Maximal displacement of branching symmetric stable processes

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Abstract. We determine the limiting distribution and the explicit tail behavior for the maximal displacement of a branching symmetric stable process with spatially inhomogeneous branching structure. Here the branching rate is a Kato class measure with compact support and can be singular with respect to the Lebesgue measure.

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1 Introduction

We studied in [26] the limiting distributions for the maximal displacement of a branching Brownian motion with spatially inhomogeneous branching structure. In this paper, we show the corresponding results for a branching symmetric stable process. Our results clarify how the tail behavior of the underlying process would affect the long time asymptotic properties of the maximal displacement.

There has been great interest in the maximal displacement of branching Brownian motions and branching random walks with light tails for which the associated branching structures are spatially homogeneous. We refer to [9] as a pioneering work, and to [8] Sections 5–7 and references therein, and [22,32] for recent developments on this research subject. On the other hand, Durrett [17] proved the weak convergence of a properly normalized maximal displacement of a branching random walk on $\mathbb{R}$ with regularly varying tails. This result in particular shows that the maximal displacement grows exponentially in contrast with the light tailed model. Bhattacharya-Hazra-Roy [3, Theorem 2.1] further showed the weak convergence of point processes associated with the scaled particle positions. We refer to [24,20] for related studies on the maximal displacement of branching stable processes or branching random walks with regularly varying tails.

Here we are interested in how the spatial inhomogeneity of the branching structure would affect the behavior of the maximal displacement. For a branching Brownian motion on $\mathbb{R}$, Erickson [18] determined the linear growth rate of the
maximal displacement in terms of the principal eigenvalue of the Schrödinger type operator. Lalley-Sellke [23] then showed that a properly shifted maximal displacement is weakly convergent to a random shift of the Gumbel distribution with respect to the limiting martingale. Bocharov and Harris [6,7] proved the results corresponding to [18,23] for a catalytic branching Brownian motion on \( \mathbb{R} \) in which reproduction occurs only at the origin. We showed in [29,30] that the results in [18,6] are valid for the branching Brownian motion on \( \mathbb{R}^d \) (\( d \geq 1 \)) in which the branching rate is a Kato class measure with compact support. Under the same setting, Nishimori and the author [26] further determined the limiting distribution of the shifted maximal displacement as in [23,7].

This result reveals that, even though reproduction occurs only on a compact set unlike [23], the spatial dimension \( d \) appears in the lower order term of the maximal displacement. We refer to [5,25] for further developments.

Carmona-Hu [14] and Bulinskaya [11,12] obtained the linear growth rate and weak convergence for the maximal displacement of a branching random walk on \( \mathbb{Z}^d \) in which reproduction occurs only on finite points and the underlying random walk has light tails. We note that the underlying random walk in [11,12,14] is irreducible and allowed to be nonsymmetric, and the so-called \( L \log L \) condition is sufficient for the validity of these results as proved by [11,12]. Recently, Bulinskaya [13] showed the weak convergence of a properly normalized maximal displacement of a branching random walk on \( \mathbb{Z} \) with regularly varying tails as in [17,3] and reproduction occurs only on finite points. As for the spatially homogeneous model, the growth rate of the maximal displacement is exponential in contrast with the light tailed model.

In this paper, we prove the weak convergence and the long time tail behavior for the maximal displacement of a branching symmetric stable process on \( \mathbb{R}^d \) with spatially inhomogeneous branching structure (Theorems 17 and 18). We will then see that the maximal displacement grows exponentially and the growth rate is determined by the principal eigenvalue of the Schrödinger type operator associated with the fractional Laplacian. The spatial dimension \( d \) also affects the limiting distribution and tail behavior of the maximal displacement. Our results are applicable to a branching symmetric stable process in which reproduction occurs only on singular sets.

Our results can be regarded as a continuous state space and multidimensional analogue of [13]. In particular, we provide an explicit form of the limiting distribution for the maximal displacement. On the other hand, since our approach is based on the second moment method as for [20], we need the second moment condition on the offspring distribution, which is stronger than the \( L \log L \) condition as imposed in [3,13,17].

We note that the functional analytic approach works well for the continuous space model. Our model of branching symmetric stable processes is closely related to the Schrödinger type operator associated with the fractional Laplacian and Kato class measure through the first moment formula on the expected population (Lemma 15). It is possible to calculate or estimate the principal eigenvalue of the Schrödinger type operator as in Subsection 3.3.
The rest of this paper is organized as follows: In Section 2, we first prove the resolvent asymptotic behaviors of a symmetric stable process. We then discuss the invariance of the essential spectra, and the asymptotics of the integral associated with the ground state, for the Schrödinger type operator. We finally determine asymptotic behaviors of the Feynman–Kac functional. In Section 3, we first introduce a model of branching symmetric stable processes. We then present our results in this paper with examples. In Section 4, we prove the weak convergence result (Theorem 17) by following the approach of [26, Theorem 2.4].

2 Symmetric stable processes

For \( \alpha \in (0, 2) \), let \( M = \langle \{X_t\}_{t \geq 0}, \{P_x\}_{x \in \mathbb{R}^d} \rangle \) be a symmetric \( \alpha \)-stable process on \( \mathbb{R}^d \) generated by \( -(-\Delta)^{\alpha/2}/2 \). Let \( p_t(x, y) \) be the transition density function of \( M \),

\[
P_x(X_t \in A) = \int_A p_t(x, y) \, dy, \quad t > 0, x \in \mathbb{R}^d, A \in \mathcal{B}(\mathbb{R}^d).
\]

Here \( \mathcal{B}(\mathbb{R}^d) \) is the family of Borel measurable sets on \( \mathbb{R}^d \). By [4, Theorem 2.1] and [37, Lemmas 2.1 and 2.2], we have

**Lemma 1** There exists a positive continuous function \( g \) on \( [0, \infty) \) such that

\[
p_t(x, y) = \frac{1}{t^{d/\alpha}} g \left( \frac{|x - y|}{t^{1/\alpha}} \right).
\]

Moreover, the function \( g \) satisfies the following.

(i) The value \( g(0) \) is given by

\[
g(0) = \frac{2^{d/\alpha} \Gamma(d/\alpha)}{\alpha 2^{d-1} \pi^{d/2} \Gamma(d/2)}.
\]

(ii) There exists \( c > 0 \) such that for any \( r \geq 0 \),

\[
0 \leq g(0) - g(r) \leq cr^2.
\]

(iii) The next equality holds.

\[
\lim_{r \to \infty} r^{d+\alpha} g(r) = \frac{\alpha \sin(\alpha \pi/2) \Gamma((d + \alpha)/2) \Gamma(\alpha/2)}{2^{2-\alpha} \pi^{1+d/2}} (=: C_{d,\alpha}).
\]

2.1 Resolvent asymptotics

In this subsection, we prove the asymptotic behaviors and the so-called 3G-type inequality for the resolvent density of \( M \). For \( \beta > 0 \), let \( G_\beta(x, y) \) denote the \( \beta \)-resolvent density of \( M \),

\[
G_\beta(x, y) = \int_0^\infty e^{-\beta t} p_t(x, y) \, dt.
\]
Define
\[ w_\beta(r) = \int_0^\infty e^{-\beta t} \frac{1}{t^{d/\alpha}} g \left( \frac{r}{t^{1/\alpha}} \right) dt \quad (r \geq 0) \]
so that \( G_\beta(x, y) = w_\beta(|x - y|) \) by (1).

**Lemma 2** Let \( \beta > 0 \).

(i) The function \( w_\beta \) satisfies
\[ \lim_{r \to \infty} r^{d+\alpha} w_\beta(r) = \beta^{-2} C_{d, \alpha} \] (4)

and
\[ w_\beta(r) \sim \begin{cases} 
\beta^{(d-\alpha)/\alpha} \Gamma((\alpha - d)/d) g(0) & (d < \alpha), \\
\alpha q(0) \log r^{-1} & (d = \alpha), \quad (r \to +0). \\
\alpha r^{\alpha-d} \int_0^\infty s^{d-\alpha-1} g(s) \, ds & (d > \alpha),
\end{cases} \] (5)

(ii) There exists \( C > 0 \) such that for any \( x, y, z \in \mathbb{R}^d \),
\[ G_\beta(x, y) G_\beta(y, z) \leq C G_\beta(x, z) (G_\beta(x, y) + G_\beta(y, z)). \]

**Proof.** (ii) follows by (i) and direct calculation. We now show (i). The relation (4) follows by (3) and the dominated convergence theorem:
\[ r^{d+\alpha} w_\beta(r) = \int_0^\infty e^{-\beta t} \left( \frac{r}{t^{1/\alpha}} \right)^{d+\alpha} g \left( \frac{r}{t^{1/\alpha}} \right) t \, dt \to \beta^{-2} C_{d, \alpha} \quad (r \to \infty). \]

We next show (5). If \( d < \alpha \), then as \( r \to +0 \),
\[ w_\beta(r) \to \int_0^\infty e^{-\beta t} t^{-d/\alpha} g(0) = \beta^{(d-\alpha)/\alpha} \Gamma \left( \frac{\alpha - d}{\alpha} \right) g(0). \]

If \( d > \alpha \), then (3) implies that as \( r \to +0 \),
\[ r^{d-\alpha} w_\beta(r) = \alpha \int_0^\infty e^{-\beta (r/s)\alpha} s^{d-\alpha-1} g(s) \, ds \to \alpha \int_0^\infty s^{d-\alpha-1} g(s) \, ds. \]

We now assume that \( d = \alpha (= 1) \). Let
\[ w_\beta(r) = \int_0^r e^{-\beta t^{-1}} g \left( \frac{r}{t^{1/\alpha}} \right) dt + \int_r^\infty e^{-\beta t^{-1}} g \left( \frac{r}{t^{1/\alpha}} \right) dt = (I) + (II). \]

Then by (3),
\[ (I) \leq \frac{c_1}{r^{2\alpha}} \int_0^r e^{-\beta t} \, dt \leq c_2. \]

Let
\[ (II) = \int_r^\infty e^{-\beta t^{-1}} \left( g \left( \frac{r}{t^{1/\alpha}} \right) - g(0) \right) \, dt + g(0) \int_r^\infty e^{-\beta t^{-1}} \, dt. \]
Then by (2),
\[
\left| \int_{r^n}^\infty e^{-\beta t} t^{-1} \left( g \left( \frac{r}{t^{1/\alpha}} \right) - g(0) \right) \, dt \right| \leq c_3 r^2 \int_{r^n}^\infty e^{-\beta t} t^{-1-2/\alpha} \, dt \leq c_4.
\]
Since
\[
\int_{r^n}^\infty e^{-\beta t} t^{-1} \, dt \sim \alpha \log \left( \frac{1}{r} \right) \quad (r \to +0),
\]
we obtain the desired assertion for \( d = \alpha \).
\( \Box \)

If \( d > \alpha \), then \( M \) is transient and the Green function
\[
G(x, y) = \int_0^\infty p_t(x, y) \, dt
\]
is given by
\[
G(x, y) = \frac{2^{1-\alpha} \Gamma((d-\alpha)/2)}{\pi^{d/2} \Gamma(\alpha/2)} |x - y|^\alpha - d.
\]
We use the notation \( G_0(x, y) = G(x, y) \) for \( d > \alpha \).

2.2 Spectral properties of Schrödinger type operators with the fractional Laplacian

In this subsection, we study spectral properties of a Schrödinger type operator associated with the fractional Laplacian and Green tight Kato class measure. We first introduce Kato class and Green tight measures, and the associated bilinear forms.

**Definition 3**

(i) Let \( \mu \) be a positive Radon measure on \( \mathbb{R}^d \). Then \( \mu \) belongs to the Kato class (\( \mu \in K \) in notation) if
\[
\lim_{\beta \to \infty} \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} G_\beta(x, y) \mu(dy) = 0.
\]
(ii) A measure \( \mu \in K \) is (1-)Green tight (\( \mu \in K_\infty(1) \) in notation) if
\[
\lim_{R \to \infty} \sup_{x \in \mathbb{R}^d} \int_{|y| > R} G_1(x, y) \mu(dy) = 0.
\]

Any Kato class measure with compact support in \( \mathbb{R}^d \) belongs to \( K_\infty(1) \) by definition.

Let \( (\mathcal{E}, \mathcal{F}) \) be a regular Dirichlet form on \( L^2(\mathbb{R}^d) \) associated with \( M \),
\[
\mathcal{F} = \left\{ u \in L^2(\mathbb{R}^d) \mid \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{(u(x) - u(y))^2}{|x - y|^{d+\alpha}} \, dx \, dy < \infty \right\},
\]
\[
\mathcal{E}(u, u) = \frac{1}{2} \mathcal{A}(d, \alpha) \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{(u(x) - u(y))^2}{|x - y|^{d+\alpha}} \, dx \, dy, \quad u \in \mathcal{F}
\]
with
\[
\mathcal{A}(d, \alpha) = \frac{\alpha 2^{d-1} \Gamma((d+\alpha)/2)}{\pi^{d/2} \Gamma(1 - \alpha/2)}
\]
(19 Example 1.4.1)). For $\beta > 0$, we let $\mathcal{E}_\beta(u, u) = \mathcal{E}(u, u) + \beta \int_{\mathbb{R}^d} u^2 \, dx$. Since any function in $\mathcal{F}$ admits a quasi continuous version (19 Theorem 2.1.3)), we may and do assume that if we write $u \in \mathcal{F}$, then $u$ denotes its quasi continuous version.

For $\mu \in \mathcal{K}$ and $\beta > 0$, let $G_\beta \mu(x) = \int_{\mathbb{R}^d} G_\beta(x, y) \mu(dy)$. Then $\|G_\beta \mu\|_\infty < \infty$ by the definition of the Kato class measure. Moreover, the Stollmann-Voigt inequality ([23 Theorem 3.1] and [19 Exercise 6.4.1]) holds:

$$\int_{\mathbb{R}^d} u^2 \, d\mu \leq \|G_\beta \mu\|_\infty \mathcal{E}_\beta(u, u), \ u \in \mathcal{F}. \quad (7)$$

In particular, any function $u \in \mathcal{F}$ belongs to $L^2(\mu)$. We also know by (7) that the embedding

$$I_\mu : (\mathcal{F}, \sqrt{\mathcal{E}_\beta}) \to L^2(\mu), \quad I_\mu f = f, \ \mu\text{-a.e.}$$

is continuous. Even if $\mu$ is a signed measure, we can define the continuous embedding $I_\mu$ as above by replacing $\mu$ with $|\mu|$.

Let $\nu$ be a signed Borel measure on $\mathbb{R}^d$ such that the measures $\nu^+$ and $\nu^-$ in the Jordan decomposition $\nu = \nu^+ - \nu^-$ belong to $\mathcal{K}$. Let $(\mathcal{E}^\nu, \mathcal{F})$ be the quadratic form on $L^2(\mathbb{R}^d)$ defined by

$$\mathcal{E}^\nu(u, u) = \mathcal{E}(u, u) - \int_{\mathbb{R}^d} u^2 \, d\nu, \ u \in \mathcal{F}. \quad (8)$$

Since $\nu$ charges no set of zero capacity ([1 Theorem 3.3]), $(\mathcal{E}^\nu, \mathcal{F})$ is well-defined. Furthermore, it is a lower bounded closed symmetric bilinear form on $L^2(\mathbb{R}^d)$ ([1 Theorem 4.1]) so that the associated self-adjoint operator $\mathcal{H}^\nu$ on $L^2(\mathbb{R}^d)$ is formally written as $\mathcal{H}^\nu = (-\Delta)^{\alpha/2}/2 - \nu$.

Let $\{p_t^\nu\}_{t \geq 0}$ be the strongly continuous symmetric semigroup on $L^2(\mathbb{R}^d)$ generated by $(\mathcal{E}^\nu, \mathcal{F})$. Let $A_t^\nu$ and $A_t^\nu$ be the positive continuous additive functionals in the Revuz correspondence to $\nu^+$ and $\nu^-$, respectively (see [19 p.401] for details). If we define $A_t^\nu = A_t^\nu - A_t^\nu$, then

$$p_t^\nu f(x) = E_x \left[ e^{A_t^\nu} f(X_t) \right], \quad f \in L^2(\mathbb{R}^d) \cap B_b(\mathbb{R}^d).$$

Here $B_b(\mathbb{R}^d)$ is the family of bounded Borel measurable functions on $\mathbb{R}^d$. Moreover, there exists a jointly continuous integral kernel $p_t^\nu(x, y)$ on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ such that $p_t^\nu f(x) = \int_{\mathbb{R}^d} p_t^\nu(x, y) f(y) \, dy$ ([1 Theorem 7.1]).

For $\beta \geq 0$, we define $G_\beta^\nu f(x) = \int_0^\infty e^{-\beta t} p_t^\nu f(x) \, dt$ provided that the right hand side above makes sense. Then

$$G_\beta^\nu f(x) = E_x \left[ \int_0^\infty e^{-\beta t + A_t^\nu} f(X_t) \, dt \right].$$

If we define $G_\beta^\nu(x, y) = \int_0^\infty e^{-\beta t} p_t^\nu(x, y) \, dt$, then $G_\beta^\nu f(x) = \int_{\mathbb{R}^d} G_\beta^\nu(x, y) f(y) \, dy$.

We next discuss the invariance of the essential spectra of $(-\Delta)^{\alpha/2}/2$ under the perturbation with respect to the finite Kato class measure. For a self-adjoint operator $\mathcal{L}$ on $L^2(\mathbb{R}^d)$, let $\sigma_{\text{ess}}(\mathcal{L})$ denote its essential spectrum.
Proposition 4 If $\nu^+$ and $\nu^-$ are finite Kato class measures, then $\sigma_{ess}(\mathcal{H}^\nu) = \sigma_{ess}((-\Delta)^{\alpha/2}/2) = [0, \infty)$. 

We refer to [2] and [10] for $\alpha = 2$. To prove Proposition 4 we follow the argument of [10, Theorem 3.1]. More precisely, we show three lemmas below for the proof of Proposition 4.

Suppose that $\nu$ is a signed Borel measure on $\mathbb{R}^d$ such that $\nu^+$ and $\nu^-$ are positive Radon measures on $\mathbb{R}^d$ charging no set of zero capacity. For $\beta > 0$, let $G_{\beta}(x) = \int_{\mathbb{R}^d} G_{\beta}(x,y) \nu(dy)$. Then for any $f \in L^2(\nu)$, $(f\nu)^+$ and $(f\nu)^-$ are positive Radon measures on $\mathbb{R}^d$ charging no set of zero capacity and

$$G_{\beta}(f\nu)(x) = E_x \left[ \int_0^\infty e^{-\beta t} f(X_t) \, dA_t^\nu \right] = E_x^\beta \left[ A_t^\nu \right].$$

Here $P_x^\beta$ is the law of the $e^{-\beta t}$-subprocess of $\mathbf{M}$ and $\zeta$ is the lifetime of this subprocess. By [16, Theorem 6.7.4], we have $G_{\beta}(f\nu) \in \mathcal{F}$ and $\mathcal{E}_\beta(G_{\beta}(f\nu), v) = \int_{\mathbb{R}^d} f \cdot I_v \nu \, d\nu$ for any $v \in \mathcal{F}$. Hence if we define $K_{\beta}f(x) = G_{\beta}(f\nu)(x)$ for $f \in L^2(\nu)$, then $K_{\beta}f \in \mathcal{F}$.

Lemma 5 If $\nu^+$ and $\nu^-$ belong to the Kato class, then for any $\beta > 0$, $I_v K_{\beta}$ is a bounded linear operator on $L^2(\nu)$. Moreover, there exists $\beta_0 > 0$ such that for any $\beta > \beta_0$, the associated operator norm $\|I_v K_{\beta}\|$ satisfies $\|I_v K_{\beta}\| < 1$.

Proof. We prove this lemma only for $\nu^- = 0$ because a similar calculation applies to the general case. By (7), we have for any $f \in L^2(\nu)$,

$$\int_{\mathbb{R}^d} (I_v K_{\beta} f)^2 \, d\nu = \int_{\mathbb{R}^d} (K_{\beta} f)^2 \, d\nu \leq \|G_{\beta}\|_{\infty} \mathcal{E}_\beta(K_{\beta} f, K_{\beta} f) < \infty$$

so that $I_v K_{\beta} f \in L^2(\nu)$. Combining this with the relation

$$\mathcal{E}_\beta(K_{\beta} f, K_{\beta} f) = \int_{\mathbb{R}^d} (I_v K_{\beta} f) f \, d\nu \leq \sqrt{\int_{\mathbb{R}^d} (I_v K_{\beta} f)^2 \, d\nu} \sqrt{\int_{\mathbb{R}^d} f^2 \, d\nu},$$

we get $\|I_v K_{\beta}\| \leq \|G_{\beta}\|_{\infty}$. Since $\nu$ is a Kato class measure, we have $\|G_{\beta}\|_{\infty} \to 0$ as $\beta \to \infty$ so that the desired assertion holds.

Lemma 5 implies that for any $\beta > \beta_0$, we can define $(1 - I_v K_{\beta})^{-1}$ as a bounded linear operator on $L^2(\nu)$.

Lemma 6 Let $\beta_0$ and $\nu^\pm$ be as in Lemma 5. Then for any $\beta > \beta_0$,

$$G_{\beta}^\nu f - G_{\beta} f = K_{\beta}((1 - I_v K_{\beta})^{-1} I_v G_{\beta} f), \quad f \in L^2(\mathbb{R}^d).$$

Proof. As in Lemma 4 we may assume that $\nu^- = 0$. Fix $\beta > \beta_0$ and $f \in L^2(\mathbb{R}^d)$. Then by Lemma 5 we can define the bounded linear operator $(1 - I_v K_{\beta})^{-1}$ on $L^2(\nu)$ and $u = (1 - I_v K_{\beta})^{-1} I_v G_{\beta} f \in L^2(\nu)$. For any $v \in \mathcal{F}$,

$$\mathcal{E}_\beta^\nu(G_{\beta} f + K_{\beta} u, v) = \mathcal{E}_\beta(G_{\beta} f + K_{\beta} u, v) - \int_{\mathbb{R}^d} I_v(G_{\beta} f + K_{\beta} u) \cdot I_v v \, d\nu$$

$$= \int_{\mathbb{R}^d} f v \, dx + \int_{\mathbb{R}^d} u \cdot I_v v \, d\nu - \int_{\mathbb{R}^d} I_v(G_{\beta} f + K_{\beta} u) \cdot I_v v \, d\nu.$$
Since $I_\nu(G_\beta f + K_\beta u) = u$, we have $E_\nu(G_\beta f + K_\beta u, v) = \int_{\mathbb{R}^d} f v \, dx$ so that the proof is complete.

**Lemma 7** Let $\nu^+$ and $\nu^-$ be finite Kato class measures, and let $\beta_0$ be as in Lemma 5.

(i) For any $\beta > 0$, $K_\beta$ is a compact operator from $L^2(\nu)$ to $L^2(\mathbb{R}^d)$.

(ii) For any $\beta > \beta_0$, $G_\beta^2 - G_\beta$ is a compact operator on $L^2(\mathbb{R}^d)$.

**Proof.** As in the previous lemmas, we may assume that $\nu^- = 0$. We first prove (i). For $n = 1, 2, 3, \ldots$, we define

$$K_\beta^{(n)} f(x) = \int_{|x-y| \geq 1/n} G_\beta(x,y)f(y) \, dy, \quad f \in L^2(\nu).$$

Since (5) yields

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} (G_\beta(x,y)1_{|x-y| \geq 1/n})^2 \, d\nu(dy) \, dx \leq c_1 \int_{\mathbb{R}^d} \frac{dx}{|x|^{2(d+\alpha)}} < \infty,$$

$K_\beta^{(n)}$ is a Hilbert-Schmidt operator so that it is compact from $L^2(\nu)$ to $L^2(\mathbb{R}^d)$ (see, e.g., [21, Corollary 4.6]).

We now assume that $d > \alpha$. Let $\varepsilon \in (0, \alpha/2)$ and

$$k_1^{(n)}(x,y) = \frac{1_{\{|x-y| < 1/n\}}}{|x-y|^{d/2-\varepsilon}}, \quad k_2^{(n)}(x,y) = \frac{1_{\{|x-y| < 1/n\}}}{|x-y|^{d/2-\alpha+\varepsilon}}.$$

By (5), there exists $c_1 > 0$ such that for any $n = 1, 2, 3, \ldots$,

$$G_\beta(x,y)1_{\{|x-y| < 1/n\}} \leq c_1 k_1^{(n)}(x,y)k_2^{(n)}(x,y), \quad x, y \in \mathbb{R}^d$$

and

$$K_\beta f(x) - K_\beta^{(n)} f(x) = \int_{|x-y| < 1/n} G_\beta(x,y)f(y) \, d\nu(y), \quad x \in \mathbb{R}^d.$$

Therefore,

$$\|K_\beta f - K_\beta^{(n)} f\|_{L^2(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^d} \left( \int_{|x-y| < 1/n} G_\beta(x,y)f(y) \, d\nu(y) \right)^2 \, dx \leq c_2 \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} k_1^{(n)}(x,y)^2 f(y)^2 \, d\nu(dy) \right) \left( \int_{\mathbb{R}^d} k_2^{(n)}(x,y)^2 \, d\nu(dy) \right) \, dx \leq c_2 \left\{ \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} k_1^{(n)}(x,y)^2 f(y)^2 \, d\nu(dy) \right) \right\} \sup_{x \in \mathbb{R}^d} \left( \int_{\mathbb{R}^d} k_2^{(n)}(x,y)^2 \, d\nu(dy) \right).$$

(8)

Let $\omega_d = 2\pi^{d/2} \Gamma(d/2)^{-1}$ be the surface area of the unit ball in $\mathbb{R}^d$. Then by the Fubini theorem,

$$\int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} k_1^{(n)}(x,y)^2 f(y)^2 \, d\nu(dy) \right) \, dx = \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} k_1^{(n)}(x,y)^2 \, dx \right) f(y)^2 \, d\nu(dy) = \frac{\omega_d \|f\|_{L^2(\nu)}^2}{2c_1 n^{2\varepsilon}}.$$  

(9)
Since \( \alpha > 2 \varepsilon \), we have by \( \mathbf{[3]} \),
\[
\int_{\mathbb{R}^d} k_2^{(n)}(x, y)^2 \nu(dy) = \int_{|x-y|<1/n} |x-y|^{-d+2\alpha-2\varepsilon} \nu(dy)
\leq \frac{1}{n^{\alpha-2\varepsilon}} \int_{|x-y|<1/n} |x-y|^{-d+\alpha} \nu(dy) \leq \frac{c_3}{n^{\alpha-2\varepsilon}} \int_{\mathbb{R}^d} G_1(x, y) \nu(dy).
\]

Hence
\[
\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} k_2^{(n)}(x, y)^2 \nu(dy) \leq \frac{c_3}{n^{\alpha-2\varepsilon}} \|G_1\nu\|_{\infty}.
\]

Combining this inequality with \( \mathbf{[9]} \), we see by \( \mathbf{[8]} \) that
\[
\|K_\beta - K_\beta^{(n)}\| := \sup_{f \in L^2(\nu), f \neq 0} \frac{\|K_\beta f - K_\beta^{(n)} f\|_{L^2(\nu)}}{\|f\|_{L^2(\nu)}} \leq \frac{c_4}{\varepsilon n^{\alpha/2}} \to 0 \quad (n \to \infty).
\]

For \( d \leq \alpha \), we also have \( \|K_\beta - K_\beta^{(n)}\| \to 0 \) as \( n \to \infty \) by \( \mathbf{[3]} \) and direct calculation. Since \( K_\beta^{(n)} \) is compact, so is \( K_\beta \) by \( \mathbf{[27]} \) Theorem VI.12. This completes the proof of (i).

Since \( G_\beta \) is a bounded linear operator from \( L^2(\mathbb{R}^d) \) to \( (\mathcal{F}, \sqrt{\mathcal{E}_\beta}) \) and \( (1 - I_\nu K)^{-1} I_\nu \) is a bounded linear operator from \( (\mathcal{F}, \sqrt{\mathcal{E}_\beta}) \) to \( L^2(\nu) \), Lemma \( \mathbf{[4]} \) and (i) imply (ii).

\( \square \)

**Proof of Proposition \( \mathbf{[3]} \)** Since \( \sigma_{ess}((-\Delta)^{\alpha/2}) = [0, \infty) \), the assertion follows by Lemma \( \mathbf{[4]} \) and \( \mathbf{[27]} \) Theorem VIII.14.

\( \square \)

**Remark 8** Let \( \nu^+ \) and \( \nu^- \) be Kato class measures such that \( \nu = \nu^+-\nu^- \) forms a signed Borel measure on \( \mathbb{R}^d \). If \( \tilde{\nu}^+ - \tilde{\nu}^- \) is the Jordan decomposition of \( \nu \), then \( \tilde{\nu}^+ \) and \( \tilde{\nu}^- \) are also Kato class measures and \( A_\nu^{\nu^+} - A_\nu^{\nu^-} = A_\nu^{\tilde{\nu}^+} - A_\nu^{\tilde{\nu}^-} \) by the uniqueness of the Revuz correspondence \( \mathbf{[19]} \) Theorem 5.13]. In particular, Proposition \( \mathbf{[3]} \) is true as it is even if \( \nu = \nu^+ - \nu^- \) is not the Jordan decomposition of \( \nu \).

We next discuss the asymptotic behavior of an integral associated with the ground state of \( \mathcal{H}' \). In what follows, we may and do assume that \( \nu \) can be decomposed as \( \nu = \nu^+-\nu^- \) for some \( \nu^+, \nu^- \in \mathcal{K}_\infty(1) \). Let \( \lambda(\nu) \) be the bottom of the \( L^2 \)-spectrum of \( \mathcal{H}' \). Then
\[
\lambda(\nu) = \inf \left\{ \mathcal{E}(u, u) - \int_{\mathbb{R}^d} u^2 \, d\nu \mid u \in C_0^\infty(\mathbb{R}^d), \int_{\mathbb{R}^d} u^2 \, dx = 1 \right\},
\]

where \( C_0^\infty(\mathbb{R}^d) \) is the totality of smooth functions with compact support in \( \mathbb{R}^d \). Moreover, if \( \lambda(\nu) < 0 \), then \( \lambda(\nu) \) is the eigenvalue and the corresponding eigenfunction, which is called the ground state, has a bounded and strictly positive continuous version \( \mathbf{[33]} \) Theorem 2.8 and Section 4). We write \( h \) for this version with \( L^2 \)-normalization \( \|h\|_{L^2(\mathbb{R}^d)} = 1 \).
By the same proof as for [35, Lemma 4.1] and [30, Lemma A.1], we see that for any positive constants $p$ and $p'$ with $p' < 1 < p$, there exist positive constants $c$ and $C$ such that

$$
\frac{c}{|x|^{(d+\alpha)p}} \leq h(x) \leq \frac{C}{|x|^{(d+\alpha)p'}} \quad (|x| \geq 1). \quad (10)
$$

If $\nu^+$ and $\nu^-$ are in addition compactly supported in $\mathbb{R}^d$, then (10) is valid for $p = p' = 1$.

Let $\lambda := \lambda(\nu)$ and

$$
c_* = \frac{C_{d,\alpha}\omega_d}{\alpha(-\lambda)^2} = \frac{\sin(\pi\alpha/2)\Gamma((d + \alpha)/2)\Gamma(\alpha/2)}{(-\lambda)^22^{1-\alpha}\pi\Gamma(d/2)}. \quad (11)
$$

The next lemma determines the asymptotic behavior of the ground state $h$ integrated outside the ball.

**Lemma 9** Suppose that $\lambda < 0$.

(i) For any $x \in \mathbb{R}^d$,

$$
h(x) = \int_{\mathbb{R}^d} G_{-\lambda}(x, y) h(y) \nu(dy).
$$

(ii) If $\nu^+$ and $\nu^-$ are compactly supported in $\mathbb{R}^d$, then $\int_{\mathbb{R}^d} h(y) \nu(dy) > 0$ and

$$
R^\alpha \int_{|y| > R} h(y) \nu(dy) \to c_* \int_{\mathbb{R}^d} h(y) \nu(dy) \quad (R \to \infty). \quad (12)
$$

**Proof.** Let $\nu^+$ and $\nu^-$ belong to $K_\infty(1)$ and $\lambda < 0$. Since [29] Lemma 3.1 (i) and its proof remain valid under the current setting, we have (i) in the same way as for the proof of [26] Lemma 3.1 (iii)].

We assume in addition that $\nu^+$ and $\nu^-$ are compactly supported in $\mathbb{R}^d$. Then by (10) with $p = p' = 1$, $\int_{|y| > R} h(y) \nu(dy)$ is convergent for any $R > 0$. Since (i) yields

$$
\int_{|y| > R} h(y) \nu(dy) = \int_{|y| > R} \left( \int_{\mathbb{R}^d} G_{-\lambda}(y, z) h(z) \nu^+(dz) \right) \nu(dy) \quad (R \to \infty),
$$

we have by (11),

$$
\int_{|y| > R} \left( \int_{\mathbb{R}^d} G_{-\lambda}(y, z) h(z) \nu^+(dz) \right) \nu(dy) \sim \frac{C_{d,\alpha}}{(-\lambda)^2} \int_{|y| > R} \left( \int_{\mathbb{R}^d} h(z) \nu^+(dz) \right) \nu(dy) \\
= \frac{c_*}{R^\alpha} \int_{\mathbb{R}^d} h(z) \nu^+(dz),
$$

whence (12) holds. Moreover, since there exist $c_1 > 0$ and $c_2 > 0$ by (10) such that $c_1 \leq R^\alpha \int_{|y| > R} h(y) \nu(dy) \leq c_2$ for any $R \geq 1$, we obtain $\int_{\mathbb{R}^d} h(y) \nu(dy) > 0$. \qed
For $\alpha = 2$, we proved in [26, Lemma 3.1 (iv)] the assertion corresponding to Lemma 3.1 (ii), but the scaling order there is exponential in contrast with the polynomial order in [12].

Suppose that $\nu^+$ and $\nu^-$ are Kato class measures with compact support in $\mathbb{R}^d$ and $\lambda < 0$. Then for any $\beta > -\lambda$, we have
\[
\inf \left\{ \mathcal{E}_\beta(u, u) - \int_{\mathbb{R}^d} u^2 \, d\nu \mid u \in C_0^\infty(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} u^2 \, dx = 1 \right\} = \beta + \lambda > 0
\]
so that by [34, Lemma 3.5],
\[
\inf \left\{ \mathcal{E}_\beta(u, u) + \int_{\mathbb{R}^d} u^2 \, d\nu^- \mid u \in C_0^\infty(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} u^2 \, dx = 1 \right\} > 1.
\]
Hence by [15, Lemma 3.5 (1), Theorems 3.6 and 5.2] with Lemma 2 (ii), there exist positive constants $c$ and $C$ for any $\beta > -\lambda$ such that
\[
cG_\beta(x, y) \leq G^\nu_\beta(x, y) \leq CG_\beta(x, y) \quad (13)
\]

Let $\lambda_2(\nu)$ be the second bottom of the spectrum for $\mathcal{H}_\nu$,
\[
\lambda_2(\nu) = \inf \left\{ \mathcal{E}(u, u) - \int_{\mathbb{R}^d} u^2 \, d\nu \mid u \in C_0^\infty(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} u^2 \, dx = 1, \int_{\mathbb{R}^d} uh \, dx = 0 \right\}.
\]
If $\lambda < 0$, then Proposition 4 implies that $(\lambda < \lambda_2(\nu) \leq 0$. Then as in [26, Lemma 3.1 (ii)], there exists $C > 0$ such that
\[
|p^\nu_t(x, y) - e^{-\lambda t}h(x)h(y)| \leq Ce^{-\lambda_2 t}, \quad t \geq 1. \quad (14)
\]

### 2.3 Asymptotic behaviors of Feynman-Kac functionals

In this subsection, we prove the asymptotic properties of the Feynman-Kac functionals for a symmetric stable process. Even though our approach is similar to that of [26, Proposition 3.2], we need calculations by taking into account the polynomial decay property of the tail distribution of the symmetric stable process in [33]. Throughout this subsection, we assume that $\nu^+$ and $\nu^-$ are the Kato class measures with compact support in $\mathbb{R}^d$ and that $\lambda < 0$.

Let
\[
q_t(x, y) = p^\nu_t(x, y) - p_t(x, y) - e^{-\lambda t}h(x)h(y) \quad (15)
\]
so that for $R > 0$,
\[
E_x[e^{A^\nu_t} ; |X_t| > R] = P_x(|X_t| > R) + e^{-\lambda t}h(x) \int_{|y| > R} h(y) \, dy + \int_{|y| > R} q_t(x, y) \, dy. \quad (16)
\]
For $r > 0$, let $B(r) = \{ y \in \mathbb{R}^d \mid |y| < r \}$ be an open ball with radius $r$ centered at the origin. Fix $M > 0$ so that the support of $|\nu|$ is included in $B(M)$. For $c > 0$, define
\[
I_c(t, R) = \begin{cases} e^{ct}/R^\alpha & (\lambda_2 < 0), \\ (tP_0(|X_t| > R - M)) \wedge (e^{ct}/R^\alpha) & (\lambda_2 = 0) \end{cases}
\]
and
\[ J(t, R) = e^{-\lambda t}(R - M)^d \int_{1/\alpha}^{\infty} e^{\lambda u} g \left( \frac{R - M}{u} \right) \frac{du}{u^{d+1}}, \]
where \( g \) is the same function as in (1).

**Proposition 10** For any \( c > 0 \) with \( c > -\lambda_2 \), there exists \( C > 0 \) such that for any \( x \in \mathbb{R}^d \), \( t \geq 1 \) and \( R > 2M \),
\[ \left| \int_{|y| > R} q_t(x, y) \, dy \right| \leq C (h(x)P_0(|X_t| > R - M) + I_c(t, R) + h(x)J(t, R)). \]

**Proof.** As for (20) (3.19), we have
\[
\int_{|y| > R} q_t(x, y) \, dy = \int_0^1 \left( \int_{\mathbb{R}^d} p^\nu_s(x, z)P_z(|X_{t-s}| > R) \nu(\,dz\,) \right) \, ds
\]
\[ + \int_1^t \left( \int_{\mathbb{R}^d} (p^\nu_s(x, z) - e^{-\lambda s}h(x)h(z))P_z(|X_{t-s}| > R) \nu(\,dz\,) \right) \, ds
\]
\[ - e^{-\lambda s}h(x) \int_{t-1}^{\infty} e^{\lambda s} \left( \int_{\mathbb{R}^d} h(z)P_z(|X_s| > R) \nu(\,dz\,) \right) \, ds
\]
\[ = (I) + (II) + (III). \tag{17} \]

For any \( s \in [0, t] \) and \( z \in \mathbb{R}^d \),
\[ P_z(|X_{t-s}| > R) \leq P_0(|X_{t-s}| > R - |z|) \leq P_0(|X_t| > R - |z|) \tag{18} \]
by the spatial uniformity and scaling property of the symmetric stable process. Then for any \( \varepsilon > 0 \), we see by (4), (10) (with \( p = p' = 1 \)) and (13) (with \( \beta = -\lambda + \varepsilon \)) that
\[
\int_0^1 \left( \int_{\mathbb{R}^d} p^\nu_s(x, z) \nu(\,dz\,) \right) \, ds \leq e^{-\lambda + \varepsilon} \int_{\mathbb{R}^d} e^{\lambda - \varepsilon s} p^\nu_s(x, z) \, ds \nu(\,dz\,)
\]
\[ \leq e^{-\lambda + \varepsilon} \int_{\mathbb{R}^d} G_{-\lambda + \varepsilon}(x, z) \nu(\,dz\,) \leq c_1 \int_{\mathbb{R}^d} G_{-\lambda + \varepsilon}(x, z) \nu(\,dz\,) \leq c_2 h(x). \]

Hence by (13),
\[ (I) \leq P_0(|X_t| > R - M) \int_0^1 \left( \int_{\mathbb{R}^d} p^\nu_s(x, z) \nu(\,dz\,) \right) \, ds
\]
\[ \leq c_3 P_0(|X_t| > R - M) h(x). \]

Fix \( c > 0 \) with \( c \geq -\lambda_2 \). Then by (13) and Lemma 2 (i),
\[
\int_1^t \left( \int_{\mathbb{R}^d} e^{-\lambda z s} P_z(|X_{t-s}| > R) \nu(\,dz\,) \right) \, ds
\]
\[ \leq |\nu(\mathbb{R}^d)| \int_1^t e^{-\lambda z s} P_0(|X_{t-s}| > R - M) \, ds
\]
\[ \leq |\nu(\mathbb{R}^d)| e^{ct} \int_{|y| > R-M} G_c(0, y) \, dy \leq c_4 e^{ct} \int_{|y| > R-M} \frac{dy}{|y|^{d+\alpha}} \leq \frac{c_5 e^{ct}}{R^\alpha}. \]
If \( \lambda_2 = 0 \), then by (18) again,
\[
\int_1^t \left( \int_{\mathbb{R}^d} P_z(|X_{t-s}| > R) \nu(\mathrm{d}z) \right) \mathrm{d}s \leq c_6 t P_0(|X_t| > R - M).
\]
Hence by (14),
\[
|\langle II \rangle| \leq \int_1^t \left( \int_{\mathbb{R}^d} |p_z^x(x, z) - e^{-\lambda s} h(x) h(z)| P_z(|X_{t-s}| > R) \nu(\mathrm{d}z) \right) \mathrm{d}s
\leq c_7 \int_1^t \left( \int_{\mathbb{R}^d} e^{-\lambda s} P_z(|X_{t-s}| > R) \nu(\mathrm{d}z) \right) \mathrm{d}s \leq c_8 I_8(t, R).
\]
Since (1) yields
\[
\lambda < \lambda_2 \leq c R^2 = 0, \quad \text{then by (18) again,}
\]
\[
\int_1^t e^{\lambda s} P_0(|X_s| > R) \mathrm{d}s = \frac{e^{\lambda t}}{-\lambda} P_0(|X_t| > R) + \frac{\omega_d}{-\lambda} R^d \int_t^\infty e^{\lambda u} g \left( \frac{R}{u} \right) \frac{\mathrm{d}u}{u^{d+1}}.
\]
We also see by (18) that
\[
\int_{t-1}^\infty e^{\lambda s} P_0(|X_s| > R - M) \mathrm{d}s = \int_t^\infty e^{\lambda (s-1)} P_0(|X_{s-1}| > R - M) \mathrm{d}s
\leq e^{-\lambda} \int_t^\infty e^{\lambda s} P_0(|X_s| > R - M) \mathrm{d}s.
\]
Therefore,
\[
|\langle III \rangle| \leq e^{-\lambda t} h(x) \int_{t-1}^\infty e^{\lambda s} P_0(|X_s| > R - M) \mathrm{d}s \left( \int_{\mathbb{R}^d} h(z) \nu(\mathrm{d}z) \right)
\leq c_9 e^{-\lambda t} h(x) \int_t^\infty e^{\lambda s} P_0(|X_s| > R - M) \mathrm{d}s
\leq c_{10} h(x) \left( P_0(|X_t| > R - M) + J(t, R) \right),
\]
which completes the proof. \( \square \)

**Remark 11** Let us take \( R = 0 \) in (17). Then by (14) and (15), there exists \( c > 0 \) such that for any \( f \in B_b(\mathbb{R}^d) \) and \( t \geq 1 \),
\[
\sup_{x \in \mathbb{R}^d} \left| e^{\lambda t} E_x \left[ e^{A_t} f(X_t) \right] - h(x) \int_{\mathbb{R}^d} f(y) h(y) \mathrm{d}y \right| \leq c \| f \|_\infty e^{\lambda t} \left( t \lor e^{-\lambda_2(\nu) t} \right).
\]
The right hand side above goes to 0 as \( t \to \infty \) because \( \lambda < \lambda_2(\nu) \leq 0 \). This result extends the assertion in [20, Remark 3.4] for the Brownian motion to the symmetric stable process, and provides a convergence rate bound in [35, (1.3)].
Let \( R(t) \) be a positive measurable function on \((0, \infty)\) such that \( R(t) \to \infty \) as \( t \to \infty \). Then by Lemma 9 (ii),

\[
\eta(t) := e^{-\lambda t} \int_{|y| > R(t)} h(y) \, dy \sim c_* \frac{e^{-\lambda t}}{R(t)^{1/A}}
\]

with

\[
c_* = c_* \int_{\mathbb{R}^d} h(y) \, \nu(dy).
\]

The next lemma reveals the exact asymptotic behavior of the Feynman-Kac semigroup conditioned that the particle at time \( t \) sits outside the ball with radius \( R(t) \).

**Lemma 12** Let \( K \) be a compact set in \( \mathbb{R}^d \). If \( R(t)/t^{1/\alpha} \to \infty \) as \( t \to \infty \), then there exist positive constants \( c_1, c_2 \) and \( T \) such that for any \( x \in K, t \geq T \) and \( s \in [0, t - 1] \),

\[
E_x \left[ e^{A_{t-s};X_t} | X_{t-s} > R(t) \right] = e^{\lambda s} h(x) \eta(t)(1 + \theta_{s,x}(t))
\]

with

\[
|\theta_{s,x}(t)| \leq c_1 e^{-c_2(t-s)}.
\]

Here \( c_1 \) and \( c_2 \) can be independent of the choice of the function \( R(t) \). In particular,

\[
\lim_{t \to \infty} \sup_{x \in K} \left| \frac{1}{h(x) \eta(t)} E_x \left[ e^{A_{t-s};X_t} | X_t > R(t) \right] - 1 \right| = 0.
\]

**Proof.** Take \( M > 0 \) so that \( B(M) \) includes both \( K \) and the support of \(|\nu|\). Then for any \( s \in [0, t-1] \),

\[
(R(t) - M)/(t-s)^{1/\alpha} \geq (R(t) - M)/t^{1/\alpha}
\]

and the right hand above goes to \( \infty \) as \( t \to \infty \). Hence by 1 and 3, there exist \( c_1 > 0, c_2 > 0 \) and \( T_1 > 1 \) such that for any \( x \in K \) and \( t \geq T_1 \) and \( s \in [0, t - 1] \),

\[
P_x(|X_{t-s}| > R(t)) \leq P_0(|X_{t-s}| > R(t) - M) = \omega_d \int_0^\infty \frac{R(t)-M}{(t-s)^{1/\alpha}} g(u) u^{d-1} \, du
\]

\[
\leq c_1 \int_0^\infty \frac{du}{u^{\alpha+1}} \leq c_2 \frac{t-s}{R(t)^{1/\alpha}} = c_2 e^{\lambda(t-s)}(t-s) e^{-\lambda(t-s)} R(t)^{1/\alpha}
\]

For any \( c > 0 \),

\[
I_c(t-s, R(t)) \leq \frac{e^{c(t-s)}}{R(t)^{1/\alpha}} = \frac{e^{(c+\lambda)(t-s)}}{R(t)^{1/\alpha}}
\]

By \( 3 \), there exists \( T_2 > 1 \) such that for all \( t \geq T_2 \),

\[
\int_{t^{1/\alpha}}^{R(t)} e^{\lambda u^{\alpha}} g \left( \frac{R(t) - M}{u} \right) \frac{du}{u^{\alpha+1}} \leq \frac{c_3}{R(t)^{d+\alpha}} \int_{t^{1/\alpha}}^\infty e^{\lambda u^{\alpha}} u^{\alpha-1} \, du \leq \frac{c_4 \lambda^{\alpha}}{R(t)^{d+\alpha}}
\]
Hence
\[
\int_{R(t)}^{\infty} e^{\lambda u^\alpha} g \left( \frac{R(t) - M}{u} \right) \frac{du}{u^{d+1}} \leq c_8 \int_{R(t)}^{\infty} e^{\lambda u^\alpha} \frac{du}{u^{d+1}} \leq \frac{c_9 e^{\lambda R(t)^\alpha}}{R(t)^{d+\alpha}} \leq \frac{c_7 e^{\lambda t}}{R(t)^{d+\alpha}}.
\]

Note that all the constants \( c_i \) can be independent of the choice of the function \( R(t) \).

Fix \( c \in (-\lambda_2(\nu), -\lambda) \). Then by combining (16) and Proposition 10 with (22), there exist positive constants \( c_9, c_{10} \) and \( c_{11} \), and \( T_\alpha \geq 1 \) such that for any \( x \in K, t \geq T \) and \( s \in [0, t - 1] \),
\[
\begin{align*}
&\left| E_x \left[ e^{A_{t-s}^\nu} | X_{t-s} > R(t) \right] - e^{\lambda t} \eta(t) h(x) \right| \\
&\leq P_x (|X_{t-s}| > R(t)) + \left| \int_{|y| > R(t)} q_{t-s}(x, y) dy \right| \\
&\leq \frac{c_9 e^{-\lambda(t-s)}}{R(t)^\alpha} \left( e^{\lambda(t-s)}(t-s) + e^{(c+\lambda)(t-s)} + e^{\lambda (t-s)} \right) \leq c_{10} e^{\lambda_s} e^{-c_{11}(t-s)} \frac{e^{-\lambda t}}{R(t)^\alpha}.
\end{align*}
\]

Then by (24), the proof is complete. \( \square \)

Recall that by [28] Lemma 3.4, we have for any \( \mu \in \mathcal{K}_\infty(1) \),
\[
\sup_{x \in \mathbb{R}^d} E_x \left[ \int_0^{\infty} e^{2\lambda u + A_{t-s}^\nu} dA_s^\mu \right] < \infty.
\] (25)

The next two lemmas will be used later for the second moment estimates of the expected population for a branching symmetric stable process.

**Lemma 13** Let \( K \) be a compact set in \( \mathbb{R}^d \) and \( \mu \) a Kato class measure with compact support in \( \mathbb{R}^d \). If \( R(t)/t^{1/\alpha} \to \infty \) as \( t \to \infty \), then there exist \( C > 0 \) and \( T > 0 \) such that for any \( t \geq T \),
\[
\sup_{x \in K} E_x \left[ \int_0^t e^{A_{t-s}^\nu} E_{X_s} \left[ e^{A_{t-s}^\nu} | X_{t-s} > R(t) \right]^2 dA_s^\mu \right] \leq C \eta(t)^2.
\]

**Proof.** Fix \( x \in K. \) For \( t \geq 1 \),
\[
\begin{align*}
E_x \left[ \int_0^t e^{A_{t-s}^\nu} E_{X_s} \left[ e^{A_{t-s}^\nu} | X_{t-s} > R(t) \right]^2 dA_s^\mu \right] \\
= E_x \left[ \int_0^{t-1} e^{A_{t-s}^\nu} E_{X_s} \left[ e^{A_{t-s}^\nu} | X_{t-s} > R(t) \right]^2 dA_s^\mu \right] \\
+ E_x \left[ \int_{t-1}^t e^{A_{t-s}^\nu} E_{X_s} \left[ e^{A_{t-s}^\nu} | X_{t-s} > R(t) \right]^2 dA_s^\mu \right] = (IV) + (V).
\end{align*}
\] (26)
If \(0 \leq s \leq t - 1\), then Lemma 12 yields for any \(z \in \text{supp}[\mu]\)

\[
E_z \left[ e^{A_{t-s}^\ast} ; |X_t - s| > R(t) \right] \leq c_1 e^{\lambda s} \eta(t)
\]

so that by (25),

\[
(IV) \leq c_2 \eta(t)^2 \sup_{x \in \mathbb{R}^d} E_x \left[ \int_0^\infty e^{2\lambda s + A_{t-s}^\ast} \, dA_s^\mu \right] \leq c_3 \eta(t)^2. \tag{27}
\]

By [1, Theorem 6.1 (i)] and (18), there exists \(c_4 > 0\) such that for any \(M > 0\), \(R > M\), \(t \in [0, 1]\) and \(x \in \mathbb{R}^d\) with \(|x| \leq M\),

\[
E_x \left[ e^{A_t^\ast} ; |X_t| > R \right] \leq c_4 P_0(|X_1| > R - M).
\]

Hence (3) implies that for any \(z \in \text{supp}[\mu]\), all sufficiently large \(t \geq 1\) and any \(s \in [t - 1, t]\),

\[
E_z \left[ e^{A_{t-s}^\ast} ; |X_{t-s}| > R(t) \right] \leq c_5 P_0(|X_1| > R(t) - M) \leq \frac{c_6}{R(t)^\alpha}. \tag{28}
\]

Since (25) yields

\[
E_x \left[ \int_{t-1}^t e^{A_s^\nu} \, dA_s^\mu \right] \leq e^{-2\lambda t} \sup_{x \in \mathbb{R}^d} E_x \left[ \int_0^\infty e^{2\lambda s + A_{t-s}^\ast} \, dA_s^\mu \right] \leq c_7 e^{-2\lambda t},
\]

we have by (20),

\[
(V) \leq \frac{c_8}{R(t)^{2\alpha}} E_x \left[ \int_{t-1}^t e^{A_s^\nu} \, dA_s^\mu \right] \leq \frac{c_9 e^{-2\lambda t}}{R(t)^{2\alpha}} \leq c_{10} \eta(t)^2.
\]

Combining this with (26) and (27), we complete the proof.

\[
\eta(t) \to c \kappa^{-1} (t \to \infty). \tag{29}
\]

**Lemma 14** Let \(K \subset \mathbb{R}^d\) be a compact set.

(i) For any \(\kappa > 0\),

\[
\lim_{t \to \infty} \sup_{x \in K} \left| \frac{\kappa}{h(x)} E_x \left[ e^{A_t^\ast} ; |X_t| > R^\kappa(t) \right] - c_4 \right| = 0.
\]

(ii) Let \(\mu\) be a Kato class measure with compact support in \(\mathbb{R}^d\). Then

\[
\lim_{\kappa \to \infty} \lim_{t \to \infty} \sup_{x \in K} \kappa E_x \left[ \int_{t-1}^t e^{A_s^\nu} E_x \left[ e^{A_{t-s}^\ast} ; |X_{t-s}| > R^\kappa(t) \right]^2 \, dA_s^\mu \right] = 0.
\]
Proof. (i) follows by Lemma \[(12)\] and \[(28)\]. We now show (ii). By Lemma \[(12)\] and \[(28)\], there exist \(c_1 > 0\) and \(T = T(\kappa) > 1\) for any \(\kappa > 0\) such that, for any \(z \in \text{supp}[\mu], t \geq T\) and \(s \in [0, t],\)

\[E_z \left[ e^{A_y }; \left| X_{t-s} \right| > R^\kappa(t) \right] \leq c_1 e^{\lambda x \eta(t)}.\]

Hence by \[(25)\], there exists \(c_2 > 0\) such that for all \(t \geq T,\)

\[\sup_{x \in K} \kappa E_x \left[ \int_0^t e^{A_y } E_{X_x} \left[ e^{A_y }; \left| X_{t-s} \right| > R^\kappa(t) \right]^2 dA_x \right] \leq \frac{c_2^2}{\kappa}.\]

Then by \[(24)\],

\[\lim_{t \to \infty} \sup_{x \in \mathbb{R}^d} \kappa E_x \left[ \int_0^t e^{A_y } E_{X_x} \left[ e^{A_y }; \left| X_{t-s} \right| > R^\kappa(t) \right]^2 dA_x \right] \leq \frac{c_2^2}{\kappa}.\]

The right hand side above goes to 0 as \(\kappa \to \infty\).

\[\square\]

3 Maximal displacement of branching symmetric stable processes

In this section, we first introduce a model of branching symmetric stable processes. We then present our main results with examples.

3.1 Branching symmetric stable processes

For \(\alpha \in (0, 2)\), let \(\mathbf{M} = (\Omega, \mathcal{F}, \{X_t\}_{t \geq 0}, \{P_x\}_{x \in \mathbb{R}^d}, \{\mathcal{F}_t\}_{t \geq 0})\) be a symmetric \(\alpha\)-stable process on \(\mathbb{R}^d\), where \(\{\mathcal{F}_t\}_{t \geq 0}\) is the minimal augmented admissible filtration.

Let us formulate the model of a branching symmetric \(\alpha\)-stable process on \(\mathbb{R}^d\) by following \[\cite{28}\] and references therein. We first define the set \(\mathbf{X}\) as follows: let \((\mathbb{R}^d)^{(0)} = \{\Delta\}\) and \((\mathbb{R}^d)^{(1)} = \mathbb{R}^d\). Let \(n \geq 2\). For \(x = (x_1, \ldots, x^n)\) and \(y = (y^1, \ldots, y^n)\) in \((\mathbb{R}^d)^n\), we write \(x \sim y\) if there exists a permutation \(\sigma\) of \(\{1, 2, \ldots, n\}\) such that \(y^i = x_i^{\sigma(i)}\) for any \(i = 1, \ldots, n\). Using this equivalence relation, we define \((\mathbb{R}^d)^{(n)} = (\mathbb{R}^d)^n/\sim\) for \(n \geq 2\) and \(\mathbf{X} = \bigcup_{n=0}^{\infty} (\mathbb{R}^d)^{(n)}\).

Let \(\mathbf{p} = \{p_n(x)\}_{n=0}^{\infty}\) be a probability function on \(\mathbb{R}^d\), \(0 \leq p_n(x) \leq 1\) and \(\sum_{n=0}^{\infty} p_n(x) = 1\) for any \(x \in \mathbb{R}^d\). We assume that \(p_0(x) + p_1(x) \neq 1\) to avoid the triviality. Fix \(\mu \in \mathcal{K}\) and \(\mathbf{p}\). We next introduce a particle system as follows: a particle starts from \(x \in \mathbb{R}^d\) at time \(t = 0\) and moves by following the distribution \(P_x\) until the random time \(U\). Here the distribution of \(U\) is given by

\[P_x(U > t \mid \mathcal{F}_\infty) = e^{-A_x^t} (t > 0).\]

At time \(U\), this particle dies leaving no offspring with probability \(p_0(X_{U-})\), or splits into \(n\) particles with probability \(p_n(X_{U-})\) for \(n \geq 1\). For the latter case, these \(n\) particles then move by following the distribution \(P_{X_{U-}}\) and repeat the
same procedure independently. If there exist \( n \) particles alive at time \( t \), then the positions of these particles determine a point in \((\mathbb{R}^d)^{(n)}\). Let \( X_t \) denote such a point,

\[ X_t = (X_t^{(1)}, \ldots, X_t^{(n)}) \in (\mathbb{R}^d)^{(n)}. \]

In this way, we can define the model of a branching symmetric \( \alpha \)-stable process \( \overline{M} = (\{X_t\}_{t \geq 0}, \{P_x\}_{x \in \mathbb{R}^d}) \) on \( \mathbb{R}^d \) with branching rate \( \mu \) and branching mechanism \( p \). Note that for \( x \in \mathbb{R}^d \), \( P_x \) denotes the law of the process such that the initial state is a single particle at \( x \).

Let \( S \) be the first splitting time of \( \overline{M} \) given by

\[ P_x(S > t | \sigma(X)) = P_x(U > t | \mathcal{F}_\infty) = e^{-A \mu t} \quad (t > 0). \]

Let \( Z_t \) be the population at time \( t \) and \( e_0 := \inf\{t > 0 \mid Z_t = 0\} \) the extinction time of \( \overline{M} \). Note that \( Z_t = 0 \) for all \( t \geq e_0 \). For \( f \in B_b(\mathbb{R}^d) \), we define

\[ Z_t(f) = \begin{cases} \sum_{k=1}^{Z_t} f(X_t^{(k)}) \quad (t < e_0), \\ 0 \quad (t \geq e_0). \end{cases} \]

For \( A \in B(\mathbb{R}^d) \), let \( Z_t(A) := Z_t(1_A) \) denote the population on \( A \) at time \( t \).

Let \( Q(x) = \sum_{n=0}^{\infty} np_n(x) \) and \( \nu_Q(dx) = Q(x)\mu(dx) \). Let \( R(x) = \sum_{n=1}^{\infty} n(n-1)p_n(x) \) and \( \nu_R(dx) = R(x)\mu(dx) \). We here recall the next lemma on the first and second moments of \( Z_t(f) \):

**Lemma 15** ([26, Lemma 2.2] and [28, Lemma 3.3]) Let \( \mu \in \mathcal{K} \) and \( f \in B_b(\mathbb{R}^d) \).

(i) If \( \nu_Q \in \mathcal{K} \), then

\[ E_x[Z_t(f)] = E_x\left[e^{A_t^{(Q-1)\mu}}f(X_t)\right]. \]

(ii) If \( \nu_R \in \mathcal{K} \), then

\[ E_x[Z_t(f)^2] = E_x\left[e^{A_t^{(Q-1)\mu}}f(X_t)^2\right] + E_x\left[\int_0^t e^{A_s^{(Q-1)\mu}}E_{X_s}\left[e^{A_{t-s}^{(Q-1)\mu}}f(X_{t-s})\right]^2 \, dA_s^\mu\right]. \]

### 3.2 Weak convergence and tail asymptotics

Let \( \overline{M} = (\{X_t\}_{t \geq 0}, \{P_x\}_{x \in \mathbb{R}^d}) \) be a branching symmetric \( \alpha \)-stable process on \( \mathbb{R}^d \) with branching rate \( \mu \in \mathcal{K} \) and branching mechanism \( p \). We impose the next assumption on \( \mu \) and \( p \):

**Assumption 16** (i) The support of \( \mu \) is compact in \( \mathbb{R}^d \).
(ii) \( \nu_R \in \mathcal{K} \).
(iii) \( \lambda((Q-1)\mu) < 0 \).
Let $\lambda = \lambda((Q - 1)\mu)$. Under Assumption 14 $\lambda$ is the principal eigenvalue of the operator $\mathcal{H}(Q-1)\mu$ on $L^2(\mathbb{R}^d)$ as mentioned in Subsection 2.2. Let $h$ denote the bounded and strictly positive continuous version of the corresponding ground state with $L^2$-normalization. We define $M_t = e^{\lambda t}Z_t(h)$. Then by [28, Lemma 3.4], \( \{M_t\}_{t \geq 0} \) is a nonnegative square integrable $P_x$-martingale so that $E_x[M_t] = h(x)$ and $M_\infty = \lim_{t \to \infty} M_t$ exists $P_x$-a.s. with $P_x(M_\infty > 0) > 0$.

Let $L_t$ denote the maximal Euclidean norm of particles alive at time $t$:

$$L_t = \begin{cases} \max_{1 \leq k \leq Z_t} |X_t^{(k)}| & (t < e_0), \\ 0 & (t \geq e_0). \end{cases}$$

Since each particle follows the law of the symmetric stable process and $P_x(Z_t < \infty) = 1$ for any $t > 0$, $L_t$ is well-defined and $P_x(L_t < \infty) = 1$ for any $t \geq 0$.

In what follows, let $c_\ast$ denote the positive constant given by (21) with $\nu = (Q - 1)\mu$. For $\kappa > 0$, let $R^{\kappa}(t) = (e^{-\lambda t}\kappa)^{1/\alpha}$. We then have

**Theorem 17** For any $\kappa > 0$,

$$\lim_{t \to \infty} P_x(L_t > R^{\kappa}(t)) = E_x[1 - \exp(-\kappa^{-1}c_\ast M_\infty)].$$

Theorem 17 extends [26, Theorem 2.4] for the branching Brownian motion to that for the branching symmetric stable process. Theorem 17 implies that $L_t$ grows exponentially fast in contrast with the linear growth for the branching Brownian motion.

Since

$$P_x(L_t > R^{\kappa}(t), e_0 < \infty) \leq P_x(t < e_0 < \infty) \to 0 \quad (t \to \infty)$$

and \( \{e_0 < \infty\} \subset \{M_\infty = 0\} \), we obtain

$$\lim_{t \to \infty} P_x(L_t > R^{\kappa}(t) \mid e_0 = \infty) = E_x[1 - \exp(-\kappa^{-1}c_\ast M_\infty) \mid e_0 = \infty].$$

If we let $Y_t = e^{\lambda t/\alpha}L_t$, then the equality above reads

$$\lim_{t \to \infty} P_x(Y_t \leq \kappa \mid e_0 = \infty) = E_x[\exp(-\kappa^{-\alpha}c_\ast M_\infty) \mid e_0 = \infty]. \quad (30)$$

Moreover, if $d = 1$ and $1 < \alpha < 2$, then [28, Remark 3.14] yields $\{e_0 = \infty\} = \{M_\infty > 0\}$, $P_x$-a.s. so that

$$\lim_{t \to \infty} P_x(Y_t \leq \kappa \mid M_\infty > 0) = E_x[\exp(-\kappa^{-\alpha}c_\ast M_\infty) \mid M_\infty > 0]. \quad (31)$$

Hence the distribution of $Y_t$ under $P_x(\cdot \mid M_\infty > 0)$ is weakly convergent to the average over the Fréchet distributions with parameter $\alpha$ scaled by $c_\ast M_\infty$ (see, e.g., [31, Theorem 1.12] and references therein for the terminologies about external distributions). On the other hand, if $d > \alpha$, then $M$ is transient so that $P_x(\{e_0 = \infty\} \cap \{M_\infty = 0\}) > 0$. In particular, we do not know the validity of (31).

For $R > 0$, let $Z_t^R = Z_t(\partial(B(R)))$. The next theorem determines the long time asymptotic behavior of the tail distribution of $L_t$:

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**Theorem 18** Let $a$ be a positive measurable function on $(0, \infty)$ such that $a(t) \to \infty$ as $t \to \infty$, and let $R(t) = (e^{-\lambda t}a(t))^{1/\alpha}$.

(i) The next equality holds locally uniformly in $x \in \mathbb{R}^d$.

$$\lim_{t \to \infty} \text{P}_x(L_t > R(t)) = 1.$$ 

(ii) For each $k \in \mathbb{N}$, the next equality holds locally uniformly in $x \in \mathbb{R}^d$.

$$\lim_{t \to \infty} \text{P}_x(Z_{R(t)}^t = k \mid L_t > R(t)) = \begin{cases} 1 & (k = 1), \\ 0 & (k \geq 2). \end{cases}$$

The statement of this theorem is similar to those of [26, Theorem 2.5 and Corollary 2.6]; however, the tail distribution of the maximal displacement for the branching symmetric stable process is completely different from that for the branching Brownian motion (see [26, (2.16), (2.17)]). In fact, combining Theorem 18 with Lemmas 12 and 15, and (20), we have as $t \to \infty$,

$$\text{P}_x(L_t > R(t)) \sim \text{E}_x[Z_{R(t)}^t] \sim c^* h(x) a(t). \quad (32)$$

We omit the proof of Theorem 18 because it is identical with those of [26, Theorem 2.5 and Corollary 2.6], respectively.

### 3.3 Examples

In this subsection, we present three examples to which the results in the previous subsection are applicable.

**Example 19** Let $d = 1$ and $\alpha \in (1, 2)$. Then $\delta_0$, the Dirac measure at the origin, belongs to the Kato class. Let $\overline{M}$ be a branching symmetric $\alpha$-stable process on $\mathbb{R}$ with branching rate $\mu = c\delta_0$ ($c > 0$) and branching mechanism $p = \{p_n(x)\}_{n=0}^\infty$.

We assume that $p_0(0) + p_2(0) = 1$ for simplicity. Then $\text{P}_x(c_0 = \infty) > 0$ if and only if $p_2(0) > 1/2$ ([28, Example 4.4]). In particular, if $m = 2p_2(0) > 1$, then

$$\lambda := \lambda((Q - 1)\mu) = \left\{ \frac{c(m - 1)^{1/\alpha}}{\alpha \sin(\pi/\alpha)} \right\}^{\alpha/(\alpha - 1)}$$

and

$$c^* = C_{1,\alpha}^* \omega_1 \sqrt{c(m - 1)} \int_{\mathbb{R}} h(y) \delta_0(dy) = \frac{2c(m - 1)c_1\alpha}{\alpha(-\lambda)^2} h(0).$$

With these $\lambda$ and $c^*$, (31) and (32) hold.

**Example 20** Let $1 < \alpha < 2$ and $d > \alpha$. For $r > 0$, let $\delta_r$ be the surface measure on $\partial B(r) = \{y \in \mathbb{R}^d \mid |y| = r\}$. Let $\overline{M}$ be a branching symmetric $\alpha$-stable process on $\mathbb{R}^d$ with branching rate $\mu = c\delta_r$ ($c > 0$) and branching mechanism
\( p = \{ p_n(x) \}_{n=0}^\infty \). We assume that \( p_0 \equiv p_0(x) \), \( p_2 \equiv p_2(x) \) and \( p_0 + p_2 = 1 \). Then \( P_x(c_0 = \infty) > 0 \) holds irrelevantly of the value of \( p_2 \) because \( M \) is transient. If we let \( m = 2p_2 \) and \( \lambda := \lambda((Q-1)\mu) \), then \( \lambda < 0 \) if and only if \( p_2 > 1/2 \) and 

\[
 r > \left\{ \frac{\sqrt{\pi} \Gamma((d + \alpha - 2)/2) \Gamma(\alpha/2)}{c(m - 1) \Gamma((d - \alpha)/2) \Gamma((\alpha - 1)/2)} \right\}^{1/(\alpha - 1)}
\]

(see [28, Example 4.7] and references therein). Under this condition, (30) and (32) hold.

Assume that \( 1 < \alpha < 2 \) and \( d > \alpha \). Let \( r > 0 \) and \( \mu_r(dx) = 1_{B(r)}(x)dx \). To present the last example, we estimate

\[
 \hat{\lambda}_\beta = \inf \left\{ E(u, u) \mid u \in F, \beta \int_{B(r)} u^2 dx = 1 \right\} \quad (\beta > 0),
\]

which is the bottom of the spectrum for the time changed Dirichlet form of \((\mathcal{E}, F)\) with respect to the measure \( \beta \mu_r \) (see, e.g., [31, Section 3] for details).

Let \( \hat{\lambda} = \hat{\lambda}_1 \), and let \( v(x) = \int_{B(r)} G(x, y) \) be the 0-potential of the measure \( \mu_r \). Then

\[
 \hat{\lambda} \leq \frac{1}{\|v\|^2_{L^2(B(r))}} \mathcal{E}(v, v) = \frac{1}{\|v\|^2_{L^2(B(r))}} \int_{B(r)} v dx \leq \frac{1}{\inf_{y \in B(r)} v(y)}. \tag{33}
\]

Let

\[
 I_{d,\alpha} = \alpha \int_0^1 u^{d-1} (1 + u)^{\alpha-d} du, \quad \kappa_{d,\alpha} = \frac{\alpha \Gamma(d/2) \Gamma(\alpha/2)}{2^{\alpha-d} \Gamma(d - \alpha/2)}.
\]

Recall that \( \omega_d = 2\pi^{d/2} \Gamma(d/2)^{-1} \) is the surface area of the unit ball in \( \mathbb{R}^d \). Then for any \( y \in B(r) \), \( |y - z| \leq |y| + |z| \leq r + |z| \) and thus

\[
 \int_{B(r)} \frac{dz}{|y - z|^{\alpha - d}} \geq \int_{B(r)} \frac{dz}{(r + |z|)^{\alpha - d}} = \omega_d \int_0^r \frac{s^{d-1}}{(r + s)^{\alpha - d}} ds = \frac{\omega_d I_{d,\alpha}}{\alpha} r^\alpha.
\]

Since this inequality and (11) yield

\[
 \inf_{y \in B(r)} v(y) \geq \frac{I_{d,\alpha} r^\alpha}{\kappa_{d,\alpha}},
\]

we get by (33),

\[
 \hat{\lambda} \leq \frac{\kappa_{d,\alpha}}{I_{d,\alpha} r^\alpha}.
\]

We also know by [31, Example 3.10] that

\[
 \hat{\lambda} \geq \frac{\kappa_{d,\alpha}}{r^\alpha}.
\]

Noting that \( \hat{\lambda}_\beta = \hat{\lambda}/\beta \), we further obtain

\[
 \frac{\kappa_{d,\alpha}}{\beta r^\alpha} \leq \hat{\lambda}_\beta \leq \frac{\kappa_{d,\alpha}}{\beta I_{d,\alpha} r^\alpha}. \tag{34}
\]
We here note that the lower bound of $\bar{\lambda}$ in [31, Example 3.10] is incorrect because of the computation error.

Let $\lambda_\beta = \lambda(\beta \mu_r)$. Then by [36, Lemma 2.2], $\lambda_\beta < 0$ if and only if $\bar{\lambda}_\beta < 1$. Hence by [34], we obtain

$$ r > \left( \frac{\kappa_d}{\beta I_{d,\alpha}} \right)^{1/\alpha} \Rightarrow \lambda_\beta < 0, \quad r \leq \left( \frac{\kappa_d}{\beta} \right)^{1/\alpha} \Rightarrow \lambda_\beta \geq 0. \quad (35) $$

We do not know if $\lambda_\beta$ is negative or not for $(\kappa_{d,\alpha}/\beta)^{1/\alpha} < r \leq (\kappa_{d,\alpha}/(\beta I_{d,\alpha}))^{1/\alpha}$.

**Example 21** Let $1 < \alpha < 2$ and $d > \alpha$. For $r > 0$, let $\mu_r(dx) = 1_{B(r)}(x) dx$. Let $\bar{M}$ be a branching symmetric $\alpha$-stable process on $\mathbb{R}^d$ with branching rate $\mu = c \mu_r$ ($c > 0$) and branching mechanism $p = \{p_n(x)\}_{n=0}^\infty$. We assume that $p_0 \equiv p_0(x)$, $p_2 \equiv p_2(x)$ and $p_0 + p_2 = 1$. Then $P_x(e_0 = \infty) > 0$ holds irreducibly of the value of $p_2$.

Let $m = 2p_2$ and $\lambda := \lambda((Q-1)\mu)$. Assume that $p_2 > 1/2$. Then by (35), we have the following: if $r > \{\kappa_{d,\alpha}/(c(m-1)I_{d,\alpha})\}^{1/\alpha}$, then $\lambda < 0$ so that (33) and (32) hold. On the other hand, if $r \leq \{\kappa_{d,\alpha}/(c(m-1))\}^{1/\alpha}$, then we have $\lambda = 0$ so that Assumption 10 fails.

### 4 Proof of Theorem 17

Once we obtain the asymptotic behaviors of the Feynman-Kac functionals as in Subsection 2.3, we can establish Theorem 17 along the way as for the proof of [26, Theorem 2.4].

Let $\bar{M} = \{(X_t)_{t \geq 0}, \{P_x\}_{x \in \mathbb{R}}\}$ be a branching symmetric $\alpha$-stable process on $\mathbb{R}^d$ with branching rate $\mu$ and branching mechanism $p$ so that Assumption 10 is fulfilled. Let $\nu = (Q-1)\mu$ and $\lambda = \lambda((Q-1)\mu)$, and let $c_\ast$ be the corresponding value in (21). Recall that for $\kappa > 0$, $R^\kappa(t) = (e^{-\lambda t})^{1/\alpha}$.

**Lemma 22** Let $K$ be a compact set in $\mathbb{R}^d$. Then

$$ \lim_{\kappa \to \infty} \limsup_{t \to \infty} \sup_{x \in K} \frac{\kappa}{h(x)} P_x(L_t > R^\kappa(t) - c_\ast) = 0 $$

and for any $c > 0$,

$$ \lim_{\gamma \to +0} \limsup_{t \to \infty} \sup_{x \in K} \frac{1}{\gamma h(x)} E_x \left[ 1 - e^{-\gamma e M_t} \right] - c = 0. $$

We omit the proof of Lemma 22 by using Lemma 13 we can show Lemma 22 in the same way as for [26, Lemma 4.1].

Let $\mathcal{L}$ be the totality of compact sets in $\mathbb{R}^d$.

**Lemma 23** The following equalities hold:

$$ \lim_{\kappa \to \infty} \sup_{L \in \mathcal{L}} \limsup_{t \to \infty} \sup_{x \in L} \left| \frac{\kappa}{h(x)} P_x(L_t > R^\kappa(t) - c_\ast) \right| = 0 \quad (36) $$
and for any \( c > 0 \),
\[
\lim \sup_{\gamma \to +0} \sup_{L \in \mathcal{L}} \sup_{x \in L} \frac{1}{\gamma h(x)} \mathbb{E}_x \left[ 1 - e^{-\gamma \epsilon t} \right] - c = 0.
\]

**Proof.** We can prove the assertion in the same way as for [26, Proposition 4.2]. We here prove (36) only. By the Chebyshev inequality and Lemma 22,
\[
\mathbb{P}_x(L_t > R^\kappa(t)) = \mathbb{P}_x \left( Z_t^R \geq 1 \right)
\leq \mathbb{E}_x \left[ Z_t^R \right] = \mathbb{E}_x \left[ e^{A_{(Q-1)\nu}/t} ; |X_t| > R^\kappa(t) \right].
\]
Hence by Lemma 14
\[
\lim \sup \sup \sup \frac{\mathbb{P}_x(L_t > R^\kappa(t))}{\kappa} \leq c_\ast.
\]
Then for the proof of (36), it suffices to show that
\[
\lim \inf \inf \lim \inf \inf \frac{\mathbb{P}_x(L_t > R^\kappa(t))}{\kappa} \geq c_\ast. \tag{37}
\]
In what follows, we give a proof of (37). Lemma 22 says that for any \( \epsilon > 0 \) and \( K \in \mathcal{L} \), there exists \( \kappa_0 = \kappa_0(\epsilon, K) > 0 \) such that for any \( \kappa \geq \kappa_0 \), there exists \( T_0 = T_0(\epsilon, K, \kappa) > 0 \) such that for any \( t > T_0 \) and \( x \in K \),
\[
|\kappa \mathbb{P}_x(L_t > R^\kappa(t)) - c_\ast h(x)| < \epsilon h(x).
\]
Let \( t > 0 \) and \( 0 \leq s < t \). Then
\[
R^\kappa(t) = (e^{-\lambda(t-s)})^{1/\alpha}(\kappa e^{-\lambda s})^{1/\alpha} = R^\kappa e^{-\lambda s}(t-s)
\]
and \( \kappa e^{-\lambda s} \geq \kappa \geq \kappa_0 \). Hence for any \( T > 0, t \geq T + T_0 \) and \( s \in [0, T] \),
\[
|\kappa \mathbb{P}_x(L_{t-s} > R^\kappa(t)) - c_\ast h(x)| < \epsilon h(x), \; x \in K. \tag{38}
\]
Fix \( K \in \mathcal{L} \) which includes the support of \( \mu \). Let \( \sigma \) be the first hitting time to \( K \) of some particle. Then \( \sigma \) is relevant to the initial particle only because particles can not branch outside \( K \). We use the same notation \( \sigma \) to denote the first hitting time to \( K \) of a symmetric \( \alpha \)-stable process \( M = \{X_t \}_{t \geq 0}, \{P_x \}_{x \in \mathbb{R}^d} \). Since \( X_\sigma \in K \), there exists \( \kappa_1 = \kappa_1(\epsilon) > 0 \) for any \( \epsilon > 0 \) such that for any \( \kappa \geq \kappa_1 \), there exists \( T_1 = T_1(\epsilon, \kappa) > 0 \) such that for any \( T > 0 \) and \( t \geq T + T_1 \), we have by the strong Markov property and (38),
\[
\kappa \mathbb{P}_x(L_t > R^\kappa(t)) \geq \mathbb{E}_x \left[ \kappa \mathbb{P}_{X_t}(L_{t-s} > R^\kappa(t)) \right]_{s=\sigma; \sigma \leq T} \geq (c_\ast - \epsilon) \mathbb{E}_x \left[ h(X_{\sigma}) ; \sigma \leq T \right], \; x \in \mathbb{R}^d. \tag{39}
\]
Since \( e^{\lambda + A_{(Q-1)\nu}/t} h(x) \) is a \( P_x \)-martingale and \( P_x(A_{(Q-1)\nu} = 0) = 1 \) for any \( t \geq 0 \), the optional stopping theorem yields
\[
\mathbb{E}_x \left[ e^{\lambda(T \wedge \sigma)} h(X_{T \wedge \sigma}) \right] = \mathbb{E}_x \left[ e^{\lambda(T \wedge \sigma)} h(X_{T \wedge \sigma}) \right] = h(x)
\]
and thus
\[ E_x[h(X_T);\sigma \leq T] \geq E_x\left[e^{\lambda\sigma}h(X_{\sigma});\sigma \leq T\right] \]
\[ = E_x\left[e^{\lambda(T\wedge\sigma)}h(X_{T\wedge\sigma})\right] - E_x\left[e^{\lambda T}h(X_T); T < \sigma\right] \]
\[ \geq h(x) - e^{\lambda T}\|h\|_{\infty}. \]

Then by (39), we have for any \( t \geq T + T_1 \),
\[ \kappa P_x(L_t > R^\kappa(t)) \geq (c_\ast - \varepsilon)(h(x) - e^{\lambda T}\|h\|_{\infty}), \ x \in \mathbb{R}^d. \]

In particular, for any \( L \in \mathcal{L} \) and \( t \geq T + T_1 \),
\[ \inf_{x \in L} \frac{\kappa}{h(x)} P_x(L_t > R^\kappa(t)) \geq (c_\ast - \varepsilon) \left(1 - \frac{e^{\lambda T}\|h\|_{\infty}}{\inf_{x \in L} h(x)}\right). \]

Letting \( t \to \infty \) and then \( T \to \infty \), we have
\[ \lim_{t \to \infty} \inf_{x \in L} \left( \frac{\kappa}{h(x)} P_x(L_t > R^\kappa(t)) \right) \geq c_\ast - \varepsilon. \]

Furthermore, since the right hand side above is independent of the choice of \( L \in \mathcal{L} \), we obtain (40) by letting \( \kappa \to \infty \) and then \( \varepsilon \to 0 \). \( \Box \)

**Proof of Theorem 17** We follow the argument of [26, Proof of Theorem 2.4]. In what follows, we write \( R(t) = R^\kappa(t) \) for simplicity. Since \( P_x(L_t < \infty) = 1 \) for any \( t \geq 0 \), there exists \( r_1 = r_1(\varepsilon, T_1) \) for any \( \varepsilon > 0 \) and \( T_1 > 0 \) such that \( P_x(L_{T_1} > r_1) \leq \varepsilon \). Hence for any \( t \geq T_1 \),
\[ P_x(L_t \leq R(t)) \leq P_x(L_t \leq R(t), L_{T_1} \leq r_1) + \varepsilon, \]
which yields
\[ P_x(L_t > R(t)) - E_x\left[1 - \exp(-\kappa^{-1}c_\ast M_t)\right] \geq E_x\left[\exp(-\kappa^{-1}c_\ast M_t) - 1_{\{L_t \leq R(t)\}}; L_{T_1} \leq r_1\right] - \varepsilon. \] (40)

Recall that \( e_0 = \inf\{t > 0 \mid Z_t = 0\} \) is the extinction time of \( \overline{M} \). Then for any \( t \geq e_0, M_t = 0 \) and \( L_t = 0 \) by definition. Therefore, by the Markov property,
\[ E_x\left[\exp(-\kappa^{-1}c_\ast M_t) - 1_{\{L_t \leq R(t)\}}; L_{T_1} \leq r_1\right] \]
\[ = E_x\left[\left\{E_{X_{T_1}}\left[\exp(-\kappa^{-1}c_\ast e^{\lambda T_1}M_{T_1})\right]\right\} - P_{X_{T_1}}(L_{t-T_1} \leq R(t))\right]; T_1 < e_0, L_{T_1} \leq r_1 \]
\[ = (VI). \]

By Lemma 23 there exists \( \kappa_0 = \kappa_0(\delta) > 0 \) for any \( \delta \in (0, c_\ast) \) such that if \( T > 0 \) satisfies \( \kappa e^{-\lambda T} \geq \kappa_0 \), then
\[ \sup_{L \in \mathcal{L}} \lim_{t \to \infty} \sup_{x \in L} \frac{\kappa e^{-\lambda T}}{h(x)} P_x(L_{t-T} > R(t)) - c_\ast < \delta. \]
Let $T_1 = T_1(\kappa_0)$ satisfy $\kappa e^{\lambda T_1} \geq \kappa_0$. Then there exists $T_2 = T_2(\varepsilon, \delta, T_1) > 0$ such that for any $y \in B(\varepsilon T_1)$ and $t \geq T_1 + T_2$,

$$\kappa^{-1}(c_\ast - \delta) e^{\lambda T_1} h(y) \leq P_y(L_{t - T_1} > R(t)) \leq \kappa^{-1}(c_\ast + \delta) e^{\lambda T_1} h(y). \quad (41)$$

Note that $1 - x \leq e^{-x}$ for any $x \in \mathbb{R}$ and there exists $r_0(\delta) > 0$ for any $\delta > 0$ such that $1 - x \geq e^{-(1+\delta)x}$ for any $x \in [0, r_0(\delta)]$. Hence if we take $T_1$ so large that $\kappa^{-1}(c_\ast + \delta) e^{\lambda T_1} \|h\|_\infty \leq r_0(\delta)$, then by (41),

$$\exp\left(-(1+\delta)\kappa^{-1}(c_\ast + \delta) e^{\lambda T_1} h(y)\right) \leq \mathbb{P}_y(L_{t - T_1} \leq R(t)) \leq \exp\left(-(1+\delta)\kappa^{-1}(c_\ast - \delta) e^{\lambda T_1} h(y)\right).$$

By Lemma 23 we also obtain

$$\exp\left(-(1+\delta)\kappa^{-1}(c_\ast + \delta) e^{\lambda T_1} h(y)\right) \leq \mathbb{E}_x \left[\exp\left(-(1+\delta)\kappa^{-1}(c_\ast + \delta) M_{T_1}\right)\right] \leq \exp\left(-(1+\delta)\kappa^{-1}(c_\ast - \delta) e^{\lambda T_1} h(y)\right)$$

so that for any $t \geq T_1 + T_2$,

$$(VI) \geq \mathbb{E}_x \left[\exp\left(-(1+\delta)\kappa^{-1}(c_\ast + \delta) M_{T_1}\right)\right] - \mathbb{E}_x \left[\exp\left(-\kappa^{-1}(c_\ast - \delta) M_{T_1}\right)\right].$$

Since the right hand side goes to 0 as $t \to \infty$, $T_1 \to \infty$ and $\delta \to +0$, we have by (10),

$$\lim_{t \to \infty} \inf \left(\mathbb{P}_x(L_t > R(t)) - \mathbb{E}_x \left[1 - \exp\left(-\kappa^{-1}(c_\ast M_t)\right)\right]\right) \geq 0.$$ 

In the same way, we also have

$$\lim_{t \to \infty} \sup \left(\mathbb{P}_x(L_t > R(t)) - \mathbb{E}_x \left[1 - \exp\left(-\kappa^{-1}(c_\ast M_t)\right)\right]\right) \leq 0$$

so that the proof is complete. \qed

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