Tropical optimization technique in bi-objective project scheduling under temporal constraints

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Abstract

Tropical (idempotent) mathematics, which investigates the theory and applications of algebraic systems with idempotent operations, finds increasing use in solving challenging problems in operations research, including temporal project scheduling problems. We consider a project that consists of a set of activities performed in parallel under temporal constraints on their start and finish times. The problem of interest is to schedule the activities to minimize both the maximum flow-time over all activities and the project makespan. We formulate and solve the problem in the framework of tropical mathematics as a tropical bi-objective optimization problem. As a result, we derive a complete Pareto-optimal solution in a direct explicit form, ready for further analysis and straightforward computation. We examine the computational complexity of the solution, and give an illustrative example.

Key-Words: tropical mathematics, idempotent semifield, bi-objective optimization problem, project scheduling under temporal constraints.

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1 Introduction

Tropical optimization deals with optimization problems that are formulated and solved in terms of tropical (idempotent) mathematics, which is concentrated on the theory and applications of algebraic systems with idempotent operations. Methods and techniques of tropical optimization find application in operations research, management science and other fields to provide new efficient solutions to both known and novel optimization problems.

Even in the pioneering works by [29, 5, 9, 15, 35, 31], which appeared in the area of tropical mathematics in the early 1960s, various optimization
problems have served to motivate and illustrate the study, including the
problems of project scheduling ([29], [5], [9]). Further successful research in
succeeding decades were frequently concerned with the analysis and solution
of optimization problems. The results obtained in the area are presented in
a number of contributed papers and books, among which are recent mono-
graphs by [10], [14], [26], [11], [4], [25].

Tropical optimization problems typically consist in minimizing or max-
imizing functions defined in the tropical mathematics setting on vectors
over idempotent semifields (semirings with idempotent addition and invert-
ible multiplication). In some cases, but not infrequently, the problems can
be solved analytically to provide a complete direct solution in an explicit
form under rather general assumptions. For other problems, only numeri-
cal techniques are known, available for a specific semifield, in the form of
computational algorithms that offer a particular solution or indicate that
no solution exists. A brief overview of tropical optimization problems and
related solution approaches can be found, e.g., in [17], [19].

The application of tropical optimization techniques frequently allows one
to obtain a direct analytic solution to various real-world problems that, in
practice, are solved numerically by using appropriate computational proce-
dures, and do not have exact explicit solutions available. Such analytical
solutions may efficiently serve to complement and supplement the existing
numerical approaches, and become the only solution when the algorithmic
solutions are infeasible or impossible to implement.

Applications of tropical mathematics include project sche-
duling problems, which appear in various settings in many publications from the early
works by [29], [5], [9] to more recent papers [36], [3], [12], [13], [32] and books [14],
[11], [4]. Models and methods of tropical optimization proved to be well suited
for solving deterministic, pure temporal project scheduling problems, also
referred to as time-dependent, time-constrained or resource-unconstrained
problems (see, e.g., [6], [23], [34]). These problems involve temporal constraints
(such as precedence relations, release times, due dates, deadlines), given
without uncertainty, temporal criteria (such as makespan, maximum devi-
ation from due dates, maximum flow-time), and do not entail direct cost
dependencies and resource requirements. The solution of the problems can
be used as an auxiliary tool in solving more general project scheduling prob-
lems, and is of independent interest.

While general scheduling problems are normally \(NP\)-hard, the tempo-
ral problems, due to the lack of cost and resource constraints imposed, can
normally be formulated as linear programs or graph (network) optimization
problems. As a result, they are numerically solved with the help of appro-
priate computational procedures of polynomial computational complexity,
such as the Karmarkar and Floyd-Warshall algorithms.

In contrast to these algorithmic solutions, tropical optimization approach
can offer complete, direct solutions to the problems, which are obtained
in a compact vector form, ready for further analysis and straightforward computations. Examples of temporal project scheduling problems and their solutions in the framework of tropical optimization are provided, e.g., in [16, 18, 20, 22, 21].

Multi-objective project scheduling, where two or more conflicting criteria have to be met [33, 1], allows the decision-maker to make more realistic and reasonable choice, and thus, is of particular importance. The common way to handle multi-objective problems is to derive the best compromise solutions, called the nondominated or Pareto-optimal solution, at which no objective can be improved without degrading another one (see, e.g., [8, 24, 27, 30, 21, 7]). A natural approach to handle multi-objective temporal problems involves the representation as multi-objective linear programs, which are then solved by appropriate numerical procedures, like multiple objective variants of the simplex algorithm and the Benson algorithm.

In this paper, we consider a project that consists of a set of activities performed in parallel under temporal constraints on their start and finish times. The problem of interest is to develop a schedule for the activities, which minimizes both the the maximum flow-time over all activities and the project makespan. We formulate and solve the problem in the framework of tropical mathematics as a tropical bi-objective optimization problem. As a result, we derive a complete Pareto-optimal solution in a direct explicit form that is suitable for both formal analysis and numerical implementation. We examine the computational complexity of the solution, and give illustrative examples.

The paper is organized as follows. In Section 2, we provide a formal description of the bi-objective temporal project scheduling problem of interest. Section 3 includes a brief compact overview of basic facts about tropical algebra, which are used in the subsequent solution of a bi-objective tropical optimization problem. The main result, which provides a complete Pareto-optimal solution to the optimization problem in an exact analytical form, is given in Section 4. We apply the result obtained to solve the project scheduling problem, and present an illustrative example in Section 5.

2 Bi-objective project scheduling problem

We start with the formal description of a bi-objective temporal project scheduling problem that serves to motivate and illustrate the solutions obtained below in the framework of tropical optimization. In order to facilitate formulation of the problem in terms of tropical algebra, we use simplified, but still rather general, model and notation, which are slightly different from those commonly adopted in the literature on project scheduling (see, e.g., [6, 28, 33, 34]).

Consider a project, which consists of \( n \) activities (jobs, tasks, operations)
to be performed in parallel under certain temporal constraints. The problem is to construct a schedule for the activities, which can be considered optimal according to two different criteria to be described later.

For each activity \( i = 1, \ldots , n \), we denote the unknown start time by \( x_i \) and the unknown finish time by \( y_i \), and assume these variables to be subject to constraints imposed due to some technological, organizational or other limitations. First, we suppose that the start time cannot exceed the interval between given release (ready, arrival) time \( g_i \) and release deadline \( h_i \), which yields the double inequality constraints

\[
g_i \leq x_i \leq h_i, \quad i = 1, \ldots , n.
\]

Furthermore, the minimum allowed time lag \( a_{ij} \) between the start of activity \( i \) and the finish of activity \( j \) is given to specify the start-finish constraints in the form of the inequalities

\[
a_{ij} + x_j \leq y_i, \quad i, j = 1, \ldots , n,
\]

where we take \( a_{ij} = -\infty \), if the lag \( a_{ij} \) is not defined. Note that the difference between the finish and start times of activity \( i \) is bounded from below by the value of \( a_{ii} \), which is normally assumed to be non-negative, and presents the duration of the activity when no other activities are taken into account.

Moreover, we assume that each activity finishes immediately as soon as all its related start-finish constraints satisfy, which yields the equality

\[
\max_{1 \leq j \leq n} (a_{ij} + x_j) = y_i, \quad i = 1, \ldots , n.
\]

To develop an optimal schedule, we consider two optimality criteria, which are frequently used in project scheduling and involve the minimization of the maximum flow-time of activities and the minimization of the project makespan. The flow-time (shop-time) is defined for activity \( i \) as the difference \( y_i - x_i \) between its finish and start times, and may directly or indirectly reflect the expenditure incurred to perform the activity. The maximum flow-time over all activities is given by

\[
\max_{1 \leq i \leq n} (y_i - x_i).
\]

The project makespan, defined as the overall project duration, presents a commonly used measure of schedule efficiency to be minimized. The makespan can be calculated as the difference between the maximum finish time and the minimum start time of activities, and represented as

\[
\max_{1 \leq i \leq n} y_i - \min_{1 \leq i \leq n} x_i = \max_{1 \leq i \leq n} y_i + \max_{1 \leq i \leq n} (-x_i).
\]
The problem of interest is formulated to find the start and finish times for all activities to minimize both maximum flow time and the makespan under the release time, release deadline and start-finish constraints. By combining the objective functions with the temporal constraints, we arrive at the bi-criteria problem of project scheduling in which, given numbers \(a_{ij}, \ g_i\) and \(h_i\) such that \(g_i \leq h_i\), one needs to find \(x_i\) and \(y_j\) for all \(i = 1, \ldots, n\) that

\[
\begin{align*}
\text{minimize} & \quad \left\{ \max_{1 \leq i \leq n} (y_i - x_i), \ \max_{1 \leq i \leq n} y_i + \max_{1 \leq i \leq n} (-x_i) \right\}; \\
\text{subject to} & \quad \max_{1 \leq j \leq n} (a_{ij} + x_j) = y_i, \\
& \quad g_i \leq x_i \leq h_i, \quad i = 1, \ldots, n.
\end{align*}
\]

Below, we apply methods and techniques of tropical optimization to obtain a complete direct Pareto-optimal solution to the problem in a compact vector form, and give illustrative examples.

3 Preliminary definitions, notation and results

In this section, we offer an overview of basic definitions, notation and preliminary results of tropical (idempotent) algebra from [17, 16, 18, 19, 20, 22, 21] to provide an appropriate analytical framework for compact formulation and complete solution of a tropical optimization problem in the next section. Further details on the theory and applications of tropical mathematics can be found, e.g., in the monographs and textbooks by [10, 14, 26, 11, 4, 25].

3.1 Idempotent semifield

Let \(\mathcal{X}\) be a set closed under two associative and commutative operations, addition \(\oplus\) and multiplication \(\otimes\), which have neutral elements, zero \(0\) and unit \(1\). Addition is idempotent, that is \(x \oplus x = x\) for any \(x \in \mathcal{X}\). Multiplication distributes over addition, has \(0\) as absorbing element, and is invertible to endow each \(x \neq 0\) with the inverse \(x^{-1}\) such that \(x \otimes x^{-1} = 1\). The system \((\mathcal{X}, 0, 1, \oplus, \otimes)\) is normally referred to as the idempotent semifield.

The integer powers are introduced on \(\mathcal{X}\) in the standard way to represent iterated products given by \(0^p = 0\), \(x^p = x \otimes x^{p-1}\), and \(x^{-p} = (x^{-1})^p\) for any \(x \neq 0\) and natural \(p\). Moreover, the semifield is assumed algebraically complete in the sense that the equation \(x^p = a\) is uniquely solvable in \(x\) for any \(a \in \mathcal{X}\) and natural \(p\), which makes the powers with rational exponents well defined. In the subsequent text, the multiplication sign \(\otimes\) is omitted to save writing.

Idempotent addition defines a partial order on \(\mathcal{X}\) by the rule that \(x \leq y\) if and only if \(x \oplus y = y\). In terms of this order, addition and multiplication are monotone in both operands, that is the inequality \(x \leq y\) results in the inequalities \(x \oplus z \leq y \oplus z\) and \(xz \leq yz\) for any \(x, y, z \in \mathcal{X}\). Inversion is
antitone, which means that, for all $x, y \neq 0$, the inequality $x \leq y$ yields $x^{-1} \geq y^{-1}$. Addition has an extremal property that the inequalities $x \leq x \oplus y$ and $y \leq x \oplus y$ hold for any $x, y \in \mathbb{X}$. Moreover, the inequality $x \oplus y \leq z$ is equivalent to the system of two inequalities $x \leq z$ and $y \leq z$. In what follows, the partial order is assumed extended to a consistent total order to make the semifield linearly ordered.

For any $a, b \in \mathbb{X}$ and non-negative integer $m$, the idempotent analogue of binomial identity is given by

$$
(x \oplus y)^m = x^m \oplus x^{m-1} y \oplus \cdots \oplus y^m = x^m \oplus \bigoplus_{i=1}^{m} x^{m-i} y^i.
$$

It follows from the identity with $m = 2$ that the inequality $x \oplus y \geq (xy)^{1/2}$ is valid as an idempotent analogue of the relation between arithmetic and geometric means of two positive numbers. For any $x_1, \ldots, x_k \in \mathbb{X}$ and integer $k > 0$, this inequality readily extends to that of the form

$$
(x_1 \cdots x_k)^{1/k} \leq x_1 \oplus \cdots \oplus x_k.
$$

As a typical example of the idempotent semifield, consider the real semifield $\mathbb{R}_{\max,+} = (\mathbb{R} \cup \{-\infty\}, -\infty, 0, \max, +)$, which is frequently called the max-plus algebra. In the semifield, the addition $\oplus$ is computed as calculation of maximum, and the multiplication $\otimes$ as arithmetic addition; the zero $0$ is defined as $-\infty$, and the unit $1$ as the arithmetic zero $0$. Furthermore, for any $x \in \mathbb{R}$, there exists the inverse $x^{-1}$, which corresponds to the opposite number $-x$ in conventional algebra. The power $x^y$ coincides with the arithmetic product $yx$, which is defined for all $x, y \in \mathbb{R}$. The order, induced by idempotent addition, conforms with the natural linear order on $\mathbb{R} \cup \{-\infty\}$.

### 3.2 Algebra of matrices and vectors

Denote the set of matrices over $\mathbb{X}$ with $m$ rows and $n$ columns by $\mathbb{X}^{m \times n}$. A matrix that has all entries equal to $0$ is the zero matrix. If a matrix has no zero columns, it is called column-regular.

Addition and multiplication of matrices, as well as multiplication by scalar, follow the standard rules, where the arithmetic addition and multiplication are replaced by $\oplus$ and $\otimes$. Specifically, for conforming matrices $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$, and a scalar $x$, the matrix operations are given by

$$
\{A \oplus B\}_{ij} = a_{ij} \oplus b_{ij}, \quad \{AC\}_{ij} = \bigoplus_k a_{ik} c_{kj}, \quad \{xA\}_{ij} = xa_{ij}.
$$

The transpose of a matrix $A$ is denoted $A^T$. The multiplicative conjugate transpose of a nonzero matrix $A = (a_{ij})$ is the matrix $A^-$ with the
entries given by the condition

\[
\{A^\perp\}_{ij} = \begin{cases} 
    a_{ji}^{-1}, & \text{if } a_{ji} \neq 0; \\
    0, & \text{otherwise.}
\end{cases}
\]

The properties of scalar addition and multiplication, which are associated with the order relation induced by idempotent addition, extend to the matrix operations, where the inequalities are considered entry-wise.

Consider the set \( \mathbb{F}^{n \times n} \) of square matrices of order \( n \). A diagonal matrix that has all diagonal entries equal to 1 and all off-diagonal entries to 0 is the identity matrix denoted by \( I \). For any nonzero matrix \( A \in \mathbb{F}^{n \times n} \) and integer \( p > 0 \), the matrix powers are given by \( A^0 = I \) and \( A^p = A^{p-1}A \).

The trace of any matrix \( A = (a_{ij}) \) is calculated as

\[
\text{tr} A = a_{11} + \cdots + a_{nn} = \bigoplus_{i=1}^{n} a_{ii}.
\]

For any matrices \( A \) and \( B \) and scalar \( x \), the following equalities hold:

\[
\text{tr}(A \oplus B) = \text{tr} A \oplus \text{tr} B, \quad \text{tr}(AB) = \text{tr}(BA), \quad \text{tr}(xA) = x \text{tr} A.
\]

Consider a function that maps any matrix \( A \in \mathbb{F}^{n \times n} \) to the scalar

\[
\text{Tr}(A) = \text{tr} A \oplus \cdots \oplus \text{tr} A^n = \bigoplus_{k=1}^{n} \text{tr} A^k.
\]

Provided that \( \text{Tr}(A) \leq 1 \), one can define the matrix (also known as the Kleene star matrix) given by

\[
A^* = I \oplus A \oplus \cdots \oplus A^{n-1} = \bigoplus_{k=0}^{n-1} A^k.
\]

Any matrix that consists of a single column (row) forms a column (row) vector. In the following, all vectors are taken as column vectors unless otherwise indicated. The set of column vectors of size \( n \) is denoted \( \mathbb{F}^n \).

A vector without zero components is called regular. The vector with all zero components (the zero vector) and the vector with all components equal to 1 are denoted by \( \mathbf{0} = (0, \ldots, 0)^T \) and \( \mathbf{1} = (1, \ldots, 1)^T \).

The conjugate transpose of a nonzero column vector \( x = (x_i) \) is the row vector \( x^\perp = (x_i^\perp) \), where \( x_i^\perp = x_i^{-1} \) if \( x_i \neq 0 \), and \( x_i^\perp = 0 \) otherwise.

A scalar \( \lambda \) is an eigenvalue of a square matrix \( A \in \mathbb{F}^{n \times n} \), if there exists a nonzero vector \( x \in \mathbb{F}^n \) that satisfies the equation \( Ax = \lambda x \) and hence is called the eigenvector. The maximum eigenvalue (in the sense of the order
induced by idempotent addition) is referred to as the spectral radius and calculated as
\[ \lambda = \text{tr} A \oplus \cdots \oplus \text{tr}^{1/n}(A^n) = \bigoplus_{k=1}^{n} \text{tr}^{1/k}(A^k). \]

For any matrix \( A \in \mathbb{X}^{m \times n} \) and vector \( x \in \mathbb{X}^n \), idempotent analogues of matrix and vector norms are given by
\[ \|A\| = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n} a_{ij} = 1^T A 1. \]
\[ \|x\| = \bigoplus_{i=1}^{n} x_i = 1^T x = x^T 1. \]

### 3.3 Vector inequalities

We now present results, which provide the basis for the solution of a tropical optimization problem below. First, suppose that, given a matrix \( A \in \mathbb{X}^{m \times n} \) and a vector \( d \in \mathbb{X}^m \), one needs to solve, with respect to the unknown vector \( x \in \mathbb{X}^n \), the inequality
\[ Ax \leq d. \] (3)

A direct solution to the problem is described as follows (see, e.g., [18]).

**Lemma 1.** For any column-regular matrix \( A \) and regular vector \( d \), all solutions to inequality (3) are given by
\[ x \leq (d - A)^-. \]

Now suppose that, given a matrix \( A \in \mathbb{X}^{n \times n} \) and vector \( b \in \mathbb{X}^n \), the problem is to find regular vectors \( x \in \mathbb{X}^n \) to satisfy the inequality
\[ Ax \oplus b \leq x. \] (4)

The next result, obtained in [19], offers a direct solution.

**Theorem 2.** For any matrix \( A \), the following statements hold:

1. If \( \text{Tr}(A) \leq 1 \), then all regular solutions to (4) are given by \( x = A^* u \), where \( u \geq b \).
2. If \( \text{Tr}(A) > 1 \), then there is only the trivial solution \( x = 0 \).

### 3.4 Identities and inequalities for traces

We conclude the overview of preliminary results with some useful matrix formulas. We start with a binomial identity that is valid for any square matrices \( A, B \in \mathbb{X}^{n \times n} \) and natural \( m \) in the following form (see also [20]):
\[ (A \oplus B)^m = A^m \oplus \bigoplus_{k=1}^{m} \bigoplus_{i_0 + i_1 + \cdots + i_k = m-k} A^{i_0} (BA)^i_1 \cdots (BA)^i_k. \] (5)
The evaluation the sum over all \( m = 0, \ldots, n - 1 \) and rearrangement of terms yield the identity

\[
(A \oplus B)^* = I \oplus \bigoplus_{m=1}^{n-1} A^m \oplus \bigoplus_{k=1}^{n-k} \bigoplus_{i_0+i_1+\cdots+i_k=m} A^{i_0}(BA^{i_1} \cdots BA^{i_k}),
\]

where the empty sums are assumed to be equal to \( 0 \).

Furthermore, after taking trace of (5), we sum up the traces over \( m = 1, \ldots, n \), and then rearrange the terms to obtain

\[
\text{Tr}(A \oplus B) = \bigoplus_{k=1}^n \text{tr} A^k \oplus \bigoplus_{k=1}^{n-k} \bigoplus_{i_0+i_1+\cdots+i_k=m} \text{tr}(BA^{i_1} \cdots BA^{i_k}).
\]

4 Bi-objective tropical optimization problem

We now in a position to formulate and solve a new bi-objective tropical optimization problem, which is used in the next section to solve the scheduling problem of interest. For the sake of compactness and generality, we represent here the tropical optimization problem and its solution in terms of a general idempotent semifield \((X, 0, 1, \oplus, \otimes)\).

Suppose that, given a square matrix \( A \in X^{n \times n} \), we need to find regular vectors \( x \in X^n \) that solve the bi-objective optimization problem

\[
\begin{align*}
\text{minimize} & \quad \{ x^T Ax, x^T 11^T Ax \}; \\
\text{subject to} & \quad g \leq x \leq h.
\end{align*}
\]

To handle the problem, we follow the approach first developed in [17, 16, 18, 19, 20, 22, 21] to solve ordinary single objective problems, and then applied to bi-objective problems in [23]. The approach involves the introduction of parameters to represent the optimal values of objective functions, and hence to reduce the optimization problem to a system of parametrized vector inequalities. We exploit the existence conditions for the solution of the system to evaluate the parameters, which describe the Pareto frontier of the problem. The complete solution of the system, which corresponds to the parameters given by the Pareto frontier, is taken as the Pareto-optimal solution of the initial bi-objective problem.

4.1 Parametrization of problem

Denote the minimum values of the scalar objective functions \( x^T Ax \) and \( x^T 11^T Ax \) in the Pareto frontier of problem (8) by \( \alpha \) and \( \beta \). Then, all solutions are defined by the system of inequalities

\[
x^T Ax \leq \alpha, \quad x^T 11^T Ax \leq \beta, \quad g \leq x \leq h.
\]
By using Lemma 1, we solve the first inequality with respect to $Ax$ and the second to $11^T Ax$ to obtain the equivalent system

$$\alpha^{-1}Ax \leq x, \quad \beta^{-1}11^T Ax \leq x, \quad g \leq x \leq h,$$

which can then be combined into one double inequality

$$(\alpha^{-1}A \oplus \beta^{-1}11^T A)x \oplus g \leq x \leq h. \quad (9)$$

According to Theorem 2, regular solutions of the left inequality at (9) exist if and only if the condition

$$\text{Tr}(\alpha^{-1}A \oplus \beta^{-1}11^T A) \leq 1$$

holds, under which all solutions are given, through a vector of parameters $u$, by

$$x = (\alpha^{-1}A \oplus \beta^{-1}11^T A)^*u, \quad u \geq g.$$

Furthermore, to provide the right inequality at (9), the vector $u$ must satisfy the condition

$$(\alpha^{-1}A \oplus \beta^{-1}11^T A)^*u \leq h.$$

We apply Lemma 1 to obtain an upper bound on $u$, which, together with the lower bound $g$, gives the double inequality

$$g \leq u \leq (h^-(\alpha^{-1}A \oplus \beta^{-1}11^T A)^*)^-.$$

The set of vectors $u$ is not empty, if the following inequality holds:

$$g \leq (h^-(\alpha^{-1}A \oplus \beta^{-1}11^T A)^*)^-,$$

which is, due to Lemma 1, equivalent to the inequality

$$h^-(\alpha^{-1}A \oplus \beta^{-1}11^T A)^*g \leq 1.$$

By collecting the existence conditions and combining the bounds on $u$, we finally conclude that inequality (9) has regular solutions if and only if both inequalities

$$\text{Tr}(\alpha^{-1}A \oplus \beta^{-1}11^T A) \leq 1, \quad (h^-(\alpha^{-1}A \oplus \beta^{-1}11^T A)^*)g \leq 1. \quad (10)$$

are satisfied, and all solutions are given in parametric form by the relations

$$x = (\alpha^{-1}A \oplus \beta^{-1}11^T A)^*u, \quad g \leq u \leq (h^-(\alpha^{-1}A \oplus \beta^{-1}11^T A)^*)^-.$$

(11)
4.2 Determination of parameters

To derive the Pareto frontier for the problem, we examine the conditions at (10). We consider the first condition and use identity (7) for calculating

\[
\text{Tr}(\alpha^{-1} A \oplus \beta^{-1} 11^T A) = \bigoplus_{k=1}^{n} \alpha^{-k} \text{tr} A^k \\
\bigoplus_{k=1}^{n} \bigoplus_{m=0}^{n-k} \bigoplus_{i_1, \ldots, i_k \geq 0} \alpha^{-m} \beta^{-k} \text{tr}(11^T A A^{i_1} \cdots 11^T A A^{i_k}).
\]

By applying properties of trace, we represent the trace in the second sum in the form

\[
\text{tr}(11^T A A^{i_1} \cdots 11^T A A^{i_k}) = (1^T A^{i_1+1} 1) \cdots (1^T A^{i_k+1} 1) = \|A^{i_1+1}\| \cdots \|A^{i_k+1}\|
\]

and then expand the condition as follows

\[
\bigoplus_{k=1}^{n} \alpha^{-k} \text{tr} A^k \bigoplus_{k=1}^{n-k} \bigoplus_{m=0}^{n-i_1-\cdots-i_k} \bigoplus_{i_1, \ldots, i_k \geq 0} \alpha^{-m} (\|A^{i_1+1}\| \cdots \|A^{i_k+1}\|) \leq 1.
\]

We replace the last inequality by the equivalent system of inequalities

\[
\alpha^{-k} \text{tr} A^k \leq 1,
\]

\[
\beta^{-k} \bigoplus_{m=0}^{n-i_1-\cdots-i_k} \bigoplus_{i_1, \ldots, i_k \geq 0} \alpha^{-m} (\|A^{i_1+1}\| \cdots \|A^{i_k+1}\|) \leq 1, \quad k = 1, \ldots, n.
\]

After rearranging terms to isolate powers of \(\alpha\) and \(\beta\) on the right-hand side, and taking roots, we rewrite the system as

\[
\text{tr}^{1/k} (A^k) \leq \alpha,
\]

\[
\bigoplus_{m=0}^{n-i_1-\cdots-i_k} \bigoplus_{i_1, \ldots, i_k \geq 0} \alpha^{-m/k} (\|A^{i_1+1}\| \cdots \|A^{i_k+1}\|)^{1/k} \leq \beta, \quad k = 1, \ldots, n.
\]

Aggregating the inequalities for \(\alpha\) and then for \(\beta\) yields the system

\[
\alpha \geq \bigoplus_{k=1}^{n} \text{tr}^{1/k} (A^k) = \lambda,
\]

\[
\beta \geq \bigoplus_{k=1}^{n} \bigoplus_{m=0}^{n-k} \bigoplus_{i_1, \ldots, i_k \geq 0} \alpha^{-m/k} (\|A^{i_1+1}\| \cdots \|A^{i_k+1}\|)^{1/k}.
\]
where $\lambda$ denotes the spectral radius of the matrix $A$.

We now simplify the sum on the right-hand side of the second inequality in the system, and consider that part of this sum, which corresponds to $k = 1$ and takes the form

$$\bigoplus_{m=0}^{n-1} \bigoplus_{i_1=m}^{n-1} \bigoplus_{i_1 \geq 0}^{n-1} \alpha^{-m} \|A^{i_1+1}\| = \bigoplus_{i=0}^{n-1} \alpha^{-i} \|A^{i+1}\|.$$

Our aim is to verify that the rest of the sum is dominated by this part, and thus can be eliminated. Indeed, with the condition that $i_1 + \cdots + i_k = m$, where $1 < k \leq n$ and $0 \leq m \leq n - k$, we apply identity (2) to obtain

$$\alpha^{-m/k}(\|A^{i_1+1}\| \cdots \|A^{i_k+1}\|)^{1/k} = (\alpha^{-i_1} \|A^{i_1+1}\| \cdots \alpha^{-i_k} \|A^{i_k+1}\|)^{1/k} \leq \alpha^{-i_1} \|A^{i_1+1}\| \oplus \cdots \oplus \alpha^{-i_k} \|A^{i_k+1}\| \leq \bigoplus_{i=0}^{n-1} \alpha^{-i} \|A^{i+1}\|.$$

It follows from this inequality that the sum of all terms corresponding to $k > 1$ satisfies the condition

$$\bigoplus_{k=2}^{n} \bigoplus_{m=0}^{n-k} \bigoplus_{i_1, \ldots, i_k \geq 0}^{n-k} \alpha^{-m/k}(\|A^{i_1+1}\| \cdots \|A^{i_k+1}\|)^{1/k} \leq \bigoplus_{i=0}^{n-1} \alpha^{-i} \|A^{i+1}\|,$$

and hence can be dropped without affecting the entire sum.

As a result, we obtain the system of inequalities in the reduced form

$$\alpha \geq \bigoplus_{k=1}^{n} \text{tr}^{1/k}(A^k), \quad \beta \geq \bigoplus_{k=0}^{n-1} \alpha^{-k} \|A^{k+1}\|.$$

(12)

Next, we examine the second condition at (10). Application of identity (4) to the Kleene star matrix on the left-hand side gives the equality

$$(\alpha^{-1} A \oplus \beta^{-1} 11^T A)^* = I \oplus \bigoplus_{k=1}^{n-1} \alpha^{-k} A^k \oplus \bigoplus_{k=1}^{n-1} \beta^{-k} \bigoplus_{m=0}^{n-k-1} \bigoplus_{i_0+i_1+\cdots+i_k = m}^{n-k-1} \alpha^{-m} A^{i_0} (11^T A^{i_1+1} \cdots 11^T A^{i_k+1}),$$

where the empty sums are understood as zero $\emptyset$ and the empty products as one $\mathbb{I}$.
We multiply this equality by $h^-$ on the left and by $g$ on the right, and then obtain the condition under consideration in the form

$$
\begin{align*}
&h^- g \oplus \bigoplus_{k=1}^{n-1} \alpha^{-k} h^- A^k g \\
&\oplus \bigoplus_{m=0}^{n-k-1} \bigoplus_{i_0+i_1+\cdots+i_k=m} \alpha^{-m} ||h^- A^i|| ||A^{i_1+1}|| \cdots ||A^{i_k+1}|| ||A^{i_k+1}g|| \leq 1.
\end{align*}
$$

We solve the inequality for $\alpha$ and $\beta$ in the same way as before. First, we observe that $h^- g \leq 1$ by assumption, and hence the term $h^- g$ can be eliminated. Next, we replace the inequality by a system of inequalties, and then solve these inequalities with respect to $\alpha$ and $\beta$.

After combining the results, we arrive at the system

$$
\begin{align*}
&\alpha \geq \bigoplus_{k=1}^{n-1} (h^- A^k g)^{1/k}, \\
&\beta \geq \bigoplus_{m=0}^{n-2} \bigoplus_{i,j \geq 0} \alpha^{-m} ||h^- A^i|| ||A^{j+1}|| \\
&\oplus \bigoplus_{k=2}^{n-1} \bigoplus_{m=0}^{n-k-1} \bigoplus_{i_0+i_1+\cdots+i_k=m} \alpha^{-m/k} ||h^- A^i|| ||A^{i_1+1}|| \cdots ||A^{i_k+1}|| ||A^{i_k+1}g||^{1/k},
\end{align*}
$$

where two sums on the right-hand side of the second inequality are obtained by separating all summands, which correspond to $k = 1$, from the others.

By coupling the last inequalities for $\alpha$ and $\beta$ with the corresponding inequalities from (12), we form the system

$$
\begin{align*}
&\alpha \geq \bigoplus_{k=1}^{n} \text{tr}^{1/k}(A^k) \oplus \bigoplus_{k=1}^{n-1} (h^- A^k g)^{1/k}, \\
&\beta \geq \bigoplus_{k=0}^{n-1} \alpha^{-k} ||A^{k+1}|| \oplus \bigoplus_{m=0}^{n-2} \bigoplus_{i,j \geq 0} \alpha^{-m} ||h^- A^i|| ||A^{j+1}|| \\
&\oplus \bigoplus_{k=2}^{n-1} \bigoplus_{m=0}^{n-k-1} \bigoplus_{i_0+i_1+\cdots+i_k=m} \alpha^{-m/k} ||h^- A^i|| ||A^{i_1+1}|| \cdots ||A^{i_k+1}|| ||A^{i_k+1}g||^{1/k}.
\end{align*}
$$

To simplify the right-hand side of the second inequality in the system, we verify that the first two sums on this side dominate the third sum. By
using (2) once again, for all \( k = 2, \ldots, n - 1, \) \( m = 0, \ldots, n - k - 1 \) and \( i_0 + i_1 + \cdots + i_k = m \), we have the inequalities

\[
\alpha^{-m/k}(\|h - A^{i_0}\|\|A^{i_1+1}\|\cdots\|A^{i_k-1+1}\|\|A^{i_k+1}g\|)^{1/k}
\leq (\alpha^{-i_0}\|A^{i_1+1}\| + \cdots + \alpha^{-i_k-1}\|A^{i_k-1+1}\| + \alpha^{-(i_0+i_k)}\|h - A^{i_0}\||A^{i_k+1}g\|)
\leq \bigoplus_{i=0}^{n-1} \alpha^{-i}\|A^{i+1}\| + \bigoplus_{m=0}^{n-2} \bigoplus_{i+j=m} \bigoplus_{i,j \geq 0} \alpha^{-m}\|h - A^{i}\||A^{j+1}g\|,
\]

which shows that each summand of the third sum is not greater than the first two sums, and hence the third sum can be eliminated.

As a result, the system of inequalities for \( \alpha \) and \( \beta \) takes the form

\[
\alpha \geq \bigoplus_{k=1}^{n} \mathrm{tr}^{1/k}(A^k) \oplus \bigoplus_{k=1}^{n-1} (h - A^k g)^{1/k},
\]

\[
\beta \geq \|A\| \oplus \|h - \|A g\| \oplus \bigoplus_{k=1}^{n-1} \alpha^{-k}\|A^{k+1}\| \oplus \bigoplus_{i+j=k} \bigoplus_{i,j \geq 0} \alpha^{-m}\|h - A^{i}\||A^{j+1}g\|.
\]

To simplify further formulas, we use the notation

\[
\lambda = \bigoplus_{k=1}^{n} \mathrm{tr}^{1/k}(A^k), \quad \mu = \bigoplus_{k=1}^{n-1} (h - A^k g)^{1/k}, \quad \nu = \|A\| \oplus \|h - \|A g\|,
\]

and introduce the functions

\[
G(s) = \bigoplus_{k=1}^{n-1} s^{-k}\|A^{k+1}\| \oplus \bigoplus_{i+j=k} \bigoplus_{i,j \geq 0} \|h - A^{i}\||A^{j+1}g\|, \quad s > 0.
\]

\[
H(t) = \bigoplus_{k=1}^{n-1} t^{-1/k}\|A^{k+1}\|^{1/k} \oplus \bigoplus_{i+j=k} \bigoplus_{i,j \geq 0} \|h - A^{i}\|^{1/k}\|A^{j+1}g\|^{1/k}, \quad t > 0.
\]

We note that both functions are monotone decreasing. Furthermore, it is not difficult to verify by direct calculation that the inequalities

\[
G(s) \leq t, \quad H(t) \leq s
\]

are equivalent in the sense that all solutions of the first inequality with respect to \( s \) are represented by the second inequality, and vice versa.

Finally, with the new notation, we represent the system (13) in the following form

\[
\alpha \geq \lambda \oplus \mu, \quad \beta \geq \nu \oplus G(\alpha).
\]
4.3 Derivation of Pareto frontier

Consider the area of points $(\alpha, \beta)$, which is given by the system of inequalities at (16). To determine the Pareto frontier, we need to examine the boundary of the area, which is bounded from the left by the vertical line

$$\alpha = \lambda \oplus \mu,$$

from the lower left by the curve

$$\beta = \nu \oplus G(\alpha),$$

and from below by the horizontal line

$$\beta = \nu.$$

It is clear that, since the function $G(\alpha)$ is monotone decreasing in $\alpha$, the Pareto frontier for the problem is a segment of the curve that lies right of the vertical line and above the horizontal line, or a single point if the curve lies below the intersection of these lines.

To describe the Pareto frontier and related Pareto-optimal solutions, we examine two cases. First, we assume that the following condition holds:

$$\lambda \oplus \mu \geq H(\nu).$$

Under this condition, it follows from the inequality $\alpha \geq \lambda \oplus \nu$ that the inequality $\alpha \geq H(\nu)$ is satisfied. Solving the last inequality with respect to $\nu$ yields $\nu \geq G(\alpha)$. As a result, system (16) becomes

$$\alpha \geq \lambda \oplus \mu, \quad \beta \geq \nu,$$

which gives the Pareto frontier reduced to a single point $(\alpha, \beta)$, where

$$\alpha = \lambda \oplus \mu = \bigoplus_{k=1}^{n} \text{tr}^{1/k}(A^{k}) \oplus \bigoplus_{k=1}^{n-1} (h^{-}A^{k}g)^{1/k}, \quad \beta = \nu = \|A\| \oplus \|h^{-}\|\|Ag\|.$$

An example of the Pareto frontier for this case is shown in Fig. 1 (left) by the thick dot in the intersection of the lines $\alpha = \lambda \oplus \mu$ and $\beta = \nu$.

Now suppose that the opposite condition is valid in the form

$$\lambda \oplus \mu < H(\nu).$$

Then, system (16) defines an area, which is given by the conditions

$$\lambda \oplus \mu \leq \alpha \leq H(\nu), \quad \beta \geq \nu \oplus G(\alpha),$$

with its lower left boundary defined as

$$\lambda \oplus \mu \leq \alpha \leq H(\nu), \quad \beta = \nu \oplus G(\alpha).$$
Let us verify that, under the condition \( \lambda \oplus \mu \leq \alpha \leq H(\nu) \), we have \( G(\alpha) \geq \nu \), and hence the last equality can be reduced to \( \beta = G(\alpha) \).

Since \( G(\alpha) \) is a monotone decreasing function of \( \alpha \), it is sufficient to verify that \( G(\alpha) \geq \nu \) for \( \alpha = H(\nu) \). We consider the equality

\[
H(\nu) = \bigoplus_{k=1}^{n-1} (\|A\| \oplus \|h^{-}\| \|Ag\|)^{-1/k} \|A^{k+1}\|^{1/k}
\]

\[
+ \bigoplus_{k=1}^{n-2} (\|A\| \oplus \|h^{-}\| \|Ag\|)^{-1/k} \bigoplus_{i+j=k} \bigoplus_{i,j \geq 0} \|h^{-}A^{i}\|^{1/k} \|A^{j+1}g\|^{1/k},
\]

and note that this equality is valid in two cases.

For the first case, we assume that there exists an index \( m \) such that

\[
H(\nu) = \bigoplus_{k=1}^{n-1} (\|A\| \oplus \|h^{-}\| \|Ag\|)^{-1/k} \|A^{k+1}\|^{1/k}
\]

\[
= (\|A\| \oplus \|h^{-}\| \|Ag\|)^{-1/m} \|A^{k+1}\|^{1/m}.
\]

Then, under the assumption that \( \alpha = H(\nu) \), we obtain

\[
G(\alpha) = \bigoplus_{k=1}^{n-1} \alpha^{-k} \|A^{k+1}\| \oplus \bigoplus_{k=1}^{n-2} \alpha^{-k} \bigoplus_{i+j=k} \bigoplus_{i,j \geq 0} \|h^{-}A^{i}\| \|A^{j+1}g\|
\]

\[
\geq \bigoplus_{k=1}^{n-1} \alpha^{-k} \|A^{k+1}\| \geq \alpha^{-m} \|A^{m+1}\| = \|A\| \oplus \|h^{-}\| \|Ag\| = \nu.
\]
The other case involves the condition that, for some $m$, the following equality holds:

\[
H(\nu) = \bigoplus_{k=1}^{n-2} \left( \|A\| \oplus \|h^\perp\|\|Ag\| \right)^{-1/k} \bigoplus_{i,j \geq 0} \|h^\perp A^i\|^{1/k} \|A^{i+1}g\|^{1/k}
= \left( \|A\| \oplus \|h^\perp\|\|Ag\| \right)^{-1/m} \bigoplus_{i,j \geq 0} \|h^\perp A^i\|^{1/m} \|A^{i+1}g\|^{1/m},
\]

which, with setting $\alpha = H(\nu)$, yields

\[
G(\alpha) \geq \bigoplus_{k=1}^{n-2} \alpha^{-k} \bigoplus_{i,j \geq 0} \|h^\perp A^i\|\|A^{i+1}g\|
\geq \alpha^{-m} \bigoplus_{i,j \geq 0} \|h^\perp A^i\|\|A^{i+1}g\| = \|A\| \oplus \|h^\perp\|\|Ag\| = \nu.
\]

Since $G(\alpha) \geq \nu$ in both cases, the description of the Pareto frontier can be reduced to the system

\[
\lambda \oplus \mu \leq \alpha \leq H(\nu), \quad \beta = G(\alpha).
\]

This case is illustrated in Fig. 1 (right), where the Pareto frontier is depicted by a thick segment that is cut by the vertical line $\alpha = \lambda \oplus \mu$ and horizontal line $\beta = \nu$.

### 4.4 Pareto-optimal solution

We are now in a position to summarize the results obtained in the form of the next statement.

**Theorem 3.** For any matrix $A$ and vectors $g$ and $h$ such that $h^\perp g \leq 1$, with the notation (14)–(15), the following statements hold:

1. Under the condition $\lambda \oplus \mu \geq H(\nu)$,
   the Pareto frontier for problem (8) degenerates into the single point
   \[
   \alpha = \lambda \oplus \mu, \quad \beta = \nu.
   \]

2. Otherwise, the Pareto frontier is the segment given by the conditions
   \[
   \lambda \oplus \mu \leq \alpha \leq H(\nu), \quad \beta = G(\alpha).
   \]
All Pareto-optimal solutions are represented in parametric form as
\[ x = (\alpha^{-1}A \oplus \beta^{-1}11^TA)^*u, \quad g \leq u \leq (h^-(\alpha^{-1}A \oplus \beta^{-1}11^TA)^*)^- . \]

We now briefly discuss the computational complexity of the solution obtained. First, we note that the computational time required to obtain a solution vector depends on the time spent on calculating \( \lambda, \mu, G \) and \( H \). As it easy to see, the most computationally demanding task in evaluating \( \lambda \) and \( \mu \) is obtaining the first \( n \) powers of the matrix \( A \). Since direct multiplication of two matrices of order \( n \) takes at most \( O(n^3) \) operations, the time to obtain the powers of \( A \), and thus to calculate \( \lambda \) and \( \mu \), can be estimated \( O(n^4) \). Similar reasoning shows that, for a given argument, both functions \( G \) and \( H \) can be computed in the same time. As result, the overall computational complexity of the solution is no more than \( O(n^4) \).

Let us find the form that the solution takes in the case of two-dimensional problems. First note that, with \( n = 2 \), the notation at (14)–(15) reduces to
\[
\lambda = \text{tr} A \oplus \text{tr}^{1/2}(A^2), \quad \mu = h^-Ag, \quad \nu = \|A\| \oplus \|h^-\| \|Ag\|,
\]
\[
G(s) = s^{-1}\|A^2\|, \quad H(t) = t^{-1}\|A^2\|.
\]

Furthermore, we can see that the inequality \( \lambda\|A\| \geq \|A^2\| \) is valid for any matrix \( A = (a_{ij}) \) of second order with spectral radius \( \lambda > 0 \). To proof the inequality, we consider the matrices
\[
A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad A^2 = \begin{pmatrix} a_{11}^2 + a_{12}a_{21} & a_{11}a_{12} + a_{12}a_{22} \\ a_{11}a_{21} + a_{21}a_{22} & a_{12}a_{21} + a_{22} \end{pmatrix},
\]
and calculate
\[
\|A\| = a_{11} \oplus a_{12} \oplus a_{21} \oplus a_{22}, \quad \lambda = a_{11} \oplus (a_{12}a_{21})^{1/2} \oplus a_{22}.
\]

It remains to verify that each entry of the matrix \( A^2 \) is less than or equal to the product \( \lambda\|A\| \), and thus this product is an upper bound for \( \|A^2\| \). Let us examine the first entry of the first row in the matrix. It follows from the inequalities \( \lambda \geq a_{11} \) and \( \|A\| \geq a_{11} \) that \( \lambda\|A\| \geq a_{11}^2 \), and from the inequalities \( \lambda \geq (a_{12}a_{21})^{1/2} \) and \( \|A\| \geq a_{12} \oplus a_{21} \geq (a_{12}a_{21})^{1/2} \) that \( \lambda\|A\| \geq a_{12}a_{21} \). By combining these results, we obtain the upper bound
\[
a_{11}^2 \oplus a_{12}a_{21} \leq \lambda\|A\|.
\]

By similar argument, we ascertain that all entries in \( A^2 \) have the same upper bound, and thus the inequality \( \lambda\|A\| \geq \|A^2\| \) is satisfied. Using this inequality, we obtain
\[
(\lambda \oplus \mu)(\|A\| \oplus \|h^-\| \|Ag\|) \geq \lambda\|A\| \geq \|A^2\|,
\]
18
which yields the inequality

\[
\lambda \oplus \mu \geq (\|A\| \oplus \|h^-\| \|Ag\|)^{-1}\|A^2\| = H(\nu).
\]

The result obtained means that for any second-order matrix \(A\) with nonzero spectral radius the first condition of Theorem 3 is always fulfilled. Finally, observing that, for the matrix \(A\), we have \(\text{tr}^{1/2}(A^2) \geq \text{tr}A\) and

\[
(\alpha^{-1}A \oplus \beta^{-1}11^T A)^* = I \oplus \alpha^{-1}A \oplus \beta^{-1}11^T A,
\]
the solution is described in the next form.

**Corollary 4.** For any \((2 \times 2)\)-matrix \(A\) and 2-vectors \(g\) and \(h\) such that \(h^- g \leq 1\), the Pareto frontier for problem (8) is the single point

\[
\alpha = \text{tr}^{1/2}(A^2) \oplus h^- Ag, \quad \beta = \|A\| \oplus \|h^-\| \|Ag\|.
\]

(17)

All Pareto-optimal solutions are given in parametric form as

\[
x = (I \oplus \alpha^{-1}A \oplus \beta^{-1}11^T A)u, \quad g \leq u \leq (h^- (I \oplus \alpha^{-1}A \oplus \beta^{-1}11^T A))^-. \quad (18)
\]

We illustrate the solution obtained with a numerical example for a two-dimensional problem.

**Example 1.** Consider the problem defined in terms of the max-plus algebra \(\mathbb{R}_{\max,+}\), where \(0 = -\infty\), with the matrix and vectors given by

\[
A = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}, \quad g = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.
\]

We start with the evaluation of the optimal values \(\alpha\) and \(\beta\) of the objective functions. First, we calculate

\[
A^2 = \begin{pmatrix} 2 & 5 \\ 0 & 4 \end{pmatrix}, \quad \text{tr}(A^2) = 4, \quad h^- = \begin{pmatrix} -1 & -2 \end{pmatrix}, \quad Ag = \begin{pmatrix} 3 \\ 2 \end{pmatrix},
\]

\[
h^- Ag = 2, \quad \|A\| = 3, \quad \|h^-\| = -1, \quad \|Ag\| = 3.
\]

Next, we apply (17) to obtain

\[
\alpha = 2, \quad \beta = 3.
\]

To find the solution according to (4), we form the matrices

\[
\alpha^{-1}A = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \quad \beta^{-1}11^T A = \begin{pmatrix} -2 & 0 \\ -2 & 0 \end{pmatrix}.
\]
and then calculate
\[ I \oplus \alpha^{-1} A \oplus \beta^{-1} \mathbf{1} \mathbf{1}^T A = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}, \]
\[ h^{-1}(I \oplus \alpha^{-1} A \oplus \beta^{-1} \mathbf{1} \mathbf{1}^T A) = \begin{pmatrix} -1 & 0 \end{pmatrix}. \]

We write the solution \( \mathbf{x} = (x_1, x_2)^T \) in the form
\[ \mathbf{x} = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} \mathbf{u}, \quad \mathbf{u}' \leq \mathbf{u} \leq \mathbf{u}'' \]
where \( \mathbf{u} = (u_1, u_2)^T \) is the vector of parameters with the bounds given by
\[ \mathbf{u}' = \mathbf{g} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \mathbf{u}'' = (h^{-1}(I \oplus \alpha^{-1} A \oplus \beta^{-1} \mathbf{1} \mathbf{1}^T A))^{-1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \]

Since the components of the vector \( \mathbf{u} \) satisfy the conditions
\[ 0 \leq u_1 \leq 1, \quad u_2 = 0, \]
the components of the solution vector \( \mathbf{x} \) reduce to
\[ x_1 = u_1 \oplus 1 = 1, \quad x_2 = (-2)u_1 \oplus 0 = 0, \]
which yields the final solution in the form of the single vector
\[ \mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \]

### 5 Application to bi-objective scheduling problem

Consider the bi-objective scheduling problem at (1) and note that the representation of both the objective functions and the constraints involves only the arithmetic operations of maximum, addition and additive inversion (subtraction). As a consequence, we can rewrite (1) in the max-plus algebra setting by changing the operation symbols to obtain a tropical optimization problem that is to

\[
\text{minimize } \left\{ \bigoplus_{i=1}^{n} y_i x_i^{-1}, \bigoplus_{i=1}^{n} y_i \bigoplus_{j=1}^{n} x_j^{-1} \right\};
\]

subject to \[ \bigoplus_{j=1}^{n} a_{ij} x_j = y_i, \]
\[ g_i \leq x_i \leq h_i, \quad i = 1, \ldots, n. \]

Furthermore, we introduce the following matrix and vectors:
\[ A = (a_{ij}), \quad \mathbf{x} = (x_i), \quad \mathbf{y} = (y_i), \quad \mathbf{g} = (g_i), \quad \mathbf{h} = (h_i), \]
and represent the problem in the vector form

\[
\begin{align*}
\text{minimize} & \quad \{ x^\top y, \ x^\top 11^T y \} ; \\
\text{subject to} & \quad Ax = y, \\
& \quad g \leq x \leq h.
\end{align*}
\]

After substitution \( y = Ax \) into the objective functions, we obtain a problem in the form of \([S]\), which has a Pareto-optimal solution given by Theorem 3.

To demonstrate application of the result obtained, we offer an example of solution of a three-dimensional problem, which, in particular, shows that, generally, the Pareto frontier may be a segment rather than a single point. Although the example deals with a somewhat artificial problem, it clearly demonstrates the proposed computational technique and its scalability to handle real-world problems of high dimension.

**Example 2.** Consider a project that consists of \( n = 3 \) activities with the start-finish constraints, release time and release deadline given by the following matrix and vectors:

\[
A = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad g = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}.
\]

To find an optimal schedule by minimizing both the maximum flow-time of activities and the project makespan, we apply Theorem 3 in the max-plus algebra setting. By convention, we represent the numerical constants (including negative integer and rational numbers) in the ordinary notation, whereas all algebraic operations are considered in terms of max-plus algebra.

First, we form the matrices

\[
A^2 = \begin{pmatrix} 3 & 3 & 4 \\ 2 & 3 & 3 \\ 1 & 1 & 2 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 4 & 5 & 5 \\ 4 & 4 & 5 \\ 2 & 3 & 3 \end{pmatrix}, \quad 11^T A = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix},
\]

and then obtain

\[
\text{tr} \ A = 1, \quad \text{tr} \ A^2 = 3, \quad \text{tr} \ A^3 = 4, \quad \|A\| = 2, \quad \|A^2\| = 4, \quad \|A^3\| = 5.
\]

Next, we calculate the vectors

\[
h^- = \begin{pmatrix} -1 & -2 & -2 \end{pmatrix}, \quad h^- A = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix},
\]

\[
Ag = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \quad A^2 g = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}, \quad A^3 g = \begin{pmatrix} 5 \\ 5 \\ 3 \end{pmatrix},
\]

21
which allow us to find the scalars
\[
\|h^{-}\| = -1, \quad \|h^{-}A\| = 1, \quad h^{-}Ag = 1, \quad h^{-}A^2g = 3,
\]
\[
\|Ag\| = 2, \quad \|A^2g\| = 4, \quad \|A^3g\| = 5.
\]

Using above results, we evaluate
\[
\lambda = \text{tr} A \oplus \text{tr}^{1/2}(A^2) \oplus \text{tr}^{1/3}(A^3) = 3/2, \quad \mu = h^{-}Ag \oplus (h^{-}A^2g)^{1/2} = 3/2,
\]
\[
\nu = \|A\| \oplus ||h^{-}|| \|Ag\| = 2.
\]

Finally, we derive the functions
\[
G(s) = s^{-1} (\|A^2\| \oplus \|h^{-}A\||\|Ag\| \oplus \|h^{-}||\|A^2g\|) \oplus s^{-2} \|A^3\| = 4s^{-1} \oplus 5s^{-2},
\]
\[
H(t) = t^{-1} (\|A^2\| \oplus \|h^{-}A\||\|Ag\| \oplus \|h^{-}||\|A^2g\|) \oplus t^{-1/2} \|A^3\|^{1/2} = 4t^{-1} \oplus (5/2)t^{-1/2}.
\]

Observing that the following equalities hold:
\[
\lambda \oplus \mu = 3/2, \quad H(\nu) = 4\nu^{-1} \oplus (5/2)\nu^{-1/2} = 2,
\]
we see that \(\lambda \oplus \mu < H(\nu)\). Thus, by Theorem 3, the Pareto frontier for the problem is the set of points \((\alpha, \beta)\) given by the conditions
\[
3/2 \leq \alpha \leq 2, \quad \beta = 4\alpha^{-1} \oplus 5\alpha^{-2}.
\]

Note that, for all \(\alpha \geq 3/2\), the inequalities \(4\alpha \geq 11/2 > 5\) hold, which yields the inequality \(4\alpha^{-1} > 5\alpha^{-2}\). As a result, we can reduce the above conditions of the Pareto frontier to those in the form
\[
3/2 \leq \alpha \leq 2, \quad \beta = 4\alpha^{-1}.
\]

A graphical illustration of the Pareto frontier is given in Fig. 2 (left).

To describe all Pareto-optimal solutions to the problem, we consider the matrix
\[
\alpha^{-1} A \oplus \beta^{-1} 11^T A = \alpha^{-1} A \oplus (-4)\alpha 11^T A.
\]

By applying the condition \(3/2 \leq \alpha \leq 2\), we represent this matrix and derive its square in the form
\[
\begin{pmatrix}
1\alpha^{-1} & 2\alpha^{-1} & 2\alpha^{-1} \\
1\alpha^{-1} & (-2)\alpha & 2\alpha^{-1} \\
(-3)\alpha & (-2)\alpha & (-2)\alpha
\end{pmatrix}, \quad
\begin{pmatrix}
3\alpha^{-2} & 0 & 4\alpha^{-2} \\
-1 & 0 & 0 \\
-1 & (-2)\alpha^2 & 0
\end{pmatrix}.
\]
Evaluation of the Kleene star matrix yields
\[
(\alpha^{-1}A \oplus \beta^{-1}11^T A)^* = I \oplus \alpha^{-1}A \oplus \beta^{-1}11^T A \oplus (\alpha^{-1}A \oplus \beta^{-1}11^T A)^2 = \begin{pmatrix}
0 & 2\alpha^{-1} & 4\alpha^{-2} \\
1\alpha^{-1} & 0 & 2\alpha^{-1} \\
-1 & (-2)\alpha & 0
\end{pmatrix}.
\]

After calculating the vector
\[
h^{-}(\alpha^{-1}A \oplus \beta^{-1}11^T A)^* = (\begin{pmatrix}
-1 & 1\alpha^{-1} & 3\alpha^{-2}
\end{pmatrix}),
\]
we obtain the solution to the problem in the form
\[
x = \begin{pmatrix}
0 & 2\alpha^{-1} & 4\alpha^{-2} \\
1\alpha^{-1} & 0 & 2\alpha^{-1} \\
-1 & (-2)\alpha & 0
\end{pmatrix} u, \quad u' \leq u \leq u'', \quad 3/2 \leq \alpha \leq 2,
\]
where \( u = (u_1, u_2, u_3)^T \) is the vector of parameters with bounds given by
\[
u' = g = \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}, \quad u'' = (h^{-}(\alpha^{-1}A \oplus \beta^{-1}11^T A)^*)^{-} = \begin{pmatrix}
1 \\
(-1)\alpha \\
(-3)\alpha^2
\end{pmatrix}.
\]

To simplify the solution, we note that the last two columns in the Kleene star matrix are collinear, and hence this matrix can be factored as follows:
\[
(\alpha^{-1}A \oplus \beta^{-1}11^T A)^* = \begin{pmatrix}
0 & 2\alpha^{-1} \\
1\alpha^{-1} & 0 \\
-1 & (-2)\alpha
\end{pmatrix} \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 2\alpha^{-1}
\end{pmatrix},
\]
where we use the notation \( 0 = -\infty \).

We introduce a new vector of parameters \( v = (v_1, v_2)^T \) by the equality
\[
v = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 2\alpha^{-1}
\end{pmatrix} u.
\]

After turning from \( u \) to \( v \), the solution takes the simpler form
\[
x = \begin{pmatrix}
0 & 2\alpha^{-1} \\
1\alpha^{-1} & 0 \\
-1 & (-2)\alpha
\end{pmatrix} v, \quad v' \leq v \leq v'', \quad 3/2 \leq \alpha \leq 2,
\]
where the boundaries for \( v \) are derived from \( u' \) and \( u'' \) to be
\[
v' = \begin{pmatrix}
0 \\
2\alpha^{-1}
\end{pmatrix}, \quad v'' = \begin{pmatrix}
1 \\
(-1)\alpha
\end{pmatrix}.
\]
Moreover, it follows from the boundary conditions
\[ 0 \leq v_1 \leq 1, \quad 2\alpha^{-1} \leq v_2 \leq (-1)\alpha \]
that \( 1\alpha^{-1}v_1 \leq 2\alpha^{-1} \leq v_2 \) and \((-1)v_1 \leq 0 \leq (-2)\alpha v_2\), which allows to write
\[ x_2 = 1\alpha^{-1}v_1 \oplus v_2 = v_2, \quad x_3 = (-1)v_1 \oplus (-2)\alpha v_2 = (-2)\alpha v_2. \]

As a result, we can finally represent the Pareto-optimal solution of the problem in the form
\[
\mathbf{x} = \begin{pmatrix} 0 & 2\alpha^{-1} \\ 0 & 0 \\ 0 & (-2)\alpha \end{pmatrix} \mathbf{v}, \quad \begin{pmatrix} 0 \\ 2\alpha^{-1} \end{pmatrix} \leq \mathbf{v} \leq \begin{pmatrix} 1 \\ (-1)\alpha \end{pmatrix}, \quad 3/2 \leq \alpha \leq 2. \tag{19}
\]

Note that the vectors \( \mathbf{v}' \) and \( \mathbf{v}'' \) provide the boundaries for \( \mathbf{x} \), given by
\[
\mathbf{x}' = \begin{pmatrix} 4\alpha^{-2} \\ 2\alpha^{-1} \\ 0 \end{pmatrix}, \quad \mathbf{x}'' = \begin{pmatrix} 1 \\ (-1)\alpha \\ (-3)\alpha^2 \end{pmatrix}.
\]

We conclude this example by calculating the solutions, which correspond to the two extreme and one inner points of the Pareto frontier.

First, consider the point \((\alpha, \beta)\), where \( \alpha = 3/2 \) and \( \beta = 4\alpha^{-1} = 5/2 \), which corresponds to the best solution with respect to the minimum of the maximum flow time. Then, the solution at (19) becomes
\[
\mathbf{x} = \begin{pmatrix} 0 & 1/2 \\ 0 & 0 \\ 0 & -1/2 \end{pmatrix} \mathbf{v}, \quad \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} \leq \mathbf{v} \leq \begin{pmatrix} 1 \\ 1/2 \end{pmatrix}.
\]

It is not difficult to verify that, in this case, we have
\[
\mathbf{x} = \mathbf{x}' = \mathbf{x}'' = \begin{pmatrix} 1 \\ 1/2 \\ 0 \end{pmatrix},
\]
and thus, the Pareto-optimal solution is the single vector with components
\[
x_1 = 1, \quad x_2 = 1/2, \quad x_3 = 0.
\]

Next, we examine the solution corresponding to \( \alpha = 2 \) and \( \beta = 4\alpha^{-1} = 2 \), which is the best with respect to the project makespan. The solution takes the form
\[
\mathbf{x} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{v}, \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix} \leq \mathbf{v} \leq \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\]
In scalar form, the solution obtained can be represented as

\[ x_1 = v_1 \oplus v_2, \quad x_2 = x_3 = v_2, \quad 0 \leq v_1, v_2 \leq 1, \]

or, equivalently, using only one parameter \( v \), as

\[ v \leq x_1 \leq 1, \quad x_2 = x_3 = v, \quad 0 \leq v \leq 1. \]

Finally, consider the inner point of the Pareto frontier with \( \alpha = 5/3 \) and \( \beta = 4\alpha^{-1} = 7/3 \). We have the solution

\[ x = \begin{pmatrix} 0 & 1/3 \\ 0 & 0 \\ 0 & -1/3 \end{pmatrix} v, \quad \begin{pmatrix} 0 \\ 1/3 \end{pmatrix} \leq v \leq \begin{pmatrix} 1 \\ 2/3 \end{pmatrix}. \]

In scalar form, the solution is written as

\[ x_1 = v_1 \oplus (1/3)v_2, \quad x_2 = v_2, \quad x_3 = (-1/3)v_2, \quad 0 \leq v_1 \leq 1, \quad 1/3 \leq v_2 \leq 2/3, \]

or, in terms of the conventional algebra, as

\[ x_1 = \max\{v_1, v_2+1/3\}, \quad x_2 = v_2, \quad x_3 = v_2-1/3, \quad 0 \leq v_1 \leq 1, \quad 1/3 \leq v_2 \leq 2/3. \]

Another equivalent representation of the solution takes the form

\[ (1/3)v \leq x_1 \leq 1, \quad x_2 = v, \quad x_3 = (-1/3)v, \quad 1/3 \leq v \leq 2/3, \]

which can also be rewritten in the usual notation as

\[ v + 1/3 \leq x_1 \leq 1, \quad x_2 = v, \quad x_3 = v - 1/3, \quad 1/3 \leq v \leq 2/3. \]

The optimal solutions obtained are shown in Fig. 2 (right). The thick dot and the big shaded triangle represent the solutions, which correspond to the extreme points of the Pareto frontier with \( \alpha = 3/2 \) and \( \alpha = 2 \), whereas the small triangle does the solution for the inner point for \( \alpha = 5/3 \).

### 6 Conclusions

In the paper, a temporal bi-objective project scheduling problem without cost dependencies and resource requirements has been examined. Given temporal constraints that include start-finish precedence relations, release time and release deadline, the problem is to develop a schedule that minimizes both the maximum flow-time over all activities and the project makespan.

We have represented the problem as a bi-objective optimization problem in terms of tropical (idempotent) mathematics, which concerns with the theory and applications of idempotent semirings and semifields. By using methods and techniques of tropical optimization, we obtain a complete
Pareto-optimal solution of the problem in an exact analytical form, which is suitable for formal analysis and computations with polynomial time.

Further investigation can include the derivation of analytical solutions to bi-objective and multiobjective problems with new criteria, including minimum deviation from due dates, minimum deviation of start times or finish times of activities, as well as with additional constraint, such as due-dates, deadlines, start-start and finish-start precedence relations.

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