Triality, Biquaternion and Vector Representation of the Dirac Equation

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Abstract

The triality properties of Dirac spinors are studied, including a construction of the algebra of (complexified) biquaternion. It is proved that there exists a vector-representation of Dirac spinors. The massive Dirac equation in the vector-representation is actually self-dual. The Dirac’s idea of non-integrable phases is used to study the behavior of massive term.

1 Introduction

The first example of a vector representation for spinors was derived by E. Cartan\cite{1}. He noticed that the group $Spin(2n)$ is the double covering group of the rotation group $SO(2n)$, i.e. this group has two basic half-spinor (semi-spinor) representations of degree $2^{n-1}$. In the special case only when $2n = 2^{n-1}$, the group $Spin(8)$ has just three irreducible representations of degree 8, all real, and the three representation spaces (vector space) $R$, (semi-spinors spaces) $S_+$ and $S_-$ are, remarkably, on an equal footing. It turns out that there is an extra automorphism, known as ‘triality’\cite{1,2,3}, which changes the spinor representations of $SO(8)$ to the vector representation and is not related
to any symmetry of other $SO(2n)$ groups. The word 'triality' is applied to the algebraic and geometric aspects of the $\Sigma_3$ symmetry which $Spin(8)$ has. One wonders what three objects does the symmetric group $\Sigma_3$ permutes; and the answer is that it permutes representations.

**Theorem** (Cartan’s principle of triality$^1$)$^2$: There exists an automorphism $J$ of order 3 of the vector space $A = R \times S_+ \times S_-$ (dimension=8+8+8) which has following properties: $J$ leaves the quadratic form $\Omega$ and the cubic form $F$ invariant; $J$ maps $R$ onto $S_+$, $S_+$ onto $S_-$, and $S_-$ onto $R$.

The law of composition in the algebra $A$, is defined in terms of the quadratic forms $\Omega$ and cubic form $F$ only. For the $SO(8)$ case the defined algebra is the algebra of Cayley octonions$^2$, and it is clear that any automorphism of the vector space $A$ which leaves the quadratic forms $\Omega$ and cubic form $F$ invariant is an automorphism of this algebra. The Brioschi’s formula in the real domain (i.e. The product of two sums of eight squares is itself the sum of eight squares.), which deduced from the above considerations by Cartan$^1$, can be considered as a normed condition for division octonions. Generally, any division algebra gives a triality$^4$$^5$, and it follows that normed trialities only occur in dimensions 1,2,4 or 8. (Here we use a generalized concept of 'triality' advocated by Adams$^4$.) This conclusion is quite deep. By comparison, Hurwitz’s classification of normed division algebras (in the real domain) is easier to prove$^6$.

The construction of the algebras from trialities has tantalizing links to physics. In the Standard Model, all particles other than the Higgs boson transform either as vectors or spinors. The interaction between matter (quarks and leptons) and the forces (gauge bosons) is described by a **trilinear** map involving two spinors and a vector. It is fascinating that the same sort of mathematics can be used both to construct a suitable algebra and to describe the interaction between matter and forces.

One prima facie problem with the above speculation is that physics uses spinors associated to Lorentz groups rather than $SO(8)$ rotational groups,
due to the fact that 4-dimension space-time has a Lorentzian rather than Euclidean metric. Luckily the dimension of spinor space, depends on both the dimension of vector space and the signature of the metric. In 4-Lorentzian dimensions, the gamma matrices $\gamma^\mu$ are (at least) $4 \times 4$, and thus the number of complex spinor components are four too, or equivalently 4+4 independent real components. Thus, we have here three spaces each of four dimensions, that of vectors, that of semi-spinors of the first type, and that of semi-spinors of the second type. The quadratic form in the vector space is defined by means of $\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$, and the quadratic form in the spinor space is defined by means of Dirac conjugate spinors ($\overline{\psi}\psi$ here $\overline{\psi} = \psi^T \gamma^0$). We will prove that there exists a double covering vector representation of Dirac spinors, and that there exists an automorphism $J$ of order 3 in the $A = M^{1+3} \times S^4_+ \times S^4$ (vector space and two half-spinor spaces), which leaves quadratic forms and the special cubic form invariant up to the sign. (In Minkowski space we are faced with spacelike and timelike vectors, thus sometimes it is convenient to use the term pseudoscalar, here the prefix 'pseudo' referring to automorphism $J$.) The situation differs from the case of $SO(8)$, and we propose to denominate this symmetry as a “ding” construction. (Chinese ding is an ancient vessel which has two loop handles and three legs. It is the metaphor of tripartite balance of forces.) The algebra associated with the above construction is the algebra of (complexified) biquaternion. In this sense the vector representation of Dirac spinor is equivalent to a biquaternion representation.

Moreover we will prove that the massive Dirac equation in the vector representation is actually self-dual.

It is important to stress that the triality transformation, i.e. the representations permutation, is not a symmetry of the theory. It maps one description of the theory to another description of the same theory.
2 Triality and ”Ding” construction

Let us first introduce trinomial unit-basis \((\varphi^\beta, f^\alpha, j^\mu)\), where the basic unit vector \(j^\mu\) and two basic unit spinors \(\varphi^\beta, f^\alpha\) are normalized so that

\[\varphi = j^\mu i\gamma^\mu f, \quad f = j^\mu i\gamma^\mu \varphi, \quad j^\mu = -\varphi i\gamma^\mu f = \Gamma i\gamma^\mu \varphi\]  

(1)

\[\varphi \varphi = -ff = -j^\mu \eta_{\mu\nu} j^\nu = 1\] 

\[\varphi f = f \varphi = j^\mu (\varphi \gamma^\mu f) = j^\mu (\Gamma \gamma^\mu f) = 0\]  

(2)

Here any two objects determine the third. (Raising and lowering indices by Lorentzian metric \(\eta_{\mu\nu}\) and \(\eta^\mu_\nu\), the summation convention is assumed for repeated indices.)

By using the quantities \((j^\mu i\gamma^\mu), (i\gamma^\mu f), (i\gamma^\mu \varphi)\) we can translate between different types of vectors and spinors indexes. In addition, we can introduce another unit vector \(k^\mu\), which is determined by the above trinomial unit-basis, that will play a very important role in our theory.

\[k^\mu \overset{def}{=} \varphi \gamma^\mu \varphi = \Gamma \gamma^\mu f, \quad k^\mu k_\mu = 1, \quad k^\mu j_\mu = 0\]  

(3)

\[k_\mu \gamma^\mu f = -f, \quad k_\mu \gamma^\mu \varphi = \varphi\]  

(4)

The existence of such objects in Dirac theory is verified by the special case (28) below.

Furthermore we need the following algebraic properties

\[\varphi \gamma^\mu \gamma^\nu \varphi = -f \gamma^\mu \gamma^\nu f = \eta^{\mu\nu} + i\epsilon^{\mu\nu\lambda\rho} j_\lambda k_\rho\]  

(5)

\[\varphi \gamma^\mu \gamma^\nu f = \Gamma \gamma^\mu \gamma^\nu \varphi = i(k^\mu j^\nu - j^\mu k^\nu)\] 

\[\varphi \gamma^\mu \gamma^\nu \gamma^\lambda \varphi = \Gamma \gamma^\mu \gamma^\nu \gamma^\lambda f = i\epsilon^{\mu\nu\lambda\rho} j_\rho + t^{\mu\nu\lambda\rho} k_\rho\] 

\[-\varphi \gamma^\mu \gamma^\nu \gamma^\lambda \varphi = \Gamma \gamma^\mu \gamma^\nu \gamma^\lambda \varphi = \epsilon^{\mu\nu\lambda\rho} k_\rho - i t^{\mu\nu\lambda\rho} j_\rho\]  

(6)

here \(\epsilon^{\mu\nu\lambda\rho}\) is the Levi-Civita symbol, with the definition \(\epsilon^{0123} = 1\), and

\[\epsilon^{\mu\nu\lambda\rho} = \frac{i}{4} tr(\gamma^5 \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\rho)\] 

\[t^{\mu\nu\lambda\rho} \overset{def}{=} \frac{i}{4} tr(\gamma^5 \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\rho) = (\eta^{\mu\nu} \eta^{\lambda\rho} + \eta^{\mu\rho} \eta^{\nu\lambda} - \eta^{\mu\lambda} \eta^{\nu\rho})\]  

(7)
(Comments: In calculating S-matrix elements and transition rates for processes involving particles of spin $\frac{1}{2}$, we often encounter traces of products of Dirac gamma matrices. The above symbols $\epsilon^{\mu\nu\lambda\rho}$ and $t^{\mu\nu\lambda\rho}$ are not new for physics, we always use them in all such calculations.)

We are now in a position to construct the suitable algebra, that are needed. Let

$$c^{\mu\nu\lambda} \overset{\text{def.}}{=} (t^{\mu\nu\lambda\rho} - i\epsilon^{\mu\nu\lambda\rho})k_\rho = (c^{\nu\lambda\mu})^*$$

$$\tilde{c}^{\mu\nu\lambda} \overset{\text{def.}}{=} (t^{\mu\nu\lambda\rho} - i\epsilon^{\mu\nu\lambda\rho})j_\rho = (\tilde{c}^{\nu\lambda\mu})^*$$

and

$$c_5^{\mu\nu} \overset{\text{def.}}{=} -c^{\mu\nu\lambda}j_\lambda = \tilde{c}^{\nu\lambda\mu}k_\nu = -c_5^{\nu\lambda}$$

The Dirac operators are defined as

$$\hat{D}^{\mu\nu} \overset{\text{def.}}{=} c^{\mu\nu\sigma}\partial_\sigma , \quad (\hat{D}^{\nu\lambda})^* \eta_\lambda (\hat{D}^{\mu\rho}) = -\eta^{\mu\nu} \Box$$

$$D^{\mu\nu} \overset{\text{def.}}{=} c^{\mu\nu\sigma}\partial_\sigma , \quad (D^{\nu\lambda})^* \eta_\lambda (D^{\mu\rho}) = \eta^{\mu\nu} \Box$$

(11)

We define a bilinear law of composition $\otimes$ (and $\tilde{\otimes}$) for complex vectors

$$(G \otimes H) \lambda \overset{\text{def.}}{=} G_\mu c^{\mu\lambda\nu} H_\nu = S^\lambda , \quad (G \tilde{\otimes} H)^\rho \overset{\text{def.}}{=} G_\mu \tilde{c}^{\mu\rho\nu} H_\nu$$

(12)

We can prove that

$$(G \otimes H) \otimes K = G \otimes (H \otimes K)$$

and

$$(G \otimes H) \tilde{\otimes} K = G \tilde{\otimes} (H \tilde{\otimes} K)$$

(13)

The dot product is defined by means of Minkowski metric, i.e. $G \cdot H \overset{\text{def.}}{=} G_\mu \eta_{\mu\nu} H^\nu$, and is not necessarily real; let alone positive.

We can prove that

$$(G \otimes H) \cdot (H \otimes H) = (G \otimes H) \cdot (G \otimes H)$$

and

$$(G \cdot G)(H \cdot H) = -(G \tilde{\otimes} H) \cdot (G \tilde{\otimes} H)$$

(14)
The algebra defined here is associative and noncommutative. The last identities look like the modified normed condition. (It is worth a passing to mention that a somewhat vague formulation of normed condition for reals is \[ \left( \sum_{i=1}^{n} a_i^2 \right) \left( \sum_{i=1}^{n} b_i^2 \right) = \sum_{i=1}^{n} c_i^2 \] in which \( c_i = \sum_{j=1}^{n} \sum_{k=1}^{n} a_j \gamma_{ijk} b_k \) where \( \gamma_{ijk} \) are constants. It is natural to ask whether or not do analogous identities involving more squares exist. It has occupied the mind of mathematicians for many years. Only in 1898 did Hurwitz\(^6\) prove that the identities of interest to us are possible only for \( n = 1, 2, 4, 8 \). This result, of course, is intimately related to the well known result, due to ...... that there exist only four normed algebras over the reals: \( \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O} \).

Because the algebra above is associative, it can be considered in terms of the matrices. Let:

\[
G = G_\mu e^\mu ; \quad (e^\mu)_{\nu}^{\lambda} \overset{\text{def.}}{=} \epsilon_{\nu\sigma\mu} \eta_{\sigma\lambda} \tag{15}
\]

where \( G_\mu \) are complex and the matrices \( e^\mu \) can be considered as the hypercomplex basic units. We can prove that a bilinear law of composition (12) is equivalent to matrix multiplication \( (G_\mu e^\mu)_{\nu}^{\lambda} (H_\nu e^\nu)_{\sigma}^{\rho} = (S_\mu e^\mu)_{\nu}^{\lambda} \). In the special coordinate system (28), where \( k^\mu = \delta^{\mu 0} \), the basis element \( e^0 = I \) and for \( \nu = 1, 2, 3 \), we have \( e^\nu = \sqrt{-1} \hat{e}^\nu \) here:

\[
\hat{e}^1 = \begin{pmatrix}
0 & -i & 0 & 0 \\
-i & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix} \tag{16}
\]

\[
\hat{e}^2 = \begin{pmatrix}
0 & 0 & -i & 0 \\
0 & 0 & 0 & 1 \\
-i & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix} \tag{17}
\]

\[
\hat{e}^3 = \begin{pmatrix}
0 & 0 & 0 & -i \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
-i & 0 & 0 & 0
\end{pmatrix} \tag{18}
\]
One notices that they obey the multiplication law:

\[ \hat{e}^1 \hat{e}^1 = \hat{e}^2 \hat{e}^2 = \hat{e}^3 \hat{e}^3 = \hat{e}^1 \hat{e}^2 \hat{e}^3 = -I \]

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"left-handed" spinor \( l \) (pure spinors) are determined in the following way

\[
\begin{align*}
\bar{\tau} l &= \bar{r} r = 2, \quad f = \frac{i}{2}(r - l), \quad \varphi = \frac{1}{2}(r + l) \\
k_{\pm}^\mu &= \frac{1}{2}(k^\mu \pm j^\mu), \quad k_\pm^\mu \eta_{\mu
u} k_\pm^\nu = 0, \quad k_\pm^\mu \eta_{\mu
u} k_\pm^\nu = \frac{1}{2}
\end{align*}
\]

(24)

It is easy to prove that

\[
\begin{align*}
 k_\mu \gamma^\mu r &= k_\mu^- \gamma^\mu r = l, \quad k_\mu^+ \gamma^\mu r = 0 \\
k_\mu \gamma^\mu l &= k_\mu^- \gamma^\mu l = r, \quad k_\mu^+ \gamma^\mu l = 0
\end{align*}
\]

(25)

The Dirac spinor can be equivalently decomposed into the sum of a "right-handed" and a "left-handed" spinors \( \Psi = R + L \), here

\[
\begin{align*}
 R &\overset{\text{def.}}{=} \frac{1}{2} G^\mu \eta_{\mu\nu} \gamma^\nu l, \quad L &\overset{\text{def.}}{=} \frac{-1}{2} G^{*\mu} \eta_{\mu\nu} \gamma^\nu r \\
 G^\mu &= \frac{1}{2} (\tau^\mu \gamma^\nu \Psi - \bar{\Psi} \gamma^\nu l) = B^\mu + i N^\mu
\end{align*}
\]

(26)

In order to understand our idea easily, it is convenient to work in a special coordinate system such that

\[
\begin{align*}
 [\varphi^\beta]^T &= [1, 0, 0, 0] \\
 [f^\alpha]^T &= [0, 0, i, 0] \\
 j^\mu &= (0, 0, 0, 1) \\
 k^\mu &= (1, 0, 0, 0)
\end{align*}
\]

(28)

(here we use the Dirac representation of gamma matrices). In this special case

\[
\Psi = \Psi_1(B) + \Psi_2(N) = \begin{pmatrix} B^3 + i N^0 \\ B^1 + i B^2 \\ B^0 + i N^3 \\ -N^2 + i N^1 \end{pmatrix}.
\]

(29)

Now let \( V^\mu \in M^{1+3} \) be a vector in Minkowski space, \( \Psi_1(B) \in S_1 \) and \( \Psi_2(N) \in S_2 \) are two half-spinors referred to the trinomial unit-basis defined above. The quadratic forms and the special cubic-form are defined as

\[
V^\mu \eta_{\mu\nu} V^\nu = V_\nu V^\nu, \quad \bar{\Psi}_1 \Psi_1 = -B_\nu B^\nu, \quad \bar{\Psi}_2 \Psi_2 = N_\mu N^\mu
\]

(30)
\[
V^\mu \eta_{\mu\nu}(\overline{\Psi}_1 \gamma^\mu \Psi_2 + \overline{\Psi}_2 \gamma^\mu \Psi_1) = 2(\epsilon_{\nu\lambda\rho\sigma}k^\sigma)V^\nu N^\lambda B^\rho
\]

One realizes that there exists a "ding" automorphism \( J \) of order 3 in \( M^{1+3} \times S_1 \times S_2 \), which leaves quadratic two-forms (30) and the trilinear-form (31) invariant up to the sign. \( J \) maps \( M \) onto \( S_1 \), \( S_1 \) onto \( S_2 \), and \( S_2 \) onto \( M \). (Or equivalently \( V \to B \to N \to V \).)

3 Bosonization of Dirac equation.

The most interesting for physicists is: by passing from ordinary spinor representation to the vector representation, one can express Dirac Lagrangian in the Bosonic form

\[
\frac{1}{2}[(\overline{\Psi} i\gamma^\mu(\partial_\mu - i eA_\mu)\Psi) - ((\partial_\mu + i eA_\mu)\overline{\Psi})i\gamma^\mu\Psi] = \frac{1}{2}[(\nabla_\mu G_\nu)^* i\bar{c}^{\nu\mu\lambda} G_\lambda - G_\rho^* i\bar{c}^{\mu\nu\lambda} \nabla_\mu G_\lambda + m(G_\rho^* G_\rho + G_\rho G_\rho)]
\]

Here \( \nabla_\mu G_\lambda \overset{def}{=} \partial_\mu G_\lambda - i eA_\mu \eta_{\lambda\rho} c_5^{\rho\sigma} G_\sigma \) and \( \bar{c}^{\nu\mu\lambda}, c_5^{\rho\sigma} \) are defined by (8).

It is important to notice that from the abstract point of view, there is an arbitrariness in our Bosonization procedure since it depends on the choice of the neutral elements \( j^\mu \) and \( k^\mu \). In the Lagrangian (32) they are arbitrary but must be fixed. The only restrictions on them are that \( k^\mu k^\mu = -j_\mu j^\mu = 1 \) and \( k_\mu j^{\mu} = 0 \). The remaining basis elements \( (\epsilon^{\mu_\nu\rho}, \bar{c}^{\mu_\nu\rho}, c_5^{\mu_\nu}, \ldots) \) can be reconstructed if desired, once \( j^\mu \) and \( k^\mu \) have been chosen.

The corresponding massive Dirac equation in the vector-representation takes the form of

\[
\bar{c}^{\mu\nu\lambda} i\nabla_\nu G_\lambda - mG^{\mu\nu} = 0.
\]

Or equivalently in the following self-dual form

\[
\partial_\mu G_\mu - im_j G^{\mu\nu} = 0
\]

\[
G_{\mu\nu} = \frac{i}{2} \epsilon_{\mu\nu\lambda\rho} G^{\lambda\rho}
\]

here (for simplicity) we take \( A_\mu = 0 \) and

\[
G_{\mu\nu} \overset{def}{=} [iA_\mu G_\nu + im_j G_{\mu\nu} - iA_\nu G_\mu + im_j G_{\nu\mu}] = 0.
\]
The Dirac equation (33) can be rewritten in the real form

\[ \begin{align*}
\bar{\epsilon}^{\mu\nu} \partial_\nu B_\lambda - i\bar{\epsilon}^{\mu\nu} \partial_\nu N_\lambda - eA_\nu(t^{\mu\nu} B_\lambda + \epsilon^{\mu\nu_\lambda} N_\lambda) - mB^\mu &= 0 \\
\bar{\epsilon}^{\mu\nu} \partial_\nu N_\lambda + i\bar{\epsilon}^{\mu\nu} \partial_\nu B_\lambda - eA_\nu(t^{\mu\nu} N_\lambda - \epsilon^{\mu\nu_\lambda} B_\lambda) + mN^\mu &= 0
\end{align*} \] (36)

(here \(\bar{\epsilon}^{\mu\nu} = \bar{\epsilon}^{\mu\nu_\sigma j_\rho}\) and \(\epsilon^{\mu\nu} = \epsilon^{\mu\nu_\sigma k_\rho}\)) or equivalently

\[ \begin{align*}
\partial_\mu N^\mu - [mB^\mu + eA_\nu(B_\lambda t^{\mu\nu} + N_\lambda \epsilon^{\mu\nu_\lambda})]j_\mu &= 0 \\
\partial_\mu B^\mu - [mN^\mu - eA_\nu(N_\lambda t^{\mu\nu} - B_\lambda \epsilon^{\mu\nu_\lambda})]j_\mu &= 0 \\
(\nabla'_\mu N_\nu - \nabla'_\nu N_\mu) &= \frac{1}{2} \bar{\epsilon}_{\mu\nu\lambda\rho}(\nabla^{\lambda\rho} B^\nu - \nabla^{\nu\rho} B^\lambda)
\end{align*} \] (37)

In addition, let us define the operator \(\nabla^c_\mu \overset{def.}{=} (\partial_\mu + imj_\mu C^*)\), such that

\[ \nabla^c_\mu G_{\nu\lambda} \overset{def.}{=} \partial_\mu G_{\nu\lambda} + imj_\mu G^*_{\nu\lambda} \] (39)

where \(C^*\) is the operator of complex conjugation: \(C^*\Phi = \Phi^*\). In this notation the identity

\[ \nabla^c_\mu G_{\nu\lambda} + \nabla^c_\lambda G_{\mu\nu} + \nabla^c_\nu G_{\lambda\mu} = 0 \] (40)

looks like the Bianchi identity, and

\[ \frac{1}{4}[G_{\mu\nu} \epsilon^{\mu\nu_\rho\lambda} G_{\lambda\rho} + G^*_{\mu\nu} \epsilon^{\mu\nu_\lambda\rho} G^*_{\lambda\rho}] = \frac{1}{2}\partial_\mu[\epsilon^{\mu\nu_\lambda\rho}(G_\nu G_{\lambda\rho} + G^*_\nu G^*_{\lambda\rho})] = 2\partial_\mu[\epsilon^{\mu\nu_\lambda\rho}(B_\nu \partial_\lambda B_\rho - N_\nu \partial_\lambda N_\rho + 2mB_\nu N_\lambda j_\rho)] \] (41)

looks like the Chern-Pontryagin density and a total derivative of the Chern-Simons density. Thus the Dirac Lagrangian can be modified by the additional total derivative of the Chern-Simons density introduced above.
4 Lorentz rotations, U(1) and Chiral transformations

Because of the lack of commutativity there are two types of $\otimes$ products in the biquaternion algebra, the left and right multiplications. Thus for a given biquaternion $G$ (and $q$ such that $q_\mu q^\mu = -1$) we can consider the two maps, i.e. two transformations\[7\]:

\[ \tilde{G} = q \otimes G \quad \text{and} \quad \tilde{G} = G \otimes q, \]

depending as we multiply $q$ on the left or the right. In the vector representation, these transformations take the form of

\[
\tilde{G}_\mu = G^\nu (\eta_{\nu \lambda} \tilde{c}^{\lambda \sigma} q_\sigma) = G^\nu [S_1(q)]_{\nu}^{\mu} \\
\tilde{H}_\mu = (q_\sigma \tilde{c}_{\sigma \lambda} \eta_{\lambda \nu}) H^\nu = [S_2(q)]_{\nu}^{\mu} G^\nu
\]

(42)

From (modified normed condition) (14) we find that under the above transformations the dot product $G_\mu q^{\mu} G_\nu$ is unchanged. Thus the two-form of the spinors $\overline{\Psi} \Psi = -\frac{1}{2} (G_\nu G^\nu + G_\nu G^\nu)$ is invariant under the above ‘complex rotations’. (Similarly, if $q_\mu q^\mu = 1$, then $\tilde{c}_{\mu \lambda}$ is replaced by $c^{\mu \lambda}$).

For the space-time (real) vector $x^\mu$ (and $\partial_\mu$) the Lorentz transformation, in quaternionic terms, is characterized by the above unit-biquaternion $q$ through the relation (considered as left-right mixed transformations\[7\]):

\[
\tilde{x}_\mu = [\Lambda(q)]^{\mu}_{\nu} x^\nu = [S_2(q^*)]^{\mu}_{\sigma} (x^\nu [S_1(q)]_{\nu}^{\sigma})
\]

(43)

The constrain $q_\mu q^\mu = -1$ (implying two real conditions) ensures that this transformation is 6-dimensional. This transformation preserves the value of the dot product and the reality of the space-time vector. Thus we have here two types of vectors with one-to-one distinct transformation operations $[S_1(q)]_{\nu}^{\mu}$ and $[\Lambda(q)]^{\mu}_{\nu}$. In contrast with an ordinary real space-time vector, we shall use the name ‘s-vector’ for $G^\mu$ which transforms as (42).

We can prove that under the above Lorentz transformations (with fixed $j^\mu$ and $k^\nu$)

\[
[\Lambda(q)]^{\nu}_{\mu}[S_2(q^*)]^{\mu}_{\sigma} \tilde{c}^{\sigma \rho \delta} [S_1(q)]^{\lambda}_{\delta} = \tilde{c}^{\mu \nu \lambda} \\
[\Lambda(q)]^{\nu}_{\mu}[S_2(q^*)]^{\mu}_{\sigma} c^{\sigma \rho \delta} [S_1(q)]^{\lambda}_{\delta} = c^{\mu \nu \lambda}
\]

(44)
and this ensures the Lorentz invariance of the Dirac Lagrangian (32) which is written in the s-vector form.

Also the Lagrangian (32) is invariant under $U(1)$ gauge transformations

$$\tilde{\Psi} = \Psi e^{ia} = \Psi (\cos \alpha + i \sin \alpha) \ , \ \tilde{A}_\mu = A_\mu + \partial_\mu \alpha \quad (45)$$

In the vector representation it is equivalent to

$$\tilde{B}_\mu = B_\mu \cos \alpha - [\xi_{\mu\nu\lambda} k^\lambda B^\nu - (k_\mu j_\nu - k_\nu j_\mu) N^\nu] \sin \alpha$$

$$\tilde{N}_\mu = N_\mu \cos \alpha - [\xi_{\mu\nu\lambda} k^\lambda N^\nu + (k_\mu j_\nu - k_\nu j_\mu) B^\nu] \sin \alpha \quad (46)$$

or in complex form

$$\tilde{G}^\mu = (\eta^{\mu\nu} \cos \alpha + ic^{\mu\nu} 5 \sin \alpha) G_\nu = (\exp i\alpha c_5)^{\mu\nu} G_\nu \quad (47)$$

It looks like a chiral transformation for $G^\mu$, here $(c_5^{\mu\sigma} \eta_{\sigma\rho} c_5^{\rho\nu}) = \eta^{\mu\nu}$, and from which the De Moivre theorem is deduced $(\cos \alpha + ic_5 \sin \alpha)^n = (\cos n\alpha + ic_5 \sin n\alpha)$.

We know that the massless Dirac Lagrangian is invariant under the chiral transformation

$$\Psi \rightarrow e^{i\alpha \gamma_5} \Psi \ ; \ \bar{\Psi} \rightarrow \bar{\Psi} e^{i\alpha \gamma_5} \quad (48)$$

In the s-vector representation it is equivalent to

$$G_\mu \rightarrow e^{i\alpha} G_\mu \quad (49)$$

It means that a chiral transformation for $\Psi$ is equivalent to a $U(1)$ transformation for $G_\mu$. The mass term $m \bar{\Psi} \Psi = -\frac{m}{2} (G_\mu^* G^{\mu} + G_\nu G^{\nu})$ is not invariant under the above chiral transformation, but is invariant under $U(1)$ transformation (45)-(47). Therefore we must distinguish between the plan wave solutions for $\Psi$ and for $G^\mu$.

Now, we define the ”dual” transformation

$$B \rightarrow N \ , \ N \rightarrow -B \ , \ m \rightarrow -m \quad (50)$$

which leaves the quadratic form $m \bar{\Psi} \Psi = m(N_\mu N^\mu - B_\mu B^\mu)$, the cubic form $V_\mu B_\nu N_\lambda \epsilon^{\nu\mu\lambda}$ and thus the above Lagrangian and Dirac equation invariant.
(It looks like the electro-magnetic duality which is accompanied by $e \rightarrow q$ and $q \rightarrow -e$). In this sense the half-spinor of the first type is dual to the half-spinor of the second type.

One may have many choices of "representations" of the neutral elements $(f, \varphi, j^\mu, k^\mu)$ and $\tilde{\epsilon}^{\mu\nu\lambda}$. For example, we can change the representations in the following way

\[
\tilde{f} = \frac{1}{2}(\overline{a} + \frac{1}{a})f + \frac{i}{2}(\overline{a} - \frac{1}{a})\varphi, \quad \tilde{\varphi} = \frac{1}{2}(\overline{a} + \frac{1}{a})\varphi - \frac{i}{2}(\overline{a} - \frac{1}{a})f, \quad \tilde{l} = \frac{1}{a} l
\]

(51)

\[
\tilde{j}^\mu = \frac{1}{2}(a\overline{a} + \frac{1}{a^2})j^\mu + \frac{i}{2}(a\overline{a} - \frac{1}{a^2})k^\mu
\]

\[
k^\mu = \frac{1}{2}(a\overline{a} + \frac{1}{a^2})k^\mu + \frac{i}{2}(a\overline{a} - \frac{1}{a^2})j^\mu
\]

(52)

here $\overline{a}$ represents the complex conjugate of $a$. Correspondingly

\[
\tilde{G}^\mu = \left[ \frac{1}{2}(a + \frac{1}{a})\eta^{\mu\nu} + \frac{1}{2}(a - \frac{1}{a})\tilde{c}^{\mu\nu}\right] G_\nu
\]

(53)

The Dirac Lagrangian (32) is invariant under this change of representation. In fact the $U(1)$ transformation (47), can be considered as the special case of (53).

5 The behavior of the mass term.

**Theorem** There exists a suitable complex vector $K^\mu$ associated with spinor $\Psi = R + L$ (and completely determined by it) such that

\[
K^\mu \gamma^\mu R = K^-\gamma^\mu R = L, \quad \overline{R}K^*\gamma^\mu = \overline{L}
\]

\[
K^\mu \gamma^\mu L = K^+\gamma^\mu L = R, \quad \overline{L}K^*\gamma^\mu = \overline{R}
\]

(54)

The explicit form of these vectors are

\[
K^\mu = \frac{\overline{R}r^\mu}{2R\overline{L}} + \frac{\overline{L}r^\mu}{2L\overline{R}} = K^+ + K^-
\]

\[
K^\mu \gamma^{\mu\nu} K^\nu = 1
\]

(55)

(Comments: the existence of such vector is due to the existence of $k_\mu = k_\mu^+ + k_\mu^-$ which has been introduced in the previous section (see eq.(25)), and thus has close relation with the "ding" construction.)
Corollary  The mass quadratic form equal to the interaction trilinear form
\[ RL = K_\mu \overline{R} \gamma^\mu R = K^*_\mu \overline{R} \gamma^\mu L \]
\[ LR = K_\mu \overline{L} \gamma^\mu L = K^*_\mu \overline{L} \gamma^\mu R \]  (56)

If \( \Psi = R + L \), then
\[ m\overline{\Psi} \Psi = m\overline{\Psi} K_\mu \gamma^\mu \Psi \]  (57)

These identities have close relation with the modified normed condition (14).

Next we use the Dirac idea of non-integrable phases[8].

Theorem  The charged massive Dirac equation can be rewritten in the following "uncharged massless" forms
\[ i\gamma^\mu (\partial_\mu - ieA_\mu) R - mL = [i\gamma^\mu \partial_\mu R_0] \Theta^{-1} = 0 \]  (58)
\[ i\gamma^\mu (\partial_\mu - ieA_\mu) L - mR = [i\gamma^\mu \partial_\mu L_0] \Theta^{-1} = 0 \]  (59)

where \( R_0 = R \cdot e^{-i \int (eA_\mu - mK_\mu) dx^\mu} = R \cdot \Theta \).

It is equivalent to
\[ i\gamma^\mu (\partial_\mu - ieA_\mu) \Psi - m\Psi = [i\gamma^\mu \partial_\mu \Psi_0] \Theta^{-1} = 0 \]  (60)

where \( \Psi_0 = \Psi \cdot e^{-i \int (eA_\mu - mK_\mu) dx^\mu} \) satisfies the massless Dirac equation.

One knows that a charged particle must be massive. Thus in our formalism the mass \( m \) was introduced in the same way as the charge \( e \)! It is mathematically beautiful and physically natural.

It is important to notice that in general \( K_\mu = [\text{Re}(K_\mu) + i\text{Im}(K_\mu)] \) is complex:
\[ i\text{Im}(K_\mu) = \frac{(K_\mu R - \overline{K_\mu L})(LR - R\overline{L})}{4(\overline{R}L)(LR)} = \frac{\pi^\nu_\mu(\overline{\Psi} \gamma^\nu \Psi)}{\pi_\nu} \]  
\[ \text{Re}(K_\mu) = \frac{(K_\mu R + \overline{K_\mu L})(RL + \overline{RL})}{4(\overline{R}L)(LR)} = \frac{\pi^\nu_\mu(\overline{\Psi} \gamma^\nu \Psi)}{\pi_\nu} \]  (61)
here
\[
\pi^\mu = \overline{\Psi} \gamma^\mu \Psi = G^\ast_{\lambda} \mathcal{C}_{\lambda \mu \nu} G^\nu
\]
\[
\pi_5^\mu = \overline{\Psi} \gamma^5 \gamma^\mu \Psi = -G^\ast_{\lambda} \mathcal{C}_{\lambda \mu \nu} G^\nu
\]  
(62)

It means that \( \text{Re}(K^\mu) \) is associated with unit vector \( k^\mu \) and \( \text{Im}(K^\mu) \) is associated with unit vector \( j^\mu \).

Only the real part \( \text{Re}(K^\mu) \) is associated with the phase, and the imaginary part \( \text{Im}(K^\mu) \) is associated with the “scale factor” \( \sigma = e^{-m \int \text{Im}(K_\mu) dx^\mu} \) of a spinor. (A similar scale change can be found in the Weyl’s earlier work.)

Furthermore
\[
[\text{Re}(K^\mu)] \cdot [\text{Im}(K^\mu)] = k^\mu \eta^\mu \nu j^\nu = 0
\]  
(63)
\[
\overline{\Psi} [\text{Im}(K^\mu)] \gamma^\mu \Psi = 0
\]
\[
\overline{\Psi} [\text{Re}(K^\mu)] \gamma^\mu \Psi = \overline{\Psi} K^\mu \gamma^\mu \Psi = \overline{\Psi} \Psi
\]  
(64)

One can realize that the trilinear form is independent of \( \text{Im}(K^\mu) \). Thus, if
\[
\Psi_0 \overset{d}{=} \Psi \cdot e^{-i \int (eA_\mu - mK_\mu) dx^\mu} = \Psi \Theta
\]  
(65)
\[
\Psi \overset{d}{=} \Psi^s T_\gamma^0 ; \quad \overline{\Psi_0} \overset{d}{=} \overline{\Psi} \cdot \Theta^{-1}
\]  
(66)
then the Lagrangian
\[
\overline{\Psi} i \gamma^\mu (\partial_\mu \Psi - ieA_\mu) \Psi - m \overline{\Psi} \Psi
\]
\[
= \overline{\Psi} i \gamma^\mu [\partial_\mu - ieA_\mu + im \text{Re}(K^\mu)] \Psi
\]  
(67)
\[
= \overline{\Psi_0} i \gamma^\mu \partial_\mu \Psi_0
\]
is independent of the “scale factor” \( \sigma = e^{-m \int \text{Im}(K_\mu) dx^\mu} \).

6 Non-integrable exponential factors

A physical spinor field \( \Psi \) which satisfies the massive charged Dirac equation (i.e. all physical solutions of massive Dirac equation) can be expressed in the following form
\[
\Psi \overset{d}{=} \Psi_0 \cdot e^{i \int (eA_\mu - mK_\mu) dx^\mu}
\]
\[
= R_0 \cdot e^{i \int (eA_\mu - mK_\mu) dx^\mu} + L_0 \cdot e^{i \int (eA_\mu - mK_\mu) dx^\mu}
\]  
(68)
Here $\Psi_0$ satisfies the massless Dirac equation and

$$K_\mu = \frac{R_0 \gamma_\mu R}{2RL} + \frac{L_0 \gamma_\mu L}{2LR} = \frac{R_0 \gamma_\mu R_0}{2R_0 L_0} + \frac{L_0 \gamma_\mu L_0}{2L_0 R_0}$$

(69)

The plane-wave solution of an electron is the most important solution in the quantum field theory. In this special case, the real time-like vector $mK_\mu$ is nothing but the energy-momentum of a massive Dirac particle.

The connection between non-integrability of phase and electromagnetic potential $A_\mu$ given here is not new, which is essentially just Weyl’s (1929) principle of gauge invariance$^{[10]}$ in its modern form. C.N.Yang has reformulated the concept of a gauge field in an integral formalism and extended Weyl’s idea to the more general non-Abelian case$^{[11]}$. The non-integrable phase for the wave functions was also discussed by Dirac in 1931$^{[8]}$, where the problem of monopole was studied. He emphasizes that “non-integrable phases are perfectly compatible with all the general principles of quantum mechanics and do not in any way restrict their physical interpretation.” Dirac conjectured that: ”The change in phase of a wave function round any closed curve may be different for different wave functions by arbitrary multiplies of $2\pi$”. There is the famous Dirac relation: $eq/(4\pi) = (n/2)$ . This means that if the quantization of electric charge (the universal unit $e$ ) is accepted, then the above relation is the law of quantization of the magnetic pole strength.

Notice, in our case $K_\mu = [\text{Re}(K_\mu) + i\text{Im}(K_\mu)]$ is complex. Thus only $i \int (eA_\mu - m(\text{Re}(K_\mu))) dx^\mu$ is associated with the phase change, and the imaginary part $\int \text{Im}(K_\mu) dx^\mu$ is associated with the scale change.

Because of the single-valued nature of a quantum mechanical wave function, we naturally conjecture that:

1. The phase change of a wave function round any closed curve must be close to $2n\pi$ where $n$ is some integer, positive or negative. This integer will be a characteristic of possible singularity in $m\text{Re}(K_\mu)$.

2. The scale change of a wave function round any closed curve must be close to zero. As being mentioned in the previous section, the Lagrangian (67) is independent of this scale change. This Lagrangian in the massless
form is conformally invariant$^{[12]}$. Thus this scale-factor can be gauged away by conformal rescaling.

This is a new (very strong) assumption, and cannot be proved, not derived. It is a conjecture of the overall consistency among all the solutions to the same equation. The existence of magnetics monopole is an open question yet, thus in our case it means that the change in $\oint mK_\mu dx^\mu$ round any closed curve, with the possibility of there being singularity in $m\text{Re}(K_\mu)$, will lead to the law of quantization of physical constants, including mass.

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