Why walking is easier than pointing: Hydrodynamics of dry active matter

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Abstract

Although human beings have known about the phenomenon of “flocking” - that is, the coherent movement of large numbers of creatures (flocks of birds, schools of fish, herds of woolly mammoths, etc.)- since prehistoric times, it is only in the last two decades that we have begun to truly understand this phenomenon. In particular, the surprising fact that a very large collection of organisms in two dimensions cannot all point in the same direction, but can quite easily move in the same direction, can now be explained. In these lectures, I’ll review one of the principle theoretical tools that made this possible: hydrodynamics. My intention is both to elucidate flocking- or, to use the specific technical mouthful, ”polar ordered dry active fluids”-, and to use flocking as an illustration of how to use the hydrodynamic approach on new and unfamiliar systems.

I. INTRODUCTION

Everyone has seen “flocking”, by which I mean the collective, coherent motion of large numbers of organisms[1]. Flocks of birds and schools of fish, and herds of wildebeest, are all familiar sights (although the latter possibly only in nature documentaries). Perhaps nowadays it is most commonly seen in the simulations used for digital cinematic special effects [1]; these have lead to the only Oscar ever given for a physics project!

In the past couple of decades, many synthetic systems of self-propelled particles have been fabricated[2 3] that also exhibit flocking. In addition to providing important experimental realizations of this phenomenon, these experiments make clear that flocking does not depend
on intelligent decision making by the flockers, but, rather, can arise spontaneously from simple short ranged interactions.

I will hereafter refer to all such collective motions - flocks, swarms, herds, collections of synthetic self-propelled objects, etc - as "flocking"; for convenience, I will also refer to the "flockers" as "birds", or, interchangeably, "boids".

Note that flocking can occur over an enormous range of length scales: from kilometers (herds of wildebeest) to microns (e.g., the microorganism Dictyostelium discoideum [4–6]).

Remarkably, despite the familiarity and widespread nature of the phenomenon, it is only in the last 24 years that many of the universal features of flocks have been identified and understood. It is my goal in these lectures to explain how we’ve come to understand one particular type of ‘flocking”, namely ”polar ordered dry active fluids”, which I’ll define soon. In the process, I hope to introduce those of you unfamiliar with it to the “hydrodynamic” approach, which is a powerful technique that can be applied to any large scale collective phenomenon.

To my knowledge, the first physicist to think about flocking - certainly the physicist who kicked off the modern field of active matter- was Thomas Vicsek [7]. He was, as far as I know, the first to recognize that flocks fall into the broad category of nonequilibrium dynamical systems with many degrees of freedom that has, over the past few decades, been studied using powerful techniques originally developed for equilibrium condensed matter and statistical physics (e.g., scaling, the renormalization group, etc). In particular, Vicsek noted an analogy between flocking and ferromagnetism: the velocity vector of the individual birds is like the magnetic spin on an iron atom in a ferromagnet. The usual “moving phase” of a flock, in which all the birds, on average, are moving in the same direction, is then the analog of the “ferromagnetic” phase of iron, in which all the spins, an average, point in the same direction. Another way to say this is that the development of a nonzero mean center of mass velocity $\langle v \rangle$ for the flock as a whole therefore requires spontaneous breaking of a continuous symmetry (namely, rotational), precisely as the development of a nonzero magnetization $\vec{M} \equiv \langle \vec{S} \rangle$ of the spins in a ferromagnet breaks the continuous spin rotational symmetry of the Heisenberg magnet [8].

To make this analogy complete obviously requires that the birds, like the spins in a ferromagnet, live in a rotation invariant environment; that is, that the spins have nothing external that tells them which direction to point, and the birds have nothing external that
tells them which way to fly.

To study this phenomenon—the spontaneous breaking of rotation invariance by spontaneous collective motion—which is what I will mean henceforth by the term “flocking”—Vicsek formulated his now famous algorithm. I will not describe this algorithm in detail—it’s probably already been described by others at this school—but will limit myself to noting the features of it that are important for a hydrodynamic theory. These features are: activity, conservation laws, symmetries, short ranged interactions, and noisiness. To elaborate on these:

1. Activity: A large number (a “flock”) of point particles (“boids”[10]) each move over time through a space of dimension \( d \) \((= 2, 3, \ldots)\), attempting at all times to “follow” (i.e., move in the same direction as) its neighbors. This motion is due to some form of self-propulsion; in Vicsek’s algorithm, the rule is that the speed of each creature is constant. Departures from this rule are not important, provided that the boids prefer to be in a state of motion, rather than at rest. This is what is meant by the word “active” in “polar ordered dry active fluids”.

This self propulsion requires an energy source; it also requires that the system be out of equilibrium. Dead birds don’t flock!

2. Conservation laws: the underlying model does not conserve momentum; the total momentum of the flock can change. Indeed, it does so every time a creature turns. We imagine this happening because the creatures move either over a fixed surface, in two dimension, or through some fixed matrix (e.g., a gel) in three dimensions. This is what is meant by the term “dry” in ”polar ordered dry active fluids”. Note that many of the systems you have heard about at this school—e.g., active nematics—are “wet”, by which we mean momentum is conserved. Note, incidentally, that real birds (and not only water birds!) are “wet” in this sense, since the sum of their momentum and the momentum of the air through which they fly is conserved. This changes the dynamics considerably. The problem of wet flocks can still be treated by a hydrodynamic approach[11], but the hydrodynamic model is different because of momentum conservation. I will not discuss that case further here.

There is one conservation law in the Vicsek algorithm, however: the number of birds is conserved. That is, birds are not being born or dying “on the wing”. You laugh,
but there are many biological situations—bacteria swarms, and tissue development to name just two—in which this is not a good approximation: bacteria or cells are born and dying on the time scale of the motion. The hydrodynamics of this case is quite interesting[12], but, again, I won’t consider that case in these lectures.

3. Symmetry: the underlying model has complete rotational symmetry: the flock is equally likely, a priori, to move in any direction. I will here consider models that do not have Galilean invariance: that is, they have a preferred Galilean frame. This frame is the one in which the background medium over or through which the boids move is stationary.

4. The interactions are purely short ranged: in Vicsek’s model, each “boid” only responds to its neighbors. In Vicsek’s model, these are defined as those “boids” within some fixed, finite distance $R_0$, which is assumed to be independent of $L$, the linear size of the “flock.” Hence, in the limit of flock size going to infinity—i.e., the “thermodynamic limit”—the range of interaction is much smaller than the size of the flock. Variants on this rule—for example, interactions whose strength falls off exponentially with distance—can also be considered short-ranged.

5. The “following” is not perfect: the “boids” make errors at all times, which are modeled as a stochastic noise. This noise is assumed to have only short ranged spatio-temporal correlations. Its role in this problem is very similar to the role of temperature in equilibrium systems: it tends to disorder the flock. As you’ll see, one of the most interesting questions in this problem is whether the ordered state can survive this noise.

In addition to these symmetries of the questions of motion, which reflect the underlying symmetries of the physical situation under consideration, it is also necessary to treat correctly the symmetries of the state of the system under consideration. These may be different from those of the underlying system, precisely because the system may spontaneously break one or more of the underlying symmetries of the equations of motion. Indeed, this is precisely what happens in the ordered state of a ferromagnet: the underlying rotation invariance of the system as a whole is broken by the system in its steady state, in which a unique direction is picked out—namely, the direction of the spontaneous magnetization.
As should be apparent from our earlier discussion, this is also what happens in a spontaneously moving flock. Indeed, the symmetry that is broken – rotational – and the manner in which it is broken - namely, the development of a nonzero expectation value for some vector (the spin $\vec{S}$ in the ferromagnetic case; the velocity $\langle \mathbf{v} \rangle$ in the flock) are precisely the same in both cases [9].

The fact that it is a unique vector that is singled out, rather than merely a unique axis, is the meaning of the word “polar” in “polar ordered dry active fluids”.

Many different “phases” [13], in this sense of the word, of a system with a given underlying symmetry are possible. Indeed, I have already described two such phases of flocks: the “ferromagnetic” or moving flock, and the “disordered,” “paramagnetic,” or stationary flock.

In equilibrium statistical mechanics, this is precisely how we classify different phases of matter: by the underlying symmetries that they break. Crystalline solids, for example, differ from fluids (liquid and gases) by breaking both translational and orientational symmetry. Less familiar to those outside the discipline of soft condensed matter physics are the host of mesophases known as liquid crystals, in some of which (e.g., nematics [14]) only orientational symmetry is broken, while in others, (e.g., smectics [14]) translational symmetry is only broken in some directions, not all.

It seems clear that, at least in principle, every phase known in condensed matter systems could also be found in flocks. In these lectures, I’m going to focus one just one phase: the ”polar ordered dry active fluid phase”, in which rotational symmetry is completely broken by the development of a non-zero average flock speed $\langle \mathbf{v} \rangle$, but all of the other symmetries of the dynamics (e.g., translation invariance) are preserved.

The first, and to my mind, still the biggest surprise in the entire field of active matter is that a ”polar ordered dry active fluid phase” is even possible in two dimensions. The reason I (and Vicsek) find this so surprising is the well-known “Mermin-Wagner Theorem” [15] of equilibrium statistical mechanics. This theorem states that in a thermal equilibrium model at nonzero temperature with short-ranged interactions, it is impossible to spontaneously break a continuous symmetry. This implies in particular that the equilibrium or “pointer” version of Vicsek’s algorithm described above, in which the birds carry a vector $\mathbf{v}_i$ whose direction is updated according to Vicsek’s algorithm, but in which the birds do not actually move, can never develop a true long-range ordered state in which all the $\mathbf{v}_i$’s point, on average, in the same direction (more precisely, in which $\langle \mathbf{v} \rangle \equiv \frac{\sum_i \mathbf{v}_i}{N} \neq \mathbf{0}$), since such a
state breaks a continuous symmetry, namely rotation invariance.

Yet the *moving* flock evidently has no difficulty in doing so; as Vicsek’s simulation shows, even two-dimensional flocks with rotationally invariant dynamics, short-ranged interactions, and noise—i.e., seemingly all of the ingredients of the Mermin-Wagner theorem—do move with a nonzero macroscopic velocity, which requires \(< \mathbf{v} \neq \mathbf{0}, \) which, in turn, breaks rotation invariance, in seeming violation of the theorem.

There are a pair of gedanken experiments that make the very paradoxical and surprising nature of this result more obvious. Both experiments start by putting a million people on a flat, featureless plane in the fog. (This school is clearly not a good place to perform this experiment: Mont Blanc provides a rather conspicuous “special direction”!) The featurelessness of the plane, and the fog, ensure rotation invariance (since they leave the people with no external indication of a preferred direction), while the fog has the further role of ensuring that each person can see only a few of her nearest neighbors.

The first experiment now consists of asking everyone to try to point in the same direction. The result is that the people cannot all point in the same direction, no matter how good a job they do at aligning with their nearest neighbors (unless, of course, the alignment is perfect). If they make the slightest errors, those will accumulate over distance, so that, even though a given person may point in roughly the same direction as others not too far away from her, widely separated people will inevitably be pointing in wildly different directions.

The second gedanken experiment consists of slightly modifying the instructions given to these million folks: now ask them to all *walk* in the same direction.

Amazingly, if this instruction is given to the same people, in the same fog, with the same errors, they *can* all walk in the same direction. Moving, apparently, is fundamentally different from pointing.

Why? That is the question I will answer in the remainder of these notes.

There is a very simple explanation for this apparent “violation” of the Mermin-Wagner theorem: one of the essential premises of the Mermin-Wagner theorem does *not* apply to movers: they are *not* systems in thermal equilibrium. The nonequilibrium aspect arises from the motion: you can’t move forever in a medium with friction unless you’re alive. And, if you’re alive, you’re not in thermal equilibrium (that’s why we say “cold and dead”).

Clearly, motion must be what stabilizes the order in \(d = 2\): as described above, the motion is the *only* difference between the pointing and moving gedanken experiments just
described.

But _how_ does motion get around the Mermin-Wagner theorem? And, more generally, how best to understand the large-scale, long-time dynamics of a very large, moving flock?

The answer to this second question can be found in the field of hydrodynamics.

Hydrodynamics is a well-understood subject. This understanding does _not_ come from solving the many (very many!) body problem of computing the time-dependent positions $\mathbf{r}_i(t)$ of the $10^{23}$ constituent molecules of a fluid subject to intermolecular forces from all of the other $10^{23}$ molecules. Such an approach is analytically intractable even if one knew what the intermolecular forces were. Trying to compute analytically the behavior of, e.g., Vicsek’s algorithm directly would be the corresponding, and equally impossible, approach to the flocking problem.

Instead, the way we understand fluid mechanics is by writing down a set of continuum equations - the Navier-Stokes equations - for continuous, smoothly varying number density $\rho(\mathbf{r}, t)$ and velocity $\mathbf{v}(\mathbf{r}, t)$ fields describing the fluid.

Although we know that fluids are made out of atoms and molecules, we can define “coarse-grained” number density $\rho(\mathbf{r}, t)$ and velocity $\mathbf{v}(\mathbf{r}, t)$ fields by averaging over “coarse-graining” volumes large compared to the intermolecular or, in the flocks, “interbird” spacing. On a large scale, even discrete systems _look_ continuous, as we all know from close inspection of newspaper photographs and television images.

In writing down the Navier-Stokes equations, one “buries one’s ignorance” [16] of the detailed microscopic dynamics of the fluid in a few phenomenological parameters, namely the mean density $\rho_0$, the bulk and shear viscosities $\eta_B$ and $\eta_S$, the thermal conductivity $\kappa$, the specific heat $c_v$, and the compressibility $\chi$. Once these have been deduced from experiment, (or, occasionally, and at the cost of immense effort, calculated from a microscopic model), one can then predict the outcomes of all experiments that probe length scales much greater than a spatial coarse-graining scale $\ell_0$ and time scales $\gg t_0$, a corresponding microscopic time, by solving these continuum equations, a far simpler task than solving the microscopic dynamics.

But how do we write down these continuum equations? The answer to this question is, in a way, extremely simple: we write down every _relevant_ term that is not ruled out by the symmetries and conservation laws of the problem. In the case of the Navier-Stokes equations, the symmetries are rotational invariance, space and time translation invariance, and Galilean
invariance (i.e., invariance under a boost to a reference frame moving at a constant velocity), while the conservation laws are conservation of particle number, momentum and energy.

“Relevant,” in this specification means terms that are important at large length scales and long timescales. In practice, this means a “gradient expansion:” we only keep in the equations of motion terms with the smallest possible number of space and time derivatives. For example, in the Navier-Stokes equations, we keep a viscous term $\eta_s \nabla^2 v$, but not a term $\gamma \nabla^4 v$, though the latter is also allowed by symmetry, because the $\gamma \nabla^4 v$ term involves more spatial derivatives, and hence is smaller, for slow spatial variation, than the viscous term we’ve already got.

Our current theoretical understanding of both dry and wet active matter are based largely on applying the hydrodynamic approach I’ve just outlined to those systems. The rest of these notes will demonstrate how that is done for the specific case of dry polar active fluids, for which the only symmetry is rotation invariance (“dry” means no momentum conservation, while energy conservation is doesn’t apply to any active system, since the very term “active” implies the existence of an energy source for each particle or flocker).

The remainder of these notes are organized as follows:

in section II, I’ll present a highly unorthodox, and extremely handwaving, dynamical “derivation” of the Mermin-Wagner theorem, to make it clear that there’s something very surprising about the stability of flocks in two dimensions. Then in section III, I’ll review the formulation such a hydrodynamic model for dry active matter (which I will sometimes refer to as “ferromagnetic flocks”) in [17–21]. In section IV, I’ll show how one solves this model, and how that solution implies, among many other results, that the Mermin-Wagner theorem does not apply to dry polar active fluids: that is, they can develop long-ranged order, even in two dimensions, even in the presence of noise. In section V I’ll give a handwaving argument in the spirit of the derivation of the Mermin-Wagner theorem in section II which explains in physical terms the mechanism that stabilizes long ranged order in two dimensions for flocks.

II. DYNAMICAL “DERIVATION” OF THE MERMIN-WAGNER THEOREM

You will not, with good reason, see anything like the following derivation in any textbook on statistical mechanics. The usual derivation involves the powerful tools of equilibrium statistical mechanics: Boltzmann weights, Hamiltonians, and the like. Since the Mermin-
Wagner theorem was derived for equilibrium systems, for which all these tools are available, it would be completely nuts (to use the technical term) not to take advantage of these tools.

However, as emphasized by Mike Cates in his lectures at this school (see chapter (??) of this book), none of those very powerful tools are available for non-equilibrium systems like flocks. It’s therefore useful, I think, to attempt the seemingly crazy stunt of deriving the Mermin-Wagner theorem in a purely dynamical way that can be generalized to non-equilibrium systems. In this way I hope to elucidate exactly what it is about moving that is fundamentally different from pointing, and in particular, how that difference makes long-ranged order literally infinitely more robust in two dimensions in a moving system than a pointing one.

So let’s think about those million pointers on the featureless plane in the fog. Consider in particular the angle $\theta_i(t)$ between the direction a given pointer labeled by $i$ is pointing at time $t$ and some fixed reference direction. A “Vicsek-like” algorithm for pointers which try to align with their neighbors is the following updating rule for $\theta_i$:

$$\theta_i(t + 1) = \langle \theta_j(t) \rangle_n + \eta_i(t)$$

(2.1)

where the symbol $\langle \rangle_n$ denotes an average over “neighbors”, which are defined as the set of pointers $j$ satisfying

$$|\mathbf{r}_j(t) - \mathbf{r}_i(t)| < R_0.$$  

(2.2)

This allows us to define “neighbors” even if the pointers are distributed in random positions, rather than on a regular lattice.

The extra term $\eta_i$ is a random noise that takes into account the fact that the pointers will inevitably make mistakes in aligning with their neighbors. We’ll assume this has zero mean (that is, the pointers are no more likely to err to the left than to the right), and variance $\Delta$, and that it is uncorrelated between pointers $(i, j)$, and between successive time steps. That is,

$$\langle \eta_i(t) \rangle = 0$$  

(2.3)

$$\langle \eta_i(t) \eta_j(t') \rangle = \Delta \delta_{ij} \delta_{tt'}$$

(2.4)

where $\langle \rangle$ without the subscript $n$ denote averages over the random distribution of the noises $\eta_i(t)$. Here the noise strength $\Delta$ will play the role of temperature, in the sense that larger $\Delta$ will lead to more fluctuations, and hence, presumably, less order.
The flock evolves through the iteration of this rule. Note that the “neighbors” of a given pointer do not change on each time step. To foreshadow where I’m ultimately going here, this is not true for movers, which can change their neighbors due to the differences in the motion of different movers within the flock. This is the fundamental difference between pointers and movers that makes the movers capable of aligning in two dimensions, while the pointers cannot.

But let’s not get ahead of ourselves here. Returning to the pointers problem, I note, as first noted by Vicsek himself, that this model is exactly a simple, relaxational dynamical model for an equilibrium ferromagnet. That is, if we interpret each unit vector $\mathbf{n}_i$ that gives the direction the $i$th pointer is pointing as a the direction that “spins” carried by each pointer, and update them according to the above rule, then the model is easily shown to be an equilibrium ferromagnet, which will relax to the Boltzmann distribution for an equilibrium Heisenberg model (albeit with the “spins” living not on a periodic lattice, as they usually do in most models and in real ferromagnets, but, rather, on a random set of points). In the absence of noise (i.e., for $\Delta = 0$), this algorithm will, unsurprisingly, lead to a “ferromagnetic” state, characterized by a non-zero “magnetization”:

$$M \equiv \langle \mathbf{n} \rangle \equiv \frac{\sum_{i=1}^{N} \mathbf{n}_i}{N}, \quad (2.5)$$

where in this expression the $<>$ mean an average over all the pointers. I’ll assume throughout these notes that this average is equal to an average over the noise; in equilibrium physics, this is sometimes called the assumption of “ergodicity”.

At zero noise, we would expect to, and do, eventually reach a state in which $|M| = 1$; i.e., perfect alignment of all the pointers. The big question is: what happens when there is noise (i.e., when $\Delta \neq 0$)?

To answer this, begin by noting that the dynamical rule (2.1) is actually a disguised version of a noisy diffusion equation. To see this, recall that one of the numerical algorithms for solving Laplace’s equation $\nabla^2 \theta = 0$ is to replace the value of the field $\theta$ at each point at each point with the average of its neighbors. Thus, in the absence of noise, the dynamics (2.1) will eventually relax the field $\theta$ to a state in which $\nabla^2 \theta = 0$, which implies that the rate of change of $\theta$ (again, in the absence of noise) is itself proportional to $\nabla^2 \theta$ (since it vanishes when $\nabla^2 \theta = 0$). Indeed, one can very simply derive this result as follows:

Consider for simplicity (although it is not necessary) a two dimensional collection of
pointers arranged on a square grid of lattice constant $a$. The pointer at position $\mathbf{r}_i = (x_i, y_i)$, where my $x$ and $y$-axes are aligned with the square grid, has four neighbors, one to its right at $\mathbf{r}_1 = (x_i + a, y_i)$, a second to its left at $\mathbf{r}_2 = (x_i - a, y_i)$, a third above at $\mathbf{r}_3 = (x_i, y_i + a)$, and the fourth below at $\mathbf{r}_4 = (x_i, y_i - a)$. Thus, the dynamical rule (2.1) can be rewritten:

$$\theta_i(x_i, y_i, t + 1) - \theta_i(x_i, y_i, t) =$$
$$\frac{1}{4}\left[\theta(x_i + a, y_i, t) + \theta(x_i - a, y_i, t) + \theta(x_i, y_i + a, t) + \theta(x_i, y_i - a, t)\right] - \theta_i(x_i, y_i, t) + \eta_i(t),$$

(2.6)

where I have subtracted the value $\theta_i(t)$ of $\theta_i$ on the last time step from both sides, so as to make the right hand side look like a discrete representation of a time derivative. In the process, I have made the left hand side into a discrete version of the Laplacian. To see this, just reorganize the right hand side as follows:

$$\theta_i(x_i, y_i, t + 1) - \theta_i(x_i, y_i, t) = \frac{1}{4}\left\{\theta(x_i + a, y_i, t) - 2\theta_i(x_i, y_i, t) + \theta(x_i - a, y_i, t)\right\}$$
$$+ \left\{\theta(x_i, y_i + a, t) - 2\theta_i(x_i, y_i, t) + \theta(x_i, y_i - a, t)\right\} + \eta_i(t).$$

(2.7)

Now note that, just as the left-hand side can be approximated as the time derivative of $\theta$ if $\theta$ varies slowly in time - that is, we can write $\theta_i(x_i, y_i, t + 1) - \theta_i(x_i, y_i, t) \approx \partial_t \theta$, the term in the first parenthesis on the right hand side can be approximated by the second derivative of $\theta$ with respect to $x$: $\theta(x_i + a, y_i, t) - 2\theta_i(x_i, y_i, t) + \theta(x_i - a, y_i, t) \approx a^2 \partial_x^2 \theta$, provided that $\theta$ varies slowly with position.Likewise, the second term can be approximated by the second derivative of $\theta$ with respect to $y$: $\theta(x_i, y_i + a, t) - 2\theta_i(x_i, y_i, t) + \theta(x_i, y_i - a, t) \approx a^2 \partial_y^2 \theta$. Hence, equation (2.7) can be approximated as

$$\partial_t \theta = \frac{a^2}{4}[\partial_x^2 \theta + \partial_y^2 \theta] + \eta = D \nabla^2 \theta + \eta,$$

(2.8)

where I’ve defined the “diffusion constant” $D \equiv \frac{a^2}{4}$.

Note that I could also have derived this result purely on symmetry grounds: $\partial_t \theta$ must be a scalar made out of $\theta$ itself and its derivatives. By rotation invariance, it must vanish if $\theta$ is spatially uniform. By the isotropy of space, it must be an isotropic operator. The only thing you can make that does this to second order in gradients of $\theta$ is $\nabla^2 \theta$. 
So what are the consequences of the fact that $\theta$ obeys a diffusion equation? There are two that are important for our discussion:

1) $\theta$ is slow, and

2) $\theta$ is conserved (in the absence of noise).

To be more precise about point 1), the form of the diffusion equation implies that an initially localized departure of $\theta(r, t = 0)$ from spatial homogeneity spreads very slowly. One can read this off by power counting from the form of the diffusion equation: a time derivative of $\theta$ can be estimated as roughly $\theta$ over a time $t$, while the Laplacian of $\theta$ can be estimated as $\theta$ divided by a distance $r$ squared. Equating these gives

$$r^2 \propto t,$$

or, equivalently,

$$r \propto \sqrt{t}.$$  \hspace{1cm} (2.9)

The exact solution (in the absence of noise) of the diffusion equation in $d$-spatial dimensions for an initially localized $\theta$, which is

$$\theta(r, t) = A \exp \left( -\frac{r^2}{4Dt} \right) / (4\pi Dt)^\frac{d}{2},$$

clearly obeys this scaling law.

This is very slow; indeed, anything moving at any constant speed, however small, will eventually outrun diffusive spreading, since $t \gg \sqrt{t}$ as $t \to \infty$. This is why you stir your coffee after adding milk to it: even a slow stir leads to far faster mixing than diffusion. We’ll see later that the reason flocks can order in two dimensions is essentially that, by their motion, they stir themselves.

Turning now to point 2), we can see that $\theta$ is conserved in the absence of noise by setting $\eta = 0$ and integrating both sides of the diffusion equation (2.8) over all $r$. This gives

$$\frac{d}{dt} \int d^d r \theta(r, t) = D \int d^d r \nabla^2 \theta(r, t) = D \int d^d r \nabla \cdot \nabla \theta(r, t) = D \int_S dA \cdot \nabla \theta(r, t),$$  \hspace{1cm} (2.12)

where in the last equality I’ve used the divergence theorem, with $S$ being the surface bounding my volume of integration. This shows that the integral of $\theta$ over any region of space can only change if there is a current (proportional to $\nabla \theta$) through the surface of that region. Basically, $\theta$ acts like the milk in your coffee: the total quantity of it is conserved; diffusion
FIG. 1: Evolution of a single error in the pointer problem. The original error $\theta_0$ gets shared evenly after a time $t$ among all of the pointers within a distance $\propto \sqrt{t}$ of the original pointer.

can only redistribute it in space. And it can only do that very slowly; i.e., the spatial spread $r(t)$ after a time $t$ only grows like $\sqrt{t}$.

The consequence of these two observations is that fluctuations decay very slowly in the pointer system. To illustrate this dramatically, consider a pointer system with no noise, and with an almost perfectly ordered initial condition: only the pointer at the center is pointing in a different direction from any of the others, and he is pointing at an angle $\theta_0$ to the left of the direction the others are pointing. (See figure (1)). Defining $\theta = 0$ as the direction the others are pointing, we have $\int d^d r \theta(r, t = 0) = \theta_0$.

What will this collection look like after time $t$, if there is no noise? Well, by point 1) above, the initial error will now be spread out over all the pointers within a distance $r(t) \propto \sqrt{t}$. By point 2), the sum of the deviations of all of these pointers (including the
original error making pointer) from the original direction of most of them must still be \( \theta_0 \), since \( \theta \) is conserved. So the original fluctuation of \( \theta_0 \) must now be distributed over all of those pointers within that distance \( r(t) \propto \sqrt{t} \). Hence, we can crudely estimate the angular deviation after a time \( t \) by assuming (as proves to be the case) that this initial error is spread roughly uniformly over all \( N(t) \) of the pointers within this distance \( r(t) \). That number \( N(t) \) is easy to estimate; it’s just the density times the volume (or hypervolume, if we’re considering \( d \neq 3 \)) of the region of radius \( r(t) \propto \sqrt{t} \). Assuming the density is roughly constant, at least over a sufficiently large region (as indeed it is for a random set of points; fluctuations in the density of a random set of points over a volume \( V \) scale as \( \frac{1}{\sqrt{V}} \to 0 \) for \( V \to \infty \)), it’s clear that

\[
N(t) \propto [r(t)]^d \propto t^{d/2},
\]

where I’ve used \( r(t) \propto \sqrt{t} \).

Since the original total error of \( \theta_0 \) is now divided among all \( N(t) \) of these pointers, the typical fluctuation \( \theta(t) \) of each of them, including the original error-maker, is now

\[
\theta(t) \approx \frac{\theta_0}{N(t)} \propto \frac{\theta_0}{t^{d/2}}.
\]

I want to call your attention to two things about this result:

1) the decay is extremely slow; specifically, it is a power law in time. Hence, it is asymptotically slower than any exponential decay. This is a consequence of the conservation law for \( \theta \), which is in turn a consequence of the underlying rotation invariance. This means \( \theta \) is a Goldstone mode of the system, a concept that may already be familiar to some of you. I’ll discuss this more in the next section.

2) The power law of this decay is dimension-dependent, with slower decay in lower dimensions. This is a general and recurring theme in statistical and condensed matter physics: fluctuations decay more slowly, and, hence, are more important, in lower dimensions. Ultimately, this is why the Mermin-Wagner theorem applies to low-dimensional systems—specifically, \( d \leq 2 \), but not higher dimensional ones.

With this result (2.14) for a single initial error in hand, let’s now go back and consider our original model with noise. Now the situation is even worse: while any initial errors are very slowly decaying according to (2.14), more errors are constantly being made. The question now becomes, can the slow decay of (2.14) keep up with the accumulation of new
errors? Given the dimension dependence of (2.14), you won’t be surprised to learn that the answer to this question is also dimension dependent: the errors can be kept under control for spatial dimensions \(d > 2\), but not for \(d \leq 2\). This is the Mermin-Wagner theorem\[15\].

To see this, consider the “blob” of pointers that can have exchanged information diffusively with some central pointer after a time \(t\). As noted earlier, this blob will have radius \(r(t) \propto \sqrt{t}\), or, equivalently, given a radius \(r\) of the blob, the time required for all parts of that blob to be able to communicate with each other is

\[
t \propto r^2. \tag{2.15}
\]

This blob will contain \(N(t) \propto r^d\) pointers. How many errors will these pointers collectively have made? Well, each of them will have made

\[
\text{# of errors/pointer} \propto t \propto r^2; \tag{2.16}
\]

hence, the full collection of \(N(t) \propto r^d\) of them will have made

\[
\text{total # of errors} \propto N(t)t \propto r^{d+2}. \tag{2.17}
\]

Since the sum of a number of independent random variables with zero mean is proportional to the square root of that number, we have

\[
\sqrt{\langle \theta^2 \rangle} \approx \frac{\sqrt{\text{total # of errors}}}{N(t)} \propto \frac{r^{d/2}}{r^d} \propto r^{1-d/2}, \tag{2.18}
\]

which diverges as \(r\) (or, equivalently, time \(t\)), goes to infinity for \(d < 2\). As often happens, the vanishing of this exponent \(1 - d/2\) in (2.18) in \(d = 2\) indicates not a constant, but a logarithm: in fact, a slightly more careful version of the reasoning used here, applied in exactly \(d = 2\), shows that

\[
\sqrt{\langle \theta^2 \rangle} \propto \sqrt{\ln(r)} \to \infty. \tag{2.19}
\]

So we’ve shown by this purely dynamical argument that, for \(d \leq 2\), fluctuations diverge in the limit of an infinitely large system. Hence, there can be no long ranged order in our system of pointers for those spatial dimensions. This is the Mermin-Wagner theorem, derived in a very unorthodox dynamical way. In the final section of these notes, I’ll show that modifying this argument to take into account motion shows that movers can order in \(d = 2\). But first, I’ll show this more formally and systematically using hydrodynamics.
III. FORMULATING THE HYDRODYNAMIC MODEL

In this section, I’ll review the derivation and analysis of the hydrodynamic model of polar ordered dry active fluids, which I’ll also refer to as “ferromagnetic flocks”. More details can be found in references[19–21].

As discussed in the introduction, the system we wish to model is any collection of a large number $N$ of organisms (hereafter referred to as “birds”) in a $d$-dimensional space, with each organism seeking to move in the same direction as its immediate neighbors.

I further assume that each organism has no “compass;” in the sense defined in the Introduction, i. e., no intrinsically preferred direction in which it wishes to move. Rather, it is equally happy to move in any direction picked by its neighbors. However, the navigation of each organism is not perfect; it makes some errors in attempting to follow its neighbors. I consider the case in which these errors have zero mean; e. g., in two dimensions, a given bird is no more likely to err to the right than to the left of the direction picked by its neighbors. I also assume that these errors have no long temporal correlations; e. g., a bird that has erred to the right at time $t$ is equally likely to err either left or right at a time $t'$ much later than $t$.

The continuum model will describe the long distance behavior of any flock satisfying the symmetry conditions I’ll specify in a moment. The automaton studied by Vicsek et al [7] described in the introduction provides one concrete realization of such a model. Adding “bells and whistles” to this model by, e.g., including purely attractive or repulsive interactions between the birds, restricting their field of vision to those birds ahead of them, giving them some short-term memory, etc., will not change the hydrodynamic model, but can be incorporated simply into a change of the numerical values of a few phenomenological parameters in the model, in much the same way that all simple fluids are described by the Navier-Stokes equations, and changing fluids can be accounted for simply by changing, e.g., the viscosity that appears in those equations.

This model should also describe real flocks of real living organisms, provided that the flocks are large enough, and that they have the same symmetries and conservation laws that, e.g., Vicsek’s algorithm does.

So, given this lengthy preamble, what are the symmetries and conservation laws of flocks? The only symmetries of the model are invariance under rotations and translations.
Translation-invariance simply means that displacing the positions of the whole flock rigidly by a constant amount has no physical effect, since the space the flock moves through is assumed to be on average homogeneous [22]. Since I am not considering translational ordering, this symmetry remains unbroken. Rotation invariance simply says the “birds” lack a compass, so that all directions of space are equivalent. Thus, the “hydrodynamic” equation of motion I write down cannot have built into it any special direction picked “a priori”; all directions must be spontaneously picked out by the motion and spatial structure of the flock. As we shall see, this symmetry severely restricts the allowed terms in the equation of motion.

Note that the model does not have Galilean invariance: changing the velocities of all the birds by some constant boost $v_b$ does not leave the model invariant. Indeed, such a boost is impossible in a model that strictly obeys Vicsek’s rules, since the speeds of all the birds will not remain equal to $v_0$ after the boost. One could image relaxing this constraint on the speed, and allowing birds to occasionally speed up or slow down, while tending an average to move at speed $v_0$. Then the boost just described would be possible, but clearly would change the subsequent evolution of the flock.

Another way to say this is that birds move through a resistive medium, which provides a special Galilean reference frame, in which the dynamics are particularly simple, and different from those in other reference frames. Since real organisms in flocks always move through such a medium (birds through the air, fish through the sea, wildebeest through the arid dust of the Serengeti), this is a very realistic feature of the model [23].

As we shall see shortly, this lack of Galilean invariance allows terms in the hydrodynamic equations of birds that are not present in, e. g., the Navier-Stokes equations for a simple fluid, which must be Galilean invariant, due to the absence of a luminiferous ether.

The sole conservation law for flocks is conservation of birds: we do not allow birds to be born or die “on the wing”.

In contrast to the Navier-Stokes equation, I here consider systems without momentum, due to the presence of the resistive background medium which breaks Galilean invariance.

Having established the symmetries and conservation laws constraining our model, we need now to identify the hydrodynamic variables.

What do I mean by “hydrodynamic”? I mean variables that evolve slowly at long wavelength. More precisely, I mean variables whose evolution rate goes to zero as the length
scale on which they are probed goes to infinity.

When one first hears this concept, it is natural to wonder why there should be any such variables. For example, in a flock consisting of millions of organisms, wouldn’t one expect all variables to relax on some “microscopic” time scale, such as the mean time scale of interaction between neighboring birds?

This reasoning is almost correct: almost any variable one can think of in any system with an enormous number of degrees of freedom will relax back, on a microscopic time scale, to a value determined by the local values of the few “slow”, or “hydrodynamic” variables. But again, why should any variable be slow?

There are two possible reasons a variable will be slow:

1) conservation laws, and
2) broken continuous symmetries.

The density $\rho$ is an example of a variable which is hydrodynamic for the first reason. Variables that are slow for second reason are called “Goldstone modes”. In our problem, rotation invariance implies that $\delta \mathbf{v}_\perp$, defined via

$$
\mathbf{v}(r, t) = v_0 \hat{x}_\parallel + \delta \mathbf{v}(r, t) = (v_0 + \delta v_\parallel)\hat{x}_\parallel + \delta \mathbf{v}_\perp(r, t)
$$

(3.1)

is a hydrodynamic variable. This is because a constant $\delta \mathbf{v}_\perp$ amounts to just a rotation, if $\delta v_\parallel$ relaxes back to the value $\delta v_\parallel = \sqrt{v_0^2 - |\delta \mathbf{v}_\perp|^2} - v_0$ required to keep $|\mathbf{v}| = v_0$. Since the system is rotation invariant, such a spatially uniform variation of $\delta \mathbf{v}_\perp$ can never relax; i.e., it has an infinite lifetime. Therefore, by continuity, if the field $\delta \mathbf{v}_\perp$ varies slowly in space, it must relax very slowly. More precisely, the relaxation time of such a distortion in $\delta \mathbf{v}_\perp$ must go to infinity as the length scale on which it varies does.

We’ve already seen an illustration of this for the pointer problem: as distance $r \to \infty$, the time $t$ required for the field $\theta$ to equilibrate over that distance $r$ diverges like $r^2$. This is because $\theta$ is the Goldstone mode, in the sense just described, for the pointer problem. The broken continuous symmetry with which $\theta$ is associated- that is, the symmetry that guarantees that $\theta$ will be “slow” at long wavelengths (which is precisely what I mean by “hydrodynamic”) is just rotation invariance.

Note that although $\delta \mathbf{v}_\perp$ is a hydrodynamic variable, $\delta v_\parallel$ is not, since there is no symmetry that forbids the speed of the flockers from relaxing back to the preferred speed $v_0$ in a finite time, even if the fluctuation of the speed away from $v_0$ is spatially uniform. Nonetheless,
because it is far simpler to see the consequences of rotation invariance for the full velocity field \( \mathbf{v} \) than it is for the perpendicular component \( \delta \mathbf{v}_\perp \) of \( \mathbf{v} \) alone, I will initially formulate hydrodynamic equations of motion for the full velocity \( \mathbf{v} \), even though this will include the non-hydrodynamic variable \( \delta \mathbf{v}_\parallel \). Once I have the equations of motion, it is then conceptually straightforward (although algebraically fairly monstrous, as we’ll see) to eliminate \( \delta \mathbf{v}_\parallel \) and rewrite the equations of motion entirely in terms of the hydrodynamic variables \( \delta \mathbf{v}_\perp \) and \( \rho \).

I will also follow the historical precedent of the Navier-Stokes\cite{16,24} equation by deriving our continuum, long wavelength description of the flock not by explicitly coarse graining the microscopic dynamics (a very difficult procedure in practice), but, rather, by writing down the most general continuum equations of motion for \( \mathbf{v} \) and \( \rho \) consistent with the symmetries and conservation laws of the problem. This approach allows us to bury our ignorance in a few phenomenological parameters, (e.g., the viscosity in the Navier-Stokes equation) whose numerical values will depend on the detailed microscopic rules of individual bird motion. What terms can be present in the EOMs, however, should depend only on symmetries and conservation laws, and not on other aspects of the microscopic rules.

To reduce the complexity of our equations of motion still further, I will perform a spatio-temporal gradient expansion, and keep only the lowest order terms in gradients and time derivatives of \( \mathbf{v} \) and \( \rho \). This is motivated and justified by our desire to consider only the long distance, long time properties of the flock. Higher order terms in the gradient expansion are “irrelevant”: they can lead to finite “renormalization” of the phenomenological parameters of the long wavelength theory, but cannot change the type or scaling of the allowed terms.

So let’s begin.

Rotation invariance implies that \( \partial_t \mathbf{v} \), being a vector itself, must equal a sum of some other vectors. So, what vectors can we make out of \( \mathbf{v} \), the scalar \( \rho \), and the gradient operator?

Well, the most obvious vector is \( \mathbf{v} \) itself. More generally, we can multiply \( \mathbf{v} \) by any scalar function of the speed \( |\mathbf{v}| \) and the density \( \rho \):

\[
(\partial_t \mathbf{v})_1 = U(|\mathbf{v}|, \rho) \mathbf{v}
\]

This looks like a conventional frictional drag coefficient, except for the crucial difference that, while the drag coefficient must always be negative (friction slows down a passive particle), for an active system, we’ll allow \( U > 0 \), at least for small \( |\mathbf{v}| \). This is how we make our system active. We don’t want \( U \) to be positive for all \( |\mathbf{v}| \); if it was, the speed of the flock
\[ \hat{v} \cdot \partial_t \vec{v} = |\vec{v}| U(|\vec{v}|) \]

**FIG. 2:** Plot of the acceleration along \( \vec{v} \) arising from the \( \vec{v} U(|\vec{v}|) \) term in the equation of motion. This is the simplest qualitative form that can lead (for sufficiently small noise) to an ordered, moving flock.

would grow without bound, which is clearly unphysical. So we will assume that \( U \), plotted as a function of the speed \(|\vec{v}|\), is positive for small speeds \(|\vec{v}|\), and turns negative for large speeds \(|\vec{v}|\). This leads to the acceleration in the direction of motion illustrated in figure (2).

The effect of such a form for \( U \) is clearly the following: an initially slowly moving flock (or region thereof) will increase its speed until it reaches the speed \( v_0 \) at which \( U(|\vec{v}|) \) vanishes. Likewise, a flock (or region thereof) that is moving faster than \( v_0 \) will slow down until its speed again reaches \( v_0 \). Thus the speed \(|\vec{v}|\) is *not* a hydrodynamic variable; it relaxes back in a finite time \( \tau = 1/(\partial U/\partial |\vec{v}|)_\rho \) to \( v_0 \).

There are, obviously, infinitely many functions of the speed \(|\vec{v}|\) and \( \rho \) that have the properties just described. Fortunately, since the speed always adjusts itself to be close
to $v_0$, there prove to be only three parameters that we need to extract from $U$ for our hydrodynamic theory: the steady state speed $v_0$, and the derivatives $\partial U/\partial |v|$ and $\partial U/\partial \rho$ evaluated at $|v| = v_0$ and $\rho = \rho_0$, where $\rho_0$ is the mean density.

One popular choice for the function $U$ (indeed, the choice Yuhai and I made in our early papers on this problem) is the “ψ^4” theory:

$$U = \alpha(\rho) - \beta(\rho)|v|^2.$$  \hfill (3.3)

The reason this is called “ψ^4” theory is that with this choice, we can write

$$UV = -\partial V(v)/\partial v,$$  \hfill (3.4)

where the “potential”

$$V(v) = -\frac{1}{2}\alpha(\rho)|v|^2 + \frac{1}{4}\beta(\rho)|v|^4$$ \hfill (3.5)

takes the form of the famous “Mexican hat”, as shown in figure (3). The dynamical effect of the $U(|v|)$ term is then simply to make the velocity $v$ evolve down towards the circular (or, in three dimensions, spherical) ring of identical minima at $|v| = v_0 = \sqrt{\alpha/\beta}$.

This form is widely used in Condensed matter physics and field theory. It is also the form most appropriate for studying the order-disorder transition. However, since here I’m just interested in the behavior of the flock deep inside its ordered phase, I will not restrict myself to this form. It is useful, however, to keep figure (3) in mind, as we can construct a potential via (3.4) for any $U$, and, if that $|v|U$ has the form plotted in figure (2), the associated $V$ will look qualitatively like figure (3). And we can imagine the flock velocity $v$ evolving by seeking the minimum of this potential, or, more precisely, the ring of minima of this potential.

Indeed, the “spontaneous symmetry breaking” of a flock can be thought of as the system’s settling into one of these degenerate minima.

Note that if we use the expansion (3.1) around the mean velocity in (3.2), we get no linear term in $\delta v$ in the equation of motion for $\delta v_\perp$. To see this, note that the speed

$$|v| = \sqrt{(v_0 + \delta v_\parallel)^2 + |\delta v_\perp(r, t)|^2} = v_0 + \delta v_\parallel + O(|\delta v_\perp(r, t)|^2).$$ \hfill (3.6)

Using this in the projection of (3.2) perpendicular to the mean velocity, we obtain

$$(\partial_t \delta v_\perp)_1 = U(v_0 + \delta v_\parallel, \rho_0)\delta v_\perp + O(|\delta v_\perp|^3)$$

$$= U(v_0, \rho_0)\delta v_\perp + (\partial U/\partial |v|) |v_0, \rho_0|\delta v_\parallel + (\partial U/\partial \rho) |v_0, \rho_0|\delta \rho \delta v_\perp + O(|\delta v_\perp|^3)$$

$$= U(v_0, \rho_0)\delta v_\perp + O(|\delta v_\perp|^3, \delta v_\parallel \delta v_\perp, \delta \rho \delta v_\perp) = O(|\delta v_\perp|^3, \delta v_\parallel \delta v_\perp, \delta \rho \delta v_\perp),$$ \hfill (3.7)
\[ U_m(\vec{v}) = -\frac{\alpha}{2} |\vec{v}|^2 + \frac{\beta}{4} |\vec{v}|^4 \]

\[ \delta v_x \]

\[ \delta v_y \]

\[ \delta v_\perp \]  

\[ \delta v_\parallel \]

\[ \delta v_\perp = \text{"Goldstone mode"} \]

FIG. 3: The “Mexican hat” potential. Note the circular ring of minima. Fluctuations \( \delta v_\perp \) of the velocity that move it around this ring are “Goldstone modes”, which experience no restoring force from this “potential”. In contrast, fluctuations \( \delta v_\parallel \) that carry one up the brim of the hat, or towards the crown (i.e., radially), are “fast”; i.e., they will relax quickly due to this potential, which simply reflects the existence of a preferred speed for the flockers’ motion through the frictional medium.

where in the last equality I have used the fact that \( U(v_0, \rho_0) = 0 \), which is just a consequence of the definition of \( v_0 \) as the steady state speed.

This vanishing of all terms linear in \( \delta v \) in this term is no coincidence; rather, it is a consequence of the fact that \( \delta v_\perp \) is the Goldstone mode for this problem, so any term in it’s time derivative must have some spatial gradients, so that it vanishes when \( \delta v_\perp \) is spatially uniform.

This means if we want to include terms that depend on \( \delta v_\perp \) (which we certainly do!), then we need to look at terms involving the gradient operator \( \nabla \).

So let’s look at:
2) Combinations of velocities and one gradient operator. We need at least two velocities. Why? Well, can we make anything with a single gradient operator and a single velocity that transforms like a vector? The answer is no, as can most easily be seen using the Einstein summation convention: if we write

$$(\partial_t v_i)_{\text{trial}} = \text{constant} \times \partial_j v_k$$

(3.8)

there’s no choice of the indices $j$ and $k$ on the right hand side that will make this equation make sense. If we take $j = i$, we have an extra index $k$ running loose on the right hand side. If we try to get rid of it by instead taking $j = k$, then we’ve made the right hand side into a scalar (in fact, into $\nabla \cdot \mathbf{v}$), which we can’t equate to a vector. So we need at least two velocities to combine with our gradient operator.

So let’s try

$$(\partial_t v_i)^2 = \text{constant} \times v_\ell \partial_j v_k$$

(3.9)

This can be made to work with a suitable choice of the three indices $\ell$, $j$, and $k$. What we need to do is make two of them equal, so that they are summed over by the Einstein summation convention. This leaves one free index, which we must choose to be $i$, the free index on the right hand side. Basically, we use the Einstein summation convention to “eat” the extra indices on the right hand side. In fact, there are three ways to make this work:

i) Take $k = i$ and $\ell = j$. The right hand side has only one free index, so it’s a vector, and it’s the same free index as on the left hand side, so the equation makes sense in the Einstein summation convention. We can write this term as

$$(\partial_t v_i)_{2.1} = -\lambda_1 \mathbf{v} \cdot \nabla v_i,$$

(3.10)

where I’ve arbitrarily defined the “constant” in (3.9) to be $-\lambda_1$. (The minus sign is chosen to make the resulting equation look as much like the Navier-Stokes equation as possible, as you’ll see). Rewriting this in full glorious vector notation,

$$(\partial_t \mathbf{v})_{2.1} = -\lambda_1 \mathbf{v} \cdot \nabla \mathbf{v}.$$  

(3.11)

ii) Take $k = j$ and $\ell = i$. Once again, the right hand side has only one free index, so it’s a vector, and its the same free index as on the left hand side, so the equation makes sense in the Einstein summation convention. We can write this term as
\[(\partial_t v)_{2,2} = -\lambda_2 v_i \nabla \cdot \mathbf{v}, \quad (3.12)\]

where I’ve called the ““constant” $-\lambda_2$. In vector notation,
\[(\partial_t \mathbf{v})_{2,2} = -\lambda_2 \mathbf{v}(\nabla \cdot \mathbf{v}). \quad (3.13)\]

iii) take $j = i$ and $\ell = k$. This also makes sense in the Einstein summation convention. We can write this term as
\[ (\partial_t v_i)_{2,3} = -2\lambda_3 v_j \partial_i v_j = -\lambda_3 \partial_i (|\mathbf{v}|^2), \quad (3.14)\]

where I’ve introduced the factor of 2 in this definition of $\lambda_3$ for convenience in writing the second equality. In vector notation,
\[ (\partial_t \mathbf{v})_{2,3} = -\lambda_3 \nabla (|\mathbf{v}|^2). \quad (3.15)\]

There are also combinations of one gradient, $\mathbf{v}$, and the density $\rho$ that do not have the structure of (3.9), particularly if we allow more than one power of $\mathbf{v}$. Note that there is no reason we should not include such higher powers: because of the spontaneous ordering, $|\mathbf{v}|$ itself is not small (in fact, it’s close to $v_0^2$, which need not be small). We do intend to expand in powers of the fluctuation $\delta \mathbf{v}$ of $\mathbf{v}$ away from its mean value $v_0 \hat{x}$, but that is not the same as an expansion in powers of $\mathbf{v}$, because of this spontaneous order.

Fortunately, it turns out that we can incorporate all such one gradient terms into five terms, namely:
I) a pressure term
\[ (\partial_t \mathbf{v})_{\text{pressure}} = -\nabla P(|\mathbf{v}|, \rho). \quad (3.16)\]

Those of you familiar with the Navier-Stokes equation will recognize this as exactly the form of the pressure term in that equation, except for the peculiarity here that the pressure can depend not only on the density $\rho$, but also on the speed $|\mathbf{v}|$. Such dependence is forbidden in the Navier-Stokes equation by Galilean invariance; since we don’t have Galilean invariance in our dry active fluid, this dependence is allowed, and hence will, in general, be present.

II-IV) density and speed dependences of the $\lambda_{1,2,3}$ terms, and
V) an anisotropic pressure term $P_2$ of the form
\[ (\partial_t \mathbf{v})_{\text{aniso pressure}} = -\mathbf{v}(\mathbf{v} \cdot \nabla)P_2(|\mathbf{v}|, \rho) \quad (3.17)\]
To see that these five terms exhaust all possibilities, consider, for example, the term

\[(\partial_t v_i)_3 = v_\ell v_k \partial_j f(|v|, \rho)\]

Again choosing the indices so that two of them are eaten by the Einstein summation convention, while the remaining one is \(i\), we see that there are two ways to do this:

i) \(j = i, k = \ell\). This choice gives

\[(\partial_t v_i)_{3,1} = |v|^2 \nabla f = \nabla(|v|^2 f) - f \nabla v^2 \equiv -\nabla \delta P - \delta \lambda_2 \nabla v^2.\]  

where I’ve defined a contribution \(\delta P(|v|, \rho) \equiv f(|v|, \rho)|v|^2\) to the “Pressure” defined above, and a contribution \(\delta \lambda_2(|v|, \rho) \equiv f(|v|, \rho)\) to \(\lambda_2(|v|, \rho)\).

ii) \(j = k, \ell = i\) (note that \(j = \ell, k = i\) gives the same term).

\[(\partial_t v_i)_{3,2} = v_i v_j \partial_j f(|v|, \rho),\]  

which in vector form is precisely the \(P_2\) term \(\mathbf{3.20}\) with \(P_2(|v|, \rho) = -f(|v|, \rho)\).

The \(\lambda_{1,2,3}, P,\) and \(P_2\) terms can between them incorporate every “relevant” (i.e., non-negligible) term that involves one gradient and arbitrary powers on \(v\) and \(\rho\). To see this, consider, for example, the following term with four velocities and one gradient:

\[(\partial_t v_i)_{\text{trial2}} = \lambda_4 v_\ell \partial_j (v_n v_m v_k)\]  

We need to “eat” four of the five indices on the right hand side, and set the remaining one equal to \(i\). Let’s consider the term we get if we choose \(m = n, \ell = j,\) and \(k = i\). This gives

\[(\partial_t v_i)_4 = \lambda_4 v \cdot \nabla (v_i |v|^2) = \lambda_4 [|v|^2 v \cdot \nabla (v_i) + v_i v \cdot \nabla |v|^2].\]  

The first term on the right hand side is immediately recognizable as a contribution to \(\lambda_1\) proportional to \(|v|^2\), while the second is a constant contribution to \(P_2\).

It is straightforward to check that all terms that involve only one gradient can likewise be incorporated into speed \(|v|\) and density \(\rho\)- dependent corrections to one of the five aforementioned quantities isotropic pressure \(P\), anisotropic pressure \(P_2\), and \(\lambda_{1,2,3}\).

3) Let’s now consider terms with two gradients. One might think that we need not keep such terms, since they have more gradients than the one gradient terms we’ve just considered. However, it turns out, as we’ll see in the next section, that none of the one gradient terms we’ve just considered damps out velocity fluctuations to linear order in the velocity.
fluctuations $\delta \mathbf{v}$, which prove to be small. Instead, they just lead to propagation without dissipation. Therefore, if we do not include any two gradient terms, our theory would (erroneously) predict that there would be no damping of the fluctuations induced by the noise, which would therefore grow without bound over time. To prevent such an unphysical result, we need to go to higher order gradient terms. Second order proves, again with hindsight, to be sufficient.

So what can we make with two gradients that transforms like a vector? As before, let’s proceed by writing out possible terms in Einstein summation convention, and figure out how the indices can get eaten. So let’s start with terms with one velocity and two gradients. Generically, this can be written:

$$(\partial_t v_i)_{\text{trial3}} = \text{constant} \times \partial_j \partial_k v_\ell$$ (3.23)

By now, you should be familiar enough with how this goes to see that there are two menu options for ”index eating”:

i) $j = k$ and $\ell = i$. This gives

$$(\partial_t v_i)_{4.1} = D_T \nabla^2 v_i$$ (3.24)

or, in vector notation,

$$(\partial_t \mathbf{v})_{4.1} = D_T \nabla^2 \mathbf{v}$$ (3.25)

ii) $j = i$ and $\ell = k$ (or, equivalently, $k = i$ and $\ell = j$), which gives

$$(\partial_t v_i)_{4.2} = D_B \partial_i \partial_j v_j$$ (3.26)

or, in vector notation,

$$(\partial_t \mathbf{v})_{4.2} = D_B \nabla (\nabla \cdot \mathbf{v})$$ (3.27)

That’s it for terms with two spatial gradients and one velocity. These two terms also occur in the Navier-Stokes equations, where the coefficients $D_T$ and $D_B$ are usually denoted as $\nu_T$ and $\nu_B$, and are called the shear and bulk viscosities, respectively.

Can we make terms with more velocities and two derivatives? Absolutely; indeed, an overabundance of them. We can, however, tremendously reduce the number of possibilities by noting (as we did for the one gradient terms above) that when we expand about the state of uniform motion via (3.1), any velocity that a gradient acts on can be replaced by $\delta \mathbf{v}$, since $v_0 \hat{x}_i$ is a constant. Since $\delta \mathbf{v}$ is small, the dominant terms will be those with only one
\( \delta \mathbf{v} \). We can therefore restrict ourselves to terms with only one full velocity \( \mathbf{v} \) acted upon by the two derivatives. Therefore, all possible “relevant” terms involving two gradients and an arbitrary number of velocities can be written

\[
(\partial_t \mathbf{v}_i)_{4\text{ general}} = \text{constant} \times [v_pv_n \cdots v_sv_u] \partial_j \partial_k v_\ell
\]  

(3.28)

where \([v_pv_n \cdots v_sv_u]\) is a product of an even number \(2m\) of components of \( \mathbf{v} \). This number must be even, so that there are an odd number of indices altogether on the right hand side. This is necessary to allow us to pair all but one of them off, thereby producing a vector.

There are now four ways we can do this pairing off:

i) Pair all of the \(v\)'s to the left of the derivatives off with themselves, and set \( j = k \) and \( \ell = i \). This gives

\[
(\partial_t v_i)_{4.1} = \text{constant} \times |\mathbf{v}|^{2m} \partial_j \partial_j v_i = \text{constant} \times |\mathbf{v}|^{2m} \nabla^2 v_i
\]  

(3.29)

or, in vector notation,

\[
(\partial_t \mathbf{v})_{4.1} = \text{constant} \times |\mathbf{v}|^{2m} \nabla^2 \mathbf{v}
\]  

(3.30)

We can absorb this into a contribution to the “shear viscosity” \( D_T \) proportional to \(|\mathbf{v}|^{2m}\).

We can therefore incorporate all possible such terms, up to arbitrary even powers of \(|\mathbf{v}|\), by making \( D_T \) a suitably chosen function of \(|\mathbf{v}|\). We can generalize this even further by making \( D_T \) depend on the density \( \rho \) as well.

ii) Pair all of the \(v\)'s to the left of the derivatives off with themselves, and set \( j = i \) and \( k = \ell \) (or, equivalently, \( k = i \) and \( \ell = j \)). This gives

\[
(\partial_t v_i)_{4.2} = \text{constant} \times |\mathbf{v}|^{2m} \partial_i \partial_j v_j
\]  

(3.31)

or, in vector notation,

\[
(\partial_t \mathbf{v})_{4.2} = \text{constant} \times |\mathbf{v}|^{2m} \nabla (\nabla \cdot \mathbf{v})
\]  

(3.32)

which we can absorb into a contribution to the “bulk viscoscity” \( D_B \) proportional to \(|\mathbf{v}|^{2m}\).

We can therefore incorporate all possible such terms, up to arbitrary even powers of \(|\mathbf{v}|\), by making \( D_B \) a suitably chosen function of \(|\mathbf{v}|\). As for \( D_T \), we can generalize this even further by making \( D_B \) depend on the density \( \rho \) as well.

iii) Pair all but two of the \(v\)'s to the left of the gradients off with themselves, and pair the remaining two \(t\) with the gradients; this forces \( \ell = i \) (since there are no other free indices...
This gives

\[(∂_t v_i)_{4.3} = \text{constant} \times |v|^{2m-2} v_j v_k \partial_j \partial_k v_i \quad (3.33)\]

or, in vector notation,

\[(∂_t \mathbf{v})_{4.3} = \text{constant} \times |\mathbf{v}|^{2m-2} (\mathbf{v} \cdot \nabla)^2 \mathbf{v} \quad (3.34)\]

This is the first genuinely new term. I’ll sum up all such terms into a function that I’ll call \(D_2(|\mathbf{v}|, \rho)\) of the speed \(|\mathbf{v}|\) and the density \(\rho\) times the combination \((\mathbf{v} \cdot \nabla)^2 \mathbf{v}\). This term makes anisotropic diffusion possible: we can now have a different diffusion constant along the direction of flock motion than perpendicular to it, as we would expect, since we’ve broken (or, rather, the flock has broken) the symmetry between the direction of flock motion and directions perpendicular to it.

iv) Pair all but two of the \(v\)’s to the left of the gradients off with themselves, and pair the one of the other with one of the gradients, and the other with the velocity to the right of the gradients; this forces one of the gradient indices to be \(i\) (since there are no other free indices left). This gives

\[(∂_t v_i)_{4.4} = \text{constant} \times |v|^{2m-2} v_j v_k \partial_i \partial_k v_i \quad (3.35)\]

This contribution proves to be negligible compared to those we’ve already kept. To see this, consider the implied sum on \(j\) in \((3.35)\). One term in this sum is that with index \(j = \parallel\); i.e., the Cartesian component along the mean direction of motion. We can replace \(v_{\parallel}\) with \(δv_{\parallel}\) to the right of the gradient in \((3.35)\), since the mean velocity contribution to this term \(v_0 \hat{\mathbf{x}}\) is a constant, and hence has zero gradient. But, as we noted earlier, \(δv_{\parallel} \ll |v_{\perp}|\) since \(δv_{\parallel}\) is not a Goldstone mode, so this contribution from the sum on \(j\) to this term is negligible compared to the two gradient, one \(v_{\perp}\) terms we found above.

The other terms in the sum on \(j\) will be proportional to two \(v_{\perp}\)’s (one to the left of the gradient, and one to the right), and so will be negligible compared to the “one \(v_{\perp}\), two gradient” terms found above if \(v_{\perp}\) is small, as it will be in an ordered state. So those terms in the sum are negligible as well. So the entire term \((3.35)\) is negligible.

v) Finally, we can pair all but four of the \(v\)’s to the left of the gradients among themselves, set one of the remaining indices on those \(v\)’s equal to \(i\), and pair off the remaining three velocities with the two gradients and the velocity to the right of the gradient. This gives

\[(∂_t v_i)_{4.5} = \text{constant} \times |v|^{2m-4} v_i v_j v_k v_l \partial_j \partial_k v_l \quad (3.36)\]
The sum on $\ell$ in this term can be shown to be negligible by an argument almost identical to the one we just used for the previous contribution (3.35). So we’ll drop this as well.

The only other term we need to include is a random noise term:

$$\left( \partial_t v \right)_{\text{noise}} = f(r, t) \quad (3.37)$$

It is assumed to be Gaussian with white noise correlations:

$$\langle f_i(r, t)f_j(r', t') \rangle = 2D\delta_{ij}\delta^d(r - r')\delta(t - t') \quad (3.38)$$

where the “noise strength” $D$ is a constant parameter of the system, and $i, j$ denote Cartesian components.

Using the dynamical RG, one can show that small departures of the noise statistics from purely Gaussian have no effect on the long-distance physics.

Dah-deeb, dah-deeb, that’s all, folks! Any other terms you construct will have more gradients, and so will be negligible at long distances compared to the terms we’ve already found.

Putting all of these terms together gives the equation of motion for $v$:

$$\partial_t v + \lambda_1(v \cdot \nabla) v + \lambda_2(\nabla \cdot v) v + \lambda_3 \nabla(|v|^2) =$$

$$U(|v|, \rho) v - \nabla P + D_B \nabla (\nabla \cdot v) + D_T \nabla^2 v + D_2 (v \cdot \nabla)^2 v + f \quad (3.39)$$

Keep in mind that this equation is (even!) more complicated than it looks, because all of the parameters $\lambda_i (i = 1 \rightarrow 3)$, $U$, the “damping coefficients” $\mu_{B,T,2}$, the “isotropic pressure” $P(\rho, v)$ and the “anisotropic Pressure” $P_2(\rho, v)$ are functions of the density $\rho$ and the magnitude $v \equiv |v|$ of the local velocity.

To close these equations of motion, we also need one for the density. The final equation (3.40) is just conservation of bird number (we don’t allow our birds to reproduce or die on the wing).

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (v \rho) = 0 \quad (3.40)$$

With these equations of motion (3.39) and (3.40) in hand, we can now use them to figure out how flocks actually behave, and in particular why they can order even in $d = 2$. 
IV. SOLVING THE HYDRODYNAMIC MODEL

A. Expanding the equations of motion to “relevant” non-linear order

The hydrodynamic model embodied in equations (3.39) and (3.40) is equally valid in both the “disordered” (i.e., non-moving) state, in which $U(|v|, \rho)$ is negative for all $|v|$, and in the moving or “ferromagnetically ordered’ state, in which $|v|U(|v|)$ looks like figure [2], with a positive region at small $|v|$, which allows for the possibility of a moving state. In this section I’ll focus on the “ferromagnetically ordered”, broken-symmetry phase; and specifically on the question of whether fluctuations around the symmetry broken ground state destroy the ordered phase (as in the analogous phase of the 2D XY model). When $|v|U(|v|)$ looks like figure [2], we can write the expand the velocity field as in (3.1), which I rewrite here for convenience:

$$v(r, t) = v_0 \hat{x}_\parallel + \delta v(r, t) = (v_0 + \delta v_\parallel) \hat{x}_\parallel + \delta v_\perp(r, t),$$

where I remind you that $v_0 \hat{x}_\parallel$ is the spontaneous average value of $v$ in the ordered phase in the absence of fluctuations, whose magnitude $v_0$ is just that at which $U(|v|, \rho) = 0$.

As I’ve discussed above, the fluctuation $\delta v_\parallel$ of the component of $v$ along the mean direction of flock motion $\hat{x}_\parallel$ away from its preferred value $v_0$ is not a hydrodynamic variable of the system; rather, it relaxes back quickly to a value determined by the true hydrodynamic variables $\rho$ and $\delta v_\perp$. It therefore behooves us to eliminate it by solving for it in terms of those variables. Doing so is rather tricky - indeed, Yuhai and I got this slightly wrong in our earlier work on this problem [17–20]- so I will go through the argument rather carefully and in some detail here. For further details, see [21].

Since we know fluctuations in the speed (i.e., the magnitude $|v|$ of $v$) will be fast, it is useful to turn our equation of motion (3.39) for the velocity into an equation of motion for that speed. This can be done by taking the dot product of both sides of equation (3.39) with $v$ itself, which gives:

$$\frac{1}{2} \left( \partial_t |v|^2 + (\lambda_1 + 2\lambda_3)(v \cdot \nabla)|v|^2 \right) + \lambda_2 (\nabla \cdot v)|v|^2 = U(|v|, \rho)|v|^2 - v \cdot \nabla P - |v|^2 v \cdot \nabla P_2 + D_1 v \cdot \nabla (\nabla \cdot v) + D_T v \cdot \nabla^2 v + D_2 v \cdot ((v \cdot \nabla)^2 v) + v \cdot f$$

(4.2)
In this hydrodynamic approach, we are interested only in fluctuations $\delta v(r, t)$ and $\delta \rho(r, t)$ that vary slowly in space and time. (Indeed, the hydrodynamic equations (3.39) and (3.40) are only valid in this limit). Hence, terms involving space and time derivatives of $\delta v(r, t)$ and $\delta \rho(r, t)$ are always negligible, in the hydrodynamic limit, compared to terms involving the same number of powers of fields without any time or space derivatives.

Furthermore, the fluctuations $\delta v(r, t)$ and $\delta \rho(r, t)$ can themselves be shown to be small in the long-wavelength limit. Hence, we need only keep terms in equation (4.2) up to linear order in $\delta v(r, t)$ and $\delta \rho(r, t)$. The $\mathbf{v} \cdot \mathbf{f}$ term can likewise be dropped, since it only leads to a term of order $\mathbf{v}_\perp \mathbf{f}_\parallel$ in the $\mathbf{v}_\perp$ equation of motion, which is negligible (since $\mathbf{v}_\perp$ is small) relative to the $\mathbf{f}_\perp$ term already there.

These observations can be used to eliminate many of the terms in equation (4.2), and solve for $U$; the solution is:

$$U = \lambda_2 \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla P_2 + \frac{\sigma_1}{v_0} \partial_\parallel \delta \rho + \frac{1}{2v_0} \left( \partial_t + \gamma_2 \partial_\parallel \right) \delta v_\parallel,$$

(4.3)

where I’ve defined

$$\gamma_2 \equiv (\lambda_1 + 2\lambda_3)v_0$$

(4.4)

and

$$\sigma_1 \equiv \left( \frac{\partial P}{\partial \rho} \right)_0.$$

(4.5)

Here and hereafter, super- or sub-scripts 0 denoting functions of $\rho$ and $|\mathbf{v}|$ evaluated at the steady state values $\rho = \rho_0$ and $|\mathbf{v}| = v_0$.

Inserting this expression (4.3) for $U$ back into equation (3.39), I find that $P_2$ and $\lambda_2$ cancel out of the $\mathbf{v}$ equation of motion, leaving

$$\partial_t \mathbf{v} + \lambda_1 (\mathbf{v} \cdot \nabla) \mathbf{v} + \lambda_3 \nabla(|\mathbf{v}|^2) = \frac{\sigma_1}{v_0} \mathbf{v}(\partial_\parallel \delta \rho) - \nabla P + D_1 \nabla (\mathbf{v} \cdot \nabla \mathbf{v}) + D_T \nabla^2 \mathbf{v} + D_2 (\mathbf{v} \cdot \nabla)^2 \mathbf{v}$$

$$+ \left\{ \frac{1}{2v_0} \left( \partial_t + \gamma_2 \partial_\parallel \right) \delta v_\parallel \right\} \mathbf{v} + \mathbf{f}.$$

(4.6)

This can be made into an equation of motion for $\mathbf{v}_\perp$ involving only $\mathbf{v}_\perp (r, t)$ and $\delta \rho(r, t)$ by projecting perpendicular to the direction of mean flock motion $\hat{\mathbf{x}}_\parallel$, and eliminating $\delta v_\parallel$ using equation (4.3) and the expansion

$$U \approx -\Gamma_1 \left( \delta v_\parallel + \frac{|\mathbf{v}_\parallel|^2}{2v_0} \right) - \Gamma_2 \delta \rho,$$

(4.7)
where I’ve defined
\[ \Gamma_1 \equiv -\left( \frac{\partial U}{\partial |v|} \right)_0^\rho, \quad \Gamma_2 \equiv -\left( \frac{\partial U}{\partial \rho} \right)_{|v|}. \] (4.8)

I’ve also used the expansion (4.1) for the velocity in terms of the fluctuations \( \delta v_\parallel \) and \( \delta v_\perp \) to write
\[ |v| = v_0 + \delta v_\parallel + \left( \frac{|v_\perp|^2}{2v_0} \right) + O(\delta v_\parallel^2, |v_\perp|^4), \] (4.9)
and kept only terms that an RG analysis shows to be relevant in the long wavelength limit. Inserting (4.7) into (4.3) gives:
\[ -\Gamma_1 \left( \delta v_\parallel + \frac{|v_\perp|^2}{2v_0} \right) - \Gamma_2 \delta \rho = \lambda_2 \nabla_\perp \cdot v_\perp + \lambda_2 \partial_\parallel \delta v_\parallel + \left( \frac{\mu_1 v_0^2 + \sigma_1}{v_0} \right) \partial_\parallel \delta \rho + \frac{1}{2v_0} \left( \partial + \gamma_2 \partial_t \right) \delta v_\perp, \] (4.10)
where I’ve kept only linear terms on the right hand side of this equation, since the non-linear terms are at least of order derivatives of \( |v_\perp|^2 \), and hence negligible, in the hydrodynamic limit, relative to the \( |v_\perp|^2 \) term explicitly displayed on the left-hand side.

This equation can be solved iteratively for \( \delta v_\parallel \) in terms of \( v_\perp, \delta \rho, \) and its derivatives. To lowest (zeroth) order in derivatives, \( \delta v_\parallel \approx -\frac{\Gamma_2}{\Gamma_1} \delta \rho \). Inserting this approximate expression for \( \delta v_\parallel \) into equation (4.10) everywhere \( \delta v_\parallel \) appears on the right hand side of that equation gives \( \delta v_\parallel \) to first order in derivatives:
\[ \delta v_\parallel \approx -\frac{\Gamma_2}{\Gamma_1} \left( \delta \rho - \frac{1}{v_0 \Gamma_1 \partial_\parallel} \delta \rho + \frac{\lambda_4 \partial_\parallel \delta \rho}{\Gamma_2} \right) - \frac{\lambda_2}{\Gamma_1} \nabla_\perp \cdot v_\perp - \frac{|v_\perp|^2}{2v_0}, \] (4.11)
where I’ve defined
\[ \lambda_4 \equiv \left( \frac{\mu_1 v_0^2 + \sigma_1}{v_0} \right) - \frac{\Gamma_2}{\Gamma_1} \left( \lambda_2 + \frac{\gamma_2}{v_0} \right) = \left( \frac{\mu_1 v_0^2 + \sigma_1}{v_0} \right) - \frac{\Gamma_2}{\Gamma_1} \left( \lambda_1 + \lambda_2 + 2\lambda_3 \right). \] (4.12)

In deriving the second equality in (4.12), I’ve used the definition (4.4) of \( \gamma_2 \).

Inserting (4.1), (4.9), and (4.11) into the equation of motion (4.6) for \( v \), and projecting that equation perpendicular to the mean direction of flock motion \( \hat{x}_\parallel \) gives, neglecting “irrelevant” terms:
\[ \partial_t v_\perp + \gamma \partial_\parallel v_\perp + \lambda_1^0 (v_\perp \cdot \nabla_\perp) v_\perp = -g_1 \delta \rho \partial_\parallel v_\perp - g_2 v_\perp \partial_\parallel \delta \rho - \frac{c_0^2}{\rho_0} \nabla_\perp \delta \rho - g_3 \nabla_\perp (\delta \rho^2) \]
\[ + D_B \nabla_\perp (v_\perp \cdot v_\perp) + D_T \nabla^2_\perp v_\perp + D_\parallel \partial_\parallel^2 v_\perp + \nu_t \partial_t \nabla_\perp \delta \rho + \nu_\parallel \partial_\parallel \nabla_\perp \delta \rho + f_\perp, \]

(4.13)

where I’ve defined
\[ D_B \equiv D_1 + \frac{2 v_0 \lambda_3^0 \lambda_2^0}{\Gamma_1}, \]

(4.14)
\[ D_\parallel \equiv D_T + D_2 v_0^2, \]

(4.15)
\[ \gamma \equiv \lambda_1^0 v_0, \]

(4.16)
\[ g_1 \equiv v_0 \left( \frac{\partial \lambda_1}{\partial \rho} \right)_0 - \frac{\Gamma_2 \lambda_1^0}{\Gamma_1}, \]

(4.17)
\[ g_2 \equiv \frac{\Gamma_2 \gamma_2^0}{\Gamma_1 v_0} - \frac{\sigma_1}{v_0}, \]

(4.18)
\[ g_3 \equiv \sigma_2 + \left( \frac{\Gamma_2}{\Gamma_1} \right)^2 \lambda_3^0 - \left( \frac{\partial \lambda_3}{\partial \rho} \right)_0 \frac{\Gamma_2 v_0}{\Gamma_1}, \]

(4.19)
\[ c_0^2 \equiv \rho_0 \sigma_1 - \frac{2 \rho_0 v_0 \lambda_3^0 \Gamma_2}{\Gamma_1}, \]

(4.20)
\[ \nu_t \equiv -\frac{2 \Gamma_2 \lambda_3^0}{\Gamma_1^2}, \]

(4.21)

and
\[ \nu_\parallel \equiv \frac{2 v_0 \lambda_3^0 \lambda_2^0}{\Gamma_1} + \frac{\Gamma_2 D_1}{\Gamma_1}. \]

(4.22)

Using (4.1) and (4.9) in the equation of motion (3.40) for \( \rho \) gives, again neglecting irrelevant terms:
\[
\partial_t \delta \rho + \rho_0 \nabla_\perp \cdot \mathbf{v}_\perp + w_1 \nabla_\perp \cdot (\mathbf{v}_\perp \delta \rho) + v_2 \partial_\parallel \delta \rho = D_{\rho_\parallel} \partial^2_\parallel \delta \rho + D_{\rho_\perp} \nabla^2_\perp \delta \rho + D_{\rho_\nu} \partial_\parallel (\nabla_\perp \cdot \mathbf{v}_\perp) \\
+ \phi \partial_t \partial_\parallel \delta \rho + w_2 \partial_\parallel (\delta \rho^2) + w_3 \partial_\parallel (|\mathbf{v}_\perp|^2),
\]

(4.23)

where I’ve defined:

\[v_2 \equiv v_0 - \frac{\rho_0 \Gamma_2}{\Gamma_1},\]

(4.24)

\[\phi \equiv \frac{\Gamma_2 \rho_0}{v_0 \Gamma_1^2},\]

(4.25)

\[w_2 \equiv \frac{\Gamma_2}{\Gamma_1},\]

(4.26)

\[w_3 \equiv \frac{\rho_0}{2v_0} ,\]

(4.27)

\[D_{\rho_\parallel} \equiv \frac{\rho_0 \lambda_1^0}{\Gamma_1} \]

\[= \frac{\rho_0}{\Gamma_1} \left( \frac{(\mu_1 v_0^2 + \sigma_1)}{v_0} - \frac{\Gamma_2}{\Gamma_1} \left( \lambda_1^0 + \lambda_2^0 + 2\lambda_3^0 \right) \right),\]

(4.28)

and, last but by no means least,

\[D_{\rho_\nu} \equiv \frac{\lambda_2^0 \rho_0}{\Gamma_1},\]

(4.29)

The parameter \(D_{\rho_\perp}\) is actually zero at this point in the calculation, but I’ve included it in equation \(4.23\) anyway, because it is generated by the nonlinear terms under the Renormalization Group, as I’ll discuss in section (IV C). Likewise, the parameter \(w_1 = 1\), but I’ve also included it for convenience in discussing the renormalization group in section (IV C).

I will henceforth focus my attention on the fluid, orientationally ordered state, in which all of the diffusion constants \(D_{\rho_\parallel}, D_{\rho_\perp}, D_{\rho_\nu}, D_B, D_\parallel,\) and \(D_T\) are positive. I’ll take them all to have their steady state values \(D_{\rho_\parallel}^0\) etc. at \(|\mathbf{v}| = v_0\) and \(\rho = \rho_0\), since fluctuations away from that can be shown to be irrelevant.
B. Linearized Theory

Expanding (4.13) and (4.23) to linear order in the small fluctuations $v_\perp$ and $\delta \rho$ gives:

$$\frac{\partial_t v_\perp + \gamma \partial_\parallel v_\perp}{\rho_0} = -\frac{c_0^2}{\rho_0} \nabla_\perp \delta \rho + D_B \nabla_\perp (\nabla_\perp \cdot v_\perp) + D_T \nabla_\perp^2 v_\perp + D_\rho_\parallel \partial_\parallel \delta \rho + \nu_t \partial_t \nabla_\perp \delta \rho + \nu_\parallel \partial_\parallel \nabla_\perp \delta \rho + f_\perp,$$

(4.30)

and

$$\frac{\partial_t \delta \rho + \rho_0 \nabla_\perp \cdot v_\perp + v_2 \partial_\parallel \delta \rho}{\rho_0} = D_\rho_\parallel \partial_\parallel^2 \delta \rho + D_\rho_\perp \nabla_\perp^2 \delta \rho + D_\rho v_\perp \nabla_\perp \nabla_\perp \delta \rho + \phi \partial_t \partial_\parallel \delta \rho.$$

(4.31)

These equations can now readily be solved for the mode structure and correlations by Fourier transforming in space and time; this gives

$$[-i(\omega - \gamma q_\parallel) + \Gamma_L(q)] v_L + \left[ \frac{i c_0^2}{\rho_0} q_\perp - v_t q_\perp \omega - v_\parallel q_\perp q_\parallel \right] \delta \rho = f_L,$$

(4.32)

$$[-i(\omega - \gamma q_\parallel) + \Gamma_T(q)] v_T = f_T,$$

(4.33)

$$[i \rho_0 q_\perp + D_\rho v_\perp q_\parallel] v_L + [-i(\omega - v_2 q_\parallel) + \Gamma_\rho(q) - \phi q_\parallel \omega] \delta \rho = 0,$$

(4.34)

and where I’ve defined the wavevector dependent longitudinal, transverse, and $\rho$ dampings $\Gamma_{L,T,\rho}$:

$$\Gamma_L(q) = D_L q_\perp^2 + D_\parallel q_\parallel^2,$$

(4.35)

$$\Gamma_T(q) = D_T q_\perp^2 + D_\parallel q_\parallel^2,$$

(4.36)

$$\Gamma_\rho(q) = D_\rho_\parallel q_\parallel^2 + D_\rho_\perp q_\perp^2,$$

(4.37)

with $D_L \equiv D_B + D_T$. I’ve also separated the velocity $v_\perp$ and the noise $f_\perp$ into components along and perpendicular to the projection $q_\perp$ of $v$ perpendicular to $<v>$ via

$$v_L \equiv v_\perp \cdot q_\perp / q_\perp, \quad v_T \equiv v_\perp - v_L \frac{q_\perp}{q_\perp}.$$
with \( f_L \) and \( f_T \) obtained from \( f \) in the same way.

These equations differ from the corresponding equations considered in \([17–20]\) only in the \( \nu_{\perp} \) terms in (4.32), and the \( D_{\rho ||} \) and \( D_{\rho v} \) terms in (4.34). These prove to lead only to minor changes in the propagation direction dependence, but not the scaling with wavelength, of the damping of the sound modes found in \([17–20]\), as I will now demonstrate.

I begin by determining the eigenfrequencies of the system, defined in the usual way as the complex, wavevector dependent frequencies \( \omega(q) \) at which the Fourier transformed hydrodynamic equations (4.32), (4.33), and (4.34) admit non-zero solutions for \( v_T, \delta \rho, \) and \( v_L \) when the noise \( f \) is set to zero. Note that \( v_T \) is decoupled from \( v_L \) and \( \rho \); this implies a pair of “longitudinal” eigenmodes involving just the longitudinal velocity \( v_L \) and \( \rho \), and an additional \( d - 2 \) “transverse” modes associated with the transverse velocity \( v_T \). The longitudinal modes are closely analogous to ordinary sound waves in a simple fluid\([16]\), while the transverse modes are the analog of the diffusive shear modes in such a fluid.

In the hydrodynamic limit (i.e., when wavenumber \( q \to 0 \)), the longitudinal eigenfrequencies become a pair of underdamped, propagating modes with complex eigenfrequencies

\[
\omega_{\pm}(q) = c_{\pm}(\theta_q)q - i\epsilon_{\pm}(q),
\]

where the direction-dependent sound speeds \( c_{\pm}(\theta_q) \) are given by exactly the same expression as found in previous work\([17–20]\):

\[
c_{\pm}(\theta_q) = \left( \frac{\gamma + v_2}{2} \right) \cos(\theta_q) \pm c_2(\theta_q),
\]

where I’ve defined

\[
c_2(\theta_q) \equiv \sqrt{\frac{(\gamma - v_2)^2 \cos^2(\theta_q)}{4} + c_0^2 \sin^2(\theta_q)},
\]

where \( \theta_q \) is the angle between \( q \) and the direction of flock motion (i.e., the \( x_\parallel \) axis).

A polar plot of this highly anisotropic sound speed is given in figure (4).

This prediction for the anisotropy of the sound speeds in flocks has recently been confirmed quantitatively in experiments on synthetic flockers (specifically, Quinke rotators) \([3]\).

As mentioned earlier, the wavevector dependent dampings \( \epsilon_{\pm}(q) \) of these propagating sound modes are altered slightly from the form found in \([17–20]\). They remain of \( O(q^2) \), as found in previous work, but with a slightly modified dependence on propagation direction \( \hat{q} \). More precisely, they are given by:
FIG. 4: Polar plot of the direction-dependent sound speeds $c_{\pm} (\theta_q)$, with the horizontal axis along the direction of mean flock motion.
\[ \epsilon_{\pm} \equiv \frac{\text{Hideous Numerator}}{(2c_{\pm}(\theta_q) - (v_2 + \gamma)\cos(\theta_q))} \]  

with

\[ \text{Hideous Numerator} \equiv (\Gamma_L(q) + \Gamma_{\rho}(q) - \phi c_{\pm}(\theta_q)\cos(\theta_q)q^2) c_{\pm}(\theta_q) - v_2 \Gamma_L(q) \cos(\theta_q) \]

\[ - \gamma(\Gamma_{\rho}(q) - \phi c_{\pm}(\theta_q)\cos(\theta_q)q^2) \cos(\theta_q) + \frac{c_0}{\rho_0} D_{\rho \mu} \frac{q_i q_j^2}{q} \]

\[ - \rho_0 q_i^2 (\nu_{\parallel} c_{\pm}(\theta_q) + \nu_{\perp} \cos(\theta_q)) \]  

(4.43)

where I remind the reader that the wavevector dependent dampings \( \Gamma_{L,\rho} \) are \( O(q^2) \), and defined earlier in equations (4.35, 4.37). Thus, the “hideous numerator”, while indeed hideous in its angular dependence, is nonetheless simple in its scaling with the magnitude \( q \) of \( q \): it scales like \( q^2 \). This implies that the dampings \( \epsilon_{\pm} \propto q^2 \) as well.

The transverse modes have the far simpler character of simply convected anisotropic diffusion:

\[ \omega_T(q) = \gamma q_{\parallel} - i \Gamma_T(q) \]  

(4.44)

with the wavevector dependent damping \( \Gamma_T \) also \( O(q^2) \), and defined earlier in equation (4.36). This corresponds to simple anisotropic diffusion in a “pseudo-comoving” frame, by which I mean a frame that moves in the direction of mean flock motion, but with a speed \( \gamma \) that differs from the speed \( v_0 \) of the flock itself.

I now turn to the correlation functions in this linearized approximation. These are easily obtained by first solving the linear algebraic equations (4.32, 4.33, and 4.34) for the fields \( v_L(q, \omega) \), \( v_T(q, \omega) \), and \( \rho(q, \omega) \) in terms of the noises \( f_L(q, \omega) \), and \( f_T(q, \omega) \). These solutions are, of course, linear in those noises. Hence, by correlating these solutions pairwise, one can obtain any two field correlation function in terms of the correlations (3.38) of \( f \). The resulting correlation function for the velocity is:

\[ C_{ij}(q, \omega) \equiv \langle v_{\perp i}(q, \omega) v_{\perp j}(-q, -\omega) \rangle = \frac{\Delta \left( \omega - v_2 q_{\parallel} \right)^2 L_{ij}}{[(\omega - c_+ (\theta_q) q)^2 + \epsilon_{\pm}^2(q)] \left[ (\omega - c_- (\theta_q) q)^2 + \epsilon_+^2(q) \right]} \]

\[ + \frac{\Delta P_{ij}^\perp}{(\omega - \gamma q_{\parallel})^2 + \Gamma_T^2(q)}. \]  

(4.45)
where I’ve defined the longitudinal (L) and transverse (T) projection operators in the ⊥ plane

\[ L_{ij}^\perp(\hat{q}) = \frac{q_i q_j}{q^2} \]
\[ P_{ij}^\perp(\hat{q}) = \delta_{ij}^\perp - L_{ij}^\perp(\hat{q}) \]

(4.46)

where \( \delta_{ij}^\perp \) is a Kronecker delta in the ⊥ plane (i.e., it is equal to the usual Kronecker delta if \( i \neq \parallel \neq j \), and zero otherwise). These operators project any vector first into the ⊥ plane, and then either along (L) or orthogonal to (P) \( q_\perp \) within the ⊥ plane.

The first term in equation (4.45) comes from the “longitudinal” component \( v_L \) while the second comes from the \( d - 2 \) “transverse” components of \( v_\perp \). Clearly, in \( d = 2 \), only the longitudinal component is present; the second (transverse) term in (4.45) vanishes in \( d = 2 \).

The density autocorrelations obtained by the procedure described above are given, to leading order in wavevector and frequency, by:

\[ C_{\rho \rho}(q,\omega) \equiv \langle \rho(q,\omega)\rho(-q,-\omega) \rangle = \rho_0 q^2 \Delta \left[ (\omega - c_+ (\theta_q) q)^2 + \epsilon_+^2(q) \right] \left[ (\omega - c_- (\theta_q) q)^2 + \epsilon_-^2(q) \right] . \]

(4.47)

Both the velocity correlations (4.45) and the density correlations (4.47) have the same form, and the same scaling with frequency and wavevector, as those reported in earlier work [17–20]. The only change from those earlier results is the slightly modified form (4.42, 4.43) of the sound dampings which appear in (4.45) and (4.47).

The same statement is true of the equal-time correlations of \( v \) and \( \rho \), which can be obtained in the usual way by integrating the spatiotemporally Fourier transformed correlations (4.45) and (4.47) over all frequency \( \omega \). These equal time correlations are important, because they determine the size of the velocity and density fluctuations. The size of the velocity fluctuations determines whether or not long ranged order can exist in these systems, while the size of the density fluctuations determines the presence or absence of giant number fluctuations [20, 25–27].

Integrating (4.45) over all \( \omega \) and tracing over the Cartesian components \( i, j \) gives the equal-time correlation of \( v \):

\[ \langle |v_\perp(q,t)|^2 \rangle = \frac{1}{2} \left( \frac{\Delta (c_+ (\theta_q) - v_2 \cos (\theta_q))^2}{\epsilon_+^2(q)} + \frac{\Delta (c_- (\theta_q) - v_2 \cos (\theta_q))^2}{\epsilon_-^2(q)} + \frac{(d - 2)\Delta}{\Gamma_T(q)} \right) . \]

(4.48)
Note that these scale like $1/q^2$ for all directions of wavevector $\mathbf{q}$. This scaling is precisely the same as that found in the linearized theory of [17] [20]; only the precise form of the dependence on the direction of $\mathbf{q}$ is slightly changed by the presence of the new linear terms $\nu_t$, $\nu_\parallel$, and $\phi$ that I’ve found here that were missed in the treatment of [17] [20].

This $1/q^2$ scaling of $\mathbf{v}_\perp$ fluctuations with $q$ in Fourier space implies that the real space fluctuations

$$\langle |\mathbf{v}_\perp (\mathbf{r}, t)|^2 \rangle = \int \frac{d^d q}{(2\pi)^d} \langle |\mathbf{v}_\perp (\mathbf{q}, t)|^2 \rangle$$

(4.49)

diverge in the infra-red ($q \to 0$ or system size $L \to \infty$) limit in all spatial dimensions $d \leq 2$. This in turn implies that long-ranged order (i.e., the existence of a non-zero $\langle \mathbf{v}_\perp (\mathbf{r}, t) \rangle$) is not possible in $d = 2$, according to the linearized theory.

This result, which is simply the Mermin-Wagner [15] theorem, is actually overturned by non-linear effects, which stabilize the long-ranged order in $d = 2$ (i.e., make the existence of a non-zero $\langle \mathbf{v} (\mathbf{r}, t) \rangle$ possible), as first noted by [17] [20]. I’ll show in subsection (IV C) that non-linear effects still stabilize long-ranged order in this way even when the additional nonlinearities I’ve found here, which were missed in [17] [20], are included.

The equal time density autocorrelations can likewise be obtained by integrating equation (4.47) over frequency $\omega$; this gives

$$\langle |\delta \rho (\mathbf{q}, t)|^2 \rangle = \frac{1}{2} \left( \frac{\Delta \rho q^2_\perp}{|c_+ (\theta_\mathbf{q}) - c_- (\theta_\mathbf{q})|^2 q^2} \right) \left( \frac{1}{\epsilon_+ (\mathbf{q})} + \frac{1}{\epsilon_- (\mathbf{q})} \right)$$

(4.50)

This also scale like $1/q^2$ for all directions of $\mathbf{q}$. This divergence implies “Giant Number Fluctuations” [20] [25] [27]: the RMS fluctuations $\sqrt{\langle \delta N^2 \rangle}$ of the number of particles within a large region of the system scale like the mean number of particles $\langle N \rangle$ faster than $\sqrt{\langle N \rangle}$; specifically, $\sqrt{\langle \delta N^2 \rangle} \propto \langle N \rangle^{\phi(d)}$, with $\phi(d) = 1/2 + 1/d$ in spatial dimension $d$. Note that this means in particular that $\sqrt{\langle \delta N^2 \rangle} \propto \langle N \rangle$ in $d = 2$.

Again, I emphasize that this is the prediction of the linearized theory. It once again coincides with the results of the linearized treatment of [17] [20].

Both the prediction that long ranged orientational order is destroyed in $d = 2$, and the value $\phi(d) = 1/2 + 1/d$ of the exponent $\phi(d)$ for $d < 4$ prove, when non-linear effects are taken into account, to be incorrect, as first noted by [17]. I now turn to the treatment of those nonlinear effects.
C. Non-linear Effects

We have seen that the linearized theory does not explain the mystery that motivated my original interest in this problem: the persistence of long-ranged order in flocks even in $d = 2$. Fortunately, it turns out that the non-linearities that we ignored in the previous section in fact completely change the scaling behavior of these systems at long distances, as first noted by [17][20]. In this section, I’ll deal with those non-linearities. While a few of the precise quantitative conclusions of [17][20] prove to be less certain than Yuhai and I originally thought, the essential conclusions that

1) non-linearities radically change the scaling behavior of these systems for all $d \leq 4$, and
2) these changes in scaling stabilize long-ranged order in $d = 2$,

remain valid.

Equally noteworthy are the non-linear terms that are missing from (4.13) and (4.23): all non-linearities arising from the anisotropic pressure $P_2$ and the $\lambda_2$ nonlinearity drop out of (4.13) and (4.23). This in particular has the very important consequence of saving the Mermin-Wagner theorem. This is because the $\lambda_2$ term is allowed even in equilibrium systems [28]. The incorrect treatment in [17][20] suggested that this term by itself could stabilize long-range order in $d = 2$. Given that this term is allowed in equilibrium, this would imply that the Mermin-Wagner theorem would fail for such an equilibrium system. The correct treatment I’ve done here shows that this is not the case: the $\lambda_2$ term by itself cannot stabilize long ranged order in $d = 2$, since the non-linearities associated with it drop out of the long-wavelength description of the ordered phase.

Returning now to the non-linearities in (4.13) and (4.23) that were missed by [17][20], I will now show that all of them become relevant, in the renormalization group (RG) sense[29], for spatial dimensions $d \leq 4$.

To assess the effect of the new non-linear terms I’ve found here, I’ll analyze equations (4.13) and (4.23) using the dynamical Renormalization Group (RG) [24].

The dynamical RG starts by averaging the equations of motion over the short-wavelength fluctuations: i.e., those with support in the “shell” of Fourier space $b^{-1}\Lambda \leq |q| \leq \Lambda$, where $\Lambda$ is an “ultra-violet cutoff”, and $b$ is an arbitrary rescaling factor. Then, one rescales lengths, time, $\delta \rho$ and $v_\perp$ in equations (4.13) and (4.23) according to $v_\perp = b^x v_\perp', \delta \rho = b^x \delta \rho', r_\perp = b^x r_\perp'$, $r'_\parallel = b^x(r_\parallel')'$, and $t = b^x t'$ to restore the ultra-violet cutoff to $\Lambda$ [30]. This leads to a new
pair of equations of motion of the same form as \((4.13)\) and \((4.23)\), but with “renormalized” values (denoted by primes below) of the parameters given by:

\[
D'_{B,T} = b^{z-2}(D_{B,T} + \text{graphs}) , \tag{4.51}
\]

\[
D'_{\parallel,\parallel} = b^{z-2\zeta}(D_{\parallel,\parallel} + \text{graphs}) , \tag{4.52}
\]

\[
\Delta' = b^{z-\zeta-2}\Delta + \text{graphs} , \tag{4.53}
\]

\[
(\lambda^0_1)' = b^{z+\chi-1}(\lambda^0_1 + \text{graphs}) , \tag{4.54}
\]

\[
g_{1,2}' = b^{z+\chi-\zeta}(g_{1,2} + \text{graphs}) , \tag{4.55}
\]

\[
g_3' = b^{z+\chi-1}(g_3 + \text{graphs}) , \tag{4.56}
\]

\[
\phi' = b^{z+\chi-1}(\phi + \text{graphs}) , \tag{4.57}
\]

\[
w'_{1,2,3} = b^{z+\chi-\zeta}(w_{1,2,3} + \text{graphs}) , \tag{4.58}
\]

where “graphs” denotes contributions from integrating out the short wavelength degrees of freedom.

I have focused on the particular linear parameters \(D_{B,T,\parallel,\parallel}\) and \(\Delta\) since, as is clear from equations \((4.48)\) and \((4.50)\), they determine the size of the fluctuations in the linearized theory.

One proceeds by seeking fixed points of these recursion relations. One simple fixed point is the linear fixed point, at which all of the non-linear coefficients \(\lambda^0_1, g_{1,2,3},\) and \(w_{1,2,3}\) are zero. At such a fixed point, the graphical corrections (denoted “graphs” in equations \((4.51)\) - \((4.58)\)) vanish, since, without nonlinearities, Fourier modes at different wavevectors and frequencies do not interact. It is then straightforward to determine from equations \((4.51)\) -
the values of the rescaling exponents $z$, $\zeta$, and $\chi$ that will keep $D_{B,T,||,\parallel}$ and $\Delta$ (and, hence, the size of the fluctuations) fixed: simply those that make the exponents in (4.51) - (4.58) vanish. That is, we must chose

$$z - 2 = 0 \quad \text{(linear fixed point)}$$

(4.59)

to keep $D_B$ and $D_T$ fixed,

$$z - 2\zeta = 0 \quad \text{(linear fixed point)},$$

(4.60)
to keep $D_\parallel$ and $D_{\rho\parallel}$ fixed, and

$$z - \zeta - 2\chi + 1 - d = 0 \quad \text{(linear fixed point)},$$

(4.61)
to keep $\Delta$ fixed under the RG. The solutions to these three conditions (4.59)-(4.61) are trivially found to be:

$$z = 2 \quad \text{(linear fixed point)},$$

(4.62)

$$\zeta = 1 \quad \text{(linear fixed point)},$$

(4.63)

and

$$\chi = (2 - d)/2 \quad \text{(linear fixed point)}.$$  

(4.64)

Let’s now consider the stability this linear fixed point against the effect of the non-linear terms $\lambda_1^0$, $g_{1,2,3}$, and $w_{1,2,3}$. Because, as mentioned earlier, I have chosen the rescaling exponents so as to keep the magnitude of the fluctuations the same on all length scales, a given non-linearity has important effects at long distances if it grows upon renormalization with this choice (4.62)-(4.64) of the rescaling exponents $z$, $\zeta$, and $\chi$ provided that it grows upon renormalization; contrariwise, if it gets smaller upon renormalization with this choice of the rescaling exponents, it is unimportant at long distances. Using the exponents (4.62)-(4.64) in the recursion relations (4.65), (4.66), (4.66), (4.57), and (4.61), and ignoring the graphical corrections, which are higher than linear order in $\lambda_1^0$, $g_{1,2,3}$, and $w_{1,2,3}$, I find that all seven of these non-linearities have identical renormalization group eigenvalues of $(4 - d)/2$ at the linearized fixed point; that is:

$$(\lambda_1^0)' = b^{4-d/2} \lambda_1^0,$$

(4.65)
Thus, for $d > 4$, all of the nonlinearities flow to zero, and so become unimportant, at long length and time scales. Hence, the linearized theory is correct at long length and time scales, for $d > 4$. For $d < 4$, however, all of these nonlinearities grow, and the linear theory breaks down at sufficiently long length and time scales.

Both this analysis, and its conclusion that non-linear effects invalidate the linear theory for $d < 4$, are almost identical to those of [17–20]. However, whereas they found only four non-linearities ($\lambda_{1,2}, w_1$, and $g_3$ in the notation I’m using here) that became relevant as $d$ is decreased below $d = 4$, I find seven such nonlinearities. More importantly, the vector structure of some of the new nonlinearities differs from that of those studied in [17–20] in crucial ways. In particular, all of the nonlinearities considered in [17–20] could, in $d = 2$, be written as total $\perp$ derivatives. This implies that these nonlinearities can only renormalize terms which themselves involved $\perp$-derivatives (i.e., $D_{B,T}$); hence, all of the terms that did not involve $\perp$-derivatives (i.e., $D_{\parallel,\rho}, \Delta$) were incorrectly argued in [17–20] to get no graphical corrections. This lead to the incorrect conclusion that, in order to obtain a fixed point, one had to choose the rescaling exponents $z$, $\zeta$, and $\chi$ to make the exponents in (4.52) and (4.53) vanish; i.e., that in $d = 2$,

$$z - 2\zeta = 0, \quad z - \zeta - 2\chi + 1 - d = z - \zeta - 2\chi - 1 = 0.$$  

(4.68)

The earlier work of [17–20] went on to incorrectly argue that there were no graphical corrections $\lambda_1^0$ either, because the equations of motion (4.13) and (4.23) have, in $d = 2$ and in the absence of the extra relevant nonlinearities $g_{1,2}$ and $w_{1,2,3}$ found here, an exact “pseudo-Galilean invariance” symmetry[32]: they remain unchanged by a pseudo-Galilean transformation:

$$r_\perp \to r_\perp - \lambda_1 v_1 t, \quad v_\perp \to v_\perp + v_1,$$

(4.69)

for arbitrary constant vector $v_1 \perp \hat{x}_j$. Note that if $\lambda = 1$, this reduces to the familiar Galilean invariance in the $x$-direction. Since such an exact symmetry must continue to hold
upon renormalization, with the same value of $\lambda_1$, $\lambda_1'$ cannot be graphically renormalized in the absence of the extra relevant nonlinearities $g_{1,2}$ and $w_{1,2,3}$ found here. Requiring that $\lambda_1' = \lambda_1$ in (4.65), and setting graphs = 0, implies that

$$\chi = 1 - z$$  \hspace{1cm} (4.70)

in $d = 2$. This and (4.68) forms three independent equations for the three unknown exponents $\chi$, $z$, and $\zeta$, whose solution in $d = 2$ is

$$z = 6/5$$  \hspace{1cm} (4.71)

$$\zeta = 3/5$$  \hspace{1cm} (4.72)

and

$$\chi = -1/5$$  \hspace{1cm} (4.73)

which are the exponents purported in [17–20] to be exact in $d = 2$.

The presence of the extra nonlinearities $g_{1,2,3}$ and $w_{1,2,3}$ invalidates every essential ingredient of the above argument: these nonlinearities are not total $\perp$-derivatives, so one can not argue that $D_{\|,\rho}$ and $\Delta$ get no graphical corrections. This invalidates the exact scaling relations (4.68), and makes it impossible to obtain exact exponents in $d = 2$.

I have been unable to come up with alternative arguments that give exact exponents in the presence of these additional terms.

Now, if these additional nonlinearities were irrelevant in $d = 2$ under a full dynamical RG, then the exact exponents of [17–20] would be correct in $d = 2$.

There is a precedent for this (that is, for terms that appear relevant by simple power counting below some critical dimension $d_c$ actually proving to be irrelevant once”graphical corrections” -i.e., nonlinear fluctuation effects - are taken into account). One example of this is the cubic symmetry breaking interaction[33] in the $O(n)$ model, which is relevant by power counting at the Gaussian fixed point for $d < 4$, but proves to be irrelevant, for sufficiently small $n$, at the Wilson-Fisher fixed point that actually controls the transition for $d < 4$, at least for $\epsilon = 4 - d$ sufficiently small.

Unfortunately, doing a similar $4 - \epsilon$ analysis of the relevance of these new nonlinearities in the flocking problem would tell us nothing about whether or not these terms are relevant in $d = 2$, since $2$ is far below the critical dimension $d_c = 4$ of the flocking problem.
Hence, whether or not the exact exponents predicted by [17–20] are correct remains an open question. They could be; numerical experiments[18–20, 26, 34, 35] suggest they are, but we really don’t know at this point.

Not all of the predictions of [17–20] become questionable in light of the existence of these new nonlinearities, however. In particular, the claim that long ranged orientational order can exist even in $d = 2$ is unaffected. I know this because the nonlinear terms clearly make positive contributions to the damping coefficient corrections to the velocity diffusion “constants” $D_B$ and $D_T$ are positive, and that they are relevant in the RG sense, which means they must change the scaling of the velocity fluctuations from that predicted by the linearized theory. I know that they are relevant by the following proof by contradiction: if all of the nonlinear effects were irrelevant, then simple power counting would suffice to determine their relevance. But simple power counting says that all of the nonlinearities are relevant for $d < d_c = 4$, which contradicts the original assumption that they’re all irrelevant. Thus, the nonlinearities must change the scaling of the velocity fluctuations. Since the effect of the nonlinearities is to renormalize the velocity diffusion “constants” $D_B$ and $D_T$ upwards, and since this tends to reduce velocity fluctuations, the growth of velocity fluctuations with length scale must be suppressed (more precisely, its scaling must be suppressed; i.e., it must grow like a smaller power of length scale $L$) than is predicted by the linearized version of the equations of motion (4.13) and (4.23). But those linearized equations predict [17–20] only logarithmic divergences of velocity fluctuations with length scale. Hence, the real fluctuations, including nonlinear effects, must be smaller than logarithmic by some power of length scale[36], which means they must be finite as $L \to \infty$. This boundedness of velocity fluctuations means that long ranged order is possible in a two-dimensional flock, in contrast to equilibrium systems with continuous symmetries.

Note that all of the troublesome nonlinearities that make it impossible to determine exact exponents in $d = 2$ involve the fluctuation $\delta \rho$ of the density $\rho$. Therefore, if these fluctuations could somehow be “frozen out”, it would be possible to determine exact exponents in $d = 2$.

There are a number of types of flocks in which precisely such a freezing out of density fluctuations occurs. Two classes of such systems, namely, flocks with birth and death [12] and incompressible systems[37] - have been treated elsewhere. In both of these systems, exact scaling exponents can be found in $d = 2$.

By simply keeping track of the rescaling done in the dynamical RG, one can derive
scaling laws for the velocity correlations. For example, the correlations of the perpendicular components \( \delta v_\perp \) of the velocity (which can be measured in both simulations and experiments, the latter by image analysis) are given by

\[
\langle \delta v_\perp(0,0) \cdot \delta v_\perp(r,t) \rangle \sim \begin{cases} 
  r^{2\chi}, & |x - \gamma t| \ll r_\perp^\zeta, |t| \ll r_\perp^z, \\
  |x - \gamma t|^{2\chi}, & |x - \gamma t| \gg r_\perp^\zeta, |x - \gamma t| \gg |t|^{1/2}, \\
  |t|^{2\chi}, & |t| \gg r_\perp^z, |t| \gg |x - \gamma t|^2
\end{cases}
\]

(4.74)

where I remind the reader that, in \( d = 2 \), the conjecture described above leads to the exact values of the scaling exponents

\[
\zeta = \frac{3}{5}, \quad z = \frac{6}{5}, \quad \chi = -\frac{1}{5},
\]

(4.75)

These scaling predictions agree extremely well with numerical simulations [18–20].

V. 20-20 HINDSIGHT HANDWAVING ARGUMENT

In this section, I will rederive the scaling results we found above by an extension of the “blob” derivation of the Mermin-Wagner theorem given in section (II).

Consider the group of birds that at time \( t \) whose velocities will be well-correlated with what some reference bird was doing at \( t = 0 \). This group will be a “blob” whose center moves along the direction of mean motion of the flock at a speed \( \gamma \).

I’ll start by proving by contradiction that, for spatial dimensions \( d \leq 4 \), the motion of the flockers implies that the width of this blob can not scale like the width of the blob of pointers in section II. Those of you who’ve been paying attention will note that the critical dimension of 4 is exactly what we found in the dynamical RG of the previous section.

If we do assume that this blob grows like the analogous blob for pointers—that is, diffusively—then it will be essentially isotropic, and have width:

\[
w(t) \propto t^{1/2}
\]

(5.1)

as illustrated in figure (5).

But unlike the pointers, this blob will now be moving. Indeed, its velocity perpendicular to the mean direction of flock motion will be \( \delta v_\perp \sim v_0 \theta \), where \( \theta \) is the deviation of the mean direction of the blob from that of the flock. This means that the blob will wander laterally relative to the rest of the flock.
FIG. 5: Evolution of the “blob” in the incorrect picture that it grows diffusively in all directions. The lateral wandering of the blob $\delta x$ actually eventually exceeds its diffusive width $w(t)$ for all spatial dimensions $d \leq 4$.

How far will it wander laterally in time $t$? Roughly

$$\delta x \sim v_0 \langle \sqrt{\theta^2} \rangle t \propto t^{3/2-d/4}. \tag{5.2}$$

Note that this wandering is purely lateral: fluctuations in the velocity along the mean direction of flock motion go like $v_0[1 - \cos(\theta)] \propto \theta^2$ for small $\theta$, and, so, are much smaller than the lateral velocity fluctuations $\delta v_\perp$.

If we now assume (and remember, we’re going to be showing that this assumption is actually self-contradictory for $d \leq 4$), that the fluctuations of $\theta$ of this blob of flockers scale just as those of the blob of pointers in section II, we have:

$$\sqrt{\langle \theta^2 \rangle} \propto r^{1-d/2} \propto t^{1/2-d/4}. \tag{5.3}$$
Using this in (5.2), I get

\[ \delta x \propto t^{3/2-d/4} \tag{5.4} \]

Comparing this to the width \( w(t) \) of the blob, I find

\[ \frac{\delta x}{w(t)} \propto \frac{t^{3/2-d/4}}{t^{1/2}} \propto t^{1-d/4} \tag{5.5} \]

Note that this ratio diverges as \( t \to \infty \) for \( d < 4 \). Therefore, \( d = 4 \) is another critical dimension, below which the assumption that information transport is dominated by diffusion breaks down. This is a "breakdown of linearized hydrodynamics": the linear theory, which ignores the convective \( \lambda_1 \mathbf{v} \cdot \nabla \mathbf{v} \) term - that is, the term that contains the physics of the lateral wandering of the "blob" just calculated- is incorrect: that non-linear term-i.e., the convective wandering of the blob- actually dominates over the linear diffusive process.

What happens for \( d < 4 \)? Well, now the blob must become anisotropic, since information is transmitted much more rapidly perpendicular to the mean direction of flock motion, while parallel to the mean direction of flock motion, it’s still diffusive, since the speed of the flockers does not fluctuate much (it isn’t a Goldstone mode), and it doesn’t vary much as the direction \( \theta \) fluctuates \( (v_\parallel \sim O(\theta^2), \text{versus } v_\perp \sim O(\theta)) \). So it looks like figure (6), where the length of the blob along the mean direction of flock motion still grows diffusively (i.e., like \( \sqrt{t} \)), while the spatial extent \( w(t) \) of the blob in all of the \( d-1 \) directions perpendicular to \( <\mathbf{v}> \) is controlled by the fluctuations of \( v_\perp \), which I will have to determine self-consistently. I’ll do so by the same sort of sloppy handwaving argument I used in section II to ”derive” the Mermin-Wagner theorem. That is, I’ll note that

\[ \text{# of errors/flocker} \propto t. \tag{5.6} \]

The number of flockers making errors is

\[ N(t) \propto \text{volume of blob} \propto [w(t)]^{d-1} \sqrt{t}. \tag{5.7} \]

Hence, the total number of errors made inside the blob after a time \( t \) is

\[ \text{total # of errors} \propto N(t)t. \tag{5.8} \]

This gives for the rms fluctuations of \( \theta \)

\[ \sqrt{<\theta^2>} \approx \frac{\sqrt{\text{total # of errors}}}{N(t)} \propto \frac{\sqrt{N(t)t}}{N(t)} \propto \frac{t}{N(t)} \propto t^{1/4} w^{(1-d)/2}. \tag{5.9} \]
FIG. 6: Correct picture of the evolving blob. It still grows diffusively in the direction of the mean flock velocity (vertical in this figure), but grows laterally at a rate dominated by motion of the flockers (i.e., convection), rather than diffusion. This convective width \( w(t) \) can be determined self-consistently via equation (5.10).

As for our earlier estimate of \( \delta x \), we can estimate the lateral width \( w(t) \) as

\[
w(t) \sim v_0 \sqrt{\langle \theta^2 \rangle} t \propto t^{5/4} w^{(1-d)/2}
\]

This is the promised self-consistent condition on \( w(t) \). It is easily solved to give

\[
w(t) \propto t^{\frac{5}{2(d+1)}}
\]

(5.11)

which can be inverted to give the dynamical exponent \( z \):

\[
t(w) \propto w^z,
\]

(5.12)

with

\[
z = \frac{2(d + 1)}{5}.
\]

(5.13)
Note that this is the same result I got from dynamical RG argument of section IV! I can get the anisotropy exponent by using the fact that the blob still grows diffusively in the parallel direction:

\[ w_\parallel \propto t(w)^{1/2} \propto w^{\zeta}, \tag{5.14} \]

with

\[ \zeta = z/2 = (d+1)/5, \tag{5.15} \]

which also agrees with the result of the RG analysis of the preceding section. (To obtain the second proportionality in (5.15), I've used the relation (5.14) to relate \( t(w) \) to \( w \).

Finally, using (5.12) in (5.9), I get

\[ \sqrt{\langle \theta^2 \rangle} \propto w^\chi \tag{5.16} \]

with

\[ \chi = z/4 + (1-d)/2 = (3-2d)/5 \tag{5.17} \]

which, once again, agrees with the RG result. It also predicts that \( \chi = -1/5 \) in \( d = 2 \); since this is \( <0 \), this implies that long-ranged orientational order is stable in two-dimensional flocks.

VI. ACKNOWLEDGMENTS

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[8] Of course, in a real crystalline ferrogmagnet, crystal symmetry breaking fields make the rotational symmetry of the spins discrete, rather than continuous, since there are only a discrete set of orientations for the spins preferred by the lattice. In flocks, there are no such symmetry breaking fields, so the rotational symmetry is continuous, as it is in the idealized 0(\(n\)) Heisenberg model of a ferromagnet. Everything I say hereafter about ferromagnetics implicitly refers to this fully rotationally invariant 0(\(n\)) model.

[9] The isotropic Heisenberg model of magnetism is invariant under uniform rotation of all the spins, without a corresponding rotation of the lattice on which they live. A flock, like an ordinary collection of interacting molecules, such as those which form liquid crystals, is invariant only under spatial rotations, which rotate both the position and the velocity vectors of the creatures of the flock.

[10] I will frequently use the term “boid” (a short form for “birdoid”), coined by C. Reynolds in the first reference in [1].

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[22] Strictly speaking, the medium through which the flock moves must be inhomogeneous (and dynamical) in order to exert a dissipative force and thus define a preferred frame. I assume here that the medium is statistically homogeneous, so that its only effect is to provide such a damping, and that a rigid, static displacement of the flock results in no restoring force. It is in this sense that I assume the medium to be translation-invariant.

[23] This lack of Galilean invariance is appropriate if only the flock is dynamical (i.e., if the background dissipative medium is static). The fully Galilean-invariant treatment of the coupled dynamics of a flock and a dynamical fluid medium through which it moves is discussed in [11] and [20].

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[30] One could more generally rescale $\delta \rho$ with a different rescaling exponent $\chi_\rho$ from the exponent $\chi$ used for $\vec{v}_\perp$. However, since fluctuations of $\delta \rho$ and $\vec{v}_\perp$ have the same scaling with distance
as time, it is most convenient to rescale them the same way. None of my conclusions would be changed if I did not make this arbitrary choice.

[31] Other choices of rescaling force one to find suitable “dimensionless couplings”; that is, ratios of the nonlinear parameters (e.g., $\lambda_0^1$) and suitable powers of the linear parameters (e.g., $\Delta, D_B$) that actually give the ratios of the renormalizations of the linear parameters by fluctuations to their linear values. These necessarily have the same RG eigenvalues as those we found here for $\lambda_0^1$, etc. alone, independent of the arbitrary choice of $z, \chi$, and $\zeta$.

[32] In order for both equations of motion ($\rho$’s and $\vec{v}_\perp$’s) to have this pseudo-Galilean invariance, it is actually also necessary that $w_1 = \lambda_0^1$. However, by a more complicated argument, one can show that the existence of a pseudo-Galilean invariance for one value of $w_1$ ensures the same relation between exponents for all values of $w_1$ as that obtained when $w_1 = \lambda^0$.

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[36] It might appear from this argument that it would be possible for the divergence to be suppressed by only a power of the logarithm of length scale. Such a thing could happen were we in precisely the critical dimension of the problem. However, here the critical dimension is $d = 4$, while the dimension under discussion here is $d = 2$. In every other problem of which I know (with the exception of problems involving quenched disorder, which are known to be a very special case), the anomalous behavior of correlation functions is algebraic in wavenumber $q$ once one goes below the critical dimension by a finite amount. This means here that the equal time averages $< |\vec{v}_\perp(\vec{q}, t)|^2 >$ must be smaller than $1/q^2$ by a power of $q$; hence, the integral of this over all $\vec{q}$, which gives the real space fluctuations $< |\vec{v}_\perp(\vec{r}, t)|^2 >$, must be finite. This leads unavoidably to the conclusion that long ranged order must be stable in $d = 2$.

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