SUMS OF SERIES INVOLVING CENTRAL BINOMIAL COEFFICIENTS & HARMONIC NUMBERS

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ABSTRACT. This paper contains a number of series whose coefficients are products of central binomial coefficients & harmonic numbers. An elegant sum involving \( \zeta(2) \) and two other nice sums appear in the last section.

1. Introduction and Preliminary results

1.1. Beginnings. Euler investigated the partial sums of the harmonic series, finding connection between them and \( \log(n) \). These sums:
\[
H_n = \sum_{k=1}^{n} \frac{1}{k} = \int_0^1 \frac{1-x^n}{1-x} \, dx
\]
are generally known as harmonic numbers. He then discovered the formula for
\[
\zeta(2m) = \sum_{n=1}^{\infty} \frac{1}{n^{2m}}
\]
and computed these values:
\[
\text{Li}_2(1) = \frac{\pi^2}{6}, \quad \text{Li}_2(-1) = -\frac{\pi^2}{12}, \quad \text{Li}_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \log 2 \log 2.
\]

Further, he investigated the double sums
\[
\sum_{n=1}^{\infty} \frac{H_n}{n^m}
\]
which have been a popular topic of research in recent years.

1.2. Generating functions for central binomial coefficients and Catalan numbers. Recall that the generating function for the sequence \( a_0, a_1, a_2, \ldots \) is defined to be the function represented by power series:
\[
G(x) := \sum_{n=0}^{\infty} a_n x^n.
\]
It is always permissible to integrate (and differentiate) a power series term by term over any closed interval lying entirely within its interval of convergence.

The binomial coefficient \( \binom{2n}{n} \), the largest coefficient of the polynomial \((1+x)^{2n}\), forms the central column of Pascal’s triangle and so is always an integer. The sequence of these numbers is generated by
\[
\frac{1}{\sqrt{1-4x}} = \sum_{n=0}^{\infty} \binom{2n}{n} x^n.
\]
The expansion is a consequence of the binomial theorem as for \( n \in \mathbb{N} \):
\[
\frac{1}{(1-x)^{1/n}} = 1 + \frac{1}{n} x + \frac{1}{n \cdot 2n} x^2 + \frac{1}{n \cdot 2n \cdot 3n} x^3 + \ldots.
\]

The series on the R.H.S. of (1) converges if \(|x| < \frac{1}{4}\). Lehmer[11] obtained some ‘interesting series’ by repeated integrations of (1).

If we integrate (1) from 0 to \( x \) then divide the result by \( x \) we get generating function for \( \frac{1}{n+1} \left( \frac{2n}{n} \right) \), known as the Catalan numbers and denoted by \( C_n \).

\[
1 - \sqrt{1 - 4x} = \sum_{n=0}^{\infty} C_n x^n.
\]

Transposing its first term of (1) to the left, dividing both sides by \( x \) and then integrating, one gets [11, (6)]:
\[
\ln(1 - x) + \frac{1}{2} \ln(1 - x) = \sum_{n=1}^{\infty} \frac{H_n}{n} x^n (x \neq 1).
\]

1.3. Generating function for harmonic numbers and few series. Using Euler’s integral representation of harmonic numbers, we have:
\[
\sum_{n=1}^{\infty} H_n x^n dx = \sum_{n=1}^{\infty} \left( \int_0^1 \frac{1 - u^n}{1 - u} du \right) x^n = \int_0^1 \frac{1}{1 - u} \left( \sum_{n=1}^{\infty} (x^n - (ux)^n) \right) du
\]
\[
= \int_0^1 \frac{1}{1 - u} \left( \frac{1}{1 - x} - \frac{1}{1 - ux} \right) du = x \int_0^1 \frac{du}{1 - ux}.
\]

Since
\[
x \int_0^1 \frac{du}{1 - ux} = -\frac{\ln(1-x)}{1-x},
\]
we get the generating function for harmonic numbers given in [5] p.54, 1.514.6):
\[
-\frac{\ln(1-x)}{1-x} = \sum_{n=1}^{\infty} H_n x^n \quad (x^2 < 1).
\]

Considering the partial sums of the alternating harmonic series (having sum \( \log 2 \)), with notation \( H'_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} = \log 2 + \frac{(-1)^n}{2} \left[ \psi \left( \frac{n+1}{2} \right) - \psi \left( \frac{n+2}{2} \right) \right] \)
where \( \psi(x) \) is the digamma function, we have this generating function:
\[
\frac{\ln(1+x)}{1-x} = \sum_{n=1}^{\infty} H'_n x^n \quad (x \neq 1).
\]

Now
\[
\int \frac{\log(1+x)}{1-x} dx = -\text{Li}_2 \left( \frac{1+x}{2} \right) - \log \left( \frac{1-x}{2} \right) \log(1+x) + C.
\]

This integral evaluated between -1 and 0 gives the sum:
\[
\sum_{n=1}^{\infty} \frac{(-1)^{n+1} H'_n}{n+1} = \frac{\pi^2}{12} - \frac{1}{2} \log^2 2.
\]

(6)
Further,

\[ \int \frac{\log(1 + x)}{x(1 - x)} \, dx = \int \frac{\log(1 + x)}{1 - x} \, dx + \int \frac{\log(1 + x)}{x} \, dx \]

\[ = -\text{Li}_2(x) - \text{Li}_2 \left( \frac{1 + x}{2} \right) - \log \left( \frac{1 - x}{2} \right) \log(1 + x) + C \]

which yields the sum valid for \(-1 \leq x < 1\):

\[ \sum_{n=1}^{\infty} \frac{H'_n}{n} x^n = \text{Li}_2 \left( \frac{1}{2} \right) - \text{Li}_2(-x) - \text{Li}_2 \left( \frac{1 + x}{2} \right) - \log \left( \frac{1 - x}{2} \right) \log(1 + x). \quad (7) \]

Its alternative form occurs in [12, p.302, A.2.8 (1)] [3, (5)]

\[ \sum_{n=1}^{\infty} \frac{H'_n}{n} x^n = \text{Li}_2 \left( \frac{1 - x}{2} \right) - \text{Li}_2 \left( \frac{1}{2} \right) - \text{Li}_2(-x) - \log(1 - x) \log 2 \]

and gives the sum:

\[ \sum_{n=1}^{\infty} (-1)^{n+1} \frac{H'_n}{n} = \frac{\pi^2}{12} + \frac{1}{2} \log^2 2. \quad (8) \]

So the sum and difference of (6) and (8) result in

\[ \sum_{n=1}^{\infty} (-1)^{n+1} \frac{H'_n}{n(n+1)} = \log^2 2; \quad \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2n+1) H'_n}{n(n+1)} = \frac{\pi^2}{6}. \]

The two integrals evaluated between 0 and 1/2 yield the sums:

\[ \sum_{n=1}^{\infty} \frac{H'_n}{n 2^n} = \frac{1}{2} \text{Li}_2 \left( \frac{1}{4} \right) + \log^2 2 = -\frac{1}{6} \text{Li}_2 \left( \frac{1}{9} \right) + \frac{\pi^2}{36} - \frac{\log^2 3}{3} + \log 2 \log 3, \quad (9) \]

and

\[ \sum_{n=1}^{\infty} \frac{H'_n}{(n+1) 2^{n+1}} = -\frac{1}{3} \text{Li}_2 \left( \frac{1}{9} \right) - \frac{\pi^2}{36} - \frac{1}{2} \log^2 2 - \frac{2}{3} \log^2 3 + 2 \log 2 \log 3. \quad (10) \]

If we multiply (10) by 2 and subtract it from (9), we get

\[ \sum_{n=1}^{\infty} \frac{(3n+4) H'_n}{n(n+1) 2^n} = \frac{2 \pi^2}{3} + (\log 2)^2 = 8 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{H'_n}{n}. \quad (11) \]
1.4. Relation between binomial coefficients and harmonic numbers. Harmonic numbers can be expressed in terms of binomial coefficients:

\[ H_n = \sum_{k=0}^{n-1} \frac{1}{k+1} = \sum_{k=0}^{n-1} \left( \int_0^1 x^k \, dx \right) = \int_0^1 \left( \sum_{k=0}^{n-1} x^k \right) \, dx \]

\[ = \int_0^1 \frac{1 - x^n}{1 - x} \, dx \quad \text{(partial sum of the G.P.)} \]

\[ = - \int_0^1 \frac{1 - (1 - y)^n}{y} \, dy \quad \text{(on setting } 1 - x = y) \]

\[ = \int_0^1 \sum_{k=1}^n \binom{n}{k} (-1)^k y^k \, dy \quad \text{(removing } 1 \text{ from numerator)} \]

\[ = \sum_{k=1}^n \binom{n}{k} (-1)^k \int_0^1 y^{k-1} \, dy = \sum_{k=1}^n \binom{n}{k} (-1)^k \frac{1}{k}. \]

No \( H_n \) is an integer for \( n > 1 \). We find in [6, p.192, (5.48)] this inversion formula:

\[ g(n) = \sum_k \binom{n}{k} (-1)^k f(k) \iff f(n) = \sum_k (\frac{n}{k}) (-1)^k g(k). \]

Since \( H_n = \sum_{k=1}^n (-1)^k \binom{n}{k} \frac{1}{k} \), hence:

\[ \frac{1}{n} = \sum_{k=1}^n (-1)^k \frac{1}{k} H_k. \]

This relation is used by Boyadzhiev [1, 2] to put in effect Euler’s transformation of series for the derivation of certain series whose coefficients are products of the central binomial coefficients and harmonic numbers, and whose sums are expressible in terms of logarithms.

We intend to obtain here series whose sums involve \( \pi, \zeta(2) \) and Catalan’s constant and thereby supplement Boyadzhiev’s work.

2. Generating function for \( \binom{2n}{n} H_n \) and known series

2.1. Using Euler’s transformation of series. We find in [10, p.469] this version of Euler’s transformation of series:

\[ \sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{k=0}^{\infty} a_k \left( \frac{y}{1-y} \right)^{k+1} = \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{n} \binom{n}{m} a_m \right\} y^{n+1}. \]

Boyadzhiev comes up with this formula for \( \alpha \in \mathbb{C} \) in [1, (2.4)] and [2, (10)]:

\[ \sum_{n=0}^{\infty} \binom{\alpha}{n} (-1)^n a_n z^n = (z+1)^{\alpha} \sum_{n=0}^{\infty} \binom{\alpha}{n} (-1)^n \left\{ \sum_{k=0}^{n} \binom{n}{k} \right\} \left( \frac{z}{z+1} \right)^n \]

Setting \( z = 4x \), \( a_k = (-1)^{k-1} H_k \), \( \alpha = -1/2 \) and using the relation we derived above, Boyadzhiev obtained for \( |x| < \frac{1}{4} \) these generating functions for the product of the harmonic numbers and the central binomial coefficients:

\[ \sum_{n=0}^{\infty} H_n \binom{2n}{n} (-1)^{n+1} x^n = \frac{2}{\sqrt{1+4x}} \log \left( \frac{2\sqrt{1+4x}}{1+ \sqrt{1+4x}} \right). \quad (12) \]

\[ \sum_{n=0}^{\infty} H_n \binom{2n}{n} x^n = \frac{2}{\sqrt{1-4x}} \log \left( \frac{1 + \sqrt{1-4x}}{2\sqrt{1-4x}} \right). \quad (13) \]
2.2. Alternative derivation. We can also derive (13) in a different way as follows:

\[
\sum_{n=1}^{\infty} \binom{2n}{n} H_n x^n = \sum_{n=1}^{\infty} \binom{2n}{n} x^n \int_0^1 \frac{1-t^n}{1-t} \, dt
\]

\[
= \int_0^1 \frac{1}{1-t} \left( \sum_{n=1}^{\infty} \binom{2n}{n} (x^n - (xt)^n) \right) \, dt
\]

\[
= \int_0^1 \frac{1}{1-t} \left( \sum_{n=1}^{\infty} \binom{2n}{n} x^n - \sum_{n=1}^{\infty} \binom{2n}{n} (xt)^n \right) \, dt
\]

\[
= \int_0^1 \frac{1}{1-t} \left( \frac{1}{\sqrt{1-4x}} - \frac{1}{\sqrt{1-4xt}} \right) \, dt \quad \text{[by using (1)].}
\]

We assumed that swapping of summation and integration is permissible here.

Now \( \frac{1}{\sqrt{1-4x}} \int_0^1 \frac{dt}{1-t} = \log(1-t) / \sqrt{1-4x} \). To evaluate the second integral we make the substitution: \( u = \sqrt{1-4xt} \) so that \( du = -\frac{2x}{\sqrt{1-4xt}} \, dt \) and \( 1-t = \frac{4x-1+u^2}{4x} \).

Then,

\[
\int \frac{dt}{(1-t)\sqrt{1-4xt}} = 2x \int \frac{du}{(1-4x) - u^2} = \frac{1}{\sqrt{1-4x}} \log \frac{\sqrt{1-4x} + \sqrt{1-4xt}}{\sqrt{1-4x} - \sqrt{1-4xt}}
\]

Thus the integral becomes:

\[
- \frac{1}{\sqrt{1-4x}} \log \frac{\sqrt{1-4x} + \sqrt{1-4xt}(1-t)}{(\sqrt{1-4x} - \sqrt{1-4xt})}
\]

\[
= - \frac{2}{\sqrt{1-4x}} \log \frac{\sqrt{1-4x} + \sqrt{1-4xt}(1-t)}{-4x(1-t)}
\]

that is,

\[
- \frac{2}{\sqrt{1-4x}} \log \frac{\sqrt{1-4x} + \sqrt{1-4xt}}{-4x}.
\]

which at \( t = 1 \) has the value \( -\frac{2}{\sqrt{1-4x}} \log \frac{\sqrt{1-4x}}{-2x} \) and at \( t = 0 \) it becomes:

\[
\frac{2}{\sqrt{1-4x}} \log \frac{1+\sqrt{1-4x}}{-4x}.
\]

Thus the definite integral becomes:

\[
\frac{2}{\sqrt{1-4x}} \log \frac{1+\sqrt{1-4x}}{-4x} - \frac{2}{\sqrt{1-4x}} \log \frac{1+\sqrt{1-4x}}{2\sqrt{1-4x}}
\]

which is the formula (13).

Integrating the power series (13), using the substitution \( 1-4x = y^2 \) for the RHS, one obtains for every \( |x| \leq \frac{1}{4} \),

\[
\sum_{n=0}^{\infty} H_n \binom{2n}{n} \frac{x^{n+1}}{n+1} = \sqrt{1-4x} \log(2\sqrt{1-4x})
\]

\[
- (1+\sqrt{1-4x}) \log(1+\sqrt{1-4x}) + \log 2.
\]

(14)

Putting \( x = 1/4 \) in (14) yields:

\[
\sum_{n=1}^{\infty} \frac{H_n \binom{2n}{n}}{(n+1)2^{2n}} = 4 \log 2.
\]

(15)
By shifting the index we get:

\[
\sum_{n=1}^{\infty} \frac{H_n \left(\frac{2n}{n}\right)}{(2n-1)2^{2n}} = \sum_{n=0}^{\infty} \frac{H_{n+1} \left(\frac{2n+2}{n+1}\right)}{(2n+1)2^{2n+2}} = \frac{1}{2} \left[ \sum_{n=0}^{\infty} \frac{(H_n + \frac{1}{n+1}) \left(\frac{2n}{n}\right)}{(n+1)2^{2n}} \right]
\]

and since (see [4, pp.251–252, §1081])

\[
\sum_{n=0}^{\infty} \frac{(2n)_n}{(n+1)^2 2^{2n}} = 4 - 4 \log 2
\]

so we obtain by using (15) and (16):  

\[
\sum_{n=1}^{\infty} \frac{H_n \left(\frac{2n}{n}\right)}{(2n-1)2^{2n}} = 2.
\]

Edwards deduced (16) via the integral  

\[ I = \int_{0}^{1} \frac{\log(1/x)}{\sqrt{1-x}} \ dx \] 

and putting  

\[ x = \sin^2 \theta. \]

He also gave (pp.252–253):

\[
\sum_{n=0}^{\infty} \frac{(2n)_n}{(2n+1)^2 2^{2n}} = \pi \ln 2
\]

It may be interesting to mention here that \( \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \approx 0.9159655941 \ldots \)

We can similarly derive with  \( x = -\frac{1}{16} \) in (14) and by shifting the index:

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1} \frac{H_n \left(\frac{2n}{n}\right)}{2^{4n}} = 16 \log(4\sqrt{5} - 8) + 8\sqrt{5} \log(10 - 4\sqrt{5}),
\]

and

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1} \frac{H_n \left(\frac{2n}{n}\right)}{2^{4n}} = (\sqrt{5} - 2) + \sqrt{5} \log \left( \frac{1}{\sqrt{5}} + \frac{1}{2} \right).
\]

By a variation on the method, we will now derive some interesting series whose sums involve \( \pi, \zeta(2) \) and Catalan’s constant.

3. New sums involving \( \pi, \zeta(2), G \)

This is the series that we aim to obtain:

\[
\sum_{n=1}^{\infty} \frac{H_n \left(\frac{2n}{n}\right)}{n2^{2n}} = \frac{\pi^2}{3}.
\]

Dividing both sides of (13) by \( x \) and integrating between limits \( x = 0 \) to \( x = 1/4 \), after using the substitution \( 1 - 4x = y^2, \ x = \frac{1-y^2}{2}, \ dx = \frac{y}{2} \ dy \) we get on the R.H.S.:

\[
I = \int_{0}^{1} \frac{2 \cdot 4}{(1-y^2)y} \log \left( \frac{1+y}{2y} \right) \cdot \frac{y}{2} \ dy = \int_{0}^{1} \frac{4}{(1-y^2)} \log \left( \frac{1+y}{2y} \right) \ dy.
\]
The software *WolframAlpha* at [https://www.wolframalpha.com](https://www.wolframalpha.com) gives:

\[
\int 4 \log \left( \frac{1+y}{2y} \right) \frac{dy}{1-y^2} = -2\text{Li}_2(1-y) - 2\text{Li}_2(-y) - 2\text{Li}_2 \left( \frac{y+1}{2} \right) \\
- \log^2(y+1) + 2\log \left( 1 + \frac{1}{y} \right) \log(1-y) \\
+ 2\log(2)\log(1-y) - 2\log(1-y)\log(y) + C,
\]

and

\[
\int_0^1 4 \log \left( \frac{1+y}{2y} \right) \frac{dy}{1-y^2} = \frac{\pi^2}{3}.
\]

Consequently,

\[
\sum_{n=1}^{\infty} \frac{H_n(2n)}{n^{2n}} = \frac{\pi^2}{3}.
\]

Let us try to evaluate the integral \(I\) by partial fractions decomposition:

\[
I = \int_0^1 \frac{2}{1-y} \log \left( \frac{1+y}{2y} \right) \frac{dy}{1-y^2} + \int_0^1 \frac{2}{1+y} \log \left( \frac{1+y}{2y} \right) \frac{dy}{1-y^2} \\
= \int_0^1 \log(1+y) \frac{1}{1-y} \frac{dy}{1-y^2} - \int_0^1 \log(1+y) \frac{1}{1+y} \frac{dy}{1-y^2} \\
+ \int_0^1 \log(1+y) \frac{1}{1+y} \frac{dy}{1-y^2} - \int_0^1 \log(1+y) \frac{1}{1+y} \frac{dy}{1-y^2}.
\]

These are the relevant indefinite integrals:

\[
I_1 = \int \frac{\log(1+y)}{1-y} \frac{dy}{1-y^2} = -\text{Li}_2 \left( \frac{1+y}{2} \right) - \log \left( \frac{1-y}{2} \right) \log(1+y) + C \\
= -\text{Li}_2 \left( \frac{1+y}{2} \right) - \log(1-y)\log(1+y) + \log 2\log(1+y) + C,
\]

\[
I_2 = \int \frac{\log(1+y)}{1+y} \frac{dy}{1-y^2} = \frac{\log^2(1+y)}{2} + C;
\]

\[
I_3 = \int \frac{\log 2}{1-y} \frac{dy}{1-y^2} = -\log 2\log(1-y) + C,
\]

\[
I_4 = \int \frac{\log 2}{1+y} \frac{dy}{1-y^2} = \log 2\log(1+y) + C,
\]

\[
I_5 = \int \frac{\log y}{1-y} \frac{dy}{1-y^2} = \text{Li}_2(1-y) + C,
\]

and

\[
I_6 = \int \frac{\log y}{1+y} \frac{dy}{1+y} = \text{Li}_2(-y) + \log(y)\log(1+y) + C.
\]

**Remark 1.** There arises a problem when one puts \(y = 1\) in the two integrals (with opposite signs) \(I_1\) and \(I_3\) two indeterminate terms: \(\log(0) \times \log(2)\) with opposite signs, but we cannot simply cancel them to get 0. Again, when we put \(y = 0\) in \(I_6\), we get an indeterminate term: \(\log(0)\log(1) = \infty \times 0\) which cannot be straightway taken to be 0; for this, we will take limit as \(y \to 0\).
For $I_1 - I_3$, we notice as in [3, (6)]
\[
\frac{d}{dx} \text{Li}_2 \left( \frac{1 - x}{2} \right) = \frac{1}{1 - x} \log \left( \frac{1 + x}{2} \right) = \frac{\log(1 + x)}{1 - x} - \frac{\log(2)}{1 - x}.
\]
Therefore,
\[
\int_0^1 \left( \frac{\log(1 + x)}{1 - x} - \frac{\log 2}{1 - x} \right) = \text{Li}_2 \left( \frac{1 - x}{2} \right) \bigg|_0^1 - \frac{\log 2}{2} - \frac{\pi^2}{12}.
\]

For $I_6$ we have: \( \lim_{t \to 0} \log t \log(1 + t) \) and to apply l’Hôpital’s rule, we write:
\[
\lim_{t \to 0} \frac{\log t}{\log(1 - t)} = \lim_{t \to 0} \left( -\log^2(1 - t) - 2 \log(1 - t) \right) = 0.
\]
Thus the indeterminate expression has value 0.

Consequently, we get (on using the values of the logarithm given by Euler) the same value of the integral $I = c^2$ as returned by the Wolfram software. □

Combining (15) and (20), we get:
\[
\sum_{n=1}^{\infty} H_n \left( \frac{2n}{n+1} \right) = \frac{\pi^2}{3} - 4 \log 2.
\]

By shift of the index, we have:
\[
\sum_{n=1}^{\infty} \frac{H_n \left( \frac{2n}{n+1} \right)}{2^{2n}} = \sum_{n=0}^{\infty} \frac{H_{n+1} \left( \frac{2n+2}{n+1} \right)}{(n+1)^2 2^{2n+2}} = \frac{1}{2} \left[ \sum_{n=0}^{\infty} \frac{H_n + \frac{1}{n+1} (2n+1) \left( \frac{2n}{n} \right)}{(n+1)^2 2^{2n}} \right]
\]
and the expression within brackets on the extreme right can be expanded as
\[
2 \sum_{n=0}^{\infty} \frac{H_n \left( \frac{2n}{n+1} \right)}{(n+1)^2 2^{2n}} - \sum_{n=0}^{\infty} \frac{H_n \left( \frac{2n}{n} \right)}{(n+1)^2 2^{2n}} + 2 \sum_{n=0}^{\infty} \frac{\left( \frac{2n}{n} \right)}{(n+1)^2 2^{2n}} - \sum_{n=0}^{\infty} \frac{\left( \frac{2n}{n} \right)}{(n+1)^3 2^{2n}}.
\]

Dividing both sides of (3) by $2x$, and then integrating it yields:
\[
\sum_{n=1}^{\infty} \frac{\left( \frac{2n}{n} \right) x^n}{2^{2n} 2^{2n}} = \int \log((1 - \sqrt{1 - 4x})/2x) dx
\]
\[
= -\text{Li}_2 \left( \frac{1 + \sqrt{1 - 4x}}{2} \right) - \frac{1}{2} \log^2(1 + \sqrt{1 - 4x})
\]
\[
- \log \left( \frac{1 - \sqrt{1 - 4x}}{2} \right) \log(1 + \sqrt{1 - 4x})
\]
\[
- \log(-4x) \log \left( \frac{1 + \sqrt{1 - 4x}}{2} \right) + \log(-4x) \log(1 + \sqrt{1 - 4x}) + C.
\]
This in turn gives us
\[
\sum_{n=1}^{\infty} \frac{\left( \frac{2n}{n} \right) 1}{2^{2n} n^2} = \zeta(2) - 2(\log 2)^2.
\]

Prof Paul Levrie drew my attention to this formula in [14, p.354, (22)]
\[
\sum_{n=1}^{\infty} \frac{\left( \frac{1}{n} \right) H_{n-1}}{n!} = \zeta(2) + 2(\log 2)^2
\]
that involves the Pochhammer symbol \( (a)_n = \prod_{k=0}^{n-1} (a + k) \) so that \( \left( \frac{1}{2} \right)_n = \frac{(2n)}{n! 2^{2n}} \).

The formula can be written as \( \sum_{n=1}^{\infty} \frac{(2n)}{n^4} H_n = \sum_{n=1}^{\infty} \frac{(2n)}{n^2 4^n} = \zeta(2) + 2(\log 2)^2 \) which follows immediately from [20] minus [22].

Now
\[
\sum_{n=1}^{\infty} \frac{(2n)}{2^{2n} n^2} = \sum_{n=0}^{\infty} \frac{(2n+2)}{(n+1)^2 2^{2n+2}} = \sum_{n=0}^{\infty} \frac{(2n)}{(n+1)^2 2^{2n}} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(2n)}{(n+1)^3 2^{2n+2}}
\]
so using [10] and [22] we deduce:
\[
\sum_{n=0}^{\infty} \frac{(2n)}{(n+1)^3 2^{2n}} = 8 - 8 \log 2 - \frac{\pi^2}{3} + 4(\log 2)^2.
\] (24)

And by using the sums [15], [10], [20] and [24], we obtain:
\[
\sum_{n=1}^{\infty} \frac{H_n (2n)}{(n+1)^2 2^{2n}} = \frac{\pi^2}{3} - 4(\log 2)^2 + 8 \log 2.
\] (25)

Further, combining [24] and [25], we get:
\[
\sum_{n=1}^{\infty} \frac{H_n (2n)}{n(n+1)^2 2^{2n}} = \frac{2\pi^2}{3} + 4(\log 2)^2 - 12 \log 2.
\] (26)

Let us replace \( x \) by \( x^2 \) in [13] and set \( \sqrt{1 - 4x^2} = y^2 \). We then take the integral from \( y = 1 \) to \( y = 0 \), which means taking \( x \) from 0 to \( 1/2 \). Assuming the value of the integral to be \( \int_1^0 \frac{dy}{\sqrt{1 - y^2}} = \frac{\pi}{2} \) we obtain:
\[
\sum_{n=1}^{\infty} \frac{H_n (2n)}{(2n+1) 2^{2n}} = 4G - \pi \log 2,
\] (27)

where \( G \) is Catalan’s constant.

Using a shift of the index, we have:
\[
\sum_{n=1}^{\infty} \frac{H_n (2n)}{(2n-1)^2 2^{2n}} = \sum_{n=0}^{\infty} \frac{H_{n+1} (2n+2)}{(2n+1)^2 2^{2n+2}}
\]
\[
= \sum_{n=0}^{\infty} \frac{H_n (2n)}{(2n+1) 2^{2n}} - \frac{1}{2} \sum_{n=0}^{\infty} \frac{H_n (2n)}{(n+1) 2^{2n}} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(2n)}{(2n+1)(n+1) 2^{2n}}.
\]

By putting \( x = 1 \) in the expansion:
\[
\arcsin x = x + \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3 \cdot x^5}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5 \cdot x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots
\]
we obtain \( \sum_{n=0}^{\infty} \frac{(2n)}{(2n+1)2^{2n}} = \frac{\pi}{2} \). We already have two sums \( \sum_{n=0}^{\infty} \frac{(2n)}{(n+1)2^{2n}} \) and 
\( \sum_{n=0}^{\infty} \frac{(2n)}{(n+1)^22^{2n}} \) so that by combining the three sums we get:
\[
\sum_{n=0}^{\infty} \frac{(2n)}{(2n+1)(n+1)^22^{2n}} = 2\pi + 4\log 2 - 8. \tag{28}
\]
And using (27) and (28) we derive this result:
\[
\sum_{n=1}^{\infty} \frac{H_n (2n)}{(2n-1)^22^{2n}} = \pi(1 - \log 2) - 4(1 - G). \tag{29}
\]
where \( G \) is Catalan’s constant.

Our integral also gives:
\[
4 \int_{1/3}^{1} \log \left( \frac{1 + y}{2y} \right) \frac{dy}{1-y^2} = 2 Li_2 \left( -\frac{1}{3} \right) + 4 Li_2 \left( \frac{2}{3} \right) - \frac{\pi^2}{6} - (\log 2)^2 + 3(\log 3)^2 - 4 \log 2 \log 3,
\]
which yields another beautiful formula:
\[
\sum_{n=1}^{\infty} \frac{2^n H_n (2n)}{n3^{2n}} = \frac{\pi^2}{6} - (\log 2)^2. \tag{30}
\]
Morris[13, p.781] notes that \( 6Li_2(3) - 3Li_2(-3) = 2\pi^2 \) using which and various relations from [12, p.283], we found:
\[
2 Li_2 \left( \frac{1}{3} \right) - Li_2 \left( -\frac{1}{3} \right) = \frac{\pi^2}{6} - \frac{1}{2} (\log 3)^2 \tag{31}\]
which we used in the derivation of (30).

It may be pointed out that Kirillov [8, p.155, (2.3)] [9, p.89, 6(i)] erroneously omits the term \(- (\log 3)^2\) on the R.H.S. of the relation: \( 6 Li_2(1/3) - Li_2(1/9) = \frac{\pi^2}{3} \).

Further, our integral, in conjunction with the relations [12, p.283, (7)&(13)]:
\[
Li_2(x) + Li_2(1-x) = \frac{\pi^2}{6} - \log(x) \log(1-x) \text{ and } Li_2(x) + Li_2(-x) = \frac{1}{2} Li_2(x) \text{, yields:}
\]
\[
4 \int_{1/2}^{1} \log \left( \frac{1 + y}{2y} \right) \frac{dy}{1-y^2} = \sum_{n=1}^{\infty} \frac{3^n H_n (2n)}{n2^{4n}} = Li_2 \left( \frac{3}{4} \right) + 2(\log 2)^2 - (\log 3)^2,
\]
while the relation [12, p.283, (12)] \( Li_2 \left( \frac{1}{1+x} \right) - Li_2(-x) = \frac{\pi^2}{6} - \frac{1}{2} \log(1 + x) \log \left( \frac{1 + x}{x^2} \right) \) \( x > 0 \) transforms it into:
\[
\sum_{n=1}^{\infty} \frac{3^n H_n (2n)}{n2^{4n}} = \frac{\pi^2}{6} + Li_2 \left( -\frac{1}{3} \right) - \frac{1}{2} (\log 3)^2\]
and using (31) we obtained this lovely result:
\[
\sum_{n=1}^{\infty} \frac{3^n H_n (2n)}{n2^{4n}} = 2 Li_2 \left( \frac{1}{3} \right). \tag{32}\]
The R.H.S. can also be written as: \(\frac{1}{3} Li_2 \left(\frac{1}{9}\right) + \frac{\pi^2}{9} - \frac{1}{3} \left(\log 3\right)^2\).

**Concluding remarks:** We have given a host of interesting sums here. The reader may try to compute the sums:

\[
\sum_{n=1}^{\infty} \frac{H_n}{n^2 2^{2n}} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{H_n}{(2n + 1)^3 2^{2n}}
\]

which we couldn’t.

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