POLYSYMPLECTIC HAMILTONIAN FORMALISM AND SOME QUANTUM OUTCOMES

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Covariant (polysymplectic) Hamiltonian field theory is formulated as a particular Lagrangian theory on a polysymplectic phase space that enables one to quantize it in the framework of familiar quantum field theory.

1 Introduction

The Hamiltonian counterpart of first-order Lagrangian formalism on a fibre bundle \(Y \to X\) has been rigorously developed since the 1970s in the multisymplectic, polysymplectic and Hamilton – De Donder variants (see [6, 7, 9, 10, 12, 16, 17, 18] and references therein). If \(X = \mathbb{R}\), we are in the case of time-dependent mechanics [19].

The relations between multisymplectic, polysymplectic, Hamilton – De Donder and Lagrangian formalisms on \(Y \to X\) are briefly the following.

- The multisymplectic phase space is the homogeneous Legendre bundle
  \[ Z_Y = T^*Y \wedge (\wedge^{n-1} T^*X), \]
  coordinated by \((x^\lambda, y^i, p^\lambda_i, p)\). It is endowed with the canonical exterior form
  \[ \Xi_Y = p\omega + p^\lambda_i dy^i \wedge \omega^\lambda, \]
  whose exterior differential \(d\Xi_Y\) is the canonical multisymplectic form, which belongs to the class of multisymplectic forms in the sense of Martin [4, 20].

- The homogeneous Legendre bundle (1) is the trivial one-dimensional bundle
  \[ \zeta : Z_Y \to \Pi \]
  over the Legendre bundle
  \[ \Pi = \wedge^n T^*X \otimes V^*Y \otimes TX = V^*Y \wedge (\wedge^{n-1} T^*X), \]
coordinated by \((x^\lambda, y^i, p^\mu_i)\). Being provided with the canonical polysymplectic form

\[ \Omega = dp^\lambda_i \wedge dy^i \wedge \omega \otimes \partial_\lambda, \]  

\(5\)

the Legendre bundle \(\Pi\) is the momentum phase space of polysymplectic Hamiltonian formalism. A Hamiltonian \(\mathcal{H}\) on \(\Pi\) is defined as a section \(p = -\mathcal{H}\) of the bundle \(\zeta\) (3). The pull-back of \(\Xi_Y\) onto \(\Pi\) by a Hamiltonian \(\mathcal{H}\) is a Hamiltonian form

\[ H = \mathcal{H}^* \Xi_Y = p^\lambda_i dy^i \wedge \omega^\lambda_i - \mathcal{H} \omega. \]  

\(6\)

In the case of mechanics, \(Z_Y = TY\) and \(\Pi = Vy\) are the homogeneous momentum phase space and the momentum phase space of time-dependent mechanics on \(Y \rightarrow \mathbb{R}\), respectively. Accordingly, \(H\) (6) is the well-known integral invariant of Poincaré–Cartan.

From the mathematical viewpoint, an essential advantage of a multisymplectic formalism is that the multisymplectic form is an exterior form. In physical applications, one however meets an additional variable \(p\) which is the energy one in homogeneous time-dependent mechanics.

It should be emphasized that multisymplectic and polysymplectic formalisms need not be related to Lagrangian one. In contrast with them, Hamilton – De Donder formalism necessarily describes Lagrangian systems as follows.

Let us consider a first order Lagrangian

\[ L = \mathcal{L} \omega : J^1 Y \rightarrow \wedge^n T^* X, \quad \omega = dx^1 \wedge \cdots \wedge dx^n, \quad n = \dim X, \]  

\(7\)

on \(J^1 Y\), the Euler–Lagrange equations

\[ (\partial^\lambda_i - d_\lambda \partial^\lambda_i) \mathcal{L} = 0, \]  

\(8\)

and the Poincaré–Cartan form

\[ H_L = L + \pi^\lambda_i \theta^i \wedge \omega_\lambda, \quad \pi^\lambda_i = \partial^\lambda_i \mathcal{L}, \quad \omega_\lambda = \partial_\lambda \omega. \]  

\(9\)

The latter is both the particular Lepagean equivalent of a Lagrangian \(L\) (7) and that of the Lagrangian

\[ \mathcal{T} = \hat{h}_0(H_L) = (\mathcal{L} + (\hat{y}^i_\lambda - y^i_\lambda) \pi^\lambda_i) \omega, \quad \hat{h}_0(dy^i) = \hat{y}^i_\lambda dx^\lambda, \]  

\(10\)

on the repeated jet manifold \(J^1 J^1 Y\). Its Euler–Lagrange equations are the Cartan equations

\[ \partial^\lambda_i \pi^\mu_j (\hat{y}^i_\mu - y^i_\mu) = 0, \quad \partial_i \mathcal{L} - d_\lambda \pi^\lambda_i + (\hat{y}^i_\lambda - y^i_\lambda) \partial_i \pi^\lambda_j = 0. \]  

\(11\)

- The Poincaré–Cartan form (9) yields the Legendre morphism

\[ \widehat{H}_L : J^3 Y \rightarrow Z_Y, \quad (p^\mu_i, p) \circ \widehat{H}_L = (\pi^\mu_i, \mathcal{L} - \pi^\mu_i y^i_\mu), \]
of $J^1Y$ to the homogeneous Legendre bundle $Z_Y$ (1). Let its image $Z_L = \widehat{H}_L(J^1Y)$ be an imbedded subbundle $i_L : Z_L \hookrightarrow Z_Y$ of $Z_Y \to Y$. Then it is provided with the pull-back De Donder form $\Xi_L = i_L^*\Xi_Y$. The Hamilton – De Donder equations for sections $\sigma$ of $Z_L \to X$ are written as

$$\sigma^*(u \rfloor d\Xi_L) = 0,$$

where $u$ is an arbitrary vertical vector field on $Z_L \to X$. Let the Legendre morphism $\widehat{H}_L$ be a submersion. Then one can show that a section $\sigma$ of $J^1Y \to X$ is a solution of the Cartan equations (11) iff $\widehat{H}_L \circ \sigma$ is a solution of the Hamilton–De Donder equations (12). In a general setting, one can consider different Lepagean forms in order to develop Hamilton – De Donder formalism [15, 16].

The relation between polysymplectic Hamiltonian and Lagrangian formalisms is based on the fact that any Lagrangian $L$ yields the Legendre map

$$\widehat{L} : J^1Y \longrightarrow \Pi, \quad p^i_\mu \circ \widehat{L} = \partial^i_\mu L,$$

whose image $N_L = \widehat{L}(J^1Y)$ is called the Lagrangian constraint space. Conversely, any Hamiltonian $\mathcal{H}$ defines the Hamiltonian map

$$\widehat{H} : \Pi \longrightarrow J^1Y, \quad y^i_\lambda \circ \widehat{H} = \partial^i_\lambda \mathcal{H}.$$

A Hamiltonian $\mathcal{H}$ on $\Pi$ is said to be associated to a Lagrangian $L$ on $J^1Y$ if it satisfies the relations

$$p^i_\mu = \partial^i_\mu L(x^\mu, y^i, \partial^j_\lambda \mathcal{H}),$$

$$p^i_\mu \partial^i_\mu \mathcal{H} - \mathcal{H} = L(x^\mu, y^i, \partial^j_\lambda \mathcal{H}).$$

If an associated Hamiltonian $\mathcal{H}$ exists, the Lagrangian constraint space $N_L$ is given by the coordinate relations (15) and $\widehat{L} \circ \widehat{H}$ is a projector of $\Pi$ onto $N_L$.

Lagrangian and polysymplectic Hamiltonian formalisms are equivalent in the case of hyperregular Lagrangians. The key point is that a degenerate Lagrangian admits different associated Hamiltonians, if any. At the same time, there is a comprehensive relation between these formalisms in the case of almost-regular Lagrangians. Recall that a Lagrangian $L$ is called almost-regular if the Lagrangian constraint space is a closed imbedded subbundle $i_N : N_L \to \Pi$ of the Legendre bundle $\Pi \to Y$ and the surjection $\widehat{L} : J^1Y \to N_L$ is a fibred manifold possessing connected fibres. In particular, the Poincaré–Cartan form (9) is the pull-back $H_L = \widehat{L}^*H$ of the Hamiltonian form $H$ (6) for any associated Hamiltonian $\mathcal{H}$.

Now let us focus on polysymplectic Hamiltonian formalism [9, 10]. Bearing in mind its quantization, we formulate it as particular Lagrangian formalism on the Legendre bundle $\Pi$ (4).
2 Polysymplectic Hamiltonian dynamics

For every Hamiltonian form $H$ (6), there exists a connection

$$\gamma = dx^\lambda \otimes (\partial_\lambda + \gamma^j_\lambda \partial_i + \gamma^\mu_\lambda \partial_\mu)$$

on $\Pi \to X$ such that

$$\gamma_\lambda^i \Omega = dH, \quad \gamma^i_\lambda = \partial_i \mathcal{H}, \quad \gamma^\lambda_\lambda = -\partial_\lambda \mathcal{H}. \quad (18)$$

The connection (17), called the Hamiltonian connection, yields the first order dynamic Hamilton equations on $\Pi$ given by the closed submanifold

$$y^i_\lambda = \partial_i \mathcal{H}, \quad p^\lambda_\lambda = -\partial_\lambda \mathcal{H}$$

(19)
of the jet manifold $J^1 \Pi$ of $\Pi \to X$.

A polysymplectic Hamiltonian system on $\Pi$ is equivalent to the above mentioned particular Lagrangian system on $\Pi$ as follows.

**Proposition 1.** The Hamilton equations (19) are equivalent to the Euler–Lagrange equations for the first-order Lagrangian

$$L_\mathcal{H} = h_0(H) = \mathcal{L}_\mathcal{H} \omega = (p^i_\lambda y^j_\lambda - \mathcal{H}) \omega. \quad (20)$$

Let $i_N : N \to \Pi$ be a closed imbedded subbundle of the Legendre bundle $\Pi \to Y$ which is regarded as a constraint space of a polysymplectic Hamiltonian field system with a Hamiltonian $\mathcal{H}$. Let $H_N = i_N^* H$ be the pull-back of the Hamiltonian form $H$ (6) onto $N$. This form defines the constrained Lagrangian

$$L_N = h_0(H_N) = (J^1 i_N)^* L_\mathcal{H}$$

(21)
on the jet manifold $J^1 N_L$ of the fibre bundle $N_L \to X$. The Euler–Lagrange equations for this Lagrangian are called the constrained Hamilton equations.

In fact, the Lagrangian $L_\mathcal{H}$ (20) is the pull-back onto $J^1 \Pi$ of the horizontal form $L_\mathcal{H}$ on the bundle product $\Pi \times J^1 Y$ by the canonical map

$$J^1 \Pi \to \Pi \times J^1 Y.$$

Therefore, the constrained Lagrangian $L_N$ (21) is simply the restriction of $L_\mathcal{H}$ to $N \times J^1 Y$.

**Proposition 2.** A section $r$ of $\Pi \to X$ is a solution of the Hamilton equations (19) iff it satisfies the condition

$$r^*(u_{\Pi} | dH) = 0$$
for any vertical vector field $u_\Pi$ on $\Pi \rightarrow X$.

**Proposition 3.** A section $r$ of the fibre bundle $N \rightarrow X$ is a solution of constrained Hamilton equations iff it satisfies the condition $r^*(u_N|dH) = 0$ for any vertical vector field $u_N$ on $N \rightarrow X$.

Propositions 2 and 3 result in the following.

**Proposition 4.** Any solution of the Hamilton equations (19) which lives in the constraint manifold $N$ is also a solution of the constrained Hamilton equations on $N$.

Forthcoming Theorems 5 - 6 establish the above mentioned relation between Lagrangian and polysymplectic Hamiltonian formalisms in the case of almost-regular Lagrangians.

**Theorem 5.** Let $L$ be an almost-regular Lagrangian and $H$ an associated Hamiltonian. Let a section $r$ of $\Pi \rightarrow X$ be a solution of the Hamilton equations (19) for $H$. If $r$ lives in the Lagrangian constraint manifold $N_L$, then $s = \pi_Y \circ r$ satisfies the Euler–Lagrange equations (8) for $L$, while $\overline{s} = \hat{H} \circ r$ obeys the Cartan equations (11). Conversely, let $\overline{s}$ be a solution of the Cartan equations (11) for $L$. If $H$ satisfies the relation

$$\hat{H} \circ \hat{L} \circ \overline{s} = J^1(\pi^1_0 \circ \overline{s}),$$

the section $r = \hat{L} \circ \overline{s}$ of the Legendre bundle $\Pi \rightarrow X$ is a solution of the Hamilton equations (19) for $H$.

If an almost-regular Lagrangian admits associated Hamiltonians $H$, they define a unique constrained Lagrangian $L_N = h_0(H_N)$ (21) on the jet manifold $J^1N_L$ of the fibre bundle $N_L \rightarrow X$. Basing on Proposition 4 and Theorem 5, one can prove the following.

**Theorem 6.** Let an almost-regular Lagrangian $L$ admit associated Hamiltonians. A section $\overline{s}$ of the jet bundle $J^1Y \rightarrow X$ is a solution of the Cartan equations for $L$ iff $\hat{L} \circ \overline{s}$ is a solution of the constrained Hamilton equations. In particular, any solution $r$ of the constrained Hamilton equations provides the solution $\overline{s} = \hat{H} \circ r$ of the Cartan equations.

Thus, one can associate to an almost-regular Lagrangian (7) a unique constrained Lagrangian system on the constraint Lagrangian manifold $N_L$ (15).

### 3 Quadratic degenerate systems

Quadratic Lagrangians provide a most physically relevant example of degenerate Lagrangian systems.
Let us consider a quadratic Lagrangian
\[ \mathcal{L} = \frac{1}{2} a_{ij}^{\lambda \mu} y_i^j y_j^\mu + b_i^\lambda y_i^\mu + c, \] (22)
where \( a, b \) and \( c \) are local functions on \( Y \). The associated Legendre map (13) reads
\[ p_i^\lambda \circ \hat{L} = a_{ij}^{\lambda \mu} y_j^\mu + b_i^\lambda. \] (23)

Let a Lagrangian \( \mathcal{L} \) be almost-regular, i.e., the matrix function \( a \) is a linear bundle morphism
\[ a : T^* X \otimes V Y \to \Pi, \quad p_i^\lambda = a_{ij}^{\lambda \mu} y_j^\mu, \] (24)
of constant rank, where \((x^\lambda, y^i, \overline{\gamma}_\lambda^i)\) are coordinates on \( T^* X \otimes V Y \). Then the Lagrangian constraint space \( N_L \) (23) is an affine subbundle of \( \Pi \to Y \). Hence, \( N_L \to Y \) has a global section. Let us assume that it is the canonical zero section \( \hat{0}(Y) \) of \( \Pi \to Y \). The kernel of the Legendre map (23) is also an affine subbundle of the affine jet bundle \( J^1 Y \to Y \). Therefore, it admits a global section
\[ \Gamma : Y \to \text{Ker} \hat{L} \subset J^1 Y, \quad a_{ij}^{\lambda \mu} \Gamma_j^\mu + b_i^\lambda = 0, \] (25)
which is a connection on \( Y \to X \). With \( \Gamma \), the Lagrangian (22) is brought into the form
\[ \mathcal{L} = \frac{1}{2} a_{ij}^{\lambda \mu} (y_i^\lambda - \Gamma_i^\lambda)(y_j^\mu - \Gamma_j^\mu) + c'. \] (26)

**Theorem 7.** There exists a linear bundle morphism
\[ \sigma : \Pi \to T^* X \otimes V Y, \quad \overline{\gamma}_\lambda \circ \sigma = \sigma_{ij}^{ij \mu} y_j^\mu, \] (27)
\[ a \circ \sigma \circ a = a, \quad a_{ij}^{\lambda \mu} \sigma_{jk}^{\mu \alpha} \sigma_{\alpha k}^{\lambda} = a_{ij}^{\lambda \nu}. \] (28)

The morphism \( \sigma \) (27) is not unique, but it falls into the sum \( \sigma = \sigma_0 + \sigma_1 \) such that
\[ \sigma_0 \circ a \circ \sigma_0 = \sigma_0, \quad a \circ \sigma_1 = \sigma_1 \circ a = 0, \] (29)
where \( \sigma_0 \) is uniquely defined. The equalities (25) and (28) give the relation
\[ (a \circ \sigma_0)_{ij}^{\lambda \mu} b_j^\mu = b_i^\lambda. \]

**Theorem 8.** There are the splittings
\[ J^1 Y = \text{Ker} \hat{L} \oplus \text{Im}(\sigma_0 \circ \hat{L}), \] (30)
\[ y_i^\lambda = S_i^\lambda + F_i^\lambda = [y_i^\lambda - \sigma_0 \rho_j^{ik} (a_{kj}^{\alpha \mu} y_j^\mu + b_k^\alpha)] + [\sigma_0 \rho_j^{ik} (a_{kj}^{\mu \alpha} y_j^\mu + b_k^\alpha)], \]
\[ \Pi = \text{Ker} \sigma_0 \oplus N_L, \] (31)
\[ p_i^\lambda = R_i^\lambda + P_i^\lambda = [p_i^\lambda - \sigma_0 \rho_j^{ik} (a_{kj}^{\mu \alpha} y_j^\mu + b_k^\alpha)] + [a_{ij}^{\lambda \mu} \sigma_0 \rho_j^{ik} (a_{kj}^{\mu \alpha} y_j^\mu + b_k^\alpha)]. \]
The relations (29) lead to the equalities

\[ \sigma_{0}^{jk} \mathcal{R}_{k}^{\alpha} = 0, \quad \sigma_{1}^{jk} \mathcal{P}_{k}^{\alpha} = 0, \quad \mathcal{R}_{i}^{\lambda} \mathcal{F}_{\lambda}^{i} = 0. \] (32)

By virtue of the equalities (29) and the relation

\[ \mathcal{F}_{\mu}^{i} = (\sigma_{0} \circ a)_{\mu}^{ij}(y_{\lambda}^{j} - \Gamma_{\lambda}^{j}), \] (33)

the Lagrangian (22) takes the form

\[ L = L_{\omega}, \quad \mathcal{L} = \frac{1}{2} a^{\mu}_{ij} \mathcal{F}_{\lambda}^{i} \mathcal{F}_{\mu}^{j} + c'. \] (34)

It admits a set of associated Hamiltonians

\[ \mathcal{H}_{\Gamma} = (\mathcal{R}_{i}^{\lambda} + \mathcal{P}_{i}^{\lambda}) \Gamma_{\lambda}^{i} + \frac{1}{2} \sigma_{0}^{ij} \mathcal{P}_{i}^{\lambda} \mathcal{P}_{j}^{\mu} + \frac{1}{2} \sigma_{1}^{ij} \mathcal{R}_{i}^{\lambda} \mathcal{R}_{j}^{\mu} - c' \] (35)

indexed by connections \( \Gamma \) (25). Accordingly, the Lagrangian constraint manifold (23) is characterized by the equalities

\[ \mathcal{R}_{i}^{\lambda} = p_{i}^{\lambda} - a^{\mu}_{ij} \sigma_{0}^{jk} \mathcal{P}_{k}^{\alpha} = 0. \] (36)

Given a Hamiltonian \( \mathcal{H}_{\Gamma} \), the corresponding Lagrangian (20) on \( \Pi \times J^{1}Y \) reads

\[ \mathcal{L}_{\mathcal{H}_{\Gamma}} = \mathcal{R}_{i}^{\lambda}(S_{\lambda}^{i} - \Gamma_{\lambda}^{i}) + \mathcal{P}_{i}^{\lambda} \mathcal{F}_{\lambda}^{i} - \frac{1}{2} \sigma_{0}^{ij} \mathcal{P}_{i}^{\lambda} \mathcal{P}_{j}^{\mu} - \frac{1}{2} \sigma_{1}^{ij} \mathcal{R}_{i}^{\lambda} \mathcal{R}_{j}^{\mu} + c'. \] (37)

Its restriction (21) to the constraint manifold \( N_{L} \times J^{1}Y \) is

\[ L_{N} = L_{N\omega}, \quad \mathcal{L}_{N} = \mathcal{P}_{i}^{\lambda} \mathcal{F}_{\lambda}^{i} - \frac{1}{2} \sigma_{0}^{ij} \mathcal{P}_{i}^{\lambda} \mathcal{P}_{j}^{\mu} + c'. \] (38)

It is independent of the choice of a Hamiltonian (35).

The Hamiltonian \( \mathcal{H}_{\Gamma} \) yields the Hamiltonian map \( \widehat{H}_{\Gamma} \) and the projector

\[ \mathcal{T} = \hat{L} \circ \widehat{H}_{\Gamma}, \quad p_{i}^{\lambda} \circ \mathcal{T} = \mathcal{T}_{ij}^{\lambda} \nu_{j}^{\mu} = a^{\lambda
u}_{ik} \sigma_{0}^{kj} \nu_{j}^{\mu} = \mathcal{P}_{i}^{\lambda}, \]

of \( \Pi \) onto its summand \( N_{L} \) in the decomposition (31). It is a linear morphism over \( \text{Id}Y \). Therefore, \( \mathcal{T} : \Pi \to N_{L} \) is a vector bundle. Let us consider the pull-back

\[ L_{\Pi} = \mathcal{T}^{*} L_{N} = \mathcal{L}_{\Pi\omega}, \quad \mathcal{L}_{\Pi} = \mathcal{P}_{i}^{\lambda} \mathcal{F}_{\lambda}^{i} - \frac{1}{2} \sigma_{0}^{ij} \mathcal{P}_{i}^{\lambda} \mathcal{P}_{j}^{\mu} + c', \] (39)

of the constrained Lagrangian \( L_{N} \) (38) onto \( \Pi \times J^{1}Y \).
4 Quantization

In order to quantize covariant Hamiltonian systems, one usually attempts to construct the multisymplectic generalization of a Poisson bracket [5, 8, 13, 14]. In a different way, we suggested to quantize covariant (polysymplectic) Hamiltonian field theory in path integral terms [21]. This quantization scheme has been modified in order to quantize a polysymplectic Hamiltonian system with a Hamiltonian \( H \) on \( \Pi \) as a Lagrangian system with the Lagrangian \( L \) in the framework of familiar quantum field theory [1, 2].

If there is no constraint and the matrix
\[
\frac{\partial^2 H}{\partial p^i \partial p^j} = -\frac{\partial^2 L}{\partial p^i \partial p^j}
\]
is positive-definite and non-degenerate on an Euclidean space-time, this quantization is given by the generating functional
\[
Z = N^{-1} \int \exp \left\{ \int (L_H + \Lambda + iJ_i y^i + iJ^i_\mu p^\mu_i)\omega \right\} \prod_x [dp(x)] [dy(x)]
\]
of Euclidean Green functions, where \( \Lambda \) comes from the normalization condition
\[
\int \exp \left\{ \int \left( \frac{1}{2} \partial_i \partial_j L_H p^i \cdot p_j + \Lambda \right) dx \right\} \prod_x [dp(x)] = 1.
\]

A constrained Hamiltonian system on a constraint manifold \( N \) can be quantized as a Lagrangian system with the pull-back Lagrangian \( L_N \). Furthermore, a closed imbedded constraint submanifold \( N \) of \( \Pi \) admits an open neighbourhood \( U \) which is a fibred manifold \( U \to N \). If \( \Pi \) is a fibred manifold \( \pi_N : \Pi \to N \) over \( N \), it is often convenient to quantize a Lagrangian system on \( \Pi \) with the pull-back Lagrangian \( L_\Pi = \pi_N^* L_N \). Since this Lagrangian possesses gauge symmetries, BV (Batalin–Vilkoviski) quantization can be called into play [3, 11].

For instance, BV quantization can be applied to Hamiltonian systems associated to Lagrangian field systems with quadratic Lagrangians \( L \). If this Lagrangian is hyper-regular (i.e., the matrix function \( a \) is non-degenerate), there exists a unique associated Hamiltonian system whose Hamiltonian \( H \) is quadratic in momenta \( p^\mu_i \), and so is the Lagrangian \( L_H \). If the matrix function \( a \) is positive-definite on an Euclidean space-time, the generating functional (40) is a Gaussian integral of momenta \( p^\mu_i (x) \). Integrating \( Z \) with respect to \( p^\mu_i (x) \), one restarts the generating functional of quantum field theory with the original Lagrangian \( L \). Using the BV quantization procedure, this result is generalized to field theories with almost-regular Lagrangians \( L \), e.g., Yang–Mills gauge theory.

The Lagrangian \( L_\Pi \) possesses gauge symmetries. By gauge transformations are meant automorphisms \( \Phi \) of the composite fibre bundle \( \Pi \to Y \to X \) over bundle automorphisms \( \phi \) of \( Y \to X \) over \( \text{Id} \, X \). Such an automorphism \( \Phi \) gives rise to the automorphism
An automorphism $\Phi$ is said to be a gauge symmetry of the Lagrangian $L_\Pi$ if

$$(\Phi, J^1 \phi)^* L_\Pi = L_\Pi.$$  

If the Lagrangian (22) is degenerate, the group $G$ of gauge symmetries of the Lagrangian $L_\Pi$ (39) is never trivial. Indeed, any vertical automorphism of the vector bundle $\text{Ker} \sigma_0 \to Y$ in the decomposition (31) is obviously a gauge symmetry of the Lagrangian $L_\Pi$ (39). The gauge group $G$ acts on the space $\Pi(X)$ of sections of the Legendre bundle $\Pi \to X$. For the purpose of quantization, it suffices to consider a subgroup $\mathcal{G}$ of $G$ which acts freely on $\Pi(X)$ and satisfies the relation

$$\Pi(X)/\mathcal{G} = \Pi(X)/G.$$  

Moreover, we need one-parameter subgroups of $\mathcal{G}$. Their infinitesimal generators are represented by projectable vector fields

$$u_\Pi = u^i(x^\mu, y^j) \partial_i + u^k(x^\mu, y^j, p^\nu_j) \partial^i_k$$  

on the Legendre bundle $\Pi \to Y$ which give rise to the vector fields

$$\pi = u^i \partial_i + u^k \partial^i_k + d_\lambda u^i \partial^i_\lambda, \quad d_\lambda = \partial_\lambda + y^i_\lambda \partial_i,$$  

on $\Pi \times J^1 Y$. A Lagrangian $L_\Pi$ is invariant under a one-parameter group of gauge transformations iff its Lie derivative

$$L_\pi L_\Pi = \pi(\mathcal{L}_\Pi)\omega$$  

along the infinitesimal generator $\pi$ (42) of this group vanishes.

Any vertical vector field $u$ on $Y \to X$ gives rise to the vector field

$$u_\Pi = u^i \partial_i - \partial_j u^i p^\lambda_j \partial^i_\lambda$$  

on the Legendre bundle $\Pi$ and to the vector field

$$\pi_\Pi = u^i \partial_i - \partial_j u^i p^\lambda_j \partial^i_\lambda + d_\lambda u^i \partial^i_\lambda$$  

on $\Pi \times J^1 Y$.

Let us assume that the one-parameter gauge group with the infinitesimal generators $u$ preserves the splitting (30), i.e., $u$ obey the condition

$$u^k \partial_k (\sigma_0^i n^m a^\mu_{mj}) + \sigma_0^i n^m a^\mu_{mk} \partial_j u^k - \partial_k u^i \sigma_0^k m^\mu a^\mu_{mj} = 0.$$  

**Proposition 9.** If the condition (45) holds, the vector field $u_\Pi$ (43) is an infinitesimal gauge symmetry of the Lagrangian $L_\Pi$ (39) iff $u$ is an infinitesimal gauge symmetry of the Lagrangian $L$ (34).
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