Spin-Locality of Higher-Spin Theories and Star-Product Functional Classes

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Abstract

The analysis of spin-locality of higher-spin gauge theory is formulated in terms of star-product functional classes appropriate for the $\beta \to -\infty$ limiting shifted homotopy proposed recently in arXiv:1909.04876 where all $\omega^2C^2$ higher-spin vertices were shown to be spin-local. For the $\beta \to -\infty$ limiting shifted contracting homotopy we identify the class of functions $H^+0$, that do not contribute to the r.h.s. of HS field equations at a given order. A number of theorems and relations that organize analysis of the higher-spin equations are derived including extension of the Pfaffian Locality Theorem of arXiv:1805.11941 to the $\beta$-shifted contracting homotopy and the relation underlying locality of the $\omega^2C^2$ sector of higher-spin equations.

Space-time interpretation of spin-locality of theories involving infinite towers of fields is proposed as the property that the theory is space-time local in terms of original constituent fields $\Phi$ and their local currents $J(\Phi)$ of all ranks. Spin-locality is argued to be a proper substitute of locality for theories with finite sets of fields for which the two concepts are equivalent.
# Contents

1 Introduction .......................................................... 4

2 Free fields ............................................................. 3

3 Spin-locality and its space-time interpretation ............... 7
   3.1 Spinor space .................................................... 8
   3.2 Space-time interpretation .................................. 9

4 Nonlinear higher-spin equations .................................. 11

5 Perturbative analysis ................................................. 12

6 Star-product functions ............................................... 13
   6.1 Higher-spin algebra $\mathcal{H}$ .............................. 14
      6.1.1 Star product ........................................... 14
      6.1.2 Inequalities ........................................... 15
      6.1.3 Class $\mathcal{H}$ ........................................ 16
      6.1.4 Ideal $\mathcal{I}$ ........................................ 16
      6.1.5 $\mathcal{H}^{0+}$ and $\mathcal{H}^{+0}$ ....................... 17
   6.2 Invariant operations ........................................... 18
      6.2.1 $\gamma$ maps ........................................... 18
      6.2.2 Integration by parts ................................... 19

7 Shifted homotopy ...................................................... 19
   7.1 General setup .................................................. 19
   7.2 Shifted homotopy .............................................. 21
   7.3 Pfaffian Locality Theorem .................................... 24

8 Limiting contracting homotopy and Factorization Lemma .... 26
   8.1 Limiting contracting homotopy ................................ 26
   8.2 Factorization Lemma and limiting cohomology projector ... 27
      8.2.1 Factorization Lemma ................................... 27
      8.2.2 Limiting cohomology projector ....................... 28

9 Specific form degree relations .................................... 29
   9.1 Star products .................................................. 29
      9.1.1 $\mathcal{H}_0 \ast \mathcal{H}_0$ ................................ 30
      9.1.2 $\mathcal{H}_0 \ast \mathcal{H}_1$ and $\mathcal{H}_1 \ast \mathcal{H}_0$ ............... 31
      9.1.3 $\mathcal{H}_1 \ast \mathcal{H}_1$ .................................... 32
   9.2 Limiting contracting homotopy ................................ 33
      9.2.1 Contracting homotopy of $\mathcal{H}_1^{0\mu}$ ............ 33
      9.2.2 Space $\tilde{\mathcal{H}}_1^{0+}$ ............................... 34
9.2.3 Contracting homotopy of $\mathcal{H}_2^{\nu\mu}$

10 Pre-ultra-locality and ultra-locality
   10.1 Pre-ultra-locality
       10.1.1 Pre-ultra-local spaces
       10.1.2 Consequences
   10.2 Ultra-locality

11 Structure relation
   11.1 Summary
   11.2 The proof

12 Example: ultra-locality of holomorphic $\Upsilon_2(\omega, \omega, C, C)$

13 Conclusion

   Appendix A. Useful formulae
   Appendix B. Contracting homotopy derivation
1 Introduction

The most symmetric vacuum solution to nonlinear field equations for 4d massless fields of all spins of \( [3, 4] \) describes \( AdS_4 \). Due to the presence of dimensionful \( AdS_4 \) radius, higher-spin (HS) interactions can contain infinite tails of higher-derivative terms. This can make the theory non-local in the standard sense, raising the question which field variables lead to the local or minimally non-local setup in the perturbative analysis as was originally discussed in \([5]\). In \([6, 7, 8]\) it was shown how nonlinear HS equations of \([4]\) reproduce local current interactions in the lowest order in interactions. More recently, in \([2, 9]\) these results were reproduced and extended to some higher-order vertices by an appropriate modification of the conventional homotopy technics of \([1]\).

However, it was not clear how the homotopy technics should be further modified to lead directly to the proper local results in the higher orders of the perturbative analysis of HS equations until recently a new type of limiting shifted homotopy was introduced in \([1]\) allowing to extend the results of the previous work to the vertices up to the fifth order (in the action counting) in the sector of equations on the one-form gauge HS fields. The resulting vertices were shown to be spin-local which means that, as explained in \([2, 8, 1]\) and in this paper, a vertex is local in the spinor space for any given set of spins. Moreover, as argued below, spin-locality implies usual space-time locality in terms of combinations of field variables (like different currents for instance) associated with the primary fields both from the boundary and from the bulk perspective.

The new class of homotopy operators exhibits remarkable properties, partially studied in \([1]\). The aim of this paper is to extend this analysis using the language of classes of functions developed in \([10]\). This will allow us to greatly simplify the formalism factoring out the structures that do not contribute to the final result. In this setup, HS equations of \([4]\) provide an extremely powerful tool for the analysis of HS gauge theories directly in the bulk with no reference to \( AdS/CFT \) allowing a systematic computation of higher-order HS vertices. We prove useful lemmas that simplify the analysis of the HS equations in general and derive an important relation underlying locality of \( \omega^2 C^2 \) vertices computed in \([1]\).

There are many reasons why it is important to elaborate the intrinsic analysis of the HS gauge theory in the bulk with no reference to the holographic duals. The simplest is that apart from free boundary theories dual to particular HS gauge theories in the bulk \([1, 2]\), the latter equally well describe CFT dual interacting Chern-Simons boundary theories \([3, 4]\) where the computation of amplitudes is more involved. More general background solutions of HS theories with more complicated boundary duals like for instance massive deformations can also be of interest.

The approach proposed in this paper is applicable with slight modifications to HS theories in \( d = 3 \) \([3]\) and any \( d \) \([5]\) as well as to a more general class of Coxeter HS theories \([6]\) some of which were conjectured to be related to String Theory upon spontaneous breakdown of HS symmetries. Another class of problems where it can be useful includes exact solutions in HS theory like the HS black hole solutions of \([7, 8, 9]\). The results of \([1]\) and of this paper demonstrate great efficiency of the limiting homotopy approach to the analysis of
equations of [4]. We are not aware of any other means that could provide a comparably efficient computational scheme in HS gauge theory.

Let us now explain the organization of the rest of the paper highlighting the main results. We start by recalling by now standard material on free HS fields in Section 2. The concept of spin-locality and its space-time interpretation are discussed in Section 3. Nonlinear HS equations and general features of their perturbative analysis are recalled in Sections 4 and 5, respectively.

The construction of the class of star-product functions $\mathcal{H}$ introduced in [10] is recalled in Section 6 where it is further extended in a way appropriate for the limiting shifted homotopy approach. In particular, $\mathcal{H}$ is represented as a span of two subspaces $\mathcal{H}^{0+}$ and $\mathcal{H}^{+0}$ such that, by Factorization Lemma (8.9), elements of $\mathcal{H}^{+0}$ do not contribute to the dynamical equations at a given order in the limiting homotopy formalism. Also, we identify the two-sided ideal $\mathcal{I} = \mathcal{H}^{0+} \cap \mathcal{H}^{+0}$ elements of which can be discarded in all expressions containing HS gauge fields $\omega$ as not contributing to dynamical field equations within the $\beta \to -\infty$ limiting homotopy procedure.

The shifted homotopy formalism is recalled in Section 7. Namely, after recalling the general setup in Section 7.1, expressions for contracting homotopy and cohomology projector are presented in Section 7.2 for a general $\beta$-shift. In Section 7.3, Pfaffian Locality Theorem (PLT) of [2] is extended to the $\beta$-dependent contracting homotopies. This will be used later in Section 9.2.3 as the instrumental tool for the proof of ultra-locality of the vertices in question.

The limit $\beta \to -\infty$ is considered in Section 8. Namely, in Section 8.1 we derive the limiting contracting homotopy formulae which underly the Pre-Ultra-Locality Theorem of Section 9.2.3. In Section 8.2 the formula for limiting cohomology projector is derived from which simple but important Factorization Lemma follows showing that elements of $\mathcal{H}^{+0}$ do not contribute to field equations on physical fields.

A priori, application of the limiting homotopy prescription to general elements of $\mathcal{H}$ may not be well defined leading to the one-forms $W$ divergent in the $\beta \to -\infty$ limit. This would not imply any divergency in the HS equations, the perturbative analysis of which is well defined for any finite $\beta < 1$, but rather the inapplicability of the limiting homotopy indicating potential non-locality of the theory. Hence, for the analysis of locality, it is important to have a sufficient criterion guaranteeing that this does not happen. Details of this analysis depend on a degree of the differential forms in the anticommuting spinorial differentials $\theta$.

Specificities of the spaces $\mathcal{H}_p$ of $p$-forms in $\theta$ are studied in Section 9. In Section 9.1 we collect useful formulae on star products of elements of various spaces $\mathcal{H}_p$ up to terms in the ideal $\mathcal{I}$. Then in Section 9.2.1 we analyse properties of the limiting homotopy applied to $\mathcal{H}_1$. It is shown that, generally, application of the limiting homotopy to $\mathcal{H}_1^{0+}$ can lead to infinity and a subspace $\tilde{\mathcal{H}}_1 \subset \mathcal{H}_1$ is identified in Section 9.2.2, such that application of the limiting homotopy to $\tilde{\mathcal{H}}_1$ has a well defined limit $\beta \to -\infty$. In turn, in Section 9.2.3 it is shown that $\Delta_{q,-\infty} \mathcal{H}_2^{+0} \subset \mathcal{H}_1$. It is also shown here that the $y$-dependent part of $\Delta_{q,-\infty} \mathcal{H}_2^{+0}$ belongs to $\mathcal{H}_1^{+0}$ which does not contribute to the field equations by Factorization Lemma.
In Section 10, we introduce the notions of ultra-locality (originally introduced in [9]) and pre-ultra-locality, that underly the analysis of spin-locality of HS equations. It is shown here that the contribution resulting from $\Delta_{q,-\infty}{\mathcal H}_{2}^{+0}$ not only is well defined in the limit $\beta \to -\infty$ but is also pre-ultra-local that, in accordance with PLT, guarantees ultra-locality in the second order in HS zero-forms $C$.

In Section 11 we prove the relation underlying analysis of HS vertices in the second order in zero-forms $C$. Structure Relation considered in Section 11 proves that the r.h.s. of the second-order part of nonlinear HS equations is indeed in $\mathcal{H}_{2}^{+0}$ meeting the conditions of Pre-Ultra-Locality Theorem. This implies that the part of the vertex bilinear in the zero-forms $C$ is ultra-local.

Finally in Section 12 the efficiency of the developed methods is illustrated by an elementary computation-free proof of spin-locality of the holomorphic part of the vertex $\Upsilon_{2}(\omega,\omega,C,C)$ evaluated in [1].

Conclusions are in Section 13. Some useful formulae are collected in Appendix A. Details of the derivation of the $\beta$-dependent shifted contracting homotopy are given in Appendix B.

2 Free fields

The formulation of [4] uses the language of spinors. Its relation to the conventional setup in terms of space-time derivatives is via unfolded equations as we briefly recall now.

Unfolded equations of 4d massless Fronsdal [20, 21] fields of all spins $s = 0, 1/2, 1, 3/2, 2, \ldots$ in $AdS_{4}$ are formulated in terms of a one-form $\omega(Y; K|x) = dx^a\omega_a(Y; K|x)$ and zero-form $C(Y; K|x)$ [22]. Klein operators $K = (k, \bar{k})$ satisfy

$$ky^a = -y^ak, \quad k\bar{y}^\dot{a} = \bar{y}^\dot{a}k, \quad \bar{k}y^a = y^a\bar{k}, \quad \bar{k}\bar{y}^\dot{a} = -\bar{y}^\dot{a}\bar{k}, \quad kk = \bar{k}\bar{k} = 1, \quad k\bar{k} = \bar{k}k. \quad (2.1)$$

To describe massless fields, the one-form $\omega(Y; K|x)$ and zero-form $C(Y; K|x)$ should be, respectively, even and odd in $k, \bar{k}$. As a result, massless fields are doubled

$$C(Y; K|x) = C^{1,0}(Y|x)k + C^{0,1}(Y|x)\bar{k}, \quad \omega(Y; K|x) = \omega^{0,0}(Y|x) + \omega^{1,1}(Y|x)k\bar{k}. \quad (2.2)$$

Unfolded field equations for free massless fields of all spins in $AdS_{4}$ are [22]

$$R_{1}(Y; K|x) = \frac{i}{4}(\bar{\eta}H^{\dot{\alpha}\dot{\beta}}A_{\alpha\beta} - \partial^2_{y^a\bar{y}^\dot{a}}C(0, \bar{y}; K|x) + \bar{\eta}H^{\alpha\beta}\partial^2_{y^a\bar{y}^\dot{a}}C(y, 0; K|x)k), \quad (2.3)$$

$$\bar{D}C(Y; K|x) = 0, \quad (2.4)$$

$A_{\alpha\beta}$ is a Majorana spinor index while $\alpha = 1, 2$ and $\dot{\alpha} = 1, 2$ are two-component ones raised and lowered by $\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}, \varepsilon_{12} = 1$: $A^\alpha = \varepsilon^{\alpha\beta}A_{\beta}$, $\bar{A}_{\alpha} = A^\beta\varepsilon_{\beta\alpha}$ and analogously for dotted indices.
where $\eta$ is a free phase parameter and

$$R_1(Y; K|x) := D^{ad} \omega(Y; K|x) := D^L \omega(Y; K|x) + \lambda h^{\alpha\beta}(y_\alpha \frac{\partial}{\partial y^\beta} + \frac{\partial}{\partial y^\beta} y_\alpha) \omega(Y; K|x),$$

(2.5)

$$\bar{D}C(Y; K|x) := D^L C(Y; K|x) - i \lambda h^{\alpha\beta}(y_\alpha \bar{y}^\beta - \frac{\partial^2}{\partial y^\alpha \partial y^\beta}) C(Y; K|x),$$

(2.6)

$$D^L f(Y; K|x) := d_x f(Y; K|x) + (\omega^{\alpha\beta} y_\alpha \frac{\partial}{\partial y^\beta} + \omega^{\alpha\beta} y_\alpha \frac{\partial}{\partial y^\beta}) f(Y; K|x), \quad d_x := dx^n \frac{\partial}{\partial x^n}. \quad (2.7)$$

Background $AdS_4$ space of radius $\lambda^{-1} = \rho$ is described by a flat $sp(4)$ connection $w = (w_{\alpha\beta}, \omega_{\alpha\beta}, h_{\alpha\beta})$ containing Lorentz connection $w_{\alpha\beta}$, $\omega_{\alpha\beta}$ and vierbein $h_{\alpha\beta}$, that obey

$$d_x w_{\alpha\beta} + w_{\alpha\gamma} w_{\beta}\gamma - \lambda^2 H_{\alpha\beta} = 0, \quad d_x \omega^{\alpha\beta} + \omega^{\alpha\gamma} \omega^{\beta}\gamma - \lambda^2 \bar{\omega}_{\alpha\beta} = 0, \quad d_x h_{\alpha\beta} + w_{\alpha\gamma} h_{\beta}\gamma + \omega_{\beta}\delta h_{\alpha\delta} = 0. \quad (2.8)$$

Here $H^{\alpha\beta} := h^{\alpha\hat{\alpha}} h^{\hat{\beta}\beta}$ and $\bar{\omega}^{\alpha\beta} := h^{\alpha\hat{\alpha}} h^{\hat{\beta}\beta}$ are the frame two-forms (wedge symbol is omitted).

In the massless sector, system (2.3), (2.4) decomposes into subsystems of different spins, with a spin $s$ described by the one-forms $\omega(y, \bar{y}; K|x)$ and zero-forms $C(y, \bar{y}; K|x)$ obeying

$$\omega(\mu y, \mu \bar{y}; K|x) = \mu^{2(s-1)} \omega(y, \bar{y}; K|x), \quad C(\mu y, \mu^{-1} \bar{y}; K|x) = \mu^{\pm 2s} C(y, \bar{y}; K|x), \quad (2.9)$$

where $+$ and $-$ correspond to helicity $h = \pm s$ selfdual and anti-selfdual parts of the generalized Weyl tensors $C(y, \bar{y}; K|x)$. For spins $s \geq 1$, equation (2.3) expresses the Weyl zero-forms $C(Y; K|x)$ via gauge invariant combinations of derivatives of the HS gauge connections. The primary-like Weyl zero-forms are just the holomorphic and antiholomorphic parts $C(y, 0; K|x)$ and $C(0, \bar{y}; K|x)$ which appear on the r.h.s. of Eq. (2.3). Those associated with higher powers of auxiliary variables $y$ and $\bar{y}$ describe on-shell nontrivial combinations of derivatives of the generalized Weyl tensors as is obvious from Eqs. (2.4), (2.6) relating second derivatives in $y, \bar{y}$ to the $x$ derivatives of $C(Y; K|x)$ of lower degrees in $Y$. Hence, higher derivatives in the nonlinear system hide in the components of $C(Y; K|x)$ of higher orders in $Y$. To see whether the resulting equations are local or not at higher orders one has to inspect the dependence of vertices on the higher components of $C(Y; K|x)$.

At the linearized level, Eq. (2.6) implies that $\frac{\partial}{\partial x}$ is equivalent to $\bar{\omega}^{\alpha\beta} \frac{\partial}{\partial y^\beta}$. Hence, at this level the analysis of spin-locality in terms of $y, \bar{y}$ variables is equivalent to that in terms of space-time derivatives. However in higher orders Eq. (2.6) acquires nonlinear corrections making the relation between the two formalisms less straightforward but still tractable as we explain now.

3 Spin-locality and its space-time interpretation

Non-linear corrections to unfolded equations on physical fields $\omega(Y)$ and $C(Y)$ extending Central-on-shell theorem (2.3), (2.4) to higher orders can be packed into the form

$$d_x \omega = -\omega \ast \omega + \Upsilon_1(\omega^2, C) + \Upsilon_2(\omega^2, C^2) + \ldots + \Upsilon_n(\omega^2, C^n) + \ldots, \quad (3.1)$$

$$d_x C = -[\omega, C] + \Upsilon_1^C(\omega, C^2) + \Upsilon_2^C(\omega, C^3) + \ldots + \Upsilon_n^C(\omega, C^n) + \ldots, \quad (3.2)$$

7
where * is the Moyal star product acting on the commuting spinor variables $Y$ in $\omega(Y; K|x)$ and $C(Y; K|x)$

$$
(f \ast g)(Y) = \int \frac{d^4U \, d^4V}{(2\pi)^4} \exp [iU^AV^BC_{AB}] f(Y + U)g(Y + V),
$$

where $C_{AB} = (\epsilon_{\alpha\beta}, \bar{\epsilon}_{\dot{\alpha}\dot{\beta}})$ is the 4d charge conjugation matrix and $U^A, V^B$ are real integration variables.

### 3.1 Spinor space

Since the spinor sector of HS equations is of fundamental importance all concepts in HS theory including locality have to admit a proper interpretation in these terms. Therefore, we regard the spin-locality of the HS theory as a fundamental concept. On the other hand, as we explain now, spin-locality in spinor variables has clear space-time interpretation leading to proper concept of minimal space-time non-locality in presence of infinite towers of fields.

As explained in [2], general exponential representation for the order-$n$ corrections in the zero-forms $C$ can be put into the form

$$
\sum_{pp} \int d\tau \hat{P}_{pp}^p(y, \bar{y}, p, \bar{p}, \tau) \hat{E}_{pp}^p(y, \bar{y}, p, \bar{p}, \tau) C(Y_1; K) \ldots C(Y_n; K)|_{Y_j=0},
$$

where $p^i_\alpha := -i \frac{\partial}{\partial y^i_\alpha}$, $\bar{p}^{\dot{i}}_{\dot{\alpha}} := -i \frac{\partial}{\partial \bar{y}^{\dot{i}}_{\dot{\alpha}}}$,

$$
\hat{P}_{pp}^p(y, \bar{y}, p, \bar{p}, \tau) \text{ is some polynomial of } y, \bar{y}, p^i \text{ and } \bar{p}^{\dot{i}} \text{ with coefficients being regular functions of some homotopy integration parameters } \tau, \text{ and}
$$

$$
\hat{E}_{pp}^p = \hat{E}_{pp}^p \hat{\bar{E}}_{pp}^p, \quad \hat{E}_{pp}^p(\hat{B}, \hat{P}, p|z, y) = \exp i \left( -\hat{B}_j(\tau)p^i_\alpha y^a_\alpha + \frac{1}{2} \hat{P}_{ij}(\tau)p^{a\alpha}p^{j\dot{\alpha}} \right) k^p,
$$

where $p = 0, 1$ while $\hat{B}_j(\tau)$ and $\hat{P}_{ij}(\tau) = -\hat{P}_{ji}(\tau)$ are some $\tau$-dependent coefficients.

Spin-locality of HS interactions is governed by the coefficients $\hat{P}_{ij}$ in $\hat{E}_{n}^p$ and $\hat{\bar{P}}_{ij}$ in $\hat{\bar{E}}_{n}$ that determine contractions between, respectively, undotted and dotted spinor arguments of different factors of $C(Y_j; K|x)$. Since the contribution of $\hat{P}_{ij}$ and $\hat{\bar{P}}_{ij}$-dependent terms is via the exponential it gives rise to a non-polynomial expansion in $p^i_\alpha p^{i\alpha}$ and $\bar{p}^{\dot{i}\dot{\alpha}} \bar{p}^{\dot{j}\dot{\alpha}}$ and, hence, via (2.4) and (2.6), to non-local expansion in space-time derivatives. In all available examples nonlinear corrections to HS equations have the form (3.4), (3.6) where at least one of the coefficients $\hat{P}_{ij}(\tau)$ and $\hat{\bar{P}}_{ij}(\bar{\tau})$ is non-zero. This is a manifestation of the fact that in presence of an infinite tower of HS fields the full theory must contain infinite tower of higher derivatives.

A less trivial question is on the locality of vertices involving particular spins $s_1, \ldots, s_n$. In accordance with (2.9), for fixed helicities, the degree in $y_i$ variables in $C(Y_i; K|x)$ is related to
that in $\bar{y}_i$. In that case the degree in $p^i\alpha p^j\bar{\alpha}$ gets related to that in $\bar{p}^i\bar{\alpha} \bar{p}^j\bar{\alpha}$ in a particular vertex. As a result, for vertices with fixed spins, polynomiality in $p^i\alpha p^j\bar{\alpha}$ implies polynomiality in $\bar{p}^i\bar{\alpha} \bar{p}^j\bar{\alpha}$ and vice versa. Hence spin-locality for any fixed set of spins is achieved if at least one of the coefficients $\hat{P}_{ij}$ or $\hat{\bar{P}}_{ij}$ is zero for any $i,j$. Once this happens in all orders, this is equivalent to all-order spin-locality of HS equations. In [9] and [1] it was shown that this happens for all $\omega C^2$ and $\omega^2 C^2$ vertices, respectively.

Equivalently, the spin-locality condition in 4d HS theory is that the rank of the second derivative matrices contracting indices between any pair of zero-forms $C(Y_j)$ does not exceed 2.

### 3.2 Space-time interpretation

Unfolded HS equations acquire nonlinear corrections (3.1), (3.2). In the lowest order, these are interactions with currents which, in the sector of zero-forms, have the structure

$$d_\chi C = -[\omega, C]_s + \sum_{s,s_1,s_2} J^s_{s_1s_2}(Y; K|x) + \ldots ,$$

(3.7)

where, schematically, $J^s_{s_1s_2} = J^s_{s_1s_2}[C_{s_1}, C_{s_2}]$ denotes the spin-$s$ current bilinear in the massless fields of spins $s_1$ and $s_2$. These currents are local [1] [4]. However, their appearance affects the relation between the fields $C$ and space-time derivatives which can schematically be written in the form

$$D^L C^s(Y; K|x) = i\lambda h^{\alpha\beta}(y_\alpha \bar{y}_\beta - \frac{\partial^2}{\partial y^\alpha \partial \bar{y}^\beta}) C^s(Y; K|x) + \sum_{s_1,s_2=0}^\infty J^s_{s_1s_2}(Y; K|x) + \ldots .$$

(3.8)

with $D^L$ (2.7).

In the nonlinear HS theory discussed in this paper the currents $J$ are built from the constituent fields $C$. As follows from the general properties of the unfolded equations [23], they form modules over space-time symmetry algebra and can be interpreted as independent current fields as well [24]. It is important that current corrections to spin-local field equations contain at most a finite number of descendants of the primary current components in the equations on the primary-like components of $C^s$. (This is most appropriately formulated in terms of $\sigma_-$-cohomology associated with primary-like dynamical fields and l.h.s.’s of their field equations [25] [26].)

Eq. (3.8) means that the interpretation of the components of $C$ in terms of space-time derivatives acquires $J$-dependent corrections. Plugging these into equation (3.8) will bring corrections to the space-time equations of the structure

$$L^{FR}_s = \sum_{s_1,s_2=0}^\infty J^s_{s_1s_2}[\phi_{s_1}, \phi_{s_2}] + \sum_{i=0}^\infty \sum_{s_1,s_2,s_3} J^s_{s_1t}[C_{s_1}, J^t_{s_2s_3}] + \ldots ,$$

(3.9)

where $\phi^s$ denotes a spin-$s$ Fronsdal field, $L^{FR}_s$ is the l.h.s. of free Fronsdal equations and, for a given $s$, at most a finite number of components (descendants) of every current
contribute. Once currents in corrections to Fronsdal equations are treated as independent fields (corresponding to independent operators of the boundary operator algebra) these terms are still local containing a finite number of derivatives of each current.

If instead, expressions for currents \( J_{s_2s_3}^t = J_{s_2s_3}^t [C_{s_2}, C_{s_3}] \) in terms of constituent fields are plugged into (3.9), this can lead to expressions with an arbitrary number of derivatives of the constituent fields because of the infinite summation over the spin label \( t \). This is because the same constituent fields \( C \) contribute to currents of different spins. For instance, two spin-zero fields generate currents \( J_{00}^t [C_0, C_0] \) of any spin \( t \) where the number of derivatives of \( C_0 \) increases with \( t \). By this mechanism, formula (3.9) can bring corrections with the infinite number of derivatives of the same constituent fields. Clearly, this phenomenon is specific for theories containing infinite towers of fields allowing an infinite summation over \( t \) in (3.9). For theories with a finite number of fields the concepts of locality and spin-locality are equivalent.

The class of spin-local HS theories obviously leads to space-time theories of the type explained above. Indeed, if at every next order of perturbation theory the field equations acquire local higher-current corrections in terms of the genuine fields \( \omega \) and \( C \), at higher orders these may lead to space-time non-local corrections in terms of constituent fields via the mechanism illustrated by (3.9) which however will be still spin-local in terms of higher currents.

On the other hand, if the r.h.s. of field equations cannot be expressed in terms of local expressions of the constituent fields and all associated currents, such a theory is not spin-local and should be treated as essentially non-local. For instance, this would happen if the expressions for bilinear currents \( J_{s_1s_2}^s [C_{s_1}, C_{s_2}] \) were nonlocal. This can happen, in particular, as a result of application of a non-local field redefinition to a spin-local (HS) theory.

To summarize, spin-locality implies that corrections to the space-time equations can contain infinite tails of derivatives of the constituent fields but acquire local form once reformulated in terms of local currents and their further generalizations as independent objects. In fact, the currents \( J_{s_1s_2}^s [C_{s_1}, C_{s_2}] \) correspond to proper conformal fields in the \( AdS/CFT \) picture and lead themselves to higher currents that appear on the r.h.s. of the their field equations which are the conservation conditions that acquire non-linear corrections by virtue of completion of the unfolded equations in the higher orders. This happens if currents and their higher extensions have local form in terms of lower fields and currents analogously to how usual currents \( J_{s_1s_2}^s [C_{s_1}, C_{s_2}] \) are local in terms of constituent fields \( C \). If HS equations are taken in the non-local frame where the nonlinear corrections are not spin-local the corrections to HS equations may contain contributions built from infinite chains of derivatives of the constituent fields and currents. These can contain infinite number of terms built of \( C \) that may or may not at all be interpretable in terms of local currents and their descendants. The resulting essentially non-local field equations may not allow meaningful interpretation both from the bulk and from the boundary perspective.

Let us note that the higher currents have clear meaning in terms of homological resolution underlying unfolded formulation of HS equations. For instance, local currents \( J_2 \) that appear on the r.h.s. of 4d massless field equations are primary fields of a rank-two module of the
4d HS algebra \([27]\). The latter can themselves be interpreted as 6d massless fields \([24]\). As such, they admit currents built from 6d massless fields that, in turn, can be interpreted as rank-four currents \(J_4 = J_2 J_2\) from the 4d perspective, having local form in terms of \(J_2\). This process continues indefinitely as was observed in \([28]\).

Corrections (3.9) are somewhat reminiscent of the current-exchange-like contributions resulting from the holographic reconstruction \([29]\) picture while the spin-local corrections to the r.h.s. of the unfolded equations, which are local at any order, are analogues of the contact terms. In other words, spin-locality implies that unfolded equations acquire only spin-local corrections while the resulting space-time equations can acquire additional current-exchange-like corrections which, may be non-local in terms of constituent fields but have local form in terms of bilinear and higher currents associated with the boundary conformal fields.

It should be stressed that spin-local unfolded equations in which spin-locality is defined directly in the spinor space as suggested in \([9]\) and in this paper contain the full information about HS theory allowing to do computations with no reference to space-time picture at all, as was for instance demonstrated in \([30]\) where the boundary OPE was computed directly in terms of spinor space.

### 4 Nonlinear higher-spin equations

4d nonlinear HS equations \([4]\) have the form

\[
d_x \mathcal{W} + \mathcal{W} \ast \mathcal{W} = i(\theta^A \theta_A + \eta B \ast \gamma + \bar{\eta} B \ast \bar{\gamma}),
\]

\[
d_x B + \mathcal{W} \ast B - B \ast \mathcal{W} = 0,
\]

where

\[
\gamma = \theta^a \theta_a \kappa k, \quad \bar{\gamma} = \bar{\theta}^\dot{a} \bar{\theta}^\dot{a} \bar{\tau} \bar{k}.
\]

\(\mathcal{W}\) and \(B\) are fields of the theory which depend both on space-time coordinates \(x^n\) and on twistor-like variables \(Y^A = (y^a, \bar{y}^{\dot{a}})\) and \(Z^A = (z^\alpha, \bar{z}^{\dot{\alpha}})\). It is convenient to introduce anticommuting \(Z\)-differentials \(\theta^A, \theta^A \bar{\theta} = -\theta^B \theta^A\). \(B\) is a zero-form, while \(\mathcal{W}\) is the one-form with respect to both \(dx^n\) and \(\theta^A\) differentials, i.e., \(\mathcal{W} = (W, S)\), where \(W(Z; Y; K|x)\) is a space-time one-form, while \(S = \theta^A S_A(Z; Y; K|x)\). As a result, equation Eq. (4.1) contains three equations

\[
d_x W + W \ast W = 0,
\]

\[
d_x S + W \ast S + S \ast W = 0,
\]

\[
S \ast S = i(\theta^A \theta_A + \eta B \ast \gamma + \bar{\eta} B \ast \bar{\gamma}),
\]

while equation Eq. (4.2) gives

\[
d_x B + W \ast B - W \ast B = 0,
\]

\[
S \ast B = B \ast S.
\]
The $Y$ and $Z$ variables provide a realization of HS algebra through the noncommutative associative star product $\ast$ acting on functions of two spinor variables

$$
(f \ast g)(Z; Y) = \int \frac{d^{4}U \, d^{4}V}{(2\pi)^{4}} \exp \left[ iU^{A}V^{B}C_{AB} \right] f(Z + U; Y + U)g(Z - V; Y + V). \tag{4.9}
$$

$1$ is unity of the star-product algebra, i.e., $f \ast 1 = 1 \ast f = f$. Star product (4.9) provides a particular realization of the Weyl algebra

$$
[Y_{A}, Y_{B}]_{\ast} = -[Z_{A}, Z_{B}]_{\ast} = 2iC_{AB}, \quad [Y_{A}, Z_{B}]_{\ast} = 0, \quad [a, b]_{\ast} := a \ast b - b \ast a. \tag{4.10}
$$

The Klein operators satisfy relations analogous to (2.1) with $y^{\alpha} \rightarrow w^{\alpha} = (y^{\alpha}, z^{\alpha}, \theta^{\alpha})$, $\bar{y}^{\dot{\alpha}} \rightarrow \bar{w}^{\dot{\alpha}} = (\bar{y}^{\dot{\alpha}}, \bar{z}^{\dot{\alpha}}, \bar{\theta}^{\dot{\alpha}})$, which extend the action of the star product to the Klein operators. Decomposing master fields with respect to the Klein-operator parity, $A^{\pm}(Z; Y; K|x) = \pm A^{\pm}(Z; Y; -K|x)$, HS gauge fields are $W^{+}, S^{+}$ and $B^{-}$, while $W^{-}, S^{-}$ and $B^{+}$ describe an infinite tower of topological fields with every $AdS_{4}$ irreducible field describing at most a finite number of degrees of freedom. (For more detail see [1, 23]).

The left and right inner Klein operators

$$
\kappa := \exp iz_{\alpha}y^{\alpha}, \quad \bar{\kappa} := \exp i\bar{z}_{\dot{\alpha}}\bar{y}^{\dot{\alpha}}, \tag{4.11}
$$

which enter Eq. (1.3), change a sign of undotted and dotted spinors, respectively,

$$
(\kappa \ast f)(z, \bar{z}; y, \bar{y}) = \exp iz_{\alpha}y^{\alpha}f(y, \bar{z}; z, \bar{y}), \quad (\bar{\kappa} \ast f)(z, \bar{z}; y, \bar{y}) = \exp i\bar{z}_{\dot{\alpha}}\bar{y}^{\dot{\alpha}}f(z, \bar{y}; y, \bar{z}), \tag{4.12}
$$

$$
\kappa \ast f(z, \bar{z}; y, \bar{y}) = f(-z, \bar{z}; -y, \bar{y}) \ast \kappa, \quad \bar{\kappa} \ast f(z, \bar{z}; y, \bar{y}) = f(z, -\bar{z}; y, -\bar{y}) \ast \bar{\kappa}, \tag{4.13}
$$

$$
\kappa \ast \kappa = \bar{\kappa} \ast \bar{\kappa} = 1, \quad \kappa \ast \bar{\kappa} = \bar{\kappa} \ast \kappa, \tag{4.14}
$$

but commute with the differentials $\theta^{A}$.

A complex parameter $\eta = |\eta| \exp i\varphi, \varphi \in [0, \pi)$, parameterizes a class of pairwise nonequivalent nonlinear HS theories. The cases of $\varphi = 0$ and $\varphi = \frac{\pi}{2}$ correspond to so called $A$ and $B$ HS models that respect parity [31]. In the original paper [1] a more general class of models was considered with an arbitrary star-product function $F_{\ast}(B) = \eta_{1}B + \eta_{2}B \ast B + \ldots$ in place of the linear one $\eta B$. As argued in [2], the nonlinear terms in $F_{\ast}(B)$ are essentially non-local and hence have no obvious holographic duals. From the perspective of this paper, the non-locality of these terms is indeed obvious as is briefly discussed in Conclusion.

## 5 Perturbative analysis

Perturbative analysis of Eqs. (1.1), (1.2) assumes their linearization around some vacuum solution. The simplest one is

$$
W_{0}(Z; Y; K|x) = w(Y; K|x), \quad S_{0}(Z; Y; K|x) = \theta^{A}Z_{A}, \quad B_{0}(Z; Y; K|x) = 0, \tag{5.1}
$$
where $w(Y|x)$ is some solution to the flatness condition

$$d_x w + w * w = 0.$$  \hfill (5.2)

A flat connection $w(Y|x)$, that describes $AdS_4$ via (2.8), is bilinear in $Y^A$

$$w(Y|x) = -\frac{i}{4}(w^\alpha\beta(x) y_\alpha y_\beta + \bar{w}^{\dot{\alpha}\dot{\beta}}(x) \bar{y}_{\dot{\alpha}}\bar{y}_{\dot{\beta}} + 2h^{\alpha\dot{\beta}}(x) y_\alpha \bar{y}_{\dot{\beta}}).$$  \hfill (5.3)

Since $S_0$ has a trivial star-commutator with the Klein operators $K$, the star-commutator with $S_0$ produces De Rham derivative in $Z$-space

$$[S_0, F(Z; Y; K|x)]_* = -2i d_Z F(Z; Y; K|x), \quad d_Z := \theta^A \frac{\partial}{\partial Z^A}.$$  \hfill (5.4)

HS equations reconstruct the dependence on $Z^A$ in terms of the zero-form $C(Y; K|x)$ and one-form $\omega(Y; K|x)$ representing the $d_Z$-cohomological parts of $B$ and $W$, respectively,

$$B(Z; Y; K|x) = C(Y; K|x) + \sum_{j=2}^\infty B_j(Z; Y; K|x),$$  \hfill (5.5)

$$W(Z; Y; K|x) = \omega(Y; K|x) + \sum_{j=1}^\infty W_j(Z; Y; K|x),$$  \hfill (5.6)

where zero-forms $B_j(Z; Y; K|x)$ and one-forms $W_j(Z; Y; K|x)$ are of order $j$ in $\omega$ and $C$ and have zero projections to $d_Z$ cohomology

$$h_{d_Z}(B_j(Z; Y; K|x)) = 0, \quad h_{d_Z}(W_j(Z; Y; K|x)) = 0 \quad \forall j$$  \hfill (5.7)

with the projector $h_{d_Z}$ defined within the chosen homotopy procedure as discussed in Section 4.

The perturbative analysis goes as follows. Suppose that an order-$n$ solution

$$W^{(n)}(Z; Y; K|x) = \omega(Y; K|x) + \sum_{j=1}^n W_j(Z; Y; K|x),$$  \hfill (5.8)

$$B^{(n)}(Z; Y; K|x) = \sum_{j=1}^n B_j(Z; Y; K|x), \quad B_1(Z; Y; K|x) = C(Y; K|x)$$  \hfill (5.9)

is found. Then, plugging it into equations (4.5), (4.6), (4.8) that contain $S$ gives equations that determine the dependence on $Z$ in the next order while equations (4.4) and (4.7) turn out to be $Z$-independent as a consequence of the consistency of the system. These produce all nonlinear corrections to the unfolded HS equations in the form (3.1), (3.2).

6 Star-product functions

To simplify presentation, in this section we confine ourselves to the holomorphic sector of unbarred variables $z^\alpha$ and $y^\alpha$. Extension to the antiholomorphic sector is straightforward.
6.1 Higher-spin algebra \( \mathcal{H} \)

6.1.1 Star product

Analysis of locality is most convenient in terms of functions \( f(z, y, \theta) \) of the form

\[
\begin{align*}
f(z, y, \theta) &= \int d\tau \phi(\tau z, (1 - \tau)y, \tau \theta, \tau) \exp[i\tau z \alpha y^\alpha] \\
&= \int d^2 \tau \delta(1 - \tau_1 - \tau_2) \exp[i\tau_1 z \alpha y^\alpha] \phi(\tau_1 z, \tau_2 y, \tau_1 \theta, \tau_1),
\end{align*}
\]

where \( \tau \)-kernel \( \phi \) is defined as

\[
\begin{align*}
\phi(\tau_1 z, \tau_2 y, \tau_1 \theta, \tau_1) &= \phi^i(\tau_1 z, \tau_2 y, \tau_1 \theta, \tau_1) + \phi^b(\tau_1 z, \tau_2 y, \tau_1 \theta, \tau_1) \\
\phi^i(\tau_1 z, \tau_2 y, \tau_1 \theta, \tau_1) &= \frac{\tau_2}{\tau_1} \psi(\tau_1 z, \tau_2 y, 1 - \tau_1 \theta, \tau_1) \\
\phi^b(\tau_1 z, \tau_2 y, \tau_1 \theta, \tau_1) &= \delta(\tau_1) \chi_0(y) + \delta(1 - \tau_1) \theta^a \theta^a \chi_2(z)
\end{align*}
\]

with regular functions \( \psi(w, u, \xi, \tau_i) \), \( \chi_0(y) \) and \( \chi_2(z) \) such that the poles in \((6.3)\) in \( \tau_1 \) and \( \tau_2 \) are fictitious taking into account that \( z \)- and \( y \)-dependencies are accompanied with \( \tau_1 \) and \( \tau_2 \), respectively:

\[
\psi(0, y, 0, 0) = 0, \quad \epsilon^{\alpha\beta} \frac{\partial^2}{\partial \theta^\alpha \partial \theta^\beta} \psi(z, 0, \theta, 1) = 0.
\]

In the sequel we will distinguish between the inner \( \tau \)-kernels \( \phi^i \) \((6.2)\) and boundary ones \( \phi^b \) \((6.3)\). Note that the decomposition of \( \phi \) \((6.2)\) into inner and boundary parts in not unique due to the freedom in partial integration over \( \tau \) (see Section 6.2.2). An important consequence of \((6.2)\) is that all inner zero-forms in \( \theta \) contain a pre-exponential factor of \( \tau_2 \) while all inner two-forms in \( \theta \) contain a pre-exponential factor of \( \tau_1 \).

Functions of the form \((6.1)\) belong to the space of fields \( \mathcal{H} \) introduced in \((10)\)

\[
\mathcal{H} := \bigoplus_{p=0}^2 \mathcal{H}_p.
\]

Here \( \mathcal{H}_p \) is spanned by such \( p \)-forms in \( \theta \) \((6.1)\) that

\[
\lim_{\tau \to 0} \tau^{1-p+\epsilon} \phi^i(w, u, \theta, \tau) = 0, \quad \lim_{\tau \to 1} (1 - \tau)^{p-1+\epsilon} \phi^i(w, u, \theta, \tau) = 0 \quad \forall \epsilon > 0.
\]

The boundary functions \( \phi^b \) associated with \( \chi_0 \) and \( \chi_2 \) belong to \( \mathcal{H}_0 \) and \( \mathcal{H}_2 \), respectively.

Space \( \mathcal{H} \) has a number of important properties. As shown in \((10)\) and is explained below, it forms an algebra with respect to the star product. To see this it is convenient to use the following formula \((10)\):

\[
f_1 \ast f_2 = \int_0^1 d\tau_1 \int_0^1 d\tau_2 \int d^2 s d^2 t \exp[i(\tau_1 \circ \tau_2 z \alpha y^\alpha + s_\alpha t^\alpha - \tau_1 \circ \tau_2 (-y^\alpha + s_\alpha t^\alpha)) \times \phi_1(\tau_1((1 - \tau_2)z - \tau_2 y + s), (1 - \tau_1)((1 - \tau_2)y - \tau_2 z + s), \tau_1 \theta, \tau_1) \\
\times \phi_2(\tau_2((1 - \tau_1)z + \tau_1 y - t), (1 - \tau_2)((1 - \tau_1)y + \tau_1 z + t), \tau_2 \theta, \tau_2),
\]

(6.8)
where
\[ \tau_1 \circ \tau_2 = \tau_1 (1 - \tau_2) + \tau_2 (1 - \tau_1) . \] (6.9)

The product law \( \circ \) is commutative and associative. Note that \( 0 \leq \tau_1 \circ \tau_2 \leq 1 \) and \( 0 \leq 1 - \tau_1 \circ \tau_2 \leq 1 \),
\[ 1 - \tau_1 \circ \tau_2 = \tau_1 \tau_2 + (1 - \tau_1)(1 - \tau_2) . \] (6.10)

This follows from the simple observation that \( \tau_1 \circ \tau_2 \) and \( 1 - \tau_1 \circ \tau_2 \) can be visualized as areas of the diagonal and off-diagonal rectangles in the unite square cut by horizontal and vertical lines going through points with coordinates \( \tau_1 \) and \( \tau_2 \).

6.1.2 Inequalities

Functions
\[ \alpha_{11}(\tau) := \frac{\tau_1 \tau_2}{1 - \tau_1 \circ \tau_2}, \quad \alpha_{22}(\tau) := \frac{(1 - \tau_1)(1 - \tau_2)}{1 - \tau_1 \circ \tau_2}, \quad \alpha_{11}(\tau) + \alpha_{22}(\tau) = 1 , \] (6.11)
\[ \alpha_{12}(\tau) := \frac{\tau_1 (1 - \tau_2)}{\tau_1 \circ \tau_2}, \quad \alpha_{21}(\tau) := \frac{(1 - \tau_1)\tau_2}{\tau_1 \circ \tau_2}, \quad \alpha_{12}(\tau) + \alpha_{21}(\tau) = 1 \]

obey obvious inequalities
\[ 0 \leq \alpha_{ij}(\tau) \leq 1 . \] (6.12)

Also one can make sure that the following useful inequalities hold by virtue of (6.9)
\[ \tau_1 (1 - \tau_1) \leq \tau_1 \circ \tau_2 (1 - \tau_1 \circ \tau_2) , \quad \tau_2 (1 - \tau_2) \leq \tau_1 \circ \tau_2 (1 - \tau_1 \circ \tau_2) . \] (6.13)

Indeed, the first inequality follows from the elementary relation
\[ \tau_1 \circ \tau_2 (1 - \tau_1 \circ \tau_2) - \tau_1 (1 - \tau_1) = \tau_2 (1 - \tau_2)(1 - 2\tau_1)^2 \geq 0 . \]

Also, note that from associativity of the product \( \circ \) an infinite chain of inequalities follows
\[ \tau_1 \circ \tau_2 (1 - \tau_1 \circ \tau_2) \leq \tau_1 \circ \tau_2 \circ \tau_3 (1 - \tau_1 \circ \tau_2 \circ \tau_3) \leq \ldots . \] (6.14)
### 6.1.3 Class $\mathcal{H}$

With notations (6.11), $f_1 \ast f_2$ can be rewritten in the form

$$f_1 \ast f_2 = \int_0^1 d\tau_1 \int_0^1 d\tau_2 \int_0^1 d\tau_{1,2} \delta(\tau_{1,2} - \tau_1 \circ \tau_2) \int d^2sd^2t \exp i[\tau_{1,2}z_\alpha y^\alpha + s_\alpha t^\alpha]$$

(6.15)

$$\phi_1(\alpha_{12}\tau_1\tau_2 - \alpha_{11}(1 - \tau_{1,2})y + \tau_1s, \alpha_{22}(1 - \tau_{1,2})y + \alpha_{21}\tau_1\tau_2 + (1 - \tau_1)s, \tau_1\theta_1, \tau_1)$$

$$\phi_2(\alpha_{21}\tau_1\tau_2 + \alpha_{11}(1 - \tau_{1,2})y - \tau_1t, \alpha_{22}(1 - \tau_{1,2})y + \alpha_{12}\tau_1\tau_2 + (1 - \tau_2)t, \tau_2\theta_2, \tau_2).$$

For instance consider $f_1$ and $f_2$ with inner $\tau$-kernels (6.3). From (6.13) it follows that $f_1^b \ast f_2^b$ is also of the form (6.1) with inner $\tau$-kernel. (The dependence on $\alpha_{ij}(\tau_k)$ in (6.15) does not affect this conclusion thanks to inequalities (6.12).) Moreover, using that $[10]$

$$\int_0^1 d\tau_1 \int_0^1 d\tau_2 \delta(\tau_1 \circ \tau_2) = -\frac{1}{2} \log((1 - 2\tau)^2),$$

(6.16)

one finds following the same reference that the class of functions (6.2) remains invariant under the star product because $-\log((1 - 2\tau)^2)$ has simple zeros both at $\tau \to 0$ and at $\tau \to 1$. Let us stress that though formula (6.3) contains negative powers of $\tau$ or $1 - \tau$, r.h.s. of (6.15) contains no divergences since, as a consequence of (6.3), the negative powers of $\tau$ and $1 - \tau$ are compensated by the $\tau$-dependence of the $z$– or $y$–dependent terms.

Formula (6.13) simplifies if at least one of functions $f_1$, $f_2$ has a boundary $\tau$-kernel. Straightforwardly one can make sure that

$$f_1^b \ast f_2^b = f_1^b, \quad f_1 \ast f_2^b = f_1^b, \quad f_1^b \ast f_2^b = f_1^b.$$  

(6.17)

In [10], a subalgebra $\mathcal{H}^{loc} \subset \mathcal{H}$ was identified such that its elements have a milder dependence at $1 - \tau$. In this paper we find it convenient to denote the same algebra $\mathcal{H}^{0+}$. Namely, for $f \in \mathcal{H}^{0+}_{p}$ of the form (6.11) the condition

$$f \in \mathcal{H}^{0+}_p: \quad \exists \varepsilon > 0: \lim_{\tau \to 1}(1 - \tau)^{p-1-\varepsilon}\phi(w, u, \theta, \tau) = 0$$

(6.18)

is obeyed. This algebra has a number of interesting properties and was interpreted in [10] as the algebra of local field redefinitions in the theory. Its interpretation in this paper is similar.

### 6.1.4 Ideal $\mathcal{I}$

The new important point not discussed in [10] is that $\mathcal{H}$ contains an ideal $\mathcal{I}$ spanned by functions that have a polynomially softer behavior of $\tau$-kernels both at $\tau \to 0$ and at $\tau \to 1$. Namely,

$$f \in \mathcal{I}: \quad \exists \varepsilon > 0: \lim_{\tau \to 0}(1 - \tau)^{1-p-\varepsilon}\phi(w, u, \theta, \tau) = 0, \quad \lim_{\tau \to 1}(1 - \tau)^{p-1-\varepsilon}\phi(w, u, \theta, \tau) = 0.$$  

(6.19)

Note that the boundary functions with $\tau$-kernels $\phi^b$ (6.4) do not belong to $\mathcal{I}$.
To show that $I$ is a two-sided ideal of $H$ we use formula (6.8). Let $f_1 \in H$, $f_2 \in I$. Every element of $I$ contains an additional factor of $\tau^{\varepsilon'}(1-\tau)^{\varepsilon'}$ with some $\varepsilon' > 0$. Hence, the product (6.8) contains an additional factor of
\[ a(\tau_2) = \tau_2^{\varepsilon'}(1-\tau_2)^{\varepsilon'}. \] (6.20)

By virtue of (6.13)
\[ a(\tau_2) \leq (\tau_1 \circ \tau_2 (1-\tau_1 \circ \tau_2))^{\varepsilon'}. \] (6.21)

This implies that $f_1 \cdot f_2 \in I$. Thus $I$ is a left ideal in $H$. The proof that $I$ is also a right, and, hence, two-sided ideal is analogous. To complete the proof, one has to check this property for the boundary terms (6.4). This is elementary as well by virtue of (6.8).

6.1.5 $H^{0+}$ and $H^{+0}$

Elements of $H^{0+}$ obey condition (6.18). Analogously, we define $H^{+0}$ as the class of functions obeying
\[ f \in H^{+0}_p : \; \exists \varepsilon > 0 : \lim_{\tau \to 0} \tau^{1-p-\varepsilon}\phi(w, u, \theta, \tau) = 0. \] (6.22)

For boundary terms (6.4) we assign
\[ \chi_0(y) \in H_0^{0+}, \; \exp[iz_\alpha y^\alpha] \chi_2(z)\theta^\alpha \theta_\alpha \in H_2^{+0}. \] (6.23)

Clearly,
\[ I = H^{+0} \cap H^{0+}, \; H^{0+} := \sum_{p=0}^2 H_p^{0+}, \; H^{+0} := \sum_{p=0}^2 H_p^{+0}. \] (6.24)

It is not difficult to make sure that
\[ H^{0+} \ast H^{0+} \subset H^{0+}, \; H^{+0} \ast H^{+0} \subset H^{0+}, \] (6.25)
\[ H^{0+} \ast H^{+0} \subset H^{+0}, \; H^{+0} \ast H^{0+} \subset H^{+0}. \] (6.26)

These relations are in agreement with the facts that $H^{0+}$ forms a subalgebra of $H$ and $I$ forms a two-sided ideal of $H$.

Any $f(z, y, \theta) \in H$ can be decomposed as
\[ f(z, y, \theta) = f^{0+}(z, y, \theta) + f^{+0}(z, y, \theta), \quad f^{0+}(z, y, \theta) \in H^{0+}, \quad f^{+0}(z, y, \theta) \in H^{+0}. \] (6.27)

This is achieved by rewriting (6.4) in the form
\[ f(z, y, \theta) = \int d^2_+ \tau \delta(1-\tau_1 - \tau_2)(\tau_1 + \tau_2) \exp[i\tau_1 z_\alpha y^\alpha]\phi(\tau_1 z, \tau_2 y, \tau_1 \theta, \tau_1) \] (6.28)
giving
\[ f^{0+}(z, y, \theta) = \int d^2_+ \tau \delta(1-\tau_1 - \tau_2)\tau_2 \exp[i\tau_1 z_\alpha y^\alpha]\phi(\tau_1 z, \tau_2 y, \tau_1 \theta, \tau_1), \] (6.29)
\[ f(z, y, \theta) = \int d^2 \tau \delta(1 - \tau_1 - \tau_2) \tau_1 \exp[i \tau z \alpha] \phi(\tau z, \tau y, \tau \theta, \tau) . \] (6.30)

Note that plugging repeatedly \( \tau_1 + \tau_2 \) into these formulae and discarding elements of the ideal one can reach any powers of \( \tau_2 \) in (6.29) or \( \tau_1 \) in (6.30). Moreover, discarding terms in the ideal \( I \) one arrives at

\[ f^0(z, y, \theta) \simeq \int_0^\varepsilon d\tau \exp[i \tau z \alpha] \phi(\tau z, y, \tau \theta, \tau) , \] (6.31)

\[ f^+(z, y, \theta) \simeq \int_1^{1-\varepsilon} d\tau \exp[i \tau z \alpha] \phi(z, (1 - \tau) y, \theta, \tau) \] (6.32)

with any \( \varepsilon > 0 \) where equivalence \( \simeq \) is up to terms in \( I \). Indeed, all terms resulting from the integration over \( \tau \) in the region disconnected from 0 and 1 belong to \( I \).

### 6.2 Invariant operations

In this section we consider two more operations that map \( H \) to itself.

#### 6.2.1 \( \gamma \) maps

Operator \( \gamma \) (4.43) belongs to \( H \) and, hence,

\[ \gamma * f \in H , \quad f * \gamma \in H \quad \forall f \in H . \] (6.33)

This is because the multiplication with \( \gamma \) adds two powers of \( \theta \) due to multiplication with \( \theta^a \theta_b \) and exchanges \( z \) and \( y \) simultaneously replacing \( \tau \rightarrow 1 - \tau \) as a consequence of (4.12), (4.13). The star product with \( \gamma \) maps zero-forms in \( \theta \) to two-forms.

A less obvious fact is that star multiplication with \( \gamma \) admits inverse \( \gamma^{-1} \)

\[ \gamma^{-1}(f) := \frac{1}{2} \epsilon^{\alpha \beta} \frac{\partial^2}{\partial \theta^a \partial \theta^b} k * \kappa * f(z, y, k; \theta | x) \] (6.34)

that leaves invariant class \( H \)

\[ \gamma^{-1}(f) \in H \quad \forall f \in H . \] (6.35)

Since the multiplication by \( \gamma \) and application of \( \gamma^{-1} \) swaps \( \tau \leftrightarrow 1 - \tau \) both of these operations swap \( H^{0+} \) and \( H^{+0} \)

\[ \gamma * H^{0+} \subset H^{+0} , \quad \gamma * H^{+0} \subset H^{0+} , \] (6.36)

\[ \gamma^{-1}(H^{0+}) \subset H^{0+} , \quad \gamma^{-1}(H^{+0}) \subset H^{+0} . \] (6.37)

As a consequence of (6.24) both of them leave ideal \( I \) invariant

\[ \gamma * I \subset I , \quad \gamma^{-1}(I) \subset I . \] (6.38)
6.2.2 Integration by parts

Analysis of HS field equations sometimes involves integration by parts over the homotopy integration parameters. It is convenient to eliminate a pre-exponential factor of $z_\alpha y^\alpha$ by partial integration over the homotopy parameter as resulting from the $\partial / \partial \tau$ derivative of the exponential in (6.1). It is important to make sure that this operation leaves invariant classes $\mathcal{H}^{0+}$ and $\mathcal{H}^{+0}$.

Consider the following element of $\mathcal{H}$

$$f(z, y, \theta) = \int d\tau \theta(\tau) \theta(1 - \tau)(1 - \tau)^2 i z_\alpha y^\alpha \exp[i \tau z_\alpha y^\alpha] \psi(\tau z, (1 - \tau)y, \tau, \theta).$$  \hspace{1cm} (6.39)

It can be represented as

$$f(z, y, \theta) = -\int d\tau \partial / \partial \tau \left[ \theta(\tau) \theta(1 - \tau)(1 - \tau)^2 \psi(\tau z, (1 - \tau)y, \tau, \theta) \right] \exp[i \tau z_\alpha y^\alpha] \hspace{1cm} (6.40)$$

giving

$$f(z, y, \theta) = f^{0+}(z, y, \theta) + f^{+0}(z, y, \theta),$$  \hspace{1cm} (6.41)

where

$$f^{0+}(z, y, \theta) = -\psi(0, y, 0, 0)$$  \hspace{1cm} (6.42)

$$f^{+0}(z, y, \theta) = \tau_2^2 \psi(z, 0, \tau_2^{-1} \theta, 1) \bigg|_{\tau_2 = 0}$$  \hspace{1cm} (6.43)

If $f(z, y, \theta) \in \mathcal{H}^{+0}$ then the boundary part of $f^{0+}$ (6.42) is zero, while the inner one $\mathcal{H}^{0+}$ contains an additional degree of $\tau_1$. Analogously, if $f(z, y, \theta) \in \mathcal{H}^{0+}$ then $f^{+0}$ (6.43) contains an additional degree of $\tau_2$. As a result, the partial integration over the homotopy parameter preserves the classes $\mathcal{H}^{0+}$ and $\mathcal{H}^{+0}$ as well as the ideal $\mathcal{I}$ allowing to freely integrate by parts within a given class.

7 Shifted homotopy

7.1 General setup

To eliminate $Z$-variables one has to repeatedly solve equations of the form

$$d_Z f(Z; Y; K|x) = g(Z; Y; K|x)$$  \hspace{1cm} (7.1)
resulting from equations (4.5), (4.6), (4.8) that contain $S$. Here $g(Z; Y; K| x)$ is built from already determined lower-order fields $B_j$ (5.3) and $W_j$ (5.6). Consistency of HS equations guarantees formal consistency of Eq. (7.1)

$$d_Zg(Z; Y; K| x) = 0.$$ (7.2)

Given homotopy operator $\partial$

$$\partial^2 = 0,$$ (7.3)

the operator

$$A := \{d_Z, \partial\}$$ (7.4)

obeys

$$[d_Z, A] = 0, \quad [\partial, A] = 0.$$ (7.5)

For diagonalizable $A$, the standard Homotopy Lemma states that cohomology $H_{d_Z}$ of $d_Z$ is in the kernel of $A$

$$H_{d_Z} \subset \text{Ker} A.$$ (7.6)

In this case, it is possible to define such projector $h$ to $\text{Ker} A$

$$h^2 = h$$ (7.7)

and the operator $A^*$ that

$$[h, d_Z] = [h, \partial] = 0, \quad A^*A = AA^* = Id - h.$$ (7.8)

The contracting homotopy operator

$$\Delta := A^*\partial = \partial A^*$$ (7.9)

gives the resolution of identity

$$\{d_Z, \Delta\} + h = Id$$ (7.10)

allowing to find a solution to equation (7.1) with $d_Z$-closed $g$ outside $H_{d_Z}$ (i.e., obeying $\hat{h}g = 0$) in the form

$$f = \Delta g + d_Z\epsilon + c,$$ (7.11)

where an exact part $d_Z\epsilon$ and $c \in H_{d_Z}$ remain undetermined. These describe solutions to the homogeneous equation (7.1) with $g = 0$.

The form of the resulting solutions depends on a chosen contracting homotopy $\Delta$. The freedom in this choice affects both the $d_z$-exact and cohomological terms in (7.11). The freedom in $\epsilon$ affects the form of gauge transformations while the form of $c(\omega, C)$ induces perturbatively nonlinear field redefinitions. The problem is to single out a specific homotopy procedure that leads to the spin-local form of the field equations. In [2, 9] we have identified a shifted homotopy that solves the problem in the lowest non-trivial order in the zero-form sector. In [1] and in this paper this construction is extended further to the class of contracting homotopy operators allowing to solve the problem in higher orders as well.
7.2 Shifted homotopy

The conventional homotopy operator
\[ \partial = Z^A \frac{\partial}{\partial \theta^A} \]  

and contracting homotopy
\[ \triangle J (Z; Y; \theta) = Z^A \frac{\partial}{\partial \theta^A} \int_0^1 dt \frac{1}{t} J (tZ; Y; t\theta) \]

were used in the perturbative analysis of HS equations since [4]. Though being simple and looking natural, they are known to lead to non-localities beyond the free field level [12, 32, 33, 8].

An obvious freedom in the definition of homotopy operator (7.12) is to replace \( Z^A \) by \( Z^A + a^A \) with some \( Z \)-independent \( a^A \),
\[ \partial \rightarrow \partial_a = (Z^A + a^A) \frac{\partial}{\partial \theta^A}, \quad \frac{\partial}{\partial Z^A} (a^B) = 0. \]  

Contracting homotopy \( \triangle_a \) and cohomology projector \( h_a \) act as follows
\[ \triangle_a \phi (Z, Y, \theta) = \int_0^1 dt \frac{1}{t} (Z + a)^A \frac{\partial}{\partial \theta^A} \phi (tZ - (1 - t)a, t\theta), \quad h_a \phi (Z, Y, \theta) = \phi (-a, Y, 0). \]

\( \triangle_0 \) is conventional contracting homotopy (7.13). The resolution of identity has standard form
\[ \{d_Z, \triangle_a\} + h_a = Id. \]  

For instance, one can set \( a^A = cY^A \) with some constant \( c \). Naively, this exhausts all Lorentz covariant options for \( a^A \). However \( a^A \) can also be composed from the derivatives with respect to the arguments of \( \omega (Y; K) \) and \( C(Y; K) \) in \( g = g(\omega, C) \) (7.1).

Let
\[ \Phi^1 (Y; K) = \omega (Y; K), \quad \Phi^0 (Y; K) = C(Y; K). \]

Various terms on the r.h.s. of HS field equations contain products
\[ \Phi^n (Y; K) = \Phi^{a_1} (Y_1; K) \Phi^{a_2} (Y_2; K) \cdots \Phi^{a_n} (Y_n; K) |_{Y_i = Y}, \quad a \in \{a_1, \ldots, a_n\}, \quad a_i = 0, 1. \]

These products is useful to treat independently for different orderings of \( \omega \) and \( C \) since HS equations are known to remain consistent with all fields valued in any associative (say, matrix) algebra [22], in which case the fields are not commuting. This implies that the terms associated with different labels \( a \) can be treated as independent.

The simplest option used in [3, 9] is
\[ a^A = c_0 (a) Y^A + \sum_j c_j (a) \partial_j A, \quad a = \{a_1, \ldots, a_n\}, \]

21
where \( \partial_i A \) is the derivative with respect to the argument of the \( i \)th factor \( \Phi^a_i(Y_i; K) \). The class of shifts (7.19) was modified in [1] by replacing the \( Y_A \)-shift by the \( \frac{\partial}{\partial Y_A} \)-shift:

\[
a^A_a = i\beta(a) \frac{\partial}{\partial Y_A} + \sum_j c_j(a) \partial_j A, \quad a = \{a_1, \ldots, a_n\}. \tag{7.20}
\]

Note that it is hard to keep the \( Y_A \)- and \( \frac{\partial}{\partial Y_A} \)-shifts simultaneously because they do not commute and, hence, the resulting shifts \( a^A_a \) would be noncommuting that is not allowed.

In fact, formula (7.15) with shift (7.20) is not convenient for practical computations. The following integral representation for the shifted contracting homotopy [1] is more useful:

\[
\Delta_0,\beta f(z, y, \theta) = \int \frac{d^2 u d^2 v}{(2\pi)^2} \exp iv_\alpha u^\alpha \int_0^1 \frac{dt}{t} (z - u)^\alpha \frac{\partial}{\partial \theta^\alpha} f(tz + (1 - t)u, \beta v + y, t\theta). \tag{7.21}
\]

(To simplify formulae we confine ourselves to the sector of left spinors with undotted indices.) More generally, for any \( z, y \)-independent spinor \( q \),

\[
\Delta q,\beta f(z, y, \theta) := \int \frac{d^2 u d^2 v}{(2\pi)^2} \exp iv_\alpha u^\alpha \int_0^1 \frac{dt}{t} (z - u + q)^\alpha \frac{\partial}{\partial \theta^\alpha} f(tz + (1 - t)(u - q), \beta v + y, t, \theta). \tag{7.22}
\]

This shifted contracting homotopy obeys resolution of identity (7.16) with the cohomology projector

\[
h_{q,\beta}(f(z, y, \theta)) = \int \frac{d^2 u d^2 v}{(2\pi)^2} \exp iv_\alpha u^\alpha f(u - q, \beta v + y, 0). \tag{7.23}
\]

Note that application of formulae (7.21) and (7.23) to functions (6.1) leads to the Gaussian integration over \( u^\alpha \) and \( v^\alpha \) which, in turn, generates nontrivial Jacobian in the integration measure, that is hard to obtain in the differential definition (7.20). This Jacobian plays crucial role in the analysis of locality in [1] and in this paper.

Formulae (7.21) and (7.23) can be easily extended to the class of homotopies (7.20) containing shifts of arguments of various fields \( f(z, y, \theta) \) is built of. For instance, for

\[
f(z, y, \theta) = F(z, y, \frac{\partial}{\partial y_i}) \Phi(y_1, K) \ldots \Phi(y_k, K) \bigg|_{y_i=0} \tag{7.24}
\]

appropriate modifications of (7.21) and (7.23) result from the replacement of \( C(y_1) \ldots C(y_k) \) by \( C(c_1 v + y_1) \ldots C(c_k v + y_k) \).

Formula (7.23) yields for \( f(z, y, \theta) \) (6.1)

\[
h_{(1-\beta)q,\beta}(f) = \int_0^1 d\tau \zeta^{-2} \int \frac{d^2 u d^2 v}{(2\pi)^2} \exp i[v_\alpha u^\alpha + \tau(1 - \beta)\zeta^{-1} y_\alpha q^\alpha] \int \phi(\tau(\beta u - (1 - \beta)q)\zeta^{-1}, (1 - \tau)(v + y\zeta^{-1}), \tau),
\]

where

\[
\zeta := (1 - \beta \tau). \tag{7.25}
\]
Note that we use a normalized shift $q(1 - \beta)$, that naturally appears in the star-exchange procedure [1] (see also Appendix A).

The contracting homotopy with $q = 0$ was presented in [1]. Derivation of the expression for contracting homotopy with any $q$ sketched in Appendix B yields

$$\Delta_{(1-\beta)q,\beta}(f) = \int \frac{d^2u d^2v}{(2\pi)^2} \int d^3\tau \delta(1 - \sum_{i=1}^{3} \tau_i) \left[ \frac{(1 - \beta)\tau_1}{1 - \beta(1 - \tau_2)} \right]^{p-1} \exp \left[ i[v_\alpha u^\alpha + \tau_1 z_\alpha y^\alpha - \tau_2 q_\alpha y^\alpha] \right] \frac{(1 - \beta \tau_1)(z + q)^\beta - \beta \tau_3(u + q)^\beta}{1 - \beta(1 - \tau_2)} \frac{\partial}{\partial \theta^\beta}$$

$$\phi \left( \tau_1 z + \frac{\tau_2 \tau_3 \beta}{1 - \beta(1 - \tau_2)} u - \tau_2 q, v + \tau_3 y, \theta, \frac{1 - \tau_3 - \beta \tau_1}{1 - \beta(1 - \tau_2)} \right)$$

where

$$d^3\tau := d\tau_1 d\tau_2 d\tau_3$$

and $p$ is the degree of $f$ in $\theta$:

$$f(w, u, \mu \theta, \tau) = \mu^p f(w, u, \theta, \tau).$$

The last argument of $\phi$ in (7.27) results from the change of integration variables (B.7).

For inner functions $\phi^i$ the contracting homotopy takes the form

$$\Delta_{(1-\beta)i,\beta}(f) = \int \frac{d^2u d^2v}{(2\pi)^2} \int d^3\tau \delta(1 - \sum_{i=1}^{3} \tau_i) \exp \left[ i[v_\alpha u^\alpha + \tau_1 z_\alpha y^\alpha - \tau_2 q_\alpha y^\alpha] \right] \frac{(1 - \beta \tau_1)(z + q)^\beta - \beta \tau_3(u + q)^\beta}{1 - \beta(1 - \tau_2)} \frac{\partial}{\partial \theta^\beta}$$

$$\psi \left( \tau_1 z + \frac{\tau_2 \tau_3 \beta}{1 - \beta(1 - \tau_2)} u - \tau_2 q, v + \tau_3 y, \theta, \frac{1 - \tau_3 - \beta \tau_1}{1 - \beta(1 - \tau_2)} \right).$$

To simplify formulae in the sequel we will use notations

$$\Delta^{\prime}_{q,\beta} := \Delta_{(1-\beta)q,\beta}, \quad h^{\prime}_{q,\beta} := h_{(1-\beta)q,\beta}.$$  

Formulae (7.25), (7.27) and (7.30) contain nontrivial prefactors and rational dependence on the integration homotopy parameters $\tau$ resulting from the substitution of the dependence on $u$ and $v$ into the exponential factor in (6.1). The final expressions are well defined for

$$-\infty < \beta < 1.$$  

In particular, the potential divergency due to the factor of $\tau^{-1}$ in (6.2) does not contribute in (7.25) because of the factor of $\tau$ in the first argument of $\phi$. The seeming divergency due to the factor of $\tau_3^{-1}$ in (7.30) is compensated due to the second regularity condition (6.5).

Beyond this region, divergencies can appear due to the degeneracy of the quadratic form in the Gaussian integral. At $\beta = 0$, these formulae reproduce those of the conventional
contracting homotopy introduced in [4] (see also [23]). Let us note that, as shown in [1], the expression $\Delta'_{q,\beta}(\gamma)$ is $\beta$-independent

$$\Delta'_{q,\beta}(\gamma) = \Delta_{q,0}(\gamma)$$

(7.33)

that, along with star-exchange formulae (see Appendix A), imply that the analysis of the $\eta\bar{\eta}$ sector of HS equations turns out to be insensitive to $\beta$ and can be performed in particular at $\beta = 0$ as in [2, 9]. These vertices are also found in [1].

### 7.3 Pfaffian Locality Theorem

A class of shifted contracting homotopies introduced in [2] was shown to reduce the degree of non-locality in all orders of the perturbation theory provided that shifts obey certain conditions prescribed by the Pfaffian Locality Theorem (PLT). Properties of these contracting homotopies were studied in [9] where they were shown to reproduce local lower-order vertex $\Upsilon(\omega, C, C)$ found originally in [6, 8] provided that the PLT conditions are respected. Here we extend PLT to the $\beta$-shifted contracting homotopies.

In [2] we have identified odd and even classes of functions as follows. General exponential representation for order-$n$ corrections in the zero-forms $C$ has the form

$$\sum_{PP} \int d\tau P^{PP}_n E^{PP}_n(\tau) C(Y_1) \ldots C(Y_n) \big|_{Y_j=0},$$

(7.34)

where $P^{PP}_n$ is some polynomial of $z, y$ and $p^i$ (3.3) and their conjugates with coefficients being regular functions of the homotopy parameters $\tau$, and

$$E^{PP}_n = E^0_n E^{Pp}_n, \quad E^P_n(T, A, B, P, p|z, y) = \exp i(T z \gamma y^\gamma - A_j p^i_j z^\gamma - B_j p^j_i y^\gamma + \frac{1}{2} P_{ij} p^i_j p^j_i) k^p,$$

(7.35)

where $p = 0, 1$ and parameters $T \in \mathbb{C}, A, B \in \mathbb{C}^n, P_{ij} = -P_{ji} \in \mathbb{C}^n \times \mathbb{C}^n$ may be $\tau$-dependent.

In the even class of $k$-equipped exponentials

$$\mathcal{E}^0_n: \quad E^P_n(T, A, B, P, p|z, y), \quad p = n|_{\text{mod } 2}, \quad n \geq 1$$

(7.36)

parameters satisfy

$$\sum_{j=1}^n (-1)^j A_j = -T, \quad \sum_{j=1}^n (-1)^j B_j = 0, \quad \sum_{i=1}^n (-1)^i P_{ij} = B_j.$$

(7.37)

In the odd class of $k$-equipped exponentials

$$\mathcal{E}^1_n: \quad E^P_n(T, A, B, P, p|z, y), \quad p = (n + 1)|_{\text{mod } 2}, \quad n \geq 0$$

(7.38)

parameters obey

$$\sum_{j=1}^n (-1)^j A_j = 0, \quad \sum_{j=1}^n (-1)^j B_j = 1 - T, \quad \sum_{i=1}^n (-1)^i P_{ij} = -A_j.$$

(7.39)
The odd and even classes form a $\mathbb{Z}_2$–graded algebra with respect to star product
\[ \mathcal{E}_n^j \ast \mathcal{E}_m^i \subseteq \mathcal{E}_{m+n}^{(j+i)2} \] (7.40)
in the sense that if conditions (7.37) or (7.39) were respected by the product factors $f$ and $g$, the same conditions of the respective parity will be respected by $f \ast g$.

Following [2] we consider the action of the contracting homotopy $\Delta'_{q_n(v),\beta}$ (7.31) with
\[ q_n(v) = v_j p^j, \quad v_j \in \mathbb{C} \] (7.41)
on $\phi_n E^p_n$, where $E^p_n$ is some $k$-equipped exponential (7.32), while $\phi(z, y, p, \theta)$ is a pre-exponential factor containing a finite number of $p^j$.

By definition (7.22), performing Gaussian integration with respect to $u$ and $v$ one has
\[ \Delta'_{q_n(v),\beta} \phi_n E^p_n(T', A', B', P'|z, y) = \int_0^1 d\sigma \phi_n(\sigma, z, y, v, \beta, \theta) E'^p_n(T'', A'', B'', P'|z, y), \] (7.42)
where $\tilde{\phi}$ is some pre-exponential factor and
\[ E'^p_n = \exp i(T'' z_\gamma y^\gamma - A_j' p_j^\gamma z^\gamma - B_j' p_j^\gamma y^\gamma + \frac{1}{2} P'_{ij} p_i^\gamma p_j^\gamma) k^p \] (7.43)
with
\[ T' = \sigma T \xi^{-1}, \quad A'_i = \sigma A_i \xi^{-1}, \quad B'_i = (B_i + (1 - \sigma)(1 - \beta) T v_i) \xi^{-1}, \] (7.44)
\[ P'_{ij} = P_{ij} + (1 - \sigma)(1 - \beta) (A_j v_i - A_i v_j) \xi^{-1} - \beta \xi^{-1} (1 - \sigma)(B_j A_i - B_i A_j), \]
\[ \xi = 1 - (1 - \sigma) T \beta \]
Elementary calculation yields the following
Lemma 3 of [2]: If
\[ \sum_{j=1}^{n} (-1)^j v_j = 1, \] (7.45)
\[ E^p_n(T, A, B, P|z, y) \in \mathcal{E}_n^1 \] (7.46)
then $k$-equipped exponential $E'^p_n(T'', A'', B'', P'')$ (7.43) belongs to $\mathcal{E}_n^1$ for any $\sigma$ and $\beta$.

Indeed, by virtue of (7.44)-(7.46), parameters $T', A', B', P'$ of $E$ (7.43) can be easily shown to satisfy (7.39) for any $\sigma$ and $\beta$. $\Box$

Contracting homotopy $\Delta'_{q_n(v),\beta}$ with $v$ satisfying (7.43) will be called odd. Note that PLT-condition (7.45) coincides with that of [2] obtained at $\beta = 0$.

Analogously, one proves the following
Lemma 4 of [2]: If
\[ \sum_{j=1}^{n} (-1)^j v_j = 0, \] (7.47)
\[ E^p_n(T, A, B, P, p|z, y) \in \mathcal{E}_n^0 \] (7.48)
then $E \in \mathcal{E}_n$ for any $\sigma$ and $\beta$.

Contracting homotopy $\Delta'_{q,(v),\beta}$ with $v$ obeying (7.47) will be called even. Note that PLT-condition (7.47) coincides with that of [2] in the absence of $y$-shifts.

In [2] we considered the odd class of zero-forms using PLT to show that the final result is local by virtue of $Z$-dominance Lemma stating that since all $Z$-dependent terms should disappear upon reduction to the cohomology sector, the part of the coefficients $P_{ij}$ responsible for contraction of derivatives between different factors of $C$ must vanish as well because they are proportional to the coefficients $A_i$ in the $z$-dependent terms in (7.35). In the even case, this argument does not work since $P_{ij}$ is related to the coefficients $B_i$ in the $y$-dependent term in (7.35). However, the latter relation is useful again since, as will be shown below, the final result turns out to be $y$-independent in the terms important for the analysis of locality. As a result, it becomes not just local, but ultra-local in terminology of [9].

8 Limiting contracting homotopy and Factorization Lemma

As argued in [1] to obtain a local frame in the HS theory one has to use contracting homotopy in the limit $\beta \to -\infty$. Our goal is to analyse when the limit $\beta \to -\infty$ is well defined. Let us stress that even if it is not, this does not mean that the theory is ill-defined but rather that it is unlikely spin-local since $\beta$ has to be kept finite. In all cases analysed so far this does not happen, however. In Section 9.2.3 we formulate a sufficient condition guaranteeing that the limit $\beta \to -\infty$ is well defined.

8.1 Limiting contracting homotopy

To analyse the limit $\beta \to -\infty$ one has to use the class of functions (6.1). The worst possibility would be if the terms in the arguments of $\phi$ in (7.27) were divergent. Fortunately, this does not happen. Since $\sum_{i=1}^{3} \tau_i = 1$, the $\beta$-dependent coefficient in $\phi$ does not exceed 1 hence being well defined at $\beta \to -\infty$. This allows us to take the limit directly in (7.27) to obtain

$$\Delta'_{q,-\infty}(f) = \int \frac{d^2 u d^2 v}{(2\pi)^2} \int d^3 \tau \delta(1 - \sum_{i=1}^{3} \tau_i) \left[ \frac{\tau_1}{\tau_1 + \tau_3} \right]^{p-1} \exp i \left[ v_\alpha u^\alpha + \tau_1 z_\alpha y^\alpha - \tau_2 q_\beta y^\beta \right]$$

$$\frac{\tau_1 (z^\beta + q^\beta) + \tau_3 (u^\beta + q^\beta)}{\tau_1 + \tau_3} \frac{\partial}{\partial \theta^\beta} \phi \left( \tau_1 z - \frac{\tau_2 \tau_3}{\tau_1 + \tau_3} u - \tau_2 q, v + \tau_3 y, \theta, \frac{\tau_1}{\tau_1 + \tau_3} \right). \quad (8.1)$$

Analogously, (7.30) gives at $\beta \to -\infty$

$$\Delta'_{q,-\infty}(f) = \int \frac{d^2 u d^2 v}{(2\pi)^2} \int d^3 \tau \delta(1 - \sum_{i=1}^{3} \tau_i) \exp i \left[ v_\alpha u^\alpha + \tau_1 z_\alpha y^\alpha - \tau_2 q_\beta y^\beta \right]$$

$$\times \left( \frac{\tau_1}{\tau_3} \right)^{p-2} \left( \frac{\tau_1 (z^\beta + q^\beta) + (u^\beta + q^\beta)}{\tau_1 + \tau_3} \right) \frac{\partial}{\partial \theta^\beta} \psi \left( \tau_1 z - \frac{\tau_2 \tau_3}{\tau_1 + \tau_3} u - \tau_2 q, v + \tau_3 y, \theta, \frac{\tau_1}{\tau_1 + \tau_3} \right). \quad (8.2)$$
The fate of potential divergency on the r.h.s. of (8.2) due to the factor of $\tau_3^{p-2}$ is discussed in Sections 9.2.1 and 9.2.3. Note that
\begin{align}
\frac{\tau_1}{\tau_1 + \tau_3} &\leq 1, \quad \frac{\tau_3}{\tau_1 + \tau_3} \leq 1. 
\end{align}

Naively, one might think that $(\frac{\tau_1}{\tau_1 + \tau_3})^n$ behaves as $\tau_1^n$ at $\tau_1 \to 0$. However, this is not the case because of the integration over $\tau_3$. Indeed
\begin{align}
\int d\tau_2 d\tau_3 \delta(1 - \sum_{i=1}^3 \tau_i) \frac{\tau_1^n}{(\tau_3 + \tau_1)^n} = \int_0^{1-\tau_1} d\tau_3 \left( \frac{\tau_1^n}{(\tau_3 + \tau_1)^n} = \frac{1}{n-1} \tau_1 (1 - \tau_1^{n-1}) \right), \quad n > 1, 
\end{align}
\begin{align}
\int d^3 \tau \delta(1 - \sum_{i=1}^3 \tau_i) \frac{\tau_1}{(\tau_3 + \tau_1)} = \int_0^{1-\tau_1} d\tau_3 \frac{\tau_1}{(\tau_3 + \tau_1)} = -\tau_1 \log \tau_1. 
\end{align}

The common feature of this expressions is that, independently of $n \geq 1$ they have simple zeros both at $\tau_1 = 0$ and at $\tau_1 = 1$. (Logarithmic corrections do not matter in our analysis.) This allows us to estimate behaviour of the integrand of (8.1) at $\tau_1 \to 0$ and $\tau_1 \to 1$ upon integration over $\tau_2$ and $\tau_3$. If the factors $\tau_1$ and $1 - \tau_1$ enter explicitly the formulae above give for the leading behaviour at $\tau_1 \to 0$ and $\tau_1 \to 1$
\begin{align}
\int_0^1 \int_0^1 d\tau_2 d\tau_3 \tau \delta(1 - \sum_{i=1}^3 \tau_i) \frac{\tau_1^{n+m} (1 - \tau_1)^k}{(\tau_3 + \tau_1)^n} = \alpha(n, m, k) \tau_1^{m+1} (1 - \tau_1)^{k+1}, \quad n > 0, 
\end{align}
where $\alpha(n, m, k)$ are some coefficients.

We conclude that expressions
\begin{align}
X = \int d^3 \tau \delta(1 - \sum_{i=1}^3 \tau_i) \frac{\tau_1^n}{(\tau_1 + \tau_3)^n} \phi(\tau_1 z, (1 - \tau_1) y, \tau_1) \exp i[\tau_1 z \alpha y^a] 
\end{align}
with $n \geq 1$ behave with respect to $\tau_1$ as
\begin{align}
X \sim \int_0^1 d\tau_1 \tau_1 (1 - \tau_1) \phi(\tau_1 z, (1 - \tau_1) y, \tau_1) \exp i[\tau_1 z \alpha y^a], 
\end{align}
i.e., independently of $n$, $\tau_3$–integration adds one power of both $\tau_1$ and $1 - \tau_1$.

**8.2 Factorization Lemma and limiting cohomology projector**

**8.2.1 Factorization Lemma**

*Factorization Lemma* states:

In the limit $\beta \to -\infty$, cohomology projector (7.25) gives zero on $H^0$:
\begin{align}
h'_{q,-\infty}(H^0) = 0. 
\end{align}
Indeed, as shown in [1], typical integrals that appear in the limit $\beta \to -\infty$ have the form

$$\lim_{\beta \to -\infty} \int_0^1 d\tau \frac{\beta(\beta \tau)^m}{(1 - \beta \tau)^{2+n}}, \quad n \geq m. \quad (8.10)$$

Obviously, it gives a finite result after the change of variables $\tau \to \tau' = -\beta \tau$. However, if there is an additional factor of $\tau^\varepsilon$ in (7.25) and hence (8.10) this is equivalent to the appearance of the factor of $(-\beta)^{-\varepsilon}$ that sends the final result to zero in the limit $\beta \to -\infty$.

Note that Factorization Lemma provides a simple interpretation of the Z-dominance Lemma of [2] which states that if the coefficients in front of the terms in the exponential responsible for contractions between different product factors in (7.18) are dominated by the coefficient in front of $iz\alpha y^\alpha$, i.e., $\tau$, then these terms do not contribute to the dynamical equations leading to local field equations. In the setup of this paper this is simply because, being dominated by $\tau$, contractions bring an extra factor of $\tau$ hence belonging to $H^{+0}$.

### 8.2.2 Limiting cohomology projector

Remarkably, the cohomology projector (7.25) remains finite in the limit $\beta \to -\infty$. Naively, it gives 0 at $\beta = -\varepsilon^{-1}$ with $\varepsilon \to 0$ since

$$\int_0^1 d\tau \frac{1}{(1 - \beta \tau)^{2+n}} = -(n + 1)^{-1} \left( \frac{\varepsilon^{n+2}}{(\varepsilon + 1)^{n+1}} - \frac{\varepsilon^{n+2}}{\varepsilon^{n+1}} \right) = (n + 1)^{-1} \varepsilon + O(\varepsilon^2). \quad (8.11)$$

However, this is not the case because, being a zero-form in $\theta, \phi$ in (7.25) contains a factor of $\tau^{-1}$ in front of the $u$-dependent terms. Indeed, by rescaling $v_\alpha \to (1 - \tau)^{-1}v_\alpha, u_\alpha \to (1 - \tau)u_\alpha$ the coefficient in front of $v$ in the argument of $\phi$ in (7.25) takes the form

$$\frac{|\tau(1 - \tau)\beta|}{(1 - \tau \beta)} \leq 1. \quad (8.12)$$

As such, it disappears in the limit $\beta \to -\infty$ by (8.11). However, once one of the factors of $\tau$ is removed, $\beta \frac{(1 - \tau)}{(1 - \tau \beta)}$ contains an extra factor of $\beta \sim \varepsilon^{-1}$ that precisely compensates the factor of $\varepsilon$ in (8.11) hence yielding a finite result in the limit. Thus, it is crucial that $\phi$ (6.1) is of the form (6.2).

The limit $\beta \to -\infty$ can be taken directly in (7.25) to obtain for any $f^i$

$$h'_{q,-\infty}(f^i) = \int_0^1 d\sigma \int \frac{d^2 u d^2 v}{(2\pi)^2} \exp i[v_\alpha u^\alpha + \sigma y_\alpha q^\alpha] \phi^i(-\sigma(q + u), v + (1 - \sigma)y, 0, 0). \quad (8.13)$$
Indeed, due to (6.2), Eq. (7.25) yields for an inner zero-form $\phi^i$

\[ h'_{q,\beta}(f^i) = \int_0^1 d\tau (1 - \beta \tau)^{-2} \int \frac{d^2 u d^2 v}{(2\pi)^2} \exp i[v_\alpha u^\alpha] \exp i((\tau - \beta \tau)(1 - \beta \tau)^{-1} y_\alpha q^\alpha) \]

\[ \phi((\tau \beta u - (\tau - \beta \tau)q)(1 - \beta \tau)^{-1}, (1 - \tau)(v + y(1 - \beta \tau)^{-1}), 0, \tau) \]

\[ \simeq -\int_0^1 d\tau (1 - \beta \tau)^{-2} \int \frac{d^2 u d^2 v}{(2\pi)^2} \exp i[v_\alpha u^\alpha] \exp i[(1 - (1 - \beta \tau)^{-1})y_\alpha q^\alpha) \]

\[ \frac{\beta(1 - \beta \tau)^{-1}}{[1 - (1 - \beta \tau)^{-1}]^2} \psi(-(q + u)[1 - (1 - \beta \tau)^{-1}], (v + y(1 - \beta \tau)^{-1}), 0, \tau). \]

Hence by virtue of (8.11) and

\[ \int_0^1 d\sigma (1 - \sigma)^n = (n + 1)^{-1} \quad (8.14) \]

\[ h'_{q,-\infty}(f) = -\int_0^1 d\sigma (1 - \sigma)^{-1} \int \frac{d^2 u d^2 v}{(2\pi)^2} \exp i[v_\alpha u^\alpha + \sigma y_\alpha q^\alpha] \psi(-(q + u)\sigma, v + y(1 - \sigma), 0, 0). \quad (8.15) \]

Whence, using (6.2), one obtains (8.13). Note that $h'_{q,-\infty}(f)$ acts as identity operator on the $z$–independent boundary term (6.4) associated with $\chi_0$ and by zero on $\chi_2$.

9 Specific form degree relations

So far we considered general relations like (6.25), (6.26) valid for $\theta$-forms of arbitrary degrees. For the practical analysis it is important to specify them further for the spaces $\mathcal{H}_p^{0+}$ and $\mathcal{H}_p^{+0}$ of $p$-forms of specific degrees. In this section we first derive star-product relations that greatly simplify computations, and then discuss the properties of limiting contracting homotopies acting on $\mathcal{H}_1$ and $\mathcal{H}_2$ resulting in the important Pre-Ultra-Locality Theorem.

9.1 Star products

In this section we compute star products for the spaces $\mathcal{H}_0$ and $\mathcal{H}_1$ up to the terms that belong to the ideal $\mathcal{I}$. The main reason for this is that such terms do not contribute to the field equations by Factorization Lemma (8.9) since $\mathcal{I}_0 \subset \mathcal{H}_0^{0+}$ and $\Delta_{q,-\infty}\mathcal{I}_1 \subset \mathcal{I}_0$ (9.25). In this section we focus on the more complicated star products of inner functions, omitting for brevity label $i$. Extension to boundary functions is evident.
9.1.1 $\mathcal{H}_0 \ast \mathcal{H}_0$

Consider product (5.8) for $f_0 \ast \tilde{f}_0$ where both $f_0$ and $\tilde{f}_0$ are zero-forms in $\theta$.

$$f_0 \ast \tilde{f}_0 = \int_0^1 d\tau_1 \int_0^1 d\tau_2 \int \frac{d^2 s d^2 t}{(2\pi)^2} \exp i[\tau_1 \circ \tau_2 z \alpha y^\alpha + s \alpha t^\alpha] \times \phi(\tau_1((1 - \tau_2)z - \tau_2 y + s), (1 - \tau_1)((1 - \tau_2)y - \tau_2 z + s), \tau_1) \times \tilde{\phi}(\tau_2((1 - \tau_1)z + \tau_1 y - t), (1 - \tau_2)((1 - \tau_1)y + \tau_1 z + t), \tau_2).$$

(9.1)

It is easy to see that

$$f_0^{+0} \ast \tilde{f}_0^{+0} \simeq 0.$$

(9.2)

Indeed, since $f_0^{+0}$ has the form (6.1) with $\phi(u, w, \tau) = \tau \phi'(u, w, \tau)$ carrying an additional factor of $\tau$, and similarly for $\tilde{f}_0^{+0}$, from (6.16) and (6.12) it follows that the product $f_0^{+0} \ast \tilde{f}_0^{+0}$ contains a factor dominated by $\tau_1 \circ \tau_2(1 - \tau_1 \circ \tau_2)^2$ sending it to $\mathcal{I}$.

Computation of $f_0^{+0} \ast \tilde{f}_0^{+0}$ and $f_0^{+0} \ast \tilde{f}_0^{+0}$ gives using (6.13)

$$\mathcal{H}^{+0} \ni \int_0^1 d\tau_1 \int_0^1 d\tau_2 \int \frac{d^2 s d^2 t}{(2\pi)^2} \exp i[\tau_1 \circ \tau_2 z \alpha y^\alpha + s \alpha t^\alpha] \times \phi(\tau_1((1 - \tau_2)z - \tau_2 y + s), (1 - \tau_1)((1 - \tau_2)y - \tau_2 z + s), \tau_1) \times \tilde{\phi}(\tau_2((1 - \tau_1)z + \tau_1 y - t), (1 - \tau_2)((1 - \tau_1)y + \tau_1 z + t), \tau_2).$$

(9.3)

Indeed, since $\tau$-kernel of $f_0^{+0} \ast \tilde{f}_0^{+0}$ (9.3) contains a factor of $\tau_2(1 - \tau_1)$, from (6.16) and (6.13) it follows that terms proportional to $\tau_1$ or $1 - \tau_2$ are dominated by $\tau_1 \circ \tau_2(1 - \tau_1 \circ \tau_2)^2$ hence belonging to $\mathcal{I}$. Analogously,

$$\mathcal{H}^{+0} \ni \int_0^1 d\tau_1 \int_0^1 d\tau_2 \int \frac{d^2 s d^2 t}{(2\pi)^2} \exp i[\tau_1 \circ \tau_2 z \alpha y^\alpha + s \alpha t^\alpha] \times \phi(\tau_1((1 - \tau_2)z - \tau_2 y + s), (1 - \tau_1)((1 - \tau_2)y - \tau_2 z + s), \tau_1) \times \tilde{\phi}(\tau_2((1 - \tau_1)z + \tau_1 y - t), (1 - \tau_2)((1 - \tau_1)y + \tau_1 z + t), \tau_2).$$

(9.4)

In $f_0^{+0} \ast \tilde{f}_0^{+0}$, we can neglect the $s$, $t$ and $y$ dependence in the first arguments of $\phi$ and $\tilde{\phi}$. Indeed, since $\tau$-kernel of (9.3) contains a factor of $(1 - \tau_1)(1 - \tau_2)$, from (6.16) and (6.13) it follows that terms proportional to $\tau_1$ or $\tau_2$ are dominated by $(1 - \tau_1 \circ \tau_2)^2$. As a result,

$$\mathcal{H}^{+0} \ni \int_0^1 d\tau_1 \int_0^1 d\tau_2 \int \frac{d^2 s d^2 t}{(2\pi)^2} \exp i[\tau_1 \circ \tau_2 z \alpha y^\alpha + s \alpha t^\alpha] \times \phi(\tau_1((1 - \tau_2)z - \tau_2 y + s), (1 - \tau_1)((1 - \tau_2)y - \tau_2 z + s), \tau_1) \times \tilde{\phi}(\tau_2((1 - \tau_1)z + \tau_1 y - t), (1 - \tau_2)((1 - \tau_1)y + \tau_1 z + t), \tau_2).$$

(9.5)

In particular, if $\phi(w, u, \tau) = \varphi(w, \tau)$, then $f_0^{+0} \ast \tilde{f}_0^{+0}$ contains no contractions

$$f_0^{+0} \ast \tilde{f}_0^{+0} \simeq \int_0^1 d\tau_1 \int_0^1 d\tau_2 \exp i[\tau_1 \circ \tau_2 z \alpha y^\alpha] \times \varphi(\tau_1((1 - \tau_2)z - \tau_2 y + s), (1 - \tau_1)((1 - \tau_2)y - \tau_2 z + s), \tau_1) \times \tilde{\varphi}(\tau_2((1 - \tau_1)z + \tau_1 y - t), (1 - \tau_2)((1 - \tau_1)y + \tau_1 z + t), \tau_2).$$

(9.6)
Analogously, if $\tilde{\phi}(w, u, \tau) = \tilde{\varphi}(w, \tau)$

$$f_0^+ * \tilde{f}_0^+ \simeq \int_0^1 d\tau_1 \int_0^1 d\tau_2 \exp i[\tau_1 \circ \tau_2 z \alpha y^\alpha]$$

$$\times \phi(\tau_1(1 - \tau_2)z, (1 - \tau_1)((1 - \tau_2)y - \tau_2 z), \tau_1) \times \tilde{\phi}(\tau_2(1 - \tau_1)z, \tau_2). \quad (9.7)$$

These formulae play the key role in the analysis of locality of HS equations because they apply to the contributions involving the space-time one-form $W_1$ in either of the combinations $W_1 * f$ or $f * W_1$ with $f$ being a zero-form in $\theta$. These terms contribute to the $r.h.s.$ of the dynamical equations

$$d\omega + \omega * W_1 * f + f * W_1 + \ldots \quad (9.8)$$

and

$$dC + [\omega, C]_s + [W_1, f]_s + \ldots \quad (9.9)$$

implying ultra-locality of the $W_1$-depended terms provided that $f$ was local in the lower orders. In particular, the contribution of $W_1 * W_1$ to the field equations for $\omega$ turns out to be local, which observation originally suggested locality of the whole deformation in this sector.

### 9.1.2 $H_0 * H_1$ and $H_1 * H_0$

Let us consider product (8.8) for inner zero- and one-forms in $\theta$, $f_0$ and $f_1$, respectively. Being a zero-form, $f_0$ should contain a factor of $\frac{1 - \tau_1}{\tau_1}$ in the pre-exponential. On the other hand, a one-form $f_1$ contains no prefactors. As a result, if the factor of $\tau_1^{-1}$ gets cancelled by one or another mechanism, taking into account the logarithmic factor (8.10), the resulting expression will belong to $I$ and can be discarded. For instance, this allows us to discard the integration variable $s$ in the first argument of $f_0$ in (8.8) giving

$$f_0 * f_1 \simeq \int_0^1 d\tau_1 \int_0^1 d\tau_2 \int d^2s d^2t \left(\frac{2\pi}{2}\right)^2 \exp i[\tau_1 \circ \tau_2 z \alpha y^\alpha + s \alpha t^\alpha]$$

$$\times \phi_0(\tau_1((1 - \tau_2)z - \tau_2 y), (1 - \tau_1)((1 - \tau_2)y + \tau_2 z + s), \tau_1 \theta, \tau_1)$$

$$\times \phi_1(\tau_2((1 - \tau_1)z + \tau_1 y - t), (1 - \tau_2)((1 - \tau_1)y + \tau_1 z + t), \tau_2 \theta, \tau_2). \quad (9.10)$$

First of all we observe that if $f_0 \in H_0^+$ containing an additional positive power of $\tau_1$ in $\tau$-kernel, then the whole result is in $I$, i.e.,

$$f_0^+ * f_1 \simeq 0. \quad (9.11)$$

Analogously,

$$f_1 * f_0^+ \simeq 0. \quad (9.12)$$

Thus, the products $f_1 * f_0$ and $f_0 * f_1$ can be nontrivial if $f_0 = f_0^+ \in H_0^+$. Elementary analysis using (8.13) and (8.16) shows

$$H_1^+ \ni f_1^+ * f_0^+ \simeq \int_0^1 d\tau_1 \int_0^1 d\tau_2 (1 - \tau_2) \int d^2s d^2t \left(\frac{2\pi}{2}\right)^2 \exp i[\tau_1 \circ \tau_2 z \alpha y^\alpha + s \alpha t^\alpha]$$

$$\times \phi_1^+(\tau_1((1 - \tau_2)z - \tau_2 y + s), (1 - \tau_1)((1 - \tau_2)y, \tau_1 \theta, \tau_1)$$

$$\times \phi_0^+(\tau_2 \tau_1, y, (1 - \tau_2)((1 - \tau_1)y + \tau_1 z + t), \tau_2). \quad (9.13)$$
\[ \mathcal{H}_1^{0+} \ni f_1^{0+} \ast f_0^{0+} \simeq \int_0^1 d\tau_1(1 - \tau_1) \int_0^1 d\tau_2(1 - \tau_2) \int \frac{d^2s^2t}{(2\pi)^2} \exp i[\tau_1 \circ \tau_2 z_\alpha y^\alpha + s_\alpha t^\alpha] \]
\[ \times \phi_1^{0+}(\tau_1(1 - \tau_2)z, (1 - \tau_1)((1 - \tau_2)y - \tau_2 z + s), \tau_1 \theta, \tau_1) \]
\[ \times \phi_0^{0+}(\tau_2(1 - \tau_1)z, (1 - \tau_2)((1 - \tau_1)y + \tau_1 z + t), \tau_2) , \] (9.14)

\[ \mathcal{H}_1^{+0} \ni f_0^{0+} \ast f_1^{+0} \simeq \int_0^1 d\tau_1(1 - \tau_1) \int_0^1 d\tau_2 \tau_2 \int \frac{d^2s^2t}{(2\pi)^2} \exp i[\tau_1 \circ \tau_2 z_\alpha y^\alpha + s_\alpha t^\alpha] \]
\[ \times \phi_0^{0+}(-\tau_1 \tau_2 y, (1 - \tau_1)((1 - \tau_2)y - \tau_2 z + s), \tau_1) \]
\[ \times \phi_1^{+0}(\tau_2(\tau_1 y + (1 - \tau_1)z - t), (1 - \tau_2)(1 - \tau_1)y, \tau_2) , \] (9.15)

\[ \mathcal{H}_1^{0+} \ni f_0^{0+} \ast f_1^{+0} \simeq \int_0^1 d\tau_1(1 - \tau_1) \int_0^1 d\tau_2(1 - \tau_2) \int \frac{d^2s^2t}{(2\pi)^2} \exp i[\tau_1 \circ \tau_2 z_\alpha y^\alpha + s_\alpha t^\alpha] \]
\[ \times \phi_0^{0+}(\tau_1(1 - \tau_2)z, (1 - \tau_1)((1 - \tau_2)y - \tau_2 z + s), \tau_1) \]
\[ \times \phi_1^{+0}(\tau_2(1 - \tau_1)z, (1 - \tau_2)((1 - \tau_1)y + \tau_1 z + t), \tau_2 \theta, \tau_2) . \] (9.16)

An important consequence of these relations is that the star product of any one-form of the form \( f_1 = \theta^\alpha z_\alpha f \) with any zero-form keeps this form modulo terms in \( \mathcal{H}_1^{+0} \).

### 9.1.3 \( \mathcal{H}_1 \ast \mathcal{H}_1 \)

To analyse star products of one-forms in \( \theta^\alpha \) we should take into account that a two-form in \( \theta^\alpha \) from \( \mathcal{H} \) should contain an overall factor of \( \frac{\tau}{1 - \tau} \) in \( \tau \)-kernels. Since \( f_1 \) and \( g_1 \) had no overall factors in \( \tau \)-kernels, \( f_1 \ast g_1 \) will be regular as well. Taking into account the contribution due to logarithm (6.16), this means that \( f_1 \ast g_1 \) in fact contains two extra powers of \( 1 - \tau_1 \circ \tau_2 \) in the \( \tau \)-kernel. Thus

\[ f_1 \ast g_1 \in \mathcal{H}_2^{0+} \quad \forall f_1, g_1 \in \mathcal{H}_1 . \] (9.17)

As a result, using (6.26), we obtain that

\[ f_1^{0+} \ast g_1^{0+} \simeq f_1^{+0} \ast g_1^{0+} \simeq 0 . \] (9.18)

The remaining two products are

\[ \mathcal{H}_2^{0+} \ni f_1^{0+} \ast g_1^{0+} \simeq \int_0^1 d\tau_1(1 - \tau_1) \int_0^1 d\tau_2(1 - \tau_2) \int \frac{d^2s^2t}{(2\pi)^2} \exp i[\tau_1 \circ \tau_2 z_\alpha y^\alpha + s_\alpha t^\alpha] \]
\[ \times \phi_1^{0+}(\tau_1(1 - \tau_2)z, (1 - \tau_1)((1 - \tau_2)y - \tau_2 z + s), \tau_1 \theta, \tau_1) \]
\[ \times \phi_1^{0+}(\tau_2(1 - \tau_1)z, (1 - \tau_2)((1 - \tau_1)y + \tau_1 z + t), \tau_2 \theta, \tau_2) , \] (9.19)

\[ \mathcal{H}_2^{+0} \ni f_1^{+0} \ast g_1^{+0} \simeq \int_0^1 d\tau_1 \tau_1 \int_0^1 d\tau_2 \tau_2 \int \frac{d^2s^2t}{(2\pi)^2} \exp i[\tau_1 \circ \tau_2 z_\alpha y^\alpha + s_\alpha t^\alpha] \]
\[ \times \phi_1^{+0}(\tau_1((1 - \tau_2)z \tau_2 y + s), -\tau_2(1 - \tau_1)z, \tau_1 \theta, \tau_1) \]
\[ \times \phi_1^{+0}(\tau_2((1 - \tau_1)z + \tau_1 y - t), \tau_1(1 - \tau_2)z, \tau_2 \theta, \tau_2) . \] (9.20)

This completes the list of star products between inner zero- and one-forms in \( \theta \). Star products \( \mathcal{H}_0 \ast \mathcal{H}_2 \) and \( \mathcal{H}_2 \ast \mathcal{H}_0 \) follow from \( \mathcal{H}_0 \ast \mathcal{H}_0 \) with the help of \( \gamma \)-maps of Section 3.2.1.
9.2 Limiting contracting homotopy

Contracting homotopies with general parameters $-\infty < \beta < 1$ do not leave the spaces $H^{\nu\mu}$ invariant. A distinguishing feature of the $\beta \to -\infty$ limiting homotopy is that, as shown in this section, it exhibits special properties when acting on the spaces $H^{\nu\mu}$, that underly spin-locality of HS interactions and allow us to formulate a sufficient condition for the limit $\beta \to -\infty$ be well defined.

9.2.1 Contracting homotopy of $H^{\nu\mu}_1$

For inner $f_1 \in H^{\nu\mu}_1$ formula (8.2) gives

$$\Delta'_{q, -\infty}(f_1) = \frac{1}{(2\pi)^2} \int d^2v d^2u \int d^3\tau \delta(1 - \sum_{i=1}^3 \tau_i) \exp i[v_\alpha u^\alpha + \tau_1 z_\alpha y^\alpha - \tau_2 q_\beta y^\beta]$$

$$\times \left( (z^\beta + q^\beta) + \frac{\tau_3}{\tau_1}(u^\beta + q^\beta) \right) \frac{\partial}{\partial \theta^3} \psi \left( \tau_1 z - \frac{\tau_2 \tau_3}{\tau_1 + \tau_3} u - \tau_2 q, v + \tau_3 y, \theta, \frac{\tau_1}{\tau_1 + \tau_3} \right). \quad (9.21)$$

Generally, this formula may have logarithmic divergency due to $\tau_1$ integration of the term $\frac{\tau_3}{\tau_1}(u^\beta + q^\beta)$. This does not happen however at least in the following two cases.

First, if $f_1^{0+} \in H^{0+}_1$ then $\psi$ contains an additional factor of $\frac{\tau_1}{\tau_1 + \tau_3}$ that cancels the divergency. Since, by virtue of (8.8), $\tau_3$ integration brings a factor of $\tau_1(1 - \tau_1)$ we find that

$$\Delta_{q, -\infty} H^{0+}_1 \subset H^{0+}_0. \quad (9.22)$$

Second, if $f_1^{0+} \in H^{0+}_1$, then $\psi$ contains an additional factor of

$$1 - \frac{\tau_1}{\tau_1 + \tau_3} = \frac{\tau_3}{\tau_1 + \tau_3}. \quad (9.23)$$

Since, up to non-essential logarithmic terms,

$$\int_0^{1-\tau_1} \frac{\tau_3}{\tau_1 + \tau_3} \sim 2(1 - \tau_1)^2 + O((1 - \tau_1)^3), \quad (9.24)$$

the resulting expression at r.h.s. of (9.21) behaves as $(1 - \tau_1)^2$ at $\tau_1 \to 1$ as it has to in $H^{0+}_0$.

From here and (9.22) follows an important fact that

$$\Delta_{q, -\infty} I_1 \subset I_0 \quad (9.25)$$

allowing to discard the contribution of $I_1$ to $H^{0+}_1$ in (9.22). As a consequence we obtain

$\omega$-Lemma: Elements of $I$ can be discarded in all terms containing space-time one-forms $\omega$.

Indeed, such terms can never contribute to the sector of two-forms in $\theta^\alpha$ via star product, allowing us to use (9.25) along with Factorization Lemma (8.9) implying that $I \subset H^{+0}$ does not contribute under the cohomology projector.

However, to belong to $H^{0+}_0$, $\Delta'_{q, -\infty}(f_1)$ (9.21) should have a fictitious pole in $\tau_1$ obeying (6.3) which is true if the whole expression consists of terms proportional to $z^\alpha$ or carrying an additional power of $\tau$ (the latter terms belong to $H^{+0}$). As we show now, this is the case if $f_1^{0+} \sim z_\alpha \theta^\alpha$. 

33
9.2.2 Space $\tilde{H}_1^{0+}$

Let $\tilde{H}_1^{0+}$ be the subspace of $H_1^{0+}$ that consists of the one-forms proportional of $z_{\alpha}\theta^{\alpha}$. In other words, $\tilde{f}_1^{0+} \in H_1^{0+}$ has $\psi$ \cite{6.2} of the form

$$\psi(w, u, \theta, \tau) = w_{\alpha}\theta^{\alpha}\tilde{\psi}(w, u, \tau).$$

(9.26)

Then formula \cite{9.21} gives

$$\Delta'_{q,-\infty}(f_1^{\prime \mu}) = \frac{1}{(2\pi)^2} \int d^2v d^2u \int d^3\tau \delta(1 - \sum_{i=1}^{3} \tau_i) \exp i[v_{\alpha}u^{\alpha} + \tau_1 z_{\alpha}y^{\alpha} - \tau_2 q_{\beta}y^{\beta}]$$

$$\times z_{\beta} \left( \frac{\tau_3}{\tau_1 + \tau_3} w^{\beta} + q^{\beta} \right) \tilde{\psi} \left( \tau_1 z - \frac{\tau_3}{\tau_1 + \tau_3} u - \tau_2 q, v + \tau_3 y, \frac{\tau_1}{\tau_1 + \tau_3} \right),$$

whence, by virtue of \cite{9.24}

$$\Delta'_{q,-\infty}\tilde{H}_1^{0+} \subset H_0^{0+}.$$  

(9.28)

On the other hand, $\Delta'_{q,-\infty}H_1^{0+} \subset H_0^{0+}$ for generic elements of $H_1^{0+}$ may be away from $H_0^{0+}$ giving rise to divergent expressions. (Recall that, since for any finite $\beta < 1$ contracting homotopies $\Delta_{q,\beta}$ give finite results, this would just mean that the limit $\beta \to -\infty$ is ill-defined.) In the next section we formulate a sufficient condition guaranteeing that this does not happen. Then in Section \cite{11} it will be shown that these conditions are indeed fulfilled at least to the order $\omega^2C^2$.

The following comment is now in order. Consider a one-form $f_1$ containing an overall factor $z_{\alpha}\theta^{\alpha}$

$$f_1(z, y, \theta) = z_{\alpha}\theta^{\alpha} \int_0^1 d\tau \tau \psi(\tau z, (1 - \tau)y) \exp i\tau z_{\alpha}y^{\alpha}. \quad (9.29)$$

Using decomposition \cite{6.27} consider its $f_1^{0+}$ part \cite{6.29}

$$f_1^{0+}(z, y, \theta) = z_{\alpha}\theta^{\alpha} \int_0^1 d\tau (1 - \tau) \psi(\tau z, (1 - \tau)y) \exp i\tau z_{\alpha}y^{\alpha}. \quad (9.30)$$

The remarkable fact is that

$$d_z f_1^{0+}(z, y, \theta) \simeq 0. \quad (9.31)$$

Indeed,

$$d_z f_1^{0+}(z, y) = \int_0^1 d\tau (1 - \tau) \exp i(\tau z_{\alpha}y^{\alpha}) \left( \theta_{\gamma} \frac{\delta}{\partial z^{\alpha}} \psi(\tau z, (1 - \tau)y) \right)$$

$$\theta_{\gamma} \int_0^1 d\tau_1 d\tau_2 \theta(\tau_1) \theta(\tau_2) \delta(1 - \tau_1 - \tau_2) \tau_1 \tau_2 (1 + \frac{1}{2} \frac{\partial}{\partial \tau_1}) \left[ \psi(\tau_1 z, \tau_2 y) \exp i(\tau_1 z_{\alpha}y^{\alpha}) \right]$$

$$= \frac{1}{2} \theta_{\gamma} \int_0^1 d\tau_1 d\tau_2 \theta(\tau_1) \theta(\tau_2) \delta(1 - \tau_1 - \tau_2) \tau_1^2 \frac{\partial}{\partial \tau_2} \left[ \tau_2 \psi(\tau_1 z, \tau_2 y) \right] \exp i(\tau_1 z_{\alpha}y^{\alpha}) \right]. \quad (9.32)$$

34
The last term belongs to \( \mathcal{I} \) because it contains an additional factor of \( \tau_1 \tau_2 \) compared to the normally assigned to a two-form in \( H_2 \) by (6.1), (6.3).

Relation (9.31) has a consequence that any element of \( \tilde{H}_1^{0+} \) is weakly \( d_z \)-closed. This is because any modification of the \( \tau \)-dependence in formula (9.29) within the class \( H_1^{0+} \) that leaves the leading \( \tau \)-dependence intact will only contribute to \( \mathcal{I} \).

It is useful to introduce the space \( \tilde{H}_1 \)

\[
\tilde{H}_1 := \text{Span}(\tilde{H}_1^{0+}, H_1^+) .
\] (9.33)

Relation (9.33) implies that

\[
d_z \tilde{H}_1 \subset H_2^+ .
\] (9.34)

\( \tilde{H}_1 \) is just the space that leads to the finite result under the action of the limiting contracting homotopy, i.e.,

\[
\Delta_{q,-\infty}' \tilde{H}_1 \subset H_0 .
\] (9.35)

From (9.13)-(9.16) it follows that the result of star product of any element of \( \tilde{H}_1 \) and any element of \( H_0 \) is either proportional to \( z^\alpha \theta_\alpha \) or belongs to \( H_1^{+0} \). Hence it holds remarkable

\( \tilde{H}_1 \) Closure Lemma

\[
\tilde{H}_1 * H_0 \subset \tilde{H}_1 , \quad H_0 * \tilde{H}_1 \subset \tilde{H}_1 .
\] (9.36)

\( \tilde{H}_1 \) Closure Lemma has an important consequence that star product of \( S \in \tilde{H}_1 \) with HS fields \( W \) or \( B \) still belongs to \( \tilde{H}_1 \).

9.2.3 Contracting homotopy of \( \mathcal{H}_2^{\nu\mu} \)

The contracting homotopy \( \Delta_{q,-\infty}' \) does not leave the spaces \( \mathcal{H}_2^{\nu\mu} \) of two-forms invariant. For instance, consider \( f_2^{+0} \) with \( \psi^{+0} \) of the form

\[
\psi^{+0}(w, u, \theta, \tau) = \theta^\alpha \theta_\alpha \tau \bar{\psi}^{+0}(w, u, \tau) .
\] (9.37)

Then, by (8.2),

\[
\Delta_{q,-\infty}'(f_2^{+0}) = \frac{1}{2(2\pi)^2} \int d^2v d^2u \int d^3 \tau \delta(1 - \sum_{i=1}^3 \tau_i) \exp i[v_\alpha u^\alpha + \tau_1 z_\alpha y^\alpha - \tau_2 q_\beta y^\beta] \\
\times \frac{\tau_1}{\tau_1 + \tau_3} \left( \frac{\tau_1}{\tau_3} (z^\beta + q^\beta) + (u^\beta + q^\beta) \right) \theta_\beta \bar{\psi}^{+0} \left( \tau_1 z - \frac{\tau_2 \tau_3}{\tau_1 + \tau_3} u - \tau_2 q, v + \tau_3 y, \frac{\tau_1}{\tau_1 + \tau_3} \right) .
\] (9.38)

Let us show that \( \Delta_{q,-\infty}' \mathcal{H}_2^{+0} \nsubseteq \tilde{H}_1^{+0} \).

Decomposing \( f_2^{+0} = f_2^{+0}_{1+0} + f_2^{+0}_b \), firstly we note that this happens already to the boundary term. Indeed, any boundary term \( f_2^b \) can be rewritten as \( f_2^b = F(y) * \gamma \) for some \( F(y) \). Using the star-exchange formulae along with the identity \( \Delta_{q,\beta}\gamma = \Delta_{q,0}\gamma \) [4], we have

\[
\Delta_{q,-\infty}' f_2^b = \Delta_{q,0} f_2^b .
\] (9.39)
One can see that $\Delta_{q,-\infty} f_{2}^{b+0}$ belongs to full $\mathcal{H}_{1}$, contributing, in particular to $\mathcal{H}_{1}^{0+}$. It is by this mechanism the non-trivial first-order contribution to HS field equations comes from the term $\gamma * C \in \mathcal{H}_{2}^{2+}$ on the $r.h.s.$ of (4.16). Moreover, from (9.39) along with (7.21) it follows that, for a zero shift $q = 0$, the result is proportional to $z_{a} \theta^{a}$, i.e.,

$$
\Delta_{0,-\infty} f_{2}^{b+0} \in \mathcal{H}_{1}.
$$

(9.40)

Let us now consider the contribution $\Delta'_{q,-\infty} f_{1}^{b+0}$ of the inner part. To see what happens, first of all note that the pole in $\tau_{3}$ is fictitious because, in agreement with (8.3), $\tilde{\psi}^{+0}(w, r, \tau)$ must be linear in the second argument (up to possible $y$-independent terms from $I$ that carry an additional power of $\frac{1}{\tau_{1} + \tau_{3}}$). Indeed, both $\tau_{3} y$ and $v_{a}$ being equivalent to $i \frac{\partial}{\partial \theta^{a}}$ then bring a factor of $\tau_{3}$ that cancels $\tau_{3}^{-1}$ in the pre-exponential. More in detail, setting

$$
\tilde{\psi}_{1}(w, r, \tau) = r_{a} \tilde{\psi}_{1}^{a}(w, r, \tau)
$$

(9.41)

and changing integration variables $u \rightarrow (\tau_{1} + \tau_{3})^{-1} u$, $v \rightarrow \tau_{3}(\tau_{1} + \tau_{3})^{-1} v$ we obtain from (9.21)

$$
f_{1} = \Delta'_{q,-\infty} (f_{2}^{b+0}) = 2 \left(\frac{1}{2\pi}\right)^{2} \int d^{2}v d^{2}u \int d^{3}\tau \delta(1 - \sum_{i=1}^{3} \tau_{i}) \exp i[v_{a} u^{a} + \tau_{1} z_{a} y^{a} - \tau_{2} q_{b} y^{b}] \right)
$$

(9.42)

$$
\times \left(\frac{\tau_{1}}{\tau_{1} + \tau_{3}} \left(\tau_{1} \tilde{z}^{\beta} + (\tau_{1} + \tau_{3})(u + q)^{\beta}\right) \theta_{\beta} \left(\frac{v_{a}}{\tau_{1} + \tau_{3}} + y_{a}^{\beta}\right) \tilde{y}_{+0 a} \left(\tau_{1} \tilde{z} - \tau_{2}(u + q), \frac{\tau_{3}}{\tau_{1} + \tau_{3}} y + \tau_{3} y, \frac{\tau_{1}}{\tau_{1} + \tau_{3}}\right)\right).
$$

It is not hard to see that most of the terms in this expression belong to $\mathcal{H}_{1}^{0+}$ except for one. Namely, $v_{a}$ in the pre-exponential can be replaced by $i \frac{\partial}{\partial \theta^{a}}$. The important contribution is from the differentiation of the first argument of $\tilde{\psi}^{+0 a}$. Neglecting terms from $\mathcal{H}_{1}^{0+}$ this gives

$$
-2i \left(\frac{1}{2\pi}\right)^{2} \int d^{2}v d^{2}u \int d^{3}\tau \delta(1 - \sum_{i=1}^{3} \tau_{i}) \exp i[v_{a} u^{a} + \tau_{1} z_{a} y^{a} - \tau_{2} q_{b} y^{b}] \right)
$$

(9.43)

$$
\times \left(\frac{\tau_{1} \tau_{2}}{(\tau_{1} + \tau_{3})^{2}} \left(\tau_{1} \tilde{z}^{\beta} + (\tau_{1} + \tau_{3})(u + q)^{\beta}\right) \theta_{\beta} \partial_{1 a} \tilde{\psi}^{+0 a} \left(\tau_{1} \tilde{z} - \tau_{2}(u + q), \frac{\tau_{3}}{\tau_{1} + \tau_{3}} y + \tau_{3} y, \frac{\tau_{1}}{\tau_{1} + \tau_{3}}\right)\right).
$$

By virtue of (8.3)–(8.5), all $z$-independent terms in the pre-exponential belong to $\mathcal{H}_{1}^{0+}$. However, by virtue of (8.4), the $z$-dependent term contributes to $\mathcal{H}_{1}^{0+}$ giving

$$
-2i \left(\frac{1}{2\pi}\right)^{2} \int d^{2}v d^{2}u \int d^{3}\tau \delta(1 - \sum_{i=1}^{3} \tau_{i}) \exp i[v_{a} u^{a} + \tau_{1} z_{a} y^{a} - \tau_{2} q_{b} y^{b}] \right)
$$

(9.44)

$$
\times \left(\frac{\tau_{1} \tau_{2}}{(\tau_{1} + \tau_{3})^{2}} \tilde{\psi}^{+0 a} \left(\tau_{1} \tilde{z} - \tau_{2}(u + q), \frac{\tau_{3}}{\tau_{1} + \tau_{3}} y + \tau_{3} y, \frac{\tau_{1}}{\tau_{1} + \tau_{3}}\right)\right).
$$

By virtue of (8.3)–(8.5), all $z$-independent terms in the pre-exponential belong to $\mathcal{H}_{1}^{0+}$. However, by virtue of (8.4), the $z$-dependent term contributes to $\mathcal{H}_{1}^{0+}$ giving

$$
-2i \left(\frac{1}{2\pi}\right)^{2} \int d^{2}v d^{2}u \int d^{3}\tau \delta(1 - \sum_{i=1}^{3} \tau_{i}) \exp i[v_{a} u^{a} + \tau_{1} z_{a} y^{a} - \tau_{2} q_{b} y^{b}] \right)
$$

(9.44)

Finally, taking into account that $\tau_{2} = 1 - \tau_{1} - \tau_{3}$ due to the delta-function $\delta(1 - \sum_{i=1}^{3} \tau_{i})$ and that any additional factor of $\tau_{1} + \tau_{3}$ or $\tau_{3}$ effectively increases the power of $\tau_{1}$ hence
sending the result to $\mathcal{H}_1^{0+}$, we can replace $\tau_2$ by one and neglect the $y$-dependent terms in the argument of $\tilde{\psi}^{+0\alpha}$, that carry an additional factor of $\tau_3$, arriving at the final result

$$\tilde{\mathcal{H}}_1^{0+} = f_1^{0+} \mid_{\mathcal{H}_1^{0+}} := f_1^{0+} = -2i \frac{\tau_3^2}{(\tau_1 + \tau_3)^2} \tau_1 \delta(1 - \sum_{i=1}^3 \tau_i) \exp \left(\tau_1 z - u - q, \frac{\tau_3}{\tau_1 + \tau_3} v, \frac{\tau_1}{\tau_1 + \tau_3}\right).$$

(9.45)

Taking into account the definition (9.33) of $\tilde{\mathcal{H}}_1$, we arrive at $\mathcal{H}_2^{0+} \text{ Homotopy Lemma}:$

$$\Delta_{q,-\infty} \mathcal{H}_2^{0+} \subset \tilde{\mathcal{H}}_1.$$  

(9.46)

Note that setting $q = 0$ in (9.46) we achieve that $\mathcal{H}_2^{0+} \text{ Homotopy Lemma} (9.46)$ holds for the whole $\mathcal{H}_2^{0+}$ including the boundary elements associated with $\chi_2 (6.2)$.

$\mathcal{H}_2^{0+} \text{ Homotopy Lemma}$ is of great importance for the analysis of HS vertices. A representative of $\tilde{\mathcal{H}}_1^{0+}$ for inner elements of $\mathcal{H}_2^{0+}$ can be chosen in the form of $f_1^{0+} (9.43)$.

The following comment is now in order. If $\Delta_{q,\beta} \mathcal{H}_2^{0+}$ is evaluated at finite $\beta = -\varepsilon^{-1}$ the result may have the form

$$\Delta_{q,\beta} \mathcal{H}_2^{0+} \in \tilde{\mathcal{H}}_1 + \varepsilon \mathcal{H}_1^{0+},$$

(9.47)

i.e., the contribution to $\mathcal{H}_0^{0+}$ may be non-zero, being suppressed by the factor of $\varepsilon$. If, following the strategy of $[1]$, one would keep $\beta$ finite till arriving to the final result containing further action of $\Delta_{q,\beta}$ and $\tilde{h}_{q,\beta}$ taking the limit $\beta \to -\infty$ in the very end, this may lead to finite but different result since, being in general singular, $\Delta_{q,\beta} \mathcal{H}_0^{0+}$ may develop the terms containing a factor of $\varepsilon^{-1}$ that can cancel $\varepsilon$ in (9.47). Nevertheless, the final result will still be ultra-local because the form of the exponentials (7.33) remains unaffected by this procedure. Each of these limiting procedures is properly defined. Generally, one can consider three limiting parameters $\beta_2$ on two-forms, $\beta_1$ on one-forms and $\beta_0$ on zero-forms (in the cohomology projector $h_{q,\beta_0}$) with $\beta_i = \alpha_i \beta$ allowing various ratios of $\alpha_i$ including $\frac{\alpha_i}{\alpha_j} \to 0$ at $i > j$ implying that the limit $\beta \to -\infty$ is taken at every step as in this paper. The procedure of $[1]$ assumes $\alpha_2 = \alpha_1 = \alpha_0 = 1$. The results of application of different limiting prescriptions may differ at most by ultra-local field redefinitions.

10 Pre-ultra-locality and ultra-locality

The results of Section 9.2 have a number of important consequences allowing to prove ultra-locality of the vertices $\mathcal{T}_2 (\omega^2, C^2)$ in equations (9.37) on space-time one-forms $\omega$.

A remarkable property of formula (9.44) is that $f_1^{0+}$ is free from $y$-dependence in the arguments of $C$ if $q$ is independent of $C$-derivatives $p_j$, in particular at $q = 0$. Elements of $\mathcal{H}$ such that arguments of zero-forms $C$ are independent of $y$, will be called pre-ultra-local. Note that for elements bilinear in the zero-forms $C$, that respect the PLT conditions, pre-ultra-locality implies ultra-locality by virtue of (9.33) from which it follows that $P_{ij} = 0$ once $B_i = 0$. 

37
A subspace of $\mathcal{H}^{\nu\mu}$ that consists of pre-ultra-local forms will be denoted $\mathcal{P}^{\nu\mu}$. The space $\mathcal{U}^{\nu\mu} \subset \mathcal{P}^{\nu\mu}$ of ultra-local forms consists of elements with at most a finite number of contractions between either holomorphic or anti-holomorphic arguments of the zero-forms $C$.

Now we consider properties of these two spaces separately.

### 10.1 Pre-ultra-locality

#### 10.1.1 Pre-ultra-local spaces

From formulae (9.6), (9.7) it follows that the space $\mathcal{P}_0^{0+} \subset \mathcal{H}_0^{0+}$ of pre-ultra-local zero-forms in $\theta$ is closed under the star product modulo terms in ideal $\mathcal{I}$

$$\mathcal{P}_0^{0+} \ast \mathcal{P}_0^{0+} \subset \text{Span}(\mathcal{P}_0^{0+}, \mathcal{I}).$$

(10.1)

Indeed, by definition of pre-ultra-locality all additional contractions in (9.6), (9.7) among the $y$-dependent terms will not affect arguments of zero-forms $C$ free from the $y$-dependence.

From relations (9.2)-(9.4) it also follows that the space

$$\mathcal{P}_0 := \text{Span}(\mathcal{P}_0^{0+}, \mathcal{H}_0^{0+})$$

(10.2)

forms a subspace of $\mathcal{H}_0$

$$\mathcal{P}_0 \ast \mathcal{P}_0 \subset \mathcal{P}_0 \subset \mathcal{H}_0.$$

(10.3)

Clearly,

$$\mathcal{P}_0^{0+} \ast \mathcal{P}_0 \subset \mathcal{P}_0, \quad \mathcal{P}_0 \ast \mathcal{P}_0^{0+} \subset \mathcal{P}_0.$$  

(10.4)

Analogously, the space of pre-ultra-local one-forms $\mathcal{P}_1^{0+} \subset \mathcal{H}_1^{0+}$ and

$$\mathcal{P}_1 := \text{Span}(\mathcal{P}_1^{0+}, \mathcal{H}_1^{0+})$$

(10.5)

form $\mathcal{P}_0^{0+}$ - and $\mathcal{P}_0$ - bi-modules up to elements in $\mathcal{I}$

$$\mathcal{P}_0^{0+} \ast \mathcal{P}_1^{0+} \subset \text{Span}(\mathcal{P}_1^{0+}, \mathcal{I}), \quad \mathcal{P}_1^{0+} \ast \mathcal{P}_0^{0+} \subset \text{Span}(\mathcal{P}_0^{0+}, \mathcal{I}),$$

(10.6)

$$\mathcal{P}_0 \ast \mathcal{P}_1 \subset \mathcal{P}_1, \quad \mathcal{P}_1 \ast \mathcal{P}_0 \subset \mathcal{P}_1.$$  

(10.7)

Introducing the space $\tilde{\mathcal{P}}_1^{0+} \subset \tilde{\mathcal{H}}_1^{0+}$ as the pre-ultra-local subspace of the space $\tilde{\mathcal{H}}_1^{0+}$ introduced in Section 9.2.2 we define a space

$$\tilde{\mathcal{P}}_1 := \text{Span}(\tilde{\mathcal{P}}_1^{0+}, \tilde{\mathcal{H}}_1^{0+})$$

(10.8)

and using again product formulae (9.14) and (9.11) obtain

$$\mathcal{P}_0^{0+} \ast \tilde{\mathcal{P}}_1^{0+} \subset \text{Span}(\tilde{\mathcal{P}}_1^{0+}, \mathcal{I}) \subset \tilde{\mathcal{P}}_1,$$

$$\tilde{\mathcal{P}}_1^{0+} \ast \mathcal{P}_0^{0+} \subset \text{Span}(\tilde{\mathcal{P}}_1^{0+}, \mathcal{I}) \subset \tilde{\mathcal{P}}_1,$$

(10.9)

$$\mathcal{P}_0 \ast \tilde{\mathcal{P}}_1 \subset \tilde{\mathcal{P}}_1, \quad \tilde{\mathcal{P}}_1 \ast \mathcal{P}_0 \subset \tilde{\mathcal{P}}_1.$$  

(10.10)
10.1.2 Consequences

Formula (9.43) along with (10.8) implies the following

Pre-Ultra-Locality Theorem:

\[ \Delta_{0,-\infty}^{'} H_{2}^{+0} \subset \tilde{P}_{1}, \]  

(10.11)

from which it follows that if the r.h.s. of equations for \( S \) is in \( H_{2}^{+0} \) then, at this order, \( S \in \tilde{P}_{1} \).

As a simple consequence of (8.13) the arguments of zero-forms \( C \) in \( h_{0,-\infty}(P_{0}^{0+}) \) are \( y \)-independent. (More generally this is true for \( h_{q,-\infty}(P_{0}^{0+}) \) with \( q \) not acting on the arguments of \( C \).) Hence, using Factorization Lemma (8.9), one has

\[ h_{0,-\infty}(P_{0}) \subset P_{0}^{0+}. \]  

(10.12)

Note that the r.h.s. here only contains terms with boundary \( \tau \)-kernels (6.4).

Analogously, from (9.27) it follows that the arguments of zero-forms \( C \) in \( \Delta_{q,-\infty} P_{1}^{0+} \) with \( C \)-derivative-independent \( q \) are \( y \)-independent. Hence, taking into account (9.28) and definition (10.8), we obtain

\[ \Delta_{0,-\infty}(\tilde{P}_{1}) \subset P_{0}. \]  

(10.13)

As a result, by virtue of (10.3), (10.10) and (10.12) along with Factorization Lemma,

\[ h_{0,-\infty}\left(\Delta_{0,-\infty}(P_{0} \ast \tilde{P}_{1}) \ast P_{0}\right) \subset P_{0}^{0+}, \quad h_{0,-\infty}\left(\Delta_{0,-\infty}(\tilde{P}_{1} \ast P_{0}) \ast P_{0}\right) \subset P_{0}^{0+}, \]  

(10.14)

\[ h_{0,-\infty}\left(P_{0} \ast \Delta_{0,-\infty} P_{0} \ast \tilde{P}_{1}\right) \subset P_{0}^{0+}, \quad h_{0,-\infty}\left(P_{0} \ast \Delta_{0,-\infty}(\tilde{P}_{1} \ast P_{0})\right) \subset P_{0}^{0+}. \]

In particular, from here it follows by virtue of Pre-Ultra-Locality Theorem (10.11)

\[ h_{0,-\infty}\left(\Delta_{0,-\infty}(\Delta_{0,-\infty}(H_{2}^{+0}) \ast P_{0}) \ast P_{0}\right) \subset P_{0}^{0+}, \quad \text{etc.} \]  

(10.15)

For expressions bilinear in the zero-forms \( C \) pre-ultra-locality implies ultra-locality by virtue of PLT. As explained in Sections 11 and 12, this proves that the vertices \( \Upsilon_{2}(\omega^{2}, C^{2}) \) in (3.1) are ultra-local.

10.2 Ultra-locality

Properties of spaces \( U_{p} \) of \( p \)-forms (6.1) with ultra-local \( \tau \)-kernels are analogous to those with pre-ultra-local ones as we describe now.

Firstly, we observe that

\[ U_{p}^{0+} \subset \text{Span}(U_{0}^{0+}, I). \]  

(10.16)

Indeed, by definition of pre-ultra-locality, formulæ (3.6), (3.7) imply that additional contractions between the \( y \)-dependent terms will not affect \( y \)-independent arguments of zero-forms \( C \). From relations (3.2)-(3.4) it also follows that the space

\[ U_{0} := \text{Span}(U_{0}^{0+}, H_{0}^{+0}) \]  

(10.17)
form a subspace of $H_0$

$$U_0 \ast U_0 \subset U_0 \subset H_0.$$  \hfill (10.18)

Using Factorization Lemma (8.9) and formula (8.13) with $q = 0$ one has

$$h_{0,-\infty}(U_0) \in U_0^{0+}.$$  \hfill (10.19)

Analogously, the ultra-local space of one-forms $U_1^{0+} \subset H_1^{0+}$ and

$$U_1 := \text{Span}(U_1^{0+}, H_1^{0+})$$  \hfill (10.20)

form, respectively, $U_0^{0+}$- and $U_0$- bi-modules (modulo elements of $I$ in the former case)

$$U_0^{0+} \ast U_1^{0+} \subset \text{Span}(U_1^{0+}, I), \quad U_1^{0+} \ast U_0^{0+} \subset \text{Span}(U_1^{0+}, I),$$  \hfill (10.21)

$$U_0 \ast U_1 \subset U_1, \quad U_1 \ast U_0 \subset U_1.$$  \hfill (10.22)

Introducing the space $\tilde{U}_1^{0+}$ as the ultra-local subspace of $\tilde{H}_1^{0+}$ and

$$\tilde{U}_1 := \text{Span}(\tilde{U}_1^{0+}, H_1^{0+}),$$  \hfill (10.23)

and using again product formulae $9.14$ and $9.16$ we obtain

$$U_0^{0+} \ast \tilde{U}_1^{0+} \subset \text{Span}(\tilde{U}_1^{0+}, I) \subset \tilde{U}_1, \quad \tilde{U}_1^{0+} \ast U_0^{0+} \subset \text{Span}(\tilde{U}_1^{0+}, I) \subset \tilde{U}_1,$$  \hfill (10.24)

$$U_0 \ast \tilde{U}_1 \subset \tilde{U}_1, \quad \tilde{U}_1 \ast U_0 \subset \tilde{U}_1.$$  \hfill (10.25)

As above, one can see that by virtue of (9.28), (10.18), (10.19), (10.23) and (10.25) along with Factorization Lemma

$$\Delta_{0,-\infty}(\tilde{U}_1) \subset U_0$$  \hfill (10.26)

and

$$h_{0,-\infty} \left( \Delta_{0,-\infty}(U_0 \ast \tilde{U}_1) \ast U_0 \right) \subset U_0^{0+}, \quad \text{etc.}$$  \hfill (10.27)

11 Structure relation

11.1 Summary

Let us briefly summarize the key facts of the analysis performed so far.

Limiting contracting homotopy $\Delta_{0,-\infty}$ maps $H_2^{+0}$ to the space $\tilde{H}_1$ that gives finite pre-ultra-local result under the action of $\Delta_{q,-\infty}$. This implies that the contribution resulting from $I$ should be kept in the $\theta^2$ terms as giving rise to nontrivial $S$ fields in $\tilde{H}_1$. Also, the parts of $S$ in $I$ should be kept to compute the contribution to $S \ast S$ giving rise to higher-order corrections to $S = S_0 + \tilde{S}$ via

$$- 2i \bar{d}_Z \tilde{S} = - \tilde{S} \ast \tilde{S} + i(\eta B \ast \gamma + \bar{\eta} B \ast \bar{\gamma}).$$  \hfill (11.1)
For this scheme to work the r.h.s. of the equation on $S$ has to be in $\mathcal{H}_2^{-0}$. This is indeed the case in the first order in $C$ since $C \ast \gamma \in \mathcal{H}_2^{-0}$. In this section we show that this property also holds true in the second order thus allowing to apply the limiting homotopy formalism to the computation of the second-order in $C$ corrections to the equations on the one-form HS fields $\omega$ leading to a spin-ultra-local result in accordance with PLT and Pre-Ultra-Locality Theorem.

The central result of this section is *structure relation* that has the form

$$R_2 := \Delta_{a,0} \Delta_{b,0}(\gamma) \ast \gamma - \Delta_{a,0}(\gamma) \ast \Delta_{b,0}(\gamma) \in \mathcal{H}_2^{-0}.$$

(11.2)

It plays the key role in the perturbative analysis of the second-order in $C$ contribution to $S$ in the (anti)holomorphic sector. Indeed, the equation on $S_2$ in the holomorphic sector has the form

$$-2i d_2 S_2 + S_1 \ast S_1 - i \eta B_2 \ast \gamma = 0.$$

(11.3)

As shown in [9], by virtue of the star-exchange formulae the last two terms turn out to be proportional to $\Delta_{a,0} \Delta_{b,0}(\gamma) \ast \gamma - \Delta_{a,0}(\gamma) \ast \Delta_{b,0}(\gamma)$. By (9.46), (11.2) implies that

$$S_2 \in \mathcal{H}_1.$$

(11.4)

As a result, the second-order part of $W_2$ generated by $S_2$ is not only well defined in the limit $\beta \rightarrow -\infty$ but ultra-local by PLT and Pre-Ultra-Locality Theorem. Note that each of the two terms on the l.h.s. of (11.2) gives divergent contributions to $W_2$ in the limit $\beta \rightarrow -\infty$. However the contribution of the whole expression is finite.

### 11.2 The proof

First, from (7.22) it follows that

$$\Delta_{a,0}(\gamma) = 2(z^a + a^a) \theta_\alpha \int d\tau \tau \exp[i(\gamma z^a - (1 - \tau) a_\alpha y^\alpha)] k.$$

(11.5)

(Recall that at the first order $\Delta_{a,0}(\gamma)$ is independent of $\beta$ [1] and $\Delta_{a,0}(\gamma) = \Delta_{a,0}(\gamma)$.) Using (6.8) it is straightforward to compute $\Delta_{a,0}(\gamma) \ast \Delta_{b,0}(\gamma)$. The only comment is that the Klein operator $k$ from the first factor of $\gamma$ moved to the right changes a sign of the shift parameter $b$ acting on the fields standing on the left from the expression $\Delta_{a,0}(\gamma) \ast \Delta_{b,0}(\gamma)$. (For more detail see [2].) The final result is

$$\Delta_{a,0}(\gamma) \ast \Delta_{b,0}(\gamma) = 2 \theta^\alpha \theta_\alpha \int_0^1 d\tau_1 \int_0^{\tau_1} d\tau_2 \tau_2 \left(2i - \tau_1 \circ \tau_2 z^2 a^\alpha \right)
+ (1 - (1 - \tau_1)(1 - \tau_2))(a_\alpha b^\alpha - (a_\alpha - b_\alpha) y^\alpha) - (\tau_1(1 - \tau_2) b_\alpha + \tau_2(1 - \tau_1) a_\alpha) z^\alpha
\exp i[\tau_1 \circ \tau_2 z^2 a^\alpha + (1 - \tau_1)(1 - \tau_2)((a_\alpha b^\alpha - (a_\alpha - b_\alpha) y^\alpha) + (\tau_1(1 - \tau_2) b_\alpha + \tau_2(1 - \tau_1) a_\alpha) z^\alpha],$$

(11.6)

where the integral over $s^a$ and $t^\alpha$ has been evaluated by virtue of

$$\frac{1}{4\pi^2} \int d^2 s d^2 t \exp is_\alpha t^\alpha = 1, \quad \frac{1}{4\pi^2} \int d^2 s d^2 t s_\alpha t^\alpha \exp is_\alpha t^\alpha = 2i.$$

(11.7)
\[ \int d^2 s d^2 t \alpha \exp i s_\alpha t^\alpha = \int d^2 s d^2 t \alpha \exp i s_\alpha t^\alpha = 0. \quad (11.8) \]

Now we single out the terms that belong to \( \mathcal{I} \). Namely, all terms containing a factor of \((1 - \tau_1)(1 - \tau_2)\) are of this type because, multiplied by \( \tau_1 \tau_2 \) from the measure, by (5.13) these are dominated by \((\tau_1 \tau_2(1 - \tau_1 \tau_2))^2\), thus bringing additional degrees both in \( \tau_1 \tau_2 \) and in \((1 - \tau_1 \tau_2)\). As a result, \( \Delta_{a,0}(\gamma) \ast \Delta_{b,0}(\gamma) \) can be represented in the form

\[ \Delta_{a,0}(\gamma) \ast \Delta_{b,0}(\gamma) = X^X_1 + X^X_2 + X, \quad (11.9) \]

where

\[ X^X_1 = -2\theta^\alpha \theta_\alpha \int_0^1 d\tau_1 \tau_1(1 - \tau_1) \int_0^1 d\tau_2 \tau_2(1 - \tau_2) (a_\alpha b^\alpha - (a_\alpha - b_\alpha) y^\alpha) \]

\[ \exp i [\tau_1 \circ \tau_2 \alpha_\alpha y^\alpha + (1 - \tau_1)(1 - \tau_2)(a_\alpha b^\alpha - (a_\alpha - b_\alpha) y^\alpha) + (\tau_1(1 - \tau_2)b_\alpha + \tau_2(1 - \tau_1)a_\alpha) z^\alpha] \]

and, using that \( \int d\sigma \frac{\partial}{\partial \sigma} f(\sigma x) = f(x) - f(0) \),

\[ X^X_2 = 2\theta^\alpha \theta_\alpha \int_0^1 d\tau_1 \tau_1(1 - \tau_1) \int_0^1 d\tau_2 \tau_2(1 - \tau_2) \int_0^1 d\sigma (a_\alpha b^\alpha - (a_\alpha - b_\alpha) y^\alpha) \]

\[ (2i - \tau_1 \circ \tau_2 \alpha_\alpha y^\alpha + a_\alpha b^\alpha - (a_\alpha - b_\alpha) y^\alpha - (\tau_1(1 - \tau_2)b_\alpha + \tau_2(1 - \tau_1)a_\alpha) z^\alpha) \]

\[ \exp i [\tau_1 \circ \tau_2 \alpha_\alpha y^\alpha + \sigma(1 - \tau_1)(1 - \tau_2)(a_\alpha b^\alpha - (a_\alpha - b_\alpha) y^\alpha) + (\tau_1(1 - \tau_2)b_\alpha + \tau_2(1 - \tau_1)a_\alpha) z^\alpha] \]

belong to \( \mathcal{I} \), while

\[ X = 2\theta^\alpha \theta_\alpha \int_0^1 d\tau_1 \tau_1 \int_0^1 d\tau_2 \tau_2 \exp i \left( \tau_1 \circ \tau_2 \alpha_\alpha y^\alpha + (\tau_1(1 - \tau_2)b_\alpha + \tau_2(1 - \tau_1)a_\alpha) z^\alpha \right) \]

\[ (2i - \tau_1 \circ \tau_2 \alpha_\alpha y^\alpha + a_\alpha b^\alpha - (a_\alpha - b_\alpha) y^\alpha - (\tau_1(1 - \tau_2)b_\alpha + \tau_2(1 - \tau_1)a_\alpha) z^\alpha) \]. \quad (11.12) \]

Now we observe that

\[ \left( (1 - \tau_1) \frac{\partial}{\partial \tau_1} + (1 - \tau_2) \frac{\partial}{\partial \tau_2} \right) \tau_1(1 - \tau_2) = (1 - \tau_1)(1 - \tau_2) - \tau_1(1 - \tau_2), \]

\[ \left( (1 - \tau_1) \frac{\partial}{\partial \tau_1} + (1 - \tau_2) \frac{\partial}{\partial \tau_2} \right) \tau_2(1 - \tau_1) = (1 - \tau_1)(1 - \tau_2) - \tau_2(1 - \tau_1) \]

and, hence,

\[ \left( (1 - \tau_1) \frac{\partial}{\partial \tau_1} + (1 - \tau_2) \frac{\partial}{\partial \tau_2} \right) \tau_1 \circ \tau_2 = 2(1 - \tau_1)(1 - \tau_2) - \tau_1 \circ \tau_2. \]

\[ \quad (11.15) \]

This implies,

\[ i \left( (1 - \tau_1) \frac{\partial}{\partial \tau_1} + (1 - \tau_2) \frac{\partial}{\partial \tau_2} \right) \exp i \left( \tau_1 \circ \tau_2 \alpha_\alpha y^\alpha + (\tau_1(1 - \tau_2)b_\alpha + \tau_2(1 - \tau_1)a_\alpha) z^\alpha \right) \]

\[ = \left( \tau_1 \circ \tau_2 \alpha_\alpha y^\alpha + (\tau_1(1 - \tau_2)b_\alpha + \tau_2(1 - \tau_1)a_\alpha) z^\alpha - (1 - \tau_1)(1 - \tau_2)(2z_\alpha y^\alpha + (a_\alpha + b_\alpha) z^\alpha) \right) \]

\[ \exp i \left( \tau_1 \circ \tau_2 \alpha_\alpha y^\alpha + (\tau_1(1 - \tau_2)b_\alpha + \tau_2(1 - \tau_1)a_\alpha) z^\alpha \right) \]

42
and hence
\[
X = 2\theta^a \theta^a \int_0^1 d\tau_1 \int_0^1 d\tau_2 \left( 2i - i(1 - \tau_1) \frac{\partial}{\partial \tau_1} + (1 - \tau_2) \frac{\partial}{\partial \tau_2} \right) + a_a b^\alpha - (a_a - b_a)y^\alpha - (1 - \tau_1)(1 - \tau_2)(2z_\alpha y^\alpha + (a_a + b_a)z^\alpha) \right) \exp i \left( \tau_1 \circ \tau_2 z_\alpha y^\alpha + (\tau_1(1 - \tau_2)b_a + \tau_2(1 - \tau_1)a_\alpha)z^\alpha \right).
\]
(11.17)

Integration by parts gives
\[
X = Y + X_3^T,
\]
(11.18)
where \(X_3^T\) belongs to \(I\),
\[
I \ni X_3^T = 2\theta^a \theta_a \int_0^1 d\tau_1 \int_0^1 d\tau_2 \left( i\tau_1 \circ \tau_2 - \tau_1 \tau_2(1 - \tau_1)(1 - \tau_2)(2z_\alpha y^\alpha + (a_a + b_a)z^\alpha) \right) \exp i \left( \tau_1 \circ \tau_2 z_\alpha y^\alpha + (\tau_1(1 - \tau_2)b_a + \tau_2(1 - \tau_1)a_\alpha)z^\alpha \right)
\]
(11.19)
and
\[
Y = 2\theta^a \theta_a \int_0^1 d\tau_1 \int_0^1 d\tau_2 (a_a b^\alpha - (a_a - b_a)y^\alpha) \exp i \left( \tau_1 \circ \tau_2 z_\alpha y^\alpha + (\tau_1(1 - \tau_2)b_a + \tau_2(1 - \tau_1)a_\alpha)z^\alpha \right)
\]
(11.20)
is the remaining term in \(\Delta_{a,b}(\gamma) * \Delta_{b,0}(\gamma)\) that does not belong to \(I\). (Note that the analysis of the holomorphic vertex in \([1]\) contained partial integration being a zero-form image of this one.)

This can now be compared with the expression for \(\Delta_{a,0}\Delta_{b,0}(\gamma) * \gamma\) obtained in \([4]\) (recall that \(\Delta_{a,0}\) of this paper coincides with \(\Delta_a\) of \([4]\))
\[
\Delta_{a,0}\Delta_{b,0}(\gamma) * \gamma = 2\theta^a \theta_a \int d\sigma_1 d\sigma_2 \theta(\sigma_1) \theta(\sigma_2)(1 - \sigma_1 - \sigma_2)(a_\alpha - b_\alpha)(a^\alpha - y^\alpha) \times \exp i \left( (\sigma_1 + \sigma_2)z_\alpha y^\alpha + (\sigma_1 a_\alpha + \sigma_2 b_\alpha)z^\alpha \right),
\]
(11.21)
that is not difficult to obtain directly. We observe that \(Y\) and \(\Delta_a \Delta_b(\gamma) * \gamma\) have similar form up to the substitution
\[
\sigma_1 \rightarrow \tau_2(1 - \tau_1), \quad \sigma_2 \rightarrow \tau_1(1 - \tau_2).
\]
(11.22)
As we show now, this implies (11.22).

Indeed, from (9.17) we see that \(S_1 * S_1\) and hence \(Y\) belong to \(\mathcal{H}_2^{0+}\). Due to the factor of \(\tau_1 \tau_2\) in the measure of \(Y\) (11.20) the dominating part of \(Y = Y^{0+}\) comes from \(\tau_1 \sim 1 - \varepsilon_1, \tau_2 \sim 1 - \varepsilon_2\) with small \(\varepsilon_{1,2}\),
\[
Y \simeq 2\theta^a \theta_a \int_0^\varepsilon d\varepsilon_1 \varepsilon_1 \int_0^\varepsilon d\varepsilon_2 \varepsilon_2 (a_a b^\alpha - (a_a - b_a)y^\alpha) \exp i \left( (\varepsilon_1 + \varepsilon_2)z_\alpha y^\alpha + (\varepsilon_2 b_\alpha + \varepsilon_1 a_\alpha)z^\alpha \right),
\]
(11.23)
where $\epsilon$ is some small parameter. This expression coincides with the part of (11.21) resulting from the integration around small $\sigma_1$ and $\sigma_2$ that proves (11.2). The precise form of the l.h.s. of (11.2), which is quite tricky, will be presented elsewhere.

Let us stress that property (11.2) has been proven for any parameters $a_\alpha$ and $b_\beta$. Further simplifications, occur in accordance with Pre-Ultra-Locality Theorem (10.11) eliminating the $y$-dependence from zero-forms $C$ upon application of the limiting contracting homotopy and, if these parameters respect PLT, by Ultra-Locality Theorem implying that the resulting contributions to the sectors of one- and zero-forms in $\theta$ are ultra-local.

12 Example: ultra-locality of holomorphic $\Upsilon_2(\omega, \omega, C, C)$

Perturbative analysis sketched in Section 5 implies that the quadratic correction to the one-form sector of the field equations is (see also [1])

$$\Upsilon_2(\omega, \omega, C, C) = -h_{0,-\infty} (d_x W_1 + d_x W_2 + W_1 * W_1 + \{\omega, W_2\}_*) ,$$

where

$$W_2 = \frac{1}{2i} \Delta_{0,-\infty} (d_x S_1 + d_x S_2 + \{W_1, S_1\}_* + \{\omega, S_2\}_*),$$

$$S_2 = \frac{i}{2} \Delta_{0,-\infty} (i\eta B_2 * \gamma - S_1 * S_1).$$

By PLT the holomorphic part of $\Upsilon_2(\omega, \omega, C, C)$ belongs to the PLT-even class. Following [4], we consider the PLT-even contracting homotopy $\Delta_{0,-\infty}$ and the respective cohomology projector $h_{0,-\infty}$ allowing to discard the terms containing space-time differential $d_x$. In agreement with [4], the remaining terms will now be shown to be ultra-local by the following computation-independent arguments.

I. The contribution of $W_1 * W_1$ is ultra-local.

Indeed, from [4] one has

$$W_1 = \frac{i\eta}{2} \int d^3 \tau \left\{ C(y_1) \bar{\omega}(w_1) t^\alpha z_\alpha \exp i(\tau_1 z_\alpha (y^\alpha + p^\alpha) + t^\alpha (\tau_1 z_\alpha + \tau_3 y_\alpha - (1 - \tau_3) p_\alpha)) \right\} k|_{y_1 = w_1 = 0} + h.c.$$

with $t = -i \frac{\partial}{\partial w_1}$, $p = -i \frac{\partial}{\partial y_1}$ and convention that $h.c.$ (Hermitean conjugation) swaps barred and unbarred variables along with dotted and undotted indices.

By definition (10.17)

$$W_1 \in U_0.$$

Hence by virtue of (10.18) and (10.19) $W_1 * W_1 \in U_0$ and $h_{0,-\infty}(W_1 * W_1) \in U_0$. This means that the contribution of $W_1 * W_1$ to the field equations is ultra-local. □

II. The contribution of $\{W_1, S_1\}_*$ is ultra-local.
Indeed, from [9] one has

\[ S_1 = \eta \theta^\alpha z_\alpha \int_0^1 d\tau \exp(i\tau z_\alpha(y_0^\alpha + p_0^\alpha))k C(y_1)|_{y_1=0} + h.c. . \]  

(12.6)

Hence, by definition (10.23), \( S_1 \in \tilde{U}_1 \) and by virtue of (12.3) \( \{W_1, S_1\}_s \in \tilde{U}_1 \). By virtue of (10.26) \( \Delta_{0,\infty}(\{W_1, S_1\}_s) \in U_0 \). Then (10.18) and (10.19) give

\[ h_{0,\infty}\left(\{\Delta_{0,\infty}(\{S_1, W_1\}_s), \omega\}_s\right) \in U_0. \]  

(12.7)

Thus, the contribution to the field equations of the PLT-even expression \( \{W_1, S_1\}_s \) is ultra-local. □

III. The contribution of \( \{\omega, S_2\}_s \) is ultra-local.

Indeed, by virtue of \( S_2 \) (12.3) has a form

\[ S_2 = -\frac{\eta}{2} \Delta_{0,\infty}(C*C*(\Delta_{a,0}\Delta_{b,0}(\gamma)*\gamma - \Delta_{a,0}(\gamma)*\Delta_{b,0}(\gamma))) \]  

(12.8)

where \( a_\alpha = p_{1\alpha} + 2p_{2\alpha} \), \( b_\alpha = p_{2\alpha} \) with \( p_j \) (3.5).

Straightforwardly one can make sure that for any zero-form \( f_0(y) \)

\[ f_0(y) * f_2^{+0} \subset \mathcal{H}_2^{+0}, \quad f_2^{+0} * f_0(y) \subset \mathcal{H}_2^{+0}. \]  

(12.9)

Thus by virtue of structure relation (11.2) and Pre-Ultra-Locality Theorem (10.11) it follows that

\[ S_2 \in \Delta_{0,\infty}' \mathcal{H}_2^{+0} \in \tilde{P}_1. \]  

(12.10)

In \[2\] it was shown in particular that \( S_2 \) is PLT-even, then by virtue of (7.37) from (12.10) it follows

\[ \Delta_{0,\infty}' \mathcal{H}_2^{+0} \subset \tilde{U}_1. \]  

(12.11)

Since \( \omega \in U_0 \) then (10.18), (10.22), (10.26) and (10.27) give

\[ h_{0,\infty}\left(\{\Delta_{0,\infty}(\{S_2, \omega\}_s), \omega\}_s\right) \in U_0 \]  

(12.12)

whence the vertex is ultra-local. □

13 Conclusion

In this paper we have analysed spin-locality of the 4d HS theory in terms of classes of star-product functions that appear in the perturbative analysis of nonlinear equations of [1] based on the \( \beta \to -\infty \) limiting homotopy introduced in [1]. The space \( \mathcal{H} \) of star-product functions that appear in the perturbative analysis was introduced in [11]. It consists of two subspaces \( \mathcal{H} = \text{Span}(\mathcal{H}_0^+, \mathcal{H}_0^{+0}) \) such that elements of the zero-form sector in spinor differentials \( \mathcal{H}_0^{+0} \subset \mathcal{H}_0^+ \) do not contribute to the dynamical equations in the limiting homotopy formalism.
This fact is referred to as *Factorization Lemma* in this paper. Elements of $\mathcal{H}_0^{\pm 0}$ give rise to nonlocal contributions to vertices in HS field equations at finite $\beta$, that fits the interpretation of $\mathcal{H}^{0+}$ as a local subalgebra in $\mathcal{H}$ suggested in [10]. Also, we identified the two-sided ideal $\mathcal{I} = \mathcal{H}^{0+} \cap \mathcal{H}^{+0}$ elements of which can be discarded within the limiting homotopy procedure in all sectors of HS field equations that contain HS gauge fields $\omega$.

* A priori, application of the limiting homotopy prescription to general elements of $\mathcal{H}$ may not be well defined leading to the HS gauge fields $W$ divergent in the $\beta \to -\infty$ limit. This does not imply any divergency in the HS equations, that are well defined for any finite $\beta < 1$, but rather that inapplicability of the limiting homotopy may indicate that the theory is essentially nonlocal. Hence, it is important to have a sufficient criterion guaranteeing that this does not happen. This is provided by the $\mathcal{H}_2^{+0}$ *Homotopy Lemma* proven in the paper, which states that the limit $\beta \to -\infty$ is well defined provided that the two-form in spinorial differential $\theta$ on the r.h.s. of HS equations on $S$ belongs to $\mathcal{H}^{+0}$. It is shown that this is indeed true in the first and second orders in the zero-forms $C$. In the first order this fact is trivial while in the second it follows from the remarkable *Structure Relation* proven in Section 11.

Another important issue is to have a sufficient criterion for spin-locality of the resulting vertices. This is also found in this paper in the form of *Pre-Ultra-Locality Theorem* following from the Pfaffian Locality Theorem of [2] and its extension to $\beta$-dependent contracting homotopies given in this paper. Using remarkable form of the limiting contracting homotopy, it is shown that if the conditions of the $\mathcal{H}_2^{+0}$ *Homotopy Lemma* are fulfilled along with PLT conditions, the resulting HS vertex is ultra-local in terminology of [9], i.e., in addition to being spin-local, arguments of the zero-forms $C$ are independent of the spinor variable $y$. Using general properties of the limiting homotopy formalism it is shown that the resulting $\omega^2C^2$ vertices must be spin-local. This is of course in agreement with the detailed analysis of [1]. The developed technique is, however, promising from the perspective of the analysis of higher-order corrections.

It should be stressed that the analysis of this paper is heavily based on the specific form of HS equations (4.1), (4.2) and star product (4.9). In particular, it follows that the only version of the HS theory that admits spin-locality is that with linear function $F_*(B) = \eta B$ with some complex parameter $\eta$. In accordance with [16], all possible nonlinear terms in $F_*(B)$ contain unremovable spin-non-local terms resulting from star products of the factors of zero-forms $C(Y)$. This explains the distinguished role of linear function $F_*(B) = \eta B$ in HS theory from the holographic perspective: the HS theories with nonlinear $F_*(B)$ have some essentially nonlocal boundary duals. (It would be interesting to see which ones, however.)

The approach of this paper, which is applicable not only to the 4d HS theory of [3] but also to HS theory in 3d of [4], any d of [14] and Coxeter HS theories of [16], provides a step towards complete analysis of the level and role of non-locality in HS gauge theory. (For instance, in the model of [13] spin-locality, demanding at most a finite number of contractions between different zero-forms, should take place with respect to Lorentz-covariant components $Y^i_AV^A$ of the auxiliary variables $Y^i_A$ where $i = 1, 2$ is the $sp(2)$ vector index, $A = 0, \ldots d$ carries the vector representation of $o(d-1,2)$ and $V^A$ is the compensator field of the model.) So far
it agrees with the conjecture of [2] that HS theory should be spin-local in all orders of the perturbation theory. The identification of the spin-local formulation of the HS gauge theory should make it possible to analyze such important issues as causality and, in the framework of Coxeter HS theory of [16], relation with analogous aspects of String Theory.

The concept of spin-locality underlying analysis of HS interactions in terms of spinors, allows a clear interpretation in terms of usual $x$-space formulation. Namely, as explained in Section 3.2, spin-local theories are space-time local in terms of the original set of fields (say $C$) extended by their non-linear local currents $J^n(C_1, \ldots, C_n)$. In other words, in spin-local theories corrections to space-time dynamical equations have a form of local operators in terms of $J^n$ with various $n$. The class of spin-local theories sharing this property is just in between local theories with local vertices expressed directly in terms of $C$ and non-local ones where the current corrections themselves can be nonlocal. Note that the difference between local and spin-local theories matters only for the theories with infinite sets of fields as is the case in HS theories. We believe that the concept of spin-local theories of infinite sets of fields is just a proper substitute for that of local theories describing finite collections of fields.

The same time we believe that results of [1] and of this paper provide a proper basis for the extension of the study of HS interactions to all higher orders and, in particular, to the $C^3$ vertex in the equations for zero-forms that includes, in particular, the scalar self-interaction vertex. This will make it possible to compare the output of the limiting homotopy prescription in the bulk with the conclusions of the papers [34], [35] obtained via holographic reconstruction as well as with the paper [36] based on the light-cone formalism.

An interesting feature of the developed formalism is that it treats differently HS one-forms $\omega$ and zero-forms $C$. In the sector of higher spins this is just what is needed given that zero-forms $C$ contain infinite tails of higher derivatives of Fronsdal fields while one-forms $\omega$ contain at most a finite number of derivatives. However, the general version of the 4d HS theory [4] contains also an infinite set of topological (Killing-like) fields, each carrying at most a finite number of degrees of freedom. In this case the roles of one-forms and zero-forms are just swapped: zero-forms $C^{\text{top}}$ contain finite numbers of derivatives of the topological fields while one-forms $\omega^{\text{top}}$ contain infinite towers of derivatives. This can affect the analysis of locality in the cases when the HS and topological sectors get interacting, that can happen if some of the topological fields acquire a nontrivial VEV. In particular this happens in the 3d HS theory of [5] where the topological sector is related to the dynamical one. From this perspective the results of [1] and of this paper demand further investigation accounting this phenomenon.

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Appendix A. Useful formulae

As shown in [1], different contracting homotopy operators anticommute

\[ \Delta_{q,\beta_i} \Delta_{q,\beta_j} = -\Delta_{q,\beta_j} \Delta_{q,\beta_i} . \]  

(A.1)

In particular, each of them squares to zero

\[ \Delta_{q,\beta_i} \Delta_{q,\beta_i} = 0 . \]  

(A.2)

Also,

\[ h_{q,\beta_i} \Delta_{q,\beta_i} = 0 , \quad \Delta_{q,\beta_i} h_{q,\beta_i} = 0 , \quad h_{q,\beta_i} h_{q,\beta_i} = h_{q,\beta_i} . \]  

(A.3)

Redefined contracting homotopy operators

\[ \Delta'_{q,\beta} := \Delta_{(1-\beta)q,\beta} , \quad h'_{q,\beta} := h_{(1-\beta)q,\beta} \]  

(A.4)

obey star-exchange formulae of [1]

\[ \Delta'_{q,\beta} (a(y) \ast f(z, y, k, \theta)) = a(y) \ast \Delta'_{q+q\alpha,\beta} (f(z, y, k, \theta)) , \]  

(A.5)

\[ \Delta'_{q,\beta} (f(z, y, k, \theta) \ast a(y)) = \Delta'_{q-q\alpha,\beta} (f(z, y, k, \theta)) \ast a(y) \]  

and

\[ h'_{q,\beta} (a(y) \ast f(z, y, k, \theta)) = a(y) \ast h'_{q+q\alpha,\beta} (f(z, y, k, \theta)) , \]  

(A.6)

\[ h'_{q,\beta} (f(z, y, k, \theta) \ast a(y)) = h'_{q-q\alpha,\beta} (f(z, y, k, \theta)) \ast a(y) , \]

where \( q\alpha \) represents the shift of the argument of \( a(y) \).

Using that \( \gamma \ast a(y) = a(y) \ast \gamma \) we obtain

\[ \Delta'_{q,\beta} (\gamma) \ast a(y) = a(y) \ast \Delta'_{q+2q\alpha,\beta} (\gamma) . \]  

(A.7)

Appendix B. Contracting homotopy derivation

Here we outline the main steps of the derivation of formula (7.28) following [1] where it was derived for the case of \( q = 0 \). Applying (7.21) to (6.1) we obtain

\[ \Delta_{q,\beta} f(z, y, \theta) = \frac{1}{(2\pi)^2} \int d^2 u d^2 v \int_0^1 d\tau \int_0^1 dt t^{p-1} \exp i[v_{\beta\beta} u^\beta + \tau (z + (1-t)(u-q))\alpha(\beta v + y)\alpha] \times (z + q - u)^\alpha \frac{\partial}{\partial \theta^\alpha} \phi(\tau (z + (1-t)(u-q)), (1-\tau)(\beta v + y), \tau\theta, \tau) . \]  

(B.1)
Now, shifting \( u \to u + q \) and introducing new integration variables,

\[
\tau_1 = t\tau, \quad \tau = \tau_1 + \tau_2, \quad 1 - \tau = \tau_3,
\]

with the Jacobian

\[
\det \left| \frac{\partial \tau, t}{\partial \tau_i} \right| = (\tau_1 + \tau_2)^{-1}
\]

we obtain

\[
\Delta_{q,\beta}f(z, y, \theta) = \int \frac{d^2u d^2v}{(2\pi)^2} \int_{\tau_1} d^3\tau \exp i[v_\beta(u^\beta + q^\beta) + (\tau_1 z + \tau_2 u)\beta(v + y)^\beta]
\]

\[
(z - u)^\alpha \frac{\partial}{\partial \theta^\alpha} \phi(\tau_1 z + \tau_2 u), \tau_3(\beta v + y), (\tau_1 + \tau_2)\theta, \tau_1 + \tau_2).
\]

(B.4)

Then, shifting the integration variables,

\[
u_\alpha \to u_\alpha + \frac{\tau_1 \beta}{1 - \tau_2 \beta} z_\alpha, \quad v_\alpha \to (1 - \tau_2 \beta)^{-1}(v_\alpha + \tau_2 y_\alpha)
\]

we have

\[
\Delta_{q,\beta}f(z, y, \theta) = \int \frac{d^2u d^2v}{(2\pi)^2} \int_{\tau_1} d^3\tau \exp i[v_\beta u^\beta + \frac{\tau_1}{1 - \beta \tau_2} z_\alpha y^\alpha + \frac{1}{1 - \tau_2 \beta}(v_\alpha + \tau_2 y_\alpha) q^\alpha]
\]

\[
((1 - (\tau_1 + \tau_2)\beta)z - (1 - \tau_2 u)\beta) \frac{\partial}{\partial \theta^\alpha} \phi \left( \frac{\tau_1}{1 - \beta \tau_2} z + \tau_2 u, \frac{\tau_3}{1 - \beta \tau_2} (y + \beta v), \theta, \tau_1 + \tau_2 \right).
\]

(B.6)

To reduce this expression to the desired form (6.1) we finally change variables to

\[
\tau_1' = \frac{\tau_1}{1 - \beta \tau_2}, \quad \tau_3' = \frac{\tau_3}{1 - \beta \tau_2}, \quad \tau_2' = \frac{(1 - \beta) \tau_2}{1 - \beta \tau_2}.
\]

(B.7)

This simplicial map preserves the class of simplices of unit perimeter in the sense that

\[
\sum_{i=1}^{3} \tau_i' = 1
\]

as a consequence of \( \sum_{i=1}^{3} \tau_i = 1 \). The Jacobian is

\[
\det \left| \frac{\partial \tau_i'}{\partial \tau_j} \right| = \frac{1 - \beta}{(1 - \beta \tau_2)^3}.
\]

(B.9)

Using also that

\[
1 - \beta \tau_2 = \frac{1 - \beta}{1 - \beta (1 - \tau_2')}
\]

(B.10)

we finally obtain (7.27) after discarding primes.
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