Fibre bundle formulation of nonrelativistic quantum mechanics

I. Introduction. The evolution transport

Bozhidar Z. Iliev * † ‡

Short title: Bundle quantum mechanics: I

Basic ideas: → March 1996
Began: → May 19, 1996
Ended: → July 12, 1996
Revised: → December 1996 – January 1997,
Revised: → April 1997, September 1998
Last update: → October 15, 1998
Composing part I: → September 23, 1997
Extracting part I: → October 4, 1997
Updating part I: → October 16, 1998
Produced: → September 14, 2018

LANL xxx archive server E-print No.: quant-ph/9803084

Subject Classes:
Quantum mechanics; Differential geometry

1991 MSC numbers: 81P05, 81P99, 81Q99, 81S99
1996 PACS numbers: 02.40.Ma, 04.60.-m, 03.65.Ca, 03.65.Bz

Key-Words:
Quantum mechanics; Geometrization of quantum mechanics;
Fibre bundles

*Department Mathematical Modeling, Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, Boul. Tzarigradsko chaussée 72, 1784 Sofia, Bulgaria
†E-mail address: bozho@inrne.bas.bg
‡URL: http://www.inrne.bas.bg/mathmod/bozhome/
Bozhidar Z. Iliev: Bundle quantum mechanics. I

Contents

1 Introduction 1

2 Evolution of pure quantum states (review) 4

3 Linear transports along paths and Hilbert fibre bundles 6

4 The Hilbert bundle description of quantum mechanics 9

5 The (bundle) evolution transport 17

6 Conclusion 18

References 19

This article ends at page 19
Abstract

We propose a new systematic fibre bundle formulation of nonrelativistic quantum mechanics. The new form of the theory is equivalent to the usual one but it is in harmony with the modern trends in theoretical physics and potentially admits new generalizations in different directions. In it a pure state of some quantum system is described by a state section (along paths) of a (Hilbert) fibre bundle. Its evolution is determined through the bundle (analogue of the) Schrödinger equation. Now the dynamical variables and the density operator are described via bundle morphisms (along paths). The mentioned quantities are connected by a number of relations derived in this work.

The present first part of this investigation is devoted to the introduction of basic concepts on which the fibre bundle approach to quantum mechanics rests. We show that the evolution of pure quantum-mechanical states can be described as a suitable linear transport along paths, called evolution transport, of the state sections in the Hilbert fibre bundle of states of a considered quantum system.
1. Introduction

Usually the standard nonrelativistic quantum mechanics of pure states is formulated in terms of vectors and operators in a Hilbert space \[\mathcal{H}\] This is in discrepancy and not in harmony with the new trends in (mathematical) physics \[\mathcal{H}\] in which the theory of fibre bundles \[\mathcal{H}\], in particular vector bundles \[\mathcal{H}\], is essentially used. This paper (and its further continuation(s)) is intended to incorporate the quantum theory in the family of fundamental physical theories based on the background of fibre bundles.

The idea of geometrization of quantum mechanics is an old one (see, e.g., \[\mathcal{H}\] and the references therein). A good motivation for such approach is given in \[\mathcal{H}\]. Different geometrical structures in quantum mechanics were introduced \[\mathcal{H}\], for instance such as inner products(s) \[\mathcal{H}\], (linear) connection \[\mathcal{H}\], symplectic structure \[\mathcal{H}\], complex structure \[\mathcal{H}\], etc. The introduction of such structures admits a geometrical treatment of some problems, for instance, the dynamics in the (quantum) phase space \[\mathcal{H}\] and the geometrical phase \[\mathcal{H}\]. In a very special case a gauge structure, i.e. a parallel transport corresponding to a linear connection, in quantum mechanics is pointed out in \[\mathcal{H}\]. For us this work is remarkable with the fact that the equation (10) found in it is a very ‘ancient’ special version of the transformation law for the matrix-bundle Hamiltonian, derived in this investigation, which, together with the bundle (analog of the) Schrödinger equation, shows that (up to a constant) with respect to the quantum evolution the Hamiltonian plays the rôle of a gauge field (connection). In \[\mathcal{H}\] one finds different (vector) bundles defined on the base of the (usual) Hilbert space of quantum mechanics or its modifications. In these works different parallel transports in the corresponding bundles are introduced too.

A general feature of all of the references above-cited is that in them all geometric concepts are introduced by using in one or the other way the accepted mathematical foundation of quantum mechanics, viz. a suitable Hilbert or projective Hilbert space and operators acting in it. The Hilbert space may be extended in a certain sense or replaced by a more general space, but this does not change the main ideas. One of the aims of this work is namely to change this mathematical background of quantum mechanics.

Separately we have to mention the approach of Prugovečki to the quantum theory, a selective summary of which can be found in \[\mathcal{H}\] (see also the references therein) and in \[\mathcal{H}\]. It can be characterized as ‘stochastic’ and ‘bundle’. The former feature will not be discussed in the present investigation; thus we lose some advantages of the stochastic quantum theory to which we shall return elsewhere. The latter ‘part’ of the Prugovečki’s approach has some common aspects with our present work but, generally, it is essentially different. For instance, in both cases the quantum evolution from point to point (in space-time) is described via a kind of (parallel or generic
linear) transport (along paths) in a suitable Hilbert fibre bundle. But the notion of a ‘Hilbert bundle’ in our and Prugovečki’s approach is different nevertheless that in both case the typical (standard) fibre is practically the same (when one and the same theory is concerned).Besides, we need not even to introduce the Poincaré (principal) fibre bundle over the space-time or the phase space which play an important rôle in Prugovečki’s theory. Also we have to notice that the used in it concepts of quantum and parallel transport are special cases of the notion of a ‘linear transport along paths’ introduced in [?, ?]. The application of the last concept, which is accepted in the present investigation, has a lot of advantages, significantly simplifies some proofs and makes certain results ‘evident’ or trivial (e.g. the last part of section 2 and the whole section 4 of [?]). At last, at the present level (nonrelativistic quantum mechanics) our bundle formulation of the quantum theory is insensitive with respect to the space-time curvature. A detail comparison of Prugovečki’s and our approaches to the quantum theory will be done elsewhere.

Another geometric approach to quantum mechanics is proposed in [?] and partially in [?], the letter of which is, with a few exceptions, almost a review of the former. These works suggest two ideas which are quite important for us. First, the quantum evolution could be described as a (kind of) parallel transport in an infinitely dimensional (Hilbert) fibre bundle over the space-time. And second, the concrete description of a quantum system should explicitly depend on (the state of) the observer with respect to which it is depicted (or who ‘investigates’ it). These ideas are incorporated and developed in our work.

From the known to the author literature, the closest to the approach developed in this work is [?] which contains an excellent motivation for applying the fibre bundle technique to nonrelativistic quantum mechanics. Generally said, in this paper the evolution of a quantum system is described as a ‘generalized parallel transport’ of appropriate objects in a Hilbert fibre bundle over the 1-dimensional manifold $\mathbb{R}_+ := \{ t : t \in \mathbb{R}, t \geq 0 \}$, interpreted as a ‘time’ manifold (space). We shall comment on reference [?] in the second part of this series, after developing the formalism required for its analysis. Besides, the paper [?] contains an excellent motivation for applying the apparatus of fibre bundle theory to quantum mechanics.

An attempt to formulate quantum mechanics in terms of a fibre bundle over the phase space is made in [?]. Regardless of some common features, this paper is quite different from the present investigation. We shall comment on it later. In particular, in [?] the gauge structure of the arising theory is governed by a non-dynamical connection related to the symplectic structure of the system’s phase space, while in this work analogous structure

---

The author thanks J.F. Coriñena (University of Zaragoza, Zaragoza, Spain) for drawing his attention to reference [?] in May 1998.
(linear transport along paths) is uniquely connected with system’s Hamiltonian, playing here the rôle of a gauge field itself.

The present work is a direct continuation of the considerations in [?] which paper, in fact, may be regarded as its preliminary version. Here we suggest a purely fibre bundle formulation of the nonrelativistic quantum mechanics. This new form of the theory is entirely equivalent to the usual one, which is a consequence or our step by step equivalent reformulation of the quantum theory. The bundle description is obtained on the base of the developed by the author theory of transports along paths in fibre bundles [?, ?, ?, ?], generalizing the theory of parallel transport, which is partially generalized here to the infinitely dimensional case.

The main object in quantum mechanics is the Hamiltonian (operator) which, through the Schrödinger equation, governs the evolution of a quantum system [?, ?, ?, ?]. In our novel approach its rôle is played by a suitable linear transport along paths in an appropriate (Hilbert) fibre bundle. It turns out that up to a constant the matrix-bundle Hamiltonian, which is uniquely determined by the Hamiltonian in a given field of bases, coincides with the matrix of the coefficients of this transport (cf. an analogous result in [?, sect. 5]). This fact, together with the replacement of the usual Hilbert space with a Hilbert fibre bundle, is the corner-stone for the possibility for the new formulation of the nonrelativistic quantum mechanics.

The present first part of our investigation is organized as follows.

In Sect. 2 are reviewed some facts from the quantum mechanics and partially is fixed our notation. Here, as well as throughout this work, we follow the established in the physical literature degree of rigor. But, if required, the present work can reformulate to meet the present-day mathematical standards. For this purpose one can use, for instance, the quantum-mechanical formalism described in [?] or in [?] (see also [?]).

In Sect. 3 we recall the notion of a linear transport along paths in vector fibre bundles and make certain remarks concerning the special case of a Hilbert bundle.

Sect. 4 begins the building of the new bundle approach to quantum mechanics. Here the concept of a Hilbert fibre bundle of the states corresponding to a quantum system is introduced. The analogue of the state vector now is the state section (along paths). We present here also some technical (mathematical) details, such as ones concerning (Hermitian) bundle metric, Hermitian and unitary maps etc.

In Sect. 5 is proved that in the new description the evolution operator of a quantum system is (equivalently) replaced by a suitable linear transport along paths, called evolution transport or a bundle evolution operator.

The paper closes with Sect. 6.
2. Evolution of pure quantum states (review)

In quantum mechanics [?], [?], [?] a pure state of a quantum system is described by a state vector \( \psi(t) \) (in Dirac’s [?] notation \( |t\rangle \)) generally depending on the time \( t \in \mathbb{R} \) and belonging to a Hilbert space \( \mathcal{F} \) (specific to any concrete system) endowed with a nondegenerate Hermitian scalar product \( \langle \cdot | \cdot \rangle : \mathcal{F} \times \mathcal{F} \to \mathbb{C} \). For any two instants of time \( t_2 \) and \( t_1 \) the corresponding state vectors are connected by the equality

\[
\psi(t_2) = U(t_2, t_1) \psi(t_1) \tag{2.1}
\]

where \( U \) is the evolution operator of the system [?, chapter IV, Sect. 3.2]. It is supposed to be linear and unitary, i.e.

\[
U(t_2, t_1)(\lambda \psi(t_1) + \mu \xi(t_1)) = \lambda U(t_2, t_1)(\psi(t_1)) + \mu U(t_2, t_1)(\xi(t_1)), \tag{2.2}
\]

\[
U^\dagger(t_1, t_2) = U^{-1}(t_2, t_1), \tag{2.3}
\]

for any \( \lambda, \mu \in \mathbb{C} \) and state vectors \( \psi(t), \xi(t) \in \mathcal{F} \), and such that for any \( t \)

\[
U(t, t) = \text{id}_\mathcal{F}. \tag{2.4}
\]

Here \( \text{id}_X \) means the identity map of a set \( X \) and the dagger (\( \dagger \)) denotes Hermitian conjugation, i.e. if \( \varphi, \psi \in \mathcal{F} \) and \( A : \mathcal{F} \to \mathcal{F} \), then \( A^\dagger \) is defined by

\[
\langle A^\dagger \varphi | \psi \rangle = \langle \varphi | A \psi \rangle. \tag{2.5}
\]

In particular \( U^\dagger \) is defined by \( \langle U^\dagger(t_1, t_2) \varphi(t_2) | \psi(t_1) \rangle = \langle \varphi(t_2) | U(t_2, t_1) \psi(t_1) \rangle \).

So, knowing \( \psi(t_0) = \psi_0 \) for some moment \( t_0 \), one knows the state vector for any moment \( t \) as \( \psi(t) = U(t, t_0) \psi(t_0) = U(t, t_0) \psi_0 \).

Let \( \mathcal{H}(t) \) be the Hamiltonian (function) of the system, i.e. its total energy operator. It generally depends on the time \( t \) explicitly\(^2\) and it is supposed to be a Hermitian operator, i.e. \( \mathcal{H}^\dagger(t) = \mathcal{H}(t) \). The Schrödinger equation (see [?, § 27] or [?, chapter V, Sec. 3.1])

\[
i \hbar \frac{d\psi(t)}{dt} = \mathcal{H}(t) \psi(t), \tag{2.6}
\]

\(^2\)For some (e.g. unbounded) states the system’s state vectors form a more general space than a Hilbert one. This is insignificant for the following presentation.

\(^3\)Of course, the Hamiltonian depends also on the observer with respect to which the evolution of the quantum system is described. This dependence is usually implicitly assumed and not written explicitly [?, ?]. This deficiency will be eliminated in a natural way further in the present work. The Hamiltonian can also depend on other quantities, such as the (operators of the) system’s generalized coordinates. This possible dependence is insignificant for our investigation and will not be written explicitly.
with \( i \in \mathbb{C} \) and \( \hbar \) being respectively the imaginary unit and the Plank’s constant (divided by \( 2\pi \)), together with some initial condition

\[
\psi(t_0) = \psi_0 \in \mathcal{F}
\]

is postulated in the quantum mechanics.

The substitution of (2.1) into (2.6) shows that there is a 1:1 correspondence between \( \mathcal{U} \) and \( \mathcal{H} \) described by

\[
\hbar i \frac{\partial \mathcal{U}(t, t_0)}{\partial t} = \mathcal{H}(t) \circ \mathcal{U}(t, t_0), \quad \mathcal{U}(t_0, t_0) = \text{id}_\mathcal{F}
\]

where \( \circ \) denotes composition of maps. If \( \mathcal{U} \) is given, then

\[
\mathcal{H}(t) = \hbar i \frac{\partial \mathcal{U}(t, t_0)}{\partial t} \circ \mathcal{U}^{-1}(t, t_0) = \hbar i \frac{\partial \mathcal{U}(t, t_0)}{\partial t} \circ \mathcal{U}(t_0, t),
\]

where we have used the equality

\[
\mathcal{U}^{-1}(t_2, t_1) = \mathcal{U}(t_1, t_2)
\]

which follows from (2.1) (see also below (2.10) or Sect. 5). Conversely, if \( \mathcal{H} \) is given, then [?, chapter VIII, § 8] \( \mathcal{U} \) is the unique solution of the integral equation

\[
\mathcal{U}(t, t_0) = \text{id}_\mathcal{F} + \frac{1}{\hbar i} \int_{t_0}^{t} \mathcal{H}(\tau) \mathcal{U}(\tau, t_0) d\tau,
\]

i.e. we have

\[
\mathcal{U}(t, t_0) = \exp \int_{t_0}^{t} \frac{1}{\hbar i} \mathcal{H}(\tau) d\tau,
\]

where \( \exp \int_{t_0}^{t} \cdots d\tau \) is the chronological (called also T-ordered, P-ordered or path-ordered) exponent (defined, e.g. as the unique solution of the initial-value problem (2.8); see also [?, equation (1.3)]). From here follows that the Hermiticity of \( \mathcal{H} \), \( \mathcal{H}^\dagger = \mathcal{H} \), is equivalent to the unitarity of \( \mathcal{U} \) (see (2.3)).

Let us note that for mathematically rigorous understanding of the derivations in (2.6), (2.8), and (2.9), as well as of the chronological (path-ordered) exponent in (2.10), one has to apply the developed in [?] mathematical apparatus, but this is out of the subject of the present work.

If \( \mathcal{A}(t) : \mathcal{F} \to \mathcal{F} \) is the (linear) operator corresponding to a dynamical variable \( \mathcal{A} \) at the moment \( t \), then the mean value (= the mathematical expectation) which it assumes at a state described by a state vector \( \psi(t) \) with a finite norm is

\[
\langle \mathcal{A}(t) \rangle^\psi := \frac{\langle \psi(t)|\mathcal{A}(t)\psi(t) \rangle}{\langle \psi(t)|\psi(t) \rangle}.
\]

\(^4\)The physical meaning of \( \mathcal{U} \) as a propagation function, as well as its explicit calculation (in component form) via \( \mathcal{H} \) can be found, e.g., in [?, § 21, § 22]
Often the operator $A$ can be chosen to be independent of the time $t$. (This is possible, e.g., if $A$ does not depend on $t$ explicitly [?, chapter VII, § 9] or if the spectrum of $A$ does not change in time [?, chapter III, sect. 13].) If this is the case, it is said that the system’s evolution is depicted in the Schrödinger picture of motion [?, § 28], [?, chapter VII, § 9].

3. Linear transports along paths and Hilbert fibre bundles

The general theory of linear transports along paths in vector bundles is developed at length in [?, ?]. In the present investigation we shall need only a few definitions and results from these papers when the bundle considered is a Hilbert one (see below definition 3.2). To their partial introduction and motivation is devoted the current section.

Let $(E,\pi,B)$ be a complex vector bundle [?] with bundle (total) space $E$, base $B$, projection $\pi: E \to B$, and isomorphic fibres $\pi^{-1}(x) \subset E$, $x \in B$. Let $E$ be the (standard, typical) fibre of the bundle, i.e. a vector space to which all $\pi^{-1}(x)$, $x \in B$ are homeomorphic (isomorphic). By $J$ and $\gamma: J \to B$ we denote, respectively, a real interval and path in $B$.

**Definition 3.1.** A linear transport along paths in the bundle $(E,\pi,B)$ is a map $L$ assigning to any path $\gamma$ a map $L^\gamma_{s\to t}$, transport along $\gamma$, such that

$$L^\gamma_{s\to t}: \pi^{-1}(\gamma(s)) \to \pi^{-1}(\gamma(t)) \quad s,t \in J,$$

(3.1)

called transport along $\gamma$ from $s$ to $t$, has the properties:

$$L^\gamma_{s\to t} \circ L^\gamma_{r\to s} = L^\gamma_{r\to t}, \quad r,s,t \in J, \quad (3.2)$$

$$L^\gamma_{s\to s} = \text{id}_{\pi^{-1}(\gamma(s))}, \quad s \in J, \quad (3.3)$$

$$L^\gamma_{s\to t}(\lambda u + \mu v) = \lambda L^\gamma_{s\to t}u + \mu L^\gamma_{s\to t}v, \quad \lambda,\mu \in \mathbb{C}, \quad u,v \in \pi^{-1}(\gamma(s)), \quad (3.4)$$

where $\circ$ denotes composition of maps and $\text{id}_X$ is the identity map of a set $X$.

**Remark 3.1.** Equations (3.2) and (3.3) mean that $L$ is a transport along paths in the bundle $(E,\pi,B)$ [?, definition 2.1], while (3.4) specifies that it is linear [?, equation (2.8)]. In the present paper only linear transports will be used.

5 All of our definitions and results hold also for real vector bundles. Most of them are valid for vector bundles over more general fields too but this is inessential for the following.
This definition generalizes the concept of a parallel transport in the theory of (linear) connections (see [?, ?] and the references therein for details and comparison).

A few comments on definition 3.1 are now in order. According to equation (3.1), a linear transport along paths may be considered as a path-depending connection: it establishes a fibre (isomorphic - see below) correspondence between the fibres over the path along which it acts. By virtue of equation (3.4) this correspondence is linear. Such a condition is a natural one when vector bundles are involved, it simply represents a compatibility condition with the vectorial structure of the bundle (see [?, sect. 2.3] for details). Equation (3.3) is a formal realization of our intuitive and naive understanding that if we ‘stand’ at some point of a path without ‘moving’ along it, then ‘nothing’ must happen with the fibre over that point. This property fixes a 0-ary operation in the set of (linear) transports along paths, defining in it the ‘unit’ transport. At last, the equality (3.2), which may be called a group property of the (linear) transports along paths, is a rigorous expression of the intuitive representation that the ‘composition’ of two (linear) transports along one and the same path must be a (linear) transport along the same path.

In general, different forms of (3.1)–(3.4) are well know properties of the parallel transports generated by (linear) connections (see [?]). By this reason these transports turn to be special cases of the general (linear) transport along paths [?, theorem 3.1]. In particular, comparing definition 3.1 with [?, definition 2.1] and taking into account [?, proposition 4.1], we conclude that special types of linear transports along paths are: the parallel transport assigned to a linear connection (covariant differentiation) of the tensor algebra of a manifold [?, ?], Fermi-Walker transport [?, ?], Fermi transport [?], Truesdell transport [?, ?], Jaumann transport [?], Lie transport [?, ?], the modified Fermi-Walker and Frenet-Serret transports [?], etc. Consequently definition 3.1 is general enough to cover a list of important transports used in theoretical physics and mathematics. Thus studying the properties of the linear transports along paths we can make corresponding conclusions for any one of the transports mentioned.

From (3.2) and (3.3) we get that $L_{s \rightarrow t}^{\gamma}$ are invertible and

$$
(L_{s \rightarrow t}^{\gamma})^{-1} = L_{t \rightarrow s}^{\gamma}, \quad s, t \in J.
$$

Hence the linear transports along paths are in fact linear isomorphisms of the fibres over the path along which they act.

---

The concept of linear transport along paths in vector bundles can be generalized to the transports along paths in arbitrary bundles [?] and to transports along maps in bundles [?]. An interesting considerations of the concept of (parallel) ‘transport’ (along closed paths) in connection with homotopy theory and the classification problem of bundles can be found in [?]. These generalizations will not be used in the present work.
The following two propositions establish the general structure of linear transports along paths.

**Proposition 3.1.** A map (3.1) is a linear transport along \( \gamma \) from \( s \) to \( t \) for every \( s, t \in J \) if and only if there exist an isomorphic with \( \pi^{-1}(x), \ x \in B \) vector space \( V \) and family of linear isomorphisms \( \{F(s; \gamma): \pi^{-1}(\gamma(s)) \to V, \ s \in J\} \) such that

\[
L^\gamma_{s \to t} = F^{-1}(t; \gamma) \circ F(s; \gamma), \quad s, t \in J. \tag{3.6}
\]

**Proof.** If (3.1) is a linear transport along \( \gamma \) from \( s \) to \( t \), then fixing some \( s_0 \in J \) and using (3.3) and (3.5), we get \( L^\gamma_{s \to t} = L^\gamma_{s_0 \to t} \circ L^\gamma_{s \to s_0} = (L^\gamma_{t \to s_0})^{-1} \circ L^\gamma_{s \to s_0} \). So (3.6) holds for \( V = \pi^{-1}(\gamma(s_0)) \) and \( F(s; \gamma) = L^\gamma_{s \to s_0} \). Conversely, if (3.6) is valid for some linear isomorphisms \( F(s; \gamma) \), then a straightforward calculation shows that it converts (3.2) and (3.3) into identities and (3.4) holds due to the linearity of \( F(s; \gamma) \). \( \square \)

**Proposition 3.2.** Let in the vector bundle \( (E, \pi, B) \) be given linear transport along paths with a representation (3.6) for some vector space \( V \) and linear isomorphisms \( F(s; \gamma): \pi^{-1}(\gamma(s)) \to V, \ s \in J \). Then for a vector space \( \ast V \) there exist linear isomorphisms \( \ast F(s; \gamma): \pi^{-1}(\gamma(s)) \to \ast V, \ s \in J \) for which

\[
L^\gamma_{s \to t} = \ast F^{-1}(t; \gamma) \circ \ast F(s; \gamma), \quad s, t \in J. \tag{3.7}
\]

iff there exists a linear isomorphism \( D(\gamma): V \to \ast V \) such that

\[
\ast F(s; \gamma) = D(\gamma) \circ F(s; \gamma), \quad s \in J. \tag{3.8}
\]

**Proof.** If (3.8) holds, then substituting \( F(s; \gamma) = D^{-1}(\gamma) \circ \ast F(s; \gamma) \) into (3.6), we get (3.7). Vice versa, if (3.7) is valid, then from its comparison with (3.6) follows that \( D(\gamma) = \ast F(t; \gamma) \circ (F(t; \gamma))^{-1} = \ast F(s; \gamma) \circ (F(s; \gamma))^{-1} \) is the required (independent of \( s, t \in J \)) isomorphism. \( \square \)

The above definition and results for linear transports along paths deal with the general case concerning arbitrary vector bundles and are therefore insensitive to the dimensionality of the bundle’s base or fibres. Below we point out some peculiarities of the case of a Hilbert bundle whose fibres are generally infinitely dimensional.

**Definition 3.2.** A Hilbert fibre bundle is a fibre bundle whose fibres are homeomorphic Hilbert spaces or, equivalently, whose (standard) fibre is a Hilbert space.

In the present investigation we shall show that the Hilbert bundles can be taken as a natural mathematical framework for a geometrical formulation of quantum mechanics. For linear transports in a Hilbert bundle are valid.
all results of [?, ?, ?] with a possible exception of the ones in which (local) bases in the fibres are involved. The cause for this is that the dimension of a Hilbert space is (generally) infinity. So, there arise problems connected with the convergence or divergence of the corresponding sums or integrals. Below we try to avoid these problems and to formulate our assertions and results in an invariant way.

Of course, propositions 3.1 and 3.2 remain valid on Hilbert bundles; the only addition is that the vector spaces $V$ and $\star V$ are now Hilbert spaces.

Below, in Sect. 4 (see below the paragraph after equation (4.18)), we shall establish a result specific for the Hilbert bundles that has no analogue in the general theory: a transport along paths is Hermitian if and only if it is unitary. This assertion is implicitly contained in [?, sect. 3] (see the paragraph after equation (3.6) in it).

In [?, sect. 3] are introduced the so-called normal frames for a linear transport along paths as a (local) field of bases in which (on some set) the matrix of the transport is unit. Further in this series [?] we shall see that the normal frames realize the Heisenberg picture of motion in the Hilbert bundle formulation of quantum mechanics.

4. The Hilbert bundle description of quantum mechanics

As we shall see in this investigation, the Hilbert bundles provide a natural mathematical framework for a geometrical formulation of quantum mechanics. In it all quantum-mechanical quantities, such as Hamiltonians, observables, wavefunctions, etc., have an adequate description. For instance, the evolution of a systems is described as an appropriate (parallel or, more precisely, linear) transport of system’s state sections along some path. We have to emphasize on the fact that the new bundle formulation of quantum mechanics and the conventional one are completely equivalent at the present stage.

Before going on, we want to mention several works in which attempts are made for a (partial) formulation of nonrelativistic quantum mechanics in terms of fibre bundles.

It seems that for the first time the real bundle approach to quantum mechanics is developed in [?] where the single Hilbert space of quantum mechanics is replaced with an infinitely many copies of it forming a bundle space over the 1-dimensional ‘time’ manifold (i.e. over $\mathbb{R}_+$). In this Hilbert fibre bundle the quantum evolution is (equivalently) described as a kind of ‘parallel’ transport of appropriate objects over the bundle’s base.

Analogous construction, a Hilbert bundle over the system’s phase space, is used in the Prugovečki’s approach to quantum theory (see, e.g. the references in [?]).
In [?] is first mentioned about the gauge, i.e. linear connection, structure in quantum mechanics. That structure is pointed to be connected with the system’s Hamiltonian. This observation will find natural explanation in our work.

Some ideas concerning the interpretation of quantum evolution as a kind of a ‘parallel’ transport in a Hilbert bundle can also be found in [?, ?].

After this introduction, we want to present some non-exactly rigorous ideas and statements whose only purpose is the motivation for applying the fibre bundle formalism to quantum mechanics. Another excellent arguments and motives confirming this approach are given in [?].

Let $M$ be a differentiable manifold, representing in our context the space in which the (nonrelativistic) quantum-mechanical objects ‘live’, i.e. the usual 3-dimensional coordinate space (isomorphic to $\mathbb{R}^3$ with the corresponding structures). Let $\gamma: J \to M$, $J$ being an $\mathbb{R}$-interval, be the trajectory of an observer describing the behaviour of a quantum system at any moment $t \in J$ by a state vector $\Psi_{\gamma}(t)$ depending on $t$ and, possibly, on $\gamma$. For a fixed point $x = \gamma(t) \in M$ the variety of state vectors describing a quantum system and corresponding to different observers form a Hilbert space $F_{\gamma(t)}$ which depends on $\gamma(t) = x$, but not on $\gamma$ and $t$ separately.

Remark 4.1. As we said above in footnote 7, the next considerations are completely valid mathematically if $M$ is an arbitrary differentiable manifold and $\gamma$ is a path in it. In this sense $M$ and $\gamma$ are free parameters in our theory and their concrete choice is subjected only to physical reasons, first of all, ones requiring adequate physical interpretation of the resulting theory. Typical candidates for $M$ are: the 3-dimensional Euclidean space $\mathbb{E}^3$ or $\mathbb{R}^3$, the 4-dimensional Minkowski space $M^4$ of special relativity or the Riemannian space $V_4$ of general relativity, the system’s configuration or phase space, the ‘time’ manifold $\mathbb{R}_+: = \{a : a \in \mathbb{R}, a > 0\}$, etc. Correspondingly, $\gamma$ obtains interpretation as particle’s trajectory, its world line, and so on. The degenerate case when $M$ consists of a single point corresponds (up to an isomorphism - see below) to the conventional quantum mechanics. Throughout this work we most often take $M = \mathbb{R}^3$ as a natural choice corresponding to the non-relativistic case investigated here but, as we said, this is not required by necessity. Elsewhere we shall see that $M = M^4$ or $M = V_4$ are

Note 1. In the following $M$ can naturally be considered also as the Minkowski space-time of special relativity. In this case the below-defined observer’s trajectory $\gamma$ is his world line. But we avoid this interpretation because only the nonrelativistic case is investigated here. It is important to be noted that mathematically all of what follows is valid in the case when by $M$ is understood an arbitrary differentiable manifold. The physical interpretation of these cases will be given elsewhere.

Note 2. In this way we introduce the (possible) explicit dependence of the description of a system’s state on the concrete observer with respect to which it is determined.

Note 3. If there exists a global time, as in the nonrelativistic quantum mechanics, the parameter $t \in J$ can be taken as such. Otherwise by $t$ we have to understand the local (‘proper’ or ‘eigen-’) time of a concrete observer.
natural choices in the relativistic region. An expanded commend on these problems will be given in the concluding part of this series. Here we want to note only that the interpretation of $\gamma$ as an observer’s (particle’s) trajectory or world line, as accepted in this work, is reasonable but not necessary one. May be more adequate is to interpret $\gamma$ as a mean (in quantum-mechanical sense) trajectory of some point particle but this does not change anything in the mathematical structure of the bundle approach proposed here.

The spaces $F_{\gamma(t)}$ must be isomorphic as, from physical view-point, they simply represent the possible variety of state vectors from different positions. In this way over $M$ arises a natural bundle structure, viz. a Hilbert bundle $(F, \pi, M)$ with a total space $F$, projection $\pi: F \to M$ and isomorphic fibres $\pi^{-1}(x) := F_x$. Since $F_x$, $x \in M$ are isomorphic, there exists a Hilbert space $F$ and (linear) isomorphisms $l_x: F_x \to F$, $x \in M$. Mathematically $F$ is the typical (standard) fibre of $(F, \pi, M)$. (Note that we do not suppose local triviality, i.e. that for any $x \in M$ there is a neighborhood $W \ni x$ in $M$ such that $\pi^{-1}(W)$ is homeomorphic to $F \times W$.)$ The maps $\Psi_{\gamma}: J \to \pi^{-1}(\gamma(J))$ can be considered as sections over any part of $\gamma$ without self-intersections (see below).

Now a natural question arises: how the quantum evolution in time in the bundle constructed is described? There are two almost ‘evident’ ways to do this. On one hand, we can postulate the conventional quantum mechanics in every fibre $F_x$, i.e. the Schrödinger equation for the state vector $\Psi_{\gamma}(t) \in F_{\gamma(t)}$ with $F_{\gamma(t)}$ being (an isomorphic copy of) the system’s Hilbert space. But the only thing one gets in this way is an isomorphic image of the usual quantum mechanics in any fibre over $M$. Therefore one can not expect some new results or descriptions in this direction (see below (4.3) and the comments after it). On the other hand, we can demand the ordinary quantum mechanics to be valid in the fibre $F$ of the bundle $(F, \pi, M)$. This means to identify $F$ with the system’s Hilbert space of states and to describe the quantum time evolution of the system via the vector $\psi(t) = l_{\gamma(t)}(\Psi_{\gamma}(t)) \in F$ (4.1) which evolves according to (2.1) or (2.6). This approach is accepted in the present investigation. What we intend to do further, is, by using the basic relation (4.1), to ‘transfer’ the quantum mechanics from $F$ to $(F, \pi, M)$ or, in other words, to investigate the quantum evolution in terms of the vector $\Psi_{\gamma}(t)$ connected with $\psi(t)$ via (4.1). Since $l_x$, $x \in M$ are isomorphisms, both descriptions are completely equivalent. This equivalence resolves a psychological problem that may arise at first sight: the single Hilbert space $F$ of standard quantum theory [{1}, {2}, {3}] is replaced with a, generally, infinite number copies $F_x$, $x \in M$ thereof (cf. {4}). In the present investigation we shall show that the merit one gains from this is an entirely geometrical reformulation of quantum mechanics in terms of Hilbert fibre bundles.
The above considerations were more or less heuristic ones. The rigorous problem we want to investigate is the following. Let there be given a quantum system described in the (nonrelativistic) quantum mechanics by a state vector $\psi(t)$ satisfying the Schrödinger equation (2.3) and belonging to the system’s Hilbert space $\mathcal{F}$ of states [?], [?]. We postulate that $(F, \pi, M)$ is a Hilbert fibre bundle with bundle space $F$, base $M$, projection $\pi: F \to M$, and (typical, standard) fibre coinciding with $\mathcal{F}$. We suppose to be fixed a set of isomorphisms $\{l_x : F \to F, x \in M\}$ between the fibres $F_x := \pi^{-1}(x)$, $x \in M$ and the typical fibre $\mathcal{F}$. The base $M$ is supposed to be a differentiable manifold which, for definiteness, we shall identify with $\mathbb{R}^3$ (or with other manifold ‘suitable’ for the physical model; see remark 4.1).

Let $\gamma: J \to M$ be a path. In the case $M = \mathbb{R}^3$ (resp. $M = M^4, V_4$) we interpret $\gamma$ as a trajectory (resp. world line) of an observer describing the behaviour of the quantum system under consideration. If $\psi(t)$ is the complex vector-valued function of time representing the system’s state vector at a moment $t$, then our goal is to describe the system’s state at some instant of time $t$ via the vector (cf. (4.1))

$$\Psi_\gamma(t) = l_{\gamma(t)}^{-1}(\psi(t)) \in \mathcal{F}_{\gamma(t)}. \quad (4.2)$$

Since $l_x$, $x \in M$ are isomorphisms, both descriptions of the quantum evolution, through $\psi(t)$ and $\Psi_\gamma(t)$, are completely equivalent.

Two important notes have to be made here. Firstly, the state vectors in the bundle description generally explicitly depend on the observer, i.e. on the reference path $\gamma$, which is depicted in the index $\gamma$ in $\Psi_\gamma(t)$. This is on the contrary to the quantum mechanics where it is almost everywhere implicitly assumed. And secondly, the bundle, as well as the conventional, description of quantum mechanics is defined up to a linear isomorphism(s). In fact, if $\iota: \mathcal{F} \to \mathcal{F}'$, $\mathcal{F}'$ being a Hilbert space, is a linear isomorphism (which may depend on the time $t$), then $\psi'(t) = \iota(\psi(t))$ equivalently describes the evolution of the quantum system in $\mathcal{F}'$. (Note that in this way, for $\mathcal{F}' = \mathcal{F}$, one can obtain the known pictures of motion in quantum mechanics — see [?].)

In the bundle case the shift from $\mathcal{F}$ to $\mathcal{F}'$ is described by the transformation $l_x \to l'_x := \iota \circ l_x$ which reflects the arbitrariness in the choice of the typical fibre (now $\mathcal{F}'$ instead of $\mathcal{F}$) of $(F, \pi, M)$. There is also arbitrariness in the choice of the fibres $F_x = \pi^{-1}(x)$ which is of the same character as the one in the case of $\mathcal{F}$, viz. if $\iota_x: F_x \to F'_x$, $x \in M$ are linear isomorphisms, then the fibre bundle $(F', \pi', M)$ with $F' := \bigcup_{x \in M} F'_x$, $\pi'|_{F'_x} := \pi \circ l^{-1}_x$, typical fibre $\mathcal{F}'$, and isomorphisms $l'_x := l_x \circ l^{-1}_x$ can equivalently be used to describe the evolution of a quantum system. In the most general case, we have a fibre bundle $(F', \pi', M)$ with fibres $F'_x = \iota_x^{-1}(F_x)$, typical fibre $\mathcal{F}' = \iota(\mathcal{F})$, and isomorphisms $l'_x := \iota \circ l_x \circ l^{-1}_x : F'_x \to \mathcal{F}'$. Further we will not be interested in such generalizations. Thus, we shall suppose that all of the mentioned isomorphisms are fixed in such a way that the evolution of a quantum system will be described in a fibre bundle $(F, \pi, M)$ with fixed isomorphisms.
the fulfillment of the Schrödinger equation for the evolving vectors in it.

Bozhidar Z. Iliev: Bundle quantum mechanics. I

role as the inertial frames in the Newtonian mechanics.  

\[
\psi(t) = \pi^{-1}(x) \rightarrow F, \quad x \in M
\]

So, in the Schrödinger picture a quantum system is described by a state vector \( \psi \) in \( F \). Generally \[ [7] \] \( \psi \) depends (maybe implicitly) on the observer with respect to which the evolution is studied\[ [7] \] and it satisfies the Schrödinger equation \[ (2.6) \]. We shall refer to this representation of quantum mechanics as a Hilbert space description. In the new (Hilbert fibre) bundle description, which will be studied below, the linear isomorphisms \( l_x: F_x = \pi^{-1}(x) \rightarrow F, \quad x \in M \) are supposed arbitrarily fixed\[ [3] \] and the quantum systems are described by a state section along paths \( \Psi \) of a fibre bundle \( (F, \pi, M) \) whose typical fibre is the Hilbert space \( F \) (the same Hilbert space as in the Hilbert space description).

Here the term (state) section along paths needs some explanations and correct definition. The proper bundle analogue of \( \psi(t) \in F \) is \( \Psi_\gamma(t) \in F_{\gamma(t)} \), given by \[ (4.2) \], which explicitly depends on the observer’s trajectory (world line in the special relativity interpretation). Let \( J' \subseteq J \) be any subinterval of \( J \) on which \( \gamma \) is without self-intersections, i.e. if \( s, t \in J' \) and \( s \neq t \), then \( \gamma(s) \neq \gamma(t) \). The map \( \Psi_{\gamma|J'}: \gamma(J') \rightarrow \pi^{-1}(\gamma(J')) \subseteq F \) given by \( \Psi_{\gamma|J'}: x \mapsto \Psi_\gamma(t), \quad x \in \gamma(J') \), for the unique \( t \in J' \) for which \( \gamma(t) = x \), is a depending on \( \gamma \) section of the restricted bundle \( (F, \pi, M)|_{\gamma(J')} \), i.e. \( \Psi_{\gamma|J'} \in \text{Sec}\left( (F, \pi, M)|_{\gamma(J')}\right) \). Generally we can put \( \Psi_\gamma: x \mapsto \{\Psi_\gamma(t) : t \in J, \gamma(t) = x\} \) for every \( x \in M \). Evidently \( \Psi_\gamma: x \mapsto \emptyset \), \( \emptyset \) being the empty set, for \( x \notin \gamma(J) \). \( \pi \circ \Psi_\gamma|_{\gamma(J)} = \text{id}_{\gamma(J)} \), and at the points of self-intersection of \( \gamma \), if any, \( \Psi_\gamma \) is multiple valued, with the number of its values being equal to one plus the number of self-intersections of \( \gamma \) at the corresponding point.

We call section along paths any map \( \Psi: \gamma \mapsto \Psi_\gamma \), where \( \Psi_\gamma: M \rightarrow F \) may be multiple valued and such that \( \pi \circ \Psi_\gamma|_{\gamma(J)} = \text{id}_{\gamma(J)} \) and \( \Psi_\gamma: x \mapsto \emptyset \) for \( x \notin \gamma(J) \). So, the above-defined object \( \Psi_\gamma \) is a section along \( \gamma \). It is single valued, and consequently a section over \( \gamma(J) \) in the usual sense \[ [7] \], iff \( \gamma \) is without self-intersections.

We want also to mention explicitly the natural interpretation of \( \Psi \) as a lifting of paths, which is suggested by the notation used (see, e.g., \[ (4.2) \]).

---

\[ ^{10} \] Note that in the mentioned context the Schrödinger picture of motion plays the same rôle as the inertial frames in the Newtonian mechanics.

\[ ^{11} \] The concrete choice of \( F \) is insignificant for the following, the only important thing is the fulfillment of the Schrödinger equation for the evolving vectors in it.

\[ ^{12} \] Usually this dependence is not written explicitly, but it is always presented as actually \( t \) is the time with respect to a given observer.

\[ ^{13} \] The particular choice of \( \{l_x\} \) (and, consequently, of the fibres \( F_x \)) is inessential for our investigation.

\[ ^{14} \] Since in the special relativity interpretation \( \gamma \) is observer’s world line, the path \( \gamma \) can not have self-intersections (for real particles and (extended) bodies). In this case the map \( \Psi_\gamma \) is a section over the whole set \( \gamma(J) \).
Actually (cf. [chapter I, sect. 16 and chapter III, sect. 7][?]), a lifting of paths (from $M$ to $F$) is a map $\Psi: \gamma \mapsto \Psi_\gamma$ assigning to any path $\gamma: J \rightarrow M$ a path $\Psi_\gamma: J \rightarrow F$, lifting of $\gamma$ (from $M$ to $F$), such that $\pi \circ \Psi_\gamma := \gamma$. Evidently, the map $\Psi_\gamma: t \mapsto \Psi_\gamma(t)$ given by (4.2) is a lifting of $\gamma$; therefore $\Psi: \gamma \mapsto \Psi_\gamma$ is lifting of $\gamma$ which is single-valued irrespectively of the existence of self-intersections of $\gamma$.

Generally, to any vector $\varphi \in F$ there corresponds a unique (global) section $\Phi \in \text{Sec}(F, \pi, M)$ defined via

$$
\Phi: x \mapsto \Phi_x := l_x^{-1}(\varphi) \in F_x, \quad x \in M, \varphi \in F.
$$

Consequently to a state vector $\psi(t) \in F$ one can assign the (global) section $\Psi(t), \Psi(t): x \mapsto \Psi_x(t) = l_x^{-1}(\psi(t)) \in F_x$ and thus obtaining in $F_x$ for every $x \in M$ an isomorphic picture of (the evolution in) $F$. But in this way one can not obtain something significantly new as the evolution in $F$ is simply replaced with the (linearly isomorphic to it) evolution in $F_x$ for any arbitrary fixed $x \in M$. This reflects the above-mentioned fact that the quantum mechanical description is defined up to linear isomorphism(s). Besides, on the contrary to the bundle description, in this way one loses the explicit dependence on the observer. So in it one can’t get something really new with respect to the Hilbert space description.

Below we are going to define some structures and maps specific to Hilbert bundles and having a relation to the Hilbert bundle description of quantum mechanics.

Denote by $\langle \cdot | \cdot \rangle_x$ the Hermitian scalar product in $F_x$. We demand the isomorphisms $l_x$ to preserve not only the linear but also the metric structure of the bundle, i.e. $\langle \varphi | \psi \rangle = \langle l_x^{-1}(\varphi) | l_x^{-1}(\psi) \rangle_x, \varphi, \psi \in F$. Consequently $l_x$ transform the metric structure from $F$ to $F_x$ for every $x \in M$ according to

$$
\langle \cdot | \cdot \rangle_x = \langle l_x \cdot | l_x \cdot \rangle, \quad x \in M
$$

and, consequently, from $F_x$ to $F$ through

$$
\langle \cdot | \cdot \rangle = \langle l_x^{-1} \cdot | l_x^{-1} \cdot \rangle_x, \quad x \in M.
$$

Defining the Hermitian conjugate map (operator) $A_x^\dagger: F \rightarrow F_x$ of a map $A_x: F_x \rightarrow F$ by

$$
\langle A_x^\dagger \varphi | \chi_x \rangle := \langle \varphi | A_x \chi_x \rangle, \quad \varphi \in F, \chi_x \in F_x,
$$

we find (see (1.4))

$$
A_x^\dagger = l_x^{-1} \circ (A_x \circ l_x^{-1})^\dagger
$$

where the dagger denotes Hermitian conjugation in $F$ (see (2.3)).
We call a map \( A_x \) unitary if
\[
A_x^\dagger = A_x^{-1}. \tag{4.8}
\]
Evidently, the isomorphisms \( l_x \) are unitary in this sense:
\[
l_x^\dagger = l_x^{-1}. \tag{4.9}
\]
Similarly, the Hermitian conjugate map to a map \( A_{x\to y} \in \{C_{x\to y}: F_x \to F_y, \ x, y \in M\} \) is a map \( A_{x\to y}^\dagger: F_x \to F_y \) defined via
\[
\langle A_{x\to y}^\dagger \Phi_x | \Psi_y \rangle_y := \langle \Phi_x | A_{y\to x} \Psi_y \rangle_x, \quad \Phi_x \in F_x, \ \Psi_y \in F_y. \tag{4.10}
\]
Its explicit form is
\[
A_{x\to y}^\dagger = l_y^{-1} \circ (l_x \circ A_{y\to x} \circ l_y^{-1})^\dagger \circ l_x. \tag{4.11}
\]
As \((A^\dagger)^\dagger \equiv A\) for any \( A: \mathcal{F} \to \mathcal{F}\), we have
\[
\left(A_{x\to y}^\dagger\right)^\dagger = A_{x\to y}. \tag{4.12}
\]
If \( B_{x\to y} \in \{C_{x\to y}: F_x \to F_y, \ x, y \in M\} \), then a simple verification shows
\[
(B_{y\to z} \circ A_{x\to y})^\dagger = A_{y\to z}^\dagger \circ B_{x\to y}^\dagger, \quad x, y, z \in M. \tag{4.13}
\]
A map \( A_{x\to y} \) is called Hermitian if
\[
A_{x\to y}^\dagger = A_{x\to y}. \tag{4.14}
\]
A simple calculation proves that the maps \( l_{x\to y} := l_y^{-1} \circ l_x \) are Hermitian.

A map \( A_{x\to y} \) is called unitary if it has a left inverse map and
\[
A_{x\to y}^\dagger = A_{y\to x}^{-1}, \tag{4.15}
\]
where \( A_{x\to y}^{-1}: F_y \to F_x \) is the left inverse of \( A_{x\to y} \), i.e. \( A_{y\to x}^{-1} \circ A_{x\to y} := \text{id}_{F_x}\).

A simple verification by means of (4.10) shows the equivalence of (4.15) with
\[
\langle A_{y\to s} \cdot | A_{y\to s'} \rangle_x = \langle | \cdot y: F_y \times F_y \to \mathbb{C}, \tag{4.15'}
\]
i.e. the unitary maps are fibre-metric compatible in a sense that they preserve the fibre scalar (inner) product. Such maps will be called fibre-isometric or simply isometric.

It is almost evident that the maps \( l_{x\to y} = l_y^{-1} \circ l_x \) are unitary, that is we have:\[15\]
\[
l_{x\to y} = l_x^\dagger = l_y^{-1} \circ l_x^\dagger, \quad l_{x\to y} := l_y^{-1} \circ l_x: \pi^{-1}(x) \to \pi^{-1}(y). \tag{4.16}
\]
\[15\]The Hermiticity and at the same time unitarity of \( l_{x\to y} \) is not incidental as they define a (flat) linear transport (along paths or along the identity map of \( M \)) in \((F, \pi, M)\) (see [3.3] and below the paragraph after (4.13)).
We call a (possibly linear) transport along paths in \((F, \pi, M)\) *Hermitian* or *unitary* if it satisfies respectively (4.14) or (4.15) in which \(x\), and \(y\) are replaced with arbitrary values of the parameter of the transportation path, i.e. if respectively
\[
(L^\gamma_{s \to t})^\dagger = L^\gamma_{s \to t}, \quad s, t \in J, \quad \gamma : J \to M, \quad (4.17)
\]
\[
(L^\gamma_{s \to t})^\dagger = (L^\gamma_{t \to s})^{-1}. \quad (4.18)
\]

A simple corollary from (3.5) is the equivalence of (4.17) and (4.18); therefore, a transport along paths in a Hilbert bundle is Hermitian if and only if it is unitary, i.e. these concepts are equivalent. For such transports we say that they are consistent or compatible with the Hermitian structure (metric (inner product)) of the Hilbert bundle \([\cdot,\cdot]\). Evidently, they are isometric fibre maps along the paths they act. Therefore, a transport along paths in a Hilbert bundle is isometric iff it is Hermitian iff it is unitary.

Let \(A\) be a bundle morphism of \((F, \pi, M)\), i.e. \(A : F \to F\) and \(\pi \circ A = \text{id}_M\), and \(A_x := A|_{F_x}\). The *Hermitian conjugate* bundle morphism \(A^\dagger\) to \(A\) is defined by (cf. (4.10))
\[
\langle A^\dagger \Phi | \Psi\rangle_x := \langle \Phi | A \Psi\rangle_x, \quad \Phi, \Psi \in F_x. \quad (4.19)
\]

Thus (cf. (4.11))
\[
A^\dagger_x := A^\dagger|_{F_x} = l_x^{-1} \circ (l_x \circ A_x \circ l_x^{-1})^\dagger \circ l_x. \quad (4.20)
\]

A bundle morphism \(A\) is called *Hermitian* if \(A^\dagger_x = A_x\) for every \(x \in M\), i.e. if
\[
A^\dagger = A, \quad (4.21)
\]
and it is called *unitary* if \(A^\dagger_x = A_x^{-1}\) for every \(x \in M\), i.e. if
\[
A^\dagger = A^{-1}. \quad (4.22)
\]

Using (4.19), we can establish the equivalence of (4.22) and
\[
\langle A \cdot | \cdot A\rangle_x = \langle | \cdot \rangle_x : F_x \times F_x \to \mathbb{C}. \quad (4.22')
\]
Consequently the unitary morphisms are fibre-metric compatible, i.e. they are isometric in a sense that they preserve the fibre Hermitian scalar (inner) product.

\(^{16}\)The author thanks prof. James Stasheff (Math-UNC, Chapel Hill, NC, USA) for suggesting in July 1998 the term “isometric transport” in the context given.
5. The (bundle) evolution transport

Using (2.1), we get
\[ \psi(t_3) = U(t_3, t_2) \psi(t_2) = U(t_3, t_2)[U(t_2, t_1) \psi(t_1)], \psi(t_3) = U(t_3, t_1) \psi(t_1), \text{ and } \psi(t_1) = U(t_1, t_1) \psi(t_1) \text{ for every moments } t_1, t_2, t_3 \text{ and arbitrary state vector } \psi. \text{ Hence }
\]
\[ U(t_3, t_1) = U(t_3, t_2) \circ U(t_2, t_1), \quad (5.1) \]
\[ U(t_1, t_1) = \text{id}_F. \quad (5.2) \]

Besides, by definition, \( U(t_2, t_1) : F \to F \) is a linear unitary operator, i.e. for \( \lambda_i \in \mathbb{C} \) and \( \psi_i(t_1) \in F, i = 1, 2 \), we have:
\[ U(t_2, t_1) \left( \sum_{i=1,2} \lambda_i \psi_i(t_1) \right) = \sum_{i=1,2} \lambda_i U(t_2, t_1) \psi_i(t_1), \quad (5.3) \]
\[ U(t_1, t_2) = U^{-1}(t_2, t_1). \quad (5.4) \]

From (5.1) and (5.2), evidently, follows
\[ U^{-1}(t_2, t_1) = U(t_1, t_2) \quad (5.5) \]
and consequently
\[ U^\dagger(t_1, t_2) = U(t_1, t_2). \quad (5.6) \]

If one takes as a primary object the Hamiltonian \( H \), then these facts are direct consequences of (2.10).

Thus the properties of the evolution operator are very similar to the ones defining a ((flat) Hermitian) linear transport along paths in a Hilbert bundle. In fact, below we show that the evolution operator is a kind of such transport. (Note that this description is not unique.)

The bundle analogue of the evolution operator \( U(t_2, t_1) : F \to F \) is a linear operator \( U_\gamma(t_2, t_1) : F_\gamma(t_1) \to F_\gamma(t_2) \), \( s, t \in J \) such that
\[ \Psi_\gamma(t) = U_\gamma(t, s) \Psi_\gamma(s) \quad (5.7) \]
for every instants of time \( s, t \in J \). Analogously to (5.1) and (5.2), now we have:
\[ U_\gamma(t_3, t_1) = U_\gamma(t_3, t_2) \circ U_\gamma(t_2, t_1), \quad t_1, t_2, t_3 \in J, \quad (5.8) \]
\[ U_\gamma(t, t) = \text{id}_{F_\gamma(t)}, \quad t \in J. \quad (5.9) \]

We call \( U \) bundle evolution operator or evolution transport (see below).

Comparing (5.7) with (2.1) and using (4.2), we find
\[ U_\gamma(t, s) = l^{-1}_\gamma(t) \circ U(t, s) \circ l_\gamma(s), \quad s, t \in J \quad (5.10) \]
or
\[
U(t, s) = l^\gamma(t) \circ U^\gamma(t, s) \circ l^{-1}_\gamma(s), \quad s, t \in J. \tag{5.11}
\]

This shows the equivalence of the description of evolution of quantum systems via \(U\) and \(U^\gamma\).

A trivial corollary of (5.10) is the linearity of \(U^\gamma\) and
\[
U^{-1}_\gamma(t, s) = U^\gamma(s, t). \tag{5.12}
\]

As \(l_x : F_x \to F, x \in M\) are linear isomorphisms, from (5.8)–(5.10) follows that \(U : \gamma \mapsto U^\gamma : (s, t) \mapsto U^\gamma(t, s) =: U^\gamma_{t \to s} : F^\gamma(t) \to F^\gamma(s)\) is a linear transport along paths in \((F, \pi, M)\).

In fact, applying (4.11) to \(U^\gamma_{t \to s}\) and using (5.10), we get
\[
U^\dagger_{\gamma}(t, s) = l^{-1}_\gamma(t) \circ U^\dagger_\gamma(s, t) \circ l_\gamma(t). \tag{5.13}
\]

So, using (5.6), once again (5.10), and (5.5), we find
\[
U^\dagger_{\gamma}(t, s) = U^\gamma(t, s) = U^{-1}_\gamma(s, t). \tag{5.14}
\]

Hence \(U^\gamma(t, s)\) is simultaneously Hermitian and unitary operator, as it should be for any Hermitian or unitary transport along paths in a Hilbert bundle (see Sect. 4). Consequently, the evolution transport is an isometric transport along paths.

In this way, we see that the bundle evolution operator \(U\) is a Hermitian (and hence unitary) linear transport along paths in \((F, \pi, M)\). Consequently, to any unitary evolution operator \(\mathcal{U}\) in the Hilbert space \(F\) there corresponds a unique isometric linear transport \(U\) along paths in the Hilbert bundle \((F, \pi, M)\) and vice versa.

6. Conclusion

In the present work we have prepared the background for a full self-consistent fibre bundle formulation of nonrelativistic quantum mechanics. For this purpose we constructed a Hilbert (vector) fibre bundle replacing now the

\footnote{In the context of quantum mechanics it is more natural to define \(U^\gamma(s, t)\) from \(F^\gamma(t)\) into \(F^\gamma(s)\) instead from \(F^\gamma(s)\) into \(F^\gamma(t)\), as is the map \(U_{s \to t}^\gamma = U^\gamma(t, s) : F^\gamma(s) \to F^\gamma(t)\). The latter notation is better in the general theory of transports along paths \([?, ?]\). Consequently, when applying results from \([?, ?]\), we have to remember that they are valid for the maps \(U_{s \to t}^\gamma\) (or \(U^\gamma\)) and then to use the connection \(U_{s \to t}^\gamma = U^\gamma(t, s) = U_{s \to t}^{-1}(s, t)\) (or \(U^\gamma = U_\gamma^{-1}\)). Some results for \(U_{s \to t}^\gamma\) and \(U^\gamma(s, t)\) coincide but this is not always the case. In short, the results for linear transports along paths are transferred to the considered in this work case by replacing \(L_{s \to t}^\gamma\) with \(U^\gamma(t, s) = U_\gamma^{-1}(s, t)\).}
conventional Hilbert space of quantum mechanics. On this scene, as was shown here, the conventional quantum evolution is described by a suitable linear transport along paths.

An advantage of the bundle description of quantum mechanics is that it does not make use of any particular model of the base $M$. But on this model depends the interpretation of ‘time’ $t$ used. For instance, if we take $M$ to be the 3-dimensional Euclidean space $\mathbb{E}^3$ of classical (or quantum) mechanics, then $t$ is natural to be identified with the absolute Newtonian (global) time. However, if $M$ is taken to be the Minkowski 4-dimensional space $\mathbb{M}_4$, then it is preferable to take $t$ to be the proper time of some (local) observer, but the global coordinate time in some frame can also play the rôle of $t$. Principally different is the situation when the pseudo-Riemannian space $V_4$ of general relativity is taken as $M$: now $t$ must be the local time of some observer as a global time does not generically exist.

Generally, the space-time model $M$ is external to (bundle) quantum mechanics and has to be determined by another theory, such as special or general relativity. This points to a possible field of research: a connection between the quantities of the total bundle space with a concrete model of $M$ may result in a completely new theory. Elsewhere we shall show that just this is the case with relativistic quantum mechanics.

The development of the bundle approach to quantum mechanics will be done in the continuation of this paper. In particular, we intend to investigate the following topics from the novel fibre bundle viewpoint: equations of motion, description of observables, pictures and integrals of motion, mixed states, interpretation of the theory and possible ways for its further development and generalizations.