Effect of inertial mass on elastic waveguide characteristics

M V Belubekyan$^1$* and S V Sarkisyan$^{1,2}$**

$^1$Institute of Mechanics of the National Academy of Sciences of Armenia, Yerevan, Armenia
$^2$Yerevan State University, Yerevan, Armenia

E-mail: *mbelubekyan@yahoo.com, **vas@ysu.am

Abstract. When studying the process of wave propagation in elastic bodies the choice of boundary conditions is important. In this paper we propose a simple model to study the effect of the inertial mass distributed over the plane of the elastic layer on the characteristics of the elastic waveguide.

1. Introduction

When studying the process of wave propagation in elastic bodies, the choice of boundary conditions plays a special role. Basically, one of the assumptions is accepted: the boundaries of the body are rigidly fixed or the boundaries of the body are free. However, in practice, there are many situations where you cannot neglect the real properties of the media surrounding the body. The elastic layer with plane-parallel facial planes is a special case of the mathematical model of the elastic waveguide. Multilayer structures are widely used in various fields of mechanics to solve many engineering problems, and in some cases the layered composite is a waveguide structure [1–12]. In this paper we propose a simple model to study the effect of the inertial mass distributed over the plane of the elastic layer on the characteristics of the elastic waveguide. The following problems are investigated [13–14]:

(i) one boundary of the elastic isotropic layer is rigidly fixed, and at the other boundary there is an inertial mass (pure shear waves);
(ii) one boundary of the elastic isotropic layer is rigidly fixed, and at the other boundary there is an inertial mass (plane deformation);
(iii) elastic isotropic layer the inertial mass (plane deformation) is distributed at the two boundaries of the layer.

In all considered problems the influence of the inertial mass on the characteristics of the elastic waveguide is shown.

2. An elastic layer with inertial mass

Consider the elastic isotropic layer with thickness $h$. A layer in a rectangular coordinate system $(x, y, z)$ occupies an area $-\infty < x < \infty$, $0 \leq y \leq h$, $-\infty < z < \infty$. The equation of pure shear waves in a layer has the form [1–3]:

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} = \rho \frac{\partial^2 w}{\partial t^2}.$$  

(1)
Stresses and strains for an isotropic elastic body are related by the following dependencies

\[ \sigma_{xz} = \mu \frac{\partial w}{\partial x}, \quad \sigma_{yz} = \mu \frac{\partial w}{\partial y}. \]  

(2)

It is assumed that the following conditions are set at the boundaries of the layer:

\[ w(x, y, t) = 0 \quad \text{for} \quad y = 0, \]

\[ \sigma_{yz} = -m_0 \frac{\partial^2 w}{\partial t^2} \quad \text{for} \quad y = h. \]  

(3)

Here \( \rho, \mu, \) and \( w(x, y, t) \) are bulk density, the shear modulus of the layer material, the component of the displacement vector points layer, \( m_0 > 0 \) is inertial mass. The second boundary condition (3) may occur due to either the presence of a thin layer of a material with characteristics other than the material of the layer [8, 10], or the presence of an inertial (concentrated) mass distributed over the plane \( y = h \).

If a periodic wave with a phase velocity propagates in the layer, according to the boundary conditions (3), we obtain the following dispersion equation with respect to the dimensionless phase velocity \( \eta \):

\[ m \tanh(kh\sqrt{1 - \eta}) - \sqrt{\frac{1 - \eta}{\eta}} = 0. \]  

(4)

Here \( \omega \) is vibration frequency, \( k = \omega/c \) is wave number, \( m = m_0 \omega/\rho c_1, \eta = c^2/c_1^2 < 1, \) finding dimensionless phase velocity, \( c_1 = \sqrt{\mu/\rho} \) the velocity of the transverse wave in the layer. Note that equation (4) can generally have a solution satisfying to the condition \( \eta < 1 \), as well as to the condition \( \eta \geq 1 \). The similar situation occurs in the problem of shear wave propagation in a two-layer medium.

Let the wavelength \( l = 2\pi/k \) is very small compared to the layer thickness \( h \). In this limited case we obtain

\[ \eta = \frac{1}{1 + m^2}. \]  

(5)

If the wavelength is big compared to the layer thickness, so for \( \eta \) we will have

\[ \eta = \frac{1}{m^2k^2h^2}. \]  

(6)

By passing to the limit \( \eta \rightarrow 1 \) it can be shown that equation (4) has the solution satisfying to the condition \( \eta < 1 \) (localized vibrations type) by the condition

\[ kh > \frac{1}{m}. \]  

(7)

Note that by \( \eta > 1 \) we obtain vibrations of an elastic isotropic layer with a thickness \( h \) with concentrated mass \( m_0 \) with a frequency \( \omega \). On the figure 1 shown the results of solving the dispersion equation (4) for different values of the \( kh \) parameter and \( m \), satisfying to the existence condition (7). In the absence of a concentrated mass \( m \), as follows from the dispersion equation (4), the dimensionless phase velocity \( \eta = 1 \). The presence of a concentrated mass leads to the formation of localized oscillations in the layer with a dimensionless phase velocity \( \eta < 1 \) (for example, by \( m = 2, \quad kh = 1, \quad \eta = 0.3624 \)). The inequality (7) is a condition for the occurrence of phase velocities satisfying the condition \( \eta < 1 \).
3. Waves in a layer with an inertial mass (plane deformation)

Consider the problem of wave propagation in an infinite elastic layer of thickness $h$. The layer is bounded by planes $y = 0$ and $y = h$. The plane $y = 0$ is rigidly fixed, and on the plane $y = h$ is given concentrated (inertial) mass $m_0 > 0$ (figure 2).

Consider the problem of plane deformation, i.e. the displacements $\vec{u}(u(x, y, t), v(x, y, t), 0)$ did not depends on variable $z$, and $w = 0$.

By means of transformations

$$u = \frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \varphi}{\partial y} - \frac{\partial \psi}{\partial x}.$$  \hspace{1cm} (8)

Dynamical equations of elasticity theory come to the autonomic wave equations regarding dynamic potentials $\varphi(x, y, t)$ and $\psi(x, y, t)$ [5].

The solutions of these equations is represented by

$$\varphi = [A \sinh(k\alpha_1 y) + B \cosh(k\alpha_1 y)] \exp[i k(x - ct)],$$
$$\psi = [C \sinh(k\alpha_2 y) + D \cosh(k\alpha_2 y)] \exp[i k(x - ct)],$$ \hspace{1cm} (9)

where $\alpha_1^2 = 1 - \eta \theta$, $\alpha_2^2 = 1 - \eta$, $\eta = c^2/c_1^2$, $\theta = c_2^2/c_1^2 < 1$, $c_1^2 = (\lambda + 2\mu)/\rho$.

The constants $A$, $B$, $C$, and $D$ should be determined from the following boundary conditions

$$\frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial y} = 0, \quad \frac{\partial \varphi}{\partial y} - \frac{\partial \psi}{\partial x} = 0 \quad \text{for } y = 0,$$

$$\frac{1 - 2\theta}{\theta} \frac{\partial^2 \varphi}{\partial x^2} + \frac{1}{\theta} \frac{\partial^2 \varphi}{\partial y^2} - 2 \frac{\partial^2 \psi}{\partial x \partial y} = 0, \quad 2 \frac{\partial^2 \varphi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x^2} - m \left( \frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial y} \right) = 0 \quad \text{for } y = h.$$ \hspace{1cm} (10)
Here \( m = m_0 k^2 c^2 \mu^{-1} \), \( \mu \) is the shear modulus of the material of the elastic layer.

In the case of the problem of plane deformation with respect to the dimensionless phase velocity of wave propagation in the elastic layer \( \eta \), we obtain the following characteristic equation:

\[
(2 - \eta) \sqrt{(1 - \eta)(1 - \eta \theta)} - [(2 - \eta)^2 + 4] \sqrt{(1 - \eta)(1 - \eta \theta)} \cosh(H \sqrt{1 - \eta \theta}) \cosh(H \sqrt{1 - \eta}) \\
+ [(2 - \eta)^2 + 4(1 - \eta \theta)(1 - \eta)] \sinh(H \sqrt{1 - \eta \theta}) \sinh(H \sqrt{1 - \eta}) \\
+ \frac{m}{k} \eta \sqrt{1 - \eta} \cosh(H \sqrt{1 - \eta \theta}) \cosh(H \sqrt{1 - \eta}) \\
- \sqrt{(1 - \eta)(1 - \eta \theta)} \cosh(H \sqrt{1 - \eta \theta}) \sinh(H \sqrt{1 - \eta \theta}) \sinh(H \sqrt{1 - \eta}) = 0,
\]

(11)

The analysis of equation (11) shows that by different values \( H = kh \), we get the given values \( m \) at which there are solutions of the type of localized oscillations with dimensionless phase velocity \( \eta < 1 \). For example, by \( \theta = 1/3 \), \( kh \equiv H_* = 3.8063 \). The table 1 shows the values of the inertial mass \( m \) at different values \( H = kh \), where \( \theta = 1/3 \).

As can be seen from the table with increasing \( H \) the given values of the concentrated mass \( m \) decreases. Note that by \( \eta > 1 \) we obtain vibrations of an elastic isotropic layer of thickness \( h \), one of the front planes of which is rigidly fixed, and the other is given an inertial mass \( m_0 \) with a frequency \( \omega = kc \) which can be determined from the equation (11).

4. Waveguide with inertial mass

Consider an elastic isotropic layer with a thickness \( 2h \). A layer in a rectangular coordinate system \( 0xyz \) occupies an area \( L = \{(x, y, z) : x, y \in (-\infty, \infty), z \in [-h, h]\} \). In this layer periodic wave propagates at a phase velocity \( c \). For the plane-stressed state of an infinite elastic layer

\[
u = \partial \varphi / \partial x - \partial \psi / \partial z, \quad w = \partial \varphi / \partial z + \partial \psi / \partial x,
\]

(12)

where \( u \) and \( w \) are projections of elastic displacements on coordinate axes \( x \) and \( z \).

Dynamical equations of elasticity theory come to the autonomic wave equations regarding dynamical potentials \( \varphi(x, z, t) \) and \( \psi(x, z, t) \). By mathematical modeling of physical phenomena the choice of boundary conditions plays an important role. Basically, when studying the process of wave propagation in elastic bodies, one of the assumptions is accepted: the boundaries of the body are rigidly fixed (Dirichlet conditions) or the boundaries of the body are free (Neumann conditions). However, in practice, there are many situations when it is impossible to neglect the real properties of the media surrounding the body [5].

Here we assume that the following boundary conditions are set on the planes \( z = \pm h \), bounding the layer:

\[
\sigma_{zz} = 0, \quad \sigma_{zx} = \mp m_0 \partial^2 u / \partial t^2,
\]

(13)

where \( m_0 > 0 \) is concentrated (insertional) mass. The second boundary condition (13) may occur due to either the presence of a thin layer of material with characteristics other than the layer material [9], or the presence of a concentrated mass distributed over the planes \( z = \pm h \). By \( m_0 = 0 \) the boundary conditions (13) corresponding to the case of free boundaries, by \( m_0 \to \infty \) we get mixed boundary conditions.
Using Hook’s law and transformation (12) the boundary conditions (13) by \( z = \pm h \) come to the form:

\[
\begin{align*}
(\lambda + 2\mu) \frac{\partial^2 \varphi}{\partial z^2} + \lambda \frac{\partial^2 \varphi}{\partial x^2} + 2\mu \frac{\partial^2 \psi}{\partial x \partial z} &= 0, \\
2 \frac{\partial^2 \varphi}{\partial x \partial z} + \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial z^2} &+ m_0 \frac{\partial^2}{\mu \partial^2} \left( \frac{\partial \varphi}{\partial x} - \frac{\partial \psi}{\partial z} \right) = 0.
\end{align*}
\]

(14)

Let’s set the following problem: find solutions of two-dimensional wave equations satisfying the boundary conditions (14). Solution can be introduced in the form [3]:

\[
\begin{align*}
\varphi &= [A \sinh(\nu_1 z) + B \cosh(\nu_1 z)] \exp[i(kx - ct)], \\
\psi &= [C \sinh(\nu_2 z) + D \cosh(\nu_2 z)] \exp[i(kx - ct)],
\end{align*}
\]

(15)

where \( A, B, C, \) and \( D \) are arbitrary constants, \( \lambda \) and \( \mu \) are elastic constants, \( \nu_1^2 = \kappa^2(1 - \eta^2) \), \( \nu_2^2 = \kappa^2/(1 - \eta^2) \), \( \eta = \omega^2/(\kappa^2 c_0^2) = v_2^2/c_2^2 \).

Each solution for \( u \) and \( w \) can be introduced as linear combination of four integrals, connected to the roots of characteristic equation \( \nu_i \) (\( i = 1, 4 \)). Substituting (15) into boundary conditions (14) we get the system of four linear homogeneous algebraic equations with respect to arbitrary constants. Equating the determinant of this system of equations to zero leads to the characteristic equation, from which the phase velocity \( c \) can be found at the given values \( \rho, \mu, \lambda, \alpha, \beta, m_0, \) and \( \omega \). For symmetric modes, we obtain the following dispersion equation with respect to the dimensionless characteristic of the square of the phase velocity \( \eta \):

\[
(2 - \eta)^2 \tanh(H\sqrt{1-\eta}) - 4\sqrt{(1-\eta)(1-\eta^2)} \tanh(H\sqrt{1-\eta}) + \frac{\alpha}{H} \eta \sqrt{1-\eta} = 0,
\]

(16)

\[
H = \kappa h, \quad \alpha = m_0 \omega^2 h \mu^{-1}.
\]

The equation (16) by \( \alpha = 0 \) coincides with the Rayleigh–Lamb dispersion equation [3]. For antisymmetric vibrations, with respect to the dimensionless characteristic of the square of the phase velocity, we obtain the following dispersion equation:

\[
(2 - \eta)^2 \coth(H\sqrt{1-\eta}) - 4\sqrt{(1-\eta)(1-\eta^2)} \coth(H\sqrt{1-\eta}) + \frac{\alpha}{H} \eta \sqrt{1-\eta} = 0.
\]

(17)

The limited cases are considered. Let the wavelength \( l = 2\pi/k \) is very big compared to the layer thickness \( 2h \). In this case by \( \mu = \lambda \) from (16) we get \( c = c_{ps} \equiv 2c_2 \sqrt{2(1-3\alpha/8H^2)/3} \).

Suppose that the wavelength is very small compared to the layer thickness of \( 2h \). Thus we have:

\[
R(\eta) \equiv (2 - \eta)^2 - 4\sqrt{(1-\eta)(1-\eta^2)} + \frac{\alpha \eta \sqrt{1-\eta}}{H} = 0.
\]

(18)

This equation by \( \alpha = 0 \) coincides to the Rayleigh classical equation [3]. Unlike Rayleigh’s equation, it is dispersion equation, i.e. the phase velocity \( \eta \) depends on the wave number \( k \). It should be noted that a similar dispersion equation is obtained for the Rayleigh problem when the boundary of the half-space is elastically constrained either in the normal direction or in the tangential direction in [12]. There are also established conditions under which the surface wave cannot exist and the conditions for the existence of the surface wave, depending on the coefficient characterizing the constraint and the wavelength. It follows from the discussion of limiting cases that for the first form of symmetric vibrations the phase velocity lies within the limits of \( c_{ps} \geq c \geq c_{ps} (c_{ps} \ is \ root \ of \ dispersion \ equation \ (18)) \). The presence of a concentrated mass and if the condition \( \alpha < 2H(1-\theta) \) is not met leads to the fact that in a layer with a given thickness, the surface wave does not propagate. From the second equation (17) it is possible to determine the phase velocity of the bending waves for the first form of antisymmetric
oscillations. On the figure 3 we can see the behavior of function $D(\eta) = R(\eta)/\eta$ by different values of $\alpha/H$. Numerical analysis shows the effect of the concentrated mass on the surface wave velocity. Increase of the value of the concentrated mass distributed over the planes of the layer $z = \pm h$, leads to the fact that the speed of the surface wave, at given values of the layer thickness, decreases.

References
[1] Pochhammer L 1876 Über die Fortpflanzungsgeschwindigkeiten kleiner Schwingungen in einem unbegrenzten isotopen Kreiszylinder J. Reine Angew. Math. 81 324–36
[2] Lamb H 1917 On waves in an elastic plate Proc. Roy. Soc. London 93 (648) 114–28
[3] Miklowitz J 1978 The Theory of Elastic Wave and Waveguides (Amsterdam: North-Holland)
[4] Newton M I, McHale G, Martin F, et al. 2002 Generalized Love waves Europhys. Lett. 58 (6) 818–22
[5] Kossovich L Y, Mukhomodyarov R R, and Parfenova Y A 2008 Wave propagation in elastic fixed isotropic layer Vestnik Sam. Gos. Univ. Estestv.-Nauchn. Ser. No. 8/2(67) 78–89
[6] Meleshko V V, Bondarenko A A, Dovgiy S A, et al. 2008 Elastic waveguides: history and contemporary Mat. Met. Fiz.-Mekh. Polya 51 (2) 86–104
[7] Vilde M V, Kaplunov Y D, and Kosovich L Y 2010 Boundary and Interface Resonance Phenomena in Elastic Bodies (Moscow: Fizmatlit)
[8] Belubekyan V M, Belubekyan M V 2015 Resonance and localized shear vibration in the layer with rectangular cross section Rep. NAS Armenia 115 (1) 40–3
[9] Belubekyan V M, Ohanyan S K, Ghazaryan K B, et al. 2017 The propagation of shear waves in the flat isotropic layer with thin coatings Probl. Fiz. Mat. Tekhn. No. 4(33), 40–3
[10] Belubekyan M V 1991 On the existence conditions of Love’s waves with nonhomogeneous layer Proc. NAS Armenia. Mech. 44 (3) 7–10
[11] Grinenko V T and Meleshko V V 1981 Harmonic Vibrations and Waves in Elastic Bodies (Kiev: Naukova Dumka)
[12] Belubekyan M V 2011 Rayleigh wave in case of elastic-rigidity fixed boundary Proc. NAS Armenia. Mech. 64 (4) 3–6
[13] Belubekyan M V and Sarkisyan S V 2019 Waves propagation in elastic layer with one boundary clumped, in with inertial mass on the another Rep. NAS Armenia 119 (1) 74–8
[14] Sarkisyan A S and Sarkisyan S V 2019 Waves in an elastic layer with inertial mass on the border Proc. NAS Armenia. Mech. 72 (1) 65–72