Existence of the transfer matrix for a class of nonlocal potentials in two dimensions

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Abstract

Evanescent waves are waves that decay or grow exponentially in regions of the space void of interaction. In potential scattering defined by the Schrödinger equation, \((-\nabla^2 + v)\psi = k^2\psi\) for a local potential \(v\), they arise in dimensions greater than one and are present regardless of the details of \(v\). The approximation in which one ignores the contributions of the evanescent waves to the scattering process corresponds to replacing \(v\) with a certain energy-dependent nonlocal potential \(\hat{V}_k\). We present a dynamical formulation of the stationary scattering for \(\hat{V}_k\) in two dimensions, where the scattering data are related to the dynamics of a quantum system having a non-self-adjoint, unbounded, and nonstationary Hamiltonian operator. The evolution operator for this system determines a two-dimensional analog of the transfer matrix of stationary scattering in one dimension which contains the information about the scattering properties of the potential. Under rather general conditions on \(v\), we establish the strong convergence of the Dyson series expansion of the evolution operator and prove the existence of the transfer matrix for \(\hat{V}_k\) as a densely-defined operator acting in \(\mathbb{C}^2 \otimes L^2(-k,k)\).

1 Introduction

Recently we have proposed a formulation of stationary scattering in two and three dimensions that is based on a multi-dimensional generalization of the transfer matrix of potential scattering in one dimension [1]. The earlier attempts in this direction [2, 3, 4, 5] involved slicing the space along the scattering axis and discretizing the transverse degrees of freedom. This led to large numerical transfer matrices which allowed for a numerical treatment of the scattering problem. The approach pursued in [1] is in sharp contrast, for it identifies the transfer matrix with a fundamental mathematical construct that is given by the time-evolution operator for an effective quantum system. Because the Hamiltonian operator for this system is an unbounded non-self-adjoint operator, the developments reported in [1] are generally formal. In the present article we take a first step towards

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providing a mathematically rigorous basis for these developments. Specifically, we offer a comprehensive analysis of various operators entering the definition of the transfer matrix and give a proof of its existence for a class of energy-dependent nonlocal potentials in two dimensions. These potentials arise in an approximate scheme that involves neglecting the contribution of the evanescent waves to the scattering data [6].

Consider the scattering problem defined by the stationary Schrödinger equation,

\[ [-\partial_x^2 - \partial_y^2 + v(x, y)]\psi(x, y) = k^2 \psi(x, y), \quad (x, y) \in \mathbb{R}^2, \tag{1} \]

where \( v : \mathbb{R}^2 \to \mathbb{C} \) is a real or complex short-range potential [7], \( k \) is a wavenumber, and we have adopted units where \( \hbar^2/2m = 1 \). Performing a partial Fourier transformation of both sides of (1) with respect to \( y \), we find

\[ -\partial_x^2 \tilde{\psi}(x, p) + v(x, i\partial_p)\tilde{\psi}(x, p) = \varpi(p)^2 \tilde{\psi}(x, p), \quad x, p \in \mathbb{R}, \tag{2} \]

where \( \tilde{\psi}(x, p) \) is the Fourier transform of \( \psi(x, y) \) with respect to \( y \), i.e.,

\[ \tilde{\psi}(x, p) := \int_\infty^{-\infty} dp e^{-ipy}\psi(x, y), \tag{3} \]

and we have introduced

\[ v(x, i\partial_p)f(p) := \frac{1}{2\pi} \int_\infty^{-\infty} dq \tilde{v}(x, p - q)f(q), \tag{4} \]

\[ \varpi(p) := \begin{cases} \sqrt{k^2 - p^2} & \text{for } |p| < k, \\ i\sqrt{p^2 - k^2} & \text{for } |p| \geq k. \end{cases} \tag{5} \]

In view of (3), we can express \( \psi \) in the form \( \psi = \psi_{os} + \psi_{ev} \), where

\[ \psi_{os}(x, y) := \frac{1}{2\pi} \int_k^{k} dp e^{ipy} \tilde{\psi}(x, p), \tag{6} \]

\[ \psi_{ev}(x, y) := \frac{1}{2\pi} \left[ \int_k^{-k} dp e^{ipy} \tilde{\psi}(x, p) + \int_k^{\infty} dp e^{ipy} \tilde{\psi}(x, p) \right]. \tag{7} \]

Suppose that there is some \( \Omega \subseteq \mathbb{R}^2 \) such that \( v(x, y) = 0 \) for \((x, y) \in \Omega \). Then the general solution of (1) in \( \Omega \) takes the form,

\[ \psi(x, y) = \int_\infty^{-\infty} \frac{dp}{4\pi^2 \varpi(p)} \left[ \mathscr{A}(p)e^{i\varpi(p)x} + \mathscr{B}(p)e^{-i\varpi(p)x} \right] e^{ipy}, \tag{8} \]

where \( \mathscr{A}, \mathscr{B} : \mathbb{R} \to \mathbb{C} \) are some coefficient functions. This in turn implies that

\[ \psi_{os}(x, y) = \int_{-k}^{k} \frac{dp}{4\pi^2 \varpi(p)} \left[ A(p)e^{i\varpi(p)x} + B(p)e^{-i\varpi(p)x} \right] e^{ipy}, \tag{9} \]

\[ \psi_{ev}(x, y) = \int_{-\infty}^{-k} \frac{dp}{4\pi^2 \varpi(p)} \left[ \mathscr{A}(p)e^{-|\varpi(p)|x} + \mathscr{B}(p)e^{i\varpi(p)x} \right] e^{ipy} + \int_{k}^{\infty} \frac{dp}{4\pi^2 \varpi(p)} \left[ \mathscr{A}(p)e^{-|\varpi(p)|x} + \mathscr{B}(p)e^{i\varpi(p)x} \right] e^{ipy}, \tag{10} \]
where \((x, y) \in \Omega\) and

\[
A(p) := \begin{cases}
\mathcal{A}(p) & \text{for } |p| < k, \\
0 & \text{for } |p| \geq k,
\end{cases}
B(p) := \begin{cases}
\mathcal{B}(p) & \text{for } |p| < k, \\
0 & \text{for } |p| \geq k.
\end{cases}
\] (11)

According to (9) and (10), \(\psi_{os}\) is a superposition of the plane-wave solutions of (1) which are oscillating functions of \(x\), whereas \(\psi_{ev}\) is the superposition of exponentially growing or decaying functions of \(x\). We therefore call \(\psi_{os}\) and \(\psi_{ev}\) the oscillating and evanescent waves, respectively.

Let \(\mathcal{F}\) denote the vector space of functions (tempered distributions) of \(p\) that possess Fourier transform, and \(\mathcal{F}_k\) be the subspace of \(\mathcal{F}\) consisting of elements whose support lies in the interval \((-k, k)\). Then \(A, B \in \mathcal{F}_k\).

If \(v\) is a short-range potential \([7]\), the bounded solutions of (1) tend to the superposition of the plane-wave solutions as \(x \to \pm \infty\), i.e., there are \(A_{\pm}, B_{\pm} \in \mathcal{F}_k\) such that

\[
\psi(x, y) \to \int_{-k}^{k} \frac{dp}{4\pi^2} \mathcal{A}(p)e^{ipx} + B_{\pm}(p)e^{-ipx} \] 
for \(x \to \pm \infty\). (12)

Ref. [1] identifies the fundamental transfer matrix for the potential \(v\) with a \(2 \times 2\) matrix \(\widehat{M}\) with operator entries \(\widehat{M}_{ij}\) that satisfy

\[
\widehat{M} \begin{bmatrix} A_- \\ B_- \end{bmatrix} = \begin{bmatrix} A_+ \\ B_+ \end{bmatrix}. \] (13)

Because \(A_{\pm}, B_{\pm} \in \mathcal{F}_k\), \(\widehat{M}_{ij}\) and \(\widehat{M}\) are respectively linear operators acting in \(\mathcal{F}_k\) and

\[
\mathcal{F}_k^{2 \times 1} := \mathbb{C}^2 \otimes \mathcal{F}_k := \left\{ \begin{bmatrix} \phi_+ \\ \phi_- \end{bmatrix} \bigg| \phi_{\pm} \in \mathcal{F}_k \right\}.
\]

Eq. (13) coincides with the defining relation for the transfer matrix in one dimension except that in one dimension \(A_{\pm}\) and \(B_{\pm}\) are complex numbers and the transfer matrix is a numerical matrix \([8, 9]\). Another common feature of the transfer matrix (13) and its one-dimensional analog is that it stores the information about the scattering properties of the potential \([1]\). We provide a brief description of the relationship between the transfer matrix and the scattering amplitude of the potential in the appendix.

Next, suppose that the \(f(q)\) appearing on the right-hand side of (11) vanishes for all \(|q| \geq k\), and that \(v \neq 0\). Then it is not difficult to show that the left-hand side of (11) does not vanish for all \(|p| \geq k\). Making use of this observation in (2), we infer that \(v\) always couples to the evanescent waves, and \(\psi_{os}\) can never solve the Schrödinger equation (11) for \(v \neq 0\). It is also clear from (6) that the requirement, \(\psi = \psi_{os}\), implies \(\tilde{\psi}(x, p) = 0\) for \(|p| \geq 0\). Such a function satisfies an equation of the form (2), if we replace \(v(x, i\partial_x)\) with the operator \(\tilde{V}_k(x)\) acting in \(\mathcal{F}\) according to

\[
(\tilde{V}_k(x)f)(p) := \frac{\chi_k(p)}{2\pi} \int_{-k}^{k} dq \tilde{v}(x, p - q)f(q),
\] (14)

where

\[
\chi_k(p) := \begin{cases}
1 & \text{for } |p| < k, \\
0 & \text{for } |p| \geq k.
\end{cases}
\]

\(^1\)The coefficient functions \(A_{\pm}\) and \(B_{\pm}\) correspond to those denoted by \(\breve{A}_{\pm}\) and \(\breve{B}_{\pm}\) in Ref. [1].
This corresponds to replacing \( v \) in (1) with the energy-dependent nonlocal potential given by

\[
(\hat{V}_k \psi)(x, y) := \frac{1}{4\pi^2} \int_{-k}^{k} dp \int_{-k}^{k} dq \ e^{ipy} \tilde{v}(x, p - q) \tilde{\psi}(x, q).
\]  

(15)

This relation defines an operator acting in \( L^2(\mathbb{R}^2) \) which satisfies

\[
\hat{V}_k = \hat{\Pi}_k v(\hat{x}, \hat{y}) \hat{\Pi}_k.
\]

(16)

Here \( \hat{\Pi}_k \) is the projection operators given by,

\[
(\hat{\Pi}_k \psi)(x, y) := \frac{1}{2\pi} \int_{-k}^{k} dp e^{ipy} \tilde{\psi}(x, p),
\]

(17)

and \( \hat{x} \) and \( \hat{y} \) are respectively the multiplication (position) operators: \( (\hat{x} \psi)(x, y) := x\psi(x, y) \) and \( (\hat{y} \psi)(x, y) := y\psi(x, y) \).

The right-hand side of (12) is a superposition of oscillating waves. This suggests that although the solutions of the Schrödinger equation (1) for \( v \neq 0 \) are different from the ones for the nonlocal potential \( \hat{V}_k \), their asymptotic form need not differ appreciably. This means that the solution to the scattering problem for \( \hat{V}_k \) may provide a reliable approximation for the solution to the scattering problem defined by \( v \) at the energy \( k^2 \). Ref. [6] offers evidence for the exactness of this approximation for potentials \( v \) satisfying \( \tilde{v}(x, p) = 0 \) for \( p \leq 0 \) or \( p \geq 0 \). This provides our basic motivation for the study of the scattering properties of the nonlocal potentials \( \hat{V}_k \).

The transfer matrix \( \hat{M} \) for the nonlocal potential \( \hat{V}_k \) turns out to be given by the following formal Dyson series [6].

\[
\hat{I} + \sum_{n=1}^{\infty} (-i)^n \int_{-\infty}^{\infty} dx_n \int_{-\infty}^{x_n} dx_{n-1} \cdots \int_{-\infty}^{x_2} dx_1 \hat{H}(x_n) \hat{H}(x_{n-1}) \cdots \hat{H}(x_1),
\]

(18)

where \( \hat{I} \) is the identity operator acting in \( F_{k}^{2 \times 1} \), \( \hat{H}(x) \) is an effective Hamiltonian operator given by

\[
\hat{H}(x) := \frac{1}{2} e^{-i\hat{x} \sigma_3} \hat{V}(x) e^{i\hat{x} \sigma_3},
\]

(19)

\[
(\hat{x} f)(p) := x f(p), \quad (\hat{p} f)(p) := p f(p),
\]

(20)

\[
\mathcal{K} := \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \sigma_3 + i\sigma_2,
\]

(21)

\( f \in F_k \), and \( \sigma_j \) are the Pauli matrices;

\[
\sigma_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.
\]

(22)

The purpose of this article is to make the above crude description of the transfer matrix \( \hat{M} \), Hamiltonian operator \( \hat{H}(x) \), and the dynamics it generates into mathematically rigorous statements.

The outline of this article is as follows. In Sec. 2, we collect a few basic mathematical results about the analogs of the nonlocal potential \( \hat{V}_k \) in one dimension. In Sec. 3, we extend these to two dimensions by confining our attention to a particular class of potentials \( v \) for which our analysis
apply. In Sec. 4 we outline the dynamical formulation of the scattering problem for \( \hat{V}_k \). Here we identify \( \hat{H}(x) \) with a densely-defined linear operator acting in the Hilbert space,

\[
\mathcal{H} := \mathbb{C}^2 \otimes L^2(-k, k) = \left\{ \begin{bmatrix} \xi_+ \\
\xi_- \end{bmatrix} \mid \xi_+ \in L^2(-k, k) \right\},
\]

establish the \( x \)-independence of the domain of \( \hat{N}(x) \), and show that the range of \( \hat{N}(x) \) lies in its domain. In Sec. 5, we establish the strong convergence of the Dyson series,

\[
\hat{1} + \sum_{n=1}^{\infty} (-i)^n \int_{x_0}^{x} dx_n \int_{x_0}^{x_n} dx_{n-1} \cdots \int_{x_0}^{x_2} dx_1 \hat{N}(x_n) \hat{N}(x_{n-1}) \cdots \hat{N}(x_1),
\]

for \( x_0, x \in \mathbb{R} \) such that \( x_0 \leq x \). This gives a densely-defined linear operator \( \hat{U}(x, x_0) \) that serves as the evolution operator for the Hamiltonian operator \( \hat{H}(x) \). Finally, we prove the existence of the strong limit of \( \hat{U}(x, x_0) \) as \( x_0 \to -\infty \) and \( x \to +\infty \), which we identify with the transfer matrix of \( \hat{V}_k \).

## 2 Analogs of nonlocal potentials \( \hat{V}_k \) in one dimension

Consider an integrable potential, \( v : \mathbb{R} \to \mathbb{C} \), in one dimension, i.e., \( v \in L^1(\mathbb{R}) \). Let \( \hat{N}_k \) be the projection operator defined on \( L^2(\mathbb{R}) \) by (17), and \( \hat{V}_k, \hat{V}_k : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) be linear operators given by

\[
\hat{V}_k := \hat{N}_k v(\hat{y}) \hat{N}_k, \quad \hat{V}_k := \mathcal{F} \hat{V}_k \mathcal{F}^{-1},
\]

where \( \hat{y} : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) is the multiplication (position) operator: \( \hat{y}(\phi)(y) := y \phi(y) \), and \( \mathcal{F}, \mathcal{F}^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) respectively label the Fourier transformation and its inverse;

\[
(\mathcal{F} \phi)(p) := \hat{\phi}(p) := \int_{-\infty}^{\infty} dy \, e^{-ipy} \phi(y), \quad (\mathcal{F}^{-1} \hat{\phi})(y) := \frac{1}{2\pi} \int_{-\infty}^{\infty} dp \, e^{ipy} \hat{\phi}(p).
\]

Then, according to (17) and (24), for every function \( \phi \) in the domain of \( \hat{V}_k \) and all \( y \in \mathbb{R} \),

\[
(\hat{V}_k \phi)(y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dq \, e^{ipq} \tilde{v}(p-q) \hat{\phi}(q).
\]

In view of (24) - (26),

\[
(\hat{V}_k \hat{\phi})(p) = (\mathcal{F} \hat{V}_k \mathcal{F}^{-1} \hat{\phi})(p) = (\mathcal{F} \hat{V}_k \phi)(p) = \frac{\chi_k(p)}{2\pi} \int_{-\infty}^{\infty} dq \, \tilde{v}(p-q) \hat{\phi}(q).
\]

Because \( v \in L^1(\mathbb{R}) \), \( \tilde{v} \) is uniformly continuous [10]. This in particular implies that \( \tilde{v} \) is bounded on \([-2k, 2k]\), i.e., there is \( \mu \in \mathbb{R}^+ \) such that for all \( p \in [-2k, 2k] \), \( |\tilde{v}(p)| \leq \mu \). This together with the Cauchy-Schwarz inequality show that, for all \( \phi \in L^2(\mathbb{R}) \) and \( p \in [-k, k] \),

\[
\left| \int_{-k}^{k} dq \, \tilde{v}(p-q) \hat{\phi}(q) \right|^2 \leq \left( \int_{-k}^{k} dq \, |\tilde{v}(p-q)|^2 \right) \left( \int_{-k}^{k} dq \, |\hat{\phi}(q)|^2 \right) \leq 2k \mu^2 \|
\hat{\phi} \|^2.
\]
(24), (27), and (28) imply that \( \hat{V}_k \) and consequently \( \hat{\mathcal{F}}_k \) are defined everywhere in \( L^2(\mathbb{R}) \);

\[ \text{Dom}(\hat{\mathcal{F}}_k) = \text{Dom}(\hat{V}_k) = L^2(\mathbb{R}). \]

**Theorem 1:** \( \hat{\mathcal{F}}_k \) and \( \hat{V}_k \) are normal bounded operators acting in \( L^2(\mathbb{R}) \).

**Proof:** According to (24), (27), and the fact that \( \mathcal{F} \) is a unitary operator,

\[ \| \hat{\mathcal{F}}_k \phi \|^2 = \| \hat{V}_k \phi \|^2 = \frac{1}{4\pi^2} \int_{-k}^k dp \int_{-k}^k dq \hat{v}(p-q)\overline{\phi(q)}^2 \leq \left( \frac{k\mu}{\pi} \right)^2 \| \phi \|^2 = \left( \frac{k\mu}{\pi} \right)^2 \| \phi \|^2. \]  

(29)

This shows that \( \hat{\mathcal{F}}_k \) and \( \hat{V}_k \) are bounded. Next, we use (26) to deduce

\[ \hat{\mathcal{F}}_k^\dagger = \hat{\Pi}_k v(\hat{y})^\dagger \hat{\Pi}_k = \hat{\Pi}_k v^*(\hat{y}) \hat{\Pi}_k, \]  

(30)

where \( v^* : \mathbb{R} \to \mathbb{C} \) is the complex-conjugate of \( v \), i.e., for all \( y \in \mathbb{R} \), \( v^*(y) := v(y)^* \). Because \( v(\hat{y}) \) and \( v^*(\hat{y}) \) commute, so do \( \hat{\mathcal{F}}_k \) and \( \hat{V}_k^\dagger \). Hence \( \hat{\mathcal{F}}_k \) is a normal operator. The same holds for \( \hat{V}_k \) by virtue of (24) and the fact that \( \mathcal{F} \) is a unitary operator. \( \square \)

Consider the embedding \( \iota : L^2(-k, k) \to L^2(\mathbb{R}) \) given by

\[ (\iota \xi)(p) := \begin{cases} \xi(p) & \text{for } p \in (-k, k), \\ 0 & \text{for } p \notin (-k, k), \end{cases} \]  

(31)

for all \( \xi \in L^2(-k, k) \) and \( p \in \mathbb{R} \). Clearly, \( \iota \) is an isometry, and

\[ \text{Dom}(\iota^{-1}) = \text{Ran}(\iota) = \{ \psi \in L^2(\mathbb{R}) \mid \psi(p) = 0 \text{ for } |p| \geq k \}. \]

In view of (27), \( \hat{V}_k \iota \) maps elements of \( L^2(-k, k) \) to \( \text{Ran}(\iota) \). Therefore,

\[ \hat{v}_k := \iota^{-1} \hat{V}_k \iota, \]  

(32)

defines a linear operator acting in \( L^2(-k, k) \). Because \( \iota \) is an isometry and \( \hat{V}_k \) is a bounded operator with domain \( L^2(\mathbb{R}) \), \( \hat{v}_k \) is a bounded operator with domain \( L^2(-k, k) \). We can use (27), (31), and (32) to show that for all \( \xi \in L^2(-k, k) \) and all \( p \in (-k, k) \),

\[ (\hat{v}_k \xi)(p) = \frac{1}{2\pi} \int_{-k}^k dq \hat{v}(p-q)\overline{\xi(q)}. \]  

(33)

Because \( \hat{v} \) is continuous, \( \hat{v}_k : L^2(-k, k) \to L^2(-k, k) \) is a Hilbert-Schmidt operator [11]. In particular, it is compact. Because \( \hat{V}_k \) is a normal bounded operator, (32) shows that the same holds for \( \hat{v}_k \). This proves the following theorem.

**Theorem 2:** \( \hat{v}_k \) is a normal Hilbert-Schmidt operator acting in \( L^2(-k, k) \).

An immediate consequence of this theorem is the existence of an orthonormal basis of \( L^2(-k, k) \) consisting of the eigenvectors \( \psi_n \) of \( \hat{v}_k \) with possibly repeated eigenvalues \( \nu_n \) so that [12, 13]

\[ \hat{v}_k = \sum_{n=0}^{\infty} \nu_n \langle \psi_n | \cdot | \psi_n \rangle \psi_n. \]  

(34)

Furthermore, because \( \hat{v}_k \) is a normal compact operator, there is \( n_* \in \mathbb{Z}^+ \) such that

\[ \| \hat{v}_k \|_0 := |\nu_{n_*}|, \]  

(35)

where \( \| \cdot \|_0 \) stands for the operator norm [12]:

\[ \| \hat{v}_k \|_0 := \sup_{\xi \neq 0} \frac{\| \hat{v}_k \xi \|}{\| \xi \|}. \]  

(36)
3 Generalization to two dimensions

**Definition:** Let $s, k \in \mathbb{R}^+$. Then a function $v : \mathbb{R}^2 \to \mathbb{C}$ is said to belong to the class $\mathcal{C}_{s,k}$ if it fulfills the following conditions.

C1: For all $x \in \mathbb{R}$, $v(x, \cdot) : \mathbb{R} \to \mathbb{C}$ is integrable, i.e., \( \int_{-\infty}^{\infty} dy \left| v(x, y) \right| < \infty \), so that its Fourier transform $\hat{v}(x, \cdot) : \mathbb{R} \to \mathbb{C}$ exists and is uniformly continuous.

C2: The function $\hat{v}_k : \mathbb{R} \to L^\infty(-2k, 2k)$ defined by
\[
(\hat{v}_k(x))(p) := \hat{v}(x, p), \quad x \in \mathbb{R}, \; p \in (-2k, 2k),
\]
is piecewise continuous, i.e., for every closed interval $I$ in $\mathbb{R}$, there are finitely many open subintervals, $I_1, I_2, \cdots, I_j$, such that $I$ is the closure of $\bigcup_{i=1}^j I_i$ and $\hat{v}_k$ is continuous on $I_i$ for all $i \in \{1, 2, \cdots, j\}$. The latter requirement means that for all $x, x' \in I_i$ and all $\epsilon \in \mathbb{R}^+$, there is some $\delta \in \mathbb{R}^+$ such that
\[
|x - x'| < \delta \Rightarrow \sup_{p \in (-2k, 2k)} |\hat{v}(x, p) - \hat{v}(x', p)| < \epsilon.
\]

C3: $v$ is bounded, and there are $\alpha, \beta, \sigma \in \mathbb{R}^+$ such that $\sigma > s$ and for all $(x, y) \in \mathbb{R}^2$,
\[
|v(x, y)| \leq \frac{\beta}{(1 + |x|)^\sigma} \text{ for } |x| \geq \alpha.
\]

Condition C1 allows us to apply the constructions of Sec. 2 by letting $v(x, \cdot)$ play the role of the potential $v$ of Sec. 2. In particular, we can use $v(x, \cdot)$ to introduce an associated one-parameter family of normal Hilbert-Schmidt operators acting in $L^2(-k, k)$. We view this as an operator function $\tilde{v}_k$ that maps $\mathbb{R}$ to the space of normal Hilbert-Schmidt operators acting in $L^2(-k, k)$; for each $x \in \mathbb{R}$, $\xi \in L^2(-k, k)$, and $p \in (-k, k)$,
\[
(\tilde{v}_k(x)\xi)(p) := \frac{1}{2\pi} \int_{-k}^k dq \; \hat{v}(x, p - q)\xi(q).
\]

As we show below, C2 implies that $\| \tilde{v}_k(\cdot) \|_0$ is a piecewise continuous function, while C3 puts an upper bound on $\| \tilde{v}_k(x) \|_0$ for $|x| \geq \alpha$.

**Lemma 1:** Let $k \in \mathbb{R}^+$ and $v : \mathbb{R}^2 \to \mathbb{C}$ be a function that satisfies conditions C1 and C2. Then $\| \tilde{v}_k(\cdot) \|_0 : \mathbb{R} \to [0, \infty)$ is a piecewise continuous function.

**Proof:** Let $I$ be any closed interval in $\mathbb{R}$. According to C2 there are open subintervals, $I_1, I_2, \cdots, I_j$, such that $I$ is the closure of $\bigcup_{i=1}^j I_i$ and for all $i \in \{1, 2, \cdots, j\}$, $\hat{v}_k$ is continuous on $I_i$. Let $i \in \{1, 2, \cdots, j\}$ and $\epsilon \in \mathbb{R}^+$ be arbitrary, and $\epsilon' := \pi \epsilon/k$. Because $\hat{v}_k$ is continuous on $I_i$ and $\epsilon' \in \mathbb{R}^+$, for all $x, x' \in I_i$, there is some $\delta \in \mathbb{R}^+$ such
\[
|x - x'| < \delta \Rightarrow \sup_{p \in (-2k, 2k)} |\hat{v}(x, p) - \hat{v}(x', p)| < \epsilon'.
\]
Let $\phi \in L^2(-k, k) \setminus \{0\}$ be arbitrary, and suppose that $x, x' \in I_i$ and $|x - x'| < \delta$. Then, (40) and (41) imply

\[
\left( [\hat{v}_k(x) - \hat{v}_k(x')] \phi \right)(p) = \frac{1}{2\pi} \left| \int_{-k}^{k} dq [\hat{v}(x, p - q) - \hat{v}(x', p - q)] \phi(q) \right| \\
\leq \frac{1}{2\pi} \int_{-k}^{k} dq |\hat{v}(x, p - q) - \hat{v}(x', p - q)| |\phi(q)| \\
\leq \frac{1}{2\pi} \sup_{p \in (-2k, 2k)} |\hat{v}(x, p) - \hat{v}(x', p)| \int_{-k}^{k} dq |\phi(q)| \\
< \frac{\epsilon}{\pi} \sqrt{\frac{k}{2}} \| \phi \| = \frac{\epsilon \| \phi \|}{\sqrt{2k}}, \quad (42)
\]

where we have made use of the fact that $\int_{-k}^{k} dq |\phi(q)| \leq \sqrt{\int_{-k}^{k} dq \int_{-k}^{k} dq |\phi(q)|^2 = \sqrt{2k} \| \phi \|}$. The following is a simple consequence of (42).

\[
\frac{\| [\hat{v}_k(x) - \hat{v}_k(x')] \phi \|}{\| \phi \|} = \frac{1}{2\pi \| \phi \|} \sqrt{\int_{-k}^{k} dp \left( [\hat{v}_k(x) - \hat{v}_k(x')] \phi \right)(p)^2} < \epsilon.
\]

Because this relation holds for all $\phi \in L^2(-k, k) \setminus \{0\}$, it implies $\| \hat{v}_k(x) - \hat{v}_k(x') \|_0 < \epsilon$. This completes the proof that for all $x, x' \in I_i$, $|x - x'| < \delta$ implies $\| \hat{v}_k(x) - \hat{v}_k(x') \|_0 < \epsilon$. Hence, $\| \hat{v}_k(\cdot) \|_0$ is continuous on $I_i$. This together with the fact that $I$ is an arbitrary closed subset of $\mathbb{R}$ and the closure of $\bigcup_{i=1}^{2} I_i$ coincides with $I$ show that $\| \hat{v}_k(\cdot) \|_0$ is piecewise continuous. $\square$

**Lemma 2:** Let $k, s \in \mathbb{R}^+$ and $v \in \mathcal{C}_{k, k}$, so that (39) holds for some $\alpha, \beta, \sigma \in \mathbb{R}^+$ with $\sigma > s$. Then,

\[
\| \hat{v}_k(x) \|_0 \leq \frac{2\pi \beta}{(1 + |x|)^\sigma} \text{ for } |x| \geq \alpha. \quad (43)
\]

**Proof:** Because for each $x \in \mathbb{R}$, $\hat{v}_k(x)$ is a compact normal operator, it has an eigenvalue $\nu_*(x)$ such that $\| \hat{v}_k(x) \|_0 = |\nu_*(x)|$, [12]. Therefore, to prove (43), it suffices to show that

\[
|\nu_*(x)| \leq \frac{2\pi \beta}{(1 + |x|)^\sigma} \text{ for } |x| \geq \alpha. \quad (44)
\]

Let $\xi_*(x) \in L^2(-k, k)$ be an eigenvector of $\hat{v}_k(x)$ with eigenvalue $\nu_*(x)$, and introduce

\[
\phi_*(x) := \mathcal{L} \xi_*(x), \quad \tilde{\phi}_* := \mathcal{F}^{-1} \phi_*(x), \quad \tilde{\phi}_*(x, y) := (\tilde{\phi}_*(x))(y),
\]

\[
\xi_*(x, p) := (\xi_*(x))(p), \quad \phi_*(x, q) := (\phi_*(x))(q),
\]

where $\mathcal{L}$ is the embedding of $L^2(-k, k)$ into $L^2(\mathbb{R})$ given by (31), $p \in (-k, k)$, $q \in \mathbb{R}$, and $y \in \mathbb{R}$. Then,

\[
\tilde{\phi}_*(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dp e^{ipy} \phi_*(x, p) = \frac{1}{2\pi} \int_{-k}^{k} dp e^{ipy} \xi_*(x, p), \quad (45)
\]

\[
\| \xi_*(x) \| = \| \phi_*(x) \| = \| \tilde{\phi}_*(x) \|, \quad (46)
\]

\[8\]
Because \( \hat{\nu}_k(x) \xi_\ast(x) \) see that them to everywhere-defined bounded self-adjoint operators acting in In particular, we take the set \( C \). In view of (40) this is equivalent to the following differential equation in
\[
\sqrt{\hat{\nu}_k(x)} \xi_\ast(x) = -i \int_{-k}^{k} dq \left( \frac{\xi_\ast(x)}{p} \right) e^{-ip(y-q)} v(x,y) \xi_\ast(x) = \int_{-\infty}^{\infty} dy e^{-ipy} v(x,y) \tilde{\phi}_\ast(x,y) \].
\]

Multiplying both sides of the latter equation by \( \xi_\ast(x,p) \ast \), integrating over \( p \), and using (39) and (41), we have
\[
|\nu_\ast(x)| \ |\xi_\ast(x)|^2 = \left| \int_{-k}^{k} dp \left( \xi_\ast(x,p) \right) \int_{-\infty}^{\infty} dy e^{-ipy} v(x,y) \tilde{\phi}_\ast(x,y) \right|
\]
\[
= 2\pi \left| \int_{-\infty}^{\infty} dy v(x,y) |\tilde{\phi}_\ast(x,y)|^2 \right| \leq 2\pi \int_{-\infty}^{\infty} dy |v(x,y)| |\tilde{\phi}_\ast(x,y)|^2
\]
\[
\leq \frac{2\pi \beta |\tilde{\phi}_\ast(x)|^2}{(1 + |x|)^\alpha} \quad \text{for} \quad |x| \geq \alpha. \tag{47}
\]
Because \( |\tilde{\phi}_\ast(x)| = |\xi_\ast(x)| \neq 0 \), this implies (44). \( \square \)

4 Dynamical formulation of stationary scattering for \( \hat{\nu}_k \)

The scattering problem for the nonlocal potential \( \hat{\nu}_k \) involves bounded solutions of the Schrödinger equation,
\[
(-\partial_x^2 - \partial_y^2) \psi(x,y) + (\hat{\nu}_k \psi)(x,y) = k^2 \psi(x,y), \quad (x,y) \in \mathbb{R}^2. \tag{48}
\]

We can express this in the form
\[
-\partial_x^2 \tilde{\psi}(x,p) + \frac{\chi_\ast(p)}{2\pi} \int_{-k}^{k} dq \tilde{\nu}(x,p-q) \tilde{\psi}(x,q) = \varpi(p)^2 \tilde{\psi}(x,p), \quad x \in \mathbb{R}, \ p \in (-k,k). \tag{49}
\]

In view of (40) this is equivalent to the following differential equation in \( L^2(-k,k) \).
\[
[-\partial_x^2 + \hat{\nu}_k(x)] \tilde{\psi}(x) = \hat{\varpi}^2 \tilde{\psi}(x), \quad x \in \mathbb{R}, \tag{50}
\]

where \( \tilde{\psi}(x) \in L^2(-k,k) \) is given by \( (\tilde{\psi}(x))(p) := \tilde{\psi}(x,p), \ p \in (-k,k), \hat{\varpi} := \sqrt{\hat{\nu}_k(x)}, \varpi(p) := \sqrt{k^2 - p^2} \), and \( \hat{\varpi} : L^2(-k,k) \to L^2(-k,k) \) is the multiplication operator defined by \( (\hat{\varpi} \xi)(p) := p \xi(p) \). In particular,
\[
(\hat{\varpi} \xi)(p) = \sqrt{k^2 - p^2} \xi(p), \quad p \in (-k,k).
\]

We take the set \( C^0(-k,k) \) of continuous functions, \( \phi : (-k,k) \to \mathbb{C} \), as the common domain of \( \hat{\varpi} \) and \( \hat{\varpi}^{-1} \), and can easily verify that they are bounded self-adjoint operators. Therefore, we can extend them to everywhere-defined bounded self-adjoint operators acting in \( L^2(-k,k) \). It is also easy to see that \( \hat{\varpi} \) is one-to-one. Its inverse, \( \hat{\varpi}^{-1} \), is an unbounded self-adjoint operator defined on the range of \( \hat{\varpi} \);
\[
\text{Dom}(\hat{\varpi}^{-1}) = \mathcal{R} := \text{Ran}(\hat{\varpi}) = \left\{ \hat{\varpi} \xi \mid \xi \in L^2(-k,k) \right\}.
\]

Because \( \hat{\varpi} \) is a one-to-one self-adjoint operator, \( \mathcal{R} \) is a dense subset of \( L^2(-k,k) \).
As we noted earlier, the condition that $v$ is a short-range potential implies that the bounded solutions of (48) satisfy (12). We can express this relation in the form:

$$
\tilde{\psi}(x) \to \frac{1}{2\pi} \hat{\omega}^{-1}(e^{ix\hat{\omega}} A_\pm + e^{-ix\hat{\omega}} B_\pm) \quad \text{for} \quad x \to \pm \infty.
$$

(51)

Clearly,

$$
\hat{\omega}^{-1}(e^{ix\hat{\omega}} A_\pm + e^{-ix\hat{\omega}} B_\pm) = \hat{\omega}^{-1}(A_\pm + B_\pm) + \hat{\omega}^{-1}(e^{ix\hat{\omega}} - \hat{1})A_\pm + \hat{\omega}^{-1}(e^{-ix\hat{\omega}} - \hat{1})B_\pm,
$$

(52)

where $\hat{1}$ stands for the identity operator acting in $L^2(-k, k)$. Because $\hat{\omega}^{-1}(e^{\pm ix\hat{\omega}} - \hat{1})$ are everywhere-defined bounded operators, (51) and (52) suggest that

$$
A_\pm + B_\pm \in \mathcal{R}.
$$

(53)

Next, let $\Psi : \mathbb{R} \to \mathcal{H} := \mathbb{C}^2 \otimes L^2(-k, k)$ and $\tilde{\mathcal{H}}(x) : \mathcal{H} \to \mathcal{H}$ be defined by

$$
\Psi(x) := \pi \left[ e^{-ix\hat{\omega}} \left[ \hat{\omega} \tilde{\psi}(x) - i \partial_x \tilde{\psi}(x) \right] + e^{ix\hat{\omega}} \left[ \hat{\omega} \tilde{\psi}(x) + i \partial_x \tilde{\psi}(x) \right] \right],
$$

(54)

$$
\tilde{\mathcal{H}}(x) := \frac{1}{2} e^{-ix\sigma_3 \hat{\omega}} \tilde{\gamma}_k(x) \hat{\omega}^{-1} \mathcal{K} e^{ix\sigma_3 \hat{\omega}},
$$

(55)

where $\tilde{\psi} : \mathbb{R} \to L^2(-k, k)$ is a given strongly differentiable function with derivative $\partial_x \tilde{\psi}$. It is easy to see that, according to (21) and (55), $\tilde{\mathcal{H}}(x)$ is an unbounded operator with (maximal) domain

$$
\mathcal{D} := \left\{ \left[ \begin{array}{c} \phi_+ \\ \phi_- \end{array} \right] \in \mathcal{H} \mid e^{ix\hat{\omega}} \phi_+ + e^{-ix\hat{\omega}} \phi_- \in \mathcal{R} \right\}.
$$

(56)

It is also clear from (54) and (55) that

$$
\Psi(x) \in \mathcal{D}.
$$

(57)

Eq. (56) gives the impression that $\mathcal{D}$ depends on $x$. This is however not true. Because

$$
e^{ix\hat{\omega}} \phi_+ + e^{-ix\hat{\omega}} \phi_- = \phi_+ + \phi_- + (e^{ix\hat{\omega}} - \hat{1})\phi_+ + (e^{-ix\hat{\omega}} - \hat{1})\phi_-$$

and $(e^{\pm ix\hat{\omega}} - \hat{1})\phi_\pm \in \mathcal{R}$, Eq. (56) implies

$$
\mathcal{D} = \left\{ \left[ \begin{array}{c} \phi_+ \\ \phi_- \end{array} \right] \in \mathcal{H} \mid \phi_+ + \phi_- \in \mathcal{R} \right\} = \mathcal{R}_- \oplus \mathcal{N}_6,
$$

(58)

where $\oplus$ marks a direct sum, and

$$
\mathcal{R}_- := \left\{ \left[ \begin{array}{c} 0 \\ \rho \end{array} \right] \mid \rho \in \mathcal{R} \right\} = \left\{ \left[ \begin{array}{c} 0 \\ 1 \end{array} \right] \hat{\omega} \phi \mid \phi \in L^2(-k, k) \right\},
$$

(59)

$$
\mathcal{N}_6 := \left\{ e^{-ix\sigma_3} \left[ \begin{array}{c} 1 \\ -1 \end{array} \right] \xi \mid \xi \in L^2(-k, k) \right\}.
$$

(60)

Observe that according to (53) and (58),

$$
\left[ \begin{array}{c} A_\pm \\ B_\pm \end{array} \right] \in \mathcal{D}.
$$

(61)
Another important property of \( D \) is that it is a dense subset of \( \mathcal{H} \). To see this, we introduce
\[
\mathcal{R}^{2 \times 1} := \left\{ \begin{bmatrix} \xi_+ \\ \xi_- \end{bmatrix} \mid \xi_\pm \in \mathcal{R} \right\},
\]
and use (58) to infer
\[
\mathcal{R}^{2 \times 1} \subseteq \mathcal{D}.
\]
Because \( \mathcal{R} \) is dense in \( L^2(-k, k) \), \( \mathcal{R}^{2 \times 1} \) is dense in \( \mathcal{H} \). In view of (63), this shows that \( \mathcal{D} \) is dense in \( \mathcal{H} \), i.e., \( \hat{\mathcal{H}}(x) \) is densely defined.

Next, we obtain an alternative direct-sum decomposition of \( \mathcal{D} \). According to (58), for every \( \begin{bmatrix} \phi_+ \\ \phi_- \end{bmatrix} \in \mathcal{D} \), there is some \( \zeta \in L^2(-k, k) \) such that \( e^{i x \hat{\sigma}_3} \phi_+ + e^{-i x \hat{\sigma}_3} \phi_- = \hat{\sigma} \zeta \). Solving this equation for \( \phi_- \) and introducing \( \xi := e^{i x \hat{\sigma}_3} \phi_+ \) and \( \zeta := e^{i x \hat{\sigma}_3} \), we have \( \phi_- = \hat{\sigma} \zeta - e^{i x \hat{\sigma}_3} \phi_- \). This calculation shows that
\[
\begin{bmatrix} \phi_+ \\ \phi_- \end{bmatrix} = \begin{bmatrix} 0 \\ \hat{\sigma} \zeta \end{bmatrix} + e^{-i x \hat{\sigma}_3} \begin{bmatrix} \xi \\ -\zeta \end{bmatrix} \in \mathcal{R} - \mathcal{N}_x.
\]
Therefore, \( \mathcal{D} \subseteq \mathcal{R} - \mathcal{N}_x \). It is also easy to see that \( \mathcal{R} \) and \( \mathcal{N}_x \) are subspaces of \( \mathcal{D} \). This together with \( \mathcal{R} \cap \mathcal{N}_x = \{0\} \) imply
\[
\mathcal{D} = \mathcal{R} - \mathcal{N}_x.
\]
An appealing property of this decomposition is that \( \mathcal{N}_x \) lies in the kernel of \( \hat{\mathcal{H}}(x) \). Therefore, the range of \( \hat{\mathcal{H}}(x) \) coincides with the image of \( \mathcal{R} \) under \( \hat{\mathcal{H}}(x) \). It has the following explicit form.
\[
\text{Ran}[\hat{\mathcal{H}}(x)] = \left\{ e^{-i x \hat{\sigma}_3} \begin{bmatrix} \hat{\nu}(x) \zeta \\ -\hat{\nu}(x)\zeta \end{bmatrix} \mid \zeta \in L^2(-k, k) \right\}.
\]
In view of (60) and (66),
\[
\text{Ran}[\hat{\mathcal{H}}(x)] \subseteq \mathcal{N}_x \subseteq \text{Ker}[\hat{\mathcal{H}}(x)] \subseteq \mathcal{D},
\]
which is consistent with the fact that \( \hat{\mathcal{H}}(x)^2 = \hat{0} \). The latter follows from (55) and the fact that \( \mathcal{K}^2 = \hat{0} \).

The main motivation for the introduction of \( \Psi(x) \) and \( \hat{\mathcal{H}}(x) \) is that whenever (50) and (51) holds, they satisfy
\[
i \partial_x \Psi(x) = \hat{\mathcal{H}}(x)\Psi(x),
\]
\[
\Psi(x) \rightarrow \begin{bmatrix} A_\pm \\ B_\pm \end{bmatrix} \text{ for } x \rightarrow \pm \infty.
\]
In view of (54), these relations suggest that we identify the fundamental transfer matrix \( \hat{\mathcal{M}} \) for the nonlocal potential \( \bar{v}_k \) with
\[
\text{s-lim}_{x_\pm \rightarrow \pm \infty} \hat{\mathcal{U}}(x_+, x_-),
\]
where “s-lim” stands for the “strong limit,” and \( \hat{\mathcal{U}}(x, x_0) \) is the time-evolution operator for the effective Hamiltonian (55), i.e., the linear operator \( \hat{\mathcal{U}}(x, x_0) : \mathcal{H} \rightarrow \mathcal{H} \) satisfying
\[
i \partial_x \hat{\mathcal{U}}(x, x_0) = \hat{\mathcal{H}}(x)\hat{\mathcal{U}}(x, x_0), \quad \hat{\mathcal{U}}(x_0, x_0) = \hat{1},
\]
where \( \hat{1} \) is the identity operator acting in \( \mathcal{H} \). Notice that (57) and (67) are necessary conditions for making sense of (68) as an equation defined in \( \mathcal{D} \). Because \( \hat{\mathcal{H}}(x) \) is an unbounded operator, we still need to establish the existence of \( \hat{\mathcal{U}}(x, x_0) \) and (70). In the next section we do this by requiring that \( v \) belongs to \( C_{3,k} \).
5 Existence of transfer matrix for nonlocal potentials $\hat{\mathcal{V}}_k$

To establish the existence of the evolution operator $\hat{u}(x, x_0)$, we express it as a Dyson series and prove its strong convergence on $\mathcal{D}$. To achieve this, first we report some preliminary results.

Let $\Phi_0 \in \mathcal{D}$ and $x \in \mathbb{R}$. Then in view of (60) and (65), there are unique elements, $\zeta(x)$ and $\xi(x)$, of $L^2(-k, k)$ such that

$$\Phi_0 = \begin{bmatrix} 0 & \xi(x) \\ -\hat{\sigma} \zeta(x) & -\xi(x) \end{bmatrix} = \begin{bmatrix} e^{-ix\hat{\sigma} \zeta(x)} & \xi(x) \\ \hat{\sigma} \zeta(x) - e^{ix\hat{\sigma}} \xi(x) & 0 \end{bmatrix}. \quad (72)$$

**Lemma 3:** Let $\Phi_0 \in \mathcal{D}$ and for all $x \in \mathbb{R}$, $\zeta(x)$ and $\xi(x)$ be the elements of $L^2(-k, k)$ satisfying (72). Then, there are $a, b \in [0, \infty)$ such that for all $x \in \mathbb{R}$,

$$\| \zeta(x) \| \leq a + b |x|. \quad (73)$$

Furthermore, the function $3_{\Phi_0} : \mathbb{R} \to [0, \infty)$ defined by

$$3_{\Phi_0}(x) := \| \zeta(x) \|, \quad (74)$$

is continuous.

**Proof:** Let $\zeta_0 := \zeta(0)$ and $\xi_0 := \xi(0)$. Then setting $x = 0$ in (72), we have

$$\Phi_0 = \begin{bmatrix} \xi_0 & \xi_0 \\ -\hat{\sigma} \zeta_0 - \xi_0 \end{bmatrix}. \quad (75)$$

Substituting (72) in the left-hand side of this equation, and solving the resulting equation for $\zeta(x)$ in terms of $\zeta_0$ and $\xi_0$, we find

$$\zeta(x) = \zeta_0 + \hat{\sigma}^{-1} \left( e^{2ix\hat{\sigma}} - 1 \right) \xi_0 = \zeta_0 + 2i\hat{\sigma}^{-1} \sin(x\hat{\sigma}) e^{ix\hat{\sigma}} \xi_0. \quad (76)$$

Let $a := \| \zeta_0 \|$ and $b := 2 \| \xi_0 \|$. Then, (76) implies

$$\| \zeta(x) \| \leq \| \zeta_0 \| + 2 \| \hat{\sigma}^{-1} \sin(x\hat{\sigma}) \| \| \xi_0 \| + e^{ix\hat{\sigma}} \xi_0 \| \leq a + b |x|,$$

where we have made use of the fact that $\hat{\sigma}^{-1} \sin(x\hat{\sigma})$ is a bounded operator, $\| \hat{\sigma}^{-1} \sin(x\hat{\sigma}) \|_0 \leq |x|$, and $e^{ix\hat{\sigma}}$ is a unitary operator. Next, let $x_1, x_2 \in \mathbb{R}$. Then, we can use (76) to show that

$$|3_{\Phi_0}(x_1) - 3_{\Phi_0}(x_2)| = \| \zeta(x_1) \| - \| \zeta(x_2) \| \leq \| \xi_0 \| + 2 \| \hat{\sigma}^{-1} \sin(x_1\hat{\sigma}) \| \| e^{ix_1\hat{\sigma}} \xi_0 \| \leq 2 \| \hat{\sigma}^{-1} \sin(x_1\hat{\sigma}) \| \| e^{ix_1\hat{\sigma}} \xi_0 \| \leq b |x_1 - x_2|. \quad (77)$$

For all $\epsilon \in \mathbb{R}^+$ let $\delta := 1$ if $b = 0$, and $\delta := \epsilon/b$ if $b \neq 0$. Then $\delta \in \mathbb{R}^+$, and in view of (77), $|x_1 - x_2| < \delta$ implies $|3_{\Phi_0}(x_1) - 3_{\Phi_0}(x_2)| < b \delta \leq \epsilon$. Hence $3_{\Phi_0}$ is continuous. \qed
Lemma 4: Let $\mathcal{K}, \hat{\nu}_k(x)$, and $\hat{\mathcal{H}}(x)$ be respectively defined by (21), (40), and (55), $\mathbf{I}$ be the $2 \times 2$ identity matrix, $\hat{\nu}_k(x) := \hat{\nu}_k(x)\mathbf{I}$, $\hat{\sigma}^{-1} := \hat{\sigma}^{-1}\mathbf{I}$,

$$\hat{\mathcal{L}}(x) := e^{-ix\hat{\sigma}^3} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} e^{ix\hat{\sigma}^3} = \begin{bmatrix} 0 & 0 \\ e^{2ix\hat{\sigma}} & 1 \end{bmatrix},$$  

(78)

$n$ be an integer such that $n \geq 2$, $x_1, x_2, \ldots, x_n$ be real numbers, for each $m \in \{1, 2, \ldots, n - 1\}$,

$$\hat{s}_m := i\hat{\sigma}^{-1}\sin[(x_{m+1} - x_m)\hat{\sigma}] \hat{\nu}_k(x_m),$$  

(79)

and

$$\hat{\mathcal{B}}(x_n, x_{n-1}, \ldots, x_1) := \frac{1}{2} e^{-ix_n\hat{\sigma}^3\hat{\nu}_k(x_n)} \hat{s}_{n-1} \hat{s}_{n-2} \ldots \hat{s}_1 \mathcal{K} e^{ix_1\hat{\sigma}^3}.$$  

(80)

Then $\hat{\mathcal{B}}(x_n, x_{n-1}, \ldots, x_1)$ is a Hilbert-Schmidt operator acting in $\mathcal{H}$, and

$$\hat{\mathcal{H}}(x_n) \hat{\mathcal{H}}(x_{n-1}) \ldots \hat{\mathcal{H}}(x_1) = \hat{\mathcal{B}}(x_n, x_{n-1}, \ldots, x_1) \hat{\sigma}^{-1} \hat{\mathcal{L}}(x_1).$$  

(81)

**Proof:** First, we recall that, according to Theorem 2, $\hat{\nu}_k(x)$ and consequently $\hat{\nu}_k(x)$ are Hilbert-Schmidt operators. It is also clear that $\hat{\sigma}^{-1}\sin[(x_{m+1} - x_m)\hat{\sigma}]$ is a bounded operator. Because products of Hilbert-Schmidt and bounded operators are Hilbert-Schmidt, $\hat{s}_m$ and $\hat{\mathcal{B}}(x_n, x_{n-1}, \ldots, x_1)$ are Hilbert-Schmidt operator. To prove (81), first we use (21), (22), (55), and (78) to establish the identities:

$$\mathcal{K} e^{i(x_{m+1} - x_m)\hat{\sigma}^3} \mathcal{K} = 2i \sin[(x_{m+1} - x_m)\hat{\sigma}] \mathcal{K},$$  

(82)

$$\hat{\mathcal{H}}(x_1) = \hat{\mathcal{H}}(x_1) \hat{\mathcal{L}}(x_1) = \frac{1}{2} e^{-ix_1\hat{\sigma}^3} \mathcal{K} \hat{\nu}_k(x_1) e^{ix_1\hat{\sigma}^3} \hat{\sigma}^{-1} \hat{\mathcal{L}}(x_1).$$  

(83)

Substituting (55) in the left-hand side of (81) to obtain the explicit form of $\hat{\mathcal{H}}(x_n) \hat{\mathcal{H}}(x_{n-1}) \ldots \hat{\mathcal{H}}(x_1)$ and using (83), we find an expression involving terms of the form $\mathcal{K} e^{i(x_{m+1} - x_m)\hat{\sigma}^3} \mathcal{K}$, Eq. (81) follows from this expression and (82). \(\square\)

Lemma 5: Let $n \in \mathbb{Z}^+$, $\hat{\mathcal{B}}(x_n, x_{n-1}, \ldots, x_1)$ be the operator defined by (80) for $n \geq 2$, and

$$\hat{\mathcal{B}}(x_1) := \hat{\mathcal{H}}(x_1) \hat{\sigma} = \frac{1}{2} e^{-ix\hat{\sigma}^3} \hat{\nu}_k(x) \mathcal{K} e^{ix\hat{\sigma}^3}.$$  

(84)

Then,

$$\| \hat{\mathcal{B}}(x_1) \|_0 \leq \| \hat{\nu}_k(x_1) \|_0,$$  

(85)

$$\| \hat{\mathcal{B}}(x_n, x_{n-1}, \ldots, x_1) \|_0 \leq \| \hat{\nu}_k(x_n) \|_0 \prod_{m=1}^{n-1} |x_{m+1} - x_m| \| \hat{\nu}_k(x_m) \|_0 \text{ for } n \geq 2.$$  

(86)

**Proof:** In (80) and (84), we can identify $\mathcal{K}$ with $\mathcal{K} \hat{\mathcal{I}}$ which is a bounded operator acting on $\mathcal{H}$ with operator norm 2. We also know that because $e^{\pm ix\hat{\sigma}^3}$ are unitary operators, they have unit operator norm. Making use of these observations, Eqs. (80) and (84), and $\| \hat{\nu}_k(x) \|_0 = \| \hat{\nu}_k(x) \|_0$, we are led to (85) and

$$\| \hat{\mathcal{B}}(x_n, x_{n-1}, \ldots, x_1) \|_0 \leq \| \hat{\nu}_k(x_n) \|_0 \| \hat{s}_{n-1} \|_0 \| \hat{s}_{n-2} \|_0 \ldots \| \hat{s}_1 \|_0 \text{ for } n \geq 2.$$  

(87)
Furthermore, according to (79),

\[
\| \hat{s}_m \|_0 = \| \hat{\omega}^{-1} \sin[(x_{m+1} - x_m)\hat{\omega}] \hat{v}_k(x_m) \|_0 \\
\leq \| \hat{\omega}^{-1} \sin[(x_{m+1} - x_m)\hat{\omega}] \hat{I} \|_0 \| \hat{v}_k(x_m) \|_0 \\
\leq |x_{m+1} - x_m| \| \hat{v}_k(x_m) \|_0,
\]

Relation (86) follows from (87) and (88). □

**Theorem 3:** Let \( \hat{\mathcal{H}}(x) \) be defined by (85), and \((x_0, x) \in \mathbb{R}^2\) such that \(x_0 \leq x\). Then the Dyson series,

\[
\hat{I} + \sum_{n=1}^{\infty} (-i)^n \int_{x_0}^{x} dx_n \int_{x_0}^{x_n} dx_{n-1} \cdots \int_{x_0}^{x_2} dx_1 \hat{\mathcal{H}}(x_n) \hat{\mathcal{H}}(x_{n-1}) \cdots \hat{\mathcal{H}}(x_1),
\]

(89)

converges strongly to a linear operator \( \hat{\mathcal{U}}(x, x_0) \) defined on \( \mathcal{D} \). In particular, (71) has a solution.

**Proof:** Let \( \Phi_0 \) be an arbitrary element of \( \mathcal{D} \), and for each \( n \in \mathbb{Z}^+ \),

\[
\Phi_n(x, x_0) := (-i)^n \int_{x_0}^{x} dx_n \int_{x_0}^{x_n} dx_{n-1} \cdots \int_{x_0}^{x_2} dx_1 \hat{\mathcal{H}}(x_n) \hat{\mathcal{H}}(x_{n-1}) \cdots \hat{\mathcal{H}}(x_1) \Phi_0.
\]

(90)

To prove the theorem it suffices to show that the series \( \sum_{n=0}^{\infty} \Phi_n(x, x_0) \) converges. To do this, we introduce

\[
\Omega(x_1) := \hat{\omega}^{-1} \hat{\mathcal{L}}(x_1) \Phi_0,
\]

(91)

and substitute \( x_1 \) for \( x \) in (72) to express \( \Phi_0 \) in the form

\[
\Phi_0 = \begin{bmatrix} 0 \\ \hat{\omega} \zeta(x_1) \end{bmatrix} + e^{-ix_1\hat{\omega}\sigma_3} \begin{bmatrix} \xi(x_1) \\ -\xi(x_1) \end{bmatrix}.
\]

(92)

In view of (78), (81), (91), and (92),

\[
\Omega(x_1) = \hat{\omega}^{-1} \begin{bmatrix} 0 \\ \hat{\omega} \zeta(x_1) \end{bmatrix} = \begin{bmatrix} 0 \\ \zeta(x_1) \end{bmatrix},
\]

(93)

\[
\hat{\mathcal{H}}(x_n) \hat{\mathcal{H}}(x_{n-1}) \cdots \hat{\mathcal{H}}(x_1) \Phi_0 = \hat{\mathcal{B}}(x_n, x_{n-1}, \cdots, x_1) \Omega(x_1).
\]

(94)

Substituting the latter equation in (90), we have

\[
\| \Phi_n(x, x_0) \| = \| \int_{x_0}^{x} dx_n \int_{x_0}^{x_n} dx_{n-1} \cdots \int_{x_0}^{x_2} dx_1 \hat{\mathcal{B}}(x_n, x_{n-1}, \cdots, x_1) \Omega(x_1) \| \\
\leq \int_{x_0}^{x} dx_n \int_{x_0}^{x_n} dx_{n-1} \cdots \int_{x_0}^{x_2} dx_1 \| \hat{\mathcal{B}}(x_n, x_{n-1}, \cdots, x_1) \|_0 \| \Omega(x_1) \|
\]

In view of (83), (86), and (93), and the fact that \( x_{m+1} \geq x_m \), this relation implies

\[
\| \Phi_1(x, x_0) \| \leq \int_{x_0}^{x} dx_1 \| \hat{v}_k(x_1) \|_0 \| \zeta(x_1) \|,
\]

(95)

and, for \( n \geq 2 \),

\[
\| \Phi_n(x, x_0) \| \leq \int_{x_0}^{x} dx_n \int_{x_0}^{x_n} dx_{n-1} \cdots \int_{x_0}^{x_2} dx_1 \| \hat{v}_k(x_n) \|_0 \| \hat{v}_k(x_m) \|_0 \| \zeta(x_1) \|.
\]

(96)
Next, we observe that, whenever \( x_0 \leq x_1 \leq x_2 \leq x \),
\[
\int_{x_0}^{x_2} dx_1 (x_2 - x_1) \| \hat{v}_k(x_1) \|_0 \| \zeta(x_1) \| \leq \int_{x_0}^{x_2} dx_1 (x_2 - x_0) \| \hat{v}_k(x_1) \|_0 \| \zeta(x_1) \|
\leq (x_2 - x_0) F_+(x_2, x_0) \leq (x_2 - x_0) F_+(x, x_0), \tag{97}
\]
where for all \((u, u_0) \in \mathbb{R}^2,\)
\[
F_+(u, u_0) := \int_{u_0}^{u} dx' \| \hat{v}_k(x') \|_0 \| \zeta(x') \|_0 = \int_{u_0}^{u} dx' \| \hat{v}_k(x') \|_0 \Phi_0(x'), \tag{98}
\]
and we have made use of (74). Notice also that (95) and (98) imply
\[
\| \Phi_1(x, x_0) \| \leq F_+(x, x_0). \tag{99}
\]

Lemmas 1 and 4 state that \( \| \hat{v}_k(\cdot) \|_0 \) and \( \Phi_0 \) are respectively piecewise continuous and continuous functions. This implies that the function \( F_+ : \mathbb{R}^2 \to \mathbb{R} \) defined by (98) is continuous. Next, we use (90) and (107) to show that, for \( n \geq 2,\)
\[
\| \Phi_n(x, x_0) \| \leq F_+(x, x_0) \int_{x_0}^{x} dx_n \int_{x_0}^{x_n} dx_{n-1} \cdots \int_{x_0}^{x_2} dx_2 \left( x_2 - x_0 \right) \| \hat{v}_k(x_2) \|_0 \times \prod_{m=2}^{n-1} |x_{m+1} - x_m| \| \hat{v}_k(x_{m+1}) \|_0
\leq F_+(x, x_0) \int_{x_0}^{x} dx_n \int_{x_0}^{x_n} dx_{n-1} \cdots \int_{x_0}^{x_2} dx_2 \prod_{m=1}^{n-1} |x_{m+1} - x_0| \| \hat{v}_k(x_{m+1}) \|_0
\leq \frac{F_+(x, x_0) G(x, x_0)^{n-1}}{(n - 1)!}, \tag{100}
\]
where for all \((u, u_0) \in \mathbb{R}^2,\)
\[
G(u, u_0) := \int_{u_0}^{u} dx' |x' - u_0| \| \hat{v}_k(x') \|_0. \tag{101}
\]

Again, because \( \| \hat{v}_k(\cdot) \|_0 \) is piecewise continuous, (101) defines a continuous function \( G : \mathbb{R}^2 \to \mathbb{R} \) on \( \mathbb{R}^2. \) It is clear from (99) and (100) that, for all \( N \in \mathbb{Z}^+,\)
\[
\sum_{n=0}^{N} \| \Phi_n(x, x_0) \| \leq \| \Phi_0 \| + F_+(x, x_0) \sum_{n=1}^{N} \frac{G(x, x_0)^{n-1}}{(n - 1)!}
\leq \| \Phi_0 \| + F_+(x, x_0) \left[ e^{G(x, x_0)} - 1 \right]. \tag{102}
\]
This shows that \( \sum_{n=0}^{\infty} \Phi_n(x, x_0) \) converges absolutely. Therefore, it converges. We identify \( \hat{U}(x, x_0) \) with the operator defined on \( \mathcal{D} \) according to \( \hat{U}(x, x_0) \Phi_0 := \sum_{n=0}^{\infty} \Phi_n(x, x_0). \) \( \square \)

Having established the existence of \( \hat{U}(x, x_0), \) we address the existence problem for its strong limit as \( x_0 \to -\infty \) and \( x \to +\infty, \) which we identify with the transfer matrix \( \hat{M}. \)

**Theorem 4:** Let \( k \in \mathbb{R}^+ \) and \( v \in \mathcal{C}_{3,k}. \) Then \( \lim_{x_\pm \to \pm \infty} \hat{U}(x_+, x_-) \) exists as a linear operator defined
on \( \mathcal{D} \).

**Proof:** First, we recall that because \( v \in \mathcal{C}_{3,k} \), there are \( \alpha, \beta, \sigma \in \mathbb{R}^+ \) such that \( \sigma > 3 \) and \( \hat{v}_k(\cdot) \) satisfies (43). We also note that using the Dyson series expansion (89) of \( \hat{U}(x,x_0) \) and its strong absolute convergence we can prove the following identity [14].

\[
\hat{U}(x_3, x_2) \hat{U}(x_2, x_1) = \hat{U}(x_3, x_1) \quad \text{for} \quad x_1 \leq x_2 \leq x_3.
\]

(103)

Repeated use of this relation, we can express \( \hat{U}(x_+, x_-) \) in the form

\[
\hat{U}(x_+, x_-) = \hat{U}(x_+, \alpha) \hat{U}(\alpha, -\alpha) \hat{U}(-\alpha, x_-) \quad \text{for} \quad \pm x_+ \geq \alpha,
\]

(104)

which reduces the existence problem for \( \text{s-lim}_{x_+ \to \pm \infty} \hat{U}(x_+, x_-) \) to that for \( \text{s-lim}_{x_+ \to \pm \infty} \hat{U}(x_+, \alpha) \) and \( \text{s-lim}_{x_- \to -\infty} \hat{U}(-\alpha, x_-) \). To prove the existence of \( \text{s-lim}_{x_+ \to \pm \infty} \hat{U}(x_+, \alpha) \), it is sufficient to show that as \( x \to +\infty \) the right-hand side of (102) with \( x_0 = \alpha \) remains bounded by a positive real number. We show this by proving the existence of \( \gamma, \delta \in \mathbb{R}^+ \) such that for all \( x_+ \geq \alpha \),

\[
F_+(x_+, \alpha) \leq \gamma, \quad G(x_+, \alpha) \leq \delta.
\]

(105)

According to Lemma 3, there are \( a, b \in [0, \infty) \) such that for all \( x \in \mathbb{R} \), \( \mathcal{Z}_{\Phi_0}(x) \leq a + b|x| \). Using this in (98), we have

\[
F_+(x_+, \alpha) \leq a f_0(x_+, \alpha) + b f_1(x_+, \alpha),
\]

(106)

where

\[
f_\ell(x, x_0) := \int_{x_0}^x dx' \left| x' \right| \ell \parallel \hat{v}_k(x') \parallel_0, \quad \ell \in \{0, 1\}.
\]

(107)

Next, we use (101) to infer that whenever \( x_+ \geq \alpha \),

\[
G(x_+, \alpha) = \int_{x_+}^{x_0} dx' (x' - \alpha) \parallel \hat{v}_k(x') \parallel_0 \leq \int_{x_+}^{x_0} dx' x' \parallel \hat{v}_k(x') \parallel_0 = f_1(x_+, \alpha).
\]

(108)

Relations (106) – (108) reduce the proof of the existence of \( \gamma \) and \( \delta \) to finding upper bounds on \( f_\ell(x_+, \alpha) \). In view of (43) and (107), and the fact that \( x_+ \geq \alpha > 0 \), \( \sigma > 3 \), and \( \ell \in \{0, 1\} \),

\[
f_\ell(x_+, \alpha) = \int_{x_+}^{x_0} dx' |x'|^\ell \parallel \hat{v}_k(x') \parallel_0 \leq 2\pi \beta \int_{x_+}^{x_0} dx' \frac{|x'|^\ell}{(1 + |x'|)^\sigma} \\
\leq 2\pi \beta \int_{x_+}^{x_0} dx' x'^{\ell - \sigma} = \frac{2\pi \beta}{\sigma - \ell - 1} \left( \frac{1}{\alpha^{\sigma - \ell - 1}} - \frac{1}{x_+^{\sigma - \ell - 1}} \right) < \frac{2\pi \beta}{(\sigma - 2) \alpha^{\sigma - 2}}.
\]

(109)

In view of (106), (108), and (109), the following choices for \( \eta, \gamma, \) and \( \delta \) satisfy (105).

\[
\gamma := \frac{2\pi \beta (a + b \alpha)}{(\sigma - 2) \alpha^{\sigma - 1}}, \quad \delta := \frac{2\pi \beta}{(\sigma - 2) \alpha^{\sigma - 2}}.
\]

This observation together with (102) show that \( \sum_{n=0}^{\infty} \parallel \Phi_n(x_+, \alpha) \parallel < \infty \) which implies the existence of \( \lim_{x_+ \to +\infty} \sum_{n=0}^{\infty} \Phi_n(x_+, \alpha) \) for every choice of \( \Phi_0 \in \mathcal{D} \). Hence \( \text{s-lim}_{x_+ \to +\infty} \hat{U}(x_+, \alpha) \) exists as a linear operator defined on \( \mathcal{D} \). To prove the existence of \( \text{s-lim}_{x_- \to -\infty} \hat{U}(-\alpha, x_-) \), we first use (96) to establish the following inequality for \( n \geq 2 \).

\[
\parallel \Phi_n(x, x_0) \parallel \leq \frac{f_0(x, x_0) F_-(x, x_0) [-G(x_0, x)]^{n-2}}{(n-2)!},
\]

(110)
where \( f_0 \), and \( G \) are respectively defined by (107) and (101), for all \((u, u_0) \in \mathbb{R}^2\),

\[
F_-(u, u_0) := \int_{u_0}^{u} dx' |u - x'| \| \hat{v}_k(x') \|_0 \| \zeta(x') \|,
\]

and we have made use of the identity,

\[
\int_{x_0}^{x} dx' (x - x') \| \hat{v}_k(x') \|_0 = - \int_{x}^{x_0} dx'|x - x'| \| \hat{v}_k(x') \|_0 = -G(x_0, x),
\]

which follows from (101). Notice that similarly to \( f_0(x, x_0) \), \( F_-(x, x_0) \) and \(-G(x_0, x)\) take nonnegative real values for \( x \geq x_0 \). According to (99) and (110),

\[
\sum_{n=0}^{N} \| \Phi_n(x, x_0) \| \leq \| \Phi_0 \| + \| \Phi_1(x, x_0) \| + f_0(x, x_0) F_-(x, x_0) \sum_{n=2}^{N} \frac{[-G(x_0, x)]^{n-2}}{n!} \]

Next, we use (73), (98), and (107) to show that

\[
0 \leq F_-(\alpha, x_-) \leq a f_0(\alpha, x_-) + b f_1(\alpha, x_-) \quad \text{for} \quad x_- \leq -\alpha.
\]

Repeating the calculations leading to (109) with \((x_+, \alpha)\) replaced with \((-\alpha, x_-)\), we obtain

\[
0 \leq f_b(\alpha, x_-) < \frac{2\pi \beta}{(\sigma - 2)\alpha^{\sigma-1}} \quad \text{for} \quad x_- \leq -\alpha.
\]

Furthermore, we can use (43), (73), (107), (111), (112), and the fact that \( x_- \leq -\alpha < 0 \), and \( \sigma > 3 \), to establish the following relations.

\[
0 \leq F_-(\alpha, x_-) = \int_{x_-}^{-\alpha} dx' (-x' - \alpha) \| \hat{v}_k(x') \|_0 \| \zeta(x') \| \leq \int_{x_-}^{-\alpha} dx' (-x') \| \hat{v}_k(x') \|_0 \| \zeta(x') \| \leq 2\pi \beta \int_{x_-}^{-\alpha} dx' (-x') (a + b|x'|) \]

\[
\leq 2\pi \beta \int_{x_-}^{-\alpha} dx' (\frac{a(\alpha + b|x'|)}{1 + |x'|^\sigma}) \leq 2\pi \beta \int_{x_-}^{-\alpha} dx' (\frac{a(\alpha + b|x'|)}{|x'|^\sigma}) \]

\[
\leq 2\pi \beta \int_{x_-}^{-\alpha} dx' [a(\alpha - x')^{1-\sigma} + b(\alpha - x')^{2-\sigma}] \leq \frac{2\pi \beta (a + b \alpha)}{(\sigma - 3)\alpha^{\sigma-2}},
\]

\[
0 \leq -G(\alpha, x_-) = \int_{x_-}^{-\alpha} dx' (-x' - \alpha) \| \hat{v}_k(x') \|_0 \leq \int_{x_-}^{-\alpha} dx' (-x') \| \hat{v}_k(x') \|_0 \leq 2\pi \beta \int_{x_-}^{-\alpha} dx' (-x')^{\sigma-1} \leq \frac{2\pi \beta}{(\sigma - 2)\alpha^{\sigma-2}}.
\]

In view of (113) – (117), \( \lim_{x_+ \to -\infty} \sum_{n=0}^{\infty} \Phi_n(\alpha, x_-) \) exists for every choice of \( \Phi_0 \in \mathcal{D} \). This implies the existence of \( \text{\mathcal{S}}\text{-}\lim \hat{U}(\alpha, x_-) \) on \( \mathcal{D} \). \( \square \)

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Appendix: Scattering amplitude and transfer matrix in 2D

When \( v \) is a short-range potential, the asymptotic expression for the scattering solutions of (1) may be put in the form

\[
\psi(r) \to \frac{1}{2\pi} \left[ e^{i k_0 r} + \sqrt{\frac{i}{kr}} e^{ikr} f(\theta_0, \theta) \right] \quad \text{for} \quad r \to \infty,
\]

where \( k_0 \in \mathbb{R}^2 \) is the incident wave vector, \( r := (x, y) \), \((r, \theta)\) are the polar coordinates of \( r \), \( \theta_0 \) is the angle between \( k_0 \) and the \( x \)-axis, and \( f(\theta_0, \theta) \) is the scattering amplitude of the potential \( v \) for the incident wavenumber \( k \). \[15\]. Let \( p_0 := k \sin \theta_0 \) and \( p := k \sin \theta \). Then \( \varpi(p) = k|\cos \theta| \), \( \varpi(p_0) = k|\cos \theta_0| \), and we have \[1\]:

1) For \( \theta_0 \in (-\frac{\pi}{2}, \frac{\pi}{2}) \): \( A_-(p) = 2\pi \varpi(p_0) \delta(p - p_0) = 2\pi \delta(\theta - \theta_0) \), \( B_+(p) = 0 \),

\[
f(\theta_0, \theta) = -\frac{i}{\sqrt{2\pi}} \begin{cases} A_+(k \sin \theta) - 2\pi \delta(\theta - \theta_0) & \text{for} \quad \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}), \\ B_-(k \sin \theta) & \text{for} \quad \theta \in (\frac{\pi}{2}, \frac{3\pi}{2}), \end{cases}
\]

\( A_+ = 2\pi \hat{M}_{11} \delta_{p_0} + \hat{M}_{12} B_- \), and \( B_- \) satisfies \( \hat{M}_{22} B_- = -2\pi \hat{M}_{21} \delta_{p_0} \), where \( \delta_{p_0} \) stands for the Dirac delta function centered at \( p_0 \), i.e., \( \delta_{p_0}(p) := \delta(p - p_0) \).

2) For \( \theta_0 \in (\frac{\pi}{2}, \frac{3\pi}{2}) \): \( A_-(p) = 0 \), \( B_+(p) = 2\pi \varpi(p_0) \delta(p - p_0) = 2\pi \delta(\theta - \theta_0) \),

\[
f(\theta_0, \theta) = \frac{i}{\sqrt{2\pi}} \begin{cases} A_+(k \sin \theta) & \text{for} \quad \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}), \\ B_-(k \sin \theta) - 2\pi \delta(\theta - \theta_0) & \text{for} \quad \theta \in (\frac{\pi}{2}, \frac{3\pi}{2}), \end{cases}
\]

\( A_+ = \hat{M}_{12} B_- \), and \( B_- \) solves \( \hat{M}_{22} B_- = 2\pi \delta_{p_0} \).

In particular, given the transfer matrix \( \hat{M} \), the determination of the scattering amplitude \( f(\theta_0, \theta) \) requires the solution of \( \hat{M}_{22} B_- = -2\pi \hat{M}_{21} \delta_{p_0} \) and \( \hat{M}_{22} B_- = 2\pi \delta_{p_0} \). The existence of a well-defined scattering problem for the potential is therefore linked with the triviality of the kernel of \( \hat{M}_{22} \).

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