GENERAL DECAY FOR A VISCOELASTIC ROTATING EULER-BERNOULLI BEAM

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Abstract. In this paper, we consider a viscoelastic rotating Euler-Bernoulli beam that has one end fixed to a rotated motor in a horizontal plane and to a tip mass at the other end. For a large class relaxation function $q$, namely, $q'(t) \leq -\zeta(t)H(q(t))$, where $H$ is an increasing and convex function near the origin and $\zeta$ is a nonincreasing function, we establish optimal explicit and general energy decay results from which we can recover the optimal exponential and polynomial decay.

1. Introduction. In this paper, we consider a viscoelastic rotating Euler-Bernoulli beam that has one end fixed to a rotated motor in a horizontal plane and to a tip mass at the other end. The dynamic of the problem is also modeled as,

$$
\begin{align*}
I_h S_{tt}(t) + \rho \int_0^L (x (xS + v(x,t)))_{tt} dx + m_p L(S(t) + v(L,t))_{tt} \\
+ J_p (S(t) + v_x(L,t))_{tt} = \tau(t), & \quad t \geq 0,
\end{align*}
$$
and the boundary conditions

$$
\begin{align*}
v(0,t) = v_x(0,t) = 0, & \quad t \geq 0, \\
EI v_{xxx} (L,t) - EI \int_0^t q(t-s)v_{xxx} (L,s) ds = m_p \left( L S(t) + v(L,t) \right)_{tt}, & \quad t \geq 0,
\end{align*}
$$
and the initial data

$$
v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), \quad S(0) = S_0, \quad S_t(0) = S_1, \quad x \in (0,L),
$$
where "(_)_t" denotes the derivative with respect to the time $t$ and "(_)_x" denotes the derivative with respect to $x$. $S(t)$ is the hub rotation angle, $v(x,t)$ is the deflection of the beam at the position $x$ for the time $t$, $\tau(t)$ is the input control torque, $\rho$ is the linear density of the beam, $L$ is the length of the beam, $EI$ is the bending stiffness of the beam, $m_p$ is the mass of the translational base, $J_p$ is the rotary inertia of the beam. $H$ is an increasing and convex function near the origin and $\zeta$ is a nonincreasing function.
rigid body at the free end of the beam, \( I_h \) is the moment of inertia of the motor, and \((q \ast f)(t)\) is defined by

\[
(q \ast f)(t) = \int_0^t q(t - s)f(x, s)ds.
\]

In the second equation of (1.1), the term \( EI(q \ast v_{xxxx})(t)\) represents the viscoelastic damping. This term appears in the constitutive relationship between the stress and the strain according to the Boltzmann Principle [21, 22]. The kernel \( q \) is called the relaxation function and will be specified later.

The first (ordinary differential) equation of (1.1) describes the dynamic of the motor, and therefore, \( S(t) \) is assumed not identically zero, and the second (integro-differential) equation describes the dynamic of the beam, and therefore, \( v(x, t) \) is assumed small and any extension is neglected. The models (1.1)-(1.3) have a wide application in mechanical engineering such as links of robot [8] and space-shuttle arms. In the last decades, the control of rotating Euler-Bernoulli beam has attracted considerable attention of many researchers, because of its numerous practical applications in mechanical engineering. Many methods have been proposed to suppression and/or reduction of the vibration of a rotating beam.

Let us mention some known results related to the rotating Euler-Bernoulli beam. Morguil [48] studied the motion of a rigid body with a flexible beam clamped to it. Precisely, he investigated the following system

\[
\begin{align*}
I_hS_{tt}(t) &= EI(-bv_{xxxx}(0, t) + v_{xx}(0, t))\tau(t), \quad t \geq 0, \\
\rho v_{tt}(x, t) + E Iv_{xxxx}(x, t) + \rho(b + x)S_{tt}(t) - \rho S_t^2(x, t) &= 0, \quad \forall (x, t) \in (0, L) \times \mathbb{R}^+,
\end{align*}
\]

with the following boundary conditions

\[
\begin{align*}
v(0, t) &= v_x(0, t) = 0, \quad t \geq 0, \\
E Iv_{xxxx}(L, t) &= -\alpha v_t(L, t), \quad t \geq 0, \\
E Iv_{xx}(L, t) &= -\beta v_x(L, t), \quad t \geq 0,
\end{align*}
\]

where \( \alpha \) and \( \beta \) are two positive constants. Using semigroup theory, the author studied the existence of the solution and showed the exponential decay of the energy under the control

\[
\tau(t) = EI(bv_{xx}(0, t) - v_x(0, t)) - k_1S_t(t), t \geq 0.
\]

Nguyen and Egeland ([55]) studied the stability of the motorized Euler-Bernoulli beam described by the following model

\[
\begin{align*}
I_hS_{tt}(t) + EIv_0v_{xxxx}(0, t) - EIv_{xx}(0, t) + (L + r_0)F_L(t) &= \tau(t), \quad t \geq 0, \\
\rho([r_0 + x]S_{tt}(t) + v_{tt}(x, t)) + EIv_{xxxx}(x, t) &= 0, \quad \forall (x, t) \in (0, L) \times \mathbb{R}^+, 
\end{align*}
\]

(1.4)

with the boundary conditions

\[
\begin{align*}
v(0, t) &= v_x(0, t) = EIv_{xx}(L, t) = 0, \quad t \geq 0, \\
E Iv_{xxxx}(L, t) &= F_L(t), \quad t \geq 0,
\end{align*}
\]

(1.5)

where \( r_0 \) is the radius of the motor and \( F_L(t) \) is the boundary control force generated by the force actuator at tip of the beam. They established an exponential stability using semigroup theory under the following boundary control and control force

\[
F_L(t) = k_d v_t(L, t), \quad t \geq 0,
\]

\[
\tau(t) = I_h \frac{d}{dt}S_d(t) + K_d \frac{d}{dt}(S_d(t) - S(t)) + K_\rho(S_d(t) - S(t))
\]
+ EIr_0(v_{xx}(0,t) - v_{xx}(0)) + (L + r_0)F_L(L), \quad t \geq 0,

where \( k_d, K_p, \) and \( K_d \) are the positive control gains, and \( \frac{d}{dt^2}S_d(t), \frac{d}{dt}S_d(t), \) and \( S_d(t) \) are given desired reference trajectories.

In Nguyen and Egeland [55], the same authors improved these results by attaching a mass \( m \) at the free end of the beam with boundary control \( F_L \) applied at the mass. They investigated the existence, uniqueness, and the exponential stability of the problems (1.4) to (1.5) with

\[
F_L(t) = -k_d v_1(L, t), \quad t \geq 0,
\]

\[
\tau(t) = I_h \frac{d}{dt^2} S_d(t) + k_d \frac{d}{dt} (S_d(t) - S(t)) + K_p (S_d(t) - S(t)), \quad t \geq 0.
\]  \hspace{1cm} (1.6)

Also, when \( g = 0 \) (i.e., in the absence of viscoelastic damping) in the second equation of (1), Gua and Song [29] considered systems (1.1)-(1.3) and they showed the well-posedness of the systems. Additionally, they established the asymptotic stability of the system if \( I_h = 0 \) under the nonlinear feedback control

\[
\tau(t) = -\alpha v_x(0, t) - f(y_{zt})(t), \quad t \geq 0,
\]  \hspace{1cm} (1.7)

where \( \alpha > 0 \) and \( f \in C(\mathbb{R}) \) is increasing such that \( f(0) = 0 \) and \( s f(s) > 0 \) for \( s \neq 0 \). Furthermore, the same author in [28] improved the result in Guo and Song [29]. For \( I_h \neq 0 \), he established the observability and controllability of the system under the control law

\[
\tau(t) = -\alpha v_x(0, t) + \beta v_{tx}(t) - kv_{xxx}(0, t), \quad \alpha > 0, \alpha \beta, k \in \mathbb{R}, \quad t \geq 0.
\]  \hspace{1cm} (1.8)

For similar problems dealing with the stability theory of the rotating Euler-Bernoulli beam with other types of dissipations, the reader is referred to previous studies [10, 11, 12].

On the other hand, the boundary stabilization and boundary control of Euler-Bernoulli beam without rotating (non rotating case) by using different boundary dampers were considered by several authors, and many results have been obtained in this regard. We would like to mention the following [13, 20, 16, 17, 23, 24, 25, 6, 35, 36, 42, 59] and a long list of references therein.

For the stabilization and the control of viscoelastic beams without rotation (non-rotational case), we may cite the works of [56, 57, 43, 34] (see also references therein). We note also that in [18], Cavalcanti et al. treated the following system

\[
\ddot{u} + \Delta^2 u - \int_0^t q(t - \tau) \Delta^2 u(\tau)d\tau = 0 \quad \text{in} \quad \Omega \times (0, \infty),
\]

with the nonlinear boundary conditions

\[
\begin{cases}
\frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \Gamma_0 \times (0, \infty), \\
\Delta u - \int_0^t q(t - \tau) \Delta u(\tau)d\tau & \text{on} \quad \Gamma_1 \times (0, \infty), \\
\frac{\partial (\Delta u)}{\partial \nu} - \int_0^t q(t - \tau) \frac{\partial (\Delta u(\tau))}{\partial \nu}d\tau = f(u) + a(t)u_t & \text{on} \quad \Gamma_1 \times (0, \infty), \\
u(x, 0) = u_0^0(x); \quad u_t(x, 0) = u_1^0(x), & x \in \Omega,
\end{cases}
\]

where \( \Omega \) is the a bounded domain of \( \mathbb{R}^n, n \geq 1 \), with smooth boundary \( \Gamma = \Gamma_0 \cup \Gamma_1 \). In addition to the well-posedness, he established a uniform decay rate of the energy...
by implementing a nonlinear and nonlocal feedback acting on the boundary. The kernel $q$ is of an exponential type. More precisely, it was assumed that

$$
\exists \xi_0, \xi_1, \xi_2 > 0 : -\xi_0 q(t) \leq q(t) \leq -\xi_1 k(t), \quad 0 < q(t) \leq \xi_2 q(t), \quad \forall t \geq 0.
$$

In [19], Cavalcanti et al. obtained an exponential rate of decay by assuming that the kernel $q$ is decaying exponentially. This work was later improved by Berrimi and Messaoudi [7] by introducing a different functional which allowed them to weaken the conditions on $q$.

For a wider class of relaxation functions, Messaoudi [46, 47] considered

$$
\ddot{u} - \Delta u + \int_0^t q(t-s) \Delta u(s) ds = b \nabla u, \quad (1.9)
$$

for $\gamma > 0$ and $b = 0$ or $b = 1$, and the relaxation function satisfies

$$
q'(t) \leq -\zeta(t) q(t), \quad (1.10)
$$

where $\zeta$ is a differentiable nonincreasing positive function. He established a more general decay result, from which the usual exponential and polynomial decay results are only special cases. Such a condition was then employed in a series of papers, see for instance [31, 50, 52, 58].

Recently, Mustafa and Messaoudi [51] studied the problem (1.9) with $b = 0$ for the relaxation functions satisfying

$$
q'(t) \leq -H(q(t)), \quad (1.11)
$$

where $H$ is a nonnegative function, with $H(0) = H'(0) = 0$ and $H$ is strictly increasing and strictly convex on $[0,k]$ for some $k_0 > 0$. The authors showed a general relation between the decay rate for the energy and that of the relaxation function $q$ without imposing restrictive assumptions on the behavior of $q$ at infinity. On the other hand, a condition of the form (1.11) where $H$ is a convex function satisfying some smoothness properties, was introduced by Alabau-Boussouira and Cannarsa [4] and used then by several authors with different approaches. We refer to [41] where not only general but also optimal result was established by Lasiecka and Wang.

As far as we are concerned, there are few papers which deal with the asymptotic dynamics to problem (1.1)-(1.3), and the present paper seems to be among the pioneer in investigating the general stability to problem (1.1)-(1.3).

Our aim is to investigate (1.1)-(1.3) for relaxation functions $q$ of more general type than the ones in (1.10) and (1.11). We consider the condition

$$
q'(t) \leq -\zeta(t) H(q(t)), \quad a.e. \ t \geq 0 \quad (1.12)
$$

where $H$ is an increasing and convex function near the origin and $\zeta$ is a non-increasing function with only these very general assumptions on the behavior of $q$ at infinity, we establish optimal explicit and general energy decay results from which we can recover the optimal exponential and polynomial rates.

To prove decay estimates, we use a perturbed energy method and some properties of convex functions. Arguments of convexity were introduced and developed in [15, 26, 39, 40, 45, 27, 3].

The outline of this paper is the following: in the next section, we prepare some mathematical preliminaries, which will be used throughout this paper. In Section 3, we formulate our assumptions on the relaxation function, state our results, and provide the proofs based on the multiplier method and some ideas introduced in Mustafa [50, 52, 53].
2. Preliminary results. To simplify the stability analysis in the next section, we start by defining the total deflection of the beam $\xi(x,t)$, as follows

$$\xi(x,t) = xS(t) + v(x,t), \quad t \geq 0. \quad (2.1)$$

Therefore, we have for all $x$ in $(0, L)$ and $t \geq 0$

$$\begin{cases}
\xi_x(0,t) = S(t), & \xi_{xx}(0,t) = S_t(t), \\
\xi_t(x,t) = xS_t(t) + v_t(x,t), & \xi_{tt}(x,t) = xS_{tt}(t) + v_{tt}(x,t), \\
\xi_{xxx}(x,t) = v_{xxx}(x,t), & \xi_{xxxx}(x,t) = v_{xxxx}(x,t), \\
\xi(x,0) = xS_0 + v_0(x) = \xi_0(x), & \xi_t(x,0) = xS_1 + v_1(x) = \xi_1(x).
\end{cases} \quad (2.2)$$

Using (1.5) and (2.2), the problem (1.1)–(1.3) is transformed into the following one with the boundary conditions

$$\begin{cases}
I_h \xi_{xtt}(0,t) + \rho \int_0^L x\xi_{ttt}(x,t)dx + m_p L\xi_{ttt}(L,t) + J_p \xi_{xttt}(L,t) = \tau(t), & t \geq 0, \\
\rho \xi_{tt}(x,t) + EL \xi_{xxxx}(x,t) - EL \int_0^t q(t-s)\xi_{xxxx}(s)ds = 0, & \forall(x,t) \in (0,L) \times \mathbb{R}^+, \\
with the boundary conditions
\end{cases} \quad (2.3)$$

$$\begin{cases}
\xi(0,t) = 0, \\
EL \xi_{xxxx}(L,t) - EL \int_0^t q(t-s)\xi_{xxxx}(L,s)ds = m_p \xi_{ttt}(L,t), \\
EL \xi_{xx}(L,t) - EL \int_0^t q(t-s)\xi_{xx}(L,s)ds = -J_p \xi_{xttt}(L,t), & \forall t \in [0, \infty),
\end{cases} \quad (2.4)$$

and the initial data

$$\xi(x,0) = \xi_0(x), \quad \xi_t(x,0) = \xi_1(x), \quad x \in (0, L). \quad (2.5)$$

Now, substituting the second equation of (2.3) to the first equation of (2.3), using integration by party and the boundary condition (2.4), the equation of motion for the hub is seen to be

$$I_h \xi_{xtt}(0,t) - EL \xi_{xx}(0,t) + EL \int_0^t q(t-s)\xi_{xx}(0,s)ds = \tau(t), \quad t \geq 0. \quad (2.6)$$

To stabilize systems (2.3)–(2.5), we propose the following control torque $\tau(t)$,

$$\tau(t) = -K \xi_{xtt}(0,t) - \xi_x(0,t), \quad t \geq 0, \quad (2.7)$$

where $K$ is a positive “control gain”.

2.1. Assumptions. To state and prove our result, we use the following assumptions:

(A1) To preserve the hyperbolicity of our system, we assume that the kernel is such that $q \in L^1(\mathbb{R}^+) \cap C^2(\mathbb{R}^+)$ with $q(0) > 0$ and

$$1 - \int_0^\infty q(s)ds := 1 - \kappa > 0.$$

Let $t_0 > 0$ be a number such that $\int_0^{t_0} q(s)ds = q_* > 0$.

(A2) There exists a $C^1$-function $H : (0, \infty) \rightarrow (0, \infty)$ which is linear or strictly increasing and strictly convex $C^2$-function on $[0, r]$ ($r \leq q(0)$) with $H(0) = H'(0) = 0$, such that

$$q'(t) \leq -\zeta(t)H(q(t)), \quad \forall t \geq 0, \quad (2.8)$$
where $\zeta$ is a positive non-increasing differentiable function.

**Remark 1.** (1) As in [51], $(A_1)$-$(A_2)$, we have $\lim_{s \to +\infty} q(s) = 0$, hence $\lim_{s \to +\infty} - q'(s)$ cannot be positive, then $\lim_{s \to +\infty} - q'(s) = 0$. Then there exists $t_1 > 0$ large enough such that $q(t_1) > 0$ and

$$\max \{ q(t), -q'(t) \} < \min \{ r, H(r), H_0(r) \}, \quad \forall t \geq t_1,$$

where $H_0(t) = H(D(t))$ provided that $D$ is a positive $C^1$ function, $D(0) = 0$, for which $H_0$ is strictly increasing and strictly convex $C^2$ function on $(0, r]$ and

$$\int_0^{+\infty} \frac{q(s)}{H^{-1}_0(-q'(s))} ds < +\infty. \quad (2.9)$$

(2) Moreover, as $q$ is non-increasing, $q(0) > 0$ and $q(t_1) > 0$, we have

$$0 < q(t_1) \leq q(t) \leq q(0), \quad \forall t \in [0, t_1].$$

A combination of this with the continuity of $H$ yields

$$a \leq H(q(t)) \leq b, \quad \forall t \in [0, t_1],$$

for two constants $a, b > 0$.

$$q'(t) \leq -\zeta(t) H(q(t)) \leq -a \zeta(t) = -\frac{a}{q(0)} \zeta(t)q(0) \leq -\frac{a}{q(0)} \zeta(t)q(t)$$

and, hence,

$$\zeta(t)q(t) \leq -\frac{q(0)}{a} q'(t), \quad \forall t \in [0, t_1]. \quad (2.10)$$

We give some examples of functions satisfying $(A_1)$ and $(A_2)$, see [33].

(1) If the relaxation function $g$ is given by $g(t) = ae^{-\lambda t}$, $t \geq 0$, $a, \lambda > 0$ are constants, and $a$ is chosen so that $(A_1)$ is satisfied, then

$$q'(t) = \lambda H(q(t)) \text{ with } H(s) = s.$$  

(2) Consider $q(t) = ae^{-1+(1+t)^\nu}$, for $t \geq 0$, $0 < \nu < 1$, and $a$ is chosen so that condition $(A_1)$ is satisfied, then

$$q'(t) = -\zeta(t) H(q(t)) \text{ with } \zeta(t) = \nu(1+t)^{\nu-1} \text{ and } H(s) = s.$$  

(3) Consider the following relaxation function, for $\nu > 1$,

$$q(t) = \frac{a}{1+t^\nu}, \quad t \geq 0$$

and $a$ is chosen so that hypothesis $(A_1)$ remains valid. Then

$$q'(t) = bH(q(t)) \text{ with } H(s) = s^p,$$

where $b$ is a fixed constant, $p = \frac{1+\nu}{\nu}$, which satisfies $1 < p < 2$.

Now, we prepare some notations and hypotheses which will be needed in the proof of our result.

In the sequel, $c$ denotes a general positive constant, which may be different in different estimates.

In the following, we will need:

**Lemma 2.1 (Young’s inequality).** We have,

$$ab \leq \delta a^2 + \frac{1}{4\delta} b^2, \quad a, b \in \mathbb{R}, \quad \delta > 0. \quad (2.11)$$

Also, we shall use the following lemma.
Lemma 2.2 (See Hardy et al. [32]). Under our boundary conditions (1.2), we have
\[ v^2(x,t) \leq L\|v_x\|_2^2, \quad v^2(x,t) \leq L^2\|v_{xx}\|_2^2, \quad v^2(x,t) \leq L\|v_{xx}\|_2^2, \quad \forall x \in [0,L], \quad (2.12) \]
and
\[ \|v\|_2^2 \leq L^2\|v_x\|_2^2 \leq L^4\|v_{xx}\|_2^2 \quad \forall x \in [0,L], \quad (2.13) \]
where \( \| \cdot \| \) is the norm in \( L^2(0,L) \).

Throughout this paper, we denote by \( \Box \) and \( \diamond \) the binary operators, respectively, defined by
\[ (q\Box v)(t) := \int_0^t q(t-s)\left(v(x,t) - v(x,s)\right)^2 ds, \quad t \geq 0, \]
and
\[ (q\diamond v)(t) := \int_0^t q(t-s)\left(v(0,t) - v(0,s)\right)^2 ds, \quad t \geq 0. \]
The next lemma gives an identity for the convolution product.

Lemma 2.3. (See Lemma 2.2, [13]) For real functions \( h, \varphi \in C^1(\mathbb{R}^+) \), we have
\[ (h \ast \varphi)(t) = -\frac{1}{2}(\varphi(t))^2 + \frac{1}{2}h' \circ \varphi - \frac{1}{2} \frac{d}{dt} \left(h \circ \varphi - \left(\int_0^t h(s)ds\right)\varphi(t)\right). \quad (2.14) \]

The identity to follow is helpful to prove our result.

Lemma 2.4. We have for \( q \in C(0,\infty) \) and \( y \in C\left((0,\infty); L^2(0,L)\right) \)
\[ \int_0^L y \int_0^t q(t-s)y(s)dsdx = \frac{1}{2} \left( \int_0^t q(s)ds \right) \|y\|_2^2 + \frac{1}{2} \int_0^t q(t-s)\|y(s)\|_2^2 ds - \frac{1}{2} \int_0^t (q\Box y) dx, \quad t \geq 0. \quad (2.15) \]

We define the energy associated to the solution of problem (2.3)-(2.5) by the means of:
\[ 2E(t) = I_h\xi_{xt}^2(0,t) + \rho\|\xi_t\|_2^2 + EI\|\xi_{xx}\|_2^2 + m_p\xi_{xt}^2(L,t) + J_p\xi_{xt}^2(L,t), \quad t \geq 0. \quad (2.16) \]
We note that
\[ 2 \int_0^L \xi_{xt}(t) \int_0^t q(t-s)\xi_{xx}(s)dsdx = \int_0^L (q'\xi_{xx})dx - q(t)\|\xi_{xx}\|_2^2 - \frac{d}{dt} \left\{ \int_0^L (q\Box \xi_{xx})dx - \left(\int_0^t q(s)ds\right)\|\xi_{xx}\|_2^2 \right\}. \]
This motivates suggesting the 'modified' energy functional
\[ E(t) = \frac{1}{2} \left\{ I_h\xi_{xt}^2(0,t) + \rho\|\xi_t\|_2^2 + EI\left(1 - \int_0^t q(s)ds\right)\|\xi_{xx}\|_2^2 \right. \]
\[ + EI \int_0^L (q\Box \xi_{xx})dx + m_p\xi_{xt}^2(L,t) + J_p\xi_{xt}^2(L,t) \right\}, \quad t \geq 0. \quad (2.17) \]
Clearly,
\[ \frac{d}{dt} E(t) = \frac{EI}{2} \int_0^L (q'\xi_{xx})dx - \frac{EI}{2} q(t)\|\xi_{xx}\|_2^2 + \tau(t)\xi_{xt}(0,t), \quad t \geq 0. \quad (2.18) \]
Lemma 2.5 (Jensen’s inequality). Let $F$ be a convex function on $[a, b]$, $f : \Omega \to [a, b]$ and $h$ are integrable functions on $\Omega$, $h(x) \geq 0$, and $\int_{\Omega} h(x)dx = k > 0$, then Jensen’s inequality states that

$$F\left[ \frac{1}{k} \int_{\Omega} f(x)h(x)dx \right] \leq \frac{1}{k} \int_{\Omega} F[f(x)]h(x)dx. \quad (2.19)$$

Now, we define a Lyapunov functional $L(t)$ by

$$L(t) := E(t) + \sum_{i=1}^{4} \lambda_i \psi_i(t), \quad t \geq 0, \quad (2.20)$$

where $\lambda_i, i = 1, \ldots, 4$ are positive constants that will be chosen later with $\lambda_3 = \lambda_4 = 1$ and

$$\psi_1(t) = \rho \int_{0}^{L} \xi_k(t)dx + I_h \xi_k(0, t) \xi_x(0, t) + m_p \xi_k(L, t) \xi_x(L, t), \quad t \geq 0, \quad (2.21)$$

$$\psi_2(t) = -\rho \int_{0}^{L} \xi_k(t) \int_{0}^{t} q(t-s)(\xi(t) - \xi(s))dsdx$$

$$- I_h \xi_k(t) \int_{0}^{t} q(t-s)(\xi_x(0, t) - \xi_x(0, s))ds$$

$$- m_p \xi_k(t) \int_{0}^{t} q(t-s)(\xi(L, t) - \xi(L, s))ds$$

$$- J_p \xi_k(t) \int_{0}^{t} q(t-s)(\xi_x(L, t) - \xi_x(L, s))ds, \quad t \geq 0, \quad (2.22)$$

$$\psi_3(t) := \frac{1}{2}(q\xi_x) + \frac{1}{2} \xi_x^2(0, t), \quad t \geq 0, \quad (2.23)$$

and

$$\psi_4(t) := \int_{0}^{t} q(s)(\xi_x(0, t) - \xi_x(0, s))^2ds, \quad t \geq 0, \quad (2.24)$$

whith

$$q_\beta(t) = e^{-\beta t} \int_{1}^{\infty} q(s)e^{\beta s}ds, \quad t \geq 0,$$

and $\beta > 0$ is a constant to be determined later.

The following result shows that $L(t)$ and $E(t) + \psi_3(t) + \psi_4(t)$ are equivalent.

Proposition 1. There exist two positive constants $\alpha_i > 0, i = 1, 2$ such that

$$\alpha_1(E(t) + \psi_3(t) + \psi_4(t)) \leq L(t) \leq \alpha_2(E(t) + \psi_3(t) + \psi_4(t)), \quad t \geq 0. \quad (2.25)$$

Proof. By using (2.11), (2.12) and (2.13), we get

$$\psi_1(t) \geq \frac{\rho}{2} \left\| \xi_k \right\|^2 + \frac{\rho}{2} \left\| \xi_x \right\|^2 + \frac{I_h}{2} \xi_x^2(0, t) + \frac{I_h}{2} \xi^2(0, t) + \frac{m_p}{2} \xi_x^2(L, t)$$

$$+ \frac{m_p}{2} \xi^2(L, t) + \frac{J_p}{2} \xi_x^2(L, t) + \frac{J_p}{2} \xi^2(L, t), \quad t \geq 0.$$

In view of (2.2), we see that $\xi(x, t) = xS(t) + v(x, t) = x\xi_x(0, t) + v(x, t)$. Therefore, using (2.12) and (2.13), we get

$$\left\| \xi \right\|^2 \leq \frac{2}{3} L^3 \xi_x^2(0, t) + 2 \left\| v \right\|^2 \leq \frac{2}{3} L^3 \xi_x^2(0, t) + 2L^4 \left\| v_{xx} \right\|^2, \quad t \geq 0,$$

$$\xi^2(L, t) \leq 2L^2 \xi_x^2(0, t) + 2v^2(L, t) \leq 2L^2 \xi_x^2(0, t) + 2L^3 \left\| v_{xx} \right\|^2, \quad t \geq 0.$$
\[
\xi_x^2(L, t) \leq 2\xi_x^2(0, t) + 2v^2_x(L, t) \leq 2\xi_x^2(0, t) + 2L\|v_x\|_2^2, \quad t \geq 0.
\]
Since \(\xi_{xx}(x, t) = v_{xx}(x, t)\), we obtain
\[
\psi_1 (t) \leq \frac{\rho}{2} \|\xi_t\|^2 + \left(\frac{\rho L^3}{3} + \frac{I_h}{2} + m_p L^2 + J_p\right)\xi_x^2(0, t) + \frac{I_h}{2} \xi_{xx}^2(0, t) + \frac{J_p}{2} \xi_{xx}^2(L, t) + \frac{m_p}{2} \xi_{xx}^2(L, t) + (\rho L^4 + m_p L^3 + J_p L)\|\xi_{xx}\|^2, \quad t \geq 0.
\]
(2.26)
For \(\psi_2(t)\), we have
\[
\psi_2 (t) \leq \frac{\rho}{2} \|\xi_t\|^2 + \frac{\rho}{2} \left(\int_0^t q(s)ds\right) \int_0^L \int_0^t (q(t - s) (\xi(t) - \xi(s))^2 \right) dsdx + \frac{J_p}{2} \xi_{xx}^2(L, t) + \frac{I_h}{2} \xi_{xx}^2(0, t) + \int_0^t q(s)ds \int_0^L \int_0^t (q(t - s) (\xi(x, t) - \xi(x, s))^2) dsdx + \frac{m_p}{2} \xi_{xx}^2(L, t) + \int_0^t q(s)ds \int_0^L \int_0^t (q(t - s) (\xi(L, t) - \xi(L, s))^2) dsdx + \frac{J_p}{2} \left(\int_0^t q(s)ds\right) \int_0^L \int_0^t (q(t - s) (\xi_x(L, t) - \xi_x(L, s))^2) dsdx, \quad t \geq 0.
\]
(2.27)
Now, using (2.12) and (2.13), we obtain
\[
\int_0^L \int_0^t (q(t - s) (\xi(x, t) - \xi(x, s))^2) dsdx \leq \frac{2}{3} \rho L^3 (q\xi_x) + 2L^5 \int_0^L (q\xi_x)dx, \quad t \geq 0,
\]
(2.28)
\[
\int_0^t q(t - s) (\xi(L, t) - \xi(L, s))^2 ds \leq 2L^2 (q\xi_x) + 2L^3 \int_0^L (q\xi_x)dx, \quad t \geq 0.
\]
(2.29)
and
\[
\int_0^t q(t - s) (\xi_x(L, t) - \xi_x(L, s))^2 ds \leq 2 (q\xi_x) + 2L \int_0^L (q\xi_x)dx, \quad t \geq 0.
\]
(2.30)
Also, the fact \(\xi_{xx}(x, t) = v_{xx}(x, t)\), together with (2.28)-(2.30), implies that
\[
\psi_2 (t) \leq \frac{\rho}{2} \|\xi_t\|^2 + \frac{I_h}{2} \xi_{xx}^2(0, t) + (\rho L^4 + m_p L^3 + J_p L) \kappa \int_0^L (q\xi_x)dx + \frac{m_p}{2} \xi_{xx}^2(L, t) + \left(\frac{\rho L^3}{3} + \frac{I_h}{2} + m_p L^2 + J_p\right) \kappa (q\xi_x) + \frac{J_p}{2} \xi_{xx}^2(L, t), \quad t \geq 0.
\]
(2.31)
Gathering (2.17), (2.26) and (2.31), we end up with
\[
\mathcal{L}(t) \leq (1 + \lambda_1 + \lambda_2) \left[\frac{\rho}{2} \|\xi_t\|^2 + \frac{I_h}{2} \xi_{xx}^2(0, t) + \frac{m_p}{2} \xi_{xx}^2(L, t) + \frac{J_p}{2} \xi_{xx}^2(L, t)\right]
+ \psi_4(t) + \left[\frac{1}{2} + \lambda_1 \left(\frac{\rho L^3}{3} + \frac{I_h}{2} + m_p L^2 + J_p\right)\right] \xi_x^2(0, t)
+ \left[\frac{E I}{2} \left(1 - \int_0^t q(s)ds\right) + \lambda_1 \left(\rho L^3 + m_p L^3 + \frac{J_p L}{2}\right)\right] \|\xi_{xx}\|^2
+ \left[\frac{1}{2} + \lambda_2 \left(\frac{\rho L^3}{3} + \frac{I_h}{2} + m_p L^2 + J_p\right)\right] (q\xi_x)
+ \left[\frac{E I}{2} + \lambda_2 \left(\rho L^4 + m_p L^3 + J_p L\right)\right] \int_0^L (q\xi_{xx}) dx, \quad t \geq 0.
\]
Hence,
\[ L(t) \leq \alpha_2 (E(t) + \psi_3(t) + \psi_4(t)), \quad t \geq 0 \]
for some positive constant \( \alpha_2 \).

On the other hand, we have
\[
2L(t) \geq (1 - \lambda_1 - \lambda_2) \left[ \rho \| \xi \|_2 + \frac{I_h}{2} \xi_{xt}(0, t) + m_p \xi_{t}(L, t) + J_p \xi_{xt}(L, t) \right] \\
+ 2\psi_4(t) + \left[ 1 - \lambda_1 \left( \frac{2\rho L^3}{3} + I_h + 2m_pL^2 + 2J_p \right) \right] \xi_x^2(0, t) \\
+ \left[ \frac{EI}{2} (1 - \kappa) - \lambda_1 \left( 2\rho L^3 + 2m_pL^3 + J_pL \right) \right] \| \xi_{xx} \|_2^2 \\
+ \left[ 1 - 2\lambda_2 \left( \frac{\rho L^3}{3} + \frac{I_h}{2} + m_pL^2 + J_p \right) \right] (q \cdot \xi_x) \\
+ \left[ EI - 2\lambda_2 \left( \rho L^4 + m_pL^3 + J_pL \right) k \right] \int_0^L (q \cdot \xi_{xx}) \, dx, \quad t \geq 0.
\]
Therefore,
\[ \alpha_1 (E(t) + \psi_3(t) + \psi_4(t)) \leq L(t) \leq \alpha_2 (E(t) + \psi_3(t) + \psi_4(t)), \quad t \geq 0, \]
for some constants \( \alpha_i > 0, i = 1, 2 \), provided that
\[ \lambda_1 < \min \left[ 1, \frac{1}{2\rho L^3 + I_h + 2m_pL^2 + 2J_p}, \frac{EI(1 - \kappa)}{2\rho L^4 + 2m_pL^3 + J_p} \right] \]
and
\[ \lambda_2 < \min \left[ 1 - \lambda_1, \frac{EI}{(2\rho L^3 + 2m_pL^3 + J_p) \kappa}, \frac{1}{\left( \frac{4\rho L^2}{3} + I_h + 2m_pL^2 + 2J_pL \right) \kappa} \right]. \]
This completes the proof.

3. Well posedness. We state, without proof the global existence and regularity result for the problem (1.1)-(1.3) which can be established by a standard Galerkin argument. We refer the reader to [2, 1, 14, 37, 42, 44, 49]. The technique based on the theory of non-linear semigroups used in Nicaise and Pignotti [54] does not seem to be applicable in the non-linear case. Throughout this paper we define
\[ V = \{ \omega \in H^2(0, L), \quad \omega(0) = \omega_x(0) = 0 \} \]
and
\[ W = \{ \omega \in V \cap H^2(0, L), \quad \omega(0) = \omega_x(0) = 0 \}. \]

**Theorem 3.1.** For \((\xi_0, \xi_1) \in V \cap H^4(0, L) \times V \) and \( q \) be a nonnegative summable kernel. Then, under the control torque \( \tau(t) \) defined in (2.7), the system (2.3)-(2.5) has a unique global (weak) solution \( \xi \) in the class
\[ \xi \in L^\infty ([0, \infty); V), \quad \xi_t \in L^\infty ([0, \infty); V), \quad \xi_{tt} \in L^\infty ([0, \infty); L^2(0, L)), \]
Lemma 4.1. Under the assumptions \((A_1)\) and \((A_2)\), the functional \(\psi_1(t)\) defined by (2.21) satisfies, along solutions of (2.3)-(2.5), the estimate

\[
\psi_1(t) \leq \rho \|\xi_t\|_2^2 - \frac{EI}{2} (1 - \kappa) \|\xi_{xx}\|_2^2 + \left( I_h + \frac{1}{4\eta_1} \right) \xi_{xx}^2 (0, t) - (1 - K^2 \eta_1) \xi_x^2 (0, t) + m_p \xi_t^2 (L, t) - J_p \xi_{xt}^2 (L, t) + \frac{EIC_\alpha}{2 (1 - \kappa)} \int_0^L (h \square \xi_{xx}) \, dx, \quad t \geq 0, \tag{4.1}
\]

for any \(0 < \alpha < 1\), and for any \(\eta_1 > 0\), where

\[
C_\alpha = \int_0^\infty \frac{q^2(s)}{\alpha q(s) - q'(s)} \, ds \quad \text{and} \quad h(t) = \alpha q(t) - q'(t). \tag{4.2}
\]

Proof. Using the equations of (2.3) and (2.7), we get

\[
\psi_1(t) = I_1 + I_2 + J_p \xi_{xt}^2 (L, t) - K \xi_{xt} (0, t) \xi_x (0, t) + \rho \|\xi_t\|_2^2 + m_p \xi_t^2 (L, t) - \xi_x^2 (0, t) + m_p \xi_{xt} (L, t) \xi_x (L, t) + J_p \xi_{xtt} (L, t) \xi_x (L, t) + I_h \xi_t^2 (0, t) + (EI \xi_{xx} (0, t) - EI \int_0^t (q(t - s) \xi_{xx} (0, s) \, ds) \xi_x (0, t), \quad t \geq 0. \tag{4.3}
\]

where

\[
I_1 = -EI \int_0^L \xi_{xxxx} \, dx \quad \text{and} \quad I_2 = EI \int_0^L \xi (t) \int_0^t (q(t - s) \xi_{xxxx} (s) \, ds) \, dx.
\]

Integrating by parts twice, we get

\[
I_1 = -EI \xi_{xx} (L, t) \xi (L, t) + EI \xi_{xx} (L, t) \xi_x (L, t)
- EI \xi_{xx} (0, t) \xi_x (0, t) - EI \|\xi_{xx}\|_2^2, \quad t \geq 0. \tag{4.4}
\]

Similarly,

\[
I_2 = EI \xi (L, t) \int_0^t (q(t - s) \xi_{xxxx} (L, s) \, ds) - EI \xi_x (L, t) \int_0^t (q(t - s) \xi_{xx} (L, s) \, ds)
+ EI \xi_x (0, t) \int_0^t (q(t - s) \xi_{xx} (0, s) \, ds) + EI \int_0^L \xi_{xx} (t) \int_0^t (q(t - s) \xi_{xx} (s) \, ds) \, dx, \quad t \geq 0. \tag{4.5}
\]

Substituting (4.4) and (4.5) in (4.3) and using the boundary conditions (2.4) after canceling out some similar terms, we obtain

\[
\psi_1(t) = \rho \|\xi_t\|_2^2 - EI (1 - \kappa) \|\xi_{xx}\|_2^2 - \xi_x^2 (0, t) + I_h \xi_t^2 (0, t) + m_p \xi_t^2 (L, t)
+ J_p \xi_{xt}^2 (L, t) + EI \int_0^L \xi_{xx} (t) \int_0^t (q(t - s) (\xi_{xx} (s) - \xi_{xx} (t)) \, ds) \, dx
- K \xi_{xt} (0, t) \xi_x (0, t), \quad t \geq 0. \tag{4.6}
\]

Now, we estimate the terms in the right-hand side of expression (4.6). We start with the seventh

\[
EI \int_0^L \xi_{xx} (t) \int_0^t (q(t - s)(\xi_{xx} (s) - \xi_{xx} (t)) \, ds) \, dx
\]
\begin{align*}
&\leq EI \frac{1 - \kappa}{2} \|\xi_{xx}\|_2^2 + \frac{EI}{2 (1 - \kappa)} \int_0^L \left( \int_0^t q(t-s) (\xi_{xx}(s) - \xi_{xx}(t)) \, ds \right)^2 \, dx, \quad t \geq 0.
\end{align*}

Using Cauchy-Schwarz inequality, we get

\begin{align*}
&\int_0^L \left( \int_0^t q(t-s) (\xi_{xx}(s) - \xi_{xx}(t)) \, ds \right)^2 \, dx \\
&= \int_0^L \left( \int_0^t q(t-s) \sqrt{\alpha q(t-s) - q'(t-s)} \, ds \right)^2 \, dx \\
&\leq \left( \int_0^t \frac{q^2(s)}{\alpha q(s) - q'(s)} \, ds \right) \int_0^L \left[ \alpha q(t-s) - q'(t-s) \right] (\xi_{xx}(s) - \xi_{xx}(t))^2 \, ds \, dx \\
&\leq C_\alpha \int_0^L (h \square \xi_{xx})(t) \, dx, \quad t \geq 0. \tag{4.7}
\end{align*}

For the eighth term, using (2.11), we see

\[ K\xi_{xt}(0,t)\xi_x(0,t) \leq \frac{1}{4\eta_1} \eta_1^2 \xi_{xt}(0,t) + \eta_1 K^2 \xi_x^2(0,t), \quad \eta_1 > 0, \quad t \geq 0. \]

Combining all above estimates gives (4.1).

**Lemma 4.2.** Under the assumptions \((A_1)\) and \((A_2)\), the functional \(\psi_2(t)\) defined by (2.22) satisfies, along solutions of (2.3)-(2.5), the estimate

\[ \psi_2(t) \leq (\eta_2 - q_*) \left[ \rho \|\xi_t\|^2_2 + m_p \xi_t^2(L,t) + J_p \xi_{xt}(L,t) \right] + \left[ \frac{K}{2} + I_h(\eta_2 - q_*) \right] \xi_{xt}^2(0,t) + \eta_3 \xi_x^2(0,t) + \eta_2 EI \|\xi_{xx}\|_2^2 \\
+ \frac{\kappa}{4\eta_3} (q\varnothing \xi_x) + C_1 \int_0^L (h \square \xi_{xx}) \, dx + C_2 (h \varnothing \xi_x), \quad t \geq 0, \tag{4.8}
\]

for all \(t \geq t_* > 0\) and for any \(\eta_2, \eta_3 > 0\), where

\[ C_1 := \frac{c}{\eta_2} (J_p L + m_p L^3) (1 + C_\alpha) + \frac{c\rho}{\eta_2} L^3 (1 + LC_\alpha) + C_\alpha \left( \frac{EIc}{\eta_2} + 1 \right) \]

and

\[ C_2 := \frac{c}{\eta_2} (m_p L^2 + I_h) (C_\alpha + 1) + \frac{c\rho}{\eta_2} L^2 (1 + LC_\alpha) + \frac{KC_\alpha}{2}. \]

**Proof.** Differentiating \(\psi_2(t)\) along solutions of (2.3)-(2.5), we find

\[ \psi_2(t) = I_3 + I_4 - \left( \int_0^t q(s) \, ds \right) \left[ \rho \|\xi_t\|^2_2 + I_h \xi_{xt}^2(0,t) + m_p \xi_t^2(L,t) + J_p \xi_{xt}^2(L,t) \right] \\
- EI \left( \xi_{xx}(0,t) - \int_0^t q(t-s) \xi_{xx}(0,s) \, ds \right) \int_0^t q(t-s) (\xi_x(0,t) - \xi_x(0,s)) \, ds \\
- \rho \int_0^L \xi_t \int_0^t q'(t-s) (\xi(t) - \xi(s)) \, ds \, dx \\
- I_h \xi_{xt}(0,t) \int_0^t q'(t-s) (\xi_x(0,t) - \xi_x(0,s)) \, ds \\
- m_p \xi_{tt}(L,t) \int_0^t q(t-s) (\xi(L,t) - \xi(L,s)) \, ds \]

and

\[ \leq EI \frac{1 - \kappa}{2} \|\xi_{xx}\|_2^2 + \frac{EI}{2 (1 - \kappa)} \int_0^L \left( \int_0^t q(t-s) (\xi_{xx}(s) - \xi_{xx}(t)) \, ds \right)^2 \, dx, \quad t \geq 0. \]
where
\[ I_3 = EI \int_0^L \xi_{xxxx}(t) \int_0^t q(t-s) \left( \xi(t) - \xi(s) \right) ds dx, \]
and
\[ I_4 = -EI \left( \int_0^t q(t-s) \xi_{xxxx}(s) ds \right) \left( \int_0^t q(t-s) \left( \xi(t) - \xi(s) \right) ds \right) dx. \]

Integration by parts in \( I_3 \) and \( I_4 \), lead to
\[ I_3 = EI \xi_{xxxx}(L, t) \int_0^t q(t-s) \left( \xi(L, t) - \xi(L, s) \right) ds \]
\[ - EI \xi_{xxxx}(L, t) \int_0^t q(t-s) \left( \xi_x(L, t) - \xi_x(L, s) \right) ds \]
\[ + EI \xi_{xxxx}(0, t) \int_0^t q(t-s) \left( \xi_x(0, t) - \xi_x(0, s) \right) ds \]
\[ + EI \int_0^L \xi_{xx}(t) \int_0^t q(t-s) \left( \xi_{xx}(t) - \xi_{xx}(s) \right) ds dx, \quad t \geq 0, \quad (4.10) \]
and
\[ I_4 = -EI \left( \int_0^t q(t-s) \xi_{xxxx}(L, s) ds \right) \left( \int_0^t q(t-s) \left( \xi(L, t) - \xi(L, s) \right) ds \right) \]
\[ + EI \left( \int_0^t q(t-s) \xi_{xx}(L, s) ds \right) \left( \int_0^t q(t-s) \left( \xi_x(L, t) - \xi_x(L, s) \right) ds \right) \]
\[ - EI \left( \int_0^t q(t-s) \xi_{xx}(0, s) ds \right) \left( \int_0^t q(t-s) \left( \xi_x(0, t) - \xi_x(0, s) \right) ds \right) \]
\[ - EI \int_0^L \left( \int_0^t q(t-s) \xi_{xx}(s) ds \right) \int_0^t q(t-s) \left( \xi_{xx}(t) - \xi_{xx}(s) \right) ds dx, \quad t \geq 0. \]

Substituting (4.10) and (4.11) in (4.9) using the boundary conditions (2.4) and the control torque \( \tau(t) \) after canceling out some similar terms, we obtain
\[ \psi'(t) = - \left( \int_0^t q(s) ds \right) \left[ \rho ||\xi(t)||^2 + I_h \xi_{xx}^2(0, t) + m_p \xi_t^2(L, t) + J_p \xi_{xx}^2(L, t) \right] \]
\[ + EI \left( 1 - \int_0^t q(s) ds \right) \int_0^L \xi_{xx}(t) \int_0^t q(t-s) \left( \xi_{xx}(t) - \xi_{xx}(s) \right) ds dx \]
\[ + EI \int_0^L \left[ \int_0^t q(t-s) \left( \xi_{xx}(t) - \xi_{xx}(s) \right) ds \right]^2 dx \]
Using Cauchy-Schwarz inequality, we get

\[ \frac{1}{c} \int_{0}^{t} q(t-s) (\xi_x(0,t) - \xi_x(0,s)) ds \]

\[ + K \xi_{xt}(0,t) \int_{0}^{t} q(t-s) (\xi_x(0,t) - \xi_x(0,s)) ds \]

\[ - \rho \int_{0}^{L} \xi_t \int_{0}^{t} q'(t-s) (\xi(t) - \xi(s)) ds dx \]

\[ - I_0 \xi_{xt}(0,t) \int_{0}^{t} q'(t-s) (\xi_x(0,t) - \xi_x(0,s)) ds \]

\[ - m_p \xi_t(L,t) \int_{0}^{t} q'(t-s) (\xi(L,t) - \xi(L,s)) ds \]

\[ - J_p \xi_{xt}(L,t) \int_{0}^{t} q'(t-s) (\xi_x(L,t) - \xi_x(L,s)) ds, \quad t \geq 0. \]  

Now, we estimate the terms on the right-hand side of expression (4.12). We start with the fifth term

\[ \left( 1 - \int_{0}^{t} q(s) ds \right) \int_{0}^{L} \xi_{xx}(t) \int_{0}^{t} q(t-s) (\xi_{xx}(t) - \xi_{xx}(s)) ds dx \]

\[ \leq \eta_2 \| \xi_{xx} \|_2^2 + \frac{c}{\eta_2} C_\alpha \int_{0}^{L} (h \square \xi_{xx}) dx, \quad t \geq 0. \]  

(4.13)

For the seventh and eighth terms, it is easy to see that

\[ \xi_x(0,t) \int_{0}^{t} q(t-s) (\xi_x(0,t) - \xi_x(0,s)) ds \leq \eta_3 \xi_x^2(0,t) + \frac{K}{4\eta_3} (q \odot \xi_x), \quad \eta_3 > 0, \quad t \geq 0. \]  

(4.14)

Then, using (2.11), we obtain

\[ \xi_{xt}(0,t) \int_{0}^{t} q(t-s) (\xi_x(0,t) - \xi_x(0,s)) ds \]

\[ \leq \frac{1}{2} \xi_{xt}(0,t) + \frac{1}{2} \left( \int_{0}^{t} q(t-s) (\xi_x(0,t) - \xi_x(0,s)) ds \right)^2, \quad t \geq 0. \]  

(4.15)

Using Cauchy-Schwarz inequality, we get

\[ \left( \int_{0}^{t} q(t-s) (\xi_x(0,t) - \xi_x(0,s)) ds \right)^2 \]

\[ = \left( \int_{0}^{t} \frac{q(t-s)}{\sqrt{\alpha q(t-s) - q'(t-s)}} \sqrt{\alpha q(t-s) - q'(t-s)} (\xi_x(0,t) - \xi_x(0,s)) ds \right)^2 \]

\[ \leq \left( \int_{0}^{t} \frac{q^2(s)}{\alpha q(s) - q'(s)} ds \right) \int_{0}^{t} \left[ \alpha q(t-s) - q'(t-s) \right] (\xi_x(0,t) - \xi_x(0,s))^2 ds \]

\[ \leq C_\alpha (h \odot \xi_x)(t) dx, \quad t \geq 0. \]  

(4.16)

Combining (4.15) and (4.16), we get

\[ \xi_{xt}(0,t) \int_{0}^{t} q(t-s) (\xi_x(0,t) - \xi_x(0,s)) ds \leq \frac{1}{2} \xi_{xx}^2(0,t) + \frac{C_\alpha}{2} (h \odot \xi_x), \quad t \geq 0. \]  

(4.17)
Since $\xi_{xx}(x,t) = v_{xx}(x,t)$, using Young’s inequality, the last 4 terms in the right-hand side of (4.12) can be handled in the following manner, by noticing that
\[
(\xi(x,t) - \xi(x,s))^2 = \left[ x(\xi_x(0,t) - \xi_x(0,s)) + (v(x,t) - v(x,s))^2 \right] \\
\leq 2x^2 (\xi_x(0,t) - \xi_x(0,s)) + 2(v(x,t) - v(x,s))^2,
\]
and using (2.13), we obtain
\[
- \int_0^L \xi_t \int_0^t q(t-s) \left( \xi(t) - \xi(s) \right) ds dx \\
= \int_0^L \xi_t \int_0^t h(t-s) \left( \xi(t) - \xi(s) \right) ds dx - \int_0^L \xi_t \int_0^t \alpha q(t-s) \left( \xi(t) - \xi(s) \right) ds dx \\
\leq \eta_2 \|\xi_t\|_2^2 + \frac{1}{2\eta_2} \int_0^L \left( \int_0^t h(t-s) \sqrt{h(t-s)} \left( \xi(t) - \xi(s) \right) ds \right)^2 dx \\
+ \frac{\alpha^2}{2\eta_2} \int_0^L \left( \int_0^t q(t-s) \left( \xi(t) - \xi(s) \right) ds \right)^2 dx \\
\leq \eta_2 \|\xi_t\|_2^2 + \frac{cL^2}{\eta_2} (1 + LC_\alpha) (h \diamond \xi_x) + \frac{cL^3}{\eta_2} (1 + LC_\alpha) \int_0^L (h \diamond \xi_x) (t) dx, \ t \geq 0,
\]
and
\[
- \xi_{xt}(0,t) \int_0^t q(t-s) \left( \xi_x(0,t) - \xi_x(0,s) \right) ds \\
= \xi_{xt}(0,t) \int_0^t h(t-s) \left( \xi_x(0,t) - \xi_x(0,s) \right) ds \\
- \alpha \xi_{xt}(0,t) \int_0^t q(t-s) \left( \xi_x(0,t) - \xi_x(0,s) \right) ds \\
\leq \eta_2 \xi_{xt}(0,t) + \frac{C_\alpha + 1}{\eta_2} (h \diamond \xi_x), \ t \geq 0.
\]
Similarly, using (2.12), the relations
\[
(\xi(L,t) - \xi(L,s))^2 = \left[ L (\xi_x(0,t) - \xi_x(0,s)) + (v(L,t) - v(L,s))^2 \right] \\
\leq 2L^2 (\xi_x(0,t) - \xi_x(0,s)) + 2(v(L,t) - v(L,s))^2,
\]
and
\[
(\xi_x(L,t) - \xi_x(L,s))^2 = \left[ L (\xi_x(0,t) - \xi_x(0,s)) + (v_x(L,t) - v_x(L,s))^2 \right] \\
\leq 2 (\xi_x(0,t) - \xi_x(0,s)) + 2(v_x(L,t) - v_x(L,s))^2,
\]
also, together with (4.21) and (4.22), imply that
\[
- \xi_t(L,t) \int_0^t q(t-s) \left( \xi(L,t) - \xi(L,s) \right) ds \\
= \xi_t(L,t) \int_0^t h(t-s) \left( \xi(L,t) - \xi(L,s) \right) ds
\]
Collecting the previous estimates (4.27) and (4.28) in (4.26), (4.25) is established.

Proof. Direct computations yield

\[-\alpha \xi_t(L, t) \int_0^t q(t-s) \left( \xi(L, t) - \xi(L, s) \right) ds\]

\[\leq \eta_2 \xi_t^2(L, t) + \frac{cL^2}{\eta_2} (1+C_\alpha) (h\circ \xi_x) + \frac{cL^3}{\eta_2} (1+C_\alpha) \int_0^L (h\boxdot \xi_{xx}) \, dx, \quad t \geq 0, \quad (4.23)\]

and

\[-\xi_{xt}(L, t) \int_0^t q(t-s) \left( \xi_x(L, t) - \xi_x(L, s) \right) ds\]

\[= \xi_{xt}(L, t) \int_0^t h(t-s) \left( \xi_x(L, t) - \xi_x(L, s) \right) ds\]

\[-\alpha \xi_{xt}(L, t) \int_0^t q(t-s) \left( \xi_x(L, t) - \xi_x(L, s) \right) ds\]

\[\leq \eta_2 \xi_{xt}^2(L, t) + \frac{cL}{\eta_2} (1+C_\alpha) \int_0^L (h\boxdot \xi_{xx}) \, dx, \quad t \geq 0. \quad (4.24)\]

Making use of (4.7) and (4.13)-(4.24) in (4.12), we establish (4.8) for \( t \geq t_* > 0. \)

Lemma 4.3. Under the assumptions (A_1) and (A_2), the functional \( \psi_3(t) \) defined by (2.23) satisfies

\[\psi_3'(t) \leq \frac{1}{2} (q' \circ \xi_x) + \eta_5 \xi_x^2(0, t) + \kappa \eta_4 (q \circ \xi_x) + \left( \frac{1}{4\eta_4} + \frac{1}{4\eta_5} \right) \xi_{xt}^2(0, t), \quad t \geq 0, \quad (4.25)\]

where \( \eta_4 \) and \( \eta_5 \) are some positive constants.

Proof. Direct computations yield

\[\psi_3(t) = \frac{1}{2} (q' \circ \xi_x) + \xi_{xt}(0, t) \int_0^t q(t-s) \left( \xi_x(0, t) - \xi_x(0, s) \right) ds + \xi_{xt}(0, t) \xi_x(0, t), \quad t \geq 0. \quad (4.26)\]

It suffices to observe that

\[\xi_{xt}(0, t) \int_0^t q(t-s) \left( \xi_x(0, t) - \xi_x(0, s) \right) ds \leq \frac{1}{4\eta_4} \xi_{xt}^2(0, t) + \kappa \eta_4 (q \circ \xi_x), \quad \eta_4 > 0, \quad t \geq 0, \quad (4.27)\]

and

\[\xi_{xt}(0, t) \xi_x(0, t) \leq \frac{1}{4\eta_5} \xi_{xt}^2(0, t) + \eta_5 \xi_x^2(0, t), \quad \eta_5 > 0, \quad t \geq 0. \quad (4.28)\]

Collecting the previous estimates (4.27) and (4.28) in (4.26), (4.25) is established.

Lemma 4.4. Under the assumptions (A_1) and (A_2), the functional \( \psi_4(t) \) defined by (2.24) satisfies

\[\psi_4'(t) \leq -\left( \beta - 2\eta_6 \eta_\beta \right) \psi_4(t) - (q \circ \xi_x) + \frac{1}{2\eta_6} \xi_{xt}^2(0, t), \quad t \geq 0. \quad (4.29)\]

for any constant \( \eta_6 > 0. \)

Proof. Direct computations yield

\[\psi_4(t) = -\beta \psi_4(t) - \int_0^t q(t-s) \left( \xi_x(0, t) - \xi_x(0, s) \right)^2 ds\]

\[+ 2 \xi_{xt}(0, t) \int_0^t Q_\beta(t-s) \left( \xi_x(0, t) - \xi_x(0, s) \right) ds, \quad t \geq 0. \quad (4.30)\]
Now, the last term in the right-hand side of (4.30) will be estimated as

\[
2\xi_x(t) \int_0^t Q_\beta(t-s) (\xi_x(0,t) - \xi_x(0,s)) \, ds
\]

\[
\leq \frac{1}{2\eta_0} \xi_x^2(0,t) + 2\eta_0 q_4(t), \quad \eta_0 > 0, \quad t \geq 0,
\]

where

\[
\overline{q}_\beta = \int_0^\infty e^{qs} q(s) \, ds.
\]

This completes the proof (4.29). \qed

**Lemma 4.5.** Under the assumptions \((A_1)\) and \((A_2)\), the functional \(\psi_5(t)\) defined by

\[
\psi_5(t) := \int_0^L \int_0^t f(t-s) \xi_x^2(s) \, ds \, dx, \quad t \geq 0,
\]

where \(f(t) = \int_t^\infty q(s) \, ds\) satisfies the estimate

\[
\psi_5(t) \leq -\frac{1}{2} \int_0^L (q \square \xi_{xx}) \, dx + 3\kappa \|\xi_{xx}\|_2, \quad t \geq 0.
\]

**Proof.** Using Young’s inequality and the fact \(f'(t) = -q(t)\), we see that

\[
\psi_5'(t) = f(0) \int_0^L \xi_x^2(t) \, dx - \int_0^L \int_0^t q(t-s) \xi_x^2(s) \, ds \, dx
\]

\[
= - \int_0^L \int_0^t q(t-s) (\xi_x(s) - \xi_x(t))^2 \, ds \, dx + f(t) \int_0^L \xi_x^2(t) \, dx
\]

\[
- 2 \int_0^L \xi_x(t) \int_0^t q(t-s) (\xi_x(s) - \xi_x(t)) \, ds \, dx, \quad t \geq 0.
\]

But

\[
- 2 \int_0^L \xi_x(t) \int_0^t q(t-s) (\xi_x(s) - \xi_x(t)) \, ds \, dx
\]

\[
\leq 2\kappa \|\xi_{xx}\|_2^2 + \frac{1}{2\kappa} \int_0^L \int_0^t q(t-s) (\xi_x(s) - \xi_x(t))^2 \, ds \, dx, \quad t \geq 0.
\]

Then, as \(f(t) \leq f(0) = \kappa\) and \(\int_0^t q(s) \, ds \leq \kappa\), we get (4.32). \qed

Using the previous lemmas we now give the proof of our main result.

**Lemma 4.6.** For suitable positive constants \(\lambda_1\) and \(\lambda_2\), that will be chosen later and with \(\lambda_3 = \lambda_4 = 1\), the functional \(\mathcal{L}(t)\) defined by (2.20) satisfies

\[
\mathcal{L}'(t) \leq -\sigma_1 \xi_x^2(0,t) - \sigma_2 \left( \rho \|\xi_t\|_2^2 + m_p \xi_x^2(L,t) + J_p \xi_x^2(t) \right) - \sigma_3 \|\xi_{xx}\|_2^2
\]

\[
- \sigma_4 \xi_{xx}^2(0,t) - \sigma_5 (q \square \xi_x) - \lambda_4 \beta \psi_4(t) + \frac{\alpha EI}{2} \int_0^L (q \square \xi_{xx}) \, dx, \quad t \geq 0,
\]

(4.33)
where
\[
\begin{align*}
\sigma_1 &= \lambda_2 (q_* - \varepsilon) \left( \frac{11}{16} - K^2 \eta_1 \right), \\
\sigma_2 &= \frac{\varepsilon}{2} \lambda_2, \\
\sigma_3 &= \frac{EI}{2} \lambda_2 \left[ q_* (1 - \kappa) - \varepsilon (2 - \kappa) \right], \\
\sigma_4 &= K \left( 1 - \frac{\lambda_2}{2} \right) + \frac{\lambda_2 I_h \varepsilon}{2} - \frac{\lambda_2 (q_* - \varepsilon)}{4 \eta_1} - \frac{8}{(q_* - \varepsilon)} \frac{(q_* - \varepsilon)}{\lambda_2} - 4 \kappa - \frac{2 \eta_3}{\beta}, \\
\sigma_5 &= \left( \frac{15}{16} - \frac{\lambda_2 \kappa}{q_* - \varepsilon} \right) - \frac{\alpha}{2}.
\end{align*}
\]

(4.34)

Proof. Let \( q_* = \int_0^t q(s) ds > 0 \) from (2.18), (4.1), (4.8), (4.25) and (4.29), recalling that \( q'(t) = \alpha q(t) - h(t) \), we obtain, for \( t \geq t_* > 0 \),

\[
\begin{align*}
\mathcal{L}'(t) &\leq - \left[ \lambda_1 \left( 1 - K^2 \eta_1 \right) - \lambda_2 \eta_3 - \eta_5 \right] \xi^2_2 (0, t) - \left[ \lambda_2 (q_* - \eta_2) - \lambda_1 \right] \left( \rho \| \xi_t \|_2 \right)^2 \\
&\quad + m_p \xi^2_t (L, t) + J_p \xi^2_{xt} (L, t)) - EI \left( \frac{\lambda_1 (1 - \kappa)}{2} - \lambda_2 \eta_2 \right) \| \xi_{xx} \|_2^2 \\
&\quad - (\beta - 2 \eta_5 \beta) \psi_4 (t) - \left\{ K \left( 1 - \frac{\lambda_2}{2} \right) + \lambda_2 I_h (q_* - \eta_2) \\
&\quad - \lambda_1 \left( I_h + \frac{1}{4 \eta_1} - \frac{1}{4 \eta_4} - \frac{1}{2 \eta_5} - \frac{1}{2 \eta_6} \right) \xi^2_{xt} (0, t) \\
&\quad - \left\{ 1 - \left[ \frac{\lambda_2}{4 \eta_3} + \frac{\alpha}{2 \kappa} + \eta_4 \right] \kappa \right\} (q \circ \xi_x) \\
&\quad - \left\{ \frac{EI}{2} - \left[ \frac{\lambda_1 EIC_\alpha}{2 (1 - \kappa)} + \lambda_2 C_1 \right] \right\} \int_0^L \langle h \square \xi_{xx} \rangle dx \\
&\quad - \left\{ \frac{1}{2} - \lambda_2 C_2 \right\} \langle h \circ \xi_x \rangle + \frac{\alpha EI}{2} \int_0^L \langle q \square \xi_{xx} \rangle dx, \quad t \geq 0.
\end{align*}
\]

Now, we start selecting the different parameters so that all the coefficients in (4.34) be positive.

First, let us pick \( \varepsilon < \frac{q_* (1 - \kappa)}{2 - \kappa} \), so \( q_* - \varepsilon > 0 \). Then, setting \( \lambda_1 = (q_* - \varepsilon) \lambda_2 \), \( \eta_2 = \varepsilon, \eta_3 = \frac{q_* - \varepsilon}{4}, \eta_4 = \frac{1}{16}, \eta_5 = \frac{1}{16} \) and \( \eta_6 = \frac{\beta}{4 \eta_3} \), we arrive at

\[
\begin{align*}
\mathcal{L}'(t) &\leq - \lambda_2 (q_* - \varepsilon) \left( \frac{11}{16} - K^2 \eta_1 \right) \xi^2_2 (0, t) \\
&\quad - \frac{\varepsilon}{2} \lambda_2 \left[ \rho \| \xi_t \|_2^2 + m_p \xi^2_t (L, t) + J_p \xi^2_{xt} (L, t) \right] - \frac{\beta}{2} \psi_4 (t) \\
&\quad - \frac{EI}{2} \lambda_2 \left[ q_* (1 - \kappa) - \varepsilon (2 - \kappa) \right] \| \xi_{xx} \|_2^2 + \frac{\alpha EI}{2} \int_0^L \langle q \square \xi_{xx} \rangle dx \\
&\quad - \left\{ K \left( 1 - \frac{\lambda_2}{2} \right) + \frac{\lambda_2 I_h \varepsilon}{2} - \frac{\lambda_2 (q_* - \varepsilon)}{4 \eta_1} - \frac{8}{(q_* - \varepsilon)} \frac{(q_* - \varepsilon)}{\lambda_2} - 4 \kappa - \frac{2 \eta_3}{\beta} \right\} \xi^2_{xt} (0, t) - \left( \frac{15}{16} - \frac{\lambda_2 \kappa}{q_* - \varepsilon} - \frac{\alpha}{2} \right) \langle q \circ \xi_x \rangle \\
&\quad - \left\{ \frac{EI}{2} - \lambda_2 \left[ C_1 + \frac{(q_* - \varepsilon) EIC_\alpha}{2 (1 - \kappa)} \right] \right\} \int_0^L \langle h \square \xi_{xx} \rangle dx.
\end{align*}
\]
Proof. We start by using (2.10), (2.18), (4.26) and (4.29) to conclude that, for any \( t \geq 0 \),

\[- \left( \frac{1}{2} - \lambda_2 C_2 \right) (h \phi \xi_x), \quad t \geq 0.\]

At this point, we select \( \lambda_2 \) small enough so that

\[\begin{align*}
&\left\{ \frac{15}{16} - \frac{\lambda_2 \kappa}{q_* - \varepsilon} \frac{\alpha}{2} > 0, \\
&\frac{EI}{2} - \lambda_2 \left[ C_1 + \frac{(q_* - \varepsilon) EIC_\alpha}{2(1 - \kappa)} \right] \right\} > 0, \\
&\frac{1}{2} - \lambda_2 C_2 > 0.
\end{align*}\]

Therefore we arrive at

\[\mathcal{L}'(t) \leq - \lambda_2 (q_* - \varepsilon) \left( \frac{11}{16} - K^2 \eta_1 \right) \xi^2_x (0, t)\]

\[\begin{align*}
&- \frac{\varepsilon}{2} \lambda_2 \left[ \rho \| \xi_x \|_2^2 + m_p \xi^2_x (L, t) + J_p \xi^2_x (L, t) \right] - \frac{\beta}{2} \phi(t) \\
&- \frac{EI}{2} \lambda_2 \left[ (1 - \kappa) - (2 - \kappa) \right] \| \xi_x \|_2^2 + \frac{\alpha EI}{2} \int_0^L (q \phi \xi_x) \, dx \\
&- \left\{ K \left( 1 - \frac{\lambda_2}{2} \right) + \frac{\lambda_2 I_h \varepsilon}{2} - \frac{\lambda_2 (q_* - \varepsilon)}{4 \eta_1} - \frac{8}{(q_* - \varepsilon) \lambda_2} \right. \\
&\left. - 4 \varepsilon - \frac{27 \beta}{8} \right\} \xi^2_{xx} (0, t) - \left( \frac{15}{16} - \frac{\lambda_2 \kappa}{q_* - \varepsilon} - \frac{\alpha}{2} \right) (q \phi \xi_x), \quad t \geq 0.
\end{align*}\]

This completes the proof. \( \square \)

The main result of the present work is to establish the general decay rate of the energy, which is given by the following theorem.

**Theorem 4.7.** Assume that (A_1) and (A_2) hold. Then there exist positive constants \( k_1 \leq 1 \) and \( k_2 \) such that the energy functional satisfies

\[E(t) \leq k_2 H^{-1}_1 \left( k_1 \int_{q^{-1}(t)}^t \zeta(s) \, ds \right), \quad (4.35)\]

where \( H_1(t) = \int_{q^{-1}(t)}^t \frac{1}{\zeta(s)} \, ds \). Here, \( H_1 \) is strictly decreasing and convex on \([0, r]\), with \( \lim_{t \to 0} H_1(t) = +\infty \).

**Proof.** We start by using (2.10), (2.18), (4.26) and (4.29) to conclude that, for any \( t \geq t_1 \),

\[\begin{align*}
&\int_0^L \int_0^{t_1} q(s) \left( \xi_{xx} (t) - \xi_{xx} (t - s) \right)^2 \, ds \, dx \\
&+ \int_0^{t_1} q(s) \left( \xi_x (0, t) - \xi_x (0, t - s) \right)^2 \, ds + \xi_{xx} (0, t) \, \xi_x (0, t) \\
&\leq \frac{1}{\zeta(t_1)} \int_0^L \int_0^{t_1} \zeta(s) q(s) \left( \xi_{xx} (t) - \xi_{xx} (t - s) \right)^2 \, ds \, dx \\
&+ \frac{1}{\zeta(t_1)} \int_0^{t_1} \zeta(s) q(s) \left( \xi_x (0, t) - \xi_x (0, t - s) \right)^2 \, ds + \xi_{xx} (0, t) \, \xi_x (0, t) \\
&\leq - \frac{q(0)}{a \zeta(t_1)} \int_0^L \int_0^{t_1} q'(s) \left( \xi_{xx} (t) - \xi_{xx} (t - s) \right)^2 \, ds \, dx
\end{align*}\]
Now, we choose $\eta$. Therefore, we have very large so that

\[- \frac{q(0)}{a \zeta(t)} \int_0^t q(s) (\xi_x(0, t) - \xi_x(0, t - s))^2 ds + \xi_{x\xi}(0, t) \xi_x(0, t) \]

\[\leq - \frac{q(0)}{a \zeta(t)} \int_0^t q(s) \int_0^L (\xi_{xx}(t) - \xi_{xx}(t - s))^2 dsdx + \xi_{xt}(0, t) \xi_x(0, t) \]

\[- \frac{q(0)}{a \zeta(t)} \int_0^t q'(s) (\xi_x(0, t) - \xi_x(0, t - s))^2 ds + \frac{15}{16} (q \xi_{xx}) \]

\[+ \frac{\beta}{2} \psi_4(t) + \left[ K - 4\kappa - \frac{2\eta_3}{\beta} \right] \xi_{x\xi}(0, t) + \frac{EI}{2} q(t) \|\xi_{xx}\|^2. \]

Making use of (4.37), we arrive at

\[\int_0^t q(s) \int_0^L (\xi_{xx}(t) - \xi_{xx}(t - s))^2 dxds + \xi_{xt}(0, t) \xi_x(0, t) \]

\[+ \int_0^t q(s) (\xi_x(0, t) - \xi_x(0, t - s))^2 ds \]

\[\leq - c \left( \mathcal{E}' + \psi_3' + \psi_4' \right) (t). \quad (4.36) \]

Now, adding and subtracting $2\xi_{xt}(0, t) \xi_x(0, t)$ and $(q \xi_{xx})$ in the right-hand side of (4.33) and using (4.28), we obtain

\[\mathcal{L}'(t) \leq - \left[ \sigma_1 - \frac{\lambda_2 (q_\ast - \varepsilon)}{16} \right] \xi_x^2 (0, t) - \sigma_2 \left( \rho \|\xi_t\|^2 + m_p \xi_t^2 \right) - \sum \|\xi_{xx}\|^2 \frac{\beta}{2} \psi_4(t) \]

\[- \sigma_3 \xi_{x\xi}(0, t) - (\sigma_4 - \frac{4}{\lambda_2 (q_\ast - \varepsilon)}) \xi_{x\xi}(0, t) - (\sigma_5 + 1) (q \xi_{xx}) \]

\[+ \frac{\alpha EI}{2} \int_0^L (q \xi_{xx}) dx + (q \xi_{xx}) + 2 \xi_{xt}(0, t) \xi_x(0, t) \quad t \geq 0. \]

Using (4.34), we get

\[\mathcal{L}'(t) \leq - \lambda_2 (q_\ast - \varepsilon) \left( \frac{5}{8} - K^2 \eta_1 \right) \xi_x^2 (0, t) - \sum \|\xi_{xx}\|^2 \frac{\beta}{2} \psi_4(t) \]

\[- \sigma_2 \left( \rho \|\xi_t\|^2 + m_p \xi_t^2 \right) - (\sigma_5 + 1) (q \xi_{xx}) \]

\[- \left( K \left( 1 - \frac{\lambda_2}{2} \right) + \frac{\lambda_2 I_{h\varepsilon}}{2} - \frac{\lambda_2 (q_\ast - \varepsilon)}{4 \eta_1} \right)^2 \xi_{x\xi}(0, t) \]

\[+ \frac{\alpha EI}{2} \int_0^L (q \xi_{xx}) dx + (q \xi_{xx}) + 2 \xi_{xt}(0, t) \xi_x(0, t) \quad t \geq 0. \]

Now, we choose $\eta_1 = \frac{1}{n^2}$ so that $\lambda_2 (q_\ast - \varepsilon) \left( \frac{5}{8} - K^2 \eta_1 \right) > 0$. After that, we pick $K$ very large so that

\[K \left( 1 - \frac{\lambda_2}{2} \right) + \frac{\lambda_2 I_{h\varepsilon}}{2} - \frac{\lambda_2 (q_\ast - \varepsilon)}{4 \eta_1} - \frac{12}{(q_\ast - \varepsilon) \lambda_2} - 4\kappa - \frac{2\eta_3}{\beta} > 0. \quad (4.37) \]

Therefore, we have

\[\mathcal{L}'(t) \leq - m (E + \psi_3 + \psi_4) (t) + c \left( \int_0^L (q \xi_{xx}) dx + (q \xi_{xx}) + 2 \xi_{xt}(0, t) \xi_x(0, t) \right) \]

\[\leq - m (E + \psi_3 + \psi_4) (t) - c (\mathcal{E}' + \psi_3' + \psi_4') (t) \]

\[+ c \int_0^t q(s) \int_0^L (\xi_{xx}(t) - \xi_{xx}(t - s))^2 dxds \].
where (4.40) allows for a constant $0 < \mu < 1$ chosen so that, for all $t \geq t_1$, 

$$0 < I(t) < 1.$$  

(4.41)
Also, we define \( \lambda(t) \) by

\[
\lambda(t) := \int_{t_1}^{t} q'(s) \int_{0}^{L} (\xi_{xx}(t) - \xi_{xx}(t-s))^2 dx ds + \int_{t_1}^{t} q'(s) (\xi_x(0,t) - \xi_x(0,t-s))^2 ds.
\]

Using (2.18), (4.26) and (4.29), we get

\[
\lambda E(t) := \frac{EI}{2} \left[ 1 + \left( \int_{0}^{L} (q' \xi_{xx} + \frac{1}{2} q' \xi_x) - \left[ K + 4\kappa - \frac{2q \beta}{\beta} \right] \xi_x^2(0,t) \right) \right] \psi_4(t) - \frac{EI}{2} q(t) \| \xi_{xx} \|^2_2 - \frac{\beta}{2} \psi_4(t) - \frac{15}{16} (q \xi_x), \quad t \geq 0.
\]

Therefore,

\[
(E' + \psi_3 + \psi_4)(t) \leq \frac{EI}{2} \int_{0}^{L} (q' \xi_{xx}) dx + \frac{1}{2} (q' \xi_x) \leq 0,
\]

which implies that

\[
\lambda(t) \leq -c (E' + \psi_3 + \psi_4)(t).
\]

Since \( H \) is strictly convex on \([0, r]\) and \( H(0) = 0 \), then

\[
H(Sx) \leq SH(x)
\]

provided \( 0 < S < 1 \) and \( x \in [0, r] \). The use of this fact, hypothesis \( (A_1) \), (4.41) and Jensen’s inequality leads to

\[
\lambda(t) = \frac{1}{\mu} \left[ \frac{1}{I(t)} \int_{t_1}^{t} I(t)(-q'(s)) \int_{0}^{L} \mu (\xi_{xx}(t) - \xi_{xx}(t-s))^2 dx ds \right.
\]

\[
\left. + \frac{1}{I(t)} \int_{t_1}^{t} \mu I(t)(-q'(s))(\xi_x(0,t) - \xi_x(0,t-s))^2 ds \right]
\]

\[
\geq \frac{1}{\mu} \left[ \frac{1}{I(t)} \int_{t_1}^{t} I(t) \xi_x(s) H(q(s)) \int_{0}^{L} \mu (\xi_{xx}(t) - \xi_{xx}(t-s))^2 dx ds \right.
\]

\[
\left. + \frac{1}{I(t)} \int_{t_1}^{t} \mu I(t) \xi_x(s) H(q(s))(\xi_x(0,t) - \xi_x(0,t-s))^2 ds \right]
\]

\[
\geq \frac{\zeta(t)}{\mu I(t)} \int_{t_1}^{t} H(I(t)q(s)) \int_{0}^{L} \mu (\xi_{xx}(t) - \xi_{xx}(t-s))^2 dx ds
\]

\[
+ \frac{\zeta(t)}{I(t)} \int_{t_1}^{t} \mu H(I(t)q(s))(\xi_x(0,t) - \xi_x(0,t-s))^2 ds,
\]

so,

\[
\lambda(t) \geq \frac{\zeta(t)}{\mu I(t)} \int_{t_1}^{t} H(I(t)q(s)) \mu \left( \int_{0}^{L} (\xi_{xx}(t) - \xi_{xx}(t-s))^2 dx \right.
\]

\[
+ (\xi_x(0,t) - \xi_x(0,t-s))^2 \right) ds
\]

\[
\geq \frac{\zeta(t)}{q I(t)} H \left( \int_{t_1}^{t} I(t)q(s) \mu \left( \int_{0}^{L} (\xi_{xx}(t) - \xi_{xx}(t-s))^2 dx \right.
\]

\[
+ (\xi_x(0,t) - \xi_x(0,t-s))^2 \right) ds \right)\]
where \( \bar{H} \) is in extension of \( H \) such that \( \bar{H} \) is strictly increasing and strictly convex \( C^2 \)-function on \([0, \infty[\).

This implies that

\[
\int_{t_1}^{t} q(s) \int_{0}^{L} (\xi_{xx}(t) - \xi_{xx}(t-s))^2 \, dx \, ds + \int_{t_1}^{t} q(s) (\xi_x(0, t) - \xi_x(0, t-s))^2 \, ds \\
\leq \frac{1}{\mu} \bar{H}^{-1} \left( \frac{q\lambda(t)}{\zeta(t)} \right),
\]

and (4.38) becomes

\[
F'(t) \leq -m (\mathcal{E} + \psi_3 + \psi_4) (t) + c\bar{H}^{-1} \left( \frac{\mu\lambda(t)}{\zeta(t)} \right), \quad t \geq t_1. \tag{4.44}
\]

For \( \varepsilon_0 < r \), using (4.43), (4.44) and the fact that \( \bar{H}' > 0 \) and \( \bar{H}'' > 0 \), we find that the functional \( F_1 \), defined by

\[
F_1(t) := \bar{H} \left( \varepsilon_0 \left( \frac{\mathcal{E} + \psi_3 + \psi_4}{\mathcal{E} + \psi_3 + \psi_4} \right) (t) \right) F(t) + \mathcal{E}(t) + \psi_3(t) + \psi_4(t)
\]

is equivalent to \( (\mathcal{E} + \psi_3 + \psi_4) (t) \) and

\[
F'_1(t) = \varepsilon_0 \left( \frac{\mathcal{E} + \psi_3 + \psi_4}{\mathcal{E} + \psi_3 + \psi_4} \right) (t) \bar{H}' \left( \frac{\varepsilon_0 \left( \frac{\mathcal{E} + \psi_3 + \psi_4}{\mathcal{E} + \psi_3 + \psi_4} \right) (t)}{\mathcal{E} + \psi_3 + \psi_4} \right) F(t) \\
+ \bar{H}' \left( \varepsilon_0 \left( \frac{\mathcal{E} + \psi_3 + \psi_4}{\mathcal{E} + \psi_3 + \psi_4} \right) (t) \right) F'(t) + \mathcal{E}'(t) + \psi'_3(t) + \psi'_4(t) \\
\leq -m (\mathcal{E} + \psi_3 + \psi_4) (t) \bar{H} \left( \varepsilon_0 \left( \frac{\mathcal{E} + \psi_3 + \psi_4}{\mathcal{E} + \psi_3 + \psi_4} \right) (t) \right) \\
+ c\bar{H} \left( \varepsilon_0 \left( \frac{\mathcal{E} + \psi_3 + \psi_4}{\mathcal{E} + \psi_3 + \psi_4} \right) (t) \right) \bar{H}^{-1} \left( \frac{\mu\lambda(t)}{\zeta(t)} \right) + \mathcal{E}'(t) + \psi'_3(t) + \psi'_4(t). \tag{4.45}
\]

Let \( \bar{H}^\ast \) be the convex conjugate of \( \bar{H} \) in the sense of Young, then

\[
\bar{H}^\ast(s) = s \left( \bar{H}^\ast \right)^{-1} (s) - \bar{H} \left[ \left( \bar{H}^\ast \right)^{-1} (s) \right] \tag{4.46}
\]
and $H^*$ satisfies the following Young’s inequality
\[
AB \leq H^*(A) + H(B).
\] (4.47)

With $A = H^* \left( \varepsilon_0 \frac{(E + \psi_3 + \psi_4)(t)}{E + \psi_3 + \psi_4}(0) \right)$ and $B = H^{-1} \left( \frac{m(t)}{c(t)} \right)$, using (2.18), (4.25), (4.29) and (4.45)-(4.47), we arrive at
\[
F'_1(t) \leq -m (E + \psi_3 + \psi_4)(t) H' \left( \varepsilon_0 \frac{(E + \psi_3 + \psi_4)(t)}{E + \psi_3 + \psi_4}(0) \right) \]
\[
+ cH^* \left( \left( \varepsilon_0 \frac{(E + \psi_3 + \psi_4)(t)}{E + \psi_3 + \psi_4}(0) \right) \right) + c\mu \lambda(t) + E'(t) + \psi'_3(t) + \psi'_4(t)
\]
\[
\leq -m (E + \psi_3 + \psi_4)(t) H' \left( \varepsilon_0 \frac{(E + \psi_3 + \psi_4)(t)}{E + \psi_3 + \psi_4}(0) \right) + c\mu \lambda(t) + E'(t) + \psi'_3(t) + \psi'_4(t). \tag{4.48}
\]
Then, we multiply by $\zeta(t)$ and use the fact that, as
\[
\varepsilon_0 \frac{(E + \psi_3 + \psi_4)(t)}{E + \psi_3 + \psi_4}(0) < r, \quad H' \left( \varepsilon_0 \frac{(E + \psi_3 + \psi_4)(t)}{E + \psi_3 + \psi_4}(0) \right) = \zeta(t) \left( \varepsilon_0 \frac{(E + \psi_3 + \psi_4)(t)}{E + \psi_3 + \psi_4}(0) \right),
\]
\[
\zeta(t) F'_1(t) \leq -m \zeta(t) (E + \psi_3 + \psi_4)(t) H' \left( \varepsilon_0 \frac{(E + \psi_3 + \psi_4)(t)}{E + \psi_3 + \psi_4}(0) \right)
\]
\[
+ c\varepsilon_0 \frac{(E + \psi_3 + \psi_4)(t)}{E + \psi_3 + \psi_4}(0) \zeta(t) H' \left( \varepsilon_0 \frac{(E + \psi_3 + \psi_4)(t)}{E + \psi_3 + \psi_4}(0) \right) + c\mu \lambda(t) + \zeta(t) (E' + \psi'_3 + \psi'_4)(t)
\]
\[
\leq -m \zeta(t) (E + \psi_3 + \psi_4)(t) H' \left( \varepsilon_0 \frac{(E + \psi_3 + \psi_4)(t)}{E + \psi_3 + \psi_4}(0) \right) + c\varepsilon_0 (E + \psi_3 + \psi_4)(t) H' \left( \varepsilon_0 \frac{(E + \psi_3 + \psi_4)(t)}{E + \psi_3 + \psi_4}(0) \right)
\]
\[
+ c\varepsilon_0 \frac{(E + \psi_3 + \psi_4)(t)}{E + \psi_3 + \psi_4}(0) \zeta(t) H' \left( \varepsilon_0 \frac{(E + \psi_3 + \psi_4)(t)}{E + \psi_3 + \psi_4}(0) \right) - c (E' + \psi'_3 + \psi'_4)(t) \tag{4.49}
\]
Consequently, with $F_2(t) = \zeta(t) F_1(t) + c (E + \psi_3 + \psi_4)(t)$, which satisfies, for some $\beta_1, \beta_2 > 0$,
\[
\beta_1 F_2(t) \leq (E + \psi_3 + \psi_4)(t) \leq \beta_2 F_2(t),
\]
and with a suitable choice of $\varepsilon_0$, we obtain, for some constant $k > 0$ and for all $t \geq t_1$,
\[
F'_2(t) \leq -k \zeta(t) \left( \frac{(E + \psi_3 + \psi_4)(t)}{E + \psi_3 + \psi_4}(0) \right) H' \left( \varepsilon_0 \frac{(E + \psi_3 + \psi_4)(t)}{E + \psi_3 + \psi_4}(0) \right) \]
\[
= -k \zeta(t) H_2 \left( \frac{(E + \psi_3 + \psi_4)(t)}{E + \psi_3 + \psi_4}(0) \right), \tag{4.49}
\]
where $H_2(t) = tH'(\varepsilon_0 t)$. Since $H'_2(t) = H'(\varepsilon_0 t) + \varepsilon_0 t H''(\varepsilon_0 t)$, then, using the strict convexity of $H$ on $[0, r]$, we find that $H'_2(t), H_2(t) > 0$ on $[0, 1]$. Thus, with
\[
R(t) = \frac{\beta_1 F_2(t)}{(E + \psi_3 + \psi_4)(0)},
\]
taking in account (4.48) and (4.49), we have
\[
R(t) \sim (E + \psi_3 + \psi_4)(t) \tag{4.50}
\]
and, for some $k_1 > 0$,

$$R'(t) \leq -k_1 \zeta(t) H_2(R(t)), \quad t \geq t_1.$$ 

Then, the integration over $[t_1, t]$ yields

$$\int_{t_1}^{t} \frac{-R'(s)}{H_2(R(s))} ds \geq k_1 \int_{t_1}^{t} \zeta(s) ds \Rightarrow \int_{t_0}^{R(t_1)} \frac{1}{sH'(s)} ds \geq k_1 \int_{t_1}^{t} \zeta(s) ds,$$

so,

$$R(t) \leq \frac{1}{k_1} H^{-1}_1 \left( k_1 \int_{t_1}^{t} \zeta(s) ds \right), \quad (4.51)$$

where $H_1(t) = \int_{t}^{t_0} \frac{1}{sH'(s)} ds$. Here, we have used, based on the properties of $H_1$ is strictly decreasing function on $[0, r]$ and $\lim_{t \to 0} H_1(t) = +\infty$. A combination of (4.50) and (4.51), estimate (4.35) is established.

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