Multichoice Minority Game: Dynamics and Global Cooperation

F. K. Chow and H. F. Chau

Department of Physics, University of Hong Kong, Pokfulam Road, Hong Kong

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In the original two-choice minority game (MG), selfish players cooperate with each other even though direct communication is not allowed. Moreover, there is a periodic dynamics in the MG whenever the strategy space size is much smaller than the number of strategies at play. Do these phenomena persist if every player has $N_c > 2$ choices where all player’s strategies are picked from a reduced strategy space? We answer this question by studying a multichoice minority game model known as MG($N_c, |S|$). Numerical simulation shows that these two models have very similar global cooperative behaviors. Nevertheless, unlike in the MG, periodic dynamics does not always appear in the MG($N_c, |S|$) even when the strategy space size is much smaller than the number of strategies at play.

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I. INTRODUCTION

In the past decade, there is an explosion of research interest in the study of complex adaptive system (CAS). In fact, examples of CAS are ubiquitous in our complex and complicated world. Ecosystems, nervous system, banking industry, organism and economic systems like stock markets are all examples of CAS. The study of CAS allowed us to investigate how rich, complicated pattern structure or behavior emerged from collection of relatively simple components.

So, what are the essential features of a complex adaptive system? According to Levin [1], CAS must have three essential elements: (i) sustained diversity and individuality of components, (ii) localized interactions among those components and (iii) an autonomous process that selects from among those components, based on the results of local interactions, a subset for replication or enhancement.

In the framework of CAS, econophysicists study the collective behavior of economic system by building up simple model using statistical mechanical and nonlinear physical methods. In particular, minority game (MG) [2, 3] is a simple model of economic system composed of adaptive and inductive reasoning agents, which successfully captures the minority seeking behavior in a free market economy. This game was proposed by Challet and Zhang under the inspiration of the El Farol bar problem introduced by the theoretical economist Arthur [4].

MG is a toy model of $N$ inductive reasoning players who have to choose one out of two alternatives independently in each turn by their current best working strategies or mental model. Those who end up in the minority side (that is, the choice with the least number of players) win. Although its rules are remarkably simple, MG shows a surprisingly rich self-organized collective behavior [3, 6, 7]. To explain the dynamics of MG, Hart et al. introduced the so-called crowd-anticrowd theory [8, 9]. Their theory stated that fluctuations arisen in the MG is controlled by the interplay between crowds of like-minded agents and their perfectly anti-correlated partners.

Numerical simulation as well as the crowd-anticrowd theory [3, 6, 7] showed that the global behavior of MG depends on three parameters: (i) the number of players $N$, (ii) the number of strategies $S$ each player has and (iii) the number of the most recent minority sides $M$ that a strategy depends on. Global cooperation, as indicated by the fact that average number of players winning the game each time is larger than that where all players make their choice randomly, is observed whenever $2^{M+1} \approx NS$ [3, 6, 7]. Is it true that global behavior of MG is determined once $N$, $S$ and $M$ are fixed? Specifically, we ask if it is possible to lock the system in a global cooperative phase for any fixed values of $N$, $S$ and $M$. We can do so by studying the multichoice minority game — a variant of the MG in which players have $N_c > 2$ choices. In our previous work, we have answered the above question by building up a multichoice minority game model called MG($N_c, N^2$) where $N_c$ is the number of player’s choice [10]. In MG($N_c, N^2$), all strategies are picked from a reduced strategy space consisting of $N^2$ mutually anti-correlated and uncorrelated strategies only. We found that it is possible to keep (almost) optimal cooperation amongst the players in almost the entire parameter space since the MG($N_c, N^2$) model can be readily extended to MG($N_c, N^k$) model for integer $k \in [3, M + 1]$.

On the other hand, it was found that the dynamics of the MG is dominated by a period-$2 \cdot 2^M$ dynamics whenever $2^{M+1} \ll NS$, i.e. in the overcrowding phase [3, 11, 12]. In this phase, the strategy space size is much smaller than the number of strategies at play and thus each strategy is used by a large number of players. It is such an overcrowding effect which leads to the period-$2 \cdot 2^M$ dynamics in this phase. However, will the periodic dynamics also appear in the overcrowding phase of a multichoice minority game?

In this paper, we would like to continue our previous study by investigating the global cooperation of players in the multichoice minority game model known as MG($N_c, |S|$). Besides, we also investigate the dynamics in the overcrowding
phase of such multichoice minority game model. In Section II, we give a general formalism for constructing the 
MG(\(N_c, |S|\)) model. Numerical simulation results on the dynamics and also global cooperation of our model are 
reported and discussed in Section III. Lastly, we sum up by giving a brief conclusion in Section IV.

II. THE MODEL

In this Section, we would like to show how to construct the MG(\(N_c, |S|\)) model. It is a generalized MG model where 
all players have \(N_c \geq 2\) choices and they all pick strategies from the reduced strategy space \(S\) with size \(|S| = N_c^k\) 
for integer \(k \in [2, M + 1]\). (Note that a reduced strategy space is only consisted of strategies which are significantly 
different from each other.) Here we assume that \(N_c\) is a prime power and we label the \(N_c\) alternatives as the \(N_c\) 
elements in the finite field \(GF(N_c)\).

In this repeated game, there are \(N\) heterogeneous inductive reasoning players whose aim is to maximize one’s own 
profit from the game. In every turn, each player has to choose one out of \(N_c\) alternatives while direct communication 
with others is not allowed. The minority choice, denoted as \(\Omega(t)\) at time \(t\), is simply the choice chosen by the least 
non-zero number of players in that turn. (In case of a tie, the minority choice is chosen randomly amongst the choices 
with the least non-zero number of players.) The players of the minority choice will gain one unit of wealth while all the 
other lose one. The only public information available to all players is the so-called history \((\Omega(t-M), \ldots, \Omega(t-1))\) 
which is the \(N_c\)-ary string of the minority choice of the last \(M\) turns. The history can only take on \(L \equiv N_c^M\) different 
states. We label these states by an index \(\mu = 1, \ldots, L\) and denote the history \((\Omega(t-M), \ldots, \Omega(t-1))\) by the index 
\(\mu(t)\). In fact, players can only interact indirectly with each other through the history \(\mu(t)\).

However, how does each player decide his/her own choice in the game using inductive reasoning? He/She does so 
by employing strategies to predict the next minority choice according to the history where a strategy is a map sending 
individual history \(\mu\) to the choice \(\{0, 1, \ldots, N_c - 1\}\). We represent a strategy \(s\) by a vector \(s \equiv (\chi_1^s, \chi_2^s, \ldots, \chi_L^s)\) where 
\(\chi^s_\mu\) is the prediction of the minority choice by the strategy \(s\) for the history \(\mu\) as illustrated in Table I.

| history \(\mu\) | prediction \(\chi^\mu\) |
|----------------|----------------|
| (0,0)          | 0              |
| (0,1)          | 1              |
| (0,2)          | 2              |
| (1,0)          | 2              |
| (1,1)          | 1              |
| (1,2)          | 0              |
| (2,0)          | 1              |
| (2,1)          | 2              |
| (2,2)          | 0              |

TABLE I: The prediction of the minority choice \(\chi^\mu\) by the strategy \(s \equiv (0,1,2,2,1,0,1,2,0)\) where \(N_c = 3\) and \(M = 2\).

In MG(\(N_c, |S|\)), each player is assigned once and for all \(S\) randomly drawn strategies from the reduced strategy 
space \(S\). At each time step, each player uses his/her own best working strategies to guess the next minority choice. 
But how do players decide which strategy is the best? They use the virtual score, which is the hypothetical profit for 
a player using a single strategy throughout the game, to evaluate the performance of a strategy. The strategy with 
the highest virtual score is considered as the best one.

As mentioned before, all the strategies in MG(\(N_c, |S|\)) are drawn from the reduced strategy space \(S\). For \(|S| = N_c^k\) 
where integer \(k \in [2, M + 1]\), \(S\) is formed by the spanning strategies \(\vec{v}_{u_a}, \vec{v}_{u_1}, \ldots, \vec{v}_{u_k}\) as follows:

\[
S = \{\lambda_u \vec{v}_u + \sum_{i=1}^{k-1} \lambda_u^{-i} \vec{v}_u : \lambda_u, \lambda_u^1, \ldots, \lambda_u^{k-1} \in GF(N_c)\}. \tag{1}
\]

Note that all arithmetical operations here are performed in the finite field \(GF(N_c)\). Since \(N_c\) is a prime power, \(S\) must 
be a linear space. Adapted from Ref. 10, the spanning strategies must satisfy the following technical conditions:

\[
v_{ai} \neq 0 \text{ for all } i, \tag{2}
\]
where \( v_{ai} \) is the \( i \)th element of the vector \( \vec{v}_a \) and by regarding \( i \) as a uniform random variable between 1 and \( L \),

\[
\begin{align*}
\Pr(v_{ai}^1 = j_1|v_{ai} = j_1) &= 1/N_c \text{ if } \Pr(v_{ai} = j_1) \neq 0 \text{ for all } j_1, j_2 \in GF(N_c), \\
\Pr(v_{ai}^2 = j_2|v_{ai} = j_2 \text{ and } v_{ai} = j_1) &= 1/N_c \text{ if } \Pr(v_{ai}^1 = j_2 \text{ and } v_{ai} = j_1) \neq 0 \text{ for all } j_1, j_2, j_3 \in GF(N_c), \\
\vdots
\end{align*}
\]

(3)

Under such conditions, we can show that

\[
d(\lambda_{a1} \vec{v}_a + \lambda_{a2} \vec{v}_a + \cdots + \lambda_{ak-1} \vec{v}_a - \lambda_{a2} \vec{v}_a + \cdots + \lambda_{ak-1} \vec{v}_a)
\equiv
\begin{cases}
0 & \text{if } \lambda_{a1} = \lambda_{a2} \text{ and } \lambda_{a1} = \lambda_{a3} \text{ and } \cdots \text{ and } \lambda_{ak-1} = \lambda_{ak-2}, \\
L & \text{otherwise.}
\end{cases}
\]

(4)

It is easy to show that the reduced strategy space \( S \) whose size equals \( N_c^k \) is composed of \( N_c^{k-1} \) distinct mutually anti-correlated strategy ensembles (namely, those with same \( \lambda_{a1}, \ldots, \lambda_{ak-1} \)); whereas the strategies of each of these ensembles are uncorrelated with each other. Indeed, the reduced strategy space \( S \) has a very beautiful structure since any two strategies randomly drawn from it are either uncorrelated or anti-correlated with each other \[\text{(10)}\]. Fig. \[\text{11}\] shows the structure of the reduced strategy space \( S \) of MG\( (N_c, N_c^2) \) for \( N_c = 5 \).

III. RESULTS

A. Global cooperation of players

We would like to investigate if there is global cooperation of players in the MG\( (N_c, |S|) \) model. If the global cooperation does exist, we want to compare it with that of the minority game. All the results in this part are taken from 1000 independent runs. In each run, we run 10000 steps starting from initialization before making any measurements. We have checked that it is already enough for the system to attain equilibrium in all the cases reported below. Then we took the average values on 150000 steps after the equilibration.

Let us start by first studying the average attendance

\[
\langle A(t) \rangle \equiv \lim_{T \to 0} \frac{1}{T} \sum_{t=1}^{T} A_i(t)
\]

(5)

where the attendance of an alternative \( A_i(t) \) is just the number of players chosen that alternative. Although the attendance of different alternatives are not the same, their long time averages must be equal since there is no prior bias for any alternative in this game. We observed that the average attendance \( \langle A_i(t) \rangle \) always converges to \( N/N_c \) in both MG\( (N_c, N_c^2) \) and MG\( (N_c, N_c^3) \) no matter what \( S \), \( S \) and \( M \) are. It is similar to the case of the MG where the average attendance always converges to \( N/2 \) \[\text{2} \].

In order to evaluate the performance of players in our model, we study the mean variance of attendance over all alternatives (or simply the mean variance)

\[
\Sigma^2 = \frac{1}{N_c} \sum_{i \in GF(N_c)} [(\langle A_i(t) \rangle^2) - \langle A_i(t) \rangle^2].
\]

(6)

(We remark that the variance of the attendance of a single alternative was studied for the MG \[\text{2} \].) In fact, the sum of the variance of all the attendance represents the total loss of players in the game. The variance \( \Sigma^2 \), to first order approximation, is a function of the control parameter \( \alpha \) alone \[\text{2} \].

Besides, we will compare the mean variance with that in the random choice game (RCG) in order to evaluate the significance of the strategies. The RCG is similar to the MG\( (N_c, |S|) \) model except that all the players in the RCG make their own decision simply by tossing their own coin. Indeed, we have showed that the variance is equal to \( N(N_c - 1)/N_c^2 \) for every alternative in the RCG \[\text{15} \].
Since the structure of the strategy space $S$ matches the assumptions of the crowd-anticrowd theory, we expect the collective behavior of MG($N_c,|S|$) should follow the predictions of the theory. Based on our previous work \[10\], we found that the crowd-anticrowd theory predicts

$$\Sigma^2 = \left\langle \frac{1}{N_{c}^{k-1}} \sum_{\bar{\xi}} \sum_{\gamma \in GF(N_{c})} \left( \frac{1}{N_{c}} \sum_{\eta \neq \gamma} (N_{\bar{\xi},\gamma} - N_{\bar{\xi},\eta}) \right)^2 \right\rangle,$$

(7)

where the mutually anti-correlated strategy ensemble $S_{\bar{\xi}} = \{\xi_1 v_{u}^1 + \ldots + \xi_{k-1} v_{u}^{k-1} + \gamma v_{u} \mid \gamma \in GF(N_{c})\}$ for $\bar{\xi} = (\xi_1, \ldots, \xi_{k-1})$ and $N_{\bar{\xi},\gamma}$ is the number of players making decision according to the strategy $(\xi_1 v_{u}^1 + \ldots + \xi_{k-1} v_{u}^{k-1} + \gamma v_{u})$. Note that $\sum_{\bar{\xi}}$ denotes the sum of the variance over all the mutually anti-correlated strategies ensembles $S_{\bar{\xi}}$. We should notice that when averaged over both time and initial choice of strategies, variance of attendance of different alternatives must be equal because there is no prior preference for any alternative in our game.

Fig. 2 displays the mean variance of attendance as a function of the control parameter $\alpha \equiv |S|/NS$, which is the ratio of the strategy space size $|S|$ to the number of strategies at play $NS$, in the MG($N_c,N_c^2$) and MG($N_c,N_c^3$) for some typical $N_c$. For MG($N_c,N_c^2$) and MG($N_c,N_c^3$), the mean variance of attendance, $\Sigma^2$, exhibits properties similar to that in the MG irrespective of $S$, $S$ and $M$. In particular, the mean variance $\Sigma^2$ is smaller than the so-called coin-tossed value (i.e. the corresponding value in the RCG) whenever $\alpha \approx 1$. It shows that players cooperate globally in this parameter range. Moreover, the mean variance predicted by the crowd-anticrowd theory is consistent with our numerical finding as shown in Fig. 2.

In both MG($N_c,N_c^2$) and MG($N_c,N_c^3$), we found that there is an indication of a second order phase transition. To check if the phase transition is second order or not, we calculate the order parameter \[10\, 14\]

$$\theta = \frac{1}{L} \sum_{\mu} \left\{ \sum_{\Omega} \left[ \langle p(\Omega|\mu) \rangle \frac{1}{N_{c}} \right]^2 \right\},$$

(8)

where $\langle p(\Omega|\mu) \rangle$ denotes the conditional time average of the probability for current minority choice $\Omega(t) = \Omega$ given that history $\mu(t) = \mu$. In fact, the order parameter measures the bias of player’s decision to any choice for individual history.

Fig. 3 shows that the order parameter always vanishes in MG($N_c,N_c^2$) and MG($N_c,N_c^3$) when the control parameter $\alpha$ is smaller than its value corresponding to minimum variance. As a result, we confirm that the phase transition is a second order one in both MG($N_c,N_c^2$) and MG($N_c,N_c^3$).

In summary, our numerical results show that both MG($N_c,N_c^2$) and MG($N_c,N_c^3$) model always exhibit global cooperative behavior similar to that in the original MG. Moreover, all these results agree with the prediction of the crowd-anticrowd theory. Thus it is reasonable for us to believe that the MG($N_c,|S|$) model has global cooperative behavior which agrees with both the original MG and crowd-anticrowd theory. Therefore, we conclude that we have successfully build up the MG($N_c,|S|$) model whenever $N_c$ is a prime power.

By using the MG($N_c,|S|$) model, we can always alter the complexity of each strategy in minority game with fixed $N$, $S$ and $M$ while the cooperative behavior still persists. As a result, we can always keep (almost) optimal cooperation amongst the players in almost the entire parameter space.

B. Dynamics of the game

In MG, there is periodic dynamics when $\alpha$ is small, i.e. in the overcrowding phase \[6\, 11\, 12\]. Thus, it is natural for us to ask whether such periodic dynamics also exists in the overcrowding phase of MG($N_c,|S|$).

To investigate the dynamics in MG($N_c,|S|$), we would like to look at the correlation in the times series of the attendance of an alternative. Therefore, we study both the conditional and unconditional autocorrelation functions:

$$C_0(i) = \frac{\langle A_0(t-i)A_0(t) \rangle - \langle A_0(t-i) \rangle \langle A_0(t) \rangle}{\langle [A_0(t)]^2 \rangle - \langle A_0(t) \rangle^2},$$

$$C_0^j(i) = \frac{\langle A_0^j(t-i)A_0^j(t) \rangle - \langle A_0^j(t-i) \rangle \langle A_0^j(t) \rangle}{\langle [A_0^j(t)]^2 \rangle - \langle A_0^j(t) \rangle^2},$$

where $A_0^j(t)$ is the attendance of the room-0 for the $j$th occurrence of the history $\mu$. Here we calculate the correlation functions for the room-0 only as the correlation should be the same for all the alternatives since there is no prior bias for any alternative in our game.
We observed that both \( C_0(i) \) and \( C_0^\mu(i) \) fluctuate greatly in different runs of the game with the same configuration. Such phenomenon is probably due to the incomplete exploration of the history space for particular initial configurations. Since we are only interested in studying the generic properties of the autocorrelation functions, we averaged \( C_0(i) \) and \( C_0^\mu(i) \) over 50 independent runs to eliminate the effect of the initial configuration. In each run, we took the average values on 10000 steps after running 50000 steps for equilibrium starting from initialization.

Before moving further on, let us introduce a concept called the prediction vector. The prediction vector \( \vec{\pi}(\mu) \) of a history \( \mu \) is simply a vector consisted of all the predictions of the minority choice for that history given by each strategies of the (full/reduced) strategy space used in MG or MG\((N_c,|S|)\). Specifically, \( \vec{\pi}(\mu) \equiv (\lambda_0^\mu, \ldots, \lambda_{|S|-1}^\mu) \) in MG\((N_c,|S|)\) where \( \tau_j = (\lambda_0^\mu \vec{v}_1^\mu + \lambda_1^\mu \vec{v}_2^\mu + \ldots + \lambda_{|S|-1}^\mu \vec{v}_{|S|}^\mu)^{-1} \) for \( j = (\lambda_0^\mu N_0^{|S|-1} + \ldots + \lambda_{|S|-1}^\mu N_{|S|-1} + \lambda_0^\mu) \). Table I shows the prediction vectors in the MG\((N_c,N_c^2)\) with \( M = 2 \) and \( N_c = 3 \) for a typical strategy space \( S \).

| history \( \mu \) | prediction vector \( \vec{\pi}(\mu) \) |
|---------------|-------------------------|
| (0,0)         | (0,1,2,0,1,2,0,1,2)     |
| (0,1)         | (0,1,2,1,2,0,2,0,1)     |
| (0,2)         | (0,1,2,1,2,0,1,2,0)     |
| (1,0)         | (0,1,2,0,1,2,0,1,2)     |
| (1,1)         | (0,1,2,1,2,0,2,0,1)     |
| (1,2)         | (0,1,2,0,1,1,2,0)       |
| (2,0)         | (0,1,2,0,1,2,0,1,2)     |
| (2,1)         | (0,1,2,1,2,0,2,0,1)     |
| (2,2)         | (0,1,2,2,0,1,1,2,0)     |

There is an interesting relationship between the prediction vector and the history under special circumstances. For some strategy spaces, the prediction vector of each history \( \mu(t) \equiv (\Omega(t-M), \ldots, \Omega(t-1)) \) in MG\((N_c,N_c^k)\) is uniquely determined by one of the following vectors alone: \( \hat{\Lambda}(t-M), \hat{\Lambda}(t-M-1), \ldots, \hat{\Lambda}(t-k+1) \) where \( \hat{\Lambda}(t) = (\Omega(t), \ldots, \Omega(t+k-2)) \). For example, suppose the spanning strategies of the strategy space \( S \) are \( \vec{v}_4 \equiv (1,1,1,1,1,1,1,1,1,0) \) and \( \vec{v}_5 \equiv (0,0,0,1,1,1,2,2,2) \) in the MG\((N_c,N_c^2)\) with \( M = 2 \) and \( N_c = 3 \). Then the prediction vector are \( (0,1,2,0,1,2,0,1,2,0) \) and \( (0,1,2,0,1,2,0,1,0,1) \) for histories in which \( \Omega(t-2) \) equals 0, 1 and 2 respectively. In this case, the prediction vector of an arbitrary history \( \mu(t) \) is uniquely determined by \( \Omega(t-2) \).

We examined the behavior of \( C_0^\mu(i) \) in the overcrowding phase for MG\((N_c,N_c^2)\) and MG\((N_c,N_c^3)\) with \( 2 \leq M \leq 4 \) and various strategy spaces. We noticed that \( C_0^\mu(i) \) exhibits periodic features only in MG\((N_c,N_c^3)\) with \( M = 2 \) no matter what strategy space \( S \) is used. Such observation indicates periodic dynamics of individual history persists in the MG\((N_c,N_c^3)\) game with \( M = 2 \). For other studied games, the values of \( C_0^\mu(i) \) \( (i > 1) \) are very small and sometimes even \( C_0^\mu(1) \) is nearly zero. These observations suggest only the dependence between consecutive \( A_0^\mu(j) \) is non-negligible in these MG\((N_c,|S|)\) games.

On the contrary, we observed that \( C_0^\mu(i) \) is a period-2 function of \( i \) in the overcrowding phase of the 2-choice MG for any history \( \mu \). It originates from the presence of a period-2 dynamics for every possible history in this game \( \left[ \right] \). In the overcrowding phase, there is a high probability for a player to employ similar strategies since the number of strategies at play is much larger than the strategy space size. As a result, for odd occurrence of each history, every player has the same probability for choosing any alternative which results in \( A_0(odd \ t) \approx N/2 \). Moreover, on the even occurrence of each history, every players must choose the alternative which is the minority choice of the last odd occurrence of the same history and thus \( A_0(even \ t) \) is either \( < N/2 \) or \( > N/2 \). If this periodic dynamics can persist, \( C_0^\mu(i) \) must be a period-2 function of \( i \) in the overcrowding phase of the 2-choice MG for any history \( \mu \). In fact, there is a similar period-\(N_c\) dynamics for each history of the \( N_c\)-choice MG with \( N_c > 2 \) in which the full strategy space is used (see fig. [I]).

We expect there is period-\(N_c\) dynamics for every possible history in the overcrowding phase of MG\((N_c,|S|)\) due to its similarity with the MG. However, why the period-\(N_c\) dynamics cannot always persist in our game? Such phenomenon is resulted from the “interference” of the periodic dynamics of individual history. In MG\((N_c,N_c^k)\), each prediction vector corresponds to more than one history if \( M > (k-1) \) since the prediction vector can only take on \( N_k^{k-1} \) to \( (N_c-1)N_c^{k-1} \) different states for \( |S| = N_c^k \). Moreover, the virtual scores of all player’s strategies must change in the same way for those histories with the same prediction vector. As a result, the periodic dynamics of individual history must “interfere” with those corresponding to the same prediction vector and thus the periodic dynamics
cannot persist. In addition, there is a one-to-one correspondence between the prediction vector and history for both the (full/reduced) strategy spaces used in MG and MG(Nc, Nc^k) with M = k – 1. Consequently, the periodic dynamics of individual history persist in these games since they no longer “interfere” with each other.

On the other hand, is it possible to have any periodic dynamics in the system of MG and MG(Nc, |S|)? Zheng and Wang revealed that period-2 · 2^M multi-peak structure appears in the unconditional autocorrelation function of the logarithm change of the attendance of an alternative in the 2-choice MG when α is small. Thus all players must have similar behavior for every 2 · 2^M time steps in such case. In other words, there is always a period-2 · 2^M dynamics in the overcrowding phase of the 2-choice MG. Fig. 3 showed that a period-Nc · Nc^M dynamics also exists in the overcrowding phase of the Nc-choice MG for Nc > 2 although it is weaker than the periodic dynamics in the 2-choice MG.

In order to study the dynamics in MG(Nc, |S|), we investigated the properties of C0(i) in MG(Nc, Nc^2) and MG(Nc, Nc^3). Figs. 4 to 12 display C0(i) as a function of i for MG(Nc, Nc^2) with different parameters. We found that the dynamics of the system in the overcrowding phase of MG(Nc, |S|) is remarkably different from that of MG. First, C0(i) does not always exhibit periodic features in MG(Nc, |S|) with arbitrary strategy space S which implies periodic dynamics does not always exist. Moreover, the periodicity of the periodic dynamics in MG(Nc, |S|) (which is equal to the periodicity of C0(i) as a function of i) is independent of the length of memory M. In particular, we found that C0(i) is a period-Nc^k function of i in MG(Nc, Nc^k) with Nc > 2 and 2 ≤ k ≤ 3 whenever the prediction vector of each history is uniquely determined by one of the following vectors alone: \( \hat{\lambda}(t) = (\Omega(t), \Omega(t + k - 2)) \). Besides, these periodic dynamics is observed to be less and less pronounced for larger Nc and κ if the prediction vector of each history is uniquely determined by \( \hat{\lambda}(t - \kappa) \) where the integer κ ∈ [k – 1, M].

To understand the dynamics in MG(Nc, |S|), let us start by considering the dynamics of the game for every possible history. As mentioned before, the virtual scores of all player’s strategies must change in the same manner for those histories with the same prediction vector and thus the dynamics of the game for these histories cannot be considered independently. We will consider the dynamics of the game for each possible effective history instead in which those histories with the same prediction vector are considered to be belonged to the same effective history. We should note that there are totally Nc^k-1 distinct effective histories in MG(Nc, Nc^k) if the prediction vector only takes on Nc^k-1 different states for |S| = Nc^k.

In fact, the effective history in MG(Nc, |S|) is analogous to the history in the Nc-choice MG. Thus there should be a similar period-Nc dynamics for each effective history of MG(Nc, |S|) when α is small. Our numerical findings validates the existence of such periodic dynamics in MG(Nc, |S|). However, do all effective histories have the same probability of occurrence in the overcrowding phase of MG(Nc, |S|)? It is true only if the prediction vector of each history is uniquely determined by one of the following vectors alone: \( \hat{\lambda}(t - M), \hat{\lambda}(t - M - 1), \ldots, \hat{\lambda}(t - k + 1) \) where \( \hat{\lambda}(t) = (\Omega(t), \ldots, \Omega(t + k - 2)) \). Under such circumstances, the current effective history is determined by the corresponding past minority choice(s). In addition, each alternative must have the same probability to be the minority choice in the overcrowding phase under any history because there is no prior bias for any alternative and all alternatives can be picked by some of the players for such case. As a result, all effective histories must have the same probability of occurrence for small α in these MG(Nc, |S|) games.

We revealed that there is a period-Nc dynamics for all Nc^k-1 effective histories in the overcrowding phase of MG(Nc, Nc^k) if the prediction vector only takes on Nc^k-1 different states. Moreover, we found that all effective histories on average appear once per Nc^k-1 time steps as they all have the same probability of occurrence if the prediction vector of each history is uniquely determined by \( \hat{\lambda}(t - \kappa) \) where the integer κ ∈ [k – 1, M]. Besides, the probability for any transition of effective history must be the same for such case which lead to no “destructive interference” between the dynamics of individual effective history. As a results, there is period-Nc^k dynamics in the overcrowding phase of MG(Nc, Nc^k) under such circumstances.

However, why the period-Nc^k dynamics in MG(Nc, Nc^k) becomes less significant when Nc increases? It is because the total number of possible evolutionary path in the effective history space increases exponentially with Nc under the period-Nc^k dynamics. Indeed, we have found that the period-Nc^k dynamics become more pronounced if we add bias to some of the effective history’s evolutionary path by manipulating the payoff function. On the other hand, we should note that the evolution in the effective history space depends only on \( \Omega(t - \kappa), \ldots, \Omega(t - 1) \) if the prediction vector of each history is uniquely determined by \( \hat{\lambda}(t - \kappa) \) where the integer κ ∈ [k – 1, M]. Thus the total number of possible evolutionary path in the effective history space also increases exponentially with κ and the periodic dynamics become less pronounced when κ increases.

Nevertheless, we found that C0(i) exhibits peculiar behavior in MG(Nc, Nc^2) with Nc = 2. In particular, C0(i) shows a period-Nc^3 multi-peak structure whenever the prediction vector of each history is uniquely determined by \( \Omega(t - 2\kappa) \) for any integer κ ∈ [1, |M/2|]. For MG(Nc, Nc^2), the strategy space is k-dimensional since the strategy space is spanned by k independent strategies (see eqn. 11 - 13). Thus the strategy space is 2-dimensional in MG(Nc, Nc^2) which give rise to the abnormal properties of C0(i). In addition, there are only 2 choices for Nc = 2 which is remarkably different.
from those games with more than two choices. It is because if a strategy does not choose room-0, then it must choose room-1 and vice-versa for a 2-choice game. We believe the above two factors lead to the marked difference of the dynamics of the system in MG($N_c, N^2_c$) with $N_c = 2$ compared to other MG($N_c, |S|$) games with $N_c > 2$.

In summary, we found that the periodic dynamics only exists in the overcrowding phase for particular reduced strategy spaces in the MG($N_c, |S|$) game. In contrast, the periodic dynamics always appears in the overcrowding phase for the original MG where all player’s strategies are drawn from the full strategy space.

IV. CONCLUSION

In conclusion, we have successfully build up the MG($N_c, |S|$) model whenever $N_c$ is a prime power. It is a generalized MG model where each players has $N_c \geq 2$ choices and all strategies are drawn from the reduced strategy space that is consisted of uncorrelated and anti-correlated strategies only. Although our MG($N_c, |S|$) model is not exactly same as the MG, its global cooperative behavior is similar to that of the MG no matter what $S$, $S$ and $M$ are.

However, the dynamics of MG($N_c, |S|$) model is remarkably different from that of the MG. For the MG($N_c, |S|$), the periodic dynamics only exists in the overcrowding phase for particular reduced strategy spaces; on the contrary, the periodic dynamics always appears in the overcrowding phase for the MG.

Indeed, we have only given explanation of the periodic dynamics in the MG($N_c, N^k_c$) model for some of the reduced strategy spaces in which a periodic dynamics exists. Thus it is instructive for us to find reasonable explanation for the periodic dynamics in other cases.

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FIG. 1: The reduced strategy space $S$ of $\text{MG}(N_c,N_c^2)$ for $N_c = 5$. 

mutually uncorrelated
FIG. 2: The mean variance $\Sigma^2$ versus the control parameter $\alpha \equiv |S|/NS$ in MG($N_c, N^2_c$) with $N_c = 37$ and MG($N_c, N^3_c$) with $N_c = 16$. The solid lines are the predictions of the crowd-anticrowd theory whereas the dashed lines indicate the coin-tossed value, i.e. the corresponding value in the random choice game.
FIG. 3: The order parameter $\theta$ versus the control parameter $\alpha \equiv |S|/NS$ in $\text{MG}(N_c,N_c^2)$ with $N_c = 37$ and $\text{MG}(N_c,N_c^3)$ with $N_c = 16$. 
FIG. 4: The conditional autocorrelation function $C^\mu_0(i)$ versus $i$ in $N_c$-choice MG with $S = 2$, $M = 2$ and $\alpha \approx 0.06$ where (a) $N_c = 2$, (b) $N_c = 3$, (c) $N_c = 4$, (d) $N_c = 5$. Our numerical results show that $C^\mu_0(i)$ exhibits similar behavior for all other histories and $M = 3, 4$. 
FIG. 5: The autocorrelation function $C_0(i)$ versus $i$ in $N_c$-choice MG with $S = 2$, $M = 2$ and $\alpha \approx 0.06$ where (a) $N_c = 2$, (b) $N_c = 3$, (c) $N_c = 4$, (d) $N_c = 5$. Our numerical results show that $C_0(i)$ exhibits similar behavior for $M = 3$ and 4.
FIG. 6: The autocorrelation function \( C_0(i) \) versus \( i \) in \( \text{MG}(N_c, N_c^2) \) with \( N_c = 2, S = 2, M = 2 \) and \( \alpha \approx 0.06 \) where the strategy space \( S \) is formed by the spanning strategies \( \vec{v}_a = (1, 1, 1) \) and (a) \( \vec{v}_{a1}^1 = (0, 1, 0, 1) \), (b) \( \vec{v}_{a2}^2 = (0, 0, 1, 1) \), (c) \( \vec{v}^3_a = (0, 1, 1, 0) \).
FIG. 7: The autocorrelation function $C_0(i)$ versus $i$ in $\text{MG}(N_c, N^2_c)$ with $N_c = 2$, $S = 2$, $M = 3$ and $\alpha \approx 0.06$ where the strategy space $S$ is formed by the spanning strategies $\vec{v}_u = (1, \ldots, 1)$ and (a) $\vec{v}_u = (0, 1)$, (b) $\vec{v}_u = (0, 1, 1)$, (c) $\vec{v}_u = (0, \ldots, 0, 1, \ldots, 1)$, (d) irregular $\vec{v}_u$. Note that the length of a strategy $L = N_c^M$ and $(\sigma)^n$ denotes the vector of $n$ consecutive segment $\sigma$. 
FIG. 8: The autocorrelation function $C_0(i)$ versus $i$ in $\text{MG}(N_c,N_c^2)$ with $N_c = 2$, $S = 2$, $M = 4$ and $\alpha \approx 0.06$ where the strategy space $S$ is formed by the spanning strategies $\vec{v}_a = (1, \ldots, 1)$ and (a) $\vec{v}_u^1 = (0, 1)^4$, (b) $\vec{v}_u^1 = (0, 0, 1, 1)^4$, (c) $\vec{v}_u^1 = (0, \ldots, 0, 1, \ldots, 1)^4$, (d) $\vec{v}_u^1 = (0, \ldots, 0, 1, \ldots, 1)$, (e) $\vec{v}_u^1 = (0, 1, 1, 0)^4$, (f) irregular $\vec{v}_u^1$ with the same notation as in Fig.7.
FIG. 9: The autocorrelation function $C_{0,0}(i)$ versus $i$ in $\text{MG}(N_c, N_s^2)$ with $N_c = 3$, $S = 2$, $M = 2$ and $\alpha \approx 0.06$ where the strategy space $S$ is formed by the spanning strategies $\vec{v}_u = (1, \ldots, 1)$ and (a) $\vec{v}_u^1 = (0, 1, 2)^3$, (b) $\vec{v}_u^1 = (0, 0, 0, 1, 1, 1, 2, 2, 2)$, (c) irregular $\vec{v}_u^1$ with the same notation as in Fig.8.
FIG. 10: The autocorrelation function \( C_0(i) \) versus \( i \) in \( \text{MG}(N_c, N_c^2) \) with \( N_c = 3, S = 2, M = 3 \) and \( \alpha \approx 0.06 \) where the strategy space \( S \) is formed by the spanning strategies \( \vec{v}_{\alpha} = (1, \ldots, 1) \) and (a) \( \vec{v}_1^a = (0, 1, 2)^3 \), (b) \( \vec{v}_1^b = (0, 0, 1, 1, 2, 2, 2)^3 \), (c) \( \vec{v}_1^c = (0, \ldots, 0, 1, \ldots, 1, 2 \ldots, 2) \), (d) irregular \( \vec{v}_1^d \) with the same notation as in Fig.7.
FIG. 11: The autocorrelation function $C_0(i)$ versus $i$ in $\text{MG}(N_c,N_c^2)$ with $N_c = 3$, $S = 2$, $M = 4$ and $\alpha \approx 0.06$ where the strategy space $\mathcal{S}$ is formed by the spanning strategies $\vec{v}_a = (1, \ldots, 1)$ and (a) $\vec{v}_a^1 = (0, 1, 2)^2$, (b) $\vec{v}_a^2 = (0, 0, 0, 1, 1, 2, 2, 2)^3$, (c) $\vec{v}_a^3 = (0, \ldots, 0, 1, \ldots, 1, 2 \ldots, 2)^3$, (d) $\vec{v}_a^4 = (0, \ldots, 0, 1, \ldots, 1, 2 \ldots, 2)$, (e) $\vec{v}_a^5 = (0, 1, 2, 2, 0, 1, 1, 2, 0)^3$, (f) irregular $\vec{v}_a^6$ with the same notation as in Fig.7.
FIG. 12: The autocorrelation function $C_0(i)$ versus $i$ in MG($N_c, N_c^2$) with $S = 2$, $M = 2$ and $\alpha \approx 0.06$ for the strategy space $S$ formed by the spanning strategies $\vec{v}_a = (1, \ldots, 1)$ and $\vec{v}_b = (0, 1, \ldots, N_c - 1)^{L/N_c}$ where (a) $N_c = 2$, (b) $N_c = 3$, (c) $N_c = 4$, (d) $N_c = 5$, (e) $N_c = 7$, (f) $N_c = 8$ with the same notation as in Fig. 7. Our numerical results show that $C_0(i)$ exhibits similar periodic behavior in MG($N_c, N_c^2$) with the same settings but $M = 3$ and 4.