The monodromy representation of Lauricella's hypergeometric function $F_C$

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Abstract

We study the monodromy representation of the system $E_C$ of differential equations annihilating Lauricella's hypergeometric function $F_C$ of $m$ variables. Our representation space is the twisted homology group associated with an integral representation of $F_C$. We find generators of the fundamental group of the complement of the singular locus of $E_C$, and give some relations for these generators. We express the circuit transformations along these generators, by using the intersection forms defined on the twisted homology group and its dual.

1. Introduction

Lauricella's hypergeometric series $F_C$ of $m$ variables $x_1, \ldots, x_m$ with complex parameters $a, b, c_1, \ldots, c_m$ is defined by

$$F_C(a, b, c; x) = \sum_{n_1, \ldots, n_m=0}^{\infty} \frac{(a, n_1 + \cdots + n_m)(b, n_1 + \cdots + n_m)}{(c_1, n_1) \cdots (c_m, n_m)n_1! \cdots n_m!} x_1^{n_1} \cdots x_m^{n_m},$$

where $x = (x_1, \ldots, x_m)$, $c = (c_1, \ldots, c_m)$, $c_1, \ldots, c_m \notin \{0, -1, -2, \ldots\}$, and $(c_1, n_1) = \Gamma(c_1 + n_1)/\Gamma(c_1)$. This series converges in the domain

$$D_C := \left\{ (x_1, \ldots, x_m) \in \mathbb{C}^m \left| \sum_{k=1}^{m} \sqrt{|x_k|} < 1 \right. \right\},$$

and admits the integral representation (2.3) of Euler type. The system $E_C(a, b, c)$ of differential equations annihilating $F_C(a, b, c; x)$ is a holonomic system of rank $2^m$ with the singular locus $S$ given in (2.1). There is a fundamental system of solutions to $E_C(a, b, c)$ in a simply connected domain in $D_C - S$, which is given in terms of Lauricella’s hypergeometric series $F_C$ with different parameters; see (2.2) for their expressions.

In the case of $m = 2$, the series $F_C(a, b, c; x)$ and the system $E_C(a, b, c)$ are called Appell’s hypergeometric series $F_4(a, b, c; x)$ and system $E_4(a, b, c)$ of differential equations. The monodromy representation of $E_4(a, b, c)$ has been studied from several different points of view; see [T80], [Kan81], [HU08], and [GM1x]. On the other hand, there were few results of the monodromy representation for general $m$. In [B1x], Beukers studies the monodromy representation of $A$-hypergeometric system and gives representation matrices for many kinds of hypergeometric systems as examples of his main theorem. However, it seems that his method is not applicable for Lauricella’s $F_C$.

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In this paper, we study the monodromy representation of $E_C(a,b,c)$ for general $m$, by using twisted homology groups associated with the integral representation (2.3) of $F_C(a,b,c;x)$ and the intersection form defined on the twisted homology groups. Our consideration is based on the method for Appell's $E_4(a,b,c)$ in [GMIx].

Let $X$ be the complement of the singular locus $S$. The fundamental group of $X$ is generated by $m+1$ loops $\rho_0, \rho_1, \ldots, \rho_m$ which satisfy

$$\rho_i \rho_j = \rho_j \rho_i \quad (1 \leq i, j \leq m), \quad (\rho_0 \rho_k)^2 = (\rho_k \rho_0)^2 \quad (1 \leq k \leq m).$$

Here, $\rho_k$ ($1 \leq k \leq m$) turns the divisor $(x_k = 0)$, and $\rho_0$ turns the divisor

$$\prod_{\varepsilon_1, \ldots, \varepsilon_m = \pm 1} \left( 1 + \sum_k \varepsilon_k \sqrt{\epsilon_k} \right) = 0$$

around the point $(\frac{1}{m^2}, \ldots, \frac{1}{m^2})$. In the appendix, we show this claim by applying the Zariski theorem of Lefschetz type. Note that for $m = 2$, an explicit expression of the fundamental group of $X$ is given in [Kan81].

We thus investigate the circuit transformations $\mathcal{M}_i$ along $\rho_i$, for $0 \leq i \leq m$. We use the $2^m$ twisted cycles $\{\Delta_I\}_{I \subset \{1, \ldots, m\}}$ constructed in [G13], which represent elements in the $m$-th twisted homology group and correspond to the solutions (2.2) to $E_C(a,b,c)$. We obtain the representation matrix of $\mathcal{M}_k$ ($1 \leq k \leq m$) with respect to the basis $\{\Delta_I\}_I$ easily. The eigenvalues of $\mathcal{M}_k$ are $\exp(-2\pi \sqrt{-1}c_k)$ and 1. Both eigenspaces are $2^{m-1}$-dimensional, and spanned by half subsets of $\{\Delta_I\}_I$. On the other hand, it is difficult to represent $\mathcal{M}_0$ directly with respect to the basis $\{\Delta_I\}_I$. Thus we study the structure of the eigenspaces of $\mathcal{M}_0$. We find out that it is quite simple; our main theorem (Theorem 4.6) is stated as follows. The eigenvalues of $\mathcal{M}_0$ are $(-1)^{m-1}\exp(2\pi \sqrt{-1}(c_1 + \cdots + c_m - a - b))$ and 1. The eigenspace $W_0$ of eigenvalue $(-1)^{m-1}\exp(2\pi \sqrt{-1}(c_1 + \cdots + c_m - a - b))$ is one-dimensional, and spanned by the twisted cycle $D_{1\cdots m}$ defined by some bounded chamber. Further, the eigenspace $W_1$ of eigenvalue 1 is characterized as the orthogonal complement of $W_0 = CD_{1\cdots m}$ with respect to the intersection form.

As a corollary, we express the linear map $\mathcal{M}_i$ by using the intersection form. Our expressions are independent of the choice of a basis of the twisted homology group. To represent $\mathcal{M}_i$ by a matrix with respect to a given basis, it is sufficient to evaluate some intersection numbers. In particular, the images of any twisted cycles by $\mathcal{M}_0$ are determined only by the intersection number with the eigenvector $D_{1\cdots m}$; see Corollary 4.7. In Section 5, we give the simple representation matrix of $\mathcal{M}_i$ with respect to a suitable basis, and write down the examples for $m = 2$ and $m = 3$.

The irreducibility condition of the system $E_C(a,b,c)$ is known to be

$$a - \sum_{i \in I} c_i, \quad b - \sum_{i \in \bar{I}} c_i \notin \mathbb{Z}$$

for any subset $I$ of $\{1, \ldots, m\}$, as is in [HT1x]. Throughout this paper, we assume that the parameters $a$, $b$, and $c = (c_1, \ldots, c_m)$ are generic, which means that we add other conditions to the irreducibility condition; for details, refer to Remark 6.7.
2. Differential equations and integral representations

In this section, we collect some facts about Lauricella’s $F_C$ and the system $E_C$ of differential equations annihilating it.

**Notation 2.1.**

(i) Throughout this paper, the letter $k$ always stands for an index running from 1 to $m$. If no confusion is possible, $\sum_{k=1}^{m}$ and $\prod_{k=1}^{m}$ are often simply denoted by $\sum$ (or $\sum_{k}$) and $\prod$ (or $\prod_{k}$), respectively. For example, under this convention $F_{C}(a, b, c; x)$ is expressed as

$$F_{C}(a, b, c; x) = \sum_{n_{1}, \ldots, n_{m}=0}^{\infty} \frac{(a, \sum n_{k})(b, \sum n_{k})}{\prod(c_{k}, n_{k}) \cdot \prod n_{k}!} \prod n_{k}. $$

(ii) For a subset $I$ of $\{1, \ldots, m\}$, we denote the cardinality of $I$ by $|I|$.

Let $\partial_{k}$ ($1 \leq k \leq m$) be the partial differential operator with respect to $x_{k}$. We set $\theta_{k} := x_{k} \partial_{k}$, $\theta := \sum_{k} \theta_{k}$. Lauricella’s $F_{C}(a, b, c; x)$ satisfies differential equations

$$[\theta_{k}(\theta_{k} + c_{k} - 1) - x_{k}(\theta + a)(\theta + b)] f(x) = 0, \quad 1 \leq k \leq m. $$

The system generated by them is called Lauricella’s hypergeometric system $E_{C}(a, b, c)$ of differential equations.

**Fact 2.2** [HTT1], [L893]. The system $E_{C}(a, b, c)$ is a holonomic system of rank $2^{m}$ with the singular locus

$$S := \left(\prod_{k} x_{k} \cdot R(x) = 0\right) \subset \mathbb{C}^{m}, \quad R(x_{1}, \ldots, x_{m}) := \prod_{\varepsilon_{1}, \ldots, \varepsilon_{m}=\pm 1} \left(1 + \sum_{k} \varepsilon_{k} \sqrt{x_{k}}\right). \quad (2.1) $$

If $c_{1}, \ldots, c_{m} \notin \mathbb{Z}$, then the vector space of solutions to $E_{C}(a, b, c)$ in a simply connected domain in $D_{C} - S$ is spanned by the following $2^{m}$ elements:

$$f_{I} := \prod_{i \in I} x_{i}^{1-c_{i}} \cdot F_{C}\left(a + |I| - \sum_{i \in I} c_{i}, b + |I| - \sum_{i \in I} c_{i}, c'; x\right), \quad (2.2) $$

where $I$ is a subset of $\{1, \ldots, m\}$, and the row vector $c' = (c'_{1}, \ldots, c'_{m})$ of $\mathbb{C}^{m}$ is defined by

$$c'_{k} = \begin{cases} 2 - c_{k} & (k \in I), \\ c_{k} & (k \notin I). \end{cases} $$

Note that the solution (2.2) for $I = \emptyset$ is $f(= f_{\emptyset}) = F_{C}(a, b, c; x)$, and $R(x)$ is an irreducible polynomial of degree $2^{m-1}$ in $x_{1}, \ldots, x_{m}$.

**Fact 2.3** Integral representation of Euler type, Example 3.1 in [AK11]. For sufficiently small positive real numbers $x_{1}, \ldots, x_{m}$, if $c_{1}, \ldots, c_{m}, a - \sum c_{k} \notin \mathbb{Z}$, then $F_{C}(a, b, c; x)$ admits the following integral representation:

$$F_{C}(a, b, c; x) = \frac{\Gamma(1 - a)}{\prod \Gamma(1 - c_{k}) \cdot \Gamma(\sum c_{k} - a - m - 1)} \cdot \int_{\Delta} \prod t_{k}^{-c_{k}} \cdot (1 - \sum t_{k})^{\sum c_{k} - a - m} \cdot \left(1 - \sum \frac{x_{k}}{t_{k}}\right)^{-b} \cdot dt_{1} \wedge \cdots \wedge dt_{m}, \quad (2.3) $$

where $\Delta$ is the twisted cycle made by an $m$-simplex, in Sections 3.2 and 3.3 of [AK11].
3. Twisted homology groups and local systems

For twisted homology groups and the intersection form between twisted homology groups, refer to [AK11], [Y97], or Section 3 of [G13].

Put $X := \mathbb{C}^m - S$ and

$$v(t) := 1 - \sum_k t_k, \quad w(t, x) := \prod_k t_k \cdot \left(1 - \sum_k \frac{x_k}{t_k}\right),$$

$$\mathcal{X} := \left\{(t, x) \in \mathbb{C}^m \times X \mid \prod_k t_k \cdot v(t) \cdot w(t, x) \neq 0\right\}.$$

There is a natural projection

$$pr : \mathcal{X} \to X; \quad (t, x) \mapsto x,$$

and we define $T_x := pr^{-1}(x)$ for any $x \in X$. We regard $T_x$ as an open submanifold of $\mathbb{C}^m$ by the coordinates $t = (t_1, \ldots, t_m)$. We consider the twisted homology groups on $T_x$ with respect to the multi-valued function

$$u_x(t) := \prod t_k^{1-c_k+b} \cdot v(t) \sum c_k - a - m + 1 w(t, x)^{-b}$$

$$= \prod t_k^{1-c_k} \cdot \left(1 - \sum t_k\right)^{\sum c_k - a - m + 1} \cdot \left(1 - \sum \frac{x_k}{t_k}\right)^{-b}$$

(the second equality holds under the coordination of branches). We denote the $k$-th twisted homology group by $H_k(T_x, u_x)$, and locally finite one by $H^lf_k(T_x, u_x)$.

**Facts 3.1 [AK11], [G13].**

(i) $H_k(T_x, u_x) = 0$, $H^lf_k(T_x, u_x) = 0$, for $k \neq m$.

(ii) $\dim H_m(T_x, u_x) = 2^m$.

(iii) The natural map $H_m(T_x, u_x) \to H^lf_m(T_x, u_x)$ is isomorphic (the inverse map is called the regularization).

Hereafter, we identify $H^lf_k(T_x, u_x)$ with $H_k(T_x, u_x)$, and call an $m$-dimensional twisted cycle by a twisted cycle simply. Note that the intersection form $I_h$ is defined between $H_m(T_x, u_x)$ and $H_m(T_x, u_x^{-1})$.

For $x, x' \in X$ and a path $\tau$ in $X$ from $x$ to $x'$, there is the canonical isomorphism

$$\tau_\tau : H_m(T_x, u_x) \to H_m(T_x', u_{x'}).$$

Hence the family

$$\mathcal{H} := \bigcup_{x \in X} H_m(T_x, u_x)$$

forms a local system on $X$.

Let $\delta$ be a twisted cycle in $T_x$ for a fixed $x$. If $x'$ is a sufficiently close point to $x$, there is a unique twisted cycle $\delta'$ such that $\int_{\delta} u_{x'}\varphi$ is obtained by the analytic continuation of $\int_{\delta} u_x\varphi$, where

$$\varphi := \frac{dt_1 \wedge \cdots \wedge dt_m}{\prod t_k \cdot (1 - \sum t_k)}.$$

Thus we regard the integration $\int_{\delta} u_x\varphi$ as a holomorphic function in $x$. Fact 2.3 means that the integral $\int_{\Delta} u_x\varphi$ represents $F_C(a, b, c; x)$ modulo Gamma factors. Let $Sol$ be the sheaf on $X$ whose
sections are holomorphic solutions to $E_C(a, b, c)$. The stalk $Sol_x$ at $x \in X$ is the space of local holomorphic solutions near $x$.

**Fact 3.2 [G13]**. For any $x \in X$,

$$\Phi_x : H_m(T_x, u_x) \to Sol_x; \, \delta \mapsto \int_\delta u_x \varphi$$

is isomorphic.

Fact 2.2 implies that $Sol_x$ is a $\mathbb{C}$-vector space of dimension $2^m$ and spanned by $f_j$'s, for $x \in D_C - S$. In [G13], we construct twisted cycles $\Delta_I$ for all subsets $I$ of $\{1, \ldots, m\}$. We regard $\{\Delta_I\}_I$ as the $2^m$ twisted cycles $\Delta_I$'s arranged as $(\Delta, \Delta_1, \Delta_2, \ldots, \Delta_m, \Delta_{12}, \Delta_{13}, \ldots, \Delta_{1\ldots m})$. For a twisted cycle $\delta$ with respect to $u_x$, we denote by $\delta^\vee$ the twisted cycle with respect to $u_x^{-1}$, which is defined by the same construction as $\delta$.

**Fact 3.3 [G13]**. We have

$$\Phi_x(\Delta_I) = \prod_{i \in I} \Gamma(c_i - 1) \cdot \prod_{j \not\in I} \Gamma(1 - c_j) \cdot \Gamma(\sum_k c_k - a - m + 1) \Gamma(1 - b) \cdot \frac{\prod_i c_i - a - |I| + 1}{\prod_i c_i - b - |I| + 1} \cdot f_I,$$

The intersection matrix $H := (I_h(\Delta_I, \Delta_j^\vee))_{I, J}$ is diagonal. Further, the $(I, I)$-entry $H_{I, I}$ of $H$ is

$$H_{I, I} = (-1)^{|I|} \cdot \frac{\prod_{j \not\in I} \gamma_j \cdot (\alpha - \prod_{i \in I} \gamma_i)(\beta - \prod_{i \in I} \gamma_i)}{\prod_i \gamma_k - 1 \cdot (\alpha - \prod_k \gamma_k)(\beta - 1)},$$

where $\alpha := e^{2\pi \sqrt{-1} a}$, $\beta := e^{2\pi \sqrt{-1} b}$, $\gamma_k := e^{2\pi \sqrt{-1} c_k}$. Therefore, $\Delta_I$'s form a basis of $H_m(T_x, u_x)$.

### 4. Monodromy representation

Put $\dot{x} := (\frac{1}{2m^2}, \ldots, \frac{1}{2m^2}) \in X$. For $\rho \in \pi_1(X, \dot{x})$ and $g \in Sol_{\dot{x}}$, let $\rho_* g$ be the analytic continuation of $g$ along $\rho$. Since $\rho_* g$ is also a solution to $E_C(a, b, c)$, the map $\rho_* : Sol_{\dot{x}} \to Sol_{\dot{x}}; \, g \mapsto \rho_* g$ is a $\mathbb{C}$-linear automorphism which satisfies $(\rho \cdot \rho')_* = \rho'_* \circ \rho_*$ for $\rho, \rho' \in \pi_1(X, \dot{x})$. Here, the composition $\rho \cdot \rho'$ of loops $\rho$ and $\rho'$ is defined as the loop going first along $\rho$, and then along $\rho'$. We thus obtain a representation

$$\mathcal{M}' : \pi_1(X, \dot{x}) \to GL(Sol_{\dot{x}})$$

of $\pi_1(X, \dot{x})$, where $GL(V)$ is the general linear group on a $\mathbb{C}$-vector space $V$. Since we can identify $Sol_{\dot{x}}$ with $H_m(T_{\dot{x}}, u_{\dot{x}})$ by Fact 3.2, the representation $\mathcal{M}'$ is equivalent to

$$\mathcal{M} : \pi_1(X, \dot{x}) \to GL(H_m(T_{\dot{x}}, u_{\dot{x}})).$$

Note that for $\rho \in \pi_1(X, \dot{x})$, the map $\mathcal{M}(\rho) : H_m(T_{\dot{x}}, u_{\dot{x}}) \to H_m(T_{\dot{x}}, u_{\dot{x}})$ coincides with the canonical isomorphism $\rho_* : H_m(T_{\dot{x}}, u_{\dot{x}}) \to H_m(T_{\dot{x}}, u_{\dot{x}})$ in the local system $\mathcal{H}$. The representation $\mathcal{M}$ (and $\mathcal{M}'$) is called the monodromy representation, which is the main object in this paper.

For $1 \leq k \leq m$, let $\rho_k$ be the loop in $X$ defined by

$$\rho_k : [0, 1] \ni \theta \mapsto \left(\frac{1}{2m^2}, \ldots, \frac{e^{2\pi \sqrt{-1} \theta}}{2m^2}, \ldots, \frac{1}{2m^2}\right) \in X,$$

where $\frac{e^{2\pi \sqrt{-1} \theta}}{2m^2}$ is the $k$-th entry of $\rho_k(\theta)$. We take a positive real number $\varepsilon_0$ so that $\varepsilon_0 < \text{...}^\text{...}$. 

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**5. Monodromy of $F_C$**

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**6. Monodromy of $F_C$**

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**7. Monodromy of $F_C$**

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**8. Monodromy of $F_C$**

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\[
\min \left\{ \frac{1}{2m^2}, \frac{1}{(m-2)^2} - \frac{1}{m^2} \right\},
\]
and we define the loop \( \rho_0 \) in \( X \) as \( \rho_0 := \tau_0 \rho'_0 \tau_0^{-1} \), where

\[
\tau_0 : [0, 1] \ni \theta \mapsto \left( (1 - \theta) \cdot \frac{1}{2m^2} + \theta \cdot \left( \frac{1}{m^2} - \varepsilon_0 \right) \right) (1, \ldots, 1) \in X,
\]

\[
\rho'_0 : [0, 1] \ni \theta \mapsto \left( \frac{1}{m^2} - \varepsilon_0 e^{2\pi \sqrt{-1} \theta} \right) (1, \ldots, 1) \in X,
\]

and \( \tau_0^{-1} \) is the reverse path of \( \tau_0 \).

Remark 4.1. The loop \( \rho_k (1 \leq k \leq m) \) turns the hyperplane \( (x_k = 0) \), and \( \rho_0 \) turns the hypersurface \( (R(x) = 0) \) around the point \( (\frac{1}{m^2}, \ldots, \frac{1}{m^2}) \), positively. Note that \( (\frac{1}{m^2}, \ldots, \frac{1}{m^2}) \) is the nearest to the origin in \( (R(x) = 0) \cap (x_1 = x_2 = \cdots = x_m) = \{ \frac{1}{m^2}(1, \ldots, 1), \frac{1}{(m-2)^2}(1, \ldots, 1), \ldots \} \).

**Theorem 4.2.** The loops \( \rho_0, \rho_1, \ldots, \rho_m \) generate the fundamental group \( \pi_1(X, \hat{x}) \). Moreover, they satisfy the following relations:

\[
\rho_i \rho_j = \rho_j \rho_i \quad (1 \leq i, j \leq m), \quad (\rho_0 \rho_k)^2 = (\rho_k \rho_0)^2 \quad (1 \leq k \leq m).
\]

**Remark 4.3.** We conjecture that \( \pi_1(X, \hat{x}) \) is the group generated by \( \rho_0, \rho_1, \ldots, \rho_m \) with the relations in Theorem 4.2. To prove this conjecture, we need to show that there are no extra relations among \( \rho_0, \rho_1, \ldots, \rho_m \). It is shown in [Kan81] that this conjecture is true for \( m = 2 \).

We show this theorem in Appendix A. By this theorem, for the study of the monodromy representation \( \mathcal{M} \), it is sufficient to investigate \( m + 1 \) linear maps

\[
\mathcal{M}_i := \mathcal{M}(\rho_i) \quad (0 \leq i \leq m).
\]

**Proposition 4.4.** For \( 1 \leq k \leq m \), the eigenvalues of \( \mathcal{M}_k \) are \( \gamma_k^{-1} \) and 1. The eigenspace of \( \mathcal{M}_k \) of eigenvalue \( \gamma_k^{-1} \) is spanned by the twisted cycles

\[
\Delta_I, \quad k \in I \subset \{1, \ldots, m\}.
\]

That of eigenvalue 1 is spanned by

\[
\Delta_I, \quad k \notin I \subset \{1, \ldots, m\}.
\]

In particular, both eigenspaces are of dimension \( 2^{m-1} \).

**Proof.** By Fact 3.3, the twisted cycle \( \Delta_I \) corresponds to the solution

\[
f_I = \prod_{i \in I} x_i^{1-c_i} \cdot F_C \left( a + |I| - \sum_{i \in I} c_i, b + |I| - \sum_{i \in I} c_i, c^I \cdot x \right)
\]
to \( E_C(a, b, c) \). Since the series \( F_C \) defines a single-valued function around the origin, we have

\[
\mathcal{M}'(\rho_k)(f_I) = \begin{cases} \gamma_k^{-1} f_I & (k \in I), \\ f_I & (k \notin I). \end{cases}
\]

Therefore, we obtain this proposition.

**Corollary 4.5.** For \( 1 \leq k \leq m \), the linear map \( \mathcal{M}_k : H_m(T_{\hat{x}}, u_{\hat{x}}) \to H_m(T_{\hat{x}}, u_{\hat{x}}) \) is expressed as

\[
\mathcal{M}_k : \delta \mapsto \delta - (1 - \gamma_k^{-1}) \sum_{I \ni k} \frac{f_k(\delta, \Delta_I^v)}{h(\Delta_I, \Delta_I^v)} \Delta_I.
\]
Further, the representation matrix $M_k$ of $M_k$ with respect to the basis $\{\Delta_I\}_I$ is the diagonal matrix whose $(I, I)$-entry is 
\[
\begin{cases} 
\gamma_k^{-1} & (I \ni k), \\
1 & (I \notni k).
\end{cases}
\]

Proof. We prove the first claim. By Proposition 4.4, $H_m(T_{\hat{x}}, u_{\hat{x}})$ is decomposed into the direct sum of the eigenspaces: $H_m(T_{\hat{x}}, u_{\hat{x}}) = (\bigoplus_{I \ni k} \mathbb{C}\Delta_I) \oplus (\bigoplus_{I \notni k} \mathbb{C}\Delta_I)$. Then it is sufficient to show that the claim holds for $\delta = \Delta_I$. This is clear by Fact 3.3 and Proposition 4.4. The second claim is obvious.

For each subset $I \subset \{1, \ldots, m\}$, we define a chamber $D_I$ which gives an element in $H_m(T_{\hat{x}}, u_{\hat{x}})$.

For $I = \emptyset$, we put 
\[D_\emptyset = D := \{(t_1, \ldots, t_m) \in \mathbb{R}^m | t_k < 0 (1 \leq k \leq m)\}.\]

For $I \notni \emptyset, \{1, \ldots, m\}$, we put 
\[D_I := \left\{(t_1, \ldots, t_m) \in \mathbb{R}^m | t_i > 0 (i \in I), t_j < 0 (j \notni I), v(t) > 0, (-1)^{m-|I|+1} w(t, \hat{x}) > 0 \right\}.
\]

The arguments of the factors of $u_{\hat{x}}(t)$ are defined as follows.

|                | $t_i(i \in I)$ | $t_j(j \notni I)$ | $v(t)$ | $w(t, \hat{x})$ |
|----------------|----------------|--------------------|--------|-----------------|
| $D_{1 \ldots m}$ | 0              | $-\pi$             | 0      | 0               |
| otherwise      | $-\pi$         | 0                  | $-(m-|I|+1)\pi$|

Note that if $m = 2$, then $D$, $D_1$, $D_2$, and $D_{12}$ are equal to $\Delta_6$, $\Delta_7$, $\Delta_8$, and $\Delta_5$ in [GM1x], respectively. We state our main theorem.

**Theorem 4.6.** The eigenvalues of $M_0$ are $(-1)^{m-1} \prod_k \gamma_k \cdot \alpha^{-1} \beta^{-1}$ and 1. The eigenspace $W_0$ of $M_0$ of eigenvalue $(-1)^{m-1} \prod_k \gamma_k \cdot \alpha^{-1} \beta^{-1}$ is spanned by $D_{1 \ldots m}$, and hence is one dimensional. The eigenspace $W_1$ of $M_0$ of eigenvalue 1 is spanned by $D_I, \ I \notni \{1, \ldots, m\}$,

and expressed as

\[W_1 = \{\delta \in H_m(T_{\hat{x}}, u_{\hat{x}}) | I_h(\delta, D_{1 \ldots m}^\vee) = 0\}.
\]

In particular, this space is $(2^m - 1)$-dimensional.

The proof of this theorem is given in Section 6.

**Corollary 4.7.** The linear map $M_0 : H_m(T_{\hat{x}}, u_{\hat{x}}) \rightarrow H_m(T_{\hat{x}}, u_{\hat{x}})$ is expressed as

\[M_0 : \delta \mapsto \delta - \left(1 + (-1)^m \prod_k \gamma_k \cdot \alpha^{-1} \beta^{-1}\right) \frac{I_h(\delta, D_{1 \ldots m}^\vee)}{I_h(D_{1 \ldots m}, D_{1 \ldots m})} D_{1 \ldots m}.
\]

Proof. By Theorem 4.6, we have $H_m(T_{\hat{x}}, u_{\hat{x}}) = W_0 \oplus W_1 = \mathbb{C}D_{1 \ldots m} \oplus W_1$. Then it is sufficient to show that the claim holds for $\delta = D_{1 \ldots m}$ and $\delta \in W_1$. This is clear by Theorem 4.6.
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Proposition 4.8. We have
\[ I_h(D_{1\ldots m}, \Delta_I^\sim) = I_h(\Delta_I, \Delta_I^\sim) = I_h(\Delta_I, D_{1\ldots m}^\sim). \]  
(4.1)

Thus we obtain
\[ D_{1\ldots m} = \sum_{I \subset \{1, \ldots, m\}} \Delta_I, \]  
(4.2)
\[ I_h(D_{1\ldots m}, D_{1\ldots m}^\sim) = \frac{\alpha \beta + (-1)^m \prod k \gamma_k}{(\beta - 1)(\alpha - \prod k \gamma_k)}. \]  
(4.3)

This proposition is also proved in Section 6. By this proposition, we obtain the following corollary.

Corollary 4.9. The linear map \( M_0 \) is expressed as
\[ M_0 : \delta \mapsto \delta - \frac{(-1)^m \prod k \gamma_k}{\alpha \beta} I_h(\delta, D_{1\ldots m}^\sim) D_{1\ldots m}. \]

Let \( M_0 \) be the representation matrix of \( M_0 \) with respect to the basis \( \{\Delta_I\} \). Then we have
\[ M_0 = E_{2^m} - \frac{(-1)^m \prod k \gamma_k}{\alpha \beta} N H, \]
where \( E_{2^m} \) is the unit matrix of size \( 2^m \), \( N \) is the \( 2^m \times 2^m \) matrix with all entries 1, and \( H = (I_h(\Delta_I, \Delta_I^\sim))_{I, I'} \) is the intersection matrix given in Fact 3.3.

Proof. The expression of \( M_0 \) follows from Corollaries 4.5 and 4.9 immediately. To obtain the representation matrix, we have to show that the representation matrix of the linear map \( \delta \mapsto I_h(\delta, D_{1\ldots m}^\sim) D_{1\ldots m} \) is given by \( N H \). By Proposition 4.8, we have
\[ I_h(\Delta_I, D_{1\ldots m}^\sim) D_{1\ldots m} = I_h(\Delta_I, \Delta_I^\sim) \Delta_I \]
\[ = (\Delta, \Delta_1, \Delta_2, \ldots, \Delta_m, \Delta_{12}, \Delta_{13}, \ldots, \Delta_{1m}) \left( \begin{array}{c} I_h(\Delta_I, \Delta_I^\sim) \\ I_h(\Delta_I, \Delta_I^\sim) \\ \vdots \\ I_h(\Delta_I, \Delta_I^\sim) \end{array} \right), \]
and hence the claim is proved. \( \square \)

Remark 4.10. Let \( \rho_\infty \) be a loop in \( X \) turning the divisor \( L_\infty \subset \mathbb{P}^m \). Because of
\[ \rho_\infty = \eta(\ell_1 \cdots \ell_m \ell_1 \cdots \ell_{\ell - 1} \cdots \ell_0)^{-1}, \]
we can express \( M(\rho_\infty) \) by Corollaries 4.5 and 4.9 equalities (A.1) and (A.2); see Appendix A for the notations \( \eta \) and \( \ell \). However, it is too complicated to write down. Here, we give the eigenvalues of \( M(\rho_\infty) \). Similarly to Section 2.3 of [Kat93], it turns out that \( x_m a f \left( \frac{x_m}{x_m}, \ldots, \frac{x_{m-1}}{x_m}, \frac{1}{x_m} \right) \) is a solution to \( E_C(a, b, c) \) if and only if \( f(\xi_1, \ldots, \xi_m) \) is a solution to \( E_C(a, a - c_m + 1, (c_1, \ldots, c_{m-1}, a - b + 1)) \) with variables \( \xi_1, \ldots, \xi_m \). Then a similar argument to Proposition 4.4 shows that the eigenvalues of \( M(\rho_\infty) \) are \( a \) and \( b \). Moreover, both eigenspaces are of dimension \( 2^{m-1} \).

5. Representation matrices

For \( 0 \leq i \leq m \), the matrix representation of \( M_i \) with respect to the basis \( \{\Delta_I\} \) is given by \( M_i \) in Corollaries 4.5 and 4.9. However, \( M_0 \) is too complicated to write down. In this section, we
give another basis \( \{\Delta_i\}_I \) of \( H_m(T_x, u_x) \), and write down the representation matrix of \( M_i \) with respect to this basis.

In this and the next sections, we use the following formulas.

**Lemma 5.1.** For a positive integer \( n \) and complex numbers \( \lambda_1, \ldots, \lambda_n \), we have

\[
\sum_{N \subset \{1, \ldots, n\}} \prod_{l \in N} \frac{\lambda_l}{1 - \lambda_l} = \prod_{l=1}^{n} \frac{1}{1 - \lambda_l}, \quad \sum_{N \subset \{1, \ldots, n\}} \prod_{l \in N} \frac{1}{\lambda_l - 1} = \prod_{l=1}^{n} \frac{\lambda_l}{\lambda_l - 1}. \tag{5.1}
\]

\[
\sum_{N \subset \{1, \ldots, n\}} \prod_{l \in N} (1 - \lambda_l) \prod_{l \notin N} \lambda_l = \sum_{N \subset \{1, \ldots, n\}} (-1)^{|N|} \prod_{l \in N} (\lambda_l - 1) \prod_{l \notin N} \lambda_l = 1, \tag{5.2}
\]

\[
\sum_{N \subset \{1, \ldots, n\}} \prod_{l \in N} (\lambda_l - 1) = \prod_{l=1}^{n} \lambda_l, \tag{5.3}
\]

**Proof.** Because of

\[
1 + \frac{\lambda_l}{1 - \lambda_l} = \frac{1}{1 - \lambda_l}, \quad 1 + \frac{1}{\lambda_l - 1} = \frac{\lambda_l}{\lambda_l - 1},
\]

we obtain (5.1) by induction on \( n \). The equalities (5.2) and (5.3) follow from the first and the second ones of (5.1), respectively.

Let \( P \) be the \( 2^m \times 2^m \) matrix whose \((N, I)\)-entry is

\[
\begin{cases}
\alpha \beta \prod_{j \notin I} \frac{\gamma_j - 1}{\gamma_j} \cdot \frac{\prod_{n \in N} \gamma_n}{(\alpha - \prod_{n \in N} \gamma_n)(\beta - \prod_{n \in N} \gamma_n)} & (N \subset I), \\
0 & (N \notin I),
\end{cases}
\]

and \( \{\Delta_i\}_I \) be the basis of \( H_m(T_x, u_x) \) defined as

\[
(\Delta', \Delta'_1, \Delta'_2, \ldots, \Delta'_m, \Delta_{12}', \Delta_{13}', \ldots, \Delta_{1 \cdot m}') = (\Delta, \Delta_1, \Delta_2, \ldots, \Delta_m, \Delta_{12}, \Delta_{13}, \ldots, \Delta_{1 \cdot m}) P.
\]

Namely, \( \Delta'_i \) is defined by

\[
\Delta'_i = \alpha \beta \prod_{j \notin I} \frac{\gamma_j - 1}{\gamma_j} \cdot \sum_{N \subset I} \frac{\prod_{n \in N} \gamma_n}{(\alpha - \prod_{n \in N} \gamma_n)(\beta - \prod_{n \in N} \gamma_n)} \Delta_N.
\]

Note that \( P \) is an upper triangular matrix.

**Lemma 5.2.** We have

\[
\frac{(\alpha - \prod_k \gamma_k)(\beta - \prod_k \gamma_k)}{\alpha \beta \prod_k \gamma_k} \Delta_{1 \cdot m}' + \sum_{I \subseteq \{1, \ldots, m\}} \left( \frac{1}{\prod_{i \in I} \gamma_i} + (-1)^{m-|I|} \frac{\prod_k \gamma_k}{\alpha \beta} \right) \Delta'_I = D_{1 \cdot m}.
\]

**Proof.** By the definition, the left-hand side is equal to

\[
\frac{(\alpha - \prod_k \gamma_k)(\beta - \prod_k \gamma_k)}{\alpha \beta \prod_k \gamma_k} \cdot \alpha \beta \sum_{N \subset \{1, \ldots, m\}} \frac{\prod_{n \in N} \gamma_n}{(\alpha - \prod_{n \in N} \gamma_n)(\beta - \prod_{n \in N} \gamma_n)} \Delta_N
\]

\[
\cdot \sum_{I \subseteq \{1, \ldots, m\}} \left[ \prod_{j \notin I} (\gamma_j - 1) \left( \frac{\alpha \beta}{\prod_k \gamma_k} + (-1)^{m-|I|} \prod_{i \in I} \gamma_i \right) \right]
\]

\[
\cdot \sum_{N \subset I} \frac{\prod_{n \in N} \gamma_n}{(\alpha - \prod_{n \in N} \gamma_n)(\beta - \prod_{n \in N} \gamma_n)} \Delta_N. \tag{5.4}
\]
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Clearly the coefficient of $\Delta_{1\ldots m}$ in (5.4) is 1. The coefficient of $\Delta_N$ ($N \neq \{1, \ldots, m\}$) is

$$\left(\frac{\alpha - \prod_{n \in N} n}{(\alpha - \prod_{n \in N} n)(\beta - \prod_{n \in N} n)}\right) \cdot \left(\frac{(\alpha - \prod_k k)(\beta - \prod_k k)}{\prod_k k} + \sum_{I \supset N \neq \emptyset, I \neq \emptyset, \gamma_i} \prod_j \gamma_j - 1 \left(\frac{\alpha \beta}{\prod \gamma_k} + (-1)^m \prod_{i \in I} \gamma_i\right)\right)$$

which equals to 1 by the equalities (5.2) and (5.3). Therefore, by using (4.2), we conclude that (5.4) is equal to

$$\sum_{I \subseteq \{1, \ldots, m\}} \Delta_I = D_{1\ldots m}.$$  

\[\square\]

**Corollary 5.3.** For $0 \leq i \leq m$, let $M'_i$ be the representation matrix of $M_i$ with respect to the basis $\{\Delta'_I\}_I$. Then we have

$$M'_0 = E_{2m} - N_0, \quad M'_k = M_k + N_k \quad (1 \leq k \leq m),$$

where $N_k$ is defined as follows. The $(I, I')$-entry of $N_0$ (resp. $N_k$) is zero, except in the case of $I' = \emptyset$ (resp. $k \in I'$ and $I = I' - \{k\}$). The $(I, \emptyset)$-entry of $N_0$ is

$$\left\{\begin{array}{ll}
\frac{(\alpha - \prod_k k)(\beta - \prod_k k)}{\alpha \beta \prod_k k} & (I = \{1, \ldots, m\}), \\
\frac{1}{\prod_{i \in I} \gamma_i} + (-1)^m \prod_{i \in I} \gamma_i & (\text{otherwise}).
\end{array}\right.$$ 

The $(I' - \{k\}, I')$-entry of $N'_k$ is 1.

In particular, $M'_k$ ($1 \leq k \leq m$) is upper triangular, $M'_0$ is lower triangular, and the $(\emptyset, \emptyset)$-entry of $M'_0$ is

$$1 - \left(1 + (-1)^m \frac{\prod \gamma_k}{\alpha \beta}\right) = (-1)^{m-1} \prod \gamma_k \cdot \alpha^{-1} \beta^{-1}.$$ 

**Proof.** We evaluate $M'_0$. By Corollary 4.9 it is sufficient to show that the matrix representation of the linear map

$$\delta \mapsto \frac{(\beta - 1)(\alpha - \prod_k k)}{\alpha \beta} I_h(\delta, D_{1\ldots m}) D_{1\ldots m}$$

is given by $N_0$. By Fact 3.3 and Proposition 4.8, we have

$$\frac{(\beta - 1)(\alpha - \prod_k k)}{\alpha \beta} I_h(\Delta'_{I'}, D_{I'\ldots m}) D_{1\ldots m} = \left(\sum_{N \subseteq p} (-1)^{|N|}\right) \prod_{i \in I'} \gamma_i - 1 \cdot D_{1\ldots m},$$

and hence we obtain

$$\frac{(\beta - 1)(\alpha - \prod_k k)}{\alpha \beta} I_h(\Delta'_{I'}, D_{I'\ldots m}) D_{1\ldots m} = \left\{\begin{array}{ll}
D_{1\ldots m} & (I' = \emptyset), \\
0 & (\text{otherwise}).
\end{array}\right.$$ 

Thus Lemma 5.2 shows the claim.

We evaluate $M'_k$ ($1 \leq k \leq m$). We have to show that

$$M_k(\Delta'_I) = \left\{\begin{array}{ll}
\Delta'_{I \setminus k} + \Delta'_{I - \{k\}} & (k \notin I), \\
\gamma_k \Delta'_{k = I} & (k \in I).
\end{array}\right.$$
If $k \not\in I$, then the subsets $N$ of $I$ also satisfy $k \not\in N$, and hence we have $\mathcal{M}_k(\Delta_N) = \Delta_N$ by Proposition 4.4. This implies that $\mathcal{M}_k(\Delta'_I) = \Delta'_I$, for $k \not\in I$. We assume $k \in I$. For a subset $N$ of $I - \{k\}$, we have

$$
\mathcal{M}_k(\Delta_N) = \Delta_N = \left(\gamma_k^{-1} + \gamma_k^{-1} \right) \Delta_N, \quad \mathcal{M}_k(\Delta_{N \cup \{k\}}) = \gamma_k^{-1} \Delta_{N \cup \{k\}}.
$$

Then we obtain

$$
\mathcal{M}_k(\Delta'_I) = \gamma_k^{-1} \Delta'_I + \frac{\gamma_k - 1}{\gamma_k} \cdot \alpha \beta \prod_{j \not\in I - \{k\}} \gamma_j^{-1} \cdot \sum_{N \subseteq I - \{k\}} \frac{\prod_{n \in N} \gamma_n}{(\alpha - \prod_{n \in N} \gamma_n)(\beta - \prod_{n \in N} \gamma_n)} \Delta_N
$$

$$
= \gamma_k^{-1} \Delta'_I + \alpha \beta \prod_{j \not\in I - \{k\}} \gamma_j^{-1} \cdot \sum_{N \subseteq I - \{k\}} \frac{\prod_{n \in N} \gamma_n}{(\alpha - \prod_{n \in N} \gamma_n)(\beta - \prod_{n \in N} \gamma_n)} \Delta_N
$$

$$
= \gamma_k^{-1} \Delta'_I + \Delta'_{I - \{k\}}.
$$

**Example 5.4.** We write down $M'_i$ ($0 \leq i \leq m$) for $m = 2, 3$.

(i) In the case of $m = 2$, the representation matrices $M'_0, M'_1, M'_2$ are as follows:

$$
M'_0 = \begin{pmatrix}
\gamma_1 \gamma_2 \\
\gamma_2 \\
\gamma_1 \gamma_2
\end{pmatrix}, \quad
M'_1 = \begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad
M'_2 = \begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}.
$$

These are equal to the transpose matrices of those in Remark 4.4 of [GM1X].

(ii) In the case of $m = 3$, the representation matrices $M'_0, M'_1, M'_2, M'_3$ are as follows:

$$
M'_0 = \begin{pmatrix}
\gamma_1 \gamma_2 \gamma_3 \\
\gamma_2 \gamma_3 \\
\gamma_1 \gamma_2 \gamma_3 \\
\gamma_1 \gamma_3 \\
\gamma_1 \gamma_2 \gamma_3
\end{pmatrix}, \quad
M'_1 = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{pmatrix}, \quad
M'_2 = \begin{pmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
$$
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\[
M'_3 = \begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & \frac{1}{\gamma_3} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{\gamma_3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{\gamma_3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\gamma_3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\gamma_3}
\end{pmatrix}.
\]

6. Proof of the main theorem

In this section, we prove Theorem 4.6. Since \( \dim H_m(T_x, u_x) = 2^m \), it is sufficient to show that \( D_I \)'s are eigenvectors and linearly independent. First, we evaluate the intersection numbers \( I_h(\Delta_I, D_I') \). Second, we show the linear independence of \( \{D_I\}_I \) by evaluating the determinant of the matrix \( (I_h(\Delta_I, D_I'))_{1,I} \). Third, we prove the properties of the eigenspace of \( M_0 \) of eigenvalue \( 1 \). Finally, we show that \( D_{1...m} \) is an eigenvector of \( M_0 \) of eigenvalue \( (-1)^{m-1} \prod_k \gamma_k \cdot \alpha^{-1} \beta^{-1} \).

**Notation 6.1.** We use same notations as Section 4 in [G13]. For fixed sufficiently small positive real numbers \( x_1, \ldots, x_m \), we denote \( M = T_x \), where \( x = (x_1, \ldots, x_m) \).

We review the construction of \( \Delta_I \) in [G13] briefly. We set \( J := I^c = \{1, \ldots, m\} - I \). We consider

\[
M_I := \mathbb{C}^m - \left( \bigcup_k \{s_k = 0\} \cup \{v_I = 0\} \cup \{w_I = 0\} \right),
\]
where \( v_I \) and \( w_I \) are polynomials in \( s_1, \ldots, s_m \) defined by

\[
v_I := \prod_{i \in I} s_i \cdot \left( 1 - \sum_{i \in I} \frac{x_i}{s_i} - \sum_{j \in J} s_j \right), \quad w_I := \prod_{j \in J} s_j \cdot \left( 1 - \sum_{i \in I} s_i - \sum_{j \in J} \frac{x_j}{s_j} \right).
\]

Let \( u_I \) be a multi-valued function on \( M_I \) defined as

\[
u_I := \prod_k s_k^{C_k} \cdot v_I^A \cdot w_I^B,
\]
where

\[
A := \sum c_k - a - m + 1, \quad B := -b,
\]

\[
C_i := c_i - 1 - A \ (i \in I), \quad C_j := 1 - c_j - B \ (j \in J).
\]

We construct the twisted cycle \( \tilde{\Delta}_I \) in \( M_I \) with respect to \( u_I \) from the direct product \( \sigma_I \subset \mathbb{R}^m \) of an \( r \)-simplex and an \((m-r)\)-simplex. The orientation of \( \sigma_I \) is induced from the natural embedding \( \mathbb{R}^m \subset \mathbb{C}^m \). Note that when we eliminate the boundary of \( \sigma_I \), we regulate the difference of branches of \( u_I \) by a different way from the usual regularization. We consider the \( \varepsilon \)-neighborhood of the divisors \( (s_1 = 0), \ldots, (s_m = 0), \ (1 - \sum_{i \in I} s_i = 0), \ (1 - \sum_{j \in J} s_j = 0) \), where \( \varepsilon \) is a positive real number satisfying \( \varepsilon < \frac{1}{m+1} \) and \( x_k < \frac{\varepsilon^2}{m} \) (we use the assumption \( \varepsilon_1 = \cdots = \varepsilon_m = \varepsilon \) in Section 4
of [G13]). By using the bijection
\[ \iota_I : M_I \to M; \quad \iota_I(s_1, \ldots, s_m) := (t_1, \ldots, t_m), \]
\[ t_i = \frac{x_i}{s_i} \quad (i \in I), \quad t_j = s_j \quad (j \in J), \]
we define the twisted cycle \( \Delta_I \) in \( M = T_x \) as \( \Delta_I := (-1)^{|I|}(\iota_I)_*(\bar{\Delta}_I) \). Note that \( \sigma_I \) is contained in the bounded domain \( \iota_I^{-1}(D_{1\ldots m}) \).

6.1 An expression of \( D_{1\ldots m} \)

We prove Proposition 4.8 by using imaginary cycles and the construction of \( \Delta_I \)'s.

Fix any \( s_0 \in \sigma_I \), and set
\[ \sqrt{-1}R^m_I := \{ s_0 + \sqrt{-1}(\eta_1, \ldots, \eta_m) \mid (\eta_1, \ldots, \eta_m) \in \mathbb{R}^m \} \subset M_I, \]
which is called an imaginary cycle. By arguments similar to those in the proof of Proposition 4.3 and Theorem 4.4 in [G13], we can prove that the integration of \( u_\varphi \) on \( (\iota_I)_*(\sqrt{-1}R^m_I) \) also gives the solution \( f_I \) to \( E_C(a, b, c) \), under some conditions for the parameters \( a, b, c \). Therefore, \( (\iota_I)_*(\sqrt{-1}R^m_I)^\vee \) is orthogonal to the cycles \( \Delta_{I'} \) (\( I' \neq I \)) with respect to \( I_h \) (cf. Proof of Lemma 4.1 in [GM1x]), and hence \( (\iota_I)_*(\sqrt{-1}R^m_I)^\vee \) is a constant multiple of \( \Delta_I^\vee \). Note that both \( D_{1\ldots m} \) and the simplex \( \iota_I(\sigma_I) \) in \( \Delta_I \) intersect \( \iota_I(\sqrt{-1}R^m_I) \) at \( \iota_I(s_0) \) transversally. Since \( D_{1\ldots m} \) and \( \iota_I(\sigma_I) \) have a same orientation (c.f. Remark 4.5 (i) in [G13]), we have
\[ I_h(D_{1\ldots m}, (\iota_I)_*(\sqrt{-1}R^m_I)^\vee) = I_h(\Delta_I, (\iota_I)_*(\sqrt{-1}R^m_I)^\vee). \]
Thus we obtain
\[ \Delta_I^\vee = \frac{I_h(\Delta_I, \Delta_I^\vee)}{I_h(D_{1\ldots m}, (\iota_I)_*(\sqrt{-1}R^m_I)^\vee)} \cdot (\iota_I)_*(\sqrt{-1}R^m_I)^\vee, \]
which implies the first equality of (4.11) because of
\[ I_h(D_{1\ldots m}, \Delta_I^\vee) = \frac{I_h(\Delta_I, \Delta_I^\vee)}{I_h(D_{1\ldots m}, (\iota_I)_*(\sqrt{-1}R^m_I)^\vee)} \cdot I_h(D_{1\ldots m}, (\iota_I)_*(\sqrt{-1}R^m_I)^\vee) = I_h(\Delta_I, \Delta_I^\vee). \]
The second equality of (4.11) is shown as
\[ I_h(\Delta_I, D_{1\ldots m}^\vee) = (-1)^m I_h(D_{1\ldots m}, \Delta_I^\vee) = (-1)^m I_h(\Delta_I, \Delta_I^\vee) = I_h(\Delta_I, \Delta_I^\vee), \]
where \( g(\alpha, \beta, \gamma_1, \ldots, \gamma_m)^\vee := g(\alpha^{-1}, \beta^{-1}, \gamma_1^{-1}, \ldots, \gamma_m^{-1}) \) for \( g(\alpha, \beta, \gamma_1, \ldots, \gamma_m) \in \mathbb{C}(\alpha, \beta, \gamma_1, \ldots, \gamma_m) \). The orthogonality of \( \Delta_I \)'s implies
\[ D_{1\ldots m} = \sum_I \frac{I_h(D_{1\ldots m}, \Delta_I^\vee)}{I_h(\Delta_I, \Delta_I^\vee)} \Delta_I = \sum_I \Delta_I, \]
which is the equality (4.12). Hence the self-intersection number of \( D_{1\ldots m} \) is
\[ I_h(D_{1\ldots m}, D_{1\ldots m}^\vee) = \sum_I I_h(\Delta_I, \Delta_I^\vee) \]
\[ = \sum_I (-1)^{|I|} \prod_{j \notin I} \gamma_j \cdot (\alpha - \prod_{i \in I} \gamma_i) \cdot (\beta - \prod_{i \in I} \gamma_i) \cdot \frac{\alpha \beta}{\prod_k (\gamma_k - 1) \cdot (\alpha - \prod_k \gamma_k)} = \alpha \beta + (-1)^m \prod_k \gamma_k. \]
At the last equality, we use (5.3). Therefore, Proposition 4.8 is proved.
6.2 Intersection numbers
For \( I, I' \subset \{1, \ldots, m\} \), we evaluate the intersection number \( I_h(\Delta_I, D'_I) \). By Proposition 4.8, we may assume \( I' \neq \{1, \ldots, m\} \). We set
\[
J := \{1, \ldots, m\} - I, \quad J' := \{1, \ldots, m\} - I',
\]
\[
I_0 := I \cap I', \quad I_1 := I \cap J', \quad J_0 := J \cap I', \quad J_1 := J \cap J'.
\]
By using \( \iota_I \), we have \( I_h(\Delta_I, D'_I) = I_h(\Delta_I, \tilde{D}'_I \iota_I) \), where \( \tilde{D}'_I := (-1)^{|I|} \cdot (\iota_I)^{-1}(D'_I) \). Note that the orientation of \( \tilde{D}'_I \) is also induced from the natural embedding \( \mathbb{R}^m \subset \mathbb{C}^m \). Thus \( \sigma_I \) and \( \tilde{D}'_I \) have the same orientation. For \( I' \neq \emptyset \), \( \tilde{D}'_I \) is a chamber
\[
\left\{(s_1, \ldots, s_m) \in \mathbb{R}^m \mid s_i > 0 \ (i \in I'), \ s_j < 0 \ (j \notin I'), \ (-1)^{|I|}v_I(s) > 0, \ (-1)^{|I|+|J'|+1}w_I(s) > 0 \right\}
\]
loaded the branch of \( u_I \) by the assignment of arguments as in the following.

| \( s_i(i \in I') \) | \( s_i(i \in I_1) \) | \( s_i(i \in J_1) \) | \( v_I(s) \) | \( w_I(s) \) |
|-------------------|-------------------|-------------------|----------|----------|
| \( 0 \)           | \( \pi \)          | \( -\pi \)        | \( |I|\pi \) | \( (|I| - (|I'| + 1))\pi \) |

In fact, the conditions for \( v_I \) and \( w_I \) are simply given by
\[
1 - \sum_{i \in I} x_i s_i - \sum_{j \in J} s_j > 0, \quad 1 - \sum_{i \in I} s_i - \sum_{j \in J} x_j s_j < 0,
\]
respectively, because of \( |J'| = |I_1| + |J_1| \). In the case of \( I' = \emptyset \) (then \( I_0 = J_0 = \emptyset \), \( \tilde{D}_0 = \tilde{D} \) is a chamber
\[
\left\{(s_1, \ldots, s_m) \in \mathbb{R}^m \mid s_k < 0 \ (1 \leq k \leq m) \right\}
\]
loaded the branch of \( u_I \) by the assignment of arguments as in the following.

| \( s_i(i \in I_1) \) | \( s_i(i \in J_1) \) | \( v_I(s) \) | \( w_I(s) \) |
|-------------------|-------------------|----------|----------|
| \( \pi \)          | \( -\pi \)        | \( |I|\pi \) | \( (|I| - m)\pi \) |

**Lemma 6.2.** If \( I' \neq \emptyset \) and \( I \subset J' \), we have \( I_h(\tilde{\Delta}_I, \tilde{D}'_I) = 0 \).

*Proof.* By the assumption, we have \( J_0 = J \cap I' = I' \neq \emptyset \). For \((s_1, \ldots, s_m) \in \tilde{D}'_I \), we show that at least one of \( s_j \)’s \((j \in J_0)\) satisfies \( 0 < s_j < mx_j \). Because of \( mx_j < m \cdot \frac{\pi}{m} \varepsilon \), it implies that the chamber \( \tilde{D}'_I \) is included in the \( \varepsilon \)-neighborhood of \( (s_j = 0) \), and hence \( \tilde{D}'_I \) does not intersect \( \tilde{\Delta}_I \). Thus, the lemma is proved. We assume that all of \( s_j \)’s \((j \in J_0)\) satisfy \( s_j \geq mx_j \). By
\[
0 > 1 - \sum_{i \in I_1} s_i - \sum_{j \in J} \frac{x_j}{s_j} = 1 - \sum_{i \in I_1} s_i - \sum_{j \in J_0} \frac{x_j}{s_j} - \sum_{j \in J_1} \frac{x_j}{s_j},
\]
\( s_i < 0 \ (i \in I_1) \) and \( s_j < 0 \ (j \in J_1) \), we have
\[
1 < 1 - \sum_{i \in I_1} s_i - \sum_{j \in J_1} \frac{x_j}{s_j} < \sum_{j \in J_0} \frac{x_j}{s_j}.
\]
However, the inequalities
\[
\sum_{j \in J_0} \frac{x_j}{s_j} \leq \sum_{j \in J_0} \frac{x_j}{mx_j} \leq \sum_{j \in J_0} \frac{1}{m} \leq 1
\]
lead to a contradiction to \( 1 < \sum_{j \in J_0} \frac{x_j}{s_j} \). \( \Box \)

We consider in the case of \( I' \neq \emptyset \). By Lemma 6.2, we may assume that \( I \not\subset J' \). If we consider \( x_1, \ldots, x_m \to 0 \), the condition \( (-1)^{|I|}v_I(s) > 0 \) may be replaced with \( 1 - \sum_{j \in J} s_j > 0 \), and
Theorem 6.3

\((-1)^{|I| + |J| + 1} w(s) > 0\) may be replaced with \(1 - \sum_{i \in I} s_i < 0\) to judge if \(s\) belongs to a central area of \(\tilde{D}_I^\vee\). This observation means that we can evaluate the intersection number \(I_h(\tilde{\Delta}_I, \tilde{D}_I^\vee)\) like that of the regularization of \(V_I\) and \(V_I^\prime, \vee\) by omitting the difference of the branches of \(u_I\), where

\[ V_I := \{ (s_1, \ldots, s_m) \in \mathbb{R}^m \mid s_k > 0, 1 - \sum_{i \in I} s_i > 0, 1 - \sum_{j \in J} s_j > 0 \} , \]

\[ V_I^\prime := \{ (s_1, \ldots, s_m) \in \mathbb{R}^m \mid s_k > 0 (k \in I'), s_k < 0 (k \in J'), 1 - \sum_{i \in I} s_i < 0, 1 - \sum_{j \in J} s_j > 0 \} . \]  

\[(6.1)\]

Note that the chamber \(V_I^\prime\) is not empty, because of \(I \not\subset J'\). In the case of \(I' = \emptyset\), we can see that the above claim is valid, by replacing \((6.1)\) with

\[ V' := \{ (s_1, \ldots, s_m) \in \mathbb{R}^m \mid s_k < 0 (1 \leq k \leq m) \} \]

(note that \(1 - \sum_{i \in I} s_i > 0\) and \(1 - \sum_{j \in J} s_j > 0\) hold clearly). Recall that when we construct the twisted cycle \(\tilde{\Delta}_I\), the exponents of \((s_i = 0), (s_j = 0), (1 - \sum_{i \in I} s_i = 0)\) and \((1 - \sum_{j \in J} s_j = 0)\) are

\[ c_i - 1, \quad 1 - c_j, \quad -b, \quad \sum_{k=1}^m c_k - a - m + 1, \]

respectively, where \(i \in I\) and \(j \in J\); see Section 4 of [G13].

**Theorem 6.3.** For \(I' \neq \emptyset\), we have

\[ I_h(\tilde{\Delta}_I, \tilde{D}_I^\vee) = (-1)^{|I| - 1} \prod_{k \in J'} \frac{1}{1 - \gamma_k} \cdot \frac{1}{1 - \beta} \cdot \left[ 1 + \sum_{K_I \subset \emptyset \atop K_J \subset J_0} \left( \prod_{i \in K_I} \frac{1}{\gamma_i - 1} \cdot \prod_{j \in K_J} \frac{\gamma_j}{1 - \gamma_j} \right) \right. \]

\[ \left. + \frac{\alpha}{\prod_{k} \gamma_k - \alpha} \sum_{K_I \subset I_0 \atop K_J \subset J_0} \left( \prod_{i \in K_I} \frac{1}{\gamma_i - 1} \cdot \prod_{j \in K_J} \frac{\gamma_j}{1 - \gamma_j} \right) \right] . \]  

\[(6.2)\]

For \(I' = \emptyset\), we have

\[ I_h(\tilde{\Delta}_I, \tilde{D}_I^\vee) = (-1)^{|I|} \prod_{k=1}^m \frac{1}{1 - \gamma_k} . \]  

\[(6.3)\]

**Proof.** Let \(s_0\) be an intersection point of \(\tilde{\Delta}_I\) and \(\tilde{D}_I^\vee\). We denote the difference of the branches of \(u_I\) at \(s_0\) by \(\chi_{I,I'}\), namely,

\[ \chi_{I,I'} := \begin{cases} \text{the value } u_I(s_0) \text{ with respect to the branch defined on } \tilde{\Delta}_I, \\ \text{the value } u_I(s_0) \text{ with respect to the branch defined on } \tilde{D}_I^\vee \end{cases} \]

Note that \(\chi_{I,I'}\) is independent of the choice of the intersection point \(s_0\). We prove the theorem by two steps.
Step 1: We show that

\[ I_h(\tilde{\Delta}_I, \tilde{\Delta}'_{I'}) = \chi_{I, I'} \cdot (-1)^{m-|J'|+1} \prod_{i \in I} \frac{1}{\gamma_i - 1} \cdot \prod_{j \in J'} \frac{1}{\gamma_j - 1} \cdot \frac{1}{\beta - 1 - 1} \]

\[ + \left[ 1 + \sum_{\substack{K_l \subseteq I_0 \subseteq I \subseteq J_0 \subseteq J \subseteq K_j \subseteq J_0}} \prod_{i \in K_l} \frac{1}{\gamma_i - 1} \cdot \prod_{j \in K_J} \frac{1}{\gamma_j - 1} \right] \]

\[ + \frac{1}{\alpha - 1} \prod_{k} \frac{1}{\gamma_k - 1} \]

\[ \times \sum_{\substack{K_l \subseteq I_0 \subseteq I \subseteq J_0 \subseteq J \subseteq K_j \subseteq J_0}} \left[ \prod_{i \in K_l} \frac{1}{\gamma_i - 1} \cdot \prod_{j \in K_J} \frac{1}{\gamma_j - 1} \right] \]

\[ (I' \neq \emptyset), \quad (6.4) \]

We prove (6.4), by using results in [KY94]. Obviously, we have

\[ V_I \cap V_{I'} = \{(s_1, \ldots, s_m) \in \mathbb{R}^m \mid s_j = 0 \ (j \in J'), \ 1 - \sum_{i \in I} s_i = 0, \ s_i > 0 \ (i \in I') \ 1 - \sum_{j \in J} s_j = 0 \}, \]

which implies that the intersection number \( I_h(\tilde{\Delta}_I, \tilde{\Delta}'_{I'}) \) is equal to the product of

\[ \chi_{I, I'} \prod_{i \in I \cap J'} \frac{1}{\gamma_i - 1} \cdot \prod_{j \in I \cap J} \frac{1}{\gamma_j - 1} \cdot \frac{1}{\beta - 1 - 1} \]

and the self-intersection number of the twisted cycle determined by the chamber

\[ \{(s_1, \ldots, s_m) \in \mathbb{R}^m \mid s_j = 0 \ (j \in J'), \ 1 - \sum_{i \in I} s_i = 0, \ s_i > 0 \ (i \in I') \ 1 - \sum_{j \in J} s_j = 0 \} \]

in the \((m - (|J'| + 1))-dimensional space \( L := \bigcap_{j \in J'} (s_j = 0) \cap (1 - \sum_{i \in I} s_i = 0) \). To evaluate this self-intersection number, we investigate non-empty intersections of \((s_i = 0) \ (i \in I') \), \((1 - \sum_{j \in J} s_j = 0) \) with \( L \).

(i) Without \((1 - \sum_{j \in J} s_j = 0)\): we choose subsets \( K \) of \( I' \) such that \( \bigcap_{k \in K} (s_k = 0) \cap L \neq \emptyset \). By the condition \( 1 - \sum_{i \in I} s_i = 0 \), we have

\[ \bigcap_{k \in K} (s_k = 0) \cap L \neq \emptyset \iff K \cap I \subseteq I \iff K = K_I \cup K_J \ (K_I \subseteq I, \ K_J \subseteq J). \]

(ii) With \((1 - \sum_{j \in J} s_j = 0)\): we choose subsets \( K \) of \( I' \) such that \( \bigcap_{k \in K} (s_k = 0) \cap (1 - \sum_{j \in J} s_j = 0) \cap L \neq \emptyset \). By the conditions \( 1 - \sum_{i \in I} s_i = 0 \) and \( 1 - \sum_{j \in J} s_j = 0 \), we have

\[ \bigcap_{k \in K} (s_k = 0) \cap (1 - \sum_{j \in J} s_j = 0) \cap L \neq \emptyset \]

\[ \iff K \cap I \subseteq I, \ K \cap J \subseteq J \iff K = K_I \cup K_J \ (K_I \subseteq I, \ K_J \subseteq J). \]
Therefore, the self-intersection number is equal to

\[
(-1)^{m-(|J'|+1)} \left[ 1 + \sum_{K_j \subseteq I_0 \atop K_j \not\subseteq J_0} \left( \prod_{i \in K_j} \frac{1}{\gamma_i - 1} \cdot \prod_{j \in K_j} \frac{1}{\gamma_j - 1} \right) \right]
\]

\[+ \frac{1}{\alpha - 1 \prod_{k} \gamma_k - 1} \sum_{K_j \subseteq I_0 \atop K_j \not\subseteq J_0} \left( \prod_{i \in K_j} \frac{1}{\gamma_i - 1} \cdot \prod_{j \in K_j} \frac{1}{\gamma_j - 1} \right),\]

and hence (6.4) is proved. We can obtain the equality (6.5) in a similar way.

Step 2: We evaluate \( \chi_{I,I'} \). We consider the differences of the branches of the factors of \( u_I \) at an intersection point of \( \tilde{\Delta}_I \) and \( \tilde{D}_{I'} \).

(i) The argument of \( s_k \) on \( \tilde{\Delta}_I \) and \( \tilde{D}_{I'} \) are given as in the following.

|      | \( k \in I' = I_0 \cup J_0 \) | \( k \in I_1 \) | \( k \in J_1 \) |
|------|-------------------------------|----------------|----------------|
| \( \Delta_I \) | 0                             | \( \pi \)     | \( \pi \)     |
| \( D_{I'} \)  | 0                             | \( \pi \)     | \( -\pi \)    |

Since the exponent of \( s_j \) (\( j \in J \)) is \( C_j = 1 - c_j + b \), the contribution by the branch of \( \prod_k s_k^{C_k} \) is \( \prod_{j \in J_1} (\gamma_j^{-1}\beta) \).

(ii) We have

\[
v_I = \prod_{i \in I} s_i \cdot \left( 1 - \sum_{j \in J} s_j - \sum_{i \in I} \frac{x_i}{s_i} \right),
\]

and the term \( \sum_{i \in I} \frac{x_i}{s_i} \) does not concern the difference of the branches. By (i) and the fact that \( s \in V_{I'}' \) satisfies \( 1 - \sum_{j \in J} s_j > 0 \), both the argument of \( v_I \) on \( \tilde{\Delta}_I \) and that on \( \tilde{D}_{I'} \) are \( |I_1|\pi \), and hence the contribution by the branch of \( v_I^A \) is 1.

(iii) We have

\[
w_I = \prod_{j \in J} s_j \cdot \left( 1 - \sum_{i \in I} s_i - \sum_{j \in J} \frac{x_j}{s_j} \right),
\]

and the term \( \sum_{j \in J} \frac{x_j}{s_j} \) does not concern the difference of the branches. By (i) and the fact that \( s \in V_{I'}' \) satisfies

\[
\begin{align*}
&\left\{ \begin{array}{l}
1 - \sum_{i \in I} s_i < 0 \quad (I' \neq \emptyset), \\
1 - \sum_{i \in I} s_i > 0 \quad (I' = \emptyset),
\end{array} \right.
\end{align*}
\]

the arguments of \( w_I \) on \( \tilde{\Delta}_I \) and \( \tilde{D}_{I'} \) at the intersection points are as follows:

\[
(\text{argument on } \tilde{\Delta}_I) = \left\{ \begin{array}{l}
(|I_1| + 1)\pi \quad (I' \neq \emptyset), \\
|I_1|\pi \quad (I' = \emptyset),
\end{array} \right.
\]

\[
(\text{argument on } \tilde{D}_{I'}) = \left\{ \begin{array}{l}
(|I_1| - |I'| - 1)\pi \quad (I' \neq \emptyset), \\
(|I_1| - m)\pi = -|J_1|\pi \quad (I' = \emptyset),
\end{array} \right.
\]

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Here, note that $m = |J'| = |I_1| + |J_1|$, if $I' = \emptyset$. Because of $|J'| = |I_1| + |J_1|$, we obtain (difference of the arguments of $w_I$)

$$
\left\{ \begin{array}{ll}
(|J_1| + 1)\pi - (|I_1| - |J'| - 1)\pi = 2(|J_1| + 1)\pi & (I' \neq \emptyset), \\
|J_1|\pi - (-|J_1|)\pi = 2|J_1|\pi & (I' = \emptyset).
\end{array} \right.
$$

Since the exponent of $w_I$ is $B = -b$, the contribution by the branch of $w^B_I$ is

$$
\left\{ \begin{array}{ll}
\beta^{-|J_1|+1} & (I' \neq \emptyset), \\
\beta^{-|J_1|} & (I' = \emptyset).
\end{array} \right.
$$

We thus have

$$
\chi_{I', I} = \prod_{j \in J_1} (\gamma_j^{-1}\beta) \cdot \beta^{-|J_1|+1} \quad (I' \neq \emptyset), \\
\chi_{I, \emptyset} = \prod_{j \in J_1} (\gamma_j^{-1}\beta) \cdot \beta^{-|J_1|}.
$$

By Step 1, we obtain (6.2) and (6.3).

To simplify the equality (6.2), we use Lemma 5.1. We summarize the results in this subsection.

**Corollary 6.4.** If $I' \neq \emptyset, \{1, \ldots, m\}$, then we have

$$
I_h(\Delta_I, D^\vee_{I'}) = (-1)^{|I'|+|I'|-1} \cdot \prod_{k=1}^m \frac{1}{1 - \gamma_k} \cdot \frac{\prod_{i \in I_0} \gamma_i - 1}{1 - \beta} \cdot \frac{\prod_{j \in J_0} \gamma_j \cdot \prod_{j \in J_0} (\gamma_j \cdot \prod_{j \in J_0} (\gamma_j - 1))}{\prod_{k \in I'} \gamma_k - \alpha}.
$$

(6.6)

This equality holds, even if $I \subset J'$. For $I' = \emptyset$, we have

$$
I_h(\Delta_I, D^\vee) = (-1)^{|I|} \cdot \prod_{k=1}^m \frac{1}{1 - \gamma_k}.
$$

(6.7)

**Proof.** Recall that $I_h(\Delta_I, D^\vee_{I'}) = I_h(\tilde\Delta_I, \tilde{D}^\vee_{I'})$. The equality (6.7) coincides with that in Theorem 6.3. If $I \subset J'$, then we have $I_0 = I \cap I' = \emptyset$, and hence $\prod_{i \in I_0} \gamma_i - 1 = 0$. Thus the right-hand side of (6.6) is 0, which is compatible with Lemma 6.2. Then we have to show that the right-hand side of (6.2) is equal to that of (6.6). By (5.1), we have

$$
1 + \sum_{K_j \subseteq I_0, K_j \subseteq J_0} \left( \prod_{i \in K_f} \frac{1}{\gamma_i - 1} \cdot \prod_{j \in K_j} \frac{\gamma_j}{1 - \gamma_j} \right) = (-1)^{|I_0|} \cdot \left( \prod_{i \in I_0} \frac{\gamma_i - 1}{1 - \beta} \right) \cdot \prod_{k \in I'} \frac{1}{1 - \gamma_k}.
$$

$$
\sum_{K_j \subseteq I_0, K_j \subseteq J_0} \left( \prod_{i \in K_f} \frac{1}{\gamma_i - 1} \cdot \prod_{j \in K_j} \frac{\gamma_j}{1 - \gamma_j} \right) = (-1)^{|I_0|} \cdot \left( \prod_{i \in I_0} \frac{\gamma_i - 1}{1 - \beta} \right) \cdot \prod_{j \in J_0} \frac{1}{1 - \gamma_j}.
$$

Therefore, we obtain

$$
I_h(\Delta_I, D^\vee_{I'}) = I_h(\tilde\Delta_I, \tilde{D}^\vee_{I'})
$$

$$
= (-1)^{|I_1|+|J_0|-1} \cdot \prod_{k \in I'} \frac{1}{1 - \gamma_k} \cdot \frac{1}{1 - \beta} \cdot (-1)^{|I_0|} \cdot \left( \prod_{i \in I_0} \frac{\gamma_i - 1}{1 - \beta} \right) \cdot \frac{1}{1 - \gamma_k} \cdot \prod_{k \in I'} \frac{1}{1 - \gamma_k}.
$$

$$
= (-1)^{|I_1|+|J_0|-1} \cdot \prod_{k=1}^m \frac{1}{1 - \gamma_k} \cdot \frac{\prod_{i \in I_0} \gamma_i - 1}{1 - \beta} \cdot \frac{\prod_{j \in J_0} (\gamma_j \cdot \prod_{j \in J_0} (\gamma_j - 1))}{\prod_{k \in I'} \gamma_k - \alpha}.
$$
Here we use \( m = |I_0| + |I_1| + |J_0| + |J_1| \). Further, since
\[
|I_1| + |J_0| = |I \cap I'| + |I^c \cap I'| = |I \cup I'| - |I \cap I'| = |I| + |I'| - 2|I \cap I'|,
\]
we have \((-1)^{|I_1|+|J_0|} = (-1)^{|I|+|I'|} \).

**Lemma 6.5.** We have \( I_h(D_{1\ldots m}, D_{0}) = 0 \), if \( I' \neq \{1, \ldots, m\} \).

**Proof.** This is obvious, since
\[
D_{1\ldots m} \subset \{(s_1, \ldots, s_m) \in \mathbb{R}^m \mid s_k > x_k (1 \leqslant k \leqslant m)\},
\]
\[
D_{0} \cap \{(s_1, \ldots, s_m) \in \mathbb{R}^m \mid s_k \geqslant x_k (1 \leqslant k \leqslant m)\} = \emptyset.
\]

### 6.3 Linear independence

Let \( \Lambda_0 \) be the matrix \( \{(I_h(\Delta_I, D_{I'}))_{I, I'}\} \) with arranging \( I, I' \) in the same way as the basis \( \{\Delta_I\} \) (see Section 3). In this subsection, we evaluate the determinant of \( \Lambda_0 \).

**Theorem 6.6.** We have
\[
\det \Lambda_0 = \begin{cases} 
-\left( \alpha \beta - \prod_{k=1}^{m} \gamma_k \right)^{m-1} \frac{(\prod_{k=1}^{m} \gamma_k + \alpha)^{2m-1} - 1}{(1 - \beta)^{2m-1} (\prod_{k=1}^{m} \gamma_k - \alpha)^{2m-1} - 1} \cdot \prod_{k=1}^{m} \frac{1}{(1 - \gamma_k)^{2m-1}} & (m \text{ : odd}), \\
\left( \alpha \beta + \prod_{k=1}^{m} \gamma_k \right)^{m-1} \frac{(\prod_{k=1}^{m} \gamma_k + \alpha)^{2m-1} - 2}{(1 - \beta)^{2m-1} (\prod_{k=1}^{m} \gamma_k - \alpha)^{2m-1} - 1} \cdot \prod_{k=1}^{m} \frac{1}{(1 - \gamma_k)^{2m-1}} & (m \text{ : even}).
\end{cases}
\]

In particular, we obtain \( \det \Lambda_0 \neq 0 \), and hence \( \{D_I\} \) is linearly independent.

**Remark 6.7.** In this paper, we assume that the parameters \( a, b, c = (c_1, \ldots, c_m) \) are generic. In fact, it is sufficient for our proof of Theorem 4.6 to assume the irreducibility condition of the system \( E_C(a, b, c) \)
\[
a - \sum_{i \in I} c_i, \quad b - \sum_{i \in I} c_i \notin \mathbb{Z} \quad (I \subset \{1, \ldots, m\}),
\]
and the conditions
\[
c_1, \ldots, c_m \notin \mathbb{Z}, \quad a - \sum_{k=1}^{m} c_k \notin \frac{1}{2} \mathbb{Z}, \quad a + b - \sum_{k=1}^{m} c_k + \frac{m+1}{2} \notin \mathbb{Z}.
\]

To compute \( \det \Lambda_0 \), we change \( \Lambda_0 \) by elementary transformations with keeping the determinant as follows. Add the first, second, \ldots, \((2^m - 1)\)-th row of \( \Lambda_0 \) to the \(2^m\)-th row of \( \Lambda_0 \), then \(2^m\)-th row becomes
\[
\left( I_h\left( \sum_{I} \Delta_I, D_{0} \right), \ldots, I_h\left( \sum_{I} \Delta_I, D_{2^m-m} \right), I_h\left( \sum_{I} \Delta_I, D_{0} \right) \right) = \left( I_h(D_{1\ldots m}, D_{0}), \ldots, I_h(D_{1\ldots m}, D_{2^m-m}), I_h(D_{1\ldots m}, D_{0}) \right),
\]
by Lemma 6.5. It means that
\[
\det \Lambda_0 = I_h(D_{1\ldots m}, D_{0}) \cdot \det \Lambda',
\]

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where $\Lambda'$ is the leading principal minor of $\Lambda_0$ of size $2^m - 1$. By Proposition \ref{prop:det} and Corollary \ref{cor:det} we have

$$
\det \Lambda_0 = \frac{\alpha \beta + (-1)^m \prod_{k} \gamma_k}{(1 - \beta)^{2^{m-1}}(\prod \gamma_k - \alpha)^{2^{m-1}}} \cdot \prod_{k=1}^{m} \frac{1}{(1 - \gamma_k)^{2^{m-1}}} \cdot \det \Lambda,
$$

where $\Lambda$ is a $(2^m - 1) \times (2^m - 1)$ matrix whose $(I, I')$-entry is

$$
\Lambda_{I, I'} := (-1)^{|I|+|I'|-1} \cdot \left( \prod_{i \in I \cap I'} \gamma_i - 1 \right) \cdot \left( \prod_{k=1}^{m} \gamma_k - \alpha \prod_{j \in I \cap I'} \gamma_j \right) \quad (I' \neq \emptyset),
$$

$$
\Lambda_{I, \emptyset} := (-1)^{|I|}.
$$

We write

$$
\Lambda = \begin{pmatrix}
\Lambda(0, 0) & \Lambda(0, 1) & \ldots & \Lambda(0, m - 1) \\
\Lambda(1, 0) & \Lambda(1, 1) & \ldots & \Lambda(1, m - 1) \\
\vdots & \vdots & \ddots & \vdots \\
\Lambda(m - 1, 0) & \Lambda(m - 1, 1) & \ldots & \Lambda(m - 1, m - 1)
\end{pmatrix},
$$

where $\Lambda(k, k')$ is the $\binom{m}{k} \times \binom{m}{k'}$ matrix. Note that the entries of $\Lambda(k, k')$ are the $(I, I')$-entries of $\Lambda$ with $|I| = k$, $|I'| = k'$.

We compute $\det \Lambda$. Put $\Lambda^{(0)} := \Lambda$. We take $\Lambda^{(n)}$ by induction on $n$ as follows: for $n \geq 1$, we define $\Lambda^{(n)}$ by replacing the columns of $I'$ ($|I'| \geq n + 1$) of $\Lambda^{(n-1)}$ with

$$
\Lambda^{(n-1)}_{*, I'} + \sum_{K' \subseteq I', |K'| = n} (-1)^{|I'|+n+1} \prod_{k} \gamma_k + (-1)^n \alpha \prod_{j \in K' \cap I'} \gamma_j \\
\prod_{k} \gamma_k + (-1)^n \alpha \cdot \Lambda^{(n-1)}_{*, K'},
$$

where $\Lambda^{(n-1)}_{*, I'}$ is the column of $I'$ of $\Lambda^{(n-1)}$. Straightforward calculations show the following lemma.

**Lemma 6.8.**
(i) $\det \Lambda^{(n)} = \det \Lambda$, $\Lambda^{(n)}_{\emptyset, \emptyset} = 1$,
(ii) if $|I'| \geq n + 1$, then

$$
\Lambda^{(n)}_{I, I'} = (-1)^{|I|+|I'|-1} \cdot \left[ \left( \prod_{i \in I \cap I'} \gamma_i - 1 \right) \cdot \left( \prod_{k=1}^{m} \gamma_k - \alpha \prod_{j \in I \cap I'} \gamma_j \right) \right] \\
- \sum_{K \subseteq I \cap I'} \sum_{0 < |K| \leq n} \left( \prod_{i \in K} \gamma_i - 1 \right) \cdot \left( \prod_{k=1}^{m} \gamma_k - (-1)^{|K|} \alpha \prod_{j \in K \cap I'} \gamma_j \right),
$$

(iii) $k \leq n \implies \Lambda^{(n)}(k, k') = O (k' > k)$,
(iv) $\Lambda^{(n)}(1, 1), \ldots, \Lambda^{(n)}(n + 1, n + 1)$ are diagonal,
(v) $1 \leq |I| \leq n + 1 \implies \Lambda^{(n)}_{I, I} = - \prod_{i \in I} (\gamma_i - 1) \cdot (\prod_{k} \gamma_k + (-1)^{|I|} \alpha)$.

Note that the columns of $I'$ ($|I'| \leq n$) and the rows of $I$ ($|I| \leq n - 1$) are equal to those of $\Lambda^{(n-1)}$. By using this lemma, we prove Theorem \ref{thm:main}.

**Proof of Theorem \ref{thm:main}** By Lemma \ref{lem:det} $\Lambda^{(m-2)}$ is the lower triangular matrix whose diagonal
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entries are given by (i) and (v). Hence we obtain

\[
\det \Lambda_0 = \frac{\alpha \beta + (-1)^m \prod_k \gamma_k}{(1 - \beta)^{2m-1}(\prod_k \gamma_k - \alpha)^{2m-1}} \cdot \prod_{k=1}^m \frac{1}{(1 - \gamma_k)^{2m-1}} \cdot \det \Lambda^{(m-2)}
\]

\[
= (-1)^m \cdot \frac{\alpha \beta + (-1)^m \prod_k \gamma_k}{(1 - \beta)^{2m-1}(\prod_k \gamma_k - \alpha)^{2m-1}} \cdot \prod_{k=1}^m \frac{1}{(1 - \gamma_k)^{2m-1}} \cdot \prod_{\emptyset \neq I \subseteq \{1, \ldots, m\}} \left( \prod_{k=1}^m \gamma_k + (-1)^{|I|} \alpha \right).
\]

If $m$ is odd, we have

\[
\prod_{\emptyset \neq I \subseteq \{1, \ldots, m\}} \left( \prod_{k=1}^m \gamma_k + (-1)^{|I|} \alpha \right) = \left( \prod_{k=1}^m \gamma_k - \alpha \right)^{2m-1} \cdot \left( \prod_{k=1}^m \gamma_k + \alpha \right)^{2^{m-1} - 1}.
\]

If $m$ is even, we have

\[
\prod_{\emptyset \neq I \subseteq \{1, \ldots, m\}} \left( \prod_{k=1}^m \gamma_k + (-1)^{|I|} \alpha \right) = \left( \prod_{k=1}^m \gamma_k - \alpha \right)^{2m-2} \cdot \left( \prod_{k=1}^m \gamma_k + \alpha \right)^{2^{m-1} - 2}.
\]

Therefore, the proof of Theorem 6.6 is completed. \(\square\)

6.4 The eigenspace of $M_0$ associated to 1

By Lemma 6.5 and Theorem 6.6, we have to show that

- $M_0(D_I) = D_I$ for $I \subseteq \{1, \ldots, m\}$,
- $M_0(D_{1 \ldots m}) = [(-1)^{m-1} \prod_k \gamma_k \cdot \alpha^{-1} \beta^{-1}] \cdot D_{1 \ldots m},$

to prove Theorem 4.6. In this subsection, we show the first claim. The second one is proved in the next subsection.

Hereafter, we use the coordinates $(s_1, \ldots, s_m) = \left(\frac{t_1}{x_1}, \ldots, \frac{t_m}{x_m}\right)$. The functions $v(t)$ and $w(t, x)$ are expressed as

\[
1 - \sum_{k=1}^m x_k s_k, \quad \prod_{k=1}^m \left(x_k s_k\right) \cdot \left(1 - \sum_{k=1}^m \frac{1}{s_k}\right),
\]

respectively. Let

\[
v'(s, x) := 1 - \sum_{k=1}^m x_k s_k, \quad w'(s) := \prod_{k=1}^m s_k \cdot \left(1 - \sum_{k=1}^m \frac{1}{s_k}\right).
\]

If $x_1, \ldots, x_m$ are positive real numbers, then we have

\[
t_k \geq 0 \Leftrightarrow s_k \geq 0, \quad v(t) \geq 0 \Leftrightarrow v'(s, x) \geq 0, \quad w(t, x) \geq 0 \Leftrightarrow w'(s) \geq 0,
\]

and hence the expressions of $D_I$'s are as follows:

- $D_{1 \ldots m} : s_k > 0 \quad \text{for} \quad 1 \leq k \leq m,$
- $v'(s, x) > 0, \quad w'(s) > 0,$
- $D : s_k < 0 \quad \text{for} \quad 1 \leq k \leq m,$
- $D_I$ (otherwise) : $s_i > 0 \quad \text{for} \quad i \in I, \quad s_j < 0 \quad \text{for} \quad j \not\in I,$
- $v'(s, x) > 0, \quad w'(s) > 0, \quad (-1)^{m-|I|+1} w'(s) > 0.$

Note that if $x = (x_1, \ldots, x_m)$ moves, then only the divisor $(v'(s, x) = 0)$ varies.
Recall that the loop $\rho_0$ is homotopic to the composition $\tau_0\rho'_0\overline{\tau_0}$, where

$$\tau_0 : [0, 1] \ni \theta \mapsto \left((1 - \theta) \cdot \frac{1}{2m^2} + \theta \cdot \left(\frac{1}{m^2} - \varepsilon_0\right)\right) (1, \ldots, 1) \in X,$$

$$\rho'_0 : [0, 1] \ni \theta \mapsto \left(\frac{1}{m^2} - \varepsilon_0 e^{2\pi \sqrt{-1} \theta}\right) (1, \ldots, 1) \in X,$$

for a sufficiently small positive real number $\varepsilon_0$. Since variations along the paths $\tau_0$ and $\overline{\tau_0}$ give trivial transformations of the cycles $D_I$'s, we have to consider the variation along $\rho'_0$ for a sufficiently small $\varepsilon_0$. Let $x \to \left(\frac{1}{m^2}, \ldots, \frac{1}{m^2}\right)$, then $(v'(s, x) = 0)$ and $(w'(s) = 0)$ are tangent at $(s_1, \ldots, s_m) = (m, \ldots, m)$. Thus $D_{1\ldots m}$ is a vanishing cycle. Each $D_I$ ($I \subset \{1, \ldots, m\}$) survives as $x \to \left(\frac{1}{m^2}, \ldots, \frac{1}{m^2}\right)$, and its variation along $\rho'_0$ is too slight to change the branch of $u_x$ on it. This implies that $M_0(D_I) = D_I$ for $I \subset \{1, \ldots, m\}$.

6.5 An eigenvector of $M_0$ associated to the eigenvalue $(-1)^{m-1} \prod k \gamma_k \cdot \alpha^{-1} \beta^{-1}$

In this subsection, we show $M_0(D_{1\ldots m}) = \left([-1]^{m-1} \prod k \gamma_k \cdot \alpha^{-1} \beta^{-1}\right) \cdot D_{1\ldots m}$. As mentioned in the previous subsection, it is sufficient to consider the variation of $D_{1\ldots m}$ along $\rho'_0$ for a sufficiently small $\varepsilon_0$. Thus we may consider that $D_{1\ldots m}$ is contained in a small neighborhood of $s = (m, \ldots, m)$ in $\mathbb{R}^m$.

Putting $x_1 = \cdots = x_m = \frac{1}{m^2} - \varepsilon_0$, we have

$$v'(s, \rho'_0(0)) = 1 - \left(\frac{1}{m^2} - \varepsilon_0\right) \sum_{k=1}^m s_k.$$

We use the coordinates system

$$(s'_1, \ldots, s'_{m-1}, s'_m) := \left(s_1 - m, \ldots, s_{m-1} - m, \sum_{k=1}^m s_k - m^2\right).$$

Note that $s_l = s'_l + m$ ($1 \leq l \leq m - 1$) and $s_m = s'_m - \sum_{l=1}^{m-1} s'_l + m$. Then the origin $(s'_1, \ldots, s'_m) = (0, \ldots, 0)$ corresponds to $(s_1, \ldots, s_m) = (m, \ldots, m)$. Let $U$ be a small neighborhood of $(s'_1, \ldots, s'_m) = (0, \ldots, 0)$ so that $s_k > 0$ ($1 \leq k \leq m$). In $U$, we have

$$v'(s, \rho'_0(0)) > 0 \iff 1 - \left(\frac{1}{m^2} - \varepsilon_0\right) (s'_m + m^2) > 0 \iff s'_m < \frac{m^2}{1 - \frac{1}{\sum_{l=1}^{m-1} s'_l + m}} \cdot \varepsilon_0,$$

$$w'(s) > 0 \iff 1 - \frac{1}{\sum_{k=1}^m s_k} > 0 \iff s'_m > \sum_{l=1}^{m-1} s'_l - m + \frac{1}{1 - \sum_{l=1}^{m-1} s'_l + m}.$$ 

Hence $D_{1\ldots m}$ is expressed as

$$\left\{(s'_1, \ldots, s'_m) \in U \left| \sum_{l=1}^{m-1} s'_l - m + \frac{1}{1 - \sum_{l=1}^{m-1} s'_l + m} < s'_m < \frac{m^2}{1 - \frac{1}{\sum_{l=1}^{m-1} s'_l + m}} \cdot \varepsilon_0\right.\right\}.$$ 

Let $\theta$ move from 0 to 1, then the arguments of $\frac{1}{m^2} - \varepsilon_0 e^{2\pi \sqrt{-1} \theta}$ at the start point and the end point are equal. Thus the argument of $\frac{m^2}{1 - \frac{1}{\sum_{l=1}^{m-1} s'_l + m}} \cdot \varepsilon_0 e^{2\pi \sqrt{-1} \theta}$ increases by $2\pi$, when $\theta$ moves from 0 to 1. Put

$$f(s'_1, \ldots, s'_{m-1}) := \sum_{l=1}^{m-1} s'_l - m + \frac{1}{1 - \sum_{l=1}^{m-1} s'_l + m}.$$
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Then $(s'_1, \ldots, s'_{m-1}) = (0, \ldots, 0)$ is a critical point of $f$, and the Hessian matrix $H_f(0, \ldots, 0)$ at this point is positive definite. The Morse lemma implies that $f$ is expressed as

$$
\sum_{l=1}^{m-1} z_l^2,
$$

with appropriate coordinates $(z_1, \ldots, z_{m-1})$ around the origin. Therefore, the claim $\mathcal{M}_0(D_{1\ldots m}) = \left(-1\right)^{m-1} \prod_k \gamma_k \cdot \alpha^{-1} \beta^{-1} \cdot D_{1\ldots m}$ is obtained from the following lemma.

**Lemma 6.9.** For $y, \lambda, \mu \in \mathbb{C}$, we put

$$
Z_y := \mathbb{C}^m - \left( \left( z_m - \sum_{l=1}^{m-1} z_l^2 = 0 \right) \cup (y - z_m = 0) \right) \subset \mathbb{C}^m,
$$

$$
\nu_y(z) := \left( z_m - \sum_{l=1}^{m-1} z_l^2 \right)^\lambda \cdot (y - z_m)^\mu,
$$

where $z_1, \ldots, z_m$ are coordinates of $\mathbb{C}^m$. We consider the twisted homology groups $H_m(Z_y, \nu_y)$ ($y \in \mathbb{C}$). Let $\delta_y \in H_m(Z_y, \nu_y)$ ($y > 0$) be expressed by the twisted cycle defined by the domain

$$
D(y) := \left\{ (z_1, \ldots, z_m) \in \mathbb{R}^m \left| \sum_{l=1}^{m-1} z_l^2 < z_m < y \right. \right\},
$$

and let $\delta'$ be the element in $H_m(Z_1, \nu_1)$, which is obtained by the deformation of $\delta_1$ along $y = e^{2\pi \sqrt{-1} \theta}$ as $\theta : 0 \to 1$. Then we have

$$
\delta' = (-1)^{m-1} e^{2\pi \sqrt{-1}(\lambda + \mu)} \cdot \delta_1.
$$

**Proof.** It is easy to see that the domain $D(y)$ is expressed by $(\xi_1, \ldots, \xi_m) \in [0,1]^m$ as

$$
z_l = (2\xi_l - 1) \sqrt{y\xi_m \prod_{j=ll+1}^{m-1} (1 - (2\xi_j - 1)^2)} \quad (1 \leq l \leq m - 1),
$$

$$
z_m = y\xi_m.
$$

The functions $z_m - \sum_{l=1}^{m-1} z_l^2$ and $y - z_m$ are expressed as

$$
y\xi_m \left( 1 - \sum_{l=1}^{m-1} (2\xi_l - 1)^2 \prod_{j=ll+1}^{m-1} (1 - (2\xi_j - 1)^2) \right), \quad y(1 - \xi_m),
$$

respectively. We consider the variation along $y = e^{2\pi \sqrt{-1} \theta}$ as $\theta : 0 \to 1$. The expression of the domain $D(1)$ by $(\xi_1, \ldots, \xi_m) \in [0,1]^m$ is changed. However, by a bijection

$$
r : \xi_l \mapsto 1 - \xi_l \quad (1 \leq l \leq m - 1), \quad \xi_m \mapsto \xi_m,
$$

the expression coincides with the original one with contributions to orientation. Further, both arguments of $z_m - \sum_{l=1}^{m-1} z_l^2$ and $y - z_m$ increase by $2\pi$, and the expressions (6.8) are invariant under the bijection $r$. Therefore, we obtain

$$
\delta' = (-1)^{m-1} e^{2\pi \sqrt{-1}(\lambda + \mu)} \cdot \delta_1.
$$

□
Appendix A. Fundamental group

In this appendix, we prove Theorem 4.2.

We regard $\mathbb{C}^m$ as a subset of $\mathbb{P}^m$ and put $L_\infty := \mathbb{P}^m - \mathbb{C}^m$. Then we can consider that $S \cup L_\infty$ is a hypersurface in $\mathbb{P}^m$, and

$$X = \mathbb{C}^m - S = \mathbb{P}^m - (S \cup L_\infty).$$

By a special case of the Zariski theorem of Lefschetz type (refer to Proposition 3.3.1 in [D92]), the inclusion $L - (L \cap (S \cup L_\infty)) \hookrightarrow X$ induces a surjection

$$\eta : \pi_1(L - (L \cap (S \cup L_\infty))) \rightarrow \pi_1(X),$$

for a generic line $L$ in $\mathbb{P}^m$. Note that generators of $\pi_1(L - (L \cap (S \cup L_\infty)))$ are given by $m + 2^{m-1}$ loops going once around each of the intersection points in $L \cap S \subset \mathbb{C}^m$. To define loops in $X$ explicitly, we specify the generic line $L$ in the following way. Let $r_1, \ldots, r_{m-1}$ be positive real numbers satisfying

$$r_1 < \frac{1}{4}, \quad r_k < \frac{r_{k-1}}{4} \quad (2 \leq k \leq m - 1),$$

and let $\varepsilon$ be a sufficiently small positive real number. We consider lines

$$L_0 : (x_1, \ldots, x_{m-1}, x_m) = (r_1, \ldots, r_{m-1}, 0) + t(0, \ldots, 0, 1) \quad (t \in \mathbb{C}),$$

$$L_\varepsilon : (x_1, \ldots, x_{m-1}, x_m) = (r_1, \ldots, r_{m-1}, 0) + t(\varepsilon, \ldots, \varepsilon) \quad (t \in \mathbb{C})$$

in $\mathbb{C}^m$. Though $L_0$ is not a generic line (for example, $L_0 \cap (x_1 \cdots x_{m-1} = 0) = \emptyset$), $L_\varepsilon$ is one of generic lines. We identify $L_\varepsilon$ with $\mathbb{C}$ by the coordinate $t$. The intersection point $L_\varepsilon \cap (x_k = 0)$ is coordinated by $t = -\frac{\varepsilon}{r_k} < 0$, for $1 \leq k \leq m - 1$. The intersection point $L_\varepsilon \cap (x_m = 0)$ is coordinated by $t = 0$. $L_\varepsilon$ and $(R(x) = 0)$ intersect at $2^{m-1}$ points. We coordinate the intersection points $L_\varepsilon \cap (R(x) = 0)$ by $t = t_{a_1 \cdots a_{m-1}}$, $(a_1, \ldots, a_{m-1}) \in \{0, 1\}^{m-1}$. The correspondence is as follows. We denote the coordinates of the intersection points $L_0 \cap (R(x) = 0)$ by

$$t_{a_1 \cdots a_{m-1}}^{(0)} := \left( 1 + \sum_{k=1}^{m-1} (-1)^{a_k} \sqrt{r_k} \right)^2.$$

By this definition, we have

$$t_{a_1 \cdots a_{m-1}}^{(0)} < t_{a'_1 \cdots a'_{m-1}}^{(0)} \quad \Leftrightarrow \quad a_1 - a'_1 = \cdots = a_{r-1} - a'_{r-1} = 0, \quad a_r = 1, \quad a'_r = 0 \quad \Leftrightarrow \quad a_1 \cdots a_{m-1} > a'_1 \cdots a'_{m-1},$$

where $a_1 \cdots a_{m-1}$ is regarded as a binary number. For example, if $m = 4$, then

$$t_{000}^{(0)} < t_{001}^{(0)} < t_{010}^{(0)} < t_{011}^{(0)} < t_{100}^{(0)} < t_{101}^{(0)} < t_{110}^{(0)} < t_{111}^{(0)}.$$

Since $L_\varepsilon$ is sufficiently close to $L_0$, $t_{a_1 \cdots a_{m-1}}$ are supposed to be arranged near to $t_{a_1 \cdots a_{m-1}}^{(0)}$.

Let $\ell_k$ be the loop in $L_\varepsilon - (L_\varepsilon \cap S)$ going once around the intersection point $L_\varepsilon \cap (x_k = 0)$, and let $\ell_{a_1 \cdots a_{m-1}}$ be the loop in $L_\varepsilon - (L_\varepsilon \cap S)$ going once around the intersection point $t_{a_1 \cdots a_{m-1}}$. Each loop approaches the intersection point through the upper half plane of $t$-space, see Figure 1.

It is easy to see that

$$\eta(\ell_k) = \rho_k \quad (1 \leq k \leq m), \quad \eta(\ell_{1 \cdots 1}) = \rho_0. \quad (A.1)$$
Further, we have
\[ \rho_i \rho_j = \rho_j \rho_i \quad (1 \leq i, j \leq m), \]
since the fundamental group of \((\mathbb{C}^\times)^m\) is abelian. To investigate relations among \(\eta(\ell_{a_1 \cdots a_{m-1}})\)'s, we may consider these loops are in \(L_0 - (L_0 \cap S)\), since \(\ell_{a_1 \cdots a_{m-1}}\)'s can be defined in \(L_0 - (L_0 \cap S)\) by a same way as in \(L_\varepsilon\).

**Lemma A.1.**

(i) \(\eta(\ell_{a_1 \cdots a_{m-1}\cdot 0 \cdot a_{m-1}}) = \rho_k \eta(\ell_{a_1 \cdots a_{k-1} a_{k+1} \cdots a_{m-1}}) \rho_k^{-1}\).

(ii) \(\eta(\ell_{1 \cdots 1}) = \rho_{m-1} \eta(\ell_{1 \cdots 1} \ell_{1 \cdots 10} \ell_{1 \cdots 1}) \rho_{m-1}^{-1}\).

Temporarily, we admit this lemma. By (i), we have
\[
\eta(\ell_{a_1 \cdots a_{m-1}}) = (\rho_1^{b_1} \cdots \rho_{m-1}^{b_{m-1}}) \cdot \eta(\ell_{1 \cdots 1}) \cdot (\rho_1^{b_1} \cdots \rho_{m-1}^{b_{m-1}})^{-1}
= (\rho_1^{b_1} \cdots \rho_{m-1}^{b_{m-1}}) \cdot \rho_0 \cdot (\rho_1^{b_1} \cdots \rho_{m-1}^{b_{m-1}})^{-1},
\]
where \((b_1, \ldots, b_{m-1}) := (1 - a_1, \ldots, 1 - a_{m-1})\). This implies that the loops \(\rho_0, \ldots, \rho_m\) generate \(\pi_1(X)\), since the images of \(\ell_k\)'s and \(\ell_{a_1 \cdots a_{m-1}}\)'s by \(\eta\) generate \(\pi_1(X)\). By (ii) and the above argument, we obtain
\[
\rho_0 = \eta(\ell_{1 \cdots 1}) = \rho_{m-1} \eta(\ell_{1 \cdots 1} \ell_{1 \cdots 10} \ell_{1 \cdots 1}) \rho_{m-1}^{-1}
= \rho_{m-1} \cdot \rho_0 \cdot \rho_{m-1} \rho_0 \rho_{m-1}^{-1} \cdot \rho_0^{-1} \cdot \rho_{m-1}^{-1},
\]
that is, \((\rho_0 \rho_{m-1})^2 = (\rho_{m-1} \rho_0)^2\). Changing the definitions of \(L_0\) and \(L_\varepsilon\), we obtain the relations
\[
(\rho_0 \rho_k)^2 = (\rho_k \rho_0)^2 \quad (1 \leq k \leq m).
\]

For example, if we put
\[
L_\varepsilon : (x_1, x_2, \ldots, x_m) = (0, r_1, \ldots, r_{m-1}) + t(1, \varepsilon, \ldots, \varepsilon) \quad (t \in \mathbb{C}),
\]
then a similar argument shows \((\rho_0 \rho_m)^2 = (\rho_m \rho_0)^2\). Therefore, the proof of Theorem 4.2 is completed.

**Proof of Lemma A.1.** For \(\theta \in [0, 1]\), let \(L(\theta)\) be the line defined by
\[
L(\theta) : (x_1, \ldots, x_k, \ldots, x_{m-1}, x_m)
= (r_1, \ldots, e^{2\pi \sqrt{-1} \theta} r_k, \ldots, r_{m-1}, 0) + t(0, \ldots, 0, 1) \quad (t \in \mathbb{C}).
\]
Note that \(L(0) = L(1) = L_0\). We identify \(L(\theta)\) with \(\mathbb{C}\) by the coordinate \(t\). It is easy to see that
the intersection points of \(L(\theta)\) and \((R(x) = 0)\) are given by the following \(2^{m-1}\) elements:

\[
\ell_{a_1\cdots a_{m-1}}^{(\theta)} := \left( 1 + \sum_{j=1}^{m-1} (-1)^{a_j} \sqrt{\tau_j} + (-1)^{a_k} \sqrt{r_k} e^{i\pi/4} \right)^2.
\]

The points \(1 + \sum_{j\neq k} (-1)^{a_j} \sqrt{\tau_j} + (-1)^{a_k} \sqrt{r_k} e^{i\pi/4} \theta\) are in the right half plane for any \(\theta \in [0, 1]\), since \(\sum_{j=1}^{m-1} \sqrt{\tau_j} < \sum_{j=1}^{m-1} 2^{-j} < 1\). Let \(\theta\) move from 0 to 1, then

(a) \(t_{a_1\cdots a_{k-1}0a_{k+1}\cdots a_{m-1}}^{(1)}\) moves in the upper half plane,
(b) \(t_{a_1\cdots a_{k-1}0a_{k+1}\cdots a_{m-1}}^{(\theta)}\) moves in the upper half plane,
(c) \(t_{a_1\cdots a_{k-1}1a_{k+1}\cdots a_{m-1}}^{(\theta)}\) moves in the lower half plane.

For example, \(t_{1023}\)'s move as Figure 2 for \(m = 4\) and \(k = 2\).

![Figure 2. \(t_{1023}\) for \(m = 4, k = 2\).](image)

Put \(P(\theta) := \mathbb{C} - \{t_{a_1\cdots a_{m-1}}^{(\theta)} \mid a_j \in \{0, 1\}\} \subset L(\theta)\). Let \(\varepsilon'\) be a sufficiently small positive real number, and we consider the fundamental group \(\pi_1(P(\theta), \varepsilon')\). As mentioned above, \(\ell_{a_1\cdots a_{m-1}}\)'s are defined as elements in \(\pi_1(P(0), \varepsilon') = \pi_1(P(1), \varepsilon')\). Let \(\theta\) move from 0 to 1, then \(\ell_{a_1\cdots a_{m-1}}\)'s define the elements in each \(\pi_1(P(\theta), \varepsilon')\) naturally. The properties (a), (b), (c) imply that \(\ell_{a_1\cdots a_{k-1}0a_{k+1}\cdots a_{m-1}} \in \pi_1(P(0), \varepsilon')\) changes into \(\ell_{a_1\cdots a_{k-1}1a_{k+1}\cdots a_{m-1}} \in \pi_1(P(1), \varepsilon')\). By this variation, the base point moves around the divisor \((x_k = 0)\), since the base point \(\varepsilon' \in P(\theta)\) corresponds to the point \((r_1, \ldots, e^{2\pi i \theta} r_k, \ldots, r_{m-1}, \varepsilon') \in L(\theta)\). It implies the conjugation by \(r_k\) in \(\pi_1(X)\). Hence we obtain the relation (i).

To prove (ii), we use a similar argument for \(k = m - 1\) and \(\ell_{1\cdots 1} \in \pi_1(P(0), \varepsilon')\). Let \(\theta\) move from 0 to 1, then \(\ell_{1\cdots 1}\) changes into a loop in \(P(1)\), which goes once around \(t_{10}^{(1)} = t_{10}^{(0)}\) and approaches this point through the lower half plane (see Figure 3). Since such a loop is homotopic to \(\ell_{1\cdots 1} \ell_{1\cdots 1}^{-1}\), we obtain (ii).  

![Figure 3. the variation of \(\ell_{1\cdots 1}\).](image)
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