COUPLING RESONANCES AND SPECTRAL PROPERTIES OF THE PRODUCT OF RESOLVENT AND PERTURBATION

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Abstract. Given a self-adjoint operator $H_0$ and a relatively compact self-adjoint perturbation $V$, we study in some detail the spectral properties of the product $(H_0 - z)^{-1}V$, $z \in \mathbb{C}$. For some numbers $r_z$, the eigenvalues of $(H_0 + sV - z)^{-1}V$, $s \in \mathbb{C}$, are $(s - r_z)^{-1}$. We study the root spaces of the eigenvalues $(s - r_z)^{-1}$ and complex-analytic properties of the functions $r_z$ such as branching points. In particular, for a generic case, we give a variety of necessary and sufficient conditions for branching. The functions $r_z$, called coupling resonances, are important in the spectral analysis of $H_0 + rV$ for any real number $r$. For instance, they afford a description of the spectral shift function (SSF) of the pair $H_0$ and $V$, as well as the absolutely continuous and singular parts of the SSF. A thorough study of real-valued coupling resonances $r_\lambda$ for real $\lambda$ outside of the essential spectrum was carried out in a recent work by the first author. Here we extend this study to the complex domain, motivated by the fact, which is well known in the case of a rank-one perturbation, that the behaviour of coupling resonances $r_z$ near the essential spectrum provides valuable information about the latter.

1. Introduction

The product
\begin{equation}
R_z(H_0)V = (H_0 - z)^{-1}V,
\end{equation}
where $H_0$ is a self-adjoint operator and $V$ is a $H_0$-compact self-adjoint operator, is ubiquitous within perturbation and spectral theories, appearing for example in the second resolvent identity, Neumann series and its numerous applications, perturbation determinants, the stationary formula for the scattering matrix, and in scattering theory more generally. A detailed study of the spectral properties of this operator is of interest, yet we feel that the subject has not received the attention it deserves. This paper arose more specifically because of the relevance of the spectral properties of (1) to the phenomenon, associated with the pair of self-adjoint operators $H_0$ and $H_1 := H_0 + V$, of the flow of singular spectrum within the aggregate of essential spectrum. Before discussing this connection and the context it provides, one straightforward way to outline the main result of this paper is as follows.

Since the operator (1) is compact, its spectrum is discrete away from zero, which is the only element of its essential spectrum. Further, since it depends holomorphically on $z$, the isolated elements of its spectrum are also holomorphic in $z$ – everywhere besides potentially any of the discrete set of exceptional points, defined as those points where the number of distinct eigenvalues within a small neighbourhood of a given isolated eigenvalue is different from its otherwise constant value. The general theory of e.g. Kato’s classic \cite{18} provides many other properties of these eigenvalues. However, we are interested here in those spectral properties that are specific to (1). Three questions which can be posed in this regard are:

- Can an eigenvalue get absorbed by, or, which is the same, emerge from, zero?
- Are there branching points of the eigenvalues?

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Do the eigenvalues have critical points, that is, points with zero derivative?
The main result of this paper addresses the second question. In a limited yet generic case, we establish various equivalent conditions to branching. One characterisation for an eigenvalue $-s^{-1} = -s(z)^{-1}$ of $H_s := H_0 + sV$ to have a branching point at $z$ is as follows. Let $\varphi$ be an eigenvector of $H_s$ corresponding to the eigenvalue $z$. Note that the eigenspaces of $H_s$ and $H_0$ corresponding to the eigenvalues $-s^{-1}$ and $z$ respectively are equal, as is easily seen by multiplying the eigenvalue equations by $H_0 - z$ and $R_z(H_0)$ respectively. Then $z$ is a branching point of $s(z)$ iff there exists a ‘conjugate’ vector $\varphi^*$, belonging to the eigenspace of $H_s = H_0 + sV$ corresponding to $\bar{z}$ and satisfying $\langle \varphi^*, \varphi \rangle = 1$, such that $\langle \varphi^*, V \varphi \rangle = 0$.

The limitation of this result is that we assume the existence of a conjugate vector as well as its regular behaviour as a function of $s$; an assumption which holds for example if the eigenvalue $z$ of $H_s$ is simple, or if $z$ is semisimple and, when considered as a function of $s$, it does not have an exceptional point at $s$. We believe that this concept of a conjugate vector may prove useful in further study of the essential spectrum.

The eigenvalues of (1) are related to the eigenvalues of

$$R_z(H_s)V = (H_0 + sV - z)^{-1}V,$$

by a simple relation: there are functions, $r_j^z$, $j = 1, 2, \ldots$, of the variable $z$ only, such that for any $s \in \mathbb{C}$ for which $z$ belongs to the resolvent set of $H_s$, the eigenvalues of (2) are

$$(s - r_j^z)^{-1}, \quad j = 1, 2, \ldots$$

This relation shows that the functions $r_j^z$ have no less bearing than the eigenvalues themselves.

If $V$ has rank one, there is only one such function, which appears in different guises, for instance in relation to Weyl $m$-functions in the theory of the Sturm-Liouville equation (see e.g. [28]) and in the theory of random Schrödinger operators (see e.g. [1] Theorem 5.3). If rank($V$) = 1 and also $V \geq 0$, the function $r_z$ is a Herglotz function, whose properties are well known to be effective in the study of the spectral properties of $H_0$ and $H_1$ (e.g. [15] [28]). Unfortunately, complications arise for more general $V$, due in a large part to the fact that the functions $r_j^z$ may have branching points.

The functions $r_j^z$ admit other equivalent descriptions. As mentioned above, they are those values of $s$ for which the energy variable $z$ is an eigenvalue of $H_s$. They are also the poles of (2) treated as a meromorphic function of the coupling variable $s$. Indeed, the second resolvent identity implies

$$R_z(H_s)V = (1 + sR_z(H_0)V)^{-1}R_z(H_0)V$$

and it follows from the analytic Fredholm alternative (see e.g. [24] Theorem VI.14)) that when considered as a function of $s$, the poles of (2) occur when $s = r_j^z$. It is convenient to represent this relation symbolically as

$$z \leftrightarrow s,$$

in which the energy and coupling variables are in general multi-valued functions of one another which can have branching points.

Following the physics literature, poles of the scattering matrix $S(z; H_1, H_0)$ treated as a function of $z$ are called energy resonance points (see e.g. [32]). For this reason, we refer to those values of $s$ such that $z \leftrightarrow s$ as coupling resonance points, since they are poles of the scattering matrix $S(z; H_s, H_0)$ treated as a function of $s$ (see [4] [5] [7]). Similarly, we refer to the functions $r_j^z$
as coupling resonance functions of the pair $H_0$ and $V$. Because we will only be concerned with coupling resonances here, we usually leave the word ‘coupling’ implicit.

In the next few pages we discuss the connection between coupling resonances and the flow of singular spectrum and related topics, hoping to give a fuller perspective and further explain our motivation. However, this discussion digresses from the focus of this paper and the upcoming sections do not depend on it.

Coupling resonances appear to contain a lot of information about the pair $H_0$ and $V$. For instance, assuming $V$ satisfies a certain trace class condition, the Lifshitz-Krein spectral shift function (SSF) $\xi(\lambda; H_1, H_0)$ can be expressed in such terms (see [8]):

$$\xi(\lambda; H_1, H_0) = \frac{1}{2\pi} \int_0^1 \sum_{j=1}^{\infty} \frac{2\beta_j^2}{|r - \alpha_j^2|^2 + |\beta_j^2|^2} \, dr + \sum_{r_\lambda \in [0,1]} \text{ind}_{\text{res}}(\lambda; H_\lambda, V),$$

where $r_{\lambda+i0} = \alpha_j^\pm + i\beta_j^\pm$ are the limits of resonance points with non-zero imaginary part $\beta_j^\pm$ and the second term is the sum of resonance indices, which are only non-zero for the discrete set of real limit points $r_\lambda = r_{\lambda+i0}$.

The resonance index, which originates in [5], measures the flow of singular spectrum and provides the previously mentioned link between this phenomenon and the spectral properties of ($H_0$). It is those eigenvalues of ($H_0$) which converge to the real axis as $z = \lambda + iy \to \lambda + i0$ that are involved in the flow of singular spectrum of $H_s$ through the point $\lambda$. More specifically, if $N_+$ and $N_-$ respectively denote the numbers of eigenvalues of ($H_0$) converging from the upper $\mathbb{C}_+$ and lower $\mathbb{C}_-$ complex half-planes to a real number $-r_\lambda^{-1}$ as $y \to 0^+$, then their difference $N_+ - N_-$ is equal to the net flux of singular spectrum through $\lambda$ as the perturbation $H_0$ crosses the operator $H_\lambda = H_0 + r_\lambda V$. The infinitesimal singular spectral flow defined in this way is the resonance index, denoted

$$\text{ind}_{\text{res}}(\lambda; H_\lambda, V) = N_+ - N_-.$$

The sum of resonance indices at points $r_\lambda$ from the compact interval $[0,1]$, only finitely many of which can be non-zero, is called the total resonance index for the pair $H_0$ and $H_1$. For a point $\lambda$ outside of the common essential spectrum $\sigma_{\text{ess}}$ of $H_0$ and $H_1$, the associated flow of singular spectrum is also known as spectral flow which has many characterisations. In this case, for $\lambda \notin \sigma_{\text{ess}}$, direct proofs can be found in [3] of the agreement of the total resonance index with four prominent definitions of spectral flow: in terms of the intersection number of eigenvalues with a point ($\mathbb{2}$), the Fredholm index of a pair of projections ($\mathbb{3}$, $\mathbb{21}$), Robbin-Salamon’s axioms ($\mathbb{26}$), and the SSF ($\mathbb{19}$). Of these definitions, the resonance index has the advantage of requiring minimal assumptions and moreover it is the only one capable of measuring the flow of singular spectrum within the essential spectrum as well.

If $\lambda$ belongs to the essential spectrum, it is convenient to assume a decomposition

$$V = F^* JF,$$

where $F$ is a closed $|H_0|^{1/2}$-compact operator and $J$ is a bounded self-adjoint operator. One choice that works for any $H_0$-compact self-adjoint $V$ is $F = |V|^{1/2}$ and $J = \text{sgn} V$. This factorisation also provides a technique for dealing with more general symmetric forms $V$ that are merely relatively form-compact with respect to $H_0$. Let the sandwiched resolvent be denoted

$$T_z(H_0) := FR_z(H_0)F^*.$$

Then instead of ($\mathbb{11}$) we can consider the compact operator

$$T_z(H_0)J,$$
which shares the same non-zero eigenvalues, but has the advantage of possibly existing in the limit as \( z = \lambda + iy \) approaches the spectrum of \( H_0 \). If the sandwiched resolvent has a norm limit

\[
T_{\lambda+iy}(H_0) := \lim_{y \to 0^+} T_{\lambda+iy}(H_0),
\]

which for example, if \( V \) satisfies a relatively trace class condition, occurs for a.e. point \( \lambda \in \mathbb{R} \) (see e.g. [11, 14, 31]), then the limits of eigenvalues of (1) are eigenvalues of (4) with \( z = \lambda + i0 \) and the resonance index is well-defined.

The total resonance index is related to the Birman-Schwinger principle (see e.g. [28, 27]), which is based on the relation \( z \leftrightarrow s \) and is used for example in the evaluation of bounds on the number of eigenvalues of Schrödinger operators. Supposing \( H_0 \) is bounded from below, \( V \leq 0 \), and \( z = \lambda - \inf \sigma(H_0) \), it states that the number of eigenvalues of \( H_1 \) which are less than \( \lambda \) is equal to the number \( N_\lambda \) of eigenvalues of \( R_\lambda(H_0)V \) which are less than \(-1\). In [30] (also see [12]), it is observed that the Birman-Schwinger principle can be extended to a representation of the SSF for \( \lambda \notin \sigma_{ess} \): again supposing \( V \leq 0 \),

\[
\xi(\lambda; H_1, H_0) = -N_\lambda.
\]

Note that the total resonance index gives the same result, since for \( V \leq 0 \) all the eigenvalues of (1) shift into \( \mathbb{C}_- \) as \( z = \lambda + i0 \to \lambda + iy \) with \( y > 0 \). The representation (5) of the flow of singular spectrum is extended to the essential spectrum in [22], where the focus is on an interval of essential spectrum in a neighbourhood of which the sandwiched resolvent \( T_z(H_0) \) is assumed to be uniformly continuous in the operator norm. The total resonance index provides a further extension, valid at any point \( \lambda \) where the limit \( T_{\lambda+i0}(H_0) \) exists.

Within the essential spectrum, the relationship of the resonance index to the SSF is clarified as follows. Assuming \( V \) is of relatively trace class type, \( \xi(\lambda; H_1, H_0) \) and its absolutely continuous \( \xi^{a}(\lambda; H_1, H_0) \) and singular \( \xi^{s}(\lambda; H_1, H_0) \) parts can be respectively defined by the formula

\[
\xi^\#(\varphi; H_1, H_0) = \int_0^1 \text{Tr} \left( E^\#_r(\text{supp } \varphi)V \varphi(H_r) \right) \, dr, \quad \varphi \in C_c(\mathbb{R}),
\]

where the placeholder \# is either removed altogether, or replaced by \((a)\), or \((s)\), in which case \( E^{a}_r \) and \( E^{s}_r \) denote the absolutely continuous and singular parts of the spectral measure \( E_r \) corresponding to the self-adjoint operator \( H_r = H_0 + rV \). The first case is the famous Birman-Solomyak formula for the SSF [13], while the variants defining its absolutely continuous and singular parts originate in [4]. To be clear, in any case this formula defines a locally finite measure via the Riesz representation theorem, which can be shown to be absolutely continuous and can therefore be identified with its density function (see e.g. [7]).

The singular SSF is integer-valued and coincides with the total resonance index (see e.g. [3, 7]): For a.e. \( \lambda \in \mathbb{R} \),

\[
\xi^{s}(\lambda; H_1, H_0) = \sum_{r_\lambda \in [0,1]} \text{ind}_{\text{res}}(\lambda; H_{r_\lambda}, V).
\]

(2) \( \xi^{a}(\lambda; H_1, H_0) \). In fact the total resonance index was the result of a search for a more tangible representation of the singular SSF, after it became clear that, as a function of \( r \in \mathbb{R} \) for fixed \( \lambda \), \( \xi^{s}(\lambda; H_{r_\lambda}, H_0) \) is locally constant with discontinuities at the real limits \( r_\lambda \) of resonance functions. This fact was derived from a representation of the singular SSF in terms of the scattering matrix, which we now outline before returning to the focus of this paper.
For our purposes, the (off-axis) scattering matrix can be defined by the ‘stationary formula’

\[ S(z; H_r, H_0) := 1 - 2i \sqrt{\text{Im} T_z(H_0)} J(1 + r T_z(H_0))^{-1} \sqrt{\text{Im} T_z(H_0)}, \]

where \( r \in \mathbb{R} \) and \( z = \lambda + iy \), for \( y \geq 0 \). The case when \( y = 0 \) is defined by the limit \( y \to 0^+ \), assuming it exists. This amounts to assuming the existence of \( T_{\lambda+i0}(H_0) \) along with \( T_{\lambda+i0}(H_r) \), and for this to be possible, it must be that \( s = 0 \) and \( s = r \) are not limit points of resonance functions. Note that a resonance function cannot take real values except in the limit as \( y \to 0 \).

An algebraic calculation shows that the scattering matrix is unitary and thus belongs to the operator-normed group

\[ \{ \text{unitary operators } S \text{ such that } S - 1 \text{ is compact} \}. \]

The eigenvalues of any such operator lie on the unit circle \( \mathbb{T} \) and can only accumulate at the common essential spectrum \( \{ 1 \} \). We now consider the limit of (7) as \( y \to +\infty \). Since in this case \( \text{Im} T_z(H_0) \to 0 \), it follows that the scattering matrix \( S(\lambda; H_1, H_0) \) for the pair \( H_0 \) and \( H_1 \) can be continuously connected to the identity operator 1 within the normed group (8). In doing so, all of its eigenvalues are deformed along the circle to 1. Following [23], we denote the net number of eigenvalues which cross a given point \( e^{i\theta} \in \mathbb{T} \), or in other words the spectral flow through \( e^{i\theta} \), by \( -\mu(\theta, \lambda; H_1, H_0) \). In this context, our preferred method of definition for the flow of discrete spectrum is based on the continuous enumeration of eigenvalues – an intuitive but non-trivial result which is presented in [10]. Assuming \( V \) is relatively trace class, it is proved by A. Pushnitski in [23] that the average of \( -\mu(\theta, \lambda; H_1, H_0) \) over \( e^{i\theta} \in \mathbb{T} \) is equal to the SSF. That is, for a.e. \( \lambda \in \mathbb{R} \),

\[ \xi(\lambda; H_1, H_0) = -\frac{1}{2\pi} \int_0^{2\pi} \mu(\theta, \lambda; H_1, H_0) \, d\theta. \]

Spectral flow is a well-known homotopy invariant, so that \( \mu(\theta, \lambda; H_1, H_0) \) is not altered by continuous deformations of the path connecting \( S(\lambda; H_1, H_0) \) with the identity. However, there is a non-equivalent such path: instead of sending the imaginary part of the energy variable \( z = \lambda + iy \) from \( y = 0 \) to \( y = +\infty \), send the coupling variable from \( r = 1 \) to \( r = 0 \). When defined by (7), it is evident (from the analytic Fredholm theorem) that the scattering matrix depends meromorphically on the coupling variable \( r \) and that its poles can only occur at real resonance points. However these turn out to be removable singularities, since the scattering matrix is unitary and thus bounded away from the discrete set of such resonance points.

The definition (7) is properly justified by the following theorem (see e.g. [7], the proof is based on an argument originating in [5]): Assume the limiting absorption principle holds, by which we mean that the set \( \Lambda \) of values of \( \lambda \) such that both limits \( T_{\lambda+i0}(H_0) \) and \( T_{\lambda+i0}(H_r) \) exist, has full measure in \( \mathbb{R} \). Then there is a spectral representation of the absolutely continuous part of \( H_0 \) as a direct integral over \( \Lambda \), on which the definition of \( S(\lambda; H_r, H_0) \) by (7) coincides with the classical definition of the scattering matrix in terms of the fibres of the product of wave operators. Since the full set \( \Lambda \) is given explicitly, it becomes possible to consider the scattering matrix as a function of the coupling variable. Assuming \( V \) satisfies a relatively trace class condition, the set \( \Lambda \) is known to have full measure for any \( r \in \mathbb{R} \) (see e.g. [11, 14, 31]). Moreover, if \( \lambda \) is such that \( T_{\lambda+i0}(H_0) \) exists, then \( T_{\lambda+i0}(H_r) \) must also exist and hence \( \lambda \in \Lambda \), as long as \( r \) is not a real resonance point.

After removing a finite number of singularities as \( r = 1 \) is deformed to \( r = 0 \), we obtain an analytic path connecting \( S(\lambda; H_1, H_0) \) with the identity. Following [5], let the spectral flow along this path be denoted by \( -\mu^{(a)}(\theta, \lambda; H_1, H_0) \). Assuming \( V \) is relatively trace class, the average of \( -\mu^{(a)}(\theta, \lambda; H_1, H_0) \) over \( e^{i\theta} \in \mathbb{T} \) is shown in (7), following an argument originating in [5], to
be equal to the absolutely continuous SSF; for a.e. \( \lambda \in \mathbb{R} \),

\[
\xi^{(a)}(\lambda; H_1, H_0) = -\frac{1}{2\pi} \int_0^{2\pi} \mu^{(a)}(\theta, \lambda; H_1, H_0) d\theta.
\]

A representation of the singular SSF is obtained by subtracting (10) from (9). By the additivity of spectral flow, the so-called singular \( \mu \)-invariant, defined by

\[
\mu^{(s)}(\lambda; H_1, H_0) := \mu(\theta, \lambda; H_1, H_0) - \mu^{(a)}(\theta, \lambda; H_1, H_0),
\]

is the spectral flow of the scattering matrix \( S(z; H_r, H_0) \), \( z = \lambda + iy \), as it traverses the loop based at the identity which results as \((y, r)\) goes from \((+\infty, 1)\) to \((0, 1)\) to \((0, 0)\). As a (finite) sum of winding numbers of eigenvalues, the singular \( \mu \)-invariant does not depend on the angle \( \theta \) and we find that for a.e. \( \lambda \in \mathbb{R} \),

\[
\xi^{(s)}(\lambda; H_1, H_0) = -\mu^{(s)}(\lambda; H_1, H_0).
\]

Given the previous two representations of the singular SSF (9) and (11), it follows that the total resonance index and singular \( \mu \)-invariant are equal for relatively trace-class \( V \) and a.e. \( \lambda \in \mathbb{R} \). Note however that a relatively trace class condition is not required for the definition of either of these representations. Suppose \( \lambda \) is such that both norm limits \( T_{\lambda+i0}(H_0) \) and \( T_{\lambda+i0}(H_1) \) exist. Then both the total resonance index and the singular \( \mu \)-invariant are well-defined at \( \lambda \) and moreover, as shown in [9], these integers are equal. This leads to the conjecture that, although the SSF does not exist in this context, the singular SSF does exist and is given by the same integer. Much more information about the decomposition of the SSF and its connections to scattering theory can be found e.g. in [4,7] and for more about the total resonance index including other descriptions see e.g. [5,9].

Returning to the focus of this paper, things are much simpler outside of the essential spectrum, as is to be expected. In its complement in \( \mathbb{R} \), the properties of resonance functions are studied in detail in [6]. Here we extend this study to the complex plane, with the aim to be able to approach the essential spectrum and allow a further study of its perturbation by relatively compact operators. Although a number of results have proofs similar to those of [6], there are some essential differences and for this reason we believe that a detailed presentation of this material is warranted.

In terms of the relation \( z \leftrightarrow s \), shifting into the complex plane entails working with non-self-adjoint operators \( H_s \). In fact, we will consider perturbing the operator \( H_s \) in another relatively compact direction \( W \). This introduces a few novel hurdles, since some arguments used in [6] are specific to the self-adjoint case. The techniques presented in this paper are not only more general, but are also simpler in some aspects. However, these new elements of the method do not void the approach of [6] which allows to prove deeper results for the self-adjoint case.

Although the eigenspaces of \( H_s \) at \( z \) and of \( \Gamma \) at \(-s^{-1}\) coincide, the root spaces do not. In the self-adjoint case of \( s \in \mathbb{R} \), the eigenvalue \( z \) is semisimple, i.e. the root space and eigenspace of \( H_s \) coincide, and for \( s \in \mathbb{C} \setminus \mathbb{R} \) this is more or less an assumption we will make. On the other hand, the root space of \( \Gamma \) can be strictly larger than its eigenspace even for real \( s \). Root vectors are called resonance vectors of the triple \((z; H_0, V)\). In Section 2 we consider the Laurent coefficients of the resolvent \( R_{\lambda}(H_s) \) as a function of the coupling variable \( s \) and their connection with the projection onto the space of resonance vectors. In particular, it is shown that the latter is equal to the product of the residue of the resolvent at a resonance point with the perturbation.

Section 1 concerns the assumptions used in later sections, which in essence amount to the following: Given a path of eigenvectors \( \varphi(s) \) corresponding to the eigenvalue \( z(s) \) of \( H_s \), we assume the existence of a non-orthogonal conjugate eigenpath \( \varphi^*(s) \) corresponding to the eigenvalue \( \bar{z} \).
of $H_s$. For any given eigenvector $\varphi$ corresponding to a fixed eigenvalue $z$ of $H_s$, the existence of an eigenvector $\varphi^*$ of $H_s$ corresponding to $\bar{z}$ which is not orthogonal to $\varphi$, is an equivalent condition to the semisimplicity of the eigenvalue $z$. In addition to this, we require a certain stability under perturbations of the coupling variable. These conditions are automatically satisfied in the self-adjoint case.

As shown in [6], the structure of the space of resonance vectors is related to the behaviour of eigenvalues of $H_s$ as they cross a real threshold value $z = \lambda$. For example, assume that an eigenvalue $\lambda$ of $H_{r\lambda}$, for $r\lambda \in \mathbb{R}$, is simple. Then, provided that $\lambda$ is not also an eigenvalue of $H_0$, the dimension of the root space of $R_\lambda(H_0)V$ corresponding to the eigenvalue $-r^{-1}$ tells us, in particular, whether the eigenvalue $z$ of $H_s$ stops instantaneously at $\lambda$, that is, whether $dz/ds$ is zero at $s = r\lambda$, and if so, in which direction it goes after stopping. That is, for a simple eigenvalue $z$ of $H_s$, the dimension of the root space is equal to the order of $s$ as a critical point of $z$. This result is generalised to the non-self-adjoint case in Section 5. We define the order of an eigenpath for $H_s$, establish various characterisations, and show that it exceeds 1 iff $z$ is a branching point of the eigenvalue $-s^{-1}$ of $H$. In particular, we prove the following theorem, in which $A_s(H_s, V)$ denotes a nilpotent operator associated to the space of resonance vectors.

**Theorem 1.1.** Suppose $z \leftrightarrow s$, that is, $-s^{-1}$ is an eigenvalue of $R_z(H_0)V$, or equivalently $z$ is an isolated eigenvalue of $H_s = H_0 + sV$. Consider $z$ as a function of $s$ in a neighbourhood of some point $s_0$. Let $\varphi(s)$ be an eigenvector function corresponding to $z(s)$. Assume $z_0 := z(s_0)$ is a simple eigenvalue. Then the following assertions are equivalent:

(i) the eigenpath $\varphi(s)$ has order $\geq 2$ at $s_0$,  
(ii) $\langle \varphi^*, V\varphi(s_0) \rangle = 0$ for some eigenvector $\varphi^*$ of $H_{s_0}$ corresponding to the eigenvalue $\bar{z}_0$, such that $\langle \varphi^*, \varphi(s_0) \rangle \neq 0$,  
(iii) $z'(s_0) = 0$,  
(iv) $(H_{s_0} - z_0)\varphi'(s_0) = -V\varphi(s_0)$,  
(v) the vector $\varphi(s_0)$ belongs to the image of $A_{s_0}(H_{s_0}, V)$,  
(vi) $A_{s_0}(H_{s_0}, V)\varphi'(s_0) = \varphi(s_0)$,  
(vii) $z_0$ is a branching point of the inverse function $s(z)$.

The equivalence (iii) $\iff$ (vii) is of course a well-known property of holomorphic functions.

The structure of the projection onto the space of resonance vectors is considered in more detail in Section 6. We establish its decomposition into a sum of projections corresponding to cycles of resonance points. This provides a natural Jordan basis of resonance vectors, in which we consider a Schmidt representation of the resonance projection.

Section 7 demonstrates a connection between order and tangency to the so-called resonance set. The resonance set is by definition the algebraic variety of operators, obtained from $H_{s_0}$ by relatively compact perturbations of the form $vW$ with $W = W^*$ and $v \in \mathbb{C}$, which share a given eigenvalue $z_0$. For a simple eigenvalue $z_0$ of $H_{s_0}$, it is shown that a path of operators of the form $H_{s_0} + vW$ is tangent to the resonance set if and only if it has a corresponding eigenpath of order $\geq 2$.

Finally, Section 8 is a short remark on the isospectrality solutions to Lax’s equation in light of these results.

2. Preliminaries

A $C^1$-function $f$ acting from an open subset $G$ of $\mathbb{C}$ to a Hilbert space $\mathcal{H}$ is anti-holomorphic, if the limit $\partial_s f(s) := \lim_{h \to 0} h^{-1}(f(s + h) - f(s))$ exists at every $s \in G$. If $f : G \to \mathcal{H}$ is anti-holomorphic and $g : G \to \mathcal{H}$ is holomorphic, then the scalar product $\langle f, g \rangle$ is holomorphic and $\partial_s \langle f, g \rangle = \langle \partial_s f, g \rangle + \langle f, \partial_s g \rangle$. If $f(s)$ is anti-holomorphic in $G$, then $f(s)$ is holomorphic in $G^*$ =
functions. For a \( z(s), \partial_0^j z(s) = 0 \) if and only if \( \partial_0^j z(s) = 0, \ j = 1, 2, \ldots \)

For a closed operator \( T \) on \( \mathcal{H} \) we denote by \( \rho(T) = \{ z \in \mathbb{C} : T - z \) has bounded inverse\} the resolvent set of \( T \), and for the resolvent we use the notation \( R_z(T) = (T - z)^{-1} \). By \( \sigma(T) = \mathbb{C} \setminus \rho(T) \) we denote the spectrum of \( T \), and by \( \sigma_d(T) \) the discrete spectrum of \( T \). That is, \( z \in \sigma_d(T) \) iff \( z \in \sigma(T) \) and \( T - z \) is Fredholm, the latter means \( \text{im}(T - z) \) is closed and both \( \ker(T - z) \), \( \text{coker}(T - z) \) are finite-dimensional. For \( z \in \sigma_d(T) \), the geometric and algebraic multiplicities of \( z \) are the dimensions of, respectively, the eigenspace \( \mathcal{V}(z, T) := \ker(T - z) \) and the root space \( \bigcup_{k=1}^\infty \ker(T - z)^k \). Elements of the root space are called root vectors or generalised eigenvectors. If the operator \( T \) is normal, then the root space and eigenspace coincide. The essential spectrum of \( T \) is \( \sigma_{ess}(T) := \sigma(T) \setminus \sigma_d(T) \). Other definitions of essential spectrum exist and this one is the largest of all, but for the operators considered here – relatively compact perturbations of self-adjoint operators, all of these definitions coincide.

If a family of closed operators \( T(v) \) is holomorphic at \( v = 0 \) in the general sense that \( v \mapsto R_z(T(v)) \) is bounded-holomorphic for some \( z \in \rho(T(0)) \), then in a neighbourhood of any finite subset of \( \sigma_d(T_0) \) there are, counting multiplicities, a constant finite number of eigenvalues of \( T(v) \) for all small enough \( v \). The number of distinct eigenvalues is also constant except for a discrete set of exceptional points. Away from such exceptional points the eigenvalues are holomorphic functions of \( v \), which constitute the branches of one or more multivalued analytic functions. These functions can have only algebraic singularities and can therefore be represented by branches of one or several Puiseux series. The same can be said of the corresponding eigenprojections, eigennilpotents, and therefore also eigenvectors.

2.1. The affine space \( \mathcal{A} \). Throughout this paper \( H_0 \) will denote a self-adjoint operator on a complex separable Hilbert space \( \mathcal{H} \). We denote by \( \mathcal{A}_0 \) the real vector space of all bounded self-adjoint \( H_0 \)-compact operators \( V \), the latter means that for some, and thus for any, \( z \) from the resolvent set \( \rho(H_0) \), the operator \( \Gamma \) is compact.

The results of this paper hold for certain unbounded perturbations too, but for simplicity we focus on bounded perturbations. Our strategy will be to deal with unbounded perturbations at the end of sections in various brief remarks, whose purpose is to describe the most significant necessary adjustments.

By \( \mathbb{C}\mathcal{A}_0 \) we denote the complexification of \( \mathcal{A}_0 \). By Weyl’s theorem (see e.g. [25]), the operators from

\[ \mathcal{A} := H_0 + \mathbb{C} \mathcal{A}_0 \]

have the same essential spectrum, denoted \( \sigma_{ess}(\mathcal{A}) \). For a fixed perturbation \( V \in \mathcal{A}_0 \), we write \( H_s = H_0 + sV \), for \( s \in \mathbb{C} \). Isolated eigenvalues of \( H_s \) we denote by \( z_\tau(s), \tau = 1, 2, \ldots \). If we work with a fixed eigenvalue function of \( H_s \), we may write \( z(s) \). A corresponding eigenvector we denote by \( \varphi(s) \):

\[ H_s \varphi(s) = z(s) \varphi(s). \]

If \( z \in \sigma_d(H_s) \), we say that \( s \) is a resonance point corresponding to \( z \). In what follows we shall often consider a particular value of the complex variable \( s \) denoted \( s_0 \), and the corresponding value of \( z(s) \) we denote \( z_0 \). We will be perturbing the operator \( H_{s_0} \) in another direction \( W \in \mathcal{A}_0 \). In such a case we use \( N_0 \), instead of \( H_{s_0} \), for an element of \( \mathcal{A} \) and write

\[ N_v = N_0 + vW. \]

Thus, \( N_0 \) is a not necessarily self-adjoint relatively compact perturbation of \( H_0 \). By the second resolvent identity, any \( W \in \mathcal{A}_0 \) is \( N_0 \)-compact. An eigenvalue function of \( N_v \) we denote \( z_\tau(v) \)
or \( z(v) \), and a corresponding eigenvector function by \( \varphi(v) : \)

\[
N_v \varphi(v) = z(v) \varphi(v).
\]

The variables \( z \) and \( v \) from the relation \( z \in \sigma_d(N_v) \) are multi-valued functions of each other which can have branching points.

We note that splitting – the situation in which a single eigenvalue splits into two or more distinct eigenvalues when perturbed, may occur with or without branching. The eigenvalues which arise from the splitting of a given eigenvalue \( z_0 \) are said to belong to the \( z_0 \)-group.

**Remark 2.1.** When it comes to unbounded perturbations, we will assume that the initial operator \( H_0 \) is semibounded. In this case if a symmetric perturbation \( V \) is relatively form-compact with respect to \( H_0 \), we may consider a decomposition of the form \( V = F^* J F \), where \( J \) is a bounded self-adjoint operator on an auxiliary Hilbert space \( K \) and \( F : \mathcal{H} \to K \) is a fixed operator which is \(|H_0|^{1/2}\)-compact (this can be taken as the definition of relative form-compactness, cf. [29, pp. 662–663]). We will assume in addition that the operator \( F \) is closed. It is also convenient to choose \( F \) to have trivial kernel and cokernel, which can be achieved without loss of generality.

In this situation, we set \( \mathcal{A}_0 = \{ V = F^* J F : J \text{ is bounded and self-adjoint} \} \), where \( V \) is interpreted as the symmetric form \((f, g) \mapsto V[f, g] := (F f, J F g)\) defined on \( \text{dom} F \). We define \( T_z(H_0) \) to be the compact operator which is the closure of the product \( F R_z(H_0) F^* \) and we study the spectral properties of the compact operator [1] in place of [1].

In case \( V \) is a \( H_0\)-compact bounded self-adjoint operator, the operators [1] and [1] share the same non-zero eigenvalues. Moreover, the eigenspaces satisfy

\[
\mathcal{V}(\sigma, T_z(H_0) J) = F \mathcal{V}(\sigma, R_z(H_0)V)
\]

and the root spaces are related similarly.

An operator \( H_s = H_0 + s V \) from the affine space \( \mathcal{A} \) is well-defined as a form-sum in the sense of the KLMN Theorem, rather than an operator-sum as in the Kato-Rellich Theorem. Note that \( H_0 \) corresponds to a semibounded form (see e.g. [18, Theorem VI-2.7]) and each form-sum \( H_s \) is sectorial ([18, Theorem VI-1.33]) and hence corresponds to a sectorial operator. It is no longer true, as it is in the case of bounded perturbations, that the operators of \( \mathcal{A} \) share a common domain, however this is true of their corresponding sectorial forms. In remarks concerning unbounded perturbations to follow, we will use square brackets to distinguish a sectorial form \( N_0[\cdot, \cdot] \) from its corresponding sectorial operator \( N_0 \in \mathcal{A} \) and the common form-domain of operators from \( \mathcal{A} \) will be denoted \( \text{dom}[\mathcal{A}] \).

### 2.2. Vector spaces \( \mathcal{Y}_{z_0}^k(N_0,W) \). Let \( N_0 \in \mathcal{A}, \ W \in \mathcal{A}_0, \) and \( N_v = N_0 + vW \). If \( z_0 \in \sigma_d(N_0) \), then for any \( v \in \mathcal{C} \) such that \( z_0 \in \rho(N_v) \), the number \( v^{-1} \) is an eigenvalue of

\[
(12) \quad R_{z_0}(N_v)W,
\]

and for any such \( v \)

\[
(13) \quad \mathcal{V}(z_0, N_0) = \mathcal{V}(v^{-1}, R_{z_0}(N_v)W).
\]

In other words, for any \( v \) such that \( z_0 \in \rho(N_v) \), a vector \( \chi \in \mathcal{V}(z_0, N_0) \) is a solution of the equation

\[
(14) \quad \left[ 1 - v R_{z_0}(N_v)W \right]^k \chi = 0
\]

with \( k = 1 \). Since the compact operator (12) is not necessarily normal, its root space corresponding to the eigenvalue \( v^{-1} \) can be larger than the eigenspace. The root space consists of \( \chi \in \mathcal{H} \) which obey (14) for some \( k \geq 1 \). For any \( k \geq 1 \) (14) holds for any \( v \) if it holds for some \( v \) with
z_0 \in \rho(N_v). A vector χ which obeys (14) we call a resonance vector of the triple \((z_0, N_0, W)\), and the smallest of positive integers \(k\) obeying (14) is the order of χ.

For \(z_0 \in \sigma_d(N_0)\), the vector space of resonance vectors of order \(\leq k\) we denote \(\Upsilon^k_{z_0}(N_0, W)\). The order \(d\) of the triple \((z_0, N_0, W)\) is the smallest of positive integers \(d\) such that

\[ \Upsilon^{d+1}_{z_0}(N_0, W) = \Upsilon^d_{z_0}(N_0, W). \]

Since \(v^{-1}\) is an eigenvalue of a compact operator, such a number \(d\) exists. We write \(\Upsilon_{z_0}(N_0, W)\) for \(\Upsilon^d_{z_0}(N_0, W)\). From the fact that a linear operator and its adjoint have kernels of the same dimension, it follows that for any \(k\) (c.f. [5, Lemma 3.1.4]),

\[ \dim \Upsilon^k_{z_0}(N_0, W) = \dim \Upsilon^k_{z_0}(N_0^*, W). \]

If we study a particular triple \((z_0, N_0, W)\) then we use notation \(n := \dim \Upsilon_{z_0}(N_0, W)\) and \(m := \dim \Upsilon^1_{z_0}(N_0, W)\). So, \(m\) is the geometric multiplicity of the eigenvalue \(z_0\) of \(N_0\). However \(n\) is not its algebraic multiplicity, although it is that of the eigenvalue \(v^{-1}\) of (12).

**Remark 2.2.** In the situation for unbounded perturbations, every occurrence of \(R_{z_0}(N_v)\) and \(W\) should be replaced by \(T_{z_0}(N_v)\) and \(J\), where \(W = F^*JF\). This is a general rule when it comes to treating the unbounded case and for many things, including the upcoming section, it is the only difference.

Instead of (13), it happens that

\[ F\mathcal{V}(z_0, N_0) = \mathcal{V}(v^{-1}, T_{z_0}(N_v)J). \]

Otherwise the above definitions are unchanged, e.g. resonance vectors of \((z_0, N_0, W)\), which in this case belong to the auxiliary space \(\mathcal{K}\), are by definition root vectors of \(T_{z_0}(N_v)J\).

For convenience we include a short proof of (15): For \(z_0 \in \rho(N_v)\) such that \(z_0 \in \sigma_d(N_0)\), suppose that \(N_0χ = z_0χ\). Then

\[(N_v - z_0)[f, χ] = vW[f, χ] = v \langle Ff, JFχ \rangle, \quad ∀f \in \text{dom}[A].\]

In particular, for any \(f = R_{z_0}(N_v^*)g \in \text{dom} N_v^*\) (which we note is a core of the form \(N_v^*[\cdot, \cdot]\)), we find

\[ \langle g, χ \rangle = v \langle FR_{z_0}(N_v^*)g, JFχ \rangle, \quad ∀g \in \mathcal{H}.\]

Thus \(χ = v(FR_{z_0}(N_v^*))^*JFχ\) and hence

\[ Fχ = vF(FR_{z_0}(N_v^*))^*JFχ = vT_{z_0}(N_v)JFχ.\]

Each of the above implications is reversible, completing the proof.

3. LAURENT COEFFICIENTS OF \(R_{z_0}(N_v)\)

Most of the material of this section can be found in [5] Section 3, but this presentation has some new elements and is shorter. We consider the resolvent \(R_{z_0}(N_v)\) as a meromorphic function of \(v\). That this is possible can be seen as a result of the analytic Fredholm alternative (see e.g. [24, Theorem VI.14]) and the resolvent identity

\[ R_{z_0}(N_v) = (1 + (v - u)R_{z_0}(N_v)W)^{-1}R_{z_0}(N_a). \]

**Lemma 3.1.** Let \(z_0 \in \sigma_d(N_0)\). The meromorphic functions \(R_{z_0}(N_v)\) and \(R_{z_0}(N_v)W\) have a pole of the same order at \(v = 0\).

**Proof.** The question is why the order of the pole \(v = 0\) of \(R_{z_0}(N_v)\) cannot decrease after multiplication by \(W\). This follows from the second resolvent identity

\[ R_{z_0}(N_v) = R_{z_0}(N_a) + (u - v)R_{z_0}(N_v)WR_{z_0}(N_a). \]
Here and below, we assume that $z_0 \in \sigma_d(N_0)$. Let
\[
R_{z_0}(N_v) = \sum_{j=-d}^{\infty} v^j K_{j+1}(z_0, N_0, W) = v^{-d} K_{-d+1} + \ldots + v^{-2} K_{-1} + v^{-1} K_0 + K_1 + v K_2 + \ldots
\]
be the Laurent expansion, so
\[
K_j(z_0, N_0, W) = \frac{1}{2\pi i} \oint_{C(0)} R_{z_0}(N_v) v^{-j} dv,
\]
and $K_{-d+1}$ is non-zero. The number $d$ appearing here is equal to the order of the triple $(z_0, N_0, W)$ by Lemma (3.1).

**Lemma 3.2.** For $k, j \geq 0$,
\[
K_{-k} W K_{-j} = K_{-k-j}.
\]

**Proof.** Using (17) we have
\[
(E) := (2\pi i)^2 K_{-k} W K_{-j} = \oint_{C_u(0)} \oint_{C_v(0)} R_{z_0}(N_v) W R_{z_0}(N_u) v^k u^j dv du,
\]
where $C_u(0)$ and $C_v(0)$ are contours enclosing zero. We choose $C_v(0)$ to be strictly inside $C_u(0)$. Applying the second resolvent identity gives
\[
(E) = \oint_{C_u(0)} \oint_{C_v(0)} [R_{z_0}(N_v) - R_{z_0}(N_u)] (u-v)^{-1} v^k u^j dv du.
\]
With the choice of the contours, the summand which contains the factor $R_{z_0}(N_u)$ is a holomorphic function of $v$ on and inside $C_v(0)$, and thus its integral vanishes. Hence, Cauchy’s theorem gives
\[
(E) = 2\pi i \oint_{C_u(0)} R_{z_0}(N_v) v^k \oint_{C_v(0)} (u-v)^{-1} u^j dv du = (2\pi i)^2 K_{-k-j}.
\]

Let
\[
P_{z_0}(N_0, W) = K_0(z_0, N_0, W) W, \quad Q_{z_0}(N_0, W) = W K_0(z_0, N_0, W),
\]
and for $k > 0$
\[
A_{z_0}^k (N_0, W) = K_{-k}(z_0, N_0, W) W, \quad B_{z_0}^k (N_0, W) = W K_{-k}(z_0, N_0, W).
\]

Lemma (3.2) implies that $P_{z_0}(N_0, W)$ and $Q_{z_0}(N_0, W)$ are idempotents, and, assuming the convention that $A_{z_0}^0 = P_{z_0}$, $B_{z_0}^0 = Q_{z_0}$, that for $k, j \geq 0$ we have $A_{z_0}^k A_{z_0}^j = A_{z_0}^{k+j}$ and $B_{z_0}^k B_{z_0}^j = B_{z_0}^{k+j}$, which justifies the notation.

From the definition (17) of $K_j$ it follows that
\[
(K_j(z_0, N_0, W))^* = K_j(z_0, N_0^*, W).
\]
This implies the equalities
\[
(P_{z_0}(N_0, W))^* = Q_{z_0}(N_0^*, W), \quad (A_{z_0}(N_0, W))^* = B_{z_0}(N_0^*, W),
\]
and
\[
\begin{align*}
R_{z_0}(N_v) &= \sum_{j=-d}^{\infty} v^j K_{j+1}(z_0, N_0, W) \\
&= v^{-d} K_{-d+1} + \ldots + v^{-2} K_{-1} + v^{-1} K_0 + K_1 + v K_2 + \ldots
\end{align*}
\]
and the following equality which will be frequently used without reference: for \( f, g \in \mathcal{H} \),
\[
\langle A_{z_0}(N_0^*, W)f, Wg \rangle = \langle f, W A_{z_0}(N_0, W)g \rangle.
\]
By Lemma 3.2 for \( k \geq 0 \)
\[
(21) \quad K_{-k} = A_{z_0}^k K_0 = K_0 B_{z_0}^k.
\]

If \( j < 0 \), then \( \int_{C_u(0)} (u - v)^{-1} u^j \, du = 0 \) for \( v \) inside \( C_u(0) \), because the residues sum to zero in this case, and so the argument from the proof of Lemma 3.2 yields

**Lemma 3.3.** If one of \( k \) or \( j \) is \( \geq 0 \) and the other \( < 0 \), then \( K_{-k} W K_{-j} = 0 \).

**Proof.** The proof is the same as that of Lemma 3.2 with the following changes: choose \( C_v(0) \) to be strictly outside \( C_u(0) \), and apply Cauchy’s theorem to the exterior of \( C_u(0) \), including \( \infty \), where \( u^{-j} \) is holomorphic. Since \( C_u(0) \) is anticlockwise oriented, this also gives the sign. \( \Box \)

**Lemma 3.4.** If \( k, j > 0 \) then \( K_k W K_j = -K_{k+j} \).

**Proof.** (19) gives \( \text{im} K_{-k} = \text{im} A_{z_0}^k \). In particular, \( \text{im} K_0 = \text{im} P_{z_0} \).

**Proof.** (19) gives \( \text{im} A_{z_0}^k \subset \text{im} K_{-k} \) and (21) gives \( \text{im} K_{-k} \subset \text{im} A_{z_0}^k \). \( \Box \)

Based on previous material, we can write the Laurent expansion of \( R_{z_0}(N_v) \) as
\[
R_{z_0}(N_v) = v^{-d} A_{z_0}^{d-1} K_0 + \ldots + v^{-2} A_{z_0} K_0 + v^{-1} K_0
+ K_1 - v \bar{A} K_1 + v^2 \bar{A}^2 K_1 - \ldots,
\]
where \( \bar{A} = K_1 W \).

By definition, for \( k \geq 1 \)
\[
(22) \quad \text{a resonance vector } \varphi \text{ has depth at least } k \text{, if } \varphi \in \text{im} A_{z_0}^k (N_0, W).
\]

The following facts come from results in [5], namely Proposition 3.2.3 and Theorem 3.4.3, whose proofs apply here more or less verbatim. The vector space of resonance vectors \( \Upsilon_{z_0}(N_0, W) \) is \( \text{im} P_{z_0}(N_0, W) \). Also, for all \( k = 1, 2, \ldots, d \)
\[
A_{z_0}(N_0, W) \Upsilon_{z_0}^k (N_0, W) = \Upsilon_{z_0}^{k-1}(N_0, W),
\]
where by agreement \( \Upsilon_{z_0}^0(N_0, W) = \{0\} \). Thus, the nilpotent operator \( A_{z_0}(N_0, W) \) decreases the order of resonance vectors by 1.

4. **Semisimple eigenvalues and the conjugate of an eigenpath**

Recall that we write \( z \leftrightarrow s \) if \( z \) is an eigenvalue of \( H_s = H_0 + sV \), that is, \( s \) is a resonance point corresponding to the triple \((z, H_0, V)\). As we shall see shortly, imposing a generic condition, with a resonance vector \( \varphi \) corresponding to \( z \leftrightarrow s \), one can naturally associate a resonance vector \( \varphi^* \) corresponding to \( \bar{z} \leftrightarrow \bar{s} \). In order to explain the nature of this condition, assume that the geometric multiplicity of \( z \) as an eigenvalue of \( H_s \) is 1. In such a case, the geometric multiplicity of \( \bar{z} \) as an eigenvalue of \( H_\bar{s} \) is also equal to 1, and thus there are eigenvectors \( \varphi \) and \( \varphi^* \) corresponding to these eigenvalues, which are unique up to scaling. The condition is that the vectors \( \varphi \) are \( \varphi^* \) are not orthogonal. In general, if the multiplicity of \( z \) is greater than 1, this condition is to be replaced by non-orthogonality of \( \varphi \) and \( \mathcal{V}(\bar{z}, H_\bar{s}) \). As reviewed below, this property is equivalent to the semisimplicity of the eigenvalue \( z \). In addition, the upcoming sections require a certain kind of stability under perturbations of \( H_s \), which in essence amounts to the regular behaviour of both \( \varphi \) and \( \varphi^* \). These assumptions are clarified in this section.
For \( z_0 \in \sigma_d(N_0) \), with the pair \((z_0, N_0)\), apart from \( P_{z_0}(N_0, W) \) defined in [18], we can associate another idempotent, the standard eigenprojection (cf. [18] (II.1.16))

\[
E(z_0, N_0) := -\frac{1}{2\pi i} \oint_{C(z_0)} (N_0 - \zeta)^{-1} d\zeta.
\]

There is also the associated nilpotent operator [18] (II.1.22)

\[
D(z_0, N_0) := -\frac{1}{2\pi i} \oint_{C(z_0)} (\zeta - z_0)(N_0 - \zeta)^{-1} d\zeta.
\]

We note

\[
(D(z_0, N_0))^* = D(\bar{z}_0, N_0^*), \quad (E(z_0, N_0))^* = E(\bar{z}_0, N_0^*).
\]

**Proposition 4.1.** Let \( z_0 \in \sigma_d(N_0) \). The following assertions are equivalent.

(i) \( \text{im } E(z_0, N_0) = \mathcal{V}(z_0, N_0) \).

(ii) Any non-zero vector \( \varphi \) from \( \mathcal{V}(z_0, N_0) \) is not orthogonal to some vector of \( \mathcal{V}(\bar{z}_0, N_0^*) \).

(iii) The orthogonal projection of \( \mathcal{V}(z_0, N_0) \) to \( \mathcal{V}(\bar{z}_0, N_0^*) \) is a linear isomorphism.

(iv) \( D(z_0, N_0) = 0 \).

(v) \( \text{im } E(z_0, N_0^*) = \mathcal{V}(\bar{z}_0, N_0^*) \).

(vi) Any non-zero vector \( \varphi \) from \( \mathcal{V}(\bar{z}_0, N_0^*) \) is not orthogonal to some vector of \( \mathcal{V}(z_0, N_0) \).

(vii) The orthogonal projection of \( \mathcal{V}(\bar{z}_0, N_0^*) \) to \( \mathcal{V}(z_0, N_0) \) is a linear isomorphism.

(viii) \( D(z_0, N_0^*) = 0 \).

**Proof.** The equivalences \([\text{viii}] \iff [\text{v}] \iff [\text{i}] \iff [\text{iv}] \iff [\text{iii}] \iff [\text{vii}] \iff [\text{vi}]\) are obvious and/or well-known. The equivalence \([\text{i}] \iff [\text{ii}]\) is also obvious, but still we give its proof.

\([\text{i}] \Rightarrow [\text{ii}]\) For \( 0 \neq \varphi \in \mathcal{V}(z_0, N_0) \) we have \( 0 \neq \langle \varphi, \varphi \rangle = \langle \varphi, E(z_0, N_0)\varphi \rangle = \langle E(\bar{z}_0, N_0^*)\varphi, \varphi \rangle \). So, the non-zero vector \( E(\bar{z}_0, N_0^*)\varphi \in \mathcal{V}(\bar{z}_0, N_0^*) \) is not orthogonal to \( \varphi \).

\([\text{ii}] \Rightarrow [\text{i}]\) If \( \text{im } E(z_0, N_0) \neq \mathcal{V}(z_0, N_0) \), then for some non-zero \( \varphi \in \mathcal{V}(z_0, N_0) \) and \( \psi \) we have \( D(z_0, N_0)\psi = \varphi \). So, for any \( \varphi^* \in \mathcal{V}(\bar{z}_0, N_0^*) \) we have \( \langle \varphi^*, \varphi \rangle = \langle D(\bar{z}_0, N_0^*)\varphi^*, \psi \rangle = 0 \).

**Definition 4.2.** \( z_0 \in \sigma_d(N_0) \) is called **semisimple**, if some and therefore any of the assertions of Proposition 4.1 hold.

There are yet other equivalent definitions of semisimplicity in the literature. Another worth noting is for the gap \( \delta(M, N) \) between the subspaces \( M = \mathcal{V}(z_0, N_0) \) and \( N = \mathcal{V}(\bar{z}_0, N_0^*) \), as defined in [18] IV-§2.1, to be less than 1 – its equivalence can be seen with items (ii) and (iv) of Proposition 4.1 taken together.

Numerical experiments easily generate pairs of self-adjoint matrices \( H_0 \) and \( V \) such that some \( z_0 \in \sigma(H_0 + s_0V) \) fails to be semisimple. If \( s_0 \) is real, then all eigenvalues are semisimple, which is one of the differences between the real case considered in [6] and the present one.

For \( N_0 \in \mathcal{A} \) and \( z_0 \in \sigma_d(N_0) \), we now reintroduce a perturbation \( W \in \mathcal{A}_0 \). Let \( z(v) \) be an eigenvalue function corresponding to \( N_v = N_0 + vW \), with \( z(0) = z_0 \). Then \( z(v) \) is holomorphic in a deleted neighbourhood of 0, but not necessarily at \( v = 0 \) if it is an exceptional point, that is, if \( v = 0 \) is one of the discrete set of points where the number of distinct eigenvalues of \( N_v \) within a small neighbourhood of \( z_0 \) is different from its otherwise constant value.

We will restrict our attention to the case when \( z(v) \) is not subject to branching at \( v = 0 \), in other words, we assume \( z(v) \) is analytic at \( v = 0 \). In this case (see e.g. [16] Chapter 18) there must exist an eigenvector function, or eigenpath, corresponding to \( z(v) \) which is also analytic in a neighbourhood of \( v = 0 \). Such an analytic eigenpath will be denoted by \( \varphi(v) \). The anti-holomorphic eigenvalue function of \( N_v^* = N_0^* + ivW \), equal to \( \bar{z}(v) \), will be denoted \( z^*(v) \).
A corresponding anti-holomorphic eigenpath will be denoted $\varphi^*(v)$. Given $\varphi(v)$, our aim is to canonically assign a non-orthogonal $\varphi^*(v)$.

Assume first that $z_0$ is a simple eigenvalue, i.e. its algebraic multiplicity is equal to 1. Then by Proposition 4.1, $\varphi(v)$ and $\varphi^*(v)$ are not orthogonal and we can normalise $\varphi^*(v)$ in such a way that

$$\langle \varphi^*(v), \varphi(v) \rangle = 1.$$ 

We call the resulting eigenpath $\varphi^*(v)$ the conjugate of $\varphi(v)$. Clearly, the conjugate of an eigenpath of geometric multiplicity 1 is unique.

In the case of arbitrary geometric multiplicity $m$, the following assumption is suggested by developments in [17] and [20]. Some terminology beforehand: Let $z_0 \in \sigma_d(N_0)$ and let $z(v)$ be an eigenvalue of $N_v$ which converges to $z_0$ (and is not necessarily analytic there). Suppose $\varphi(v)$ is a continuous normalised eigenvector corresponding to $z(v)$. Then, following [17], $\varphi(0)$ will be called a generating eigenvector of $N_v$ corresponding to $z(v)$.

A generating eigenvector is necessarily an eigenvector, but the converse is not true in general. The comparison gives some indication about the presence of branching at $z_0$. Consider for example the following results taken from [17]: If branching occurs at $z_0$, then eigenvectors $\varphi_\tau(v)$, $j = 1, 2, \ldots$ corresponding to different branches $z_\tau(v)$, $j = 1, 2, \ldots$ of the same Puiseux series can be chosen so that they determine the same generating eigenvector $\varphi_1(0) = \varphi_2(0) = \ldots$ ([17, Lemma 3.2]). On the other hand, if $z_0$ is semisimple and generating eigenvectors span the eigenspace $V(z_0, N_0)$, then there can be no branching at $z_0$ ([17, Proposition 3.5]).

**Assumption 4.3.** Let $z(v)$ be an eigenvalue of $N_v$ converging to $z_0 \in \sigma_d(N_0)$. For every generating eigenvector $\varphi$ of $N_v$ corresponding to $z(v)$, there exists a generating eigenvector $\varphi^*$ of $N^*_v$ corresponding to $z^*(v)$ which is not orthogonal to $\varphi$.

The next theorem, here for convenience, is a reproduction of [17, Theorem 3.6].

**Theorem 4.4.** If Assumption 4.3 holds for an eigenvalue $z(v)$ of $N_v$, then $z(v)$ is analytic in a neighbourhood of $v = 0$ and semisimple in a deleted neighbourhood of $v = 0$. Suppose the eigenvalue $z_0$ has arbitrary (algebraic) multiplicity $m$ and Assumption 4.3 holds for all eigenvalues of the $z_0$-group. Then for all small enough $v$, the $z_0$-group consists of $m$ eigenvalues $z_\tau(v)$, $\tau = 1, 2, \ldots, m$, with any repeated eigenvalues being semisimple. All eigenvalues and corresponding eigenvectors can be chosen to be analytic. In particular, $z_0$ is semisimple, so that its geometric multiplicity is $m$.

**Proof.** The proof proceeds by contradiction and makes use of ([17, Lemma 3.1]): For distinct eigenvalue functions $z_\tau(v)$ and $z_\mu(v)$ of $N_v$, any continuous eigenvectors $\varphi_\tau(v)$ and $\varphi^*_\mu(v)$ corresponding respectively to the eigenvalues $z_\tau(v)$ of $N_v$ and $z^*_\mu(v)$ of $N^*_\mu$, must satisfy

$$\langle \varphi^*_\mu(0), \varphi_\tau(0) \rangle = 0. \tag{25}$$

Assume there is an eigenvalue $z_\tau(v)$ of $N_v$ for which Assumption 4.3 holds and yet $z_\tau(v)$ tends to $z_0$ as $v \to 0$ and is not analytic. Then $z_\tau(v)$ must be a branch of some Puiseux series. Let $z_\tau(v)$ be any different branch of the same algebraic function. Let $\varphi_\tau(v)$ and $\varphi^*_\tau(v)$ be corresponding continuous eigenvector functions, which it is possible to choose such that $\varphi_\tau(0) = \varphi^*_\tau(0) =: \varphi_0$ (see [17, Lemma 3.2]). Let $\varphi^*_\mu(v)$ be any eigenvector corresponding to some eigenvalue $z^*_\mu(v)$ of $N^*_\mu$ such that both $z^*_\mu(v)$ and $\varphi^*_\mu(v)$ are continuous. Since $z_\mu(v)$ can not be identical to both $z_\tau(v)$ and $z_\tau(v)$, it follows from (25) that

$$\langle \varphi^*_\mu(0), \varphi_0 \rangle = 0,$$

which is a contradiction. Hence $z_\tau(v)$ must be analytic.
Suppose \( z_\tau(v) \) is not semisimple for \( v = v_j, j \in \mathbb{N} \), such that \( v_j \to 0 \) as \( j \to \infty \). Choosing an eigenvector \( \varphi_\tau(v) \) corresponding to \( z_\tau(v) \) such that \( \varphi_\tau(v_j) \) belongs to a Jordan chain of length greater than 1, we get for any \( \varphi^*_\tau(v_j) \in \mathcal{V}(z^*_\tau(v_j), N^*_\tau) \),

\[
\langle \varphi_\tau(v_j), \varphi^*_\tau(v_j) \rangle = 0.
\]

This is because, in general, if an eigenvalue \( z_0 \) of \( N_0 \) is not semisimple, with a Jordan chain \( \varphi_0, \varphi_1, \ldots \), then for any \( \varphi^*_0 \in \mathcal{V}(z_0, N^*_0) \),

\[
0 = \langle \varphi_0 + (z_0 - N_0)\varphi_1, \varphi^*_0 \rangle = \langle \varphi_0, \varphi^*_0 \rangle + \langle \varphi_1, (z_0 - N_0)\varphi^*_0 \rangle = \langle \varphi_0, \varphi^*_0 \rangle.
\]

The vector function \( \varphi^*_\tau(v) \) can be chosen to be continuous, so that sending \( j \to \infty \) in (26) gives \( \langle \varphi_\tau(0), \varphi^*_\tau(0) \rangle = 0 \). Since by (25), \( \langle \varphi^*_\tau(0), \varphi_\tau(0) \rangle = 0 \) for any \( \varphi^*_\mu(v) \) corresponding to \( z^*_\mu(v) \) such that \( z_\tau \neq z_\mu \), we get a contradiction.

Proof of the remaining parts of the second statement is omitted (see [17, Theorem 3.6]). \( \square \)

**Proposition 4.5.** For Assumption 4.3 to hold for all eigenvalues of the \( z_0 \)-group is equivalent to the statement:

\( (P) \) The eigenvalue \( z_0 \) is semisimple and generating eigenvectors span the eigenspace \( \mathcal{V}(z_0, N_0) \).

**Proof.** Suppose Assumption 4.3 holds for the \( z_0 \)-group. Then \( z_0 \) must be semisimple by Theorem 4.4. Moreover, by the same theorem the spectrum of the \( z_0 \)-group is given for all small \( v \) by \( m \) (counting multiplicities) analytic eigenvalues \( z_\tau(v), j = 1, \ldots, m \), for which we can find analytic normalised eigenvectors \( \varphi_\tau(v), j = 1, \ldots, m \). It follows that the eigenspace \( \mathcal{V}(z_0, N_0) \) is spanned by the generating eigenvectors \( \varphi_\tau(0), j = 1, \ldots, m \).

Suppose \( (P) \) holds. We note that by [17, Proposition 3.5] there can be no branching at \( z_0 \), so all eigenvalues of the \( z_0 \)-group are analytic. For any eigenvalue \( z_\tau(v) \) of the \( z_0 \)-group, there exists a corresponding analytic eigenpath \( \varphi_\tau(v) \). Since \( z_0 \) is semisimple, by Proposition 4.4 there must be an eigenvector \( \varphi^*_0 \) corresponding to \( z_0 \) such that \( \langle \varphi^*_0, \varphi_\tau(0) \rangle \neq 0 \). It follows from \( (P) \) that generating eigenvectors span the eigenspace \( \mathcal{V}(z_0, N^*_0) \). Thus there exists an eigenpath \( \varphi^*_\tau(v) \) corresponding to an eigenvalue \( z^*_\tau(v) \) of \( N^*_0 \) such that \( \varphi^*_\tau(0) = \varphi^*_0 \). The eigenvalue \( z^*_\tau(v) \) cannot be different from \( z^*_\mu(v) \) by (25) (or in this case also Lemma 6.1 below), which completes the proof. \( \square \)

It follows that Assumption 4.3 holds at least when \( v = 0 \) is not an exceptional point and \( z_0 \) is semisimple. It also clearly holds if \( z_0 \) is a simple eigenvalue. This assumption is therefore generic in the sense that for random matrices it occurs with probability zero.

**Lemma 4.6.** Given Assumption 4.3 for the eigenvalues \( z_\tau(v), \tau = 1, 2, \ldots, m \), which split from \( z_0 \in \sigma_d(N_0) \), let \( \varphi_\tau(v), \tau = 1, 2, \ldots, m \), be corresponding holomorphic eigenvector functions. Then there exist anti-holomorphic eigenvector functions \( \varphi^*_\tau(v) \), taking values in \( \mathcal{V}(z^*_\tau(v), N^*_0) \), such that

\[
\langle \varphi^*_\tau(v), \varphi_\tau(v) \rangle = \delta_{\mu \tau},
\]

which are unique up to rearrangement for repeated eigenvalues.

**Proof.** The existence of such eigenpaths follows from Theorem 4.4, so it remains to prove their uniqueness. Clearly, the vectors \( \varphi^*_\mu(v) \) are linearly independent. So, if \( \psi^*_\mu(v) \) is another set with the same property then for some scalar anti-holomorphic functions \( \bar{\psi}_\mu(v) \) we have

\[
\psi^*_\mu(v) = \varphi^*_\tau(v)\bar{\psi}_\mu(v).
\]
Taking the scalar product of both sides with $\varphi_\sigma(v)$, we infer from (27) that $c^\sigma_\mu(v) = \delta_{\mu\sigma}$. \qed

For a basis of vectors $\varphi_1(0), \ldots, \varphi_m(0)$ of $V(z_0, N_0)$, we say that the vectors $\varphi_1^*(0), \ldots, \varphi_m^*(0)$ from Lemma 4.6 are their conjugates.

Let’s recap the situation when Assumption 4.3 holds for all eigenvalues $z_\tau(v)$, $\tau = 1, 2, \ldots$ which split from a given eigenvalue $z_0 \in \sigma_d(N_0)$. The geometric multiplicity of $z_0$ is equal to $m = \dim \mathcal{Y} \sigma_{z_0}(N_0, W)$ and since it follows from Theorem 4.4 that $z_0$ is semisimple, its algebraic multiplicity is also equal to $m$. The eigenvalue $z_0 = z(0)$ is non-branching and in a neighbourhood of 0, it splits into single-valued functions of $v$ only, namely $z_\tau(v)$. On the other hand, the inverse function $v(z)$ may consist of multi-valued functions with $z_0$ as a branching point. Corresponding to $z_\tau(v)$ are analytic eigenpaths $\varphi_\tau(v)$. Counting multiplicities, there are $m$ eigenvalues and $m$ corresponding eigenpaths. The vectors $\varphi_\tau(0)$, $\tau = 1, \ldots, m$, can be chosen to form a basis of $V(z_0, N_0)$. Their conjugates $\varphi_\tau^*(0)$, $\tau = 1, \ldots, m$, form a basis of the eigenspace $V(\bar{z}_0, N_0^*)$.

**Remark 4.7.** For unbounded perturbations, no changes to this section are required.

5. **Order of an eigenpath**

Let $\varphi(v)$ be an eigenpath of $N_v$ corresponding to an eigenvalue function $z(v)$. Following [6, Definition 3.1.2], we say that $\varphi(v)$ has order at least $k$ at $v = 0$, if the vectors

$$W\varphi(0), \ W\partial_v\varphi(0), \ldots, \ W\partial_v^{k-2}\varphi(0)$$

are orthogonal to $V(\bar{z}_0, N_0^*)$. If $\varphi^*(v)$ is an eigenpath for $N_v^*$ corresponding to an eigenvalue function $z^*(v)$, then we also say that $\varphi^*(v)$ has order at least $k$ at $v = 0$ if the vectors

$$W\varphi^*(0), \ W\partial_v\varphi^*(0), \ldots, \ W\partial_v^{k-2}\varphi^*(0)$$

are orthogonal to $V(z_0, N_0)$.

**Lemma 5.1.** Given Assumption 4.3 for an eigenvalue function $z(v)$ of $N_v$, let $\varphi(v)$ be an analytic eigenvector function corresponding to $z(v)$. Then the following assertions are equivalent (for $k \geq 1$).

(i) $\varphi(v)$ has order at least $k$ at $v = 0$.

(ii) each vector $W\varphi(0)$, $W\partial_v\varphi(0)$, $\ldots$, $W\partial_v^{k-2}\varphi(0)$ is orthogonal to a vector $\varphi^* \in \mathcal{V}(\bar{z}_0, N_0^*)$.

(iii) $\partial_v z(0) = 0$, $\ldots$, $\partial_v^{k-1}z(0) = 0$.

(iv) for all $j = 1, \ldots, k-1$,

$$\left(N_0 - \bar{z}_0\right)\partial_v^j\varphi(0) = -jW\partial_v^{j-1}\varphi(0).$$

**Proof.** This proof follows that of [6, Lemma 3.1.4] with some adjustments.

First, we remark that the existence of a vector $\varphi^* \in \mathcal{V}(\bar{z}_0, N_0^*)$ not orthogonal to $\varphi(0)$ is given by Assumption 4.3. It also follows from Theorem 4.4 that $z(v)$ is holomorphic at $v = 0$, which guarantees the existence of a holomorphic eigenpath $\varphi(v)$.

The implication (i) $\Rightarrow$ (ii) is obvious. The implication (iv) $\Rightarrow$ (i) is also obvious since $\mathcal{V}(\bar{z}_0, N_0^*)$ is the kernel of $(N_0 - \bar{z}_0)^*$.

(ii) $\Rightarrow$ (iii): $k - 1$ times differentiating the eigenvalue equation $N_v \varphi(v) = z(v)\varphi(v)$ gives the equality

$$k-1)W\varphi^{(k-2)}(v) + N_v\varphi^{(k-1)}(v) = \sum_{j=0}^{k-1} \binom{k-1}{j} z^{(j)}(v)\varphi^{(k-1-j)}(v).$$
Here we let \( v = 0 \) and take the scalar product of both sides with \( \varphi^* \). After cancelling the second summand of the left hand side with the first summand of the right hand side, we obtain the equality

\[
(k - 1) \left\langle \varphi^*, W \varphi^{(k-2)}(0) \right\rangle = \sum_{j=1}^{k-1} \binom{k-1}{j} z(j)(0) \left\langle \varphi^*, \varphi^{(k-j-1)}(0) \right\rangle.
\]

If \( k = 2 \) then

\[
\left\langle \varphi^*, W \varphi(0) \right\rangle = z'(0) \left\langle \varphi^*, \varphi(0) \right\rangle.
\]

Since by the premise \( \left\langle \varphi^*, \varphi(0) \right\rangle \neq 0 \), this equality implies the assertion in the case \( k = 2 \). Assume that the claim holds for \( k < n \). Then from (30) with \( k = n \), using the induction assumption, we get

\[
(n - 1) \left\langle \varphi^*, W \varphi^{(n-2)}(0) \right\rangle = z^{(n-1)}(0) \left\langle \varphi^*, \varphi(0) \right\rangle.
\]

Since by the premise \( \left\langle \varphi^*, W \varphi^{(n-2)}(0) \right\rangle = 0 \) and \( \left\langle \varphi^*, \varphi(0) \right\rangle \neq 0 \), this gives \( z^{(n-1)}(0) = 0 \).

(iii) \( \Rightarrow \) (iv): We prove the claim for \( j = k - 1 \) but the same proof clearly works for the other values of \( j \). Letting \( v = 0 \) in (29) gives the equality

\[
(k - 1)W\varphi^{(k-2)}(0) + N_0\varphi^{(k-1)}(0) = \sum_{j=0}^{k-1} \binom{k-1}{j} z(j)(0)\varphi^{(k-j-1)}(0).
\]

By the premise, the right hand side simplifies to \( z(0)\varphi^{(k-1)}(0) = z_0\varphi^{(k-1)}(0) \). Hence,

\[
(N_0 - z_0)\varphi^{(k-1)}(0) = -(k - 1)W\varphi^{(k-2)}(0).
\]

\[\square\]

**Lemma 5.2.** Let \( k \geq 2 \), and vectors \( \varphi_0, \varphi_1, \ldots, \varphi_{k-1} \) be such that for all \( j = 1, 2, \ldots, k - 1 \)

\[
(N_0 - z_0)\varphi_j = -jW\varphi_{j-1},
\]

and \( \varphi_0 \) is an eigenvector of \( N_0 \) corresponding to \( z_0 \). Then

(i) \( \varphi_0, \varphi_1, \ldots, \varphi_{k-1} \) are resonance vectors of orders respectively \( 1, 2, \ldots, k \),

(ii) \( A_{z_0}(N_0, W)\varphi_j = j\varphi_{j-1} \), for any \( j = 1, 2, \ldots, k - 1 \),

(iii) \( \varphi_0 \) is a resonance vector of depth at least \( k - 1 \).

**Proof.** For \( v \) such that \( z_0 \notin \sigma_d(N_v) \), the equality (31) can be rewritten as

\[
(1 - vR_{z_0}(N_v)W)\varphi_j = -jR_{z_0}(N_v)W\varphi_{j-1}.
\]

The vector \( \varphi_0 \) has order 1 since it is an eigenvector. It follows from the definition (14) of vectors of order \( k \) that the operator \( R_{z_0}(N_v)W \) preserves the order of vectors, while the operator \( (1 - vR_{z_0}(N_v)) \) decreases the order of vectors by 1. Therefore the item (i) follows by induction from (32). According to the definitions (19) of \( A_{z_0}(N_0, W) \) and (18) of \( P_{z_0}(N_0, W) \), the equality (32) also implies (ii) after taking contour integrals over a small circle enclosing 0 in the \( v \)-plane. Finally, (ii) plainly implies (iii) by the definition of depth (22). \[\square\]

**Lemma 5.3.** Given Assumption 4.3 for an eigenvalue function \( z(v) \) of \( N_v \), with a corresponding analytic eigenvector function \( \varphi(v) \), if the path \( \varphi(v) \) has order at least \( k \geq 2 \) at 0, then

(i) \( \varphi(0), \varphi'(0), \ldots, \varphi^{(k-1)}(0) \) are resonance vectors of orders respectively \( 1, 2, \ldots, k \),

(ii) for any \( j = 1, \ldots, k - 1 \), \( A_{z_0}(N_0, W)\partial_v^j\varphi(0) = j\partial_v^{j-1}\varphi(0) \),

(iii) \( \varphi(0) \) has depth at least \( k - 1 \).
Proof. By Lemma 5.1, a path $\varphi(v)$ of order $\geq k$ at $0$ obeys (28). So, the claim follows from Lemma 5.2. \qed

Proposition 5.4. Suppose Assumption 4.3 holds for an eigenvalue function $z(v)$ of $N_v$, and $\varphi(v)$ is a corresponding analytic eigenpath. Let also $\varphi^*(v)$ be an anti-holomorphic eigenpath corresponding to the anti-holomorphic eigenvalue function $z^*(v)$ of $N_v^*$. Then the following assertions are equivalent.

(i) $\varphi(v)$ has order at least $k$ at $v = 0$.
(ii) $W\varphi^*(0), W\bar{\partial}_v\varphi^*(0), \ldots, W\bar{\partial}_v^{k-2}\varphi^*(0)$ are orthogonal to a vector $\varphi \in \mathcal{V}(z_0, N_0)$, such that $\langle \varphi^*(0), \varphi \rangle \neq 0$.
(iii) $\varphi^*(v)$ has order at least $k$ at $0$.
(iv) $\bar{\partial}_v z^*(0) = 0, \ldots, \bar{\partial}_v^{k-1}z^*(0) = 0$.
(v) $(N_0^* - \bar{z}_0)\bar{\partial}_v \varphi^*(0) = -jW\bar{\partial}_v^{k-1}\varphi^*(0)$, for all $j = 1, \ldots, k - 1$.

Proof. Since $z^*(v) = \bar{z}(v)$, it follows that $\bar{\partial}_v z(0) = 0$ if and only if $\bar{\partial}_v z^*(0) = 0$. Hence, (iv) is equivalent to Lemma 5.1(iii), and thus also to (i). The equivalence of the other items is proved as in Lemma 5.1, considering that, by conjugate invariance, Assumption 4.3 holds for the eigenvalue $z^*(v)$ of $N_v^*$. \qed

It is worth emphasising that an anti-holomorphic eigenpath $\varphi^*(v)$ corresponding to $\varphi(v)$ as in this proposition may be non-unique, where by non-uniqueness we of course mean that another such eigenpath may exist which is not a scaling of $\varphi^*(v)$. Proposition 5.4 holds for all such eigenpaths.

Lemma 5.5. Under the premise of Theorem 5.7, let $k \geq 2$ and let $z(v)$ be an eigenvalue function of $N_v$ with a corresponding eigenvector function $\varphi(v)$. If $\varphi(0)$ has depth at least $k - 1$, then $\varphi(v)$ has order at least $k$ at $0$.

Proof. This proof is an adjustment to this setting of the proofs of [6, Lemma 3.1.6] and the implication $(ii) \Rightarrow (i)$ of [6, Theorem 2.3.1]. One of the differences is the use of a conjugate vector.

We agree to write $A_{z_0}$ for $A_{z_0}(N_0, W)$ and $A_{\bar{z}_0}$ for $A_{\bar{z}_0}(N_0^*, W)$. Let $k = 2$. By the definition (22) of depth, there exists a vector $\chi$ such that $A_{z_0}\chi = \varphi(0)$. So, for any $f \in \mathcal{V}(z_0, N_0^*)$, we have

$$\langle f, W\varphi(0) \rangle = \langle f, WA_{z_0}\chi \rangle = \langle A_{z_0}f, W\chi \rangle = \langle 0, W\chi \rangle = 0,$$

and so, $\varphi(v)$ has order $\geq 2$ at $0$.

Assume the claim for integers less than $k$ and let $\varphi(0)$ be of depth $\geq k - 1$. Then by the induction assumption the path $\varphi(v)$ at $0$ has order $\geq k - 1$. Choose an eigenpath $\varphi^*(v)$ of $N_v^*$, corresponding to $z^*(v)$, so that

$$C := \langle \varphi^*(0), \varphi(0) \rangle \neq 0,$$

which can be done according to Assumption 4.3. By Proposition 5.4, the eigenpath $\varphi^*(v)$ also has order $\geq k - 1$. In particular,

$$\bar{\partial}_v z^*(0) = \ldots = \bar{\partial}_v^{k-1}z^*(0) = 0.$$

We combine this with an equation similar to (30), which is obtained by $k-1$ times differentiating the eigenvalue equation $N_v^*\varphi^*(v) = z^*(v)\varphi^*(v)$ with respect to $\bar{v}$ at the point $\bar{v} = 0$ and taking the scalar product with $\varphi(0)$. The result is

$$(k - 1) \langle \bar{\partial}_v^{k-2}\varphi^*(0), W\varphi(0) \rangle = \langle \bar{\partial}_v^{k-1}z^*(0)\varphi^*(0), \varphi(0) \rangle = Cz^{(k-1)}(0),$$

$$\bar{\partial}_v z^*(0) = \ldots = \bar{\partial}_v^{k-1}z^*(0) = 0.$$
where in the last equality we used the fact that $\bar{\partial}_v \bar{z}$ is the conjugate of $\partial_v z$. Since $\varphi(0)$ has depth at least $k - 1$, there is some $f$ so that $A_{z_0}^{k-1} f = \varphi(0)$. Hence,

$$C_z^{(k-1)}(0) = (k-1) \left< \bar{\partial}_v^{k-2} \varphi^*(0), W A_{z_0}^{k-1} f \right>$$

$$= (k-1) \left< A_{z_0}^{k-1} \bar{\partial}_v^{k-2} \varphi^*(0), W f \right> .$$

Since $\varphi^*(v)$ at 0 has order $\geq k - 1$, the vector $\bar{\partial}_v^{k-2} \varphi^*(0)$ has order $k - 1$ by Lemma 5.3(i). Therefore $A_{z_0}^{k-1} \bar{\partial}_v^{k-2} \varphi^*(0) = 0$. Combining this with the previous display gives $C_z^{(k-1)}(0) = 0$, and since $C$ is non-zero, we have $z^{(k-1)}(0) = 0$. Hence, by Lemma 5.1 the path $\varphi(v)$ at 0 has order $\geq k$.

**Corollary 5.6.** Suppose Assumption 4.3 holds for an eigenvalue function $z(v)$ of $N_v$. Let $\varphi(v)$ be a holomorphic eigenpath corresponding to $z(v)$ and let $\varphi^*(v)$ be an anti-holomorphic eigenpath corresponding to the eigenvalue $z^*(v)$ of $N_v^*$. The following are equivalent statements.

(i) The order of $\varphi(v)$ at $v = 0$ is equal to $k$.
(ii) The order of $\varphi^*(v)$ at $v = 0$ is equal to $k$.
(iii) The depth of $\varphi(0)$ is equal to $k - 1$.
(iv) The depth of $\varphi^*(0)$ is equal to $k - 1$.

**Proof.** By Proposition 5.4 the eigenpaths $\varphi(v)$ and $\varphi^*(v)$ have the same order. By Lemma 5.3(iii), both $\varphi_\tau(0)$ and $\varphi^*_\tau(0)$ have depth at least $k - 1$. Their depth is at most $k - 1$ by Lemma 5.5. \□

Recall that we have $m := \dim \Upsilon_z^1(N_0, W)$ and $n := \dim \Upsilon_z(N_0, W)$. The vector space $\Upsilon_z(N_0, W)$ is invariant for the nilpotent operator $A_{z_0}(N_0, W)$. The sizes of its Jordan blocks we denote $d_\tau$, $\tau = 1, \ldots, m$.

A Jordan block is determined by an eigenpath $\varphi_\tau(v)$ corresponding to an eigenvalue $z_\tau(v)$ which splits from $z_0$ and satisfies Assumption 4.3. Indeed, $\varphi_\tau(v)$ determines a set of vectors

$$\frac{1}{j!} \varphi^{(j)}_\tau(0), \quad j = 0, 1, \ldots, \tilde{d}_\tau - 1 ,$$

where $\tilde{d}_\tau$ is the order of the eigenpath $\varphi_\tau(v)$ at $v = 0$. These vectors form a Jordan chain for $A_{z_0}(N_0, W)$ by Lemma 5.3(ii), which is of maximal length by Corollary 5.6 hence $\tilde{d}_\tau = d_\tau$.

Although we assume $z_0$ is non-branching, the inverse function $v(z)$ may consist of multivalued functions with $z_0$ as a branching point. As $z$ makes one round around $z_0$ these different values of $v(z)$ undergo a permutation. This permutation gives rise to a partition of the values of $v(z)$ into cycles. There is a correspondence (unique up to multiplicity) between the cycles and the eigenpaths, namely, if $N_v \varphi_\tau(v) = z_\tau(v) \varphi_\tau(v)$ then the inverse of $z_\tau(v)$ defines the cycle corresponding to $\varphi_\tau(v)$. Let the period of the cycle corresponding to $\varphi_\tau(v)$ be denoted $\tilde{d}_\tau$. The equivalence of items (i) and (iii) in Lemma 5.1 shows that $\hat{d}_\tau = \tilde{d}_\tau$.

Thus, the following analogue of [6] Theorem 5.2.3 holds.

**Theorem 5.7.** Suppose Assumption 4.3 holds for all eigenvalue functions $z_\tau(v)$ of $N_v$ which split from $z_0 \in \sigma_d(N_0)$. Let $\varphi_\tau(v)$ be corresponding eigenpaths. Then the following numbers are equal:

(i) the order $d_\tau$ of the eigenpath $\varphi_\tau(v)$ at 0,
(ii) the period of the $\tau$th cycle of resonance points of the group of 0,
(iii) the size of the $\tau$th Jordan block for the nilpotent operator $A_{z_0}(N_0, W)$.
Moreover, the vectors
\[
\frac{1}{J^\tau} \varphi_j^{(\tau)}(0), \quad \tau = 1, \ldots, m, \quad j = 0, 1, \ldots, d_\tau - 1,
\]
form a Jordan basis for \( A_{z_0}(N_0, W) \).

We combine Lemmas 5.1, 5.3 and 5.5 in the following theorem which includes Theorem 1.1 as a special case.

**Theorem 5.8.** Let Assumption 4.3 be given for an eigenvalue function \( z(v) \) of \( N_\nu \) which splits from \( z_0 \in \sigma_d(N_0) \). Let \( k \geq 1 \) and let \( \varphi(v) \) be an eigenvector function corresponding to \( z(v) \). Then the following assertions are equivalent:

(i) the path \( \varphi(v) \) has order \( k \) at 0.

(ii) the vectors \( W\varphi(0), W\varphi'(0), \ldots, W\varphi^{(k-2)}(0) \) are orthogonal to some \( \varphi^* \in \mathcal{V}(z_0, N_0) \), \( \langle \varphi^*, \varphi(0) \rangle \neq 0 \), while \( W\varphi^{(k-1)}(0) \) is not.

(iii) \( z'(0) = 0, \ldots, z^{(k-1)}(0) = 0 \), and \( z^{(k)}(0) \neq 0 \).

(iv) for all \( j = 1, \ldots, k-1 \), \( (N_0 - z_0)\varphi(j)(0) = -j W\varphi^{(j-1)}(0) \), and this fails for \( j = k \).

(v) the vector \( \varphi(0) \) has depth \( k-1 \).

(vi) for any \( j = 1, 2, \ldots, \), \( A_{z_0}(N_0, W)\varphi(j)(0) = j\varphi^{(j-1)}(0) \), and this fails for \( j = k \).

Moreover, if these conditions hold, then

(vii) for \( j = 1, 2, \ldots, k \), the vector \( \varphi^{(j)(-1)}(0) \) has order \( j \).

In [6 Lemma 3.1.5] the equality of the item (vi) in this theorem was obtained by a different method, which is specific to the case of real \( z_0 \) (outside essential spectrum) and self-adjoint \( N_0 \), and it seems unlikely that this method can be adjusted to the current setting.

We conjecture that item (vii) is also equivalent to the other items of this theorem. For an eigenvalue \( z_0 = z(0) \) of geometric multiplicity 1 this is true, since in this case (vii) obviously implies (v).

**Corollary 5.9.** \( \varphi(v) \) cannot have order greater than rank(\( W \)).

*Proof.* This follows from Theorem 5.8(ii), since for \( \varphi(v) \) of order \( k \) at 0 the vectors \( W\varphi(0), \ldots, W\varphi^{(k-1)}(0) \) are linearly independent. \( \square \)

**Remark 10.** For an unbounded perturbation \( W = F^*JF \), the definition of the order of an eigenpath is interpreted as follows: \( \varphi(v) \) has order at least \( k \) at \( v = 0 \) if

\[
JF\varphi(0), JF\varphi'(0), \ldots, JF\varphi^{(k-2)}(0)
\]

are orthogonal to \( F\mathcal{V}(z_0, N_0) \).

Items (ii) and (iv) in the statement of Lemma 5.1 are replaced by the following:

(ii') the vectors \( JF\varphi(0), JF\varphi'(0), \ldots, JF\varphi^{(k-2)}(0) \) are orthogonal to a vector \( F\varphi^* \in F\mathcal{V}(z_0, N_0) \), such that \( \langle \varphi^*, \varphi(0) \rangle \neq 0 \).

(iv') for all \( j = 1, \ldots, k-1 \), and any \( f \in \text{dom}[A] \),

\[
(N_0 - z_0)[f, \partial^j_v\varphi(0)] = -j \left\langle Ff, JF\partial^{j-1}_v\varphi(0) \right\rangle.
\]

The changes to the proof are then straightforward.

The statement of Lemma 5.2 is altered so that (31) reads

\[
(N_0 - z_0)[f, \varphi_j] = -j \left\langle Ff, JF\varphi_{j-1} \right\rangle, \quad \forall f \in \text{dom}[A],
\]

and within each item of the conclusion \( F\varphi_j \) appears instead of \( \varphi_j \).
In the proof, instead of (32), we obtain

\[
(34) \quad \left[1 - vT_{20}(N_0)J\right]F\varphi_j = -jT_{20}(N_0)JF\varphi_{j-1}
\]

using a similar technique as in Remark 2.2. The equality (33) can be rewritten as

\[
(N_v - z_0)[f, \varphi_j] - v\langle FF, JF\varphi_j\rangle = -j\langle FF, JF\varphi_{j-1}\rangle,
\]

which for \(f = R_{z_0}(N_0^*)\psi\) and any \(\psi \in \mathcal{K}\) implies the equality of vectors

\[
\varphi_j - v(FR_{z_0}(N_0^*))^*JF\varphi_j = -j(FR_{z_0}(N_0^*))^*JF\varphi_{j-1}.
\]

Equality (34) follows after applying F. Given (34), the rest of the argument is virtually unchanged.

The necessary changes to the rest of the results in this document are minimal and should be clear given those previous and so this will be the last remark concerning the case of unbounded perturbations.

6. Structure of the resonance projection

6.1. Decomposition of \(P_z\). Here we show that the resonance projection \(P_{z_0}(N_0, W)\) can be decomposed into a sum of projections corresponding to cycles of resonance points.

**Lemma 6.1.** Assume that \(z_\tau(v)\) and \(z_\mu(v)\) are two different holomorphic eigenvalue functions of \(N_v\), and \(\varphi_\tau(v)\) and \(\varphi_\mu^*(v)\) are eigenvector functions corresponding to \(z_\tau(v)\) and \(z_\mu^*(v)\) respectively. Then \(\langle \varphi_\mu^*(v), \varphi_\tau(v) \rangle = 0\).

**Proof.** This proof uses a standard argument used to prove orthogonality of eigenvectors of self-adjoint operators. Using the eigenvalue equations for \(\varphi_\tau(v)\) and \(\varphi_\mu^*(v)\), we have

\[
z_\mu(v)\langle \varphi_\mu^*(v), \varphi_\tau(v) \rangle = \langle z_\mu^*(v)\varphi_\mu^*(v), \varphi_\tau(v) \rangle = \langle N_v\varphi_\mu^*(v), \varphi_\tau(v) \rangle = \langle \varphi_\mu^*(v), N_v\varphi_\tau(v) \rangle = \langle \varphi_\mu^*(v), z_\tau(v)\varphi_\tau(v) \rangle = z_\tau(v)\langle \varphi_\mu^*(v), \varphi_\tau(v) \rangle.
\]

Since the functions \(z_\tau(v), z_\mu(v)\) and \(\langle \varphi_\mu^*(v), \varphi_\tau(v) \rangle\) are holomorphic and \(z_\tau(v) \neq z_\mu(v)\), the scalar product must vanish. \(\square\)

**Theorem 6.2.** Let Assumptions 4.1–4.3 hold for two distinct eigenvalue functions \(z_\mu(v)\) and \(z_\tau(v)\), with corresponding eigenvector functions \(\varphi_\mu(v)\) and \(\varphi_\tau(v)\). If the functions \(\varphi_\mu(v)\) and \(\varphi_\tau(v)\) have orders \(d_\mu\) and \(d_\tau\) respectively at 0, then for any \(j = 0, 1, 2, \ldots, d_\mu - 1\) and \(k = 0, 1, 2, \ldots, d_\tau - 1\),

\[
(35) \quad \langle \tilde{\partial}_v^j\varphi_\mu^*(0), W\tilde{\partial}_v^k\varphi_\tau(0) \rangle = 0.
\]

**Proof.** (cf. [6, Theorem 3.1.12]) From Lemma 6.1 we have

\[
\langle \varphi_\mu^*(v), N_v\varphi_\tau(v) \rangle = z_\tau(v)\langle \varphi_\mu^*(v), \varphi_\tau(v) \rangle = 0,
\]

which when differentiated \(m + 1 := j + k + 1\) times at \(v = 0\) using the Leibniz rule gives

\[
(36) \quad 0 = \sum_{l=0}^{m+1} \binom{m + 1}{l} \langle \tilde{\partial}_v^l\varphi_\mu^*(0), N_v\tilde{\partial}_v^{m+1-l}\varphi_\tau(0) \rangle + \sum_{l=0}^{m} (m+1) \binom{m}{l} \langle \tilde{\partial}_v^l\varphi_\mu^*(0), W\tilde{\partial}_v^{m-l}\varphi_\tau(0) \rangle.
\]
We transform the first summand as

\[(E) := \sum_{l=0}^{m+1} \left( \begin{array}{c} m+1 \\ l \end{array} \right) \langle \partial_v^l \phi^*_\mu(0), N_0 \partial_v^{m+1-l} \phi_\tau(0) \rangle \]

\[= \sum_{l=0}^{j} \left( \begin{array}{c} m+1 \\ l \end{array} \right) \langle N_0^* \partial_v^l \phi^*_\mu(0), \partial_v^{m+1-l} \phi_\tau(0) \rangle + \sum_{l=j+1}^{m+1} \left( \begin{array}{c} m+1 \\ l \end{array} \right) \langle \partial_v^l \phi^*_\mu(0), N_0 \partial_v^{m+1-l} \phi_\tau(0) \rangle \]

and apply the equalities

\[N_0^* \partial_v^l \phi^*_\mu(0) = z_0 \partial_v^l \phi^*_\mu(0) - lW \partial_v^{m-l} \phi^*_\mu(0),\]

\[N_0 \partial_v^{m+1-l} \phi_\tau(0) = z_0 \partial_v^{m+1-l} \phi_\tau(0) - (m + 1 - l)W \partial_v^{m-l} \phi_\tau(0),\]

from Lemma 5.1 and Proposition 5.4 to obtain

\[(E) = z_0 \sum_{l=0}^{m+1} \left( \begin{array}{c} m+1 \\ l \end{array} \right) \langle \partial_v^l \phi^*_\mu(0), \partial_v^{m+1-l} \phi_\tau(0) \rangle - \sum_{l=1}^{j} l \left( \begin{array}{c} m+1 \\ l \end{array} \right) \langle W \partial_v^{m-l} \phi^*_\mu(0), \partial_v^{m+1-l} \phi_\tau(0) \rangle \]

\[= z_0 \partial_v^{m+1} \phi^*_\mu(v), \phi_\tau(v) \mid_{v=0} - \sum_{l=0}^{j-1} \frac{(m+1)!}{l!(m-l)!} \langle W \partial_v^l \phi^*_\mu(0), \partial_v^{m+1-l} \phi_\tau(0) \rangle \]

\[= \sum_{l=j+1}^{m+1} \frac{(m+1)!}{l!(m-l)!} \langle \partial_v^l \phi^*_\mu(0), W \partial_v^{m+1-l} \phi_\tau(0) \rangle.\]

The first summand here is zero by Lemma 6.1 and hence returning to (36), we find (35). \(\square\)

In the next theorem we consider the total projection corresponding to a cycle of resonance points.

**Theorem 6.3.** Let Assumption 4.3 be given for all eigenvalue functions \(z_\tau(v)\) of \(N_0\) which split from \(z_0 \in \sigma_d(N_0)\) and let \(\phi_\tau(v)\) be corresponding eigenvector functions for \(\tau = 1, \ldots, m\). For each \(\tau = 1, \ldots, m\), let \(v_\tau^{(j)}(z)\), \(j = 0, \ldots, d_\tau - 1\), be the corresponding cycle of resonance points which constitute the multivalued inverse of \(z_\tau(v)\). Then the following assertions hold:

(i) The function

\[P^{[\tau]}_z := \sum_{j=0}^{d_\tau-1} P^{[\tau]}_{z_{\tau v}^{(j)}(z)}(N_{v_\tau^{(j)}(z)}, W)\]

of \(z\) is holomorphic in a neighbourhood of \(z_0\).

(ii) The range

\[\Upsilon^{[\tau]}_{z_0} := \text{im} P^{[\tau]}_{z_0}\]

of the idempotent \(P^{[\tau]}_{z_0}\) (which exists by item (1)) has dimension \(d_\tau\) and the vectors

\[\phi_\tau(0), \phi'_\tau(0), \ldots, \phi^{(d_\tau-1)}_\tau(0)\]

form its basis.

(iii) The operators \(P^{[\tau]}_{z_0}(N_0, W)\) and \(A_{z_0}(N_0, W)\) commute and thus the vector space \(\Upsilon^{[\tau]}_{z_0}\) is invariant with respect to \(A_{z_0}(N_0, W)\).
Theorem 6.4. Let Assumption 4.3 hold for all eigenvalue functions there is a correspondence of cycles and eigenvectors, which proves (iv).

6.2. Schmidt representation for coefficients on both sides of the equality and \( z \) bounded in a neighbourhood of \( P \).

Using these equalities and the Laurent expansion of \( R \) terms with negative powers of \( z \) the eigenvalue it follows from previous material that the total projection \( \sum_{j=1}^{d_r} P_z^{[\tau]} \) converges to the linear span of the independent vectors \( \varphi^{(j)}(0), j = 0, \ldots, d_r - 1 \).

Each \( P_z^{[\tau]}, \tau = 1, \ldots, m, \) is by construction single-valued and its Laurent expansion can only have finitely many terms with negative powers of \((z - z_0)\) (see e.g. [18, Theorem II-1.8]). Since the total projection \( \sum_{\tau=1}^{m} P_z^{[\tau]} \) converges to \( P_z(N_0, W) \) and is therefore bounded as \( z \to z_0 \), any terms with negative powers of \( P_z^{[\tau]}, \tau = 1, \ldots, m, \) must cancel one another as \( z \to z_0 \). However, they cannot interact in the sense that \( P_z^{[\tau]} P_z^{[\mu]} = \delta_{\tau\mu} P_z^{[\tau]} \). Hence it must be that each \( P_z^{[\tau]} \) is also bounded in a neighbourhood of \( z_0 \). This proves (i).

We have

\[
P_z(N_0, W) = \sum_{\tau=1}^{m} P_z^{[\tau]}
\]

Using these equalities and the Laurent expansion of \( R_z(N_\tau) \), (iii) is proved by comparing coefficients on both sides of the equality \( P_z(N_0, W) = R_z(N_\tau) W P_z^{[\tau]} \).

It follows from (iii) that the dimension of \( Y \cap Y_0 \) is at least 1, but it cannot be more since there is a correspondence of cycles and eigenvectors, which proves (iv).

Finally, (v) follows directly from (iii) and (iv). \( \square \)

6.2. Schmidt representation for \( P_z \).

Theorem 6.4. Let Assumption 4.3 hold for all eigenvalue functions \( z_\tau(v) \) of \( N_\tau \) which split from the eigenvalue \( z_0 \) of \( N_0 \) of geometric multiplicity \( m \). Let \( \varphi_\tau(v), \tau = 1, \ldots, m, \) be corresponding eigenpaths. Then the idempotent operator \( P_z(N_0, W) \) can be written in the form

\[
P_z(N_0, W) = \sum_{\tau=1}^{m} \sum_{k,j=0}^{d_r-1} \alpha_{\tau}^{kj} \left< W \delta_{v} \varphi_\tau^*(0), \right> \left< \partial_{v} \varphi_\tau(0), \right>
\]

where \( \alpha_{\tau} \) is a skew-upper-triangular Hankel matrix of size \( d_r \times d_r \).

This theorem is an analogue of [6, Theorem 5.9.1], where in this case the matrix \( \alpha \) is not necessarily real.

Proof. By Theorem 6.3, a natural Jordan basis for the vector space \( \text{im} P_z(N_0, W) \) is given by

\[
\frac{1}{j!} \partial_{v}^j \varphi_\tau(0), \quad \tau = 1, \ldots, m, \ j = 0, 1, \ldots, d_r - 1.
\]

It also follows from previous material that

\[
\frac{1}{k!} W \delta_{v}^k \varphi_\mu^*(0), \quad \mu = 1, \ldots, m, \ k = 0, 1, \ldots, d_\mu - 1,
\]
is a basis of \( \text{im}(P_{z_0}(N_0, W))^* = W \text{ im } P_{z_0}(N_0^*, W) \). Therefore, there must exist a unique invertible \( n \times n \) matrix \( \alpha = (\alpha_{\mu \tau}^{kj}) \) such that

\[
P_{z_0}(N_0, W) = \sum_{\mu=1}^{m} \sum_{k=0}^{d_\mu-1} \sum_{\tau=1}^{d_\tau-1} \sum_{j=0}^{d_j-1} \frac{1}{k!j!} \alpha_{\mu \tau}^{kj} \langle W \partial_v \varphi_{\mu}^*(0), \partial_v \varphi_{\tau}(0) \rangle.
\]

It remains to see that \( \alpha \) is a direct sum of skew-upper-triangular Hankel matrices.

Since \( P_{z_0} \) leaves the vectors \( \partial_v \varphi_{\tau}(0) \) invariant, we have

\[
\frac{1}{j!} \partial_{v}^{j} \varphi_{\tau}(0) = \frac{1}{j!} P_{z_0}(N_0, W) \partial_{v}^{j} \varphi_{\tau}(0) = \sum_{\tau,j} \left( \sum_{\mu,k} \alpha_{\mu \tau}^{kj} \beta_{\mu \tau}^{kj} \right) \frac{1}{j!} \partial_{v}^{j} \varphi_{\tau}(0),
\]

where the matrix \( \beta \) is given by

\[
\beta_{\mu \tau}^{kj} = \frac{1}{k!j!} \langle \partial_v^k \varphi_{\mu}^*(0), W \partial_v^j \varphi_{\tau}(0) \rangle.
\]

Hence it must be that \( \sum_{\mu,k} \alpha_{\mu \tau}^{kj} \beta_{\mu \tau}^{kj} = \delta_{\tau\tau'} \delta_{j j'} \), that is, \( \alpha \) is the inverse transpose of \( \beta \).

Theorem 6.2 implies \( \beta_{\mu \tau}^{kj} = \delta_{\mu \mu'} \beta_{\tau \tau'}^{kj} \), or in other words, \( \beta \) is a direct sum of \( m \) matrices of size \( d_\tau \times d_\tau \). Moreover, using Theorem 6.8(vi),

\[
\beta_{\tau \tau}^{kj} = \frac{1}{k!j!} \langle \partial_v^k \varphi_{\tau}^*(0), W \partial_v^j \varphi_{\tau}(0) \rangle
\]

\[
= \frac{1}{k!(j+1)!} \langle \partial_v^k \varphi_{\tau}^*(0), W A_{z_0}(N_0, W) \partial_v^{j+1} \varphi_{\tau}(0) \rangle
\]

\[
= \frac{1}{k!(j+1)!} \langle A_{z_0}(N_0^*, W) \partial_v^k \varphi_{\tau}^*(0), W \partial_v^{j+1} \varphi_{\tau}(0) \rangle
\]

\[
= \frac{1}{(k-1)!(j+1)!} \langle \partial_v^{k-1} \varphi_{\tau}^*(0), W \partial_v^{j+1} \varphi_{\tau}(0) \rangle
\]

\[
= \beta_{\tau \tau}^{k-1,j+1}.
\]

By the same argument, if \( k + j + 1 \leq d_\tau - 1 \) then

\[
\beta_{\tau \tau}^{kj} = \beta_{\tau \tau}^{0,j+k} = \frac{1}{(j+k)!} \langle \varphi_{\tau}^*(0), W \partial_v^{j+k} \varphi_{\tau}(0) \rangle
\]

\[
= \frac{1}{(j+k)!} \langle A_{z_0}(N_0^*, W) \varphi_{\tau}^*(0), W \partial_v^{j+k+1} \varphi_{\tau}(0) \rangle = 0.
\]

Thus the matrix \( \beta \) is a direct sum of Hankel matrices with zeros above the main skew-diagonal. It follows that \( \alpha = \beta^{-1} \) is a direct sum of Hankel matrices with zeros below the main skew-diagonal. \( \square \)

7. Order and tangency to the resonance set

For a given eigenvalue \( z_0 \in \sigma_d(N_0) \), we say a direction \( W \) has order \( k \) if the triple \((z_0, N_0, W)\) has order \( k \). In this section we demonstrate a direct connection between high order directions \( W \) and those that are tangent to the complex variety of operators in \( \mathcal{A} \) which have \( z_0 \) as an eigenvalue. This connection was observed in [6] in the case of a real isolated eigenvalue \( z_0 \) and self-adjoint \( N_0 \).

Given a fixed number \( z_0 \) outside the essential spectrum of the affine space \( \mathcal{A} \), the resonance set \( \mathcal{R}(z_0) \) is defined by

\[
\mathcal{R}(z_0) := \{ N \in \mathcal{A} : z_0 \in \sigma_d(N) \}.
\]
We say that an analytic path of operators \( N(v) \) in \( \mathcal{A} \) is resonant (at \( z_0 \)), if \( N(v) \in \mathcal{R}(z_0) \) for all \( v \). If an analytic path is not resonant, then the following trivial lemma holds.

**Lemma 7.1.** For an analytic curve \( \gamma \) in \( \mathcal{A} \) which is not contained in \( \mathcal{R}(z_0) \), the set \( \mathcal{R}(z_0) \cap \gamma \) is discrete.

Let \( z_0 \in \sigma_d(N_0) \), where \( N_0 \in \mathcal{A} \), and let \( k \in \mathbb{N} \). We say that a direction \( W \in \mathcal{A}_0 \) is tangent to order at least \( k \) at \( N_0 \), if there exists a resonant path \( N(v) \) such that for some numbers \( c_2, c_3, \ldots, c_{k-1} \),

\[
N(v) = N_0 + vW + \sum_{j=2}^{k-1} c_j v^j W + O(v^k), \quad v \to 0.
\]

(37)

We say that \( W \) is tangent at \( N_0 \) if it is tangent to order at least two. Otherwise, we say that \( W \) is transversal at \( N_0 \). By changing the parameter \( v \), the coefficients \( c_j, j = 2, \ldots, k-1 \), can be eliminated so that

\[
N(v) = N_0 + vW + O(v^k), \quad v \to 0.
\]

A resonant path of this form is called standard.

### 7.1. Tangent directions have high orders

The next theorem is an analogue of [6, Theorem 4.1.2].

**Theorem 7.2.** Let \( k \geq 2 \), let \( z_0 \) be a semisimple eigenvalue of \( N_0 \) and let \( W \) be a direction at \( N_0 \) for which Assumption 4.3 holds for all eigenvalues of the \( z_0 \)-group. If \( N(v) \) is a resonant path tangent to \( W \) at \( N_0 \) to order at least \( k \) and if \( \chi(v) \) is a corresponding analytic eigenpath, then

(i) the vectors \( \chi(0), \chi'(0), \ldots, \chi^{(k-1)}(0) \) have orders respectively 1, 2, \ldots, \( k \),
(ii) the direction \( W \) has order at least \( k \),
(iii) for any \( l = 1, 2, \ldots, k \)

\[
\mathbf{A}_{z_0}(N_0, W)\chi^{(l-1)}(0) = (l-1)\chi^{(l-2)}(0) + \sum_{j=2}^{l-1} j! \binom{l-1}{j} c_j \chi^{(l-1-j)}(0),
\]

where the numbers \( c_j \) are as in (37), and
(iv) the eigenvector \( \chi(0) \) has depth at least \( k - 1 \).

**Proof.** (cf. [6, Theorem 4.1.2]) We prove the first item by induction. The vector \( \chi(0) \) has order 1 since it is an eigenvector. Now assume the assertion holds for all numbers up to but not including \( l \). Since the path \( N(v) \), which is tangent to \( W \) at \( N_0 \) to order at least \( l - 1 \), is also tangent to order at least \( l - 1 \), it follows from this assumption that the vectors \( \chi(0), \chi'(0), \ldots, \chi^{(l-2)}(0) \) have orders 1, 2, \ldots, \( l - 1 \) respectively. Differentiating the eigenvalue equation \( N(v)\chi(v) = z_0 \chi(v) \) \( l - 1 \) times at \( v = 0 \) gives

\[
z_0\chi^{(l-1)}(0) = \sum_{j=0}^{l-1} \binom{l-1}{j} N^{(j)}(0)\chi^{(l-1-j)}(0)
\]

\[
= N_0\chi^{(l-1)}(0) + (l-1)W\chi^{(l-2)}(0) + \sum_{j=2}^{l-1} j! \binom{l-1}{j} c_j W\chi^{(l-1-j)}(0),
\]

for any \( l = 1, 2, \ldots, k \).
where the second line follows from the tangency of $N(v)$ to $W$ at $N_0$ to order $l$. Choosing $v$ so that $z_0$ belongs to the resolvent set of $N_v = N_0 + vW$, the above equality can be rewritten as

$$(N_v - z_0)\chi^{(l-1)}(0) - vW\chi^{(l-1)}(0) = -(l - 1)W\chi^{(l-2)}(0) - \sum_{j=2}^{l-1} j! \binom{l-1}{j} c_j W\chi^{(l-1-j)}(0),$$

which we multiply by $R_{z_0}(N_v)$ in order to obtain

$$(38) \quad (1 - vR_{z_0}(N_v)W)\chi^{(l-1)}(0) = -R_{z_0}(N_v)W \left( (l - 1)\chi^{(l-2)}(0) + \sum_{j=2}^{l-1} j! \binom{l-1}{j} c_j \chi^{(l-1-j)}(0) \right).$$

Recall that $R_{z_0}(N_v)W$ preserves the order of resonance vectors whereas $(1 - vR_{z_0}(N_v)W)$ decreases it by 1. Therefore, since the vector on the right of (38) has order $l - 1$, the vector $\chi^{(l-1)}(0)$ must have order $l$. This completes the proof of (i).

(ii) follows immediately from (i).

To prove (iii), we integrate (38) with respect to $v$ along a contour enclosing 0 and refer to the definitions (17), (18), and (19) of $P_{z_0}(N_0,W)$ and $A_{z_0}(N_0,W)$.

(iv) follows immediately from (iii). \hfill \Box

An immediate corollary is the following analogue of [6, Proposition 4.1.4].

Corollary 7.3. Under the assumptions of Theorem 7.2, the path $N(v)$ is standard iff

$$A_{z_0}(N_0,W)\chi^{(l-1)}(0) = (l - 1)\chi^{(l-2)}(0), \quad l = 1, \ldots, k.$$

7.2. High order directions are tangent. The following is an off-real-axis analogue of [6, Theorem 4.2.1].

Theorem 7.4. Let $z_0$ be a simple eigenvalue of $N_0$, $\chi$ a corresponding normalised eigenvector of $N_0$ and $W_0 = \langle \chi, \cdot \rangle \chi$. Then the intersection of the affine space

$$\alpha := N_0 + CW + CW_0$$

with a sufficiently small neighbourhood of the point $N_0$ in $\mathcal{R}(z_0)$ consists of one and only one simple complex analytic curve $\gamma$.

Proof. First we show that any deleted neighbourhood of $N_0$ in the affine plane $\alpha$ has a resonance point. Assume the contrary: there exists a convex neighbourhood $O$ of $N_0$ in $\alpha$ which does not contain resonance points other than $N_0$. Choose positive $s_0, \varepsilon_0$ so that

$$N_0 + \varepsilon_0 \mathbb{D}W + s_0 \mathbb{D}W_0 \subset O,$$

where $\mathbb{D}$ denotes the closed unit disk. Soon we will also impose another condition on $\varepsilon_0$. The eigenvalue of an operator $N_0 + vW + sW_0$ from $\alpha$ which is obtained by perturbation of $z_0$ we denote $z(v,s)$. Since $z_0$ is simple, there can be no branching of the function $z(v,s)$ which is therefore single-valued in a small enough neighbourhood of $(0,0)$.

The set $z(0, s_0 \partial \mathbb{D})$ is a circle in the $z$-plane with centre at $z_0 = z(0,0)$ and radius $s_0$. We choose $\varepsilon_0$ small enough so that all $z(\varepsilon_0 \mathbb{D}, 0)$ are inside this circle. Now, we fix $v \in \mathbb{D}$ and deform $s$ in

$$z(\varepsilon_0 v, s \partial \mathbb{D})$$

from $s_0$ to zero. Clearly, for all $v \in \mathbb{D}$ these are closed loops. For $s = s_0$ and small enough fixed $v$ this loop encloses $z_0$ and for $s = 0$ this loop degenerates to a point $z(\varepsilon_0 v, 0)$ different (by
assumption) from $z_0$. Hence, for some $s$ between $s_0$ and $0$ the loop has to cross $z_0$. The claim is proved.

The intersection of $\alpha$ with the resonance set must be a union of smooth curves. Since $z_0$ has geometric multiplicity 1 at $N_0$, such a curve is unique and simple near $N_0$.

The eigenvalue $z = z(v,s)$ depends analytically on $v$ and $s$ and $\frac{\partial z}{\partial s}(0,0) = 1 \neq 0$. Therefore, the implicit equation $z_0 = z(v,s(v))$ defines a single-valued analytic function $s(v)$ with $s(0) = 0$.

For a simple eigenvalue $z_0 = z(0)$ the operator $W_0$ is uniquely determined, and therefore, in this case so is the curve $\gamma$ from Theorem 7.4 which thus depends only on the triple $(z_0, N_0, W)$. We denote this simple curve by

$$\gamma(z_0, N_0, W).$$

Proof of the next theorem follows that of [6, Theorem 4.3.1] with some adjustments. In this theorem we assume that the geometric multiplicity $m$ is equal to 1, and so we can and do write $z(v)$ etc., instead of $z_\tau(v)$ etc. However we note that this assumption is not necessary for the proof as long as we have the curve $\gamma$, the existence of which is implied by the condition $m = 1$ according to Theorem 7.4.

**Theorem 7.5.** Let $k$ be an integer $\geq 2$. Let $z(v)$ be a simple eigenvalue function of $N_v$, $\varphi(v)$ a corresponding eigenvector function, and let $\chi_0 = \varphi(0)$. Let $\gamma$ be the curve (39). If $\chi_0$ is an eigenvector of depth $\geq k - 1$ for the triple $(z_0; N_0, W)$, then

(i) the direction $W$ is tangent to the curve $\gamma$ to order at least $k$,

(ii) in the Taylor expansion

$$\chi(v) = \sum_{j=0}^{\infty} v^j \chi_j$$

of an eigenpath $\chi(v)$ corresponding to $\gamma(v)$, the vectors $\chi_0, \chi_1, \ldots, \chi_{k-1}$ are resonance vectors of orders respectively 1, 2, ..., $k$,

(iii) for any eigenpath (40) corresponding to $\gamma$ and for all $j = 1, 2, \ldots, k - 1$ the vector $A_{z_0}(N_0, W)\chi_j$ is a linear combination of vectors $\chi_0, \chi_1, \ldots, \chi_{j-1}$. Moreover, if the parametrisation of the curve $\gamma$ is standard then $A_{z_0}(N_0, W)\chi_j = \chi_{j-1}$.

(iv) the vectors $\chi_0, \chi_1, \ldots, \chi_{k-2}$, have depth at least one and are $W$-orthogonal to $\mathcal{V}(z_0, N_0^*)$.

**Proof.** Let $N(v)$ be a parametrisation of the resonance curve $\gamma$. Since the curve $\gamma$ is the intersection of the resonance set with the affine plane $N_0 + CW + CW_0$, where $W_0 = \langle \chi_0, \cdot \rangle \chi_0$, the Taylor expansion of $N(v)$ has the form

$$N(v) = N_0 + \sum_{j=1}^{\infty} v^j (\alpha_j W + \beta_j W_0).$$

A path of eigenvectors $\chi(v)$ corresponding to this path has Taylor series

$$\chi(v) = \chi_0 + v\chi_1 + v^2 \chi_2 + \ldots$$

which starts at the vector $\chi_0$. Comparing the coefficients of $v^1$ on both sides of the eigenvalue equation

$$N(v)\chi(v) = z_0\chi(v),$$

gives

$$(N_0 - z_0)\chi_1 = - (\alpha_1 W + \beta_1 W_0)\chi_0.$$
The vector \((N_0 - z_0)\chi_1\) is orthogonal to the eigenspace \(\mathcal{V}(z_0, N_0^*)\) and in particular it is orthogonal to
\[ \chi_0^* := \varphi^*(0). \]

Hence,
\[ \langle \chi_0^*, (\alpha_1 W + \beta_1 W_0)\chi_0 \rangle = 0. \]

Since by the premise \(\chi_0\) has depth \(\geq 1\), we have
\[ \langle \chi_0^*, W\chi_0 \rangle = 0, \]
and combining this equality with previous one gives \(\beta_1 = 0\). Combining this with (41) we see that \(N(v)\) is tangent to \(W\) at \(N_0\) (to order at least 2), and
\[ (N_0 - z_0)\chi_1 = -\alpha_1 W\chi_0. \]

So, by Lemma 5.2, the vector \(\chi_1\) is a resonance vector, has order 2 and \(A_{z_0}(N_0, W)\chi_1 = \alpha_1 \chi_0\). Further, if the parametrisation of \(\gamma(v)\) is standard, then \(\alpha_1 = 1\).

This proves the theorem in the case of \(k = 2\). We proceed by induction on \(k\). So, assume that the claim holds for \(k \leq l\) and let \(\chi_0\) be an eigenvector of depth \(\geq l\). By \(l\) times differentiating the eigenvalue equation (42), we obtain
\[ \sum_{j=0}^{l} \binom{l}{j} N^{(j)}(v)\chi^{(l-j)}(v) = z_0 \chi^{(l)}(v). \]

Letting \(v = 0\) and replacing \(\chi^{(j)}(0)/j!\) by \(\chi_j\) gives
\[ N_0\chi_l + \sum_{j=1}^{l} (\alpha_j W + \beta_j W_0)\chi_{l-j} = z_0 \chi_l. \]

By the induction assumption, we have
\[ \beta_1 = \ldots = \beta_{l-1} = 0. \]

Hence,
\[ (N_0 - z_0)\chi_l + \sum_{j=1}^{l-1} \alpha_j W\chi_{l-j} + (\alpha_l W + \beta_l W_0)\chi_0 = 0. \]

Since \(\chi_0\) has depth at least \(l\), by Corollary 5.6 so does \(\chi_0^*\) and therefore, for some vector \(g\) we have \(\chi_0^* = A_{z_0}^l(N_0^*, W)g\). Since, by the induction assumption, \(\chi_j, j = 0, 1, \ldots, l-1\), is a vector of order \(j + 1\), it follows that for all \(j = 0, 1, \ldots, l-1\)
\[ \langle \chi_0^*, W\chi_j \rangle = \langle A_{z_0}^l g, W\chi_j \rangle = \langle g, W A_{z_0}^l \chi_j \rangle = 0. \]

Hence, it follows from (41) (and \(\langle \chi_0^*, \chi_0 \rangle \neq 0, \) see (27)) by taking the scalar product with \(\chi_0^*\) that
\[ \beta_l = 0 \]
and so
\[ (N_0 - z_0)\chi_l + \sum_{j=1}^{l} \alpha_j W\chi_{l-j} = 0. \]

It follows from (41), (43) and (45) that \(W\) is tangent to \(\gamma\) to order at least \(l + 1\). Further, the vector \((N_0 - z_0)\chi_l\) is orthogonal to \(\mathcal{V}(z_0, N_0^*)\) and the vectors \(W\chi_0, \ldots, W\chi_{l-2}\) are also orthogonal to \(\mathcal{V}(z_0, N_0^*)\) by the induction assumption, since the vectors \(\chi_0, \ldots, \chi_{l-2}\) have depth
at least one. Hence, according to \([40]\), so is the vector \(W \chi_{l-1}\). Using an argument from the proof of Lemma \([5, 2]\) one can infer from \([16]\) that \(\chi_l\) is a resonance vector and

\[
A_{z_0}(N_0, W)\chi_l = \sum_{j=1}^l \alpha_j \chi_{l-j}.
\]

It also follows from this that \(\chi_l\) has order \(l + 1\). Since, by the induction assumption, the vectors \(\chi_0, \ldots, \chi_{l-2}\) have depth \(\geq 1\), the last equality implies that \(\chi_{l-1}\) is also a resonance vector and has the same property.

Further, if the parametrisation of \(\gamma(v)\) is standard, then \(\alpha_2 = \ldots = \alpha_l = 0\) and \(\alpha_1 = 1\) and therefore \(A_{z_0}(N_0, W)\chi_l = \chi_{l-1}\).

\[\square\]

8. **Tangency of the Vector Field Generated by a Commutator**

To conclude we remark on the well-known fact of the isospectrality of a solution to a differential equation with a Lax representation, from the point of view of this paper. Let a pair of operators \(N(t)\) and \(W(t)\), depending on time \(t\), satisfy Lax’s equation

\[
\frac{d}{dt}N(t) = [N(t), W(t)],
\]

where \([N, W] = NW - WN\) is the commutator. As suggested by the notation, we consider the case when \(N\) and \(W\) respectively take values in the affine space \(A\) and the vector space of directions \(A_0\).

Given \(W\), the isospectrality of a solution \(N\) of \([47]\) can be viewed as the tangency of the vector field \(N \mapsto [N, W]\), hence the curve \(N\), to the resonance set \(R(z_0)\) for any simple eigenvalue \(z_0\) of \(N(t_0)\). Indeed, if for any \(t = t_0\) with \(N_0 := N(t_0)\) we have \(N_0 \varphi = z_0 \varphi\), then

\[
\langle \varphi^*, [N_0, W(t_0)] \varphi \rangle = \langle N_0^* \varphi^*, W(t_0) \varphi \rangle - \langle \varphi^*, W(t_0) N_0 \varphi \rangle = 0.
\]

Thus by Theorem \([5, 8]\) \(\varphi\) has depth at least one and equivalently \(z'(t_0) = 0\). Therefore \(N(t)\) is tangent at \(t_0\) to the resonance set \(R(z_0)\) by Theorem \([7, 5]\). Since \(t_0\) was chosen generically, it follows that the eigenvalue \(z(t)\) remains fixed at \(z_0\) and that \(N(t)\) remains tangent to the resonance set \(R(z_0)\).

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