Initial Blow-up of Solutions of Semilinear Parabolic Inequalities

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Abstract

We study classical nonnegative solutions $u(x, t)$ of the semilinear parabolic inequalities

$$0 \leq u_t - \Delta u \leq u^p \quad \text{in} \quad \Omega \times (0, 1)$$

where $p$ is a positive constant and $\Omega$ is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 1$.

We show that a necessary and sufficient condition on $p$ for such solutions $u$ to satisfy an a priori bound on compact subsets $K$ of $\Omega$ as $t \to 0^+$ is $p \leq 1 + 2/n$ and in this case the a priori bound on $u$ is

$$\max_{x \in K} u(x, t) = O(t^{-n/2}) \quad \text{as} \quad t \to 0^+.$$

If in addition, $u$ satisfies Dirichlet boundary conditions $u = 0$ on $\partial \Omega \times (0, 1)$ and $p < 1 + 2/(n + 1)$, then we obtain a uniform a priori bound for $u$ on the entire set $\Omega$ as $t \to 0^+$.

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1 Introduction

It is not hard to prove that if $u$ is a nonnegative solution of the heat equation

$$u_t - \Delta u = 0 \quad \text{in} \quad \Omega \times (0, 1),$$

where $\Omega$ is an open subset of $\mathbb{R}^n$, $n \geq 1$, then for each compact subset $K$ of $\Omega$, we have

$$\max_{x \in K} u(x, t) = O(t^{-n/2}) \quad \text{as} \quad t \to 0^+.$$  

The exponent $-n/2$ in (1.2) is optimal because the Gaussian

$$\Phi(x, t) = \begin{cases} 
\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}, & t > 0 \\
0, & t \leq 0
\end{cases}$$

is a nonnegative solution of the heat equation in $\mathbb{R}^n \times \mathbb{R} - \{(0, 0)\}$ and

$$\Phi(0, t) = (4\pi t)^{-n/2} \quad \text{for} \quad t > 0.$$  

(1.3)
It is also not hard to prove that if $u$ is a nonnegative solution of the heat equation with Dirichlet boundary conditions

\[
\begin{align*}
  u_t - \Delta u &= 0 & \text{in} & \quad \Omega \times (0,1), \\
  u &= 0 & \text{on} & \quad \partial \Omega \times (0,1), \\
\end{align*}
\]

where $\Omega$ is a $C^2$ bounded domain in $\mathbb{R}^n$, $n \geq 1$, then there exists a positive constant $C$ such that

\[
  u(x,t) \leq C \frac{\rho(x)}{\sqrt{t}} \wedge \frac{1}{\sqrt{n+1}} \quad \text{for all} \quad (x,t) \in \Omega \times (0,1/2) \tag{1.6}
\]

where $\rho(x) = \text{dist}(x, \partial \Omega)$.

Note that (1.6) is an a priori bound for $u$ on the entire set $\Omega$ rather than on compact subsets of $\Omega$. As we discuss and state precisely in the paragraph after Theorem 1.3, the estimate (1.6) is optimal for $x$ near the boundary of $\Omega$ and $t$ small.

In this paper, we generalize these results to nonnegative solutions $u(x,t)$ of the inequalities

\[
  0 \leq u_t - \Delta u \leq f(u) \quad \text{in} \quad \Omega \times (0,1) \tag{1.7}
\]

when the continuous function $f: [0, \infty) \to [0, \infty)$ is not too large at infinity. Note that solutions of the heat equation (1.1) satisfy (1.7). Our first result deals with nonnegative solutions $u$ of (1.7) when no boundary conditions are imposed on $u$.

**Theorem 1.1.** Suppose $u(x,t)$ is a $C^{2,1}$ nonnegative solution of

\[
  0 \leq u_t - \Delta u \leq (u + 1)^{1+2/n} \quad \text{in} \quad \Omega \times (0,1), \tag{1.8}
\]

where $\Omega$ is an open subset of $\mathbb{R}^n$, $n \geq 1$. Then, for each compact subset $K$ of $\Omega$, $u$ satisfies (1.2).

We proved Theorem 1.1 in [21] with the strong added assumption that,

\[
  \text{for some} \quad x_0 \in \Omega, \quad u \text{ is continuous on} \quad (\Omega \times [0,1)) - \{(x_0,0)\}. \tag{1.9}
\]

Theorem 1.1 is optimal in two ways. First, the exponent $-n/2$ on $t$ in (1.2) cannot be improved because, as already pointed out, the Gaussian (1.3) is a $C^\infty$ nonnegative solution of the heat equation in $\mathbb{R}^n \times \mathbb{R} - \{(0,0)\}$ satisfying (1.4).

And second, the exponent $1 + 2/n$ in (1.8) cannot be increased by the following theorem in [21].

**Theorem 1.2.** Let $p > 1 + 2/n$ and $\psi: (0,1) \to (0,\infty)$ be a continuous function. Then there exists a $C^\infty$ nonnegative solution $u(x,t)$ of

\[
  0 \leq u_t - \Delta u \leq u^p \quad \text{in} \quad (\mathbb{R}^n \times \mathbb{R}) - \{(0,0)\}
\]

satisfying $u \equiv 0$ in $\mathbb{R}^n \times (-\infty,0)$ and

\[
  u(0,t) \neq O(\psi(t)) \quad \text{as} \quad t \to 0^+.
\]

Our next result deals with nonnegative solutions of (1.7) satisfying Dirichlet boundary conditions.
Theorem 1.3. Suppose $u \in C^{2,1}(\Omega \times (0,1))$ is a nonnegative solution of
\begin{align}
0 \leq u_t - \Delta u &\leq (u + 1)^p & \text{in } & \Omega \times (0,1) \\
u & = 0 & \text{on } & \partial\Omega \times (0,1),
\end{align}
(1.10)
where $1 < p < 1 + 2/(n + 1)$ and $\Omega$ is a $C^2$ bounded domain in $\mathbb{R}^n$, $n \geq 1$. Then there exists a positive constant $C$ such that $u$ satisfies (1.6).

Note that the bound (1.6) for $u$ in Theorem 1.3 is, like $u$, zero on $\partial\Omega \times (0,1)$. Furthermore, the estimate (1.6) is optimal for $x$ near the boundary of $\Omega$ and $t$ small. More precisely, let $x_0 \in \partial\Omega$, $G(x, y, t)$ be the heat kernel of the Dirichlet Laplacian in $\Omega \times (0,1)$, and $\eta$ be the unit inward normal to $\Omega$ at $x_0$. Then using the lower bound for $G$ in [24], it is easy to show that
\[ u(x, t) := \lim_{r \to 0^+} \frac{G(x, x_0 + r\eta, t)}{r} \]
is a nonnegative solution of (1.5), and hence of (1.10), such that for some $T > 0$
\[ \frac{u(x, t)}{(\rho(x) \sqrt{t} \wedge 1)/\sqrt{t^{n+1}}} \]
is bounded between positive constants for all $(x, t) \in \Omega \times (0, T)$ satisfying $|x - x_0| < \sqrt{t}$.

By modifying the proof of Theorem 1.2, it can be shown that Theorem 1.3 is not true for $p > 1 + 2/n$. An open question is for what values of $p \in [1 + 2/(n + 1), 1 + 2/n]$ is Theorem 1.3 true.

Philippe Souplet communicated to us a proof of Theorem 1.3 in the special case that the differential inequalities in problem (1.10) are replaced with the equation $u_t - \Delta u = u^p$. However, his method of proof does not seem to work for Theorem 1.3 as stated. See also [19, Theorem 26.14(i)].

Theorems 1.1 and 1.3 can be strengthened by replacing the term 1 on the right sides of (1.8) and (1.10) with a term which tends to infinity as $t \to 0^+$. We state and prove these strengthened versions of Theorems 1.1 and 1.3 in Sections 3 and 4 respectively.

The proofs of Theorems 1.1 and 1.3 rely heavily on Lemmas 2.1 and 2.2, respectively, which we state and prove in Section 2. We are able to prove Theorem 1.1 without condition (1.9) because we do not impose this kind of condition on the function $u$ in Lemma 2.1.

As in [21], a crucial step in the proofs of Theorems 1.1 and 1.3 is an adaptation and extension to parabolic inequalities of a method of Brezis [4] concerning elliptic equations and based on Moser’s iteration. This method is used to obtain an estimate of the form
\[ \|u_j\|_{L^{q+2/q}(\Omega')} \leq C\|u_j\|_{L^{q}(\Omega)} \]
where $q > 1$, $\Omega' \subset \Omega$, $C$ is a constant which does not depend on $j$, and $u_j$, $j = 1, 2, \ldots$, is obtained from the function $u$ in Theorem 1.1 or 1.3 by appropriately scaling $u$ about $(x_j, t_j)$ where $(x_j, t_j) \in \Omega \times (0,1)$ is a sequence such that $t_j \to 0^+$ and for which (1.2) or (1.6) is violated.

Our proofs also rely on upper and lower bounds for the heat kernel of the Dirichlet Laplacian. We use the upper bound in [10] and the lower one in [24].

The blow-up of solutions of the equation
\[ u_t - \Delta u = u^p \]
(1.11)
has been extensively studied in \[1, 2, 3, 5, 6, 7, 8, 9, 11, 13, 14, 15, 16, 17, 18, 20, 23\] and elsewhere. The book \[19\] is an excellent reference for many of these results. However, other than \[21\], we know of no previous blow-up results for the inequalities

\[0 \leq u_t - \Delta u \leq u^p.\]

Also, blow-up of solutions of \(au^p \leq u_t - \Delta u \leq u^p\), where \(a \in (0, 1)\), has been studied in \[22\].

2 Preliminary lemmas

For the proof in Section \ref{sec:3} of Theorem \ref{thm:1.1}, we will need the following lemma.

**Lemma 2.1.** Suppose \(u\) is a \(C^{2,1}\) nonnegative solution of

\[Hu \geq 0 \quad \text{in} \quad B_4(0) \times (0, 3) \subset \mathbb{R}^n \times \mathbb{R}, \quad n \geq 1,\]

where \(Hu = u_t - \Delta u\) is the heat operator. Then

\[u, Hu \in L^1(B_2(0) \times (0, 2)) \quad (2.2)\]

and there exist a finite positive Borel measure \(\mu\) on \(B_2(0)\) and \(h \in C^{2,1}(B_1(0) \times (-1,1))\) satisfying

\[Hh = 0 \quad \text{in} \quad B_1(0) \times (-1,1) \quad (2.3)\]

\[h = 0 \quad \text{in} \quad B_1(0) \times (-1,0) \quad (2.4)\]

such that

\[u = N + v + h \quad \text{in} \quad B_1(0) \times (0,1) \quad (2.5)\]

where

\[N(x,t) := \int_0^2 \int_{|y|<2} \Phi(x-y,t-s)Hu(y,s) \, dy \, ds, \quad (2.6)\]

\[v(x,t) := \int_{|y|<2} \Phi(x-y,t) \, d\mu(y), \quad (2.7)\]

and \(\Phi\) is the Gaussian \((1.3)\).

**Proof.** Let \(\varphi_1 \in C^2(B_3(0))\) and \(\lambda > 0\) satisfy

\[-\Delta \varphi_1 = \lambda \varphi_1 \quad \varphi_1 > 0 \quad \varphi_1 = 0 \quad \text{for} \quad |x| < 3 \quad \text{for} \quad |x| = 3.\]

Then for \(0 < t \leq 2\), we have by \((2.1)\) that

\[0 \leq \int_{|x|<3} [Hu(x,t)] \varphi_1(x) \, dx \]

\[= \int_{|x|<3} u_t(x,t) \varphi_1(x) \, dx + \lambda \int_{|x|<3} u(x,t) \varphi_1(x) \, dx + \int_{|x|=3} u(x,t) \frac{\partial \varphi_1(x)}{\partial \eta} \, dS_x \]

\[\leq U'(t) + \lambda U(t)\]
where $U(t) = \int_{|x|<3} u(x,t)\varphi_1(x)\,dx$. Thus $(U(t)e^{\lambda t})' \geq 0$ for $0 < t \leq 2$ and consequently for some $U_0 \in [0, \infty)$ we have

$$U(t) = (U(t)e^{\lambda t})e^{-\lambda t} \to U_0 \quad \text{as} \quad t \to 0^+. \quad (2.8)$$

Thus $u\varphi_1 \in L^1(B_3(0) \times (0, 2))$. Hence, since for $0 < t \leq 2$,

$$\int_t^2 \int_{|x|<3} H_u(x,\tau)\varphi_1(x)\,dx\,d\tau = \int_{|x|<3} \left( \int_t^2 u_t(x,\tau)\,d\tau \right)\varphi_1(x)\,dx - \int_{|x|<3} \int_t^2 (\Delta u(x,\tau))\varphi_1(x)\,dx\,d\tau$$

$$= \int_{|x|<3} u(x,2)\varphi_1(x)\,dx - \int_{|x|<3} u(x,t)\varphi_1(x)\,dx$$

$$+ \int_t^2 \int_{|x|=3} u(x,\tau)\frac{\partial \varphi_1(x)}{\partial \eta} \,dS_x\,d\tau$$

$$+ \lambda \int_t^2 \int_{|x|<3} u(x,\tau)\varphi_1(x)\,dx\,d\tau, \quad (2.9)$$

we see that $(H_u)\varphi_1 \in L^1(B_3(0) \times (0, 2))$. So (2.2) holds. By (2.8),

$$\int_{|x|\leq 2} u(x,t)\,dx \quad \text{is bounded for} \quad 0 < t \leq 2. \quad (2.10)$$

Hence there exists a finite positive Borel measure $\hat{\mu}$ on $\overline{B_2(0)}$ and a sequence $t_j$ decreasing to 0 such that for all $g \in C(\overline{B_2(0)})$ we have

$$\int_{|x|\leq 2} g(x)u(x,t_j)\,dx \to \int_{|x|\leq 2} g(x)\,d\hat{\mu} \quad \text{as} \quad j \to \infty.$$

In particular, for all $\varphi \in C_0^\infty(B_2(0))$ we have

$$\int_{|x|<2} \varphi(x)u(x,t_j)\,dx \to \int_{|x|<2} \varphi(x)\,d\mu \quad \text{as} \quad j \to \infty. \quad (2.11)$$

where we define $\mu$ to be the restriction of $\hat{\mu}$ to $B_2(0)$.

For $(x,t) \in \mathbb{R}^n \times (0, \infty)$, let $v(x,t)$ be defined by (2.7). Then $v \in C^{2,1}(\mathbb{R}^n \times (0, \infty))$, $Hv = 0$ in $\mathbb{R}^n \times (0, \infty)$, and

$$\int_{\mathbb{R}^n} v(x,t)\,dx = \int_{|y|<2} d\mu(y) < \infty \quad \text{for} \quad t > 0. \quad (2.12)$$

Thus $v \in L^1(\mathbb{R}^n \times (0, 2))$.

For $\varphi \in C_0^\infty(B_2(0))$ and $t > 0$ we have

$$\int_{|x|<2} \varphi(x)v(x,t)\,dx = \int_{|y|<2} \left( \int_{|x|<2} \Phi(x-y,t)\varphi(x)\,dx \right) d\mu(y) \to \int_{|y|<2} \varphi(y)\,d\mu(y) \quad \text{as} \quad t \to 0^+,$$

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and hence it follows from (2.11) that
\[
\int_{|x|<2} \varphi(x)(u(x,t_j) - v(x,t_j)) \, dx \to 0 \quad \text{as} \quad j \to \infty.
\] (2.13)

Let
\[
f := \begin{cases} 
Hu, & \text{in } B_2(0) \times (0,2) \\
0, & \text{elsewhere in } \mathbb{R}^n \times \mathbb{R}.
\end{cases}
\]

Then by (2.2),
\[
f \in L^1(\mathbb{R}^n \times \mathbb{R}).
\] (2.14)

Let
\[
w := \begin{cases} 
 u - v, & \text{in } B_2(0) \times (0,2) \\
0, & \text{elsewhere in } \mathbb{R}^n \times \mathbb{R}.
\end{cases}
\]

Then
\[
w \in C^{2,1}(B_2(0) \times (0,2)) \cap L^1(\mathbb{R}^n \times \mathbb{R}),
\]
\[Hw = f \quad \text{in } B_2(0) \times (0,2),
\] (2.15)

and
\[
\int_{|x|<2} |w(x,t)| \, dx \quad \text{is bounded for } 0 < t < 2
\] (2.16)

by (2.10) and (2.12). Let \(\Omega = B_1(0) \times (-1,1)\) and define \(\Lambda \in \mathcal{D}'(\Omega)\) by \(\Lambda = -Hw + f\), that is
\[
\Lambda \varphi = \int wH^* \varphi + \int f \varphi \quad \text{for } \varphi \in C_0^\infty(\Omega),
\]
where \(H^* \varphi := \varphi_t + \Delta \varphi\). We now show \(\Lambda = 0\). Let \(\varphi \in C_0^\infty(\Omega)\), let \(j\) be a fixed positive integer, and let \(\psi_\varepsilon : \mathbb{R} \to [0,1] \), \(\varepsilon\) small and positive, be a one parameter family of smooth nondecreasing functions such that
\[
\psi_\varepsilon(t) = \begin{cases} 
1, & t > t_j + \varepsilon \\
0, & t < t_j - \varepsilon.
\end{cases}
\]
where \(t_j\) is as in (2.11). Then
\[
-\int f \varphi \psi_\varepsilon = -\int (Hw)\varphi \psi_\varepsilon = \int wH^*(\varphi \psi_\varepsilon)
\]
\[= \int w(\varphi_t \psi_\varepsilon + \varphi \psi'_\varepsilon + \psi_\varepsilon \Delta \varphi)
\]
\[= \int w\psi_\varepsilon H^* \varphi + \int w\varphi \psi'_\varepsilon.
\]
Letting \(\varepsilon \to 0^+\) we get
\[
-\int_{t_j}^1 \int_{|x|<1} f \varphi \, dx \, dt = \int_{t_j}^1 \int_{|x|<1} wH^* \varphi \, dx \, dt + \int w(x,t_j) \varphi(x,t_j) \, dx.
\] (2.17)
Also, it follows from (2.16) and (2.13) that
\[ \int_{|x|<1} w(x,t_j)\varphi(x,t_j) \, dx = \int_{|x|<1} w(x,t_j)[\varphi(x,t_j) - \varphi(x,0)] \, dx + \int_{|x|<1} w(x,t_j)\varphi(x,0) \, dx \]
\[ \to 0 \quad \text{as} \quad j \to \infty. \]

Thus letting \( j \to \infty \) in (2.17) and using (2.14) and (2.15) we get
\[ -\int f\varphi = \int wH^*\varphi. \]
So \( \Lambda = 0 \).

For \((x,t) \in \mathbb{R}^n \times \mathbb{R}\), let \( N(x,t) \) be defined by (2.6). Then
\[ N(x,t) = \iint_{\mathbb{R}^n \times \mathbb{R}} \Phi(x-y,t-s)f(y,s) \, dy ds \]
and \( N \equiv 0 \) in \( \mathbb{R}^n \times (-\infty,0) \). By (2.11), we have \( N \in L^1(\Omega) \) and \( HN = f \) in \( \mathcal{D}'(\Omega) \). Thus
\[ H(w - N) = -\Lambda + f - f = 0 \quad \text{in} \quad \mathcal{D}'(\Omega) \]
which implies
\[ w - N = h \quad \text{in} \quad \mathcal{D}'(\Omega) \]
for some \( C^2 \), \( 1 \) solution \( h \) of (2.3) and (2.4). Hence (2.5) holds.

For the proof in Section 4 of Theorem 1.3, we will need the following lemma.

**Lemma 2.2.** Suppose \( u \in C^{2,1}(\Omega \times (0,2T)) \) is a nonnegative solution of \( Hu \geq 0 \) in \( \Omega \times (0,2T) \), where \( Hu = u_t - \Delta u \) is the heat operator, \( T \) is a positive constant, and \( \Omega \) is a bounded \( C^2 \) domain in \( \mathbb{R}^n \), \( n \geq 1 \). Then
\[ u, \rho Hu \in L^1(\Omega \times (0,T)), \quad (2.18) \]
where \( \rho(x) = \text{dist}(x,\partial \Omega) \). Moreover, there exists \( C > 0 \) such that
\[ 0 \leq u(x,t) - \int_0^t \int_{\Omega} G(x,y,t-s)Hu(y,s) \, dy ds \]
\[ \leq C \frac{\rho(x)}{t^{\frac{n+1}{2}}} + \sup_{\partial \Omega \times (0,T)} u \quad \text{for all} \quad (x,t) \in \Omega \times (0,T), \quad (2.19) \]
where \( G \) is the Dirichlet heat kernel for \( \Omega \).

**Proof.** For \( \varphi \in C^2(\Omega) \cap C^1(\overline{\Omega}) \), \( \varphi = 0 \) on \( \partial \Omega \), and \( 0 < t < T \) we have
\[ \int_t^T \int_{\Omega} [Hu(y,\tau)]\varphi(y) \, dy d\tau = \int_{\Omega} u(y,T)\varphi(y) \, dy - \int_{\Omega} u(y,t)\varphi(y) \, dy \]
\[ - \int_t^T \int_{\Omega} u(y,\tau)\Delta \varphi(y) \, dy d\tau + \int_t^T \int_{\partial \Omega} u(y,\tau) \frac{\partial \varphi(y)}{\partial n} \, dS_y d\tau \quad (2.20) \]

Let \( \varphi_1 \in C^2(\Omega) \cap C^1(\overline{\Omega}) \) and \( \lambda > 0 \) satisfy
\[ \begin{cases} -\Delta \varphi_1 = \lambda \varphi_1 \\ 0 < \varphi_1 < 1 \\ \varphi_1 = 0 \end{cases} \quad \text{in} \quad \Omega \\
\quad \text{on} \quad \partial \Omega. \]
Then for $0 < t < 2T$ we have
\[
0 \leq \int_{\Omega} Hu(y,t)\phi_1(y) \, dy = U'(t) + \lambda U(t) + \int_{\partial\Omega} u(y,t) \frac{\partial\phi_1(y)}{\partial\eta} \, dS_y \leq U'(t) + \lambda U(t),
\]
where $U(t) = \int_{\Omega} u(y,t)\phi_1(y) \, dy$. Thus $(U(t)e^{\lambda t})' \geq 0$ for $0 < t < 2T$ and hence for some $U_0 \geq 0$ we have
\[
U(t) = (U(t)e^{\lambda t})e^{-\lambda t} \to U_0 \quad \text{as} \quad t \to 0^+.
\]
Consequently $u\phi_1 \in L^1(\Omega \times (0,T))$. So taking $\phi = \phi_1$ in (2.20) we have
\[
\phi_1 Hu \in L^1(\Omega \times (0,T)), \quad (2.22)
\]
and taking $\phi = \phi_2$ in (2.20) we obtain $u|\nabla\phi_1|^2 \in L^1(\Omega \times (0,T))$. Thus, since $\phi_1/\rho$ is bounded between positive constants on $\Omega$, it follows from (2.22) that (2.18) holds, and by (2.21) we have
\[
\int_{\Omega} u(y,t)\rho(y) \, dy \quad \text{is bounded for} \quad 0 < t \leq T. \quad (2.23)
\]
Let $x \in \Omega$ and $0 < \tau < t < T$ be fixed. Then for $\varepsilon > 0$ we have
\[
\begin{aligned}
&\int_{\Omega} G(x,y,\varepsilon)u(y,t) \, dy - \int_{\tau}^t \int_{\Omega} G(x,y,t+\varepsilon-s)Hu(y,s) \, dy \, ds \\
&= \int_{\Omega} G(x,y,\varepsilon)u(y,\tau) \, dy - \int_{\tau}^t \int_{\partial\Omega} u(y,s) \frac{\partial G(x,y,t+\varepsilon-s)}{\partial\eta_y} \, dS_y \, ds \\
&\geq 0.
\end{aligned}
\]
Since $\int_{\Omega} G(x,y,\zeta) \, dy \leq 1$ for $\zeta > 0$, we have
\[
0 \leq -\int_{\tau}^t \int_{\partial\Omega} \frac{\partial G(x,y,t+\varepsilon-s)}{\partial\eta_y} \, dS_y \, ds \\
= \int_{\Omega} G(x,y,\varepsilon) \, dy - \int_{\Omega} G(x,y,t+\varepsilon-\tau) \, dy \leq 1
\]
and
\[
\int_{\Omega} G(x,y,t+\varepsilon-s)Hu(y,s) \, dy \leq \max_{\Omega \times [\tau,t]} Hu < \infty
\]
for $\varepsilon > 0$ and $\tau \leq s \leq t$. Thus, letting $\varepsilon \to 0^+$ in (2.24) and using the fact that the function $(y,\zeta) \to G(x,y,\zeta)$ is continuous for $(y,\zeta) \in \overline{\Omega} \times (0,\infty)$ we get
\[
0 \leq u(x,t) - \int_{\tau}^t \int_{\Omega} G(x,y,t-s)Hu(y,s) \, dy \, ds \\
\leq v(x,t,\tau) + \sup_{\partial\Omega \times (0,T)} u \quad (2.25)
\]
where
\[
v(x,t,\tau) := \int_{\Omega} G(x,y,t-\tau)u(y,\tau) \, dy \leq C \frac{\sqrt{t-\tau}}{\sqrt{t} \wedge 1} \int_{\Omega} u(y,\tau)\rho(y) \, dy
\]
because, as shown by Hui [10, Lemma 1.3], there exists a positive constant \( C = C(n, \Omega, T) \) such that if
\[
\hat{G}(r, t) = \frac{C}{t^{n/2}} e^{-r^2/(Ct)} \quad \text{for} \quad r \geq 0 \quad \text{and} \quad t > 0
\]
then the heat kernel \( G(x, y, t) \) for \( \Omega \) satisfies
\[
G(x, y, t) \leq \left( \frac{\rho(x)}{\sqrt{t}} \land 1 \right) \left( \frac{\rho(y)}{\sqrt{t}} \land 1 \right) \hat{G}(|x - y|, t) \quad \text{for} \quad x, y \in \Omega \quad \text{and} \quad 0 < t \leq T. \tag{2.26}
\]
Hence, letting \( \tau \to 0^+ \) in (2.25) and using (2.23) and the monotone convergence theorem we obtain (2.19).

For the proofs in Sections 3 and 4 of Theorems 1.1 and 1.3 respectively we will need the following lemma whose proof is an adaptation to parabolic inequalities of a method of Brezis [4] for elliptic equations.

**Lemma 2.3.** Suppose \( T > 0 \) and \( \lambda > 1 \) are constants, \( B \) is an open ball in \( \mathbb{R}^n \), \( E = B \times (-T, 0) \), and \( \varphi \in C_0^\infty(B \times (-T, \infty)) \). Then there exists a positive constant \( C \) depending only on \( n, \lambda \), and \( \sup_E (|\varphi|, |\nabla \varphi|, |\frac{\partial \varphi}{\partial t}|, |\Delta \varphi|) \) such that if \( \Omega \) is a \( C^2 \) bounded domain in \( \mathbb{R}^n \), \( \Omega \cap B \neq \emptyset \), \( D = \Omega \times (-T, 0) \), and \( u \in C^{2,1}(D) \) is a nonnegative solution of
\[
Hu \geq 0 \quad \text{in} \quad \Omega \times (-T, 0) \\
u = 0 \quad \text{on} \quad (\partial \Omega \cap B) \times (-T, 0)
\]
then
\[
\left( \iint_{E \cap D} (u\lambda \varphi^2)^\frac{n+2}{n+1} \, dx \, dt \right)^{\frac{n}{n+2}} \leq C \left( \iint_{E \cap D} (Hu)u^{\lambda-1} \varphi^2 \, dx \, dt + \iint_{E \cap D} u\lambda \, dx \, dt \right). \tag{2.28}
\]

**Proof.** Let \( u \) be as in the lemma. Since
\[
\nabla \cdot \nabla (u^{\lambda-1} \varphi^2) = \frac{4(\lambda - 1)}{\lambda^2} |\nabla (u^{\lambda/2} \varphi)|^2 - \frac{\lambda - 2}{\lambda^2} \nabla u \cdot \nabla \varphi^2 - \frac{4(\lambda - 1)}{\lambda^2} u\lambda |\nabla \varphi|^2 \tag{2.29}
\]
we have for \(-T < t < 0\) that
\[
\int_{B \cap \Omega} (-\Delta u)u^{\lambda-1} \varphi^2 \, dx = \int_{B \cap \Omega} \nabla u \cdot \nabla (u^{\lambda-1} \varphi^2) \, dx \\
\geq \frac{4(\lambda - 1)}{\lambda^2} \int_{B \cap \Omega} |\nabla (u^{\lambda/2} \varphi)|^2 \, dx - C \int_{B \cap \Omega} u\lambda \, dx \tag{2.30}
\]
where \( C \) is a positive constant depending only on the quantities (2.27) whose value may change from line to line. Also, for \( x \in B \cap \Omega \) we have
\[
\int_{-T}^0 u_t u^{\lambda-1} \varphi^2 \, dt = \frac{1}{\lambda} \int_{-T}^0 \frac{\partial u^\lambda}{\partial t} \varphi^2 \, dt \\
= \frac{1}{\lambda} \left[ u(x, 0) \lambda \varphi(x, 0)^2 - \int_{-T}^0 u^\lambda \frac{\partial \varphi^2}{\partial t} \, dt \right] \\
\geq -C \int_{-T}^0 u^\lambda \, dt. \tag{2.31}
\]
Integrating inequality (2.30) with respect to $t$ from $-T$ to 0, integrating inequality (2.31) with respect to $x$ over $B \cap \Omega$, and then adding the two resulting inequalities we get

$$C(I + B) \geq \int \int_{E \cap D} |\nabla (u^{\lambda/2}\varphi)|^2 \, dx \, dt$$

(2.32)

where

$$I = \int \int_{E \cap D} (Hu)u^{\lambda-1}\varphi^2 \, dx \, dt \quad \text{and} \quad B = \int \int_{E \cap D} u^\lambda \, dx \, dt.$$ 

Multiplying (2.32) by

$$M := \max_{-T \leq t \leq 0} \left( \int_{B \cap \Omega} u^\lambda \varphi^2 \, dx \right)^{2/n}$$

and using the parabolic Sobolev inequality (see [12, Theorem 6.9]) we obtain

$$C(I + B)M \geq A := \int \int_{E \cap D} (u^\lambda \varphi^2)^{\frac{n+2}{n}} \, dx \, dt.$$  

(2.33)

Since

$$\frac{\partial}{\partial t} (u^\lambda \varphi^2) = \lambda u^{\lambda-1} u_t \varphi^2 + 2u^\lambda \varphi \varphi_t$$

$$= \lambda u^{\lambda-1} \varphi^2 (\Delta u + Hu) + 2u^\lambda \varphi \varphi_t$$

it follows from (2.30) that for $-T < t < 0$ we have

$$\frac{\partial}{\partial t} \int_{B \cap \Omega} u^\lambda \varphi^2 \, dx \leq C \int_{B \cap \Omega} u^\lambda \, dx + \lambda \int_{B \cap \Omega} u^{\lambda-1} \varphi^2 Hu \, dx$$

and thus

$$M \frac{\partial}{\partial t} \leq C(I + B).$$

(2.34)

Substituting (2.34) in (2.33) we get

$$A \leq C(I + B)^{\frac{n+2}{n}}$$

which implies (2.28).

3 Proof of Theorem 1.1

In this section we prove the following theorem which clearly implies Theorem 1.1.

**Theorem 3.1.** Suppose $u$ is a $C^{2,1}$ nonnegative solution of

$$0 \leq u_t - \Delta u - b \left( u + \frac{1}{\sqrt{t}} \right)^{1+2/n} \quad \text{in} \quad \Omega \times (0, T),$$

(3.1)

where $T$ and $b$ are positive constants and $\Omega$ is an open subset of $\mathbb{R}^n$, $n \geq 1$. Then, for each compact subset $K$ of $\Omega$, we have

$$\max_{x \in K} u(x, t) = O(t^{-n/2}) \quad \text{as} \quad t \to 0^+.$$  

(3.2)
Proof. To prove Theorem 3.1, we claim it suffices to prove Theorem 3.1′ where Theorem 3.1′ is the theorem obtained from Theorem 3.1 by replacing (3.1) with

$$0 \leq u_t - \Delta u \leq \left( u + \frac{b}{\sqrt{t}} \right)^{1+2/n} \quad \text{in} \quad B_4(0) \times (0,3)$$

and replacing (3.2) with

$$\max_{|x| \leq \frac{1}{2}} u(x,t) = O(t^{-n/2}) \quad \text{as} \quad t \to 0^+.$$  \hspace{1cm} (3.4)

To see this, let $u$ be as in Theorem 3.1 and let $K$ be a compact subset of $\Omega$. Since $K$ is compact there exist finite sequences \( \{r_j\}_{j=1}^N \subset (0, \sqrt{T}/4) \) and \( \{x_j\}_{j=1}^N \subset K \) such that

$$K \subset \bigcup_{j=1}^N B_{r_j/2}(x_j) \subset \bigcup_{j=1}^N B_{4r_j}(x_j) \subset \Omega.$$  \hspace{1cm} (3.3)

Let \( v_j(y,s) = r_j^n b^{n/2} u(x,t) \), where \( x = x_j + r_j y \) and \( t = r_j^2 s \). Then

$$0 \leq Hv_j \leq \left( v_j + \frac{b^{n/2}}{\sqrt{s}} \right)^{1+2/n} \quad \text{for} \quad |y| < 4, \quad 0 < s < 16,$$

where \( Hv_j := \frac{\partial v_j}{\partial s} - \Delta_y v_j \). Hence by Theorem 3.1 there exist \( s_j \in (0,16) \) and \( C_j > 0 \) such that

$$\max_{|y| \leq \frac{1}{2}} v_j(y,s) \leq C_j s^{-n/2} \quad \text{for} \quad 0 < s < s_j.$$  \hspace{1cm} (3.5)

That is

$$\max_{|x-x_j| \leq r_j/2} u(x,t) \leq C_j b^{-n/2} t^{-n/2} \quad \text{for} \quad 0 < t < t_j := r_j^2 s_j.$$  \hspace{1cm} (3.6)

So for \( 0 < t < \min_{1 \leq j \leq N} t_j \) we have

$$\max_{x \in K} u(x,t) \leq \max_{1 \leq j \leq N} \max_{|x-x_j| \leq r_j/2} u(x,t) \leq \left( \max_{1 \leq j \leq N} C_j \right) b^{-n/2} t^{-n/2}.$$  \hspace{1cm} (3.7)

That is, (3.2) holds.

We now complete the proof of Theorem 3.1 by proving Theorem 3.1′. Suppose \( u \) is a \( C^{2,1} \) nonnegative solution of (3.3). By Lemma 2.1

$$u, Hu \in L^1(B_2(0) \times (0,2))$$

and

$$u = N + v + h \quad \text{in} \quad B_1(0) \times (0,1)$$

where \( N, v, \) and \( h \) are as in Lemma 2.1. Suppose for contradiction that (3.4) does not hold. Then there exists a sequence \( \{(x_j,t_j)\} \subset B_{1/2}(0) \times (0,1/4) \) such that for some \( x_0 \in B_{1/2}(0) \) we have \( (x_j,t_j) \to (x_0,0) \) as \( j \to \infty \) and

$$\lim_{j \to \infty} t_j^{n/2} u(x_j, t_j) = \infty.$$

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Clearly
\[(4\pi t)^{n/2} \nu(x,t) \leq \int_{|y|<2} d\mu(y) < \infty \quad \text{for} \quad (x,t) \in \mathbb{R}^n \times (0,\infty). \tag{3.8}\]

For \((x,t) \in \mathbb{R}^n \times \mathbb{R}\) and \(r > 0\), let
\[E_r(x,t) := \{(y,s) \in \mathbb{R}^n \times \mathbb{R} : |y-x| < \sqrt{r} \quad \text{and} \quad t-r < s < t\}.\]

In what follows, the variables \((x,t)\) and \((\xi,\tau)\) are related by
\[x = x_j + \sqrt{t_j} \xi \quad \text{and} \quad t = t_j + t_j \tau \tag{3.9}\]
and the variables \((y,s)\) and \((\eta,\zeta)\) are related by
\[y = x_j + \sqrt{t_j} \eta \quad \text{and} \quad s = t_j + t_j \zeta. \tag{3.10}\]

For each positive integer \(j\), define
\[f_j(\eta,\zeta) = \sqrt{t_j^{n+2}} Hu(y,s) \quad \text{for} \quad (y,s) \in E_{t_j(x_j,t_j)} \tag{3.11}\]
and define
\[u_j(\xi,\tau) = \sqrt{t_j^n} \int\int_{E_{t_j(x_j,t_j)}} \Phi(x-y,t-s) Hu(y,s) \, dy \, ds \quad \text{for} \quad (x,t) \in \mathbb{R}^n \times (0,\infty). \tag{3.12}\]

By (3.5) we have
\[\int\int_{E_{t_j(x_j,t_j)}} Hu(y,s) \, dy \, ds \to 0 \quad \text{as} \quad j \to \infty \tag{3.13}\]
and thus making the change of variables (3.10) in (3.13) and using (3.11) we get
\[\int\int_{E_{t_j(x_j,t_j)}} f_j(\eta,\zeta) \, d\eta \, d\zeta \to 0 \quad \text{as} \quad j \to \infty. \tag{3.14}\]

Since
\[\Phi(x-y,t-s) = \frac{1}{\sqrt{t_j^n}} \Phi(\xi-\eta,\tau-\zeta)\]
it follows from (3.12) and (3.11) that
\[u_j(\xi,\tau) = \int\int_{E_{t_j(x_j,t_j)}} \Phi(\xi-\eta,\tau-\zeta) f_j(\eta,\zeta) \, d\eta \, d\zeta. \tag{3.15}\]

It is easy to check that for \(1 < q < \frac{n+2}{n}\) and \((\xi,\tau) \in \mathbb{R}^n \times (-1,0]\) we have
\[\left( \int_{\mathbb{R}^n \times (-1,0]} \Phi(\xi-\eta,\tau-\zeta)^q \, d\eta \, d\zeta \right)^{1/q} < C(n,q) < \infty. \tag{3.16}\]

Thus for \(1 < q < \frac{n+2}{n}\) we have by (3.15) and standard \(L^p\) estimates for the convolution of two functions that
\[\|u_j\|_{L^q(E_{t_j(x_j,t_j)})} \leq C(n,q) \|f_j\|_{L^1(E_{t_j(x_j,t_j)})} \to 0 \quad \text{as} \quad j \to \infty. \tag{3.17}\]
by (3.14). If 

\( (x, t) \in E_{t_j/4}(x_j, t_j) \) and \( (y, s) \in \mathbb{R}^n \times (0, \infty) - E_t(x_j, t_j) \)

then

\[ \Phi(x - y, t - s) \leq \max_{0 \leq \tau < \infty} \Phi \left( \frac{\sqrt{t_j}}{2}, \tau \right) \leq \frac{C(n)}{\sqrt{t_j}^n}. \]

Thus for \( (x, t) \in E_{t_j/4}(x_j, t_j) \) we have

\[ \int \int_{B_2(0) \times (0,2) - E_j(x_j, t_j)} \Phi(x - y, t - s) Hu(y, s) \, dy \, ds \leq \frac{C(n)}{\sqrt{t_j}^n} \int \int_{B_2(0) \times (0,2)} Hu(y, s) \, dy \, ds. \]

It follows therefore from (3.6), (3.8), (3.5), and (3.12) that

\[ u(x, t) \leq \frac{u_j(\xi, \tau) + C}{\sqrt{t_j}^n} \text{ for } (x, t) \in E_{t_j/4}(x_j, t_j) \]

where \( C \) is a positive constant which does not depend on \( j \) or \( (x, t) \).

Substituting \( (x, t) = (x_j, t_j) \) in (3.19) and using (3.7) we obtain

\[ u_j(0, 0) \to \infty \text{ as } j \to \infty. \]

For \( (\xi, \tau) \in E_1(0, 0) \) we have by (3.12) that

\[ Hu_j(\xi, \tau) = \sqrt{t_j}^{n+2} Hu(x, t). \]

Hence for \( (\xi, \tau) \in E_1(0, 0) \) we have by (3.11) that

\[ Hu_j(\xi, \tau) = f_j(\xi, \tau) \]

and for \( (\xi, \tau) \in E_1/4(0, 0) \) we have by (3.3) and (3.19) that

\[ Hu_j(\xi, \tau) \leq \sqrt{t_j}^{n+2} \left( u(x, t) + \sqrt{\frac{4}{3}} b \frac{1}{\sqrt{t_j}^n} \right)^{\frac{n+2}{n}} \]

\[ \leq \sqrt{t_j}^{n+2} \left( \frac{u_j(\xi, \tau) + C}{\sqrt{t_j}^n} \right)^{\frac{n+2}{n}} \]

\[ = (u_j(\xi, \tau) + C)^{\frac{n+2}{n}} \]

\[ =: v_j(\xi, \tau)^{\frac{n+2}{n}} \]

where the last equation is our definition of \( v_j \). Thus

\[ v_j(\xi, \tau) = u_j(\xi, \tau) + C \text{ for } (\xi, \tau) \in E_1/4(0, 0) \]

where \( C \) is a positive constant which does not depend on \( (\xi, \tau) \) or \( j \). Hence in \( E_1/4(0, 0) \) we have

\( Hu_j = Hv_j \) and

\[ \left( \frac{Hu_j}{v_j} \right)^{\frac{n+2}{2}} = Hu_j \left( \frac{Hu_j}{v_j} \right)^{n/2} < Hu_j = f_j. \]

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by (3.22) and (3.21). Thus

\[ \int \int_{E_{1/4}(0,0)} \left( \frac{Hv_j}{v_j} \right)^{\frac{n+2}{n}} d\eta d\zeta \to 0 \quad \text{as} \quad j \to \infty \]  

by (3.14).

Let \( 0 < R < 1/8 \) and \( \lambda > 1 \) be constants and let \( \varphi \in C_0^\infty(\sqrt{2R}B_0(0) \times (-2R, \infty)) \) satisfy \( \varphi \equiv 1 \) on \( E_R(0,0) \) and \( \varphi \geq 0 \) on \( \mathbb{R}^n \times \mathbb{R} \). Then

\[ \int \int_{E_{2R}(0,0)} (Hv_j)^{\lambda-1} \varphi^2 d\xi d\tau = \int \int_{E_{2R}(0,0)} \frac{Hv_j}{v_j} \left( \frac{\varphi}{v_j} \right)^{\lambda} d\xi d\tau \leq \left( \int \int_{E_{2R}(0,0)} \left( \frac{Hv_j}{v_j} \right)^{\frac{n+2}{n}} d\xi d\tau \right)^{\frac{n}{n+2}} \left( \int \int_{E_{2R}(0,0)} \left( v_j^{\lambda} \varphi^2 \right)^{\frac{n+2}{n}} d\xi d\tau \right)^{\frac{n}{n+2}} \].

Hence, using (3.21) and applying Lemma 2.3 with \( T = 2R, B = \Omega = B_{\sqrt{2R}}(0), E = E_{2R}(0,0), \) and \( u = v_j \) we have

\[ \int \int_{E_{2R}(0,0)} \left( v_j^{\lambda} \varphi^2 \right)^{\frac{n+2}{n}} d\xi d\tau \leq C \left( \int \int_{E_{2R}(0,0)} v_j^{\lambda} d\xi d\tau \right)^{\frac{n+2}{n}} \]

where \( C \) does not depend on \( j \). Therefore

\[ \int \int_{E_{R}(0,0)} v_j^{\frac{n+2}{n}} d\xi d\tau \leq C \left( \int \int_{E_{2R}(0,0)} v_j^{\lambda} d\xi d\tau \right)^{\frac{n+2}{n}}. \]  

(3.25)

Starting with (3.17) with \( q = \frac{n+1}{n} \) and applying (3.25) a finite number of times we find for each \( p > 1 \) there exists \( \varepsilon > 0 \) such that the sequence \( v_j \) is bounded in \( L^p(E_{\varepsilon}(0,0)) \) and thus the same is true for the sequence \( f_j \) by (3.22) and (3.21). Thus by (3.16) and Hölder’s inequality we have

\[ \lim \sup_{j \to \infty} \int \int_{E_{\varepsilon}(0,0)} \Phi(-\eta, -\zeta) f_j(\eta, \zeta) d\eta d\zeta < \infty \]  

(3.26)

for some \( \varepsilon > 0 \). Also by (3.14)

\[ \lim_{j \to \infty} \int \int_{E_{\varepsilon}(0,0)-E_{\varepsilon}(0,0)} \Phi(-\eta, -\zeta) f_j(\eta, \zeta) d\eta d\zeta = 0. \]

(3.27)

Adding (3.26) and (3.27), and using (3.15), we contradict (3.20).

4 Proof of Theorem 1.3

In this section we prove the following theorem which clearly implies Theorem 1.3.
Theorem 4.1. Suppose $u \in C^{2,1}(\Omega \times (0,2T))$ is a nonnegative solution of
\[
\begin{align*}
0 \leq u_t - \Delta u &\leq b \left(u + \frac{1}{\sqrt{t+1}}\right)^p \quad \text{in } \Omega \times (0,2T) \\
u &\leq b \quad \text{on } \partial \Omega \times (0,2T)
\end{align*}
\] (4.1)
where $T$ and $b$ are positive constants, $1 < p < 1 + 2/(n+1)$, and $\Omega$ is a $C^2$ bounded domain in $\mathbb{R}^n$, $n \geq 1$. Then there exists a positive constant $C$ such that
\[
u(x,t) \leq C \rho(x) \sqrt{t} \wedge 1 + \sup_{\partial \Omega \times (0,T)} u \quad \text{for all } (x,t) \in \Omega \times (0,T),
\] (4.2)
where $\rho(x) = \text{dist}(x, \partial \Omega)$.

Proof. Suppose for contradiction that (4.2) does not hold. Then there exists a sequence $\{(x_j, t_j)\} \subset \Omega \times (0,T)$ such that $t_j \to 0$ as $j \to \infty$ and
\[
u(x_j, t_j) - \sup_{\partial \Omega \times (0,T)} u \left(\frac{\rho(x_j)}{\sqrt{t_j}} \wedge 1\right) / \sqrt{t_j}^{n+1} \to \infty \quad \text{as } j \to \infty.
\] (4.3)
For $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ and $r > 0$, let
\[E_r(x,t) = \{(y,s) \in \mathbb{R}^n \times \mathbb{R}: |y - x| < \sqrt{r} \quad \text{and} \quad t - r < s < t\}.
\]
In what follows the variables $(x,t)$ and $(\xi, \tau)$ are related by
\[x = x_j + \sqrt{t_j} \xi \quad \text{and} \quad t = t_j + t_j \tau
\] (4.4)
and the variables $(y,s)$ and $(\eta, \zeta)$ are related by
\[y = x_j + \sqrt{t_j} \eta \quad \text{and} \quad s = t_j + t_j \zeta.
\] (4.5)
For each positive integer $j$, define
\[ho_j(\eta) = \frac{\rho(y)}{\sqrt{t_j}} \quad \text{and} \quad f_j(\eta, \zeta) = \sqrt{t_j}^{n+3} Hu(y, s) \quad \text{for } (y, s) \in \Omega \times (0,2T)
\] (4.6)
and define
\[
u_j(\xi, \tau) = \sqrt{t_j}^{n+1} \int_{E_{t_j}(x_j, t_j) \cap (\Omega \times (0,T))} G(x, y, t - s) Hu(y, s) dy ds \quad \text{for } (x, t) \in \Omega \times (0,2T)
\] (4.7)
where $Hu$ and $G$ are as in Lemma 2.2 and we define $G(x, y, \tau) = 0$ if $\tau \leq 0$. By (2.18) we have
\[
u_j(\xi, \tau) = \sqrt{t_j}^{n+1} \int_{E_{t_j}(x_j, t_j) \cap (\Omega \times (0,T))} \rho(y) Hu(y, s) dy ds \to 0 \quad \text{as } j \to \infty,
\] (4.8)
and thus making the change of variables (4.5) in (4.8) we get
\[
u_j(\eta, \zeta) \rho_j(\eta) \int_{E_{t_j}(x_j, t_j) \cap (\Omega \times (0,T))} f_j(\eta, \zeta) \rho_j(\eta) d\eta d\zeta \to 0 \quad \text{as } j \to \infty.
\] (4.9)
where \( D_j = \Omega_j \times (-1, 0) \) and \( \Omega_j = \{ \eta : y \in \Omega \} \).

Since, by (2.26) and (4.6),

\[
G(x, y, t - s) \leq \left( \frac{\rho(x)}{\sqrt{t - s}} \wedge 1 \right) \left( \frac{\rho(y)}{\sqrt{t - s}} \wedge 1 \right) \hat{G}(|x - y|, t - s)
\]

\[
= \left( \frac{\rho_j(\xi)}{\sqrt{\tau - \xi}} \wedge 1 \right) \left( \frac{\rho_j(\eta)}{\sqrt{\tau - \eta}} \wedge 1 \right) \frac{1}{\sqrt{t_j}} \hat{G}(|\xi - \eta|, \tau - \zeta),
\]

it follows from (4.7) and (4.6) that for \((\xi, \tau) \in (\Omega \times (-1, 0))\) we have

\[
|u_j(\xi, \tau)| \leq \iint_{E_j(0,0) \cap D_j} \left( \frac{\rho_j(\xi)}{\sqrt{\tau - \xi}} \wedge 1 \right) \left( \frac{\rho_j(\eta)}{\sqrt{\tau - \eta}} \wedge 1 \right) \hat{G}(|\xi - \eta|, \tau - \zeta) f_j(\eta, \zeta) \, d\eta d\zeta
\]  

(4.10)

where we define \( \hat{G}(r, \tau) = 0 \) if \( \tau \leq 0 \). It is easy to check that for \( 1 < q < \frac{n+2}{n+1} \) and \((\xi, \tau) \in \mathbb{R}^n \times (-1, 0)\) we have

\[
\left( \iint_{\mathbb{R}^n \times (-1, 0)} \left( \frac{1}{\sqrt{\tau - \xi}} \hat{G}(|\xi - \eta|, \tau - \zeta) \right)^q \, d\eta d\zeta \right)^{\frac{1}{q}} \leq C(n, q, \Omega, T) < \infty.
\]

Thus, for \( 1 < q < \frac{n+2}{n+1} \), we have by (4.10) and standard \( L^p \) estimates for the convolution of two functions that

\[
\|u_j\|_{L^q(E_j(0,0) \cap D_j)} \leq C(n, q, \Omega, T) \|f_j\|_{L^1(E_j(0,0) \cap D_j)} \to 0 \quad \text{as} \quad j \to \infty
\]

(4.12)

by (4.9).

If

\[
(x, t) \in \overline{E_{t_j/4}(x_j, t_j)} \cap (\Omega \times (0, T)) \quad \text{and} \quad (y, s) \in \Omega \times (0, t) - E_{t_j}(x_j, t_j)
\]

(4.13)

then

\[
|x - y| \geq \sqrt{t_j}/2
\]

(4.14)

and hence by (2.26) we have

\[
G(x, y, t - s) \leq \left( \frac{\rho(x)}{\sqrt{t - s}} \wedge 1 \right) \frac{\rho(y)}{\sqrt{t - s}} \hat{G} \left( \frac{\sqrt{t_j}}{2}, t - s \right)
\]

\[
\leq \rho(y) \max_{0 < \tau < \infty} \left( \frac{\rho(x)}{\sqrt{\tau}} \wedge 1 \right) \frac{1}{\sqrt{\tau}} \hat{G} \left( \frac{\sqrt{t_j}}{2}, \tau \right) = \frac{C(n, \Omega, T) \rho(y)}{\sqrt{t_j}} \left( \frac{\rho(x)}{\sqrt{t_j}} \wedge 1 \right).
\]

Thus for \((x, t) \in \overline{E_{t_j/4}(x_j, t_j)} \cap (\Omega \times (0, T))\) we have

\[
\iint_{\Omega \times (0, t) - E_{t_j}(x_j, t_j)} G(x, y, t - s) Hu(y, s) \, dy \, ds \leq \frac{C(n, \Omega, T)}{\sqrt{t_j^{n+1}}} \left( \frac{\rho(x)}{\sqrt{t_j}} \wedge 1 \right) \int_{\Omega \times (0, T)} \rho(y) Hu(y, s) \, dy \, ds.
\]

It follows therefore from Lemma 2.22 and (4.7) that

\[
u(x, t) \leq \frac{u_j(\xi, \tau) + C \left( \frac{\rho(x)}{\sqrt{t_j}} \wedge 1 \right)}{\sqrt{t_j^{n+1}}} + \sup_{\partial \Omega \times (0, T)} u \quad \text{for} \quad (x, t) \in \overline{E_{t_j/4}(x_j, t_j)} \cap (\Omega \times (0, T))
\]

(4.15)

where \( C \) is a positive constant which does not depend on \( j \) or \( (x, t) \).
Substituting \((x, t) = (x_j, t_j)\) in (4.15) and using (4.3) we obtain
\[
\frac{u_j(0, 0)}{\rho_j(0) \wedge 1} \geq \frac{u(x_j, t_j) - \sup_{\partial \Omega \times (0, T)} u}{\left(\frac{\rho(x_j)}{\sqrt{T_j}} \wedge 1\right)/\sqrt{T_j}^{n+1}} - C \to \infty \quad \text{as} \quad j \to \infty.
\] (4.16)

For \((\xi, \tau) \in E_1(0, 0) \cap D_j\) we have by (4.7) that
\[
(Hu_j)(\xi, \tau) = \sqrt{T_j}^{n+3}(Hu)(x, t).
\] (4.17)
Hence for \((\xi, \tau) \in E_1(0, 0) \cap D_j\) we have by (4.6) that
\[
(Hu_j)(\xi, \tau) = f_j(\xi, \tau)
\] (4.18)
and for \((\xi, \tau) \in E_{1/4}(0, 0) \cap D_j\) we have by (4.1) and (4.15) that
\[
Hu_j(\xi, \tau) \leq \sqrt{T_j}^{n+3} b \left(\frac{u_j(\xi, \tau) + C}{\sqrt{T_j}^{n+1}}\right)^p
\]
\[
= \sqrt{T_j}^3 b(u_j(\xi, \tau) + C)^p \quad \text{where} \quad \alpha = (n + 1)\left(\frac{n + 3}{n + 1} - p\right) > 0
\]
\[
=: \sqrt{T_j}^3 bv_j(\xi, \tau)^p,\] (4.19)
where the last equation is our definition of \(v_j\). Thus
\[
v_j(\xi, \tau) = u_j(\xi, \tau) + C
\] (4.20)
where \(C\) is a positive constant which does not depend on \((\xi, \tau)\) or \(j\). Hence in \(E_{1/4}(0, 0) \cap D_j\) we have
\[
\left(\frac{H u_j}{v_j}\right)^{\frac{n+2}{2}} \leq \left(\sqrt{T_j}^\alpha b v_j^{p-1}\right)^{\frac{n+2}{2}} \leq \sqrt{T_j}^{\alpha(n+2)/2} b^{\frac{n+2}{2}} v_j^q,
\]
where \(q = (p - 1)\frac{n+2}{2} < \frac{2}{n+1}\frac{n+2}{2} = \frac{n+2}{n+1}\). Thus
\[
\int_{E_{1/4}(0,0) \cap D_j} \left(\frac{H u_j}{v_j}\right)^{\frac{n+2}{2}} d\eta d\zeta \leq \sqrt{T_j}^{\alpha(n+2)/2} b^{\frac{n+2}{2}} \|v_j\|_{L^q(E_1(0,0) \cap D_j)} \to 0 \quad \text{as} \quad j \to \infty
\] (4.21)
by (4.12).

Let \(0 < R < 1/8\) and \(\lambda > 1\) be constants and let \(\varphi \in C_0^\infty(B_\sqrt{2R}(0, 0) \times (-2R, \infty))\) satisfy \(\varphi \equiv 1\) on \(E_R(0, 0)\) and \(\varphi \geq 0\) on \(\mathbb{R}^n \times \mathbb{R}\). Then using (4.20) we have
\[
v_j^\lambda \varphi^2 = (u_j + C)^\lambda \varphi^2 \leq 2^\lambda (u_j^\lambda \varphi^2 + C^\lambda \varphi^2) \quad \text{in} \quad E_{1/4}(0,0) \cap D_j
\]
and hence
\[
\int_{E_{2R}(0,0) \cap D_j} (Hu_j)u_j^{-1} \varphi^2 \, d\tau \leq \int_{E_{2R}(0,0) \cap D_j} (Hu_j)u_j^{-1} \varphi^2 \, d\tau
\]
\[
= \int_{E_{2R}(0,0) \cap D_j} \frac{Hu_j}{v_j} \varphi^2 \, d\tau
\]
\[
\leq \left( \int_{E_{2R}(0,0) \cap D_j} \left( \frac{Hu_j}{v_j} \right)^{\frac{n+2}{n+2}} \varphi^2 \, d\tau \right)^{\frac{n}{n+2}} \left[ \int_{E_{2R}(0,0) \cap D_j} \left( \frac{v_j^\lambda \varphi^2}{\lambda} \right)^{\frac{n+2}{n}} \, d\xi \, d\tau \right]^{\frac{n}{n+2}}
\]
\[
\leq C \left( \int_{E_{2R}(0,0) \cap D_j} \left( \frac{Hu_j}{v_j} \right)^{\frac{n+2}{n+2}} \varphi^2 \, d\tau \right)^{\frac{n}{n+2}} \left[ \int_{E_{2R}(0,0) \cap D_j} \left( \frac{u_j^\lambda \varphi^2}{\lambda} \right)^{\frac{n+2}{n}} \, d\xi \, d\tau \right]^{\frac{n}{n+2}} + 1
\]
(4.22)

where \( C \) is a positive constant which does not depend on \( j \) and whose value may change from line to line. Thus using (4.21) and applying Lemma 2.3 with \( T = 2R \), \( B = B_{2R}(0) \), \( E = E_{2R}(0,0) \), \( \Omega = \Omega_j \), and \( u = u_j \), we have

\[
\left( \int_{E_{2R}(0,0) \cap D_j} \left( \frac{u_j^\lambda \varphi^2}{\lambda} \right)^{\frac{n+2}{n}} \, d\xi \, d\tau \right)^{\frac{n}{n+2}} \leq C \left( \int_{E_{2R}(0,0) \cap D_j} \frac{u_j^\lambda \varphi^2}{\lambda} \, d\xi \, d\tau + 1 \right),
\]

Consequently,

\[
\int_{E_{2R}(0,0) \cap D_j} \frac{u_j^\lambda \varphi^2}{\lambda} \, d\xi \, d\tau \leq C \left( \int_{E_{2R}(0,0) \cap D_j} \frac{u_j^\lambda \varphi^2}{\lambda} \, d\xi \, d\tau + 1 \right)^{\frac{n+2}{n}}
\]
(4.23)

By (4.12),

\[
\lim_{j \to \infty} \int_{E_{2R}(0,0) \cap D_j} u_j^{\frac{n+3}{n+2}} \, d\xi \, d\tau = 0.
\]
(4.24)

Starting with (4.21) and using (4.23) a finite number of times we find that for each \( p > 1 \) there exists \( \varepsilon > 0 \) such that the sequence \( u_j \) is bounded in \( L^p(E_{2R}(0,0) \cap D_j) \) and thus the same is true for the sequences \( v_j, Hu_j, \) and \( f_j \) by (4.20), (4.19), and (4.18).

Thus by (4.10), there exists \( \varepsilon > 0 \) such that

\[
\limsup_{j \to \infty} \frac{u_j(0,0)}{\rho_j(0)} \leq \limsup_{j \to \infty} \int_{E_{1}(0,0) \cap D_j} \frac{1}{\sqrt{-\zeta}} \left( \frac{\rho_j(\eta)}{\sqrt{-\zeta}} \right)^{\lambda} \tilde{G}(\eta - \zeta, \zeta) f_j(\eta, \zeta) \, d\eta \, d\zeta
\]
\[
\leq \limsup_{j \to \infty} \left( \int_{E_{2}(0,0) \cap D_j} \frac{1}{\sqrt{-\zeta}} \tilde{G}(\eta - \zeta, \zeta) f_j(\eta, \zeta) \, d\eta \, d\zeta \right) + \int_{(E_{1}(0,0) - E_{2}(0,0)) \cap D_j} \frac{1}{\zeta} \tilde{G}(\eta - \zeta, \zeta) f_j(\eta, \zeta) \rho_j(\eta) \, d\eta \, d\zeta < \infty
\]

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where we have estimated the first integral using (4.11) and Hölder’s inequality and the second integral using (4.9). Also by (4.10),

\[
\limsup_{j \to \infty} u_j(0,0) \leq \limsup_{j \to \infty} \int\int_{E_1(0,0) \cap D_j} \left( \frac{\rho_j(\eta)}{\sqrt{-\zeta}} \wedge 1 \right) \hat{G}(|-\eta|, -\zeta) f_j(\eta, \zeta) \, d\eta \, d\zeta
\]

\[
\leq \limsup_{j \to \infty} \int\int_{E_1(0,0) \cap D_j} \hat{G}(|-\eta|, -\zeta) f_j(\eta, \zeta) \, d\eta \, d\zeta
\]

\[
+ \int\int_{(E_1(0,0) - E_\varepsilon(0,0)) \cap D_j} \frac{1}{\sqrt{-\zeta}} \hat{G}(|-\eta|, -\zeta) f_j(\eta, \zeta) \rho_j(\eta) \, d\eta \, d\zeta < \infty.
\]

Hence

\[
\limsup_{j \to \infty} \frac{u_j(0,0)}{\rho_j(0) \wedge 1} < \infty
\]

which contradicts (4.16) and completes the proof of Theorem 4.1.

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