On a $p$-Kirchhoff-type problem arising in ecosystems

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Abstract In this article, we discuss the existence of positive solutions for an ecological model of the form:

$$
\begin{cases}
-M(\int_{\Omega} |\nabla u|^p \, dx) \Delta_p u = \frac{au^{p-1} - bu^{q-1} - c}{u^z}, & x \in \Omega, \\
u = 0, & x \in \partial \Omega,
\end{cases}
$$

where $\Omega$ is a bounded domain with smooth boundary, $\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)$, $1 < p < \gamma$, $M : [0, \infty) \to (0, \infty)$ is a continuous and increasing function, $a > 0$, $b > 0$, $c \geq 0$, and $z \in (0, 1)$. This model describes the steady states of a logistic growth model with grazing and constant yield harvesting. We also discuss the dynamics of the fish population with natural predation and constant yield harvesting. We discuss the existence of a positive solution for given $a, b, \gamma$ and small values of $c$.

Keywords Positive solutions · Sub-supersolutions · $p$-Kirchhoff-type problems

Mathematics Subject Classification 35J55 · 35J65

Introduction

In this paper, we are interested in the existence of positive solutions for the $p$-Kirchhoff-type problems

$$
\begin{cases}
-M(\int_{\Omega} |\nabla u|^p \, dx) \Delta_p u = \frac{au^{p-1} - bu^{q-1} - c}{u^z}, & x \in \Omega, \\
u = 0, & x \in \partial \Omega,
\end{cases}
$$

where $M : [0, \infty) \to (0, \infty)$ is a continuous and increasing function, $c \geq 0$, $a, b > 0$, $\Omega$ is a bounded domain with smooth boundary, $\Delta_p$ denotes the $p$-Laplacian operator defined by $\Delta_p z = \text{div}(|\nabla z|^{p-2} \nabla z)$, $1 < p < \gamma$ and $x \in (0, 1)$.

Here $u$ is the population density and $au^{p-1} - bu^{q-1}$ represents logistics growth. This model describes grazing of a fixed number of grazers on a logistically growing species (see [11]). The herbivore density is assumed to be a constant which is a valid assumption for managed grazing systems and the rate of grazing is given by $\frac{c}{u}$. At high levels of vegetation density this term saturates to $c$ as the grazing population is a constant. This model has also been applied to describe the dynamics of fish populations (see [15]). In the case of the fish population the term $\frac{c}{u}$ corresponds to natural predation.

In recent years, problems involving Kirchhoff-type operators have been studied in many papers, we refer to [3, 4, 6, 10, 14] in which the authors have used the variational and topological methods to get the existence of solutions. In this article, we are motivated by the ideas introduced in [7, 12, 13] and properties of Kirchhoff-type operators in [3, 4, 6], we study problem (1) in semipositone case (i.e., $\lim_{s \to 0^+} f(s) = -\infty$; $f(s) = \frac{as^{p-1} - bs^{q-1} - c}{s^z}$; see [5, 7–9]). Using sub-supersolution techniques, we prove the existence of a positive solution for the problem.

To precisely state our existence result we consider the eigenvalue problem

$$
\begin{cases}
-\Delta_p \phi = \lambda \phi |\phi|^{p-2} \phi, & x \in \Omega, \\
\phi = 0, & x \in \partial \Omega.
\end{cases}
$$

Let $\phi$ be the eigenfunction corresponding to the first eigenvalue $\lambda_1$ of (3) such that $\phi(x) > 0$ in $\Omega$ and $||\phi||_{\infty} = 1$. It can be shown that $\frac{\partial \phi}{\partial n} < 0$ on $\partial \Omega$. Here $n$ is the outward normal. Let $m, \delta > 0$ and $\mu > 0$ be such that:
\[ \mu \leq \phi \leq 1, \quad x \in \Omega - \Omega_{\delta}, \quad (3) \]
\[ |\nabla \phi|^{p} \geq m, \quad x \in \Omega_{\delta}, \quad (4) \]
with \( \Omega_{\delta} := \{ x \in \Omega | d(x, \partial \Omega) \leq \delta \} \). This is possible since \( |\nabla \phi|^{p} \neq 0 \) on \( \partial \Omega \) while \( \phi = 0 \) on \( \partial \Omega \). We will also consider the unique solution \( e \in W_{0}^{1,p}(\Omega) \) of the boundary value problem
\[
\begin{cases}
-\Delta_{p} e = 1, & x \in \Omega, \\
e = 0, & x \in \partial \Omega,
\end{cases}
\]
to discuss our existence result, it is known that \( e > 0 \) in \( \Omega \) and \( \frac{\partial e}{\partial \nu} < 0 \) on \( \partial \Omega \).

**Existence results**

In this section, we shall establish our existence result via the method of sub-supersolution. A function \( \psi \) is said to be a subsolution of (1), if it is in \( W_{0}^{1,p}(\Omega) \) such that
\[
-M \left( \int_{\Omega} |\nabla \psi|^{p} \, dx \right) \int_{\Omega} |\nabla \psi|^{p-2} \nabla \psi \cdot \nabla w \, dx \\
\leq \int_{\Omega} \left[ a|\nabla \phi|^{p-2} - b|\nabla \phi|^{p-2} - c \right] w \, dx,
\]
and \( z \) is said supersolution of (1), if it is in \( W_{0}^{1,p}(\Omega) \) such that
\[
-M \left( \int_{\Omega} |\nabla z|^{p} \, dx \right) \int_{\Omega} |\nabla z|^{p-2} \nabla z \cdot \nabla w \, dx \\
\geq \int_{\Omega} \left[ a|\nabla \phi|^{p-2} - b|\nabla \phi|^{p-2} - c \right] w \, dx,
\]
for all \( w \in W = \{ w \in C_{0}^{\infty}(\Omega) | w \geq 0, x \in \Omega \} \). Then the following result holds:

**Lemma 2.1** (See [1, 2, 8]) Suppose there exist sub and supersolutions \( \psi \) and \( z \) respectively of (1) such that \( \psi \leq z \). Then (1) has a solution \( u \) such that \( \psi \leq u \leq z \).

Now we state our main result.

**Theorem 2.2** Let there exist constants \( M_{0} > 0 \) and \( M_{\infty} \geq 0 \) such that \( M_{0} \leq M(t) \leq M_{\infty} \) for all \( t \in [0, \infty) \). Given \( a, b > 0, 1 < p < \gamma, \) and \( \alpha \in (0, 1) \), there exists a constant \( c_{1} = c_{1}(a, b, \alpha, \gamma, p, \Omega) > 0 \) such that for \( c < c_{1} \), (1) has a positive solution.

**Remark 2.3** In the nonsingular case (\( \alpha = 0 \)), positive solutions exist only when \( a > \lambda_{1} \) (the principle eigenvalue) (see [12, 13]). But in the singular case, we establish the existence of a positive solution for any \( a > 0 \).

**Proof of Theorem 2.2** We start with the construction of a positive subsolution for (1). Fix \( \beta \in (1, \frac{p}{p-1+\gamma}) \). Define \( \psi = k\phi^{{\beta}} \), where \( k > 0 \) is such that \( a \geq 2b\phi^{p-\gamma} + M_{\infty}\beta^{-1}\lambda_{1}k^{2} \). Define
\[
c_{1} := \min \left\{ M_{\infty}\lambda_{1}^{-1} \beta^{-1}(\beta - 1)(p - 1)M^{p}, \right. \\
\left. \frac{1}{2} M_{\infty}\lambda_{1}^{-1} \mu \beta^{-1}(a - \beta^{2}\lambda_{1}k^{2}) \right\},
\]
Note that \( c_{1} > 0 \) by the choice of \( k \) and \( \beta \). A calculation shows that
\[
\nabla \psi = k\beta \phi^{\beta-1},
\]
and
\[
-M \left( \int_{\Omega} |\nabla \psi|^{p} \right) \Delta \psi \\
= M \left( \int_{\Omega} |\nabla \psi|^{p} \right) \int_{\Omega} |\nabla \psi|^{p-2} \nabla \psi \cdot \nabla \psi \, dx \\
= k^{-1} \beta^{-1} M \left( \int_{\Omega} |\nabla \psi|^{p} \right) \int_{\Omega} |\nabla \phi|^{p-2} \nabla \phi \cdot \nabla \phi \, dx \\
= k^{-1} \beta^{-1} M \left( \int_{\Omega} |\nabla \psi|^{p} \right) \int_{\Omega} |\nabla \phi|^{p-2} \nabla \phi \left( |\nabla \phi|^{(\beta - 1)(p - 1)} - w \nabla \phi^{(\beta - 1)(p - 1)} \right) \, dx \\
= k^{-1} \beta^{-1} M \left( \int_{\Omega} |\nabla \psi|^{p} \right) \int_{\Omega} |\nabla \phi|^{p-2} \nabla \phi \left( |\nabla \phi|^{(\beta - 1)(p - 1)}w \right) \, dx \\
- k^{-1} \beta^{-1} M \left( \int_{\Omega} |\nabla \psi|^{p} \right) \int_{\Omega} |\nabla \phi|^{p-2} \nabla \phi \left( |\nabla \phi|^{(\beta - 1)(p - 1)} \right) \, dx \\
= k^{-1} \beta^{-1} M \left( \int_{\Omega} |\nabla \psi|^{p} \right) \int_{\Omega} |\nabla \phi|^{p-2} \nabla \phi \left( |\nabla \phi|^{(\beta - 1)(p - 1)}w \right) \, dx \\
- (\beta - 1)(p - 1)|\nabla \phi|^{p} \phi^{p+\beta(p-1)} \, dx.
\]

Thus \( \psi \) is a subsolution of (1) if
\[
M_{\infty}\lambda_{1}^{-1} \beta^{-1} \left[ \lambda_{1} \phi^{\beta(p-1)} - (\beta - 1)(p - 1)|\nabla \phi|^{p} \phi^{p+\beta(p-1)} \right] \\
\leq ak^{p-1+\gamma} \phi^{(p-1)\gamma} - bk^{p-1+\gamma} \phi^{(\gamma - 1)(p-1)\gamma} - \frac{c}{k^{2} \phi^{\beta}}.
\]
For this, we have to show the following three inequalities:
\[ -k^{p-1-z} \phi^{p(1-z)} (a - M_{\infty} k^z \beta^{-1} \lambda_1 \phi^2) \]
\[ \leq -2k^{p-1-z} \phi^{p(1-z)-z}, \quad x \in \Omega, \]
\[ -\frac{1}{2} k^{p-1-z} \phi^{p(1-z)} (a - M_{\infty} k^z \beta^{-1} \lambda_1 \phi^2) \]
\[ \leq -\frac{c}{k^2 \phi^2}, \quad x \in \Omega - \Omega_0, \]
\[ -M_{\infty} k^p \beta^{-1} (p-1)(\beta - 1) \left| \frac{\nabla \phi}{\phi^{p-\beta}(p-1)} \right| \]
\[ \leq -\frac{c}{k^2 \phi^2}, \quad x \in \Omega_0, \]
by the choice of \( k \), we have:
\[ -\frac{1}{2} k^{p-1-z} \phi^{p(1-z)} (a - M_{\infty} k^z \beta^{-1} \lambda_1 \phi^2) \]
\[ \leq -bk^{p-1-z} \phi^{p(1-z)} \]
\[ \leq -bk^{p-1-z} \phi^{p(1-z)}. \] (5)

Now, we have in \( \Omega_0 \), \( |\nabla \phi|^p \geq m \), and \( c < M_{\infty} k^{p+1} \beta^{-1} (\beta - 1)(p-1)m^p \), then the following inequalities hold:
\[ -M_{\infty} k^{p-1} \beta^{-1} (p-1)(\beta - 1) \left| \frac{\nabla \phi}{\phi^{p-\beta}(p-1)} \right| \]
\[ \leq -M_{\infty} k^{p-1} \beta^{-1} (p-1)(\beta - 1)(p-1)m^p \]
\[ \leq -\frac{c}{k^2 \phi^2}, \quad x \in \Omega_0. \] (6)

On the other hand, since \( p - \beta(p - 1 + \varepsilon) > 0 \),
\[ -\frac{c}{k^2 \phi^2} \frac{\phi^{p-\beta(p-1)-z}}{\phi^{p-\beta(p-1)-z}} \leq -\frac{c}{k^2 \phi^2}. \] Hence
\[ -M_{\infty} k^{p-1} \beta^{-1} (p-1)(\beta - 1) \left| \frac{\nabla \phi}{\phi^{p-\beta}(p-1)} \right| \leq -\frac{c}{k^2 \phi^2}. \] (6)

Finally, in \( \Omega - \Omega_0 \) using \( \phi \geq \mu \) and \( c < \frac{1}{2} M_{\infty} k^{p-1} \mu^{p-1}(a - \beta^{-1} \lambda_1 k^2) \), we have:
\[ -\frac{1}{2} k^{p-1-z} \phi^{p(1-z)} (a - M_{\infty} k^z \beta^{-1} \lambda_1 \phi^2) \]
\[ \leq -k^{p-1-z} \phi^{p(1-z)} (a - M_{\infty} k^z \beta^{-1} \lambda_1 \phi^2) \]
\[ \leq -\frac{c}{k^2 \phi^2}. \] (7)

For \( c < c_1 \), by (6) and (7) the Eq. (5) holds. Thus \( \psi \) is a subsolution of (1).

Now for a supersolution choose \( \varepsilon := N \varepsilon \), where \( N > 0 \) is such that \( N \varepsilon \geq \psi \) and
\[ au^{p-1} - bu^{q-1} - c \leq N^{p-1}, \]
for all \( u > 0 \). We have
\[ -M \left( \int_{\Omega} |\nabla z|^p \, dx \right) \triangle_p z = M \left( \int_{\Omega} |\nabla z|^p \, dx \right) N^{p-1} \]
\[ \geq M \varepsilon N^{p-1} \]
\[ \geq \frac{a^{p-1} - b^{q-1} - c}{z^2}. \]

i.e., \( z \) is a supersolution of (1) with \( z \geq \psi \) for \( N \) large (note \( \nabla e \neq 0; \varepsilon \Omega \)). Thus, there exists a positive solution \( u \) of (1) such that \( \psi \leq u \leq z \). This completes the proof of Theorem 2.2. \( \square \)

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