Research Article

Products of Composition and Differentiation between the Fractional Cauchy Spaces and the Bloch-Type Spaces

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The operators $D_{C_{\alpha}}$ and $C_{\alpha}D$ are defined by $D_{C_{\alpha}}(f) = (f \circ \Phi)'$ and $C_{\alpha}D(f) = f' \circ \Phi$ where $\Phi$ is an analytic self-map of the unit disc and $f$ is analytic in the disc. A characterization is provided for boundedness and compactness of the products of composition and differentiation from the spaces of fractional Cauchy transforms $F_{\alpha}$ to the Bloch-type spaces $B^\beta$, where $\alpha > 0$ and $\beta > 0$. In the case $\beta < 2$, the operator $D_{C_{\alpha}} : F_{\alpha} \longrightarrow B^\beta$ is compact $\Leftrightarrow$ $D_{C_{\alpha}} \Phi \in B^\beta$ is bounded $\Leftrightarrow$ $\Phi' \in B^\beta$ and $||\Phi||_{C1} < 1$. For $\beta < 1$, $C_{\alpha}D : F_{\alpha} \longrightarrow B^\beta$ is compact $\Leftrightarrow$ $C_{\alpha}D \Phi \in B^\beta$ is bounded $\Leftrightarrow$ $\Phi \in B^\beta$ and $||\Phi||_{C2} < 1$.

1. Introduction

Let $U = \{ z \in C : |z| < 1 \}$ and let $H(U)$ denote the family of functions analytic on $U$. Let $M$ denote the Banach space of complex Borel measures on $T = \{ x \in C : |x| = 1 \}$, endowed with the total variation norm. For $\alpha > 0$, the space $F_{\alpha}$ of fractional Cauchy transforms is the family of functions of the form

$$f(z) = \frac{1}{(1-z \bar{z})^\alpha} \frac{1}{2 \pi i} \int_{T} \frac{d\mu(x)}{|x|^2}$$

where $\mu \in M$. The principal branch of the logarithm is used here. The space $F_{\alpha}$ is a Banach space, with norm

$$||f||_{F_{\alpha}} = \inf ||\mu||,$$

where $\mu$ varies over all measures in $M$ for which (1) holds. The families $F_{\alpha}$ have been studied extensively [1, 2]. Interest in these spaces was first established in connection with the classical family $S$ of normalized univalent functions. It is known that $S \subseteq F_{\alpha}$ for any $\alpha > 2$ [2]. The reference [2] also includes MacGregor’s construction of a function $f \in S$ with $f \notin F_2$.

Let $\beta > 0$. The Bloch-type space $B^\beta$ is the Banach space of functions analytic in $U$ such that $\sup_{z \in U} |1 - |z|^2|^\beta |f'(z)| < \infty$, with norm

$$||f||_{B^\beta} = |f(0)| + \sup_{z \in U} |1 - |z|^2|^\beta |f'(z)|.$$

The relation (1) implies that $F_{\alpha} \subset B^{\alpha + 1}$, and there is a constant $C$ depending only on $\alpha$ such that $||f||_{B^{\alpha + 1}} \leq C ||f||_{F_{\alpha}}$ for all $f \in F_{\alpha}$.

Let $\Phi$ be an analytic self-map of $U$. The composition operator $C_{\alpha}$ is defined by $C_{\alpha}(f) = f \circ \Phi$ for $f \in H(U)$. The differentiation operator $D$ is defined by $D(f) = f'$. In this paper, the products $C_{\alpha}D(f) = f' \circ \Phi$ and $D_{C_{\alpha}}(f) = \Phi' \circ f' \circ \Phi$ are studied. Conditions on $\Phi$ are given, necessary and sufficient to imply boundedness or compactness of $C_{\alpha}D : F_{\alpha} \longrightarrow B^\beta$ and $D_{C_{\alpha}} : F_{\alpha} \longrightarrow B^\beta$.

Products of composition and differentiation on the Bloch space were studied by Ohno in [3]. In [4], Li and Stević studied $C_{\alpha}D$ and $D_{C_{\alpha}}$ acting between the weighted Bergman spaces and the Bloch-type spaces. In [5], Hibschweiler and Portnoy studied these operators between Bergman and Hardy spaces.
2. Preliminary Results

Fix $\alpha > 0$. For fixed $z \in U$ and for $n = 0, 1, \ldots$, the relation (1) yields a constant $C$ depending only on $n$ such that $|f^{(n)}(z)| \leq C\|f\|_{F_n}/(1 - |z|^2)^{an}$ [2].

For each $w \in U$, $\|1/(1 - wz)^{an}\|_{F_n} = 1$ [2].

We follow the convention that $C$ denotes a positive constant, the precise value of which will differ from one appearance to the next.

Lemma 1 and Lemma 2 will be used to develop test functions for $F_n$. Proofs appear in [6].

**Lemma 1.** Fix $\alpha > 0$. For $w \in U$, define

$$h_w(z) = \frac{1 - |w|^2}{(1 - wz)^{\alpha n}} (z \in U).$$

Then, $h_w \in F_n$, and there is a constant $C$ such that $\|h_w\|_{F_n} \leq C$ for all $w \in U$.

**Lemma 2.** Fix $\alpha > 0$. For $w \in U$, define

$$k_w(z) = \frac{(1 - |w|^2)^2}{(1 - wz)^{\alpha n}} (z \in U).$$

Then, $k_w \in F_n$, and there is a constant $C$ such that $\|k_w\|_{F_n} \leq C$ for all $w \in U$.

3. The Operator $DC_\Phi : F_\alpha \longrightarrow B^\beta$

In [7], Shapiro proved that the condition $\|\Phi\|_{C_\Phi} < 1$ is necessary for $C_\Phi : X \longrightarrow X$ to be compact, for Banach spaces $X$ obeying boundary regularity and M"{o}bius invariance. In particular, Shapiro’s result applies to the Lipschitz spaces and thus, to the space $B^\beta$ when $\gamma < 1$ [8].

**Theorem 3.** Fix $\alpha > 0$ and $0 < \beta < 2$. Let $\Phi$ be an analytic self-map of $U$.

$DC_\Phi : F_\alpha \longrightarrow B^\beta$ is bounded $\Leftrightarrow$

$$\Phi' \in B^\beta, \Phi\Phi' \in B^\beta \text{ and } \|\Phi\|_{C_\Phi} < 1 \Leftrightarrow$$

$DC_\Phi : F_\alpha \longrightarrow B^\beta$ is compact.

**Proof.** First, assume that $DC_\Phi : F_\alpha \longrightarrow B^\beta$ is bounded, that is, there is a constant $C$ such that $\|DC_\Phi(f)\|_{B^\beta} \leq C\|f\|_{F_n}$ for all $f \in F_\alpha$. It is clear that $\Phi' = DC_\Phi(z) \in B^\beta$ and $\Phi\Phi' = DC_\Phi(z^2)$ is in $B^\beta$. Thus,

$$(1 - |z|^2)^{\beta}\Phi^{(n)}(z) \leq C,$$  

and

$$(1 - |z|^2)^{\beta}\left|\Phi(z)\Phi^{(n)}(z) + \left(\Phi'(z)\right)^2\right| \leq C,$$

for all $z \in U$. It follows that

$$\sup_{z \in U} (1 - |z|^2)^{\beta}\left|\Phi'(z)\right|^2 < \infty,$$

and thus, $\Phi \in B^\beta$. Let $w \in U$ and define

$$g_w(z) = \frac{\alpha + 1}{(1 - \Phi(w)z)^{\alpha n}} - \frac{\alpha(1 - |\Phi(w)|^2)}{(1 - \Phi(w)z)^{\alpha n}} (z \in U).$$

By Lemma 1 and the preliminary results, there is a constant $C$ independent of $w$ such that $\|g_w\|_{F_n} \leq C$, and thus, $\|DC_\Phi(g_w)\|_{B^\beta} = \|(g_w \circ \Phi)\Phi'\|_{B^\beta} \leq C$. It follows that

$$\sup_{z \in U} (1 - |z|^2)^{\beta}\left|g_w'(\Phi(z))\left(\Phi'(z)\right)^2 + g_w'(\Phi(z))\Phi^{(n)}(z)\right| \leq C,$$

for all $w \in U$. Calculations yield $g_w'(\Phi(z)) = 0$ and

$$g_w''(\Phi(z)) = \frac{-\alpha(\alpha + 1)\Phi(w)^2}{(1 - |\Phi(w)|^2)^{\alpha n}}.$$

The substitution $z = w$ in (11) now yields

$$\sup_{w \in U} (1 - |w|^2)^{\beta}\left|\Phi(w)^2\left(\Phi'(w)^2\right) + \Phi'(w)\Phi^{(n)}(w)\right| \leq C,$$

and thus,

$$\sup_{|\Phi(w)| < 1/2} (1 - |w|^2)^{\beta}\left|\Phi'(w)^2\right| < \infty.$$  

By the relation (9),

$$\sup_{w \in U} (1 - |w|^2)^{\beta}\left|\Phi'(w)^2\right| < \infty.$$  

Theorem 3 follows. By Xiao’s result [9], $C_\Phi : B^{\beta_2} \longrightarrow B^{\beta_{22}}$ is bounded. Furthermore, (16) yields
\[
\frac{(1 - |w|^2)^{\beta/2} |\Phi'(w)|}{(1 - |\Phi(w)|^2)^{\beta/2}} \leq C(1 - |\Phi(w)|^2)^{(\alpha-\beta+2)/2} \to 0, \tag{18}
\]
as \(|\Phi(w)| \to 1\). Thus, \(C_{\Phi} : B^{\beta/2} \to B^{\beta/2}\) is compact [9], and it follows as in [7] that \(||\Phi||_\infty < 1\). It has been established that the conditions \(\Phi' \in B^\beta, \Phi\Phi' \in B^{\beta}\), and \(||\Phi||_\infty < 1\) are necessary if \(DC_\Phi : F_a \to B^\beta\) is bounded.

Next, assume that \(\Phi' \in B^\beta, \Phi\Phi' \in B^\beta\), and \(||\Phi||_\infty < 1\). To show that \(DC_\Phi : F_a \to B^\beta\) is compact, let \((f_n)\) be a bounded sequence in \(F_a\) with \(f_n \to 0\) uniformly on compact subsets of \(U\) as \(n \to \infty\). It is enough to prove that \(||DC_\Phi(f_n)||_{B^\beta} \to 0\) as \(n \to \infty\). First, note that \(|f_n'(\Phi(0))\Phi'(0)| \to 0\) as \(n \to \infty\). For \(z \in U\), (9) yields
\[
(1 - |z|^2)^{\beta/2} |(DC_\Phi f_n)'(z)| = (1 - |z|^2)^{\beta/2} |f_n'(\Phi(z))\Phi'(z)|^2 + f_n'(\Phi(z))\Phi'(z) \\
\leq C \max_{|w| \geq |\Phi||_\infty} |f_n'(w)| \\
+ \|\Phi'\|_{B^\beta} \max_{|w| \geq |\Phi||_\infty} |f_n'(w)|.
\]
(19)

Since \(f_n' \to 0\) and \(f_n'' \to 0\) uniformly on compact subsets as \(n \to \infty\), the argument shows that \(\sup_{z \in U}(1 - |z|^2)^{\beta/2} |(DC_\Phi f_n)'(z)| \to 0\) as \(n \to \infty\). Thus, \(||DC_\Phi(f_n)||_{B^\beta} \to 0\) as \(n \to \infty\), and \(DC_\Phi : F_a \to B^\beta\) is compact, as required.

The remaining implication is clear, and the proof is complete.

**Theorem 4.** Fix \(\alpha > 0\) and \(\beta \geq 2\). Let \(\Phi\) be an analytic self-map of \(U\). Then,
\[
DC_\Phi : F_a \to B^\beta \text{ is bounded } \iff \sup_{z \in U} \frac{(1 - |z|^2)^{\beta} |\Phi'(z)|^2}{(1 - |\Phi(z)|^2)^{\alpha+1}} < \infty, \tag{20}
\]
and
\[
\sup_{z \in U} \frac{(1 - |z|^2)^{\beta} |\Phi'(z)|^2}{(1 - |\Phi(z)|^2)^{\alpha+2}} < \infty. \tag{22}
\]

**Proof.** Fix \(\alpha, \beta\) and \(\Phi\) as described.

First, assume (21) and (22). Let \(f \in F_a\). By (21) and the introductory remarks in Section 2,
\[
(1 - |z|^2)^\beta |f'(\Phi(z))| |\Phi'(z)| \leq (1 - |z|^2)^\beta |\Phi'(z)| \frac{C||f||_{F_a}}{(1 - |\Phi(z)|^2)^{\alpha+1}} \\
\leq C||f||_{F_a}. \tag{23}
\]
A similar argument using (22) yields
\[
(1 - |z|^2)^\beta |f''(\Phi(z))| |\Phi''(z)|^2 \leq C||f||_{F_a}, \tag{24}
\]
for all \(z \in U\). Thus, \(\sup_{z \in U}(1 - |z|^2)^{\beta} |(DC_\Phi f)'(z)| \leq C||f||_{F_a}\).

Since \(||DC_\Phi(f)'(0)|| \leq C||f||_{F_a}\), it now follows that \(||DC_\Phi(f)||_{B^\beta} \leq C||f||_{F_a}\) as required.

For the converse, assume that \(||DC_\Phi(f)||_{B^\beta} \leq C||f||_{F_a}\) for a constant \(C\) independent of \(f \in F_a\). In particular, \(\Phi' \in B^\beta\).

The argument leading to (16) remains valid for \(\beta \geq 2\). Thus, (22) holds. It remains to prove (21). First, note that
\[
\sup_{|\Phi(z)| \leq \frac{1}{2}} \frac{(1 - |w|^2)^{\beta} |\Phi'(w)|^2}{(1 - |\Phi(w)|^2)^{\alpha+1}} \leq \left(\frac{4}{3}\right)^{\alpha+1} ||\Phi'||_{B^\beta} < \infty.
\]
(25)

For \(w \in U\), define
\[
H_w(z) = (\alpha + 1) \frac{(1 - |\Phi(w)|^2)^2}{(1 - |\Phi(w)|^2)^{\alpha+1}} - (\alpha + 1) (1 - |\Phi(w)|^2)^2 \\
(1 - |\Phi(w)|^2)^{\alpha+2}, \tag{26}
\]
for \(z \in U\). By Lemma 1 and Lemma 2, there is a constant \(C\) independent of \(w\) such that \(||H_w||_{F_a} \leq C\). Thus, \(||DC_\Phi(H_w)||_{B^\beta} \leq C\) for all \(w\). It follows that
\[
\sup_{z \in U}(1 - |z|^2)^{\beta} |H_w'(\Phi(z))\Phi''(z) + (\Phi'(z))^2 H_w''(\Phi(z))| < C, \tag{27}
\]
for all \(w \in U\). An argument using \(H_w'(\Phi(w)) = (\alpha + 1) \Phi(\Phi(w)/(1 - |\Phi(w)|^2)^{\alpha+1})\) and \(H_w''(\Phi(w)) = 0\) yields
\[
\sup_{1/2 < |\Phi(z)|} \frac{(1 - |w|^2)^{\beta} |\Phi'(w)|^2}{(1 - |\Phi(w)|^2)^{\alpha+1}} < \infty. \tag{28}
\]

The relations (25) and (28) establish relation (21), and the proof is complete.

**Theorem 5.** Fix \(\alpha > 0\) and assume \(\beta \geq 2\). Let \(\Phi\) be a self-map of \(U\) for which \(DC_\Phi : F_a \to B^\beta\) is bounded.
\[
DC_\Phi : F_a \to B^\beta \text{ is compact } \iff \lim_{|\Phi(z)| \to 1} \frac{(1 - |z|^2)^{\beta} |\Phi'(z)|^2}{(1 - |\Phi(z)|^2)^{\alpha+2}} = 0, \tag{29}
\]
and
\[
\lim_{|\Phi(z)| \to 1} \frac{(1 - |z|^2)^{\beta} |\Phi'(z)|^2}{(1 - |\Phi(z)|^2)^{\alpha+1}} = 0. \tag{30}
\]
Proof. First, assume that $DC_\Phi : F_a \to B^\beta$ is bounded and relations (30) and (31) hold. Let $(f_n)$ be a bounded sequence in $F_a$ such that $f_n \to 0$ uniformly on compact subsets of $U$. As previously noted, there is a constant $C$ depending only on $\alpha$ such that

$$\left(1 - |z|^2\right)^\beta |f_n'(\Phi(z))||\Phi'(z)|^2 \leq C \frac{\left(1 - |z|^2\right)^\beta |\Phi'(z)|^2}{\left(1 - |\Phi(z)|^2\right)^{\alpha+1}},$$

for $n = 1, 2, \ldots$ and $z \in U$. Relation (31) now implies that given $\varepsilon > 0$, there exists $r_0, 0 < r_0 < 1$, such that

$$\sup_{|\Phi(z)| > r_0} \left(1 - |z|^2\right)^\beta |f_n''(\Phi(z))||\Phi'(z)|^2 < \varepsilon,$$

for all $n$. Since $DC_\Phi : F_a \to B^\beta$ is bounded, relation (9) holds, and thus

$$\left(1 - |z|^2\right)^\beta |f_n''(\Phi(z))||\Phi'(z)|^2 < C |f_n''(\Phi(z))|,$$

for all $z \in U$. Since $f_n'' \to 0$ uniformly on $\{w : |w| \leq r_0\}$, there exists $N > 0$ such that

$$\sup_{|\Phi(z)| \leq r_0} \left(1 - |z|^2\right)^\beta |f_n''(\Phi(z))||\Phi'(z)|^2 < \varepsilon,$$

for all $n > N$. The relations (33) and (35) yield

$$\sup_{z \in U} \left(1 - |z|^2\right)^\beta |f_n''(\Phi(z))||\Phi'(z)|^2 < \varepsilon,$$

for $n > N$. A similar argument using $\Phi' \in B^\beta$ and (30) yields $N_1 > 0$ such that

$$\sup_{z \in U} \left(1 - |z|^2\right)^\beta |f_n''(\Phi(z))||\Phi'(z)| < \varepsilon,$$

for $n > N_1$. The relations (36) and (37) yield

$$\sup_{z \in U} \left(1 - |z|^2\right)^\beta |DC_\Phi f_n''(\Phi(z))| < \varepsilon,$$

as $n \to \infty$. Since $\|DC_\Phi f_n\|(0) \to 0$ as $n \to \infty$, the argument yields $\|DC_\Phi (f_n)\|_{B^\beta} \to 0$ as $n \to \infty$ for any sequence $(f_n)$ as described, and therefore, $DC_\Phi : F_a \to B^\beta$ is compact.

For the converse, assume that $DC_\Phi : F_a \to B^\beta$ is compact. We may assume that $\|\Phi\|_{\infty} = 1$. Let $(z_n)$ be any sequence in $U$ with $|\Phi(z_n)| \to 1$ as $n \to \infty$. For $z \in U$, define

$$f_n(z) = \frac{(\alpha + 3) (1 - |\Phi(z_n)|^2)}{(1 - |\Phi(z_n)z|^2)^{\alpha+1}} - \frac{(\alpha + 1) (1 - |\Phi(z_n)|^2)}{(1 - |\Phi(z_n)z|^2)^{\alpha+2}}.$$

(39)

By the lemmas above, $\|f_n\|_{F_a} \leq C$. Also, $f_n \to 0$ uniformly on compact subsets. Therefore, $\|DC_\Phi (f_n)\|_{B^\beta} \to 0$ and

$$\sup_{z \in U} \left(1 - |z|^2\right)^\beta |f_n'(\Phi(z))||\Phi'(z)|^2 \to 0,$$

as $n \to \infty$. Calculations yield $f_n''(\Phi(z_n)) = 0$ and

$$f_n'(\Phi(z_n)) = \frac{(\alpha + 1) |\Phi(z_n)|}{(1 - |\Phi(z_n)|^2)^{\alpha+1}}.$$

(41)

Substitution into (40) yields

$$\frac{(1 - |z|^2)^\beta |\Phi(z)| |\Phi'(z)|}{(1 - |\Phi(z)|^2)^{2\alpha+1}} \to 0,$$

as $n \to \infty$. Since $(z_n)$ is a generic sequence with $|\Phi(z_n)| \to 1$ as $n \to \infty$, this yields the relation (30).

A similar argument using the functions

$$g_n(z) = \frac{(\alpha + 2) (1 - |\Phi(z)|^2)}{(1 - |\Phi(z)z|^2)^{\alpha+1}} - \frac{(\alpha + 1) (1 - |\Phi(z)|^2)}{(1 - |\Phi(z)z|^2)^{\alpha+2}},$$

(43)

yields the relation (31). The details are omitted.

Theorem 3 implies that if $DC_\Phi : F_a \to B^\beta$ is bounded for fixed $\alpha, \beta$ with $\beta < \gamma < 2$, then $DC_\Phi : F_{\gamma} \to B^\beta$ is compact for all $\gamma > 0$. The next corollary gives a related result when $\beta \geq 2$.

**Corollary 6.** Fix $\alpha > 0$ and $\beta \geq 2$. Let $\Phi$ be a self-map of $U$ and assume that $DC_\Phi : F_a \to B^\beta$ is bounded. Then, $DC_\Phi : F_{\gamma} \to B^\beta$ is compact for any $\gamma, 0 < \gamma < \alpha$.

Proof. By assumption, there is a constant $C$ such that $\|DC_\Phi(f)\|_{B^\beta} \leq C \|f\|_{F_a}$ for all $f \in F_a$. Fix $\gamma$ with $0 < \gamma < \alpha$ and let $f \in F_{\gamma}$. Then, $f \in F_a$ and $\|f\|_{F_a} \leq \|f\|_{F_{\gamma}}$. Therefore, $DC_\Phi : F_{\gamma} \to B^\beta$ is compact. Since $DC_\Phi : F_a \to B^\beta$ is bounded and Theorem 5 applies.

Since $DC_\Phi : F_a \to B^\beta$ is bounded, (21) yields

$$\frac{(1 - |z|^2)^\beta |\Phi'(z)|}{(1 - |\Phi(z)|^2)^{\alpha+1}} \leq C (1 - |\Phi(z)|^2)^{\alpha-\gamma},$$

(44)
and therefore,
\[
\lim_{|\Phi(z)| \to 1} \frac{(1 - |z|^2)^\beta |\Phi'(z)|}{(1 - |\Phi(z)|^2)^{1/2}} = 0. \tag{45}
\]

A similar argument using (22) yields
\[
\lim_{|\Phi(z)| \to -1} \frac{(1 - |z|^2)^\beta |\Phi'(z)|^2}{(1 - |\Phi(z)|^2)^{1/2}} = 0. \tag{46}
\]

Theorem 5 now yields \( DC_\Phi : F_\gamma \to B^\beta \) is compact.

4. The Operator \( C_{qD} \)

In this section, characterizations are given for self-maps \( \Phi \) for which \( C_{qD} : F_n \to B^\beta \) is bounded or compact. The proofs are similar to those in Section 3, so details are kept to a minimum.

**Theorem 7.** Fix \( \alpha > 0 \) and \( 0 < \beta < 1 \).

\( C_{qD} : F_n \to B^\beta \) is bounded \( \iff \)

\[ \Phi \in B^\beta \text{ and } \sup_{w \in U} |(\Phi(z) - \Phi(w))| < \infty \]

\[ \iff \]

\( C_{qD} : F_n \to B^\beta \) is compact.

**Proof.** First, assume that there is a constant \( C \) independent of \( f \) in \( F_n \) such that \( \|C_{qD}(f)\|_{B^\beta} \leq C \|f\|_{F_n} \). In particular, \( \Phi \in B^\beta \). For \( w \in U \), define
\[
g_w(z) = \frac{1}{1 - |\Phi(w)|^2} (z \in U). \tag{48}
\]

There is a constant \( C \) independent of \( w \in U \) such that \( \|g_w\|_{F_n} \leq C \), and it follows that
\[
\sup_{z \in U} (1 - |z|^2)^\beta |g_w'(z)|/|\Phi'(z)| < C, \tag{49}
\]

for all \( w \in U \). The substitution \( z = w \) yields
\[
(1 - |w|^2)^\beta (\alpha + 1) |\Phi'(z)|/|\Phi'(w)| < C, \tag{50}
\]

for all \( w \in U \). Therefore,
\[
\sup_{1/2 < |\Phi(w)|} (1 - |w|^2)^\beta |\Phi'(w)|/|\Phi'(z)| < \infty. \tag{51}
\]

Since \( \Phi \in B^\beta \),
\[
\sup_{1/2 < |\Phi(w)|} (1 - |w|^2)^\beta |\Phi'(w)| < \infty. \tag{52}
\]

It follows that
\[
\sup_{w \in U} (1 - |w|^2)^\beta |\Phi'(w)| < \infty \tag{53}
\]

and therefore
\[
\sup_{w \in U} (1 - |w|^2)^\beta |\Phi'(w)| < \infty. \tag{54}
\]

By [9], \( C_{qD} : B^\beta \to B^\beta \) is bounded. A further argument as in the proof of Theorem 3 yields that \( C_{qD} : B^\beta \to B^\beta \) is compact. Since \( \beta < 1 \), Shapiro’s result [7] applies and yields \( \|C_{qD}\|_{B^\beta} < 1 \) are necessary in order for \( C_{qD} : F_n \to B^\beta \) to be bounded.

Next, assume \( \Phi \in B^\beta \) and \( \sup_{w \in U} |(\Phi(z) - \Phi(w))| < \infty \). Let \( (f_n) \) be a bounded sequence in \( F_n \) with \( f_n \to 0 \) uniformly on compact subsets of \( U \). First, note that \( \|f'_n(\Phi(0))\| \to 0 \) as \( n \to \infty \).

For \( z \in U \),
\[
(1 - |z|^2)^\beta \left| f'_n(\Phi(z)) \right| \leq \sup_{w \in U} |\Phi'(w)| \leq \sup_{w \in U} |\Phi'(w)| = C \|f\|_{F_n}. \tag{55}
\]

Since \( f'_n \to 0 \) uniformly on compact subsets, the argument yields \( \|C_{qD}(f_n)\|_{B^\beta} \to 0 \) and \( C_{qD} : F_n \to B^\beta \) is compact.

The remaining implication is trivial, and the proof is complete.

**Theorem 8.** Fix \( \alpha > 0 \) and \( \beta \geq 1 \). Let \( \Phi \) be a self-map of \( U \).

\( C_{qD} : F_n \to B^\beta \) is bounded \( \iff \)

\[ \sup_{z \in U} (1 - |z|^2)^\beta |\Phi'(z)| < \infty \]

\[ \iff \]

\( \sup_{z \in U} (1 - |\Phi(z)|^2)^\beta < \infty \).

**Proof.** First, assume that the supremum is finite.

Let \( f \in F_n \). By previous remarks, \( |f' \Phi(0)| \leq C \|f\|_{F_n} \). By an argument as in the proof of Theorem 4,
\[
(1 - |z|^2)^\beta |f'(\Phi(z))| \leq (1 - |z|^2)^\beta |f'(\Phi(z))||\Phi'(z)| \leq C \|f\|_{F_n} \leq C \|f\|_{F_n}, \tag{56}
\]

and thus, \( \|C_{qD}(f)\|_{B^\beta} \leq C \|f\|_{F_n} \) as required.

To complete the proof, assume that \( \|C_{qD}(f)\|_{B^\beta} \leq C \|f\|_{F_n} \) for a constant \( C \) independent of \( f \). The argument leading to (53) remains valid for \( \beta \geq 1 \). This proves the opposite implication, and the proof is complete.
Theorem 9. Fix $\alpha > 0$ and $\beta \geq 1$. Let $\Phi$ be a self-map of $U$ and assume that $C_\alpha D : F_\alpha \rightarrow B^\beta$ is bounded.

$$C_\alpha D : F_\alpha \rightarrow B^\beta \text{ is compact } \Leftrightarrow \lim_{|\Phi(z)| \to 1} \frac{(1 - |z|^\beta)|\Phi'(z)|}{(1 - |\Phi(z)|^2)^{\beta/2}} = 0.$$  \hspace{1cm} (58)

Proof. First, assume that $C_\alpha D : F_\alpha \rightarrow B^\beta$ is bounded and the limit condition holds. Let $(f_n)$ be a bounded sequence in $F_\alpha$ with $f_n \to 0$ uniformly on compact subsets as $n \to \infty$. Clearly, $|f_n'(\Phi(0))| \to 0$ as $n \to \infty$. As in previous arguments,

$$(1 - |z|^\beta)|f_n''(\Phi(z))||\Phi'(z)| \leq C \frac{(1 - |z|^\beta)|\Phi'(z)|}{(1 - |\Phi(z)|^2)^{\beta/2}},$$  \hspace{1cm} (59)

for all $z \in U$. The hypothesis now implies that, given $\varepsilon > 0$, there exists $r_0, 0 < r_0 < 1$, such that

$$\sup_{|\Phi(z)| > r_0} (1 - |z|^\beta)|f_n''(\Phi(z))||\Phi'(z)| < \varepsilon,$$  \hspace{1cm} (60)

for all $n$. Since $\Phi \in B^\beta$ and since $f_n'' \to 0$ uniformly on compact subsets,

$$\sup_{|\Phi(z)| \leq r_0} (1 - |z|^\beta)|f_n''(\Phi(z))||\Phi'(z)| \to 0,$$  \hspace{1cm} (61)

as $n \to \infty$. By (60) and (61),

$$\sup_{z \in U} (1 - |z|^\beta)|\left(f_n \circ \Phi\right)'(z)| \to 0,$$  \hspace{1cm} (62)

as $n \to \infty$. The argument yields $\|f_n' \circ \Phi\|_{B^\beta} \to 0$ as $n \to \infty$ for any sequence $(f_n)$ as described above. Thus, $C_\alpha D : F_\alpha \rightarrow B^\beta$ is compact.

Now, assume that $C_\alpha D : F_\alpha \rightarrow B^\beta$ is compact. We may assume that $\|\Phi\|_{\infty} = 1$. Let $(z_n)$ be any sequence in $U$ with $|\Phi(z_n)| \to 1$ as $n \to \infty$. For $n = 1, 2, \ldots$, define

$$f_n(z) = \frac{1 - |\Phi(z_n)|^2}{(1 - |\Phi(z_n)|^2)^{\beta/2}},$$  \hspace{1cm} (63)

for $z \in U$. By Lemma 1, $\|f_n\|_{F_\alpha} \leq C$ for all $n$. Also, $f_n \to 0$ uniformly on compact subsets. Therefore, $\|C_\alpha D(f_n)\|_{B^\beta} \to 0$ as $n \to \infty$. Given $\varepsilon > 0$, there exists $N > 0$ such that

$$\sup_{z \in U} (1 - |z|^\beta)|f_n''(\Phi(z))||\Phi'(z)| < \varepsilon,$$  \hspace{1cm} (64)

for all $n > N$. In particular, $(1 - |z_n|^\beta)|f_n''(\Phi(z_n))||\Phi'(z_n)| < \varepsilon$ for $n > N$. Calculations yield

$$\frac{(1 - |z_n|^\beta)(\alpha + 1)(\alpha + 2)|\Phi(z_n)|^2|\Phi'(z_n)|}{(1 - |\Phi(z_n)|^2)^{\alpha + \beta}} < \varepsilon,$$  \hspace{1cm} (65)

for $n > N$. Since $(z_n)$ is a generic sequence with $|\Phi(z_n)| \to 1$, it follows that

$$\lim_{|\Phi(z)| \to 1} \frac{(1 - |z|^\beta)|\Phi'(z)|}{(1 - |\Phi(z)|^2)^{\beta/2}} = 0.$$  \hspace{1cm} (66)

The proof is complete.

Assume that $C_\alpha D : F_\alpha \rightarrow B^\beta$ is bounded for fixed $\alpha > 0$ and $\beta < 1$. By Theorem 7, $\Phi \in B^\beta$ and $\|\Phi\|_{\infty} < 1$. It follows that $C_\alpha D : F_\gamma \rightarrow B^\beta$ is compact for any $\gamma > 0$. Corollary 10 gives a related result in the case $\beta \geq 1$.

Corollary 10. Fix $\alpha > 0$, $\beta > 1$ and assume that $C_\alpha D : F_\alpha \rightarrow B^\beta$ is bounded. Then, $C_\alpha D : F_\gamma \rightarrow B^\beta$ is compact for any $\gamma$, $0 < \gamma < \alpha$.

Proof. Fix $0 < \gamma < \alpha$ and let $f \in F_\gamma$. Then, $f \in F_\alpha$ and $\|f\|_{F_\alpha} \leq \|f\|_{F_\gamma}$ [2]. Therefore, $C_\alpha D : F_\gamma \rightarrow B^\beta$ is bounded and Theorem 9 applies.

By Theorem 8, there is a constant $C$ with

$$\frac{(1 - |z|^\beta)|\Phi'(z)|}{(1 - |\Phi(z)|^2)^{\beta/2}} \leq C,$$  \hspace{1cm} (67)

for all $z \in U$. Therefore,

$$\frac{(1 - |z|^\beta)|\Phi'(z)|}{(1 - |\Phi(z)|^2)^{\beta/2}} \leq C (1 - |\Phi(z)|^2)^{-\gamma/2} \rightarrow 0,$$  \hspace{1cm} (68)

as $|\Phi(z)| \to 1$. By Theorem 9, $C_\alpha D : F_\gamma \rightarrow B^\beta$ is compact.

Data Availability

This manuscript does not contain any data.

Conflicts of Interest

The author declares that there is no conflict of interest regarding the publication of this paper.

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