Quantum radiation force on a moving mirror for a thermal and a coherent field

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Abstract.
We consider a real massless scalar field in a two-dimensional spacetime, satisfying Dirichlet or Neumann boundary condition at the instantaneous position of a moving boundary. For a relativistic particular motion proposed by Walker-Davies (J. Phys. A, \textbf{15}, L477 (1982)), we obtain the exact formulas for the radiation force on the boundary for thermal and coherent states.

1. Introduction
Thirty years ago, several authors started the investigation about the moving mirror radiation problem (see Refs. [1–7]). In the context of a real scalar field in a two dimensional spacetime, with the Dirichlet boundary condition imposed to the field at the moving boundary position $x = z(t)$, Fulling and Davies [3] obtained an exact formula for the finite physical part of the expected value of the energy-momentum tensor, assuming the initial state as the vacuum. Several mirror trajectories considered in the literature, including those investigated in Ref. [3], assumed the boundary acceleration starting or dropping abruptly, generating an infinite energy density, or a delta pulse of energy emitted from the discontinuity point. Desiring to study an exactly soluble model without these discontinuities, Walker and Davies [8] proposed a law of motion for which the mirror accelerates along a smooth and asymptotically static trajectory. Then, they obtained an exact solution for the radiation moving mirror problem. Their asymptotically static motion brought the advantage of avoid certain pathologies related to the radiation emitted by a moving mirror with abrupt acceleration.

In the present paper, we investigated the model proposed by Walker and Davies [8], but considering the initial field state as thermal or a coherent one, and also Dirichlet and Neumann boundary conditions imposed to the field. The present work is based on the exact results obtained in Ref. [9].

2. Thermal and Coherent Forces for a Particular Moving Mirror Law
In Ref. [8], Walker and Davies proposed the following law of motion of the mirror:

$$t = -z \pm A \left(e^{-2z/B} - 1\right)^{1/2},$$

where $A$ and $B$ are constants, with $A > B$ so that $|\dot{z}| < 1$. This is a smooth and asymptotically static motion (see Fig. 1) at $t \to \pm \infty$. The mirror velocity can be relativistic (see Fig. 2) for
the parameters $A = 2$ and $B = 1$. This kind of motion enables us to write the expression for the mirror velocity as:

$$
\dot{z} = -\frac{BP(u)}{BP(u) + p(u)^2 + A^2},
$$

(2)

where the function $p(u)$ is obtained from $p(u) = 2\tau(u) - u$ and $\tau(u)$ from $u = \tau(u) - z[\tau(u)]$.

From Eq. (1) we get $u = B\ln(p(u)^2/A^2 + 1) + p(u)$ [8], and also:

$$
z = -\frac{B}{2}\ln(p(u)^2/A^2 + 1).
$$

(3)

**Figure 1.** The smooth and asymptotically motion proposed by Walker and Davies, with A=2 and B=1.

Let us now examine the radiation force on the moving mirror when a thermal bath with temperature $T$ is considered as the initial field state. For this case, we have: $\langle \hat{a}_\omega^\dagger \hat{a}_\omega \rangle = \bar{n}(\omega)\delta(\omega - \omega')$ where $\bar{n}(\omega) = 1/(e^{\hbar\omega/T} - 1)$, with the Boltzmann constant equal to 1. Hereafter we consider the averages $\langle ... \rangle$ taken over an initial field state, assumed here, for simplicity, as being the same one for both sides of the mirror. Starting from the expected value of the energy density operator $T = \langle T_{00}(t, x) \rangle$, we can write the net force $F(t)$ acting on the moving boundary defined by (since $T_{00} = T_{11}$ in this model):

$$
F(t) = T[t, z(t)]^{(-)} - T[t, z(t)]^{(+)},
$$

(4)

where the superscript “$+$” indicates that the force acts on the right side of the boundary, whereas “$-$” indicates the force acting on the left side. We can split the net force in the following manner:

$$
F = F_{\text{vac}} + F^{(T)},
$$

(5)
Figure 2. The relativistic mirror velocity for the parameters $A = 2$ and $B = 1$.

where $F_{\text{vac}}$ stands for the vacuum contribution to the total force. For the thermal contribution, $F^{(T)}$, we get the exact formula [9]:

$$F^{(T)} = -\sigma T \left[ \frac{z (1 + \dot{z}^2)}{(1 - \dot{z}^2)^2} \right] = \sum_{n=0}^{\infty} F^{(T)}_{(n)}, \quad (6)$$

where

$$F^{(T)}_{(n)} = -\sigma T \sum_{n=0}^{\infty} (2n + 1) \dot{z}^{2n+1} \quad (7)$$

and $\sigma^{(T)} = 2\pi T^2/3$ is the viscosity coefficient. From Eq. (2) and (6) we get:

$$F^{(T)} = \sigma^{(T)} \left\{ \frac{Bp(u)}{(2Bp(u))^3 + 2Bp(u)A^2 + p(u)^4 + 2p(u)^2 A^2 + A^4)} \times \left[ \frac{Bp(u)}{2Bp(u)^2 + 2Bp(u)A^2 + p(u)^4 + 2p(u)^2 A^2 + A^4)} \right] \right\}, \quad (8)$$

In Eq. (6), truncating the series in $n = N (N = 0, 1, 2, ...)$, we get the approximate force $\tilde{F}^{(T)}$:

$$\tilde{F}^{(T)} = \sum_{n=0}^{N} F^{(T)}_{(n)}. \quad (9)$$

For $n = 0$ we get:

$$F^{(T)} \approx F^{(T)}_{(0)} = -\sigma^{(T)} \dot{z} = \sigma^{(T)} \frac{Bp(u)}{Bp(u) + p(u)^2 + A^2}. \quad (10)$$

This formula shows that the force is proportional to the boundary velocity (the approximate thermal force depending on velocity was obtained by Jaekel and Reynaud [10]). In Fig. 3 we show the exact force (solid line) and the approximate one (dashed line). We see that for
higher velocities (see also Fig. 2) a bigger discrepancy between exact and approximate values occurs. But for \( t = 10 \), the boundary velocity is around 0.1 of the light velocity and, besides this relativistic velocity, both exact and approximate formulas are in good agreement in this region. It is also remarkable that the thermal force is the same for Dirichlet or Neumann boundary condition [9].

![Figure 3.](image)

**Figure 3.** The exact thermal force \( F^{(T)} \) (solid line) and the approximate thermal force \( F^{(T)} \approx F^{(0)} \) (dashed line), for \( T = 1 \).

Now, let us investigate the coherent state as the initial field state. The coherent state of amplitude \( \alpha \) is defined as an eigenstate of the annihilation operator: \( \hat{a}_\omega \left| \alpha \right> = \alpha \delta \left( \omega - \omega_0 \right) \left| \alpha \right> \), where \( \alpha = |\alpha| \exp(i\theta) \) and \( \omega_0 \) is the frequency of the excited mode. For this case, we can split the net force in the following manner:

\[
F = F_{\text{vac}} + F^{(\alpha)},
\]

where:

\[
F^{(\alpha)} = F^{(\alpha)}_{\langle \hat{a}^\dagger \hat{a} \rangle} + F^{(\alpha)}_{\langle \hat{a} \hat{a} \rangle},
\]

\[
F^{(\alpha)}_{\langle \hat{a}^\dagger \hat{a} \rangle} = -\frac{4}{\pi} \omega_0 |\alpha|^2 \hat{z} \left( 1 + \hat{z}^2 \right) / \left( 1 - \hat{z}^2 \right)^2,
\]

\[
F^{(\alpha)}_{\langle \hat{a} \hat{a} \rangle} = \pm \frac{\omega_0}{4\pi} |\alpha|^2 e^{-2i(\omega_0 t - \theta)} \left\{ e^{2i\omega_0 z(t)} \left( \frac{1 - \hat{z}}{1 + \hat{z}} \right)^2 - e^{-2i\omega_0 z(t)} \right\}
\]

\[
= \frac{\omega_0}{4\pi} |\alpha|^2 e^{2i(\omega_0 t - \theta)} \left( \frac{1 + \hat{z}}{1 - \hat{z}} \right)^2 \left\{ e^{2i\omega_0 z(t)} - e^{-2i\omega_0 z(t)} \right\} + c.c.,
\]

and the sign “+” refers to Dirichlet and “-” to Neumann boundary conditions. Using Eq. (2) and (3) in Eqs.(13) and (14), we get the specific formulas:

\[
F^{(\alpha)}_{\langle \hat{a}^\dagger \hat{a} \rangle} = \sigma^{(\alpha)} \left( \frac{Bp(u)(Bp(u) + p(u)^2 + A^2)}{(2Bp(u)^3 + 2Bp(u) A^2 + p(u)^4 + 2p(u)^2 A^2 + A^4)^2} \right.
\]

\[
\times \left( 2B^2 p(u)^2 + 2Bp(u)^3 + 2Bp(u) A^2 + p(u)^4 + 2p(u)^2 A^2 + A^4 \right),
\]

4
\[ F^{(\alpha)}_{(\hat{a}\hat{a})} = \pm \sigma^{(\alpha)} e^{-2i(\omega_0 t - \theta)} \left\{ \frac{(p(u)^2 + A^2)^2 + (2Bp(u) + p(u)^2 + A^2)^2}{(p(u)^2 + A^2)^2} \left( \frac{p(u)^2 + A^2}{A^2} \right)^{-i\omega_0 B} \right\} \]

\[ = \frac{16}{\pi} |\alpha|^2 \omega_0 \left\{ \frac{(p(u)^2 + A^2)^2 + (2Bp(u) + p(u)^2 + A^2)^2}{(2Bp(u) + p(u)^2 + A^2)^2} \left( \frac{p(u)^2 + A^2}{A^2} \right)^{i\omega_0 B} \right\} + c.c. \]  

(16)

Where \( \sigma^{(\alpha)} = 4 |\alpha|^2 \omega_0 / \pi \). From Eq. (15) we see that the force \( F^{(\alpha)}_{(\hat{a}\hat{a})} \) is the same for Dirichlet or Neumann conditions. On the other hand, in Eq. (16) (and also in Eq. (17)), the sign “+” refers to Dirichlet and “−” to Neumann boundary conditions.

If we consider simultaneously nonrelativistic velocities and small displacements (in the sense considered in Ref. [11]), the force \( F^{(\alpha)} \) can be approximated as [9]:

\[ F^{(\alpha)} \approx \tilde{F}^{(\alpha)} = -\frac{4\omega_0}{\pi} |\alpha|^2 \left\{ \dot{z}(t) \pm [\cos(2(\omega_0 t - 2\theta)) \dot{z}(t) - \sin(2(\omega_0 t - 2\theta)) \omega_0 z(t)] \right\}. \]  

(17)

Using Eq. (2) and (3) in Eqs.(17), we get:

\[ \tilde{F}^{(\alpha)} = \sigma^{(\alpha)} \left\{ \frac{Bp(u)}{Bp(u) + p(u)^2 + A^2} \left[ 1 \pm \cos(2(\omega_0 t - \theta)) \right] \right. \]

\[ \left. \mp \frac{\omega_0 B}{2} \ln \left( \frac{p^2/A^2 + 1}{\sin(2(\omega_0 t - \theta))} \right) \right\}. \]  

(18)

In Fig. 4 we show the exact force (solid line) and the approximate one (dashed line) for the coherent case with Dirichlet boundary condition. Again we see that for higher velocities (see also Fig. 2) a bigger discrepancy between exact and approximate values occurs. But, here, the discrepant part presents the approximate coherent forces bigger than the exact one in absolute values. The inverse behavior was observed in the thermal case. It is also remarkable that for Neumann boundary condition, the coherent force oscillates in a different manner (see Fig. 5).

**Figure 4.** The exact coherent force \( F^{(\alpha)} \) (solid line) and approximate coherent force \( F^{(\alpha)} \approx \tilde{F}^{(\alpha)} \) (dashed line), for the Dirichlet case, with \( \alpha = 1, \theta = \pi/2 \) and \( \omega_0 = 10 \).
In Fig. 3 we see that $|F^{(T)}| \geq |\tilde{F}^{(T)}|$. In Eq. (6) we can see that $F^{(T)}$ is given in terms of a series in odd powers of $\dot{z}$, with each term in this series having the same sign. This enable us to write:

$$|F^{(T)}| = \sum_{n=0}^{\infty} |F^{(T)}_{(n)}|.$$  \hspace{1cm} (19)

From the above equation, we can conclude that:

$$|F^{(T)}| \geq |\tilde{F}^{(T)}|. \hspace{1cm} (20)$$

The result in Eq. (20) is valid for any law of motion (not only for the one considered in this paper). In this context, the inequality in Eq. (20) explains the graphical behavior of the curves in Fig. 3. Note that in Figs. 2 and 3 $F^{(T)}$ is always opposite to the boundary motion, what can also be concluded via Eq. (6).

In Fig. 4 we can observe an inverse behavior: $|F^{(\alpha)}| \leq |\tilde{F}^{(\alpha)}|$. In fact, the formula (17) is obtained after considering small velocities and also small displacements. The latter consideration originates a linear dependence on $z(t)$ in Eq. (17). Then, when $t \to \pm \infty$, $|z| \to \infty$ and the amplitude of $\tilde{F}^{(\alpha)}$ grows up.

In summary, we studied the force exerted by the field on one moving mirror which imposes on a real massless scalar field in a two-dimensional spacetime a Dirichlet or Neumann boundary condition. We extended to the thermal and coherent initial field states the analytical results obtained by Walker and Davies for the force on the moving boundary. Numerical calculations were done, resulting in the graphical behavior of these forces.

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