NONRATIONAL COVERS OF \( \mathbb{CP}^m \times \mathbb{CP}^n \)

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1. Introduction

The aim of this article is to provide further examples of higher dimensional varieties which are rationally connected, but not rational and not even ruled. The original methods of [Iskovskikh-Manin71; Clemens-Griffiths72] were further developed by many authors (for instance, [Beauville77; Iskovskikh80; Bardelli84]) and they give a quite complete picture in dimension three. [Corti96] is a recent overview. For a while only special examples have been known in higher dimensions [Artin-Mumford72; Sarkisov81,82; Pukhlikov87; CTO89]. For hypersurfaces in \( \mathbb{P}^n \) the rationality question was considered in [Kollár95]. There I proved the following:

1.1 Theorem. [Kollár95] Let \( X_d \subset \mathbb{CP}^{n+1} \) be a very general hypersurface of degree \( d \). Assume that

\[
\frac{2}{3} n + 3 \leq d \leq n + 1.
\]

Then \( X_d \) is not rational. \( \square \)

Here “very general” means that the result holds for hypersurfaces corresponding to a point in the complement of countably many closed subvarieties in the space of all hypersurfaces.

The method can be applied to hypersurfaces in \( \mathbb{P}^m \times \mathbb{P}^{n+1} \). Let \( X_{c,d} \) be such a hypersurface of bidegree \((c,d)\). Via the projection \( X_{c,d} \rightarrow \mathbb{P}^m \) it can be viewed as a family of degree \( d \) hypersurfaces in \( \mathbb{P}^{n+1} \) parametrized by \( \mathbb{P}^m \). \( X_{c,d} \) is rationally connected if \( d \leq n + 1 \), no matter what \( c \) is (cf. [Kollár96, IV.6.5]).

A straightforward application of the method of [Kollár95] provides an analog of (1.1) for hypersurfaces when \( c \geq m + 3 \) and \( d,n \) satisfy the inequalities (1.1.1). It is, however, of more interest to study some cases where the fibers of \( X_{c,d} \rightarrow \mathbb{P}^m \) are rational. Here I propose to work out two cases: conic bundles and families of cubic surfaces.

The method of [Kollár95] works naturally for cyclic covers, and it is easier to formulate the results that way.

Fix a prime \( p \) and let \( X_{ap,bp} \rightarrow \mathbb{CP}^m \times \mathbb{CP}^n \) be a degree \( p \) cyclic cover ramified along a very general hypersurface of bidegree \((ap,bp)\). [Kollár95] shows that \( X_{ap,bp} \) is not rational and not even ruled if \( ap > m + 1 \) and \( bp > n + 1 \). In this paper I study the case when \( bp = n + 1 \) and \( n = 1,2 \). The main results are the following:
1.2 Theorem. Let \( X_{2a,2} \to \mathbb{CP}^m \times \mathbb{CP}^1 \) be a double cover ramified along a very general hypersurface of bidegree \((2a,2); m \geq 2\). Then \( X_{2a,2} \) is not rational if \( 2a > m + 1\).

More precisely, if \( Y \) is any variety of dimension \( m \) and \( \phi : Y \times \mathbb{P}^1 \to X_{2a,2} \) a dominant map then \( 2|\deg \phi \).

1.2.1 Remarks. (1.2.1.1) \( X_{2a,2} \to \mathbb{CP}^n \) is a conic bundle. For conic bundles we have the very strong results of [Sarkisov81,82] which say that a conic bundle is not rational if the locus of singular fibers plus 4 times the canonical class of the base is effective. In our case the locus of singular fibers is a divisor in \( \mathbb{P}^m \) of degree \( 4a > 2m + 2 \). This is lower than the Sarkisov bound \( 4m + 4 \). Sarkisov’s normal crossing assumptions are also not satisfied by \( X_{2a,2} \to \mathbb{CP}^n \). Thus some of our cases are not covered by the results of [Sarkisov81,82]. This suggests the possibility that the bounds of [Sarkisov81,82] can be improved considerably. This is not clear even for conic bundles over surfaces.

(1.2.1.2) It is natural to ask if the projection \( X_{2a,2} \to \mathbb{CP}^m \) is the only conic bundle structure of \( X_{2a,2} \) or not. The proof gives such examples in characteristic 2, but not over \( \mathbb{C} \).

For families of cubic surfaces the result is weaker:

1.3 Theorem. Let \( X_{3a,3} \to \mathbb{CP}^m \times \mathbb{CP}^2 \) be a cyclic triple cover ramified along a very general hypersurface of bidegree \((3a,3); m \geq 1\). Then \( X_{3a,3} \) is not rational and not even ruled if \( 3a > m + 1\).

1.3.1 Remark. Similar results for \( m = 1 \) were proved by [Bardelli84].

For hypersurfaces in \( \mathbb{CP}^m \times \mathbb{CP}^{m+1} \) these imply the following:

1.4 Theorem. (1.4.1) Let \( X_{c,2} \subset \mathbb{CP}^m \times \mathbb{CP}^2 \) be a very general hypersurface of bidegree \((c,2); m \geq 2\). Then \( X_{c,2} \) is not rational if \( c \geq m + 3 \).

(1.4.2) Let \( X_{c,3} \subset \mathbb{CP}^m \times \mathbb{CP}^3 \) be a very general hypersurface of bidegree \((c,3); m \geq 1\). Then \( X_{c,3} \) is not rational if \( c \geq m + 4 \).

Proof. By [Kollár96, V.5.12–13] it is sufficient to find an algebraically closed field \( k \) and a single example of a nonruled hypersurface \( X_{c',2} \subset \mathbb{CP}^m \times \mathbb{CP}^2 \) resp. \( X_{c',3} \subset \mathbb{CP}^m \times \mathbb{CP}^3 \) over \( k \) for some \( c' \leq c \). We proceed to construct such examples in characteristic two for conic bundles and in characteristic three for families of cubic surfaces.

Consider \( \mathbb{A}^m \times \mathbb{A}^n \) with coordinates \((u_1, \ldots, u_m; v_1, \ldots, v_n)\). Let \( f(u,v) \) be a polynomial of bidegree \((c, n + 1)\). We have a hypersurface

\[
Z := (y^{n+1} - f(u,v) = 0) \subset V := \mathbb{A}^1 \times \mathbb{A}^m \times \mathbb{A}^n,
\]

where \( y \) is a coordinate on \( \mathbb{A}^1 \). There are several natural ways to associate a projective variety to \( Z \):

(1.4.3.1) We can view \( V \) as a coordinate chart in \( \mathbb{P}^m \times \mathbb{P}^{n+1} \). The closure \( \bar{Z}_1 \) of \( Z \) is a hypersurface of bidegree \((c, n + 1)\). For \( n = 1, 2 \) this gives our examples \( X_{c,2} \) and \( X_{c,3} \).

(1.4.3.2) We can compactify \( \mathbb{A}^m \times \mathbb{A}^n \) to \( \mathbb{P}^m \times \mathbb{P}^n \) and view \( f \) as a section of the line bundle \( \mathcal{O}_{\mathbb{P}^m \times \mathbb{P}^n}(c, n + 1) \). Assume that \( n + 1 |c \) and let \( L = \mathcal{O}(c/(n + 1), 1) \). The corresponding cyclic cover \( \bar{Z}_2 = \mathbb{P}^m \times \mathbb{P}^{n+1}[\sqrt[2]{f}] \) (defined in (2.3)) gives another compactification of \( Z \).
(1.2–3) show using the second representation that $Z_2$ is not ruled if $f$ is very general and $c > n + 1$. This implies (1.4). □

1.5 Generalizations. It is clear from the proof that it applies to many different cases. The main point is to have a family of conics or cubic surfaces whose branch divisor is sufficiently large, but I found it hard to write down a reasonably general statement.

1.6 Outline of the proof of (1.2–3). By [Kollár96, V.5.12–13] it is sufficient to find an algebraically closed field $k$ and a single example of a nonruled cyclic cover $X_{a',2} \to \mathbb{P}^m \times \mathbb{P}^1$ (resp. $X_{3a',3} \to \mathbb{P}^m \times \mathbb{P}^2$) for some $a' \leq a$. We proceed to construct such examples in characteristic two for conic bundles and in characteristic three for families of cubic surfaces.

Fix an algebraically closed field $k$ of characteristic $p$. Let $\pi : X_{ap,bp} \to \mathbb{P}^m \times \mathbb{P}^n$ be a degree $p$ cyclic cover ramified along a very general hypersurface of bidegree $(ap,bp)$. $\pi$ is purely inseparable, thus $X_{ap,bp}$ is (purely inseparably) unirational. We intend to show that it is frequently not ruled.

Section 2 is a review of the machinery of inseparable cyclic covers and their applications to nonrationality problems. $X_{ap,bp}$ has isolated singularities by (2.1.4) and (2.2.3). These can be resolved, and if $\pi_Y : Y \to X_{ap,bp} \to \mathbb{P}^m \times \mathbb{P}^n$ is a resolution then by (2.5) there is a nonzero map

$$\pi_Y^* \mathcal{O}(ap - m - 1, bp - n - 1) \to \wedge^{m+n-1} \Omega^1_Y \cong \Omega^{m+n-1}_Y.$$ 

Thus [Kollár95] shows that $X_{ap,bp}$ is not rational and not even ruled if $ap > m + 1$ and $bp > n + 1$. In this paper I study the case when $ap > m + 1$ and $bp = n + 1$.

Section 3 contains a nonruledness criterion. Let $f : Y \to \mathbb{P}^m$ be the composite of $\pi_Y$ with the first projection. (3.2) shows that if $ap > m + 1$ and $bp = n + 1$, then $Y$ is ruled iff the generic fiber of $f$ is ruled over the function field $k(\mathbb{P}^m)$. This is the main technical departure from [Kollár95]. There I used varieties in positive characteristic which were shown to be not even separably uniruled. Here the varieties in question are separably uniruled, but we are able to get a description of all separable unirulings.

Rationality or ruledness over nonclosed fields is a very interesting question. Unfortunately I can not say much, except for $n = 1, 2$.

When $n = 1$, the generic fiber of $f$ is a plane conic. Conics over nonclosed fields are considered in section 4. We get a complete description of their unirulings in (4.1). Theorem (1.2) is implied by (2.5), (3.1) and (4.2).

If $n = 2$, the generic fiber of $f$ is a cubic surface $S$ given by an equation $u^3 = f_3(x, y, z)$. Since $X_{ap,bp}$ has only isolated singularities, $S$ is nonsingular over $k(\mathbb{P}^m)$, though we will see that it is not smooth (5.2). Thus we are led to investigate nonsingular Del Pezzo surfaces over arbitrary fields in section 5. We are forced to study the situation over nonperfect fields; this introduces several new features. The main result is (5.7) which generalizes results of Segre and Manin to nonperfect fields.

1.7 Terminology. I follow the terminology of [Kollár96].

If a scheme $X$ is defined over a field $F$, and $E$ is a field extension then $X_E$ denotes the scheme obtained by base extension.

For a field $E$, $\overline{E}$ denotes an algebraic closure.
$X_F$ is called rational, if it is birational to $\mathbb{P}^n_F$, and the birational map is defined over $F$. Sometimes for emphasis I say that $X_F$ is rational over $F$. The same convention applies to other notions (ruled, uniruled, irreducible etc.).

If $X_F$ is rational then I say that $X_F$ is geometrally rational. Similarly for other notions (ruled, uniruled, irreducible etc.).

Following standard terminology, a scheme is called nonsingular if all of its local rings are regular. Over nonperfect fields this is not the same as being smooth.

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2. INSEPARABLE CYCLIC COVERS

First we recall the definitions and basic properties of critical points of sections of line bundles in positive characteristic. For proofs see [Kollár95] or [Kollár96, V.5].

2.1 Definition. Let $X$ be a smooth variety over an algebraically closed field $k$ and $f$ a function on $X$. Let $x \in X$ be a closed point and assume that $f$ has a critical point at $x$. Choose local coordinates $x_1, \ldots, x_n$ at $x$.

(2.1.1) $f$ has a nondegenerate critical point at $x$ iff $\partial f/\partial x_1, \ldots, \partial f/\partial x_n$ generate the maximal ideal of the local ring $\mathcal{O}_{x,X}$. This notion is independent of the local coordinates chosen.

If $\text{char } k \neq 2$ or $\text{char } k = 2$ and $n$ is even then $f$ has a nondegenerate critical point at $x$ iff in suitable local coordinates $f$ can be written as

$$f = c + \begin{cases} x_1x_2 + x_3x_4 + \cdots + x_{n-1}x_n + f_3, & \text{if } n \text{ is even,} \\ x_1^2 + x_2x_3 + \cdots + x_{n-1}x_n + f_3, & \text{if } n \text{ is odd,} \end{cases}$$

where $f_3 \in m^3_x$.

(2.1.2) If $\text{char } k = 2$ and $\dim X$ is odd, then every critical point is degenerate.

(2.1.3) Assume that $\text{char } k = 2$ and $\dim X$ is odd. A critical point of $f$ is called almost nondegenerate iff $\text{length } \mathcal{O}_{x,X}/(\partial f/\partial x_1, \ldots, \partial f/\partial x_n) = 2$. Equivalently, in suitable local coordinates $f$ can be written as

$$f = c + ax_1^2 + x_2x_3 + \cdots + x_{n-1}x_n + bx_3^3 + f_4 \text{ where } b \neq 0.$$ 

(2.1.4) Assume that $\text{char } k | d$. Then the hypersurface $Z = (y^d - f(x_1, \ldots, x_n) = 0)$ is singular at the point $(y,x) \in Z$ iff $x \in X$ is a critical point of $f$.

(2.1.5) Let $L$ be a line bundle on $X$ and $s \in H^0(X,L^d)$ a section. Let $U \subset X$ be an open affine subset such that $L|U \cong \mathcal{O}_U$. Choose such an isomorphism. Then $s|U$ can be viewed as section of $\mathcal{O}_U^\otimes d \cong \mathcal{O}_U$. Thus it makes sense to talk about its critical points. If $\text{char } k | d$ then this is independent of the choice of $U$ and of the trivialization $L|U \cong \mathcal{O}_U$. (This fails if the characteristic does not divide $d$.)

The usual Morse lemma can be generalized to positive characteristic. We use it in a somewhat technical form.

2.2 Proposition. Let $X$ be a smooth variety over a field of char $p$ and $L$ a line bundle on $X$. Let $d$ be an integer divisible by $p$ and $W \subset H^0(X,L^d)$ a finite dimensional subvector space. Let $m_x$ denote the ideal sheaf of $x \in X$. Assume that:

(2.2.1) For every closed point $x \in X$ the restriction map $W \to (\mathcal{O}_X/m^2_x) \otimes L^k$ is surjective.
(2.2.2) For every closed point \( x \in X \) there is an \( f_x \in W \) which has an (almost) nondegenerate critical point at \( x \).

Then a general section \( f \in W \) has only (almost) nondegenerate critical points.

**Proof.** This is a simple constant count. Fix \( x \in X \) and let \( W_x \subset W \) be the set of functions with a critical point at \( x \). By (2.2.1), \( W_x \) has codimension \( n \). In \( W_x \) the functions with an (almost) nondegenerate critical point at \( x \) form an open subset \( W_x^0 \) which is nonempty by (2.2.2). Thus the set of functions with a degenerate critical point is \( \bigcup_x (W_x - W_x^0) \) and it has codimension at least one in \( W \). □

**2.2.3 Lemma.** Let \( X_1, X_2 \) be smooth varieties over a field of \( \text{char} \; p \) and \( L_i \) very ample line bundles on \( X_i \). Let \( L = p_1^*L_1 \otimes p_2^*L_2 \) be the corresponding line bundle on \( X = X_1 \times X_2 \). If \( p \not| d \) then a general section \( f \in H^0(X, L) \) has only (almost) nondegenerate critical points.

**Proof.** Pick a point \( x = (x_1, x_2) \). The condition (2.2.1) is clearly satisfied. In order to check (2.2.2) choose global sections \( u_i \in H^0(X_1, L_1) \) and \( v_j \in H^0(X_2, L_2) \) such that they give local coordinates at \( x_1 \) resp. \( x_2 \).

If \( p \not| d \) then \( \sum u_i^2 + \sum v_j^2 \) gives a section of \( L^d \) with a nondegenerate critical point at \( x \).

If \( p = 2 \) we need to consider a few cases. Set \( n_i = \dim X_i \). We plan to use the function
\[
g = \sum_{1 \leq i \leq n_1/2} u_{2i-1}u_{2i} + \sum_{1 \leq j \leq n_2/2} v_{2j-1}v_{2j}.
\]

In both of the \( n_i \) are even, then we can take \( f = g \). If both of the \( n_i \) are odd, then we can use \( f = g + u_{n_1}v_{n_2} \). Otherwise we may assume that \( n_1 \) is odd and \( n_2 \) is even. Then we use
\[
f = g + u_{n_1}v_{n_2-1} + u_{n_1}^2v_{n_2}.
\]

Explicit computation shows that \( f \) has an almost nondegenerate critical point. □

**2.3 Definition.** Cyclic covers

Let \( X \) be a scheme, \( L \) a line bundle on \( X \) and \( s \in H^0(X, L^d) \) a section. Assume for simplicity that the divisor of its zeros (\( s = 0 \)) is reduced. The cyclic cover of \( X \) obtained by taking a \( d^{\text{th}} \)-root of \( s \), denoted by \( X[\sqrt[d]{s}] \) is a scheme locally constructed as follows:

Let \( U \subset X \) be an open set such that \( L|U \cong \mathcal{O}_U \). Then \( s|U \) can be identified with a function \( f \in H^0(U, \mathcal{O}_U) \). Let \( V \subset \mathbb{A}^1 \times U \) be the closed subset defined by the equation \( y^d - f = 0 \) where \( y \) is the coordinate on \( \mathbb{A}^1 \). The resulting schemes can be patched together in a natural way to get a scheme \( X[\sqrt[d]{s}] \); cf. [Kollár96, II.6.1].

We are interested in it only up to birational equivalence, so the precise definitions are unimportant.

The only result about cyclic covers we need is the following special case of [Kollár96, V.5.10]:

**2.4 Proposition.** Let \( X \) be a smooth variety of dimension \( n \) over a field \( k \) of \( \text{char} \; p \), \( L \) a line bundle on \( X \) and \( d \) an integer divisible by \( p \). Let \( s \in H^0(X, L^d) \) be a section with (almost) nondegenerate critical points. Let \( \pi : Y \to X \) be a smooth projective model of \( X[\sqrt[d]{s}] \) (\( Y \) always exists).

Then there is a nonzero map
\[
\pi^*(K_{X} \otimes L^d) \otimes \mathcal{O}_{\mathbb{P}^1}^{n-1} \cong \mathcal{O}_{\mathbb{P}^1}^{n-1}.
\] □
Applied to the cyclic covers $Z_2 = \mathbb{P}^m \times \mathbb{P}^n[\sqrt{s}]$ from the introduction, we get the following:

**2.5 Corollary.** Fix a prime $p$ and let $k$ be an algebraically closed field of characteristic $p$. Let $s \in H^0(\mathbb{P}^m \times \mathbb{P}^n, \mathcal{O}(a,b)^{\otimes p})$ be a general section and $q : Y \to \mathbb{P}^m \times \mathbb{P}^n$ a smooth projective model of $\mathbb{P}^m \times \mathbb{P}^n[\sqrt{s}]$.

Then there is a nonzero map

$$q^*\mathcal{O}(ap - m - 1, bp - n - 1) \to \wedge^{m+n-1}\Omega^1_Y \cong \Omega^{m+n-1}_Y. \quad \square$$

### 3. A Nonruledness Criterion

In this section we prove the following generalization of [Kollár96, V.5.11].

**3.1 Theorem.** Let $X, Y$ be smooth proper varieties and $f : Y \to X$ a surjective morphism, $n = \dim Y$. Let $M$ be a big line bundle on $X$ and assume that for some $i > 0$ there is a nonzero map

$$h : f^*M \to \wedge^i\Omega^1_Y.$$

(3.1.1) Let $Z$ be an affine variety of dimension $n - 1$ and $\phi : Z \times \mathbb{P}^1 \to Y$ a dominant and separable morphism. Then there is a morphism $\psi : Z \to X$ which fits in the commutative diagram

$$
\begin{array}{ccc}
Z \times \mathbb{P}^1 & \to & Y \\
\downarrow & & \downarrow f \\
Z & \psi & \to X.
\end{array}
$$

(3.1.2) Let $F = k(X)$ be the field of rational functions on $X$ and $Y_F$ the generic fiber of $f$. There is a one-to-one correspondence

$$\{\text{degree } d \text{ separable unirulings of } Y\} \leftrightarrow \{\text{degree } d \text{ separable unirulings of } Y_F\}.$$  

In particular, $Y$ is ruled iff $Y_F$ is ruled over $F$.

(3.1.3) Assume that for any two general points $y_1, y_2 \in Y_F$ there is a morphism $f = f(y_1, y_2) : \mathbb{P}^1 \to Y_F$ such that, $y_1, y_2 \in \text{im } f$, $Y_F$ is smooth along $\text{im } f$ and $f^*T_{Y_F}$ is semi positive. Then any birational selfmap of $Y$ preserves $f$. Hence there is an exact sequence

$$1 \to \text{Bir}(Y_F) \to \text{Bir}(Y) \to \text{Bir}(X).$$

**3.1.4 Remark.** The assumption (3.1.3) is satisfied if $Y_F$ is separably rationally connected (cf. [Kollár96,IV.3.2]). More generally, it also holds for the cyclic covers of $\mathbb{P}^n$ that we are considering, [ibid,V.5.19].

**Proof.** $M$ is big, hence there is an open set $U \subset X$ such that sections of $M^k$ separate points of $U$ for $k \gg 1$. In particular, if $g : C \to X$ is a nonconstant morphism from a smooth proper curve to $X$ whose image intersects $U$, then $\deg g^*M > 0$. 


Let \( g : C \to Y \) be a morphism such that \( g^*\Omega^1_{Y} \) is semi negative. We have a map
\[
g^*h : g^*f^*M \to \wedge^i g^*\Omega^1_{Y}.
\]
Thus either \((f \circ g)(C) \subset X - U\) or \((f \circ g)(C)\) is a single point. This will allow us to identify the fibers of \( f \).

In order to prove (3.1.1) pick a general point \( z \in Z \) and let \( \phi_z : \mathbb{P}^1 \to Y \) be the restriction of \( \phi \) to \( \{z\} \times \mathbb{P}^1 \).
\[
\Omega^1_{Z \times \mathbb{P}^1}|\{z\} \times \mathbb{P}^1 \cong \mathcal{O}^n_{\mathbb{P}^1} + \mathcal{O}_{\mathbb{P}^1}(-2),
\]
and \( \phi \) gives a map
\[
\Phi : (f \circ \phi_z)^*M \xrightarrow{\phi^*_h} \phi_z^* \wedge^i \Omega^1_{Y} \xrightarrow{\wedge^id\phi} \wedge^i(\mathcal{O}^n_{\mathbb{P}^1} + \mathcal{O}_{\mathbb{P}^1}(-2)),
\]
which is nonzero for general \( z \) since \( \phi \) is separable. Thus \( \text{deg}(f \circ \phi_z)^*M \leq 0 \). By the above remarks, this implies that \( f \circ \phi_z \) is a constant morphism.

Pick a point \( 0 \in \mathbb{P}^1 \) and define \( \psi : Z \to X \) by \( \psi(z) := f \circ \phi(z, 0) \). This shows (3.1.1).

Let \( Z_F \) be the generic fiber of \( \psi \). We obtain a dominant \( F \)-morphism \( \phi_F : Z_F \times \mathbb{P}^1 \to Y_F \) which is birational (resp. separable) iff \( \phi \) is birational (resp. separable).

Conversely, if \( W_F \) is any variety and \( W_F \to Y_F \) a morphism then it extends to a map \( W \to Y \) of the same degree. This shows (3.1.2).

Finally assume (3.1.3). Then there is an open set \( Y' \subset Y \) such that if \( y_1, y_2 \in Y' \) and \( f(y_1) = f(y_2) \) then there is a morphism \( f = f_{(y_1, y_2)} : \mathbb{P}^1 \to Y \) such that, \( y_1, y_2 \in \text{im} f \), \( Y \) is smooth along \( \text{im} f \) and \( f^*T_Y \) is semi positive.

Let \( \phi : Y \to Y \) be a birational selfmap of \( Y \); \( \phi \) is defined outside a codimension 2 set \( Z \subset Y \). By [Kollár96, II.3.7], the image of the general \( f_{(y_1, y_2)} \) is disjoint from \( Z \). Thus we have an injection
\[
f_{(y_1, y_2)}^*T_Y \hookrightarrow (\phi \circ f_{(y_1, y_2)})^*T_Y,
\]
which shows that the latter is also semi positive. Thus \( \phi(y_1) \) and \( \phi(y_2) \) are in the same fiber of \( f \). Therefore \( \phi \) preserves \( f \), which gives the exact sequence
\[
1 \to \text{Bir}(Y_F) \to \text{Bir}(Y) \to \text{Bir}(X). \quad \Box
\]

Applying (3.1) to the projection \( Y \to \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^m \) of (2.5) we obtain:

**3.2 Corollary.** Fix a prime \( p \) and let \( k \) be an algebraically closed field of characteristic \( p \). Let \( s \in H^0(\mathbb{P}^m \times \mathbb{P}^n, \mathcal{O}(a, b)^{\otimes p}) \) be a general section. Assume that \( ap - m - 1 > 0 \) and \( bp - n - 1 = 0 \).

Then \( Y' := \mathbb{P}^m \times \mathbb{P}^n[\sqrt{p}s] \) is ruled iff the generic fiber of \( Y' \to \mathbb{P}^m \) is ruled over the field \( k(x_1, \ldots, x_m) \).

Furthermore, \( Y' \) has a degree \( d \) separable uniruling iff the generic fiber of \( Y' \to \mathbb{P}^m \) has a degree \( d \) separable uniruling over the field \( k(x_1, \ldots, x_m) ). \quad \Box
\]

(3.2) naturally leads to the following:

**3.3 Question.** Let \( X \) be a variety over a field. When is \( X \) ruled over \( F^2 \)?
The problem is mainly interesting when $X_F$ is geometrically ruled. I can not say much in general, so I consider only two simple examples: conics and Del Pezzo surfaces. One should keep in mind that in our applications $F$ is the function field of a variety in positive characteristic, so $F$ is not perfect. Also, we need these results in characteristic 2 for conics and in characteristic 3 for cubic surfaces. These are the most unusual cases.

3.3.1 Example. If $X_F$ is an arbitrary variety which has no $F$-points, then $X_F$ is not rational, but it can happen that it is ruled. For instance, if $Y_F$ has no $F$-points then $Y \times \mathbb{P}^1$ is ruled and has no $F$-points.

This can happen even for quadrics in $\mathbb{P}^3$. For example, choose $a, b \in F$ such that $C = (x_0^2 + ax_1^2 + bx_2^2 = 0)$ has no $F$-points (say $F = \mathbb{R}$ and $a = b = 1$). Then $C \times \mathbb{P}^1$ is birational to the quadric $Q = (y_0^2 + ay_1^2 + by_2 + aby_3^2 = 0)$ via the map
\[
\phi : (x_0 : x_1 : x_2, s : t) \mapsto (sx_0 + atx_1 : sx_1 - tx_0 : sx_2 : tx_2).
\]

$Q$ is has no $F$-points.

4. Conics over nonclosed fields

The aim of this section is to study conics over arbitrary fields. We study when they are ruled or uniruled. The main result is (4.1), but for the applications we need (4.2).

4.1 Theorem. Let $F$ be a field and $C_F \subset \mathbb{P}^2_F$ an irreducible and reduced conic. The following are equivalent:

1. $C_F$ has a point in $F$.
2. $C_F$ is ruled.
3. $C_F$ has an odd degree uniruling.

If $C_F$ is smooth then these are also equivalent to

1. $C_F \cong \mathbb{P}^1_F$.

Proof. Let $P$ be an $F$-point of $C$. If $C$ is geometrically irreducible, then projecting it from $P$ gives a birational map $C \to \mathbb{P}^1_F$, hence $C$ is ruled. Otherwise, projection exhibits $C$ as a cone over a length two subscheme of $\mathbb{P}^1_F$, thus again $C$ is ruled. This shows that (4.1.1) $\Rightarrow$ (4.1.2) while (4.1.2) $\Rightarrow$ (4.1.3) is clear.

The proof of (4.1.3) $\Rightarrow$ (4.1.1) is longer. Let $A$ be a zero dimensional $F$-scheme and $\phi_F : A \times_F \mathbb{P}^1 \to C_F$ an odd degree uniruling. Let $A_i \subset A$ be the irreducible components. $\deg \phi_F = \sum_i \deg(\phi_F|A_i \times \mathbb{P}^1)$, thus one of the $\deg(\phi_F|A_i \times \mathbb{P}^1)$ is odd. Thus we may assume that $A$ is irreducible. $\deg(\phi_F|\text{red } A \times \mathbb{P}^1)$ divides $\deg \phi_F$, hence we may also assume that $A = \text{Spec}_F F'$ where $F' \supset F$ is a field extension.

By base change to $F'$ we obtain
\[
\mathbb{P}^1_{F'} \leftrightarrow \text{Spec}_{F'}(F' \otimes_F F') \times \mathbb{P}^1_F \to C_{F'}.
\]

Thus by the Lüroth theorem, every irreducible component of $\text{red } C_{F'}$ is birational to $\mathbb{P}^1_{F'}$. Passing to the algebraic closure we obtain
\[
\phi_{\bar{F}} : \text{Spec}_{\bar{F}}(\bar{F} \otimes_F F') \times \mathbb{P}^1 \to C_{\bar{F}}.
\]

The left hand side may have several irreducible components, conjugate to each other. Let $\bar{\phi} : \mathbb{P}^1 \to \mathbb{P}^1$ be the induced map between any of the irreducible and reduced components. By counting degrees we obtain the following:
4.1.5 Claim. Notation as above. Then
(4.1.5.1) \( \deg \phi_F = \deg(F'/F) \deg(\phi) \) if \( C_F \) is a smooth conic, and
(4.1.5.2) \( 2 \deg \phi_F = \deg(F'/F) \deg(\phi) \) if \( C_F \) is a singular conic.
Thus \( \deg(F'/F) \) is odd in the first case and is not divisible by 4 in the second case. \( \square \)

Assume first that \( C_F \) is a smooth conic. \( A \times F \) Spec \( F' \) has a closed point, thus
we get a morphism \( \mathbb{P}^1_{F'} \to C_{F'} \). Hence \( C_{F'} \) has a point in \( F' \). This
in turn gives an odd degree point on \( C_F \); let \( L \) be the corresponding line bundle. The
restriction of \( \mathcal{O}_{\mathbb{P}^2}(1) \) to \( C_F \) is a line bundle of degree 2. We conclude that \( C_F \) has a line bundle
of degree 1. Any of its sections gives an \( F \)-point on \( C_F \).

If \( C_F \) is a pair of intersecting lines, then the intersection point is defined over \( F \).

Finally consider the case when \( C_F \) is a double line; this can happen only in
characteristic 2. The equation of \( C_F \) is \( \sum b_i x_i^2 = 0 \).

4.1.6 Claim. Let \( C_F = (\sum b_i x_i^2 = 0) \) be an irreducible conic over a field of characteristic 2. Let \( E/F \) be a separable extension. Then any \( E \)-point of \( C_F \) is an \( F \)-point.

Proof. We may assume that \( E/F \) is Galois. Assume that \( P \) is an \( E \)-point which
is not an \( F \)-point. Conjugates of \( P \) over \( F \) also give \( E \)-points, thus we obtain
that \( \text{red} C_F \) is defined over \( E \). \( \text{red} C_F \) is also defined over the purely inseparable
extension \( F^i = F(\sqrt{b_0}, \sqrt{b_1}, \sqrt{b_2}) \), hence also over the intersection \( F'' \cap F^i = F \) (cf.
[Kollár96, I.3.5]). This is a contradiction. \( \square \)

As in the irreducible case we know that \( C_{F'} \) has an \( F' \)-point. Thus by (4.1.6) \( F'/F \) is not separable. In view of (4.1.5.2) we conclude that there is a subextension
\( F' \supset F'' \supset F \) such that \( F'/F \) has odd degree (hence separable) and \( F' = F''(\sqrt{s}) \)
for some \( s \in F'' \). By (4.1.6) it is enough to show that \( C \) has an \( F'' \)-point.

As we mentioned earlier, \( \text{red} C_{F'} \) is birational to \( \mathbb{P}^1_{F'} \), thus \( \text{red} C_{F'} \) is a line in
\( \mathbb{P}^2 \). Therefore \( C_{F'} \) is a double line with equation \( \sum a_i x_i^2 = 0 \) where \( a_i \in F' \).
\( a_i^2 \in F'' \), thus the equation of \( C \) over \( F'' \) is \( \sum a_i^2 x_i^2 = 0 \). Write \( a_i = c_i + sd_i \) where
\( c_i, d_i \in F'' \). The equation of \( C \) is

\[ \sum a_i^2 x_i^2 = (\sum c_i x_i)^2 + s^2(\sum d_i x_i)^2 = 0. \]

The solution of \( \sum c_i x_i = \sum d_i x_i = 0 \) gives an \( F'' \)-point on \( C \).

The equivalence with (4.1.4) was established in the course of the proof. \( \square \)

4.1.7 Remark. If \( C_F \) is a smooth conic, then \( C_F \) has a degree 2 separable uniruling.
Take any general line in \( \mathbb{P}^2 \). Its intersection points with \( C_F \) are in a degree 2
separable extension of \( E \supset F \). Thus \( C_E \cong \mathbb{P}^1_E \).

It remains to establish that the generic fibers appearing in (3.1) are not ruled.
There should be a general result about conics over function fields, but I could not
find a simple proof. For our applications the following is sufficient:

4.2 Proposition. Let \( k \) be a field of characteristic 2 and \( F = k(x_1, \ldots, x_m) \) the
field of rational functions in \( m \geq 2 \) variables. Fix an even integer \( d \geq 2 \) and let
\( a, b, c \in k[x_1, \ldots, x_m] \) be general polynomials of degree \( d \). Then the conic

\[ C = (a x_1^2 + a x_2^2 + by_1 + cy_2) \subset \mathbb{P}^2. \]
is not ruled (over $F$). Moreover $C$ does not have any odd degree uniruling.

4.2.1 Comments. It is worth while to remark that for (4.2) to hold it is essential that $k(x_1, \ldots, x_m)$ is not perfect. A point on $C$ is given by $P = (\sqrt{a}, 1, 0)$. If $F$ is perfect of characteristic $2$, then $P$ is an $F$-point and $C$ is rational.

**Proof.** By (4.1) it is sufficient to establish that $C$ has no $F$-points. We can identify $ay^2 + by_1y_2 + cy_2^2$ with a section $s$ of $\mathcal{O}_{\mathbb{P}^{-1}}(d, 2)$. Let $Y := \mathbb{P}^{-1} \times \mathbb{P}^1[\sqrt{s}]$ be the corresponding double cover. The generic fiber of $\pi : Y \to \mathbb{P}^{m-1}$ is $C$, thus it is sufficient to prove that $\pi$ has no sections. By (2.2.3) $Y$ has only isolated singularities. The fibers of $\pi$ over $b = 0$ are double lines. This shows that $\pi$ does not even have local sections at the generic point of $b = 0$.  

5. Del Pezzo surfaces over nonclosed field

In order to complete the proof of (1.3) we need to study the Del Pezzo surfaces:

(5.1.1) $S_F$ with equation $u^3 = f_3(x, y, z)$ in characteristic $3$.

Although we do not need it, it is very natural to study also the surfaces:

(5.1.2) $T_F$ with equation $u^2 = f_4(x, y, z)$ in characteristic $2$.

To get an idea of the geometry of these surfaces, we study them first over perfect fields.

5.2 Remark: The case of perfect fields. Assume first that our base field is algebraically closed. It is easy to see that we can write

$$f_3 = l_1l_2l_3 + l_4^3, \quad \text{and} \quad f_4 = l_1l_2l_3l_4 + q^2$$

where the $l_i$ are linear and $q$ quadratic. Thus we can make coordinate changes $u = u - l_4$ (resp. $u = u - q$) and then a suitable coordinate change among the $x, y, z$ to reduce the equations to the form

$$u^3 = xyz, \quad \text{and} \quad u^2 = xyz(x + y + z).$$

From this we see that the cubic $S$ has three singular points of type $A_2$. The degree two Del Pezzo $T$ has seven singular points of type $A_1$ at the seven points of $\mathbb{F}_2\mathbb{P}^2$.

Next consider the case when the base field $F$ is perfect. Then these singular points are defined over $F$. For the cubic $S_F$ resolve the singular points, and contract the birational transforms of the 3 coordinate axes to get a degree 6 Del Pezzo surface. Over a perfect field $F$ a degree 6 Del Pezzo surface is rational iff it has an $F$-point [Manin72,IV.7.8]. Our surface $S_F$ does have $F$-points over perfect fields of characteristic $3$, for example $P = (1, 0, 0, \sqrt[3]{f_3(1, 0, 0)})$.

For the surface $T_F$ resolve the singular points, and contract the birational transforms of the seven lines in $\mathbb{F}_2\mathbb{P}^2$ to get a Brauer-Severi variety. It has a point in a degree 7 extension, hence it is isomorphic to $\mathbb{P}^2$. Thus our surfaces $S_F$ and $T_F$ are rational over any perfect field.

In our case the field $F$ is not perfect, the surfaces $S_F$ and $T_F$ are nonsingular over $F$ but they are not smooth. In order to show that they are not ruled, we have to understand how the presence of nonsingular but nonsmooth points effects the birational geometry of a surface.

The crucial result is the following:
5.3 Theorem. Let $F$ be a field and $S, T$ nonsingular and proper surfaces over $F$. Assume that $T$ is smooth except possibly at finitely many points. Let $f : S \dasharrow T$ be a birational map.

Then $f$ is defined at all nonsmooth points of $S$.

Proof. There is a sequence of blowing ups of closed points $p : S' \rightarrow S$ such that $f \circ p : S' \rightarrow T$ is a morphism. Let $P \in S$ be a closed nonsmooth point, and assume that $f$ is not defined at $P$. Then $p^{-1}(P)$ is 1-dimensional and there is an irreducible component $E \subset p^{-1}(P)$ such that $p \circ f$ is a local isomorphism at the generic point of $E$. This implies that $S'$ is smooth at the generic point of $E$. This contradicts the following lemma. □

5.3.1 Lemma. Let $F$ be a field and $S, S'$ nonsingular surfaces over $F$. Let $p : S' \rightarrow S$ be a proper and birational morphism and $P \in S$ a closed nonsmooth point.

Then $S$ is not smooth at all points of $p^{-1}(P)$.

Proof. By induction it is sufficient to consider the case when $p$ is the blow up of $P$. We may also assume that $S$ is an affine neighborhood of $P$ such that the maximal ideal $m_P$ is generated by two global sections $u, v \in m_P \subset O_S$. The blow up can be described by two affine charts, one of them is

$$S'_1 := (u - vs = 0) \subset S \times \mathbb{A}^1,$$ where $s$ is a global coordinate on $\mathbb{A}^1$.

By assumption $S$ is not smooth at $P$, so $S \times \mathbb{A}^1$ is not smooth along $P \times \mathbb{A}^1$. $S'_1$ is a Cartier divisor on $S \times \mathbb{A}^1$, thus it also not smooth along $P \times \mathbb{A}^1$. This was to be proved. □

In the course of the proof we used the following results about birational transformations of nonsingular surfaces. For a proof see, for instance, [Zariski58].

5.3.2 Proposition. Let $S, T$ be proper and nonsingular surfaces over a field $F$ and $\phi : S \dasharrow T$ a birational map. Then $\phi$ is a composite of blow ups and blow downs (of closed points). In particular, $h^i(S, O_S) = h^i(T, O_T)$. □

5.3.3 Corollary. Let $F$ be a field and $S$ a nonsingular and proper surface over $F$. Assume that $S$ is smooth except possibly at finitely many points. Let $f : S \dasharrow S$ be a birational map.

Then $f$ is a local isomorphism at all nonsmooth points of $S$.

Proof. Factor $f$ as

$$f : S \xrightarrow{p} S' \xrightarrow{p'} S$$

where $p, p'$ are birational morphisms. By (5.3) we see that $p$ and $p'$ are both local isomorphisms at the nonsmooth points of $S$. □

This gives the following rationality criterion:

5.4 Corollary. Let $F$ be a field and $S$ a nonsingular and proper surface over $F$. Assume that $S$ is generically smooth. The following are equivalent:

(5.4.1) $S$ is rational (over $F$).

(5.4.2) There is a two dimensional linear system $L = |C|$ on $S$ with (infinitely near) base points $P_i$ of multiplicity $m_i$ such that

(5.4.2.1) a general $C \in L$ is birational to $\mathbb{P}^1$;

(5.4.2.2) $S$ is smooth along a general $C \in L$.
\[(5.4.2.3) \quad C \cdot K_S + \sum m_i = -3 \quad \text{and} \quad C^2 - \sum m_i^2 = 1.\]

**Proof.** Assume that there is a birational map \( f : S \rightarrow \mathbb{P}^2 \). Let \( Z \subset S \) be the locus of nonsmooth points. By (5.3), \( f \) is defined along \( Z \) and \( f(Z) \) is zero dimensional. Set \( L = f^{-1}|O_{\mathbb{P}^2}(1)| \). (5.4.2.1–2) are clear, and (5.4.2.3) is the usual equalities (5.4.3.1).

Conversely, assume (5.4.2). Resolve the base points of \( L \) to obtain a base point free linear system \( L' \) on \( S' \). From (5.4.2.3) we obtain that

\[ C' \cdot K_{S'} = -3 \quad \text{and} \quad C'^2 = 1 \quad \text{for} \quad C' \in L'. \]

Thus the linear system \( L' \) maps \( S' \) birationally to \( \mathbb{P}^2 \). \( \square \)

What we really need is a ruledness criterion. If \( S \) is ruled, it can be birational to a surface which is the product of \( \mathbb{P}^1 \) with a conic which has no \( F \)-points. Thus the natural linear system obtained on \( S \) is two dimensional and its general member is geometrically reducible. I found it clearer to concentrate instead on a single curve, but we may also have some smooth points assigned with multiplicity one.

If \( C \subset S \) is a curve on a smooth surface with assigned multiplicities \( m_i \) at the points \( P_i \) and \( \phi : S \rightarrow S' \) is a birational map, there is a natural birational transform \( C' \subset S' \) with assigned multiplicities \( m'_i \) at the points \( P'_i \). In order to define this we need only the case when \( \phi \) is the blow up of a point or its inverse, where the definition is the obvious one. The values of the expressions

\[(5.4.3.1) \quad C \cdot K_S + \sum m_i \quad \text{and} \quad C^2 - \sum m_i^2 \]

are birational invariants, cf. [Hudson27, p.5].

**5.5 Corollary.** Let \( F \) be a field and \( S \) a nonsingular and proper surface over \( F \) such that \( S \) is generically smooth, geometrically irreducible and \( h^1(S, O_S) = 0 \). Assume that \( S \) is ruled (over \( F \)).

Then there is a rational curve \( C \subset S_F \) with assigned (infinitely near) multiple points \( P_i \) of multiplicity \( m_i \) such that

\[(5.5.1) \quad S_F \text{ is smooth along } C.\]

\[(5.5.2) \quad C \cdot K_S + \sum m_i = -2 \quad \text{and} \quad C^2 - \sum m_i^2 = 0.\]

\[(5.5.3) \quad O_{S_F}(2C) \in \text{Pic}(S).\]

**Proof.** By assumption there is a nonsingular, geometrically integral curve \( D \) and a birational map \( f : S \rightarrow D \times \mathbb{P}^1 \). By (5.3.2),

\[ 0 = h^1(S, O_S) = h^1(D \times \mathbb{P}^1, O_{D \times \mathbb{P}^1}) = h^1(D, O_D). \]

Thus \( D \) is isomorphic to a smooth conic.

Let \( d \in D_F \) be a general point, \( C' = d \times \mathbb{P}^1 \) and \( C = f^{-1}_*(C') \subset S_F \) the corresponding birational transform. \( S \) is smooth along \( C \) by (5.3), and (5.5.2) is the usual equalities (5.4.3.1).
Finally, $D$ has a degree 2 point defined over $F$, thus the line bundle $\mathcal{O}_D(\mathbb{P}^2)(2C')$ is defined over $F$. Therefore $\mathcal{O}_{S_F}(2C)$ is also defined over $F$. \hfill \Box

The main result of this section is (5.7). Over perfect fields it is a special case of more general results of Segre and Manin (see [Manin72]). The proofs in [Manin72] use the structure of the Picard group of smooth Del Pezzo surfaces to compute the action of certain involutions. In our case the Picard groups are small (5.6), and it is easier to use the geometric ideas of [Segre43,51] to analyze the involutions. (5.3) essentially says that all the relevant geometry takes place inside the smooth locus, where the geometric description of the involutions works well.

5.6 Lemma. Let $S_F$ denote an integral cubic surface $u^3 = f_3(x, y, z) \subset \mathbb{P}^3$ for char $F = 3$, or an integral double plane with equation $u^2 = f_4(x, y, z)$ for char $F = 2$.

Then Pic$(S_F) = \mathbb{Z}[-K_S]$.

Proof. Let $\pi : S_F \to \mathbb{P}^2$ be the projection to the $(x, y, z)$-plane. $\pi$ is purely inseparable, thus if $C \subset S_F$ is any divisor then $\pi^*(\pi_*(C)) = (\deg \pi)C$. Therefore $-K_S = \pi^*(\mathcal{O}_{\mathbb{P}^2}(1))$ generates Pic$(S_F) \otimes \mathbb{Q}$. Since $(K_S^2) \leq 3$, we get that $K_S$ is not divisible in Pic$(S_F)$, hence $-K_S$ generates Pic$(S_F)/(\text{torsion})$. Thus we are left to prove that Pic$(S)$ has no torsion.

Let $[C] \in \text{Pic}(S)$ be a numerically trivial Cartier divisor. By Riemann-Roch,

$$h^0(\mathcal{O}_S(C)) + h^0(\mathcal{O}_S(K_S - C)) \geq \chi(\mathcal{O}_S) = 1.$$  

$-K_S$ is ample, so $h^0(\mathcal{O}_S(K_S - C)) = 0$. Thus $\mathcal{O}_S(C)$ has a section and $\mathcal{O}_S(C) \cong \mathcal{O}_S$. \hfill \Box

5.7 Theorem. Let $F$ be a field and $S$ a nonsingular Del Pezzo surface over $F$. Assume that $S_F$ is integral, $\chi(\mathcal{O}_S) = 1$, Pic$(S) = \mathbb{Z}[-K_S]$ and $K_S^2 = 1, 2, 3$.

Then $S$ is not ruled (over $F$).

5.7.1 Remark. Let $S$ be a nonsingular Del Pezzo surface over $F$ such that $S_F$ is integral, $\chi(\mathcal{O}_S) = 1$, and $K_S^2 = 1, 2, 3$. The structure theory of Del Pezzo surfaces shows that $S$ is a cubic surface for $K_S^2 = 1$, a double plane for $K_S^2 = 2$ and as expected for $K_S^2 = 3$ (cf. [Kollár96, III.3]).

[Reid94] contains examples of nonnormal Del Pezzo surfaces with $\chi(\mathcal{O}_S) < 1$. Some of these may have nonsingular models over nonperfect fields. I have not checked if this indeed happens or what can be said about their arithmetic properties.

Proof. Assuming that $S$ is ruled, we derive a contradiction. (5.5) guarantees the existence of a rational curve $C \subset S_F$ satisfying the properties (5.5.1-3). By (5.5.3) $\mathcal{O}(2C)$ is in Pic$(S)$. Since $K_S^2 \leq 3$, $K_S$ is not divisible by 2 in Pic$(S_F)$ and therefore $\mathcal{O}(C)$ is in Pic$(S)$. This implies that $C \in \{-dK_S\}$ for some $d$. We show that there can not be such a curve:

5.8 Proposition. Let $S$ be an integral Del Pezzo surface over an algebraically closed field such that $\chi(\mathcal{O}_S) = 1$, and $K_S^2 = 1, 2, 3$. Then $S$ does not contain any curve $C$ satisfying the following properties:

(5.8.1) $C$ is birational to $\mathbb{P}^1$;
(5.8.2) $S$ is smooth along $C$;
(5.8.3) $C \subset \{-dK_S\}$ for some $d$. 

(5.8.4) \( \sum_i m_i = d(K_S^2) - 2 \) and \( \sum_i m_i^2 = d^2(K_S^2) \), where \( P_i \) are the (assigned and infinitely near) multiple points of \( C \) with multiplicity \( m_i \).

**Proof.** If \( m_i \leq d \) for every \( i \) then
\[
\sum_i m_i^2 \leq d \sum_i m_i = d^2(K_S^2) - 2d < d^2(K_S^2),
\]
a contradiction. Thus there is a point \( P = P_1 \) such that \( m = m_1 > d \).

Let \( P \in L \subset S \) be a line (that is, \( -K_S \cdot L = 1 \)). Then \( m \leq (C \cdot L) = d \). Therefore \( P \) can not lie on any line.

If \( (K_S^2) = 1 \) then any member of \( | -K_S | \) is a line, thus there is a line through any point. Hence we are done if \( (K_S^2) = 1 \).

Next consider the case when \( (K_S^2) = 3 \). That is, \( S \subset \mathbb{P}^3 \) is a cubic surface.

Let \( D \subset S \) be the intersection of \( S \) with the tangent plane at \( P \). Since there is no line through \( P \), \( D \) is an irreducible cubic whose unique singular point is at \( P \). In particular, \( D \) is contained in the smooth locus of \( S \).

The point \( P \) determines a birational selfmap \( \tau \) of \( S \) as follows. Take any point \( Q \in S \), connect \( P, Q \) with a line and let \( \tau(Q) \) be the third intersection point of the line with \( S \). \( \tau \) is an automorphism of \( S - D \). Another way of describing \( \tau \) is the following. Projecting \( S \) from \( P \) gives a diagram
\[
B_P S \xrightarrow{q} \mathbb{P}^2
\]
\[
p \downarrow
\]
\[
S
\]
Let \( E \subset B_P S \) be the exceptional curve; \( C', D' \subset B_P S \) the birational transforms of \( C, D \). \( q \) is a degree two morphism and \( \tau \) is the involution interchanging the two sheets. (Computing with the local equation at \( P \) shows that \( q \) is separable in characteristic \( 2 \) if \( D \) is irreducible, thus \( \tau \) always exists.) Furthermore, \( \tau(E) = D' \) and \( p^*\mathcal{O}_S(1)(-E) = q^*\mathcal{O}_{\mathbb{P}^2}(1) \). Thus
\[
C' + (m - d)E \in |q^*\mathcal{O}_{\mathbb{P}^2}(d)|, \quad \text{hence} \quad \tau(C' + (m - d)E) \in |q^*\mathcal{O}_{\mathbb{P}^2}(d)|.
\]
Pushing this down to \( S \) we obtain that
\[
\tau(C) + (m - d)D \in |\mathcal{O}_S(d)|, \quad \text{hence} \quad \tau(C) \in |\mathcal{O}_S(d - (m - d))|.
\]
Thus \( \tau(C) \) satisfies all the properties (5.8.1–4) and its degree is lower than the degree of \( C \). We obtain a contradiction by induction on \( d \).

A similar argument works if \( (K_S^2) = 2 \), but the details are a little more complicated. I just outline the arguments, leaving out some simple details.

We already proved that \( P \) is not on any line, and a similar argument shows that \( P \) can not be a singular point of a member of \( | -K_S | \).

Since \( h^0(S, -2K_S) = 7 \), there is a curve \( D \in | -2K_S | \) which has a triple point at \( P \). In fact, \( D \) is unique, it is a rational curve and \( P \) is its only singular point. Thus \( S \) is smooth along \( D \). As before we look at the blow up diagram
\[
B_P S \xrightarrow{q} Q \subset \mathbb{P}^3
\]
\[
p \downarrow
\]
where $Q$ is a quadric cone; the image of $B_P S$ by the linear system $|-2K_{B_P S}|$, $q$ is a degree two morphism and $\tau$ is the involution interchanging the two sheets. (Again one can see that $q$ is separable in characteristic 2 if $| -K_S|$ does not have a member which is singular at $P$.) Let $E \subset B_P S$ be the exceptional curve; $C', D' \subset B_P S$ the birational transforms of $C, D$. $\tau(E) = D'$ and $p^*O_S(2)(-2E) = q^*O_{\mathbb{P}^3}(1)$. As before we obtain that

$$\tau(C) \in |O_S(d - 2(m - d))|.$$ 

We obtain a contradiction by induction on $d$. $\square$

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