Non-perturbative Renormalization of Four-Fermion Operators Relevant to $B_K$ with Staggered Quarks

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We present matching factors for the four-fermion operators obtained using the non-perturbative renormalization (NPR) method in RI-MOM scheme for improved staggered fermions on the MILC asqtad lattices ($N_f = 2 + 1$).

Using $20^3 \times 64$ lattice ($a \approx 0.12\text{fm}, am_\ell/am_s = 0.01/0.05$), we obtain the matching factor of $B_K$ operator.

We compare NPR results with those of one-loop perturbative matching.
\( \tilde{p} \) is the momentum in reduced Brillouin zone.

\[
p \in \left(-\frac{\pi}{a}, \frac{\pi}{a}\right]^4, \quad \tilde{p} \in \left(-\frac{\pi}{2a}, \frac{\pi}{2a}\right]^4, \quad p = \tilde{p} + \pi_B
\]

where \( \pi_B \equiv \frac{\pi}{a}B \) is cut-off momentum in hypercube.

- \( a \) : lattice spacing.
- \( B \) : vector in hypercube. Each element is 0 or 1
  - ex) \( B = (0, 0, 1, 1) \)
The one-color trace four-fermion operator

\[ O_I(x) = \left[ \bar{\chi}_{c_1}(x_A)(\gamma S_1 \otimes \xi F_1)_{AB}\chi_{c_2}(x_B) \right] \left[ \bar{\chi}_{c_3}(x_C)(\gamma S_2 \otimes \xi F_2)_{CD}\chi_{c_4}(x_D) \right] \times U_{AD;c_1c_4}(x)U_{BC;c_2c_3}(x) \quad (1) \]

The two-color trace four-fermion operator

\[ O_{II}(x) = \left[ \bar{\chi}_{c_1}(x_A)(\gamma S_1 \otimes \xi F_1)_{AB}\chi_{c_2}(x_B) \right] \left[ \bar{\chi}_{c_3}(x_C)(\gamma S_2 \otimes \xi F_2)_{CD}\chi_{c_4}(x_D) \right] \times U_{AB;c_1c_2}(x)U_{CD;c_3c_4}(x) \quad (2) \]

- **U**: link variable.
- **A, B, C, D**: hypercube index
- **c_1, c_2, c_3, c_4**: color index
Four-Fermion Operator Renormalization

(a) one-color trace

(b) two-color trace

**Figure**: One-color trace and two-color trace four-fermion operators
Red Circle : 1PI Diagram.

(a) Unamputated Green’s Function

(b) Amputated Green’s Function
(c) Amputated Green’s Function : $\Lambda^\alpha$

(d) Projected Amputated Green’s Function : $\Gamma^{\alpha\beta}$

- $\alpha, \beta$ : the indices to represent different operators.
  - ex) $\alpha = (V \otimes P)(V \otimes P)_I$, $\beta = (A \otimes P)(A \otimes P)_{II}$
- momentum conservation in reduced Brillouin zone. $\tilde{k} = \tilde{p}_1 - \tilde{p}_2 + \tilde{p}_3 - \tilde{p}_4$
The one-color trace projection operator is

$$\hat{P}^\beta = (\gamma_{S'} \otimes \xi_{F'})_{BA} (\gamma_{S''} \otimes \xi_{F''})_{DC} \delta_{c_4 c_1} \delta_{c_3 c_2}$$

The two-color trace projection operator is

$$\hat{P}^\beta = (\gamma_{S'} \otimes \xi_{F'})_{BA} (\gamma_{S''} \otimes \xi_{F''})_{DC} \delta_{c_2 c_1} \delta_{c_4 c_3}$$

- $A, B, C, D$: hypercube index
- $c_1, c_2, c_3, c_4$: color index
For simplicity, we define the following notations.

\[
\begin{align*}
O_{V1} & \equiv O[V \otimes P][V \otimes P], I \\
O_{V2} & \equiv O[V \otimes P][V \otimes P], II \\
O_{A1} & \equiv O[A \otimes P][A \otimes P], I \\
O_{A2} & \equiv O[A \otimes P][A \otimes P], II
\end{align*}
\] (3)

The tree level \(B_K\) operator is

\[
O_{B_K}^{\text{tree}} = O_{V1}^{\text{tree}} + O_{V2}^{\text{tree}} + O_{A1}^{\text{tree}} + O_{A2}^{\text{tree}}
\] (4)

The matching formula of \(B_K\) operator is

\[
O_{B_K}^{R} = z_1 O_{V1}^{B} + z_2 O_{V2}^{B} + z_3 O_{A1}^{B} + z_4 O_{A2}^{B} + \sum_{\alpha \in (D)} z_{\alpha} O_{\alpha}^{B}
\] (5)

- The superscription \(R(B)\) denotes renormalized(bare) quantity.
- \(z_1, z_2, z_3, z_4\) and \(z_{\alpha}\) are the renormalization factors.
- We classify operators as follows.
  - (C): \{V1, V2, A1, A2\} (diagonal operators)
  - (D): remaining operators (off-diagonal operators)
The renormalization of quark field is

$$\chi_R = Z_q^{1/2} \chi_B$$  \hfill (6)

where $z_q$ is wave function renormalization factor for quark field. The renormalized amputated Green’s function of $B_K$ operator is

$$\Lambda_{B_K}^R = \frac{z_1}{z_q^2} \Lambda_{V_1}^B + \frac{z_2}{z_q^2} \Lambda_{V_2}^B + \frac{z_3}{z_q^2} \Lambda_{A_1}^B + \frac{z_4}{z_q^2} \Lambda_{A_2}^B + \sum_{\alpha \in (D)} \frac{z_\alpha}{z_q^2} \Lambda_{\alpha}^B,$$  \hfill (7)

For simplicity, the projection operators are defined as follows.

$$\hat{P}_{V_1} = \overline{(V_\mu \otimes P)}_{BA} \overline{(V_\mu \otimes P)}_{DC} \delta_{c_4 c_1} \delta_{c_3 c_2}$$

$$\hat{P}_{V_2} = \overline{(V_\mu \otimes P)}_{BA} \overline{(V_\mu \otimes P)}_{DC} \delta_{c_2 c_1} \delta_{c_4 c_3}$$

$$\hat{P}_{A_1} = \overline{(A_\mu \otimes P)}_{BA} \overline{(A_\mu \otimes P)}_{DC} \delta_{c_4 c_1} \delta_{c_3 c_2}$$

$$\hat{P}_{A_2} = \overline{(A_\mu \otimes P)}_{BA} \overline{(A_\mu \otimes P)}_{DC} \delta_{c_2 c_1} \delta_{c_4 c_3}$$  \hfill (8)
We apply the projection operator to tree level amputated Green’s function.

\[ \text{tr}[\Lambda_{BK}^\text{tree} \hat{P}_V] = \text{tr}[\Lambda_{V1}^\text{tree} \hat{P}_V] + \text{tr}[\Lambda_{V2}^\text{tree} \hat{P}_V] + \text{tr}[\Lambda_{A1}^\text{tree} \hat{P}_V] + \text{tr}[\Lambda_{A2}^\text{tree} \hat{P}_V] \]
\[ = (16 \times 16 \times 3 \times 3) + (16 \times 16 \times 3) = 3072 \equiv N \quad (9) \]

\[ \text{tr}[\Lambda_{BK}^\text{tree} \hat{P}_{V2}] = \text{tr}[\Lambda_{V1}^\text{tree} \hat{P}_{V2}] + \text{tr}[\Lambda_{V2}^\text{tree} \hat{P}_{V2}] + \text{tr}[\Lambda_{A1}^\text{tree} \hat{P}_{V2}] + \text{tr}[\Lambda_{A2}^\text{tree} \hat{P}_{V2}] \]
\[ = (16 \times 16 \times 3) + (16 \times 16 \times 3 \times 3) = 3072 \equiv N \quad (10) \]

\[ \text{tr}[\Lambda_{BK}^\text{tree} \hat{P}_{A1}] = \text{tr}[\Lambda_{V1}^\text{tree} \hat{P}_{A1}] + \text{tr}[\Lambda_{V2}^\text{tree} \hat{P}_{A1}] + \text{tr}[\Lambda_{A1}^\text{tree} \hat{P}_{A1}] + \text{tr}[\Lambda_{A2}^\text{tree} \hat{P}_{A1}] \]
\[ = (16 \times 16 \times 3 \times 3) + (16 \times 16 \times 3) = 3072 \equiv N \quad (11) \]

\[ \text{tr}[\Lambda_{BK}^\text{tree} \hat{P}_{A2}] = \text{tr}[\Lambda_{V1}^\text{tree} \hat{P}_{A2}] + \text{tr}[\Lambda_{V2}^\text{tree} \hat{P}_{A2}] + \text{tr}[\Lambda_{A1}^\text{tree} \hat{P}_{A2}] + \text{tr}[\Lambda_{A2}^\text{tree} \hat{P}_{A2}] \]
\[ = (16 \times 16 \times 3) + (16 \times 16 \times 3 \times 3) = 3072 \equiv N \quad (12) \]

\[ \text{tr}[\Lambda_{BK}^\text{tree} \hat{P}_{(D)}] = 0 \quad (13) \]

\[ \text{tr}[\Lambda_{(D)}^\text{tree} \hat{P}_{(C)}] = 0, \quad (14) \]

Here,
(C): \{V1, V2, A1, A2\} (diagonal operators)
(D): remaining operators (off-diagonal operators)
The RI-MOM scheme prescription is

\[
tr[\Lambda^R(\tilde{p}, \tilde{p}, \tilde{p}, \tilde{p})\hat{P}] = tr[\Lambda^{tree}(\tilde{p}, \tilde{p}, \tilde{p}, \tilde{p})\hat{P}]
\] (15)

Therefore,

\[
N = tr[\Lambda_{BK}^R \hat{P}_{V1}]
\] (16)

\[
N = tr[\Lambda_{BK}^R \hat{P}_{V2}]
\] (17)

\[
N = tr[\Lambda_{BK}^R \hat{P}_{A1}]
\] (18)

\[
N = tr[\Lambda_{BK}^R \hat{P}_{A2}]
\] (19)

\[
0 = tr[\Lambda_{BK}^R \hat{P}_{(D)}]
\] (20)

\[
0 = tr[\Lambda_{(D)}^R \hat{P}_{(C)}]
\] (21)

(22)

Here,

(C): \{V1, V2, A1, A2\} (diagonal operators)

(D): remaining operators (off-diagonal operators)
We define the projected amputated Green's function as follows.

\[ \Gamma^B_{\alpha\beta} \equiv \frac{1}{Nz_q^2} tr[\Lambda^B_{\alpha} \hat{P}_\beta] \] (23)

where \( \alpha, \beta \) are operator such as \( V_1, V_2, A_1, A_2, \ldots \) and \( z_q \) is obtained from conserved vector current channel.

\[ 1 = (z_1 \Gamma^B_{VV1} + z_2 \Gamma^B_{VV2} + z_3 \Gamma^B_{AA1} + z_4 \Gamma^B_{AA2} + \sum_{\alpha \in (D)} z_\alpha \Gamma^B_{\alpha V1}) \] (24)

\[ 1 = (z_1 \Gamma^B_{V1V2} + z_2 \Gamma^B_{V2V2} + z_3 \Gamma^B_{A1V2} + z_4 \Gamma^B_{A2V2} + \sum_{\alpha \in (D)} z_\alpha \Gamma^B_{\alpha V2}) \] (25)

\[ 1 = (z_1 \Gamma^B_{V1A1} + z_2 \Gamma^B_{V2A1} + z_3 \Gamma^B_{A1A1} + z_4 \Gamma^B_{A2A1} + \sum_{\alpha \in (D)} z_\alpha \Gamma^B_{\alpha A1}) \] (26)

\[ 1 = (z_1 \Gamma^B_{V1A2} + z_2 \Gamma^B_{V2A2} + z_3 \Gamma^B_{A1A2} + z_4 \Gamma^B_{A2A2} + \sum_{\alpha \in (D)} z_\alpha \Gamma^B_{\alpha A2}) \] (27)

\[ 0 = (z_1 \Gamma^B_{V1\beta} + z_2 \Gamma^B_{V2\beta} + z_3 \Gamma^B_{A1\beta} + z_4 \Gamma^B_{A2\beta} + \sum_{\alpha \in (D)} z_\alpha \Gamma^B_{\alpha \beta}), \quad \beta \in (D) \] (28)
We can express these equations as a matrix equation.

\[ \vec{z}_{\text{tree}} = \vec{z} \Gamma^B, \]  

(29)

where

\[ \vec{z}_{\text{tree}} = (1, 1, 1, 1, 0, \ldots, 0) \]  

(30)

\[ \vec{z} = (z_1, z_2, z_3, z_4, z_\alpha, z_\beta, \ldots) \]  

(31)

Then, we can obtain the \( \vec{z} \) from the following equation.

\[ \vec{z} = \vec{z}_{\text{tree}} (\Gamma^B)^{-1} \]  

(32)
We can make the matrix \( \Gamma^B \) as a block matrix.

\[
\Gamma^B = \begin{pmatrix} X_{4 \times 4} & Y_{4 \times M} \\ Z_{M \times 4} & W_{M \times M} \end{pmatrix}
\]  

(33)

Here, \( X \) is \( 4 \times 4 \) matrix.

\[
X = \begin{pmatrix} 
\Gamma^B_{V1V1} & \Gamma^B_{V1V2} & \Gamma^B_{V1A1} & \Gamma^B_{V1A2} \\
\Gamma^B_{V2V1} & \Gamma^B_{V2V2} & \Gamma^B_{V2A1} & \Gamma^B_{V2A2} \\
\Gamma^B_{A1V1} & \Gamma^B_{A1V2} & \Gamma^B_{A1A1} & \Gamma^B_{A1A2} \\
\Gamma^B_{A2V1} & \Gamma^B_{A2V2} & \Gamma^B_{A2A1} & \Gamma^B_{A2A2} 
\end{pmatrix}
\]  

(34)

The matrix \( Y \) is \( 4 \times M \).

\[
Y = \begin{pmatrix} 
\Gamma^B_{V1\alpha} & \Gamma^B_{V1\beta} & \cdots \\
\Gamma^B_{V2\alpha} & \Gamma^B_{V2\beta} & \cdots \\
\Gamma^B_{A1\alpha} & \Gamma^B_{A1\beta} & \cdots \\
\Gamma^B_{A2\alpha} & \Gamma^B_{A2\beta} & \cdots 
\end{pmatrix}
\]  

(35)

- \( \{ \alpha, \beta, \cdots \} = \{(S \otimes V)(S \otimes V)_I, (S \otimes V)(S \otimes V)_II, (S \otimes A)(S \otimes A)_I, \cdots \} \).
- We calculate the complete set of off-diagonal mixing matrix \( Y \).
- We assume that \( Z_{ij} \approx Y_{ji} \). It is an approximation within factor of 2.
- \( W \approx 1 \).
\[ \Gamma^B = \begin{pmatrix} X_{4\times4} & Y_{4\times M} \\ Z_{M\times4} & W_{M\times M} \end{pmatrix} \]  

(36)

The inverse of block matrix is

\[ (\Gamma^B)^{-1} = \begin{pmatrix} (X - YW^{-1}Z)^{-1} & -X^{-1}Y(W - ZX^{-1}Y)^{-1} \\ -W^{-1}Z(X - YW^{-1}Z)^{-1} & (W - ZX^{-1}Y)^{-1} \end{pmatrix} \]  

(37)

\[ \approx \begin{pmatrix} X^{-1} + X^{-1}YW^{-1}ZX^{-1} & -X^{-1}Y(W^{-1} + W^{-1}ZX^{-1}YW^{-1}) \\ -W^{-1}Z(X^{-1} + X^{-1}YW^{-1}ZX^{-1}) & W^{-1} + W^{-1}ZX^{-1}YW^{-1} \end{pmatrix} \]  

(38)

with \( Z_{ij} \approx Y_{ji} \) and \( W \approx 1 \)

\[ \approx \begin{pmatrix} X^{-1} + X^{-1}YY^TX^{-1} & -X^{-1}Y(1 + Y^TX^{-1}Y) \\ -Y^T(X^{-1} + X^{-1}YY^TX^{-1}) & 1 + Y^TX^{-1}Y \end{pmatrix} \]  

(39)
Simulation Detail

- $20^3 \times 64$ MILC asqtad lattice ($a \approx 0.12\text{fm}$, $am_{\ell}/am_s = 0.01/0.05$).
- HYP smeared staggered fermions as valence quarks.
- The number of configurations is 30.
- 5 valence quark masses (0.01, 0.02, 0.03, 0.04, 0.05)
- 9 external momenta in the units of \(\left(\frac{2\pi}{L_s}, \frac{2\pi}{L_s}, \frac{2\pi}{L_t}\right)\).
- We use the jackknife resampling method to estimate statistical errors.

| \(n(x, y, z, t)\) | \(|a\tilde{p}|\) | GeV |
|-------------------|---------------|-----|
| (2, 2, 2, 7)      | 1.2871        | 2.1332 |
| (2, 2, 2, 8)      | 1.3421        | 2.2243 |
| (2, 2, 2, 9)      | 1.4018        | 2.3233 |
| (2, 3, 2, 7)      | 1.4663        | 2.4302 |
| (2, 3, 2, 8)      | 1.5148        | 2.5106 |
| (2, 3, 2, 9)      | 1.5680        | 2.5987 |
| (3, 2, 3, 8)      | 1.6698        | 2.7674 |
| (3, 3, 3, 7)      | 1.7712        | 2.9355 |
| (3, 3, 3, 9)      | 1.8562        | 3.0764 |
Matrix X analysis

\[ \Gamma^B = \begin{pmatrix} X_{4\times4} & Y_{4\times M} \\ Z_{M\times4} & W_{M\times M} \end{pmatrix} \]  

(40)

As an example, we consider the data analysis of \( \Gamma^B_{V1V1} \).

\[ X = \begin{pmatrix} \Gamma^B_{V1V1} & \Gamma^B_{V1V2} & \Gamma^B_{V1A1} & \Gamma^B_{V1A2} \\ \Gamma^B_{V2V1} & \Gamma^B_{V2V2} & \Gamma^B_{V2A1} & \Gamma^B_{V2A2} \\ \Gamma^B_{A1V1} & \Gamma^B_{A1V2} & \Gamma^B_{A1A1} & \Gamma^B_{A1A2} \\ \Gamma^B_{A2V1} & \Gamma^B_{A2V2} & \Gamma^B_{A2A1} & \Gamma^B_{A2A2} \end{pmatrix} \]  

(41)
$\Gamma_{V_1V_1}^B$ analysis

We convert the scale of raw data in the RI-MOM scheme from $\mu(=|\vec{p}|)$ to $\mu_0(=2\text{ GeV} \text{ or } 3\text{ GeV})$ using two-loop RG evolution factor $U_{B_K}^{\text{RI-MOM}}(\mu_0, \mu)$.

\[ \vec{z}_{\text{RI-MOM}}(\mu_0) = U_{B_K}^{\text{RI-MOM}}(\mu_0, \mu) \vec{z}_{\text{RI-MOM}}(\mu) \] (42)

\[ \vec{z}_{\text{RI-MOM}} = \vec{z}_{\text{tree}}(\Gamma^B)^{-1} \] (43)

\[ \Gamma_{V_1V_1}(\mu_0) = 1/U_{B_K}^{\text{RI-MOM}}(\mu_0, \mu)\Gamma_{V_1V_1}^B(\mu) \] (44)
m-fit (fitting with respect to quark mass)

We fit the data with respect to quark mass for a fixed momentum to the following function $f_{V1V1}$. [RBC, PRD66, 2002]

$$f_{V1V1}(m, a, \tilde{p}) = a_1 + a_2 \cdot am + a_3 \cdot \frac{1}{(am)} + a_4 \cdot \frac{1}{(am)^2},$$

where $a_i$ is a function of $\tilde{p}$. We call this m-fit. After m-fit, we take the chiral limit values which corresponds to $a_1(a, \tilde{p})$. Because of the sea quark determinant contributions ($a_3, a_4 \propto m_\ell^2 m_s$), $a_3$ and $a_4$ term contribution vanishes in the chiral limit.

| $\mu_0$ | $a_1$       | $a_2$       | $a_3$       | $a_4$       | $\chi^2$/dof |
|---------|-------------|-------------|-------------|-------------|--------------|
| 2GeV    | 0.17457(19) | -0.0946(15) | 0.0006907(83) | -0.0000000766(35) | 0.00194(40) |
| 3GeV    | 0.17991(20) | -0.0975(15) | 0.0007118(85) | -0.0000000790(36) | 0.00194(40) |
m-fit plot

(e) $\mu_0 = 2\text{ GeV}$

(f) $\mu_0 = 3\text{ GeV}$
p-fit (fitting with respect to reduced momentum)

We fit \( a_1(a, \tilde{p}) \) to the following fitting function.

\[
f_{V_1V_1}(am = 0, a\tilde{p}) = b_1 + b_2(a\tilde{p})^2 + b_3((a\tilde{p})^2)^2 + b_4(a\tilde{p})^4 + b_5((a\tilde{p})^2)^3
\]

To avoid non-perturbative effects at small \((a\tilde{p})^2\), we choose the momentum window as \((a\tilde{p})^2 > 1\). Because we assume that those terms of \(O((a\tilde{p})^2)\) and higher order are pure lattice artifacts, we take the \(b_1\) as \(X_{11}\) value at \(\mu = 2\) GeV and \(\mu = 3\) GeV in the RI-MOM scheme.

| \(\mu_0\) | \(b_1\)   | \(b_2\)   | \(b_3\)   | \(b_4\)   | \(b_5\)   | \(\chi^2/dof\) |
|---------|----------|----------|----------|----------|----------|----------------|
| 2GeV    | 1.056(15)| -0.500(18)| 0.0925(72)| 0.0019(63)| -0.00643(88)| 0.08(17)       |
| 3GeV    | 1.088(16)| -0.515(18)| 0.0953(74)| 0.0020(65)| -0.00663(91)| 0.08(17)       |
**Preliminary Results**

**p-fit plot**

(g) $p$-fit ($\mu_0 = 2 \text{ GeV}$)

(h) $p$-fit ($\mu_0 = 3 \text{ GeV}$)
$\Gamma^{-1}(2\text{GeV})$ matrix result

$$(\Gamma^B)^{-1} = \begin{pmatrix}
X^{-1} + X^{-1}YY^TX^{-1} & -X^{-1}Y(1 + Y^TX^{-1}Y) \\
-Y^T(X^{-1} + X^{-1}YY^TX^{-1}) & 1 + Y^TX^{-1}Y
\end{pmatrix}$$

$$X^{-1} = \begin{pmatrix}
1.374(33) & -0.818(47) & 0.347(21) & 0.007(32) \\
-0.748(45) & 1.999(43) & -0.009(31) & -0.049(34) \\
0.352(31) & 0.033(47) & 1.260(33) & -0.623(39) \\
0.019(46) & -0.086(54) & -0.656(40) & 1.590(37)
\end{pmatrix}$$

$$X^{-1}YY^TX^{-1} = \begin{pmatrix}
0.0127(15) & -0.0079(11) & 0.0020(17) & -0.0001(10) \\
-0.00713(92) & 0.0064(11) & -0.0010(11) & -0.00045(73) \\
0.0020(18) & 0.0000(15) & 0.0200(90) & -0.0103(48) \\
0.0001(12) & -0.0012(11) & -0.0109(52) & 0.0059(28)
\end{pmatrix}$$

$$-X^{-1}Y \approx 1\% \rightarrow 0.07\% \text{ by wrong taste suppression}$$
Result of renormalization factors of diagonal operators

We obtain $\vec{z}$ in RI-MOM scheme at 2GeV and 3GeV.

$$\vec{z} = \vec{z}_{\text{tree}}(\Gamma^B)^{-1}, \quad (47)$$

where

$$\vec{z}_{\text{tree}} = (1, 1, 1, 1, 0, \cdots, 0) \quad (48)$$

$$\vec{z} = (z_1, z_2, z_3, z_4, z_{\alpha}, z_{\beta}, \cdots) \quad (49)$$

We convert the scheme from RI-MOM $\rightarrow \overline{\text{MS}}$ using two-loop matching factor.

|      | RI-MOM(2GeV) | $\overline{\text{MS}}$(2GeV) | RI-MOM(3GeV) | $\overline{\text{MS}}$(3GeV) |
|------|--------------|----------------|--------------|----------------|
| $z_1$ | 0.9962(80)   | 1.0139(82)    | 0.9666(78)   | 0.9812(79)    |
| $z_2$ | 1.128(31)    | 1.148(32)     | 1.095(30)    | 1.111(31)     |
| $z_3$ | 0.9418(75)   | 0.9586(77)    | 0.9139(73)   | 0.9277(74)    |
| $z_4$ | 0.926(29)    | 0.942(30)     | 0.898(28)    | 0.912(29)     |
Systematic Error

We estimate two different systematic errors.

- The size of $X^{-1}YY^TX^{-1}$.

$$
(\Gamma^B)^{-1} = \begin{pmatrix}
X^{-1} + X^{-1}YY^TX^{-1} & -X^{-1}Y(1 + Y^TX^{-1}Y) \\
-Y^T(X^{-1} + X^{-1}YY^TX^{-1}) & 1 + Y^TX^{-1}Y
\end{pmatrix} \quad (50)
$$

- The uncertainty comes from truncated higher order of the two-loop matching factor (RI-MOM→ $\overline{MS}$): $O(\alpha_s^3)$

$$
\text{(sys. err. of } z_i) = z_i \cdot \alpha_s^3 \quad (51)
$$

We add these systematic errors in quadrature.
Off-diagonal renormalization factors

\[
(\Gamma^B)^{-1} = \begin{pmatrix}
X^{-1} + X^{-1}YY^TX^{-1} & -X^{-1}Y(1 + Y^TX^{-1}Y)
\end{pmatrix}

- Y^T(X^{-1} + X^{-1}YY^TX^{-1})

1 + Y^TX^{-1}Y
\]

(52)

\[-X^{-1}Y(2 \text{ GeV}) = \begin{pmatrix}
-0.013(12) & 0.0075(64) & 0.013(12) & \cdots \\
0.0155(38) & 0.0068(32) & -0.0008(31) & \cdots \\
0.083(35) & 0.078(22) & -0.0162(45) & \cdots \\
-0.048(19) & -0.046(12) & 0.0104(13) & \cdots \\
\end{pmatrix}
\]

(53)
Off-diagonal renormalization factors

\[ O_{B_K}^R = z_1 O_{V1}^B + z_2 O_{V2}^B + z_3 O_{A1}^B + z_4 O_{A2}^B + \sum_{\alpha \in (D)} z_\alpha O_{\alpha}^B \]  

\begin{array}{|c|c|c|c|c|}
\hline
z_\alpha & \text{RI-MOM(2GeV)} & \underline{\text{MS}(2GeV)} & \text{RI-MOM(3GeV)} & \underline{\text{MS}(3GeV)} \\
\hline
z(S \otimes V)(S \otimes V)_{I} & 0.037(28) & 0.038(28) & 0.037(28) & 0.038(28) \\
\hline
z(S \otimes V)(S \otimes V)_{II} & 0.047(17) & 0.047(17) & 0.047(17) & 0.047(17) \\
\hline
z(V \otimes S)(V \otimes S)_{I} & -0.00257(73) & -0.00257(73) & -0.00252(71) & -0.00256(72) \\
\hline
z(V \otimes S)(V \otimes S)_{II} & 0.00039(11) & 0.00039(11) & 0.00038(10) & 0.00039(10) \\
\hline
z(T \otimes V)(T \otimes V)_{I} & -0.0030(14) & -0.0030(14) & -0.0029(13) & -0.0029(14) \\
\hline
z(T \otimes V)(T \otimes V)_{II} & 0.0096(42) & 0.0096(42) & 0.0094(42) & 0.0096(42) \\
\hline
z(A \otimes S)(A \otimes S)_{I} & -0.0007(43) & -0.0007(44) & -0.0007(43) & -0.0007(44) \\
\hline
z(A \otimes S)(A \otimes S)_{II} & 0.0027(14) & 0.0027(14) & 0.0027(13) & 0.0027(14) \\
\hline
z(P \otimes V)(P \otimes V)_{I} & -0.0685(17) & -0.0685(17) & -0.0673(17) & -0.0683(17) \\
\hline
z(P \otimes V)(P \otimes V)_{II} & -0.0317(23) & -0.0317(23) & -0.0311(22) & -0.0316(23) \\
\hline
\end{array}
The effect of off-diagonal operators

The matrix element of the wrong taste operator with external $K$ meson ($P \otimes P$) is suppressed. Ref.[Thesis, Weonjong Lee]

The wrong taste channel with same normalization with $B_K$ is.

\[
\frac{\langle K_0 | O_{(P \otimes V)(P \otimes V)} | K_0 \rangle}{\frac{8}{3} \langle K_0 | A_\mu | 0 \rangle \langle 0 | A_\mu | K_0 \rangle} \leq 1\% \quad (55)
\]

The size of wrong taste channel is smaller than 1% of $B_K$ operator.

For example, the effect of $(P \otimes V)(P \otimes V)_I$ channel to $B_K$ is

\[
z_{(P \otimes V)(P \otimes V)_I} \times 1\% \leq 0.069\% \quad (56)
\]

Hence, this effect of wrong taste channel is neglected.
We compare the NPR result ($\overline{\text{MS}}$ [NDR]) with those of one-loop perturbative matching.

|       | NPR(2GeV)      | one-loop(2GeV) | NPR(3GeV)      | one-loop(3GeV) |
|-------|----------------|----------------|----------------|----------------|
| $z_1$ | 1.0139(82)(274)| 1.080(95)      | 0.9812(79)(163)| 1.035(62)      |
| $z_2$ | 1.148(32)(30)  | 1.168(102)     | 1.111(31)(17)  | 1.120(67)      |
| $z_3$ | 0.9586(77)(269)| 1.088(95)      | 0.9277(74)(171)| 1.043(63)      |
| $z_4$ | 0.942(30)(25)  | 0.994(87)      | 0.912(29)(14)  | 0.953(57)      |

We estimate the systematic error of one-loop result as two-loop uncertainty ($\alpha_s^2$).

\[
(\text{sys. err. of } z_i) = z_i \cdot \alpha_s^2
\]  \hspace{1cm} (57)
We have done the first round data analysis for the $B_K$ operator.
We plan to analyse the BSM operators in near future.