REPRESENTING INTEGERS AS LINEAR COMBINATIONS OF POWER PRODUCTS

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Abstract. Let \( P \) be a finite set of at least two prime numbers, and \( A \) the set of positive integers that are products of powers of primes from \( P \). Let \( F(k) \) denote the smallest positive integer which cannot be presented as sum of less than \( k \) terms of \( A \). In a recent paper Nathanson asked to determine the properties of the function \( F(k) \), in particular to estimate its growth rate. In this paper we derive several results on \( F(k) \) and on the related function \( F_\pm(k) \) which denotes the smallest positive integer which cannot be presented as sum of less than \( k \) terms of \( A \cup (-A) \).

1. Introduction

Let \( P \) be a nonempty finite set of at least two prime numbers, and \( A \) the set of positive integers that are products of powers of primes from \( P \). Put \( A_\pm = A \cup (-A) \). Then there does not exist an integer \( k \) such that every positive integer can be represented as a sum of at most \( k \) elements of \( A_\pm \). This follows e.g. from Theorem 1 of Jarden and Narkiewicz [6], cf. [14, 14]. At a conference in Debrecen in 2010 Nathanson announced the following stronger result (see also [7]):

For every positive integer \( k \) there exist infinitely many integers \( n \) such that \( k \) is the smallest value of \( l \) for which \( n \) can be written as

\[
n = a_1 + a_2 + \cdots + a_l \quad (a_1, a_2, \ldots, a_l \in A_\pm).
\]

Let \( F(k) \) be the smallest positive integer which cannot be presented as a sum of less than \( k \) terms of \( A \) and \( F_\pm(k) \) the smallest positive integer which cannot be presented as a sum of less than \( k \) terms of

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Problem 2 of [7], Nathanson is to give estimates for $F(k)$. (The notation in [7] is different from ours.) Problem 1 is the corresponding question for $F(k)$ in case $A$ consists of the pure powers of 2 and of 3. In [5], two of the authors considered Problem 1 in the more general setting of powers of any finite set of positive integers. They gave lower and upper bounds for $F(k)$ and $F(k)$. In the present paper we consider Problem 2. We give lower and upper bounds for $F(k)$ and $F(k)$ for $A$ as defined above.

We show that there exists an effectively computable number $c$ depending only on $P$, an effectively computable number $C$ depending only on $\varepsilon$ and an effectively computable constant $C$ such that $k^{c} < F(k) < (kt)^{(1+\varepsilon)k}$ and $k^{c} < F(k) < \exp((kt)^{C})$. The method of proof is an adaptation of that in [5], but in the case of the lower bound an additional argument is needed. For the upper bound we need an extended version of a theorem of Ádám, Hajdu and Luca [1] in which a result of Erdős, Pomerance and Schmutz [2] plays an important part. We state the result of Erdős, Pomerance and Schmutz and its refinement in Section 2 and our generalization of the result of Ádám, Hajdu and Luca in Section 3. In Section 4 we derive the lower and upper bounds for $F(k)$ and $F(k)$. In Section 5 we apply the Qualitative Subspace Theorem to prove that for some number $c^{*}$ depending only on $P$, $k$ and $\varepsilon$ the inequality $F(k) \leq (kt)^{(1+\varepsilon)k}$ holds for $k > c^{*}$.

2. An extension of a theorem of Erdős, Pomerance and Schmutz

Let $\lambda(m)$ be the Carmichael function of the positive integer $m$, that is the least positive integer for which

$$b^{\lambda(m)} \equiv 1 \pmod{m}$$

for all $b \in \mathbb{Z}$ with $\gcd(b, m) = 1$. Theorem 1 of [2] gives the following information on small values of the Carmichael function.

For any increasing sequence $(n_{i})_{i=1}^{\infty}$ of positive integers, and any positive constant $C_{1} < 1/\log 2$, one has

$$\lambda(n_{i}) > (\log n_{i})^{C_{1}\log \log \log n_{i}}$$

for $i$ sufficiently large. On the other hand, there exist a strictly increasing sequence $(n_{i})_{i=1}^{\infty}$ of positive integers and a positive constant $C_{2}$, such that, for every $i$,

$$\lambda(n_{i}) < (\log n_{i})^{C_{2}\log \log \log n_{i}}.$$
This nice result does not give any information on the size of $n_i$. For our purposes the following quantitative version will be needed.

Lemma 1 ([5], Theorem 1). There exist positive constants $C_3, C_4$ such that for every large integer $i$ there is an integer $m$ with

$$\log m \in [\log i, (\log i)^{C_3}]$$

and $\lambda(m) < (\log m)^{C_4 \log \log \log m}$.

3. An extension of a theorem of Ádám, Hajdu and Luca

Let $k$ be a positive integer. Put

$$H_{P,k} = \{n \in \mathbb{Z} : n = \sum_{i=1}^{l} a_i \text{ with } l \leq k\}$$

where $a_i \in A (i = 1, 2, \ldots, k)$. For $H \subseteq \mathbb{Z}$ and $m \in \mathbb{Z}, m \geq 2$, we write $\sharp H$ for the cardinality of the set $H$ and

$$H \pmod{m} = \{i : 0 \leq i < m, h \equiv i \pmod{m} \text{ for some } h \in H\}.$$

The next theorem is a generalization of a result from [1].

Theorem 1. Let $C_3, P$ and $k$ be given as above. There is a constant $C_5$ such that for every sufficiently large integer $i$ there exists an integer $m$ with $\log m \in [\log i, (\log i)^{C_3}]$ and

$$\sharp H_{P,k} \pmod{m} < (\log m)^{C_5 kt \log \log \log m}.$$

In the proof of Theorem 1 the following lemma is used.

Lemma 2. ([1], Lemma 1). Let $m = q_1^{\alpha_1} \cdots q_z^{\alpha_z}$ where $q_1, \ldots, q_z$ are distinct primes and $\alpha_1, \ldots, \alpha_z$ positive integers, and let $b \in \mathbb{Z}$. Then

$$\sharp\{b^u \pmod{m} : u \geq 0\} \leq \lambda(m) + \max_{1 \leq j \leq z} \alpha_j.$$

Proof of Theorem 1. Let $i$ be a large integer. Choose $m$ according to Lemma 1. Write $m$ as a product of powers of distinct primes as in Lemma 2. Lemma 2 implies that

$$\sharp\{h \pmod{m} : h \in H_{P,k}\} \leq \left(\lambda(m) + \max_{1 \leq j \leq z} \alpha_j + 1\right)^{kt}.$$

On the other hand, with the constant $C_4$ from Lemma 1

$$\lambda(m) + \max_{1 \leq j \leq z} \alpha_j < (\log m)^{C_4 \log \log \log m} + \frac{\log m}{\log 2}.$$  

The combination of both inequalities yields the theorem.  $\square$
4. Effective results on combinations of power products

Suppose we want to express the positive integer \( n \) as a finite sum of elements of \( A \). For this we apply the greedy algorithm. If we subtract the largest element of \( A \) not exceeding \( n \) from \( n \), we are left with a rest which is less than \( n/(\log n)^{c_1} \) for some number \( c_1 > 0 \) depending only on the two smallest elements of \( P \) according to [8]. We can iterate subtracting the largest element of \( A \) not exceeding the rest from the rest and as long as the rest exceeds \( \exp(\sqrt{\log n}) \) reduce the rest each time by a factor at least \((\log n)^{c_1/2}\). If the rest is smaller than \( \exp(\sqrt{\log n}) \) we can reduce the rest each step by a factor larger than some constant \( c_2 > 1 \), with \( c_2 \) depends only on the smallest prime from \( P \). Thus we find that the sum of \( k \leq 2\log n \frac{c_1}{c_1 \log \log n} \frac{\sqrt{\log n}}{\log c_2} \) elements of \( A \) suffices to represent \( n \). This implies the lower bound \( k^{c_2 k} \) for \( F(k) \) in Theorem 2(i) below. Of course, \( F(k) \leq F_\pm(k) \) for all \( k \).

For an upper bound for \( F(k) \) we study the number of representations of positive integers up to \( n \) as \( \sum_{j=1}^{l} a_j \) with \( a_j \in A, l \leq k \). Since the number of elements of \( A \cup \{0\} \) not exceeding \( n \) is at most \((C_6 \log n)^t \), the number of represented integers is at most \((C_6 \log n)^k \). If this number is less than \( n \), then we are sure that some positive integer \( \leq n \) is not represented. This is the case if

\[
kt < \frac{\log n}{\log \log n + \log C_6}.
\]

Suppose \( n > (kt)^{(1+\varepsilon)kt} \). Then it follows from the monotonicity of the function \( \log x/(\log \log x + C_6) \) for large \( x \) that

\[
\frac{\log n}{\log \log n + C_6} > \frac{(1 + \varepsilon)kt \log kt}{\log(kt) + \log((1 + \varepsilon) \log(kt)) + C_6} > kt
\]

for \( kt \) sufficiently large. By choosing \( C_7 \) suitably for the smaller values of \( kt \), it suffices for all values of \( kt \) that \( n \geq C_7(kt)^{(1+\varepsilon)kt} \). Thus

\[
F(k) \leq C_7(kt)^{(1+\varepsilon)kt}.
\]

Next we consider representations by sums of elements from \( A_{\pm} \). We write \( H_{P,k} = \{ n \in \mathbb{Z} : n = \sum_{j=1}^{l} a_j \text{ with } a_j \in A_{\pm}, l \leq k \} \). Choose the smallest positive integer \( i > 10 \) such that \( j > (\log j)^{C_5 kt \log \log \log j} \) for \( j \geq i \). Then \( i < 2(\log i)^{C_5 kt \log \log \log i} \). It follows that

\[
\log i < C_8 kt(\log \log i)(\log \log \log i)
\]
for some constant $C_8$. Hence $\log i < C_9 kt (\log(kt))(\log \log(kt))$ for some constant $C_9$. According to Theorem 2 there exists an integer $m$ with $\log i \leq \log m \leq (\log i)^{C_3}$ such that all representations in $H^*_{P,k}$ are covered by at most $(\log m)^{C_5 kt\log\log\log m}$ residue classes modulo $m$. By the definition of $i$ and the inequality $i \leq m$, we see that this number of residue classes is less than $m$, therefore at least one positive integer $n \leq m$ has no representation of the form $\sum_{j=1}^{k} a_j$ with $a_j \in A \cup \{0\}$ for $j = 1, \ldots, k$. Hence

$$\log n \leq \log m \leq (\log i)^{C_3} < (C_9 kt (\log kt)(\log \log kt))^{C_3} < (kt)^{C_{10}}$$

for some constant $C_{10}$. Thus $F_\pm(k) < \exp((kt)^{C_{10}})$.

So we have proved the following result.

**Theorem 2.** Let $P = \{p_1, \ldots, p_t\}$ be a finite set of primes with $t \geq 2$. Let $A$ be the set of integers composed of numbers from $P$. Let $k$ be a positive integer. Denote by $F(k)$ the smallest positive integer which cannot be represented in the form $\sum_{i=1}^{k} a_i$ with $a_i \in A \cup \{0\}$ for all $i$ and by $F_\pm(k)$ the smallest positive integer which cannot be represented in the form $\sum_{i=1}^{k} a_i$ with $a_i \in A \pm \cup \{0\}$ for all $i$. Then, for every $\varepsilon > 0$ there are a number $c$ depending only on the two smallest elements of $P$, a number $C$ depending only on $\varepsilon$ and an absolute constant $C_\pm$ such that

(i) $F(k) > k^ck$ for all $k > 1$,
(ii) $F(k) \leq C(kt)^{(1+\varepsilon)kt}$ for all $k > 1$,
(iii) $F_\pm(k) < \exp((kt)^{C_\pm})$ for all $k > 1$.

**Remark 1.** In Section 5 we shall use an ineffective method to show that $C_\pm = 16$ suffices.

**Remark 2.** Following the proof of Theorem 3(iv) of [3] it can be shown that there are infinitely many positive integers $k$ for which $F_\pm(k) \leq \exp(C_*^{kt \log(kt)} \log\log(kt))$ for some suitable effectively computable constant $C_*$. In Section 5 we derive the better upper bound $(kt)^{(1+\varepsilon)kt}$ for $F_\pm(k)$ for all but finitely many $k$. However, it cannot be deduced from the proof from which value of $k$ on this bound holds.

**Remark 3.** Using the above methods similar bounds can be derived if $P$ is replaced by any finite set of positive integers.
5. Application of the ineffective Subspace theorem

By applying another version of the Subspace Theorem we derive an estimate for $F_\pm(k)$ which is much better than the bound in Theorem 2(iii) and holds for all but finitely many $k$’s.

**Theorem 3.** Under the conditions of Theorem 2 for every $\varepsilon > 0$ there is a number $c_\pm^*$ depending only on $P, k$ and $\varepsilon$ such that

$$F_\pm(k) \leq (kt)^{(1+\varepsilon)kt}$$

whenever $k > c_\pm^*$.

In the proof we apply the following result of Evertse. Here the $p$-adic value $|x|_p$ is defined as $|x|p^{-r}$ where $p^r||x$.

**Lemma 3** ([3], Corollary 1). Let $c, d$ be constants with $c > 0, 0 \leq d < 1$. Let $S_0$ be a finite set of primes and let $l$ be a positive integer. Then there are only finitely many tuples $(x_0, x_1, \ldots, x_l)$ of rational integers such that

$$x_0 + x_1 + \cdots + x_l = 0;$$
$$x_{i_0} + x_{i_1} + \cdots x_{i_s} \neq 0$$

for each proper, non-empty subset $\{i_0, i_1, \ldots, i_s\}$ of $\{0, 1, \ldots, l\}$;

$$\gcd(x_0, x_1, \ldots, x_l) = 1;$$

$$\prod_{j=0}^l \left( |x_j| \prod_{p \in S_0} |x_j|_p \right) \leq c \left( \max_{0 \leq j \leq l} |x_j| \right)^d.$$

**Proof of Theorem 4.** Let $n$ be an integer which is not divisible by any prime from $P$. Suppose $n = a_1 + a_2 + \cdots + a_l$ with $a_j \in A_\pm$ for $j = 1, 2, \ldots, l$ with $l \leq k$. Without loss of generality we may assume that $l$ is minimal, hence $a_1 + a_2 + \cdots + a_l$ has no proper subsums which vanish. Moreover, we know that $\gcd(a_1, a_2, \ldots, a_l) = 1$. We apply Lemma 3 with $c = 1, d = 1/2, S_0 = P$ to the equation $a_0 + a_1 + \cdots + a_l = 0$ with $a_0 = -n$. It follows that given $k, P$ there only finitely many tuples $(n, a_1, a_2, \ldots, a_l)$ with $\gcd(n, p_1, \ldots, p_l) = 1$ and $l \leq k$ such that $n = a_1 + a_2 + \cdots + a_l$ with $a_j \in A_\pm$ for $j = 1, 2, \ldots, l$ and

$$n \leq \left( \max_{0 \leq j \leq l} |a_j| \right)^{1/2},$$

hence

$$n^2 \leq \max_{1 \leq j \leq l} |a_j|.$$ 

Let $N_0$ be the maximum of $|n|$ for all such tuples, where $N_0 = 0$ if there are no such tuples.
Next consider positive integers \( n > N_0 \) which are not divisible by any prime from \( P \). Then, for any representation \( n = a_1 + a_2 + \cdots + a_l \) with \( a_j \in A_\pm \) for \( j = 1, 2, \ldots, l \) and \( l \leq k \), we have \( |a_j| < n^2 \) for \( j = 1, 2, \ldots, l \). Writing \( a_j = \pm p_{i_1}^{s_1} \cdots p_{i_t}^{s_t} \) we obtain \( s_j \leq 3 \log n - 1 \). The number of possible tuples \((a_1, \ldots, a_l)\) for \( l \) is therefore at most \( 2^l (3 \log n)^t \). Then the number of all possible tuples \((a_1, \ldots, a_j)\) with \( j \leq k \) is at most \( 2 \cdot 2^k (3 \log n)^{kt} \). Thus for \( N > N_0 \) there are at most \( N_0 + 2 \cdot 2^k (3 \log N)^{kt} \) integers \( n \leq N \) coprime to \( P \) such that \( n \) is representable as sum of at most \( k \) integers from \( A_\pm \). The number of positive integers \( n \leq N \) coprime to \( P \) is at least \( N \prod_{p \in P} (1 - 1/p) - 2^t > 2^{-t}N - 2^t \). Hence for finding an \( n \) with \( n \leq N \) such that \( n \) is not representable in the desired form, it suffices that

\[
2^{-t}N - 2^t > N_0 + 2 \cdot 2^k (3 \log N)^{kt}.
\]

As in the proof of Theorem 2(ii) it follows that for every \( \varepsilon > 0 \) there is an unspecified number \( c^*_\pm \) depending only on \( k, P \) and \( \varepsilon \) such that

\[
F_\pm(k) \leq (kt)^{(1+\varepsilon)kt}
\]

whenever \( k > c^*_\pm \). \( \Box \)

Remark 4. Theorem 4 is also an improvement of Theorem 3.4(iv) of [5] where, only for sums of perfect powers, a weaker bound is given.

References

[1] Zs. Ádám, L. Hajdu, F. Luca, Representing integers as linear combinations of \( S \)-units, Acta Arith. 138 (2009), 101–107.
[2] P. Erdös, C. Pomerance, E. Schmutz, Carmichael’s lambda function, Acta Arith. 58 (1991), 365–385.
[3] J.-H. Evertse, On sums of \( S \)-units and linear recurrences, Compositio Math. 53 (1984), 225–244.
[4] L. Hajdu, Arithmetic progressions in linear combinations of \( S \)-units, Period. Math. Hungar. 54 (2007), 175–181.
[5] L. Hajdu, R. Tijdeman, Representing integers as linear combinations of powers, Publ. Math. Debrecen 79 (2011), 461–468.
[6] M. Jarden, W. Narkiewicz, On sums of units, Monatsh. Math. 150 (2007), 327–332.
[7] M. B. Nathanson, Geometric group theory and arithmetic diameter, Publ. Math. Debrecen 79 (2011), 563–572.
[8] R. Tijdeman, On the maximal distance of integers composed of small primes, Compos. Math. 28 (1974), 159–162.
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