Emergence of cosmic space in Tsallis modified gravity from equilibrium and non-equilibrium thermodynamic perspective

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Abstract
In the context of Tsallis entropy, we explore the connection between the law of emergence and the thermodynamic laws from a more accurate non-equilibrium perspective. Here, the equilibrium Clausius relation does not conform to the standard energy-momentum conservation. Therefore, an effective gravitational coupling is introduced to rewrite the field equation similar to general relativity, and the corresponding generalized continuity equation is obtained. As a result, thermodynamic laws were modified with the non-equilibrium energy dissipation and entropy production terms, using which we derive the law of emergence. The investigation of the law of emergence and the entropy maximization principle with Tsallis entropy in the non-equilibrium perspective shows that both result in the same constraints as obtained in other gravity theories and the equilibrium context of Tsallis entropy, except for an additional constraint on the Tsallis parameter as a result of extra entropy production. Consequently, the thermodynamic interpretation of the expansion of the universe stays valid even with quantum corrections to the horizon entropy since the correction terms in Tsallis entropy can be treated as the quantum corrections to Bekenstein-Hawking entropy.

1. Introduction

The developments in black hole thermodynamics in the last decades of the twentieth century led to the discovery of the profound connection between horizon thermodynamics and gravity. Hawking [1] showed that when two black holes merged, the area of the horizon of the new black hole would exceed the total of the areas of the horizons of the original black holes. Following this, Bekenstein [2–4] put forth that the black hole horizon area is a measure of its entropy. Meanwhile, the laws governing the black hole dynamics were formulated [5] and were found similar to thermodynamic laws. Later, Hawking [6, 7] discovered that black holes have a temperature, which is related to surface gravity. Soon after, Davies [8] and Unruh [9] generalized Hawking’s idea to any observer accelerating in flat space-time and showed that the horizon perceived by such an observer could have a temperature proportional to the observer’s acceleration. In extending this idea to cosmology, Gibbons and Hawking [10] have shown that the horizon of an expanding universe possesses both entropy and temperature. Jacobson [11], considering these results on a local Rindler horizon, and using Clausius relation obtained the Einstein field equation. These results motivate further research on the connection between the fundamental laws of thermodynamics and gravity.

Padmanabhan [12] restructured Einstein’s equation of gravity on a spherically symmetric space-time as a thermodynamic identity. This result indicates that Einstein’s equation describes the thermodynamics of space-time [13–17]. Paranjape et al [18] further expanded this outcome to more general Lanczos-Lovelock gravity. This result firmly pointed out that the field equation can have thermodynamic interpretation even if the Einstein–Hilbert action has quantum corrections [19]. This is because, in Lanczos-Lovelock gravity, the higher derivative terms can be treated as the quantum corrections to general relativity [19]. Around the same time, for
Lovelock gravity, Cai and Cao [20] showed that the Clausius relation remains valid, where the apparent horizon entropy is the corresponding black hole entropy and an effective energy-momentum tensor is defined using the higher derivative terms in Lovelock theory.

The thermodynamic connection of Einstein’s equation of gravity has implications in cosmology. In Einstein’s gravity, Cai and Kim [21] used the area law of entropy and the Clausius relation to obtain the Friedmann equation. The authors extended these results to Gauss-Bonnet theory and Lovelock gravity by employing the corresponding black hole entropy. At a later stage, Akbar and Cai [22] restructured the Friedmann equation in Einstein, Gauss-Bonnet, and Lovelock gravity theories into the first law of thermodynamics. However, the same authors [23] later found that in the case of scalar-tensor gravity and \( f(R) \) theory, it is impossible to obtain the Friedmann equations from the Clausius relation using the corresponding black hole entropy relation. Nevertheless, in these gravity theories, the Friedmann equations can be obtained from the Clausius relation by defining an effective energy-momentum tensor and using the conventional area law of entropy.

A significant turn in treating the horizon thermodynamics in gravity theories with higher-order curvature contributions to entropy was brought by Eling et al [24]. The authors suggested that while accounting for corrections in the entropy, like the higher-order curvature corrections, a non-equilibrium thermodynamic perspective is essential, rather than an equilibrium perspective, for the successful derivation of the field equation [24]. Following this intriguing idea, Akbar and Cai [25] have rewritten the \( f(R) \) gravity field equation in the FRW setup into the first law of thermodynamics in the non-equilibrium context. Later, Cai and Cao [20] showed that scalar-tensor gravity requires a non-equilibrium thermodynamic perspective. These results indicate that theories of gravity having entropy with higher order curvature contributions or other than area law form for black hole entropy might require a non-equilibrium thermodynamic perspective.

The vital link between thermodynamics and gravity directly implies that the space-time continuum is thermodynamic and can carry heat like a fluid [26, 27]. The macroscopic geometrical properties like metric and curvature are analogous to the emergent variables like temperature and pressure in thermodynamics, which have no existence in the microscopic realm [26–29]. Following this, Padmanabhan claimed ‘gravity as an emergent phenomenon’ [30]. In extending the emergent paradigm to cosmology, Padmanabhan, in 2012, proposed that ‘cosmic space is emergent as cosmic time progresses’ [31, 32]. According to this, the difference in the degrees of freedom on the horizon, \( N_{\text{sur}} \), and that on the bulk enclosed by the horizon, \( N_{\text{bulk}} \), is responsible for the expansion of the universe. As the universe expands, the disparity in these degrees of freedom will decrease and finally attain zero, at which it achieves a static equilibrium. At the final phase, when the disparity is zero, the degrees of freedom become equal, and that equality is called the holographic equipartition condition [31]. Thus, the emergence of space is the quest of the universe for achieving holographic equipartition condition [31]. Accordingly, Padmanabhan [31] proposed a fundamental principle to describe the expansion of the universe, given by \( dV/dt = L_f^2 (N_{\text{sur}} - N_{\text{bulk}}) \), where \( V \) is the Hubble volume of a flat \((3 + 1)\)-dimensional FRW universe and \( t \) is the cosmic time. In the context of Einstein gravity, Padmanabhan [31] used the above fundamental principle known as the law of emergence to obtain the Friedmann equation in a spatially flat universe.

The Friedmann equations for a flat FRW universe were obtained in the context of \((n + 1)\) Einstein gravity, Gauss-Bonnet, and Lovelock gravity from the law of emergence by extending Padmanabhan’s idea to these gravity theories [33]. In the \((n + 1)\) Einstein gravity, the author used modified surface degrees of freedom for a flat universe to propose the law of emergence. The authors have considered an effective volume, distinct from the areal volume, in addition to the modified surface degrees of freedom while extending the law to Gauss-Bonnet and Lovelock gravity theories. On the other hand, Yang et al [34] proposed an alternative generalization to emergent cosmology by introducing a function, \( f(\Delta N, N_{\text{sur}}) \), where \( \Delta N = N_{\text{sur}} - N_{\text{bulk}} \) such that increase in horizon volume is proportional to that function. Even though this generalization retains the horizon volume as the areal volume, the function \( f(\Delta N, N_{\text{sur}}) \) is intricate except in Einstein’s gravity. Later, Padmanabhan’s proposal was extended to \((n + 1)\)-dimensional Einstein, Gauss-Bonnet, and Lovelock geometries by Sheykhi [35] and the corresponding Friedmann equations for a non-flat universe were derived. Subsequently, Yuan and Huang [36] derived the modified Friedmann equation for a non-flat universe from the law of emergence, incorporating logarithmic and power-law corrected entropy relations. Later, Eune and Kim [37] extended Padmanabhan’s idea for a non-flat universe by considering the invariant volume of the horizon. However, this work has been criticized [28, 38] for using a time-dependent Planck length. In the meantime, Ali [39] modified the law of emergence using a generic entropy in \((3 + 1)\) Einstein’s gravity and Gauss-Bonnet gravity and derived the Friedmann equation. While Ai et al [40] proposed another modified law of emergence in deformed Horava-Lifshitz theory and \( f(R) \) theory, where they utilized the fundamental relation \( N_{\text{sur}} = 4S \) to express the growth rate of the horizon volume as related to \( 1/H (dN_{\text{sur}}/dt) \). Later, Young and Lee [41] pointed out that the Friedmann equation for the non-flat universe emerges from Padmanabhan’s proposal only when the areal volume is used and argued that the holographic principle gives a clue about the flatness of our universe. The literature also
shows the application of the concept of emergence of space to braneworld scenarios [42, 43] and the derivation of the corresponding cosmological equations.

Later, Dezaki and Mirza [44] introduced the link between thermodynamics and the law of emergence in Einstein’s gravity by obtaining the law of emergence for a (3 + 1)-flat FRW universe from the first law of thermodynamics. However, Mahith et al [28] suggested a more direct way to derive the law of emergence using the unified first law of thermodynamics in (n + 1) Einstein, Gauss-Bonnet, and Lovelock gravities. Another meaningful thermodynamic connection to the law of emergence was obtained later by Krishna and Mathew [29–45, 47]. The authors investigated the association between the entropy maximization and the law of emergence in Einstein, Gauss-Bonnet, and Lovelock gravities. They found that the cosmic space emerges to maximize the horizon entropy [45]. Meanwhile, Hareesh et al [38] found the reason for preferring areal volume while using the unified first law and the Clausius relation for obtaining the law of emergence. It has been shown that the unified first law and the Clausius relation are defined properly in a non-flat universe, only with areal volume. Recently, Basari et al [48, 49] introduced a generic derivation of the law of emergence from the thermodynamic laws in Gauss-Bonnet and Lovelock gravity. The same idea is then extended to f(R) gravity [48], to find the law of emergence, where the authors used a non-equilibrium thermodynamic perspective, as the theory demands it.

The thermodynamic laws are the fundamental laws that remain intact in any gravity theory. Hence, while exploring the thermodynamic link of the law of emergence of cosmic space, we look at the entropy of the black hole in the particular gravity theory of interest. In the case of gravitating systems whose partition function diverges, the Boltzmann-Gibbs theory is invalid, as mentioned by Gibbs around 1902 [50]. Tsallis [51] in 1988 proposed a generalization to the Boltzmann-Gibbs entropy for the multifractal structures. The well-known Bekenstein-Hawking results [2–4] regarding the black hole entropy is that it is related to the area A and not to its volume. Similarly, in the case of strongly quantum-entangled d-dimensional systems with linear dimension L and for d > 1, entropy is related to $L^{d-1}$ [50]. The extensivity of the thermodynamic entropy is violated in these cases. Hence, to remove this thermodynamic inconsistency, Tsallis and Cirto [30] in 2013 introduced a generalized non-additive entropy proportional to $A^\beta$, which is extensive for $\beta = d/(d - 1)$, $\beta$ is the non-additive parameter. The correction terms in Tsallis entropy can also be treated as the quantum corrections to Bekenstein-Hawking entropy [52–55]. Tsallis entropy is given by $S_T = \gamma A^{\beta} [53, 56]$, where $\gamma$ is a positive unknown constant [57]. For independent systems I and II, we have $S_T(I + II) = \gamma \left[ \frac{S_T(I)}{\gamma} + \frac{S_T(II)}{\gamma} \right]^{\beta}$ [50], showing that the total entropy of the composite system is not the sum of the entropies. The form of the Tsallis entropy reveals its non-extensive property. When we consider the Tsallis entropy as the horizon entropy of the universe, this peculiar property can be explained as due to strong correlations arising from the interactions on the large scale of the universe, or from the strong quantum entanglement, or both [50, 58–60].

Inquisitive to know the implications of the non-extensive entropy in cosmology, Sheykhi [56] assumed Tsallis entropy for the horizon and derived the Friedmann equations from the unified first law of thermodynamics for a non-flat FRW universe. In this derivation, the author considers the cosmic components as perfect fluids that obey the standard conservation law. Further, the author obtained the Friedmann equations from a newly proposed modified law of emergence. Later, this was extended to (n + 1)-dimensional non-flat FRW universe by Chen [61]. Recently, Basari et al [49] obtained the law of emergence through a unified expansion law derived from the equilibrium first law of thermodynamics, for the Tsallis entropy given by $S = (A_0/(4L_{PL}^3))(A/A_0)\tilde{\alpha}$, where $A_0$ is a constant and $L_{PL}^3$ is the Planck length. Earlier to this, Lymeris and Saridakis [62] extracted the modified Friedmann equations, using Tsallis entropy of the form $S = (\alpha/(4G))A^3$, where $\alpha$ is an unknown constant, from the equilibrium Clausius relation. They have rewritten the modified Friedmann equation to the standard form in the (3 + 1) Einstein gravity by extracting some form of effective dark energy and studied cosmology. Nojiri et al [63] also extracted the modified Friedmann equations from the equilibrium Clausius relation using Tsallis entropy given by $S = (A_0/(4G))(A/A_0)^\beta$, where $A_0$ is an unknown constant but the exponent $\beta$ as the varying, and studied the evolution in the late and early phase. Later, Nojiri et al [64] defined an effective fluid equivalent to dark energy to obtain the Friedmann equation similar to that in Einstein gravity, using the Tsallis entropy, and explained the late acceleration.

In the above works, the authors have obtained the Friedmann equations using Tsallis entropy, either directly from the law of thermodynamics or by proposing the law of emergence, assuming the conventional conservation law. However, the equilibrium Clausius relation with the standard energy-momentum conservation may not hold together for entropies other than Bekenstein-Hawking entropy [24, 65]. Eling et al [24] have revealed the contradiction between the equilibrium Clausius relation and the standard energy-momentum conservation for entropy with higher-order curvature corrections. The authors resolved it by adding an entropy production term $dS$, which is the extra entropy developed due to the irreversible processes, to the entropy balance relation. The resulting relation, $dS = dQ/T + dS$, is the non-equilibrium entropy balance relation that can be used to obtain the field equation by considering a divergence-free matter stress tensor [24].
Following this approach, Asghari and Sheykhi [65] have shown that, with Tsallis entropy, a non-extensive entropy, one needs the non-equilibrium thermodynamic description to restore the standard energy-momentum conservation. These authors used the non-equilibrium entropy balance relation with Tsallis entropy and derived the field equation by considering a divergence-free matter stress tensor. Further, they have obtained the corresponding Friedmann equations, which differ from the one obtained by Sheykhi [56] with equilibrium thermodynamics using Tsallis entropy.

As mentioned earlier, the law of emergence with Tsallis entropy has been proposed, and the Friedmann equations are derived from an equilibrium thermodynamic perspective. However, Tsallis entropy demands a non-equilibrium treatment. Hence, obtaining the law of emergence from the non-equilibrium thermodynamic perspective is essential. That motivates us to obtain the law of emergence using the Tsallis entropy from the non-equilibrium thermodynamic perspective. We accomplish our purpose using a unified formulation of Tian and Booth [66], which was proposed to derive the Friedmann equations from non-equilibrium thermodynamics in the context of modified gravity theories. Unlike the previous works in Tsallis entropy, we first restructure the Tsallis entropy $S = \gamma A^\gamma$ into a form, $S = A/(4G_{\text{eff}})$, similar to Bekenstein-Hawking entropy, where $G_{\text{eff}}$ firmly depends on the nature of the parameter $\gamma$. The $G_{\text{eff}}$ used here, is different from the idea of varying gravitational constant in Dirac’s large numbers hypothesis [67]. The field equation corresponding to the above entropy form can be expressed in a form, $G_{\mu\nu} = 8\pi G_{\text{eff}}T_{\mu\nu}^{\text{eff}}$, which is similar to the Einstein field equation. The covariant derivative of this field equation results in a continuity equation $\dot{\rho}_{\text{eff}} + nH(\rho_{\text{eff}} + p_{\text{eff}}) = -G_{\text{eff}}\rho_{\text{eff}}(G_{\text{eff}})^{-1}$ [66]. Unlike the conventional continuity equation for a perfect fluid, a non-zero term $-G_{\text{eff}}\rho_{\text{eff}}(G_{\text{eff}})^{-1}$ appearing in the above equation effectively balances the energy flow due to the non-equilibrium situation [66]. The corresponding non-equilibrium energy dissipation, $\dot{E}$, results in the additional entropy production, $dS/dt$. Taking account of this additional energy and entropy, the Clausius relation and the unified first law of thermodynamics modifies to $dQ = T_{\mu\nu}dS + dE = A\psi + WdV + E$ respectively, which then constitute the non-equilibrium thermodynamic relations. From these non-equilibrium thermodynamic relations, we derive the law of emergence. We also examine whether entropy maximization is deductible from the resulting law of emergence.

The structure of the present paper is as follows: We derive the law of emergence from the equilibrium thermodynamic laws using Tsallis entropy in section 2. In section 3, we derive the law of emergence from the non-equilibrium thermodynamic perspective using the unified first law of thermodynamics and the Clausius relation. We also see whether the law indicates entropy maximization. In section 4, we conclude our results.

2. Emergence of cosmic space in Tsallis modified gravity from equilibrium perspective

In the equilibrium thermodynamic perspective, we obtain the law of emergence with Tsallis entropy in this section. The line element [68]

$$ds^2 = h_{\mu\nu}dx^\mu dx^\nu + \tilde{r}^2d\Omega_{n-1}^2,$$

represents the spatially homogeneous and isotropic $(n + 1)$-dimensional universe. Here, $h_{\mu\nu} = \text{diag}(-1, a^2(t)/(1 - kr^2))$ is the 2-dimensional metric with coordinates $t$ and $r$, $a(t)$ is the scale factor, $\tilde{r} = a(t)r$, and $k$ is the curvature index. The line element $d\Omega_{n-1}^2$ represents the $(n - 1)$-dimensional unit sphere. We assume the universe is bounded by the apparent horizon of radius $[68]

$$\tilde{r}_A = \frac{1}{\sqrt{H^2 + \frac{k}{a^2}}},$$

which satisfies the relation, $h^{\mu\nu}\tilde{r}_A \partial_\mu \tilde{r}_A \partial_\nu \tilde{r} = 0$, and is equivalent to a thermodynamic system. Here, $H$ is the Hubble parameter. We consider the generalized non-additive Tsallis entropy $[53]

$$S_A = \gamma A^\beta,$$

for the apparent horizon. Here $\gamma$ is a positive unknown constant $[57]$, $\beta$ is the Tsallis parameter, and the apparent horizon area is given by $A = n\Omega_\alpha r_A^{n-1}$, where $\Omega_\alpha = \pi^\frac{n}{2}/\Gamma(\frac{n}{2} + 1)$ is the volume of an $n$-dimensional unit sphere $[21]$. When $\beta = 1$ and $\gamma = 1/(4G)$, where $G$ represents the gravitational constant, Tsallis entropy reduces to Bekenstein-Hawking entropy. In literature $[56, 62-65]$, different authors have assumed slightly different forms for the proportionality constant $\gamma$ for convenience. However, dimensionally, $\gamma$ is inversely proportional to the gravitational constant $G$. The exact form of the $\gamma$ can only be obtained by specifying the cosmology or the gravity theory. We use units with $\hbar = 1 = c = k$ and Planck area $L_p^2 = G$. The temperature of the apparent horizon $[69, 70]$ is related to surface gravity, $\kappa$, as follows
\[ T_A = \frac{\kappa}{2\pi} = \frac{1}{2\pi r_A} \left( 1 - \frac{\dot{r}_A}{2Hr_A} \right), \]

where over-dot designates the time derivative. Following these, we derive the law of emergence from the equilibrium unified first law of thermodynamics and Clausius relation.

In the equilibrium perspective, for a perfect fluid, the continuity equation is given by

\[ \dot{\rho} + nH(\rho + P) = 0, \]

where \( \rho \) and \( P \) are the energy density and pressure of the matter and energy content of the universe. Then the universal thermodynamic identity, the unified first law of thermodynamics [71], is given by the expression

\[ dE = A\psi + WdV, \]

where \( E = \rho V \) serve as the total energy content inside the horizon of radius \( r_A \), \( W = \frac{1}{2}(\rho - P) \) is the work density, and volume, \( V = \Omega_a r^3_A \). Here the energy flux density \( \psi \) gives the total energy passing through the apparent horizon [21, 23, 72] and is defined as the vector \( \psi = T^\mu_\nu \partial_\mu \tilde{r} + W \partial_\mu \tilde{r} \) [20, 66, 71], where \( T^\mu_\nu \) is the energy-momentum tensor. For a perfect fluid, it is given by \[20, 66\]

\[ \psi = \psi_t dt + \psi_A d\tilde{r}_A = -(\rho + P)H\tilde{r}_A dt + \frac{1}{2}(\rho + P) d\tilde{r}_A. \]

Here \( \psi_t \) and \( \psi_A \) are the time part and space part of \( \psi \). For an infinitesimal time interval, \( dt \), the entropy change \( dS_A \), associated with the heat flow \( \delta Q \), across the apparent horizon is given by the Clausius relation

\[ \delta Q = T_A dS_A, \]

In this case, the horizon is assumed to be stationary; hence, its radius is fixed. Therefore, the apparent horizon temperature during this interval is \( \tilde{T}_A = 1/(2\pi r_A) \). Here \( \delta Q \) is equivalent to \( -dE|_{\psi_{\text{constant}}} = -A\psi dt \), the energy change inside the apparent horizon during the infinitesimal time interval [23]. Hence we have,

\[ \tilde{T}_A dS_A = \frac{1}{2\pi r_A} dS_A = -A\psi dt. \]

Substituting equations (7), (9) and the work density \( W \) in (6), we can express the unified first law [73] as

\[ dE = -\frac{1}{2\pi r_A} dS_A + \rho dV. \]

Let us now multiply the above equation (10) throughout by \( 1 - \dot{r}_A/(2Hr_A) \), and on simplification will leads to,

\[ dE = -\frac{1}{2\pi r_A} \left( 1 - \frac{\dot{r}_A}{2Hr_A} \right) dS_A + WdV = -T_A dS_A + WdV. \]

The above equation reduces to \( dE = -T_A dS_A - PdV \), for pure de Sitter universe (i.e., \( \rho = -P \)).

To derive the law of emergence with Tsallis entropy from the unified first law of thermodynamics (11), we can use the continuity equation (5), and on integration, we get

\[ \frac{\rho^{(n-1)\beta-(n+1)}}{(n + 1)\beta - (n + 1)} = -\frac{2\pi \Omega_a \rho}{\gamma \beta (n \Omega_a)^\beta (n - 1)}. \]

Here, we have omitted the integration constant \(^1\). Both sides of the equation (12) are multiplied by \( a^2 \) and then differentiated with respect to time. Later the resulting equation is divided throughout by \( 2na \), and it leads to

\[ \dot{r}_A = \frac{2Hr_A}{[n + 1] - (n + 1) \beta} - \frac{4\pi \Omega_a H^{\beta + 2 - (n - 1)\beta}}{\gamma \beta (n \Omega_a)^\beta (n - 1)} \left( \frac{\rho}{2H} + \rho \right). \]

The equation (13) can be further simplified by applying continuity equation (5) and multiplying throughout by \( 4\alpha \gamma n \Omega_a \rho_A^{\beta - 1} \), where \( \alpha = (n - 1)/(2(n - 2)) \), then we get

\[ 4\alpha \gamma n \Omega_a \rho_A^{\beta - 1} d\tilde{V}/dt = \dot{r}_A H \left[ \frac{8\alpha \gamma A}{[(n + 1) - (n + 1) \beta]} + \frac{4\pi \Omega_a H^{2\beta - (n - 1)\beta}}{\beta (n \Omega_a)^\beta - 1} ([n - 2] \rho + nP) \right]. \]

Hence, whatever the form of \( \gamma \), the rate of change of volume cannot be related to the degrees of freedom as Padmanabhan proposed [31, 32]. Now, if we consider \( A^{\beta} \) in the Tsallis entropy relation as the effective area of the horizon, i.e., \( A = (n \Omega_a \rho_A^{\beta - 1})^{\beta} \) [56, 61], then the rate of change of effective volume will be \( d\tilde{V}/dt = (\dot{r}_A/(n - 1)) (d\tilde{A}/d\tilde{t}) \) such that for \( \beta = 1 \), the effective area \( \tilde{A} \) and volume \( \tilde{V} \) reduce to area \( A \) and the volume \( V \). Subsequently, simplifying the equation (13) by multiplying both sides with \( 4\alpha \gamma \beta (n \Omega_a \rho_A^{\beta - 1})^\beta \) and using continuity equation (5), we get

\(^1\) Retaining the integration constant will imply adding a dark energy with constant density. Since \( \rho \) and \( P \) represent the energy density and pressure of all the cosmic components, including the dark energy, we have omitted the integration constant to avoid redundancy.
The above equation on integration will result in equation (15).

\[
4\alpha y \frac{d\tilde{V}}{dt} = r_A H \left[ \frac{8\alpha y\beta A}{(n + 1) - (n - 1)\beta} + \frac{4\pi \Omega_{\gamma}^{n+1}}{n - 2}[(n - 2)\rho + nP] \right].
\] (15)

The first term \(\frac{8\alpha y\beta A}{(n + 1) - (n - 1)\beta}\) is a function of effective area, assuming that the form of \(\gamma\) will reduce all its coefficients, we can identify that term as surface degrees of freedom, i.e., \(N_{sur} = \frac{(\alpha \beta y\beta A)}{(n + 1) - (n - 1)\beta}\). The positive surface degrees of freedom implies \(\beta < (n + 1)/(n - 1)\).

In the above equation, the second term on the right-hand side is the bulk degrees of freedom, i.e., \(N_{bulk} = (-4\pi \Omega_{\gamma}^{n+1})(n - 2)^{-1}[(n - 2)\rho + nP]\). Hence, with the above assumptions the equation (15) can be recast as the law of emergence

\[
4\alpha y \frac{d\tilde{V}}{dt} = r_A H (N_{sur} - N_{bulk}).
\] (16)

This equation is identical to the law of emergence of cosmic space proposed by Chen [61] with Tsallis entropy. For \(n = 3\), the equation (16) reduces to

\[
4\gamma \frac{d\tilde{V}}{dt} = r_A H (N_{sur} - N_{bulk}),
\] (17)

where \(N_{sur} = (4\gamma\beta A)/(2 - \beta)\) and \(N_{bulk} = -\frac{16\pi \gamma}{\beta}[(2\rho + 3P)].\) This equation is identical to the law of emergence of cosmic space proposed by Sheykhi [56] for \((3 + 1)\)-dimensional space-time using Tsallis entropy.

From the Clausius relation (8) also, we can derive the law of emergence with Tsallis entropy. By relating the heat flux across the horizon to the energy change outside the horizon during the infinitesimal time interval, the equation (8) becomes (9) and substituting (7) into it, and using the continuity equation (5), will yield

\[
\frac{\gamma}{\beta}((n\Omega_{\gamma})^{(n - 1)}dA = -\frac{2\pi \Omega_{\gamma} d\rho}{\gamma\beta (n\Omega_{\gamma})^{(n - 1)}}.
\] (18)

The above equation on integration will result in equation (12). Then by following the same steps as mentioned above, we can obtain the law of emergence given in equation (16). Hence, the same law of emergence of cosmic space can be obtained from the unified first law and the Clausius relation. Note that in the Clausius relation (8), we have considered the temperature derived from the metric [20]. Our approach to temperature is through surface gravity\(\gamma\), just like in the case of black holes [6, 7], since we study the horizon thermodynamics of the universe using black hole thermodynamics. In the case of non-extensive systems, the temperature may be defined as the one from ‘the inverse of the Lagrange multiplier’ associated with the constraint on the internal energy’ [74], where the non-extensive entropy considered becomes the Clausius entropy [74] and the other based on the generalized zeroth law of thermodynamics, where the Clausius definition of entropy is changed [75]. Here, we have not considered the two different usual approaches to temperature in non-extensive systems [75] as divisibility of the system cannot be assumed when there can be strong correlations [74, 76]. Thus, investigation of the Clausius relation on this aspect [75] is beyond the scope of this paper.

3. Emergence of cosmic space in Tsallis modified gravity from non-equilibrium perspective

Previously, from the equilibrium thermodynamic perspective, we obtained the law of emergence with Tsallis entropy. However, the necessity of using a non-equilibrium thermodynamic perspective in the context of Tsallis entropy has been pointed out. Asghari and Sheykhi [65] showed that when Tsallis entropy is used as the horizon entropy, the conservation law, \(\nabla^\mu T_{\mu\nu} = 0\), is satisfied only if \(dS = \delta Q/T + dS\). The additional entropy, \(dS\), in the above non-equilibrium entropy balance relation, is the entropy produced inside the horizon due to irreversible processes. This motivates the use of non-equilibrium thermodynamic relations with Tsallis entropy as the horizon entropy in analyzing the evolution of the universe. Here, the law of emergence with Tsallis entropy is derived using the non-equilibrium thermodynamic relations. For this, we adopt a general formulation developed by Tian and Booth [66], through which the Friedmann equations were obtained in different modified theories of gravity from thermodynamics with non-equilibrium considerations. In order to implement the formulation, we first restructure the Tsallis entropy given in equation (3) into a form similar to Bekenstein-Hawking entropy, that is,

\[
S_A = \frac{A}{4G_{\text{eff}}},
\] (19)

Here, \(G_{\text{eff}} = ((n\Omega_{\gamma})^{1 - \beta}(\gamma^{n - 1})^{(n - 1)\beta})/(4\gamma)\). The field equation for any modified gravity theory with horizon entropy as given in equation (19) can be recast into a concise field equation similar to that in general relativity [66, 73, 77, 78], that is,
Here,  is the same as in equation (19), which can be taken as an effective gravitational coupling. Unlike the field equation in Einstein’s gravity, the effective gravitational coupling in the present case is time-dependent. However, the dependence on time differs from the idea of the time-varying gravitational constant over cosmic time. The effective energy-momentum tensor, , represents the energy-momentum tensor for the content of the universe, and arises due to the effects of Tsallis entropy. If we assume the cosmic component of the universe to be of perfect fluid type, then , where , and are the density and pressure arising due to the effects of Tsallis entropy. If we assume the cosmic component of the universe, and are the additional density and pressure arising due to the effects of Tsallis entropy. From the contracted Bianchi identity, the non-zero term, , has the dimension of the effective density. This implies a generalized continuity equation [66],

\[
\dot{\rho}_{\text{eff}} + nH (\rho_{\text{eff}} + P_{\text{eff}}) = -\frac{G_{\text{eff}}}{\rho_{\text{eff}}} \rho_{\text{eff}}.
\]

The non-zero term, , has the dimension of the effective density and balances the energy flow. The corresponding non-equilibrium energy dissipation term is \( \dot{E} = -\Omega n^2 G_{\text{eff}} (G_{\text{eff}})^{-1} \dot{\rho}_{\text{eff}} dt \) [66]. Following this, the unified first law [71] modifies to [66]

\[
dE_{\text{eff}} = A V_{\text{eff}} + W_{\text{eff}} dV + \dot{E}.
\]

For a sphere of radius , the effective total energy content is \( E_{\text{eff}} = \rho_{\text{eff}} V \) and the effective work density is \( W_{\text{eff}} = (\rho_{\text{eff}} - P_{\text{eff}})/2 \). In terms of the effective density and pressure, the effective energy flux density is

\[
\psi_{\text{eff}} = \psi_{r\text{eff}} dt + \psi_{\text{eff}} d\tau = -\left(\rho_{\text{eff}} + P_{\text{eff}}\right) H\dot{\tau}_{\text{A}} dt + \frac{1}{2} (\rho_{\text{eff}} + P_{\text{eff}}) d\tau_{\text{A}}.
\]

Substituting the expressions for \( \psi_{\text{eff}} \) and \( W_{\text{eff}} \) in the unified first law (22) results in

\[
dE_{\text{eff}} = A \psi_{\text{eff}} dt + A \psi_{\text{eff}} d\tau_{\text{A}} + \frac{\rho_{\text{eff}} - P_{\text{eff}}}{2} dV + \dot{E}
\]

Here, for an infinitesimal interval of time, \( dt \), the energy flow across the horizon is \( (A \psi_{\text{eff}} dt + \dot{E}) \), and \( \dot{\tau}_{\text{A}} \) can be a constant. Then, following the Clausius relation, the above energy flow is the heat flow \( \delta Q \) within an interval \( dt \) across the horizon. As mentioned, the term \( \dot{E} \) corresponds to the non-equilibrium process, which will generate an additional entropy, \( dS_{\text{A}} \). With these considerations, we can write the Clausius relation as [66]

\[
\delta Q = - A \psi_{\text{eff}} dt - \dot{E} = \dot{S}_{\text{A}} (dS_{\text{A}} + d\rho_{\text{A}}).
\]

This can be considered as the non-equilibrium extension of the Clausius relation [66] with an irreversible extra entropy production term \( dS_{\text{A}} \). These results simplify the unified first law in the non-equilibrium perspective (24) to the form [73]

\[
dE_{\text{eff}} = -\frac{1}{2\pi \dot{\tau}_{\text{A}}} (dS_{\text{A}} + d\rho_{\text{A}}) + \rho_{\text{eff}} dV.
\]

Multiply throughout the above equation (26) with \( (1 - \dot{\tau}_{\text{A}}/(2H\dot{\tau}_{\text{A}})) \), and on further simplification, we arrive at another form of the unified first law given by

\[
dE_{\text{eff}} = -T_{\text{A}} (dS_{\text{A}} + d\rho_{\text{A}}) + \rho_{\text{eff}} dV + \frac{\dot{E}_{\text{A}}}{2H\dot{\tau}_{\text{A}}},
\]

To proceed further, we have to extract the entropy production term \( d\rho_{\text{A}} \). By rearranging the unified first law (27), the extra entropy production term can be written as

\[
T_{\text{A}} d\rho_{\text{A}} = -T_{\text{A}} dS_{\text{A}} + \rho_{\text{eff}} dV + \frac{\dot{E}_{\text{A}}}{2H\dot{\tau}_{\text{A}}} - dE_{\text{eff}}.
\]

Applying the continuity equation (21) on equation (28) and using the following Friedmann equations [66], which is obtained from the field equation (20) by substituting the FRW metric (1) and the effective energy-momentum tensor,

\[
H^2 + \frac{k}{a^2} = \frac{16\pi G_{\text{eff}}}{n(n - 1)} \rho_{\text{eff}}
\]

and

\[
\dot{H} = -\frac{8\pi G_{\text{eff}}}{(n - 1)} (\rho_{\text{eff}} + P_{\text{eff}}),
\]
we will obtain the entropy production term as
\[ dpS_A = \frac{n(n + 1)\Omega_A p_A^{n-1} G_{\text{eff}}}{8G_{\text{eff}}^2} dt. \]  
(31)

The entropy production term can also be calculated from the non-equilibrium extension of the Clausius relation (25) by adopting a similar procedure as shown above. We will derive the law of emergence with these results in the following subsections.

### 3.1. Emergence of cosmic space from the non-equilibrium description of the unified first law of thermodynamics

In this section, we will consider the non-equilibrium description of the unified first law of thermodynamics (27) to derive the law of emergence. Substituting the expressions for the temperature \( T_A \) (4), the horizon entropy \( S_A \) (19), the entropy production term \( dPS_A \) (31), the effective energy \( E_{\text{eff}} = \rho_{\text{eff}} V \), the effective work density \( W_{\text{eff}} \), and the non-equilibrium energy dissipation term \( \dot{E} \) into the unified first law (27) and applying the continuity equation (29), we get

\[ -2 \frac{\dot{r}_A dt}{r_A^2} = \frac{16\pi (G'_{\text{eff}} \rho_{\text{eff}} + G_{\text{eff}} \dot{\rho}_{\text{eff}})}{n(n - 1)}. \]  
(32)

On integrating the above expression, we obtain

\[ \frac{1}{r_A^2} = \frac{16\pi G_{\text{eff}} \rho_{\text{eff}}}{n(n - 1)}. \]  
(33)

We have omitted the integration constant \(^2\). Both sides of the equation (33) are multiplied by \( a^2 \) and then differentiated with respect to time. Then dividing both sides of the resultant equation by \( 2\alpha d\dot{a} \), we get

\[ -\frac{\dot{\alpha}}{\alpha} + \frac{1}{\alpha^2} = \frac{16\pi}{n(n - 1)} \left[ \frac{G_{\text{eff}} \rho_{\text{eff}} + G_{\text{eff}} \dot{\rho}_{\text{eff}}}{2H} + G_{\text{eff}} \dot{\rho}_{\text{eff}} \right]. \]  
(34)

Rearranging the equation and using the continuity equation will result in

\[ \dot{\alpha} = \dot{\alpha} H - \frac{16\pi}{n(n - 1)} \left[ \frac{nG_{\text{eff}}(\rho_{\text{eff}} + \rho_{\text{ef}})}{2} + G_{\text{eff}} \dot{\rho}_{\text{eff}} \right]. \]  
(35)

Multiplying (35) by \( \alpha n\Omega_A p_A^{n-1} \) can reduce the expression to

\[ \alpha n\Omega_A p_A^{n-1} \dot{\alpha} = \alpha n\Omega_A p_A^{n} H + \frac{4\pi\Omega_A p_A^{n+1} G_{\text{eff}} H}{(n - 2)} [(n - 2) \rho_{\text{eff}} + nP_{\text{eff}}]. \]  
(36)

The above expression can be written as

\[ \frac{dV}{dt} = \dot{\alpha} H G_{\text{eff}} \left[ \frac{\alpha A}{G_{\text{eff}}} + \frac{4\pi\Omega_A p_A^{n+1}}{(n - 2)} [(n - 2) \rho_{\text{eff}} + nP_{\text{eff}}] \right]. \]  
(37)

Identifying the terms inside the square bracket on the right-hand side of equation (37) as surface degrees of freedom \( N_{\text{sur}} = (\alpha A)/G_{\text{eff}} \) and bulk degrees of freedom \( N_{\text{bulk}} = (-4\pi\Omega_A p_A^{n+1})(n - 2)^{-1}[(n - 2) \rho_{\text{eff}} + nP_{\text{eff}}] \) will result in

\[ \frac{dV}{dt} = \dot{\alpha} H G_{\text{eff}} (N_{\text{sur}} - N_{\text{bulk}}). \]  
(38)

This is the law of emergence obtained from the unified first law of thermodynamics in the non-equilibrium perspective using Tsallis entropy. Interestingly, we can see that in the law (38), the volume appearing on the left-hand side is the volume associated with the area of the apparent horizon. On the other hand, in the equilibrium case, the corresponding form of the law contains the effective volume of the horizon. Transferring the modifications in the curvature part of the field equation into the matter part appendix B simplifies the law of emergence in the non-equilibrium case. Therefore, it is evident that the features of the laws we have derived are due to the different approaches used. Here, we focus on the correctness of the law of emergence obtained from the non-equilibrium approach using Tsallis entropy rather than the merit of the law of emergence obtained from the above-mentioned approaches.

\(^2\) Retaining the integration constant in equation (33) will manifest as a dynamical dark energy component. The dark energy component coming through the integration constant would have the same effect as the already existing dark energy component whose contribution is considered in \( \rho_{\text{ms}} \) and \( P_{\text{ms}} \) and is hence omitted to avoid redundancy.
3.2. Emergence of cosmic space from non-equilibrium description of the Clausius relation

In this section, we will show that from the Clausius relation (25), the law of emergence (38) can be obtained in the non-equilibrium approach. Substituting the expressions for the horizon entropy (19), the entropy production term (31), and the non-equilibrium energy dissipation term into the Clausius relation (25) in the non-equilibrium perspective will result in

\[
\left[ n(\rho_{\text{eff}} + P_{\text{eff}})H + \frac{G_{\text{eff}}}{2G_{\text{eff}}} \rho_{\text{eff}} \right] \Omega_n \dot{R}_A^\alpha \frac{dt}{\Omega_n \dot{R}_A^\alpha} = \frac{\dot{P}_A}{\dot{R}_A} - \frac{n(1-\Omega_n \dot{R}_A^{n-2}) dt}{2 \pi G_{\text{eff}}}.
\]

(39)

The above equation can be reduced to the equation (32) using the continuity equation (21) and the Friedmann equation (29). Then, by following the steps mentioned in section 3.1, we can obtain the law of emergence of cosmic space in Tsallis modified gravity (38). Hence, obtaining the same law of emergence in Tsallis modified gravity is possible from both the non-equilibrium Clausius relation and unified first law. In case of \( \beta = 1 \), \( G_{\text{eff}} \), \( \rho_{\text{eff}} \) and \( P_{\text{eff}} \) reduces to \( G \), \( \rho_m \) and \( P_m \), consequently the equation (38) reduces to

\[
\frac{\alpha dV}{dt} = \dot{\rho}_H G(N_{\text{sur}} - N_{\text{bulk}}),
\]

(40)

where \( N_{\text{sur}} = (\alpha A)/G \) and \( N_{\text{bulk}} = -4\pi \Omega_n \dot{R}_A^{n+1}(n - 2)^{-1}((n - 2) \rho_m + n P_m) \). Sheykhi’s proposed law of emergence in the context of Einstein’s gravity for a \((n + 1)\)-dimensional non-flat FRW universe [35] is similar to the above equation. Moreover, the equation (38) reduces to

\[
\frac{dV}{dt} = G(N_{\text{sur}} - N_{\text{bulk}}),
\]

(41)

for \((3 + 1)\)-dimensional flat universe and \( \beta = 1 \). Here, \( N_{\text{sur}} = A/G \) and \( N_{\text{bulk}} = -16\pi^2 H^{-4}(\rho_m + 3P_m) \). This is identical to Padmanabhan’s original law of emergence [31] in Einstein’s gravity for a \((3 + 1)\)-dimensional flat FRW universe. In the above limits, the equation (38) reduces to the standard results. In equation (38), the surface and the bulk degrees of freedom are not the mere identification of the terms in the expression. The profound thermodynamic connection of Einstein gravity provides the surface term in the Einstein–Hilbert action a clear thermodynamic interpretation [13, 79–83]. Similarly, the surface degrees of freedom have a clear thermodynamic interpretation [64] provided by the profound thermodynamic connection of the law of emergence. Another interesting result is that the law of emergence given in (38) can be treated as the general formula for any modified gravity theories, provided the horizon entropy in it can be written in a form \( S_A = A/(4G_{\text{eff}}) \).

3.3. Emergence of cosmic space and maximization of entropy from the non-equilibrium perspective

Any isolated system will achieve stability by evolving towards a maximum entropy state by satisfying the generalized second law (GSL), \( S \geq 0 \), and convexity condition \( S < 0 \), at least in the final stage of evolution [85, 86]. In Einstein gravity, our universe tends to achieve a maximum entropy state [86]. This is also true from the emergent perspective of cosmic space [45]. According to the law of emergence, the universe expands to achieve the state of holographic equipartition, \( N_{\text{sur}} = N_{\text{bulk}} \), where the universe is in de Sitter epoch and the corresponding entropy becomes maximum. This is also true in the context of Gauss-Bonnet and Lovelock gravity [46]. These are in the framework of equilibrium thermal evolution. Apart from the other gravity theories, the law of emergence in the context of Tsallis entropy is also consistent with entropy maximization [61], but in the equilibrium perspective. This motivates us to check whether the law of emergence, obtained from the non-equilibrium perspective (38), is consistent with the maximization of entropy in the case of Tsallis entropy. Here, the constraints that satisfy the principle of entropy maximization are obtained. The feasibility of these constraints is checked at the end de Sitter stage of an expanding universe. Here, the horizon entropy and the entropy of all the matter content of the universe constitute the total entropy. We must incorporate the extra non-equilibrium entropy production since we consider Tsallis entropy as the horizon entropy. In this paper, we neglect the entropy of the matter content since the horizon entropy is considerably more significant than the matter entropy [10, 47]. To obtain the constraints for entropy maximization, let us consider the rate of change of cosmic volume with respect to time in \((n + 1) - \) dimensional universe, which is given by

\[
\frac{dV}{dt} = n\Omega_n \dot{R}_A^{n-1}\dot{R}_A.
\]

(42)

The time derivative of the total entropy using the equations (19) and (31) is

\[
\dot{S} = \dot{S}_A + dP_{\text{eff}} = \frac{(n\Omega_n)^{\gamma}(n - 1)(n + 1 - \beta(n - 1))\dot{R}_A^{\gamma(n-1)(n-1)}\dot{R}_A}{2}.
\]

(43)
Comparing the equations (42) and (43), we can write
\[
d\frac{dV}{dt} = \frac{2(nM_{\ast})^3 - \beta p}{(n - 1)(n + 1 - \beta(n - 1))} S. \tag{44}\]

Using the law of emergence (38), the equation (44) can be written as
\[
\dot{S} = \frac{H(n - 2)(n + 1 - \beta(n - 1))}{4} (N_{\text{sur}} - N_{\text{bulk}}). \tag{45}\]

Here, \(N_{\text{sur}} - N_{\text{bulk}} \geq 0\), which implies that the total entropy is non-decreasing for \(n + 1 - \beta(n - 1) > 0\). Thus, the generalized second law remains valid for \(\beta < (n + 1)/(n - 1)\). We assume that the constraint for the parameter \(\beta\) obtained from the maximization condition is true for the following. Now, we consider the second time derivative of total entropy to check the convexity condition, which can be obtained from the above equation by differentiating with the cosmic time, and we get as,
\[
\dot{S} = S_{\lambda} + d\beta S = (n + 1 - \beta(n - 1))(n - 2) \left[ H(N_{\text{sur}} - N_{\text{bulk}}) + H(N_{\text{sur}} - N_{\text{bulk}}) \right]. \tag{46}\]

During the de Sitter epoch, \(N_{\text{sur}} = N_{\text{bulk}}\), hence the first term inside the square brackets will vanish and for \(\dot{S} < 0\), \(N_{\text{sur}} - N_{\text{bulk}}\) should be less than zero. From the equation (38) and the Friedmann equations (29) and (30), we know that
\[
N_{\text{sur}} - N_{\text{bulk}} = \frac{2(nM_{\ast})^3 - \beta p_{\ast}^2}{(n - 2)H}. \tag{47}\]

Substituting the above equation (47) and its derivative in to (46) results to
\[
\dot{S} = \frac{2(nM_{\ast})^3 - \beta p_{\ast}^2}{2} \left[ ((n - 1)\beta - 1)\ddot{A} + \ddot{A} \ddot{A} \right]. \tag{48}\]

For \(\beta = 1\), the above equations (43) and (48) reduce to the corresponding derivatives of entropy in Einstein gravity [47]. For the above equation to be less than zero, at least in the final stage, the condition
\[
((n - 1)\beta - 1)\ddot{A} < -\ddot{A} \ddot{A}. \tag{49}\]

should be satisfied. To check the validity of the above condition, let us consider the relation of \(\ddot{A}\) in terms of the effective equation of state parameter \(\omega_{\ast\text{eff}}\), taken from the equation of state \(p_{\ast\text{eff}} = \omega_{\ast\text{eff}}\rho_{\ast\text{eff}}\) written using the continuity equation (21) and the Friedmann equations (29) and (30), in the following way
\[
\ddot{A} = \frac{nH\dot{A}(1 + \omega_{\ast\text{eff}})}{2}. \tag{50}\]

Taking the derivative of the above equation will give a relation connecting \(\ddot{A}\) and \(\omega_{\ast\text{eff}}\) as follows
\[
\ddot{A} = \frac{n\ddot{A}}{2} \left[ (1 + \omega_{\ast\text{eff}}) \left( \dot{H} + \frac{nH^2(1 + \omega_{\ast\text{eff}})}{2} \right) + H\dot{\omega}_{\ast\text{eff}} \right]. \tag{51}\]

During the de Sitter epoch, \(\omega_{\ast\text{eff}}\) tends to \(-1\), then \(\ddot{A} = 0\), which reduces the condition (49) to \(-\ddot{A} \ddot{A} > 0\). Then for \(\dot{S} < 0\), \(\ddot{A}\) should be negative. When \(\omega_{\ast\text{eff}}\) tends to \(-1\), the first term inside the curly bracket of equation (51) vanishes, and the negativity of \(\ddot{A}\) will guarantee \(\ddot{A} < 0\), thereby showing \(\dot{S} < 0\) in the final stage.

From the emergent perspective (equation (45) and (47)), the validity of the GSL demands \(\ddot{A} > 0\) and the same constraint is obtained from the time derivative of total entropy (43) indicating that the entropy maximization is deducible from the law of emergence. The equation (50) guarantees that \(\ddot{A} > 0\) for an universe approaching end de Sitter stage with \(\omega_{\ast\text{eff}} > -1\). For \(\dot{S} < 0\) in the end stage we obtained the constraint
\[
((n - 1)\beta - 1)\ddot{A} < -\ddot{A} \ddot{A} \text{ from the emergent perspective and interestingly same constraint is obtained from the second derivative of total entropy (19), thereby again indicating that the entropy maximization is deducible from the law of emergence. The entropy maximization constraints obtained using Tsallis entropy through the equilibrium approach [61] are similar to the above results. However, through the non-equilibrium approach, we obtained an additional constraint, an upper bound for the Tsallis parameter } \beta < (n + 1)/(n - 1). The additional entropy production term in the non–equilibrium context has resulted in the additional constraint on the Tsallis parameter. In the equilibrium thermodynamic approach, such an upper bound for the Tsallis parameter \(\beta\) arises from the surface degrees of freedom (16). In \((3 + 1)\)-dimensional universe, the constraint reduces to \(\beta < 2\) and is found to be true in Tsallis modified gravity from the observational constraints [65]. Hence, all these results show that using Tsallis entropy, it is possible to state that the emergence of space is due to the disparity between the degrees of freedom residing on the surface and the bulk and is in pursuit of holographic equipartition or, in other words, an equilibrium state. These outcomes are consistent with the results in general relativity [45], Gauss–Bonnet, and Lovelock gravity [46]. These results imply that the important link between the law of emergence and the fundamental thermodynamic laws, as well as the thermodynamic interpretation of the expansion of the
universe, remains valid even with the inclusion of quantum corrections to the horizon entropy since the correction terms in Tsallis entropy can be treated as the quantum corrections to Bekenstein–Hawking entropy [52–55].

4. Conclusions

The link between gravity and thermodynamics conveys ‘gravity as an emergent phenomenon’ [30]. That made Padmanabhan propose that the space could also be an emergent structure [31, 32]. Based on this, the emergence of space with cosmic time adds a new perspective to the expansion of the universe. The new perspective in terms of the degrees of freedom on the boundary and the bulk is given by the law of emergence

\[ dv/\gamma dt = \gamma L^2_\gamma (\rho_{\text{sur}} - \rho_{\text{bulk}}) \]

A link exists between the law of emergence and the fundamental thermodynamic laws, as one can derive the former from the latter. In this paper, we explored this link in the background of Tsallis modified gravity from the non-equilibrium perspective and compared the results with that from the equilibrium perspective. The best-known way to understand the universe from the thermodynamic perspective is by applying the black hole thermodynamics to the horizon thermodynamics of the universe. In such a case, Tsallis entropy \( S_A = \gamma A^{3/4} \) is the best choice for horizon entropy as it is the generalized entropy introduced to mitigate the violation of the extensivity of thermodynamic entropy. Considerable applications of Tsallis entropy in gravitation and cosmology in recent years also add to the interest [54, 56, 57, 62, 63, 65, 88–90]. Hence, we have considered it as the apparent horizon entropy and derived the law of emergence from the fundamental thermodynamic laws.

Firstly, we obtained the law of emergence using an equilibrium approach from both the unified first law of thermodynamics and the Clausius relation. The law of emergence thus obtained substantiates the proposal of Sheykhi [56] and Chen [61]. It describes the expansion of the space as the one that emerges in proportion to the rate of change of the effective volume since the effects of Tsallis entropy are on the geometrical part. In the case of Tsallis entropy, there is a contradiction between the equilibrium Clausius relation and the standard energy-momentum conservation [65], which can be understood when we try to obtain the field equation. Despite that, in appendix A, the field equation has been obtained from the equilibrium Clausius relation with Tsallis entropy by assuming the horizon area to be constant as it is the only possible way for the Clausius relation to hold. The corresponding Friedmann equation obtained from the field equation is the same as obtained from the equilibrium Clausius relation. However, \( \gamma \) differs by a factor of \((2 - \beta)\) in the \((3 + 1)\) dimension. The lack of uniqueness in the form of \( \gamma \) poses a problem in the equilibrium approach. The apparent horizon area can be considered constant only for the infinitesimal time interval for an expanding universe. Hence, the equilibrium Clausius relation does not hold in the case of Tsallis entropy [65].

In the present paper, we restructured the Tsallis entropy, leading to the concise field equation and a generalized continuity equation for a perfect fluid, following the formulation of Tian and Booth [66]. Consequently, the corresponding thermodynamic relations will hold with the respective energy-momentum conservation.

We restructured the Tsallis entropy to \( S_A = A/(4G_{\text{eff}}) \), similar to the standard form of the Bekenstein–Hawking entropy, by defining \( G_{\gamma \text{eff}} = ((m\omega_0)^{3/4} - \beta A^{3/4} - \gamma A^{3/4})/(\beta A) \), which is not the time-varying gravitational constant. Consequently, the gravitational field equation of Tsallis modified gravity takes a concise form

\[ G_{\gamma \mu \nu} = 8\pi G_{\text{eff}} T_{\mu \nu}^{\text{therm}} \]

similar to the Einstein ‘s gravity. The resulting continuity equation has an extra term that depends on the \( G_{\gamma \text{eff}} \) and \( \rho_{\text{eff}} \). That introduces the corresponding non-equilibrium energy dissipation, \( \dot{\epsilon}_C \), implying the need for a non-equilibrium thermodynamic perspective. An additional entropy, \( dS_{\gamma} \), is generated to counteract the created energy difference, which is evaluated using Friedmann equations. By observing the entropy production term (31) and the equations (43), (50), (48) and (51), we can understand that the rate of entropy production will decrease during the late acceleration stage as the derivative of the rate of entropy production is less than zero in the end stage of the evolution (i.e., for \( \ddot{A} < 0 \)). That shows that the universe is approaching thermal equilibrium. Whereas the results from the entropy predictions to the early universe [91–93] indicate that ‘the rate of production of entropy increases during inflation’ [91]. That shows that the universe is receding from thermal equilibrium. This comparison does provide us with a picture of the thermal evolution of the universe, though the realms discussed here are different.

We have obtained the law of emergence through the non-equilibrium approach by incorporating the non-equilibrium energy dissipation and entropy production term into the unified first law and the Clausius relation. In the non-equilibrium perspective, the law of emergence describes the rate of change of the areal volume as the volume increase due to the accelerated expansion of the universe. In the non-equilibrium approach, the corrections due to the Tsallis entropy are on the effective energy-momentum tensor rather than on the curvature part in the field equation. That is to formulate a general law of emergence, which is impossible in the latter case (appendix B). In the non-equilibrium case, the law of emergence is lucid mainly due to the introduction of \( G_{\gamma \text{eff}} \) and the resulting effective energy-momentum tensor, \( T_{\gamma \mu \nu}^{\text{therm}} \). Another advantage of this non-equilibrium
approach is that the ambiguity in $\gamma$ does not cause a problem like in equilibrium since the modifications due to Tsallis entropy are transferred to the matter part. Using this approach, the thermodynamic perspective of the expansion of the universe is achievable in the case of Tsallis entropy. It can be concluded that the law of emergence (38) will hold for any modified gravity theories for which the horizon entropy is of the form $S_A = A/(4G_{\text{eff}})$. This result can be obtained from the generalized law of emergence (28 in [48]) derived using a generic method that holds for any modified gravity theories with the above entropy form.

Considering the fact that the Tsallis entropy demands non-equilibrium thermodynamic treatment, our intention through this paper is to direct attention to the accurate non-equilibrium approach to obtain the law of emergence with Tsallis entropy rather than pointing out the merit of the law of emergence obtained from these approaches. The effects of Tsallis entropy were present in the curvature part in the equilibrium case. As a result, the increase in the volume of emerged space is given by the rate of change of effective volume, and the number of surface degrees of freedom is proportional to the effective area. In the non-equilibrium case, the effects of Tsallis entropy are transferred to the matter part. As a result, the increase in the volume of emerged space is given by the rate of change of areal volume, and the number of surface degrees of freedom is proportional to area, and the number of bulk degrees of freedom is in terms of effective pressure and density. The vital point is that the modifications to the (bulk or surface) degrees of freedom depend only on the entropy-area relation [39], as shown here.

Using a non-equilibrium perspective, we have analyzed that the entropy maximization with Tsallis entropy is deducible from the law of emergence (38). We obtained the same constraints from both. Our results were similar to the constraints obtained in Einstein theory, Gauss-Bonnet theory, and Lovelock gravity [47] and in the equilibrium context of Tsallis entropy, except there is an extra constraint on the Tsallis parameter as $\beta < (n + 1)/(n - 1)$. The additional entropy production term in the non-equilibrium context has resulted in the additional constraint on the Tsallis parameter. Consequently, considering the evolution of the universe as the proclivity to achieve maximum entropy state of equilibrium remains valid even with the inclusion of quantum corrections to the horizon entropy since the correction terms in Tsallis entropy can be treated as the quantum corrections to Bekenstein–Hawking entropy [32–55]. In future investigations, we expect to link the results in the present paper with the observational data [94–96], as that is possible with Tsallis entropy, unlike in the case of Rényi [97] and $q$-additive entropy [98, 99]. It will be interesting to see the existence of thermodynamic and holographic connections of the surface and bulk terms in the action functional of Tsallis modified gravity, just like in other theories of gravity [81, 82] in future investigations.

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Data availability statement

No new data were created or analysed in this study.

Appendix A. Gravitational field equation and Friedmann equation based on Tsallis entropy from equilibrium thermodynamic perspective

In this section, we would like to obtain the field equation based on Tsallis entropy from the equilibrium Clausius relation and obtain the Friedmann equation using the approach of Jacobson [11], Eling et al [24], Asghari and Sheykhi [65], and Gennaro [100]. Here, we consider the $(3 + 1)$-dimensional universe for convenience. Following this approach, the heat flow across the horizon $H$ is expressed as [11]

$$
\delta Q = -\kappa \int_{H} \lambda T_{\mu \nu} k^{\mu} k^{\nu} dA,
$$

(A.1)

where $k^{\mu}$ is the tangent vector to the horizon generators for an affine parameter $\lambda$, and $dA$ is the area element on a cross-section of the horizon. Assuming the Tsallis entropy as the horizon entropy, $dS_A$ can be written as [11, 65]

$$
dS_A = \gamma \beta \int_{H} -\lambda R_{\mu \nu} A^{\beta - 1} k^{\mu} k^{\nu} d\lambda dA,
$$

(A.2)

where $R_{\mu \nu}$ is the Ricci tensor. Substituting equations (A.1) and (A.2) into the equilibrium Clausius relation $\delta Q = T_{\lambda} dS_{A}$, we get [65]
\[ \int_{\mathcal{H}} \left( -\lambda T_{\mu\nu} + \frac{\gamma \beta}{2\pi} R_{\mu\nu} A^{\beta-1} \right) k^\nu d\lambda A = 0 \quad (A.3) \]

For all null \( k^\nu \), and for a scalar \( f \), we have
\[ -T_{\mu\nu} + \frac{\gamma \beta}{2\pi} R_{\mu\nu} A^{\beta-1} = f g_{\mu\nu}. \quad (A.4) \]

Then imposing the energy-momentum conservation \( \nabla^\mu T_{\mu\nu} = 0 \) and using Bianchi identity, we obtain
\[ \frac{\gamma \beta}{4\pi} (\partial_\mu R) A^{\beta-1} + \frac{\gamma \beta}{2\pi} R_{\mu\nu} \partial^\mu A^{\beta-1} = \partial_\mu f. \quad (A.5) \]

The second term on the left-hand side of the equation is inconsistent with the right-hand side, revealing that the equilibrium Clausius relation and the energy-momentum conservation do not hold together. However, if we assume the horizon area to be constant \([100]\), that is, \( \partial^\mu A^{\beta-1} = 0 \), then we have
\[ f = \frac{\gamma \beta}{4\pi} R A^{\beta-1}. \quad (A.6) \]

Substituting the expression for \( f \) in equation \((A.4)\) will result in the following field equation in the case of Tsallis entropy
\[ \left( R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) A^{\beta-1} = \frac{2\pi}{\gamma \beta} T_{\mu\nu}. \quad (A.7) \]

The corresponding Friedmann equation in the \((3 + 1)\) FLRW universe from the time component of the field equation is
\[ H^2 + \frac{k}{a^2} = \frac{8\pi G}{3} \rho, \quad (A.8) \]

if \( \gamma = (4\pi)^{1-\beta}/(4G\beta) \). This is identical to the Friedmann equation obtained using the equilibrium approach by Sheykhi \([56]\) from the unified first law of thermodynamics by assuming \( \gamma = (2 - \beta)(4\pi)^{1-\beta}/(4G\beta) \). From the Clausius relation using Sheykhi’s equilibrium approach, the same Friedmann equation can be derived by assuming \( \gamma = (2 - \beta)(4\pi)^{1-\beta}/(4G\beta) \). However, here we have derived the Friedmann equation \((A.8)\) by assuming \( \gamma = (4\pi)^{1-\beta}/(4G\beta) \) from the field equation that is obtained from the Clausius relation using Jacobson’s approach \([11]\). The form of the \( \gamma \) differs in both equilibrium approaches, but they arrive at the same Friedmann equation. For \( \beta = 1 \), i.e., for Bekenstein-Hawking entropy, the two relations of \( \gamma \) are equivalent. Thus, the lack of uniqueness in the form of \( \gamma \) is a problem in the equilibrium approach. In the non-equilibrium approach used in section 3, \( \gamma \) is related to \( G_{\text{eff}} \) such that different forms of \( \gamma \) lead to different dynamical equations.

**Appendix B. Tsallis modified gravity with curvature corrections from non-equilibrium thermodynamic perspective**

Using the non-equilibrium extension of the Clausius relation, Asghari and Sheykhi \([65]\) have derived the gravitational field equation based on Tsallis entropy as follows
\[ R_{\mu\nu} A^{\beta-1} - \nabla_\mu \nabla_\nu A^{\beta-1} - \frac{1}{2} R A^{\beta-1} g_{\mu\nu} + \Box A^{\beta-1} g_{\mu\nu} = \frac{2\pi}{\gamma \beta} T^{\text{int}}_{\mu\nu}. \quad (B.1) \]

Here \( \Box = \nabla^\alpha \nabla_\alpha \) is the covariant d’Alembertian operator. It is clear from equation \((B.1)\) that the curvature part of the field equation is modified. The Friedmann equation for a flat \((3 + 1)\)-dimensional universe using the field equation in Tsallis modified gravity \((B.1)\) is given by \([65]\)
\[ H^{4-2\beta} = \frac{1}{4\beta - 3} \frac{8\pi G}{3} \rho_{\text{mc}}. \quad (B.2) \]

Here, \( \gamma = (4\pi)^{1-\beta}/(4G\beta) \). The rate of change of effective volume, or areal volume, in this case, will be
\[ 4\gamma \frac{dV}{dt} = \ddot{\varphi} A \left[ \frac{4\gamma \beta \ddot{A}}{2 - \beta} + \frac{16\pi^2 \ddot{A}^2}{3(2 - \beta)(4\beta - 3)} (\rho_m + 3P_m) \right], \quad (B.3) \]

and
\[ \frac{dV}{dt} = \ddot{\varphi} H G \left[ \frac{A}{(2 - \beta)G} + \frac{16\pi^2 \ddot{A}^2}{3(2 - \beta)(4\beta - 3)} (\rho_m + 3P_m) \right], \quad (B.4) \]

respectively. Thus, we cannot relate the emerged space to the surface and bulk degrees of freedom in Tsallis gravity with modifications in the curvature part of the field equation. However, this can be achieved if we can
modify the matter part of the field equation. Re-writing the equation (B.1) in a compact, general relativity form is possible by moving all the correction terms to the matter part of the field equation, as shown below,

\begin{equation}
R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = 8\pi G_{\text{eff}} \left[ \frac{\mathcal{T}^{(m)}_{\mu\nu}}{\beta} + \frac{\gamma \nabla_{\lambda} A^{\lambda - 1}}{2\pi} - \frac{\gamma \nabla_{\lambda} A^{\lambda - 1} g_{\mu\nu}}{2\pi} \right].
\end{equation}

(B.5)

We have followed the latter way in section 3 to express the effects of Tsallis entropy on the universe and obtained the law of emergence.

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