A New Quantum Algorithm for the Random Subset Sum Problem

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Abstract Solving random subset sum instances plays an important role in constructing cryptographic systems. For the random subset sum problem, in 2013 Bernstein et al. proposed a quantum algorithm with heuristic time complexity $\tilde{O}(2^{0.241n})$, where the “$\tilde{O}$” symbol is used to omit poly($\log n$) factors. In 2018, Helm and May proposed another quantum algorithm that reduces the heuristic time and memory complexity to $\tilde{O}(2^{0.226n})$. In this paper, a new quantum algorithm is proposed, with heuristic time and memory complexity $\tilde{O}(2^{0.209n})$.

Keywords Random subset sum problem, Quantum algorithm, Quantum walk, Representation technique.

1 Introduction

The subset sum problem (SSP) is a fundamental problem in theoretical computer science, and is one of the most famous NP-hard problems[12]. Due to its difficulty, the subset sum problem is popular in designing cryptosystems[8, 10, 17, 21, 22].

Given data $(a_1, a_2, \ldots, a_n, s) \in (\mathbb{Z}_2^n)^{n+1}$, called an instance, there exist two forms of SSP. The first is the decision SSP: decide whether there exists a subset of \{a_1, a_2, \ldots, a_n\} summing up to $s$; or in vector form, decide whether there exists a vector $e \in \{0, 1\}^n$ such that

$$\langle a, e \rangle \equiv s \mod 2^n.$$  \hspace{1cm} (1)

The second is the computational SSP: find a subset of \{a_1, a_2, \ldots, a_n\} summing up to $s$. The decision SSP is NP-complete. Given access to an oracle that solves the decision SSP, the computational SSP can be solved with $n$ calls to this oracle.

If the data $(a_1, a_2, \ldots, a_n, s) \in (\mathbb{Z}_2^n)^{n+1}$ is given randomly, the subset sum problem becomes the random subset sum problem (RSSP).

Definition 1.1 Let $a \in (\mathbb{Z}_2^n)^n$ be chosen at random uniformly. For a random $e \in \{0, 1\}^n$ with $|e| = \frac{n}{2}$, let $s \equiv \langle a, e \rangle \mod 2^n$, where $|e|$ stands for the Hamming weight of $e$. Then $(a, s) \in (\mathbb{Z}_2^n)^{n+1}$ is called a random subset sum instance. The density of the random subset sum instance is

$$d := \frac{n}{\log(\max_i a_i)}.$$  \hspace{1cm} (2)
Every \( f \in \{0, 1\}^n \) satisfying \( \langle a, f \rangle \equiv s \mod 2^n \) is called a solution to the random subset sum instance. The random subset sum problem (RSSP) refers to the problem of finding a solution to a random subset sum instance.

**SSP and RSSP algorithms running on classical computer.**

For **SSP**, enumerating all possible \( e \in \{0, 1\}^n \) and checking whether \( \langle a, e \rangle \equiv s \mod 2^n \) can solve this problem in time \( \tilde{O}(2^n) \). In 1974, Horowitz and Sahni (HS) [16] introduced a Meet-in-the-Middle algorithm with time and space complexity \( \tilde{O}(2^{n/2}) \). In the HS algorithm, enumerating all \( e_1 \in \{0, 1\}^{n/2} \times \{0\}^{n/2} \), \( e_2 \in \{0\}^{n/2} \times \{0, 1\}^{n/2} \) and setting up two ordered lists \( L_1, L_2 \), that sorted by \( \langle a, e_1 \rangle \), \( s - \langle a, e_2 \rangle \) accordingly. Then, for each \( (e_1, \langle a, e_1 \rangle) \in L_1 \), looking for \( (e_2, s - \langle a, e_2 \rangle) \in L_2 \) that satisfies \( s - \langle a, e_2 \rangle \equiv \langle a, e_1 \rangle \mod 2^n \) by binary search. If there is a collision \( \langle a, e_1 \rangle \equiv s - \langle a, e_2 \rangle \mod 2^n \), then \( e_1 + e_2 \) is a solution.

In 1981, Schroeppel and Shamir (SS) [27] improved this to time complexity \( O(2^{n/2}) \) with only space complexity \( \tilde{O}(2^{n/2}) \). These algorithms are still the fastest known for solving general instances of subset sum.

For **RSSP**, Brickell [4], Lagarias and Odlyzko [20] showed that random subset sum instances can be solved with density \( d < 0.64 \), by giving an oracle solving the shortest vector problem (SVP) in lattices.

In 1991 this bound was improved by Coster et al. [7] and Joux, Stern [18] to \( d < 0.94 \). Note that this transformation does not rule out the hardness of subset sum problem in the low-density regime, since solving SVP is known to be NP-hard [1]. In the high-density regime with \( d = \Omega(\frac{1}{\log n}) \) dynamic programming solves subset sum problem efficiently [13].

However, for the case \( d \approx 1 \) only exponential time algorithms are known. In a breakthrough paper, Howgrave-Graham and Joux (HGJ) [14] at Eurocrypt 2010 showed that random subset sum instances can be solved in time \( \tilde{O}(2^{0.317n}) \). The main technique used is called representation technique. Recall that the HS algorithm splits \( e \) as \( e_1 \in \{0, 1\}^{n/2} \times \{0\}^{n/2} \) and \( e_2 \in \{0\}^{n/2} \times \{0, 1\}^{n/2} \). The main idea of HGJ is to represent \( e \) in a different, ambiguous way as a 4-sum \( e_1 + e_2 + e_3 + e_4 \) with \( e_i \in \{0, 1\}^n, |e_i| = \frac{n}{8}, 1 \leq i \leq 4 \). As a consequence, the HGJ technique is called in the literature representation technique. The HGJ algorithm first constructs 4 lists of candidates \( c_i \in \{0, 1\}^n \) for \( e_i \) by enumerating all candidates \( c_i \in \{0, 1\}^n \) with \( |c_i| = \frac{n}{8} \). It then computes 2-sums \( c_1 + c_2, c_3 + c_4 \in \{0, 1, 2\}^n \) and filters out all sums that contain 2-entries. To control the list sizes (which in turn determine the run time), some constraints are introduced. At the same time, these constraints reduce the number of representations. The key observation is that finding only one representation of \( e \) is sufficient to solve the random subset sum problem. Therefore, the parameters in HGJ need to be optimized based on this objective.

At Eurocrypt 2011, Becker, Coron and Joux (BCJ) [5] proposed a modification to the HGJ algorithm with heuristic run time \( \tilde{O}(2^{0.291n}) \). The core idea of the BCJ algorithm is to represent \( e \) as an 8-sum \( e_1 + \cdots + e_8 \) with \( e_i \in \{-1, 0, 1\}^n, 1 \leq i \leq 8 \). The BCJ algorithm as well as the HGJ algorithm proceed in a divide-and-conquer fashion. The BCJ algorithm first uses enumeration to construct 8 lists of candidates \( c_i \in \{-1, 0, 1\}^n \) with a certain predefined (optimized) number of \(-1\)’s, \( 0\)’s and \( 1\)’s. It then computes 2-sums \( c_1 + c_2, \ldots, c_7 + c_8 \in \{0, 1\}^n \).
\{-2, -1, 0, 1, 2\}^n and filters out all sums that contain \(\pm 2\)-entries, and in addition filters out among all remaining vectors those that do not possess another pre-defined (optimized) number of \(-1\)'s, 0's and 1's. As above, the parameters in BCJ need to be optimized to make sure that one representation of \(e\) can be found.

In 2019, Esser and May (EM) [9] proposed a new heuristic algorithm based on representation and sampling technique with run time \(\widetilde{O}(2^{0.255n})\). While the initial lists in HGJ and BCJ are constructed by enumeration, the initial lists in EM are constructed by sampling from a Bernoulli distribution. Sampling technique introduces variance that increases the amount of representations and brings more optimization flexibility. Note that all lists in EM form a tree. A remarkable property is that the complexity of the EM algorithm improves with increasing tree depth.

**SSP and RSSP algorithms running on quantum computer.**

In 2013, Bernstein, Jeffery, Lange and Meurer [6] constructed quantum subset sum algorithms, inspired by the HS algorithm, the SS algorithm and the HGJ algorithm. In detail, Bernstein et al. showed that the quantum HS algorithm achieve run time \(\widetilde{O}(2^{n/3})\). Moreover, a first quantum version of the SS algorithm with Grover search [11] runs in time \(\widetilde{O}(2^{3n/8})\) using only space \(\widetilde{O}(2^{n/8})\). A second quantum version of the SS algorithm using quantum walks [2, 3] achieves time \(\widetilde{O}(2^{0.3n})\). Eventually, Bernstein et al. used the quantum walk framework of Magniez et al. [23] to achieve a quantum version of the HGJ algorithm with time and space complexity \(\widetilde{O}(2^{0.241n})\). In 2018, Helm and May [15] achieve a quantum version of the BCJ algorithm with time and space complexity \(\widetilde{O}(2^{0.226n})\), which is the best known quantum random subset sum algorithm.

Quantum algorithms based on the quantum walk framework are designed in the following three steps:

1. Start with a classic algorithm.
2. Generalize to a lower-probability algorithm and build a data structure that expresses the entire computation of the lower-probability algorithm.
3. Apply a quantum walk.

The key point of quantum HGJ and quantum BCJ algorithms is that we no longer enumerate the initial lists, but only start with random subsets of the initial lists with some fixed size that has to be optimized. On the one hand, subsets of the leaves lists yields small list sizes, which speeds up the construction of lists. On the other hand, subsets of the leaves lists reduces the probability that the corresponding classical algorithms succeed. The quantum algorithms achieve the acceleration of the corresponding classical algorithms because quantum walks amplify the probability of success.

**Contribution of this paper.**

We propose a new quantum algorithm with running time down to \(\widetilde{O}(2^{0.209n})\). Our algorithm is actually a quantum version of the EM algorithm. Note that the initial lists in EM are constructed by sampling from a Bernoulli distribution. Recall that the initial lists in HGJ are
constructed by enumerating all candidates \( c_i \in \{0, 1\}^n \) with \(|c_i| = \frac{n}{8}\). Now consider how the initial lists are built in EM. All we know is that the elements of the initial lists belong to \( \{0, 1\}^n \). The existence of randomness prevents us from using quantum walks directly. One simple way to solve this problem is firstly sampling to give us the initial lists. Next, carry out quantum walks. Moreover, we need to define an appropriate quantum walk for the EM algorithm within the framework of Magniez et al. [23].

Note that, whereas the complexity of the EM algorithm improves with increasing tree depth, our quantum algorithm is optimal when the search depth is 4.

The paper is organized as follows. In Section 2 we outline the quantum walk technology and the EM classical algorithm. In Section 3 we firstly describe the connection between random subset sum problem and graph search problem. Then we define an appropriate data structure and give our quantum algorithm.

2 Preliminaries

By \( H(\cdot) \) we refer to the binary entropy function, which is defined on input \( 0 \leq \alpha \leq 1 \) as \( H(\alpha) := -\alpha \log_2 \alpha - (1 - \alpha) \log_2 (1 - \alpha) \), where \( \log \) is the logarithmic function with base 2 and we use the convention \( 0 \log 0 := 0 \). We approximate binomial coefficients by the entropy function, derived from Stirlings formula \( \binom{n}{m} = \widetilde{\Theta}(2^{nH(m/n)}) \).

Let \( X \sim D \) be a discrete random variable following the distribution \( D \), which is defined on a finite alphabet \( \Lambda \). For \( x \in \Lambda \) let \( p_X(x) := \Pr[X = x] \). We define the entropy of a random variable or equivalently its distribution as \( H(X) = H(D) := -\sum_{x \in \Lambda} p_X(x) \log p_X(x) \).

For \( 0 \leq \alpha \leq 1 \) we refer by \( B(\alpha) \) to the Bernoulli distribution with parameter \( \alpha \), that is for \( X \sim B(\alpha) \) we have \( \Pr[X = 1] = \alpha \) and \( \Pr[X = 0] = 1 - \alpha \). The sum of \( m \) iid \( B(\alpha) \)-distributed random variables is binomially distributed with parameters \( m \) and \( \alpha \), which we denote by \( \text{Bin}_{m, \alpha} \). Let \( x \sim \text{Bin}^n_{m, \alpha} \) denote a vector of \( n \) iid random variables, thus \( x = (x_1, \ldots, x_n) \) with \( x_i \sim \text{Bin}_{m, \alpha} \) and therefore \( x_i \in \{0, \ldots, m\} \). Note that the entropy of such a vector is \( H(x) = H(\text{Bin}^n_{m, \alpha}) = H(\text{Bin}_{m, \alpha}) n \).

2.1 Quantum walks

Problem 2.1 (Graph Search Problem) Given a graph \( G = (V, E) \) and a set of vertices \( M \subset V \), called the set of all marked vertices, find a marked vertex \( u \in M \).

The graph search problem can be solved using a quantum walk on graph \( G \). A state of the walk will correspond to a vertex \( u \in V \), and a data structure \( d(u) \) associated to each state \( u \) will help us to decide whether \( u \) is marked. Three types of cost are associated with \( d(u) \). The setup cost \( T_s \) is the cost to set up the data structure \( d(u) \) for a given vertex \( u \in V \). The update cost \( T_u \) is the cost to update the data structure, i.e., the cost needed to convert \( d(u) \) into \( d(v) \) for two given connected vertices \( u, v \in V \). The checking cost \( T_c \) is the cost of checking with high probability whether \( u \) is marked, given \( u \in V \) and \( d(u) \).
Several quantum walks algorithms have been proposed by many authors, notably Ambainis [2], Szegedy [24], and Magniez et al. [23]. A survey of these results can be found in [25, 26]. The following theorem is important and useful.

**Theorem 2.2** (Magniez et al. [23]) Let $G = (V, E)$ be a regular graph with spectral gap $\delta$, and let $\varepsilon > 0$ be a lower bound on the probability that a vertex chosen randomly of $G$ is marked. Let $T_s, T_u, T_c$ be the setup, update and checking cost. Then there exists a quantum algorithm that with high probability finds a marked vertex with cost

$$T = T_s + \frac{1}{\sqrt{\varepsilon}} \left( \frac{1}{\sqrt{\delta}} T_u + T_c \right).$$

Note that $T$ is time (memory) complexity of this quantum algorithm if $T_s, T_u, T_c$ are measured in time (memory).

**Definition 2.3** (Johnson Graph) Given a set $L$ with $|L| = N$, the Johnson graph $J = (N, r)$ is an undirected graph whose vertices are the subsets of $L$ containing $r$ elements, where $0 \leq r \leq N$. An edge between two vertices $S$ and $S'$ exists iff $|S \cap S'| = r - 1$. That is, two vertices are adjacent iff they differ in only one element that belongs to $L$.

**Definition 2.4** (Cartesian Product of Graphs) Let $G_1 = (V_1, E_1), G_2 = (V_2, E_2)$ be undirected graphs. The Cartesian product $G_1 \times G_2 = (V, E)$ is defined via

$$V = V_1 \times V_2 = \{v_1v_2 | v_1 \in V_1, v_2 \in V_2\}$$

and

$$E = \{(u_1u_2, v_1v_2) | (u_1 = v_1 \land (u_2, v_2) \in E_2) \lor ((u_1, v_1) \in E_1 \land u_2 = v_2)\}.$$

For Johnson graphs it is well-known that $\delta(J(N, r)) = \Omega(1/r)$. The following lemma gives us the spectral gap of the cartesian product of Johnson graphs.

**Lemma 2.5** (Kachigar, Tillich [19]) Let $J(N, r)$ be a Johnson graph, and let $J^m(N, r) := \times^m J(n, r)$. Then $\delta(J^m(N, r)) \geq \frac{1}{m} \delta(J(N, r))$.

**2.2 The EM Classical Algorithm**

Denote by $EM^{(d)}$ the EM algorithm with tree depth $d$. Whereas run time of the EM algorithm decreases with increasing tree depth $d$, our quantum algorithm is optimal at the tree depth 4. Thus, we only describe $EM^{(4)}$.

Let $\langle a, s \rangle \in (\mathbb{Z}_2^n)^{n+1}$ be a subset sum instance with a solution $\mathbf{e} \in \{0, 1\}^n$ with $|\mathbf{e}| = \frac{n}{2}$. That is, $\langle a, \mathbf{e} \rangle \equiv s \mod 2^n$.

The basic idea of representation is to represent the solution $\mathbf{e}$ as a sum $\mathbf{e}_1 + \mathbf{e}_2$ where $\mathbf{e}_1$ and $\mathbf{e}_2 \in \{0, 1\}^n$. If two lists $L_1$ and $L_2$ whose elements are candidates for $\mathbf{e}_1, \mathbf{e}_2$ accordingly can effectively (respectively $\overline{O}(L_1), \overline{O}(L_2)$) be constructed, then using the list join operator above, we can effectively get the solution $\mathbf{e}$ via the joined list $L_0 = \{\mathbf{e} = \mathbf{e}_1 + \mathbf{e}_2 | (\mathbf{e}_1, \mathbf{e}_2) \in L_1 \times L_2 \land \langle a, \mathbf{e}_1 \rangle + \langle a, \mathbf{e}_2 \rangle \equiv s \mod 2^n\}$. Note that sorting $L_1$ and searching are performed with respect to $\langle a, \mathbf{e}_1 \rangle$ and $\langle a, \mathbf{e}_2 \rangle$, where $\mathbf{e}_1 \in L_1, \mathbf{e}_2 \in L_2$. Generally speaking, this effectiveness of constructing base lists $L_1, L_2$ is not guaranteed. Thus, splitting the solution $\mathbf{e}$ several times leads to improve the complexity of the EM algorithm. That is, represent the solution $\mathbf{e}$ as a sum $\sum_j \mathbf{e}_j$ for some $j$ and construct lists $L_j$ whose elements are candidates for $\mathbf{e}_j$ accordingly.
**Definition 2.6** (The Level-\(i\) Representation) \((\mathbf{e}_1^{(i)}, \mathbf{e}_2^{(i)}, \ldots, \mathbf{e}_{2^{4-i}}^{(i)})\) is called a level-\(i\) representation if every \(\mathbf{e}_j^{(i)} \in \{0, 1\}^n, 1 \leq j \leq 2^{4-i}\) and \(\sum_j \mathbf{e}_j^{(i)} = \mathbf{e}\).

In \(EM^{(4)}\), the solution \(\mathbf{e}\) is represented as a sum \(\sum_j \mathbf{e}_j^{(i)}\) with \(\mathbf{e}_j^{(i)} \in \{0, 1\}^n, 1 \leq j \leq 2^{4-i}\). Construct lists \(L_j^{(i)}\) whose elements are candidates for \(\mathbf{e}_j^{(i)}\) accordingly. Then all lists form a tree. The tree structure of \(EM^{(4)}\) is shown in Figure 1. Define the join operator for \(k \in \mathbb{N}, 0 \leq k \leq n\) and \(s_j^{(i)} \in \mathbb{Z}_2^k\) as \(L_j^{(i)} = L_{2j-1} \bowtie_k L_{2j}^{(i)} := \{x_1 + x_2 | (x_1, x_2) \in L_{2j-1} \times L_{2j}^{(i-1)} \land \langle a, x_1 + x_2 \rangle \equiv s_j^{(i)} \mod 2^k\},\) where \(1 \leq i \leq 4, 1 \leq j \leq 2^{4-i}\). Also write \(L_j^{(i)} = L_{2j-1} \bowtie_k L_{2j}^{(i-1)}\).

Consider the join operator. On the one hand, these constraints \(\langle a, x_1 + x_2 \rangle \equiv s_j^{(i)} \mod 2^k\) reduce the search space \(|L_j^{(i)}|\). On the other hand, these constraints reduce the number of level-\(i\) representations. The crucial observation is that it is sufficient to construct a single level-4 representation \((\mathbf{e}_1^{(4)})\) of \(\mathbf{e}\) in list \(L_1^{(4)}\) for solving the original problem.

![Figure 1](image_url)  
**Figure 1** Tree structure of \(EM^{(4)}\). The portion covered by the slash represents the part of the lists that meet the constraints accordingly.

**The Process of \(EM^{(4)}\)**

To construct on expectation a single level-4 representation, initially construct the level 0 lists \(L_{2j-1}^{(0)}, L_{2j}^{(0)}\), where \(1 \leq j \leq 8\). For each \(j\), sample iid vectors \((x, 0^{n/2}) \in L_{2j-1}^{(0)}, \) and \((0^{n/2}, y) \in L_{2j}^{(0)}\), where \(x, y \sim \mathcal{B}^2(\alpha) \in \{0, 1\}^{n/2}\). Then, construct the level 1 lists \(L_j^{(1)} = L_{2j-1} \bowtie_1 L_{2j}^{(0)}\).
where \(1 \leq j \leq 8\). Choose random \(s^{(1)}_1, s^{(1)}_2, \ldots, s^{(1)}_7 \in \mathbb{Z}_{2^j}\), and let \(s^{(1)}_8 = s - \sum_{j=1}^{7} s^{(1)}_j \mod 2^i\). By the definition of the join operator, on level 1 we get only those candidates \(c^{(1)}_j \in L^{(1)}_j\) satisfying \(\langle a, c^{(1)}_j \rangle \equiv s^{(1)}_j \mod 2^i\) for some \(0 \leq l_1 \leq n, 1 \leq j \leq 8\). Note that all level-1 candidates \(c^{(1)}_j\) are vectors from \(\{0, 1\}^n\).

Similarly, construct the level 2 lists \(L^{(2)}_j = L^{(1)}_{2j-1} \bowtie_{l_1+l_2} L^{(1)}_{2j}\), where \(1 \leq j \leq 4\). Note that \(s^{(2)}_j\) be chosen randomly on \(\mathbb{Z}_{2^j+2}\) satisfying \(s^{(2)}_j \equiv s^{(1)}_{2j-1} + s^{(1)}_{2j} \mod 2^i, 1 \leq j \leq 3\) and \(s^{(2)}_4 \equiv s - s^{(2)}_1 - s^{(2)}_2 - s^{(2)}_3 \mod 2^i+l_2\). Then, construct the level 3 lists \(L^{(3)}_j = L^{(2)}_{2j-1} \bowtie_{l_1+l_2+l_3} L^{(2)}_{2j}\), where \(1 \leq j \leq 2\). Note that \(s^{(3)}_1\) be chosen randomly on \(\mathbb{Z}_{2^j+2+l_3}\) satisfying \(s^{(3)}_1 \equiv s^{(2)}_1 + s^{(2)}_2 \mod 2^i+l_2\) and \(s^{(3)}_2 \equiv s - s^{(3)}_1 \mod 2^i+l_2+l_3\). Finally, construct \(L^{(4)}_1 = L^{(3)}_{(1)} \bowtie_{l_3} L^{(3)}_{(2)}\) by setting \(s^{(4)} = s\). If \(\exists c^{(4)} \in L^{(4)}_1\) satisfying \(|c^{(4)}| = \frac{n}{2}\), then \(c^{(4)}\) is a solution of the original random subset sum instance.

Note that any non-binary \(c^{(i)}_j \in L^{(i)}_j\) cannot be part of a valid representation of \(e\), and may safely be filtered out. Therefore, after constructing each \(L^{(i)}_j\), immediately eliminate all non-binary vectors.

A pseudocode description of the \(EM^{(4)}\) algorithm is given by Algorithm 1.

**Algorithm 1 \(EM^{(4)}\)**

**Input:** subset sum instance \((a, s) \in (\mathbb{Z}_{2^n})^{n+1};\) parameters \(\alpha \in (0, 1)\), and \(l_1, l_2, l_3 \in \mathbb{N}\) with \(l_1 + l_2 + l_3 \leq n\).

**Output:** solution \(e \in \{0, 1\}^n\) with \(|e| = \frac{n}{2}\), or \(\perp\) if no solution is found.

1: Sample all level-0 lists \(L^{(0)}_i\) for \(i = 1, \ldots, 16\) as following:

(1.1) Initially set \(L^{(0)}_1 = \emptyset, i = 1, \ldots, 16\).

(1.2) for \(i = 1\) to 8, do the following:

(1.2.1) Repeat \(L^{(0)}_{2i-1} \leftarrow L^{(0)}_{2i-1} \cup \{(x, 0^{n/2}) \in \{0, 1\}^n | x \sim B^{\perp} (\alpha)\}\) until \(|L^{(0)}_{2i-1}| \geq (\frac{n}{2})^{\alpha/2}\).

(1.2.2) Repeat \(L^{(0)}_{2i} \leftarrow L^{(0)}_{2i} \cup \{(0^{n/2}, x) \in \{0, 1\}^n | x \sim B^{\perp} (\alpha)\}\) until \(|L^{(0)}_{2i}| \geq (\frac{n}{2})^{\alpha/2}\).

2: Compute all level-i lists \(L^{(i)}_j\) for \(i = 1, \ldots, 4, j = 1, \ldots, 2^{4-i}\):

for \(i = 1\) to 4, do the following:

(2.1) Compute \(k_i = \sum_{j=1}^{i-1} l_j\).

(2.2) Choose random \(s^{(i)}_j \in \mathbb{Z}_{2^{k_i+i}}\) satisfying \(s^{(i)}_j \equiv s^{(i-1)}_{2j-1} + s^{(i-1)}_{2j-1} \mod 2^{k_i}\), for all \(j = 1, \ldots, 2^{4-i-1}\).

(2.3) Compute \(s^{(i)}_{2^i} \equiv s - \sum_{j=1}^{2^{i-1}-1} s^{(i)}_j\).

(2.4) for \(j = 1\) to \(2^{4-i}\), do

(2.4.1) Compute \(L^{(i)}_j = \text{filter}(L^{(i-1)}_{2j-1}) \bowtie_{k_i+l_i} \text{filter}(L^{(i-1)}_{2j})\).

(2.4.2) Compute \(\text{filter}(L^{(i)}_j)\).

3: if \(\exists e \in L^{(4)}\) such that \(|e| = \frac{n}{2}\), then return \(e\), else return \(\perp\).

**Lemma 2.7** [9] The run time for sampling the level-0 lists in Algorithm 1 is \(\widetilde{O}(2^{H(\alpha)n/2})\).

**Heuristic 1** [9].

Heuristically assume that the random variable that counts the number of representations
per run of the \( EM^{(d)} \), \( d \geq 3 \) algorithm is sharply centered around its expectation to conclude that a single run (or at most polynomially many runs) suffices to find a solution with good probability.

This treatment is similar to that in Wagners original \( k \)-tree algorithm \[28\] and its applications \[5,14\].

**Lemma 2.8** \[9\] Denote

\[
EMC1 : \quad 7l_1 + 3l_2 + l_3 \leq (8H(\alpha) + \frac{1}{2} \log(8\alpha(1 - \alpha)^{15}))n, \quad (4)
\]

\[
EMC2 : \quad 4l_1 \geq (8H(\alpha) - 4H(Bin_2,\alpha))n, \quad (5)
\]

\[
EMC3 : \quad 2l_2 \geq (8H(\alpha) - 2H(Bin_4,\alpha))n. \quad (6)
\]

Then

- **constraint EMC1** and Heuristic 1 guarantees that in a single run (or at most polynomially many runs) of \( EM^{(4)} \), the expected number of returned representations of the solution is at least one.

- **Constraints EMC2, EMC3** are necessary to ensure that Heuristic 1 does not fail.

**Theorem 2.9** \[9\] Under Heuristic 1, \( EM^{(4)} \) solves the random subset sum problem in time and memory \( \tilde{O}(2^{0.266n}) \); \( EM^{(13)} \) reduces the time and memory complexity to \( \tilde{O}(2^{0.255n}) \).

3 The Quantum Algorithm

Now we use \( EM^{(4)} \) as a bridge connecting random subset sum problem and graph search problem. Recall the tree structure of \( EM^{(4)} \) in Figure 2.2. Denote by \( L_j^{(i)} \) the \( j \)-th list of the level \( i \) lists in \( EM^{(4)} \), where \( 1 \leq i \leq 4, 1 \leq j \leq 2^{4-i} \).

Consider the graph

\[
G_{\text{search}}(V_{\text{search}},E_{\text{search}}) := J(|L_1^{(0)}|,r) \times J(|L_2^{(0)}|,r) \times \cdots \times J(|L_{16}^{(0)}|,r), \quad (7)
\]

which is the cartesian product of Johnson graphs.

- The vertices of \( G_{\text{search}} \) are \((U_1^{(0)},U_2^{(0)},\ldots,U_{16}^{(0)})\), with \( U_j^{(0)} \subseteq L_j^{(0)}, |U_j^{(0)}| = r, 1 \leq j \leq 16 \).
- Denote by \( U_j^{(i)} \) the \( j \)-th list of the level \( i \) lists that constructed from \( U_1^{(0)},U_2^{(0)},\ldots,U_{16}^{(0)} \) according to \( EM^{(4)} \), where \( 1 \leq i \leq 4, 1 \leq j \leq 2^{4-i} \).
- For a vertex \((U_1^{(0)},U_2^{(0)},\ldots,U_{16}^{(0)}) \in V_{\text{search}}\), its data structure contains all \( U_j^{(i)} \), where \( 1 \leq i \leq 4, 1 \leq j \leq 2^{4-i} \).
- A vertex \((U_1^{(0)},U_2^{(0)},\ldots,U_{16}^{(0)}) \in V_{\text{search}}\) belongs to the marked set if and only if \( U_1^{(4)} \) contains a solution to the original random subset sum instance.
When we use quantum walk to find a marked vertex, then based on the data structure of the marked vertex, we can solve the original random subset sum problem.

In order to implement the quantum walk, it is necessary to build the Johnson graphs $J(|L_j^{(0)}|, r)$ for $1 \leq j \leq 16$. So before we go on a walk, we need to build $L_j^{(0)}$ for $1 \leq j \leq 16$ by a classical sample, which is the same as the first step of the EM$(^4)$ algorithm.

From now on, let
\[ L^{(i)} := \mathbb{E}[|L_j^{(i)}|], \quad U^{(i)} := \mathbb{E}[|U_j^{(i)}|] \]  
(8)
be the expected size of a list on level $i$ (before filtering) for $0 \leq i \leq 4$. Let
\[ L_j^{(i)} := \mathbb{E}[|\text{filter}(L_j^{(i)})|], \quad U_j^{(i)} := \mathbb{E}[|\text{filter}(U_j^{(i)})|] \]  
(9)
denote the expected size of filtered lists for $0 \leq i \leq 4$.

According to Lemma 2.8, under Heuristic 1, and the constraints (4), (5) and (6), if $r = U^{(0)} = L^{(0)}$, then $M_{\text{search}} = V_{\text{search}}$. That is, if $U_j^{(0)}$ contains all vertices of $L_j^{(0)}$ for $1 \leq j \leq 16$, then all vertices of $G_{\text{search}}$ are marked. Thus, $\left(\frac{U^{(0)}}{L^{(0)}}\right)^{16}$ is the lower bound on the probability that a vertex chosen randomly of $G_{\text{search}}$ is marked.

Let
\[ \varepsilon = \left(\frac{U^{(0)}}{L^{(0)}}\right)^{16}. \]  
(10)
By Lemma 2.5, $\delta = \Omega(1/U^{(0)})$. The remaining task is to determine $T_s, T_c, T_u$.

**Data Structure.**

We use augmented radix trees [6] to store the data structure of vertices in $V_{\text{search}}$. Augmented radix trees allow the three operations search, insertion and deletion in time logarithmic in the number of stored elements. Since our lists have exponential size and we ignore polynomials in the run time analysis, the cost of search, insertion and deletion operation can be ignored.

Recall that EM$(^4)$ level-0 lists are of the form $L_j^{(0)} = \{(c_j^{(0)}, \langle a, c_j^{(0)} \rangle)\}$ for $1 \leq j \leq 16$. For our $U_j^{(0)} \subseteq L_j^{(0)}$ we store the $c_j^{(0)}$ and their inner products with $a$ separately in $A_j^{(0)} = \{c_j^{(0)} | c_j^{(0)} \in U_j^{(0)}\}$ and $B_j^{(0)} = \{\langle a, c_j^{(0)} \rangle, c_j^{(0)} \} | c_j^{(0)} \in U_j^{(0)}\}$, where in $B_j^{(0)}$ elements are addressed via their first datum $\langle a, c_j^{(0)} \rangle$, $1 \leq j \leq 16$. Similarly, for $U_j^{(i)}$, $1 \leq i \leq 3, 1 \leq j \leq 2^{4-i}$, we also build separate $A_j^{(i)}$ and $B_j^{(i)}$. For $U_1^{(4)}$, it suffices to build $A_1^{(4)}$. So, we store 61 sets in augmented radix trees.

Before computing $T_s, T_c, T_u$, we recall a classical list join operator that we will use in the analysis of complexity. The join operator performs the following task: given two lists of numbers $L_1$ and $L_2$ of respective sizes $|L_1|$ and $|L_2|$, together with two integers $M$ and $R$, the algorithm computes the list $L_0$ such that: $L_0 = \{x_1 + x_2 | (x_1, x_2) \in L_1 \times L_2 \land x_1 + x_2 \equiv R \mod M\}$. The list $L_0$ can be constructed as follows. Sort $L_1$ and then for every $x_2 \in L_2$ we find via binary search all elements $x_1 \in L_1$ such that $x_1 + x_2 \equiv R \mod M$. The complexity of this method is $O(\max(|L_1|, |L_2|, |L_0|))[28]$. Moreover, assuming that the values of the initial lists modulo $M$ are randomly distributed, the expected of $|L_0|$ is $\frac{|L_1||L_2|}{M}[5]$. 


Now, we compute $T_s, T_c, T_u$. Our goal is to obtain the optimal time complexity of our quantum algorithm, and then, under the optimal time complexity, to compute the memory complexity. We don’t consider space-time tradeoffs. Thus, $T_s, T_c, T_u$ will be computed below are time complexity.

**Setup.**

Start with analyzing the run time for sampling the level-0 lists in Algorithm 1. Note that sampling is stopped when $\left(\frac{n}{\alpha n/2}\right) = \tilde{O}(2^{H(\alpha)n/2})$ different list elements have been found. Conclude by Lemma 2.7 that this has only $\tilde{O}(2^{H(\alpha)n/2})$ time complexity. So, $L^{(0)} = \tilde{O}(2^{H(\alpha)n/2})$

Definition $\beta$ is the parameter that satisfies $U^{(0)} = (L^{(0)})^{\beta}$. Then, $U^{(0)} = \tilde{O}(2^{\beta H(\alpha)n/2})$.

Now turn to the computation of the level-1 to level-4 lists.

Denote by $\gamma_i$ the probability that a level-$i$ element gets filtered, that is $L^{(i)} = \gamma_i L^{(i)}$. Note that $L^{(1)} = L^{(1)}$, since level-1 elements are by construction in $\{0, 1\}^n$. For completeness, also define $L^{(0)} = L^{(0)}$. So, $\gamma_0 = \gamma_1 = 1$. Let $l_4 = n - l_1 - l_2 - l_3$.

Compute $\gamma_i$ by definition. The result is

$$\gamma_i = (1 - \frac{2^2(i - 1)\alpha^2}{((2^{i-1} - 2\alpha + 2)^2})^n) for 2 \leq i \leq 3.$$  

$U^{(i)}$ is constructed in the list join manner as above. The result is

$$U^{(i)} = \frac{(U^{(i-1)})^2}{2i}, \quad \gamma_i = \frac{(U^{(i-1)})^2}{2i}, \quad for 1 \leq i \leq 4. \quad \text{(11)}$$

Now solve for the $U^{(i)}$ from equation (11). The result is

$$U^{(1)} = \tilde{O}(2^{\beta H(\alpha)n-l_1}),$$

$$U^{(2)} = \tilde{O}(2^{2\beta H(\alpha)n-2l_1-l_2}),$$

$$U^{(3)} = \tilde{O}(2^{4\beta H(\alpha)n-4l_1-2l_2-l_3+2\log \gamma_2}),$$

$$U^{(4)} = \tilde{O}(2^{8\beta H(\alpha)n-n-7l_1-3l_2-l_3+4\log \gamma_2+2\log \gamma_3}). \quad \text{(12)}$$

Thus, the expected setup time complexity is

$$E[T_s] = \max(U^{(0)}, U^{(1)}, U^{(2)}, U^{(3)}, U^{(4)}).$$

**Lemma 3.1** [9] Under Heuristic 1, $|U^{(i)}| = \tilde{O}(U^{(i)})$ is true for any $0 \leq i \leq 4$ and $1 \leq j \leq 2^{4-i}$.

Proof. Denote by $R^{(i)}_j$ the list made up of all representations in $U^{(i)}$, where $0 \leq i \leq 4, 1 \leq j \leq 2^{4-i}$ and denote $R^{(i)} := E[|R^{(i)}_j|]$. By Heuristic 1, $|R^{(i)}_j| = \tilde{O}(R^{(i)}).$

The elements in the leaf lists $U^{(0)} \subseteq L^{(0)}$ are sampled from $\mathcal{B}^\gamma(\alpha) \times 0^\gamma$ and $0^\gamma \times \mathcal{B}^\gamma(\alpha)$. As a consequence, the elements of the level-1 lists $U^{(1)}_j$ are from $\mathcal{B}^0(\alpha)$. 
Let \( \mathbf{x} = \mathbf{c}_1^{(1)} + \cdots + \mathbf{c}_8^{(1)} \), where \( \mathbf{c}_j^{(1)} \in U_j^{(1)}, 1 \leq j \leq 8 \). Then for each coordinate \( x_i \) of \( \mathbf{x} \), \( \Pr[x_i = 0] = (1 - \alpha)^6 \) and \( \Pr[x_i = 1] = 8\alpha(1 - \alpha)^7 \). Hence a candidate \((\mathbf{c}_1^{(1)}, \ldots, \mathbf{c}_8^{(1)}) \in U_1^{(1)} \times \cdots \times U_8^{(1)} \) is a representation of the \( n/2 \)-weight solution \( \mathbf{e} \) with probability

\[
p := \Pr[x_i = 0]^2 \Pr[x_i = 1]^2 = (8\alpha(1 - \alpha)^{15})^2.
\]

So \( |R_j^{(1)}| = |U_j^{(1)}|p, 1 \leq j \leq 8 \) and \( R^{(1)} = U^{(1)}p \). Similarly, \( |R_j^{(i)}| = |U_j^{(i)}|p, 2 \leq i \leq 4, 1 \leq j \leq 8 \) and \( R^{(i)} = U^{(i)}p \). Furthermore, \( |U_j^{(0)}| = \tilde{O}(U^{(0)}) = \tilde{O}(2^{\beta H(n/2)}), 1 \leq j \leq 16 \). So \( |U_j^{(i)}| = \tilde{O}(U^{(i)}) \) for \( 0 \leq i \leq 4 \) and \( 1 \leq j \leq 2^{4-i} \).

This finishes the proof.

By Lemma 3.1, the setup time complexity is

\[
T_s = \tilde{O}(\max(U^{(0)}, U^{(1)}, U^{(2)}, U^{(3)}, U^{(4)})).
\]

**Checking and Update**

Checking whether a vertex \((U_1^{(0)}, U_2^{(0)}, \ldots, U_{16}^{(0)}) \) is marked can be done easily by looking at \( U_1^{(4)} \). And the data of \( U_1^{(4)} \) is already stored in \( A_1^{(4)} \) after the setup subroutine. Thus, \( T_s = \tilde{O}(1) \).

One step of our random walk replaces a list item in exactly one of the leaf lists. We can perform one update by first deleting the replaced item and update the path to the root accordingly, and second adding the new item and again updating the path to the root.

We denote the operators that are used in the update subroutine as follows. *Insert*\( (A_j^{(i)}, \mathbf{x}) \) inserts \( \mathbf{x} \) into \( A_j^{(i)} \), and *Delete*\( (A_j^{(i)}, \mathbf{x}) \) deletes \( \mathbf{x} \) from \( A_j^{(i)} \). Furthermore, \( \{\mathbf{x}\} \leftarrow \text{Search}(B_j^{(i)}, \langle \mathbf{a}, \mathbf{y} \rangle) \) returns the list of all second datum \( \mathbf{c}_j^{(i)} \) with first datum \( \langle \mathbf{a}, \mathbf{c}_j^{(i)} \rangle = \langle \mathbf{a}, \mathbf{y} \rangle \), where \( \langle \mathbf{a}, \mathbf{c}_j^{(i)} \rangle, \mathbf{c}_j^{(i)} \in B_j^{(i)} \).

Now, we describe the deleting an element subprogram. Without loss of generality, we assume the deleted element \( \mathbf{x} \in U_1^{(0)} \).

1. *Delete*\( (A_1^{(0)}, \mathbf{x}) \).
2. \( \{\mathbf{x}^{(0)}\} \leftarrow \text{Search}(B_2^{(0)}, s_1^{(0)} - \langle \mathbf{a}, \mathbf{x} \rangle \mod 2^i) \).
3. For all \( \mathbf{x}_1 = \mathbf{x} + \mathbf{x}' \) with \( \mathbf{x}' \in \{\mathbf{x}^{(0)}\} \), do the following:
   (a) *Delete*\( (A_1^{(1)}, \mathbf{x}_1) \).
   (b) \( \{\mathbf{x}^{(1)}\} \leftarrow \text{Search}(B_2^{(1)}, s_1^{(1)} - \langle \mathbf{a}, \mathbf{x}_1 \rangle \mod 2^{i_1 + i_2}) \).
   (c) For all \( \mathbf{x}_2 = \mathbf{x}_1 + \mathbf{x}' \) with \( \mathbf{x}' \in \{\mathbf{x}^{(1)}\} \), do the following:
      i. *Delete*\( (A_1^{(2)}, \mathbf{x}_2) \).
      ii. \( \{\mathbf{x}^{(2)}\} \leftarrow \text{Search}(B_2^{(2)}, s_1^{(2)} - \langle \mathbf{a}, \mathbf{x}_2 \rangle \mod 2^{i_1 + i_2 + i_3}) \).
      iii. For all \( \mathbf{x}_3 = \mathbf{x}_2 + \mathbf{x}' \) with \( \mathbf{x}' \in \{\mathbf{x}^{(2)}\} \), do the following:
         A. *Delete*\( (A_1^{(3)}, \mathbf{x}_3) \).
         B. \( \{\mathbf{x}^{(3)}\} \leftarrow \text{Search}(B_2^{(3)}, s_1^{(3)} - \langle \mathbf{a}, \mathbf{x}_3 \rangle \mod 2^n) \).
C. For all $x_4 = x_3 + x'$ with $x' \in \{x^{(3)}\}$, $\text{Delete}(A_4^{(4)}, x_4)$.

Since
\[
E[|\{x^{(0)}\}|] = \frac{U^{(0)}}{2^{l_1}}, \quad E[|\{x^{(1)}\}|] = \frac{U^{(1)}}{2^{l_2}}, \quad E[|\{x^{(2)}\}|] = \frac{U^{(2)}}{2^{l_3}}, \quad E[|\{x^{(3)}\}|] = \frac{U^{(3)}}{2^{n-l_1-l_2-l_3}},
\]

the expected cost of deleting an element is
\[
\max(1, \frac{U^{(0)}}{2^{l_1}}, \frac{U^{(0)}U^{(1)}}{2^{l_1+l_2}}, \frac{U^{(0)}U^{(1)}U^{(2)}}{2^{l_1+l_2+l_3}}, \frac{U^{(0)}U^{(1)}U^{(2)}U^{(3)}}{2^n}).
\]

Inserting an element is analogous to deleting an element. Simply replace the deletion operator in the deleting subprogram with the insertion operator to enable insertion of an element. Thus, the expected update time complexity is
\[
\mathbb{E}[T_u] = \max(1, \frac{U^{(0)}}{2^{l_1}}, \frac{U^{(0)}U^{(1)}}{2^{l_1+l_2}}, \frac{U^{(0)}U^{(1)}U^{(2)}}{2^{l_1+l_2+l_3}}, \frac{U^{(0)}U^{(1)}U^{(2)}U^{(3)}}{2^n}).
\]

From Lemma 3.1, the update time complexity is
\[
T_u = \tilde{O} \left( \max \left( 1, \frac{U^{(0)}}{2^{l_1}}, \frac{U^{(0)}U^{(1)}}{2^{l_1+l_2}}, \frac{U^{(0)}U^{(1)}U^{(2)}}{2^{l_1+l_2+l_3}}, \frac{U^{(0)}U^{(1)}U^{(2)}U^{(3)}}{2^n} \right) \right).
\]

**Stopping unusually long updates.**

The update time complexity is determined by the maximum cost over all vertices in a superposition. Therefore, even one node with an unusually slow update time complexity can disrupt our runtime. To prevent this problem, we modify our quantum walk algorithm by imposing an upper bound of $\kappa = \text{poly}(n)$ steps for updating the data structure. After $\kappa$ steps, we simply stop the update of all nodes and proceed as if the update has been completed. We will use the following hypothesis.

**Heuristic 2 [15].**

Let $\varepsilon$ be the fraction of marked states and $\delta$ be the spectral gap of the quantum walk $W$. Denote by $W_{\text{stop}}$ the quantum walk that forces $W$ to stop after $\kappa$ steps. Then the fraction of marked states in $W_{\text{stop}}$ is at least $\varepsilon_{\text{stop}} = \tilde{\Omega}(\varepsilon)$, and the spectral gap of $W_{\text{stop}}$ is at least $\delta_{\text{stop}} = \tilde{\Omega}(\delta)$. Moreover, the stationary distribution of $W_{\text{stop}}$ is close to the distribution of its setup. Namely, we obtain with high probability a random node which can be superposition of the Johnson graph with correctly built data structure.

In summary, we use $EM^{(4)}$ as a bridge connecting random subset sum problem and graph search problem. Then, we give a new algorithm, which starts with a classic sample, next implements a quantum walk over the cartesian product of Johnson graphs. By reasonably constructing the data structure, we calculate the $T_s, T_c, T_u$. From Theorem 2.2 and Lemma 2.8, we know that under Heuristic 1, Heuristic 2 and constraints (4), (5) and (6) there exists an algorithm that with high probability finds a marked vertex in time
\[
T = L^{(0)} + T_s + \frac{1}{\sqrt{\delta}} \left( \frac{1}{\sqrt{\delta}} T_u + T_c \right).
\]
where the time complexity of classic sampling is \(L(0)\) and the time complexity of quantum walk is \(T_s + \frac{1}{\sqrt{\delta}} (\frac{1}{\sqrt{\delta}} T_u + T_c)\).

We will give the optimal value of all parameters later. Now let us give our quantum EM\((4)\) algorithm.

**Algorithm 2 Quantum EM\((4)\) Algorithm.**

**Input:** subset sum instance \((a, s) \in (\mathbb{Z}_2^n)^{n+1}\);

parameters \(\alpha, \beta \in [0, 1], l_1, l_2, l_3 \in \mathbb{N}\) and \(l_1 + l_2 + l_3 \leq n\).

**Output:** solution \(e \in \{0, 1\}^n\) with \(|e| = \frac{n}{2}\), or \(\perp\) if no solution is found.

1: [Classical Sampling] Sample all level-0 lists \(L_i(0)\) for \(i = 1, \ldots, 16:\)

2: [Quantum Setup] Prepare the initial state:

\[
|\pi\rangle = \frac{1}{c} \left( \sum_{(U_j(0) \times \cdots \times U_{16}(0)) \in V_{search}} \bigotimes_{j=1}^{16} |U_j(0)\rangle |\text{coin}\rangle |\text{data}\rangle \right),
\]

where \(c\) is the normalization factor, \(|\text{coin}\rangle\) is the superposition of vertices adjacent to the current vertex, and \(|\text{data}\rangle\) is the data structure of the current vertex.

Note that parameters \(\alpha, \beta, l_1, l_2, l_3\) are used in the process of building \(|\text{data}\rangle\).

3: Do the following \(O(1/\sqrt{\varepsilon})\) times:

3.1 [Quantum Checking]

\[
\bigotimes_{j=1}^{16} |U_j(0)\rangle |\text{coin}\rangle |\text{data}\rangle \mapsto \begin{cases} 
- \bigotimes_{j=1}^{16} |U_j(0)\rangle |\text{coin}\rangle |\text{data}\rangle, & \text{if } (U_1(0) \times \cdots \times U_{16}(0)) \in M, \\
\bigotimes_{j=1}^{16} |U_j(0)\rangle |\text{coin}\rangle |\text{data}\rangle, & \text{else.}
\end{cases}
\]

3.2 [Quantum Update] Do the following \(O(1/\sqrt{\delta})\) times:

3.2.1 Take a quantum step of the walk.

3.2.2 Update the data structure accordingly, stop after \(\kappa\) steps.

4: Measure the final state, obtain a state \(\bigotimes_{j=1}^{16} |U_j(0)\rangle |\text{coin}\rangle |\text{data}\rangle\).

If \(\exists e \in A^{(4)}_1\) such that \(|e| = \frac{n}{2}\), then return \(e\), else return \(\perp\).

**Theorem 3.2** Under Heuristics 1,2 and the constraints (4), (5) and (6), Algorithm 2 gives with high probability the solutions of random subset sum instances in time and with memory \(2^{0.209n}\).

Proof. By the analysis of section 3, the time complexity of Algorithm 2 is

\[
T = L(0) + T_s + \frac{1}{\sqrt{\varepsilon}} \left( \frac{1}{\sqrt{\delta}} T_u + T_c \right),
\]

(16)
where

\[ L^{(0)} = \tilde{O}(2^{H(\alpha)n/2}), \]

\[ U^{(0)} = \tilde{O}(2^{3H(\alpha)n/2}), \]

\[ \varepsilon = \left( \frac{U^{(0)}}{L^{(0)}} \right)^{16}, \]

\[ \delta = \Omega(1/U^{(0)}), \]

\[ T_c = \tilde{O}(1), \]

\[ T_s = \tilde{O}(\max(U^{(0)}, U^{(1)}, U^{(2)}, U^{(3)}, U^{(4)})), \]

\[ T_u = \tilde{O}(\max(1, \frac{U^{(0)}}{2l_1}, \frac{U^{(0)}U^{(1)}}{2l_1+l_2}, \frac{U^{(0)}U^{(1)}U^{(2)}}{2l_1+l_2+l_3}, \frac{U^{(0)}U^{(1)}U^{(2)}U^{(3)}}{2n})). \]

Furthermore,

\[ U^{(1)} = \tilde{O}(2^{3H(\alpha)n-l_1}), \]

\[ U^{(2)} = \tilde{O}(2^{H(\alpha)n-2l_1-l_2}), \]

\[ U^{(3)} = \tilde{O}(2^{4H(\alpha)n-4l_1-2l_2-l_3+2\log \gamma_2}), \]

\[ U^{(4)} = \tilde{O}(2^{8H(\alpha)n-n+7l_1-3l_2-l_3+4\log \gamma_2+2\log \gamma_3}), \]

\[ \gamma_0 = 1, \]

\[ \gamma_i = 1 - \frac{2^{2(i-1)}(\alpha^2)}{(2^{i-1} - 2)(\alpha + 2)^2}, \text{ for } i = 1, 2, 3. \]

So

\[ T = \tilde{O}\left( \max\left( \left( \frac{U^{(0)}}{2l_1} \right)^{\frac{8}{2}}, \left( \frac{U^{(0)}^2}{2l_1+l_2} \right)^{4}, \left( \frac{U^{(0)}^{\frac{8}{2}}}{2l_1+l_2+l_3} \right)^{2}, \left( \frac{U^{(0)}^{8+\frac{4}{5}}}{2n+7l_1+3l_2+l_3} \right)^{2}, \left( \frac{U^{(0)}^{16+\frac{7}{2}}}{2n+7l_1+3l_2+l_3} \right)^{2} \right) \right). \]

Under the constraints (4), (5) and (6), the numerical optimization for minimizing \( T \) gives

\[ \alpha = 0.188, \quad \beta = 0.941, \quad l_1 = 0.184n, \quad l_2 = 0.209n, \quad l_3 = 0.188n. \]

So \( T = \tilde{O}(2^{0.209n}) \) and \( L^{(0)} = \tilde{O}(2^{0.209n}). \)

Obviously, \( L^{(0)} \leq C_{\text{memory}} \leq T \), where \( C_{\text{memory}} \) represents the memory cost. Thus, under Heuristic 1, and constraints (4), (5), (6), Algorithm 2 runs in time \( T = 2^{0.209n} \) using \( L^{(0)} = 2^{0.209n} \) memory.

This finishes the proof.

**Remark 3.3** In a similar way to quantum EM\(^{(d)}\) algorithm, we can quantize EM\(^{(d)}\) algorithm, for \( d \geq 3 \). When analyzing varying depths, we could not improve over the run time. For \( 3 \leq d \leq 8 \), our results are listed in Table 1. If \( \beta = 1 \), then our quantum EM\(^{(d)}\) algorithm is the classic EM\(^{(d)}\) algorithm, for \( d \geq 3 \). As observed from the Table 1, \( \beta \) is getting closer to 1 as the depth increases. We conjecture that \( |T_{\text{quantum EM}}(d) - T_{\text{EM}}(d)| \) converges for \( d \to \infty \),
where $T_{\text{quantum } EM}(d)$ is the run time of our quantum $EM^{(d)}$ algorithm and $T_{EM}(d)$ is the run time of $EM^{(d)}$ algorithm.

**Table 1** Comparison of the run times of quantum and classical $EM$ algorithm by depth $d$.

| $d$ | $T_{\text{quantum } EM}$ | $\beta$ | $T_{EM}$ |
|-----|--------------------------|---------|----------|
| 3   | $2^{0.2531}$             | 0.8889  | $2^{0.2960}$ |
| 4   | $2^{0.2090}$             | 0.9412  | $2^{0.2659}$ |
| 5   | $2^{0.2194}$             | 0.9697  | $2^{0.2616}$ |
| 6   | $2^{0.2326}$             | 0.9846  | $2^{0.2584}$ |
| 7   | $2^{0.2417}$             | 0.9922  | $2^{0.2565}$ |
| 8   | $2^{0.2473}$             | 0.9961  | $2^{0.2558}$ |

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