On monochromatic representation of sums of squares of primes
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Abstract
When the sequences of squares of primes is coloured with $K$ colours, where $K \geq 1$ is an integer, let $s(K)$ be the smallest integer such that each sufficiently large integer can be written as a sum of no more than $s(K)$ squares of primes, all of the same colour. We show that $s(K) \ll K \exp \left( \frac{(3 \log 2 + o(1)) \log K}{\log \log K} \right)$ for $K \geq 2$. This improves on $s(K) \ll \epsilon K^{2+\epsilon}$, which is the best available upper bound for $s(K)$.

1. Introduction
A subset $\mathcal{D}$ of $\mathbb{N}$ is said to be an asymptotic basis of finite order if there exists a positive integer $m$ such that every sufficiently large integer can be written as a sum of at most $m$ elements of $\mathcal{D}$. The smallest $m$ for which the above property holds is called the order of the asymptotic basis $\mathcal{D}$. There are two classical examples of asymptotic bases of finite order. The first example is provided by Lagrange’s four squares theorem, which tells us that the set of squares is an asymptotic basis of order four (in fact, it is a basis of order four). The second one is the set of primes. A classical 3-primes theorem of Vinogradov asserts that every sufficiently large odd integer can be written as a sum of three primes; it follows from this theorem that the set of primes is an asymptotic basis of order at most four.

Given an asymptotic basis $\mathcal{D}$, one can ask the following “chromatic” question. Let $K \geq 1$ be an integer and let $\{\mathcal{D}_i : 1 \leq i \leq K\}$ be any colouring of $\mathcal{D}$ in $K$ colours. Let $s_{\mathcal{D}}(K)$ denote the smallest integer $n$ (if it exists) such that every sufficiently large integer is expressible as a sum of at most $n$ elements of $\mathcal{D}$, all of the same colour. Then what is the optimal bound for $s_{\mathcal{D}}(K)$ in terms of $K$?

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Sárközy [5] was the first to ask this question in the case when $D$ is either the set of squares or the set of primes. N. Hegyváry and F. Hennecart [2] attacked the problems of Sárközy using elementary methods. They obtained the bounds $s_D(K) \ll (K \log K)^5$ when $D$ is the set of squares and $s_D(K) \ll K^3$ when $D$ is the set of primes. D. S. Ramana and O. Ramaré [3] obtained the bound $s_D(K) \ll K \log \log 2K$ when $D$ is the set of primes, improving on the bound given by N. Hegyváry and F. Hennecart. Until recently, the best known bound on $s_D(K)$ when $D$ is the set of squares was due to P. Akhilesh and D. S. Ramana [1], namely $s_D(K) \ll \varepsilon K^{2+\varepsilon}$. Very recently Gyan Prakash, D. S. Ramana and O. Ramaré [8] obtained the bound $s_D(K) \ll K \exp \left( \frac{(3 \log 2 + o(1)) \log K}{\log \log K} \right)$ when $D$ is the set of squares, which is the best possible up to a constant.

A classical theorem of Hua asserts that the set of squares of primes is an asymptotic basis of finite order. In this paper, we consider a “chromatic” version of Hua’s theorem, raised by F. Hennecart, and following ideas in [8] obtain the following theorem.

**Theorem 1.1.** For any integer $K \geq 2$ we have $s(K) \leq K \exp \left( \frac{(3 \log 2 + o(1)) \log K}{\log \log K} \right)$.

Here $o(1) \ll \frac{\log \log K}{\log \log K}$ for all large enough $K$. This improves on the bound $s(K) \ll \varepsilon K^{2+\varepsilon}$ given by Theorem 1.1, page 181 of Guohua Chen [7].

Our path to Theorem 1.1 passes through the theorem below, which we state with the help of the following notation. For any subset $S$ of the squares of primes in the interval $(N, 4N]$, we shall write

$$E_6(S) = \sum_{p_1^2 + p_2^2 + \ldots + p_{12}^2 = \text{prime squares}} \log p_1 \log p_2 \ldots \log p_{12}. \quad (1)$$

**Theorem 1.2.** Let $A \geq e^{e^2}$ be real number. Then for all sufficiently large integers $N$, depending only on $A$, and any subset $S$ of the squares of primes in the interval $(N, 4N]$ with $|S| \geq \frac{N^2}{A \log N}$ we have

$$E_6(S) \ll \frac{|S|^{11} (\log N)^{11}}{N^{\frac{11}{2}}} \exp \left( \frac{(3 \log 2 + o(1)) \log A}{\log \log A} \right), \quad (2)$$

where $o(1) \ll \frac{\log \log \log A}{\log \log A}$.

We prove Theorem 1.2 in Section 3. Indeed, we reduce the problem of bounding $E_6(S)$ to the
finite problem using the circle method: On the minor arcs estimations we use known exponential sum estimates over prime squares. Dealing with the major arcs we use methods of Sam Chow [9] to get the asymptotics on major arcs and reduce to the finite problem. We deal with the finite problem in Section 2 and resolve the finite problem using a slightly modification of Gyan Prakash, D. S. Ramana and O. Ramaré [8]. Theorem 1.1 is deduced from Theorem 1.2 by means of a classical application of the Cauchy-Schwarz inequality followed by an appeal to the argument from [2], involving the finite addition theorem of Sárközy [6]. We give details of this deduction in Section 4.

Throughout this article we use $e(z)$ to denote $e^{2\pi iz}$, for any complex number $z$ and write $e_p(z)$ for $e^{2\pi iz}$ when $p$ is a prime number. Further, all constants implied by the symbols $\ll$ and $\gg$ are absolute except when dependencies are indicated, either in words or by subscripts to these symbols. The Fourier transform of an integrable function $f$ on $\mathbb{R}$ is defined by $\hat{f}(u) = \int_{\mathbb{R}} f(t)e(-ut)dt$. Finally, the notations $[a,b], (a,b]$ etc. will denote intervals in $\mathbb{Z}$, rather than $\mathbb{R}$, with end points $a, b$ unless otherwise specified.

2. The Finite Problem

The main result of this section is Theorem 2.1, which is a slightly modification of the Theorem 2.1 of [8]. For the sake of completeness of our arguments we will provide proof of Theorem 2.1 in full details. Before state our main theorem in this section, we introduce the terms involving in this theorem. We shall suppose that $A \geq e^{e^2}$ and let $U = \prod_{p \leq w} p$, where $w = A^{25}$. In addition, we let $Z$ be a set of primes in the interval $(\sqrt{N}, 2\sqrt{N}]$ with

$$|Z| \geq \frac{\sqrt{N}}{A \log N} \quad \text{and} \quad |\{z \in Z | z \equiv a \mod U\}| \leq \frac{3\sqrt{N}}{\phi(U) \log N},$$

for all classes $a$ in $\mathbb{Z}/U\mathbb{Z}$. Also, we denote by $c = \{c(i)\}_{i \in I}$ a given finite sequence of integers and let $R_U(Z, c)$ denote the set of triples $(x, y, i)$ in $Z \times Z \times I$ such that $x^2 + y^2 + c(i)$ is an invertible square modulo $U$. Finally, let $\tau(U) = 2^{\pi(w)}$ be the number of divisors of $U$. Now we can state the theorem as follows.

**Theorem 2.1.** We have
\[ |R_U(\mathbb{Z}, c)| \leq \left( \frac{U}{\phi(U)} \right)^2 \frac{|\mathbb{Z}|^2 |I|}{\tau(U)} \exp \left( \frac{(3 \log 2 + o(1)) \log 3A}{\log \log 3A} \right), \quad (4) \]

where \( o \ll \frac{(\log \log \log 3A)}{\log \log 3A} \).

We prove Theorem 2.1 in Subsection 2.4. The proof begins by using the optimisation principle given by Lemma 2.5 to pass to a problem in \((\mathbb{Z}/U\mathbb{Z})^*)\, considered in Theorem 2.4. Two applications Hölder’s inequality and the Chinese Remainder Theorem reduce the proof of Theorem 2.4 to the solution of a problem in \((\mathbb{Z}/p\mathbb{Z})^*)\ for a given prime \( p | U \). This problem is treated by the Corollary 2.3 of the following subsection.

2.1 A Sum in \( \mathbb{Z}/p\mathbb{Z} \)

We write \( G_p \) for the ring \( \mathbb{Z}/p\mathbb{Z} \) when \( p \) is a prime number. Also, \( \lambda_p(x) \) shall denote the Legendre symbol \( (\frac{x}{p}) \), for any \( x \) in \( G_p \).

**Lemma 2.2.** Let \( p \) be a prime number and \( c \) an element of \( G_p \). Then for any an even integer \( t \geq 2 \) we have

\[
\sum_{(y_1, y_2, \ldots, y_{t/2}) \in G_{p}^{t/2}} \prod_{1 \leq i, j \leq t/2} \left( 1 + \lambda_p(x_i^2 + y_j^2 + c) \right) \leq p^{3t/2} \exp \left( \frac{4t^5 2^t}{p} \right). \quad (5)
\]

**Proof.**— See the proof of Proposition 2.2 of [8], for example.

Before stating a corollary of the above lemma, let us introduce some additional notation. Let \( p \) be a fixed prime number and let \( c \) be a given element of \( G_p \). For any \( (x, y) \) in \( G_p^2 \) we set \( \delta_p(x, y) = \lambda_p(x^2 + y^2 + c) \) and \( \epsilon_p(x, y) = 1 + \delta_p(x, y) \). We endow \( G_p \setminus \{0\} \), and likewise \( (G_p \setminus \{0\})^t \) for any integer \( t \geq 1 \), with their uniform probability measures and write \( E_x \) and \( E_{x_1, x_2, \ldots, x_t} \) respectively in place of \( \frac{1}{p-1} \sum_{x \in G_p \setminus \{0\}} \) and \( \frac{1}{(p-1)^t} \sum_{x_1, x_2, \ldots, x_t \in G_p \setminus \{0\}} \). Finally, we define \( \mathcal{E}_p(k, t) \) for any integer \( k \) with \( 1 \leq k \leq t \) by

\[
\mathcal{E}_p(k, t) = E_{y_1, y_2, \ldots, y_t} E_{x_1, x_2, \ldots, x_t} \prod_{1 \leq i \leq t, 1 \leq j \leq k} \epsilon_p(x_i, y_j). \quad (6)
\]

Using this notation we state our corollary as follows.

**Corollary 2.3.** For any even integer \( t \geq 2 \) we have
\[ \mathcal{E}_p(t/2, t) \leq \left( \frac{p}{p - 1} \right)^{2t} \exp \left( \frac{4t^5 2^t}{p} \right). \] (7)

**Proof.**— Since \( t \geq 2 \) is an even integer. By taking \( k = t/2 \) in (6) we get that

\[ \mathcal{E}_p(t/2, t) = \mathbb{E}_{y_1, y_2, \ldots, y_t} \prod_{1 \leq i \leq t, 1 \leq j \leq t/2} \epsilon_p(x_i, y_j). \] (8)

Observing that the summands in the sum are independent of the variables \( y_i \) for \( t/2 < i \leq t \), and allowing the sum over full \( G_p \) we get

\[ \mathcal{E}_p(t/2, t) \leq p^{t/2} \sum_{(y_1, y_2, \ldots, y_{t/2}) \in G_p^{t/2}} \prod_{1 \leq i \leq t} \left( 1 + \lambda_p(x_i^2 + y_j^2 + c) \right). \] (9)

Substituting the upper bound on the sum in the right of (9) by Lemma 2.3, we conclude the inequality (7) holds.

### 2.2 The Problem Modulo U

Let, as above, \( A \geq e^2 \) be real number and \( U = \prod_{p \leq w} p \), where \( w = A^{25} \). Suppose further that \( \mathcal{X} \) and \( \mathcal{Y} \) are subsets of \((\mathbb{Z}/U\mathbb{Z})^*\) of density at least \( \frac{1}{A} \). That is,

\[ |\mathcal{X}| \text{ and } |\mathcal{Y}| \geq \frac{\phi(U)}{A}. \] (10)

For a given element \( c \) of \( \mathbb{Z}/U\mathbb{Z} \), let \( T_c(\mathcal{X}, \mathcal{Y}) \) denote the set of pairs \((x, y) \in \mathcal{X} \times \mathcal{Y}\) such that \( x^2 + y^2 + c \) is an invertible square in \( \mathbb{Z}/U\mathbb{Z} \).

**Theorem 2.4.** For all \( A, U, \mathcal{X}, \mathcal{Y} \) and \( c \) as above, we have

\[ |T_c(\mathcal{X}, \mathcal{Y})| \leq \left( \frac{U}{\phi(U)} \right)^2 |\mathcal{X}| |\mathcal{Y}| \exp \left( \frac{3 \log 2 + O \left( \frac{\log \log \log A}{\log \log A} \right)}{\log \log A} \right). \] (11)

**Proof.**— We shall write \( G \) for the set \((\mathbb{Z}/U\mathbb{Z})^*\) and continue to use \( G_p \) for \( \mathbb{Z}/p\mathbb{Z} \). Also, for any \( x \) in \( \mathbb{Z}/U\mathbb{Z} \) and \( p | U \) we denote the canonical image of \( x \) in \( \mathbb{Z}/p\mathbb{Z} \) by \( x_p \) and, to be consistent with the notation of preceding subsection, write \( \lambda_p(x) \) for the Legendre symbol \((\frac{x}{p})\). Then we have that
\[ |T_c(\mathcal{X}, \mathcal{Y})| \leq \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} \prod_{p | U} \left( \frac{1 + \lambda_p(x^2 + y^2 + c)}{2} \right), \tag{12} \]

since \(0 \leq 1 + \lambda_p(x^2 + y^2 + c) \leq 2\) for any pair \((x, y)\) in \(\mathcal{X} \times \mathcal{Y}\), with equality in the upper bound for every prime \(p | U\) when \(x^2 + y^2 + c\) is an invertible square in \(\mathbb{Z}/U\mathbb{Z}\). On extending the definitions of \(\delta_p\) and \(\epsilon_p\) from Subsection 2.1 by setting \(\delta_p(x, y) = \lambda_p(x^2 + y^2 + c)\) and \(\epsilon_p(x, y) = 1 + \delta_p(x, y)\) for any \((x, y)\) in \((\mathbb{Z}/U\mathbb{Z})^2\) and \(p | U\), we may rewrite (12) as

\[ |T_c(\mathcal{X}, \mathcal{Y})| \leq \frac{1}{\tau(U)} \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} \prod_{p | U} \epsilon_p(x, y). \tag{13} \]

Let \(t \geq 2\) be an even integer. Then an interchange of summations followed by an application of Hölder’s inequality to exponent \(t\) to the right hand side of (13) gives

\[ |T_c(\mathcal{X}, \mathcal{Y})| \leq \frac{|\mathcal{Y}|^{1 - \frac{1}{t}}}{\tau(U)} \left( \sum_{y \in \mathcal{Y}} \left( \sum_{x \in \mathcal{X}} \prod_{p | U} \epsilon_p(x, y) \right)^{t} \right)^{\frac{1}{t}}. \tag{14} \]

To bound the sum over \(y \in \mathcal{Y}\) on the right hand side of the inequality above, we first expand the summand in this sum and extend the summation to all \(y \in G\). By this we see that

\[ \sum_{y \in \mathcal{Y}} \left( \sum_{x \in \mathcal{X}} \prod_{p | U} \epsilon_p(x, y) \right)^{t} \leq \sum_{y \in \mathcal{Y}} \sum_{(x_1, x_2, \ldots, x_t) \in \mathcal{X}^t} \prod_{1 \leq i \leq t} \prod_{p | U} \epsilon_p(x_i, y). \tag{15} \]

Interchanging the summations over \(G\) and \(\mathcal{X}^t\) on the right hand side of the above relation and applying Hölder’s inequality again, this time to exponent \(\frac{1}{2}\), we obtain that the right hand side of (15) does not exceed

\[ |\mathcal{X}|^{t-2} \left( \sum_{(x_1, x_2, \ldots, x_t) \in \mathcal{X}^t} \left( \sum_{y \in \mathcal{Y}} \prod_{1 \leq i \leq t} \prod_{p | U} \epsilon_p(x_i, y) \right)^{\frac{1}{2}} \right)^{\frac{t}{2}}. \tag{16} \]

Finally, on expanding the summand in the sum over \(\mathcal{X}^t\) in (16) and extending the summation to all of \(G^t\) we conclude using (15) and (14) and a rearrangement of terms that

\[ |T_c(\mathcal{X}, \mathcal{Y})| \leq \frac{|\mathcal{X}||\mathcal{Y}|}{\tau(U)} \left( \frac{\phi(U)^3}{|\mathcal{X}^2||\mathcal{Y}|} \right)^{\frac{1}{2}} \mathcal{E} \left( \frac{t}{2}, t \right)^{\frac{t}{2}}, \tag{17} \]

where for any integer \(k\) with \(1 \leq k \leq t\) we have set
\[ E(k, t) = \frac{1}{\phi(U)^{2t}} \sum_{(y_1, y_2, \ldots, y_t) \in G^t} \sum_{x_1, x_2, \ldots, x_t \in G^t} \prod_{1 \leq i \leq t, 1 \leq j \leq k} \epsilon_p(x_i, y_j). \] (18)

The Chinese Remainder Theorem gives \( G = \prod_{p \mid U} (G_p \setminus \{0\}) \). Moreover, for all \( p \mid U \) and \((x, y)\) in \((\mathbb{Z}/U\mathbb{Z})^2\) we have \( \epsilon_p(x, y) = \epsilon_p(x_p, y_p) \). It follows that \( E(k, t) = \prod_{p \mid U} E_p(k, t) \), where \( E_p(k, t) \) is as defined by (6). Using (7) with \( k = \frac{t}{2} \), valid on account of Corollary 2.3, and recalling that \( U = \prod_{p \leq A^{25}} p \) we then obtain

\[
\left( E(k, t) \right)^2 = \left( \prod_{p \mid U} E_p(k, t) \right)^{1/t} \leq \left( \frac{U}{\phi(U)} \right)^{4/t} \exp \left( 8 t^3 2^t \sum_{p \leq A^{25}} \frac{1}{p} \right). \] (19)

From (3.20) on page 70 of [4] we deduce that \( \sum_{p \leq A^t} \frac{1}{p} \leq (\log 50) \log \log A \), since \( A \geq 4 \). On combining this remark with (19), (10) and (17) we then conclude that for any even integer \( t \geq 2 \) we have

\[
|T_c(\mathcal{X}, \mathcal{Y})| \leq \left( \frac{U}{\phi(U)} \right)^2 \frac{|\mathcal{X}| |\mathcal{Y}|}{\tau(U)} \exp \left( \frac{3 \log A}{t} + 8 (\log 50) t^3 2^t \log \log A \right). \] (20)

Let us now set \( v \log 2 = \log \left( \frac{\log A}{(\log \log A)^w} \right) \) and suppose that \( A_0 \geq e^e \) is such that we have \( \frac{\log A}{\log \log A} \geq 12 \) and \( v \geq 4 \) for all \( A > A_0 \). For such \( A \) we take \( t \) in (20) to be an even integer satisfying \( v \leq t \leq v + 2 \). Also, with \( w = \frac{6 \log \log \log A}{\log A} \) we have \( w \leq \frac{1}{2} \) and \( v = \frac{(1-w) \log \log A}{\log 2} \). Thus \( \frac{1}{t} \leq 1 \leq \frac{(\log 2)(1+2v)}{\log \log A} \) and \( t^3 2^t \leq 32 v^3 2^v \leq \frac{32 \log A}{(\log 2)^3 (\log \log A)^3} \). Substituting these inequalities in (20) we obtain (11) for \( A > A_0 \). To obtain (11) for \( e^{e^2} \leq A \leq A_0 \) it suffices to take \( t = 2 \) in (20).

2.3 An Optimisation Principle

This subsection summarizes Subsection 2.3 of [3]. Suppose that \( n \geq 1 \) is an integer and let \( P \) and \( D \) be real numbers \( > 0 \). Further, assume that the subset \( \mathcal{K} \) of points \( x = (x_1, x_2, \ldots, x_n) \) in \( \mathbb{R}^n \) satisfying the conditions

\[
\sum_{1 \leq i \leq n} x_i = P \quad \text{and} \quad 0 \leq x_i \leq D \quad \text{for all } i.
\] (21)

is not empty. Then \( \mathcal{K} \) is a non-empty, compact and convex subset of \( \mathbb{R}^n \) and we have the following fact.
Lemma 2.5. If \( f : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \) a bilinear form with real coefficients \( \alpha_{ij} \) defined by \( f(x, y) = \sum_{1 \leq i, j \leq n} \alpha_{ij} x_i y_j \) then

(i) There are extreme points \( x^* \) and \( y^* \) of \( K \) so that \( f(x, x) \leq f(x^*, y^*) \) for all \( x \in K \).

(ii) If \( x^* = (x^*_1, x^*_2, \ldots, x^*_n) \) is an extreme point of \( K \) then, excepting at most one \( i \), we have either \( x^*_i = 0 \) or \( x^*_i = D \) for each \( i \). Also, if \( m \) is the number of \( i \) such \( x^*_i \neq 0 \) then \( mD \geq P > (m - 1)D \).

**Proof.**— See the proof of Proposition 2.2 of [3], for example.

### 2.4 Proof of Theorem 2.1

Let \( a, b \) be any elements of \( \mathbb{Z}/U\mathbb{Z} \). For any \( i \) in \( I \) we set \( \alpha_i(a, b) = 1 \) if \( a^2 + b^2 + c(i) \) is an invertible square in \( \mathbb{Z}/U\mathbb{Z} \) and 0 otherwise. Further, we write \( m(a) \) for the number of \( z \) in \( \mathbb{Z} \) such that \( z \equiv a \mod U \). Then if \( \tilde{\mathbb{Z}} \) denotes the image of \( \mathbb{Z} \) in \( \mathbb{Z}/U\mathbb{Z} \) we have

\[
|R_U(\mathbb{Z}, c)| = \sum_{i \in I} \sum_{(a, b) \in \tilde{\mathbb{Z}}^2} \alpha_i(a, b) m(a)m(b) . \tag{22}
\]

Moreover, on account of the second assumption in (3) we have that

\[
\sum_{a \in \tilde{\mathbb{Z}}} m(a) = |\mathbb{Z}| \text{ and } 0 \leq m(a) \leq D, \tag{23}
\]

where \( D = \frac{3\sqrt{N}}{\phi(U) \log N} \). For large values of \( N \), depending on \( A \); \( \tilde{\mathbb{Z}} \) is contained in \( (\mathbb{Z}/U\mathbb{Z})^* \). Let us bound the inner sum on the right hand side of (22) for a fixed \( i \) in \( I \). By means of Lemma 2.5 and (23) we obtain

\[
\sum_{(a, b) \in \tilde{\mathbb{Z}}^2} \alpha_i(a, b) m(a)m(b) \leq \sum_{(a, b) \in \tilde{\mathbb{Z}}^2} \alpha_i(a, b) x^*_a y^*_b , \tag{24}
\]

for some \( x^*_a \) and \( y^*_b \), with \( a \) and \( b \) varying over \( \tilde{\mathbb{Z}} \), satisfying the following conditions. All the \( x^*_a \), and similarly all the \( y^*_b \), are either 0 or \( D \) excepting at most one, which must lie in \((0, D)\). Moreover, if \( \mathcal{X} \) and \( \mathcal{Y} \) are, respectively, the subsets of \( \tilde{\mathbb{Z}} \) for which \( x^*_a \neq 0 \) and \( y^*_b \neq 0 \) then \( |\mathcal{X}|D \geq |\mathcal{Z}| > (|\mathcal{X}| - 1)D \). From the first condition in (3) we then get \( |\mathcal{X}| \geq \frac{|\mathcal{Z}|}{D} \geq \frac{\phi(U)}{D} \geq 2 \). Consequently, we also have \( D \leq \frac{|\mathcal{Z}|}{|\mathcal{X}|-1} \leq \frac{2|\mathcal{Z}|}{|\mathcal{X}|} \). The same inequalities hold with \( |\mathcal{X}| \) replaced by \( |\mathcal{Y}| \). Then with \( T_{c(i)}(\mathcal{X}, \mathcal{Y}) \) as in Subsection 2.2 we have that \( \sum_{(a, b) \in \mathcal{X} \times \mathcal{Y}} \alpha_i(a, b) = |T_{c(i)}(\mathcal{X}, \mathcal{Y})| \)
and therefore that the right hand side of (24) does not exceed \(\frac{4|T_{c(i)}(X,Y)||Z|^2}{|X||Y|}\). Using this in (22) together with the bound supplied by (11) for \(|T_{c(i)}(X,Y)|\), applicable since \(3A \geq e^2\), we then conclude that (4) holds.

\[ \square \]

3. An Application of the Circle Method

We prove Theorem 1.2 in this section. As stated in Section 1, we will first reduce the problem of bounding \(E_6(S)\) to the finite problem. This is carried out in Subsections 3.1 through 3.3 starting with the preliminaries given below. We then complete the proof of Theorem 1.2 in Subsection 3.4 by applying Theorem 2.1.

We suppose that \(A \geq e^2\) are real number and assume that \(N\) is a sufficiently large integer depending only on \(A\), its actual size varying to suit our requirements at various stages of the argument. We set \(\alpha(t) = 1 - \frac{2t}{5N}\) when \(|t| \leq \frac{5N}{2}\) and 0 for all other \(t \in \mathbb{R}\) and set \(\beta(t) = \alpha(t - \frac{5N}{2})\). Thus \(\beta(t) \geq 0\) for all \(t\) in \(\mathbb{R}\) and \(\beta(t) \geq \frac{2}{3}\) when \(t \in [N, 4N]\). Finally, we set

\[
\psi(t) = \sum_n \mathbb{1}_P(n) 2n \log n \beta(n^2) e(n^2t)
\]

and write \(\hat{S}(t) = \sum_{p^2 \in S} \log p e(p^2t)\) for any \(t \in \mathbb{R}\) for a given subset \(S\) of the squares in \((N, 4N]\) satisfying the hypotheses of Theorem 1.2. We observe that

\[
\frac{4}{3} \sqrt{N} E_6(S) \leq \int_0^1 \hat{S}(t)^6 \hat{S}(-t)^5 \psi(-t) dt.
\]

Indeed,
We apply the circle method to estimate the integral on the right hand side of (26). From (27) we call the interval \([a, b]\) disjoint, it then follows that the right hand side of (26) is sufficiently large depending only on \(A\). We denote by \(M\) the union of the family of major arcs \(M(a/q)\). Each interval in the complement of \(M\) in \([0, 1)\) is called a minor arc. We denote the union of the minor arcs by \(m\).

We have

\[
E_0(S) = \sum_{p_1^2 + p_2^2 + \ldots + p_k^2 = \log p_1 \log p_2 \ldots \log p_{12}}^{p_1^2 + p_2^2 + \ldots + p_k^2 = \log p_1 \log p_2 \ldots \log p_{11} q^2, \quad p_i \in S; q^2 \text{ is a prime square in } (N, 4N].}
\]

\[
\leq \sum_{p_1^2 + p_2^2 + \ldots + p_k^2 = \log p_1 \log p_2 \ldots \log p_{11} \log q, \quad p_i \in S: q^2 \text{ is a prime square in } (N, 4N].}
\]

\[
\leq \frac{1}{2} \sqrt{N} \sum_{p_1^2 + p_2^2 + \ldots + p_k^2 = \log p_1 \log p_2 \ldots \log p_{11} 2q \log q}
\]

\[
\leq \frac{5}{4} \sqrt{N} \sum_{p_1^2 + p_2^2 + \ldots + p_k^2 = \log p_1 \log p_2 \ldots \log p_{11} 2q \log q \beta(q^2),
\]

\[
\leq \frac{5}{4} \sqrt{N} \int_{0}^{1} \hat{S}(t)^6 \hat{S}(-t)^5 \psi(-t) dt,
\]

in above inequalities: from (29) to (30) we use the lower bound 2/5 on \(\beta(t^2)\) in the interval \((N, 4N]\) and from (30) to (31) we use the orthogonality of the functions \(t \rightarrow e(nt)\) on \([0, 1]\).

We apply the circle method to estimate the integral on the right hand side of (26). To this end, we set \(Q = (\log N)^B A^{48}, \quad M = \frac{N}{(\log N)^{2M}}, \) where \(B\) is a large absolute constant and, for any integers \(a\) and \(q\) satisfying

\[
0 \leq a \leq q \leq Q \quad \text{and} \quad (a, q) = 1,
\]

we call the interval \([a/q - 1/M, a/q + 1/M]\) the major arc \(M(a/q)\). It is easily checked that distinct major arcs are in fact disjoint when \(M > 2Q^2\), which holds when \(N\) is sufficiently large depending only on \(A\). We denote by \(M\) the union of the family of major arcs \(M(a/q)\). Each interval in the complement of \(M\) in \([0, 1)\) is called a minor arc. We denote the union of the minor arcs by \(m\).

We have

\[
\int_{0}^{1} \hat{S}(t)^6 \hat{S}(-t)^5 \psi(-t) dt = \int_{-1/M}^{1/M} \hat{S}(t)^6 \hat{S}(-t)^5 \psi(-t) dt,
\]

by the periodicity of the integrand. From the definitions given above it is easily seen that the interval \([-1/M, 1 - 1/M]\) is the union of \(m\) and \(M \setminus [1 - 1/M, 1 + 1/M]\). Since distinct major arcs are disjoint, it then follows that the right hand side of (33) is the same as
\[
\sum_{1 \leq q \leq Q} \sum_{0 \leq a < q, (a, q) = 1} \int_{\mathfrak{M}(\frac{2\pi}{q})} \hat{S}(t)^6 \hat{S}(-t)^5 \psi(-t) \, dt + \int_{\mathfrak{M}} \hat{S}(t)^6 \hat{S}(-t)^5 \psi(-t) \, dt.
\] (34)

We shall presently estimate each of the two terms in (34). We begin by observing that

\[
\int_{0}^{1} |\hat{S}(t)|^{11} \, dt \ll |S|^{9} (\log N)^{9} A^{3}.
\] (35)

In effect, the integral in (35) does not exceed \( |S| \log N E_{5}(S) \). Thus (35) follows from \( |S| \geq N^{\frac{5}{2}} / A (\log N) \) and

\[
E_{5}(S) = \sum_{p_{i}^{2} + p_{j}^{2} + \cdots + p_{k}^{2} = n, p_{i}^{2} \in S} \log p_{1} \log p_{2} \cdots \log p_{10} \leq (\log N)^{10} \sum_{1 \leq n \leq 20N} R_{5}^{2}(n) \ll N^{\frac{9}{2}} (\log N)^{5} |S|^{5},
\] (37)

where \( R_{5}(n) \) denotes the number of representations of an integer \( n \) as a sum of five elements of \( S \). To verify (37) we note that \( R_{5}(n) = 0 \) when \( n > 20N \) and \( R_{5}(n) \leq r_{5}(n) \), the number of representations of \( n \) as a sum of five squares of prime numbers, and we have that \( r_{5}(n) \ll n^{\frac{3}{2}} / (\log n)^{5} \) [10, Theorem 11], by an application of the circle method. As a consequence of (35) we have

\[
\sum_{1 \leq q \leq Q} \sum_{0 \leq a < q, (a, q) = 1} \int_{\mathfrak{M}(\frac{2\pi}{q})} |\hat{S}(t)|^{11} \, dt \leq \int_{-\frac{1}{M\pi}}^{\frac{1}{M\pi}} |\hat{S}(t)|^{11} \, dt \ll |S|^{9} (\log N)^{9} A^{3}.
\] (38)

3.1 The Minor Arc Contribution

Here we bound the second term in (34). Let us first verify that for all \( t \in \mathfrak{m} \) we have

\[
|\psi(t)| \ll \frac{N}{A^{6}},
\] (39)

when \( N \) is large enough, depending only on \( A \). Indeed, for any real \( t \) Dirichlet’s approximation theorem gives a rational number \( \frac{a}{q} \) satisfying \( |t - \frac{a}{q}| \leq \frac{1}{qM} \) together with \( 1 \leq q \leq M \) and \( (a, q) = 1 \). When \( t \) is in \( \mathfrak{m} \) we see that \( \frac{a}{q} \) is in \([0, 1]\) since \( \mathfrak{m} \subseteq [\frac{1}{M}, 1 - \frac{1}{M}] \). Consequently, we also have \( 0 \leq a \leq q \). Since, however, \( t \) is not in \( \mathfrak{M} \), we must then have \( Q < q \) on account (32). We then conclude using \( q^{2} \leq qM \) that for each \( t \) in \( \mathfrak{m} \) there are integers \( a \) and \( q \neq 0 \) with \( (a, q) = 1 \)
satisfying

$$|t - \frac{a}{q}| \leq \frac{1}{q^2} \quad \text{and} \quad Q < q \leq M. \quad (40)$$

To get a bound on $\psi(t)$ when $t \in \mathfrak{m}$, we appeal to the following lemma.

**Lemma 3.1.** Let $\alpha$ be a real number such that

$$\alpha = \frac{a}{b} + \lambda, \ (a, q) = 1, \ |\lambda| \leq \frac{1}{q^2}, \ Q < q \leq M.$$

and let $T(u) = \sum_{0 \leq n \leq u} \mathbb{1}_P(n) \log n e(n^2 \alpha)$. Then we have

$$\max_{0 \leq u \leq \sqrt{5}N} |T(u)| \ll \frac{\sqrt{N}}{A^6}. \quad (41)$$

**Proof.**— On the assumption on $\alpha$, we have

$$\sum_{x < n \leq 2x} \mathbb{1}_P(n) \log n e(n^2 \alpha) \ll x (\log x)^c \left( \frac{1}{Q} + \frac{1}{x^{1/2}} + \frac{M}{x^2} \right)^{\frac{1}{8}}, \quad (42)$$

for some absolute constant $c > 0$; see [7, Lemma 2.1], for example. From this it follows that

$$T(x) = \sum_{0 \leq n \leq x} \mathbb{1}_P(n) \log n e(n^2 \alpha) \ll x (\log x)^{c+1} \left( \frac{1}{Q} + \frac{1}{x^{1/2}} + \frac{M}{x^2} \right)^{\frac{1}{8}}, \quad (43)$$

by dividing the interval $[0, x]$ into dyadic intervals $\left( \frac{x}{2^{j+1}}, \frac{x}{2^j} \right]$; $j = 0, 1, \ldots, \log x$ and using the fact that the right of (42) is increasing function of $x$.

Again using the fact that the right of (43) is increasing function and recalling the values of $Q, M$; we get that

$$\max_{0 \leq u \leq \sqrt{5}N} |T(u)| \ll \sqrt{N} (\log N)^{c+1} \left( \frac{1}{(\log N)^B A^{48}} + \frac{1}{N^{1/4}} + \frac{1}{(\log N)^{2B}} \right)^{\frac{1}{8}}. \quad (44)$$

For large values of $N$ depends on $A$, large absolute value of $B$ depends on $c$, we then get

$$\max_{0 \leq u \leq \sqrt{5}N} |T(u)| \ll \frac{\sqrt{N}}{A^6}, \quad (45)$$

this proves the lemma.
Now we return to bound $\psi(t)$ on minor the arcs. By the Properties of Riemann-Stieltjes integral we have

$$\psi(t) = \int_0^{\sqrt{5N}} 2u \beta(u^2) dT(u),$$

where $T(u)$ defined as in Lemma 3.1. Thus, on integrating by parts and using the inequality (41), we have

$$|\psi(t)| \ll N_{\max} \leq u \leq \sqrt{5N} |T(u)| \ll N_{\max},$$

on remarking that $2u \beta(u^2)$ is piecewise monotonic in the interval $[0, \sqrt{5N}]$.

From (39) and (35) it now follows that for all $N$ large enough, depending only on $A$, we have

$$\int_m |\hat{S}(t)|^{11} |\psi(t)| \ dt \ll \frac{N}{A^6} \int_0^1 |\hat{S}(t)|^{11} dt \ll \frac{N|S|^9 (\log N)^9}{A^3} \ll \frac{|S|^{11} (\log N)^{11}}{A},$$

since $|S| \geq N^{\frac{1}{2}}/A \log N$. An application of the triangle inequality now allows us to conclude that

$$\int_m \hat{S}(t)^6 \hat{S}(-t)^5 \psi(-t) \ dt \ll \frac{|S|^{11} (\log N)^{11}}{A}.$$ (49)

### 3.2 The Function $\psi$ on a Major Arc

Let us set $W = 2U$, where $U$ is as defined staring of Section 3. For any integers $a, q$ and $r$, with $q > 0$, we set $V_q(a, r) = \sum_{0 \leq m < q, (r+mW,qW) = 1} e\left(\frac{a(r+mW)^2}{q}\right)$.

**Proposition 3.2.** Let $a$ and $q$ be any integers satisfying (32). Then for all $t$ in the major arc $\mathfrak{M}(\frac{a}{q})$ we have

$$\psi(t) = \frac{1}{\phi(qW)} \sum_{0 \leq r < W, (r,W) = 1} V_q(a, r) \beta\left(t - \frac{a}{q}\right) + O\left(\phi(W) N \exp\left(-c\sqrt{\log N}\right)\right).$$

**Proof.**— Let $\theta = t - \frac{a}{q}$ and $f(u) = 2u \log u \beta(u^2) e(u^2 \theta)$ for any real $u$. we have
\[
\psi(t) = \sum_{0 \leq r < W, \ (r,W) = 1} \sum_{n \equiv r \mod W} \mathbb{1}_P(n) \ n \log n \beta(n^2)e(n^2t) + O\left(9^4t\right), \tag{51}
\]

on recalling expression of \(\psi(t)\), noticing the fact that all primes more than \(U\) are co-prime to \(W\), and trivially estimating contribution to the sum over the interval \([0, U]\) using an upper bound \(3^4t\) on \(U\).

We now find an asymptotic formula for the inner sum on right of (51). To this end, we let \(X = \sqrt{5N}\), and for \(n \in [1, X]\) let

\[
S_n = \sum_{m \leq n, \ m \equiv r \mod W} \mathbb{1}_P(n) e\left(\frac{am^2}{q}\right). \tag{52}
\]

Using the fact that \(q \leq Q\), we get

\[
S_n = \sum_{0 \leq m \leq q, \ (r+mW, qW) = 1} e\left(\frac{a(r + mW)^2}{q}\right) \sum_{d, \ (r+mW+dqW = 1, \ (r+mW+dqW) \ is \ a \ prime} 1 + O(Q). \tag{53}
\]

As \(n \leq \sqrt{5N}\) and \(qW \leq (\log N)^{B+1}\) for large values of \(N\), by appealing to the Siegel-Walfisz theorem, we get the asymptotic expression

\[
S_n = \frac{\text{Li}(n)}{\phi(qW)} \sum_{0 \leq m \leq q, \ (r+mW, qW) = 1} e\left(\frac{a(r + mW)^2}{q}\right) + O\left(\sqrt{N} \exp\left(-c\sqrt{\log N}\right)\right), \tag{54}
\]

where \(\text{Li}(n) = \int_2^n \frac{dt}{\log t}\), and \(c\) is a positive absolute constant.

Using the functions \(f(u)\) and \(S_n\), we have

\[
\sum_{n \equiv r \mod W} \mathbb{1}_P(n) \ n \log n \beta(n^2)e(n^2t) = \sum_{n=2}^X (S_n - S_{n-1}) f(n) = S_X f(X + 1) + \sum_{n=2}^X S_n (f(n) - f(n + 1)). \tag{55}
\]

As \(|\theta| \leq \frac{(\log N)^2B}{N}\), the Mean-Value theorem implies that

\[
f(n) - f(n + 1) \ll (\log N)^{2B+1}. \tag{57}
\]
Hence the sum on left of (55) becomes

\[
\frac{V_q(a, r)}{\phi(qW)} \left[ L_i(X) f(X + 1) + \sum_{n=2}^{X} L_i(n) (f(n) - f(n + 1)) \right] + O \left( N \exp \left( -c\sqrt{\log N} \right) \right) .
\]  

(58)

As \( L_i(2) = 0 \), we then rewrite (58) as

\[
\frac{V_q(a, r)}{\phi(qW)} \sum_{n=3}^{X} \frac{f(n)}{\log x} dx + O \left( N \exp \left( -c\sqrt{\log N} \right) \right) .
\]  

(59)

When \( n - 1 < x < n \), the Mean-value theorem reveals that

\[
f(n) = f(x) + O \left( (\log N)^{2B+1} \right) ,
\]  

(60)

and so

\[
\sum_{n=3}^{X} \int_{n-1}^{n} \frac{f(n)}{\log x} dx = \int_{2}^{X} 2x \beta(x^2) e(\theta x^2) dx + O \left( \sqrt{N} (\log N)^{2B+1} \right) .
\]  

(61)

Note that the integral on right of above is nothing but \( \overline{\beta}(\theta) \). Thus, we have an asymptotic formula for the inner sum on right of (51) as follows

\[
\sum_{n \equiv r \mod W} \sum_{n=3}^{X} \frac{f(n)}{\log x} dx = \int_{2}^{X} 2x \beta(x^2) e(\theta x^2) dx + O \left( \sqrt{N} (\log N)^{2B+1} \right) .
\]  

(62)

for large enough \( N \) depends on only on \( A \). Substituting this into (51) we get an asymptotic formula for \( \psi(t) \) as in (50).

We need the following proposition, which provides information about \( V_q(a, r) \).

**Proposition 3.3.** Let \( a \) and \( q \) be integers satisfying (32) and \( r \) any integer co-prime to \( W \). Then we have

(i) \( V_q(a, r) = 0 \) unless \( q | 2W \) or there is a prime \( p > w \) such that \( p | q \).

(ii) \( \left| \frac{1}{\phi(qW)} V_q(a, r) \right| \ll \frac{1}{\phi(W) \omega^{2B+1/2n}} \) when \( q \) does not divide \( 2W \).

We prove this Proposition with the help of following lemma.
Lemma 3.4. Let $P(z) = c_0z^2 + c_1z + c_2$ be a polynomial with integer coefficients and let $d$ be a positive integer with $d = d_1d_2$ and $d_2$ divides $c_0$. Then

$$
\sum_{0 \leq m < d} e\left(\frac{P(m)}{d}\right) = \sum_{0 \leq m_1 < d_1} e\left(\frac{P(m_1)}{d_1}\right) \sum_{0 \leq m_2 < d_2} e\left(\frac{c_1m_2}{d_2}\right). \tag{63}
$$

**Proof.**— See [8, page 26], for example.

We now give a proof of the above Proposition with the aid of this lemma. Since $(r, W) = 1$, the condition $(r + mW, qW) = 1$ in the definition of $V_q(a, r)$ can be replaced by the condition $(r + mW, q) = 1$, thus we have

$$
V_q(a, r) = \sum_{0 \leq m < q, \atop (r + mW, q) = 1} e\left(\frac{a(r + mW)^2}{q}\right). \tag{64}
$$

Let $q = UV$, where $U$ is $w$-smooth and $(V, W) = 1$ and let $a = a_2U + a_1V, (a_1, U) = 1$ and $(a_2, V) = 1$. Then from (64) follows that

$$
e(\frac{-ar^2}{q}) V_q(a, r) = \left(\sum_{0 \leq m_2 < V, \atop (r + m_2W, V) = 1} e\left(\frac{a_2W^2m_2^2 + 2a_2Wr_2}{V}\right)\right) \left(\sum_{0 \leq m_1 < U, \atop (r + m_1W, U) = 1} e\left(\frac{a_1W^2m_1^2 + 2a_1Wr_1}{U}\right)\right). \tag{65}
$$

Now we analyze the second term in the product of right of the above equation. Since $U$ is $w$-smooth and $(r, W) = 1$, the condition $(r + m_1W, U) = 1$ is always holds, thus we get

$$
\sum_{0 \leq m < U, \atop (r + mW, U) = 1} e\left(\frac{a_1W^2m^2 + 2a_1Wr}{U}\right) = \sum_{0 \leq m < U} e\left(\frac{a_1W^2m^2 + 2a_1Wr}{U}\right). \tag{66}
$$

We write $U = U_1U_2$, where $U_1 = \frac{U}{(U, W^2)}, U_2 = (U, W^2)$. Note that $U_2 | a_1W^2$, thus applying the Lemma 3.4, we get that

$$
\sum_{0 \leq m < U} e\left(\frac{a_1W^2m^2 + 2a_1Wr}{U}\right) = \left(\sum_{0 \leq m < U_1} e\left(\frac{a_1W^2m^2_1 + 2a_1Wr_1}{U_1}\right)\right) \left(\sum_{0 \leq m_2 < U_2} e\left(\frac{2a_1Wr_2}{U_2}\right)\right). \tag{67}
$$
We can conclude from (67), (66) and (65) that $V_q(a, r) = 0$ unless $U_2|2a_1 W r$. That is, unless $(U, W^2)|2a_1 W r$ we have $V_q(a, r) = 0$.

Since $(a_1, U) = 1$ and $(r, W) = 1$, it follows that $V_q(a, r) = 0$ unless $(U, W^2)|2W$. We note that $(U, W^2) = (q, W^2)$ and that $(q, W^2)|2W$ is equivalent to $\inf(v_p(q), 2v_p(W)) \leq v_p(2W)$ for all primes $p|2W$. From the definition of $W$ we have $2v_p(W) > v_p(2W)$ for all primes $p|2W$. Consequently, $V_q(a, r) = 0$ unless $v_p(q) \leq v_p(2W)$ for all primes $p|2W$, which is the same as (i).

To prove (ii), we may assume that $(q, W^2)|2W$ and $(q, W^2)|2W$ and $V > 1$. We can conclude from $(q, W^2) = (U, W^2)$ and $(q, W^2)|2W$ that $U|2W$, in particular $U|W^2$. Thus, we have $(U, W^2) = U$. It follows that

$$\sum_{0 \leq m_1 < U, (r + m_1 W, U) = 1} e\left(\frac{a_1 W^2 m_1^2 + 2a_1 W r m_1}{U}\right) = U,$$  \hspace{1cm} (68)

again using the same fact that $(r + m_1 W, U) = 1$ is always holds, as $U$ is a $w$-smooth number.

On combining (65) and (68) we get that

$$e\left(\frac{-ar^2}{q}\right) V_q(a, r) = U \sum_{0 \leq m_2 < V, (r + m_2 W, V) = 1} e\left(\frac{a_2 W^2 m_2^2 + 2a_2 W r m_2}{V}\right).$$  \hspace{1cm} (69)

Multiplying by a function $e\left(\frac{-ax^2}{V}\right)$ both side of above equation and change the variable $r + m_2 W \mapsto x$ in the summation on right of (69) and using the fact that $(V, W) = 1$, we get

$$e\left((-r^2(a_2/V + a/q))\right) V_q(a, r) = U \sum_{0 \leq x < V, (x, V) = 1} e\left(\frac{a_2 x^2}{V}\right).$$  \hspace{1cm} (70)

We have a following bound on the summation on right of (70)

$$\sum_{0 \leq x < V, (x, V) = 1} e\left(\frac{a_2 x^2}{V}\right) \ll \epsilon V^{\frac{1}{2} + \epsilon},$$  \hspace{1cm} (71)

see [10, Lemma 8.5], for example.

From (71) and (70), it follows that
\[
\frac{1}{\phi(qW)} |V_q(a, r)| \ll_{\epsilon} \frac{U V^{\frac{1}{2} + \epsilon}}{\phi(qW)} = \frac{U V^{\frac{1}{2} + \epsilon}}{\phi(V) \phi(UW)} 
\]

(72)

here we use the fact that \((V, W) = 1\) in the equality on the right of the above equation. Since \(U|2W\), we have \(\phi(UW) = U \phi(W)\) and we have the lower bound on \(\phi(V)\), namely \(\phi(V) \gg V/\log \log V\). From this and (72) we get

\[
\frac{1}{\phi(qW)} |V_q(a, r)| \ll_{\epsilon} \frac{\log \log V}{\phi(W) V^{\frac{1}{2} - \epsilon}}. 
\]

(73)

Taking \(\epsilon = 1/200\) and using the bound \(\log \log V \leq V^{1/100}\) in (73) we conclude (ii).

3.3 The Major Arc Contribution

In this subsection we reduce the problem of bounding \(E_6(S)\) to a finite problem. Let us first dispose of the first term in (34), which we denote here by \(T\). Then on writing \(T_1\) for

\[
\sum_{0 \leq r < W, \ (r, W) = 1} \sum_{1 \leq q \mid 2W} \frac{1}{\phi(qW)} \sum_{0 \leq a < q, \ (a, q) = 1} V_q(-a, r) \int_{\mathfrak{M}(\frac{a}{q})} \hat{\beta} \left( t - \frac{a}{q} \right) \hat{S}(t) \hat{S}(-t)^5 dt 
\]

(74)

we deduce by substituting the complex conjugate of right hand side of (50) for \(\psi(-t) = \overline{\psi(t)}\) in \(T\) and using the triangle inequality together with (38) that

\[
T - T_1 \ll \phi(W) N \exp (-c \sqrt{\log N}) \int_0^1 |\hat{S}(t)|^{11} dt \ll \phi(W) N \exp (-c \sqrt{\log N}) |S|^9 (\log N)^9 A^3. 
\]

(75)

If we now set

\[
T(W) = \sum_{0 \leq r < W, \ (r, W) = 1} \sum_{1 \leq q \mid 2W} \frac{1}{\phi(qW)} \sum_{0 \leq a < q, \ (a, q) = 1} V_q(-a, r) \int_{\mathfrak{M}(\frac{a}{q})} \hat{\beta} \left( t - \frac{a}{q} \right) \hat{S}(t) \hat{S}(-t)^5 dt. 
\]

(76)

then by (ii) of Lemma 3.3 combined with the triangle inequality and (38) we get

\[
T_1 - T(W) \ll \frac{\| \hat{\beta} \|_{\infty} |S|^9 (\log N)^9 A^3}{w^{97/200}} \ll \frac{A^3 |S|^9 (\log N)^9 N}{w^{97/200}}, 
\]

(77)

since \(\| \hat{\beta} \|_{\infty} = \sup_{t \in \mathbb{R}} |\hat{\beta}(t)| \leq \frac{5N}{2}\). From (77), (75) and on recalling that \(|S| \geq \frac{\sqrt{N}}{A \log N}\) and \(w = A^{25}\) we conclude that
where \( f \) of the change of variable 

Finally, on interchanging summations and remarking that we conclude that the right hand side of (78) is the same as the left hand side of

\[
T = T(W) + O \left( \frac{|S^{11}|}{A} \right),
\]

when \( N \) is sufficiently large, depending only on \( A \). Let us now estimate \( T(W) \). When \( q | 2W \) we have \( \phi(qW) = q\phi(W) \) and \( (r + mW)^2 \equiv r^2 \) modulo \( q \) for all integers \( m \) and the condition \( (r + mW) = 1 \) holds always. Therefore we have \( V_q(a, r) = qe \left( -\frac{ar^2}{q} \right) \) when \( q | 2W \), for all \( 0 \leq a < q \). Furthermore, since \( r \mapsto r + W \) is a bijection from the integers co-prime to \( 2W \) in \([0, W)\) to those in \((W, 2W]\) co-prime to \( 2W \), we obtain

\[
\frac{1}{\phi(qW)} \sum_{0 \leq r < W, \ (r, W) = 1} V_q(-a, r) = \frac{1}{2\phi(W)} \sum_{0 \leq r < 2W, \ (r, 2W) = 1} e \left( -\frac{ar^2}{q} \right)
\]

for any \( q | 2W \) and all \( 0 \leq a < q \). Also, we have \( \hat{S}(t)^6 \hat{S}(-t)^5 = \sum_{x \in S^{11}} \log x_1 \log x_2 \ldots \log x_{11} e(f(x)t) \)

where \( f(x) \) denotes \( x_1 + x_2 + \ldots + x_6 - x_7 - \ldots - x_{11} \) for any \( x = (x_1, \ldots, x_{11}) \in S^{11} \). By means of the change of variable \( t - \frac{a}{q} \) \( \mapsto t \) in the integrals in (76) we then see that

\[
T(W) = \frac{1}{2\phi(W)} \sum_{0 \leq r < 2W, \ q | 2W, \ (r, 2W) = 1} \sum_{0 \leq a < q, \ (a, q) = 1} \sum_{r<M} \hat{\beta}(t) \sum_{x \in S^{11}} \prod_{i=1}^{11} \log x_i e(f(x)) e \left( \frac{a(f(x) - r^2)}{q} \right) dt.
\]

Finally, on interchanging summations and remarking that

\[
\frac{1}{2W} \sum_{q | 2W} \sum_{0 \leq a < q, \ (a, q) = 1} e \left( \frac{a(f(x) - r^2)}{q} \right) = \frac{1}{2W} \sum_{0 \leq a < 2W} e \left( \frac{a(f(x) - r^2)}{2W} \right)
\]

we conclude that the right hand side of (80) is the same as the left hand side of

\[
\frac{W}{\phi(W)} \sum_{0 \leq r < 2W, \ (r, 2W) = 1} \sum_{x \in S^{11}} \log x_1 \log x_2 \ldots \log x_{11} \int_{-\frac{1}{W}}^{\frac{1}{W}} \hat{\beta}(t) e(f(x)) dt 
\]

\[
\leq \frac{W (log N)^{11}}{\phi(W)} \sum_{0 \leq r < 2W, \ (r, 2W) = 1} \sum_{x \in S^{11}} 1,
\]

where we have used \( |\int_{-\frac{1}{W}}^{\frac{1}{W}} \hat{\beta}(t) e(f(x)) dt| \leq \int_{R} \hat{\alpha}(t) dt = 1 \), since \( |\hat{\beta}(t)| = \hat{\alpha}(t) \) for all \( t \in R \).

For each invertible square \( b \) in \( \mathbb{Z}/2W\mathbb{Z} \), the number of \( r \) in \([0, 2W)\) co-prime to \( 2W \) and such
that \( r^2 \equiv b \) modulo 2W is \( 2\tau(U) \). Then it follows from (82) and (80) that

\[
T(W) \leq \frac{2W\tau(U)(\log N)^{11}}{\phi(W)} \left| \{ x \in S^{11} \mid f(x) \text{ an invertible square mod 2W} \} \right|.
\] (83)

On combining (83) with (78), (49) and recalling that (34) is the same as the integral in (26), we get

\[
E_6(S) \ll \frac{2W\tau(U)(\log N)^{11}}{\phi(W)} \left| \{ x \in S^{11} \mid f(x) \text{ an invertible square mod 2W} \} \right| + O \left( \frac{|S|^{11}(\log N)^{11}}{A} \right).
\] (84)

### 3.4 Proof of Theorem 1.2 Completed

It remains only to bound the cardinality of the set \( \{ x \in S^{11} \mid f(x) \text{ an invertible square mod 2W} \} \).

We find an upper bound the cardinality of this set using Theorem 2.1. Let \( Z \) be the set of integers \( n > 0 \) such that \( n^2 \in S \). The set \( Z \) is contained in \([\sqrt{N}, 2\sqrt{N}]\) and satisfies \( |Z| \geq \frac{\sqrt{N}}{A\log N} \) and \( |\{ z \in Z \mid z \equiv a \mod U \}| \leq \frac{3\sqrt{N}}{\phi(U)\log N} \), when \( N \) is sufficiently large depending on \( A \). Finally, let \( I = S^9 \) and for any \( x = (x_1, x_2, \ldots, x_9) \in S^9 \) we set \( c(x) = x_1 + \ldots + x_4 - x_5 - \ldots - x_9 \).

Then with \( R_U(Z, c) \) as in Theorem 2.1 we have that

\[
\left| \{ x \in S^{11} \mid f(x) \text{ an invertible square modulo 2W} \} \right| \leq |R_U(Z, c)|,
\] (85)

since \( U \mid 2W \). On combining the bound for \( |R_U(Z, c)| \) given by Theorem 2.1 with (85) and (84) and after noticing that \( \frac{W}{\phi(W)} \left( \frac{U}{\phi(U)} \right)^2 \ll (\log A)^3 \) we finally obtain (2), as required.

### 4. Monochromatic Representation

Here we deduce Theorem 1.1 from Theorem 1.2. Before this deduction we give a lemma with the following notation. For any subset \( S \) of the integers, we write \( e_6(S) \) for the number of tuples \((x_1, x_2, \ldots, x_{12})\) in \( S^{12} \) satisfying \( x_1 + \ldots + x_6 = x_7 + \ldots + x_{12} \). We observe that if \( S \subset [N, 4N] \) satisfying the hypothesis of Theorem 1.2, then we can conclude from (2) that

\[
e_6(S) \ll \frac{|S|^{11}}{N^{\frac{1}{2}} \log N} \exp \left( \frac{(3 \log 2 + o(1)) \log A}{\log \log A} \right).
\] (86)
Now we state the lemma as follows.

**Lemma 4.1.** Let $N$ be positive integer and let $D \geq 1$ be a real number satisfying the condition $N \geq 72D + 12$. If $S$ be a subset of the interval $(N, 4N]$ such that

$$e_{6}(S) \leq \frac{|S|^{12}D}{3N} \quad (87)$$

and if $S$ contains an integer that is not divisible by any prime $p \leq [6D]$ then every integer $n \geq 30N(2[6D] + 1)$ is a sum of no more than $\frac{n}{N}$ elements of $S$.

**Proof.**— See [8, Lemma 1.2], for example.

We now give the proof of Theorem 1.1. Since $s(K)$ is increasing with $K$, it suffices to prove Theorem 1.1 for all $K$ sufficiently large. For such a $K$, let $\cup_{1 \leq i \leq K} \Omega_{i}$ be a partition of the set of square of primes $\Omega$ into $K$ disjoint subsets.

We set $N_{0} = 2K^{50}$. Let $N$ be an integer $\geq N_{0}$. There is an $i, 1 \leq i \leq K$, such that $\Omega_{i} \cap (N, 4N]$ contains atleast $\frac{\sqrt{N}}{K \log N}$ elements of $\Omega$. For such $i$ we set $S = \Omega_{i} \cap (N, 4N]$. Then $S$ is set of square of primes in $(N, 4N]$ with $|S| \geq \frac{\sqrt{N}}{K \log N}$ and no integer in $S$ is divisible by a prime $p \leq K^{25}$.

It now follows from (86) that (87) holds with $D \ll K \exp \left(\frac{(3 \log 2 + o(1)) \log K}{\log \log K}\right)$. Since elements of $S$ does not divisible by any prime $p \leq [6D]$ when $N$ is large enough, we may apply Lemma 4.1 to $S$ to deduce that every integer $n \geq (288D + 72)N$ is a sum of no more than $\frac{n}{N}$ elements of $S$.

In particular, there is a $C_{1} > 0$ such that every integer in $I(N) = [(288D + 72)N, (288D + 73)N]$ is a sum of at most $C_{1}D$ squares of primes all belonging to $S$ and therefore to $\Omega_{i}$. Thus for all large enough $N$, every integer in the interval $I(N)$ can be expressed as a sum of no more than $C_{1}D$ squares of primes all of the same colour. On remarking that the interval $I(N)$ meets $I(N + 1)$ for all large enough $N$, we obtain that $s(K) \leq C_{1}D$. This yields the conclusion of Theorem 1.1 since $C_{1}D \ll K \exp \left(\frac{(3 \log 2 + o(1)) \log K}{\log \log K}\right)$.

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