Twisted smooth Deligne cohomology

Daniel Grady and Hisham Sati

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Abstract

Deligne cohomology can be viewed as a differential refinement of integral cohomology, hence captures both topological and geometric information. On the other hand, it can be viewed as the simplest nontrivial version of a differential cohomology theory. While more involved differential cohomology theories have been explicitly twisted, the same has not been done to Deligne cohomology, although existence is known at a general abstract level. We work out what it means to twist Deligne cohomology, by taking degree one twists of both integral cohomology and de Rham cohomology. We present the main properties of the new theory and illustrate its use with examples and applications. Given how versatile Deligne cohomology has proven to be, we believe that this explicit and utilizable treatment of its twisted version will be useful.

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1 Introduction

Twistings form an interesting feature of (generalized) cohomology theories. While on general grounds existence is established at the axiomatic/abstract level via parametrized spectra (see [MS06] [ABGHR14]), constructing such theories explicitly is often a nontrivial task (see [ABG10] [SW15] [LSW16] for recent illustrations of this). By the same token, differential refinements of (generalized) cohomology theories are also established at the general abstract level (see [BNV16] [BS10] [Sc13]). However, again, explicit utilizable constructions require considerable work (see [GS16a] [GS16b]). Combining the above two aspects, one also has twisted differential cohomology theories at the general yet somewhat abstract level [BNI4]. The goal of this paper is to work out an explicit case of such a differential twisted theory.

We will be interested mainly in twists of differential refinements of ordinary, i.e. integral, cohomology. This cohomology theory has smooth extension with various different realizations, including
those of [De71] [CS85] [Ga97] [Br93] [DL05] [HS05] [BKS10] [BB14]. All of these realizations are in fact isomorphic (see [SS08] [BS10]). We will use the smooth Deligne cohomology incarnation [De71] [Be85] [Gi84] [Ja88] [EV88] [Ga97]. In terms of machinery, we will use the approach of simplicial presheaves and higher stacks [FSS12] [Sc13] [FSS13] [HQ15] [FSS15a]. This has the virtue of being amenable to generalizations and allowing the use of powerful algebraic machinery.

Twisted Deligne cohomology is understood to exist from general constructions [Bu12] [BN14] and should be in some sense one of the simplest twisted differential cohomology theories. However, we have not seen this theory discussed in detail anywhere in the literature. This, together with the versatility and utility of Deligne cohomology, motivated us to believe that this would be a very useful task. Furthermore, we view this, in some sense, as the toy model and the prototype example for more involved twisted differential spectra described this way. From the fact that constructing this theory was not as straightforward as one might have thought, with unexpected structures and subtleties, the involved task of unraveling the details turned out to be worthwhile.

The study of the twisted de Rham cohomology is essential for understanding the confluent hypergeometric integral which can be regarded as a pairing of the twisted de Rham cohomology and a twisted cycle. For the case of 1-forms there has been a considerable amount of work in this direction, e.g. Deligne, who gives the twisted de Rham theorem in [De70] II: 6.3 [DM86] as well as [Ki93] [Ki94] [AKOT97] [AS97]. Deligne’s work has also had other incarnations, such as in Witten’s approach to Morse theory [Wi82]. Together with twisted integral cohomology, discussed in Sec. 2, 1-form twisted de Rham cohomology lead to compatible twistings of Deligne cohomology in Sec. 4. Appropriately, the theory turns out to have pleasant computability properties that, again, are there axiomatically and abstractly, but that we do explicitly verify and cast in utilizable form in Sec. 5. We give several examples in Sec. 6 to illustrate the constructions.

We now highlight directly-related constructions in the literature. In [Bu12] differential refinements of integral cohomology are considered, leading to a sheaf-theoretic definition of smooth Deligne cohomology. In [BKS10] a bordism model for the differential extension of ordinary integral cohomology is given in which one has integration and products and a simple verification the projection formula. Twistings of integral cohomology, as explained in [Fr01], a priori are 1-dimensional, determined by a local system $\mathbb{Z} \to M$, which is a bundle of groups isomorphic to $\mathbb{Z}$. This is then determined up to isomorphism by an element of $H^1(M; \text{Aut}(\mathbb{Z})) \cong H^1(M; \mathbb{Z}/2)$. The twisted integral cohomology $H^*(M; \mathbb{Z})$ may be defined using a cochain complex. This also admits a Čech description. We will make use of a generalization of this degree 1 local system setup in Sec. 4. An extensive discussion of the degree 1 case can be found in [Fa04].

Note that the Deligne complex can be viewed from more than one angle. From an algebraic point of view, it is the resolution of the group $\mathbb{Z}$. From a geometric (and topological) point of view, the starting point is the de Rham complex and this is viewed as the resolution of some sheaf. In the simplest case, this is $\mathbb{R}$, while more involved situations corresponding to twists, will require more delicate local systems. Note also that there is a related concept of $\mathbb{Z}/2$-twisted de Rham forms [HZ98], whereby one can also twist by $\mathbb{Z}/2$ classes related to orientation; $\mathcal{O}_X$ is taken as the principal $\mathbb{Z}/2$-bundle of orientations of $TX$, and form $\mathcal{O}$-twisted $k$-forms as sections of $\mathcal{O} \otimes_{\mathbb{Z}/2} \Lambda^k T^* X$. This is used to study twisted currents associated to Stiefel-Whitney classes.

We will be in the setting of homotopy sheaves [Ja15] (see also [Du99] for a very readable account). In Sec. 4 we show that pulling back the universal bundle over a map which classifies a twist, we get a bundle $\mathcal{H}^q \to M$ over $M$. We define the $\omega$-twisted Deligne cohomology of $M$ of degree $q$ to be the the connected components $\pi_0 \Gamma(M, \mathcal{H}^q)$. We will mostly be dealing with the category of
sheaves of chain complexes, while occasionally considering twisted differential cohomology within smooth sheaves of spectra.

Chain complexes provide a useful way to present $HZ$-module spectra [Sh07]. Let Ch be the ordinary symmetric monoidal category of chain complexes of vector spaces. Formally invert the class of quasi-isomorphisms in Ch gives an $\infty$-category $\text{Ch}_\infty := N(\text{Ch})[W^{-1}]$, the stable $\infty$-category obtained by localization of the $\infty$-category at quasi-isomorphisms. The natural map $\iota : \text{Ch} \to \text{Ch}_\infty$ is a lax symmetric monoidal functor. Furthermore, there is an equivalence of symmetric monoidal categories [Lu11]

$$H : \text{Ch}_\infty \xrightarrow{\sim} \text{Mod}_{HZ},$$

where $\text{Mod}_{HZ}$ denotes the module spectra over the Eilenberg-MacLane spectrum $HZ$. Consider the group of integers $\mathbb{Z}$ as an object in $N(\text{Ch})[W^{-1}]$ by viewing it as a chain complex concentrated in degree zero. Being a stable $\infty$-category, $\text{Ch}_\infty$ is enriched over spectra. Hence the Eilenberg-MacLane spectrum can be defined by the mapping spectrum $HZ := \text{Map}(\mathbb{Z}, \mathbb{Z})$. Moreover, $H$ can be chosen such that $H(\mathbb{Z}) = HZ$. As explained in [Bu12], this can be considered as a commutative algebra in $N(\text{Sp})[W^{-1}]$ (inverting stable equivalences on spectra) so that we can form its module category $\text{Mod}_{HZ}$. The homotopy groups of $HZ$ are given by

$$\pi_*(HZ) \cong \begin{cases} \mathbb{Z}, & * = 0 \\ 0, & * \neq 0. \end{cases}$$

A differential refinement of a commutative ring spectrum $R$ is a triple $(R, A, c)$ consisting of a CDGA $A$ over $\mathbb{R}$ together with an equivalence

$$c : R \wedge H\mathbb{R} \xrightarrow{\sim} HA$$

in $\text{CAlg}(\text{Mod}_{H\mathbb{R}})$. Shipley has shown that one can model every $H\mathbb{R}$-algebra by a CDGA [Sh07]. When $R$ has a differential refinement $\hat{R}$ whose underlying CDGA is the graded ring $\pi_*(R) \otimes \mathbb{R}$ with trivial differential, $R \wedge H\mathbb{R}$ is called formal. In this case, there is an equivalence $c$ which is uniquely determined up to homotopy by the property that it induces the canonical identification on homotopy groups [Bu12]. This occurs for $HZ$, for which one can choose a real model whose underlying CDGA is $\mathbb{R}$ concentrated in degree 0 [BN14]. Ordinary cohomology is an example of what Bunke and Nikolaus [BN14] call a differentially simple spectrum. These have the property that there is a very good choice of a differential extension as well as control of differential twists. Assume that $R$ is a differentially simple spectrum and $(R, A, c)$ is the canonical differential extension with $A = \pi_*(R) \otimes \mathbb{R}$ and equivalence $c : R \wedge H\mathbb{R} \xrightarrow{\sim} HA$ in $\text{CAlg}(\text{Mod}_{H\mathbb{R}})$. Then every topological $R$-twist $E$ on a manifold $M$ admits a differential refinement which is unique up to canonical equivalence [BN14 Theorem 9.5].

For ordinary integral cohomology, the twists are classified by $B\mathbb{Z}/2$. The pullback of a map $\eta : M \to B\mathbb{Z}/2$ by the universal $\mathbb{Z}$-bundle over $B\mathbb{Z}/2$ gives a $\mathbb{Z}$-bundle over $M$. The space of sections of this bundle is an infinite loop space and represents the $\eta$-twisted cohomology. Alternatively, we can think of the pullback bundle as a locally constant sheaf on $M$. From this point of view, the constant stack $B\mathbb{Z}/2 \simeq B\mathbb{Z}/2$ classifies all locally trivial $\mathbb{Z}$-bundles over $M$. The local sections of the bundle in this case form a sheaf of spectra over $M$. The cohomology represented by this spectrum can be calculated explicitly by replacing a smooth manifold $M$ with its Čech nerve $C(\{U_\alpha\})$. We can identify the group of connected components of the mapping space $\text{Map}(C(\{U_\alpha\}), B\mathbb{Z}/2)$ with the Čech cohomology group $H^1(M; \mathbb{Z}/2)$ as the group classifying the twists. To obtain cocycle data for a twist, we unravel the bundle data coming from the action of $\mathbb{Z}/2$ on $\mathbb{Z}$.
It has been brought to the authors’ attention that a very interesting variation on the concept of twisted Deligne cohomology has been considered in the algebraic setting. Deligne-Beilinson cohomology with coefficients in a unipotent variations of mixed Hodge structure (VMHS) are considered in [CH89]. In [Ha15] Hain, motivated by Hodge theory and motives, develops Deligne-Beilinson cohomology of affine groups with a mixed Hodge structure. Kapranov has described MHS’s in more geometric terms via certain categories of bundles with connections [Kap2]. Our paper should be considered as an approach to twisting Deligne cohomology from the point of view of differential geometry and algebraic topology.

The construction of surface holonomy of a bundle gerbe on unoriented surfaces and orientifolds can be described using twists for gerbes [SSW07] [GSW01] [G05] [HMSV16]. Translating our constructions to the language of gerbes by writing out the Čech double complex corresponding to our sheaves should recover the cocycle data discussed in the above works. More directly using our language of stacks is the much more general model for “higher orientifolds” given in [FSS15b, Sec. 4.4] to describe involutions arising from M-theory on a manifold with boundary. In the same way that Deligne cohomology is equivalent to gerbes with connections upon unraveling simplicial and cocycle data, our description of twisted Deligne cohomology should likewise be equivalent to twisted gerbes with connection in all the above works. The main point is that these two points of view give different models for twisted differential integral cohomology.

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2 Twists of integral cohomology

Integral cohomology appears as part of the data of ordinary differential cohomology. Therefore, twisting the former might give us some insight into the latter. Note, however, that finding the right way to do the twist is highly nontrivial, as was demonstrated in [BN14]. In this section, we review several models for twisted integral cohomology. The final approach uses the machinery of smooth stacks and fits in to the general machinery for twisted differential cohomology established in [BN14]. This last approach will be used in the discussion for twisted Deligne cohomology in subsequent sections.

We consider twists of integral cohomology at the level of the Eilenberg-MacLane spectrum $HZ$ as follows [MQ177] [ABG10]. The corresponding infinite loop space is $\Omega^\infty HZ \simeq K(\mathbb{Z}, 0) \simeq \mathbb{Z}$. This implies that the group of units is $GL_1(HZ) \simeq \{ \pm 1 \} \simeq \mathbb{Z}/2$, the invertible elements in $\mathbb{Z}$. Delooping then gives $BGL_1(HZ) \simeq B\mathbb{Z}/2 \simeq K(\mathbb{Z}/2, 1)$. Now consider a space $X$ with a twist given by a map $X \to BGL_1(HZ)$. This can be viewed as an obstruction to orientation of a vector bundle $E$ over $X$ with respect to singular cohomology. This is given by the composite map that factors via the $J$-homomorphism through $BGL_1(S)$. Here $S$ is the sphere spectrum, which is the unit for any spectrum, including $HZ$, so that there is always a map $S \to HZ$. That composite obstruction map is

$$X \xrightarrow{E} BO \xrightarrow{BJ} BGL_1(S) \xrightarrow{BGL_1(HZ)} BGL_1(HZ) \simeq K(\mathbb{Z}/2, 1).$$

This class $X \to K(\mathbb{Z}/2, 1)$ is, in fact, the first Stiefel-Whitney class $w_1(X)$. 

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The above homotopic description of the twist can be described geometrically as follows [Fr01].

One-dimensional twists of integral cohomology is given by a local system $Z \to M$, which is a bundle of groups isomorphic to $Z$. Hence, the twists are determined by an element of

$$H^1(M; \text{Aut}(Z)) \cong H^1(M; Z_2),$$

since the only nontrivial automorphism of $Z$ is multiplication by $-1$. Twisted integral cohomology may be thought of as the sheaf cohomology $H^*(M; Z)$, taken with respect to the local system $Z$. The Čech description goes as follows (see [Fr01]). Let $\{U_\alpha\}$ be an open covering of $M$ and $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \{\pm 1\}$ be a cocycle defining the local system $Z$. Then an element of $H^q(M; Z)$ is represented by a collection of $q$-cochains $a_i \in Z^q(U_\alpha)$ which satisfy

$$a_\beta = g_{\alpha\beta} a_\alpha \quad \text{on} \quad U_{\alpha\beta} = U_\alpha \cap U_\beta. \quad (2.1)$$

One can also describe the situation in spaces via Eilenberg-MacLane spaces (see [Fr01]). More precisely, one uses a model of cochains as maps to Eilenberg-MacLane space $K(Z, q)$. The automorphisms $\text{Aut}(Z) \cong \{\pm 1\}$ act on $K(Z, q)$ for each $q \geq 0$. For example, $K(Z, 0) \cong Z$ on which $-1$ acts by multiplication, $K(Z, 1) \cong S^1$ on which $-1$ acts by reflection. The action of $\text{Aut}(Z)$ on $K(Z, q)$ and the cocycle $g_{\alpha\beta}$ gives rise to an associated bundle $\mathcal{H}^q \to M$ with fiber $K(Z, q)$. Eq. (2.1) says that twisted cohomology classes are represented by sections of $\mathcal{H}^q \to M$. The twisted cohomology group $H^q(M; Z)$ is the set of homotopy classes of sections of $\mathcal{H}^q \to M$.

A third approach uses smooth stacks and unifies the two previous perspectives: namely, the Čech cocycle approach and the approach via Eilenberg-MacLane spaces (see [PSSt12] [PSSt13] [PSSt15a] for detailed discussions and applications). This approach is both extremely general and versatile, and we will rely on it when dealing with Deligne cohomology in subsequent sections. To that end, let $\mathcal{C}artSp$ denote the category with objects that are convex open subsets of $\mathbb{R}^n$ with $n \geq 0$ and morphisms that are smooth maps between them. As a coverage on this small category we take the covering families to be good open covers (i.e. covers with contractible finite intersections). Now let $\mathcal{B}Z/2$ denote the smooth stack on this site obtained by delooping the constant sheaf $\mathbb{Z}/2$. For example, the Dold-Kan image of the sheaf of chain complex $\mathbb{Z}[2][1]$ can serve as a model.

It turns out that $\mathcal{B}Z/2$ is a fibrant object in the local projective model structure. That is, it is objectwise a Kan-complex and satisfies descent with respect to Čech hypercovers [DHI04]. Therefore, the connected components of the mapping space $\text{Map}(M, \mathcal{B}Z/2)$ can be calculated by replacing the manifold $M$ by the homotopy colimit over the nerve of an open good open cover $\{U_\alpha\}$ (see [PSSt12] for details). Let $\mathcal{C}(\{U_\alpha\})$ denote this homotopy colimit. A map

$$\xymatrix{ M \ar[d] \ar[r] & \mathcal{B}Z/2 \ar@{<-}[d]^q \\
\mathcal{C}(\{U_\alpha\}) \ar[r]^g & \mathbb{Z}/2 \quad (2.1)$$

determines and is uniquely determined by a Čech 1-cocycle with coefficients in $\mathbb{Z}/2$. We, therefore, have an isomorphism

$$\pi_0 \text{Map}(\mathcal{C}(\{U_\alpha\}), \mathcal{B}Z/2) \cong \hat{H}^1(\{U_\alpha\}; \mathbb{Z}/2).$$

The stacky perspective also makes it very transparent how the local system $Z$ and the bundles $\mathcal{H}^q$ are related. Indeed, we can similarly define the locally constant, smooth stacks $\mathcal{B}^qZ$, which model higher integral Čech-cocycles of degree $q$. The action of $\text{Aut}(Z) \cong \mathbb{Z}/2$ on these stacks gives
rise to an action groupoid $\mathcal{B}^q\mathbb{Z}/(\mathbb{Z}/2)$, and in turn one can take the homotopy orbit stack associated to this groupoid (see [NSS15] for details). The resulting smooth stack models $\mathbb{Z}/2$-bundles with fiber the ‘stacky Eilenberg-MacLane spaces’ $\mathcal{B}^q\mathbb{Z}$. Moreover, this bundle is universal in the sense that any such bundle over $M$ fits into a pullback square

$$
\begin{array}{c}
\mathcal{H}^q \\
\pi \\
M
\end{array} \longrightarrow \begin{array}{c}
\mathbb{B}^q\mathbb{Z}/(\mathbb{Z}/2) =: \text{Tw}(\mathbb{B}^q\mathbb{Z}) \\
\pi' \\
\mathbb{B}\mathbb{Z}/2
\end{array}
$$

(2.2)

defining the twisting stack $\text{Tw}(\mathbb{B}^q\mathbb{Z})$. Here the right vertical map is the canonical map which projects out $\mathbb{B}^q\mathbb{Z}$ and we have identified the homotopy orbit stack $*/(\mathbb{Z}/2)$ with $\mathbb{B}\mathbb{Z}/2$. Note that we can compute this stack as the homotopy colimit over the nerve of the action

$$
\text{Tw}(\mathbb{B}^q\mathbb{Z}) \simeq \text{hocolim}\{ \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathcal{B}^q\mathbb{Z} \to \mathbb{Z}/2 \times \mathcal{B}^q\mathbb{Z} \to \mathcal{B}^q\mathbb{Z} \}. 
$$

**Definition 1.** Define the $q$-th $\eta$-twisted cohomology as the connected components of the simplicial space of sections $\pi_0\Gamma(M, \mathcal{H}^q)$.

The smooth stacks $\mathcal{H}^q$ are relatively easy to describe using descent. Indeed, fix a cover $\{U_\alpha\}$ of $M$ and observe that on each $k$-fold intersection, any map $\eta : U_{\alpha_0...\alpha_k} \to \mathbb{B}\mathbb{Z}/2$ factors through the basepoint, since $\mathbb{B}\mathbb{Z}/2$ is constant on covers. Consequently, we have

$$
\mathcal{H}^q_{U_{\alpha_0...\alpha_k}} \simeq \mathbb{B}^q\mathbb{Z} \times U_{\alpha_0...\alpha_k}.
$$

Now since $\mathbb{B}\mathbb{Z}/2$ is a 1-type, descent implies that we can compute $\mathcal{H}^q$ as the coequalizer

$$
\mathcal{H}^q \simeq \text{coeq}\{ \bigsqcup_{\alpha, \beta} \mathbb{B}^q\mathbb{Z} \times U_{\alpha \beta} \xrightarrow{i_\alpha \eta_{\alpha \beta}} \bigsqcup_{\alpha} \mathbb{B}^q\mathbb{Z} \times U_{\alpha} \}
$$

where $i$ is the inclusion and $\eta_{\alpha \beta}$ is the Čech cocycle determined by $\eta : M \to \mathbb{B}\mathbb{Z}/2$. Then the global sections of $\mathcal{H}^q$ can be identified with a choice of Čech cocycle $a_\alpha$ on each open set $U_\alpha$ such that $a_\alpha = \eta_{\alpha \beta} a_\beta$ on intersections of the cover. We, therefore, recover Freed’s description in [Fr01].

We can also recover the local system $Z$ via the sheaf of sections of the bundle $\mathcal{H}^0$. Furthermore, we can see that the sheaf cohomology of $Z$ can be computed as the components of simplicial space of sections of the bundles $\mathcal{H}^q$. We thus have the following equivalent characterizations of twisted integral cohomology.

**Proposition 2.** Let $\eta : M \to \mathbb{B}\mathbb{Z}/2$ be a twist for integral cohomology and let $Z$ be the locally constant sheaf associated to the twist. Then twisted integral cohomology is given by the isomorphism

$$
H^q_{\eta}(M; Z) := \pi_0\Gamma(M, \mathcal{H}^q) \cong H^q(M; Z).
$$

### 3 Deligne cohomology

We begin by recalling the definition of smooth Deligne cohomology (see [De71] [Be85] [Gi84] [Ja88] [EV88] [Ga97]). Let $\mathcal{D}(n)$ denote the sheaf of chain complexes

$$
\mathcal{D}(n) := ( \ldots \to 0 \to \mathbb{Z} \to \Omega^0 \to \Omega^1 \to \ldots \to \Omega^{n-1} ).
$$
with differential \((n-1)\)-forms in degree 0 and locally constant integer valued functions in degree \(n\). For a smooth manifold \(M\), the Deligne cohomology group of degree \(n\) is defined to be the sheaf cohomology group \(\hat{H}^n(M;\mathbb{Z}) := H^0(M;\mathcal{D}(n))\). These cohomology groups can be explicitly calculated via a Čech resolution. More precisely, if \(\{U_\alpha\}\) is a good open cover of \(M\), then we can form the Čech-Deligne double complex

\[
\begin{array}{ccccccc}
\mathbb{Z}(U_{\alpha_0\ldots\alpha_n}) & \xrightarrow{2\pi i} & \Omega^0(U_{\alpha_0\ldots\alpha_n}) & \xrightarrow{d} & \Omega^1(U_{\alpha_0\ldots\alpha_n}) & \xrightarrow{d} & \ldots & \xrightarrow{d} & \Omega^{n-1}(U_{\alpha_0\ldots\alpha_n}) \\
\downarrow{(1)^{n-1}\delta} & & \downarrow{(1)^{n-1}\delta} & & \downarrow{(1)^{n-1}\delta} & & \downarrow{(1)^{n-1}\delta} \\
\mathbb{Z}(U_{\alpha_0\ldots\alpha_{n-1}}) & \xrightarrow{2\pi i} & \Omega^0(U_{\alpha_0\ldots\alpha_{n-1}}) & \xrightarrow{d} & \Omega^1(U_{\alpha_0\ldots\alpha_{n-1}}) & \xrightarrow{d} & \ldots & \xrightarrow{d} & \Omega^{n-1}(U_{\alpha_0\ldots\alpha_{n-1}}) \\
\downarrow{(1)^{n-2}\delta} & & \downarrow{(1)^{n-2}\delta} & & \downarrow{(1)^{n-2}\delta} & & \downarrow{(1)^{n-2}\delta} \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\mathbb{Z}(U_{\alpha_0\alpha_1}) & \xrightarrow{2\pi i} & \Omega^0(U_{\alpha_0\alpha_1}) & \xrightarrow{d} & \Omega^1(U_{\alpha_0\alpha_1}) & \xrightarrow{d} & \ldots & \xrightarrow{d} & \Omega^{n-1}(U_{\alpha_0\alpha_1}) \\
\downarrow{\delta} & & \downarrow{\delta} & & \downarrow{\delta} & & \downarrow{\delta} \\
\mathbb{Z}(U_{\alpha_0}) & \xrightarrow{2\pi i} & \Omega^0(U_{\alpha_0}) & \xrightarrow{d} & \Omega^1(U_{\alpha_0}) & \xrightarrow{d} & \ldots & \xrightarrow{d} & \Omega^{n-1}(U_{\alpha_0}) 
\end{array}
\tag{3.1}
\]

where \(U_{\alpha_0\ldots\alpha_k}\) denotes the \(k\)-fold intersection. The total operator on the double complex is the Čech-Deligne operator \(D := d + (-1)^p\delta\), where \(d\) and \(\delta\) is the de Rham and Čech differentials, respectively, acting on elements of degree \(p\). The sheaf cohomology group \(H^0(M;\mathcal{D}(n))\) can be identified with the group of diagonal elements \(\alpha_{k,k}\) in the double complex which are Čech-Deligne closed in the sense that \((d + (-1)^p\delta)\alpha_{k,k} = 0\), modulo those which are Čech-Deligne exact.

Deligne cohomology satisfies most of the properties that an ordinary cohomology theory satisfies (such as functoriality and the Mayer-Vietoris sequence). However, one needs to be careful when using these properties. For example, the form that the Mayer-Vietoris sequence takes is slightly different from what one might expect. The following proposition is fairly classical – a proof for each property can be found in the more modern treatment via differential cohomology in [Bu12].

**Proposition 3 (Properties of Deligne cohomology).** Deligne cohomology satisfies the following properties:

(i) (Functoriality) For a smooth map between manifolds \(M \to N\), we have an induced map

\[
\hat{H}^n(N;\mathbb{Z}) \to \hat{H}^n(M;\mathbb{Z}).
\]

(ii) (Additivity) For \(M = \bigsqcup M_\alpha\) a disjoint union of smooth manifolds, we have an isomorphism

\[
\hat{H}^n(M;\mathbb{Z}) \cong \bigoplus_\alpha \hat{H}^n(M_\alpha;\mathbb{Z}).
\]

(iii) (Mayer-Vietoris) For an open cover of \(M\) by open smooth manifolds \(U\) and \(V\), we have a sequence

\[
\begin{array}{cccccccc}
\ldots & \to & H^{*-2}(U \cap V;\mathbb{R}/\mathbb{Z}) & \to & \hat{H}^*(M;\mathbb{Z}) & \to & \hat{H}^*(U;\mathbb{Z}) \oplus \hat{H}^*(U;\mathbb{Z}) & \to & \\to & \hat{H}^*(U \cap V;\mathbb{Z}) & \to & H^{*-1}(M;\mathbb{Z}) & \to & \ldots
\end{array}
\]
Note that the Mayer-Vietoris sequence has ordinary integral cohomology on the right and cohomology with $\mathbb{R}/\mathbb{Z}$-coefficients on the left. This effect is an artefact of the way the theory is constructed. More precisely, the Deligne complex depends on an integer $n$, which indexes the degree of the underlying cohomology group. The Mayer-Vietoris sequence comes from the sheaf cohomology of a fixed Deligne complex $D(n)$ and this is why we see the full differential cohomology group in only one degree. In fact, Deligne cohomology is really a mixture of three different cohomology theories (integral, $\mathbb{R}/\mathbb{Z}$-coefficients, and de Rham) and captures the interactions between these theories. These interactions can be understood via the “differential cohomology diagram”

\[
\begin{array}{ccc}
\Omega^{*}(\text{im}(d)) & \xrightarrow{d} & \Omega_{\text{cl}}^{*}(M) \\
H^{*}_{\text{dR}}(M) & \xrightarrow{a} & \tilde{H}^{*}(M; \mathbb{Z}) \\
H^{*}_{\text{dR}}(M) & \xrightarrow{\beta} & H^{*}(M; \mathbb{Z}) \\
H^{*}_{\text{dR}}(M) & \xrightarrow{j} & H^{*}(M; \mathbb{Z}) \\
\end{array}
\]

where $d$ is the de Rham differential, $R$ is the curvature map, $I$ is the forgetful map, $j$ is the rationalization, and $\beta$ is the Beckstein associated with the exponential coefficient sequence. This diamond (or hexagon) diagram was originally introduced and emphasized by Simons and Sullivan in [SS08] and for more generalized theories, a full characterization via this diamond was proved in [BNV16]. Parts of it appear in the foundational work of Cheeger and Simons [CS85].

**4 Twists of Deligne cohomology**

We now discuss the twists for Deligne cohomology. Since Deligne cohomology is a combination of de Rham cohomology and integral cohomology, the twists will be some sort of combination of the integral twists and twists via differential forms. Note that we have an obvious inclusion of the integral twists

\[\text{Aut}(\mathbb{Z}) \cong \mathbb{Z}/2 \hookrightarrow \mathbb{R}^{\times} \cong \text{Aut}(\mathbb{R}).\]

Delooping this map gives a map between the classifying stacks of units

\[r : B\mathbb{Z}/2 \hookrightarrow B\mathbb{R}^{\times}.\]

In spaces this map would be an equivalence, but since we are in stacks the geometry prevents this map from defining an equivalence of smooth stacks. This map instead simply rationalizes (or realifies) the twists and can be viewed as a first approximation to the differential refinement.

The stack $B\mathbb{R}^{\times}$ classifies smooth, locally trivial, real line bundles over a smooth manifold and this bundle is determined (up to isomorphism of bundles) by the pullback square

\[
\begin{array}{ccc}
\mathcal{L}_{\eta} & \xrightarrow{\eta} & \mathbb{R}/\mathbb{R}^{\times} \\
\downarrow & \downarrow & \downarrow \\
M & \xrightarrow{\eta} & B\mathbb{R}^{\times}. \\
\end{array}
\]

Moreover, given a homotopy equivalence between two maps $\eta : M \rightarrow B\mathbb{R}^{\times}$ and $\xi : M \rightarrow B\mathbb{R}^{\times}$, we have an induced isomorphism of line bundles $\mathcal{L}_{\eta} \cong \mathcal{L}_{\xi}$. This correspondence gives rise to an equivalence of $\infty$-groupoids

\[\text{Map}(M, B\mathbb{R}^{\times}) \cong \text{Line}(M),\]
where on the right we have the \(\infty\) groupoid of line bundles on \(M\). In a similar way, maps to \(\mathbf{BZ}/2\) classify \(\mathbb{Z}\)-bundles on \(M\) and, given a map \(\eta : M \to \mathbf{BZ}/2\), we have a commutative diagram

\[
\begin{array}{ccc}
\mathcal{L}_\eta & \xrightarrow{j} & \mathbb{R}/\mathbb{R}^x \\
\downarrow & & \downarrow \\
Z/\mathbb{Z}/2 & \xrightarrow{\eta} & \mathbf{B} \mathbb{R}^x \\
\end{array}
\]

where \(Z := \mathcal{H}^0\) is the \(\mathbb{Z}\)-bundle classified by the map \(\eta\). The map \(j : Z \to \mathcal{L}_\eta\) is the bundle map which includes the \(\mathbb{Z}\)-bundle classified by \(\eta\) as a subbundle of the real line bundle classified by the same twist.

To arrive at the twists of the differential refinement, we need to include the differential form data for the rationalization. The crucial ingredient for forming the twisted theory is provided by the Riemann-Hilbert correspondence. Fix a flat connection \(\nabla\) on a line bundle \(\mathcal{L} \to M\) and let \(\ker(\nabla)\) denote the sheaf of solutions to the parallel transport equation \(\nabla s = 0\). The flat connection allows us to make an identification of sheaves

\[
\ker(\nabla) \simeq \Gamma(-; \mathcal{L}^\delta),
\]

where the sheaf on the right is the sheaf of sections of the bundle obtained by equipping the fibers of \(\mathcal{L} \to M\) with the discrete topology and taking transition functions induced by the differences of parallel section on intersections. This identification is at the core of the construction and we will revisit it in detail later. We begin by introducing the moduli stack of line bundles with flat connection. Let \(\mathcal{L}^\delta := \mathfrak{C}^{\infty}(-; \mathbb{R}^x)\) be the sheaf of smooth plots of the punctured real line and let \(\Omega^1_{cl}(-)\) be the sheaf of closed 1-forms on the small site of cartesian spaces. The logarithm map \(d\log : \mathbb{R}^x \to \Omega^1_{cl}(-)\) gives a smooth action of the sheaf of groups \(\mathbb{R}^x\) on \(\Omega^1_{cl}(-)\), via the assignment \(\omega \mapsto \omega + d\log(f)\), for sections \(\omega \in \Omega^1_{cl}(-)\) and \(f \in C^\infty(-; \mathbb{R}^x)\).

**Definition 4.** We define the moduli stack of line bundles with flat connection as the smooth stack

\[
\mathbf{bB} \mathbb{R}^x_{\nabla} := \Omega^1_{cl}(-) \//\mathbb{R}^x,
\]

where the homotopy orbit stack on the right is taken with respect to the action. Alternatively, this smooth stack is presented as the image of the positively graded sheaf of chain complexes

\[
\begin{array}{cccccccc}
\ldots & \to & 0 & \to & \mathbb{R}^x & \xrightarrow{d\log} & \Omega^1_{cl}(-) & \to & \ldots
\end{array}
\]

under the Dold-Kan functor \(DK : \text{Ch}^+ \to \text{sAb} \to \text{sSet}\).

One can show (see for example [FSSt12]) that we have an equivalence of \(\infty\)-groupoids

\[
\text{Map}(M, \mathbf{bB} \mathbb{R}^x_{\nabla}) \simeq \mathbf{bLine}(M),
\]

where the \(\infty\)-groupoid on the right is that of line bundles, equipped with flat connection. Let \(\mathbb{R}^x\) denote the constant smooth stack which associates each element of a good open cover \(U_\alpha \in \{U_\alpha\}\)

\[\text{(Note that we are working over the small site of Cartesian spaces, so stackification of this prestack is not necessary.)}\]
to the group $\mathbb{R}^\times$. \footnote{Note that this is different than the smooth stack $\mathbb{R}^\times : = C^\infty (\mathbb{R}^\times)$. It is obtained by regarding $\mathbb{R}^\times$ as a discrete group.} This stack classifies discrete bundles with discrete fiber $\mathbb{R}^\delta$, where here $\mathbb{R}$ is equipped with the discrete topology. By the Poincaré Lemma, the morphism
\[
\mathcal{B}\mathbb{R}^\times \rightarrow \mathcal{B}\mathbb{R}^\times^\delta,
\]
from the classifying stack of discrete $\mathbb{R}$-bundles to the classifying stack of $\mathbb{R}$-bundles with flat connection, defines an equivalence on every element of a good open cover. By descent, this implies that we have an equivalence of smooth stacks. In summary, we have an induced diagram
\[
\begin{array}{ccc}
\text{Map}(M, \mathcal{B}\mathbb{R}^\times) & \longrightarrow & \delta\text{Line}(M) \\
\downarrow & & \downarrow \\
\text{Map}(M, \mathcal{B}\mathbb{R}^\times^\delta) & \longrightarrow & \delta\text{Line}(M),
\end{array}
\]
where the right vertical map is defined by the composition and all the maps involved are homotopy equivalences of Kan-complexes. It is well-known that the Riemann-Hilbert correspondence defines a functor
\[
\text{RH} : \delta\text{Line}(M) \longrightarrow \delta\text{Line}(M),
\]
which associated each line bundle with flat connection to its corresponding local system. Such a local system is equivalently an $\mathbb{R}^\delta$-bundle over $M$ and it is easy to see that RH defines a homotopy inverse to the right vertical map in (4.1). Restricting to Cartesian spaces, this map induces a morphism of smooth stacks
\[
\text{RH} : \mathcal{B}\mathbb{R}^\times^\delta \longrightarrow \mathcal{B}\mathbb{R}^\times. \tag{4.2}
\]
This map will help us to identify the twists of the differential refinement, as we shall see.

Essentially the story we are spelling out is that of a twisted de Rham theorem, provided by the Riemann-Hilbert correspondence. More concretely, we have the following.

**Proposition 5.** Let $\nabla$ be a flat connection on a line bundle $\mathcal{L} \rightarrow M$ and let $\Omega^*(\mathcal{L})$ denote the corresponding de Rham complex with coefficients in $\mathcal{L} \rightarrow M$ and differential $\nabla$. Let $\mathcal{L}^\delta \rightarrow M$ be the $\mathbb{R}^\delta$-bundle obtained by taking $\mathcal{L} \rightarrow M$ to have discrete fibers and constant transition functions defined via local parallel sections. Then we have a resolution
\[
j : \mathcal{L}^\delta \longrightarrow \Omega^*(\mathcal{L}).
\]
In particular, since $\Omega^*(\mathcal{L})$ are fine sheaves, this implies that we have an isomorphism
\[
H^k(M; \mathcal{L}^\delta) \cong \frac{\ker(\nabla : \Omega^k(M; \mathcal{L}) \rightarrow \Omega^{k+1}(M; \mathcal{L}))}{\text{im}(\nabla : \Omega^{k-1}(M; \mathcal{L}) \rightarrow \Omega^k(M; \mathcal{L}))}. \tag{4.3}
\]

**Proof.** Since $\mathcal{L}^\delta$ is a locally constant sheaf of vector spaces, the sheaf gluing condition implies that for every $U \in \text{Open}(M)$, the sections $\Gamma(U; \mathcal{L}^\delta)$ appear as the kernel
\[
\Gamma(U; \mathcal{L}^\delta) \subset \prod_{\alpha} \mathbb{R}^{\iota_{\alpha\beta} - \eta_{\alpha\beta} + \iota_{\alpha\beta} \alpha} \prod_{\alpha\beta} \mathbb{R}. \tag{4.4}
\]
for some good open cover $\{U_\alpha\}$ of $U$, where $\eta_{\alpha\beta} \in \mathbb{R}^\times$ are the transition functions of the bundle $\mathcal{L}^\delta$. By the fundamental theorem of ODE’s, there are nonvanishing local solutions $e_\alpha$ to the equation...
\( \nabla(e_\alpha) = 0 \) and all such solutions are parametrized by the fiber \( \mathbb{R} \). The \( e_\alpha \)'s define local trivializations \( \phi_\alpha : \mathbb{R} \times U_\alpha \to \mathcal{L}|_{U_\alpha} \) by the assignment \((r, x) \mapsto re_\alpha(x)\). From these observations, we see that the induced map

\[ j_\alpha : \mathbb{R} \to \Gamma(U_\alpha; \mathcal{L}), \]

which maps an element \( r \in \mathbb{R} \) to the corresponding unique solution \( e_\alpha \), exhibits \( \Omega^*(U_\alpha; \mathcal{L}) \) as a resolution of \( \mathbb{R} \). Now consider the diagram

\[
\begin{array}{ccccccccc}
0 & \to & \Gamma(U; \mathcal{L}^\delta) & \overset{j}{\to} & \Omega^0(U; \mathcal{L}) & \overset{\nabla}{\to} & \Omega^1(U; \mathcal{L}) & \to & \cdots \\
0 & \to & \prod_\alpha \mathbb{R} & \overset{j_\alpha}{\to} & \prod_\alpha \Omega^0(U_\alpha; \mathcal{L}) & \to & \prod_\alpha \Omega^1(U_\alpha; \mathcal{L}) & \to & \cdots \\
0 & \to & \prod_\alpha \mathbb{R} & \overset{j_\alpha}{\to} & \prod_\alpha \Omega^0(U_\alpha; \mathcal{L}) & \to & \prod_\alpha \Omega^1(U_\alpha; \mathcal{L}) & \to & \cdots \\
0 & \to & 0 & \to & 0 & \to & 0 & \to & 0
\end{array}
\]

where the vertical sequences are short exact and the bottom two horizontal rows are exact. The map \( j \) is the map induced by the universal property of the kernel. The diagram commutes as the transition functions \( \eta_{\alpha\beta} \) were defined via the local sections \( e_\alpha \). The top horizontal sequence is natural in \( U \) and we have a sequence of sheaves

\[
0 \to \mathcal{L}^\delta \overset{j}{\to} \Omega^0(-; \mathcal{L}) \overset{\nabla}{\to} \Omega^1(-; \mathcal{L}) \to \cdots
\]

Note that the above diagram \( \text{(4.5)} \) holds for all refinements of the chosen cover. Using this diagram, along with the fact that the image sheaf is the sheafification of the image in presheaves, a quick diagram chase reveals that the above sequence is indeed a resolution of sheaves. \( \square \)

**Remark 1** (Twisted de Rham theorem). We define the \( \nabla \)-twisted de Rham cohomology groups, \( H^*(M; \nabla) \), as the quotient on the right hand side of \( \text{(4.3)} \). Thus, the above proposition states the we have a twisted de Rham isomorphism theorem

\[
H^*(M; \mathcal{L}^\delta) \cong H^*(M; \nabla).
\]

Note that we could also define the associated discrete bundle \( \mathcal{L}^\delta \) as the bundle associated to the monodromy representation of the flat connection \( \rho : \pi_1(M) \to \mathbb{R}^\times \). For the twists of the differential refinement, we need to require that this monodromy representation factors through the units of \( \mathbb{Z} \). That is, we have

\[
\rho : \pi_1(M) \to \mathbb{Z}/2 \to \mathbb{R}^\times.
\]

This imposes a restriction on the types of flat connections we can choose on the bundle \( \mathcal{L} \). If we start with a twist for integral cohomology \( \eta : M \to \mathbb{BZ}/2 \) giving the transition functions of a real line bundle \( \mathcal{L} \to M \), then \( \nabla \) must be compatible with this structure. Slight modifications of the proof of Proposition \( \text{5} \) yield the following.

**Proposition 6.** Let \( \eta : M \to \mathbb{BZ}/2 \) be a twist of integral cohomology and let \( \mathcal{L}_\eta \to M \) be the real line bundle classified by \( \eta \). Let \( \rho : \pi_1(M) \to \mathbb{Z}/2 \) be the map corresponding to the homotopy class of \( \eta \) and let \( \nabla \) be a flat connection associated to this monodromy representation. Then we have a resolution

\[
j : \mathcal{L}_\eta^\delta \to \Omega^*(-; \mathcal{L}_\eta),
\]

where \( \mathcal{L}_\eta^\delta \) is the locally constant sheaf obtained via the sheaf of sections of the discrete bundle \( \mathcal{L}_\eta^\delta \to M \), obtained by regarding the bundle \( \mathcal{L}_\eta \to M \) as having fiber \( \mathbb{R} \), equipped with the discrete topology.

\[\text{The sheafification is computed as a limit over refinements of covers and here we know that the sequences are exact.}\]

\[\text{The associated map here is provided via the adjunction } [M, \mathbb{BZ}/2] \cong [\pi_1(M), \mathbb{BZ}/2] \cong \hom(\pi_1(M), \mathbb{Z}/2).\]
Given the information in Proposition [4] i.e. a smooth map \( \eta : M \to \mathbb{B} \mathbb{Z}/2 \) classifying a line bundle \( \mathcal{L}_\eta \to M \), a flat connection \( \nabla \) on \( \mathcal{L}_\eta \) and a resolution \( j : \mathcal{L}_\eta^\delta \hookrightarrow \Omega^*(-;\mathcal{L}_\eta) \), we can define the twisted Deligne complex as follows.

**Definition 7.** (i) Given a triple \( \nabla := (\eta, \nabla, j) \) as described above, we define the twisted Deligne complex as the sheaf of chain complexes on \( M \)

\[
\mathcal{D}_\nabla(n) := \left( \ldots \to 0 \to \mathcal{Z} \xrightarrow{\eta^j} \Omega^0(-;\mathcal{L}_\eta) \xrightarrow{\nabla} \Omega^1(-;\mathcal{L}_\eta) \xrightarrow{\nabla} \ldots \to \Omega^{n-1}(-;\mathcal{L}_\eta) \right) ,
\]

where \( \Omega^k(-;\mathcal{L}_\eta) := \Omega(-;\Lambda^k(T^*M) \otimes \mathcal{L}_\eta) \) denotes the sheaf of local sections of the bundle and \( Z \) is the local system associated to the \( \mathbb{Z} \)-bundle classified by \( \eta \).

(ii) We define the \( \nabla \)-twisted Deligne cohomology of \( M \) to be the sheaf hypercohomology group

\[
\hat{H}^n(M;\nabla) := H^0(M;\mathcal{D}_\nabla(n)) .
\]

The twists for the Deligne complex can be organized into a smooth stack themselves.

**Definition 8.** We define the stack of twists for the Deligne complex as the \((\infty, 1)\)-pullback

\[
\begin{CD}
\mathcal{B}(\mathbb{Z}/2)_{\nabla} @>>> \mathcal{B}(\mathbb{B} \mathbb{R}^2_{\nabla}) \\
\@VVV @VV{\text{RH}}V \\
\mathbb{B} \mathbb{Z}/2 @>>> \mathbb{B} \mathbb{R}^2 .
\end{CD}
\]

This smooth stack indeed defines the necessary information.

**Proposition 9.** A map \( M \to \mathcal{B}(\mathbb{Z}/2)_{\nabla} \) determines and is uniquely determined by the following data:

1. A discrete \( \mathbb{R}^d \)-bundle \( \mathcal{L}_\eta^\delta \to M \) classified by a map \( \eta : M \to \mathbb{B} \mathbb{Z}/2 \) and a \( \mathbb{Z} \)-subbundle \( Z \to M \).
2. A flat connection \( \nabla \) on a line bundle \( \mathcal{L} \to M \).
3. An isomorphism of local systems \( j : \mathcal{L}_\eta^\delta \cong \ker(\nabla) \) giving rise to a resolution \( j : \mathcal{L}_\eta^\delta \to \Omega^*(-;\mathcal{L}) \).

**Proof.** A map \( M \to \mathcal{B}(\mathbb{Z}/2)_{\nabla} \) can be identified with a pair of maps \( \eta : M \to \mathbb{B} \mathbb{Z}/2 \) and \( \nabla : M \to \mathcal{B}(\mathbb{B} \mathbb{R}^2_{\nabla}) \) such that \( \text{RH}(\nabla) \) and \( r(\eta) \) are connected by an edge in \( \text{Map}(M, \mathcal{B} \mathbb{R}^2) \). An edge in this mapping space can be identified with an isomorphism of corresponding \( \mathbb{R}^d \)-bundles. Thus, we must have an isomorphism of corresponding local systems

\[
j : \mathcal{L}_\eta^\delta \to \ker(\nabla) .
\]

By definition, \( \Omega^*(-;\mathcal{L}) \) resolves \( \ker(\nabla) \) and therefore the isomorphism gives rise to the desired resolution.\( \square \)

**Remark 2.** Since the map \( \text{RH} \) in definition [8] is an equivalence, the induced map \( \mathcal{B}(\mathbb{Z}/2)_{\nabla} \to \mathbb{B}(\mathbb{Z}/2) \) is also an equivalence. Thus, up canonical equivalence, there is a unique differential refinement of any topological twist \( \eta \).

We illustrate the definition with the following example.

**Example 1** (Punctured complex plane). Let \( M \) be the punctured complex plane \( \mathbb{C} - \{0\} \). There are two isomorphism classes of real line bundles on \( \mathbb{C} - \{0\} \), classified by \( H^1(\mathbb{C} - \{0\}; \mathbb{Z}/2) \cong \mathbb{Z}/2 \): the trivial bundle and the Möbius bundle. Let \( \mathcal{L} \to \mathbb{C} - \{0\} \) denote the Möbius bundle and \( \nabla \) be a flat connection compatible with the monodromy representation defined by sending \( 1 \in \mathbb{Z} \cong \pi_1(S^1) \) to \( -1 \in \mathbb{Z}/2 \). Notice that this representation also defines the principal \( \mathbb{Z}/2 \)-bundle associated to the Möbius bundle over \( \mathbb{C} - \{0\} \). Consider the open cover \{U, V\} of \( \mathbb{C} - \{0\} \) obtained by removing the rays \( x > 0 \) and \( x < 0 \), where \( z = x + iy \). Let \( s_U \) be the local section traversing one edge of Möbius strip on \( U \) and \( s_U \) be the local section traversing the same edge on \( V \).
Then \( s_U \) and \( s_V \) define local trivializations, in which \( \nabla = d \). In this case, the twisted Deligne complex takes the form

\[
D_{\nabla}(n) := \left( \ldots \longrightarrow 0 \rightarrow \mathbb{Z} \xrightarrow{j} \Omega^0(\; ; \mathcal{L}) \xrightarrow{\nabla} \Omega^1(\; ; \mathcal{L}) \xrightarrow{\nabla} \ldots \xrightarrow{\nabla} \Omega^n(\; ; \mathcal{L}) \right)
\]

where \( Z \) is the sheaf of sections of the \( Z \)-subbundle of the Möbius bundle. Of course, for dimension reasons, we only need to consider \( D_{\nabla}(n) \) up to degree \( n = 2 \). Locally this complex is isomorphic to the untwisted Deligne complex and the isomorphism is defined by \( s_U \) and \( s_V \). For a general nonvanishing local sections \( \sigma_U \) and \( \sigma_V \), the complex will be isomorphic (over \( U \) for example) to

\[
D_{\nabla}(n) := \left( \ldots \longrightarrow 0 \rightarrow \mathbb{Z} \xrightarrow{j} \Omega^0(\; ; \mathcal{L}) \xrightarrow{d + df_U} \Omega^1(\; ; \mathcal{L}) \xrightarrow{d + df_U} \ldots \xrightarrow{d + df_U} \Omega^n(\; ; \mathcal{L}) \right),
\]

with \( f_U \) a smooth function such that \( f_U s_U = \sigma_U \).

**Remark 3.** Note that there is a canonical map from the twisted Deligne cohomology groups to the twisted de Rham cohomology groups. Indeed, for any line bundle \( \mathcal{L} \rightarrow M \) and connection \( \nabla \), we have a morphism of complexes

\[
\ldots \longrightarrow 0 \rightarrow \mathbb{Z} \xrightarrow{j} \Omega^0(\; ; \mathcal{L}) \xrightarrow{\nabla} \Omega^1(\; ; \mathcal{L}) \xrightarrow{\nabla} \ldots \xrightarrow{\nabla} \Omega^n(\; ; \mathcal{L}) \]

(4.6)

since \( \nabla^2 = 0 \). Here, \( \Omega^n(\; ; \mathcal{L}) \) denotes the subsheaf \( \ker(\nabla : \Omega^n(\; ; \mathcal{L}) \rightarrow \Omega^{n+1}(\; ; \mathcal{L})) \).

**Remark 4.** The map (4.6) then induces a map \( R : \widehat{H}^n(M; \nabla) \rightarrow H^d_{\text{dR}}(M; \nabla) \). Note that \( R \) is in fact natural in \( M \) in the following sense. Let \( f : M \rightarrow N \) be a smooth map and fix a twist \( \nabla \) on \( N \). Let \( f^*(\nabla) \) be the pullback of the connection on the line bundle \( f^*(\mathcal{L}) \). Via functoriality of sheaf cohomology, we get an induced map

\[
f^* : \widehat{H}^n(N; \nabla) \rightarrow \widehat{H}^n(M; f^*(\nabla)).
\]

Then we have a commutative diagram

\[
\begin{array}{ccc}
\widehat{H}^n(N; \nabla) & \xrightarrow{R} & H^d_{\text{dR}}(N; \nabla) \\
\downarrow f^* & & \downarrow f^* \\
\widehat{H}^n(M; f^*(\nabla)) & \xrightarrow{R} & H^d_{\text{dR}}(M; f^*(\nabla)).
\end{array}
\]

The next proposition shows that twisted Deligne cohomology indeed reduces to ordinary cohomology when the twist is trivial. When \( \nabla \simeq 0 \) the local systems \( Z \) and \( R \) trivialize: \( Z \simeq \mathbb{Z} \) and \( R \simeq \mathbb{R} \). Moreover, \( \omega = 0 \), and we recover the usual calculation for cohomology with coefficients in the Deligne complex.

**Proposition 10.** Let \( \nabla : M \rightarrow B(\mathbb{Z}/2) \) be a twist of Deligne cohomology which is trivial in the sense that \( \nabla \) factors through the basepoint \( 0 : * \rightarrow B(\mathbb{Z}/2) \) up to homotopy equivalence. Then we have a natural isomorphism of functors

\[
\widehat{H}^n(\; ; \mathbb{Z}) \cong \widehat{H}^n(\; ; \nabla).
\]

**Proof.** If \( \nabla \simeq 0 \), then in particular \( \eta \simeq 0 \) and the bundle \( \mathcal{L} \rightarrow M \) is trivializable as a bundle with flat connection. Moreover, the homotopy gives rise to a preferred choice of trivialization for each structure. These trivializations gives rise to a quasi-isomorphism of complexes

\[
\ldots \longrightarrow 0 \rightarrow \mathbb{Z} \xrightarrow{i} \Omega^0(\; ; \mathcal{L}) \xrightarrow{d} \Omega^1(\; ; \mathcal{L}) \xrightarrow{d} \ldots \xrightarrow{d} \Omega^n(\; ; \mathcal{L}),
\]

and the claim follows. \( \square \)
5 Properties of twisted Deligne cohomology

In this section we discuss the properties of basic twisted Deligne cohomology. Several of these properties have familiar counterparts in ordinary cohomology, while others have properties which are analogous to ordinary differential cohomology. We start with calculating the sheaf cohomology groups with coefficients the local systems associated with the twists.

Lemma 11. Let $\nabla : M \to \mathcal{B}(\mathbb{Z}/2)_{\nabla}$ be a twist for the Deligne complex on a smooth manifold $M$ and let $\eta : M \to \mathcal{B}\mathbb{Z}/2$ denote the underlying topological twist. The sheaf cohomology groups of degree $k \neq 0$ of $\mathcal{D}_\nabla(n)$ (see Def. [2]) are given by

$$H^k(M; \mathcal{D}_\nabla(n)) \cong \begin{cases} H^{n+k}(M; Z), & k > 0 \\ H^{n+k-1}(M; L^d/Z), & k < 0. \end{cases}$$

Proof. In what follows, the complex $\Omega(-; \mathcal{L})^*$ is equipped with the differential $\nabla$. The complex $\mathcal{D}_\nabla(n)$ is quasi-isomorphic to the shifted cone $\text{cone}(Z \otimes \tau_{\leq 0} \Omega(-; \mathcal{L})^* [n] \to \Omega(-; \mathcal{L})^* [n])[-1]$, where the morphism is given by the assignment $(z, x) \mapsto j(z) - x$. Here $\tau_{\leq 0}$ is the truncation functor to positive degrees. Thus, we have an exact triangle

$$\text{cone}(Z \otimes \tau_{\leq 0} \Omega(-; \mathcal{L})^* [n] \to \Omega(-; \mathcal{L})^* [n])[-1] \to \Omega(-; \mathcal{L})^* [n] \oplus Z \to \text{cone}(Z \otimes \tau_{\leq 0} \Omega(-; \mathcal{L})^* [n] \to \Omega(-; \mathcal{L})^* [n]).$$

(5.1)

Note that we also have a short exact sequence

$$0 \to \text{cone}(Z \to \Omega(-; \mathcal{L})^* [n])[-1] \to \mathcal{D}_\nabla(n) \to \text{cone}(\tau_{\leq 0} \Omega(-; \mathcal{L})^* [n]) \to 0 \to 0$$

(5.2)

induced by the map $j$ and the projection. Since $\Omega^*(-; \mathcal{L})$ resolves the locally constant sheaf $L^d$, the mapping cone on the left is quasi-isomorphic to $L^d / Z[n - 1]$. The mapping cone on the right has trivial sheaf cohomology in negative degrees. The long exact sequence in sheaf cohomology gives an isomorphism

$$H^k(M; \mathcal{D}_\nabla(n)) \cong H^k(M; L^d / Z[n - 1]) \cong H^{n+k-1}(M; L^d / Z),$$

for $k < 0$. For $k > 0$, the long exact sequence obtained from the exact triangle (5.1) gives the desired isomorphism.

Abstractly, a twisted differential cohomology theory should satisfy certain axioms and properties. We now verify an explicit diagrammatic characterization of twisted Deligne cohomology, refining diagram (5.2).

Proposition 12 (Twisted Deligne cohomology diamond). Let $M$ be a smooth manifold and let $\nabla : M \to \mathcal{B}(\mathbb{Z}/2)_{\nabla}$ be a twist for the Deligne complex on $M$. Then the twisted Deligne cohomology groups fit into the diamond diagram

\begin{align*}
\Omega(M; \mathcal{L})^* / \text{im}(\nabla) & \to \Omega(M; \mathcal{L})^* \\
H^*_{dR}(M; \nabla) & \to H^*(M; \nabla) \\
H^*_{dR}(M; \nabla) & \to H^*(M; Z)
\end{align*}

\begin{align*}
\nabla & \to \Omega(M; \mathcal{L})^* \\
\beta & \to H^*(M; \mathcal{L}^d/Z)
\end{align*}

Proof. The right square follows from the sequence (5.1), after passing to sheaf cohomology. The two diagonal sequences are induced from the two short exact sequences,

$$0 \to \tau_{\leq 0} \Omega(-; \mathcal{L})^* [n] \to \mathcal{D}_\nabla(n) \to Z[n] \to 0$$
and (5.2). The commutativity of the top part of the diagram follows from the web of short exact sequences

\[
\begin{array}{cccccc}
\text{cone}(0 \to \tau_{\geq 0}\Omega(-;\mathcal{L})^*[n])[-1] & \text{cone}(Z \to \Omega(-;\mathcal{L})^*[n])[-1] & \text{cone}(Z \to \tau_0\Omega(-;\mathcal{L})^*[n])[-1] \\
\downarrow & \downarrow & \downarrow \\
\text{cone}(\tau_{\geq 0}\Omega(-;\mathcal{L})^*[n] \to \tau_{\geq 0}\Omega(-;\mathcal{L})^*[n])[-1] & \Rightarrow & \text{cone}(Z \oplus \tau_0\Omega(-;\mathcal{L})^*[n] \to \Omega(-;\mathcal{L})^*[n])[-1] & \xrightarrow{\pi} & D_{\mathcal{V}}(n) \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\text{cone}(\tau_{\geq 0}\Omega(-;\mathcal{L})^*[n] \to 0)[-1] & \text{cone}(\tau_0\Omega(-;\mathcal{L})^*[n] \to 0)[-1] \\
\text{cone}(\tau_{\geq 0}\Omega(-;\mathcal{L})^*[n] \to 0)[-1] & \text{cone}(\tau_0\Omega(-;\mathcal{L})^*[n] \to 0)[-1] & 0
\end{array}
\]

along with the fact that the connecting homomorphisms \(\delta_v\) and \(\delta_h\) for the vertical and horizontal sequences, respectively, obey

\[
\delta_v R = -\delta_h \pi.
\]

For commutativity of the bottom part of the diagram, consider the long exact sequence associated to the cone \(\text{cone}(Z \to \Omega^*(-;\mathcal{L})^*[n])[-1]\). This sequence is just the Bockstein sequence associated to the short exact sequence

\[
0 \to Z \to \mathcal{L}^\delta \to \mathcal{L}^\delta/Z \to 0,
\]

shifted down by 1. Taking long exact sequences associated to cones, the first map in sequence (5.2) induces a commutative diagram

\[
\begin{array}{ccccccc}
H^{n-1}(M;Z) & \xrightarrow{\delta} & H^{n-1}(M;\mathcal{L}^\delta) & \xrightarrow{\beta} & H^n(M;Z) & \xrightarrow{\delta} & H^n(M;\mathcal{L}^\delta) \\
\downarrow & \cong & \downarrow & \cong & \downarrow & \cong & \downarrow \\
H^{n-1}(M;Z) & \xrightarrow{\delta} & H^{n-1}(M;\nabla) & \xrightarrow{\hat{\beta}} & \hat{H}^n(M;\nabla) & \xrightarrow{I \oplus R} & H^n(M;Z) \oplus \Gamma^\eta_0(M;\mathcal{L}) & \xrightarrow{\delta} & H^n(M;\nabla) .
\end{array}
\]

Therefore, the result follows.

Besides the differential cohomology diamond, we also have a Mayer-Vietoris sequence at our disposal. Combined with Lemma 11 this takes the following form.

**Proposition 13** (Mayer-Vietoris for twisted Deligne cohomology). Let \(M\) be a smooth manifold with open cover \(\{U, V\}\) and let \(\nabla : M \to B(Z/2)\mathcal{V}\) be a twist with underlying topological twist \(\eta\). There is a Mayer-Vietoris type sequence

\[
\begin{array}{ccccccc}
\cdots & \xrightarrow{\delta} & H^{*-2}(U \cap V;\mathcal{L}^\delta/Z) & \xrightarrow{\hat{\delta}} & \hat{H}^*(M;\nabla) & \xrightarrow{\hat{\delta}} & \hat{H}^*(U;\nabla) \oplus \hat{H}^*(V;\nabla) \\
\cdots & \xrightarrow{\delta} & \hat{H}^*(U \cap V;\nabla) & \xrightarrow{\delta} & H^{*+1}(M;Z) & \cdots .
\end{array}
\]

**Proof.** The Mayer-Vietoris sequence holds for sheaf cohomology, hence for \(D_{\mathcal{V}}(n)\). The claim then follows from the characterization of Lemma 11.



6 Computations and examples

In this section we compute the twisted Deligne cohomology for various spaces, illustrating the constructions and computational techniques developed earlier. We start with the simplest case.

---

\[5\] It is a straightforward exercise to show that this follows in general for any such web of short exact sequences.
Example 2 (Twisted Deligne cohomology of $\mathbb{R}^n$). Since $H^1(\mathbb{R}^n; \mathbb{Z}/2) \cong 0$, every real line bundle is trivializable over $\mathbb{R}^n$. A flat connection on a trivial line bundle is simply a closed differential 1-form $\omega$. Since the de Rham cohomology of $\mathbb{R}^n$ is also trivial, $\omega = d\beta$, and multiplication by the exponential map gives a quasi-isomorphism of complexes

$$e^\beta \times : (\Omega^*(-), d + d\beta) \longrightarrow (\Omega^*(-), d).$$

Thus, we see that the twisted Deligne cohomology groups reduce to the ordinary Deligne cohomology groups. These in turn are easily computed as

$$\hat{H}^0(\mathbb{R}^n; \mathbb{Z}) \cong \mathbb{Z}, \quad \hat{H}^1(\mathbb{R}^n; \mathbb{Z}) \cong C^\infty(\mathbb{R}^n, \mathbb{R}/\mathbb{Z}), \quad \text{and} \quad \hat{H}^k(\mathbb{R}^n; \mathbb{Z}) \cong \Omega^{k-1}(\mathbb{R}^n)/\text{im}(d).$$

Similar effects occur for the punctured Euclidean space.

Example 3 (Twisted Deligne cohomology of $\mathbb{R}^n - \{0\}$, $n > 1$). For $n > 1$, the punctured real $n$-space $\mathbb{R}^n - \{0\}$ is simply connected, we have $H^1(\mathbb{R}^n - \{0\}; \mathbb{Z}/2) \cong 0$. Therefore, every real line bundle over $\mathbb{R}^n - \{0\}$ is trivializable. Similarly, every closed differential 1-form is exact. As in the calculation of the differential cohomology of $\mathbb{R}^n$ (Example 2), it follows that the twisted Deligne cohomology groups reduce to the ordinary Deligne cohomology groups. These are readily calculated via the diamond (Prop. 12) as

$$\hat{H}^0(\mathbb{R}^n - \{0\}; \mathbb{Z}) \cong \mathbb{Z} \quad \text{and} \quad \hat{H}^1(\mathbb{R}^n - \{0\}; \mathbb{Z}) \cong C^\infty(\mathbb{R}^n - \{0\}, \mathbb{R}/\mathbb{Z}).$$

For $k \neq n - 1, n$, we have

$$\hat{H}^k(\mathbb{R}^n - \{0\}; \mathbb{Z}) \cong \Omega^{k-1}(\mathbb{R}^n - \{0\})/\text{im}(d).$$

For $k = n - 1$, we have the identification, via the Hodge decomposition,

$$\hat{H}^{n-1}(\mathbb{R}^n - \{0\}; \mathbb{Z}) \cong \Omega^{n-2}(\mathbb{R}^n - \{0\})/\text{im}(d) \oplus \langle \omega \rangle,$$

where $\langle \omega \rangle$ is the $\mathbb{Z}$-linear span of a normalized harmonic $(n - 1)$-form, restricting to a volume form $(n - 1)$-sphere. Note also that the identification depends on a choice of metric. For $k = n$ and a choice of metric, the Hodge decomposition now gives

$$\hat{H}^n(\mathbb{R}^n - \{0\}; \mathbb{Z}) \cong d\Omega^n(\mathbb{R}^n - \{0\}) \oplus \mathbb{R}/\mathbb{Z},$$

where the copy of $\mathbb{R}/\mathbb{Z}$ is identified with the group $\langle \omega \rangle_{\mathbb{R}} \langle \omega \rangle_{\mathbb{Z}}$, with $\omega$ the harmonic form extending the normalized volume form of the $(n - 1)$-sphere and the subscripts indicate that we are taking the $\mathbb{R}$ and $\mathbb{Z}$-linear spans, respectively.

Passing to the complex setting allows for some additional information in the twists.

Example 4 (Twisted Deligne cohomology of the punctured complex plane). Let $\nabla$ be the flat connection on the Möbius bundle $\mathcal{L} \to \mathbb{C} - \{0\}$ as in example 3. Recall that we have nontrivial monodromy arising from the map

$$\pi_1(\mathbb{C} - \{0\}) \cong \mathbb{Z} \longrightarrow \mathbb{Z}/2 \longrightarrow \mathbb{R}^\times,$$

sending $1 \mapsto -1$. Let $Z$ denote the local system corresponding to the $\mathbb{Z}$-subbundle of the Möbius bundle. We can compute the cohomology with coefficients in $Z$ via Čech cohomology, as follows. Consider the covering of $\mathbb{C} - \{0\}$ by three open sets as in the following picture:
where the boundary of each of the open sets $V, U$ and $W$ is colored red, blue and black, respectively. Note that, because the plane is punctured, there are no three-fold intersections of the open sets. The two-fold intersections are the wedge regions in between the colors. Now we can choose a cocycle representative for the twist defined by the assignment of integers $(-1, -1, -1)$ on the double intersections depicted above. In integral cohomology, the cocycle $(-1, -1, -1)$ is a representative for the generator of $H^1(C - \{0\}; \mathbb{Z}) \cong \mathbb{Z}$ and is not an integral Čech coboundary. However, when viewed as a Čech cocycle in the sheaf $Z_2(-1, -1, -1) = (-2, -2, -2)$ is a coboundary. Indeed, the transition functions for the bundle modify the restriction maps for $Z$ and we are reduced to showing that the system of equations

$$n_U - (-n_V) = -2, \quad n_U - (-n_W) = -2, \quad n_V - (-n_W) = -2$$

has a solution. But this is easy to see, for example $n_U = -1$, $n_V = -1$, $n_W = -1$. The since the analogous system for the cocycle $(-1, -1, -1)$ has no solutions, we conclude that

$$H^1(C - \{0\}; \mathbb{Z}) \cong \mathbb{Z}/2.$$  

To see what a global section of $Z$ looks like, we attempt to find solutions the analogous equations

$$n_U - (-n_V) = 0, \quad n_U - (-n_W) = 0, \quad n_V - (-n_W) = 0.$$ 

But these imply that $n_V = -n_W$, $n_U = n_W$ and $2n_U = 0$. Hence, $n_U = n_V = n_W = 0$ and so

$$H^0(C - \{0\}; Z) \cong \Gamma(C - \{0\}, Z) \cong 0.$$  

The twisted de Rham cohomology is now easy to compute from the twisted de Rham theorem. Indeed, the calculations in Čech cohomology apply equally well to the discrete bundle $\mathcal{L}$. The presence of 2-torsion in degree 1 kills $H^1_{\text{dR}}(C - \{0\}; \nabla)$ and we have the identifications

$$H^0_{\text{dR}}(C - \{0\}; \nabla) \cong H^1_{\text{dR}}(C - \{0\}; \nabla) \cong 0.$$  

From the differential cohomology diamond diagram (Prop. [12]), we have $\hat{H}^0(C - \{0\}; \nabla) \cong \hat{H}^0(C - \{0\}; Z) \cong \mathbb{Z}$. Since the global sections of $\mathcal{L}$ are divisible as an abelian group, they form an injective module. By the differential cohomology diamond diagram (Prop. [12]), we conclude that

$$\hat{H}^1(C - \{0\}; \nabla) \cong \mathbb{Z}/2 \oplus \Gamma(C - \{0\}; \mathcal{L}) \quad \text{and} \quad \hat{H}^0(C - \{0\}; \nabla) \cong 0.$$  

Example 5 (Orientation line bundle). Let $M$ be a closed, smooth manifold of dimension $n$, with orientation bundle $\Lambda^n M \to M$. If $M$ is simply-connected then $\Lambda^n M \to M$ is trivializable and every closed 1-form is exact. In this case, the twisted Deligne cohomology groups reduce to the usual Deligne cohomology groups.

Let $M$ be a non-orientable manifold and equip $M$ with a Riemannian metric so that $TM \to M$ has orthogonal structure. Consider the Levi-Civita connection $\nabla$ on $TM$. Taking the determinant of the transition functions of $TM$ and the trace of the connection gives the orientation bundle $\Lambda^n M \to M$, equipped with the zero connection.

Let $U_\alpha$ be a local chart of $M$, with coordinates $\{x_i\}_{i=1}^n$. Write a differential $n$-form on a patch $U_\alpha$ as $f_\alpha dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n$, with $f_\alpha \in C^\infty(U_\alpha; \mathbb{R})$. From the definitions, we see that a flat section of this bundle is locally of the above form with $f_\alpha \equiv C \in \mathbb{R}$. Under orthonormal coordinate transformations, we see that these constants differ by the determinant $\det(g_{\alpha\beta}) = \pm 1$. Thus, the sheaf of local sections can be regarded as sections of the bundle $\Lambda^n M^\delta \to M$, where the $\delta$ indicates that we have taken the fibers $\mathbb{R}$ to have the discrete topology. The local system $Z \subset \Lambda^n M^\delta$ in this case is the sheaf of sections which are locally of the form $n_\alpha dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n$, with $n_\alpha \in \mathbb{Z}$.

In this case, the $\nabla = d_{\Lambda^n M}$-twisted Deligne complex is given by the hypercohomology of the complex

$$Z \xleftarrow{d} \Omega^0(-; \Lambda^n M) \xrightarrow{d} \Omega^1(-; \Lambda^n M) \xrightarrow{d} \ldots \xrightarrow{d} \Omega^{n-1}(-; \Lambda^n M).$$
Example 6 (Twisted Deligne cohomology of real projective space). Let \( M = \mathbb{R}P^n \) be the \( n \)-dimensional real projective space. The restriction of the first Stiefel-Whitney class \( w_1 \in H^1(\mathbb{R}P^n; \mathbb{Z}/2) \cong \mathbb{Z}/2 \) to \( \mathbb{R}P^n \) classifies the tautological bundle on \( \mathbb{R}P^n \). This tautological bundle is a special case of the previous example. Hence, the \( w_1 \)-twisted Deligne complex reduces to

\[
D_{w_1}(k) = \left( \begin{array}{c}
Z \\
\Gamma^0(-; L_{w_1}) \\
\Gamma^1(-; L_{w_1}) \\
\vdots \\
\Gamma^{n-1}(-; L_{w_1})
\end{array} \right).
\]

We first calculate the sheaf cohomology of \( Z \) and \( L_{w_1} \), via the Mayer-Vietoris sequence. Let \( \mathcal{N} \) be an open tubular neighborhood of the equator in \( S^n \) and let \( W \) be the complement of the closure \((1 - \epsilon)\mathcal{N} \). In the quotient, this gives a cover \( \{ U, V \} \) of \( \mathbb{R}P^n \) with \( U \cong B^n \) and \( V \cong \mathbb{R}P^{n-1} \), where \( B^n \) is the \( n \)-dimensional ball. The intersection \( U \cap V \cong S^{n-1} \) and we have the Mayer-Vietoris sequence with local coefficients

\[
\ldots \to H^k(\mathbb{R}P^n; Z) \to H^k(B^n; Z) \oplus H^k(\mathbb{R}P^{n-1}; Z) \to H^k(S^{n-1}; Z) \to \ldots .
\]

Since the restriction of the tautological bundle to \( B^n \) and \( S^{n-1} \) trivializes, and the restriction to \( \mathbb{R}P^{n-1} \) is the tautological bundle over \( \mathbb{R}P^{n-1} \), the sequence reduces to

\[
\ldots \to H^k(\mathbb{R}P^n; Z) \to H^k(\mathbb{R}P^{n-1}; Z) \to H^k(S^{n-1}; Z) \to \ldots .
\]

Thus, for \( n > 1 \) and \( 1 \leq k \neq n, n - 1 \), we have an isomorphism

\[
H^k(\mathbb{R}P^n; Z) \cong H^k(\mathbb{R}P^{n-1}; Z).
\]

We also have the sequence

\[
0 \to H^{n-1}(\mathbb{R}P^n; Z) \to H^n(\mathbb{R}P^n; Z) \to \mathbb{Z} \to H^n(\mathbb{R}P^n; Z) \to 0. \tag{6.1}
\]

We have already shown that \( H^1(S^1; Z) \cong \mathbb{Z}/2 \). The sequence \((6.1)\) implies that \( H^1(\mathbb{R}P^2; Z) \cong \mathbb{Z}/2 \) and \( H^2(\mathbb{R}P^2; Z) \cong \mathbb{Z} \). We claim that for even \( n \)

\[
H^k(\mathbb{R}P^n; Z) = \begin{cases} 
Z, & k = n \\
\mathbb{Z}/2, & 0 < k < n \text{ odd} \\
0, & \text{otherwise},
\end{cases}
\]

while for odd \( n \)

\[
H^k(\mathbb{R}P^n; Z) = \begin{cases} 
\mathbb{Z}/2, & 0 < k \leq n \text{ odd} \\
0, & \text{otherwise}.
\end{cases}
\]

Note the shifts in degrees compared to integral coefficients. To prove the claim, we proceed by induction on the dimension \( n \). The only nontrivial part of the induction is to show that \( H^n(\mathbb{R}P^n; Z) \cong \mathbb{Z}/2 \) when \( n \) is odd. To prove this, we use the sequence \((6.1)\). By the induction hypothesis, this reduces to the exact sequence

\[
0 \to H^{n-1}(\mathbb{R}P^n; Z) \to \mathbb{Z} \to H^n(\mathbb{R}P^n; Z) \to 0. \tag{6.2}
\]

and we need to identify the map between the copies of \( \mathbb{Z} \). By the induction by hypothesis, the generator of \( H^{n-1}(\mathbb{R}P^{n-1}; Z) \) maps to the generator of \( H^{n-1}(S^{n-1}; Z) \) under the quotient \( q : S^{n-1} \to \mathbb{R}P^{n-1} \). Consider the commutative diagram

\[
\begin{array}{ccc}
S^{n-1} & \xrightarrow{id} & S^{n-1} \\
\downarrow{q} & & \downarrow{q} \\
N & \xrightarrow{id} & S^{n-1}
\end{array}
\]

where \( N \) is a tubular neighborhood of \( S^{n-1} \) in \( S^n \) and \( -id \) the antipodal map. In ordinary cohomology the left map in \((6.3)\) induces the map \((x, y) \mapsto x - y \). However, with \( L \)-twisted coefficients, the map is modified by a local trivialization of \( L \) and the resulting map sends \((x, y) \mapsto x + y \). The top map induces the map which sends
the generator $x \in H^{n-1}(S^{n-1}; \mathbb{Z})$ to $(x, x) \in H^{n-1}(S^{n-1}; \mathbb{Z}) \oplus H^{n-1}(S^{n-1}; \mathbb{Z})$. Thus, the commutativity of the diagram implies that the restriction must send the generator of $H^{n-1}(\mathbb{R}P^{n-1}; \mathbb{Z})$ to twice the generator of $H^{n-1}(S^{n-1}; \mathbb{Z})$. Hence the map $\mathbb{Z} \to \mathbb{Z}$ in (6.2) is the $\times 2$ map and $H^n(\mathbb{R}P^n; \mathbb{Z}) \cong \mathbb{Z}/2$. This proves the claim.

Similar computations hold for coefficients in $L$, where the torsion is killed. Using the diagonal sequence

$$H^{k-1}(M; \mathbb{Z}) \xrightarrow{\Omega} \Omega^{k-1}(M)/\text{im}(d) \xrightarrow{\hat{H}^k(M; \nabla)} H^k(M; \mathbb{Z}) \xrightarrow{0},$$

in the differential cohomology diamond diagram, we compute

$$\hat{H}^{k}(\mathbb{R}P^n; w_1) = \begin{cases} Z \oplus \Omega^{k-1}(\mathbb{R}P^n; L)/\text{im}(d) & k = n \\ Z/2 \oplus \Omega^{k-1}(\mathbb{R}P^n; L)/\text{im}(d) & 0 < k < n \text{ odd} \\ \Omega^{k-1}(\mathbb{R}P^n; L)/\text{im}(d) & \text{otherwise} \end{cases}$$

for $n$ even and

$$\hat{H}^{k}(\mathbb{R}P^n; w_1) = \begin{cases} Z/2 \oplus \Omega^{k-1}(\mathbb{R}P^n; L)/\text{im}(d) & 0 < k \leq n \text{ odd} \\ \Omega^{k-1}(\mathbb{R}P^n; L)/\text{im}(d) & \text{otherwise} \end{cases}$$

for $n$ odd.

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