Perturbed Laguerre Unitary Ensembles, Painlevé V and Information Theory

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Abstract

In this review, based mainly on [3], we investigate a key information-theoretic performance metric in multiple-antenna wireless communications, the so-called outage probability. This quantity may be expressed in terms of a moment generating function, for which we present two separate integral representations, one involving a particular $\sigma$– form of Painlevé V. We also present a representation involving a non-linear second order difference equation.

1 Introduction

In Random Matrix Theory (RMT) many statistical quantities can be described as determinants. This is especially true in the cases of what are known as the classical ensembles, for example, the Gaussian Unitary Ensemble (GUE) or the Laguerre Ensembles. Since the groundbreaking work of Tracy and Widom [20], which characterized the largest eigenvalue distribution for the GUE, one common approach is to write some statistical quantity as a determinant and then express the determinant as something involving a Painlevé transcendent, a solution to one of the classical Painlevé second order non-linear differential equations.

Techniques to find the determinant and then the resulting differential equation are quite complicated. Tracy and Widom used more of an operator theory approach, others have used an integrable system approach, whilst others still have used a stochastic equation approach.

This paper highlights a technique known as the “ladder operator” approach and is, in some sense, one long example illustrating the technique and describing the application of interest. The specific application relates to the performance of multiple-input multiple-output (MIMO) wireless communication systems, in which both the transmitter and receiver devices are equipped with multiple antennas. Such systems have been the subject of intense
interest since the key papers [17] and [5], and now form the cornerstone of most modern day wireless systems (Wi-Fi, cellular networks, etc). Here, a key fundamental performance measure is considered, the so-called “outage probability”, and this is shown to involve the probability distribution of a certain “linear statistic” in a Laguerre random matrix ensemble. The problem thus falls naturally within the realm of the ladder operator framework.

Here is the idea of the approach. For the example at hand, the linear statistic of interest can be described through its moment generating function as a Hankel determinant, which using the theory of orthogonal polynomials (for a certain nonstandard weight generally) can be computed via a product of norms of monic orthogonal polynomials. Now it is well-known that orthogonal polynomials satisfy three term recurrence equations. In fact, the two coefficients in the recurrence combined with initial conditions, completely determine the polynomials. Thus information about the coefficients in the recurrence equations should yield information about the Hankel determinants.

The path to this information is from the ladder operators, two formulas that connect the polynomials one index apart to their derivatives. These yield, using basic complex analysis, a set of equations in the coefficients along with two additional auxiliary quantities that arise. The story would end here, except for the fact that often there is a “time” parameter implicit in the original weight and thus in the polynomials themselves. Using “time” evolution one can then, using only elementary means, find a pair of coupled Ricatti equations in the two auxiliary quantities. These then lead directly to a Painlevé equation. It should be pointed out that this method works at least in principle, if the “time” parameter is present and if the derivative of the logarithm of the weight is a rational function. Then one can in many cases follow the steps illustrated in this paper.

Here is an outline of the paper. The next section contains the preliminaries of the theory including the ladder operator equations. Section III shows how the application of interest, the outage probability performance measure which arises in the application of MIMO wireless communication, can be described using the RMT framework. Section IV provides the details of the path to the differential equation solutions, which are presented Theorems 1 and 2.

2 Preliminaries

2.1 Linear Statistics of Hermitian Random Matrices and Hankel Determinants

For the MIMO capacity application, it will be seen that the problem of interest falls within the general theory of linear statistics of Hermitian random matrices, with a close connection to the theory of orthogonal polynomials. Here a brief introduction to the general theory is given, and preliminaries are established for later use.
We will require the distribution of a certain linear statistic
\[ \sum_{k=1}^{N} f(x_k) \] (2.1)
in the eigenvalues \( \{x_k\} \) of a \( N \times N \) Hermitian random matrix, with joint eigenvalue density of the form
\[ p(x_1, \ldots, x_N) \propto \prod_{k=1}^{N} w_0(x_k) \prod_{1 \leq i < j \leq N} (x_j - x_i)^2, \quad x_k \in (a, b) \] (2.2)
for some weight function \( w_0(\cdot) \). It is convenient to attempt to characterize the distribution of the linear statistic (2.1) through its moment generating function\(^1\),
\[ M(\lambda) = E \left[ \exp \left( \lambda \sum_{k=1}^{N} f(x_k) \right) \right] = E \left[ \prod_{k=1}^{N} e^{\lambda f(x_k)} \right] \] (2.3)
which upon substituting for (2.2) gives
\[ M(\lambda) = \frac{1}{N!} \int_{(a,b)^N} \prod_{1 \leq i < j \leq N} (x_j - x_i)^2 \prod_{k=1}^{N} w(x_k)dx_k \] (2.4)
where
\[ w(x) := w_0(x)e^{\lambda f(x)} \]
denotes the deformed version of the reference weight \( w_0(\cdot) \). Application of the Andrefief-Heine identity \[16\] now directly leads to
\[ M(\lambda) = D_N[w] \] \[ \frac{D_N[w_0]}{D_N[w]} = \frac{\det \left( \int_{a}^{b} x^{i+j-2} w(x)dx \right)_{i,j=1}^{N}}{\det \left( \int_{a}^{b} x^{i+j-2} w_0(x)dx \right)_{i,j=1}^{N}}, \] (2.5)
which is a ratio of Hankel determinants. By virtue of the Selberg integral, for most “classical” weight functions of interest, the Hankel determinant in the denominator of (2.4) admits an explicit closed-form (non determinantal) representation. The numerator, on the other hand, is much more difficult to characterize, since it involves the more complicated deformed weight \( w \). To proceed, methods based on orthogonal polynomials will be introduced in the sequel.

\(^1\)The parameter \( \lambda \) is an indeterminate which generates the random variable \( \sum_{k=1}^{N} f(x_k) \).
2.2 Orthogonal Polynomials and their Ladder Operators

We start by noting that

$$\prod_{1 \leq i < j \leq N} (x_j - x_i) = \det \left( x_j^{i-1} \right)_{i,j=1}^N = \det \left( P_{i-1}(x)_j \right)_{i,j=1}^N$$  \hspace{1cm} (2.6)

where $P_j(\cdot)$ represents any monic polynomial of degree $j$,

$$P_j(z) = z^j + p_1(j) z^{j-1} + ...$$  \hspace{1cm} (2.7)

Applying this in (2.4) and once again integrating via the Andreief-Heine identity, the numerator evaluates to

$$D_N[w] = \det \left( \int_a^b P_{i-1}(x)P_{j-1}(x)w(x)dx \right)_{i,j=1}^N.$$  \hspace{1cm} (2.8)

If we orthogonalize the polynomial sequence $\{P_n(x)\}$ with respect to $w(x)$ over the interval $[a,b]$, i.e.,

$$\int_a^b P_j(x)P_k(x)w(x)dx = h_j \delta_{j,k}, \quad j, k = 0, 1, 2, ...$$  \hspace{1cm} (2.9)

with $h_j$ denoting the square of the $L^2$ norm of $P_j$ over $[a,b]$, then (2.8) reduces to

$$D_N[w] = \prod_{k=0}^{N-1} h_k.$$  \hspace{1cm} (2.10)

The key challenge is how to characterize the class of polynomials which obey the orthogonality constraints in (2.9) or, more importantly, the norms of such polynomials required to evaluate (2.10).

If all the moments of the weight $w$ exist, then the theory of orthogonal polynomials states that the $P_n(z)$ for $n = 0, 1, 2, ...$ satisfy the three term recurrence relations,

$$zP_n(z) = P_{n+1}(z) + \alpha_n P_n(z) + \beta_n P_{n-1}(z).$$  \hspace{1cm} (2.11)

The above sequence of polynomials can be generated from the orthogonality conditions (2.9), the recurrence relations (2.11), and the initial conditions,

$$P_0(z) = 1, \quad \beta_0 P_{-1}(z) = 0.$$  \hspace{1cm} (2.12)

Substituting (2.7) into the recurrence relations, an easy computation shows that

$$p_1(n) - p_1(n + 1) = \alpha_n,$$  \hspace{1cm} (2.13)
with \( p_1(0) := 0 \). A telescopic sum of (2.13) gives

\[
p_1(n) = - \sum_{j=0}^{n-1} \alpha_j.
\]  

(2.14)

¿From the recurrence relation (2.11) and the orthogonality relations (2.9), we find

\[
\beta_n = \frac{h_n}{h_{n-1}}.
\]  

(2.15)

We shall see that \( p_1(n) \) plays an important role in later developments. For more information on orthogonal polynomials, we refer the reader to Szegő’s treatise [16].

Next, we present three Lemmas which are concerned with the “ladder operators” associated with orthogonal polynomials, as well as certain supplementary conditions. Note that these have been known for quite sometime; we reproduce them here for the convenience of the reader using the notation of [2], where one can also find a list of references to the literature. We also mention that Magnus [9] was perhaps the first to apply these lemmas—albeit in a slightly different form—to random matrix theory and the derivation of Painlevé equations. Tracy and Widom also made use of the compatibility conditions in their systematic study of finite \( n \) matrix models [19]. See also [4].

**Lemma 1** Suppose \( v(x) = -\log w(x) \) has a derivative in some Lipshitz class with positive exponent. The lowering and raising operators satisfy the differential-difference formulas:

\[
P'_n(z) = -B_n(z)P_n(z) + \beta_n A_n(z)P_{n-1}(z) \tag{2.16}
\]

\[
P'_{n-1}(z) = [B_n(z) + v'(z)]P_{n-1}(z) - A_{n-1}(z)P_n(z), \tag{2.17}
\]

where

\[
A_n(z) := \frac{1}{h_n} \int_a^b \frac{v'(z) - v'(y)}{z - y} P_n^2(y)w(y)dy \tag{2.18}
\]

\[
B_n(z) := \frac{1}{h_{n-1}} \int_a^b \frac{v'(z) - v'(y)}{z - y} P_n(y)P_{n-1}(y)w(y)dy. \tag{2.19}
\]

A direct computation produces two fundamental supplementary or compatibility conditions valid for all \( z \in \mathbb{C} \cup \{\infty\} \). These are stated in the next Lemma.

**Lemma 2** The functions \( A_n(z) \) and \( B_n(z) \) satisfy the supplementary conditions:

\[
B_{n+1}(z) + B_n(z) = (z - \alpha_n)A_n(z) - v'(z) \tag{S_1}
\]

\[
1 + (z - \alpha_n)[B_{n+1}(z) - B_n(z)] = \beta_{n+1}A_{n+1} - \beta_nA_{n-1}(z). \tag{S_2}
\]

It turns out that there is an equation which gives better insight into the coefficients \( \alpha_n \) and \( \beta_n \), if (\( S_1 \)) and (\( S_2 \)) are suitably combined to produce a “sum rule” on \( A_n(z) \). We state this in the next lemma. The sum rule, we shall see later, provides important information
Lemma 3 The functions $A_n(z)$, $B_n(z)$, and the sum
\[ \sum_{j=0}^{n-1} A_j(z), \]
satisfy the condition:
\[ B_n^2(z) + \nu'(z)B_n(z) + \sum_{j=0}^{n-1} A_j(z) = \beta_n A_n(z) A_{n-1}(z). \] (S'_2)

3 Information Theory of MIMO Wireless Systems

In this section we introduce the wireless communication problem of interest, and connect it with the general linear statistics framework introduced previously.

We consider a MIMO communication system in which a transmitter equipped with $n_t$ antennas communicates with a receiver equipped with $n_r$ antennas. Denoting the transmitted signal vector as $x \in \mathbb{C}^{n_t}$ and the received signal vector as $y \in \mathbb{C}^{n_r}$, under a certain assumption on the channel (known as “flat fading”), these signals are related via the linear model
\[ y = Hx + n, \] (3.1)
where $n \in \mathbb{C}^{n_r}$ the receiver noise vector, is complex Gaussian with zero mean and covariance $E(nn^\dagger) = I_{n_r}$. The matrix channel matrix, $H \in \mathbb{C}^{n_r \times n_t}$, represents the wireless fading coefficients between each transmit and receive antenna. The channel is modeled stochastically, with distribution depending on the specific wireless environment. Under the realistic assumption that there are sufficient scatterers surrounding the transmit and receive terminals, the channel matrix $H$ is well modeled by a complex Gaussian distribution with independent and identically distributed (i.i.d.) elements having zero mean and unit variance. This matrix is assumed to be known at the receiver\(^2\), but the transmitter only has access to its distribution. The transmitted signal $x$ is designed to meet a power constraint:
\[ E(x^\dagger x) \leq P. \] (3.2)

Our objective is to study the fundamental capacity limits of a MIMO communication system. Such limits are described by the field of information theory, founded by Claude Shannon in 1948 [14]. Specifically, information-theoretic measures allow one to precisely determine the highest data rate that can be communicated with negligible errors by any transmission scheme. Consequently, information theory offers a benchmark for the design

\(^2\)In practice, this information can be obtained using standard estimation techniques.
of practical transmission technologies, and has become an indispensable tool for modern communication system design.

The capacity of a communication link is determined by the so-called “mutual information” between the input and output signals. For the MIMO model (3.1) it is given by:

\[ I(x; y|H) = H(y|H) - H(n) \]  

with \( H(y|H) \) denoting the conditional entropy of \( y \),

\[ H(y|H) = -\int_{C_{nr}} p(y|H) \log p(y|H) dy, \]  

where \( p(y|H) \) denotes the conditional density of \( y \) given \( H \). This formula represents the maximum amount of information that can be reliably transported between the transmitter and receiver (i.e., it represents the rate which is “supportable” by a given realization of the MIMO channel). It was proved in [17] that the conditional mutual information \( I(x; y|H) \) is maximized by choosing the input signal vector \( x \) according to a zero-mean circularly-symmetric complex Gaussian distribution with covariance \( Q_x = E(xx^\dagger) \) satisfying \( \text{tr}(Q_x) \leq P \). In this case, the mutual information (3.3) was shown to be

\[ I(x; y|H) = \log \det \left( I_{nr} + HQ_x H^\dagger \right). \]  

In this paper, we will consider a scenario in which the channel is selected randomly at the beginning of a transmission, and remains fixed during the transmission. In this scenario, it is impossible to guarantee that the communication will be completely reliable, since no matter what transmission rate \( R \) we choose (which is assumed fixed) there is always a non-zero probability that the rate may not be supportable by the channel. In other words, there is always a chance that the mutual information \( I(x; y|H) \) falls below \( R \), and thus communicating at rate \( R \) becomes impossible. This is referred to as an “outage event”, and the probability of this occurring is called the outage probability,

\[ P_{out}(R) := \Pr (I(x; y|H) < R). \]  

Here, we will make the common assumption that

\[ Q_x = \frac{P}{n_t} I_{n_t}, \]  

corresponding to sending independent complex Gaussian signals from each transmit antenna, each with power \( P/n_t \). Hence, with this input signal covariance, the quantity \( P (> 0) \) will also represent the signal-to-noise ratio (SNR). With \( Q_x \) given by (3.7), the mutual information \( I(x; y|H) \) becomes

\[ I(x; y|H) = \log \det \left( I_{nr} + \frac{1}{t} HH^\dagger \right), \quad t := \frac{n_t}{P}. \]
To fix notation, let \( M := \max\{n_r, n_t\}, N := \min\{n_r, n_t\}, \alpha := M - N \) and define
\[
W := \begin{cases} 
HH^\dagger, & n_r < n_t \\
H^\dagger H, & n_r \geq n_t 
\end{cases}
\]
with positive eigenvalues \( \{x_k\}_{k=1}^N \). With these definitions, and with \( \det(I + AB) = \det(I + BA) \), we can further evaluate
\[
I(x; y|H) = \sum_{k=1}^N \log \left(1 + \frac{1}{t} x_k\right) = -N \log t + \sum_{k=1}^N \log (t + x_k).
\]
(3.9)

Computation of the outage probability (3.6) requires the probability distribution of \( I(x; y|H) \). From (3.9), this is clearly a linear statistic in the eigenvalues of the Hermitian random matrix \( W \) (with a constant shift of \( -N \log t \)). Moreover, \( W \) is complex Wishart distributed [13], thus the eigenvalues \( \{x_k\}_{k=1}^N \) are well-known to admit the joint density
\[
p(x_1, x_2, ..., x_N) \propto \prod_{l=1}^N w_{\text{Lag}}(x_l) \prod_{1 \leq j < k \leq N} (x_k - x_j)^2, \quad x_l \in [0, \infty)
\]
(3.10)
where
\[
w_{\text{Lag}}(x) := x^\alpha e^{-x}, \quad \alpha > -1, \quad x \in [0, \infty)
\]
is the classical Laguerre weight.

Our aim will be to compute the moment generating function of the linear statistic,
\[
\bar{M}(\lambda) := E_H \left( e^{\lambda I(x; y|H)} \right) = t^{-N \lambda} \mathcal{M}(\lambda)
\]
(3.12)
where \( \mathcal{M}(\lambda) \) is identified by (2.3) but with the following particularizations:
\[
(f(x), w_0(x), w(x), a, b) \implies (\log(t + x), w_{\text{Lag}}(x), w_{\text{dLag}}(x, t), 0, \infty)
\]
(3.13)
where \( w_{\text{dLag}}(x, t) \) is a deformed Laguerre weight,
\[
w_{\text{dLag}}(x, t) := (x + t)^\lambda w_{\text{Lag}}(x), \quad t > 0 \quad x > 0.
\]
(3.14)
Thus, (2.5) immediately gives
\[
\mathcal{M}(\lambda) = \frac{D_N(t, \lambda)}{D_N[w_{\text{Lag}}]}
\]
(3.15)
where
\[
D_N(t, \lambda) = \det (\mu_{i+j-2}(t, \lambda))_{i,j=1}^N
\]
(3.16)
is the Hankel determinant generated from \( w_{d\text{Lag}}(x, t) \) with moments
\[
\mu_k(t, \lambda) := \int_0^\infty x^k w_{d\text{Lag}}(x) dx, \quad k = 0, 1, 2, \ldots .
\]
(3.17)

The quantity \( D_N[w_{\text{Lag}}] \) in the denominator of (3.15) is the Hankel determinant generated
from the classical Laguerre weight, \( w_{\text{Lag}}(x) \), and can be computed in terms of the Barnes
\( G \)–function as
\[
D_N[w_{\text{Lag}}] = \frac{G(N + 1)G(N + \alpha + 1)}{G(\alpha + 1)}, \quad G(1) = 1.
\]
(3.18)

Our next objective will be to compute a non-determinantal representation for the (scaled)
moment generating function (3.15). This, in turn, will require evaluation of the Hankel
determinant \( D_N(t, \lambda) \) in (3.16). We will address this problem in the sequel by appealing to
the orthogonal polynomial framework introduced in Section 2.2. We should like to mention
here that, unlike the classical ladder operators, the “coefficients” in our ladder operators are
“\( x \)” dependent, as we shall see later.

4 Painlevé V and the Continuous and Discrete \( \sigma \)-Form

4.1 Main Results

The following two theorems present the main results of the paper:

**Theorem 1**  The Hankel determinant \( D_N(t, \lambda) \) admits the following integral representation:
\[
\frac{D_N(t, \lambda)}{D_N[w_{\text{Lag}}]} = t^{N\lambda} \exp \left( \int_t^\infty f(y(s), y'(s), s) ds \right)
\]
(4.1)

where
\[
f(y(s), y'(s), s) := \frac{\lambda^2 + 2(s + \alpha - \lambda)y + (4Ns + (s + \alpha)^2 - 2(s + 2\alpha)\lambda + \lambda^2)y^2}{4y(y - 1)^2}
\]
\[
+ \frac{-2(2Ns + \alpha(s + \alpha - \lambda))y^3 + \alpha^2y^4 - [s y'(s)]^2}{4y(y - 1)^2}.
\]
(4.2)

**Theorem 2**  Equivalently, the Hankel determinant \( D_N(t, \lambda) \) also admits the following integral representation:
\[
\frac{D_N(t, \lambda)}{D_N[w_{\text{Lag}}]} = t^{N\lambda} \exp \left( \int_0^\infty \frac{H_N(x) - N\lambda}{x} dx \right)
\]
(4.4)

where \( H_N(t) \) satisfies the Jimbo-Miwa-Okamoto \( \sigma \)–form of Painlevé V:

\[
(tH_N''(t))^2 = \left[ tH_N' - H_N + H_N'(2N + \alpha + \lambda) + N\lambda \right]^2
\]
\[
- 4(tH_N' - H_N + \delta_N) \left[ (H_N')^2 + \lambda H_N'' \right]
\]
(4.5)
with $\delta_N := N(N + \alpha + \lambda)$.

Before presenting the proof of these results, we would like to point out that Painlevé equations first appeared in the early 1900’s through the work of Painlevé and his collaborators \[6\]. In the mid 1970’s, Painlevé equations first appeared in characterizing the correlation function of an Ising model through the pioneering work of Barouch, McCoy, Tracy and Wu, see \[10\]. The 1-particle reduced density matrix was shown in 1980 to satisfy a particular Painlevé V, see \[7\]. For a recent review on this and other related problems in matrix ensembles, see \[18\]. Another Painlevé V appeared in the Hankel determinant associated with the “time evolved” Jacobi polynomials, see \[1\].

### 4.2 Proof of Theorems 1 and 2

#### 4.2.1 Compatibility Conditions, Recurrence Coefficients and Discrete Equations

For the purpose of applying the ladder operator framework introduced in Lemmas 1–3, an easy computation shows that

$$v(z, t) := -\log w_{\text{dLag}}(z, t) = -\alpha \log z - \lambda \log(z + t) + z,$$

$$v'(z, t) = -\frac{\alpha}{z} - \frac{\lambda}{z + t} + 1$$

and therefore

$$\frac{v'(z, t) - v'(y, t)}{z - y} = \frac{\alpha}{zy} + \frac{\lambda}{(z + t)(y + t)}.$$

Substituting the above into \eqref{2.18} and \eqref{2.19}, followed by integration by parts, we obtain

$$A_n(z) = \frac{1 - R_n(t)}{z} + \frac{R_n(t)}{z + t} \quad \text{(4.6)}$$

$$B_n(z) = -\frac{n + r_n(t)}{z} + \frac{r_n(t)}{z + t} \quad \text{(4.7)}$$

where we have introduced the auxiliary quantities:

$$R_n(t) := \frac{\lambda}{h_n} \int_0^\infty \frac{[P_n(x)]^2}{x + t} w_{\text{dLag}}(x, t) dx \quad \text{(4.8)}$$

$$r_n(t) := \frac{\lambda}{h_{n-1}} \int_0^\infty \frac{P_n(x)P_{n-1}(x)}{x + t} w_{\text{dLag}}(x, t) dx. \quad \text{(4.9)}$$

The next Lemma gives a representation of the recurrence coefficients $\alpha_n$, $\beta_n$, $\sum_j R_j$, and $p_1(n)$—the coefficient of $z^{n-1}$ of $P_n(z)$—in terms of the auxiliary variables $r_n$ and $R_n$. Note that $p_1(n)$ also depends on $t$ but we do not display this if there is no confusion.

**Lemma 4** The recurrence coefficients $\alpha_n$ and $\beta_n$ are expressed in terms of the auxiliary quantities $r_n$ and $R_n$ as:

$$\alpha_n = 2n + 1 + \alpha + \lambda - tR_n \quad \text{(4.10)}$$

$$\beta_n = \frac{1}{1 - R_n} \left[ r_n(2n + \alpha + \lambda) + \frac{r_n^2 - \lambda r_n}{R_n} + n(n + \alpha) \right]. \quad \text{(4.11)}$$
Furthermore,

\[ t \sum_{j=0}^{n-1} R_j = n(n + \alpha + \lambda) - \beta_n - tr_n, \]  
\[ p_1(n) = -\beta_n - tr_n. \]  

**Proof** We start from \((S_1)\). Equating the coefficients of \( z^{-1} \) and \((z + t)^{-1}\), we obtain the following difference equations relating \( \alpha_n \) to \( r_n \) and \( R_n \):

\[-(2n + 1 + r_{n+1} + r_n) = \alpha - \alpha_n(1 - R_n) \]  
\[ r_{n+1} + r_n = \lambda - R_n(t + \alpha_n). \]  

To proceed further, we take note of \((4.6)-(4.9)\), and derive identities based on the supplementary condition \((S'_2)\), which will be of particular interest. A straightforward (but long) computation shows that the r.h.s. of \((S'_2)\) becomes

\[ B^2_n(z) + \nu'(z)B_n(z) + \sum_{j=0}^{n-1} A_j(z) \]
\[ = z^{-2}[(n + r_n)^2 + \alpha(n + r_n)] \]
\[ + z^{-1}\left\{ n - \sum_{j=0}^{n-1} R_j + r_n[\lambda - \alpha - t - 2(n + r_n)]/t + (n - \lambda)/t \right\} \]
\[ + (z + t)^{-1}\left\{ \sum_{j=0}^{n-1} R_j + r_n[t + \alpha + 2(n + r_n)]/t - n\lambda/t \right\} \]
\[ + (z + t)^{-2}[r_n^2 - \lambda r_n]. \]

Now focusing on \((S'_2)\) as presented above and equating the coefficients of \( z^{-2} \), \( z^{-1} \), \((z + t)^{-1}\), \((z + t)^{-2}\), give rise to the following difference equations involving \( \beta_n \), \( r_n \), \( R_n \) and \( \sum_j R_j \):

\[ (n + r_n)^2 + \alpha(n + r_n) = \beta_n(1 - R_n)(1 - R_{n-1}) \]  
\[ n - \sum_{j=0}^{n-1} R_j + \frac{r_n}{t}[(\lambda - \alpha - t - 2(n + r_n)] + \frac{n(\lambda - t)}{t} = \frac{\beta_n}{t}[(1 - R_{n-1})R_n + (1 - R_{n-1})R_n] \]
\[ \sum_{j=0}^{n-1} R_j + \frac{r_n}{t}[t + \alpha - \lambda + 2(n + r_n)] - \frac{n\lambda}{t} = -\frac{\beta_n}{t}[(1 - R_n)R_{n-1} + (1 - R_{n-1})R_n] \]
\[ r_n^2 - \lambda r_n = \beta_nR_nR_{n-1}. \]
Observe that (4.17) and (4.18) are equivalent. We shall see later that (4.17), when combined with certain identities, performs the sum \( \sum_{j=0}^{n-1} R_j \) automatically in closed-form. This sum will provide an important link between the logarithmic derivative of the Hankel determinant with respect to \( t \), and \( \beta_n, r_n \), which is an essential step in establishing the Painlevé equation. Whilst the difference relations (4.14)–(4.16) and (4.19) look rather complicated, these can be manipulated to give us insight into the recurrence coefficients \( \alpha_n \) and \( \beta_n \). To this end, summing (4.14) and (4.15) gives us a simple expression for the recurrence coefficient \( \alpha_n \) in terms of \( R_n \):

\[
\alpha_n = 2n + 1 + \alpha + \lambda - tR_n. \tag{4.20}
\]

From (4.16) and (4.19) we find after a minor re-arrangement

\[
\beta_n(R_n + R_{n-1}) = \beta_n - n(n + \alpha) - r_n(\alpha + \lambda + 2n). \tag{4.21}
\]

Now substituting (4.19) and (4.21) into either (4.17) or (4.18) to eliminate \( R_n \) and \( R_{n-1} \) leaves us the following very simple form for \( \sum_{j=0}^{n-1} R_j \), which will play a crucial role later,

\[
t \sum_{j=0}^{n-1} R_j = n(n + \alpha + \lambda) - \beta_n - tr_n. \tag{4.22}
\]

But in view of (4.20), we also have an alternative representation of \( \sum_{j=0}^{n-1} R_j \), namely,

\[
t \sum_{j=0}^{n-1} R_j = n(n + \alpha + \lambda) - \sum_{j=0}^{n-1} \alpha_j = n(n + \alpha + \lambda) + p_1(n). \tag{4.23}
\]

In summary, we have obtained two different ways to express \( \sum_{j=0}^{n-1} R_j \), and comparing (4.22) with (4.23) gives us the important relation (4.13). linking \( p_1(n) \) to \( \beta_n \) and \( r_n \). We are now in the position to find an expression for \( \beta_n \) in terms of \( r_n \) and \( R_n \). This is found by eliminating \( R_{n-1} \) from (4.21) and (4.19) resulting in (4.11). End of Proof.

4.2.2 \( t \) Evolution and Painlevé V

In the next stage of the development, we vary \( t \) and \( n \). The differential-difference relations generated here when combined with the difference relations obtained previously will give us the desired Painlevé equation. A straightforward computation shows that

\[
\frac{d}{dt} \log h_n = R_n. \tag{4.24}
\]

But, from (2.15), it follows that

\[
\frac{d\beta_n}{dt} = \beta_n(R_n - R_{n-1}) \tag{4.25}
\]

\[
= \beta_n R_n - \frac{r_n^2 - \lambda r_n R_n}{R_n}, \tag{4.26}
\]

12
where the last equality follows from (4.19).

Differentiating
\[ 0 = \int_{0}^{\infty} x^\alpha (x + t)^{\lambda - 1} e^{-x} P_n(x) P_{n-1}(x) dx \]
with respect to \( t \) produces
\[ 0 = \lambda \int_{0}^{\infty} x^\alpha (x + t)^{\lambda - 1} e^{-x} P_n(x) P_{n-1}(x) dx + \int_{0}^{\infty} x^\alpha (x + t)^{\lambda - 1} e^{-x} \left[ \frac{d}{dt} p_1(n) x^{n-1} + ... \right] P_{n-1}(x) dx \]
resulting in
\[ \frac{d}{dt} p_1(n) = -r_n. \] (4.27)

Upon noting (2.13), this implies
\[ \frac{d\alpha_n}{dt} = r_{n+1} - r_n. \] (4.28)

Now differentiating (4.13) with respect to \( t \) and noting (4.27), we find
\[ \frac{d}{dt} p_1(n) = -\frac{d\beta_n}{dt} - \frac{d}{dt}(tr_n) = -\frac{d\beta_n}{dt} - r_n - t \frac{dr_n}{dt} = -r_n. \]

The above result combined with (4.26) gives
\[ \frac{d\beta_n}{dt} = -t \frac{dr_n}{dt} = \beta_n R_n - \frac{r_n^2 - \lambda r_n}{R_n}. \] (4.29)

We now come to a key Lemma which gives the first order derivative of \( r_n(t) \) and \( R_n(t) \) with respect to \( t \), and where \( n \) appears as a parameter.

Lemma 5 The auxiliary variables \( r_n \) and \( R_n \) satisfy the following coupled Riccati equations,
\[ t \frac{dr_n}{dt} = \frac{r_n^2 - \lambda r_n}{R_n} - \frac{R_n}{1-R_n} \left[ r_n(2n + \alpha + \lambda) + \frac{r_n^2 - \lambda r_n}{R_n} + n(n+\alpha) \right], \] (4.30)
and
\[ 2r_n = t \frac{dR_n}{dt} + \lambda - R_n (t + 2n + \alpha + \lambda - t R_n). \] (4.31)

Furthermore,
\[ y(t) = y(t, n) := 1 - \frac{1}{1 - R_n(t)}, \]
satisfies the following second order non-linear ode,
\[ y'' = \frac{3y - 1}{2y(y - 1)} (y')^2 - \frac{y'}{t} + \frac{(y - 1)^2}{t^2} \left( \frac{\alpha^2}{2} y - \frac{\lambda^2}{2y} \right) + \frac{(2n + 1 + \alpha + \lambda)}{t} y - \frac{y(y + 1)}{2(y - 1)}, \]
which is recognized to be a
\[ PV \left( \frac{\alpha^2}{2}, -\frac{\lambda^2}{2}, 2n + 1 + \alpha + \lambda, -1/2 \right). \]

**Proof** Because (4.11) expresses \( \beta_n \) as a quadratic in \( r_n \), we see that \( r_n \) satisfies the Riccati equation (4.30). Eliminating \( r_{n+1} \) from (4.15) and (4.28), and upon referring to (4.11), we obtain (4.31). Next, we simply substitute \( r_n(t) \) from (4.31) into (4.30), to see that \( R_n(t) \) satisfies a second order non-linear ode in \( t \), in which \( n, \alpha, \) and \( \lambda \) appear as parameters. A further linear fractional change of variable
\[ R_n(t) = 1 - \frac{1}{1 - y(t)} \quad \text{or} \quad y(t) = 1 - \frac{1}{1 - R_n(t)}, \]
establishes that \( y(t) \) satisfies the Painlevé V displayed in the Lemma. **End of Proof.**

We begin here a series of computations which ultimately give rise two integral representations of the Hankel determinant of interest, i.e.,
\[ D_N(t, \lambda) = \det \left( \int_0^\infty x^{j+k-2}(x + t)^\lambda x^\alpha e^{-x} dx \right)_{1 \leq j, k \leq N}, \]
given in Theorems 1 and 2. To this end, an easy computation shows that
\[ H_N(t) := t \frac{d}{dt} \log D_N(t, \lambda) = t \frac{d}{dt} \sum_{j=0}^{N-1} \log h_j = t \sum_{j=0}^{N-1} R_j \]
\[ = N(N + \alpha + \lambda) - \beta_N - tr_N \quad \text{(4.33)} \]
\[ = N(N + \alpha + \lambda) + p_1(N), \quad \text{(4.34)} \]
where the last two equations follow from (4.12) and (4.13) of Lemma 4. Integrating (4.33) with respect to \( t \), while noting (4.11), (4.31) and \( R_N(t) = 1 - 1/(1 - y(t)) \), we obtain the result stated in Theorem 1.

To obtain the second integral representation for \( D_n(t, \lambda) \) stated in Theorem 2 (i.e., in terms of \( H_n(t) \)), we note that from (4.27), (4.33), and (4.34), we obtain expressions for \( \beta_N \) and \( r_N \) in terms of \( H_N \) and \( H'_N \),
\[ \beta_N = N(N + \alpha + \lambda) + tH'_N - H_N \quad \text{(4.35)} \]
\[ r_N = -H'_N. \quad \text{(4.36)} \]
What we need to do is to eliminate $R_N$ to find a functional equation satisfied by $H_N$, $H'_N$, and $H''_N$. For this purpose, we examine two quadratic equations satisfied by $R_N$, one of which is simply a rearrangement of (4.11) and reads

$$\frac{r_N^2 - \lambda r_N}{R_N} + \beta_N R_N = \beta_N - r_N(2N + \alpha + \lambda) - N(N + \alpha).$$

(4.37)

The other follows from a derivative of the first equation of (4.35) with respect to $t$ and

$$\beta_N R_N - \frac{r_N^2 - \lambda r_N}{R_N} = tH'_N.$$

(4.38)

Solving for $R_N$ and $1/R_N$ from the linear system (4.37) and (4.38), we find

$$2R_N = 1 + \frac{tH''_N - (2N + \lambda + \lambda)R_N - N(N + \alpha)}{tH'_N - H_N + N(N + \alpha + \lambda)}$$

(4.39)

$$\frac{2}{R_N} = \frac{-tH''_N + (t + 2N + \alpha + \lambda)H'_N - H_N + N\lambda}{(H'_N)^2 + \lambda H'_N},$$

(4.40)

where we have replaced $\beta_N$ and $r_N$ in terms of $H_N$, $H'_N$ and $H''_N$ with (4.35). The product (4.39) and (4.40) gives us the desired $\sigma-$ form (4.5).

It is finally worth noting that with $D_N(t, \lambda) = t^{\delta_N} \tilde{D}_N$, we find, after a little computation that $\tilde{D}_N$ satisfies the Toda molecule equation [21]

$$\frac{d^2}{dt^2} \log \tilde{D}_N = \frac{\tilde{D}_{N+1} \tilde{D}_{N-1}}{\tilde{D}_N^2}.$$  

(4.41)

### 4.3 The Discrete $\sigma-$ Form

As an alternative to the Jimbo-Miwa-Okamoto $\sigma-$ form satisfied by $H_N$ in Theorem 2, we state (without proof) a non-linear difference equation satisfied by $H_N$. We believe this to be new. Specifically, we find that the logarithmic derivative of the Hankel determinant in (3.16), generated by the deformed Laguerre weight in (3.14), satisfies a second order non-linear difference equation, which we call the discrete $\sigma-$ form,

$$
\left[ \frac{N(N + \alpha)t + (\Delta^2 H_N + t)(H_N - \delta_N)}{\Delta^2 H_N + 2N + \alpha + \lambda + t} \right]^2 - \lambda \frac{N(N + \alpha)t + (\Delta^2 H_N + t)(H_N - \delta_N)}{\Delta^2 H_N + 2N + \alpha + \lambda + t}
$$

$$= \left[ \delta_N - H_N + \frac{N(N + \alpha)t + (\Delta^2 H_N - t)(H_N - \delta_N)}{\Delta^2 H_N + 2N + \alpha + \lambda + t} \right] (H_{N+1} - H_N)(H_N - H_{N-1})
$$

where $\Delta^2 H_N := H_{N+1} - H_{N-1}$. The initial conditions are $H_1(t) = \frac{d}{dt} \log D_1(t, \lambda)$, $H_2(t) = \frac{d}{dt} \log D_2(t, \lambda)$ with $D_1(t, \lambda) = \mu_0(t)$, $D_2(t, \lambda) = \mu_0(t)\mu_2(t) - \mu_1^2(t)$, and the moments are defined in (3.17).
We do not further elaborate on this result, but present it here since we believe that it may provide a useful tool for efficiently computing $H_N$ in an iterative manner, and give further insight into the mutual information of a single-user MIMO system. A thorough analysis of the implications of this non-linear difference equation in regards to wireless communication applications is the subject of on-going work.

5 Concluding Remarks

This is a short review is about Hankel determinants that arise in the information-theoretic study of MIMO communication systems. We consider the capacity of single-user MIMO systems, in which case the determinant of interest is generated from a certain deformed Laguerre weight. We have obtained two exact integral representations for this Hankel determinant, one of which is described in terms of the $\sigma-$ form of a particular Painlevé V differential equation. We have also stated an alternative representation, involving a second-order non-linear difference equation.

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