**Abstract**

We study the problem of the phonon–induced electron–electron interaction in a solid. Starting with a Hamiltonian that contains an electron–phonon interaction, we perform a similarity renormalization transformation to calculate an effective Hamiltonian. Using this transformation singularities due to degeneracies are avoided explicitly. The effective interactions are calculated to second order in the electron–phonon coupling. It is shown that the effective interaction between two electrons forming a Cooper pair is attractive in the whole parameter space. For a simple Einstein model we calculate the renormalization of the electronic energies and the critical temperature of superconductivity.

**Keywords:** Electron–phonon coupling, effective interactions, superconductivity
1 Introduction

BCS–theory [1] is one of the most successful and most popular theories in condensed matter physics. It is based on a Hamiltonian with an attractive interaction between the electrons. Phonons are not explicitly present in this model. The phonon frequencies enter in the explicit form of the phonon–induced effective electron–electron interaction. Most of the properties of superconductors can be understood using BCS–theory. But one of the problems of BCS–theory is that the critical temperature $T_c$ of the superconductor cannot be calculated. The famous BCS–formula $T_c = 1.13 \Theta \exp(-1/N(\epsilon_F)V)$, where $\Theta$ is the Debye temperature and $V$ is the strength of the interaction, has to be used to determine $N(\epsilon_F)V$ from $T_c$. If one uses for $N(\epsilon_F)V$ values that have been obtained using standard perturbative treatments of the electron–phonon interaction, the calculated critical temperature is too large. Reliable results for $T_c$ can be obtained in the framework of the Eliashberg theory [2]. For a review of this theory and a critical discussion of various formulas for the critical temperature we refer to [3, 4], see also [5].

In their discussion of BCS–theory, Allen et al [4] argued that the BCS–equation yields a wrong result for $T_c$ because retardation effects in the interaction are neglected. This point of view is possible in a formulation of the theory in terms of Green’s functions like in the Eliashberg–theory [2] or equivalently in a field–theoretic formulation of a model containing electrons and phonons, where it is possible to integrate out the phonons. In both cases one obtains an effective phonon–induced electron–electron interaction that depends on time or frequency. But in the standard formulation of BCS–theory one starts with a Hamiltonian. The effective electron–electron interaction Bardeen et al [1] had in mind was the phonon–induced interaction of Fröhlich [6] or Bardeen and Pines [7]. In these approaches the phonon–induced interaction is obtained by applying a unitary transformation to eliminate the electron–phonon interaction in the Hamiltonian in lowest order. It is clear that in a Hamiltonian formulation of the theory, the interaction cannot be frequency dependent, since the Hamiltonian is a hermitean operator. From the viewpoint of a Hamiltonian formulation the problem of BCS–theory with an interaction of the Fröhlich type is that the interaction is singular and does not contain the correct energy scale. If one performs a single unitary transformation to eliminate the electron–phonon interaction, one attempts to treat all energy scales in the problem at once. This usually fails, even in perturbation theory. Instead one should break the problem into pieces, dealing with each energy scale in sequence. This is the usual approach of renormalization theory.

About two years ago Glazek and Wilson [8] proposed a new renormalization scheme for Hamiltonians, called similarity renormalization. The main idea of this approach is to perform a continuous unitary transformation that yields a band–diagonal effective Hamiltonian. In the effective Hamiltonian the dependence on the original ultra–violet cutoff is removed, but it will contain additional interactions. In the present paper we apply this renormalization scheme to the classical problem of interacting electrons and phonons, modelled by a Hamiltonian

$$H = \sum_k \epsilon_k : c_k^\dagger c_k : + \sum_q \omega_q : b_q^\dagger b_q : + \sum_{k,q} (g_q c_k^\dagger c_{k+q}^\dagger b_q^\dagger + g_q^* c_{k+q}^\dagger c_k b_q).$$

(1.1)

c_k^\dagger$ and $c_k$ are the usual creation and annihilation operators for electrons, $b_q^\dagger$ and $b_q$ for phonons, and the colons denote normal ordering. We calculate the renormalized Hamiltonian to second order in the coupling constant $g_q$ of the electron–phonon interaction. It contains an additional effective electron–electron interaction, that is responsible for superconductivity.

Recently, the construction of the effective phonon–induced electron–electron interaction has been studied by Lenz and Wegner [9]. They used flow equations for Hamiltonians, a method proposed by Wegner [10] to block–diagonalize a given Hamiltonian. They obtained an expression
for the effective interaction that differs from Fröhlich’s result, in fact it is less singular. If we write the induced electron–electron interaction in the form

\[
\frac{1}{2} \sum_{k,k',q} V_{k,k',q} : e_{k+q}^\dagger e_{k'}^\dagger :c_{k'} c_k :,
\]

(1.2)

the coefficient for the interaction of two electrons forming a Cooper pair obtained by Fröhlich is

\[
V_{k,-k,q} = |g_q|^2 \frac{2\omega_q}{(\epsilon_{k+q} - \epsilon_k)^2 - \omega_q^2}
\]

(1.3)

whereas Lenz and Wegner obtained

\[
V_{k,-k,q} = -|g_q|^2 \frac{2\omega_q}{(\epsilon_{k+q} - \epsilon_k)^2 + \omega_q^2}.
\]

(1.4)

There are two remarkable differences between these results. First, the interaction obtained by Lenz and Wegner has no singularity, second it is attractive for all \(k\) and \(q\).

Recently Wegner’s flow equations for Hamiltonians have been applied to the Anderson impurity model. For that model a similar problem occurs. It can be mapped onto the Kondo model using the well known Schrieffer–Wolff transformation. The Schrieffer–Wolff transformation eliminates the hybridization between the impurity state and the electronic band states to first order. Using this transformation, the induced spin–spin interaction contains a singularity if the impurity orbital lies in the electronic band. Furthermore, the induced interaction does not have the correct energy scale built in, therefore the Kondo temperature can not be calculated correctly. The flow equations yield a less singular expressions which has a different sign in some regions of the parameter space. In the case of the Anderson impurity model it was possible to compare the results with renormalization group calculations, and it could be shown that the flow equations yield the correct behaviour. For instance, the energy scale which sets the scale for the Kondo temperature comes out correctly, whereas the Schrieffer–Wolff result does not have this property.

Unfortunately, a similar comparison with renormalization group calculations was not possible in the case of the electron–phonon coupling. The primary purpose of the present paper is to fill this gap. Using the above mentioned similarity renormalization scheme, we will calculate the effective electron–electron interaction. We will show that the interaction between two electrons forming a Cooper pair has the sign structure of the expression obtained by Lenz and Wegner, it is attractive for all \(k\) and \(q\). This clearly supports the result by Lenz and Wegner. The difficulties of the \(T_c\) formula of BCS–theory can be traced back to the wrong effective electron–electron interaction. We will come back to this point in Sect. 4.

For a reader who is familiar with Eliashberg theory, it is probably unclear how an approach using a Hamiltonian framework can lead to quantitative results without a fit parameter like in BCS–theory. Neither the interactions nor the interaction obtained using similarity renormalization contain retardation effects. The reason that the procedure yields reliable results is that similarity renormalization yields the correct low energy scale of the problem. Different energy scales are automatically separated. How this happens will be made explicit in Sect. 2. A second point is that since the renormalization procedure consists of a continuous unitary transformation applied to the Hamiltonian, one has to transform the observables as well. Retardation, i.e. the decay of single particle states and the broadening of peaks in spectral functions can be obtained within the present approach, if the transformation of the observables is taken into account. In a recent paper on time–dependent equilibrium correlation functions in dissipative quantum systems it has been demonstrated, how the transformation of the observables can be
done \cite{13}. In that case Wegner’s flow equation \cite{10} have been used, but the transformation of the observables is very similar in both approaches. It was possible to obtain very accurate quantitative results in that case. In the present paper we only calculate static properties of the system, therefore we will not discuss this point further.

The outline of this paper is as follows. In the next section we give a brief description of the similarity renormalization scheme. For details we refer to \cite{8}. This method has been applied to quantum electrodynamics \cite{14} and to quantum chromodynamics \cite{15,16} (and the references therein). Many details concerning the method are explained in these papers as well. In Sect. 3 we apply the method to the electron–phonon coupling. We derive the renormalization group equations and calculate the renormalized Hamiltonian to second order in $g_q$. Mainly for illustrational purpose we derive some explicit results for the simple Einstein model for phonons in Sect. 4. First we calculate the renormalization of the electronic single particle energies. They show a typical logarithmic renormalization, which leads to the well known renormalization of the density of states at the Fermi surface. Next we calculate $T_c$ within the Einstein model using our effective interaction. We compare our results with those obtained using \eqref{1.4} or using the $T_c$–formula of BCS-theory. Finally we give some conclusions including a comparison between the similarity renormalization scheme and Wegner’s flow equations.

## 2 Similarity renormalization

This method yields renormalization equations for a given arbitrary Hamiltonian $H_\lambda$ as a function of the cutoff $\lambda$. The goal of the method is to transform the initial Hamiltonian (with a large cutoff $\Lambda$) into an effective Hamiltonian that has no matrix elements providing energy jumps large compared to the small cutoff $\lambda$. The initial Hamiltonian contains usually multiple energy scales and couplings between these energy scales. The renormalized Hamiltonian $H_\lambda$ has a band diagonal form. The different energy scales in the initial Hamiltonian are decoupled in $H_\lambda$.

Let us write $H_\lambda$ in the form

$$H_\lambda = H_{0\lambda} + H_{I\lambda}. \quad (2.1)$$

$H_{0\lambda}$ is the free Hamiltonian, $H_{I\lambda}$ contains interactions and counter terms. The eigenvalues of $H_{0\lambda}$ are $E_{i\lambda} \geq 0$. We introduce cutoff functions $u_{ij\lambda}$ for the matrix elements. To assure a band diagonal structure of the renormalized Hamiltonian, we choose $u_{ij\lambda} = 1$ if $|E_{i\lambda} - E_{j\lambda}|$ is small compared to lambda and $u_{ij\lambda} = 0$ if $|E_{i\lambda} - E_{j\lambda}|$ is large compared to $\lambda$. The detailed form of $u_{ij\lambda}$ is not important. One possible choice, which has been used in the treatment of QED on the light front \cite{14} is

$$u_{ij\lambda} = \theta(\lambda - |E_{i\lambda} - E_{j\lambda}|). \quad (2.2)$$

In other situations one has to introduce a smooth cutoff function. For a discussion of this point we refer to \cite{8,15}. Furthermore we introduce

$$r_{ij\lambda} = 1 - u_{ij\lambda} \quad (2.3)$$

The renormalization of $H_\lambda$ can be written down as an infinitesimal unitary transformation

$$\frac{dH_\lambda}{d\lambda} = [\eta_\lambda, H_\lambda] \quad (2.4)$$

with a generator $\eta_\lambda$. $\eta_\lambda$ has to be chosen so that the matrix elements of $H_\lambda$ obey

$$H_{ij\lambda} = u_{ij\lambda} Q_{ij\lambda}. \quad (2.5)$$
This assures that $H_\lambda$ is band diagonal. For the matrix elements $Q_{ij\lambda}$ we obtain
\[
\frac{d u_{ij\lambda}}{d \lambda} Q_{ij\lambda} + u_{ij\lambda} \frac{d Q_{ij\lambda}}{d \lambda} = \eta_{ij\lambda} (E_{i\lambda} - E_{j\lambda}) + [\eta_{\lambda}, H_{I\lambda}]_{ij}.
\] (2.6)

One defines
\[
G_{ij\lambda} \overset{\text{def}}{=} [\eta_{\lambda}, H_{I\lambda}]_{ij} - \frac{d u_{ij\lambda}}{d \lambda} Q_{ij\lambda}
\] (2.7)

and chooses
\[
\eta_{ij\lambda}(E_{i\lambda} - E_{j\lambda}) = -r_{ij\lambda} G_{ij\lambda}.
\] (2.8)

This yields the final equations for the matrix elements of the generator of the infinitesimal unitary transformation
\[
\eta_{ij\lambda} = \frac{r_{ij\lambda}}{E_{i\lambda} - E_{j\lambda}} \left( [\eta_{\lambda}, H_{I\lambda}]_{ij} - \frac{d u_{ij\lambda}}{d \lambda} \frac{H_{ij\lambda}}{u_{ij\lambda}} \right),
\] (2.9)

and for the renormalization of the Hamiltonian
\[
\frac{d H_{ij\lambda}}{d \lambda} = u_{ij\lambda} [\eta_{\lambda}, H_{I\lambda}]_{ij} + r_{ij\lambda} \frac{d u_{ij\lambda}}{d \lambda} \frac{H_{ij\lambda}}{u_{ij\lambda}}.
\] (2.10)

Since $r_{ij\lambda} = 0$ if $|E_{i\lambda} - E_{j\lambda}| \lesssim \lambda$, the denominator $E_{i\lambda} - E_{j\lambda}$ in (2.9) is bounded from below. These equations are exact, but it is clear that one is not able to solve them explicitly. Suitable approximations are necessary. Since small energy denominators are avoided in this approach one possibility is a systematic expansion in a coupling constant.

Before we proceed to the application of the similarity renormalization, let us take a closer look at the equations (2.9) and (2.10). The main point is that due to (2.5) the renormalized Hamiltonian contains only couplings between states with an energy difference less than $\lambda$. Starting with an initial cutoff $\Lambda$, couplings between very different energy scales are eliminated first, couplings between states belonging to the same energy scale are not eliminated or are eliminated later for smaller values of $\lambda$. As a consequence, different energy scales are separated during the renormalization process.

### 3 Application to the electron–phonon coupling

The Hamiltonian of the model is
\[
H_\lambda = H_{0\lambda} + H_{I\lambda}
\] (3.1)

with
\[
H_{0\lambda} = \sum_k \epsilon_{k,\lambda} : c_k^\dagger c_k : + \sum_q \omega_{q,\lambda} : b_q^\dagger b_q :,
\] (3.2)

\[
H_{I\lambda} = \sum_{k,q} g_{k,q,\lambda} c_k^\dagger c_{k+q} b_q^\dagger + g_{k,q,\lambda}^* c_k^\dagger c_{k+q} b_q : + O(g^2).
\] (3.3)

$c_k^\dagger$ and $c_k$ are the electron creation and annihilation operators. The index $k$ is a shorthand notation for $(k, \sigma)$ and $-k$ denotes $(-k, \sigma)$. $b_q^\dagger$ and $b_q$ are the creation and annihilation operators for phonons. Let us assume $\omega_{q,\lambda} = \omega_{-q,\lambda}$, $\epsilon_{k,\lambda} = \epsilon_{-k,\lambda}$. For a large cutoff $\Lambda$ we let $g_{k,q,\lambda} = g_{q,\lambda} = g_{-q,\lambda}$ independent of $k$. The terms $O(g^2)$ in $H_{I\lambda}$ contain induced interactions. For simplicity we
do not include a Coulomb repulsion in our model Hamiltonian. We have introduced a normal ordering for the fermions and for the bosons,

\[
: c^\dagger_k c_k := c^\dagger_k c_k - n_k, \tag{3.4}
\]

\[
: b^\dagger_q b_q := b^\dagger_q b_q - \bar{n}_q, \tag{3.5}
\]

where \( n_k \) is the occupation number for electrons in the state \( k \) and \( \bar{n}_q \) is the occupation number for the bosons, respectively. The Fermi energy is set to zero.

The model contains several energy scales. The largest energy scale is given by the band width of the electronic single particle states, which is usually a few eV. Typical phonon energies are given by the Debye frequency, they are about two orders of magnitude smaller. Due to the phonon induced interaction between the electrons, Cooper pairs are formed. Their energy scale is given by the critical temperature of superconductivity, which is again about two orders of magnitude smaller than typical phonon energies. One has to resolve an energy scale in a Hamiltonian that is dominated by energies being four orders of magnitude larger. Such a situation often occurs when a given problem has a marginal relevant operator. In our case this is the attractive phonon–induced electron–electron interaction. When this interaction is calculated within a perturbative scheme like the one used by Fröhlich, all energy scales are treated at once. This usually fails.

We now investigate this model using (2.9, 2.10). All quantities will be calculated as series expansions in \( g \). Due to (2.9) the generator of the infinitesimal unitary transformation can be written as

\[
\eta_\lambda = \sum_{k,q} : \eta_{k,q,\lambda} c^\dagger_k c_{k+q} b^\dagger_q - \eta^*_{k,q,\lambda} c^\dagger_{k+q} c_k b_q : + O(g^2) \tag{3.6}
\]

with \( \eta_{k,q,\lambda} = O(g) \). In order to have a definite choice for \( u_{k,q,\lambda} \), we let

\[
u_{k,q,\lambda} = u(|\epsilon_{k,\lambda} - \epsilon_{k+q,\lambda} + \omega_{q,\lambda}|/\lambda) \tag{3.7}
\]

with a smooth cutoff function \( u(x) \) that drops fast from 1 to zero in the vicinity of \( x = 1 \). Then (2.9) yields

\[
\eta_{k,q,\lambda} = - \frac{r_{k,q,\lambda}}{\epsilon_{k,\lambda} - \epsilon_{k+q,\lambda} + \omega_{q,\lambda}} \frac{d \ln u_{k,q,\lambda}}{d \lambda} g_{k,q,\lambda} + O(g^3). \tag{3.8}
\]

Let us now calculate the effective Hamiltonian. The first term in (2.10) is of order \( g^2 \). This term contains the renormalization of the single particle energies \( \epsilon_{k,\lambda} \) and \( \omega_{q,\lambda} \) and some new, induced interaction in \( H_{I,\lambda} \). The second term in (2.10) is of order \( g \) and contributes to the renormalization flow of the electron–phonon coupling. The renormalization of this coupling is determined by the equation

\[
\frac{dg_{k,q,\lambda}}{d \lambda} = r_{k,q,\lambda} \frac{d \ln u_{k,q,\lambda}}{d \lambda} g_{k,q,\lambda} + O(g^3). \tag{3.9}
\]

It can be solved using the ansatz

\[
g_{k,q,\lambda} = g_{q,\lambda} \frac{\epsilon_{k,q,\lambda}}{\epsilon_{q,\lambda}} \tag{3.10}
\]

with \( \epsilon_{k,q,\lambda} = e(|\epsilon_{k,\lambda} - \epsilon_{k+q,\lambda} + \omega_{q,\lambda}|/\lambda) \). \( e(x) \) obeys \( \frac{1}{e} \frac{de}{dx} = \frac{r}{1 - u} \frac{du}{dx} \). Using \( r = 1 - u \) it can be written in the form

\[
e(x) = u(x) \exp(r(x)). \tag{3.11}
\]
Using (3.8) and the definition of \( \lambda \) we integrate this flow equation from a small cutoff \( \epsilon \) to the initial large cutoff \( \Lambda \). On the right hand side we perform an integration by parts. Then the remaining integral contains derivatives of the single particle energies and is therefore of fourth order in \( g_{k,q,\lambda} \). Neglecting this term we obtain

\[
\omega_{q,\lambda} = \omega_{q,\Lambda} + \sum_k \frac{n_k - n_{k+q}}{\epsilon_{k,\lambda} - \epsilon_{k+q,\lambda} + \omega_{q,\lambda}} |g_{k,q,\lambda}|^2 \left( 1 - \epsilon_{k,\lambda}^2 \right).
\]

A similar calculation can be performed for \( \epsilon_{k,\lambda} \). The renormalization of \( \epsilon_{k,\lambda} \) is determined by

\[
\frac{d\epsilon_{k,\lambda}}{d\lambda} = \sum_q \left( \eta_{k,q,\lambda} g_{k,q,\lambda}^* (n_{k+q} + \bar{n}_q) + \eta_{k,q,\lambda}^* g_{k,q,\lambda} (n_{k+q} + \bar{n}_q) \right) - \eta_{k+q,q-\lambda} g_{k+q,q-\lambda} (1 - n_{k+q} + \bar{n}_q) - \eta_{k+q,q-\lambda}^* g_{k+q,q-\lambda} (1 - n_{k+q} + \bar{n}_q) - \frac{1}{\epsilon_{k+q,\lambda} - \epsilon_{k,\lambda} + \omega_{q,\lambda}} \frac{d|g_{k+q,q,\lambda}|^2}{d\lambda} + \sum_q \frac{1}{\epsilon_{k+q,\lambda} - \epsilon_{k,\lambda} + \omega_{q,\lambda}} \frac{d|g_{k+q,q,\lambda}|^2}{d\lambda}.
\]
and with the same assumptions as above we obtain

$$
\epsilon_{k,\lambda} = \epsilon_{k,\lambda} + \sum_q \frac{n_{k+q} + \bar{n}_q}{\epsilon_{k,\lambda} - \epsilon_{k+q,\lambda} + \omega_{q,\lambda}} |g_{q,\lambda}|^2 \left(1 - \frac{2}{\epsilon_{k,q,\lambda}^2}\right)
$$

$$
- \sum_q \frac{1 - n_{k+q} + \bar{n}_q}{\epsilon_{k+q,\lambda} - \epsilon_{k,\lambda} + \omega_{q,\lambda}} |g_{q,\lambda}|^2 \left(1 - \frac{2}{\epsilon_{k+q,-q,\lambda}^2}\right)
$$

(3.18)

The two equations (3.16) and (3.18) can be used to determine the single particle energies self consistently. The result is correct up to second order in $g$. The main point is that due to the factors $(1 - \epsilon_{k,q,\lambda}^2)$ or $(1 - \epsilon_{k+q,-q,\lambda}^2)$ small energy denominators are avoided explicitly. Without these factors and with the renormalized energies in the second and the third term on the right hand side replaced by the unrenormalized energies, the renormalization equations (3.16) and (3.18) are well known and can be found in various textbooks on solid state theory (see e.g. [18]).

In the next section we show that (3.18) yields the correct renormalization of the density of states at the Fermi surface, which is usually calculated within the framework of Eliashberg theory.

$[\eta, H_f]$ contains further terms generating new couplings in the Hamiltonian $H_f$. It can be written as

$$
H_{I,\lambda} = \sum_{k,q} : g_{k,q,\lambda} c^\dagger_k c_k + g_{k,q,\lambda}^* c^\dagger_k c_k b_q + \frac{1}{2} \sum_{k,k'q} V_{kk'q,\lambda} : c^\dagger_{k+q} c^\dagger_{k'} c_k c_k' : + \sum_q (c^\dagger_q b_q b_{-q} + c_q b_q b_{-q}) + \text{couplings between electrons and two bosons } + O(g^3).
$$

(3.19)

Let us calculate $V_{k,k'q}$ and $c_q$ to second order in $g$. The other couplings can be obtained similarly. Following (2.10) the renormalization equation of $V_{kk'q,\lambda}$ is

$$
\frac{dV_{kk'q,\lambda}}{d\lambda} = -u_{k,k'q,\lambda}(\eta_{k+q,-q,\lambda} g_{k,q,\lambda}^* + \eta_{k,-q,\lambda} g_{k+q,-q,\lambda}) + u_{k,k'q,\lambda} \frac{d\ln u_{k,k'q,\lambda}}{d\lambda} V_{kk'q,\lambda}.
$$

(3.20)

In the first term in the right hand side the induced electron–electron interaction is generated due to the elimination of the electron–phonon interaction. The second term eliminates the induced electron–electron interaction again. This elimination then yields higher interactions which are of fourth or higher order in the electron–phonon coupling. The problem is now that we expect to obtain an attractive induced interaction between the electrons. Such an attractive interaction is known to be a marginal relevant operator for a fermionic system. The elimination of such a term can cause difficulties. Within the framework of Wegner’s flow equations, this has already been observed for the one–dimensional problem [10]. Therefore we modify the renormalization scheme at this point. We simply choose $u_{k,k'q,\lambda} = 1$. Then the second term in the renormalization equation for the induced interaction vanishes. Using the above expression for $g_{k,q,\lambda}$ and $\eta_{k,q,\lambda}$, this equations becomes

$$
\frac{dV_{kk'q,\lambda}}{d\lambda} = \frac{\epsilon_{k',-q,\lambda}}{\epsilon_{k+q,\lambda} - \epsilon_{k,\lambda} + \omega_{q,\lambda}} \frac{d\epsilon_{k+q,-q,\lambda}}{d\lambda} + \frac{\epsilon_{k+q,q,\lambda}}{\epsilon_{k'+q,-q,\lambda} + \omega_{q,\lambda}} \frac{d\epsilon_{k'+q,-q,\lambda}}{d\lambda} + \frac{\epsilon_{k',q,\lambda}}{\epsilon_{k,q,\lambda} - \epsilon_{k',\lambda} + \omega_{q,\lambda}} \frac{d\epsilon_{k,q,\lambda}}{d\lambda} + \frac{\epsilon_{k'-q,q,\lambda}}{\epsilon_{k'+q,q,\lambda}} \frac{d\epsilon_{k'+q,q,\lambda}}{d\lambda}.
$$

(3.21)
Defining

\[ f_{k,k',q,\lambda} = \int_{\lambda}^{\Lambda} \! ds e^{k' - q, s} \frac{d c_{k,q,s}}{ds} \]

we can write \( V_{k,k',q,\lambda} \) in the form

\[ V_{k,k',q,\lambda} = - |g_{q,\lambda}|^2 \left( \frac{f_{k,k',q,\lambda}}{\epsilon_{k,\lambda} - \epsilon_{k+q,\lambda} + \omega_{q,\lambda}} + \frac{f_{k' - q,k+q,\lambda}}{\epsilon_{k' - q,\lambda} - \epsilon_{k,\lambda} + \omega_{q,\lambda}} \right) \]

\[ + \frac{f_{k' + q,k,q,\lambda}}{\epsilon_{k+q,\lambda} - \epsilon_{k,\lambda} + \omega_{q,\lambda}} + \frac{f_{k,k',q,\lambda}}{\epsilon_{k',\lambda} - \epsilon_{k' - q,\lambda} + \omega_{q,\lambda}} + O(g^4). \]  

(3.23)

The terms \( O(g^4) \) arise again due to the fact that the derivative of the single particle energies is of order \( g^2 \). Further corrections \( O(g^4) \) occur since in (3.21) terms of \( O(g^4) \) have been neglected. (3.23) is our main result. Note that \( f_{k,k',q,\lambda} \geq 0 \). Let us discuss two interesting cases:

a) \( k' = k + q \) (the diagonal part of the interaction)

We have \( f_{k,k+q,\lambda} = \frac{1}{2} (1 - e^2_{k,q,\lambda}) \). This yields

\[ V_{k,k+q,\lambda} = |g_{q,\lambda}|^2 \left( \frac{2\omega_{q,\lambda}}{(\epsilon_{k+q,\lambda} - \epsilon_{k,\lambda})^2 - \omega_{q,\lambda}^2} \right. \]

\[ - \left. \frac{\epsilon_{k+q,\lambda}^2}{\epsilon_{k+q,\lambda} - \epsilon_{k,\lambda} + \omega_{q,\lambda}} - \frac{\epsilon_{q,\lambda}^2}{\epsilon_{k,\lambda} - \epsilon_{k+q,\lambda} + \omega_{q,\lambda}} \right) \]  

(3.24)

The first term is the well known expression already obtained by Fröhlich [3]. The singularity is cancelled by the two other terms. These terms occur because the electron–phonon coupling has not been eliminated completely.

b) \( k' = -k \) (the interaction of two electrons forming a Cooper pair)

Using \( f_{k,-k,\lambda} = f_{-k,k,-\lambda} \) we obtain

\[ V_{k,-k,\lambda} = -2 |g_{q,\lambda}|^2 \left( \frac{f_{k,-k,\lambda}}{\epsilon_{k,\lambda} - \epsilon_{k+q,\lambda} + \omega_{q,\lambda}} + \frac{f_{k+q,-k,q,\lambda}}{\epsilon_{k+q,\lambda} - \epsilon_{k,\lambda} + \omega_{q,\lambda}} \right). \]  

(3.25)

In the case \( \omega_{q,\lambda} > |\epsilon_{k+q,\lambda} - \epsilon_{k,\lambda}| \) both terms are negative, the interaction is thus attractive. If \( \omega_{q,\lambda} < |\epsilon_{k+q,\lambda} - \epsilon_{k,\lambda}| \) the sign of the two terms is different. Let us consider the case \( \epsilon_{k+q,\lambda} - \epsilon_{k,\lambda} > \omega_{q,\lambda} \). Under this condition one has \( |\epsilon_{k+q,\lambda} - \epsilon_{k,\lambda} + \omega_{q,\lambda}| > |\epsilon_{k,\lambda} - \epsilon_{k+q,\lambda} + \omega_{q,\lambda}| \). For some large cutoff \( \Lambda \), both, \( |\epsilon_{k+q,\lambda} - \epsilon_{k,\lambda} + \omega_{q,\lambda}| \) and \( |\epsilon_{k,\lambda} - \epsilon_{k+q,\lambda} + \omega_{q,\lambda}| \) become smaller than \( \lambda \). Lowering the cutoff, it happens that \( |\epsilon_{k+q,\lambda} - \epsilon_{k,\lambda} + \omega_{q,\lambda}| \) becomes larger than \( \lambda \) whereas \( |\epsilon_{k,\lambda} - \epsilon_{k+q,\lambda} + \omega_{q,\lambda}| \) is still smaller than \( \lambda \). This means that \( f_{k+q,-k,q,\lambda} \) is nonzero, whereas \( f_{k,-k,\lambda} \) remains zero. In that case only the second term in (3.23) contributes. This term is negative, the interaction is thus attractive. Similarly, if \( \epsilon_{k,\lambda} - \epsilon_{k+q,\lambda} > \omega_{q,\lambda} \), the first term starts to contribute first and is negative. This shows that for sufficiently large \( \lambda \) and \( V_{k,-k,q,\lambda} = 0 \) one has \( V_{k,-k,\lambda} < 0 \). This property holds for general \( \lambda \) as well if \( e(x) \) (or \( u(x) \)) falls of sufficiently fast in the interval around \( x = 1 \). To see this, we calculate the integral in the definition of \( f_{k,-k,q,\lambda} \). The integration from \( \lambda \) to \( \Lambda \) can be replaced by an integration from \( \lambda_1 \) to \( \Lambda \), where \( \lambda_1 = \lambda \) if \( \epsilon_{k,q,\lambda} \epsilon_{k+q,-q,\lambda} > 0 \), otherwise \( \lambda_1 \) is the supremum of all \( s \) with \( e_{k,q,s} e_{k+q,-q,s} = 0 \). In a second step we use that
$e_{k,q,s}$, and $e_{k+q,-q,s}$ are monotonic increasing functions of $s$. Then the second mean value theorem for integrals yields

$$f_{k,-k,q,\lambda} = \int_{\lambda_1}^{\Lambda} \frac{d \ln e_{k,q,s}}{ds} = - \ln e_{k,q,\lambda_2}$$

(3.26)

with some $\lambda_2 > \lambda_1$. This finally yields

$$V_{k,-k,q,\lambda} = 2 \frac{|g_{q,\lambda}|^2}{\lambda_2} \left( \frac{\ln e_{k+q,-q,\lambda_2}}{x_{k+q,-q,\lambda_2}} + \frac{\ln e_{k,q,\lambda_2}}{x_{k,q,\lambda_2}} \right)$$

(3.27)

where $x_{k,q,\lambda} = \frac{(\epsilon_{k,\lambda} - \epsilon_{k+q,\lambda} + \omega_{q,\lambda})}{\lambda}$. Whenever $|\ln e(x)| / |x|$ is a monotonously increasing function of $|x|$, the right hand side is negative. This condition can easily be satisfied. Let us write $e(x)$ in the form $e(x) = \exp(-|x| h(|x|))$. Then the condition is satisfied if $h(x)$ increases monotonously as a function of $x$. This means that the functions $e(x)$ and $u(x)$ have to decay sufficiently fast.

To obtain an explicit expression for the interaction within a Cooper pair, one has to choose a specific form of $e(x)$. If we assume that $e(x)$ drops rapidly from 1 to 0, we can take for simplicity $e(x) = \theta(1 - x) \theta(1 + x)$. This choice of a step function is useful for doing analytical calculations, but as mentioned above it can lead to pathologies in higher orders in $g$. With this choice of $e(x)$ we obtain

$$f_{k,-k,q,\lambda} = \theta(\epsilon_{k,\lambda} - \epsilon_{k+q,\lambda}) \theta(|\epsilon_{k,\lambda} - \epsilon_{k+q,\lambda} + \omega_{q,\lambda} - \lambda|).$$

(3.28)

This yields

$$V_{k,-k,q,\lambda} = - \frac{2 |g_{q,\lambda}|^2}{\epsilon_{k+q,\lambda} - \epsilon_{k,\lambda} + \omega_{q,\lambda}} \theta(|\epsilon_{k+q,\lambda} - \epsilon_{k,\lambda} + \omega_{q,\lambda} - \lambda|).$$

(3.29)

The form (3.29) of the phonon–induced electron–electron interaction is similar to the result obtained by Lenz et al [9], see (1.4), except for the cutoff function. It differs from the result by Fröhlich (1.3). The three expressions agree for $|\epsilon_{k+q,\lambda} - \epsilon_{k,\lambda}| = 0$.

The coupling $c_{q,\lambda}$ can be calculated likewise. We obtain

$$\frac{dc_{q,\lambda}}{d\lambda} = - \frac{1}{2} u(x_{q,\lambda}) \sum_k n_k (\eta_{k,q,\lambda} g_{k+q,-q,\lambda} + \eta_{k+q,-q,\lambda} g_{k,q,\lambda} + \eta_{k-q,\lambda} g_{k-q,q,\lambda} + \eta_{k-q,q,\lambda} g_{k-q,-q,\lambda})$$

$$+ r(x_{q,\lambda}) \frac{d \ln u(x_{q,\lambda})}{d\lambda} c_{q,\lambda}$$

(3.30)

where

$$x_{q,\lambda} = \frac{2 \omega_{q,\lambda}}{\lambda}.$$

(3.31)

With a similar ansatz as above,

$$c_{q,\lambda} = e(x_{q,\lambda}) \tilde{c}_{q,\lambda}$$

(3.32)

we have

$$\frac{d \tilde{c}_{q,\lambda}}{d\lambda} = - \frac{1}{2} \exp(-r(x_{q,\lambda})) \sum_k n_k (\eta_{k,q,\lambda} g_{k+q,-q,\lambda} + \eta_{k+q,-q,\lambda} g_{k,q,\lambda})$$

$$+ \eta_{k-q,\lambda} g_{k-q,q,\lambda} + \eta_{k-q,q,\lambda} g_{k-q,-q,\lambda}).$$

(3.33)
\( c_{q,\lambda} \) can be calculated using the same approximations as above. We obtain

\[
\tilde{c}_{q,\lambda} = \frac{1}{2} |g_{q,\Lambda}|^2 \sum_k n_k \left( \frac{f_{k+q,q,\lambda}}{\epsilon_{k,\lambda} - \epsilon_{k+q,\lambda} - \omega_{q,\lambda}} + \frac{f_{k+q,-q,\lambda}}{\epsilon_{k+q,\lambda} - \epsilon_{k,\lambda} + \omega_{q,\lambda}} \right.
\]

\[
+ \frac{f_{k,-q,q,\lambda}}{\epsilon_{k,\lambda} - \epsilon_{k-q,\lambda} + \omega_{q,\lambda}} + \frac{f_{k-q,q,\lambda}}{\epsilon_{k-q,\lambda} - \epsilon_{k,\lambda} + \omega_{q,\lambda}} \right) \quad (3.34)
\]

where

\[
f_{k,q,\lambda} = \int_{\Lambda} ds \exp(-r(x_{q,s})) \frac{d\epsilon_{k,q,s}}{\epsilon_{k,q,s}}. \quad (3.35)
\]

In the final Hamiltonian these couplings can be treated perturbatively. They yield a renormalization of the phonon frequencies in fourth order in \( g_{q,\Lambda} \). Similarly, the additional terms in the Hamiltonian \( H_I \) which contain couplings of an electron to two phonons yield an additional contribution to the electron–electron interaction of fourth order in \( g \).

### 4 Some explicit results for the Einstein model

The above results are very general and can be applied to any type of electron–phonon interaction, even to any type of interaction between electrons and other bosonic degrees of freedom. In this section we apply our results to the simple Einstein model of phonons. Furthermore we will assume a constant density of states for the electrons near the Fermi surface, and we neglect renormalization effects of the phonon energies. The motivation is to show the reader how well known results can be obtained using the above procedure. We will also show what are the main effects of the new phonon–induced electron–electron interaction (3.29) on the critical temperature.

In the Einstein model one takes phonons with \( \omega_q = \omega_0 \) and \( g_{q,\Lambda} = g_0 V^{-1/2} \). These assumptions yield considerable simplifications in the above formulae. Let us first study the renormalization of the electronic energies \( \epsilon_{k,\lambda} \). From (3.18) we obtain

\[
\epsilon_{k,\lambda} = \epsilon_{k,\Lambda} + g_0^2 V^{-1} \sum_{k'} \frac{n_{k'}}{\epsilon_{k,\lambda} - \epsilon_{k',\lambda} + \omega_0} \theta(|\epsilon_{k,\lambda} - \epsilon_{k',\lambda} + \omega_0| - \lambda)
\]

\[
- g_0^2 V^{-1} \sum_{k'} \frac{1 - n_{k'}}{\epsilon_{k',\lambda} - \epsilon_{k,\lambda} + \omega_0} \theta(|\epsilon_{k',\lambda} - \epsilon_{k,\lambda} + \omega_0| - \lambda), \quad (4.1)
\]

at zero temperature, where the phonon occupation number \( \bar{n}_q = 0 \). As usual we replace the summation over \( k' \) by an integration over \( \epsilon_{k'} \). Furthermore we assume that the density of states is constant, we denote it by \( N(0) \). The integral over \( \epsilon_{k'} \) extends over the whole band and we assume that the band width \( D \) is large compared to \( \omega_0 \) and \( \lambda \). The renormalization of band energies \( \epsilon_{k,\lambda} \) with \( |\epsilon_{k,\lambda}| < \omega_0 \) is then given by

\[
\epsilon_{k,\lambda} = \epsilon_{k,\Lambda} + g_0^2 N(0) \ln \frac{\omega_0 - \epsilon_{k,\lambda} + \max(0, \lambda + \epsilon_{k,\lambda} - \omega_0)}{\omega_0 + \epsilon_{k,\lambda} + \max(0, \lambda - \epsilon_{k,\lambda} - \omega_0)}. \quad (4.2)
\]

For a cutoff \( \lambda \) that is smaller than the phonon frequency \( \omega_0 \) and for \( \epsilon_k \ll \omega_0 \) this yields the well known expression (see e.g. [3])

\[
\epsilon_{k,\lambda} = \frac{\epsilon_{k,\Lambda}}{1 + 2g_0^2 N(0)/\omega_0}. \quad (4.3)
\]

The last expression can be derived in the framework of Eliashberg theory, which is valid under the same conditions as used for the derivation of (4.2), namely \( \omega_0 \ll D \). An important point in
then no longer depends on the cutoff $\lambda$ unity. This is similar to the behaviour of the single particle energies. This part of the interaction $\Delta$ the electron–electron interaction (3.29) has to be taken, Hamiltonian as in BCS-theory. This leads to the usual BCS equation for the energy gap, where

$$\theta = \frac{\epsilon_{k+q} - \epsilon_{k,\lambda} + \omega_0 - \lambda}{2}$$

This factor arises due the fact that for a given $\lambda$ only a part of the electron–phonon interaction has been eliminated. If we choose $\lambda$ to be less than $\omega_0$, this factor is unity. This is similar to the behaviour of the single particle energies. This part of the interaction then no longer depends on the cutoff $\lambda$.

The energy gap and the critical temperature can be estimated if we treat the renormalized Hamiltonian as in BCS-theory. This leads to the usual BCS equation for the energy gap, where the electron–electron interaction (3.29) has to be taken, Hamiltonian as in BCS-theory. This leads to the usual BCS equation for the energy gap, where

$$\Delta(\epsilon_k) = \Delta_k$$

In the isotropic case we can replace the sum over $q$ by an integral over $\epsilon_{k+q}$. Introducing $\Delta(\epsilon_k) = \Delta_k$ and using (3.29) we obtain

$$\Delta(\epsilon) = 2g_0^2 \int d\epsilon' \rho(\epsilon') \frac{1}{\epsilon - \epsilon' + \omega_0} \frac{\Delta(\epsilon')}{2\sqrt{\epsilon'^2 + \Delta(\epsilon')^2}} \tanh\left(\frac{\beta}{2} \sqrt{\epsilon'^2 + \Delta(\epsilon')^2}\right).$$

Here $\rho(\epsilon')$ is the renormalized density of states since (3.3) has been derived from the renormalized Hamiltonian and therefore we have to use renormalized energies. It can be approximated by a constant renormalized density of states $\rho(0) = N(0)/(1 + 2g_0^2 N(0)/\omega_0)$. Using dimensionless quantities $x = \epsilon/\omega_0$, $x' = \epsilon'/\omega_0$, $\tau = (\beta \omega_0)^{-1}$, $\Delta(x) = \Delta(\epsilon)/\omega_0$ we obtain

$$\Delta(x) = \frac{2g_0^2 N(0)/\omega_0}{1 + 2g_0^2 N(0)/\omega_0} \int dx' \frac{\Delta(x')}{|x - x'| + 1} \frac{\tanh(\sqrt{x'^2 + \Delta(x')^2}/\tau)}{\sqrt{x'^2 + \Delta(x')^2}}.$$ (4.6)

This equation can be compared to the usual BCS equation with a constant interaction in an energy interval given by $\omega_0$,

$$1 = 2g_0^2 N(0)/\omega_0 \int_{|x'|<1} dx' \frac{\tanh(\sqrt{x'^2 + \Delta^2}/\tau)}{\sqrt{x'^2 + \Delta^2}}.$$ (4.7)

The main difference between our result (4.6) and the BCS result is the renormalization of the density of states $N(0)$. It is well known (see e.g. the discussion by Allen et al. [4, 5]) that this renormalization is important when the coupling $2g_0^2 N(0)/\omega_0$ is not small. Another important point is the different way the cutoff is introduced. In BCS theory one has to introduce a cutoff by hand, and this is usually done by restricting the electronic energies to an energy interval of width $2\omega_0$ around the Fermi surface. The main motivation for this is that an interaction of the Fröhlich–type (1.3) contains a singularity and becomes repulsive for larger energies. In our case we use a renormalization procedure which leads automatically to a cutoff $\omega_0$ due to the factor $(|x - x'| + 1)$ in the denominator.

(4.6) cannot be solved analytically, but it is easy to solve it numerically. In Fig. 1 we show the solution for $\Delta(x)$ for $2g_0^2 N(0)/\omega_0 = 1/3$ and $2g_0^2 N(0)/\omega_0 = 1.4$ for different temperatures. One
clearly sees that the gap becomes small for large \( x \) and decreases with increasing temperature. The scaled curves in Fig. 1 show that the scaling relation
\[
\Delta(\omega, T) = \Delta(\omega, 0) \frac{\Delta(0, T)}{\Delta(0, 0)}
\]
is satisfied within the numerical accuracy. We have checked this scaling law numerically for values of \( 2g_0^2 N(0)/\omega_0 \) up to 2.5. The scaling law has been found by Scalapino et al [17] (see also [3]) for weak coupling superconductors, whereas in the strong coupling case deviations occurred. But their calculations have been done for more complicated phonon spectra so that a direct comparison with our results is not possible. Scalapino [3] also mentions that even for strong coupling the reduced energy gap \( \Delta(0, T)/\Delta(0, 0) \) as a function of \( T/T_c \) follows very closely the BCS curve for this quantity. We have plotted the reduced energy gap in Fig. 2, again for \( 2g_0^2 N(0)/\omega_0 = 1/3 \) and \( 2g_0^2 N(0)/\omega_0 = 1.4 \). For other values of the coupling strength the curve is the same, it agrees with the BCS curve. The data shown in Fig. 2a show that for small values of the coupling the ratio \( 2\Delta(0, 0)/T_c \) is 3.53 as predicted by BCS-theory. For \( 2g_0^2 N(0) = 1.4 \) we obtain \( \Delta(0, 0) = 0.298\omega_0 \) and \( T_c = 0.161\omega_0 \). In that case the ratio \( 2\Delta(0, 0)/T_c = 3.70 \). This tendency to larger values of \( 2\Delta(0, 0)/T_c \) for strongly coupled systems is well known. The behaviour of \( \Delta(\omega, T) \) for other values of \( 2g_0^2 N(0)/\omega_0 \) is similar. The main difference is the value of \( \Delta(0) \), which shows a strong dependence on \( 2g_0^2 N(0)/\omega_0 \) as expected.

An important point is that the energy gap \( \bar{\Delta}(0) \) is about an order of magnitude (for small couplings \( 2g_0^2 N(0)/\omega_0 \lesssim 0.5 \) a factor of 5) smaller than the value one obtains using (4.7). As a consequence the calculated critical temperature is smaller by roughly the same factor. We show this general feature in Fig. 3, where \( T_c \) is plotted as a function of \( 2g_0^2 N(0)/\omega_0 \) together with the BCS-result. A comparison with the value of the critical temperature calculated with the interaction (1.4) from Lenz and Wegner [9] is not so easy, since in their paper the renormalization of the electronic single particle energies was not taken into account. But the flow equation for \( \epsilon_k \) in their paper is similar to what we obtained, so that one should expect a similar renormalization of the single particle energies. Therefore we put this in by hand. The values for the gap calculated with the same assumptions as above show a behaviour that is very similar to our results. Only for larger values of \( 2g_0^2 N(0)/\omega_0 \) deviations occur. For \( 2g_0^2 N(0)/\omega_0 \gtrsim 1.2 \) the gap for the interaction (1.4) lies below our curve. Furthermore it is interesting to compare our result with results for \( T_c \) obtained in the framework of Eliashberg theory. One famous result is the McMillan–Dynes equation [19, 20], which has in our case the form
\[
T_c = \frac{\omega_0}{1.2} \exp \left( -\frac{1.04(1 + 2g_0^2 N(0)/\omega_0)}{2g_0^2 N(0)/\omega_0} \right).
\]

This curve has also been plotted in Fig. 3. The plot shows that for small and intermediate coupling the agreement is very good, whereas for strong coupling \( (2g_0^2 N(0)/\omega_0 \gtrsim 1.2) \) deviations occur. It is well known that in the strong coupling case the McMillan–Dynes equation is not valid. But in the region where the McMillan–Dynes equation is applicable, we obtain a good agreement. These results clearly show that a renormalization procedure for Hamiltonians like similarity renormalization or flow equations treats the different energy scales in the problem in a satisfactory way.

5 Discussion of the results

Starting from an initial Hamiltonian with an electron–phonon interaction, we calculated an effective Hamiltonian to second order in the electron–phonon coupling. Due to the special structure
of the similarity renormalization scheme, the effective Hamiltonian has a band–diagonal structure. It contains still a small part of the electron–phonon interaction, in addition an effective electron–electron interactions and other couplings. The renormalized single particle energies and the induced interactions are free of divergencies. The actual values of the renormalized quantities depend on the special choice of the infinitesimal unitary transformation, i.e. on the cutoff \( \lambda \) and on the special choice of the function \( u(x) \). On the other hand it is clear that measurable quantities like expectation values of observables should not depend on \( \lambda \) or \( u(x) \). This is not a contradiction, since the unitary transformation applied to the Hamiltonian has to be applied to the observables as well. We have only calculated the effective Hamiltonian, thus we are not able to check this explicitly. But we have shown that the structure of the Hamiltonian does not depend on \( u(x) \). The renormalization of the single particle energies shows no essential dependence on \( u(x) \) if the cutoff \( \lambda \) is sufficiently small. With some additional weak condition on \( u(x) \) (\( u(x) \) has to decay sufficiently fast), \( \text{the induced interaction between two electrons forming a Cooper pair is attractive in the whole parameter space.} \)

Comparing our results for the induced electron–electron interaction with Fröhlich’s result (1.3) shows that the effective interaction obtained by Fröhlich has not only the problem that it contains a divergence, even the sign of the interaction is wrong in some part of the parameter space. In contrast the effective interaction obtained by Lenz and Wegner shows the correct behaviour. Another difference is that in the perturbative expression of Fröhlich the unrenormalized single particle energies enter. As a consequence, the renormalization of the electronic density of states, which has an important effect when one calculates \( T_c \), is not taken into account. As already mentioned in the introduction, a similar behaviour has been observed in the case of induced interactions for other problems as well [11]. The fact that the induced interaction in the general form (3.25) is always attractive can be traced back to the correct treatment of the different energy scales. Whenever large energy differences are treated first, the interaction (3.25) is attractive. If, as in the treatment by Fröhlich [6] or by Bardeen and Pines [7] all energy scales are treated at the same time, \( f_{k,-k,q,\lambda} \) does not depend on \( k \) and \( q \). In that case (3.25) yields a result similar to the one obtained by Fröhlich.

The discussion of the simple model in section 4 has shown, that the similarity renormalization scheme can be successfully applied to the electron–phonon problem. It yields the correct energy scale for single particle excitations and an effective interaction that yields reasonable values for the energy gap or for \( T_c \). But several problems have been left open. The reason was that the simple example was only presented to illustrate how the method works. Applying the similarity renormalization scheme to a more realistic model is possible and in that case a more detailed study should be done. For instance, the renormalization of the phonon frequencies has not been take into account. Furthermore, the renormalization procedure has been performed only to second order in the electron–phonon coupling. The similarity renormalization used here is a perturbative renormalization procedure. Higher corrections to the electron–electron interaction can be calculated systematically. The non–linear electron–phonon interaction in (3.19) e.g. yields a contribution to the induced electron–electron interaction in fourth order. A detailed discussion of such higher terms may be important, since the attractive interaction is known to be a marginal relevant operator in a problem of interacting fermions. Furthermore it is possible to include the initial Coulomb repulsion in the Hamiltonian. It will be modified by the electron–phonon interaction as well, but it does not introduce a new energy scale to the problem. In this way it should be possible to obtain a more complete expression for the electron–electron interaction. We leave these problems for future work.

Let us now compare the similarity renormalization scheme with Wegner’s flow equations for Hamiltonians. A common feature of both approaches is that they work in a Hamiltonian...
framework. As a consequence, we will never be able to obtain effective interactions containing retardation effects. The Hamiltonian is always a hermitian operator and induced interactions never depend on frequency. This is in contrast to methods using path integrals. In a path integral formulation of the present problem one is able to integrate out the phonons completely. The effective theory then contains a time–dependent or frequency–dependent electron–electron interaction. The interaction obtained in this way has nothing to do with the effective interaction in our approach. But this does not mean that one is not able to describe the decay of metastable states or the lifetime of some bound state in a Hamiltonian framework. In a recent paper on dissipative quantum systems this point has been discussed in great detail [13].

Both approaches, flow equations and the similarity renormalization scheme use a continuous unitary transformation to construct an effective Hamiltonian from a given initial Hamiltonian. This is typical for renormalization procedures. Already in the old formulation of renormalization Wilson [21] used a sequence of unitary transformations to calculate an effective Hamiltonian with a small cutoff. But the goal is different in both cases. In the flow equation approach, the Hamiltonian is transformed into a block–diagonal form. In the present case of the electron–phonon coupling the number of phonons is conserved in each block. But the Hamiltonian may still contain off–diagonal elements between two states with a large energy difference. In impurity models as in the Anderson impurity model [11], or in dissipative quantum systems, where a small system is coupled to a bath [13], this does not happen, but there is no way to exclude such matrix elements in general. On the other hand, Wegner’s flow equations can be used to diagonalize a given Hamiltonian approximately [10]. If in that case one performs the integration of the flow equations only to a finite value of the flow parameter, one obtains a Hamiltonian with a band–diagonal structure. The far off–diagonal matrix elements do not vanish, but they become exponentially small.

When a Hamiltonian is transformed into a block–diagonal form, some interactions like the electron–phonon coupling in our case are eliminated completely. It is even possible to choose the continuous unitary transformation in such a way that some of the possible new induced interactions do not occur. This never happens in the similarity renormalization scheme. It has been designed to obtain an effective Hamiltonian with a band diagonal structure. As a consequence, the final Hamiltonian will contain any possible induced interaction. An example is the coupling $c_{q,\lambda}$ that occurs in our effective Hamiltonian. It has been avoided explicitly in the paper by Lenz and Wegner. On the other hand, all the induced couplings in the similarity renormalization scheme are free of divergencies. As a consequence, a systematic expansion of matrix elements in a coupling constant is always possible and well defined. The induced interactions one obtains in the flow equation approach are less singular than similar expression obtained perturbatively, but they may still contain some weak divergencies. An expansion in a small coupling constant is therefore not automatically well defined. But due to the simpler structure of the final Hamiltonian, it is much easier to calculate expectation values of observables or dynamical properties in the flow equation approach. Therefore, depending on the problem one wants to treat, one or the other approach will have advantages.

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Figure captions

Fig. 1. The energy gap as a function of $\omega$ for $2g_0^2N(0)/\omega_0 = 1/3$ (a) and $2g_0^2N(0)/\omega_0 = 1.4$ (b) and different values of $T$. In Fig. 1a the temperatures are $T = 0$ (solid line), $T = 0.0075\omega_0$ (dashed line), and $T = 0.01\omega_0$ (long dashed line). In Fig. 2a the temperatures are $T = 0$ (solid line), $T = 0.12\omega_0$ (dashed line), and $T = 0.15\omega_0$ (long dashed line). The scaled curves (using (4.8)) lie on top of the solid line.

Fig. 2. The gap at $\omega = 0$ as a function of temperature for $2g_0^2N(0)/\omega_0 = 1/3$ (a) and $2g_0^2N(0)/\omega_0 = 1.4$ (b).

Fig. 3. $T_c$ as a function of the coupling $2g_0^2N(0)/\omega_0$. The solid line shows the result from (4.6). The dashed line is the result obtained with the interaction of Lenz and Wegner, as explained in the text. The long dashed line shows the result of BCS–theory, obtained from (4.7), and the dashed-dotted line is a plot of the McMillan–Dynes formula.
Fig. 1a

\[ \frac{\Delta(\omega, T)}{\omega_0} \] versus \[ \frac{\omega}{\omega_0} \]
Fig. 1b

The figure shows a graph of $D(\omega, T)/\omega_0$ as a function of $\omega/\omega_0$. The x-axis represents $\omega/\omega_0$ ranging from 0.01 to 10.00, and the y-axis represents $D(\omega, T)/\omega_0$ ranging from 0.00 to 0.40.

The graph includes multiple curves, each for different values of $T$, demonstrating the behavior of $D(\omega, T)/\omega_0$ over the specified range of $\omega/\omega_0$. The curves show a decreasing trend as $\omega/\omega_0$ increases, indicating a decrease in $D(\omega, T)/\omega_0$.
Fig. 2a
Fig. 2b

\[ \Delta(0,T) / \omega_0 \]

\[ T / \omega_0 \]
Fig. 3