Mathematical methods for solving problems of electrodynamics in a layered environment

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Abstract. The article considers methods for solving applied problems of electrodynamics in the case of layered and vertically inhomogeneous environments. This article suggests the algorithms of primary processing of GPR data. GPR data at a fixed observation point are investigated via the “selecting” method. The analysis of the frequency response of the signal energy distribution by frequency components is being processed. We proposed software implementation for determining the depth of objects and the relative dielectric permittivity of the subsurface environment. To perform this task, we use a module to determine the depth and conductivity of subsurface objects. The computational part was performed by the computationally common method of the approximate solution. As a result of the research, we have developed numerical methods for solving direct electrodynamics problems.

1. Introduction
The actual necessity to apply analytical and electrodynamics problems is to determine the urgency of V.A. Goncharova and G.G. Markova’s consideration in the case of layered and vertically inhomogeneous environments. The original problem of electrodynamics, in the case of a special choice of the source of electromagnetic disturbance, is reduced to a series of one-dimensional problems for the geoelectric equation. Based on the “fitting” method, fixed measurement points for a class of calculated physical fields define a class of functions that describe the response of the medium.

To solve such applied problems, it is necessary to have a dependence of the signal amplitude on the depth of its reflection, and the initial radarogram will express the dependence of the signal amplitude on the reflection time. Then it is necessary to get rid of various interference hiding the useful signal.

For this purpose, we should consider algorithms of noise reduction in the radargram using different wavelets. We used Haar’s wavelet and Do be shi’s wavelets of order 4. The main field of application of wavelet transforms is the analysis and processing of signals and functions that are non-stationary in time and inhomogeneous in space. The results of such analysis should contain the frequency response of the signal distribution of signal energy by frequency components. In comparison with signal decomposition into the Fourier series, wavelets can represent the local features of signals with a higher accuracy [1].
2. Results and Discussion

As a result of the investigation of the subsurface environment, we obtain a set of signals obtained from the receiving antenna for each measurement by GPR. The set of such traces is visualized by the variable density method in the form of an image. Automation of calculation processes should be realized based on the software module that will allow determining the depth of objects and the relative dielectric permittivity of the subsurface. The location of a subsurface object is defined by a top of a hyperbola which is built on points of maximum values of each trace amplitude [2].

For the realization of such types of tasks, the algorithm for the interpretation of GPR data on determination of dielectric permittivity and conductivity of the medium is being developed. For this purpose, two practical inverse problems are solved. First, the parameters of the source generated by GPR are determined. From the known source, the relative permittivity and conductivity of the medium. The problems are solved in a known environment, which has been specially prepared at the test site. For the first problem, a homogeneous medium is created, and for the second one, a local inhomogeneity is located in a horizontally layered homogeneous medium. For the second problem, the software module for determining the depth of occurrence and conductivity of subsurface objects was described [3].

Experimental studies were carried out on a polygon with a geological section containing perfectly pure sand and an inhomogeneous inclusion "salt dome" of artificial origin. The numerical algorithm makes it possible to determine the secondary source excited by an inhomogeneous inclusion and, subsequently, to determine the dielectric permittivity of this inclusion. The response of the medium obtained from GPR was cleaned from noise and interference using filtering and wavelet algorithms. The tabular representation of the environmental response was used as additional information to solve the inverse problem of determining the geophysical properties of the localized object. The obtained results demonstrate both the adequacy of the mathematical model and the possibility of the practical application of the considered method for the interpretation of radagrams.

The initial problem of electrodynamics, in the case of a special choice of the source of electromagnetic perturbation, is reduced to a series of one-dimensional problems for the geoelectric equation [4]. We consider the vertically inhomogeneous environment as cases of inclined environment, as well as combinations of inclined and layered environment, i.e., we consider continuous (smoothed environment). For this case, there are no discontinuities of the coefficients of physical characteristics of the environment, which, in the case of discontinuities, lead to certain difficulties when constructing the algorithm for the solution of direct problems and the algorithms for the solution of inverse problems, in the latter case when constructing the gradient of the residual functional. The method of "selection" of the fixed measurement points for the class of the calculated physical fields defines the class of functions describing the response of the medium [5]. From the condition of a minimum of the square deviation of the observed field (GPR data at a fixed point) and the calculated physical field, by under the singularity theorem, we obtain the desired structure of the medium, corresponding to the GPR data.

Let us consider the algorithm of the numerical method for solving the electrodynamics problem in a layered medium. The "fitting" method is a widespread method in the computational practice of approximate solution of the equation of the form:

\[ A_z u = 0, \quad u \in U, \quad z \in F, \]

where \( U, F \) are metric spaces.

While using this method, with a sufficiently wide class of possible environment, the corresponding calculated physical fields are calculated and, as a solution to the problem, some possible structure of the medium is selected for which the calculated physical field differs little from the observed field. An operator \( A_z \) is calculated for an element \( z \) of some predetermined subclass of possible solutions \( M(M \subset F) \), i.e., the direct problem is being solved. An element \( z_0 \) from the set \( M \), on which the residual \( \rho_u(Az_0,u) = \inf_{z \in M} \rho_u(Az,u) \) reaches a minimum is taken as an approximate solution [6]. In
our case, the “selection” method is implemented as follows: let $M^{(1)}(e_n^1, \sigma_n^1, h_n^1), M^{(2)}(e_n^2, \sigma_n^2, h_n^2), M^{(3)}(e_n^3, \sigma_n^3, h_n^3)$ be the classes of possible environment structures (geoelectric section), where $h_n^i = h_n^{i-1} + \delta h_n^i, \sigma_n^i = \sigma_n^{i-1} + \delta \sigma_n^i, \delta = 0, 1, 2 ...$ is the variation parameter, $h$ is the width of the model layers. As noted above, we solve a series of direct problems $A\zeta_j = u_j, j = 1, n$. From where it is easy to determine the class of responses of the environment at a fixed observation point, i.e. $z_j(x,t) = g^j(t)$. Let the readings of the device (GPR) at the observation point be known, i.e., $f^j(t)$. According to the "fitting" method, we calculate the discrepancy $\rho_u^j(A\zeta_j, u) = \inf_{z \in M} \cdot \rho_u(A\zeta; u_j)$ of the element that delivers the minimum discrepancy being the response of the environment, from where the environment from the class of environment is determined $M^j$, thereby solving the problem of interpreting the GPR data.

A set of algorithm solution data was created to construct a class of possible calculated physical fields for a set of geological section models in the case of a layered environment. Here are the ways of forming a class of possible structures of environment, as shown in Figure 1.

![Figure 1. Variations in depth of layer thickness and variations in the parameter-dielectric constant.](image)

By varying the main characteristics of the environment (dielectric constant, conductivity of environment, layer powers), a fairly wide class of possible environment was created. Next, a series of direct problems were solved for each class of possible environment structures.

For clarity of reasoning, we present the formulation of the direct problem of electrodynamics, which consists of the following: on the day surface, an external current source $J^{cm}$ is switched on, which has a bell-shaped form in time $r(t)$. During about 30-50 nanoseconds, the response of the medium is measured, which is the solution of the direct problem at the point of observation (measurement).

We assume that the dielectric constant $\varepsilon$ and conductivity $\sigma$ depend on the depth $x_3$. Let us choose as a source of external current a sufficiently long cable located in the center and stretched along the axis $x_2$.

Under such assumptions, the system of Maxwell’s equations is reduced to a system of one-dimensional problems in the constructed class and satisfies the following equations:

$$\varepsilon^{(s)}V^{(k)}_{x_3} + \sigma V^{(k)}_{t} = \frac{1}{\mu} (V^{(k)}_{x_3} - \lambda^2 V^{(k)}) - P_j q(x_2) r'(t)$$

(1)
\[ V^{(k)}|_{\eta=0} = 0, \quad V^{(k)}|_{\eta=0} = 0. \quad (2) \]

Here: \( \varepsilon = \varepsilon_0 \cdot \varepsilon_{rel} \) - dielectric constant, \( \mu = \mu_0 \cdot \mu_{rel} \) - magnetic permeability, \( \sigma \) - medium conductivity \( p(x_i), q(x_i) \) - functions describing the transverse dimensions of the source, \( s \) - variation parameter, \( V^{(k)} \) - solutions corresponding to the classes \( M^{(k)}(\varepsilon, \sigma, h), k = 1,2,3,m,e \).

\[ V^{(k)} = F_2 \left[ E^{(k)}(x_1, x_2, t) \right], \quad p \lambda = F \left[ p(x_i) \right], \quad \lambda \) - parameter of the Fourier transforms in the variable \( x_3 \):

\[ \vartheta = p_\lambda q(x_3) r(t), \quad \varepsilon = \varepsilon_0 \varepsilon_{rel}, \quad \mu = \mu_0 \mu_{rel}, \quad \varepsilon_0 = 8.854 \cdot 10^{-12} \frac{\Phi}{M}, \quad \mu_0 = 1.257 \cdot 10^{-6} \frac{\Gamma}{M} \]

Consider the case of a layered medium with known interfaces. In this case, we add to system (1) and (2), the continuity conditions for the horizontal component \( E_{2x} \) at the interfaces \( x_3^m \):

\[ \left[ V^{(k)} \right]_{x_3 = x_3^m} = 0, \left[ V^{(k)} \right]_{x_3 = x_3^m} = 0, \quad m \) - break node number. \quad (3)

Formulation of the direct problem: Using the known values of piecewise constant functions \( \varepsilon^{(s)}(x_i), \sigma^{(s)}(x_i) \) and a positive constant \( \mu \) we determine the function \( V^{(k)} \) as a solution to the generalized Cauchy problem from relations (1) and (3). When using the class \( M^{(3)}(\varepsilon, \sigma, h) \), we consider a system of one-dimensional problems:

\[ \varepsilon V^{(3)} + \sigma V^{(3)} = \frac{1}{\mu} \left( V^{(3)}_{x_3} - \lambda^2 V^{(3)} \right) - P_\lambda q(x_3) r(t) \]

Accordingly, when using the class \( M^{(1)}(\varepsilon, \sigma, h) \), we consider the following equations:

\[ \varepsilon V^{(1)} + \sigma V^{(1)} = \frac{1}{\mu} \left( V^{(1)}_{x_3} - \lambda^2 V^{(1)} \right) - P_\lambda q(x_3) r(t) \]

Here we carry out a variation in the thickness of the environment layers and \( \delta h^{(s)} = h^{(s-1)} + \delta h^{(s)} \).

Let us present a numerical algorithm for solving the direct problem, constructed according to the general theory of difference schemes [7].

Next, we introduce a change of variables \( \tau = \beta \tau, \beta \) - nondimensionalization coefficient. We suppose that \( \beta = 10^6 \), then the solution to the problem in new variables \( (\tau, x_3) U \) will take the form:

\[ b^{(s)} U^{(s)} + a^{(s)} U^{(s)} = U^{(s)}_{x_3} - \lambda^2 U^{(s)} - \gamma \vartheta, \quad x_3 \neq x_3^k \quad (4) \]

\[ U^{(s)}|_{\tau=0} = 0, \quad U^{(s)}|_{\tau=0} = 0 \]

\[ U^{(s)}|_{x_3 = x_3^k} = 0, \quad U^{(s)}|_{x_3 = x_3^k} = 0 \quad (6) \]

where:

\[ b^{(s)} = c \cdot \varepsilon^{(s)}_{rel}, \quad a^{(s)} = \gamma \cdot \sigma^{(s)}_{rel}, \quad \gamma = 1.256 \cdot 10^6, \quad c = 8.8541.256 \cdot 0.01, \quad \vartheta = p_\lambda q(x_3) r(t) \]

Let us determine the size of the area of calculations by the variables \( x_3 \), and \( \tau \). For this purpose, we calculate the travel times of the direct and reflected waves in the environment. We calculate the wave velocity by layers as follows:

\[ V^{(k)} = \frac{1}{\lambda} \sqrt{\varepsilon_0 \mu_0 \varepsilon_{rel}^{(k)}}, \quad (k \) is the layer number). \]

The difference scheme for equation (4) has the form:
\[ h_i(t) \cdot \hat{y}_i(t) - 2y_i(t) + y_{i+1}(t) + a_i(t) \cdot \hat{y}_{i+1}(t) - y_i(t) = \frac{1}{2\tau_0} \left( y_{i+1}(t) - y_i(t) \right) - \lambda^2 \gamma y_i(t) - \gamma \theta, \text{ if } \]

\[ i \neq i^k, \text{ and } i = -N_1, -N_1 + 1, ... , 0, 1, ..., N_1, \quad j = 3, 4, ..., N_2. \quad (7) \]

A difference analogue of the initial conditions (5):

\[ y_i^{0}_i = 0, \quad y_i^{0}_j = 0, \quad i = -N_1, -N_1 + 1, ... , 0, 1, ..., N_1. \quad (8) \]

For calculations in a finite region, it follows from condition (6) that:

\[ y_{-N_1}^{0} = 0, \quad y_{N_1}^{0} = 0, \quad j = 3, 4, ..., N_2. \quad (9) \]

Resolving equation (7), we have:

\[ \hat{y}_i(t) = \left( \frac{r_2 y_{i}^{0} + r_1 y_{i+1}^{0}}{r_2 + r_1 + \lambda^2} \right) y_{i}^{0} - \left( b_i^{0} - 0.5\tau_0 a_i^{0} \right) y_{i}^{0} - \gamma \theta, \quad \text{if } \]

\[ i \neq i^k, \quad j = 3, 4, ..., N_2. \quad (10) \]

We indicated: \( r_i = \tau_i^2 / t^k + 1, \quad r_i = \tau_i^2 / t^k \) at the rupture nodes, i.e. at \( i \neq i^k \). Based on the conjugation conditions (6), we have:

\[ \hat{y}_i(t) = \left( \frac{1}{h_{i+1}^k} \hat{y}_{i+1}^k + \frac{1}{h_i^k} \hat{y}_i^k \right) + \left( \frac{1}{h_{i+1}^k} + \frac{1}{h_i^k} \right), \quad j = 3, 4, ..., N_2. \quad (11) \]

Let us approximate the source \( \hat{\theta} = q(x_i) r(\tau) \). Let us put:

\[ q(x_i) = \begin{cases} \cos(\pi(x_i / l_0 + 1)), & x_i \in \left[ -l_0, 0 \right] \\ 0, & x_i \notin \left[ -l_0, 0 \right] \end{cases} \]

\[ r(\tau) = \begin{cases} (\pi / 2t_0 \sin(\pi \tau / t_0)), & \tau \in \left[ 0, t_0 \right] \\ 0, & \tau \notin \left[ 0, t_0 \right] \end{cases} \quad (12) \]

The values \( t_0 \) are determined from the condition of the problem, i.e. if the source duration is \( 2t_0 \), then for a real model \( 2t_0 = 2\tau_0 \) and in the shapeless form it will be 0.2 units. In our case, the Courant conditions have the form:

\[ \tau_0 < \sqrt{l_0 / c}, \text{ where } h_0 = \min_{-N_1 < i < N_1} h_i, \quad \bar{c} = \max_{-N_1 < i < N_1} c_i. \]

The area in time \( \tau \) is approximated by a uniform mesh:

\[ \omega = \{ \tau = (j - 1)\tau_0, \quad j = 1, 2, ..., N_2 \} \quad \text{where} \quad N_2 = \frac{T}{\tau_0} + 1. \quad \bar{T} - \text{travel time of direct and reflected waves.} \]

We approximate the region in terms of a variable \( x_3 \) with a non-uniform mesh so that the nodes of the discontinuities \( x_3^k \) coincide with its nodes

\[ \bar{x}_h = \left\{ x_3^k, \quad i = -N_1, -N_1 + 1, ..., 0, 1, ..., N_1 \right\}. \]

To demonstrate the operation of algorithms (7) and (12), Figure 2 shows the numerical solution of direct problems for a possible structure of a layered medium.
3. Conclusion

The obtained GPR readings at a fixed observation point using the "picking" method were compared with the class of possible structures created for more than 10,000 variants. Thus, the interpretation of radargrams was carried out. Figures 6 and 7 clearly show the work of this algorithm. In Fig. 6 the bold black line shows the solution of the direct problem for the possible structure of the layered environment, the thin black line shows the obtained GPR readings at the fixed observation point, obtained on the 5002nd iteration of the "picking" method. A similar result obtained at the 6752nd iteration is shown in Figure 3.

As a result of the study, numerical methods for solving direct electrodynamics problems (layered environment) have been developed. Algorithms and programs of numerical methods for solving direct electrodynamics problems were developed. To compare the GPR data with the results of model problem calculations for the geoelectric equation in the case of a layered environment, the "fitting" method was used. In the class of finite parametric environment, an algorithm and software for determining the class of calculated physical fields have been constructed. Then, comparing the measurement data with this class, we restore the geological section.

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Figure 2. Numerical solution of direct problems for a possible layered environment structure.

Figure 3. The result of the "selection" method for the case of layered environment.
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