CYCLES OF EVEN-ODD DROP PERMUTATIONS AND CONTINUED FRACTIONS OF GENOCCHI NUMBERS

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Abstract. Recently, Lazar and Wachs (arXiv:1910.07651) showed that the (median) Genocchi numbers play a fundamental role in the study of the homogenized Linial arrangement and obtained two new permutation models (called D-permutations and E-permutations) for (median) Genocchi numbers. They further conjecture that the distributions of cycle numbers over the two models are equal. In a follow-up, Eu et al. (arXiv:2103.09130) further proved the gamma-positivity of the descent polynomials of even-odd descent permutations, which are in bijection with E-permutations by Foata’s fundamental transformation. This paper merges the above two papers by considering a general moment sequence which encompasses the number of cycles and number of drops of E-permutations. Using the combinatorial theory of continued fraction, the moment connection enables us to confirm Lazar-Wachs’ conjecture and obtain a natural \((p, q)\)-analogue of Eu et al.’s descent polynomials. Furthermore, we show that the \(\gamma\)-coefficients of our \((p, q)\)-analogue of descent polynomials have the same factorization flavor as the \(\gamma\)-coefficients of Brändén’s \((p, q)\)-Eulerian polynomials.

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1. Introduction

The Genocchi numbers \(G_{2n}\) and median Genocchi numbers \(H_{2n+1}\) are well studied, and have seen recent attention and new combinatorial interpretations, see [2, 3, 8, 11, 13, 21, 2010 Mathematics Subject Classification. 05A05, 05A15, 05A19, 33C45.
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These two allied sequences of numbers can be easily defined by the Seidel triangle [27] through a boustrophedon algorithm as follows:

![Seidel triangle](https://oeis.org/A005439) =

\[
\begin{array}{cccccccc}
1 & 1 &  &  &  &  &  & \\
1 &  &  &  &  &  &  & \\
2 & 1 &  &  &  &  &  & \\
2 & 3 & 3 &  &  &  &  & \\
8 & 6 & 3 &  &  &  &  & \\
8 & 14 & 17 & 17 &  &  &  & \\
56 & 48 & 34 & 17 & 56 & 104 & 138 & 155 & 155
\end{array}
\]

**Figure 1.** The first values of the Genocchi numbers \(G_{2n}\), median Genocchi numbers \(H_{2n+1}\) and normalized median Genocchi numbers \(H_{2n+1}/2^n\) are tabulated in the right table.

In 2019 Hetyei [21] introduced a hyperplane arrangement (called the homogenized Linial arrangement) and showed that its number of regions is a median Genocchi number. Lazar and Wachs [23] refined Hetyei’s result by obtaining a combinatorial interpretation of the Möbius function of this lattice in terms of variants of the Dumont permutations.

A permutation \(\sigma \in S_{2n}\) is a D-permutation (resp. E-permutation) if \(i \leq \sigma(i)\) whenever \(i\) is odd and \(i \geq \sigma(i)\) whenever \(i\) is even (resp. if \(i > \sigma(i)\) implies \(i\) is even and \(\sigma(i)\) is odd). An E-permutation is also called even-odd drop permutation. Introduce the set notations

\[
\begin{align*}
\mathcal{D}_{2n} &= \{\text{D-permutations on } [2n]\}; \\
\mathcal{C}_{2n} &= \{\text{D-cycles on } [2n]\}; \\
\mathcal{E}_{2n} &= \{\text{E-permutations on } [2n]\}; \\
\mathcal{C}_{2n} &= \{\text{E-cycles on } [2n]\}.
\end{align*}
\]

For example,

\[
\mathcal{D}_4 = \{(1)(2)(3)(4), \ (1, 2)(3)(4), \ (1, 4)(2)(3), \ (3, 4)(1)(2), \\
(1, 2)(3, 4), \ (1, 3, 4)(2), \ (1, 4, 2)(3), \ (1, 3, 4, 2)\};
\]

and

\[
\mathcal{E}_4 = \{(1)(2)(3)(4), \ (1, 2)(3)(4), \ (1, 4)(2)(3), \ (3, 4)(1)(2), \\
(1, 2)(3, 4), \ (1, 3, 4)(2), \ (1, 2, 4)(3), \ (1, 2, 3, 4)\}.
\]

Here the permutations are written in cycle notations.

**Theorem 1** (Lazar and Wachs [23]). For integer \(n \geq 1\) we have

\[
\begin{align*}
H_{2n+1} &= \left|\mathcal{D}_{2n}\right| = \left|\mathcal{E}_{2n}\right|, \\
G_{2n} &= \left|\mathcal{C}_{2n}\right|.
\end{align*}
\]
Lazar and Wachs [23, Conjecture 6.5] also make the following conjecture.

**Conjecture 2** (Lazar and Wachs). The number of $D$-permutations on $[2n]$ with $k$ cycles equals the number of $E$-permutations on $[2n]$ with $k$ cycles for all $k$. Consequently

$$G_{2n} = |\mathcal{EC}_{2n}|.$$  

In this paper we will take a different approach to enumerate the cycles and drops of the $E$-permutations and prove Lazar-Wachs’ conjecture. A descent of a permutation $\sigma$ on a finite set of positive integers is a pair $(\sigma_i, \sigma_{i+1})$ for which $\sigma_i > \sigma_{i+1}$. We call $\sigma_i$ (resp. $\sigma_{i+1}$) the descent top (resp. descent bottom). A descent pair $(\sigma_i, \sigma_{i+1})$ is called even-odd if $\sigma_i$ is even and $\sigma_{i+1}$ is odd. The number of descents of $\sigma$ is denoted by $\text{des} \, \sigma$. Moreover, the value $\sigma_i$ is said to be

- a left-to-right maxima, if $\sigma_j < \sigma_i$ for all $j < i$;
- a right-to-left minimum, if $\sigma_i < \sigma_j$ for all $j > i$.

Let $\text{lma} \, \sigma$ and $\text{rmi} \, \sigma$ be the numbers of left-to-right maxima and right-to-left minima. Let $\text{lem} \, \sigma$ and $\text{loma} \, \sigma$ the numbers of even and odd left-to-right maxima. Similarly we define $\text{remi} \, \sigma$ and $\text{romi} \, \sigma$ to be the numbers of even and odd right-to-left minima.

Let $X_{2n}$ be the set of permutations in $S_{2n}$ that contain only even-odd descents. A permutation in $X_{2n}$ is called an even-odd descent permutation. For $n = 1, 2$, we have $X_2 = \{12, 21\}$ and

$$X_1 = \{1234, 2134, 1243, 3412, 4123, 2341, 2413, 2143\}.$$

Note that one version of Foata’s first fundamental transformation $\varphi$ takes drops to descents and maxima of cycles to left-to-right maxima, i.e.,

$$(\text{drop, cyc}) \sigma = (\text{des, lma}) \varphi(\sigma) \quad \text{for} \quad \sigma \in S_n. \quad (1.1)$$

Here drop $\sigma$ and cyc $\sigma$ denote the numbers of drops and cycles of $\sigma \in S_n$, respectively. Hence, by restriction $\varphi$ sets up a bijection from $\mathcal{E}_{2n}$ to $X_{2n}$, and the cardinality of $X_{2n}$ equals the median Genocchi number $H_{2n+1}$.

An even-odd descent permutation $\sigma \in X_{2n}$ is called $E$-permutation if for any integer $j \in \{1, \ldots, n\}$ the two entries $2j - 1$ and $2j$ are not simultaneously a descent bottom and a descent top of $\sigma$.\footnote{It is easy to see that our definition of $E$-permutations is equivalent to the primary even-odd descent permutations in Eu et al. [15].} Let $\overline{X}_{2n}$ be the set of $E$-permutations in $X_{2n}$ and $\overline{X}_{2n,k}$ be the subset consisting of permutations with $k$ descents. The first few $E$-permutations are

$$\overline{X}_{2,0} = \{12\}, \quad \overline{X}_{4,0} = \{1234\}, \quad \overline{X}_{4,1} = \{3412, 4123, 2341, 2413\}$$

$$\overline{X}_{6,0} = \{123456\}, \quad \overline{X}_{6,1} = \{124563, 124635, 125634, 126345, 234156, 241356, 234156, 261345, 361245, 461235, 561234, 236145, 246135, 256134, 346125, 356124, 456123, 234615, 235614, 245613, 345612, 234561, 61234\}.$$  

Note that the even-odd descent permutation $\sigma = 24163785$ is not an $E$-permutation.
Eu-Fu-Lai-Lo [15, Theorem 1.2] recently studied the descent polynomials of even-odd descent permutations, and proved, among other things, the following result.

**Theorem 3** (Eu et al. [15]). We have

\[ X_n(t) := \sum_{\sigma \in X_{2n}} t^{\text{des} \sigma} = \sum_{k=0}^{\lfloor n/2 \rfloor} |X_{2n,k}| t^k (1 + t)^{n-2k}. \tag{1.2} \]

An expansion of type \( 1.2 \) is called the \( \gamma \)-expansion of the polynomial \( X_n(t) \) in the literature [1, 7]. Indeed, a polynomial with real coefficients \( h(t) = \sum_{i=0}^{n} h_i t^i \) is said to be palindromic if \( h_i = h_{n-i} \) for \( 0 \leq i \leq n/2 \). It is known that any palindromic polynomial can be written uniquely in the form \( \sum_{i=0}^{[(n-1)/2]} \gamma_i t^i (1 + t)^{n-2i} \). The coefficients \( \gamma_i \) are called the \( \gamma \)-coefficients of \( h(t) \). If the \( \gamma \)-coefficients \( \gamma_i \) are all nonnegative then we say that \( h(t) \) is \( \gamma \)-positive. Note that the \( \gamma \)-positivity of \( h(t) \) implies the symmetry and unimodal property of the coefficients \( (h_0, \ldots, h_n) \). A prototype of \( \gamma \)-positive polynomials is the Eulerian polynomials \( A_n(t) \), which are the descent polynomials of permutations in \( \mathfrak{S}_n \).

The following expansion of Eulerian polynomials is known [17, 25]:

\[ A_n(t) := \sum_{\sigma \in \mathfrak{S}_n} t^{\text{des} \sigma} = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} 2^k d_{n,k} t^k (1 + t)^{n-2k-2k}, \tag{1.3} \]

where \( d_{n,k} \) is the number of André permutations in \( \mathfrak{S}_n \) with \( k \) descents.

This paper stems from the observation that the generating function of the descent polynomials \( X_n(t) \) has a similar continued fraction expansion as the Eulerian polynomials and the cardinality \( |X_{2n,k}| \) in \( 1.2 \) is divisible by \( 4^k \), consequently \( 1.2 \) would imply a refinement of the normalized median Genocchi numbers

\[ H_{2n+1}/2^n = \sum_{k=0}^{\lfloor n/2 \rfloor} |X_{2n,k}|/4^k. \tag{1.4} \]

In 2008 Brändén [6] studied a \((p, q)\)-analogue of Eulerian polynomials and proved that his generalized Eulerian polynomials have a \( \gamma \)-positive expansion by modifying Foata and Strehl’s valley-hopping action. At the end of his paper Brändén [6] speculated that the corresponding \( \gamma \)-coefficients have a factor \((p + q)^k\), which is a \((p, q)\)-analogue of \( 2^k \). Shin and Zeng [28] confirmed the conjectured divisibility of \( \gamma \)-coefficients by using Flajolet-Viennot’s combinatorial theory of orthogonal polynomials [16, 30] and proved that the corresponding quotient \( d_{n,k}(p, q) \) is a polynomial in \( p \) and \( q \) with nonnegative integer coefficients. Finally, motivated by an open problem of Han [19] about \( q \)-analogue of Euler numbers, the two authors [25] came up with a combinatorial interpretation for \( d_{n,k}(p, q) \) by refining the André permutation interpretation for \( d_{n,k} \).

In this paper we shall consider a natural \((p, q)\)-analogue of the descent polynomials \( X_n(p, q, t) \) and prove that the \( \gamma \)-coefficient \( \gamma_{n,k}(p, q) \) of \( X_n(p, q, t) \) has a factor \((p + q)^{2k}\), and provide a combinatorial interpretation for \( \gamma_{n,k}(p, q)/(p + q)^{2k} \), see Theorem 13. It turns out that our approach can catch more permutation statistics over even-odd descent
permutations. For $j \in [n]$, the doubleton $\{2j - 1, 2j\}$ is called a domino of $\sigma \in \mathcal{X}_{2n}$ if $2j$ and $2j - 1$ are descent top and descent bottom of $\sigma$, respectively. Denote by $\text{dom}(\sigma)$ the number of dominos of $\sigma$. For $\sigma = \sigma_1 \sigma_2 \ldots \sigma_n \in \mathfrak{S}_n$, the statistic $(31-2)\sigma$ is the number of pairs $(i, j)$ such that $2 \leq i < j \leq n$ and $\sigma_{i-1} > \sigma_j > \sigma_i$. The pair $(\sigma_{i-1}, \sigma_i)$ is called a left-embracing of $\sigma_j$. Similarly, the statistic $(2-31)\sigma$ is the number of pairs $(i, j)$ such that $1 \leq i < j \leq n - 1$ and $\sigma_j > \sigma_i > \sigma_{j+1}$. The pair $(\sigma_j, \sigma_{j+1})$ is called a right-embracing of $\sigma_i$.

We refine the median Genocchi numbers by the octuple-variable polynomials

$$X_n := X_n(a, \bar{a}, b, \bar{b}, p, q, y, t) = \sum_{\sigma \in \mathcal{X}_{2n}} a^{\text{lema}_\sigma} \bar{a}^{\text{loma}_\sigma} b^{\text{romi}_\sigma} \bar{b}^{\text{romi}_\sigma} p^{(2-31)\sigma} q^{(31-2)\sigma} y^{\text{dom}_\sigma} t^{\text{des}_\sigma},$$

which reduces to $X_n(t)$ when $a = \bar{a} = b = \bar{b} = y = 1$. For example,

$$X_1 = ab(\bar{a} + t),$$
$$X_2 = a^2a^2b^2b^2 + 2a^2ab^2\bar{b}yt + a\bar{a}b\bar{b}pqt + ab^2\bar{b}q^2t + a^2\bar{a}b^2pt + a^2b^2pqt + a^2b^2t^2y^2.$$ We will prove an explicit J-fraction expansion (Theorem 4) for the ordinary generating function of $X_n$, and prove $(p, q)$-analogue of Theorem 3 and Lazar-Wachs’ conjecture as applications.

The Genocchi numbers $G_{2n}$ can also be defined by the exponential generating function $\sum_{n=1}^{\infty} G_{2n} \frac{x^{2n}}{(2n)!} = x \tan \frac{x}{2}$, while the median Genocchi numbers do not seem to have a reasonable exponential generating function. Our method relies on the S-fraction expansions of the ordinary generating functions of Genocchi and median Genocchi numbers (see [14, 29]):

$$\sum_{n=0}^{\infty} G_{2n+2}x^n = \frac{1}{\left(1 - \frac{1^2 \cdot x}{1 - \frac{1 \cdot 2 \cdot x}{1 - \frac{2 \cdot 3 \cdot x}{\cdots}}}\right) / 1} = 1 + x + 3x^2 + 17x^3 + \cdots, \quad (1.6a)$$

$$1 + \sum_{n=0}^{\infty} H_{2n+1}x^{n+1} = \frac{1}{\left(1 - \frac{1^2 \cdot x}{1 - \frac{2 \cdot 3 \cdot x}{\cdots}}\right) / 1} = 1 + x + 2x^2 + 8x^3 + \cdots. \quad (1.6b)$$
By the general theory of orthogonal polynomials the above continued fraction expansions mean that the Genocchi and median Genocchi numbers are moments of orthogonal polynomials. More precisely, as shown in [18], they are indeed moments of special or shifted continuous dual Hahn polynomials, see also [5,9,10,18] for recent papers on combinatorial aspects related to the moments of Askey-Wilson polynomials.

For reader’s convenience, we recall two standard contraction formulae transforming an S-fraction to J-fractions, see [14].

**Lemma 1** (Contraction formula). For any sequence \( \{\alpha_n\} \) of elements in an arbitrary ring containing \( \mathbb{Z} \), the following holds

\[
\begin{align*}
\frac{1}{1 - \alpha_1 x} &= \frac{1}{1 - b_0 x - \frac{\lambda_1 x^2}{1 - b_1 x - \frac{\lambda_2 x^2}{1 - b_2 x - \frac{\lambda_3 x^2}{\ddots}}}} \quad (1.7) \\
\frac{1}{1 - \alpha_1 x} &= 1 + \frac{\alpha_1 x}{1 - b_0 x - \frac{\lambda_1 x^2}{1 - b_1 x - \frac{\lambda_2 x^2}{1 - b_2 x - \frac{\lambda_3 x^2}{\ddots}}}} \quad (1.8)
\end{align*}
\]

if \( b_0 = \alpha_1 \), \( b_n = \alpha_{2n} + \alpha_{2n+1} \), \( \lambda_n = \alpha_{2n-1} \alpha_{2n} \) for \( n \geq 1 \); and

\[
\begin{align*}
\frac{1}{1 - \alpha_1 x} &= \frac{1}{1 - 2 \cdot 1^2 x - \frac{1^2 \cdot 2^2 x^2}{1 - 2 \cdot 2^2 x - \frac{2^2 \cdot 3^2 x^2}{1 - 2 \cdot 3^2 x - \frac{3^2 \cdot 4^2 x^2}{\ddots}}}} \quad (1.9) \\
\frac{1}{1 - \alpha_1 x} &= \frac{1}{1 - 2^2 x - \frac{(\frac{1}{2})^2 \cdot x^2}{1 - 3^2 x - \frac{(\frac{1}{2})^2 \cdot x^2}{\ddots}}} \quad (1.10)
\end{align*}
\]

From (1.6b) we derive the J-fraction

\[
\sum_{n=0}^{\infty} H_{2n+1} x^n = \frac{1}{1 - 2 \cdot 1^2 x - \frac{1^2 \cdot 2^2 x^2}{1 - 2 \cdot 2^2 x - \frac{2^2 \cdot 3^2 x^2}{1 - 2 \cdot 3^2 x - \frac{3^2 \cdot 4^2 x^2}{\ddots}}}} \quad (1.9)
\]

Let \( h_n = H_{2n+1}/2^n \) (\( n \in \mathbb{N} \)) be the normalized median Genocchi numbers. By replacing \( x \) by \( x/2 \), we obtain

\[
\sum_{n=0}^{\infty} h_n x^n = \frac{1}{1 - 1^2 x - \frac{1^2 \cdot x^2}{1 - 2^2 x - \frac{(\frac{1}{2})^2 \cdot x^2}{1 - 3^2 x - \frac{(\frac{1}{2})^2 \cdot x^2}{\ddots}}}} \quad (1.10)
\]
Dumont [11] initiated the combinatorial characterization of Genocchi numbers. A permutation \( \sigma \in S_{2n} \) is a Dumont permutation if \( \sigma(2i - 1) \geq 2i - 1 \) and \( \sigma(2i) < 2i \) for \( i \in [n] \). A permutation \( \sigma \in S_{2n} \) is a Dumont derangement if \( \sigma(2i - 1) > 2i - 1 \) and \( \sigma(2i) < 2i \) for \( i \in [n] \). Dumont [11] and Dumont-Randrianarivony [13] proved respectively that the Dumont permutations in \( S_{2n} \) is the Genocchi number \( G_{2n} \) and the Dumont derangements in \( S_{2n} \) is the median Genocchi number \( H_{2n+1} \). Kitaev and Remmel [22] conjectured and Burstein, Josuat-Vergès, and Stromquist [8] proved that the set of permutations in \( S_{2n} \) with only even-even descents has cardinality equal to \( G_{2n} \). Clearly the set of even-even descent permutations in \( S_{2n} \) is in bijection with the set of odd-odd descent permutations in \( S_{2n-1} \).

Let \( Y_{2n+1} \) be the set of permutations in \( S_{2n+1} \) that contain only odd-odd descents, notice that \( |Y_{2n+1}| = G_{2n+2} \). Let \( Y_{2n+1}^* \) be the subset of \( Y_{2n+1} \) consisting of the permutations ending with an odd element. The first three sets are \( Y_1^* = \{1\} \), \( Y_3^* = \{123, 231\} \) and

\[
Y_5^* = \{12345, 12453, 23451, 24531, 24513, 23145, 31245, 45123\}.
\]

Eu et al. [15] proved that the cardinality of \( Y_{2n+1}^* \) is equal to the median Genocchi number \( H_{2n+1} \). They also considered a \( q \)-analogue of the descent polynomials of \( Y_{2n+1}^* \) and proved a similar \( \gamma \)-expansion for the \( q \)-descent polynomials. We show Section 7 that it is possible to refine their results and prove similar results as for the descent polynomials of even-odd descent permutations.

2. Main results

For any integer \( n \geq 1 \) we define the generalized \((p,q)\)-analogue of \( n \) by

\[
[x, n, y]_{p,q} = x p^{n-1} + \sum_{i=1}^{n-2} p^{n-1-i} q^i + y q^{n-1},
\]

\[
[x, n]_{p,q} = x p^{n-1} + \sum_{i=1}^{n-1} p^{n-1-i} q^i,
\]

with \([x, 1, y]_{p,q} = xy\) and \([x, 1]_{p,q} = x\). In particular,

\[
[n]_{p,q} = [1, n]_{p,q} = \frac{p^n - q^n}{p - q},
\]

and the \((p,q)\)-analogue of binomial coefficient \( \binom{n}{k} \) is defined by

\[
\binom{n}{k}_{p,q} = \frac{[n]_{p,q} \cdots [n-k+1]_{p,q}}{[1]_{p,q} \cdots [k]_{p,q}} \quad (1 \leq k \leq n).
\]
Theorem 4. We have

\[ 1 + \sum_{n=1}^{\infty} X_n(a, \bar{a}, b, \bar{b}, p, q, y, t) x^n = \frac{1}{1 - ab(\bar{a} b + ty) x - \frac{abt(\bar{a} p + b q)(\bar{a} p + b q) x^2}{1 - \frac{ab(\bar{a} b + ty) x - \frac{abt(\bar{a} p + b q)(\bar{a} p + b q) x^2}{\ldots}}}} \]  

(2.2)

with coefficients for \( n \geq 1 \)

\[ \begin{cases} b_{n-1} = [a, n, b]_{p,q}[a, n, b]_{p,q} + ty[a, n]_{p,q}[b, n]_{q,p}, \\ \lambda_n = t[a, n]_{p,q}[a, n + 1, \bar{b}]_{p,q} [b, n]_{q,p} [a, n + 1, b]_{p,q}. \end{cases} \]

We first record some special cases of Theorem 4. When \( b = \bar{b} = 1 \) we have the more compact formula.

Corollary 5.

\[ 1 + \sum_{n=1}^{\infty} X_n(a, \bar{a}, 1, 1, p, q, y, t) x^n = \frac{1}{1 - (\bar{a} + ty)ax - \frac{at(\bar{a} p + q)(ap + q)x^2}{1 - (\bar{a}, 2)_{p,q} + ty[2]_{p,q} [a, 2]_{p,q} x - \frac{t[a, 2]_{p,q}[a, 3]_{p,q}[2]_{p,q}[\bar{a}, 3]_{p,q} x^2}{\ldots}}} \]  

(2.3)

with coefficients for \( n \geq 1 \)

\[ \begin{cases} b_{n-1} = ([a, n]_{p,q} + ty[n]_{p,q})[a, n]_{p,q}, \\ \lambda_n = t[a, n]_{p,q}[a, n + 1]_{p,q} [n]_{p,q} [a, n + 1]_{p,q}. \end{cases} \]

Let \( y = 1 \) in Theorem 4, by the contraction formula we can rewrite the above J-fraction as an S-fraction.

Corollary 6. We have the S-fraction expansion

\[ 1 + a\bar{a} \sum_{n=0}^{\infty} X_n(a, \bar{a}, 1, 1, p, q, 1, t) x^{n+1} = \frac{1}{1 - \frac{a\bar{a} \cdot x}{a\bar{a} \cdot x}} \]  

(2.4)

with \( X_0 = 1 \) and coefficients

\[ \begin{cases} \alpha_{2n-1} = [a, n]_{p,q}[a, n]_{p,q}, & n \geq 1, \\ \alpha_{2n} = [a, n]_{p,q}[n]_{p,q} t. \end{cases} \]
By Foata’s fundamental transformation (see (1.1)) we have

\[ P_n(t, z) := \sum_{\sigma \in \mathcal{X}_{2n}} z^{\text{des}} \sigma z^{\text{loma}} = \sum_{\sigma \in \mathcal{E}_{2n}} z^{\text{drop}} \sigma z^{\text{cyc}} \sigma. \]

Thus, letting \( a = \bar{a} = z, p = q = y = 1 \) in Corollary 5 we derive the continued fraction for the ordinary generating function of \( P_n(t, z) \).

**Corollary 7.**

\[
1 + \sum_{n=1}^{\infty} P_n(t, z)x^n
\]

\[
= \frac{1}{1 - z(z + t)x - \frac{z(z + 1)^2 t x^2}{1 - (z + n)(z + n + (n + 1)t)x - \frac{(n + 1)(z + n)(z + n + 1)^2 t x^2}{\cdots}}}.
\]

(2.5)

Taking \( y = 0 \) in Theorem 4 we obtain the corresponding formula for \( \mathcal{E} \)-permutations.

**Corollary 8.** We have

\[
1 + \sum_{n=1}^{\infty} \sum_{\sigma \in \mathcal{X}_{2n}} a^{\text{loma}} \sigma \bar{a}^{\text{loma}} \sigma p^{(2)} q^{(3)} \sigma t^{\text{des}} \sigma x^n =
\]

\[
= \frac{1}{1 - a\bar{a} x - \frac{at[2]_{p,q}[a, 2]_{p,q} x^2}{1 - (ap + q)(\bar{a}p + q)x - \frac{t[2]_{p,q}[a, 2]_{p,q}[3]_{p,q}[\bar{a}, 3]_{p,q} x^2}{\cdots}}}.
\]

(2.6)

with coefficients for \( n \geq 1 \)

\[
\begin{cases}
  b_{n-1} = [a, n]_{p,q}[\bar{a}, n]_{p,q}, \\
  \lambda_n = t[a, n]_{p,q}[a, n + 1]_{p,q}[n]_{p,q}[\bar{a}, n + 1]_{p,q}.
\end{cases}
\]

We shall prove Theorem 4 in Section 3 by constructing a bijection from the even-odd descent permutations to lattice path diagrams. Combining Theorem 4, the surjective pistol theory developed by Dumont-Randrianarivony and Lazar-Wachs and Randrianarovony-Zeng’s continued fraction expansion for the ordinary generating function of generalized Dumont-Foata polynomials we prove the conjecture of Lazar-Wachs [23, Conjecture 6.4] in section 4.
Theorem 9. Conjecture 2 is true, i.e., for all \( n \geq 1 \),
\[
\sum_{\sigma \in \mathcal{E}_{2n}} z^{\text{cyc} \sigma} = \sum_{\sigma \in \mathcal{D}_{2n}} z^{\text{cyc} \sigma}.
\]  
(2.7)

Consequently \( |\mathcal{E}_{2n}| = |\mathcal{D}_{2n}| = G_{2n+2} \).

Our third result is a quasi-gamma decomposition of \( X_n(a, 1, b, 1, p, q, y, t) \), which provides a neat \((p, q)\)-analogue of the \(\gamma\)-formula (1.2) by setting \( a = b = y = 1 \).

Theorem 10. We have
\[
1 + \sum_{n=1}^{\infty} X_n(a, 1, b, 1, p, q, y, t) x^n = \frac{1}{1 - ab(\bar{a} \bar{b} + ty) x} - \frac{abt (\bar{a} p + \bar{q}) (\bar{a} p + b q) x^2}{\cdots} \tag{2.8}
\]
with coefficients for \( n \geq 1 \)
\[
\begin{cases}
    b_{n-1} & = (1 + ty)[a, n]_{p, q}[b, n]_{q, p}, \\
    \lambda_n & = t[a, n]_{p, q} [a, n+1]_{p, q} [b, n]_{q, p} [b, n+1]_{q, p}.
\end{cases}
\]

Moreover the following \(\gamma\)-formula holds
\[
X_n(a, 1, b, 1, p, q, y, t) = \sum_{k=0}^{|n/2|} \gamma_{n,k}(a, b, p, q) t^k(1 + yt)^{n-2k}, \tag{2.9a}
\]
with coefficients
\[
\gamma_{n,k}(a, b, p, q) = \sum_{\sigma \in \mathcal{X}_{2n,k}} a^{\text{len} \sigma} y^{\text{romi} \sigma} p^{(2-3) \sigma} q^{(31-2) \sigma}. \tag{2.9b}
\]

Clearly Theorem 4 reduces to (2.8) by setting \( \bar{a} = \bar{b} = 1 \). By comparing Corollary 8 with formula (2.9b) we derive (2.9a). We shall give another proof of (2.9a) using \textit{Inter-hopping} action on \( \mathcal{X}_{2n} \) in Section 5.

A \textit{weak signature} of order \( k \) in \([2n]\) is a subset \( S \subseteq [2n] \) consisting of \( k \) odd and \( k \) even numbers such that for each \( i \in [2n] \), the number of odd numbers is greater than or equal to the number of even numbers in \( S \cap \{1, \ldots, i\} \). Note that our weak signature is different with the \textit{signature} in [8,15]. Denote by \( S_o \) (resp. \( S_e \)) the set of odd (resp. even) elements of \( S \). Let \( S_o(i) \) and \( S_e(i) \) be the numbers of odd and even elements in \( S \) less than \( i \). The associated signature function \( s \) has domain \([2n]\) and is defined by
\[
s(x) = S_o(x) - S_e(x) + (1 \text{ if } x \in S_o \text{ or } x \notin S). \tag{2.10}
\]
We use $l_{\sigma}(i)$ (resp. $r_{\sigma}(\sigma_i)$) to denote the number of left-embracings (resp. right-embracings) of $\sigma_i$ in $\sigma$, thus

\[(31-2)\sigma = \sum_{i=1}^{n} l_{\sigma}(\sigma_i); \quad (2.11a)\]

\[(2-31)\sigma = \sum_{i=1}^{n} r_{\sigma}(\sigma_i). \quad (2.11b)\]

**Lemma 2.** For $\sigma \in \mathcal{X}_{2n}$, if $S$ be the set of descent tops and bottoms of $\sigma$, then $S$ is a week signature of $[2n]$ and the associated signature function $s$ is given by

\[s(i) = l_{\sigma}(i) + r_{\sigma}(i) + 1 \quad \forall i \in [2n]. \quad (2.12)\]

**Proof.** Let $\sigma \in \mathcal{X}_{2n}$. For any $i \in [2n]$, there is an injection $\phi : S_e \cap [i] \to S_o \cap [i]$ which maps each descent top $t \in S_e \cap [i]$ to a descent bottom $b \in S_o \cap [i]$ such that $(t, b)$ forms a descent pair of $\sigma$. Hence $S_o(i) \geq S_e(i)$ and $S$ is a week signature of $[2n]$.

Next we compute the embracing numbers $l_{\sigma}(i) + r_{\sigma}(i)$ of $i \in [2n]$.

1. If $i \in S$ is odd or $i \notin S$, the number of even-odd descents $(t, b)$ with $t < i$ is equal to $S_e(i)$, so the embracing number of $i$ is $S_o(i) - S_e(i)$.
2. If $i \in S$ is even, the number of even-odd descents $(t, b)$ with $t \leq i$ is equal to $S_e(i) + 1$, so $i$ has $S_o(i) - S_e(i) - 1$ embraces.

Combining the two cases and comparing with (2.10) we obtain (2.12). \hfill $\square$

In what follows, for any $\sigma \in \mathcal{X}_{2n}$ we use $S_{\sigma}$ and $s_{\sigma}$ to denote the weak signature of $\sigma$ and the associated signature function, respectively.

**Definition 11.** An $\mathcal{E}$-permutation $\sigma \in \mathcal{X}_{2n}$ is normalized if it satisfies the following conditions: for $j \in [n]$,

(a) if $(2j - 1, s(2j - 1)) \in S_{\sigma} \times (2\mathbb{N} + 1)$, then $l_{\sigma}(2j)$ is even;
(b) if $(2j - 1, s(2j - 1)) \in S_{\sigma} \times 2\mathbb{N}$, then $l_{\sigma}(2j - 1)$ is even;
(c) if $(2j, s(2j)) \in S_{\sigma} \times (2\mathbb{N} + 1)$, then $r_{\sigma}(2j - 1)$ is even;
(d) if $(2j, s(2j)) \in S_{\sigma} \times 2\mathbb{N}$, then $r_{\sigma}(2j)$ is even.

Let $\mathcal{X}_{2n}$ be the set of normalized $\mathcal{E}$-permutations in $\mathcal{X}_{2n}$ and let

$\mathcal{X}_{2n,k} = \{\sigma \in \mathcal{X}_{2n} : \text{des } \sigma = k\}$.

**Example 1.** The following are the first few normalized $\mathcal{E}$-permutations:

$\mathcal{X}_2 = \{12\}$

$\mathcal{X}_4 = \{1234, 2413\}$

$\mathcal{X}_6 = \{123456, 124635, 241356, 234615, 261345, 236145, 246135\}$.

**Proposition 12.** For $\sigma \in \mathcal{X}_{2n,k}$, we have $(31-2)\sigma \geq k$ and $(2-31)\sigma \geq k$. 
Proof. As $\sigma \in \hat{X}_{2n,k}$, according to the definition of normalized $E$-permutations,
(a) if $2i - 1 \in S$ and $s(2i - 1)$ is odd, then $s(2i) = s(2i - 1) + 1$, so by Lemma 2, we have $r(2i)$ is odd.
(b) if $2i - 1 \in S$ and $s(2i - 1)$ is even, then $l(2i - 1)$ is even, by Lemma 2, $r(2i - 1)$ is odd.
(c) if $2i \in S$ and $s(2i)$ is odd then $r(2i)$ is even, and $s(2i - 1) = s(2i) + 1$ is even, by Lemma 2, $l(2i)$ is odd.
(d) if $2i \in S$ and $s(2i)$ is even, then $\gamma(2i)$ is even, also by Lemma 2, we have $l(2i)$ is odd.

So, for a descent bottom $2i - 1$ of $\sigma$, either $2i - 1$ or $2i$ contributes at least one $(2-31)$ pattern. And for a descent top $2j$ of $\sigma$, either $2j$ or $2j - 1$ contributes at least one $(31-2)$ pattern. Therefore, $(31-2) \sigma \geq k$ and $(2-31) \sigma \geq k$. □

Theorem 13. We have the $(p, q)$-analogue of (1.2)

$$X_n(p, q, t) := \sum_{\sigma \in \hat{X}_{2n}} p^{(2-31)}q^{(31-2)}t^{\operatorname{des} \sigma} = \sum_{k=0}^{\lfloor n/2 \rfloor} \gamma_{n,k}(p, q)t^k(1 + t)^{n - 2k},$$

(2.13a)

with coefficients

$$\gamma_{n,k}(p, q) = \sum_{\sigma \in \hat{X}_{2n,k}} p^{(2-31)}q^{(31-2)}.$$ 

(2.13b)

Moreover, the $\gamma$-coefficients $\gamma_{n,k}(p, q)$ have the factorization

$$\gamma_{n,k}(p, q) = (p + q)^{2k}\sum_{\sigma \in \hat{X}_{2n,k}} q^{(31-2)\sigma_k}p^{(2-31)\sigma_k}.$$ 

(2.13c)

Clearly Theorem 10 implies (2.13a). We shall prove Theorem 13 (2.13c) in Section 6. From Theorem 13 and Corollary 8 we derive the following J-fraction.

Corollary 14. We have

$$\sum_{n \geq 0} \sum_{\sigma \in \hat{X}_{2n}} p^{(2-31)}q^{(31-2)}t^{\operatorname{des} \sigma} x^n = \frac{1}{1 - [1]_{p,q}^2x - \frac{(3)^2_{p,q}t \cdot x^2}{1 - [2]_{p,q}^2x - \frac{(3)^2_{p,q}t \cdot x^2}{1 - [3]_{p,q}^2x - \ldots}}}.$$ 

(2.14)

Note that (2.14) reduces to (1.10) when $p = q = t = 1$. Thus $|\hat{X}_{2n}| = h_n$ for $n \geq 1$. 


3. Even-odd permutations and labelled Motzkin paths

A Motzkin path of length \( n \) is a sequence of points \( \omega := (\omega_0, \ldots, \omega_n) \) in the integer plane \( \mathbb{Z} \times \mathbb{Z} \) such that

- \( \omega_0 = (0, 0) \) and \( \omega_n = (n, 0) \),
- \( \omega_i - \omega_{i-1} \in \{(1, 0), (1, 1), (1, -1)\} \),
- \( \omega_i := (x_i, y_i) \in \mathbb{N} \times \mathbb{N} \) for \( i = 0, \ldots, n \).

Even-odd permutations and labelled Motzkin paths

In other words, a Motzkin path of length \( n \geq 0 \) is a lattice path in the right quadrant \( \mathbb{N} \times \mathbb{N} \) starting at \( (0, 0) \) and ending at \( (n, 0) \), each step \( s_i = \omega_i - \omega_{i-1} \) is a rise \( U = (1, 1) \), fall \( D = (1, -1) \) or level \( L = (1, 0) \). A 2-Motzkin path is a Motzkin path with two types of level-steps \( L_1 \) and \( L_2 \). Let \( \mathcal{MP}_n \) be the set of 2-Motzkin paths of length \( n \). Clearly we can identify 2-Motzkin paths of length \( n \) with words \( w \) on \( \{U, L_1, L_2, D\} \) of length \( n \) such that all prefixes of \( w \) contain no more \( D \)'s than \( U \)'s and the number of \( D \)'s equals the number of \( U \)'s. The height of \( \omega_i \) is the ordinate of \( \omega_i \) and denoted by \( h(\omega_i) \), which is also called the height of the step \( s_i \).

Let \( \mathbf{a} = (a_i)_{i \geq 0}, \mathbf{b} = (b_i)_{i \geq 0}, \mathbf{b}' = (b'_i)_{i \geq 0} \) and \( \mathbf{c} = (c_i)_{i \geq 1} \) be four sequences of indeterminates; we will work in the ring \( \mathbb{Z}[\mathbf{a}, \mathbf{b}, \mathbf{b}', \mathbf{c}] \). To each Motzkin path \( \omega \) we assign a weight \( W(\omega) \) that is the product of the weights for the individual steps, where a up-step (resp. down-step) at height \( i \) gets weight \( a_i \) (resp. \( \lambda_i \)), and a level-step of type 1 (resp. 2) at height \( i \) gets weight \( b_i \) (resp. \( b'_i \)). The following result of Flajolet [16] is folklore.

Lemma 3 (Flajolet). We have

\[
\sum_{n=0}^{\infty} \left( \sum_{\omega \in \mathcal{MP}_n} W(\omega) \right) x^n = \frac{1}{1 - (b_0 + b'_0)x - \frac{a_0c_1x^2}{1 - (b_1 + b'_1)x - \frac{a_1c_2x^2}{1 - (b_2 + b'_2)x - \cdots}}}
\]

Definition 15. A path diagram of length \( n \) is a triplet \((\omega, (\xi, \xi'))\), where \( \omega \) is a Motzkin path of length \( n \), \( \xi = (\xi_1, \ldots, \xi_n) \) and \( \xi' = (\xi'_1, \ldots, \xi'_n) \) are two integer sequences satisfying the following conditions:

- If the \( k \)-th step of \( \omega \) is a rise with height \( h \geq 0 \), then
  \( (\xi_i, \xi'_i) \in \{0, \ldots, h\} \times \{0, \ldots, h + 1\} \).

- If the \( k \)-th step of \( \omega \) is a fall with height \( h > 0 \), then
  \( (\xi_i, \xi'_i) \in \{0, \ldots, h\} \times \{0, \ldots, h - 1\} \).

- If the \( k \)-th step of \( \omega \) is a level of type 1 or 2 with height \( h \geq 0 \), then
  \( (\xi_i, \xi'_i) \in \{0, \ldots, h\} \times \{0, \ldots, h\} \).

We denote by \( \mathcal{PD}_n \) the set of path diagrams of length \( n \).
For \( \sigma \in \mathcal{X}_{2n} \), we construct the path diagram \( \Phi(\sigma) = (\omega, (\xi, \xi')) \) in \( \mathcal{PD}_n \) as in the following. Let \( \overline{S}_\sigma = \{2n\} \setminus S_\sigma \). For \( j \in \{1, \ldots, n\} \) we define the \( j \)-th step \( s_j = \omega_j - \omega_{j-1} \) of the path \( \omega = (\omega_0, \omega_1, \ldots, \omega_n) \) by

\[
\begin{align*}
s_j = \begin{cases}
U & \text{if } (2j - 1, 2j) \in S_\sigma \times \overline{S}_\sigma \\
D & \text{if } (2j - 1, 2j) \in \overline{S}_\sigma \times S_\sigma \\
L_1 & \text{if } (2j - 1, 2j) \in S_\sigma \times \overline{S}_\sigma \\
L_2 & \text{if } (2j - 1, 2j) \in \overline{S}_\sigma \times S_\sigma
\end{cases}
\end{align*}
\tag{3.2a}
\]

and the bi-sequence \( (\xi, \xi') \) by

\[
(\xi_j, \xi'_j) = (r_\sigma(2j - 1), r_\sigma(2j)).
\tag{3.2b}
\]

The restriction of \( \sigma \) on \( [2j] \) is the word \( \pi_j \) obtained from \( \sigma \) by replacing each subword of \( \sigma \) consisting of consecutive letters greater than \( 2j \) by a slot \( \omega \). It is easier to describe the above construction by looking at the successive restrictions of \( \sigma \) on \( \{1, \ldots, 2j\} \) for \( j \in [n] \). For example, if \( \sigma = 2 \ 6 \ 8 \ 1 \ 4 \ 7 \ 14 \ 9 \ 10 \ 12 \ 3 \ 5 \ 11 \ 15 \ 16 \ 13 \in \mathcal{X}_{16} \), then \( S = \{1, 3, 9, 13\} \cup \{8, 12, 14, 16\} \). Hence \( \pi_0 = \omega \), the successive restrictions of \( \sigma \) read as follows:

| \( i \) | \( \pi_j \) | \( (2j - 1, 2j) \) | \( (\xi_j, \xi'_j) \) | \( (s_\sigma(2j - 1), s_\sigma(2j)) \) |
|---|---|---|---|---|
| 1 | 2 1 1 | (1, 2) \( \in S_\sigma \times \overline{S}_\sigma \) | (0,1) | (1,2) |
| 2 | 2 1 4 1 3 | (3, 4) \( \in S_\sigma \times \overline{S}_\sigma \) | (0,1) | (2,3) |
| 3 | 2 6 1 4 3 5 | (5, 6) \( \in \overline{S}_\sigma \times S_\sigma \) | (0,2) | (3,3) |
| 4 | 2 6 8 1 4 7 3 5 | (7, 8) \( \in \overline{S}_\sigma \times S_\sigma \) | (1,1) | (3,2) |
| 5 | 2 6 8 1 4 7 9 10 3 5 | (9, 10) \( \in S_\sigma \times \overline{S}_\sigma \) | (1,1) | (2,3) |
| 6 | 2 6 8 1 4 7 9 10 12 3 5 11 | (11, 12) \( \in \overline{S}_\sigma \times S_\sigma \) | (0,0) | (3,2) |
| 7 | 2 6 8 1 4 7 14 9 10 12 3 5 11 | (13, 14) \( \in S_\sigma \times S_\sigma \) | (0,1) | (2,2) |
| 8 | 2 6 8 1 4 7 14 9 10 12 3 5 11 15 16 13 | (15, 16) \( \in \overline{S}_\sigma \times S_\sigma \) | (1,0) | (2,1) |

**Figure 3.** The restrictions \( \pi_j \) of \( \sigma \in \mathcal{X}_{16} \) for \( j \in [8] \).
Hence the corresponding step sequence is \((U, U, L_1, D, U, D, L_2, D)\) with height sequence \((0, 1, 2, 2, 1, 2, 1, 1)\). The corresponding path diagram \((\omega, (\xi, \xi'))\) is illustrated in Figure 2.

**Lemma 4.** The mapping \(\Phi : \sigma \mapsto (\omega, (\xi, \xi'))\) is a bijection from \(X_{2n}\) to \(PD_n\) such that

\[
\begin{align*}
\text{dom } \sigma &= |L_2|_\omega, \\
\text{des } \sigma &= |U|_\omega + |L_2|_\omega, \\
(2-31) \sigma &= \sum_{i=1}^{n} (\xi_i + \xi'_i), \\
(31-2) \sigma &= |U|_\omega + |D|_\omega + \sum_{i=1}^{n} (h(\omega_i) - \xi_i + h(\omega_i) - \xi'_i), \\
(h(\omega_{j-1}), h(\omega_j)) &= (s_\sigma(2j-1) - 1, s_\sigma(2j) - 1) \quad j \in [n],
\end{align*}
\]

where \(|A|_\omega\) is the number of steps of type \(A\) in the path \(\omega\).

**Proof.** First of all, Lemma 2 ensures that \((\omega, (\xi, \xi'))\) is a path diagram in \(PD_n\). The trivial verification of (3.3a)-(3.3e) is omitted. Note that

\[
0 \leq \xi_j \leq s_\sigma(2j - 1) - 1, \quad 0 \leq \xi'_j \leq s_\sigma(2j) - 1
\]

To show that \(\Phi\) is a bijection, we construct the inverse mapping \(\Phi^{-1}\). Starting from a path diagram \((\omega, (\xi, \xi'))\) \(\in PD_n\), by \(\omega\) and (3.2a), we determine the weak signature \(S_\sigma\). For \(j \in [n]\) we construct the permutation \(\sigma\) step by step by inserting \(2j-1\) and \(2j\) in the word \(\pi_{j-1}\) with \(\pi_0 = \underline{\omega}\) as follows:

1. if the \(j\)-th step is *rise*, we replace the \(\xi_j + 1\)-th slot from right to left in \(\pi_{j-1}\) by \(\underline{(2j-1)}\) and then replace the \(\xi'_j + 1\)-th slot by \(2j\).
2. if the \(j\)-th step is *fall*, we replace the \(\xi_j + 1\)-slot in \(\pi_{j-1}\) by \((2j-1)\underline{\omega}\) and then the \(\xi'_j + 2\)-th slot in by \((2j)\);
3. if the \(j\)-th step is *level of type 1*, we replace the \(\xi_j + 1\)-th slot in \(\pi_{j-1}\) by \((2j-1)\underline{\omega}\) and then replace the \(\xi'_j + 1\)-slot by \(2j\underline{\omega}\);
4. if the \(j\)-th step is *level of type 2*, we replace the \(\xi_j + 1\)-th slot in \(\pi_{j-1}\) by \(\underline{(2j-1)}\) and then replace the \(\xi'_j + 2\)-th slot by \(2j\).

The permutation \(\sigma\) is obtained by deleting the final slot \(\underline{\omega}\) in \(\pi_{n}\). \(\square\)
Lemma 5. For $\sigma \in \mathcal{X}_{2n}$ let $\Phi(\sigma) = (\omega, (\xi, \xi'))$ be the path diagram in $\mathcal{PD}_n$. Then

$$\text{lema } \sigma = \sum_{i=1}^{n} \chi(\xi_i = h(w_i)), \quad (3.4a)$$

$$\text{loma } \sigma = \sum_{i=1}^{n} \chi(\xi_i = h(w_i))(s_i \in \{D, L_1\}), \quad (3.4b)$$

$$\text{remi } \sigma = \sum_{i=1}^{n} \chi(\xi_i = 0)(s_i \in \{U, L_1\}), \quad (3.4c)$$

$$\text{romi } \sigma = \sum_{i=1}^{n} \chi(\xi_i = 0)(s_i \in \{U, D, L_1\}), \quad (3.4d)$$

Proof. The trivial verification is omitted. $\square$

Proof of Theorem 4. For $\sigma \in \mathcal{X}_{2n}$, if $\Phi(\sigma) = (\omega, (\xi, \xi')) \in \mathcal{PD}_n$, we derive from the above lemma

$$a_{\text{lema } \sigma} a_{\text{loma } \sigma} a_{\text{romi } \sigma} a_{\text{remi } \sigma} q^{(2-31)} s q^{(31-2)} s y_{\text{dom } \sigma} y_{\text{des } \sigma} = \prod_{i=1}^{n} w(s_i, (\xi_i, \xi_i')) \quad (3.5)$$

with $h$ being the height of the step $s_i$,

$$w(s_i, (\xi_i, \xi_i')) = \begin{cases} a^{\chi(\xi_i = h)} b^{\chi(\xi_i = 0)} p_{\xi_i} q^{h-\xi_i} \cdot p_{\xi_i} q^{h+1-\xi_i} \cdot t & \text{if } s_i = U; \\ a^{\chi(\xi_i = h)} b^{\chi(\xi_i = 0)} p_{\xi_i} q^{h-\xi_i} \cdot p_{\xi_i} q^{h+1-\xi_i} \cdot t & \text{if } s_i = D; \\ a^{\chi(\xi_i = h)} b^{\chi(\xi_i = 0)} p_{\xi_i} q^{h-\xi_i} \cdot p_{\xi_i} q^{h+1-\xi_i} \cdot t & \text{if } s_i = L_1; \\ a^{\chi(\xi_i = h)} b^{\chi(\xi_i = 0)} p_{\xi_i} q^{h-\xi_i} \cdot p_{\xi_i} q^{h+1-\xi_i} \cdot t & \text{if } s_i = L_2. \end{cases} (3.6)$$

Therefore, the corresponding polynomial and weights become

$$X_n(a, \bar{a}, b, \bar{b}, p, q, y, t) = \sum_{\sigma \in \mathcal{X}_{2n}} a_{\text{lema } \sigma} a_{\text{loma } \sigma} a_{\text{romi } \sigma} a_{\text{remi } \sigma} q^{(2-31)} s q^{(31-2)} s y_{\text{dom } \sigma} y_{\text{des } \sigma}$$

$$= \sum_{\omega \in \mathcal{MP}_{n}} \sum_{(\xi_i, \xi_i')} w(s_i, (\xi_i, \xi_i'))$$

with

$$\sum_{(\xi_i, \xi_i')} w(s_i, (\xi_i, \xi_i')) = \begin{cases} [b, h + 1]_{q, p} \cdot [a, h + 2, \bar{b}]_{p, q} \cdot t & \text{if } s_i = U; \\ [\bar{a}, h + 2, b]_{p, q} \cdot [a, h + 1]_{p, q} & \text{if } s_i = D; \\ [\bar{a}, h + 1, b]_{p, q} \cdot [a, h + 1, \bar{b}]_{p, q} & \text{if } s_i = L_1; \\ [a, h + 1]_{p, q} \cdot [b, h + 1]_{p, q} t y & \text{if } s_i = L_2. \end{cases} (3.7)$$

with $[x, 1, y]_{p, q} = xy$ and

$$[x, n, y]_{p, q} = x p^{n-1} + \sum_{i=1}^{n-2} p^{n-1-i} q^i + y q^{n-1}.$$
By Lemma 3 we derive the corresponding continued fraction for the generating function of \(X_n(a, \bar{a}, b, \bar{b}, p, q, y, t)\).

### 4. Continued fractions for cycles in D-permutations and E-permutations

A **surjective pistol** on \([2n]\) is the graph of a surjective mapping \(f : [2n] \to 2[2n]\) such that

\[f(i) \geq i \quad \text{for} \quad i \in [2n].\]

Let \(\mathcal{P}_{2n}\) be the set of surjective pistols on \([2n]\). A surjective pistol is depicted in Figure 4. A point \((k, f(k))\) of a pistol \(f \in \mathcal{P}_{2n}\) is

- maximum if \(f(k) = 2n\) and \(k \leq 2n - 2\),
- fixed if \(f(k) = k\) and \(k < 2n\);
- surfixed if \(f(k) = k + 1 < 2n\);
- doubled if \(\exists j \neq k\) such that \(f(j) = f(k)\).

A maximum \((k, f(k))\) is even (resp. odd) if \(k\) is even (resp. odd). The number of even (resp. odd) maxima of \(f\) is denoted by \(\text{me}(f)\) (resp. \(\text{mo}(f)\)). The number of doubled (resp. isolated) fixed points is denoted by \(\text{fd}(f)\) (resp. \(\text{fi}(f)\)). The number of doubled (resp. isolated) surfixed points is denoted by \(\text{sd}(f)\) (resp. \(\text{si}(f)\)). The six statistics of the pistol \(f\) in Figure 4 are:

\[\text{mo}(f) = 1, \quad \text{me}(f) = 2, \quad \text{fd}(f) = 2, \quad \text{fi}(f) = 1, \quad \text{sd}(f) = 2, \quad \text{si}(f) = 1.\]

Let

\[\Gamma_n(\alpha, \beta, \gamma, \bar{\alpha}, \bar{\gamma}, y, \bar{y}) = \sum_{f \in \mathcal{P}_{2n}} \alpha^{\text{mo}(f)} \beta^{\text{fd}(f)} \gamma^{\text{si}(f)} \bar{\alpha}^{\text{me}(f)} \bar{\beta}^{\text{fi}(f)} \bar{\gamma}^{\text{sd}(f)}.\]

Dumont [12] conjectured, and Randrianarivony [26] and Zeng [31] proved the following result.
Lemma 6 (Randrianarivony-Zeng). We have
\[
\sum_{n \geq 0} \Gamma_{n+1}(\alpha, \bar{\alpha}) x^n = \frac{1}{1 - (\alpha \bar{\beta} + \beta \bar{\gamma} + \gamma \bar{\alpha}) x - \frac{(\alpha + \beta)(\beta + z)(\gamma + \alpha)x^2}{1 - (\alpha \bar{\beta} + \beta \bar{\gamma} + \gamma \bar{\alpha}) x - \frac{(\alpha + \beta)(\beta + z)(\gamma + \alpha)x^2}{\cdots}}}
\] (4.1)
where the coefficients under the \((n+1)\)-th row of fraction is
\[
1 - [(\alpha + n)(\bar{\beta} + n) + (\beta + n)(\bar{\gamma} + n) + (\gamma + n)(\bar{\alpha} + n) - n(n + 1)] x \\
- \frac{(n + 1)(\bar{\alpha} + \beta + n)(\bar{\beta} + \gamma + n)(\bar{\gamma} + \alpha + n)x^2}{1 - (\alpha \bar{\beta} + \beta \bar{\gamma} + \gamma \bar{\alpha}) x - \frac{(\alpha + \beta)(\beta + z)(\gamma + \alpha)x^2}{\cdots}}}
\] (4.2)

Lazar-Wachs [23, Lemma 5.2] proved the following result.

Lemma 7 (Lazar-Wachs). There is a bijection
\[\phi : \mathcal{D}_{2n} \mapsto \{ f \in \mathcal{P}_{2n+2} : f \text{ has no even maximal}\}\]
such that for all \(\sigma \in \mathcal{D}_{2n}\) and \(j \in [2n]\), the following properties hold:
(1) \(j\) is an even cycle maximum of \(\sigma\) iff it is a fixed point of \(\phi(\sigma)\),
(2) \(j\) is an even fixed point of \(\sigma\) iff it is an isolated fixed point of \(\phi(\sigma)\),
(3) \(j\) is an odd fixed point of \(\sigma\) iff it is an odd maximum of \(\phi(\sigma)\).

Proof of Theorem 9. By Lemma 7 we obtain immediately
\[
\sum_{\sigma \in \mathcal{D}_{2n}} x_0^{\text{fix}_e(\sigma)} x_1^{\text{fix}_o(\sigma)} z^{\text{cyc}(\sigma)} = \Gamma_{n+1}(x_1 z, z, 1, 0, x_0 z, 1) \quad \text{for } n \geq 1,
\]
where \(\text{fix}_e(\sigma)\) (resp. \(\text{fix}_o(\sigma)\)) is the number of even (resp. odd) fixed points of \(\sigma\).

It follows from Lemma 6 that
\[
1 + \sum_{n \geq 1} \sum_{\sigma \in \mathcal{D}_{2n}} x_0^{\text{fix}_e(\sigma)} x_1^{\text{fix}_o(\sigma)} z^{\text{cyc}(\sigma)} x^n = \frac{1}{1 - (x_0 x_1 z^2 + z)x - \frac{z(x_0 z + 1)(x_1 z + 1)x^2}{1 - (x_0 x_1 z^2 + z)x - \frac{z(x_0 z + 1)(x_1 z + 1)x^2}{\cdots}}}
\] (4.3a)
where the coefficients under the \((n+1)\)-th row of fraction is
\[
1 - [(x_1 z + n)(x_0 z + n) + (z + n)(1 + n)] x \\
- \frac{(n + 1)(z + n)(x_0 z + 1 + n)(x_1 z + 1 + n)x^2}{1 - (x_0 x_1 z^2 + z)x - \frac{z(x_0 z + 1)(x_1 z + 1)x^2}{\cdots}}}
\] (4.3b)

Comparing Corollary 7 with \(t = 1\) and (4.3a) with \(x_0 = x_1 = 1\) we see that \(\sum_{\sigma \in \mathcal{D}_{2n}} z^{\text{cyc}(\sigma)} = \sum_{\sigma \in \mathcal{D}_{2n}} z^{\text{cyc}(\sigma)}\). \qed
Note that setting \(x_0 = x_1 = 0\) in (4.3a) yields the S-fraction

\[
1 + \sum_{n \geq 1} \sum_{\sigma \in \mathcal{D}_{2n}^*} z^{\text{cyc}(\sigma)} t^n = \frac{1}{1 - \frac{1 \cdot z t}{1 - \frac{1^2 t}{1 - \frac{2 \cdot (z + 1) t}{1 - \frac{2^2 t}{1 - \ldots}}}}
\]  

(4.4)

where \(D_{2n}^*\) is the set of derangements in \(\mathcal{D}_{2n}\).

5. Gamma-decomposition and group actions

We give another proof of (2.9a) using Inter-hopping action on \(X_{2n}\), see [8, 15]. For \(\sigma \in X_{2n}\) and \(j \in [n]\), a doubleton \(\{2j - 1, 2j\}\) is called free (in \(\sigma\)) if both \(2j - 1\) and \(2j\) are in \(S_\sigma\), or neither \(2j - 1\) nor \(2j\) is in \(S_\sigma\), namely \(\{2j - 1, 2j\} \in (S_\sigma \times S_\sigma) \cup (\overline{S_\sigma} \times \overline{S_\sigma})\).

Let \(w = w_1 \cdots w_{2n} \in X_{2n}\) with a free pair \(\{w_i, w_j\} = \{2r - 1, 2r\}\) for some \(i < j\). We define the action \(\phi_r\) on \(w\) such that \(|\text{des } w - \text{des } \phi_r(w)| = 1\).

**Inter-hopping action \(\phi_r\)**

(A1) \((2j - 1, 2j) \in \overline{S_\sigma} \times S_\sigma\). If the elements \(2r - 1\) and \(2r\) are adjacent in \(w\) then \(\phi_r(w)\) is obtained by switching \(2r - 1\) and \(2r\). Otherwise, we factorize \(w\) as

\[w = \cdots \beta_0 w_{i_1} \alpha_1 \beta_1 \alpha_2 \beta_2 \cdots \alpha_d \beta_d w_j \alpha_{d+1} \cdots,\]

where \(\alpha_j\) (resp. \(\beta_j\)) is a maximal sequence of consecutive entries greater than \(2r\) (resp. less than \(2r - 1\)). Note that neither \(\alpha_j\) nor \(\beta_j\) is empty for \(1 \leq j \leq d\), but \(\beta_0\) and \(\alpha_{d+1}\) are possibly empty.

- If \(w_i = 2r\) and \(w_j = 2r - 1\), then
  \[\phi_r(w) := \cdots \beta_0 \alpha_1 (2r - 1) \alpha_2 \beta_2 \cdots \alpha_d \beta_d (2r) \beta_d \alpha_{d+1} \cdots.\]

- If \(w_i = 2r - 1\) and \(w_j = 2r\), then
  \[\phi_r(w) := \cdots \beta_0 (2r) \beta_1 \alpha_1 \beta_2 \alpha_2 \cdots \beta_d \alpha_d (2r - 1) \alpha_{d+1} \cdots.\]

(A2) \((2j - 1, 2j) \in S_\sigma \times S_\sigma\). Then \(\phi_r(w)\) is obtained by reversing the process of (A1).

For \(\sigma \in X_{2n}\), let \(\text{Orb}(\sigma) = \{g(\sigma) : g \in \mathbb{Z}_2^n\}\) be the orbit of \(\sigma\) under the Inter-hopping action \(\phi\). Let \(\bar{\sigma}\) be the unique \(\mathcal{E}\)-permutation in \(\text{Orb}(\sigma)\). We have the following lemma.

**Lemma 8.** For any \(\bar{\sigma} \in X_{2n}\), we have

\[
\sum_{\sigma \in \text{Orb}(\bar{\sigma})} p^{(31-2) \sigma} q^{(23-2) \sigma} t^{\text{des } \sigma} y^{\text{dom}(\sigma)} = p^{(31-2) \bar{\sigma}} q^{(23-2) \bar{\sigma}} t^{\text{des } \bar{\sigma}} (1 + yt)^{n - 2 \text{des } \bar{\sigma}}.
\]

(5.1)

**Proof.** For \(w \in X_{2n}\) with a free pair \(\{2r - 1, 2r\}\), if \(2r - 1\) and \(2r\) are adjacent in \(w\), then obviously,

\[(23-2) \phi_r(w) = (23-1) w, \quad (31-2) \phi_r(w) = (31-2) w.\]
Otherwise, we factorize $w$ as

$$w = \cdots \beta_0 w_i \alpha_1 \beta_1 \alpha_2 \beta_2 \cdots \alpha_d \beta_d w_j \alpha_{d+1} \cdots.$$ 

We assume that $w_i = 2r$ and $w_j = 2r - 1$ with $i < j$ (the $i > j$ case is similarly). Then

$$\phi_r(w) := \cdots \beta_0 \alpha_1 (2r - 1) \alpha_2 \beta_1 \alpha_3 \beta_2 \cdots \alpha_d \beta_{d-1} (2r) \beta_d \alpha_{d+1} \cdots.$$ 

We see that the Inter-hopping action $\phi_r$ does not change the relative orders of $\alpha_i$ and $\beta_j$ respectively, so

(i) if there is a descent pair in $\alpha_i$ ($\beta_j$, respectively), then this descent pair can only form (2-31) or (31-2) patterns with a letter in some $\alpha_k$ ($\beta_l$, respectively);

(ii) in $w$, for $a \in \alpha_i$, $a$ forms a (2-31) pattern with some descent pair say (last($\alpha_j$), first($\beta_j$)), if and only if in $\phi_r(w)$, $a$ forms a (2-31) pattern either with descent (last($\alpha_j$), first($\beta_{j-1}$)) or (last($\alpha_j$), first($\beta_j$)).

For $b \in \beta_i$, $b$ formed a (2-31) pattern with some descent pairs (last($\alpha_j$), first($\beta_j$)) if and only if $b$ forms a (2-31) pattern either with descent pair (last($\alpha_{j-1}$), first($\beta_j$)) or with descent pair (last($\alpha_j$), first($\beta_{j-1}$));

(iii) the number of (2-31) patterns formed by $2r - 1$ and $2r$ with some descent pairs in $w$ is equal to the number of (2-31) patterns formed by $2r - 1$ and $2r$ with some descent pairs in $\phi_r(w)$.

Combining (i), (ii) and (iii), we obtain (2-31) $w = (2-31) \phi_r(w)$. Similarly, we have (31-2) $w = (31-2) \phi_r(w)$. Setting each free pair of $w$ not in $S_w$ yields the unique $\mathcal{E}$-permutation, say $\tilde{\sigma}$, in $\text{Orb}(\sigma)$. Moreover, if $(2j, 2j - 2k - 1)$ is a descent pair, then the doubletons $\{2j, 2j - 1\}$ and $\{2j - 2k, 2j - 2k - 1\}$ are not free. So, the number of free doubletons in $\tilde{\sigma}$ is $n - 2 \text{des}(\tilde{\sigma})$, which implies the identity (5.1). \hfill \Box

For convenience, we call an element changed its type, if it changed from an even left-to-right maxima (resp. odd right-to-left minima) to a non even left-to-right maxima (resp. a non odd right-to-left minima), or from a non even left-to-right maxima to an even left-to-right maxima (resp. an odd right-to-left minima).

**Involution $\theta_r$**

For $\sigma \in X_{2n}$, and $\{2r - 1, 2r\}$ is free with $(2r - 1, 2r) \in \overline{S}_\sigma \times \overline{S}_\sigma$. Then $\theta_r(\sigma) = \tau$, where $\tau \in X_{2n}$ is obtained by exchanging $2r - 1$ and $2r$ in $\sigma$.

**Lemma 9.** For $w \in X_{2n}$, if $\{2r - 1, 2r\}$ is free with $(2r - 1, 2r) \in \overline{S}_\sigma \times \overline{S}_\sigma$, then

$$\text{lema} \theta_r(w) = \text{lema} \phi_r(w) \quad \text{(5.2)}$$

$$\text{romi} \theta_r(w) = \text{romi} \phi_r(w), \quad \text{(5.3)}$$

and

$$\text{lema}(\phi_r \circ \theta_r)(w) = \text{lema} w \quad \text{(5.4)}$$

$$\text{romi}(\phi_r \circ \theta_r)(w) = \text{romi} w. \quad \text{(5.5)}$$
If \( \{2r - 1, 2r\} \) is a free pair with \( (2r - 1, 2r) \in S_\sigma \times S_\sigma \), then

\[
\text{lema} (\theta_r \circ \phi_r)(w) = \text{lema} w
\]

\[
\text{romi} (\theta_r \circ \phi_r)(w) = \text{romi} w.
\]

**Proof.** We take the factorization of \( w \) as in (A1), i.e.,

\[
w = \cdots \beta_0 w_1 \alpha_1 \beta_1 \alpha_2 \beta_2 \cdots \alpha_d \beta_d w_j \alpha_{d+1} \cdots,
\]

(5.8)

here we assume \( w_i = 2r - 1 \), \( w_j = 2r \). Then,

\[
\phi_r(w) = \cdots \beta_0(2r) \beta_1 \alpha_1 \beta_1 \alpha_2 \beta_2 \cdots \beta_d \alpha_d (2r - 1) \alpha_{d+1} \cdots
\]

(5.9)

\[
\theta_r(w) = \cdots \beta_0(2r) \alpha_1 \beta_1 \alpha_2 \beta_2 \cdots \alpha_d \beta_d (2r - 1) \alpha_{d+1} \cdots.
\]

(5.10)

Since all the letters in \( \beta_i \) are smaller than \( 2r - 1 \) and all the letters in \( \alpha_j \) are larger than \( 2r \).

So, the even left-to-right maxima element, if exists, it must be in one of \( \alpha_i \) or the element before \( \beta_0 \), or it is \( 2r \). And the odd right-to-left minima element, if exists, it must be in one of \( \beta_j \) or the element after \( \alpha_{d+1} \) or it is \( 2r - 1 \). Compare (5.9) with (5.8) and (5.10) with (5.8), we can see that under the actions \( \theta_r \) and \( \phi_r \), if some elements changed their type, they must in \{2r - 1, 2r\}. And compare (5.9) and (5.10), we have (5.2) and (5.3).

By the same reason, we can derive (5.4), (5.5), (5.6) and (5.7).

Note that for \( \sigma \in \mathcal{X}_{2n} \), neither \( \theta_r \) and \( \phi_r \) changes the number of (31-2) or (2-31) patterns when them action on \( \sigma \). Then combine Lemma 8 and Lemma 9, we can finally derive Theorem 10.

6. FACTORIZATION OF \( \gamma \)-COEFFICIENTS AND GROUP ACTIONS

We prove Theorem 13. Clearly it suffices to prove

\[
\sum_{\sigma \in \mathcal{X}_{2n,k}} p^{(2-31) \sigma} q^{(31-2) \sigma} = (p + q)^{2k} \sum_{\sigma \in \mathcal{X}_{2n,k}} q^{(31-2) \sigma - k} p^{(2-31) \sigma - k}
\]

(6.1)

with \( \mathcal{X}_{2n,k} = \{ \sigma \in \mathcal{X}_{2n} : \text{des} \, \sigma = k \} \) and \( \hat{\mathcal{X}}_{2n,k} = \{ \sigma \in \hat{\mathcal{X}}_{2n} : \text{des} \, \sigma = k \} \). We define an action \( \varphi_x \) on \( \hat{\mathcal{X}}_{2n} \) as the following.

For \( \sigma \in \mathcal{X}_{2n} \), and \( x \in S_\sigma \), we define \( \varphi_x(\sigma) \) as follows:

(A) If \( x = 2i - 1 \) and \( s_\sigma(2i - 1) \) is odd, we factorize \( \sigma = \tau_3' \tau_2' \tau_1' (2i) \tau_2 \tau_3 \), where \( \tau_1 \) and \( \tau_2' \) is the maximal sequence of consecutive entries greater than \( 2i \), \( \tau_2 \) and \( \tau_1' \) is the maximal sequence of consecutive entries smaller than \( 2i \). We define

\[
\varphi_x(\sigma) = \begin{cases} 
\tau_3' \tau_2' \tau_1' \tau_1 \tau_2 (2i) \tau_3 & \text{if } l_\sigma(2i) \text{ is even;} \\
\tau_3' (2i) \tau_2' \tau_1' \tau_1 \tau_2 \tau_3 & \text{if } l_\sigma(2i) \text{ is odd.}
\end{cases}
\]

(B) If \( x = 2j \), and \( s_\sigma(2j) \) is odd, we factorize \( \sigma = \tau_3' \tau_2' \tau_1' (2j - 1) \tau_1 \tau_2 \tau_3 \), where \( \tau_1 \) and \( \tau_2' \) is the maximal sequence of consecutive entries greater than \( 2j - 1 \), \( \tau_2 \) and \( \tau_1' \) is
the maximal sequence of consecutive entries smaller than $2j - 1$. Define

$$
\varphi_x(\sigma) = \begin{cases} 
\tau'_3(2j - 1)\tau'_2\tau'_1\tau_2\tau_3 & \text{if } r_\sigma(2j - 1) \text{ is even;} \\
\tau'_3\tau'_2\tau'_1\tau_2(2j - 1)\tau_3 & \text{if } r_\sigma(2j - 1) \text{ is odd.}
\end{cases}
$$

(C) If $x = 2i - 1$ and $s_\sigma(2i - 1)$ is even,

(i) if $l_\sigma(2i - 1) = 2a$ for some $a \in \mathbb{N}$, then

$$
\varphi_x(\sigma) = w \in \overline{X}_{2n},
$$

where $w$ is constructed as follows.

Let $w_0$ be the subsequence of $\sigma$ consisting of those elements which are smaller than $2i - 1$ in $S_\sigma$. And $w_1$ ($\sigma_1$, respectively) is the subsequence of $w$ ($\sigma$, respectively) which consists of all the elements in $S_\sigma$. We obtain $w_{01}$ by inserting $2i - 1$ in $w_0$ such that the number of elements at the left side of $2i - 1$ in $w_{01}$ minus two times of the number of even-odd descents at the left side of $2i - 1$ in $w_{01}$ is $2a + 1$. Then we insert the rest elements of $S_\sigma$ in $w_{01}$ one by one in increasing order to obtain $w$, such that the order of the rest elements in $S_\sigma$ appear in $w_1$ is the same as in $\sigma_1$. (The same order means when restrict $w_1$ and $\sigma_1$ on the elements $\{1, 2, \ldots, k\}$ for $|S_\sigma| \geq k \geq 2i$ then the elements at left side of $k$ minus two times of the number of even-odd descents are the same in the restrictions of $w_1$ and $\sigma_1$) Last, we insert the elements of $[2n] \setminus S_\sigma$ one by one in increasing order in $w_1$ to obtain $w$ such that $(l_w(i), r_w(i)) = (l_\sigma(i), r_\sigma(i))$.

(ii) if $l_\sigma(2i - 1) = 2a + 1$ for some $a \in \mathbb{N}$, then

$$
\varphi_x(\sigma) = w \in \overline{X}_{2n},
$$

where $w$ is constructed by following almost the process in (i), except when obtain $w_{01}$ by inserting $2i - 1$ in $w_0$ such that the number of the elements at the left side of $2i - 1$ in $w_{01}$ minus two times the number of even-odd descents at the left side of $2i - 1$ in $w_{01}$ is $2a$.

(D) If $x = 2j$ and $s_\sigma(2j)$ is even,

(i) if $r_\sigma(2j) = 2a$ for some $a \in \mathbb{N}$, then

$$
\varphi_x(\sigma) = w \in \overline{X}_{2n},
$$

Let $w_0$ be the subsequence of $\sigma$ which consists of the elements in $S_\sigma$ and smaller than $2j$. And $w_1$ ($\sigma_1$, respectively) be the subsequence of $w$ ($\sigma$, respectively) which consists of all the elements in $S_\sigma$. We obtain $w_{01}$ by inserting $2j$ in $w_0$ such that the number of elements at the right side of $2j$ in $w_{01}$ minus two times of the number of even-odd descents at the right side of $2j$ in $w_{01}$ is $2a + 1$. Then we insert the rest elements of $S_\sigma$ in $w_{01}$ one by one in increasing order to obtain $w_1$, such that the order of the rest elements in $S_\sigma$ appear in $w_1$ is the same as in $\sigma_1$. Last, we insert the elements of $[2n] \setminus S_\sigma$
one by one in increasing order in $w_1$ to obtain $w$ such that $(l_w(i), r_w(i)) = (l_\sigma(i), r_\sigma(i))$.

(ii) if $r_\sigma(2j) = 2a + 1$ for some $a \in \mathbb{N}$, then

$$\varphi_x(\sigma) = w \in \mathcal{X}_{2n},$$

where $w$ is constructed by following almost the process in (i), except when obtain $w_{01}$ by inserting $2j$ in $w_0$ such that the number of the elements at the right side of $2j$ in $w_{01}$ minus two times of the number of even-odd descents at the right side of $2j$ in $w_{01}$ is $2a$.

**Example 2.** For $\sigma = 56347812910 \in \mathcal{X}_{10}$, the set of descent tops and descent bottoms is $S_\sigma = \{1, 3, 6, 8\}$. So we have, $\varphi_1(\sigma) = 25634781910$, $\varphi_3(\sigma) = 56124783910$, $\varphi_6(\sigma) = 57834612910$, $\varphi_8(\sigma) = 56348127910$. We illustrate the $\varphi$ act on 1 and 8 in Figure 5.

![Figure 5. $\varphi_1(\sigma) = 25634781910$ and $\varphi_8(\sigma) = 56348127910$](image)

We extend $\varphi_x$ to all $x \in [2n]$ by $\varphi_x(\sigma) = \sigma$ for $x \notin S_\sigma$. Clearly the action $\varphi_x$ is an involution on $\mathcal{X}_{2n}$, besides two actions $\varphi_x$ and $\varphi_y$ commute for all $x, y \in [2n]$. Hence for any subset $X \subset [2n]$ we may define the action $\varphi_X$ on $\sigma \in \mathcal{X}_{2n}$ by

$$\varphi_X(\sigma) = \prod_{x \in X} \varphi_x(\sigma).$$ (6.2)

**Lemma 10.** Let $\sigma \in \mathcal{X}_{2n}$ with weak signature $S_\sigma$. For any $x \in S_\sigma$, let

$$\Delta_{\varphi_x}(\sigma) = ((2-31)\sigma', (31-2)\sigma') - ((2-31)\sigma, (31-2)\sigma)$$

with $\sigma' := \varphi_x(\sigma)$, then $S_\sigma = S_{\sigma'}$ and

(a) If $x$ and $s_\sigma(x)$ are odd, then

$$\Delta_{\varphi_x}(\sigma) = \begin{cases} (-1, 1) & \text{if } l_\sigma(x + 1) \text{ is even}, \\ (1, -1) & \text{if } l_\sigma(x + 1) \text{ is odd}; \end{cases}$$

(b) If $x$ is even but $s_\sigma(x)$ is odd, then

$$\Delta_{\varphi_x}(\sigma) = \begin{cases} (1, -1) & \text{if } l_\sigma(x - 1) \text{ is even}, \\ (-1, 1) & \text{if } l_\sigma(x - 1) \text{ is odd}; \end{cases}$$
(c) If \( x \) is odd but \( s_\sigma(x) \) is even, then
\[
\Delta_{\varphi_x}(\sigma) = \begin{cases} 
(1,1) & \text{if } l_\sigma(x) \text{ is even} \\
(-1,-1) & \text{if } l_\sigma(x) \text{ is odd};
\end{cases}
\]

(d) If \( x \) and \( s_\sigma(x) \) are even, then
\[
\Delta_{\varphi_x}(\sigma) = \begin{cases} 
(1,1) & \text{if } r_\sigma(x) \text{ is even} \\
(-1,-1) & \text{if } r_\sigma(x) \text{ is odd}.
\end{cases}
\]

**Proof.** For case (a), as the action \( \varphi_x \) moves \( 2i \) to the slot between two adjacent descent pairs, so the number of (2-31) patterns reduces by 1 and the number of (31-2) patterns increases by 1 when \( 2i \) moves to the slot between a descent pair at right. And the number of (2-31) patterns increase by 1 and the number of (31-2) patterns decreases by 1 when \( 2i \) moves to the slot between a descent pair at left. Case (b) is similar to case (a).

For case (c), it corresponds to (C) of the action \( \varphi_x \). In this case, it is easy to see that
\begin{itemize}
  \item if \( l_\sigma(x) \) is even, the construction of \( \sigma' = \varphi_x(\sigma) \) decreases the right embracing number of \( x \) by 1 and increases the left embracing number of \( x \) by 1;
  \item if \( l_\sigma(x) \) is odd, it increases the right embracing number of \( x \) by 1 and decreases the left embracing number of \( x \) by 1.
\end{itemize}

Case (d) is similar to case (c).

**Lemma 11.** For each \( \hat{\sigma} \in \hat{X}_{2n,k} \) let \( \text{Orb}(\hat{\sigma}) \) be the orbit of \( \hat{\sigma} \) under \( \varphi_S \). Then
\[
\sum_{\sigma \in \text{Orb}(\hat{\sigma})} p^{(2-31)} \sigma q^{(31-2)} \sigma = (p+q)^{2k} \cdot q^{(31-2)} \hat{\sigma} p^{(2-31)} \hat{\sigma}^{-k}.
\]  

(6.3)

**Proof.** We rewrite (6.3) as
\[
\sum_{\sigma \in \text{Orb}(\hat{\sigma})} p^{(2-31)} \sigma q^{(31-2)} \sigma = (1+p^{-1})^k \cdot (1+pq^{-1})^k \cdot q^{(31-2)} \hat{\sigma} p^{(2-31)} \hat{\sigma}.
\]  

(6.4)

The four cases (a)-(3.4a), (b)-(3.4c), (c)-(3.4e) and (d)-(3.4g) correspond to the four cases of normalized \( E \)-permutations in Definition 11. If \( \hat{\sigma} \in \hat{X}_{2n} \), by Lemma 10, for any \( X \subset S_\hat{\sigma} \), applying action \( \varphi_X \) on a descent top (resp. bottom) of \( \hat{\sigma} \) decreases the number of (31-2) (resp. (2-31)) patterns by 1 and increases the number of (2-31) (resp. (31-2)) patterns by 1, namely, it modifies the weight of \( \hat{\sigma} \) by the factor \( pq^{-1} \) (resp. \( qp^{-1} \)). Since \( \hat{\sigma} \) has exactly \( k \) descent tops (resp. bottoms), we obtain (6.4).

**Example 3.** If \( \sigma = 124635 \in \hat{X}_{6,1} \) with \( S = \{3,6\} \), then \( \varphi_3(\sigma) = 126345, \varphi_6(\sigma) = 124563 \) and \( \varphi_S(\sigma) = 125634. \) Thus
\[
\text{Orb}(\sigma) = \{124635, 126345, 124563, 125634\}.
\]

By Lemma 11, summing over all the orbits yields (6.1).
7. OTHER INTERPRETATIONS OF GENOCCHI NUMBERS

For $\sigma = \sigma_1 \sigma_2 \ldots \sigma_n \in S_n$, the statistics (2-13) $\sigma$ is the number of pairs $(i, j)$ such that $1 \leq i < j \leq n - 1$ and $\sigma_{j+1} > \sigma_i > \sigma_j$.

**Definition 16.** For $\sigma = \sigma_1 \ldots \sigma_n \in S_n$ with $\sigma_0 = \sigma_{n+1} = 0$, the value $\sigma_i$ is

- a peak if $\sigma_{i-1} < \sigma_i$ and $\sigma_i > \sigma_{i+1}$;
- a valley if $\sigma_{i-1} > \sigma_i$ and $\sigma_i < \sigma_{i+1}$;
- a double ascent if $\sigma_{i-1} < \sigma_i < \sigma_{i+1}$;
- a double descent if $\sigma_{i-1} > \sigma_i > \sigma_{i+1}$.

Let $\text{dd} \sigma$ be the number of double descents in $\sigma$. Define the enumerative polynomials

$$Y_n(a, p, q, y, t) = \sum_{\sigma \in \mathcal{Y}_{2n+1}^*} a^{b \text{m} \sigma} p^{(2-13)} q^{(31-2)} y^{\text{dd} \sigma} t^{\text{des} \sigma}$$  \hspace{1cm} (7.1)

**Theorem 17.** For $n \geq 1$ we have

$$Y_n(a, p, q, y, t) = X_n(a, 1, 1, 1, p, q, y, t).$$ \hspace{1cm} (7.2)

**Proof.** It suffices to show that the two polynomials have the same ordinary generating functions. In view of Corollary 5 we need only to prove that

$$\sum_{n=0}^{\infty} Y_n(a, p, q, y, t)x^n = \frac{1}{1 - (1 + yt)[a, 1]_{p,q} x - \frac{t[a, 1]_{p,q}[a, 2]_{p,q}[2]_{p,q} \cdot x^2}{\ldots}}$$ \hspace{1cm} (7.3)

with coefficients

$$\begin{cases}
    b_{n-1} = (1 + ty)[a, n]_{p,q}[n]_{p,q} & \text{for } n \geq 1, \\
    \lambda_n = t[a, n]_{p,q}[a, n + 1]_{p,q}[n]_{p,q}[n + 1]_{p,q}.
\end{cases}$$

As the proof is similar to that of Theorem 4, we just indicate the mapping $\psi : \mathcal{Y}_{2n+1}^* \to \mathcal{PD}_n$ with the associated weight and omit the details.

Let $\sigma \in \mathcal{Y}_{2n+1}^*$ and $\psi(\sigma) = (w, (\xi, \xi'))$. For $j \in [n]$ we define the steps $s_j = w_j - w_{j-1}$ of the path $w = (w_0, w_1, \ldots, w_n)$ as follows:

$$s_j = \begin{cases}
    U & \text{if } 2j - 1 \text{ is a valley of } \sigma; \\
    D & \text{if } 2j - 1 \text{ is a peak of } \sigma; \\
    L_1 & \text{if } 2j - 1 \text{ is a double ascent of } \sigma; \\
    L_2 & \text{if } 2j - 1 \text{ is a double descent of } \sigma.
\end{cases}$$ \hspace{1cm} (7.4)

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2The (2-13) statistic is called *artificial statistic* and denoted by “art” in [15].
and let \((\xi_j, \xi'_j) = (l_\sigma(2j) - 1, l_\sigma(2j))\). Thus, the weight of the path diagram \((w,(\xi, \xi')) = \prod_{i=1}^n w(w_i, (\xi_i, \xi'_i))\) is defined by
\[
w(\omega_i, (\xi_i, \xi'_i)) = \begin{cases} 
  a x(\xi'_i=0) q^\xi_i p^{h-\xi_i} q^\xi_i' p^{h+1-\xi_i'} t & \text{if } \omega_i = U; \\
  a x(\xi'_i=0) q^\xi_i p^{h+1-\xi_i} q^\xi_i' p^{h-\xi_i'} & \text{if } \omega_i = D; \\
  a x(\xi'_i=0) q^\xi_i p^{h-\xi_i} q^\xi_i' p^{h-\xi_i'} & \text{if } \omega_i = L_1; \\
  a x(\xi'_i=0) q^\xi_i p^{h-\xi_i} q^\xi_i' p^{h-\xi_i'} y t & \text{if } \omega_i = L_2.
\end{cases}
\] (7.5)

The J-fraction (7.3) follows then from Lemma 3. \(\square\)

A permutation \(\sigma \in Y_{2n+1}^\ast\) is called a \(\mathcal{F}\)-permutation if \(\sigma\) contains no double descent, and the last entry of \(\sigma\) is a peak, i.e., \(\sigma_{2n} < \sigma_{2n+1}\). Denote \(\bar{Y}_{2n+1}\) as the set of \(\mathcal{F}\)-permutations with length \(\mathcal{G}_{2n+1}\), and \(\bar{Y}_{2n+1,k}\) the subset of permutations in \(\bar{Y}_{2n+1}\) with \(k\) descents.

**Theorem 18.** We have
\[
Y_n(a, p, q, y, t) = \sum_{k=0}^{\lfloor n/2 \rfloor} \gamma_{n,k}(a, p, q)t^k(1 + yt)^{n-2k},
\] (7.6)

with coefficients
\[
\gamma_{n,k}(a, p, q) = \sum_{\sigma \in \bar{Y}_{2n+1,k}} a^{\text{VOP}\sigma} p^{(2-13)\sigma} q^{(31-2)\sigma}.
\]

**Proof.** Note that \(Y_n(a, p, q, 0, t) = \sum_{k=0}^{\lfloor n/2 \rfloor} \gamma_{n,k}(a, p, q)t^k\) and
\[
\sum_{n=0}^{\infty} Y_n(a, p, q, 0, t)x^n = \frac{1}{1 - [a, 1]_{p,q} x - t[a, 1]_{p,q}[a, 2]_{p,q}[2]_{p,q} \cdot x^2} \ldots
\] (7.7)

with coefficients
\[
\begin{cases}
  b_{n-1} = [a, n]_{p,q}[n]_{p,q} & \text{for } n \geq 1, \\
  \lambda_n = t[a, n]_{p,q}[a, n + 1]_{p,q}[n]_{p,q}[n + 1]_{p,q}.
\end{cases}
\]

We derive (7.6) by comparing (7.3) and (7.7). \(\square\)

**Remark 1.** We can also prove the above theorem by applying Brändén’s modified Foata-Strehl action (a.k.a. MFS-action) on \(Y_{2n+1}^\ast\) [6]. More precisely, we move every double descent to a double ascent position. It is known that that MFS-action does not change the number of (2-13) patterns and the number of (31-2) patterns. It is also clear that MFS-action does not change the even left-to-right maxima.

For \(\sigma \in \bar{Y}_{2n+1}\) and \(i \in [n]\), the doubleton \(\{2i - 1, 2i\}\) is called a VOP pair of \(\sigma\) if \(2i - 1\) is either a valley or peak of \(\sigma\). As for \(\mathcal{E}\)-permutations, we associate a sequence \((a(1), a(2), \ldots, a(2n))\) with \(\sigma\) by
\[
a(j) = f(j) - g(j) + 1 \quad \text{for } j \in [2n].
\]
Definition 19. A $F$-permutation $\sigma \in \mathcal{Y}^*_2$ is called normalized, if for any VOP pair $x = \{2i - 1, 2i\}$ of $\sigma$, the embracing number $l_\sigma(x)$ is even. Let $\mathcal{Y}^*_{2n+1}$ be the set of normalized $F$-permutations in $\mathcal{Y}^*_2$ and $\mathcal{Y}^*_{2n+1,k}$ the subset of normalized $F$-permutations in $\mathcal{Y}^*_{2n+1}$ with $k$ descents by.

**Action $\varphi_x$ on $\mathcal{Y}^*_{2n+1}$.** For a VOP pair $x = \{2i - 1, 2i\}$ of $\sigma \in \mathcal{Y}^*_{2n+1,k}$. We define $\varphi_x$ as follows.

1. If $a(2i)$ is even then we factorize $\sigma = \tau'_3\tau'_2\tau'_1(2i)\tau_1\tau_2\tau_3$, where $\tau_1$ and $\tau'_2$ is the maximal sequence of consecutive entries greater than $2i$, $\tau_2$ and $\tau'_1$ is the maximal sequence of consecutive entries smaller than $2i$. We define

$$\varphi_x(\sigma) = \begin{cases} 
\tau'_3\tau'_2\tau'_1\tau_1\tau_2(2i)\tau_3 & \text{if } l_\sigma(x) \text{ is even;} \\
\tau'_3(2i)\tau'_2\tau'_1\tau_1\tau_2\tau_3 & \text{if } l_\sigma(x) \text{ is odd.}
\end{cases}$$

2. If $a(2i - 1)$ is even.

(a) If $l_\sigma(2i - 1) = 2b$ for some $b \in \mathbb{N}$, then

$$\varphi_{2i-1}(\sigma) = w \in \mathcal{Y}^*_2$$

where $w$ is constructed as follows.

If $2i - 1$ is a valley (resp. peak) of $\sigma$, then insert $2i - 1$ in $w_0$ such that at the left side of $2i - 1$ in $w_{01}$, the number of valleys minus the numbers of peaks equals to $l_{2i-1}(\sigma) + 1$ (here, valleys and peaks are those element smaller than $2i - 1$ in $\sigma$). Then we insert one by one the rest peaks and valleys are greater than $2i - 1$ in $\sigma$ into $w_{01}$ in increasing order to obtain $w_1$, such that those peaks and valleys say 2j − 1's are still peaks and valleys in $w_1$ and for every $l_\sigma(2j - 1) = l_{w_1}(2j - 1)$. Last we insert the rest element of $\sigma$ in $w_1$ one by one in increasing order such that $l_{w_1}(i) = l_\sigma(i)$ for all $[2n + 1] \setminus \{2i - 1\}$.

(b) If $l_\sigma(2i - 1) = 2b + 1$ for some $b \in \mathbb{N}$, then

$$\varphi_{2i-1}(\sigma) = w \in \mathcal{Y}^*_2$$

where $w$ is constructed by following almost the process in (a), except when obtain $w_{01}$ by inserting $2i - 1$ in $w_0$ such that at the left side of $2i - 1$ in $w_{01}$, the number of valleys minus the number of peaks equals to $l_{2i-1}(\sigma) - 1$. 

where $f(i)$ (resp. $g(i)$) is the number of valleys (resp. peaks) less than $i$ in $\sigma$.

**Fact.** If $\{2i - 1, 2i\}$ is a VOP pair of $\sigma \in \mathcal{Y}^*_{2n+1}$, then there is one and only one even element in $\{a(2i - 1), a(2i)\}$.

For convenience, we define the embracing number of a VOP pair $x = \{2i - 1, 2i\}$ of $\sigma$ by

$$l_\sigma(x) = \begin{cases} 
(2-31)(2i - 1) & \text{if } a(2i - 1) \text{ is even;} \\
(31-2)(2i) & \text{if } a(2i - 1) \text{ is odd.}
\end{cases}$$
Clearly the action $\overline{\phi}_x$ is an involution on $\tilde{Y}_{2n+1}^*$, besides two actions $\overline{\phi}_x$ and $\overline{\phi}_y$ commute for all $x, y$ are VOP pairs. Hence for any subset $X \subset H$ where $H$ is the set of VOP pairs of $\sigma$, we may define the action $\phi_X$ on $\sigma \in \tilde{X}_{2n}$ by
\[
\overline{\phi}_X(\sigma) = \prod_{x \in X} \overline{\phi}_x(\sigma).
\] (7.8)

Following the similar proof of Lemma 10 and Lemma 11, we also have the following two Lemmas for $F$-permutations.

**Lemma 12.** Let $\sigma \in \tilde{Y}_{2n+1}^*$, for any $x$ a VOP pair of $\sigma$, the permutation $\overline{\phi}_x(\sigma)$ has the same set of valleys and peaks as $\sigma$. Let
\[
\Delta_{\overline{\phi}_x}(\sigma) = ((2-31) \overline{\phi}_x(\sigma), (31-2) \overline{\phi}_x(\sigma)) - ((2-31) \sigma, (31-2) \sigma).
\]
Then we have,
\[
\Delta_{\overline{\phi}_x}(\sigma) = \begin{cases} 
(1, -1) & \text{if } l_\sigma(x) \text{ is odd} \\
(-1, 1) & \text{if } l_\sigma(x) \text{ is even}
\end{cases}
\]

**Lemma 13.** For each $\hat{\sigma} \in \hat{Y}_{2n+1}^*$, let $\text{Orb}(\hat{\sigma})$ be the orbit of $\hat{\sigma}$ under $\overline{\phi}_H$. Then
\[
\sum_{\sigma \in \text{Orb}(\hat{\sigma})} p^{(2-13)} \sigma q^{(31-2)} \sigma = (p + q)^{2k} p^{(2-13)} \sigma - 2k q^{(31-2)} \sigma.
\]

Combine Lemma 12 and Lemma 13, we obtain the following Theorem.

**Theorem 20.** We have
\[
\gamma_{n,k}(1, p, q) = (p + q)^{2k} \sum_{\sigma \in \tilde{Y}_{2n+1}^*} p^{(2-13)} \sigma - 2k q^{(31-2)} \sigma.
\]

Let
\[
\overline{Y}_n(a, p, q, t) = \sum_{\sigma \in \tilde{Y}_{2n+1}^*} a^{\text{lema } \sigma} p^{(2-13)} \sigma q^{(31-2)} \sigma t^{\text{des } \sigma},
\] (7.9a)
\[
\tilde{Y}_n(p, q, t) = \sum_{\sigma \in \tilde{Y}_{2n+1}^*} p^{(2-13)} \sigma q^{(31-2)} \sigma t^{\text{des } \sigma}.
\] (7.9b)

From Theorems 17, 18 and 20 we derive immediately the following identities.

**Corollary 21.** We have
\[
\overline{Y}_n(a, p, q, t) = \overline{X}_n(a, 1, p, q, t),
\] (7.9c)
\[
\tilde{Y}_n(p, q, t) = \tilde{X}_n(1, 1, p, q, t).
\] (7.9d)

**Remark 2.** The normalized $E$-permutations and normalized $F$-permutations are two new models for the normalized median Genocchi numbers. It would be interesting to establish connections of these models with the well-known Dellac configurations for normalized Genocchi numbers in [4, 20].


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