Newtonian Hydrodynamics with General Relativistic Pressure

Jai-chan Hwang
Department of Astronomy and Atmospheric Sciences, Kyungpook National University, Daegu 702-701, Republic of Korea
E-mail: jchan@knu.ac.kr

Hyerim Noh
Korea Astronomy and Space Science Institute, Daejeon 305-348, Republic of Korea
E-mail: hr@kasi.re.kr

Abstract. We present the general relativistic pressure correction terms in Newtonian hydrodynamic equations to the nonlinear order: these are equations (1)-(3). The derivation is made in the zero-shear gauge based on the fully nonlinear formulation of cosmological perturbation in Einstein’s gravity. The correction terms differ from many of the previously suggested forms in the literature based on hand-waving manners. We confirm our results by comparing with (i) the nonlinear perturbation theory, (ii) the first order post-Newtonian approximation, and (iii) the special relativistic limit, and by checking (iv) the consistency with full Einstein’s equation.

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1. Introduction

The general relativistic pressure corrections in Newtonian hydrodynamic equations have attracted some attentions in the cosmology literature. Previously the pressure correction terms were guessed in hand-waving manner without properly based on relativistic gravity \[1, 2, 3, 4\]. The proper study should begin with Einstein’s gravity and take Newtonian limit except for keeping the relativistic and gravitating pressure terms. This is the aim of our present work.

Based on the formulation of fully nonlinear cosmological perturbation theory in [5], recently we have derived the exact Newtonian hydrodynamic equations as the infinite speed-of-light limit (weak gravity, nonrelativistic speed, subhorizon, and negligible pressure compared with the energy density) in two gauge conditions [6]: the zero-shear gauge and the uniform-expansion gauge. Here, by relaxing only the condition on the pressure term we derive the Newtonian hydrodynamic equations in the presence of relativistic and gravitating pressure correction terms in the zero-shear gauge. The cosmological hydrodynamic equations include the ordinary nonlinear hydrodynamic equations as a case.

Our main result is the Newtonian hydrodynamic equations with the general relativistic pressure corrections. In the presence of relativistic pressure the mass conservation, momentum conservation, and the Poisson’s equation, respectively, become:

\[
\dot{\rho} + \nabla \cdot \left[ \left( \rho + \frac{\rho^2}{c^2} \right) v \right] = \frac{2}{c^2} v \cdot \nabla \rho,
\]

\[
\dot{v} + v \cdot \nabla v - \nabla U = -\frac{1}{\rho + \rho^2/c^2} \left( \nabla \rho + \rho^2/c^2 \right),
\]

\[
\Delta U = -4\pi G \rho,
\]

where \(\rho, \rho^2, v\), and \(U\) are the mass density (including the internal energy), the pressure, the velocity, and the gravitational potential, respectively. In the context of flat Friedmann background the equations become equations \(14\)–\(16\).

The derived corrections are new in the sense that some of the pressure corrections differ from the previously guessed forms in the literature based on pseudo-Newtonian manner: compare our equations \(1\)–\(3\) with the ones in equations \(10\), \(11b\) and \(13\) in [3], equations \(10.9.1a-c\) in [4]. Many of the subsequent works in the literature were based on these incorrect equations. The comparison reveals that the pressure term in the right-hand-side of equation \(1\) was not recognized in the previous studies. For example, equations \(10.9.1a-c\) in a textbook [4] differ completely from ours in all three equations. Based on semi-post-Newtonian treatment the author of [7] has shown equation \(2\) but failed to show equation \(1\): see equations \(23\), \(25\) and \(26\) in [7]. Equation \(2\) can be found in equation \(2.10.16\) of [8] based on the special relativistic hydrodynamics (thus without assuming nonrelativistic speed) in the absence of gravity.

More remarkably, notice the absence of pressure correction term in the Poisson’s equation contrary to the common belief made in the literature \[1, 3, 4, 9\]. See
comments below equation \((24)\).

Our aim in this work is to derive equations \((1)-(3)\) from the fully nonlinear cosmological perturbation equations in Einstein’s gravity (section \(4\)). We will check the validity of our pressure correction terms by comparing our equations with the perturbation theory up to the third order (section \(5\)) and with the first-order post-Newtonian hydrodynamic equations (section \(6\)). For an easy comparison we present the fully nonlinear cosmological perturbation equations in Einstein’s gravity in the Appendix.

2. Notation

We consider the scalar- and vector-type perturbations in a flat background with the metric convention \([10,5]\)

\[
d s^2 = -(1 + 2\alpha) c^2 dt^2 - 2\chi_i cdtdx^i + a^2 (1 + 2\varphi) \delta_{ij} dx^i dx^j,
\]

where \(a(t)\) is the cosmic scale factor; \(\alpha, \varphi\) and \(\chi_i\) are functions of spacetime with arbitrary amplitudes; index of \(\chi_i\) is raised and lowered by \(\delta_{ij}\) as the metric. The spatial part of the metric is simple because we already have taken the spatial gauge condition without losing any generality to the fully nonlinear order \([10,5]\), and have ignored the transverse-tracefree part of the metric perturbation.

We consider a fluid without anisotropic stress. The energy momentum tensor is

\[
\tilde{T}_{ab} = \tilde{\rho} c^2 \tilde{u}_a \tilde{u}_b + \tilde{p} (\tilde{g}_{ab} + \tilde{u}_a \tilde{u}_b),
\]

where tildes indicate the covariant quantities; \(\tilde{u}_a\) is the normalized fluid four-vector. We set

\[
\tilde{u}_i \equiv a \frac{v_i}{c},
\]

where the index of \(v_i\) is raised and lowered by \(\delta_{ij}\) as the metric. We introduce the fluid three-velocity \(\hat{v}_i\) measured by the Eulerian observer with the normal four-vector \(\tilde{n}^c\). It is related to \(v_i\) as \([5]\)

\[
v_i \equiv \hat{\gamma} \hat{v}_i,
\]

with

\[
\hat{\gamma} \equiv \sqrt{1 + \frac{v^k v_k}{c^2 (1 + 2\varphi)}} = \frac{1}{\sqrt{1 - c^2 (1 + 2\varphi) \hat{v}_k \hat{v}_k}},
\]

the Lorentz factor; the index of \(\hat{v}_i\) is raised and lowered by \(\delta_{ij}\).

We can decompose \(\chi_i\) and \(\hat{v}_i\) into the scalar- and vector-type perturbations to the nonlinear order as \([5]\)

\[
\chi_i = c \chi_i^a + a \Psi_i^{(v)}, \quad \hat{v}_i \equiv -\hat{v}_i^a + \hat{v}_i^{(v)},
\]

with \(\Psi_i^{(v)} \equiv 0 \equiv \hat{v}_i^{(v)}\). Dimensions are presented in \([5,6]\).

The complete set of fully nonlinear perturbation equations without taking the temporal gauge was derived in Equations (28)-(38) of \([5]\). The basic equations in \([5]\)
were presented using $v_i$ as the perturbed velocity. In the Appendix we present the same equations now using $\hat{v}_i$ as the perturbed velocity; we recover $c$, and $\tilde{\rho}$ includes the internal energy; in explicit presence of the internal energy density we should replace

$$\tilde{\rho} \rightarrow \tilde{\rho} \left( 1 + \frac{1}{c^2} \tilde{\Pi} \right),$$

where $\tilde{\rho}$ in the right-hand-side is the rest-mass density \([11]\).

Our basic set has seven equations presented in the Appendix: these are (i) the definition of $\kappa$ which is the perturbed part of trace of extrinsic curvature, (ii) the ADM energy constraint, (iii) the ADM momentum constraint, (iv) the trace of ADM propagation, (v) the tracefree ADM propagation, (vi) the covariant energy conservation, and (vii) the covariant momentum conservation. In the following we will call these equations using the names assigned above.

3. Infinite speed-of-light limit except for pressure

In order to show the Newtonian limit, in \([6]\) we have taken the infinite speed-of-light limit. Here we relax the nonrelativistic condition on pressure, thus we do not assume $\tilde{p} \ll \tilde{\rho}c^2$; we also do not assume $\tilde{\Pi}/c^2 \ll 1$. Thus, as the non-relativistic limit we consider the weak-gravity and the slow-motion limits

$$\alpha \ll 1, \quad \varphi \ll 1, \quad \frac{\hat{v}^k \hat{v}_k}{c^2} \ll 1.$$  \((11)\)

Under this slow-motion limit we have $v^i = \hat{v}^i$. We identify

$$\alpha = -\frac{1}{c^2} U, \quad \varphi = \frac{1}{c^2} V, \quad \hat{v}^k = v^k,$$  \((12)\)

where $v$ is the perturbed Newtonian velocity; $U$ and $V$ correspond to the Newtonian and the post-Newtonian perturbed gravitational potentials, respectively \([11] [12]\); later we will show $\varphi = -\alpha$ in our case, thus $V = U$. As the subhorizon limit, we take the dimensionless quantity

$$c^2k^2 \gg \frac{a^2}{H^2},$$  \((13)\)

where $k$ is the comoving wave-number with $\Delta = -k^2$; $H \equiv \dot{a}/a$; in the presence of the cosmological constant $\Lambda$, we consider $H^2 \sim 8\pi G \rho$.

4. Proof in the zero-shear gauge

In this section we will derive the following equations from the fully nonlinear cosmological perturbation equations in Einstein’s gravity presented in the Appendix.

$$\dot{\tilde{\rho}} + 3 \frac{\dot{a}}{a} \left( \tilde{\rho} + \frac{\tilde{p}}{c^2} \right) + \frac{1}{a} \nabla \cdot \left[ \left( \tilde{\rho} + \frac{\tilde{p}}{c^2} \right) \mathbf{v} \right] = \frac{2}{c^2 a} \mathbf{v} \cdot \nabla \tilde{p},$$  \((14)\)

$$\dot{\mathbf{v}} + \frac{\dot{a}}{a} \mathbf{v} + \frac{1}{a} \mathbf{v} \cdot \nabla \mathbf{v} - \frac{1}{a} \nabla U = -\frac{1}{\tilde{\rho} + \tilde{p}/c^2} \left( \frac{1}{a} \nabla \tilde{p} + \frac{\dot{\tilde{p}}}{c^2} \mathbf{v} \right),$$  \((15)\)

$$\frac{\Delta}{a^2} U = -4\pi G \left( \tilde{\rho} - \varphi \right),$$  \((16)\)
where we decompose the mass density \( \bar{\rho} \) and the pressure \( \bar{p} \) into the background and perturbation as
\[
\bar{\rho} = \rho + \delta \rho, \quad \bar{p} = p + \delta p.
\]
Equation (14) can be decomposed to the background and perturbation orders as
\[
\dot{\rho} + 3 \frac{\dot{a}}{a} (\rho + \frac{p}{c^2}) = 0,
\]
\[
\dot{\delta} + \frac{1}{c^2 a} \left( \frac{\delta p}{\rho} - \frac{p}{\delta} \right) + \frac{1}{a} \nabla \cdot \left[ (1 + \delta) \mathbf{v} \right] = \frac{1}{c^2} \frac{1}{a \rho} (\mathbf{v} \cdot \nabla \bar{p} - \bar{p} \nabla \cdot \mathbf{v}),
\]
where \( \delta \equiv \delta \rho / \rho \).

By taking the infinite-speed-of-light limit, the above equations properly reduce to the Newtonian hydrodynamic equations \[6\]; for the Newtonian derivation, see sections 7 and 9 in \[13\].

In the Minkowski background, the background order quantities become \( a = 1 \) and \( \rho = 0 = p \), thus \( \delta \rho = \bar{\rho} \) and \( \delta p = \bar{p} \). Equations (14)-(16) properly become equations (1)-(3).

Now, the derivation of equations (14)-(16) goes as the following. We consider the zero-shear gauge (\( \chi \equiv 0 \)). In \[6\] we have shown that compared with \( \tilde{v}_i^{(v)} \), \( \Psi_i^{(v)} \) is suppressed by the small-scale condition in equation (13). Thus, \( \chi_i = c \chi, \) while keeping \( \tilde{v}_i^{(v)} \). Thus, in the subhorizon limit and in the zero-shear gauge we have
\[
\chi_i = 0.
\]

The tracefree ADM propagation equation gives
\[
\varphi = -\alpha.
\]

The ADM momentum constraint equation in the zero-shear gauge gives
\[
\kappa = -\frac{12 \pi G a}{c^2} \Delta^{-1} \nabla \cdot \left[ \left( \bar{\rho} + \frac{\bar{p}}{c^2} \right) \mathbf{v} \right],
\]
where \( \kappa \) is a perturbed part of the trace of extrinsic curvature (equivalently, a perturbed part of the expansion scalar of the normal-frame vector with a minus sign).

Using equations (21) and (22), the covariant energy conservation and the covariant momentum conservation equations give the conservation equations in (14) and (15), respectively.

The Poisson’s equation in (16) follows from either the ADM energy constraint equation or the trace of ADM propagation equation. The derivations deserve special comments. Under our limits, the ADM energy constraint equation and the trace of ADM propagation equation, respectively, become
\[
+ c^2 \frac{\Delta}{a^2} \varphi + 4 \pi G \delta \rho = -\frac{\dot{a}}{a} \kappa,
\]
\[
- c^2 \frac{\Delta}{a^2} \alpha + 4 \pi G \left( \delta \rho + \frac{3 \delta p}{c^2} \right) = \dot{\kappa} + 2 \frac{\dot{a}}{a} \kappa.
\]
Using equation (22) we can see the \( \kappa \) terms in the right-hand-sides are negligible. However, it is essentially important to keep \( \dot{\kappa} \) term. Using equations (14), (15) and (22)
we can show that the $\dot{\kappa}$ term becomes $12\pi G\delta p/c^2$, thus exactly cancels the $12\pi G\delta p/c^2$ term in the left-hand-side of equation (24). Thus, both equations (23) and (24) give equation (15). The left-hand-side of equation (24), the Raychaudhuri equation, shows apparent presence of the pressure contribution to the gravity often emphasized in the literature [3][4][7]. However, as we just showed $\dot{\kappa}$ term exactly cancels the pressure term in our limits. We can also check the above statements by using the linear perturbation solutions in the zero-shear gauge for an ideal fluid with constant $w = p/(\rho c^2)$ and $\delta p = w\delta \rho c^2$; the complete subhorizon scale solutions in all fundamental gauges are presented in Table 9 of [14]. For a similar analysis in the case of the post-Newtonian approximation, see below equation (37).

Here we would like to add further comments on this subtle issue. Equations (23) and (24) corresponds to the $\tilde{G}^{00}$ and $\tilde{R}^{00}$ parts of Einstein’s equation, respectively. In the presence of relativistic pressure we no longer have the conventional Newtonian limit argument; for example, in the presence of pressure, we no longer have $\tilde{G}^{00} = 2\tilde{R}^{00}$ as is often available in the Newtonian limit argument used, for example, in equation (1.81) of [9]. Notice that whereas $\tilde{G}^{00}$-equation is a constraint equation, $\tilde{R}^{00}$-equation gives a propagation equation. In the presence of the relativistic pressure the constraint equation gives a Poisson’s equation without pressure whereas the propagation equation apparently have the pressure correction term: see equations (23) and (24). At this point, we have to carefully examine the time-derivative term ($\dot{\kappa}$-term in perturbation theory, and $\ddot{U}$-term in the post-Newtonian approximation) in the propagation equation, which turns out to exactly cancel the pressure term as we have explained in the previous paragraph.

For the proof of equations (16)-(18) we have used six equations in our basic set consisting of seven equations. Thus, we still have one more equation (the definition of $\kappa$) in Einstein’s gravity which needs to be checked for consistency. This demands a careful examination of the equations with an issue which will be resolved in the following. In our limits, the definition of $\kappa$ equation leads to

$$-\dot{\phi} + H\alpha = \frac{1}{3}\kappa. \quad (25)$$

By naïvely using equations (14), (15), (21)-(24) this equation could lead to an inconsistency by a term $c^{-4}8\pi Ga\Delta^{-1}(v \cdot \nabla \tilde{p})$ present in the left-hand-side. However, by using the identifications in equation (12) and equation (22), we notice that both sides of this equation are suppressed by the small-scale limit factor in equation (13). Thus, as both sides are higher order in the small-scale expansion, the equation cannot be relied in our limits. For a similar issue addressed in the context of proving the Newtonian limit, see section 5 in [6].

This completes the proof of equations (16)-(18), thus consequently equations (11)-(3) as well.

‡ In the Table 9 of [14] we wish to correct an error by replacing $S'$ to $x(S/x)'$. 

5. Confirmation from perturbation theory

Using the gauge-ready form perturbation equations presented to the third order in section 4 of [5], we have checked the validity of equations (14)-(16) to the third order perturbation in the zero-shear gauge; this is natural (even a tautology), though, because in the previous section we already have proved these equations based on the fully nonlinear equations where the third-order perturbation equations are the simple consequence of those.

Here, as we have some issues to clarify, we would like to examine the linear order case more closely. To the linear order equations (14)-(16) give

\[
\dot{\delta} + \frac{3}{a} \frac{1}{c^2} \left( \frac{\delta p}{\dot{\rho}} - \frac{p}{\dot{\rho}} \delta \right) + \frac{1}{a} \left( 1 + \frac{p}{\rho c^2} \right) \nabla \cdot \mathbf{v} = 0, \tag{26}
\]

\[
\dot{\mathbf{v}} + \frac{2}{a} \mathbf{v} - \frac{1}{a} \nabla U = -\frac{1}{\dot{\rho} + p/c^2} \left( \frac{1}{a} \nabla \delta p + \frac{\dot{p}}{c^2} \mathbf{v} \right), \tag{27}
\]

\[
\frac{\Delta}{a^2} U = -4\pi G \delta \dot{\rho}. \tag{28}
\]

Once again, we can check that these are the same as linear perturbation equations in the zero-shear gauge in the small-scale (subhorizon) limit.

From equations (26)-(28) we can derive

\[
\ddot{\delta} + \left( 2 + 3 \frac{c_s^2}{c^2} - 6w \right) \dot{\delta} + \left[ 3H \frac{\partial}{\partial t} \left( \frac{c_s^2}{c^2} - w \right) + 3\dot{H} \left( \frac{c_s^2}{c^2} - w \right) + 3H^2 \left( \frac{c_s^2}{c^2} - w \right) \left( 2 - 3w \right) \right. \\
-4\pi G \rho (1 + w) \right] \delta = c_s^2 \frac{\Delta}{a^2} \delta + \frac{\Delta}{a^2} \frac{e}{\dot{\rho}} - 3H \frac{\dot{e}}{\rho c^2} - 4\pi G \left( 7 - 3w \right) \frac{e}{c^2}, \tag{29}
\]

where

\[
\delta p \equiv c_s^2 \delta \dot{\rho} + e, \quad c_s^2 \equiv \frac{\dot{p}}{\dot{\rho}}, \quad w \equiv \frac{p}{\rho c^2}. \tag{30}
\]

This equation is valid only in the subhorizon limit. Thus, applying the subhorizon limit condition, we have

\[
\ddot{\delta} + \left( 2 + 3 \frac{c_s^2}{c^2} - 6w \right) \dot{\delta} = c_s^2 \frac{\Delta}{a^2} \delta + \frac{\Delta}{a^2} \frac{e}{\dot{\rho}}. \tag{31}
\]

This \textit{coincides} with the density perturbation equation in the comoving gauge \((v \equiv 0)\) in the subhorizon limit [15, 16], see equation (45) in [17].

In the zero-pressure limit, equation (29) properly reduces to

\[
\ddot{\delta} + 2\dot{H} \dot{\delta} - 4\pi G \rho \delta - c_s^2 \frac{\Delta}{a^2} \delta = 0, \tag{32}
\]

which is the well known equation in the comoving gauge \((v \equiv 0)\) and the synchronous gauge \((\alpha \equiv 0)\) available for a zero-pressure fluid [18, 15]; we have ignored \(c_s^2/c^2\), but have kept the bare \(c_s^2\) term, and similarly have kept \(4\pi G \rho \) term which was ignored in equation (31) due to the small-scale limit in the presence of substantial pressure with \(c_s^2 \sim c^2\). Notice that equation (32) is valid in the zero-shear gauge in the subhorizon limit, whereas the same equation is valid in all scales in the comoving gauge [15, 16, 14, 17].
6. Confirmation from post-Newtonian approximation

The cosmological post-Newtonian (PN) approximation presented in [12] can provide another confirmation of equations (14)-(16). The mass conservation, the momentum conservation, and the Poisson’s equations valid to the first PN (1PN) order without taking the (temporal PN) gauge condition are presented in equations (114), (115) and (119) of [12]. The correspondence between the two (perturbation versus PN) approaches was studied in [19, 20] with identifications

\[ \alpha = -\frac{1}{c^2} \left[ U - \frac{1}{c^2} (U^2 - 2\Phi) \right], \quad \varphi = \frac{1}{c^2} V, \quad \kappa = -\frac{1}{c^2} \left( 3\frac{\dot{a}}{a} U + 3\dot{V} + \frac{1}{a} P^k_{\cdot k} \right), \]

\[ \chi_i = \frac{1}{c^3} a P_i, \quad v_i = \bar{v}_i + \frac{1}{c^2} \left[ \bar{v}_i \left( \frac{1}{2} \nabla^2 + U + 2V \right) - P \right], \quad (33) \]

where the left- and right-hand-sides correspond to the perturbation and the PN notations, respectively; we have \( V = U \), and \( \bar{v}_i \) indicates the \( v_i \) in [12]; \( \bar{v}_i \) is the fluid coordinate three velocity introduced in [5] as

\[ \frac{1}{a} \frac{\bar{v}_i}{c} = \frac{\bar{u}_i}{\bar{u}_0}, \quad (34) \]

where the index of \( \bar{v}_i \) is raised and lowered by \( \delta_{ij} \); in our metric convention in equation (4), an index 0 indicates \( x^0 = ct \). Compared with the Newtonian identifications in equation (12), the exact 1PN identifications above look more complicated; we have not taken the temporal gauge condition in the 1PN notation. By strictly applying the conditions in equation (11), equations (114), (115) and (119) in [12] give

\[ \dot{\bar{\rho}} + \frac{3}{a} \frac{\dot{a}}{a} \bar{\rho} + \frac{1}{a} \nabla \cdot (\bar{\rho} \bar{v}) = \frac{1}{c^2} \left[ \frac{1}{\bar{a}} \nabla \cdot \nabla \bar{p} - \left( 3\frac{\dot{a}}{a} + \frac{1}{\bar{a}} \nabla \cdot \bar{v} \right) \bar{p} \right], \quad (35) \]

\[ \dot{\bar{v}} + \frac{\dot{a}}{a} \bar{v} + \frac{1}{a} \nabla \cdot \bar{v} \nabla U + \frac{1}{a} \nabla ^2 - \frac{1}{c^2} \left( \bar{p} \nabla \bar{p} - \bar{p} \bar{v} \right), \quad (36) \]

\[ \frac{\Delta}{a^2} U = -4\pi G \delta \bar{\rho}. \quad (37) \]

These equations coincide exactly with equations (14)-(16) expanded to the 1PN order.

The absence of pressure correction term in equation (37) again may deserve special comments. It follows from equation (119) of [12]; in our notation

\[ \frac{\Delta}{a^2} U = -4\pi G \left[ \bar{\rho} - \bar{\rho} + 3\frac{1}{c^2} (\bar{p} - p) \right] - \frac{1}{c^2} \left\{ 3\ddot{U} + 9\frac{\dot{a}}{a} \dot{U} + 6\frac{\dot{a}}{a} \dot{U} \right. \]

\[ + 8\pi G \bar{\rho} \bar{v} \bar{v}_i + \frac{1}{a^2} \left[ 2\nabla \Phi - 2U \Delta U + \left( a P^i_{\cdot i} \right) \right] \}. \quad (38) \]

In order to show equation (37) it is important to keep \( \ddot{U}/c^2 \) term in equation (38) compared with \( \Delta U/a^2 \), whereas \( (\dot{a}/a)\ddot{U}/c^2 \) and \( (\ddot{a}/a)U/c^2 \) terms are negligible due to the small-scale condition. In the PN approximation the zero-shear gauge corresponds to the transverse-shear gauge taking \( P^k_{\cdot k} \equiv 0 \) [12]. The \( \ddot{U}/c^2 \) term in equation (38) cancels exactly with the pressure correction term which otherwise could have contributed to \( -12\pi G \delta p \) term in the right-hand side of equation (37). In order to show the cancelation
we need the momentum-constraint equation presented in equation (120) of [12] as

$$0 = \frac{1}{a^2} \left( P^k_{\left| k_i \right|} - \Delta P_i \right) - 16\pi G \tilde{\nu} v_i + \frac{4}{a} \left( \dot{U} + \frac{\dot{a}}{a} U \right).$$

(39)

This follows from the ADM momentum constraint equation [20]. In our case we can keep $\dot{U}_i = 4\pi G a \tilde{\nu} v_i$ and using equations (35) and (36) we can show $\ddot{U} = -4\pi G(\tilde{\nu} - P)$, thus the $\dddot{U}$ term cancels the pressure term in equation (38) exactly. Exactly similar analysis was made in the nonlinear perturbation approach: see comments made in the two paragraphs below equation (24), and the relation between $\kappa$ and $U$ in equation (33).

7. Discussion

In this work we have derived Newtonian hydrodynamic equations in the presence of relativistic pressure corrections: these are equations (1)-(3) in general, and equations (14)-(16) in the cosmological context. We have derived these from the fully nonlinear cosmological perturbation equations in Einstein’s gravity [3]. As our pressure correction terms differ from the commonly known ones often used in the literature [3, 4, 7], in sections 5 and 6 we have confirmed these equations by comparing with (i) the perturbation theory to the third order and (ii) the cosmological 1PN approximation. In section 4 we also have (iii) checked the self consistency of our equations using the full set of equations in Einstein’s gravity.

The referee, Dr. Donghui Jeong, has suggested yet another independent check of our two conservation equations by comparing them with the ones in the special relativistic limit available in the literature. Indeed, we can show that (iv) our conservation equations in (1) and (2) correctly reproduce the equations in the special relativistic limit (thus ignoring gravity, but keeping the relativistic speed and pressure): these are equation (2.10.16) in [8] and equations (2.3) and (2.4) in [9]; these equations are valid for the relativistic speed whereas we assume the nonrelativistic speed. This indicates that the previously suggested pressure correction terms in the literature [3, 4, 7] even fail to have correct special relativistic limit.

Applications of our hydrodynamic equations (1)-(3) with general relativistic pressure are left for future studies.

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References

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Here we summarize the complete set of fully nonlinear perturbation equations without taking the temporal gauge using $\tilde{v}_i$ as the fluid three-velocity \[5\]. We recovered $c$ and $\tilde{\rho} \equiv \tilde{\mu}/c^2$ includes the internal energy density.

Definition of $\kappa$:
\[
\kappa \equiv 3 \frac{\dot{a}}{a} \left(1 - \frac{1}{N}\right) - \frac{1}{N(1 + 2\varphi)} \left[ 3\dot{\varphi} + \frac{c}{a^2} \left( \chi^k_{,k} + \frac{\chi^{k\varphi;k}}{1 + 2\varphi} \right) \right].
\]

ADM energy constraint:
\[
- \frac{3}{2} \left( \frac{\dot{a}^2}{a^2} - \frac{8\pi G}{3} \tilde{\rho} - \frac{\Lambda c^2}{3} \right) + \frac{\dot{a}}{a} \kappa + \frac{c^2 \Delta \varphi}{a^2 (1 + 2\varphi)^2}
= \frac{1}{6} \kappa^2 - 4\pi G \left( \tilde{\rho} + \frac{\tilde{p}}{c^2} \right) \left( \tilde{\gamma}^2 - 1 \right) + \frac{3}{2} \frac{c^2 \varphi^i \varphi_{,i}}{a^2 (1 + 2\varphi)^3} - \frac{c^2}{4} K^j_i K^i_j.
\]

ADM momentum constraint:
\[
\frac{2}{3} \kappa_{,i} + \frac{c}{2a^2 N(1 + 2\varphi)} \left( \Delta \chi_i + \frac{1}{3} \chi^k_{,i;k} \right) + 8\pi G \left( \tilde{\rho} + \frac{\tilde{p}}{c^2} \right) a^2 \frac{\dot{\rho}}{c^2}
= \frac{c}{a^2 N(1 + 2\varphi)} \left\{ \left( N_{,j}^i - \frac{\varphi_{,j}}{1 + 2\varphi} \right) \left[ \frac{1}{2} (\chi^j_{,i} + \chi_i^{j}) - \frac{1}{3} \delta^j_i \chi^k_{,k} \right] + \frac{N}{1 + 2\varphi} \nabla_j \left[ \frac{1}{N} \left( \chi^j \varphi_{,i} + \chi_i \varphi^j - \frac{2}{3} \delta^j_i \chi^k\varphi_{,k} \right) \right] \right\}.
\]

Trace of ADM propagation:
\[
- 3 \frac{1}{N} \left( \frac{\dot{a}}{a} \right)' - 3 \frac{\dot{a}^2}{a^2} - 4\pi G \left( \tilde{\rho} + \frac{\tilde{p}}{c^2} \right) + \Lambda c^2 + \frac{1}{N} \tilde{\kappa} + 2 \frac{\dot{\kappa}}{a} + \frac{c^2 \Delta N}{a^2 N(1 + 2\varphi)}
\]
\[ \frac{1}{3} \kappa^2 + 8\pi G \left( \bar{\varrho} + \frac{\bar{p}}{c^2} \right) \left( \bar{\gamma}^2 - 1 \right) - \frac{c}{a^2 \mathcal{N}(1 + 2\varphi)} \left( \chi^i \kappa_{,i} + c \frac{\varphi^i N_{,i}}{1 + 2\varphi} \right) + c^2 \mathcal{K}_j^i \mathcal{K}_i^j \] (43)

Tracefree ADM propagation:
\[
\left( \frac{1}{\mathcal{N}} \frac{\partial}{\partial t} + \frac{3}{a} \dot{a} - \kappa + \frac{c \chi^k}{a^2 \mathcal{N}(1 + 2\varphi)} \nabla_k \right) \left\{ \frac{c}{a^2 \mathcal{N}(1 + 2\varphi)} \times \left[ \frac{1}{2} \left( \chi^i \varphi_{,j} + \chi^j \varphi_{,i} \right) - \frac{1}{3} \delta^i_j \chi^k_{,k} - \frac{1}{1 + 2\varphi} \left( \chi^i \varphi_{,j} + \chi^j \varphi_{,i} - \frac{2}{3} \delta^i_j \chi^k \varphi_{,k} \right) \right] \right\} \\
- \frac{c^2}{a^2(1 + 2\varphi)} \left[ \frac{1}{1 + 2\varphi} \left( \nabla^i \nabla_j - \frac{1}{3} \delta^i_j \Delta \right) \varphi + \frac{1}{\mathcal{N}} \left( \nabla^i \nabla_j - \frac{1}{3} \delta^i_j \Delta \right) \mathcal{N} \right] \\
= 8\pi G \left( \bar{\varrho} + \frac{\bar{p}}{c^2} \right) \left[ \frac{\bar{\gamma}^2 \bar{\nu}^i \bar{\nu}_i}{c^2(1 + 2\varphi)} - \frac{1}{3} \delta^i_i \left( \bar{\gamma}^2 - 1 \right) \right] + \frac{c^2}{a^4 \mathcal{N}^2(1 + 2\varphi)^2} \times \left[ \frac{1}{2} \left( \chi^i \varphi_{,j} + \chi^j \varphi_{,i} \right) + \frac{1}{1 + 2\varphi} \left( \chi^i \varphi_{,j} + \chi^j \varphi_{,i} \right) - \chi^j \varphi_{,j} \right] \\
+ \frac{3}{1 + 2\varphi} \left( \varphi^i \varphi_{,j} - \frac{1}{3} \delta^i_j \varphi^k \varphi_{,k} \right) + \frac{1}{\mathcal{N}} \left( \varphi^i N_{,j} + \varphi_{,j} N^i - \frac{2}{3} \delta^i_j \varphi^k N_{,k} \right) \right]. \] (44)

Covariant energy conservation:
\[
\left[ \frac{\partial}{\partial t} + \frac{1}{a(1 + 2\varphi)} \left( \nabla^k \chi^k + c \frac{\chi^k}{a} \right) \nabla_k \right] \bar{\varrho} + \left( \bar{\varrho} + \frac{\bar{p}}{c^2} \right) \left\{ \mathcal{N} \left( 3 \frac{\dot{a}}{a} - \kappa \right) + \frac{\mathcal{N} \bar{\nu}^k \varphi_{,k}}{a(1 + 2\varphi)} + \frac{1}{\mathcal{N}} \left[ \frac{\partial}{\partial t} + \frac{1}{a(1 + 2\varphi)} \left( \nabla^k + c \frac{\chi^k}{a} \right) \nabla_k \right] \bar{\gamma} \right\} = 0. \] (45)

Covariant momentum conservation:
\[
\frac{1}{a \hat{\gamma}} \left[ \frac{\partial}{\partial t} + \frac{1}{a(1 + 2\varphi)} \left( \nabla^k \chi^k + c \frac{\chi^k}{a} \right) \nabla_k \right] \left( a \hat{\gamma} \bar{\nu}_i \right) + \bar{\nu}^i \nabla_i \left( c \frac{\chi^k}{a^2(1 + 2\varphi)} \right) \\
+ \frac{c^2}{a^2} N_{,i} - \left( 1 - \frac{1}{\hat{\gamma}^2} \right) \frac{c^2 \varphi_{,i}}{a(1 + 2\varphi)} \\
+ \frac{1}{\bar{\varrho} + \frac{\bar{p}}{c^2}} \left\{ \frac{\mathcal{N}}{a \hat{\gamma}^2} \bar{\nu}^i \bar{\nu}_i \right\} \frac{\partial}{\partial t} + \frac{1}{a(1 + 2\varphi)} \left( \nabla^k + c \frac{\chi^k}{a} \right) \nabla_k \bar{p} \right\} = 0, \] (46)

where
\[
\mathcal{N} \equiv \sqrt{1 + 2\alpha + \frac{\chi^k \chi_k}{a^2(1 + 2\varphi)}}, \quad \mathcal{K}_j^i \mathcal{K}_i^j = \frac{1}{a^4 \mathcal{N}^2(1 + 2\varphi)^2} \left\{ \frac{1}{2} \chi^{i,j}(\chi_{i,j} + \chi_{j,i}) - \frac{1}{3} \chi^i \chi^j \varphi_{,j} \right\} \\
- \frac{4}{1 + 2\varphi} \left\{ \frac{1}{2} \chi^{i,j}(\chi_{i,j} + \chi_{j,i}) - \frac{1}{3} \chi^i \chi^j \varphi_{,j} \right\} + \frac{2}{(1 + 2\varphi)^2} \left( \chi^i \chi^j \varphi_{,j} + \frac{1}{3} \chi^j \chi^i \varphi_{,i} \right). \] (47)