How to simulate global cosmic strings with large string tension

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Abstract. Global string networks may be relevant in axion production in the early Universe, as well as other cosmological scenarios. Such networks contain a large hierarchy of scales between the string core scale and the Hubble scale, $\ln(f_a/H) \sim 70$, which influences the network dynamics by giving the strings large tensions $T \simeq \pi f_a^2 \ln(f_a/H)$. We present a new numerical approach to simulate such global string networks, capturing the tension without an exponentially large lattice.

Keywords: axions, Cosmic strings, domain walls, monopoles, dark matter theory

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1 Introduction

Cosmic strings are extended solitonic objects which arise in field theories when a symmetry
G is broken, \( G \to H \), and the quotient space \( G/H \) has nontrivial \( \pi_1 \) homotopy \([1–3]\). The simplest example is when a complex scalar field \( \varphi = \frac{\varphi_r + i \varphi_i}{\sqrt{2}} \) possesses a U(1) symmetry, \( \varphi \to e^{i \theta} \varphi \), but the potential leads to spontaneous symmetry breaking,\(^1\)

\[
- \mathcal{L} = \partial_\mu \varphi^* \partial^\mu \varphi + \frac{m^2}{8f_a^2} \left( 2 \varphi^* \varphi - f_a^2 \right)^2 .
\]

(1.1)

Here \( m \) is the mass of the radial (Higgs) excitation and \( f_a \) is the vacuum value of the scalar
field. Under this Lagrangian, the vacuum state is of form \( \varphi = \sqrt{2} f_a e^{i \theta} \), and the arbitrary
angle \( \theta \) breaks the U(1) symmetry completely. If the field initially makes this choice of sym-
metry breaking direction randomly and independently at widely separated points in space,
then the initial conditions generically contain string type defects (the Kibble mechanism \([1] \)). Qualitatively, the same lessons are true if \( \varphi \) is charged under a U(1) gauge symmetry. We
will refer to the case with gauge symmetry as a theory of local or abelian strings, and the theory where the U(1) symmetry is a global symmetry as a theory of global or scalar-only
strings.

If a network of such defects exists in the Universe, it could influence the development
of cosmic structure \([4–6]\). A more physically interesting scenario, in light of strong limits on
the role of strings in cosmic structure \([7–11]\), is the possibility that the QCD axion \([12, 13]\)
makes up the Dark Matter of the Universe \([14–16]\). The QCD axion’s Lagrangian is the same
as eq. (1.1), with the addition of a small temperature-dependent explicit symmetry breaking
term \( \chi(T) (\cos[\text{Arg} \varphi] - 1) \), which becomes important around the QCD scale. If the axion
field starts out with randomly different values for the symmetry-breaking angle \( \theta \) at different
space points, which is plausible and maybe even likely \([17]\), then there would be an initial
network of (axionic) global cosmic strings. This network is destroyed around the QCD scale
when the potential’s tilt becomes important, but it could play a dominant role in establishing
the density of axions produced \([18]\).

If we could understand the efficiency of axion production in the early Universe, then
assuming the axion makes up the dark matter, we could predict the axion mass \([17]\). The
problem is that we don’t understand the string network evolution well enough. There is no

\(^1\)We use natural units and \([–+++]\) metric signature.
consensus in the literature for the efficiency of axion production [19–36]. Recent large-scale numerical simulations [36, 37] have not resolved this problem, because no simulation to date can correctly treat the tension of a global string. As we will now explain, numerical simulations of global string networks typically study networks where the strings have a tension $O(10)$ times smaller than the physically relevant value. This may be dramatically misrepresenting the density, longevity, and role of the strings in these simulations, and therefore the amount of axions produced [37].

To understand the problem, consider the field solution for a string under the Lagrangian of eq. (1.1). Choosing the string to lie along the $z$ axis in polar coordinates ($z, r, \phi$), the string solution is

$$\varphi(z, r, \phi) = e^{i\phi} f(mr) f_a \sqrt{2}, \quad (1.2)$$

$$f''(x) = - \frac{xf'(x) + f(x)}{x^2} - \frac{f(x)(1 - f(x)^2)}{2}, \quad (1.3)$$

$$f(0) = 0, \quad \lim_{x \to \infty} f(x) = 1.$$  

Here we have shown the differential equation and boundary values which the radial function $f(mr)$ should obey. This solution should be valid out to a value of $r$ of order the distance to the next string, which is parametrically $Hr \sim 1$ (with $H$ the Hubble parameter). In terms of the parameter $x = mr$, this is a value of $x \sim m/H \gg 1$. For instance, for an axion near the QCD scale, $H \sim \Lambda^2_{\text{QCD}}/m_{\text{pl}} \sim 10^{-19}$ GeV while $m \sim f_a \sim 10^{11}$ GeV (for a typical estimate of $f_a$ [38]). To see why this is a problem for simulations, we estimate the energy per length, or tension $T$, stored in the string:

$$T = \int_0^{2\pi} d\phi \int_0^{\sim H^{-1}} r \, dr \left[ \frac{1}{2} |\partial_r \varphi|^2 + \frac{1}{2r^2} |\partial_\phi \varphi|^2 \right] \quad (1.4)$$

$$\simeq 2\pi \int_{\sim m^{-1}}^{\sim H^{-1}} r \, dr \frac{1}{2r^2} f_a^2 \simeq \pi f_a^2 \ln \frac{m}{H} \equiv \pi f_a^2 \kappa.$$  

The string tension is logarithmically small-$r$ divergent, because the derivative in the $\phi$ direction is proportional to $1/r$. Therefore the tension contains a logarithmically large factor $\ln(m/H)$, which for the values quoted above is $\ln(10^{30}) \simeq 70$. If the goal is to study global strings playing a role in structure formation in the modern Universe, and they arise from a GUT or other high-scale theory, then the logarithm may be more like $\ln(10^{15} \text{ GeV} \times 10^{10} \text{ year}) \sim 113$. We have named the magnitude of this logarithm $\kappa$.

On the other hand, existing numerical simulations of global string networks rely on modifying eq. (1.1) to incorporate Hubble expansion, in comoving coordinates and conformal time, and solving it numerically as a function of (conformal) time on a spatial lattice. To properly treat the string defects, it is necessary that the lattice resolves the string cores, that is, the lattice spacing $a$ must satisfy $ma \lesssim 1$. At the same time, the lattice box-size must be larger than the typical inter-string spacing, and is typically larger by a factor of at least a few. Therefore, in the numerical simulation, the ratio of string separation-to-core is constrained to be at most a few hundred, and the logarithm of interest is at most 5 or 6. In other words, in existing simulations [34, 37], the global strings have a string tension at least an order of magnitude too small.

Does this matter? Probably it does. Global string networks differ from the much better-studied local string networks [20, 39–43] in two important ways:
1. There is a massless field coupled to the string, namely the Goldstone boson mode $\theta$ (the phase of $\varphi$). The strings can radiate away their energy to this field. The strength of the interaction is governed by $f_2^2$. Radiation should make it easier for strings to lose energy, leading to a less-dense network of smoother strings.

2. The massless field also communicates inter-string forces. These could help the strings to find each other and annihilate, again leading to a smaller string density. The strength of the interactions is again governed by $f_2^2$.

These effects compete against string-tension effects which scale as $T = \pi \kappa f_4^2$. Therefore the global-string effects are suppressed by a factor of $1/\kappa$, and are $\sim 10$ times less important in true string network evolutions than in those which we can simulate.

We know that these effects play a major role in string network evolution, because the density of the string network in scalar-only global string simulations is about a factor of 4–8 smaller than in local string network simulations [26, 37, 40–45]. Therefore, the correct inclusion of the high tension of string cores may lead to a substantial change in the string network dynamics and the network density. Indeed, in the limit $\kappa \gg 1$, we expect the global string networks to become indistinguishable from local strings. Probably $\kappa = 70$ is not enough to achieve this limit, but it should certainly give different string dynamics than $\kappa = 5$. In the context of axion production, the denser network with a larger string tension means there is more energy available for the production of axions. But the large tension makes the strings more robust to external forces, so the network should be more persistent once the potential tilts, and will take longer to annihilate away.

It is difficult to extrapolate the consequences of the high string tension on axion production [36]. Therefore, a method to simulate global string networks with $\kappa \sim 70$ is clearly well motivated. One of us recently presented such a method and implemented it in 2 space dimensions [46]. The results indicated that the produced string density is a strong function of $\kappa$, but the axion density rises rather modestly with string tension. However, 2 dimensions may show quite different physics than 3, and we have not (yet) been able to extend the method proposed in [46] to 3 space dimensions. Instead, in this paper we will propose another approach to simulate a global string network with enhanced string core tension. Essentially, we will present a model in which each global string has an abelian-Higgs string bound onto its core. The abelian Higgs string provides most of the string tension, and the long-range interactions are controlled by the global Goldstone fields. The ratio of tensions is tunable and can be chosen to make $\kappa \sim 70$ without much difficulty. The next section, section 2, explains the model which does this and justifies that the relevant infrared physics should be correct. We study the resulting string networks numerically in section 3, and present a discussion and conclusions in section 4. Very briefly, we find that as $\kappa$ is increased, the resulting string network grows denser and its properties (velocity, kinkiness) change from those of a global network towards those of an abelian-Higgs network. Applications to axion production are postponed to a follow-up paper.

2 How to get large string tensions in a global model

We are interested in the large-scale structure of string networks and the infrared behavior of any (pseudo)Goldstone modes they radiate. For these purposes it is not necessary to keep track of all physics down to the scale of the string core. Rather, it is sufficient to describe the desired IR behavior with an effective theory of the strings and the Goldstone modes
around them. This consists of removing the physics very close to the string core with an equivalent set of physics. It has long been known how to do this [20]. The string cores are described by the Nambu-Goto action [47–49], which describes the physics generated by the string tension arising close to the string core. The physics of the Goldstone mode is described by a Lagrangian containing the scalar field’s phase. And they are coupled by the Kalb-Ramond action [50, 51]:

$$L = L_{NG} + L_{GS} + L_{KR},$$

$$L_{NG} = \kappa \pi f_a^2 \int d\sigma \sqrt{y^2(\sigma)(1 - \dot{y}^2(\sigma))},$$

$$L_{GS} = f_a^2/2 \int d^3x \partial_\mu \theta \partial^\mu \theta,$$

$$L_{KR} = \int d^3x A_{\mu\nu} j^{\mu\nu},$$

$$H_{\mu\nu\alpha} = f_a \epsilon_{\mu\nu\alpha\beta} \partial^\beta \theta = \partial_\mu A_{\nu\alpha} + \text{cyclic},$$

$$j^{\mu\nu} = -2\pi f_a \int d\sigma (v^\mu y^\nu - v^\nu y^\mu) \delta^3(x - y(\sigma)).$$

Here $\sigma$ is an affine parameter describing the string’s location $y^\mu(\sigma, t)$, $v^\mu = (1, \dot{y}) = dy^\mu/dt$ is the string velocity, and $H_{\mu\nu\alpha}$ and $A_{\mu\nu}$ are the Kalb-Ramond field strength and tensor potential, which are a dual representation of $\theta$. Effectively $L_{NG}$ tracks the effects of the string tension, which we name $\kappa \pi f_a^2$, stored locally along its length. Next, $L_{GS}$ says that the axion angle propagates under a free wave equation, as expected for a Goldstone boson, and its decay constant is $f_a$. And $L_{KR}$ incorporates the interaction between strings and axions, also controlled by $f_a$. The interaction can be summarized by saying that the string forces $\theta$ to wind by $2\pi$ in going around the string (in the same sense that the $eJ_\mu A^\mu$ interaction in electrodynamics can be summarized by saying that it enforces that the electric flux emerging from a charge is $e$).

It should be emphasized that in writing these equations, we are implicitly assuming a separation scale $r_{\min}$; at larger distances from a string $r > r_{\min}$ we consider $\nabla \varphi$ energy to be associated with $\theta$; for $r < r_{\min}$ the gradient energy is considered as part of the string tension [20].

Any other set of UV physics which reduces to the effective description of eq. (2.1) would present an equally valid way to study this string network. Our plan is to find a model without a large scale hierarchy, such that the IR behavior is also described by eq. (2.1) with a large value for the string tension. Optimally, we want a model which is easy to simulate on the lattice with a spacing not much smaller than $r_{\min}$. Reading eq. (2.2) through eq. (2.6) in order, the model must have Goldstone bosons with a decay constant $f_a$ and strings with a large and tunable tension $T = \kappa \pi f_a^2$, with $\kappa \gg 1$. There can be other degrees of freedom, but only if they are very heavy (with mass $m \sim r_{\min}^{-1}$), and we will be interested in the limit that their mass goes to infinity. Finally, the string must have the correct Kalb-Ramond charge. Provided everything is derived from an action, this will be true if the Goldstone boson mode always winds by $2\pi$ around a loop which circles a string.

We do this by writing down a model of two scalar fields $\varphi_1, \varphi_2$, each with a U(1) phase symmetry. A linear combination of the phases is gauged; specifically, the fields are given electrical charges $q_1 \in \mathbb{Z}$ and $q_2 = q_1 - 1$ under a single U(1) gauge field. The orthogonal
phase combination represents a global U(1) symmetry which will give rise to our Goldstone bosons. The role of the gauge symmetry will be to attach an abelian-Higgs string onto every global string, which will enhance the string tension. The added degrees of freedom are all massive off the string, achieving our intended effective description. The model falls under the general rubric of “frustrated cosmic strings” [52], but our motivation and some specifics (particularly our initial conditions) are different.

Specifically, the Lagrangian is
\[
-L(\varphi_1, \varphi_2, A_\mu) = \frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \left| (\partial_\mu - iq_1 A_\mu) \varphi_1 \right|^2 + \left| (\partial_\mu - iq_2 A_\mu) \varphi_2 \right|^2 \\
+ \frac{m_1^2}{8v_1^2} (2\varphi_1^* \varphi_1 - v_1^2)^2 + \frac{m_2^2}{8v_2^2} (2\varphi_2^* \varphi_2 - v_2^2)^2 \\
+ \frac{\lambda_{12}}{2} (2\varphi_1^* \varphi_1 - v_1^2) (2\varphi_2^* \varphi_2 - v_2^2) .
\]

For simplicity we will specialize to the case
\[
\lambda_{12} = 0, \quad m_1 = m_2 = \sqrt{e^2(q_1^2v_1^2 + q_2^2v_2^2)} \equiv m_e .
\]

The model has 6 degrees of freedom; two from each scalar and two from the gauge boson. Symmetry breaking, \( \varphi_1 = e^{i\theta_1} v_1 \sqrt{2} \) and \( \varphi_2 = e^{i\theta_2} v_2 \sqrt{2} \), spontaneously breaks both U(1) symmetries and leaves five massive and one massless degrees of freedom. Specifically, expanding about a vacuum configuration, the fluctuations and their masses are
\[
v_1 \rightarrow v_1 + h_1 , \quad m = m_1 \quad (2.9) \\
v_2 \rightarrow v_2 + h_2 , \quad m = m_2 \quad (2.10) \\
A_\mu \neq 0 , \quad (\theta_1, \theta_2) \rightarrow (\theta_1, \theta_2) + \omega(q_1, q_2) , \quad m = \sqrt{e^2(q_1^2v_1^2 + q_2^2v_2^2)} \equiv m_e \quad (2.11) \\
(\theta_1, \theta_2) \rightarrow (\theta_1, \theta_2) + \theta \left( \frac{q_2}{q_1^2 + q_2^2}, \frac{-q_1}{q_1^2 + q_2^2} \right) \quad m = 0 .
\]

We see that the choices in eq. (2.8) have made all heavy masses equal.\(^2\) To clarify, note that a gauge transformation \( A_\mu \rightarrow A_\mu + \partial_\mu \omega \) changes \( \theta_1 \rightarrow \theta_1 + q_1 \omega \) and \( \theta_2 \rightarrow \theta_2 + q_2 \omega \). Therefore the linear combination of \( \theta_1, \theta_2 \) fluctuations with \( \delta \theta_1 \propto q_1 \) and \( \delta \theta_2 \propto q_2 \) is precisely the combination which can be shifted into \( A_\mu \) by a gauge change, and is therefore the combination which is “eaten” by the \( A \)-field to become the third massive degree of freedom. The remaining phase difference \( q_2 \theta_1 - q_1 \theta_2 \) is gauge invariant,
\[
q_2 \theta_1 - q_1 \theta_2 \rightarrow \omega q_2 (\theta_1 + q_1 \omega) - q_1 (\theta_2 + q_2 \omega) = q_2 \theta_1 - q_1 \theta_2 + 0 \omega \quad (2.14)
\]
and represents a global, Goldstone-boson mode.

\(^2\) We set \( \lambda_{12} = 0 \) so that the fluctuations in \( |\varphi_1| \) and \( |\varphi_2| \) are unmixed; our other choices ensure that all heavy fields have the same mass. We could consider other cases but we see no advantage in doing so if the goal is to implement the model on the lattice. The lattice spacing is limited by the inverse of the heaviest particle mass, while the size of the string core and the mass of extra degrees of freedom off the string will be set by the inverse of the lightest particle mass. So we get a good continuum limit with the thinnest strings, and therefore the best resolution of the network, by having all heavy masses equal.
The model breaks two U(1) symmetries, so we must describe strings in terms of a double $(m,n)$ representing the winding number of each scalar field around the string. In the absence of gauge interactions (for $e \to 0$) the $\varphi_1, \varphi_2$ fields would not interact and the tension of the $(1,1)$ string would be the sum of the tensions of a $(1,0)$ and a $(0,1)$ string. Therefore the $(1,1)$ string would be neutrally stable to splitting into $(1,0)$ and $(0,1)$ strings. Gauge interactions strongly change this, such that $(1,0)$ and $(0,1)$ strings strongly attract and $(1,1)$ strings are stable. To show this we analyze the form of a $(j,k)$ string. For the Ansatz

$$\sqrt{2}\varphi_1(r,\phi) = e^{ijr}f_1(r)v_1, \quad \sqrt{2}\varphi_2(r,\phi) = e^{ikr}f_2(r)v_2,$$

we find the equations of motion from eq. (2.7) are

$$g'' - \frac{g'}{r} = e^{2v_1^2f_1^2}f_1^2q_1(q_1g - j) + e^{2v_2^2f_2^2}f_2^2q_2(q_2g - k), \quad (2.16)$$

$$f_1'' + \frac{f_1'}{r} = f_1^2(j - q_1g)^2 + \frac{m_1^2}{2}f_1(f_1^2 - 1), \quad (2.17)$$

$$f_2'' + \frac{f_2'}{r} = f_2^2(k - q_2g)^2 + \frac{m_2^2}{2}f_2(f_2^2 - 1). \quad (2.18)$$

Here $f_1, f_2$ represent the progress of the two scalar fields towards their large-radius asymptotic vacuum values, while $2\pi g(r)$ is the magnetic flux enclosed by a loop at radius $r$, which trends at large $r$ towards the total enclosed magnetic flux. The large-$r$ behavior is well behaved only if $f_1 \to 1, f_2 \to 1$, and

$$\lim_{r \to \infty} g(r) = \frac{jq_1v_1^2 + kq_2v_2^2}{q_1^2v_1^4 + q_2^2v_2^4} = \frac{1}{2\pi} \left(\text{enclosed magnetic flux}\right). \quad (2.19)$$

The magnetic flux is therefore a compromise between the value $j/q_1$, which cancels large-distance gradient energies for the first field, and $k/q_2$, which cancels large-distance gradient energies for the second field. Unless $q_2j - q_1k = 0$, the gradient energies are not canceled at long distance. Indeed, we can understand the difference $(q_2j - q_1k)$ as the global (axionic) charge of the string. The gradient energy at large distance is given by

$$T \simeq 2\pi \int r \, dr \left( |D_\phi \varphi_1|^2 + |D_\phi \varphi_2|^2 \right)$$

$$\simeq \pi \int r \, dr \left( \frac{v_1^2}{r^2} (j - q_1g)^2 + \frac{v_2^2}{r^2} (k - q_2g)^2 \right),$$

$$\simeq \pi \int \frac{dr}{r} \frac{v_1^2v_2^2(jq_2 - q_1k)^2}{q_1^2v_1^4 + q_2^2v_2^4}, \quad (2.20)$$

which is proportional to the squared global charge of the string. Naming $q_{\text{global}} = jq_2 - kq_1$, comparing eq. (1.4) with eq. (2.20), we identify the Goldstone-mode decay constant as

$$f_a^2 = \frac{v_1^2v_2^2}{q_1^2v_1^4 + q_2^2v_2^4}. \quad (2.21)$$
Because $q_1$ and $q_2$ are of the same sign, the resulting large-distance energy is smaller for the $(1,1)$ string than for the sum of the $(1,0)$ and $(0,1)$ strings, and therefore there is an attractive interaction between $(1,0)$ and $(0,1)$ strings, which like to bind into a $(1,1)$ string. Alternatively we could say that the global charge of the $(1,0)$ string is $q_2$ and the $(0,1)$ string is $-q_1$, so they are strongly attracted by the Goldstone-mediated interaction and bind into a $(1,1)$ string.

For a more intuitive explanation, consider figure 1. It shows the set of possible phases $(\theta_1, \theta_2)$ for the two scalar fields, in the case $(q_1, q_2) = (4,3)$. The figure includes a dotted line to indicate which phase choices are gauge-equivalent. Moving along the dotted line corresponds to changing the gauge, or moving through space along a gauge field; a vector potential of the right size can cancel a gradient energy along this field direction. The orthogonal direction, which is unaffected by a gauge field, is the global (axion) field direction. A change in this direction from one blue dotted line to the next represents a full $2\pi$ rotation in the (axial) Goldstone direction, which explains the value of $f_a$ found in eq. (2.21). Figure 2 then shows how gradient energies behave in a $(1,0)$ or $(1,1)$ string. As the scalar field value varies around a loop, the gradient energy in the dotted-blue direction is partly canceled by $A_\phi$ gauge field. The extent of cancellation depends on the enclosed flux. As one goes from a small loop within the string core to a large loop outside the core, more and more of the scalar gradient along the dotted-line direction is canceled by the enclosed flux. For the innermost loop there is no enclosed flux, and the gradient energy is given by the distance between the point $(\theta_1, \theta_2) = (0,0)$ to the point $(2\pi,0)$ or $(2\pi,2\pi)$ for the $(1,0)$ or the $(1,1)$ string respectively. For a loop enclosing the entire flux, all gradient energy arising from the gauge-direction is canceled. This nearly completely removes the gradient energy for the $(1,1)$ string, but the reduction is modest for the $(1,0)$ string. Therefore the $(1,1)$ string has a small gradient energy outside the core, representing a small residual coupling to the (axion) Goldstone field, but the $(1,0)$ string has a large gradient energy and a large coupling.

For our simulation to correspond with the global string model, it should have Goldstone modes between a network of strings with charge-1 under the Goldstone fields. That is, we want a network of $(1,1)$ strings only, with no other string types. Because the $(1,1)$ string is
Figure 2. Left: cross-section of a string, showing the magnetic field strength “bundle” and three possible loops one can take around the center of the string. Right: path through \((\phi_1, \phi_2)\) space taken along each loop, for a (1,0) string (top) and a (1,1) string (bottom). As more magnetic flux is enclosed, the component of \((\Delta \theta_1, \Delta \theta_2)\) along the gauge-direction is canceled, but the component in the “global” direction is not. This component is small for the (1,1) string.

stable, achieving this is simply a matter of choosing the right initial conditions. We choose \(\theta_1\) randomly and independently at each point, \(A_{\mu} = 0 = E_i = \dot{\phi}_1 = \dot{\phi}_2\) initially, and \(\theta_2 = \theta_1\). In this case only (1,1) strings are present initially, and the network never develops any other sort of string or structure. Since \(q_2\theta_1 - q_1\theta_2\) is initially random, there are no long-range correlations in the Goldstone field and the Kibble mechanism ensures a network of (1,1) strings, whose evolution should approach the scaling behavior of a network with a tension set by the (1,1) string tension and the Goldstone mode interaction strength found above. Our initial conditions also obey Gauss’ Law, which is then preserved by the evolution.

Now let us estimate the effective value of \(\kappa\), the ratio of the string tension to the string interaction via Goldstone modes, for such a (1,1) string. The energy of the string’s core is the energy of an abelian Higgs string with \(m_h = m_e\) and with \(f^2 = v_1^2 + v_2^2\), which is

\[
T_{\text{abelian}} \simeq \pi(v_1^2 + v_2^2).
\] (2.22)

The value of \(\kappa\) is therefore

\[
\kappa = \frac{T}{\pi f_a^2} \simeq \frac{v_1^2 + v_2^2}{v_1^2 v_2^2} = \frac{(v_1^2 + v_2^2)(q_1^2 v_1^2 + q_2^2 v_2^2)}{v_1^2 v_2^2} \rightarrow_{v_1=v_2} 2(q_1^2 + q_2^2).
\] (2.23)

This is not quite correct; the solution only coincides with the abelian-Higgs solution for large \(q_1 \gg 1\). For finite \(q_1\) we must compute the true solution, and account for the \(1/v^2\) tail of
energy arising from the long-distance Goldstone-mode content of the string. Therefore we solve eq. (2.16), eq. (2.17), eq. (2.18) numerically by multiparameter shooting to establish the string solution and its energy. Artificially separating the short and long distance energy contributions by choosing \( r_{\text{min}} = \pi/m \) and writing the energy-per-length stored in fields out to radius \( R \) as

\[
T_R = \int_0^R dT \frac{dT}{dr} \equiv (\bar{\kappa} + \ln(mR/\pi)) \pi f_1^2,
\]

(2.24)

(or equivalently, \( \bar{\kappa} = \lim_{R \to \infty} T_R/\pi f_1^2 - \ln mR/\pi \)), we find the values of \( \kappa \) shown in table 1. The table shows that eq. (2.23) is quite accurate, for our choice of \( r_{\text{min}} \). For future use, the table also records the small-distance behavior of \( f_1, f_2, \) and \( g \), defined as

\[
\begin{align*}
f_1(r) &= c_1 mr + O(r^3), \\
f_2(r) &= c_2 mr + O(r^3), \\
g(r) &= d(mr)^2 + O(r^4).
\end{align*}
\]

(2.25)

Note that all results in the table are for the case of equal masses and equal vacuum values; we could achieve \( \bar{\kappa} \) values intermediate between those shown by considering asymmetric vacuum values \( v_1 > v_2 \).

To summarize, the model of eq. (2.7) has an infrared description consisting of one Goldstone mode and strings. The strings have a tension \( \kappa \pi f_1^2 \) with tunable \( \kappa \) given by eq. (2.23). The Goldstone phase winds by \( 2\pi \) in circling the string, so the Kalb-Ramond charge of the string is correct. This description breaks down at a scale \( m_1 = m_2 \), which is both the scale setting the thickness of the string and the scale of massive excitations off the strings. The model can be implemented numerically with only a little more work than the abelian-Higgs model.

### 3 Numerical implementation and results

#### 3.1 Numerical implementation

For an FRW spacetime in comoving coordinates and conformal time, the action is

\[
S = \int_0^t dt \int d^3x \; t^k \mathcal{L}[\text{eq. (2.7)}]
\]

(3.1)

where \( k \) is determined by the expansion rate; \( k = 2 \) for radiation (which we will study), \( k = 4 \) for matter, etc.\(^3\) Our approach is to write a spacetime-lattice discretization of the action and

\[^3k\text{ is the power of the conformal time which appears in } g^{\mu\nu} \sqrt{-g}. \text{ For an FRW universe with equation of state } P = \omega \varepsilon, \text{ we have } k = 4/(1 + 3\omega). \text{ The potential terms would scale as } t^{2k} \text{ if we kept the masses fixed in physical} \]
to determine the update rule by extremization of this action. This leads automatically to a
leapfrog update rule. We use noncompact formulation of U(1), recording gauge fields \(A_i(x)\)
(which “live” on links) directly and computing link variables
\[
U_i(x) = \text{Pexp} \int_x^{x + a \hat{i}} -iA_i dl = e^{-iaA_i(x)}
\]  
when needed. In the following, when directly expressing the gauge field, we write \(aA_i\)
as simply \(A_i\), absorbing the factor of the lattice spacing. We use an \(a^2\) improved action, both
for the scalar fields and the gauge fields. To our knowledge this has not been done correctly
before in simulating abelian Higgs fields for cosmic string networks. Our implementation
is almost the same as a previous attempt to \(a^2\)-improve the abelian Higgs mechanism [53],
except that we correctly modify the electric field part, see below. The improved scalar
field “hopping” term is \((4/3)\) times a nearest-neighbor term minus \((1/12)\) a next-nearest
neighbor term,
\[
S_{\nabla \varphi} = \sum_{t=\delta n, a} \sum_{x=a \tilde{n}_t} \left[ \sum_{i=1,2,3} t^k \left( \frac{4|U_i^q(x,t)\varphi(x+a \hat{i},t) - \varphi(x,t)|^2}{3} \right. \right.
\]
\[
- \left. \frac{|U_i^q(x,t)U_i^q(x+a \hat{i},t)\varphi(x+2a \hat{i},t) - \varphi(x,t)|^2}{12} \right) 
\]
\[
- (t + \delta a/2)^k \frac{|U_0^q(x,t)\varphi(x,t + \delta a) - \varphi(x,t)|^2}{\delta^2}, \tag{3.3}
\]
with \(q\) the charge for the specific field considered, and \(\delta\) the ratio of temporal to spatial
discretization; we typically use \(\delta = 1/6\) which is adequate [37]. In practice we fix to temporal
gauge, \(U_0 = 1\), which is numerically convenient but not very relevant as long as we ask only
gauge invariant questions.

The noncompact magnetic field action is \((5/3)\) a square plaquette term minus \((1/12)\) a sum on rectangular plaquettes (the abelian version of the tree-level \(a^2\) improved or Symanzik
action [54, 55]),
\[
S_B = \sum_{t,x,i>j} \frac{5t^k}{6e^2} \left( A_i(x) + A_j(x+a \hat{i}) - A_i(x+a \hat{j}) - A_j(x) \right)^2
\]
\[
- \sum_{t,x,i\neq j} \frac{t^k}{24e^2} \left( A_i(x) + A_i(x+a \hat{i}) + A_j(x+2a \hat{i}) - A_i(x+a \hat{j}) - A_i(x+a \hat{j}) - A_j(x) \right)^2 
\]  
while the electric field action is
\[
S_E = - \sum_{t,x,i}(t+a\delta/2)^k \left[ \frac{2(A_i(x,t+\delta a) - A_i(x,t))^2}{\delta^2} \right. \right.
\]
\[
- \left. \frac{1}{24} \left( A_i(x,t+a \delta) + A_i(x+a \hat{i},t+a \delta) - A_i(x,t) - A_i(x+a \hat{i},t) \right)^2 \right]. \tag{3.5}
\]  
units. But since the masses are an artifice to regulate string thickness, we keep them fixed in lattice units and scale them in the same way as the other terms.
This last modification, explained in detail in [56], is necessary to make the evolution truly improved — for instance, without it the gauge boson dispersion relation has \( a^2 \) corrections. Unfortunately it causes the \( A \)-field update to be implicit. To see this, first define \( E_i(x,t) = A_i(x,t + a\delta) - A_i(x,t) \). Then \( S_E \) can be written as \( \sum_{x_i} (7/12) E_i^2(x) - (1/12) E_i(x) E_i(x+a\delta) \). Because the Lagrangian is not simply diagonal in the \( E \sim A \), there is a difference between the time derivative of \( A \) and the canonical momentum of \( A \). It is convenient to define the quantity \( P_i(x) = -(1/12) E_i(x-a\delta) + (7/6) E_i(x) - (1/12) E_i(x+a\delta) \), which in the continuous-time limit is the canonical momentum of the \( A \) field. Its time-update is simple, but the relation between \( P \) and \( E \) must be inverted to update the \( A \) field. Because the relation is nearly diagonal, this inversion can be done perturbatively and proves not to be a large numerical overhead.

(In ref. [56] this was the dominant cost, because the reference works with SU(2) where \( E \) is a gauge non-singlet and parallel transportation is involved in inverting the \( E\)-\( P \) relation.) Improvement increases the operation count by roughly a factor of 2.5, and most of the update effort is spent on the scalar field update. We have implemented the resulting equations of motion in \( c \) using MPI and AVX2, obtaining \( 2 \times 10^7 \) site updates per second on an i5 core duo (two physical cores), and with MPI and AVX512, obtaining \( 5 \times 10^8 \) site updates per second on a compute node with 2 64-core KNL Xeon Phi processors communicating through Infiniband. A 2048³ lattice fits in approximately 512G of memory. Relative to the simple complex-scalar model of eq. (1.1), the memory demand is 3.5x as large and the compute time is 6x larger on the Xeon Phi and 16x larger on the i5 core.

We identify the plaquettes pierced by a string using the gauge-invariant definition of Kajantie et al. [57], applied to one of the fields (we use \( \varphi_1 \)). We measure string velocity by a slight generalization of the method of ref. [37]; we use the small-\( r \) expansion of \( f_1(r) \):

\[
f_1(r) = c_1 mr + e_1 (mr)^3 + \ldots \quad e_1 = -\frac{c_1 (1 + 4 dq d)}{16},
\]

and we use the string velocity estimate that near the center of a string core, \([37]\)

\[
\gamma^2 v^2 = \frac{|\partial t \varphi|^2}{2 m^2 c_1^2 v^2} \left( 1 - \frac{4 e_1 \varphi^* \varphi}{c_1^2 v^2} \right) - \frac{4 e_1 (\varphi^* \partial_t \varphi + \varphi \partial_t \varphi^*)^2}{m^2 c_1^4 v^4} + O \left( \frac{\varphi^4 (\partial_t \varphi)^2}{m^2 v^6} \right),
\]

which should converge to the correct velocity in the small-\( a \) limit as \((ma)^4\). The values of \( c_1 \) and \( d \) are in table 1. For each plaquette pierced by a string, we average \( \gamma^2 v^2 \) over the plaquette’s four corners, and use this average to determine \( v, \gamma \). Finally, we interpolate the position within a plaquette where the string pierces it, by fixing to the gauge where each link is \( \pm 1/4 \) the value of the magnetic flux through the plaquette, and interpolating the \( \varphi_1 \) field to find its zero \([58]\). We construct strings as the series of straight segments connecting these interpolating points \([58]\). The overhead to identify and record strings is a small fraction of the numerical effort. We have compared the results using the other field \( \varphi_2 \) for string
identification, and consistently find string length and mean velocity to agree within 1%. We also check the average distance between a point on the $\varphi_2$ string network and the nearest point on the $\varphi_1$ string network: for $q_1 = 4$ and $m_1 = 512$ we find an average distance of 0.065/m. Therefore each scalar describes essentially the same string network; in particular our procedure for getting only (1,1) type strings is successful.

### 3.2 Results

We will present some very preliminary results obtained on 2048 $\times$ 2016 $\times$ 2000 lattices with relatively coarse spacing $m\alpha = 1$. We compare the 2-scalar model at a few values of $q_1$ with the abelian-Higgs model on the one hand, and a scalar-only global model on the other. In each case we take $m\alpha = 1.0$, except for the scalar-only model, which has thicker strings relative to the mass ($c_1$ of eq. (2.25) is $c_1 = 0.412$) and which is therefore less sensitive to the mass value. Therefore for the scalar-only case we used $m\alpha = 1.5$. We have not extrapolated to finer lattice spacing. Our preliminary study indicates that doing so will increase the network density by a few percent, and increase string velocities by closer to 10 percent. We intend to make a more complete study, including continuum extrapolation, in the future. Our main goals are to show that the numerics are relatively straightforward to conduct, and that many properties of the string networks evolve smoothly from their behavior in the global theory with low tension towards the behavior observed in local (abelian-Higgs) networks as the string tension is increased. But not at the same rate; the string velocity shifts rather quickly, while the network density takes a much larger tension to become more abelian-Higgs-like. We intend to make a more comprehensive study in the future.

We will focus on the density of the string network, the mean string velocity, and how “kinky” the strings are. The general expectation is that the network should evolve towards a scaling solution, where the density of strings and other string properties scale with the system age (see for instance [59, 60]). We introduce the scaled network density $\xi$ and mean inter-string distance $L_{\text{sep}}$, defined as

$$L_{\text{sep}}^{-2} \equiv V^{-1} \int_{\text{all string}} \gamma dl, \quad \xi \equiv \frac{t^2}{(1 + k/2)^2 L^2}. \tag{3.8}$$

Here $\int \gamma dl$ is the invariant string length, $V$ is the space volume and $t$ the time, all in comoving conformal coordinates. The factor $(1 + k/2)^{-2}$ converts from conformal-time based to physical-time based normalization, which is common usage in some of the literature. Note that different authors define the string density in different ways, often with the same symbol. For instance, a recent study of abelian-Higgs networks\(^6\) [45, 61] uses the symbol $\xi$ to represent the quantity we call $L_{\text{sep}}$.

We will also consider the orientation autocorrelator of the string; defining $\mathbf{s}$ as the unit tangent vector of the string, this is defined as\(^7\)

$$D(\Delta l) = \frac{\int_{\text{all string}} \mathbf{s}(l) \cdot \mathbf{s}(l + \Delta l) \, dl}{\int_{\text{all string}} dl}; \tag{3.9}$$

more details will be given in a future publication [58].

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\(^6\)Our abelian-Higgs simulations are generally in good agreement with this reference, but the reference is much more systematic, and achieves higher statistics and better extrapolation towards the continuum.

\(^7\)Note that we have used rest-frame string lengths without $\gamma$ factors in this definition. This can be improved but we have not yet done so.
Our goal is to understand the scaling behavior of string networks, and how it depends on $\kappa$. In practice we will never precisely observe scaling, and in some cases we may be quite far away. Three effects can cause scaling violations at a finite time $t$:

1. Scaling occurs when the string core has negligible size compared to the inter-string spacing. Therefore it is difficult for the network to show good scaling at early times, before $mt \gg 1$.

2. The string tension does have a residual contribution from the Goldstone field, which increases as the string separation increases with time. Since the mean inter-string spacing is expected to grow linearly with time, we expect $\kappa(t) \simeq \kappa(t_0) + \ln(t/t_0)$.

3. Initial conditions may start the network out as denser or thinner than the scaling form, and it takes time for the network to adjust.

The severity of the first problem should scale as $(mt)^{-1}$ and therefore becomes less severe as we achieve larger lattices which can be run to later times\(^8\) (and if we can use coarser grids, $ma \sim 1$, which is why we use an improved action). But the severity of this problem also depends on how "kinky" the strings are — how many places the string has an abrupt discontinuity or rapid change in its tangent vector (a kink), and how “sharp” these kinks are. After all, such kinks can occur on much smaller scales than the inter-string spacing, and we expect that any kink structure on scales smaller than roughly the string thickness will be lost to heavy-mode radiation, which is unphysical from the viewpoint of the thin-string limit. Kinks should tend to dissipate through Goldstone mode radiation. Parametrically, because radiation involves $\pi f_s^2$ while the energy available is set by the tension $\pi \kappa f_s^2$, string features with length scale $l$ should dissipate in time $t \sim l/\kappa$. Therefore smaller-$\kappa$ strings should lose their short-scale structure, and the necessary separation-to-core hierarchy should scale roughly linearly with $\kappa$. So larger-$\kappa$ networks should demand larger lattices and later times before correct scaling sets in. In the abelian-Higgs limit, it is not obvious if any lattice spacing is sufficient to capture true scaling.

The second problem is most severe when the $\kappa$ contribution from our abelian degrees of freedom is small; for larger $q$-values the abelian contribution overwhelms any small scale-dependence in the Goldstone contribution. Therefore this problem is mostly an issue for purely global simulations, and perhaps for $(q_1, q_2) = (2, 1)$.

The issue of initial conditions requires that, if we want to say with any confidence that we are close to scaling, we need to see a variety of network initial conditions, with different initial string densities, converge to a common string density. As stated above, we take as initial conditions that $A_i = 0$ and $\varphi_1 = v_1 e^{i \theta_1(x)}$, $\varphi_2 = v_2 e^{i \theta_1(x)}$ (same phase as $\varphi_1$) at time $t = 0$ — actually at time $t = a$ with $a$ the lattice spacing. This choice leads however to a string network which starts out quite dilute compared to the scaling solution. Therefore we modify the evolution by setting $k = k_{\text{start}}$ until some time $t_{\text{start}}$. A large value of $k_{\text{start}}$ represents strongly overdamped evolution, leading to slow string motion and a much denser starting network, without excessive fluctuations. By varying $k_{\text{start}}$ and $t_{\text{start}}$, we can vary the early-time density of the string network.

Figure 3 shows how the network density varies with the string tension. For each type of network, we have run two groups of simulations, one which starts with a somewhat underdense

\(^8\)We stop the evolution at $t = L/2$, $L$ the box length, to ensure that the lattice’s periodicity is invisible under causal dynamics.
network \((k_{\text{start}} = 20 \text{ and } t_{\text{start}} = 40)\) and one which starts with a somewhat overdense network \((k_{\text{start}} = 50 \text{ and } t_{\text{start}} = 80\) except for the abelian-Higgs case, where we used \(k_{\text{start}} = 80\) and \(t_{\text{start}} = 100\). In every case the different densities converge towards each other with time. For the pure scalar simulations and the case with the smallest added tension, the curves become approximately flat, indicating good approach to scaling. For higher tensions and for the abelian Higgs case, the network density is still rising at the last time studied. Even in the cases where the network started overdense, the density reaches a minimum and then rises. Therefore we have no evidence that these cases have achieved scaling, and the true scaling network densities are presumably higher than the last value in the plot. However the good scaling for \((q_1, q_2) = (2, 1)\), and the expectation that the required \(mt\) value should grow linearly with \(\kappa\), suggest that the other global strings considered may at least be nearing their scaling regimes.

While many authors consider only the length of “long” strings, neglecting short loops, our results are based on summing over all string lengths. However, in all of our simulations, small loops make up a relatively small fraction of the total string length: for each simulation type, we find that loops satisfying \(\int \gamma dl < 2\pi \Lambda_{\text{sep}}\) make up less than 10% of the total string length. This is expected for global strings, since loops generally start small, frequently self-intersect, and radiate away power on a time scale at most \(\kappa\) larger than the string curvature-radius scale. The same is not true in the abelian Higgs model, where, at least in the small core-thickness limit, non-selfintersecting string loops should be persistent.

For the abelian Higgs case, the network density and particularly the low abundance of string loops is consistent with previous field-theoretical findings [45, 61] but is in contrast
to expectations from Nambu-Goto simulations [40, 41, 44]. The largest scale Nambu-Goto simulation, counting only long strings, finds\(^9\) \(\xi \simeq 11\) [44], about a factor of 2 larger than the value we find for the abelian Higgs model. This simulation also finds a large contribution from string loops. This difference probably arises because the Nambu-Goto simulations are sensitive to very short-scale phenomena which abelian-Higgs field simulations cannot resolve, even with the largest field theoretical simulations to date [45]. For instance, the largest string loops found to be of relevance in ref. [44] had invariant length \(l < t/20\) with \(t\) the comoving system age. Since the loops are moving and noncircular, the curvature radius is at least another order of magnitude smaller. But if a string loop ever develops points with curvature radius comparable to the string core thickness, these points will strongly radiate massive modes, leading to loss of string length. Therefore the required scale hierarchy needed to study abelian-Higgs string networks is enormous. On the other hand, we emphasize that the same should not be true of global networks. Small global-string loops should radiate away their energy quickly, and long global strings should lose their short-scale structure, so the required hierarchy is not as severe. It appears that we see the scaling solution for \((q_1, q_2) = (2, 1)\) strings, so if \(m_t \propto \kappa\) then scaling should be fairly close at hand for the larger \(\tilde{\kappa}\) values we study.

Figure 4 shows the mean string velocity, squared velocity, and gamma factor, each defined as

\[
\frac{\langle v \rangle}{\langle v^2 \rangle} \equiv \frac{\int \gamma dl \times \left\{ \begin{array}{c} v \\ v^2 \\ \gamma \end{array} \right\}}{\int \gamma dl}.
\]

For the overdense network the mean values are always slightly higher than for the underdense network; the difference exceeds the statistical error in either measurement. Therefore, rather than statistical error bars, we have plotted the mean values of the overdense and underdense network for the latest time we achieved, \(m_t = 1024\). We stress that the velocity measurements are not extrapolated to the continuum (in the sense of small \(m_a\)); preliminary indications are that all values will rise when we do so. However the qualitative feature, that the scalar-only theory has a higher velocity and that it then comes down rather quickly towards the abelian-Higgs value as the string tension is increased, appears to be robust.

Figure 5 shows the string-direction autocorrelator for each string type. The \(x\)-axis is a separation distance along a string, normalized by the system age. That is, an \(x\)-axis value of \(\ell/t = 0.2\) means that we consider all pairs of points \((x, y)\) separated along a string by \(\int_x^y dl = 0.2t\). The \(y\)-axis is the dot product of their unit tangent vectors. We see that the strings with a larger coupling to Goldstone modes are systematically straighter (larger correlator) than the strings with smaller or no Goldstone coupling. The effect is especially clear at very small separation. A string with everywhere differentiable tangent vector would have vanishing slope at \(\ell/t = 0\), while a string with perfectly sharp kinks would have a nonzero slope at \(\ell/t = 0\) set by the density and angle of the kinks. This kinky behavior is consistent with the abelian-Higgs curve, but not with the scalar-only curve. Enhanced-tension strings lie in between, though closer to the abelian-Higgs case. We expect that the abelian-Higgs string and the highest-tension 2-field strings are not yet displaying their large-\(m_t\) asymptotic behavior.

\(^9\)Their \(\gamma\) is our \(L_{sep}/t\), and therefore \(\xi = 1/(4\gamma^2)\) relates our and their notations. They find \(\gamma = 0.15\) which becomes \(\xi = 11\).
Figure 4. Mean gamma factor, velocity and squared velocity. The upper and lower bar indicates the mean values for the overdense network (upper) and for the underdense network (lower).

Figure 5. Autocorrelation of the string-direction for different string tensions.

4 Discussion and conclusions

We have presented a new algorithm for simulating global string networks, which makes it possible to consider networks with a large value of $\kappa$, the ratio of the string tension to the coupling to Goldstone modes. This makes it possible to simulate global string networks with a very large hierarchy between the Hubble scale and the microscopic string core scale,
without actually resolving the hierarchy numerically. Preliminary numerical studies find that high-tension global strings behave similarly to abelian Higgs networks, for the lattice sizes we have achieved. In particular we very clearly see that the density of string networks smoothly increases from the value observed in scalar-only simulations towards the value observed in abelian Higgs simulations, as the string tension is increased. Physically, we expect that the needed lattice resolution to properly capture small scale string structure should grow linearly with $\kappa$, meaning that large lattices are needed. But with our approach, the lattice size need only grow as the logarithm of $f_a/H$, not with $f_a/H$ itself.

Our approach has clear applications to the physics of axion production in the early Universe. We have shown that existing simulations must underestimate the network density, because they fail to capture the larger string tension. Roughly, existing axion simulations are comparable to the scalars-only curve in figure 3, while the physical tension is somewhat above the (4,3) line in the figure — which itself is probably not yet scaling, but should display a still higher network density. Therefore the string density in simulations of axion production is underestimated by at least a factor of 3. This could certainly be important in establishing axion production. The large difference in string tension could also be important. Therefore one should revisit the question of axion production from axionic string network breakup, using our approach. We intend to do so in the near future. It might also be interesting to revisit the study of the possible role of global cosmic strings in cosmology.

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