Do Fresnel coefficients exist?

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Abstract

The starting point of the article is the puzzling fact that one cannot recover the Fresnel coefficients by letting tend the width of a slab to infinity. Without using the so-called limiting absorption principle, we show by a convenient limit analysis that it is possible to define rigorously the field diffracted by a semi-infinite periodic medium.

I. INTRODUCTION

Over the past decades, the numerical resolution of Maxwell equations has made outstanding advances, thanks to the various rigorous methods developed to solve them. One may think for instance, in the theory of gratings, of the integral method, the differential method or the method of fictitious sources. The tremendous progress of computer capacities has been equally important. Generally speaking, it seems that we now have very efficient tools to solve any diffraction problem (the 3D is merely a problem of computer power, and not a theoretical one). Nevertheless, quoting R. Thom, the famous mathematician, Field medalist and inventor of Catastrophe Theory, ”predicting is not explaining”. And although we are able to compute almost everything, we cannot explain much. This paper points out that a simple problem, here a one dimensional one, may lead to deep questions and somewhat interesting answers. Our starting point is a basic remark concerning diopters. The diffraction of a plane wave, in normal incidence, by a diopter separating the vacuum from a homogeneous medium of optical index $\nu$ gives rise to Fresnel coefficients as is well known.

$$r = \frac{1 - \nu}{1 + \nu}, \quad t = \frac{2}{1 + \nu}$$  \hspace{1cm} (1)
This is indeed a fundamental result, commonly used by the opticists. Unfortunately diopters do not exist and, consequently, the famous Fresnel coefficients should not exist either. The problem may be formulated in the following way: let us consider a slab of index $\nu$ and width $h$. This gives rise to a reflection coefficient $r_h$. A natural question is: do we get $r$ when letting $h$ tend to infinity in $r_h$? When we consider a lossless material ($\nu$ is therefore real), the answer is no, obviously. So, how can one measure Fresnel coefficients as there is no such thing as a semi-infinite medium in practice? Two kinds of answers are generally put forward. The first answer that seems to come naturally to many is to evoke the absorption, and the fact that transparent media do not exist. Let us say that it is a poor answer. Indeed, consider for instance the diffraction of a beam, in optical range, by a homogeneous layer (for example, silice for which the refractive index is $\nu = 1.47 + i10^{-5}$), let’s say of a few millimeters (namely the substrate). We can hardly say that there is any absorption in that case. But we have here a medium that is almost thousand wavelengths wide. Another way out generally put forward is to evoke coherence length. This argument is probably correct for the natural light for which coherence length is about micrometer, i.e. less than the depth of materials commonly used in optics (See [1] for an exhaustive review on this subject). However for a beam whose coherence length is greater than a few millimeters one has to find another explanation (it is the case for He-Ne laser beam, for which coherence length is about 20 cm). Without using the two above quoted arguments, we propose a rigorous answer to this problem, in the more general case of a semi-infinite periodic medium. More precisely, we show that it is possible to define the field reflected by a semi-infinite periodic medium. We demonstrate that there is no need to use the so-called limiting absorption principle nor to use explicitly the notion of coherence length. We proceed in two stages. First, we study the diffraction of a plane wave by a finite medium. In this case, the diffracted field is characterized by a reflection coefficient. In a second stage, we study the limit behavior of the reflection coefficient as the width of the medium tends to infinity. All the fields that we consider are $z$-independent and therefore we may reduce the diffraction problem to the basic cases of polarization: $E//\parallel$ (electric field parallel to $z$) and $H//\parallel$ (magnetic field parallel to $z$).

II. REFLECTION BY A FINITE ONE-DIMENSIONAL MEDIUM

We consider a one dimensional structure made of $N$ identical layers (See [2]). A layer is characterized by its relative permittivity $\varepsilon(x)$, which is assumed to be real and positive. For convenience up to the section 4 included, the thickness of a layer is equal to 1. The structure is illuminated by a wavepacket of the following form:

$$U^i(x, y, t) = \int_{P} A(k, \theta) \exp(ik(\cos \theta x + \sin \theta y)) \exp(-i\omega t)d\mu,$$

where $\mu$ is some measure over the set $P$ of parameters $(k, \theta)$:

$$P = \left\{ (k, \theta) \in \mathbb{R}^+ \times [-\frac{\pi}{\mu}, \frac{\pi}{\mu}] \right\}.$$

Explicit forms of $\mu$ will be given in section 4. By means of Fourier analysis, the diffraction of such a wavepacket by a finite structure may be reduced to the study of the diffraction of a plane wave, which we develop in the following paragraph.
We consider a plane wave of wave vector $k_0$ illuminating the structure under the incidence angle $\theta$ ($k_0$ is assumed to belong to the $(xOy)$ plane, we denote $\beta_0 = k_0 \cos \theta$). The total field is described by a single variable function $U_N(x)$. When the electric field is parallel to the $z$-axis ($E//$ case) $U_N(x) \exp(ik_0y \sin \theta)$ represents the $z$-component of the electric field and when the magnetic field is parallel to the $z$-axis ($H//$ case), it represents the $z$-component of magnetic field. Denoting: $\beta(x) = k_0 \left[ \varepsilon(x) - \sin^2(\theta) \right]^{1/2}$, the total field $U_N$ verifies the following equation:

$$0 \leq x \leq N : \left(q^{-1}(x)U'_N\right)' + q^{-1}(x)\beta^2 U_N = 0$$

and the radiation conditions lead to the following equations

$$\begin{cases} x \leq 0 : U_N(x) = \exp(i\beta_0 x) + r_N(k, \theta) \exp(-i\beta_0 x) \\ x \geq N : U_N(x) = t_N(k, \theta) \exp(i\beta_0 x) \end{cases}$$

with: $q \equiv 1$ for $E//$ polarization, $q \equiv \varepsilon$ for $H//$ polarization. Let $\chi_1$ and $\chi_2$ be the solutions of equation (4) verifying

$$\begin{cases} \chi_1(0) = 1, \chi'_1(0) = 0 \\ \chi_2(0) = 0, \chi'_2(0) = 1 \end{cases}$$

The fundamental matrix of the system is then

$$V(x) := \begin{pmatrix} \chi_1(x) & \chi_2(x) \\ \chi'_1(x) & \chi'_2(x) \end{pmatrix}$$

and the resolvent matrix is:

$$R(x, y) = V(y)V^{-1}(x)$$

It is the matrix linking the value of one solution at point $x$ to its value at point $y$. The monodromy matrix is finally defined as:

$$T = R(0, 1) = \begin{pmatrix} \chi_1(1) & \chi_2(1) \\ \chi'_1(1) & \chi'_2(1) \end{pmatrix}$$

This matrix characterizes a layer as it allows the writing of the matching between the boundary conditions. Indeed, taking into account the propagation conditions (5), we derive the following relation:

$$T^N \begin{pmatrix} 1 + r_N \\ i\beta_0(1 - r_N) \end{pmatrix} = t_N \begin{pmatrix} 1 \\ i\beta_0 \end{pmatrix}$$

which permits to obtain both values of $r_N$ and $t_N$. When dealing with a wavepacket, we write the reflected field under the form:

$$U_N^d(x, y, t) = \int_{P} r_N(k, \theta) \exp(ik(- \cos \theta x + \sin \theta y)) \exp(-i\omega t)d\mu.$$  \hspace{1cm} (7)

Our aim is now to study the limit behavior of $r_N$ as $N$ tends to infinity.
A. Some properties of matrix $T$. Connection with Bloch wave theory

The determinant of matrix $T$ is the value of the wronskian of solutions $w_1$ and $w_2$ at point 1. As in that case the wronskian is constant, it is equal to 1. Consequently, $T$ is a unimodular matrix and its characteristic polynomial is $X^2 - tr(T)X + 1$. Therefore the eigenvalues of $T$ are real if and only if $\frac{1}{2} |tr(T)| > 1$. This suggests a splitting of the set of parameters:

$$G = \{(k, \theta) \in \mathbb{P}, \frac{1}{2} |tr(T)| > 1\}$$
$$B = \{(k, \theta) \in \mathbb{P}, \frac{1}{2} |tr(T)| < 1\}$$
$$\Delta = \{(k, \theta) \in \mathbb{P}, \frac{1}{2} |tr(T)| = 1\}$$

When $(k, \theta) \in B$, the eigenvalues of $T$ are conjugate complex numbers of modulus 1, whereas when $(k, \theta) \in G$, $T$ has two real eigenvalues $\gamma(k, \theta)$, where, by convention, $|\gamma(k, \theta)| < 1$. If $(k, \theta) \in \Delta$, then $T$ has an eigenvalue $\gamma(k, \theta)$ of multiplicity 2 with either $\gamma(k, \theta) = 1$ or $\gamma(k, \theta) = -1$. We denote by $\Delta_0$ the subset of $\Delta$ where $T$ or $-T$ is the identity matrix.

Bloch wave theory is the convenient tool when dealing with propagation equations with periodic coefficients. Given a periodic medium with period $Y$, it consists in searching solutions of Schrödinger or wave equation under the form $u_{\rightarrow k}(\rightarrow x)e^{i\rightarrow k \cdot \rightarrow x}$, where $u_{\rightarrow k}(\rightarrow x)$ is a $Y$-periodic function and $\rightarrow k$ belongs to the so-called first Brillouin zone $Y'$. For one-dimensional media, with period $Y = [0,1]$, the theory is quite simple, for in that case $Y' = [-\pi, +\pi]$. Therefore, Bloch solutions write $v_{\phi}(x) = u_{\phi}(x) e^{i\phi x}$, where the so-called Bloch frequency $\phi$ belongs to $Y'$, and $u_{\phi}(x+1) = u_{\phi}(x)$. Thus we have $v_{\phi}(x+1) = e^{i\phi}v_{\phi}(x)$. From the definition of $T$ this means that $e^{i\phi}$ is an eigenvalue of $T$, and the above remarks shows that $tr(T) = 2 \cos \phi$, which provides us with the dispersion relation of the medium and leads us to define $\phi$ a function on $B$ by:

$$\phi(k, \theta) = \arccos \left(\frac{1}{2} tr(T)\right)$$

(8)

Obviously, for couples $(k, \theta)$ belonging to $B$, it is possible to define a Bloch frequency $\phi$, and therefore there exists propagating waves in the medium: such couples $(k, \theta)$ thus define a conduction band. If $(k, \theta)$ belongs to $G$ then there are only evanescent waves and $(k, \theta)$ belongs to a forbidden band.

B. Explicit expression of the reflection coefficient

We suppose that $(k, \theta)$ belongs to $\mathbb{P} \setminus \Delta$. Denoting $(v, w)$ a basis of eigenvectors of $T$ we write in the canonical basis of $\mathbb{R}^2$: $v = (v_1, v_2), w = (w_1, w_2)$. Eigenvector $v$ (resp. $w$) is associated to eigenvalue $\gamma(k, \theta)$ (resp. $\gamma^{-1}(k, \theta)$). It is of course always possible to choose $(v, w)$ such that $\det(v, w) = 1$. After tedious but easy calculations, we get $r_N$ in closed form from (6):

4
\[ r_N(k, \theta) = \frac{\left( \gamma^{2N} - 1 \right) f}{\gamma^{2N} - g^{-1} f}, \quad (9) \]

\[ t_N(k, \theta) = \frac{(1 - g^{-1} f) \gamma^N}{\gamma^{2N} - g^{-1} f}, \quad (10) \]

denoting \( q(x_1, x_2) = \frac{i \beta_0 x_2 - x_1}{i \beta_0 x_2 + x_1} \), functions \( f \) and \( g \) are defined by

\[
\begin{align*}
\text{if } (k, \theta) \in G & \quad g(k, \theta) = q(v), f(k, \theta) = q(w) \\
\text{if } (k, \theta) \in B & \quad \begin{cases} 
  g(k, \theta) = q(v), f(k, \theta) = q(w) \text{ if } |q(v)| < |q(w)| \\
  g(k, \theta) = q(w), f(k, \theta) = q(v) \text{ if } |q(w)| < |q(v)|
\end{cases}
\end{align*} \tag{11}
\]

**Remark:** We have \(|g| < |f|\) and in a conduction band \( f = \overline{g}^{-1} \) so that we always have \(|g(k, \theta)| \leq 1\).

Let us denote finally denote:

\[
\begin{align*}
H_r(z) &= f z^2 - 1 \\
H_t &= (1 - g^{-1} f) \frac{z^2}{z^2 - g^{-1} f},
\end{align*} \tag{12}
\]

We immediately see that the reflection and transmission coefficients are obtained through \( H_r \) and \( H_t \) by:

\[ r_N(k, \theta) = H_r(\gamma^N), t_N(k, \theta) = H_t(\gamma^N). \]

### III. ASYMPTOTIC ANALYSIS OF THE REFLECTION COEFFICIENT

The reflection coefficient defines a sequence of points belonging to the closed unit disc \( \mathbb{D} \) of the complex plane. In order to have a clear understanding of the behavior of \( \{r_N, t_N\} \), we interpret this sequence as a discrete dynamical system. As \( N \) increases, we want to study how the orbits of this system spread over \( \mathbb{D} \).

When \((k, \theta)\) belongs to \( G \cup (\Delta \setminus \Delta_0) \) the behavior of the dynamical system is trivial as \( \{r_N, t_N\} \) admits one cluster point, situated on \( \mathbb{U} = \{F, |F| = |F|\} \). Indeed, in that case \( \gamma \) belongs to \( \mathbb{R} \setminus \{-i\varepsilon, i\varepsilon\} \). Recalling that by convention \(|\gamma| < 1\), we see that \( H_r(\gamma^N) \) tends to \( g(k, \theta) \). As the eigenvectors of \( T \) are real, we conclude that \( g(k, \theta) \) belongs to \( \mathbb{U} \). The second easy case is for \((k, \theta)\) belonging to \( \Delta_0 \), indeed \( \{r_N, t_N\} \) is constant and equal to 0 whatever \( N \).

These two cases hand the gaps as well as the edges of the gaps. The case of the conduction bands, i.e. for \((k, \theta)\) belonging to \( B \) is much more complicated and interesting. In the following we will skip the mathematical rigor and stress on the physical meaning of the results. The interested reader will find a complete and rigorous mathematical discussion in \cite{3}.

Dealing with a couple \((k, \theta)\) belonging to \( B \), the eigenvalues of \( T \) now belong to \( \mathbb{U} \) and we may write \( \gamma = e^{i\phi} \). Obviously, \( r_N = H_r(e^{iN\phi}) \) has no pointwise limit as \( N \) tends to
infinity. From a geometrical point of view, it is easy to show that the image of \( U \) through \( H \) is a circle \( V(k, \theta) \) passing through the origin and whose cartesian equation writes:

\[
(x - \Re \{z_0\})^2 + (y - \Im \{z_0\})^2 = |z_0|^2, \quad \text{with} \quad z_0 = (f^{-1} + g^{-1})^{-1}
\]  

(13)

Therefore, \( \{r_N, t_N\} \) describes a set of point on \( V(k, \theta) \). As \( \{r_N, t_N\} \) does not converge pointwise, we turn to another notion of convergence, in some average meaning. An easy computation in the case \(|g(k, \theta)| < |f(k, \theta)|\), shows that

\[
r_N(k, \theta) = g + g \sum_{k=1}^{\infty} \gamma^{2Nk} \left[(gf^{-1})^k - (gf^{-1})^{k-1}\right]
\]

(14)

\[
t_N(k, \theta) = (1 - gf^{-1}) \gamma^N \sum_{k=0}^{\infty} \gamma^{2Nk} (gf^{-1})^k
\]

(15)

then the reflected and transmitted fields write

\[
U^d_N(x, y, t) = \int_{\mathcal{P}} g(k, \theta) \exp(i(k \cdot r - \omega t)) \, d\mu
\]

\[
+ \sum_{k=1}^{\infty} \int_{\mathcal{P}} \gamma^{2Nk} g(k, \theta) \left[(gf^{-1})^k - (gf^{-1})^{k-1}\right] \exp(i(k \cdot r - \omega t)) \, d\mu
\]

\[
U^t_N(x, y, t) = \sum_{k=1}^{\infty} \int_{\mathcal{P}} \gamma^{(2k+1)N} (1 - gf^{-1}) (gf^{-1})^k \exp(i(k \cdot r - \omega t)) \, d\mu
\]

**Definition 1:** We say that a sequence of functions \( \psi_N(k, \theta) \) converges weakly towards \( \psi_\infty(k, \theta) \) if \( \lim_{N \to +\infty} \int_B \psi_N(k, \theta) \varphi(k, \theta) \, d\mu = \int_B \psi_\infty(k, \theta) \varphi(k, \theta) \, d\mu \) for every \( \varphi \) belonging to \( L^1(B, \mu) \).

We want to pass to the limit \( N \to +\infty \) in the preceding expressions. What we expect is some averaging over the set \( Y' \). Clearly, the limit behavior relies on the properties of \( \mu \) and \( \phi \). Let us define a convenient class of measures. Denoting \( C_\#(Y') \) the space of continuous \( Y' \)-periodic functions, we put:

**Definition 2:** A measure \( \mu \) is said admissible, if \( \exp(iN\phi(k, \theta)) \) tends weakly to 0.

Of course, this looks like an ad hoc property as it allows to get directly the limits of interest, but indeed this is a correct way of handling the problem, as it can be shown that the measures of interest for our problem, i.e. that of physical significance, will prove to be admissible. From the above expression, we conclude that

\[
U^d_N(x, y, t) \to \int_{\mathcal{P}} g(k, \theta) \exp(i(k \cdot r - \omega t)) \, d\mu
\]

\[
U^t_N(x, y, t) \to 0
\]

We can now conclude by collecting the above results.

**Proposition 1:** As \( N \) tends to infinity, \( r_N \) converges weakly towards \( r_\infty(k, \theta) = g(k, \theta) \), \( t_N \) tends weakly to 0.
IV. DIFFRACTION OF A WAVEPACKET BY A SEMI-INFINITE MEDIUM

Now, let us choose the incident field as a wavepacket of the form (2) where \( \mu \) is of one of the following forms:

I : \( \mu = p(k, \theta) dk \otimes d\theta, p \in L^1(\mathbb{P}, dk \otimes d\theta) \)

II : \( \mu = p(\theta) \delta_k \otimes d\theta, p \in L^1(\mathbb{R}, \delta_k \otimes d\theta) \)

III : \( \mu = p(k) dk \otimes \delta_\theta, p \in L^1(\mathbb{R}^+, dk \otimes \delta_\theta) \)

These measures define the most commonly used incident fields. Indeed measures of type I correspond to a general wavepacket, measures of type II to a monochromatic beam, and measures of type III to a temporal pulse. In order to apply the above results we have the following fundamental result:

**Proposition 2**: Measures of type I, II, III are admissible.

We can conclude that, for these measures \( \mu \), the diffracted field \( U^d_N(x, y, t) \) corresponding to \( N \) layers converges uniformly towards \( U^d_\infty(x, y, t) \) given by:

\[
U^d_\infty(x, y, t) = \int_{\mathbb{P}} r_\infty(k, \theta) \exp(ik(s\sin \theta x + \cos \theta y)) \exp(-i\omega t) d\mu
\]

That way, we have obtained a rigorous formulation for the diffraction of a wavepacket by a semi-infinite medium.

V. REFLECTION OF A MONOCHROMATIC BEAM BY A LOSSLESS SLAB OF INFINITE THICKNESS

A very special and interesting case is the case of a simple slab of optical index \( \nu \): the reflection coefficient \( r_h \) is therefore well known:

\[
r_h = r_{dio} \frac{1 - \exp(2i\beta h)}{1 - r_{dio}^2 \exp(2i\beta h)}
\]

where \( r_{dio} \) is the reflection coefficient for the diopter \( r_{dio} = \frac{\beta_0 - \beta}{\beta_0 + \beta} \). When the slab is filled with a lossy material \( N \) is therefore element of \( \mathbb{C} - \{\mathbb{R}, \mathbb{R} \} \), and in that case:

\[
\forall (k_0, \theta) \in \mathbb{P}, r_h \longrightarrow r_{dio}
\]

On the contrary, when we are dealing with lossless materials, \( \nu \) is real. In that case, \( r_h \) has an oscillating behavior and does not converge (and \textit{a fortiori} does not converge to \( r_{dio} \)). However, if we consider a limited monochromatic incident beam described by a function \( u_{inc}(x, y) \):

\[
u_{inc}(x, y) = \int_{-k_0}^{k_0} p(\alpha) e^{i(\alpha x + \beta(\alpha)y)} d\alpha
\]

where \( p(\alpha) \in L^1(-k_0, k_0, d\alpha) \) and characterizes the shape of the incident beam. We are therefore in the case where the measure \( \mu \) is of the form II (cf. paragraph V) i.e. :
\[ \mu = p(\theta)\delta_k \otimes d\theta \]  

(20)

with \( p \in L^1(\pi/2, \pi/2, d\theta) \). In these conditions the diffracted (reflected) field \( u_h(x, y) \) corresponding to a slab of finite thickness \( h \) follows as such:

\[ u_h(x, y) = \int_{-k_0}^{k_0} r_h(\alpha)p(\alpha)e^{i(-\alpha x + \beta(\alpha)y)} d\alpha \]  

(21)

and the diffracted field \( u_{dio}(x, y) \) corresponding to the diopter:

\[ u_{dio}(x, y) = \int_{-k_0}^{k_0} r_{dio}\ h(\alpha)p(\alpha)e^{i(-\alpha x + \beta(\alpha)y)} d\alpha \]  

(22)

and we have the fundamental result:

\[ \forall (x, y) \in \mathbb{R} \times \mathbb{R}_+, u_h(x, y) \rightarrow u_\infty(x, y) = \int_{-k_0}^{k_0} r_\infty(\alpha)p(\alpha)e^{i(-\alpha x + \beta(\alpha)y)} d\alpha \]  

(23)

It is very easy to calculate the reflection coefficient \( r_\infty \) in that case. The monodromy matrix is:

\[ T = \begin{pmatrix} \cos(\beta h) & \frac{1}{2}\sin(\beta h) \\ -\beta \sin(\beta h) & \cos(\beta h) \end{pmatrix} \]  

(24)

so that \( \text{tr}(T) = 2\cos \beta h \), therefore every \((k, \theta)\) belongs to \( B \cup \Delta_0 \) and \( \phi = \arccos(1/2\text{tr}(T)) = \beta h \). After elementary calculations we find:

\[ g = \frac{\beta_0 - \beta}{\beta_0 + \beta} = r_{dio} \quad f = \left(\frac{\beta_0 - \beta}{\beta_0 + \beta}\right)^{-1} \]  

(25)

Taking everything into account, we find the weak convergence \( r_\infty = r_{dio} \) and therefore \( u_h \rightarrow u_{dio} \).

**VI. ELECTROMAGNETIC PEROT FABRY**

In numerous experiments in Optics it appears that the light concentrates round particular areas of the overall space, namely the rays. In this paragraph, we establish a link between the amplitude associated with a ray and the amplitude associated with the electric field.

**A. Simple Perot Fabry**

It is well known that many reflected rays appear when dealing with the reflection coefficient of a monochromatic beam by a “simple” Perot-Fabry, we are in the same case as in the precedent paragraph, for which the reflected field corresponding to the finite thickness slab \( u_h(x, y) \) writes as follows:

\[ u_h(x, y) = \int_{-k_0}^{k_0} r_h(\alpha)p(\alpha)e^{i(-\alpha x + \beta(\alpha)y)} d\alpha \]  

(26)
From equation (17), the reflection coefficient \( r_h \) can be expressed as a series in the following manner:

\[
r_h = r_{dio} + \sum_{l=1}^{+\infty} (r_{dio}^{2l+1} - r_{dio}^{2l-1}) e^{2i\beta hl}
\]  

(27)

Consequently, \( u_h \) can also be expressed as a series (the exponential decreasing of \( r_{dio} \) allows us to reverse the signs \( \sum \) and \( \int \)):

\[
u_h(x, y) = u_{dio}(x, y) + \sum_{l=1}^{+\infty} u_{h,l}(x, y)
\]  

(28)

with

\[
u_{h,l}(x, y) = \int_{k_0}^{k_0} r_{dio}^{2l-1} (r_{dio}^2 - 1) e^{2i\beta(\alpha)hl} p(\alpha) e^{i(-\alpha x + \beta(\alpha)y)} d\alpha
\]  

(29)

Finally, introducing the two transmission coefficients \( t_{12} = \frac{2\beta_0}{\beta_0 + \beta} \) and \( t_{21} = \frac{2\beta}{\beta_0 + \beta} \), we find an expression of \( u_{h,l} \) that the optician can interpret at first glance:

\[
\begin{align*}
  u_{h,l}(x, y) &= -\int_{k_0}^{k_0} r_{dio}^{2l-1} t_{12} t_{21} e^{2i\beta(\alpha)hl} p(\alpha) e^{i(-\alpha x + \beta(\alpha)y)} d\alpha, l \geq 1 \\
  u_{h,0}(x, y) &= u_{dio}(x, y)
\end{align*}
\]  

(30)

Each function \( u_{h,l}, l \in \{0, \cdots, +\infty\} \) is interpreted as the complex amplitude associated with a ray labeled by \( l \) (cf. figure 1):

- the term \( r_{dio}^{2l-1} t_{12} t_{12} \) corresponds to the amplitude associated with a ray \( l \).
- the term \( e^{2i\beta(\alpha)hl} \) corresponds to a term of phase which expresses the delay of the reflected ray \( l \) with respect to the first reflected ray \( (l = 0) \)
- the minus sign before the integral may appear as suspicious. However, for the optician, the interpretation of this sign is easy. The reflection of all rays are of the same nature (reflection of a ray from a medium of index \( \nu \) on a medium of index 1) except for the first ray (reflection of a ray from a medium of index 1 on a medium of index \( \nu \)), which implies a change of sign in the reflection coefficient.

Besides, expression (28) calls for two remarks:

1. the optical interpretation of functions \( u_{h,l} \) is all the more clear as the following conditions are better fulfilled: a) the supports of the rays \( l \) are actually separated, i.e. the function \( p \) has a narrow support, b) the incident field is sufficiently slanted and c) the depth of the slab is sufficiently large with respect to the wavelength.

2. in the precedent paragraph, we have shown that \( u_h \) tends to \( u_{dio} \) as \( h \) tends to infinity, consequently the series in equation (28) has to tend to zero when \( h \) tends to infinity. Therefore, each function \( u_{h,l} \) behaves as a corrector for the diffracted field (each ray except the first one does vanish when the depth of the slab tends to infinity).
B. Generalized Perot Fabry

We consider now a one dimensional structure made of \( N \) identical layers. Writing \( u_N(x, y) \) the field diffracted by this structure, we have:

\[
    u_N(x, y) = \int_{-k_0}^{k_0} r_N(\alpha) p(\alpha) e^{i(-\alpha x + \beta(\alpha)y)} \, d\alpha
\]

(31)

Using the expression of \( r_N \) in (4) and the same techniques used above, we find :

\[
    u_N(x, y) = u_\infty(x, y) + \sum_{l=1}^{+\infty} u_{N,l}(x, y)
\]

(32)

with

\[
    u_\infty(x, y) = \int_{-k_0}^{k_0} r_\infty(\alpha) p(\alpha) e^{i(-\alpha x + \beta(\alpha)y)} \, d\alpha
\]

(33)

and

\[
    u_{N,l}(x, y) = \int_{-k_0}^{k_0} \gamma^{2N_l} g(k, \theta) \left[ (gf^{-1})^l - (gf^{-1})^{l-1} \right] p(\alpha) e^{i(-\alpha x + \beta(\alpha)y)} \, d\alpha
\]

(34)

The same comments as in the preceding section can be made.

C. Speed of convergence : numerical experiments

Up to now a special attention has been drawn about theoretical aspects of electromagnetic diffraction. More precisely, we have proved that, in any case encountered in physics, the function \( u_N(x, y, t) \) which represents the diffracted field converges to a function \( u_\infty(x, y, t) \) which one can easily calculate. But it remains to be seen how the function \( u_N(x, y, t) \) converges to its limit. For instance, in harmonic regime with a pulsantance \( \omega \), it is of practical prime importance to know the number of layers \( N_\eta \) from which one can replace, with a given precision \( \eta \), \( u_N(x, y, \omega) \) by \( u_\infty(x, y, \omega) \). From a theoretical point of view, it is very difficult to answer such questions even if, in some respects, we outlined an answer in the precedent paragraph. We are thus “doomed” to make numerical experiments and to leave the general insights of the Theory. Of course, in this paragraph, we do not aspire to the exhaustiveness and we only aim at giving some rough estimates.

In what follows, we are only dealing with a single Perot-Fabry made of silice for which the refractive index is \( \nu = 1.47 \) illuminated by a monochromatic gaussian beam solely characterized by the wavelength (\( \lambda = \frac{2\pi c}{\omega} \)), the size \( \Delta x_w \) and the height \( x_w \) of the waist and the mean incidence angle \( \theta \) (cf. figure ??). For such a beam the function \( p(\alpha) \) introduced above writes as follows:

\[
p(\alpha) = \frac{1}{2\pi} \exp \left( - \left( \frac{\alpha - \alpha_0}{2\Delta x_w} \right)^2 \right) \exp \left( -i(\alpha_0 x_w - \beta_0 y_w) \right)
\]

(35)
where $\alpha_0 = \frac{2\pi}{\lambda} \sin (\theta)$ and $y_w = -\tan (\theta) x_w$. In our numerical experiments, we have taken the following values: $\theta = \frac{\pi}{4}$, $x_w = 200$ and $\frac{H}{\lambda} = 200$, where $H$ is the “height of observation” (cf. figure ??). For these values, we have drawn, in the same figure $|u_h(H,y)|^2$, $|u_\infty(H,y)|^2$ and $|u_{h,1}(H,y)|^2$ as functions of $y$ (shifted of $\frac{\alpha_0}{\lambda}$) for the following normalized depths ($\frac{h}{\lambda} = 2, 20, 200, 2000, 5000$) and for two normalized waists ($\frac{\Delta x_w}{\lambda} = 5, 50$) (cf. figures ?? to ?? for a waist of 5, and figures 8 to 12 for a waist of 50). Geometrical Optics predicts the following locations of the maxima (cf. figure ??):

| $\frac{h}{\lambda}$ | 2 | 20 | 200 | 2000 | 5000 |
|---------------------|---|----|-----|------|------|
| $\frac{\Delta x_w}{\lambda}$ | 100 | 100 | 100 | 100 | 100 |
| $\frac{\Delta y}{\lambda}$ | 102.2 | 121.9 | 319.5 | 2295 | 5587 |

Finally, we have drawn in figure ?? (resp. fig. ??) the total field map $|u_h(x,y) + u_{inc}(x,y)|$ for $\frac{\Delta x_w}{\lambda} = 5$ (resp. for $\frac{\Delta x_w}{\lambda} = 50$) and for depth $\frac{h}{\lambda} = 200$.

In this example, we see that a width of about 1000 wavelengths (for $\lambda = 0.5\mu m$ this means a 0.5 mm width for the substrate) is necessary to obtain the rays described in classical optics. In practical experiments, the width of the substrate is usually of the order of one millimeter or more, and therefore the Fresnel coefficients are indeed measured.

VII. CONCLUSION

What is classical Optics (or coherent Optics)? Since Maxwell, at the end of the nineteenth century, the answer seems to be easy: classical optics is the study of the diffraction of an electromagnetic field by a body whose size is very large compared with the wavelength (mean wavelength). That means that we have to consider classical optics as a limit for small wavelengths of electromagnetic optics. Unlike the doxa, we consider that this limit is far from being clear. For instance, it is well known that in electromagnetic optics, the diffracted field is very sensitive to the polarization of the incident field. On the contrary, for small wavelengths, in a lot of applications, the diffracted field is independent of the polarization. In many cases, this remarkable property remains mysterious. In this paper, we only dealt with one dimensional problem and we demonstrated that the limit analysis (the path from the electromagnetic problem to the optical one) is generally ill posed for a plane wave (Fortunately, this case does not exist!) but is well posed for the cases encountered in physics. Besides, the limit analysis that we have given above does not take into account the roughness of the layers. For the sake of simplicity, we did not think fit to describe in a realistic way the process of measuring the field: this would have lead us to use a spatial convolution process. These remarks are not at all limitations of our study. On the contrary, both these phenomena would have improved the convergence of the involved sequences. Nevertheless we do think that this Limit Analysis is far more fundamental for a good understanding of classical optics than the usual explanations. Take the case of spectacles, which are usually intended to improve the vision. Can one be satisfied by such
an explanation as that involving absorption? The challenge is now to extend these results to the bidimensional or even tridimensional case, for which the very mathematical formulation itself is far from being clear.

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**Figures Captions**

Figure 1: Experimental device, position of the waist and height of observation.

Figure 2: Location of the maxima foreseen by geometrical optics.

Figure 3: Wavelength ($\lambda = 1$), size of the waist ($\Delta x_w = 5$), position of the waist ($x_w = 200, y_w = -200$), mean incidence angle ($\theta = \pi/4$), height of observation ($H = 100$), depth of layer $h = 2$.

Figure 4: Wavelength ($\lambda = 1$), size of the waist ($\Delta x_w = 5$), position of the waist ($x_w = 200, y_w = -200$), mean incidence angle ($\theta = \pi/4$), height of observation ($H = 100$), depth of layer $h = 20$.

Figure 5: Wavelength ($\lambda = 1$), size of the waist ($\Delta x_w = 5$), position of the waist ($x_w = 200, y_w = -200$), mean incidence angle ($\theta = \pi/4$), height of observation ($H = 100$), depth of layer $h = 200$.

Figure 6: Wavelength ($\lambda = 1$), size of the waist ($\Delta x_w = 5$), position of the waist ($x_w = 200, y_w = -200$), mean incidence angle ($\theta = \pi/4$), height of observation ($H = 100$), depth of layer $h = 2000$.

Figure 7: Wavelength ($\lambda = 1$), size of the waist ($\Delta x_w = 5$), position of the waist ($x_w = 200, y_w = -200$), mean incidence angle ($\theta = \pi/4$), height of observation ($H = 100$), depth of layer $h = 5000$.

Figure 8: Wavelength ($\lambda = 1$), size of the waist ($\Delta x_w = 50$), position of the waist ($x_w = 200, y_w = -200$), mean incidence angle ($\theta = \pi/4$), height of observation ($H = 100$), depth of layer $h = 2$.

Figure 9: Wavelength ($\lambda = 1$), size of the waist ($\Delta x_w = 50$), position of the waist ($x_w = 200, y_w = -200$), mean incidence angle ($\theta = \pi/4$), height of observation ($H = 100$), depth of layer $h = 20$.

Figure 10: Wavelength ($\lambda = 1$), size of the waist ($\Delta x_w = 50$), position of the waist ($x_w = 200, y_w = -200$), mean incidence angle ($\theta = \pi/4$), height of observation ($H = 100$), depth of layer $h = 200$.

Figure 11: Wavelength ($\lambda = 1$), size of the waist ($\Delta x_w = 50$), position of the waist ($x_w = 200, y_w = -200$), mean incidence angle ($\theta = \pi/4$), height of observation ($H = 100$), depth of layer $h = 2000$.

Figure 12: Wavelength ($\lambda = 1$), size of the waist ($\Delta x_w = 50$), position of the waist ($x_w = 200, y_w = -200$), mean incidence angle ($\theta = \pi/4$), height of observation ($H = 100$), depth of layer $h = 5000$.

Figure 13: The total field map $|u_h(x, y) + u_{inc}(x, y)|^2$ with wavelength ($\lambda = 1$), size of the waist ($\Delta x_w = 50$), position of the waist ($x_w = 200, y_w = -200$), mean incidence angle ($\theta = \pi/4$), depth of layer $h = 200$. 

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\[|u_\infty(H,y)|^2 \quad |u_h(H,y)|^2 \quad |u_{h,1}(H,y)|^2\]
$|u_{\infty}(H,y)|^2$

$|u_h(H,y)|^2$

$|u_{h,1}(H,y)|^2$
This figure "figure_13.jpg" is available in "jpg" format from:

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