NONLINEAR DEVELOPMENT OF THE R-MODE INSTABILITY AND THE MAXIMUM ROTATION RATE OF NEUTRON STARS

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ABSTRACT

We describe how the nonlinear development of the R-mode instability of neutron stars influences spin up to millisecond periods via accretion. When nearly resonant interactions of the \( \ell = m = 2 \) R-mode with pairs of “daughter modes” are included, the R-mode saturates at the lowest amplitude which leads to significant excitation of a pair of modes. The lower bound for this threshold amplitude is proportional to the damping rate of the particular daughter modes that are excited parametrically. We show that if dissipation occurs in a very thin boundary layer at the crust–core boundary, the R-mode saturation amplitude is too large for angular momentum gain from accretion to overcome loss to gravitational radiation. We find that lower dissipation is required to explain spin up to frequencies much higher than 300 Hz. We conjecture that if the transition from the fluid core to the crystalline crust occurs over a distance much longer than 1 cm, then a sharp viscous boundary layer fails to form. In this case, damping is due to shear viscosity dissipation integrated over the entire star. We estimate the lowest parametric instability threshold from first principles. The resulting saturation amplitude is low enough to permit spin up to higher frequencies. The requirement to allow continued spin up imposes an upper bound to the frequencies attained via accretion that plausibly may be about 750 Hz. Within this framework, the R-mode is unstable for all millisecond pulsars, whether accreting or not.

Key words: stars: evolution – stars: interiors – stars: neutron – stars: oscillations (including pulsations) – stars: rotation

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1. THE R-MODE INSTABILITY VERSUS THE SPIN-UP LINE

The fastest spinning radio pulsar has a rotational frequency \( \nu = 716 \text{ Hz} \) (Hessels et al. 2006) and 39 have been detected with \( \nu > 400 \text{ Hz} \) (Manchester et al. 2005). See the ATNF Pulsar Catalogue at http://www.atnf.csiro.au/research/pulsar/psrcat/. Moreover, there are 14 pulsars in X-ray binaries with inferred \( \nu > 400 \text{ Hz} \), but none convincingly demonstrated to be faster \( \nu > 620 \text{ Hz} \) (Watts 2012; Patruno & Watts 2012); Chakrabarty 400 Hz, but none convincingly demonstrated to be faster than 620 Hz (Watts 2012; Patruno & Watts 2012); Chakrabarty 2005, 2008, 2012. In the standard picture, millisecond pulsars are thought to be spun up via accretion (Alpar et al. 1982) and the \( P - P \) diagram for radiopulsars is consistent with the idea that accreting neutron stars reach spin equilibrium (e.g., Bildsten et al. 1997) in that there appear to be no neutron stars outside the boundary set by the “spin-up line” (e.g., Arzoumanian et al. 1999),

\[
\nu_{eq} = \frac{\alpha_s}{2\pi} \sqrt{\frac{GM}{R^{3}_{\text{acc}}}} \approx 760 \text{ Hz} \frac{\alpha_s \mu^{3/2}_{30} M_{1.4}^{3/4}}{\eta_{\text{acc}}^{3/5} \mu_{26}^{6/5}}, \quad (1)
\]

where the accretion radius is \( R_{\text{acc}} \approx 2 \eta_{\text{acc}}^{1/2} M_{1.4}^{1/2} M_{13}^{-2/3} \text{ km} \), \( M = 10^{-9} M_9 M_2 \text{ yr}^{-1} \) is the mass accretion rate, \( M = 1.4 M_{1.4} M_{12} \) is the stellar mass, \( \mu = 10^{26} \mu_{26} \text{ G cm}^{2} \) is the stellar magnetic moment, and \( \alpha_s \simeq 1 \) and \( \eta_{\text{acc}} \simeq 1 \) are parameters that are determined by the magnetohydrodynamics of disk accretion. However, there is no particular reason for there to be a spin frequency cutoff as low as 730 Hz; although there are exceptions, for most representative equations of state of dense nuclear matter accretion can spin-up a neutron star from \( M = 1.4 M_{\odot} \) and \( \nu = 0 \) to \( \nu \approx 1000 - 1500 \text{ Hz} \) before instability ensues (e.g., Cook et al. 1994).

The R-mode instability can prevent neutron stars from spinning up to such high frequencies that either dynamical instability or viscosity-driven secular instability occurs. The R-mode instability, which is driven slowly by gravitational radiation but stabilized by viscosity, is reviewed briefly below. However, in the presence of a crust–core boundary layer, the R-mode prevents spin up too efficiently; instability sets in at \( \nu \approx 300 \text{ Hz} \) (see Equation (3)), which is too low to allow for observed frequencies of up to 716 Hz since nonlinear effects prevent substantial spin up while the star is unstable. We call this the “Spin-Up Problem.” Phenomenologically, the absence of millisecond pulsars outside the spin-up line up to at least 660 Hz and the inference that some low-mass X-ray binaries (LMXBs) are spinning faster than 500 Hz suggest that spin up via slow equilibrium accretion is responsible for the highest spin frequencies observed, but the R-mode instability appears to suppress spin up beyond about 300 Hz.

2. R-MODE DYNAMICS AND THE SPIN-UP PROBLEM

The inertial modes of a rotating star may be thought of as zero frequency “gauge modes” of a nonrotating star (\( \delta \rho = -\nabla (\rho \Omega) = 0 \)) which acquire frequencies of \( |\omega| \lesssim 2 \Omega \) in a rotating star to \( \Omega(\Omega) \) (Papaloizou & Pringle 1978; Friedman & Schutz 1978; Lee & Strohmayer 1996; Schenk et al. 2002); the \( \ell = |m| \) R-modes are a subset that are axial to \( \Omega(1) \) for Newtonian stars, and have (rotating frame) frequencies, \( |\omega| = 2 \Omega/(\ell + 1) \). (Relativistic modifications have been discussed by Lockitch et al. 2000.) For the R-modes, \( \Omega(1) \) displacement fields can be expressed in terms of a single (magnetic) vector...
spherical harmonic; decompositions of the other inertial modes are more complicated even at \( \ell = 1 \), generally involving a sum of vector spherical harmonics up to a maximum \( \ell \) (e.g., Lock & Friedman 1999; Yoshida & Lee 2000a, 2001; Lock et al. 2000, 2003).

The \( R \)-modes of rotating neutron stars are destabilized by the emission of gravitational radiation because their rotating and inertial frame frequencies have opposite signs, implying that the rotating frame energy increases as the star radiates energy and angular momentum in the inertial frame (Chandrasekhar 1970; Friedman & Schutz 1978; Andersson 1998; Friedman & Morsink 1998; Lindblom et al. 1998; Bildsten 1998; Andersson et al. 1999). For the most unstable \( \ell_R = m_R = 2 \) \( R \)-mode, the instability grows at a rate (numerical coefficients are for the Newtonian \( N = 1 \) polytrope)

\[
y_{GR} \approx \frac{M_{1.4} R_{10}^4 \nu_{500}^4}{2900 \text{ s}} \approx 1.6 \times 10^{-7} M_{1.4} R_{10}^4 v_{500} \nu_R,
\]

where \( M = 1.4 \times 10^{34} \) \( M_{\odot} \) and \( R = 10R_{10} \) km are the stellar mass and radius, and \( v = 500v_{500} \) Hz; \( \nu = 2\pi/3 = 4\pi v/3 \) (Andersson 1998; Friedman & Morsink 1998; Lindblom 1998; Bildsten 1998; Andersson et al. 1999).

Viscous effects (and other forms of dissipation) act against the instability; the “Chandrasekhar-Friedman-Schutz (CFS) stability curve” in the frequency–temperature (\( \nu - T \)) plane separates stable and unstable states (e.g., Lindblom et al. 1998; Andersson et al. 1999; Bildsten & Ushomirsky 2000; Bildsten & Owen 2002; Nayyar & Owen 2006; Haskell et al. 2009). For accreting neutron stars spinning up toward the CFS stability curve, balancing accretional heating (e.g., Brown 2000) against neutrino cooling implies an internal temperature of \( T \approx 10^9 \text{ K} \) (e.g., Yakovlev & Pethick 2004; Yakovlev et al. 2008; Page et al. 2009). At such low temperatures, dissipation in a viscous boundary layer at the interface between the stellar crust and core is thought to dominate for the \( R \)-mode (e.g., Bildsten & Ushomirsky 2000), implying that the mode first becomes unstable at a spin frequency

\[
\nu_5 \approx 340 \text{ Hz} \left( \frac{\rho_b,14 / T_{S,8}}{2} \right)^{2/11} \times \left[ 10\nu_R (\nu_5/0.9\nu_R) / M_{1.4} R_{10} \right]^{4/11} \nu_{4/11},
\]

where \( \rho_b = 10^{14} \rho_{b,14} \) g cm\(^{-3} \) is the density at the crust–core boundary, which is at radius \( r_b \), and \( \nu_b = 10^5 K (k_B)^2 \nu_{500} \) cm s\(^{-1} \) is the kinematic viscosity at \( r_b \), and \( T_S = 10^8 T_{S,8} \text{ K} \) is the temperature. The quantity \( S_R \) measures the imperviousness of the crust to penetration by the \( R \)-mode; it depends primarily on the shear modulus of the crust, but may also be altered by magnetic effects and compressibility (Levin & Ushomirsky 2001; Mendell 2001; Kinney & Mendell 2003; Glampedakis & Andersson 2006). Levin & Ushomirsky (2001) estimate that for the (most unstable) \( R \)-mode \( S_R \approx 0.1c_{1,8} / R_{10}^3 \)) where \( c_1 = 10^8 c_{1,8} \) cm s\(^{-1} \) is the speed of crustal shear waves. The various input parameters are somewhat uncertain. For example, recent calculations of the shear viscosity in the core of a neutron star with improved treatment of dynamical screening change \( \eta_b \) by a factor of a few and also alter its temperature scaling compared to “traditional” expressions (Cutler et al. 1990; Andersson et al. 2005; Shertin & Yakovlev 2008). While these refinements slightly alter Equation (3), the weak viscosity dependence, \( K^{4/11}_4 \), still implies that \( \nu_5 \) is well below 716 Hz.

However, detailed nonlinear three mode evolutions using representative input physics do not support this; the stellar frequency changes little (Bondarescu et al. 2007).

Two basic principles emerged from our work on multimode (Schenk et al. 2002; Brink et al. 2004, 2005) and three mode (Bondarescu et al. 2007, 2009) nonlinear models for saturation of the \( R \)-mode instability.

1. The \( R \)-mode amplitude does not grow beyond the first or second lowest parametric instability threshold amplitude \( |C_{RPI}|^2 \) for interactions with a pair of daughter modes. \( |C_{RPI}|^2 \) depends on the detuning \( \delta \omega = \omega_R - \omega_2 - \omega_3 \) between the \( R \)-mode (\( \omega_R \)) and daughter (\( \omega_2,3 \)) frequencies, the damping rates of the daughters \( (\gamma_2,3) \), and the three mode coupling \( \kappa \),

\[
|C_{RPI}|^2 = \frac{y_2 y_3}{4k^2_2\nu_3^2} \left[ 1 + \left( \frac{\delta \omega}{\gamma_2 + \gamma_3} \right)^2 \right] = \frac{9}{4(k_2\nu_3^2)^2} \left[ \frac{\gamma_2}{\gamma_2 + \gamma_3} \right]^2.
\]

Parity and triangle selection rules for the interactions require that the principal mode numbers of the daughters satisfy the constraint \( n_1 = n_2 \pm 1, \) and for large \( n \) we expect the viscous damping rates of the daughter modes to have similar values \( \gamma_i \approx \gamma_D \) (\( i = 2, 3 \)); we have also defined \( \delta \omega = \omega_2 - \omega_3 \) (\( i = 2, 3 \)).

2. Dissipation of the multitude of daughter modes heats the star at a rate

\[
H_R = 2MR^2\Omega^2 y_{GR} |C_R|^2.
\]
gravitational radiation spin down balances accretional spin up. But even in the latter case, Bondarescu et al. (2007) found that, when damping in a shearing boundary layer dominates the dissipation, evolutionary tracks never wander very far from $v_S$. Moreover, once accretion ceases, the star spins down along the curve where heating and cooling balance, so the end point is virtually the same as if there were no spin up in the unstable regime.

Thus, we have two aspects of the spin-up problem.

1. The star crosses into the unstable regime at a spin frequency of about 300 Hz.
2. Saturation of the $R$-mode instability prevents spin up to higher frequency.

The physical reasons that the evolution is constrained so tightly can be understood from considering three characteristic $R$-mode amplitudes.

1. From Equation (4), the lowest parametric instability threshold is $|C_R|\text{PTT} \gtrsim 3\gamma_D/2\Omega$. For damping in a shearing boundary layer this inequality implies

$$|C_R|\text{PTT} \gtrsim \frac{3\gamma_D}{2\Omega} \approx \frac{3\gamma_D^2 \ell (dE_D/dr)_b}{4\kappa_D E_D} \approx \frac{1.2 \times 10^{-6} S_D^2 R_4^{1/2} (R dE_D/dr)_b}{\kappa_D T_{\text{b}} v_{500}^{1/2} R_{10} E_D}. \quad (6)$$

Here $S_D < 1$ is the fractional velocity jump across the crust–core boundary for daughter mode $D$, and $\ell = (\eta_D/\Omega)^{1/2}$ is the boundary layer thickness. At principal mode numbers $n_D \gtrsim 30 n_{500} R_{10}/c_{s,8}$, where $c_s$ is the transverse shear speed in the crust, we expect $S_D \approx 1$; for lower $n_D$, $S_D < 1$. The fractional velocity jump for the $n = 3$ $R$-mode is $S_D \approx 0.1$. The lowest $|C_R|\text{PTT}$ arises from modes with $\delta \omega \lesssim \gamma_D$, which are likeliest at large $n$. Explicit evaluation for modes of an incompressible star as well as WKB calculations for a compressible star imply that $(R/\ell)(dE_D/dr)_b$, is independent of $n_D$ for $n_D \gg 1$ (see Appendix A.4). For incompressible stars, calculations by Brink (2005) show that $|\kappa| \lesssim 1$ is insensitive to $n_D$, although larger values are likelier at large $n_D$; moreover, $\kappa_D$ is independent of $\Omega$ (see Schenk et al. 2002; Arras et al. 2003). Thus, the lower bound in Equation (6) is roughly independent of $n_D$. In fact, because the lowest expected $\delta \omega$ decreases with $n$ while $\gamma_D$ is roughly independent of $n_D$, ultimately, $|C_R|\text{PTT} \approx 3\gamma_D/2\Omega$ is the lowest parametric instability threshold for damping in a shearing boundary layer.

2. Gravitational radiation spins the star down at a rate $-J_{\text{GR}} = 6M^2 R^2 \Omega_{\text{GR}} |C_R|^2$. If $-J_{\text{GR}} < -J_{\text{acc}}$, the star spins up until $J_{\text{GR}} = -J_{\text{acc}}$. Otherwise, if $-J_{\text{GR}} > -J_{\text{acc}}$, the star will spin down and re-enter the region in which the $R$-mode is stable. In spin equilibrium, $J = I \Omega_{\text{eq}} \propto 1 M^{5/7} \mu^{-6/7}$, where $\Omega_{\text{eq}} = 2 \pi v_{\text{eq}}$ and $v_{\text{eq}}$ is given by Equation (1). As accretion proceeds, the magnetic moment decreases and $v_{\text{eq}}$ increases (e.g., Shibazaki et al. 1989; Zhang & Kojima 2006). A good approximation is $\mu \propto (\Delta M)^{3/7}$, where $\Delta M$ is the total mass accreted, and so $J \propto \Delta M^{5/7} (\Delta M)^{3/7}$; Shibazaki et al. (1989) originally suggested $\beta = 1$, but Zhang & Kojima (2006) advocate $\beta = 7/4$ until $\mu$ “bottoms out” at $\mu_{\text{25}} \approx 1$ (see also Wang et al. 2011). In general, the accretion torque is defined to be $J_{\text{acc}} = M dJ/dM$, which can be written as $J_{\text{acc}}/J = \sigma_J M/M$, where $\sigma_J = 6\beta M/7\Delta M + 5/7 + 4 \ln 1/\mu M$; spin up is faster before $\mu$ bottoms out, and $\beta \to 0$ and slows as mass accretes and $\nu$ increases. As a simple model, we adopt $J_{\text{acc}}/J = \gamma_{\text{acc}}(v_0/v)^2$; for numerical estimates, we take $\gamma_{\text{acc}}(v_0/v)^2 = 10^{-8} \nu^{-1} v_{\text{acc},8}^3 S_D^2$, where $s \approx 1/3$ and $s \approx 0.7$ before and after $\mu$ bottoms out, respectively. The parameter $\gamma_{\text{acc},8}$ is different for each accreting neutron star. With this simplified model, $J_{\text{GR}} = -J_{\text{acc}}$ at an $R$-mode amplitude

$$|C_R| \approx \frac{2.1 \times 10^{-7} (\sigma_J \gamma_{\text{acc},8})^{1/2}}{v_{500}^3 M_{1.4}^2 R_{10}^3}, \quad (7)$$

where the moment of inertia of the star is $I = 0.37M R^2$. For numerical estimates, we shall use $s = 1/3$, since most of the spin up occurs in this regime. Comparing Equation (7) with Equation (6) we see that $|C_R| \lesssim 0.1 |C_R|\text{PTT}$, which means that $J_{\text{GR}} > -J_{\text{acc}}$, and spin up is prevented.

3. The amplitude at which heating by the $R$-mode balances heating via accretion, $H_{\text{acc}} = \epsilon_{\text{acc}} M c^2$ with $\epsilon_{\text{acc}} = 10^{-3} \epsilon_{\text{acc},3}$ (Brown 2000), is

$$|C_R| H \approx \frac{5 \times 10^{-8} (\epsilon_{\text{acc},3} M_{1.4})^{1/2}}{v_{500}^2 M_{1.4} R_{10}^2}. \quad (8)$$

For $|C_R|\text{PTT} > |C_R| H$, heating by the $R$-mode dominates. Comparing Equation (8) with Equation (6) implies that heating by the $R$-mode is more important than accretional heating for damping in a shearing boundary layer.

A fourth important amplitude comes from equating gravitational radiation spin down with $-J_{\text{B}} = \eta_{\text{mag}} \mu^2 \Omega^2 / 3 \epsilon^3$, the rate of pulsar spin down,

$$|C_R| B \approx \frac{1.5 \times 10^{-8} \mu_{20} \eta_{\text{mag}}^{1/2}}{v_{500}^3 M_{1.4} R_{10}^2}. \quad (9)$$

This is relevant to the evolution after accretion ceases. If $|C_R|\text{PTT} > |C_R| B$, then spin down via gravitational radiation is faster than pulsar spin down.

For damping in a shearing boundary layer, $|C_R|\text{PTT} > |C_R| j$ and so accretion spin up is limited to about 300 Hz, which is inconsistent with observations of pulsars spinning up to 716 Hz. In Figure 1, the left panel illustrates a typical evolution sequence in this case. Because $|C_R|\text{PTT} > |C_R| j$, the evolutionary track in the $v – T$ plane is a rather tight cycle that is confined to a small range of frequencies $\approx v_S$, the frequency at which the mode first becomes unstable, given in Equation (3).

For spin up substantially beyond 300 Hz to be possible, $|C_R|\text{PTT}$ must remain below $|C_R| j$ up to frequencies well above $v_S$. Equation (6) shows that $|C_R|\text{PTT} \approx 3\gamma_D/2\Omega$ but that for damping within a viscous shearing boundary layer, $\gamma_D$ is too large to allow significant spin up. However, Equation (6) also suggests that lower $\gamma_D$ would permit prolonged spin up. In Section 3 we examine what happens if a thin viscous shearing boundary layer does not form near the core–crust boundary, so that $\gamma_D$ is due to shear viscosity damping distributed over the entire star. In Figure 1, the right panel illustrates the sort of evolution that would become possible in this case. As can be seen from the figure, prolonged spin up is possible, but even in this case, there is a maximum attainable spin frequency. We conjecture—but do not prove—that if the transition from core
If these conditions can be met, the star heats modestly after spin up, but may still dominate over accretional heating. Otherwise, the spun up neutron star would simply spin down too rapidly via gravitational radiation, roughly retracing its steps down to the characteristic of the fastest millisecond pulsars. Similarly extended region. We do not present a rigorous calculation of how this happens, since there are many uncertainties, principally in how the shear modulus grows within the transition region. The crude toy model developed in Appendix B illustrates the salient features. Figure 2 shows how the displacement field evolves smoothly across the layer in this toy model. The dissipation associated with this smooth transition of how the shear modulus grows within the transition zone from crust to core, with a characteristic length scale \( \Delta r \), the thickness of the zone.

Suppose that instead of an abrupt transition, the shear modulus of the star grows from zero near \( r_i \) to its value at the inner edge of the crust over a radial zone of thickness, \( \Delta r \gg \ell \). Then we expect the velocity jump to occur over this relatively extended region. We do not present a rigorous calculation of how this happens, since there are many uncertainties, principally in how the shear modulus grows within the transition region. The crude toy model developed in Appendix B illustrates the salient features. Figure 2 shows how the displacement field evolves smoothly across the layer in this toy model. The dissipation associated with this smooth transition is

\[
E \sim 4\pi r_i^2 \Delta r (\Delta v)^2 \times \frac{\eta}{\ell^2} \propto (\Delta r)^{-1}.
\]

(For the specific case of the toy model computed in Appendix B, the constant of proportionality is about two.) For \( \Delta r \gg \ell \), the dissipation rate

\[
\eta \approx \frac{4\pi r_i^2 \ell^2 (\Delta v)^2}{(\Delta r)^{-1}} \propto (\Delta r)^{-1}.
\]
Equations (B7) and (10), with $\delta\omega$ does not rise systematically with $n_D$ with $n_D$. As was mentioned above, explicit calculations by Brink et al. (2004) (see also Brink 2005) indicate that $\kappa_D$ does not rise systematically with $n_D$, the principal mode number of the daughters, although larger values become likelier as $n_D$ increases. We use statistical arguments for the expected smallest value of $\delta\omega$ as a function of $n_D$. We use WKB calculations presented in Appendix A.5 plus explicit numerical evaluations (Brink et al. 2004; Brink 2005) for $\gamma_D$. The upshot is that $\delta\omega$ tends to decrease with $n_D$, whereas $\gamma_D$ tends to increase, so there is a minimum value of $|C_R|^2_{\text{PT}}$ at large values of $n_D$. We shall demonstrate that the minimum occurs at $n_D \simeq 100$, considerably beyond the ranges computed explicitly even for the modes of an incompressible star.

### 3.1. Permeable Crust

Let us consider the non-rigid case first. The WKB calculation detailed in Appendix A.5 implies that

$$
\gamma_D \simeq \frac{p_D^2 \eta_{\text{core}} - \eta_{\text{crust}}(r_b/R)^2}{R^2 \sqrt{1 - (r_b/R)^2}} + \frac{2p_D^2 \eta_{\text{crust}}}{3R^2},
$$

(10)

assuming different kinematic viscosities $\eta_{\text{core}}$ and $\eta_{\text{crust}}$ in the core ($r < r_b$) and crust ($r_b < r < 1$), respectively; here, $p_D = \sqrt{\eta_D (n_D + 1) - |m_D| |m_D| + 1} \simeq n_D$. The second term dominates for sufficiently large values of $n_D$, but since $\eta_{\text{crust}} \ll \eta_{\text{core}}$ (Shternin & Yakovlev 2008), for practical purposes almost all of the dissipation occurs in the core. (For uniform $\eta_{\text{core}} = \eta_{\text{crust}}$ the second term dominates, and Equation (10) agrees with results in Brink et al. 2004.) Moreover, comparing Equations (B7) and (10), with $n_b \sim n_{\text{core}}$, we see that dissipation in the bulk of the star dominates over dissipation in the transition region as long as $p_D \simeq n_D \gtrsim S_D \sqrt{\Delta \Omega} = 10^5 S_D \sqrt{R/100 \Delta \Omega}$. Tentatively, we assume that this inequality holds for the daughter modes involved in the lowest $|C_R|^2_{\text{PT}}$; we shall see that this is likely to be true. Thus, we adopt $\gamma_D = \gamma_D^0$ for estimating the lowest $|C_R|^2_{\text{PT}}$; from the first term in Equation (10) with $p_D \simeq n_D$ (as WKB requires),

$$
\gamma_D^0 = \frac{\eta_{\text{core}}(r_b/R)^2}{\Omega^2 R^2 \sqrt{1 - (r_b/R)^2}} = \frac{5.9 \times 10^{-12} K_4}{\nu_{500} T_8^2 R_{10}^2},
$$

(11)

where $\eta_{\text{core}} = 10^4 K_4 T_8^{-2}$ cm$^2$ s$^{-1} = \eta_0$ and we set $r_b = 0.9 R$.

The other factor in Equation (4) is the detuning. The minimum $\delta\omega/\Omega$ up to principal quantum number $n$ is expected to be approximately $2\sqrt{2}/N(< n)$, where $N(< n) \approx 1/n^2$ is the number of couplings to the $R$-mode, consistent with selection rules for the transitions (Brink 2005). Since we are seeking an estimate of the lowest $|C_R|^2_{\text{PT}}$, we substitute this into Equation (4) to get

$$
|C_R|^2_{\text{PT}} = \frac{9}{4\pi^2} \left( \frac{\gamma_D^0(\eta_{\text{core}}/\Omega^2)^2 + 72}{n_D^5} \right),
$$

(12)

recalling that $\kappa_D$ is relatively insensitive to $n_D$, we minimize the quantity in brackets over $n_D$ and find the lowest value of the threshold at

$$
n_D = 1.5 \left( \frac{\Omega}{\gamma_0} \right)^{1/6} \approx 110 \frac{1/6}{500} T_8^{1/3} R_{10}^{1/3},
$$

(13)

where we have used $r_b = 0.9 R$, and therefore

$$
|C_R|^2_{\text{PT, min}} = \frac{4.2}{\kappa_D} \left( \frac{\gamma_0}{\Omega} \right)^{2/3} \approx 1.4 \times 10^{-7} K_4^{2/3} \nu_{500}^{-4/3} T_8^{-4/3} R_{10}^{-4/3}.
$$

(14)

Requiring that $|C_R|^2_{\text{PT}} < |C_R|^2_{\text{PT, min}}$ implies that nonlinear dynamics limits spin up to frequencies

$$
v \lesssim \frac{590 \nu_{500}^{3/8} T_8^{1/2} (T_{0.3})^{1/2} \gamma_{\text{acc}, 8}^{3/16}}{K_4^{1/4} M_{1.4}^{3/4} R_{10}^{1/4}},
$$

(15)

where we have used $s = 1/3$ to obtain the numerical constant. The existence of a maximum spin frequency limit for spin up via accretion is a generic feature of the dynamics: $|C_R|^2_{\text{PT, min}}$ is determined by a competition between the decrease of the smallest expected detuning $\delta\omega$ and the increase of the dissipation $\gamma_D$ with increasing $n_D$.

Including other physical features that we have neglected here will not do away with this key feature of the dynamics. Two physical features we shall subsequently study are buoyancy and relativistic corrections. Buoyancy shifts mode frequencies, but not that of the $R$-mode (Saio 1982; Yoshida & Lee 2000a), and also activates the $n \neq |m| + 1r$ modes in the star (Saio 1982; Yoshida & Lee 2000b). Relativistic corrections also shift mode frequencies (e.g., Lockitch et al. 2000, 2003), and may also generate non-axial contributions to the $R$-mode eigenfunction which permit additional couplings that would be forbidden non-relativistically (see, e.g., Lockitch et al. 2000, 2003). Studies that combine buoyancy and relativity are tricky (e.g., Kojima 1998; Kojima & Hosonuma 1999; Boutloukos & Nollert 2007; Passamonti et al. 2008) but detailed calculations seem to support the existence of a mode structure very similar to the Newtonian case (Lockitch et al. 2001, 2003, 2004; Pons et al. 2005; Villain et al. 2005). In any event, including both buoyancy and relativistic corrections will not alter the key feature of the network of interacting modes, namely, that there exists a dense set of frequencies bounded above and below, which permits an increasing number of near resonances as $n_D$ increases. Moreover, additional couplings may become possible that would otherwise be forbidden, which could lower the value of $|C_R|^2_{\text{PT, min}}$, thus permitting spin up to larger $v$. Differential rotation and magnetic fields (Rezania & Morsink 2002; Rezzolla et al. 2000, 2001a, 2001b) and mutual friction (Haskell et al. 2013) may play an important role in limiting the $R$-mode amplitude. More work is needed to investigate such effects in detail.

The existence of a maximum frequency dictated by the nonlinear dynamics is a basic conclusion of this paper. The actual value of the maximum frequency depends on the external variable, $\gamma_{\text{acc}, 8}$, even though the dependence is weak. Each
individual neutron star has its own value of $\gamma_{\text{acc,8}}$, so the maximum spin rate that is attainable is not the same for all neutron stars. We emphasize that our estimate of $|CR|_{\text{PT, min}}$ is the key to determining the value of the maximum spin rate. The saturation amplitude of the R-mode is not an adjustable parameter, but is determined by the nonlinear hydrodynamics of the network of interacting modes.

It is reassuring that the value of the maximum frequency in Equation (15) is close to 700 Hz, but to go further we need the value of $T_8$ in particular; this is determined from balancing heating and neutrino cooling. We determine $T$ from the relationship

$$L_v = H_{\text{acc}} + H_R,$$

where $L_v$ is the neutrino cooling rate. We assume that cooling is primarily via the Cooper pair process, with $L_v \sim 10^{13} f_4 T_8^3 \text{ erg s}^{-1}$, where $f_4 \sim 1$ may depend on $M$ and $R$ (e.g., Gusakov et al. 2004; Page et al. 2004, 2011; Shetynin et al. 2011). We evaluate $H_R$ using $|CR|_{\text{PT, min}}$ from Equation (14); with $H_R = \epsilon_{\text{acc}} M c^2$, we find that thermal balance implies

$$f_4 T_8^3 = 57 M_0 \epsilon_{\text{acc,3}} + 360 \frac{20^3}{500} K_4 \frac{M_1^2}{M_{10}^{10/3}} \frac{\kappa_D^3}{f_4^2 T_8^{10/3}},$$

which defines a curve in the $v - T$ plane along which the star evolves during accretion. Equation (17) shows that $H_{\text{acc}}$ dominates at low $v_{500}$ (e.g., where the instability ensues), and $T_8 \approx 1.7 (M_0 \epsilon_{\text{acc,3}}/f_4)^{1/8}$ in this regime; $H_R$ dominates at large $v_{500}$ (i.e., where the upper spin limit is fixed) and

$$T_8 \approx 1.7 \frac{v_{500} M_4^{3/4}}{K_{D,4}^{3/32}} \frac{M_{14}^{3/32}}{R_{10}^{3/16}} f_4 \frac{1/8}{\kappa_D^{3/16}}.$$

Using Equation (18) in Equation (15) implies a more precise upper bound

$$v \lesssim 950 \text{ Hz} \left( \frac{\kappa_D}{K_{D,4}} \right)^{9/22} \left( \frac{\epsilon_{\text{acc,8}}}{0.3} \right)^{3/11} \left( \frac{f_4}{3/44} \right)^{11/3} \left( \frac{M_{14} R_{10}}{M_{10}^{10/3}} \right)^{11/2} \equiv v_{\text{max}}.$$

The full solution of Equation (17) would give a slightly lower value.

Because $\gamma_D$ is low, spin up begins at a significantly lower frequency than Equation (3), typically $v_5 \lesssim 100-150 \text{ Hz}$, so prolonged spin up via accretion is required. Throughout much of this evolution, the R-mode plays almost no role because of the strong frequency dependences of $H_R/H_{\text{acc}}$ and $|CR|_{\text{PT, min}}/|CR|$. Spin up ends either because accretion ceases or because spin equilibrium $|CR|_{\text{PT}} = |CR|/f$ is achieved. In the former case, spin-up proceeds almost as if there were no R-mode instability. At its maximum spin frequency, a neutron star is in spin balance, with equal and opposite gravitational radiation and accretion torques, and remains in that state until accretion ends. Depending on the detailed evolution, spin equilibrium can occupy a substantial fraction of the time during which a neutron star accretes. In thermal balance, Equation (17) shows that the neutron star’s internal temperature is an increasing function of frequency, but also depends on $M_0$, which is different for each accreting neutron star, and $\kappa_D$. Although we expect similar values of $\kappa_D$ for different neutron stars, they need not be identical, because $\eta_D$ is not the same for all neutron stars affected by the R-mode instability. Thus, there may be some variability in internal and effective temperatures for neutron stars in the unstable domain.

Intermittent accretion is unlikely to affect these conclusions, cooling timescales are $\sim 100-1000 \text{ yr}$ so if the heating rate fluctuates at much shorter timescales, the time averaged heating rate is all that matters. Similarly, the detailed time dependent dynamical evolution of the R-mode proceeds on timescales that are too short, $\sim 1/\delta \nu \sim 1/|CR|_{\text{PT}} \Omega$, to be important for the secular evolution of spin and internal temperature; whether there are any observable effects of the dynamics is beyond the scope of this paper.

Once accretion ends, the fast rotating neutron star cools and spins down. Because the cooling timescale is short compared with the spin-down timescale (\(\gtrsim 10^8 \gamma_{\text{acc,8}} \text{ yr}\)) at the end of spin up, the neutron star first cools at fixed spin frequency. Cooling ends when $H_R$ is balanced by cooling. Once this point is reached, the neutron star spins down along the curve given by Equation (18), and

$$|CR|_{\text{PT, min}} \approx \frac{6.6 \times 10^{-8} K_4^{1/2}}{\nu_{500}^3 f_4^{1/8} M_{14}^{1/4} R_{10}^{3/16}}.$$

Slow evolution along this curve is driven by spin down: if there were no change in $v$, then the star would remain at a single point in the $v - T$ plane. The total spin-down rate is the sum of contributions from gravitational radiation and electromagnetic radiation $-\dot{J} = \dot{J}_{\text{GR}} + \dot{J}_B$. The spin-down rate at the lowest parametric instability threshold given by Equation (20) is

$$\frac{-\dot{J}}{I \Omega} \approx \frac{v/\nu_{\text{max}}} {10^8 y^{2/3} \nu_{\text{max,500}}} + \frac{\eta_{\text{mag}} H_{20}^2 \nu_{500}} {14.5 \times 10^6 y I_{3,1} R_{10}^{1/4} \nu_{\text{max}}},$$

where we have used $\nu_{\text{max}} = \eta_{\text{mag}} / 500 \text{ Hz}$. Spin-down ages $t_{\text{sd}} = -I \Omega / 2 \dot{J}$ for radiopulsars with $v \gtrsim 400 \text{ Hz}$ range between $1.64 \times 10^8 y$ and $14.3 \times 10^{10} y$; see Manchester et al. (2005), http://www.atnf.csiro.au/research/pulsar/psrcat/. If we require a spin-down age $\gtrsim 10^9 y$ at $\nu_{\text{max}}$, Equation (21) implies that $\gamma_{\text{acc,8}} \lesssim 0.05$. Inserting this into Equation (19) lowers $v_{\text{max}}$, keeping all other parameters fixed. Since $v_{\text{max}} \propto (\kappa_D \gamma_{\text{acc,8}} / K_4^{1/4} \nu_{\text{max}}^{1/2})$, the bound can still be around 750 Hz if $\kappa_D / K_4^{1/4} \lesssim 4.1$; values of $\kappa_D$ large but not unheard of for incompressible stars (Brink et al. 2004; Brink 2005), and it is conceivable that $K_4 \lesssim 1$. This scenario for millisecond pulsar formation requires that all of the fastest spinning pulsars are in the unstable domain. Equation (21) predicts spin-down indices $n = \nu^{2} / v^2 > 3$. Determinations of $\nu$ for millisecond pulsars are contaminated by timing noise so there is no conclusive evidence against this picture.

3.2. Impermeable Crust

Calculations for the rigid case closely follow the methodology of Section 3.1 but there is an important difference: because the daughter modes are confined to the core, for practical purposes $R$ is replaced by $r_b$ in the WKB solutions. We regard this as an extreme limit, and that more realistically, for the values of $n_D$ we estimate below, the daughter modes penetrate the crust incompletely with a fractional velocity jump $S_D \lesssim 1$. In this case, we get a damping rate

$$\gamma_D = \frac{2 p^3_{\gamma_0} \gamma_{\text{core}}}{3 r_b^2},$$

i.e., we get the second term in Equation (10) with $R \rightarrow r_b$ and $\gamma_{\text{rest}} \rightarrow \gamma_{\text{core}}$. There is no need to include the effect of the transition region, since it is already included (and
partly responsible for the stronger scaling with $p_D$). Instead of Equation (12), we get

$$|C_R|^2_{\text{PT}} = \frac{9}{4\kappa_D^2} \left( \frac{n_D^6}{\Omega^2} + \frac{72}{n_D^4} \right)$$

(23)

but with (letting $r_b = 9.0_{-0.9}^{+1.9}$ km)

$$\frac{\gamma_0}{\Omega} = \frac{\eta_{\text{core}}}{3\gamma\kappa_D^2} = \frac{2.6 \times 10^{-12} K_4}{v_{500} T^{13/8}_{9/7} r_b^{1/9}}.$$  

(24)

Neglecting variations in $\kappa_D$ as before, Equation (23) is minimized at

$$n_D = 1.4 \left( \frac{\Omega}{\gamma_0} \right)^{1/7} \approx \frac{63 v_{500} T_{9/7} r_b^{1/9}}{K_4^{1/7}}$$

(25)

and

$$|C_R|^2_{\text{PT, min}} = \frac{5.3}{\kappa_D} \left( \frac{\eta_{\text{core}}}{\Omega} \right)^{4/7}.$$  

(26)

Following the same procedure as led to Equations (19) and (20) leads to the final results

$$v \lesssim \frac{360 \text{ Hz}}{K_{4/7}^{7/16} (K_{9/7}^3)^{14/89} J_{1/4,14}^{2/3} M_{1/4}^{2/3} R_{10}^{2/3}} = v_{\text{max}}$$

(27)

and

$$|C_R|^2_{\text{PT, min}} \approx \frac{4.1 \times 10^{-7}}{K_{4/7}^{1/9} J_{1/4}^{1/9} v_{\text{max}}} \left( \frac{v_{500} T_{9/7} r_b^{1/9}}{K_4} \right)^{4/7}.$$  

(28)

Equation (27) requires $\kappa_D J_{1/4,14}^{-1/89} K_{9/7}^{4/7} \gtrsim 6.6$ for $v_{\text{max}} \approx 750 \text{ Hz}$, holding all other parameters fixed. Equation (28) implies a spin-down rate

$$-\frac{j}{I\Omega} \approx \left( \frac{v_{\text{max}}}{10^5} \right)^4 \frac{\gamma_{\text{acc},8}}{v_{\text{acc},8}} + \frac{\eta_{\text{mag}}}{10^5} \frac{v_{500}^2}{M_{1/4} R_{10}^2}.$$  

(29)

Just as we found for the nonrigid case, we need to cut down the gravitational radiation contribution in order to be consistent with pulsar data; requiring a spin-down timescale due to gravitational radiation $\gtrsim 10^9$ yr near $v_{\text{max}}$ implies $\gamma_{\text{acc},8} \lesssim 0.05$, and for $v_{\text{max}} \approx 750 \text{ Hz}$ we would then require $\kappa_D / K_4^{4/7} \gtrsim 45$, which is a more stringent constraint than we found in Section 3.1.

4. CONCLUSIONS

Our examination of the nonlinear dynamics of rotational modes of a neutron star suggests that in the conventional picture, where modes damp in a thin viscous boundary layer, spin up beyond about 300 Hz is not possible; not only is the frequency at which the $R$-mode first destabilizes about $300 \text{ Hz}$ (see Equation (3)), but the $R$-mode amplitude saturates at a level large enough that gravitational radiation spin down prevents significant spin up subsequently (see Equation (6)). Thus, we consider what happens if a thin shearing boundary layer cannot form. We conjecture that this may happen if the transition between core and crust occurs in a region thicker than $\sim 1-2$ cm, and justify that assumption partially with the toy model in Appendix B. If a thin boundary layer does not form, damping of all modes is dominated by the distributed effects of shear viscosity throughout the star, which naturally leads to a lower $R$-mode saturation amplitude.

Using scaling relations found by a combination of exact calculations for incompressible stars (Brink et al. 2004; Brink 2005), statistical arguments (Brink 2005), and approximate WKB calculations (Appendix A) we estimate the lowest parametric instability threshold $|C_R|^2_{\text{PT, min}}$ analytically for coupling of the $R$-mode to pairs of daughters. We find that the daughter modes for which this occurs are typically at principal mode quantum $n_D \approx 100$; this is beyond the range for which explicit calculations exist, even for incompressible stars. We stress that the lowest parametric instability threshold sets the amplitude at which the $R$-mode amplitude saturates during evolution of a network of rotational modes of a neutron star (Brink et al. 2005). Thus, our estimate of $|C_R|^2_{\text{PT}}$ represents a first principles calculation of the saturation amplitude. We stress that this is not an adjustable parameter (Heyl 2002), but rather arises from the nonlinear hydrodynamics. Although it may seem counterintuitive, when there are many nearly resonant modes, as is the case for a rotating neutron star, nonlinear effects become important at low amplitude and lead to saturation.

With the lower $|C_R|^2_{\text{PT, min}}$ that arises when shear viscosity dominates the damping, prolonged spin up to frequencies above $500 \text{ Hz}$ is possible. A basic conclusion is that the nonlinear development of the $R$-mode instability naturally gives rise to an upper spin frequency limit. This bound arises from the requirement that $|C_R|^2_{\text{PT, min}}$ be smaller than $|C_R|^2_{\text{PT}, \text{max}}$, the amplitude where gravitational radiation spin down balances accretion spin up. Equations (19) and (27) provide rough estimates for the maximum spin frequency $v_{\text{max}}$ that can be attained under the assumption that the crust is permeable and impermeable to small scale modes, respectively. It is plausible that $v_{\text{max}} \approx 750 \text{ Hz}$, but consistency with observations of millisecond pulsars requires relatively strong (but not outrageously strong) coupling $\kappa_D$; smaller values are allowed for the permeable case, which may argue in its favor. This suggests that nonlinear interactions among the rotational modes of a neutron star may naturally imply a maximum spin frequency below what one might expect from dynamical instabilities of the star. This conclusion is compatible with studies that suggest that LMXBs are not spun up beyond about $730 \text{ Hz}$ (Chakrabarty 2005, 2008, 2012) as well as the fact that the fastest spinning neutron star yet discovered spins at $716 \text{ Hz}$.

A second conclusion of our study is that after accretion ceases, fast spinning millisecond pulsars cool until they reach a balance between neutrino cooling and heating that results from the energy sent to smaller scale modes from the unstable $R$-mode. The result is slow evolution along a curve in the $v-T$ plane (Equation 18). Spin-down timescales are sufficiently long that once spun up, a millisecond pulsar ought to remain close to the upper part of this curve. This means that millisecond pulsars remain stuck in the domain where the $R$-mode is unstable, and are therefore radiating gravitational radiation. However, the emission rate is very low, and strain amplitudes at Earth are correspondingly low, $\sim 10^{-26} / D_{\text{pc}} K_{\text{psd},9}$ for a source at $D = D_{\text{kpc}} \text{ kpc}$ with a spin-down time $10^9 T_{\text{psd},9} \text{ yr}$. Although gravitational radiation may dominate the spin down, because the accretion spin-up rate generally sets torque amplitudes, we expect millisecond pulsars to be near the conventional spin-up line but possibly slightly above it.

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APPENDIX A

ASSORTED WKB RESULTS

A1. Preliminaries: Coordinates

The Bryan coordinates \( x_{1,2} \) for a mode with \( \omega = 2\Omega|\mu| \equiv 2\Omega \cos \theta_{|\mu|} \ll 2\Omega \) are

\[
\sigma = \sqrt{x^2 + y^2} = \sqrt{\frac{(1-x_1^2)(1-x_2^2)}{1-\mu^2}} = \frac{\sin \theta_1 \sin \theta_2}{\sqrt{1-\mu^2}}
\]

\[
z = \frac{x_1 x_2}{|\mu|} = \frac{\cos \theta_1 \cos \theta_2}{|\mu|}
\]

\( x_1 \in [|\mu|, 1], x_2 \in [-|\mu|, |\mu|] \)

\( \theta_1 \equiv \cos^{-1}(x_1) \in [0, \theta_{|\mu|}], \theta_2 \equiv \cos^{-1}(x_2) \in [\theta_{|\mu|}, \pi - \theta_{|\mu|}] \)

We use units in which the radius of the star is \( R = 1 \). The following are useful definitions: with \( \theta_\pm = \theta \pm \theta_{|\mu|} \)

\[
\begin{align*}
\cos(\theta_2 - \theta_1) &= z|\mu| + \sigma \sqrt{1-\mu^2} = r \cos \theta_-
\cos(\theta_2 + \theta_1) &= z|\mu| - \sigma \sqrt{1-\mu^2} = r \cos \theta_+
\end{align*}
\]

(A2)

For finding mode displacements, we will want derivatives of \( \theta_{1,2} \) with respect to coordinates. In cylindrical coordinates, Equation (A1) implies

\[
\begin{align*}
\sqrt{1-\mu^2} d\sigma &= \cos \theta_1 \sin \theta_2 d\theta_1 + \sin \theta_1 \cos \theta_2 d\theta_2 \\
|\mu| d\z &= -\sin \theta_1 \cos \theta_2 d\theta_1 - \cos \theta_1 \sin \theta_2 d\theta_2 \\
d\theta_1 &= \frac{\cos \theta_1 \sin \theta_2 d\sigma \sqrt{1-\mu^2} + \sin \theta_1 \cos \theta_2 d\z |\mu|}{\cos^2 \theta_1 \sin^2 \theta_2 - \sin^2 \theta_1 \cos^2 \theta_2} \\
d\theta_2 &= \frac{\cos \theta_2 \sin \theta_1 d\sigma \sqrt{1-\mu^2} + \sin \theta_2 \cos \theta_1 d\z |\mu|}{\cos^2 \theta_2 \sin^2 \theta_1 - \sin^2 \theta_2 \cos^2 \theta_1}.
\end{align*}
\]

(A3)

Using Equation (A3), we find the area element

\[
dA = \sigma d\sigma d\z = \frac{\sin \theta_1 \sin \theta_2 (\cos \theta_1^2 - \cos \theta_2^2) d\theta_1 d\theta_2}{|\mu|(1-\mu^2)} = \frac{(x_1^2 - x_2^2) dx_1 dx_2}{|\mu|(1-\mu^2)}.
\]

(A4)

(The integral \( \int dA \) over the ranges of \( \theta_{1,2} \) or \( x_{1,2} \) is 2/3.) The stellar surface \( r = 1 \) is patched together in the following way:

\[
\begin{align*}
\theta_1 &= \theta_{|\mu|} \quad \text{and} \quad \theta_2 = \theta \in [\theta_{|\mu|}, \pi - \theta_{|\mu|}] \\
\theta_2 &= \theta_{|\mu|} \quad \text{and} \quad \theta_1 = \theta \in [0, \theta_{|\mu|}] \\
\theta_1 &= \pi - \theta_{|\mu|} \quad \text{and} \quad \theta_2 = \theta \in [0, \theta_{|\mu|}].
\end{align*}
\]

(A5)

There are special points where \( x_1^2 - x_2^2 = 0 = dA \); at these points, \( \cos \theta_1 = \pm \cos \theta_2 = |\mu| \).

A2. WKB Approximation to Displacements

From Arras et al. (2003) Section 3.2, we take the WKB Eulerian enthalpy perturbation to be\(^4\)

\[
\Psi \approx \frac{P_{nm}(x_1) P_{nm}(x_2) \exp[i(m \phi + \omega t)]}{\sqrt{\rho}} \approx \frac{\cos(\rho \theta_1 + \alpha_1) \cos(\rho \theta_2 + \alpha_1) \exp[i(m \phi + \omega t)]}{\sqrt{\rho \sin \theta_1 \sin \theta_2}},
\]

(A6)

where \( \rho = \rho(r) \) is the density profile and \( \rho = \sqrt{\rho(n+1) - m(m+1)} \approx n \). The first approximation assumes that the density scale height is large compared with characteristic scales on which \( \Psi \) varies. The second approximation is for the associated Legendre

\[^4\] \text{However, we use the convention that the mode is proportional to} \exp(i \omega t); \text{Arras et al. (2003)} \text{employed modes} \propto \exp(-i \omega t).
functions, and holds at sufficiently large values of \( r \). The values of the phases \( \alpha_i \) depend on the parity of the mode; based on asymptotic properties of the \( P_{nm}(z) \), Arras et al. (2003) adopted \( \alpha_1 = \alpha_2 = -p \pi/2 \) or \( \alpha = -(p + 1) \pi/2 \) even or odd parity, respectively, but Ivanov & Papaloizou (2010), using a more delicate treatment of boundary conditions, argued for \( \alpha_1 \neq \alpha_2 \). The exact phases should not matter for computing most quantities and we adopt the values used by Arras et al. (2003).

Mode displacements are computed from the equation\(^5\)

\[
\left( 1 - \frac{1}{\mu^2} \right) \xi = \nabla \Psi - \frac{\xi \cdot \nabla \Psi}{\mu} + \frac{i \hat{z} \times \nabla \Psi}{\mu} \tag{A7}
\]

up to an overall normalization factor. With the approximation that the density scale height is large, we do not include derivatives of \( \rho \) in computing the displacements; thus, we write

\[
\sqrt{\rho} \left( 1 - \frac{1}{\mu^2} \right) \xi \approx \exp(+i \omega t) \left\{ \nabla [P_{nm}(x_1)P_{nm}(x_2) \exp(i \phi)] - \frac{\xi \cdot \nabla [P_{nm}(x_1)P_{nm}(x_2) \exp(i \phi)]}{\mu} \right. \\
\left. + \frac{i \hat{z} \times \nabla [P_{nm}(x_1)P_{nm}(x_2) \exp(i \phi)]}{\mu} \right\}. \tag{A8}
\]

For evaluating the derivatives, we use

\[
\nabla P_{nm}(x_i) = \frac{dP_{nm}}{dx_i} \nabla x_i \\
\nabla x_i = -\sin \theta_i \nabla \theta_i = -\sqrt{1 - x_i^2} \nabla \theta_i, \tag{A9}
\]

where \( \nabla \theta_i \) were computed in Equation (A3). If we further invoke the large \( p \) approximation to the associated Legendre polynomials, then we ignore the variation of the sin \( \theta_i \) factors in computing derivatives; in this approximation,

\[
\sqrt{\rho} \sin \theta_1 \sin \theta_2 \left( 1 - \frac{1}{\mu^2} \right) \xi \approx \exp(+i \omega t) \left\{ \nabla [\cos(p \theta_1 + \alpha_1) \cos(p \theta_2 + \alpha_2) \exp(i \phi)] - \frac{\xi \cdot \nabla [\cos(p \theta_1 + \alpha_1) \cos(p \theta_2 + \alpha_2) \exp(i \phi)]}{\mu} \right. \\
\left. + \frac{i \hat{z} \times \nabla [\cos(p \theta_1 + \alpha_1) \cos(p \theta_2 + \alpha_2) \exp(i \phi)]}{\mu} \right\}. \tag{A10}
\]

In Equation (A10), gradients are computed via

\[
\nabla_a [\cos(p \theta_1 + \alpha_1)] = -p \nabla_a \theta_i \sin(p \theta_1 + \alpha_1), \tag{A11}
\]

where \( \nabla \theta_i \) is computed from Equation (A3). The components of the displacement are

\[
\left( 1 - \frac{1}{\mu^2} \right) \xi_{\sigma} = \frac{\partial \Psi}{\partial \sigma} + \frac{i m \Psi}{\mu \sigma} \approx \frac{\partial \Psi}{\partial \sigma} \\
\xi_{\sigma} = \frac{\partial \Psi}{\partial \sigma} \\
\left( 1 - \frac{1}{\mu^2} \right) \xi_{\phi} = \frac{i m \Psi}{\sigma} + \frac{i \partial \Psi}{\mu \sigma} \approx \frac{i \xi_{\sigma}}{\mu}, \tag{A12}
\]

where the approximations are valid within the WKB limit. The necessary derivatives are

\[
\frac{\partial \Psi}{\partial \sigma} = \frac{p \sqrt{1 - \frac{1}{\mu^2} e^{i(m \phi + \text{cut})}}}{2 \sqrt{\rho} \sin \theta_1 \sin \theta_2} \left( \sin \eta_+ - \sin \eta_- \right) \\
\frac{\partial \Psi}{\partial \sigma} = \frac{p |\mu| e^{i(m \phi + \text{cut})}}{2 \sqrt{\rho} \sin \theta_1 \sin \theta_2} \left( \sin \eta_+ + \sin \eta_- \right), \tag{A13}
\]

where \( \eta_{\pm} = p(\theta_2 \pm \theta_1) + \alpha_2 \pm \alpha_1 \) and \( \tilde{\theta}_{\pm} = \theta_2 \pm \theta_1 \).

---

5 The sign of the last term here is opposite to Equation (29) in Arras et al. (2003) because of the different sign convention for frequency used here.
A3. Normalization Integral

Define

$$N \equiv \int d^3 x \rho |\xi|^2; \quad (A14)$$

using Equations (A12) and (A13) as well as Equation (A4), we get

$$N = \frac{\pi |\mu|^2}{(1 - \mu^2) \sin \theta_1 \sin \theta_2} \left\{ \sin^2[p \cos^{-1}(r \cos \theta_+)] + \frac{\sin^2[p \cos^{-1}(r \cos \theta_-)]}{1 - r^2 + r^2 \sin^2 \theta_-} \right\}$$

$$- \frac{\sigma \mu^2 \sin[p \cos^{-1}(r \cos \theta_+)] - \mu \pi \sin[p \cos^{-1}(r \cos \theta_-)]}{(1 - \mu^2) \sin \theta_1 \sin \theta_2 \sin^2[r \cos \theta_+ \cos \theta_- \cos \theta_+ \cos \theta_-]}. \quad (A15)$$

We replace the rapidly oscillating terms $\sin^2 \eta_+ \to 1/2$ and $\sin \eta_+ \sin \eta_- \to 0$. Judiciously substitute $\sin \theta_+ = \sin(\theta_+ \pm 2\theta_0) = \sin \theta _+ \cos 2\theta_0 \pm \cos \theta _+ \sin 2\theta_0$, with which the integral becomes

$$N = \frac{\pi p^2 |\mu|}{2(1 - \mu^2) \sin \theta_1 \sin \theta_2} \int_{\theta_1}^{\theta_2} \int_{\theta_1}^{\theta_2} 2 \cos 2\theta_0 + \sin 2\theta_0 \left( \frac{\cos \theta_+}{\sin \theta_-} - \frac{\cos \theta_-}{\sin \theta_+} \right). \quad (A16)$$

The remaining integrals may all be done analytically; the result is

$$N = \frac{\pi^2 p^2 \mu^2}{(1 - \mu^2)^3/2}. \quad (A17)$$

See Arras et al. (2003), Equation (47); the exact result for Bryan modes is in Brink et al. (2004), Section II.D.

A4. Damping in a Viscous Boundary Layer

For evaluating damping via boundary layer viscosity, we will need to compute the surface integral of $\rho |\xi|^2$. We use Equation (A2) to write $\tilde{\eta}_\pm = \theta_2 \mp \theta_1 = \cos^{-1}(r \cos \theta_\pm)$ and therefore

$$\rho |\xi|^2 \approx \frac{\pi |\mu|^2}{2(1 - \mu^2) \sin \theta_1 \sin \theta_2} \left\{ \sin^2[p \cos^{-1}(r \cos \theta_+)] + \frac{\sin^2[p \cos^{-1}(r \cos \theta_-)]}{1 - r^2 + r^2 \sin^2 \theta_-} \right\}$$

$$- \frac{\sigma \mu^2 \sin[p \cos^{-1}(r \cos \theta_+)] - \mu \pi \sin[p \cos^{-1}(r \cos \theta_-)]}{(1 - \mu^2) \sin \theta_1 \sin \theta_2 \sin^2[r \cos \theta_+ \cos \theta_- \cos \theta_+ \cos \theta_-]}, \quad (A18)$$

where $\sigma = +1$ for even parity and $\sigma = -1$ for odd parity, and in the first two terms we used $\sin^2[r \cos \theta_\pm] = 1 - r^2 \cos^2 \theta_\pm = 1 - r^2 + r^2 \sin^2 \theta_\pm$. Using Equation (A1) to eliminate $\sin \theta_1 \sin \theta_2$, we get

$$\frac{r^2}{2N} \int d\Omega \rho |\xi|^2 \approx \frac{r}{\pi} \int_0^{\pi} d\theta \left\{ \frac{\sin^2[p \cos^{-1}(r \cos \theta_+)]}{1 - r^2 + r^2 \sin^2 \theta_+} + \frac{\sin^2[p \cos^{-1}(r \cos \theta_-)]}{1 - r^2 + r^2 \sin^2 \theta_-} \right\}$$

$$- \frac{\sigma \pi}{\pi} \int_0^{\pi} d\theta \sin[p \cos^{-1}(r \cos \theta_+)] \sin[p \cos^{-1}(r \cos \theta_-)] \sin^2[r \cos \theta_+ \cos \theta_- \cos \theta_+ \cos \theta_-], \quad (A19)$$

where $\sigma = +1$ for even parity and $\sigma = -1$ for odd parity. In the notation of Equation (6), $R(dE_D/dr)_b/E_D$ is twice Equation (A19).

In Equation (A19), the integrands in $\cdots$ have large values near $\theta_+ = \pi$ and $\theta_- = 0$, respectively. Writing $\theta_+ = \pi + \delta$ and $\theta_- = \delta$, we get $1 - r^2 + r^2 \sin^2 \theta_\pm \approx 1 - r^2 + r^2 \delta^2$, and $\cos^{-1}(r \cos \theta_+) = -\sqrt{1 - r^2 + r^2 \delta^2}$ and $\cos^{-1}(r \cos \theta_-) = \sqrt{1 - r^2 + r^2 \delta^2}$, respectively, near those points; in both cases, then

$$\frac{\sin^2[p \cos^{-1}(r \cos \theta_+)]}{1 - r^2 + r^2 \sin^2 \theta_+} \approx \frac{\sin^2[p \sqrt{1 - r^2 + r^2 \delta^2}]}{1 - r^2 + r^2 \delta^2} = \frac{\sin^2[p \sqrt{1 - r^2 + (1 + u^2)^{1/2}}]}{(1 - r^2)(1 + u^2)}, \quad (A20)$$

where $u^2 = r^2 \delta^2/(1 - r^2)$. If $1 - r^2 \ll 1$, we can approximate the integrals by

$$\frac{1}{r \sqrt{1 - r^2}} \int_{-\infty}^{\infty} d\nu \sin^2[p \sqrt{1 - r^2 + (1 + u^2)^{1/2}}] = \frac{\pi K(p \sqrt{1 - r^2})}{r \sqrt{1 - r^2}}; \quad (A21)$$

$$K(z) \approx \begin{cases} z & \text{if } z \ll 1 \frac{1}{z} & \text{if } z \gg 1. \end{cases} \quad (A22)$$

Since the contribution to the integral from the cross term is smaller, we find that

$$\frac{r^2}{2N} \int d\Omega \rho |\xi|^2 \approx \frac{K(p \sqrt{1 - r^2})}{r \sqrt{1 - r^2}} \quad (A23)$$

for $1 - r^2 \ll 1$. The damping rate from a viscous shearing boundary layer is therefore proportional to $R(dE_D/dr)_b/2E_D \simeq K(p \sqrt{1 - r^2})/\sqrt{1 - r^2}$; at large $p$, the damping rate is proportional to $1/2 \sqrt{1 - r^2}$, which is independent of $p$. 
A5. Shear Viscosity Damping within $r = r_b$

For computing shear viscosity damping, we need the square of the shear tensor:

$$
\sigma_{ab} = \frac{\partial \xi_a}{\partial x_b} + \frac{\partial \xi_b}{\partial x_a} - \frac{2}{3} \delta_{ab} \nabla \cdot \xi \approx \frac{\partial \xi_a}{\partial x_b} + \frac{\partial \xi_b}{\partial x_a} = S_{ab} + S_{ba},
$$

(A24)

where the approximation $\nabla \cdot \xi = 0$ holds in WKB. Using Equation (A12), we can compute the shear tensor components; then

$$
\sigma^2 \equiv \sum_{ab} \sigma_{ab} \sigma_{ab} = \left[ \frac{8\mu^4 + 2\mu^2}{(1 - \mu^2)^2} \right] \left( \frac{\partial^2 \Psi}{\partial \sigma^2} \right)^2 + \frac{2(1 - 3\mu^2 + 4\mu^4)}{(1 - \mu^2)^2} \left( \frac{\partial^2 \Psi}{\partial \sigma \partial \zeta} \right)^2;
$$

(A25)

In the WKB limit,

$$
\frac{\partial^2 \Psi}{\partial \sigma \partial \nu} = -\frac{p^2}{\sqrt{\rho} \sin \theta_1 \sin \theta_2} \left[ \cos(p\theta_1 + \alpha_1) \cos(p\theta_2 + \alpha_2) \left( \frac{\partial \theta_1}{\partial \nu} + \frac{\partial \theta_2}{\partial \nu} \right) \right]
$$

$$
-\sin(p\theta_1 + \alpha_1) \sin(p\theta_2 + \alpha_2) \left( \frac{\partial \theta_1}{\partial \nu} + \frac{\partial \theta_2}{\partial \nu} \right);
$$

(A26)

using Equation (A3) and $\sin^2(\theta_1 \pm \theta_2) = 1 - r^2 \cos^2 \theta_\pm$, we get

$$
\sigma^2 = \frac{p^4 \mu^2}{2\rho r \sin \theta (1 - \mu^2)^{3/2}} \left[ \cos^2 [p(\theta_1 + \theta_2) + \alpha_1 + \alpha_2] + \cos^2 [p(\theta_1 - \theta_2) + \alpha_1 - \alpha_2] \right]
$$

$$
+ \frac{4\mu^2(3 - 4\mu^2) \cos^2 [p(\theta_1 + \theta_2) + \alpha_1 + \alpha_2] \cos^2 [p(\theta_1 - \theta_2) + \alpha_1 - \alpha_2] }{(1 - r^2 \cos^2 \theta_\pm)(1 - r^2 \cos^2 \theta_\mp)};
$$

(A27)

the terms on the second line of Equation (A27) oscillate rapidly and will be dropped in our detailed calculation of the damping rate.

We will assume that the main contribution to the damping rate is from the core of the neutron star, $r \ll r_b < 1$; this means that we will never encounter the exact two points on the surface where Equation (A27) is singular. Figure 6 in Shternin & Yakovlev (2008) suggests that the shear viscosity perhaps grows linearly in the core of a neutron star, so we let $\eta = \eta_{\text{core}} \rho$ in the core, where $\eta_{\text{core}}$ is independent of density; this roughly cancels the $1/\rho$ factor in Equation (A27) from the WKB form of the modes. For large values of $\rho$, we approximate the first two terms in Equation (A27) by replacing $\cos^2 [p(\theta_1 \pm \theta_2) + \alpha_1 \pm \alpha_2] \rightarrow \frac{1}{2}$, and drop the cross term entirely. Then

$$
\int d^3r \rho \sigma^2 \approx \frac{\pi p^4 \mu^2}{2(1 - \mu^2)^{3/2}} \int_0^\pi d\theta \int_0^{r_b} dr \frac{1}{r^2 \cos \theta} \left[ (1 - r^2 \cos^2 \theta \mp) + (1 - r^2 \cos^2 \theta \pm) \right]
$$

$$
- \frac{\pi p^4 \mu^2}{4(1 - \mu^2)^{3/2}} \sum_{s=\pm} \int_0^{2\pi} d\theta \frac{1}{1 - r_b \cos \theta};
$$

(A28)

The integrals involved are all $2\pi/\sqrt{1 - r_b^2}$, so the final result is

$$
\int d^3r \rho \sigma^2 = \frac{\pi p^4 \mu^2}{(1 - \mu^2)^{3/2} \sqrt{1 - r_b^2}}.
$$

(A29)

Dividing by $N$ and multiplying by $\eta_{\text{core}}$ gives the damping rate

$$
\gamma_{\text{core}} = \frac{\eta_{\text{core}} p^2 (r_b/R)^2}{R^2 \sqrt{1 - (r_b/R)^2}},
$$

(A30)

where we have restored dimensional units. Note that Equation (A30) would diverge as $r_b \rightarrow R$. That case requires a more careful treatment.

Brink et al. (2004) included the entire star in the calculation of shear damping; Equation (31) in that paper is an accurate analytic fit, which we reproduce here: for kinematic viscosity $\eta$,

$$
\frac{\gamma}{\eta} R^2 = \frac{2n + 1}{3} \left[ (n + 3)(n - 2) - \frac{m(m - 2\mu)}{1 - \mu^2} \right],
$$

(A31)

which is approximately $\gamma/\eta = 2n^3/3$ for $n \gg |m|$, which is typical of couplings of large $n$ modes to the $R$-mode. We have also done a WKB calculation that gives $\gamma R^2/\eta \approx 2p^3/3$. That calculation is rather complicated because the result is dominated by contributions from near the special points on the surface where $\cos \theta_\pm = \pm \cos \theta_\mp = |\mu|$. The procedure is to return to the displacement field $\xi$, introduce approximations valid near the special points, and then compute the shear tensor by direct differentiation. This last step deviates from the strict WKB approximation in that if $k_i = \nabla \theta_i$, it includes terms arising from $\nabla k_i$ which would ordinarily be discarded. The expression that results can then be integrated analytically, and the result is what we quoted above.

To get the expression for shear viscosity damping in the main text, we divide the star into a core out to $r_b$ and crust outside $r_b$, with separate viscosities $\eta_{\text{core}}$ and $\eta_{\text{crust}}$, respectively.
APPENDIX B

TOY MODEL FOR DISPLACEMENT EVOLUTION AS SHEAR MODULUS RISES

We consider the transition region of thickness, $\Delta r$, within which the shear modulus $\mu(x)$ rises from zero to its value in the crystalline crust. We ignore density variation and consider planar displacement fields only with $\nabla \cdot \mathbf{\xi} = 0$. We orient the radial direction along $x$ and define $c_i^2(x) = \mu(x)/\rho$.

We assume that displacements are proportional to functions of $x$ times $\exp[i(k_x x + k_y y)]$; the divergenceless condition implies that $\xi_x$ and $\xi_z$ are both $O(|\partial \xi_z/\partial x|)$, and hence much larger than $\xi_r$. Then if we systematically ignore $k_y z$ compared with $\partial/\partial x$ in this region, both $\xi_r$ and $\xi_z$ obey the approximate linear differential equation

\[- \omega^2 \xi = \frac{\partial}{\partial x} \left( c_i^2 \frac{\partial \xi}{\partial x} \right). \tag{B1} \]

This equation describes the evolution of the jumps in these displacement components. Let $c_i^2 = c_{i,S}^2 f(u)$, where within the layer $x = x_{\text{inner}} + u \Delta r$ and in the solid $c_i^2 = c_{i,S}^2$. The function $f(u)$ may be determined from microphysics. Written in terms of $u$, Equation (B1) is

\[0 = \frac{\partial}{\partial u} \left[ f(u) \frac{\partial \xi}{\partial u} \right] + \frac{\omega^2 (\Delta r)^2 \xi}{c_{i,S}^2} \equiv \frac{\partial}{\partial u} \left[ f(u) \frac{\partial \xi}{\partial u} \right] + q^2 \xi. \tag{B2} \]

Provided that both $f(u)$ and $\xi(u)$ are monotonic, we can regard $\xi$ as a function of $f$.

Realistically, we would solve Equation (B2) for a specified $f(u)$. To get a rough idea of what a solution might look like, we pursue an illustrative toy calculation: let $\xi = f^p$, where $p$ is some powerlaw index, to get

\[0 = \frac{d^2 f^{1+p}}{du^2} + q^2(1+p)f^p; \tag{B3} \]

rescale so that $g = Af^{1+p}$ to get

\[0 = \frac{d^2 g}{du^2} + g^{\frac{g}{p}}; \tag{B4} \]

where we have chosen the scaling constant so that $q^2(1+p)A^{\frac{1}{1+p}} = 1$. We solve Equation (B4) with $g = 0$ at $u = 0$ but $(dg/du)_0 \neq 0$; we impose the condition that $(dg/du)_1 = 0$ at $u = 1$, which follows since $(df/du)_1 = 0$ for $p > 0$. Consequently, $(dg/du)_0$ is an eigenvalue. Since we require that $f(1) = 1$, it follows that $g(1) = A$, so we have $q^2(1+p)[g(1)]^{\frac{1}{1+p}} = 1$, which determines $q^2$. Thus, it should be clear that this choice of $\xi(f)$ is hardly general, and would only hold for a very specific $f(u)$ and $q^2$.

Note that Equation (B4) can be integrated once to yield

\[\frac{dg}{du} = \left( \frac{dg}{du} \right)_0 \sqrt{1 - \left( \frac{g}{g_0} \right)^{\frac{1}{1+p}}}, \tag{B5} \]

where $g^{\frac{1}{1+p}} = (1 + 2p)(dg/du)^2_0/2(1+p)$. Equation (B5) can be solved via quadrature. An acceptable solution has $g(1) = g_0$.

The viscous dissipation rate within the layer is

\[\dot{E} = 4\pi \rho_h r_h^2 \omega^2 \int dx \xi \frac{\partial}{\partial x} \left( \eta \frac{\partial \xi}{\partial x} \right) = \frac{4\pi \rho_h r_h^2 \omega^2}{\Delta r} \left[ \xi \eta(u) \frac{\partial \xi}{\partial u} \bigg|_0^1 - \int_0^1 du \eta(u) \left( \frac{\partial \xi}{\partial u} \right)^2 \right]. \tag{B6} \]

Let the kinematic viscosity be $\eta(u) = \eta_{\text{core}} \hat{\eta}(u)$, where $\hat{\eta}(0) = 1$ and $\hat{\eta}(1) = \eta_{\text{crust}}/\eta_{\text{core}} \ll 1$; we assume $\hat{\eta}(u) \ll 1$ to get an upper bound on $\dot{E}$. Since $\omega \xi = \Delta \omega [g(1)]^{\frac{1}{1+p}}$ in our toy model,

\[\dot{E} = -\frac{4\pi \rho_h r_h^2 (\Delta \omega)^2}{\Delta r (1+p)^2 [g(1)]^{\frac{1}{1+p}}} \left\{ \left[ \frac{(1+p)g^{\frac{1}{1+p}}}{p} \frac{dg}{du} \right]_0^1 + \int_0^1 du \hat{\eta}(u) \left( \frac{dg}{du} \right)^2 \right\}. \tag{B7} \]

Since $g(1) \approx u$ at $u \ll 1$, the integral diverges for $p \leq 1$.

We have solved Equation (B4) for $p = 2$; the eigenvalue is $(dg/du)_0 \approx 0.135164405635$, and $g(1) \approx 0.0811944$. Consequently, $3\omega^2 (\Delta r)^2 [g(1)]^{3/2}/c_{i,S}^2 = 1$, or $c_{i,S} = 1.14\omega \Delta r = 3.5 \times 10^5 \text{ cm s}^{-1} \nu_{\text{geom}}(\omega/\Omega)(\Delta r/100 \text{ m})$, which is a plausible value but cannot be right for all modes, each of which has its own value of $\omega$. For a given $f(u)$, the function $\xi(f)$ must differ among modes, and generally, the problem does not scale as it does when $\xi(f) = f^p$. Nevertheless, this toy model illustrates the salient features of how a transition might occur. The solution is shown in Figure 2. The dissipation rate in this model is $\dot{E} \leq 4\pi \eta_{\text{core}} \rho_h r_h^2 (\Delta \omega)^2/\Delta r \times 1.99$. 

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