Robustness of Cohesion in Group Formation

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Cohesion is fundamental for the function of a social group. To study the interplay between group growth and cohesion, we propose a model where a group grows by a noisy admission process of new members who can be of two different types. Cohesion is defined as the fraction of members of the same type. The model can reproduce the empirically reported decrease in cohesion with the group size. However, when admissions require a consensus of several members, we find a phase transition belonging to the mean-field universality class. Below a critical noise level, the growing group can remain cohesive.

A social group is an organized collection of individuals who share common interests or goals [1]. Social groups are the building blocks of our society: they emerge naturally in larger social networks [2]. The formation and growth of social groups have been studied in both offline [3–5] and online [6–9] environments.

In the past two decades, physicists working on complex networks focused on how growing processes influence the group's network structure [10–15]. Here, instead, we study how group growth affects cohesion. Defined as “the tendency of a group to stick together and remain united” [16], cohesion is crucial for the survival of a group, and hence must be considered before any social structure. Cohesion is a much-studied subject in psychology and sociology (see [17] for a recent review) with important implications. For example, it has been shown that cohesive groups perform better in their tasks [16–18]. Among the main factors that influence cohesion is the admission process of new members [3, 19]. At the same time, empirical studies show that groups with more members tend to be less cohesive [20, 21]. A successful group that attracts new members is thus in danger of gradually losing its cohesion, worsening its performance, and possibly even fragmenting into smaller cohesive groups. Yet, the underlying mechanism of the interplay between the admission process, group size, and cohesion is unknown.

Motivated by this lack of knowledge, we formulate a minimalist model for the evolution of a social group. In the model, a small homogeneous group gradually grows by adding new members through an admission process. We assume candidate members of two types: fit for the group (sharing goals or values with the founder members) or unfit for the group otherwise. Candidates are admitted or rejected based on positive or negative evaluations received from the group members. Since member similarity importantly contributes to group cohesion [22–24], in our framework we define cohesion as the fraction of fit group members. More complicated definitions capturing other aspects of group cohesion [25, 20], such as the group’s ability to work toward a given task, can be studied in the future. We assume that individual evaluations of candidates are at odds with homophily (a group member appreciating a candidate of the same type) with some small probability (which we refer to as the evaluation noise). Our goal is to study the long-term effects of different admission processes on group cohesion. We find that if each candidate is evaluated by one group member, group cohesion inevitably decreases with the group size until it reaches the cohesion of a random group. The outcome of the admission process is fundamentally different when admissions require a consensus of two or more group members. We find analytically that a critical evaluation noise level exists, below which the group cohesion is always better than the random group cohesion. As the number of members involved in the admission process increases, the critical noise level increases. We show that at the critical point, the model exhibits a continuous phase transition that belongs to the mean-field universality class.

The model is defined as follows. Consider individuals of two different types, +1 (fit for the group) and −1 (unfit for the group), respectively. Denote the type of group member $i$ by $\sigma_i$. The group initially consists of $N_0$ founder members that are all of type +1: $\sigma_i = +1$ for $i = 1, \ldots, N_0$. The growth of the group proceeds in discrete steps. In each step, one candidate member is considered to be admitted or rejected. The type of each candidate is drawn at random with equal probability. Denote the type of the candidate in step $j$ as $c_j$. We assume that group members tend to positively evaluate candidates of the same type. Specifically, if group member $i$ is asked to evaluate candidate $j$ and $c_j\sigma_i = +1$, the evaluation is positive with probability $1 - \eta$ and negative otherwise. If $c_j\sigma_i = -1$, the evaluation of candidate $j$ by member $i$ is positive with probability $\eta$ and negative otherwise. Here $\eta \in [0, 0.5]$ is a parameter that characterizes the level of randomness in the evaluation of candidates. We assume that each candidate is evaluated by $m$ group members ($m \leq N_0$ to ensure a sufficient number of evaluators). A candidate is admitted only if all evaluations are positive. Within this framework, we consider two different ways to choose the evaluating group mem-
bers: (1) uniformly (uniform case, UC) and (2) proportionally to the number of admissions the member has already contributed to (preferential attachment, PA). The PA case mimics the accumulation of social capital by influential group members who thus evaluate more group members and leads to a scale-free distribution of social capital in the group [27]. As an additional benchmark, we consider the case where one founder member evaluates each candidate (dictatorship, DS).

The growth of the group continues until a given group size, $N$, is reached. We then evaluate the resulting group cohesion, $C$, as the fraction of fit group members,

$$C = \frac{|\{i : \sigma_i = +1\}|}{N}. \quad (1)$$

When $N = N_0$, $C = 1$ as all founder members are fit for the group. When $\eta = 0$, unfit candidates always receive negative evaluations from fit group members. As the group is initially composed of only fit members, all unfit candidates are rejected and the group cohesion remains one for any $N$. When $\eta = 0.5$, the evaluations are not informative of the candidate types and the probability of admitting an unfit candidate is thus $1/2$. As $N$ grows, group cohesion then approaches $1/2$ which is the cohesion level of a random group.

Let us first consider the case where the candidate is evaluated by only one group member. To study cohesion analytically, we count the time in the growing process by the number of admitted members. In this way, the $N_0$ founder members are in the group at time $t = 0$, while the $N$-th member is admitted at time $t = N - N_0$. We introduce the probability $P(t)$ that the $t$-th admitted member is fit for the group. As the group is assumed to initially consist of $N_0$ fit members, $P(0) = 1$. If member $i$ admits a candidate, the candidate is fit with the probability

$$W(i) = P(i)(1 - \eta) + [1 - P(i)]\eta, \quad (2)$$

where the two terms correspond to $\sigma_i = +1$ ($i$ is fit) and $\sigma_i = -1$ ($i$ is unfit), respectively. For the dictatorship case, only the founder members are allowed to evaluate, so $i = 0$ and $P(t) = W(0) = 1 - \eta$. Cohesion is then given by $(1 - \eta) + \eta N_0/N$. As $N \to \infty$, group cohesion thus approaches $1 - \eta$, i.e., it decreases linearly with noise.

For the uniform case, the $t$-th admitted member can be evaluated by a random group member $i$ with $i < t$; $P(t)$ is hence obtained from $W(i)$ by averaging over all $i < t$ [$W(0)$ contributes $N_0$ times as there are $N_0$ founder members]. We get

$$P(t + 1) = \frac{N_0}{t + N_0} W(0) + \frac{1}{t + N_0} \sum_{i=1}^{t} W(i) \quad (3)$$

which is a recursive equation for $P(t)$ with the initial condition $P(0) = 1$. Eq. (3) can be rearranged as

$$P(t + 1) = \left(1 - \frac{2\eta}{t}\right) P(t) + \frac{\eta}{t} \quad (4)$$

which can be solved analytically. Averaging the solution over $t = 0, \ldots, N - N_0$ [again, the weight of $P(0)$ is $N_0$], the expected cohesion reads

$$\bar{C}(N, N_0, \eta) = \frac{1}{2} \left(1 + \frac{\Gamma(N_0 + 1)\Gamma(N + 1 - 2\eta)}{\Gamma(N_0 + 1 - 2\eta)\Gamma(N + 1)}\right). \quad (5)$$

When $\eta = 0$, $\bar{C}(N, N_0, \eta) = 1$ for any $N$ and $N_0$, as only fit candidates can be admitted. However, the expected cohesion decreases fast with $\eta$ and is always lower than that of the dictatorship case (Fig. 1a). When $N_0$ is fixed and $N \gg N_0$, Eq. (5) implies

$$\bar{C}(N, N_0, \eta) \approx \frac{1}{2} + \frac{\Gamma(N_0 + 1)}{2\Gamma(N_0 + 1 - 2\eta)} N^{-2\eta}. \quad (6)$$

This means that, with only one evaluating member, the group cohesion tends to the cohesion of a random group as the group grows, regardless of how small is the level of noise and how many are the founder members (Fig. 1a). Notably, the model with one evaluating member and uniform choice (UC) can be mapped on a recent opinion formation model [28] where a result analogous to Eq. (6) has been derived using the master equation formalism.

To study the impact of introducing social capital in the model, we now consider choosing the evaluating member by the classical PA mechanism. To this end, we keep an
activity counter, $k$, for all group members. This counter increases by 1 for each involvement in admitting a new member. Without loss of generality, we set the initial activity counter to $a_0 := N_0 - 1$ and 1 for all founder members and the later admitted group members, respectively (when $N_0 = 1$, we set $a_0 = 1$ to avoid zero counter of the sole founder member). The probability to choose a member is assumed to be directly proportional to the activity counter. Using the continuum approximation [24] for the dynamics of the activity counter, we can write

$$k_i(t + 1) = k_i(t) + \frac{k_i(t)}{2t + N_0 a_0}, \quad (7)$$

where the second term on the rhs reflects the PA selection mechanism.

Analogously to Eq. (3), the probability that the $t$-th admitted member is fit for the group, $P(t)$, for the PA case reads

$$P(t + 1) = N_0 \frac{W(0) k_0(t)}{2t + N_0 a_0} + \sum_{i=1}^{t} \frac{W(i) k_i(t)}{2t + N_0 a_0}, \quad (8)$$

where $W(i)$ is defined by Eq. (2). With the solution of Eq. (7) and the initial condition $P(0) = 1$, Eq. (8) can be rearranged to a simpler form (similar to Eq. (4))

$$P(t + 1) = \left(1 - \frac{2\eta}{2t + N_0 a_0}\right) P(t) + \frac{\eta}{2t + N_0 a_0}. \quad (9)$$

Similarly to the uniform case, the expected group cohesion is obtained by averaging the obtained $P(t)$ over $t = 0, \ldots, N - N_0$ (again assigning weight $N_0$ to $t = 0$). We see (Fig. [3]) that the expected cohesion still decreases rapidly with $\eta$, yet it differs from the uniform case. When $N \to \infty$ and $N_0$ is fixed, the leading contribution to cohesion is

$$\overline{C}(N, N_0, \eta)_{P_A} \approx \frac{1}{2} \frac{(1 - 2\eta) \Gamma(l_0 + 1)}{2(1 - \eta) \Gamma(l_0 + 1 - \eta)} N^{-\eta}, \quad (10)$$

where $l_0 := N_0(N_0 - 1)/2$. We find that group cohesion in the PA case is not only higher than in the uniform case (Fig. [3]), it also decays with $N$ slower (Fig. [3]): the scaling exponent is $\eta$ instead of $2\eta$. This higher robustness is due to preferential attachment effectively giving more power to early group members who, on average, are more likely to be fit than group members who join the group later. At the same time, the limit cohesion remains unchanged: as the group grows, its cohesion approaches the cohesion of a random group.

Finally, we study the model behaviour when $m > 1$ members evaluate each candidate who is admitted only when all evaluations are positive. Because of the strong non-linearity of the process, we only treat the $N \to \infty$ limit. Consider first the uniform case with $m = 2$ and denote the evaluating members $i$ and $j$. For large $N$, the correlation between the types of $i$ and $j$ can be neglected, and the probabilities of admitting fit and unfit candidates can be factorized as $W(i)W(j)$ and $[1 - W(i)][1 - W(j)]$, respectively. As fit and unfit candidates are equally likely, the probability that an admitted member is fit is

$$W_{ij} = \frac{W(i)W(j)}{W(i)W(j) + [1 - W(i)][1 - W(j)]}, \quad (11)$$

where $W(i)$ is given by Eq. (2). We build the solution again on the probability $P(t)$. As for Eq. (3), $P(t)$ is obtained by averaging $W_{ij}$ over all possible pairs of evaluating members, each of whose is equally likely in the case of uniform selection. Thus, we can write

$$P(t + 1) = \left(\frac{N_0}{2}\right) W_{00} + N_0 \sum_{i=1}^{t} W_{i0} + \sum_{j \neq i}^{t} W_{ij},$$

$$\quad (12)$$

where the initial condition remains $P(0) = 1$. If $\lim_{t \to \infty} P(t)$ exists, then $\lim_{t \to \infty} \overline{C}(N) = \lim_{t \to \infty} P(t) := P$. To find the expected cohesion for $N \to \infty$, it thus suffices to obtain the stationary solution of Eq. (12). For large $t$, the main contribution to the numerator of Eq. (12) is from $W_{ij}$ where $i$ and $j$ are large. Denoting $W := P(1 - \eta) + (1 - P)\eta$, we obtain

$$P = \lim_{i, j \to \infty} W_{ij} = \frac{W^2}{W^2 + (1 - W)^2}; \quad (13)$$

Besides the trivial solution $P = 1/2$, this equation has two non-trivial solutions when $\eta < \eta_0 = 1/4$, representing the group composed of mostly fit and mostly unfit members, respectively. As we assume that the founder members are fit, the second solution is not physical, and we can write

$$P = \begin{cases} \frac{1}{2}, & \text{if } \eta < 1/4, \\ \frac{1}{2} + \frac{\sqrt{1 - 4\eta}}{2(1 - \eta)}, & \text{if } \eta \geq 1/4. \end{cases} \quad (14)$$

As we said before, this $P$ is equal to the expected group cohesion in the limit $N \to \infty$. The expected cohesion...
thus undergoes a second-order phase transition at the critical noise $\eta_c = 1/4$: from an “ordered phase”, where most of the group’s members are fit, to a “disordered phase”, where the group is equally composed of fit and unfit members. Fig. 2 shows that the numerical simulations converge consistently to Eq. (14) as $N$ increases, thus confirming our analytical results.

The above results can be generalized to $m > 2$, leading to

$$P = \frac{W^m}{W^m + (1 - W)^m},$$

which is a generalization of Eq. (13). The solution again undergoes a second-order phase transition, this time at

$$\eta_c = \frac{1}{2} - \frac{1}{2m}.$$ 

as confirmed in Fig. 3a. This phase transition can be further characterized by studying its critical exponents. Expanding Eq. (15) around the critical value, we find $P - 1/2 \propto (\eta_c - \eta)^{\beta}$, where $\beta = 1/2$ for any $m \geq 2$ (Fig. 3b). This shows that our model belongs to the mean-field universality class [30]. Note that Eq. (16) is in principle valid also for $m = 1$, but in this case, we have a first-order phase transition at $\eta_c = 0$ where the limit cohesion immediately drops from 1 (for $\eta = 0$) to 1/2 (for $\eta > 0$).

Taken together, we find that more evaluating members dramatically improve the group’s robustness to noise compared to the case of only one evaluating member. When $\eta$ is sufficiently small, two evaluating members are already enough to outperform the dictatorship case where the limit cohesion is also above 1/2. For $\eta > \eta_c$, instead, cohesion decreases with the group size until it reaches the cohesion of a random group independently of $m$.

It is straightforward to show that when the choice of the $m$ evaluating members is driven by preferential attachment, the expected cohesion in the limit $N \to \infty$ coincides with that of the uniform case derived above. However, the PA mechanism makes the convergence to the solution of Eq. (15) slower, as the weight of the founder members in Eq. (12) is larger. This is analogous to Eq. (10) converging to 1/2 slower than Eq. (6) for $m = 1$.

In conclusion, we introduced a simple, yet not trivial, model of group formation to study the dynamics of group cohesion. We show that the number of members involved in the evaluation of new candidates is crucial to determine cohesion in large groups. As the level of randomness in the admission process increases, the system undergoes a phase transition. Above a critical noise level, large groups cannot remain cohesive. In the extreme case of only one evaluating member, the critical noise is zero, meaning that cohesion decreases with the group size regardless of how small is the noise. However, the more members evaluate each candidate, the higher the critical noise level.

In reality, the factors that make a group incohesive are multifaceted and not easily identifiable (e.g., psychological factors). Nonetheless, our work focuses on a fundamental group growth mechanism suggesting that group size, randomness, and the admission processes jointly affect group cohesion in a non-trivial way.

Among possible extensions of our work are different admissions rules (e.g., majority voting) and the introduction of node sign dynamics as a result of peer influence. Moreover, fit and unfit nodes can be interpreted as individuals with different characteristics whereas the evaluation noise then represents a tolerance to diversity. Our framework can be then used to study the relationship between group size, cohesion, and diversity. As our model is closely related to a recent model of opinion formation [28], our results can also show how to prevent the formation of unreliable opinions or erroneous inference from complex network data [31].

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