EIGENVALUES OF NON–HERMITIAN RANDOM MATRICES
AND BROWN MEASURE OF NON-NORMAL OPERATORS:
HERMITIAN REDUCTION AND LINEARIZATION METHOD

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ABSTRACT. We study the Brown measure of certain non–Hermitian operators arising from Voiculescu’s free probability theory. Usually those operators appear as the limit in $\star$-moments of certain ensembles of non–Hermitian random matrices, and the Brown measure gives then a canonical candidate for the limit eigenvalue distribution of the random matrices. A prominent class for our operators is given by polynomials in $\star$-free variables. Other explicit examples include $R$–diagonal elements and elliptic elements, for which the Brown measure was already known, and a new class of triangular–elliptic elements. Our method for the calculation of the Brown measure is based on a rigorous mathematical treatment of the Hermitian reduction method, as considered in the physical literature, combined with subordination and the linearization trick.

1. INTRODUCTION

1.1. Eigenvalues of non-Hermitian random matrices. The study of the eigenvalue distribution of non-Hermitian random matrices is regarded as an important and interesting problem, especially in the mathematical physics literature. Unfortunately, most of the methods used for the study of Hermitian random matrices fail in the non-Hermitian case, which makes the latter very difficult.

1.2. Convergence of $\star$–moments. Free probability theory. Usually we are interested in the behavior of the random matrix eigenvalues in the limit as the size of the matrix tends to infinity. It is therefore natural to ask: does a given sequence of random matrices converge in one sense or another to some (infinite-dimensional) object as the size of the matrix tends to infinity? It would be very tempting to study this limit instead of the sequence of random matrices itself.

1991 Mathematics Subject Classification. 15B52 (Primary), 46L54, 46L10 (Secondary).

Key words and phrases. eigenvalues of non-Hermitian random matrices, Brown measure, non-normal operators.
In order to perform this program, we will use the notion of a $W^*$-probability space (which is a von Neumann algebra $\mathcal{A}$ equipped with a tracial, faithful, normal state $\phi : \mathcal{A} \to \mathbb{C}$). The algebra $\mathcal{A}_N = \mathcal{L}^\infty(\Omega, \mathcal{M}_N)$ of $N \times N$ random matrices with all moments finite equipped with a tracial state $\phi_N(x) = \frac{1}{N} \mathbb{E} \text{Tr} x$ for $x \in \mathcal{A}_N$ fits well into this framework (to be very precise: the definition of a von Neumann algebra requires its elements to be bounded which is not the case for the most interesting examples of random matrices, but this small abuse of notation will not cause any problems in the following).

We say that a sequence of random matrices $(A_N)$, where $A_N \in \mathcal{A}_N$, converges in $\ast$-moments to some element $x \in \mathcal{A}$ if for every choice of $s_1, \ldots, s_n \in \{1, \ast\}$ we have
\[
\lim_{N \to \infty} \phi_N(A_{s_1}^N \cdots A_{s_n}^N) = \phi(x^{s_1} \cdots x^{s_n}).
\]

It turns out that many classes of random matrices have a limit in a sense of $\ast$-moments and the limit operator can be found by the means of free probability theory [41, 24, 28].

1.3. Brown measure. The Brown measure [10] is an analogue of the density of eigenvalues for elements of $W^*$-probability spaces. Its great advantage is that it is well-defined not only for self-adjoint or normal operators; furthermore for random matrices it coincides with the mean empirical eigenvalue distribution. We recall the exact definition in Section 2.1.

1.4. The main tools: (i) Hermitian reduction method. In this article we study rigorously the idea of Janik, Nowak, Papp and Zahed [26] which was later used in the papers [15, 13, 14] under the name Hermitian reduction method.

As we shall see in Section 2.1 the Brown measure $\mu_x$ of an element $x$ is closely related to the Cauchy transform $G_x(\lambda) = \phi((x - \lambda)^{-1})$ of $x$. The asymptotic expansion of $G_x(\lambda)$ at infinity is given by the sequence of moments of $x$ and for this reason it can be computed explicitly by the means of free probability theory for many operators $x$. However, given a sequence of moments $\{m_n = \int t^n \, d\mu_x(t)\}_{n \in \mathbb{N}}$, there usually are many probability measures supported in $\mathbb{C}$ which have $\{m_n\}$ as their sequence of moments. In particular, if $x$ is not Hermitian then the series $G_x(\lambda) = \sum_{n=0}^{\infty} \phi(x^n)\lambda^{-n-1}$ alone does not determine the distribution of $x$.

The idea is to arrange the operator $x$ and its Hermitian conjugate $x^\ast$ into a $2 \times 2$ matrix and to consider a matrix–valued Cauchy transform
\[
\mathbf{G}_x(\lambda) = \phi \left( \begin{bmatrix} i\epsilon & \lambda - x^\ast \\ \overline{\lambda - x} & i\epsilon \end{bmatrix} \right)^{-1} \in \mathcal{M}_2(\mathbb{C})
\]
which depends on an additional parameter $\epsilon > 0$. The above function makes sense for all such $\epsilon > 0$; it will be viewed as a restriction of an analytic map defined on a matricial upper half-plane. As we shall see, the limit $\epsilon \to 0$ gives us access to the original Cauchy transform $G_x(\lambda)$ and therefore to the Brown measure of $x$.

This method appears to be extremely simple and indeed its applications in the physics literature involved very short and simple calculations. However, from a mathematical point of view they are often far from being rigorous. In this article we would like to put the Hermitian reduction method on solid ground.

The main ingredient in our approach is a recent progress in [5] on the analytic description of operator-valued free convolutions, relying on the idea of subordination. Roughly speaking, subordination usually yields an analytic description of the relevant equations (say, for the operator-valued Cauchy transforms) which are not only valid in some neighborhood of infinity, but everywhere in the (operator-valued) complex upper half plane. Since recovering the desired distribution relies on the knowledge of the Cauchy transform close to the real axis the analytic description everywhere in the complex upper half plane is crucial for a rigorous treatment. As examples for such explicit calculations we will present a detailed analysis of two interesting classes of non-Hermitian random matrices and the corresponding non-Hermitian operators (namely, so-called $R$-diagonal and elliptic-triangular operators).

1.5. The main tools: (ii) linearization method. It seems that by mimicking the methods presented in this article it should be possible to calculate the Brown measure of virtually any operator described in terms of free probability. As a very general class of such operators we will in particular consider the problem of arbitrary (in general, non-self-adjoint) polynomials in free variables. In the above mentioned paper [5] the corresponding problem for self-adjoint polynomials in self-adjoint free variables was solved by invoking what is known as linearization trick, which allows us to reduce the polynomial problem to an operator-valued linear problem. The same reduction works in the non-self-adjoint case, and we will show how the combination of the Hermitization and the linearization methods will result in an algorithm for calculating the Brown measure of a polynomial in $\star$-free variables out of the $\star$-distributions of its variables; in particular, if the variables are normal then this gives a way to calculate the Brown measure of the polynomial out of the Brown measures of the variables.

We will then also start an investigation on the qualitative features of the simplest polynomial, namely the sum of two $\star$-free operators. We will, in particular, address the question how eigenvalues in the sum can arise
from eigenvalues in summands. This is related (but not equivalent) to the question of atoms in the corresponding Brown measures.

1.6. Overview of this article and statement of results. In Section 2 we recall the definitions of the Brown measure and the Cauchy transform as well as their regularized versions and their connection with the Hermitian reduction method. We also recall some basic tools of Voiculescu’s free probability theory.

In Section 3 we present the linearization trick and show how it combines with the Hermitian reduction method to yield an algorithm for the computation of the Brown measure of polynomials in free variables.

In Section 4 we will then address in more detail polynomials in free variables; in particular, we present analytic properties of the Brown measure of the sum of two $\star$-free variables.

The next two sections will then deal with important special classes of non-normal operators and we will (re)derive explicit expressions for their Brown measures.

In Section 5 we study the class of $R$–diagonal operators which are limits of bi-unitarily invariant random matrices. We calculate by our methods the Brown measure of $R$–diagonal operators and rederive in this way the results of Haagerup and Larsen [20]. That this Brown measure is actually the limit of the eigenvalue distributions of the corresponding random matrix models was recently proved by Guionnet, Krishnapur, and Zeitouni [18]. We also note that results of the present paper are used in [6] to determine the asymptotics of the eigenvector overlap for those random matrices.

In Section 6 we study certain non–Hermitian Gaussian random matrices the entries of which above the diagonal, informally speaking, have a different covariance than the entries below the diagonal. We describe the operators which arise as limits of such random matrices, and we use the Hermitian reduction method to calculate their Brown measures. By a result of Śniady [32] we know that this agrees in this case with the limit distribution of the eigenvalues of these random matrices.

In Section 7 we then briefly discuss the problem of discontinuity of the Brown measure. Roughly speaking, the eigenvalues of non–Hermitian matrices do not depend on the matrix $\star$-moments in a continuous way and therefore the Brown measure of some operator might not be related to the eigenvalues of matrices which converge in $\star$-moments to this operator. However, one expects that for natural choices of random matrices such a convergence should hold.

2. Preliminaries

2.1. Cauchy transform and Brown measure.
2.1.1. **Cauchy transform.** Let $\mu$ be a probability measure on the complex plane $\mathbb{C}$. We define its Cauchy transform as the analytic function

$$G_\mu(\lambda) = \int_{\mathbb{C}} \frac{1}{\lambda - z} d\mu(z)$$

for $\lambda \notin \text{supp} \mu$. It is known \[17\] that the integral $\int_{\mathbb{C}} \frac{1}{\lambda - z} d\mu(z)$ is in fact well-defined everywhere outside a set of $\mathbb{R}^2$-Lebesgue measure zero, and thus $G_\mu$ will be viewed from now on as a function on all of $\mathbb{C}$, whose analyticity will, however, be guaranteed only outside the support of $\mu$. The measure $\mu$ can be extracted from its Cauchy transform by the formula

$$\mu = \frac{1}{\pi} \frac{\partial}{\partial \lambda} G_\mu(\lambda),$$

where as usual

$$\frac{\partial}{\partial \lambda} = \frac{1}{2} \left( \frac{\partial}{\partial (\Re \lambda)} - i \frac{\partial}{\partial (\Im \lambda)} \right), \quad \frac{\partial}{\partial \bar{\lambda}} = \frac{1}{2} \left( \frac{\partial}{\partial (\Re \lambda)} + i \frac{\partial}{\partial (\Im \lambda)} \right)$$

denote the derivatives in the Schwartz distribution sense; thus, (2) should be understood in the distributional sense too.

2.1.2. **Spectral measure of self-adjoint operators.** Let $\mathfrak{A}$ be a von Neumann algebra equipped with a normal faithful tracial state $\phi$. Every self-adjoint operator $x \in \mathfrak{A}$ can be written as a spectral integral

$$x = \int_{\mathbb{R}} \lambda dE(\lambda),$$

where $E$ denotes the operator–valued spectral measure of $x$. It is natural to consider a probability measure $\mu_x$ on $\mathbb{C}$ given by

$$\mu_x(Z) = \phi(E(Z))$$

for any Borel set $Z \subseteq \mathbb{C}$.

In a full analogy with (1) we consider the Cauchy transform of $x$ given by

$$G_x(\lambda) = \phi\left((\lambda - x)^{-1}\right).$$

Then the spectral measure $\mu_x$ as defined by (3) can be recovered from (4) via (2).

2.1.3. **Spectral measure of matrices.** Let $\mathfrak{A} = \mathcal{M}_N$ be the matrix algebra equipped with the tracial state $\phi = \text{tr}$, where $\text{tr} x = \frac{1}{N} \text{Tr} x$ is a normalized trace. Let $\lambda_1, \ldots, \lambda_N \in \mathbb{C}$ be the eigenvalues (counted with multiplicities) of a given matrix $x \in \mathcal{M}_N$. We define $\mu_x$ to be the (normalized) counting measure of the set of eigenvalues

$$\mu_x = \frac{\delta_{\lambda_1} + \cdots + \delta_{\lambda_N}}{N}.$$
The Cauchy transform of $x$ defined by (4) is well-defined on the set $\mathbb{C} \setminus \{\lambda_1, \ldots, \lambda_N\}$ and it again satisfies (2).

2.1.4. Empirical eigenvalue distribution. Let $\mathfrak{A} = \mathcal{L}^\infty(\Omega, \mathcal{M}_N)$ be the algebra of random matrices having all moments finite. We equip it with a state $\phi(x) = \mathbb{E} \text{tr} x$. For $x \in \mathfrak{A}$ consider a random variable $\Omega \ni \omega \mapsto \mu_x(\omega)$, called empirical eigenvalue distribution, the values of which are probability measures on $\mathbb{C}$ (where $\mu_x(\omega)$ is to be understood as in Section 2.1.3). We define the mean eigenvalue distribution $\mu_x$ by

\begin{equation}
\mu_x = \mathbb{E} \mu(\omega(x)).
\end{equation}

2.1.5. Brown measure. Let $\mathfrak{A}$ be a von Neumann algebra equipped with a normal faithful tracial state $\phi$. Inspired by the above examples we might try to define the Cauchy transform of $x \in \mathfrak{A}$ by the formula (4) and then define its spectral measure $\mu_x$ by (2). However, in general this is not possible because formula (4) requires $\lambda$ to lie outside of the spectrum of $x$. In the non-Hermitian case, the spectrum might be a large set; in fact, it can be an arbitrary compact subset of $\mathbb{C}$. For such arbitrary subsets of $\mathbb{C}$, the moment problem is not well-defined. Thus, knowing the Cauchy transform on the resolvent set of $x$ might very well not be sufficient. For this reason we need a more elaborate definition of the spectral measure.

The Fuglede–Kadison determinant $\Delta(x)$ of $x \in \mathfrak{A}$ is defined in [16] by

$$\log \Delta(x) = \frac{1}{2} \phi\left( \log(x x^*) \right).$$

If $x$ is not invertible, the above definition should be understood as $\Delta(x) = \lim_{\epsilon \to 0} \Delta_\epsilon(x)$, where $\Delta_\epsilon$ denotes the regularized Fuglede–Kadison determinant

$$\log \Delta_\epsilon(x) = \frac{1}{2} \phi\left( \log(x x^* + \epsilon^2) \right)$$

for $\epsilon > 0$.

The Brown measure of $x \in \mathfrak{A}$ is defined in [10] by

\begin{equation}
\mu_x = \frac{1}{2\pi} \left( \frac{\partial^2}{\partial(\Re \lambda)^2} + \frac{\partial^2}{\partial(\Im \lambda)^2} \right) \log \Delta(x - \lambda) = \frac{2}{\pi} \frac{\partial}{\partial \lambda} \frac{\partial}{\partial \bar{\lambda}} \log \Delta(x - \lambda).
\end{equation}

The Brown measure $\mu_x$ as defined in (7) could be a priori a Schwartz distribution but one can show that the map $\lambda \mapsto \log \Delta(x - \lambda)$ is subharmonic and hence $\mu_x$ is a positive measure on $\mathbb{C}$. In fact $\mu_x$ is a probability measure supported on a subset of the spectrum of $x$. One can show that for the examples from Sections 2.1.2, 2.1.4 the above definition gives the correct values for (3), (5), and (6).
Following (1) we define the Cauchy transform of x as

\[ G_x(\lambda) = \int_{\mathbb{C}} \frac{1}{\lambda - z} d\mu_x(z). \]

2.1.6. Regularized Cauchy transform and regularized Brown measure. For every \( \epsilon > 0 \) the regularized Cauchy transform

\[ G_{\epsilon,x}(\lambda) = \phi((\lambda - x)^* ((\lambda - x)(\lambda - x)^* + \epsilon^2)^{-1}) \]

is well-defined for every \( \lambda \in \mathbb{C} \), but is not an analytic function. In fact it was shown by Larsen [27] (see also [2, Lemma 4.2]) and can be verified through direct arithmetic (here it is essential to remember that \( \phi \) is tracial!), that

\[ G_{\epsilon,x}(\lambda) = 2 \frac{\partial}{\partial \lambda} \log \Delta_{\epsilon}(x - \lambda). \]

The function \( \lambda \mapsto \log \Delta_{\epsilon}(x - \lambda) \) is subharmonic, hence the regularized Brown measure defined by

\[ \mu_{\epsilon,x} = \frac{1}{\pi} \frac{\partial}{\partial \lambda} G_{\epsilon,x}(\lambda) = \frac{2}{\pi} \frac{\partial}{\partial \lambda} \frac{\partial}{\partial \bar{\lambda}} \log \Delta_{\epsilon}(x - \lambda) \]

is a positive measure on the complex plane. Integration by parts shows that for \( \epsilon \to 0 \) the regularized Brown measure \( \mu_{\epsilon,x} \) converges (in the weak topology of probability measures) towards the Brown measure \( \mu_x \) as defined by (7). The comparison of (10) and (2) is a heuristic argument that the definition of the Brown measure is a reasonable extension of the cases from Sections 2.1.2–2.1.4.

We should probably point the reader to the fact that the measure \( \mu_{\epsilon,x} \) has full support. Indeed, if \( \lambda = u + iv \), then,

\[ \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) \log \Delta_{\epsilon}(x - \lambda) = \phi \left( \left[ 2 - (\lambda - x) ((\lambda - x)(\lambda - x)^* + \epsilon^2)^{-1} (\lambda - x)^* \right. \\
- (\lambda - x)^* ((\lambda - x)(\lambda - x)^* + \epsilon^2)^{-1} (\lambda - x) \right. \\
\times \left. ((\lambda - x)(\lambda - x)^* + \epsilon^2)^{-1} \right) . \]

For \( \epsilon = 0 \) and \( \lambda \) outside the spectrum of \( x \), it follows straightforwardly that the term on the third row above (second row in the expression on the right of =) is equal to one. Grouping the first row with the last and applying traciality of \( \phi \) allows us to conclude that \( \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) \log \Delta(x - \lambda) = 0 \) for \( \lambda - x \) invertible. But it makes equally clear that, since in general \( x^* xx^* + \epsilon^2 x^* x \leq 1 \), with no equality for \( \epsilon \neq 0 \), the above is always positive when \( \epsilon > 0 \) (we have used here also the faithfulness of the trace).
Equation (10) implies that

\[ G_{\epsilon,x}(\lambda) = \int_{\mathbb{C}} \frac{1}{\lambda - z} d\mu_{\epsilon,x}(z), \]

with the equality defined a priori only \( \mathbb{R}^2 \)-Lebesgue almost everywhere, but extended by continuity to all \( \lambda \in \mathbb{C} \). Since for \( \epsilon \to 0 \) the measures \( \mu_{\epsilon,x} \) converge weakly to \( \mu_x \), (8) and (11) imply that the regularized Cauchy transforms \( G_{\epsilon,x} \) converge to \( G_x \) in the local \( L^1 \) norms; in particular \( G_{\epsilon,x}(\lambda) \to G_x(\lambda) \) for almost all \( \lambda \in \mathbb{C} \). It should be mentioned that in fact the limit

\[ \lim_{\epsilon \to 0} G_{\epsilon,x}(\lambda) = G_x(\lambda) \in \mathbb{C} \]

exists for all \( \lambda \in \mathbb{C} \) for which

\[ \lim_{\epsilon \to 0} \phi \left( \left( (\lambda - x)(\lambda - x)^* + \epsilon^2 \right)^{-1} \right) < \infty \]

(this limit always exists and is strictly positive, unless \( x \) is a multiple of the identity, but may very well be infinite). Unfortunately, finiteness of the limit can usually only be guaranteed for \( \lambda \) outside the spectrum of \( x \). Indeed, since

\[ G_{\epsilon,x}(\lambda) = \phi \left( (\lambda - x)^* \left( (\lambda - x)(\lambda - x)^* + \epsilon^2 \right)^{-1} \right), \]

we consider the decomposition of \( (\lambda - x)^* \) into four positive operators. For any operator \( v \geq 0 \),

\[ 0 \leq \left( (\lambda - x)(\lambda - x)^* + \epsilon^2 \right)^{-1/2} v \left( (\lambda - x)(\lambda - x)^* + \epsilon^2 \right)^{-1/2} \]
\[ \leq \|v\| \left( (\lambda - x)(\lambda - x)^* + \epsilon^2 \right)^{-1}. \]

Applying \( \phi \) to the above inequalities and the monotonicity of the correspondence \( \epsilon \mapsto (\lambda - x)(\lambda - x)^* + \epsilon^2)^{-1} \) allows us to conclude.

2.2. Hermitian reduction method. Following the idea of the Janik, Nowak, Papp, Zahed [26], for fixed \( x \in \mathfrak{A} \) let

\[ X = \begin{bmatrix} 0 & x \\ x^* & 0 \end{bmatrix} \in \mathcal{M}_2(\mathfrak{A}). \]

This is trivially a self-adjoint element in \( \mathcal{M}_2(\mathfrak{A}) \). We equip the algebra \( \mathcal{M}_2(\mathfrak{A}) \) with a positive conditional expectation \( \mathbb{E} : \mathcal{M}_2(\mathfrak{A}) \to \mathcal{M}_2(\mathbb{C}) \) given by

\[ \mathbb{E} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \phi(a_{11}) & \phi(a_{12}) \\ \phi(a_{21}) & \phi(a_{22}) \end{bmatrix}. \]
Following [40], we can define a fully matricial $\mathcal{M}_2(\mathbb{C})$-valued Cauchy-Stieltjes transform: for any $b \in \mathcal{M}_2(\mathbb{C})$ which satisfies the condition that $\Im b := (b - b^*)/2i > 0$, the map

$$G_X(b) = \mathbb{E} \left[ (b - X)^{-1} \right]$$

(15)

is well defined and analytic on the set of elements $b$ for which $\Im b > 0$. In particular, for $\epsilon > 0$ the element

$$\Lambda_{\epsilon} = \begin{bmatrix} i\epsilon & \lambda \\ \lambda & i\epsilon \end{bmatrix} \in \mathcal{M}_2(\mathbb{C})$$

belongs to the domain of $G_X$, and

$$G_{\epsilon}(\lambda) = G_X(\Lambda_{\epsilon}) = \begin{bmatrix} g_{\epsilon,11} & g_{\epsilon,12} \\ g_{\epsilon,21} & g_{\epsilon,22} \end{bmatrix} = \mathbb{E}((\Lambda_{\epsilon} - X)^{-1}).$$

(16)

Note that the element $\Lambda_0 - X$ is self-adjoint, and for this reason $\Im \Lambda_{\epsilon} = \epsilon 1$, making the element $\Lambda_{\epsilon} - X$ invertible whenever $\epsilon \neq 0$, guarantees that (16) makes sense. One can easily check that

$$G_{\epsilon}(\lambda) = G_X(\Lambda_{\epsilon}) = \begin{bmatrix} -i\epsilon((\lambda - x)(\lambda - x)^* + \epsilon^2)^{-1} & (\lambda - x)((\lambda - x)^*(\lambda - x) + \epsilon^2)^{-1} \\ (\lambda - x)^*((\lambda - x)(\lambda - x)^* + \epsilon^2)^{-1} & -i\epsilon((\lambda - x)^*(\lambda - x) + \epsilon^2)^{-1} \end{bmatrix}.$$

(17)

Equations (9) and (17) show that two of the entries of $G_{\epsilon}(\lambda)$ carry important information, namely they coincide with the regularized Cauchy transform, and its adjoint, respectively:

$$g_{\epsilon,21}(\lambda) = g_{\epsilon,22}(\lambda) = G_{\epsilon,x}(\lambda).$$

It is therefore very tempting to ask what kind of information is being carried by the other two entries $g_{\epsilon,11}(\lambda) = g_{\epsilon,21}(\lambda)$. It was shown by Janik et al. [25] that if $x$ is a random matrix then in the limit $\epsilon \to 0$ these two entries provide information about the interplay between the bases of the left and the right eigenvectors.

More generally, we record for future use that

$$\begin{bmatrix} a & b - x \\ c - x^* & d \end{bmatrix}^{-1} = \begin{bmatrix} -d[(b - x)(c - x^*) - ad]^{-1} & (b - x)[(c - x^*)(b - x) - ad]^{-1} \\ (c - x^*)[(b - x)(c - x^*) - ad]^{-1} & -a[(c - x^*)(b - x) - ad]^{-1} \end{bmatrix}.$$

(18)

The reader will probably find a great disconnect between the above and the linearization procedure described below. Indeed, the correspondence described above $(\lambda, \epsilon) \mapsto G_X(\Lambda_{\epsilon})$ has the significant disadvantage of being profoundly non-analytic. Below we will work only with analytic maps,
hence our argument will never be \( \Lambda_c \). However, as we are free to evaluate the analytic map \( f(z, w) = zw \) in the numbers \((z, \overline{z})\), we shall take the liberty of evaluating certain analytic functions (like \( G_X \)) in the matrix \( \Lambda_c \), without viewing it as an argument in two variables. However, for \( b = \lambda, c = \overline{\lambda} \) fixed, we can consider the analytic map \((z, w) \mapsto G_X \left( \begin{bmatrix} z & \lambda \\ \overline{\lambda} & w \end{bmatrix} \right) \). A simple calculation shows that the matrix \( \Im \begin{bmatrix} z & \lambda \\ \overline{\lambda} & w \end{bmatrix} \) is strictly positive if and only if \( \Im z > 0 \) and \( \Im z \Im w > |\lambda - \overline{\lambda}|^2 = 0 \), i.e. if and only if \( z \) and \( w \) are in the upper half plane \( \mathbb{C}^+ \) of \( \mathbb{C} \).

Since \( G_X \) maps the matricial upper half-plane into the matricial lower half-plane, \( F_X(b) := G_X(b)^{-1} \) is well defined and \( \Im F_X(b) \geq \Im b \). In the case that \( x \) is not a multiple of the identity, this inequality is in fact strict.

One can show [5, Proposition 2.15] that the strict inequality \( \Im F_X(b) > \Im b \) can fail only when there exists a non-zero projection \( p \in \mathcal{M}_2(\mathbb{C}) \) so that \( pX = p \mathbb{E}[X] \). Since nontrivial projections in \( \mathcal{M}_2(\mathbb{C}) \) are necessarily of the form
\[
\begin{bmatrix}
\alpha & j \sqrt{\alpha(1-\alpha)} \\
-j \sqrt{\alpha(1-\alpha)} & 1 - \alpha
\end{bmatrix},
\]
this would require that both of \( \alpha x = \alpha \phi(x) \) and \( (1 - \alpha)x^* = (1 - \alpha) \overline{\phi(x)} \), as equality of operators, hold. This contradicts the assumption \( x \) is not a multiple of the identity.

3. Linearization

3.1. \( \star \)-distribution of \( x \) out of \( G_X @ 1_n \). In the previous section we have shown that \( G_X \) includes all the information about the Brown measure of \( x \). But we will be interested in knowing the Brown measure of any polynomial \( P(x_1, \ldots, x_k) \) in \( \star \)-free random variables \( x_1, \ldots, x_k \) in terms of the Brown measures (maybe less than full \( \star \)-distributions) of the individual random variables. In order to be able to encapsulate all that information and efficiently manipulate it, we will use what is called the fully matricial extension [40] of \( G_X \):
\[
G_{X @ 1_n}(b) = (\mathbb{E} \otimes \text{Id}_{1_n}) \left[ (b - X \otimes 1_n)^{-1} \right],
\]
where \( b \in \mathcal{M}_n(\mathcal{M}_2(\mathbb{C})) \) is so that \( b - X \otimes 1_n \) is invertible (in particular, this holds true if \( \Im b > 0 \)). It is known [40] that \( G_{X @ 1_n}, n \in \mathbb{N} \) encodes all \( \mathcal{M}_2(\mathbb{C}) \)-moments of \( X \), and hence all \( \star \)-moments of \( x \).

3.2. Linearization. The work in [5] introduces an iterative method to compute spectra of self-adjoint polynomials in free variables. This is based on a linearization trick introduced in [1]. (Versions of this linearization trick
have a long history in different settings, see [9, 22, 21]. In this subsection we shall show how the Hermitian reduction method combines with the linearization trick to allow for the computation of the Brown measure of any polynomial in free random variables.

To begin with, let us linearize an arbitrary monomial: assume we desire to compute the Brown measure of \(x_1 x_2 \cdots x_k\). We will assume that any two neighbouring elements are \(*\)-free from each other. However, it is not necessary that all \(x_1, x_2, \ldots, x_k\) are free. The Hermitian reduction requires us to build

\[
\begin{bmatrix}
\lambda & \lambda - x_1 x_2 \cdots x_k \\
-i\epsilon & i\epsilon
\end{bmatrix}^{-1}.
\]

In order to be able to use the freeness of the elements involved, we will have to separate them in sums of (possibly quite large) matrices. Observe first that

\[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
0 & x_2 \\
x_2^* & 0
\end{bmatrix}
\begin{bmatrix}
0 & x_1^* \\
x_1 & 0
\end{bmatrix}
= \begin{bmatrix}
0 & x_2 x_1 \\
x_2^* x_1^* & 0
\end{bmatrix}.
\]

Thus, by induction, if we have a matrix

\[
\begin{bmatrix}
0 & x_2 \cdots x_k \\
x_2^* \cdots x_k^* & 0
\end{bmatrix},
\]

we will obtain

\[
\begin{bmatrix}
0 & x_1 x_2 \cdots x_k \\
(x_1 x_2 \cdots x_k)^* & 0
\end{bmatrix}
\]

as the product

\[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
0 & x_2 \cdots x_k \\
x_2^* \cdots x_k^* & 0
\end{bmatrix}
\begin{bmatrix}
0 & x_1^* \\
x_1 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}.
\]

For the linearization trick, it would however be convenient to write this product explicitly: if \(X_j = \begin{bmatrix} 0 & x_j \\ x_j^* & 0 \end{bmatrix}, \tilde{X}_j = \begin{bmatrix} 0 & 1 \\ x_j & 0 \end{bmatrix}\), and \(\mathcal{J} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\), then

\[
\begin{bmatrix}
0 & x_1 \cdots x_k \\
(x_1 \cdots x_k)^* & 0
\end{bmatrix}
= \mathcal{J} \tilde{X}_1 \cdots \mathcal{J} \tilde{X}_{k-2} \mathcal{J} \tilde{X}_{k-1} X_k \tilde{X}_{k-1}^* \mathcal{J} \tilde{X}_{k-2}^* \mathcal{J} \cdots \tilde{X}_1^* \mathcal{J}.
\]

Now we shall linearize the right-hand monomial as a separate entity over \(\mathcal{M}_2(\mathbb{C})\). For simplicity, we shall denote \(Y_j = \mathcal{J} \tilde{X}_j\). The linearization of
$Y_1Y_2\cdots Y_{k-1}X_kY_{k-1}^*\cdots Y_2^*Y_1^*$ is then performed by the matrix

$$
X := \begin{bmatrix}
Y_1 & & & \\
& Y_2 & \cdots & \\
& \ddots & \ddots & \\
& & \ddots & \ddots \\
& & & Y_k \\
& & & \ddots \\
& & & & Y_2^* \\
& & & & \ddots \\
& & & & & Y_1^*
\end{bmatrix}.
$$

This is a $(4k-2) \times (4k-2)$ matrix, and the entries shown are $2 \times 2$ matrices, with $1_2$ being the $2 \times 2$ identity matrix. All empty spaces correspond to zero entries.

Let

$$b_1 = b \otimes e_{1,1} = \begin{bmatrix}
b & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{bmatrix}.$$

Then the matrix

$$b_1 - X = \begin{bmatrix}
b & & & Y_1 \\
& & Y_2 & \cdots & \\
& \ddots & \ddots & \ddots & \\
& & \ddots & \ddots & \ddots \\
& & & \ddots & \ddots \\
& & & & Y_2^* \\
& & & & \ddots \\
& & & & & Y_1^*
\end{bmatrix}$$

is invertible, and

$$(b_1 - X)^{-1} = \begin{bmatrix}
(b - Y_1Y_2\cdots Y_{k-1}X_kY_{k-1}^*\cdots Y_2^*Y_1^*)^{-1} & * & \cdots & * \\
* & * & \ddots & * \\
\cdots & \cdots & \ddots & \cdots \\
* & * & \cdots & *
\end{bmatrix},$$

where $*$ represents some unspecified entries. Putting

$$b = \begin{bmatrix} i\epsilon & \lambda \\ \overline{\lambda} & i\epsilon \end{bmatrix}^{-1}$$

in the last equation and picking then from $(b_1 - X)^{-1}$ the $(1, 1), (1, 2), (2, 1), (2, 2)$ entries will provide (19), as desired. Note that if $P = \sum_j q_j x_{i_1,j}x_{i_2,j}\cdots x_{i_k(j),j}$,
then

\[
\begin{bmatrix}
0 & P \\
P^* & 0
\end{bmatrix} = \sum_j \begin{bmatrix}
x_{i_1,j}x_{i_2,j} \cdots (q_j x_{i_k,j})^* \\
x_{i_1,j}x_{i_2,j} \cdots (q_j x_{i_k,j})
\end{bmatrix}.
\]

In order to avoid having to deal with the scalars when writing the linearization matrix, we will “merge” them into \(x_{i_k,j}\), so that now \(X_{k(j)} = \begin{bmatrix} 0 & q_j x_{i_k,j}^* \\ q_j x_{i_k,j} & 0 \end{bmatrix}\). As shown in the proof of [5, Proposition 3.4], the linearization of \(\begin{bmatrix} 0 & P \\ P^* & 0 \end{bmatrix}\) will then be obtained simply by “stacking” the corresponding linearizations of the (it should be noted!) self-adjoint 2 \(\times\) 2-matrix monomials: we shall denote \(u_j = (0, \ldots, 0, Y_{i_1,j})\) and \(Y_j\) the matrix

\[
Y_j = \begin{bmatrix}
Y_{i_2,j} & -1_2 \\
X_{i_k(j),j} & -1_2 \\
\vdots & \vdots \\
Y_{i_2,j} & -1_2 \\
-1_2 & \end{bmatrix},
\]

so that \(X_j\) is obtained from \(Y_j\) by adding one first row \(u_j\) and one first column \(u_j^*\):

\[
X_j = \begin{bmatrix} 0 & u_j \\ u_j^* & Y_j \end{bmatrix}.
\]

(The reader is warned to remember that \(u_j\) is not properly speaking a row, but two: it is a 2 \(\times\) \((4k(j) - 4)\) matrix and 0 in the upper right corner of \(X_j\) is the 2 \(\times\) 2 zero matrix.) Then [5, Proposition 3.4] informs us that the linearization of \(\begin{bmatrix} 0 & P \\ P^* & 0 \end{bmatrix}\) is the matrix

\[
\mathbb{L}_P = \begin{bmatrix} 0 & u_1 & u_2 & \cdots & u_k \\ u_1^* & Y_1 & \cdots & \cdots \\ u_2^* & \cdots & Y_2 & \cdots \\ \vdots & & \vdots & & \vdots \\ u_k^* & \cdots & \cdots & \cdots & \end{bmatrix}.
\]
To conclude,

\[(b_1 - \mathbb{1}_P)^{-1} = \left( b - \begin{bmatrix} 0 & P \\ P^* & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} * & \cdots & * \\ * & \cdots & * \\ \vdots & \vdots & \vdots \\ * & \cdots & * \end{bmatrix} \].

Here we should again remember that the only condition required is that

\[ b - \begin{bmatrix} 0 & P \\ P^* & 0 \end{bmatrix} \]

is invertible, so, since \[ \begin{bmatrix} 0 & P \\ P^* & 0 \end{bmatrix} \] is self-adjoint, the requirement that \( \Re b > 0 \) will do.

In order to apply the iteration procedure from [5], we only need now to split \( \mathbb{1}_P \) into a sum in which elements coming from one algebra are grouped in one matrix. Since self-adjointness is preserved by this procedure, the subordination result of [5] applies.

4. Brown measure of polynomials in free variables

4.1. Identifying the Brown measure. The linearization procedure described above guarantees that the Brown measure of a polynomial \( P \) in free variables can be expressed in terms of the \( \star \)-distributions of its variables in an explicit manner. However, we would like to emphasize that the knowledge of a significant part of the \( \star \)-distributions of the variables in question is needed: we cannot hope to obtain in general the Brown measure of \( P \) out of the Brown measures of its variables. The knowledge of these \( \star \)-distributions is however guaranteed, as noted in Section 3.1, by the knowledge of the \( G_{X \otimes 1_n}, n \in \mathbb{N} \). As one can see easily, it is not in fact necessary to know \( G_{X \otimes 1_n} \) for all \( n \in \mathbb{N} \), but just up to a certain \( n_0 \) depending on the degree of the polynomial \( P \).

An important special case is of course when all the variables \( x_i \) are normal (for example, self-adjoint or unitary): in this case the \( \star \)-distribution of \( x_i \) is determined in terms of its Brown measure (which is now nothing but the trace applied to the spectral distribution according to the spectral theorem). Thus our linearization procedure gives us, in particular, a way to calculate the Brown measure of any polynomial in free self-adjoint variables out of the distribution of the variables.

Having provided the general machinery for dealing with Brown measures there are now various obvious questions to address:

- Are there special cases where we can derive explicit solutions?
- How can we implement our algorithm to calculate numerically Brown measures for general polynomials? Can we control the speed or accuracy of these calculations?
• Can we derive qualitative analytic features of the Brown measures?

The second question, on numerical implementation, will be addressed somewhere else (see [36, 23] for some preliminary results); here, we want to concentrate on the more analytic questions and show how we can indeed get some quite non-trivial statements out of our general method.

As already in the self-adjoint case, there are actually not many non-trivial cases which allow an explicit description of Brown measures. One prominent example where one has indeed some explicit analytic formula is the case of \( R \)-diagonal elements. The Brown measure for those was calculated by Haagerup and Larsen in [20]. We will show in the next section how their formula can be rederived in our framework. In Section 5 we will address another situation where an explicit calculation is possible. There we will consider elliptic triangular operators, which describe the limit of special Gaussian random matrix models. Only special cases of this were known before.

In this section, however, we want to start the analytic investigation of the simplest polynomial, namely the sum of two variables. Thus we want to address the question: what can we say about the Brown measure of \( x + y \) where \( x \) and \( y \) are \(*\)-free, given the \(*\)-distribution of \( x \) and the \(*\)-distribution of \( y \).

In the case where \( x \) and \( y \) are self-adjoint this is one of the first fundamental questions which has been treated quite exhaustively in free probability theory, with a long list of contributions, see for example [8]. One should note that already in the case where \( x = a \) and \( y = ib \), with \( a \) and \( b \) self-adjoint (thus we are asking for the Brown measure of \( a + ib \) where the real part \( a \) and the imaginary part \( b \) are free) there have been up to now no general results on the Brown measure.

4.2. Brown measure of the sum of two \(*\)-free variables. The Hermitization and linearization method from the last section show that the treatment of an arbitrary polynomial in two \(*\)-free variables requires the analysis of the matrix \( \mathbb{L}_P \) from (20), split into the sum of the two terms corresponding to the two free variables.

At the moment we are not able to analyze the analytic features of this general framework; what we can and will do here is to treat in some detail the case where \( k = 1 \), i.e., the case of a linear polynomial, corresponding to the Brown measure of the sum of two \(*\)-free random variables \( x \) and \( y \). Note that in this situation there is no need for a linearization and we just have to understand the \( \mathcal{M}_2(\mathbb{C}) \)-valued distribution of

\[
\begin{bmatrix}
0 & x + y \\
(x + y)^* & 0
\end{bmatrix}
= X + Y =
\begin{bmatrix}
0 & x \\
x^* & 0
\end{bmatrix}
+ 
\begin{bmatrix}
0 & y \\
y^* & 0
\end{bmatrix}
\]
in terms of the $\mathcal{M}_2(\mathbb{C})$-valued distributions of $X$ and of $Y$. Note that $X$ and $Y$ are free over $\mathcal{M}_2(\mathbb{C})$

Let us first remind the reader of the result from [40,5] related to subordination: there exist two analytic self-maps of the upper half-plane of $\mathcal{M}_2(\mathbb{C})$, called $\omega_1, \omega_2$, so that

$$\omega_1(b) + \omega_2(b) - b)^{-1} = G_X(\omega_1(b)) = G_Y(\omega_2(b)) = G_{X+Y}(b),$$

for all $b \in \mathcal{M}_2(\mathbb{C})$ with $\exists b > 0$. We shall be concerned with a special type of $b$, namely $b = \left[ \begin{array}{cc} z & \lambda \\ \bar{z} & w \end{array} \right]$. As mentioned before, while the correspondences in $z$ and $w$ are analytic, the correspondence in $\lambda$ is not, so we shall view it as a parameter, on which however there is a continuous correspondence when $z$ and $w$ in the upper half plane of $\mathbb{C}$ are fixed. Thus, we denote below $\omega_j(b), G(b), F(b)$ by $\omega_j(z, w), G(z, w), F(z, w)$, sometimes adding the parameter $\lambda$, when it becomes important in terms of our analysis (it will usually be fixed).

For a fixed $\lambda \in \mathbb{C}$, we are interested in the behavior of $G_{X+Y}(z, w)$ close to $z = w = 0$. For our purposes, it will be enough to consider the case $z = w$ and view the functions involved as single-variable holomorphic functions. For obvious practical purposes, we would like to argue that $\lim_{\epsilon \to 0} G_{X+Y}(i\epsilon, i\epsilon)$ exists for all $\lambda \in \mathbb{C}$. Sadly, this is, quite trivially, not true. We shall analyze this problem in several steps, obtaining along the way side results which we believe interesting in their own right.

4.2.1. The question of left and right invariant projections. We remind the reader of some general facts about elements in finite von Neumann algebras (i.e., when we have a faithful normal trace) (see [34]).

- If $a \in \mathfrak{A}$ is an arbitrary, non-Hermitian, element, and $p = p^* = p^2 \in \mathfrak{A} \setminus \{0\}$ is a projection so that $ap = \lambda p$, then $p \leq \ker((a-\lambda)^*(a-\lambda))$ (we denote here by $\ker(b)$ both the (closed) space on which $b$ is zero and the orthogonal projection onto this space). Indeed, recalling that $\phi$ is a faithful trace and $p(a-\lambda)^*(a-\lambda)p \geq 0$,

$$0 = \phi((a-\lambda)^*(a-\lambda)p) = \phi(p(a-\lambda)^*(a-\lambda)p),$$

implies that $\ker((a-\lambda)^*(a-\lambda)) \geq p$. Similarly, if $p \leq \ker((a-\lambda)^*(a-\lambda))$, then by the above equality $(a-\lambda)p = 0$.

- If $a = v(a^*a)^{1/2} = v|a|$ is the polar decomposition of $a$, then the partial isometry $v$ can be completed to a unitary operator in the von Neumann algebra generated by $a$.

- If $p = \ker(a-\lambda)$, then there exists $q = q^* = q^2 \in \mathfrak{A}$ such that $\phi(p) = \phi(q)$ and $qa = \lambda q$. Indeed, by replacing $a$ with $a - \lambda$, we
may assume that $\lambda = 0$. We assume that the partial isometry in the polar decomposition of $a$ is (completed to) a unitary operator. Then

$$ap = 0 \implies |a|p = 0 \text{ and } p|a| = 0.$$  

In particular, if we let $q = vvp^*$, then $qa = vvp^*a = vvp^*v|a| = v|a| = 0$. Since $q^2 = vvp^*vvp^* = vvp^* = q$ and $\phi(q) = \phi(vvp^*) = \phi(v^*v) = \phi(p)$, we conclude that $q$ is a projection equivalent to $p$.

• Thus, if $p = \ker(a - \lambda), a = v|a|, q = vvp^*$, then

\begin{equation} \tag{22} \label{eq:22}
ap = \lambda p, \quad qa = \lambda q, \quad pa^* = \overline{\lambda}p, \quad a^*q = \overline{\lambda}q. \end{equation}

Now we are in the position to carry on almost to the letter the analysis from $[8]$. We shall denote $G_{X + Y}(i\epsilon, i\epsilon)$ simply by $G_{X + Y}(i\epsilon)$, and similarly for the other functions involved. We have

$$\lim_{\epsilon \downarrow 0} G_{X + Y}(i\epsilon) = \begin{bmatrix} \lim_{\epsilon \downarrow 0} \text{g}_{X + Y,11}(i\epsilon) & \lim_{\epsilon \downarrow 0} \text{g}_{X + Y,12}(i\epsilon) \\ \lim_{\epsilon \downarrow 0} \text{g}_{X + Y,21}(i\epsilon) & \lim_{\epsilon \downarrow 0} \text{g}_{X + Y,22}(i\epsilon) \end{bmatrix},$$

where we remind the reader that

$$\text{g}_{X + Y,11}(i\epsilon) = -i\epsilon\phi\left(\left((\lambda - x - y)(\lambda - x - y)^* + \epsilon^2\right)^{-1}\right)$$

$$\text{g}_{X + Y,12}(i\epsilon) = \phi\left((\lambda - x - y)((\lambda - x - y)^*(\lambda - x - y) + \epsilon^2)^{-1}\right)$$

$$\text{g}_{X + Y,21}(i\epsilon) = \phi\left((\lambda - x - y)^*((\lambda - x - y)(\lambda - x - y)^* + \epsilon^2)^{-1}\right)$$

$$\text{g}_{X + Y,22}(i\epsilon) = -i\epsilon\phi\left(\left((\lambda - x - y)^*(\lambda - x - y) + \epsilon^2\right)^{-1}\right).$$

Since $\phi$ is a trace, $\text{g}_{X + Y,11}(i\epsilon) = \text{g}_{X + Y,22}(i\epsilon) \in -i(0, +\infty)$ and $\text{g}_{X + Y,12}(i\epsilon) = \text{g}_{X + Y,21}(i\epsilon)$. Clearly, the same holds if $X + Y$ is replaced by $X$ or $Y$. We assume that $0 < p = \ker(\lambda - x - y) < 1$, where $x, y \in \mathcal{A}\setminus \mathbb{C}$ are $\ast$-free. Recall that the last hypothesis makes $\mathcal{M}_2(\mathbb{C} \langle x^* \rangle)$ and $\mathcal{M}_2(\mathbb{C} \langle y, y^* \rangle)$ free over $\mathcal{M}_2(\mathbb{C})$. For convenience, we shall temporarily denote $a = \lambda - x - y$, so that $ap = a^*ap = 0$ and

$$\text{g}_{X + Y,11}(i\epsilon) = -i\epsilon\phi\left((aa^* + \epsilon^2)^{-1}\right), \quad \text{g}_{X + Y,12}(i\epsilon) = \phi\left(a(a^*a + \epsilon^2)^{-1}\right),$$

$$\text{g}_{X + Y,21}(i\epsilon) = -i\epsilon\phi\left((a^*a + \epsilon^2)^{-1}\right), \quad \text{g}_{X + Y,22}(i\epsilon) = \phi\left(a^*(aa^* + \epsilon^2)^{-1}\right).$$

The weak limits

$$\lim_{\epsilon \downarrow 0} i\epsilon \left(-i\epsilon(a^*a + \epsilon^2)^{-1}\right) = \lim_{\epsilon \downarrow 0} \epsilon^2(a^*a + \epsilon^2)^{-1}$$

$$= 1 - \lim_{\epsilon \downarrow 0} a^*a(a^*a + \epsilon^2)^{-1}$$

$$= \ker(a^*a)$$

$$= p,$$
and
\[
\lim_{\epsilon \downarrow 0} i\epsilon (a^*a + \epsilon^2)^{-1} = \lim_{\epsilon \downarrow 0} \epsilon^2 (a^*a + \epsilon^2)^{-1} = \ker(a^*) = q
\]
hold, so that in particular
\[
\lim_{\epsilon \downarrow 0} i\epsilon g_{X+Y,11}(i\epsilon) = \lim_{\epsilon \downarrow 0} i\epsilon g_{X+Y,22}(i\epsilon) = \phi(p) = \phi(q).
\]
For the $(1, 2)$ and $(2, 1)$ entries, the situation is slightly more delicate: for the polar decomposition $a = v(a^*a)^{\frac{1}{2}}$, using the generalized Schwarz inequality $|\phi(x^*y)|^2 \leq \phi(x^*x)\phi(y^*y)$ applied to $x = v$ and $y = (a^*a)^{\frac{1}{2}}(a^*a + \epsilon^2)^{-1}$, we write
\[
|\phi(a(a^*a + \epsilon^2)^{-1})| \leq |\phi(1)|^{\frac{1}{2}} \left[|\phi((a^*a(a^*a + \epsilon^2)^{-1})|^{\frac{1}{2}}.
\]
Since $2e^2(t + \epsilon^2)^{-2} < 1$ for all $t \geq 0$, we obtain by applying continuous functional calculus to $\epsilon^2 a^*a(a^*a + \epsilon^2)^{-2}$ and by the positivity of $\phi$ that
\[
|\epsilon\phi(a(a^*a + \epsilon^2)^{-1})| \leq |\phi(1)|^{\frac{1}{2}} \left[|\phi(\epsilon^2 a^*a(a^*a + \epsilon^2)^{-2})|^{\frac{1}{2}} < 1.
\]
On the other hand, observe that $\lim_{\epsilon \downarrow 0} \epsilon^2 t(t + \epsilon^2)^{-2} = 0$ pointwise for $t \in [0, +\infty)$, so, if $\theta$ denotes the distribution of $a^*a$ with respect to $\phi$, then by dominated convergence we obtain
\[
\lim_{\epsilon \downarrow 0} \phi(\epsilon^2 a^*a(a^*a + \epsilon^2)^{-2}) = \lim_{\epsilon \downarrow 0} \int_{[0, +\infty)} \epsilon^2 t(t + \epsilon^2)^{-2} d\theta(t) = 0.
\]
We conclude that, as expected,
\[
\lim_{\epsilon \downarrow 0} i\epsilon g_{X+Y,21}(i\epsilon) = \lim_{\epsilon \downarrow 0} i\epsilon g_{X+Y,12}(i\epsilon) = 0.
\]
Recall from (21) that
\[
(23) \quad \omega_1(i\epsilon) + \omega_2(i\epsilon) = \begin{bmatrix} \frac{i\epsilon}{\lambda} & \lambda \\ \lambda & i\epsilon \end{bmatrix} + F_{X+Y}(i\epsilon).
\]
The function $F_{X+Y}$ is easily obtained by inverting $G_{X+Y}$:
\[
F_{X+Y} = \frac{1}{G_{X+Y,11}G_{X+Y,22} - G_{X+Y,12}G_{X+Y,21}} \begin{bmatrix} G_{X+Y,22} & -G_{X+Y,12} \\ -G_{X+Y,21} & G_{X+Y,11} \end{bmatrix}.
\]
It follows from the corresponding property of the matrix-valued Cauchy transform $G$ that $f_{X+Y,11}(i\epsilon) = f_{X+Y,22}(i\epsilon) \in i(0, +\infty)$, and $f_{X+Y,12}(i\epsilon) = \overline{f_{X+Y,21}(i\epsilon)}$, whith the same holding when $X + Y$ is replaced by $X$ or $Y$. 

Consider the entrywise limits \( \lim_{\epsilon \downarrow 0} \frac{f_{X+Y}(i\epsilon)}{i\epsilon} \) in the above. Using the previously obtained estimates,

\[
\lim_{\epsilon \downarrow 0} \frac{f_{X+Y,11}(i\epsilon)}{i\epsilon} = \lim_{\epsilon \downarrow 0} \frac{[i\epsilon g_{X+Y,21}(i\epsilon)]}{[i\epsilon g_{X+Y,11}(i\epsilon)]} = \frac{\phi(p)}{\phi(q) \phi(p) - 0} = 1
\]

and

\[
\lim_{\epsilon \downarrow 0} \frac{f_{X+Y,12}(i\epsilon)}{i\epsilon} = \lim_{\epsilon \downarrow 0} \frac{[i\epsilon g_{X+Y,12}(i\epsilon)]}{[i\epsilon g_{X+Y,11}(i\epsilon)]} - \lim_{\epsilon \downarrow 0} \frac{i\epsilon g_{X+Y,12}(i\epsilon)}{i\epsilon g_{X+Y,11}(i\epsilon)} = \frac{-1}{\phi(q) \phi(p) - 0} = 0.
\]

Identical computations provide \( \lim_{\epsilon \downarrow 0} \frac{f_{X+Y,11}(i\epsilon)}{i\epsilon} = 1/\phi(p) = 1/\phi(q) \) and \( \lim_{\epsilon \downarrow 0} \frac{f_{X+Y,12}(i\epsilon)}{i\epsilon} = 0 \) in \( \mathcal{M}_2(\mathbb{C}) \)-norm, we obtain all of

\[
\lim_{\epsilon \downarrow 0} F_{X+Y}(i\epsilon) = 0, \quad \lim_{\epsilon \downarrow 0} \frac{1}{i\epsilon} F_{X+Y}(i\epsilon) = \lim_{\epsilon \downarrow 0} \frac{1}{i\epsilon} \Im F_{X+Y}(i\epsilon) = \phi(p)^{-1} 1_2.
\]

Recall that \( \omega_1, \omega_2 \) map the upper half-plane of \( \mathcal{M}_2(\mathbb{C}) \) into itself, increase the imaginary part (meaning that \( \Im \omega_k(b) \geq \Im b \) whenever \( b \in \mathcal{M}_2(\mathbb{C}), \Im b > 0 \), and \( \Im \omega_k(b) \leq \Im F_{X+Y}(b) \) if \( \Im b > 0, k = 1, 2 \) - see [40, 5, 7]). This, together with the behavior of \( F_{X+Y} \) near zero, allows us to conclude that

\[
\lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \Im \omega_1(i\epsilon), \quad \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \Im \omega_2(i\epsilon) \in \left[ 1_2, \frac{1}{\phi(p)} 1_2 \right].
\]

and Ky Fan’s operator generalization of the Julia-Carathéodory Theorem from [12] allows us to conclude that \( \lim_{\epsilon \downarrow 0} \omega_j(i\epsilon), j = 1, 2, \) exist and are selfadjoint \( 2 \times 2 \) matrices. Recall that both \( F_X \) and \( F_Y \) preserve the set of matrices of the form \( \begin{bmatrix} i\epsilon & \lambda \\ \bar{\lambda} & i\epsilon \end{bmatrix} \), \( \epsilon > 0, \lambda \in \mathbb{C} \). Then, according to [5, Theorem 2.7], so do \( \omega_1 \) and \( \omega_2 \). We conclude that

\[
\lim_{\epsilon \downarrow 0} \omega_1(i\epsilon) = \begin{bmatrix} 0 & u_1 \\ \bar{u}_1 & 0 \end{bmatrix}, \quad \lim_{\epsilon \downarrow 0} \omega_2(i\epsilon) = \begin{bmatrix} 0 & u_2 \\ \bar{u}_2 & 0 \end{bmatrix},
\]

where \( u_1 + u_2 = \lambda \), and

\[
\omega_k := \lim_{\epsilon \downarrow 0} \frac{\Im \omega_k(i\epsilon)}{\epsilon} = \lim_{\epsilon \downarrow 0} \frac{1}{i\epsilon} \left( \omega_k(i\epsilon) - \begin{bmatrix} 0 & u_k \\ \bar{u}_k & 0 \end{bmatrix} \right) \in \left[ 1_2, \frac{1}{\phi(p)} 1_2 \right],
\]

and

\[
(26) \quad w_k := \lim_{\epsilon \downarrow 0} \frac{\Im \omega_k(i\epsilon)}{\epsilon} = \lim_{\epsilon \downarrow 0} \frac{1}{i\epsilon} \left( \omega_k(i\epsilon) - \begin{bmatrix} 0 & u_k \\ \bar{u}_k & 0 \end{bmatrix} \right) = \lim_{\epsilon \downarrow 0} \frac{1}{i\epsilon} \left( \omega_k(i\epsilon) - \begin{bmatrix} 0 & u_k \\ \bar{u}_k & 0 \end{bmatrix} \right) \in \left[ 1_2, \frac{1}{\phi(p)} 1_2 \right],
\]
for \( k \in \{1, 2\} \), all being norm limits. Subtracting \( \begin{bmatrix} 0 & \lambda \\ \overline{\lambda} & 0 \end{bmatrix} \) from both sides of (23) then dividing by \( i\epsilon \) and then letting \( \epsilon \) tend to zero guarantees that

\[
\lim_{\epsilon \downarrow 0} \frac{(\omega_1(i\epsilon) + \omega_2(i\epsilon))_{12} - \lambda}{i\epsilon} = 0,
\]

with a similar result for the \((2, 1)\) entry, except that \( \lambda \) is replaced by \( \overline{\lambda} \).

We use next the coalgebra morphism property of the conditional expectation proved by Voiculescu in [40]: it is known that whenever \( X, Y \) are free over \( B \), one has

\[
\mathbb{E}_B(B) \left( B - X - Y \right)^{-1} = (\omega_1(b) - X)^{-1}. \]

We apply this to our \( 2 \times 2 \) matrices \( X, Y \) and \( B = \mathcal{M}_2(\mathbb{C}) \) to write

\[
\mathbb{E}_{\mathcal{M}_2(\mathbb{C})}(X) \left[ \left( \begin{bmatrix} i\epsilon & \lambda \\ \overline{\lambda} & i\epsilon \end{bmatrix} - X - Y \right)^{-1} \right] = \left[ \begin{bmatrix} (\omega_1(i\epsilon))_{11} & (\omega_1(i\epsilon))_{12} - x \\ (\omega_1(i\epsilon))_{21} - x^* & (\omega_1(i\epsilon))_{22} \end{bmatrix} \right]^{-1}.
\]

The formulas for the \((1,1)\) and \((1,2)\) entries of the right-hand matrix are

\[
(\omega_1(i\epsilon))_{22} \left( (\omega_1(i\epsilon))_{11} (\omega_1(i\epsilon))_{22} - ((\omega_1(i\epsilon))_{12} - x)(\omega_1(i\epsilon))_{21} - x^* \right]^{-1}
\]

and

\[
\left[ ((\omega_1(i\epsilon))_{12} - x)((\omega_1(i\epsilon))_{21} - x^*) - (\omega_1(i\epsilon))_{11}(\omega_1(i\epsilon))_{22} \right]^{-1}((\omega_1(i\epsilon))_{12} - x).
\]

The assumption \( \ker(x + y - \lambda) = p \) implies that

\[
\left( X + Y - \begin{bmatrix} 0 & \lambda \\ \overline{\lambda} & 0 \end{bmatrix} \right) \left[ \begin{bmatrix} q & 0 \\ 0 & p \end{bmatrix} \right] = \left[ \begin{bmatrix} q & 0 \\ 0 & p \end{bmatrix} \right] \left( X + Y - \begin{bmatrix} 0 & \lambda \\ \overline{\lambda} & 0 \end{bmatrix} \right) = 0.
\]

Thus, the weak limit

\[
\lim_{\epsilon \downarrow 0} \begin{bmatrix} i\epsilon & 0 \\ 0 & i\epsilon \end{bmatrix} \left( \begin{bmatrix} i\epsilon & \lambda \\ \overline{\lambda} & i\epsilon \end{bmatrix} - X - Y \right)^{-1} = \begin{bmatrix} q & 0 \\ 0 & p \end{bmatrix}
\]

holds. Recalling the weak continuity of the conditional expectation (w.r.t. \( \phi \)) onto a von Neumann subalgebra of \( \mathcal{A} \), we obtain

\[
\lim_{\epsilon \downarrow 0} \begin{bmatrix} i\epsilon & 0 \\ 0 & i\epsilon \end{bmatrix} \left[ \begin{bmatrix} (\omega_1(i\epsilon))_{11} & (\omega_1(i\epsilon))_{12} - x \\ (\omega_1(i\epsilon))_{21} - x^* & (\omega_1(i\epsilon))_{22} \end{bmatrix} \right]^{-1} = \begin{bmatrix} \mathbb{E}_{C,(x,x^*)}[q] & 0 \\ 0 & \mathbb{E}_{C,(x,x^*)}[p] \end{bmatrix}.
\]

(28)
A similar relation is deduced for \( \omega_2 \) and \( y \). Using the formulas for the entries of the term under the limit, we obtain

\[
\mathbb{E}_{C(x,x^*)}[q] = \lim_{\epsilon \downarrow 0} \epsilon (\omega_1(\epsilon))_{22} \\
\times \left[ (\omega_1(\epsilon))_{11}(\omega_1(\epsilon))_{22} - ((\omega_1(\epsilon))_{12} - x)((\omega_1(\epsilon))_{21} - x^*) \right]^{-1}
\]

Thus,

\[
(w_1)_{11}\mathbb{E}_{C(x,x^*)}[q] = -\lim_{\epsilon \downarrow 0} \left[ (\omega_1(\epsilon))_{11}(\omega_1(\epsilon))_{22} \times \right.
\]

\[
\left. \left[ ((\omega_1(\epsilon))_{12} - x)((\omega_1(\epsilon))_{21} - x^*) - (\omega_1(\epsilon))_{11}(\omega_1(\epsilon))_{22} \right]^{-1} \right]
\]

The operator \( \mathbb{E}_{C(x,x^*)}[q] \) is nonzero, nonnegative and bounded from above by 1. Similarly, considering the (1,2) entry, we have the weak limit

\[
\lim_{\epsilon \downarrow 0} \epsilon \left[ ((\omega_1(\epsilon))_{12} - x)((\omega_1(\epsilon))_{21} - x^*) - (\omega_1(\epsilon))_{11}(\omega_1(\epsilon))_{22} \right]^{-1} \\
\times ((\omega_1(\epsilon))_{12} - x) = 0.
\]

Observe that

\[
\|(\omega_1(\epsilon))_{12} - x)((\omega_1(\epsilon))_{21} - x^*) - (u_1 - x)(u_1 - x)^*\|
\leq ((\omega_1(\epsilon))_{12} + |u_1| + 2\|x\|)((\omega_1(\epsilon))_{12} - u_1).
\]

Equations (24) and (26) guarantee that \( \lim_{\epsilon \downarrow 0} |(\omega_1(\epsilon))_{12} - u_1|/\epsilon < 1/\phi(p) \).

As \( (\omega_1(\epsilon))_{12} \) converges to \( u_1 \) and \( x \) is constant, the above allows us to conclude that \( \mathbb{E}_{C(x,x^*)}[q](u_1 - x) = 0 \). A similar computation, in which we use entries (2,2) and (2,1) of the corresponding matrices, provides \( (u_1 - x)\mathbb{E}_{C(x,x^*)}[p] = 0 \). Considering the support projection of \( \mathbb{E}_{C(x,x^*)}[q] \) and \( p_1 \) of \( \mathbb{E}_{C(x,x^*)}[p] \), we find that \( x - u_1 \) has nonzero left and right invariant projections (with \( p_1 \) being the right-invariant projection). A similar argument shows that \( y - u_2 \) has left and right invariant projections, with \( p_2 \) being the right-invariant projection.

The (sum of the) length of these projections is deduced the following way: it is clear that \( p \geq p_1 \land p_2 \). Recalling the subordination relation and (28), we obtain

\[
\phi(p)^{-1} = (w_1)_{11} + (w_2)_{11} - 1
\]
(recall that $\omega_k(i\epsilon)$ have equal entries on the diagonal, so picking $p$ and $(1, 1)$ or $q$ and $(2, 2)$ makes no difference). On the other hand, we recall that (by definition), $E_{\mathbb{C}(x,x^*)}$ preserves the trace: $\phi(p) = \phi(E_{\mathbb{C}(x,x^*)}[p])$. Since $0 \leq E_{\mathbb{C}(x,x^*)}[p] \leq 1$, it follows immediately from elementary functional calculus that the trace of the support of $E_{\mathbb{C}(x,x^*)}[p]$ cannot be less than $\phi(E_{\mathbb{C}(x,x^*)}[p])$, so $\phi(p_1), \phi(p_2) \geq \phi(p)$. Also, by applying $\phi$ in relation (29) it follows that

$$\phi(p)(\omega'_1(0))_{11} = \phi(E_{\mathbb{C}(x,x^*)}[p])(w_1)_{11} \leq \phi(p_1).$$

Clearly, a similar relation holds for $p_2, y$ and $(w_2)_{11}$. Thus,

$$\frac{1}{\phi(p)} = (w_1)_{11} + (w_2)_{11} - 1 \leq \frac{\phi(p_1) + \phi(p_2)}{\phi(p)} - 1,$$

which is equivalent to

$$\phi(p_1) + \phi(p_2) \geq \phi(p) + 1.$$

This, together with the relation $p \geq p_1 \wedge p_2$, implies $\phi(p_1) + \phi(p_2) = \phi(p) + 1$.

Thus we have proved the following result, paralleling [8, Theorem 7.4].

**Proposition 1.** If $x, y \in A \setminus \mathbb{C}1$ are *-free with respect to $\phi$ and there exist a projection $p \in A \setminus \{0\}, \lambda \in \mathbb{C}$, so that $(x+y)p = \lambda p$, then

1. $p = \ker((x+y-\lambda)^*(x+y-\lambda))$;
2. there exist $p_1, p_2$ projections in $A$ and $u_1, u_2 \in \mathbb{C}$ so that
   - $xp_1 = u_1 p_1$ and $yp_2 = u_2 p_2$,
   - $u_1 + u_2 = \lambda$,
   - $\phi(p_1) + \phi(p_2) = \phi(p) + 1$.

Conversely, if the three conditions of item (2) above hold, then $p := p_1 \wedge p_2$ satisfies $(x+y)p = (u_1 + u_2)p$.

**Remark.** In [8] it is shown that under the same hypotheses, if $x = x^*$ and $y = y^*$, then $\omega'_1(\lambda)E_{\mathbb{C}(x)}[p]$ is a projection. It would be interesting to determine whether $E_{\mathbb{C}(x,x^*)}[q]$ and $E_{\mathbb{C}(x,x^*)}[p]$ are themselves multiples of projections.

**Remark.** Let us note that, regrettably, the limits $\lim_{\epsilon \downarrow 0} G_{x+y}(i\epsilon)$ in general will not provide the value of the Cauchy-Stieltjes transform in $\lambda$. Indeed, assume that $x = x^*$ and $y = y^*$, neither a multiple of the identity. Then it is known that, roughly speaking, $G_{x+y}$ extends continuously to the real line and $G_{x+y}(r) \in \mathbb{C}$ (the lower half plane of $\mathbb{C}$) has an analytic extension around $r$ for most points in the spectrum of $x+y$. However, for $\lambda = r$
being one of those points in the spectrum of $x + y$, we have
\[
\lim_{\epsilon \downarrow 0} G_{X+Y;21}(i\epsilon) = \lim_{\epsilon \downarrow 0} \phi((r - x - x)((r - x - y)^2 + \epsilon^2)^{-1})
\]
\[
= \lim_{\epsilon \downarrow 0} \int_{\mathbb{R}} \frac{r - t}{(r - t)^2 + \epsilon^2} d\mu_{x+y}(t),
\]
which is simply the Hilbert transform of $\mu_{x+y}$ evaluated at $r$, a real number. On the bright side, note that this limit does exist for all $r \in \mathbb{R}$, with the exception of those points $r$ where $G_{x+y}$ is infinite (and it is known that there are only finitely many such points). On the even brighter side, the $(1, 1)$ entry will provide through the same argument the imaginary part of $G_{x+y}(r)$. Of course, it is natural that it should be so, because otherwise we would have $\frac{\partial}{\partial \lambda} G_{x+y}(r) = 0$ for most points $r$ in the spectrum of $x + y$. This simple example shows us that $\lambda \mapsto G_{X+Y}(0)$ has little chance of being generally continuous, and, in particular, that [3, Theorem 3.3 (3)] does not hold in the operator-valued context.

We dare nevertheless to make the following conjecture.

**Conjecture.** If $x, y \in \mathfrak{A}$ are $\star$-free w.r.t. the trace state $\phi$ and the spectrum of each contains more than one point, then the function $\mathbb{C} \ni \lambda \mapsto G_{X+Y}(0)$ is continuous when restricted to each component of the spectrum of $x + y$. The points of discontinuity of $\mathbb{C} \ni \lambda \mapsto G_{X+Y}(0)$ belong to the closure of the resolvent of $x + y$.

**5. BROWN MEASURE OF R–DIAGONAL ELEMENTS**

In this section we will show that $R$-diagonal operators fit very nicely in the Hermitization and subordination frame and that one can recover from this point of view in a quite systematic way the result of Haagerup and Larsen on the Brown measure of $R$-diagonal operators.

Let us first recall, for later use, the definition of the $S$-transform. Recall that for a random variable $y$ with $\phi(y) \neq 0$ we put [41]
\[
\psi_y(\lambda) := \phi \left( (1 - \lambda y)^{-1} \right) - 1;
\]
once one can show that
\[
S_y(\lambda) := \frac{\lambda + 1}{\lambda} \psi_y(-1)(\lambda)
\]
is well-defined in some neighborhood of 0; it is called Voiculescu’s $S$-transform of $y$. It was proved in [27, 20] that if $y$ is a positive operator then its $S$-transform has an analytic continuation to an interval $(\mu_y \{ \mu_y \} - 1, 0]$; it has a strictly negative derivative on this interval and
\[
S_y(\mu_y \{ \mu_y \} - 1) = \phi \left( y^{-2} \right), \quad S_y(0) = \frac{1}{\phi(y^2)}.
\]
Let now $x = ua \in (\mathcal{A}, \phi)$ be $R$-diagonal \textsuperscript{29}. Recall that this means that $u$ and $a$ are $\ast$-free with respect to $\phi$, and that $u$ is a Haar unitary and $a \geq 0$. Note that $R$-diagonal operators are, in $\ast$-moments, the limits of an important class of random matrices, namely of bi-unitarily invariant random matrices.

We shall denote by $D$ the commutative $C^\ast$-algebra of diagonal matrices in $\mathcal{M}_2(\mathbb{C})$, where we consider the same inclusion of $\mathcal{M}_2(\mathbb{C})$ in $\mathcal{M}_2(\mathcal{A})$ as in the previous sections. Let us recall from \textsuperscript{30} that an element $x$ in a tracial $W^*$-noncommutative probability space is $R$-diagonal if and only if the matrix \begin{bmatrix} 0 & x \\ x^\ast & 0 \end{bmatrix} is free over $D$ from $\mathcal{M}_2(\mathcal{C})$ with respect to the expectation $E_D: \mathcal{M}_2(\mathcal{A}) \to D$, \begin{align*} E_D \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \phi(a_{11}) & 0 \\ 0 & \phi(a_{22}) \end{bmatrix}. \end{align*}

Fix now a $\lambda$ in the upper half plane of $\mathbb{C}$. With the notation $D \langle X \rangle$ for the $\ast$-algebra generated by $D$ and $X$, it is obvious that
\begin{align*} D \langle \begin{bmatrix} 0 & \lambda \\ \lambda & 0 \end{bmatrix} \rangle = \mathcal{M}_2(\mathbb{C}). \end{align*}

(The reader can easily verify this by, for example, constructing all matrix units out of the two nontrivial projections of $D$ and the element $\begin{bmatrix} 0 & \lambda \\ \lambda & 0 \end{bmatrix}$.)

The fundamental result \textsuperscript{40, Theorem 3.8} of Voiculescu is written in this context as:
\begin{align} E \begin{bmatrix} z & \lambda \\ \lambda & w \end{bmatrix} - \begin{bmatrix} 0 & x \\ x^\ast & 0 \end{bmatrix}^{-1} = \begin{bmatrix} \omega_1(z, w) & \lambda \\ \lambda & \omega_2(z, w) \end{bmatrix}^{-1}, \end{align}

where
\begin{align*} E = E_D \langle \begin{bmatrix} 0 & \lambda \\ \lambda & 0 \end{bmatrix} \rangle = E_{\mathcal{M}_2(\mathbb{C})} \end{align*}
denotes, as above, the unique conditional expectation onto $\mathcal{M}_2(\mathbb{C})$ which preserves the trace, and which is given by evaluation of $\phi$ on the entries of the matrix. As before, the $(2, 1)$ entry is the object of interest $G_{\mu_x}$, determined by the functions $\omega_1, \omega_2$ via a straightforward algebraic relation. (It is obvious that the functions $\omega_1$ and $\omega_2$ depend also on $\lambda$, and this dependence is relevant to us. While it might be unfair to the reader, we will follow tradition and suppress this dependence in notations.)

On the other hand, the subordination function $\omega = \begin{bmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{bmatrix}$, obtained by applying $E_D$ in (30), is determined by \textsuperscript{5, Theorem 2.2} via the iteration procedure, if desired, or via straightforward direct computation in terms of (functions derived from) the Cauchy-Stieltjes transform of $a^2$. Let $w = z$. 

\textsuperscript{29} S. Belinschi, P. Śniady, and R. Speicher

\textsuperscript{30} S. Belinschi, P. Śniady, and R. Speicher

\textsuperscript{40} S. Belinschi, P. Śniady, and R. Speicher
Performing the inversion in the right hand side of (30) gives

\[
\left[ \frac{\omega_1(z, z)}{\lambda} \right]^{-1} = \left[ \frac{\omega_2(z, z)}{\lambda} \right]^{-1} = \left[ \frac{\omega_1(z, z) - |\lambda|^2}{\lambda} \right]^{-1} = \left[ \frac{\omega_2(z, z) - |\lambda|^2}{\lambda} \right]^{-1}.
\]

Inverting under the expectation in the left hand side of (30) and taking the expectation gives

\[
\begin{bmatrix}
\omega_1(z, z) & \lambda & \omega_2(z, z)
\end{bmatrix}^{-1}
= \left[ \frac{\omega_1(z, z) - |\lambda|^2}{\lambda} \right]^{-1}
= \left[ \frac{\omega_2(z, z) - |\lambda|^2}{\lambda} \right]^{-1}
= \left[ \frac{\omega_1(z, z) - |\lambda|^2}{\lambda} \right]^{-1}.
\]

Traciability of \( \phi \) easily implies that the (1, 1) and (2, 2) entries of the above matrix are equal, guaranteeing thus that \( \omega_1(z, z) = \omega_2(z, z) = \omega(z) \). Since the above matrix must be equal to the one in (31), solving a quadratic equation and recalling that the asymptotics at infinity of \( \omega(z) \) is of order \( z \) allows us to write

\[
\omega(z) = \frac{1 + \sqrt{1 + 4z^2|\lambda|^2\phi (\lambda - x)^{-1}} - \phi (\lambda - x)^{-1}}{2z\phi (\lambda - x)^{-1}}.
\]

The regularized Cauchy-Stieltjes transform \( G_{\mu_x, \epsilon} \) of \( x \), namely the (2, 1) entry \( \phi (\lambda - x)^{-1} \), is then given by

\[
\begin{align*}
G_{\mu_x, \epsilon} (\lambda) &= \frac{\lambda}{|\lambda|^2 - \omega(i\epsilon)^2} \\
&= \frac{\lambda}{|\lambda|^2 - \left( 1 + \sqrt{1 + 4|\lambda|^2\phi (|\epsilon^2 + (\lambda - x)^{-1})^{-1}}} \\
&= \frac{2z\phi (\lambda - x)^{-1}}{2z\phi (\lambda - x)^{-1}}.
\end{align*}
\]

Quite trivially, if \( \lambda \) does not belong to the spectrum of \( x \) and we let \( \epsilon \) go to zero, then \( G_{\mu_x} (\lambda) = \frac{\lambda}{|\lambda|^2} \). In particular, this holds for \( |\lambda| > ||x|| \).

Now, a direct computation shows that applying the fixed point equation determining the subordination functions allows us to write

\[
\begin{align*}
\omega_1(z, w) &= \frac{|\lambda|^2}{\omega_2(z, w)} + \frac{1}{\phi \left( \frac{w - |\lambda|^2}{\omega_2(z, w)} \right) - \frac{\lambda^2}{\omega_1(z, w)}}, \\
\omega_2(z, w) &= \frac{|\lambda|^2}{\omega_1(z, w)} + \frac{1}{\phi \left( \frac{w - |\lambda|^2}{\omega_1(z, w)} \right) - \frac{\lambda^2}{\omega_2(z, w)}}.
\end{align*}
\]
In particular, for $z = w$ we have seen that $\omega_1(z, z) = \omega_2(z, z) = \omega(z)$, and

\begin{equation}
\omega(z) = \frac{|\lambda|^2}{\omega(z)} + \frac{1}{\phi \left( \frac{1}{\omega(z)} \right)} = \frac{|\lambda|^2}{\omega(z)} + \frac{1}{\phi \left( \frac{1}{\omega(z)} \right)}.
\end{equation}

From this expression and the relation (31) we recognize the (well-known) fact that the distribution of our $R$-diagonal operator $x$ depends only on its positive part. This functional equation guarantees at the same time that the dependence of $\omega(z)$ on the argument of $\lambda$ is constant, thus guaranteeing (via (32)) that the distribution of $x$ has radial symmetry. Now we shall describe the precise dependence of this distribution on the distance from zero: by simple algebraic manipulations, relation (33) becomes

\begin{equation}
\phi \left( \frac{a^2}{(z - |\lambda|^2 \omega(z))^2 - a^2} \right) = \frac{z - \omega(z)}{\omega(z) - |\lambda|^2 \omega(z)}.
\end{equation}

In terms of the analytic transform $\psi$ we can write (34) in the form

\begin{equation}
\psi_{\mu, 2} \left( \left( z - \frac{|\lambda|^2}{\omega(z)} \right)^{-2} \right) = \frac{z - \omega(z)}{\omega(z) - |\lambda|^2 \omega(z)}.
\end{equation}

We recall that $\omega(i \epsilon) \in i \mathbb{R}_+$, and in fact $\Im \omega(i \epsilon) > \epsilon$. We define

\[ A_\epsilon(\lambda) = \Im \left( \frac{1}{i \epsilon - \frac{|\lambda|^2}{\omega(i \epsilon)}} \right) = \frac{1}{\epsilon + \frac{|\lambda|^2}{\omega(i \epsilon)}} \quad \epsilon > 0. \]

This transforms (35) into

\begin{equation}
\psi_{\mu, 2} \left( -A_\epsilon(\lambda)^2 \right) = \frac{\epsilon A_\epsilon(\lambda) - (\epsilon^2 + |\lambda|^2) A_\epsilon(\lambda)^2}{(\epsilon^2 + |\lambda|^2) A_\epsilon(\lambda)^2 - 2 \epsilon A_\epsilon(\lambda) + 1}.
\end{equation}

Recall from the formula of $\omega(z)$ that

\[ \lim_{\epsilon \to 0} \Im \omega(i \epsilon) = \lim_{\epsilon \to 0} \frac{1 + \sqrt{1 - 4 \epsilon^2 |\lambda|^2 \phi \left( [\epsilon^2 + (\lambda - x)^*(\lambda - x)]^{-1} \right)^2}}{2 \phi \left( [\epsilon^2 + (\lambda - x)^*(\lambda - x)]^{-1} \right)}. \]

This quantity (while depending on $\lambda$) is necessarily positive and finite, zero almost everywhere in the spectrum of $x$. For all these $\lambda$, we take the limit as $\epsilon \to 0$ in (36) to obtain

\[ \psi_{\mu, 2} \left( -A_0(\lambda)^2 \right) = \frac{|\lambda|^2 (-A_0(\lambda)^2)}{1 - |\lambda|^2 (-A_0(\lambda)^2)}. \]

We should note that this functional equation is quite trivially solvable on $(-\infty, 0)$. Indeed, with the obvious notations, $\psi_{\mu, 2}(f(r)) + 1 = \frac{1}{1 - r^2 f(r)}$ is
equivalent to $r^2 = \frac{\eta_{\mu^2}(f(r))}{f(r)}$ (recall that $\eta = \frac{\psi}{1+\psi}$), and $g: v \mapsto \frac{\eta_{\mu^2}(v)}{v} = \frac{1}{v} - F_{\mu^2} \left( \frac{1}{v} \right)$ is known \[4\] to be injective, in fact strictly increasing whenever $a \not\in \mathbb{C} \cdot 1$, on $(-\infty, 0)$ with $0^- \mapsto \phi(a^2)$. This might be a more convenient way to express $\omega$: we would have

\[
G_{\mu,\epsilon}(\lambda) = \frac{\overline{\lambda}}{|\lambda|^2 \left( 1 + |\lambda|^2 A_0(\lambda)^2 \right)},
\]

and when $\epsilon A_0(\lambda) \to 0$ as $\epsilon \to 0$,

\[
G_{\mu}(\lambda) = \frac{\overline{\lambda}}{|\lambda|^2 \left( 1 + |\lambda|^2 A_0(\lambda)^2 \right)}.
\]

Since

\[
A_0(\lambda) = \sqrt{-g^{-1}(|\lambda|^2)},
\]

we write

\[
(37) \quad G_{\mu}(\lambda) = \frac{1}{\lambda \left( 1 - |\lambda|^2 g^{-1}(|\lambda|^2) \right)}.
\]

The border $|\lambda|^2 = \phi(a^2)$ follows from the above remark on the behavior next to zero of $g$. However, equally easily in terms of the $S$-transform, we have

\[
S_{\mu}(\lambda) = \frac{|\lambda|^2 (-A_0(\lambda)^2)}{1 - |\lambda|^2 (-A_0(\lambda)^2)} = \frac{\omega(0)^2}{|\lambda|^2 - \omega(0)^2}.
\]

Then

\[
(38) \quad G_{\mu}(\lambda) = \frac{1}{\lambda} \left( 1 + S_{\mu}(\lambda)^{-1} \right),
\]

with the exact same restrictions on $\lambda$ as above.

So we have finally

\[
G_{\mu}(\lambda) = \begin{cases} 
\frac{1}{\lambda} & \text{for } |\lambda| \geq \phi(a^2), \\
1 + S_{\mu}(\lambda)^{-1} & \text{for } |\lambda|^2 \leq \phi(a^2), 
\end{cases}
\]

which allows us to recover the following theorem of Haagerup and Larsen \[20\].

**Theorem 2.** The Brown measure $\mu_x$ of the $R$-diagonal operator $x = ua$ is the unique rotationally invariant probability measure such that

\[
\mu_x\{ \lambda \in \mathbb{C} : |\lambda| \leq z \} = \begin{cases} 
0 & \text{for } z \leq \frac{1}{\sqrt{\phi((x^*)^{-1})}}, \\
1 + S_{xx^*}(z^{-2}) & \text{for } \frac{1}{\sqrt{\phi((x^*)^{-1})}} \leq z \leq \sqrt{\phi((x^*)^2)}, \\
1 & \text{for } z \geq \sqrt{\phi((x^*)^2)}. 
\end{cases}
\]
Let us recall that Guionnet, Krishnapur, and Zeitouni showed in [18] that this Brown measure describes indeed the asymptotic eigenvalue distribution of the corresponding bi-unitarily invariant random matrices.

6. Brown measure of elliptic–triangular operators

6.1. Elliptic and triangular–elliptic ensembles. Consider a random matrix $A_N = (a_{ij})_{1 \leq i,j \leq N}$ such that the joint distribution of random variables $(\Re a_{ij}, \Im a_{ij})_{1 \leq i,j \leq N}$ is centered Gaussian with the covariance given by

$$E a_{ij} a_{kl} = \begin{cases} \frac{\alpha}{N} \delta_{ik} \delta_{jl} & \text{if } i < j, \\ \frac{\alpha + \beta}{2N} \delta_{ik} \delta_{jl} & \text{if } i = j, \\ \frac{\beta}{N} \delta_{ik} \delta_{jl} & \text{if } i > j, \end{cases}$$

$$E a_{ij} a_{kl} = \gamma \frac{\sqrt{\alpha \beta}}{N} \delta_{il} \delta_{jk},$$

where $\alpha, \beta \geq 0$ and $\gamma \in \mathbb{C}$ are such that $|\gamma| \leq \sqrt{\alpha \beta}$. Informally speaking: there is a correlation between $a_{ij}$ and $a_{ji}$ and random variables $(\Re a_{ij}, \Im a_{ij})_{1 \leq i,j \leq N}$ are as independent as it is possible to fulfill this requirement. Furthermore, the entries above the diagonal have the same variance; also the entries below the diagonal have the same variance (but these two variances need not to coincide).

We call $A_N$ triangular–elliptic random matrix. It is a natural generalization of some important random matrix ensembles: for $\alpha = \beta$ it coincides with the elliptic ensemble and in particular for $\alpha = \beta = 1$, $\gamma = 0$ it coincides with the Wigner matrix (i.e. $(\Re a_{ij}, \Im a_{ij})$ is a family of iid Gaussian variables) and for $\alpha = \beta = \gamma = 1$ it is a random Hermitian matrix which coincides with the Gaussian Unitary Ensemble. For $\alpha = 1$ and $\beta = \gamma = 0$ the sequence $(A_N)$ converges in $\ast$–moments to a very interesting quasinilpotent operator $T$ (see, e.g., [11, 33, 2]) and for $\alpha = \sqrt{1 + t^2}$, $\beta = t$, $\gamma = 0$ the sequence $(A_N)$ converges in $\ast$–moments to $T + tY$, where $Y$ is the Voiculescu circular element such that $T$ and $Y$ are free. The Brown measure of the latter operator was computed by Aagaard and Haagerup [2].

One can show [11] that the sequence of random matrices $A_N$ converges in $\ast$–moments to a certain generalized circular element $x$ which will be described precisely below in Section 6.2.

6.2. Elliptic triangular operators. In this section we will use the notions of operator–valued free probability; the necessary notions can be found in [35].

6.2.1. Preliminaries. Let $\alpha, \beta \geq 0$, $\gamma \in \mathbb{C}$ such that $|\gamma| \leq \sqrt{\alpha \beta}$ be fixed. Let $\mathcal{B} = \mathcal{L}^\infty(0,1)$, let $(\mathcal{B} \subset \mathcal{A}, \overline{E} : \mathcal{A} \rightarrow \mathcal{B})$ be an operator–valued
probability space and let \( x \in \mathfrak{A} \) be a generalized circular element \( x \) the only nonzero free cumulants of which are given by

\[
(39) \quad k(x, fx)(t) = \gamma \int_0^1 f(s) ds,
\]

\[
(40) \quad k(x^*, fx^*)(t) = \bar{\gamma} \int_0^1 f(s) ds,
\]

\[
(41) \quad k(x, fx^*)(t) = \alpha \int_t^1 f(s) ds + \beta \int_0^t f(s) ds,
\]

\[
(42) \quad k(x^*, fx)(t) = \alpha \int_0^t f(s) ds + \beta \int_t^1 f(s) ds
\]

for every \( f \in \mathfrak{B} \). The reader may find the details of this construction in the case \( \alpha = 1, \beta = \gamma = 0 \) in (33).

In order to be able to consider the Brown measure of \( x \) we need to define a tracial state \( \phi : \mathfrak{A} \to \mathbb{C} \). We do this by setting

\[
\phi(f) = \int_0^1 f(s) ds
\]

for \( f \in \mathfrak{B} \) and in the general case \( \phi(y) = \phi(\widetilde{\mathbb{E}}(y)) \). One can show [11] that the sequence of random matrices \( A_N \) considered in Section 6.1 converges in \( \star \)-moments to \( x \) and therefore \( \phi \) is indeed a tracial state.

6.2.2. Computation of regularized Cauchy transform. Since we are dealing with an operator–valued case it is useful to define \( \mathbb{E} : \mathcal{M}_2(\mathfrak{A}) \to \mathcal{M}_2(\mathfrak{B}) \) as

\[
\mathbb{E} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \widetilde{\mathbb{E}}(a_{11}) & \widetilde{\mathbb{E}}(a_{12}) \\ \widetilde{\mathbb{E}}(a_{21}) & \widetilde{\mathbb{E}}(a_{22}) \end{bmatrix}
\]

instead of the definition (14).

The relation between the free cumulants of \( x \) and the free cumulants of \( X \), as defined in (13), implies that \( X \) is an operator-valued semicircular element whose \( R \)-transform is explicitly given by

\[
R_X \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} k(x, a_{22}x^*) & k(x, a_{21}x) \\ k(x^*, a_{12}x^*) & k(x^*, a_{11}x) \end{bmatrix}.
\]

Thus the general equation \( G(\Lambda) = (\Lambda - R(G(\Lambda)))^{-1} \) for an operator-valued semicircular element gives in our case

\[
(43) \quad G_x(\lambda) = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \frac{1}{d} \begin{bmatrix} i \epsilon - k(x^*, g_{11}x) & -\lambda + k(x, g_{21}x) \\ -\bar{\lambda} + k(x^*, g_{12}x^*) & i \epsilon - k(x, g_{22}x^*) \end{bmatrix},
\]
where

\[
\det \begin{bmatrix}
  i\epsilon - k(x, g_{22}x^*) & \lambda - k(x, g_{21}x) \\
  \overline{\lambda} - k(x^*, g_{12}x^*) & i\epsilon - k(x^*, g_{11}x)
\end{bmatrix} = \\
(\epsilon - k(x^*, g_{11}x))(i\epsilon - k(x, g_{22}x^*)) - (\lambda - k(x, g_{21}x))(\overline{\lambda} - k(x^*, g_{12}x^*)).
\]

For simplicity, we suppressed the dependence on \(\epsilon\) and \(\lambda\) of \(g_{ij} = g_{\epsilon,\lambda,ij} \in \mathcal{B}\) and \(d = d_{\epsilon,\lambda} \in \mathcal{B}\).

Observe that for fixed \(\epsilon\) and \(\lambda\) by (39)–(42) the second summand on the right–hand side of (44) is a constant function in \(\mathcal{B}\) and

\[
d' = (\epsilon - k(x^*, g_{11}x))' (\epsilon - k(x, g_{22}x^*)) + (\epsilon - k(x^*, g_{11}x)) (i\epsilon - k(x, g_{22}x^*)') = \\
- (\alpha - \beta) g_{11} (\epsilon - k(x, g_{22}x^*)) - (\epsilon - k(x^*, g_{11}x)) (\beta - \alpha) g_{22} = 0,
\]

where the last equality follows from the comparison of the matrix entries in (43). It follows that \(d \in \mathcal{B}\) is in fact a constant function. By comparing the entries of (43) we see that also \(g_{12}, g_{21} \in \mathcal{B}\) are constant. Equation (17) implies that \(d \in \mathbb{R}\).

The comparison of upper–left corners of (43) gives us a simple integral equation for the function \(g_{11}\) which has a unique solution

\[
g_{11}(t) = \begin{cases} 
  \frac{i\epsilon(\alpha - \beta)}{\alpha - \beta e^\frac{\epsilon - \alpha}{d + \alpha}} e^{\frac{\beta - \alpha}{d} t} & \text{for } \alpha \neq \beta, \\
  \frac{i\epsilon}{d + \alpha} & \text{for } \alpha = \beta.
\end{cases}
\]

In fact, the case \(\alpha = \beta\) may be regarded as a special case of \(\alpha \neq \beta\) by taking the limit \(\alpha \to \beta\). Similarly, we find

\[
g_{22}(t) = \begin{cases} 
  \frac{i\epsilon(\alpha - \beta)}{\alpha - \beta e^\frac{\epsilon - \alpha}{d + \alpha}} e^{\frac{\beta - \alpha}{d} (1-t)} & \text{for } \alpha \neq \beta, \\
  \frac{i\epsilon}{d + \alpha} & \text{for } \alpha = \beta.
\end{cases}
\]

We also find

\[
g_{12} = -\frac{\gamma \overline{\lambda} - \lambda d}{d^2 - |\gamma|^2},
\]

\[
g_{21} = -\frac{\gamma \lambda - \overline{\lambda} d}{d^2 - |\gamma|^2}.
\]

Hence

\[
\frac{1}{d} = \det \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = -\frac{e^2(\alpha - \beta)^2}{(\alpha - \beta e^{\frac{\epsilon - \alpha}{d}})^2} \frac{e^{\frac{\beta - \alpha}{d} (1-t)}}{d^2 - |\gamma|^2} \quad \text{for } \alpha \neq \beta.
\]
for $\alpha \neq \beta$ and

$$\frac{1}{d} = \det \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = -\frac{\epsilon^2}{(d + \alpha)^2} - \frac{|\gamma \bar{\lambda} + \lambda d|^2}{(d^2 - |\gamma|^2)^2} \tag{50}$$

if $\alpha = \beta$.

A priori, all above statements hold only for $\epsilon$ in some neighborhood of infinity, but it is easy to check from the definition that $g_{\epsilon,\lambda,ij}(s)$ is an analytic function of $\epsilon$ (in the region $\Re \epsilon \neq 0$) for fixed values of $0 \leq s \leq 1$, $i, j \in \{1, 2\}$ and $\lambda \in \mathbb{C}$. Similarly, $d = (g_{11}g_{22} - g_{12}g_{21})^{-1}$ is an analytic function of $\epsilon$. It follows that the above identities hold for all $\epsilon > 0$.

6.2.3. Existence and continuity of Cauchy transform. Since the function $x \mapsto e^x$ is convex it follows that

$$\frac{\alpha - \beta}{\log \alpha - \log \beta} = \frac{1}{\log \alpha - \log \beta} \int_{\log \beta}^{\log \alpha} e^x \, dx \geq \frac{1}{\log \alpha - \log \beta} \int_{\log \alpha}^{\log \beta} x \, dx = \sqrt{\alpha \beta}.$$ 

It is easy to check from (17) that for fixed $\lambda \in \mathbb{C}$ and $\epsilon \to \infty$ we have $d \to -\infty$. In the case $\alpha \neq \beta$ equation (46) implies that $d \neq \frac{\beta - \alpha}{\log \alpha - \log \beta}$ for all $\epsilon > 0$. From the Darboux property it follows that $d < \frac{\beta - \alpha}{\log \alpha - \log \beta} \leq -|\gamma|$ for $\epsilon > 0$. Similarly, if $\alpha = \beta$ one can show that $d < -\alpha \leq -|\gamma|$.

Observe that for fixed $\lambda \in \mathbb{C}$ and $d \in \mathbb{R}$ there is at most one $\epsilon > 0$ for which (49) holds true, therefore the continuous function $\epsilon \mapsto d_{\lambda,\epsilon}$ must be monotone. Since $\lim_{\epsilon \to \infty} d_{\epsilon,\lambda} = -\infty$ hence $\epsilon \mapsto d_{\lambda,\epsilon}$ must be decreasing and the limit $d_{0,\lambda} := \lim_{\epsilon \to 0} d_{\epsilon,\lambda}$ exists and is finite. Furthermore, except for the trivial case $\alpha = \beta = \gamma = 0$ we have $d_{0,\lambda} < 0$. Equation (49) implies that $d = d_{0,\lambda}$ is a solution of the equation

$$\left(\alpha - \beta e^{\frac{\beta - \alpha}{\pi}}\right)^2 \left(\frac{d^2 - |\gamma|^2}{d}\right) + |\gamma \bar{\lambda} + \lambda d|^2 = -\frac{\epsilon^2(\alpha - \beta)^2 e^{\frac{\beta - \alpha}{\pi}}(d^2 - |\gamma|^2)^2}{d^2} \tag{51}$$

with $\epsilon = 0$.

It is easy to see that for each $\lambda \in \mathbb{C}$ and $\epsilon = 0$ there are only finitely many (at most 5) solutions $d \in \mathbb{R}$ of the above equation, therefore for every $\lambda_0 \in \mathbb{C}$ and $\delta_0 > 0$ we can find $0 < \delta < \delta_0$ such that the triples $\lambda_0$, $\epsilon = 0$, $d = d_{0,\lambda_0} \pm \delta$ are not the solutions. It follows that there exists $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$ and $|\lambda - \lambda_0| < \epsilon_0$ the triples $\lambda, \epsilon$, $d_{0,\lambda_0} \pm \delta$ are not the solutions. The function $(\epsilon, \lambda) \mapsto d_{\epsilon,\lambda}$ is continuous for $\epsilon > 0$ hence from Darboux property it follows that for $|\lambda - \lambda_0| < \epsilon_0$ and $0 < \epsilon < \epsilon_0$ we have $|d_{\epsilon,\lambda} - d_{0,\lambda_0}| < \delta$ hence $|d_{0,\lambda} - d_{0,\lambda_0}| \leq \delta$. It follows that the function $\lambda \mapsto d_{0,\lambda}$ is continuous. Equation (48) implies that also
the Cauchy transform $\lambda \mapsto G_x(\lambda)$ is continuous—possibly except for the case $\alpha = \beta = |\gamma| = -d_{0,\lambda}$.

6.2.4. Computation of non–regularized Cauchy transform. Equation (46) implies that in the limit as $\epsilon \to 0$, either $g_{11}$ tends uniformly to zero or $\alpha - \beta e^{\frac{\beta - \alpha}{\alpha - \beta}} \to 0$ for $\alpha \neq \beta$ or $d \to -\alpha$ for $\alpha = \beta$.

Suppose $g_{11} \to 0$; then also $g_{22} \to 0$ and the comparison of bottom–left corners of (43) together with (44) gives us

\begin{equation}
(52) \quad \gamma (G_x(\lambda))^2 - \lambda G_x(\lambda) + 1 = 0,
\end{equation}

hence

\begin{equation}
(53) \quad G_x(\lambda) = \frac{-\lambda \pm \sqrt{\lambda^2 - 4\gamma}}{-2\gamma} = \frac{2}{\lambda \mp \sqrt{\lambda^2 - 4\gamma}}
\end{equation}

for $\gamma \neq 0$ and $G_x(\lambda) = \lambda^{-1}$ for $\gamma = 0$; note that (53) is still valid in the latter case.

Suppose now that $\alpha - \beta e^{\frac{\beta - \alpha}{\alpha - \beta}} \to 0$ if $\alpha \neq \beta$ or $d \to -\alpha$ of $\alpha = \beta$; it follows that

\begin{equation}
(54) \quad G_x(\lambda) = \frac{-\lambda d + \gamma \lambda}{d^2 - |\gamma|^2},
\end{equation}

where $d$ is given by

\begin{equation}
(55) \quad d = d_{0,\lambda} = \begin{cases} \frac{\beta - \alpha}{\log \alpha - \log \beta} & \text{for } \alpha \neq \beta, \\ -\alpha & \text{for } \alpha = \beta. \end{cases}
\end{equation}

Let us summarize the above discussion: we showed that $\lambda \mapsto G_x(\lambda)$ is a continuous function given at each $\lambda$ either by (53) or by (54). These two solutions coincide on the ellipse given by the system of equations (52), (54). Therefore on each of the connected components of the complement of the ellipse the Cauchy transform is given either by (53) or by (54).

At infinity, the Cauchy transform satisfies $\lim_{|\lambda| \to \infty} G_x(\lambda) = 0$ therefore (53) is the correct choice on the outside of the ellipse.

On the other hand, for $\lambda = 0$ the factor (except, possibly, for the case $\alpha = \beta = |\gamma|$)

\begin{equation}
(56) \quad \frac{(d^2 - |\gamma|^2)^2}{d} + |\gamma \lambda + \lambda d|^2
\end{equation}

is non–zero and therefore equation (51) implies that (54) is the correct choice of the solution on the inside of the ellipse.

Having computed the Cauchy transform, we can easily compute the Brown measure.
Theorem 3. The Brown measure of the operator $x$ is the uniform probability measure on the inner part of the ellipse given by the system of equations (52), (53).

A careful reader might object that for the case $\alpha = \beta = e^{2i\tau} \gamma$ with $\tau \in \mathbb{R}$ our proof has a gap since we cannot guarantee that the Cauchy transform $G_x(\lambda)$ is continuous. However, in this case the operator $e^{-i\tau}x$ is self-adjoint and coincides with Voiculescu’s semicircular operator, and the calculation of its spectral measure is trivial. On the other hand one can easily check that in the case $\alpha = \beta = |\gamma|$ our ellipse degenerates to an interval and the uniform measure on such a degenerated ellipse coincides with the semicircular measure. Therefore our result is true also in this case. That the Brown measure of the operator $x$ is, at least for $\beta \neq 0$, indeed also the asymptotic eigenvalue distribution of the elliptic-triangular random matrices $A_N$ follows from the result of Śniady [32] that a small Gaussian deformation of a random matrix ensemble does not change the Brown measure in the limit, but makes the convergence of Brown measure continuous; in our case, a small Gaussian deformation does not change the nature of the considered ensemble.

7. Final remarks: Discontinuity of Brown spectral measure

Following the program from Section 1.2 we would like to find the connection between the eigenvalue density of the random matrices $A_N$ and the Brown measure $\mu_x$ of their limit; in particular one would hope that the Brown spectral measure is continuous with respect to the topology of the convergence of $\kappa$-moments and hence the empirical eigenvalue distributions $\mu_{A_N}$ converge to $\mu_x$. Unfortunately, in general this is not true; a very simple counterexample is presented in [19, 32]. The reason for this phenomenon is that the definition of the Fuglede–Kadison determinant uses the logarithm, a function unbounded from below on any neighborhood of zero.

Let us consider some sequence $(A_N)$ of random matrices which converges in $\kappa$-moments to some $x$. Even though there exist such sequences with a property that the eigenvalue densities $\mu_{A_N}$ do not converge to $\mu_x$, there is a growing evidence that such examples are very rare. In particular, it was shown by Haagerup [19] and later by Śniady [32] that every such sequence can be perturbed by a certain small random correction in such a way that the new sequence $(A'_N)$ still converges to $x$ and furthermore the Brown measures converge: $\mu_{A'_N} \to \mu_x$. In general, adding a small perturbation changes the nature of the considered random matrix model, so these results do not apply directly to the original ensemble. (An exception from this is the case of elliptic-triangular random matrices, considered in Section 6.)
There has been a lot of research in this context in the last ten years; in particular, controlling the discontinuity of the Brown measure and thus showing that the asymptotic eigenvalue distribution of the random matrices is indeed given by the Brown measure of the limit operator was achieved in considerable generality in the circular law for Wigner matrices and then, more generally, also for $R$-diagonal operators and bi-unitarily random matrices in [18]. At the moment it is not clear whether the ideas from those investigations apply also to situations like polynomials in Gaussian (or even Wigner matrices). However, we believe the following conjecture to be true. We will address this question in forthcoming investigations.

**Conjecture.** Let $p$ be a (not necessarily self-adjoint) polynomial in $m$ non-commuting variables. Consider $m$ independent self-adjoint Gaussian (or, more general, Wigner) random matrices $X_N^{(1)}, \ldots, X_N^{(m)}$. One knows that they converge in $\ast$-moments to a free semicircular family $s_1, \ldots, s_m$. Consider now the polynomial $p$ evaluated in the random matrices and in the semicircular family, respectively; i.e.,

$$A_N := p(X_N^{(1)}, \ldots, X_N^{(m)}) \quad \text{and} \quad x = p(s_1, \ldots, s_m).$$

The convergence in $\ast$-moments of $X_N^{(1)}, \ldots, X_N^{(m)}$ to $s_1, \ldots, s_m$ implies then that also the (in general, non-normal) $A_N$ converge in $\ast$-moments to the operator $x$. We conjecture that also the eigenvalue distributions $\mu_{A_N}$ of the random matrices $A_N$ converge to the Brown measure $\mu_x$ of the limit operator $x$.

It would be interesting to see whether methods from [31], where the special case of a product of independent elliptic random matrices is treated, can be extended to more general polynomials.

8. **Acknowledgments**

For part of the duration of the work on this project, S.T.B. has been supported by a Discovery Grant from NSERC and by an Alexander von Humboldt Research Fellowship.

Research of P.Ś. supported by *Narodowe Centrum Nauki*, grant number 2014/15/B/ST1/00064.

R.S. has been supported by the ERC Advanced Grant “Non-commutative distributions in free probability” (grant no. 339760).

We thank Tobias Mai for several helpful discussions in the context of this work. We also thank the referee for many suggestions which improved the readability.
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