PHOTON DAMPING OF WAVES IN ACCRETION DISKS

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ABSTRACT

MHD turbulence is generally believed to have two important functions in accretion disks: it transports angular momentum outward, and the energy in its shortest wavelength modes is dissipated into the heat that the disks radiate. In this paper we examine a pair of mechanisms that may play an important role in regulating the amplitude and spectrum of this turbulence: photon diffusion and viscosity. We demonstrate that in radiation pressure–dominated disks, photon damping of compressive MHD waves is so rapid that it likely dominates all other dissipation mechanisms.

Subject headings: accretion, accretion disks — MHD — radiation mechanisms: nonthermal — turbulence — waves

1. INTRODUCTION

Turbulence is widely thought to be central to the dynamics of accretion disks. A combination of magnetic and Reynolds turbulent stresses may be responsible for the outward transport of angular momentum without which no accretion could occur (Shakura & Sunyaev 1973, Balbus, Gammie, & Hawley 1994). The energy put into this turbulence is ultimately deposited as heat and is therefore the energy source for the radiation by which we observe accretion disks. Although much effort has gone into identifying mechanisms that excite turbulence (Balbus & Hawley 1991), far less attention in the literature has been given to how the turbulence dissipates. In most instances, it is simply assumed that nonlinear couplings transfer energy from long wavelengths to short, and that some dissipative mechanism eventually damps very short wavelength motions.

One reason why little thought has been given to the specifics of dissipation is that, as matter drifts inward through an accretion disk, if the disk is in a time-steady state its lost potential energy is transformed into heat and kinetic energy at a rate that is entirely fixed by global properties. If the gravitational potential is dominated by the mass $M$ of the central object, the heating rate per unit area is

$$ Q = \frac{3}{4\pi} \frac{GM^3}{r^3} R_g(r). $$

$R_g$ (≈ 1 at large radii) describes the reduction of the local heating due both to the kinetic energy carried outward with the angular momentum flux, and relativistic effects should the central object be a neutron star or black hole (Novikov & Thorne 1973).

It is a great simplification to calculations of disk equilibria that the heating rate should depend only on global quantities. However, this fact leaves open the question of how exactly the energy lost by the accretion flow is transformed into heat, and there are strong observational consequences that depend on just how this happens. For example, the existence of weakly radiative disks (Ichimaru 1977; Rees et al. 1982; Narayan & Yi 1995) depends critically on the assumption that most of the heat goes to the ions, not the electrons. There have been other suggestions that a significant part of the heat goes into nonthermal particle distributions (e.g., Ferrari 1984; Stecker et al. 1991). Alternatively, the energy can be lost in magnetic fields that escape the disk, forming a corona or outflow (Galeev, Rosner, & Vaiana 1979).

Balbus & Hawley (1991) pointed out that MHD fluctuations should be linearly unstable in weakly magnetized accretion disks. Fully nonlinear simulations (Hawley, Gammie, & Balbus 1995; Brandenburg et al. 1995; Stone et al. 1996) have shown that these fluctuations grow until the field energy density approaches the pressure in the disk, and that nonlinear couplings create fluctuations on shorter and shorter wavelengths. Most recent work on how the energy in these fluctuations is dissipated has concentrated on plasma physics effects that work on modes of very short wavelength (e.g., Bisnovatyi-Kogan & Lovelace 1997; Quataert 1998; Blackman 1998; Gruzinov 1998), especially in low-density, high-temperature disks.

Although this focus is well grounded in reality in the context of MHD turbulence in laboratory plasmas, it ignores the fact that accretion disks are often extremely bright and can contain such high photon densities that radiation dominates the total pressure. In this paper we point out that photon diffusion and viscosity can, in radiation-dominated accretion disks, dominate all other mechanisms of dissipation. When that is so, compressive modes whose wavelengths are almost as great as a disk thickness can be rapidly damped. Significant consequences follow for the amplitude of MHD turbulence, the rate at which angular momentum may be transported, and the way in which the energy associated with the turbulence is dissipated into heat.

The structure of this paper is as follows: we first extend the theory of MHD modes interacting with a background photon gas by substituting a time-dependent radiation transfer solution for the conventional description in terms of a photon viscosity (§ 2). Our procedure is similar in character to the one adopted to treat photon diffusion damping of perturbations in the early universe (“Silk damping”: Silk 1968; Hu & Sugiyama 1996). We then apply this improved theory to conventional accretion disk models (§ 3). In § 4 we discuss the impact of photon damping on both advection-dominated accretion disks and disks in which the dissipation is segregated into a corona. Finally, in § 5 we summarize our results and discuss their significance.

We close this Introduction with some notes of distinction. There were earlier suggestions by Loeb & Loeb (1992) and Tsuribe & Umemura (1997) that photon viscosity due to an
external radiation field might explain the radial angular momentum transport in some accretion disks. We do not make that claim; in this paper we consider only how photon kinetic effects help regulate the amplitude of the MHD turbulence that is responsible for angular momentum transport. The effects of photon damping we consider are for a scattering-dominated plasma, and thus the relations we derive are different from those found by Bogdan & Knöller (1989) and Mihalas & Mihalas (1983), who derived the dispersion relation for a radiation field in LTE, ignoring both scattering and radiation viscosity, which we include. Our problem also differs from that treated by Cassen & Woolum (1996), who considered only optically thick spiral waves that lose angular momentum through radiation. Our equations are very similar to those of Jedamzik, Katalinić & Olinto (1998) and Subramanian & Barrow (1997) in the diffusion and free-streaming limits; however, we have bridged the two regimes by truncating the radiation field moment expansion above quadrupole moment. We also note that Thompson & Blaes (1998) have considered radiation damping for waves in the context of gamma-ray bursts.

2. EQUATIONS

Our aim in this section is to derive a dispersion relation for MHD waves in the presence of a background radiation field. In a sense, this is not a fully self-consistent approach, since the linearized equations are only appropriate when the turbulent velocities are small in the fluid frame, yet the dissipation of significant turbulent motions is the source of energy for the radiation. Nonetheless, we believe our approach should lead to a reasonable approximation of the truth. Simulations show that when the only damping is numerical, the turbulence spectrum declines sharply toward shorter wavelengths. Thus, the short-wavelength modes are legitimately in the linear regime relative to the “equilibrium” background provided by larger amplitude, longer wavelength fluctuations, except that there exist nonlinear couplings that cause the cascade of energy to smaller scales. A linear dispersion relation should at least provide a qualitative indication of the major effects.

2.1. Photon Damping

We begin with a qualitative description of the different regimes of photon damping. When radiation pressure in a fluid is significant compared to gas pressure, momentum and energy can be transported by radiation in such a way as to damp out perturbations in the fluid. There are two relevant length scales: $k_T^{-1} = 1/n_e\sigma_T$ ($n_e$ is the electron number density and $\sigma_T$ is the Thomson scattering cross section), the photon mean free path, and $k_D^{-1} \approx k_T^{-1}c/\sigma_T$, the diffusion length ($\sigma_T$ is the phase speed of long-wavelength acoustic perturbations). These two wavelengths define three characteristic regimes:

1.—Optically thin regime: When the wavenumber $k > 2\pi k_T$, photons can travel freely across a wavelength. The Doppler shift due to fluid motion creates a flux in the fluid rest frame that acts as a headwind for the electrons. As we will show later, this effect leads to a damping rate that is independent of $k$.

2.—Nondiffusive regime: This is the range of wavenumbers $k_D < k < 2\pi k_T$. In this regime, although a single wavelength is optically thick, photons can diffuse out of a fluctuation in a single wave period. This effect will prove especially important to compressive waves.

3.—Optically thick diffusive regime: When $k < k_D$, photons are effectively dragged along with the fluid oscillations. Their diffusion can be described well by conventional transport coefficients (Weinberg 1972). If one thinks of the system as a single fluid, these correspond to shear viscosity and (a version of) heat conduction. Mihalas & Mihalas (1984) and references therein have derived these coefficients in the diffusion approximation.

With these wavelength distinctions in mind, we now derive the exact dispersion relations for MHD modes damped by radiation transport.

2.2. Radiation Transfer Equation

Mihalas & Mihalas (1984) derived the equations of radiation viscosity in the limit of time-steady, diffusive behavior. They additionally assumed pure absorptive opacity and LTE, in contrast to our assumption of pure isotropic scattering; however, this does not affect radiation viscosity. Our case involves time-dependent behavior and gradients that may be so sharp as to completely invalidate the diffusion approximation. Consequently, we must rederive the equations of radiation viscosity in a way that is appropriate for our regimes of interest.

We write down the radiation transfer equation in a quasi-inertial “lab” frame that travels along with the local mean orbital velocity. We neglect rotation because we will be interested only in fluctuation wavelengths very short compared to a radius (in fact, for some purposes to make rotation negligible requires a stronger constraint to wavelengths very short compared to a disk thickness). We also neglect the thermal source function, absorption opacity, and stimulated scattering. The source function is then solely due to electron scattering. Evaluated in the lab frame and averaged over frequency, it is (Pomraning 1973, eq. [1])

$$\delta \ell (\mathbf{n}, \nu) = \frac{1}{\sigma_T} \int dv_f dv_i d\Omega_i \frac{v_i}{v_f} \frac{d\sigma_T}{dv_i} (v_i \to v_f, \mathbf{n}_i \to \mathbf{n}_f) I(\mathbf{n}_i, \nu)$$

(2)

where $I(\mathbf{n}, \nu)$ is the specific intensity in the direction $\mathbf{n}$ at frequency $\nu$, $i$ and $f$ subscripts indicate the initial and final photon respectively, $\sigma_T$ is the Thomson scattering cross section, and $\mathbf{n}$ is the direction of photon motion in the lab frame. If the fluid moves with velocity $\mathbf{b}$ (in units of $c$) relative to the lab frame, we have the following relations, correct to first order in $\mathbf{b}$:

$$\frac{v_f}{v_i} = 1 - \beta \cdot (\mathbf{n}_i - \mathbf{n}_f),$$

(3)

$$\frac{d\sigma_T}{d\Omega_i} (v_i \to v_f, \mathbf{n}_i \to \mathbf{n}_f) = [1 + \beta \cdot (\mathbf{n}_i - \mathbf{n}_f)]$$

$$\times \delta [v_f(1 - \beta \cdot \mathbf{n}_f) - v_i(1 - \beta \cdot \mathbf{n}_i)] \sigma_T \frac{\sigma_T}{4\pi},$$

(4)

where the first relation is the familiar frequency shift due to Compton scattering, in which we have neglected terms of order $\hbar v/m_e c^2$; the second is the transformation between frames for the Thomson scattering cross section, where we have made the approximations of isotropic scattering and negligible electron recoil.
The first four moments of the frequency-integrated specific intensity are:

\[
J = \frac{1}{4\pi} \int d\Omega I(n) \quad (5)
\]

\[
H = \frac{1}{4\pi} \int d\Omega n I(n) \quad (6)
\]

\[
K_{ij} = \frac{1}{4\pi} \int d\Omega n_i n_j I(n) \quad (7)
\]

\[
L_{ijk} = \frac{1}{4\pi} \int d\Omega n_i n_j n_k I(n) \quad (8)
\]

where \(d\Omega = \sin \theta \, d\theta \, d\phi\), \(n\) is the unit vector pointing in the \((\theta, \phi)\) direction, and \(I(n)\) is the frequency-integrated specific intensity.

Integrating the source function (2) over solid angle and frequency and keeping only terms of first order in \(T\), we get

\[
S_i(n) = (1 + 3\beta \cdot n)J - 2\beta \cdot H \quad . (9)
\]

The first four moments of the frequency-integrated specific intensity are:

\[
1 + \frac{\partial I(n)}{\partial t} + \frac{n \cdot \nabla I(n)}{k_T} = (1 + 3\beta \cdot n)J - 2\beta \cdot H - (1 - n \cdot \beta)I(n) \quad , (10)
\]

The last term on the right-hand side of this equation is due to electron-scattering opacity, boosted from the fluid frame to the lab frame. This equation agrees with Psaltis & Lamb (1997), except that we have dropped terms of second order in \(\beta\) and have ignored the temperature of the electrons. Taking the first moment of this equation \([(1/4\pi) \int d\Omega]\), we get

\[
1 \frac{\partial J}{\partial t} + \frac{1}{k_T} \nabla \cdot H = -\beta \cdot H \quad . (11)
\]

Next, taking the second moment \([(1/4\pi) \int d\Omega n I(n)]\) gives

\[
1 \frac{\partial H_i}{\partial t} + \frac{1}{k_T} \nabla_i K_{ji} = \beta_i J + \beta_j K_{ji} - H_i \quad . (12)
\]

Finally, taking the third moment of the transfer equation, we find

\[
1 \frac{\partial K_{ij}}{\partial t} + \frac{1}{k_T} \nabla_k L_{ijk} = \delta_{ij} J - \frac{2}{3} \delta_{ij} \beta \cdot H - K_{ij} + \beta_k L_{ijk} \quad . (13)
\]

Now, to close these equations, we must make some assumption about the form of the radiation field. The Eddington approximation is equivalent to setting the quadrupole and higher moments to zero. However, we want to consider the effect of radiation viscosity, which is present only if there is shear, and this requires a term of quadrupole order or higher in the radiation field. We therefore set all higher moments to zero but retain the monopole, dipole, and quadrupole moments:

\[
I(n) = I_1 + I_2 n \cdot n_D + \frac{I_4}{2} [3(n \cdot n_\phi)^2 - 1] \quad , (14)
\]

where \(n_D\) is the direction of the dipole moment and \(n_\phi\) is the direction of the quadrupole moment, and \(I_{1,2,4}\) are independent of \(n\). Using this multipole expansion for the intensity, the fourth moment of the radiation field (eq. [8]) can be expressed in terms of the flux

\[
L_{ijk} = \frac{1}{2} (\delta_{ij} H_k + \delta_{ik} H_j + \delta_{jk} H_i) \quad , (15)
\]

using the relation \(\int (d\Omega/4\pi) n_i n_j n_k = (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{lj} + \delta_{jk} \delta_{li} + \delta_{il} \delta_{jk})/15\). This result allows us to express the third moment of the transfer equation (13) as

\[
1 \frac{\partial K}{\partial t} + \frac{1}{5k_T} (\delta_{ij} \partial \cdot V + \nabla_i H_j + \nabla_j H_i) = \frac{\delta_{ij} (J - 2\beta \cdot H) - K_{ij} + \frac{1}{5} (\delta_{ij} \beta \cdot H + \beta_i H_j + \beta_j H_i)}{I_1(\beta_i, \phi) / h} \quad , (16)
\]

Finally, we need to calculate the effect of the photons on the electrons. The rate of momentum transfer from the photons to the electrons via Compton scattering is

\[
C = \int dv \, d\Omega \, d\Omega' d^3\beta' \Delta p_n \frac{d\tau}{d\Omega} f(\beta')(1 - \beta' \cdot n') \times \frac{I(\beta', \phi')}{h} \quad , (17)
\]

where primed quantities refer to the particles before scattering in the lab frame, \(\beta'\) is the electron velocity, \(f(\beta)\) is the electron distribution function, and \(\Delta p\) is the momentum transferred during the scattering. We make the assumption that all electrons move with the fluid velocity \(\beta\), i.e., \(f(\beta) = \delta^3(\beta - \beta)\). In the limit of nonrelativistic electron speeds, the momentum equation would be unchanged if we had instead averaged over a finite-width velocity distribution. We again assume the scattering is isotropic and we ignore terms of order \(hv/m_e c^2\) and higher. Performing the integral in equation (17) and keeping only terms of order \(\beta\), we find

\[
C_i = -\frac{4\pi k_T}{c} (\beta_i J + \beta_j K_{ji} - H_i) \quad , (18)
\]

which is proportional to the right-hand side of equation (12). We can then use this electron-photon momentum transfer rate in the fluid momentum equation. Ignoring all other forces, the fluid momentum equation is

\[
\rho \frac{\partial v}{\partial t} = C \quad , (19)
\]

where \(v = c\beta\). In most cases, \(C \cdot \beta\) is negative, and thus there is generally a drag on the fluid due to collisions with photons.

We assume that in the equilibrium state the radiation is uniform, time independent, and isotropic so that \(J = I\). The higher order moments of the unperturbed radiation field are then simply \(H_i = 0\), \(K_{ij} = \delta_{ij} J/3\), and \(L_{ijk} = 0\). We also assume that the unperturbed fluid is at rest in the lab frame. These assumptions greatly simplify the equations and retain most of the physics of the waves. We assume that all perturbations vary with spacetime dependence \(e^{ik \cdot x - \omega t}\), e.g., that the perturbed mean intensity is \(J + \delta J e^{ik \cdot x - \omega t}\); for this reason we must also restrict attention to modes with \(kh \gg 1\). The perturbed radiation transfer equations are

\[
\delta J = c \frac{k \cdot \delta H}{\omega} \quad , (20)
\]
modes with \( k \), the momentum transfer rate is frequency, optically thick limit), the momentum transfer for incompressive waves, in the limit that (low-

\[ u_6 \]

\[ \frac{\delta J}{3} \delta J - \delta K_{ij} \cdot J - \delta H_i, \]  

Solving these equations for \( \delta J \) in terms of \( \delta \beta \) yields

\[ \delta J = \frac{4}{3} J (1 - i \omega) \left[ \frac{(1 - i \omega)}{4} - i \frac{\delta \beta}{k T} \left( 1 - \frac{9}{5} i \omega \right) \right] \delta \beta, \]  

where we have defined normalized variables \( \tilde{k} \equiv k/k_T \), and \( \tilde{\omega} \equiv \omega/(k T) \), so that the optically thin regime corresponds to \( \tilde{k} > 2 \pi \). The perturbed mean intensity disappears for modes with \( k \perp \delta \beta \), since there is no compression of the radiation field.

Next, the perturbed flux is

\[ \delta H = \frac{4}{3} J (1 - i \omega) \left[ \frac{(1 - i \omega)}{4} - i \frac{\delta \beta}{k T} \left( 1 - \frac{9}{5} i \omega \right) \right] \delta \beta, \]  

The perturbed collision integral is

\[ \delta C = - \frac{4 \pi k T}{c} \left[ \frac{4}{3} \delta \beta J - \delta H \right]. \]  

For incompressive waves, in the limit that \( \tilde{\omega} \ll \tilde{k}^2 \ll 1 \) (low-frequency, optically thick limit), the momentum transfer rate is

\[ \delta C = - \frac{4 \pi k T}{c} \delta \beta, \]  

Thus, the photon-fluid friction is proportional to \(- k^2 \delta \beta \), which looks like a \( V^2 \eta \) term, with a constant of proportionality \( 4 \pi k T \). This is the same as the radiation viscosity term derived by multiple authors, e.g., Mihalas & Mihalas (1984). Indeed, the photon viscosity computed by Loeb & Laor (1992) for a steady shear flow gives exactly this viscosity term, except with a factor of 10/9 from considering the exact differential Thomson cross section (rather than assuming isotropic scattering as we did above). Including polarization introduces another small correction factor (Hu & Sugiyama 1996). In the large \( k/k_T \) (optically thin) limit, the friction approaches a constant, which is what one would expect for electrons in a uniform radiation field; since the wavelength is much shorter than the photon mean free path, each electron sees the averaged radiation from several wavelengths. The friction changes when \( \omega \neq 0 \) to take into account time-dependent diffusion of the radiation as the wave oscillates.

2.3. MHD Equations

With the perturbed radiation quantities in hand, we now write down the perturbed MHD equations of motion. We define the \( z \)-axis as the direction of the magnetic field and ignore both gravitational potential gradients and rotation, in keeping with our restriction to modes with \( k h \gg 1 \). Ignoring rotation is valid for \( \omega > \Omega \) or \( k h > c_s/v_A \). By omitting rotation effects, we restrict our attention to wavenumbers short enough that the Balbus-Hawley instability does not operate. The effects of vertical gravity on radiation waves in accretion disks has been considered by Gammie (1998), who found unstable “photon bubble” modes. Our equations ignore these modes; however, we do include radiation viscosity, which Gammie (1998) ignored. The MHD equations are

\[ \rho \frac{\partial v}{\partial t} + (v \cdot \nabla) v + \nabla \left( P_{\text{gas}} + \frac{B^2}{8\pi} \right) - \frac{1}{4\pi} (B \cdot \nabla) B - C = 0, \]  

\[ \frac{\partial P_{\text{gas}}}{\partial t} + (v \cdot \nabla) P_{\text{gas}} = 0, \]  

where \( P_{\text{gas}} = N k_B T \) and \( k_B \) is Boltzmann’s constant.

Assuming an equilibrium state with \( \rho \) and \( B \) constant and \( v = 0 \), the perturbed equations are

\[ - i \omega \rho \delta v + ik \delta \rho_{\text{gas}} + \frac{i}{4\pi} \left[(k \cdot \delta B) - (B \cdot \delta k)B\right] = -C = 0, \]  

\[ \omega \delta B = (k \cdot \delta v) - (B \cdot \delta k)B, \]  

We will assume that \( \delta P_{\text{gas}}/\delta \rho = c_s^2 \) is constant in what follows. Combining these equations gives

\[ \omega^2 \delta v - \left( \frac{c_s^2}{c} \right)^2 \left( k \cdot \delta v \right) k - \left( v_A/c \right)^2 \times \left[(k \cdot \delta v) k + k^2 z \delta v - k_z \cdot \delta v \right] \]  

\[ \omega \delta B = (k \cdot \delta v) - (B \cdot \delta k)B, \]  

where \( v_A = (B^2/4\pi\rho)^{1/2} \), the Alfvén speed.

Now, define \( k = (\sin \theta \hat{x} + \cos \theta \hat{z}) \). Setting the determinant of equation (29) to zero, and using equations (24) and (25), we find the following dispersion relation:

\[ A_1 [(A_1 + A_3 \cos \theta)^2 + (A_1 A_2 - A_3^2 k^2)] = 0, \]  

where we have defined the auxiliary quantities

\[ A_1 = (\omega^2 - v_A^2 k^2 \cos^2 \theta + i \omega \Gamma)D_1 - i \omega \Gamma(1 - i \omega), \]  

\[ A_2 = -(\tilde{v}_A^2 + \tilde{u}_A^2)D_1 - i \omega \Gamma(5 - 6i\omega)(1 - i \omega)D_2, \]  

\[ A_3 = \tilde{v}_A^2 k \cos \theta D_1, \]  

\[ D_1 = (1 - i \omega)^2 + \frac{1}{2} k^2, \]  

\[ D_2 = 15(1 - i \omega)^2 + ik^2(5 - 9i \omega), \]  

\[ \tilde{v}_A = v_A/c, \]  

\[ \tilde{c}_g = c_g/c, \]  

\[ \Gamma = \Gamma/(k T c) = 4P_{\text{rad}}/(\rho c^2) = 3c_s^2/c^2. \]  

In the limit \( P_{\text{rad}} = 0 \), this dispersion relation becomes the MHD dispersion relation for Alfvén modes and magnetosonic modes (e.g., Jackson 1975). For nonzero \( P_{\text{rad}} \), this dispersion relation admits a variety of modes: modified versions of Alfvén modes (incompressive); fast and slow magnetosonic modes (compressive); and radiative (electromagnetic) modes. On account of the complexity of the full dispersion relation, we will discuss some simplified
limits. The dispersion relation for Alfvén-like modes factors out separately, \( A_1 = 0 \). We will discuss this branch in the next subsection. When \( \theta = 0 \) or \( \theta = \pi/2 \), the part of the dispersion relation in brackets simplifies considerably; we will look at the compressive modes for these propagation directions in \$2.5 \).

2.4. Dispersion Relation for Incompressive Waves

The case of \( A_1 = 0 \) yields the modified Alfvén modes, for which \( k \cdot \delta v = 0 \) and \( \delta E = 0 \). Since these modes are incompressive, \( \delta \lambda = 0 \) and \( \delta H \) simplifies drastically. The equation \( A_1 = 0 \) becomes

\[
\omega^2 - k^2 v_A^2 \cos^2 \theta + i \Delta_\omega \omega = 0 ,
\]

where

\[
\Delta_i = \frac{k^2/5 - \omega^2}{(1-i\omega)^2 + k^2/2} .
\]

If \( v_A \ll c \), then \( |\omega| \ll k \) (since \( |\omega| \approx kv_A \)) and we can expand:

\[
\Delta_i = \frac{k^2/5 - \omega^2}{(1-i\omega)^2 + k^2/2} + O\left(\frac{\omega^2}{k}\right) .
\]

Substituting this expression into equation (43), we can solve for \( \omega \):

\[
\omega = \pm \sqrt{k^2 v_A^2 \cos^2 \theta + \Gamma^2 \left(\frac{k^2}{5 + k^2}\right)^2 k^2 - i \Gamma \frac{k^2 A_k}{2} \frac{5 + k^2}{5 + k^2}} ,
\]

where \( A_k = [1 + 5\Gamma(5 - k^2)/(5 + k^2)]^{-1} \). This dispersion relation is valid for any \( \Gamma \) and for \( v_A \ll c \). For \( \Gamma \ll v_A \), it simply describes damped Alfvén waves, and it agrees with Jedamzik et al. (1998) in the diffusion, optically thin, and overdamped limits. Note that the damping rate we have computed bridges the optically thin and thick limits, which have been examined separately by other authors (Subramanian & Barrow 1997; Jedamzik et al. 1998).

We also find higher frequency modes with \( |\omega| \gg \Gamma \), illustrated in Figure 1. These modes are nonpropagating in the optically thick limit and damped on a timescale \( \sim (k_T c)^{-1} \), but travel at \( c/(5^{1/2}) \) when their wavelengths are shorter than a photon scattering length. Their dispersion relation is to first order the solution of \( D_i = 0 \), which means that \( \delta \theta \to 0 \), so that these modes are simply electromagnetic. The speed of these modes is likely an artifact of the particular form taken for the moment hierarchy closure; their speed increases when we include moments higher than quadrupole.

2.5. Dispersion Relation for Compressive Waves

Next, we consider the opposite limit of the dispersion relation: strongly compressive waves for which \( k \perp \delta \theta \). This condition permits two sorts of waves: \( \theta = 0 \) (\( k \parallel B \)), which is simply a radiation-damped sound wave; and \( \theta = \pi/2 \) (\( k \perp B \)), the fast magnetosonic wave. The slow magnetosonic wave disappears when we impose \( k \perp \delta \theta \), since it has a velocity component perpendicular to the wavevector. The damping rate of the slow magnetosonic wave is intermediate between the Alfvén wave and the fast magnetosonic wave, which is why we do not treat it here. The fast wave also has a small \( k \perp \delta \theta \) component when it propagates at an angle \( 0 < \theta < \pi/2 \); we ignore these waves here since they will have damping rates in between the \( \theta = 0 \) and \( \theta = \pi/2 \) cases. The dispersion relation for the \( \theta = \pi/2 \) case is

\[
\omega^2 - k^2 (v_g^2 + v_A^2) + i \Delta_\omega \omega = 0 ,
\]

where we have defined

\[
\Delta_\omega = \frac{1 - 1 - i\omega}{D_1} \left[ 1 - \frac{k^2}{D_2} (5 - 6i\omega) \right] .
\]

In the diffusive limit, \( k \ll k_T \), the fast magnetosonic dispersion relation becomes

\[
\omega^2 \left(1 + 3 \frac{c_g^2}{c_T^2}\right) - k^2 (v_g^2 + v_A^2 + c_T^2)
\]

\[
+ i\Gamma \omega \left(\frac{2}{5} k^2 + \frac{k^4}{9\omega^2} + \omega^2\right) + O\left(\frac{1}{k_T^3}\right) = 0 ,
\]

where \( c_T^2 = 4P_{rad}/(3\rho) \) is the sound speed due to radiation and we have assumed that all velocities are smaller than \( c \).

Since \( k \ll k_T \), \( k \) is very much smaller than \( k_T \), it makes sense to expand in terms of \( k/k_T \). In terms of this expansion, the zeroth order solution to equation (38) is

\[
\omega_0 = \frac{c_g^2 + c_A^2 + c_T^2}{1 + (3c_g^2/c_T^2)} k .
\]

We now substitute this approximate solution into the next higher order term in equation (38) and obtain a quadratic equation for \( \omega \) with the solution:

\[
\omega = \pm \omega_0 - i \frac{ck^2}{6k_A(1 + R)^2[1 + R(c_g^2 + v_A^2)/c_T^2]}
\]

\[
\times \left[ R^2 + \frac{4}{5} (1 + R) + \frac{4}{5} \left(\frac{R}{5}\right) \frac{c_g^2 + v_A^2}{c_T^2}\right] R
\]

\[
+ \frac{9(c_g^2 + v_A^2)R^2}{c_T^2} ,
\]

where \( R = 1/\Gamma = \rho c^2/4P_{rad} = c_T^2/(3c_g^2) \), and we expect \( R \gg 1 \) in all applications of interest here. In the limit \( c_g =
v_\Lambda = 0$, this equation has the same form as equation (52) in Peebles & Yu (1970). In the limit that all velocities are smaller than $c$, this equation becomes

$$\omega = \pm k \sqrt{c_g^2 + v_A^2 + c_r^2} - i \frac{ck^2 c_r^2}{2k_1(3c_g^2 + c_r^2 + v_A^2)}. \quad (41)$$

Thus, in the diffusive limit, the damping rate has the usual $\propto k^2$ dependence.

In the optically thin limit ($k \gg k_1$), the dispersion relation becomes

$$\omega = \pm k \sqrt{c_g^2 + v_A^2} - \frac{i\Gamma}{2}, \quad (42)$$

giving the same damping rate as for incompressive waves in the optically thin limit. Compressive waves are damped at the optically thin rate when the time for photons to diffuse across a wavelength is comparable to or shorter than the wave oscillation period. We do not have an analytic formula for the dispersion relation for $k_D \lesssim k \lesssim k_1$, so the full dispersion relation must be solved numerically, as shown in Figure 2. A more accurate formula for $k_D$ can be found by equating the diffusive and nondiffusive damping rates:

$$k_D = \sqrt{3(3c_r^2 + c_g^2 + v_A^2)k_1/c}, \quad (43)$$

which is where the compressive damping rate reaches its near-maximum.

When $\rho c^2 \gg P_{\text{rad}} \gg P_{\text{gas}}$ (i.e., $c_r \ll c, c_g \ll c$) and $v_A \ll c_r$, the damping rate for magnetosonic waves at small $k$ is $ck^2/(6k_1)$. The damping rate for Alfvén modes, however, for small $k$ is $\sim \Gamma k^2/10 = 3(c_r/c)^2ck^2/(10k_1)$; i.e., it is smaller by a factor $\sim (c_r/c)^2$. Thus, magnetosonic modes are damped much more strongly than Alfvén modes in the small-$k$, small-$\Gamma$ limit. The reason is that compressional waves continually compress the radiation field, which diffuses out of the wave, causing the wave to lose its pressure support, and thus damping it out. Since there is no compression in the Alfvén modes, photons diffuse out of the wave only perpendicular to $\partial_\theta$, creating a quadrupole moment in the radiation field that leads to viscous damping, and that is much weaker than diffusive damping. A comparison of the dispersion relation for Alfvén modes and magnetosonic modes is shown in Figure 2, with $v_A = c_r = 0.1c$ and $c_g = 0$ ($\Gamma = 0.03$). In the optically thin limit, both waves are damped at a rate $\Gamma/2$, since the radiation is isotropic and uniform, and thus the damping is just due to the dipole moment of Doppler-shifted photons in the fluid frame. The phase speed of magnetosonic waves with these parameters is somewhat greater in the diffusion ($k < k_D$) limit than at larger $k$ because the radiation adds to the restoring force. As a corollary, the waves are mildly dispersive for $k \sim k_D$. Also shown are the analytic approximations to the damping rates, equations (35) and (40).

More complicated behavior can occur when $v_A \ll c_r$ because the removal of radiation pressure support when $k$ exceeds $k_0$ is a relatively more important effect. In Figure 3, we show the compressive dispersion relation (eq. [36]) for $v_A = 0.001c, c_r = 0.01c$, and $c_g = 0$. There are three separate cases to consider for the fast magnetosonic modes:

1. In the diffusion limit, the modes are simply sound waves supported by radiation pressure, since the diffusion timescale is much greater than the wave period.

2. In that portion of the nondiffusive regime in which $k_D < k < \Gamma/(2v_A)$, the drag due to radiation escaping from the waves is greater than the magnetic restoring force, so the wave becomes overdamped. This is the region with $\text{Re}(\omega) = 0$ in Figure 3.

3. For $k > \Gamma/(2v_A)$, the radiation isotropizes on a timescale much shorter than the wave period. In this range of wavenumbers, fast magnetosonic waves propagate with phase speed $(v_A^2 + c_g^2)^{1/2}$, but damp owing to radiation drag.

There are additional overdamped modes that occur when the velocity perturbation is so small that the fluid reaches a terminal velocity and the mode thus damps out before it can

![Fig. 2.—Plot of the real and imaginary parts for Alfvén (solid lines) and magnetosonic (dotted lines) with $v_A = c_r = 0.1c$, $c_g = 0$. The dot-dashed line in the top panel displays eq. (40); the dashed line (which overlaps the solid line) is for eq. (35).](image1)

![Fig. 3.—Plot of the real and imaginary parts for magnetosonic waves with $v_A = 0.001c$, $c_r = 0.01c$, $c_g = 0$. The dotted lines are overdamped waves, while the solid lines are magnetosonic waves.](image2)
oscillate (Subramanian & Barrow 1997). In the diffusive limit when Re (ω) = 0, the radiation always has time to isotropize, so the photon damping rate is the same as in the optically thin case. The magnetic restoring term in equation (29), with k ∥ δβ, balances the optically thin radiation collision term when

\[ \omega = -i \frac{k^2 c}{3kT} \frac{v_A^2}{c^2 + v_A^2}. \]  

(44)

This agrees exactly with the damping rate for the lower dotted curve in the upper panel of Figure 3. For k ≈ 0.1kT, the upper dotted curve is the solution of D^0 ≈ 0 (for very large k, ω ≈ −i5kT c/9), which means δβ → 0, so that this mode becomes electromagnetic (but nonpropagating) in the optically thin limit.

We also find propagating electromagnetic modes (not plotted) whose velocities approach c/(5)^1/2 and c(3/5)^1/2 in the optically thin limit. These speeds are again artifacts of our closure relation.

2.6. The Nature of the Transport

Why is it that photon transport is so much more effective in damping compressive waves than incompressive ones? One way to understand the contrast is to take a closer look at equations (24) and (25). In the incompressive case (k ∥ δβ = 0), δH ≈ (4/3)J δβ when ω and k are both ≪ 1. That is, the perturbed flux is simply 4/3 times the mean intensity shifted by β. However, the force felt by the electrons is the difference between δH and (4/3)J δβ, so the two very nearly cancel. The remainder after this near-cancellation is the retarding force due to the photon shear viscosity and has magnitude ∼ (ω + k^2)J. However, the relative importance of the photon drag is characterized by the ratio Γ, the ratio of the photon inertia to the fluid inertia, and this is ≪ 1.

On the other hand, in the compressive case, there is an additional contribution to δH (and therefore δC) that is, in the very low-frequency limit, ∼ (4/3)J δβ, much greater than the contribution of shear viscosity when ω and k are ≪ 1. This new contribution is the result of photon diffusion. In a single-fluid picture this can be thought of as a sort of thermal conductivity (Weinberg 1972), but with respect to a peculiar equation of state (see eq. [23]). It damps the waves much faster than shear viscosity because the diffusing particles in this case are exactly those responsible for the wave’s restoring force.

In principle, photon-electron scattering could also lead to a magnetic diffusivity by creating a new source of electrical resistivity. In the optically thin limit, the resistivity is ηe ≈ 1.4 × 10^{-18} (c/e)^2 s. In most parameter regimes this photon resistivity is small compared to the resistivity due to ordinary electron-ion Coulomb scattering, ηp ≈ 1.4 × 10^{-7} T^{-3/2} s; however, in the corona, where T_e ≈ 10^9 K and c_i ≈ 0.1c, photon resistivity may compete with Coulomb resistivity. Owing to flux-freezing, the damping of the turbulent motions also damps the magnetic field fluctuations; however, owing to the small resistivity, magnetic flux is still conserved.

3. APPLICATIONS TO CONVENTIONAL ACCRETION DISKS

3.1. Context

In the preceding section we characterized the effects of photon diffusion and viscosity in terms of the rate at which they cause damping of linear MHD waves. In this section we will evaluate how effective these processes may be in dissipating fluctuations in accretion disks. To gain a sense of scale, we begin this section by estimating the corresponding damping rate for several other proposed dissipation mechanisms. Although the natural unit of time for the dispersion relation was the photon scattering time, in the context of disks the natural unit is (the inverse of) the orbital frequency Ω, so we will quote all rates in that unit. As a further set of reference rates, in this subsection we will also establish the relevant standards of comparison for several different questions of interest.

Ordinary molecular viscosity (due to ion-ion collisions) creates a damping rate

\[ \Gamma_{\text{mol}} = \frac{1}{3} \left( \frac{\eta}{\sigma} \right)^2 \frac{c_s}{c_e} \Omega, \]  

(45)

where h is the (half) disk thickness and σ is the collision cross section. If σ is the Coulomb cross section, for example, \[ \sigma = \frac{k_m T m_i c^2}{\ln \Lambda}, \] where \ln Λ is the usual Coulomb logarithm, ≈ 30. Π_{mol}/Ω is generally a very small number. Ordinary viscosity is rendered even less effective because the magnetic field suppresses transport perpendicular to the field.

Transit-time damping (and the associated Landau damping) has been suggested by Quataert (1998) as the dissipational mechanism in accretion disks, particularly when the ion temperature is much greater than the electron temperature, as in advection-dominated accretion flows. As a fiducial point, we quote its rate (as calculated by Quataert 1998) for a single-temperature plasma:

\[ \Gamma_{\text{nt}} ≈ 0.2 \cos \theta \sin^{2/3} \frac{\delta (kh)^{5/3}}{\Omega} \frac{\Omega}{\bar{\Omega}} \frac{c_s}{c_e}, \]  

(46)

where \bar{\Omega} is the ion Larmor frequency. When k is greater than an inverse ion Larmor radius, Π_{nt} ∝ (kh)^{5/2}. Unless the magnetic field is exceedingly strong, Ω/\bar{\Omega} ≈ 1, so that Π_{nt} ≪ Ω.

Depending on the question being asked, any candidate damping rate should be compared to one of these fiducial rates: the growth rate (absent dissipation) of the MHD waves (as, for example, due to magnetorotational instability, as in Balbus & Hawley 1991); the inverse time for waves to cross a disk scale height; and the “nonlinear frequency” or inverse “eddy turnover time,” the rate at which energy moves between modes because of nonlinear coupling. If the damping rate exceeds the nondissipative growth rate, the fluctuations are unable to grow at all. This is a strong statement, for the growth rate of the magnetorotational instability is generally Ω.

When the damping time is short compared to a disk scale-height crossing time, waves cannot carry significant energy from the midplane to the disk surface (see § 4.2). This, too, may require very rapid damping, for the wave crossing time can be as short as Ω^{-1} (for diffusive regime fast magnetosonic modes). The time for other modes to traverse a disk thickness is somewhat slower: \[ \sim (\Omega t/\bar{c}_e)^{-1} \] for pure Alfvén modes, \[ \sim \frac{1}{2} \frac{1}{\Omega (c_s^2 + v_A^2)^{1/2}} \] for fast magnetosonic modes with k ≫ k_p, and ω ≈ 0.

Thirdly, as emphasized by Gruzinov (1998), exceeding the nonlinear frequency at some wavenumber is the relevant criterion for deciding whether the damping can cut off the “inertial range” of turbulence at short wavelengths. The
nonlinear frequency is defined by \( \omega_{nl}(k) \equiv \epsilon/(kE_k) \), where \( E_k \) is the energy density per unit wavenumber in the turbulent spectrum. The rate of energy dissipation per unit volume, \( \epsilon \), is determined solely by the accretion rate (eq. [1]) and disk height, while the total energy in fluctuations may be related to the accretion rate if we know the ratio between the trace of the fluctuations' stress tensor and its \( r - \phi \) component (under the assumption that it is this last quantity that accounts for angular momentum transport in the disk). That is, the total energy density in fluctuations, volume averaged, is

\[
\frac{1}{2} \langle \langle \delta T \rangle \rangle = \int_{k_{min}}^{k_{max}} dk E_k = \frac{T_r(\pi h)^{5/3}}{8\pi h},
\]

where \( T_r = \langle \rho (\delta v_r, \delta v_r, \delta v_r) \rangle \), and angle brackets denotes volume averaging (Balbus et al. 1994). Suppose the fluctuation spectrum is a power law \( E_k \propto k^{-n} \) from \( k_{min} = \pi/h \) to \( k_{max} \). Then,

\[
\omega_{nl} = \frac{3}{n-1} \frac{T_{\phi}^{\pi h}}{T_r(\pi h)} \left( \frac{k_{max}}{k_{min}} \right)^{n-1} \Omega.
\]

If \( k_{max} \gg k_{min} \) and \( n > 1 \), then

\[
\omega_{nl} \approx \frac{3}{n-1} \frac{T_{\phi}^{\pi h}}{T_r(\pi h)} \left( \frac{k_{max}}{k_{min}} \right)^{n-1} \Omega.
\]

In the simulations of Brandenburg et al. (1995) and Stone et al. (1996), \( T_{\phi}/T_r(\pi h) \sim 0.1 \), very roughly.

3.2. Radiation Pressure-dominated Disk

First consider radiation pressure-dominated disks. In this case, the Shakura-Sunyaev solution (in which \( T_{\phi} \) is set equal to \( \pi h \)) yields two important results about the equilibrium. The disk aspect ratio is

\[
h \approx \frac{3}{2} \frac{\dot{m}}{x},
\]

where \( \dot{m} \) is the accretion rate in Eddington units (for unit efficiency) and \( x = r c^2/\Omega M \) is the radius in gravitational units. We have ignored all relativistic factors. In addition, the (half) optical depth is

\[
\tau = \frac{2c}{\alpha \Omega h} = \frac{4}{3} \frac{x^{3/2}}{\dot{m} h},
\]

where, in consonance with the results of simulations, we have ignored any contribution of magnetic pressure to disk vertical support.

We may now use these facts to evaluate the rate of photon damping. In this case,

\[
\frac{c_e}{c} = \frac{3}{2} \frac{\dot{m} h^{-3/2}}{x}.
\]

Typically \( c_e \ll c \) \( (\Gamma \ll 1) \) in thin accretion disks. For \( k < k_D \) and radiation pressure larger than magnetic or gas pressures, compressive modes damp at a rate

\[
\Gamma_{comp}^d = \frac{x}{12} (kh)^2 \Omega.
\]

Incompressive modes damp more slowly at small \( k \):

\[
\frac{\Gamma_{inc}^d}{\Gamma_{comp}^d} \approx 4 \frac{\dot{m} h}{x^3}.
\]

So only for large \( \dot{m} \) and very small radii will the incompressive damping rate equal or exceed the compressive. In the optically thin \( (k > 2\pi k_D) \) limit, both damping rates are constant and equal to \( \Gamma_{inc}^d = 3 \pi^{-1} \Omega \).

Thus, we immediately see that compressive modes damp extremely rapidly. At the longest wavelengths the damping time is, not surprisingly, the same as the thermal time, \( (\pi h)^{-1} \). They are, in fact, the same process—photon diffusion out of a region \( \sim h \) in size. These modes damp so quickly because in this regime photons provide most of the pressure; consequently, it is their diffusion rate, not that of the ions, which controls the damping rate.

To see just how rapid the photon damping is, we may compare it to, e.g., the rate of transit-time damping. In this context of radiation-dominated accretion disks,

\[
\Gamma_{nd} \approx 6 \times 10^{-13} \cos \theta \sin^{2/3} \theta (kh)^{5/3} (c_e/c_t)^{2/3} M_8^{1/2} \Omega,
\]

where \( M_8 \) is the mass of the central black hole in units of \( 10^8 M_\odot \). Because \( c_e/c_t \ll 1 \) when radiation pressure is dominant, \( \Gamma_{nd} \) is very slow indeed compared to even \( \Gamma_{comp}^d \).

So long as \( x < 1 \), the damping rate for compressive modes does not exceed the Balbus-Hawley growth rate. However, the damping rate may well exceed the nonlinear frequency even for the longest wavelength modes, for

\[
\frac{\Gamma_{comp}^d}{\omega_{nl}} = \frac{\pi (n-1)}{36} \frac{T_{\phi}^{\pi h}}{T_r(\pi h)} \left( \frac{k_{max}^{3/2}}{k_{min}^{5/2}} \right) \Omega.
\]

This expression follows from the fact that the Shakura-Sunyaev parameter \( \alpha = [T_{\phi}/T_r(\pi h)]/[Tr(\pi h)/p] \), where \( p \) is the total pressure. If the spectrum has the Kolmogorov slope \( n = 5/3 \), the photon damping rate is greater than the nonlinear frequency for wavenumbers not much greater than \( \pi/h \) unless \( Tr(\pi h)/p \) is very much less than unity (in the simulations of Stone et al. 1996 this quantity was \( \sim 0.01-0.1 \)).

Even if photon damping does not overcome the fluctuations at longer wavelengths, it is still likely to end the inertial range of turbulence. The maximum photon damping rate (the optically thin limit) is achieved at \( k_D \), where the damping rate is \( \Gamma_{comp}^d \approx 3 \Omega/\alpha \). Comparing this rate to \( \omega_{nl} \), we find

\[
\frac{\Gamma_{comp}^d}{\omega_{nl}(k_D)} = \frac{\pi (n-1)}{12} \left( \frac{x}{\dot{m} h} \right)^{3/2} \frac{\alpha}{\Omega} \frac{T_{\phi}^{\pi h}}{T_r(\pi h)} \left( \frac{T_r(\pi h)}{p} \right)^{-2}.
\]

So long as \( n < 2 \), it is almost guaranteed that \( \Gamma_{comp}^d > \omega_{nl} \) at some wavenumber. For instance, for \( n = 5/3 \), we find that

\[
\frac{\Gamma_{comp}^d}{\omega_{nl}(k_D)} = 43 \left( \frac{T_{\phi}^{\pi h}}{T_r(\pi h)} \right)^{-4/3} \left( \frac{100 T_r(\pi h)}{p} \right)^{-1/3},
\]

where we have normalized to fiducial values in the ballpark of what is seen in simulations. We also emphasize that this equation is independent of \( x, \dot{m} \), and the mass of the black hole. Thus, radiation damping of compressive modes can be quite strong whenever radiation pressure dominates.

If there is any significant azimuthal field, i.e., \( \nu_{\phi} \approx c_e \), all long-wavelength \( (\pi \ll k) \) modes are at least partly compressive (Blaes & Balbus 1996). Simulations indicate that \( \nu_{\phi} \ll c_e \), so the Balbus-Hawley-unstable modes are very nearly incompressible. However, there can still be significant coupling between incompressible and compressive modes. In simulations of nonlinear MHD turbulence by Stone et al. (1996), the density fluctuations \( \langle \delta \rho^2 \rangle^{1/2}/\rho \approx 5\%-8\% \), while the
velocity fluctuations \( \langle \delta v^2 \rangle^{1/2}/c_s \approx 15\% \). Since \( \delta p/\rho = k \cdot \delta \omega/\omega \approx k \cdot \delta v/c_s \), then \( k \cdot \delta v \approx 0.5 \). Thus, the waves in the turbulence spectrum have a rather large compressive component. Also, in these simulations, pressure waves are seen that are not present in the non-turbulent state, indicating that the turbulence does create compressive waves (J. Hawley 1998, private communication). Another way to quantify the fraction of compressive turbulence is to take the power spectrum of the vortical and compressive components of the velocity, \( \mathbf{v} = v_{\text{vort}} + v_{\text{comp}} \) such that \( \mathbf{V} \cdot v_{\text{vort}} = 0 \) and \( \mathbf{V} \times v_{\text{comp}} = 0 \). MHD shearing box simulations by Brandenburg et al. (1995) show that the power spectrum amplitude of \( v_{\text{comp}} \) is about 10% of that of \( v_{\text{vort}} \) (Brandenburg 1998).

In addition to the canonical solution for radiation-dominated disks, there are also several extensions of this solution to which photon damping is relevant. At high accretion rates, \( m \gg 1 \), the radiation pressure causes the disk to puff up, creating a “slim disk” (Abramowicz et al. 1988). Slim accretion disks have larger luminosities than thin accretion disks, making the effects of radiation damping much stronger. The standard thin-disk equations cannot be applied to slim disks, since some of the radiation is carried radially inward through the disk rather than being radiated locally. However, the slim-disk solutions look similar to the Shakura & Sunyaev (1973) solution in the limit of large \( m \) (Szuszkiewicz, Malkan, & Abramowicz 1996), so we expect our criterion for radiation dissipation to apply, under the assumption that slim disks can be approximately described by the thin-disk equations.

Radiation pressure–dominated disks in which \( T_{\text{rad}} = \alpha T_{\text{tot}} \) are viscously and thermally unstable (Shakura & Sunyaev 1976, Lightman & Eardley 1974). To cure this, some have suggested that the viscosity is proportional to the gas pressure rather than the total pressure, i.e., \( T_{\phi} = \alpha P_{\text{gas}} \). Indeed, as we will discuss in \S 5, photon diffusion may decouple the radiation pressure from the MHD fluctuations, leading to just this sort of result. If so, such disks effectively have a much smaller \( \alpha \), and consequently radiation viscosity is much more efficient at damping perturbations, since (1) the turbulent velocities are much smaller, reducing the nonlinear frequency, and (2) the optical depth of the disk is much larger, so the radiation pressure at disk center is larger. Since the disk is still supported by radiation pressure, the height is the same as for \( \alpha T_{\text{tot}} \) disks, and thus \( \Gamma \) also remains the same. However, the nonlinear frequency is reduced by a factor of \( P_{\phi}/P_{\text{tot}} \), which is given by

\[
\frac{P_{\phi}}{P_{\text{tot}}} = \frac{32 x^{2/3}}{27 \pi \pi}.
\]

(59)

In these disks, even the incompressive damping rate can beat the nonlinear frequency at \( k \sim k_T \), the wavenumber at which \( \Gamma_{\text{thin}}^h \) reaches its maximum value. For example, for \( n = 5/3 \) and \( T(T)/P_{\text{tot}} = 10 \),

\[
\frac{\Gamma_{\text{thin}}^h}{\omega_{\text{nl}}(k_T)} = 9 t^{4/3} m^2 x^{-3}.
\]

(60)

The optical depth in these disks is given by

\[
\tau = 5 \times 10^5 x^{-4/5} m^{3/5} M_8^{1/5} x^{-3/5}.
\]

(61)

Using this expression for \( \tau \), we see that the incompressive damping rate beats the nonlinear frequency out to a radius of

\[
x < 178 m^{14/19} M_8^{4/57} a^{-16/57}.
\]

(62)

Since the compressive damping rate is always greater than the incompressive, radiation damping will be important for a large range of radii if the viscous stress scales with gas pressure rather than radiation pressure.

Our discussion of accretion disks so far has neglected the vertical stratification of density and radiation pressure, since we have been using values computed from a one-zone model. Because the radiation damping rate is proportional to the radiation pressure, we would expect the damping to be relatively more important in the interior of the disk, so that more dissipation occurs deep inside the disk (provided the nonlinear frequency is independent of height).

3.3. Gas Pressure–-dominated Disks

The damping criterion we have discussed applies only to the \( P_{\text{rad}} \gg P_{\text{gas}} \) regions of the accretion disk. When \( P_{\text{gas}} \) or \( P_{\text{mag}} \gg P_{\text{rad}} \), the damping rate becomes \( (ck^2/2k_T c_s^2)(3c_s^2 + c_g^2 + v_T^2) \) (see eq. [40]) in the diffusive regime, so radiation damping will not compete with the nonlinear frequency. Thus, we expect that the radiation damping will be important only in the radiation pressure–dominated part of an accretion disk. The radius at which radiation pressure equals gas pressure is given by

\[
x_{\text{trans}} = 188 (x M_8)^{2/21} m^{16/21} (1 - f)^{6/7},
\]

(63)

where \( M_8 \) is the black hole mass in units of \( 10^8 M_\odot \), \( f \) is the fraction of energy lost to a corona, and we have assumed Thomson scattering opacity. Since this radius is very insensitive to the black hole mass, we expect radiation dissipation to be important in the range of radii in which most of the luminosity is created for black hole X-ray binaries, Seyfert galaxies, and quasars. There is a rather strong dependence on the luminosity relative to Eddington, so radiation dissipation will not play a role for objects with small \( m \). We have computed a disk model that includes both radiation and gas pressure, and compared the nonlinear frequency with the numerical root of the dispersion relation for compressive waves. We find that the radius at which the damping rate exceeds the nonlinear frequency (for some \( k \)) is typically at \( P_{\text{gas}} \approx \) a few times \( P_{\text{rad}} \).

3.4. Growth of the Radiation Field

So far our discussion of accretion disks has assumed that they are already radiating. Since the radiation is derived from the dissipation of kinetic energy into electron thermal energy or photon energy density, we have only shown that radiation damping provides a dissipation mechanism that gives a self-consistent disk solution. Another stronger question to ask is: if a disk is in a state in which radiation pressure is small relative to gas pressure, for what parameters will the radiation dissipation cause growth of the radiation field, causing the disk to find a radiation pressure–dominated equilibrium?

The rate of change of the radiation field is given approximately by

\[
\frac{\partial P_{\text{rad}}}{\partial t} = \frac{Q}{2\hbar} \min \left[ 1, \left( \frac{\Gamma_{\delta}}{\omega_{\text{nl}}/\omega_{\text{max}}} \right)^{-c P_{\text{rad}}/\hbar(1 + \tau)} \right],
\]

(64)

where the first term on the right-hand side is the rate of creation of radiation due to photon dissipation of turbulence (\( Q \) is given by eq. [1]), and the second term is the rate
of escape of radiation from the disk. Now, $Q \approx cP_{\text{rad}}/(1 + \tau)$ in equilibrium, so a steady state radiation field can be achieved only for $(\Gamma_d/\omega_m) \gtrsim 1$. In general, the maximum damping for compressive waves $\Gamma/2$ occurs for $k \sim k_d \sim k_T c/s$. For a general disk, the maximum ratio of radiation damping to nonlinear frequency is given by

$$\frac{\Gamma_d^{\text{comp}}}{\omega_m(k_d)} \propto k^2 \frac{n}{r_g} \left(\frac{h}{r_g}\right)^{1-n} x^{(3n-6)/2} \frac{\text{Tr}(T)}{T_{\text{rad}}},$$

where $r_g = GM/c^2$, and the constant of proportionality is of order unity. For incompressive modes, the maximum damping rate occurs for $k \sim k_T$, where

$$\frac{\Gamma_d^{\text{inc}}}{\omega_m(k_T)} \propto k^2 x^{-2n/3} \frac{n}{r_g} \frac{\text{Tr}(T)}{T_{\text{rad}}}. \tag{66}$$

These expressions are valid for optically thick disks that may be radiation pressure- or gas pressure-supported and have $Q$ given by equation (1). Whether either of these is greater than unity depends on what state the disk begins in. We consider one such starting state in the next section: an advection-dominated disk.

4. UNCONVENTIONAL ACCRETION DISKS

4.1. Advection-dominated Disks

In an advection-dominated disk, the equilibrium depends on the fact that the cooling timescale is much longer than the accretion timescale and thus the heat is advected inward rather than being radiated locally. If radiation damping is strong enough to cause growth of the radiation field, then the radiation will damp out the turbulence and most of the heat will go into radiation rather than proton thermal energy that gets advected.

To estimate when radiation pressure is subject to growth, we assume a steady state disk with electrons of a constant temperature in which the viscous stress is generated by magnetic fields that create a turbulent cascade to smaller wavelengths. Using the criterion of Narayan & Yi (1995) for the existence of an advection-dominated solution ($m < 0.5 x^3$), we find $\tau < (x/0.1)^{1/2}$, so advection-dominated disks are usually in the optically thin regime. Since the disk is optically thin, the radiation damping is given by $\Gamma_{\text{rad}}^{\text{thin}}$ for either compressive or incompressive modes. Comparing the damping rate to the slowest nonlinear frequency (at $kh = n$), we find that the criterion for radiation growth using equation (64) is $x < 0.45[k^2 \text{Tr}(T)/T_{\text{rad}}]^{2/3}$, which means that radiation viscosity will not cause optically thin advection-dominated disks to cool and radiate. This also means that whatever radiation is produced in an advection-dominated accretion flow cannot be produced by the photon damping mechanism.

4.2. Corona-dominated Accretion Disks

An alternative disk equilibrium has been proposed by Svensson & Zdziarski (1994), in which all of the angular momentum transport occurs within the accretion disk, while a fraction $f$ of the associated heat released occurs above the disk in a corona. Their equilibrium relies on the idea that the energy can be efficiently transported from the disk to the corona somehow, presumably through magnetic or acoustic waves. Since there is no outgoing radiation flux within the disk, its equilibrium density and gas pressure are much greater than in the radiation-supported case. However, unless $f$ is very close to unity, there will still be a significant region of the disk in which radiation pressure dominates (see Fig. 2 of Svensson & Zdziarski 1994). In this case equations (64) and (65) still apply, and the radiation damping time for compressive waves is less than the wave crossing time, $2\pi \Omega^{-1}$, for

$$kh \gtrsim \sqrt{\frac{12}{\pi(1-f)}}. \tag{67}$$

For incompressive modes, the crossing time $2\pi c_s/(\Omega v_\lambda)$ is greater than the damping time for

$$kh \gtrsim \sqrt{\frac{3x^4v_\lambda}{\rho m^2(1-f) v_s^2}}. \tag{68}$$

Thus, only a limited range of wavelengths can successfully carry energy to the corona.

When gas pressure dominates, if $v_\lambda \gtrsim c_s$, the crossing time for hydromagnetic waves is about $2\pi/\Omega$. For incompressive waves with wavelengths less than the mean free path of a photon, or for compressive waves with $k > k_d$, the damping rate will be $\Gamma_{\text{rad}}^{\text{thin}}$. For $f \approx 1$ disks,

$$k_d h = 2 \times 10^5 \frac{\dot{m}}{\rho} \frac{x^{-9/8}}{a^2} \frac{(1-f/2)^{-1/3}}{M_8^{1/8}}, \tag{69}$$

while

$$k_T h = \tau = 1.8 \times 10^4 \frac{\dot{m}}{\rho} \frac{x^{-3/4}}{a^2} \frac{(1-f/2)^{-1/3}}{M_8^{4/3}(\kappa/\kappa_1)}. \tag{70}$$

Now for waves to be damped by photon viscosity before they can escape from the disk requires $\Gamma/2\Omega > 1$. This ratio is

$$\frac{\Gamma}{2\Omega} = 4aT^4 \rho \frac{k_T}{\Omega} \frac{hc}{2} \frac{1}{\kappa} \frac{\dot{m}}{\rho} \left(\frac{x}{10}\right)^{-3/2}, \tag{71}$$

neglecting relativistic factors, where $T$ and $\rho$ are the density and temperature inside the disk, and we have used the equations from the appendix of Sincell & Krolik (1997) to evaluate the disk parameters for $f \approx 1$. Thus, only extremely short wavelength waves may be damped rapidly enough, and then only in rather extreme conditions (relatively large $\dot{m}$ and small $x$). If $v_\lambda/c_s \lesssim 1$, the requirements for damping incompressive Alfven waves may be relaxed somewhat, but unless this ratio is very small, the qualitative conclusion is unlikely to be altered.

5. DISCUSSION

We have shown in the previous sections that the effectiveness of radiation in damping fluid motions depends strongly on the ratio of radiation to gas pressure. As noted in equation (63), radiation tends to be most important in the inner parts of accretion disks, which are, of course, the most important for energy release. At least some part of the disk is radiation-dominated when

$$\dot{m} > 1.0 \times 10^{-3} x_{\text{min}}^{21/16} a^{-1/8} M_8^{-1/8}, \tag{72}$$

where $x_{\text{min}}$ is the inner radius of the disk. If the central object is a black hole or a weakly magnetized neutron star, we may expect $x_{\text{min}}$ to be the radius of the marginally stable orbit, equal to 6 in the limit of a spinless black hole, and approaching 1 as the spin of the black hole approaches its maximum possible value. However, if the disk does not...
extend in so far, because the central mass is either a strongly magnetized neutron star or a larger object such as a white dwarf, the minimum accretion rate for which at least part of the disk is radiation-dominated rises, and may become impossibly high.

The remainder of this section, in which we outline the consequences of radiation damping in accretion disks, is divided according to consequences applicable to radiation-dominated disks and those applicable to the gas pressure-dominated case. Whether one set or the other is relevant to a given disk depends on how it fares according to the criterion of equation (72).

5.1. Radiation pressure-dominated disks

Two qualitative physical consequences follow from the strength of radiation damping in photon pressure-dominated disks. First, dissipative heating is delivered to the electrons and photons through radiation scattering, and not to the ions. Because it is the electrons that cool the gas through the creation and upscattering of photons, the only energy exchange process involving the ions is Coulomb scattering. This mechanism should keep the ion temperature very close to the electron temperature. If the scattering. This mechanism should keep the ion temperature, if not to the ions. Because it is the electrons that cool the gas through the creation and upscattering of photons, the only energy exchange process involving the ions is Coulomb scattering. This mechanism should keep the ion temperature very close to the electron temperature. If the average energy of photons is less than $\beta^2 m_e c^2 / 3 + 4k_B T_e$, where $T_e$ is the electron temperature, then the photons will receive most of the energy from scattering (Psaltis & Lamb 1997). The $\beta^2 m_e c^2$ term represents a modification of the Compton temperature due to bulk Comptonization.

Second, the process by which these disks shine may be thought of as a sort of “bootstrap”: if the disk were initially free of radiation, any initial photon creation by the electrons and photons would lead to wave dissipation that heats the electrons and therefore leads to more radiation. The question of what makes near-Eddington accretion disks shine has a tautological answer: bright accretion disks shine because they are so bright.

That MHD fluctuations should be present at all is likely due to the operation of the magnetorotational instability identified by Balbus & Hawley (1991). This instability grows at a rate $\sim k \nu_A$ for wavenumbers $k \leq (3)^{1/2} \Omega / \nu_A$ when the magnetic field is weak (i.e., $v_A < c_s$). The compressibility of the growing modes is slight, so the corresponding radiation damping rate should be a fraction of the pure compressive rate, as given by equation (54). If most of the torque in the disk is due to magnetic fluctuations, the ratio between the magnetorotational growth rate and the radiation damping rate is then at least $\sim 10(c_s/v_A)(kh)$. We therefore expect the linear growth of MHD fluctuations to proceed unaffected by radiation damping.

However, shorter wavelength waves are not amplified by the magnetorotational instability. Instead, they are pumped by nonlinear coupling with the longer wavelength, growing modes. Because the radiation damping rate is $\propto k^2$ in the diffusive regime, compressive modes excited by nonlinear coupling will be strongly damped. In other words, provided only that the nonlinear coupling between incompressive and compressive modes is reasonably strong, the “inner scale” of the MHD turbulence will be not much shorter than its “outer scale.” Any turbulent “inertial range” will be severely limited.

This fact leads to several other results. At a purely technical level, if short wavelengths are all severely damped, the life of the numerical simulator is made much easier, for there is no need to strive for very fine spatial resolution.

More physically, radiation damping may play an important role in regulating the value of the “viscosity” parameter $\alpha$. The magnetic part of the stress causing angular momentum transport may be written in the form

$$T_{\phi} = -\frac{1}{4\pi} \int d^3k \delta \tilde{B}_x(k) \tilde{B}_y^*(k).$$

(73)

If there is little power in the fluctuations at wavenumbers much more than $\sim 1/h$, the angular momentum transport is reduced below what it would otherwise be. Disks in this situation would then maintain rather larger surface densities. Increased optical depth also leads to greater radiation pressure for fixed emergent flux.

Another consequence for turbulence in radiative disks is that the ratio of the sound speed to the Alfvén speed changes with wavelength. In the diffusive regime, $c_s \sim c_e \gg v_A$, leading to a large plasma $\beta = P_{\text{tot}}/P_{\text{mag}}$. When the plasma $\beta$ is large, MHD fluctuations are generally close to incompressive because pressure waves can travel rapidly enough to smooth out density disturbances. However, for short wavelengths, the radiation field decouples from the fluid, and $c_s \sim c_e \ll v_A$, which means the plasma $\beta$ becomes effectively quite small. For these short wavelengths, then, we can expect the turbulence to exhibit much greater compressibility. In the compressible regime, the speeds of the magnetosonic and Alfvén waves are comparable, so they may couple much more easily. A similar effect happens for Alfvén waves near recombination, as discussed by Subramanian & Barrow (1997).

The slope and inertial range of the turbulent spectrum will also be affected by the plasma $\beta$ parameter, which is usually held fixed in compressive MHD simulations (Matthaeus et al. 1996). Analytic theory and simulations show that for compressible MHD, $\delta \rho/\rho \sim (\delta v_e/c_s)^2$ [where $\delta v_e \equiv |\delta \mathbf{B}|/(4\pi \rho v_z)^{1/2}$], so when $c_s$ drops dramatically in the nondiffusive regime, compressive damping will become very effective. Simulations of turbulent cascades with small $\beta$ but with incompressive stirring will show how much energy can be transferred to compressive modes. Current simulations of compressible turbulence in the ISM (C. Gammie 1998, private communication) show that shocks form when $v_A \gg c_s$, so that if the incompressive cascade does not transfer energy to compressive modes before reaching the nondiffusive scale, the energy may be dissipated in shocks at that scale. The dissipation in these shocks may be partly due to ordinary plasma processes and partly due to radiation scattering. Thus, we expect that $k_{\text{max}}$ will never be much greater than $k_D$. As the radiation pressure varies with disk radius, $k_0$ changes and thus $k_{\text{max}}$ changes, so the value of $\alpha$ may become a function of radius.

Although certain consequences of radiation damping are relatively clear (at least qualitatively), consideration of this process also raises a number of questions:

1. What is the nature of the coupling between compressive and incompressive modes? Is it large enough to allow the radiation damping rate to compete with the nonlinear frequency? Are the analytic estimates we have made useful in the nonlinear regime?

2. In the simulations done to date, in which radiation pressure and transport are equally ignored, the magnetic energy density is an interesting fraction of the pressure and the associated fluctuations lead to a stress that is also proportional to the pressure. The question naturally arises...
whether, in radiation pressure–dominated disks, the \( r-\phi \) stress and the energy in the magnetic field scale with the total pressure or just with the gas pressure. The photon bubble instability (Arons 1992) will likely affect the disk structure and stress (Gammie 1998). With explicit consideration of the quality of dynamical coupling between radiation fluctuations and fluid fluctuations, as outlined here, simulations should now be able to answer these questions.

3. Can thermal or viscous instabilities be suppressed by radiation damping? Or does the dependence of dissipation on the radiation pressure exacerbate these instabilities? In both cases, the most important modes have radial wave-numbers \(<h^{-1}\), so the calculation here does not directly bear on them. However, one might expect that some of the same effects will qualitatively carry over.

4. The relativistic portions of accretion disks may trap a number of long-wavelength (i.e., \( kh < 1\)) normal modes (Nowak & Wagoner 1991, 1992). Some of these \( g \) grow in amplitude owing to viscous dissipation (Nowak & Wagoner 1992). Modulo the caveat of point 3, will radiation damping enhance (or destroy) these modes?

5. Many seek the origin of disk coronal heating in the dissipation of rising MHD waves (e.g., Rosner, Tucker, &Vaiana 1978; Heyvaerts & Priest 1989; Tout &Pringle 1996). If radiation damping quenches short-wavelength fluctuations, will this affect the rate at which magnetic flux rises to the disk surface?

5.2. Gas Pressure–dominated Disks

When gas pressure dominates over radiation pressure, radiation damping does not compete with the nonlinear frequency. The question of what causes the heating of the disk therefore remains open. This conclusion is equally true of conventional gas pressure–dominated disks and unconventional ones like advection-dominated accretion flows.

Finally, the contrast between the radiation pressure–dominated and gas pressure–dominated regimes may mean that interesting observable effects occur in disks whose accretion rate fluctuates around the critical value of equation (72). If the value of \( \alpha \) and the radiative efficiency depend on whether radiation damping plays a role, there could be significant modulations in the luminosity and spectrum on a viscous timescale.

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REFERENCES

Abramowicz, M., Czerny, B., Lasota, J. P., & Szuszkiewicz, E. 1988, ApJ, 332, 646

Arons, J. 1992, ApJ, 388, 561

Balbus, S. A., & Hawley, J. F. 1991, ApJ, 376, 214

Balbus, S. A., Gammie, C. F., & Hawley, J. F. 1994, MNRAS, 271, 197

Bisnovatyi-Kogan, G. S., & Loelalec, K. V. E. 1997, ApJ, 486, L43

Blackman, E. 1998, preprint (astro-ph/9710137)

Blaes, O. M., & Balbus, S. A. 1994, ApJ, 421, 163

Bogdan, T. J., & Knöllker, M. 1989, ApJ, 339, 579

Brandenburg, A. 1998. Nonlinear Phenomena in Accretion Discs, ed. M. Abramowicz, G. Björnsson, & J.E. Pringle (Cambridge: Cambridge Univ. Press)

Brandenburg, A., Nordlund, A., Stein, R. F., & Torkelsson, U. 1995, ApJ, 446, 741

Cassen, P., & Woolum, D. S. 1996, ApJ, 472, 789

Ferrari, A. 1984, Adv. Sp. Res., 4, 345

Galeev, A. A., Rosner, R., & Vaiana, G. S. 1979, ApJ, 229, 318

Gammie, C. F. 1998, MNRAS, 297, 929

Gruzinov, A. V. 1998, ApJ, 501, 978

Hawley, J. F., Gammie, C. F., & Balbus, S. A. 1995, ApJ, 440, 742

Heyvaerts, J. F., & Priest, E. R. 1989, A & A, 216, 230

Hu, W., & Sugiyama, N. 1996, ApJ, 471, 542

Ichimaru, S. 1977 ApJ, 214, 840

Jackson, J. D. 1975, Classical Electrodynamics (New York: Wiley)

Jedamzik, K., Katalinčič, V., & Olinto, A. V. 1998, Phys. Rev. D, 52, 3264

Lightman, A. P., & Eardley, D. M. 1974, ApJ, 187, L1

Loeb, A., & Laor, A. 1992, ApJ, 384, 115

Matthaeus et al. 1996, J. Geophys. Res. A, 101, 9619

Mihalas, D., & Mihalas, B. W. 1983, ApJ, 273, 355

1984, Foundations of Radiation Hydrodynamics (New York: Oxford Univ. Press)