WEIGHTED PSEUDO ALMOST PERIODICITY OF 
MULTI-PROPORTIONAL DELAYED SHUNTING INHIBITORY 
CELLULAR NEURAL NETWORKS WITH D OPERATOR 

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Abstract. Taking into account the effects of multi-proportional delays and 
D operator, this paper investigates the stability issue of a general class of 
networks (shunting inhibitory neural networks). With the 
help of fixed point theorem and some novel differential inequality techniques, 
we derive a new sufficient conditions to ensure the existence, uniqueness and 
exponential stability of weighted pseudo almost periodic solutions (WPAPS) of 
the considered model. The obtained main results are totally new and generalize 
some published results. At the end of this work, we also give some numerical 
simulations to support the proposed approach and demonstrate the correctness 
of the main conclusions.

1. Introduction. Differential equations modeling is greatly applied in the fields of 
natural and engineering process[11, 12, 15, 27, 35], and it is also a basic tool in 
the characterization of neural network[32, 34, 36, 43, 44]. Because the shunting 
lateral inhibition can enhance edges and contrast greatly and plays a key role in vision, 
Bouzerdoum and Pinter creatively proposed SICNNs in the early 1990s [2, 3, 4]. So far, theoretical and applied investigation of cellular neural networks 
basically on shunting lateral inhibition have been successful implemented in vision, 
psychophysics, robotics, pattern recognition, speech, perception, image processing 
tasks and so on. Many applications mainly concerned the long time dynamical behaviors [28, 30, 39, 40], such as the asymptotic convergence of periodic solutions, the 
existence, uniqueness and exponential stability of equilibrium point, dissipativity 
and global attracting, multi-stability, robust stabilization and stabilization of the 
designed models [5, 14, 16, 17].

In real-world networks the interactions between network elements are inherently 
time-delayed[13, 18]. As is known to all, delay is unavoidable in diverse areas of
physics, population biology, neural networks [19, 20, 21, 23, 24, 29, 31, 33], etc.

These time delays can not only slow the network but can have a destabilizing effect on the network’s dynamics leading to poor performance. Furthermore, the large-scale and nonlinearity of neurons lead to more complicated dynamic behavior of the system. There is a growing concern that there are still some drawbacks in considering only the influences of time delays incorporated in the states. It might be a better candidate to take into account the effects of the delays in the derivation term [37], which will extend to the case of neutral-type SICNNs with delays[37]. On the other hand, time delays are not mandatorily a constant, they may change over time and/or depends on system parameters. Neural networks with proportional delays have remarkable predictability and controllability [22, 25, 26]. In the past decades, by introducing unbounded proportional delays, the investigation on convergence and almost periodicity of the following neutral-type SCNNs with \( D \) operators have been the new world-wide focus [42].

\[
[x_{ij}(t) - p_{ij}(t)x_{ij}(r_{ij}t)]' = -d_{ij}(t)x_{ij}(t) - \sum_{C_{kl} \in N_r(i,j)} C_{kl}^{ij}(t) f(x_{kl}(q_{kl}t)) x_{ij}(t) + L_{ij}(t), \quad t \geq t_0 > 0, \quad ij \in J = \{11, 12, \cdots, 1n, \cdots, m1, m2, \cdots, mn\}.
\] (1)

Part of the reason is that the model has been fruitful applied in engineering fields such as combination optimization, fault diagnosis, automatic control engineering, etc.(see [6, 37, 42] and the references therein). Here, the specific meaning of each parameter of system (1) is as follows: the number of neural units is represented by \( mn \), \( C_{ij} \) expresses the cell at the lattice \( (i, j) \), the \( r \)-neighborhood of \( C_{ij} \) can be described by

\[
N_r(i,j) = \{C_{kl} : \max(|k - i|, |l - j|) \leq r, 1 \leq k \leq m, 1 \leq l \leq n\},
\]

\( x_{ij}(t) \) is neuron’s state, \( d_{ij}(t) \) is the decay rate, \( f \) represents the behavior function, \( q_{ij}, r_{ij} \in (0, 1) \) are proportional delay factors, and \( L_{ij}(t) \) is corresponding to the outside input.

Recently, the authors in [1] had pointed out that the definition of WPAPS is a certain generalization of pseudo-almost periodicity, and this weighted phenomenon is more common than almost periodic phenomenon. Thus, studying the WPAPS is more realistic and significant on nonlinear dynamic models [8, 9, 38, 41]. In particular, when \( p_{ij}(t) \equiv 0 \), for the following proportional delayed SICNNs:

\[
x_{ij}'(t) = -d_{ij}(t)x_{ij}(t) - \sum_{C_{kl} \in N_r(i,j)} C_{kl}^{ij}(t) f(x_{kl}(q_{kl}t)) x_{ij}(t) + L_{ij}(t), \quad t \geq t_0 > 0, \quad ij \in J,
\] (2)

the authors of [7] investigated the existence, uniqueness and exponential stability of WPAPS under the following conditions:

\[
-\gamma_{ij} = \sup_{t \in R} \{-\hat{d}_{ij}(t) + K_{ij} \left[ \sum_{C_{kl} \in N_r(i,j)} |C_{kl}^{ij}(t)|(M_f + L_f \frac{I}{1-\delta}) \right] \} < 0,
\] (3)
where $\delta <1$, $ij \in J$,
\[
M_f := \sup_{u \in \mathbb{R}} |f(u)|, \quad \delta = \max_{ij \in J} \left\{ K_{ij} \frac{\sum_{C_{ij} \in N_r(i,j)} \sup_{t \in \mathbb{R}} |C_{ij}^{(i)}(t)|M_f}{d_{ij}} \right\} \geq 0,
\]
and
\[
I = \max_{ij \in J} \left\{ \frac{\sup_{t \in \mathbb{R}} |L_{ij}(t)|}{d_{ij}} \right\}.
\]
It should be mentioned that the authors in [7] used (3) to evidence that there exist a positive constant $\lambda \in (0, \min \{ \lambda_0, \min_{ij \in J} \inf_{t \geq t_0} \hat{d}_{ij}(t) \})$ and get the below inequality
\[
\sup_{t \geq t_0} \left\{ \lambda - \hat{d}_{ij}(t) + K_{ij} \left[ \sum_{C_{ij} \in N_r(i,j)} |C_{ij}^{(i)}(t)||M_f + L_f \frac{I}{1-\delta} e^{\lambda(1-q_i)t}) \right] \right\} <0, \quad ij \in J.
\]
Obviously, it follows from \( \lim_{t \to +\infty} e^{\lambda(1-q_i)t} = +\infty \) that (3) can not lead to (4). This wrong is also found in Examples 4.1 and 4.2 of [7], where
\[
\{ C_{ij} \} = \left[ \begin{array}{c} \frac{1}{10} | \cos t | \\ \frac{1}{20} | \cos t | \\ \frac{1}{20} | \cos t | \end{array} \right],
\]
and
\[
\hat{d}_{ij}(t) = 1, \quad L_f = \frac{1}{10}, \quad M_f = \frac{\pi}{20}, \quad K_{ij} = e^{\pi}, \quad i, j = 1, 2,
\]
can not meet (4). Thus, the results in [7] leave some spaces for improvement. In addition, to the authors’ knowledge, no existing work reported on the existence and generalized exponential stability of WPAPS for the model (1) we concerned.

Inspired by the aforementioned works, we aim to build up some sufficient conditions to check the existence, uniqueness and exponential stability of WPAPS for the addressed system in this paper. Our results correct the errors in [7] and ameliorate the corresponding main conclusions of the literature of [18, 42], one can find Remark 2.1 for extensive information.

In the sequel, we denote $U$ as the set of functions (weights) defined from $\mathbb{R}$ to $\to (0, +\infty)$ obeying
\[
\lim_{r \to +\infty} \mu([-r, r]) = +\infty, \quad \text{where} \quad \mu([-r, r]) := \int_{-r}^{r} \mu(x)dx \quad (r > 0).
\]
Furthermore, we employ the following notations:
\[
x = \{ x_{ij} \} = (x_{ij})_{1 \times mn} \in \mathbb{R}^{mn}, \quad |x| = \{ |x_{ij}| \}, \quad \| x(t) \| = \max_{ij \in J} \| x_{ij}(t) \|, \quad \| x \| = \max_{ij \in J} \| x_{ij} \|, \quad \| x(t) \| = \max_{ij \in J} \| x_{ij}(t) \|,
\]
\[
\| x \| = \max_{ij \in J} | x_{ij} |, \quad h^+ = \sup_{t \in \mathbb{R}} | h(t) |, \quad h^- = \inf_{t \in \mathbb{R}} | h(t) |,
\]
\[
\quad \mathbb{U}_\infty := \{ \mu | \mu \in \mathbb{U}, \inf_{x \in \mathbb{R}} \mu(x) = \mu_0 > 0 \},
\]
and
\[
\mathbb{U}_a := \{ \mu | \mu \in \mathbb{U}_\infty, \lim_{|x| \to +\infty} \sup_{r \to +\infty} \frac{\mu(ax)}{\mu(x)} < +\infty, \lim_{r \to +\infty} \sup_{|x| \to +\infty} \frac{\mu([-ar, ar])}{\mu([-r, r])} < +\infty, \forall \alpha \in (0, +\infty) \}.
\]

Let $BC(\mathbb{R}, \mathbb{R}^{mn})$ represent the collection of bounded continuous functions defined from $\mathbb{R}$ to $\mathbb{R}^{mn}$, which is a Banach space equipped with the norm $\| f \|_\infty :=$
sup \|f(t)\|$. Also, we designate the collection of the almost periodic functions from $\mathbb{R}$ to $\mathbb{R}^m$ by $AP(\mathbb{R}, \mathbb{R}^m)$. Moreover, we set

$$PAP^u_0(\mathbb{R}, \mathbb{R}^m) = \{ \varphi \in BC(\mathbb{R}, \mathbb{R}^n) | \lim_{r \to +\infty} \frac{1}{\mu([-r, r])} \int_{-r}^{r} \mu(t)|\varphi(t)|dt = 0 \}.$$ 

If a function $f \in BC(\mathbb{R}, \mathbb{R}^m)$ satisfies

$$f = h + \varphi,$$

here $h \in AP(\mathbb{R}, \mathbb{R}^m)$ and $\varphi \in PAP^u_0(\mathbb{R}, \mathbb{R}^m)$, then $f$ is a weighted pseudo almost periodic function. We designate the collection of such functions by $PAP^n(\mathbb{R}, \mathbb{R}^m)$. In addition, fixed $\mu \in \mathbb{U}_+^s$, it is easy to see that $(PAP^n(\mathbb{R}, \mathbb{R}^m), \| \cdot \|_\infty)$ is a Banach space and $AP(\mathbb{R}, \mathbb{R}^m)$ is a appropriate subspace of $PAP^n(\mathbb{R}, \mathbb{R}^m)$ [8, 9].

As usual, the initial values of system (1) take the form as below

$$x_{ij}(s) = \varphi_{ij}(s), \ s \in [\rho_{ij}t_0, t_0], \ \varphi_{ij} \in C([\rho_{ij}t_0, t_0], \mathbb{R}), \ \rho_{ij} = \min\{r_{ij}, \min_{kl \in J}q_{kl}\}, \ i,j \in J. \quad (5)$$

Throughout this paper, for all $ij$, $kl \in J$, we always suppose $C_{ij}^{kl}, L_{ij}, p_{ij}, r_{ij} \in PAP^n(\mathbb{R}, \mathbb{R})$, $d_{ij} \in AP(\mathbb{R}, \mathbb{R})$, and

$$M[d_{ij}] = \lim_{T \to +\infty} \frac{1}{T} \int_{t}^{t+T} d_{ij}(s)ds > 0. \quad (6)$$

We also make the following technical assumptions.

(H$_1$) we can find a function $d_{ij} : \mathbb{R} \to (0, +\infty)$, which is bounded and continuous, and there is a constant $K_{ij} > 0$, such that the following inequality holds:

$$e^{-f_s^t d_{ij}(u)du} \leq K_{ij} e^{-f_s^t d_{ij}(u)du} \text{ for all } t, s \in \mathbb{R} \text{ and } t - s \geq 0.$$

(H$_2$) for all $u, v \in \mathbb{R}$, there are constants $M^f$ and $L^f$ such that

$$|f(u) - f(v)| \leq L^f|u - v|, \ |f(u)| \leq M^f.$$

(H$_3$) for all $\mu \in \mathbb{U}_+^s$, denote $L_{ij} = \max\{\sup_{i \in J} |\int_{t}^{t+\infty} e^{-f_s^t d_{ij}(u)du} L_{ij}(s)ds|\}$, one can find a positive constant $\kappa$ satisfying

$$E_{ij} = \sup_{t \in \mathbb{R}} \frac{1}{d_{ij}(t)} K_{ij} \left[ |d_{ij}(t)p_{ij}(t)| + \sum_{C_{kl} \in N_r(i,j)} |C_{kl}^{ij}(t)(L^f(\kappa + L) + |f(0)|)\right] < \frac{\kappa}{\kappa + L} - p_{ij}^+,$$

$$F_{ij} = \sup_{t \in \mathbb{R}} \frac{1}{d_{ij}(t)} K_{ij} \left[ |d_{ij}(t)p_{ij}(t)| + \sum_{C_{kl} \in N_r(i,j)} |C_{kl}^{ij}(t)(M^f + L^f(\kappa + L))\right] < 1 - p_{ij}^+,$$
2. Main results. Lemma 2.1. From Lemma 2.1, we obtain
\[ \int_{t_0}^{t_1} e^{-f_s d_{ij}(s)\mu} \left[ -d_{ij}(s)p_{ij}(s)\varphi_{ij}(r_{ij},s) \right. \]
\[ \left. - \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl}(s)f(\varphi_{kl}(q_{kl},s))\varphi_{ij}(s) \right) ds, \quad \varphi \in PAP^\mu(\mathbb{R}, \mathbb{R}^n). \]

Remark 1.1. According to the assumptions \((H_1)\) and \((H_2)\), one can use the similar method as that employed in Lemma 2.2 of \([42]\) to show the existence of solutions on \([t_0, +\infty)\) for system (1) with the initial value (5).

2. Main results. Lemma 2.1 (please refer to [41]). Let \(\beta \in \mathbb{R} \setminus \{0\}\) and \(w \in PAP^\mu(\mathbb{R}, \mathbb{R})\), then we have \(w(\beta t) \in PAP^\mu(\mathbb{R}, \mathbb{R})\).

With the help of Lemma 2.2 in [7], using a similar method as shown in Lemma 2.3 of [41], it is easy to have the below lemma.

Lemma 2.2. If \((H_1)\) and \((H_2)\) hold, then the nonlinear operator \(\Pi\) maps \(PAP^\mu(\mathbb{R}, \mathbb{R}^m)\) into itself,
\[ (\Pi \varphi)(t) = \int_{-\infty}^{t} e^{-f_s d_{ij}(u)\mu} \left[ -d_{ij}(s)p_{ij}(s)\varphi_{ij}(r_{ij},s) \right. \]
\[ \left. - \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl}(s)f(\varphi_{kl}(q_{kl},s))\varphi_{ij}(s) \right) ds, \quad \varphi \in PAP^\mu(\mathbb{R}, \mathbb{R}^n). \]

Theorem 2.1. Under assumptions \((H_1)\), \((H_2)\) and \((H_3)\), then we can conclude that the WPAPS \(x^*(t) \in PAP^\mu(\mathbb{R}, \mathbb{R}^n)\) of system (1.1) is exist and unique. More specifically, the solution \(x^*(t)\) is globally exponentially stable, i.e., for all solution \(x(t)\) of system (1) with respect to initial condition (5), one can has a constant \(\sigma \in (0, \min_{ij \in J} \hat{d}_{ij})\) fulfilling following
\[ x_{ij}(t) - x^*_{ij}(t) = O\left(\left(\frac{1}{1+t}\right)^{\sigma}\right) \text{ as } t \to +\infty, \]
where \(ij \in J\).

Proof. From Lemma 2.1, we obtain
\[ Q(t) = \{Q_{ij}(t)\} = \left\{ \int_{-\infty}^{t} e^{-f_s a_{ij}(u)\mu}L_{ij}(s)ds \right\} = \Pi(0) \subset PAP^\mu(\mathbb{R}, \mathbb{R}^n), \quad L = \|Q\|_{\infty}. \]

Set \(\Omega = \{\varphi \in PAP^\mu(\mathbb{R}, \mathbb{R}^n), \|\varphi - Q\|_{\infty} \leq \kappa\}\), and \(\varphi \in \Omega\), then we have the following inequalities
\[ \|\varphi\|_{\infty} \leq \|\varphi - Q\|_{\infty} + \|Q\|_{\infty} \leq \kappa + L. \]

Furthermore, we can determine a mapping \(W : \Omega \to \Omega\) by setting
\[ (W \varphi)(t) = \{p_{ij}(t)\varphi_{ij}(r_{ij},t)\} + (\Pi \varphi)(t), \quad \forall \varphi \in \Omega. \]
Next, we arrive the position to prove for any $\varphi \in \Omega$, it can be obtained $W\varphi \in \Omega$. Clearly, according to (7), $(H_1)$, $(H_2)$ and $(H_3)$ we have

$$
\|(W\varphi)(t) - Q(t)\|
= \{ |p_{ij}(t)\varphi_{ij}(r_{ij}t) + \int_{-\infty}^{t} e^{-\int_{t}^{s} d_{ij}(u)du}[-d_{ij}(s)p_{ij}(s)\varphi_{ij}(r_{ij}s)]
- \sum_{C_{ij}\in N_{(r_{ij}, ij)}} C^{ij}_{ij}(s)f(\varphi_{kl}(q_{kl}s))\varphi_{ij}(s)ds| \}
\leq \{ p_{ij}^{+}\|\varphi\|_{\infty} + \int_{-\infty}^{t} e^{-f_{ij}(u)du}K_{ij}[\|d_{ij}(s)p_{ij}(s)\|\varphi\|_{\infty} + \sum_{C_{ij}\in N_{(r_{ij}, ij)}} |C^{ij}_{ij}(s)|(L^{f}\|\varphi\|_{\infty} + |f(0)|)\|\varphi\|_{\infty}]ds \}
\leq \{ p_{ij}^{+} + \int_{-\infty}^{t} e^{-f_{ij}(u)du}K_{ij}[\|d_{ij}(s)p_{ij}(s)\|\varphi\|_{\infty} + \sum_{C_{ij}\in N_{(r_{ij}, ij)}} |C^{ij}_{ij}(s)|(L^{f}(\kappa + L) + |f(0)|)\}(\kappa + L) \}
\leq \{ p_{ij}^{+} + \int_{-\infty}^{t} e^{-f_{ij}(u)du}[(\frac{\kappa}{\kappa + L} - p_{ij}^{+})d_{ij}(s)ds](\kappa + L) \}
\leq \{ \kappa \}, \text{ for all } t \in \mathbb{R},
$$

which suggests that $W\varphi \in \Omega$.

Furthermore, we evidence that $W$ is a contract operator. Indeed, again from (7), $(H_1)$, $(H_2)$ and $(H_3)$, we get

$$
\|(W\varphi)(t) - (W\psi)(t)\|
= \{|((T\varphi)(t)) - (T\psi)(t))_{ij}| \}
= \{ |p_{ij}(t)[\varphi_{ij}(r_{ij}t) - \psi_{ij}(r_{ij}t)]
+ \int_{-\infty}^{t} e^{-\int_{t}^{s} d_{ij}(u)du}[-d_{ij}(s)p_{ij}(s)(\varphi_{ij}(r_{ij}s) - \psi_{ij}(r_{ij}s))]
- \sum_{C_{ij}\in N_{(r_{ij}, ij)}} C^{ij}_{ij}(s)f(\varphi_{kl}(q_{kl}s))\varphi_{ij}(s) - f(\psi_{kl}(q_{kl}s))\psi_{ij}(s)ds| \}
\leq \{ p_{ij}^{+}\|\varphi - \psi\|_{\infty} + \int_{-\infty}^{t} e^{-f_{ij}(u)du}K_{ij}[\|d_{ij}(s)p_{ij}(s)\|\varphi - \psi\|_{\infty} + \sum_{C_{ij}\in N_{(r_{ij}, ij)}} |C^{ij}_{ij}(s)|(\|f(\varphi_{kl}(q_{kl}s))\|\varphi_{ij}(s) - \psi_{ij}(s)\| + |f(\varphi_{kl}(q_{kl}s)) - f(\psi_{kl}(q_{kl}s))|\psi_{ij}(s))ds \}
$$
From (9) and for all \( t \) can easy to obtain \( x \) which implies \( x \) as we provide the detail verification as below.

Then \( H(0) \) is globally generalized exponentially stable. In order to enhance readability, we denote the solution of system (1) with respect to the initial value condition (5) such that

\[
\begin{align*}
\{ x_{ij}^*(t) \} &= x_{ij}(t) = (Wx^*) (t) = \left\{ p_{ij}(t)x_{ij}^* (r_{ij}t) \right\} + (\Pi x^*) (t),
\end{align*}
\]

which implies \( x^* (t) \) is a weighted pseudo almost periodic solution of system (1).

Lastly, we can use the similar approach applied in Theorem 3.1 of [42] to prove \( x^* (t) \) is globally generalized exponentially stable. In order to enhance readability, we provide the detail verification as below.

We denote the solution of system (1) with respect to the initial value condition (5) as \( x(t) = \{ x_{ij}(t) \} \), and \( x_{ij}(t) = \varphi_{ij}(t) = \varphi_{ij}(\rho_{ij} t_0) \) for \( t \in [r_{ij}\rho_{ij} t_0, \rho_{ij} t_0], ij \in J \).

We also denote

\[
\begin{align*}
z_{ij}(t) &= x_{ij}(t) - x_{ij}^*(t), \quad Z_{ij}(t) = z_{ij}(t) - p_{ij}(t)z_{ij}(r_{ij}t), ij \in J.
\end{align*}
\]

Then

\[
\begin{align*}
Z_{ij}'(t) &= - d_{ij}(t)Z_{ij}(t) - d_{ij}(t)p_{ij}(t)z_{ij}(r_{ij}t) - \sum_{C_{kl} \in N_{r, (i,j)}} C_{ij}^{kl}(t) f(x_{kl}(q_{kl}t))x_{ij}(t) - f(x_{ij}^* (q_{kl}t))x_{ij}^* (t), \quad ij \in J.
\end{align*}
\]

According to assumption (\( H_3 \)), it is obvious that there do exists a constant \( \sigma \in (0, \min_{ij \in J} \frac{d_{ij}}{1}) \) such that \( e^{\sigma \ln(\frac{1}{p_{ij}})} p_{ij} < 1 \), furthermore, we have the following inequality

\[
\begin{align*}
\begin{aligned}
&\sup_{t \geq \rho_{ij} t_0} \left\{ \sigma - d_{ij}(t) + K_{ij} \left[ d_{ij}(t)p_{ij}(t) \right] e^{\sigma \ln(\frac{1}{p_{ij}})} \frac{1}{1 - e^{\sigma \ln(\frac{1}{p_{ij}})} p_{ij}^+} \right\} + \\
&\sum_{C_{kl} \in N_{r, (i,j)}} \left( C_{ij}^{kl}(t) \left[ M_{ij}^f \frac{1}{1 - e^{\sigma \ln(\frac{1}{p_{ij}})} p_{ij}^+} \right] + L^f (\kappa + L) \frac{1}{1 - e^{\sigma \ln(\frac{1}{p_{ij}})} p_{ij}^+} e^{\sigma \ln(\frac{1}{p_{ij}^+})} \right)) < 0, \quad ij \in J.
\end{aligned}
\end{align*}
\]

From (9) and for all \( t \geq 0 \), we have the following three inequalities

\[
\begin{align*}
\frac{\sigma}{1 + t} &\leq \sigma, \quad \ln(\frac{1 + t}{1 + r_{ij}t}) \leq \ln\frac{1}{r_{ij}}, \\
\ln(\frac{1 + t}{1 + q_{kl}t}) &\leq \ln\frac{1}{q_{kl}},
\end{align*}
\]
therefore, we obtain

\[
\sup_{t \geq \rho_{ij}, t_0} \left\{ \sigma \cdot \frac{1}{1 + t} - \tilde{d}_{ij}(t) + K_{ij} \left[ \left| d_{ij}(t)p_{ij}(t) \right| e^{\sigma \ln \left( \frac{\tilde{d}_{ij}(t) + 1}{\tilde{d}_{ij}(t)} \right)} \right] \right\} 
\leq \sup_{t \geq \rho_{ij}, t_0} \left\{ \sigma \cdot \frac{1}{1 + t} - \tilde{d}_{ij}(t) + K_{ij} \left[ \left| d_{ij}(t)p_{ij}(t) \right| e^{\sigma \ln \left( \frac{\tilde{d}_{ij}(t) + 1}{\tilde{d}_{ij}(t)} \right)} \right] \right\} < 0, \quad i, j \in J.
\]

For any \( \varepsilon > 0 \), we can define

\[
V(t) = M(\|\varphi\|_X + \varepsilon) e^{-\sigma \ln \left( \frac{\tilde{d}_{ij}(t) + 1}{\tilde{d}_{ij}(t)} \right)}, \quad M = 1 + \max_{i, j} K_{ij}, \quad t \geq t_0,
\]

where

\[
\|\varphi\|_X = \max_{i, j} \left[ \sup_{t \in [\rho_{ij}, t_0]} \left| \varphi_{ij}(t) - x_{ij}^*(t) \right| - p_{ij} \left| \varphi_{ij}(r_{ij}t) - x_{ij}^*(r_{ij}t) \right| \right].
\]

Obviously, for all \( t \geq t_0, i, j \in J \), we have

\[
V(q_{ij}t) = M(\|\varphi\|_X + \varepsilon) e^{-\sigma \ln \left( \frac{\tilde{d}_{ij}(t) + 1}{\tilde{d}_{ij}(t)} \right)} \leq V(t) e^{\sigma \ln \frac{1}{\tilde{d}_{ij}}},
\]

and

\[
|Z_{ij}(t_0)| < (\|\varphi\|_X + \varepsilon) \leq M(\|\varphi\|_X + \varepsilon) = V(t_0), \quad i, j \in J.
\]

Afterward, for all \( t > t_0 \), one can certificate that

\[
\|Z(t)\| < V(t).
\]

If the hypothesis is not valid, we can choose \( i, j \in J \) and \( \theta \in (t_0, +\infty) \) follows that

\[
|Z_{ij}(\theta)| = V(\theta) = M(\|\varphi\|_X + \varepsilon) e^{-\sigma \ln \left( \frac{\tilde{d}_{ij}(t) + 1}{\tilde{d}_{ij}(t)} \right)},
\]

and

\[
|Z_{kl}(t)| < V(t) \quad \text{for all} \quad t \in [\rho_{kl}t_0, \theta], \quad k, l \in J.
\]
Furthermore,

\[
e^{\sigma \ln \left( \frac{1 + p_{kl} + z_{kl}(\nu)}{1 + p_{kl}} \right)} \left| z_{kl}(\nu) \right|
\]

\[
\leq e^{\sigma \ln \left( \frac{1 + p_{kl} + z_{kl}(\nu)}{1 + p_{kl}} \right)} \left| z_{kl}(\nu) - p_{kl}(\nu)z_{kl}(r_{kl}) \right|
\]

\[
+ e^{\sigma \ln \left( \frac{1 + p_{kl} + z_{kl}(\nu)}{1 + p_{kl}} \right)} \left| p_{kl}(\nu)z_{kl}(r_{kl}) \right|
\]

\[
= e^{\sigma \ln \left( \frac{1 + p_{kl} + z_{kl}(\nu)}{1 + p_{kl}} \right)} \left| p_{kl}(\nu) \right| e^{\sigma \ln \left( \frac{1 + p_{kl} + z_{kl}(\nu)}{1 + p_{kl}} \right)} \left| z_{kl}(r_{kl}) \right|
\]

\[
+ e^{\sigma \ln \left( \frac{1 + p_{kl} + z_{kl}(\nu)}{1 + p_{kl}} \right)} \left| Z_{kl}(\nu) \right|
\]

\[
\leq e^{\sigma \ln \left( \frac{1 + p_{kl} + z_{kl}(\nu)}{1 + p_{kl}} \right)} \sup_{s \in [r_{kl}p_{kl}t, r_{kl}t]} e^{\sigma \ln \left( \frac{1 + p_{kl} + z_{kl}(\nu)}{1 + p_{kl}} \right)} \left| z_{kl}(s) \right|
\]

\[
+ e^{\sigma \ln \left( \frac{1 + p_{kl} + z_{kl}(\nu)}{1 + p_{kl}} \right)} \left| Z_{kl}(\nu) \right|
\]

\[
\leq e^{\sigma \ln \left( \frac{1 + p_{kl} + z_{kl}(\nu)}{1 + p_{kl}} \right)} \sup_{s \in [r_{kl}t, t]} e^{\sigma \ln \left( \frac{1 + p_{kl} + z_{kl}(\nu)}{1 + p_{kl}} \right)} \left| z_{kl}(s) \right|
\]

\[
+ e^{\sigma \ln \left( \frac{1 + p_{kl} + z_{kl}(\nu)}{1 + p_{kl}} \right)} \left| Z_{kl}(\nu) \right|
\]

(17)

and

\[
e^{\sigma \ln \left( \frac{1 + p_{kl} + z_{kl}(\nu)}{1 + p_{kl}} \right)} \left| z_{kl}(t) \right| \leq \sup_{s \in [r_{kl}t, t]} e^{\sigma \ln \left( \frac{1 + p_{kl} + z_{kl}(\nu)}{1 + p_{kl}} \right)} \left| z_{kl}(s) \right|
\]

\[
\leq \frac{M(\|\phi\| + \varepsilon)}{1 - e^{\sigma \ln \left( \frac{1 + p_{kl} + z_{kl}(\nu)}{1 + p_{kl}} \right)} p_{kl}^{+}}
\]

(18)

where \( \nu \in [\rho_{kl}t_0, t_1], t \in [t_0, \theta], \ kl \in J \).

In view of (10), (13), (16), (18) and the fact that

\[
\left| Z_{ij}(\theta) \right|
\]

\[
= \left| Z_{ij}(t_0) - \int_{t_0}^{\theta} d_{ij}(u)du \right|
\]

\[
+ \int_{t_0}^{\theta} e^{\sigma \ln \left( \frac{1 + p_{kl} + z_{kl}(\nu)}{1 + p_{kl}} \right)} \left[ - a_{ij}(t)p_{ij}(t)z_{ij}(r_{ij}t) \right.
\]

\[
- \sum_{C_{kl} \in N_{r}(i,j)} C_{ij}^k(t) \left( f(x_{kl}(q_{kl}t))x_{ij}(t) - f(x_{kl}(q_{kl}t))x_{ij}^*(t) \right) \right] dt,
\]

we have

\[
\left| Z_{ij}(\theta) \right|
\]

\[
\leq \left| Z_{ij}(t_0) \right| K_{ij} e^{- \int_{t_0}^{\theta} \tilde{d}_{ij}(u)du}
\]

\[
+ \int_{t_0}^{\theta} e^{\sigma \ln \left( \frac{1 + p_{kl} + z_{kl}(\nu)}{1 + p_{kl}} \right)} \left[ - d_{ij}(t)p_{ij}(t)z_{ij}(r_{ij}t) \right.
\]

\[
- \sum_{C_{kl} \in N_{r}(i,j)} C_{ij}^k(t) \left( f(x_{kl}(q_{kl}t))x_{ij}(t) - f(x_{kl}(q_{kl}t))x_{ij}^*(t) \right) \right] dt
\]
\[ \leq (\| \varphi \| X + \varepsilon) K_{ij} e^{-\int_{0}^{\theta} \overline{d}_{ij}(u) du} + \int_{0}^{\theta} e^{-\int_{0}^{u} \overline{d}_{ij}(v) dv} K_{ij} \left[ |d_{ij}(t)p_{ij}(t)||z_{ij}(r_{ij}t)| + \sum_{C_{kl} \in \mathcal{N}_{x(i,j)}} |C^{kl}_{ij}(t)(|M^{f}_{1}z_{ij}(t)| + L^{f}(k + L)|z_{kl}(r_{kl}t)|) |ds \right] \]

\[ \leq (\| \varphi \| X + \varepsilon) K_{ij} e^{-\int_{0}^{\theta} \overline{d}_{ij}(u) du} + \int_{0}^{\theta} e^{-\int_{0}^{u} \overline{d}_{ij}(v) dv} K_{ij} \left[ |d_{ij}(t)p_{ij}(t)||e^{-\sigma \ln(1+r_{ij}t)} \frac{1}{1 - e^{-\sigma \ln(\frac{1}{r_{ij}t})}} p_{ij}^{+} \right. \]

\[ + L^{f}(k + L)e^{-\sigma \ln(1+q_{kl}t)} \frac{1}{1 - e^{-\sigma \ln(\frac{1}{q_{kl}t})}}] dt \]

\[ \times M(\| \varphi \| X + \varepsilon) e^{-\sigma \ln(\frac{1}{r_{ij}t})} \]

\[ = (\| \varphi \| X + \varepsilon) K_{ij} e^{-\int_{0}^{\theta} \overline{d}_{ij}(u) du} e^{-\sigma \ln(\frac{1+\theta}{1+0})} \]

\[ + \int_{0}^{\theta} e^{-\int_{0}^{u} \overline{d}_{ij}(v) dv} K_{ij} \left[ |d_{ij}(t)p_{ij}(t)| \right. \]

\[ \left. \left(\frac{1}{1 - e^{-\sigma \ln(\frac{1}{r_{ij}t})}} e^{\sigma \ln(\frac{1+\theta}{1+0})} \right) \right] dt \]

\[ \times M(\| \varphi \| X + \varepsilon) e^{-\sigma \ln(\frac{1}{r_{ij}t})} \]

\[ \leq (\| \varphi \| X + \varepsilon) K_{ij} e^{-\int_{0}^{\theta} \overline{d}_{ij}(u) du} e^{-\sigma \ln(\frac{1+\theta}{1+0})} \]

\[ + \int_{0}^{\theta} e^{-\int_{0}^{u} \overline{d}_{ij}(v) dv} K_{ij} \left[ |d_{ij}(t)p_{ij}(t)| \right. \]

\[ \left. \left(\frac{1}{1 - e^{-\sigma \ln(\frac{1}{r_{ij}t})}} e^{\sigma \ln(\frac{1+\theta}{1+0})} \right) \right] dt \]

\[ \times M(\| \varphi \| X + \varepsilon) e^{-\sigma \ln(\frac{1}{r_{ij}t})} \]

\[ \leq (\| \varphi \| X + \varepsilon) K_{ij} e^{-\int_{0}^{\theta} \overline{d}_{ij}(u) du} e^{-\sigma \ln(\frac{1+\theta}{1+0})} \]

\[ + \int_{0}^{\theta} e^{-\int_{0}^{u} \overline{d}_{ij}(v) dv} K_{ij} \left[ |d_{ij}(t)p_{ij}(t)| \right. \]

\[ \left. \left(\frac{1}{1 - e^{-\sigma \ln(\frac{1}{r_{ij}t})}} e^{\sigma \ln(\frac{1+\theta}{1+0})} \right) \right] dt \]

\[ \times M(\| \varphi \| X + \varepsilon) e^{-\sigma \ln(\frac{1}{r_{ij}t})} \]
This is a clear contradiction of the fact that $|Z_{ij}(\theta)| = M(\|\varphi\|_X + \varepsilon)e^{-\sigma \ln(\frac{1+\theta}{1+\theta})}$, and proves (14). Letting $\varepsilon \to 0^+$ yields
\[
\|Z(t)\| \leq M\|\varphi\|_X e^{-\sigma \ln(\frac{1+\theta}{1+\theta})} \text{ for all } t > t_0.
\] (19)

Then, through the proof methods of (17) and (18), similarly, we have (19), which indicates
\[
e^{-\sigma \ln(\frac{1+\theta}{1+\theta})}|z_{ij}(t)| \leq \sup_{s \in [p_j, t_0, t]} e^{-\sigma \ln(\frac{1+\theta}{1+\theta})}|z_{ij}(s)| \leq \frac{M\|\varphi\|_X}{1 - e^{-\sigma \ln(\frac{1+\theta}{1+\theta})} p_{ij}^+},
\]
and
\[
|z_{ij}(t)| \leq \frac{M\|\varphi\|_X}{1 - e^{-\sigma \ln(\frac{1+\theta}{1+\theta})} p_{ij}^+} \left(1 + \frac{t_0}{1 + t}\right)^\sigma, \forall t > t_0, \ ij \in J,
\]

This completes the proof of Theorem 2.1.

**Remark 2.1.** Clearly, one can see that all results in [18, 42] can be concluded from Theorem 2.1 in this present paper. At the same time, all results on the pseudo almost periodicity for (2) in [7] are special cases of our results. It is worth mentioning that we successfully corrected the mistakes appearing in (4). This implies that our results supplement some previously known researches in [7, 18, 42].

3. **An numerically illustrative example. Example 3.1.** Consider the following delayed SICNNs with $D$ operators:

\[
\begin{align*}
&[x_{ij}(t) - \frac{\sin(i + j)t}{50} + \frac{1}{100} e^{-t^2} x_{ij}(\frac{1}{t + j})']' \\
&= -a_{ij}(t)x_{ij}(t) - \sum_{C_{kl} \in N(i, j)} C_{ijkl}(t) \frac{1}{10} \arctan(x_{kl}(\frac{1}{k + 2l})x_{ij}(t)) + L_{ij}(t),
\end{align*}
\]

where

\[
C_{ijkl}(t) = 0.01 \cos 2t + 0.01 e^{-2|t|}, \quad L_{ij}(t) = 0.02 \sin t + 0.01 e^{-|t|}, \quad i, j = 1, 2,
\]

and

\[
\begin{bmatrix}
a_{11}(t) & a_{12}(t) \\
a_{21}(t) & a_{22}(t)
\end{bmatrix} = \begin{bmatrix}
0.8 + \cos 100t & 1 + 1.1 \sin 100t \\
0.8 + 1.3 \cos 100t & 1 + 1.2 \sin 100t
\end{bmatrix}.
\]

Take

\[
\begin{bmatrix}
\tilde{a}_{11} & \tilde{a}_{12} \\
\tilde{a}_{21} & \tilde{a}_{22}
\end{bmatrix} = \begin{bmatrix}
0.8 & 1 \\
0.8 & 1
\end{bmatrix},
\]

and
Figure 1. Numerical solutions of model (3.1), take the initial value (0.5,-0.7,1,-0.5), (0.6,-0.3,0.4,-0.6), (-0.5,0.7,-0.2,0.4).

Table 1. Numerical solutions of model (3.1), take the initial value (0.5,-0.7,1,-0.5), (0.6,-0.3,0.4,-0.6), (-0.5,0.7,-0.2,0.4).

Remark 3.1. It should be mentioned that, in the earlier publications, there are no similar results on weighted pseudo almost periodic dynamics of the model addressed in this paper. One can easily check that all the results in the [1, 5, 7, 8, 9, 18, 37, 38, 41, 42] cannot be straightly used to prove the fact that every solution of system (20) globally converges to WPAPS.

4. Conclusions. This paper mainly deals with the stability of weighted pseudo almost periodic solutions (WPAPS) for a class of neutral-type delayed SICNNs with D operator and multi-proportional delays. By a key role of fixed point theorem and some novel differential inequality techniques, some new criteria on the existence and exponential stability of WPAPS are established. These criteria extend and complement earlier ones. It should be pointed out that the method employed here provides us some new lights for investigating the weighted pseudo dynamics of other types of neural networks such as Cohen-Grossberg neural networks, high-order Hopfield neural networks, inertial neural networks involving D operator and unbounded time-varying delays. Further studies will be made in our next research.

Conflict of interests. We confirm that we have no conflict of interest and this paper hasn’t been published anywhere else.

REFERENCES

[1] N. S. Al-Islam, S. M. Alsulami and T. Diagana, Existence of weighted pseudo anti-periodic solutions to some non-autonomous differential equations, Applied Mathematics and Computation, 218 (2012), 6536–6548.
[2] A. Bouzerdoum and R. B. Pinter, Analysis and analog implementation of directionally sensitive shunting inhibitory cellular neural networks, in: Visual Information Processing: From Neurons to Chips, in: SPIE, (1991), 29–38.

[3] A. Bouzerdoum and R. B. Pinter, Nonlinear lateral inhibition applied to motion detection in the fly visual system, in: R.B. Pinter, B. Nabet (Eds.), Nonlinear Vision, CRC Press, Boca Raton, FL, (1992), 423–450.

[4] A. Bouzerdoum and R. B. Pinter, Shunting inhibitory cellular neural networks: Derivation and stability analysis, IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications, 40 (1993), 215–221.

[5] F. Chérif, Existence and global exponential stability of pseudo almost periodic solution for SICNNs with mixed delays, Journal of Applied Mathematics and Computing, 39 (2012), 235–251.

[6] D. Chen, W. Zhang, J. Cao and C. Huang, Fixed time synchronization of delayed quaternion-valued memristor-based neural networks, Advances in Difference Equations, 2020 (2020), Paper No. 92, 16 pp.

[7] Z. Chen and A. Zhang, Weighted pseudo almost periodic shunting inhibitory cellular neural networks with multi-proportional delays, Neural Processing Letters, 50 (2019), 1831–1843.

[8] T. Diagana, Weighted pseudo almost periodic functions and applications, Comptes Rendus Mathematique, 343 (2006), 643–646.

[9] T. Diagana, Weighted pseudo-almost periodic solutions to some differential equations, Nonlinear Analysis: Theory, Methods and Applications, 68 (2008), 2250–2260.

[10] J. K. Hale, Ordinary Differential Equations, Robert E. Krieger Publishing Co., Inc., Huntington, N.Y., Florida, 1980.

[11] H. Hu and X. Zou, Existence of an extinction wave in the fisher equation with a shifting habitat, Proceedings of the American Mathematical Society, 145 (2017), 4763–4771.

[12] H. Hu, T. Yi and X. Zou, On spatial-temporal dynamics of a Fisher-KPP equation with a shifting environment, Proceedings of the American Mathematical Society, 148 (2020), 213–221.

[13] H. Hu, X. Yuan, L. Huang and C. Huang, Global dynamics of an SIRS model with demographics and transfer from infectious to susceptible on heterogeneous networks, Mathematical Biosciences and Engineering, 16 (2019), 5729–5749.

[14] C. Huang and B. Liu, New studies on dynamic analysis of inertial neural networks involving non-reduced order method, Neurocomputing, 325 (2019), 283–287.

[15] C. Huang, L. Yang and J. Cao, Asymptotic behavior for a class of population dynamics, AIMS Mathematics, 5 (2020), 3378–3390.

[16] C. Huang, H. Zhang and L. Huang, Almost periodicity analysis for a delayed Nicholson's blowflies model with nonlinear density-dependent mortality term, Communications on Pure and Applied Analysis, 18 (2019), 3337–3349.

[17] C. Huang, B. Liu, X. Tian, L. Yang and X. Zhang, Global convergence on asymptotically almost periodic sircns with nonlinear decay functions, Neural Processing Letters, 49 (2019), 625–641.

[18] C. Huang, J. Cao, F. Wen and X. Yang, Stability analysis of sir model with distributed delay on complex networks, Plos One, 11 (2016), e0158813.

[19] C. Huang, Z. Yang, T. Yi and X. Zou, On the basins of attraction for a class of delay differential equations with non-monotone bistable nonlinearities, Journal of Differential Equations, 256 (2014), 2101–2114.

[20] C. Huang, H. Zhang, J. Cao and H. Hu, Stability and Hopf bifurcation of a delayed prey-predator model with disease in the predator, International Journal of Bifurcation and Chaos, 29 (2019), 1950091, 23 Pages.

[21] C. Huang, X. Long, L. Huang and S. Fu, Stability of almost periodic Nicholson’s blowflies model involving patch structure and mortality terms, Canadian Mathematical Bulletin, 63 (2020), 405–422.

[22] C. Huang, X. Long and J. Cao, Stability of anti-periodic recurrent neural networks with multi-proportional delays, Mathematical Methods in the Applied Sciences, 43 (2020), 6093–6102.

[23] C. Huang, X. Yang and J. Cao, Stability analysis of Nicholson’s blowflies equation with two different delays, Mathematics and Computers in Simulation, 171 (2020), 201–206.

[24] C. Huang, Y. Qiao, L. Huang and R. Agarwal, Dynamical behaviors of a food-chain model with stage structure and time delays, Advances in Difference Equations, 2018 (2018), Paper No. 186, 26 pp.
[25] C. Huang, S. Wen and L. Huang, Dynamics of anti-periodic solutions on shunting inhibitory cellular neural networks with multi-proportional delays, *Neurocomputing*, 357 (2019), 47–52.

[26] C. Huang, R. Su, J. Cao and S. Xiao, Asymptotically stable high-order neutral cellular neural networks with proportional delays and D operators, *Mathematics and Computers in Simulation*, 171 (2020), 127–135.

[27] S. Kumari, R. Chugh, J. Cao and C. Huang, On the construction, properties and Hausdorff dimension of random Cantor one pn set, *AIMS Mathematics*, 5 (2020), 3138–3155.

[28] X. Li, X. Yang and T. Huang, Persistence of delayed cooperative models: Impulsive control method, *Applied Mathematics and Computation*, 342 (2019), 130–146.

[29] X. Longn and S. Gong, New results on stability of Nicholson’s blowflies equation with multiple pairs of time-varying delays, *Applied Mathematics Letters*, 100 (2020), 106027, 6 pp.

[30] C. Qian and Y. Hu, Novel stability criteria on nonlinear density-dependent mortality Nicholson’s blowflies systems in asymptotically almost periodic environments, *Journal of Inequalities and Applications*, 2020 (2020), Paper No. 13.

[31] G. Rajchakit, A. Pratap, R. Raja, J. Cao, J. Alzabut and C. Huang, Hybrid control scheme for projective lag synchronization of riemann–liouville sense fractional order memristive bam neural networks with mixed delays, *Mathematics*, 7 (2019), 759.

[32] C. Song, S. Fei, J. Cao and C. Huang, Robust synchronization of fractional-order uncertain chaotic systems based on output feedback sliding mode control, *Mathematics*, 7 (2019), 599.

[33] Y. Tan, C. Huang, B. Sun and Tao. Wang, Dynamics of a class of delayed reaction-diffusion systems with Neumann boundary condition, *Journal of Mathematical Analysis and Applications*, 458 (2018), 1115–1130.

[34] Y. Tang, Pseudo almost periodic shunting inhibitory cellular neural networks with multi-proportional delays, *Neural Processing Letters*, 48 (2018), 167–177.

[35] J. Wang, C. Huang and L. Huang, Discontinuity-induced limit cycles in a general planar piecewise linear system of saddle-focus type, *Nonlinear Analysis: Hybrid Systems*, 33 (2019), 162–178.

[36] W. Wang, C. Huang, C. Huang, J. Cao, J. Lu and L. Wang, Bipartite formation problem of second-order nonlinear multi-agent systems with hybrid impulses, *Applied Mathematics and Computation*, 370 (2020), 124926, 17 pp.

[37] S. Xiao, Global Exponential Convergence of HCNNs with Neutral Type Proportional Delays and D Operator, *Neural Processing Letters*, 49 (2019), 347–356.

[38] Y. Xu, Exponential stability of weighted pseudo almost periodic solutions for HCNNs with mixed delays, *Neural Processing Letters*, 46 (2017), 507–519.

[39] D. Yang, X. Li and J. Qiu, Output tracking control of delayed switched systems via state-dependent switching and dynamic output feedback, *Nonlinear Analysis: Hybrid Systems*, 32 (2019), 294–305.

[40] X. Yang, X. Li, Q. Xi and P. Duan, Review of stability and stabilization for impulsive delayed systems, *Mathematical Biosciences and Engineering*, 15 (2018), 1495–1515.

[41] G. Yang and W. Wan, Weighted Pseudo Almost Periodic Solutions for Cellular Neural Networks with Multi-proportional Delays, *Neural Processing Letters*, 49 (2019), 1125–1138.

[42] A. Zhang, Almost periodic solutions for SICNNs with neutral type proportional delays and D operators, *Neural Processing Letters*, 47 (2018), 57–70.

[43] J. Zhang and C. Huang, Dynamics analysis on a class of delayed neural networks involving inertial terms, *Advances in Difference Equations*, 2020 (2020), Paper No. 120.

[44] Y. Zhou, X. Wan, C. Huang and X. Yang, Finite-time stochastic synchronization of dynamic networks with nonlinear coupling strength via quantized intermittent control, *Applied Mathematics and Computation*, 376 (2020), 125157.

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