SEIBERG-WITTEN INVARIANTS AND SURFACE SINGULARITIES III
(SPLICINGS AND CYCLIC COVERS)

ANDRÁS NÉMETHI AND LIVIU I. NICOLAESCU

ABSTRACT. We verify the conjecture formulated in [31] for suspension singularities of type \( g(x, y, z) = f(x, y) + z^n \), where \( f \) is an irreducible plane curve singularity. More precisely, we prove that the modified Seiberg-Witten invariant of the link \( M \) of \( g \), associated with the canonical spin\(^c\) structure, equals \(-\sigma(F)/8\), where \( \sigma(F) \) is the signature of the Milnor fiber of \( g \). In order to do this, we prove general splicing formulae for the Casson-Walker invariant and for the sign refined Reidemeister-Turaev torsion (in particular, for the modified Seiberg-Witten invariant too). These provide results for some cyclic covers as well. As a by-product, we compute all the relevant invariants of \( M \) in terms of the Newton pairs of \( f \) and the integer \( n \).

1. INTRODUCTION

The present article is a natural continuation of [31] and [32], and it is closely related to [28, 29, 30]. In [31], the authors formulated a very general conjecture which connects the topological and the analytical invariants of a complex normal surface singularity whose link is a rational homology sphere.

Even if we restrict ourselves to the case of hypersurface singularities, the conjecture is still highly non-trivial. The “simplified” version for this case reads as follows.

1.1. Conjecture. [31] Let \( g : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0) \) be a complex analytic germ which defines an isolated hypersurface singularity. Assume that its link \( M \) is a rational homology sphere. Denote by \( sw^0_M(\sigma_{\text{can}}) \) the modified Seiberg-Witten invariant of \( M \) associated with the canonical spin\(^c\) structure \( \sigma_{\text{can}} \) (cf. 2.7). Moreover, let \( \sigma(F) \) be the signature of the Milnor fiber \( F \) of \( g \). Then

\[
sw^0_M(\sigma_{\text{can}}) = \frac{\sigma(F)}{8}.
\] (1)

The goal of the present paper is to verify this conjecture for suspension hypersurface singularities. More precisely, in 6.15 we prove the following.

1.2. Theorem. Let \( f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0) \) be an irreducible plane curve singularity. Fix an arbitrary positive integer \( n \) such that the link \( M \) of the suspension singularity \( g(x, y, z) := f(x, y) + z^n \) is a rational homology sphere (cf. 6.4(3)). Then (1) holds.

The numerical identity (1) covers a very deep qualitative analytic-rigidity phenomenon.

From topological point of view, any normal two-dimensional analytic singularity \((X, 0)\) is completely characterized by its link \( M \), which is an oriented 3-manifold. Moreover, by
a result of Neumann [34], any decorated plumbing (or resolution) graph of \((X, 0)\) carries the same information as \(M\). A property of \((X, 0)\) will be called topological if it can be determined from any of these graphs.

A very intriguing issue, which has generated intense research efforts, is the possibility of expressing the analytic invariants of \((X, 0)\) (like the geometric genus \(p_g\), multiplicity, etc.) or the smoothing invariants (if they exist, like the signature \(\sigma(F)\) or the topological Euler-characteristic \(\chi(F)\) of the Milnor fiber \(F\)) in terms of the topology of \(M\).

1.3. A short historical survey. M. Artin proved in [3, 4] that the rational singularities (i.e. the vanishing of \(p_g\)) can be characterized completely from the resolution (or plumbing) graph. In [18], H. Laufer extended Artin’s results to minimally elliptic singularities, showing that Gorenstein singularities with \(p_g = 1\) can be characterized topologically. Additionally, he noticed that the program breaks for more complicated singularities (see also the comments in [31] and [27]). On the other hand, the first author noticed in [27] that Laufer’s counterexamples do not signal the end of the program. He conjectured that if we restrict ourselves to the case of those Gorenstein singularities whose links are rational homology spheres then \(p_g\) is topological. This was carried out explicitly for elliptic singularities in [27] (partially based on some results of S. S.-T. Yau, cf. e.g. with [44]).

For Gorenstein singularities which have smoothing (with Milnor fiber \(F\)), the topological invariance of \(p_g\) can be reformulated in terms of \(\sigma(F)\) and/or \(\chi(F)\). Indeed, via some results of Laufer, Durfee, Wahl and Steenbrink, any of \(p_g, \sigma(F)\) and \(\chi(F)\) determines the remaining two modulo \(K^2 + \#V\) (for the precise identities see e.g. [24] or [31]). Here, \(K^2 + \#V\) is defined as follows. For a given resolution, one takes the canonical divisor \(K\), and the number \(\#V\) of irreducible components of the exceptional divisor of the resolution. Then \(K^2 + \#V\) can be deduced from the resolution graph, and is independent of the choice of the graph. In particular, it is an invariant of the link \(M\).

For example, the identity which connects \(p_g\) and \(\sigma(F)\) is

\[
8p_g + \sigma(F) + K^2 + \#V = 0. \tag{2}
\]

This connects the above facts about \(p_g\) with the following list of results about \(\sigma(F)\).

Fintushel and Stern proved in [11] that for a hypersurface Brieskorn singularity whose link is an integral homology sphere, the Casson invariant \(\lambda(M)\) of the link \(M\) equals \(\sigma(F)/8\). This fact was generalized by Neumann and Wahl in [36]. They proved the same statement for all Brieskorn-Hamm complete intersections and suspensions of irreducible plane curve singularities (with the same assumption about the link). Moreover, they conjectured the validity of the formula for any isolated complete intersection singularity whose link is an integral homology sphere.

In [31] the authors extended the above conjecture for smoothing of Gorenstein singularities with rational homology sphere link. Here the Casson invariant \(\lambda(M)\) is replaced by a certain Seiberg-Witten invariant \(\text{sw}_M^0(\sigma_{\text{can}})\) of the link associated with the canonical spin\(^c\) structure of \(M\). \(\text{sw}_M^0(\sigma_{\text{can}})\) is the difference of a certain Reidemeister-Turaev sign-refined torsion invariant and the Casson-Walker invariant, for details see [27]. If \(M\) is an integral homology sphere, then the torsion invariant vanishes.

In fact, the conjecture in [31] is more general. A part of it says that for any \(\mathbb{Q}\)-Gorenstein singularity whose link is a rational homology sphere, one has

\[
8p_g - 8\text{sw}_M^0(\sigma_{\text{can}}) + K^2 + \#V = 0. \tag{3}
\]
Notice that (in the presence of a smoothing and of the Gorenstein property, e.g. for any hypersurface singularity) (3) via (2) is exactly (1). The identity (3) was verified in [31] for cyclic quotient singularities, Brieskorn-Hamm complete intersections and some rational and minimally elliptic singularities. [32] contains the case when \((X, 0)\) has a good \(\mathbb{C}^*\)-action.

Finally we mention, that recently Neumann and Wahl initiated in [37] an all-embracing program about those \(\mathbb{Q}\)-Gorenstein singularities whose link is rational homology sphere. They conjecture that the universal abelian cover of such a singularity is an isolated complete intersection, and its Milnor fiber can be recovered from \(M\) together with the action of \(H_1(M)\) (modulo some equisingular and equivariant deformation). In particular, this implies the topological invariance of \(p_g\) as well. We hope that our efforts in the direction of the conjecture formulated in [31] will contribute to the accomplishment of this program as well.

On the other hand, the theory of suspension hypersurface singularities also has its own long history. This class (together with the weighted-homogeneous singularities) serve as an important “testing and exemplifying” family for various properties and conjectures. For more information, the reader is invited to check [29, 30] and the survey paper [28], and the references listed in these articles.

1.4. A few words about the proof. If \(M\) is the link of \(g = f + z^n\) (as in 1.2), then \(M\) has a natural splice decomposition into Seifert varieties of type \(\Sigma(p,a,m)\). Moreover, in [28] (3.2) the first author established an additivity formula for \(\sigma(F)\) compatible with the geometry of this decomposition. On the other hand, for any Brieskorn singularity \((x, y, z) \mapsto x^p + y^a + z^m\) (whose link is \(\Sigma(p,a,m)\)) the conjectured identity (1) is valid by [31, 32]. Hence it was natural to carry out the proof of 1.2 by proving an additivity result for \(\text{sw}_M^0(\sigma_{\text{can}})\) with respect to the splice decomposition of \(M\) into Seifert varieties.

This additivity result is proved in 6.14 (as an outcome of all the preparatory results of the previous sections) but its proof contains some surprising steps.

Our original plan was the following. First, we identify the splicing data of \(M\). Then, for such splicing data, we establish splicing formulas for the Casson-Walker invariant and for the Reidemeister-Turaev sign-refined torsion with the hope that we can do this in the world of topology without going back for some extra restrictions to the world of singularities. This program for the first invariant was straightforward, thanks to the results of Fujita [12] and Lescop [21] (see section 3). But when dealing with the torsion we encountered some serious difficulties (and finally we had to return back to singularities for some additional properties).

The torsion-computations require the explicit description of the supports of all the relevant characters, and then the computation of some sophisticated Fourier-Dedekind sums. The computation turned out to be feasible because these sums are not arbitrary. They have two very subtle special features which follow from various properties of irreducible plane curve singularities. The first one is a numerical inequality (5.1(6)) (measuring some strong algebraic rigidity). The second (new) property is the alternating property of their Alexander polynomial (5.2).

In section 4 we establish different splicing formulas for \(\text{sw}_M^0(\sigma_{\text{can}})\), and we show the limits of a possible additivity. We introduce even a new invariant \(D\) which measures the non-additivity property of \(\text{sw}_M^0(\sigma_{\text{can}})\) with respect to (some) splicing (see e.g. 4.9) or (some) cyclic covers (see 4.11). This invariant vanishes in the presence of the alternating property of the involved Alexander polynomial.
This shows clearly (and rather surprisingly) that the behavior of \( \text{sw}_M^0(\sigma_{\text{can}}) \) with respect to splicing and cyclic covers (constructions topological in nature) definitely prefers some special algebraic situations. For more comments, see 4.12, 5.4 and 6.8.

Section 5 contains the needed results for irreducible plane curve singularities and the Algebraic Lemma used in the summation of the Fourier-Dedekind sums mentioned above.

In section 6 we provide a list of properties of the link of \( f + z^n \). Here basically we use almost all the partial results proved in the previous sections. Most of the formulae are formulated as inductive identities with respect to the number of Newton pairs of \( f \).

1.5. Notations. All the homology groups with unspecified coefficients are defined over the integers. In section 3, \( s(\cdot, \cdot) \) denotes Dedekind sums, defined by the same convention as in \([12, 21]\) or \([31]\). For definitions and detailed discussions about the involved invariants, see \([31]\). For different properties of hypersurface singularities, the reader may consult \([2]\) as well.

2. Preliminaries and notations

2.1. Oriented knots in rational homology spheres. Let \( M \) be an oriented 3-manifold which is a rational homology sphere. Fix an oriented knot \( K \subset M \), denote by \( T(K) \) a small tubular neighborhood of \( K \) in \( M \), and let \( \partial T(K) \) be its oriented boundary with its natural orientation. The natural oriented meridian of \( K \), situated in \( \partial T(K) \), is denoted be \( m \). We fix an oriented parallel \( \ell \) in \( \partial T(K) \) (i.e. \( \ell \sim K \) in \( H_1(T(K)) \)). If \( (\cdot, \cdot) \) denotes the intersection form in \( H_1(\partial T(K)) \), then \( (m, \ell) = 1 \) (cf. e.g. Lescop’s book \([21]\), page 104; we will use the same notations \( m \) and \( \ell \) for some geometric realizations of the meridian and parallel as primitive simple curves, respectively for their homology classes in \( H_1(\partial T(K)) \)).

Obviously, the choice of \( \ell \) is not unique. In all our applications, \( \ell \) will be characterized by some precise additional geometric construction.

Assume that the order of the homology class of \( K \) in \( H_1(M) \) is \( o > 0 \). Consider an oriented surface \( F_{oK} \) with boundary \( oK \), and take the intersection \( \lambda := F_{oK} \cap \partial T(K) \). \( \lambda \) is called the longitude of \( K \). The homology class of \( \lambda \) in \( H_1(\partial T(K)) \) can be represented as \( \lambda = o\ell + km \) for some integer \( k \). Set \( \gcd(o, |k|) = \delta > 0 \). Then \( \lambda \) can be represented in \( \partial T(K) \) as \( \delta \) primitive torus curves of type \( (o/\delta, k/\delta) \) with respect to \( \ell \) and \( m \).

2.2. Dehn fillings. Let \( T(K)^o \) be the interior of \( T(K) \). For any homology class \( a \in H_1(\partial T(K)) \), which can be represented by a primitive simple closed curve in \( \partial T(K) \), one defines the Dehn filling of \( M \setminus T(K)^o \) along \( a \) by

\[
(M \setminus T(K)^o)(a) = M \setminus T(K)^o \bigsqcup_{f} S^1 \times D^2,
\]

where \( f : \partial(S^1 \times D^2) \to \partial T(K) \) is a diffeomorphism which sends \( \{\ast\} \times \partial D^2 \) to a curve representing \( a \).

2.3. Linking numbers. Consider two oriented knots \( K, L \subset M \) with \( K \cap L = \emptyset \). Fix a Seifert surface \( F_{oK} \) of \( oK \) (cf. \([21]\)) and define the linking number \( Lk_M(K, L) \in \mathbb{Q} \) by the “rational” intersection \( (F_{oK} \cdot L)/o \). In fact, \( Lk_M(K, \cdot) : H_1(M \setminus K, \mathbb{Q}) \to \mathbb{Q} \) is a well-defined homeomorphism and \( Lk_M(K, L) = Lk_M(L, K) \). For any oriented knot \( L \) on \( \partial T(K) \) one has (see e.g. \([21]\) 6.2.B)):

\[
Lk_M(L, K) = \langle L, \lambda \rangle / o. \tag{1}
\]
Seiberg-Witten invariants and surface singularities III

For any oriented knot $K \subset M$ one has the obvious exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\alpha} H_1(M \setminus T(K)) \xrightarrow{j} H_1(M) \to 0,$$

where $\alpha(1_\mathbb{Z}) = m$. If $K \subset M$ is homologically trivial then this sequence splits. Indeed, let $\phi$ be the restriction of $Lk_M(K, \cdot)$ to $H_1(M \setminus K) = H_1(M \setminus T(K))$. Then $\phi$ has integer values and $\phi \circ \alpha = 1_\mathbb{Z}$. This provides automatically a morphism $s : H_1(M) \to H_1(M \setminus T(K))$ such that $j \circ s = 1$ and $\alpha \circ \phi + s \circ j = 1$; in particular with $\phi \circ s = 0$ too. In fact, $s(H_1(M)) = \text{Tor}_1 H_1(M \setminus T(K))$. Moreover, under the same assumption $o = 1$, one has the isomorphisms

$$H_2(M, K) \xrightarrow{\partial} H_1(K) = \mathbb{Z} \text{ and } H_1(M) \to H_1(M, K).$$

Sometimes, in order to simplify the notations, we write $H$ for the group $H_1(M)$.

The finite group $H$ carries a natural symmetric bilinear form $b_M : H \otimes \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ defined by $b_M([K], [L]) = Lk_M(K, L) \pmod{\mathbb{Z}}$, where $L$ and $K$ are two representatives with $K \cap L = \emptyset$. If $\tilde{H}$ denotes the Pontryagin dual $\text{Hom}(H, S^1)$ of $H$, then $[K] \mapsto b_M([K], \cdot)$ is an isomorphism $H \to \tilde{H}$.

2.4. $(M, K)$ represented by plumbing. The main application of the present article involves algebraic links $(M, K)$ which can be represented by plumbing. We recall the notations briefly (for more details, see e.g. [11] or [23]).

We will denote by $\Gamma(M, K)$ the plumbing graph of a link $K \subset M$. The vertices $v \in \mathcal{V}$ are decorated by the Euler numbers $e_v$ of the $S^1$-bundles over $E_v \approx S^1$ used in the plumbing construction. The components of the link $K$ are represented by arrows in $\Gamma(M, K)$: if an arrow is attached to the vertex $v$ then the corresponding component of $K$ is a fixed fiber of the $S^1$-bundle over $E_v$ (We think about an arrow as an arrowhead connected to $v$ by an edge.) If we delete the arrows then we obtain a plumbing graph $\Gamma(M)$ of $M$. Let $\delta_v$ (resp. $\overline{\delta}_v$) be the degree (i.e. the number of incident edges) of the vertex $v$ in $\Gamma(M)$ (resp. in $\Gamma(M, K)$). Evidently $\delta_v - \overline{\delta}_v$ is exactly the number of arrows supported by the vertex $v$.

Since $M$ is a rational homology sphere, $\Gamma$ is a tree.

Let $\{I_{uv}\}_{u, v \in \mathcal{V}}$ be the intersection matrix associated with $\Gamma$; i.e. $I_{uu} = e_v$, and for $u \neq v$ the entry $I_{uv} = 0$ or 1 depending that $u$ and $v$ are connected or not in $\Gamma$. Since $M$ is rational homology sphere, $I$ is non-degenerate. In fact

$$|\det(I)| = |\mathcal{H}|.$$  \hspace{1cm} (4)

The generic fiber of the $S^1$-bundle over $E_v$ is denoted by $g_v$, and we use the same notation for its homology class in $H_1(M)$ as well. By the above discussion, if $u \neq v$ then $Lk_M(g_u, g_v)$ is well-defined. If $u = v$ then we write $Lk_M(g_u, g_u)$ for $Lk_M(g_u, g_u')$ where $g_u$ and $g_u'$ are two different fibers of the $S^1$-bundle over $E_v$.

For any fixed vertex $u \in \mathcal{V}$, we denote by $\vec{b}(u)$ the column vector with entries $=1$ on the place $u$ and zero otherwise. We define the column vector $\vec{w}(u)$ (associated with the knot $g_u \subset M$ and its order $o(u)$) as the solution of the (non-degenerate) linear system $I \cdot \vec{w}(u) = -o(u)\vec{b}(u)$. The entries $\{w_v(u)\}_{v \in \mathcal{V}}$ are called the weights associated with $g_u$. Then the inverse matrix $I^{-1}$ of $I$, the set of weights $\{w_v(u)\}_{v \in \mathcal{V}}$, and the linking pairing $Lk_M$ satisfy:

$$-I_{uv}^{-1} = \frac{w_v(u)}{o(u)} = Lk_M(g_u, g_v) \text{ for any } u \text{ and } v \in \mathcal{V}.$$  \hspace{1cm} (5)
In particular, \( w_v(u) \in \mathbb{Z} \) for any \( u \) and \( v \). In fact, if \( I \) is negative definite, then the integers \( w_v(u) \) are all positive.

2.5. \((M,K)\) represented by splice diagram. If \( M \) is an integral homology sphere, and \((M,K)\) has a plumbing representation, then there is an equivalent graph-codification of \((M,K)\) in terms of the splice (or Eisenbud-Neumann) diagram, for details see [10].

The splice diagram preserves the “shape” of the plumbing graph (e.g. there is a one-to-one correspondence between those vertices \( v \) with \( \delta_v \neq 2 \) of the splice, respectively of the plumbing graphs), but in the splice diagram one collapses into an edge each string of the plumbing graph. Moreover, the decorations are also different. In the splice diagram, each vertex has a sign \( \epsilon = \pm 1 \), which in all our cases will be \( \epsilon = +1 \), hence we omit them. Moreover, if an end of an edge is attached to a vertex \( v \) with \( \delta_v \geq 3 \), then it has a positive integer as its decoration. The arrows have the same significance.

One of the big advantages of the of the splice diagram is that it codifies in an ideal way the splicing decomposition of \( M \) into Seifert pieces. (In fact, the numerical decorations are exactly the Seifert invariant of the corresponding Seifert splice-components.) Therefore, in some cases it is much easier and more suggestive to use them. (Nevertheless, we will use them only in those cases when we really want to emphasize this principle, e.g. in the proof of 4.11 or when it is incomparably easier to describe a construction with them, e.g. in 4.13.) The reader is invited to consult the book of Eisenbud and Neumann [10] for the needed properties. The criterion which guarantees that \((M,K)\) is algebraic is given in 9.4; the equivalence between the splice and plumbing graphs is described in sections 20-22; the splicing construction appears in section 8.

2.6. The Alexander polynomial. Assume that \( K \) is a homologically trivial oriented knot in \( M \). Let \( V \) be the Alexander matrix of \( K \subset M \), and \( V^* \) its transposed (cf. [21], page 26).

In the literature one can find different normalizations of the Alexander “polynomial”. The most convenient for us, which makes our formulae the simplest possible, is

\[
\Delta^\natural_M(K)(t) := \det(t^{1/2}V - t^{-1/2}V^*). 
\]

In the surgery formula 3.4 we will need Lescop’s normalization [21], in this article denoted by \( \Delta^L_M(K)(t) \). They are related by the identity (cf. [21], 2.3.13]):

\[
\Delta^L_M(K)(t) = \Delta^\natural_M(K)(t)/|H|. 
\]

Then (see e.g. [21], 2.3.1) one has:

\[
\Delta^\natural_M(K)(1) = \Delta^L_M(K)(1)/|H| = 1. \tag{6}
\]

We also prefer to think about the Alexander polynomial as a characteristic polynomial. For this notice that \( V \) is invertible over \( \mathbb{Q} \), hence one can define the “monodromy operator” \( \mathcal{M} := V^{-1}V^* \). Then set:

\[
\Delta_M(K)(t) := \det(I - t\mathcal{M}) = \det(V^{-1}) \cdot t^r \cdot \Delta^\natural_M(K)(t). \tag{7}
\]

If \((M,K)\) can be represented by a negative definite plumbing graph, then by a theorem of Grauert [13] \((M,K)\) is algebraic, hence by Milnor fibration theorem, it is fibrable. In this case, \( \mathcal{M} \) is exactly the monodromy operator acting on the first homology of the (Milnor) fiber. Moreover, \( \Delta_M(K)(t) \) can be computed from the plumbing graph by A’Campo’s...
theorem [4] as follows (see also [11]). Assume that $K = g_u$ for some $u \in \mathcal{V}$, then

$$\frac{\Delta_M(g_u)(t)}{t-1} = \prod_{v \in \mathcal{V}} (t^{w_v(u)} - 1)^\delta_v - 2.$$  \hfill (8)

Notice that (6) guarantees that $\Delta_M(K)(1) = \text{det}(V^{-1})$, from (8) we get $\Delta_M(K)(1) > 0$, and from the Wang exact sequence of the fibration $|\Delta_M(K)(1)| = |H|$. Therefore:

$$\Delta_M(K)(1) = \text{det}(V)^{-1} = |H|. \quad \hfill (9)$$

More generally, if $(M, K)$ has a negative definite plumbing representation, and $K = g_u$ for some $u$, then for any character $\chi \in \hat{H}$ we define $\Delta_{M,\chi}(g_u)(t)$ via the identity

$$\frac{\Delta_{M,\chi}(g_u)(t)}{t-1} := \prod_{v \in \mathcal{V}} (t^{w_v(u)} \chi(g_v) - 1)^\delta_v - 2,$$  \hfill (10)

and we write

$$\Delta_H^M(g_u)(t) := \frac{1}{|H|} \cdot \sum_{\chi \in \hat{H}} \Delta_{M,\chi}(g_u)(t). \quad \hfill (11)$$

In section 6 we will need the following analog of (9) in the case when $K = g_u \subset M$ is not homologically trivial (but $(M, K)$ has a negative definite plumbing representation):

$$\lim_{t \to 1} (t - 1) \prod_{v \in \mathcal{V}} (t^{w_v(u)} - 1)^\delta_v - 2 = |H|/o(u). \quad \hfill (12)$$

This follows e.g. from [11] A10(b). (In fact, $|H|/o(u)$ has the geometric meaning of $|\text{Tors}H_1(M \setminus K)|$, and (12) can also be deduced from the Wang exact sequence of the monodromy, similarly as above.)

### 2.7. The Seiberg-Witten invariant.

If $M$ is a rational homology 3-sphere, then the set $\text{Spin}^c(M)$ of the $\text{spin}^c$ structures of $M$ is a $H$-torsor. If $M$ is the link of a normal surface singularity (or, equivalently, if $M$ has a plumbing representation with a negative definite intersection matrix), then $\text{Spin}^c(M)$ has a distinguished element $\sigma_{\text{can}}$, called the canonical $\text{spin}^c$ structure (cf. [31]).

To describe the Seiberg-Witten invariants one has to consider an additional geometric data belonging to the space of parameters

$$\mathcal{P} = \{ u = (g, \eta); \quad g = \text{Riemann metric}, \quad \eta = \text{closed two-form} \}.$$  

Then for each $\text{spin}^c$ structure $\sigma$ on $M$ one defines the $(\sigma, g, \eta)$-Seiberg-Witten monopoles. For a generic parameter $u$, the Seiberg-Witten invariant $\text{sw}_M(\sigma, u)$ is the signed monopole count. This integer depends on the choice of the parameter $u$ and thus it is not a topological invariant. To obtain an invariant of $M$, one needs to alter this monopole count. The additional contribution is the Kreck-Stolz invariant $K_S M(\sigma, u)$ (associated with the data $(\sigma, u)$), cf. [23] (or see [14] for the original “spin version”). Then, by [3, 23, 25], the rational number

$$\frac{1}{8} K_S M(\sigma, u) + \text{sw}_M(\sigma, u)$$

is independent of $u$ and thus it is a topological invariant of the pair $(M, \sigma)$. We denote this modified Seiberg-Witten invariant by $\text{sw}_M^0(\sigma)$. In general, it is very difficult to compute
\[ sw^0_M(\sigma) \text{ using this definition. Fortunately, it has another realization as well. For any spin}^c \text{ structure } \sigma \text{ on } M, \text{ we denote by} \]
\[ T_{M,\sigma} = \sum_{h \in H} T_{M,\sigma}(h) h \in \mathbb{Q}[H] \]
the sign refined Reidemeister-Turaev torsion associated with \( \sigma \) (for its detailed description, see [10]). It is convenient to think of \( T_{M,\sigma} \) as a function \( H \to \mathbb{Q} \) given by \( h \mapsto T_{M,\sigma}(h) \).

The augmentation map \( \text{aug} : \mathbb{Q}[H] \to \mathbb{Q} \) is defined by \( \sum a_h h \mapsto \sum a_h \). It is known that \( \text{aug}(T_{M,\sigma}) = 0 \).

Denote the Casson-Walker invariant of \( M \) by \( \lambda_W(M) \) [42]. It is related with Lescop’s normalization \( \lambda(M) \) [21, §4.7] by \( \lambda_W(M) = 2\lambda(M)/|H| \). Then by a result of the second author [38], one has:
\[ sw^0_M(\sigma) = T_{M,\sigma}(1) - \lambda_W(M)/2. \]

Below we will present a formula for \( T_{M,\sigma} \) in terms of Fourier transform. Recall that a function \( f : H \to \mathbb{C} \) and its Fourier transform \( \hat{f} : \hat{H} \to \mathbb{C} \) satisfy:
\[ \hat{f}(\chi) = \sum_{h \in H} f(h) \chi(h); \quad f(h) = \frac{1}{|H|} \sum_{\chi \in \hat{H}} \hat{f}(\chi) \chi(h). \]
Here \( \hat{H} \) denotes the Pontryagin dual of \( H \) as above. Notice that \( \hat{f}(1) = \text{aug}(f) \), in particular \( \hat{T}_{M,\sigma}(1) = \text{aug}(T_{M,\sigma}) = 0 \). Therefore,
\[ T_{M,\sigma}(1) = \frac{1}{|H|} \sum_{\chi \in \hat{H}\setminus\{1\}} \hat{T}_{M,\sigma}(\chi). \]

Now, assume that \( M \) is represented by a negative definite plumbing graph. Fix a non-trivial character \( \chi \in \hat{H}\setminus\{1\} \) and an arbitrary vertex \( u \in V \) with \( \chi(g_u) \neq 1 \). Then set
\[ \hat{P}_{M,\chi,u}(t) := \prod_{v \in V} (t^{w_v(u)}(g_v)\chi(g_v) - 1)^{\delta_v-2}, \]
where \( t \in \mathbb{C} \) is a free variable. Then, by [31] (5.8), the Fourier transform \( \hat{T}_{M,\sigma,\text{can}} \) of \( T_{M,\sigma,\text{can}} \) is given by
\[ \hat{T}_{M,\sigma,\text{can}}(\chi) = \lim_{t \to 1} \hat{P}_{M,\chi,u}(t). \]
This limit is independent of the choice of \( u \), as long as \( \chi(g_u) \neq 1 \). In fact, by [31], even if \( \chi(g_u) = 1 \), but \( u \) is adjacent to some vertex \( v \) with \( \chi(g_v) \neq 1 \), \( u \) does the same job.

3. Some general splicing formulae

3.1. The splicing data. We will consider the following geometric situation. We start with two oriented 3-manifolds \( M_1 \) and \( M_2 \), both rational homology spheres. For \( i = 1, 2 \), we fix an oriented knot \( K_i \) in \( M_i \), and we use the notations of [24] with the corresponding indices \( i = 1, 2 \). In this article we will consider a particular splicing, which is motivated by the geometry of the suspension singularities. The more general case will be treated in a forthcoming paper.

On the pair \( (M_2, K_2) \) we impose no additional restrictions. But, for \( i = 1 \), we will consider the following working assumption:
WA1: Assume that \( \sigma_1 = 1 \), i.e. \( K_1 \) is homologically trivial in \( M_1 \). Moreover, we fix the parallel \( \ell_1 \) exactly as the longitude \( \lambda_1 \). Evidently, \( k_1 = 0 \).

Finally, by splicing, we define a 3-manifold \( M \) (for details, see e.g. [12]):

\[
M = M_1 \setminus T(K_1)^\circ \coprod_A M_2 \setminus T(K_2)^\circ,
\]

where \( A \) is an identification of \( \partial T(K_2) \) with \( -\partial T(K_1) \) determined by

\[
A(m_2) = \lambda_1, \quad \text{and} \quad A(\ell_2) = m_1.
\]

3.2. The closures \( \overline{M}_i \). Once the splicing data is fixed, one can consider the closures \( \overline{M}_i \) of \( M_i \setminus T(K_i)^\circ \) (\( i = 1, 2 \)) with respect to \( A \) (cf. [3] or [12]) by the following Dehn fillings:

\[
\overline{M}_2 = (M_2 \setminus T(K_2)^\circ)(A^{-1}(y_1)), \quad \overline{M}_1 = (M_1 \setminus T(K_1)^\circ)(A(y_2)),
\]

where \( \delta_i y_i := \lambda_i \) (\( i = 1, 2 \)). Using (1) one has \( A^{-1}(y_1) = m_2 \), hence

\[
\overline{M}_2 = M_2.
\]

Moreover, \( A(y_2) = A((o_2 \ell_2 + k_2 m_2)/\delta_2) = (o_2 m_1 + k_2 \lambda_1)/\delta_2 \), hence:

\[
\overline{M}_1 = (M_1 \setminus T(K_1)^\circ)(\mu), \quad \text{where} \quad \mu := (o_2 m_1 + k_2 \lambda_1)/\delta_2.
\]

In fact, \( \overline{M}_1 \) can be represented as a \((p, q)\)-surgery of \( M_1 \) along \( K_1 \). The integers \((p, q)\) can be determined as in [21], page 8: \( \mu \) is homologous to \( q K_1 \) in \( T(K_1) \), hence \( q = k_2/\delta_2 \). Moreover, \( p = Lk_{M_1}(\mu, K_1) \), which via 2.3(1) equals \( (o_2 m_1 + k_2 \lambda_1)/\delta_2, \lambda_1) = o_2/\delta_2 \). Therefore,

\[
\overline{M}_1 = M_1(K_1, p/q) = M_1(K_1, o_2/k_2).
\]

3.3. Fujita’s splicing formula for the Casson-Walker invariant. Using the above expressions for the closures, (1.1) of [12], in the case of the above splicing (with \( A = f^{-1} \)), reads as

\[
\lambda_W(M) = \lambda_W(M_2) + \lambda_W(M_1(K_1, o_2/k_2)) + s(k_2, o_2).
\]

Additionally, if we assume that \( K_2 \) is homologically trivial in \( M_2 \) (i.e. \( o_2 = 1 \)), and we fix \( \ell_2 \) as \( \lambda_2 \) (i.e. \( k_2 = 0 \)), then (3) transforms into:

\[
\lambda_W(M) = \lambda_W(M_1) + \lambda_W(M_2).
\]

3.4. Walker-Lescop surgery formula. Now, we will analyze the manifold \( M_1(K_1, p/q) \) obtained by \( p/q \)-surgery, where \( p = o_2/\delta_2 > 0 \) and \( q = k_2/\delta_2 \) (not necessarily positive).

First notice (cf. [21], 1.3.4) that \( |H_1(M_1(K_1, p/q))| = p \cdot |H_1(M_1)| \). Using this, the surgery formula (T2) from [21], page 13, and the identification \( \lambda_W(\cdot) = 2\lambda(\cdot)/|H_1(\cdot, \mathbb{Z})| \), one gets:

\[
\lambda_W(M_1(K_1, p/q)) = \lambda_W(M_1) + \text{Cor},
\]

where the correction term \( \text{Cor} \) is

\[
\text{Cor} := \frac{q}{p} \cdot \frac{\Delta_{M_1}(K_1)^\circ(1)}{|H_1(M_1)|} - \frac{p^2 + 1 + q^2}{12pq} + \text{sign}(q) \left( \frac{1}{4} + s(p, q) \right).
\]

Using (3), (5) and the reciprocity law of the Dedekind sums (for \( p > 0 \)):

\[
s(q, p) + \text{sign}(q)s(p, q) = -\frac{\text{sign}(q)}{4} + \frac{p^2 + 1 + q^2}{12pq},
\]

one gets the following formula:
3.5. Theorem. The splicing formula for Casson-Walker invariant. Consider a splicing manifold M characterized by the data described in \[7.4\]. Then:

\[ \lambda_W(M) = \lambda_W(M_1) + \lambda_W(M_2) + \frac{k_2}{\alpha_2} \cdot \Delta_{M_1}(K_1)^{\epsilon}(1). \]

3.6. The splicing property of the group \( H_1(M, \mathbb{Z}) \). In the next paragraphs we analyze the behavior of \( H_1(\cdot, \mathbb{Z}) \) under the splicing construction \[3.1\].

First notice that by excision, for any \( q \), one has:

\[ H_q(M, M_2 \setminus T(K_2)^o) = H_q(M_1 \setminus T(K_1)^o, \partial T(K_1)) = H_q(M_1, K_1). \]

Therefore, the long exact sequence of the pair \((M, M_2 \setminus T(K_2)^o)\) reads as:

\[ 0 \to H_2(M_1, K_1) \xrightarrow{\partial} H_1(M_2 \setminus T(K_2)^o) \to H_1(M) \to H_1(M_1, K_1) \to 0. \]

Using the isomorphisms \[2.3(3)\], \( \partial \) can be identified with \( \partial (1_{Z}) = m_2 \), hence \( \ker \partial = H_1(M_2) \), cf. \[2.3(2)\]. Therefore, \[2.3(3)\] gives the exact sequence

\[ 0 \to H_1(M_2) \xrightarrow{i} H_1(M) \xrightarrow{p} H_1(M_1) \to 0. \] (6)

This exact sequence splits. Indeed, let \( s \) be the composition of \( s_1 : H_1(M_1) \to H_1(M_1 \setminus T(K_1)) \) (cf. \[2.3\] and \( H_1(M_1 \setminus T(K_1)) \to H_1(M) \) (induced by the inclusion). Then \( p \circ s = 1 \). In particular,

\[ H_1(M) = im(i) \oplus im(s) \approx H_1(M_2) \times H_1(M_1). \] (7)

Notice that any \([K] \in H_1(M_1)\) can be represented (via \( s_1 \)) by a representative \( K \) in \( M_1 \setminus T(K_1) \) providing a class in \( \text{Tors } H_1(M_1 \setminus T(K_1)) \). Write \( o \) for its order, and take a Seifert surface \( F \), sitting in \( M_1 \setminus T(K_1) \) with \( \partial F = oK \). If \( L \subset M_1 \setminus T(K_1) \) with \( L \cap K = \emptyset \) then obviously \( Lk_M(K, L) = Lk_{M_1}(K, L) \). Moreover, since \( F \) has no intersection points with any curve \( L \subset M_2 \setminus T(K_2) \), for such an \( L \) one gets:

\[ Lk_M(K, L) = 0, \text{ hence } b_M(im(s), im(i)) = 0. \] (8)

By a similar argument,

\[ b_M(s(x), y) = b_{M_1}(x, p(y)) \text{ for any } x \in H_1(M_1) \text{ and } y \in H_1(M). \] (9)

The point is that in the next sections we need \( Lk_M(K, \cdot) \) for general \( K \subset M_1 \setminus K_1 \) which is not a torsion element in \( H_1(M_1 \setminus K_1) \).

To compute this linking number consider another oriented knot \( L \subset M_1 \setminus K_1 \) with \( K \cap L = \emptyset \). Our goal is to compare \( Lk_{M_1}(K, L) \) and \( Lk_M(K, L) \). Assume that the order of the class \( K \) in \( H_1(M_1) \) is \( o \). Let \( F_{oK} \) be a Seifert surface in \( M_1 \) with \( \partial F_{oK} = oK \) which intersects \( L \) and \( K_1 \) transversally. It is clear that \( F_{oK} \) intersects \( L \) exactly in \( o \cdot Lk_{M_1}(K, L) \) points (counted with sign). On the other hand, it intersects \( K_1 \) in \( o \cdot Lk_{M_1}(K_1, K_1) \) points. For each intersection point with sign \( \epsilon = \pm 1 \), we cut out from \( F_{oK} \) the disc \( F_{oK} \cap T(K_1) \), whose orientation depends on \( \epsilon \). Its boundary is \( em_1 \) which, by the splicing identification, corresponds to \( \ell_2 \) in \( M_2 \setminus T(K_2) \). Using rational coefficients, \( \ell_2 = (1/o_2)\lambda_2 - (k_2/o_2)m_2 \). Some multiple of \( \lambda_2 \) can be extended to a surface in \( M_2 \setminus K_2 \) which clearly has no intersection with \( L \). \( m_2 \) by splicing is identified with \( \lambda_1 \) which has a Seifert surface in \( M_1 \setminus K_1 \) which intersect \( L \) in \( Lk_{M_1}(K_1, L) \) points. This shows that for \( K, L \subset M_1 \setminus T(K_1) \):

\[ Lk_M(K, L) = Lk_{M_1}(K, L) - Lk_{M_1}(K, K_1) \cdot Lk_{M_1}(L, K_1) \cdot k_2/o_2. \] (10)
Assume now that $K \in M_2 \setminus T(K_2)$, and let $o$ be the order of its homology class in $H_1(M_2)$. Assume that the Seifert surface $F$ of $oK$ intersects $K_2$ transversally in $t_2$ points. Then $F \cap \partial T(K_2) = t_2 m_2$, and after the splicing identification this becomes $t_2 \lambda_1$. Let $F_1$ be the Seifert surface of $\lambda_1$ in $M_1 \setminus T(K_1)^o$. Then (after some natural identifications) $F \setminus T(K_2)^o \bigsqcup t_2 F_1$ is a Seifert surface of $oK$ in $M$. Therefore, for any $L \subset M_1 \setminus T(K_1)$ one has:

$$Lk_M(K, L) = Lk_{M_2}(K, K_2) \cdot Lk_{M_1}(K_1, L) \quad \text{for} \quad K \subset M_2 \setminus T(K_2) \quad \text{and} \quad L \subset M_1 \setminus T(K_1).$$

(11)

By a similar argument:

$$Lk_M(K, L) = Lk_{M_2}(K, L) \quad \text{for} \quad K, L \subset M_2 \setminus T(K_2).$$

(12)

3.7. Splicing plumbing manifolds. Some of the results of this article about Reidemeister-Turaev torsion can be formulated and proved in the context of general (rational homology sphere) 3-manifolds, and arbitrary spin$^c$ structures. Nevertheless, in this article we are mainly interested in algebraic links, therefore we restrict ourselves to plumbed manifolds.

This can be formulated in the second working assumption:

**WA2:** $(M_i, K_i) \ (i = 1, 2)$ can be represented by a negative definite plumbing graph. Moreover, if $M$ is the result of the splicing (satisfying WA1, cf. 3.1), then $M$ can also be represented by a negative definite plumbing graph.

Assume that the plumbing graphs $\Gamma(M_1, K_1)$ and $\Gamma(M_2, K_2)$ have the following schematic form (with $g_{v_1} = K_1$ and $g_{v_2} = K_2$):

Then it is not difficult to see that (a possible) plumbing graph $\Gamma(M)$ for $M$ has the following form (where $v_1$ and $v_2$ are connected by a string):

If $V_i \ (i = 1, 2)$, respectively $V$, represent the set of vertices of $\Gamma(M_i)$, resp. of $\Gamma(M)$, then $V = V_1 \cup V_2$ modulo some vertices with $\delta = 2$. (In particular, in any formula like 3.6(15), $V$ behaves as the union $V_1 \cup V_2$.)

3.8. Remark. If one wants to compute $\tau_{M, \sigma_{can}}$ for such plumbed manifolds, then one can apply 3.7(15) and (16). For this, one has to analyze the supports of the characters, and the corresponding weights $w_\nu(u)$. These weights are closely related with the corresponding linking numbers $Lk_M(g_\nu, g_\nu)$ (cf. 3.4(5)), hence the relations 3.6(10)-(11)-(12) are crucial. For characters of type $\chi \in \hat{p}(H_1(M_1))$ (cf. 3.6(6) or the next proof) $u \in V_1$, hence 3.6(10) should be applied for any $v \in V_1$. But this is rather unpleasant due to the term $Lk_{M_1}(g_\nu, K_1) \cdot Lk_{M_1}(g_\nu, K_1) \cdot k_2/o_2$. The description is more transparent if either $H_1(M_1) = 0$ or $k_2 = 0$.

Therefore, we will consider first these particular cases only. They, as guiding examples, already contain all the illuminating information and principles we need to proceed. For the link of $\{ f(x, y) + z^n = 0 \}$, the splicing formula will be made very explicit in 3.13 (based on a detailed and complete classification of the characters and the regularization terms $\hat{P}$, which is rather involved).
3.9. **Theorem.** Some splicing formulae for the Reidemeister-Turaev torsion.

Assume that $M$ satisfy WA1 [(3.1)] and WA2 [(3.7)]. Then the following hold:

(A) Assume that $K_2 \subset M_2$ is also homologically trivial (i.e. $\ell_2 = 1$), and $\ell_2 = \chi_2$ (i.e. $k_2 = 0$). Then

$$\mathcal{F}_{M, \sigma_{\text{can}}}(1) = \mathcal{F}_{M_1, \sigma_{\text{can}}}(1) + \mathcal{F}_{M_2, \sigma_{\text{can}}}(1).$$

(B) Assume that $M_1$ is an integral homology sphere (i.e. $H_1(M_1) = 0$). Then

$$\mathcal{F}_{M, \sigma_{\text{can}}}(1) = \sum_{\chi_2 \in H_1(M_2) \setminus \{1\}} \frac{\hat{\mathcal{F}}_{M_2, \sigma_{\text{can}}} (\chi_2)}{|H_1(M_2)|} \cdot \Delta_{M_1}(K_1)(\chi_2(K_2)).$$

In particular, if $K_2 \subset M_2$ is homologically trivial, then $\chi_2(K_2) = 1$ for any $\chi_2$, hence by [(2.6)(9)] one gets

$$\mathcal{F}_{M, \sigma_{\text{can}}}(1) = \mathcal{F}_{M_2, \sigma_{\text{can}}}(1) \quad \text{(and evidently} \quad \mathcal{F}_{M_1, \sigma_{\text{can}}}(1) = 0).$$

This is true for any choice of $\ell_2$, i.e. even if $k_2$ is non-zero.

**Proof.** The theorem is a consequence of [(2.7)(14)-(15)-(16)] and [(3.6)]. For this, we have to analyze the characters $\chi$ of $H_1(M)$. The dual of the exact sequence [(3.6)(6)] is

$$0 \to H_1(M) \xrightarrow{\hat{\rho}} H_1(M) \xrightarrow{\hat{i}} H_1(M_2) \to 0.$$  

First, consider a character $\chi$ of $H_1(M)$ of the form $\chi = \hat{\rho}(\chi_1)$ for some $\chi_1 \in H_1(M_1)$. Since any $\chi_1 \in H_1(M_1)$ can be represented as $b_{\chi_1}(x, \cdot)$ for some $x \in H_1(M_1)$ (cf. [(2.3)]), and $\hat{\rho}(b_{\chi}(x, \cdot)) = b_{\chi}(\hat{s}(x), \cdot)$ (cf. [(3.6)(9)]), property [(3.6)(8)] guarantees that $\chi(g_v) = 1$ for any $v \in V_2$. In particular, for $\chi = \hat{\rho}(\chi_1)$ with $\chi_1 \in H_1(M_1) \setminus \{1\}$, and for some $u \in V_1$ with $\chi_1(g_u) \neq 1$ (which works for $M$ as well), one gets:

$$\hat{\rho}_{M, \chi, u}(t) = \prod_{v \in V_1} (t^{w_v(u)}\chi_1(g_v) - 1)^{\delta_v - 2} \cdot \prod_{v \in V_2} (t^{w_v(u)} - 1)^{\delta_v - 2}.$$  

Here, for $v \in V_1$, $\delta_v$ means the number of adjacent edges of $v$ in $\Gamma(M_i, K_i)$ ($i = 1, 2$), cf. [(2.4)].

By [(3.6)(11)], for any $v \in V_2$ one has

$$w_v(u) = o(u)Lk_M(g_u, g_v) = o(u)Lk_{M_1}(g_u, g_v) \cdot Lk_{M_2}(g_v, g_v),$$  

hence by [(2.6)(8)]

$$\hat{\rho}_{M, \chi, u}(t) = \hat{\rho}_{M_1, \chi_1, u}(t) \cdot \Delta_{M_2}(g_v)(t^{o(u)Lk_{M_1}(g_u, g_v)}).$$

Taking the limit $t \to 1$, and using [(2.6)(9)], one gets:

$$\hat{\mathcal{F}}_{M, \sigma_{\text{can}}} (\chi) = \hat{\mathcal{F}}_{M_1, \sigma_{\text{can}}} (\chi_1) \cdot |H_1(M_2)|.$$  

(13)

Now, we prove (A). In this case $M_1$ and $M_2$ are symmetric, hence there is a similar term as in (13) for characters $\chi_2 \in H_1(M_2)$.

On the other hand, if $\chi = \chi_1\chi_2$ for two non-trivial characters $\chi_i \in H_1(M_i)$ ($i = 1, 2$), then one can show that $\hat{\rho}_{M, \chi, u}(t)$ has a root (of multiplicity at least two) at $t = 1$, hence $\hat{\mathcal{F}}_{M, \sigma_{\text{can}}} (\chi) = 0$. Now, use [(2.7)(14)] and $|H_1(M)| = |H_1(M_1)| \cdot |H_1(M_2)|$ (cf. [(3.3)(7)]).

For (B), fix a non-trivial character $\chi_2 \in H_1(M_2)$. The relations in [(3.6)] guarantee that if we take $u \in V_2$ with $\chi_2(g_u) \neq 1$ (considered as a property of $M_2$) then $\chi(g_u) \neq 1$ as well for $\chi = \hat{i}^{-1}(\chi_2)$. Moreover, for any $v \in V_1$ one has: $\chi(g_v) = \chi_2(K_2)\cdot Lk_{M_1}(K_1, g_v)$ (cf. [(3.6)(11)]). Therefore:

$$\hat{\rho}_{M, \chi, u}(t) = \prod_{v \in V_1} (t^{w_v(u)}\chi(g_v) - 1)^{\delta_v - 2} \cdot \prod_{v \in V_2} (t^{w_v(u)}\chi(g_v) - 1)^{\delta_v - 2}.$$
Seiberg-Witten invariants and surface singularities III

\[ = \hat{P}_{M_2, \chi_2, u}(t) \cdot \Delta_{M_1}(g_{v_1})(e(u) Lk_{M_2}(g_0, g_{v_2}) \chi_2(K_2)). \]

3.10. Remarks. (1) A similar proof provides the following formula as well (which will be not used later). Assume that \( M \) satisfies WA1 and WA2, and \( k_2 = 0 \). Then

\[ T_{M, \sigma_{can}}(1) = T_{M, \sigma_{can}}(1) + \sum_{x_2 \in H_1(M_2) \setminus \{1\}} \frac{\hat{f}_{M_2, \sigma_{can}}(\chi_2)}{|H_1(M_2)|} \cdot \Delta_{M_1}(K_1)(\chi_2(K_2)). \]

(2) The obstruction term (see also [3, 1]) which measures the non-additivity of the Casson-Walker invariant (under the splicing assumption WA1) is given by \( \frac{k_2}{o_2} \cdot \Delta^2_{M_1}(K_1)''(1) \) (cf. [3]). On the other hand, if \( H_1(M_1) = 0 \), then the obstruction term for the non-additivity of the Reidemeister-Turaev torsion (associated with \( \sigma_{can} \)) is

\[ \sum_{x_2 \in H_1(M_2) \setminus \{1\}} \frac{\hat{f}_{M_2, \sigma_{can}}(\chi_2)}{|H_1(M_2)|} \cdot \left( \Delta_{M_1}(K_1)(\chi_2(K_2)) - 1 \right). \]

Notice they “look rather different” (even under this extra-assumption \( H_1(M_1) = 0 \)). In fact, even their nature are different: the first depends essentially on the choice of the parallel \( e_2 \) (see the coefficient \( k_2 \) in its expression), while the second not. In particular, one cannot really hope (in general) for the additivity of the modified Seiberg-Witten invariant.

Therefore, it is really remarkable and surprising, that in some of the geometric situations discussed in the next sections, the modified Seiberg-Witten invariant is additive (though the invariants \( \lambda_W \) and \( T_{\sigma_{can}}(1) \) are not additive, their obstruction terms cancel each other).

4. The basic topological example

4.1. Recall that in our main applications (for algebraic singularities) the involved 3-manifolds are plumbed manifolds. In particular, they can be constructed inductively from Seifert manifolds by splicing (cf. also with 2.3). The present section has a double role. First, we work out explicitly the splicing results obtained in the previous section for the case when \( M_2 \) is a Seifert manifold (with special Seifert invariants). On the other hand, the detailed study of this splicing formulae provides us a better understanding of the subtlety of the behavior of the (modified) Seiberg-Witten invariant with respect to splicing and cyclic covers. They will be formulated in some “almost–additivity” properties, where the non-additivity will be characterized by a new invariant \( D \) constructed from the Alexander polynomial of \((M_1, K_1)\).

4.2. The splicing component \( M_2 \). Assume that \( M_2 \) is the link

\[ \Sigma = \Sigma(p, a, n) := \{(x, y, z) \in \mathbb{C}^3 : x^p + y^a + z^n = 0, |x|^2 + |y|^2 + |z|^2 = 1\} \]

of the Brieskorn hypersurface singularity \( X_2 := \{(x, y, z) \in \mathbb{C}^3 : x^p + y^a + z^n = 0\} \), where \( \gcd(n, a) = 1 \) and \( \gcd(p, a) = 1 \). Set \( d := \gcd(n, p) \). Then \( M_2 \) is a rational homology sphere with the following data (for more details, and for a more complete list of the relevant invariants, see e.g. [31], section 6; cf also with [3, 6]).

(a) the Seifert invariants are: \( n/d, p/d, a, a, \ldots, a \) (a appearing \( d \) times, hence all together there are \( d + 2 \) special fibers); these numbers also give (up to a sign) the determinants of the corresponding arms of the plumbing graph of \( \Sigma \);

(b) the orbifold Euler characteristic is \( e = -d^2/(npa) \);
4.3. The link $K_2$. In this section we will modify slightly the construction of 3.1: we will fix a link with $d$ connected components (instead of a knot), and we will perform splicing along each connected component.

This link $K_2 \subset M_2$ is given by the equation \{y = 0\} in $M_2$. In other words, $K_2$ is the union of the $d$ special (Seifert) fibers corresponding to the Seifert invariant $a$. The components of $K_2$ will be denoted by $K_2^{(i)}$ ($1 = 1, \ldots , d$), and their tubular neighborhood by $T(K_2^{(i)})$ as in 2.1.

The link-components $\{K_2^{(i)}\}$ generate $H_1(M_2)$ (see, e.g. [31], section 6); in fact $H_1(M_2)$ has the following presentation (written additively):

$$H_1(M_2) = \langle [K_2^{(1)}], \ldots , [K_2^{(d)}] : a[K_2^{(i)}] \text{ for each } i, \text{ and } [K_2^{(1)}] + \cdots + [K_2^{(d)}] = 0 \rangle. \quad (1)$$

Besides $K_2$, there are two more special orbits in $M_2 = \Sigma$, namely $Z := \{z = 0\}$ and $X := \{x = 0\}$. Moreover, let $O$ be the generic fiber of the Seifert fibration of $\Sigma$ (i.e. $g_v$ associated with the central vertex $v$). Then one has the following linking numbers (use 2.4(5) and [31] (5.5)(1)):

\begin{itemize}
  \item[(a)] $Lk_{M_2}(K_2^{(i)}, K_2^{(j)}) = np/(d^2a)$ for any $i \neq j$;
  \item[(b)] $Lk_{M_2}(K_2^{(i)}, O) = np/d^2$; $Lk_{M_2}(K_2^{(i)}, Z) = p/d^2$ for any $i$; \hfill (L\#)
  \item[(c)] $Lk_{M_2}(O, O) = npa/d^2$ and $Lk_{M_2}(O, Z) = pa/d$;
  \item[(d)] $Lk_{M_2}(X, Z) = a$.
\end{itemize}

Notice that $K_2 \subset M_2$ is fibrable. Indeed, $K_2 = \{y = 0\}$ is the link associated with the algebraic germ $y : (X_2, 0) \to (\mathbb{C}, 0)$, hence one can take its Milnor fibration. Let $F$ be the fiber with $\partial F = K_2$ (equivalently, take any minimal Seifert surface $F$ with $\partial F = K_2$). Then for each $i = 1, \ldots , d$, we define the parallel $\ell_2^{(i)}$ in $\partial T(K_2^{(i)})$ by $F \cap \partial T(K_2^{(i)})$.

Let $\lambda_2^{(i)}$ be the longitude of $K_2^{(i)} \subset M_2$, and consider the invariants $o_2^{(i)}, k_2^{(i)}$, etc. as in 2.1 with the corresponding sub- and superscripts added.

4.4. Lemma. For each $i = 1, \ldots , d$ one has:

$$o_2^{(i)} = a \quad \text{and} \quad k_2^{(i)} = \frac{np(d-1)}{d^2}.$$

Proof. The first identity is clear (cf. 1.3(1)). For the second, notice that (cf. 2.3(1))

$$-k_2^{(i)} / o_2^{(i)} = \langle \ell_2^{(i)}, \lambda_2^{(i)} \rangle / o_2^{(i)} = Lk_{M_2}(\ell_2^{(i)}, K_2^{(i)}).$$

Moreover,

$$Lk_{M_2} \sum_j \ell_2^{(j)}, K_2^{(i)} = 0,$$

hence

$$k_2^{(i)} / a = \sum_{j \neq i} Lk_{M_2}(K_2^{(j)}, K_2^{(i)}).$$

Then use 4.3(L#a). \qed
4.5. The manifold $M$. Next, we consider $d$ manifolds $M_1^{(i)}$ with knots $K_1^{(i)} \subset M_1^{(i)}$ ($i = 1, \ldots, d$), each satisfying the assumption WA1 (i.e. $o_1^{(i)} = 1$, $\ell_1^{(i)} = \lambda_1^{(i)}$, and $k_1^{(i)} = 0$, cf. [3.1]). Then, for each $i = 1, \ldots, d$, we consider the splicing identification of $\partial T(K_2^{(i)})$ with $-\partial T(K_1^{(i)})$ (similarly as in [3.1]):

$$A^{(i)}(m_2^{(i)}) = \lambda_1^{(i)} \text{ and } A^{(i)}(\ell_2^{(i)}) = m_1^{(i)}.$$  

Schematically:

```
```

In the sequel we denote by WA1’ the assumption which guarantees that the manifold $M$ is constructed by this splicing procedure. Moreover, WA2’ guarantees that all the involved 3-manifolds have negative definite plumbing representations.

Here some comments are in order.

(1) Assume that for some $i$, $M_1^{(i)} = S^3$, and $K_1^{(i)}$ is the unknot $S^1$ in $S^3$. Then performing splicing along $K_2^{(i)}$ with $(S^3, S^1)$ is equivalent to put back $T(K_2^{(i)})$ unmodified, hence it has no effect. In this case, one also has $\Delta_{S^3}(S^1)(t) \equiv \Delta_{S^3}^S(S^1)(t) \equiv 1$.

(2) Assume that we already had performed the splicing along the link-components $K_2^{(i)}$ for $i \leq k - 1$, but not along the other ones. Let us denote the result of this partial modification by $M^{(k-1)}$. Consider $K_2^{(k)}$ in $M^{(k-1)}$ (in a natural way). Then all the invariants (e.g. $o_2^{(k)}$, $\lambda_2^{(k)}$, $k_2^{(k)}$, etc.) associated with $K_2^{(k)}$ in $M_2$ or in $M^{(k-1)}$ are the same. (This follows from the discussion in [3.4], and basically, it is a consequence of WA1’.)

In particular, performing splicing at place $i$ does not effect the splicing data of the place $j$ ($j \neq i$). Therefore, using induction, the computation of the invariants can be reduced easily to the formulae established in the previous section.

4.6. Definitions/Notations.

(1) In order to simplify the exposition, for any 3-manifold invariant $\mathcal{I}$, we write

$$\mathcal{O}(\mathcal{I}) := \mathcal{I}(M) - \mathcal{I}(M_2) - \sum_{i=1}^{d} \mathcal{I}(M_1^{(i)}),$$

for the “additivity obstruction” of $\mathcal{I}$ (with respect to the splicing construction WA1’). E.g., using [3.6] and [3.2] one has $\mathcal{O}(\log |H_1(\cdot)|) = 0$. Moreover, in all our Alexander invariant notations (e.g. in $\Delta_{M_1^{(i)}}(K_1^{(i)})(t)$), we omit the link $K_1^{(i)}$ (e.g. we simply write $\Delta_{M_1^{(i)}}(t)$).

When we will compare $\mathcal{O}(\mathcal{I}, \sigma_{can})$ with $\mathcal{O}(\lambda_W(\cdot))$, the next terminology will be helpful.
(2) For any set of integers $c_1, \ldots, c_r$ define
\[
D(c_1, \ldots, c_r) := \sum_{i,j=1}^{r} c_i c_j \min(i, j) - \sum_{i=1}^{r} i c_i.
\]

(3) Define $D(\Delta^2(t))$ by $D(c_1, \ldots, c_r)$ for any symmetric polynomial
\[
\Delta^2(t) = 1 + \sum_{i=1}^{r} c_i (t^i + t^{-i} - 2) \quad (\text{for some } c_i \in \mathbb{Z}).
\]

(4) A set $\{c_i\}_{i \in I}$ ($I \subset \mathbb{N}$) is called alternating if $c_i \in \{-1, 0, +1\}$ for any $i \in I$; and if $c_i \neq 0$ then $c_i = (-1)^{n_i}$, where $n_i = \#\{j : j > i, \text{ and } c_j \neq 0\}$.

4.7. Corollary. Assume that $M$ satisfies WA1'. Then
\[
O(\lambda_W) = \frac{np(d-1)}{ad^2} \sum_{i=1}^{d} (\Delta^2_{M^i(\alpha)})''(1).
\]

Proof. Use \[\text{[5]} \text{ L4 and } \text{[4.5]}(2). \]

4.8. Corollary. Assume that $M$ satisfies WA1' and WA2', and $M^i$ is an integral homology sphere for any $i$. Identify $\mathbb{Z}_a := \{\xi \in \mathbb{C} : \xi^a = 1\}$ and write $\mathbb{Z}_a^* := \mathbb{Z}_a \setminus \{1\}$. Then
\[
O(\sigma, \sigma_{\text{can}}(1)) = \frac{np}{ad^2} \sum_{i \neq j} \sum_{\xi \in \mathbb{Z}_a^*} \frac{\Delta_{M^i(\alpha)}(\xi) \cdot \Delta_{M^j(\alpha)}(\xi) - 1}{(\xi-1)(\xi-1)}.
\]

Proof. We recall first how one computes the torsion for the manifold $M_2$ (for a detailed presentation, see \[\text{[4]}\]). The point is (cf. also with the last sentence of \[\text{[2]}\]), that for any character $\chi \in H_1(M_2) \setminus \{1\}$, one can choose the central vertex of the star-shaped graph for the vertex $u$ in order to generate the weights in $\hat{P}$. Then, by \[\text{[31]} \text{ or } \text{[4.3]}(L\#), \text{ one gets:}
\]
\[
\hat{P}_{M_2, \chi, u}(t) = \frac{(t^a - 1)^d}{(t^d a/n - 1)(t^d a/p - 1) \prod_{i} (t^a a \chi(K^{(i)}_2) - 1)},
\]

where $\alpha := npa/d^2$. The limit of this expression, as $t \to 1$, always exists. In particular
\[
\#\{i : \chi(K^{(i)}_2) \neq 1\} \geq 2 \quad (\text{cf. also with } \text{[4.3]}(1)).
\]

If this number is strict greater than 2, then the limit is zero. If $\chi(K^{(i)}_2) \neq 1$ exactly for two indices $i$ and $j$, then using \[\text{[4.3]}(1) \text{ clearly}
\]
\[
\chi(K^{(i)}_2) \chi(K^{(j)}_2) = 1.
\]

Since there are exactly $d(d-1)/2$ such pairs, one gets:
\[
\sigma_{\text{can}}(1) = \frac{1}{|H_1(M_2)|} \lim_{t \to 0} \left(\frac{(t^a - 1)^d}{(t^d a/n - 1)(t^d a/p - 1) (t^a a - 1)^{d-2}} \cdot \sum_{i \neq j} \sum_{\xi \in \mathbb{Z}_a^*} \frac{1}{(\xi-1)(\xi-1)}\right) = \frac{np}{ad^2} \frac{d(d-1)}{2} \cdot \frac{1}{(\xi-1)(\xi-1)} = \frac{np(d-1)(a^2-1)}{24 ad}.
\]

(*)
\[
\sum_{\xi \in \mathbb{Z}_a^*} \frac{1}{(\xi-1)(\xi-1)} = \frac{a^2-1}{12}.
\]

(**)
Consider now the manifold $M$. Then using (3.3)(B) (and/or its proof), by the same argument as above, one obtains:

$$\mathcal{T}_{M, \sigma_{\text{can}}}(1) = \frac{np}{ad^2} \sum_{i \neq j} \sum_{\xi \in \mathbb{Z}_a^*} \Delta_{M_1(i)}(\xi) \cdot \Delta_{M_1(j)}(\xi) \frac{1}{(\xi - 1)(\xi - 1)}.$$  

$$(***)$$

Finally, making the difference between (***) and (*) one gets the result. $lacksquare$

4.9. Example/Discussion. Assume that $M$ satisfies WA1' and WA2', and additionally $(M_1^{(i)}, K_1^{(i)}) = (M_1, K_1)$ for some integral homology sphere $M_1$. Then

$$\mathcal{O}(\lambda_{W})/2 = \frac{np(d - 1)}{2ad} \cdot (\Delta_{M_1}^z)^\prime(1);$$

$$\mathcal{O}(\mathcal{T}_{M, \sigma_{\text{can}}}(1)) = \frac{np(d - 1)}{2ad} \cdot \sum_{\xi \in \mathbb{Z}_a^*} \Delta_{M_1}(\xi) \cdot \Delta_{M_1}(\xi - 1) \frac{1}{(\xi - 1)(\xi - 1)}.$$  

Recall that the modified Seiberg-Witten invariants $sw_0^0(M, \sigma_{\text{can}})$ is defined by the difference $\mathcal{T}_{M, \sigma_{\text{can}}}(1) - \lambda_{W}(M)/2$ (cf. [2.7](13)). Notice the remarkable fact that in the above expressions the coefficients before the Alexander invariants became the same. Hence

$$\mathcal{O}(sw_0^0(\sigma_{\text{can}})) = \frac{np(d - 1)}{2ad} \mathcal{D}_a,$$  

where $\mathcal{D}_a := \sum_{\xi \in \mathbb{Z}_a^*} \frac{\Delta_{M_1}(\xi) \cdot \Delta_{M_1}(\xi - 1)}{(\xi - 1)(\xi - 1)} - (\Delta_{M_1}^z)^\prime(1)$.  

$(D)$

Recall that $\Delta_{M_1}^z(t)$ is a symmetric polynomial (cf. [21, 2.3.1]) with $\Delta_{M_1}^z(1) = 1$ (cf. [2.6](6)). In the sequel we will compute explicitly $\mathcal{D}_a$, provided that $a$ is sufficiently large, in terms of the coefficients $\{c_i\}_{i=1}^r$ of $\Delta_{M_1}^z(t)$ (cf. [1.6](3)).

The contribution $(\Delta_{M_1}^z)^\prime(1)$ is easy: it is $\sum_{i=1}^r 2i^2 c_i$. By [2.4](9), $\Delta_{M_1}(t) = t^r \cdot \Delta_{M_1}^z(t)$. In particular, $\Delta_{M_1}(\xi) \cdot \Delta_{M_1}(\xi) = \Delta_{M_1}^z(\xi) \cdot \Delta_{M_1}^z(\xi)$. Then write

$$\Delta_{M_1}^z(t) = \frac{1}{1 - t} - \sum_{i=1}^r c_i(1 + t + \cdots + t^{i-1}) + \sum_{i=1}^r c_i(t^{i-1} + \cdots t^i).$$

Using the identity $\sum_{\xi \in \mathbb{Z}_a^*} 1/(1 - \xi) = (a - 1)/2$, an elementary computation gives

$$\sum_{\xi \in \mathbb{Z}_a^*} \frac{\Delta_{M_1}(\xi) \cdot \Delta_{M_1}(\xi - 1)}{(1 - \xi)(1 - \xi)} = \sum_{i=1}^r 2i^2 c_i + 2a \cdot \mathcal{D}(\Delta_{M_1}^z(t)), \text{ provided that } a \geq 2r.$$

In particular, if $a \geq 2r$, then $\mathcal{D}_a = 2a \cdot \mathcal{D}(\Delta_{M_1}^z(t))$. Hence

$$\mathcal{O}(sw_0^0(\sigma_{\text{can}})) = \frac{np(d - 1)}{d} \cdot \mathcal{D}(\Delta_{M_1}^z(t)).$$

This raises the following natural question: for what Alexander polynomials the expression $\mathcal{D}(\Delta_{M_1}^z(t))$ is zero? The next lemma provides such an example (the proof is elementary and it is left to the reader).
4.10. **Lemma.** If \{c_i\}_{i=1}^r is an alternating set then \( \mathcal{D}(c_1, \ldots, c_r) = 0 \).

The above discussions have the following topological consequence:

Fix two relative prime positive integers \( p \) and \( a \). Let \( K \) be the primitive simple curve in \( \partial \mathcal{T}(L_1) \) with homology class \( am_1 + p\lambda_1 \). Let \( M \) denote the \( n \)-cyclic cover of \( N_1 \) branched along \( K \). Set \( d := \gcd(n, p) \), and let \( M_1 \) be the \((n/d)\)-cyclic cover of \( N_1 \) branched along \( L_1 \). Denote by \( K_1 \) the preimage of \( L_1 \) via this cover. Finally, let \( \Delta_{M_1}(t) \) be the normalized Alexander polynomial of \((M_1, K_1)\).

4.11. **Corollary.** Consider the above data. Additionally, assume that WA2’ is satisfied and \( M \) is a rational homology sphere. Then:

(A) \( (d - 1) \cdot (\gcd(n, a) - 1) = 0 \).

(B) If \( d = 1 \), then
\[
\text{sw}^0_M(\sigma_{\text{can}}) = \text{sw}^0_{M_1}(\sigma_{\text{can}}) + \text{sw}^0_{\Sigma(p, a, n)}(\sigma_{\text{can}}).
\]

(C) If \( \gcd(n, a) = 1 \), \( a \geq \deg \Delta_{M_1}(t) \), and \( M_1 \) is an integral homology sphere, then
\[
\text{sw}^0_M(\sigma_{\text{can}}) = \frac{n(p(d - 1))}{d} \cdot \mathcal{D}(\Delta_{M_1}(t)).
\]

If the coefficients of \( \Delta_{M_1}(t) \) form an alternating set, then \( \mathcal{D}(\Delta_{M_1}(t)) = 0 \).

**Proof.** Consider the following schematic splicing of splice diagrams (cf. 2.3):

The result of the splicing can be identified with \( N_1 \) (and under this identification \( L_1 \) is identified with \( K'_2 \)). The advantage of this splicing representation is that it emphasizes the position of the knot \( K \) in the Seifert component \( \Sigma(p, a, 1) \). If \( M \) is a rational homology sphere then the \( n \)-cyclic cover of \( \Sigma(p, a, 1) \) branched along \( K \) (which is \( \Sigma(p, a, n) \)) should be rational homology sphere, hence (A) follows. If \( d = 1 \) then \( M \) has a splice decomposition of the following schematic plumbing diagrams (where at the right \( M_2 = \Sigma(p, a, n) \) and the dots mean \( \gcd(n, a) \) arms):

Here \( o_1 = o_2 = 1 \) and \( k_1 = k_2 = 0 \), and \( A \) is the identification \( \lambda_2 = m_1, m_2 = \lambda_1 \). Therefore, part (B) follows from 3.3(4) and 3.9(A). The last case corresponds exactly to the situation treated in 4.9. \( \square \)

4.12. **Remarks.** (1) Our final goal (see the following sections) is to prove the additivity result \( \mathcal{O}(\text{sw}^0(\sigma_{\text{can}})) = 0 \) for any \((M_1, K_1)\), which can be represented as a cyclic cover of \( S^3 \) branched along the link \( K_f \subset S^3 \) of an arbitrary irreducible (complex) plane curve singularity (even if \( M_1 \) is not an integral homology sphere), provided that \( a \) is sufficiently large. This means that from the above Corollary, part (C), we will need to eliminate the assumption about the vanishing of \( H_1(M_1) \). The assumption about \( a \) will follow from the
special property 5.1(6)) of irreducible plane curve singularities (cf. also with 5.4 especially with (7)).

The proof of the vanishing of the $D$-correction will take up most of the last section of the paper. It relies in a crucial manner on the alternating nature of the Alexander polynomial $\Delta_{S^3}(K_f)(t)$ of any irreducible plane curve singularity $f$, fact which will be establish in Proposition 5.4.

(2) It is really interesting and remarkable, that the behavior of the modified Seiberg-Witten invariant with respect to (some) splicing and cyclic covers (constructions, which basically are topological in nature) definitely gives preference to the Alexander polynomials of some algebraic links. The authors hope that a better understanding of this phenomenon would lead to some deep properties of the Seiberg-Witten invariant.

4.13. Example. In general, in 4.11, the invariant $D(\Delta_{M_1}(K_1)(t))$ does not vanish. In order to see this, start for example with a pair $(N_1, L_1)$ with non-zero $D(\Delta_{N_1}(L_1)(t))$, and consider the case when $d|n$. (If $d \neq 1$, then the coefficient of $D(\Delta_{N_1}(L_1)(t))$ in 4.11(C) will be non-zero as well.)

Next, we show how one can construct a pair $(N, L)$ which satisfies WA2, $H_1(N) = 0$, but $D(\Delta_{N}(L)(t)) \neq 0$. First, we notice the following fact.

If the Alexander polynomial $\Delta_{M}(K)(t)$ is realizable for some pair $(M, K)$ (satisfying WA2 and $H_1(M) = 0$), then the $k$-power of this polynomial is also realizable for some pair $(M^k, K^k)$ (satisfying WA2 and $H_1(M^k) = 0$). Indeed, assume that $(M, K)$ has a schematic splice diagram of the following form:

```
Γ →
```

Then let $(M^k, K^k)$ be given by the following schematic splice diagram:

```
Γ →
```

Here, we take $q$ sufficiently large (in order to assure that the new edges will also satisfy the algebraicity condition 10(9.4)), and also $q$ should be relative prime with some integers which appear as decorations of $\Gamma$ (see [loc. cit.]). Obviously, by construction, $M^k$ is an integral homology sphere. Then, by [loc. cit.] 12.1, one can easily verify that

$$\Delta_{M^k}(K^k)(t) = \Delta_{M}(K)(t)^k.$$  

For example, if $(M, K) = (S^3, K_f)$, where $K_f$ is the $(2,3)$-torus knot (or, equivalently, the knot of the plane curve singularity $f = x^2 + y^3$, cf. 5.4), then $\Delta_{S^3}(K_f)(t) = t - 1 + 1/t$ (see 5.1(5)). Now, if we take $k = 2$ and $q = 7$ then $(M^2, K^2)$ has the following splice, respectively plumbing graph:
Then \((M^2, K^2)\) is algebraic, \(H_1(M^2) = 0\). But \(\Delta_{M^2}^2(K^2)(t) = (t - 1 + 1/t)^2\) whose coefficients are not alternating. In fact, \(r = 2, c_1 = -2\) and \(c_2 = 1\); in particular \(\mathcal{D}(\Delta_{M^2}^2(K^2)(t)) = 2\).

We end this section with the following property which is needed in the last section.

4.14. **Lemma.** Assume that \(M\) satisfies WA1’ and WA2’ with \((M^{(i)}_1, K^{(i)}_1) = (M_1, K_1)\). Let \(\Gamma\) denote the plumbing graph of \(M\). Let \(v\) be the central vertex of \(M_2\) considered in \(M\), and let \(\Gamma_\ast\) be that connected component of \(\Gamma \setminus \{v\}\) which contains the vertices of \(M^{(1)}_1\). Then \(|\det(\Gamma_\ast)| = a \cdot |H_1(M_1)|\).

**Proof.** If \(I\) denotes the intersection matrix of \(M\), then

\[-I_{vv}^{-1} = Lk_M(g_v, g_v) = Lk_{\Sigma(p,a,n)}(O,O)\]

On the other hand, \(I_{vv}^{-1}\) can be computed from the determinants of the components of \(\Gamma \setminus \{v\}\), hence (cf. also with \(\ref{(12)}\))

\[-|H_1(M)| \cdot I_{vv}^{-1} = |\det(\Gamma_\ast)|^d \cdot pm/d^2.\]

By \(\ref{(7)}\) and \(\ref{(e)}\) \(|H_1(M)| = |H_1(M_1)|^d \cdot a^{d-1}\), hence the result follows. \(\square\)

5. **Properties of irreducible plane curve singularities**

5.1. **The topology of an irreducible plane curve singularity.** Consider an irreducible plane curve singularity \(f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)\) with Newton pairs \((p_k, q_k)\) \(k = 1\) (cf. \(\ref{[10]}\), page 49). Clearly \(\gcd(p_k, q_k) = 1\) and \(p_k \geq 2\) and \(q_k \geq 2\). Define the integers \(\{a_k\}_{k=1}^s\) by

\[a_1 = q_1\] and \(a_{k+1} = q_{k+1} + p_{k+1}p_ka_k\) if \(k \geq 1\). \(\quad(1)\)

Then again, \(\gcd(p_k, a_k) = 1\) for any \(k\). The minimal (good) embedded resolution graph of the pair \((\mathbb{C}^2, \{f = 0\})\) has the following schematic form:

\[
\begin{array}{cccccc}
\bar{v}_0 & v_1 & v_2 & \cdots & v_{s-1} & v_s \\
\bar{v}_1 & \bar{v}_2 & \bar{v}_{s-1} & \bar{v}_s & K_f
\end{array}
\]

This can be identified with the plumbing graph \(\Gamma(S^3, K_f)\), where \(K_f\) is the link of \(f\) (with only one component) in the Milnor sphere \(S^3\). In the above diagram we emphasized only those vertices \(\{\bar{v}_k\}_{k=0}^s\) and \(\{v_k\}_{k=1}^s\) which have \(\delta \neq 2\). We denote the set of these vertices by \(\mathcal{V}^*\). The dash-line between two such vertices replaces a string \(\cdots\). In our discussion the corresponding self-intersection (or Euler) numbers will be not important (the interested reader can find the complete description of the graph in \(\ref{[10]}\) section 22, or in \(\ref{[30]}\)). The above numerical data \(\{(p_k, a_k)\}_k\) and the set of vertices \(\mathcal{V}^*\) is codified in the splice diagram (cf. \(\ref{(1)}\)).
It is clear that the coefficients of a polynomial of this form are alternating. Recall that the characteristic polynomial $\Delta(f)(t) := \Delta_{S^3}(K_f)(t)$ of the monodromy acting on the first homology of the Milnor fiber of $f$ is given by A’Campo’s formula 2.6(8):
\[
\frac{\Delta_{S^3}(K_f)(t)}{t - 1} = \prod_{v \in \mathbb{V}^*} (t^{\omega_v} - 1)^{\delta_v - 1}. \tag{3}
\]

In inductive proofs and constructions (over the number of Newton pairs of $f$), it is convenient to use the notation $f(l)$ for an irreducible plane curve singularity with Newton pairs $\{(p_k, q_k)\}_{k=1}^l$, where $1 \leq l \leq s$. Evidently $f(s) = f$, and $f(1)$ can be taken as the Brieskorn singularity $x^{p_1} + y^{q_1}$. We write $\Delta(f(l))$ for the characteristic polynomial associated with $f(l)$. Then from (2) and (3) one gets
\[
\Delta(f(l))(t) = \Delta(x^{p_1} + y^{q_1})(t) \cdot \Delta(f(l-1))(t^{p_1}) \quad \text{for } l \geq 2, \tag{4}
\]
where
\[
\Delta(x^{p_1} + y^{q_1})(t) = \frac{(t^{p_1} - 1)(t - 1)}{(t^{p_1} - 1)(t - 1)}. \tag{5}
\]

By induction, using the identities (1), one can prove (see e.g. [29, 5.2])
\[
a_l > p_l \cdot \deg \Delta(f(l-1)) \quad \text{for any } l \geq 2. \tag{6}
\]

5.2. **Proposition.**

(a) $\Delta(f)(0) = \Delta(f)(1) = 1$, and the degree of $\Delta(f)(t)$ is even (say $2r$). 
(b) If $\Delta(f)(t) = \sum_{i=0}^{2r} b_i t^i$, then the set $\{b_i\}_{i=0}^{2r}$ is alternating (cf. [4.7(4)]). 
(c) The coefficients $\{c_i\}_{i=1}^{2r}$ of $\Delta^s(f)(t) := t^{-r} \Delta(f)(t)$ (cf. [4.7(3)]) are alternating as well.

**Proof.** (a) is clear from (4) and (5), and (c) follows easily from (b). We will prove (b) by induction over $s$. For each $1 \leq l \leq s$ we verify that there exist

(i) $a_l$-residue classes $\{r_1, \ldots, r_t\} \subset \{1, 2, \ldots, a_l - 1\}$ (where $t$ may depend on $l$); and 
(ii) integers $n_1, \ldots, n_t \in \mathbb{N}$, such that
\[
\Delta(f(l))(t) = 1 + \sum_{i=1}^{t} \sum_{j=0}^{n_i} t^{r_i + j a_i} \cdot (t - 1).
\]

It is clear that the coefficients of a polynomial of this form are alternating.

Let us start with the case $l = 1$. Write $(p_1, a_1) = (p, a)$. Then (cf. (5))
\[
\Delta(x^p + y^a)(t) = (t^{p(a-1)} + \cdots + t^p + 1)/Q(t), \quad Q(t) := t^{a-1} + \cdots + t + 1.
\]
5.3. R are all different from the residue classes \( \{Z_i\}_i \) are torsion of reducible plane curve singularity will be crucial in the computation of the Reidemeister-Turaev expression reads as

\[
\sum_{i=0}^{a-1} t^{p_i} = Q(t) + \sum_{i: p \mid r_i} t^{r_i} (t^{x_i - 1}) = Q(t) \cdot \left[ 1 + \sum_{i: p \mid r_i} t^{r_i + ja_i}(t - 1) \right].
\]

Now we prove that \( \Delta(f(t)) \) has a similar form. By the inductive step, assume that

\[
\Delta(f(t-1))(t) = 1 + \sum_{i=1}^{t} \sum_{j=0}^{n_i} t^{r_i + ja_i}(t - 1).
\]

Then, using 5.3(4) and (5), one gets for \( \Delta(f(t)) \):

\[
\frac{(tp_{ai} - 1)(t - 1)}{(tp_i - 1)(ta_i - 1)} + \sum_{i=1}^{t} \sum_{j=0}^{n_i} t^{r_i + ja_i}(p_i) \cdot \frac{(tp_{ai} - 1)(t - 1)}{ta_i - 1}.
\]

Let \( \{s_j\}_{j=0}^{a_i-1} \) be the set of \( a_i \)-residues classes. Then (using the result of case \( l = 1 \)) the above expression reads as

\[
1 + \sum_{j: p_i \mid s_j} \sum_{k=0}^{x_j - 1} t^{s_j + ka_i}(t - 1) + \sum_{i=1}^{t} \sum_{j=0}^{n_i} \sum_{k=0}^{p_i - 1} t^{r_i + ja_i}(p_i + ka_i) \cdot (t - 1).
\]

Notice that 5.1(6) guarantees that for each \( i \) and \( j \) one has the inequality \( (r_i + ja_i)p_i < a_i \), hence these numbers can be considered as (non-zero) \( a_i \)-residues classes. Moreover, they are all different from the residue classes \( \{s_j : p_i \mid s_j\} \) since they are all divisible by \( p_i \).

The “alternating property” of the coefficients of the Alexander polynomial of any irreducible plane curve singularity will be crucial in the computation of the Reidemeister-Turaev torsion of \( \{f + z^n = 0\} \). The key algebraic fact is summarized in the next property:

5.3. Algebraic Lemma. In the next expressions \( t \) is a free variable and \( a \) is a positive integer. \( \mathbb{Z}_a \) is identified with the \( a \)-roots of unity. Assume that the coefficients of a polynomial \( \Delta(t) \in \mathbb{Z}[t] \) form an alternating set, \( \Delta(1) = 1 \), and \( a \geq \deg \Delta \). Then:

(a) For an arbitrary complex number \( A \) one has:

\[
\frac{1}{a} \sum_{\xi \in \mathbb{Z}_a} \Delta(\xi^t) \left( \frac{1}{1 - \xi^t} \right) \cdot \frac{\Delta(\xi^t) \cdot \Delta(At^2)}{1 - \xi^t} = \frac{(1 - A^{a}t^{2a}) \cdot \Delta(At^2)}{(1 - t^a)(1 - A^a t^a)(1 - At^2)}.
\]

(b) For arbitrary integers \( d \geq 2 \) and \( k \geq 1 \) one has:

\[
\frac{1}{a^{d-1}} \left[ \sum_{\xi \in \mathbb{Z}_a, \xi^{d-1} \neq 1} \Delta(\xi^t) \right] \cdot \left[ \sum_{\xi \in \mathbb{Z}_a, \xi^{d-1} \neq 1} \Delta(\xi^d - 1) \right] \cdot \left[ \sum_{\xi \in \mathbb{Z}_a, \xi^{d-1} \neq 1} \Delta(\xi^d - 1) \right] = \frac{(1 - t^{a(d+k-1)}) \cdot \Delta(t^{d+k-1})}{(1 - t^a)(1 - t^a)(1 - t^{d+k-1})}.
\]

Proof. The assumption about \( \Delta(t) \) guarantees that one can write \( \Delta(t) = 1 - R(t)(1 - t) \) for some \( R(t) = \sum_{j \geq 1} b_j t^j \) with \( b_j \in \{0, 1\} \) for all \( j \). Then the left hand side of (a) is

\[
\frac{1}{a} \sum_{\xi \in \mathbb{Z}_a} \frac{1}{1 - \xi^t} - \frac{1}{a} \sum_{\xi \in \mathbb{Z}_a} \frac{R(\xi^t)}{1 - \xi^t} - \frac{1}{a} \sum_{\xi \in \mathbb{Z}_a} \frac{R(\xi^t)}{1 - \xi^t} + \frac{1}{a} \sum_{\xi \in \mathbb{Z}_a} R(\xi^t) \cdot R(\xi^t).
\]
The first sum (with the coefficient 1/a) can be written in the form
\[
\frac{1}{a} (1 + \xi t + \xi^2 t^2 + \cdots)(1 + \bar{\xi} A t + \bar{\xi}^2 A^2 t^2 + \cdots) = \frac{1}{a} \sum_{n \geq 0} \sum_{j=0}^{n} \xi^{n-2j} A^j t^n.
\]
This, by an elementary computation gives:
\[
\frac{1 - A^a t^{2a}}{(1 - t^a)(1 - A^a t^a)(1 - A t^2)}.
\]
The second term gives \( R(At^2)/(1 - t^a) \). In order to prove this, first notice that the formula is additive in the polynomial \( R \), hence it is enough to verify for \( R(t) = t^k \) for all \( 0 \leq k < a \). The case \( k = 0 \) is easy, it is equivalent with the identity
\[
\frac{1}{a} \sum_{\xi} \frac{1}{1 - \xi t} = \frac{1}{1 - t^a}.
\]
If \( 1 \leq k < a \), then write
\[
\frac{1}{a} \sum_{\xi} \bar{\xi}^k A^k t^k = \frac{A^k t^{2k}}{a} \left[ \sum_{\xi} \frac{1}{1 - \xi t} + \sum_{\xi} (\xi t)^{-1} + \cdots + (\xi t)^{-k} \right].
\]
Since \( k < a \) the last sum is zero (here \( k < a \) is crucial!), hence (*) gives the claimed identity.

By similar method, the third term is \( R(At^2)/(1 - A^a t^a) \). Finally, the forth is \( R(At^2) \) (here one needs to apply the alternating property, namely that \( \tilde{b}_j = \tilde{b}_j \) for any \( j \)).

For part (b), use (a) and induction over \( d \). For this, write \( \xi_d \) for \( \xi_1 \cdots \xi_{d-1} \) and use (a) for \( \xi = \xi_{d-1} \) and \( A = \xi_1 \cdots \xi_{d-2} t^{k-1} \). Then apply the inductive step. \( \Box \)

5.4. **Remarks.**

1. Let \( \Delta(t) \) and \( a \) be as in §3. The expression in §3(a), with \( A = 1 \), has a pole of order 2 at \( t = 1 \). This comes from the pole of the summand given by \( \xi = 1 \). Therefore:
\[
\frac{1}{a} \sum_{\xi \in \mathbb{Z}_n} \frac{\Delta(\xi)}{1 - \xi} \cdot \frac{\Delta(\bar{\xi})}{1 - \bar{\xi}} = \lim_{t \to 1} \left[ \frac{(1 - t^{2a}) \cdot \Delta(t^2)}{(1 - t^a)^2 (1 - t^2)} - \frac{\Delta(t)^2}{a(1 - t)} \right]
\]
\[
= \frac{a^2 - 1}{12a} + \frac{1}{a} \left[ \Delta'(1) - \Delta'(1)^2 + \Delta''(1) \right],
\]
where the first equality follows from §3, the second by a computation.

2. Assume that \( \Delta(t) \) is an arbitrary symmetric polynomial of degree \( 2r \), and write \( \Delta(t) = t^{-r} \Delta(t) \). Then it is easy to show that
\[
\Delta'(1) = r \Delta(1), \quad (\Delta^2)'(1) = 0, \quad \text{and} \quad (\Delta^2''(1) = (r - r^2) + \Delta'(1)/\Delta(1).
\]

3. If one combines (1) and (2), then for a symmetric polynomial \( \Delta(t) \) with alternating coefficients and with \( \Delta(1) = 1 \) one gets
\[
\sum_{\xi \in \mathbb{Z}_n} \frac{\Delta(\xi)}{1 - \xi} \cdot \frac{\Delta(\bar{\xi})}{1 - \bar{\xi}} = \frac{a^2 - 1}{12} + (\Delta^2''(1) (a \geq \deg \Delta).
\]
This reproduces the vanishing of \( D_a \) in §4.9(D) for such polynomials (cf. also with §4.3(**)).

4. Although the Alexander polynomial \( \Delta(f) \) of the algebraic knot \( (S^3, K_f) \) (\( f \) irreducible plane curve singularity) is known since 1932 \([7, 43]\), and it was studied intensively (see e.g. §20 [1, 16]), the property §2 remained hidden (to the best of the author’s knowledge).
On the other hand, similar properties were intensively studied in number theory: namely in 40's, 50's and 60's a considerable large number of articles were published about the coefficients of cyclotomic polynomials. Here we mention only a few results. If $\phi_n$ denotes the $n^{th}$-cyclotomic polynomial, then it was proved that the coefficients of $\phi_n$ have values in $\{-1, 0, +1\}$ for $n = 2^\alpha p^\beta q^\gamma$ ($p$ and $q$ distinct odd primes) (result which goes back to the work of I. Schur); if $n$ is a product of three distinct primes $pqr$ ($p < q < r$ and $p + q > r$) then the coefficient of $t^r$ in $\phi_n$ is $-2$ (result of V. Ivanov); later Erdős proved interesting estimates for the growth of the coefficients; and G.S. Kazandzidis provided exact formula for them. The interested reader can consult [22], pages 404-411, for a large list of articles about this subject. (Reading these reviews, apparently the alternating property was not perceived in this area either.)

Clearly, the above facts are not independent of our problem: by 5.1(3) the Alexander polynomial $\Delta(f)$ is a product of cyclotomic polynomials.

(5) In fact, there is a recent result [14] in the theory of singularities, which implies the alternating property 5.2(b). For any irreducible curve singularity, using its normalization, one can define a semigroup $S \subset \mathbb{N}$, with 0 $\in$ $S$ and $\mathbb{N}\setminus S$ finite. Then, in [14], based on some results of Zariski, for an irreducible plane curve singularities is proved that $\Delta(f)(t)/(1-t) = \sum_{i \in S} t^i$. This clearly implies the alternating property.

(6) Are the irreducible plane curve singularities unique with the alternating property? The answer is negative. In order to see this, consider the Seifert integral homology sphere $\Sigma = \Sigma(a_1, a_2, \ldots, a_{k+1})$ (where $\{a_i\}$ are pairwise coprime integers). Let $K$ be the special orbit associated with the last arm (with Seifert invariant $a_{k+1}$). Then the Alexander polynomial $\Delta_{\Sigma}(K)$ has alternating coefficients. Indeed, write $a := a_1 \cdots a_k$, $a_i' := a/a_i$ for any $1 \leq i \leq k$, and let $S \subset \mathbb{N}$ be the semigroup (with 0 $\in$ $S$) generated by $a_1', \ldots, a_k'$. Then

$$\Delta_{\Sigma}(K)(t) = \frac{(1 - t^a)^{k-1}(1-t)}{\prod_{i=1}^{k}(1-t^{a_i})} = (1-t) \sum_{i \in S} t^i.$$  

The first equality follows e.g. by [10], section 11; the second by an induction over $k$. This implies the alternating property as above.

(7) We can ask the following natural question: what is that property which distinguishes $(M, K_f)$ (where $f$ is an irreducible plane curve singularity), or $(\Sigma, K)$ given in (6), from the example described in 4.13? Why is it in the first case the $D$-invariant zero and in the second case not? Can this be connected with some property of the semigroups $S$ associated with the curve whose link is $K$? (We believe that the validity of an Abhyankar-Azevedo type theorem for this curve plays an important role in this phenomenon.)

(8) Examples show that the assumptions of 5.3 are really essential (cf. also with 6.8(2)).

6. The link of $\{f(x, y) + z^n = 0\}$

6.1. Preliminaries. The present section is more technical than the previous ones, and some of the details are left to the reader, which might cost the reader some work.

Fix an irreducible plane curve singularity $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ and let $K_f \subset S^2$ be its link as in the previous section. Fix an integer $n \geq 1$, and consider the “suspension” germ $g : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ given by $g(x, y, z) = f(x, y) + z^n$. Its link (i.e. $\{g = 0\} \cap S^2$ for $\epsilon \ll 1$) will be denoted by $M$. We will assume that $M$ is a rational homology sphere, cf. 3.2(c).

First, we recall/fix some numerical notations. We set:

- the Newton pairs $\{(p_k, q_k)\}_{k=1}^s$ of $f$;
- the integers $\{a_k\}_{k=1}^s$ defined as in 5.1(1); recall that $\gcd(p_k, a_k) = 1$ for any $k$.
\begin{itemize}
\item $d_k := \gcd(n, pk+1p_{k+2} \cdots p_s)$ for $0 \leq k \leq s - 1$, and $d_s := 1$;
\item $h_k := d_{k-1}/d_k = \gcd(p_k, n/d_k)$ for $1 \leq k \leq s$;
\item $\tilde{h}_k := \gcd(a_k, n/d_k)$ for $1 \leq k \leq s$.
\end{itemize}

For any integer $1 \leq l \leq s$, let $M(l)$ be the link of the suspension singularity $g(l)(x, y, z) := f(l)(x, y) + z^{n/d_l}$. Evidently, $M(s) = M$, and $M(1) = \Sigma(p_1, a_1, n/d_1)$.

### 6.2. Some properties of the 3-manifolds \{M(l)\}_l

(a) \cite{10, 11, 13, 14} For each $1 \leq l \leq s$, $M(l)$ is the $(n/d_l)$-cyclic cover of $S^3$ branched along $\{f_l = 0\}$. Let $K(l) \subset M(l)$ be the preimage of $\{f_l = 0\}$ with respect to this cover.

(b) $(M(l), K(l))$ can be represented by (a “canonical”) plumbing graph (or resolution graph) which is compatible with the above cover. This is made explicitly in \cite{28} (based on the idea of \cite{15}), see also \cite{3} here. Using this one obtains the following inductive picture.

For any $2 \leq l \leq s$, $M(l)$ can be obtained by splicing, as it is described in section 4, the 3-manifold $M_2 = \Sigma(p_l, a_l, n/d_l)$ along $K_2 = \{y = 0\}$ with $h_l$ copies of $M(l-1)$ along the link $K(l-1)$ (with the same splicing data $\{A^{(i)}\}_i$ as in section 4). (In order to prove this, one needs to determine the invariant $M_n$ used in \cite{28}; this is done in \cite{3}, in the proof of (3.2).)

(c) Part (b) assures that $M$ is a rational homology sphere if and only if for each $1 \leq l \leq s$ the Seifert 3-manifold $\Sigma(p_l, a_l, n/d_l)$ is a rational homology sphere. Since $\gcd(p_l, a_l) = 1$, this is happening if and only if

$$(h_l - 1)(\tilde{h}_l - 1) = 0,$$

for any $l$ (cf. \cite{0} or \cite{28}).

(d) Using (b) and \cite{3}, one has:

$$|H_1(M(l))| = |H_1(\Sigma(p_l, a_l, n/d_l))| + h_l \cdot |H_1(M(l-1))|$$

for any $2 \leq l \leq s$.

or

$$|H_1(M)| = \sum_{l=1}^{s} d_l \cdot |H_1(\Sigma(p_l, a_l, n/d_l))|.$$

In fact, one can give a complete description of the group $H_1(M)$ and the character group $H_1(M)$ using \cite{5} (we will come back to this in \cite{3}).

(e) As a parallelism, let us recall some similar formulae for other numerical invariants: let $\mu(l)$, respectively $\sigma(l)$, be the Milnor number, respectively the signature of the Milnor fiber of $g(l)$. Similarly, let $\mu(p_l, a_l, n/d_l)$ and $\sigma(p_l, a_l, n/d_l)$ be the Milnor number and the signature of the Brieskorn singularity $x^{p_l} + y^{a_l} + z^{n/d_l}$. Then, by \cite{29}, for any $2 \leq l \leq s$:

$$\sigma(l) = \sigma(p_l, a_l, n/d_l) + h_l \cdot \sigma(l-1) \quad \text{or} \quad \sigma(s) = \sum_{l=1}^{s} d_l \cdot \sigma(p_l, a_l, n/d_l).$$

By contrast, for the Milnor numbers one has $\mu(l) = \mu(p_l, a_l, n/d_l) + p_l \cdot \mu(l-1)$ (involved $p_l$ versus $h_l$, fact which follows e.g. from \cite{5, 4}).

Our goal is to establish an inductive formula for $\text{sw}_M^0(\sigma_{\text{can}})$, similar to $|H_1|$ or to the signature $\sigma(l)$. (For $\lambda_{\text{geom}}$ or $\mathcal{M}_{\text{geom}}(1)$ such a formula does not hold, see below.)

(f) \cite{0, 10, 13, 14} $(S^3, K(l))$ is fibable. Let $\mathcal{M}_{\text{geom},(l)} : F(l) \to F(l)$ (respectively $\mathcal{M}(l)$) be a geometric (respectively the algebraic) monodromy acting on the Milnor fiber $F(l)$ (respectively on $H_1(F(l))$. Then $(M(l), K(l))$ is also fibable, whose open book decomposition has the same fiber $F(l)$ and geometric monodromy $\mathcal{M}_{\text{geom},(l)}^{n/d_l}$. In particular, the (normalized)
Alexander polynomial of \((M(t), K(t))\) is (the normalization of)

\[
\Delta_{M(t)}(t) = \Delta_{M(t)}(K(t))(t) = \det(1 - tM^{n/d}(t)).
\]

Therefore, using 2.6(7) and 2\(r_{i} := \text{rank} H_{1}(F(t)):\)

\[
\Delta_{M(t)}^{z}(t) = \frac{1}{|H_{1}(M(t))|} \cdot t^{-n} \cdot \det(1 - tM^{n/d}(t)) \quad \text{with} \quad |H_{1}(M(t))| = |\det(1 - M^{n/d}(t))|.
\]

Notice that \(\Delta_{M(t)}^{z}(t)\) can be deduced from the Alexander polynomial

\[
\Delta(f(t))(t) = \det(1 - tM(t)).
\]

of \(f(t)\) (cf. section 5). Indeed, for any polynomial \(\Delta(t)\) of degree \(2r\) and of the form

\[
\Delta(t) = \prod_{v} (1 - t^{m_{v}})^{n_{v}},
\]

and for any positive integer \(k\), define

\[
\Delta^{c(k)}(t) := \prod_{v} (1 - t^{m_{v}/\gcd(m_{v}, k)})^{n_{v}} \cdot \gcd(m_{v}, k).
\]

Let \(\Delta^{c(k), z}(t) = t^{-r} \Delta^{c(k)}(t)/\Delta^{c(k)}(1)\) denote the normalization of \(\Delta^{c(k)}(t)\). An eigenvalue-argument then proves:

\[
\Delta_{M(t)}(t) = \Delta(f(t))^{c(n/d)}(t) \quad \text{and} \quad \Delta_{M(t)}^{\natural}(t) = \Delta(f(t))^{c(n/d)}(1).
\]

(g) The inductive formula 5.4(4) reduces the computation of the Alexander invariants to the Seifert case. Clearly (from 5.1(5), 11.2(c) and (f) above):

\[
\Delta(x^{p_{1}} + y^{a_{1}})^{c(n/d)}(t) = \frac{1}{a_{l}^{h-1} p_{l}^{-1}} \cdot t^{-a_{l} - 1} \cdot \left(\frac{(tp^{a_{1}}/(h_{l} h_{l})) - 1}{(tp^{a_{1}}/(h_{l}) - 1)} \right)^{h_{l}} \cdot (m_{l} - 1).\]

(Recall that \((h_{l} - 1)(h_{l} - 1) = 0\) for any \(l\).) Then, by a computation, one can show that

\[
(\Delta(x^{p_{1}} + y^{a_{1}})^{c(n/d)}(t))^{\nu}(1) = \frac{1}{12} \left(\frac{a_{l}^{2}}{h_{l} - 1} - 1\right) \left(\frac{p_{l}^{2}}{h_{l} - 1}\right).
\]

(h) Using (f) and 5.4(4) one gets:

\[
\Delta_{M(t)}^{\natural}(t) = \Delta(x^{p_{1}} + y^{a_{1}})^{c(n/d)}(t) \cdot \left[\Delta_{M(t-1)}^{\natural}(t^{p_{1}/h_{1}})\right]^{h_{l}}.
\]

Then, using \((\Delta^{\natural})^{\nu}(1) = 0\) (cf. 5.4(2)) and the result from (g), one obtains:

\[
(\Delta_{M(t)}^{\natural})^{\nu}(1) = \frac{1}{12} \left(\frac{a_{l}^{2}}{h_{l} - 1} - 1\right) \left(\frac{p_{l}^{2}}{h_{l} - 1}\right) + \frac{p_{l}^{2}}{h_{l}} \cdot (\Delta_{M(t-1)}^{\natural})^{\nu}(1).
\]

Therefore:

\[
(\Delta_{M(t)}^{\natural})^{\nu}(1) = \sum_{k=1}^{l} \frac{1}{12} \left(\frac{a_{k}^{2}}{h_{k} - 1} - 1\right) \left(\frac{p_{k}^{2}}{h_{k} - 1}\right) \cdot \frac{(p_{k+1} \cdots p_{l})^{2}}{h_{k+1} \cdots h_{l}}.
\]

(i) For any \(2 \leq l \leq s\), one has:

\[
\lambda_{W}(M(t)) = \lambda_{W}(\Sigma(p_{1}, a_{l}, n/d_{l})) + h_{l} \cdot \lambda_{W}(M(t-1))
\]

\[
+ \frac{n_{pl} h_{l} - 1}{d_{l} a_{l} h_{l}} \cdot \sum_{k=1}^{l-1} \frac{1}{12} \left(\frac{a_{k}^{2}}{h_{k} - 1} - 1\right) \left(\frac{p_{k}^{2}}{h_{k} - 1}\right) \cdot \frac{(p_{k+1} \cdots p_{l-1})^{2}}{h_{k+1} \cdots h_{l}}.
\]

Indeed, if \(h_{l} = 1\) then we have additivity as in 3.3(4), cf. also with the proof of 4.11, part B. If \(h_{l} = 1\), then apply 4.17 (see also 4.19) and (h) above.
For the value of the Casson-Walker invariant \(\lambda_W(\Sigma(p, a, n))\) of a Seifert manifold, see [21, 6.1.1] or [21, 5.4].

(j) Below in (k), we will compute \((\Delta_{M_{(l)}})''(1)\) in terms of \(\{ (\Delta(f_{(k)})^2)''(1) \}_{k \leq l}\). Clearly, one can obtain similar inductive formula for these \((\Delta(f_{(k)})^2)''(1)\) as that one in (h) by taking \(n = 1\). More precisely, for any \(2 \leq l \leq s\):

\[
(\Delta(f_{(l)})^2)''(1) = (a_l^2 - 1)(p_l^2 - 1)/12 + p_l^2 \cdot (\Delta(f_{(l-1)})^2)''(1).
\]

(k) The next (rather complicated) identity looks very artificial, but it is one of the most important formulae in this list. Basically, its validity is equivalent with the fact that the two correction terms \(O(T)\) and \(O(\lambda_W/2)\) are the same, cf. 6.13 and 6.14. For any \(2 \leq l \leq s\) one has:

\[
(\Delta_{M_{(l)}})''(1) = (\Delta(f_{(l)})^2)''(1) - \sum_{k=1}^{l} \frac{a_k^2 p_k^2}{h_k h_{k+1} \cdots h_1} \cdot A_k,
\]

where

\[
A_k := \frac{h_k(h_k - 1)}{a_k^2} \cdot \left[ \frac{a_k^2 - 1}{12} + (\Delta(f_{(k-1)})^2)''(1) \right] + \frac{\tilde{h}_k(\tilde{h}_k - 1)}{p_k^2} \cdot \frac{p_k^2 - 1}{12}.
\]

For the proof proceed as follows. Let \(E_{(l)}\) be the difference between the left and the right hand side of the identity. Then, using the inductive formulae (h) and (j) and the property \((h_k - 1)(\tilde{h}_k - 1) = 0\), by an elementary computation one can verify that \(E_{(l)} = -(p_l^2/h_l) \cdot E_{(l-1)} = 0\), and \(E_{(1)} = 0\). Then \(E_{(l)} = 0\) by induction.

(l) The last invariant we wish to determine is \(\mathcal{T}_{M_{(l)}, \sigma_{can}}(1)\). The computation is more involved and it is separated in the next subsections. The inductive formula for \(\mathcal{T}_{M_{(l)}, \sigma_{can}}(1)\) is given in 6.13.

6.3. Characters of \(H_1(M)\). In the computation of \(\mathcal{T}_{M, \sigma_{can}}(1)\) we plan to use 2.7(16). For this, we need to describe the characters \(\chi \in \hat{H} = H_1(M)\).

The group \(H\) can be determined in many different ways. For example, using the monodromy operator \(M = M_{(a)}\) of \(f\), \(H\) can be identified, as an abstract group, with \(\text{coker}(1 - M^n)\). The homology of the Milnor fiber of \(f\) and \(M\) have a direct sum decomposition with respect to the splicing (see e.g. [10] or [23]). This can be used to provide an inductive description of \(H\).

Nevertheless, we prefer to use 3.6. The main reason is that, in fact, we have to understand \(\hat{H}\) (rather than \(H\)) together with the description of the supports \(\{ v : \chi(g_v) \neq 1 \}\) for each character \(\chi \in \hat{H}\). For this, the discussion from section 3 is more suitable.

We consider again the “canonical” plumbing graph \(\Gamma(M)\) of \(M\) provided by the algorithm 2.8, cf. 3.2(b) here. In fact, that algorithm provides \(\Gamma := \Gamma(M, K_z)\), the plumbing graph of the 3-dimensional link \(M = \{ f(x, y) + z^n = 0\}\) with the knot \(K_z := \{ z = 0\}\) in it. Again, if we replace the strings by dash-lines, then one can represent \(\Gamma\) as a covering graph of \(\Gamma(S^3, K_f)\); for details, see [loc. cit.], cf. also with 6.2(b). If we denote this graph-projection by \(\pi\), then

\[
\#\pi^{-1}(v_k) = h_{k+1} \cdots h_s, \quad 1 \leq k \leq s,
\]

\[
\#\pi^{-1}(\tilde{v}_k) = \tilde{h}_k h_{k+1} \cdots h_s, \quad 1 \leq k \leq s, \quad \text{and}
\]

\[
\#\pi^{-1}(\tilde{v}_0) = h_1 \cdots h_s.
\]
(see §1 for notations about $\Gamma(S^3, K_f)$). In fact, there is a \( \mathbb{Z}_n \)-action on $\Gamma$ which acts transitively on each fiber of $\pi$, hence all the vertices above a given vertex $v \in \mathcal{V}$ of $\Gamma(S^3, K_f)$ are symmetric in $\Gamma$. In particular, their decorations and their numerical invariants (computed out of the graph $\Gamma$) are the same. Therefore, $\Gamma$ has the following schematic form:

Now, we consider the Seifert manifold $\Sigma := \Sigma(p, a, n)$ with $\gcd(p, a) = 1$.

If $\gcd(a, n) = 1$ and $\gcd(p, n) = d$, by §3, $H_1(\Sigma)$ is generated by the homology classes of $\{K_2^{(i)}\}_{i=1}^d$. For an arbitrary character $\chi$, we write $\chi(K_2^{(i)}) = \xi_i$. Here $\xi_i$ is an $a$-root of unity in $\mathbb{C}$ (shortly $\xi_i \in \mathbb{Z}_a$). Then $\chi$ is completely characterized by the collection $\{\xi_i\}_{i=1}^d$ which satisfy $\xi_i \in \mathbb{Z}_a$ for any $i$ and $\xi_1 \cdots \xi_d = 1$.

Notice that $\chi(O) = 1$, and $\chi$ is supported by those $d$ “arms” of the star-shaped graph which have Seifert invariant $a$ (i.e. $\chi(g_v) = 1$ for the vertices $v$ situated on the other arms).

In §1 is proved that for each non-trivial character $\chi$, in $\hat{P}_{\Sigma, \chi, u}(t)$ one can take for $u$ the central vertex $O$. Moreover, $\lim_{t \to 1} \hat{P}_{\Sigma, \chi, u}(t)$ can be non-zero only if $\chi$ is supported exactly on two arms, i.e. $\xi_i \neq 1$ exactly for two values of $i$, say for $i_1$ and $i_2$ (hence $\xi_{i_1} = \bar{\xi}_{i_2}$).

If $\gcd(n, p) = 1$, but $\gcd(n, a) \neq 1$, then clearly we have a symmetric situation; in this case we use the notation $\eta$ instead of $\xi$.

In both situation, for any character $\chi$, $\chi(g_v) = 1$ for any $v$ situated on the arm with Seifert invariant $n/\gcd(n, ap)$.

These properties proved for the building block $\Sigma$ will generate all the properties of $H = H_1(M)$ via the splicing properties §6 and linking relations §3 (L#).

For example, one can prove by induction, that for any character $\chi$ of $H$, $\chi(g_v) = 1$ for any vertex $v$ situated on the string which supports the arrow of $K_z$. 
Consider the splicing decomposition
\[ M = M_{(s)} = h_s M_{(s-1)} \prod \Sigma(p_s, a_s, n). \]

As in the previous inductive arguments, assume that we understand the characters \( \chi \) of \( H_1(M_{(s-1)}) \). By \( 3.6 \), they can be considered in a natural way as characters of \( H_1(M_{(s)}) \) satisfying additionally \( \chi(g_v) = 1 \) for any vertex \( v \) of \( \Sigma(p_s, a_s, n) \) (cf. also with the first paragraph of the proof of \( 3.9 \)). We say that these characters do not “propagate” from \( M_{(s-1)} \) into \( \Sigma(p_s, a_s, n) \).

In the “easy” case when \( h_s = 1 \) (even if \( \tilde{h}_s \neq 1 \)), the splicing invariants are \( o_1 = o_2 = 1 \) and \( k_1 = k_2 = 0 \), hence \( H \) (together with its linking form) is a direct sum in a natural way, hence the characters of \( \Sigma(p_s, a_s, n) \) (described above) will not propagate into \( M_{(s-1)} \) either.

On the other hand, if \( h_s > 1 \), then the non-trivial characters of \( \Sigma(p_s, a_s, n) \) do propagate into the \( h_s \) copies of \( M_{(s-1)} \). If \( \chi \) is such a character, with \( \chi(K_2^{(i)}) = \xi_i \), then analyzing the properties of the linking numbers in \( 3.6 \), we deduce that \( \chi \) can be extended into \( M \) in such a way that \( \chi(g_v(i)) = \xi_i \) for any vertex \( v(i) \) of \( M_{(s-1)} \), where \( l = Lk_{M_{(s-1)}}(g_v(i), K_1^{(i)}) \).

This fact can be proved as follows. Assume that \( \chi = \exp(Lk_\Sigma(L, \cdot)) \) for some fixed \( L \subset \Sigma \). Then \( \xi_i = \chi(K_2^{(i)}) = \exp(Lk_\Sigma(L, K_2^{(i)})) \). Therefore, \( \chi \) extended into \( M \) as \( \exp(Lk_M(L, \cdot)) \) satisfies (cf. \( 3.6 \) (11))
\[ \chi(g_v(i)) = \exp(Lk_M(L, g_v(i))) = \exp(Lk_\Sigma(L, K_2^{(i)}) \cdot Lk_{M_{(s-1)}}(K_1^{(i)}, g_v(i))) = \xi_i. \]

Clearly, all these linking numbers \( Lk_{M_{(s-1)}}(g_v(i), K_1^{(i)}) \) can be determined inductively by the formulae provided in \( 3.6 \) and \( 4.3 \) (L#).

Moreover, if we multiply such an extended character \( \chi \) with a character \( \chi' \) supported by \( M_{(s-1)} \), then \( \chi \chi'(g_v) = \chi(g_v) \) for any vertex \( v \) situated above the vertices \( v_s \) or \( v_{s-1} \) or above any vertex on the edge connecting \( v_{s-1} \) with \( v_s \) (since the support of \( \chi' \) does not contain these vertices). We will say that such a character \( \chi \chi' \) is “born at level \( s \)” (provided that the original \( \chi \) of \( \Sigma \) was non-trivial).

(The interested reader can reformulate the above discussion in the language of an exact sequence of dual groups, similar to one used in the first paragraph of the proof of \( 3.3 \).)

Below we provide some examples. In this diagrams, for any fixed character \( \chi \), we put on the vertex \( v \) the complex number \( \chi(g_v) \).

6.4. Example. Assume that \( s = 2 \). Then basically one has two different cases: \( \tilde{h}_1 = h_2 = 1 \), or \( \tilde{h}_1 = \tilde{h}_2 = 1 \) (since the first pairs \((p_1, a_1)\) can be permuted).

In the first (“easy”) case, the schematic diagram of the characters is:

In the second case \( \tilde{h}_1 = \tilde{h}_2 = 1 \), with the notation \( p'_1 := p_1/h_1 \), the characters \( \tilde{H} \) are as follows.
6.5. Using the discussion \[8\], the above example for \(s = 2\) can be generalized inductively to arbitrary \(s\). For the convenience of the reader, we make this explicit. In order to have a uniform notation, we consider the case \(\hat{h}_k = 1\) for any \(1 \leq k \leq s\). The interested reader is invited to write down similar description of the characters in those cases when \(\hat{h}_k \neq 1\) for some \(k\), using the present model and \[6.4\].

More precisely, for any character \(\chi\), we will indicate \(\chi(g_v')\) for any vertex \(v' \in \pi^{-1}(\mathcal{V}^*)\).

It is convenient to introduce the index set \((i_1, i_2, \ldots, i_s)\), where \(1 \leq i_l \leq h_l\) for any \(1 \leq l \leq s\). As we already mentioned, this set can be considered as the index set of \(\pi^{-1}(\mathcal{V}_0)\). Moreover, for any \(1 \leq k \leq s - 1\), \((i_{k+1}, \ldots, i_s)\) (where \(1 \leq i_l \leq h_l\) for any \(k \leq s \leq l \leq s\)) is the index set of \(\pi^{-1}(\mathcal{V}_k)\) (respectively of \(\pi^{-1}(\mathcal{V}_0)\) since \(\hat{h}_k = 1\)). Moreover, for any \(l\) we write \(p'_l := p_l/h_l\).

Next, we consider a system of roots of unity as follows:

- \(\xi_{i_s} \in \mathbb{Z}_{a_s}\) with \(\prod_{1 \leq i_s \leq h_s} \xi_{i_s} = 1\);
- for any fixed \(i_s\) a collection \(\xi_{i_{s-1}i_s} \in \mathbb{Z}_{a_{s-1}}\) with \(\prod_{1 \leq i_{s-1} \leq h_{s-1}} \xi_{i_{s-1}i_s} = 1\);
- and, more generally, if \(1 \leq k \leq s - 1\):
- for any fixed \((i_{k+1}, \ldots, i_s)\) a collection \(\xi_{i_k \ldots i_s} \in \mathbb{Z}_{a_k}\) with \(\prod_{1 \leq i_k \leq h_k} \xi_{i_k \ldots i_s} = 1\).

Then, any character \(\chi\) can be characterized by the following properties:

- \(\pi^{-1}(\mathcal{V}_s)\) contains exactly one vertex, say \(v'\). Then \(\chi(g_v') = 1\). The same is valid for \(\pi^{-1}(\mathcal{V}_k)\) and for that vertex in \(\Gamma\) which supports the arrow \(\{z = 0\}\).
- For any \(1 \leq k \leq s - 1\), if \(v'(i_{k+1}, \ldots, i_s)\) is the vertex in \(\pi^{-1}(\mathcal{V}_k)\) corresponding to the index \((i_{k+1}, \ldots, i_s)\), then
  \[
  \chi(g_v'(i_{k+1}, \ldots, i_s)) = \xi_{i_{k+1} \ldots i_s}^{a_k p_k'} \xi_{i_{k+2} \ldots i_s}^{a_k p'_k} \cdots \xi_{i_s}^{a_k p'_{s-1}}.
  \]
- Similarly, if \(v'(i_{k+1}, \ldots, i_s)\) is the vertex in \(\pi^{-1}(\mathcal{V}_k)\) corresponding to the index \((i_{k+1}, \ldots, i_s)\), then
  \[
  \chi(g_v'(i_{k+1}, \ldots, i_s)) = \xi_{i_{k+1} \ldots i_s}^{a_k p_k'} \xi_{i_{k+2} \ldots i_s}^{a_k p'_k} \cdots \xi_{i_s}^{a_k p'_{s-1}}.
  \]
- Finally, if \(v'(i_1, \ldots, i_s)\) is the vertex in \(\pi^{-1}(\mathcal{V}_0)\) corresponding to the index \((i_1, \ldots, i_s)\), then
  \[
  \chi(g_v'(i_1, \ldots, i_s)) = \xi_{i_{1} \ldots i_s}^{p_1'} \xi_{i_{2} \ldots i_s}^{p'_1} \cdots \xi_{i_s}^{p'_{s-1}}.
  \]

If \(\xi_{i_s} \neq 1\) for some \(i_s\) then \(\chi\) is “born at level \(s\)”. If \(\xi_{i_s} = 1\) for all \(i_s\), but \(\chi_{i_{s-1}i_s} \neq 1\) for some \((i_{s-1}, i_s)\), then \(\chi\) is “born at level \(s - 1\)”, etc.
In general, a character $\chi$ is “born at level $k$” ($1 \leq k \leq s$) if for any $l \geq k$ and $v' \in \pi^{-1}(v_l)$, one has $\chi(g_{v'}) = 1$, but there exists at least one vertex $v' \in \pi^{-1}(v_k)$ which is adjacent (in the graph $\Gamma$) with the support of $\chi$.

The next result will be crucial when we apply the Fourier inversion formula \eqref{10}. It is a really remarkable property of the links associated with irreducible plane curve singularities. It is the most important qualitative ingredient in our torsion computation (see also \ref{6.8} for another powerful application).

6.6. Proposition. Consider $(S^3, K_f)$ associated with an irreducible plane curve singularity $f$. Let $\Delta_{S^3}(K_f)(t)$ be its Alexander polynomial. For any integer $n \geq 1$, consider $(M, K_z)$, i.e. the link $M$ of $\{f(x, y) + z^n = 0\}$ and the knot $K_z := \{z = 0\}$ in it. Let $\Delta_{M,z}^H(K_z)(t)$ be the Alexander invariant defined in \eqref{4} \eqref{11} with $H = H_1(M)$. Then

$$\Delta_{M,z}^H(K_z)(t) = \Delta_{S^3}(K_f)(t).$$

Proof. First we notice that in \eqref{6}(10), $g_u = K_z$ and $o(u) = 1$. Then, by \eqref{3}, $m_{v'}(u) = Lk_M(g_{v'}, K_z)$ for any vertex $v'$ of $\Gamma$. By the algorithm \eqref{28}, for any vertex $v' \in \pi^{-1}(v)$, where $v \in V^*(\Gamma(S^3, K_f))$, the linking number $Lk_M(g_{v'}, K_z)$ is given by $w_{v'} = w_{v}/\gcd(w_{v}, n)$. Recall, that for any $v \in V^*(\Gamma(S^3, K_f))$, the corresponding weights $w_v$ are given in \eqref{5}(2). In particular, this discussion provides all the weight $w_{v'}(u)$ needed in the definition \eqref{6}(10) of $\Delta_{M,z}^H(K_z)(t)$. One the other hand, for characters $\chi \in H$ we can use the above description. Then the proposition follows (by some computation) inductively using the Algebraic Lemma, part (b), for the Alexander polynomials $\Delta(f_l)(0)$ ($0 \leq l \leq s - 1$, where $\Delta(f_0)(t) \equiv 1$). Notice that this lemma can be applied thanks to the proposition \ref{5} (which assures that the coefficients of $\Delta(f_l)(t)$ are alternating), and to the inequality \eqref{6}(6) (which assures that $a$ “is sufficiently large”). (For an expression of $\Delta(f_l)(t)$, see \ref{5}.)

In the next example we make this argument explicit for the case $s = 2$ and $h_1 = h_2 = 1$. Using this model, the reader can complete the general case easily.

6.7. Example. Assume that $s = 2$ and $h_1 = h_2 = 1$. Then, with the notation of \ref{4.1}, $\Delta_{M,z}^H(K_z)(t)$ equals

$$\prod_{\tilde{t}_2} \frac{1}{1 - t^{\tilde{p}_2} \xi_{\tilde{t}_2}} \prod_{\tilde{t}_2} \frac{(1 - t^{\tilde{a}_1} p_1 \tilde{p}_2 \xi_{\tilde{t}_2})^{\tilde{a}_1} p_1^{h_1}}{1 - t^{\tilde{a}_1} \tilde{p}_2 \xi_{\tilde{t}_2}} \cdot \frac{(1 - t^{\tilde{a}_2} \tilde{p}_2) h_2 \cdot (1 - t)}{1 - t^{\tilde{a}_2}}.$$

First, for each fixed index $\tilde{t}_2$, we make a sum over $\xi_{\tilde{t}_2} \in \mathbb{Z}_{a_1}$. Using \ref{5}(b) for $\Delta \equiv 1$, $t = t^{\tilde{p}_2} \tilde{p}_2 \xi_{\tilde{t}_2}$ and $a = a_1$ and $d = h_1$, the above expression transforms (after some simplifications) into:

$$\prod_{\tilde{t}_2} \frac{1 - t^{\tilde{a}_1} p_1 \tilde{p}_2 \xi_{\tilde{t}_2}}{(1 - t^{\tilde{a}_1} \tilde{p}_2 \xi_{\tilde{t}_2}) \cdot (1 - t^{\tilde{a}_1} \xi_{\tilde{t}_2})} \cdot \frac{(1 - t^{\tilde{a}_2} \tilde{p}_2) h_2 \cdot (1 - t)}{1 - t^{\tilde{a}_2}}.$$

The expression in the product is exactly $\Delta(f_{(1)})(\tilde{p}_2)/(1 - \tilde{p}_2)$. Therefore, \ref{5}(b) can be applied again, now for $\Delta = \Delta(f_{(1)})$, $t = \tilde{p}_2$, $a = a_2$ and $d = h_2$. Then the expression transforms into $\Delta(f_{(2)})$.

6.8. Remarks. (1) If the link $M$ of $\{f(x, y) + z^n = 0\}$ is a rational homology sphere, then in \eqref{26} we prove the following facts. Using the combinatorics of the plumbing graph of $M$, one can recover the knot $K_z$ in it. Then, by the above proposition, from the pair $(M, K_z)$ one can recover the Alexander polynomial $\Delta_{S^3}(K_f)$ of $f$. It is well-known that this is
For any vertex $v \in (\text{asy})$ case:
properties.

\[ \chi \]

of the graph of \( (S^3, M) \) of $g$ with its integral Seifert matrix.

(2) \[ \chi \]
suggests the following question. Let \( N \) be an integral homology sphere, \( L \subset N \) a knot in it such that \( (N, L) \) can be represented by a (negative definite) plumbing. Let \( (M, K) \) be the $n$-cyclic cover of \( (N, L) \) (branched along \( L \)) such that $M$ is a rational homology sphere with $H_1(M) = H$. Then is it true that $\Delta_M^H(K)(t) = \Delta_N(L)(t)$?

The answer is negative: one can construct easily examples (satisfying even the algebraicity condition) when the identity \[ \chi \]
 fails. For example, consider \( (N, L) \) given by the following splice, respectively plumbing diagram:

\[
\begin{align*}
2 & \quad 7 & \quad 3 & \quad 1 & \quad -2 & \quad -1 & \quad -4 \\
& \quad 5 & & & -5 & & -2
\end{align*}
\]

Then one can show that e.g. for $n = 2$ the identity $\Delta_M^H(K)(t) = \Delta_N(L)(t)$ fails.

This example also shows that in the Algebraic Lemma 5.3 the assumption $a \geq \deg \Delta$ is crucial. Indeed, in this example $H = \mathbb{Z}_3$; and in order to determine $\Delta_M^H(K)(t)$, one needs to compute a sum like in 5.3(a) with $a = 3$, $A = 1$ and $\Delta = t^4 - t^3 + t^2 - t + 1$ (i.e. with $a < \deg \Delta$). But for these data, the identity in 5.3(a) fails.

6.9. The Reidemeister-Turaev sign-refined torsion. Now we will start to compute $\mathcal{T}_{M, \sigma_{\text{can}}}(1)$ associated with $M = M(\sigma)$ and the canonical spin$^c$-structure $\sigma_{\text{can}}$ of $M$. Similarly as above, we write $H = H_1(M)$. Using 2.7(14), $\mathcal{T}_{M, \sigma_{\text{can}}}(1)$ can be determined by the Fourier inversion formula from $\{ \mathcal{T}_{M, \sigma_{\text{can}}} : \chi \in \hat{H} \setminus \{1\} \}$. On the other hand, each $\mathcal{T}_{M, \sigma_{\text{can}}} (\chi)$ is given by the limit $\lim_{t \to 1} \mathcal{P}_{M, \chi, u}(t)$ for some convenient $u$, cf. 2.7(16).

In the next discussion, the following terminology is helpful. Fix an integer $1 \leq k \leq s$ and a vertex $v'(I) := v'(i_{k+1}, \ldots, i_s) \in \pi^{-1}(v_k)$. Consider the graph $\Gamma \setminus \{v'(I)\}$. If $\tilde{h}_k = 1$ then it has $h_k + 2$ connected components: $h_k$ (isomorphic) subgraphs $\Gamma_{\tilde{h}}(v'(I))$ (1 $\leq i_k \leq h_k$) which contain vertices at level $k - 1$, a string $\Gamma_{\text{st}}(v'(I))$ containing a vertex above $\tilde{v}_k$, and the component $\Gamma_+(v'(I))$ which supports the arrow $\{z = 0\}$. Similarly, if $h_k = 1$, then $\Gamma \setminus \{v'(I)\}$ has $\tilde{h}_k + 2$ connected components, $\Gamma_-(v'(I))$ contains vertices at level $k - 1$, $\Gamma_+(v'(I))$ supports the arrow $\{z = 0\}$, and $\tilde{h}_k$ other (isomorphic) components $\Gamma_{\text{st}}(v'(I))$ (1 $\leq j_k \leq \tilde{h}_k$), which are strings, and each of them contains exactly one vertex staying above $\tilde{v}_k$.

Similarly as for the Seifert manifold $\Sigma(p, n, n)$ (see the discussion in 5.3 after the diagram), for a large number of characters $\chi$, the limit $\lim_{t \to 1} \mathcal{P}_{M, \chi, u}(t)$ is zero. Analyzing the structure of the graph of $M$ and the supports of the characters, one can deduce that a non-trivial character $\chi$, with the above limit nonzero, should satisfy one of the following structure properties.

\textbf{E(asy) case:} The character $\chi$ is born at level $k$ (for some $1 \leq k \leq s$) with $\tilde{h}_k > 1$. For any vertex $v' \in \pi^{-1}(v_k)$ one has $\chi(g_{v'}) = 1$, but there is exactly one vertex $v'(I) :=$
v'(i_{k+1}, \ldots, i_s) \in \pi^{-1}(v_k) which is adjacent with the support of χ. Moreover, χ is supported by exactly two components of type Γ^\text{v'} \in \pi^{-1}(v_k). Let v'(j'_k) be the unique vertex in Γ^\text{v'}(v(I)) ∩ π^{-1}(i_k) (similarly for j''_k). Then v'(j'_k) and v'(j''_k) are the only vertices v' of the graph of M with δ_{w'} \neq 2 and χ(g_{w'}) \neq 1. Moreover, χ(g_{w'}(\tilde{v}')) = \tilde{χ}(g_{w'}(\tilde{v}''_k)) = \eta \in Z_{p_k}^1. Therefore, with fixed (i_{k+1}, \ldots, i_s) and (j'_k, j''_k), there are exactly p_k - 1 such characters.

\textbf{D(ifficult) case:} The character χ is born at level k (for some 1 ≤ k ≤ s) with h_k > 1. For any vertex v' ∈ π^{-1}(v_k), χ(g_{v'}) = 1, but there is exactly one vertex v'(i_{k+1}, \ldots, i_s) ∈ π^{-1}(v_k) which is adjacent with the support of χ. The character χ is supported by exactly two components of type Γ^\text{v'}(v(I)), say for indices i'_k and i''_k. Using the previous notations, this means that ξ_{i_k,i_{k+1} \ldots i_s} = 1 excepting for i_k = i'_k or i_k = i''_k. (Evidently, ξ_{t \ldots i_s} = 1 for any t > k.) For t < k, the values ξ_{t \ldots i_s} are un-obstructed. In particular, with indices (i_{k+1}, \ldots, i_s) and (i'_k, i''_k) fixed, there are exactly (a_k - 1) |H_1(M_{k-1})|^2 such characters. Here, a_k - 1 stands for ξ_{i'_k,i_{k+1} \ldots i_s} = ξ_{i''_k,i_{k+1} \ldots i_s} \in Z^*_{a_k}, and |H_1(M_{k-1})|^2 for the un-obstructed characters born at level k on the two branches corresponding to (i'_k, i_{k+1}, \ldots, i_s) and (i''_k, i_{k+1}, \ldots, i_s).

In both cases (E) or (D), if such a character χ is born at level k (i.e. if satisfies the above characterization for k), then we write χ ∈ B_k.

Now, we fix a nontrivial character χ. Let S(χ) be the support of χ and S(χ) its complement. Then
\[
\frac{1}{|H|} \cdot \hat{\mathcal{T}}_{M,\sigma_{\text{can}}} (\tilde{\chi}) = \text{Loc}(\tilde{\chi}) \cdot \text{Reg}(\tilde{\chi}),
\]
where
\[
\text{Loc}(\tilde{\chi}) := \prod_{v' \in S(\chi)} (\chi(g_{v'}) - 1)^{\delta_{v'} - 2} \quad \text{and} \quad \text{Reg}(\tilde{\chi}) := \frac{1}{|H|} \cdot \lim_{t \to 1} \prod_{v' \in S(\chi)} (t^{u_{v'}(w)} - 1)^{\delta_{v'} - 2}.
\]
We will call \text{Loc}(\tilde{\chi}) the “local contribution”, while \text{Reg}(\tilde{\chi}) the “regularization contribution”.

By the above discussion, \text{Reg}(\tilde{\chi}) = 0 unless χ is not of the type (E) or (D) described above. If χ is of type (E) or (D) described as above, then in \hat{\mathcal{P}}_{M,\chi,u}(t) (cf. [27], (15)) one can take v'(I). Moreover, if χ ∈ B_k, then by the symmetry of the plumbing graph of M, \text{Reg}(\tilde{\chi}) does not depend on the particular choice of χ, but only on the integer k. We write \text{Reg}(k) for \text{Reg}(\chi) for some (any) χ ∈ B_k.

In particular,
\[
\mathcal{T}_{M,\sigma_{\text{can}}} (1) = \sum_{k=1}^{s} \text{Reg}(k) \cdot \sum_{\chi \in B_k} \text{Loc}(\tilde{\chi}). \tag{7}
\]

6.10. \textbf{Proposition.} For any fixed 1 ≤ k ≤ s one has:

\textbf{(E)} If h_k = 1 then
\[
\sum_{\chi \in B_k} \text{Loc}(\tilde{\chi}) = d_k \cdot \frac{\hat{h}_k(h_k - 1)}{2} \cdot \frac{p_k^2 - 1}{12}.
\]

\textbf{(D)} If \hat{h}_k = 1 then
\[
\sum_{\chi \in B_k} \text{Loc}(\tilde{\chi}) = d_k \cdot \frac{h_k(h_k - 1)}{2} \cdot |H_1(M_{k-1})|^2 \cdot \left[ \frac{a_k^2 - 1}{12} + (\Delta(\bar{f}_{k-1}))''(1) \right].
\]
Proof. (E) $d_k = h_{k+1} \cdots h_s$ is the cardinality of the index set $(i_{k+1}, \ldots, i_s)$, $\hat{h}_k(\hat{h}_k - 1)/2$ is the number of possibilities to choose the indices $(j'_k, j''_k)$. The last term comes from a formula of type 4.8(\text{**}), where the sum is over $\eta \in \mathbb{Z}_{p_k}$.

In the case (D), $d_k h_k(h_k - 1)/2$ has the same interpretation. Fix the branch $(i'_k, i_{k+1}, \ldots, i_s)$ and consider the sum over all the characters born at level $< k$. Then 6.6, applied for $(M_{(k-1)}, K_{(k-1)})$ as a covering of $(S^3, f_{(k-1)} = 0)$, provides $|H_1(M_{(k-1)})|^2 \cdot \Delta(f_{(k-1)}(t))/(t-1)$ evaluated at $t = \xi_{i'_k, i_{k+1} \cdots i_s}$. The same is true for the other index $i''_k$. Then apply 5.4(3) for $\Delta = \Delta(f_{(k-1)})$ and $a = a_k$. This can be applied because of 5.2 and 5.1(6). \hfill \square

6.11. The “regularization contribution” $Reg(k)$. Fix a character $\chi \in B_k$ of type (E) or (D) as in 6.1. Recall that one can take $u = v'(I)$. Consider the connected components of $\Gamma \setminus \{v'(I)\}$ (as in 5.3), where we add to each component an arrow corresponding to the edge which connects the component to $v'(I)$. For these graphs, if one applies 2.6(12), one gets that $Reg(k)$ is

$$Reg(k) = \begin{cases} - \det(\Gamma_-) \cdot \det(\Gamma_{st})^\hat{h}_k - 2 \cdot \det(\Gamma_+)/\det(\Gamma) & \text{case (E)} \\ - \det(\Gamma_-)^{h_k} \cdot \det(\Gamma_{st}) \cdot \det(\Gamma_+)/\det(\Gamma) & \text{case (D)}. \end{cases}$$

Let $I$ be the intersection matrix of $M$, and $I_k^{-1} := I_{v'v'}^{-1}$ for any $v' \in \pi^{-1}(v_k)$. Then, by the formula which provides the entries of an inverse matrix, one gets that

$$\det \Gamma \cdot I_{v'v'}^{-1} = \begin{cases} \det(\Gamma_-) \det(\Gamma_{st})^\hat{h}_k \det(\Gamma_+) & \text{case (E)} \\ \det(\Gamma_-)^{h_k} \det(\Gamma_{st}) \det(\Gamma_+) & \text{case (D)}. \end{cases}$$

These two facts combined show that

$$Reg(k) = \begin{cases} -I_k^{-1}/\det(\Gamma_{st})^2 = -I_k^{-1}/p_k^2 & \text{case (E)} \\ -I_k^{-1}/\det(\Gamma_-)^2 = -I_k^{-1}/(a_k \cdot |H_1(M_{(k-1)})|^2) & \text{case (D)} \end{cases}$$

since in case (E) $|\det(\Gamma_{st})| = p_k$ e.g. from 4.3(i)), and in case (D) $|\det(\Gamma_-)| = a_k \cdot |H_1(M_{(k-1)})|$ by 4.14.

This, together with 6.9 (F) and 6.10 transform into

$$\tau_{M, \sigma, \text{can}}(1) = - \sum_{k=1}^s I_k^{-1} \cdot d_k \cdot A_k/2,$$

where $A_k$ is defined in 5.2(k) in terms of the numerical invariants of $f_k$.

6.12. The computation of $I_k^{-1}$. For any $l \geq k$, let $I_k^{-1}(M(l))$ be the $(v', v')$-entry of the inverse of the intersection form $I(M(l))$ associated with $M(l)$, where $v'$ is any vertex above $v_k$. E.g. $I_k^{-1}(M_{(s)})$ is $I_k^{-1}$ used above.

By 4.3(5), $-I_k^{-1}(M(l)) = Lk_{M(l)}(g_{v'}, g_{v'})$. If $l = k$, by 3.6(12) this is $Lk_{\Sigma(p_k, a_k, n/d_k)}(O, O)$, hence by 1.3 (L\#) it is $np_k a_k/(d_k h_k^2 \hat{h}_k^2)$.

Next, assume that $l > k$. If $h_l = 1$, then the splicing

$$M(l) = h_l M_{(l-1)} \amalg \Sigma(p_l, a_l, n/d_l)$$

is trivial (with $o_1 = o_2 = 1$ and $k_1 = k_2 = 0$), hence by 3.6(10) one gets

$$-I_k^{-1}(M(l)) = -I_k^{-1}(M_{(l-1)}).$$
Seiberg-Witten invariants and surface singularities III

If $h_l > 1$, then 4.4 and an iterated application of 3.6(10) and 3.3 (L#) give

$$-I_k^{-1}(M(l)) = -I_k^{-1}(M(l-1)) - \left(\frac{a_k p_k}{h_k h_{k+1}} \cdots \frac{p_{l-1}}{h_{l-1}}\right)^2 \cdot \frac{np_l(h_l - 1)}{d a_l h_l^2}.$$  

Indeed, by 2.4(5) and 3.6

$$-I_k^{-1}(M(l)) = -I_k^{-1}(M(l-1)) - \left(L_k M(l-1)(g_\nu, K(l-1))\right)^2 \cdot \frac{np_l(h_l - 1)}{d a_l h_l^2},$$

and, again by 3.4, $L_k M(l-1)(g_\nu, K(l-1))$ equals

$$L_k \Sigma(p_{k+1}, a_{k+1}, n/d_k) (O, Z) \cdot L_k \Sigma(p_{k+1}, a_{k+1}, n/d_k + 1) (K_2^{(i)} , Z) \cdots L_k \Sigma(p_{l-1}, a_{l-1}, n/d_{l-1}) (K_2^{(i)} , Z).$$

6.13. The splicing formula for $\mathcal{M}_{\sigma_{can}}(1)$. Using 6.11(*) and 6.12 (and $d_s = 1$), we can write

$$\mathcal{M}_{\sigma_{can}}(1) - h_s \cdot \mathcal{M}_{\sigma_{can}}(1) = \frac{n a_s p_s}{2 h_s^2 h_s^2} \cdot A_s$$

$$+ \sum_{k=1}^{s-1} (-I_k^{-1}(M(s))) \cdot h_{k+1} \cdots h_s \cdot A_k/2 - h_s \sum_{k=1}^{s-1} (-I_k^{-1}(M(s-1))) \cdot h_{k+1} \cdots h_s \cdot A_k/2$$

$$= \frac{n a_s p_s}{2 h_s^2 h_s^2} \cdot A_s - \sum_{k=1}^{s-1} \frac{a_k^2 p_k^2 \cdots p_{s-1}^2}{h_k^2 h_{k+1}^2 \cdots h_{s-1}^2} \cdot \frac{np_s(h_s - 1)}{a_s h_s^2} \cdot A_k/2.$$  

But, by 4.2(e) one has

$$\frac{n a_s p_s}{2 h_s^2 h_s^2} \cdot A_s = \mathcal{T}_{\Sigma(p_s, a_s, n), \sigma_{can}}(1) + \frac{n a_s p_s}{2 h_s^2 h_s^2} \cdot \frac{h_s(h_s - 1)}{a_s^2} \cdot (\Delta(f(s-1))^2)''(1).$$

Therefore, (using also $(h_s - 1)/\tilde{h}^2 = h_s - 1$) one gets

$$\mathcal{O}(\mathcal{T}_{\sigma_{can}}(1)) = \mathcal{T}_{\sigma_{can}}(1) - h_s \cdot \mathcal{T}_{\sigma_{can}}(1) - \mathcal{T}_{\sigma_{can}}(1)$$

$$= \frac{np_s(h_s - 1)}{2 h_s a_s} \times \left[(\Delta(f(s-1))^2)''(1) - \sum_{k=1}^{s-1} \frac{a_k^2 p_k^2 \cdots p_{s-1}^2}{h_k^2 h_{k+1}^2 \cdots h_{s-1}^2} \cdot A_k\right].$$

Notice that this inductive formula (together with 4.2(e)) shows that the numerical function $n \mapsto \mathcal{T}(n)$ (where $\mathcal{T}(n)$ denotes the sign-refined Reidemeister-Turaev torsion of the link of $f + z^n$, associated with $\sigma_{can}$) can be written in the form $n \mapsto np_1(n) + p_2(n)$, where $p_i(n)$ are periodic functions. Similar result for the signature of the Milnor fiber was obtained by Neumann [33].

Now, using 4.9 for $M = M_{(s)}$, and 6.2(k) for $l = s - 1$, one has the following consequences:

6.14. Theorem. The additivity obstruction $\mathcal{O}(\text{sw}^0(\sigma_{can})) = 0$, in other words:

$$\text{sw}^0_{M_{(s)}}(\sigma_{can}) = h_s \cdot \text{sw}^0_{M_{(s)}}(\sigma_{can}) + \text{sw}^0_{\Sigma(p_s, a_s, n)}(\sigma_{can}).$$

In particular, by induction for $M = M_{(s)}$ one gets:

$$\text{sw}^0_{M_{(s)}}(\sigma_{can}) = \sum_{k=1}^{s} d_k \cdot \text{sw}^0_{\Sigma(p_k, a_k, n/d_k)}(\sigma_{can}).$$
6.15. **Corollary.** Consider the hypersurface singularity $g(x, y, z) = f(x, y) + z^n$, where $f$ is an irreducible plane curve singularity. Assume that its link $M$ is a rational homology sphere. If $\sigma(g)$ denotes the signature of the Milnor fiber of $g$, then

$$-\text{sw}^0_M(\sigma_{\text{can}}) = \sigma(g)/8.$$  

In particular, the geometric genus of $\{g = 0\}$ is topological and it is given by

$$p_g = \text{sw}^0_M(\sigma_{\text{can}}) - (K^2 + \#\mathcal{V})/8,$$

where the invariant $K^2 + \#\mathcal{V}$ (associated with any plumbing graph of $M$) is defined in the introduction.

**Proof.** By 6.14 an [29] (3.2) (cf. also with 6.2(e)), we only have to show that

$$-\text{sw}^0_{\Sigma(p_k, a_k, n/d_k)}(\sigma_{\text{can}}) = \sigma(p_k, a_k, n/d_k)/8$$

for any $k$. But this follows from [31] (section 6).

For an explicit formula of $\text{sw}^0_{\Sigma(p_k, a_k, n/d_k)}(\sigma_{\text{can}})$, see [31] or [32].

**References**

[1] N. A’Campo: *La fonction zeta d’une monodromy*, Com. Math. Helvetici 50 (1975), 233-248.

[2] V.I. Arnold, S.M. Gusein-Zade, S.M. and A.N. Varchenko: *Singularities of Differentiable Mappings*, Vol. 2, Birkhauser, Boston, 1988.

[3] M. Artin: *Some numerical criteria for contractibility of curves on algebraic surfaces*, Amer. J. of Math., 84, 485-496 (1962).

[4] _______ *On isolated rational singularities of surfaces*, Amer. J. of Math., 88, 129-136 (1966).

[5] S. Boyer and A. Nicas: *Varieties of group representations and Casson’s invariant for rational homology 3-spheres*, Trans. Am. Math. Soc. 332 (2) , 507-522 (1990).

[6] E. Brieskorn: *Beispiele zur Differentialtopologie von Singularit¨ aten*, Inventiones math. 2 (1966), 1-14.

[7] W. Burau: *Kennzeichnung der Schlauchknoten*, Abh. Math. Sem. Hamburg, 9 (1932), 125-133.

[8] W. Chen: *Casson invariant and Seiberg-Witten gauge theory*, Turkish J. Math., 21 (1997), 61-81.

[9] A. Durfee and L. Kauffman: *Periodicity of Branched Cyclic Covers*, Math. Ann., 218 (1975), 157-174.

[10] Eisenbud, D. and Neumann, W.: *Three-dimensional link theory and invariants of plane curve singularities*, Ann. of Math. Studies 110, Princeton Univ. Press (1985).

[11] R. Fintushel and R. Stern: *Instanton homology of Seifert fibered homology three spheres*, Proc. London Math. Soc., 61, 109-137 (1990).

[12] G. Fujita: *A splicing formula for the Casson-Walker’s invariant*, math. Annalen, 296, 327-338 (1993).

[13] H. Grauert: *Über Modifikationen und exceptionelle analytische Mengen*, Math. Ann., 146 (1962), 331-368.

[14] S.M. Gusein-Zade, F. Delgado and A. Campillo: *On the Monodromy of a Plane Curve Singularity and the Poincaré Series of the Ring of Functions on the Curve*, Functional Analysis and Its Applications, Vol. 33, No. 1 (1999), 56-57.

[15] M. Jankins and W. D. Neumann: *Lectures on Seifert Manifolds*, Brandeis Lecture Notes, 1983.

[16] L.H. Kauffman: *Branched coverings, open books and knot periodicity*, Topology 13 (1974), 143-160.

[17] M. Kreck and S. Stolz: *Nonconnected moduli spaces of positive sectional curvature metrics*, J. of the AMS., 6-4, 825-850 (1993).

[18] H. Laufer: *On minimally elliptic singularities*, Amer. J. of Math., 99, 1257-1295 (1977).

[19] H.B. Laufer: *Normal two-dimensional singularities*, Annals of Math. Studies 71, Princeton University Press 1971.

[20] Lé Dung Tráng: *Sur les noeuds algébriques*, Composition Math., 25(3) (1972), 281-321.

[21] C. Lescop: *Global Surgery Formula for the Casson-Walker Invariant*, Annals of Math. Studies, vol. 140, Princeton University Press, 1996.

[22] W. J. LeVeque (editor): *Reviews in Number Theory*, Volume 1, AMS, Providence, 1974.

[23] Y. Lim: *Seiberg-Witten invariants for 3-manifolds in the case $b_1 = 0$ or 1*, Pacific J. of Math., 195(2000), 179-204.
Seiberg-Witten invariants and surface singularities III

[24] E. Looijenga and J. Wahl: Quadratic functions and smoothing surface singularities, Topology, 25(1986), 261-291.
[25] M. Marcolli and B.L. Wang: Exact triangles in monopole homology and the Casson-Walker invariant [math.DG/0101127], Comm. Contem. Math, to appear.
[26] R. Mendris and A. Némethi: On the resolution graph of $f(x, y)+z^n=0$, OSU preprint.
[27] A. Némethi: “Weakly” elliptic Gorenstein singularities of surfaces, Invent. math. 137, 145-167 (1999).
[28] _______The signature of $f(x, y)+z^n$, Proceedings of Real and Complex Singularities, C.T.C Wall’s 60th birthday meeting, Liverpool (England), August 1996; London Math. Soc. Lecture Note Series, 263, 131-149 (1999).
[29] _______Dedekind sums and the signature of $z^N+f(x, y)$, Selecta Math., 4(1998), 361-376.
[30] _______Dedekind sums and the signature of $z^N+f(x, y)$, II, Selecta Math., 5(1999), 161-179.
[31] A. Némethi, L. I. Nicolaescu: Seiberg-Witten invariants and surface singularities, preprint [math.AG/0111298].
[32] _______Seiberg-Witten invariants and surface singularities II (singularities with good $C^*$-action), preprint [math.AG/0201120].
[33] W.D. Neumann: Cyclic suspension of knots and periodicity of signature for singularities, Bull. AMS 80(5) (1974), 977-981.
[34] _______A calculus for plumbing applied to the topology of complex surface singularities and degenerating complex curves, Transactions of the AMS, 268 Number 2, 299-344 (1981)
[35] W.D. Neumann and F. Raymond: Seifert manifolds, plumbing, $\mu$-invariant and orientation preserving maps, Algebraic and Geometric Topology (Proceedings, Santa Barbara 1977), Lecture Notes in Math. 664, 161-196.
[36] W. Neumann and J. Wahl: Casson invariant of links of singularities, Comment. Math. Helvetici, 65(1990), 58-78.
[37] _______Universal abelian covers of surface singularities, arXiv:math.AG/0110167.
[38] L.I. Nicolaescu: Seiberg-Witten invariants of rational homology spheres, math.DG/0103020.
[39] J. Stevens: Periodicity of Branched Cyclic Covers of Manifolds with Open Book Decomposition, Math. Ann., 273 (1986), 227-239.
[40] V.G. Turaev: Torsion invariants of $Spin^c$-structures on 3-manifolds, Math. Res. Letters, 4 (1997), 679-695.
[41] _______Surgery formula for torsions and Seiberg-Witten invariants of 3-manifolds, math.GT/0101108.
[42] K. Walker: An extension of Casson’s invariant, Annals of Mathematics Studies, 126, Princeton University Press, Princeton 1992.
[43] O. Zariski: On the Topology of Algebroid Singularities, Amer. J. Math., 54 (1932), 453-465.
[44] S. S.-T. Yau: On maximally elliptic singularities, Transact. AMS, 257, (2) (1980), 269-329.

DEPARTMENT OF MATHEMATICS, OHIO STATE UNIVERSITY, COLUMBUS, OH 43210
E-mail address: nemethi@math.ohio-state.edu
URL: http://www.math.ohio-state.edu/~nemethi/

UNIVERSITY OF NOTRE DAME, NOTRE DAME, IN 46556
E-mail address: nicolaescu.1@nd.edu
URL: http://www.nd.edu/~nicolaescu/