vertex partition $V_1, V_2$ such that there are no edges with both ends in $V_i$, $i = 1, 2$. A graph $H$ is a subgraph of $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For $X \subseteq V(G)$, we write $G[X]$ for the subgraph of $G$ induced by $X$, that is, the subgraph with vertex set $X$ and all edges with both ends in $X$.  

REGULAR PARTITIONS OF GENTLE GRAPHS 5
The graph $H$ with vertex set $X$ is an induced subgraph of $G$ if $H = G[X]$. For an infinite graph $G$, we call the class $\text{Age}(G)$ of all finite induced subgraphs of $G$ the \textit{age} of $G$. For disjoint subsets $X, Y$ of $V(G)$, we write $G[X, Y]$ for the subgraph of $G$ \textit{semi-induced} by $X$ and $Y$, that is, the subgraph with vertex set $X \cup Y$ and all the edges with one endpoint in $X$ and one endpoint in $Y$. A bipartite graph $H$ is a \textit{semi-induced} subgraph of $G$ if $H = G[X, Y]$ for some disjoint subsets $X$ and $Y$ of $V(G)$. A class $\mathcal{C}$ of graphs is called \textit{monotone} if it is closed under taking subgraphs and \textit{hereditary} if it is closed under taking induced subgraphs. For a graph $H$, a class $\mathcal{C}$ is called \textit{$H$-free} if no $G \in \mathcal{C}$ contains $H$ as an induced subgraph. A set $X \subseteq V(G)$ is called \textit{homogeneous} if either all distinct vertices of $X$ are adjacent (induce a clique) or non-adjacent (induce an independent set). More generally, a pair $(A, B)$ of subsets of vertices is \textit{homogeneous} if $G[A, B]$ is either complete bipartite or edgeless. A graph $G$ is \textit{$d$-degenerate} if every non-empty induced subgraph of $G$ has minimum degree at most $d$. A class $\mathcal{C}$ is \textit{degenerate} if there is an integer $d$ such that all the graphs in $\mathcal{C}$ are $d$-degenerate. For a graph $G$ we denote by $\bar{d}(G)$ the \textit{average degree} of $G$, that is, the average of the degrees of the vertices of $G$.

2.1. \textbf{Sparse graph classes.} We refer the reader to [58] for an in-depth study of the notions outlined here. The \textit{$r$-subdivision} of a graph $G$ is the graph $G^{(r)}$ obtained by subdividing each edge of $G$ exactly $r$ times. A $\leq r$-\textit{subdivision} of $G$ is a graph obtained by subdividing each edge of $G$ at most $r$ times. A graph $H$ is a \textit{topological minor} of a graph $G$ at depth $r$ if a $\leq 2r$-subdivision of $H$ is a subgraph of $G$. We denote by $G \bar{\triangledown} r$ the set of all the topological minors of $G$ at depth $r$. The two key notions in the theory of sparsity [58] are the notions of \textit{bounded expansion} and \textit{nowhere denseness}.

\textbf{Definition 6.} A class $\mathcal{C}$ of graphs has \textit{bounded expansion} if there exists a function $f : \mathbb{N} \to \mathbb{N}$ with

\begin{equation}
\forall G \in \mathcal{C}. \forall H \in G \bar{\triangledown} r. \; \bar{d}(H) \leq f(r).
\end{equation}

\textbf{Definition 7.} A class $\mathcal{C}$ of graphs is \textit{nowhere dense} if there exists a function $f : \mathbb{N} \to \mathbb{N}$ with

\begin{equation}
\forall G \in \mathcal{C}. \forall H \in G \bar{\triangledown} r. \; \omega(H) \leq f(r).
\end{equation}

Note that every class with bounded expansion is nowhere dense.

Bounded expansion and nowhere dense classes enjoy numerous characterizations and applications (see [58]). In fact, most graph invariants ($\alpha, \chi, \chi_f, \omega$, etc.) lead to characterizations of these classes [18, 57, 60]. It also appears that for monotone classes of graphs these definitions provide a natural link to model theory (see e.g. [2, 65, 67]). The notions of stability, monadic stability, dependence and monadic dependence mentioned in the next theorem are fundamental notions from model theory, which will be formally recalled later in Definition 20 and Definition 13.
Theorem 8 (Podewski, Ziegler [67], Adler, Adler [2]). If a class $\mathcal{C}$ of graphs is monotone, then the following are equivalent.

(i) $\mathcal{C}$ is nowhere dense,
(ii) $\mathcal{C}$ is stable,
(iii) $\mathcal{C}$ is monadically stable,
(iv) $\mathcal{C}$ is dependent,
(v) $\mathcal{C}$ is monadically dependent.

Corollary 1. Every nowhere dense class $\mathcal{C}$ is monadically stable.

Proof. According to (2) the monotone closure $\overline{\mathcal{C}}$ of a nowhere dense class $\mathcal{C}$ is nowhere dense. Hence $\overline{\mathcal{C}}$ is monadically stable (by Theorem 8), and so is $\mathcal{C} \subseteq \overline{\mathcal{C}}$. □

2.2. Rankwidth and linear rankwidth. The notion of rankwidth was introduced in [64] as an efficient approximation to cliquewidth. For a graph $G$ and a subset $X \subseteq V(G)$ we define the cut-rank of $X$ in $G$, denoted $\rho_G(X)$, as the rank of the $|X| \times |V(G) \setminus X|$ matrix $A_X$ over the binary field $\mathbb{F}_2$, where the entry of $A_X$ on the $i$-th row and $j$-th column is 1 if and only if the $i$-th vertex in $X$ is adjacent to the $j$-th vertex in $V(G) \setminus X$. If $X = \emptyset$ or $X = V(G)$, then we define $\rho_G(X)$ to be zero.

A subcubic tree is a tree where every node has degree 1 or 3. A rank decomposition of a graph $G$ is a pair $(T, L)$, where $T$ is a subcubic tree with at least two nodes and $L$ is a bijection from $V(G)$ to the set of leaves of $T$. For an edge $e \in E(T)$, the connected components of $T - e$ induce a partition $(X, Y)$ of the set of leaves of $T$. The width of an edge $e$ of $(T, L)$ is the cut-rank $\rho_G(L^{-1}(X))$. The width of $(T, L)$ is the maximum width over all edges of $T$ (and at least 0). The rankwidth $\text{rw}(G)$ of $G$ is the minimum width over all rank decompositions of $G$. When the graph has at most one vertex then there is no rank decomposition and the rankwidth is defined to be 0.

The linear rankwidth of a graph is a linearized variant of rankwidth, similarly as pathwidth is a linearized variant of treewidth: Let $G$ be an $n$-vertex graph and let $v_1, \ldots, v_n$ be an order of $V(G)$. The width of this order is $\max_{1 \leq i \leq n-1} \rho_G(\{v_1, \ldots, v_i\})$. The linear rankwidth of $G$, denoted $\text{lrw}(G)$, is the minimum width over all linear orders of $G$. If $G$ has less than 2 vertices we define the linear rankwidth of $G$ to be zero. An alternative way to define the linear rankwidth is to define a linear rank decomposition $(T, L)$ to be a rank decomposition such that $T$ is a caterpillar and then define linear rankwidth as the minimum width over all linear rank decompositions. Recall that a caterpillar is a tree in which all the vertices are within distance 1 of a central path.

Chudnovsky and Oum [14] observed that classes of graphs with rankwidth at most $k$ have the strong Erdős-Hajnal property: Indeed, an $n$-vertex graph $G$ of rankwidth at most $k$ has a vertex set $X$ such that the cut-rank of $X$ is at most $k$ and $|X|, |V(G)| - |X| > n/3$. Then one can partition each of $X$ and $V(G) - X$ into at most $2^k$ subsets such that each part of $X$ is complete or anti-complete to each part of $V(G) - X$. It is thus natural to ask whether classes with bounded rankwidth are distal-defined. We leave this question as a problem (Problem 3).
2.3. **Low complexity classes.** Structurally sparse classes are classes that are transductions of sparse classes, or, in other words, classes that can be encoded in a sparse class by means of a coloring and a simple first-order interpretation [29, 60, 62] (see Figure 1).

For instance, classes with *bounded shrubdepth* [32, 31] are the transductions of classes of bounded height trees, structurally bounded expansions classes are transductions of classes with bounded expansion, structurally nowhere dense classes are transductions of nowhere dense classes.

Alternatively, classes with bounded shrubdepth can be defined using a graph invariant called *SC-depth*, which is inductively defined as follows: the class $\mathcal{SC}_1$ of all graphs of SC-depth 1 is $\{K_1\}$ and for $t > 1$, the class $\mathcal{SC}_t$ of all graphs of SC-depth at most $t$ is the class of all graphs $G$ such that there exists graphs $G_1, \ldots, G_k$ in $\mathcal{SC}_{t-1}$ (with disjoint vertex sets) and $A \subseteq V(G_1) \cup \cdots \cup V(G_k)$, such that $G$ is obtained from the disjoint union of $G_1, \ldots, G_k$ by complementing the adjacency of the pairs of vertices in $A \times A$. Then a class $\mathcal{C}$ has bounded shrubdepth if and only if it is included in some class $\mathcal{SC}_t$, i.e. if it has bounded SC-depth.

Graphs with SC-depth $t$ are special instances of $m$-partite cographs (for $m = 2^t$). An *$m$-partite cograph* is a graph that can be encoded in a tree semilattice $(T, \wedge)$, that is, the meet-semilattice defined by the least common ancestor operation $\wedge$ in the rooted tree $T$, as follows: the leaves of $T$ (i.e. the maximal elements of $T$) are the vertices of $G$ and are colored by $c: V(G) \to [m]$, where $[m] = \{1, \ldots, m\}$, while each internal vertex $v$ of $T$ (i.e. each non-maximal element $v$ of $T$) is
assigned a symmetric function $f_v : [m] \times [m] \to \{0, 1\}$ in such a way that two vertices $u, v$ of $G$ are adjacent if and only if $f_{u \wedge v}(c(u), c(v)) = 1$. Hence, cographs are exactly 1-partite cographs (only one color of vertices). Note that $m$-partite cographs are clearly transductions of tree-orders, that is of partial orders defined by the ancestor relation in a rooted tree. However, not every transduction of the class of tree-orders is a class of $m$-partite cographs for some $m$. As proved by Colcombet [15], a class is a transduction of a class of tree-orders if and only if it has bounded rankwidth, and it is a transduction of a class of linear orders if and only if it has bounded linear rankwidth.

**Definition 9.** A class $\mathcal{C}$ is $p$-covered by a class $\mathcal{D}$ if there exists an integer $K(p) \geq p$ such that every $G \in \mathcal{C}$ has a vertex partition $V_1, \ldots, V_{K(p)}$ with $G[V_{i_1} \cup \cdots \cup V_{i_p}] \in \mathcal{D}$ for all $1 \leq i_1 < i_2 < \cdots < i_p \leq K(p)$. The minimum integer $K(p)$ is the magnitude of the $p$-cover. If a class $\mathcal{C}$ is $p$-covered by a class $\mathcal{D}$ for each integer $p$, we say that $\mathcal{C}$ has low $\mathcal{D}$-covers.

We have the following non-trivial characterizations of classes with bounded expansion and of classes with structurally bounded expansion.

**Theorem 10 ([57]).** A class $\mathcal{C}$ has bounded expansion if and only if for every integer $p$ the class $\mathcal{C}$ is $p$-covered by a class with bounded treedepth.

**Theorem 11 ([29]).** A class $\mathcal{C}$ has structurally bounded expansion if and only if for every integer $p$ the class $\mathcal{C}$ is $p$-covered by a class with bounded shrubdepth.

Classes with low bounded rankwidth covers have been considered in [49] and with low bounded linear rankwidth covers are discussed in [61]. In particular, it is proved in [49] that interval graphs and permutations graphs are not 3-covered by any class with bounded rankwidth.

### 3. Random-free regularity lemmas

We first survey regularity properties of hereditary graph classes that are defined by excluding semi-induced bipartite graphs. Common feature of these results is that the regular pairs fail to be random-like bipartite graphs. The different types of regularity considered here are summarized in Table 1.

#### 3.1. VC-dimension

We start with graph classes of bounded VC-dimension. A hereditary class of graphs has bounded VC-dimension if and only if it excludes some bipartite graph as an induced subgraph. Usually, VC-dimension is defined for set families [80], however, in the context of graph theory the following equivalent definition is more convenient.

**Definition 12.** The VC-dimension of a graph $G$ is the largest integer $d$ such that there exist vertices $a_1, \ldots, a_d \in V(G)$ and vertices $b_j \in V(G)$ for $J \subseteq \{1, \ldots, d\}$ such that $\{a_i, b_j\} \in E(G) \iff i \in J$.

Note that a hereditary class $\mathcal{C}$ has bounded VC-dimension if and only if the number of graphs in $\mathcal{C}$ on $n$ vertices is at most $2^{2^{\epsilon n^2}}$ for some $\epsilon > 0$, as proved by Alon, Balogh, Bollobás, and Morris [3].
Variants of Szemerédi’s Regularity Lemma

General case: all pairs but an \(\varepsilon\)-fraction are \(\varepsilon\)-regular. This means that for most pairs \((A,B)\) of parts, the density of edges between subsets of \(A\) and \(B\) (with at least \(\varepsilon\) relative size) differ from the density of edges between \(A\) and \(B\) by at most \(\varepsilon\).

Bounded VC-dimension, NIP: all pairs but an \(\varepsilon\)-fraction are \(\varepsilon\)-homogeneous. This means that their densities are either \(<\varepsilon\) or \(>1-\varepsilon\). The number of parts is polynomial.

Bounded order-dimension, stable: all parts are \(\varepsilon\)-excellent and all the pairs are \(\varepsilon\)-uniform. In particular, every part \(A\), all the vertices of \(G\) have degree either \(<\varepsilon|A|\) in \(A\) or at least \((1-\varepsilon)|A|\) in \(A\), and that for every pair \((A,B)\) of parts at least \((1-\varepsilon)\) proportion of the vertices in \(A\) have similar degree in \(B\). The number of parts is polynomial.

Induced subgraph of a graph definable in a distal structure, semi-algebraic: all pairs but an \(\varepsilon\)-fraction are homogeneous. This means that between non exceptional pairs either all edges are present or no edge is present. The number of parts is polynomial.

### Table 1.

| Summary of the different types of regularity lemmas considered in this section. Hatched zones correspond to irregular parts or pairs. |
|---|

The notion of VC-dimension is strongly related to the model theoretic notion of dependence (or NIP) \([72, 74]\).
Definition 13. A class $\mathcal{C}$ of structures is \textit{dependent} if every graph class definable in $\mathcal{C}$ has bounded VC-dimension.

The class $\mathcal{C}$ is \textit{monadically dependent} if every graph class definable in the class $\{\mathcal{M}^+ | \mathcal{M}^+ \text{ monadic lift of } \mathcal{M} \in \mathcal{C}\}$ of all monadic lifts of structures from $\mathcal{C}$ has bounded VC-dimension.

Theorem 14 (\cite{9}, see also \cite{7}). A class $\mathcal{C}$ is monadically dependent if and only if every transduction of $\mathcal{C}$ is dependent.

Definition 15. Let $\varepsilon > 0$, let $G$ be a graph and let $A, B \subseteq V(G)$ be two disjoint non-empty subsets of vertices. The pair $(A, B)$ is called $\varepsilon$-\textit{homogeneous} if $\text{dens}(A, B) < \varepsilon$ or $\text{dens}(A, B) > 1 - \varepsilon$.

Remark 16. Assume $(A, B)$ is $\varepsilon^3$-homogeneous, let $A' \subseteq A$ and $B' \subseteq B$ with $|A'| \geq \varepsilon |A|$ and $|B'| \geq \varepsilon |B|$. Then, if $\text{dens}(A, B) < \varepsilon^3$ we have

$$\text{dens}(A', B') = \frac{|E(A', B')|}{|A'||B'|} \leq \frac{|E(A, B)|}{\varepsilon^3 |A||B|} < \varepsilon.$$ 

Similarly, considering the complement graph, if $\text{dens}(A, B) > 1 - \varepsilon^3$ we have $\text{dens}(A', B') > 1 - \varepsilon$. We deduce that an $\varepsilon^3$-homogeneous pair is $\varepsilon$-regular.

The following theorem is also known as the \textit{ultra-strong regularity lemma} for graphs with bounded VC-dimension \cite{5, 52, 27}. The presented bounds come from \cite{27}.

Theorem 17 (Bounded VC-dimension regularity lemma; Fox, Pach and Suk \cite{27}). Let $0 < \varepsilon < 1/4$ and let $G$ be a graph of VC-dimension at most $d$. Then there exists an equipartition of $V(G)$ into $k$ classes $V_1, \ldots, V_k$, where $k$ satisfies $8/\varepsilon \leq k \leq c \cdot (1/\varepsilon)^{2d+1}$ for some constant $c$ depending only on $d$, such that all but at most $\varepsilon k^2$ of the pairs $(V_i, V_j)$ are $\varepsilon$-homogeneous.

Erdős and Hajnal \cite{21} proved that for every proper hereditary graph class $\mathcal{C}$ there exists a constant $c$ such that every $n$-vertex graph $G \in \mathcal{C}$ contains a homogeneous set of size $e^{c \sqrt{\log n}}$. They conjectured that for every proper hereditary graph class $\mathcal{C}$ there exists a constant $\delta$ such that every $n$-vertex graph $G \in \mathcal{C}$ must contain a homogeneous set of size $n^\delta$. A graph class with this property is said to have the \textit{Erdős-Hajnal property}. Fox, Pach and Suk \cite{27} also proved that graphs of bounded VC-dimension \textit{almost} have the Erdős-Hajnal property:

Theorem 18 (Fox, Pach and Suk \cite{27}). Every $n$-vertex graph with bounded VC-dimension contains a homogeneous set of size at least $e^{{(\log n)}^{1-o(1)}}$.

Following \cite{24}, we say that a graph class $\mathcal{C}$ has the \textit{strong Erdős-Hajnal property} if there exists a constant $\delta > 0$ such that every $n$-vertex graph $G \in \mathcal{C}$ contains a homogeneous pair $(A, B)$, where $A$ and $B$ are two disjoint sets of at least $\delta n$ vertices each.

3.2. \textbf{Order-dimension}. The study of structures without $k$-\textit{order property} or, equivalently with bounded order dimension, has been initiated by Shelah in his study of stability \cite{71}. Order-dimension is also related to Littlestone-dimension, which is a combinatorial parameter that characterizes error bounds in online learning (see \cite{10}).
Definition 19. The order-dimension of a graph $G$ is the largest integer $\ell$ such that there exist vertices $a_1, \ldots, a_\ell, b_1, \ldots, b_\ell \in V(G)$ with $\{a_i, b_j\} \in E(G) \iff i \leq j$.

The notion of order-dimension is strongly related to the model theoretic notion of stability.

Definition 20. A class $\mathcal{C}$ of structures is stable if every graph class definable in $\mathcal{C}$ has bounded order-dimension.

The class $\mathcal{C}$ is monadically stable if every graph class definable in the class $\{M^+ | M^+ \text{ monadic lift of } M \in \mathcal{C}\}$ of all monadic lifts of structures from $\mathcal{C}$ has bounded order-dimension.

Theorem 21 ([9]). A class $\mathcal{C}$ is monadically stable if and only if every transduction of $\mathcal{C}$ is stable.

Definition 22. Let $\varepsilon > 0$, let $G$ be a graph and let $A \subseteq V(G)$. The set $A$ is called $\varepsilon$-good when for every $b \in V(G)$ either

$$|\{a \in A \mid \{a, b\} \in E(G)\}| \leq \varepsilon |A|$$

or

$$|\{a \in A \mid \{a, b\} \in E(G)\}| \geq (1 - \varepsilon) |A|.$$

In the following, we write $E(a, b)$ for the boolean value true if $\{a, b\} \in E(G)$ and false otherwise. Then the above reads as: for every $b \in V(G)$ there exists a boolean value $t(b/A)$ such that

$$|\{a \in A \mid E(b, a) \neq t(b/A)\}| \leq \varepsilon |A|.$$

The set $A$ is $\varepsilon$-excellent if $A$ is $\varepsilon$-good and if $B \subseteq V(G)$ is $\varepsilon$-good, then there exists a boolean value $t(A/B)$ such that

$$|\{a \in A \mid t(a/B) \neq t(A/B)\}| \leq \varepsilon |A|.$$

A pair $(A, B)$ satisfying this latter condition is called $\varepsilon$-uniform. In other words, a pair $(A, B)$ is $\varepsilon$-uniform if all but at most $\varepsilon |A|$ vertices of $A$ have a degree in $B$ that is smaller than $\varepsilon |B|$ or all but at most $\varepsilon |A|$ vertices of $A$ have a degree in $B$ that is greater than $(1 - \varepsilon) |B|$.

Theorem 23 (Stable regularity lemma; Malliaris and Shelah [54]). For every $\ell$ and every $\varepsilon > 0$ there exist $M$ and $N$ such that for every graph $G$ with $n \geq N$ vertices of order-dimension at most $\ell$, there is an equipartition of the vertex set into $k$ classes $V_1, \ldots, V_k$, $k \leq M$, where each of the pieces is $\varepsilon$-excellent, all of the pairs are $\varepsilon$-uniform and if $\varepsilon < 1/2^\ell$, then $M(\varepsilon, \ell) \leq (3 + \varepsilon) \left(\frac{2}{3}\right)^{2^\ell}$.

Malliaris and Shelah also showed that graphs of bounded order-dimension have the Erdős-Hajnal property.

Theorem 24 ([54, 13]). For every integer $\ell$ there is a constant $\delta > 0$ such that every $n$-vertex graph $G$ of order-dimension at most $\ell$ contains a homogeneous subset of size at least $n^\delta$. 
3.3. **Weakly-sparse classes.** Forbidding a biclique (i.e. a complete bipartite graph $K_{s,s}$) as a semi-induced subgraph is equivalent, by a standard Ramsey argument, to forbidding a clique and an induced biclique, which is in turn equivalent to excluding some biclique as a (non induced) subgraph. A monotone class has bounded VC-dimension if and only if it excludes a biclique. Excluding a biclique as a subgraph implies strong properties (see for instance [17, 48, 62]). A class $C$ that excludes a biclique as a subgraph is called *weakly sparse* [62].

**Observation 1.** For all integers $s$ and $k$ and every $\varepsilon > 0$ there exists an integer $n$ such that every $K_{s,s}$-free graph $G$ of order at least $n$ has the property that every equipartition of $V(G)$ in $k$ parts is $\varepsilon$-uniform.

**Proof.** Fix integers $s,k$ and $\varepsilon > 0$. According to [46], there exist a constant $C$ such that $\text{ex}(n,K_{s,s}) \leq Cn^{2^s-1}$. Thus if $n$ is sufficiently large, the average degree $\bar{d}(G)$ of $G$ is at most $\varepsilon^2 n/k^2$. It follows that $G$ contains at most $\varepsilon n/k$ vertices of degree greater than $\varepsilon n/k$. It follows that every equipartition of $V(G)$ into $k$ parts is $\varepsilon$-uniform. □

However, it is not clear whether one can require that all the parts are $\varepsilon$-excellent in some partition of size $(1/\varepsilon)^c$, for some universal constant $c$.

**Classes in the age of an infinite structure.** We now consider classes included in the age of infinite structures. Precisely, starting from some “nice” infinite structure $M$ (like the real field $(\mathbb{R},+,-,0,1)$, the dense linear order $(\mathbb{Q},\prec)$, or the infinite set $\mathbb{N}$) we first construct an infinite graph $U$ definable in $M$, which is an infinite graph whose vertex set is $M^d$ for some $d$, and whose adjacency is given by a definable relation. We then consider classes $C$ of graphs with $C \subseteq \text{Age}(U)$. Examples of this general scheme are abundant in both model theory and combinatorics.

3.4. **Semi-algebraic and distal-defined classes.** A class $C$ is *semi-algebraic* if there are polynomials

$$f_1, \ldots, f_t \in \mathbb{R}[x_1, \ldots, x_d, y_1, \ldots, y_d]$$

and a Boolean function $\Phi$ such that for every graph $G \in C$ there exists a mapping $p: V(G) \to \mathbb{R}^d$ with

$$\{u, v\} \in E(G) \iff \Phi(f_1(p(u),p(v)) \geq 0; \ldots; f_t(p(u),p(v)) \geq 0) = 1.$$ 

In other words, if we let $U$ to be the graph with vertex set $\mathbb{R}^d$ and edge set $E(U) = \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \mid \Phi(f_1(x,y) \geq 0; \ldots; f_t(x,y) \geq 0) = 1\}$, then every graph in $C$ is a finite induced subgraph of $U$. Note that real closed fields have quantifier elimination and hence the above is equivalent to stating that the graph $U$ is definable in $(\mathbb{R}, +,-,0,1)$. We say that $C$ has complexity $(t,D)$ if each polynomial $f_i$ with $1 \leq i \leq t$ has degree at most $D$.

**Example 25.** Intersection graphs of segments and intersection graphs of balls in $\mathbb{R}^d$ are examples of semi-algebraic classes.
Alon, Pach, Pinchasi, Radoičić, and Sharir [6] proved that for semi-algebraic graphs with bounded description complexity, the pairs in the regularity lemma can be required to be homogeneous instead of $\varepsilon$-regular. This result has been extended by Fox, Gromov, Lafforgue, Naor, and Pach [22] to $k$-uniform hypergraphs and, in this more general setting, Fox, Pach, and Suk [25] proved that a polynomial number of parts are sufficient and that semi-algebraic classes have the strong Erdős-Hajnal property.

**Theorem 26** (Semi-algebraic regularity lemma; Fox, Pach, and Suk [25]). *For all integers $d, D, t \geq 1$ there exists a constant $c$ such that for every $0 < \varepsilon < 1/2$ and every semi-algebraic graph $G$ in $\mathbb{R}^d$ with complexity $(t, D)$ there is an equipartition $V_1, \ldots, V_k$ of the vertex set into $k$ classes with $k \leq (1/\varepsilon)^c$ such that all but at most $\varepsilon k^2$ pairs are homogeneous.*

The real field $(\mathbb{R}, +, \cdot, <, 0, 1)$ is an example of so-called distal structures, and the above results have then been extended to classes of graphs included in the age of a graph definable in a distal structure [12], which we call *distal-defined* classes.

The notion of distal theories was defined in [73] to isolate the class of “purely unstable” dependent theories. The original definition is in terms indiscernible sequences, but a more combinatorial characterization can be found in [11]. While stability allows a short combinatorial definition, distality is a lot more complicated and we refrain from giving a formal definition here. Apart from real closed fields, an example of distal theories is the theory of dense linear orders without endpoints, $p$-adic fields with valuation, and Presburger arithmetic. We now state the graph version of [12, Theorem 5.8] in our setting.

**Definition 27.** Let $G$ be an $n$-vertex graph and let $\varepsilon > 0$. A partition $V_1, \ldots, V_k$ of $V(G)$ is $\varepsilon$-nice if

$$\sum_{\text{non-homogeneous } (V_i, V_j)} \frac{|V_i||V_j|}{n^2} < \varepsilon.$$  

Note that if a partition $V_1, \ldots, V_k$ is an equipartition, then it is $\varepsilon$-nice if and only if all pairs but an $\varepsilon$-fraction are homogeneous.

**Theorem 28** (Distal regularity lemma; Chernikov and Starchenko [12]). *For every distal-defined class $\mathcal{C}$ there is a constant $c$ such that for every $\varepsilon > 0$ and for every $n$-vertex graph $G \in \mathcal{C}$, there exists an $\varepsilon$-nice partition $V = V_1 \cup \cdots \cup V_k$ of $V(G)$ with $k \leq (1/\varepsilon)^c$.***

The distal regularity lemma stated above is similar in its form to the Frieze-Kannan (weak) regularity lemma [28]. However, it is easy to deduce a version with an equipartition from Theorem 28, by applying the next lemma.

**Lemma 29.** Assume $G$ has an $\varepsilon$-nice partition into $k$ classes. Then $V(G)$ has an equipartition into $k/\varepsilon$ classes such that all but (at most) a $3\varepsilon$-fraction of the pairs are homogeneous.

**Proof.** Let $V_0 \cup \cdots \cup V_k$ be an $\varepsilon$-nice partition of $G$ into $k$ parts and let $\Sigma \subseteq [k] \times [k]$ be the set of all pairs $(i, j)$ such that $(V_i, V_j)$ is not homogeneous. Let $K = [k/\varepsilon]$. For the sake of simplicity we assume that $K$ divides $n$. We split each part $V_i$ into $W_{i,0}, W_{i,1}, \ldots, W_{i,a_i}$ with $|W_{i,0}| \leq |W_{i,1}| = \cdots = |W_{i,a_i}| = n/K$. 


Let \( I = \{(i, s) \mid 1 \leq i \leq k \text{ and } 1 \leq s \leq a_i\} \). Note that each pair \( (W_{i,s}, W_{j,t}) \) with \( (i, j) \notin \Sigma \) is homogeneous. Let \( \Sigma' \) be the set of the pairs \( (i, s), (j, t) \in I \) such that \( (i, j) \in \Sigma \). Let \( Z = \bigcup W_{i,0} \). Note that \( |Z| \leq k n/K \), say, \( |Z| = k' n/K \) for some \( k' \leq k \). We now consider an equipartition of \( Z \) into sets \( Z_1, \ldots, Z_{k'} \) of size \( n/K \). As \( (V_1, \ldots, V_k) \) is \( \varepsilon \)-nice we have \( \sum_{(i, j) \in \Sigma} |V_i| |V_j| < \varepsilon n^2 \). It follows that
\[
\sum_{(i, s), (j, t) \in \Sigma'} |W_{i,s}| |W_{j,t}| < \varepsilon n^2.
\]
As \( |W_{i,s}| = n/K \) we get \( |\Sigma'| < \varepsilon K^2 \). It follows that the global number of non-homogeneous pairs is bounded by \( |\Sigma'| + Kk' + k^2 \leq 3k^2/\varepsilon \). Hence the proportion of non-homogeneous pairs is at most \( (3k^2/\varepsilon)/(k^2/\varepsilon^2) = 3\varepsilon \).

It is also proved in [12] that distal-defined classes of graphs have the strong Erdős-Hajnal property.

Part 2. Regularity for gentle graphs

4. SET-DEFINED CLASSES: BOTH STABLE AND SEMI-ALGEBRAIC REGULARITY

We call a class \( \mathcal{C} \) set-defined if it is included in the age of a graph definable in \( \mathbb{N} \), considered as a model of an infinite set. Note that every set-defined class is obviously semi-algebraic. However, as \( \mathbb{N} \) is stable, every set-defined class not only has bounded VC-dimension, but also has bounded order-dimension. It follows that set-defined classes enjoy both stable regularity (Theorem 23) and semi-algebraic regularity (Theorem 26). For this reason, it seems that it is worth studying these classes. Moreover, set-defined classes possess properties that make them a dense analog of degenerate classes (Theorem 35).

Example 30. The class of cographs is not set-defined. Indeed, the order-dimension of cographs is unbounded.

Example 31. The shift-graph \( S(n, k) \) has vertex set \( V = \{\bar{x} \in [n]^k \mid x_1 < x_2 < \cdots < x_k\} \) and edge set \( E = \{\{\bar{x}, \bar{y}\} \mid \bigland_{i=1}^{k-1} (x_i = y_{i+1}) \lor \bigland_{i=1}^{k-1} (y_i = x_{i+1})\} \). It follows that for fixed \( k \) the class \( \{S(n, k) \mid n \in \mathbb{N}\} \) is set-defined. Note that, however, this class is not \( \chi \)-bounded, as shift-graphs are triangle-free and \( \chi(S(n, k)) = (1 + o(1)) \log \ldots \log n \) [20].

Lemma 32. Every graph class with bounded shrubdepth is set-defined.

Proof. A class \( \mathcal{C} \) of graphs has bounded shrubdepth if and only if it has bounded SC-depth. The notion of SC-depth leads to a natural notion of SC-decompositions. An SC-decomposition of a graph \( G \) of SC-depth at most \( d \) is a rooted tree \( T \) of depth \( d \) with leaf set \( V(G) \), equipped with unary predicates \( A_1, \ldots, A_d \) on the leaves. Each child \( s \) of the root in \( T \) corresponds to one of the subgraphs \( G_1, \ldots, G_d \) of SC-depth \( d - 1 \), such that \( G \) is obtained from the disjoint union of the \( G_i \) by complementing the adjacency of the pairs of vertices in \( A_1 \times A_1 \). We continue recursively with the subgraphs \( G_i \) using the predicate \( A_j \) at level \( j \) of the tree \( T \).
We now show that for fixed \(d\) the class of graphs of SC-depth \(d\) is set-defined by a formula \(\varphi(\bar{x}, \bar{y})\) with \(|\bar{x}| = d + 1\). Let
\[
\varphi(\bar{x}, \bar{y}) := \neg(x_{d+1} = y_{d+1}) \land \bigoplus_{i=1}^{d} (x_i = y_i).
\]

Let \(G \in \mathcal{C}\) and fix an SC-decomposition \(T\) of \(G\), together with some injection \(l: V(T) \to \mathbb{N}\). For \(v \in V(G)\) we call \(l(v)\) the label of \(v\). Now we map each vertex \(v \in V(G)\) to the tuple \((x_1, \ldots, x_d, x_{d+1})\), where \(x_{d+1}\) is the label of \(v\) and for \(1 \leq i \leq d\), \(x_i\) is either the label of the ancestor of \(v\) at depth \(i\) in \(T\), if the vertex belongs to the complemented set \(A_i\) at level \(i\), or the label of \(v\) otherwise. It is easy to verify this mapping induces an isomorphism of \(G\) and its image in the infinite graph \(U\) defined by \(\varphi\) on \(\mathbb{N}^{d+1}\).

We will use the lemma to prove in Corollary 2 that a class has structurally bounded expansion if and only if it has low linear rankwidth covers and is set-defined.

Our motivation to introduce set-defined classes is that they have bounded order-dimension and are semi-algebraic. This naturally leads to the following problem.

**Problem 1.** Is there a variant of the regularity lemma for set-defined classes, which would imply (for set-defined classes) both the semi-algebraic version (Theorem 26) and the stable version (Theorem 23)?

We now show that set-defined classes can be seen as a dense analog of degenerate classes.

**Lemma 33.** Every degenerate class is set-defined.

**Proof.** Let \(\mathcal{C}\) be a \(d\)-degenerate class of graphs. We consider the following formula \(\varphi(\bar{x}, \bar{y})\), where \(\bar{x}\) and \(\bar{y}\) are \(d+1\)-tuples.
\[
\varphi(\bar{x}, \bar{y}) := \bigvee_{i=1}^{d+1} (x_{d+1} = y_i) \lor (y_{d+1} = x_i).
\]
This defines the adjacency of an infinite graph \(U\) with vertex set \(\mathbb{N}^{d+1}\).

For every graph \(G \in \mathcal{C}\) there is a numbering \(\ell: V(G) \to \{1, \ldots, |G|\}\) such that every vertex \(v\) has at most \(d\) neighbors \(u\) with \(\ell(u) < \ell(v)\). We define \(f: V(G) \to \mathbb{N}^{d+1}\) as follows: for every vertex \(v \in V(G)\) such that \(u_1, \ldots, u_k\) are the neighbors of \(v\) with \(\ell(u_i) < \ell(v)\) we let
\[
f(v) = (\ell(u_1), \ldots, \ell(u_k), \ell(v), \ldots, \ell(v)).
\]
Then it is easily checked that \(f\) induces an isomorphism of \(G\) and its image in \(U\).

**Lemma 34.** Let \(s, k\) be positive integers. Let \(E(G) = A \times B \setminus \bigcup_{i=1}^{k} E_i\), where each \(E_i\) is a vertex-disjoint union of complete bipartite graphs. Then there exists \(D = D(s, k)\) such that either \(K_{s,s}\) is a subgraph of \(G\) or \(\delta(G) \leq D\).

**Proof.** We let \(D = D(s, k) := 2^{s^2k}\) and let \(G\) be as in the statement. By König’s theorem, either \(G\) contains a vertex cover \(X\) with at most \(D\) vertices or a
matching $M$ with at least $D$ edges. In the first case, $G$ contains a vertex of degree at most $D$ (consider a vertex not in $X$, all its neighbors must be in $X$).

In the second case, let $M = \{\{a_i, b_i\}: i = 1, \ldots, D\}$. Using Ramsey’s theorem for pairs and 4 colors we get that $M$ contains a matching $\overline{M}$, $|\overline{M}| \geq 2s$ such that for any two edges $\{a_i, b_i\}$ and $\{a_j, b_j\}$ in $\overline{M}$ the graph induced on the set $\{a_i, b_i, a_j, b_j\}$ is the same graph $H$. There are 4 possibilities for $H$: either $|E(H)| = 4$, or $|E(H)| = 3$ (two possibilities for this case: either $\{a_i, b_i\} \in E(G)$ or $\{b_i, a_j\} \in E(G)$) or $|E(H) = 2|$ (in which case $\overline{M}$ is an induced matching).

In the first three cases, $K_{s,s}$ is a subgraph of $G$. The last case is impossible as by [50] the complement of any graph containing an induced matching of size $D$ cannot covered by $\log D$ bipartite equivalences (i.e. disjoint unions of complete bipartite graphs). (In [50, Proposition 5.3] this is phrased in the language of the product dimension of the graph.) Hence it suffices to put $D = 2^{2s^k}$ (we do not optimize here).

\[ \square \]

**Theorem 35.** A class $\mathcal{C}$ is degenerate if and only if it is both weakly sparse and set-defined.

**Proof.** If a class $\mathcal{C}$ is degenerate, then it is weakly sparse, and it is set-defined by Lemma 33.

Conversely, let $\mathcal{C}$ be a weakly sparse set-defined class. As $\mathcal{C}$ is weakly sparse, there exists an integer $s$ such that $K_{s,s} \not\subseteq G$ for all $G \in \mathcal{C}$. As $\mathcal{C}$ is set-defined, there is an integer $k$ and a formula $\varphi(x, y)$ with $|x| = |y| = k$, such that every graph in $\mathcal{C}$ is an induced subgraph of the graph $U$ with vertex set $\mathbb{N}^k$ and adjacency relation defined by $\varphi$. We consider the formula $\varphi'(\bar{x}', \bar{y}')$ with $|\bar{x}'| = |\bar{y}'| = k + 1$ defined by

\[
\varphi'(\bar{x}', \bar{y}') := \varphi(x_1, \ldots, x_k, y_1, \ldots, y_k) \land \neg(x_{k+1} = y_{k+1})
\]

and we let $U'$ be the corresponding graph definable on $\mathbb{N}^{k+1}$.

Assume towards a contradiction that $\mathcal{C}$ is not degenerate. For every graph $G \in \mathcal{C}$ and every bipartition $A, B$ of $V(G)$, the bipartite subgraph of $G$ semi-induced by $A$ and $B$ belongs to the age of $U'$. Moreover, if $G$ is not $d$-degenerate, then $G$ has such a semi-induced subgraph that is not $d/2$-degenerate. It follows that the class

\[
\mathcal{C}' = \{H \in \text{Age}(U') \mid K_{s,s} \not\subseteq H \text{ and } H \text{ bipartite}\}
\]

is also a counterexample in the sense that it is weakly sparse, set-defined and is not degenerate. Considering a disjunctive normal form of $\varphi'$, we get that there exists a family $F$ of sets of pairs of integers in $\{1, \ldots, k + 1\}$ such that $\varphi'(\bar{x}', \bar{y}')$ is logically equivalent to

\[
\bigvee_{P \in F} \bigwedge_{(i,j) \in P} (x_i = y_j) \land \bigwedge_{(i,j) \notin P} \neg(x_i = y_j).
\]

This means that the edge set of the graphs in $\mathcal{C}'$ are the union of at most $(k + 1)^2$ sets of edges, each being defined by a formula of the type

\[
\varphi_P(\bar{x}', \bar{y}') = \bigwedge_{(i,j) \in P} (x_i = y_j) \land \bigwedge_{(i,j) \notin P} \neg(x_i = y_j).
\]
It follows that there exists $P \in \mathcal{F}$ such that the class $C_P$ of the subgraphs of the graphs in $C'$ with edge defined by $\varphi_P$ is non-degenerate. It follows that we can assume without loss of generality that the formula $\varphi'$ has the form $\bigwedge_{(i,j) \in P} (x_i = y_j) \land \bigwedge_{(i,j) \notin P} \neg (x_i = y_j)$ for some set $P$ of pairs of integers in $\{1, \ldots, k+1\}$. As the graphs we consider are bipartite we can consider an embedding in $\mathbb{N}^{(k+1)^2}$ instead of $\mathbb{N}^{k+1}$ by duplicating the $i$th coordinate of the vertices in the first part and the $j$th coordinate of the vertices in the second part to the coordinate $(i, j)$. This way we can assume that the formula has the form $\bigwedge_{i \in I} (x_i = y_i) \land \bigwedge_{i \notin I} \neg (x_i = y_i)$. As imposing $x_i = y_i$ allows only to create a disjoint union of induced subgraphs, it is useless in our setting. Thus we can assume that our graph is defined on some $N^{k'}$ by the formula

$$\psi(\vec{x}'', \vec{y}'') := \bigwedge_{i=1}^{k'} \neg (x_i = y_i).$$

Thus our bipartite graphs contradict Lemma 34. □

5. ORDER-DEFINED CLASSES

We now consider a notion sandwiched between semi-algebraic and set-defined. We call a class $C$ order-defined if it is included in the age of a graph definable in $(\mathbb{Q}, <)$, the countable dense linear order without endpoints. It is immediate that order-defined classes are semi-algebraic. Both set-defined classes and order-defined classes are strongly related to the constructions introduced in [35] for strongly polynomial sequences.

Example 36. The class of circle graphs is order-defined. It follows that the class of cographs, and more generally the class of permutation graphs and the class of distance-hereditary graphs (which are both contained in the class of circle graphs) are order-defined.

However, we are not aware of any example of a semi-algebraic class that is not order-defined.

The notion of order-defined classes is very similar to the notion of Boolean dimension considered by Gambosi, Nešetril, and Talamo [30] and by Nešetril and Pudlák [63]. The Boolean dimension of a poset $P = (X, \leq)$ is the minimum number of linear orders on $X$ a Boolean combination of which gives $\leq$. Thus we have the following property:

Observation 2. If a class $\mathcal{P}$ of posets has bounded Boolean dimension then the class of the comparability graphs of the posets in $\mathcal{P}$ is order-defined.

The conjecture on boundedness of Nešetril and Pudlák [63] on the boundedness of the Boolean dimension of planar posets can thus be weakened as follows.

Conjecture 1. The class of comparability graphs of planar posets is order-defined.

Every set-defined class has bounded order-dimension and is order-defined. The converse might be true.

Problem 2. Is every order-defined class with bounded order-dimension set-defined?
6. Inherited regularity of 2-covered classes

In this section, we show that when a class $C$ is 2-covered by a class $D$ we can deduce that $C$ inherits many properties of the class $D$, including some regularity properties.

**Theorem 37.** Assume $D$ is a distal-defined (resp. semi-algebraic, order-defined, set-defined) class and that the class $C$ is 2-covered by $D$. Then the class $C$ is also distal-defined (resp. semi-algebraic, order-defined, set-defined).

**Proof.** Let $U$ be a graph definable in a distal structure (resp. a semi-algebraic graph, a graph definable in $(\mathbb{Q},<)$, a graph definable in $\mathbb{N}$) such that $D \subseteq \text{Age}(U)$. Assume that $C$ is 2-covered by $D$ with magnitude $p$ and let $q := \left(\frac{p}{2}\right) + 1$. We consider the graph $H$ whose vertex set is the set of all tuples $\bar{v} \in V(U)^q$ (we address the elements of the tuples by indices 0 and $\{i,j\}$ for $1 \leq i < j \leq p$) and whose edges are defined by the formula

$$\varphi(\bar{x}, \bar{y}) := \bigwedge_{1 \leq i < j \leq p} (x_{\{i,j\}} = x_0) \lor (y_{\{i,j\}} = y_0) \lor E(x_{\{i,j\}}, y_{\{i,j\}}).$$

Let $G \in C$. By assumption there exists a partition $V_1, \ldots, V_p$ of $V(G)$ such that for every $1 \leq i < j \leq p$ we have $G[V_i \cup V_j] \in D$. We denote by $f_{i,j}$ the embedding of $G[V_i \cup V_j]$ in $U$. Let $g: V(G) \to V(U)^q$ be defined as follows: let $u \in V_i$ and let $z(u)$ be an arbitrary vertex of $V(U) \setminus \{f_{i,j}(u) : j \in [p] \setminus \{i\}\}$. We let $g(v) = \bar{x}$, where $x_0 = z(u)$ and

$$x_{\{k,\ell\}} = \begin{cases} z(u) & \text{if } i \notin \{k, \ell\} \\ f_{\{k,\ell\}}(u) & \text{otherwise} \end{cases}$$

Then it is easily checked that $g$ induces an isomorphism of $G$ and its image in $H$. As $H$ is definable in $U$ we infer that the class $C$ is distal-defined (resp. semi-algebraic, order-defined, set-defined).

**Corollary 2.** A class has structurally bounded expansion if and only if it has low linear rankwidth covers and is set-defined.

**Proof.** Every class $C$ with structurally bounded expansion has low shrubdepth covers (by Theorem 11), thus, has low linear rankwidth covers. As classes with bounded shrubdepth are set-defined by Lemma 32, it follows from Theorem 37 that $C$ is set-defined.

Conversely, assume $C$ has low linear rankwidth covers and is set-defined. As $C$ is set-defined, it has bounded order-dimension. According to [62], a graph with bounded linear rankwidth covers and bounded order-dimension is a transduction of a graph with bounded pathwidth covers. It follows that $C$ has structurally bounded expansion.

The following theorem is possibly the first purely model theoretical characterization of bounded expansion classes.

**Theorem 38.** A hereditary class $C$ has bounded expansion if and only if $C$ is weakly sparse and if all transductions of $C$ are set-defined.
Proof. If \( C \) has bounded expansion, then it is weakly sparse and all transductions of \( C \) are set-defined by Corollary 2.

Conversely, assume towards a contradiction that \( C \) is weakly sparse and all transductions of \( C \) are set-defined and that \( C \) fails to have bounded expansion. By [17] there exists an integer \( p \) such that \( C \) includes the \( p \)-subdivision of graphs with arbitrarily large average degree. According to [47], \( C \) also includes the \( p \)-subdivision of \( C_4 \)-free graphs with arbitrarily large average degree. By an easy transduction we obtain from \( C \) a class \( D \) of \( C_4 \)-free graphs with unbounded average degree. This class \( D \) being weakly sparse and (by assumption) set-defined, contradicting Theorem 35. □

**Theorem 39.** Assume \( D \) is a class with bounded VC-dimension (resp. bounded order-dimension) and that the class \( C \) is 2-covered by \( D \). Then the class \( C \) also has bounded VC-dimension (resp. bounded order-dimension).

**Proof.** Let \( ES_n \) be the bipartite graph with vertex set \([n] \cup 2^n\) and adjacency given by \( \{i, I\} \in E(ES_n) \) if \( i \in I \). Consider any \( p \)-coloring of the vertices of \( ES_n \). Let \( A_1, \ldots, A_p \) be the color classes of \([n]\) and let \( B_1, \ldots, B_p \) be the color classes of \( 2^n \). We have

\[
2^n = |\{N(b) \mid b \in 2^n\}| = \sum_{i=1}^{p} |\{N(b) \mid b \in B_i\}| \leq \sum_{i=1}^{p} \sum_{j=1}^{p} |\{N(b) \cap A_j \mid b \in B_i\}|
\]

Hence there exists \( i, j \in [p] \) such that \( |\{N(b) \cap A_j \mid b \in B_i\}| \geq 2^n/p^2 \). It follows from the Sauer-Shelah lemma that \( ES_n \) contains a bichromatic induced subgraph with VC-dimension \( \Omega\left(\frac{1}{\log p} \cdot \frac{n}{p \log n}\right) \). In particular, if a class \( C \) has unbounded VC-dimension and is 2-covered by a class \( D \), then \( D \) has unbounded VC-dimension.

Now consider any \( p \)-coloring of the half-graph \( H_n \) with vertices \( a_1, \ldots, a_n \) and \( b_1, \ldots, b_n \) where \( a_i \) is adjacent to \( b_j \) if \( i \leq j \). By the pigeon-hole principle, there exists a pair of colors \( (c_1, c_2) \) and a subset \( I \subseteq [n] \) of size at least \( n/p^2 \) such that for every \( i \in I \) the vertex \( a_i \) has color \( c_1 \) and the vertex \( b_i \) has color \( c_2 \). It follows that \( H_n \) contains a bi-chromatic induced \( H_{n/p^2} \). Consequently, if a class \( C \) has unbounded order-dimension and is 2-covered by a class \( D \), then \( D \) has unbounded order-dimension. □

Recall the definition of \( \varepsilon \)-nice partitions from Definition 27: A partition \( V_1, \ldots, V_k \) of the vertex set of a graph \( G \) is \( \varepsilon \)-nice if

\[
\sum_{\text{non-homogenous } (V_i,V_j)} \frac{|V_i||V_j|}{n^2} < \varepsilon.
\]

**Theorem 40** (Regularity preservation for 2-covered classes). Let \( \mathcal{D} \) be a class such that for every \( \varepsilon > 0 \) every graph in \( \mathcal{D} \) has an \( \varepsilon \)-nice partition with \( f(\varepsilon) \) parts (for some function \( f \)). Let \( \mathcal{C} \) be a class 2-covered by \( \mathcal{D} \) with magnitude \( p \). Then, for every \( \varepsilon > 0 \) every graph \( G \in \mathcal{C} \) has an \( \varepsilon \)-nice partition with \( K \leq pf\left(\frac{\varepsilon}{p-1}\right)^{p-1} \) parts.
Proof. Assume \( V(G) = \bigcup_{i=1}^{p} V_i \) and \( G[V_i \cup V_j] \in \mathcal{D} \) for all \( 1 \leq i < j \leq p \). Let \( G_{i,j} = G[V_i \cup V_j] \). According to the assumptions, there is an \( \frac{\varepsilon}{p-1} \)-nice partition \( \mathcal{P}_{i,j} = (W_{i,j}^1, W_{i,j}^2, \ldots, W_{i,j}^{K_{i,j}}) \), \( K_{i,j} \leq f(\frac{\varepsilon}{p-1}) \), of \( V(G_{i,j}) \) satisfying

\[
\sum_{\text{non-homogeneous } (W_{i,j}^s, W_{i,j}^t)} \frac{|W_{i,j}^s| |W_{i,j}^t|}{|V(G_{i,j})|^2} < \frac{\varepsilon}{p-1}.
\]

For convenience we define \( W_{j,i} = W_{i,j} \). For \( i \in [p] \) let

\[ I_i = [K_{i,1}] \times \ldots [K_{i,i-1}] \times \{0\} \times [K_{i,i+1}] \times [K_{i,p}] \]

Define a partition \( \mathcal{P} = (P_{i}^\alpha)_{1 \leq i \leq p, \alpha \in I_i} \) by letting

\[ P_{i}^\alpha = \bigcap_{j \neq i} W_{i,j}^\alpha. \]

Then

\[ |\mathcal{P}| \leq pf \left( \frac{\varepsilon}{p-1} \right)^{p-1}. \]

As cutting homogeneous pairs gives only homogeneous pairs, the only non homogeneous pairs are pairs \((P_{i}^\alpha, P_{j}^\beta)\), where \( \alpha \in I_i, \beta \in I_j \) and \((W_{i,j}^s, W_{i,j}^t)\) is not homogeneous. Thus we have

\[
\sum_{\text{non-homogeneous } (P_{i}^\alpha, P_{j}^\beta)} \frac{|P_{i}^\alpha| |P_{j}^\beta|}{n^2} \leq \sum_{\text{non-homogeneous } (W_{i,j}^s, W_{i,j}^t)} \sum_{\alpha \in I_i; \alpha_j = s} \sum_{\beta \in I_j; \beta_i = t} \frac{|P_{i}^\alpha| |P_{j}^\beta|}{n^2} \leq \sum_{\text{non-homogeneous } (W_{i,j}^s, W_{i,j}^t)} \frac{|W_{i,j}^s| |W_{i,j}^t|}{n^2} \leq \sum_{\text{non-homogeneous } (W_{i,j}^s, W_{i,j}^t)} \frac{|G_{i,j}|^2 |W_{i,j}^s| |W_{i,j}^t|}{n^2 |G_{i,j}|^2} \leq \left( \sum_{\text{non-homogeneous } (W_{i,j}^s, W_{i,j}^t)} \frac{|G_{i,j}|^2}{n^2} \right) \frac{\varepsilon}{p-1} < \varepsilon.
\]

\( \square \)

The following corollary is then a direct consequence of Theorem 37 and Lemma 29.

**Corollary 3.** Let \( \mathcal{D} \) be a class such that for every \( \varepsilon > 0 \) every graph in \( \mathcal{D} \) has an \( \varepsilon \)-nice partition with \( f(\varepsilon) \) parts (for some function \( f \)). Let \( \mathcal{C} \) be a class 2-covered by \( \mathcal{D} \) with magnitude \( p \). Then, for every \( \varepsilon > 0 \) every graph \( G \in \mathcal{C} \) has an equipartition with \( K \leq \frac{2p f(\frac{\varepsilon}{2(p-1)})^{p-1}}{\varepsilon} \) parts such that all pairs but an \( \varepsilon \)-fraction are homogenous.
In this section, we show that classes 2-covered by a class of embedded \( m \)-partite cographs are order-defined (Corollary 5), hence, satisfy the semi-algebraic regularity lemma (Theorem 26). Moreover, we give an explicit construction of an \( \varepsilon \)-nice partition with explicit bound for the number of parts (Corollary 6) in the style of the regularity lemma for distal-defined classes (Theorem 28).

A \textit{cograph}, or complement-reducible graph, is a graph that can be generated from \( K_1 \) by complementations and disjoint unions. The tree representation of a cograph \( G \) is a rooted tree \( T \) (called \textit{cotree}), whose leaves are the vertices of \( G \) and whose internal nodes represent either disjoint unions or complete joins operations.

Some generalizations of cographs have been proposed; e.g. bi-cographs [34], \( k \)-cographs [42], or \( m \)-partite cographs [32]. The following extension we present here is very natural:

**Definition 41.** An \textit{embedded \( m \)-partite cograph} is a graph that can be obtained from a plane tree \( T \) with a coloring \( \gamma_L : L(T) \to \{1, \ldots, m\} \) and an assignment \( v \in I(T) \mapsto f_v \), with \( f_v : [m] \times [m] \to \{0, 1\} \), as follows: the vertex set of \( G \) is \( L(T) \) and two vertices \( u \) and \( v \) are adjacent if (assuming that the branch from \( u \wedge v \) to \( u \) is to the left of the branch from \( u \wedge v \) to \( v \)) we have \( f_{u \wedge v}(\gamma_L(u), \gamma_L(v)) = 1 \).

By similarity with the case of cographs, the colored plane tree \( T \) is called an \textit{embedded cotree} of \( T \).

Embedded \( m \)-partite cographs share some nice properties with \( m \)-partite cographs: for instance, they are well quasi-ordered for induced subgraph inclusion, and they have bounded rankwidth, which makes their recognition fixed-parameter tractable. Moreover, they are linearly \( \chi \)-bounded, as for every embedded \( m \)-partite cograph \( G \) the \( m \) color classes of \( G \) induce cographs thus \( \chi(G) \leq m \omega(G) \).

Embedded \( m \)-partite cographs also generalize graphs with bounded shrub-depth [32] and graphs with bounded embedded shrubdepth [61]. It follows that classes 2-covered by a class of embedded \( m \)-partite cographs include structurally bounded expansion classes [29] and, more generally, class 2-covered by a class with bounded linear rankwidth [61]. As an example, this includes the class of unit-interval graphs [61, 62].

Although embedded \( m \)-partite cographs fail to be set-defined in general (as witnessed by cographs), we have the following.

**Lemma 42.** The class of embedded \( m \)-partite cographs is order-defined.

*Proof.* Consider an embedded cotree \( T \) of an embedded \( m \)-partite cograph \( G \). Then the subgraph of \( G \) induced by any two colors is obviously an embedded 2-partite cograph. Hence, the class of embedded \( m \)-partite cograph is 2-covered by the class of 2-partite cographs. To each embedded cotree \( T \) of an embedded 2-partite cograph we associate the linear order \( <_T \) on the leaves of \( T \) defined by a left-to-right traversal of \( T \) and two cotrees \( T_1 \) and \( T_2 \) (on the same rooted tree as \( T \)), where an internal node \( x \) defines a complete join in \( T_1 \) (resp. \( T_2 \)) if \( f_x(1, 2) = 1 \) (resp. \( f_x(2, 1) = 1 \)) and a disjoint union, otherwise. These cotrees define two graphs \( G_1 \) and \( G_2 \) on the vertex set of \( G \). Let \( M_1 \) and \( M_2 \) mark the
vertices with color 1 and color 2, respectively. Then if $u$ has color 1 and $v$ has color 2, the vertices $u$ and $v$ are adjacent in $G$ if and only if $u <_T v$ and $u$ and $v$ are adjacent in $G_1$, or $u >_T v$ and $u$ and $v$ are adjacent in $G_2$. Let $g_1$ (resp. $g_2$) be an embedding of $G_1$ (resp. $G_2$) in a graph definable in $(\mathbb{Q}, <)$, and let index be the index of vertices in the linear order $<_T$. Then to each vertex $v$ we associate the vector $(1, \text{color}(v), \text{index}(v), g_1(v), g_2(v))$. It is easily checked that this allows to define embedded 2-partite cographs as induced subgraph of a graph definable in $(\mathbb{Q}, <)$.

Using Theorem 40 we deduce:

**Corollary 4.** Every class $C$ that is 2-covered by a class of embedded $m$-partite cographs is order-defined.

As each class with bounded linear rankwidth is 2-covered by a class of embedded $m$-partite cographs (and even by a class of bounded embedded shrubdepth [61]) we deduce

**Corollary 5.** Every class $C$ that is 2-covered by a class with bounded linear rankwidth is also 2-covered by a class of embedded $m$-partite cographs, thus order-defined.

Note that classes 2-covered by a class with bounded linear rankwidth include in particular structurally bounded expansion classes, as these classes admit low shrubdepth covers [29].

We now give an explicit construction and proof for a weakened statement of the distal regularity lemma in the special case of embedded $m$-partite cographs.

**Definition 43.** A plane tree is a rooted tree in which the children of each vertex are ordered from left to right.

Each plane tree $T$ with root $r$ defines a partial order $\leq$ on its vertex set by $u \leq v$ if the path linking $r$ to $v$ in $T$ goes through $u$. For vertices $u, v$ in $V(T)$ we denote by $u \wedge v$ the least common ancestor of $u$ and $v$ in $T$, that is the maximum $z \in V(T)$ with $z \leq u$ and $z \leq v$. We denote by $L(T)$ the set of all leaves of $T$ and by $I(T)$ the set $V(T) \setminus L(T)$ of all internal nodes of $T$. For a vertex $v \in V(T)$ we denote by $T_v$ the plane subtree of $T$ rooted at $v$. We also denote by $F_v$ the ordered set of the children of $v$ in $T$. We shall consider two coloring functions on $V(T)$, $\gamma_L : L(T) \to \{1, \ldots, c_L\}$ and $\gamma_I : I(T) \to \{1, \ldots, c_I\}$.

**Definition 44.** Let $T$ be a plane tree, let $\mu$ be a probability measure on $V(T)$, and let $\varepsilon \geq \max_{v \in V(T)} \mu(v)$. Then $v \in T$ is called

- $\varepsilon$-light if $\mu(T_v) \leq \varepsilon$;
- $\varepsilon$-terminal if $v$ is not $\varepsilon$-light but all the children of $v$ are $\varepsilon$-light;
- $\varepsilon$-singular if $v$ has exactly 1 child that is not $\varepsilon$-light and the sum of the $\mu$-measures of the $T_u$ for $\varepsilon$-light children $u$ of $v$ is strictly greater than $\varepsilon$;
- $\varepsilon$-chaining if $v$ has exactly 1 child that is not $\varepsilon$-light and $v$ is not $\varepsilon$-singular;
- $\varepsilon$-branching if $v$ has at least 2 children that are not $\varepsilon$-light.
Definition 45. Let $T$ be a plane tree, let $\mu$ be a probability measure on $V(T)$, and let $\varepsilon \geq \max_{v \in V(T)} \mu(v)$. A partition $\mathcal{P}$ of $V(T)$ is an $\varepsilon$-partition of $T$ if

- each part is of one of the following types:
  1. $\mathcal{P} = \{v\} \cup \bigcup_{x \in F} T_x$ for some non-empty interval $F \subseteq F_v$,
  2. $\mathcal{P} = \bigcup_{x \in F} T_x$ for some interval $F \subseteq F_v$,
  3. $\mathcal{P} = T_v \setminus T_w$ for some $w \in T_v$ distinct from $v$ ($w$ is called the cut vertex of $P$, and the path from $v$ to the father of $w$ is called the spine of $P$),

where $v$ (which is easily checked to be the infimum of $\mathcal{P}$) is called the attachment vertex of $P$ and is denoted by $A_P(P)$;

- each attachment vertex of a part of type 2 is also the attachment vertex of some part of type 1;

- every part has $\mu$-measure at most $\varepsilon$.

We now prove that every plane tree has a small $\varepsilon$-partition.

Lemma 46. Let $T$ be a plane tree, let $\mu$ be a probability measure on $V(T)$, and let $\varepsilon > 0$. We assume that no vertex has measure more than $\varepsilon$. Then there exists an $\varepsilon$-partition $\mathcal{P}$ of $T$ with $8/\varepsilon$ vertices.

Proof. When considering a partition of $V(T)$, a part $X$ is thin if $\mu(X) \leq \varepsilon$ and thick if $\mu(X) > \varepsilon$. Note that the number of thick parts in a partition is at most $1/\varepsilon$ (as they are disjoint and have global measure at most 1).

We define a first partition $\mathcal{P}_0$ of $V(T)$ into atoms, where an atom is

- $T_v \setminus T_w$ if $v$ is $\varepsilon$-chaining, the atom corresponding to $v$ has measure at most $\varepsilon$, and $w$ is the unique child of $v$ that is not $\varepsilon$-light,
- $T_v$ if $v$ is $\varepsilon$-light and the parent of $v$ is not $\varepsilon$-light,
- or $\{v\}$ if $v$ is neither $\varepsilon$-branching nor $\varepsilon$-light.

We then define a coarser partition $\mathcal{P}_1$, in which atoms are gathered into groups, where a group is of the following types:

1. $T_v \setminus T_w$ where $v$ is $\varepsilon$-chaining and $w$ is the (unique) maximum descendant of $v$ that is neither $\varepsilon$-light nor $\varepsilon$-chaining,
2. the union of $\{u\}$ (for non $\varepsilon$-light and non $\varepsilon$-chaining $u$) and the maximal (possibly empty) union of $T_{v_i}$ where the $v_i$’s are consecutive $\varepsilon$-light children of $u$ starting from the leftmost one;
3. a maximal union of $T_{v_i}$, where the $v_i$’s are consecutive $\varepsilon$-light children of a non $\varepsilon$-light and non $\varepsilon$-chaining vertex $u$ not including the leftmost $\varepsilon$-light child of $u$.

Denote by $n_s$ the number of singular vertices of $T$, by $n_t$ the number of terminal vertices of $T$, and by $n_b$ the number of branching vertices $T$. All the atoms corresponding to $\varepsilon$-terminal or $\varepsilon$-singular vertices are disjoint and thick thus $n_t + n_s \leq 1/\varepsilon$.

We consider a reduced plane tree $T_1$ obtained from $T$ by removing all $\varepsilon$-light vertices and contacting all $\varepsilon$-chaining groups. Note that the leaves of $T_1$ are exactly the $\varepsilon$-terminal vertices of $T$ hence $T_1$ has $n_t$ leaves. As every $\varepsilon$-branching vertex of $T$ corresponds to a vertex of $T_1$ with at least two children, we have $n_b < n_t$. The number $n_1$ of groups of type (1) is less than the number of edges of $T_1$ hence less than $n_t + n_s + n_b$. The number $n_2$ of groups of type (2) is at most $n_t + n_s + n_b$.
(direct from the definition). The number $n_3$ of groups of type (3) is at most the number of edges of $T_1$ hence less than $n_t + n_s + n_h$. Altogether, we have $|\mathcal{P}_1| \leq 3(n_t + n_s + n_h) = 6n_t + 3n_s < 6/\varepsilon$.

The partition $\mathcal{P}$ is sandwiched between $\mathcal{P}_1$ and $\mathcal{P}_0$. It is obtained by splitting thick groups into maximal parts of consecutive atoms with global measure at most $\varepsilon$. If a group $X$ is split into $P_1, \ldots, P_k$ then, by maximality we have $\mu(P_i) + \mu(P_{i+1}) > \varepsilon$ for every $i$ in $1, \ldots, k - 1$. Summing up we get $\mu(P_i) + 2\mu(P_2) + \cdots + 2\mu(P_{k-1}) + \mu(P_k) > (k - 1)\varepsilon$ thus $k - 1 < 2\mu(X)/\varepsilon$. Summing over all thick parts, we get $|\mathcal{P}| - |\mathcal{P}_1| < 2/\varepsilon$. Thus $|\mathcal{P}| < 8/\varepsilon$. □

**Theorem 47** (Regularity lemma for embedded $m$-partite cographs). For every $\varepsilon > 0$, every (sufficiently large) embedded $m$-partite cograph $G$ of order $n$ has a vertex partition into $V_1, \ldots, V_\ell$ with $\ell \leq \frac{128}{\varepsilon}m2^{m^2} = O\left(\frac{1}{\varepsilon}\right)$, such that

$$\sum_{\text{non-homogeneous } (V_i, V_j)} |V_i| |V_j| \frac{n^2}{|V(G)|} < \varepsilon.$$ 

**Proof.** We consider the embedded cotree $T$ of $G$ with two coloring functions on $V(T)$, $\gamma_L: L(T) \to \{1, 2, \ldots, c_L\}$ and $\gamma_I: I(T) \to \{1, 2, \ldots, c_I\}$, where $c_I = m$ and $c_L \leq 2^{m^2}$. Let $\mu$ be a probability measure on $V(T)$ such that for every vertex $v \in T$,

$$\mu(v) = \begin{cases} \frac{1}{|V(G)|}; & \text{if } v \in L(T); \\ 0, & \text{otherwise}. \end{cases}$$

We say a partition $\mathcal{P}^*$ of $G$ is a refinement of a partition $P$ of $T$, if

- every part is consecutive,
- for every part $P^*$ of $\mathcal{P}^*$, there exists a part $P$ of $\mathcal{P}$ such that $P^* \subseteq P$.
- for each pair of parts $P_i^* \subseteq P$ and $P_j^* \subseteq P'$ ($P \neq P'$),

$$|\{\gamma_I(u \land v) \mid (u,v) \in P_i^* \times P_j^*\}| = 1.$$ 

- each part $P^*$ has $|\gamma_L(P^*)| = 1$.

Now we consider a refinement $\mathcal{P}^* = (V_1, V_2, \ldots, V_\ell)$ of an $\frac{\varepsilon}{8}$-partition $\mathcal{P}$. By Lemma 46, $|\mathcal{P}| < \frac{24}{\varepsilon}$. Then $\ell \leq 2|\mathcal{P}|c_Ic_L < \frac{128}{\varepsilon}m2^{m^2}$.

Observe that $V_i$ and $V_j$ are homogeneous for any $V_i \subseteq P, V_j \subseteq P'$ ($P \neq P'$) by definition of $\mathcal{P}^*$. Then the only possible case that $V_i$ and $V_j$ are not homogeneous if $V_i, V_j \subseteq P$, which implies that

$$\sum_{\text{non-homogeneous } (V_i, V_j)} \mu(V_i)\mu(V_j) = \sum_{\text{non-homogeneous } (V_i, V_j)} \frac{|V_i||V_j|}{n^2} \leq \left(\frac{\varepsilon}{8}\right)^2 |\mathcal{P}| < \varepsilon.$$

The next corollary then follows from the application of Theorem 37.

**Corollary 6** (Regularity lemma for classes 2-covered by embedded $m$-partite cographs). Let $\mathcal{D}$ be a class of embedded $m$-partite cographs and let $\mathcal{C}$ be a class of 2-covered by $\mathcal{D}$ with magnitude $p \geq 2$. Then, for every $\varepsilon > 0$ every graph $G \in \mathcal{C}$ has an $\varepsilon$-nice partition with at most $p\left(\frac{128m2^{m^2}(p-1)}{\varepsilon}\right)^{p-1}$ parts.

Applying Lemma 29 we also get
Corollary 7. Let $\mathcal{D}$ be a class of embedded $m$-partite cographs and let $\mathcal{C}$ be a class 2-covered by $\mathcal{D}$ with magnitude $p \geq 2$. Then, for every $\varepsilon > 0$ every graph $G \in \mathcal{C}$ has an equipartition into at most $2p(m2^{m^2+8}(p-1))^{p-1}p^{-p}$ parts, such that all pairs but an $\varepsilon$-fraction are homogenous.

Note that classes 2-covered by a class of embedded $m$-partite cographs include classes 2-covered by a class with bounded linear rankwidth \cite{61} and structurally bounded expansion classes (as follows from \cite{29}).

A natural question is whether the distal regularity lemma applies for classes for bounded rankwidth (and thus to classes 2-covered by a class with bounded rankwidth). The following problem would be a way to get a positive answer.

Problem 3. Is every class with bounded rankwidth distal-defined?

8. Regularity and non-regularity for nowhere dense classes

In this section, we study regularity properties of nowhere dense classes, especially through the regularity properties of the $d$-powers of the graphs in the class (Theorem 48). On a negative side, we prove that there exists a nowhere dense class that not only fails to be distal-defined, but also does not allow any $\varepsilon$-nice partition (Corollary 8).

Proposition 1. For a weakly sparse hereditary class $\mathcal{C}$ the following are equivalent:

(i) $\mathcal{C}$ is nowhere dense,

(ii) for every integer $d \geq 1$ there exists $n$ such that $\mathcal{C}$ excludes the $d$-subdivision of $K_n$ as an induced subgraph,

(iii) for every integer $d$ the hereditary closure of the class $\mathcal{C}^d = \{G^d \mid G \in \mathcal{C}\}$ does not contain all graphs,

(iv) for every integer $d$ the class $\mathcal{C}^d = \{G^d \mid G \in \mathcal{C}\}$ has bounded order dimension.

Proof. (i) $\iff$ (ii) was proved by Dvořák \cite{17}. (iii)$\implies$(ii) (by contrapositive): assume that for some integer $d \geq 1$ the class $\mathcal{C}$ contains the $d$-subdivision of all complete graphs, then it also contains the $d$-subdivision of all graphs, hence the hereditary closure of $\mathcal{C}^{d+1}$ contains all graphs. (iv)$\implies$(iii) as for each integer $d$ some half-graph is not in the hereditary closure of $\mathcal{C}^d$. (i)$\implies$(iv) by Corollary 1.

The following theorem gives yet another characterization of nowhere dense classes.

Theorem 48 (Nowhere dense by regularity lemma). Let $\mathcal{C}$ be a hereditary class of graphs. Then the following are equivalent:

(i) the class $\mathcal{C}$ is nowhere dense;

(ii) For every $d, K \in \mathbb{N}$ and $\varepsilon > 0$ there exists an integer $s$ such that for every graph $G \in \mathcal{C}$ there exists a subsets $S \subseteq V(G)$ of size at most $s$ and a equipartition $V_1, \ldots, V_K$ of $V(G) \setminus S$ with the property that for every $1 \leq i \leq K$, each vertex $u \notin S$ is at distance more than $d$ than at least $(1-\varepsilon)$ proportion of $V_i$ in $G - S$;
For every $d \in \mathbb{N}$ and $\varepsilon > 0$ there exist integers $K$ and $s$ such that for every graph $G \in \mathcal{C}$ there exists a subset $S \subseteq V(G)$ of size at most $s$ and an equipartition $V_1, \ldots, V_K$ of $V(G) \setminus S$ with the property that for every $1 \leq i, j \leq K$, each vertex $u$ of a subset of $V_i$ of size at least $(1 - \varepsilon)|V_i|$ is at distance more than $d$ than at least $(1 - \varepsilon)$ proportion of $V_j$ (which may depend on $u$) in $G - S$. 

Proof. (i)⇒(ii): By [60, Theorem 42] there exists an integer $s$ such that for every graph $G \in \mathcal{C}$ there exists a subset $S$ of at most $s$ vertices such that no ball of radius $d$ in $G - S$ contains more than an $\varepsilon/K$ proportion of the vertices. This means that the maximum degree of $(G - S)^d$ is at most $(\varepsilon/K)|G|$. Consider any equipartition of $V(G) - S$ into $K$ classes $V_1, \ldots, V_K$. Then the degree of every vertex of $(G - S)^d$ in $V_i$ is at most $(\varepsilon/K)|G| = \varepsilon|V_i|$. 

(ii)⇒(iii) is trivial.

(iii)⇒(i): The class $\mathcal{C}$ is nowhere dense as no ball of radius $d/2$ in $G - S$ contains more than $\varepsilon$-proportion of the vertices. (Otherwise many vertices have many vertices in their ball of radius $d$.)

Answering a question of P. Simon, we now prove that nowhere dense classes are not, in general, distal-defined. As distal-defined classes have the strong Erdős-Hajnal property it will be sufficient to prove that some nowhere dense class does not have this property.

Let $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{n-1}$ be the eigenvalues of the adjacency matrix of a graph $G$. If $G$ is connected and $d$-regular, the eigenvalues satisfy $d = \lambda_0 > \lambda_1 \geq \cdots \geq \lambda_{n-1} \geq -d$. Let $\lambda(G) = \max_{|\lambda_i| < d} |\lambda_i|$. A $d$-regular graph $G$ is a Ramanujan graph if $\lambda(G) \leq 2\sqrt{d-1}$. Ramanujan graphs are regular graphs with almost optimal spectral gap, which are thus excellent expanders (see [53]). Expanders have the property that they behave like “random graphs”; in particular, the number of edges between two subsets $A$ and $B$ of vertices is close to the number expected from the sizes of $A$ and $B$ (see e.g. [41]).

Lemma 49 (Expander mixing lemma). Let $G$ be a $d$-regular graph on $n$ vertices with $\lambda = \lambda(G)$. For any two subsets $S, T$ of vertices, let $e(S, T) = \{ (x, y) \in S \times T \mid \{x, y\} \in E(G) \}$. Then

$$e(S, T) - \frac{d|S||T|}{n} < \lambda \sqrt{|S||T|(1 - |S|/n)(1 - |T|/n)}.$$ 

Theorem 50. There exists a nowhere dense class $\mathcal{R}$ that does not have the strong Erdős-Hajnal property, hence it is not distal-defined.

Proof. We consider here the construction of Ramanujan graphs due to Morgenstern [55]. Let $q$ be an odd prime power and let $g(x) \in \mathbb{F}_q[x]$ be irreducible of even degree $d$. Morgenstern constructs a $(q + 1)$-regular Ramanujan graph $\Omega_q$ of order $n = q^{3d} - q^d$ or $(q^{3d} - q^d)/2$, and girth at least $2/3 \log_q n$. Consider $g(x)$ of degree $d > 2q$. Then $n \geq q^{3q}$ and girth($\Omega_q$) > $3q$. Denote this graph by $G_q$. Let

$$\mathcal{R} = \{ G_q \mid q \in \mathbb{N} \}.$$
Claim 50.1. The class \( \mathcal{R} \) is nowhere dense.

Proof. Assume towards a contradiction that \( \mathcal{R} \) is not nowhere dense. Then there exists an integer \( p \) such that for every integer \( n \) there is a graph \( G_q \in \mathcal{R} \) that contains the \( p \)-subdivision of \( K_n \) as a subgraph. As triangles in \( K_n \) appear as cycles of length \( 3p + 3 \) in \( G_q \) it follows that the girth of \( G_q \) is at most \( 3p + 3 \) hence \( q \leq p \). As \( G_q \) is \( (q+1) \)-regular, it follows that all the vertices in the subgraph of \( G_p \) (which is a \( p \)-subdivision of \( K_n \)) have degree at most \( q+1 \) hence \( n \leq q+2 \leq p+2 \). Hence choosing \( n > p+2 \) leads to a contradiction.

Claim 50.2. The class \( \mathcal{R} \) does not have the strong Erdős-Hajnal property.

Proof. Assume towards a contradiction that there exists \( \delta > 0 \) such that every graph \( n \)-vertex graph \( G \in \mathcal{R} \) contains a homogenous pair \( (A,B) \) with \( \min(|A|,|B|) > \delta n \).

Let \( q > 4/\delta^2 \) and let \( G = G_q \). Assume for contradiction that \( G \) contains a homogenous pair \( (A,B) \) with both parts of size at least \( \delta n \). Let \( z \) be the number of edges between \( A \) and \( B \). Then, according to the expander mixing lemma

\[
|z - \frac{(q+1)|A||B|}{n}| < \lambda \sqrt{|A||B|(1-|A|/n)(1-|B|/n)}.
\]

By shrinking \( A \) and \( B \) if necessary we can assume \( |A| = |B| = \delta n \). Hence

\[
|z - (q+1)\delta^2 n| < \lambda n \delta (1-\delta).
\]

Hence \( z > 0 \) if

\[
(q+1)\delta > \lambda (1-\delta),
\]

i.e. \( (q+1)/\lambda > (1/\delta - 1) \). But \( (q+1)/\lambda \approx \sqrt{q}/2 \). As \( q > 4/\delta^2 \) we deduce that no pair \( (A,B) \) of subsets size \( \delta n \) can be homogenous. (Note that they cannot indeed form a complete bipartite subgraph as the degrees are at most \( q \ll \delta n \).)

The theorem now directly follows from the previous two claims. \( \square \)

We conclude with a negative result related to just constructed class \( \mathcal{R} \):

Corollary 8. There is no \( 0 < \varepsilon < 1 \) and no integer \( K \) such that for every \( n \)-vertex graph \( G \in \mathcal{R} \), there exists an \( \varepsilon \)-nice partition \( V = V_1 \cup \cdots \cup V_k \) of \( V(G) \) with \( k \leq K \).

Proof. Let \( G \in \mathcal{R} \) be an \( n \)-vertex graph. We prove that the existence of an \( \varepsilon \)-nice partition of size \( k \) would imply that \( G \) contains a homogenous pair \( (A,B) \) of size at least \( \delta n \), where \( \delta = (1-\varepsilon)/k^2 \), contradicting the fact that \( \mathcal{R} \) does not have the strong Erdős-Hajnal property.

Assume \( V_1, \ldots, V_k \) is an \( \varepsilon \)-nice partition, and let \( \Sigma \) be the set of all pairs \( (i,j) \) with \( (V_i, V_j) \) homogenous. As the partition is \( \varepsilon \)-nice we have \( \sum_{(i,j) \in \Sigma} |V_i||V_j| > (1-\varepsilon)n^2 \). It follows that there exists a pair \( (i,j) \in \Sigma \) such that \( |V_i||V_j| > \frac{1}{k^2}n^2 \).

Then either \( i = j \) and by splitting \( V_i \) into two parts one gets a homogenous pair \( (A,B) \) of vertices, with each of \( A \) and \( B \) of size at least \( \sqrt{\frac{n^2}{2k}} \). Otherwise, \( i \neq j \) and both \( V_i \) and \( V_j \) have size at least \( \frac{1}{k^2}n \). \( \square \)
In this paper we surveyed and studied regularity properties of various graph classes. Our work highlights a strong and fruitful connection between graph theory and model theory. Figure 2 displays the studied concepts and examples of graph classes.

We introduced the new notions of order-defined and set-defined graph classes, which are special cases of semi-algebraic graph classes. These classes nicely fit into a hierarchy of classes defined from the age of well-behaved infinite structures. Our study of structurally sparse graph classes is mainly based on the notion of

**CONCLUSION**
2-covers, and we showed that covers nicely transport various properties of graph classes, including regularity lemmas. A natural follow-up of this work will be to consider hypergraphs and relational structures in full generality. One major question that remained untouched in this paper are algorithmic versions of the regularity lemmas and of 2-covers.

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