On the Algorithmic Power of Spiking Neural Networks

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Abstract

Spiking Neural Networks (SNN) are mathematical models in neuroscience to describe the dynamics among a set of neurons that interact with each other by firing instantaneous signals, a.k.a., spikes. Interestingly, a recent advance in neuroscience [Barrett-Denève-Machens, NIPS 2013] showed that the neurons’ firing rate, i.e., the average number of spikes fired per unit of time, can be characterized by the optimal solution of a quadratic program defined by the parameters of the dynamics. This indicated that SNN potentially has the computational power to solve non-trivial quadratic programs. However, the results were justified empirically without rigorous analysis.

We put this into the context of natural algorithms and aim to investigate the algorithmic power of SNN. Especially, we emphasize on giving rigorous asymptotic analysis on the performance of SNN in solving optimization problems. To enforce a theoretical study, we first identify a simplified SNN model that is tractable for analysis. Next, we confirm the empirical observation in the work of Barrett et al. by giving an upper bound on the convergence rate of SNN in solving the quadratic program. Further, we observe that in the case where there are infinitely many optimal solutions, SNN tends to converge to the one with smaller $\ell_1$ norm. We give an affirmative answer to our finding by showing that SNN can solve the $\ell_1$ minimization problem under some regular conditions.

Our main technical insight is a dual view of the SNN dynamics, under which SNN can be viewed as a new natural primal-dual algorithm for the $\ell_1$ minimization problem. We believe that the dual view is of independent interest and may potentially find interesting interpretation in neuroscience.

1 Introduction

The theory of natural algorithms is a framework that bridges the algorithmic thinking in computer science and the mathematical models in biology. Under this framework, biological systems are viewed as algorithms to efficiently solve specific computational problems. Seminal works such as bird flocking [Cha09, Cha12], slime systems [NYT00, TKN07, BMV12], and evolution [LPR+14, LP16] successfully provide algorithmic explanations for different natural objects. These works

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give rigorous theoretical results to confirm empirical observations, shed new light on the biological systems through computational lens, and sometimes lead to new biologically inspired algorithms.

In this work, we investigate *Spiking Neural Networks (SNNs)* as natural algorithms for solving convex optimization problems. SNNs are mathematical models for biological neural networks where a network of neurons transmit information by *firing spikes* through their synaptic connections (i.e., edges between two neurons). Our starting point is a seminal work of Barrett, Denève, and Machens [BDM13], where they showed that the *firing rate* (i.e., the average number of spikes fired by each neuron) of a certain class of *integrate-and-fire* SNNs can be characterized by the optimal solutions of a quadratic program defined by the parameters of SNN. Thus, the SNN can be viewed as a natural algorithm for the corresponding quadratic program. However, no rigorous analysis was given in their work.

We bridge the gap by showing that the firing rate converges to an optimal solution of the corresponding quadratic program with an explicit polynomial bound on the convergent rate. Thus, the SNN indeed gives an *efficient algorithm* for solving the quadratic program. To the best of our knowledge, this is the first result with an explicit bound on the convergent rate. Previous works [SRH13, SZHR14, TLD17] on related SNN models for optimization problems are either heuristic or only proving convergence results when the time goes to infinity (see Section 1.4 for full discussion on related works).

We take one step further to ask what other optimization problems can SNNs efficiently solve. As our main result, we show that when configured properly, SNNs can solve the $\ell_1$ minimization problem\(^1\) in polynomial time\(^2\). Our main technical insight is interpreting the dynamics of SNNs in a *dual space*. In this way, SNNs can be viewed as a new primal-dual algorithm for solving the $\ell_1$ minimization problem.

In the rest of the introduction, we will first briefly introduce the background of spiking neural networks (SNNs) and formally define the mathematical model we are working on. Next, our results will be presented and compared with other related works. Finally, we wrap up this section with potential future research directions and perspectives.

### 1.1 Spiking Neural Networks

Spiking neural networks (SNNs) are mathematical models for the dynamics of biological neural networks. An SNN consists of neurons, and each of them is associated with an intrinsic electrical charge called *membrane potential*. When the potential of a neuron reaches a certain level, it will fire an instantaneous signal, i.e., spike, to other neurons and increase or decrease their potentials.

Mathematically, the dynamic of neuron’s membrane potential in an SNN is typically described by a *differential equation*, and there are many well-studied models such as the *integrate-and-fire model* [Lap07], the *Hodgkin-Huxley model* [HH52], and their variants [Fit61, Ste65, ML81, HR84, Ger95, KGH97, BL03, FTHVVB03, I+03, TMS14]. In this work, we focus on the integrate-and-fire model defined as follows. Let $n$ be the number of neurons and $\mathbf{u}(t) \in \mathbb{R}^n$ be the vector of membrane potentials where $\mathbf{u}_i(t)$ is the potential of neuron $i$ at time $t$ for any $i \in [n]$ and $t \geq 0$. The dynamics

\[^1\]The problem is defined as given matrix $A \in \mathbb{R}^{m \times n}$, vector $\mathbf{b} \in \mathbb{R}^m$, and guaranteed that there is a solution to $Ax = b$. The goal is finding a solution $x$ with the smallest $\ell_1$ norm. See Section 2 for formal definition.

\[^2\]The running time is polynomial in a parameter depending on the inputs. In some cases, this parameter might cause the running to be quasi-polynomial or sub-exponential. See Section 3.3 for more details.
of $u(t)$ can be described by the following differential equation: for each $i \in [n]$ and $t \geq 0$

$$\frac{d}{dt} u_i(t) = \sum_{j \in [n]} -C_{ji}(t)s_j(t) + I_i(t)$$

(1)

where the initial value of the potentials are set to 0, i.e., $u_i(0) = 0$ for each $i \in [n]$. There are two terms that determine the dynamics of membrane potentials as shown in (1). The simpler term is the input charging\(^3\) $I(t) \in \mathbb{R}^n$, which can be thought of as an external effect on each neuron. The other term models the instantaneous spike effect among neurons. Specifically, the $-C_{ji}(t)s_j(t)$ term models the effect on the potential of neuron $i$ when neuron $j$ fires a spike. Here $C(t) \in \mathbb{R}^{n \times n}$ is the connectivity matrix that encodes the synapses between neurons, where $C_{ji}(t)$ describes the connection strength from neuron $j$ to neuron $i$. $s(t) \in \mathbb{R}^n$ is the spike train that records the spikes of each neuron, and $s_i(t)$ can be thought of as indicating whether neuron $i$ fires a spike at time $t$. To sum up, the $-C_{ji}(t)s(t)$ term decreases\(^4\) the potential of neuron $i$ by $C_{ji}(t^*)$ whenever neuron $j$ fires a spike at time $t^*$.

The spike train $s(t)$ is determined by the spike events, which are in turn determined by the spiking rule. A typical spiking rule is the threshold rule. Specifically, let $\eta > 0$ be the spiking threshold, the threshold rule simply says that neuron $i$ fires a spike at time $t$ if and only if $u_i(t) > \eta$. Next, record the timings when neuron $i$ fires a spike as $0 \leq t_1^{(i)} < t_2^{(i)} < \ldots$ and let $k_i(t)$ be the number of spikes within time $[0, t]$. An important statistics of the dynamics is the firing rate defined as $x_i(t) := k_i(t)/t$ for neuron $i \in [n]$ at time $t$, namely, the average number of spikes of neuron $i$ up to time $t$. The last thing we need for specifying $s(t)$ is the spike shape, which can be modeled as a function $\delta : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$. Intuitively, the spike shape describes the effect of a spike, and standard choices of $\delta$ could be the Dirac delta function or a pulse function with an exponential tail. Now we can define $s_i(t) = \sum_{1 \leq s \leq k_i(t)} \delta(t - t_s^{(i)})$ to be the spike train of neuron $i$ at time $t$.

We provide the following concrete example to illustrate the SNN dynamics introduced above.

**Example 1.1.** Let $n = 2$, $\eta = 1$, and $\delta$ be the Dirac delta function such that for any $\epsilon > 0$, $\int_0^\epsilon \delta(t)dt = 1$ and $\delta(t) \geq 0$ for any $t \geq 0$. Let both input charging and connectivity matrix be static, i.e., $I(t) = I$ and $C(t) = C$ for any $t \geq 0$, and consider

$$C = \begin{pmatrix} 1 & 0 \\ -0.1 & 1 \end{pmatrix}, \quad I = \begin{pmatrix} 0.1 \\ 0 \end{pmatrix}, \quad \text{and} \quad u(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$  

In Figure 1, we simulate this SNN for 500 seconds. We can see that neuron 1 fires a spike every ten seconds while neuron 2 fires a spike every one hundred seconds. As a result, the firing rate of neuron 1 will gradually converge to 0.1 and that of neuron 2 will go to 0.01.

In general, both the input charging vector $I(t)$ and the connectivity matrix $C(t)$ can evolve over time, in which the change of $I(t)$ models the variation of the environment and the change of $C_{ji}(t)$ captures the adaptive learning behavior of the neurons to the environmental change. Understanding how synapses evolve over time (i.e., synapse plasticity) is a very important subject in neuroscience. However, in this work, we follow the choice of Barrett et al. [BDM13] and consider static SNN dynamics, where both the input charging $I(t)$ and the synapses $C(t)$ are constants. Although this is a special case compared to the general model in (1), we justify the choice of static

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\(^3\)Also known as input signal or input current.

\(^4\)If $C_{ji}(t^*) < 0$, then the potential of neuron $i$ actually increases by $|C_{ji}(t^*)|$. 

Figure 1: The example of SNN with two neurons. In (a), we describe the dynamic of this SNN. Note that the effect of spikes is the negation of the synapse encoded in the connectivity matrix $C$. In (b), we plot the membrane potential vectors $\mathbf{u}(t)$. In (c), we plot the timings when neurons fire a spike. One can see that neuron 1 fires a spike every ten seconds while neuron 2 fires a spike every one hundred seconds. In (d), we plot the firing rate vector $\mathbf{x}(t)$. One can see that the firing rate of neuron 1 will gradually converge to 0.1 and that of neuron 2 will go to 0.01.

SNN by showing that SNN already exhibits non-trivial computational power even in this restricted model.

As in Barrett et al. [BDM13], we focus on static SNN and view it as a natural algorithm for optimization problems. Specifically, given an instance to the optimization problem, the goal is to configure a static SNN (by setting its parameters) so that the firing rate converge to an optimal solution efficiently. In this sense, the result of Barrett et al. [BDM13] can be interpreted as a natural algorithm for certain quadratic programs. In our eyes, the solution being encoded as the firing rate is an interesting and peculiar feature of the SNN dynamics. Also, the dynamics of a static SNN can be viewed as a simple distributed algorithm with a simple communication pattern. Specifically, once the dynamics is set up, each neuron only needs to keep track of its potential and communicate with each other through spikes.

1.2 Our Results

Barrett et al. [BDM13] gave a clean characterization of the firing rates by the network connectivity and input signal. Concretely, they considered static SNN where both the connectivity matrix $C \in \mathbb{R}^{n \times n}$ and the external charging $\mathbf{I} \in \mathbb{R}^n$ do not change with time. They argued that the firing rate would converge to the solution of the following quadratic program.

$$\begin{align*}
\min_{\mathbf{x} \in \mathbb{R}^n} & \quad \| C \mathbf{x} - \mathbf{I} \|_2^2 \\
\text{subject to} & \quad x_i \geq 0, \forall i \in [n].
\end{align*}$$

They supported this observation by giving simulations on the so called tightly balanced networks and yielded pretty accurate predictions in practice. Also, they heuristically explained the reason...
how they came up with the quadratic program. However, no rigorous theorem had been proved on
the convergence of firing rate to the solution of this quadratic program.

To give a theoretical explanation for the discovery of [BDM13], we start with a simpler SNN
to enable the analysis.

**The simple SNN model** In the simple SNN model, we make two simplifications on the general
model in (1).

First, we pick the shape of spike to be the Dirac delta function. That is, let
\( \delta(t) = 1_{t=0} \) and thus \( s_i(t) = 1_{u_i(t) > \eta} \). This simplification saves us from complicated calculation while the Dirac
delta function still captures the instantaneous behavior of a spike.

Second, we consider the connectivity matrix \( C \) in the form
\( C = \alpha \cdot A^\top A \) where \( \alpha > 0 \) is the
spiking strength and \( A \in \mathbb{R}^{m \times n} \) is the Cholesky decomposition of \( C \). The reason for introducing \( \alpha \)
is to model the height of the Dirac delta function. Mathematically, it is redundant to have both
\( \alpha \) and \( C \) since the model remains the same when combining \( \alpha \) with \( C \). However, as we will see
in the next subsection, separating \( \alpha \) and \( C \) is meaningful as \( C \) corresponds to the input
of the computational problem and \( \alpha \) is the parameter that one can choose to configure an SNN to solve
the problem.

In this work, we focus on the algorithmic power of SNN in the following sense. Given a problem
instance, one configures a SNN and sets the firing rate \( x(t) \) to be the output at time \( t \). We say this
SNN solves the problem if \( x(t) \) converges to the solution of the problem.

**Simple SNN solves the non-negative least squares.** As mentioned, Barrett et al. [BDM13]
identified a connection between the firing rate of SNN with integrate-and-fire neurons and a
quadratic programming problem (2). They gave empirical evidence for the correctness of this
connection, however, no theoretical guarantee had been provided. Our first result confirms their
observation by giving the first theoretical analysis. Specifically, when \( C = A^\top A \) and \( I = A^\top b \), the
firing rate will converge to the solution of the following non-negative least squares problem.

\[
\begin{align*}
\text{minimize} & \quad \|Ax - b\|_2^2 \\
\text{subject to} & \quad x_i \geq 0, \quad \forall i \in [n].
\end{align*}
\]

**Theorem 1** (informal). Given \( A \in \mathbb{R}^{m \times n}, \ b \in \mathbb{R}^{m} \), and \( \epsilon > 0 \). Suppose \( A \) satisfies some regular
conditions\(^5\). Let \( x(t) \) be the firing rate of the simple SNN with \( 0 < \alpha \leq \alpha(A) \) where \( \alpha(A) \) is a
function depending on \( A \). When \( t \geq \Omega(\sqrt{\frac{n}{\epsilon \|b\|_2}}) \), \( x(t) \) is an \( \epsilon \)-approximate solution\(^7\) for the non-negative least squares problem of \((A, b)\).

See Theorem 6 in Section 4 for the formal statement of this theorem. To the best of our
knowledge, this is the first\(^8\) theoretical result on the analysis of SNN with an explicit bound on
the convergence rate and shows that SNN can be implemented as an efficient algorithm for an
optimization problem.

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\(^5\)More details about the regular conditions will be discussed in Section 3.3.

\(^6\)The \( \Omega(\cdot) \) and the \( O(\cdot) \) later both hide the dependency on some parameters of \( A \). See Section 3.3.

\(^7\)See Definition 1 for the formal definition of \( \epsilon \)-approximate solution.

\(^8\)See Section 1.4 for comparisons with related works.
Simple SNN solves the $\ell_1$ minimization problem. In addition to solving the non-negative least squares problem, as our main result, we also show that the simple SNN is able to solve the $\ell_1$ minimization problem, which is defined as minimizing the $\ell_1$ norm of the solutions of $Ax = b$. $\ell_1$ minimization problem is also known as the basis pursuit problem proposed by Chen et al. [CDS01]. The problem is widely used for recovering sparse solution in compressed sensing, signal processing, face recognition etc.

Before the discussion on $\ell_1$ minimization, let us start with a digression on the two-sided simple SNN for the convenience of future analysis.

$$\frac{d}{dt} u(t) = -\alpha \cdot A^T As(t) + A^T b$$

where $s(t) = 1_{u(t) > \eta} - 1_{u(t) < -\eta}$. Note that the two-sided SNN is a special case of the one-sided SNN in the sense that one can use the one-sided SNN to simulate the two-sided SNN as follows. Given a two-sided SNN described above with connectivity matrix $C = A^T A$ and external charging $I = A^T b$. Let $C' = (A^T A, -A^T A)$ and $I' = (-A^T b, b)$. Intuitively, this can be thought of as duplicating each neuron and flip its connectivities with other neurons.

To solve the $\ell_1$ minimization problem, we simply configure a two-sided SNN as follows. Given an input $(A, b)$, let $C = A^T A$ and $I = A^T b$. Now, we have the following theorem.

**Theorem 2** (informal). Given $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $\epsilon > 0$. Suppose $A$ satisfies some regular conditions. Let $x(t)$ be the firing rate of the two-sided simple SNN with $0 < \alpha \leq \alpha(A)$ where $\alpha(A)$ is a function depending on $A$. When $t \geq \Omega(n^3 \epsilon^2)$, $x(t)$ is an $\epsilon$-approximate solution for the $\ell_1$ minimization problem of $(A, b)$.

See Theorem 5 for the formal statement of this theorem. As we will discuss in the next subsection, under the dual view of the SNN dynamics, the simple two sided SNN can be interpreted as a new natural primal-dual algorithm for the $\ell_1$ minimization problem.

**1.3 A Dual View of the SNN Dynamics**

The main techniques in this work is the discovery of a dual view of SNN. Recall that the dynamics of a static SNN can be described by the following differential equation.

$$\frac{d}{dt} u(t) = -\alpha \cdot C s(t) + I$$

where $u(0) = 0$ the parameters $C$ and $I$ can be represented as $C = A^T A$ and $I = A^T b$ for some $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. For simplicity, we pick the firing threshold $\eta = 1$ here. Let us call the dynamics of $u(t)$ the primal SNN. Now, the dual SNN, can be defined as follows.

$$\frac{d}{dt} v(t) = -\alpha \cdot As(t) + b$$

where $v(0) = 0$ and $s(t)$ defined as the usual way. At first glance, this merely looks like a simple linear transformation. Nevertheless, the dual SNN provides a nice geometric view for the SNN dynamics as follows.

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9See Definition 2 for the formal definition of $\epsilon$-approximate solution.
Figure 2: These are examples of the geometric interpretation of the dual SNN. In (a), we have one neuron where $A_1 = \begin{bmatrix} 1/2 & 1 \end{bmatrix}^T$. In this case, neuron $i$ would not fire as long as the dual SNN $v(t)$ stays in the gray area. In (b), we consider a SNN with 3 neurons where $A_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$, $A_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$, and $A_3 = \begin{bmatrix} 2/3 & 2 \end{bmatrix}^T$. One can see that the effect of spikes on dual SNN is a jump in the direction of the normal vector of the wall(s).

At each update in the dynamics, there are two terms affecting the dual SNN $v(t)$: the external charging $b \cdot dt$ and the spiking effect $-\alpha \cdot As(t)$. First, one can see that the external charging $b \cdot dt$ can be thought of as a constant force that drags that dual SNN in the direction $b$.

To explain the effect of spikes in the dual view, let us start with an geometric view for the spiking rule. Recall that neuron $i$ fires a spike at time $t$ if and only if $u_i(t) > 1$. In the language of dual SNN, this condition is equivalent to $A_i^T v(t) > 1$. Let $W_i = \{ v \in \mathbb{R}^m : A_i^T v = 1 \}$ be the wall of neuron $i$, the above observation is saying that neuron $i$ will fire a spike once it penetrates the wall $W_i$ from the half-space $\{ v \in \mathbb{R}^m : A_i^T v \leq 1 \}$. See Figure 2a for an example. After neuron $i$ fires a spike, the spiking effect on the dual SNN $v(t)$ would be a $-\alpha \cdot A_i$ term, which corresponds to a jump in the normal direction of $W_i$. See Figure 2b for an example.

The geometric interpretation described above is the main advantage of using dual SNN. Specifically, this gives us a clear picture of how spikes affect the SNN dynamics. Namely, neuron $i$ fires a spike if and only if the dual SNN $v(t)$ penetrates the $W_i$ and then $v(t)$ jumps back in the normal direction of $W_i$. Note that this connection would not hold in the primal SNN. In primal SNN $u(t)$, neuron $i$ fires a spike if and only if $u_i(t) > 1$ while the effect on $u(t)$ is moving in the direction $-A_i^T A_i$. See Table 1 for a comparison.

**Dual view of SNN as a primal-dual algorithm for $\ell_1$ minimization problem**

First, let us write down the $\ell_1$ minimization problem and its dual.
### Table 1: Comparison of the geometric view of primal and dual SNNs.

|                | Primal SNN $u(t)$ | Dual SNN $v(t)$ |
|----------------|-------------------|-----------------|
| Spiking rule   | $u_i(t) > 1$      | $A_i^T v(t) > 1$|
| Spiking effect | $-\alpha \cdot A^T A_i$ | $-\alpha \cdot A_i$ |

Now we observe that the dual dynamics can be viewed as a variant of the projected gradient descent algorithm to solve the dual program. Before the explanation, recall that for the $\ell_1$ minimization problem, we are considering the two-sided SNN for convenience. Indeed, without the spiking term, $v(t)$ simply moves towards the gradient direction $b$ of the dual objective function $b^T v$. For the spike term $-\alpha \cdot A s_i(t)$, note that $s_i(t) \neq 0$ (i.e., neuron $i$ fires) if and only if $|A_i^T v(t)| = |u_i(t)| > 1$, which means that $v(t)$ is outside the feasible polytope $\{v : \|A^T v\|_\infty \leq 1\}$ of the dual program. Therefore, one can view the role of the spike term as projecting $v(t)$ back to the feasible polytope. That is, when the dual SNN $v(t)$ becomes infeasible, it triggers some spikes, which maintains the dual feasibility and updates the primal solution (the firing rate). To sum up, we can interpret the simple SNN as performing a non-standard projected gradient descent algorithm for the dual program of $\ell_1$ minimization in the dual view of SNN.

With this primal-dual view in mind, we analyze the SNN algorithm by combining tools from convex geometry and perturbation theory as well as several non-trivial structural lemmas on the geometry of the dual program of $\ell_1$ minimization. One of the key ingredients here is identifying a potential function that (i) upper bounds the error of solving $\ell_1$ minimization problem and (ii) monotonously converges to 0. More details will be provided in Section 3.

### 1.4 Related Work

We compare this research with other related works in the following four aspects.

**Computational power of SNN** Recognized as the third generation of neural networks [Maa97b], the theoretical foundation for the computability of SNN had been built in the pioneering works of Maass et al. [Maa96, Maa97b, Maa99, MB01] in which SNN was shown to be able to simulate standard computational models such as Turing machines, random access machines (RAM), and threshold circuits.

However, this line of works focused on the universality of the computational power and did not consider the efficiency of SNN in solving specific computational problems. In recent years, a line of exciting research have reported the efficiency of SNN in solving specific computational problems such as sparse coding [ZMD11, Tan16, TLD17], dictionary learning [LT18], pattern recognition [DC15, KGM16, BMF17], and quadratic programming [BDM13]. These works indicated the advantage of SNN in handling sparsity as well as being energy efficient and inspired real-world applications [BT09]. However, to the best of our knowledge, no theoretical guarantee on the efficiency of SNN had been provided. For instance, Tang et al. [Tan16, TLD17] only proved the convergence...
in the limit result for SNN solving sparse coding problem instead of giving an explicit convergence rate analysis. The main contribution in this work is giving a rigorous guarantee on the convergence rate of the computational power of SNN.

**The number of spikes versus the timing of spikes** In this work, we mainly focused on the firing rate of SNN. That is, we only study the computational power with respect to the number of spikes. Another important property of SNN is the timing of spikes.

The power of the timing of spikes had been reported since the 90s from some experimental evidences indicating that neural systems might use the timing of spikes to encode information [Abe91, Hop95, RW99]. From then on, a bunch of works have been focused on the aspect of time as a basis of information coding both from theoretical [OF96, Maa97b, MB01, TDVR01] and experimental [Hei91, BRVSW91, KS93] sides. It is generally believed that the timing of spikes is more powerful then the firing rate [TFM96, RT01, PMB12]. Other than the capacity of encoding information, the timing of spikes has also been studied in the context of computational power [TFM96, Maa97a, Maa97b, GM08] and learning [BtN05, Ban16, SS17]. See the survey by Paugam et al. [PMB12] for a thorough discussion.

While the timing of spikes is conceived as an important source of the power of SNN, in this work we simply focus on the firing rate and already yield some non-trivial findings in terms of the computational power of SNN. Investigating how does the timing of spikes play a role is an interesting and important future direction. Immediate open questions here would be how could the timing of spikes fit into our study? What’s the dual view of the timing of spikes? Can the timing of spikes solve the optimization problems more efficiently? Can the timing of spikes solve more difficult problems?

**SNN with randomness** While most of the literature focus on deterministic SNN, there is also an active line of works studying the SNN model with randomness\(^{10}\) [AS94, SN94, FSW08, BBNM11, JHM14, Maa15, JHM16, LMP17a, LMP17b, LMP17c, LM18].

Buesing et al. [BBNM11] used noisy SNN to implement MCMC sampling and Jonke et al. [JHM14, Maa15, JHM16] further instantiated the idea to attack NP-hard problems such as traveling salesman problem (TSP) and constraint satisfaction problem (CSP). Concretely, their noisy SNN has a randomized spiking rule and the firing pattern would form a distribution over the solution space whereas the closer a solution is to the optimal solution, the higher the probability it is sampled. They got nice experimental performance in terms of solving empirical instance approximately. They also pointed out that their noisy SNN has the potential to be implemented energy-efficiently in practice.

Lynch, Musco, and Parter [LMP17b] studied the stochastic SNNs with a focus on the Winner-Take-All (WTA) problem. Their sequence of works [LMP17a, LMP17b, LMP17c, LM18] gave the first asymptotic analysis for stochastic SNN in solving WTA, similarity testing, and neural coding. They view SNNs as distributed algorithms and derived computational tradeoff in running time and network size.

In this work, we consider the SNN model without randomness and thus is incomparable with the above SNN models with randomness. It is an interesting direction to apply the dual view of

\(^{10}\)SNN model with noise is also known as stochastic SNN or noisy SNN depending on how the randomness involves in the model.
deterministic SNN to SNN with randomness.

**Locally competitive algorithms**  Inspired by the dynamics of biological neural networks, Ruzell et al. designed the *locally competitive algorithms* (LCA) [RJBO08] for solving the Lasso (least absolute shrinkage and selection operator) optimization problem\(^{11}\), which is widely used in statistical modeling. Roughly speaking, LCA is also a dynamics among a set of *artificial neurons* which continuously signal their potential values (or a function of the values) to their neighboring neurons. There are two main differences between SNN and LCA. First, the neuron in SNN fires discrete spikes while the artificial neuron in LCA produces continuous signal. Next, the neurons’ potentials in LCA will converge to a fixed value, which is the output of the algorithm. In contrast, in SNN, only the neurons’ firing rates may converge instead of their potentials.

Nevertheless, there is a *spikified* version of LCA introduced by Shapero et al. [SRH13, SZHR14] called *spike LCA* (*S-LCA*) in which the continuous signals are replaced with discrete spikes. S-LCA is almost the same as the SNN we are considering except a shrinkage term\(^{12}\). Recently, Tang et al. [TLD17] showed that the firing rate of S-LCA indeed converges to a variant of Lasso problem\(^{13}\) in the limit. These works also experimentally demonstrated the efficient convergence of S-LCA and its advantage of fast identifying sparse solutions with potentially competitive practical performance to other Lasso algorithms (e.g., FISTA [BT09]). However, there is no proof of convergence rate, and thus no explicit complexity bound of S-LCA.

### 1.5 Future Works and Perspectives

In this work, we give a theoretical study on the algorithmic power of SNN. Specifically, we focus on the firing rate of SNN and confirm an empirical analysis by Barrett et al. [BDM13] with a convergence theorem (*i.e.*, Theorem 1). Furthermore, we discover a dual view of SNN and show that SNN is able to solve the \(\ell_1\) minimization problem (*i.e.*, Theorem 2). In the following, we give interpretations to our results and point out future research directions.

First, how to interpret the dual dynamics of SNN? In this work, we discover the dual SNN based on mathematical convenience. Is there any biological interpretation?

Second, push further the analysis of simple SNN. We believe the parameters we get in the main theorems are not optimal. Is it possible to further sharpen the upper bound? We think this is both theoretically and practically interesting because both non-negative least squares and \(\ell_1\) minimization are important problems that have been well-studied in the literature. Comparing the running time complexity or parallel time complexity of SNN algorithm with other algorithms could also be of theoretical interest and might inspire new algorithm with better complexity. Also, for practical purpose, having better parameters would give more confidence in implementing SNN as a natural algorithm.

Third, further investigate the potential of SNN dynamics as natural algorithms. The question is two-folded: (i) What algorithms can SNN implement? (ii) What computational problems can SNN solve? It seems that SNN is good at dealing with sparsity. Could it be helpful in related computational tasks such as fast Fourier transform (FFT) or sparse matrix-vector multiplication?

---

\(^{11}\) Note that Lasso is equivalent to the Basis Pursuit De-Noising (BPDN) program under certain parameters transformation.

\(^{12}\) That is, the potential of each neuron will drop with rate proportional to the current potential value.

\(^{13}\) In this variant, all the entries in matrix \(A\) is non-negative.
It is interesting to identify optimization problems and class of instances where SNN algorithm can outperform other algorithms. Finally, explore the practical advantage of SNN dynamics as natural algorithms. The potential practical time efficiency, energy efficiency, and simplicity for hardware implementation have been suggested in several works [MMI15, BIP15, BPLG16]. It would be exciting to see whether SNN has nice performance on practical applications such as compressed sensing, Lasso, and etc.

2 Preliminaries

In Section 2.1, we build up some notations for the rest of the paper. In Section 2.2, we define two optimization problems and the corresponding convergence guarantees.

2.1 Notations

For any \( n \in \mathbb{N} \), denote \([n] = \{1, 2, \ldots, n\}\) and \([\pm n] = \{\pm 1, \pm 2, \ldots, \pm n\}\). Let \( x, y \in \mathbb{R}^n \) be two vectors. \(|x| \in \mathbb{R}^n\) denotes the entry-wise absolute value of \( x \), i.e., \(|x|_i = |x_i|\) for any \( i \in [n] \). \( x \preceq y \) refers to entry-wise comparison, i.e., \( x_i \leq y_i \) \( \forall i \in [n] \).

Let \( A \) be an \( m \times n \) real matrix. For any \( i \in [n] \), denote the \( i \)-th column of \( A \) as \( A_i \) and its negation to be \( A_{-i} \), i.e., \( A_{-i} = -A_i \). When \( A \) is positive semidefinite, we define the \( A \)-norm of a vector \( x \in \mathbb{R}^n \) to be \( \|x\|_A = \sqrt{x^T A x} \). Let \( A^\dagger \) to be the pseudo-inverse of \( A \). Define the maximum eigenvalue of \( A \) as \( \lambda_{\max}(A) := \max_{x \in \mathbb{R}^n: \|x\|_2 = 1} \|x\|_A \), the minimum non-zero eigenvalue of \( A \) to be \( \lambda_{\min}(A) := 1/(\max_{x \in \mathbb{R}^n: \|x\|_2 = 1} \|x\|_A) \), and the condition number of \( A \) to be \( \kappa(A) := \lambda_{\max}(A)/\lambda_{\min}(A) \). If we do not specified, the following \( \lambda_{\max}, \lambda_{\min}, \) and \( \kappa \) are the eigenvalues and condition number of the connectivity matrix \( C = A^\dagger A \). For any \( b \in \mathbb{R}^m \), we denote \( b_A \) to be the projection of \( b \) on the range space of \( A \).

2.2 Optimization problems

In this subsection, we are going to introduce two optimization problems: non-negative least squares and \( \ell_1 \) minimization.

2.2.1 Non-negative least squares

**Problem 1** (non-negative least squares). Let \( m, n \in \mathbb{N} \). Given \( A \in \mathbb{R}^{m \times n} \) and vector \( b \in \mathbb{R}^m \), find \( x \in \mathbb{R}^n \) that minimizes \( \|b - Ax\|_2^2 / 2 \) subject to \( x_i \geq 0 \) for all \( i \in [n] \).

**Remark 1.** Recall that the least squares problem is defined as finding \( x \) that minimize \( \|b - Ax\|_2^2 \). That is, the non-negative least squares is a restricted version of the least squares problem. Nevertheless, one can use a non-negative least squares solver to solve the least squares problem by setting \( A' = (A^\dagger A, -A^\dagger A) \) and \( b' = (b, 0) \) where \((A, b)\) is the instance of least squares and \((A', b')\) is the instance of non-negative least squares.

The SNN algorithm might not solve the non-negative least squares problem exactly and thus we define the following notion of solving the non-negative least squares problem approximately.

**Definition 1** (\( \epsilon \)-approximate solution to non-negative least squares). Let \( m, n \in \mathbb{N} \) and \( \epsilon > 0 \). Given \( A \in \mathbb{R}^{m \times n} \) and \( b \in \mathbb{R}^m \). We say \( x \) is an \( \epsilon \)-approximate solution to the non-negative least squares problem of \((A, b)\) if \( \|Ax - Ax*\|_2 \leq \epsilon \|b\|_2 \) where \( x^* \) is an optimal solution.
2.2.2 ℓ1 minimization

**Problem 2** (ℓ1 minimization). Let \( m, n \in \mathbb{N} \). Given \( A \in \mathbb{R}^{m \times n} \) and \( b \in \mathbb{R}^m \) such that there exists a solution to \( Ax = b \). The goal of ℓ1 minimization is to solve the following optimization problem.

\[
\begin{align*}
\text{minimize} & \quad \|x\|_1 \\
\text{subject to} & \quad Ax = b.
\end{align*}
\]

Similarly, we do not expect SNN algorithm to solve the ℓ1 minimization exactly. Thus, we define the notion of solving the ℓ1 minimization problem approximately as follows.

**Definition 2** (\( \epsilon \)-approximate solution to ℓ1 minimization). Let \( m, n \in \mathbb{N} \) and \( \epsilon > 0 \). Given \( A \in \mathbb{R}^{m \times n} \) and \( b \in \mathbb{R}^m \). Let \( \text{OPT}^\ell_1 \) denote the optimal value of the ℓ1 minimization problem of \((A, b)\). We say \( x \in \mathbb{R}^n \) is an \( \epsilon \)-approximate solution of the ℓ1 minimization problem of \((A, b)\) if

\[
\|b - Ax\|_2 \leq \epsilon \cdot \|b\|_2 \quad \text{and} \quad \|x\|_1 - \text{OPT}^\ell_1 \leq \epsilon \cdot \text{OPT}^\ell_1.
\]

2.3 Karush-Kuhn-Tucker conditions

Karush-Kuhn-Tucker (KKT) conditions are necessary and sufficient conditions for the optimality of optimization problems under some regular assumptions. Consider the following optimization program.

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad g_i(x) \leq 0, \quad \forall i = 1, 2, \ldots, m, \\
& \quad h_j(x) = 0, \quad \forall j = 1, 2, \ldots, k,
\end{align*}
\]

where \( f, g_1, \ldots, g_m, h_1, \ldots, h_k \) are convex and differentiable. Let \( v \in \mathbb{R}^m \) and \( \mu \in \mathbb{R}^k \) be the dual variables. KKT conditions give necessary and sufficient conditions for \((x, v, \mu)\) be a pair of primal and dual optimal solutions.

**Theorem 3** (KKT conditions). \((x, v, \mu)\) are a pair of primal and dual optimal solutions for (4) if and only if the following conditions hold.

- \( x \) is primal feasible, i.e., \( g_i(x) \leq 0 \) for all \( i \in [m] \) and \( j \in [k] \).
- \((v, \mu)\) is dual feasible, i.e., \( v_i \geq 0 \) for all \( i \in [m] \).
- The Lagrange multiplier vanishes, i.e., \( \nabla f(x) + \sum_{i \in [m]} v_i \nabla g_i(x) + \sum_{j \in [k]} \mu_j \nabla h_j(x) = 0 \).
- \((x, v, \mu)\) satisfy complementary slackness, i.e., \( v_i f_i(x) \geq 0 \) for all \( i \in [m] \).

For more details about KKT conditions, please refer to standard textbook such as Chapter 5.5.3 in [BV04].

2.4 Perturbation theory

Perturbation theory, sometimes known as sensitivity analysis, for optimization problems concerns the situation where the optimization program is perturbed and the goal is to give a good estimation for the optimal solution. See a nice survey by Bonnans and Shapiro [BS08]. In the following we state a special case for convex optimization program with strong duality.
Theorem 4 (perturbation, Chapter 5.6 in [BV04]). Given the following two optimization programs where the strong duality holds and there exists feasible dual solution.

\[
\begin{align*}
\text{minimize} \quad & f(x) \\
\text{subject to} \quad & g_i(x) \leq 0, \quad \forall i = 1, 2, \ldots, m, \\
& h_j(x) = 0, \quad \forall j = 1, 2, \ldots, k.
\end{align*}
\]

\[
\begin{align*}
\text{minimize} \quad & f(x) \\
\text{subject to} \quad & g_i(x) \leq a_i, \quad \forall i = 1, 2, \ldots, m, \\
& h_j(x) = b_j, \quad \forall j = 1, 2, \ldots, k.
\end{align*}
\]

Let \( \text{OPT}^{\text{original}} \) be the optimal value of the original program (5) and \( \text{OPT}^{\text{perturbed}} \) be the optimal value of the perturbed program (6). Let \((v^*, \mu^*) \in \mathbb{R}^m \times \mathbb{R}^k \) be the optimal dual solution of the perturbed program (6). We have

\[
\text{OPT}^{\text{original}} \geq \text{OPT}^{\text{perturbed}} + a^\top v^* + b^\top \mu^*.
\]

3 A simple SNN algorithm for \( \ell_1 \) minimization

In this section, we focus on the discovery of the dual view of simple SNN and how it can be viewed as a primal-dual algorithm for solving the \( \ell_1 \) minimization problem.

Recall that for the \( \ell_1 \) minimization problem, we are working on the two-sided simple SNN for the convenience of future analysis. That is,

\[
\frac{d}{dt} u(t) = -\alpha \cdot A^\top As(t) + A^\top b,
\]

where \( s_i(t) = 1_{u_i(t) > \eta} - 1_{u_i(t) < -\eta} \). To solve the \( \ell_1 \) minimization problem, we configure a two-sided simple SNN as follows. Given an input \((A, b)\), let \( C = A^\top A \) and \( I = A^\top b \). However, currently it is unclear how does the above simple SNN dynamics relate to the \( \ell_1 \) minimization problem.

Let us try to make some geometric observations on (8). First, the objective of the dual program is to maximize the inner product with \( b \), which is quite related to the external charging of SNN since we take \( I = A^\top b \). Next, the feasible region of the dual program is a polytope (or a polyhedron) defined by the intersection of half-spaces \( \{v \in \mathbb{R}^m : A_i^\top v \leq 1\} \) and \( \{v \in \mathbb{R}^m : -A_i^\top v \leq 1\} \) for each \( i \in [n] \) where \( A_i \) denotes the \( i \)-th column of \( A \).

It will be convenient to introduce the following notation before we move on. For \( i \in [n] \), let \( A_{-i} = -A_i \). Let \( [\pm n] = \{\pm 1, \pm 2, \ldots, \pm n\} \). Thus, the feasible polytope of the dual program

\[14\text{Note that we switch the original and perturbed programs in the statement in [BV04].}\]
is defined by the intersection of half-spaces defined by $A_j^\top v \leq 1$ for all $j \in [\pm n]$. We call this polytope the *dual polytope*. Moreover, for each $j \in [\pm n]$, we call the hyperplane $\{v : A_j^\top v = 1\}$ the *wall* $W_j$ of the dual polytope. See Figure 3b for examples.

![Figure 3](image)

(a) An example of one neuron.  
(b) An example of three neurons.

**Figure 3:** This is examples of the geometric interpretation of the dual program of $\ell_1$ minimization problem. In (a), we have $n = 1$ where $A_1 = [\frac{1}{3} 1]^\top$. In this case, the gray area, *i.e.*, the feasible region of the dual program, is unbounded. In (b), we have $n = 3$ where $A_1 = [1 0]^\top$, $A_2 = [0 1]^\top$, and $A_3 = [\frac{2}{3} \frac{2}{3}]^\top$. In this case, the gray area is bounded and thus called dual polytope.

Now, the key observation is that by a linear transformation, the dynamics of simple SNN has a natural interpretation in the dual space. We call it the *dual SNN* defined as follows.

### 3.1 Dual SNN

We first recall the simple SNN dynamics which we call the *primal SNN* from now on. For convenience, we set the threshold parameter $\eta = 1$ (and make the spiking strength parameter $\alpha$ explicit). For any $t \geq 0$,

$$u(t + dt) = u(t) - \alpha \cdot A^\top A \cdot s(t) + A^\top b \cdot dt. \quad (9)$$

Now, we define the dual SNN $v(t) \in \mathbb{R}^m$ as follows. Let $v(0) = 0$ and for each $t \geq 0$, define

$$v(t + dt) = v(t) - \alpha \cdot A s(t) + b \cdot dt. \quad (10)$$

Let us make some remarks about the connection between the primal and dual SNNS. First, it can be immediately seen that $u(t) = A^\top v(t)$ for each $t \in \mathbb{N}$ from (9) and (10). That is, given $v(t)$, it is easy to get $u(t)$ by multiplying $u(t)$ with $A^\top$ on the left. It turns out that the other direction also holds. For each $t \in \mathbb{N}$, we have $v(t) = (A^\top)^\dagger u(t)$, where $(A^\top)^\dagger$ is the pseudo-inverse of $A^\top$. The reason is because the primal SNN $u(t)$ lies in the column space of $A$. Thus, the two dynamics are in fact *isomorphic* to each other.

---

In the case where the feasible region of the dual program is not bounded, it is a dual *polyhedron*. For the convenience of the presentation, we usually assume the feasible region is bounded.
Now let us understand the dynamics of dual SNN in the dual space $\mathbb{R}^m$. At each timestep, there are two terms, i.e., the external charging $b \cdot dt$ and the spiking effect $-\alpha As(t)$, that affect the dual SNN $v(t)$. The external charging can be thought of as a constant force that drags that dual SNN in the direction $b$. See Figure 4a. This coincides with the objective function of the dual program (8) and thus the external charging can then be viewed as taking a gradient step towards solving (8).

Nevertheless, to solve (8), one need to make sure the solution $v$ is feasible, i.e., $v$ should lie in the dual polytope. Interestingly, this is exactly what the spike is doing! Recall that neuron $i$ fires a spike if $|u_i(t)| > 1$ (recall that we set $\eta = 1$), which corresponds to $|A^T_i v(t)| > 1$ in the dual space. Thus, the spike term has the following nice geometric interpretation: if $v(t)$ “exceeds” the wall $W_j$ for some $j \in \{\pm n\}$, then neuron $|j|$ fires a spike and $v(t)$ is “bounced back” in the normal direction of the wall $W_j$ in the sense that $v(t)$ is subtracted by $\alpha \cdot A_j$. See Figure 4b for example.

![Figure 4](image)

(a) The effect of external charging on dual SNN.  
(b) The effect of both the external charging and spikes on dual SNN.

Figure 4: This is examples of the geometric interpretation of the dual We consider the same matrix $A$ as in Figure 3b and $b = [0.1 \ 0.4]^T$. In (a), one can see that the external charging $b$ points the direction that dual SNN is moving. In (b), one can see that the effect of spikes on dual SNN is a jump in the direction of the normal vector of the wall.

Therefore, one can view the dual SNN as performing a variant of projected gradient descent algorithm for the dual program of $\ell_1$ minimization problem. Specifically, to maintain the feasibility, the vector is not projected back to the feasible region as usual, but is “bounced back” in the normal direction of the wall $W_j$ corresponding to the violated constraint $A^T_j v \leq 1$. An advantage of this variant is that the “bounced back” operation is simply subtraction of $\alpha \cdot A_j$, which is significantly more efficient than the orthogonal projection back to the feasible region. On the other hand, note that the dynamics might not exactly converge to the optimal dual solution $v^{\text{OPT}}$. Intuitively, the best we can hope for is that $v(t)$ will converge to a small neighboring region of $v^{\text{OPT}}$(assuming the spiking strength $\alpha$ is sufficiently small). The above intuition of viewing dual SNN as a projected gradient descent algorithm for the dual program of the $\ell_1$-minimization problem will be formally proved in the later subsections.
The primal-dual connection. So far we have informally seen that the dual SNN can be viewed as solving the dual program of the $\ell_1$-minimization problem. However, this does not immediately give us a reason why the firing rate would converge to the solution of the primal program. It turns out that there is a beautiful connection between the dual SNN and firing rate through the Karush-Kuhn-Tucker (KKT) conditions (see Section 2.3) and perturbation theory (see Section 2.4).

We now discuss some intuitions about how the dual solution translates to the primal solution. To jump into the core idea, let us consider an ideal scenario where the dual SNN $v(t)$ is already very close to the optimal dual solution $v^{OPT}$ for the dual program of the $\ell_1$ minimization problem. Since $v^{OPT}$ is the optimal solution and thus it must lie on the boundary of the dual polytope. Let $\Gamma \subseteq \{\pm n\}$ be the set of walls that $v^{OPT}$ touches. That is, $j \in \Gamma$ if and only if $A_j^T v^{OPT} = 1$.

Now, let $x^{OPT}$ denote the optimal primal solution of the $\ell_1$ minimization problem. Observe that by the complementary slackness in the KKT conditions, for each $i \in [n]$, we have $x^{OPT}_i > 0$ (resp. $x^{OPT}_i < 0$) if $i \in \Gamma$ (resp. $-i \in \Gamma$) and $x^{OPT}_i = 0$ if $i, -i \notin \Gamma$. To summary, this is saying that $\Gamma$ contains the coordinates that are non-zero in the primal optimal solution $x^{OPT}$. See Figure 5 for an example.

![Figure 5](image.png)

Figure 5: This is an example based on Figure 3b and Figure 4b. In this example, $A_1 = [1 0]^T$, $A_2 = [0 1]^T$, $A_3 = [\frac{3}{2} \frac{3}{2}]^T$, and $b = [0.1 0.4]^T$. The optimal dual solution is $v^{OPT} = [\frac{3}{2} 1]^T$ as shown in the figure. Thus, by the above definition we have $\Gamma = \{2, 3\}$. By the KKT conditions, we then know that only the 2nd and 3rd coordinate of the optimal primal solution are non-zero. Indeed, the optimal primal solution is $x^{OPT} = [0 \frac{3}{10} \frac{3}{20}]^T$.

With this observation, once the dual SNN $v(t)$ is very close to the optimal dual solution $v^{OPT}$ and stays nearby, only those neurons correspond to $\Gamma$ would fire spikes. In other words, the firing rate of the non-zero coordinates in the primal optimal solution $x^{OPT}$ will remain non-zero due to the spikes while the other coordinates will gradually go to zero.

At this point, we have seen that (i) the dual SNN can be viewed as a projected gradient descent algorithm for the dual program of $\ell_1$ minimization problem and (ii) the dual solution (resp. dual SNN) and primal solution (resp. firing rate) have a natural connection through the KKT conditions. The explanations so far are rather informal and focus on intuition. From now on, everything will start to be more and more formal and rigorous. Before that, let us state the main theorem of this section about simple SNN solving $\ell_1$ minimization problem.
Theorem 5. Given \( A \in \mathbb{R}^{m \times n} \) and \( b \in \mathbb{R}^m \) where all the row of \( A \) has unit norm. Let \( \gamma(A) \) be the niceness parameter of \( A \) defined later in Definition 4. Suppose \( \gamma(A) > 0 \) and there exists a solution for \( Ax = b \). There exists a polynomial \( \alpha(\cdot) \) such that for any \( t \geq 0 \), let \( x(t) \) be the firing rate of the simple SNN with \( C = A^\top A, I = A^\top b, \eta = 1, 0 < \alpha \leq \alpha\left(\frac{\gamma(A)}{m \lambda_{\max}}\right) \). Let \( OPT^{16} \) be the optimal value of the \( \ell_1 \) minimization problem. For any \( \epsilon > 0 \), when \( t \geq \Omega\left(\frac{m^2 n \cdot \|b\|^2_2}{\epsilon^2 \lambda_{\min} \cdot OPT^{16}}\right) \), then \( x(t) \) is an \( \epsilon \)-approximate solution for the \( \ell_1 \) minimization problem for \((A, b)\).

Two remarks on the statement of Theorem 5. First, we consider the continuous SNN instead of the discrete SNN, which is of interest for simulation on classical computer. In discrete SNN, the step size is some non-negligible \( \Delta t > 0 \) instead of \( dt \). The main reason for considering continuous SNN is that this significantly simplify the proof by avoiding a huge amount of nasty calculations. We suspect that the proof idea would hold for discrete SNN with discretization parameter \( \Delta t \leq \Delta t\left(\frac{\gamma(A)}{m \lambda_{\max}}\right) \) for some polynomial \( \Delta t(\cdot) \). Second, the parameters in Theorem 5 have not been optimized and we believe all the dependencies can be improved. Since the parameters highly affect the efficiency of SNN as an algorithm for \( \ell_1 \) minimization problem, we pose it as an interesting open problem to study what are the best dependencies one can get.

3.2 Overview of the proof for Theorem 5

The proof of Theorem 5 consists of two main steps as mentioned in the previous subsection. The first step argues that the dual SNN \( v(t) \) would converge to the neighborhood of the optimal dual solution \( v^{OPT} \). The second step is connecting the dual solution (i.e., the dual SNN) to the primal solution (i.e., the firing rate).

In the first step, we try to identify a potential function\(^{16}\) that captures how close is \( v(t) \) to the optimal dual solution \( v^{OPT} \). It turns out that this is not an easy task since the effect of spikes makes the behavior of dual SNN very non-monotone. We conquer the difficulty via a technique that we call ideal coupling (see Definition 6 and Figure 7). The idea is associating the dual SNN \( v(t) \) with an ideal SNN \( v^{ideal}(t) \) for every \( t \geq 0 \) such that the ideal SNN would have smoother behavior comparing to the spiking phenomenon in the dual SNN. We will formally define the ideal SNN in Section 3.4. There are two advantages of using ideal SNN instead of handling dual SNN directly: (i) Ideal SNN is smoother than dual SNN in the sense that it would not change after spikes (see Lemma 3.5). Further, by introducing some auxiliary processes (i.e., the auxiliary SNNs defined in Definition 8), we are able to identify a potential function that is strictly improving at any moment and measures how well the dual SNN has been solving the \( \ell_1 \) minimization problem (see Lemma 3.8). (ii) ideal SNN is naturally associated with an ideal solution (defined in Definition 7) which is easier to analyze than the firing rate. Using these good properties of ideal SNN, we can prove in Lemma 3.11 that the \( \ell_2 \) residual error of the ideal solution will converge to 0.

After we are able to show the convergence of the \( \ell_2 \) residual error in Lemma 3.11, we move to the second step where the goal is showing that the \( \ell_1 \) norm of the solution is also small. We look at the KKT conditions of the \( \ell_1 \) minimization problem and observe that the primal and dual solutions of SNN satisfy the KKT conditions of a perturbed program of the \( \ell_1 \) minimization problem. Finally, combine tools from perturbation theory, we can upper bound the \( \ell_1 \) error of the ideal solution by its \( \ell_2 \) residual error in Lemma 3.12.

\(^{16}\)Potential function is widely used in the analysis of many gradient-descent based algorithm. The difficulty lies in the search of a good potential function for the algorithm.
Theorem 5 then follows from Lemma 3.11 and Lemma 3.12 with some special cares on how to transform everything for ideal solution to the firing rate. See Figure 6 for an overall structure of the proof for Theorem 5.

| Theorem 5          | (SNN solves $\ell_1$ minimization problem) |
|--------------------|-------------------------------------------|
| Definition 4       | (niceness of input matrix)                |
| Lemma 3.5          | (unchanged after spikes)                  |
| Definition 6       | (ideal coupling)                          |
| Lemma 3.8          | (strict improvement)                      |
| KKT conditions     |                                           |
| Perturbation       |                                           |
| Lemma 3.11         | (convergence of $\ell_2$ error)           |
| Lemma 3.12         | ($\ell_2$ error upper bounds $\ell_1$ error) |

Figure 6: Overview of the proof for Theorem 5.

In the rest of this section, we are going to start from some definitions on the nice conditions we need for the input matrix in Section 3.3. Next, we define the ideal coupling in Section 3.4 and prove Lemma 3.5 and Lemma 3.8 in Section 3.5 and Section 3.6 respectively. Finally, we wrap up the proof for Theorem 5 in Section 3.7.

### 3.3 Some nice conditions on the input matrix

We need some nice conditions for the input matrix as follows.

**Definition 3** (non-degeneracy). Let $A \in \mathbb{R}^{m \times n}$ where $m \leq n$. We say $A$ is non-degenerate if for any size $m \times m$ submatrix of $A$ has full rank. For any $\gamma > 0$, we say $A$ is $\gamma$-non-degenerate if for any $\Gamma \subseteq [n]$, $|\Gamma| = m$, and $i \in \Gamma$, $\|A_i - \Pi_{A \backslash \{i\}} A_i\|_2 \geq \gamma$ where $\Pi_{A \backslash \{i\}} v$ is the projection of $v$ onto subspace spanned by $\{A_j : j \in \Gamma \backslash \{i\}\}$ for any $v \in \mathbb{R}^m$.

Note that if $A$ is non-degenerate, then for any $S \subseteq [n]$ and $|S| = m$ and $b \in \{-1, 1\}^m$, there exists an unique solution $v \in \mathbb{R}^m$ to $A_S^T v = b$ where $A_S$ is the submatrix of $A$ restricted to columns in $S$. We call such $v$ a vertex of the polytope $\mathcal{P}_{A,1}$. An important parameter for future analysis is the minimum distance between two distinct vertices of $\mathcal{P}_{A,1}$.

**Definition 4** (nice input matrix). Let $A \in \mathbb{R}^{m \times n}$ and $\gamma \geq 0$. We say $A$ is $\gamma$-nice if all of the following conditions hold.

1. $A$ is $\gamma$-non-degenerate.
2. The distance between any two distinct vertices of $\mathcal{P}_{A,1}$ is at least $\gamma$.
3. For any $b \in \{-1, 1\}^m$, $\Gamma \subseteq [n]$, and $|\Gamma| = m$, let $x = (A_i^T)^{-1} b$. For any $i \in [m]$, $|x_i| \geq \gamma$.

Define $\gamma(A)$ to be the largest $\gamma$ such that $A$ is $\gamma$-nice. We say $A$ is nice if $\gamma(A) > 0$. 18
To motivate the definition of niceness, the following lemma shows that the $\ell_1$ minimization problem defined by matrix $A$ has unique solution if $\gamma(A) > 0$.

**Lemma 3.1.** Let $A \in \mathbb{R}^{m \times n}$. If $\gamma(A) > 0$, then for any $b \in \mathbb{R}^m$, the $\ell_1$ minimization problem for $(A, b)$ has unique solution.

**Proof.** We prove the lemma by contradiction. Suppose there exists $b \in \mathbb{R}^m$ such that there are two distinct solutions $x_1 \neq x_2$ to the $\ell_1$ minimization problem for $(A, b)$. Let $v^*$ be the optimal solution of the dual program as in equation (8). By the complementary slackness in the KKT condition, for any optimal solution $x$ to the primal program, supp$(x) \subseteq \{i \in [n] : |A_i^T v^*| = 1\}$. Let $S = \{i \in [n] : |A_i^T v^*| = 1\}$, then both $x_1$ and $x_2$ are solution to $A_S x = b_S$ where $A_S$ and $b_S$ are restrictions to index set $S$. As $\gamma(A) > 0$, we have $|S| \leq m$. By the non-degeneracy of $A$, $A_S$ has full rank and thus $A_S x = b_S$ has unique solution. That is, $x_1 = x_2$, which is a contradiction.

We conclude that if $A$ is non-degenerate and $\gamma(A) > 0$, then for any $b \in \mathbb{R}^m$, the $\ell_1$ minimization problem for $(A, b)$ has unique solution. $\square$

In general, it is easy to find a matrix $A$ such that $\gamma(A) = 0$. However, we would like to argue that most of the matrices are actually nice. The following lemma shows that random matrix $A$ sampled from the **rotational symmetry model (RSM)** is nice. In RSM, each column of $A$ is an uniform vector on the unit sphere of $\mathbb{R}^m$. Note that such matrix for $\ell_1$ minimization problem is commonly used in practice such as compressed sensing.

**Lemma 3.2.** Let $A \in \mathbb{R}^{m \times n}$ be a random matrix samples from RSM, then $\gamma(A) > 0$ with high probability.

**Proof.** First, we show that $A$ is non-degenerate with high probability. For any $\Gamma \subseteq [n]$ and $i \in \Gamma$, denote the event where $A_i = \Pi_{A_{\Gamma \setminus \{i\}}} A_i$ as $E_{\Gamma,i}$. Note that this event is measured zero for all choice of $\Gamma$ and $i$ and thus by union bound, we have $A$ being non-degenerate with high probability. For the other two properties, similar arguments hold. $\square$

We remark that giving a lower bound in terms of $m$ and $n$ for $\gamma(A)$ would result in a better asymptotic bound for our main theorem and could have applications in other problems too. Since the goal of this paper is giving a provable analysis, we do not intend to optimize the parameter. Note that for $A$ sampled from RSM, $\gamma(A)$ has an inverse exponential lower bound directly from union bound when $n$ and $m$ are polynomially related. As for upper bound, there are inverse quasi-polynomial upper bound if $n \geq \text{polylog}(m) \cdot m$ and inverse exponential upper bound if $n \geq m^{1+\Omega(1)}$ as pointed out by the anonymous reviewer from ITCS 2019. See Appendix B. for more details. We leave it as an open question to understand the correct asymptotic behavior of $\gamma(A)$ when $A$ is sampled from RSM.

### 3.4 Ideal coupling

Ideal coupling is a technique to keeping track of the dual SNN $v(t)$ by associating any point in the dual polytope to a point in a smaller polytope. Concretely, let $P_{A,1} = \{v \in \mathbb{R}^m : \|A^T v\|_\infty \leq 1\}$ be the dual polytope and $P_{A,1-\tau}$ be the **ideal polytope** where $\tau \in (0, 1)$ is an important parameter that will be properly chosen\textsuperscript{17} in the end of the proof. Observe that $P_{A,1-\tau} \subsetneq P_{A,1}$. The idea of

\textsuperscript{17} The choice of $\tau$ depends on $A$ and 1 and will be discussed later.
ideal coupling is associating each \( v \in P_{A,1} \) with a point \( v^{\text{ideal}} \) in \( P_{A,1-\tau} \). In the analysis, we will then focus on the dynamics of \( v^{\text{ideal}} \) instead of that of \( v \).

Before we formally define the coupling, we have to define a partition of \( P_{A,1} \) with respect to \( P_{A,1-\tau} \) as follows.

**Definition 5** (partition of \( P_{A,1} \)). Let \( P_{A,1} \) and \( P_{A,1-\tau} \) be defined as above. For each \( v^{\text{ideal}} \in P_{A,1-\tau} \), define

\[
S_{v^{\text{ideal}}} = \{ v^{\text{ideal}} + C_{A,\Gamma(v^{\text{ideal}})} \} \cap P_{A,1},
\]

where \( \Gamma(v^{\text{ideal}}) = \{ i \in [\pm n] : (A_i, v^{\text{ideal}}) = 1 - \tau \} \) is the active walls of \( v^{\text{ideal}} \) and \( C_{A,\Gamma(v^{\text{ideal}})} = \{ \sum_{i \in \Gamma(v^{\text{ideal}})} a_i A_i, \forall a_i \geq 0 \} \) is the cone spanned by the column of \( A \) indexed by \( \Gamma(v^{\text{ideal}}) \).

**Example 3.3.** Consider the example where \( A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) and \( \tau \in (0,1) \). The dual polytope (resp. ideal polytope) is the square with vertices in the form \((\pm 1, \pm 1)\) (resp. \((\pm 1-\tau, \pm 1-\tau)\)). For an arbitrary \( v^{\text{ideal}} = (x, y) \in P_{A,1-\tau} \), let us see what \( S_{v^{\text{ideal}}} \) is:

- When \( |x|, |y| < 1-\tau, \) i.e., \( v^{\text{ideal}} \) strictly inside \( P_{A,1-\tau} \), \( \Gamma(v^{\text{ideal}}) = \emptyset \) and thus \( C_{A,\Gamma(v^{\text{ideal}})} = \emptyset \). Namely, \( S_{v^{\text{ideal}}} = v^{\text{ideal}} \).
- When \( |x| = 1 - \tau \) and \( |y| < 1 - \tau \), i.e., \( v^{\text{ideal}} \) lies on an edge of the ideal polytope, \( \Gamma(v^{\text{ideal}}) = \{ \text{sgn}(x) \cdot 1 \} \) and thus \( C_{A,\Gamma(v^{\text{ideal}})} = \{ (a,0) : a \geq 0 \} \). Namely, \( S_{v^{\text{ideal}}} = \{(a,y) : a \in [-1,1] \} \).
- When \( |x| < 1 - \tau \) and \( |y| = 1 - \tau \), i.e., \( v^{\text{ideal}} \) lies on an edge of the ideal polytope, \( \Gamma(v^{\text{ideal}}) = \{ \text{sgn}(y) \cdot 2 \} \) and thus \( C_{A,\Gamma(v^{\text{ideal}})} = \{ (0,b) : b \geq 0 \} \). Namely, \( S_{v^{\text{ideal}}} = \{(x,b) : b \in [1-\tau,1] \} \).
- When \( |x| = |y| = 1 - \tau \), i.e., \( v^{\text{ideal}} \) lies on a vertex of the ideal polytope, \( \Gamma(v^{\text{ideal}}) = \{ \text{sgn}(x) \cdot 1, \text{sgn}(y) \cdot 2 \} \) and thus \( C_{A,\Gamma(v^{\text{ideal}})} = \{ (a,b) : a,b \geq 0 \} \). Namely, \( S_{v^{\text{ideal}}} = \{(a,b) : a,b \in [1-\tau,1] \} \).

The following lemma checks that Definition 5 does give a partition for \( P_{A,1} \).

**Lemma 3.4.** \( \{ S_{v^{\text{ideal}}} \}_{v^{\text{ideal}} \in P_{A,1-\tau}} \) is a partition for \( P_{A,1} \).

**Proof of Lemma 3.4.** The proof is basically doing case analysis and using some basic properties from linear algebra. See Section A.1 for details. \( \square \)

**Definition 6** (ideal coupling). Let \( P_{A,1} \) and \( P_{A,1-\tau} \) be defined as above. For any \( v \in P_{A,1} \), define \( v^{\text{ideal}}(v) \) be the unique \( v^{\text{ideal}} \) such that \( v \in S_{v^{\text{ideal}}} \). We denote \( v^{\text{ideal}}(v) \) as \( v^{\text{ideal}} \) when the context is clear. Specifically, for any \( t \geq 0 \), we denote \( v^{\text{ideal}}(t) = v^{\text{ideal}}(v(t)) \) as the ideal SNN at time \( t \).

See Figure 7 for an example of the ideal coupling.

Note that Definition 6 is well-defined due to Lemma 3.4. With the ideal coupling, we are then switching to analyze the ideal SNN \( v^{\text{ideal}}(t) \) instead of the dual SNN \( v(t) \). In the following, we are going to show that the ideal SNN is indeed tractable for analysis, though it is highly non-trivial and is very sensitive to the choice of parameters.

To show the convergence of ideal SNN, we need a notion to measure how close \( v^{\text{ideal}}(t) \) and the optimal point is. To do so, we define the ideal solution of ideal SNN at time \( t \) as follows.
Figure 7: This is an example of ideal coupling in $\mathbb{R}^2$ where $\tau = 0.4$, $A_1 = [1, 0]^T$, $A_2 = [0, 1]^T$, and $A_3 = [\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]^T$. The dots (i.e., $v_1, v_2, v_3$) are dual SNN and the stars (i.e., $v_{\text{ideal}}^1, v_{\text{ideal}}^2, v_{\text{ideal}}^3$) are the corresponding ideal SNN. The whole gray area is the dual polytope $P_{A,1}$ and the gray area in the middle is the ideal polytope $P_{1-\tau}$.

**Definition 7 (ideal solution).** For any $t \geq 0$, define the ideal solution $x_{\text{ideal}}(t)$ at time $t$ as

$$x_{\text{ideal}}(t) = \arg \min_{x \geq 0, x_i = 0, \forall i \in \Gamma(v_{\text{ideal}}(t))} \|b - Ax\|_2.$$ 

Also, let $\Gamma^*(v_{\text{ideal}}(t)) = \{i \in \Gamma(v_{\text{ideal}}(t)) : x_{\text{ideal}}(t) \neq 0\}$ to be the set of super active neurons.

In the later proof, we need one more definition on a variant of ideal SNN called the super SNN. Similar to Definition 7, we define the super ideal SNN $v_{\text{super}}(t)$ as the projection of $v(t)$ to the ideal polytope without those non-super ideal neurons. Formally, define $v_{\text{super}}(t)$ to be the unique solution of the following equations: $v = v(t) - A_i^T (v_{\text{ideal}}(t)) z$ and $A_i^T v = 1 - \tau$ for each $i \in \Gamma^*(v_{\text{ideal}}(t))$. See Figure 8 for example. Note that the uniqueness of the solution is guaranteed by the non-degeneracy of $A$.

It is indeed unclear why we need these definitions at this stage of the proof. It would be clearer why we need them in the next two subsections once we go into the main analysis. Before we move on to more details, see Figure 7 and Figure 8 again to familiarize with the definitions.

### 3.5 Ideal SNN remains unchanged after firing spikes

In this subsection, we are going to prove the following important lemma saying that the dual SNN would not change its ideal SNN after firing spikes.

**Lemma 3.5 (ideal SNN remains unchanged after firing spikes).** There exists a polynomial $\alpha(\cdot)$ such that if $A$ is nice and $0 < \alpha \leq \alpha(\frac{\tau \gamma(A)}{n \lambda_{\max}^A})$, then $v(t) - \alpha A s(t) \in S_{\text{ideal}}(t)$ for each $t \geq 0$. 

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As a result, if $z \in R$ whenever neuron $i$ fires after the neurons fire spikes. The main reason is that neuron $i$ is defined as the projection of $v$ onto the ideal polytope that only contains neuron 2.

**Proof of Lemma 3.5.** First, note that for each $v \in S_{v^{ideal}(t)}$, by the property of dual polytope, there exists an unique $z \in R_{\geq 0}^{\Gamma(v^{ideal}(t))}$ such that $v = v^{ideal}(t) + A_{\Gamma(v^{ideal}(t))}z$ where $z$ can be thought of as the coordinates of $v$ in $S_{v^{ideal}(t)}$. With this concept in mind, it is then sufficient to show that whenever neuron $i$ fires, $z_{i} > \alpha$. The reason is that

$$v(t) - \alpha A_{\Gamma(v^{ideal}(t))}z = v^{ideal}(t) + \sum_{i \in \Gamma(v^{ideal}(t))} \alpha A_{i}$$

$$= v^{ideal}(t) + \sum_{i \in \Gamma(v^{ideal}(t)) \setminus \Gamma(v(t))} z_{i} A_{i} + \sum_{i \in \Gamma(v(t))} (z_{i} - \alpha) A_{i}.$$  

(11)

As a result, if $z_{i} - \alpha > 0$ for every $i \in \Gamma(v(t))$, then we have $v(t) - \alpha A_{\Gamma(v^{ideal}(t))}z \in S_{v^{ideal}(t)}$ because every new coordinates are still non-negative. See Figure 9 for an example.

**Claim 3.5.1.** There exists a polynomial $\alpha(\cdot)$ such that when $0 < \alpha \leq poly(\frac{\tau \gamma_{max}(A)}{m})$ and $v(t) = v^{ideal}(t) + A_{\Gamma(v^{ideal}(t))}z \in S_{v^{ideal}(t)}$ for some $t \geq 0$, if $i \in \Gamma(v(t))$, then $z_{i} > \alpha$.

**Proof of Claim 3.5.1.** The proof consists of two steps. First, we are going to show that for any $t \geq 0$, $v(t)$ is close to $v^{ideal}(t)$. Concretely, if $\alpha \leq \frac{\tau}{m}$, then $\|v(t) - v^{ideal}(t)\|_{2} \leq \tau \lambda_{max}$. Second, we are going to show that once we pick $\alpha$ small enough, then for any $i \in \Gamma(v^{ideal}(t))$, the wall $W_{i}$ is far away from the $\alpha$-level set in $S_{v^{ideal}(t)}$. Thus, whenever neuron $i$ fires, $z_{i} > \alpha$.

The first step is a key observation that the distance between $v(t)$ and $v^{ideal}(t)$ would not increase after the neurons fire spikes. The main reason is that neuron $i$ fires at time $t$ if and only if $A_{i}^{T}v(t) > 1$. As a result,

$$\|v(t) - \alpha A_{\Gamma(v^{ideal}(t))}z - v^{ideal}(t)\|_{2}^{2} = \|v(t) - v^{ideal}(t)\|_{2}^{2} + \alpha^{2}\|A_{\Gamma(v^{ideal}(t))}z\|_{2}^{2} - 2\alpha\langle A_{\Gamma(v^{ideal}(t))}z, v(t) - v^{ideal}(t)\rangle$$
Figure 9: This is an example of coordinates of $S_{v_{\text{ideal}}}$ in $\mathbb{R}^2$ where $\tau = 0.4$, $A_1 = [0 - 1]^T$ and $A_2 = [\frac{1}{\sqrt{2}} \quad -\frac{1}{\sqrt{2}}]^T$. The light gray area is the ideal polytope and the dark gray area is the dual polytope. In this example, the dot lines are the level set of each coordinates in $S_{v_{\text{ideal}}}$. For instance, the $v$ in the figure has coordinate $z = [0.1 \ 0.2]^T$ and thus we have $v = v_{\text{ideal}} + Az$.

$$= \|v(t) - v_{\text{ideal}}(t)\|_2^2 + \alpha^2 \|As(t)\|_2^2 - 2\alpha \sum_{i \in \Gamma(v(t))} A_i^T (v(t) - v_{\text{ideal}}(t))$$

$$\leq \|v(t) - v_{\text{ideal}}(t)\|_2^2 + \alpha^2 \|\Gamma(v(t))\|_2^2 - 2\alpha \tau \|\Gamma(v(t))\|_2.$$ 

That is, if $\alpha \leq \frac{\tau}{m}$, then $\| (v(t) - \alpha As(t)) - v_{\text{ideal}}(t)\|_2$ would not increase after some neurons fire. Furthermore, the longest distance between $v(t)$ and $v_{\text{ideal}}(t)$ would then be $\tau \lambda_{\text{max}}$.

The second step is rather complicated. Let us start with some definitions. Recall that for any $i \in [\pm n]$, the wall $i$ is defined as $W_i = \{ v \in \mathbb{R}^m : A_i^T v = 1 \}$. Now, define the $\alpha$-level set of $i$ in $S_{v_{\text{ideal}}(t)}$ as

$$L_{v_{\text{ideal}}(t), i, \alpha} = \{ v \in S_{v_{\text{ideal}}(t)} : v = v_{\text{ideal}}(t) + A_{\Gamma(v_{\text{ideal}}(t))} z, \ z_i = \alpha \}.$$ 

That is, $L_{v_{\text{ideal}}(t), i, \alpha}$ consists of the set of points in $S_{v_{\text{ideal}}(t)}$ that has the $i$th coordinate to be $\alpha$.

**Claim 3.5.2** (farthest point in $S_{v_{\text{ideal}}}$. For any $t \geq 0$, let $v_{\Gamma(v_{\text{ideal}}(t))}$ be the unique point $v \in S_{v_{\text{ideal}}(t)}$ such that for any $i \in \Gamma(v_{\text{ideal}}(t))$, $A_i^T (v - v_{\text{ideal}}(t)) = \tau$. Then, we have $\|v_{\Gamma(v_{\text{ideal}}(t))} - v_{\text{ideal}}(t)\|_2 = \max_{v \in S_{v_{\text{ideal}}(t)}} \|v - v_{\text{ideal}}(t)\|_2.$

**Proof of Claim 3.5.2.** Let us prove by contradiction. Suppose $v^* \in S_{v_{\text{ideal}}(t)}$ such that $\|v_{\Gamma(v_{\text{ideal}}(t))} - v_{\text{ideal}}(t)\|_2 < \|v^* - v_{\text{ideal}}(t)\|_2$. To simplify the notations, let $v_\Gamma = v_{\Gamma(v_{\text{ideal}}(t))} - v_{\text{ideal}}(t)$ and $v = v^* - v_{\text{ideal}}(t)$.

By definition, we have $A_i^T v_\Gamma = \tau$ for all $i \in \Gamma(v_{\text{ideal}}(t))$ and $v = A_{\Gamma(v_{\text{ideal}}(t))} z_\Gamma$ for some $z_\Gamma \in \mathbb{R}_{>0}$. On the other hand, we also have $0 \leq A_i^T v \leq \tau$ for all $i \in \Gamma(v_{\text{ideal}}(t))$.

Now, look at the quantity $v_\Gamma^T (v - v_\Gamma)$. Note that since $\|v\|_2 > \|v_\Gamma\|_2$, we have $v_\Gamma^T (v - v_\Gamma) > 0$. Also, for any $i \in \Gamma(v_{\text{ideal}}(t))$, we have $A_i^T (v - v_\Gamma) \leq 0$. Using the fact that $v = A_{\Gamma(v_{\text{ideal}}(t))} z_\Gamma$ for some $z_\Gamma \in \mathbb{R}_{>0}$, we have

$$0 < v_\Gamma^T (v - v_\Gamma) = z_\Gamma^T A_{\Gamma(v_{\text{ideal}}(t))}^T (v - v_\Gamma)$$

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\[= z_i^\top u \leq 0,\]

where \(u = A_{\Gamma(v_{\text{ideal}}(t))}^\top (v - v_{\Gamma}) \in \mathbb{R}^{\Gamma(v_{\text{ideal}}(t)) \leq 0}\). That is, we reach a contradiction and thus \(\|v\|_2 \leq \|v_{\Gamma}\|_2\) and we conclude that \(v_{\Gamma(v_{\text{ideal}}(t))}\) is the furtherest point from \(v_{\text{ideal}(t)}\) in \(S_{v_{\text{ideal}(t)}}\). \(\Box\)

Claim 3.5.3 (intersection of wall and \(\alpha\)-level set is far). When \(0 < \alpha \leq \tau \cdot \gamma(A)^3\), for any \(t \geq 0\) and \(i \in \Gamma(v_{\text{ideal}}(t))\), we have

\[
\min_{v: v \in W_i \cap L_{v_{\text{ideal}}(t), i, \alpha}} \|v - v_{\text{ideal}(t)}\|_2 > \|v_{\Gamma(v_{\text{ideal}}(t))} - v_{\text{ideal}(t)}\|_2.
\]

Proof of Claim 3.5.3. First, let us write \(v_{\Gamma(v_{\text{ideal}}(t))} = v_{\text{ideal}(t)} + \sum_{i \in \Gamma(v_{\text{ideal}}(t))} z_i A_i\) where \(z_i \geq \tau \cdot \gamma(A)\) by Definition 4. Furthermore, for any \(i \in \Gamma(v_{\text{ideal}}(t))\), we have

\[
\text{dist}(v_{\Gamma(v_{\text{ideal}}(t))}, \text{span}(A_\{i\})) \geq |z_i| \cdot \text{dist}(v_{\Gamma(v_{\text{ideal}}(t))}, \text{span}(A_\{i\})) \geq \tau \cdot \gamma(A)^2,
\]

where the last inequality follows Definition 4. Namely, if we pick \(0 < \alpha < \tau^2 \cdot \gamma(A)^3\), then

\[
\text{dist}(v_{\Gamma(v_{\text{ideal}}(t))}, L_{v_{\text{ideal}}(t), i, \alpha}) > 0
\]

and \(v_{\Gamma(v_{\text{ideal}}(t))} \in \text{Cone}(A_i, L_{v_{\text{ideal}}(t), i, \alpha})\) because \(z_i \geq \gamma(A)\). Finally, observe that for any \(v \in W_i \cap L_{v_{\text{ideal}}(t), i, \alpha}\), we have \(v_{\Gamma(v_{\text{ideal}}(t))}^\top (v - v_{\Gamma(v_{\text{ideal}}(t))}) > 0\). This completes the proof of Claim 3.5.3. \(\Box\)

Combine Claim 3.5.2 and Claim 3.5.3, we know that when neuron \(i\) fires, the corresponding coordinate \(z_i\) will be at least \(\alpha\). This completes the proof of Claim 3.5.1. \(\Box\)

Now, Lemma 3.5 follows from Claim 3.5.1 and equation (11). \(\Box\)

3.6 Strict convergence of ideal SNN and auxiliary SNNs

In this subsection, the goal is to characterize the dynamics of both ideal and auxiliary SNN. Before defining auxiliary SNN, let us first see the following lemma about the dynamics of ideal SNN.

Lemma 3.6 (dynamics of ideal SNN). If \(A\) is nice, then for any \(t \geq 0\), we have

\[v_{\text{ideal}}(t + dt) = v_{\text{ideal}}(t) + (b - \Pi_{A_{\Gamma(v_{\text{ideal}}(t))}} b) dt.\]

Proof of Lemma 3.6. We consider two cases: (i) there is no neuron fires any spike and (ii) there is a neuron fires a spike.

Case (i): By Definition 6, \(v(t) = v_{\text{ideal}} + A_{\Gamma(v_{\text{ideal}}(t))} z\) for some \(z \geq 0\). Also, rewrite the updates \(b\) as

\[b = (b - \Pi_{A_{\Gamma(v_{\text{ideal}}(t))}} b) + \Pi_{A_{\Gamma(v_{\text{ideal}}(t))}} b.\]

First, \(A_i^\top (b - \Pi_{A_{\Gamma(v_{\text{ideal}}(t))}} b) = 0\) for each \(i \in \Gamma(v_{\text{ideal}}(t))\). Next, since there is no neuron fires at time \(t\), observe that \(v(t) + \Pi_{A_{\Gamma(v_{\text{ideal}}(t))}} b \in S_{v_{\text{ideal}(t)}}\). Finally, since \(b - \Pi_{A_{\Gamma(v_{\text{ideal}}(t))}} b\) is orthogonal to the subspace spanned by the active neurons, we then have \(v(t) + b dt \in S_{v_{\text{ideal}(t)} + (b - \Pi_{A_{\Gamma(v_{\text{ideal}}(t))}} b) dt}.\)
Case (ii): To handle spikes, the idea is to focus on the spike term first, and once \(v(t)\) goes back to the interior of the dual polytope, then it becomes case (i). Here, we use an assumption that if there are some neurons fire at time \(t\) and they trigger consecutive firing, we add the external charging after the consecutive firing. As a result, it suffices to show that \(v(t) - \alpha As(t) \in S_{\text{ideal}(t)}\), which immediately follows from Lemma 3.5.

We conclude that for any \(t \geq 0\), \(v^{\text{ideal}}(t + dt) = v^{\text{ideal}}(t) + \left(b - \Pi_{A_{\bar{v}}(\text{ideal}(t))} b\right) dt\).

From Lemma 3.6, one can see that the improvement of ideal SNN is not proportional to the residual error when the \(\Pi_{A_{\bar{v}}(\text{ideal}(t))} \neq A^{\text{ideal}}(t)\). As a result, we have to design a bunch of auxiliary SNN to make sure that at least one of them has improvement proportional to the residual error. The auxiliary SNNs \(\{v_{d}^{\text{auxiliary}}(t)\}_{d \in [m-1]}\) is defined as follows.

**Definition 8** (auxiliary SNNs). For each \(t \geq 0\), and \(d \in [m-1]\), define \(v^{\text{auxiliary}}(0) = 0\) and

\[
v_{d}^{\text{auxiliary}}(t + dt) = \begin{cases} 
   v_{d}^{\text{auxiliary}}(t) + (b - Ax^{\text{ideal}}(t)) dt, & \text{if } |\Gamma^{*}(v^{\text{ideal}}(t + dt))] = d \\
   v^{\text{super}}(t + dt), & \text{if } |\Gamma^{*}(v^{\text{ideal}}(t + dt))] = d, \\
   v_{d}^{\text{auxiliary}}(t), & \text{else.}
\end{cases}
\]

The auxiliary SNNs have the following important property that is crucial in the proof of the Lemma 3.8 which gives the strict improvement guarantee.

**Lemma 3.7** (auxiliary SNNs jump). Suppose \(A\) is nice and \(\tau = O\left(\frac{\gamma(A)}{n^{2} \lambda_{\text{max}}}\right)\). For any \(t > 0\) and \(d \in [m-1]\), if \(|\Gamma^{*}(y^{\text{ideal}}(t))] \neq |\Gamma^{*}(v^{\text{ideal}}(t + dt))| = d\), then \(b^{T} (v_{d}^{\text{auxiliary}}(t + dt) - v_{d}^{\text{auxiliary}}(t)) > 0\).

**Proof of Lemma 3.7.** By the definition of auxiliary SNNs, we have three observations. First, \(|v_{d}^{\text{auxiliary}}(t + dt) - v^{\text{ideal}}(t)||_{2} = |v^{\text{super}}(t + dt) - v^{\text{ideal}}(t)||_{2} = O(\tau \cdot n \cdot \lambda_{\text{max}})\). Second, there exists \(0 \leq t' < t\) such that \(v_{d}^{\text{auxiliary}}(t) = v^{\text{super}}(t')\) and \(\Gamma(v^{\text{ideal}}(t')) \neq \Gamma(v^{\text{ideal}}(t))\). That is, we also have \(|v_{d}^{\text{auxiliary}}(t) - v^{\text{ideal}}(t)||_{2} = (v^{\text{super}}(t') - v^{\text{ideal}}(t)||_{2} = O(\tau \cdot n \cdot \lambda_{\text{max}})\). Finally, since \(\Gamma(v^{\text{ideal}}(t')) \neq \Gamma(v^{\text{ideal}}(t))\), by Lemma 3.6, we have \(b^{T} (v^{\text{ideal}}(t) - v^{\text{ideal}}(t')) = \Omega(|b||_{2} \cdot \frac{\gamma(A)}{n \cdot \lambda_{\text{max}}}\).

Combine the three we have

\[
b^{T} (v_{d}^{\text{auxiliary}}(t + dt) - v_{d}^{\text{auxiliary}}(t)) \geq b^{T} (v^{\text{ideal}}(t) - v^{\text{ideal}}(t')) - O(|b||_{2} \cdot \tau \cdot n \cdot \lambda_{\text{max}}) \\
\geq \Omega(|b||_{2} \cdot \frac{\lambda(A)}{n \cdot \lambda_{\text{max}}} - O(|b||_{2} \cdot \tau \cdot n \cdot \lambda_{\text{max}}) > 0,
\]

where the last inequality holds when we pick \(\tau = O\left(\frac{\gamma(A)}{n^{2} \lambda_{\text{max}}}\right)\).

Now, we are able to prove the main lemma about identifying a potential function that is strictly improving as long as \(x^{\text{ideal}}(t)\) is not the optimal solution for \(\ell_{1}\) minimization problem.

**Lemma 3.8** (strict improvement). For any \(t > 0\), we have

\[
\frac{d}{dt} b^{T} \left(v^{\text{ideal}}(t) + \sum_{d \in [m-1]} v_{d}^{\text{auxiliary}}(t)\right) \geq b^{T} Ax^{\text{ideal}}(t).
\]

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**Proof of Lemma 3.8.** The proof is based on case analysis on the size of $|\Gamma^*(v_{\text{ideal}}(t))|$. We consider three cases:

(i) $\Gamma^*(v_{\text{ideal}}(t)) = \Gamma(v_{\text{ideal}}(t))$,

(ii) $\Gamma^*(v_{\text{ideal}}(t)) \subseteq \Gamma(v_{\text{ideal}}(t))$ and $\vert \Gamma^*(v_{\text{ideal}}(t))\vert = \vert \Gamma^*(v_{\text{ideal}}(t + dt))\vert$, and

(iii) $\Gamma^*(v_{\text{ideal}}(t)) \subseteq \Gamma(v_{\text{ideal}}(t))$ and $\vert \Gamma^*(v_{\text{ideal}}(t))\vert \neq \vert \Gamma^*(v_{\text{ideal}}(t + dt))\vert$.

In each case, we are going to show that at least one of $v_{\text{ideal}}(t)$ or $v_{\text{auxiliary}}(t)$ for some $d \in [m - 1]$ has the desired improvement. Also, we need to show that all of them would not get worse. Formally, we state it as the following claim.

**Claim 3.8.1.** For any $t > 0$ and $d \in [m - 1]$, $\frac{d}{dt}b^Tv_{\text{ideal}}(t), b^Tv_{\text{auxiliary}}(t) \geq 0$.

**Proof of Claim 3.8.1.** From Lemma 3.6, we already have $b^Tv_{\text{ideal}}(t) \geq 0$. For any $d \in [m - 1]$, consider three cases as in Definition 8.

If $\vert \Gamma^*(v_{\text{ideal}}(t))\vert = \vert \Gamma^*(v_{\text{ideal}}(t + dt))\vert = d$, then $\frac{d}{dt}b^Tv_{\text{auxiliary}}(t) = b^T(A - x_{\text{ideal}}(t)) \geq 0$.

If $\vert \Gamma^*(v_{\text{ideal}}(t))\vert \neq \vert \Gamma^*(v_{\text{ideal}}(t + dt))\vert = d$, then by Lemma 3.7 we have $b^T(v_{\text{auxiliary}}(t + dt) - v_{\text{auxiliary}}(t)) > 0$ and thus $\frac{d}{dt}b^Tv_{\text{auxiliary}}(t) \geq 0$.

Finally, when none of the above happen, we simply have $\frac{d}{dt}b^Tv_{\text{auxiliary}}(t) = 0$. $\square$

With Claim 3.8.1, it suffices to show that at least one of $v_{\text{ideal}}(t)$ or $v_{\text{auxiliary}}(t)$ for some $d \in [m - 1]$ has the desired improvement in all of the above three cases.

**Case (i):** In this case, $Ax_{\text{ideal}}(t) = \Pi_{\Gamma(v_{\text{ideal}}(t))}b$. Thus, by Lemma 3.6, we have $\frac{d}{dt}b^Tv_{\text{ideal}}(t) = b^T(b - Ax_{\text{ideal}}(t))$.

**Case (ii):** In this case, let $d = \vert \Gamma^*(v_{\text{ideal}}(t + dt))\vert$. By Definition 8, we have $\frac{d}{dt}b^Tv_{\text{auxiliary}}(t) = b^T(b - Ax_{\text{ideal}}(t))$.

**Case (iii):** In this case, let $d = \vert \Gamma^*(v_{\text{ideal}}(t + dt))\vert$. By Lemma 3.7, we have $b^T(v_{\text{auxiliary}}(t + dt) - v_{\text{auxiliary}}(t)) > 0$ and thus $\frac{d}{dt}b^Tv_{\text{auxiliary}}(t) \geq b^T(b - Ax_{\text{ideal}}(t))$.

This completes the proof of Lemma 3.8. $\square$

Finally, before we go into the final proof for Theorem 5, we need the following lemma about some properties about the ideal solution defined in Definition 7.

**Lemma 3.9** (properties of ideal solution). For any $t \geq 0$, we have the following.

1. $b^TAx_{\text{ideal}}(t) = \|Ax_{\text{ideal}}(t)\|^2_2$,

2. $\|b - Ax_{\text{ideal}}(t)\|^2_2 = \|b\|^2_2 - \|Ax_{\text{ideal}}(t)\|^2_2$, and

**Proof of Lemma 3.9.** The lemma is directly followed by the following property of conic projection. For any $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $\Gamma \subseteq [\pm n]$ be a valid set, we have $b^TAx_{A,b,\Gamma} = \|Ax_{A,b,\Gamma}\|^2_2$. In the following, we are going to first prove this property of conic projection and then use it to prove the lemma.
Let us rewrite the definition of conic projection as an optimization program.

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2} \|b - Ax\|_2^2 \\
\text{subject to} & \quad x_j \geq 0, \ j \in \Gamma, \\
& \quad x_i = 0, \ i, -i \notin \Gamma.
\end{align*}
\]

Let \( y \) be the dual variable of (12) and \( y^* \) be the optimal dual value, the Lagrangian of (12) is

\[
L(x) = \frac{1}{2} \|b - Ax\|_2^2 - y^T x,
\]

and its gradient is

\[
\nabla_x L(x) = A^T Ax - A^T b - y.
\]

By the KKT condition, we know that the optimal primal solution \( x_{A,b,\Gamma} \) and the optimal dual solution \( y^* \) make the gradient of the Lagrangian diminish.

\[
\nabla_x L(x_{A,b,\Gamma}) = A^T Ax_{A,b,\Gamma} - A^T b - y^* = 0,
\]

and the complementary slackness

\[
x_{A,b,\Gamma}^T y^* = 0.
\]

By (13) and (14), we have

\[
(Ax_{A,b,\Gamma})^T (b - Ax_{A,b,\Gamma}) = 0.
\]

As a result, \( b^T Ax_{A,b,\Gamma} = \|Ax_{A,b,\Gamma}\|_2^2 \).

This completes the proof of Lemma 3.9.

\[\square\]

### 3.7 The convergence of dual SNN

In this subsection, we are going to prove the main convergence theorem of the dual SNN using ideal and auxiliary SNN. The following lemma says that at least one of ideal SNN or auxiliary SNN improves at each step.

The following lemma shows the monotonicity of the residual error \( \|b - Ax_{\text{ideal}}\|_2 \).

**Lemma 3.10** (monotonicity of residual error). *There exists a polynomial \( \alpha(\cdot) \) such that when \( 0 < \alpha \leq \alpha\left(\frac{\gamma(\mathcal{A})}{n \lambda_{\max}}\right) \), we have \( \|b - Ax_{\text{ideal}}(t)\|_2 \) is non-increasing and \( \|Ax_{\text{ideal}}(t)\|_2 \) is non-decreasing in \( t \).*

**Proof of Lemma 3.10.** Consider two cases.

1. When there is a new index joins the active set. Clearly that \( \|Ax_{\text{ideal}}(t)\|_2 \) won’t decrease since the new cone contains the old one. By Lemma 3.9, we know that \( \|b - Ax_{\text{ideal}}(t)\|_2 \) is non-increasing.

2. When there is an index leaves the the active set. Without loss of generality, assume \( j \in [\pm n] \) leaves the active set. In the following, we want to show that \( x_{\text{ideal}}(t) \) is an index.

As the direction of \( v_{\text{ideal}}(t) \) is \( b - Ax_{\text{ideal}}(t) \), it means that \( A_j^T (b - Ax_{\text{ideal}}(t)) < 0 \). Suppose \( x_{\text{ideal}}(t) \neq 0 \) for contradiction. Since \( j \) was in the active set, it is the case that \( x_j(\text{ideal}) > 0 \). Take
0 < \epsilon < \min\{x_j^{\text{ideal}}(t)/2, -(b - Ax^{\text{ideal}}(t))^\top{A_j}/\|A_j\|_2\} and define x' = x^{\text{ideal}}(t) - \epsilon \cdot A_j/\|A_j\|_2. Note that x' lies in the original active cone. Observe that
\[
\|b - Ax'\|_2^2 = \|b - Ax^{\text{ideal}}(t) + \epsilon \cdot A_j/\|A_j\|_2\|_2^2 \\
= \|b - Ax^{\text{ideal}}(t)\|_2^2 + \epsilon^2 \cdot (b - Ax^{\text{ideal}}(t))^\top{A_j}/\|A_j\|_2 + 2 \epsilon \cdot (b - Ax^{\text{ideal}}(t))^\top{A_j}/\|A_j\|_2 \\
\leq \|b - Ax^{\text{ideal}}(t)\|_2^2 + \epsilon^2 \\
< \|b - Ax^{\text{ideal}}(t)\|_2^2
\]
which contradicts to the optimality of x^{\text{ideal}}(t) since x' is also a feasible solution. We conclude that x_j^{\text{ideal}}(t) = 0. As a result, Ax^{\text{ideal}}(t) remains the same and \|Ax^{\text{ideal}}(t)\|_2 won’t decrease. □

The next lemma upper bounds the \ell_2 residual error of x^{\text{ideal}}(t).

**Lemma 3.11** (convergence of residual error). There exists a polynomial \alpha(\cdot) such that when 0 < \alpha \leq \alpha(\gamma(A)/\|A\|_{\max}) , we have for any \epsilon > 0, when \( t \geq \frac{m \cdot \text{OPT}^{\ell_1}}{\epsilon \cdot \|b\|_2}, \|b - Ax^{\text{ideal}}(t)\|_2 \leq \epsilon \cdot \|b\|_2.

**Proof of Lemma 3.11.** Assume the statement is wrong, i.e., \|b - Ax^{\text{ideal}}(t)\|_2 > \epsilon \cdot \|b\|_2. Then by Lemma 3.10, for any 0 \leq s \leq t,
\[
\|b - Ax^{\text{ideal}}(s)\|_2 = \|b\|_2^2 - \|Ax^{\text{ideal}}(s)\|_2^2 \\
\geq \|b\|_2^2 - \|Ax^{\text{ideal}}(t)\|_2^2 \\
= \|b - Ax^{\text{ideal}}(t)\|_2^2 > \epsilon^2 \cdot \|b\|_2^2.
\]
Since \( t \geq \frac{\text{OPT}^{\ell_1}}{\epsilon \cdot \|b\|_2}, \) by Lemma 3.8,
\[
b^\top\left( v^{\text{ideal}}(t) + \sum_{d \in [m-1]} v_d^{\text{auxiliary}}(t) \right) = \int_0^t b^\top d v^{\text{ideal}}(t) + \sum_{d \in [m-1]} \int_0^t b^\top d v_d^{\text{auxiliary}}(t) \\
> t \cdot \epsilon \cdot \|b\|_2 \geq m \cdot \text{OPT}^{\ell_1},
\]
which is a contradiction to the optimality of \text{OPT}^{\ell_1} since \( b^\top v^{\text{ideal}}(t), b^\top v_d^{\text{auxiliary}}(t) \leq \text{OPT}^{\ell_1} \) for all \( d \in [m-1]. \) As a result, we conclude that \|b - Ax^{\text{ideal}}(t)\|_2 \leq \epsilon \cdot \|b\|_2. □

Finally, the following lemma shows that the \ell_1 error of x^{\text{ideal}}(t) can be upper bounded by the \ell_2 error via the strong duality of \ell_1 minimization problem and perturbation trick.

**Lemma 3.12** (convergence of \ell_1 error). For any \( t \geq 0, \)
\[
\|x^{\text{ideal}}(t)\|_1 - \text{OPT}^{\ell_1} \leq \sqrt{\frac{n}{\lambda_{\min}}} \cdot \|b - Ax^{\text{ideal}}(t)\|_2 \tag{15}
\]

**Proof sketch.** The proof of Lemma 3.12 consists of two steps. First, we show that the primal and the dual solution pair of ideal SNN at time \( t \) is the optimal solution pair of a perturbed \ell_1 minimization problem defined as shifting the \( b \) in the constraint \( Ax = b \) to \( Ax^{\text{ideal}}(t). \) See (18) for the definition of the perturbed program. Next, by the standard perturbation theorem from optimization, we can upper bound \|x^{\text{ideal}}(t)\|_1 with the distance between the original program and the perturbed
program. Specifically, the difference induced by the perturbation is related to the $\ell_2$ norm of the difference between $b$ and $A\text{x}^{\text{ideal}}(t)$, which is exactly the residual error. As a result, we know that the difference between the optimal value of the original $\ell_1$ minimization program and that of the perturbed program will converge to 0. Namely, we yield a convergence of $\|x^{\text{ideal}}(t)\|_1$ to $\text{OPT}^{\ell_1}$.

See Section A.2 for more details. □

Finally, we can prove the main theorem in this section as follows.

**Proof of Theorem 5.** Pick $t_0 = \Theta(\frac{m \cdot \sqrt{n} \cdot \|b\|_2}{\epsilon \cdot \sqrt{\lambda_{\text{min}} \cdot \text{OPT}^{\ell_1}}})$. By Lemma 3.11, for any $t \geq t_0$, we can upper bound the $\ell_2$ residual error by

$$\|b - A\text{x}^{\text{ideal}}(t)\|_2 \leq \sqrt{\frac{\lambda_{\text{min}}}{n}} \cdot \frac{\epsilon}{10} \cdot \text{OPT}^{\ell_1}.$$ 

Next, by Lemma 3.12, we can then upper bound the $\ell_1$ error by

$$\left|\|x^{\text{ideal}}(t)\|_1 - \text{OPT}^{\ell_1}\right| \leq \sqrt{\frac{n}{\lambda_{\text{min}}} \cdot \|b - A\text{x}^{\text{ideal}}(t)\|_2} \leq \frac{\epsilon}{10} \cdot \text{OPT}^{\ell_1}.$$ 

Now, the only thing left is connecting the ideal solution $x^{\text{ideal}}(t)$ to the firing rate $x(t)$. First, divide $x(t)$ into two parts: the firing rate $x^{[0,t_0]}$ before time $t_0$ and the firing rate $x^{(t_0,t]}$ from time $t_0$ to $t$.

That is, $x(t) = \frac{t_0}{t} \cdot x^{[0,t_0]} + \frac{t - t_0}{t} \cdot x^{(t_0,t]}$.

Note that after $t_0 \geq \Omega(\frac{m \cdot \sqrt{n} \cdot \|b\|_2}{\epsilon \cdot \sqrt{\lambda_{\text{min}} \cdot \text{OPT}^{\ell_1}}})$, the ideal solution has $\ell_1$ norm at most $(1 + \epsilon) \cdot \text{OPT}^{\ell_1}$.

Thus, $\|x^{[t_0,t]}\|_1 \leq \left(1 + (1 + O(\frac{1}{t}))\right) \cdot \frac{\epsilon}{10} \cdot \text{OPT}^{\ell_1} \leq \left(1 + \frac{\epsilon}{5}\right) \cdot \text{OPT}^{\ell_1}$. As for $x^{[0,t_0]}$, from Lemma 3.12, we have $\|x^{[0,t_0]}\|_1 \leq \text{OPT}^{\ell_1} + \sqrt{\frac{n}{\lambda_{\text{min}}} \cdot \|b\|_2}$. Combine the two, we have

$$\left|\|x(t)\|_1 - \text{OPT}^{\ell_1}\right| \leq \frac{\epsilon \cdot \text{OPT}^{\ell_1}}{5} + \frac{t_0 \cdot \left(\text{OPT}^{\ell_1} + \sqrt{\frac{n}{\lambda_{\text{min}}} \cdot \|b\|_2}\right)}{t} \leq \epsilon \cdot \text{OPT}^{\ell_1},$$

where the last inequality holds since $t \geq \Omega(\frac{m^2 \cdot \sqrt{n} \cdot \|b\|_2}{\epsilon^2 \cdot \lambda_{\text{min}} \cdot \text{OPT}^{\ell_1}})$. This completes the proof for Theorem 5. □

## 4 A simple SNN algorithm for the non-negative least squares

In the introduction, we claim that we can show that the firing rate of one-sided SNN will converge to the solution of non-negative least squares problem. In this section, we are going to formally prove this Theorem 1. Recall that the non-negative least squares is defined as follows.

$$\begin{align*}
\text{minimize} \quad & \|Ax - b\|_2 \\
\text{subject to} \quad & x_i \geq 0, \, \forall i \in [n],
\end{align*}$$

(16)

We start with formally state the result into the following theorem.
Theorem 6. Given $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ where all the row of $A$ has unit norm. Let $\gamma(A) \geq 0$ be the niceness parameter of $A$ defined later in Definition 4. Suppose $\gamma(A) > 0$. There exists a polynomial $\alpha(\cdot)$ such that for any $t \geq 0$, let $x(t)$ be the firing rate of a simple continuous SNN with $C = A^\top A$, $I = A^\top b$, $\eta = 1$, and $0 < \alpha \leq \alpha(\frac{\gamma(A)}{\lambda_{\min}^2})$. For any $\epsilon > 0$, when $t \geq \frac{\sqrt{\lambda_{\max} n}}{\epsilon \lambda_{\min} \|b\|_2}$, then $x(t)$ is an $\epsilon$-approximation solution to the non-negative least squares problem.

Here, we say $x$ is an $\epsilon$-approximation\(^{18}\) solution if for any optimal solution $x^*$ of the above program $\|Ax - Ax^*\|_2 \leq \epsilon \cdot \|b\|_2$.

The proof follows from similar idea of ideal coupling. With the dual SNN view from Section 3.1, we can use Lemma 3.5 to control the behavior of dual SNN and thus have a good control on its dynamics.

Proof of Theorem 6. Given $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, we first define the conic projection of $b$ on the cone spanned by the column of $A$ as follows.

$$x^+ = \arg \min_{x \in \mathbb{R}^n_{\geq 0}} \|Ax - b\|_2^2.$$  

Here, $x^+$ is the optimal solution of (16) and we let $b^+ = Ax^+$ which is the conic projection of $b$ on the cone spanned by the column of $A$. Note that $x^+$ is also the optimal solution of the following optimization program with minimum value to be 0.

$$\min_{x} \|Ax - b^+\|_2^2. \tag{17}$$

Given a simple SNN with $C = A^\top A$ and $I = A^\top b$, the dual SNN as defined in Section 3.1 would be

$$v(t) = t \cdot (b - Ax(t)).$$

Define $v^+(t) = v(t) - t \cdot (b - b^+) = t \cdot (b^+ - Ax(t))$. It turns out that $\|v^+(t)\|_2^2 = \|b^+ - Ax(t)\|_2^2$ is the residual error of $x(t)$ in solving (17). That is, to prove Theorem 6, it suffices to show that $\|v^+(t)\|_2^2$ converges to 0. We put this into the lemma below.

Lemma 4.1. With the conditions stated in Theorem 1, for any $t \geq 0$, we have

$$v^+(t) \in \left\{ v \in \mathbb{R}^m : A_i^\top v \leq 1, \forall i \in [n] \right\} \cap \left\{ v = \sum_{i \in [n]} \alpha_i A_i : \alpha_i \geq 0, \forall i \in [n] \right\}.$$

Especially, we have $\|v^+(t)\|_2 \leq \frac{\sqrt{\lambda_{\max}}}{\lambda_{\min}}$.

From Lemma 4.1, we have $\|b^+ - Ax(t)\|_2 \leq \frac{\sqrt{\lambda_{\max}}}{\lambda_{\min}}$. Let $x^*$ be the optimal solution of (16) (which is also the optimal solution of (17) as we argued before), we have $\|b^+ - Ax^*\|_2 = 0$. By triangle inequality, we have $\|Ax(t) - Ax^*\|_2 \leq \frac{\sqrt{\lambda_{\max}}}{\lambda_{\min}}$. When $t \geq \frac{\sqrt{\lambda_{\max}}}{\epsilon \lambda_{\min} \|b\|_2}$, we have $\|Ax - Ax^*\|_2 \leq \epsilon \cdot \|b\|_2$. This completes the proof of Theorem 6. \(\square\)

\(^{18}\)The reason why we define in this way is to handle the case where the program has many solutions. In such case the only unique thing is that $\|Ax - b\|_2$ are all the same among these optimal solutions.
4.1 Proof of Lemma 4.1

Proof of Lemma 4.1. The proof is based on induction on $t \geq 0$. For the base case where $t = 0$, the lemma is trivially true. Suppose the lemma holds for some $t \geq 0$, consider $t + dt$. Note that

$$v^+(t + dt) = v^+(t) - \alpha A s(t) + b^+ dt.$$

By Lemma 3.5, we have

$$v^+(t) - \alpha A s(t) \in \{ v \in \mathbb{R}^m : A_i^T v \leq 1, \forall i \in [n] \} \cap \left\{ v = \sum_{i \in [n]} \alpha_i A_i : \alpha_i \geq 0, \forall i \in [n] \right\}$$

from the induction hypothesis. As $b^+ \in \left\{ v = \sum_{i \in [n]} \alpha_i A_i : \alpha_i \geq 0, \forall i \in [n] \right\}$ and $A_i^T (v^+(t) - \alpha A s(t)) < 1$ due to the spiking rule, we have

$$v^+(t + dt) \in \{ v \in \mathbb{R}^m : A_i^T v \leq 1, \forall i \in [n] \} \cap \left\{ v = \sum_{i \in [n]} \alpha_i A_i : \alpha_i \geq 0, \forall i \in [n] \right\}.$$

Note that the largest $\ell_2$ norm in the above intersection is at most the largest $\ell_2$ norm in the dual polytope $\{ v : \| A^T v \|_\infty \leq 1 \}$. Thus, $\| v^+(t) \|_2 \leq \frac{\sqrt{\lambda_{\text{max}}} \cdot n}{\lambda_{\text{min}}}$. \hfill \qed

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A Missing proofs for Theorem 5

A.1 Proofs for the properties of ideal and auxiliary SNN

Proof of Lemma 3.4. Let us start with an observation on Definition 5 about the points on the boundary of the ideal polytope $P_{A,1-\tau}$.

Claim A.0.1. If $A$ is non-degenerate, then for any $v^{\text{ideal}} \in \partial P_{A,1-\tau}$, $\text{rank}(A_{\Gamma(v^{\text{ideal}})}) = |\Gamma(v^{\text{ideal}})|$.

Thus, $A_{\Gamma(v^{\text{ideal}})}^T A_{\Gamma(v^{\text{ideal}})}$ is positive definite.

Next, let us show that for $v_1^{\text{ideal}} \neq v_2^{\text{ideal}} \in P_{A,1-\tau}$, $S_{v_1^{\text{ideal}}} \cap S_{v_2^{\text{ideal}}} = \emptyset$. It is trivially true when at least one of them does not lie on the boundary of $P_{A,1-\tau}$. Now, consider the case where both of them lie on the boundary of $P_{A,1-\tau}$ and denote their active set as $\Gamma_1 = \Gamma(v_1^{\text{ideal}})$ and $\Gamma_2 = \Gamma(v_2^{\text{ideal}})$. To prove from contradiction, suppose there exists $v \in S_{v_1^{\text{ideal}}} \cap S_{v_2^{\text{ideal}}}$. By definition, we have

$$v = v_1^{\text{ideal}} + A_{\Gamma_1}^T z_1 = v_2^{\text{ideal}} + A_{\Gamma_2}^T z_2,$$

where $z_1, z_2 \geq 0$. Let $\Gamma = \Gamma_1 \cap \Gamma_2$. Consider the following cases.

- $(\Gamma = \Gamma_1 = \Gamma_2)$ By Definition 5, we have $A_{\Gamma_1}^T v_1^{\text{ideal}} = A_{\Gamma_2}^T v_1^{\text{ideal}} = 1$ and thus

$$\begin{align*}
(z_2 - z_1)^T A_{\Gamma_1}^T A_{\Gamma_1}(z_2 - z_1) &= (z_2 - z_1)^T A_{\Gamma_1}(v_1^{\text{ideal}} - v_2^{\text{ideal}}) = 0.
\end{align*}$$

As $A_{\Gamma_1}^T A_{\Gamma_1}$ is positive definite by Claim A.0.1, we have $z_1 = z_2$ and $v_1^{\text{ideal}} = v_2^{\text{ideal}}$, which is a contradiction.

- $(\Gamma_1 \neq \Gamma_2)$ Without loss of generality, assume $\Gamma_1 \setminus \Gamma \neq \emptyset$ and $z_1 \neq 0$. By Definition 5, we have

$$\begin{align*}
A_{\Gamma_1 \setminus \Gamma_2}^T (v_1^{\text{ideal}} - v_2^{\text{ideal}}) &> 0, \\
A_{\Gamma_2 \setminus \Gamma_1}^T (v_1^{\text{ideal}} - v_2^{\text{ideal}}) &\leq 0, \\
A_{\Gamma}^T (v_1^{\text{ideal}} - v_2^{\text{ideal}}) &= 0.
\end{align*}$$

Note that $v^{\text{ideal}}$ does not lie on the boundary of $P_{A,1-\tau}$ if and only if $\Gamma(v^{\text{ideal}}) = \emptyset$.  

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As $z_1 \neq 0$, we then have

$$\|A_{\Gamma_2}z_2 - A_{\Gamma_1}z_1\|^2 = (A_{\Gamma_2}z_2 - A_{\Gamma_1}z_1)^\top (v_{1\text{ideal}} - v_{2\text{ideal}})$$

$$= (-A_{\Gamma_1\setminus \Gamma}z_1 |_{\Gamma_1\setminus \Gamma})^\top (v_{1\text{ideal}} - v_{2\text{ideal}})$$

$$+ (A_{\Gamma_2\setminus \Gamma}z_2 |_{\Gamma_2\setminus \Gamma})^\top (v_{1\text{ideal}} - v_{2\text{ideal}})$$

$$+ (A_{\Gamma}z_2 |_{\Gamma} - A_{\Gamma}z_1 |_{\Gamma})^\top (v_{1\text{ideal}} - v_{2\text{ideal}})$$

$$< 0.$$ 

Note that the reason why the last inequality holds is because $(-A_{\Gamma_1\setminus \Gamma}z_1 |_{\Gamma_1\setminus \Gamma})^\top (v_{1\text{ideal}} - v_{2\text{ideal}}) < 0$.

Finally, it is easy to see that $\{S_{v_{\text{ideal}}}| v_{\text{ideal}} \in \mathcal{P}_{A,1-\tau}\}$ covers $\mathcal{P}_{A,1}$ and thus we conclude that it is indeed a partition for $\mathcal{P}_{A,1}$.

\[ \square \]

A.2 Proofs for the convergent analysis of solving $\ell_1$ minimization

Proof. Lemma 3.12 For any $t \geq 0$, define the following perturbed program of (7) and its dual.

$$\begin{align*}
\text{minimize} & \quad \|x\|_1 \\
\text{subject to} & \quad Ax - Ax^{\text{ideal}}(t) = 0
\end{align*}$$

(18)$$

$$\begin{align*}
\text{maximize} & \quad (Ax^{\text{ideal}}(t))^\top v \\
\text{subject to} & \quad \|A^\top v\|_\infty \leq 1.
\end{align*}$$

(19)

Note that $x^{\text{ideal}}(t)$ is treated as a given constant to the optimization program. It turns out that the ideal algorithm optimizes this primal-dual perturbed program at time $t$ with the following parameters.

Lemma A.1. For any $t \geq 0$, $(x^*, v^*) = (x^{\text{ideal}}(t), v^{\text{ideal}}(t))$ is the optimal solutions of (18).

Proof. Proof of Lemma A.1 We simply check the KKT condition. Since the program can be rewritten as a linear program, it satisfies the regularity condition of the KKT condition.

First, the primal and the dual feasibility can be verified by the dynamics of ideal algorithm. That is, $Ax^* - Ax^{\text{ideal}}(t) = 0$ and $\|A^\top v^*\|_\infty \leq 1$. Next, consider the Lagrangian of (18) as follows.

$$\mathcal{L}(x, v) = \|x\|_1 - v^\top (Ax - Ax^{\text{ideal}}(t)),$$

$$\nabla_x \mathcal{L}(x, v) = \nabla \|x\|_1 - A^\top v.$$ 

Now, let’s verify that the gradient of the Lagrangian over $x$ is vanishing at $(x^*, v^*) = (x^{\text{ideal}}(t), v^{\text{ideal}}(t))$. That is, $0 \in \nabla_x \mathcal{L}(x^*, v^*) = \nabla \|x\|_1 - A^\top v$. Consider two cases as follows. For any $i \in [n],$

1. When $i, -i \not\in \Gamma^{\text{ideal}}(t)$. We have $(x^{\text{ideal}}(t))^i = 0$, i.e., the sub-gradient of the $i$th coordinate of $\|x^{\text{ideal}}(t)\|_1$ lies in $[-1, 1]$. As $A_i^\top v^{\text{ideal}}(t) \in [-1, 1]$, we have $A_i^\top v^{\text{ideal}}(t) \in \partial_{x_i} \|x^{\text{ideal}}(t)\|_1$.
2. When $i \in \Gamma^{\text{ideal}}(t)$ (or $-i \in \Gamma^{\text{ideal}}(t)$). We have $A_i^\top v^{\text{ideal}}(t) = 1$ (or $A_i^\top v^{\text{ideal}}(t) = -1$). As $\text{sgn}(x^{\text{ideal}}(t))^i = 1$ (or $\text{sgn}(x^{\text{ideal}}(t))^i = -1$), we have $A_i^\top v^{\text{ideal}}(t) = \text{sgn}(x^{\text{ideal}}(t))^i = \partial_{x_i} \|x^{\text{ideal}}(t)\|_1$. 

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Finally, the complementary slackness is satisfied because $Ax^* - Ax^{\text{ideal}}(t) = 0$. As a result, we conclude that $(x^*, v^*)$ is the optimal solution of (18).

Next, we are going to use the perturbation lemma in the Chapter 5.6 of [BV04] stated as follows.

**Lemma A.2** (perturbation lemma). Given the following two optimization programs.

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad h(x) = 0.
\end{align*}
\]

(20) \hspace{1cm} \begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad h(x) = y.
\end{align*}

Let $\text{OPT}^{\text{original}}$ be the optimal value of the original program (20) and $\text{OPT}^{\text{perturbed}}$ be the optimal value of the perturbed program (21). Let $v^*$ be the optimal dual value of the perturbed program (21). We have

\[
\text{OPT}^{\text{original}} \geq \text{OPT}^{\text{perturbed}} + y^\top v^*.
\]

Now, think of (7) as the original program and (18) as the perturbed program. Namely, $f(x) = \|x\|_1$, $h(x) = Ax - b$, and $y = Ax^{\text{ideal}}(t) - b$. By the perturbation lemma, we have

\[
\text{OPT}^{\ell_1} \geq \|x^{\text{ideal}}(t)\|_1 + \left(Ax^{\text{ideal}}(t) - b\right)^\top v^{\text{ideal}}(t).
\]

As a result, the following upper bound holds.

\[
\|x^{\text{ideal}}(t)\|_1 \leq \text{OPT}^{\ell_1} + \|v^{\text{ideal}}(t)\|_2 \cdot \|b - Ax^{\text{ideal}}(t)\|_2.
\]

Finally, as $v^{\text{ideal}}(t)$ lies in the feasible region $\{v : A^\top v\|_\infty \leq 1\}$ and the range space of $A$, we can upper bound the $\|v^{\text{ideal}}(t)\|_2$ term in (23) as follows.

**Lemma A.3.** For any $v$ in the range space of $A$ and $\|A^\top v\|_\infty \leq 1$, $\|v\|_2 \leq \sqrt{\frac{n}{\lambda_{\min}}}$.\n
**Proof.** Proof of Lemma A.3 As $v$ lies in the range space of $A$, we have $\|A^\top v\|_2 \geq \sqrt{\lambda_{\min}} \|v\|_2$. Also, because $\|A^\top v\|_\infty \leq 1$, we have $\|A^\top v\|_2 \leq \sqrt{n}$. As a result,

\[
\|v\|_2 \leq \frac{\|A^\top v\|_2}{\sqrt{\lambda_{\min}}} \leq \sqrt{\frac{n}{\lambda_{\min}}}.
\]

By (23) and Lemma A.3, Lemma 3.12 holds.

### B An inverse quasi-polynomial upper bound for the $\gamma$ of RSM

In Lemma 3.2, we saw that $\gamma(A) > 0$ with high probability when $A$ is sampled from the rotational symmetry model (RSM). As the choice of parameters (e.g., the spiking strength $\alpha$ and the discretization size $\Delta t$) in Theorem 5 has a polynomial dependency on $\gamma(A)$, it would be nice if $\gamma(A)$ is as large as possible. However, in this section, we are going to show that for $A$ sampled from RSM, $\gamma(A)$ is upper bounded by inverse quasi-polynomial in $m$ if $n \geq \text{polylog}(m) \cdot m$ and is upper bounded by inverse exponential in $m$ if $n \geq m^{1+\Omega(1)}$. We thank the anonymous reviewer from ITCS 2019 for pointing out the analysis of these upper bounds.

**Lemma B.1.** For any $m \in \mathbb{N}$ large enough, $0 < \tau \leq \frac{m}{4}$, and $n \geq (2\log m/e^{-\tau}) \cdot m$. Let $A \in \mathbb{R}^{m \times n}$ be a random matrix samples from RSM. Then, $\gamma(A) \leq e^{-\Omega(\tau \log m)}$ with high probability.
Proof of Lemma B.1. The high-level idea of the analysis is iteratively looking at the correlation between the first column of $A$ and the other columns. In particular, divide the rest of columns of $A$ into $m$ buckets each of size $k = \left\lfloor \frac{n}{m} \right\rfloor$. This will give us $m$ buckets of $k$ independent unit vectors in $\mathbb{R}^m$. The idea is then projecting $A_1$ to the subspace spanned by each bucket one by one and argue that the length of the projection decreases by a non-trivial factor. Before doing the formal analysis, let us first prove the following claim about the distribution of the inner product of two random unit vectors in $\mathbb{R}^m$.

Claim B.1.1. Let $v_1, v_2$ be two independent random unit vector in $\mathbb{R}^m$. For any $m$ large enough and $z \in [0, \frac{1}{4}]$, $\Pr[(v_1, v_2)^2 \leq z] \leq 1 - e^{-mz}$.

Proof of Claim B.1.1. Let $Z = (v_1, v_2)^2$, the probability of $Z = z$ for any $z \in [0, 1]$ can be computed by the equation for the surface area on an unit ball in $\mathbb{R}^m$. Concretely, $\Pr[Z = z]$ is proportional to $(1 - z)^{\frac{m-3}{2}}$. Thus, the probability of $Z \leq z$ is

$$\Pr[Z \leq z] = \frac{\int_0^z (1 - t)^{\frac{m-3}{2}} dt}{\int_1^1 (1 - t)^{\frac{m-3}{2}} dt} = 1 - (1 - z)^{\frac{m-1}{2}}.$$  

When $z \in [0, \frac{1}{4}]$, we can use the approximation $e^{-2z} \leq (1 - z) \leq e^{-z}$ and get $\Pr[Z \leq z] \leq 1 - e^{-mz}$. \hfill $\square$

Now, let us start with the first bucket of $k$ independent random unit vectors in $\mathbb{R}^m$. From Claim B.1.1, we know that the probability of existing $v_1$ in the bucket such that $(A_1, v_1)^2 > \frac{\tau}{m}$ is at least $1 - (1 - e^{-\tau})^k$. Let $\Gamma_1 = \{v_1\}$, with probability at least $1 - (1 - e^{-\tau})^k$,

$$\|A_1 - \Pi_{\Gamma_1} A_1\|_2^2 \leq (1 - (A_1, v))^2) \cdot \|A_1\|_2 \leq (1 - \frac{\tau}{m}) \cdot \|A_1\|_2.$$  

For the second bucket, we consider the subspace of $\mathbb{R}^m$ orthogonal to $\Gamma_1$. Using the same argument, we can find $v_2$ in the second bucket such that with probability at least $1 - (1 - e^{-m \cdot \frac{\tau}{m-1}})^k$, $\frac{\langle \Pi_{\Gamma_1} A_1, v_2 \rangle}{\|\Pi_{\Gamma_1} A_1\|_2 \cdot \|v_2\|_2} > \frac{\tau}{m-1}$ and thus

$$\|A_1 - \Pi_{\Gamma_2} A_1\|_2^2 \leq (1 - \frac{\tau}{m-1}) \cdot \|\Pi_{\Gamma_1} A_1\|_2^2 \leq (1 - \frac{\tau}{m-1}) \cdot (1 - \frac{\tau}{m}) \cdot \|A_1\|_2^2.$$  

Repeat the above argument for $m - 1$ times and apply union bound, we have

$$\|A_1 - \Pi_{\Gamma_s} A_1\|_2^2 \leq \prod_{i=1}^{s} \left(1 - \frac{\tau}{m - i + 1}\right) \cdot \|A_1\|_2^2 \leq e^{-\Omega(\tau \cdot \log m)} \cdot \|A_1\|_2^2$$  

with probability at least

$$1 - \sum_{i=1}^{m-1} (1 - e^{-m \cdot \frac{\tau}{m-i+1}})^k \geq 1 - (m - 1) \cdot e^{-\kappa e^{-\tau}} \geq 1 - e^{-\kappa e^{-\tau} + \log m}.$$  

By our choice of $\tau$ and $n$, we have $\|A_1 - \Pi_{\Gamma_s} A_1\|_2 \leq e^{-\Omega(\tau \cdot \log m)} \cdot \|A_1\|_2$ with probability $1 - o(1)$. \hfill $\square$