In this paper, we propose a new calculation method for the twist-3 gluon Sivers effect within the collinear factorization approach. The method called pole calculation has been used to derive the cross section formula for the single transverse-spin asymmetry (SSA) as a standard method. We point out that we encounter a problem when we try to apply the pole calculation to the SSA in the heavy quarkonium production whose hadronization mechanism is described by non-relativistic QCD (NRQCD) framework. We show that the new calculation method solves this problem and successfully reproduces known results derived by the pole calculation. Our new method extends the applicability of the twist-3 calculation technique to heavy quarkonium productions which are ideal observables for the investigation of the gluon Sivers effect.

I. INTRODUCTION

Unveiling a novel structure of hadron through the study of the single transverse-spin asymmetry (SSA) has been one of major research directions in high energy hadron physics in recent decades. Much experimental effort has been devoted to the measurement of SSAs in many different processes and different kinematic regions since the first observation of large SSAs in the late 70s [1, 2]. Sivers effect generated by a transversely polarized proton is a possible source of the large SSA and it has drawn much attention in the context of the nucleon structure because it gives a new perspective on the orbital spin structure inside the proton. The quark Sivers effect has been well understood in the past couple of decades through the study of SSAs in many processes [3–10]. On the other hand, the gluon Sivers effect has been hardly understood because available data is limited. The understanding of the gluon Sivers effect is the central problem in the high energy spin physics.

Electron-ion collider (EIC), the next-generation collider experiment, could measure SSAs in many processes including heavy flavored meson productions which are ideal observables for the investigation of the gluon Sivers effect. The project of EIC has motivated much theoretical work on the development of perturbative QCD techniques in recent years. Two perturbative QCD frameworks have been mainly developed in order to study the origin of the large SSA in different kinematic regions. The transverse-momentum-dependent (TMD) factorization is valid in the low transverse momentum region of a produced hadron, and in contrast the collinear twist-3 factorization is applicable in the high transverse momentum region. The EIC experiment is expected to measure SSAs in a wide range of the transverse momentum as the RHIC experiment has done. Developments of both two frameworks are required for the complete understanding of the SSAs measured in EIC. The SSA in the heavy charmonium production is one of desired observables for the investigation of the gluon Sivers effect because a heavy quark pair is mainly produced by the gluon fusion process. The gluon Sivers effect on the $J/\psi$ SSA has been well discussed within the TMD factorization [11–20]. However, no calculation has been done within the collinear twist-3 factorization although the basic technique was already formulated in D-meson production [21, 22]. The main reason why it has not been done yet is that there is a conflict between the conventional twist-3 calculation for the gluon Sivers effect and non-relativistic QCD (NRQCD) framework which describes the hadronization mechanism of $J/\psi$. The purpose of this paper is that we resolve the conflict by proposing a new calculation method for the twist-3 gluon Sivers effect. We will show that the new method can reproduce known results derived by the conventional calculation in [22] and does not have any conflicts with the NRQCD framework. Our result in this paper will give a theoretical basis to the application of the twist-3 framework to SSAs in heavy quarkonium productions.
The remainder of this paper is organized as follows: In section II, we introduce definitions of the twist-3 gluon distribution functions relevant to the present study and show some relations among them. In section III, we briefly review the conventional pole calculation and show the problem caused when we apply the pole calculation to the $J/\psi$ SSA. In section IV (and Appendix A), we show the new calculation technique for twist-3 gluon Sivers effect in detail. Section V is devoted to a summary of the present study.

II. DEFINITIONS OF THE Twist-3 GLUON FUNCTIONS FOR THE TRANSVERSELY POLARIZED PROTON

The twist-3 cross section is in general expressed by three types of nonperturbative functions\[23\]. Two of them, kinematical and dynamical functions, are relevant in the case of the proton Sivers effect. Here we recall definitions of those two types of functions for the gluon and some relations between them derived in \[24\].

(1) kinematical twist-3 gluon distribution functions

Kinematical functions are often referred to as the first $k_T^2/M^2$-moment of corresponding TMD functions\[25\]. Exact definitions of the kinematical twist-3 gluon distribution functions are given by

$$\Phi_0^{\alpha\beta}(x) = \int \frac{d\lambda}{2\pi} e^{i\lambda x} (pS_{\perp}|F^\beta(0)F^\alpha(\lambda)|pS_{\perp})(i\partial^\perp)^2$$

$$\equiv \lim_{\xi_{\perp}\to 0} \int \frac{d\lambda}{2\pi} e^{i\lambda x} (pS_{\perp}|(F^\beta(0)|0,\infty\rangle) a_i \frac{d}{d\xi_{\perp}^\perp} (|\infty + \xi_{\perp},\lambda n + \xi_{\perp}|F^\alpha(\lambda n + \xi_{\perp})\rangle a_p S_{\perp})$$

$$= \frac{M_N}{2} g_\perp^{\alpha\beta} e^{\rho n S_{\perp} G_T^{(1)}(x)} + i \frac{M_N}{2} e^{\rho n S_{\perp} S_{\perp}^\perp \Delta G_T^{(1)}(x)} \frac{M_N}{8} \left( e^{\rho n S_{\perp} (g_\perp^{\alpha\beta})} + e^{\rho n S_{\perp} (g_\perp^{\alpha S_{\perp}^\perp})} \right) \Delta H_T^{(1)}(x) + \cdots$$\hspace{1cm}\hspace{1cm}\hspace{1cm}(1)$$

where $p, S_{\perp}$ and $M_N$ represent the proton’s momentum, spin and mass respectively. We use a shorthand notation $e^{\rho n S_{\perp} G_T^{(1)}(x)} \equiv e^{\rho n S_{\perp} p}$ throughout this paper. $[0, \lambda n]$ denotes the gauge-link operator $[0, \lambda n] = P \exp\left( ig \int_0^1 dt A^n(t) \right)$ which guarantees the gauge-invariance of the matrix element. $n$ is a light-like vector satisfying $p \cdot n = 1, n^2 = 0$. The twist-3 functions $G_T^{(1)}(x)$ and $\Delta H_T^{(1)}(x)$ are relevant to the study of the SSA, whereas the contribution from $\Delta G_T^{(1)}(x)$ is canceled because it gives a pure imaginary contribution.

(2) Dynamical twist-3 gluon distribution functions

Dynamical twist-3 gluon distribution functions are defined by a matrix element of three gluon field strength tensors. There are two different types, C-even function $N(x_1, x_2)$ and C-odd function $O(x_1, x_2)$, reflecting the fact that there are two structure constants $i f^{abc}$ and $d^{abc}$ in SU($N_c$) group,

$$N^{\alpha\beta\gamma}(x_1, x_2) = i \int \frac{d\lambda}{2\pi} \int \frac{d\mu}{2\pi} e^{i\lambda x_1} e^{i\mu x_2-x_1} (pS_{\perp}|i f^{\rho\alpha\alpha} F^\gamma_{\rho\alpha}(0) F^\alpha_{\beta}(\mu)|pS_{\perp})$$

$$= 2i M_N \left[ g_\perp^{\alpha\beta} e^{\rho n S_{\perp} N(x_1, x_2)} - g_\perp^{\beta\gamma} e^{\rho n S_{\perp} N(x_2, x_2 - x_1)} - g_\perp^{\alpha\gamma} e^{\rho n S_{\perp} N(x_1, x_1 - x_2)} \right] + \cdots$$\hspace{1cm}\hspace{1cm}\hspace{1cm}(2)$$

$$O^{\alpha\beta\gamma}(x_1, x_2) = i \int \frac{d\lambda}{2\pi} \int \frac{d\mu}{2\pi} e^{i\lambda x_1} e^{i\mu x_2-x_1} (pS_{\perp}|d^{\rho\alpha} F^\gamma_{\rho\alpha}(0) F^\alpha_{\beta}(\mu)|pS_{\perp})$$

$$= 2i M_N \left[ g_\perp^{\alpha\gamma} e^{\rho n S_{\perp} O(x_1, x_2)} + g_\perp^{\beta\gamma} e^{\rho n S_{\perp} O(x_2, x_2 - x_1)} + g_\perp^{\alpha\gamma} e^{\rho n S_{\perp} O(x_1, x_1 - x_2)} \right] + \cdots$$\hspace{1cm}\hspace{1cm}\hspace{1cm}(3)$$

where we omitted gauge-links for simplicity. These dynamical functions have the following symmetries.

$$O(x_1, x_2) = O(x_2, x_1), \hspace{1cm} N(x_1, x_2) = N(x_2, x_1),$$

$$N(x_1, x_2) = N(x_2, x_1), \hspace{1cm} O(x_1, x_2) = O(-x_1, -x_2),$$

$$N(x_1, x_2) = -N(-x_1, -x_2).$$\hspace{1cm}\hspace{1cm}\hspace{1cm}(4)$$

Each kinematical function has a relation with the C-even dynamical function by the equivalence between matrix elements (1) and (2) as

$$G_T^{(1)}(x) = -4\pi (N(x, x) - N(x, 0)), \hspace{1cm} \Delta H_T^{(1)}(x) = 8\pi N(x, 0).$$\hspace{1cm}\hspace{1cm}\hspace{1cm}(5)$$

We will show that these relations are required to reproduce known results calculated by the conventional pole calculation.
III. POLE CALCULATION

A. Introduction to the conventional pole calculation

We introduce the conventional pole calculation in order to compare it with the new method in the next section and to see a problem in applying it to $J/\psi$ production. It is known that a na"ively $T$-odd observable like the SSA has to be canceled without a nontrivial imaginary phase. When we apply the collinear framework to the proton Sivers effect, the imaginary phase is given by the imaginary part of a propagator. A parton propagator with the momentum $k$ is given by

$$
\frac{1}{k^2 + i\epsilon} = P \frac{1}{k^2} - i\pi\delta(k^2).
$$

(6)

The second term with the delta function gives the imaginary phase and this is called pole contribution. In the conventional pole calculation, we replace a propagator with the delta function term at the beginning of the calculation. We here briefly review the pole calculation in semi-inclusive deep inelastic scattering (SIDIS),

$$
e(\ell) + p^T(p, S_\perp) \rightarrow e(\ell') + \pi(P_h) + X.
$$

(7)

The polarized cross section formula for the pion production is given in the hadron frame [25] as

$$
\frac{d^6\Delta\sigma}{dx_{bj}dQ^2dz_fdp_H^2d\phi d\chi} = \frac{\alpha_{em}^2}{128\pi^4}z_f^2S^2_{\perp p}\frac{z_f^2Q^2}{p_{bj}^2}W^\sigma(\ell, \ell')W^\rho_{\sigma\rho}(q, P_h),
$$

(8)

where $\alpha_{em} = e^2/4\pi$ is the QED coupling constant, $\chi$ and $\phi$ are the azimuthal angles of the hadron plane and the lepton plane respectively, the leptonic tensor is given by $L^\sigma(\ell, \ell') = 2(\ell^\rho\ell^\sigma + \ell^\sigma\ell^\rho) - Q^2g^{\rho\sigma}$ and we use the following Lorentz invariant variables.

$$
S_{cp} = (p + \ell)^2, \quad Q^2 = -q^2 = -\ell(\ell - \ell')^2, \quad x_{bj} = \frac{Q^2}{2p \cdot q}, \quad z_f = \frac{p \cdot P_h}{p \cdot q}.
$$

(9)

The hadronic tensor $W_{\rho\sigma}(p, q, P_h)$ describes the scattering of the proton and the virtual photon. The twist-3 effect from the proton is convoluted with the parton fragmentation function of the pion $D(z)$,

$$
W_{\rho\sigma}(p, q, P_h) = \int \frac{dz}{z^2} D(z)w_{\rho\sigma}(p, q, \frac{P_h}{z}).
$$

(10)

We show how to calculate the twist-3 contribution from $w_{\rho\sigma}(p, q, \frac{P_h}{z})$ in the conventional method. The imaginary phase in [6] requires an interference with one coherent gluon as shown in FIG. 1. The coherent gluon line generates the additional propagator indicated by the red bar and this propagator is replaced with the delta function by using the formula (6). The corresponding contribution to FIG. 1 is mathematically given by

$$
w_{\rho\sigma}(p, q, \frac{P_h}{z}) = \int \frac{dk_1}{(2\pi)^4} \int \frac{dk_2}{(2\pi)^4} \int \frac{d\xi}{2\pi} e^{ik_1\cdot\xi}e^{ik_2\cdot(k_2-k_1)}(pS_\perp|A_\nu'\bar{g}(0)gA_\sigma^\lambda(\eta)A_\mu^\rho(\xi)|pS_\perp)S_{\mu\nu\lambda,\rho\sigma}^{abc}(k_1, k_2).
$$

(11)

The hard part $S_{\mu\nu\lambda,\rho\sigma}^{abc}(k_1, k_2)$ is given by the sum of the diagrams in FIG. 1. We omit the virtual photon indices $\rho, \sigma$ for simplicity. The proton matrix element consists of three gluon fields which are not gauge invariant and $w(p, q, \frac{P_h}{z})$ includes all twist effects. A systematic way to extract the twist-3 contribution from (11) was developed in [22] and we can derive the gauge-invariant expression as

$$
w(p, q, \frac{P_h}{z})
\begin{equation}
= \omega^\mu_\nu\omega^\sigma_\rho\omega^\gamma_\lambda \int \frac{dx_1}{x_1} \int \frac{dx_2}{x_2} \left( -if_{abc} \frac{N_\alpha\beta\gamma(x_1, x_2)}{N_c(N_c^2-1)} + N_c\frac{d_{abc}}{(N_c^2-4)(N_c^2-1)}O^{\alpha\beta\gamma}(x_1, x_2) \right) \frac{\partial}{\partial k_\lambda^\nu}S_{\mu\nu\lambda,\rho\sigma}^{abc}(k_1, k_2) \bigg|_{k_i=x_i,p},
\end{equation}
$$

(12)

where $\omega^\mu_\nu = g^\mu_\nu - p^\mu n_\nu$. A reader can use this formula without following the derivation in detail if the hard part $S_{\mu\nu\lambda}^{abc}(k_1, k_2)$ satisfies four required conditions. Three of them are Ward-Takahashi identities (WTIs),

$$
(k_2 - k_1)^\lambda S_{\mu\nu\lambda,\rho\sigma}^{abc}(k_1, k_2) = 0,
$$

(13)

$$
k_1^\mu S_{\mu\nu\lambda,\rho\sigma}^{abc}(k_1, k_2) = 0,
$$

(14)

$$
k_2^\nu S_{\mu\nu\lambda,\rho\sigma}^{abc}(k_1, k_2) = 0.
$$

(15)
FIG. 1. $S^{abc}_{\mu\nu\rho}(k_1, k_2)$ is given by the sum of these diagrams and their complex conjugate diagrams. The coherent gluon with the momentum $k_2 - k_1$ generates the additional propagator indicated by the red bar. The red barred propagator is replaced with the delta function $-i\pi\delta\left(\frac{P_z}{z} - (k_2 - k_1)^2\right)$ by using (6).

One can easily check that the diagrams in FIG. 1 satisfy these conditions by taking into account the delta function given by the pole contribution. There is one more required condition

$$\left(\frac{\partial}{\partial k_2^\lambda} S^{abc}_{\mu\nu\rho}(k_1, k_2) + \frac{\partial}{\partial k_1^\lambda} S^{abc}_{\mu\nu\rho}(k_1, k_2)\right)_{k_i = x_i, p} = 0.$$  \hspace{1cm} (16)

Note that this condition cannot be derived by taking the derivative of $(k_2 - k_1)^\lambda S^{abc}_{\mu\nu\rho}(k_1, k_2) = 0$ with respect to $k_1, 2$ because the pole contribution gives $x_1 = x_2$ in the collinear limit. Although the WTI (13) - (15) are followed by the gauge-invariance, the condition (16) is not clearly based on such a symmetry of the field theory. Therefore, we have to verify that the sum of diagrams satisfies the condition (16) case by case with a direct inspection when we apply the pole method. Here we check that the pair of diagrams in FIG. 2 satisfies the condition by introducing the explicit form as follows:

$$S^{abc}_{\mu\nu\rho}(k_1, k_2)\bigg|_{\text{FIG. 2}} = -\text{Tr}\left[\frac{P_h}{z} \gamma^\rho \left(\frac{P_h}{z} - (z_2 - z_1)\right) \gamma_\mu \left(\frac{P_h}{z} - (z_2 - z_1) - \frac{q}{z}\right) \gamma_\nu \left(\frac{P_h}{z} + \frac{q}{z} - \frac{P_h}{z}\right) \gamma_\rho\right] \times \frac{1}{(\frac{P_h}{z} - (k_2 - k_1) - q)^2} \frac{1}{(\frac{P_h}{z} - q)^2} \left(-i\pi\delta\left(\frac{P_h}{z} - (k_2 - k_1)^2\right)\right) 2\pi\delta\left((k_2 + q - \frac{P_h}{z})^2\right)$$

$$-\text{Tr}\left[\frac{P_h}{z} \gamma^\rho \left(\frac{P_h}{z} - \frac{q}{z}\right) \gamma_\mu (z_1 + q - \frac{P_h}{z}) \gamma_\nu \left(\frac{P_h}{z} + (z_2 - z_1) - \frac{q}{z}\right) \gamma_\rho \left(\frac{P_h}{z} + (z_2 - z_1)\right) \gamma_\rho\right] \times \frac{1}{(\frac{P_h}{z} + (k_2 - k_1) - q)^2} \frac{1}{(\frac{P_h}{z} - q)^2} \left(i\pi\delta\left(\frac{P_h}{z} + (k_2 - k_1)^2\right)\right) 2\pi\delta\left((k_1 + q - \frac{P_h}{z})^2\right),$$  \hspace{1cm} (17)
FIG. 2. A pair of the diagram which satisfies the condition (16).

where we omitted the common color factor. We can find that two diagrams are canceled at \(x_1 = x_2\) in the collinear limit because they are topologically the same and the imaginary phases caused by (6) have opposite signs. The \((k_2 - k_1)\)-dependent parts obviously satisfies (16). We find that the terms given by \(k_2\)-derivative acting on \((k_2 + q - P_h)\)-dependence are canceled with the terms given by \(k_1\)-derivative acting on \((k_1 + q - P_h)\)-dependence in the collinear limit \(x_1 = x_2\), which means the condition (16) is satisfied. One can check that other pairs in FIG. 1 satisfy (16) in the same way. We have confirmed that (16) is still satisfied when we take into account a heavy quark mass \[22\] as long as an amplitude and its complex conjugate are connected by a single trace factor \(\text{Tr} \cdots\) like (17). As stated above, the condition (16) is not based on fundamental symmetries of the theory and it is not guaranteed to hold in the general case. As we see in the next subsection, NRQCD is a counter-example to (16) and it hinders an application of the pole calculation to the SSA in the heavy quarkonium production when we use NRQCD framework to describe the hadronization mechanism. Elimination of the condition (16) extends the applicability of the twist-3 calculation method which is necessary for the understanding of SSAs in heavy quarkonium productions.

B. A problem in the extension to the heavy quarkonium production

NRQCD is a widely accepted theoretical framework for the description of the hadronization mechanism of a heavy quarkonium\[27, 28\]. The \(J/\psi\) production in SIDIS is illustrated as

\[
e(\ell) + p^+(p, S_{\perp}) \rightarrow e(\ell') + \sum_n c\bar{c}[n](P_h) + X,
\]

(18)

where \(n = 3S_1^{[1]}, 1S_0^{[8]}, 3S_1^{[8]}, \ldots\) denotes possible Fock states of the charm quark pair hadronizing into \(J/\psi\). Here we focus on the color-singlet contribution\(3S_1^{[1]}\) to the \(J/\psi\) production as one of the examples so that one can see the breakdown of the condition (16). A typical diagram which gives the color-singlet contribution is shown in FIG. 3. We insert the coherent gluon line to the diagram in order to give the imaginary phase as shown in FIG. 4. The condition (16) should be satisfied in the pair of diagrams in FIG. 4 in analogy with the pion production. We can show the
FIG. 3. A typical diagram which gives the color-singlet contribution.

FIG. 4. Corresponding pair of diagrams to the pair in FIG. 2.

explicit form as follows:

\[
S^{abc}_{\mu \nu \rho \tau}(k_1, k_2)_{\text{FIG.4}} = -\Tr[\Pi^{\gamma \delta}_1\gamma_\mu(p_\gamma - (k_2 - k_1) + m_c)\gamma_\nu(p_\nu - (k_2 - k_1) - q + m_c)\gamma_\tau(p_\tau - k_2 - q + m_c)]
\]

\[
\times \Pi^{\rho \sigma}_1[\gamma_\rho(p_\rho - k_2 - q + m_c)\gamma_\sigma(p_\sigma - q + m_c)]
\]

\[
\times \Pi^{\gamma \delta}_1[\gamma_\gamma(k_2 + q - 2p_c)]
\]

\[
\times \delta((k_2 + q - 2p_c)^2).
\]

(19)
where $p_c$ and $m_c$ are respectively the momentum and the mass of the charm quark and $g_1^{\gamma^5}(k_{1,2} + q - 2p_c)$ is the polarization tensor of the unobserved gluon. The color-singlet hadronization gives $\Pi_{\alpha\beta}$ and $\Pi_1^\alpha$ which are defined as

$$\Pi_{\alpha\beta} = -g_{\alpha\beta} + \frac{P_{ha}P_{h\beta}}{P^2_h}, \quad \Pi_1^\alpha = \frac{1}{\sqrt{8m_c^2}}(p_c - m_c)\gamma^\alpha(p_c + m_c).$$

The important difference from the pion production is that the trace factors $\text{Tr}[\cdots]$ are separately closed in the amplitude level. When the coherent gluon line with momentum $k_2 - k_1$ is attached to different closed charm quark loops, the two diagrams in FIG. 4 are topologically different from each other and thus they are not canceled at $x_1 = x_2$ in the collinear limit. As a result, some terms from $k_{1,2}$-derivatives are not canceled, which means that the condition (16) is not satisfied within NRQCD framework. In the next section, we will show that the new calculation technique does not rely on the condition (16) and thus it can be applied to the heavy quarkonium production.

IV. NEW APPROACH TO THE TWIST-3 GLUON SIVERS EFFECT

In this section, we show the new calculation technique for the twist-3 gluon Sivers effect. We never use the identity (6), which is the main difference from the conventional pole calculation. We take into account the normal $2 \to 2$-scattering hard part $S_{\mu\nu}^{ab}(k)$ in FIG. 5 without the coherent gluon because we do not separate the real and the imaginary contributions at the beginning,

$$w(p, q, \frac{P_h}{z}) = \int \frac{d^4k}{(2\pi)^4} \int d^4\xi e^{ik\cdot\xi} \langle pS_{\perp}|A_0^a(0)A_0^b(\xi)|pS_{\perp}\rangle S_{\mu\nu}^{ab}(k)$$

$$+ \frac{1}{2} \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \int d^4\eta e^{ik_1\cdot\xi} e^{i\eta(k_2 - k_1)} \langle pS_{\perp}|A_0^a(0)gA_0^\lambda(\eta)A_0^b(\xi)|pS_{\perp}\rangle S_{\mu\nu}^{abc}(k_1, k_2).$$

FIG. 5. $S_{\mu\nu}^{ab}(k)$ is given by the sum of these diagrams.
FIG. 6. $S_{\mu\nu\lambda}^{abc}(k_1, k_2)$ is given by the sum of these diagrams and their complex conjugate diagrams. The coherent gluon line with the momentum $k_2 - k_1$ is connected to each black dot and thus there are 12 diagrams in this figure.

where the factor $1/2$ in the second term in the RHS is needed from the interchange symmetry of the external gluon lines. The hard part $S_{\mu\nu\lambda}^{abc}(k_1, k_2)$ is given by the sum of diagrams in FIG. 6. We can show WTIs for hard parts $S_{\mu\nu}^{ab}(k)$ and $S_{\mu\nu\lambda}^{abc}(k_1, k_2)$ [31] as

\begin{align*}
  k^\mu S_{\mu\nu}(k) &= k^\nu S_{\mu\nu}(k) = 0, \quad (22) \\
  (k_2 - k_1)^\lambda S_{\mu\nu\lambda}^{abc}(k_1, k_2) &= i f^{abc} (S_{\mu\nu}(k_2) - S_{\mu\nu}(k_1)), \quad (23) \\
  k_1^\mu S_{\mu\nu\lambda}^{abc}(k_1, k_2) &= -i f^{abc} S_{\lambda\nu}(k_2), \quad (24) \\
  k_2^\nu S_{\mu\nu\lambda}^{abc}(k_1, k_2) &= -i f^{abc} S_{\mu\lambda}(k_1), \quad (25)
\end{align*}

where we used the color averaged hard part $S_{\mu\nu}(k) = \frac{1}{N_c^2 - 1} \delta^{ab} S_{\mu\nu}^{ab}(k)$. Note that WTIs (23) - (25) have additional terms in the RHS compared to (13) - (15) because the delta function from the pole contribution does not exist in the present case. We summarize the whole derivation of the gauge-invariant twist-3 contribution in Appendix A because it is technically complicated and lengthy. The twist-3 contribution from $w_{\rho\sigma}(p, q, \frac{P_h}{z})$ is given in terms of gauge-invariant matrix elements,

\begin{equation}
  w_{\rho\sigma}(p, q, \frac{P_h}{z}) = \omega^\mu \omega^\nu \int \frac{dx}{x^2} \Phi^{\alpha\beta}(x) S_{\mu\nu,\rho\sigma}(xp) + \omega^\mu \omega^\nu \omega^\lambda \int \frac{dx}{x^2} \Phi^{\alpha\beta\gamma}(x) \frac{\partial}{\partial k^\lambda} S_{\mu\nu,\rho\sigma}(k) \bigg|_{k=px} \\
  - \frac{1}{2} \omega^\mu \omega^\nu \omega^\lambda \int dx_1 \int dx_2 \left( -i f^{abc} N_c (N_c^2 - 1)^2 O^{\alpha\beta\gamma}(x_1, x_2) + \frac{N_c d^{abc}}{(N_c^2 - 4)(N_c^2 - 1)} O^{\alpha\beta\gamma}(x_1, x_2) \right) \\
  \times \frac{1}{x_1 - i \epsilon} \frac{1}{x_2 + i \epsilon} \frac{1}{x_2 - x_1 - i \epsilon} S_{\mu\nu\lambda,\rho\sigma}(x_1 p, x_2 p), \quad (26)
\end{equation}
where we restored the virtual photon indices \( \rho, \sigma \). We would like to emphasize that we only used WTI s [22]–[25] in the derivation and no additional conditions like \( \rho \equiv \sigma \rightarrow \) were needed. The contribution from the matrix element
\[
\Phi^{\alpha \beta}(x) = \int \frac{d\lambda}{2\pi^2} \lambda x \langle pS_\perp | F^{\beta n}(0) F^{\alpha n}(\lambda) | pS_\perp \rangle,
\]
is canceled in the SSA due to the PT-invariance. The hard part \( S_{\mu \lambda, \rho \sigma}^{abc}(x_1 p, x_2 p) \) can be separated into two parts,
\[
S_{\mu \lambda, \rho \sigma}^{abc}(x_1 p, x_2 p) = H_{\mu \lambda, \rho \sigma}^L(x_1 p, x_2 p) 2\pi \delta((x_2 p + q - P_h z) + H_{\mu \lambda, \rho \sigma}^R(x_1 p, x_2 p) 2\pi \delta((x_1 p - q - P_h z)^2),
\]
where the superscript \( L(R) \) means that the coherent gluon is connected to the left(right) side of the cut. We can check that each hard part has the following pole structure (see discussion around Eq. (47) in [27]).
\[
H_{\mu \lambda, \rho \sigma}^L(x_1 p, x_2 p) = H_{\mu \lambda, \rho \sigma}^{L0,abc}(x_1 p, x_2 p) + \frac{x_2}{x_1 - \iota \epsilon} H_{\mu \lambda, \rho \sigma}^{L1,abc}(x_1 p, x_2 p) + \frac{x_2}{x_2 - x_1 - \iota \epsilon} H_{\mu \lambda, \rho \sigma}^{L2,abc}(x_1 p, x_2 p),
\]
\[
H_{\mu \lambda, \rho \sigma}^R(x_1 p, x_2 p) = H_{\mu \lambda, \rho \sigma}^{R0,abc}(x_1 p, x_2 p) + \frac{x_1}{x_2 + \iota \epsilon} H_{\mu \lambda, \rho \sigma}^{R1,abc}(x_1 p, x_2 p) + \frac{x_1}{x_2 - x_1 - \iota \epsilon} H_{\mu \lambda, \rho \sigma}^{R2,abc}(x_1 p, x_2 p).
\]
Substituting (1), (2) and (3) into (26), we can show the general form of the twist-3 contribution from \( w_{\rho \sigma}(p, q, \frac{P_h}{z}) \),
\[
w_{\rho \sigma}(p, q, \frac{P_h}{z}) = 2\pi \int \frac{dx}{x^2} \delta((xp + q - \frac{P_h}{z})^2)
\]
\[
\left[ \left( \frac{d}{dx} G^{(1)}_T(x) - 2G^{(1)}_T(x) \right) H_{\rho \sigma}^{G1} + G^{(1)}_T(x) H_{\rho \sigma}^{G2} + \left( \frac{d}{dx} \Delta H^{(1)}_T(x) - 2\Delta H^{(1)}_T(x) \right) H_{\rho \sigma}^{H1} + \Delta H^{(1)}_T(x) H_{\rho \sigma}^{H2} \right.
\]
\[
\left. + \int \frac{dx}{x^2} \sum_i \left[ \left( \frac{1}{x - \iota \epsilon} H_{1, \rho \sigma}^{N1} + \frac{1}{x - x' - \iota \epsilon} H_{2, \rho \sigma}^{N1} + \frac{x}{(x' - \iota \epsilon)^2} H_{3, \rho \sigma}^{N1} + \frac{x}{(x - x' - \iota \epsilon)^2} H_{4, \rho \sigma}^{N1} \right) N^i(x', x) \right.
\]
\[
\left. + \frac{1}{x' + \iota \epsilon} H_{1, \rho \sigma}^{O1} + \frac{1}{x - x' + \iota \epsilon} H_{2, \rho \sigma}^{O1} + \frac{x}{(x' + \iota \epsilon)^2} H_{3, \rho \sigma}^{O1} + \frac{x}{(x - x' + \iota \epsilon)^2} H_{4, \rho \sigma}^{O1} \right) O^i(x', x) \right] \right),
\]
where we used shorthand notations
\[
N^{1,2,3}(x', x) = \{ N(x', x), N(x, x - x'), N(x', x - x') \}, \quad O^{1,2,3}(x', x) = \{ O(x', x), O(x, x - x'), O(x', x - x') \}.
\]
Furthermore, the dynamical part can be simplified by interchanging the integral variable as \( x' \rightarrow x - x' \) and using the symmetries [4],
\[
\int \frac{dx}{x^2} \sum_i \left[ \left( \frac{1}{x - x' - \iota \epsilon} H_{S\perp, \rho \sigma}^{N1} + \frac{x}{(x - x' - \iota \epsilon)^2} H_{D\perp, \rho \sigma}^{N1} \right) N^i(x', x) \right.
\]
\[
\left. + \frac{1}{x - x' + \iota \epsilon} H_{S\perp, \rho \sigma}^{O1} + \frac{x}{(x - x' + \iota \epsilon)^2} H_{D\perp, \rho \sigma}^{O1} \right) O^i(x', x) \right]
\]
\[
+ \frac{1}{x - x' - \iota \epsilon} H_{S\perp, \rho \sigma}^{O1} + \frac{x}{(x - x' + \iota \epsilon)^2} H_{D\perp, \rho \sigma}^{O1} \right) O^i(x', x). \]
\]
The hard parts were recombined as
\[
H_{S\perp, \rho \sigma}^{N1} = H_{S\perp, \rho \sigma}^{N1} + H_{S\perp, \rho \sigma}^{N2}, \quad H_{D\perp, \rho \sigma}^{N1} = H_{D\perp, \rho \sigma}^{N1} + H_{D\perp, \rho \sigma}^{N2},
\]
\[
H_{S\perp, \rho \sigma}^{N2} = H_{S\perp, \rho \sigma}^{N2} + H_{S\perp, \rho \sigma}^{N1}, \quad H_{D\perp, \rho \sigma}^{N2} = H_{D\perp, \rho \sigma}^{N2} + H_{D\perp, \rho \sigma}^{N1},
\]
\[
H_{S\perp, \rho \sigma}^{N3} = H_{S\perp, \rho \sigma}^{N3} + H_{S\perp, \rho \sigma}^{N1}, \quad H_{D\perp, \rho \sigma}^{N3} = H_{D\perp, \rho \sigma}^{N3} + H_{D\perp, \rho \sigma}^{N1},
\]
\[
H_{S\perp, \rho \sigma}^{N1} = H_{S\perp, \rho \sigma}^{N1} + H_{S\perp, \rho \sigma}^{N1}, \quad H_{D\perp, \rho \sigma}^{N1} = H_{D\perp, \rho \sigma}^{N1} + H_{D\perp, \rho \sigma}^{N1},
\]
\[
H_{S\perp, \rho \sigma}^{N2} = H_{S\perp, \rho \sigma}^{N2} + H_{S\perp, \rho \sigma}^{N1}, \quad H_{D\perp, \rho \sigma}^{N2} = H_{D\perp, \rho \sigma}^{N2} + H_{D\perp, \rho \sigma}^{N1},
\]
\[
H_{S\perp, \rho \sigma}^{N3} = H_{S\perp, \rho \sigma}^{N3} + H_{S\perp, \rho \sigma}^{N1}, \quad H_{D\perp, \rho \sigma}^{N3} = H_{D\perp, \rho \sigma}^{N3} + H_{D\perp, \rho \sigma}^{N1},
\]
\[
H_{S\perp, \rho \sigma}^{N1} = H_{S\perp, \rho \sigma}^{N1} + H_{S\perp, \rho \sigma}^{N1}, \quad H_{D\perp, \rho \sigma}^{N1} = H_{D\perp, \rho \sigma}^{N1} + H_{D\perp, \rho \sigma}^{N1},
\]
\[
H_{S\perp, \rho \sigma}^{N2} = H_{S\perp, \rho \sigma}^{N2} + H_{S\perp, \rho \sigma}^{N1}, \quad H_{D\perp, \rho \sigma}^{N2} = H_{D\perp, \rho \sigma}^{N2} + H_{D\perp, \rho \sigma}^{N1},
\]
\[
H_{S\perp, \rho \sigma}^{N3} = H_{S\perp, \rho \sigma}^{N3} + H_{S\perp, \rho \sigma}^{N1}, \quad H_{D\perp, \rho \sigma}^{N3} = H_{D\perp, \rho \sigma}^{N3} + H_{D\perp, \rho \sigma}^{N1}.
\]
The cross section formula is derived by calculating \( L^{\rho\sigma} w_{\rho\sigma}(p, q, \frac{P_h}{z}) \). The lepton tensor \( L^{\rho\sigma} \) is conventionally expanded as

\[
L^{\rho\sigma} = \sum_{k=1}^{9} (L^{\mu\nu} V_{k\mu\nu}) \tilde{V}_{k}^{\rho\sigma}, \tag{35}
\]

where tensors \( V_{k\mu\nu} \) and \( \tilde{V}_{k}^{\mu\nu} \) are given in [22, 30]. Symmetric tensors \( k = 1, 2, 3, 4, 8, 9 \) give nonzero contributions to the cross section. From a direct calculation, we have observed

\[
\begin{align*}
V_{k}^{1(01)} H_{\Sigma_L\rho\sigma} &= -\tilde{V}_{k}^{\rho\sigma} H_{\Sigma_L\rho\sigma}^{1(01)}, & V_{k}^{2(02)} H_{\Sigma_L\rho\sigma} &= -\tilde{V}_{k}^{\rho\sigma} H_{\Sigma_L\rho\sigma}^{2(02)}, \\
V_{k}^{3(01)} H_{\Sigma_L\rho\sigma} &= -\tilde{V}_{k}^{\rho\sigma} H_{\Sigma_L\rho\sigma}^{1(01)}, & V_{k}^{4(02)} H_{\Sigma_L\rho\sigma} &= -\tilde{V}_{k}^{\rho\sigma} H_{\Sigma_L\rho\sigma}^{2(02)},
\end{align*}
\]

for any \( k \). Thus we can perform contour integrations as

\[
\begin{align*}
\frac{1}{2} \int dx' \left( \frac{1}{x-x'-i\epsilon} - \frac{1}{x-x'+i\epsilon} \right) \left( N(x', x) + N(x, x') \right) &= 2\pi i N(x, x), \tag{37} \\
\int dx' \left( \frac{1}{x-x'-i\epsilon} - \frac{1}{x-x'+i\epsilon} \right) N(x, x-x') &= 2\pi i N(x, 0), \tag{38} \\
\int dx' \left( \frac{1}{x-x'-i\epsilon} - \frac{1}{x-x'+i\epsilon} \right) N(x', x-x') &= 2\pi i N(x, 0), \tag{39} \\
\frac{1}{2} \int dx' \left( \frac{1}{(x-x'-i\epsilon)^2} - \frac{1}{(x-x'+i\epsilon)^2} \right) \left( N(x', x) + N(x, x') \right) &= -\pi i \frac{d}{dx} N(x, x), \tag{40} \\
\int dx' \left( \frac{1}{x-x'-i\epsilon} - \frac{1}{x-x'+i\epsilon} \right) \left( N(x', x-x') + N(x', x-x') \right) &= -2\pi i \frac{d}{dx} N(x, 0), \tag{41}
\end{align*}
\]

and the same results for \( O'(x', x) \). The kinematical functions in [31] can be eliminated by using [5]. Then we can finally express the cross section formula only in terms of \( O(x, x), O(x, 0), N(x, x) \) and \( N(x, 0) \) as shown by the pole calculation [22],

\[
\frac{d^6 \Delta \sigma}{dx_{bj} dQ^2 dz_f dP^2_k d\phi d\chi} = \frac{\alpha_s^2 \alpha_s}{16 \pi^2 z_f x_{bj} S_{ep}^2} \left( -\frac{\pi M_N}{2} \right) \sum_a c_a^2 \sum_k A_k(\phi - \chi) S_k(\Phi_s - \chi) \int \frac{dz}{z^2} D_a(z) \int \frac{dx}{x} \delta \left( x p + q - \frac{P_h}{z} \right)^2 \left[ \left( \frac{d}{dx} O(x, x) + \frac{d}{dx} N(x, x) \right) \Delta \delta_k^1 + \left( \frac{O(x, x)}{x} + \frac{N(x, x)}{x} \right) \Delta \delta_k^2 - 2 \Delta \delta_k^3 \right] + \left( \frac{d}{dx} O(x, 0) - \frac{d}{dx} N(x, 0) \right) \Delta \delta_k^2 + \left( \frac{O(x, 0)}{x} - \frac{N(x, 0)}{x} \right) \Delta \delta_k^3 - 2 \Delta \delta_k^2, \tag{42}
\]

where forms of \( A_k(\phi), S_k(\Phi_s) \) are given in [22, 30] and the index \( a = u, d, s, \ldots \) denotes the flavor of the quark fragmenting into the pion. We have confirmed that all hard cross sections for \( k = 1, 2, 3, 4, 8, 9 \) are consistent with those in [22] in the massless limit \( m_{q_i} = 0 \). Our calculation shows that the condition (16) in the conventional pole calculation is an unnecessary condition caused by the separation of the pole part in [6]. All necessary conditions are only WTI s [22, 23] followed by the gauge invariant structure in the amplitude level. The gauge invariance is the fundamental property of a scattering amplitude and it should be satisfied in reasonable perturbative QCD frameworks including NRQCD. Our new method extends the applicability of the twist-3 technique for the gluon Sivers effect, which is of great importance in the analysis of SSAs in heavy quarkonium productions.

The unknown nonperturbative functions \( O(x, x), O(x, 0), N(x, x) \) and \( N(x, 0) \) can be determined through the standard global analysis of data. The cross section formula for the SSA has been derived in some processes (D-meson production in SIDIS [21, 22] and \( pp \) [32, 33]), direct photon production and Drell-Yan in \( pp \) [34]) and the experimental investigation has just begun [35, 36]. We expect that the C-odd contribution from \( O(x, x) \) and \( O(x, 0) \) is canceled in the quarkonium production and thus the SSAs in heavy quarkonium productions could play a role in the independent determination of C-even functions \( N(x, x) \) and \( N(x, 0) \).

V. SUMMARY

In this paper, we have proposed the new calculation method for the twist-3 gluon Sivers effect in the collinear factorization approach and confirmed that known results calculated by the conventional pole method can be successfully
reproduced. Our calculation has clarified that the pole calculation was using an unnecessary condition which hinders the application of the twist-3 framework to heavy quarkonium productions. Our new method just requires basic WTs which are satisfied in reasonable perturbative QCD frameworks including NRQCD for the heavy quarkonium production. SSAs in heavy meson productions are ideal observables for the study of the gluon Sivers effect. The future EIC experiment is expected to measure these SSAs in the high transverse momentum region of produced hadrons as the RHIC experiment has done in the past couple of decades. Our method gives a theoretical basis to the analysis of SSAs observed in a wide range of the transverse momentum.

Appendix A: Derivation of Eq. (26)

We decompose a momentum vector into the longitudinal part and the other part as
\[ k^\mu = (k \cdot n)p^\mu + \omega^\mu p^\rho, \]  
(A1)
in order to extract twist-3 contributions from \( w(p, q, \frac{p}{z}) \). Consequently the WTI (22) is rewritten as
\[ k^\mu S_{\mu\nu}(k) = (k \cdot n)S_{\mu\nu}(k) + \omega^\mu p^\rho S_{\mu\nu}(k) = 0, \quad k^\nu S_{\mu\nu}(k) = (k \cdot n)S_{\mu\nu}(k) + \omega^\nu k^\rho S_{\mu\nu}(k) = 0. \]
(A2)

We can calculate the first term in (21) as
\[
\int \frac{d^4k}{(2\pi)^4} \int d^4\xi e^{ik \cdot \xi} \langle pS_\perp | A_\alpha^\perp(0)A_\alpha^\perp(\xi) | pS_\perp \rangle g^\rho_\sigma \rho^\nu_\sigma S_{\mu\nu}(k)
= \int \frac{d^4k}{(2\pi)^4} \int d^4\xi e^{ik \cdot \xi} \langle pS_\perp | A_\alpha^\perp(0)A_\alpha^\perp(\xi) | pS_\perp \rangle (p^\mu n_\rho + \omega^\rho n_\mu + \omega^\nu n_\sigma + \omega^\sigma n_\nu) S_{\mu\nu}(k)
= \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k \cdot n)^2} \omega^\mu \omega^\nu \int \frac{d^4\xi e^{ik \cdot \xi} \langle pS_\perp | A_\alpha^\perp(0)A_\alpha^\perp(\xi) | pS_\perp \rangle (k^\tau n_\rho - (k \cdot n) g^\rho_\tau)(k^\delta n_\sigma - (k \cdot n) g^\delta_\sigma) S_{\mu\nu}(k)
= \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k \cdot n)^2} \omega^\mu \omega^\nu \int \frac{d^4\xi e^{ik \cdot \xi} \langle pS_\perp | F_\alpha^{(0)\delta\tau}(0)F_\alpha^{(0)\tau\nu}(\xi) | pS_\perp \rangle S_{\mu\nu}(k),
\]  
(A3)
where \( F_\alpha^{(0)\tau\nu}(\xi) = \partial^\tau A_\alpha^\perp(\xi) - \partial^\nu A_\alpha^\perp(\xi) \) and we used (A2) in the second equality. We perform the collinear expansion for the hard part,
\[ S_{\mu\nu}(k) \approx S_{\mu\nu}((k \cdot n)p) + \frac{\partial}{\partial k^\lambda} S_{\mu\nu}(k) \bigg|_{k=(k-n)p} \omega^\lambda k^\kappa. \]  
(A4)

As a result, we obtain the twist-3 contribution from the first term in (21),
\[
\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k \cdot n)^2} \omega^\mu \omega^\nu \int \frac{d^4\xi e^{ik \cdot \xi} \langle pS_\perp | F_\alpha^{(0)\delta\tau}(0)F_\alpha^{(0)\tau\nu}(\xi) | pS_\perp \rangle \left(S_{\mu\nu}((k \cdot n)p) + \frac{\partial}{\partial k^\lambda} S_{\mu\nu}(k) \bigg|_{k=(k-n)p} \omega^\lambda k^\kappa\right)
= \omega^\mu \omega^\nu \int \frac{dx}{2\pi} \int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle pS_\perp | F_\alpha^{(0)\delta\tau}(0)F_\alpha^{(0)\tau\nu}(\lambda n) | pS_\perp \rangle S_{\mu\nu}((x,p))
+ i\omega^\mu \omega^\nu \int \frac{dx}{2\pi} \int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle pS_\perp | F_\alpha^{(0)\delta\tau}(0)F_\alpha^{(0)\tau\nu}(\lambda n) | pS_\perp \rangle \frac{\partial}{\partial k^\lambda} S_{\mu\nu}(k) \bigg|_{k=x,p}. \]  
(A5)

We next calculate the second term in (21). We rewrite (23)–(25) as,
\[
(k_2 - k_1)^\lambda S^{abc}_{\mu\nu\lambda}(k_1, k_2) = (k_2 \cdot n - k_1 \cdot n)S^{abc}_{\mu\nu\rho}(k_1, k_2) + \omega^\lambda (k_2 - k_1)^\phi S^{abc}_{\mu\nu\lambda}(k_1, k_2) = i f^{abc} \left(S_{\mu\nu}(k_2) - S_{\mu\nu}(k_1) \right), \]  
(A6)
\[
k^{\mu}_{S} S^{abc}_{\mu\nu\lambda}(k_1, k_2) = k_1 \cdot n S^{abc}_{\mu\nu\lambda}(k_1, k_2) + \omega^\mu k^{\rho}_{\perp} S^{abc}_{\mu\nu\lambda}(k_1, k_2) = -i f^{abc} S_{\mu\nu}(k_2), \]  
(A7)
\[
k^{\nu}_{S} S^{abc}_{\mu\nu\lambda}(k_1, k_2) = k_2 \cdot n S^{abc}_{\mu\nu\lambda}(k_1, k_2) + \omega^\nu k^{\rho}_{\perp} S^{abc}_{\mu\nu\lambda}(k_1, k_2) = -i f^{abc} S_{\mu\nu}(k_1). \]  
(A8)
Then we can show

\[ \frac{1}{2} \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \int d^4\xi \int d^4\eta \, e^{ik_1 \cdot \xi} e^{i\eta \cdot (k_2 - k_1)} \langle pS_\perp | A^a_\perp (0) g A^b(\eta) A^c(\xi) | pS_\perp \rangle \]
\[ \times (p^\mu n_\rho + \omega^\mu_\rho)(p^\nu n_\sigma + \omega^\nu_\sigma)(p^\lambda n_\phi + \omega^\lambda_\phi) S^{abc}_{\mu \nu \lambda}(k_1, k_2) \]
\[ = \frac{1}{2} \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \int d^4\xi \int d^4\eta \, e^{ik_1 \cdot \xi} e^{i\eta \cdot (k_2 - k_1)} \langle pS_\perp | A^a_\perp (0) g A^b(\eta) A^c(\xi) | pS_\perp \rangle \]
\[ \times \left[ -\frac{1}{k_1 \cdot n - i\epsilon} \frac{1}{k_2 \cdot n + i\epsilon} \frac{1}{k_2 \cdot n - i\epsilon} \frac{1}{k_1 \cdot n - i\epsilon} \omega^\mu_\rho \omega^\nu_\sigma \omega^\lambda_\phi \right] S^{abc}_{\mu \nu \lambda}(k_1, k_2) \]
\[ \times \left[ -i f^{abc}_{\mu \nu \lambda}(k_1, k_2) + \frac{1}{k_1 \cdot n - i\epsilon} n_\rho \int \frac{d^4k}{(2\pi)^4} \frac{1}{k_2 \cdot n + i\epsilon} \omega^\mu_\sigma \omega^\lambda_\phi (k_2^\rho n_\sigma - (k_2 \cdot n) g^\rho_\sigma) S^\nu_\phi (k_2) \right] \]
\[ + \frac{1}{k_1 \cdot n - i\epsilon} n_\rho \int \frac{d^4k}{(2\pi)^4} \frac{1}{k_2 \cdot n + i\epsilon} \omega^\mu_\sigma \omega^\lambda_\phi (k_2^\rho n_\sigma - (k_2 \cdot n) g^\rho_\sigma) S^\nu_\phi (k_1) \]
\[ + \frac{1}{k_1 \cdot n - i\epsilon} n_\rho \int \frac{d^4k}{(2\pi)^4} \frac{1}{k_2 \cdot n + i\epsilon} \omega^\mu_\sigma \omega^\lambda_\phi (k_2^\rho n_\sigma - (k_2 \cdot n) g^\rho_\sigma) S^\nu_\phi (k_2) \]
\[ - \frac{1}{k_1 \cdot n - i\epsilon} n_\rho \int \frac{d^4k}{(2\pi)^4} \frac{1}{k_2 \cdot n + i\epsilon} \omega^\mu_\sigma \omega^\lambda_\phi (k_2^\rho n_\sigma - (k_2 \cdot n) g^\rho_\sigma) S^\nu_\phi (k_1) \right] \]  \quad \text{(A9)}

We keep \( i\epsilon \)-terms in order to correctly perform contour integrations. The signs of \( i\epsilon \) are uniquely determined from the fact that the diagrams in FIG. 6 have only the final state interaction. We perform the collinear expansion for \( S^{abc}_{\mu \nu \lambda}(k_1, k_2) \).

\[ S^{abc}_{\mu \nu \lambda}(k_1, k_2) \approx S^{abc}_{\mu \nu \lambda}((k_1 \cdot n)p, (k_2 \cdot n)p) \]. \quad \text{(A10)}

The first term in the bracket \([\cdots]\) in \text{(A9)} can be calculated as

\[ \frac{1}{2} \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \int d^4\xi \int d^4\eta \frac{e^{ik_1 \cdot \xi}}{2\pi^2} \frac{e^{i\eta \cdot (k_2 - k_1)}}{2\pi^2} \langle pS_\perp | F^{(0)\delta n}_b (0) g F^{(0)\kappa \sigma}_a (\mu n) F^{(0)\tau \eta}_c (\lambda n) | pS_\perp \rangle \]
\[ \times \omega^\mu_\rho \omega^\nu_\sigma \omega^\lambda_\phi \int dx_1 \int dx_2 \frac{d\lambda}{2\pi} \frac{d\mu}{2\pi} \frac{d\nu}{2\pi} \frac{d\sigma}{2\pi} \frac{d\tau}{2\pi} \frac{d\kappa}{2\pi} \frac{d\rho}{2\pi} \frac{d\phi}{2\pi} \]
\[ \times \frac{1}{x_1 - i\epsilon x_2 + i\epsilon x_2 - x_1 - i\epsilon} S^{abc}_{\mu \nu \lambda}(x_1 p, x_2 p) \]. \quad \text{(A11)}

This term gives \( g^1 \)-term in the dynamical matrix elements \[2\] and \[3\]. We use the collinear expansion \[A4\] for other terms in \text{(A9)}. We first show contributions from the first term of \text{(A4)}. The second term in the bracket \([\cdots]\) in \text{(A9)}
where we used the integral representation of the theta function,

$$\int \frac{dx}{x-i\epsilon} = 2\pi i \theta(q).$$

(A13)

The remaining terms in (A9) can be calculated in the same way as

$$\frac{1}{2} \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \int d^4\xi \int d^4\eta e^{ik_1\cdot\xi} e^{i(k_2-k_1)} \langle pS_{\perp}|A_{\eta}^f(0)gA_{\phi}^\phi(\eta)A_{\alpha}^\alpha(\xi)|pS_{\perp}\rangle \times \left[ -if^{abc} \frac{1}{(k_1\cdot n)\cdot i\epsilon(n_\rho - (k_1\cdot n)g^\rho)\frac{1}{(k_2\cdot n)\cdot i\epsilon(n_\rho - (k_2\cdot n)g^\rho)} S_{\mu\nu}((k_1\cdot n)p) \right]$$

(A14)
\[
\frac{1}{2} \int \frac{d^4 k_1}{(2\pi)^4} \int \frac{d^4 k_2}{(2\pi)^4} \int d^4 \xi \, e^{i k_1 \cdot \xi} e^{i (k_2 - k_1) \cdot \eta} \langle p S_\perp | A_\rho^a(0) g A_\sigma^a(\eta) A_\tau^a(\xi) | p S_\perp \rangle
\]
\[
\times \left[ - \frac{i f_{abc}}{k_2 \cdot n - k_1 \cdot n - i\epsilon} \frac{1}{k_1 \cdot n - i\epsilon} \frac{1}{k_2 \cdot n + i\epsilon} \omega^\mu_\nu^\sigma_\delta (k_1^\lambda n_\rho - (k_1 \cdot n) g^\rho_\nu) (k_2^\sigma n_\tau - (k_2 \cdot n) g^\delta_\tau) \delta_{\mu\nu}(k_1 \cdot n) \right]
\]
\[
= \frac{1}{2} \int \frac{dx}{x^2} \int \frac{d\lambda}{2\pi} e^{i x \lambda} \omega^\mu_\nu^\sigma_\delta \left[ i f_{abc} \int_0^\infty d\mu \langle p S_\perp | F_b^{(0)\delta_n}(\mu) | F_c^{(0)\tau_n}(\lambda) | p S_\perp \rangle - i f_{abc} \int_0^\infty d\mu \langle p S_\perp | F_b^{(0)\delta_n}(\mu) | g A_{\rho}^a(\mu) F_c^{(0)\tau_n}(\lambda) | p S_\perp \rangle \right] S_{\mu\nu}(x p).
\] (A16)

Combining (A12)–(A16), we obtain
\[
\omega^\mu_\nu^\sigma_\delta \int \frac{dx}{x^2} \int \frac{d\lambda}{2\pi} e^{i x \lambda} \left[ \langle p S_\perp | F_b^{(0)\delta_n}(\mu) | F_c^{(0)\tau_n}(\lambda) | p S_\perp \rangle + \langle p S_\perp | F_b^{(0)\delta_n}(\mu) | g A_{\rho}^a(\mu) F_c^{(0)\tau_n}(\lambda) | p S_\perp \rangle \right] S_{\mu\nu}(x p).
\] (A17)

We find that they are \(g^1\)-terms of the gauge invariant form \(\langle p S_\perp | F_b^{(0)\delta_n}(\mu) | 0, \lambda_n \rangle_a F_c^{(0)\tau_n}(\lambda) | p S_\perp \rangle\). The calculation for the first derivative term in the collinear expansion (A4) can be simply carried by performing additional partial integration with respect to \(k_1, 2\) in (A12)–(A16). Thus we can show
\[
\omega_\mu^\gamma \omega_\nu^\delta \omega_\sigma^\lambda \int \frac{dx}{x^2} \int \frac{d\lambda}{2\pi} e^{i x \lambda} \left[ \langle p S_\perp | F_b^{(0)\delta_n}(\mu) | F_c^{(0)\tau_n}(\lambda) | p S_\perp \rangle + \langle p S_\perp | F_b^{(0)\delta_n}(\mu) | g A_{\rho}^a(\mu) F_c^{(0)\tau_n}(\lambda) | p S_\perp \rangle \right] \frac{\partial}{\partial k^\lambda} S_{\mu\nu}(k) \right|_{k=x p}
\] (A18)

which are \(g^1\)-terms of the matrix element of the kinematical functions \(1\).

ACKNOWLEDGMENTS

This research is supported by National Natural Science Foundation in China under grant No. 1195010495, Guangdong Natural Science Foundation under No. 2020A1515010794 and research startup funding at South China Normal University.

[1] R. D. Klem, J. E. Bowers, H. W. Courant, H. Kagan, M. L. Marshak, E. A. Peterson, K. Ruddick and W. H. Dragoset et al., Phys. Rev. Lett. 36, 929 (1976).
[2] G. Bunce et al., Phys. Rev. Lett. 36, 1113 (1976).
[3] M. Anselmino, M. Boglione, U. D’Alesio, A. Kotzinian, S. Melis, F. Murgia, A. Prokudin and C. Turk, Eur. Phys. J. A 39, 89-100 (2009) doi:10.1140/epja/i2008-10697-y [arXiv:0805.2677 [hep-ph]].
[4] M. Anselmino, M. Boglione and S. Melis, Phys. Rev. D 86, 014028 (2012) doi:10.1103/PhysRevD.86.014028 [arXiv:1204.1239 [hep-ph]].
[5] M. Anselmino, M. Boglione, U. D’Alesio, S. Melis, F. Murgia and A. Prokudin, Phys. Rev. D 88, no.5, 054023 (2013) doi:10.1103/PhysRevD.88.054023 [arXiv:1304.7691 [hep-ph]].
[6] M. G. Echevarria, A. Idilbi, Z. B. Kang and I. Vitev, Phys. Rev. D 89, 074013 (2014) doi:10.1103/PhysRevD.89.074013 [arXiv:1401.5078 [hep-ph]].
[7] A. Martin, F. Bradamante and V. Barone, Phys. Rev. D 95, no.9, 094024 (2017) doi:10.1103/PhysRevD.95.094024 [arXiv:1701.08283 [hep-ph]].
[8] M. G. Echevarria, Z. B. Kang and J. Terry, JHEP 01, 126 (2021) doi:10.1007/JHEP01(2021)126 [arXiv:2009.10710 [hep-ph]].
[9] M. Bury, A. Prokudin and A. Vladimirov, Phys. Rev. Lett. 126, no.11, 112002 (2021) doi:10.1103/PhysRevLett.126.112002 [arXiv:2012.05135 [hep-ph]].
[10] M. Bury, A. Prokudin and A. Vladimirov, JHEP 05, 151 (2021) doi:10.1007/JHEP05(2021)151 [arXiv:2103.03270 [hep-ph]].
[11] R. M. Godbole, A. Kaushik, A. Misra and V. S. Rawoot, Phys. Rev. D 91, no.1, 014005 (2015) doi:10.1103/PhysRevD.91.014005 [arXiv:1405.3560 [hep-ph]].
[12] A. Mukherjee and S. Rajesh, Eur. Phys. J. C 77, no.12, 854 (2017) doi:10.1140/epjc/s10052-017-5406-4 [arXiv:1609.05596 [hep-ph]].

[13] U. D’Alesio, F. Murgia, C. Pisano and P. Taels, Phys. Rev. D 96, no.3, 036011 (2017) doi:10.1103/PhysRevD.96.036011 [arXiv:1705.04169 [hep-ph]].

[14] S. Rajesh, R. Kishore and A. Mukherjee, Phys. Rev. D 98, no.1, 014007 (2018) doi:10.1103/PhysRevD.98.014007 [arXiv:1802.10359 [hep-ph]].

[15] U. D’Alesio, C. Flore, F. Murgia, C. Pisano and P. Taels, Phys. Rev. D 99, no.3, 036013 (2019) doi:10.1103/PhysRevD.99.036013 [arXiv:1811.02970 [hep-ph]].

[16] H. Sun, Tichouk and X. Luo, Phys. Rev. D 100, no.1, 014007 (2019) doi:10.1103/PhysRevD.100.014007 [arXiv:1906.04880 [hep-ph]].

[17] U. D’Alesio, F. Murgia, C. Pisano and P. Taels, Phys. Rev. D 100, no.9, 094016 (2019) doi:10.1103/PhysRevD.100.094016 [arXiv:1908.00446 [hep-ph]].

[18] R. Kishore, A. Mukherjee and S. Rajesh, Phys. Rev. D 101, no.5, 054003 (2020) doi:10.1103/PhysRevD.101.054003 [arXiv:1908.03698 [hep-ph]].

[19] U. D’Alesio, F. Murgia, C. Pisano and S. Rajesh, Eur. Phys. J. C 79, no.12, 1029 (2019) doi:10.1140/epjc/s10052-019-7551-4 [arXiv:1910.09640 [hep-ph]].

[20] U. D’Alesio, L. Maxia, F. Murgia, C. Pisano and S. Rajesh, Phys. Rev. D 102, no.9, 094011 (2020) doi:10.1103/PhysRevD.102.094011 [arXiv:2007.03353 [hep-ph]].

[21] Z. B. Kang and J. W. Qiu, Phys. Rev. D 78, 034005 (2008) doi:10.1103/PhysRevD.78.034005 [arXiv:0806.1970 [hep-ph]].

[22] H. Beppu, Y. Koike, K. Tanaka and S. Yoshida, Phys. Rev. D 82, 054005 (2010) doi:10.1103/PhysRevD.82.054005 [arXiv:1007.2034 [hep-ph]].

[23] K. Kanazawa, Y. Koike, A. Metz, D. Pitonyak and M. Schlegel, Phys. Rev. D 93, no. 5, 054024 (2016) doi:10.1103/PhysRevD.93.054024 [arXiv:1512.07233 [hep-ph]].

[24] Y. Koike, K. Yabe and S. Yoshida, Phys. Rev. D 101, no.5, 054017 (2020) doi:10.1103/PhysRevD.101.054017 [arXiv:1912.11199 [hep-ph]].

[25] F. J. Mulders and J. Rodrigues, Phys. Rev. D 63, 094021 (2001) doi:10.1103/PhysRevD.63.094021 [hep-ph/0009343].

[26] Y. Koike, K. Tanaka and S. Yoshida, Phys. Rev. D 83, 114014 (2011) doi:10.1103/PhysRevD.83.114014 [arXiv:1104.0798 [hep-ph]].

[27] W. E. Caswell and G. P. Lepage, Phys. Lett. B 167, 437-442 (1986) doi:10.1016/0370-2693(86)91297-9.

[28] G. T. Bodwin, E. Braaten and G. P. Lepage, Phys. Rev. D 51, 1125-1171 (1995) [erratum: Phys. Rev. D 55, 5853 (1997)] doi:10.1103/PhysRevD.55.5853 [hep-ph/9407339 [hep-ph]].

[29] H. Xing and S. Yoshida, Phys. Rev. D 100, no.5, 054024 (2019) doi:10.1103/PhysRevD.100.054024 [arXiv:1904.02287 [hep-ph]].

[30] R. b. Meng, F. I. Olness and D. E. Soper, Nucl. Phys. B 371, 79 (1992). doi:10.1016/0550-3213(92)90230-9.

[31] Y. Hatta, K. Kanazawa and S. Yoshida, Phys. Rev. D 88, no.1, 014037 (2013) doi:10.1103/PhysRevD.88.014037 [arXiv:1305.7001 [hep-ph]].

[32] Z. B. Kang, J. W. Qiu, W. Vogelsang and F. Yuan, Phys. Rev. D 78, 114013 (2008) doi:10.1103/PhysRevD.78.114013 [arXiv:0810.3333 [hep-ph]].

[33] Y. Koike and S. Yoshida, Phys. Rev. D 84, 014026 (2011) doi:10.1103/PhysRevD.84.014026 [arXiv:1104.3943 [hep-ph]].

[34] Y. Koike and S. Yoshida, Phys. Rev. D 85, 034030 (2012) doi:10.1103/PhysRevD.85.034030 [arXiv:1112.1161 [hep-ph]].

[35] U. A. Acharya et al. [PHENIX], Phys. Rev. Lett. 127, no.16, 162001 (2021) doi:10.1103/PhysRevLett.127.162001 [arXiv:2102.13585 [hep-ex]].

[36] U. A. Acharya et al. [PHENIX]. [arXiv:2204.12899 [hep-ex]].