Madelung, Gross–Pitaevskii and Korteweg

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Abstract
This paper surveys various aspects of the hydrodynamic formulation of the nonlinear Schrödinger equation obtained via the Madelung transform in connection to models of quantum hydrodynamics and to compressible fluids of the Korteweg type.

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1. Introduction

In his seminal work [37] (see also [33]), E Madelung introduced the so-called Madelung transform in order to relate the linear Schrödinger equation to a hydrodynamic type system. This system takes (slightly) different forms according to the context: a linear or nonlinear equation with various nonlinearities. For the semi-classical nonlinear Schrödinger equation (NLS):

\[ i \epsilon \partial_t \psi^\epsilon + \frac{\epsilon^2}{2} \Delta \psi^\epsilon = f \left( |\psi^\epsilon|^2 \right) \psi^\epsilon, \]

the Madelung transform amounts to setting

\[ \psi^\epsilon (t, x) = \sqrt{\rho(t, x)} e^{i \phi(t, x)/\epsilon}, \]

so as to obtain the following system for \( \rho \) and \( v = \nabla \phi \):

\[ \partial_t v + v \cdot \nabla v + \nabla f(\rho) = \frac{\epsilon^2}{2} \nabla \left( \frac{\Delta \left( \sqrt{\rho} \right)}{\sqrt{\rho}} \right), \]
\[ \partial_t \rho + \text{div} (\rho v) = 0. \]

This system is referred to as the hydrodynamic form of NLS because of its similarity with the compressible Euler equation (which corresponds to \( \epsilon = 0 \)). The additional term on the right-hand side is the so-called quantum pressure.
The Madelung transform is crucial to investigate qualitative properties of the nonlinear Schrödinger equation with non-zero boundary conditions at infinity whenever the solution is not expected to vanish ‘too often’. Of particular interest is the so-called Gross–Pitaevskii equation

\[ i\varepsilon \partial_t \psi^\varepsilon + \frac{\varepsilon^2}{2} \Delta \psi^\varepsilon = (|\psi^\varepsilon|^2 - 1) \psi^\varepsilon, \]

which corresponds to \( f(r) = r - 1 \) or, more generally, the case where \( f \) is a smooth function vanishing at some \( r_0 \) with \( f'(r_0) < 0 \). This covers, in particular, the ‘cubic-quintic’ NLS \( f(r) = -\alpha_1 + \alpha_3 r - \alpha_5 r^2 \) with \( \alpha_1, \alpha_3, \alpha_5 > 0 \).

Using the Madelung transform is also rather popular to study the semi-classical limit of NLS. This requires \( \rho \) to be non-vanishing, though. We shall see below to what extent the presence of vacuum (that is of points where \( \rho \) vanishes) is merely a technical problem due to the approach related to the Madelung transform.

The hydrodynamic form of NLS may be seen as a particular case of the system of quantum fluids (QHD) with a suitable choice of the pressure law, which, in turn, enters in the class of Korteweg (or capillary) fluids. Thus, (1.1) and the QHD have been used in different frameworks, by different mathematical and physical communities: to investigate the semi-classical limit in Schrödinger equations, to study travelling waves in the Gross–Pitaevskii equation and the transonic limit and to describe capillary fluids. We aim here at further investigating the link between these three a priori disjoint domains, through various uses of the Madelung transform, so that connections appear more explicitly. In passing, we will review some more or less known facts pertaining to the theories of quantum fluids and Korteweg fluids, and provide a self-contained proof of a recent result of Antonelli and Marcati [3, 4], on the global existence of weak solutions to a quantum fluids system.

1.1. Organization of the paper

The paper is organized as follows. The second section is a review of the connections of the Madelung transform with the semi-classical limit of the nonlinear Schrödinger equation. In passing we briefly present the state-of-the-art for the Cauchy problem for the Gross–Pitaevskii equation and the compressible Euler equation consisting of setting \( \varepsilon = 0 \) in (1.1). The main message that we want to communicate in this section is that WKB analysis is more sensible than the use of the Madelung transform in many cases. In particular, we discuss the role of the vacuum. In section 3, we solve the quantum hydrodynamical system (1.1) by a direct method based on the use of an extended formulation, and explain how it may be adapted to tackle general Korteweg fluids. In section 4, we review the use of the Madelung transform and of the hydrodynamical form of the Gross–Pitaevskii equation to study the existence and properties of its travelling wave solutions and of their transonic limit, both in the steady and unsteady cases. We give in section 5 a proof of the aforementioned result of Antonelli and Marcati. Finally, we list in the appendix the basic conservation laws for the Schrödinger, the QHD and the compressible Euler equations, and show that some of these laws naturally carry over to general Korteweg fluids.

1.2. Notations

- We denote by \( |\cdot|_p \) (\( 1 \leq p \leq \infty \)) the standard norm of the Lebesgue spaces \( L^p(\mathbb{R}^d) \).
- The standard \( H^s(\mathbb{R}^d) \) Sobolev norm will be denoted \( \|\cdot\|_s \). We use the Fourier multiplier notation: \( f(D)u \) is defined as \( \mathcal{F}(f(D)u)(\xi) = f(\xi)\widehat{u}(\xi) \), where \( \mathcal{F} \) and \( \widehat{\cdot} \) stand for the Fourier transform.
The operator $\Lambda = (1 - \Delta)^{1/2}$ is equivalently defined using the Fourier multiplier notation to be $\Lambda = (1 + |D|^2)^{1/2}$.

The partial derivatives will be denoted with a subscript, e.g. $u_t, u_x, \ldots$ or $\partial_t u, \partial_x u, \ldots$ or even $\partial_j$ (to designate $\partial_x$).

$C$ will denote various non-negative absolute constants, the meaning of which will be clear from the context.

2. Madelung transform and the semi-classical limit of NLS

In this section, we analyze the relevance of the Madelung transform when the semi-classical limit of the nonlinear Schrödinger equation is studied. We point out that some technical difficulties which may be present in the QHD formulation vanish when the WKB analysis point of view is adopted instead. It also turns out that the boundary conditions at infinity play an important role in the discussion.

As already mentioned in the introduction, we consider the equation

$$i\varepsilon \partial_t \psi^\varepsilon + \frac{\varepsilon^2}{2} \Delta \psi^\varepsilon = f(|\psi^\varepsilon|^2) \psi^\varepsilon, \quad \psi^\varepsilon(0,x) = \sqrt{\rho_0(x)} e^{i\phi_0(x)/\varepsilon}. \quad (2.1)$$

Here we have in mind the limit $\varepsilon \to 0$. The space variable $x$ belongs to $\mathbb{R}^d$ in this paragraph.

The periodic case $x \in T^d$ being of particular interest for numerical simulations however (see e.g. [6, 23] and references therein), we will state some analogous results in that case, which turn out to be a little simpler than in the case $x \in \mathbb{R}^d$. In this section, we shall focus on two types of nonlinearity:

- Cubic, defocusing nonlinearity:
  $$f(|\psi^\varepsilon|^2) \psi^\varepsilon = |\psi^\varepsilon|^2 \psi^\varepsilon.$$

  The Hamiltonian associated with (2.1) then reads
  $$\mathcal{H}^{\varepsilon}_{NLS}(\psi^\varepsilon) = \|\varepsilon \nabla \psi^\varepsilon\|^2_{L^2} + \|\psi^\varepsilon\|^4_{L^4}.$$

  It is well defined on the Sobolev space $H^1(\mathbb{R}^d)$ in dimension $d \leq 4$.

- Gross–Pitaevskii equation:
  $$f(|\psi^\varepsilon|^2) \psi^\varepsilon = (|\psi^\varepsilon|^2 - 1) \psi^\varepsilon.$$

  In this case, the natural Hamiltonian associated with (2.1) is
  $$\mathcal{H}^{\varepsilon}_{GP}(\psi^\varepsilon) = \|\varepsilon \nabla \psi^\varepsilon\|^2_{L^2} + \||\psi^\varepsilon|^2 - 1\|^2_{L^2}.$$

In both cases, the Hamiltonian defines an energy space, in which existence and uniqueness for the Cauchy problem (2.1) have been established.

The case of a defocusing cubic nonlinearity is now well understood. In dimension $d \leq 4$ the corresponding NLS equation is globally well-posed in $H^s(\mathbb{R}^d)$, and the additional $H^s(\mathbb{R}^d)$ regularity ($s \geq 1$) is propagated (see the textbooks [18, 35, 44] and references therein).

The situation is more complicated for the Gross–Pitaevskii equation, where the finite energy solutions $\psi^\varepsilon$ cannot be expected to be in $L^2(\mathbb{R}^d)$, since $|\psi^\varepsilon|^2 - 1 \notin L^2(\mathbb{R}^d)$. As noticed in [49, 50], and extended in [24], a convenient space to study the Gross–Pitaevskii equation is the Zhidkov space:

$$X^s(\mathbb{R}^d) = \{ \psi \in L^\infty(\mathbb{R}^d) ; \nabla \psi \in H^{s-1}(\mathbb{R}^d) \} \quad \text{with} \quad s > d/2.$$
Remark 2.1. In the case \( x \in \mathbb{T}^d \), the spaces \( H^s(\mathbb{T}^d) \) and \( X^s(\mathbb{T}^d) \) (with obvious definitions) are the same.

In the case \( x \in \mathbb{R}^d \), as a consequence of the Hardy–Littlewood–Sobolev inequality (see e.g. [32, theorem 4.5.9] or [28, lemma 7]), one may show that if \( d \geq 2 \) and \( \psi \in \mathcal{D}'(\mathbb{R}^d) \) is such that \( \nabla \psi \in L^p(\mathbb{R}^d) \) for some \( p \in [1, d[ \), then there exists a constant \( \gamma \) such that \( \psi - \gamma \in L^q(\mathbb{R}^d) \), with \( 1/p = 1/q + 1/d \). Morally, \( \gamma \) is the limit of \( \psi \) at infinity. If \( d \geq 3 \) then we can take \( p = 2 \), so every function in \( X^s(\mathbb{R}^d) \) satisfies the above property; for \( d = 2 \), the above assumption requires a little more decay on \( \nabla \psi \) than general functions in \( X^s(\mathbb{R}^2) \).

The well-posedness issue in the natural energy space

\[
E(\mathbb{R}^d) = \{ \psi \in H^1_{loc}(\mathbb{R}^d), \nabla \psi \in L^2(\mathbb{R}^d), |\psi|^2 - 1 \in L^2(\mathbb{R}^d) \}
\]

associated with the Gross–Pitaevskii equation has been investigated only very recently by Gallo in [25] and Gérard in [28, 29], in dimension \( d \leq 4 \).

We emphasize that in the \( \mathbb{R}^d \) case, the energy space is no longer a linear space (contrary to the case of zero boundary condition a infinity where it is \( H^1(\mathbb{R}^d) \)), hence solving the Gross–Pitaevskii equation in \( E(\mathbb{R}^d) \) is more complicated. However, if \( d = 3, 4 \), one may show that \( E(\mathbb{R}^d) \) coincides with

\[
\{ \psi = c(1 + \epsilon), c \in \mathbb{S}^1, \psi \in H^1(\mathbb{R}^d), 2 \text{Re}\psi + |\psi|^2 \in L^2(\mathbb{R}^d) \},
\]

which allows us to endow it with a structure of a metric space. The case \( d = 2 \) is slightly more technical.

2.1. Some issues related to the use of the Madelung transform

In the semi-classical context, the Madelung transform consists in seeking

\[
\psi^\varepsilon(t, x) = \sqrt{\rho(t,x)}e^{i\phi(t,x)/\varepsilon}
\]

for some \( \rho \geq 0 \) and real-valued function \( \phi \). Plugging (2.2) into (2.1) and separating real and imaginary values yields

\[
\begin{cases}
\sqrt{\rho} \left( \partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + f(\rho) \right) = \frac{\rho^2}{2} \Delta \sqrt{\rho}, & \phi|_{t=0} = \phi_0, \\
\partial_v \sqrt{\rho} + \nabla \phi \cdot \nabla \sqrt{\rho} + \frac{1}{2} \sqrt{\rho} \Delta \phi = 0, & \rho|_{t=0} = \rho_0.
\end{cases}
\]

Two comments are in order at this stage: the first equation shows that \( \phi \) depends on \( \varepsilon \) and the second equation shows that so does \( \rho \) in general. We shall underscore this fact using the notation \( (\phi^\varepsilon, \rho^\varepsilon) \). Second, the equation for \( \phi^\varepsilon \) can be simplified, provided that \( \rho^\varepsilon \) has no zero.

Introducing the velocity \( v^\varepsilon = \nabla \phi^\varepsilon \), (2.3) yields the system of quantum hydrodynamics (1.1) presented in the introduction.

To study the limit \( \varepsilon \to 0 \) (the Euler limit), it is natural to consider the following compressible Euler equation:

\[
\begin{cases}
\partial_t v + v \cdot \nabla v + \nabla f(\rho) = 0, & \quad v|_{t=0} = \nabla \phi_0, \\
\partial_t \rho + \text{div}(\rho v) = 0, & \quad \rho|_{t=0} = \rho_0.
\end{cases}
\]

Note that the solution to (2.4) must not be expected to remain smooth for all time, even if the initial data are smooth. In [38] (see also [47, corollary 1.4]), it is shown that compactly supported initial data lead to the formation of singularities in finite time, no matter how small they are. In [42], the author constructs a solution developing singularities in finite time, in the absence of vacuum. In [2], it is shown that for rotationally invariant two-dimensional data that are a perturbation of size \( \delta \ll 1 \) of a rest state, blow-up occurs at time \( T\delta \sim \tau/\delta^2 \).
Remark 2.2. In the framework of this section, we have \( f' = 1 > 0 \). As pointed out above, the case \( f'(r) = r^\sigma, \sigma \in \mathbb{N} \), could be considered as well. On the other hand, if \( f' < 0 \) (corresponding to the semi-classical limit for a focusing nonlinearity), (2.4) becomes an elliptic system which may be solved locally in time for analytic data (see [27, 46]). At the same time, in the case \( d = 1 \) and \( f' = -1 \), it has been shown in [40] that there are smooth initial data for which the Cauchy problem (2.4) has no solution. In short, working with analytic data in this context is not only convenient, it is mandatory.

We emphasize that the hydrodynamical formulations for the cubic NLS and the Gross–Pitaevskii equations are exactly the same as in both cases we have \( \nabla f(\rho) = \nabla \rho \). From this point of view, studying either of the equations is mainly a matter of boundary conditions at infinity: \( H^s(\mathbb{R}^d) \) is the appropriate space for the cubic NLS equation whereas \( X^s(\mathbb{R}^d) \) is adapted to the Gross–Pitaevskii equation.

In the following, \( Z^s \) denotes either \( H^s(\mathbb{R}^d) \) or \( X^s(\mathbb{R}^d) \). In addition, we set
\[
Z^\infty = \bigcap_{s > d/2} Z^s.
\]

Theorem 2.3. Let \( \rho_0, \phi_0 \in C^\infty(\mathbb{R}^d) \) with \( \sqrt{\rho_0}, \nabla \phi_0 \in Z^s \) for some \( s > d/2 + 1 \). There exists a unique maximal solution \( (v, \rho) \in C([0, T_{\text{max}}]; Z^s) \) to (2.4). In addition, \( T_{\text{max}} \) is independent of \( s > d/2 + 1 \) and
\[
T_{\text{max}} < +\infty \implies \int_0^{T_{\text{max}}} \| (v, \sqrt{\rho})'(t) \|_{W^{1,\infty}} \, dt = +\infty.
\]

Finally, if \( \nabla \phi_0 \) and \( \rho_0 \) are smooth, non-zero and compactly supported then \( T_{\text{max}} \) is finite.

We investigate the following natural questions:

**Question 1.** Assume that \( \rho_0(x) > 0 \) for all \( x \in \mathbb{R}^d \). Can we say that \( \rho'(t, x) > 0 \) for all \( x \in \mathbb{R}^d \) and \( t \in [0, T_{\text{max}}] \)? If not, what is the maximal interval allowed for \( t \)?

We will also recall that despite the appearance, the presence of vacuum (existence of zeros of \( \psi \)) is merely a technical problem: the Madelung transform ceases to make sense (pointwise), but a rigorous WKB analysis is available, regardless of the presence of vacuum (see section 2.3). More precisely, WKB analysis can be justified on a time interval \([0, T]\), with \( T > 0 \) independent of \( \varepsilon \), assuming only that the initial data are sufficiently smooth. We recall that in the linear case, such an approximation must be expected to be valid only locally in time, due to the formation of caustics. In the present nonlinear case, the WKB approximation must be expected to break down in finite time as well (as soon as the solution to (2.4) ceases to be smooth). The term *caustic* may not be appropriate, however, since it designates a geometric phenomenon, while the formation of singularities for (2.4) may be due to different effects (which are not fully understood so far, see [17] for a discussion on this issue). Note also that after singularities have appeared in (2.4), the decomposition (2.2) ceases to be relevant to describe oscillations, in the sense that \( \rho \) becomes \( \varepsilon \)-oscillatory, and so there is no reason to single out the phase \( \phi \) given by (1.1); in other words, the quantum pressure must not be expected to go to zero with \( \varepsilon \) for large time. From the results in [38], vacuum may be a cause of breakdown for the WKB analysis, since the compact support of the data ruins the regularity of the solution to (2.4) in finite time. However, in view of the above discussion, this cause can appear as rather indirect.

**Question 2.** Let \( \rho_0, \phi_0 \in C^\infty(\mathbb{R}^d) \) with \( \rho_0, \nabla \phi_0 \in H^s(\mathbb{R}^d) \) for some \( s > d/2 + 1 \). Suppose that the solution \((v, \rho)\) to (2.4) satisfies \( \rho(t, x) > 0 \) for \((t, x) \in [0, \tau] \times \mathbb{R}^d \). Can we construct a solution to (1.1) in \( C([0, \tau_*]; H^s) \) (possibly with \( 0 < \tau_* < \tau \))?
We will see that, in general, the answer for this question is no. Even though from the answer to the first question, (1.1) makes sense formally, the analytical properties associated with (1.1) are not as favourable as for (2.4). Typically, the right-hand side of the equation for the quantum velocity need not belong to $L^2(\mathbb{R}^d)$.

**Question 3.** Let $\rho_0, \phi_0 \in C^\infty(\mathbb{R}^d)$ with $\rho_0, \nabla \phi_0 \in X^s(\mathbb{R}^d)$ for some $s > d/2 + 1$. Suppose that the solution $(v, \rho)$ to (2.4) satisfies $\rho(t, x) > 0$ for $(t, x) \in [0, \tau] \times \mathbb{R}^d$. Can we construct a solution to (1.1) in $C([0, \tau_*]; X^s)$ (possibly with $0 < \tau_* < \tau$)?

### 2.2. Proof of theorem 2.3

We shall simply give the main ideas of the proof of theorem 2.3. Complete proofs can be found in [38] for the cubic defocusing NLS equation in Sobolev spaces, and in [1] for the Gross–Pitaevskii equation in Zhidkov spaces.

In the framework of this paper, we have

$$\nabla f(\rho) = \nabla \rho.$$  

Introduce formally the auxiliary function $a = \sqrt{\rho}$. This nonlinear change of variable makes (2.4) hyperbolic symmetric:

$$\begin{aligned}
\partial_t v + v \cdot \nabla v + 2a \nabla a &= 0, \\
\partial_t a + v \cdot \nabla a + \frac{1}{2} a \nabla \cdot v &= 0,
\end{aligned}$$

where $a|_{t=0} = \sqrt{\rho}_0$. (2.5)

This system is of the form

$$\partial_t u + \sum_{j=1}^d A_j(u) \partial_j u = 0,$$

and the matrices $A_j$ are symmetrized by the constant multiplier

$$S = \begin{pmatrix} I_d & 0 \\ 0 & 4 \end{pmatrix}.$$  

Standard analysis (see e.g. [45]) shows that (2.5) has a unique maximal solution $(v, a)$ in $C([0, T_{\text{max}}]; Z')$, provided that $s > d/2 + 1$ and that, in addition,

$$T_{\text{max}} < +\infty \implies \int_0^{T_{\text{max}}} \| (v, a)(t) \|_{W^{1,\infty}} \, dt = +\infty.$$  

We can then define $\rho$ by the linear equation

$$\partial_t \rho + \text{div} (\rho v) = 0,$$

where $\rho|_{t=0} = \rho_0$.

By uniqueness for this linear equation, $\rho = a^2$ (a is real-valued, so $\rho$ is non-negative), and $(v, \rho)$ solves (2.4).

We now briefly explain why compactly supported initial data lead to the formation of singularities in finite time. The first remark is that in this case, the solution to (2.4) has a finite speed of propagation, which turns out to be zero: so long as $(v, \rho)$ is smooth, it remains supported in the same compact as its initial data. To see this, consider the auxiliary system (2.5): the first equation is a Burgers’ equation with source term $2a \nabla a$; the second equation is an ordinary differential equation along the trajectories of the particles. Define the trajectory by

$$\frac{d}{dt} x(t, y) = v(t, x(t, y)), \quad x(0, y) = y.$$  

(2.6)
For $0 \leq t < T_{\text{max}}$, this is a global diffeomorphism of $\mathbb{R}^d$, as shown by the equation
\[
\frac{d}{dt} \nabla_x(t, y) = \nabla v(t, x(t, y)) \nabla_x(t, y), \quad \nabla_x(0, y) = \text{Id},
\]
and Gronwall lemma. Therefore, for a smooth function $f$,
\[
(\partial_t f + v \cdot \nabla f) (t, x(t, y)) = \partial_t (f(t, x(t, y))),
\]
and (2.5) can be viewed as a system of ordinary differential equations.

Once the non-propagation of the support of smooth solutions is established, the end of the proof relies on a virial computation (like in [48, 30], see also [18]). This computation shows that global in time smooth solutions to (2.5) are dispersive (see also [41]). This is incompatible with the zero propagation speed of smooth compactly supported solutions. Therefore, singularities have to appear in finite time.

2.3. A review of WKB analysis associated with (2.1)

We consider initial data which are a little more general than in (2.1), namely
\[
\psi^\epsilon(0, x) = a_0^\epsilon(x) e^{i\phi^\epsilon(x)/\epsilon},
\]
where the initial amplitude $a_0^\epsilon$ is assumed to be smooth, complex-valued, and possibly depending on $\epsilon$. Typically, we assume that there exist $a_0, a_1 \in Z^\infty$ independent of $\epsilon$ such that
\[
a_0^\epsilon = a_0 + \epsilon a_1 + \mathcal{O}(\epsilon^2) \text{ in } Z^s, \quad \forall s > d/2. \tag{2.8}
\]

2.3.1. First order approximation.

Introduce the solution to the quasilinear system
\[
\begin{cases}
\partial_t \psi + \frac{i}{2} |\nabla \psi|^2 + f(|\psi|^2) = 0, & \psi|_{t=0} = \psi_0, \\
\partial_t a + \nabla \psi \cdot \nabla a + \frac{1}{2} a \Delta \psi = 0, & a|_{t=0} = a_0.
\end{cases} \tag{2.9}
\]

Theorem 2.3 shows that (2.9) has a unique, smooth solution with $a, \nabla \psi \in Z^\infty$. The main remark consists in noticing that (2.9) implies that $(\nabla \psi, |a|^2)$ has to solve (2.4) ($a_0$ may be complex-valued): theorem 2.3 yields $v, \rho \in Z^\infty$. We can then define $a$ as the solution to the linear transport equation
\[
\partial_t a + v \cdot \nabla a + \frac{1}{2} a \nabla \cdot v = 0, \quad a|_{t=0} = a_0.
\]
Now $|a|^2$ and $\rho$ solve the same linear transport equation, with the same initial data, hence $\rho = |a|^2$. Using this information in the equation for the velocity, define
\[
\psi(t) = \psi_0 - \int_0^t \left( \frac{1}{2} |v(\tau)|^2 + f(|a(\tau)|^2) \right) d\tau.
\]

We easily check that $\partial_t (\nabla \psi - v) = \nabla \partial_t \psi - \partial_t v = 0$, and that $(\psi, a)$ solves (2.9).

Introduce the solution to the linearization of (2.9), with an extra source term:
\[
\begin{cases}
\partial_t \psi^{(1)} + \nabla \psi \cdot \nabla \psi^{(1)} + 2 \text{Re}(\overline{a}^{(1)} f'(|a|^2)) = 0, & \psi^{(1)}|_{t=0} = 0, \\
\partial_t a^{(1)} + \nabla \psi \cdot \nabla a^{(1)} + \nabla \psi^{(1)} \cdot \nabla a + \frac{1}{2} a \Delta \psi^{(1)} + \frac{i}{2} a \Delta a^{(1)} + \frac{i}{2} a \Delta \psi^{(1)} = \frac{i}{2} \Delta a, & a^{(1)}|_{t=0} = a_1.
\end{cases}
\]

We also check that it has a unique smooth solution, with $a^{(1)}, \nabla \psi^{(1)} \in Z^\infty$. The main result that we will invoke is the following:

Proposition 2.4. Let $a_0^\epsilon, a_0, a_1, \phi_0$ be smooth, with $a_0^\epsilon, a_0, a_1, \nabla \phi_0 \in Z^\infty$. Assume that (2.8) holds. Then
\[
\psi^\epsilon = \left( a e^{i\phi^\epsilon} + \mathcal{O}(\epsilon) \right) e^{i\psi^\epsilon/\epsilon} \text{ in } L^\infty([0, \tau]; Z^s), \quad \forall \tau < T_{\text{max}} \text{ and } \forall s \geq 0. \tag{2.10}
\]
This result was established in [31] when $Z' = H^s(\mathbb{R}^d)$, and in [1] when $Z' = X^s(\mathbb{R}^d)$. Note the shift between the order of the approximation between the initial data (known up to $O(\varepsilon^2)$) and the approximation (of order $O(\varepsilon)$ only): this is due to the fact that we consider a regime which is super-critical as far as WKB analysis is concerned (see e.g. [17]). In particular, the phase modulation $\psi^{(1)}$ is a function of $\psi_0$, $a_0$ and $a_1$. It is non-trivial in general, and since we are interested here in real-valued $a_0$, we shall merely mention two cases (see [17, pp 69–70]):

- If $a_1 \neq 0$ is real-valued, then $\psi^{(1)} \neq 0$ in general.
- If $a_1 = 0$ (or more generally if $a_1 \in i\mathbb{R}$), then $\psi^{(1)} = 0$.

To see the first point, it suffices to note that the equation for $\psi^{(1)}$ gives (recall that $f' = 1$):

$$\partial_t \psi^{(1)}(\varepsilon t) = -2a_0a_1.$$  

This shows that for (1.1)–(2.2) to yield a relevant description of the solution to (2.1), we have to assume $a_1 = 0$. Note finally that even though $a_1 = 0$, the corrector of order $O(\varepsilon)$ admits a corrector of order $O(\varepsilon^2)$ which is super-critical as far as WKB analysis is concerned (see e.g. [17]). In particular, the shift between the order of the approximation between the initial data to $O(\varepsilon^2)$ only: this is due to the fact that we consider a regime which is super-critical as far as WKB analysis is concerned (see e.g. [17]).

This is a linear, homogeneous system, with zero initial data, so its solution is identically zero.

To check the second point of the above assertion, we set $\alpha = \text{Re}(\overline{\alpha}a^{(1)})$. Direct computations show that $(\psi^{(1)}, \alpha)$ solves, as soon as $a_0 \in \mathbb{R}$ and $a_1 \in i\mathbb{R}$:

$$\begin{cases}
\partial_t \psi^{(1)} + \nabla \psi \cdot \nabla \psi^{(1)} + 2\alpha = 0, \\
\partial_t \alpha + \nabla \psi \cdot \nabla \alpha = -\frac{1}{2} \text{div} (|a|^2 \nabla \psi^{(1)}) - \alpha \Delta \psi, \\
\psi^{(1)}|_{t=0} = 0, \\
\alpha|_{t=0} = 0.
\end{cases}$$  

This is a linear, homogeneous system, with zero initial data, so its solution is identically zero.

To conclude this paragraph, we briefly outline the proof of proposition 2.4. The approach in [1] is the same as in [31], with slightly different estimates. For simplicity, and in view of the above discussion, we assume $a_0 = a_0$ independent of $\varepsilon$. We write the solution $\psi^\varepsilon$ as $\psi^\varepsilon = a^\varepsilon \psi^\varepsilon|_{\varepsilon=1}$ (exact formula), where we impose

$$\begin{cases}
\partial_t \psi^\varepsilon + \frac{1}{2} \nabla \psi^\varepsilon|^2 + f (|a^\varepsilon|^2) = 0, \\
\partial_t a^\varepsilon + \nabla \psi^\varepsilon \cdot \nabla a^\varepsilon + \frac{1}{2} a^\varepsilon \Delta \psi^\varepsilon = \frac{\varepsilon}{2} \Delta a^\varepsilon, \\
\psi^\varepsilon|_{t=0} = \psi_0, \\
a^\varepsilon|_{t=0} = a_0.
\end{cases}$$  

(2.11)

Note that both $\psi^\varepsilon$ and $a^\varepsilon$ depend on $\varepsilon$, because the right-hand side of the equation for $a^\varepsilon$ depends on $\varepsilon$, and because of the coupling between the two equations. Note also that with this approach, one abandons the possibility of considering a real-valued amplitude $a^\varepsilon$.

It is not hard to construct a solution to (2.11) in $Z^s$, for $s > d/2 + 2$, and then check that asymptotic expansions are available in $Z^s$:

$$a^\varepsilon = a + \varepsilon a^{(1)} + O(\varepsilon^2), \quad \psi^\varepsilon = \psi + \varepsilon \psi^{(1)} + O(\varepsilon^2).$$

Back to $\psi^\varepsilon$, this yields proposition 2.4. We see that the general loss in the precision (from $O(\varepsilon^2)$ in the initial data to $O(\varepsilon)$ in the approximation for $t > 0$) is due to the division of $\psi^\varepsilon$ by $\varepsilon$. Note finally that even though $a_1 = 0$, one has $a^{(1)} \neq 0$: the corrector $a^{(1)}$ solves a linear equation, with a purely imaginary (non-trivial) source term, and so $a^{(1)} \in i\mathbb{R}$ is not trivial, while $\psi^{(1)} = 0$ since $\text{Re}(\overline{\alpha}a^{(1)}) = 0$.

### 2.3.2. Higher order approximation and formal link with quantum hydrodynamics

One can actually consider an asymptotic expansion to arbitrary order,

$$a^\varepsilon = a + \varepsilon a^{(1)} + \cdots + \varepsilon^N a^{(N)} + O(\varepsilon^{N+1}) \ , \quad \psi^\varepsilon = \psi + \varepsilon \psi^{(1)} + \cdots + \varepsilon^N \psi^{(N)} + O(\varepsilon^{N+1}) \ , \quad \forall N \in \mathbb{N}.$$
For $j \geq 1$, the coefficient $a^{(j)}$ is given by a linear system for $(\varphi^{(j)}, a^{(j)})$, with source terms involving $(\varphi^{(k)}, a^{(k)})_{0 \leq k < j - 1}$. In the case $a_0 \in \mathbb{R}$ and $a_1 = 0$, we know that $\varphi^{(1)} = 0$, and $a^{(2)}$ is given by

$$\partial_0 a^{(2)} + \nabla \varphi \cdot \nabla a^{(2)} + \frac{1}{2} a^{(2)} \Delta \varphi + \nabla \varphi^{(2)} \cdot \nabla a + \frac{1}{2} a \Delta \varphi^{(2)} = \frac{i}{2} \Delta a^{(1)}, \quad a^{(2)}_{\mu=0} = 0.$$  

We check that in the case $a_0 = a_0 \in \mathbb{R}$ (which includes the case where the Madelung transform is used, $a_0 \geq 0$), all the profiles $a$ and $a^{(2j)}$, $j \geq 1$, are real-valued, while $a^{(2j+1)}$, $j \geq 0$, are purely imaginary. Moreover, $\varphi^{(2j+1)} = 0$ for all $j \in \mathbb{N}$. This is formally in agreement with (1.1): indeed, (1.1) suggests that $\varphi$ and $\rho$ have asymptotic expansions of the form

$$\varphi = \phi + \varepsilon^2 \phi^{(2)} + \ldots + \varepsilon^{2j} \phi^{(2j)} + \ldots, \quad \rho = \rho_0 + \varepsilon^2 \rho^{(2)} + \ldots + \varepsilon^{2j} \rho^{(2j)} + \ldots. \quad (2.12)$$

On the other hand, we have

$$\rho = |a|^2 \approx |a + \varepsilon a^{(1)} + \ldots|^2 \approx (a + \cdots + \varepsilon^2 a^{(2j)} + \ldots)^2 - (\varepsilon a^{(1)} + \cdots + \varepsilon^{2j+1} a^{(2j+1)} + \ldots)^2,$$

since the $a^{(2j+1)}$s are purely imaginary. This is in agreement with the second formal asymptotics in (2.12). We can check similarly that (2.12) is in agreement with the higher order generalization of (2.10), in view of the special properties of the $\varphi^{(j)}$s and $a^{(j)}$s pointed out above.

### 2.4. Absence of vacuum before shocks

**Lemma 2.5.** In theorem 2.3, assume that $\rho_0(x) > 0$ for all $x \in \mathbb{R}^d$ (absence of vacuum). Then $\rho > 0$ on $[0, T_{\max}) \times \mathbb{R}^d$.

**Proof.** As in section 2.2, we use the fact that on $[0, T_{\max})$, the equation for the density is just an ordinary differential equation. Introduce the Jacobi determinant

$$J_r(y) = \det \nabla_y x(t, y),$$

where $x(t, y)$ is given by (2.6). We have seen that $J_r(y) > 0$ for $(t, y) \in [0, T_{\max}) \times \mathbb{R}^d$.

Change the unknown $\rho$ to $r$, with

$$r(t, y) = \rho (t, x(t, y)) J_r(y).$$

Then for $0 \leq t < T_{\max}$, the continuity equation is equivalent to: $\partial_t r = 0$. Therefore,

$$\rho(t, x) = \frac{1}{J_r(y(t, x))} \rho_0(y(t, x)),$$

where $x \mapsto y(t, x)$ denotes the inverse mapping of $y \mapsto x(t, y)$.

We infer:

**Proposition 2.6.** Under the assumptions of theorem 2.3, assume that $\rho_0(x) > 0$ for all $x \in \mathbb{R}^d$ (absence of vacuum). Let $0 < T < T_{\max}$, and $K$ be a compact set in $\mathbb{R}^d$. There exists $\varepsilon(T, K) > 0$ such that for $0 < \varepsilon \leq \varepsilon(T, K)$, $|\varphi^\varepsilon| > 0$ on $[0, T] \times K$.

**Proof.** Proposition 2.4 shows that

$$|\varphi^\varepsilon| = |a| + \mathcal{O}(\varepsilon) \quad \text{ in } L^\infty ([0, T] \times \mathbb{R}^d).$$

Note that the constant involved in this $\mathcal{O}(\varepsilon)$ depends on $T$ in general. Recalling that $a = \sqrt{\rho}$, lemma 2.5 shows that

$$\min_{(t, x) \in [0, T] \times K} a(t, x) = c(T, K) > 0.$$
Now for $0 < \varepsilon \leq \varepsilon(T, K) \ll 1$,
\[ ||\psi^\varepsilon| - |a|| \leq \frac{1}{\varepsilon} c(T, K) \quad \text{in } L^\infty([0, T] \times \mathbb{R}^d), \]
and the result follows.

In the case $x \in \mathbb{T}^d$, this shows that before the formation of shocks in the Euler equation, and provided that $\varepsilon$ is sufficiently small, the amplitude remains positive: the right-hand side of (1.1) makes sense. This point was remarked initially in [27]. Note that the result of [26] in the one-dimensional case $x \in [0, 1]$ shows that suitable boundary conditions lead to the existence of finite time blow-up for (1.1). Therefore, the above result is qualitatively sharp (qualitatively only, for it might happen that the solution to (1.1) remains smooth longer than the solution to (2.4)).

Finally, we show that the compactness assumption in proposition 2.6 can be removed in the case of the Gross–Pitaevskii equation. As regards the nonlinear Schrödinger equation on $\mathbb{R}^d$, this issue seems much more delicate and will not be addressed in this paper. Assume that the Gross–Pitaevskii equation is associated with the boundary condition at infinity
\[ |\psi^\varepsilon(t, x)| \to 1 \quad \text{as } |x| \to \infty. \]
Such a condition is used frequently in physics, possibly with a stronger one, of the form (see e.g. [36] and references therein)
\[ \psi^\varepsilon(t, x) - e^{-i\frac{v^\infty \cdot x}{\varepsilon}} \to 0 \quad \text{as } |x| \to \infty, \]
for some fixed asymptotic ‘velocity’ $v^\infty \in \mathbb{R}^d$.

Now, as regards the Gross–Pitaevskii equation, putting together the continuity of $\psi^\varepsilon$ over $[0, T] \times \mathbb{R}^d$, the compactness of the time interval $[0, T]$ and proposition 2.6, we obtain

**Corollary 2.7.** Under the assumptions of theorem 2.3, assume that $\rho_0(x) > 0$ for all $x \in \mathbb{R}^d$ (absence of vacuum). Assume moreover that the Gross–Pitaevskii equation is associated with the boundary condition at infinity\(^5\)
\[ |\psi^\varepsilon(t, x)| \to 1 \quad \text{as } |x| \to \infty. \]
Let $0 < T < T_{\max}$. There exist $\varepsilon(T), c(T) > 0$ such that
\[ |\psi^\varepsilon| \geq c(T) \quad \text{on } [0, T] \times \mathbb{R}^d, \quad \text{for all } 0 < \varepsilon \leq \varepsilon(T). \]

2.5. **Functional spaces associated with the Madelung transform**

It is rather easy to see that the answer to question 2 is no, in general. Consider for $\rho_0$ the function in the Schwartz class
\[ \rho_0(x) = e^{-|x|^{2k}}, \quad k \geq 1. \]
At time $t = 0$, the quantum pressure (right-hand side of (1.1)) grows like $|x|^{2k-1}$ hence the velocity $v^\varepsilon$ has no chance to belong to $H^s$ for general initial data in $H^\infty$. Thus, working in Sobolev spaces for general initial data does not make sense for (1.1), while the results in [31] show that it is a fairly reasonable framework to study the semi-classical limit of (2.1).

On the other hand, like for the absence of vacuum, the answer to question 3 is positive, at least if we consider some special boundary conditions at infinity.

\(^5\) Note that this implies that there exists $c > 0$ such that $\rho_0(x) \geq c$ for all $x \in \mathbb{R}^d$. 

3. Solving the QHD system by a direct approach

In this section, we describe an efficient method to solve directly the hydrodynamic form of (2.1) given by (1.1), once having performed the Madelung transform. This method enables us to study the corresponding initial value problem for (1.1) with data \((v_0, \rho_0)\) such that \((v_0, \nabla \rho_0)\) has a high order Sobolev regularity and \(\rho_0\) is positive and bounded away from zero. In addition to local-in-time well-posedness results, we obtain (see theorems 3.1 and 3.3) non-trivial lower bounds on the first appearance of a zero for the solution, which are of particular interest for the study of long-wavelength asymptotics if the data are a perturbation of a constant state of modulus one.

Here, we closely follow the approach that was initiated in [8]. To help the reader to compare the present results with those of the previous section, however, we continue using the semi-classical scaling given by (2.1) (whereas \(\epsilon = 1\) in [8]).

The use of a suitable extended formulation for (1.1) and of weighted Sobolev estimates is the key to our approach. We stress that, recently, similar extended formulations have proved to be efficient in other contexts for both numerical (see [21]) and theoretical purposes. To exemplify this, in the last paragraph of this section, we briefly explain how our approach based on such an extended formulation carries over to the more complicated case of Korteweg fluids.

3.1. Solving the QHD system by means of an extended formulation

The ‘improved’ WKB method that has been described in the previous section amounts to writing the sought solution \(\psi^\epsilon\) as

\[
\psi^\epsilon = a^\epsilon e^{i\varphi^\epsilon/\epsilon}
\]

for some complex-valued function \(a^\epsilon\) and real-valued function \(\varphi^\epsilon\). In this section, we instead start from the Madelung transform

\[
\psi^\epsilon = \sqrt{\rho^\epsilon} e^{i\phi^\epsilon}
\]

where \(\sqrt{\rho^\epsilon} = |\psi^\epsilon|\),

then write

\[
\psi^\epsilon = e^{i\Phi^\epsilon/\epsilon} \text{ with } \Phi^\epsilon = \phi^\epsilon - i\frac{\epsilon}{2} \log \rho^\epsilon,
\]

and consider the redundant system that is satisfied by both \(\rho^\epsilon\) and \(z^\epsilon = \nabla \phi^\epsilon = v^\epsilon + iw^\epsilon\) with \(v^\epsilon = \nabla \phi^\epsilon\) and \(w^\epsilon = -\frac{i\epsilon}{2} \nabla \rho^\epsilon\).

In order to obtain the system for \(z^\epsilon\), we first differentiate the density equation in (1.1). This yields

\[
\partial_t w^\epsilon + \nabla (v^\epsilon \cdot w^\epsilon) = \frac{\epsilon}{2} \nabla \text{div} v^\epsilon.
\]

Next, we note that

\[
\frac{\epsilon^2}{2} \Delta \frac{\sqrt{\rho^\epsilon}}{\rho^\epsilon} = -\frac{\epsilon}{2} \text{div} w^\epsilon + \frac{1}{2} |w^\epsilon|^2.
\]

In consequence, the equation for \(v^\epsilon\) rewrites

\[
\partial_t v^\epsilon + v^\epsilon \cdot \nabla v^\epsilon - \frac{1}{2} \nabla |w^\epsilon|^2 + \frac{\epsilon}{2} \nabla \text{div} w^\epsilon + \nabla f(\rho^\epsilon) = 0.
\]

Of course, \(z^\epsilon\) is a potential vector-field, hence \(\nabla \text{div} z^\epsilon = \Delta z^\epsilon\) so that we eventually obtain the following ‘extended’ system for \((\rho^\epsilon, z^\epsilon)\):

\[
\begin{align*}
\partial_t z^\epsilon + \frac{1}{2} \nabla (z^\epsilon \cdot z^\epsilon) + \nabla f(\rho^\epsilon) &= \frac{i\epsilon}{2} \Delta z^\epsilon, \\
\partial_t \rho^\epsilon + \text{div}(\rho^\epsilon v^\epsilon) &= 0,
\end{align*}
\]

where we agree that \(a \cdot b := \sum_{j=1}^d a_j b_j\) for \(a\) and \(b\) in \(\mathbb{C}^d\).
We now explain how Sobolev estimates may be derived for $(\rho^\varepsilon, z^\varepsilon)$ in the case where $\rho^\varepsilon = 1 + b^\varepsilon$ for some $b^\varepsilon$ going to 0 at infinity, and $f(\rho^\varepsilon) = \rho^\varepsilon$ (to simplify). The following computations are borrowed from [8]. For notational simplicity, we omit the superscripts $\varepsilon$.

In order to obtain the basic energy estimate, we compute $\varepsilon$:

$$\frac{d}{dt} \int_{\mathbb{R}^d} ((1 + b)|z|^2 + b^2) = 2 \left( \int_{\mathbb{R}^d} (1 + b)(z, \partial_t z) + \int_{\mathbb{R}^d} b \partial_t b \right) + \int_{\mathbb{R}^d} \partial_t b |z|^2,$$

where the notation $(a, b) = \sum_{j=1}^{d} (\text{Re} a_j \text{Re} b_j + \text{Im} a_j \text{Im} b_j)$ has been used in $I_1$.

Further computations yield $I_1 = I_{1,1} + I_{1,2} + I_{1,3} + I_{1,4} + I_{1,5}$ and $I_2 = I_{2,1} + I_{2,2}$ with

$$I_{1,1} = - \int (z, \nabla b), \quad I_{2,1} = - \int b \text{div} v,$$
$$I_{1,2} = - \int b(z, \nabla b), \quad I_{2,2} = - \int b \text{div}(b v).$$
$$I_{1,3} = \int \left( z, i \frac{\varepsilon}{2} \Delta z \right),$$
$$I_{1,4} = \int b \left( z, i \frac{\varepsilon}{2} \Delta z \right),$$
$$I_{1,5} = - \frac{1}{2} \int \rho(z, \nabla (z \cdot z)).$$

Using obvious integrations by parts we readily obtain

$$I_{1,1} + I_{2,1} = 0, \quad I_{1,2} + I_{2,2} = 0 \quad \text{and} \quad I_{1,3} = 0.$$

Therefore, integrating by parts in $I_{1,4}$ also, we obtain

$$\frac{d}{dt} \int ((1 + b)|z|^2 + b^2) = - \int \varepsilon (z, i \nabla z \cdot \nabla b) - 2 \int \rho(z, \nabla z \cdot z) + \int \partial_t b |z|^2.$$

For ‘general’ functions $b$ and $z$, the appearance of the terms $\nabla b$ and $\nabla z$ would preclude any attempt to ‘close’ the estimates. In our case, however, as the algebraic relation $-\varepsilon \nabla b = 2 \rho w$ holds true, one may avoid this loss of one derivative for one may write

$$-\varepsilon (z, i \nabla z \cdot \nabla b) - 2 \rho(z, \nabla z \cdot z) = -2 \rho(z, \nabla z \cdot v).$$

Now, integrating by parts an ultimate time, we conclude that

$$\frac{d}{dt} \int (\rho|z|^2 + b^2) = - \int \rho v \cdot \nabla |z|^2 + \int \partial_t b |z|^2 = 0.$$

Hence, $\int (\rho|z|^2 + b^2)$ is a conserved quantity.

The same algebraic cancellations may be used for obtaining higher order Sobolev (or Besov) estimates. Indeed, consider an ‘abstract’ pseudo-differential operator $A(D)$ (for instance a differential, a fractional derivatives or a spectral localization operator). Then one may write

$$\frac{d}{dt} \int ((1 + b)|A(D)z|^2 + (A(D)b)^2)$$

$$= 2 \left( \int_{l_1} (1 + b)(A(D)z, \partial_t A(D)z) + \int_{l_2} A(D)b \partial_t A(D)b \right) + \int_{l_3} \partial_t b |A(D)z|^2.$$

6 The method may seem uselessly complicated. However, the algebraic cancellations that are going to be used remain the same when estimating higher order Sobolev norms.
We note that \( I_1 = I_{1,1} + I_{1,2} + I_{1,3} + I_{1,4} + I_{1,5} \) and \( I_2 = I_{2,1} + I_{2,2} \) with
\[
I_{1,1} = - \int (A(D)z, \nabla A(D)b), \quad I_{2,1} = - \int A(D)b \text{ div} A(D)v,
\]
\[
I_{1,2} = - \int b(A(D)z, \nabla A(D)b), \quad I_{2,2} = - \int A(D)b A(D) \text{ div}(bv).
\]
\[
I_{1,3} = \int \left( A(D)z, i \frac{\varepsilon}{2} \Delta A(D)z \right).
\]
\[
I_{1,4} = \int b(A(D)z, i \frac{\varepsilon}{2} \Delta A(D)z).
\]
\[
I_{1,5} = - \frac{1}{2} \int \rho\langle A(D)z, A(D)\nabla (\cdot z) \rangle.
\]
As above, obvious integrations by parts ensure that \( I_{1,1} + I_{2,1} = 0 \) and \( I_{1,3} = 0 \). Next, using again integrations by parts, we note that
\[
I_{1,2} + I_{2,2} = \int A(D)b \text{ div}[b, A(D)]v.
\]
Finally, integrating by parts in \( I_{1,4} \) and using the fact that \(-\varepsilon \nabla b = 2\rho w\) yields
\[
I_{1,4} = \int \langle A(D)z, i(\nabla A(D)z) \cdot (\rho w) \rangle,
\]
and we have
\[
I_{1,5} = - \int (A(D)z, A(D)\nabla \cdot (\rho v + i\rho w)) + \int \rho (A(D)z, \nabla A(D)z \cdot z - A(D)(\nabla z \cdot z)).
\]
Therefore, using the fact that \( \partial_t \rho + \text{div}(\rho v) = 0 \), we conclude that
\[
2(I_{1,4} + I_{1,5}) + I_3 = 2 \int \rho (A(D)z, \nabla A(D)z \cdot z - A(D)(\nabla z \cdot z)).
\]
Putting all the above equalities together, we thus obtain
\[
\frac{1}{2} \frac{d}{dt} \int (\rho |A(D)z|^2 + (A(D)b)^2) = \int A(D)b \text{ div}[b, A(D)]v
\]
\[
+ \int \rho (A(D)z, \nabla A(D)z \cdot z - A(D)(\nabla z \cdot z)).
\]
If, say, \( A(D) \) is a fractional derivatives operator, then one may show by means of classical commutator estimates that the right-hand side may be bounded by
\[
C \| \rho \|_{L^\infty} \| (Db, Dz) \|_{L^\infty} \| (A(D)b, A(D)z) \|_{L^2}^2.
\]
Therefore,
\[
\frac{d}{dt} \int (\rho |A(D)z|^2 + (A(D)b)^2) \leq C \| \rho \|_{L^\infty} \| (Db, Dz) \|_{L^\infty} \| (A(D)b, A(D)z) \|_{L^2}^2.
\]
Denoting by \( E_A(t) \) the left-hand side and resorting to Gronwall lemma, we thus obtain
\[
E_A(t) \leq E_A(0) \exp \left( C \int_0^t \| \rho \|_{L^\infty} \| \rho^{-1} \|_{L^\infty} \| (Db, Dz) \|_{L^\infty} \right). \tag{3.2}
\]
It is now clear that whenever \( Db \) and \( Dz \) are bounded in \( L^1([0, T]; L^\infty) \) and \( \rho \) is bounded from below and from above then we obtain a control of \( A(D)b \) and \( A(D)z \) in \( L^\infty([0, T]; L^2) \).

Taking \( A(D) = A^\delta \) and performing a time integration, (3.2) implies that
\[
\| (b, z)(t) \|_{H^s} \leq C \left( \| (b_0, z_0) \|_{H^s} + \int_0^T \| (Db, Dz) \|_{L^\infty} \| (b, z) \|_{H^s} \right) \tag{3.3}
\]
for some constant \( C = C(s, d, \| \rho^{1/2} \|_{L^\infty}) \).
So assuming that $s > 1 + d/2$ and using Sobolev embedding and Gronwall’s inequality, one may conclude by elementary methods to the following statement.

**Theorem 3.1.** Let $s > 1 + d/2$. Assume that $\rho_0 = 1 + b_0$ for some $b_0 \in H^{s+1}(\mathbb{R}^d)$ such that $1 + b_0 > 0$, and that $v_0 \in H^s(\mathbb{R}^d)$ is curl free. Then there exists a time

$$T \geq T_0 := \frac{C}{\|b_0\|_{H^s}^2 + \varepsilon \|\nabla b_0\|_{H^s} + \|v_0\|_{H^{s+1}}}$$

with $C = C(s, d, \|\rho_0\|_{L^\infty})$ such that (1.1) has a unique solution $(v', \rho')$ on $[0, T] \times \mathbb{R}^d$ with $\rho' = 1 + b'$ bounded away from 0 and $(v', b') \in C([0, T]; H^s \times H^{s+1}) \cap C^1([0, T]; H^{s-2} \times H^{s-1})$.

**Remark 3.2.** Combining basic energy estimates for the wave equation with the above result, one may control the discrepancy between $(b, v)$ and the solution to the acoustic wave equation

$$\begin{aligned}
\partial_t v + \nabla \dot{b} &= 0, \\
\partial_t \dot{b} + \text{div} \dot{v} &= 0,
\end{aligned}$$

(3.4)

We have, up to time $T_0$,

$$
\| (v' - \dot{v}, b' - b)(t) \|_{H^{s-1}} \leq C \left( t \| (b_0, u_0) \|_{H^{s+1} \times H^s} + \varepsilon t \| (b_0, u_0) \|_{H^{s+1} \times H^s} \right).
$$

Note also that (3.3) provides a blow-up criterion involving the $W^{2, \infty}$ norm of $b$ and the Lipschitz norm of $v$. In particular, this implies that for given data in $H^s$ ($s > 1 + d/2$), the lifespan in $H^s$ is the same as the lifespan in $H^{s'}$, for any $1 + d/2 < s' < s$.

### 3.2. Dispersive properties and improved lower bounds for the lifespan

The system for $(b, v)$ reads

$$\begin{cases}
\partial_t v + v \cdot \nabla v + \nabla b = \frac{\varepsilon^2}{4} \nabla \left( \frac{1}{\rho} \Delta b - \frac{1}{2\rho^2} |\nabla b|^2 \right), \\
\partial_t b + \text{div} v = -\text{div}(\dot{b}v).
\end{cases}$$

(3.5)

Therefore the linearized system about $(0, 0)$ is not (3.4) but rather

$$\begin{cases}
\partial_t \dot{v} + \nabla \dot{b} = \frac{\varepsilon^2}{4} \nabla \Delta \dot{b}, \\
\partial_t \dot{b} + \text{div} \dot{v} = 0.
\end{cases}$$

(3.6)

A straightforward spectral analysis (based on the Fourier transform) shows that the above linear system behaves as the wave equation with speed 1 for frequencies small with respect to $1/\varepsilon$, and as the Schrödinger equation with coefficient $\varepsilon/2$ in the high frequency regime. In fact, in dimension $d > 2$, it is possible to prove Strichartz inequalities (related to the wave and Schrödinger equations for low and high frequencies, respectively) for (3.6). In the case of small data $(b_0, v_0)$, these inequalities allow us to improve the lower bound for the lifespan (see also [22] where a similar idea has been used in the context of the incompressible limit for compressible flows). For the sake of simplicity, we just state the result in dimension $d \geq 4$ (the reader is referred to [8] for the case $d = 2, 3$ and for more details concerning the approximation of the solution by (3.6)).

**Theorem 3.3.** Under the assumptions of theorem 3.1 with $s > 2 + d/2$, the lifespan $T$ may be bounded from below by

$$T_1 := \frac{C}{(\|b_0\|_{H^s} + \varepsilon \|\nabla b_0\|_{H^s} + \|v_0\|_{H^{s-1}})^2},$$

and the discrepancy between $(b, v)$ and the solution $(\dot{b}, \dot{v})$ to (3.6) with the same data may be bounded in terms of $t$ and of the data, up to time $T_1$. 

3.3. Extended formulation for Korteweg fluids

Compared with the ‘improved’ WKB method, the main drawback of the direct approach based on an extended formulation for solving (1.1) is that vanishing solutions cannot be handled. On the other hand, the direct method is robust enough so as to be used to solve locally more complicated models such as the following system governing the evolution of inviscid capillary fluids:

\[ \begin{aligned}
\partial_t v + v \cdot \nabla v &+ \nabla f(\rho) = \nabla (\kappa(\rho) \Delta \rho + \frac{1}{2} \kappa'(\rho) |\nabla \rho|^2), \\
\partial_t \rho + \text{div}(\rho v) &= 0.
\end{aligned} \]  

(3.7)

Physically, the function \( \kappa \) correspond to the capillary coefficient. Obviously, system (1.1) is included in (3.7) (take \( \kappa(\rho) = \varepsilon^2 / (4\rho) \)). In the general case, introducing \( a(\rho) = \sqrt{\rho \kappa(\rho)} \), \( w = -\frac{\kappa(\rho)}{\rho} \nabla \rho \) and \( z = v + iw \), we obtain the following extended formulation for (3.7):

\[ \begin{aligned}
\partial_t z + v \cdot \nabla z + i \nabla z \cdot w + \nabla f(\rho) &= i \nabla (a(\rho) \text{div} z), \\
\partial_t \rho + \text{div}(\rho v) &= 0.
\end{aligned} \]  

(3.8)

Note that in the potential case (namely \( \text{curl} z = 0 \)) then \( v \cdot \nabla z + i \nabla z \cdot w = \frac{1}{2} \nabla (z \cdot z) \). Note also that in the general case, the second order term \( \nabla (a(\rho) \text{div} z) \) is degenerate.

The case of system (1.1) is particularly simple inasmuch as \( a \) is the constant function \( \varepsilon / 2 \) and \( \nabla \text{div} z = \Delta z \).

For general capillarity coefficients, one may prove a local well-posedness result, similar to that of theorem 3.1. This has been done in [7]. The proof relies on the use of weighted Sobolev estimates, with a weight depending both on \( \rho \) and on the order of differentiation.

As for the QHD system, we expect the potential part of the solution to system (3.7) to have dispersive properties in dimension \( d \geq 2 \). The general situation is much more complicated, however, because those properties are related to those of the quasilinear Schrödinger equation. To the best of our knowledge, this aspect has been investigated only very recently in a work by Audiard [5] that concerns potential flows.

4. Asymptotics for the Gross–Pitaevskii equation

This section is concerned with the existence and asymptotics of travelling wave solutions for the Gross–Pitaevskii equation

\[ i \psi_t + \Delta \psi + (1 - |\psi|^2) \psi = 0, \]  

(4.1)

which may be obtained from (2.1) (with \( f(r) = r - 1 \)), up to the factor \( \frac{1}{2} \), after performing the change of unknown:

\[ \psi^\varepsilon(t, x) = \psi(\varepsilon^{-1} t, \varepsilon^{-1} x). \]

Equation (4.1) is associated with the Ginzburg–Landau energy (or Hamiltonian):

\[ \mathcal{H}(\psi) = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla \psi|^2 + \frac{1}{4} \int_{\mathbb{R}^d} (1 - |\psi|^2)^2. \]  

(4.2)

As a consequence, in contrast with the cubic NLS equation, the natural energy space for (4.1) is not \( H^1(\mathbb{R}^d) \) but rather

\[ E(\mathbb{R}^d) = \{ \psi \in H^1_{\text{loc}}(\mathbb{R}^d), \text{ s.t. } \mathcal{H}(\psi) < +\infty \}. \]
As pointed out before, for \( \mathcal{H}(\psi) \) to be finite, \( |\psi| \) must, in some sense, tend to 1 at infinity. This non-trivial boundary condition provides (4.1) with a richer dynamics than in the case of the null condition at infinity which, for a defocusing NLS type equation, is essentially governed by dispersion and scattering. For instance, in nonlinear optics, the ‘dark solitons’ are localized nonlinear waves (or ‘holes’) which exist on a stable continuous wave background. The boundary condition \( |\psi(t, x)| \rightarrow 1 \) at infinity is due to this non-zero background. In the context of superfluids, 1 is the density of the fluid at infinity.

Similarly to the energy, the momentum

\[
P(\psi) = \frac{1}{2} \int_{\mathbb{R}^d} (i\nabla \psi \cdot \psi)
\]

is formally conserved. This quantity is well defined for \( \psi \in H^1(\mathbb{R}^d) \) but not for solutions with a finite Ginzburg–Landau energy. A major difficulty in the theory of the Gross–Pitaevskii equation is to find an appropriate definition of the momentum which leads to a conserved quantity. A natural definition would be

\[
P(\psi) = \frac{1}{2} \int_{\mathbb{R}^d} (i\nabla \psi \cdot \psi - 1),
\]

but this would require for instance that \( \psi - 1 \in L^2(\mathbb{R}^d) \).

In any case we will denote by \( p \) the (scalar) first component of \( P \) which will play an important role in this section.

Recall that the use of the Madelung transform\(^7\)

\[
\psi = \varrho e^{i\phi},
\]

leads to the following hydrodynamic form of the equation for \( \varrho \) and \( v = 2\nabla \phi \):

\[
\partial_t \varrho + v \cdot \nabla \varrho + 2\nabla \varrho^2 = 2\nabla \left( \frac{\Delta \varrho}{\varrho} \right),
\]

\[
\partial_t \varrho^2 + \text{div} (\varrho^2 v) = 0.
\]  

As pointed out before, if we discard the right-hand side of the first equation and look at \( \varrho^2 \) as the density of a fluid with velocity \( v \), then the above system coincides with the Euler equations for a compressible fluid with pressure law \( P(\rho) = \rho^2 \). In particular, the speed of sound waves near the constant solution \( v = 1 \) is given by

\[ c_s = \sqrt{2}. \]

As we will see below, this sound speed (the value and relevance of which is not so obvious if looking at the original Gross–Pitaevskii equation (4.1)), plays an important role in various aspects of the dynamics of (4.1). The value of \( c_s \) may be also found by neglecting the quantum pressure term and linearizing for a perturbation \( \psi = (1 + \tilde{\varrho}) \exp(i\tilde{\phi}) \). This leads to the wave equation:

\[
\partial_t^2 \tilde{\varrho} - 2\Delta \tilde{\varrho} = 0.
\]

Note that if the quantum pressure is included (as in (3.6)) then the linearization reads

\[
\partial_t^2 \tilde{\varrho} - 2\Delta \tilde{\varrho} - \Delta^2 \tilde{\varrho} = 0,
\]

which, roughly, is the factorization of two linear Schrödinger operators.

The Madelung transform and the hydrodynamic form of (4.1) turn out to be of great interest for studying the Gross–Pitaevskii equation with finite Ginzburg–Landau energy since the solution is expected to have very few ‘vortices’ (or cancellations). Even in the ‘Euler limit’

\[^7\text{In this section, we use the normalization of [12, 13], instead of } \psi = \sqrt{\rho} e^{i\phi}.\]
that has been presented in section 2, one can use it outside the vortices [14, 15] to study the travelling waves of sufficiently small velocities. We also stress that the hydrodynamic form of the (one dimensional) Gross–Pitaevskii equation is needed in order to define a generalized momentum in the context of the orbital stability of the black solitons (such solitary waves have zeros...); see [9].

In this section, we shall concentrate on the transonic limit of solutions to the Gross–Pitaevskii equation. We shall first present a result pertaining to the asymptotics of travelling waves with speed \( c \) tending to the sound speed \( c_s \), in the case \( d = 2 \), in connection with the (KP I) equation (see below). Next, for the one-dimensional case, we give an accurate description of the transonic long-wave limit of (4.1) in terms of solutions to the KdV equation.

4.1. The transonic limit of finite energy travelling waves

Finite energy travelling wave solutions of (4.1) are solutions of the form \( \psi(x, t) = \theta(x_1 - ct, x^\perp) \) where \( \mathcal{H}(\theta) < +\infty \) and \( x^\perp \) denotes the transverse variables \( x_2, \ldots, x_d \). The profile \( \theta \) satisfies the following equation:

\[
 i c \frac{\partial \theta}{\partial x_1} + \Delta \theta + \theta(1 - |\theta|^2) = 0.
\]  

(4.4)

A suitable functional setting for the study of such travelling waves is the space:

\[
 W(\mathbb{R}^d) = \{ 1 \} + V(\mathbb{R}^d),
\]

with

\[
 V(\mathbb{R}^d) = \{ \psi : \mathbb{R}^d \mapsto \mathbb{C}, (\nabla \psi, \text{Re} \psi) \in L^2(\mathbb{R}^d)^2, \text{Im} \psi \in L^4(\mathbb{R}^d), \nabla \text{Re} \psi \in L^{4/3}(\mathbb{R}^d) \}.
\]

Indeed, given that \( W(\mathbb{R}^d) \) is a subset of the energy space \( E(\mathbb{R}^d) \), for any data in \( W(\mathbb{R}^d) \), equation (4.1) admits a unique solution. In addition, one may show that this solution stays in \( W(\mathbb{R}^d) \) (see [28, 29]). Furthermore, the quantity \( (i \partial_1 \psi, \psi - 1) \) is integrable whenever \( \psi \in W(\mathbb{R}^d) \), so that the scalar momentum \( p(\psi) \) is well defined. This is a consequence of the identity

\[
 (i \partial_1 \psi, \psi - 1) = \partial_1(\text{Re} \psi) \text{Im} \psi - \partial_1(\text{Im} \psi)(\text{Re} \psi - 1),
\]  

(4.5)

and various Hölder’s inequalities.

So finally, \( \mathcal{H} \) and \( p \) are continuous on \( W(\mathbb{R}^d) \) and all finite energy subsonic solutions to (4.4) have to belong to \( W(\mathbb{R}^d) \). Moreover, if \( \psi \in W(\mathbb{R}^d) \) may be lifted as \( \psi = \varrho \exp(i\phi) \) then

\[
 p(\psi) = \frac{1}{2} \int_{\mathbb{R}^d} (i \partial_1 \psi, \psi - 1) = \frac{1}{2} \int_{\mathbb{R}^d} (1 - \varrho^2) \partial_1 \phi.
\]  

(4.6)

Note that for maps which may be lifted, with \( \varrho \geq \frac{1}{2} \), the last integral makes sense, even if we assume that \( \psi \) only belongs to the energy space \( E(\mathbb{R}^d) \).

To simplify the presentation, we shall focus on the simpler case \( d = 2 \). We shall denote \( x = x_1 \) and \( y = x_2 \).

It is proven in [12] that, for any \( p > 0 \), the following minimization problem

\[
 \mathcal{H}_{\text{min}}(p) = \inf \{ \mathcal{H}(\psi), \psi \in W(\mathbb{R}^2), p(\psi) = p \},
\]  

(4.7)

has a solution \( u_\varrho \) which is a non-trivial travelling wave. We call it a ground state.

In the rest of this subsection, we focus on the asymptotics \( p \) going to 0 for \( u_\varrho \), in connection with the Kadomtsev–Petviashvili I (KP I) equation

\[
 u_t + uu_x + u_{xxx} - \partial_y^{-1} u_{yy} = 0,
\]

where the anti-derivative is defined in Fourier variables by \( \partial_x^{-1} f(\xi) = \frac{1}{i\xi} \hat{f}(\xi) \).
The following proposition states that the corresponding speed $c(u_p)$ tends to $c_*$ and gives the first term in the asymptotic expansion for both $\mathcal{H}_{\text{min}}(p)$ and $c(u_p)$.

**Proposition 4.1.** There exist positive constants $p_1$, $K_0$ and $K_1$ such that we have the asymptotic behaviours

$$\frac{48\sqrt{2}}{\mathcal{S}_{\text{KP}}} p^3 - K_0 p^4 \leq \sqrt{2} p - \mathcal{H}_{\text{min}}(p) \leq K_1 p^3, \quad \forall 0 \leq p \leq p_1,$$

where $\mathcal{S}_{\text{KP}}$ stands for the action of the KP I ground state $N$ of velocity 1, that is

$$\mathcal{S}_{\text{KP}} = \frac{1}{2} \int_{\mathbb{R}^2} (\partial_x N)^2 + \frac{1}{2} \int_{\mathbb{R}^2} (\partial_x^{-1}(\partial_y N))^2 - \frac{1}{6} \int_{\mathbb{R}^2} N^3 + \frac{1}{2} \int_{\mathbb{R}^2} N^2.$$

Moreover, the map $u_p$ has no zeros and there exist some positive constants $p_2$, $K_2$ and $K_3$ such that

$$K_2 p^2 \leq \sqrt{2} - c(u_p) \leq K_3 p^2, \quad \forall 0 \leq p < p_2.$$

Actually, it was established in [12] that if $\psi$ is a finite energy travelling wave of sufficiently small energy, then

$$\frac{1}{2} < |\psi| < 1.$$ 

This implies that small energy travelling waves have no zeros and thus can be lifted according to the Madelung transformation. This is, in particular, the case of minimizers $u_p$ corresponding to small enough values of $p$.

The transonic limit to KP I for travelling waves is obtained through the following change of scales:

$$\tilde{x} = \epsilon(\psi)x, \quad \tilde{y} = \frac{\epsilon(\psi)^2}{\sqrt{2}} y.$$ 

We then set

$$\eta = 1 - |\psi|^2, \quad N_\psi(x, y) = \frac{6}{\epsilon(\psi)^3} \eta \left( \frac{x}{\epsilon(\psi)}, \frac{\sqrt{2} y}{\epsilon(\psi)^2} \right).$$

It turns out that $N_\psi$ converges to a travelling wave solution of the (differentiated) KP I equation as $\epsilon(\psi) \to 0$, that is, it is an approximate solution of the equation

$$-w_{xx} - w_{yy} + w_{xxxx} + (w w_x)_x = 0.$$ 

More precisely, denoting $N_p = 1 - |u_p|^2$ where $u_p = |u_p|e^{i\phi_p}$ is a minimizer of the energy with fixed momentum $p$, and

$$\Theta_p(x, y) = \frac{6\sqrt{2}}{\epsilon_p} \Phi_p \left( \frac{x}{\epsilon_p}, \frac{\sqrt{2} y}{\epsilon_p^2} \right),$$

we have (see [13]):

**Theorem 4.2.** There exist a subsequence $(p_n)_n \in \mathbb{N}$ tending to 0, as $n \to +\infty$, and a ground state $w$ of the KP I equation such that both $N_{p_n}$ and $\Theta_{p_n}$ tend to $w$ in $W^{k,q}(\mathbb{R}^2)$ (for any $k \in \mathbb{N}$ and $q \in (1, +\infty]$) as $n$ goes to $+\infty$.

**Remark 4.3.** Those results on the transonic limit of solitary waves have recently been extended in [19] to the three-dimensional case with general nonlinearities (see also [39] for other existence results). The additional serious difficulty is that the Gross–Pitaevskii ground states solutions are no longer global minimizers of the energy with fixed momentum and thus not expected to be stable.

---

8 Actually, to a ground state solution of the KP I equation, that is a minimizer of the Hamiltonian with fixed $L^2$ norm.
4.2. The unsteady transonic long-wave limit of the Gross–Pitaevskii equation

The Madelung transform is also crucial to derive and justify the transonic (weak amplitude, long wave) limit of the one-dimensional Gross–Pitaevskii equation.

With the scaling which is used in this section and for data which are perturbations of order \( \varepsilon^3 \) of a constant state with modulus 1, theorem 3.1 and the remark that follows ensure that the linear wave equation gives a good approximation of the solution for \( t = o(\varepsilon^{-3}) \). In this paragraph, we describe what happens at next order. We shall see, in particular, that, up to times of order \( \varepsilon^{-3} \), the Korteweg–de Vries (KdV) equation

\[
\partial_t u + uu_x + u_{xxx} = 0,
\]

gives an accurate approximation of the one-dimensional Gross–Pitaevskii equation

\[
i \partial_t \psi + \partial_x \psi + (1 - |\psi|^2)\psi = 0 \tag{4.12}
\]

with data which are small long-wave perturbations of the constant one, namely \( \psi = \varrho e^{i\phi} \) with

\[
\varrho_0 = \left(1 - \frac{\varepsilon^2}{6} N_0^0(\varepsilon x)\right)^{1/2}, \quad \phi_0 = \frac{\varepsilon}{6\sqrt{2}} \Theta_1^0(\varepsilon x), \tag{4.13}
\]

for \( 0 < \varepsilon \ll 1 \), and \( N_0^0 \) and \( W_0^0 = \partial_x \Theta_1^0 \) are uniformly bounded in some Sobolev space \( H^k(\mathbb{R}) \) for sufficiently large \( k \).

We refer to [10, 11] for a detailed analysis and will only summarize the limit to (long) waves propagating in two directions and following a coupled system of KdV equations.

Recall that (4.12) is globally well-posed in the Zhidkov type spaces \( Y^k \),

\[
Y^k(\mathbb{R}) = \left\{ \psi \in L^1_{\text{loc}}(\mathbb{R}; \mathbb{C}), \quad 1 - |\psi|^2 \in L^2(\mathbb{R}), \quad \partial_x \psi \in H^{k-1}(\mathbb{R}) \right\},
\]

for any integer \( k \geq 1 \).

Moreover the Ginzburg–Landau energy \( \mathcal{H}(\psi(t)) \) is conserved by the flow and, provided \( \mathcal{H}(\psi_0) < \frac{1}{2\varepsilon^2} \), the corresponding solution \( \psi(t) \) does not vanish so that one may write \( \psi = \varrho \exp(i\phi) \), for some continuous function \( \phi \).

We consider data as in (4.13) with small enough \( \varepsilon \) and assume, in addition, that

\[
\|N_0^0\|_{M(\mathbb{R})} + \|\partial_x \Theta_1^0\|_{M(\mathbb{R})} < +\infty.
\]

Here, \( \| \cdot \|_{M(\mathbb{R})} \) denotes the norm defined on \( L^1_{\text{loc}}(\mathbb{R}) \) by

\[
\|f\|_{M(\mathbb{R})} = \sup_{(a,b) \in \mathbb{R}^2} \left| \int_a^b f(x) \, dx \right|.
\]

We next introduce the slow coordinates

\[
x^- = \varepsilon(x + \sqrt{2} t), \quad x^+ = \varepsilon(x - \sqrt{2} t) \quad \text{and} \quad \tau = \frac{\varepsilon^3}{2\sqrt{2} t}.
\]

The definition of the coordinates \( x^- \) and \( x^+ \) corresponds to reference frames travelling to the left and to the right, respectively, with speed \( \sqrt{2} \) in the original coordinates \( (t, x) \). We define accordingly the rescaled functions \( N_\varepsilon^\pm \) and \( \Theta_\varepsilon^\pm \) as follows:

\[
\begin{align*}
N_\varepsilon^\pm (\tau, x^\pm) &= \frac{6}{\varepsilon^2} \eta(t, x) = \frac{6}{\varepsilon^2} \eta\left(\frac{2\sqrt{2}\tau}{\varepsilon^3}, \frac{x^\pm}{\varepsilon} \pm \frac{4\tau}{\varepsilon^3}\right) \quad \text{with} \quad \eta = 1 - \varrho^2, \\
\Theta_\varepsilon^\pm (\tau, x^\pm) &= \frac{6\sqrt{2}}{\varepsilon} \phi(t, x) = \frac{6\sqrt{2}}{\varepsilon} \phi\left(\frac{2\sqrt{2}\tau}{\varepsilon^3}, \frac{x^\pm}{\varepsilon} \pm \frac{4\tau}{\varepsilon^3}\right) \quad \tag{4.14}
\end{align*}
\]
the main result is (see [11] for details):

**Theorem 4.4.** Let \( k \geq 0 \) and \( \varepsilon > 0 \) be given. Assume that the initial data \( \psi_0 \) belongs to \( Y_{k+6}(\mathbb{R}) \) and satisfies the assumption

\[
\| N_0 \varepsilon \|_{M(\mathbb{R})} + \| \partial_x \Theta_1 \varepsilon \|_{M(\mathbb{R})} + \| N_0 \varepsilon \|_{H^{k+5}(\mathbb{R})} + \varepsilon \| \partial_{k+6} N_0 \varepsilon \|_{L^2(\mathbb{R})},
\]

the following estimates hold for any \( \tau \in \mathbb{R} \):

\[
\| U_\varepsilon^- (\tau, \cdot) - U^- (\cdot, \tau) \|_{H^{k+5}(\mathbb{R})} + \| U_\varepsilon^+ (\tau, \cdot) - U^+ (\cdot, \tau) \|_{H^{k+5}(\mathbb{R})} \leq K_1 \varepsilon^2 \exp (K_1 |\tau|),
\]

for any \( \varepsilon \leq \varepsilon_1 \).

We now turn to the two (or higher) dimensional case, which is studied in [20] for a general nonlinear Schrödinger equation of the form similar to (4.1)

\[
\partial_t \psi + i \Delta \psi = f(|\psi|^2) \psi,
\]

where \( f \) is smooth and satisfies \( f(1) = 0, \quad f'(1) > 0 \).

One also uses a ‘weakly transverse transonic’ scaling, namely

\[
T = c \varepsilon^3 t, \quad X_1 = \varepsilon (x_1 - ct), \quad X_j = \varepsilon^2 x_j, \quad j = 2, \ldots, d.
\]

After performing the ansatz

\[
\psi(t, X) = (1 + \varepsilon^3 A(t, X)) \exp (i \varepsilon f(t, X)),
\]

the hydrodynamic reformulation of the Gross–Pitaevskii equation is used to recast the problem as a singular limit for an hyperbolic system in the spirit of [31]. Then smooth \( H^s \) solutions are proven to exist on an interval independent of the small parameter \( \varepsilon \). Passing to the limit by a compactness argument yields the convergence of the solutions to that of the KPI equation.

In comparison, for such data, theorem 3.3 would ensure that the linear system (3.6) gives a good description of the solution only for times that are \( o(\varepsilon^{-3}) \) (see the introduction of [8] for more details).

5. Global existence of weak solutions to a quantum fluids system

This section is devoted to the proof of global finite energy weak solutions to the QHD system: given data \((\rho_0, v_0)\) originating from some wave function \( \psi_0 \in H^1 \) through the Madelung transform, we want to establish that the QHD system has a global solution in a weak meaning that will be specified below. This result has been established recently in dimension \( d = 2, 3 \) (also in the collisionless case) by Antonelli and Marcati in [3, 4]. The proof therein consists
of smoothing out the original equation NLS (by convolution) then passing to the limit, which involves several technical steps.

Here we propose instead to first consider smooth initial data $\psi_0$, in which case we obtain a global smooth solution $\psi$ to NLS, and then justify the equivalence with the weak hydrodynamic formulation. The second step deals with general $H^1$ initial wave functions $\psi_0$. Smoothing them out allows us to construct a sequence of smooth solutions $\psi_n$ to NLS, that also satisfy the weak hydrodynamic formulation. This is then only a matter of passing to the limit, which may be done by quite standard weak convergence arguments. We do not treat the collisional case nor the coupling with the Poisson equation (as done in [3, 4]) which are beyond the scope of this paper. However, the same weak approach would allow us to treat this more general situation.

We now state the main result of this section.

**Theorem 5.1.** Let the initial data $(\rho_0, \Lambda_0) \in W^{1,1} \times L^2$ be ‘well-prepared’ in the sense that there exists some wave function $\psi_0 \in H^1$ such that

$$\rho_0 = |\psi_0|^2 \quad \text{and} \quad J_0 := \sqrt{\rho_0} \Lambda_0 = \text{Im}(\bar{\psi}_0 \nabla \psi_0).$$

Assume that $f(r) = r^\sigma$ for some integer $\sigma$ such that $W^{1,1}(\mathbb{R}^d) \hookrightarrow L^{\sigma+1}(\mathbb{R}^d)$.

There exist some vector-field $\Lambda \in L^{\infty}(\mathbb{R}^d; L^2)$ and some non-negative function $\rho \in L^{\infty}(\mathbb{R}; L^1 \cap L^{\sigma+1})$ with $\nabla \sqrt{\rho} \in L^{\infty}(\mathbb{R}; L^2)$ such that the following system holds true in the distributional sense\(^9\):

$$
\partial_t \rho + \text{div} J = 0,
\partial_t J + \text{div} (\Lambda \otimes \Lambda) + \nabla (P(\rho)) = \frac{1}{4} \Lambda \nabla \rho - \text{div} (\nabla \sqrt{\rho} \otimes \nabla \sqrt{\rho}),
\partial_j J^k - \partial_k J^j = 2\Lambda^j \partial_j \sqrt{\rho} - 2\Lambda^j \partial_k \sqrt{\rho} \quad \text{for all } (j, k) \in \{1, \ldots, d\}^2,
(\rho, J)|_{t=0} = (\rho_0, J_0),
$$

with $J := \sqrt{\Lambda} \Lambda$ and $P(\rho) := \rho f(\rho) - F(\rho)$ with $F(\rho) = \int_0^\rho f(\rho') \, d\rho'$.

In addition, the energy

$$
\int_{\mathbb{R}^d} \left( \frac{1}{2} |\Lambda|^2 + \frac{1}{2} |\nabla \sqrt{\rho}|^2 + F(\rho) \right)
$$

is conserved for all time.

**Proof.** We first prove the statement in the smooth case, namely we assume that the data $\psi_0$ is in $H^s$ for some large enough $s$. It is well known that

$$i \partial_t \psi + \frac{1}{2} \Delta \psi = f(|\psi|^2) \psi, \quad \psi|_{t=0} = \psi_0 \quad (5.2)$$

has a unique solution $\psi$ in $C(\mathbb{R}; H^s)$ whenever $s \geq 1$ (see e.g. [18, 35, 44]) and that

$$\forall t \in \mathbb{R}, \quad \int_{\mathbb{R}^d} \left( \frac{1}{2} |\nabla \psi(t)|^2 + F(|\psi(t)|^2) \right) = \int_{\mathbb{R}^d} \left( \frac{1}{2} |\nabla \psi_0|^2 + F(|\psi_0|^2) \right). \quad (5.3)$$

Even though the solution $\psi$ is smooth, the meaning of the ‘phasis’ in the Madelung transform (2.2) is unclear on the closed set $\psi^{-1}([0])$. In order to overcome this difficulty, we define, as originally done in [3], $\rho := |\psi|^2$ and

$$\phi(x) := \begin{cases} |\psi(x)|^{-1} \psi(x) & \text{if } \psi(x) \neq 0, \\ 0 & \text{if } \psi(x) = 0. \end{cases} \quad (5.4)$$

\(^9\) In the smooth non-vanishing case, the right-hand side of the second equation coincides with that of the velocity equation multiplied by $\rho$ in (1.1) with $\varepsilon = 1$, and the third equation just means that there exists some function $\phi$ such that $J = \rho \nabla \phi$. 
We claim that
\[ \nabla \sqrt{\rho} = \text{Re}(\bar{\phi} \nabla \psi) \quad \text{a.e.} \] (5.5)
Indeed, for \( \epsilon > 0 \), we set \( \phi_{\epsilon} := \psi / \sqrt{|\psi|^2 + \epsilon^2} \). Then \( (\phi_{\epsilon}) \) converges pointwise to \( \phi \) and an easy computation shows that
\[ \phi_{\epsilon} \psi \xrightarrow{\epsilon \to 0} \sqrt{\rho} \quad \text{uniformly.} \] (5.6)
Next, we compute
\[ \nabla (\bar{\phi}_{\epsilon} \psi) = \text{Re}(\bar{\phi}_{\epsilon} \nabla \psi) + \text{Re}(\psi \nabla \phi_{\epsilon}). \] (5.7)
The first term in the right-hand side converges pointwise to \( \text{Re}(\bar{\phi} \nabla \psi) \) hence in \( L^1_{\text{loc}} \) owing to Lebesgue’s theorem as it is bounded by \( |\nabla \psi| \).
As for the last term, we have
\[ \text{Re}(\psi \nabla \phi_{\epsilon}) = \frac{\epsilon^2}{(\epsilon^2 + |\psi|^2)^{3/2}} \text{Re}(\bar{\psi} \nabla \psi). \]
It is clear that the right-hand side converges pointwise to 0 and is bounded by \( |\nabla \psi| \). Hence it also converges to 0 in \( L^1_{\text{loc}} \).
So finally, putting these two results together with (5.6) and (5.7), one may conclude to (5.5).
Let \( \Lambda := \text{Im}(\bar{\phi} \nabla \psi) \) and \( J := \sqrt{\rho} \Lambda \). We claim that \( (\rho, \Lambda, J) \) satisfies (5.1). Indeed, from (5.2), we see that
\[ \partial_t (|\psi|^2) = 2 \text{Re}(\bar{\psi} \partial_t \psi) = -\text{Im}(\bar{\psi} \partial_t \psi) = -\text{Im} \text{div}(\bar{\psi} \nabla \psi), \]

hence
\[ \partial_t \rho + \text{div} J = 0. \] (5.8)
Next, we compute
\[ \partial_j J^k - \partial_k J^j = \text{Im}(\partial_j (\bar{\psi} \partial_k \psi) - \partial_k (\bar{\psi} \partial_j \psi)) = 2 \text{Im}(\partial_j \bar{\psi} \partial_k \psi). \]
Recall (see e.g. theorem 6.19 in [34]) that
\[ \nabla \psi = 0 \quad \text{a.e. on } \psi^{-1}(\{0\}) \] (5.9)
whenever \( \psi \) is locally in \( W^{1,1} \).
Therefore, given that \( |\psi| = 1 \) on \( \psi^{-1}(C \setminus \{0\}) \), one may write a.e.
\[ \text{Im}(\partial_j \bar{\psi} \partial_k \psi) = \text{Im}(\bar{\phi} \partial_j \bar{\psi} \partial_k \psi) = \text{Re}(\bar{\phi} \partial_j \bar{\psi} \partial_k \psi) \]
\[ + \text{Re}(\bar{\phi} \partial_k \bar{\psi} \partial_j \psi). \]
So we eventually obtain
\[ \partial_j J^k - \partial_k J^j = 2 \partial_j \sqrt{\rho} \Lambda^k - 2 \partial_k \sqrt{\rho} \Lambda^j. \] (5.10)
Next, we compute
\[ \partial_t J = \text{Im}(\partial_t \bar{\psi} \nabla \psi + \bar{\psi} \nabla \partial_t \psi). \]
Hence, using the equation satisfied by \( \psi \), we obtain
\[ \partial_t J = \frac{1}{2} \text{Re}(\bar{\psi} \nabla \Delta \psi - \Delta \bar{\psi} \nabla \psi) + \text{Re}(f(|\psi|^2) \bar{\psi} \nabla \psi - \bar{\psi} \nabla (f(|\psi|^2) \psi)). \]
If \( \psi \) is \( C^1 \) and the function \( f \) has a derivative at every point of \( \mathbb{R}^+ \), then straightforward computations show that
\[ B = -\rho f'(\rho) \nabla \rho = -\nabla (P(\rho)). \] (5.11)
Next, we see that (still in the smooth case)
\[ \nabla \Delta |\psi|^2 = 4 \text{Re}(\nabla^2 \psi : \nabla \bar{\psi}) + 2 \text{Re}(\nabla \psi \Delta \bar{\psi}) + 2 \text{Re}(\bar{\psi} \nabla \Delta \psi). \]
Hence
\[ A = \frac{1}{2} \nabla \Delta |\psi|^2 - 2 \text{Re}(\Delta \bar{\psi} \nabla \psi) - 2 \text{Re}(\nabla^2 \psi : \nabla \bar{\psi}). \]
So we obtain
\[ \frac{1}{2} A = \frac{1}{2} \nabla \Delta |\psi|^2 - \text{div} \text{Re}(\nabla \bar{\psi} \otimes \nabla \psi). \] (5.12)
Now, using again (5.9), one may write at almost every point of \( \mathbb{R}^d \),
\[ \text{Re}(\nabla \bar{\psi} \otimes \nabla \psi) = \text{Re}(\phi \nabla \bar{\psi} \otimes \phi \nabla \psi), \]
\[ = \text{Re}(\bar{\phi} \nabla \psi) \otimes \text{Re}(\phi \nabla \psi) + \text{Im}(\bar{\phi} \nabla \psi) \otimes \text{Im}(\phi \nabla \psi). \]
Therefore, we have
\[ \text{Re}(\nabla \bar{\psi} \otimes \nabla \psi) = \nabla \sqrt{\rho} \otimes \nabla \sqrt{\rho} + \Lambda \otimes \Lambda. \] (5.13)
Putting this together with (5.11) and (5.12), one may conclude that
\[ \partial_t J + \text{div}(\Lambda \otimes \Lambda) + \nabla(P(\rho)) = \frac{1}{2} \Delta \nabla \rho - \text{div}(\nabla \sqrt{\rho} \otimes \nabla \sqrt{\rho}). \] (5.14)
Of course, as owing to (5.9)
\[ |\nabla \psi|^2 = (\text{Re}(\phi \nabla \psi))^2 + (\text{Im}(\phi \nabla \psi))^2 = |\nabla \sqrt{\rho}|^2 + |\Lambda|^2 \quad \text{a.e.}, \]
the energy equality for \( \psi \) recasts in
\[ \int_{\mathbb{R}^d} \left( \frac{1}{2} |\Lambda(t)|^2 + \frac{1}{2} |\nabla \sqrt{\rho(t)}|^2 + F(\rho(t)) \right) dt = \int_{\mathbb{R}^d} \left( \frac{1}{2} |\Lambda_0|^2 + \frac{1}{2} |\nabla \sqrt{\rho_0}|^2 + F(\rho_0) \right) \]
This completes the proof in the smooth case.

We now treat the rough case where \( \psi_0 \) belongs only to \( H^1 \). Then we fix some sequence \( (\psi_{0,n})_{n \in \mathbb{N}} \) of functions in \( H^s \) (with \( s \) large) converging to \( \psi_0 \) in \( H^1 \). We denote by \( \psi_n \) the solution of (5.2) in \( C(\mathbb{R}; H^s) \) corresponding to the data \( \psi_{0,n} \), and by \( \psi \in C(\mathbb{R}; H^1) \) the solution to (5.2) with data \( \psi_0 \).

From the first part of the proof, we know that there exists some sequence \( (\phi_n)_{n \in \mathbb{N}} \) of functions with modulus at most 1 such that if we set \( \Lambda_n := \text{Im}(\phi_n \nabla \psi_n) \), \( \rho_n := |\psi_n|^2 \) and \( J_n := \sqrt{\rho_n} \Lambda_n \), then
\[ \nabla \sqrt{\rho_n} = \text{Re}(\phi_n \nabla \psi_n) \quad \text{in} \ L^2, \]
and \( (\rho_n, \Lambda_n, J_n) \) satisfies (5.1) with data \( (\rho_{0,n}, J_{0,n}) \), together with the energy equality
\[ \int_{\mathbb{R}^d} \left( \frac{1}{2} |\Lambda_n(t)|^2 + \frac{1}{2} |\nabla \sqrt{\rho_n(t)}|^2 + F(\rho_n(t)) \right) dt = \int_{\mathbb{R}^d} \left( \frac{1}{2} |\Lambda_{0,n}|^2 + \frac{1}{2} |\nabla \sqrt{\rho_{0,n}}|^2 + F(\rho_{0,n}) \right). \] (5.17)
We now have to prove the convergence of \( (\rho_n, \Lambda_n, J_n) \) to some solution \( (\rho, \Lambda, J) \) of (5.1) satisfying the energy equality.

On the one hand, standard stability estimates (based on Strichartz inequalities) guarantee that
\[ \psi_n \rightarrow \psi \quad \text{in} \ L^\infty_{\text{loc}}(\mathbb{R}; H^1). \] (5.18)
On the other hand, because \( (\phi_n)_{n \in \mathbb{N}} \) is bounded by 1, it converges (up to extraction) in \( L^\infty \) weak * to some function \( \phi \) such that \( \|\phi\|_{L^\infty} \leq 1 \). As \( \nabla \psi_n \) converges strongly to \( \nabla \psi \) in \( L^2 \), this implies that
\[ \phi_n \nabla \psi_n \rightharpoonup \phi \nabla \psi \quad \text{in} \ L^2. \] (5.19)
In turn, as obviously $\sqrt{\rho_n} \to \sqrt{\rho}$ in $L^2$ and as $\nabla \sqrt{\rho_n} = \text{Re}(\bar{\phi}_n \nabla \psi_n)$, we deduce that

$$\nabla \sqrt{\rho} = \text{Re}(\bar{\phi} \nabla \psi)$$

and $\nabla \sqrt{\rho_n} \to \nabla \sqrt{\rho}$ in $L^2$.

Given that $\Lambda_n = \text{Im}(\bar{\phi}_n \nabla \psi_n)$, (5.19) also ensures that

$$\Lambda_n \rightharpoonup \Lambda := \text{Im}(\bar{\phi} \nabla \psi)$$

in $L^2$.

In order to establish that strong convergence in $L^2$ holds true, it suffices to show that (up to an omitted extraction)

$$\|\nabla \sqrt{\rho_n}\|_{L^2} \to \|\nabla \sqrt{\rho}\|_{L^2}$$

and

$$\|\Lambda_n\|_{L^2} \to \|\Lambda\|_{L^2},$$

(5.20)

On the one hand, the weak convergence ensures that

$$\|\nabla \sqrt{\rho}\|_{L^2} \leq \liminf \|\nabla \sqrt{\rho_n}\|_{L^2}$$

and

$$\|\Lambda\|_{L^2} \leq \liminf \|\Lambda_n\|_{L^2};$$

on the other hand, given that $(\nabla \psi_n)_{n \in \mathbb{N}}$ converges strongly to $\nabla \psi$ in $L^2$ and that (5.15) holds true for $\psi$ and $\psi_n$, we may write

$$\|\nabla \sqrt{\rho}\|_{L^2}^2 + \|\Lambda\|_{L^2}^2 = \|\nabla \psi\|_{L^2}^2,$$

$$= \lim_{n \to +\infty} \|\nabla \psi_n\|_{L^2}^2,$$

$$= \lim_{n \to +\infty} (\|\nabla \sqrt{\rho_n}\|_{L^2}^2 + \|\Lambda_n\|_{L^2}^2),$$

$$\geq (\liminf \|\nabla \sqrt{\rho_n}\|_{L^2})^2 + (\liminf \|\Lambda_n\|_{L^2})^2.$$

Therefore (5.20) is satisfied. As a conclusion, we thus have established that

$$\sqrt{\rho_n} \to \sqrt{\rho}$$

in $H^1$ and

$$\Lambda_n \rightharpoonup \Lambda$$

in $L^2$.

Of course, this implies that $J_n \to J := \sqrt{\rho} \Lambda$ in $L^1$ so it is easy to pass to the limit in (5.1) and in the energy equality (5.16). The details are left to the reader. \hfill $\square$

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Appendix A. Conservation laws

In this appendix we review conservation laws for the nonlinear Schrödinger, QHD and compressible Euler equations. Even though most of the results are classical (as concerns the Schrödinger equation, they may be found in the textbooks [43, 44] for instance; links between Schrödinger and Euler conservation laws may be found in [17]), we believe the relationships between the aforementioned equations to be of interest. In addition, those conservation laws are still meaningful for the less classical framework of general Korteweg fluids.

A.1. The case of Schrödinger, QHD and compressible Euler equations

For the time being, we consider the following system:

$$\begin{cases}
\partial_t v + v \cdot \nabla v + \nabla f (\rho) = \frac{\varepsilon^2}{2} \nabla \left( \frac{\Lambda (\sqrt{\rho})}{\sqrt{\rho}} \right), & v_{|t=0} = v_0, \\
\partial_t \rho + \text{div} (\rho v) = 0, & \rho_{|t=0} = \rho_0,
\end{cases}$$

(A.1)
which is the QHD system if $\varepsilon > 0$, and the compressible Euler equation if $\varepsilon = 0$, and the nonlinear Schrödinger equation:

$$i\varepsilon \partial_t \psi + \frac{\varepsilon^2}{2} \Delta \psi = f(|\psi|^2)\psi, \quad \psi|_{t=0} = \psi_0.$$  \hspace{1cm} \text{(A.2)}

Recall that for $\varepsilon > 0$, one may pass formally from (A.2) to (A.1) by setting $
\psi = \sqrt{\rho} e^{\phi/\varepsilon}$ and $v = \nabla \phi$.

In what follows, the function $f$ is assumed to be continuous on $\mathbb{R}^+$ and, say, $C^1$ on $(0, +\infty[$, standard cases being $f(r) = r^\sigma$ and $f(r) = r^{-1}$. We denote by $F$ the anti-derivative of $f$ which vanishes at 0, and set $P(\rho) = \rho f(\rho) - F(\rho)$. As pointed out before, from a physical viewpoint, $P$ is the pressure.

The first part of the appendix aims at listing (and deriving formally) the classical conservation laws for (A.1) and (A.2).

Phase invariance. For every $\alpha \in \mathbb{R}$, one has

$\psi$ solution of (A.2) $\iff$ $e^{i\alpha} \psi$ solution of (A.2).

The phase invariance is not seen at the level of (A.1) (this amount to changing $\phi$ into $\phi + \varepsilon \alpha$).

By Noether’s theorem or by an easy computation, this leads to the conservation of mass:

$$M := \int |\psi|^2 \, dx = \int \rho \, dx.$$  \hspace{1cm} \text{(A.3)}

Time translation invariance. For every $\tau \in \mathbb{R}$, one has

$\psi(t, x)$ solution of (A.2) $\iff$ $\psi(t + \tau, x)$ solution of (A.2).

By the same time translation, this is expressed at the level of (A.1) and leads to the conservation of energy (or Hamiltonian):

$$H := \int \left( \frac{\varepsilon^2}{2} |\nabla \psi|^2 + F(|\psi|^2) \right) \, dx = \int \left( \frac{1}{2} \rho |v|^2 + \frac{\varepsilon^2}{2} |\nabla \sqrt{\rho}|^2 + F(\rho) \right) \, dx.$$  \hspace{1cm} \text{(A.4)}

Space translation invariance. For every $x_0 \in \mathbb{R}^d$, one has

$\psi(t, x)$ solution to (A.2) $\iff$ $\psi(t, x + x_0)$ solution to (A.2).

By the same space translation, this is expressed at the level of (A.1) and leads to the conservation of momentum:

$$P := \text{Im} \int \bar{\psi} \nabla \psi \, dx = \int \rho v \, dx.$$  \hspace{1cm} \text{(A.5)}

Invariance by spatial rotation. Let $R$ be a spatial rotation. Then

$\psi(t, x)$ solution of (A.2) $\iff$ $\psi(t, Rx)$ solution of (A.2).

By the same spatial rotation on a solution of (A.1) this leads to the conservation of angular momentum, which we write in the case of $\mathbb{R}^3$ for the sake of simplicity:

$$A := \text{Im} \int x \wedge \bar{\psi} \nabla \psi \, dx = \int x \wedge \rho v \, dx.$$  \hspace{1cm} \text{(A.6)}
**Galilean invariance.** For every $\xi_0 \in \mathbb{R}^d$, one has

$$\psi(t, x) \text{ solution of } (A.2) \iff e^{-i \xi_0 \cdot x} e^{-i \xi_0 \cdot \xi_0 t} \psi(t, x + \varepsilon \xi_0 t) \text{ solution of } (A.2).$$

For (A.1), this implies

$$(v, \rho)(t, x) \text{ solution } \iff (v(t, x + \varepsilon \xi_0 t) - \varepsilon \xi_0, \rho(t, x + \varepsilon \xi_0 t)) \text{ solution},$$

and leads to

$$\frac{dX}{dt} = \mathcal{P} \quad \text{with } X := \int x|\psi|^2 \, dx = \int x \rho \, dx. \quad (A.7)$$

**Scale invariance.** When $f(r) = r^\sigma$, one has for all $\lambda > 0$,

$$\psi(t, x) \text{ solution of } (A.2) \iff \lambda^{1/\sigma} \psi(\lambda^2 t, \lambda x) \text{ solution of } (A.2),$$

which implies for (A.1)

$$(v, \rho)(t, x) \text{ solution } \iff (\lambda v, \lambda^{2/\sigma} \rho)(\lambda^2 t, \lambda x) \text{ solution}.$$

The associated conservation law is (recall that $d$ stands for the space dimension):

$$\frac{d\mathcal{F}}{dt} = \int \left( \varepsilon^2 |\nabla \psi|^2 + dP(|\psi|^2) \right) \, dx = 2\mathcal{H} + \int (dP - 2F(|\psi|^2)) \, dx$$

or

$$\frac{d\mathcal{F}}{dt} = \int \left( \rho |v|^2 + dP(\rho) \right) \, dx = 2\mathcal{H} + \int (dP - 2F(\rho)) \, dx$$

with

$$\mathcal{F} := \text{Im} \int \varepsilon \bar{\psi} \cdot \nabla \psi = \int \rho x \cdot v \, dx$$

and

$$P(r) := rf(r) - F(r). \quad (A.8)$$

Equality (A.8) remains formally true if the nonlinearity is not a pure power.

**Momentum of inertia or virial.** A direct computation shows that the momentum of inertia (or virial)

$$I := \frac{1}{2} \int |x|^2 |\psi|^2 \, dx = \frac{1}{2} \int |x|^2 \rho \, dx \quad (A.9)$$

satisfies $\frac{dI}{dt} = \mathcal{F}.$

**Pseudo-conformal invariance.** Let $\varphi$ the pseudo-conformal transform of $\psi$ defined by

$$\varphi(t, x) = \frac{e^{u(t/\varepsilon)}}{\int_{\mathbb{R}^d} e^{u(t/\varepsilon)} \, dx} \tilde{\psi} \left( \frac{\varepsilon^2 t}{I} \cdot \frac{x}{I} \right).$$

One notices that

$$\left( i \varepsilon \partial_t \varphi + \frac{\varepsilon^2}{2} \Delta \varphi \right)(t, x) = \frac{\varepsilon^2}{I^2} \frac{e^{u(t/\varepsilon)}}{\int_{\mathbb{R}^d} e^{u(t/\varepsilon)} \, dx} \left( i \varepsilon \partial_t \psi + \frac{\varepsilon^2}{2} \Delta \psi \right) \left( \frac{\varepsilon^2 t}{I} \cdot \frac{x}{I} \right).$$

Thus

$$i \varepsilon \partial_t \varphi + \frac{\varepsilon^2}{2} \Delta \varphi = \frac{\varepsilon^2}{I^2} \frac{d}{d\left( \frac{I}{\varepsilon} \right)} |\varphi|^2 \varphi.$$
For the $L^2$-critical power nonlinearity $f(r) = r^{2/d}$, one checks that

$$\psi(t,x) \text{ solution of (A.2)} \iff \varphi(t,x) \text{ solution of (A.2)}.$$ 

For (A.1), this yields

$$(v, \rho)(t, x) \text{ solution } \iff \left( x - t \frac{v}{t}, x - t \frac{\epsilon}{t} \right) \text{ solution.}$$

Using the conservation of energy for $\varphi$, one deduces after a lengthy computation that

$$\frac{dZ}{dt} + t \int (dP - 2F)(|\psi|^2) \, dx = 0 \quad (A.10)$$

with

$$Z(t) := \int \left( \frac{1}{2} |x + i \epsilon t \nabla \psi|^2 + t^2 F(|\psi|^2) \right) \, dx,$$

This equality can be proven more simply using the fact that the operators $i \epsilon \partial_t + \frac{\epsilon^2}{2} \Delta$ and $x + i \epsilon t \nabla$ commute. The conservation law for $Z$ is not independent from the preceding ones: by expanding the square of the modulus, one observes that

$$Z(t) = t^2 \mathcal{H} - t \mathcal{F} + \mathcal{I}.$$ 

Consequently,

$$\frac{dZ}{dt} = 2t \mathcal{H} - \mathcal{F} - t \frac{dF}{dt} + \frac{d\mathcal{I}}{dt} = 2t \mathcal{H} - t \frac{d\mathcal{F}}{dt},$$

and one recovers (A.10).

The Carles–Nakamura conservation law. Formally, the quantity

$$\mathcal{U}(t) := \text{Re} \int \bar{\psi} (x + i \epsilon t \nabla) \psi \, dx$$

is constant. This can be checked by an indirect way (see [16]) or by a direct computation. The interpretation in terms of (A.1) is

$$\frac{d}{dt} \mathcal{U} = 0 \quad \text{with} \quad \mathcal{U}(t) = \int \rho(x - tv) \, dx. \quad (A.11)$$

This conservation law also results from the conservation of momentum and from the conservation law for $X$ since in both cases, (A.1) or (A.2), one has

$$\mathcal{U} = X - t \mathcal{P}.$$

A.2. Conservation laws for general capillary fluids

We here study to what extent the conservation laws listed in the previous subsection are relevant for general inviscid capillary fluids, that are governed by system (3.7). Throughout, we assume the capillarity $\kappa$ to be a differentiable function on $\mathbb{R}^+$. For smooth solutions with a non-vanishing density, System (3.7) recasts in the following conservative form\textsuperscript{10}:

$$\partial_t (\rho v) + \text{div} (\rho v \otimes v) + \nabla P(\rho) = \text{div} K,$$

$$\partial_t \rho + \text{div}(\rho v) = 0,$$

with $K(\rho, \nabla \rho) := \left( \frac{1}{4} (\kappa(\rho) + \rho \kappa'(\rho)) |\nabla \rho|^2 + \rho \kappa(\rho) \Delta \rho \right) I_d - \kappa(\rho) \nabla \rho \otimes \nabla \rho.$

\textsuperscript{10} With the convention $(\text{div} K)^i := \sum \partial_i K_{ij}$
Theorem A.1. If \((\rho, v)\) is a sufficiently smooth solution of (3.7) which decays at infinity, with \(\rho\) non-vanishing, then the equalities (A.3), (A.5), (A.6), (A.7) and (A.9) are still valid, and the energy
\[
\mathcal{H} := \int \left( \frac{1}{2} \rho |v|^2 + \frac{\kappa(\rho)}{2} |\nabla \rho|^2 + F(\rho) \right) \, dx
\]
is conserved.

Furthermore, the quantity \( \mathcal{F} \) defined in (A.8) satisfies
\[
\frac{d}{dt} \mathcal{F} = 2 \mathcal{H} + \int \left( dP - 2F + \frac{d}{2} (\rho \kappa) |\nabla \rho|^2 \right) \, dx.
\]

Proof. It is very likely that most of those conservation laws could be derived from Noether theorem. They can also be obtained by pedestrian computations.

For (A.3), (A.7) and (A.9), there is no difference with the QHD since the only equation on \(\rho\) is concerned.

For (A.5) the proof is obvious from the conservative form (A.12).

In order to prove the conservation of energy, the simplest method is to take the \(L^2\) scalar product with \(\rho v\) of the first equation of (3.7). The treatment of the terms on the left-hand side of the equation for \(v\) is the same as in the compressible Euler equation, and one obtains after several integrations by parts and the use of the equation for \(\rho\),
\[
\frac{d}{dt} \int \left( \frac{1}{2} \rho |v|^2 + F(\rho) \right) \, dx = \int \rho v \cdot \nabla \left( \kappa \Delta \rho + \frac{1}{2} \kappa' |\nabla \rho|^2 \right) \, dx,
\]
\[
= \int \partial_t \rho \left( \kappa \Delta \rho + \frac{1}{2} \kappa' |\nabla \rho|^2 \right) \, dx,
\]
\[
= - \int \kappa \nabla \rho \cdot \partial_t \nabla \rho \, dx - \frac{1}{2} \int \kappa' \partial_t \rho |\nabla \rho|^2 \, dx,
\]
\[
= - \frac{d}{dt} \int \frac{\kappa}{2} |\nabla \rho|^2 \, dx.
\]

To prove (A.6), one writes (using the Einstein summation convention),
\[
A_i = \int \varepsilon_{ijk} x^j \rho v^k
\]
with \(\varepsilon_{ijk} = 0\) if two of the indices are equal, and equal to the signature of \((i j k)\) otherwise.

Using the conservative form of the equation for \(\rho v\) and integrating by parts, one thus deduces
\[
\frac{d}{dt} A_i = \int \varepsilon_{ijk} x^j \partial_t K_{ik} \, dx - \int \varepsilon_{ijk} x^j \partial_k P - \int \varepsilon_{ijk} x^j \partial_t (\rho v^k v^l) \, dx,
\]
\[
= - \int \varepsilon_{ijk} K_{jk} \, dx + 0 - \int \varepsilon_{ijk} \rho v^l v^k \, dx.
\]
The tensors \(K\) and \(\rho v \otimes v\) being symmetric, the right-hand member of the previous equality vanishes.

The simplest way to prove the conservation law on \(\mathcal{F}\), is to use the second equation in (A.12). Integrating by parts the capillary term and treating the other terms as in the compressible Euler equation one obtains
\[
\frac{d \mathcal{F}}{dt} = \int x \cdot \text{div} \, K \, dx - \int x \cdot \left( \nabla P + \text{div} (\rho v \otimes v) \right) \, dx,
\]
\[
= \int x^i \partial_i K_{ij} \, dx + \int (dP + \rho |v|^2) \, dx.
\]
Using the expression of the tensor $K$ and integrating by parts, one obtains

$$\int x^j \partial_i K_{ij} \, dx = - \int \text{tr } K \, dx,$$

$$= - d \int \left( \frac{1}{2} (\kappa + \rho \kappa') |\nabla \rho|^2 + \rho \kappa \Delta \rho \right) \, dx + \int \kappa |\nabla \rho|^2,$$

$$= \frac{d}{2} \int (\rho \kappa') |\nabla \rho|^2 \, dx + \int \kappa |\nabla \rho|^2 \, dx,$$

from which the claimed equality results. □

**Remark A.2.** The case of QHD corresponds to $(\rho \kappa)' = 0$. For capillary fluids, the presence of extra terms in the conservation law for $\mathcal{F}$ seems relatively harmless if $\rho \mapsto \rho \kappa (\rho)$ is an increasing function. One should be able to prove without difficulty that with a pressure law such that $dP - 2F \geq 0$, one has

$$I(t) \sim \mathcal{H} t^2.$$ 

In the case where $\rho \mapsto \rho \kappa$ is decreasing, we expect the conservation law for $\mathcal{F}$ to be the key to proving finite time blow-up results similar to those of [42].

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