ALGEBRAIC EQUATIONS AND CONVEX BODIES

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Dedicated to Oleg Yanovich Viro on the occasion of his sixtieth birthday

Abstract. The well-known Bernstein-Kušnirenko theorem from the theory of Newton polyhedra relates algebraic geometry and the theory of mixed volumes. Recently the authors have found a far-reaching generalization of this theorem to generic systems of algebraic equations on any quasi-projective variety. In the present note we review these results and their applications to algebraic geometry and convex geometry.

Key words: Bernstein-Kušnirenko theorem, convex body, mixed volume, Alexandrov-Fenchel inequality, Brunn-Minkowski inequality, Hodge index theorem, intersection theory of Cartier divisors, Hilbert function.

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1. Introduction

The famous Bernstein-Kušnirenko theorem from the Newton polyhedra theory relates algebraic geometry (mainly the theory of toric varieties) with the theory of mixed volumes in convex geometry. This relation is useful in both directions. On
one hand it allows one to prove Alexandrov-Fenchel inequality (the most important and the hardest result in the theory of mixed volumes) using Hodge inequality from the theory of algebraic surfaces. On the other hand it suggests new inequalities in the intersection theory of Cartier divisors analogous to the known inequalities for mixed volumes.

Recently the authors have found a far-reaching generalization of Kushnirenko theorem in which instead of complex torus \((\mathbb{C}^*)^n\) we consider any quasi-projective variety \(X\) and instead of a finite dimensional space of functions spanned by monomials in \((\mathbb{C}^*)^n\), we consider any finite dimensional space of rational functions on \(X\).

To this end, firstly we develop an intersection theory for finite dimensional subspaces of rational functions on a quasi-projective variety. It can be considered as a generalization of the intersection theory of Cartier divisors for a (non-complete) variety \(X\). We show that this intersection theory enjoys all the properties of mixed volumes [Kaveh-Khovanskii2]. Secondly, we introduce the Newton convex body which is a far generalization of the Newton polyhedron of a Laurent polynomial. Our construction of Newton convex body depends on a fixed \(\mathbb{Z}^n\)-valued valuation on the field of rational functions on \(X\). It associates to any finite dimensional space \(L\) of rational functions on \(X\), its Newton convex body \(\Delta(L)\). We obtain a direct generalization of the Kushnirenko theorem in this setting (see Theorem 11.2).

This construction then allows us to give a proof of the Hodge inequality using elementary geometry of planar convex domains and (as a corollary) an elementary proof of Alexandrov-Fenchel inequality. In general our construction does not imply a generalization of the Bernstein theorem. Although we also obtain a generalization of this theorem for some cases when the variety \(X\) is equipped with a reductive group action.

In this paper we present a review of the results mentioned above. We omit the proofs in this short note. A preliminary version together with proofs can be found at [Kaveh-Khovanskii1]. The paper [Kaveh-Khovanskii2] contains a detailed version of the first half of the preprint [Kaveh-Khovanskii1], and a detailed version of the second half of [Kaveh-Khovanskii1] will appear very soon.

After posting of these results in the arXiv, we learned that we were not the only ones to have been working in this direction. Firstly, A. Okounkov was the pioneer to define (in passing) an analogue of Newton polytope in general situation in his interesting papers [Okounkov1, Okounkov2] (although his case of interest was when \(X\) has a reductive group action). Secondly, R. Lazarsfeld and M. Mustata, based on Okounkov’s previous works, and independently of our preprint, have come up with closely related results [Lazarsfeld-Mustata]. Recently, following [Lazarsfeld-Mustata], similar results/constructions have been obtained for line bundles on arithmetic surfaces [Yuan].

2. Mixed volume

By a convex body we mean a convex compact subset of \(\mathbb{R}^n\). There are two operations of addition and scalar multiplication for convex bodies: let \(\Delta_1, \Delta_2\) be convex bodies, then their sum

\[
\Delta_1 + \Delta_2 = \{x + y \mid x \in \Delta_1, y \in \Delta_2\},
\]
is also a convex body called the *Minkowski sum* of $\Delta_1$, $\Delta_2$. Also for a convex body $\Delta$ and a scalar $\lambda \geq 0$, 
$$\lambda \Delta = \{ \lambda x \mid x \in \Delta \},$$
is a convex body.

Let $\Vol$ denotes the $n$-dimensional volume in $\mathbb{R}^n$ with respect to the standard Euclidean metric. Function $\Vol$ is a homogeneous polynomial of degree $n$ on the cone of convex bodies, i.e. its restriction to each finite dimensional section of the cone is a homogeneous polynomial of degree $n$. More precisely: for any $k > 0$ let $\mathbb{R}_+^k$ be the positive octant in $\mathbb{R}^k$ consisting of all $\lambda = (\lambda_1, \ldots, \lambda_k)$ with $\lambda_1 \geq 0, \ldots, \lambda_k \geq 0$. Then polynomiality of $\Vol$ means that for any choice of convex bodies $\Delta_1, \ldots, \Delta_k$, the function $P_{\Delta_1, \ldots, \Delta_k}$ defined on $\mathbb{R}_+^k$ by

$$P_{\Delta_1, \ldots, \Delta_k}(\lambda_1, \ldots, \lambda_k) = \Vol(\lambda_1 \Delta_1 + \cdots + \lambda_k \Delta_k),$$
is a homogeneous polynomial of degree $n$.

By definition the *mixed volume* of $V(\Delta_1, \ldots, \Delta_n)$ of an $n$-tuple $(\Delta_1, \ldots, \Delta_n)$ of convex bodies is the coefficient of the monomial $\lambda_1 \cdots \lambda_n$ in the polynomial $P_{\Delta_1, \ldots, \Delta_n}$ divided by $n!$.

This definition implies that the mixed volume is the polarization of the volume polynomial, that is, it is a function on the $n$-tuples of convex bodies satisfying the following:

(i) (Symmetry) $V$ is symmetric with respect to permuting the bodies $\Delta_1, \ldots, \Delta_n$.

(ii) (Multi-linearity) It is linear in each argument with respect to the Minkowski sum. Linearity in the first argument means that for convex bodies $\Delta'_1, \Delta''_1$ and $\Delta_2, \ldots, \Delta_n$ we have:

$$V(\Delta'_1 + \Delta''_1, \ldots, \Delta_n) = V(\Delta'_1, \ldots, \Delta_n) + V(\Delta''_1, \ldots, \Delta_n).$$

(iii) (Relation with volume) On the diagonal it coincides with volume, i.e. if $\Delta_1 = \cdots = \Delta_n = \Delta$, then $V(\Delta_1, \ldots, \Delta_n) = \Vol(\Delta)$.

The above three properties characterize the mixed volume: it is the unique function satisfying (i)-(iii).

The following two inequalities are easy to verify:

1) Mixed volume is non-negative, that is, for any $n$-tuple of convex bodies $\Delta_1, \ldots, \Delta_n$ we have

$$V(\Delta_1, \ldots, \Delta_n) \geq 0.$$

2) Mixed volume is monotone, that is, for two $n$-tuples of convex bodies $\Delta'_1 \subset \Delta_1, \ldots, \Delta'_n \subset \Delta_n$ we have

$$V(\Delta_1, \ldots, \Delta_n) \geq V(\Delta'_1, \ldots, \Delta'_n).$$

The following inequality attributed to Alexandrov and Fenchel is important and very useful in convex geometry. All its previously known proofs are rather complicated (see Burago-Zalgaller).

**Theorem 2.1** (Alexandrov-Fenchel). Let $\Delta_1, \ldots, \Delta_n$ be convex bodies in $\mathbb{R}^n$. Then

$$V(\Delta_1, \Delta_2, \ldots, \Delta_n)^2 \geq V(\Delta_1, \Delta_1, \Delta_3, \ldots, \Delta_n)V(\Delta_2, \Delta_2, \Delta_3, \ldots, \Delta_n).$$

Below we mention a formal corollary of Alexandrov-Fenchel inequality. First we need to introduce a notation for when we have repetition of convex bodies in the mixed volume. Let $2 \leq m \leq n$ be an integer and $k_1 + \cdots + k_r = m$ a partition of $m$ with $k_i \in \mathbb{N}$. Denote by $V(k_1 \ast \Delta_1, \ldots, k_r \ast \Delta_r, \Delta_{m+1}, \ldots, \Delta_n)$ the mixed
volume of the $\Delta_i$ where $\Delta_1$ is repeated $k_1$ times, $\Delta_2$ is repeated $k_2$ times, etc. and $\Delta_{m+1}, \ldots, \Delta_n$ appear once.

**Corollary 2.2.** With the notation as above, the following inequality holds:

$$V^m(k_1 \ast \Delta_1, \ldots, k_r \ast \Delta_r, \Delta_{m+1}, \ldots, \Delta_n) \geq \prod_{1 \leq j \leq r} V^{k_j}(m \ast \Delta_j, \Delta_{m+1}, \ldots, \Delta_n).$$

3. **Brunn-Minkowski inequality**

The celebrated *Brunn-Minkowski inequality* concerns volume of convex bodies in $\mathbb{R}^n$.

**Theorem 3.1** (Brunn-Minkowski). Let $\Delta_1, \Delta_2$ be convex bodies in $\mathbb{R}^n$. Then

$$\Vol^{1/n}(\Delta_1) + \Vol^{1/n}(\Delta_2) \leq \Vol^{1/n}(\Delta_1 + \Delta_2).$$

The inequality was first found and proved by Brunn around the end of 19th century in the following form.

**Theorem 3.2.** Let $V_\Delta(h)$ be the $n$-dimensional volume of the section $x_{n+1} = h$ of a convex body $\Delta \subset \mathbb{R}^{n+1}$. Then $V_\Delta^{1/n}(h)$ is a concave function in $h$.

To obtain Theorem 3.1 from Theorem 3.2, one takes $\Delta \subset \mathbb{R}^{n+1}$ to be the convex combination of $\Delta_1$ and $\Delta_2$, i.e.

$$\Delta = \{(x, h) \mid 0 \leq h \leq 1, \ x \in h\Delta_1 + (1-h)\Delta_2\}.$$ 

The concavity of the function

$$V_\Delta(h) = \Vol(h\Delta_1 + (1-h)\Delta_2),$$

then readily implies Theorem 3.1.

For $n = 2$ Theorem 3.2 is equivalent to the Alexandrov-Fenchel inequality (see Theorem 4.1). Below we give a sketch of its proof in the general case.

**Sketch of proof of Theorem 3.2.** 1) When the convex body $\Delta \subset \mathbb{R}^{n+1}$ is rotationally symmetric with respect to the $x_{n+1}$-axis, Theorem 3.2 is obvious.

2) Now suppose $\Delta$ is not rotationally symmetric. Fix a hyperplane $H$ containing the $x_{n+1}$-axis. Then one can construct a new convex body $\Delta'$ which is symmetric with respect to the hyperplane $H$ and such that the volume of sections of $\Delta'$ is the same as that of $\Delta$. To do this, just think of $\Delta$ as the union of line segments perpendicular to the plane $H$. Then shift each segment, along its line, in such a way that its center lies on $H$. The resulting body is then symmetric with respect to $H$ and has the same volume of sections as $\Delta$. The above construction is called the *Steiner symmetrization process*.

3) Consider the set of all convex bodies inside a bounded closed domain equipped with the Hausdorff metric. One checks that this set is compact.

4) Take the collection of all convex bodies that can be obtained from $\Delta$ by a finite number of symmetrizations with respect to hyperplanes $H$ containing the $x_{n+1}$-axis. The closure $C(\Delta)$ of this collection with respect to the Hausdorff metric is compact.

5) Take the body $\Delta_{rot}$ which is rotationally symmetric with respect to the $x_{n+1}$-axis and with the following property: the volume of any section of $\Delta_{rot}$ by a horizontal hyperplane is equal to the volume of the section of $\Delta$ by the same hyperplane. A priori we do not know that $\Delta_{rot}$ is convex.
6) Consider the continuous function \( f \) on \( C(\Delta) \) defined by
\[
f(\Delta_1) = V(\Delta_1 \setminus \Delta_{rot}) + V(\Delta_{rot} \setminus \Delta_1),
\]
for any \( \Delta_1 \in C(X) \). Since \( f \) is continuous it has a minimum. Take a body \( \Delta_0 \in C(\Delta) \) at which \( f \) attains a minimum. Let us show \( \Delta_0 = \Delta_{rot} \).

7) If \( \Delta_0 \neq \Delta_{rot} \) then there is a horizontal hyperplane \( L \) and points \( a, b \in L \) such that \( a \in \Delta_{rot}, a \notin \Delta_0 \) and \( b \notin \Delta_{rot}, b \in \Delta_0 \) (note that \( V(\Delta_{rot} \cap L) = V(\Delta_0 \cap L) \)). Consider the line \( \ell \) passing through the points \( a \) and \( b \). Take the hyperplane \( H \) orthogonal to \( \ell \) and containing \( x_{n+1} \)-axis. Denote by \( \Delta_0' \) the result of the Steiner symmetrization of \( \Delta_0 \) with respect to \( H \). It is easy to check that \( f(\Delta_0') < f(\Delta_0) \), which contradict the minimality of \( f(\Delta_0) \). The contradiction proves that \( \Delta_0 = \Delta_{rot} \) and thus \( \Delta_{rot} \) is convex. By Step 1) we then have the required inequality and the proof is finished. \( \square \)

4. BRUNN-MINKOWSKI AND ALEXANDROV-FENCHEL INEQUALITIES

Let us remind the classical isoperimetric inequality whose origins date back to the antiquity. According to this inequality if \( P \) is the perimeter of a simple closed curve in the plane and \( A \) is the area enclosed by the curve then
\[
4\pi A \leq P^2.
\]
The equality is obtained when the curve is a circle. To prove (1) it is enough to prove it for convex regions. The Alexandrov-Fenchel inequality for \( n = 2 \) implies the isoperimetric inequality (1) as a particular case and hence has inherited the name.

**Theorem 4.1** (Isoperimetric inequality). If \( \Delta_1 \) and \( \Delta_2 \) are convex regions in the plane then
\[
\text{Area}(\Delta_1)\text{Area}(\Delta_2) \leq \text{A}(\Delta_1, \Delta_2)^2,
\]
where \( \text{A}(\Delta_1, \Delta_2) \) is the mixed area.

When \( \Delta_2 \) is the unit disc in the plane, \( \text{A}(\Delta_1, \Delta_2) \) is 1/2 times the perimeter of \( \Delta_1 \). Thus the classical form (1) of the inequality (for convex regions) follows from Theorem 4.1.

**Proof of Theorem 4.1.** It is easy to verify that the isoperimetric inequality is equivalent to the Brunn-Minkowski for \( n=2 \). Let us check this in one direction, i.e. the isoperimetric inequality follows from Brunn-Minkowski for \( n = 2 \):
\[
\text{Area}(\Delta_1) + 2\text{A}(\Delta_1, \Delta_2) + \text{Area}(\Delta_2) = \text{Area}(\Delta_1 + \Delta_2)
\geq (\text{Area}^{1/2}(\Delta_1) + \text{Area}^{1/2}(\Delta_2))^2
= \text{Area}(\Delta_1) + 2\text{Area}(\Delta_1^{1/2}\text{Area}(\Delta_2)^{1/2}) + \text{Area}(\Delta_2),
\]
which readily implies the isoperimetric inequality. \( \square \)

The following generalization of Brunn-Minkowski inequality is a corollary of Alexandrov-Fenchel inequality.
Corollary 4.2. (Generalized Brunn-Minkowski inequality) For any \( 0 < m \leq n \) and for any fixed convex bodies \( \Delta_{m+1}, \ldots, \Delta_n \), the function \( F \) which assigns to a body \( \Delta \), the number

\[
F(\Delta) = V^{1/m}(m \cdot \Delta, \Delta_{m+1}, \ldots, \Delta_n),
\]

is concave, i.e. for any two convex bodies \( \Delta_1, \Delta_2 \) we have

\[
F(\Delta_1) + F(\Delta_2) \leq F(\Delta_1 + \Delta_2).
\]

On the other hand, the usual proof of Alexandrov-Fenchel inequality deduces it from the Brunn-Minkowski inequality. But this deduction is the main part (and the most complicated part) in the proof ([Burago-Zalgaller]). Interestingly, The main construction in the present paper (using algebraic geometry) allows us to obtain Alexandrov-Fenchel inequality as an immediate corollary of the simplest case of the Brunn-Minkowski, i.e. isoperimetric inequality.

5. Generic systems of Laurent polynomial equations in \((\mathbb{C}^*)^n\)

In this section we recall the famous results due to Kušnirenko and Bernstein on the number of solutions of a generic system of polynomials in \((\mathbb{C}^*)^n\).

Let us identify the lattice \( \mathbb{Z}^n \) with Laurent monomials in \((\mathbb{C}^*)^n\): to each integral point \( k \in \mathbb{Z}^n \), \( k = (k_1, \ldots, k_n) \) we associate the monomial \( z^k = z_1^{k_1} \cdots z_n^{k_n} \) where \( z = (z_1, \ldots, z_n) \). A Laurent polynomial \( P = \sum_k c_k z^k \) is a finite linear combination of Laurent monomials with complex coefficients. The support \( \text{supp}(P) \) of a Laurent polynomial \( P \), is the set of exponents \( k \) for which \( c_k \neq 0 \). We denote the convex hull of a finite set \( A \subset \mathbb{Z}^n \) by \( \Delta_A \subset \mathbb{R}^n \). The Newton polytope \( \Delta(P) \) of a Laurent polynomial \( P \) is the convex hull \( \Delta_{\text{supp}(P)} \) of its support. With each finite set \( A \subset \mathbb{Z}^n \) one associates a vector space \( L_A \) of Laurent polynomials \( P \) with \( \text{supp}(P) \subset A \).

**Definition 5.1.** We say that a property holds for a generic element of a vector space \( L \) if there is a proper algebraic set \( \Sigma \) such that the property holds for all elements in \( L \setminus \Sigma \).

**Definition 5.2.** For a given \( n \)-tuple of finite sets \( A_1, \ldots, A_n \subset \mathbb{Z}^n \) the intersection index of the \( n \)-tuple of spaces \( [L_{A_1}, \ldots, L_{A_n}] \) is the number of solutions in \((\mathbb{C}^*)^n\) of a generic system of equations \( P_1 = \cdots = P_n = 0 \), where \( P_1 \in L_{A_1}, \ldots, P_n \in L_{A_n} \).

**Problem:** Find the intersection index \( [L_{A_1}, \ldots, L_{A_n}] \), that is, for a generic element \( (P_1, \ldots, P_n) \in L_{A_1} \times \cdots \times L_{A_n} \) find a formula for the number of solutions in \((\mathbb{C}^*)^n\) of the system of equations \( P_1 = \cdots = P_n = 0 \).

Kušnirenko found the following important result which answers a particular case of the above problem [Kushnirenko]:

**Theorem 5.3.** When the convex hulls of the sets \( A_i \) are the same and equal to a polytope \( \Delta \) we have

\[
[L_{A_1}, \ldots, L_{A_n}] = n!\text{Vol}(\Delta),
\]

where \( \text{Vol} \) is the standard \( n \)-dimensional volume in \( \mathbb{R}^n \).

According to Theorem 5.3 if \( P_1, \ldots, P_n \) are sufficiently general Laurent polynomials with given Newton polytope \( \Delta \), the number of solutions in \((\mathbb{C}^*)^n\) of the system \( P_1 = \cdots = P_n = 0 \) is equal to \( n!\text{Vol}(\Delta) \).

The problem was solved by Bernstein in full generality [Bernstein]:

**Theorem 5.4.** In the general case, i.e. for arbitrary finite subsets \( A_1, \ldots, A_n \subset \mathbb{Z}^n \) we have

\[
[L_{A_1}, \ldots, L_{A_n}] = n!V(\Delta_{A_1}, \ldots, \Delta_{A_n}),
\]
where $V$ is the mixed volume of convex bodies in $\mathbb{R}^n$.

According to Theorem 5.4 if $P_1, \ldots, P_n$ are sufficiently general Laurent polynomials with Newton polyhedra $\Delta_1, \ldots, \Delta_n$ respectively, the number of solutions in $(\mathbb{C}^*)^n$ of the system $P_1 = \cdots = P_n = 0$ is equal to $n!V(\Delta_1, \ldots, \Delta_n)$.

6. Convex Geometry and Bernstein-Kušnirenko theorem

Let us examine Theorem 5.4 (which we will call Bernstein-Kušnirenko theorem) more closely. In the space of regular functions on $(\mathbb{C}^*)^n$ there is a natural family of finite dimensional subspaces, namely the subspaces which are stable under the action of the multiplicative group $(\mathbb{C}^*)^n$. Each such subspace is of the form $L_A$ for some finite set $A \subseteq \mathbb{Z}^n$ of monomials.

For two finite dimensional subspaces $L_1, L_2$ of regular functions in $(\mathbb{C}^*)^n$, let us define the product $L_1L_2$ as the subspace spanned by the products $fg$, where $f \in L_1$, $g \in L_2$. Clearly multiplication of monomials corresponds to the addition of their exponents, i.e. $z^{k_1}z^{k_2} = z^{k_1+k_2}$. This implies that $L_{A_1}L_{A_2} = L_{A_1+ A_2}$.

The Bernstein-Kušnirenko theorem defines and computes the intersection index $\langle L_{A_1}, L_{A_2}, \ldots, L_{A_n} \rangle$ of the $n$-tuples of subspaces $L_{A_i}$ for finite subsets $A_i \subseteq \mathbb{Z}^n$. Since this intersection index is equal to the mixed volume, it enjoys the same properties, namely: 1) Positivity; 2) Monotonicity; 3) Multi-linearity; and 4) Alexandrov-Fenchel inequality and its corollaries. Moreover, if for a finite set $A \subseteq \mathbb{Z}^n$ we let $A = \Delta_A \cap \mathbb{Z}^n$, then 5) the spaces $L_A$ and $L_{\overline{A}}$ have the same intersection indices. That is, for any $(n-1)$-tuple of finite subsets $A_2, \ldots, A_n \in \mathbb{Z}^n$,

$$[L_{A}, L_{A_2}, \ldots, L_{A_n}] = [L_{\overline{A}}, L_{A_2}, \ldots, L_{A_n}].$$

This means that (surprisingly!) enlarging $L_A \mapsto L_{\overline{A}}$ does not change any of the intersection indices we considered. And hence in counting the number of solutions of a system, instead of support of a polynomial, its convex hull plays the main role. Let us denote the subspace $L_{\overline{A}}$ by $\overline{L_A}$ and call it the completion of $L_A$.

Since the semi-group of convex bodies with Minkowski sum has cancelation property, the following cancelation property for the finite subsets of $\mathbb{Z}^n$ holds: if for finite subsets $A, B, C \in \mathbb{Z}^n$ we have $A + C = B + \overline{C}$ then $A = B$. And we have the same cancelation property for the corresponding semi-group of subspaces $L_A$. That is, if $\overline{L_A} \overline{L_C} = \overline{L_B} \overline{L_C}$ then $\overline{L_A} = \overline{L_B}$.

Bernstein-Kušnirenko theorem relates mixed volume in convex geometry with intersection index in algebraic geometry. In algebraic geometry the following inequality about intersection indices on a surface is well-known.

**Theorem 6.1** (Hodge inequality). Let $\Gamma_1, \Gamma_2$ be algebraic curves on a smooth irreducible projective surface. Assume that $\Gamma_1, \Gamma_2$ have positive self-intersection indices. Then

$$(\Gamma_1, \Gamma_2)^2 \geq (\Gamma_1, \Gamma_1)(\Gamma_2, \Gamma_2)$$

where $(\Gamma_i, \Gamma_j)$ denotes the intersection index of the curves $\Gamma_i$ and $\Gamma_j$.

On one hand Theorem 5.4 allows one to prove Alexandrov-Fenchel inequality algebraically using Theorem 6.1 (see [Khovanskii1, Teissier]). On the other hand Hodge inequality suggests an analogy between the mixed volume theory and the intersection theory of Cartier divisors on a projective algebraic variety.

We will return back to this discussion after statement of our main theorem (Theorem 11.2) and its corollary which is a version of Hodge inequality.
7. Analog of the Intersection theory of Cartier divisors for non-complete varieties

Now we discuss general results inspired by Bernstein-Kušnirenko theorem which can be considered as an analogue of the intersection theory of Cartier divisors for non-complete varieties ([Kaveh-Khovanskii1]). Instead of $(\mathbb{C}^*)^n$ we take any irreducible $n$-dimensional quasi-projective variety $X$ and instead of a finite dimensional space of functions spanned by monomials we take any finite dimensional space of rational functions. For these spaces we define an intersection index and prove that it enjoys all the properties of the mixed volume of convex bodies.

Consider the collection $\mathbf{K}_{rat}(X)$ of all finite dimensional subspaces of rational functions on $X$. The set $\mathbf{K}_{rat}(X)$ has a natural multiplication: product $L_1L_2$ of two subspaces $L_1, L_2 \in \mathbf{K}_{rat}(X)$ is the subspace spanned by all the products $fg$ where $f \in L_1, g \in L_2$. With respect to this multiplication, $\mathbf{K}_{rat}(X)$ is a commutative semi-group.

**Definition 7.1.** The intersection index $[L_1, \ldots, L_n]$ of $L_1, \ldots, L_n \in \mathbf{K}_{rat}(X)$ is the number of solutions in $X$ of a generic system of equations $f_1 = \cdots = f_n = 0$, where $f_1 \in L_1, \ldots, f_n \in L_n$. In counting the solutions, we neglect solutions $x$ at which all the functions in some space $L_i$ vanish as well as solutions at which at least one function from some space $L_i$ has a pole.

More precisely, let $\Sigma \subset X$ be a hypersurface which contains: 1) all the singular points of $X$; 2) all the poles of functions from any of the $L_i$; 3) for any $i$, the set of common zeros of all the $f \in L_i$. Then for a generic choice of $(f_1, \ldots, f_n) \in U \subset \mathbf{K}_{rat}(X)$ such that for any $(f_1, \ldots, f_n) \in U$ the number of solutions $x \in X \setminus \Sigma$ of the system $f_1(x) = \cdots = f_n(x) = 0$ is the same (and hence equal to $[L_1, \ldots, L_n]$). Moreover the above number of solutions is independent of the choice of $\Sigma$ containing 1)-3) above.

The following properties of the intersection index are easy consequences of the definition:

**Theorem 7.2.** The intersection index $[L_1, \ldots, L_n]$ is well-defined. That is, there is a Zariski open subset $U$ in the vector space $L_1 \times \cdots \times L_n$ such that for any $(f_1, \ldots, f_n) \in U$ the number of solutions $x \in X \setminus \Sigma$ of the system $f_1(x) = \cdots = f_n(x) = 0$ is the same (and hence equal to $[L_1, \ldots, L_n]$). Moreover the above number of solutions is independent of the choice of $\Sigma$ containing 1)-3) above.

The next two theorems contain the main properties of the intersection index.

**Proposition 7.3.** 1) $[L_1, \ldots, L_n]$ is a symmetric function of the $n$-tuples $L_1, \ldots, L_n \in \mathbf{K}_{rat}(X)$, i.e. takes the same value under a permutation of $L_1, \ldots, L_n$; 2) The intersection index is monotone, i.e. if $L'_1 \subseteq L_1, \ldots, L'_n \subseteq L_n$, then $[L_1, \ldots, L_n] \geq [L'_1, \ldots, L'_n]$; and 3) The intersection index is non-negative, i.e. $[L_1, \ldots, L_n] \geq 0$.

**Theorem 7.4** (Multi-linearity). 1) Let $L_1', L_2', \ldots, L_n' \in \mathbf{K}_{rat}(X)$ and put $L_1 = L_1'L_1''$. Then $[L_1, \ldots, L_n] = [L_1', \ldots, L_n] + [L_1'', \ldots, L_n]$. 2) Let $L_1, \ldots, L_n \in \mathbf{K}_{rat}(X)$ and take 1-dimensional subspaces $L_1', \ldots, L_n' \in \mathbf{K}_{rat}(X)$. Then $[L_1, \ldots, L_n] = [L_1'L_1, \ldots, L_n']$.

Let us say that $f \in \mathbb{C}(X)$ is integral over a subspace $L \in \mathbf{K}_{rat}(X)$ if $f$ satisfies an equation $f^m + a_1f^{m-1} + \cdots + a_m = 0$. 

where \( m > 0 \) and \( a_i \in L^i \), for each \( i = 1, \ldots, m \). It is well-known that the collection \( \mathcal{T} \) of all integral elements over \( L \) is a vector subspace containing \( L \). Moreover if \( L \) is finite dimensional then \( \mathcal{T} \) is also finite dimensional (see [Zariski-Samuel, Appendix 4]). It is called the completion of \( L \). For two subspaces \( L, M \in K_{rat}(X) \) we say that \( L \) is equivalent to \( M \) (written \( L \sim M \)) if there is \( N \in K_{rat}(X) \) with \( LN = MN \). One shows that the completion \( \mathcal{T} \) is in fact the largest subspace in \( K_{rat}(X) \) which is equivalent to \( L \). The enlarging \( L \rightarrow \mathcal{T} \) is analogous to the geometric operation \( A \mapsto \Delta(A) \) which associates to a finite set \( A \) its convex hull \( \Delta(A) \).

**Theorem 7.5.** 1) Let \( L_1 \in K_{rat}(X) \) and let \( G_1 \in K_{rat}(X) \) be the subspace spanned by \( L_1 \) and a rational function \( g \) integral over \( L_1 \). Then for any \((n-1)\)-tuple \( L_2, \ldots, L_n \in K_{rat}(X) \) we have

\[
[L_1, L_2, \ldots, L_n] = [G_1, L_2, \ldots, L_n].
\]

2) Let \( L_1 \in K_{rat}(X) \) and let \( \overline{L}_1 \) be its completion as defined above. Then for any \((n-1)\)-tuple \( L_2, \ldots, L_n \in K_{rat}(X) \) we have

\[
[L_1, L_2, \ldots, L_n] = [\overline{L}_1, L_2, \ldots, L_n].
\]

As with any other commutative semi-group, there corresponds a Grothendieck group to the semi-group \( K_{rat}(X) \). For a commutative semi-group \( K \), the Grothendieck group \( G(K) \) is the unique abelian group defined with the following universal property: there is a homomorphism \( \phi : K \to G(K) \) and for any abelian group \( G' \) and homomorphism \( \phi' : K \to G' \), there exist a homomorphism \( \psi : G(K) \to G' \) such that \( \phi' = \psi \circ \phi \). The Grothendieck group can be defined constructively also: for \( x, y \in K \) we say \( x \sim y \) if there is \( z \in K \) with \( xz = yz \). Then \( G(K) \) is the group of formal quotients of equivalence classes of \( \sim \). From the multi-linearity of the intersection index it follows that the intersection index extends to the the Grothendieck group of \( K_{rat}(X) \).

The Grothendieck group of \( K_{rat}(X) \) can be considered as an analogue (for a non-complete variety \( X \)) of the group of Cartier divisors on a projective variety, and the intersection index on the Grothendieck group of \( K_{rat}(X) \) as an analogue of the intersection index of Cartier divisors.

The intersection theory on the Grothendieck group of \( K_{rat}(X) \) enjoys all the properties of mixed volume. Some of such properties we already discussed in the present section. The others will be discussed later (see Theorem 12.3 and Corollary 12.4 below).

8. **Proof of Bernstein-Kušnirenko theorem via Hilbert theorem**

Let us recall the proof of the Bernstein-Kušnirenko theorem from [Khovanskii2] which will be important for our generalization.

For each space \( L \in K_{rat}(X) \) let us define the Hilbert function \( H_L(k) = \dim(L^k) \). For sufficiently large values of \( k \), the function \( H_L(k) \) is a polynomial in \( k \), called the Hilbert polynomial of \( L \).

With each space \( L \in K_{rat}(X) \), one associates a rational Kodaira map from \( X \) to \( \mathbb{P}(L^*) \), the projectivization of the dual space \( L^* \): to any \( x \in X \) there corresponds a functional in \( L^* \) which evaluates \( f \in L \) at \( x \). The Kodaira map sends \( x \) to the image of this functional in \( \mathbb{P}(L^*) \). It is a rational map, i.e. defined on a Zariski open subset in \( X \). We denote by \( Y_L \) the subvariety in the projective space \( \mathbb{P}(L^*) \) which is equal to the closure of the image of \( X \) under the Kodaira map.
The following theorem is a version of the classical Hilbert theorem on the degree of a subvariety of the projective space.

**Theorem 8.1** (Hilbert’s theorem). The degree of the Hilbert polynomial of the space $L$ is equal to the dimension of the variety $Y_L$, and its leading coefficient $c$ is the degree of $Y_L \subset \mathbb{P}(L^*)$ divided by $m!$.

Let $A$ be a finite subset in $\mathbb{Z}^n$ with $\Delta(A)$ its convex hull. Denote by $k \ast A$ the sum $A + \cdots + A$ of $k$ copies of the set $A$, and by $(k\Delta(A))_C$ the subset in $k\Delta(A)$ containing points whose distance to the boundary $\partial(k\Delta(A))$ is bigger than $C$. The following combinatorial theorem gives an estimate for the set $k \ast A$ in terms of the set of integral points in $k\Delta(A)$.

**Theorem 8.2** ([Khovanskii2]). 1) One has $k \ast A \subset k\Delta(A) \cap \mathbb{Z}^n$. 2) Assume that the differences $a - b$ for $a, b \in A$ generate the group $\mathbb{Z}^n$. Then there exists a constant $C$ such that for any $k \in \mathbb{N}$ we have $(k\Delta(A))_C \cap \mathbb{Z}^n \subset k \ast A$.

**Corollary 8.3.** Let $A \subset \mathbb{Z}^n$ be a finite subset satisfying the condition in Theorem 8.2(2). Then

$$\lim_{k \to \infty} \frac{\#(k \ast A)}{k^n} = \text{Vol}_n(\Delta(A)).$$

Corollary 8.3 together with the Hilbert theorem (Theorem 8.1) proves the Kuśnirenko theorem for sets $A$ such that the differences $a - b$ for $a, b \in A$ generate the group $\mathbb{Z}^n$. The Kuśnirenko theorem for the general case easily follows from this. The Bernstein theorem 5.4 follows from the Kuśnirenko theorem 5.3 and the identity $L_{A+B} = L_AL_B$.

9. Graded semigroups in $\mathbb{N} \oplus \mathbb{Z}^n$ and Newton convex body

Let $S$ be a subsemigroup of $\mathbb{N} \oplus \mathbb{Z}^n$. For any integer $k > 0$ we denote by $S_k$ the section of $S$ at level $k$, i.e. the set of elements $x \in \mathbb{Z}^n$ such that $(k, x) \in S$.

**Definition 9.1.** A subsemigroup $S$ of $\mathbb{N} \oplus \mathbb{Z}^n$ is called:

1) a graded semi-group if for any $k > 0$, $S_k$ is finite and non-empty;
2) an ample semi-group if there is a natural $m$ such that the differences $a - b$ for $a, b \in S_m$ generate the group $\mathbb{Z}^n$;
3) a semi-group with restricted growth if there is constant $C$ such that for any $k > 0$ we have $\#(S_k) \leq Ck^n$.

For a graded semi-group $S$, let $\text{Con}(S)$ denote the convex hull of $S \cup \{0\}$. It is a cone in $\mathbb{R}^{n+1}$. Denote by $M_S$ the semi-group $\text{Con}(S) \cap (\mathbb{N} \oplus \mathbb{Z}^n)$. The semigroup $M_S$ contains the semigroup $S$.

**Definition 9.2.** For a graded semi-group $S$, define the Newton convex set $\Delta(S)$ to be the section of the cone $\text{Con}(S)$ at $k = 1$, i.e.

$$\Delta(S) = \{x \mid (1, x) \in \text{Con}(S)\}.$$

**Theorem 9.3** (Asymptotic of graded semi-groups). Let $S$ be an ample graded semi-group with restricted growth in $\mathbb{N} \oplus \mathbb{Z}^n$. Then:

1) the cone $\text{Con}(S)$ is strictly convex, i.e. the Newton convex set $\Delta(S)$ is bounded;
2) Let $d(k)$ denote the maximum distance of the points $(k, x)$ from the boundary of $\text{Con}(S)$ for $x \in M_S(k) \setminus S(k)$. Then
\[
\lim_{k \to \infty} \frac{d(k)}{k} = 0.
\]

Theorem 9.3 basically follows from Theorem 8.2. For the sketch of its proof see [Kaveh-Khovanskii1].

**Corollary 9.4.** Let $S$ be an ample graded semi-group with restricted growth in $\mathbb{N} \oplus \mathbb{Z}^n$. Then
\[
\lim_{k \to \infty} \frac{\#(S_k)}{k^n} = \text{Vol}_n(\Delta(S)).
\]

10. **Valuations on the field of rational functions**

We start with the definition of a pre-valuation. Let $V$ be a vector space and let $I$ be a set totally ordered with respect to some ordering $\prec$.

**Definition 10.1.** A pre-valuation on $V$ with values in $I$ is a function $v : V \setminus \{0\} \to I$ satisfying the following: 1) For all $f, g \in V$, $v(f + g) \geq \min(v(f), v(g))$; 2) For all $f \in V$ and $\lambda \neq 0$, $v(\lambda f) = v(f)$; 3) If for $f, g \in V$ we have $v(f) = v(g)$ then there is $\lambda \neq 0$ such that $v(g - \lambda f) > v(g)$.

It is easy to verify that if $L \subset V$ is a finite dimensional subspace then $\dim(L)$ is equal to $\#(v(L))$.

**Example 10.2.** Let $V$ be a finite dimensional vector space with basis $\{e_1, \ldots, e_n\}$ and let $I = \{1, \ldots, n\}$ with usual ordering of numbers. For $f = \sum_i \lambda_i e_i$ define
\[
v(f) = \min\{i \mid \lambda_i \neq 0\}.
\]

**Example 10.3** (Schubert cells in the Grassmannian). Let $\text{Gr}(n, k)$ be the Grassmannian of $k$-dimensional planes in $\mathbb{C}^n$. In Example 10.2 take $V = \mathbb{C}^n$ with standard basis. Under the pre-valuation $v$ above each $k$-dimensional subspace $L \subset \mathbb{C}^n$ goes to a subset $M \subset I$ containing $k$ elements. The set of all $k$-dimensional subspaces which are mapped onto $M$ forms the Schubert cell $X_M$ in the Grassmannian $\text{Gr}(n, k)$.

In a similar fashion to Example 10.3 the Schubert cells in the variety of complete flags can also be recovered from the pre-valuation $v$ above on $\mathbb{C}^n$.

Next we define the notion of a valuation with values in a totally ordered abelian group.

**Definition 10.4.** Let $K$ be a field and $\Gamma$ a totally ordered abelian group. A pre-valuation $v : K \setminus \{0\} \to \Gamma$ is a valuation if moreover it satisfies the following: for any $f, g \in K$ with $f, g \neq 0$, we have
\[
v(fg) = v(f) + v(g).
\]

The valuation $v$ is called faithful if its image is the whole $\Gamma$.

We will only be concerned with the field $\mathbb{C}(X)$ of rational functions on an $n$-dimensional irreducible variety $X$, and $\mathbb{Z}^n$-valued valuations on it (with respect to some total order on $\mathbb{Z}^n$).
Example 10.5. Let $X$ be an irreducible curve. Take the field of rational functions $\mathbb{C}(X)$ and $\Gamma = \mathbb{Z}$. Take a smooth point $a$ on $X$. Then the map

$$v(f) = \text{ord}_a(f)$$

defines a faithful valuation on $\mathbb{C}(X)$.

Example 10.6. Let $X$ be an irreducible $n$-dimensional variety. Take a smooth point $a \in X$. Consider a local system of coordinates with analytic coordinate functions $x_1, \ldots, x_n$ and with the origin at the point $a$. Let $\Gamma = \mathbb{Z}^n$ be the semigroup in $\mathbb{Z}^n$ of points with non-negative coordinates. Take any well-ordering $\prec$ which respects the addition, i.e. if $a \prec b$ then $a + c \prec b + c$. For a germ $f$ at the point $a$ of an analytic function in $x_1, \ldots, x_n$ let $cx_1^{a_1} \cdots x_n^{a_n}$ be the term in the Taylor expansion of $f$ with minimum exponent $\alpha(f) = (\alpha_1, \ldots, \alpha_n)$, with respect to the ordering $\prec$. For a germ $F$ at the point $a$ of a meromorphic function $F = f/g$ define $v(F)$ as $\alpha(f) - \alpha(g)$. This function $v$ induces a faithful valuation on the field of rational functions $\mathbb{C}(X)$.

Example 10.7. Let $X$ be an irreducible $n$-dimensional variety and $Y$ any variety birationally isomorphic to $X$. Then fields $\mathbb{C}(X)$ and $\mathbb{C}(Y)$ are isomorphic and thus any faithful valuation on $\mathbb{C}(Y)$ gives a faithful valuation on $\mathbb{C}(X)$ as well.

11. Main construction and theorem

Let $X$ be an irreducible $n$-dimensional variety. Fix a faithful valuation $v : \mathbb{C}(X) \rightarrow \mathbb{Z}^n$, where $\mathbb{Z}^n$ is equipped with any total ordering respecting addition.

Let $L \in \mathbf{K}_{rat}(X)$ be a finite dimensional subspace of rational functions. Consider the semi-group $S(L)$ in $\mathbb{N} \oplus \mathbb{Z}^n$ defined by

$$S(L) = \bigcup_{k > 0} \{(k, v(f)) \mid f \in L^k\}.$$

It is easy to see that $S(L)$ is a graded semi-group. Moreover by the Hilbert theorem $S(L)$ is contained in a semi-group of restricted growth.

Definition 11.1 (Newton convex body for a subspace of rational functions). We define the Newton convex body for a subspace $L$ to be the convex body $\Delta(S(L))$ associated to the semi-group $S(L)$.

Denote by $s(L)$ the index of the subgroup in $\mathbb{Z}^n$ generated by all the differences $a - b$ such that $a, b$ belong to the same set $S_m(L)$ for some $m > 0$. Also let $Y_L$ be the closure of the image of the variety $X$ (in fact image of a Zariski open subset of $X$) under the Kodaira rational map $\Phi_L : X \rightarrow \mathbb{P}(L^*)$. If $\dim(Y_L)$ is equal to $\dim(X)$ then the Kodaira map from $X$ to $Y_L$ has finite mapping degree. Denote this mapping degree by $d(L)$.

Theorem 11.2 (Main theorem). Let $X$ be an irreducible $n$-dimensional quasi-projective variety and let $L \in \mathbf{K}_{rat}(X)$ with the Kodaira map $\Phi_L : X \rightarrow \mathbb{P}(L^*)$. Then:

1) Complex dimension of the variety $Y_L$ is equal to the real dimension of the Newton convex body $\Delta(S(L))$.

2) If $\dim(Y_L) = n$ then

$$[L, \ldots, L] = \frac{n!d(L)}{s(L)} \text{Vol}_n(\Delta(S(L))).$$
3) In particular, if $\Phi_L : X \to Y_L$ is a birational isomorphism then
$$[L, \ldots, L] = n!\text{Vol}(\Delta(S(L))).$$

4) For any two subspaces $L_1, L_2 \in K_{rat}(X)$ we have
$$\Delta(S(L_1)) + \Delta(S(L_2)) \subseteq \Delta(S(L_1 L_2)).$$

The proof of the main theorem is based on Theorem 9.3 (which describes the asymptotic behavior of an ample graded semigroup of restricted growth) and the Hilbert (Theorem 8.1). The sketch of proof can be found in [Kaveh-Khovanskii1].

12. Algebraic analogue of Alexandrov-Fenchel inequalities

Part (2) of the main theorem (Theorem 11.2) can be considered as a wide-reaching generalization of the Kušnirenko theorem, in which instead of $(\mathbb{C}^*)^n$ one takes any $n$-dimensional irreducible variety $X$, and instead of a finite dimensional space generated by monomial one takes any finite dimensional space $L$ of rational functions. The proof of Theorem 11.2 is an extension of the arguments used in [Khovanskii2] to prove Kušnirenko theorem (see also Section 8). As we mentioned the Bernstein theorem (Theorem 5.4) follows immediately from the Kušnirenko theorem and the identity
$$L_{A+B} = L_AL_B.$$  

Thus the Bernstein-Kušnirenko theorem is a corollary of our Theorem 11.2.

Note that although the Newton convex body $\Delta(S(L))$ depends on a choice of a faithful valuation, its volume depends on $L$ only: after multiplication by $n!$ it equals the self-intersection index $[L, \ldots, L]$.

Our generalization of the Kušnirenko theorem does not imply the generalization of the Bernstein theorem. The point is that in general we do not always have an equality $\Delta(S(L_1)) + \Delta(S(L_2)) = \Delta(S(L_1 L_2))$. In fact, by Theorem 11.2(4), what is always true is the inclusion
$$\Delta(S(L_1)) + \Delta(S(L_2)) \subseteq \Delta(S(L_1 L_2)).$$

This inclusion is sufficient for us to prove the following interesting corollary:

Let us call a subspace $L \in K_{rat}(X)$ a big subspace if for some $m > 0$ the Kodaira rational map of $L^m$ is a birational isomorphism between $X$ and its image. It is not hard to show that the product of two big subspaces is again a big subspace and thus the big subspaces form a subsemi-group of $K_{rat}(X)$.

**Corollary 12.1** (Algebraic analogue of Brunn-Minkowski). Assume that $L, G \in K_{rat}(X)$ are big subspaces. Then
$$[L, \ldots, L]^{1/n} + [G, \ldots, G]^{1/n} \leq [L, G, \ldots, L, G]^{1/n}.$$  

**Proof.** Since replacing $L$ and $G$ by $L^m$ and $G^m$ does not change the inequality, without loss of generality, we can assume that the Kodaira maps of $L$ and $G$ are birational isomorphisms onto their images. From Part (4) in Theorem 11.2 we have $\Delta(S(L)) + \Delta(S(G)) \subseteq \Delta(S(LG))$. So $\text{Vol}(\Delta(S(L)) + \Delta(S(G))) \leq \text{Vol}(\Delta(S(LG)))$. Also from Part (3) in the same theorem we have
$$[L, \ldots, L] = n!\text{Vol}(\Delta(S(L))),$$
$$[G, \ldots, G] = n!\text{Vol}(\Delta(S(G))),$$
$$[L, G, \ldots, L, G] = n!\text{Vol}(\Delta(S(LG))).$$

To complete the proof it is enough to use the Brunn-Minkowski inequality. \qed
Corollary 12.2 (A version of Hodge inequality). If \( L, G \in K_{\text{rat}}(X) \) are big subspaces and \( X \) is an algebraic surface then
\[
[L, L][G, G] \leq [L, G]^2.
\]

Proof. From Corollary 12.1, for \( n = 2 \), we have
\[
[L, L] + 2[L, G] + [G, G] = [LG, LG] \geq \left( \frac{[L, L]}{2} + \frac{[G, G]}{2} \right)^{1/2} = [L, L] + 2[L, L]^{1/2}[G, G]^{1/2} + [G, G],
\]
which readily implies Hodge inequality. \( \square \)

Thus Theorem 11.2 immediately enables us to reduce the Hodge inequality to the isoperimetric inequality. This way, we can easily prove an analogue of Alexandrov-Fenchel inequality and its corollaries for intersection index:

Theorem 12.3 (Algebraic analogue of Alexandrov-Fenchel inequality). Let \( X \) be an irreducible \( n \)-dimensional quasi-projective variety and let \( L_1, \ldots, L_n \in K_{\text{rat}}(X) \) be big subspaces. Then the following inequality holds
\[
[L_1, L_2, L_3, \ldots, L_n]^2 \geq [L_1, L_1, L_3, \ldots, L_n] [L_2, L_2, L_3, \ldots, L_n].
\]

Corollary 12.4 (Corollaries of the algebraic analogue of Alexandrov–Fenchel inequality). Let \( X \) be the algebraic analogue of Alexandrov–Fenchel inequality. Let \( X \) be an \( n \)-dimensional irreducible quasi-projective variety.

1) Let \( 2 \leq m \leq n \) and \( k_1 + \cdots + k_r = m \) with \( k_i \in \mathbb{N} \). Take big subspaces of rational functions \( L_1, \ldots, L_n \in K_{\text{rat}}(X) \). Then
\[
[k_1 * L_1, \ldots, k_r * L_r, L_{m+1}, \ldots, L_n]^m \geq \prod_{1 \leq j \leq r} [m * L_j, L_{m+1}, \ldots, L_n]^{k_j}.
\]

2) (Generalized Brunn-Minkowski inequality) For any fixed big subspaces \( L_{m+1}, \ldots, L_n \in K_{\text{rat}}(X) \), the function
\[
F : L \mapsto [m * L, L_{m+1}, \ldots, L_n]^{1/m},
\]
is a concave function on the semi-group of big subspaces.

As we saw above, Bernstein-Kušnirenko theorem follows from the main theorem. Applying algebraic analogue of Alexandrov-Fenchel inequality to the situation considered in Bernstein-Kušnirenko theorem one can prove Alexandrov-Fenchel inequality for convex polyhedra with integral vertices. By homogeneity it implies Alexandrov-Fenchel inequality for convex polyhedra with rational vertices. But since each convex body can be approximated by polyhedra with rational vertices, by continuity we obtain a proof of Alexandrov-Fenchel inequality in complete generality.

Thus Bernstein-Kušnirenko theorem and Alexandrov-Fenchel inequality in algebra and in geometry can be considered as corollaries of our Theorem 11.2.

13. ADDITIVITY OF THE NEWTON CONVEX BODY FOR VARIETIES WITH REDUCTIVE GROUP ACTION

While the additivity of the Newton convex body does not hold in general but, as mentioned in Section 8, it holds for the subspaces \( L_A \) of Laurent polynomials on \((\mathbb{C}^*)^n\) spanned by monomials. The subspaces \( L_A \) are exactly the subspaces which are stable under the natural action of the multiplicative group \((\mathbb{C}^*)^n\) on Laurent
polynomials (induced by the natural action of $(\mathbb{C}^*)^n$ on itself). We will see that the additivity generalizes to some classes of varieties with a reductive group action.

Let $G$ be a connected reductive algebraic group over $\mathbb{C}$, i.e. the complexification of a connected compact real Lie group. Also let $X$ be a $G$-variety, that is a variety equipped with an algebraic action of $G$.

The group $G$ naturally acts on $\mathcal{C}(X)$ by $(g \cdot f)(x) = f(g^{-1} \cdot x)$. A subspace $L \in K_{\text{rat}}(X)$ is $G$-stable if for any $f \in L$ and $g \in G$ we have $g \cdot f \in L$.

**Theorem 13.1.** Let $X$ be an $n$-dimensional variety with an algebraic action of $G$. Then there is a naturally defined faithful valuation $v : \mathcal{C}(X) \to \mathbb{Z}^n$ such that for any $G$-stable subspace $L \in K_{\text{rat}}(X)$, the Newton convex body $\Delta(S(L))$ is in fact a polytope.

**Definition 13.2.** Let $V$ be a finite dimensional representation of $G$. Let $v = v_1 + \ldots + v_k$ be a sum of highest weight vectors in $V$. The closure of the $G$-orbit of $v$ in $V$ is called an $S$-variety.

Affine toric varieties are $S$-varieties for $G = (\mathbb{C}^*)^n$.

**Theorem 13.3.** Let $X$ be an $S$-variety for one of the groups $G = \text{SL}(n, \mathbb{C})$, $\text{SO}(n, \mathbb{C})$, $\text{SP}(2n, \mathbb{C})$, $(\mathbb{C}^*)^n$, or a direct product of them. Then for the valuation in Theorem 13.1 and for any choice of $G$-stable subspaces $L_1, L_2$ in $K_{\text{rat}}(X)$ we have

$$\Delta(S(L_1L_2)) = \Delta(S(L_1)) + \Delta(S(L_2)).$$

**Corollary 13.4** (Bernstein theorem for $S$-varieties). Let $X$ be an $S$-variety for one of the groups $G = \text{SL}(n, \mathbb{C})$, $\text{SO}(n, \mathbb{C})$, $\text{SP}(2n, \mathbb{C})$, $(\mathbb{C}^*)^n$, or a direct product of them. Let $L_1, \ldots, L_n \in K_{\text{rat}}(X)$ be $G$-stable subspaces. Then, for the valuation in Theorem 13.1, we have

$$[L_1, \ldots, L_n] = n!V(\Delta(S(L_1)), \ldots, \Delta(S(L_n))),$$

where $V$ is the mixed volume.

Another class of $G$-varieties for which the additivity of the Newton polytope holds is the class of symmetric homogeneous spaces.

**Definition 13.5.** Let $\sigma$ be an involution of $G$, i.e. an order 2 algebraic automorphism. Let $H = G^\sigma$ be the fixed point subgroup of $\sigma$. The homogeneous space $G/H$ is called a symmetric homogeneous space.

**Example 13.6.** The map $M \mapsto (M^{-1})^t$ is an involution of $G = \text{SL}(n, \mathbb{C})$ with fixed point subgroup $H = \text{SO}(n, \mathbb{C})$. The symmetric homogeneous space $G/H$ can be identified with the space of non-degenerate quadrics in $\mathbb{C}P^{n-1}$.

Any symmetric homogeneous space is an affine $G$-variety (with left $G$-action).

Under mild conditions on the $L_i$, analogues of Theorem 13.3 and Corollary 13.4 hold for symmetric varieties.

Finally, the above theorems extend to subspaces of sections of $G$-line bundles.

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