A holomorphic vertex operator algebra of central charge 24 whose weight one Lie algebra has type $A_{6,7}$

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Abstract. In this article, we describe a construction of a holomorphic vertex operator algebras of central charge 24 whose weight one Lie algebra has type $A_{6,7}$.

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1. Introduction

In 1993, Schellekens [19] obtained a list of 71 possible Lie algebra structures for the weight one subspaces of holomorphic vertex operator algebras (VOAs) of central charge 24. However, only 39 of the 71 cases in his list had been constructed explicitly at that time. In recent years, many new holomorphic VOAs of central charge 24 have been constructed. In [9,10], 17 holomorphic VOAs were constructed using the theory of framed VOAs. In addition, three holomorphic VOAs were constructed in [15,18] using $\mathbb{Z}_3$-orbifold constructions associated with lattice VOAs. Recently, van Ekeren, Møller and Scheithauer [7] have established the general $\mathbb{Z}_n$-orbifold construction for elements of arbitrary orders. In particular, constructions of five holomorphic VOAs were discussed. In [11], five other holomorphic VOAs were constructed using $\mathbb{Z}_2$-orbifold constructions associated with inner automorphisms. Based on these results, there are two remaining cases in Schellekens’ list which have not been discussed yet. The corresponding Lie algebras have type $A_{6,7}$ and $F_{4,6}A_{2,2}$.
In this article, we will describe a construction of a holomorphic VOA of central charge 24 whose weight one Lie algebra has type $A_{6,7}$. Since the level of $A_{6,7}$ is 7, it is natural to hope that a $\mathbb{Z}_7$-orbifold construction will bear fruit. It is indeed correct and we will show that the desired VOA can be constructed by applying the $\mathbb{Z}_7$-orbifold construction to the Leech lattice VOA $V_\Lambda$ and an order 7 automorphism of $V_\Lambda$; however, the choice of the automorphism is somewhat tricky. It is the product of (a lift of) an order 7 isometry of $\Lambda$ and an order 7 inner automorphism of $V_\Lambda$. Combining the explicit construction of the twisted $V_\Lambda$-modules for an isometry of $\Lambda$ in [4] and the modification by Li’s $\Delta$-operator in [14], we obtain the irreducible twisted $V_\Lambda$-modules for the product of these two order 7 automorphisms. Using some explicit information, we will then show that the weight one subspace of the resulting orbifold VOA has dimension 48 and it is a Lie algebra of type $A_6$.

Let us explain the two automorphisms in more detail. It is known (see [2]) that the Leech lattice has exactly two conjugacy classes of isometries of order 7. One class acts fixed-point freely on $\Lambda$ and the other class acts on $\Lambda$ with trace 3. By the explicit construction given in [4], for an element in the former (resp. latter) class, the irreducible twisted $V_\Lambda$-module has lowest $L(0)$ weight $8/7$ (resp. $6/7$) and the resulting orbifold VOA will have a trivial (resp. 24-dimensional) weight one subspace. In either case, the weight one subspace is too small. The main trick for our construction is to modify the irreducible $\tau$-twisted $V_\Lambda$-module associated with an order 7 isometry $\tau$ with trace 3 by Li’s $\Delta$-operator (cf. Proposition 2.2), so that the resulting irreducible twisted $V_\Lambda$-module has lowest $L(0)$-weight 1. It is equivalent to find a vector $f \in (1/7)\Lambda$ such that $f$ is fixed by $\tau$ and $(f|f) = 2/7$ (see (3.3)). Then, if we set $g = \sigma_f \cdot \tau$, where $\sigma_f$ is the inner automorphism associated with $f$, then $g$ is the desired order 7 automorphism of $V_\Lambda$ and the modified module is an irreducible $g$-twisted $V_\Lambda$-module.

Our argument is, in fact, very similar to that in [11, Section 10], in which a construction of a holomorphic VOA of central charge 24 such that the corresponding weight one Lie algebra has type $A_{2,5}^2$ is discussed.

The organization of this article is as follows. In Section 2, we recall some basic facts about VOAs and review Li’s $\Delta$-operator. In Section 3, we review some basic facts about the isometry group of the Leech lattice $\Lambda$ and describe an order 7 isometry of $\Lambda$. In addition, we find a suitable vector $f \in \Lambda$ and prove some lemmas about lattices. In Section 4, we discuss a construction of a holomorphic vertex operator algebra of central charge 24 such that the weight one Lie algebra has type $A_{6,7}$.

2. Preliminary

In this section, we will review some fundamental results about VOAs. We adopt the terminology and notation used in [11].
Let $V$ be a VOA of CFT-type. Then, the weight one space $V_1$ has a Lie algebra structure via the 0-th mode, which we often call the weight one Lie algebra of $V$. Moreover, the $n$-th modes $v(n)$, $v \in V_1$, and $n \in \mathbb{Z}$, define an affine representation of the Lie algebra $V_1$ on $V$. For a simple Lie subalgebra $\mathfrak{a}$ of $V_1$, the level of $\mathfrak{a}$ is defined to be the scalar by which the canonical central element acts on $V$ as the affine representation. When the type of the root system of $\mathfrak{a}$ is $X_n$ and the level of $\mathfrak{a}$ is $k$, we denote the type of $\mathfrak{a}$ by $X_n,k$.

**Proposition 2.1** ([6, (1.1), Theorem 3 and Proposition 4.1]). Let $V$ be a strongly regular, holomorphic VOA of central charge 24. If the Lie algebra $V_1$ is neither $\{0\}$ nor abelian, then $V_1$ is semisimple, and the conformal vectors of $V$ and the subVOA generated by $V_1$ are the same. In addition, for any simple ideal of $V_1$ at level $k$, the identity

$$h^\vee = \frac{\dim V_1 - 24}{24}$$

holds, where $h^\vee$ is the dual Coxeter number.

Let $V$ be a VOA of CFT-type. Let $h \in V_1$ such that $h(0)$ acts semisimply on $V$. We also assume that $\text{Spec } h(0) \subset (1/T)\mathbb{Z}$ for some positive integer $T$, where $\text{Spec } h(0)$ denotes the set of spectra of $h(0)$ on $V$. Let $\sigma_h = \exp(-2\pi \sqrt{-1} h(0))$ be the (inner) automorphism of $V$ associated with $h$. Then by the assumption on $\text{Spec } h(0)$, we have $\sigma_h^T = 1$. Let $\Delta(h, z)$ be Li’s $\Delta$-operator defined in [14], i.e.,

$$\Delta(h, z) = z^{h(0)} \exp \left( \sum_{n=1}^{\infty} \frac{h(n)}{-n} (-z)^{-n} \right).$$

**Proposition 2.2** ([14, Proposition 5.4]). Let $\sigma$ be an automorphism of $V$ of finite order and let $h \in V_1$ be as above such that $\sigma(h) = h$. Let $(M, Y_M)$ be a $\sigma$-twisted $V$-module and define $(M^{(h)}, Y_M^{(h)}(\cdot, z))$ as follows:

- $M^{(h)} = M$ as a vector space;
- $Y_M^{(h)}(a, z) = Y_M(\Delta(h, z)a, z)$ for any $a \in V$.

Then, $(M^{(h)}, Y_M^{(h)}(\cdot, z))$ is a $\sigma h \sigma$-twisted $V$-module. Furthermore, if $M$ is irreducible, then so is $M^{(h)}$.

Assume that $V$ is self-dual. Then there exists a symmetric invariant bilinear form $\langle \cdot | \cdot \rangle$ on $V$, which is unique up to scalar ([13]). We normalize it so that $\langle 1 | 1 \rangle = -1$. Then for $a, b \in V_1$, we have $\langle a | b \rangle 1 = a(1)\overline{b}$. 
For a $\sigma$-twisted $V$-module $M$ and $a \in V$, we denote by $a^{(h)}_{(i)}$ the operator which corresponds to the coefficient of $z^{-i-1}$ in $Y_{M^{(h)}}(a, z)$, i.e.,

$$Y_{M^{(h)}}(a, z) = \sum_{i \in \mathbb{Z}} a^{(h)}_{(i)} z^{-i-1} \quad \text{for} \quad a \in V.$$ 

The 0-th mode of an element $x \in V_1$ on $M^{(h)}$ is given by

$$x_{(0)}^{(h)} = x_{(0)} + \langle h | x \rangle \text{id}. \quad (2.1)$$

Let us denote by $L^{(h)}(n)$ the $(n + 1)$-th mode of the conformal vector $\omega \in V$ on $M^{(h)}$. Then the $L(0)$-weights on $M^{(h)}$ are given by

$$L^{(h)}(0) = L(0) + h_{(0)} + \frac{\langle h | h \rangle}{2} \text{id}. \quad (2.2)$$

3. Leech Lattice and its Isometry Group

We review some notions and certain basic properties of the Leech lattice $\Lambda$ and its isometry group $O(\Lambda)$, which is also known as $Co_0$, a perfect group of order $2^{22} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$.

3.1. Hexacode Balance and the Leech Lattice

Let $\Omega = \{1, 2, 3, \ldots, 24\}$ be a set of 24 elements and let $G$ be the (extended) Golay code of length 24 indexed by $\Omega$. A subset $S \subset \Omega$ is called a $G$-set if $S = \text{supp} \alpha$ for some codeword $\alpha \in G$. We will identify a $G$-set with the corresponding codeword in $G$. A sextet is a partition of $\Omega$ into six 4-element sets of which the union of any two forms a $G$-set.

For explicit calculations, we use the notion of hexacode balance to denote the codewords of the Golay code and the vectors in the Leech lattice (cf. [3, Chapter 11] and [8, Chapter 5]). First, we arrange the set $\Omega$ into a $4 \times 6$ array, such that the six columns form a sextet. For each codeword in $G$, 0 and 1 are marked by a blanked and non-blanked space, respectively, at the corresponding positions in the array. For example, $(1^80^6)$ is denoted by the array

\[
\begin{array}{cccccc}
* & * & & & & \\
* & * & & & & \\
* & * & & & & \\
* & * & & & & \\
\end{array}
\]  

(3.1)

Let $(\cdot | \cdot)$ be the inner product of $\mathbb{R}^{24}$ and let $e_1, e_2, \ldots, e_{24} \in \mathbb{R}^{24}$ be an orthogonal basis of squared norm 2, that is, $(e_i | e_j) = 2 \delta_{i,j}$ for $1 \leq i, j \leq 24$. Denote $e_X := \sum_{i \in X} e_i$ for $X \in G$. The following is a standard construction of the Leech lattice.
DEFINITION 3.1 ([3]). The (standard) Leech lattice \( \Lambda \) is a lattice of rank 24 generated by the vectors:

\[
\begin{align*}
\frac{1}{2}e_X, & \quad \text{where } X \text{ is a generator of the Golay code } G; \\
\frac{1}{4}e_{\Omega} - e_1; \\
e_i \pm e_j, & \quad i, j \in \Omega.
\end{align*}
\]

Remark 3.2. By arranging the set \( \Omega \) into a \( 4 \times 6 \) array, every vector in the Leech lattice \( \Lambda \) can be written in the form

\[X = \frac{1}{\sqrt{8}} [X_1 X_2 X_3 X_4 X_5 X_6], \quad \text{juxtaposition of column vectors.}\]

For example,

\[
\begin{array}{cccccc}
2 & 2 & 0 & 0 & 0 & 0 \\
2 & 2 & 0 & 0 & 0 & 0 \\
2 & 2 & 0 & 0 & 0 & 0 \\
2 & 2 & 0 & 0 & 0 & 0
\end{array}
\]

denotes the vector \( \frac{1}{2} e_{(18016)} \), where \( (18016) \) is the codeword corresponding to (3.1).

3.2. AN ISOMETRY OF THE LEECH LATTICE OF ORDER 7

In this subsection, we study a certain automorphism \( \tau \) of \( \Lambda \) of order 7.

By the character table (cf. [2, Page 184]), there exist exactly two conjugacy classes of elements of order 7 in \( O(\Lambda) \). Let \( \tau \) be an isometry of \( \Lambda \) of order 7 such that the trace of \( \tau \) on \( \Lambda \) is 3. Such elements form a unique conjugacy class in \( O(\Lambda) \) and the set of fixed points of \( \tau \) in \( \Lambda \) is a sublattice of rank 6. For the simplicity of calculation, we fix \( \tau \) such that (see [3, Figure 11.21])

\[
\tau = 
\begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots 
\end{array}
\]

Let \( \mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} \Lambda \). We extend the form \( \langle \cdot | \cdot \rangle \) to \( \mathfrak{h} \mathbb{C}\)-bilinearly. We also extend the isometry \( \tau \) to \( \mathfrak{h} \mathbb{C}\)-linearly. Let \( \mathfrak{h}_{(0)} \) be the fixed-point subspace of \( \tau \) in \( \mathfrak{h} \) and let \( P_0 \) be the orthogonal projection from \( \mathfrak{h} \) to \( \mathfrak{h}_{(0)} \).

LEMMA 3.3. Let \( M = ((1 - P_0)h) \cap \Lambda \ and \ r \in \{1, 2, \ldots, 6\}. Then, \ M = (1 - \tau^r)\Lambda. \)

Proof. We note that \( (1 - P_0)h = h_{(0)}^\perp = \{ y \in \mathfrak{h} | \langle y | x \rangle = 0 \text{ for all } x \in \mathfrak{h}_{(0)} \} \). Hence, \( M = \{ \alpha \in \Lambda | \langle \alpha | x \rangle = 0 \text{ for all } x \in \mathfrak{h}_{(0)} \} \). It is clear that \( (1 - \tau^r)\Lambda \subset M \), since \( (1 - \tau^r)\Lambda \) is orthogonal to \( \mathfrak{h}_{(0)} \).
By the definition of $\tau$, we also know that the fixed-point subspace $h_0$ of $\tau^r$ is given by

$$\begin{bmatrix} a_0 & b_0 & b & c & c \\ a & a & b & b & c & c \\ a & a & b & b & c & c \\ a & a & b & b & c & c \end{bmatrix} a_0, b_0, c, a, b, c \in \mathbb{C}$$

and $M$ is given by

$$\begin{bmatrix} 0 & a_1 & 0 & b_1 & 0 & c_1 \\ a_2 & a_3 & b_2 & b_3 & c_2 & c_3 \\ a_4 & a_5 & b_4 & b_5 & c_4 & c_5 \\ a_6 & a_7 & b_6 & b_7 & c_6 & c_7 \end{bmatrix} \in \Lambda \quad \text{such that } \quad \sum_{i=1}^7 a_i = \sum_{i=1}^7 b_i = \sum_{i=1}^7 c_i = 0.$$

Note that the indexes in $\Omega$ corresponding to $a_0, b_0, c_0$ are 1, 9, and 17, respectively.

Let $G_0 = \{(c_1, \ldots, c_{24}) \in G \mid c_1 = c_9 = c_{17} = 0\}$. Then the dimension of $G_0$ is 9. Clearly, $(1 - \tau^r)G \subset G_0$. Using explicit generators of $G$ (see for example [8, (5.35)]), it is straightforward to show that $(1 - \tau^r)G$ is also a 9-dimensional subcode of $G_0$ and hence we have

$$(1 - \tau^r)G = G_0.$$  \hspace{1cm} (3.2)

For $Y \subset \Omega$, let $\varepsilon_Y$ denote the involution in $O(\Lambda)$ that acts by $-1$ on the coordinates corresponding to the elements in $Y$. Let $E^1, E^2$ and $E^3$ be the $G$-sets corresponding to the codewords $(1^8, 0^8, 0^8)$, $(0^8, 1^8, 0^8)$, and $(0^8, 0^8, 1^8)$, respectively.

By the generators of $\Lambda$ given in Definition 3.1, $M$ is generated by

$$\varepsilon_Y \frac{1}{2} e_X, \text{ where } X \text{ is a generator of the code } G_0$$

and $Y \subset \Omega$ with $|X \cap Y \cap E^i| = |X \cap E^i|/2$ for $i = 1, 2, 3$;

$$e_i - e_j, \text{ where } i, j \in E^1 \setminus \{1\}, \quad i, j \in E^2 \setminus \{9\} \text{ or } \quad i, j \in E^3 \setminus \{17\}.$$

By (3.2), one can easily see that $M \subset (1 - \tau^r)\Lambda$. Thus, $M = (1 - \tau^r)\Lambda$ as desired.

3.3. CALCULATIONS IN $P_0(\Lambda)$

In this subsection, we prove some lemmas about $P_0(\Lambda)$, which will be used later. Recall that $P_0$ is the orthogonal projection from $h$ to $h_0$. 


LEMMA 3.4. The lattice \( P_0(\Lambda) \) is spanned over \( \mathbb{Z} \) by the following vectors:

\[
\begin{array}{cccc}
1/7\sqrt{8} & 0 & 4 & 0 \\
& 4 & 4 & 4 \\
& 4 & 4 & 4 \\
& 4 & 4 & 4 \\
\end{array} \quad \begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 4 & 0 & 0 \\
0 & 0 & 4 & 0 \\
0 & 0 & 4 & 0 \\
\end{array}
\]

\[
\begin{array}{cccc}
1/7\sqrt{8} & 0 & 0 & -4 \\
& 4 & 4 & -4 \\
& 4 & 4 & -4 \\
& 4 & 4 & -4 \\
\end{array} \quad \begin{array}{cccc}
-14 & 2 & 0 & 0 \\
2 & 2 & 0 & 0 \\
2 & 2 & 0 & 0 \\
2 & 2 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{cccc}
1/7\sqrt{8} & 0 & 0 & 2 \\
& 0 & 2 & 2 \\
& 0 & 2 & 2 \\
& 0 & 2 & 2 \\
\end{array} \quad \begin{array}{cccc}
-7 & 1 & -7 & 1 \\
1 & 1 & -3 & 1 \\
1 & 1 & -3 & 1 \\
1 & 1 & -3 & 1 \\
\end{array}
\]

Proof. Note that \( P_0 = \frac{1}{7} \sum_{i=0}^{6} \tau_i \). The result now follows easily by a direct calculation using Definition 3.1 and a basis for the Golay code (cf. [8, (5.35)]). \( \square \)

We now set

\[
f = \frac{1}{7\sqrt{8}} \begin{bmatrix}
5 & 1 & 1 & 1 & 3 & 3 \\
1 & 1 & 1 & 1 & 3 & 3 \\
1 & 1 & 1 & 1 & 3 & 3 \\
1 & 1 & 1 & 1 & 3 & 3 \\
\end{bmatrix} \in \mathfrak{h}(0). \tag{3.3}
\]

Then, \( (f | f) = 2/7 \) and \( f \notin P_0(\Lambda) \).

LEMMA 3.5. For \( r \in \{\pm 1, \pm 2, \pm 3\} \), we set

\[
\mathcal{S}^r = \left\{ a + rf \mid a \in P_0(\Lambda) \text{ and } |a + rf|^2 = \frac{2}{7} \right\}.
\]

Then,

\[
\mathcal{S}^1 = \{ \beta_0, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6 \}, \quad \mathcal{S}^{-1} = -\mathcal{S}^1,
\]

\[
\mathcal{S}^2 = \{ \beta_0 + \beta_1, \beta_1 + \beta_2, \beta_2 + \beta_3, \beta_3 + \beta_4, \beta_4 + \beta_5, \beta_5 + \beta_6, \beta_6 + \beta_0 \}, \quad \mathcal{S}^{-2} = -\mathcal{S}^2,
\]

\[
\mathcal{S}^3 = \{ \beta_0 + \beta_1 + \beta_2, \beta_1 + \beta_2 + \beta_3, \beta_2 + \beta_3 + \beta_4, \ldots, \beta_6 + \beta_0 + \beta_1 \}, \quad \mathcal{S}^{-3} = -\mathcal{S}^3.
\]
where $\beta_1 = f$.

\[
\begin{array}{cccccccc}
\beta_2 &=& \frac{1}{7\sqrt{8}} & \begin{bmatrix}
-2 & -2 & -6 & -2 & -4 & 0 \\
-2 & -2 & -2 & -2 & 0 & 0 \\
-2 & -2 & -2 & -2 & 0 & 0 \\
-2 & -2 & -2 & -2 & 0 & 0 \\
5 & -3 & 1 & 1 & 3 & -1 \\
-3 & -3 & 1 & 1 & -1 & -1 \\
-3 & -3 & 1 & 1 & -1 & -1 \\
-3 & -3 & 1 & 1 & -1 & -1 \\
5 & 1 & -3 & 3 & -1 & -1 \\
1 & 1 & -3 & -3 & -1 & -1 \\
1 & 1 & -3 & -3 & -1 & -1 \\
1 & 1 & -3 & -3 & -1 & -1 \\
\end{bmatrix} \\
\beta_3 &=& \frac{1}{7\sqrt{8}} & \begin{bmatrix}
-2 & 2 & 8 & 0 & -4 & 0 \\
2 & 2 & 0 & 0 & 0 & 0 \\
2 & 2 & 0 & 0 & 0 & 0 \\
2 & 2 & 0 & 0 & 0 & 0 \\
-2 & 2 & -6 & 2 & -4 & 0 \\
2 & 2 & 2 & 2 & 0 & 0 \\
2 & 2 & 2 & 2 & 0 & 0 \\
2 & 2 & 2 & 2 & 0 & 0 \\
-9 & -1 & 1 & 1 & 3 & -1 \\
-1 & -1 & 1 & 1 & -1 & -1 \\
-1 & -1 & 1 & 1 & -1 & -1 \\
-1 & -1 & 1 & 1 & -1 & -1 \\
\end{bmatrix} \\
\beta_4 &=& \frac{1}{7\sqrt{8}} & \begin{bmatrix}
-2 & -2 & -2 & -2 & 0 & 0 \\
-2 & -2 & -2 & -2 & 0 & 0 \\
-2 & -2 & -2 & -2 & 0 & 0 \\
-2 & -2 & -2 & -2 & 0 & 0 \\
5 & -3 & 1 & 1 & 3 & -1 \\
-3 & -3 & 1 & 1 & -1 & -1 \\
-3 & -3 & 1 & 1 & -1 & -1 \\
-3 & -3 & 1 & 1 & -1 & -1 \\
5 & 1 & -3 & 3 & -1 & -1 \\
1 & 1 & -3 & -3 & -1 & -1 \\
1 & 1 & -3 & -3 & -1 & -1 \\
1 & 1 & -3 & -3 & -1 & -1 \\
\end{bmatrix} \\
\beta_5 &=& \frac{1}{7\sqrt{8}} & \begin{bmatrix}
-2 & 2 & -6 & 2 & -4 & 0 \\
2 & 2 & 2 & 2 & 0 & 0 \\
2 & 2 & 2 & 2 & 0 & 0 \\
2 & 2 & 2 & 2 & 0 & 0 \\
-9 & -1 & 1 & 1 & 3 & -1 \\
-1 & -1 & 1 & 1 & -1 & -1 \\
-1 & -1 & 1 & 1 & -1 & -1 \\
-1 & -1 & 1 & 1 & -1 & -1 \\
\end{bmatrix} \\
\beta_6 &=& \frac{1}{7\sqrt{8}} & \begin{bmatrix}
-2 & 2 & 8 & 0 & -4 & 0 \\
2 & 2 & 0 & 0 & 0 & 0 \\
2 & 2 & 0 & 0 & 0 & 0 \\
2 & 2 & 0 & 0 & 0 & 0 \\
-2 & 2 & -6 & 2 & -4 & 0 \\
2 & 2 & 2 & 2 & 0 & 0 \\
2 & 2 & 2 & 2 & 0 & 0 \\
2 & 2 & 2 & 2 & 0 & 0 \\
-9 & -1 & 1 & 1 & 3 & -1 \\
-1 & -1 & 1 & 1 & -1 & -1 \\
-1 & -1 & 1 & 1 & -1 & -1 \\
-1 & -1 & 1 & 1 & -1 & -1 \\
\end{bmatrix} \\
\beta_0 &=& \frac{1}{7\sqrt{8}} & \begin{bmatrix}
-2 & -2 & -2 & -2 & 0 & 0 \\
-2 & -2 & -2 & -2 & 0 & 0 \\
-2 & -2 & -2 & -2 & 0 & 0 \\
-2 & -2 & -2 & -2 & 0 & 0 \\
5 & -3 & 1 & 1 & 3 & -1 \\
-3 & -3 & 1 & 1 & -1 & -1 \\
-3 & -3 & 1 & 1 & -1 & -1 \\
-3 & -3 & 1 & 1 & -1 & -1 \\
5 & 1 & -3 & 3 & -1 & -1 \\
1 & 1 & -3 & -3 & -1 & -1 \\
1 & 1 & -3 & -3 & -1 & -1 \\
1 & 1 & -3 & -3 & -1 & -1 \\
\end{bmatrix}
\end{array}
\]

Proof. We discuss the case $r = 1$ only; the other cases can be proved by a similar argument.

Since $|f|^2 = 2/7$, it follows that $|a + f|^2 = 2/7$ if and only if $(a|f) = -\frac{1}{2}|a|^2$. By the Schwarz inequality, we also have

$$|(a|f)| \leq \sqrt{\frac{2}{7}|a|}.$$ 

Hence, $|a + f|^2 = 2/7$ implies $|a|^2 \leq 8/7$. Now using Lemma 3.4, it is straightforward to determine $S^1$. \square

Remark 3.6. We also note that $\beta_i$’s satisfy the relation

$$(\beta_i|\beta_j) = \begin{cases} 
\frac{2}{7} & \text{if } i = j, \\
-\frac{1}{7} & \text{if } i - j \equiv \pm 1 \pmod{7}, \\
0 & \text{otherwise.}
\end{cases}$$

4. Holomorphic VOA of Central Charge 24 with Lie Algebra $A_{6,7}$

In this section, we describe how to construct a holomorphic VOA whose weight one Lie algebra has type $A_{6,7}$.

4.1. IRREDUCIBLE TWISTED $V_{\Lambda}$-MODULES

Let $\Lambda$ be the Leech lattice and $\tau$ the isometry of $\Lambda$ of order 7 given in Sect. 3.2. Let $V_{\Lambda}$ be the lattice VOA associated with $\Lambda$. Note that the restriction of the invariant form $\langle \cdot | \cdot \rangle$ of $V_{\Lambda}$ to $(V_{\Lambda})_1$ coincides with the form $(\cdot | \cdot)$ on $\mathfrak{h} = \mathbb{C} \otimes \mathbb{Z} \Lambda$ via the canonical injective map

$$\mathfrak{h} \to (V_{\Lambda})_1, \quad h \mapsto h(-1) \cdot 1.$$  (4.1)
Since the order of $\tau$ is odd, there exists a lift of $\tau$ in $V_{\Lambda}$ of order 7; we also denote it by the same symbol $\tau$. Let $V_{\Lambda}[\tau^r]$, $(r = \pm 1, \pm 2, \pm 3)$, be the unique irreducible $\tau^r$-twisted $V_{\Lambda}$-module ([5]). Such a module was constructed in [4] explicitly (see [18, Section 2.2] for a review). As a vector space, it is given by

$$V_{\Lambda}[\tau^r] \cong M(1)[\tau^r] \otimes \mathbb{C}[P_0(\Lambda)] \otimes T_r,$$

where $M(1)[\tau^r]$ is the "$\tau^r$-twisted" free bosonic space and $T_r$ is the unique irreducible module of $\hat{M}/(1-\tau^r)\Lambda$ satisfying certain conditions (see [12, Propositions 6.1 and 6.2] and [4, Remark 4.2] for details). It follows from $(1-\tau^r)\Lambda = M$ (see Lemma 3.3) that $\dim T_r = 1$ for $r \in \{\pm 1, \pm 2, \pm 3\}$.

Let $f$ be the vector of $\Lambda$ defined in (3.3). We regard $f$ as a vector in $(V_{\Lambda})_1$ via (4.1) and define $\sigma_f = \exp(-2\pi \sqrt{-1}f(0))$. Then, $\sigma_f$ is an automorphism of $V_{\Lambda}$ of order 7 because $f \in \frac{1}{7}\Lambda \setminus \Lambda$. By the equality $\tau(f) = f$, $\sigma_f$ commutes with $\tau$, and the automorphism

$$g = \sigma_f \cdot \tau \in \text{Aut}(V_{\Lambda})$$

has order 7. Note that $g^r = \sigma_f \cdot \tau^r$.

By Proposition 2.2, we obtain the irreducible $g^r$-twisted $V_{\Lambda}$-module $V_{\Lambda}[\tau^r]^{(rf)}$ for $r \in \{\pm 1, \pm 2, \pm 3\}$. For convenience, we fix a non-zero vector $t_r \in T_r$. Then, $T_r = \mathbb{C}t_r$. By (2.2), we have

$$L^{(rf)}(0) = L(0) + rf(0) + \frac{|rf|^2}{2} - id. \quad (4.2)$$

**Lemma 4.1.** For $r \in \{\pm 1, \pm 2, \pm 3\}$,

$$\left\{ e^a \otimes t_r \mid a \in P_0(\Lambda), \ |a+rf|^2 = \frac{2}{7} \right\}$$

is a basis of $(V_{\Lambda}[\tau^r]^{(rf)}_1)$. Moreover, the dimension of $(V_{\Lambda}[\tau^r]^{(rf)}_1)$ is 7.

**Proof.** Let $w \otimes e^x \otimes t_r \in V_{\Lambda}[\tau^r]^{(rf)}$ ($w \in M(1)[\tau^r]$, $x \in P_0(\Lambda)$) be a vector whose $L(0)$-weight is 1. By [4, (6.28)], it is straightforward to show that the $L(0)$-weight of $t_r \in V_{\Lambda}[\tau^r]$ is $6/7$. Let $\ell$ be the $L(0)$-weight of $w$ in $M(1)[\tau^r]$, which belongs to $\frac{1}{7}\mathbb{Z}_{\geq 0}$. Then by (4.2), the $L(0)$-weight of $w \otimes e^x \otimes t_r$ in the twisted module $V_{\Lambda}[\tau^r]^{(rf)}$ is

$$\ell + \frac{|x|^2}{2} + \frac{6}{7} + rf(x) + \frac{|rf|^2}{2} = \ell + \frac{|x+rf|^2}{2} + \frac{6}{7}, \quad (4.3)$$

which is equal to 1 by the assumption. Hence by $x+rf \neq 0$, we have $\ell = 0$, and we may assume that $w = 1$. In addition, we obtain

$$|x+rf|^2 = \frac{2}{7}. \quad (4.4)$$

Thus, we have the first assertion. The latter assertion follows from Lemma 3.5. \(\square\)
Remark 4.2. For any \( r \in \{ \pm 1, \pm 2, \pm 3 \} \) and \( x \in P_0(\Lambda) \), we have \( |x + rf|^2 \geq 2/7 \). Hence by (4.3), the lowest \( L(0) \)-weight of any irreducible \( g^r \)-twisted \( V_\Lambda \)-module is 1.

### 4.2. \( \mathbb{Z}_7 \)-Orbifold Construction

In this subsection, we discuss the \( \mathbb{Z}_7 \)-orbifold construction from \( V_\Lambda \) and \( g \). First, we consider the following \( V_\Lambda^g \)-module:

\[
\tilde{V}_{\Lambda,g} = V_\Lambda^g \oplus (V_\Lambda[\tau](f))_\mathbb{Z} \oplus (V_\Lambda[\tau^2](2f))_\mathbb{Z} \oplus (V_\Lambda[\tau^3](3f))_\mathbb{Z} \\
\quad \oplus (V_\Lambda[\tau^{-3}](3f))_\mathbb{Z} \oplus (V_\Lambda[\tau^{-2}](2f))_\mathbb{Z} \oplus (V_\Lambda[\tau^{-1}](1f))_\mathbb{Z},
\]

where \( (V_\Lambda[\tau^r](r))_\mathbb{Z} \) is the subspace of \( V_\Lambda[\tau^r](r) \) with integral weight. By Remark 4.2, we can apply [7, Theorem 5.15] to our case under the assumption that \( V_\Lambda^g \) is regular (see also [1, 16, 17]).

**PROPOSITION 4.3** (cf. [7, Theorem 5.15]). The space \( \tilde{V}_{\Lambda,g} \) defined as above is a strongly regular, holomorphic VOA of central charge 24.

**PROPOSITION 4.4.** Let \( \tilde{V}_{\Lambda,g} \) be defined as above. Then, \( \dim(\tilde{V}_{\Lambda,g})_1 = 48 \) and the Lie algebra \( (\tilde{V}_{\Lambda,g})_1 \) has type \( A_6,7 \).

**Proof.** Recall that \( (V_\Lambda)_1 \) is an abelian Lie algebra of dimension 24. Viewing \( \mathfrak{h}_{(0)} \) as a subspace of \( (V_\Lambda)_1 \), we know that \( (V_\Lambda^g)_1 = \mathfrak{h}_{(0)} \) is an abelian Lie algebra of dimension 6. By Lemma 4.1, we have \( \dim(\tilde{V}_{\Lambda,g})_1 = 6 + 7 \times 6 = 48 \). By Proposition 2.1, the Lie algebra \( (\tilde{V}_{\Lambda,g})_1 \) is semisimple.

Now, let \( x \in \mathfrak{h}_{(0)} \subset (V_\Lambda^g)_1 \). Then,

\[
x^{(rf)}(0) = x(0) + (x|rf)id
\]

on \( V_\Lambda[\tau^r](r) \). Notice that \( x(0) = 0 \) on \( M(1)[\tau^r] \) and on \( T_r \) by the explicit description of vertex operators in [4, 12] (cf. [18]). Hence for \( w \otimes e^a \otimes t_r \in (V_\Lambda[\tau^r](r))_1 \),

\[
x^{(rf)}(0)(w \otimes e^a \otimes t_r) = (x|a + rf)w \otimes e^a \otimes t_r,
\]

which shows that \( x \) is semisimple in \( (\tilde{V}_{\Lambda,g})_1 \). Since \( x, a + rf \in \mathfrak{h}_{(0)} \) and \( (\cdot|\cdot) \) is non-degenerate on \( \mathfrak{h}_{(0)} \), the Equation (4.5) also implies that the centralizer of \( \mathfrak{h}_{(0)} \) in \( (\tilde{V}_{\Lambda,g})_1 \) is again \( \mathfrak{h}_{(0)} \). Hence, \( \mathfrak{h}_{(0)} \) is a Cartan subalgebra of \( (\tilde{V}_{\Lambda,g})_1 \). Thus, \( (\tilde{V}_{\Lambda,g})_1 \) is a 48-dimensional semisimple Lie algebra with Lie rank 6, and the only possibility of its type is \( A_6 \). We remark that, up to a scaling, \( \{ \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6 \} \) forms a set of simple roots for a root system of type \( A_6 \) (see Remark 3.6).

By Proposition 2.1, we have the ratio \( h^\vee/k = 1 \) and hence the level \( k \) is 7. Therefore, the Lie algebra \( (\tilde{V}_{\Lambda,g})_1 \) has type \( A_6,7 \).  \( \Box \)
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