On a Broken Formal Symmetry between Kinetic and Gravitational Energy

Armin Nikkhah Shirazi
University of Michigan, Ann Arbor
armin@umich.edu

Essay written for the Gravity Research Foundation 2010 Awards for Essays on Gravitation
March 25, 2010

Abstract
Historically, the discovery of symmetries has played an important role in the progress of our fundamental understanding of nature [1]. This paper will demonstrate that there exists in Newtonian theory in a spherical gravitational field a formal symmetry between the kinetic (KE) and gravitational potential energy (GPE) of a test mass. Put differently, there exists a way of expressing GPE such that the form of the mathematical expression remains invariant under an interchange of KE and GPE. When extended to relativity by a suitable assumption, it leads to a framework that bridges the general relativistic and Newtonian conceptions of gravitational energy, even though the symmetry is broken except in the infinitesimal limit. Recognizing this symmetry at infinitesimal scales makes it possible to write a relativistic equation of an individual graviton, the properties of which under one interpretation may be unexpected.
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1 Introduction

Historically, the discovery of symmetries has played an important role in the progress of our fundamental understanding of nature [1]. This paper will demonstrate that there exists in Newtonian theory in a spherical gravitational field a formal symmetry between the kinetic (KE) and gravitational potential energy (GPE) of a test mass. Put differently, there exists a way of expressing GPE such that the form of the mathematical expression remains invariant under an interchange of KE and GPE. When extended to relativity by a suitable assumption, it leads to a framework that bridges the general relativistic and Newtonian conceptions of gravitational energy, even though the symmetry is broken except in the infinitesimal limit. Recognizing this symmetry at infinitesimal scales makes it possible to write a relativistic equation of an individual graviton, the properties of which under one interpretation may be unexpected.

2 The Formal Symmetry in Newtonian Physics

We take the gravitational force to be mediated by the gravitational field. Let us define the force exerted by the field on the test mass by $F$ and that exerted by the test mass on the field by $F_g$. Then, Newton’s third law says

$$\mathbf{F} = -\mathbf{F}_g$$

(1)

We also know that $\mathbf{F}$ is the negative gradient of the GPE

$$\mathbf{F} = -\nabla U = -m\nabla \phi$$

(2)

Where $U$ is the GPE and $\phi$ is the gravitational potential. In Newtonian theory there exists no finite limit on motion, so fields are nothing more than mathematical artifacts: one simply has ”action at a distance” which due to the infinite transmission speed of force is really no different from a contact interaction, making the field concept physically hollow. It is therefore not usual in Newtonian theory to associate momentum with a gravitational field. However, to demonstrate the formal symmetry between KE and GPE we will need to use
this concept. Let $p_g$ thus be defined as the momentum stored in a gravitational field acting on the test mass. Then,

$$F_g = \frac{dp_g}{dt} = m\nabla \phi$$

(3)

Where we used (2) to relate the gradient of the field’s potential energy to the momentum stored in it. Let us now assume the simplest field configuration, spherical symmetry. Using the chain rule on the left and writing the gradient in terms of a change in the radial direction $\hat{r}$ only gives

$$\frac{dp_g}{dt} = \frac{dr}{dt} \frac{dp_g}{dr} = m \phi \hat{r}$$

(4)

Multiplying both sides by $mdr$ and defining $m \frac{dr}{dt} \equiv m v_r \equiv p_r$ gives

$$p_r dp_g = m^2 d\phi \hat{r}$$

(5)

We wish to integrate this in such a way as to obtain an expression for GPE purely in terms of $p_g$. To do so, first recall that (1) applied to this situation can be rewritten as

$$\frac{dp_r}{dt} = - \frac{dp_g}{dt}$$

(6)

Or, considering just the differential momenta,

$$dp_r = - dp_g$$

(7)

Which upon integration yields

$$p_r = - p_g + p_0$$

(8)

Where $p_0$ is an integration constant with dimensional units of momentum. The interpretation of this constant depends on the boundary conditions we impose. We assume the usual BCs, namely that at infinity the test mass is at rest and that the potential is set to zero, which means the stored momentum of the field there is also set to zero. Since due to our BCs there exists a region where both momentum variables are zero, $p_0$ must vanish, leaving

$$p_r = - p_g$$

(9)

This implies

$$p_r = p_g$$

(10)

Substituting (10) into (5) and integrating both sides in the radial direction from zero to infinity then gives

$$\frac{p_g^2}{2} = m^2 \phi$$

(11)
where the integration constant is again suppressed by our previous BCs. Dividing through by \( m^2 \) gives

\[
\frac{p_g^2}{2m^2} = \frac{v_g^2}{2} = \phi
\]

(12)

where

\[
v_g \equiv \frac{p_g}{m}
\]

(13)

is the stored momentum per test mass \( m \) and will be defined as *the motion stored in the gravitational field*. It can in the Newtonian context be thought of as the ’potential motion’ of the test mass i.e. motion which is ’gained’ by the field as the test mass moves from the source to infinity. Notice that since \( \phi < 0 \), \( v_g \) must be interpreted to be imaginary. We shall see at the end of the next section that this is merely a mathematical artifact. For a potential given by Newton’s law of gravitation

\[
v_g = \sqrt{2\phi} = \sqrt{-\frac{2GM}{r}}
\]

(14)

By defining \( v_g \) we can write the total test mass energy in Newtonian theory as

\[
E = K + U = \frac{1}{2}mv_r^2 - G\frac{Mm}{r} = \frac{1}{2}mv_r^2 + \frac{1}{2}mv_g^2
\]

(15)

making the formal symmetry between classical KE and GPE for the spherical field explicit. Due to our BCs \( E = 0 \). For different BCs, the value of \( E \) might differ due to non-vanishing integration constants, but the symmetry remains as long as \( E \) is constant, which is just a statement of conservation of energy.

### 3 The Formal Symmetry in Relativity

To develop the same idea in relativity we assume that the formal symmetry as shown in (15) has its origin in a corresponding formal symmetry in relativistic momentum. Formally, we assume for point-like particles only

**Axiom:** \( mv_g \) (Classical) \( \rightarrow \gamma_g mv_g \) (Relativity) (16)

where \( v_g \) in relativity is assumed to be real and

\[
\gamma_g \equiv \frac{1}{\sqrt{1 - \frac{v_g^2}{c^2}}} = \frac{dt}{d\tau(K = 0, U \neq 0)}
\]

(17)

Is the gravitational analog to the Lorentz factor: for \( K = 0, U = 0, \gamma_g \rightarrow 1 \) and for \( K > 0, U = 0, \gamma_g \rightarrow \gamma \). To preserve the formal symmetry, we set \( K = p = 0 \), throughout. In a rest frame in a gravitational field, by formal symmetry with KE, we write

\[
E^2 = (mc^2 + U)^2 = m^2c^4 + p_g^2c^2
\]

(18)

3
We emphasize that by treating gravitational energy as a property of the test mass, we are adopting a perspective radically different from that of canonical GR, which views it as a property of the spacetime region in which the mass finds itself. We adopt this perspective to explore this formal symmetry in relativity. To obtain the relativistic expression for $U$, rewrite (18) as

$$U^2 + 2Umc^2 - p_g^2c^2 = 0 \quad (19)$$

This quadratic equation in $U$ has two distinct roots. The first is

$$U_+(K = 0) = mc^2(\gamma_g - 1) \quad (20)$$

This solution is exactly symmetric in form to relativistic KE. It is non-negative and decreases with increasing distance to zero infinitely far from the gravitational source. Further, given that

$$ds^2(K = 0, U > 0) = \frac{c^2dt^2}{\gamma_g^2} \quad (21)$$

where $ds^2$ is the spacetime interval, we should expect a large $U_+$ to be associated with a change in the geometry of spacetime, even though quantifying the change requires a re-expression of $U_+$ in terms of Energy density. Nevertheless, this solution is close to the general relativistic conception of gravitational Energy. The second solution is

$$U_-(K = 0) = -mc^2(\gamma_g + 1) \quad (22)$$

This solution differs from (20) by a sign in $\gamma_g$. It cannot be associated with the same geometric interpretation as $U_+$ because it is negative and increases with increasing distance. These are characteristics of Newtonian GPE. Notice that infinitely far from the source $U_-$ increases to a maximum value of $-2mc^2$, so that in a zero field the total energy of the test mass becomes $E = mc^2 + (-2mc^2) = -mc^2$. But since the total energy here describes a mass at rest in a zero field, this is equivalent to considering the 'net' rest mass to be $-m$. By taking the square root of (18) using the second solution and expanding to first order we get

$$U_- \approx \frac{p_g^2}{2m} = \frac{1}{2}mv_g^2 \quad (23)$$

where both sides are negative: The left side because $U_-$ is always negative, and the right side because the choice of $U_-$ requires the 'net' rest mass to be negative. But since in the Newtonian approximation we consider $m$ to be positive, $v_g$ must now be interpreted to be imaginary in order for the sign of both sides to match. Hence, the imaginary appearance of $v_g$ is just an artifact, as mentioned above. As a final check, use the fact known from canonical GR that for a static spherical field

$$\gamma_g = \frac{1}{\sqrt{1 - \frac{2GM}{rc^2}}} \quad (24)$$
And take the negative gradient of (22) using (24):

\[- \nabla \left( -mc^2(\gamma_g + 1) \right) = -G \frac{Mm}{r^2} \left( 1 - \frac{2GM}{rc^2} \right)^2 \hat{r} = -\gamma_g^3 \frac{GMm}{r^2} \hat{r} \]  

(25)

for a weak field this reduces to Newton’s law of gravitation. The gradient operator is non-relativistic because it communicates changes in the potential instantaneously, but the time-independence of \( \gamma_g \) mitigates the importance of this distinction. Equation (16) thus leads to a framework that conceptually links the seemingly very different notions of gravitational Energy in Newtonian theory and General Relativity: their relation to each other, in a sense, is like that between the branch of a hyperbolic curve and its reflection over the horizontal axis.

Alas, in relativity the formal symmetry between \( K \) and \( U \) is broken at all but infinitesimal scales: The symmetry is a consequence of the axiom and holds when the total energy of a test mass divided by its volume describes its energy density throughout, so that its energy and momentum can be represented by a four-vector. Since spherical fields diverge in direction, the test mass must therefore be infinitesimally small to achieve constant energy density. If it has finite extent, its total energy is no longer conserved, (16) becomes inapplicable and one must resort to a description in terms of the energy-momentum tensor instead, as prescribed by canonical GR. In Newtonian theory this is not a problem because test masses are taken to be point-like from the outset.

4 The Graviton Equation

Given (16), in the infinitesimal limit and for \( m = 0 \) in analogy to the photon equation, (18) reduces to

\[ E = p_g c \]  

(26)

Like a photon, this object has zero inertial mass but unlike a photon its momentum and energy are purely gravitational. It is therefore natural to identify this as a graviton, a particle of gravitation. This equation is not evident from within standard GR because GR views GPE as a property not of a test mass but of spacetime. Also, GR is most applicable when the formal symmetry is broken.

At first glance, the form of (26) suggests a particle which travels in space at the speed of light, consistent with our current ideas about the properties of gravitons. That requires, however, that in the limit of \( c, p_g \rightarrow p \), which means that the graviton’s momentum and energy are then kinetic. This may well turn out to be the correct interpretation, but at least in this author’s view, there is another possibility which should not be dismissed out of hand until compelled by empirical evidence to do so.

By the second possibility, \( p_g \) is interpreted in the limit of \( c \) the same as it is within the context of massive particles. In that case, the graviton’s momentum describes its position, which is at a horizon. This appears to be the case in
general since $v_g = c$ is generically singular. While the singularities associated with horizons can be transformed away, the underlying spacetime feature which gave rise to them under the choice of certain coordinate systems cannot, and the second interpretation directly associates this feature with the presence of gravitons. How does the graviton mediate the field under the second interpretation? Classically, at least, the mechanism is clear: Since $ds^2 = 0$ corresponds to an expanding spherical surface, given enough time the graviton’s presence is eventually communicated to any region in space by virtue of the structure of spacetime alone without requiring KE! It is clear that the second interpretation would force a significant reassessment of some of our current ideas about quantum gravity.

5 Conclusion

This paper demonstrated that under the assumption of a formal symmetry between relativistic kinetic and gravitational momentum there is a formal symmetry between $K$ and $U$ which clarifies the relation between the general relativistic and Newtonian conceptions of gravitational Energy, yielding for spherical fields the Newtonian description as an approximation, and which, while broken at non-infinitesimal scales, yields a relativistic equation for the graviton. As in the photon’s case, it does not admit definite energy and momentum values, but it offers one the very few pieces of evidence currently available that a quantum theory of gravity exists, even if such a theory may differ from our current expectations of what it should look like.

References

[1] A Zee Fearful Symmetry: The Search for Beauty in Modern Physics Princeton University Press, Princeton 2nd Ed., 2007