FROM COMPLETENESS OF DISCRETE TRANSLATES TO PHASELESS SAMPLING OF THE SHORT-TIME FOURIER TRANSFORM

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ABSTRACT. This article bridges and contributes to two important research areas, namely the completeness problem for systems of translates in function spaces and the short-time Fourier transform (STFT) phase retrieval problem. As a first main contribution, we show that a complex-valued, compactly supported function can be uniquely recovered from samples of its spectrogram if certain density properties of an associated system of translates hold true. Secondly, we derive new completeness results for systems of discrete translates in spaces of continuous functions on compact sets. We finally combine these findings to deduce several novel recovery results from spectrogram samples. Our results hold for a large class of window functions, including Gaussians, all Hermite functions, as well as the practically highly relevant Airy disk function. To the best of our knowledge, our results constitute the first recovery guarantees for the sampled STFT phase retrieval problem with a non-Gaussian window.

Keywords. phase retrieval, discrete translates, completeness, sampling, short-time Fourier transform, lattice

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1. INTRODUCTION

The main aim of the present paper is to bridge and contribute to two active research areas in pure and applied mathematics: the problem of completeness of discrete translates and the discretization of the short-time Fourier transform (STFT) phase retrieval problem. Both problems have received significant attention in recent years:

• The completeness of translates problem constitutes a classical problem at the intersection of approximation theory, functional analysis,

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Recent contributions in the field of harmonic analysis have seen a surge in the mathematical community due to its relevance in a wide-ranging number of applications. The two recent survey papers contain an extensive list of results and references on this problem. More recent contributions on both the theoretical and numerical aspects can be found in.

We start by introducing both the completeness of translates problem as well as the STFT phase retrieval problem, followed by a detailed derivation of their interrelations.

1.1. Completeness of discrete translates. Let \((X, \| \cdot \|)\) be a Banach space over \(\mathbb{C}\), let \(\Lambda\) be an index set and consider a subset \(\{f_\lambda : \lambda \in \Lambda\}\subseteq X\). Then \(\{f_\lambda : \lambda \in \Lambda\}\) is said to be complete in \(X\) if its linear span is dense in \(X\). Determining whether a given system is complete lies at the very heart of approximation theory and numerical analysis alike. The traditionally most notable result is due to Weierstrass, later generalized by Stone. For a more modern exposition we refer to Pinkus’ overview article on the completeness problem in approximation theory. The practicality aspect further prompts one to look for structured complete sets. Indeed, suppose from now on that \(X\) is a function space on \(\mathbb{R}^d\) and let \(f_\lambda = f(\cdot - \lambda)\) be the \(\lambda\)-shift of a function \(f \in X\). We define

\[
\Sigma(f, \Lambda) := \{f(\cdot - \lambda) : \lambda \in \Lambda\}
\]

and call \(\Sigma(f, \Lambda)\) the system of translates of \(f\) by \(\Lambda\). The completeness of translates problem asks for assumptions on \(X, f\) and \(\Lambda\) so that \(\Sigma(f, \Lambda)\) is complete in \(X\). The investigation of this problem produced a vast literature, a possible starting point being the classical results of Wiener. In recent years, close attention was drawn to the situation where \(X\) is a space of functions defined on the entire real line and \(\Lambda\) is a discrete set, see for instance. However, the case where \(X = C(K)\), \(K \subseteq \mathbb{R}^d\) compact, is the space of continuous functions on \(K\) seems much less studied, notable exceptions include results of Landau, Zalik and Akhiezer. In this situation, the choice \(\Lambda = \mathbb{R}^d\) leads to the theory of mean-periodic functions, a concept which will play a rather subtle but indispensable role in this paper.

**Definition 1.1.** A function \(f \in C(\mathbb{R}^d)\) is said to be mean-periodic if there exists a compactly supported complex Radon measure \(\mu\), \(\mu \neq 0\), such that

\[
f * \mu = 0.
\]

An application of the Hahn-Banach theorem implies that if \(f\) is not mean-periodic, then \(\Sigma(f, \mathbb{R}^d)\) is complete in \(C(K)\) for every compact set \(K \subseteq \mathbb{R}^d\). It is precisely this compactness assumption that will be of central importance in the remainder of the present article.

1.2. Phaseless sampling of the short-time Fourier transform. Moving on to the second half of the title, we again consider a Banach space...
\((X, \| \cdot \|)\), typically a Hilbert space \((X, \langle \cdot, \cdot \rangle)\). In this setting, we are given a linear operator acting as
\[
A : X \to \mathbb{C}^\Gamma, \quad f \mapsto (Af(\gamma))_{\gamma \in \Gamma}.
\]
Assuming \(A\) is injective, the phase retrieval problem asks whether the phaseless measurements
\[
|A| : X \to \mathbb{C}^\Gamma, \quad f \mapsto (|Af(\gamma)|)_{\gamma \in \Gamma}
\]
still define an injective operator. Since \(A\) is linear, for all complex numbers \(\tau \in \mathbb{T}\) of modulus one and all \(f \in X\) it holds that \(|A(\tau f)| = |Af|\). Hence, we introduce the equivalence relation for \(f, h \in X\)
\[
f \sim h :\iff \exists \tau \in \mathbb{T} : f = \tau h.
\]
Then the injectivity of \(|A|\) is to be understood on the quotient space \(X/\sim\).

The fundamental operator we are investigating here is the short-time Fourier transform (STFT) with respect to a window function \(g \in L^2(\mathbb{R}^d), g \neq 0\), given by
\[
(V_g f)(x, \omega) := \int_{\mathbb{R}^d} f(t) \overline{g(t-x)} e^{-2\pi i \omega \cdot t} dt = \langle f, M_{x, \omega} g \rangle, \quad (x, \omega) \in \mathbb{R}^{2d},
\]
where \(T_x g(t) = g(t-x)\) and \(M_{x, \omega} g(t) = e^{2\pi i \omega \cdot t} g(t)\) denote the time resp. frequency shift, and \(\langle v, w \rangle := \int_{\mathbb{R}^d} v(t) \overline{w(t)} dt\) denotes the \(L^2\)-inner product of two functions \(v, w \in L^2(\mathbb{R}^d)\), while \(\omega \cdot t = \sum_{j=1}^d \omega_j t_j\) denotes the standard scalar product on \(\mathbb{R}^d\). The square of the absolute value of the STFT is called the spectrogram of \(f\),
\[
\text{SPEC}_g f(x, \omega) = |V_g f(x, \omega)|^2.
\]
With respect to (STFT) phase retrieval, for a subspace \(X \subseteq L^2(\mathbb{R}^d)\), subset \(\Gamma \subseteq \mathbb{R}^{2d}\) and window function \(g \in L^2(\mathbb{R}^d)\), we ask when
\[
\text{SPEC}_g : X \to \mathbb{C}^\Gamma, \quad f \mapsto (|V_g f(\gamma)|^2)_{\gamma \in \Gamma}
\]
is injective on \(X/\sim\). Note that \(|V_g f| \geq 0\), so having the samples of the spectrogram is equivalent to having the phaseless STFT samples. This setup fits within the abstract framework since the STFT is injective on \(L^2(\mathbb{R}^d)\). In fact, there exists an associated Calderón-type reproducing formula. While this is a good starting point, the separability of \(L^2(\mathbb{R}^d)\) motivates the search for discretizations of this representation. In particular, it would be advantageous to be able to uniquely determine \(f\) from the evaluation of its STFT (and its spectrogram, respectively) on a separated set \(\Gamma\), i.e., a set with
\[
\inf_{\gamma, \gamma' \in \Gamma \atop \gamma \neq \gamma'} \|\gamma - \gamma'\|_2 > 0.
\]
A particular type of separated sets are lattices, that is, sets of type \(\Lambda = AZ^{2d}\) for some invertible generating matrix \(A \in \text{GL}(2d, \mathbb{R})\). The special structure comes with additional nice properties that allow us to understand the STFT sampling problem better through a range of duality theorems \cite{26,27}. Unavoidably, one encounters the reciprocal lattice
\[
\Gamma^* := \{ \lambda \in \mathbb{R}^{2d} : \gamma \cdot \lambda \in \mathbb{Z} \quad \text{for all} \quad \gamma \in \Gamma \}\]
and the fundamental domain of $\Gamma$, which is a measurable cross-section of $\mathbb{R}^{2d}/\Gamma$. The *volume* of the lattice $\text{vol}(\Gamma)$ is the Lebesgue measure of a fundamental domain of $\Gamma$ and its *density* is its reciprocal $D(\Lambda) = \text{vol}(\Gamma)^{-1}$.

With regards to the discretization of the STFT, the following meta-theorem holds [36].

**Theorem 1.2.** Let $g$ be a window function that is sufficiently well-localized in the time-frequency domain. Then for all lattices $\Gamma$ of sufficient density, the restriction

$$L^2(\mathbb{R}^d) \to \ell^2(\Gamma), \quad f \mapsto (V_{g^\Lambda}f(\gamma))_{\gamma \in \Gamma}$$

is a well-defined, bounded injective operator with a bounded inverse from its range to $L^2(\mathbb{R}^d)$.

The theorem not only ensures the injectivity of the restriction operator but also guarantees a stable reconstruction of the sampled STFT. In stark contrast to this, phase retrieval in infinite-dimensional Hilbert or Banach spaces is never uniformly stable [7,18]. Moreover, the uniqueness properties of the operator $\text{SPEC}$ when sampled on a lattice differs fundamentally from the setting above: if $X = L^2(\mathbb{R}^d)$, then for every $g \in L^2(\mathbb{R}^d)$ and all lattices $\Gamma \subseteq \mathbb{R}^{2d}$ which frequently appear in time-frequency analysis the operator $\text{SPEC}_g : X \to \mathbb{C}^\Gamma$ is not injective on $X/\sim$ [8,30,31].

### 1.3. Objectives of this paper

In an attempt to circumvent the discretization barrier, in the present paper we restrict ourselves to proper subspaces of $L^2(\mathbb{R}^d)$, namely various function spaces consisting of compactly supported functions. The goal of the present exposition is threefold:

- We derive new completeness results of systems of discrete translates for spaces of continuous functions on compact sets.
- We develop a systematic link between the problem of completeness of discrete translates and the phaseless sampling problem of the short-time Fourier transform. More precisely, given a certain function space $\mathcal{C} \subseteq L^2(\mathbb{R}^d)$ of compactly supported functions and a window function $g \in L^2(\mathbb{R}^d)$, we show how completeness of translates of a (suitably chosen) family of related functions $\{g_\omega : \omega \in \Omega\}$, $\Omega$ an index set, implies that a function $f \in \mathcal{C}$ is uniquely determined by phaseless the samples $|V_{g^\Lambda}f(\Gamma)|$, provided that $\Gamma$ is a sufficiently dense lattice. Generalizations to non-uniform sampling sets $\Gamma$ are also presented.
- Leveraging these results allows us to deduce several new sampling theorems for the STFT phase retrieval problem, among them the first ever sampling result for the Airy disk window function, which is of particular practical importance [16 Chapter 8.5.2].

The proofs are based on a combination of a duality principle with the theory of mean-periodic functions and uniqueness sets of multivariate entire functions of exponential type.

### 1.4. Main results

In this section we shall state the main results of the present article.
1.4.1. From completeness to phaseless sampling. We start with an abstract result that highlights an intimate relation between completeness properties of discrete translates and the property of every function \( f \in L^2(K) \), \( K \subseteq \mathbb{R}^d \) compact, being uniquely determined up to a global phase by phaseless STFT samples. As in previous publications, we use the tensor-product notation \( g_\omega = (T_\omega g) \mathcal{T} \), \( \omega \in \mathbb{R}^d \). Additionally, \( \mathcal{T}(f, \Lambda) \) denotes the system of \( \Lambda \)-translates of a function \( f \) as defined in Equation (1). Finally, \( A \pm B \) denotes the standard set sum resp. set difference of \( A, B \subseteq \mathbb{R}^d \),

\[
A + B = \{ a + b : a \in A, b \in B \}, \quad A - B = \{ a - b : a \in A, b \in B \},
\]

and \( PW^0_S(\mathbb{R}^d) \) denotes the Paley-Wiener spaces subjected to a compact set \( S \subseteq \mathbb{R}^d \),

\[
PW^0_S(\mathbb{R}^d) = \left\{ f \in L^p(\mathbb{R}^d) : \text{supp } \hat{f} \subseteq S \right\}
\]

Finally, \( V_g f \) denotes the short-time Fourier transform as defined in Equation (2). To state the first main result, we recall the notion of uniqueness sets. Throughout this paper, by a uniqueness set for a function space \( X \) of continuous functions on \( \mathbb{R}^d \) (or \( \mathbb{C}^d \)) we understand a subset \( \Lambda \subseteq \mathbb{R}^d \) such that the samples of a function \( f \in X \) at \( \Lambda \) uniquely determine \( f \), that is, the following implication holds:

\[
f(\lambda) = 0 \quad \forall \lambda \in \Lambda \quad \implies \quad f \equiv 0.
\]

**Theorem 1.3.** Let \( \Lambda, \Gamma \subseteq \mathbb{R}^d \) and let \( K \subseteq \mathbb{R}^d \) be a compact set. Let \( g \in C(\mathbb{R}^d) \) be a window function such that for every \( \omega \in K - K \) the system of translates \( \mathcal{T}(g_\omega, \Lambda) \) is complete in \((C(K), \| \cdot \|_\infty)\). If \( \Gamma \subseteq \mathbb{R}^d \) is a uniqueness set for \( PW^1_{K - K}(\mathbb{R}^d) \), then every \( f \in L^2(K) \) is determined up to a global phase by \( |V_g f(\Lambda \times \Gamma)| \). In particular, this holds whenever \( \Gamma \) is a lattice such that the difference set \( K - K \) is contained in a fundamental domain of the reciprocal lattice \( \Gamma^* \).

The previous theorem is restated and proven in Section 3 as Theorem 3.2.

In terms of discretization of STFT phase retrieval, it states the following: if one can pick a discrete set \( \Lambda \) so that the window function \( g \) has the property that \( \mathcal{T}(g_\omega, \Lambda) \) is complete in \((C(K), \| \cdot \|_\infty)\) for every \( \omega \in K - K \), then every \( f \in L^2(K) \) is automatically determined by discrete samples of its spectrogram. If \( \Lambda \) can be chosen as a lattice, so can the discrete set that determines \( f \) by spectrogram samples as well. It is worth mentioning that STFT phase retrieval on \( L^2(K) \) is equivalent to STFT phase retrieval on \( PW^2_K(\mathbb{R}^d) \): by the fundamental identity of time-frequency analysis, for all \( x, \omega \in \mathbb{R}^d \) holds

\[
V_g f(x, \omega) = e^{-2\pi i x \omega} V_{F^{-1} g} F^{-1} f(\omega, -x).
\]

Therefore, any compactly supported function \( f \in L^2(K) \) can be uniquely determined by \( |V_g f(\Lambda \times \Gamma)| \) if and only if any band-limited function \( h \in PW^2_K(\mathbb{R}^d) \) can be uniquely determined by \( |V_{F^{-1} g} h(\Gamma \times (-\Lambda))| \). Notice that while \( \mathcal{T}(g, \Lambda) \) may be complete, it is, in general, unclear how \( \mathcal{T}(g_\omega, \Lambda) \) behaves. Therefore, as windows, we consider a class of functions that satisfies completeness properties and is in addition closed under tensor products.
Figure 1. Left plot: A lattice $\Gamma$ (empty circles) and its dual lattice $\Gamma^*$ (full circles). Right plot: The boundary of a compact set $K$ (full), the boundary of the set difference $K-K$ (dashed), the dual lattice $\Gamma^*$ as in the left plot, and a fundamental domain of $\Gamma^*$ containing $K-K$ (gray).

1.4.2. The function class $P_{\alpha,\beta}(\mathbb{C}^d)$. The prime objective of the present section is to propose a unifying approach to known completeness results of discrete translates with Gaussian generators [72] and band-limited generators [55]. We provide both an extension and a synthesis of previous completeness results. This will lead to new uniqueness statements for the STFT phase retrieval problem.

Recall that an entire function $F: \mathbb{C}^d \rightarrow \mathbb{C}$ is said to be of exponential type no greater than $\sigma < \infty$ if for every $\varepsilon > 0$ it holds that

$$|F(z)| \lesssim e^{(\sigma+\varepsilon)\|z\|_1}, \quad z \in \mathbb{C}^d, \quad \|z\|_1 = \sum_{j=1}^d |z_j|.$$  

The collection of all entire functions of exponential type no greater than $\sigma$ is denoted by $\mathcal{E}_\sigma(\mathbb{C}^d)$. Among various properties, uniqueness sets have been explored in various setups [37, 49, 56, 58]. Observe that uniqueness sets for functions of exponential type can be chosen to be discrete, in particular, lattices. We now define a related function space, denoted with $P_{\alpha,\beta}(\mathbb{C}^d)$, where we will draw window functions from.

**Definition 1.4.** Let $0 \leq \alpha, \beta < \infty$. We define the class of functions $P_{\alpha,\beta}(\mathbb{C}^d)$ via

$$P_{\alpha,\beta}(\mathbb{C}^d) := \left\{ z \mapsto p(z)e^{-z^T Az - b^T z} : p \in \mathcal{E}_\alpha(\mathbb{C}^d), A \in \mathbb{C}^{d \times d}, b \in \mathbb{C}^d, \|A + A^T\|_1 \leq \beta \right\}.$$  

We refer to Section 4 for the proofs of all forthcoming results related to $P_{\alpha,\beta}(\mathbb{C}^d)$. Note that this class contains, for instance, (generalized) Gaussians, Hermite functions, band-limited functions and products of these. We will often use the same symbol for a function on $\mathbb{R}^d$ and its holomorphic extension (provided it exists) on $\mathbb{C}^d$. Conversely, we do not make a notational difference between a function on $\mathbb{C}^d$ and its restriction on $\mathbb{R}^d$. In addition to the previous concepts, we define the $\ell^\infty$-diameter of a compact set $K$ by

$$\Delta(K) := \max_{t, t' \in K} \|t - t'\|_\infty,$$
and arrive at the following completeness statement.

**Theorem 1.5.** Let \( K \subseteq \mathbb{R}^d \) be a compact set, \( \varphi \in \mathcal{P}_{\alpha,\beta}(\mathbb{C}^d) \) not mean-periodic, and

\[
\sigma \geq \alpha + \frac{\beta}{2} \Delta(K).
\]

If \( \Lambda \subseteq \mathbb{R}^d \) is a uniqueness set for \( \mathcal{E}_\sigma(\mathbb{C}^d) \), then \( \mathcal{T}(\varphi, \Lambda) \) is complete in \( \mathcal{C}(K) \).

There exists a vast literature on uniqueness sets for functions of exponential type. Consider for example the following classical result of Carlson [37] which easily transfers to the multivariate setting: \( \mathbb{N} \) is a uniqueness set for the class of entire functions \( F : \mathbb{C} \rightarrow \mathbb{C} \) which satisfy the growth condition \( |F(z)| \lesssim e^{\beta |z|} \) for some \( \beta < \pi \). With a standard projection to the univariate case, one can show that \( \alpha \mathbb{N}^d \) is a uniqueness set for \( \mathcal{E}_\sigma(\mathbb{C}^d) \) whenever \( \alpha \sigma < \pi \). We refer to Section 4.2 for a discussion on uniqueness sets. We now arrive at the application to STFT phase retrieval in which we assume that the window function \( \varphi \) is square-integrable. Note that this is the standard assumption for the definition of the STFT.

**Theorem 1.6.** Let \( K \subseteq \mathbb{R}^d \) be a compact set, let \( \varphi \in \mathcal{P}_{\alpha,\beta}(\mathbb{C}^d) \cap L^2(\mathbb{R}^d) \), \( \varphi \neq 0 \), and let

\[
\sigma \geq 2\alpha + \beta \Delta(K).
\]

If \( \Lambda \subseteq \mathbb{R}^d \) is a uniqueness set for \( \mathcal{E}_\sigma(\mathbb{C}^d) \) and if \( \Gamma \) is a uniqueness set for \( \text{PW}_{K-K}^1(\mathbb{R}^d) \), then every \( f \in L^2(K) \) is determined up to a global phase by \( |V_{\varphi} f(\Lambda \times \Gamma)| \). In particular, this holds if \( K \subseteq [-\kappa, \kappa]^d \) for some \( \kappa > 0 \) and if \( \sigma \geq 2\alpha + 2\beta \kappa \).

The sets \( \Lambda \) and \( \Gamma \) in the previous theorem can be chosen to be lattices where \( \Lambda \) depends on the growth variables \( \alpha, \beta \) as well as the size of \( K \) which is governed by its diameter. For Hermite functions, we derive the following consequence.

**Corollary 1.7.** Let \( K \subseteq \mathbb{R}^d \) be a compact set, let \( \psi \) be a Hermite function and let

\[
\sigma \geq 2\pi \Delta(K).
\]

If \( \Lambda \subseteq \mathbb{R}^d \) is a uniqueness set for \( \mathcal{E}_\sigma(\mathbb{C}^d) \) and if \( \Gamma \subseteq \mathbb{R}^d \) is a uniqueness set for \( \text{PW}_{K-K}^1(\mathbb{R}^d) \), then every \( f \in L^2(K) \) is determined up to a global phase by \( |V_{\varphi} f(\Lambda \times \Gamma)| \).

1.4.3. **Band-limited windows and STFT-induced Paley-Wiener subspaces.** Let \( K' \subseteq \mathbb{R}^d \) be a compact set and let \( p \in \text{PW}_{K'}^2(\mathbb{R}^d) \). It is evident that for \( \alpha > 0 \) sufficiently large, we have that \( p \in \mathcal{E}_\alpha(\mathbb{C}^d) \). Hence, for such \( \alpha \) it holds that \( \text{PW}_{K'}^2(\mathbb{R}^d) \subseteq \mathcal{P}_{\alpha,0}(\mathbb{C}^d) \). Since band-limited functions are not mean-periodic, Theorem 1.5 applies and with a bit more effort we obtain the following statement (see Theorem 5.4).

**Theorem 1.8.** Let \( K, K' \subseteq \mathbb{R}^d \) be compact subsets and \( g \in \text{PW}_{K'}^2(\mathbb{R}^d) \), \( g \neq 0 \). Then every \( f \in L^2(K) \) is determined up to a global phase by \( |V_{\varphi} f(\mathcal{L})| \) provided that \( \mathcal{L} \subseteq \mathbb{R}^{2d} \) is a lattice such that \( (K' - K') \times (K - K) \) is contained in some fundamental domain of \( \mathcal{L}^* \).
Notice that a band-limited window function that is omnipresent in the application side of the STFT phase retrieval problem is the so-called Airy disk function (see, for instance, [16, Chapter 8.5.2]), which is given by
\[
A_a : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}, \quad a > 0, \quad A_a(t) = \left(\frac{a J_1(2\pi \|t\|_2 a)}{\|t\|_2}\right)^2,
\]
where \(J_1\) denotes the Bessel function of the first kind. For every \(a > 0\), the map \(A_a\) is an element of \(PW_2^k(\mathbb{R}^2)\) with \(K = \{x \in \mathbb{R}^2 : \|x\|_2 \leq 2a\}\). We discuss this function in more detail in Section 5.

Furthermore, Theorem 1.8 highlights an intimate relation between STFT phase retrieval with band-limited window functions and phase retrieval theory in Paley-Wiener spaces: if \(f \in L^2(K)\) for some compact set \(K \subseteq \mathbb{R}^d\) and \(g \in PW_2^k(K')'\) for some compact set \(K' \subseteq \mathbb{R}^d\), then \(V_gf\) defines a band-limited function of \(2d\) variables, namely \(V_gf \in PW_2^k(K' \times (-K'))(\mathbb{R}^{2d})\) (see Lemma 5.3). Before we progress with the discussion, let us make the following definition.

**Definition 1.9 (STFT-induced Paley-Wiener subspace).** Let \(K, K' \subseteq \mathbb{R}^{2d}\) be compact sets and let
\[
J := K' \times (-K).
\]
We say that a subspace \(S \subseteq PW_2^1(\mathbb{R}^{2d})\) is STFT-induced by a band-limited window if there exists a band-limited function \(g \in PW_2^k(\mathbb{R}^d)\) such that
\[
S = V_gL^2(K) = \{V_gf : f \in L^2(K)\}.
\]
We can now interpret the assertion of Theorem [18] as a result on phase retrieval in Paley-Wiener spaces: if \(S = V_gL^2(K)\) is an STFT-induced Paley-Wiener subspace, then every \(F \in S\) is determined up to a global phase by \(|F(L)|\), provided that \(L \subseteq \mathbb{R}^{2d}\) is a lattice such that the Cartesian product \((K' - K') \times (K - K)\) is contained in a fundamental domain of \(L^*\). The question of phase retrieval in Paley-Wiener spaces was considered by several authors during the last years [6,9,15,41,46,66]. Notice that usual phase retrieval in Paley-Wiener spaces is impossible in the following sense: for every \(d \in \mathbb{N}\) and every compact \(C \subseteq \mathbb{R}^d\) there exists \(f, h \in PW_2^1(\mathbb{R}^d)\) such that \(|f(t)| = |h(t)|\) for every \(t \in \mathbb{R}^d\) but \(f \not\sim h\). The entirety of functions \(f, h\) with the previous property can be constructed via the classical zero-flipping approach, as the fundamental papers of Akutowicz [2,3], Hofstetter [39], and Walther [67] reveal. On the other hand, it was shown in [6,66] that a restriction to real-valued functions in \(PW_2^1(\mathbb{R}^d)\) yields uniqueness. However, it is unclear which infinite-dimensional subspaces consisting of complex-valued function obey the uniqueness property. To the best of our knowledge, STFT-induced Paley-Wiener subspaces constitute the first example of such subspaces.

1.4.4. **Gaussian windows.** A classical choice of a window function are Gaussians. Rather remarkably, no density assumption is required for Gaussian windows, but only a semigroup structure. This is hard to be expected from other windows as Proposition 4.2 suggests, adding to the Gaussian’s special status within time-frequency analysis. We recall that the STFT with respect
to the Gaussian window is (up to a weight factor) the Bargmann transform, providing a unitary transformation from $L^2(\mathbb{R}^d)$ to the Fock space $\mathcal{F}$ [36]. Moreover, the Gaussian is the only window with an analytic model space [10]. This connection to analytic functions leads to the most extensive results in STFT phase retrieval with Gaussian windows [4,5,8,29,32,34,35] and to the best of our knowledge, there are no results available for window functions varying from a Gaussian.

**Theorem 1.10.** Let $g \in C(\mathbb{R}^d)$ be a multivariate complex-valued Gaussian

$$g(x) = \exp \left( -(x - \nu)^T A (x - \nu) \right),$$

where $\nu \in \mathbb{C}^d$ and $A \in \mathbb{C}^{d \times d}$ is a Hermitian matrix with invertible real part $\text{Re} A$. If $K \subseteq \mathbb{R}^d$ is a compact set and $\Lambda$ is a semigroup in $\mathbb{R}^d$ that contains a spanning set, then the system $\mathcal{S}(g, \Lambda)$ is complete in $(C(K), \| \cdot \|_{\infty})$. In particular, this holds when $\Lambda$ is a lattice.

The proof can be found in Section 6. An immediate consequence is the following uniqueness result for Gabor phase retrieval.

**Corollary 1.11.** Let $g \in C(\mathbb{R}^d)$ be a multivariate complex-valued Gaussian

$$g(x) = \exp \left( -(x - \nu)^T A (x - \nu) \right),$$

where $\nu \in \mathbb{C}^d$ and where $A \in \mathbb{C}^{d \times d}$ is a positive definite Hermitian matrix. Suppose that $K \subseteq \mathbb{R}^d$ is a compact set. If $\Lambda, \Gamma \subseteq \mathbb{R}^d$ are lattices such that $K - K$ is contained in a fundamental domain of the reciprocal lattice $\Gamma^*$, then every $f \in L^2(K)$ is determined up to a global phase by $|V_g f(\Lambda \times \Gamma)|$.

We now combine the completeness properties of discrete Gaussian translates with the completeness theory of complex exponentials. To that end, we recall a result of Kahane which states the existence of a sequence $K = \{\kappa_n : n \in \mathbb{Z}\} \subseteq \mathbb{R}$ with the property that the system of complex exponentials $\{e^{2\pi i \kappa_n x} : n \in \mathbb{Z}\}$ is complete in $C(I)$ for every compact interval $I \subseteq \mathbb{R}$ while $K$ has density zero. We refer to $K$ as Kahane’s sequence. In the context of STFT phase retrieval, the following can be said.

**Theorem 1.12.** Let $g \in C(\mathbb{R}^d)$ be a real, multivariate Gaussian window function. Let $K \subseteq \mathbb{R}^d$ be Kahane’s sequence and let $\Lambda \subseteq \mathbb{R}^d \setminus \{0\}$ be a sequence of distinct real numbers of density zero such that $\sum_{\lambda \in \Lambda} \frac{1}{|\lambda|} = \infty$. Further, let

$$\mathcal{V} := \bigcup_{\kappa > 0} L^d[-\kappa, \kappa]^d.$$

Then the following holds:

i) $\mathcal{V}$ is dense in $L^2(\mathbb{R}^d)$,

ii) every $f \in \mathcal{V}$ is determined up to a global phase by $|V_g f(\Lambda^d \times K^d)|$,

iii) $\Lambda^d \times K^d$ has density zero.

In particular, the theorem’s conclusion holds true if $\Lambda$ is the set of prime numbers.

Note that there exists no window function such that every $f \in L^2(\mathbb{R}^d)$ is determined up to a global phase by spectrogram samples on equidistant parallel hyperplanes of the form $\Lambda \times \mathbb{R}^d$, $\Lambda \subseteq \mathbb{R}^d$ lattice [31]. In particular,
there exists no window function and no separable lattice such that every \( f \in L^2(\mathbb{R}^d) \) is determined up to a global phase by spectrogram samples on the lattice, irrespective of the density of the lattice. By contrast, Theorem 1.12 states that uniqueness can be achieved on a dense subset \( V \subseteq L^2(\mathbb{R}^d) \) from sampling sets which have density zero. This riveting conundrum once more underlines the sensitivity of the phase retrieval problem and the high informational value of the phase factor in time-frequency analysis.

1.4.5. Semi-discrete sampling sets. The results presented so far deal with discrete sampling sets and require some form of growth control on the window function. If one transfers to semi-discrete sampling sets as addressed by the discretization barriers mentioned above, then very weak assumptions on a function yield uniqueness. This is due to the fact that the problem of \( T(g_{\omega}, \Lambda) \) being complete in \((C(K), \|\cdot\|_\infty)\) significantly simplifies if \( \Lambda = \mathbb{R}^d \) (in this case Theorem 1.3 yields a uniqueness result for semi-discrete sampling sets which are for \( d = 1 \) parallel lines in the time-frequency plane). For semi-discrete sampling sets, we have the following statement.

**Theorem 1.13.** Let \( K \subseteq \mathbb{R}^d \) be a compact set. Suppose that \( g : \mathbb{R}^d \to \mathbb{C} \) is a window function which satisfies the following two properties:

i) \( g \in C(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \),

ii) for every \( \omega \in K - K \) it holds that \( g_{\omega} \) does not vanish identically.

If \( \Gamma \) is a lattice such that \( K - K \) is contained in a fundamental domain of the reciprocal lattice \( \Gamma^* \), then every \( f \in L^2(K) \) is determined up to a global phase by \( |V_g f(\mathbb{R}^d \times \Gamma)| \).

1.5. Outline. The paper is divided as follows. In Section 2, we go over several elementary notions. In Section 3 we derive the connection between completeness of translates and STFT phase retrieval of compactly supported, square-integrable functions. Section 4 is dedicated to a detailed study of the function class \( \mathcal{P}_{\alpha,\beta}(\mathbb{C}^d) \), its completeness properties, and consequentially, STFT phase retrieval results tied to \( \mathcal{P}_{\alpha,\beta}(\mathbb{C}^d) \). We then consider STFT-induced Paley Wiener spaces and band-limited window functions in Section 1.4.3. We conclude with Section 6 which deals with the Gaussian window function and sampled STFT phase retrieval on dense subspaces of \( L^2(\mathbb{R}^d) \).

2. Preliminaries and basic definitions

In this section, we shall provide some basic definitions and settle the notation which is used throughout the remainder of the article.

2.1. **(Short-time) Fourier transform.** The Fourier transform of an integrable function \( f \in L^1(\mathbb{R}^d) \) is given by

\[
\mathcal{F} f(\omega) = \hat{f}(\omega) = \int_{\mathbb{R}^d} f(t) e^{-2\pi i \omega \cdot t} dt.
\]

This defines a bounded operator from \( L^1(\mathbb{R}^d) \) to \( L^\infty(\mathbb{R}^d) \) and extends from \( L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \) to a unitary operator on \( L^2(\mathbb{R}^d) \) by standard density arguments. The short-time Fourier transform (STFT) \( V_g f : \mathbb{R}^{2d} \to \mathbb{C} \) of a
function \( f \in L^2(\mathbb{R}^d) \) with respect to a window function \( g \in L^2(\mathbb{R}^d) \) is defined as in Equation \([2]\). The STFT defines a uniformly continuous function on \( \mathbb{R}^{2d} \) and satisfies

\[
V_{\gamma}f(x, \omega) = \mathcal{F}(f \overline{\mathcal{F}_xg})(\omega) = (\mathcal{F}f * (M_{-x} \mathcal{R}g))(\omega),
\]

where \( \mathcal{R}u(t) = u(-t) \) denotes the reflection operator. Further, we denote by \( \langle v, w \rangle \) the integral

\[
\langle v, w \rangle = \int_{\mathbb{R}^d} v(t) \overline{w(t)} \, dt, \quad v \in L^p(\mathbb{R}^d), \quad w \in L^q(\mathbb{R}^d),
\]

where \( p, q \in [1, \infty] \) are Hölder conjugate exponents.

2.2. Paley-Wiener spaces. A central role in this paper is played by Paley-Wiener spaces and uniqueness sets for these spaces. We primarily consider the Paley-Wiener spaces of integrable and square-integrable functions. Therefore, we define for \( p \in \{1, 2\} \) and for a compact set \( K \subseteq \mathbb{R}^d \) the Paley-Wiener space \( \text{PW}^p_K(\mathbb{R}^d) \) via

\[
\text{PW}^p_K(\mathbb{R}^d) = \left\{ f \in L^p(\mathbb{R}^d) : \text{supp}(\mathcal{F}f) \subseteq K \right\}.
\]

The inclusion \( \text{PW}^1_K(\mathbb{R}^d) \subseteq \text{PW}^2_K(\mathbb{R}^d) \) is well-known \([38]\). Recall that the classical theorem of Shannon, Whittaker and Kotel’nikov asserts that if the subset \( \Gamma \subseteq \mathbb{R}^d \) is a lattice, \( K \subseteq \mathbb{R}^d \) a compact set, and \( f \in \text{PW}^2_K(\mathbb{R}^d) \), then \( f \) can be uniquely retrieved from its samples at \( \Gamma \) if and only if \( K \) is contained in a fundamental domain \( D \) of the reciprocal lattice \( \Gamma^* \) (see, for instance, \([38]\) Theorem 14.4) or \([45]\) Theorem 4.4). In particular, lattices of the previous type are examples of uniqueness sets for \( \text{PW}^2_K(\mathbb{R}^d) \). As a matter of fact, \( f \) can be reconstructed via

\[
f(t) = \sum_{\gamma \in \Gamma} f(\gamma) r(t - \gamma),
\]

where the reconstruction function \( r \) is given by

\[
r(t) := \text{vol}(\Gamma^*) \int_D e^{2\pi i x \cdot t} \, dx.
\]

The inclusion \( \text{PW}^2_K(\mathbb{R}^d) \subseteq \mathcal{E}_{2\pi C}(\mathbb{C}^d) \), where \( C = \max_{x \in K} \|x\|_\infty \), relates Paley-Wiener spaces to the spaces of entire functions of exponential type and uniqueness sets for \( \mathcal{E}_{2\pi C}(\mathbb{C}^d) \) to uniqueness sets for \( \text{PW}^2_K(\mathbb{R}^d) \).

2.3. Complex measures. For a locally compact Hausdorff space \( X \) we denote by \( M(X) \) the set of all Radon measures on \( X \). With the usual definition of total variation \( |\mu| \) of \( \mu \in M(X) \) \([59]\) Chapter 6], the space \( M(X) \) is a Banach space when endowed with the total variation norm given by \( \|\mu\|_{TV} = |\mu|(X) \). Let \( C_0(X) = (C_0(X), \|\cdot\|_\infty) \) be the Banach space of all complex-valued, continuous functions on \( X \) which vanish at infinity, that is, for every \( \varepsilon > 0 \) there exists a compact set \( K \subseteq X \) such that \( |f(x)| \leq \varepsilon \) for every \( x \in K^* \). The classical Riesz representation theorem states that \( C_0(X)^* \cong M(X) \) in the following sense: \( \Psi \in C_0(X)^* \) if and only if it can be represented in the form

\[
\Psi(f) = \int_X f \, d\mu,
\]
where $\mu \in M(X)$ is the unique Radon measure corresponding to $\Psi$. Moreover, the operator norm of $\Psi$ is the total variation of $\mu$, i.e., $\|\Psi\| = \|\mu\|_{TV}$. If $K \subseteq \mathbb{R}^d$ is a compact set, then $C_0(K) = C(K)$ and therefore its dual is $C(K)^* \simeq M(K)$.

3. Abstract theory

3.1. Convolution operators and systems of translates. We dedicate this section to a conceptual take on STFT phase retrieval. Instead of showing that explicit functions together with (separated) sets can retrieve compactly supported functions, we provide general sufficient conditions in terms of completeness properties of discrete translates that can be tested on diverse families of window functions.

**Theorem 3.1.** Let $p \in [1, \infty]$, $\Lambda, \Gamma \subseteq \mathbb{R}^d$ and $K \subseteq \mathbb{R}^d$ be a compact set. Let $g \in L^\infty_{\text{loc}}(\mathbb{R}^d)$ be a window function which has the property that for every $\omega \in K - K$ the linear operator

$$
C_\omega : L^p(K) \to \mathbb{C}^\Lambda,
$$

is injective. Further, let $\Gamma$ be a uniqueness set for $\text{PWV}_{K-K}(\mathbb{R}^d)$. Then every $f \in L^{2p}(K)$ is determined up to a global phase by $|V_g f (\Lambda \times \Gamma)|$. In particular, this holds whenever $\Gamma \subseteq \mathbb{R}^d$ is a lattice such that $K - K$ is contained in a fundamental domain of the reciprocal lattice $\Gamma^*$.

**Proof.** Let $f \in L^{2p}(K)$ be a $2p$-integrable function on $K$. Since $p \geq 1$, and $K$ has bounded measure, we have $f \in L^2(K)$. Additionally, $g \in L^\infty_{\text{loc}}(\mathbb{R}^d)$, so it follows that $f T_s g \in L^2(K)$ for every $x \in \mathbb{R}^d$. Using the convolution identity of the Fourier transform and the fact that $\mathcal{F} h = \mathcal{F}^{-1} \mathcal{R} h$, $\mathcal{F} h = \mathcal{F}^{-1} \mathcal{R} h$ for every $h \in L^2(\mathbb{R}^d)$, we can represent $|V_g f (x, \cdot)|^2$ as

$$
|V_g f (x, \cdot)|^2 = \mathcal{F} (f \ T_s g) \mathcal{F} (f \ T_s g) = \mathcal{F}^{-1} \mathcal{R} (f \ T_s g) \mathcal{F}^{-1} (\mathcal{F} T_s g)
$$

$$
= \mathcal{F}^{-1} \left( \mathcal{R} \left( f \mathcal{F} \ T_s g \right) \ast \left( \mathcal{F} T_s g \right) \right).
$$

Since $f$ has support in $K$, the latter convolution can be written as

$$
\mathcal{R} \left( f \mathcal{F} \ T_s g \right) \ast \left( \mathcal{F} T_s g \right) (s) = \int_K f(-(s-t)) g(-(s-t)-x) f(t) g(t-x) dt
$$

$$
= \int_K f_s(t) T_s g_s(t) dt.
$$

Therefore, we obtain the identity

$$
\mathcal{F} \left( |V_g f (x, \cdot)|^2 \right)(\omega) = \int_K f_\omega(t) T_s g_\omega(t) dt.
$$

Suppose that $\omega \in \mathbb{R}^d$ is a vector such that $(K + \omega) \cap K \neq \emptyset$. Then there exist $k, k' \in K$ such that $k + \omega = k'$, implying that $\omega \in K - K$. By contraposition, we obtain the implication

$$
\omega \notin K - K \implies (K + \omega) \cap K = \emptyset.
$$

Since $\text{supp}(T_\omega f) \subseteq K + \omega$ and $\text{supp}(f) \subseteq K$, the previous implication shows that $f_\omega = (T_\omega f) \mathcal{F} = 0$ almost everywhere provided $\omega \notin K - K$. In view of
Equation (\ref{eq:sampling}), the map

\[ \omega \mapsto \mathcal{F} \left( |V_g f(x, \cdot)|^2 \right) (\omega) \]

is supported on \( K - K \). Further, since \( f \mathcal{T}_x g \in L^2(K) \) and the fact that the Fourier transform is an \( L^2 \)-isomorphism, we have

\[ |\mathcal{F}(f \mathcal{T}_x g)|^2 = |V_g f(x, \cdot)|^2 \in L^1(\mathbb{R}^d), \]

and therefore, \( |V_g f(x, \cdot)|^2 \in PW^1_{K-K}(\mathbb{R}^d) \). Now let \( h \in L^{2p}(K) \) be a second function such that the spectrogram samples of \( f \) and \( h \) agree on \( \Lambda \times \Gamma \), i.e.,

\[ |V_g f(\Lambda \times \Gamma)| = |V_g h(\Lambda \times \Gamma)|. \]

Since \( |V_g f(x, \cdot)|^2, |V_g h(x, \cdot)|^2 \in PW^1_{K-K}(\mathbb{R}^d) \) and since \( \Gamma \) is a uniqueness set for \( PW^1_{K-K}(\mathbb{R}^d) \), we conclude

\[ |V_g f(\Lambda \times \mathbb{R}^d)| = |V_g h(\Lambda \times \mathbb{R}^d)|. \]

In particular, Identity (\ref{eq:sampling}) yields

\[ \int_K f_\omega(t) \mathcal{T}_x g_\omega(t) \, dt = \int_K h_\omega(t) \mathcal{T}_x g_\omega(t) \, dt \quad \forall \lambda \in \Lambda, \forall \omega \in \mathbb{R}^d. \]

Given that \( f, h \in L^{2p}(K) \), we have \( f_\omega - h_\omega \in L^p(K) \) by Hölder’s inequality. The injectivity assumption on the operator \( C_\omega \) gives \( f_\omega = h_\omega \) a.e. for all \( \omega \in K - K \). Since \( f_\omega = h_\omega = 0 \) a.e. for all \( \omega \notin K - K \), it follows that \( f_\omega = h_\omega \) a.e. for every \( \omega \in \mathbb{R}^d \). Further we have \( f_\omega, h_\omega \in L^1(\mathbb{R}^d) \) and obtain \( \mathcal{F}(f_\omega)(x) = \mathcal{F}(h_\omega)(x) \) for every \( x \in \mathbb{R}^d \) and every \( \omega \in \mathbb{R}^d \). Equivalently, it holds that \( V_f f \) agrees with \( V_g h \) everywhere in \( \mathbb{R}^{2d} \), whence \( f \sim h \) by standard results on time-frequency analysis (see for instance \cite{12,36}).

If we assume that \( \Gamma \) is a lattice and \( K - K \) is contained in a fundamental domain of the reciprocal lattice \( \Gamma^* \), then Shannon’s sampling theorem implies that \( \Gamma \) is a sampling set for \( PW^1_{K-K}(\mathbb{R}^d) \supseteq PW^1_{K-K}(\mathbb{R}^d) \), in particular, a uniqueness set for \( PW^1_{K-K}(\mathbb{R}^d) \).

Notice that confirming the injectivity of the family of convolution operators \( (C_\omega)_{\omega \in K-K} \) is not a trivial matter. By Steinhaus’ Theorem \cite{63,69}, the set \( K - K \) contains an open neighborhood of zero whenever the compactum \( K \) has positive Lebesgue measure, implying that one should guarantee the injectivity of uncountably many operators. If one aims at retrieving square-integrable functions, then the injectivity of \( C_\omega \) can be achieved via a completeness property of translates of \( (g_\omega)_{\omega \in K-K} \). Recall that for a continuous function \( g \in C(\mathbb{R}^d) \) and a set \( \Lambda \subseteq \mathbb{R}^d \) we denote by \( \Xi(g, \Lambda) := \{ \mathcal{T}_x g : \lambda \in \Lambda \} \) the system of \( \Lambda \)-translates of \( g \). The relation between the injectivity conditions of Theorem \ref{thm:injectivity} to completeness properties of translates reads as follows.

**Theorem 3.2.** Let \( p \in [1, \infty] \), let \( \Lambda, \Gamma \subseteq \mathbb{R}^d \) and let \( K \subseteq \mathbb{R}^d \) be a compact set. Let \( g \in C(\mathbb{R}^d) \) be a window function such that for every \( \omega \in K - K \) the system of translates \( \Xi(g_\omega, \Lambda) \) is complete in \( (C(K), \| \cdot \|_{\infty}) \). If \( \Gamma \) is a uniqueness set for \( PW^1_{K-K}(\mathbb{R}^d) \), then every \( f \in L^1(K) \) is determined up to a global phase by \( |V_g f(\Lambda \times \Gamma)| \). In particular, this holds whenever \( \Gamma \) is
a lattice such that $K - K$ is contained in a fundamental domain of the reciprocal lattice $\Gamma^*$. 

**Proof.** According to Theorem 3.1, it suffices to prove that the convolution operator

$$C_\omega : L^1(K) \to \mathbb{C}^\Lambda, \quad f \mapsto \left( \int_K f(t) T_{\lambda} g_\omega(t) \, dt \right)_{\lambda \in \Lambda}$$

is injective for all $\omega \in K - K$. To that end, we define for $f \in L^1(K)$ the operator

$$\Phi_f : C(K) \to \mathbb{C}, \quad \Phi_f(q) = \int_K f(t) q(t) \, dt.$$ 

Then $\Phi_f$ is a continuous linear functional on $C(K)$ and by the Riesz representation theorem, there exists a unique $\mu \in M(K)$ such that the functional is given by $\Phi_f(q) = \int_K q \, d\mu$ and $\|\Phi_f\| = \|\mu\|_{TV}$. By uniqueness, we have $d\mu = f \, dm$, where $m$ denotes the Lebesgue measure on $\mathbb{R}^d$, and therefore $\|\mu\|_{TV} = \|f\|_{L^1(K)}$. Thus, the completeness of $\mathcal{F}(g_\omega, \Lambda)$ in $(C(K), \|\cdot\|_\infty)$ gives rise to the implication

$$\Phi_f(T_{\lambda} g_\omega) = 0 \quad \forall \lambda \in \Lambda \implies \Phi_f = 0 \implies f = 0 \text{ a.e.}.$$ 

This shows that $C_\omega$ is injective on $L^1(K)$. The statement follows from Theorem 3.2. $\square$

**Remark 3.3** (Multiplication by periodic functions). Consider the operator $C_\omega : L^p(K) \to \mathbb{C}^\Lambda$ as defined in Theorem 3.1

$$C_\omega(f) = \left( \int_K f(t) T_{\lambda} g_\omega(t) \, dt \right)_{\lambda \in \Lambda},$$

with $g \in L^\infty_{loc}(\mathbb{R}^d)$ and $\Lambda \subseteq \mathbb{R}^d$. Suppose that $\Lambda \subseteq \mathbb{R}^d$ is a lattice, i.e., $\Lambda = A \mathbb{Z}^d$ for some $A \in \text{GL}(d, \mathbb{R})$, and let $\chi \in L^\infty(\mathbb{R}^d)$ be an $\Lambda$-periodic function (i.e. $T_{\lambda} \chi = \chi$ for all $\lambda \in \Lambda$) such that $\chi(t) \neq 0$ for almost every $t \in \mathbb{R}^d$. Then the injectivity of $C_\omega$ is unaffected if $g$ is multiplied by $\chi$. For, if $C_\omega$ is injective, then

$$0 = \int_K f(t) T_{\lambda} (g\chi)_\omega(t) \, dt = \int_K f(t) \chi(t) T_{\lambda} g_\omega(t) \, dt \quad \forall \lambda \in \Lambda$$

and therefore, $f \chi_\omega = 0$, where we used that $\chi_\omega$ is $\Lambda$-periodic and $f \chi_\omega \in L^p(K)$ provided $f \in L^p(K)$ and $\chi \in L^\infty(\mathbb{R}^d)$. Since for every $\omega \in \mathbb{R}^d$ it holds that $\chi_\omega(t) \neq 0$ for almost every $t \in \mathbb{R}^d$, we conclude that $f = 0$.

**Remark 3.4** (Hermitian functions). Let $K \subseteq \mathbb{R}^d$ be a measurable set. A function $f \in L^2(K)$ is called Hermitian if $\mathcal{R} f = \overline{f}$. In the following, we shall denote the space of Hermitian functions by $L^2_H(K)$. It is evident that a function is Hermitian if and only if its Fourier transform is real-valued. Now suppose that $K \subseteq \mathbb{R}^d$ is a compact set and let $g \in L^2_H(\mathbb{R}^d)$ be a Hermitian window function such that $\hat{g} \in L^1(\mathbb{R}^d)$. If $f \in L^2_H(K)$, then the fundamental identity of time-frequency analysis (see for instance, [36, Equation 3.10]) shows that for every $\omega \in \mathbb{R}^d$ one has

$$|V_g f(0, \omega)| = |\hat{V}_g f(\omega, 0)| = \left| (\mathcal{R} \hat{g} * f)(\omega) \right|,$$
where we used that $\hat{g}$ is real-valued. The function $R\hat{g} * \hat{f}$ is real-valued, square-integrable (Young’s inequality), and its Fourier transform has support in $K$. In other words, $R\hat{g} * \hat{f}$ is a real-valued function belonging to $PW_K^2(\mathbb{R}^d)$. It is well-known that a function of this type is determined up to a global sign-factor by its modulus sampled on a sufficiently dense lattice $\Gamma \subseteq \mathbb{R}^d$, see Thakur’s theorem for the univariate setting and corresponding multivariate extensions by Daubechies et al. [5, 66]. Consequently, if $h \in L_2^{\mathcal{H}}(\mathbb{R}^d)$ is a second function such that $|V_{g}f(\{0\} \times \Gamma)| = |V_{g}h(\{0\} \times \Gamma)|$, then $R\hat{g} * \hat{f} \sim R\hat{g} * \hat{h}$. The assumption $Rg(t) \neq 0$ for almost every $t \in K$ implies that $f \sim h$. The uniqueness of STFT phase retrieval problem for the space $L_2^{\mathcal{H}}(K)$ follows therefore from the uniqueness of a phaseless deconvolution problem. A similar observation was done in [9] for the univariate setting. The uniqueness of this problem is a simple application of uniqueness results in real Paley-Wiener spaces.

3.2. Mean-periodicity. The notion of mean-periodicity goes back to Delporte [20, 21] and was later further developed by Schwartz [60]. For an extensive overview of mean-periodic functions, we refer to Kahane’s exposition [43]. In order to motivate the notation of mean-periodicity, assume that $f \in C(\mathbb{R})$ and let $a > 0$. Suppose that $f$ is an $a$-periodic function, that is, $f(x) = f(x - a)$ for all $x \in \mathbb{R}$. Clearly, an $a$-periodic function cannot be an $a\mathbb{Z}$-generator for $C(K)$, $K \subseteq \mathbb{R}$ compact. The $a$-periodicity of $f$ can be equivalently written as the requirement that $f \ast \mu(x) := \int_{\mathbb{R}} f(x - y) \, d\mu(y) = 0$ for all $x \in \mathbb{R}$, where $\mu = \delta_0 - \delta_a$ is a sum of two Dirac measures. This motivates the following definition.

**Definition 3.5.** A function $f \in C(\mathbb{R}^d)$ is said to be mean-periodic if there exists a compactly supported complex Radon measure $\mu$, $\mu \neq 0$, such that $f \ast \mu = 0$.

Besides the motivating example, the set of mean-periodic functions is rather sparse. One can easily verify that exponential polynomials, i.e., products of type $z \mapsto p(z)e^{\omega^T z}$, where $p$ is a multivariate polynomial and $\omega \in \mathbb{C}^d$, are also mean-periodic. In fact, any mean periodic function can be given in terms of related exponential polynomials [13]. Several equivalent definitions of mean-periodicity are known. For our purposes, the most striking one asserts that $f$ is mean-periodic if and only if $\mathcal{F}(f, \mathbb{R}^d)$ is not dense in $C(\mathbb{R}^d)$ with respect to the topology of uniform convergence on compact sets [43]. The perhaps most direct way to eliminate the property of a function being mean-periodic is to require an integrability condition.

**Proposition 3.6.** Let $f \in C(\mathbb{R}^d)$ such that $f \neq 0$. If $f \in \mathcal{S}(\mathbb{R}^d)$ and its Fourier transform $\mathcal{F}f$ is induced by a measurable function, then $f$ is not mean-periodic. In particular, if $f \in C(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ for some $p \in [1, 2]$, then $f$ is not mean-periodic.
Proof. A function is understood as a tempered distribution via the induced functional
\[ L_f : S(\mathbb{R}^d) \to \mathbb{C}, \quad h \mapsto \int_{\mathbb{R}^d} h(t)f(t) \, dt. \]
The definition is compatible with the Fourier transform in the sense that, whenever the Fourier transform of \( f \) exists, e.g., when \( f \in L^1(\mathbb{R}^d) \cup L^2(\mathbb{R}^d) \), then the Fourier transform of the distribution is given by \( \mathcal{F}L_f = L_{\mathcal{F}f} \). The analogous (in restricted form) holds for products and convolutions and can be found in the textbooks [14,25,40].

To prove the above Proposition, we assume for the sake of contradiction that there exists a non-trivial compactly supported measure \( \mu \in M(\mathbb{R}^d) \) such that \( f \ast \mu = 0 \). Due to the compact support and the bounded variation of \( \mu \), its Fourier transform \( \mathcal{F}\mu \) is a smooth function, has a holomorphic extension on \( \mathbb{C}^d \) and all its derivatives on \( \mathbb{R}^d \) have at most polynomial growth at infinity (see [40, Theorem 7.3.1]). Since \( f \) induces the tempered distribution \( L_f \), it also holds \( L_f \ast \mu = 0 \) as a distribution. The Convolution Theorem for distributions [14, Theorem 8.5.1] implies
\[ 0 = \mathcal{F}(L_f \ast \mu) = \mathcal{F}L_f \cdot \mathcal{F}\mu = L_{\mathcal{F}f} \cdot \mathcal{F}\mu. \]
While this is initially to be understood in the distributional sense, since we already know that \( \mathcal{F}L_f = L_{\mathcal{F}f} \), we can also interpret this in the pointwise a.e. sense, leading us to conclude
\[ 0 = L_{\mathcal{F}f} \cdot \mathcal{F}\mu = L_{\mathcal{F}f} \cdot \mathcal{F}\mu. \]
This holds if and only if
\[ \mathcal{F}f \cdot \mathcal{F}\mu = 0 \text{ a.e.} \]
Since \( f \) does not vanish everywhere and the Fourier transform is an automorphism on \( S'(\mathbb{R}^d) \), \( \mathcal{F}f \) also does not vanish identically, implying there exists a set \( \Omega \subseteq \mathbb{R}^d \) of positive Lebesgue measure such that \( \mathcal{F}\mu|_{\Omega} = 0 \). Considering that \( \mathcal{F}\mu \) has a holomorphic extension, \( \mathcal{F}\mu \) must vanish identically [30, Lemma 3.4], thereby proving that also \( L_\mu = 0 \). This is the desired contradiction.

For the second part of the statement, suppose that \( q \in [2, \infty] \) is the Hölder conjugate of \( p \in [1,2] \). Hausdorff-Young’s inequality states that the Fourier transform has a continuous extension \( L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d) \). In particular, the tempered distribution induced by \( f \) has a Fourier transform induced by a measurable function.

We note that the special case when \( f \in C(\mathbb{R}^d), \) \( d = 1, \) is an integrable function that does not vanish identically was already proven in [55, Proposition 6.3]. Furthermore, this cannot be trivially extended for \( L^p(\mathbb{R}^d), \) \( p > 2 \). The key role in the proof of Proposition 3.6 is the fact that the support of \( \mathcal{F}f \) has positive Lebesgue measure. If \( p > 2 \), then there are \( f \in L^p(\mathbb{R}^d) \) whose (distributional) Fourier transform is not induced by a measurable function. As a matter of fact, there are functions whose Fourier transform is supported on a set of measure zero. A nice example is the Fourier transform of the surface measure \( \sigma_d \) on the sphere \( S^{d-1} \). In that case holds (cf. [25, Appendix
f(\omega) = \mathcal{F}^{-1}\sigma_d(\nu) = \mathcal{F}\sigma_d(\omega) = 2\pi \|\omega\|_2^{-\frac{d-2}{2}} J_{\frac{d-2}{2}}(2\pi \|\omega\|_2),

where \(J_\nu\) denotes the Bessel function of first kind and order \(\nu\). With standard asymptotic estimates on Bessel functions (see [25, Appendix B.6, Appendix B.8]), one can easily show \(f \in L^p(\mathbb{R}^d)\) as long as it holds \((p - 2)(d - 1) > 2\).

The previous considerations immediately yield a phase retrieval result related to unions of hyperplanes.

**Theorem 3.7.** Let \(K \subseteq \mathbb{R}^d\) be a compact set. Suppose that \(g : \mathbb{R}^d \to \mathbb{C}\) is a window function that satisfies the following two properties:

i) \(g \in C(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)\),

ii) for every \(\omega \in K - K\) it holds that \(g_\omega\) does not vanish identically.

If \(\Gamma\) is a lattice such that \(K - K\) is contained in a fundamental domain of the reciprocal lattice \(\Gamma^*\), then every \(f \in L^2(K)\) is determined up to a global phase by \(|V_g f(\mathbb{R}^d \times \Gamma)|\).

**Proof.** Since \(g \in C(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)\), it follows that for every \(\omega \in \mathbb{R}^d\) we have \(g_\omega \in C(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)\). By assumption, \(g_\omega \neq 0\) for every \(\omega \in K - K\). In view of Proposition 3.6, \(\mathcal{F}(g_\omega, \mathbb{R}^d)\) is complete in \(C(K)\) for every \(\omega \in K - K\). Theorem 3.2 yields the statement. \(\square\)

**Remark 3.8** (Non-uniqueness theory in \(L^2(\mathbb{R}^d)\)). It was recently shown in [31, Theorem 1.1] that for every lattice \(\Gamma \subseteq \mathbb{R}^d\) and every window function \(g \in L^2(\mathbb{R}^d)\) there exist \(f, h \in L^2(\mathbb{R}^d)\) such that

\[ |V_g f(\mathbb{R}^d \times \Gamma)| = |V_g h(\mathbb{R}^d \times \Gamma)| \quad \text{and} \quad f \sim h. \]

Hence, without a support condition on the underlying signal space, the corresponding phase retrieval problem always fails to be unique if one samples on \(\mathbb{R}^d \times \Gamma\), irrespective of the choice of \(\Gamma\) and \(g\). In contrast, Theorem 3.7 shows that mild conditions on \(g\) imply unique recovery of compactly supported signals from sampling sets of the form \(\mathbb{R}^d \times \Gamma\).

**Remark 3.9** (Landau’s theorem). A theorem due to Landau [47] asserts that if \(\varphi \in L^1(\mathbb{R})\) extends to an entire function and if \(\Lambda \subseteq [-\delta, \delta] - \kappa\), \(0 < \kappa < \delta < \infty\), is a sequence which converges to zero, then \(\mathcal{F}(\varphi, \Lambda)\) is complete in \(C[-\kappa, \kappa]\). Now suppose that \(g \in L^2(\mathbb{R})\) extends to an entire function. Then for every \(\omega \in \mathbb{R}\) it holds that \(g_\omega \in L^1(\mathbb{R})\). Further, for every \(t \in \mathbb{R}\) we have \(g_\omega(t) = g(t - \omega)\), where the right-hand side of the previous equation extends to an entire function. Consequently, if \(\Lambda, \delta, \kappa\) are given as in Landau’s theorem and if \(K = [-\delta, \delta] - \kappa\), then according to Theorem 3.2, every \(f \in L^2(K)\) is determined up to a global phase by \(|V_g f(\Lambda \times \Gamma)|\), provided that \(\Gamma\) is a uniqueness set for \(PW^1_{K-\kappa}(\mathbb{R})\). Note however that in this situation \(\Lambda\), hence \(\Lambda \times \Gamma\), is not separated.

3.3. **Completeness via duality.** In view of Theorem 3.2 and the search for separated sampling sets, we concern ourselves with criteria for completeness of discrete translates. One option is to use Stone-Weierstrass-type arguments, which will be pursued in Section 5 dedicated to the Gaussian window function. However, the algebra structure required in the Stone-Weierstrass
theorem is quite a strong one. We, therefore, consider a duality strategy. To that end, recall that for all compact sets $K \subseteq \mathbb{R}^d$ there is a one-to-one correspondence between the complex Radon measures $\mu \in M(K)$ and the dual space $C(K)^*$ via

$$L_\mu(f) = \int_K f \, d\mu, \quad \|L_\mu\|_{C(K)^*} = \|\mu\|_{TV}.$$  

To relate this to the problem of completeness of discrete translates, we define an associated function space $\mathcal{O}_\varphi(K)$. For this, let $\varphi : \mathbb{C}^d \to \mathbb{C}$ be a continuous function and $K \subseteq \mathbb{R}^d$ compact. Then $\mathcal{O}_\varphi(K)$ is defined by

$$\mathcal{O}_\varphi(K) := \left\{ F : \mathbb{C}^d \to \mathbb{C} : F(z) = L_\mu(T_z\varphi) = \int_K T_z\varphi \, d\mu, \ \mu \in M(K) \right\}.$$  

**Proposition 3.10.** Let $K \subseteq \mathbb{R}^d$ be a compact set, $\Lambda \subseteq \mathbb{R}^d$ and $\varphi : \mathbb{C}^d \to \mathbb{C}$ be a continuous function which is not mean-periodic. Then the following are equivalent:

i) $\mathcal{S}(\varphi, \Lambda)$ is complete in $C(K)$,

ii) for every $\mu \in M(K)$ holds the implication

$$L_\mu(T_{\lambda}\varphi) = 0 \ \forall \lambda \in \Lambda \implies L_\mu = 0,$$

iii) $\Lambda$ is a uniqueness set for $\mathcal{O}_\varphi(K)$.

**Proof. Step 1: Equivalence of i) and ii).** Suppose that $\mathcal{S}(\varphi, \Lambda)$ is complete in $C(K)$. It follows that if $g \in C(K)$, then there exists a sequence $(g_k)_{k \in \mathbb{N}} \subseteq \text{span}_c \mathcal{S}(\varphi, \Lambda)$ such that $g_k \to g$ in $C(K)$. Hence, if $L_\mu(T_{\lambda}\varphi) = 0$ for all $\lambda \in \Lambda$, then by linearity, $L_\mu(g_k) = 0$ for all $k \in \mathbb{N}$. Since $L_\mu$ is a continuous linear functional, we have $L_\mu(g) = 0$. As $g$ was arbitrary, it follows that $L_\mu = 0$. Suppose on the other hand that ii) holds, but $\mathcal{S}(\varphi, \Lambda)$ is not complete. Then there exists a function $g \in C(K)$ such that

$$g \notin S := \text{span}_c \mathcal{S}(\varphi, \Lambda).$$

By Hahn-Banach theorem, it follows that there exists a continuous linear functional $\Psi \in C(K)^*$ such that $\|\Psi\| = 1$ and $\Psi|_S = 0$. Since $\Psi = \Psi_\mu$ for some $\mu \in M(X)$, we arrive at a contradiction.

**Step 2: Equivalence of ii) and iii).** Suppose that condition ii) holds. Let $F \in \mathcal{O}_\varphi(K)$ with

$$F(z) = \int_K T_z\varphi(t) \, d\mu(t)$$

for some $\mu \in M(K)$ and assume that $F(\lambda) = 0$ for every $\lambda \in \Lambda$. Since $F(\lambda) = L_\mu(T_{\lambda}\varphi)$, it follows that $L_\mu = 0$. Observing that $T_z\varphi \in C(K)$ for every $z \in \mathbb{C}^d$ shows that $L_\mu(T_z\varphi) = 0$ for every $z \in \mathbb{C}^d$. Therefore, the function $F$ vanishes identically. Suppose on the other hand that $\Lambda$ is a uniqueness set for $\mathcal{O}_\varphi(K)$ and let $\mu \in M(K)$ such that $L_\mu(T_{\lambda}\varphi) = 0$ for every $\lambda \in \Lambda$. If $F \in \mathcal{O}_\varphi(K)$ is given as in Equation (4), then the uniqueness property of $\Lambda$ implies that $F$ vanishes identically. Now observe that $F = \mathcal{R}\varphi \ast \mu$. Since the class of not mean-periodic functions is invariant under reflections, it follows that $\mathcal{R}\varphi$ is not mean-periodic. Therefore, $\mu = 0$, which in turn shows that $L_\mu = 0$ and the statement is proved. \qed
Remark 3.11. We note that the equivalence (i)$\Leftrightarrow$(ii) is independent of the mean-periodicity of the generator $f$. However, the valuable criterium for the following section is the third statement where the non-mean-periodicity is decisive for the equivalence. As a matter of fact, without assuming anything on the relationship between $f$ and $K$, then $f$ not being mean-periodic is both a necessary and sufficient condition for the equivalence chain.

4. The function space $\mathcal{P}_{\alpha,\beta}(\mathbb{C}^d)$

This section builds up on the equivalences derived in Theorem 3.10. The main aim is to present the proofs of the results stated in Section 1.4.2.

4.1. Completeness. Recall that an entire function $F: \mathbb{C}^d \to \mathbb{C}$ is said to be of exponential type no greater than $\sigma < \infty$ if for every $\varepsilon > 0$ it holds that

$$|F(z)| \lesssim e^{(\sigma + \varepsilon)\|z\|_1}, \quad z \in \mathbb{C}^d.$$ 

The collection of all entire functions of exponential type no greater than $\sigma$ is denoted by $\mathcal{E}_\sigma(\mathbb{C}^d)$. For $0 \leq \alpha, \beta < \infty$, we define the class of functions $\mathcal{P}_{\alpha,\beta}(\mathbb{C}^d)$ via

$$\mathcal{P}_{\alpha,\beta}(\mathbb{C}^d) := \left\{ z \mapsto p(z)e^{-z^T A z - b^T z} : p \in \mathcal{E}_\alpha(\mathbb{C}^d), A \in \mathbb{C}^{d \times d}, b \in \mathbb{C}^d, \| A + A^T \|_1 \leq \beta \right\}.$$ 

This class is intended to cover a spectrum of functions, ranging from band-limited functions $f(z)e^{-z^T \alpha z}$, $f \in \text{PW}^2_K(\mathbb{R}^d)$, to exponential polynomials $p(z)e^{\alpha z^T z - z^T \beta z}$, $p: \mathbb{C}^d \to \mathbb{C}$ a polynomial, Hermite functions, and further. Notice that for all $z \in \mathbb{C}^d$ and $A \in \mathbb{C}^{d \times d}$ holds

$$z^T A z = \frac{1}{2} z^T A z + \frac{1}{2} \left( z^T A z \right)^T = \frac{1}{2} z^T (A + A^T) z,$$

so that we can without loss of generality assume that $A$ is symmetric (with bound $2 \| A \|_1 \leq \beta$). The last required parameter is the diameter of the domain of interest. For a compact set $K \subseteq \mathbb{R}^d$ denote its $\ell^\infty$-diameter by

$$\Delta(K) := \max_{t,t' \in K} \| t - t' \|_\infty.$$

With the previous concepts in hand, we are equipped to prove one of the main theorems on completeness of discrete translates of the present paper.

Theorem 4.1. Let $K \subseteq \mathbb{R}^d$ be a compact set, $\varphi \in \mathcal{P}_{\alpha,\beta}(\mathbb{C}^d)$ not mean-periodic, and

$$\sigma \geq \alpha + \frac{\beta}{2} \Delta(K).$$

If $\Lambda \subseteq \mathbb{R}^d$ is a uniqueness set for $\mathcal{E}_\sigma(\mathbb{C}^d)$, then $\mathcal{T}(\varphi, \Lambda)$ is complete in $C(K)$.

Proof. Let $\varphi(t) = p(t)e^{-t^T A t - b^T t}$ for some $p \in \mathcal{E}_\alpha(\mathbb{C}^d)$, $b \in \mathbb{C}^d$ and $A \in \mathbb{C}^{d \times d}$ a matrix which satisfies the upper bound $\| A + A^T \|_1 \leq \beta$. According to Theorem 3.10, it suffices to show that $\Lambda$ is a uniqueness set for $\mathcal{O}_\varphi(K)$. To that end, let $F \in \mathcal{O}_\varphi(K)$ with

$$F(z) = \int_K T_z \varphi(t) \, d\mu(t)$$

where $T_z$ is the translation operator.
for some \( \mu \in M(K) \). We have to show that \( F \) vanishes identically, provided that \( F \) vanishes at \( \Lambda \). Let \( t_0 \in \mathbb{R}^d \) such that \( K - t_0 \) lies in the centered cube

\[
\left[ -\frac{\Delta(K)}{2}, \frac{\Delta(K)}{2} \right]^d.
\]

Since the diameter is invariant under translations, this is the optimal centered cube that fits \( K \). Observe that

\[
T_z \varphi(t) = p(t - z)e^{-(t-z)^T A(t-z) - b^T (t-z)}
= p(t - z)e^{(t-t_0)^T(A + A^T)z}e^{-X(z)}e^{-t^T At - b^T t},
\]

where \( X(z) = z^T Az - t_0^T (A + A^T)z - b^T z \) is an entire function independent of \( t \). Hence, if \( G \) is defined by

\[
G(z) = \int_K p(t - z)e^{(t-t_0)^T(A + A^T)z} d\tilde{\mu}(t)
\]

with \( \tilde{\mu}(t) = e^{-t^T At - b^T t} \mu(t) \in M(K) \), then \( e^X F = G \). For every \( \varepsilon > 0 \) the map \( G \) satisfies the estimate

\[
|G(z)| \leq \|\tilde{\mu}\|_{TV} \max_{t \in K} \left| p(t - z)e^{(t-t_0)^T(A + A^T)z} \right|
\lesssim \|\tilde{\mu}\|_{TV} \max_{t \in K} \left( e^{(\alpha + \varepsilon)\|t-t_0\|_1} \cdot e^{\|t-t_0\|_\infty \|A + A^T\|_1} \right)
\lesssim \|\tilde{\mu}\|_{TV} e^{(\alpha + \varepsilon)\|t\|_1} \cdot e^{\frac{\beta}{2} \Delta(K)\|t\|_1}.
\]

Since \( \|\tilde{\mu}\|_{TV} \) is independent of \( z \), it follows that \( G \) is an entire function of exponential type no great than \( \sigma = \alpha + \frac{\beta}{2} \Delta(K) \). We have therefore shown that \( e^X F \in \mathcal{E}_\sigma(\mathbb{C}^d) \). Since \( e^X(z) \neq 0 \) for every \( z \), it follows that \( F \) vanishes on \( \Lambda \) if and only if \( e^X F \) vanishes on \( \Lambda \). The assumption on \( \Lambda \) implies that \( e^X F = 0 \), whence \( F = 0 \), thereby proving the assertion.

At this stage, it is meaningful to reflect upon the proof approach presented in Theorem \ref{thm:uniqueness}. To that end, we consider a minimal example with a univariate Gaussian \( \varphi(t) = e^{-t^2} \), a centered interval \( K = [-\kappa, \kappa] \), and a set \( \Lambda \subseteq \mathbb{R} \). Since the Gaussian is integrable, hence not mean-periodic, by Theorem \ref{thm:uniqueness} the completeness is equivalent to showing

\[
\forall \mu \in M([-\kappa, \kappa]) : L_\mu(T_{\lambda} \varphi) = 0 \ \forall \lambda \in \Lambda \implies L_\mu = 0.
\]

A key step in the proof of Theorem \ref{thm:uniqueness} is the derivation of equation \eqref{eq:uniqueness}. In the setting of the univariate Gaussian, the functional equation of the exponential transforms the problem to considering for all \( \lambda \in \Lambda \) the integral

\[
0 = \sigma(\lambda) \int_{-\kappa}^{\kappa} e^{2\lambda t} \psi(t) \ d\mu(t) = \sigma(\lambda) \int_{-\kappa}^{\kappa} e^{2\lambda t} \ d\tilde{\mu}(t),
\]

where \( \sigma(\lambda) = e^{-\lambda^2} \), \( \psi(t) = e^{-t^2} \), and \( d\tilde{\mu}(t) = \psi(t) d\mu(t) \). The Gaussian is a bounded function on the subset \( [-\kappa, \kappa] \subseteq \mathbb{R} \), so \( \tilde{\mu} \in M([-\kappa, \kappa]) \). The factor \( \sigma \) is not zero, so we are left with investigating the real zeroes of

\[
F(z) = \int_{-\kappa}^{\kappa} e^{2zt} \ d\tilde{\mu}(t), \quad z \in \mathbb{C}.
\]

This function is well-defined, holomorphic, and of exponential type. The problem, therefore, reduces to the search for uniqueness sets for functions of
exponential type. We point out that this method can only be used with the Gaussian due to its functional equation, as the following statement confirms.

**Proposition 4.2.** Let \( \varphi : \mathbb{C}^d \to \mathbb{C} \) be a non-zero entire function. There exist entire functions \( \psi, \sigma : \mathbb{C}^d \to \mathbb{C} \) and a matrix \( A \in \mathbb{C}^{d \times d} \) satisfying

\[
\varphi(z + \lambda) = \psi(z) \sigma(\lambda)e^{-2z^T A \lambda}, \quad z, \lambda \in \mathbb{C}^d
\]

if and only if \( A \) is symmetric and there exist \( b \in \mathbb{C}^d \) and \( c \in \mathbb{C} \) such that \( \varphi \) is given by

\[
\varphi(z) = e^{-z^T A z + z^T b + c}.
\]

**Proof.** See Appendix. \( \square \)

4.2. **Uniqueness sets for functions of exponential type.** We present two distinct criteria for uniqueness sets due to Carlson [37] and Ronkin [57]. Carlson’s theorem does assume a semigroup structure on the sampling set, while Ronkin’s theorem holds for arbitrary separated sets. However, the former allows for a considerably better density condition on the point configuration.

**Theorem 4.3 (Carlson [37]).** Suppose that \( F : \mathbb{C} \to \mathbb{C} \) is an entire function which satisfies the growth condition \( |F(z)| \lesssim e^{\beta |z|} \) where \( \beta < \pi \). Then \( F|_{\mathbb{N}} = 0 \) implies that \( F = 0 \).

Notice that the density assumption \( \beta < \pi \) in Carlson’s theorem is sharp as evidenced by \( \sin(\pi z) \) which vanishes at \( \mathbb{N} \) but does not vanish identically. Further observe that Carlson’s theorem readily transfers to the multivariate setting, leading to the statement that \( \alpha \mathbb{N}^d \) is a uniqueness set for \( E_{\sigma}(\mathbb{C}^d) \) provided that \( \alpha \sigma < \pi \). The same holds for \( \alpha \mathbb{Z}^d \) and this cannot be improved due to the same counterexample.

Now let \( \Lambda \subseteq \mathbb{R}^d \) be an arbitrary countable set. Then it comes with two distinct quantities related to its separability and its density. Firstly, the minimum gap of \( \Lambda \) is defined via

\[
\delta := \inf_{\lambda, \lambda' \in \Lambda} \|\lambda - \lambda'\|_2.
\]

In view of this definition, \( \Lambda \) is called \( \delta \)-separated if \( \delta > 0 \). On the other end, we denote by \( \mu_\Lambda \) the number of points of \( \Lambda \) which lie in the cube \( T_r = (-r, r)^d \). The lower resp. upper density of a set \( \Lambda \) is given by

\[
D^-(\Lambda) := \liminf_{r \to \infty} \frac{\mu_\Lambda(r)}{(2r)^d}, \quad D^+(\Lambda) := \limsup_{r \to \infty} \frac{\mu_\Lambda(r)}{(2r)^d}.
\]

For the prototypical example of a lattice \( \Lambda = A \mathbb{Z}^d, A \in \text{GL}_d(\mathbb{R}) \), we have

\[
D^-(\Lambda) = D^+(\Lambda) = \text{vol}(\Lambda)^{-1}.
\]

**Theorem 4.4 (Ronkin [57]).** Let the countable set \( \Lambda \subseteq \mathbb{R}^d \) be \( \delta \)-separated.

Furthermore, let the entire function \( F : \mathbb{C}^d \to \mathbb{C} \), satisfy the condition \( F(\lambda) = 0 \) for all \( \lambda \in \Lambda \). If \( F(z) \) is of exponential type no greater than \( \sigma < A_d \delta^{d-1} D^+(\Lambda) \), where

\[
A_d = \frac{2}{d!} \left( \frac{\pi}{2} \right)^{d-1} \left( \frac{d - 1}{2} + e + \sum_{k=0}^{d-2} \left( \frac{29}{3} \right)^{d-2-k} \left( \frac{13}{3} k + \frac{25}{3} e \right) \right)^{-1},
\]
then $F$ vanishes identically.

4.3. Implications for sampled STFT phase retrieval. We are now in a position to cross over from complete systems of translates to phase retrieval results. We begin with a technical observation.

Lemma 4.5. Let $\varphi \in \mathcal{P}_{\alpha, \beta}(\mathbb{C}^d)$, $\varphi \neq 0$, and let $\omega \in \mathbb{R}^d$. Then $\varphi_{\omega}$ is the restriction to $\mathbb{R}^d$ of a non-zero entire function which belongs to $\mathcal{P}_{2\alpha, 2\beta}(\mathbb{C}^d)$.

Proof. Let $\omega \in \mathbb{R}^d$ and let $\varphi \in \mathcal{P}_{\alpha, \beta}(\mathbb{C}^d)$, i.e., $\varphi(t) = p(t)e^{-iTb^\top t}$ for a suitable non-zero $p \in \mathcal{E}_{\alpha}(\mathbb{C}^d)$, $b \in \mathbb{C}^d$ and $A \in \mathbb{C}^{d \times d}$. For every real $t \in \mathbb{R}^d$, we have

$$\varphi_{\omega}(t) = p(t - \omega) p(T) e^{-(t - \omega)^\top A(t - \omega) - b^\top (t - \omega)} e^{-t^\top \pi t - b^\top t}$$

$$= p(t - \omega) p(T) e^{-2T \Re A \omega} e^{(\omega^\top (A^\top - \Re b) - (\Re b)^\top) t} e^{-\omega^\top A \omega + b^\top \omega}.$$  

Replacing $t \in \mathbb{R}^d$ with $z \in \mathbb{C}^d$ we obtain for every $\varepsilon > 0$ the growth estimate

$$|e^{-\omega^\top A \omega + b^\top \omega} p(z - \omega) p(\overline{z})| \lesssim e^{(\alpha + \varepsilon) \|z - \omega\|_1 e^{(\alpha + \varepsilon) \|z\|_1}} \lesssim e^{(2\alpha + 2\varepsilon) \|z\|_1},$$

Note that the implicit constant in the estimate (8) depends only on $\omega$. Since $p(z - \omega) p(\overline{z}) \in \mathcal{E}_{2\alpha}(\mathbb{C}^d)$, the last estimate shows that $\varphi_{\omega} \in \mathcal{P}_{2\alpha, 2\beta}(\mathbb{C}^d)$. Considering the fact that $p$ does not vanish identically, neither does the map $z \mapsto p(z - \omega) p(\overline{z})$. $\square$

With Theorem 3.2 in hindsight, we arrive at the following statement on STFT phase retrieval with window functions belonging to the class $\mathcal{P}_{\alpha, \beta}(\mathbb{C}^d)$.

Theorem 4.6. Let $K \subseteq \mathbb{R}^d$ be a compact set, $\varphi \in \mathcal{P}_{\alpha, \beta}(\mathbb{C}^d) \cap L^2(\mathbb{R}^d)$, $\varphi \neq 0$, and

$$\sigma \geq 2\alpha + \beta \Delta(K).$$

If $\Lambda \subseteq \mathbb{R}^d$ is a uniqueness set for $\mathcal{E}_{\sigma}(\mathbb{C}^d)$ and if $\Gamma$ is a uniqueness set for $\text{PW}_K^\perp(\mathbb{R}^d)$, then every $f \in L^2(K)$ is determined up to a global phase by $|V_\varphi f(\Lambda \times \Gamma)|$. In particular, this holds if $K \subseteq [-\kappa, \kappa]^d$ for some $\kappa > 0$ and if $\sigma \geq 2\alpha + 2\beta \kappa$.

Proof. Let $\omega \in \mathbb{R}^d$. According to Lemma 4.5, it holds that $\varphi_{\omega} \neq 0$. In addition, the assumption that $\varphi \in L^2(\mathbb{R}^d)$ implies that $\varphi_{\omega} \in L^1(\mathbb{R}^d)$. In particular, $\varphi_{\omega}$ is not mean-periodic. Now let $\omega \in K - K$. According to Lemma 4.1, the map $\varphi_{\omega}$ extends from $\mathbb{R}^d$ to an entire function which belongs to $\mathcal{P}_{2\alpha, 2\beta}(\mathbb{C}^d)$. Applying Theorem 4.1 shows that $\Sigma(\varphi_{\omega}, \Lambda)$ is complete in $C(K)$. Since $\omega \in K - K$ was arbitrary, an application of Theorem 3.2 implies the assertion. $\square$

The Hermite functions on $\mathbb{R}$ are defined by

$$h_n(t) = \frac{2^{1/4}}{\sqrt{n!}} \left( -\frac{1}{\sqrt{2\pi}} \right)^n e^{\pi t^2} \left( \frac{d}{dt} \right)^n e^{-2\pi t^2}, \quad n \in \mathbb{N}_0.$$  

The multivariate Hermite functions $\{h_k : k \in \mathbb{N}_0^d \} \subseteq L^2(\mathbb{R}^d)$ are defined via tensorisation,

$$h_k(t_1, \ldots, t_d) := \prod_{j=1}^d h_{k_j}(t_j).$$
Every Hermite function $\psi = \psi_k$, $k \in \mathbb{N}_0^d$, arises as the product of a polynomial with the Gaussian $2^{d/4}e^{-\pi \|t\|_2^2}$. Since every polynomial is of exponential type zero it holds that $\psi \in \mathcal{P}_{0,2\pi}(\mathbb{C}^d)$. This implies the next corollary.

**Corollary 4.7.** Let $K \subseteq \mathbb{R}^d$ be a compact set and $\psi = \psi_k$ be an arbitrary $d$-dimensional Hermite function. If $\Lambda \subseteq \mathbb{R}^d$ is a uniqueness set for $\mathcal{L}_\sigma(\mathbb{R}^d)$ with $\sigma \geq 2\pi \Delta(K)$ and $\Gamma \subseteq \mathbb{R}^d$ is a uniqueness set for $PW_{K-K}(\mathbb{R}^d)$, then every $f \in L^2(K)$ is determined up to a global phase by $|V_g f(\Lambda \times \Gamma)|$. In particular, this holds if $K \subseteq [-\kappa, \kappa]^d$ for some $\kappa > 0$ and if $\sigma \geq 4\pi \kappa$.

5. **Band-limited window functions**

This section is dedicated to the proofs of the statements of Section 1.4.3. We start by showing that the discrete translates of a band-limited function form a complete set. If $d = 1$, then this is essentially known [55, Proposition 5.5]. A similar argument as in [55] yields a respective result for the multivariate case. For the convenience of the reader, we shall prove this statement in the following.

**Proposition 5.1.** Let $K' \subseteq \mathbb{R}^d$ be a compact set and $\varphi \in PW_{K'}^2(\mathbb{R}^d)$, $\varphi \neq 0$. If $\Lambda \subseteq \mathbb{R}^d$ is a uniqueness set for $PW_{K'}^2(\mathbb{R}^d)$, then $\Sigma(\varphi, \Lambda)$ is complete in $C(\mathbb{R}^d)$ with respect to the topology of uniform convergence on compact sets. In other words, $\Sigma(\varphi, \Lambda)$ is complete in $C(K)$ for every compact set $K \subseteq \mathbb{R}^d$.

**Proof.** Let $K \subseteq \mathbb{R}^d$ be a compact set and let $\mu \in M(K)$. In view of Proposition [3.10], we have to show that the function

$$F(z) = \int_K T_z \varphi(t) \, d\mu(t)$$

vanishes identically provided that $F$ vanishes at $\Lambda$. Since $\varphi \in PW_{K'}^2(\mathbb{R}^d)$, we have $\varphi = \mathcal{F}g$ for some $g \in L^2(-K') \subseteq L^1(-K')$. Using Fubini’s theorem, we have

$$F(z) = \int_K \mathcal{F}g(t - z) \, d\mu(t) = \int_K \int_{-K'} g(s) e^{-2\pi is \cdot (t-z)} \, ds \, d\mu(t)$$

$$= \int_{-K'} g(s) \mathcal{F} \mu(s) e^{2\pi is \cdot z} \, ds = \mathcal{F}(g \mathcal{F} \mu)(-z) = \mathcal{F}^{-1}(g \mathcal{F} \mu)(z).$$

Since $\mathcal{F} \mu$ is an entire function, it holds that $g \mathcal{F} \mu \in L^2(-K')$ and therefore $F \in PW_{K'}^2(\mathbb{R}^d)$. The statement follows from the assumption of $\Lambda$ being a uniqueness set for $PW_{K'}^2(\mathbb{R}^d)$.

An immediate consequence of the previous statement is a phase retrieval result for band-limited window functions.

**Corollary 5.2.** Let $K, K' \subseteq \mathbb{R}^d$ be compact sets and let $g \in PW_{K'}^2(\mathbb{R}^d)$, $g \neq 0$. If $\Gamma \subseteq \mathbb{R}^d$ is a uniqueness set for $PW_{K-K}^1(\mathbb{R}^d)$ and $\Lambda \subseteq \mathbb{R}^d$ is a uniqueness set for $PW_{K-K'}^2(\mathbb{R}^d)$, then every $f \in L^2(K)$ is determined up to a global phase by $|V_g f(\Lambda \times \Gamma)|$.

**Proof.** If $g \in PW_{K'}^2(\mathbb{R}^d)$ does not vanish identically, then for every $\omega \in \mathbb{R}^d$, the map $\varphi_\omega$ does not vanish identically and extends from $\mathbb{R}^d$ to an entire
function on $\mathbb{C}^d$ which satisfies

$$g_\omega \in PW_{K' - K'}^2(\mathbb{R}^d) \subseteq PW_{(K' - K')}(\mathbb{R}^d).$$

Combining Proposition 5.1 with Theorem 3.2 yields the statement. \hfill \Box

We now consider function spaces of the form

$$\mathcal{S} = V_g L^2(K) = \{V_g f : f \in L^2(K)\},$$

where $K, K' \subseteq \mathbb{R}^d$ are compact sets and $g \in PW_{K'}^2(\mathbb{R}^d)$. According to the notion settled in Section 1.4.3, we call $\mathcal{S}$ an STFT-induced Paley-Wiener subspace. This notion is due to the following observation.

**Lemma 5.3.** Let $K, K' \subseteq \mathbb{R}^d$ be compact sets and let $g \in PW_{K'}^2(\mathbb{R}^d)$, $g \neq 0$. If $J = K' \times (-K)$, then $V_g L^2(K) \subseteq PW_{2}^2(J)$.  

**Proof.** First observe that $V_g f \in L^2(\mathbb{R}^{2d})$ since both $f$ and $g$ are square integrable. Since $V_g f(x, \omega) = \mathcal{F}(\mathcal{T}_x g)(\omega)$, we have

$$\mathcal{F}(V_g f(x, \cdot))(\omega) = \mathcal{R}(\mathcal{T}_x g)(\omega) = g(-\omega - x) f(-\omega).$$

and the right-hand side vanishes provided that $\omega \in -K$. If $\psi(x) = g(-\omega - x)$, then we further have that

$$\mathcal{F}\psi(y) = e^{2\pi i \omega \cdot y} \mathcal{F}g(y).$$

Therefore, the 2$d$-dimensional Fourier transform of $V_g f$ is given by

$$\mathcal{F}(V_g f)(y, \omega) = \mathcal{F}(x \mapsto \mathcal{F}(V_g f(x, \cdot))(\omega))(y)$$

$$= \mathcal{F}(x \mapsto g(-\omega - x) f(-\omega))(y)$$

$$= e^{2\pi i \omega \cdot y} f(-\omega) \mathcal{F}g(y).$$

Since $g \in PW_{K'}^2(\mathbb{R}^d)$, this implies that the support of the 2$d$-dimensional Fourier transform of $V_g f$ is contained in $K' \times (-K)$, thereby proving the assertion. \hfill \Box

The previous considerations imply uniqueness results for STFT phase retrieval with band-limited window functions, and equivalently, for STFT-induced Paley-Wiener subspaces.

**Theorem 5.4.** Let $K, K' \subseteq \mathbb{R}^d$ be compact sets and let $g \in PW_{K'}^2(\mathbb{R}^d)$, $g \neq 0$. Further, let $U = (K' - K') \times (K - K)$ and $\mathcal{L} \subseteq \mathbb{R}^{2d}$ be a uniqueness set for $PW_{\mathcal{L}}^2(\mathbb{R}^d)$. Then every $F \in \mathcal{S} = V_g L^2(K)$ is determined up to a global phase by $|F(\mathcal{L})|$. 

**Proof.** Let $F \in \mathcal{S}$ such that $F = V_g f$ for some $f \in L^2(K)$. According to Lemma 5.3 it holds that $F \in PW_{J}^2(\mathbb{R}^{2d})$ where $J = K' \times (-K)$. With a similar argument as in the previous sections, we deduce that $|F|^2 \in PW_{J'}^2(\mathbb{R}^{2d})$. Since $J - J = U$ and since $\mathcal{L}$ is a uniqueness set for $PW_{\mathcal{L}}^2(\mathbb{R}^{2d})$, it follows that if $H = V_g h \in \mathcal{S}$, $h \in L^2(K)$, is a second function such that $|F(\mathcal{L})| = |H(\mathcal{L})|$, then $|F| = |H|$ everywhere on $\mathbb{R}^{2d}$. Corollary 5.2 shows that $f \sim h$, and since $V_g$ is a linear transformation, we obtain $F \sim H$. \hfill \Box
Example 5.5 (Airy disk). Let \( D_a := \{ x \in \mathbb{R}^2 : |x| \leq a \} \) be a circular disk of radius \( a > 0 \) and center \( 0 \) in \( \mathbb{R}^2 \). The square of the Fourier transform of the characteristic function of \( D_a \) is called the Airy disk of radius \( a \). We have

\[
\mathcal{F}(1_{D_a})(\omega) = \int_0^a r \int_0^{2\pi} e^{-2\pi i r \|\omega\|_2 \cos(\theta)} d\theta dr = 2\pi \int_0^a r J_0(2\pi \|\omega\|_2 r) dr,
\]

where we used polar coordinates as well as the definition of the Bessel function

\[
J_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(x \sin \tau - n\tau)} d\tau.
\]

The identity

\[
\int_0^a x J_0(x) dx = a J_1(a)
\]

implies that the Airy disk of radius \( a \) is then given by the radial function

\[
A_a(\omega) = (\mathcal{F}(1_{D_a})(\omega))^2 = \left( \frac{a J_1(2\pi \|\omega\|_2 a)}{\|\omega\|_2} \right)^2.
\]

As shown in Figure 2, this map behaves approximately like a Gaussian (except for the decay at infinity). The Airy disk frequently appears in diffraction imaging, namely in the situation where an incoming wavefront hits on a circular aperture (a pinhole) before getting diffracted at an object of interest. See, for instance, [16, Chapter 8.5.2]. The STFT phase retrieval problem with Airy window functions appears naturally in this context.

6. Gaussian windows: Gabor phase retrieval

6.1. Completeness of discrete Gaussian translates. Any Gaussian belongs to the class \( P_{0,\beta}(\mathbb{C}^d) \) for a suitable \( \beta > 0 \), hence the completeness Theorem 4.1, and consequentially Theorem 3.7 on phase retrieval, apply with appropriate density assumptions on the generator set \( \Lambda \). As already evidenced by the Bargmann transform [50, 61, 62], the uncertainty principles and the current progress on phase retrieval [1, 5, 8, 29, 32, 34, 35], the Gaussian is an exceptional window function. We underline this once more by showing that the completeness of \( \mathcal{T}(g, \Lambda) \) does not depend on the density of the lattice \( \Lambda \) when \( g \) is a Gaussian generator.
Theorem 6.1. Let \( g \in C(\mathbb{R}^d) \) be a multivariate complex-valued Gaussian
\[
g(x) = \exp \left( -(x - \nu)^T A (x - \nu) \right),
\]
where \( \nu \in \mathbb{C}^d \) and \( A \in \mathbb{C}^{d \times d} \) is a Hermitian matrix with invertible real part \( \text{Re} A \). If \( K \subseteq \mathbb{R}^d \) is a compact set and \( \Lambda \) is a semigroup in \( \mathbb{R}^d \) that contains a spanning set, then the system \( \Sigma(g, \Lambda) \) is complete in \( (C(K), \| \cdot \|_\infty) \). In particular, this holds when \( \Lambda \) is a lattice.

Proof. By expanding the term \( -(x - \lambda - \nu)^T A (x - \lambda - \nu) \), we obtain the factorization
\[
T_{\lambda} G(x) = c(\lambda) a(x) B_{\lambda}(x)
\]
with
\[
c(\lambda) = \exp \left( -(\lambda + \nu)^T A (\lambda + \nu) \right),
a(x) = \exp \left( -x^T Ax + 2 (\text{Re} A \, x)^T \nu \right),
B_{\lambda}(x) = \exp \left( 2 (\text{Re} A \, x)^T \lambda \right),
\]
where we used that \( A \) is Hermitian. Notice that the completeness of \( \Sigma(g, \Lambda) \) is unaffected if each \( T_{\lambda} G \) is multiplied by a non-zero constant. Since \( c(\lambda) \neq 0 \), \( \Sigma(g, \Lambda) \) is complete if and only if the set \( \{ aB_{\lambda} : \lambda \in \Lambda \} \) is complete. Moreover, \( a \) is a smooth, non-vanishing weighting factor independent of \( \lambda \). Hence, \( \{ aB_{\lambda} : \lambda \in \Lambda \} \) is complete if and only if \( \{ B_{\lambda} : \lambda \in \Lambda \} \) is complete. Now set
\[
S := \text{span}_\mathbb{C} \{ B_{\lambda} : \lambda \in \Lambda \}.
\]
We show that \( S \) satisfies the assumptions of the Stone-Weierstrass theorem. Since \( B_{\lambda} \) is real-valued, the set \( S \) is invariant under complex conjugation. Further, for \( \lambda, \lambda' \in \Lambda \) holds
\[
B_{\lambda}(x) B_{\lambda'}(x) = B_{\lambda + \lambda'}(x),
\]
which is an element of \( S \) since \( \lambda + \lambda' \in \Lambda \). It remains to show that \( S \) separates points. To that end, let \( x \neq y \in K \) and \( \lambda \in \Lambda \). Then the equality \( B_{\lambda}(x) = B_{\lambda}(y) \) holds if and only if
\[
\langle (\text{Re} A)(x - y) \rangle^T \lambda = 0.
\]
Because of the spanning property of \( \Lambda \), if this equality holds for all \( \lambda \in \Lambda \), then it holds for all \( \lambda \in \mathbb{R}^d \), implying that
\[
(\text{Re} A)(x - y) = 0.
\]
Since \( \text{Re} A \) is invertible, we have \( x = y \). In conclusion, the Stone-Weierstrass theorem \cite[Theorem 4.51]{StWe} implies that the \( \| \cdot \|_\infty \)-closure of \( S \) is either equal to \( C(K) \) or equal to \( \{ f \in C(K) : f(x_0) = 0 \} \) for a unique \( x_0 \in K \). The later can be excluded since \( B_{\lambda}(x_0) > 0 \) for all \( x_0 \in K \). \( \square \)

6.2. Implications for sampled Gabor phase retrieval. As an immediate consequence, we can retrieve compactly supported functions \( f \in L^2(K) \) from the phaseless samples \( |V_g f(\mathcal{L})| \) with respect to a Gaussian window \( g \) on sets \( \mathcal{L} \subseteq \mathbb{R}^{2d} \) of arbitrary density.
Theorem 6.2. Let \( g \in C(\mathbb{R}^d) \) be a multivariate complex-valued Gaussian
\[
g(x) = \exp \left( -(x - \nu)^T A (x - \nu) \right),
\]
where \( \nu \in \mathbb{C}^d \) and where \( A \in \mathbb{C}^{d \times d} \) is a positive definite Hermitian matrix.
Suppose that \( K \subseteq \mathbb{R}^d \) is a compact set. If \( \Lambda \) is a semigroup in \( \mathbb{R}^d \) that contains a spanning set and \( \Gamma \) is a uniqueness set in \( \text{PW}^1_{K-K}(\mathbb{R}^d) \), then every \( f \in L^2(K) \) is determined up to a global phase by \( |V_g f(\Lambda \times \Gamma)| \).

Proof. Let \( \omega \in K - K \). Since \( \text{Re} \, A \) is positive definite, the window \( g \) is square-integrable. Following Theorem 3.2, we have to show that the system \( \mathcal{T}(g_\omega, \Lambda) \) is complete in \( (C(K), \| \cdot \|_\infty) \). The map \( g_\omega \) is given by
\[
g_\omega(x) = C \cdot \exp \left( -2 (x - (\text{Re} \, \nu) - \frac{\omega}{2})^T (\text{Re} \, A) (x - (\text{Re} \, \nu) - \frac{\omega}{2}) \right),
\]
where \( C = C(A, \omega, \nu) \) is a constant depending only on \( A, \omega \) and \( \nu \). In view of Theorem 6.1, \( \mathcal{T}(g_\omega, \Lambda) \) is complete in \( C(K) \). The statement follows from that.

Since every lattice is a semigroup that contains a spanning set, we obtain the following important consequence of Theorem 6.2.

Corollary 6.3. Let \( g \in C(\mathbb{R}^d) \) be a multivariate complex-valued Gaussian
\[
g(x) = \exp \left( -(x - \nu)^T A (x - \nu) \right),
\]
where \( \nu \in \mathbb{C}^d \) and where \( A \in \mathbb{C}^{d \times d} \) is a positive definite Hermitian matrix.
Suppose that \( K \subseteq \mathbb{R}^d \) is a compact set. If \( \Lambda, \Gamma \subseteq \mathbb{R}^d \) are lattices such that \( K - K \) is contained in a fundamental domain of the reciprocal lattice \( \Gamma^* \), then every \( f \in L^2(K) \) is determined up to a global phase by \( |V_g f(\Lambda \times \Gamma)| \).

Remark 6.4 (Zalik’s theorem). Let \( g(t) = e^{-at^2 + bt + c}, a > 0, b, c \in \mathbb{R}, \) be a univariate Gaussian, let \( K \subseteq \mathbb{R} \) be a compact interval and let \( \Lambda \subseteq \mathbb{R} \setminus \{0\} \) be a countable set. A theorem due to Zalik shows that \( \mathcal{T}(g, \Lambda) \) is complete in \( L^2(K) \) if and only if
\[
\sum_{\lambda \in \Lambda} \frac{1}{|\lambda|} = \infty,
\]
see [72, Theorem 4]. For every \( \omega \in \mathbb{R} \) it holds that \( g_\omega \) is again a Gaussian of the form \( t \mapsto e^{-at^2 + \beta t + \gamma} \) for some \( \alpha > 0, \beta, \gamma \in \mathbb{R} \). By standard tensor arguments, one can show that a multivariate Gaussian of the form \( g(t) = e^{-at^2 + bt + c} \), the system \( \mathcal{T}(g, \Lambda^d) \) is complete in \( L^2(K) \) for all Cartesian products of intervals \( K \). Consequently, \( \mathcal{T}(g_\omega, \Lambda^d) \) is complete in \( L^2(K) \) for every \( \omega \in K - K \subseteq \mathbb{R}^d \). A combination of Zalik’s theorem with Theorem 6.1 therefore implies a uniqueness statement for functions in \( L^4(K) \). We refer to the articles [32, 70] for further references on the appearance of Zalik’s theorem in STFT phase retrieval.

6.3. Uniqueness in dense subspaces. Note that Corollary 6.3 allows uniqueness when sampling the phaseless Gabor transform of a compactly supported \( f \in L^2(K) \subseteq L^2(\mathbb{R}^d) \) on arbitrarily sparse lattices \( \Lambda \times \Gamma \), since \( D(\Lambda \times \Gamma) = \text{vol}(\Lambda \times \Gamma)^{-1} = \text{vol}(\Lambda)^{-1} \text{vol}(\Gamma)^{-1} \), and we have complete freedom when choosing \( \Lambda \). This is an improvement from band-limited windows or
Hermite windows, where both \( \Lambda \) and \( \Gamma \) need to be sufficiently dense. However, the sampling locations still depend on the compact set \( K \) through the choice of \( \Gamma \). We now propose a trade-off for this dependency with a minor additional assumption on the compactly supported function \( f \in L^2(\mathbb{R}^d) \).

We recall the following theorem due to Kahane [44].

**Theorem 6.5** (Kahane). There exists a sequence \( K = \{ \kappa_n : n \in \mathbb{Z} \} \subseteq \mathbb{R} \) such that the following three properties hold:

i) \( K \) is symmetric, that is, \( \kappa_n = \kappa_{-n} \) for every \( n \in \mathbb{Z} \),

ii) \( \lim_{n \to \infty} \frac{n}{\kappa_n} = 0 \),

iii) the system of complex exponentials \( \{ e^{2\pi i \kappa_n \cdot} : n \in \mathbb{Z} \} \) is complete in \( C(I) \) for every compact interval \( I \subseteq \mathbb{R} \).

The proof is constructive and the set \( K \) defined in [44] will from now on be referred to as Kahane’s sequence. Notice that property ii) in Theorem 6.5 asserts that Kahane’s sequence has density zero, where the density of a set \( S \subseteq \mathbb{R}^d \) is defined via

\[
D(S) := \lim_{r \to \infty} \frac{\#(S \cap B_r(0))}{m^d(B_r(0))},
\]

assuming that the limit exists. Here \( m^d \) denotes the \( d \)-dimensional Lebesgue measure. By a tensor argument as with Zalik’s theorem above, the system of multivariate complex exponentials \( \{ e^{2\pi i \kappa \cdot} : \kappa \in K^d \} \) is complete in \( C(K) \) for every compact set \( K \subseteq \mathbb{R}^d \). To that, the density of the Cartesian product \( K^d \) is zero;

\[
D(K^d) = \lim_{r \to \infty} \frac{\#(K^d \cap B_r(0))}{m^d(B_r(0))} \lesssim \lim_{r \to \infty} \prod_{j=1}^d \frac{\#(K \cap [-r, r])}{2r} = 0.
\]

We are now in a position to construct a sampling set of density zero which achieves phase retrieval in a dense subspace of \( L^2(\mathbb{R}^d) \).

**Theorem 6.6.** Let \( g \in C(\mathbb{R}^d) \) be a real multivariate Gaussian window function. Let \( K \subseteq \mathbb{R}^d \) be Kahane’s sequence and \( \Lambda \subseteq \mathbb{R} \setminus \{0\} \) a sequence of distinct real numbers such that \( \sum_{\lambda \in \Lambda} \frac{1}{|\lambda|} = \infty \) and such that \( D(\Lambda) = 0 \). Further, let

\[
\mathcal{V} := \bigcup_{\kappa > 0} L^d[-\kappa, \kappa]^d.
\]

Then the following holds:

i) \( \mathcal{V} \) is dense in \( L^2(\mathbb{R}^d) \),

ii) every \( f \in \mathcal{V} \) is determined up to a global phase by \( \left| V_g f(\Lambda^d \times K^d) \right| \),

iii) \( D(\Lambda^d \times K^d) = 0 \).

In particular, the theorem’s conclusion holds true if \( \Lambda \) is the set of prime numbers.

**Proof.** It is evident that \( \mathcal{V} \) is dense in \( L^2(\mathbb{R}^d) \). Therefore, we continue with the proof of the second statement. To that end, let \( K \subseteq \mathbb{R}^d \) be an arbitrary compact set. Since \( g \) is a Gaussian, so is \( g_\omega \) for every \( \omega \in \mathbb{R}^d \). In view
of Zalik’s theorem (see Remark 6.4), the system \( \Sigma(g_\omega, \Lambda^d) \) is complete in \( L^2(K) \). In particular, the operator

\[
C_\omega : L^2(K) \to \mathbb{C}^{\Lambda^d}, \quad f \mapsto \left( \langle f, \overline{\tau_\lambda g_\omega} \rangle_{L^2(K)} \right)_{\lambda \in \Lambda^d}
\]
is injective for every \( \omega \in K - K \). According to Kahane’s theorem, the system \( \left\{ e^{2\pi i \kappa z} : \kappa \in \mathcal{K}^d \right\} \) is complete in \( C(K - K) \). In particular, it is complete in \( L^2(K - K) \), which in turn implies that \( \mathcal{K}^d \) is a uniqueness set for \( PW^2_{K-K}(\mathbb{R}^d) \supseteq PW^1_{K-K}(\mathbb{R}^d) \). The statement follows from the observation that \( \Lambda^d, \mathcal{K}^d \) are independent of \( K \) and if \( f, h \in \mathcal{V} \), then there exists a compact set \( K \subseteq \mathbb{R}^d \) such that \( \text{supp}(f), \text{supp}(h) \subseteq K \) and \( f, h \in L^4(K) \). The property that \( D(\Lambda^d \times \mathcal{K}^d) = 0 \) follows from

\[
\frac{\#((\Lambda^d \times \mathcal{K}^d) \cap B_r(0))}{m^{2d}(B_r(0))} \lesssim \frac{\#(\Lambda^d \cap B_r(0)) \#(\mathcal{K}^d \cap B_r(0))}{r^{2d}} \to 0
\]
as \( r \to \infty \) (where on the left-hand side and right-hand side \( B_r(0) \) denotes a ball in \( \mathbb{R}^{2d} \) and in \( \mathbb{R}^d \), respectively).

\[
\square
\]

7. Appendix: proof of Proposition 4.2

**Proof.** We divide the proof into six steps. The main part considers the case \( A = I \). From there, we first handle non-singular matrices and finish with the singular case. Observe that if \( \varphi(z) = e^{-z^T A z + z^T b + c} \), then the statement holds true, so it suffices to show that the existence of a factorization of the form (6) for \( \varphi \) implies that \( \varphi \) has the form (7).

**Step 1.** We start by showing that \( \varphi \) is zero-free. To that end, assume the contrary, that is, the function \( \varphi \) has a zero at \( z_0 \in \mathbb{C}^d \). Then \( z_0 \) is also a zero of \( \psi \) or \( \sigma \). Without loss of generality, \( \psi(z_0) = 0 \). Then for all \( z \in \mathbb{C}^d \) it holds that

\[
\varphi(z) = \varphi(z_0 + z - z_0) = \psi(z_0) \sigma(z - z_0) e^{-2z_0^T A (z - z_0)} = 0,
\]
contradicting the assumption that \( \varphi \) is not the zero function. Therefore, none of the functions \( \varphi, \psi, \sigma \) has a zero. Furthermore, observe that it holds \( \varphi = \sigma(0) \psi = \psi(0) \sigma \). Hence, \( \psi \) and \( \sigma \) are entire functions. This also implies

\[
\varphi(z) \varphi(\lambda) e^{-2z^T A \lambda} = \sigma(0) \psi(0) \varphi(z + \lambda) = \varphi(\lambda) \varphi(z) e^{-2\lambda^T A z},
\]
which allows us to conclude that all \( \lambda, z \in \mathbb{C}^d \) satisfy \( 1 = e^{2z^T (A^T - A) \lambda} \).

**Step 2:** \( A = \text{Id} \). Assume in the following that \( A \) is the identity matrix. Since

\[
e^{-2z^T \lambda} \psi(z) \sigma(\lambda) = \varphi(z + \lambda) = \varphi(\lambda + z) = \psi(\lambda) \sigma(z) e^{-2\lambda^T z} = \psi(\lambda) \sigma(z) e^{-2z^T \lambda},
\]
we conclude that there exists a constant \( \eta \in \mathbb{C} \setminus \{0\} \) such that \( \psi = \eta \sigma \). Therefore,

\[
\varphi(z + \lambda) = \psi(z) \sigma(\lambda) e^{-2z^T \lambda} = \frac{1}{\eta} \psi(z) \psi(\lambda) e^{-2z^T \lambda}
\]

\[
= \frac{1}{\sqrt{\eta}} \psi(z) \left( \frac{1}{\sqrt{\eta}} \psi(\lambda) e^{-2z^T \lambda} \right).
\]
Replacing $\psi, \sigma$ by $\frac{1}{\sqrt{\eta}} \psi$ we can assume without loss of generality that $\eta = 1$. This implies that for each $z \in \mathbb{C}^d$ we have

$$\varphi(0) = \varphi(z - z) = \psi(z)\psi(-z)e^{2z^T z} = \Psi(z)\Psi(-z),$$

where $\Psi(z) := \psi(z)e^{z^T z}$.

**Step 3: Projection on the coordinate axes.** Let $e_j$ be the $j$-th unit vector in $\mathbb{C}^d$. Further, let $\Psi_j(t) := \Psi(te_j)$ be the restriction of $\Psi$ to the $j$-th coordinate axis. Since $\Psi_j$ is zero-free and entire, there exists an entire function $g_j$ such that $\Psi_j = e^{g_j}$. Equation (6) together with the previous considerations implies that for all $t, \xi \in \mathbb{C}$ holds

$$\varphi((t + \xi)e_j) = \Psi_j(t)\Psi_j(\xi)e^{-(t+\xi)^2} = e^{g_j(t)}e^{g_j(\xi)}e^{-(t+\xi)^2}.$$  

On the other hand, it follows from the identity $\varphi = \psi(0)\sigma = \psi(0)\psi$ and the definition of $\Psi_j$ that

$$\varphi((t + \xi)e_j) = \psi(0)\Psi_j(t + \xi)e^{-(t+\xi)^2} = \psi(0)e^{g_j(t+\xi)}e^{-(t+\xi)^2}.$$  

Now fix $\xi \in \mathbb{C}$. Combining Equation (10) and (11) shows that there exists a constant $s \in \mathbb{C}$ such that

$$s = g_j(t) + g_j(\xi) - g_j(t + \xi)$$

for all $t \in \mathbb{C}$. Differentiating with respect to $t$ yields

$$0 = g_j'(t) + 0 - g_j'(t + \xi).$$

Thus, $g_j'$ is $\xi$-periodic, and since $\xi$ was arbitrary, $g_j'$ must be a constant function. Therefore, there exist constants $b_j, c_j \in \mathbb{C}$ satisfying

$$g_j(t) = b_j t + c_j, \quad t \in \mathbb{C}.$$  

In particular, the map $\Psi_j$ is given by

$$\Psi_j(t) = e^{b_j t + c_j}.$$  

Since $\Psi(0) = \Psi_j(0) = e^{c_j}$, it holds $c_j - c_k \in 2\pi i \mathbb{Z}$ for all $j, k \in \{1, \ldots, d\}$. Therefore, we can assume that $c := c_j$ for all $j \in \{1, \ldots, d\}$.

**Step 4: Derivation of $\varphi$ for $A = \mathrm{Id}$.** Decomposing the vector $z = (z_1, \ldots, z_d)^T \in \mathbb{C}^d$ with respect to the standard unit vectors $e_1, \ldots, e_d \in \mathbb{C}^d$, and using the factorization derived in Equation (9) with $\eta = 1$ shows that

$$\varphi(z) = \varphi \left( z_1 e_1 + \sum_{j=2}^d z_j e_j \right) = \psi(z_1 e_1)\psi \left( \sum_{j=2}^d z_j e_j \right).$$

Since $\psi(z_1 e_1) = e^{-z_1^2 + b_1 z_1} e^c$ and $\varphi = \psi(0)\psi = e^c\psi$,

$$\varphi(z) = e^{-z_1^2 + b_1 z_1} \varphi \left( \sum_{j=2}^d z_j e_j \right).$$

It follows via induction that

$$\varphi(z) = \psi_d(z_d e_d) \prod_{j=1}^{d-1} e^{-z_j^2 + b_j z_j} = e^c \prod_{j=1}^d e^{-z_j^2 + b_j z_j} = \exp \left( -z^T z + z^T b + c \right).$$
Step 5: Non-singular $A$. Now assume $A$ is an arbitrary invertible symmetric matrix. By Takagi’s factorization theorem, there exists an invertible matrix $B \in \mathbb{C}^{d \times d}$ satisfying $B^T B = A$. This transforms the problem to

$$\varphi(z + \lambda) = \psi(z) \sigma(\lambda) e^{-2z^T B^T B \lambda} = \psi(B^{-1} B z) \sigma(B^{-1} B \lambda) e^{-2(Bz)^T (B \lambda)}.$$ 

If we define the auxiliary functions $\varphi_B$, $\psi_B$, and $\sigma_B$ via

$$\varphi_B := \varphi \circ B^{-1}, \quad \psi_B := \psi \circ B^{-1}, \quad \sigma_B := \sigma \circ B^{-1},$$

then it holds that

$$\varphi_B(z + \lambda) = \psi(B^{-1} B z) \sigma(B^{-1} B \lambda) e^{-2z^T B^T B \lambda}.$$ 

By Step 4 above, there is a vector $b' \in \mathbb{C}^d$ and a constant $c \in \mathbb{C}$ such that

$$\varphi_B(z) = \exp \left( -z^T z + z^T b' + c \right)$$

for all $z \in \mathbb{C}^d$. Therefore, denoting $b = B^T b'$, for all $z \in \mathbb{C}^d$ we have

$$\varphi(z) = \varphi_B(Bt) = \exp \left( -z^T Az + z^T b + c \right).$$

Step 6: General symmetric matrices. Finally, suppose that $A$ is an arbitrary symmetric matrix. Let $A = U \Sigma U^T$ be the singular value decomposition of $A$ with $U \in \mathbb{C}^{d \times d}$ unitary and $\Sigma \in \mathbb{R}^{d \times d}$ diagonal, $\Sigma_{i,i} \geq \Sigma_{i+1,i+1}$. Further, let $\Omega$ be the diagonal matrix defined by

$$\Omega_{i,i} := \begin{cases} 0, & \Sigma_{i,i} \neq 0, \\ 1, & \Sigma_{i,i} = 0. \end{cases}$$

Then the matrix

$$C := A + U \Omega U^T = U (\Sigma + \Omega) U^T$$

is invertible and it holds

$$\varphi(z + \lambda) e^{-(z+\lambda)^T U \Omega U^T (z+\lambda)} = \psi(z) e^{-z^T U \Omega U^T z} \sigma(\lambda) e^{-\lambda^T U \Omega U^T \lambda} e^{-2z^T C \lambda}.$$ 

By the previous step, there exist $b \in \mathbb{C}^d$ and $c \in \mathbb{C}$ such that

$$\varphi_0(z) = \exp \left( -z^T C z + z^T b + c \right).$$

By the definition of the map $\varphi_0$ and the matrix $C$, we obtain

$$\varphi(z) = e^{-z^T A z + z^T b + c}.$$ 

This completes the proof of the statement. □

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