ON IMPROVEMENT OF SUMMABILITY PROPERTIES IN NONAUTONOMOUS KOLMOGOROV EQUATIONS

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ABSTRACT. Under suitable conditions, we obtain some characterizations of supercontractivity, ultraboundedness and ultracontractivity of the evolution operator \(G(t,s)\) associated to a class of nonautonomous second order parabolic equations with unbounded coefficients defined in \(I \times \mathbb{R}^d\), where \(I\) is a right-halfline. For this purpose, we establish an Harnack type estimate for \(G(t,s)\) and a family of logarithmic Sobolev inequalities with respect to the unique tight evolution system of measures \(\{\mu_t : t \in I\}\) associated to \(G(t,s)\). Sufficient conditions for the supercontractivity, ultraboundedness and ultracontractivity to hold are also provided.

1. Introduction. Let \(A\) be an autonomous second order uniformly elliptic operator with unbounded coefficients defined in \(\mathbb{R}^d\). It is well known that, under suitable assumptions on its coefficients, a Markov semigroup \(T(t)\) can be associated in \(C_b(\mathbb{R}^d)\) to the operator \(A\). More precisely, for any \(f \in C_b(\mathbb{R}^d)\), \(T(t)f\) is the value at \(t\) of the (unique) bounded classical solution of the Cauchy problem

\[
\begin{cases}
D_t u(t,x) = Au(t,x), & (t,x) \in (0, +\infty) \times \mathbb{R}^d, \\
u(0,x) = f(x), & x \in \mathbb{R}^d.
\end{cases}
\]

Under somehow stronger assumptions on the coefficients of the operator \(A\), an invariant measure \(\mu\) can be associated to the semigroup \(T(t)\) which can be extended to a contractive semigroup in \(L^p(\mathbb{R}^d, \mu)\) for any \(p \in [1, +\infty)\).

It is also well known that in some cases \(T(t)\) improves summability, i.e., it maps \(L^p(\mathbb{R}^d, \mu)\) into \(L^q(\mathbb{R}^d, \mu)\) for some \(q > p\) and \(t > \tilde{t}(p,q) \geq 0\), and

\[
C_{p,q}(t) := \|T(t)\|_{L^p(\mathbb{R}^d, \mu), L^q(\mathbb{R}^d, \mu)} < +\infty. \tag{1.1}
\]

This property is called \textit{hypercontractivity} if \(p,q \in (1, +\infty)\), \(\tilde{t}(p,q) > 0\) and \(C_{p,q}(t) = 1\), \textit{supercontractivity} if \(p,q \in (1, +\infty)\) and \(\tilde{t}(p,q) = 0\), \textit{ultraboundedness} if \(p \in (1, +\infty)\), \(q = +\infty\) and \(\tilde{t}(p,q) = 0\). If \(p = 1\) this last property is called \textit{ultracontractivity}.

Estimate (1.1) is equivalent to the occurrence of some functional inequalities satisfied by the invariant measure \(\mu\). We refer to [9], the pioneering work on such
topics, where a characterization of the hypercontractivity and the supercontractivity
of the semigroup \( T(t) \) is given in terms of some logarithmic Sobolev inequalities.
Ultraboundedness and ultracontractivity have been widely studied in the au-
tonomous setting, mainly in the symmetric case (where they are equivalent). The
first result in this direction is due to Davies and Simon [6, 7] that, following the
idea of Gross and requiring some additional integrability conditions, connect ul-
tracontractivity with a family of logarithmic Sobolev inequalities. Other different
approaches to study ultracontractivity have been also suggested by [4] and, more
recently, by [18].

On the other hand, to the best of our knowledge, results on summability impro-
v ing have been not yet studied in the nonautonomous case.

In the recent paper [3] we have dealt with hypercontractivity and we have ex-
tended the connection with logarithmic Sobolev inequalities in a nonautonomous
setting, where the semigroup \( T(t) \) and the invariant measure \( \mu \) are replaced, re-
spectively, by a Markov evolution operator \( G(t, s) \) and an evolution system of measures
\( \{ \mu_t \} \).

In this paper we are interested in exploiting some regularizing properties, stronger
than hypercontractivity, for the evolution operator \( G(t, s) \), and in characterizing
them in terms of suitable inequalities satisfied by an evolution system of measures
\( \{ \mu_t \} \).

Let \( I \) be an open right halfline and for every \( t \in I \) consider the nonautonomous
second order differential operator \( A(t) \) defined on smooth functions \( \zeta \) by
\[
(A(t)\zeta)(x) = \text{Tr}(Q(t)D^2\zeta(x)) + (b(t, x), \nabla\zeta(x)), \quad x \in \mathbb{R}^d.
\]
We assume some smoothness on \( Q = [q_{ij}]_{i,j=1,...,d} \) and \( b = (b_1, \ldots, b_d) \), defined
in \( I \) and \( I \times \mathbb{R}^d \), respectively. Moreover, we require that the coefficients \( q_{ij} \) are
bounded and that the operators \( A(t) \) are uniformly elliptic, i.e., there exists a
positive constant \( \eta_0 \) such that
\[
\langle Q(t)\xi, \xi \rangle \geq \eta_0|\xi|^2, \quad t \in I, \quad \xi \in \mathbb{R}^d.
\]
Assuming the existence of a Lyapunov function, for every \( s \in I \) and \( f \in C_b(\mathbb{R}^d) \),
the nonautonomous Cauchy problem
\[
\begin{cases}
D_t u(t, x) = A(t)u(t, x), & (t, x) \in (s, +\infty) \times \mathbb{R}^d, \\
u(s, x) = f(x), & x \in \mathbb{R}^d,
\end{cases}
\]
admits a unique bounded classical solution \( u = G(\cdot, s)f \), where \( G(t, s) \) is a Markov
evolution operator. The function \( G(\cdot, s)f \) belongs to \( C^{1+\alpha/2,2+\alpha}_{\text{loc}}((s, +\infty) \times \mathbb{R}^d) \) and
admits the following representation formula
\[
(G(t, s)f)(x) = \int_{\mathbb{R}^d} g_{t,s}(x, y)f(y)dy, \quad s < t, \quad x \in \mathbb{R}^d, \quad f \in C_b(\mathbb{R}^d),
\]
where \( g_{t,s} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) is a positive function such that \( \|g_{t,s}(x, \cdot)\|_{L^1(\mathbb{R}^d)} = 1 \) for
any \( t, s \in I \), with \( t > s \), and any \( x \in \mathbb{R}^d \).

The existence of a Lyapunov function such that
\[
\lim_{|x| \to +\infty} \varphi(x) = +\infty \quad \text{and} \quad (A(t)\varphi)(x) \leq a - \gamma \varphi(x), \quad (t, x) \in I \times \mathbb{R}^d,
\]
for some positive constants \( a \) and \( \gamma \), allows to prove (see [11]) the existence of
tight evolution systems of measures \( \{ \mu_t : t \in I \} \), i.e., families of Borel probability
measures such that \( \mu_t(B(0, R)) \) tends to 1 as \( R \to +\infty \), uniformly with respect to \( t \in I \), and
\[
\int_{\mathbb{R}^d} (G(t, s)f)(y)d\mu_t(y) = \int_{\mathbb{R}^d} f(y)d\mu_s(y), \quad t > s \in I, \ f \in C_b(\mathbb{R}^d).
\] (1.3)

Here, \( B(0, R) \) denotes the ball centered at 0 with radius \( R \). The interest in evolution systems of measures is due to the good properties that the evolution operators enjoy in the \( L^p \)-spaces related to these systems. Indeed, using (1.3) and the density of \( C_0^\infty(\mathbb{R}^d) \) in \( L^p(\mathbb{R}^d, \mu_t) \) for every \( t \in I \), the evolution operator can be extended to a contraction (still denoted by \( G(t, s) \)) from \( L^p(\mathbb{R}^d, \mu_s) \) to \( L^p(\mathbb{R}^d, \mu_t) \) for every \( p \in [1, +\infty) \).

In this context the Sobolev embedding theorems fail to hold in general, so that the summability improving properties of the evolution operator \( G(t, s) \) are not always true (see e.g., [15]). Therefore, a generalization to the nonautonomous case of the definitions of hypercontractivity, supercontractivity, ultracontractivity and ultraboundedness (see Definition 2.6) and of their characterizations is significant and interesting.

As it has been already remarked, in [3] hypercontractivity of the evolution operator \( G(t, s) \) has been studied, assuming some stronger assumption than the minimal ones that guarantee the basic properties of \( G(t, s) \) and the existence of an evolution system of measures \( \{\mu_t : t \in I\} \). In fact, if the dissipativity condition
\[
\langle \nabla_x b(t, x)\xi, \xi \rangle \leq r_0|\xi|^2, \quad t \in I, \ x, \xi \in \mathbb{R}^d
\] (1.4)
is satisfied for some \( r_0 < 0 \), then the logarithmic Sobolev inequality (in short LSI) for the unique tight evolution system of measures \( \{\mu_s : s \in I\} \)
\[
\int_{\mathbb{R}^d} f^2 \log \left( \frac{|f|}{\|f\|_{L^2(\mathbb{R}^d, \mu_s)}} \right) \, d\mu_s \leq C \int_{\mathbb{R}^d} \|\nabla f\|^2 \, d\mu_s,
\] (LSI)
holds for any \( s \in I, f \in H^1(\mathbb{R}^d, \mu_s) \) and some positive constant \( C \), independent of \( f \) and \( s \). The hypercontractivity of \( G(t, s) \) in \( L^p \) spaces related to the unique tight evolution system of measures, is obtained as a consequence of the (LSI).

In general, evolution systems of measures are infinitely many (see e.g., [8]). Among all of them, the unique tight system has a prominent role. Indeed, it is related to the asymptotic behaviour of \( G(t, s) \) as \( t \to +\infty \). As it has been proved in [3], under condition (1.4)
\[
\lim_{t \to +\infty} \int_{\mathbb{R}^d} |G(t, s)f - m_s(f)|^p \, d\mu_t = 0,
\]
uniformly with respect to \( f \in L^p(\mathbb{R}^d, \mu_s), p \in [1, +\infty) \), where \( m_s(f) \) denotes the average of \( f \) with respect to the measure \( \mu_s \).

In this paper, we assume that condition (1.4) holds true and consider the unique tight evolution system of measures \( \{\mu_s : s \in I\} \).

We first prove that the supercontractivity property of the evolution operator \( G(t, s) \) is equivalent to the validity of the following family of logarithmic Sobolev inequalities (in short LSI\( \varepsilon \))
\[
\int_{\mathbb{R}^d} f^2 \log \left( \frac{|f|}{\|f\|_{L^2(\mathbb{R}^d, \mu_s)}} \right) \, d\mu_s \leq \varepsilon \|\nabla f\|^2_{L^2(\mathbb{R}^d, \mu_s)} + \beta(\varepsilon)\|f\|^2_{L^2(\mathbb{R}^d, \mu_s)},
\] (1.5)
for every \( s \in I, f \in H^1(\mathbb{R}^d, \mu_s), \varepsilon > 0 \) and some decreasing function \( \beta \), blowing up as \( \varepsilon \to 0^+ \). We follow the method of [16] that, on a Riemann manifold \( M \), deals with
the diffusion semigroup \(T(t)\) generated by the autonomous operator \(L = \Delta + Z\nabla\) with Neumann boundary conditions on \(\partial M\), where \(Z\) is a \(C^1\)-vector field satisfying a curvature condition. The condition on the curvature is used to deduce the following logarithmic Sobolev inequality satisfied by \(T(t)\)

\[
T(t)(f^2 \log f^2) \leq \frac{2(e^{2KTt} - 1)}{K} T(t)|\nabla f|^2 + (T(t)f^2) \log(T(t)f^2),
\]

which holds for every \(f \in C_0^\infty(M)\), \(t > 0\) and some positive constant \(K > 0\).

The starting point of our analysis is the analogue of (1.6) in the nonautonomous case; we prove a logarithmic Sobolev inequality satisfied by the probability measures \(g_{t,s}(x, dy) = g_{t,s}(x, y) dy\) defined in (1.2). More precisely, we show that

\[
G(t, s)(f^2 \log f^2) \leq \frac{4\Lambda}{r_0^2}(1 - e^{2r_0(t-s)})G(t, s)(|\nabla f|^2) + (G(t, s)f^2) \log(G(t, s)f^2),
\]

for every \(f \in C_b^1(\mathbb{R}^d)\) and \(s, t \in I\) such that \(t \geq s\). The key tool for the proof of estimate (1.7) (and of many results in the paper) is the pointwise gradient estimate

\[
|\langle \nabla_x G(t, s)f \rangle(x)| \leq e^{r_0(t-s)}(G(t, s)|\nabla f|)(x), \quad t > s, \quad x \in \mathbb{R}^d, \quad f \in C_b^1(\mathbb{R}^d),
\]

that has been proved in [11] under the assumption (1.4) (which is equivalent to the condition considered in [16]). As in the autonomous case (see [19]), estimate (1.8) does not hold when the diffusion coefficients depend on \(x\) and they do not satisfy the following condition

\[
D_t g_{jk}(t, x) + D_j q_{ik}(t, x) + D_k q_{ij}(t, x) = 0, \quad (t, x) \in I \times \mathbb{R}^d, \quad i, j, k = 1, \ldots, d,
\]

(see [1]) This is the reason why we confine ourself to the case of diffusion coefficients depending only on \(t\).

Another important consequence of (1.8) is the Harnack type estimate

\[
|(G(t, s)f)(x)|^2 \leq (G(t, s)|f|^2)(y) \exp\left(\frac{|x - y|^2}{2r_0(t-s)}\right), \quad t > s, \quad x, y \in \mathbb{R}^d, \quad (1.9)
\]
satisfied by any \(f \in C_b(\mathbb{R}^d)\). Estimate (1.9) and LSI allow us to prove a second criterion for supercontractivity: we show that the integrability with respect to the measures \(\{\mu_t : t \in I\}\) (uniform in \(t\)) of the Gaussian functions \(\varphi_\lambda\), defined by

\[
\varphi_\lambda(x) := e^{\lambda|x|^2}, \quad x \in \mathbb{R}^d, \quad (1.10)
\]

for every \(\lambda > 0\), is another condition equivalent to the supercontractivity of \(G(t, s)\). This second characterization is useful in order to provide a sufficient condition for the evolution operator \(G(t, s)\) to be supercontractive as stated in Theorem 3.9.

The Harnack type estimate (1.9) is also the key tool to prove that, if \(G(t, s)\varphi_\lambda \in L^\infty(\mathbb{R}^d)\) for every \(t > s \in I\) and \(\lambda > 0\), and

\[
\sup_{t, s \in I, t > s} ||G(t, s)\varphi_\lambda||_\infty < +\infty, \quad \delta, \lambda > 0, \quad (1.11)
\]

then \(G(t, s)\) is ultrabounded. We provide a sufficient condition for \(G(t, s)\varphi_\lambda\) to be bounded for every \(t > s \in I\) and every \(\lambda > 0\) (see Theorem 4.1).

Actually, condition (1.11) is also necessary to get ultraboundedness. We prove the necessity of this condition using the characterization of the supercontractivity property in terms of the family of inequalities (1.5).

We provide some explicit conditions on the coefficients of the operator \(A(t)\) in order to prove that the evolution operator \(G(t, s)\) improves summability. These
conditions are given in terms of the behaviour at infinity of the inner product between the drift $b(t,x)$ and $x$. For instance the following estimate
\[
\langle b(t,x), x \rangle \leq -K_1|x|^2(\log |x|)^\alpha, \quad t \in I, \ |x| \geq R,
\]
for some positive constants $K_1$, $\alpha > 1$ and $R > 1$, represents a quite sharp condition for the ultraboundedness of $G(t,s)$. A stronger condition than (1.12) allows us to prove that $G(t,s)$ is bounded from $L^1(\mathbb{R}^d, \mu_s)$ to $L^2(\mathbb{R}^d, \mu_t)$ (hence it is ultrahypercontractive) and that it is also $L^2$-uniformly integrable.

Finally, we extend supercontractivity, ultraboundedness and ultracontractivity to evolution operators associated to nonautonomous operators with a non zero potential term.

The paper is organized as follows. First, in Section 2, we state our main assumptions, we collect some known results on the evolution operator $G(t,s)$ and we give the definition of supercontractivity, ultraboundedness and ultracontractivity in our nonautonomous setting. Section 3 is devoted to prove two criteria for the supercontractivity property of $G(t,s)$. In Section 4 we provide a characterization of ultraboundedness for $G(t,s)$ in terms of the boundedness of the function $G(t,s)\varphi_\lambda$. Section 5 deals with the $L^1$-$L^2$ boundedness of $G(t,s)$, the ultracontractivity property and with the $L^2$-uniform integrability property of $G(t,s)$.

**Notation.** Let $k \in \mathbb{N} \cup \{0, +\infty\}$, we consider the usual space $C^k(\mathbb{R}^d)$, as well as $C^k_b(\mathbb{R}^d)$, the subspace of $C^k(\mathbb{R}^d)$ consisting of bounded functions with bounded derivatives up to the $k$-th order. We use the subscript “c” instead of “b” for the subsets of the above spaces consisting of functions with compact support.

If $J \subset \mathbb{R}$ is an interval and $\alpha \in (0, 1)$, $C^{\alpha/2,\alpha}(J \times \mathbb{R}^d)$ denotes the usual parabolic Hölder space. We use the subscript “loc” to denote the space of all $f \in C(J \times \mathbb{R}^d)$ which are $(\alpha/2, \alpha)$-Hölder continuous in any compact set of $J \times \mathbb{R}^d$.

Let $\mu$ be a probability measure on $\mathbb{R}^d$ and $1 \leq p < \infty$. We denote by $L^p(\mathbb{R}^d, \mu)$ the set of $\mu$-measurable functions $f : \mathbb{R}^d \to \mathbb{R} \cup \{\pm \infty\}$ such that $\|f\|_{L^p} := \int_{\mathbb{R}^d} |f|^p d\mu < +\infty$. If $p = +\infty$ then $L^\infty(\mathbb{R}^d, \mu) = L^\infty(\mathbb{R}^d)$ is endowed with the sup-norm $\| \cdot \|_\infty$. The space $H^1(\mathbb{R}^d, \mu)$ consists of all the functions which belong to $L^2(\mathbb{R}^d, \mu)$ together with their first order distributional derivatives.

Let $T$ be a bounded operator mapping $L^p(\mathbb{R}^d, \mu)$ to $L^q(\mathbb{R}^d, \nu)$ for $1 \leq p \leq q \leq +\infty$ where $\mu, \nu$ are two probability measures on $\mathbb{R}^d$. If no confusion may arise, we denote by $\|T\|_{L^p(\mathbb{R}^d, \mu) \to L^q(\mathbb{R}^d, \nu)}$ the operator norm $\|T\|_{L^p(\mathbb{R}^d, \mu) \to L^q(\mathbb{R}^d, \nu)}$.

About partial derivatives, the notations $D_i f := \frac{\partial f}{\partial t}$, $D_j f := \frac{\partial f}{\partial x_i}$, $D_{ij} f := \frac{\partial^2 f}{\partial x_i \partial x_j}$ are extensively used.

About matrices and vectors, we denote by $\text{Tr}(Q)$ and $\langle x, y \rangle$ the trace of the square matrix $Q$ and the inner product of the vectors $x, y \in \mathbb{R}^d$, respectively. Finally, we set $0 \log 0 := 0$.

2. Assumptions, definitions and a review of some properties of $G(t,s)$. Let $I$ be an open right halfline. For $t \in I$ we consider linear second order differential operators $A(t)$ defined on smooth functions $\zeta$ by
\[
(A(t)\zeta)(x) = \sum_{i,j=1}^d q_{ij}(t)D_{ij}\zeta(x) + \sum_{i=1}^d b_i(t,x)D_i\zeta(x) = \text{Tr}(Q(t)D^2\zeta(x)) + \langle b(t,x), \nabla \zeta(x) \rangle, \quad x \in \mathbb{R}^d,
\]
under the following assumptions on their coefficients.
Hypotheses 2.1.  
(i) \(q_{ij} \in C^{\alpha/2}_\text{loc}(I)\) and \(b_i \in C^{\alpha/2,\alpha}_\text{loc}(I \times \mathbb{R}^d)\) \((i, j = 1, \ldots, d)\) for some \(\alpha \in (0, 1)\); 
(ii) the matrix \(Q(t) = [q_{ij}(t)]_{i,j=1,\ldots,d}\) is symmetric for every \(t \in I\) and there exist \(0 < \eta_0 < \Lambda\) such that 
\[
\eta_0|\xi|^2 \leq \langle Q(t)\xi, \xi \rangle \leq \Lambda|\xi|^2, 
\]
\((t, \xi) \in I \times \mathbb{R}^d\); 
(iii) there exists \(\varphi \in C^{2}(\mathbb{R}^d)\) with positive values such that 
\[
\lim_{|x| \to +\infty} \varphi(x) = +\infty \quad \text{and} \quad \langle A(t)\varphi(x) \rangle \leq a - \gamma \varphi(x), \quad (t, x) \in I \times \mathbb{R}^d, 
\]
for some positive constants \(a\) and \(\gamma\); 
(iv) the first order spatial derivatives of \(b_i\) exist, belong to \(C^{\alpha/2,\alpha}_\text{loc}(I \times \mathbb{R}^d)\) for any \(i = 1, \ldots, d\), and there exists \(r_0 < 0\) such that 
\[
\langle \nabla b(t, x)\xi, \xi \rangle \leq r_0|\xi|^2, \quad (t, x) \in I \times \mathbb{R}^d, \quad \xi \in \mathbb{R}^d.
\]
Remark 2.2. Assumption (2.3) implies that for any \([a, b] \subset I\) there exists a positive constant \(C_{a,b}\) such that 
\[
\langle b(t, x), x \rangle \leq C_{a,b}|x|, \quad t \in [a, b], \quad x \in \mathbb{R}^d.
\]
Indeed, condition (2.3) is equivalent to 
\[
\langle b(t, x) - b(t, y), x - y \rangle \leq r_0|x - y|^2, \quad t \in I, \quad x, y \in \mathbb{R}^d.
\]
Taking \(y = 0\) and observing that \(b\) is continuous, we get 
\[
\langle b(t, x), x \rangle \leq \|b(\cdot, 0)\|_{L^\infty(a,b)}|x| + r_0|x|^2, \quad t \in [a, b], \quad x \in \mathbb{R}^d,
\]
which implies (2.4) since \(r_0 < 0\).

Hypotheses 2.1 yield the existence of a Markov evolution operator \(G(t, s)\) and a unique ([3, Rem. 2.8]) tight evolution system of measures \(\{\mu_t : t \in I\}\) associated to the evolution operator \(G(t, s)\) (where tight means that for any \(\varepsilon > 0\) there exists \(R > 0\) such that \(\mu_t(B(0, R)) \geq 1 - \varepsilon\) for any \(t \in I\)). More precisely, as it has been remarked in the Introduction, for every \(s \in I\) and \(f \in C_b(\mathbb{R}^d)\), \(G(\cdot, s)f\) is the unique bounded classical solution of the Cauchy problem 
\[
\begin{align*}
D_t u(t, x) &= A(t)u(t, x), \quad (t, x) \in (s, +\infty) \times \mathbb{R}^d, \\
u(s, x) &= f(x), \quad x \in \mathbb{R}^d,
\end{align*}
\]
it belongs to \(C_b((s, +\infty) \times \mathbb{R}^d) \cap C^{1+\alpha/2,2+\alpha}_\text{loc}((s, +\infty) \times \mathbb{R}^d)\) and it can be represented by (1.2) ([11, Prop. 2.4]). From this formula the following result follows at once.

**Lemma 2.3.** For any \(I \ni s < t\) and any nonnegative and non identically vanishing function \(f \in C_b(\mathbb{R}^d)\), \(G(t, s)f\) is everywhere positive in \(\mathbb{R}^d\). In particular, \(|G(t, s)g| \leq G(t, s)|g|\) for any \(g \in C_b(\mathbb{R}^d)\).

By Lemma 2.3, formula (1.3) and the density of \(C_b(\mathbb{R}^d)\) in \(L^p(\mathbb{R}^d, \mu_s)\) we have 
\[
\|G(t, s)f\|_{p, \mu_t} \leq \|f\|_{p, \mu_s},
\]
for every \(t > s\), \(p \in [1, +\infty)\) and \(f \in L^p(\mathbb{R}^d, \mu_s)\). Therefore, \(G(t, s)\) may be extended to a contraction (still denoted by \(G(t, s)\)) from \(L^p(\mathbb{R}^d, \mu_s)\) to \(L^p(\mathbb{R}^d, \mu_t)\).

The dissipativity condition (2.3) yields the pointwise gradient estimate 
\[
|\langle \nabla G(t, s)f(x) \rangle|^p \leq e^{pr_0(t-s)}(G(t, s)|\nabla f|^p)(x),
\]
which holds for every \(f \in C^1_b(\mathbb{R}^d)\), \(t \geq s\), \(x \in \mathbb{R}^d\) and \(p \in [1, +\infty)\) ([11, Thm. 4.5]).
Throughout the paper, if not otherwise specified, we assume that all the conditions
(i) The definitions of supercontractivity, ultraboundedness and ul-
Remark 2.7.
(ii) “ultrabounded” if it maps $L^p(\mathbb{R}^d, \mu)$ into $L^\infty(\mathbb{R}^d)$ for every $p > 1$ and $t > s$, and there exists a decreasing function $C_{p,\infty} : (0, +\infty) \to (0, +\infty)$ such that $\lim_{r \to 0^+} C_{p,\infty}(r) = +\infty$ and
\[ \|G(t,s)f\|_{p \to \infty} \leq C_{p,\infty}(t-s), \quad I \ni s < t; \]
(iii) “ultracontractive” if it maps $L^1(\mathbb{R}^d, \mu_s)$ into $L^\infty(\mathbb{R}^d)$ and (2.6) holds for $p = 1$.

Remark 2.7. (i) The definitions of supercontractivity, ultraboundedness and ultracontractivity given in Definition 2.6, where the functions $C_{p,q}$ depend on $t - s$, seem to be the most natural. Indeed, we recall that, if $T(t)$ is a semigroup, then $T(t-s)$ is an evolution operator and the definitions above are the natural extension of those given in the autonomous case.
(ii) If $G(t,s)$ is ultracontractive then it is also ultrabounded.
(iii) The strong Feller property enjoyed by the evolution operator ([11, Cor. 4.3]) states that $G(t,s)$ maps $L^\infty(\mathbb{R}^d)$ into $C_b(\mathbb{R}^d)$ for every $t > s$ and it is a contraction, i.e., for every $f \in L^\infty(\mathbb{R}^d)$
\[ \|G(t,s)f\|_\infty \leq \|f\|_\infty, \quad I \ni s < t. \]

Therefore, if $G(t,s)$ is ultrabounded (resp. ultracontractive) then, in fact, it maps $L^p(\mathbb{R}^d, \mu_s)$ into $C_b(\mathbb{R}^d)$ for every $p > 1$ (resp. $p = 1$) and $t > s$.

Throughout the paper, if not otherwise specified, we assume that all the conditions in Hypotheses 2.1 are satisfied.
3. Supercontractivity and LSI. In this section we provide two criteria to characterize the supercontractivity of the evolution operator $G(t,s)$ by means of a family of logarithmic Sobolev inequalities.

3.1. The first criterion. In this subsection we are devoted to prove the following result.

**Theorem 3.1.** The following properties are equivalent.

(i) The evolution operator $G(t,s)$ is supercontractive;

(ii) the family of logarithmic Sobolev inequalities

$$
\int_{\mathbb{R}^d} f^2 \log \left( \frac{|f|}{\|f\|_{L^2}} \right) \, d\mu_s \leq \varepsilon \| \nabla f \|_{L^2}^2 + \beta(\varepsilon) \| f \|_{L^2}^2
$$

(LSI)

holds for every $f \in H^1(\mathbb{R}^d, \mu_s)$, $s \in I$, $\varepsilon > 0$ and some decreasing function $\beta : (0, +\infty) \rightarrow \mathbb{R}$, blowing up as $\varepsilon \rightarrow 0^+$.

The proof of Theorem 3.1 is based on the following two propositions. In the first, we prove a logarithmic Sobolev inequality satisfied by the evolution operator $G(t,s)$, namely a LSI type estimate satisfied by the probability measures $g_{t,s}(x,y)\, dy$.

**Proposition 3.2.** For every $f \in C^1_b(\mathbb{R}^d)$, $p \in [2, +\infty)$ and $t,s \in I$, with $t \geq s$, we have

$$
G(t,s)(|f|^p \log |f|^p) \leq \frac{p^2 A}{\|f\|_0} (1 - e^{2\varepsilon(t-s)}) G(t,s)(|f|^{p-2} |\nabla f|^2) + (G(t,s)|f|^p \log G(t,s)|f|^p).
$$

(3.1)

**Proof.** We can limit ourselves to proving (3.1) for $p = 2$. Indeed, for every $p > 2$ and $f \in C^1_b(\mathbb{R}^d)$, the claim can be obtained applying (3.1) with $p = 2$ to the function $|f|^{p/2}$. Moreover, it is enough to prove (3.1), with $p = 2$, for nonnegative functions $f \in C^1_b(\mathbb{R}^d)$ with $\sup_{\mathbb{R}^d} f \leq 1$, taking into account that (by (1.2)) $G(t,s) = c$ for every $c \in \mathbb{R}$. To this aim, in order to apply Proposition 2.4 we introduce a standard sequence of cut-off functions

$$
\theta_n(x) = \eta \left( \frac{|x|}{n} \right), \quad x \in \mathbb{R}^d, \ n \in \mathbb{N},
$$

(3.2)

where $\eta \in C^\infty(\mathbb{R})$ and $\chi_{(-\infty,1]} \leq \eta \leq \chi_{(-\infty,2]}$.

Fix $x \in \mathbb{R}^d$, $s,t \in I$, with $s \leq t$, and a nonnegative function $f \in C^1_b(\mathbb{R}^d)$ with $\|f\|_{L^1} \leq 1$, and consider the functions

$$
F_n(r) = \{ G(t,r)\theta_n(G(r,s)f)^2 \log(G(r,s)f)^2 \} (x), \quad s \leq r \leq t, \ n \in \mathbb{N},
$$

which are well defined by Lemma 2.3. For any $s \leq r \leq t$, $F_n(r)$ converges to $F(r) = \{ G(t,r)(G(r,s)f)^2 \log(G(r,s)f)^2 \} (x)$ as $n \rightarrow +\infty$, by the monotone convergence theorem (see (1.2)). Moreover, since the function $\theta_n(G(r,s)f)^2 \log(G(r,s)f)^2$ belongs to $C^2_b(\mathbb{R}^d)$ for every $r > s$ and it vanishes outside $B(0,2n)$, by Proposition 2.4 and the formula

$$
A(r)(g^2 \log g^2) = 2g(1 + \log g^2)A(r)g + 2(3 + \log g^2)\langle Q(r)\nabla g, \nabla g \rangle,
$$

which holds for every positive function $g \in C^2(\mathbb{R}^d)$ and every $r \in I$, we get

$$
F_n(r) = - \left\{ G(t,r) \left[ 2\theta_n(3 + \log(G(r,s)f)^2) \langle Q(r)\nabla G(r,s)f, \nabla G(r,s)f \rangle + (G(r,s)f)^2 \log(G(r,s)f)^2 A(r)\theta_n \right] \right\}.
$$


for every $r \in [s,t]$. Using the dominated convergence theorem, we have
\[
\lim_{n \to +\infty} I_{1,n}(r) = -2\{G(t,r)[(3 + \log(G(r,s)f))^2](Q(r)\nabla_x G(r,s)f, \nabla_x G(r,s)f)]\}(x),
\]
Similarly, since $\nabla \theta_n$ vanishes uniformly in $\mathbb{R}^d$, as $n \to +\infty$, we conclude that $I_{3,n}(r)$ tends to 0 as $n \to +\infty$. Now, let us consider the term $I_{2,n}$; by (2.4), we can estimate
\[
(A(r)\theta_n)(x) \geq \frac{1}{n} \left[ C_{s,t} \eta\left(\frac{|x|}{n}\right) - C \right],
\]
for any $n \in \mathbb{N}$, $x \in \mathbb{R}^d$ and any $r \in [s,t]$, where $C = \Lambda(d\|\eta\|_\infty + \|\eta^\prime\|_\infty)$. Therefore, recalling that $(G(r,s)f)^2 \log(G(r,s)f)^2 \leq 0$, we conclude that
\[
\liminf_{n \to +\infty} I_{2,n}(r) \geq 0.
\]
Summing up, we have proved that
\[
\liminf_{n \to +\infty} F_n'(r) \geq -2\{G(t,r)[(3 + \log(G(r,s)f))^2](Q(r)\nabla_x G(r,s)f, \nabla_x G(r,s)f)]\}(x)
\geq -6\Lambda \{G(t,r)(|\nabla_x G(r,s)f|^2]\}(x),
\]
for any $r \in [s,t]$ and any $x \in \mathbb{R}^d$. Since the sequence $F_n'$ is bounded from below by a constant, from the Fatou lemma we can conclude that
\[
F(t) - F(s) = \lim_{n \to +\infty} (F_n(t) - F_n(s))
\geq \int_s^t \liminf_{n \to +\infty} F_n'(\sigma)d\sigma
\geq -6\Lambda \int_s^t \{G(t,r)(|\nabla_x G(r,s)f|^2]\}(x)dr.
\]
Using the gradient estimate (2.5) we get
\[
F(t) - F(s) \geq -6\Lambda(G(t,s)|\nabla f|^2)(x) \int_s^t e^{2\rho_0(r-s)}dr
= \frac{3\Lambda}{|\rho_0|}(e^{2\rho_0(t-s)} - 1)(G(t,s)|\nabla f|^2)(x),
\]
and (3.1) follows.

Next proposition shows that the boundedness of $G(t,s)$ from $L^p(\mathbb{R}^d, \mu_s)$ into $L^q(\mathbb{R}^d, \mu_t)$, for any $t > s$, yields a family of logarithmic Sobolev inequalities satisfied by the system of invariant measures $\{\mu_t : t \in I\}$. The key tools used in the proof are estimate (3.1) and the Riesz-Thorin interpolation theorem.

**Proposition 3.3.** Assume that, for every $s \in I$, $t > s$ and $1 < p < q < +\infty$, $\tilde{C}_{p,q}(t,s) := \|G(t,s)\|_{p \to q} < +\infty$. Then,
\[
\int_{\mathbb{R}^d} f^2 \log \left(\frac{|f|}{\|f\|_{L^2,\mu_s}}\right) d\mu_s \leq \frac{2\Lambda p(q - 1)}{|\rho_0|(q - p)}(1 - e^{2\rho_0(t-s)})\|\nabla f\|_{L^2,\mu_s}^2 + \frac{pq}{2(q - p)}\log(\tilde{C}_{p,q}(t,s))\|f\|_{L^2,\mu_s}^2,
\]
for every $s \in I$, $t > s$, $f \in H^1(\mathbb{R}^d, \mu_s)$, where $\rho_0$ is the constant in (2.3).
Proof. We split the proof into two steps. In the first one we show that it suffices to prove (3.4) for functions \( f \in C^1_c(\mathbb{R}^d) \) such that \( \|f\|_{2,\mu_s} = 1 \). In the second step, we get estimate (3.4) for such functions.

Step 1. For notational convenience, we set
\[
M_1(t, s) = \frac{2Ap(q-1)}{|r_0|[q-p]}(1 - e^{-2\sigma_0(t-s)}), \quad M_2(t, s) = \frac{pq}{2(q-p)} \log(\tilde{C}_{p,q}(t, s)).
\]
We assume that inequality (3.4) holds for any function \( f \in C^1_c(\mathbb{R}^d) \) such that \( \|f\|_{2,\mu_s} = 1 \), and we show that it actually holds for any \( f \in H^1(\mathbb{R}^d, \mu_s) \). For this purpose, let \( f \in H^1(\mathbb{R}^d, \mu_s) \) satisfy \( \|f\|_{2,\mu_s} = 1 \), and consider a sequence \( (f_n)_n \subset C^1_c(\mathbb{R}^d) \) such that \( \|f_n - f\|_{H^1(\mathbb{R}^d, \mu_s)} \to 0 \) as \( n \to +\infty \) (see [3, Lemma 2.5]). Without loss of generality, we can assume that \( \|f_n\|_{2,\mu_s} = 1 \) for any \( n \in \mathbb{N} \). Up to a subsequence, \( f_n(x) \) converges to \( f(x) \) for almost every \( x \in \mathbb{R}^d \) as \( n \to +\infty \) and
\[
\int_{\mathbb{R}^d} f_n^2 \log |f_n|d\mu_s 
\leq M_1(t, s) \int_{\mathbb{R}^d} |\nabla f_n|^2d\mu_s + M_2(t, s),
\]
for every \( n \in \mathbb{N} \). Let us split \( f_n^2 \log |f_n| = f_n^2 \log_+ |f_n| - f_n^2 |\log |f_n||\chi_{\{|f_n|\leq 1\}} \),
where \( \log_+(r) = \max\{\log(r), 0\} \) for any \( r > 0 \). Since \( f_n^2 \log |f_n| \leq (2e)^{-1} \sup_{r \in (0,1]} x^2 \log x \) for any \( n \in \mathbb{N} \), the dominated convergence theorem yields
\[
\lim_{n \to +\infty} \int_{\mathbb{R}^d} f_n^2 \log |f_n| \chi_{\{|f_n| \leq 1\}} d\mu_s = \int_{\mathbb{R}^d} f^2 \log |f| \chi_{\{|f| \leq 1\}} d\mu_s.
\]
Thus, by Fatou lemma we deduce that
\[
\int_{\mathbb{R}^d} f^2 \log_+ |f| d\mu_s 
\leq \liminf_{n \to +\infty} \left( M_1(t, s) \int_{\mathbb{R}^d} |\nabla f_n|^2d\mu_s + M_2(t, s) + \int_{\mathbb{R}^d} f_n^2 \log |f_n| \chi_{\{|f_n| \leq 1\}} d\mu_s \right)
= M_1(t, s) \int_{\mathbb{R}^d} |\nabla f|^2d\mu_s + M_2(t, s) + \int_{\mathbb{R}^d} f^2 \log |f| \chi_{\{|f| \leq 1\}} d\mu_s,
\]
which leads to (3.4).

Finally, the condition \( \|f\|_{2,\mu_s} = 1 \) can be removed applying (3.4) to the function \( f/\|f\|_{2,\mu_s} \).

Step 2. Let us prove the claim for \( f \in C^1_c(\mathbb{R}^d) \) such that \( \|f\|_{2,\mu_s} = 1 \). The starting point is formula (3.1) with \( p = 2 \) which yields
\[
G(t, s)(f^2 \log f^2) \leq \frac{4\Lambda}{|r_0|}(1 - e^{-2\sigma_0(t-s)})G(t, s)|\nabla f|^2 + (G(t, s)f^2) \log(G(t, s)f^2),
\]
for any \( s, t \in I, \) with \( s < t \) and any \( f \in C^1_c(\mathbb{R}^d) \). Integrating (3.5) in \( \mathbb{R}^d \) with respect to the measure \( \mu_t \) and using (1.3), we get
\[
\int_{\mathbb{R}^d} f^2 \log f^2 d\mu_s \leq \frac{4\Lambda}{|r_0|}(1 - e^{-2\sigma_0(t-s)}) \int_{\mathbb{R}^d} |\nabla f|^2 d\mu_s
+ \int_{\mathbb{R}^d} (G(t, s)f^2) \log(G(t, s)f^2)d\mu_t.
\]
Let us fix \( 1 < p < q < +\infty \). By assumptions, \( G(t, s)\|_{p \to q} = \tilde{C}_{p,q}(t, s) < +\infty \), for every \( t, s \in I \) such that \( t > s \). Since \( G(t, s)|_{1 \to 1} \leq 1 \), from the Riesz-Thorin
interpolation theorem we get that
\[ \|G(t,s)f\|_{\|p_{h,t}\|} \leq (\tilde{C}_{p,q}(t,s))^{r_h} \|f\|_{p_{h,t}}, \tag{3.7} \]
for every \( f \in L^p(\mathbb{R}^d, \mu_s) \) and \( h \in (0, 1 - 1/p) \), where
\[ r_h = \frac{ph}{p-1} \in (0, 1), \quad \frac{1}{ph} = 1 - r_h + \frac{r_h}{p}, \quad \frac{1}{qh} = 1 - r_h + \frac{r_h}{q}. \]
Fix \( f \in C_b(\mathbb{R}^d) \) such that \( \|f\|_{2,\mu_s} = 1 \). Then, from (3.7) and, since \( p_h = (1-h)^{-1} \), we have
\[ \int_{\mathbb{R}^d} (G(t,s)|f|^{2(1-h)})^{q_h} d\mu_t \leq (\tilde{C}_{p,q}(t,s))^{r_h q_h}, \quad t > s, \]
which holds also for \( h = 0 \). Consequently,
\[ \frac{1}{h} \left( \int_{\mathbb{R}^d} (G(t,s)|f|^{2(1-h)})^{q_h} d\mu_t - \int_{\mathbb{R}^d} G(t,s)|f|^{2q} d\mu_t \right) \]
\[ = \frac{1}{h} \left( \int_{\mathbb{R}^d} (G(t,s)|f|^{2(1-h)})^{q_h} d\mu_t - 1 \right) \]
\[ \leq \frac{1}{h} (\tilde{C}_{p,q}(t,s))^{r_h q_h} - 1. \tag{3.8} \]
The first and the last sides of (3.8) represent respectively the incremental ratio at \( h = 0 \) of the functions \( h \mapsto \|G(t,s)|f|^{2(1-h)}\|_{p_{h,t}} \) and \( h \mapsto (\tilde{C}_{p,q}(t,s))^{r_h q_h} \). Since these two functions are differentiable at \( h = 0 \), we deduce that
\[ \frac{p(q-1)}{q(p-1)} \int_{\mathbb{R}^d} (G(t,s)f^2) \log(G(t,s)f^2) d\mu_t - \int_{\mathbb{R}^d} G(t,s)(f^2 \log f^2) d\mu_t \]
\[ \leq \frac{p}{p-1} \log(\tilde{C}_{p,q}(t,s)), \]
or, equivalently, since \( \{\mu_t : t \in I\} \) is an evolution system of measures,
\[ \int_{\mathbb{R}^d} (G(t,s)f^2) \log(G(t,s)f^2) d\mu_t \leq \frac{q(p-1)}{p(q-1)} \int_{\mathbb{R}^d} f^2 \log f^2 d\mu_s + \frac{q}{q-1} \log(\tilde{C}_{p,q}(t,s)), \]
which, replaced into (3.6), yields
\[ \int_{\mathbb{R}^d} f^2 \log |f| d\mu_s \leq \frac{2\Lambda}{|r_0|} \frac{p(q-1)}{q-p} (1 - e^{2r_0(t-s)}) \int_{\mathbb{R}^d} |\nabla f|^2 d\mu_s \]
\[ + \frac{pq}{2(q-p)} \log(\tilde{C}_{p,q}(t,s)), \]
and the claim is proved. \( \square \)

**Proof of Theorem 3.1.** “(i) \( \Rightarrow \) (ii)” By Proposition 3.3, if \( G(t,s) \) is supercontractive, then the following family of logarithmic Sobolev inequalities
\[ \int_{\mathbb{R}^d} f^2 \log f d\mu_s \leq \tilde{\beta}(t-s), \quad \|\nabla f\|^2 d\mu_s + \tilde{\beta}(t-s), \tag{3.9} \]
holds for every \( s \in I, t > s, \) and \( f \in C_b^1(\mathbb{R}^d) \) with \( \|f\|_{2,\mu_s} = 1 \). Since \( \tilde{C}_{p,q}(t,s) \leq C_{p,q}(t,s) \), in formula (3.9) we have
\[ r(t-s) = \frac{2\Lambda p(q-1)}{|r_0|(q-p)} (1 - e^{2r_0(t-s)}) \]
\[ \tilde{\beta}(t-s) = \frac{pq}{2(q-p)} \log(\tilde{C}_{p,q}(t,s)). \]
Inverting the function \( r \) we obtain
\[
t - s = \frac{1}{2r_0} \log \left( 1 + \frac{r_0(q - p)}{2\Lambda p(q - 1)} \right), \quad r \in [0, \tau),
\]
where \( \tau = \frac{2\Lambda p(q-1)}{r_0(q-p)} \). Thus, (LSI) holds for every \( \varepsilon \in (0, \tau) \) with
\[
\beta(\varepsilon) = \frac{pq}{2(q-p)} \log \left[ \frac{1}{2r_0} \log \left( 1 + \frac{r_0(q - p)}{2\Lambda p(q - 1)} \right) \right]. \tag{3.10}
\]
Clearly we can extend (LSI) to any \( \varepsilon > 0 \) and any \( f \in H^1(\mathbb{R}^d, \mu_s) \) by setting \( \beta(\varepsilon) = \lim_{r \to \tau} \beta(r) \) for any \( \varepsilon \geq \tau \), and using a standard approximation argument.

"(ii) \Rightarrow (i)" Assume that estimate (LSI) holds for every \( f \in H^1(\mathbb{R}^d, \mu_s) \), \( s \in I \), \( \varepsilon > 0 \) and a decreasing function \( \beta : (0, +\infty) \to \mathbb{R} \). Fix a function \( f \in C^1_0(\mathbb{R}^d) \) with positive infimum, \( p \in (1, +\infty) \) and \( s \in I \). Writing (LSI) for the function \( |f|^{p/2} \), we get
\[
\int_{\mathbb{R}^d} |f|^p \log \left( \frac{|f|}{\|f\|_{p,\mu_s}} \right) d\mu_s \leq \frac{p}{2} \int_{\mathbb{R}^d} |f|^{p-2} |\nabla f|^2 d\mu_s + \frac{2\beta(\varepsilon)}{p} \|f\|^p_{p,\mu_s}, \tag{3.11}
\]
for any \( \varepsilon > 0 \). Now, set
\[
q(t) := e^{2p_0s^{-1}(t-s)}(p-1) + 1, \quad m(t) := 2\beta(\varepsilon) \left( p^{-1} - (q(t))^{-1} \right), \tag{3.12}
\]
for any \( t \geq s \). To prove that \( G(t,s) \) is supercontractive, we show that \( H \) is a non increasing function. We would like to differentiate the function \( H \) and show that its derivative is nonpositive in \((s, +\infty)\). Unfortunately, we can differentiate functions of the type \( t \to \int_{\mathbb{R}^d} \psi d\mu_t \) only when \( \psi \) is constant outside a compact set, which, in general, is not our case. For this purpose we use an approximation argument and introduce the functions \( H_n \) \((n \in \mathbb{N})\) defined by
\[
H_n(t) := e^{-m(t)} \left( \int_{\mathbb{R}^d} \theta_n(G(t,s)f)^{q(t)} d\mu_t \right)^{\frac{1}{q(t)}}, \quad t \geq s,
\]
where \( \theta_n \) is defined in (3.2). From Proposition 2.5, for every \( n \in \mathbb{N} \), the function \( H_n \) is differentiable for \( t > s \) with derivative given by
\[
H'_n(t) = H_n(t)(-m'(t) + \varphi_n(t)), \quad t > s,
\]
where
\[
\varphi_n(t) = \left( \int_{\mathbb{R}^d} \theta_n(G(t,s)f)^{q(t)} d\mu_t \right)^{-1} \left\{ \frac{q'(t)}{q(t)} \int_{\mathbb{R}^d} \theta_n(G(t,s)f)^{q(t)} \log(G(t,s)f) d\mu_t 
\right.
- \frac{q'(t)}{(q(t))^2} \int_{\mathbb{R}^d} \theta_n(G(t,s)f)^{q(t)} d\mu_t \log \left( \int_{\mathbb{R}^d} \theta_n(G(t,s)f)^{q(t)} d\mu_t \right)
- (q(t) - 1) \int_{\mathbb{R}^d} \theta_n(G(t,s)f)^{q(t)-2} (Q(t) \nabla_x G(t,s)f, \nabla_x G(t,s)f) d\mu_t 
- \frac{2}{q(t)} \int_{\mathbb{R}^d} \langle Q(t) \nabla_x ((G(t,s)f)^{q(t)} \nabla T_n) \rangle d\mu_t 
- \frac{1}{q(t)} \int_{\mathbb{R}^d} (G(t,s)f)^{q(t)} A(t) \theta_n d\mu_t \right\}.
\]
Using (3.3) we can show that \( \limsup_{n \to +\infty} \varphi_n(t) \leq \psi(t) \) for every \( t > s \), where

\[
\psi(t) := \left( \int_{\mathbb{R}^d} (G(t, s)f)^{q(t)} \, d\mu_t \right)^{-1} \left\{ \frac{q'(t)}{q(t)} \int_{\mathbb{R}^d} (G(t, s)f)^{q(t)} \log(G(t, s)f) \, d\mu_t - \frac{q'(t)}{(q(t))^2} \int_{\mathbb{R}^d} (G(t, s)f)^{q(t)} \, d\mu_t \right\} - (q(t) - 1) \int_{\mathbb{R}^d} (G(t, s)f)^{q(t) - 1} \left( \eta(\xi(t)) \nabla_x G(t, s)f, \nabla_x G(t, s)f \right) \, d\mu_t \right\}.
\]

Writing

\[
H_n(t) - H_n(s) = \int_s^t H_n(s)(-m'(\sigma) + \varphi_n(\sigma)) \, d\sigma
\]

and letting \( n \to +\infty \) yields

\[
H(t) - H(s) \leq \int_s^t H(s)(-m'(\sigma) + \psi(\sigma)) \, d\sigma. \tag{3.13}
\]

From (2.1) we get

\[
-m'(\sigma) + \psi(\sigma) \leq \left[ e^{m(\sigma) H(\sigma)} \right]^{-q(\sigma)} \frac{q'(\sigma)}{q(\sigma)} \left\{ -m'(\sigma) \frac{q(\sigma)}{q'(\sigma)} \int_{\mathbb{R}^d} (G(\sigma, s)f)^{q(\sigma)} \, d\mu_\sigma \right\} + \int_{\mathbb{R}^d} (G(\sigma, s)f)^{q(\sigma)} \log(G(\sigma, s)f) \, d\mu_\sigma - \left( \int_{\mathbb{R}^d} (G(\sigma, s)f)^{q(\sigma)} \, d\mu_\sigma \right) \log \left( \int_{\mathbb{R}^d} (G(\sigma, s)f)^{q(\sigma)} \, d\mu_\sigma \right) \frac{q(\sigma)}{q'(\sigma)} - \eta_0 \frac{q(\sigma)(q(\sigma) - 1)}{q'(\sigma)} \int_{\mathbb{R}^d} (G(\sigma, s)f)^{q(\sigma) - 1} \left( \nabla_x G(\sigma, s)f \right)^2 \, d\mu_\sigma \right\}.
\]

Now, applying the logarithmic Sobolev inequality (3.11) with \( G(\sigma, s)f \) (which belongs to \( C^1_\mu(\mathbb{R}^d) \) and has positive infimum in view of formula (1.2) and estimate (2.5)), \( q(\sigma) \) and \( \mu_\sigma \) in place of \( f, p \) and \( \mu_\sigma \) respectively, we get

\[
-m'(\sigma) + \psi(\sigma) \leq \left[ e^{m(\sigma) H(\sigma)} \right]^{-q(\sigma)} \frac{q'(\sigma)}{q(\sigma)} \left\{ \left( \varepsilon \frac{q(\sigma)}{2} - \eta_0 \frac{q(\sigma)(q(\sigma) - 1)}{q'(\sigma)} \right) \int_{\mathbb{R}^d} (G(\sigma, s)f)^{q(\sigma)} \, d\mu_\sigma \right\} \times \int_{\mathbb{R}^d} (G(\sigma, s)f)^{q(\sigma) - 2} \left( \nabla_x G(\sigma, s)f \right)^2 \, d\mu_\sigma + \left( \frac{2\beta(\varepsilon)}{q(\sigma)} - m'(\sigma) \frac{q(\sigma)}{q'(\sigma)} \right) \int_{\mathbb{R}^d} (G(\sigma, s)f)^{q(\sigma)} \, d\mu_\sigma \right\} = 0,
\]

by the definition of \( q \) and \( m \) given in (3.12). Therefore, from (3.13) we deduce that \( H(t) \leq H(s) \), so that \( H \) is nonincreasing, i.e,

\[
\|G(t, s)f\|_{q(t), \mu_t} \leq e^{2\eta_0(\varepsilon)(\frac{1}{p} - \frac{1}{q(\sigma)})} \|f\|_{p, \mu_\sigma}, \tag{3.14}
\]

Now, for any \( q > p \) and \( t > s \), we fix \( \varepsilon = 2\eta_0(t - s)(\log((q - 1)/(p - 1)))^{-1} \). We thus deduce that \( q(t) = q \) and, from (3.14) we obtain

\[
\|G(t, s)f\|_{q, \mu_t} \leq C_{p, q}(t - s)\|f\|_{p, \mu_\sigma}, \tag{3.15}
\]

with

\[
C_{p, q}(r) = \exp \left[ \frac{2(q - p)}{pq} \beta \left( 2\eta_0 r \left( \log \left( \frac{q - 1}{p - 1} \right) \right)^{-1} \right) \right], \quad r > 0,
\]
which is a decreasing function since $\beta$ is decreasing as well.

By a density argument, we can extend the validity of (3.15) to any $f \in L^p(\mathbb{R}^d, \mu_s)$ which is almost everywhere nonnegative in $\mathbb{R}^d$ (with respect to the measure $\mu_s$ and, hence, with respect to the Lebesgue measure). Finally, since $|G(t, s)| \leq G(t, s) |f|$ for any $f \in L^p(\mathbb{R}^d, \mu_s)$, (3.15) follows in its full generality.

3.2. A second criterion. Here we show that the integrability of the gaussian functions $\varphi_\lambda$ (defined in (1.10)) for every $\lambda > 0$, with respect to the measures $\{\mu_t : t \in I\}$, together with the fact that $\sup_{t \in I} \|\varphi_\lambda\|_{1, \mu_s} < +\infty$, is another condition equivalent to the supercontractivity of $G(t, s)$. To this aim we first prove some preliminary results. The next proposition, whose proof is an adaption of Ledoux’s method [12] to our setting, provides some exponential integrability result.

**Proposition 3.4.** The function $x \mapsto e^{\lambda|x|}$ belongs to $L^1(\mathbb{R}^d, \mu_s)$ for every $\lambda > 0$. More precisely,

$$\sup_{s \in I} \int_{\mathbb{R}^d} e^{\lambda|x|} d\mu_s(x) < +\infty, \quad \lambda > 0.$$ 

Moreover, if the inequality (LSI) holds, then $\varphi_\lambda \in L^1(\mathbb{R}^d, \mu_s)$ for every $\lambda > 0$ and

$$\sup_{s \in I} \int_{\mathbb{R}^d} \varphi_\lambda d\mu_s < +\infty, \quad \lambda > 0.$$

**Proof.** For every $n \in \mathbb{N}$, let $\psi_n : [0, +\infty) \to \mathbb{R}$ be a smooth increasing function such that $\psi_n(t) = t$ for any $t \in [0, n)$, $\psi_n(t) = n + 1$ for any $t \geq n + 2$ and $0 \leq \psi_n'(t) \leq 1$ for any $t \geq 0$. The functions $f_n(x) := \psi_n(|x|)$ are bounded and satisfy $\|\nabla f_n\|_\infty \leq 1$ for any $n \in \mathbb{N}$. Moreover, $f_n(x)$ converges increasingly to $f(x) := |x|$ for any $x \in \mathbb{R}^d$, as $n \to +\infty$. Fix $s \in I$, $\lambda > 0$ and $n \in \mathbb{N}$. We set $H_{n,\lambda}(r) := \int_{\mathbb{R}^d} e^{\lambda r f_n} d\mu_s$ for any $r > 0$, and observe that

$$H'_{n,\lambda}(r) = \lambda \int_{\mathbb{R}^d} e^{\lambda r f_n} f_n d\mu_s. \quad (3.16)$$

Applying the logarithmic Sobolev inequality (LSI) to the function $e^{\lambda r f_n/2}$ and using (3.16), we get

$$r H'_{n,\lambda}(r) - H_{n,\lambda}(r) \log H_{n,\lambda}(r) \leq \frac{C\lambda^2 r^2}{2} \int_{\mathbb{R}^d} e^{\lambda r f_n} |\nabla f_n|^2 d\mu_s \leq \frac{C\lambda^2 r^2}{2} H_{n,\lambda}(r),$$

for every $n \in \mathbb{N}$. Now, dividing by $r^2 H_{n,\lambda}(r)$ we have

$$\left(\frac{1}{r} \log H_{n,\lambda}(r)\right)' = \frac{1}{r} H'_{n,\lambda}(r) - \frac{1}{r^2} \log H_{n,\lambda}(r) \leq \frac{C\lambda^2}{2}. \quad (3.17)$$

Integrating (3.17) over the interval $(1, 2)$ we deduce that

$$H_{n,\lambda}(2) \leq e^{C\lambda^2} (H_{n,\lambda}(1))^2.$$

Since the evolution system of measures $\{\mu_t : t \in I\}$ is tight, we can choose $M > 0$ such that $\mu_s(\mathbb{R}^d \setminus B(0, M\lambda^{-1})) \leq (4e^{C\lambda^2})^{-1}$ for every $s \in I$. This fact and the monotonicity of $\psi_n$ imply that

$$\mu_s(\{\lambda f_n \geq M\}) \leq \mu_s(\{\lambda f \geq M\}) = \mu_s(\mathbb{R}^d \setminus B(0, M\lambda^{-1})) \leq (4e^{C\lambda^2})^{-1},$$

for every $s \in I$. Now,

$$\int_{\mathbb{R}^d} e^{\lambda f_n} d\mu_s = \int_{\{\lambda f_n \geq M\}} e^{\lambda f_n} d\mu_s + \int_{\{\lambda f_n < M\}} e^{\lambda f_n} d\mu_s.$$
\[
\leq (\mu_b(\{|\lambda f_n| \geq M\})^\frac{1}{2} \left(\int_{\mathbb{R}^d} e^{2\lambda f_n} d\mu_s\right)^\frac{1}{2} + e^M
\]
\leq (4e^{C\lambda^2})^{-\frac{1}{2}} (H_{n,\lambda}(2))^{\frac{1}{2}} + e^M
\leq 2^{-1} H_{n,\lambda}(1) + e^M.
\]

Hence, \(\int_{\mathbb{R}^d} e^{\lambda f_n} d\mu_s \leq 2e^M\) for any \(s \in I\) and, letting \(n \to +\infty\), we get the first part of the claim.

In order to prove the second part of the claim assume that (LSI) holds and, for brevity, we set \(H_n := H_{n,1}\). Arguing as before and applying (LSI) to the function \(e^{rf_n/2}\), we get
\[
\left(\frac{1}{r} \log H_n(r)\right)' = \frac{1}{r} \frac{H_n'(r)}{H_n(r)} - \frac{1}{r^2} \log H_n(r) \leq \frac{\varepsilon}{2} + 2\frac{\beta(\varepsilon)}{r^2}, \tag{3.18}
\]
for every \(\varepsilon > 0\) and \(n \in \mathbb{N}\). Integrating (3.18) from \(\gamma\) to \(\sigma\) we deduce that
\[
\frac{1}{\sigma} \log H_n(\sigma) - \frac{1}{\gamma} \log H_n(\gamma) \leq \frac{\varepsilon}{2} (\sigma - \gamma) + 2\beta(\varepsilon) \left(\frac{1}{\gamma} - \frac{1}{\sigma}\right).
\]
Therefore, for every \(0 < \gamma < \sigma\) and \(\varepsilon > 0\),
\[
H_n(\sigma) \leq \exp \left(\frac{\varepsilon}{2} \sigma^2 + \sigma \left(\frac{\log H_n(\gamma)}{\gamma} - \frac{\varepsilon}{2} + 2\beta(\varepsilon) \frac{2}{\gamma}\right) - 2\beta(\varepsilon)\right). \tag{3.19}
\]
Now, we observe that
\[
2\sqrt{\lambda} \pi \int_{\mathbb{R}^d} e^{\lambda f_n^2} d\mu_s = \int_{\mathbb{R}^d} \int_{\mathbb{R}} e^{\sigma f_n - \frac{\sigma^2}{2}} d\sigma d\mu_s = \int_{\mathbb{R}} H_n(\sigma) e^{-\frac{\sigma^2}{2}} d\sigma
\]
and, by (3.19),
\[
\int_{\mathbb{R}} H_n(\sigma) e^{-\frac{\sigma^2}{2}} d\sigma \leq \int_{\mathbb{R}} e^{\gamma(\frac{\sigma^2}{2} - \frac{\sigma^2}{2}) + \sigma \left(\log \|\sigma\|_{1,M_n}^{1/\gamma} - \frac{\sigma^2}{2} \gamma + \frac{\sigma^2}{2} \beta(\varepsilon)\right) - 2\beta(\varepsilon)} d\sigma, \tag{3.20}
\]
which is finite for every \(0 < \lambda < \frac{1}{\varepsilon^2}\) and \(n \in \mathbb{N}\). Since \(\varepsilon > 0\) is arbitrary and \(\sup_{s \in I} \|\sigma\|_{1,M_n}^{1/\gamma} < +\infty\), by the first part of the proof we deduce that
\[
\int_{\mathbb{R}^d} e^{\lambda f_n^2} d\mu_s \leq K, \quad \lambda > 0, \ n \in \mathbb{N}, \tag{3.21}
\]
for some positive constant \(K\), independent of \(s\). Finally, we get the claim by the monotone convergence theorem, letting \(n \to +\infty\) in (3.21).

\textbf{Remark 3.5.} (i) Actually, formula (3.20) shows that, just assuming the validity of the estimate (LSI), one can deduce that the functions \(\varphi_\lambda\) belong to \(L^1(\mathbb{R}^d, \mu_s)\) and \(\sup_{s \in I} \|\varphi_\lambda\|_{1,M_n} < +\infty\) for every \(\lambda < (2C)^{-1}\) where \(C\) is the constant in (LSI).

(ii) We point out that in the proof of Proposition 3.4 we have not used the invariance of the measures \(\{\mu_t : t \in I\}\). Therefore, this result holds true for any tight system of measures.

Now, we prove that the evolution operator \(G(t,s)\) satisfies an Harnack-type estimate, which will be used later on to prove that the integrability of \(\varphi_\lambda\), uniform with respect to \(\{\mu_t : t \in I\}\), is a sufficient condition for the supercontractivity.
Proposition 3.6 (An Harnack-type inequality). For every $f \in C_b(\mathbb{R}^d)$, $p > 1$, $t > s$ and $x, y \in \mathbb{R}^d$ we have

$$|(G(t, s)f)(x)|^p \leq (G(t, s)|f|^p)(y) \exp \left( \frac{p|x-y|^2}{4(p-1)t_0(t-s)} \right). \quad (3.22)$$

Proof. Since $|G(t, s)f| \leq G(t, s)|f|$ for every $f \in C_b(\mathbb{R}^d)$ and $t > s$, it suffices to prove (3.22) for nonnegative functions $f$.

We split the proof into two steps. In the first one we prove (3.22) for nonnegative functions $f \in C^1_b(\mathbb{R}^d)$. In the second step, by an approximation argument we extend (3.22) to every nonnegative function $f \in C_b(\mathbb{R}^d)$.

Step 1. We extend the method used in [17] to the nonautonomous case. For this purpose, let $f \in C^1_b(\mathbb{R}^d)$ be a nonnegative function. Fix $t > s$, $x, y \in \mathbb{R}^d$ and set

$$\Phi_n(r) := \{G(t, r)[\theta_n(G(r, s)f)^p]\} \psi(r), \quad s \leq r \leq t,$$

where $\theta_n$ is the sequence of cut-off functions defined in (3.2) and

$$\psi(r) := \left( \frac{t-r}{t-s} \right) y + \left( \frac{r-s}{t-s} \right) x, \quad s \leq r \leq t.$$ 

By Lemma 2.3 and Proposition 2.4, the function $\log \Phi_n$ is well defined, it belongs to $C^1((s, t))$ for every $n \in \mathbb{N}$ and there exist $n_0 \in \mathbb{N}$ and a positive constant $C_\Phi$ such that $\Phi_n(r) \geq C_\Phi$ for every $n > n_0$ and $r \in [s, t]$. This last assertion follows since $\Phi_n(r) > 0$ for every $r < t$ and $\Phi_n(t) = (\theta_n(G(t, s)f)^p)(x)$ for every $n \in \mathbb{N}$. Hence, choosing $n$ large enough such that $x \in \text{supp} \theta_n$, we conclude.

Differentiating the functions $r \mapsto \log \Phi_n(r)$ ($n \in \mathbb{N}$) in $(s, t)$ we get

$$\frac{d}{dr} \log \Phi_n(r) = \frac{1}{\Phi_n(r)} \left\{ -\{G(t, r)[A(r)(\theta_n(G(r, s)f)^p)]\}(\psi(r)) + \{G(t, r)[\theta_nD_r(G(r, s)f)^p]\}(\psi(r)) + \frac{1}{t-s} \{\nabla_x[G(t, r)(\theta_n(G(r, s)f)^p)](\psi(r)), x - y\} \right\}. \quad (3.23)$$

Let observe that, if $g = G(\cdot, s)f$, then

$$D_r g^p - A(r)g^p = -p(p-1)g^{p-2} \langle Q(r)\nabla_x g, \nabla_x g \rangle$$

and

$$|\nabla_x G(t, r)(\theta_n g^p)| \leq G(t, r)|\nabla_x (\theta_n g^p)| \leq G(t, r)(|\nabla \theta_n| g^p + p\eta_0^{-1/2} \theta_n g^{p-1} |Q^{1/2}(r)\nabla_x g|),$$

where in the last inequality we have used (2.7) and (2.1). From (3.23) we get

$$\frac{d}{dr} \log \Phi_n(r) \leq -\frac{1}{\Phi_n(r)} \left\{ (G(t, r)[(G(r, s)f)^pA(r)\theta_n + 2p(G(r, s)f)^p\langle Q(r)\nabla \theta_n, \nabla_x G(r, s)f \rangle + p(p-1)\theta_n(G(r, s)f)^p - Q(r)\nabla_x G(r, s)f, \nabla_x G(r, s)f)] \right.$$

$$- \frac{|x - y|}{t-s} G(t, r)[p\eta_0^{-1/2} \theta_n(G(r, s)f)^p - Q^{1/2}(r)\nabla_x G(r, s)f]$$

$$+ |\nabla \theta_n|(G(r, s)f)^p \right\} \psi(r),$$

where $\eta_0$ is a fixed constant.
hence,
\[
\frac{d}{dr} \log \Phi_n(r) \leq \frac{1}{\Phi_n(r)} \left\{ G(t,r) \left[ g_n(r) - p\theta_n(G(r,s)f)^p \left( (p-1)h^2(r) - \frac{|x-y|}{\sqrt{\eta_0(t-s)}} \right) \right] \right\} \psi(r),
\]
where
\[
g_n(r) = (G(r,s)f)^p \left( |\text{Tr}(Q(r)D^2\theta_n)| - \langle b(r,\cdot), \nabla\theta_n \rangle \right)
+ 2p(G(r,s)f)^{p-1}|(Q(r)\nabla\theta_n, \nabla_x G(r,s)f) + \frac{|x-y|}{t-s} \nabla\theta_n|(G(r,s)f)^p
\]
and \( h(r) = (G(r,s)f)^{-1}|Q^{1/2}(r)\nabla_x G(r,s)f| \). Since
\[
\langle b(r,x), \nabla\theta_n(x) \rangle \geq \eta' \left( \frac{|x|}{n} \right) \frac{C_{s,t}}{n}, \quad r \in [s,t],
\]
\( C_{s,t} \) being the constant in (2.4), we can estimate
\[
g_n(r) \leq \frac{C_1}{n^2} \|f\|_\infty^p + \frac{1}{n} \left( 2p\Lambda \|f\|_\infty^{p-1} \|\nabla f\|_\infty + \frac{|x-y|}{t-s} \|f\|_\infty^p \right) =: C(n), \quad (3.24)
\]
for every \( r \in [s,t] \), where \( C_1 = d\Lambda (2\|\eta\|_\infty + \|\eta''\|_\infty) + \|\eta\|_\infty C_{s,t} \).

Recalling that \( \gamma^2 - \beta \gamma \geq -\beta^2/4 \) for every \( \beta, \gamma \in \mathbb{R} \) and \( G(t,s)g_1 \geq G(t,s)g_2 \) for every \( t \geq s \) if \( g_1 \geq g_2 \) (see Lemma 2.3), from (3.24) we deduce that
\[
\frac{d}{dr} \log \Phi_n(r) \leq \frac{C(n)}{C_{\Phi}} + \frac{p|x-y|^2}{4(p-1)\eta_0(t-s)^2},
\]
for every \( n > n_0 \). We now integrate with respect to \( r \) between \( s \) and \( t \) to get
\[
\log \Phi_n(t) - \log \Phi_n(s) \leq \frac{C(n)}{C_{\Phi}}(t-s) + \frac{p|x-y|^2}{4(p-1)\eta_0(t-s)^2}, \quad n > n_0,
\]
and (3.22) follows letting \( n \to +\infty \).

Step 2. Let \( f \in C_b(\mathbb{R}^d) \) be a nonnegative function; we can consider a sequence \( (f_n)_n \subset C^2_b(\mathbb{R}^d) \) of nonnegative functions converging to \( f \) uniformly on compact sets of \( \mathbb{R}^d \) and such that \( \|f_n\|_\infty \leq \|f\|_\infty \). Then, by Step 1 we have
\[
|\langle G(t,s)f_n(x) \rangle|^p \leq |\langle G(t,s)f_n \rangle|^p \exp \left( \frac{p|x-y|^2}{4(p-1)\eta_0(t-s)} \right),
\]
for every \( t > s \in I \) and \( x, y \in \mathbb{R}^d \). Taking into account formula (1.2), this yields the claim by the dominated convergence theorem. 

The second announced characterization of the supercontractivity of \( G(t,s) \) is given in the following theorem. Its proof is based on Propositions 3.4, 3.6 and also on the first criterion given in Theorem 3.1.

**Theorem 3.7.** The following properties are equivalent.

(i) The evolution operator \( G(t,s) \) is supercontractive;

(ii) the function \( \varphi_\lambda \) belongs to \( L^1(\mathbb{R}^d, \mu_\lambda) \) for every \( \lambda > 0 \) and \( s \in I \). Moreover,
\[
\sup_{s \in I} \|\varphi_\lambda\|_{1, \mu_\lambda} < +\infty, \quad \lambda > 0. \quad (3.25)
\]
Proof. “(i) ⇒ (ii)” It suffices to apply Theorem 3.1 and Proposition 3.4.

“(ii) ⇒ (i)” Let us assume that (3.25) holds true and denote by $M_{\lambda}$ the supremum in the left hand side of such an inequality. Integrating the Harnack inequality (3.22) with respect to $d\mu_t(y)$ and recalling that $\{\mu_t : t \in I\}$ is an evolution system of measures, we get

$$
\int_{\mathbb{R}^d} |f(y)|^p d\mu_s(y) = \int_{\mathbb{R}^d} (G(t,s)|f|^p)(y)d\mu_t(y)
\geq \|(G(t,s)f)(x)|^p\int_{\mathbb{R}^d} e^{-\frac{p|x-y|^2}{2m_0(s-t)^{1-\delta}}} d\mu_t(y)
\geq \|(G(t,s)f)(x)|^p\mu_t(B(0,r)) e^{-\frac{p(r^2+|x|^2)}{2m_0(s-t)^{1-\delta}}},
$$

(3.26)

for every $t > s$, $r > 0$, $x \in \mathbb{R}^d$, and $f \in C_b(\mathbb{R}^d)$. Hence,

$$
|(G(t,s)f)(x)| \leq 2 \exp\left(\frac{R^2 + |x|^2}{2\eta_0(p-1)(t-s)}\right)\|f\|_{p,\mu_s}, \quad t > s, \quad x \in \mathbb{R}^d, \quad (3.27)
$$

where $R$ is such that $\mu_t(B(0,R)) > 2^{-p}$, for any $t \in I$. Let us now fix $q > p$ and set $\lambda_0 = (2\eta_0(p-1)(t-s))^{-1}q$. By (3.27) we can estimate

$$
\int_{\mathbb{R}^d} |G(t,s)f|^q d\mu_t \leq 2^q \exp\left(\frac{qR^2}{2\eta_0(p-1)(t-s)}\right)\|\varphi_{\lambda_0}\|_{1,\mu_s}\|f\|_{q,\mu_s}^q
\leq 2^q \exp\left(\frac{qR^2}{2\eta_0(p-1)(t-s)}\right) M_{\lambda_0}\|f\|_{q,\mu_s}^q
=: (C_{p,q}(t-s))^q\|f\|_{q,\mu_s}^q, \quad (3.28)
$$

for any $I \ni s < t$. Clearly the function $r \mapsto C_{p,q}(r)$ is decreasing. By density, we can extend the previous inequality to any $f \in L^p(\mathbb{R}^d,\mu_s)$. This completes the proof.

We now provide a concrete condition on the coefficients of the operator $A(t)$ which yields the supercontractivity of the evolution operator $G(t,s)$. First we prove a preliminary lemma.

Lemma 3.8. Assume that there exist $K, \beta > 0$ and $R > 1$ such that

$$
\langle b(t,x), x \rangle \leq -K|x|^2(\log |x|^2)^{\beta}, \quad t \in I, \quad |x| \geq R.
$$

Then, any positive function $\psi_{\lambda,\delta} \in C^2(\mathbb{R}^d)$ which agrees with the function $x \mapsto e^{\lambda|x|^2(\log |x|^2)^{\delta}}$ for any $x \in \mathbb{R}^d \setminus B(0,R)$, is a Lyapunov function satisfying (2.2) for every $\lambda < K(2\Lambda)^{-1}$ when $\delta = \beta$, and for every $\lambda > 0$ when $\delta \in [0,\beta]$.

Proof. A straightforward computation yields

$$
(A(t)\psi_{\lambda,\delta})(x) = 2\lambda \psi_{\lambda,\delta}(x) \left\{ 2\lambda(\log |x|^2)^{\beta} \langle Q(t)x, x \rangle + 2\delta^2 \lambda(\log |x|^2)^{2\beta-2} \langle Q(t)x, x \rangle 
\right.
\left. + 4\delta \lambda(\log |x|^2)^{2\beta-1} \langle Q(t)x, x \rangle + \text{Tr}(Q(t)))(\log |x|^2)^\delta 
+ 2\delta(\log |x|^2)^{\delta-1} \frac{\langle Q(t)x, x \rangle}{|x|^2} + \delta \text{Tr}(Q(t)))(\log |x|^2)^{\delta-1} 
\right.
\left. + 2\delta(\delta - 1)\lambda(\log |x|^2)^{\delta-2} \frac{\langle Q(t)x, x \rangle}{|x|^2} 
+ \langle b(t,x), x \rangle(\log |x|^2)^\delta + \delta \langle b(t,x), x \rangle(\log |x|^2)^{\delta-1} \right\}
$$

for every $\lambda < K(2\Lambda)^{-1}$ when $\delta = \beta$.
It thus follows that there exist two positive constants $a$ and $b$ such that
\[
|\langle b(t,x), x \rangle| \leq -K_1 |x|^2 \log |x|, \quad t \in I, \quad |x| \geq R.
\]
(3.29)

Then, the evolution operator $G(t,s)$ is supercontractive.

**Proof.** In view of [11, Thm. 5.4], the proof follows from Theorem 3.7 and Lemma 3.8.

**Remark 3.10.** The condition (3.29) is quite optimal. Indeed, the autonomous operator $\langle A(t)\xi, \xi \rangle = \Delta \xi - \langle x, \nabla \xi \rangle$ does not satisfy it and, in fact, it is well known that the associated Ornstein-Uhlenbeck semigroup is not supercontractive with respect to the Gaussian invariant measure $d\mu(x) = (2\pi)^{-d/2}e^{-|x|^2/2}dx$, as proved in [15].

4. Ultraboundedness. In this section we provide a condition equivalent to the ultraboundedness property of the evolution operator $G(t,s)$. As in [5, 16, 17, 20], which deal with the autonomous case, a key tool to prove such a result is the Harnack type estimate (3.22). However, this estimate is not enough and we need to strengthen assumption (2.2), as next theorem shows.

**Theorem 4.1.** Assume that, for any $\lambda > 0$, there exist $R = R(\lambda) > 0$, a convex increasing function $h_\lambda : [0, +\infty) \to \mathbb{R}$ such that $1/h_\lambda \in L^1(c, +\infty)$ for large $c = c(\lambda)$ and

\[
(A(t)\varphi_\lambda)(x) \leq -h_\lambda(\varphi_\lambda(x)), \quad t \in I, \quad |x| \geq R,
\]

(4.1)

where $\varphi_\lambda$ is defined in (1.10). Then, $G(t,s)$ is ultrabounded.

**Proof.** We prove the claim for $p \in (1, 2]$. For $p > 2$, estimate (2.6) will follow from the Hölder inequality.

We split the proof into two steps. First, we consider the case $p = 2$ and, then, the case $p \in (1, 2]$.

**Step 1.** An insight into the proof of [13, Thm. 3.3] (see also [2, Thm. 4.3] for further details) shows that, under our assumptions, the function $t \mapsto (G(t,s)\varphi_\lambda)(x)$ is well defined for each $t > s, \ x \in \mathbb{R}^d, \ \lambda > 0$, and $G(t,s)\varphi_\lambda \in L^\infty(\mathbb{R}^d)$ for such $s, t$ and $\lambda$. Let us set

\[
M_{\delta,\lambda} := \sup_{x \in \mathbb{R}^d} (G(t,s)\varphi_\lambda)(x),
\]

(4.2)

for every $\delta, \lambda > 0$. Integrating both sides of estimate (3.22) (with $p = 2$) with respect to $d\mu(y)$ and arguing as in the proof of Theorem 3.7, we get

\[
|G(t,s)f(x)| \leq 2 \exp\left(\frac{R + |x|^2}{2R_0(t-s)}\right) \|f\|_{2,\mu}, \quad t > s, \ x \in \mathbb{R}^d,
\]
where \( R \) is such that \( \mu_\ell(B(0, R)) > \frac{1}{2} \). Hence we obtain

\[
\|G(t, s)f\|_\infty = \|G(t, (t + s)/2)G((t + s)/2, s)f\|_\infty \\
\leq 2e^{\frac{R}{2(t-s)}}\|f\|_{\mu_\ell}\|G(t, (t + s)/2)\varphi_{\lambda_0}\|_\infty,
\]

for every \( f \in C_b(\mathbb{R}^d) \) and for \( \lambda_0 = \frac{1}{2(t-s)} \). Formulas (4.2) and (4.3) yield

\[
\|G(t, s)f\|_\infty \leq C_{2, \infty}(t-s)\|f\|_{\mu_\ell}, \quad t > s, \quad f \in C_b(\mathbb{R}^d),
\]

with

\[
C_{2, \infty}(t-s) = 2e^{\frac{R}{2(t-s)}}M_{\frac{1}{2}, \frac{1}{2}}, \quad (4.5)
\]

The monotonicity of the function \( r \mapsto C_{2, \infty}(r) \) follows from noticing that \( M_{\delta_2, \lambda_1} \leq M_{\delta_1, \lambda_2} \), for every \( 0 < \delta_1 \leq \delta_2 \) and \( 0 < \lambda_1 \leq \lambda_2 \).

Now, let \( f \in L^2(\mathbb{R}^d, \mu_\ell) \) and consider \( f_n \in C_b(\mathbb{R}^d) \) converging to \( f \) in \( L^2(\mathbb{R}^d, \mu_\ell) \) as \( n \rightarrow +\infty \). Since \( G(t, s) \) is a contraction from \( L^2(\mathbb{R}^d, \mu_\ell) \) to \( L^2(\mathbb{R}^d, \mu_\ell) \), \( G(t, s)f_n \) converges to \( G(t, s)f \) in \( L^2(\mathbb{R}^d, \mu_\ell) \) as \( n \rightarrow +\infty \). Moreover,

\[
\|G(t, s)f_n - G(t, s)f_m\|_{\infty} \leq C_{2, \infty}(t-s)\|f_n - f_m\|_{\mu_\ell},
\]

for every \( t > s \), and \( n, m \in \mathbb{N} \). Formula (4.6) yields that the sequence \( G(t, s)f_n \) converges uniformly in \( \mathbb{R}^d \) to some function \( g \in C_b(\mathbb{R}^d) \) and that \( g = G(t, s)f \).

Then, we conclude writing (4.4) for \( f_n \) and letting \( n \rightarrow +\infty \).

Step 2. To prove (2.6) when \( p \in (1, 2) \) we observe that

\[
\int_{\mathbb{R}^d} \varphi_{\lambda, n} d\mu_s = \int_{\mathbb{R}^d} G(s + 1, s)\varphi_{\lambda, n} d\mu_{s+1},
\]

for any \( \lambda > 0 \), any \( s \in I \) and any \( n \in \mathbb{N} \), where \( \varphi_{\lambda, n} = \min\{\varphi_{\lambda}, n\} \). Letting \( n \rightarrow +\infty \) and using (4.2) with \( \delta = 1 \), we obtain

\[
\int_{\mathbb{R}^d} \varphi_{\lambda} d\mu_s = \int_{\mathbb{R}^d} G(s + 1, s)\varphi_{\lambda} d\mu_{s+1} \leq M_{1, \lambda}, \quad s \in I.
\]

Hence, condition (3.25) is satisfied, and Theorem 3.7 shows that the evolution operator \( G(t, s) \) is supercontractive. Therefore,

\[
\|G(t, s)f\|_\infty = \|G(t, (t + s)/2)G((t + s)/2, s)f\|_\infty \\
\leq \|G(t, (t + s)/2)\|_{2 \rightarrow \infty}\|G((t + s)/2, s)f\|_{\mu_\ell((t+s)/2)} \\
\leq C_{2, \infty}((t-s)/2)C_p\|G((t + s)/2, s)f\|_{\mu_\ell},
\]

for any \( f \in L^p(\mathbb{R}^d, \mu_\ell) \) and any \( s, t \in I \) with \( s < t \). This completes the proof.

Remark 4.2. Each function \( \varphi_{\lambda} \), as in Theorem 4.1, satisfies Hypothesis 2.1(iii), i.e., it is a Lyapunov function for the nonautonomous elliptic operators \( A(t) \). Indeed, since \( h_{\lambda} \) is a convex function which tends to \( +\infty \) as \( r \rightarrow +\infty \), there exist \( a_{\lambda} > 0 \) and \( b_{\lambda} \in \mathbb{R} \) such that \( h_{\lambda}(r) \geq a_{\lambda} r + b_{\lambda} \) for any \( r \geq 0 \). From (4.1) it thus follows that \( (A(t)\varphi_{\lambda})(x) \leq -a_{\lambda}\varphi_{\lambda}(x) - b_{\lambda} \) for any \( t \in I \) and any \( x \in \mathbb{R}^d \setminus B(0, R) \). Up to replacing \( b_{\lambda} \) with a larger constant, if needed, we can assume that the previous inequality is satisfied by any \( x \in \mathbb{R}^d \), so that (2.2) is satisfied.

From [11, Thm. 5.4], we deduce that \( \sup_{s \in I} \|\varphi_{\lambda}\|_{1, \mu_\ell} < +\infty \) for any \( \lambda > 0 \), and this gives an alternative proof of the first part of Step 2 in Theorem 4.1.

Using the results in Theorem 4.1, we provide a quite sharp sufficient condition on the coefficients of the operator \( A(t) \) for \( G(t, s) \) to be ultrabounded.
**Theorem 4.3.** Suppose that there exist three positive constants $K_2$, $\alpha > 1$ and $R_0 > 1$ such that
\[
(b(t, x), x) \leq -K_2 |x|^2 (\log |x|)^\alpha, \quad t \in I, \ |x| \geq R_0.
\] (4.7)
Then, $G(t, s)$ is ultrabounded.

**Proof.** A straightforward computation shows that
\[
(A(t)\varphi_\lambda)(x) = 2\lambda \varphi_\lambda(x) [\text{Tr}(Q(t))] + 2\lambda (Q(t)x, x) + (b(t, x), x)
\]
\[
\leq -2\lambda \varphi_\lambda(x) [K_2|x|^2 (\log |x|)^\alpha - 2\lambda \Lambda |x|^2 - \Lambda d]
\]
for any $t \in I$ and any $x \in \mathbb{R}^d \setminus B(0, R_0)$. Let now $C_\alpha$ be a positive constant such that
\[
2\lambda \Lambda y^2 \leq \frac{K_2}{2} y^2 (\log y)^\alpha + C_\alpha, \quad y \geq R_0.
\]
Then,
\[
(A(t)\varphi_\lambda)(x) \leq -\lambda \varphi_\lambda(x) [K_2|x|^2 (\log |x|)^\alpha - 2C_\alpha - 2\Lambda d] = -g_\lambda(\varphi_\lambda(x))
\]
for any $t \in I$ and any $x \in \mathbb{R}^d \setminus B(0, R_0)$. Here,
\[
g_\lambda(y) = y \left[ K_2 2^{-\alpha} \log y (\log(\log y))^\alpha - 2\lambda C_\alpha - 2\Lambda d \right], \quad y \geq e^\lambda.
\]
$g_\lambda$ is a convex function in the interval $[e^\lambda, +\infty)$ and, since $g_\lambda(y) \sim y \log y (\log(\log y))^\alpha$ as $y \to +\infty$, $1/g_\lambda$ is integrable in a neighborhood of $+\infty$. On the other hand, $g_\lambda$ is not increasing in $[e^\lambda, +\infty)$ since $g'_\lambda(e^\lambda) = -2(C_\alpha + \Lambda d)$. To overcome this difficulty, let us introduce the function $h_\lambda = g_\lambda(y_{0,\lambda}) \chi_{[0,y_{0,\lambda}]} + g_\lambda(\chi_{[y_{0,\lambda}, +\infty)})$, where $y_{0,\lambda} > e^\lambda$ is the point where the minimum of the function $g_\lambda$ is attained. Clearly, $h_\lambda$ is a convex and increasing function in $[0, +\infty)$ which equals $g_\lambda$ in $[y_{0,\lambda}, +\infty)$. Moreover, $h_\lambda \leq g_\lambda$ in $[e^\lambda, +\infty)$, therefore $(A(t)\varphi_\lambda)(x) \leq -h_\lambda(\varphi_\lambda(x))$ for any $t \in I$ and any $|x| \geq R_0$. We can thus apply Theorem 4.1. □

**Remark 4.4.** The condition (4.7) is rather sharp. Indeed in [10], the authors consider the autonomous operator $(\mathcal{A}\zeta)(x) = \Delta \zeta(x) - \langle \nabla \Phi(x), \nabla \zeta(x) \rangle$, where $\Phi$ is such that $e^{-\Phi} \in L^1(\mathbb{R}^d)$, and prove that, if $\Phi(x) \sim |x|^2 \log |x|$ as $|x| \to +\infty$, then the semigroup $T(t)$ associated to $\mathcal{A}$ in $C_b(\mathbb{R}^d)$ is not ultrabounded in the Lebesgue spaces with respect the invariant measure $d\mu(x) = \|e^{-\Phi}\|^{-1} e^{-\Phi(x)}dx$.

The Harnack type estimate (3.22) and the fact that $G(t, s)\varphi_\lambda \in L^\infty(\mathbb{R}^d)$ for every $\lambda > 0$ and $t > s$ represent the key tools used in the proof of Theorem 4.1 to get ultraboundedness. Hypotheses 2.1 are enough to prove the Harnack formula (3.22). On the other hand, to prove that $G(t, s)\varphi_\lambda \in L^\infty(\mathbb{R}^d)$ for every $\lambda > 0$ and $t > s$ we have strengthened our assumptions requiring the additional condition (4.1). The condition $G(t, s)\varphi_\lambda \in L^\infty(\mathbb{R}^d)$ for every $\lambda > 0$ and $t > s$ is optimal to get ultraboundedness of $G(t, s)$ for every $t > s$. The proof of this fact is based on the occurrence of the family of logarithmic Sobolev inequalities (3.4) and the consequent measure concentration result proved in Proposition 3.4.

**Theorem 4.5.** The evolution operator $G(t, s)$ is ultrabounded if and only if, for every $\lambda > 0$ and $t > s$, the function $G(t, s)\varphi_\lambda$ belongs to $L^\infty(\mathbb{R}^d)$ and, for any $\delta, \lambda > 0$, there exists a positive constant $K_{\delta, \lambda}$ such that
\[
\|G(t, s)\varphi_\lambda\|_\infty \leq K_{\delta, \lambda}, \quad s, t \in I, \ t - s \geq \delta.
\] (4.8)
Proof. In view of the proof of Theorem 4.1 the “if” part of the statement is true.

Conversely, if \( G(t, s) \) is ultrabounded, then it is bounded from \( L^p(\mathbb{R}^d, \mu_s) \) into \( L^q(\mathbb{R}^d, \mu_t) \) for every \( t > s \) and \( 1 < p < q < +\infty \), and
\[
\|G(t, s)\|_{p \rightarrow q} \leq \|G(t, s)\|_{p \rightarrow \infty} < +\infty.
\]

By Proposition 3.3, the logarithmic Sobolev inequality (LSI) holds. Consequently, from Proposition 3.4 we deduce that \( \varphi_\lambda \in L^1(\mathbb{R}^d, \mu_s) \) for every \( \lambda > 0 \) and \( s \in I \), and sup \( s \in I \|\varphi_\lambda\|_{1,\mu_s} < +\infty \). Therefore,
\[
\|G(t, s)\|_\infty \leq \|G(t, s)\|_{2 \rightarrow \infty}\|\varphi_\lambda\|_{2,\mu_s} \leq C_{2,\infty}(t-s)\|\varphi_\lambda\|^2_{1,\mu_s} < +\infty,
\]
for any \( t > s \) and \( \lambda > 0 \).

Now, fix \( \delta > 0 \) and let \( t - s \geq \delta \). Since the function \( r \mapsto C_{2,\infty}(r) \) is decreasing, we get (4.8) with \( K_{\delta,\lambda} = C_{2,\infty}(\delta) \sup_{s \in I} \|\varphi_\lambda\|_{1,\mu_s}^{1/2} \).

\section{Ultracontractivity}

In this section we assume the following additional assumption on the drift term of the operators \( \mathcal{A}(t) \).

**Hypotheses 5.1.** There exist three positive constants \( K_3, R \) and \( \kappa > 2 \) such that
\[
(b(t, x), x) \leq -K_3|x|^{\kappa}, \quad t \in I, \quad x \in \mathbb{R}^d \setminus B(0, R).
\]

**5.1. \( L^1 - L^2 \) integrability.** To begin with, let us give an estimate of the asymptotic behaviour of the function \( \beta \) defined in (LSI) near zero.

**Proposition 5.2.** It holds that \( \beta(\varepsilon) = O(\varepsilon^{-\frac{3-\kappa}{\kappa}}) \) as \( \varepsilon \to 0^+ \).

**Proof.** First of all, let us prove that the function \( x \mapsto \varphi_{\delta,\kappa}(x) = e^{\delta|x|^\kappa} \) belongs to \( L^1(\mathbb{R}^d, \mu_s) \) for any \( s \in I \) and any \( \delta < K_3/(\kappa \Lambda) \) (see (2.1)) and that there exists a positive constant \( M \), independent of \( s \), such that \( \|\varphi_{\delta,\kappa}\|_{1,\mu_s} \leq M \) for any \( s \in I \). For this purpose, in view of [11, Thm. 5.4] we can limit ourselves to proving that \( (\mathcal{A}(t)\varphi_{\delta,\kappa})(x) \leq a_1 - \gamma_1\varphi_{\delta,\kappa}(x) \) for any \( t \in I, \quad x \in \mathbb{R}^d \) and some positive constants \( a_1 \) and \( \gamma_1 \). It is easy to compute and to estimate \( \mathcal{A}(t)\varphi_{\delta,\kappa} \) in the following way:
\[
\langle (\mathcal{A}(t)\varphi_{\delta,\kappa})(x) \rangle = \delta \kappa \varphi_{\delta,\kappa}(x) \left[ |\delta \kappa |x|^{2\kappa - 4} + (\kappa - 2)|x|^{\kappa - 4} \right] \langle Q(t) \rangle (t, x, x) + \text{Tr}(Q(t))|x|^{\kappa - 2} + (b(t, x), x)|x|^{\kappa - 2} \leq \delta \kappa \varphi_{\delta,\kappa}(x) |\delta \kappa |x|^{2\kappa - 4} + \Lambda(d + \kappa - 2)|x|^{\kappa - 2} - K_3|x|^{2\kappa - 2} = g_1(x)\varphi_{\delta,\kappa}(x)
\]
for any \( (t, x) \in I \times \mathbb{R}^d \), where \( g_1(x) \) tends to \(-\infty \) as \( |x| \to +\infty \). Hence, the claim follows at once.

We now observe that, for any \( \lambda > 0 \) and any \( t \geq 0 \), we have
\[
\delta \lambda t^\kappa - \lambda t^2 \geq \left( \frac{2}{\kappa \delta} \right) \frac{2 - \kappa}{\kappa} \lambda \pi^{\frac{\kappa}{2}} =: -c_1 \lambda \pi^{\frac{\kappa}{2}}.
\]

It thus follows that
\[
\|\varphi_\lambda\|_{1,\mu_s} \leq e^{c_1 \lambda \pi^{\frac{\kappa}{2}}} \|\varphi_{\delta,\kappa}\|_{1,\mu_s} \leq e^{c_1 \lambda \kappa \pi^{\frac{\kappa}{2}}} \sup_{r \in I} \|\varphi_{\delta,\kappa}\|_{1,\mu_r} =: c_2 e^{c_1 \lambda \kappa \pi^{\frac{\kappa}{2}}}.
\]

Writing (3.10) with \( p = 2 \) and \( q = 3 \) yields
\[
\beta(\varepsilon) = 3 \log \left[ C_{2,3} \left( \frac{1}{2r_0} \log \left( 1 + \frac{r_0}{8\Lambda} \varepsilon \right) \right) \right],
\]
for any $\varepsilon < 8A|r_0|^{-1}$. From (3.28), with the same choice of $p$ and $q$, we get

$$C_{2,3}(r) = 2 \exp \left( \frac{R^2}{2\eta_0 r} \right) \|\varphi_{\lambda_0}\|_{L^1_{\mu_2}}, \quad r > 0,$$

where $\lambda_0 = 3(2\eta_0 r)^{-1}$. From estimate (5.2) we thus conclude that

$$C_{2,3}(r) \leq c_3 \exp \left( \frac{R^2}{2\eta_0 r} \right) \exp \left( \frac{c_4}{r^{\varepsilon/2}} \right), \quad r > 0,$$

for some positive constants $c_3$ and $c_4$, so that

$$\log(C_{2,3}(r)) \leq \log(c_3) + c_4 r^{-\varepsilon/2} + c_5 r^{-1}, \quad r > 0.$$

Now, it is immediate to check that

$$\beta(\varepsilon) \leq 3 \left\{ \log(c_3) + c_4 \left[ \frac{1}{2r_0} \log \left( 1 + \frac{r_0}{8A} \varepsilon \right) \right]^{-\frac{\varepsilon}{3\eta_0}} + c_5 \left[ \frac{1}{2r_0} \log \left( 1 + \frac{r_0}{8A} \varepsilon \right) \right]^{-1} \right\},$$

for any $\varepsilon < 8A|r_0|^{-1}$, and the assertion follows at once. \qed

We can now prove the boundedness of $G(t,s)$ from $L^1(\mathbb{R}^d, \mu_s)$ into $L^2(\mathbb{R}^d, \mu_t)$ adapting to our case the basic ideas in [14, Thm. 3.4], which deals with the autonomous case. We stress that the nonautonomous setting gives rise to some additional technical difficulties.

**Theorem 5.3.** For any $s, t \in I$, with $s < t$, the operator $G(t,s)$ is bounded from $L^1(\mathbb{R}^d, \mu_s)$ into $L^2(\mathbb{R}^d, \mu_t)$.

**Proof.** As a first step we observe that, for any $s \in I$ and any nonnegative function $g \in C_b(\mathbb{R}^d)$,

$$2 \|g\|_2^2, \log \|g\|_2, \mu_s \|g\|_1, \mu_s \leq \int_{\mathbb{R}^d} g^2 \log g \, d\mu_s. \quad (5.3)$$

It suffices to prove (5.3) for functions with $\|g\|_1, \mu_s = 1$, which reduces to

$$2 \|g\|_2^2, \log \|g\|_2, \mu_s \leq \int_{\mathbb{R}^d} g^2 \log g \, d\mu_s, \quad (5.4)$$

since (5.3), in the general case, will follow from applying (5.4) to the function $\|g\|_1, \mu_s, g$.

To prove estimate (5.4) we observe that the measure $d\nu_s = g \, d\mu_s$ is a probability measure and the function $\psi(x) = x \log x$ is convex in $(0, +\infty)$. Therefore, Jensen inequality yields

$$\psi \left( \int_{\mathbb{R}^d} g \, d\nu_s \right) \leq \int_{\mathbb{R}^d} \psi(g) \, d\nu_s,$$

which is (5.4).

We now fix a positive function $f \in C_c^\infty(\mathbb{R}^d)$, with $\|f\|_1, \mu_s = 1$. Applying the logarithmic Sobolev inequality (LSI), with $f$ and $\mu_s$ being replaced respectively by $\theta_n G(r,s) f$ and $\mu_r$ (where $\theta_n$ is defined in (3.2)), and taking (5.3) (with $g = \theta_n G(r,s) f$) into account, we obtain

$$\|\theta_n G(r,s) f\|_2^2, \mu_r \log \left( \frac{\|\theta_n G(r,s) f\|_2, \mu_r}{\|\theta_n G(r,s) f\|_{1, \mu_r}} \right) \leq \varepsilon \| \nabla_x (\theta_n G(r,s) f) \|_2^2, \mu_r + \beta(\varepsilon) \| \theta_n G(r,s) f \|_{1, \mu_r}^2, \quad (5.5)$$
for every \( s, r \in I \) with \( s \leq r \). Since \( \{ \mu_r : r \in I \} \) is a tight evolution system of measures, we can fix \( R > 0 \) such that \( \mu_r(B(0, R)) \geq 1/2 \) for every \( r \in I \). Now, let us fix \( t > s \) and set

\[
\zeta_n(r) := \log \| \theta_n G(r, s) f \|_{2, \mu_r}^2, \quad n \geq R, \ r \in [s, t].
\]

Note that the function \( \zeta_n \) is well defined since \( G(r, s) f \) is a smooth and everywhere positive function in \( \mathbb{R}^d \) for any \( r \geq s \), (see Lemma 2.3). Hence, \( \| \theta_n G(r, s) f \|_{2, \mu_r} \geq \delta/\sqrt{2} \) for any \( r \in [s, t] \) and \( n \geq R \), where \( \delta \) denotes the minimum of the function \( G(t, s) f \) in \( [s, t] \times B(0, R) \). From Proposition 2.5 we deduce that the function \( \zeta_n \) is differentiable in \( [s, t] \) and

\[
\| \theta_n G(r, s) f \|_{2, \mu_r}^2 \zeta_n'(r) = 2 \int_{\mathbb{R}^d} \theta_n^2(G(r, s) f) A(r) G(r, s) f d\mu_r
\]

\[
- \int_{\mathbb{R}^d} A(r) [\theta_n^2(G(r, s) f)]^2 d\mu_r
\]

\[
= -2 \int_{\mathbb{R}^d} \langle Q(r) \nabla_x (\theta_n G(r, s) f), \nabla_x (\theta_n G(r, s) f) \rangle d\mu_r
\]

\[
- 4 \int_{\mathbb{R}^d} \theta_n G(r, s) f \langle Q(r) \nabla \theta_n, \nabla G(r, s) f \rangle d\mu_r
\]

\[
- 2 \int_{\mathbb{R}^d} \theta_n G(r, s) f^2 A(r) d\mu_r.
\]

Using (5.1) we can estimate

\[
-(A(r) \theta_n)(x) \leq \frac{d\Lambda}{n^2} (2||\eta'||_{\infty} + ||\eta''||_{\infty}) - \eta' \left( \frac{|x|}{n} \right) \frac{\langle b(r, x), x \rangle}{n|x|}
\]

\[
\leq \frac{d\Lambda}{n^2} (2||\eta'||_{\infty} + ||\eta''||_{\infty}).
\]

It thus follows that

\[
\| \theta_n G(r, s) f \|_{2, \mu_r}^2 \zeta_n'(r) \leq -2 \eta_0 \| \nabla_x (\theta_n G(r, s) f) \|_{2, \mu_r}^2 + \frac{4\Lambda}{n} ||\eta'||_{\infty} ||f||_{\infty} \| \nabla f \|_{\infty}
\]

\[
+ \frac{2d\Lambda}{n^2} (2||\eta'||_{\infty} + ||\eta''||_{\infty}) ||f||_{\infty}^2,
\]

for any \( r \geq s \) and \( n \in \mathbb{N} \). Hence, from (5.5) and observing that \( \| \theta_n G(r, s) f \|_{1, \mu_r} \leq \| G(r, s) f \|_{1, \mu_r} \leq ||f||_{1, \mu_r} = 1 \), we deduce that

\[
\zeta_n'(r) \leq -\frac{\eta_0}{\varepsilon} \zeta_n(r) + \frac{2\eta_0 \beta(\varepsilon)}{\varepsilon} + \frac{2C(n)}{\delta^2}, \quad r \in [s, t], \quad (5.6)
\]

where

\[
C(n) = \frac{4\Lambda}{n} ||\eta'||_{\infty} ||f||_{\infty} \| \nabla f \|_{\infty} + \frac{2d\Lambda}{n^2} (2||\eta'||_{\infty} + ||\eta''||_{\infty}) ||f||_{\infty}^2.
\]

Fix \( m > 2/(\kappa - 2) \) and take \( \varepsilon = \eta_0(r-s)/(m+1) \) in the previous inequality. Multiplying both sides of (5.6) by \( (r-s)^{m+1} \) and integrating between \( s \) and \( t \) we get

\[
\int_s^t (r-s)^m \zeta_n(r) dr \leq -\frac{1}{m+1} \int_s^t (r-s)^{m+1} \zeta_n'(r) dr
\]

\[
+ 2 \int_s^t (r-s)^m \beta \left( \frac{\eta_0(r-s)}{m+1} \right) dr + \frac{2C(n)(t-s)^{m+2}}{\delta^2(m+1)(m+2)}.
\]
Note that the last integral term in the right-hand side of the previous inequality is finite due to Proposition 5.2. An integration by parts shows that
\[
\frac{1}{m+1} \int_s^t (r-s)^{m+1} \zeta_n(r) dr = \frac{1}{m+1} (t-s)^{m+1} \zeta_n(t) - \int_s^t (r-s)^m \zeta_n(r) dr.
\]
Hence,
\[
(t-s)^{m+1} \zeta_n(t) \leq 2(m+1) \int_s^t (r-s)^m \beta \left( \frac{\eta_0(r-s)}{m+1} \right) dr + \frac{2C(n)}{\delta^2(m+2)} (t-s)^{m+2}
\]
\[
\leq \frac{2(m+1)^{m+2}}{\eta_0^{m+1}} \int_0^{\eta_0(t-s)} \sigma^m \beta(\sigma) d\sigma + \frac{2C(n)}{\delta^2} (t-s)^{m+1}.
\]
Then, letting \( n \to +\infty \) it follows that
\[
(t-s)^{m+1} \log(\|G(t,s)f\|_{2,\mu}) \leq \frac{2(m+1)^{m+2}}{\eta_0^{m+1}} \int_0^{\eta_0(t-s)} \sigma^m \beta(\sigma) d\sigma
\]
\[
\leq C(\kappa, \eta_0)(t-s)^{m+1} \frac{\kappa}{\kappa - \sigma},
\]
for some positive constant \( C(\kappa, \eta_0) \). Thus we get
\[
\|G(t,s)f\|_{2,\mu} \leq e^{\frac{C(\kappa, \eta_0)}{\kappa - (\kappa - 2)}} = e^{\frac{C(\kappa, \eta_0)}{\kappa - (\kappa - 2)}} \|f\|_{1,\mu,s}. \tag{5.7}
\]
By homogeneity, we can extend estimate (5.7) to any positive and smooth function \( f \) with \( \|f\|_{1,\mu,s} \neq 1 \). Next, for a general \( f \in C_0^\infty(\mathbb{R}^d) \), we write (5.7) with \( f_n = (f^2 + n^{-1})^{1/2} \). Observing that \( \|G(t,s)f_n\|_{2,\mu} \) converges to \( \|G(t,s)f\|_{2,\mu} \) as \( n \to +\infty \), for every \( t \geq s \), and recalling that \( |G(t,s)f| \leq G(t,s)|f| \), we get (5.7) letting \( n \to +\infty \).

Finally, by density we can extend (5.7) to any \( f \in L^1(\mathbb{R}^d, \mu,s) \) and complete the proof. \( \square \)

As a consequence of Theorems 4.3 and 5.3 we get the announced ultracontractivity property of \( G(t,s) \).

**Theorem 5.4.** The evolution operator \( G(t,s) \) is ultracontractive. Moreover,
\[
\|G(t,s)\|_{1,\rightarrow,\infty} \leq \exp \left( C \max \left\{ (t-s)^{-1}, (t-s)^{-\frac{\kappa}{\kappa - 2}} \right\} \right), \quad t > s, \tag{5.8}
\]
where the constant \( C \) depends on \( \kappa, K_3, R, \Lambda, d, \eta_0 \).

**Proof.** Using the evolution property, we can estimate
\[
\|G(t,s)\|_{1,\rightarrow,\infty} \leq \|G(t, (t+s)/2)\|_{2,\rightarrow,\infty}\|G((t+s)/2, s)\|_{1,\rightarrow,\infty}, \tag{5.9}
\]
for every \( t > s \). Now, using (4.5) and (5.7), from estimate (5.9) we get
\[
\|G(t,s)\|_{1,\rightarrow,\infty} \leq 2 \exp \left( \frac{R}{\eta_0(t-s)} + \frac{C_1}{(t-s)^{\frac{\kappa}{\kappa - 2}}} \right) M^{-\frac{1}{\eta_0(t-s)}}, \tag{5.10}
\]
for \( t > s \), where \( C_1 \) is a positive constant depending on \( \kappa \) and \( \eta_0 \), and \( M_{\delta, \kappa} = \sup \{ \|G(t,s)\varphi\|_{\infty} : t \geq s + \delta, \ s \in I \} \).

To conclude the proof, we need to estimate \( M_{(t-s)/4, (\eta_0(t-s))^{-1}}\). For this purpose, arguing as in the proof of Theorem 4.3, we deduce that condition (4.1) is satisfied with \( h_{\lambda} = g_\lambda(y_0, \lambda) \chi_{[0, y_0, \lambda]} + g_\lambda \chi_{(y_0, \lambda, +\infty)} \), where \( y_0, \lambda > 1 \) denotes the minimum of the function \( g_\lambda \) defined by
\[
g_\lambda(y) = \lambda^{-1/2} y \left( K_3(\log y)^{\frac{2}{\kappa}} - 2\lambda^{\frac{\kappa}{\kappa - 2}} C_\kappa - 2\lambda^{\frac{\kappa}{\kappa - 2}} \Lambda d \right), \quad y \geq 1,
\]
and $C_\kappa$ is any positive constant such that
\[ 2\lambda\Lambda y^2 \leq \frac{K_3}{2} y^\kappa + C_\kappa y^{\frac{\kappa - 2}{2}}, \quad y \geq 0. \]

Let us observe that
\[ K_3 (\log y)^\frac{\kappa - 2}{2} - 2\lambda\frac{\kappa - 2}{\kappa} C_\kappa - 2\lambda \Lambda d \geq \frac{K_3}{2} (\log y)^\frac{\kappa}{2} \]
if and only if
\[ y \geq \exp \left( \left( C_1\lambda\frac{\kappa - 2}{\kappa} \right)^{\frac{1}{\kappa}} + C_2\lambda^{\frac{2}{2}} \right) =: P_\lambda, \]
where $C_1 = 4C_\kappa/K_3$ and $C_2 = 4\Lambda d/K_3$. Clearly, if $y \geq P_\lambda$ we can estimate
\[ h_\lambda(y) \geq \frac{K_3}{2} (\log y)^{\frac{\kappa - 2}{2}} y^{\kappa - 2}. \]

Note that $h_\lambda(y) > 0$ for every $y \geq P_\lambda$ and, consequently, $P_\lambda > y_{0,\lambda}$. For any $r \geq 1$, let us set $P_{\lambda,r} = rP_\lambda$. Then, it follows that
\[ \int_{P_{\lambda,r}}^{+\infty} \frac{1}{h_\lambda(s)} ds \leq \frac{2}{K_3} \lambda^{\frac{\kappa - 2}{2}} \int_{P_{\lambda,r}}^{+\infty} \frac{1}{y (\log y)^{\frac{\kappa}{2}}} dy \]
\[ = \frac{4}{(\kappa - 2)K_3} \lambda^{\frac{\kappa - 2}{2}} (\log P_{\lambda,r}) \lambda^{1 - \frac{\kappa}{2}} \]
\[ = \frac{4}{(\kappa - 2)K_3} \lambda^{\frac{\kappa - 2}{2}} \left[ \left( C_1\lambda\frac{\kappa - 2}{\kappa} + C_2\lambda^{\frac{2}{2}} \right)^{\frac{1}{\kappa}} + \log r \right]^{1 - \frac{\kappa}{2}}. \quad (5.11) \]

Taking formula (5.11) into account we deduce that, for any $\delta > 0$, the inequality
\[ \frac{2}{K_3} \lambda^{\frac{\kappa - 2}{2}} \int_{P_{\lambda,r}}^{+\infty} \frac{1}{y (\log y)^{\frac{\kappa}{2}}} dy \leq \delta, \]
is satisfied when
\[ \log r \geq K_0 \delta^{\frac{2}{\kappa - 2}} \lambda - \left( C_1\lambda\frac{\kappa - 2}{\kappa} + C_2\lambda^{\frac{2}{2}} \right)^{\frac{1}{\kappa}}, \]
where $K_0 = [(\kappa - 2)K_3/4]^{2/(2-\kappa)}$. Hence, if
\[ r = \max \left\{ \exp \left[ K_0 \delta^{\frac{2}{\kappa - 2}} \lambda - \left( C_1\lambda\frac{\kappa - 2}{\kappa} + C_2\lambda^{\frac{2}{2}} \right)^{\frac{1}{\kappa}} \right], 1 \right\}, \]
then
\[ \int_{P_{\lambda,r}}^{+\infty} \frac{1}{h_\lambda(s)} ds \leq \delta. \quad (5.12) \]

From [2, Thm. 4.4] we know that there exists a positive constant $\tilde{M}_{\delta,\lambda} \geq M_{\delta,\lambda}$ such that
\[ \int_{\tilde{M}_{\delta,\lambda}}^{+\infty} \frac{1}{h_\lambda(s)} ds = \delta. \quad (5.13) \]
Comparing (5.12) and (5.13), we conclude that $\tilde{M}_{\delta,\lambda} \leq P_{\lambda,r}$, i.e.,
\[ M_{\delta,\lambda} \leq \tilde{M}_{\delta,\lambda} \leq \exp \left[ \max \left\{ K_0 \delta^{\frac{2}{\kappa - 2}} \lambda, \left( C_1\lambda\frac{\kappa - 2}{\kappa} + C_2\lambda^{\frac{2}{2}} \right)^{\frac{1}{\kappa}} \right\} \right], \quad \delta, \lambda > 0. \quad (5.14) \]
Hence, estimate (5.8) follows from (5.10) and (5.14).

We conclude this section by proving the $L^2$-uniform integrability property of $G(t, s)$.

**Proposition 5.5.** For any $s, t \in I$, with $s < t$, the operator $G(t, s)$ is $L^2(\mathbb{R}^d, \mu_t)$-uniformly integrable, i.e.,

$$\lim_{r \to +\infty} \sup_{t > s} \sup_{f \in L^2(\mathbb{R}^d, \mu_s), \|f\|_{L^2} \leq 1} \int_{\{G(t, s)f \geq r\}} |G(t, s)f|^2 \, d\mu_t = 0.$$

**Proof.** To begin with, let us prove that there exists a positive constant $C$, independent of $f$, such that

$$\int_A |G(t, s)f|^2 \, d\mu_t \leq C \int_A \varphi_{\lambda_0} \, d\mu_t,$$

for any $f \in L^2(\mathbb{R}^d, \mu_s)$, with $\|f\|_{L^2} \leq 1$, and any Borel set $A \subset \mathbb{R}^d$, where $\lambda_0 = (\eta_0(t - s)^{-1}$. Note that our assumptions imply that $\varphi_{\lambda_0} \in L^1(\mathbb{R}^d, \mu_t)$ (see (5.2)). We first assume that $f \in C_c(\mathbb{R}^d)$ satisfy $\|f\|_{L^2} \leq 1$. Integrating (3.22) (with $p = 2$) with respect to $d\mu_t(y)$ and taking (3.26) into account, we get

$$|G(t, s)f(x)|^2 \leq 2 e^{\frac{R^2}{\eta_0(t - s)^2}} e^{\frac{|x|^2}{\eta_0(t - s)^2}} =: C \varphi_{\lambda_0}(x), \quad x \in \mathbb{R}^d,$$

where $R$ is any positive constant such that $\mu_t(B(0, R)) \geq 1/2$ for any $t \in I$. From this estimate, (5.15) follows at once.

Since any function $f \in L^2(\mathbb{R}^d, \mu_s)$, with $\|f\|_{L^2} \leq 1$, can be approximated by a sequence $(f_n)_n \subset C_c(\mathbb{R}^d)$ satisfying $\|f_n\|_{L^2} \leq 1$ for any $n \in \mathbb{N}$, estimate (5.15) can be extended by density to any $f \in L^2(\mathbb{R}^d, \mu_s)$ with $\|f\|_{L^2} \leq 1$.

Now, recalling that $G(t, s)$ is a contraction from $L^2(\mathbb{R}^d, \mu_s)$ to $L^2(\mathbb{R}^d, \mu_t)$, applying Chebyshev and Hölder inequalities, from (5.15) we deduce that

$$\int_{\{G(t, s)f \geq r\}} |G(t, s)f|^2 \, d\mu_t \leq C \int_{\{G(t, s)f \geq r\}} \varphi_{\lambda_0} \, d\mu_t$$

$$\leq C \int \varphi_{2\lambda_0} \, d\mu_t(\{G(t, s)f \geq r\})$$

$$\leq \frac{C}{r} \sup_{t \in I} \|\varphi_{2\lambda_0}\|_{L^1}.$$

The claim now follows at once.

5.2. Nonautonomous elliptic operators with a non zero potential term.

All the regularizing properties in the previous sections can be extended to nonautonomous operators with a non zero potential term, i.e., operators defined on smooth functions $\zeta$ by

$$(A_c(t)\zeta)(x) = (A(t)\zeta)(x) - c(t, x)\zeta(x), \quad t \in I, \ x \in \mathbb{R}^d,$$

when the following additional assumption is satisfied.

**Hypothesis 5.6.** $c \in C^{\alpha/2, \alpha}_{\text{loc}}(I \times \mathbb{R}^d)$ and $c_0 := \inf_{I \times \mathbb{R}^d} c > -\infty$.

Let $\varphi$, $a$ and $\gamma$ be the function and the constants in Hypothesis 2.1(iii). Then,

$$A_c(t)\varphi = A(t)\varphi - c(t, x)\varphi \leq a - (\gamma + c_0)\varphi, \quad t \in I.$$

Hence, we can determine a positive constant $\lambda$ such that $A_c(t)\varphi - \lambda \varphi \leq 0$ for any $t \in I$. We can thus apply the results in [2] which show that a Markov evolution operator $G_c(t, s)$ can be associated to the operator $A_c(t)$. More precisely, for every
The evolution operator $G$ is the unique solution of the Cauchy problem
\[
\begin{align*}
D_t u(t,x) &= A_c(t) u(t,x), \\
 u(s,x) &= f(x),
\end{align*}
\]
which satisfies $\|u(t, \cdot)\|_{\infty} \leq e^{-c_0(t-s)}\|f\|_{\infty}$. In the next theorem we establish the connection between the summability improving properties of the evolution operators $G(t,s)$ and $G_c(t,s)$. Here, $\{\mu_t : t \in I\}$ is the unique tight evolution system of measures for the evolution operator $G(t,s)$, considered in the previous sections.

**Theorem 5.7.** If $G(t,s)$ is supercontractive (resp. ultrabounded/ultracontractive), then $G_c(t,s)$ is supercontractive (resp. ultrabounded/ultracontractive) and
\[
\|G_c(t,s)\|_{p\to q} \leq C_{p,q}(t-s) e^{-c_0(t-s)},
\]
for any $t > s$, $1 < p < q < +\infty$ (resp. $1 < p < q = +\infty$ / $p = 1$, $q = +\infty$), where $C_{p,q}$ is given in Definition 2.6.

**Proof.** The proof follows from a comparison argument based on [11, Thm. 2.1] which shows that $G_c(t,s)f \leq e^{-c_0(t-s)}G(t,s)f$, for any $t \geq s$ and any nonnegative function $f \in C_b(R^d)$.

**Remark 5.8.** If $c_0 \geq 0$, $\{\mu_t : t \in I\}$ is a sub-invariant system of measures for the evolution operator $G_c(t,s)$. Indeed, since $G_c(t,s)f \leq e^{-c_0(t-s)}G(t,s)f$ for any $t \geq s$ and any nonnegative function $f \in C_b(R^d)$, we can estimate
\[
\int_{R^d} G_c(t,s)f \, d\mu_t \leq e^{-c_0(t-s)} \int_{R^d} G(t,s)f \, d\mu_t \leq \int_{R^d} f \, d\mu_s,
\]
for any $t > s$.

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