Abstract. We consider the linear system of viscoelasticity with the homogeneous Dirichlet boundary condition. First we prove a Carleman estimate with boundary values of solutions of viscoelasticity system. Since a solution $u$ under consideration is not assumed to have compact support, in the decoupling of the Lamé operator by introducing $\text{div } u$ and $\text{rot } u$, we have no boundary condition for them, so that we have to carry out arguments by a pseudodifferential operator. Second we apply the Carleman estimate to an inverse source problem of determining a spatially varying factor of the external source in the linear viscoelasticity by extra Neumann data on the lateral subboundary over a sufficiently long time interval and establish the stability estimate.

1. Introduction and main results

Let $T$ be a positive constant, $x' = (x_1, ..., x_n) \in \mathbb{R}^n$, and $\Omega$ be a bounded domain in $\mathbb{R}^n$ with $\partial \Omega \in C^\infty$, let $\vec{\nu} = \vec{\nu}(x')$ be the unit outward normal vector at $x'$ to $\partial \Omega$. Let $x' \in \mathbb{R}^n$ be the spatial variable, $x_0$ be the time variable, and we set $x = (x_0, x') = (x_0, x_1, ..., x_n)$. Here and henceforth $\cdot^T$ denotes the transposes of vectors and matrices under consideration, and $D = (D_0, D')$, $D_0 = \frac{1}{i}\partial_{x_0}$, $D' = (\frac{1}{i}\partial_{x_1}, \ldots, \frac{1}{i}\partial_{x_n})$, $i = \sqrt{-1}$, $\nabla' = (\partial_{x_1}, \ldots, \partial_{x_n})$, $\nabla = (\partial_{x_0}, \nabla')$.

In the cylinder domain $Q := (-T, T) \times \Omega$, we consider the system of viscoelasticity

\begin{align}
(1.1) \quad & P(x, D)u \equiv \rho\partial_{x_0}^2 u - L_{\lambda, \mu}(x, D')u + \int_0^{x_0} L_{\lambda, \mu}(x, \tilde{x}_0, D')u(\tilde{x}_0, x')d\tilde{x}_0 = F, \\
(1.2) \quad & u|_{\partial \Omega([-T, T] \times \partial \Omega)} = 0, \quad u(T, \cdot) = \partial_{x_0} u(T, \cdot) = u(-T, \cdot) = \partial_{x_0} u(-T, \cdot) = 0,
\end{align}

where $u(x) = (u_1(x), \ldots, u_n(x))^T$ is the displacement and $F(x) = (F_1(x), \ldots, F_n(x))^T$ is an external force. For the functions $\lambda(x)$ and $\mu(x)$, the partial differential operator $L_{\lambda, \mu}(x, D')$
is defined by
\[ L_{\lambda,\mu}(x, D') u = \mu(x)\Delta u + (\mu(x) + \lambda(x))\nabla'\text{div } u \]
(1.3)
\[ + (\text{div } u)\nabla'\lambda + (\nabla' u + (\nabla' u)^T)\nabla'\mu, \quad x \in Q, \]
while \( L_{\tilde{\lambda},\tilde{\mu}}(x, \tilde{x}_0, D') \) is defined by
\[ L_{\tilde{\lambda},\tilde{\mu}}(x, \tilde{x}_0, D') u = \tilde{\mu}(x, \tilde{x}_0)\Delta u + (\tilde{\mu}(x, \tilde{x}_0) + \tilde{\lambda}(x, \tilde{x}_0))\nabla'\text{div } u \]
(1.4)
\[ + (\text{div } u)\nabla'\lambda + (\nabla' u + (\nabla' u)^T)\nabla'\mu, \quad (x, \tilde{x}_0) \in Q \times (-T, T), \]

The coefficients \( \rho, \lambda, \mu, \tilde{\lambda}, \tilde{\mu} \) are assumed to satisfy
(1.5) \( \rho, \lambda, \mu \in C^2(Q), \quad \rho(x) > 0, \mu(x) > 0, \lambda(x) + \mu(x) > 0 \) for \( x \in \overline{Q} \)
and
(1.5) \( \tilde{\lambda}, \tilde{\mu} \in C^2(\overline{Q} \times [-T, T]). \)

The equation (1.1) is a model equation for the viscoelasticity. The viscoelasticity indicates a mixed physical property of the viscosity and the elasticity. The research for the viscoelasticity dates back to Maxwell, Boltzmann, Kelvin. There are some important applications of the viscoelasticity materials related to modern technology. For example for medical diagnosis, one has to take into consideration that much of human tissues are viscoelastic (see works of de Buhan [5], Sinkus, Tanter, Xydeas, Catheline, Bercoff and Fink [36] and a monograph Lakes [30], and the references therein). Finally we mention that in the theory of viscoelasticity some other equations was proposed (see the monograph of Renardi, Hrusa and Nohel [34]), and the equation (1.1) is one of them and called the equation of linear viscoelasticity.

The purpose of this paper is to establish the following for (1.1):

1. a Carleman estimate for functions without compact supports;
2. the Lipschitz stability in an inverse source problem of determining spatially varying factor of the external force \( F \).

A Carleman estimate is an \( L^2 \)-weighted estimate of solution \( u \) to (1.1) which holds uniformly in large parameter. Carleman estimates have been well studied for single equations (e.g., Hörmander [13], Isakov [25]). A Carleman estimate yields several important results such as the unique continuation, the energy estimate called an observability inequality and the stability in inverse problems. However for systems whose principal part is coupled, for example even for isotropic Lamé system (that is, (1.1) with \( \tilde{\lambda} = \tilde{\mu} \equiv 0 \)), the Carleman estimate is difficult to be proved for functions \( u \) whose supports are not compact in \( Q \). In particular, for (1.1), no such Carleman estimates are not known.

In establishing a Carleman estimate for our system (1.1) for non-compactly supported \( u \), we can emphasize the two main difficulties:

- The principal part \( \rho \partial^2_{\tilde{x}_0} - L_{\lambda,\mu}(x, D') \) is strongly coupled.
- The Lamé operator \( L_{\tilde{\lambda},\tilde{\mu}}(x, \tilde{x}_0, D') \) appears as an integral.

In particular, because of the first difficulty, in all the existing papers de Buhan [5], de Buhan and Osses [6], Lorenzi, Messina and Romanov [32], Lorenzi and Romanov [33], Romanov and Yamamoto [35], it is assumed that functions under consideration have compact
supports or some special conditions are satisfied in proving Carleman estimates, so that observation data for the inverse problems have to be taken over the whole lateral boundary $(0, T) \times \partial \Omega$.

As for Carleman estimates for functions without compact supports and applications to inverse problems for the Lamé system without the integral terms, we refer to Imanuvilov and Yamamoto [19] - [22]. For the Carleman estimate for with Lamé system with $L_{\tilde{\lambda}, \tilde{\mu}}(x, \tilde{x}_0, D') = 0$, see Bellassoued, Imanuvilov and Yamamoto [2], Bellassoued and Yamamoto [3], Imanuvilov, Isakov and Yamamoto [24]. In this paper, we modify the arguments in those papers and establish a Carleman estimate for (1.1) for $u$ not having compact supports. Then we apply the Carleman estimate for an inverse source problem by modifying the method in Imanuvilov and Yamamoto [16] - [18] and Beilina, Cristofol, Li and M. Yamamoto [1] discussing scalar hyperbolic equations. As for the methodology for applying Carleman estimates to inverse problems, we refer to a pioneering paper Bukhgeim and Klibanov [7].

For the statement of the Carleman estimate for (1.1), we need to introduce notations and definitions.

We define the Poisson bracket by the formula

\[ \{ \varphi, \psi \} = \sum_{j=0}^{n} \partial_{\xi_j} \varphi \partial_{x_j} \psi - \partial_{\xi_j} \psi \partial_{x_j} \varphi. \]

By $\bar{z}$ we denote the complex conjugate of $z \in \mathbb{C}$, and we set $< a, b > = \sum_{k=0}^{n} a_k \bar{b}_k$ for $a = (a_0, \ldots, a_n), b = (b_0, \ldots, b_n) \in \mathbb{C}^n, \xi = (\xi_0, \ldots, \xi_n), \xi' = (\xi_1, \ldots, \xi_n), \tilde{\xi} = (\xi_0, \ldots, \xi_{n-1}), \tilde{\nabla} = (\partial_{\xi_0}, \ldots, \partial_{\xi_{n-1}}), \tilde{s} = (\xi_0, \ldots, \xi_{n-1}, \tilde{s}), \tilde{D} = (D_0, \ldots, D_{n-1}).$

For $\beta \in C^2(\Omega)$, we introduce the symbol:

\[ p_{\beta}(x, \xi) = \rho(x)\xi_0^2 - \beta(x)|\xi'|^2. \]

Let $\Gamma_0$ be some relatively open subset on $\partial \Omega$. We set $\tilde{\Gamma} = \partial \Omega \setminus \Gamma_0$ and $\Sigma_0 = (-T, T) \times \Gamma_0$, $\tilde{\Sigma} = (-T, T) \times \tilde{\Gamma}$.

In order to prove the Carleman estimate for the viscoelastic Lamé system, we assume the existence of the real-valued function $\psi$ which is pseudoconvex with respect to the symbols $p_\mu(x, \xi)$ and $p_{\lambda+2\mu}(x, \xi)$. More precisely,

**Condition 1.1.** There exists a function $\psi \in C^3(\Omega)$ such that $\nabla \psi(x) \neq 0$ on $\overline{\Omega}$,

\[ p_{\beta}(x, \xi - is\nabla \psi(x)), p_{\beta}(x, \xi + is\nabla \psi(x)) \}/2is > 0, \quad \forall \beta \in \{ \mu, \lambda + 2\mu \} \]

if $(x, \xi, s) \in \overline{\Omega} \times (\mathbb{R}^{n+1} \setminus \{0\}) \times \mathbb{R}_+ \setminus \{0\}$ satisfies $p_{\beta}(x, \xi + is\nabla \psi(x)) = 0$. On the lateral boundary, we assume

\[ p_{\mu}(x, \nabla \psi)|_{\Sigma_0} < 0, \quad \text{and} \quad \partial_{\nu} \psi|_{\Sigma_0} < 0. \]

Moreover we assume that

\[ \partial_{x_0} \psi(x) < 0 \quad \text{on} \quad (0, T] \times \overline{\Omega} \quad \text{and} \quad \partial_{x_0} \psi(x) > 0 \quad \text{on} \quad [-T, 0) \times \tilde{\Omega}. \]
If the function \( \psi \) satisfies Condition [1.1] then for any constant \( C \) the function \( \psi + C \) also satisfies Condition [1.1]. Hence, without loss of generality, we can assume that \( \psi \) is strictly positive on \( \bar{Q} \). Moreover, let us assume

\[
\nabla' \psi(x) \neq 0 \quad \text{on} \quad \bar{Q}, \quad \psi(x) > 0 \quad \text{on} \quad \bar{Q}.
\]

Using the function \( \psi \), we introduce the function \( \varphi(x) \) by

\[
\varphi(x) = e^{\tau \psi(x)}, \quad \tau > 1,
\]

where the parameter \( \tau > 0 \) will be fixed below.

For any function \( f = (f_1, \ldots, f_n) \) we introduce the differential form \( \omega_f = \sum_{j=1}^n f_j dx_j \).

\[
d\omega_f = \sum_{i<j}^n (\frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i}) dx_i \wedge dx_j.
\]

We identify the differential form \( d\omega_f \) with the vector-function:

\[
d\omega_f = \left( \frac{\partial f_1}{\partial x_2} - \frac{\partial f_2}{\partial x_1}, \ldots, \frac{\partial f_1}{\partial x_n} - \frac{\partial f_n}{\partial x_1}, \ldots, \frac{\partial f_{n-1}}{\partial x_n} - \frac{\partial f_n}{\partial x_{n-1}} \right).
\]

Denote

\[
\|u\|_{\mathcal{B}(\psi,s,\tau,Q)}^2 = \int_Q \left( \sum_{|\alpha| = 0}^2 (s\varphi)^{4-2|\alpha|} \tau^6 \|\partial_\alpha u\|^2 \right. + s\varphi \tau^2 |\nabla d\omega u|^2 \left. + (s\varphi)^3 \tau^4 |\nabla \div u|^2 \right) e^{2\varphi \psi} dx,
\]

\[
\|u\|_{\mathcal{X}(\psi,s,\tau,\Omega)}^2 = \int_\Omega \left( \sum_{|\alpha| = 0}^2 (s\varphi)^{4-2|\alpha|} \tau^{5-2|\alpha'|} \|\partial_\alpha' u\|^2 \right. + s\varphi \tau^2 |\nabla' d\omega u|^2 \left. + (s\varphi)^3 \tau^4 |\nabla' \div u|^2 \right) e^{2\varphi \psi} dx',
\]

where \( \mathbb{N} := \{1,2,3,...\}, \alpha = (\alpha_0, \alpha') = (\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n), \alpha' = (\alpha_1, \alpha_2, \ldots, \alpha_n), \alpha_j \in \mathbb{N} \cup \{0\}, \partial_\alpha = \partial_{\alpha_0} \partial_{\alpha_1} \partial_{\alpha_2} \ldots \partial_{\alpha_n} \).

Finally, we introduce the norm

\[
\|Fe^{s\varphi}\|_{\mathcal{Y}(\psi,s,\tau,Q)}^2 = \|\div F e^{s\varphi}\|_{L^2(Q)}^2 + \|d\omega F e^{s\varphi}\|_{L^2(Q)}^2 + s\tau^2 \|\varphi^{\frac{3}{2}} Fe^{s\varphi}\|_{L^2(\Sigma)}^2 + \|Fe^{s\varphi}\|_{L^2(Q)}^2.
\]

Now we are ready to state our first main result, a Carleman estimate as follows.

**Theorem 1.1.** Let \( F, \div F, d\omega_F \in L^2(Q), u \in H^1(Q) \cap L^2(0,T; H^2(\Omega)) \) satisfy [1.1], [1.2]. Moreover let [1.4] - [1.9] be satisfied and let the function \( \varphi \) be determined by [1.10]. Then there exist \( \tau_0 > 0 \) and \( s_0 > 0 \) such that for any \( \tau > \tau_0 \) and any \( s > s_0 \) the following estimate holds true:

\[
\|u\|_{\mathcal{B}(\psi,s,\tau,Q)}^2 + s\tau \|\varphi^{\frac{3}{2}} \nabla \partial_\tau u e^{s\varphi}\|_{L^2(\Sigma)}^2 + s^3 \tau^3 \|\varphi^{\frac{3}{2}} \partial_\tau u e^{s\varphi}\|_{L^2(\Sigma)}^2 \leq C_1 (\|Fe^{s\varphi}\|_{\mathcal{Y}(\psi,s,\tau,Q)}^2 + s^3 \tau^3 \|\varphi^{\frac{3}{2}} \partial_\tau u e^{s\varphi}\|_{L^2(\Sigma)}^2 + s\tau \|\varphi^{\frac{3}{2}} \nabla \partial_\tau u e^{s\varphi}\|_{L^2(\Sigma)}^2),
\]

where the constant \( C_1 > 0 \) is independent of \( s \) and \( \tau \).
The proof of Theorem 1.1 is given in Sections 2-6.

Next we apply Carleman estimate (1.11) to an inverse source problem of determining a spatially varying factor of source term of the form $F(x) := R(x)f(x')$. Now we assume that $\rho, \lambda, \mu$ are independent of $x_0$: $\rho(x) = \rho(x'), \lambda(x) = \lambda(x'), \mu(x) = \mu(x')$ for $x \in (0, T) \times \Omega$. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain. We consider

\begin{equation}
(1.12) \quad \rho(x')\frac{\partial^2}{\partial x_0^2}y = L_{\lambda, \mu}(x, D')y - \int_0^{x_0} L_{\tilde{\lambda}, \tilde{\mu}}(x, x_0, D')y(x, x_0)d\tilde{x}_0 + R(x)f(x') \quad \text{in} \quad (0, T) \times \Omega,
\end{equation}

\begin{equation}
(1.13) \quad y(0, \cdot) = \partial_{x_0}y(0, \cdot) = 0, \quad y|_{(0, T) \times \partial \Omega} = 0.
\end{equation}

Here $R(x)$ is an $n \times n$ matrix function and $f(x')$ is an $\mathbb{R}^n$-valued function.

We further assume

\begin{equation}
(1.14) \quad \tilde{\lambda}, \tilde{\mu} \in C^2([0, T] \times \overline{\Omega} \times [0, T]).
\end{equation}

We consider

**Inverse source problem.** Let the function $R$ be given and $\Gamma \subset \partial \Omega$. Determine $f(x')$ by $\partial_{\vec{\nu}}y|_{(0, T) \times \Gamma}$.

We state our main result on the inverse source problem.

**Theorem 1.2.** We assume that there exists a function $\psi$ which satisfies Condition (1.1), (1.7) - (1.9),

\begin{equation}
(1.15) \quad \{x'; \ x \in [-T, T] \times \partial \Omega, \ \partial_{\vec{\nu}}\psi(x) \geq 0\} \subset \Gamma
\end{equation}

and

\begin{equation}
(1.16) \quad \inf_{x' \in \Omega} \psi(0, x') > \max\{\inf_{x' \in \Omega} \psi(T, x'), \inf_{x' \in \Omega} \psi(-T, x')\}.
\end{equation}

The Lamé coefficients $\rho, \lambda, \mu$ satisfy (1.4) and $\tilde{\lambda}, \tilde{\mu}$ satisfy (1.14). Let

\begin{equation}
(1.17) \quad |\det R(0, x')| \geq \delta_0 > 0, \ x' \in \overline{\Omega}, \ R \in C^{2,1}([0, T] \times \Omega), \ \partial_{x_0}R(0, x') = 0
\end{equation}

with some constant $\delta_0 > 0$, and $y, \nabla'y, \partial_{x_0} \nabla'y \in H^1((0, T) \times \Omega)$ satisfy (1.12) and (1.13), and $\partial_{x_0}^2 y \in C([0, T_1]; H^{\frac{3}{2}+\epsilon_0}(\Omega))$ with some positive $\epsilon_0$ and $T_1$.

Then there exists a constant $C_2 > 0$ such that

$$
\|f\|_{H^1(\Omega)} \leq C_2 \left( \sum_{j=0}^1 \|\nabla \partial_{x_0}^j \partial_{\vec{\nu}}y\|_{L^2((0, T) \times \Gamma)} + \|\partial_{\vec{\nu}} \partial_{x_0}^2 y(0, \cdot)\|_{L^2(\Gamma)} \right)
$$

for all $f \in H^1_0(\Omega)$. 

Our viscoelasticity equation has a finite propagation speed and so for determination of \( f \) over the whole domain \( \Omega \), the observation time \( T \) must be longer than some critical value \( T_0 \).

The proof of Theorem \[ \text{1.1} \] is provided in Section \[ 7 \].

There are other works on inverse problems related to the viscoelasticity and we refer to Cavaterra, Lorenzi and Yamamoto \[ 3 \], Grasselli \[ 12 \], Janno \[ 26 \], Janno and von Wolfersdorf \[ 27 \], Loreti, Sforza and Yamamoto \[ 31 \], von Wolfersdorf \[ 40 \].

2. Proof of Theorem \[ \text{1.1} \]

Sections \[ 2-6 \] are devoted to the proof of Theorem \[ \text{1.1} \] and in Section \[ 8 \] we collect necessary lemmata for the proof.

For the function \( \beta \) we introduce the operator

\[
\Box_{\rho, \beta}(x, D) = \rho(x) \partial^2_{x_0} - \beta(x) \Delta.
\]

It is well known that the functions \( d\omega \), \( \div \mathbf{u} \) satisfy the equations

\[
\Box_{\rho, \mu}(x, D)d\omega - \int_0^{x_0} \tilde{\mu}(x, \tilde{x}_0) \Delta d\omega \, d\tilde{x}_0 = q_1 \quad \text{in } Q,
\]

\[
\Box_{\rho, \lambda + 2\mu}(x, D)d\omega - \int_0^{x_0} (\tilde{\lambda} + 2\tilde{\mu})(x, \tilde{x}_0) \Delta d\omega \, d\tilde{x}_0 = q_2 \quad \text{in } Q,
\]

where \( K_j(x, D), K_j(x, \tilde{x}_0, D) \) are first-order differential operators with \( C^1 \) coefficients. Now we introduce a new unknown function \( \mathbf{v} = (v_1, v_2) \) by formulae

\[
v_1 = d\omega \mathbf{u} - \int_0^{x_0} \frac{\tilde{\mu}(x, \tilde{x}_0)}{\mu(x)} d\omega \mathbf{u} \, d\tilde{x}_0, \quad v_2 = \div \mathbf{u} - \int_0^{x_0} \frac{(\tilde{\lambda} + 2\tilde{\mu})(x, \tilde{x}_0)}{(\lambda + 2\mu)(x)} \div \mathbf{u} \, d\tilde{x}_0.
\]

More specifically

\[
\mathbf{v}_1 = (v_{1,2}, \ldots, v_{n-1,n}), \quad v_{k,j} = \frac{\partial u_k}{\partial x_j} - \frac{\partial u_j}{\partial x_k} - \int_0^{x_0} \frac{\tilde{\mu}(x, \tilde{x}_0)}{\mu(x)} \left( \frac{\partial u_k}{\partial x_j} - \frac{\partial u_j}{\partial x_k} \right)(\tilde{x}_0, x') \, d\tilde{x}_0.
\]

Then

\[
\mathbf{P}(x, D) \mathbf{v} = \left( \Box_{\rho, \mu}(x, D) \mathbf{v}_1, \Box_{\rho, \lambda + 2\mu}(x, D) \mathbf{v}_2 \right) = \mathbf{q} \quad \text{in } Q,
\]

where \( \mathbf{q} = (q_3, q_4) = (q_1 + \tilde{q}_1, q_2 + \tilde{q}_2) \):

\[
\tilde{q}_1 = - \int_0^{x_0} \left( 2(\nabla \tilde{\mu}(x, \tilde{x}_0), \nabla' d\omega \mathbf{u}) + \Delta \frac{\tilde{\mu}(x, \tilde{x}_0)}{\mu(x)} d\omega \mathbf{u} \right) \, d\tilde{x}_0 - \rho \partial^2_{x_0} \int_0^{x_0} \frac{\tilde{\mu}(x, \tilde{x}_0)}{\mu(x)} d\omega \mathbf{u} \, d\tilde{x}_0,
\]

\[
\tilde{q}_2 = - \int_0^{x_0} \left( 2(\nabla'(\tilde{\lambda} + 2\tilde{\mu})(x, \tilde{x}_0), \nabla' \div \mathbf{u}) + \Delta \frac{\tilde{\mu}(x, \tilde{x}_0)}{\mu(x)} \div \mathbf{u} \right) \, d\tilde{x}_0
\]

\[
- \rho \partial^2_{x_0} \int_0^{x_0} \frac{(\tilde{\lambda} + 2\tilde{\mu})(x, \tilde{x}_0)}{(\lambda + 2\mu)(x)} \div \mathbf{u} \, d\tilde{x}_0.
\]
We have
\[
\|q e^{s\varphi}\|_{L^2(Q)} \leq C_1(\|\nabla d\omega u e^{s\varphi}\|_{L^2(Q)} + \|\nabla \text{div} u e^{s\varphi}\|_{L^2(Q)} + \|d\omega u e^{s\varphi}\|_{L^2(Q)} + \|\text{div} u e^{s\varphi}\|_{L^2(Q)} + \|d\omega e^{s\varphi}\|_{L^2(Q)} + \|\text{div} e^{s\varphi}\|_{L^2(Q)}).
\]
(2.5)

As for the boundary conditions we have
\[
\mathcal{B}(x, D')v = g \quad \text{on } \Sigma,
\]
where
\[
g = (g_1, \ldots, g_{n^2+n}), \quad \mathcal{B}(x, D') = (\tilde{B}(x, D'), \mathcal{C}(x')),
\]
and
\[
b_j(x, D')v = - \sum_{j=1, j \neq \hat{j}}^n \text{sign}(\hat{j} - j) \partial_x_j v_{j, \hat{j}} - \frac{\lambda + 2\mu}{\mu} \partial_x \varepsilon_{j, \hat{j}}
\]
for \(1 \leq \hat{j} \leq n\), and \(\mathcal{C}(x)\) is the smooth matrix constructed in the following way: Consider a matrix of size \(n \times n\) such that on main diagonal we have \(\nu_n(x')\), the \(n\)th row is \((\nu_1(x'), \ldots, \nu_n(x'))\) and the first \(n-1\) elements of the last column are \(-\nu_1(x'), \ldots, -\nu_{n-1}(x')\), otherwise the element of such a matrix is zero. If \(\nu_n(x') \neq 0\), then the determinant of such a matrix is not equal zero. Denote the inverse to this matrix as \(\mathcal{C}_0\) and set \(\mathbf{r} = \mathcal{C}_0 \mathbf{v}, \quad \mathbf{v} = (v_1, n, \ldots, v_2, n, \ldots, v_{n-1}, n, v_2, \ldots)\). Then \(\mathcal{C}(x')v = \mathbf{v} - (v_2r_1 - v_1r_2, \ldots, v_nr_1 - v_1r_n, \ldots, v_{n-1}r_n, \sum_{j=1}^n v_{j, \hat{j}})\). Without loss of generality we can assume that
\[
\mathbf{v}(0) = -\varepsilon_n.
\]
Let all the components of the function \(g\) starting from \(n + 1\) be equal to zero:
\[
g_k = 0 \quad k \geq n + 1.
\]

**Proposition 2.1.** Let \(v \in H^1(Q)\) satisfy (2.3), (2.6) and \(\text{supp } v \subset [-T + \epsilon_1, T - \epsilon_1] \times \bar{\Omega}\) for some positive \(\epsilon_1\), \(g \in L^2(\Sigma)\). There exist \(\hat{\tau} > 1\) and \(s_0 > 1\) such that for any \(\tau > \tau_0\) and all \(s \geq s_0\)
\[
\tau^2 \|\varphi^s \nabla v e^{s\varphi}\|_{L^2(Q)}^2 + \tau^4 \|\varphi^s \nabla v e^{s\varphi}\|_{L^2(Q)}^2 + \int_\Sigma (s\tau \varphi |\nabla v|^2 + s^3 \tau^3 \varphi^3 |v|^2) e^{2s\varphi} d\Sigma \leq C_2 \|P(x, D)v e^{s\varphi}\|_{L^2(Q)}^2 + \int_\Sigma (s\tau \varphi |\nabla v|^2 + s^3 \tau^3 \varphi^3 |v|^2) d\Sigma + s \int_\Sigma \tau \varphi |g|^2 e^{2s\varphi} d\Sigma,
\]
where \(C_2\) is independent of \(s\) and \(\tau\).

First, by an argument based on the partition of unity (e.g., Lemma 8.3.1 in [13]), it suffices to prove the inequality (2.3) locally, by assuming that
\[
\text{supp } v \subset B(y^*, \delta),
\]
where \(B(y^*, \delta)\) is the ball of the radius \(\delta > 0\) centered at some point \(y^*\).
Otherwise, without loss of generality we may assume that \( y^* = (y_0^*, 0, \ldots, 0) \). Let \( \theta \in C_0^\infty(\frac{1}{2}, 2) \) be a nonnegative function such that

\[
\sum_{\ell = -\infty}^{\infty} \theta(2^{-\ell} t) = 1 \quad \text{for all } t \in \mathbb{R}^1.
\]

Set \( v_\ell(x) = v(x)\kappa_\ell(x) \), \( g_\ell(x) = g(x)\kappa_\ell(x) \) where

\[
\kappa_\ell(x) = \theta(2^{-\ell} e^{t\psi(x)}).
\]

Observe that it suffices to prove the Carleman estimate (2.9) for the function \( v_\ell \) instead of \( v \) provided that the constants \( C_2, \tau_0 \) and the function \( s_0 \) are independent of \( \ell \). Observe that if \( G \subset \mathbb{R}^m \) is a bounded domain and \( g \in L^2(G) \), then there exist positive constants \( C_3 \) and \( C_4 \) which are independent of \( g \) (see e.g. [37]) such that

\[
C_3 \sum_{\ell = -\infty}^{\infty} \|\kappa_\ell g\|^2_{L^2(G)} \leq \|g\|^2_{L^2(G)} \leq C_4 \sum_{\ell = -\infty}^{\infty} \|\kappa_\ell g\|^2_{L^2(G)}.
\]

Denote the norm on the left-hand side of (2.9) as \( \| \cdot \|_* \). By (2.11) and (2.12), we have

\[
\|ve^{s\varphi}\|_* = \| \sum_{\ell = -\infty}^{\infty} v_\ell e^{s\varphi}\|_* \leq \sum_{\ell = -\infty}^{\infty} \|v_\ell e^{s\varphi}\|_* \leq C_5 \sum_{\ell = -\infty}^{\infty} (\|\kappa_\ell P(x, D)ve^{s\varphi}\|^2_{L^2(Q)} + \|[\kappa_\ell, P(x, D)]ve^{s\varphi}\|^2_{L^2(Q)} + \int_\Sigma (s\tau|\nabla v_\ell|^2 + s^3\tau^3\varphi^3|v_\ell|^2 + s\tau|g_\ell|^2)e^{2s\varphi}d\Sigma)
\]

\[
+ s\tau \|\varphi\chi_{\supp\kappa_\ell, \nabla} e^{s\varphi}\|^2_{L^2(\Sigma)} \frac{1}{2}.
\]

By (2.13), from the above inequality we have

\[
\|ve^{s\varphi}\|_* \leq C_6 (\|P(x, D)ve^{s\varphi}\|^2_{L^2(Q)} + \sum_{\ell = -\infty}^{\infty} \int_\Sigma (s\tau|\nabla v_\ell|^2 + s^3\tau^3\varphi^3|v_\ell|^2)e^{2s\varphi}d\Sigma + \sum_{\ell = -\infty}^{\infty} \|[\kappa_\ell, P(x, D)]ve^{s\varphi}\|^2_{L^2(Q)} + s\tau \|\varphi\chi_{\supp\kappa_\ell, \nabla} e^{s\varphi}\|^2_{L^2(\Sigma)} \frac{1}{2}.
\]

Let us estimate some terms on the right-hand side of inequality (2.15).

\[
\sum_{\ell = -\infty}^{\infty} s\tau \|\varphi\chi_{\supp\kappa_\ell, \nabla} e^{s\varphi}\|^2_{L^2(\Sigma)} \leq C_7 \sum_{\ell = -\infty}^{\infty} s\tau^5 \|\varphi\chi_{\supp\kappa_\ell, \nabla} ve^{s\varphi}\|^2_{L^2(\Sigma)} \leq C_8 s\tau^5 \|\varphi\chi_{\supp\kappa_\ell, \nabla} ve^{s\varphi}\|^2_{L^2(\Sigma)}.
\]

Estimating the commutator \( [\kappa_\ell, P(x, D)] \) we obtain

\[
\sum_{\ell = -\infty}^{\infty} \|[\kappa_\ell, P(x, D)]ve^{s\varphi}\|^2_{L^2(Q)} \leq C_9 \sum_{\ell = -\infty}^{\infty} (\tau^4 \|\chi_{\supp\kappa_\ell, \nabla} ve^{s\varphi}\|^2_{L^2(Q)} + \tau^8 \|\chi_{\supp\kappa_\ell, \nabla} ve^{s\varphi}\|^2_{L^2(Q)})
\]
From (2.14), (2.17), (2.16) and (2.15) we obtain (2.9).

Now, thanks to (2.18), without loss of generality we assume that

\[ \text{supp } \nu_\ell \subset B(y^*, \delta) \cap \text{supp } \kappa_\ell, \]

where \( B(y^*, \delta) \) is the ball of the radius \( \delta > 0 \) centered at some point \( y^* = (y_0^*, 0, \ldots, 0) \).

Assume that near \( (0, \ldots, 0) \), the boundary \( \partial \Omega \) is locally given by an equation \( x_n - \bar{\ell}(x_1, \ldots, x_{n-1}) = 0 \) and if \( (x_1, \ldots, x_n) \in \Omega \), then \( x_n - \bar{\ell}(x_1, \ldots, x_{n-1}) > 0 \) where \( \bar{\ell} \in C^2 \) and \( \bar{\ell}(0) = 0 \). Since \( \nu(0) = -\bar{c}_n \)

\[ \nabla' \bar{\ell}(0) = 0. \]

Denote \( F(x) = (x_0, \ldots, x_{n-1}, x_n - \bar{\ell}(x_1, \ldots, x_{n-1})) \). We set

\[ \Delta_{\bar{\ell}} u = \sum_{j=1}^{n-1} (\partial^2_{y_j} u - 2\partial_{x_j} \bar{\ell}) F^{-1}(y) \partial^2_{y_j y_n} u + (1 + |\nabla \bar{\ell}|^2) \partial^2_{y_n} u. \]

Henceforth we set \( y = (y_0, y') = (y_0, y_1, \ldots, y_n) \). After the change of variables, the equations (2.3) have the forms

\[ \mathbf{P}(y, D)\mathbf{v} = (\rho \partial^2_{y_0} \mathbf{v}_1 - \mu \Delta_{\bar{\ell}} \mathbf{v}_1, \rho \partial^2_{y_0} \mathbf{v}_2 - (\lambda + 2\mu) \Delta_{\bar{\ell}} \mathbf{v}_2) = \mathbf{q}, \quad \mathbf{Q} \triangleq \mathbb{R}^n \times [0, \gamma], \]

where \( \gamma \) is some positive constant. Here for functions \( \rho \circ F^{-1}(y), \mu \circ F^{-1}(y) \) and \( \lambda \circ F^{-1}(y) \), we use the notations \( \rho, \mu, \lambda \) and by \( \mathbf{q}_3, \mathbf{q}_4 \) denote the functions \( q_3, q_4 \) after the change of variables, respectively. The operator \( \mathbf{B}(y', D) \) is obtained form the operator \( \mathbf{B}(y', D) \) by the change of variables.

Now we introduce operators

\[ P_\mu(y, D, \bar{s}, \tau) = e^{s|\varphi|} P_\mu(y, D) e^{-s|\varphi|}, \quad P_{\lambda+2\mu}(y, D, \bar{s}, \tau) = e^{s|\varphi|} P_{\lambda+2\mu}(y, D) e^{-s|\varphi|}, \]

where

\[ \bar{s} = s\tau \varphi(y^*). \]

We denote the principal symbols of the operators \( P_\mu(y, D, \bar{s}, \tau) \) and \( P_{\lambda+2\mu}(y, D, \bar{s}, \tau) \) by \( p_\mu(y, \xi, \bar{s}, \tau) = p_\mu(y, \xi + i|s|\nabla \varphi) \) and \( p_{\lambda+2\mu}(y, \xi, \bar{s}, \tau) = p_{\lambda+2\mu}(y, \xi + i|s|\nabla \varphi) \) respectively.

The principal symbol of the operator \( P_\beta(y, D, \bar{s}, \tau) \) has the form

\[ p_\beta(y, \xi, \bar{s}, \tau) = \rho(y) (\xi_0 + i|s|\varphi_{y_0})^2 - \beta \sum_{j=1}^{n-1} (\xi_j + i|s|\varphi_{y_j})^2 \]

\[ -2(\nabla' \bar{\ell}, (\xi' + i|s|\nabla' \varphi))(\xi_n + i|s|\varphi_{y_n}) + (\xi_n + i|s|\varphi_{y_n})^2 G, \]

where \( G(y_1, \ldots, y_{n-1}) = 1 + |\nabla \bar{\ell}(y_1, \ldots, y_{n-1})|^2 \). The zeros of the polynomial \( p_\beta(y, \xi, \bar{s}, \tau) \) with respect to variable \( \xi_n \) for \( |(\xi, \bar{s})| \geq 1 \) and \( y \in B(y^*, \delta) \cap \text{supp } \kappa_\ell \) are

\[ \Gamma_\delta^\pm(y, \xi_0, \ldots, \xi_{n-1}, \bar{s}, \tau) = (-i|\bar{s}| \bar{\mu}(y) \psi_{\xi_n}(y) \kappa(\bar{\varphi}, \bar{s}) \pm \alpha_\delta(y, \xi_0, \ldots, \xi_{n-1}, \bar{s}, \tau)), \]
where $\tilde{\psi} = (\psi_0, \ldots, \psi_n), \psi_j = \frac{\varphi(y)}{\varphi(y^*)} \psi_{y_j}(y),$

\begin{equation}
\mu_\ell = \eta_\ast \sum_{k=0}^{\ell+2} \kappa_\ell, \quad \eta_\ast \in C_0^\infty(B(y^*, 2\delta)), \quad \eta_\ast|_{B(y^*, \delta)} = 1,
\end{equation}

the function $\kappa_\ell$ is given by (2.12),

\begin{equation}
\alpha_\beta^s(y, \tilde{\xi}, \tilde{s}, \tau) = \tilde{\mu}_\ell \left( -\kappa(|(\tilde{\xi}, \tilde{s})|) \sum_{j=1}^{n-1} (\xi_j + i|\tilde{s}| \psi_j) \partial_{y_j} \tilde{\psi}(y_1, \ldots, y_n) + \sqrt{\rho(\xi_0 + i|\tilde{s}| \psi_0) - \beta \sum_{j=1}^{n-1} (\xi_j + i|\tilde{s}| \psi_j)^2} G + \beta(\xi' + i|\tilde{s}| \psi', \nabla' \tilde{\psi})^2 \right) / \beta G^2,
\end{equation}

\kappa \in C^\infty(\mathbb{R}^1), \kappa(t) \geq 0, \kappa(t) = 1 \text{ for } t \geq 1 \text{ and } \kappa(t) = 0 \text{ for } t \in [0, 1/2], \tilde{\xi} = (\xi_0, \ldots, \xi_{n-1}).

Let $\chi_\nu \in C_0^\infty(S^n)$ on $\mathbb{R}^n = \{(\hat{\xi}, \hat{s}) \in S^n \mid \pm 1\}$ such that $\chi_\nu$ is identically equal to 1 in some neighborhood of $\hat{\xi} \in \mathbb{R}^n$ and supp $\chi_\nu \subset O(\hat{\xi}, \delta).$ Assume that

\begin{equation}
\kappa(|(\hat{\xi}, \hat{s})|)|\text{supp}\chi_\nu = 1, \quad \sup \kappa(|(\hat{\xi}, \hat{s})|) \subset O(\hat{\xi}, \delta).
\end{equation}

We extend the function $\chi_\nu$ on $\mathbb{R}^n$ as follows: $\chi_\nu(\zeta/(\hat{\xi}, \hat{s}))$ for $|(\hat{\xi}, \hat{s})| > 1$ and $\chi_\nu(\zeta/(\hat{\xi}, \hat{s})) \subset (\hat{\xi}, \hat{s})$ for $|(\hat{\xi}, \hat{s})| < 1.$ Denote by $\chi_\nu(y, \tilde{D}, \tilde{s})$ the pseudodifferential operator with the symbol $\eta(y) \chi_\nu$ and $\eta(y) = \kappa_{\ell-1}(y) + \kappa_\ell(y) + \kappa_{\ell+1}(y),$ where $\kappa_\ell$ is given by (2.12).

We set $w = v_\nu e^{\kappa_\nu}$ and $w_\nu = \chi_\nu(y, \tilde{D}, \tilde{s}) w, w_{i,j,\nu} = \chi_\nu(y, \tilde{D}, \tilde{s}) (v_{i,j} \circ F^{-1}(y)).$

Let $\mathcal{O}$ be a domain in $\mathbb{R}^n.$

**Definition.** We say that the symbol $a(\tilde{\psi}, \tilde{s}) \in C^k(\mathcal{O} \times \mathbb{R}^{n+1})$ belongs to the class $C^k_{cl}S^s(\mathcal{O})$ if

A) There exists a compact set $K \subset \subset \mathcal{O}$ such that $a(\tilde{\psi}, \tilde{s})|_{\mathcal{O} \setminus K} = 0; \quad$

B) For any $\beta = (\beta_0, \ldots, \beta_n)$ there exists a constant $C_\beta$ such that

\begin{equation}
\left\| \partial_{\tilde{s}_0}^\beta \cdots \partial_{\tilde{s}_{n-1}}^\beta \partial_{s}^\beta a(\cdot, \tilde{s}) \right\|_{C^k(\mathcal{O})} \leq C_\beta \left( s^2 + \sum_{i=0}^{n-1} \xi_i^2 \right)^{\kappa - |\beta| / 2},
\end{equation}

where $|\beta| = \sum_{j=0}^{n} \beta_j$ and $|(\tilde{\xi}, \tilde{s})| \geq 1;$

C) For any $N \in \mathbb{N}$ the symbol $a$ can be represented as

\begin{equation}
a(\tilde{\psi}, \tilde{s}) = \sum_{j=1}^{N} a_j(\tilde{\psi}, \tilde{s}) + R_N(\tilde{\psi}, \tilde{s}),
\end{equation}

where the functions $a_j$ have the following properties

\begin{equation}
a_j(\tilde{\psi}, \lambda \tilde{s}) = \lambda^{n-j} a_j(\tilde{\psi}, \tilde{s}) \quad \forall \lambda > 1, \forall (\tilde{\psi}, \tilde{s}) \in \{ (\tilde{\psi}, \tilde{s}) | \tilde{\psi} \in K, |(\tilde{\xi}, \tilde{s})| > 1 \}.
\end{equation}
\[
\left\| \partial_{\xi_0}^{\beta_0} \cdots \partial_{\xi_{n-1}}^{\beta_{n-1}} \partial_{y}^{\beta_n} a_j (\cdot, \tilde{\xi}, \tilde{s}) \right\|_{C^k(\mathcal{O})} \leq C_\beta \left( s^2 + \sum_{i=0}^{n-1} \xi_i^2 \right)^{\frac{\kappa - j - |\beta|}{2}}
\]
for all \( \beta \) and \( (\tilde{\xi}, \tilde{s}) \) satisfying \( |(\tilde{\xi}, \tilde{s})| \geq 1 \) and the term \( R_N \) satisfies the estimate
\[
\| R_N (\cdot, \tilde{\xi}, \tilde{s}) \|_{C^k(\mathcal{O})} \leq C_N (s^2 + \sum_{i=0}^{n-1} \xi_i^2)^{\frac{\kappa - N}{2}} \forall (\tilde{\xi}, \tilde{s}) \) satisfying \( |(\tilde{\xi}, \tilde{s})| \geq 1 \).

Obviously
\[
(2.29) \quad \pi_{C^k(B(0, \delta(y^*)))}(\chi_\nu) \leq C_{10} \tau^{2k} \quad \forall k \in \mathbb{N}_+.
\]
Obviously the pseudodifferential operators with the symbols \( \Gamma_{\beta}^\pm \) belongs to the class \( C^{2^s,1-s}_d \beta(B(0, \delta(y^*)) \) and
\[
(2.30) \quad \pi_{C^{2^s,1-s}_d(B(0, \delta(y^*)))(\Gamma_{\beta}^\pm)} \leq C_{11} \tau^4.
\]

We have

**Proposition 2.2.** Let \( w \in H^1(Q) \), \( \text{supp } w \subset B(y^*, \delta) \cap \text{supp } \eta_k \) and \( P_\beta(y, D, \tilde{s}, \tau) \chi_\nu w \in L^2(Q) \). Then there exist positive constants \( \delta(y^*), C_{12}, C_{13} \) independent of \( s \) and \( \tau \) and independent constants \( s_0, \tau_0 \) independent of \( s \) such that for all \( \tau \geq \tau_0 \) and \( s \geq s_0 \) we have

\[
(2.31) \quad C_{12} \int_Q (|s|^{\tau^2} \varphi |\nabla \chi_\nu w|^2 + |s|^{3\tau^2} \varphi^3 |\chi_\nu w|^2) dy + \Xi(\chi_\nu w) 
\]
\[
\leq \| P_\beta(y, D, \tilde{s}, \tau) \chi_\nu w \|^2_{L^2(Q)} + C_{13} \epsilon(\tau_0) \int_{\mathbb{R}^n} (|s|^{\tau} \varphi(y^*) |\nabla \chi_\nu w|^2 + |s|^{3\tau} \varphi^3(y^*) |\chi_\nu w|^2) \chi_\nu w (\tilde{y}, 0) d\tilde{y},
\]
where \( \epsilon(\tau_0) \to +0 \) as \( \tau_0 \to +\infty \) and

\[
\Xi(\beta) = \sum_{j=1}^3 \mathcal{J}_j(\beta, w), \quad \mathcal{J}_1(\beta, w) = \int_{\mathbb{R}^n} (|\tilde{s}|^2 \beta^2(y^*) \psi_{y_n}(y^*) |\partial_{y_n} w|^2 + |\tilde{s}|^3 \beta^2(y^*) \psi_{y_n}^3(y^*) |w|^2) d\tilde{y},
\]
\[
(2.32) \quad \mathcal{J}_2(\beta, w) = -\frac{1}{2} Re \int_{\mathbb{R}^n} 2|\tilde{s}| \beta(y^*) |\partial_{y_n} w (\nabla \chi_\nu p_\beta(y^*, \nabla w, 0), \nabla \psi(y^*))| d\tilde{y},
\]
\[
(2.33) \quad \mathcal{J}_3(\beta, w) = \int_{\mathbb{R}^n} |\tilde{s}| ^2 \beta(y^*) \psi_{y_n}(y^*) (p_\beta(y^*, \nabla w, 0) - \tilde{s}^2 p_\beta(y^*, \nabla \psi(y^*), 0) |w|^2) d\tilde{y}.
\]

**Proof.** It suffices to prove the statement of the lemma separately for \( \text{Re } w_\nu, \text{Im } w_\nu \). Let \( u_\nu = \text{Re } w_\nu \) or \( u_\nu = \text{Im } w_\nu \). For simplicity instead of \( p_\beta(y, \xi) \) we use the notation \( p(y, \xi) = \sum_{k,j=0}^{n} p_{kj}(y) \xi_k \xi_j \), where \( p_{kj} = p_{jk} \) for all \( k, j \in \{0, \ldots, n\} \). We set

\[
p_j(y, \xi) = \partial_{y_j} p(y, \xi), p^{(j)}(y, \xi) = \partial_{\xi_j} p(y, \xi), p^{(j,m)}(y, \xi) = \partial_{\xi_j \xi_m} p(y, \xi), p(y, \eta, \xi) = \sum_{k,j=0}^{n} p_{kj}(y) \eta_k \xi_j.
\]

Observe that since \( y \in B(y^*, \delta) \cap \text{supp } \eta_k \) then either \( \frac{1}{2} \leq 2^{-\ell} e^{\tau_2 \psi(y)} \leq 2 \) or \( \frac{1}{2} \leq 2^{-\ell+1} e^{r_2 \psi(y)} \leq 2 \).
This is equivalent to
\[(\ell - 2) \ln 2/\tau^2 \leq \psi(y) \leq (\ell + 2) \ln 2/\tau^2 \quad \forall y \in B(y^*, \delta) \cap \text{supp } \eta_\ell.\]

Hence, for any \( \epsilon \) there exists \( \tau_0(\epsilon) \) such that for all \( \tau \geq \tau_0 \)
\[(2.35) \quad |\varphi(y) - \varphi(y^*)| \leq \epsilon \varphi(y^*) \quad \forall y \in B(y^*, \delta) \cap \text{supp } \eta_\ell.
\]

Indeed, since \( \psi(y) > C_{14} > 0 \) on \( B(0, 2\delta) \), by (2.34), there exists an independent constant \( C_{15} \) such that
\[(2.36) \quad |\ell|/\sqrt{\tau} \leq C_{15} \quad \text{if } B(y^*, \delta) \cap \text{supp } \eta_\ell \neq \{0\}.
\]

Then estimate (2.35) follows from the inequality
\[(2.37) \quad |\varphi(y) - \varphi(y^*)| = e^{\psi(y^*)}|1 - e^{\psi(y^*) - \psi(y)}| \leq \varphi(y^*)|1 - e^{-\psi(y^*)}|.
\]

In order to get the last two inequalities in (2.37) we used (2.36) and (2.34).

We introduce the operators
\[(2.38) \quad L_1(y, D, \tilde{s}, \tau)v_\nu = -\sum_{k=0}^n s\varphi_{yk}p^{(k)}(y, \nabla v_\nu), \quad L_2(y, D, \tilde{s}, \tau)v_\nu = P(y, D)v_\nu + s^2p(y, \nabla \varphi, \nabla \varphi)v_\nu.
\]

Denote \( f_\nu = P(y, D, \tilde{s}, \tau)v_\nu - sv_\nu P(y, D)\varphi \). Then we have
\[(2.39) \quad ||f_\nu||_2^2 = ||L_1(y, D, \tilde{s}, \tau)v_\nu||_2^2 + 2\text{Re} (L_1(y, D, \tilde{s}, \tau)v_\nu, L_2(y, D, \tilde{s}, \tau)v_\nu)_{L^2(Q)}.
\]

The following equality is proved in Imanuvilov [15]:
\[(2.40) \quad \text{Re} (L_1(y, D, \tilde{s}, \tau)v_\nu, L_2(y, D, \tilde{s}, \tau)v_\nu)_{L^2(Q)} = -\text{Re} \int_{\partial Q} p(y, \tilde{e}_n, \nabla v_\nu)L_1(y, D, \tilde{s}, \tau)v_\nu d\Sigma - s \int_{\partial Q} p(y, \tilde{e}_n, \nabla \varphi)p(y, \nabla v_\nu, \nabla v_\nu) d\Sigma + s^3 \int_{\partial Q} p(y, \nabla \varphi, \nabla \varphi)p(y, \tilde{e}_n, \nabla \varphi)v_\nu^2 d\Sigma + s \int_{Q} G(y, \tilde{s}, \tau, v_\nu) dy + \int_{Q} \frac{s}{2} \sum_{k,m=0}^n p_{k,m}^{(k)}(y, \nabla v_\nu)\partial_{y_m}\varphi p^{(m)}(y, \nabla v_\nu) - \theta(y)(p(y, \nabla v_\nu, \nabla v_\nu) - s^2p(y, \nabla \varphi, \nabla \varphi)v_\nu^2) dy,
\]

where
\[
G(y, \tilde{s}, \tau, w) = \{p, \{p, \varphi\}\}(y, \nabla w) + s^2 \sum_{k,j=0}^np_j(y, \nabla \varphi)w^2 + s^2 \sum_{k,j=0}^n \partial^2_{y_jy_j}\varphi p^{(k)}(y, \nabla \varphi)p^{(j)}(y, \nabla \varphi)w^2
\]

and
\[
\theta(y) = \sum_{k,m=0}^n (\partial^2_{y_ky_j}\varphi p^{(k,m)}(y, \nabla \varphi) + \partial_{y_k}\varphi p^{(k,m)}(y, \nabla v_\nu)).
\]

Observe that the function \( \theta(y) \) is independent of \( v_\nu \) and
\[
(2.41) \quad \sup_{y \in \text{supp } \eta_\ell} |\theta(y)| \leq C_{16} \tau^2 \varphi(y^*).
\]
Without loss of generality we may assume that $\partial_{y_n} \psi(y^*) \neq 0$. We introduce the form $\mathcal{G}(\tilde{s}, \tau, \nabla v_\nu)$ in the following way: In the function $\mathcal{G}(y^*, \tilde{s}, \tau, \nabla v_\nu)$ we replace $\partial_{y_n} v_\nu$ by $-\frac{1}{\sum_{j=0}^{n-1} p_{jn}(y^*)} \sum_{j,k=0}^{n-1} \partial_{y_j} \varphi(y^*) p_{jk}(y^*) \partial_{y_k} v_\nu$. Since $p_{kj}(y^*) = 0$ for $k \neq j$

$$\sum_{j=0}^{n} p_{jn}(y) \partial_{y_j} \varphi(y) \neq 0 \quad \forall y \in B(y^*, \delta) \cap \text{supp } \eta_\ell. \tag{2.42}$$

By (2.35) we have the inequality

$$\left| \int_{\Omega} \mathcal{G}(y, \tilde{s}, \tau, \nabla v_\nu) - \mathcal{G}(\tilde{s}, \tau, \nabla v_\nu) dy \right| \leq \frac{C_{17}}{s^2} \left\| L_1(y, D, \tilde{s}, \tau) v_\nu \right\|_{L^2(\Omega)}^2 + \epsilon(\tau_0) \| v_\nu \|_{H^{1, \tilde{s}}(\Omega)}^2, \tag{2.43}$$

where $\epsilon(\tau_0) \to +0$ as $\tau_0 \to +\infty$.

Let

$$\xi = \left( \tilde{\xi}, -\frac{1}{\sum_{j=0}^{n} p_{jn}(y^*)} \sum_{j,k=0}^{n-1} \partial_{y_j} \varphi(y^*) p_{jk}(y^*) \xi_k \right). \tag{2.44}$$

We set

$$q(\tilde{\xi}, s) = \sum_{k,j=0}^{n} \partial_{y_j y_k} \varphi(y^*) p^{(k)}(y^*, \xi + is \nabla \varphi(y^*)) p^{(j)}(y^*, \xi + is \nabla \varphi(y^*)) + \frac{1}{s} \text{Im} \sum_{k=0}^{n} p_k(y^*, \xi + is \nabla \varphi(y^*)) p^{(k)}(y^*, \xi + is \nabla \varphi(y^*)).$$

Observe that

$$\int_{\mathbb{R}^{n+1} \setminus \Omega} q(\tilde{\xi}, s)|F_{\tilde{y} \to \tilde{\xi}} v_\nu|^2 d\tilde{\xi} dy_n = \int_{\Omega} \mathcal{G}(\tilde{s}, \tau, \nabla v_\nu) dy. \tag{2.45}$$

Denote $\tilde{w}(\tilde{\xi}, y_n) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \text{sign}(\text{Re} p(y^*, \xi + is \nabla \varphi(y^*))) F_{\tilde{y} \to \tilde{\xi}} v_\nu e^{i\tilde{\xi} \tilde{y}} d\tilde{\xi}$, where $F_{\tilde{y} \to \tilde{\xi}}$ is the Fourier transform given by

$$F_{\tilde{y} \to \tilde{\xi}} u = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i \sum_{j=0}^{n-1} y_j \xi_j} u(y_0, \ldots, y_{n-1}) d\tilde{y}.$$

Taking the scalar product of the function $L_2(y, D, \tilde{s}, \tau) v_\nu$ and $\tilde{w}$ in $L^2(\Omega)$ we have

$$\int_{\Omega} (-p(y, \nabla v_\nu, \nabla \tilde{w}) + s^2 p(y, \nabla \varphi, \nabla \varphi) v_\nu \tilde{w} dy = - \int_{\mathbb{R}^n} \partial_{\tilde{y}_j} v_\nu(\tilde{y}, 0) \tilde{w}(\tilde{y}, 0) d\tilde{y} - \sum_{j,k=0}^{n} \int_{\Omega} \partial_{y_k} p_{kj} \partial_{y_j} v_\nu \tilde{w} dy + (L_2(y, D, \tilde{s}, \tau) v_\nu, \tilde{w})_L^2(\Omega), \tag{2.46}$$

where $\partial_{\tilde{y}_j} w = - \sum_{j=0}^{n} p_{nj} \partial_{y_j} w$.

By (2.35) for any $\epsilon \in (0, 1)$ there exists $\tau_0(\epsilon)$ such that for all $\tau \geq \tau_0$ and all $s \geq 1$

$$\int_{\Omega} \tilde{s} \tau s^2 |p(y^*, \nabla \varphi(y^*), \nabla \varphi(y^*)) - p(y, \nabla \varphi, \nabla \varphi)| |v_\nu|^2 dy \leq \epsilon \tilde{s} \tau \| v_\nu \|_{H^{1, \tilde{s}}(\Omega)}^2$$

$$+ \int_{\Omega} \tilde{s} \tau |p(y, \nabla v_\nu, \nabla \tilde{w}) - p(y^*, \nabla v_\nu, \nabla \tilde{w})| dy \leq \epsilon \tilde{s} \tau \| v_\nu \|_{H^{1, \tilde{s}}(\Omega)}. \tag{2.47}$$
The inequalities (2.46) and (2.47) imply that for any positive $\epsilon \in (0, 1)$ there exists a constant $C_{18}(\epsilon)$ such that

$$
(2.48) \quad \tilde{s}\tau \left| \int_{\mathcal{Q}} (-p(y^*, \nabla v_\nu, \nabla \tilde{w}) + s^2 p(y^*, \nabla \varphi(y^*), \nabla \varphi(y^*)) v_\nu \tilde{w} dy \right| 
\leq C_{19} \left| \tilde{s}\tau \int_{\mathbb{R}^n} \partial_{\tilde{p}_j} v_\nu(y, 0) \overline{\tilde{w}(y, 0)} dy \right| + \epsilon \left\| L_2(y, D, \tilde{s}, \tau) v_\nu \right\|_{L^2(\mathcal{Q})}^2 + (\epsilon \tilde{s}\tau + C_{18}\tau^2) \left\| v_\nu \right\|_{H^{1,2}(\mathcal{Q})}^2.
$$

We set $\hat{\tilde{w}} = (\partial_{y_n} \tilde{w}, \ldots, \partial_{y_{n-1}} \tilde{w})$, $\sum_{j=0}^{n-1} p j \partial_{y_j} \varphi(y^*) \sum_{j,k=0}^{n-1} \partial_{y_j} \varphi(y^*) p_{jk}(y^*) \partial_{y_k} \tilde{w})$. Hence, if $\xi$ is given by (2.41), then we have

$$
(2.49) \quad \tilde{s}\tau \left| \int_{\mathbb{R}^n} (-p(y^*, \hat{\tilde{w}} v_\nu, \nabla \tilde{w}) + s^2 p(y^*, \nabla \varphi(y^*), \nabla \varphi(y^*)) v_\nu \tilde{w} dy \right| 
\leq C_{20} \tilde{s}\tau \left| \int_{\mathbb{R}^n} \partial_{\tilde{p}_j} v_\nu(y, 0) \overline{\tilde{w}(y, 0)} dy \right| + \epsilon \left\| L_2(y, D, \tilde{s}, \tau) v_\nu \right\|_{L^2(\mathcal{Q})}^2 + (\epsilon \tilde{s}\tau + C_{21}\tau^2) \left\| v_\nu \right\|_{H^{1,2}(\mathcal{Q})}^2.
$$

Observe that pseudoconvexity Condition (1.1) implies that there exists a positive constant $C_{22}$ such that

$$
(2.50) \quad q(\tilde{\xi}, s) + \tau^2 \varphi(y^*) \left| p(y^*, \tilde{\xi} + i s \nabla \varphi(y^*)) \right| \geq C_{22}\tau^2 \varphi(y^*) \left| (\tilde{\xi}, \tilde{s}) \right|^2 \quad \forall (\tilde{\xi}, \tilde{s}) \in \mathbb{S}^n.
$$

Therefore, from (2.50) and (2.49), for some positive constant $C_{23}$ for all $\tau \geq \tau_0$ and all $s \geq 1$ we have the inequality

$$
C_{23} \tilde{s}\tau \int_{\mathcal{Q}} \left( |\nabla v_\nu|^2 + \tilde{s}^2 |v_\nu|^2 \right) dy \leq C_{24}(\tilde{s}\tau) \int_{\mathbb{R}^n} \left| \partial_{\tilde{p}_j} v_\nu(y, 0) \overline{\tilde{w}(y, 0)} \right| dy
$$

$$
(2.51) + \epsilon \left\| L_2(y, D, \tilde{s}, \tau) v_\nu \right\|_{L^2(\mathcal{Q})}^2 + \int_{\mathbb{R}^n} s \mathcal{S}(\tilde{s}, \tau, \nabla v_\nu) dy + (\epsilon \tilde{s}\tau + C_{25}\tau^2) \left\| v_\nu \right\|_{H^{1,2}(\mathcal{Q})}^2.
$$

We take $\epsilon \in (0, \min\left\{ \frac{C_{23}}{4}, \frac{1}{8\tilde{s}\tau} \right\})$. Thanks to (2.40), (2.43), (2.51) for some positive constant $C_{26}$ for all $\tau \geq \tau_0$ and all $s \geq 1$ we have the inequality

$$
C_{26} \tilde{s}\tau \int_{\mathcal{Q}} \left( |\nabla v_\nu|^2 + \tilde{s}^2 |v_\nu|^2 \right) dy \leq C_{27} \tilde{s}\tau \int_{\mathbb{R}^n} \left| \partial_{\tilde{p}_j} v_\nu(y, 0) \overline{\tilde{w}(y, 0)} \right| dy + \frac{1}{8} \left\| L_2(y, D, \tilde{s}, \tau) w_\nu \right\|_{L^2(\mathcal{Q})}^2
$$

$$
+ 2(L_1(y, D, \tilde{s}, \tau) v_\nu, L_2(y, D, \tilde{s}, \tau) v_\nu)_{L^2(\mathcal{Q})} + \Re \int_{\partial\mathcal{Q}} p(y, \tilde{\nu}, \nabla v_\nu) L_1(y, D, \tilde{s}, \tau) v_\nu d\Sigma
$$

$$
- s \int_{\partial\mathcal{Q}} p(y, \tilde{\nu}, \nabla \varphi) p(y, \nabla v_\nu, \nabla v_\nu) d\Sigma + s^3 \int_{\partial\mathcal{Q}} p(y, \nabla \varphi, \nabla \varphi) p(y, \tilde{\nu}, \nabla \varphi)|v_\nu|^2 d\Sigma
$$

$$
(2.52) - \int_{\mathcal{Q}} \frac{s}{2} \sum_{k,m=0}^{n} p^{(k)}(y, \nabla v_\nu) p^{(m)}(y, \nabla v_\nu) \theta(y)(p(y, \nabla v_\nu) - s^2 p(y, \nabla \varphi)|v_\nu|^2) dy.
$$

Thanks to (2.42) and (2.38), the following identity holds true

$$
\partial_{y_n} v_\nu = \frac{L_1(y, D, \tilde{s}, \tau) v_\nu / s - \sum_{k=0}^{n} \sum_{j=0}^{n-1} p_{k,j} \partial_{y_k} \varphi \partial_{y_j} v_\nu}{\sum_{k=0}^{n} p_{j,n} \partial_{y_k} \varphi}.
$$
Therefore there exists a constant $C_{28}$ independent of $\tilde{s}$, $\tau$ such that
\begin{equation}
\tilde{s}\tau \int_Q |\partial_y, v_\nu|^2 dy \leq C_{28}\left( \int_Q \tilde{s}\tau(|\tilde{\nabla} v_\nu|^2 + \tilde{s}^2|v_\nu|^2)dy + \|L_1(y, D, \tilde{s}, \tau)v_\nu\|^2_{L^2(Q)} \right).
\end{equation}

Using (2.39), from (2.53) and (2.52) for any $\tau \geq \tau_0$ and $s > 1$ we obtain the estimate
\begin{equation}
C_{29}\tilde{s}\tau \int_Q (|\nabla v_\nu|^2 + \tilde{s}^2|v_\nu|^2)dy \leq C_{30}\tilde{s}\tau \left| \int_{\mathbb{R}^n} \partial_{\tilde{\nu}} v_\nu(y, 0)\tilde{w}(y, 0)dy \right|
+ \int_{\partial Q} p(y, \tilde{e}_n, \nabla v_\nu)L_1(y, D, \tilde{s}, \tau)v_\nu d\Sigma

-s \int_{\partial Q} p(y, \tilde{e}_n, \nabla \varphi)p(y, \nabla v_\nu, \nabla v_\nu) d\Sigma + s^3 \int_{\partial Q} p(y, \nabla \varphi, \nabla \varphi)p(y, \tilde{e}_n, \nabla \varphi)|v_\nu|^2 d\Sigma

- \int_Q \frac{s}{2} \left( \sum_{k,m=0}^n P^{(k)}(y, \nabla v_\nu, \nabla \varphi) p^{(m)}(y, \nabla v_\nu) - \theta(y) (p(y, \nabla v_\nu, \nabla v_\nu) - s^2 p(y, \nabla \varphi, \nabla \varphi)|v_\nu|^2) \right) dy

+ \|f_{\nu}\|^2_{L^2_0(Q)} - \frac{1}{2}\|L_1(y, D, \tilde{s}, \tau)v_\nu\|^2_{L^2_0(Q)} - \frac{1}{2}\|L_2(y, D, \tilde{s}, \tau)v_\nu\|^2_{L^2_1(Q)}.
\end{equation}

Now we estimate some integrals on the right hand side of (2.54): There exist a constant $C_{31}$ such that
\begin{equation}
\left| \int_Q \frac{s}{2} \sum_{k,m=0}^n P^{(k)}(y, \nabla v_\nu, \nabla \varphi) p^{(m)}(y, \nabla v_\nu) dy \right| \leq C_{31} \int_Q |\nabla v_\nu|^2 dy + \frac{1}{8} \|L_1(y, D, \tilde{s}, \tau)v_\nu\|^2_{L^2_0(Q)}.
\end{equation}

Integrating by parts we have
\begin{align*}
\int_Q \theta(y) (p(y, \nabla v_\nu, \nabla v_\nu) - s^2 p(y, \nabla \varphi, \nabla \varphi)|v_\nu|^2) dy &= - \int_Q \theta(y)L_2(y, D, \tilde{s}, \tau)v_\nu v_\nu dy

+ \int_{\mathbb{R}^n} \theta \partial_{\tilde{\nu}} v_\nu(y, 0)v_\nu(y, 0)dy + \sum_{j,k=0}^n \int_Q (\theta p^{(j)}(y, \nabla v_\nu)v_\nu + p^{(j)}(y, \nabla v_\nu)\partial_{\tilde{\nu}} \theta v_\nu) dy.
\end{align*}

Therefore (2.31) yields
\begin{equation}
s \left| \int_Q \theta(y) (p(y, \nabla v_\nu, \nabla v_\nu) - s^2 p(y, \nabla \varphi, \nabla \varphi)|v_\nu|^2) dy \right| \leq \frac{1}{8} \|L_2(y, D, \tilde{s}, \tau)v_\nu\|^2_{L^2_0(Q)}

+ C_{32}\tau^3 \|v_\nu\|^{2}_{H^{1,2}(\mathbb{R}^{n})} + C_{33}\|((\partial_{\tilde{\nu}} v_\nu(\cdot, 0), v_\nu(\cdot, 0))\|^2_{L^2_0(\mathbb{R}^{n}) \times H^{1,2}(\mathbb{R}^{n})}.
\end{equation}

Using (2.56) and (2.55), from (2.54) for any $\tau \geq \tau_0$ and $s > 1$ we obtain the estimate
\begin{equation}
C_{34}\tilde{s}\tau \int_Q (|\nabla v_\nu|^2 + \tilde{s}^2|v_\nu|^2)dy \leq C_{35}\|((\partial_{\tilde{\nu}} v_\nu(\cdot, 0), v_\nu(\cdot, 0))\|^2_{L^2_0(\mathbb{R}^{n}) \times H^{1,2}(\mathbb{R}^{n})}.
\end{equation}

Therefore (2.31) yields
\begin{align*}
\int_Q \theta(y) (p(y, \nabla v_\nu, \nabla v_\nu) - s^2 p(y, \nabla \varphi, \nabla \varphi)|v_\nu|^2) dy &= - \int_Q \theta(y)L_2(y, D, \tilde{s}, \tau)v_\nu v_\nu dy

+ \int_{\mathbb{R}^n} \theta \partial_{\tilde{\nu}} v_\nu(y, 0)v_\nu(y, 0)dy + \sum_{j,k=0}^n \int_Q (\theta p^{(j)}(y, \nabla v_\nu)v_\nu + p^{(j)}(y, \nabla v_\nu)\partial_{\tilde{\nu}} \theta v_\nu) dy.
\end{align*}
Next we estimate the difference between the boundary integrals and \( \sum_{j=1}^{3} \mathcal{J}_j(\beta, v_\nu) \). Using (2.35) we have

\[
\left| \int_{\partial Q} p(y, \vec{e}_n, \nabla v_\nu) L_1(y, D, \tilde{s}, \tau) v_\nu d\Sigma - s \int_{\partial Q} p(y, \vec{e}_n, \nabla \varphi) p(y, \nabla v_\nu, \nabla v_\nu) d\Sigma + s^3 \int_{\partial Q} p(y, \nabla \varphi, \nabla \varphi) p(y, \vec{e}_n, \nabla \varphi) |v_\nu|^2 d\Sigma - \sum_{j=1}^{3} \mathcal{J}_j(\beta, v_\nu) \right|
\]

(2.58)

\[
\leq C_{36} \epsilon(\tau_0) \int_{\mathbb{R}^n} (|s| r \varphi(y^*) |\nabla v_\nu|^2 + |s|^3 \tau^3 \varphi^3(y^*) |v_\nu|^2) (\tilde{y}, 0) d\tilde{y},
\]

where \( \epsilon(\tau_0) \to +0 \) as \( \tau_0 \to +\infty \). From (2.57) and (2.58), we obtain (2.31). \( \square \)

In some cases, we can represent the operator \( P_\beta(y, D, \tilde{s}, \tau) \) as a product of two first order pseudodifferential operators.

**Proposition 2.3.** Let \( \beta \in \{\mu, \lambda + 2\mu\} \), \( r_\beta(y^*, \zeta^*, \tau) \neq 0 \) and \( \text{supp} \chi_\nu \subset \mathcal{O}(\delta_1) \). Then we can factorize the operator \( P_\beta(y, D, \tilde{s}, \tau) \) into the product of two first order pseudodifferential operators:

\[
P_\beta(y, D, \tilde{s}, \tau) w_\nu = \beta G^{\frac{1}{2}} (\frac{1}{i} \partial_{y_0} - \Gamma_{\beta}(y, \tilde{D}, \tilde{s}, \tau)) (\frac{1}{i} \partial_{y_n} - \Gamma^+_\beta(y, \tilde{D}, \tilde{s}, \tau)) w_\nu + T_\beta w,
\]

where \( T_\beta : H^{1, \tilde{s}}(Q) \to L^2(Q) \) is a continuous operator and there exists a constant \( C_{37} \) independent of \( \tilde{s} \) and \( \tau \) such that

\[
\|T_\beta w_\nu\|_{L^2(Q)} \leq C_{37} \tau^2 \|w\|_{H^{1, \tilde{s}}(Q)}.
\]

**Proof.** Let

\[
\tilde{R}(y, \tilde{D}, \tilde{s}, \tau) = \left[ \frac{\rho(\frac{1}{i} \partial_{y_0} + i|\tilde{s}| \psi_0)^2}{\beta G} - \frac{\sum_{j=1}^{n-1}(\frac{1}{i} \partial_{y_j} + i|\tilde{s}| \psi_j)^2}{G} \right]
\]

and \( \Gamma(y, \tilde{D}, \tilde{s}, \tau) \) is the operator with symbol \( \Gamma_\beta(y, \tilde{\xi}, \tilde{s}, \tau) \Gamma^+_\beta(y, \tilde{\xi}, \tilde{s}, \tau) : \)

\[
\Gamma(y, \tilde{\xi}, \tilde{s}, \tau) = (-|\tilde{s}|^2 (\tilde{\mu}_\ell \psi_\kappa) \kappa^2 + \alpha_\beta \tilde{\mu}_\ell \psi_\kappa \kappa + \alpha_\beta \tilde{\mu}_\ell \psi_\kappa \kappa + \alpha_\beta \tilde{\mu}_\ell \psi_\kappa \kappa) = \kappa^2 (|\tilde{\xi}_0|) \left[ \frac{\rho(\xi_0 + i|\tilde{s}| \psi_0)^2}{\beta G} - \frac{\sum_{j=1}^{n-1}(\xi_j + i|\tilde{s}| \psi_j)^2}{G} \right].
\]

Functions \( \tilde{\mu}_\ell \) and \( \kappa \) are given by (2.25) and (2.28) respectively. We set \( \Upsilon_\ell = B(y^*, 2\delta) \cap \text{supp} \tilde{\mu}_\ell \). Then

\[
\Gamma(y, \tilde{D}, \tilde{s}, \tau) w_\nu = \left[ \Gamma, \eta_\ell \right] \chi_\nu(\tilde{D}, \tilde{s}) w + \eta_\ell \Gamma(y, \tilde{D}, \tilde{s}, \tau) \chi_\nu(\tilde{D}, \tilde{s}) w = \left[ \Gamma, \eta_\ell \right] \chi_\nu(\tilde{D}, \tilde{s}) w + \eta_\ell \tilde{R}(y, \tilde{D}, \tilde{s}, \tau) \chi_\nu(\tilde{D}, \tilde{s}) w = \left[ \Gamma, \eta_\ell \right] \chi_\nu(\tilde{D}, \tilde{s}) w + \tilde{R}(y, \tilde{D}, \tilde{s}, \tau) w + \eta_\ell \tilde{R}(y, \tilde{D}, \tilde{s}, \tau) \chi_\nu(\tilde{D}, \tilde{s}) w.
\]
In order to obtain the second equality in (2.61) we used (2.28). The short computations imply
\begin{align*}
(2.62) \quad \frac{1}{i} \partial_{y_n} - \Gamma^{\beta}_{\beta}(y, \bar{D}, \bar{s}, \tau)) (\frac{1}{i} \partial_{y_n} - \Gamma^{\beta}_{\beta}(y, \bar{D}, \bar{s}, \tau))
&= -\partial_{y_n}^2 - \frac{1}{i}[\partial_{y_n}, \Gamma^{\beta}_{\beta}(y, \bar{D}, \bar{s}, \tau)] + \Gamma^{\beta}_{\beta}(y, \bar{D}, \bar{s}, \tau) \Gamma^{\beta}_{\beta}(y, \bar{D}, \bar{s}, \tau) \\
&+ i\Gamma^{\beta}_{\beta}(y, \bar{D}, \bar{s}, \tau) \partial_{y_n} + i\Gamma^{\beta}_{\beta}(y, \bar{D}, \bar{s}, \tau) \partial_{y_n}.
\end{align*}

By Lemma 8.3 we have
\[ \Gamma^{\beta}_{\beta}(y, \bar{D}, \bar{s}, \tau) \Gamma^{\beta}_{\beta}(y, \bar{D}, \bar{s}, \tau) = \Gamma(y, \bar{D}, \bar{s}, \tau) + R_0, \]
where
\[ \| R_0 \|_{L(H^1_0(\tau), L^2(\tau))} \leq C_{38} \pi C^1(\tau) (\Gamma^{\beta}_{\beta}) \pi C^1(\tau) (\Gamma^{\beta}_{\beta}) \leq C_{39} \tau^4. \]

The commutator \([\partial_{y_n}, \Gamma^{\beta}_{\beta}(y, \bar{D}, \bar{s}, \tau)]\) is the pseudodifferential operator with the symbol \(\partial_{y_n} \Gamma^{\beta}_{\beta}(y, \bar{D}, \bar{s}, \tau)\). By Lemma 8.1 we have
\[ \| [\partial_{y_n}, \Gamma^{\beta}_{\beta}(y, \bar{D}, \bar{s}, \tau)] \|_{L(H^1_0(\tau), L^2(\tau))} \leq C_{40} \pi C^1(\tau) ([\partial_{y_n}, \Gamma^{\beta}_{\beta}(y, \bar{D}, \bar{s}, \tau)]) \leq C_{41} \tau^4. \]

Denote
\[ R(y, \bar{D}, \bar{s}, \tau) = \left( 2|\bar{s}| \psi_n + \sum_{j=1}^{n-1} \varphi_j \eta_j (y_1, \ldots, y_{n-1}) (\partial_{y_n} - |\bar{s}| \psi_j) \right). \]

By (2.28) - (2.26), (2.28) and the fact that \(\bar{\mu} \eta \bar{\xi} = \eta \) the following is true:
\begin{align*}
(2.63) \quad (i\Gamma^{\beta}_{\beta}(y, \bar{D}, \bar{s}, \tau) \partial_{y_n} + i\Gamma^{\beta}_{\beta}(y, \bar{D}, \bar{s}, \tau) \partial_{y_n} \kappa^2(\bar{D}, \bar{s}) w_\nu = \bar{\mu} R(y, \bar{D}, \bar{s}, \tau) \partial_{y_n} \kappa^2(\bar{D}, \bar{s}) w_\nu
&= \bar{\mu}[R(y, \bar{D}, \bar{s}, \tau) \partial_{y_n} \kappa^2, \eta \chi(\bar{D}, \bar{s})] + \bar{\mu} \bar{\eta} R(y, \bar{D}, \bar{s}, \tau) \partial_{y_n} \kappa^2, \eta \chi(\bar{D}, \bar{s})
&= \bar{\mu}[R(y, \bar{D}, \bar{s}, \tau) \partial_{y_n} \kappa^2, \eta \chi(\bar{D}, \bar{s})] + \bar{\mu} \bar{\eta} R(y, \bar{D}, \bar{s}, \tau) \partial_{y_n} \kappa^2, \eta \chi(\bar{D}, \bar{s})
\end{align*}

Since \(-\partial_{y_n}^2 w_\nu + R(y, \bar{D}, \bar{s}, \tau) \partial_{y_n} w_\nu + \bar{R}(y, \bar{D}, \bar{s}, \tau) \partial_{y_n} w_\nu = \frac{1}{\bar{\mu}} \bar{P}_\beta(y, \bar{D}, \bar{s}, \tau) w_\nu\), setting
\[ T^{\beta} = -R_0 + [\partial_{y_n}, \Gamma^{\beta}_{\beta}(y, \bar{D}, \bar{s}, \tau)] \chi(\bar{D}, \bar{s}) - [\Gamma, \eta \chi(\bar{D}, \bar{s})] + [\eta, \bar{R}(y, \bar{D}, \bar{s}, \tau)] \chi(\bar{D}, \bar{s}) - \bar{\mu}[R(y, \bar{D}, \bar{s}, \tau) \partial_{y_n} \kappa^2, \eta \chi(\bar{D}, \bar{s}) - [\eta, R(y, \bar{D}, \bar{s}, \tau)] \partial_{y_n} \chi(\bar{D}, \bar{s})
\]
and using (2.61) - (2.63), we obtain (2.59). Now we prove (2.60). Lemma 8.4 yields
\begin{align*}
(2.64) \quad \| [\Gamma, \eta \chi(\bar{D}, \bar{s}, \tau)] \|_{L(H^1_0(\tau), L^2(\tau))} \leq C_{42} (\pi C^0(\tau)) (\Gamma \pi C^0(\tau) (\eta \chi(\bar{D}, \bar{s})) + \pi C^0(\tau) (\Gamma \pi C^1(\tau) (\eta \chi(\bar{D}, \bar{s})))
&+ \pi C^1(\tau) (\Gamma \pi C^1(\tau) (\eta \chi(\bar{D}, \bar{s})) \leq C_{43} \tau^2.
\end{align*}

For differential operators \(R\) and \(\bar{R}\), we obtain the estimates
\begin{align*}
(2.65) \quad \| [\mu \partial_{y_n} \bar{R}(y, \bar{D}, \bar{s}, \tau)] \|_{L(H^1_0(\tau), L^2(\tau))} \leq C_{44} \tau^2,
\end{align*}
and
\begin{align*}
(2.66) \quad \| [\mu \partial_{y_n} R(y, \bar{D}, \bar{s}, \tau) \partial_{y_n} \kappa^2, \eta \chi(\bar{D}, \bar{s})] \|_{L(H^1_0(\tau), L^2(\tau))} \\
&\leq \| [R(y, \bar{D}, \bar{s}, \tau) \partial_{y_n} \kappa^2, \eta \chi(\bar{D}, \bar{s})] \|_{L(H^1_0(\tau), L^2(\tau))} \leq C_{45} \tau^2,
\end{align*}
and
\begin{equation}
\| \eta \|_{L^2(\Omega)} \leq C_{45} \tau^2. \tag{2.67}
\end{equation}

Form (2.64)-(2.67) we obtain (2.60). The proof of the proposition is complete. \( \square \)

Now we apply the Lemma 8.5 to estimate \((1 - \eta)\chi(y, \bar{s}, \bar{\tau})w\). Set \(O_1 = \text{supp} \kappa_\tau \cap B(y^*, \delta)\), \(O = B(0, 2R)\) there radius \(R\) is sufficiently large, \(O_2 = \text{supp}(1 - \eta) \cap B(0, R)\). By (2.18) and definition of the function \(w\) \(\text{supp} w \subset O_1\). Since \(\eta = 1\) on \(\text{supp} \kappa_\tau\) we have \(O_1 \cap O_2 = \emptyset\). Then for some positive constant \(C_{55} \\text{dist}(O_1, O_2) \geq \frac{C_{55}}{\tau^2}\). By (8.15)

\begin{equation}
\| \bar{s} \|_{L^2(\Omega)} \leq C_{47} \tau^{4n+6} \| w \|_{L^2(\Omega)}. \tag{2.68}
\end{equation}

By arguments, same as in proposition 8.5 we obtain

\begin{equation}
\| \bar{s} \|_{L^2(\Omega)} \leq C_{48} \| w \|_{L^2(\Omega)}. \tag{2.69}
\end{equation}

Hence the inequalities (2.68) and (2.69) imply

\begin{equation}
\| \bar{s} \|_{L^2(\Omega)} \leq C_{49} \tau^{4n+6} \| w \|_{L^2(\Omega)}. \tag{2.70}
\end{equation}

Denote

\[
V_{\mu, j}^\pm(k, j) = \left( \frac{1}{\ell} \sum_{-\infty}^{\infty} \partial_{y_n} - \Gamma_{\mu}^\pm(\bar{y}, \bar{\bar{s}}, \bar{\bar{\tau}}) \right) w_{k, j, \nu}, \quad V_{\lambda+2\mu, j}^\pm = \left( \frac{1}{\ell} \sum_{-\infty}^{\infty} \partial_{y_n} - \Gamma_{\lambda+2\mu}^\pm(\bar{y}, \bar{\bar{s}}, \bar{\bar{\tau}}) \right) w_{2, \nu}.
\]

Let us consider the equation
\begin{equation}
(\ell \partial_n - \Gamma_{\beta}^\pm(y, \bar{D}, \bar{s}, \bar{\tau}))V = q, \quad V|_{y_n = \gamma} = 0. \tag{2.71}
\end{equation}

For solutions of this problem, we can prove an a priori estimate.

**Proposition 2.4.** Let \(\beta \in \{\mu, \lambda + 2\mu\}, r_\beta(y^*, \zeta^*, \tau) \neq 0\) and \(V = V_{\mu}^+(k, j)\) if \(\beta = \mu\) and \(V = V_{\lambda+2\mu}^+(k, j)\) if \(\beta = \lambda + 2\mu\). There exists a constant \(C_{50} > 0\) such that
\begin{equation}
\sqrt{|\bar{s}|} \| V \|_{L^2(\Omega)} \leq C_{50} (\tau^{4n+6} \| w \|_{H^{1, \frac{1}{2}}(\Omega)} + \| q \|_{L^2(\Omega)}). \tag{2.72}
\end{equation}

**Proof.** We taking the scalar product of the equation (2.71) and the function \(-i\bar{V}\) in \(L^2(\Omega)\) and integrating by parts we obtain
\begin{equation}
\int_{0}^{\gamma} (\Gamma_{\beta}^\pm(y, \bar{D}, \bar{s}, \bar{\tau}) \bar{V})_{L^2(\Omega)}dy_n = -\int_{0}^{\gamma} (q, V)_{L^2(\Omega)}dy_n. \tag{2.73}
\end{equation}

If \(s^* \neq 0\) or \(\text{Im} r_\beta(y^*, \zeta^*, \tau) \neq 0\). Then for some positive constant \(C_{51}\)
\begin{equation}
-\text{Im} \left| \bar{s} \right| \text{Im} \left| \Gamma_{\beta}^\pm(y, \bar{D}, \bar{s}, \bar{\tau}) \bar{V} \right|_{L^2(\Omega)}dy_n \geq C_{51} \left| \bar{s} \right| \text{Im} \left| \bar{V} \right|_{L^2(\Omega)}dy_n.
\end{equation}

By (2.70) and Gårding’s inequality (8.18) there exists a positive constant \(C_{52}\) such that
\begin{equation}
-\int_{0}^{\gamma} \left| \text{Im} \left( \Gamma_{\beta}^\pm(y, \bar{D}, \bar{s}, \bar{\tau}) \bar{V} \right) \right|_{L^2(\Omega)}dy_n \geq C_{52} \int_{0}^{\gamma} \left| \text{Im} \left( \bar{V} \right) \right|_{L^2(\Omega)}dy_n - C_{53} \tau^{4n+6} \left| \bar{V} \right|_{L^2(\Omega)}dy_n.
\end{equation}

This inequality, (2.73) and Proposition 2.3 imply (2.72). Now let \(s^* = \text{Im} r_\beta(y^*, \zeta^*, \tau) = 0\). Denote \(\bar{s} = \text{Im} r_\beta(y, \zeta, \tau) = a(y, \zeta, \tau) = \text{Re} r_\beta(y, \zeta, \tau) = \zeta = (\zeta_0, \ldots, \zeta_{n-1}, \bar{s})\). If \(\text{Im} r_\beta(y^*, \zeta^*, \tau) = 0\), then we have \(a(y^*, \zeta^*, \tau) > 0\). In that case, since near \((y^*, \zeta^*)\), we
have \( r_\beta(y, \zeta, \tau) = a(y, \zeta, \tau) \left( 1 + \frac{i|s|b(y, \zeta, \tau)}{a(y, \zeta, \tau)} \right) \) and \( \left| \frac{i|s|b(y, \zeta, \tau)}{a(y, \zeta, \tau)} \right| < 1 \) on \( \Theta_\ell \times \mathcal{O}(y^*, \delta(y^*)) \) we may define the function \( \sqrt{r_\beta(y, \zeta, \tau)} \) by the infinite series for the function \( (1 + t)^{\frac{1}{2}} = \sum_{n=0}^{\infty} c_n t^n \), \( c_n = \frac{\sqrt{n}(1 - 1)(\frac{1}{2} - 2)\cdots(\frac{1}{2} - (n - 1))}{n!} \) which holds true for \( |t| < 1 \). On \( \Theta_\ell \times \mathcal{O}(y^*, \delta(y^*)) \) we set

\[
(2.75) \quad \sqrt{r_\beta(y, \zeta, \tau)} = \sqrt{a} \sum_{n=0}^{\infty} c_n \left( \frac{i|s|b}{a} \right)^n = \sqrt{a} + \frac{i}{2} \sqrt{\frac{b}{a}} + \sqrt{a} \sum_{n=2}^{\infty} c_n \left( \frac{i|s|b}{a} \right)^n.
\]

Since \( \partial_{y_n} \varphi(y^*) < 0 \) the exists a constant \( C_{54} \) such that

\[
\text{Re} \left\{ i \int_0^{\gamma} (\Gamma_\beta(y, \tilde{D}, \tilde{s}, \tau)V, V)_{L^2(\mathbb{R}^n)} dy_n \right\} \\
\geq \text{Re} \left\{ i \int_0^{\gamma} (\alpha_\beta(y, \tilde{D}, \tilde{s}, \tau)V, V)_{L^2(\mathbb{R}^n)} dy_n \right\} \\
+ C_{54} |\varphi||V|^2_{L^2(\mathbb{Q})}.
\]

Let \( A \) be the pseudodifferential operator with the symbol \( a(y, \zeta, \tau) \) and \( A^\frac{1}{2} \) the pseudodifferential operator with the symbol \( a^\frac{1}{2}(y, \zeta, \tau) \). By Lemma 8.3, \( A = A^\frac{1}{2} A^\frac{1}{2} + R_0 \) where \( \|R_0\|_{L^2(\mathbb{Q}), L^2(\mathbb{Q})} \leq \tau^2 C_{55} \). By Lemma 8.2, \( (A^\frac{1}{2})^* = A^\frac{1}{2} + R_1 \) where \( \|R_1\|_{L^2(\mathbb{Q}), L^2(\mathbb{Q})} \leq \tau^2 C_{56} \). Hence

\[
\text{Re} \left\{ i \int_0^{\gamma} (A(y, \tilde{D}, \tilde{s}, \tau)V, V)_{L^2(\mathbb{R}^n)} dy_n \right\} = \text{Re} \left\{ i \int_0^{\gamma} (A^\frac{1}{2}(y, \tilde{D}, \tilde{s}, \tau)V, A^\frac{1}{2}(y, \tilde{D}, \tilde{s}, \tau)V)_{L^2(\mathbb{R}^n)} dy_n \right\} \\
+ \text{Re} \left\{ i \int_0^{\gamma} (A^\frac{1}{2}(y, \tilde{D}, \tilde{s}, \tau)V, (2R_0 + R_1)V)_{L^2(\mathbb{R}^n)} dy_n \right\} \\
\geq -\tau^2 C_{57} ||V||^2_{L^2(\mathbb{Q})}.
\]

Let \( B \) be the pseudodifferential operator with the symbol \( \frac{i|s|b}{\sqrt{a}} + \sqrt{a} \sum_{n=2}^{\infty} c_n \left( \frac{i|s|b}{a} \right)^n \). By Gårding’s inequality (8.18) for any positive \( \epsilon \) there exists a constant \( C_{58}(\epsilon) \) such that

\[
(2.78) \quad \text{Re} \left\{ |\bar{s}| i \int_0^{\gamma} (B(y, \tilde{D}, \tilde{s}, \tau)V, V)_{L^2(\mathbb{R}^n)} dy_n \right\} \geq -\epsilon|\bar{s}|^2 ||V||^2_{L^2(\mathbb{Q})} - C_{58} \tau^{4n+6} ||V||^2_{L^2(\mathbb{Q})}.
\]

Let \( C(x, D, \tilde{s}, \tau) \) be the pseudodifferential operator with the symbol \( \frac{-\kappa((\xi, \tilde{s}))}{|\xi|} \sum_{j=1}^{n} (\xi_j + i|\tilde{s}|\psi_j)dy_j \). By (2.19) and Gårding’s inequality (8.18) for any positive \( \epsilon \) taking the raious of the ball \( \delta \) sufficiently small one can find a constant \( C_{59}(\epsilon, \delta) \) such that

\[
(2.79) \quad \text{Re} \left\{ |\bar{s}| i \int_0^{\gamma} (C(y, \tilde{D}, \tilde{s}, \tau)V, V)_{L^2(\mathbb{R}^n)} dy_n \right\} \geq -\epsilon|\bar{s}|^2 ||V||^2_{L^2(\mathbb{Q})} - C_{59} \tau^{4n+6} ||V||^2_{L^2(\mathbb{Q})}.
\]

Inequalities (2.77), (2.76), (2.78) and (2.79) imply (2.72).

We will separately consider the two cases \( r_\mu(y^*, \zeta^*, \tau) = 0 \) in Section 3 and \( r_{\lambda+2\mu}(y^*, \zeta^*, \tau) = 0 \) in Section 4.
3. Case \( r_\mu(y^*, \zeta^*, \tau) = 0 \)

In this section, we mainly treat the case when \( \text{supp} \chi_\nu \subset \mathcal{O}(y^*, \delta_1(y^*)) \), and \((y^*, \zeta^*)\) be a point on \( \mathbb{R}^{n+1} \times \mathbb{S}^n \) such that \( r_\mu(y^*, \zeta^*, \tau) = 0 \). By (2.24)–(2.27) there exists \( C_1 > 0 \) such that

\[
|p_\mu(y^*, \tilde{\zeta}, 0) - \tilde{s}^2 p_\mu(y^*, \vec{\nabla} \psi(y^*), 0)| + |\tilde{s}(\nabla_{\vec{\zeta}} p_\mu(y^*, \tilde{\zeta}, 0), \vec{\nabla} \psi(y^*))| \\
\leq \delta_1 C_1 |(\tilde{\zeta}, \tilde{s})|^2, \quad \forall \zeta \in \mathcal{O}(\zeta^*, \delta_1(y^*)).
\]

(3.1)

Hence, by (3.1) and (2.70) for some independent constants \( C_2, C_3 \)

\[
|\mathcal{J}_3(\mu, w_{k,j,\nu})| \leq C_2 \delta_1 |\tilde{s}| \|\partial y_n w_\nu(\cdot, 0), w_\nu(\cdot, 0)\|_{L^2(\mathbb{R}^n)}^2 \\
+ C_3 \tau^{4n+6} \|\partial y_n w(\cdot, 0), w(\cdot, 0)\|_{L^2(\mathbb{R}^n)}^2 \\
\]
Applying the Cauchy-Bunyakovskii inequality and using (3.5) and (3.2), we see that there exists some positive constants $C_{11}, C_{12}$ such that

$$
\Xi_{\mu}(w_{j,n,\nu}) \geq \int_{\mathbb{R}^n} \left( |\bar{s}| \mu^2(\gamma) \psi_{y_n}(y^*) |\partial_{y_n} w_{j,n,\nu}|^2 + |\bar{s}|^3 \mu(y^*) \psi_{y_n}(y^*) |w_{j,n,\nu}|^2 \right) (\bar{y}, 0) d\bar{y}
$$

(3.6)

$$
- \frac{C_{11} \delta_1 \mu(\gamma^*)}{|\bar{s}|} \left| \bar{s} \right| \left\| (\partial_{y_n} w_{\nu}(\cdot, 0), w_{\nu}(\cdot, 0)) \right\|^2_{L^2(\mathbb{R}^n) \times H^{1,\bar{s}}(\mathbb{R}^n)}
$$

$$
- C_{12} \tau^{4n+6} \left\| (\partial_{y_n} w(\cdot, 0), w(\cdot, 0)) \right\|^2_{L^2(\mathbb{R}^n) \times H^{1,\bar{s}}(\mathbb{R}^n)}.
$$

If in addition $r_{\lambda+2\mu}(y^*, \zeta^*, \tau) = 0$, then

$$
\xi^*_0 = \psi_{y_n}(y^*) = 0 \quad \text{and} \quad \bar{s}^* \neq 0.
$$

Hence, similarly to (3.6), we obtain

(3.7)

$$
\Xi_{\lambda+2\mu}(w_{2,\nu}) \geq C_{13} \int_{\mathbb{R}^n} \left( |\bar{s}| \left\| \nabla w_{2,\nu} \right\|^2 + |\bar{s}|^3 \left\| w_{2,\nu} \right\|^2 \right) (\bar{y}, 0) d\bar{y}
$$

$$
- \epsilon(\delta) \left| \bar{s} \right| \left\| (\partial_{y_n} w_{\nu}(\cdot, 0), w_{\nu}(\cdot, 0)) \right\|^2_{L^2(\mathbb{R}^n) \times H^{1,\bar{s}}(\mathbb{R}^n)} - C_{14} \tau^{4n+6} \left\| (\partial_{y_n} w(\cdot, 0), w(\cdot, 0)) \right\|^2_{L^2(\mathbb{R}^n) \times H^{1,\bar{s}}(\mathbb{R}^n)}.
$$

Combining estimates (3.6) and (3.7), we have

$$
\left| \bar{s} \right| \left\| (\partial_{y_n} w_{\nu}(\cdot, 0), w_{\nu}(\cdot, 0)) \right\|^2_{L^2(\mathbb{R}^n) \times H^{1,\bar{s}}(\mathbb{R}^n)} + \left| \bar{s} \right| \epsilon(\delta) \left\| w_{\nu}(\cdot, 0) \right\|^2_{H^{1,\bar{s}}(\mathbb{Q})} \leq C_{15} \left( \tau^{4n+6} \left\| w \right\|^2_{H^{1,\bar{s}}(\mathbb{Q})}
$$

$$
+ \left| \bar{s} \right| \left\| g e^{\gamma^*} \right\|^2_{L^2(\mathbb{R}^n)} + \left\| P_{\mu}(y, D, \bar{s}, \tau) w_{1,\nu} \right\|^2_{L^2(\mathbb{R}^n)} + \left\| P_{\lambda+2\mu}(y, D, \bar{s}, \tau) w_{2,\nu} \right\|^2_{L^2(\mathbb{R}^n)}
$$

$$
+ \tau^{4n+6} \left\| (\partial_{y_n} w(\cdot, 0), w(\cdot, 0)) \right\|^2_{L^2(\mathbb{R}^n) \times H^{1,\bar{s}}(\mathbb{R}^n)}
$$

(3.8)

If $r_{\lambda+2\mu}(y^*, \zeta^*, \tau) \neq 0$, then by Proposition 2.4 there exists a constant $C_{16}$ independent of $s$ and $\tau$ such that

$$
\sqrt{1 + \left| \bar{s} \right|} \left\| \frac{1}{\bar{s}} \partial_{y_n} w_{2,\nu} - \Gamma_{\lambda+2\mu}^+ (y, \tilde{D}, \bar{s}, \tau) w_{2,\nu} \right\|_{L^2(\mathbb{R}^n)} \leq C_{16} \left( \left\| P_{\lambda+2\mu}(y, D, \bar{s}, \tau) w_{2,\nu} \right\|_{L^2(\mathbb{Q})} + \tau^{2n+3} \left\| w_{2,\nu} \right\|_{H^{1,\bar{s}}(\mathbb{Q})} \right).
$$

On the other hand, on $\mathbb{R}^n$ from the boundary condition (2.6), $b_n(y, D) w = g_n$ we have

(3.10)

$$
((\lambda + 2\mu)(y^*) (\partial_{y_n} w_{2,\nu} - |\bar{s}| \psi_{y_n}(y^*) w_{2,\nu}) - \mu(y^*) \sum_{j=1}^{n-1} (\partial_{y_j} w_{j,n,\nu} - |\bar{s}| \psi_{y_j}(y^*) w_{j,n,\nu})) (\cdot, 0) = r,
$$

where the function $r$ satisfies

$$
\left\| r \right\|^2_{L^2(\mathbb{R}^n)} \leq \epsilon(\delta) \left\| (\partial_{y_n} w_{\nu}(\cdot, 0), w_{\nu}(\cdot, 0)) \right\|^2_{L^2(\mathbb{R}^n) \times H^{1,\bar{s}}(\mathbb{R}^n)}
$$

$$
+ \frac{C_{17} \tau^{4n+6}}{1 + \left| \bar{s} \right|} \left\| (\partial_{y_n} w(\cdot, 0), w(\cdot, 0)) \right\|^2_{L^2(\mathbb{R}^n) \times H^{1,\bar{s}}(\mathbb{R}^n)} + C_{18} \left\| g e^{\gamma^*} \right\|^2_{L^2(\mathbb{R}^n)}
$$

(3.11)

with some constants $C_{17}, C_{18}$ independent of $\bar{s}$ and $\tau$. 

From (3.10) and (3.6), it follows that

\[
|\tilde{s}| \| (\lambda + 2 \mu)(y^*) (\partial_{y^*} w_{2,\nu} - |\tilde{s}| \psi_{y^*}(y^*) w_{2,\nu}) (\cdot, 0) \|_{L^2(\mathbb{R}^n)}^2 
\leq C_{19} \sum_{j=1}^{n-1} \Xi_\mu (w_{j,n,\nu}) + \epsilon(\delta) |\tilde{s}| \| (\partial_{y^*} w_{\nu}(\cdot, 0), w_{\nu}(\cdot, 0)) \|_{L^2(\mathbb{R}^n) \times H^{1,\tilde{s}}(\mathbb{R}^n)}^2 
+ C_{20} \tau^{4n+6} \| (\partial_{y^*} w(\cdot, 0), w(\cdot, 0)) \|_{L^2(\mathbb{R}^n) \times H^{1,\tilde{s}}(\mathbb{R}^n)}^2 + C_{21} |\tilde{s}| \| \psi_{|s|\nu} \|_{L^2(\mathbb{R}^n)}^2.
\]

Then this estimate, the Gårding inequality (8.18) and (3.9) imply

\[
|\tilde{s}|^2 \| \psi_{y^*}(y^*) w_{2,\nu}(\cdot, 0) \|_{L^2(\mathbb{R}^n)}^2 + |\tilde{s}| \| \psi_{y^*}(y^*) \nabla w_{2,\nu}(\cdot, 0) \|_{L^2(\mathbb{R}^n)}^2 
\leq C_{22} \sum_{j=1}^{n-1} \Xi_\mu (w_{j,n,\nu}) + \epsilon(\delta) |\tilde{s}| \| (\partial_{y^*} w_{\nu}(\cdot, 0), w_{\nu}(\cdot, 0)) \|_{L^2(\mathbb{R}^n) \times H^{1,\tilde{s}}(\mathbb{R}^n)}^2 + C_{23} |\tilde{s}| \| \psi_{|s|\nu} \|_{L^2(\mathbb{R}^n)}^2 
+ C_{24} \tau^{4n+6} \| (\partial_{y^*} w(\cdot, 0), w(\cdot, 0)) \|_{L^2(\mathbb{R}^n) \times H^{1,\tilde{s}}(\mathbb{R}^n)}^2 + C_{25} \| P_{\lambda+2\mu}(y, D, \tilde{s}, \tau) w_{2,\nu} \|_{L^2(Q)}^2 + \tau^2 \| w_{2,\nu} \|_{H^{1,\tilde{s}}(Q)}^2.
\]

Inequalities (3.6) and (3.12) imply (3.8).

**Case B.** Let \( \tilde{s}^* = 0 \). Then \( \xi_0^* \neq 0 \) and therefore \( r_{\lambda+2\mu}(y^*, \xi^*, \tau) \neq 0 \). By Proposition 2.4 there exists a constant \( C_{16} \) independent of \( s \) and \( \tau \) such that estimate (3.9) holds true. Let \( \delta_1 \) be sufficiently small, so that there exists a constant \( C_{26} > 0 \) such that

\[
|\xi_0|^2 \leq C_{26} (\sum_{j=1}^{n-1} |\xi_j|^2 + \tilde{s}^2) \quad \forall \xi \in \mathcal{O}(\xi^*, \delta_1(y^*)).
\]

Now we need again to estimate \( \Xi_\mu(j, n) \). We start from the term \( \mathcal{J}_2(\mu, w_{j,n,\nu}) \). By (3.10) we have

\[
\mathcal{J}_2(\mu, w_{j,n,\nu}) = -\frac{1}{2} \text{Re} \int_{\mathbb{R}^n} 2|\tilde{s}| (\lambda + 2 \mu)(y^*) (\partial_{y^*} w_{2,\nu} - |\tilde{s}| \psi_{y^*}(y^*) w_{2,\nu}) (\nabla \xi \mu(y^*, \nabla w_{j,n,\nu}, 0), \nabla \psi(y^*)) d\tilde{y}
- \frac{1}{2} \text{Re} \int_{\mathbb{R}^n} 2|\tilde{s}| (\lambda + 2 \mu)(y^*) (|\tilde{s}| \psi_{y^*}(y^*) w_{2,\nu} + r_{\tau}^2) (\nabla \xi \mu(y^*, \nabla w_{j,n,\nu}, 0), \nabla \psi(y^*)) d\tilde{y}
= -\frac{1}{2} \text{Re} \int_{\mathbb{R}^n} 2|\tilde{s}| (\lambda + 2 \mu)(y^*) (\partial_{y^*} w_{2,\nu} - |\tilde{s}| \psi_{y^*}(y^*) w_{2,\nu}) (\nabla \xi \mu(y^*, \nabla w_{j,n,\nu}, 0), \nabla \psi(y^*)) d\tilde{y}
- \frac{1}{2} \text{Re} \int_{\mathbb{R}^n} 2|\tilde{s}| (\lambda + 2 \mu)(y^*) r_{\tau}^2 (\nabla \xi \mu(y^*, \nabla w_{j,n,\nu}, 0), \nabla \psi(y^*)) d\tilde{y}.
\]

Integrating by parts we have

\[
I_2(\tilde{s}) = -\text{Re} \int_{\mathbb{R}^n} |\tilde{s}| (\lambda + 2 \mu)(y^*) (\partial_{y^*} w_{2,\nu} - |\tilde{s}| \psi_{y^*}(y^*) w_{2,\nu}) (\nabla \xi \mu(y^*, \nabla w_{j,n,\nu}, 0), \nabla \psi(y^*)) d\tilde{y}
= \text{Re} \int_{\mathbb{R}^n} |\tilde{s}| (\lambda + 2 \mu)(y^*) \partial_{y^*} w_{2,\nu} (\nabla \xi \mu(y^*, \nabla w_{2,\nu}, 0), \nabla \psi(y^*)) d\tilde{y}
+ \text{Re} \int_{\mathbb{R}^n} |\tilde{s}| (\lambda + 2 \mu)(y^*) |\tilde{s}| \psi_{y^*}(y^*) w_{2,\nu} (\nabla \xi \mu(y^*, \nabla w_{j,n,\nu}, 0), \nabla \psi(y^*)) d\tilde{y}.
\]
Denote
\[ M_j = \text{Re} \int_{\mathbb{R}^n} 2|s|(\lambda+2\mu)(y^*)|\bar{s}|\psi_{y_j}(y^*)w_{2,\nu}(\mu(y^*)) \sum_{j=1}^{n-1} \partial_{y_j} w_{j,n,\nu} \psi_{y_j}(y^*) - \partial_{y_0} w_{j,n,\nu}(\rho y_{y_0})(y^*)|d\bar{y}. \]

The simple computations imply
\[ (3.17) \quad |M_j| \leq C_{27} \int_{\mathbb{R}^n} |s|p_\mu(y^*, \bar{\xi} + i\bar{s}\nabla \psi, 0)||\hat{w}_{2,\nu}\bar{w}_{j,n,\nu}|d\bar{\xi} \]
\[ \leq C_{28} \delta_1 |s| \|
(\partial_{y_n} w_\nu(\cdot, 0), w_\nu(\cdot, 0))\|_{L^2(\mathbb{R}^n) \times H^{1,\bar{s}}(\mathbb{R}^n)} + C_{29} \tau^{2n+6} \|
(\partial_{y_n} w(\cdot, 0), w(\cdot, 0))\|_{L^2(\mathbb{R}^n) \times H^{1,\bar{s}}(\mathbb{R}^n)} \cdot \]

Here in order to obtain the last equality, we used \((3.11)\).

From \((3.9)\) and \((3.11)\) on \(\{y_n = 0\}\), for some function \(r\) we obtain
\[ (3.18) \quad i(\lambda + 2\mu)(y^*)\alpha_+^{\lambda + 2\mu}(\bar{y}, 0, \bar{D}, \bar{s}, \tau)w_{2,\nu} - \mu(y^*) \sum_{j=1}^{n-1} (\partial_{y_j} w_{j,n,\nu} - \bar{s}\psi_{y_j}(y^*)w_{j,n,\nu}) = r, \]
where \(r\) satisfies estimate \((3.11)\).

Using this equation, we transform \(\sum_{j=1}^{n-1} I_2(j)\) as
\[ (3.19) \quad \sum_{j=1}^{n-1} I_2(j) = \text{Re} \int_{\mathbb{R}^n} 2|s|(\lambda + 2\mu)(y^*)i(\lambda + 2\mu)\mu(y^*) \sum_{j=1}^{n-1} |\bar{s}|\psi_{y_j}(y^*)w_{j,n,\nu} \nabla \varphi_{y_0}(y^*, \bar{\psi}, 0) \bar{\nabla}w_{2,\nu} |d\bar{y} \]
\[ -\text{Re} \int_{\mathbb{R}^n} 2|s|(\lambda + 2\mu)\mu(y^*)r \sum_{j=1}^{n-1} |\bar{s}|\psi_{y_j}(y^*)w_{j,n,\nu} \nabla \varphi_{y_0}(y^*, \bar{\psi}, 0) \bar{\nabla}w_{2,\nu} |d\bar{y} \]
\[ +\text{Re} \int_{\mathbb{R}^n} 2|s|(\lambda + 2\mu)(y^*) \sum_{j=1}^{n-1} |\bar{s}|\psi_{y_j}(y^*)w_{j,n,\nu} \nabla \varphi_{y_0}(y^*, \bar{\psi}, 0) \bar{\nabla}w_{2,\nu} |d\bar{y} + \sum_{j=1}^{n-1} M_j. \]

Since \(r_\mu(y^*, \zeta^*, \tau) = 0\), we have \(r_{\lambda + 2\mu}(y^*, \zeta^*, \tau) = -\frac{\lambda + \mu(y^*)}{\mu(y^*)} (\xi_0 + i|\bar{s}|\psi_{y_0}(y^*))^2\). This implies that \(\text{Re}\{(\nabla \varphi_{y_0}(y^*, \bar{\psi}, 0), \xi_0)\alpha_+^{\lambda + 2\mu}(y^*, \zeta^*, \tau)\} = 0\).

Therefore by \((3.19), (3.17), (3.11)\) and the Gårding inequality \((3.18)\), we have
\[ (3.20) \quad \sum_{j=1}^{n-1} I_2(j) \geq -\epsilon(\delta)|s|\|
(\partial_{y_n} w_\nu(\cdot, 0), w_\nu(\cdot, 0))\|^2_{L^2(\mathbb{R}^n) \times H^{1,\bar{s}}(\mathbb{R}^n)} - C_{31}|s|\|g\|^2_{L^2(\mathbb{R}^n)}. \]

By \((3.20), (3.2), (3.16), (3.11)\) we obtain
\[ \sum_{j=1}^{n-1} \Xi_\mu(w_{j,n,\nu}) \geq C_{32} \int_{\mathbb{R}^n} (|s| \sum_{j=1}^{n-1} |\partial_{y_n} w_{j,n,\nu}|^2 + |s|^3 |w_{j,n,\nu}|^2)(\bar{g}, 0)|d\bar{y} \]
\[ -\epsilon(\delta)|s|\|
(\partial_{y_n} w_\nu(\cdot, 0), w_\nu(\cdot, 0))\|^2_{L^2(\mathbb{R}^n) \times H^{1,\bar{s}}(\mathbb{R}^n)} - C_{33}\tau^{4n+6} \|
(\partial_{y_n} w(\cdot, 0), w(\cdot, 0))\|^2_{L^2(\mathbb{R}^n) \times H^{1,\bar{s}}(\mathbb{R}^n)} - C_{34}|s|\|g\|^2_{L^2(\mathbb{R}^n)}. \]
By (3.10), (3.11) and (3.21), we have

$$\sum_{j=1}^{n-1} \varepsilon_j(w_{j,n,\nu}) \geq C_{35} \int_{\mathbb{R}^n} (|\tilde{s}| \sum_{j=0}^{n-1} |\partial_{y_j} w_{2,\nu}|^2 + |\tilde{s}| |\partial_{y_n} w_{2,\nu} - |\tilde{s}| \psi_{y_n} w_{2,\nu}|^2)(\tilde{y}, 0) d\tilde{y}$$

(3.22)

$$-\epsilon(\delta) |\tilde{s}| \| (\partial_{y_n} w_{\nu}(\cdot, 0), w_{\nu}(\cdot, 0)) \|_{L^2(\mathbb{R}^n) \times H^1,\tilde{x}(\mathbb{R}^n)}^2 - C_{36} |\tilde{s}| \| g e^{s|\varphi|} \|_{L^2(\mathbb{R}^n)}^2$$

$$\geq C_{37} \int_{\mathbb{R}^n} \left( |\tilde{s}| \sum_{j=0}^{n-1} |\partial_{y_j} w_{2,\nu}|^2 + |\tilde{s}|^3 |w_{2,\nu}|^2 \right) d\tilde{y} - \epsilon(\delta) |\tilde{s}| \| (\partial_{y_n} w_{\nu}(\cdot, 0), w_{\nu}(\cdot, 0)) \|_{L^2(\mathbb{R}^n) \times H^1,\tilde{x}(\mathbb{R}^n)}^2$$

$$- C_{38} \tau^{4n+6} \| (\partial_{y_n} w(\cdot, 0), w(\cdot, 0)) \|_{L^2(\mathbb{R}^n) \times H^1,\tilde{x}(\mathbb{R}^n)}^2 + C_{39} |\tilde{s}| \| g e^{s|\varphi|} \|_{L^2(\mathbb{R}^n)}^2.$$ 

Now we estimate the tangential derivatives of $w_{j,n,\nu}$. By (3.18) and (3.22), we see that there exists a function $p$ such that

$$\sum_{j=1}^{n-1} \partial_{y_j} w_{j,n,\nu}(\tilde{y}, 0) = p(\tilde{y}) \quad \text{in} \quad \mathbb{R}^n,$$

(3.23)

$$|\tilde{s}| \| p \|_{H^1,\tilde{x}(\mathbb{R}^n)}^2 \leq \epsilon(\delta) |\tilde{s}| \| (\partial_{y_n} w_{\nu}(\cdot, 0), w_{\nu}(\cdot, 0)) \|_{L^2(\mathbb{R}^n) \times H^1,\tilde{x}(\mathbb{R}^n)}^2 + C_{40} \sum_{j=1}^{n-1} \varepsilon_j(w_{j,n,\nu})$$

$$+ C_{41} \tau^{4n+6} \| (\partial_{y_n} w(\cdot, 0), w(\cdot, 0)) \|_{L^2(\mathbb{R}^n) \times H^1,\tilde{x}(\mathbb{R}^n)}^2 + C_{42} |\tilde{s}| \| g e^{s|\varphi|} \|_{L^2(\mathbb{R}^n)}^2.$$ 

Taking the Fourier transform of the first equality in (3.23) we have

$$\sum_{j=1}^{n-1} i \xi_j \hat{w}_{j,n,\nu}(\tilde{\xi}, 0) = \hat{p} \quad \forall \tilde{\xi} \in \mathbb{R}^n.$$ 

By (3.10) for $1 \leq k, j \leq n - 1$, there exist $p_{kj}(\tilde{\xi})$ such that

$$\xi_k \hat{w}_{j,n,\nu}(\tilde{\xi}, 0) - \xi_j \hat{w}_{k,n,\nu}(\tilde{\xi}, 0) = p_{kj},$$

(3.25)

$$|\tilde{s}| \| p_{kj} \|_{H^1,\tilde{x}(\mathbb{R}^n)}^2 \leq \epsilon(\delta) |\tilde{s}| \| (\partial_{y_n} w_{\nu}(\cdot, 0), w_{\nu}(\cdot, 0)) \|_{L^2(\mathbb{R}^n) \times H^1,\tilde{x}(\mathbb{R}^n)}^2$$

$$+ C_{43} \tau^{4n+6} \| (\partial_{y_n} w(\cdot, 0), w(\cdot, 0)) \|_{L^2(\mathbb{R}^n) \times H^1,\tilde{x}(\mathbb{R}^n)}^2 + C_{44} |\tilde{s}| \| g e^{s|\varphi|} \|_{L^2(\mathbb{R}^n)}^2.$$ 

If $\xi_j^2 \neq 0$ from (3.25), then we have

$$\hat{w}_{k,n,\nu}(\tilde{\xi}, 0) = \xi_k \hat{w}_{j,n,\nu}(\tilde{\xi}, 0) - \frac{p_{kj}(\tilde{\xi})}{\xi_j} \quad \forall \tilde{\xi} \in \mathcal{O}(\xi^*, \delta_1(y^*)).$$ 

Substituting this equality into (3.24), we obtain

$$\hat{w}_{j,n,\nu}(\tilde{\xi}, 0) \sum_{k=1}^{n-1} \xi_k^2 = \xi_j \hat{p}(\tilde{\xi}) + \sum_{k=1, k \neq j}^{n-1} p_{kj}(\tilde{\xi}) \quad \forall \tilde{\xi} \in \mathcal{O}(\xi^*, \delta_1(y^*)).$$
Inequalities (3.24), (3.23) and (3.25) yield
\[
(3.27) \quad |\bar{s}| \int_{\mathbb{R}^n} \sum_{j=1}^{n-1} |\partial_{y_j} w_{j,n,\nu}(\bar{y}, 0)|^2 d\bar{y} \leq \epsilon(\delta)|\bar{s}|||\partial_{y_n} w_{\nu}(\cdot, 0), w_{\nu}(\cdot, 0)||^2_{L^2(\mathbb{R}^n) \times H^{1,\bar{s}}(\mathbb{R}^n)} + C_{45} \sum_{j=1}^{n-1} \Xi_{\mu}(w_{j,n,\nu}) + C_{46} \tau^{4n+6}|||\partial_{y_n} w_{\nu}(\cdot, 0), w_{\nu}(\cdot, 0)||^2_{L^2(\mathbb{R}^n) \times H^{1,\bar{s}}(\mathbb{R}^n)} + C_{47} |\bar{s}||g^e|s|^2||^2_{L^2(\mathbb{R}^n)}.
\]
By (3.24), (3.25) and (3.28), we obtain from (3.29)
\[
(3.28) \quad |\bar{s}| \int_{\mathbb{R}^n} \sum_{j=1}^{n-1} \sum_{k=1}^{n-1} |\partial_{y_j} w_{j,k,n,\nu}(\bar{y}, 0)|^2 d\bar{y} \leq \epsilon(\delta)|\bar{s}|||\partial_{y_n} w_{\nu}(\cdot, 0), w_{\nu}(\cdot, 0)||^2_{L^2(\mathbb{R}^n) \times H^{1,\bar{s}}(\mathbb{R}^n)} + C_{48} \sum_{j=1}^{n-1} \Xi_{\mu}(w_{j,n,\nu}) + \tau^{4n+6}|||\partial_{y_n} w_{\nu}(\cdot, 0), w_{\nu}(\cdot, 0)||^2_{L^2(\mathbb{R}^n) \times H^{1,\bar{s}}(\mathbb{R}^n)} + |\bar{s}||g^e|s|^2||^2_{L^2(\mathbb{R}^n)}.
\]
Form this inequality and (3.14), we have
\[
(3.29) \quad |\bar{s}| \int_{\mathbb{R}^n} \sum_{k=1}^{n-1} |\nabla^j w_{k,n,\nu}(\bar{y}, 0)|^2 d\bar{y} \leq \epsilon(\delta)|\bar{s}|||\partial_{y_n} w_{\nu}(\cdot, 0), w_{\nu}(\cdot, 0)||^2_{L^2(\mathbb{R}^n) \times H^{1,\bar{s}}(\mathbb{R}^n)} + C_{49} \sum_{j=1}^{n-1} \Xi_{\mu}(w_{j,n,\nu}) + \tau^{4n+6}|||\partial_{y_n} w_{\nu}(\cdot, 0), w_{\nu}(\cdot, 0)||^2_{L^2(\mathbb{R}^n) \times H^{1,\bar{s}}(\mathbb{R}^n)} + |\bar{s}||g^e|s|^2||^2_{L^2(\mathbb{R}^n)}.
\]
Inequalities (3.29), (3.21) and (3.22) imply (3.8).

4. Case $r_{\lambda+2\mu}(y^*, \zeta^*, \tau) = 0$.

Let $(y^*, \zeta^*)$ be a point on $\mathbb{R}^{n+1} \times S^n$ such that $r_{\lambda+2\mu}(y^*, \zeta^*, \tau) = 0$ and $\supp \chi_{\nu} \subset O(y^*, \delta_1(y^*))$. We note that if $r_{\mu}(y^*, \zeta^*, \tau) = 0$ then $\tilde{s}^* = 0, \nabla^j \psi(y^*) = 0$. This case was treated in the previous section. Therefore we may assume $r_{\mu}(y^*, \zeta^*, \tau) \neq 0$. By (2.24)-(2.27) there exists $\delta_0 > 0$ and $C_1 > 0$ such that for all $\delta_1 \in (0, \delta_0)$ we have
\[
(4.1) \quad |\xi_0|^2 \leq C_1 \left( \sum_{j=1}^{n-1} \xi_j^2 + \tilde{s}^2 \right) \quad \forall \zeta \in O(y^*, \delta_1(y^*)).
\]
By (2.24)-(2.27) there exists $C_2 > 0$ such that
\[
|p_{\lambda+2\mu}(y^*, \tilde{\zeta}, 0) - \tilde{s}^2 p_{\lambda+2\mu}(y^*, \nabla^j \psi(y^*), 0)| + |\tilde{s}(\nabla^j p_{\lambda+2\mu}(y^*, \tilde{\zeta}, 0), \nabla^j \psi(y^*))| \leq \delta_1 C_2 |(\tilde{\zeta}, \tilde{s})|^2, \quad \forall \zeta \in O(\zeta^*, \delta_1(y^*)).
\]
By (2.31) there exists $C_3 > 0$ independent of $s, \tau$ such that
\[
(4.3) \quad \Xi_{\lambda+2\mu}(w_{2,\nu}) + C_3 (|\bar{s}||w_{2,\nu}|^2_{H^1(Q)} + |\bar{s}|^3 \tau ||w_{2,\nu}|^2_{L^2(Q)}) \leq C_4 (||P_{\lambda+2\mu}(y, D, \bar{s}, \tau)w_{2,\nu}|^2_{L^2(Q)} + \tau^{4n+6}||w||^2_{H^1,\bar{s}(Q)}) + \epsilon |\bar{s}|||\partial_{y_n} w_{\nu}(\cdot, 0), w_{\nu}(\cdot, 0)||^2_{L^2(\mathbb{R}^n) \times H^{1,\bar{s}}(\mathbb{R}^n)}.
\]
where \( \epsilon(\delta) \to 0 \) as \( \delta \to +0 \).

We consider several cases.

Case A. Let \( \widetilde{s}^* = 0 \). Since \( \widetilde{s}^* = 0 \), by decreasing the parameter \( \delta_1 \) we can assume that for some constant \( C_5 > 0 \)

\[
|\xi_0|^2 + \widetilde{s}^2 \leq C_5 \sum_{j=1}^{n-1} \xi_j^2 \quad \forall \xi \in \mathcal{O}(y^*, \delta_1(y^*)).
\]

By (4.2)

\[
P_\mu(y, \tilde{\zeta}, 0) - |\tilde{s}|^2 P_\mu(y, \tilde{\nabla} \psi, 0) \geq -\epsilon(\delta, \delta_1)|(|\xi_0|, \tilde{s})|^2 \quad (y, \zeta) \in \Upsilon_\epsilon \times \mathcal{O}(y^*, \delta_1(y^*)).
\]

If \( \lim_{\zeta \to \zeta^*} \text{Im} r_\mu(y^*, \zeta, \tau)/|s| \neq 0 \), then we set \( \text{Im} r_\mu(y^*, \zeta, \tau)/|s| \neq 0 \). For all \( (y, \zeta) \in \Upsilon_\epsilon \times \mathcal{O}(y^*, \delta_1(y^*)) \) we have

\[
\Gamma_\mu^+(y^*, \zeta^*, \tau) = \text{Im} r_\mu(y^*, \zeta^*, \tau) \quad \text{if } \lim_{\zeta \to \zeta^*} \text{Im} r_\mu(y^*, \zeta, \tau)/|s| \neq 0
\]

and

\[
\Gamma_\mu^+(y^*, \zeta^*, \tau) = 0 \quad \text{if } \lim_{\zeta \to \zeta^*} \text{Im} r_\mu(y^*, \zeta, \tau)/|s| = 0.
\]

By (4.6) and (4.7), there exists \( \epsilon(\delta, \delta_1) > 0 \) such that

\[
- (\tilde{\nabla} \mu_\mu_\mu(y^*, \tilde{\zeta}, 0, \tilde{\nabla} \psi(y^*))\Gamma_\mu^+(y^*, \tilde{\zeta}, \tau) \geq \epsilon(\delta, \delta_1)|\tilde{\zeta}|^2 \quad (y, \tilde{\zeta}) \in \Upsilon_\epsilon \times \mathcal{O}(y^*, \delta_1(y^*)).
\]

Since \( r_\mu(y^*, \zeta^*, \tau) \neq 0 \) by Proposition 2.4, we have

\[
|s| \left( \frac{1}{l} \partial_{y_n} - \Gamma_\mu^+(y, \tilde{D}, \tilde{s}, \tau) \right) w_{k,n,n} \leq C_6 \left( \||P_\mu(y, D, \tilde{s}, r, \tau) w_{k,j,n}||_{L^2(\mathbb{R}^n)} \right) \leq C_6 \left( \||P_\mu(y, D, \tilde{s}, r, \tau) w_{k,j,n}||_{L^2(\mathbb{R}^n)} + \tau^{2n+3} \right) \||w||_{H^{1,2}(\mathbb{R}^n)}.
\]

Let us consider formula (3.3) from the previous section. By (4.8), Lemmata 8.3 and 8.6, we have

\[
\begin{align*}
3_2(\mu, w_{k,n,n}) &= -\text{Re} \int_{\mathbb{R}^n} 2|s| \mu(y^*) \partial_{y_n} w_{k,n,n} (\tilde{\nabla} \mu_\mu_\mu(y^*, \tilde{\nabla} \psi(y^*))) d\tilde{y} \\
&= -\text{Re} \int_{\mathbb{R}^n} 2|s| \mu(y^*) i\Gamma_\mu^+(\tilde{y}, 0, \tilde{D}, \tilde{s}, \tau) w_{k,n,n} (\tilde{\nabla} \mu_\mu_\mu(y^*, \tilde{\nabla} \psi(y^*))) d\tilde{y} \\
&\quad + \text{Re} \int_{\mathbb{R}^n} |s| \mu(y^*) iV_{\mu_\mu_\mu}^+(k, n)(r, 0) (\tilde{\nabla} \mu_\mu_\mu(y^*, \tilde{\nabla} \psi(y^*))) d\tilde{y} \\
&\geq -C_7(\epsilon(\delta, \delta_1)|s| + \tau^{4n+6}) \left( \||\partial_{y_n} w_{k,n,n}(\cdot, 0)\||_{L^2(\mathbb{R}^n)} \right)^2 \left( \||w||_{H^{1,2}(\mathbb{R}^n)} \right)^2 \\
&\quad + C_8(\epsilon(\delta, \delta_1)|s| + \tau^{4n+6}) \left( \||w||_{L^2(\mathbb{R}^n)} \right)^2 \left( \||w||_{H^{1,2}(\mathbb{R}^n)} \right)^2.
\end{align*}
\]

Therefore from (4.10) and (4.9), we obtain

\[
3_2(\mu, w_{k,n,n}) \geq -C_9(\epsilon(\delta, \delta_1)|s| + \tau^{4n+6}) \left( \||\partial_{y_n} w_{k,n,n}(\cdot, 0)\||_{L^2(\mathbb{R}^n)} \right)^2 \left( \||w||_{L^2(\mathbb{R}^n)} \right)^2 + \||w||_{H^{1,2}(\mathbb{R}^n)}^2.
\]
Inequalities (4.11) and (4.5) imply

\[ \Xi_\mu(w_{k,n,\nu}) \geq \int_{\mathbb{R}^n} (|\tilde{s}|^{2} |\partial_{y_n} w_{k,n,\nu}|^2 + |\tilde{s}|^{3} |w_{k,n,\nu}|^2) (\tilde{y}, 0) d\tilde{y} \]

\[ -C_{10} \epsilon(\delta, \delta_1) \| (\partial_{y_n} w_{\nu}(\cdot, 0), w_{\nu}(\cdot, 0)) \|_{L^2(\mathbb{R}^n) \times H^1;\tilde{\nu}(\mathbb{R}^n)}^2 \]

\[ -C_{11} \tau^{4n+6} \| (\partial_{y_n} w(\cdot, 0), w(\cdot, 0)) \|_{L^2(\mathbb{R}^n) \times H^1;\tilde{\nu}(\mathbb{R}^n)}^2 \]

(4.12)

\[ -C_{12}(\delta, \delta_1)(\| P_\mu(y, D, \tilde{s}, \tau) w_{k,n,\nu} \|_{L^2(\Omega)}^2 + \tau^{4n+6} \| w \|_{H^1;\tilde{\nu}(\mathbb{Q})}^2). \]

By (2.6) for any \( k \in \{1, \ldots, n-1\} \), we obtain

\[ \mu(y^*) (\partial_{y_n} w_{k,n,\nu} - |\tilde{s}| \psi_{y_n}(y^*) w_{k,n,\nu})(\cdot, 0) \]

\[ = (\lambda + \mu)(y^*) (\partial_{y_n} w_{2,\nu} - |\tilde{s}| \psi_{y_n}(y^*) w_{2,\nu})(\cdot, 0) + r \text{ in } \mathbb{R}^n, \]

where the function \( r \) satisfies estimate (3.11). By (4.13), (4.12) and (4.1), we derive

\[ \sum_{j=1}^{n-1} \Xi_\mu(w_{j,n,\nu}) \geq \int_{\mathbb{R}^n} (|\tilde{s}|^{2} |\partial_{y_n} w_{j,n,\nu}|^2 + |\tilde{s}|^{3} |w_{1,\nu}|^2 + |\tilde{s}| \| w_{2,\nu} \|_{L^2(\Omega)}^2) (\tilde{y}, 0) d\tilde{y} \]

\[ -C_{13}(\epsilon(\delta, \delta_1) |\tilde{s}| + 1) \| (\partial_{y_n} w_{\nu}(\cdot, 0), w_{\nu}(\cdot, 0)) \|_{L^2(\mathbb{R}^n) \times H^1;\tilde{\nu}(\mathbb{R}^n)}^2 \]

\[ -C_{14} \tau^{4n+6} \| (\partial_{y_n} w(\cdot, 0), w(\cdot, 0)) \|_{L^2(\mathbb{R}^n) \times H^1;\tilde{\nu}(\mathbb{R}^n)}^2 \]

(4.14)

By (4.9) and Lemma 8.6, (8.18), there exists a constant \( C_{16} > 0 \) such that

\[ \sum_{j=1}^{n} \Xi_\mu(w_{j,n,\nu}) \geq C_{16} |\tilde{s}| \sum_{k,j=1, k<j}^{n} \| (\partial_{y_n} w_{k,j,\nu}(\cdot, 0), w_{k,j,\nu}(\cdot, 0)) \|_{L^2(\mathbb{R}^n) \times H^1;\tilde{\nu}(\mathbb{R}^n)}^2 \]

\[ -C_{17} \tau^{4n+6} \| (\partial_{y_n} w(\cdot, 0), w(\cdot, 0)) \|_{L^2(\mathbb{R}^n) \times H^1;\tilde{\nu}(\mathbb{R}^n)}^2 \]

\[ -C_{18}(\delta, \delta_1)(\| P_\mu(y, D, \tilde{s}, \tau) w_{1,\nu} \|_{L^2(\Omega)}^2 + \tau^{4n+6} \| w \|_{H^1;\tilde{\nu}(\mathbb{Q})}^2). \]

This inequality and (3.10) imply that

\[ \sum_{k,j=1, k<j}^{n} \Xi_\mu(w_{k,j,\nu}) \geq C_{19} |\tilde{s}| \| (\partial_{y_n} w_{\nu}(\cdot, 0), w_{\nu}(\cdot, 0)) \|_{L^2(\mathbb{R}^n) \times H^1;\tilde{\nu}(\mathbb{R}^n)}^2 \]

\[ -C_{20} \tau^{4n+6} \| (\partial_{y_n} w(\cdot, 0), w(\cdot, 0)) \|_{L^2(\mathbb{R}^n) \times H^1;\tilde{\nu}(\mathbb{R}^n)}^2 \]

\[ -C_{21}(\delta, \delta_1)(\| P_\mu(y, D, \tilde{s}, \tau) w_{1,\nu} \|_{L^2(\Omega)} + |\tilde{s}| \| \psi_{y_{1,\nu}} \|_{L^2(\mathbb{R}^n)}^2 + \tau^{4n+6} \| w \|_{H^1;\tilde{\nu}(\mathbb{Q})}^2) \]

(4.15)

with some positive constant \( C_{19} \). From (4.15) and (2.31), we obtain (3.8).

**Case B.** Let \( \tilde{s}^* \neq 0 \). If \( \delta_1 > 0 \) is small enough, then there exists a constant \( C_{22} > 0 \) such that for all \( (\tilde{\xi}, \tilde{s}) \in \mathcal{O}(y^*, \delta_1(y^*)) \)

\[ |\rho(y^*) \xi_0 \psi_{y_{0}}(y^*) - (\lambda + 2\mu)(y^*) \sum_{j=1}^{n-1} \xi_j \psi_{y_{j}}(y^*)|^2 \leq \delta_1^2 C_{22} \sum_{j=1}^{n-1} |\xi_j|^2 + \tilde{s}^2. \]

(4.16)
By (2.32), (2.33) and (1.16), we have
\[
|\mathcal{J}_2(\lambda + 2\mu, w_{2,\nu}) + \mathcal{J}_3(\lambda + 2\mu, w_{2,\nu})| \leq C_{23}\delta_1|\tilde{s}|||(\partial_{y_n} w_{2,\nu}(\cdot, 0), w_{2,\nu}(\cdot, 0)||^2_{L^2(\mathbb{R}^n) \times H^{1,\tilde{s}}(\mathbb{R}^n)} + C_{24}r^{4n+6}|||\partial_{y_n} w_2(\cdot, 0), w_2(\cdot, 0)||^2_{L^2(\mathbb{R}^n) \times H^{1,\tilde{s}}(\mathbb{R}^n)}.
\] (4.17)

By (1.17), there exists a constant $C_{25} > 0$ such that
\[
\Xi_{\lambda+2\mu}(w_{2,\nu}) \geq C_{25} \int_{\mathbb{R}^n} \left( |\tilde{s}|(\lambda + 2\mu)^2(y^*)\psi_{y_n}(y^*)|\partial_{y_n} w_{2,\nu}(\cdot, 0), w_{2,\nu}(\cdot, 0)||^2_{L^2(\mathbb{R}^n) \times H^{1,\tilde{s}}(\mathbb{R}^n)} - C_{26}r^{4n+6}|||\partial_{y_n} w_2(\cdot, 0), w_2(\cdot, 0)||^2_{L^2(\mathbb{R}^n) \times H^{1,\tilde{s}}(\mathbb{R}^n)}.
\]

Since $\tilde{s}^* \neq 0$, we have
\[
\Xi_{\lambda+2\mu}(w_{2,\nu}) \geq C_{27} \int_{\mathbb{R}^n} \left( |\tilde{s}|(\lambda + 2\mu)^2(y^*)|\nabla w_{2,\nu}(\cdot, 0), w_{2,\nu}(\cdot, 0)||^2_{L^2(\mathbb{R}^n) \times H^{1,\tilde{s}}(\mathbb{R}^n)} - C_{28}r^{4n+6}|||\partial_{y_n} w_2(\cdot, 0), w_2(\cdot, 0)||^2_{L^2(\mathbb{R}^n) \times H^{1,\tilde{s}}(\mathbb{R}^n)}.
\]

By (4.13)
\[
\Xi_{\lambda+2\mu}(w_{2,\nu}) \geq C_{29} \int_{\mathbb{R}^n} \left( |\tilde{s}|(\lambda + 2\mu)^2(y^*)|\nabla w_{2,\nu}(\cdot, 0), w_{2,\nu}(\cdot, 0)||^2_{L^2(\mathbb{R}^n) \times H^{1,\tilde{s}}(\mathbb{R}^n)} - C_{30}r^{4n+6}|||\partial_{y_n} w_2(\cdot, 0), w_2(\cdot, 0)||^2_{L^2(\mathbb{R}^n) \times H^{1,\tilde{s}}(\mathbb{R}^n)}.
\]

From (4.19), (3.11) inequality (2.2) for $V_\mu(i,j)(\cdot, 0)$, we obtain the estimate
\[
|\tilde{s}|||\alpha_{\mu}(\tilde{y}, 0, \tilde{D}, \tilde{s}, \tau)w_{k,j,i}(\cdot, 0)||^2_{L^2(\mathbb{R}^n)} \leq C_{31}(\Xi_{\lambda+2\mu}(w_{2,\nu})
+|\tilde{s}|||\partial_{y_n} w_{2,\nu}(\cdot, 0), w_{2,\nu}(\cdot, 0)||^2_{L^2(\mathbb{R}^n) \times H^{1,\tilde{s}}(\mathbb{R}^n)} + P_\mu(y, D, \tilde{s}, \tau)w_{1,\nu}(\cdot, 0)||^2_{L^2(\mathbb{Q})} + r^{4n+6}||w||^2_{H^{1,\tilde{s}}(\mathbb{Q})}
(4.20)
+R_{\mu}(\tilde{y}, 0, \tilde{D}, \tilde{s}, \tau)w_{k,j,i}(\cdot, 0)||^2_{L^2(\mathbb{R}^n)} + |\tilde{s}|||\alpha_{\mu}(\tilde{y}, 0, \tilde{D}, \tilde{s}, \tau)w_{k,j,i}(\cdot, 0)||^2_{L^2(\mathbb{R}^n)}\right).
\]

Since by (2.19) $\nabla\tilde{\ell}(0) = 0$ then $|\alpha_{\mu}(y^*, \zeta^*, \tau)| = |r\mu(y^*, \zeta^*, \tau)| \neq 0$. By Lemma 8.2
\[
||\alpha_{\mu}(\tilde{y}, 0, \tilde{D}, \tilde{s}, \tau)w_{k,j,i}(\cdot, 0)||^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} A(\tilde{y}, \tilde{D}, \tilde{s}, \tau)w_{k,j,i}(\cdot, 0)||^2_{L^2(\mathbb{Q})} + \int_{\mathbb{R}^n} R_{\mu}(\tilde{y}, 0, \tilde{D}, \tilde{s}, \tau)w_{k,j,i}(\cdot, 0)||^2_{L^2(\mathbb{R}^n)}d\tilde{y}.
\]

Here $A(\tilde{y}, \tilde{D}, \tilde{s}, \tau)$ is the pseudodifferential operator with the symbol $|\alpha_{\mu}(\tilde{y}, 0, \zeta, \tau)|^2$ and $R \in L(H^{1,\tilde{s}}(\mathbb{Y}_\ell); L^2(\mathbb{Y}_\ell))$ with the norm $||R|| \leq C_{32}r^{4n+6}|\alpha_{\mu}| \leq C_{33}r^2$. Therefore
\[
||\alpha_{\mu}(\tilde{y}, 0, \tilde{D}, \tilde{s}, \tau)w_{k,j,i}(\cdot, 0)||^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} A(\tilde{y}, \tilde{D}, \tilde{s}, \tau)w_{k,j,i}(\cdot, 0)||^2_{L^2(\mathbb{Q})} - C_{34}r^{4n+6}||w_{k,j,i}(\cdot, 0)||^2_{L^2(\mathbb{Q})}.
\]

Applying the Gårding inequality (8.13) we obtain
\[
(4.21)
||\alpha_{\mu}(\tilde{y}, 0, \tilde{D}, \tilde{s}, \tau)w_{k,j,i}(\cdot, 0)||^2_{L^2(\mathbb{R}^n)} \geq C_{35}||w_{k,j,i}(\cdot, 0)||^2_{H^{1,\tilde{s}}(\mathbb{Q})} - C_{36}r^{4n+6}||w_{k,j,i}(\cdot, 0)||^2_{L^2(\mathbb{Q})}.
\]
Inequalities (4.9), (4.18), (4.21) and (4.20) imply
\[
\Xi_{\lambda+2\mu}(w_{2,\nu}) \geq C_{37} |\tilde{\tau}| \| \tilde{\tau} \|_{L^2(\mathbb{R}^n)}^2 (\partial_{y_{i}} w_{\nu}(\cdot, 0), w_{\nu}(\cdot, 0)) \|
\]
\[
- C_{38}(\| P_\mu(y, D, \tilde{s}, \tau) w_{1,\nu} \|^2_{L^2(\mathbb{R}^n)} + |\tilde{\tau}| \| g e^{s|\tau|} \|_{L^2(\mathbb{R}^n)}^2 + \tau^{4n+6} \| w \|^2_{H^{1,2}(\mathbb{R}^n)})
\]
\[
- C_{39} \tau^{4n+6} (\| \partial_{y_{i}} w_{2}(\cdot, 0), w_{2}(\cdot, 0) \|^2_{L^2(\mathbb{R}^n)} \times H^{1,2}(\mathbb{R}^n)),
\]
where $C_{37} > 0$. From (4.3) and (4.22), we obtain (3.8). \[\blacksquare\]

5. **Case $r_\mu(y^*, \zeta^*, \tau) \neq 0$ and $r_{\lambda+2\mu}(y^*, \zeta^*, \tau) \neq 0$.**

In this section we consider the conic neighborhood $\mathcal{O}(y^*, \delta_1(y^*))$ of the point $(y^*, \zeta^*)$ such that
\[
|r_\mu(y^*, \zeta^*, \tau)| \neq 0 \quad \text{and} \quad |r_{\lambda+2\mu}(y^*, \zeta^*, \tau)| \neq 0.
\]

In that case, thanks to (5.1) and Proposition 2.3, factorization (2.59) holds true for $\beta = \mu$ and $\beta = \lambda + 2\mu$. Then Proposition 2.4 yields the a priori estimate
\[
(1 + |\tilde{\tau}|) (\sum_{k,j=1, k<j}^n \| V^+_{\mu}(k, j)(\cdot, 0) \|_{L^2(\mathbb{R}^n)}^2 + \| V^+_{\lambda+2\mu}(\cdot, 0) \|_{L^2(\mathbb{R}^n)}^2)
\]
\[
\leq C_1(\| P_{\lambda+2\mu}(y, D, \tilde{s}, \tau) w_{2,\nu} \|^2_{L^2(\mathbb{Q})} + \| P_\mu(y, D, \tilde{s}, \tau) w_{1,\nu} \|^2_{L^2(\mathbb{Q})} + \tau^{4n+6} \| w \|^2_{H^{1,2}(\mathbb{Q})}).
\]

Using (2.6), we rewrite (3.10) as
\[
\frac{\lambda + 2\mu}{\mu}(y^*)(\partial_{y_{i}} w_{2,\nu} - |\tilde{\tau}| \psi_{y_{i}} w_{2,\nu}) - i \alpha^+_{\mu}(\tilde{y}, 0, \tilde{D}, \tilde{s}, \tau) w_{j, n, \nu} = V^+_{\mu}(i, n)(\cdot, 0) - r_{j, n, \nu},
\]
where $i \in \{1, \ldots, n-1\}$ and
\[
\sum_{k=1}^{n-1} \frac{\mu}{\lambda + 2\mu}(y^*) (-\partial_{y_{k}} w_{k, n, \nu} + |\tilde{\tau}| \psi_{y_{k}} w_{k, n, \nu})
\]
\[
- i \alpha^+_{\lambda+2\mu}(\tilde{y}, 0, \tilde{D}, \tilde{s}, \tau) w_{2,\nu} = V^+_{\lambda+2\mu}(\cdot, 0) - r_{2,\nu},
\]
where the function $r = (r_{1, n, \nu}, \ldots, r_{n-1, n, \nu}, r_{2, \nu})$ satisfies estimate (3.11). Let $B(\tilde{y}, \tilde{D}, \tilde{s}, \tau)$ be the matrix pseudodifferential operator with the symbol
\[
B(\tilde{y}, \zeta, \tau) = \begin{pmatrix}
- i \alpha^+_{\mu}(\tilde{y}, 0, \zeta, \tau) & 0 & \cdots & \frac{\lambda + 2\mu}{\mu}(i \xi_1 - |\tilde{\tau}| \psi_{y_{1}}) \\
0 & \cdots & \cdots & \cdots \\
0 & - i \alpha^+_{\mu}(\tilde{y}, 0, \zeta, \tau) & \cdots & \frac{\lambda + 2\mu}{\mu}(i \xi_n - |\tilde{\tau}| \psi_{y_{n}}) \\
\cdots & \cdots & \cdots & \cdots \\
\frac{\mu}{\lambda + 2\mu}(-i \xi_1 + |\tilde{\tau}| \psi_{y_{1}}) & \cdots & \frac{\mu}{\lambda + 2\mu}(-i \xi_{n-1} + |\tilde{\tau}| \psi_{y_{n-1}}) & - i \alpha^+_{\lambda+2\mu}(\tilde{y}, 0, \zeta, \tau)
\end{pmatrix}
\]

We have
Proposition 5.1. The following formula is true:

\[
\det B(y^*, \zeta, \tau) = (-i)^n (\alpha_{\mu}^+ (y^*, \zeta, \tau))^{n-1} \alpha_{\lambda+2\mu}^+ (y^*, \zeta, \tau)
\]
\[
+(-1)^{n-1} (-i)^{n-2} (\alpha_{\mu}^+ (y^*, \zeta, \tau))^{n-2} \sum_{j=1}^{n-2} (-i \xi_j + |s| \psi_{y_j}(y^*))^2.
\]

(5.6)

Proof. By \(B_n\) we denote the matrix determined by (5.5) of the size \(n \times n\) and \(B_{i,j,n}\) be the minor obtained from the matrix \(B_n\) by crossing out the \(i\)-th row and the \(j\)-th column. Our proof is based on the induction method. Except the formula (5.6), we claim

\[
|B_{1,n-1,n}(y^*, \zeta, \tau)| = (-i)^n (\alpha_{\mu}^+ (y^*, \zeta, \tau))^{n-2} (-i \xi_1^* + |\bar{s}| \psi_{y_1}(y^*))^2.
\]

(5.7)

For \(n = 2, 3\), we can easily verify the formulae by direct computations. Suppose that (5.6) and (5.7) are true for \(n - 1\). Then

\[
\det B_{n-1}(y^*, \zeta, \tau) = (-i)^{n-1} (\alpha_{\mu}^+ (y^*, \zeta, \tau))^{n-2} \alpha_{\lambda+2\mu}^+ (y^*, \zeta, \tau)
\]
\[
+(-1)^{n-1} (-i)^{n-3} (\alpha_{\mu}^+ (y^*, \zeta, \tau))^{n-3} \sum_{j=1}^{n-2} (-i \xi_j + |s| \psi_{y_j}(y^*))^2
\]

(5.8)

and

\[
|B_{1,n-1,n}(y^*, \zeta, \tau)| = (-i)^{n-1} \frac{\mu}{\lambda + 2\mu} (y^*)(\alpha_{\mu}^+ (y^*, \zeta, \tau))^{n-2} (-i \xi_1^* + |\bar{s}| \psi_{y_1}(y^*))^2.
\]

(5.9)

Since \(\det B_n(y^*, \zeta, \tau) = -i \alpha_{\mu}^+ (y^*, \zeta, \tau)|B_{1,1,n}| + (-1)^{1+n} \frac{\lambda + 2\mu}{\mu} (y^*)(i \xi_1 - |\bar{s}| \psi_{y_1}(y^*))|B_{1,n,n}|\), by (5.6) and (5.9) we have

\[
\det B_n(y^*, \zeta^*, \tau) = -i \alpha_{\mu}^+ (y^*, \zeta^*, \tau)((-i)^{n-1} (\alpha_{\mu}^+ (y^*, \zeta, \tau))^{n-2} \alpha_{\lambda+2\mu}^+ (y^*, \zeta, \tau)
\]
\[
+(-1)^{n-1} (-i)^{n-3} (\alpha_{\mu}^+ (y^*, \zeta, \tau))^{n-3} \sum_{j=2}^{n-1} (-i \xi_j + |s| \psi_{y_j}(y^*))^2
\]
\[
+(-1)^{1+n} \frac{\lambda + 2\mu}{\mu} (y^*)(-i \alpha_{\mu}^+ (y^*, \zeta^*, \tau))^n (-i \xi_1 + |\bar{s}| \psi_{y_1}(y^*))
\]
\[
= (-i)^n (\alpha_{\mu}^+ (y^*, \zeta, \tau))^{n-1} \alpha_{\lambda+2\mu}^+ (y^*, \zeta, \tau)
\]
\[
+(-i)^{n-2} (-1)^{n-1} (\alpha_{\mu}^+ (y^*, \zeta, \tau))^{n-2} \sum_{j=2}^{n-2} (-i \xi_j + |s| \psi_{y_j}(y^*))^2
\]
\[
+(-1)^{1+n} (-i \alpha_{\mu}^+ (y^*, \zeta, \tau))^n (-i \xi_1 + |\bar{s}| \psi_{y_1}(y^*))^2.
\]

(5.10)

The proof of the proposition is complete. ■

By (2.24)-(2.27) if \(\det B(y^*, \zeta^*, \tau) = 0\) and (5.1) holds true, then

\[
\zeta^* \in \mathcal{U} = \left\{ \zeta \in \mathbb{R}^{n+1}; \sum_{j=1}^{n-1} (\xi_j + i|\bar{s}| \psi_{y_j}(y^*))^2 = \frac{\rho(y^*)(\xi_0 + i|\bar{s}| \psi_{y_0}(y^*))^2}{\lambda + 3\mu(y^*)} \right\}.
\]

(5.11)

Now we consider the two cases.
Case A. Let $\zeta^* \notin \mathcal{U}$.

In that case there exists a parametrix of the operator $B(\tilde{y}, \tilde{D}, \tilde{s}, \tau)$ which we denote as $B^{-1}(\tilde{y}, \tilde{D}, \tilde{s}, \tau)$. Then

$$
\sum_{k,j=1}^{n} \Xi_{\mu}(w_{k,j}, w_{2,\nu}) + \Xi_{\lambda+2\mu}(w_{2,\nu}) \geq C_2|\tilde{s}|^2\|\partial_{x_{y}\psi} w_{\nu}(\cdot, 0)\|_{L^2(\mathbb{R}^n)}^2 - C_3(\|P_{\mu}(y, D, \tilde{s}, \tau)w_{1,\nu}\|_{L^2(\Omega)}^2 + \|P_{\lambda+2\mu}(y, D, \tilde{s}, \tau)w_{2,\nu}\|_{L^2(\Omega)}^2)
$$

By (5.13), (5.12) and (5.13) for some positive constant $C_2$, we have

$$
\sum_{k,j=1}^{n} \Xi_{\mu}(w_{k,j}, w_{2,\nu}) + \Xi_{\lambda+2\mu}(w_{2,\nu}) \geq C_2|\tilde{s}|^2\|\partial_{x_{y}\psi} w_{\nu}(\cdot, 0)\|_{L^2(\mathbb{R}^n)}^2 - C_3(\|P_{\mu}(y, D, \tilde{s}, \tau)w_{1,\nu}\|_{L^2(\Omega)}^2 + \|P_{\lambda+2\mu}(y, D, \tilde{s}, \tau)w_{2,\nu}\|_{L^2(\Omega)}^2)
$$

By (5.13), (5.14) and (2.31), we obtain (3.8).

Case B. Let $\zeta^* \in \mathcal{U}$.

By (2.24)-(2.27) there exists a constant $C_4 > 0$ such that

$$
|p_{\lambda+3\mu}(y, \tilde{\xi}, 0) - |\tilde{s}|^2 p_{\lambda+3\mu}(y, \tilde{\psi}, 0)| \leq C_4\delta_1|\zeta|^2 \quad \forall(y, \zeta) \in \mathcal{U}_\ell \times \mathcal{O}(y^*, \delta_1(y^*)).
$$

If $\tilde{s}^* = 0$, then using (5.15), we obtain

$$
p_{\beta}(y, \tilde{\xi}, 0) - \tilde{s}^2 p_{\beta}(y, \tilde{\psi}, 0) = \sum_{j=1}^{n-1} (\lambda + 3\mu - \beta)(y)(\xi_j^2 - \tilde{s}^2 \psi_{y_j}^2(y^*)) + p_{\lambda+3\mu}(y, \tilde{\xi}, 0) - \tilde{s}^2 p_{\lambda+3\mu}(y, \tilde{\psi}, 0)
$$

$$
\geq \sum_{j=1}^{n-1} (\lambda + 3\mu - \beta)(y)(\xi_j^2 - \tilde{s}^2 \psi_{y_j}^2(y^*)) - C_5\delta_1|\zeta|^2 \quad \forall(y, \zeta) \in \mathcal{U}_\ell \times \mathcal{O}(y^*, \delta_1(y^*)).
$$

Therefore for all sufficiently small $\delta_1$ there exists $C_6$ such that

$$
p_{\beta}(y, \tilde{\xi}, 0) - \tilde{s}^2 p_{\beta}(y, \tilde{\psi}, 0) \geq C_6\delta_1|\zeta|^2 \quad \forall(y, \zeta) \in \mathcal{U}_\ell \times \mathcal{O}(y^*, \delta_1(y^*)).
$$

Now let us consider the case when $\tilde{s}^* \neq 0$.

By the definition of the set $\mathcal{U}$, there exists positive constants $C_7, C_8$ such that

$$
|\rho(y^*)\xi_0^2 - \tilde{s}^2 \rho(y^*)\psi_{y_0}^2(y^*) - (\lambda + 3\mu)(y^*)\sum_{j=1}^{n-1} (\xi_j^2 - \tilde{s}^2 \psi_{y_j}^2(y^*))| \leq C_7\delta|\zeta|^2,
$$

$$
|\xi_0(\rho\psi_{y_0})(y^*) - (\lambda + 3\mu)(y^*)\sum_{j=1}^{n-1} \xi_j\psi_{y_j}(y^*)| \leq C_8\delta|\zeta|.
$$

Let $\psi_{y_0}(y^*) = 0$. By (5.17) we have

$$
\sum_{j=1}^{n-1} (\xi_j^2 - \tilde{s}^2 \psi_{y_j}^2(y^*)) \geq -C_9\delta|\zeta|^2 \quad \forall \zeta \in \mathcal{O}(y^*, \delta_1(y^*)).$$
This inequality and (1.7) imply
\begin{equation}
\Gamma(y^*) \geq \frac{1}{2}(\lambda + 3\mu)(y^*)
\end{equation}

By (5.11), we see
\begin{equation}
\rho(y^*)(\tilde{s}^* \psi_{y_0}(y^*))^2 = \frac{1}{2}(\lambda + 3\mu)(y^*)
\end{equation}

This inequality and (1.7) imply
\begin{equation}
\psi_{y_j}(y^*) > \sqrt{\frac{(\lambda + 2\mu)(y^*)}{\mu}(y^*)} \sum_{j=1}^{n-1} |\psi_{y_j}(y^*)|^2 > \sqrt{\frac{\rho(y^*)(\lambda + 2\mu)(y^*)}{\mu(\lambda + 3\mu)(y^*)}} |\psi_{y_0}(y^*)|.
\end{equation}

Inequality (5.19) yields
\begin{align*}
p_\beta(y^*, \tilde{\xi}, 0) - (\tilde{s}^*)^2 p_\beta(y^*, \tilde{\nabla} \psi, 0) &= \frac{\rho(\lambda + 2\mu - \beta)}{\lambda + 3\mu} (y^*)((\xi_0^*)^2 - (\tilde{s}^*)^2 (\psi_{y_0}(y^*))^2)
\end{align*}

Hence there exists a positive constant $C_{15}$ such that
\begin{equation}
\mathcal{J}_1(\mu, w_{k,n,\nu}) + \mathcal{J}_3(\mu, w_{k,n,\nu}) \geq |\tilde{s}| C_{15}\delta \| (\partial_{y_n} w_\nu(\cdot, 0), w_\nu(\cdot, 0)) \|_{L^2(\mathbb{R}^n) \times H^{1,2}(\mathbb{R}^n)}
\end{equation}

By $\text{Re} \ r_\beta(y^*, \xi, \tau) > 0$ for $\xi^* \in U$, short computations yield
\begin{equation}
\Gamma_\beta(y^*, \xi^*, \tau) = \mathcal{I}_\beta \text{Re} \ r_\beta(y^*, \xi^*, \tau), \quad \mathcal{I}_\beta = \text{sign}(\text{Im} \xi \to \xi^* \text{ Im} r_\mu(y^*, \xi, \tau) / |\tilde{s}|).
\end{equation}

Therefore for any $(\tilde{y}, \tilde{\xi})$ from $\Upsilon_{\ell} \times \mathcal{O}(y^*, \delta_1(y^*))$
\begin{equation}
- \text{Re} \{\Gamma_\beta(\tilde{y}, 0, \tilde{\xi}, \tilde{\xi})(\tilde{\nabla} \psi^\prime_{\beta}(y^*, \tilde{\xi}, 0, \tilde{\nabla} \psi(y^*))\} \geq \epsilon(\delta_1, \delta_2) |(\tilde{\xi}, \tilde{\xi})|^2.
\end{equation}
By (5.21), Lemmata 8.3 and 8.6 (the Gårding inequality (8.18)) in Section 8, we have

\[ \mathcal{J}_2(\mu, w_{k,n,\nu}) = -\text{Re} \int_{\mathbb{R}^n} 2|\tilde{s}|\mu(y^*)i\Gamma^+_{\mu}(\tilde{y}, 0, \tilde{D}, \tilde{s}, \tau)w_{k,n,\nu}(\nabla_{\tilde{y}}p_{\mu}(y^*, \tilde{D}, \tilde{s}, \tau), \tilde{\psi}(y^*))d\tilde{y} \]

\[ = -\text{Re} \int_{\mathbb{R}^n} |\tilde{s}|(\lambda + 2\mu)(y^*)i\Gamma^+_{\lambda+2\mu}(\tilde{y}, 0, \tilde{D}, \tilde{s}, \tau)w_{2,\nu}(\nabla_{\tilde{y}}p_{\lambda+2\mu}(y^*, \tilde{D}, \tilde{s}, \tau), \tilde{\psi}(y^*))d\tilde{y} \]

(5.23)

\[ \geq -C_{19}(\epsilon(\delta, \delta_1)|\tilde{s}| + \tau^{4n+6})\|\partial_{y^*}w_\nu(\cdot, 0), w_\nu(\cdot, 0)\|_{L^2(\mathbb{R}^n) \times H^{1,\tilde{s}}(\mathbb{R}^n)}^2 \]

\[ -C_{20}(\delta, \delta_1)(\|P_\nu(y, D, \tilde{s}, \tau)w_{k,n,\nu}\|_{L^2(\mathbb{Q})}^2 + P_{\lambda+2\mu}(y, D, \tilde{s}, \tau)w_{2,\nu}\|_{L^2(\mathbb{Q})}^2 + \tau^{4n+6}\|w\|_{H^{1,\tilde{s}}(\mathbb{Q})}^2) \]

and

\[ \mathcal{J}_2(\lambda + 2\mu, w_{2,\nu}) \]

(5.24)

By (5.20), (5.23), (5.24) and (3.3), there exists a constant \( C_{24} > 0 \) such that

\[ \sum_{k,j=1,k<j}^{n} \Xi_\mu(w_{k,j,\nu}) + \Xi_\mu(w_{2,\nu}) \geq C_{24}|\tilde{s}|\|\partial_{y^*}w_\nu(\cdot, 0), w_\nu(\cdot, 0)\|_{L^2(\mathbb{R}^n) \times H^{1,\tilde{s}}(\mathbb{R}^n)}^2 \]

\[ -C_{25}\tau^{4n+6}\|\partial_{y^*}w(\cdot, 0), w_\nu(\cdot, 0)\|_{L^2(\mathbb{R}^n) \times H^{1,\tilde{s}}(\mathbb{R}^n)}^2 \]

\[ -C_{26}(\delta, \delta_1)(\|P_\nu(y, D, \tilde{s}, \tau)w_{1,\nu}\|_{L^2(\mathbb{Q})}^2 + P_{\lambda+2\mu}(y, D, \tilde{s}, \tau)w_{2,\nu}\|_{L^2(\mathbb{Q})}^2 + \tau^{4n+6}\|w\|_{H^{1,\tilde{s}}(\mathbb{Q})}^2) \]

(5.25)

From (5.25) and (2.31), we obtain (3.8). Thus in each case, we established the estimate (3.8).

6. End of the proof.

First we finish the proof of the proposition 2.1. Now let us take the covering of the sphere \( S^n \) by conical neighborhoods \( \mathcal{O}(\zeta^*, \delta_1(\zeta^*)) \). From this covering we take the finite subcovering \( \bigcup_{m=1}^{N} \mathcal{O}(\zeta^*_m, \delta_1(\zeta^*_m)) \). Let \( \chi_m \) be the partition of unity associated to this subcovering. Hence
\[
\sum_{m=0}^{N} \chi_m(\tilde{\xi},s) \equiv 1. \text{ Then by (3.8)}
\]
\[
(6.1) \quad |\tilde{s}|\left\| (\partial_{y_n}w(\cdot,0),w(\cdot,0)) \right\|_{L^2(\mathbb{R}^n) \times H^{1,\tilde{\alpha}}(\mathbb{R}^n)}^2 + |\tilde{s}|\tau\left\| w \right\|_{H^{1,\tilde{\alpha}}(\mathbb{Q})}^2
\]
\[
\leq \sum_{m=0}^{N} \left( |\tilde{s}|\eta_\ell(\partial_{y_n}(\chi_mw)(\cdot,0),\chi_mw(\cdot,0)) \right)_{L^2(\mathbb{R}^n) \times H^{1,\tilde{\alpha}}(\mathbb{R}^n)}^2 + |\tilde{s}|\tau\left\| \eta_\ell \chi_mw \right\|_{H^{1,\tilde{\alpha}}(\mathbb{Q})}^2
\]
\[
+ C_1 \sum_{m=0}^{N} \left( |\tilde{s}|\left\| (1-\eta_\ell)(\partial_{y_n}(\chi_mw)(\cdot,0),\chi_mw(\cdot,0)) \right\|_{L^2(\mathbb{R}^n) \times H^{1,\tilde{\alpha}}(\mathbb{R}^n)}^2 + |\tilde{s}|\tau\left\| (1-\eta_\ell)\chi_mw \right\|_{H^{1,\tilde{\alpha}}(\mathbb{Q})}^2 \right)
\]
\[
\leq C_2 \left( \tau^{4n+6}\left\| w \right\|_{H^{1,\tilde{\alpha}}(\mathbb{Q})}^2 + |\tilde{s}|\left\| g e^{s|\varphi|} \right\|_{L^2(\mathbb{R}^n)}^2 + \sum_{m=0}^{N} \left\| \partial_{y_n}(y,D,\tilde{s},\tau)w_{1,m} \right\|_{L^2(\mathbb{Q})}^2 \right)
\]
\[
+ \left| \sum_{m=0}^{N} \left( |\tilde{s}|\left\| (1-\eta_\ell)(\partial_{y_n}(\chi_mw)(\cdot,0),\chi_mw(\cdot,0)) \right\|_{L^2(\mathbb{R}^n) \times H^{1,\tilde{\alpha}}(\mathbb{R}^n)}^2 + |\tilde{s}|\tau\left\| (1-\eta_\ell)\chi_mw \right\|_{H^{1,\tilde{\alpha}}(\mathbb{Q})}^2 \right)
\]
\[
\leq C_4 \left( \tau^{4n+6}\left\| (\partial_{y_n}w(\cdot,0),w(\cdot,0)) \right\|_{L^2(\mathbb{R}^n) \times H^{1,\tilde{\alpha}}(\mathbb{R}^n)}^2 + \tau^{4n+6}\left\| w \right\|_{H^{1,\tilde{\alpha}}(\mathbb{Q})}^2 \right)
\]

Using inequality (6.2) in order to estimate the last terms in (6.1) we obtain
\[
|\tilde{s}|\left\| (\partial_{y_n}w(\cdot,0),w(\cdot,0)) \right\|_{L^2(\mathbb{R}^n) \times H^{1,\tilde{\alpha}}(\mathbb{R}^n)}^2 + |\tilde{s}|\tau\left\| w \right\|_{H^{1,\tilde{\alpha}}(\mathbb{Q})}^2
\]
\[
\leq C_5 \left( \tau^{4n+6}\left\| w \right\|_{H^{1,\tilde{\alpha}}(\mathbb{Q})}^2 + |\tilde{s}|\left\| g e^{s|\varphi|} \right\|_{L^2(\mathbb{R}^n)}^2 \right)
\]
\[
+ \left| \sum_{m=0}^{N} |\chi_m|\left( |\partial_{y_n}(y,D,\tilde{s},\tau)w_{1} |_{L^2(\mathbb{Q})}^2 + \left| \partial_{y_n}(y,D,\tilde{s},\tau)w_{2} \right|_{L^2(\mathbb{Q})}^2 \right) \right)
\]
\[
+ \left| \sum_{m=0}^{N} \left| \partial_{y_n}(y,D,\tilde{s},\tau)w_{1} |_{L^2(\mathbb{Q})}^2 + \left| \partial_{y_n}(y,D,\tilde{s},\tau)w_{2} \right|_{L^2(\mathbb{Q})}^2 \right) \right)
\]
\[
\leq C_7 \left( \tau^{4n+6}\left\| w \right\|_{H^{1,\tilde{\alpha}}(\mathbb{Q})}^2 + |\tilde{s}|\left\| g e^{s|\varphi|} \right\|_{L^2(\mathbb{R}^n)}^2 \right)
\]
\[
+ \left| \sum_{m=0}^{N} \left( |\tilde{s}|\left\| (\partial_{y_n}w(\cdot,0),w(\cdot,0)) \right\|_{L^2(\mathbb{R}^n) \times H^{1,\tilde{\alpha}}(\mathbb{R}^n)}^2 + \tau^{4n+6}\left\| w \right\|_{H^{1,\tilde{\alpha}}(\mathbb{Q})}^2 \right) \right)
\]
\[
\leq C_8 \left( \tau^{4n+6}\left\| (\partial_{y_n}w(\cdot,0),w(\cdot,0)) \right\|_{L^2(\mathbb{R}^n) \times H^{1,\tilde{\alpha}}(\mathbb{R}^n)}^2 + \tau^{4n+6}\left\| w \right\|_{H^{1,\tilde{\alpha}}(\mathbb{Q})}^2 \right) .
\]
Hence there exists $\tau_0 > 1$ independent of $s$ such that for all $\tau \geq \tau_0$ and $s \geq 1$, we see

$$|\bar{s}|\left\| (\partial_n w(\cdot, 0), w(\cdot, 0)) \right\|_{L^2(\mathbb{R}^n) \times H^{1, \bar{s}}(\mathbb{R}^n)} + |\bar{s}|\left\| w \right\|_{H^{1, \bar{s}}(\mathbb{Q})}^2 \leq C_9 \left( \tau^{4n+6} \left\| w \right\|_{H^{1, \bar{s}}(\mathbb{Q})}^2 + |\bar{s}|\left\| ge^{s\varphi} \right\|_{L^2(\mathbb{R}^n)}^2 \right) \tag{6.4}$$

If in the inequality (6.4) we return to the function $v$, then we obtain

$$\int_{\Sigma} \left( s\tau |\nabla v|^2 + s^3 \tau^3 |v|^2 \right)e^{2s\varphi} d\Sigma + \int_Q \left( s\tau^2 |\nabla v|^2 + s^3 \tau^4 |v|^2 \right)e^{2s\varphi} dx \leq C_{10} \left( s\tau |\sqrt{g} e^{s\varphi} |_{L^2(\Sigma)}^2 + \left\| q e^{s\varphi} \right\|_{L^2(\mathbb{Q})}^2 \right) + \int_{\Sigma} \left( s\tau |\nabla v|^2 + s^3 \tau^2 |v|^2 \right)e^{2s\varphi} d\Sigma \right). \tag{6.5}$$

Proof of the proposition 2.1 is complete.

Now we show that the functions $v$ given by (2.2) satisfies the estimate (2.9).

First we show that function $v$ satisfies the boundary conditions (2.6). By (1.1) and the zero Dirichlet boundary conditions for the function $u$ we have

$$-L_{\lambda, \mu}(x, D')u + \int_0^{x_0} L_{\lambda, \mu}(x, \bar{x}_0, D')u(\bar{x}_0, x')d\bar{x}_0 = F \quad \text{on } \Sigma \tag{6.6}$$

Next we move all terms containing the first derivatives of the function $u$ in the right hand side, divide the both sides by $\mu$ and denote the right hand side of obtained equality as $g_k$.

$$-\Delta u - \frac{(\lambda + \mu)}{\mu} \nabla' \text{div } u + \int_0^{x_0} \left( \frac{\mu(x, \bar{x}_0)}{\mu(x)} \Delta u(\bar{x}_0, x') + \frac{\tilde{\lambda}(x, \bar{x}_0) + \tilde{\mu}(x, \bar{x}_0)}{\mu(x)} \nabla' \text{div } u \right) dx_0 \tag{6.7}$$

For any index $\hat{i} \in \{1, \ldots, n\}$ the short computations imply

$$-\Delta u_{\hat{i}} = \sum_{j=1, j \neq \hat{i}}^n -\partial_{x_j} (\partial_{x_j} u_{\hat{i}} - \partial_{x_{\hat{i}}} u_j) - \partial_{x_{\hat{i}}}^2 u_{\hat{i}} - \sum_{j=1, j \neq \hat{i}}^n \partial_{x_j} \partial_{x_{\hat{i}}} u_j$$

$$= -\sum_{j=1, j \neq \hat{i}}^n \partial_{x_j} (\partial_{x_j} u_{\hat{i}} - \partial_{x_{\hat{i}}} u_j) - \partial_{x_{\hat{i}}} \text{div } u \quad \text{in } \Omega. \tag{6.8}$$

Using the equality (6.8) we rewrite $\tilde{j}$-th equation in (6.1) as

$$b_j(x, D)v = -\sum_{j=1, j \neq \hat{i}}^n \text{sign}(j - \hat{i}) \partial_{x_j} v_{ij} - \frac{\lambda + 2\mu}{\mu} \partial_{x_j} v_2 = g_j$$

Construction of the operator $\tilde{B}(x, D)$ is complete. Next

$$v_{i,j}(x) = \nu_j(x') \frac{\partial u_j}{\partial y_j} - \nu_i(x') \frac{\partial u_i}{\partial y_i}, i < j.$$
Set \( \tilde{v} = (v_{1,n}, \ldots, v_{2,n}, \ldots v_{n-1,n}, v_{2}) \). Obviously in the small neighborhood of \( y^* \) there exists a smooth matrix \( C_0 \) such that

\[
\left( \frac{\partial u_1}{\partial x_n}, \ldots, \frac{\partial u_n}{\partial x_n} \right) = C_0(x') \tilde{v} \quad \forall x \in \Sigma \cap B(y^*, \delta).
\]

Then \( C(x')v = v - (\nu_2 \frac{\partial u_2}{\partial x_2} - \nu_1 \frac{\partial u_1}{\partial x_1}, \ldots, \nu_n \frac{\partial u_n}{\partial x_n} - \nu_1 \frac{\partial u_1}{\partial x_1} - \nu_n \frac{\partial u_n}{\partial x_n} - \nu_1 \frac{\partial u_1}{\partial x_1} + \sum_{j=1}^{n} \nu_j \frac{\partial u_j}{\partial x_j}) = 0. \)

Still we can not apply the proposition 2.1 directly since the traces of the function \( v \) and its time derivative at moment \( x_0 = \pm T \) are not equal zero. On the other hand \( v(\pm T, \cdot), \partial_{x_0} v(\pm T, \cdot) \in H^2(\Omega) \). So for sufficiently small positive \( \epsilon \) we extend the function \( v \) on \( Q_\epsilon = (-T - \epsilon, T + \epsilon) \times \Omega \) by the formula \( v = \gamma(x_0)(v(T, \cdot) + (x_0 - T)\partial_{x_0} v(T, \cdot)) \) for \( x_0 \in (T, T + \epsilon) \) and \( v = \gamma(x_0)(v(-T, \cdot) + (x_0 + T)\partial_{x_0} v(-T, \cdot)) \) for \( x_0 \in (T - \epsilon, T) \), where \( \gamma \in C^\infty_0[-T - \epsilon, T + \epsilon] \) and \( \gamma||_{[-T-\epsilon/2,T+\epsilon/2]} = 1 \). Provided that \( \epsilon \) is sufficiently small we extend the functions \( \rho, \mu, \lambda \) on \( Q_\epsilon \) such that they are positive, keeping the same regularity and we extend the function \( \psi \) on \( Q_\epsilon \) such that Condition 1.1 holds on \( Q_\epsilon \), inequality (1.7) holds true on \([-T-\epsilon, T + \epsilon] \times \Gamma_0 \) and

\[
(6.9) \quad \partial_{x_0} \psi(x) < 0 \quad \text{on} \ (0, T + \epsilon) \times \tilde{\Omega} \quad \text{and} \quad \partial_{x_0} \psi(x) > 0 \quad \text{on} \ [-T - \epsilon, 0) \times \tilde{\Omega}.
\]

To this extended function \( v \) we can now apply the proposition 2.1.

In Klibanov [29], the following inequality is proved: let \( R(x, \tilde{x}_0, \tilde{D}) \) be a second-order differential operator with smooth coefficients. Then

\[
(6.10) \quad \left\| e^{s\varphi} \int_0^{x_0} R(x, \tilde{x}_0, \tilde{D}) u(\tilde{x}_0, \cdot) d\tilde{x}_0 \right\|_{L^2([-T, T], \Sigma)} \leq o(\frac{1}{s}) \sum_{|\alpha| \leq 2} ||e^{s\varphi} \partial_x^\alpha u||_{L^2([-T, T], \Sigma)}.
\]

We set \( \Sigma_\epsilon = (-T - \epsilon, T + \epsilon) \times \partial \Omega \). By (6.9)

\[
(6.11) \quad s\tau \| \tilde{F} e^{s\varphi} \|_{L^2(\Sigma_\epsilon)} \leq \frac{C_{11}}{s} \sum_{k=0}^{1} \| \tilde{\varphi}(v((-1)^k T, \cdot), \partial_{x_0} v((-1)^k T, \cdot)) e^{s\varphi((-1)^k T, \cdot)} \|_{H^1(\Omega)}^2 + \| s\tau \tilde{\varphi}^2(v((-1)^k T, \cdot), \partial_{x_0} v((-1)^k T, \cdot)) e^{s\varphi((-1)^k T, \cdot)} \|_{L^2(\Omega)}^2 + C_{12} \left( \int_{\Sigma} (s\tau |\nabla v|^2 + s^3 \tau^2 \tilde{\varphi}^3 |v|^2) e^{2s\varphi} d\sigma \right) + s\tau \| \tilde{\varphi} e^{s\varphi} \|_{L^2(\Sigma_\epsilon)}^2.
\]

and

\[
(6.12) \quad \| e^{s\varphi} \|_{L^2(Q_\epsilon)} \leq \frac{C_{13}}{s} \sum_{k=0}^{1} \| \Delta(v((-1)^k T, \cdot), \partial_{x_0} v((-1)^k T, \cdot)) e^{s\varphi((-1)^k T, \cdot)} \|_{L^2(\Omega)}^2 + \| e^{s\varphi} \|_{L^2(Q_\epsilon)}^2.
\]

By (1.2) and (2.2)

\[
(6.13) \quad v(\pm T, \cdot) = - \left( \int_0^{\pm T} \frac{\tilde{\mu}(\pm T, x', \tilde{x}_0)}{\mu(\pm T, x')} d\omega u d\tilde{x}_0, \int_0^{\pm T} \frac{\tilde{\lambda} + 2\tilde{\mu}}{(\lambda + 2\mu)(\pm T, x')} \text{div} u d\tilde{x}_0 \right),
\]
and
\[ (\int_0^{\pm T} \partial_{x_0} \left( \frac{\tilde{\mu}(\pm T, x', \tilde{x}_0)}{\mu(\pm T, x')} \right) \ d\omega u \ d\tilde{x}_0, \int_0^{\pm T} \partial_{x_0} \left( \frac{\tilde{\lambda} + 2\tilde{\mu}}{\lambda + 2\mu}(\pm T, x', \tilde{x}_0) \right) \div \ n d\tilde{x}_0) \].

We have

Proposition 6.1. There exists a constant $C_{14}$ independent of $s$ and $\tau$ such that

\[ \| \Delta v(\pm T, \cdot) e^{\pm s\varphi(T, \cdot)} \|_{L^2(\Omega)} + \| \partial_{x_0} \Delta v(\pm T, \cdot) e^{\pm s\varphi(T, \cdot)} \|_{L^2(\Omega)} \]

\[ \leq C_{14} (s\tau)^{\frac{1}{2}} \| \nabla v e^{s\varphi} \|_{L^2(\Omega)} + (s\tau)^{\frac{3}{2}} \| \nabla v e^{s\varphi} \|_{L^2(\Omega)} + \| q e^{s\varphi} \|_{L^2(\Omega)}. \]

Proof. Let $\theta(x)$ be a smooth function. Integrating the first equation in (2.11) in $x_0$ on $[0, \tilde{T}]$ we obtain

\[ \int_0^{\tilde{T}} \theta \mu \Delta v u d\tilde{x}_0 - \int_0^{\tilde{T}} \theta [\tilde{T} - \int_0^{\tilde{T}} \mu(\tilde{x}_0, x') Y(\tilde{x}_0, x') d\tilde{x}_0]. \]

We set $\theta(x) = \tilde{\mu}/\mu$, $Y(\tilde{T}, x') = \int_0^{\tilde{T}} \mu \Delta v u d\tilde{x}_0$, and $\alpha(\tilde{T}, x') = (\partial_{x_0} \Delta v u) (\tilde{T}, x') - (\partial_{x_0} \Delta v u) (0, x') - \int_0^{\tilde{T}} \theta \mu(\tilde{x}_0, x') Y(\tilde{x}_0, x') d\tilde{x}_0$. Then $Y(T, x') = -\int_0^{\tilde{T}} \mu(\tilde{x}_0, x') Y(\tilde{x}_0, x') d\tilde{x}_0 + \alpha(\tilde{T}, x')$. By Gronwall's inequality from (6.16) we have

\[ Y(\tilde{T}, x') \leq \sup_{x_0 \in [0, T]} \alpha(x_0, x') \exp(\int_0^{\tilde{T}} \mu(\tilde{x}_0, x') d\tilde{x}_0). \]

Then

\[ v(\pm T, \cdot) e^{\pm s\varphi(T, x')} = Y(T, \cdot) e^{\pm s\varphi(T, x')} \leq \sup_{x_0 \in [0, T]} \alpha(x_0, x') \exp(\int_0^{\tilde{T}} \mu(\tilde{x}_0, x') d\tilde{x}_0) e^{\pm s\varphi(T, x')} \]

So

\[ \| v(\pm T, \cdot) e^{\pm s\varphi(T, \cdot)} \|_{L^2(\Omega)} \leq C_{15} \left( \sup_{x_0 \in [0, T]} \| \partial_{x_0} w(x_0, \cdot) \|_{L^2(\Omega)} + s\tau \| \varphi w(x_0, \cdot) \|_{L^2(\Omega)} \right) + C_{16} \| q e^{s\varphi} \|_{L^2(\Omega)} \]

\[ \leq C_{17} (s\tau)^{\frac{1}{2}} \| \nabla v e^{s\varphi} \|_{L^2(\Omega)} + (s\tau)^{\frac{3}{2}} \| \nabla v e^{s\varphi} \|_{L^2(\Omega)} + \| q e^{s\varphi} \|_{L^2(\Omega)}. \]

Estimate for $\partial_{x_0} \Delta v(\pm T, \cdot)$ proved similarly. \[\square\]

Using proposition 2.1 and (6.11)-(6.12) we obtain
\begin{align}
(6.17) \quad \int_{\Sigma} (s\tau \varphi |\nabla v|^2 + s^3 \tau^3 \varphi^3 |v|^2) e^{2\varphi} d\Sigma + \int_{Q_\epsilon} (s\tau^2 \varphi |\nabla v|^2 + s^3 \tau^4 \varphi^3 |v|^2) e^{2\varphi} dx \\
&\leq C_{18} \left( s\tau \sqrt{\tau} e^{s\varphi} \|v\|^2_{L^2(\Sigma)} + \|\eta e^{s\varphi}\|^2_{L^2(\epsilon)} + \int_{\Sigma} (s\tau \varphi |\nabla v|^2 + s^3 \tau^2 \varphi^3 |v|^2) e^{2\varphi} d\Sigma \right) \\
&\leq C_{19} \left( s\tau \sqrt{\tau} e^{s\varphi} \|v\|^2_{L^2(\Sigma)} + \|\eta e^{s\varphi}\|^2_{L^2(\epsilon)} + \int_{\Sigma} (s\tau \varphi |\nabla v|^2 + s^3 \tau^2 \varphi^3 |v|^2) e^{2\varphi} d\Sigma \right) \\
&\quad + \frac{C_{20}}{s} \sum_{k=0}^{1} \|\Delta(v((-1)^k T, \cdot), \partial_{x_0} v((-1)^k T, \cdot)) e^{s\varphi((-1)^k T, \cdot)} \|^2_{L^2(\Omega)} \\
&\quad + \frac{C_{21}}{s} \sum_{k=0}^{1} \|\sqrt{\varphi}(v((-1)^k T, \cdot)), \partial_{x_0} v((-1)^k T, \cdot)) e^{s\varphi((-1)^k T, \cdot)} \|^2_{H^1(\partial\Omega)} \\
&\quad + \|s\tau \varphi^2(v((-1)^k T, \cdot)), \partial_{x_0} v((-1)^k T, \cdot)) e^{s\varphi((-1)^k T, \cdot)} \|^2_{L^2(\partial\Omega)}. \nonumber
\end{align}

Applying the proposition \textbf{6.1} we have
\begin{align}
(6.18) \quad \int_{\Sigma} (s\tau \varphi |\nabla v|^2 + s^3 \tau^3 \varphi^3 |v|^2) e^{2\varphi} d\Sigma + \int_{Q_\epsilon} (s\tau^2 \varphi |\nabla v|^2 + s^3 \tau^4 \varphi^3 |v|^2) e^{2\varphi} dx \\
&\leq C_{22} \left( s\tau \sqrt{\tau} e^{s\varphi} \|v\|^2_{L^2(\Sigma)} + \|\eta e^{s\varphi}\|^2_{L^2(\epsilon)} + \int_{\Sigma} (s\tau \varphi |\nabla v|^2 + s^3 \tau^2 \varphi^3 |v|^2) e^{2\varphi} d\Sigma \right). 
\end{align}

Next we prove the following:

\textbf{Proposition 6.2.} Let positive \( \delta \) be sufficiently small. There exist \( \tau_0 > 0 \) and \( s_0 > 0 \) such that for all \( s \geq s_0 \) and \( \tau \geq \tau_0 \) there exists a constant \( C_{23} > 0 \) independent of \( s \) and \( \tau \) such that

\begin{align}
(6.19) \quad s\tau \|\varphi^2 \nabla \partial_{x_0} e^{s\varphi}\|^2_{L^2(\Sigma_0)} + s^3 \tau^3 \|\varphi^3 \partial_{x_0} e^{s\varphi}\|^2_{L^2(\Sigma_0)} \\
&\leq C_{23} (s\tau \|\varphi_{1/2} \nabla v e^{s\varphi}\|^2_{L^2(\Sigma_0)} + s^3 \tau^3 \|\varphi_{1/2} \nabla v e^{s\varphi}\|^2_{L^2(\Sigma_0)} + s\tau \|\sqrt{\tau} e^{s\varphi}\|^2_{L^2(\Sigma_0)}) 
\end{align}

and

\begin{align}
(6.20) \quad \|s^{1/2} \tau \varphi^2 \nabla d\omega u e^{s\varphi}\|^2_{L^2(\epsilon)} + \|(s^{1/2} \tau \varphi^2 \nabla \nabla v u) e^{s\varphi}\|^2_{L^2(\epsilon)} \leq C_{23} \left( \int_{Q} (s\tau^2 \varphi |\nabla v|^2 + s^3 \tau^4 \varphi^3 |v|^2) e^{2\varphi} dx \right).
\end{align}

\textbf{Proof.} The inequality (6.20) follows from (6.10), (1.8) and (2.2). By (6.10) there exists a constant \( C_{24} \) independent of \( s \) and \( \tau \) such that

\begin{align}
(6.21) \quad s\tau \|\varphi_{1/2} \nabla (d\omega u, \nabla v) e^{s\varphi}\|^2_{L^2(\Sigma_0)} + s^3 \tau^3 \|\varphi_{1/2} (d\omega u, \nabla v) e^{s\varphi}\|^2_{L^2(\Sigma_0)} \\
&\leq C_{24} (s\tau \|\varphi_{1/2} \nabla v e^{s\varphi}\|^2_{L^2(\Sigma_0)} + s^3 \tau^3 \|\varphi_{1/2} \nabla v e^{s\varphi}\|^2_{L^2(\Sigma_0)}) 
\end{align}
Thanks to the zero Dirichlet boundary conditions on \( \Sigma_0 \) there exists a smooth matrix \( M(x) \) such that \( \partial_\nu u = M(x)(d\omega, \text{div } u) \). Therefore

\[
\begin{align*}
(6.22) \quad \partial_\nu^2 u &= A(x, \tilde{D})\partial_\nu u + B(x)(F + \int_0^{x_0} \tilde{L}_{x,\tilde{D}}(x, \tilde{x}_0, D')u(\tilde{x}_0, x')d\tilde{x}_0),
\end{align*}
\]

where \( A(x, \tilde{D}) \) is a first order differential operator and \( B \) is a \( C^1 \)-matrix function. From \( (6.23) \) and \( (6.22) \) we have

\[
\begin{align*}
(6.24) \quad s\tau \| \varphi^{1/2} \partial_\nu^2 u e^{s\varphi} \|_{L^2(\Sigma_0)}^2 &\leq C_{25}(s\tau \| \varphi^{1/2} A(x, \tilde{D})\partial_\nu u \|_{L^2(\Sigma_0)}^2 + s\tau \| \sqrt{\varphi} F e^{s\varphi} \|_{L^2(\Sigma_0)}^2 + s\tau \| \varphi^{1/2} B(x) \int_0^{x_0} \tilde{L}_{x,\tilde{D}}(x, \tilde{x}_0, D')u(\tilde{x}_0, x')d\tilde{x}_0 \|_{L^2(\Sigma_0)}^2) \\
&\leq C_{27}(s\tau \| \varphi^{1/2} \nabla (d\omega u, \text{div } u) e^{s\varphi} \|_{L^2(\Sigma_0)} + s^3\tau^3 \| \varphi^{1/2} (d\omega u, \text{div } u) e^{s\varphi} \|_{L^2(\Sigma_0)} + s^3\tau^3 \| \varphi^{1/2} \partial_\nu u e^{s\varphi} \|_{L^2(\Sigma_0)}^2.
\end{align*}
\]

In order to get the last inequality we used (6.10). From (6.24), (6.21) and (6.22) we obtain (6.19). The proof of Proposition 6.2 is complete. ■

Next we prove

**Proposition 6.3.** Let \( u \in H^1(Q), u|_{\partial Q} = 0 \). There exist \( \tau > 1 \) and \( s_0 > 1 \) such that for any \( \tau > \tau \) and for all \( s > s_0 \)

\[
(6.25) \quad \int_Q (s^2\tau^4 \varphi^2 |\nabla u|^2 + s^4\tau^6 \varphi^4 |u|^2)e^{2s\varphi}dx \leq C_{28}(\| s^{1/2} \tau \varphi^{1/2} \nabla d\omega u e^{s\varphi} \|_{L^2(Q)}^2 + \| s\tau \varphi^{1/2} \nabla \text{div } u e^{s\varphi} \|_{L^2(Q)}^2 + \| s^2\tau^3 \varphi^{1/2} \partial_{x_0} \partial_\nu u e^{s\varphi} \|_{L^2(\Sigma)}^2 + \| s^2\tau^3 \varphi^{-1/2} \partial_\nu^2 u e^{s\varphi} \|_{L^2(\Sigma)}^2),
\]

where \( C_{28} \) is independent of \( s \) and \( \tau \).

**Proof.** Let \( x_0 \in (-T, T) \) be arbitrarily fixed. For any index \( \tilde{i} \in \{1, \ldots, n\} \) consider the equality (6.8). Then the Carleman estimate with boundary for the Laplace operator implies

\[
(6.26) \quad \int_Q (s^2\tau^4 \varphi^2 |\nabla u|^2 + s^4\tau^6 \varphi^4 |u|^2)e^{2s\varphi}dx \leq C_{29}(\| s^{1/2} \tau \varphi^{1/2} \nabla d\omega u e^{s\varphi} \|_{L^2(Q)}^2 + \| s\tau \varphi^{1/2} \nabla \text{div } u e^{s\varphi} \|_{L^2(Q)}^2 + \| s^2\tau^3 \varphi^{1/2} \partial_{x_0} \partial_\nu u e^{s\varphi} \|_{L^2(\Sigma)}^2 + \| s^2\tau^3 \varphi^{-1/2} \partial_\nu^2 u e^{s\varphi} \|_{L^2(\Sigma)}^2 + \int_\Sigma s^2\tau^3 \varphi^{-1/2} \partial_\nu u |e^{s\varphi}d\Sigma).}
\]

We differentiate both sides of equation (6.8) with respect to the variable \( x_0 \) and take the \( H^{-1} \)-Carleman estimate by authors in [22]:
Combination of (6.26) and (6.27) implies (6.25). The proof of the proposition is complete.

By (6.20), Proposition 6.2 and Proposition 6.3 from (6.18) we obtain the estimate

\[
\int \Sigma (s\tau^2 \varphi^2) \partial \omega d\omega + \int Q (s\varphi^2 \omega u) e^{s\varphi} d\omega d\omega u + s\varphi^2 \partial \nu \partial \omega d\omega d\omega u + s\varphi^2 \partial \nu \partial \omega d\omega d\omega u + (s\varphi)^2 \partial \nu \partial \omega d\omega d\omega u + (s\varphi)^2 \partial \nu \partial \omega d\omega d\omega u \leq C_{31} \left( s\tau^2 \varphi^2 \omega u + \| \varphi \| \right)_L^2(\Sigma) + \| \varphi \| \omega u e^{s\varphi} d\omega d\omega u + s\varphi^2 \partial \nu \partial \omega d\omega d\omega u + s\varphi^2 \partial \nu \partial \omega d\omega d\omega u + (s\varphi)^2 \partial \nu \partial \omega d\omega d\omega u + (s\varphi)^2 \partial \nu \partial \omega d\omega d\omega u \]

for all \( s \geq s_0 \) and for all \( \tau \geq \tau_0 \). By (2.5) and (6.28) we obtain the estimate (1.11). Thus the proof of Theorem 1.2. is finished.

7. PROOF OF THEOREM 1.2

Denote \( Q_\pm = (0, \pm T) \times \Omega \). We extend functions \( y(x), F(x), \tilde{\lambda}(x), \tilde{\mu}(x) \) on \( Q_+ \) as \( y(x) = y(-x_0, x') \), \( F(x) = F(-x_0, x') \), \( \tilde{\lambda}(x) = \tilde{\lambda}(-x_0, x') \), \( \tilde{\mu}(x) = \tilde{\mu}(-x_0, x') \) for \( Q_- \). By (1.17) the functions \( F, \partial x_0 F \) belong to \( C([-T, T]; H^1(\Omega)) \) and \( L^\infty(-T, T; H^1(\Omega)) \) respectively and \( \lambda, \tilde{\mu} \in C^1(Q_+ \times [0, T]) \cap C^1(Q_- \times [-T, 0]) \). We set

\[
a_{\lambda, \mu}(z(x_0, \cdot), v(x_0, \cdot)) = \int_\Omega (\lambda(x') (\varphi z(x), \varphi v(x)) + 2 \mu(x') \sum_{\ell,j=1}^n \varepsilon_{\ell j}(z(x)) \varepsilon_{\ell j}(v(x)) dx',
\]

\[
\varepsilon_{\ell j}(y) = \frac{1}{2} (\partial_{x_\ell} y_j + \partial_{x_j} y_\ell), \quad 1 \leq \ell, j \leq n.
\]

Denote

\[
E_k(x_0) = \int_\Omega (\lambda(x') |\partial_{x_0}^k \varphi z(x)|^2 + 2 \mu(x') \sum_{\ell,j=1}^n |\varepsilon_{\ell j}(\partial_{x_0}^k z(x))|^2 + \rho(x') |\partial_{x_0}^{k+1} y(x)|^2) dx'.
\]

The identity (6.194) established in (10) page 612 for solution of the system \( 1.12, 1.13 \) yields

\[
\frac{d}{dx_0} E_k(x_0) = (\partial_{x_0}^k \int_0^{x_0} L_{\tilde{\lambda}, \tilde{\mu}}(x', \tilde{x}_0, D') y(\tilde{x}_0, x') d\tilde{x}_0 + \partial_{x_0}^k F, \partial_{x_0}^{k+1} y)L^2(\Omega).
\]

Integrating the right-hand side of this equality for any \( k \in \{0, 1, 2\} \) we have

\[
\frac{d}{dx_0} E_k(x_0) = \int_0^{x_0} a_{\partial_{x_0}^k \tilde{\lambda}, \partial_{x_0}^k \tilde{\mu}}(y(\tilde{x}_0, \cdot), \partial_{x_0}^k y(x_0, \cdot)) d\tilde{x}_0
\]
where \( C_p \) are some constants. Integrating the above equality on the interval \((0, x_0)\) we obtain
\[
E_k(x_0) \leq E_k(0) + \| \partial_{x_0}^k \mathbf{F} \|_{L^2(Q_+)}^2 + C_1 \int_0^{x_0} E_k(\tilde{x}_0) d\tilde{x}_0 - \int_0^{x_0} a_{\lambda(x, \tilde{x}_0)} \partial_{x_0}^k \mathbf{y}(\tilde{x}_0, \cdot), \partial_{x_0}^k \mathbf{y}(x_0, \cdot) \right|_{t=x_0} d\tilde{x}_0
\]
\[
- \sum_{p=0}^{k-1} C_p a_{\lambda_p(x, \tilde{x}_0)} \partial_{x_0}^k \mathbf{y}(x_0, \cdot), \partial_{x_0}^k \mathbf{y}(x_0, \cdot) \right|_{t=x_0}
\]
\[
+ \sum_{p=0}^{k-1} C_p a_{\lambda_p(x, \tilde{x}_0)} \partial_{x_0}^k \mathbf{y}(x_0, \cdot), \partial_{x_0}^k \mathbf{y}(x_0, \cdot) \right|_{t=x_0}
\]
\[
+ \int_0^{x_0} \frac{d}{dy_0} \int_0^{y_0} a_{\lambda(y_0, x', \tilde{x}_0)} \partial_{y_0}^k \mathbf{y}(y_0, x', \tilde{x}_0), \partial_{y_0}^k \mathbf{y}(y_0, x', \tilde{x}_0) \right|_{y=y_0} d\tilde{x}_0 dy_0
\]
\[
- \int_0^{x_0} \frac{d}{dy_0} \sum_{p=0}^{k-1} C_p a_{\lambda_p(y_0, x', y_0)} \partial_{y_0}^k \mathbf{y}(y_0, x', y_0), \partial_{y_0}^k \mathbf{y}(y_0, x', y_0) \right|_{y=y_0} d\tilde{x}_0 dy_0
\]
\[
\leq E_k(0) + \| \partial_{x_0}^k \mathbf{F} \|_{L^2(Q_+)}^2 + C_2 \int_0^{x_0} E_k(\tilde{x}_0) d\tilde{x}_0 + \frac{1}{2} E_k(x_0).
\]
Thus the Gronwall inequality and the Korn inequality yield

**Lemma 7.1.** There exists a constant \( C_3 > 0 \) such that for any \( x_0 \) from \([0, T]\)

\[
(7.1) \quad \sum_{k=0}^{2} \| \nabla \partial_{x_0}^k \mathbf{y}(x_0, \cdot) \|_{L^2(\Omega)}^2 \leq C_3(\sum_{k=0}^{2} \| \partial_{x_0}^k \mathbf{F} \|_{L^2(Q_+)}^2 + \| \mathbf{F}(0, \cdot) \|_{H^1_0(\Omega)}^2 + \| \partial_{x_0} \mathbf{F}(0, \cdot) \|_{L^2(\Omega)}^2),
\]

for \( \mathbf{y} \) satisfying (1.12)-(1.13).

From (1.12), Lemma 7.1 and the classical apriori estimates for the stationary Lamé system, it follows that

\[
(7.2) \quad \| \mathbf{y} \|_{L^2(0, T; H^2(\Omega))} + \| \partial_{x_0} \mathbf{y} \|_{L^2(0, T; H^2(\Omega))} \leq C_4(\sum_{k=0}^{2} \| \nabla \partial_{x_0}^k \mathbf{y} \|_{L^2(Q_+)}^2 + \| \mathbf{F} \|_{L^2(Q_+)}^2 + \| \partial_{x_0} \mathbf{F} \|_{L^2(Q_+)}^2).
\]

Observe that by (1.16) there exists a positive \( \delta_1 \) such that

\[
(7.3) \quad \inf_{x' \in \Omega} \varphi(0, x') > \sup_{x \in ([T-\delta_1, T]\cup[-T,-T+\delta_1]) \times \Omega} \varphi(x).
\]

Let \( \tilde{\gamma} \in C_0^\infty[-T, T] \) satisfy \( \tilde{\gamma}|_{|t|\leq T-\delta_1/2} = 1 \), and the function \( \mathbf{v} \) be given by (2.2). Then, taking the even extension of the function \( \mathbf{v} \) on \( Q_- \) we have
(7.4) \( P(x, D)v = (\Box_{\rho, \mu}(x, D)v_1, \Box_{\rho, \lambda+2\mu}(x, D)v_2) = q \) in \( Q \), \( B(x', D)v = g \).

We claim \( \partial_{x_0}q \in L^2(Q) \). Indeed, since \( F, \partial_{x_0}F \in L^2(-T, T; H^1(\Omega)) \), we have \( \partial_{x_0}d\omega_F, \partial_{x_0}\text{div}F \in L^2(Q) \). By (1.13), the regularity assumption on \( y \) and time independence of the Lamé coefficients \( \lambda, \mu \), we see that the \( x_0 \)-derivative of the extensions of the terms \( \beta \partial_{x_j}y_i, \beta \partial_{x_j}x_ky_i \) with \( \beta \in \{ \lambda, \mu, \nabla \mu, \nabla \lambda \} \) belong to \( L^2(Q) \). By the same argument the \( x_0 \)-derivative of the extension of the functions \( \int_0^{x_0} \beta \partial_{x_j}y_i d\bar{x}_0, \int_0^{x_0} \beta \partial_{x_j}x_ky_i d\bar{x}_0 \) with \( \beta \in \{ \lambda, \mu, \nabla \mu, \nabla \lambda \} \) belongs to \( L^2(Q) \). Hence our claim follows from (2.4).

Setting \( \tilde{v} = \gamma v \) we have

(7.5) \( P(x, D)\tilde{v} = q + [\gamma, P(x, D)]v \) in \( Q \), \( B(x', D)v = \tilde{g}, \tilde{v}(0, \cdot) = \partial_{x_0}\tilde{v}(0, \cdot) = 0 \).

Taking the time derivative of (7.5), we have

(7.6) \( P(x, D)\partial_{x_0}\tilde{v} = \partial_{x_0}q + \partial_{x_0}[\gamma, P(x, D)]v \) in \( Q \),

\( B(x', D)v = \partial_{x_0}(\gamma g), \partial_{x_0}^2\tilde{v}(0, \cdot) = (\rho d\omega_F/\rho(0, \cdot), \text{div}(F/\rho)(0, \cdot)) \).

Since \( f \in H^1_0(\Omega) \), we have

(7.7) \( |g(x)| \leq C|\nabla'y(x)| \) and \( |\partial_{x_0}g(x)| \leq C|\nabla'\partial_{x_0}y(x)| \) on \( \Sigma \).

We claim that \( \partial_{x_0}\tilde{v} \in H^1(Q) \). By (1.13), the even extensions of the functions \( d\omega_y, \text{div}y \) on \( Q_- \) belongs to \( H^2(Q) \). Set \( \tilde{w} = w_+ \) for \( x_0 > 0 \) and \( \tilde{w} = w_- \) for \( x_0 < 0 \), \( w_{\pm} = (w_{1, \pm}, w_{2, \pm}) \), where

\[
\begin{align*}
w_{1, \pm} &= \pm \int_0^{x_0} \frac{\tilde{\mu}(\pm x_0, x', \pm \bar{x}_0)}{\mu(x')} d\omega_y d\bar{x}_0, \\
w_{2, \pm} &= \pm \int_0^{x_0} \frac{(\tilde{\lambda} + 2\tilde{\mu})(\pm x_0, x', \pm \bar{x}_0)}{(\lambda + 2\mu)(x')} \text{div}y d\bar{x}_0.
\end{align*}
\]

By our regularity assumptions \( w_{\pm}, \nabla'w_{\pm} \in H^1(Q_{\pm}) \) and \( w_{\pm}(0, \cdot) = \nabla'w_{\pm}(0, \cdot) = 0 \). Therefore \( \tilde{w}, \nabla'\tilde{w} \in H^1(Q) \). Note that

\[
\partial_{x_0}w_{\pm} = \pm \frac{\tilde{\mu}(\pm x_0, x', \pm \bar{x}_0)}{\mu(x')} d\omega_y, \frac{(\tilde{\lambda} + 2\tilde{\mu})(\pm x_0, x', \pm \bar{x}_0)}{(\lambda + 2\mu)(x')} \text{div}y d\bar{x}_0
\]

(\( \int_0^{x_0} \frac{\partial}{\partial x_0} \left[ \frac{\tilde{\mu}(\pm x_0, x', \pm \bar{x}_0)}{\mu(x')} \right] d\omega_y d\bar{x}_0, \int_0^{x_0} \frac{\partial}{\partial x_0} \left[ \frac{(\tilde{\lambda} + 2\tilde{\mu})(\pm x_0, x', \pm \bar{x}_0)}{(\lambda + 2\mu)(x')} \right] \text{div}y d\bar{x}_0 \)).

Observe that by (1.13) \( \partial_{x_0}w_{\pm}(0, \cdot) = \nabla'\partial_{x_0}w_{\pm}(0, \cdot) = 0 \) and by our regularity assumptions on function \( y \) the functions \( \partial_{x_0}w_{\pm}, \nabla'\partial_{x_0}w_{\pm} \) belong to the Sobolev space \( H^1(Q_{\pm}) \). Therefore \( \partial_{x_0}w \in H^1(Q) \). Then \( \tilde{v}, \partial_{x_0}\tilde{v} \in H^1(Q) \) by (2.2). Then applying to the problems (7.6) and
the Carleman estimate (2.9) and using (7.7), we obtain

\[
\sum_{k=0}^{1} (s \tau \| \varphi^2 \nabla x_0 \vec{v} e^{s \varphi} \|^2_{L^2(Q)} + s^3 \tau^4 \| \varphi^3 \partial x_0 \vec{v} e^{s \varphi} \|^2_{L^2(Q)} + \int_{\Sigma} (s \tau \varphi | \nabla \partial x_0 \vec{v}|^2 + s^3 \tau^3 \varphi^3 | \partial x_0 \vec{v}|^2) e^{2s \varphi} d\Sigma) \leq C_5 \sum_{k=0}^{1} (\| P(x, D) \partial x_0 \vec{v} e^{s \varphi} \|^2_{L^2(Q)} + \int_{\Sigma} (s \tau \varphi | \nabla \partial x_0 \vec{v}|^2 + s^3 \tau^3 \varphi^3 | \partial x_0 \vec{v}|^2) e^{2s \varphi} d\Sigma).
\]

Taking the scalar product of (7.6) with \( \partial x_0 \vec{v} e^{2s \varphi} \) in \( L^2(Q_+) \) and integrating by parts, using the inequality (7.8) we obtain the estimate

\[
\tau \int_{\Omega} | \partial x_0 \vec{v}(0, \cdot)|^2 e^{2s \varphi(0, \cdot)} dx' \leq C_6 (\| P(x, D) \partial x_0 \vec{v} e^{s \varphi} \|^2_{L^2(Q)} + \int_{\Sigma} (s \tau \varphi | \nabla \partial x_0 \vec{v}|^2 + s^3 \tau^3 \varphi^3 | \partial x_0 \vec{v}|^2) e^{2s \varphi} d\Sigma).
\]

By assumption (1.9), \( \nabla' \psi(0, x') \) is not equal to zero on \( \bar{\Omega} \). By Proposition 6.3, (1.15) and (1.17) we obtain

\[
\frac{\tau}{s} \int_{\Omega} \varphi(0, x') \| \nabla' f \|^2 e^{2s \varphi(0,x')} dx' \leq C_7 \left( \int_{\Omega} | \partial x_0 \vec{v}(0, \cdot)|^2 e^{2s \varphi(0, \cdot)} dx' + \int_{\Gamma} s \tau \varphi(0, x') | \partial \varphi | \partial x_0 \vec{y}(0, x') |^2 e^{2s \varphi(0,x')} d\sigma \right).
\]

Using (1.17) we have

\[
\int_{\Omega} | \partial x_0 \vec{v}(0, \cdot)|^2 e^{2s \varphi(0, \cdot)} dx' + \int_{\Gamma} s \tau \varphi(0, x') | \partial \varphi | \partial x_0 \vec{y}(0, x') |^2 e^{2s \varphi(0,x')} d\sigma \leq \sum_{k=0}^{1} \left( \| \partial x_0 (d\omega F, \div F + s \varphi) \|^2_{L^2(Q)} + \| \partial x_0 (d\omega F(0, \cdot), \div F(0, \cdot)) e^{s \varphi} \|^2_{L^2(\Omega)} + \| \partial x_0 (d\omega F(0, \cdot), \div F(0, \cdot)) e^{s \varphi} \|^2_{L^2(\Omega)} \right)
\]

By (1.8) and assumption \( \partial x_0 \psi(0, x') \neq 0 \) on \( \bar{\Omega} \), we obtain

\[
\| (d\omega F(0, \cdot), \div F(0, \cdot)) e^{s \varphi} \|^2_{L^2(\Omega)} = O \left( \frac{1}{s^4} \right) \| (d\omega F(0, \cdot), \div F(0, \cdot)) e^{s \varphi(0, \cdot)} \|^2_{L^2(\Omega)} \quad \text{as} \quad s \to +\infty.
\]
Observe that by (1.17) there exist functions \(g_0, g_1 \in C_0^1[-T, T]\) such that \(g_0(0) = g_1(0) = 0\) and

\[
\sum_{k=0}^{\infty} \left\| \partial_x^k (d\omega \mathbf{F} \cdot \mathbf{F}(0, \cdot)) \right\|_{L^2(Q)} e^{s\phi} \leq C_8 \int_{-T}^{T} \int_\Omega (g_1^2(x_0)|\nabla f|^2 + g_0^2(x_0)|f|^2)(2s\sigma \tau \partial_x \psi) dx_0 dx' \frac{1}{s^{1/2}} = C_9 \int_{-T}^{T} \int_\Omega (g_1^2(x_0)|\nabla f|^2 + g_0^2(x_0)|f|^2)(2s\sigma \tau \partial_x \psi) dx_0 dx' \frac{1}{s^{1/2}} = C_{10} \left( \frac{d}{dx_0} \left( \frac{g_1^2(x_0)}{2s\sigma \tau \partial_x \psi} \right) |\nabla f|^2 + \frac{d}{dx_0} \left( \frac{g_0^2(x_0)}{2s\sigma \tau \partial_x \psi} \right) |f|^2 \right) e^{2s\phi(x)} dx_0 dx' \frac{1}{s^{1/2}} = o(\frac{1}{s^{1/2}}).
\]

Combining (7.8), (7.9) and (7.11)-(7.13), we see that there exist functions \(c_j, \tilde{c}_j, c, \tilde{c} \in L^\infty(Q) \cap C^4(\bar{Q}_\pm)\), and \(s_0\) independent of \(y\) such that for all sufficiently large \(s\)

\[
\frac{T}{s} \int \Omega |\nabla f|^2 e^{2s\phi(x)} dx' \leq \frac{T}{s} \int \Omega |\nabla f|^2 e^{2s\phi(0, x')} dx' \leq C_{11} \sum_{k=0}^{\infty} \left( \left| \partial_x^k \nabla \psi \right|^2 + \left| \partial_x^k \psi \right|^2 \right)
\]

\[
+ \sum_{j=1}^{n} \left| \partial_x^k \int_0^{x_0} c_j(x, \tilde{x}_0) \partial_x \psi \int_0^{x_0} \tilde{c}_j(x, \tilde{x}_0) \partial_x \psi \int_0^{x_0} c(x, \tilde{x}_0) \partial_x \psi \right|^2 dx_0 dx_0 dx_0 \leq C_{12} \sum_{k=0}^{\infty} \left( \left| \partial_x^k \nabla \psi \right|^2 + \left| \partial_x^k \psi \right|^2 \right)
\]

\[
+ \sum_{j=1}^{n} \left| \partial_x^k \int_0^{x_0} c_j(x, \tilde{x}_0) \partial_x \psi \int_0^{x_0} \tilde{c}_j(x, \tilde{x}_0) \partial_x \psi \int_0^{x_0} c(x, \tilde{x}_0) \partial_x \psi \right|^2 dx_0 dx_0 dx_0 \leq C_{12} \sum_{k=0}^{\infty} \left( \left| \partial_x^k \nabla \psi \right|^2 + \left| \partial_x^k \psi \right|^2 \right)
\]

By (7.3), we have

\[
\inf_{y' \in \Omega} \phi(0, x') > \sup_{\partial \Omega} \gamma \times \Omega \phi(x).
\]

Hence, by (7.14), (7.2) and Lemma (7.1) there exists a \(s_1\) such that

\[
\frac{T}{s} \int \Omega |\nabla f|^2 e^{2s\phi(y(0, x')) dx'} \leq C_{12} \sum_{k=0}^{\infty} \left( \left| \partial_x^k \nabla \psi \right|^2 + \left| \partial_x^k \psi \right|^2 \right)
\]

\[
+ \sum_{j=1}^{n} \left| \partial_x^k \int_0^{x_0} c_j(x, \tilde{x}_0) \partial_x \psi \int_0^{x_0} \tilde{c}_j(x, \tilde{x}_0) \partial_x \psi \int_0^{x_0} c(x, \tilde{x}_0) \partial_x \psi \right|^2 dx_0 dx_0 dx_0 \leq C_{12} \sum_{k=0}^{\infty} \left( \left| \partial_x^k \nabla \psi \right|^2 + \left| \partial_x^k \psi \right|^2 \right)
\]

\[
+ \sum_{j=1}^{n} \left| \partial_x^k \int_0^{x_0} c_j(x, \tilde{x}_0) \partial_x \psi \int_0^{x_0} \tilde{c}_j(x, \tilde{x}_0) \partial_x \psi \int_0^{x_0} c(x, \tilde{x}_0) \partial_x \psi \right|^2 dx_0 dx_0 dx_0 \leq C_{12} \sum_{k=0}^{\infty} \left( \left| \partial_x^k \nabla \psi \right|^2 + \left| \partial_x^k \psi \right|^2 \right)
\]

\[
+ \sum_{j=1}^{n} \left| \partial_x^k \int_0^{x_0} c_j(x, \tilde{x}_0) \partial_x \psi \int_0^{x_0} \tilde{c}_j(x, \tilde{x}_0) \partial_x \psi \int_0^{x_0} c(x, \tilde{x}_0) \partial_x \psi \right|^2 dx_0 dx_0 dx_0 \leq C_{12} \sum_{k=0}^{\infty} \left( \left| \partial_x^k \nabla \psi \right|^2 + \left| \partial_x^k \psi \right|^2 \right)
\]

\[
+ \sum_{j=1}^{n} \left| \partial_x^k \int_0^{x_0} c_j(x, \tilde{x}_0) \partial_x \psi \int_0^{x_0} \tilde{c}_j(x, \tilde{x}_0) \partial_x \psi \int_0^{x_0} c(x, \tilde{x}_0) \partial_x \psi \right|^2 dx_0 dx_0 dx_0 \leq C_{12} \sum_{k=0}^{\infty} \left( \left| \partial_x^k \nabla \psi \right|^2 + \left| \partial_x^k \psi \right|^2 \right)
\]

\[
+ \sum_{j=1}^{n} \left| \partial_x^k \int_0^{x_0} c_j(x, \tilde{x}_0) \partial_x \psi \int_0^{x_0} \tilde{c}_j(x, \tilde{x}_0) \partial_x \psi \int_0^{x_0} c(x, \tilde{x}_0) \partial_x \psi \right|^2 dx_0 dx_0 dx_0 \leq C_{12} \sum_{k=0}^{\infty} \left( \left| \partial_x^k \nabla \psi \right|^2 + \left| \partial_x^k \psi \right|^2 \right)
\]
for all \( s \geq s_1 \). From (7.15), (7.2), (2.2) and (7.14) we obtain
\[
\frac{\tau}{s} \int_\Omega |\nabla f|^2 e^{2s\inf_{x' \in \Omega} \varphi(0, x')} \, dx' \leq C_{13} \left( \int_\Gamma s\tau^2 \varphi(0, x') |\partial_x \partial_{x_0}^2 y(0, x')|^2 e^{2s\varphi(0, x')} \, d\sigma \right.
\]
\[\left. + \sum_{k=0}^1 \int_\Sigma (s\tau \varphi |\nabla \partial_{x_0}^k v|^2 + s^3 \tau^3 \varphi^3 |\partial_{x_0}^k v|^2) e^{2s\varphi} \, d\Sigma \right).\]

The proof of Theorem 1.2 is completed. \( \square \)

8. Appendix

In the appendix we prove several lemmata which are used for the proof of Theorem 1.1 and represent the standard properties of the pseudodifferential operators with symbols of limited smoothness, keeping the dependence of the norms on parameter \( \tau \).

Let \( \{\omega_j\}_{j=1}^\infty \) be a sequence of the eigenfunctions of the operator \(-\Delta\) on \( S^{n+2} \) and let \( \{\lambda_j\}_{j=1}^\infty \) be a sequence of the corresponding eigenvalues of \(-\Delta\). Assume that
\[
(\omega_k, \omega_j)_{L^2(S^{n+2})} = \delta_{kj} \quad \forall k, j \in \mathbb{N}.
\]
The following asymptotic formula is known (e.g., Courant and Hilbert [1]):
\[
\lambda_j = cj^{\frac{2}{n}} + o(j^{2/n}) \quad \text{as } j \to +\infty.
\]
For each \( k \), thanks to the standard elliptic estimate for the Laplace operator, we have
\[
\|\omega_j\|_{H^{2k}(S^{n+2})} \leq C_k \lambda_j^k.
\]
Therefore the Sobolev embedding theorem yields
\[
\|\omega_j\|_{C_0(S^{n+2})} \leq C \lambda_j^n, \quad j \in \mathbb{N}.
\]
We extend the function \( \omega_j \) on the set \( \{|\xi| \leq 1\} \) as a smooth function and we set
\[
\omega_j(\xi) = \omega_j(\xi/|\xi|) \quad \text{for } |\xi| \geq 1.
\]
We introduce the pseudodifferential operator
\[
\tilde{\omega}_j(D)w = \int_{\mathbb{R}^{n+1}} \omega_j(\xi) \tilde{w}(\xi) e^{i<y, \xi>} \, d\xi, \quad \tilde{w}(\xi) = \frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R}^{n+1}} w(y) e^{-i<y, \xi>} \, dy.
\]
Here we recall that, in order to distinguish the Fourier transforms with respect to different variables, we will use the following notations
\[
\tilde{u}(\tilde{\xi}) := F_{y \to \xi} u = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i \sum_{j=0}^{n-1} y_j \xi_j} u(y_0, \ldots, y_{n-1}) \, dy,
\]
\[
\tilde{u}(\xi_n) := F_{y_n \to \xi_n} u = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}} e^{-i \xi_n y_n} u(y_n) \, dy_n.
\]
We recall that we set \( \tilde{\xi} = (\xi_0, \ldots, \xi_{n-1}) \) and \( \tilde{y} = (y_0, \ldots, y_{n-1}) \).

First we define the operator \( A(\tilde{\gamma}, \tilde{D}, s) \) for functions in \( C_0^\infty(\mathcal{O}) \):
\[
A(\tilde{\gamma}, \tilde{D}, s)u = \int_{\mathbb{R}^{n+1}} a(\tilde{\gamma}, \xi', s) F_{\tilde{\gamma} \to \xi} u e^{i \sum_{j=0}^{n-1} y_j \xi_j} \, d\tilde{y}.
\]
Lemma 8.1. Let \( a(\tilde{y}, \tilde{\xi}, s) \in C^0_{cl} S^{1,s}(\mathcal{O}) \). Then \( A \in \mathcal{L}(H^1_0(\mathcal{O}); L^2(\mathcal{O})) \) and \( \| A \|_{\mathcal{L}(H^1_0(\mathcal{O}); L^2(\mathcal{O}))} \leq C(\pi^{\infty}(a)) \).

**Proof.** Thanks to the assumption \( C \), it suffices to consider the case where
\[
(8.4) \quad a(\tilde{y}, \tau \tilde{\xi}, \tau s) = \tau a(\tilde{y}, \tilde{\xi}, s) \quad \forall \tau > 1.
\]

The operator
\[
\tilde{A}(\tilde{y}, D)v = \int_{|\xi| \leq 1} a(\tilde{y}, \xi) F_{y \rightarrow \xi} v(\xi) e^{i<y, \xi>} d\xi
\]
is a continuous operator from \( L^2(\mathcal{O} \times \mathbb{R}) \) into \( L^2(\mathcal{O} \times \mathbb{R}) \) with the norm estimated as
\[
\| \tilde{A}(\tilde{y}, D) \|_{\mathcal{L}(L^2(\mathcal{O} \times \mathbb{R}), L^2(\mathcal{O} \times \mathbb{R}))} \leq C(\pi^{\infty}(a)).
\]

Consider the symbol \( b(\tilde{y}, \xi) = a(\tilde{y}, \xi)/|\xi| \). Then (8.1) implies
\[
b(\tilde{y}, \tau \xi) = b(\tilde{y}, \xi) \quad \forall \tau \geq 1.
\]

We can represent the symbol \( b \) as
\[
b(\tilde{y}, \xi) = \sum_{j=1}^{\infty} b_j(\tilde{y}) \omega_j(\xi/|\xi|), \quad b_j(\tilde{y}) = (b(\tilde{y}, \xi), \omega_j(\xi))_{L^2(\mathbb{R}^{n+2})}.
\]

Observe that \( b_j(\tilde{y}) = (\Delta^k \tilde{y}, \xi, \omega_j(\xi))_{L^2(\mathbb{R}^{n+2})}/\lambda^k_j \). Therefore
\[
(8.5) \quad \| b_j \|_{C^0(\mathcal{O})} \leq C_m \lambda_j^{-m} \quad \forall m \in \{1, \ldots, \infty\}.
\]

By (8.2) and (8.5), we have
\[
\| B(\tilde{y}, D)v \|_{L^2(\mathcal{O} \times \mathbb{R})} \leq \sum_{j=1}^{\infty} \| b_j \|_{C^0(\mathcal{O})} \| \omega_j(D) \|_{\mathcal{L}(L^2(\mathcal{O} \times \mathbb{R}), L^2(\mathcal{O} \times \mathbb{R}))} \| v \|_{L^2(\mathcal{O} \times \mathbb{R})}
\]
\[
\leq \sum_{j=1}^{\infty} C_m \lambda_j^{-m} \lambda_j^n \| v \|_{L^2(\mathcal{O} \times \mathbb{R})}.
\]

Taking \( m = 3n \) we have
\[
\| B(\tilde{y}, D)v \|_{L^2(\mathcal{O} \times \mathbb{R})} \leq \sum_{j=1}^{\infty} C_m \lambda_j^{-2n} \| v \|_{L^2(\mathcal{O} \times \mathbb{R})}.
\]
Therefore the operator

\[ A^b(\tilde{y}, D)v = \int_{|\xi| \geq 1} a(\tilde{y}, \xi) F_{y \to \xi} v(\xi) e^{i<y, \xi>} d\xi \]

is a continuous operator from \( H^1_0(O \times \mathbb{R}) \) into \( L^2(O \times \mathbb{R}) \) with the norm satisfying

\[ \| A^b \|_{L^2(O \times \mathbb{R})} \leq \sum_{j=1}^{\infty} C_m \lambda_j^{-2n}. \]

Next we observe that

\[ \| \tilde{A}(\tilde{y}, D)v \|_{L^2(O \times \mathbb{R})} = \sqrt{2\pi} \| A(\tilde{y}, D_0, \ldots, D_{n-1}, \xi_n)u F_{y_n \to \xi_n} w \|_{L^2(O \times \mathbb{R})} \]

\[(8.6) \]

\[ \leq C(\pi c_0(\xi))(a) \left( \int_{-\infty}^{\infty} \| u \|_{H^1, \xi_n(O)}^2 \| F_{y_n \to \xi_n} w \|^2 d\xi_n \right)^{\frac{1}{2}} \]

for each function \( v(y) = u(\tilde{y}) w(y_n) \).

We take a sequence \( \{ w_j(x_n) \} \) such that \( F_{y_n \to \xi_n} w_j(\xi_n) \) has a compact support and \( |F_{y_n \to \xi_n} w_j|^2 \to \delta(\xi_n - s) \) for arbitrary \( s \in \mathbb{R} \). Since the function \( \xi_n \to \| A(\tilde{y}, \tilde{D}, \xi_n)u \|_{L^2(O)} \) is continuous, we have

\[ \| A(\tilde{y}, \tilde{D}, \xi_n)u w_j \|_{L^2(O \times \mathbb{R})} \]

\[ = \int_{\mathbb{R}} \| A(\tilde{y}, \tilde{D}, \xi_n)u \|_{L^2(O)}^2 \| F_{y_n \to \xi_n} w_j \|^2 d\xi_n \to \| A(\tilde{y}, \tilde{D}, s)u \|_{L^2(O)}^2. \]

This fact and (8.6) imply

\[ \| A(\tilde{y}, \tilde{D}, s)u \|_{L^2(O)} \leq C(\pi c_0(\xi))(a) \| u \|_{H^1, \xi_n(O)} \]

for almost all \( s \). Since the norm of the operator \( A \) is a continuous function of \( s \), we have this inequality for all \( s \). ♦

The following theorem provides an estimate for a commutator of a Lipschitz continuous function and the pseudodifferential operator \( \tilde{\omega}_j \).

**Proposition 8.1.** Let \( f \in W^1_{\infty}(O) \) be a function with compact support. Then

\[ \| [f, \tilde{\omega}_j] \|_{L^2(O), H^1, \xi(O)} \leq C \| f \|_{W^1_{\infty}(O)} \lambda_j^{4n}, \]

where the constant \( C \) is independent of \( j \).

From this proposition we have immediately

**Corollary 8.1.** Let \( f \in C^\xi(O) \) be a function with compact support. Then

\[ \| [f, \tilde{\omega}_j] \|_{L^2(O), H^1, \xi(O)} \leq C \| f \|_{W^1_{\infty}(O)} \lambda_j^{4n}, \]

where the constant \( C \) is independent of \( j \).

The proof of this theorem is similar to the proof of Corollary in [38], p. 309.

Let \( M(\xi) = \mu(|\xi|)|\xi| \), where \( \mu \in C^\infty(\mathbb{R}) \), \( \mu(t) = 1 \) for \( t \geq 1 \) and \( \mu(t) = 0 \) for \( t \in [0, \frac{1}{2}] \).
Lemma 8.2. Let \( a(\tilde{y}, \tilde{\xi}, s) \in C^t_d S^{t,s}(O) \). Then \( A(\tilde{y}, D, s)^* = A^*(\tilde{y}, D, s) + R \), where \( A^* \) is the pseudodifferential operator with symbol \( a(\tilde{y}, \xi, s) \) and \( R \in \mathcal{L}(H^0_{t-1,s}(O), L^2(O)) \) satisfies

\[
\|R\|_{\mathcal{L}(H^0_{t-1,s}(O), L^2(O))} \leq C \pi C^{t}(O)(a).
\]

Proof. Thanks to the assumption \( C \) it suffices to consider the case when

\[
a(\tilde{y}, \tau \tilde{\xi}, \tau s) = \tau^t a(\tilde{y}, \tilde{\xi}, s) \quad \forall \tau > 1.
\]

The symbol \( a(\tilde{y}, \xi) \) can be represented as

\[
a(\tilde{y}, \xi) = \sum_{j=1}^{\infty} a_j(\tilde{y}) M^t(\xi) \omega_j(\xi).
\]

Consider the operator

\[
\tilde{A}(\tilde{y}, D) = \sum_{j=1}^{\infty} a_j(\tilde{y}) M^t(D) \tilde{\omega}_j(D), \quad M^t(D)w = \int_{\mathbb{R}^{n+1}} M^t(\xi) \tilde{w} e^{i\langle y, \xi \rangle} d\xi.
\]

Then we find the formal adjoint operator:

\[
\tilde{A}(\tilde{y}, D)^* = \sum_{j=1}^{\infty} (a_j(\tilde{y}) M^t(D) \tilde{\omega}_j(D))^* = \sum_{j=1}^{\infty} M^t(D) \tilde{\omega}_j(D) \overline{a_j(\tilde{y})}
\]

\[
= \sum_{j=1}^{\infty} \overline{a_j(\tilde{y})} M^t(D) \tilde{\omega}_j(D) + \sum_{j=1}^{\infty} [M^t(D) \tilde{\omega}_j(D), \overline{a_j(\tilde{y})}],
\]

where \([A, B] := AB - BA\).

Observe that \( \sum_{j=1}^{\infty} a_j(\tilde{y}) M^t(D) \tilde{\omega}_j(D) \) is the operator with symbol \( \overline{a(\tilde{y}, \xi_0, \ldots, \xi_n)} \in C^t_d S^{t,s}(O) \).

Let us estimate the norm of the operator \( \sum_{j=1}^{\infty} [M^t(D) \tilde{\omega}_j(D), \overline{a_j(\tilde{y})}] \). Proposition 8.1 implies

\[
\left\| \sum_{j=1}^{\infty} \left[ a_j(\tilde{y}), M^t(D) \tilde{\omega}_j(D) \right] \right\|_{\mathcal{L}(H^0_{t-1}(O \times \mathbb{R}), L^2(O \times \mathbb{R}))} \leq C_m \sum_{j=1}^{\infty} \|a_j\|_{C^t(\tilde{\xi}^{(n)})} \lambda_j^{\tilde{\xi}(n)} \leq C_m \sum_{j=1}^{\infty} \lambda_j^{\tilde{\xi}(n)} \pi C^t(O)(a) \lambda_j^{-m} \leq C \pi C^{t}(O)(a).
\]

Denote \( v = u(\tilde{y})w(y_n), \tilde{v} = \tilde{u}(y_0, \ldots, y_{n-1})\tilde{w}(y_n) \). We have

\[
(\tilde{A}(\tilde{y}, D)v, \tilde{v})_{L^2(O \times \mathbb{R})} = (v, \tilde{A}^*(\tilde{y}, D)\tilde{v})_{L^2(O \times \mathbb{R})} = (v, \tilde{A}^*(\tilde{y}, D)\tilde{v})_{L^2(O \times \mathbb{R})} + (v, R\tilde{v})_{L^2(O \times \mathbb{R})}.
\]

By (8.7), we have

\[
(\tilde{A}(\tilde{y}, D)v, \tilde{v})_{L^2(O \times \mathbb{R})} = \pi C^t(O)(a).
\]

On the other hand

\[
(\tilde{A}(\tilde{y}, D)v, \tilde{v})_{L^2(O \times \mathbb{R})} = 2\pi \int_{\mathbb{R}} \left( A(\tilde{y}, D, \xi_n)u, \tilde{u} \right)_{\overline{\mathcal{L}^2(O)}} w d\xi_n
\]

\[
= 2\pi \int_{\mathbb{R}} \left( u, A(\tilde{y}, D, \xi_n)^* \tilde{u} \right)_{\overline{\mathcal{L}^2(O)}} w d\xi_n.
\]
Taking into account that \( (v, A^*(\tilde{y}, D)\tilde{v})_{L^2(\Omega \times \mathbb{R})} = \int_{\mathbb{R}} (u, A^*(\tilde{y}, \tilde{D}, \xi_n)\tilde{u})_{L^2(\Omega)} \tilde{w}d\xi_n \), we have
\[
\left| \int_{\mathbb{R}} (u, (A(\tilde{y}, \tilde{D}, \xi_n)^* - A^*(\tilde{y}, \tilde{D}, \xi_n))\tilde{u})_{L^2(\Omega)} \tilde{w}d\xi_n \right| = |(v, R\tilde{v})_{L^2(\Omega \times \mathbb{R})}|
\leq C\pi C^1(a)\|v\|_{L^2(\Omega \times \mathbb{R})} \|\tilde{v}\|_{H^{-1}_0(\Omega \times \mathbb{R})}.
\]
We take a sequence \( \{w_j\}_{j=1}^{\infty} \) such that \( F_{\gamma_n \to \xi_n} w_j \) have compact supports and \( |F_{\gamma_n \to \xi_n} w_j|^2 \to \delta(\xi_n - s) \) for arbitrary \( s \in \mathbb{R} \). Since the function \( \xi_n \to \|A(\tilde{y}, \tilde{D}, \xi_n)u\|_{L^2(\Omega)} \) is continuous, we have
\[
\left| (u, (A(\tilde{y}, \tilde{D}, s)^* - A^*(\tilde{y}, \tilde{D}, s))\tilde{u})_{L^2(\Omega)} \right| \leq C\pi C^1(\Omega)(a)\|u\|_{L^2(\Omega)} \|\tilde{u}\|_{H^{-1}_0(\Omega)},
\]
the lemma is proved. \( \blacksquare \)

**Lemma 8.3.** Let \( a(\tilde{y}, \tilde{\xi}, s) \in C^1_{d\sigma} S^{j,s}(\mathcal{O}) \) where \( \tilde{j} = 0,1 \) and \( b(\tilde{y}, \tilde{s}, s) \in C^1_{d\sigma} S^{j,s}(\mathcal{O}) \). Then \( A(\tilde{y}, \tilde{D}, s)B(\tilde{y}, \tilde{D}, s) = C(\tilde{y}, \tilde{D}, s) + R_0 \) where \( C(\tilde{y}, \tilde{D}, s) \) is the operator with symbol \( a(\tilde{y}, \tilde{\xi}, s) b(\tilde{y}, \tilde{s}, s) \) and \( R_0 \in \mathcal{L}(H^{j+\tilde{s},s}_0(\mathcal{O}), H^{\tilde{s}+1,s}(\mathcal{O})) \) for any \( \tilde{s} \in [-1,0] \) if \( \tilde{j} = 0 \) and \( R_0 \in \mathcal{L}(H^{j,s}_0(\mathcal{O}), L^2(\mathcal{O})) \) if \( \tilde{j} = 1 \). Moreover we have
\[
\|R_0\|_{\mathcal{L}(H^{j,s}_0(\mathcal{O}), L^2(\mathcal{O}))} \leq C\pi C^1(\mathcal{O})(a)\pi C^1(\mathcal{O})(b) \quad \text{for} \quad \tilde{j} = 1,
\]
\[
\|R_0\|_{\mathcal{L}(H^{j+s,s}_0(\mathcal{O}), H^{\tilde{s}+1,s}(\mathcal{O}))} \leq C\pi C^1(\mathcal{O})(a)\pi C^1(\mathcal{O})(b) \quad \text{for} \quad \tilde{j} = 0.
\]

**Proof.** We set
\[
A(\tilde{y}, D) = \sum_{j=1}^{\infty} a_j(\tilde{y}) M^j(D) \tilde{\omega}_j(D), \quad B(\tilde{y}, D) = \sum_{j=1}^{\infty} b_j(\tilde{y}) M^\mu(D) \tilde{\omega}_j(D).
\]
Observe that
\[
A(\tilde{y}, D) B(\tilde{y}, D) = \sum_{m,k=1}^{\infty} a_{m}(\tilde{y}) b_{k}(\tilde{y}) M^{j+\mu}(D) \tilde{\omega}_m(D) \tilde{\omega}_k(D) + \sum_{m,k=1}^{\infty} a_{m}(\tilde{y}) [M^j \tilde{\omega}_m, b_k] M^\mu(D) \tilde{\omega}_k(D).
\]
Since \( C(\tilde{y}, D) = \sum_{m,k=1}^{\infty} a_{m}(\tilde{y}) b_{k}(\tilde{y}) M^{j+\mu}(D) \tilde{\omega}_m(D) \tilde{\omega}_k(D) \), and for \( \tilde{j} = 1 \),
\[
\|R_0\|_{\mathcal{L}(H^{j,s}_0(\mathcal{O}), L^2(\mathcal{O}))} = \left\| \sum_{m,k=1}^{\infty} a_{m}(\tilde{y}) [M\tilde{\omega}_m, b_k] M^\mu(D) \tilde{\omega}_k(D) \right\|_{\mathcal{L}(H^{j,s}_0(\mathcal{O}), L^2(\mathcal{O}))}
\leq \sum_{m,k=1}^{\infty} \|a_{m}\|_{C^1(\mathcal{O})} \| [M\tilde{\omega}_m, b_k] \|_{\mathcal{L}(L^2, L^2)} \| \tilde{\omega}_k(D) \|_{\mathcal{L}(L^2(\mathcal{O}), L^2(\mathcal{O}))}
\leq C_1 \pi C^0(\mathcal{O})(a) \sum_{m,k=1}^{\infty} \lambda_m^{-1} \| [M\tilde{\omega}_m, b_k] \|_{\mathcal{L}(L^2(\mathcal{O}), L^2(\mathcal{O}))} \tilde{\omega}_k(D) .
Applying Proposition 8.1, we obtain

\[ R_0 \|_{L(H_0^\mu(\gamma),L^2(\gamma))} \leq \sum_{m,k=1}^{\infty} \| a_m \|_{C^1(\gamma)} \| [M\tilde{\omega}_m, b_k] \|_{L(L^2(\gamma),L^2(\gamma))} \| \tilde{\omega}_k(D) \|_{L(L^2(\gamma),L^2(\gamma))} \]

\[ \leq C_1 \pi_{C^1(\gamma)}(a) \sum_{m,k=1}^{\infty} \lambda_m^{-l} \| b_k \|_{C^1(\gamma)} \lambda_k \tilde{\omega}_m(\gamma) \lambda_k \tilde{\omega}_m(\gamma) \]

\[ \leq C_1 \pi_{C^1(\gamma)}(a) \pi_{C^1(\gamma)}(b) \sum_{m,k=1}^{\infty} \lambda_m^{-l} \lambda_k \tilde{\omega}_m(\gamma) \lambda_k \tilde{\omega}_m(\gamma) \]

\[ \leq C_1 \pi_{C^1(\gamma)}(a) \pi_{C^1(\gamma)}(b) \sum_{k=1}^{\infty} \lambda_k \sum_{m=1}^{\infty} \lambda_m \tilde{\omega}_m(\gamma) < \infty. \]

Let \( v = v_j = u(\tilde{y})w_{j}(y) \). We take a sequence \( \{w_j\}_{j=1}^{\infty} \) such that \( F_{y_n\to\xi_n}w_j, j \in \mathbb{N}, \) have compact supports and \( |F_{y_n\to\xi_n}w_j|^2 \to \delta(\xi_n - s) \) for arbitrary \( s \in \mathbb{R} \).

Then

\[ \| A(\tilde{y}, \tilde{D}, \xi_n) B(\tilde{y}, \tilde{D}, \xi_n) - C(\tilde{y}, \tilde{D}, \xi_n)u \|_{L^2(\gamma)}^2 |F_{y_n\to\xi_n}w_j|^2 d\xi_n \leq C \| v_j \|_{H_0^\mu(\mathbb{R}^{n+1})}^2 \]

for any \( u \in H_0^{1+\mu}(\gamma) \).

Passing to the limit in (8.10) as \( j \to +\infty \), we obtain

\[ \| (A(\tilde{y}, \tilde{D}, s) B(\tilde{y}, \tilde{D}, s) - C(\tilde{y}, \tilde{D}, s))u \|_{L^2(\gamma)}^2 \leq C \| u \|_{H_0^\mu(\mathbb{R}^{n+1})}^2. \]

Let \( \tilde{j} = 0 \). Then

\[ R_0 \|_{L(H_0^{\mu+1}(\gamma),H_0^{\mu}(\gamma))} \leq \sum_{m,k=1}^{\infty} \| a_m [\tilde{\omega}_m, b_k] M^{-s} \|_{L(L^2(\gamma),H_0^{\mu+1}(\gamma))} \| \tilde{\omega}_k(D) \|_{L(L^2(\gamma),L^2(\gamma))} \]

In order to estimate \( \| a_m [\tilde{\omega}_m, b_k] M^{-s} \|_{L(L^2(\gamma),H_0^{\mu+1}(\gamma))} \), we observe that \( M^{s+1}a_m[\tilde{\omega}_m, b_k] M^{-s} = a_m M^{s+1}[\tilde{\omega}_m, b_k] M^{-s} + [M^{s+1}, a_m][\tilde{\omega}_m, b_k] M^{-s} \). For the second term in this equality, we have

\[ [\tilde{\omega}_m, b_k] M^{-s} \|_{L(L^2(\gamma),L^2(\gamma))} \leq \| b_k \|_{C^1(\gamma)} \| \tilde{\omega}_m \|_{L(L^2(\gamma),L^2(\gamma))}, \]

\[ [M^{s+1}, a_m] \|_{L(L^2(\gamma),L^2(\gamma))} \leq C \| a_m \|_{C^1(\gamma)}. \]

An interpolation argument yields

\[ [\tilde{\omega}_m, b_k] \in L(H^{-\gamma}(\mathbb{R}^n),H^{1-\gamma}(\mathbb{R}^n)) \quad \forall \gamma \in [0, 1] \]

and

\[ \| [\tilde{\omega}_m, b_k] \|_{L(H^{-\gamma}(\mathbb{R}^n),H^{1-\gamma}(\mathbb{R}^n))} \leq \| [\tilde{\omega}_m, b_k] \|_{L(L^2(\mathbb{R}^n),H^1(\mathbb{R}^n))} \quad \forall \gamma \in [0, 1]. \]
Applying (8.12)-(8.14) to (8.11) we obtain
\[
\| R_0 \|_{\mathcal{L}(H^0(\mathcal{O}),H^1(\mathcal{O}))} \leq C_1 \sum_{m,k=1}^{\infty} \lambda_m^{-1} \| b_k \|_{C^1(\mathcal{O})} \lambda_k^{(n)} \lambda_m^{(n)} \\
\leq C_{t,1} \pi_{C^1(\mathcal{O})}(b) \sum_{m,k=1}^{\infty} \lambda_m^{-1} \lambda_k^{(n)} \lambda_k^{(n)} \leq C_{t,1} \pi_{C^1(\mathcal{O})}(b) \sum_{m=1}^{\infty} \lambda_m^{-1} \lambda_m^{(n)} \sum_{m=1}^{\infty} \lambda_k^{(n)} < \infty.
\]
We finish the proof of lemma using similar arguments as in case \( \tilde{j} = 1 \). -

The direct consequence of Lemma 8.3 is the following commutator estimate.

**Lemma 8.4.** Let \( a(\tilde{y}, \tilde{\xi}, s) \in C^1_{c}(S^1, \mathcal{O}) \) and \( b(\tilde{y}, \tilde{\xi}, s) \in C^1_{c}(S^1, \mathcal{O}) \).

Then the commutator \([A, B]\) belongs to the space \( \mathcal{L}(H^{1.5}(\mathcal{O}); L^2(\mathcal{O})) \) and
\[
\| [A, B] \|_{\mathcal{L}(H^{1.5}(\mathcal{O}); L^2(\mathcal{O}))} \leq C(\pi_{C^1(\mathcal{O})}(a)\pi_{C^1(\mathcal{O})}(b) + \pi_{C^1(\mathcal{O})}(a)\pi_{C^1(\mathcal{O})}(b) + \pi_{C^1(\mathcal{O})}(a)\pi_{C^1(\mathcal{O})}(b)).
\]

**Proof.** By Lemma 8.3 we have
\[
A(\tilde{y}, \tilde{D}, s)B(\tilde{y}, \tilde{D}, s) = C(\tilde{y}, \tilde{D}, s) + R_0, \quad B(\tilde{y}, \tilde{D}, s)A(\tilde{y}, \tilde{D}, s) = C(\tilde{y}, \tilde{D}, s) + \tilde{R}_0,
\]
where \( R_0, \tilde{R}_0 \in \mathcal{L}(H^{1.5}(\mathcal{O}), L^2(\mathcal{O})) \) and \( C(\tilde{y}, \tilde{D}, s) \) is the pseudodifferential operator with the symbol \( c(\tilde{y}, \tilde{\xi}, s) = a(\tilde{y}, \tilde{\xi}, s)b(\tilde{y}, \tilde{\xi}, s) \). Since \([A, B] = R_0 - \tilde{R}_0\), we immediately obtain the statement of the lemma.

**Lemma 8.5.** Let \( a(\tilde{y}, \tilde{\xi}, s) \in C^1_{c}(S^1, \mathcal{O}) \) be a symbol with compact support in \( \mathcal{O} \). Let \( \mathcal{O}_1 \subset \subset \mathcal{O} \) and \( \mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset \). Suppose that \( u \in H^{1.5}(\mathcal{O}) \) and \( \text{supp} u \subset \mathcal{O}_1 \). Then there exists a constant \( C \) such that
\[
\| A(\tilde{y}, \tilde{D}, s)u \|_{H^{1.5}(\mathcal{O}_2)} \leq \frac{C \pi_{C^1(\mathcal{O})}(a)}{\text{dist}(\mathcal{O}_2, \mathcal{O}_1)^{2n+3}} \| u \|_{H^{1.5}(\mathcal{O})}.
\]

**Proof.** By lemma it suffices to prove the inequality (8.15) only for \( u \in C_0^\infty(\mathcal{O}_1) \). Let \( b(t) \in C_0^\infty(-2, 2) \), and \( b_{[-1,1]} = 1 \) and \( \tilde{y} \in \mathcal{O}_2 \). We have
\[
A(\tilde{y}, \tilde{D}, s)u = \lim_{\epsilon \to +0} \int_{\mathbb{R}^n} b(\epsilon |\tilde{\xi}|)a(\tilde{y}, \tilde{\xi}, s)e^{-i <\tilde{y}, \tilde{\xi}>} \tilde{u}(\tilde{\xi})d\tilde{\xi} =
\]
\[
\frac{1}{(2\pi)^{2n}} \lim_{\epsilon \to +0} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} b(\epsilon |\tilde{\xi}|)a(\tilde{y}, \tilde{\xi}, s)e^{i <\tilde{y}, \tilde{\xi}>} u(\tilde{x})d\tilde{\xi}d\tilde{x} =
\]
\[
\frac{1}{i^{2k}(2\pi)^{2n}} \lim_{\epsilon \to +0} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{b(\epsilon |\tilde{\xi}|)a(\tilde{y}, \tilde{\xi}, s)}{|\tilde{x} - \tilde{y}|^{2k}} \Delta^k e^{i <\tilde{x}, \tilde{y}>} u(\tilde{x})d\tilde{x}d\tilde{x} =
\]
\[
\frac{1}{i^{2k}(2\pi)^{2n}} \int_{\mathcal{O}_1} \int_{\mathbb{R}^n} \frac{\Delta^k a(\tilde{y}, \tilde{\xi}, s)}{|\tilde{x} - \tilde{y}|^{2k}} e^{i <\tilde{x}, \tilde{y}>} u(\tilde{x})d\tilde{x}d\tilde{x}.\]
Let $k = n$. Then

\begin{equation}
|\partial_{y_i} A(\tilde{y}, \tilde{D}, s)u| = \left| \frac{1}{i^{2k}(2\pi)^{\frac{1}{2}}} \int_{\Omega} \int_{\mathbb{R}^n} \partial_{y_i} \left( \frac{\Delta_{\xi}^{k} a(\tilde{y}, \tilde{\xi}, s)}{|\tilde{x} - \tilde{y}|^{2k}} e^{i\tilde{\xi} - \tilde{y}} \right) u(\tilde{x}) d\tilde{\xi} d\tilde{x} \right| \\
\leq C \pi C^{1}(\mathcal{O})(a) \int_{\Omega} \int_{\mathbb{R}^n} \frac{|u(\tilde{x})|}{|\tilde{x} - \tilde{y}|^{2n+3}(1 + |\xi|^{2n+1} + s^{2n+1})} d\tilde{\xi} d\tilde{x} \\
\leq \frac{C \pi C^{1}(\mathcal{O})(a)}{(s^{n+1} + 1) \text{dist}(\partial_{\mathcal{O}})} \|u\|_{H^{1,s}(\mathcal{O})}.
\end{equation}

Proof of the lemma is complete. ■

We shall use the following variant of the Gårding inequality:

**Lemma 8.6.** Let $p(\tilde{y}, \tilde{\xi}, s) \in C_{\alpha}^{2,S^{2,s}}(\mathcal{O})$ be a symbol with compact support in $\mathcal{O}$. Let $u \in H^{1,s}(\mathcal{O})$ and $supp u \subset \mathcal{O}_{1}$. Let $\mathcal{O}_{1} \subset \mathcal{O}_{2} \subset \mathcal{O}_{3} \subset \mathcal{O}$ and $\tilde{\gamma} \in C_{\alpha}^{1}(\mathcal{O}_{3})$ be a function such that $\tilde{\gamma}|_{\mathcal{O}_{2}} = 1$ be such that $\text{Re} p(\tilde{y}, \tilde{\xi}, s) > \hat{C} |(\tilde{\xi}, s)|^{2}$ for any $\tilde{y} \in \mathcal{O}_{3}$. Then

\begin{equation}
\text{Re} (P(\tilde{y}, \tilde{D}, s)u, u)_{L^{2}(\mathcal{O})} \geq \frac{\hat{C}}{2} \|u\|_{H^{1,s}(\mathcal{O})}^{2} \left( \sum_{k=0}^{2} (\pi C^{1}(\mathcal{O}_{3})(p) + 1) \pi C^{2,k}(\mathcal{O}_{3})(\tilde{\gamma}) + (\pi C^{1}(\mathcal{O}_{3})(p) + 1) \pi C^{2,k}(\mathcal{O}_{3})(\tilde{\gamma}) \right)^{2} \\
+ \frac{1}{\text{dist}(\mathcal{O}_{1}, \mathbb{R}^{n} \setminus \mathcal{O}_{2})^{2n+3}} \|u\|_{L^{2}(\mathcal{O})}^{2}.
\end{equation}

**Proof.** Consider the pseudodifferential operator $A(\tilde{y}, \tilde{D}, s)$ with symbol $A(\tilde{y}, \tilde{\xi}, s) = (\tilde{\gamma} \text{Re} p(\tilde{y}, \tilde{\xi}, s) - \hat{C}^{\frac{1}{2}} M^2(\tilde{\xi}, s))^{\frac{1}{2}} \in C_{\alpha}^{2,S^{1,s}}(\mathcal{O})$. Then, according to Lemma 8.3

\begin{equation}
A(\tilde{y}, \tilde{D}, s)^{*} A(\tilde{y}, \tilde{D}, s) = \tilde{\gamma} \text{Re} p(\tilde{y}, \tilde{D}, s) - \hat{C}^{\frac{1}{2}} M^2(\tilde{D}, s) + R,
\end{equation}

where $R \in \mathcal{L}(H^{1,s}(\mathcal{O}); L^{2}(\mathcal{O}))$ and

\begin{equation}
\|R\|_{\mathcal{L}(H^{1,s}(\mathcal{O}); L^{2}(\mathcal{O}))} \leq C(\pi C^{2}(\mathcal{O}_{3})(a) + \pi C^{2}(\mathcal{O}_{3})(a)) \\
\leq C \sum_{k=0}^{2} (\pi C^{1}(\mathcal{O}_{3})(p) + 1) \pi C^{2,k}(\mathcal{O}_{3})(\tilde{\gamma}) + (\pi C^{1}(\mathcal{O}_{3})(p) + 1) \pi C^{2,k}(\mathcal{O}_{3})(\tilde{\gamma})).
\end{equation}

Therefore

\begin{equation}
\text{Re} (P(\tilde{y}, \tilde{D}, s)u, u)_{L^{2}(\mathcal{O})} = \|A(\tilde{y}, \tilde{D}, s)\|_{L^{2}(\mathcal{O})}^{2} - ((1 - \tilde{\gamma}) M^2(\tilde{D}, s)u, u)_{L^{2}(\mathbb{R}^{n})} \\
+ \frac{\hat{C}}{2} \|u\|_{H^{1,s}(\mathcal{O})}^{2} + (Ru, u)_{L^{2}(\mathcal{O})}.
\end{equation}
Observing that

\[
|(Ru, u)_{L^2(\mathcal{O})}| \leq C \sum_{k=0}^2 (\pi_{C^k(\mathcal{O}_3)}(p) + 1)\pi_{C^{2-k}(\mathcal{O}_3)}(\tilde{\gamma})
\]

and since by Lemma 8.5 we have

\[
|((1 - \tilde{\gamma})M^2(\tilde{D}, s)u, u)_{L^2(\mathbb{R}^n)}| \leq \frac{C}{\text{dist}(\mathcal{O}_1, \mathbb{R}^n \setminus \mathcal{O}_2)^{2n+3}} \|u\|_{L^2(\mathcal{O})} \|u\|_{H^{1,2}(\mathcal{O})},
\]

we obtain the statement of the lemma. □

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