Dynamical Quantum Groups - The Super Story

Gizem Karaali

Abstract. We review recent results in the study of quantum groups in the super setting. In particular, we provide an overview of results about solutions of the Yang-Baxter equations in the super setting and develop the super analog of the theory of dynamical quantum groups.

1. Introduction

1.1. Background and Goals. Algebraists have been studying Hopf algebras since the 1960’s. Written first in 1969, [68] is still one of the classical references in the subject. The quantum Yang-Baxter equation (QYBE) was already known in 1950s, at least within the mathematical physics community. [41] contains some of the earliest papers on the subject. However, the discovery of the particular noncommutative noncocommutative Hopf algebras called quantum groups relating these two concepts has undoubtedly intensified the research in both areas over the last two decades. Along the way, a full theory of quantum groups has been developed, to the extent that there are currently various textbooks on the subject. See, for instance, [11, 14, 26, 39, 49, 56].

Super structures have been of interest to mathematicians and physicists alike. Besides providing the mathematical framework for supersymmetry, they have proved to be mathematically rich structures.

In this paper we collect together results and ideas that can be helpful in the pursuit of a full theory of quantum groups in the super setting. Currently, there are only partial results in this direction. Although we do mention some new results, our main goal is to provide a clear exposition of the present state of affairs in the theory of super quantum groups with a strong emphasis on dynamical structures. In particular, we focus on the solutions of the classical and quantum dynamical Yang-Baxter equations, with the standard theory for the non-dynamical equations as our guide.
For Sections 2 and 3, we will assume some familiarity with the theory of Lie superalgebras. \cite{42} has the first comprehensive study of these structures. \cite{66} and \cite{70} provide some relevant background on the subject. A concise summary of sign conventions used in the study of super structures can be found in \cite{17}. For Sections 4 and 5, some familiarity with quantum groups and Hopf algebras at the level of a text like \cite{49} will be sufficient.

Acknowledgments. The author thanks P. Etingof, L. Fehér, A. Isaev, E. Koelink, M. Kotchetov, J-H. Lu, S. Montgomery, H. Rosengren and R. Wisbauer for suggestions and constructive comments during the work that led to this paper, and the two referees whose recommendations improved this paper significantly. It is also a pleasure to thank the organizers L. Kauffman, D. Radford and F. Souza, of the AMS Special Session on Hopf Algebras at the Crossroads of Algebra, Category Theory, and Topology, October 23-24, 2004, where the author had the opportunity to present her results in \cite{44}.

1.2. Plan of this Paper. Section 1 is introductory. In \S 1.3 we give a brief overview of the results from \cite{43} and \cite{44}. We explain their relevance to our ultimate goal without going into too much detail. In \S 1.4 we explain our motivation for the emphasis of this paper on the dynamical picture.

In the first (classical) part of the paper, consisting of Sections 2 and 3, we study the classical dynamical Yang-Baxter equation (CDYBE) and its solutions. In \S 2.1 we give a brief overview of the historical development of the subject of dynamical Yang-Baxter equations. \S 2.2 provides the precise definitions of the terms involved. \S 2.3 is a concise but explicit summary of the classification results for the non-graded case. Section 3 is concerned with various super analogues for the results from Section 2. We make the appropriate definitions in \S 3.1 and present some construction and classification results for the super solutions of the CDYBE in \S 3.2 and \S 3.3. The material in \S 3.2 appeared elsewhere \cite{45}, but the main result of \S 3.3 is new.

Sections 4 and 5 make up the second (quantum) part of the paper. In Section 4 we describe the general theory of dynamical quantum groups. In order to provide a comprehensible exposition, we begin, in \S 4.1, with the definitions of groupoids, bialgebroids, and Hopf algebroids. In \S 4.2 we discuss dynamical quantum groups in more detail. We consider the categorical picture in \S 4.3. In Section 5 we begin our study of the super analogue of the theory of dynamical quantum groups. We consider the super versions of the basic definitions in \S 5.1. In \S 5.2 we discuss the QDYBE and its solutions in the super setting.

Section 6 closes the present exposition with a brief discussion of some open problems and a possible plan of action for them.

\footnote{Another beautiful reference with a more geometric flavor is \cite{58}. Here, Manin develops projective algebraic geometry in the context of supermanifolds. In this note, we will need neither the strength nor the sophistication of the tools provided there.}
1.3. Super Solutions of the Classical Yang-Baxter Equation.
The classical Yang-Baxter equation (CYBE) for an element $r \in \mathfrak{g} \otimes \mathfrak{g}$ where $\mathfrak{g}$ is a Lie algebra with a nondegenerate $\mathfrak{g}$-invariant bilinear form is:

\[ [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0 \]

In [4] and [5], Belavin and Drinfeld classified the solutions of the CYBE, which are called $r$-matrices. In particular, their work provides us with an explicit construction of non-skewsymmetric $r$-matrices associated to certain discrete data (the Belavin-Drinfeld triples).

In [43], the author proved a similar construction result in the case of simple Lie superalgebras $\mathfrak{g}$ with nondegenerate forms. Hence, the constructive part of the result of Belavin and Drinfeld mentioned above was extended to the framework of superalgebras. However, the discrete data that directly generalize the Belavin-Drinfeld triples were seen to be insufficient to fully classify all (non-skewsymmetric) solutions to the CYBE in the super setting. In particular, the author introduced (in [43]) and studied in detail (in [44]) a particular non-skewsymmetric $r$-matrix on $\mathfrak{sl}(2,1)$ that cannot be distinguished from the standard $r$-matrix on the basis of the Belavin-Drinfeld type data alone. Such examples, in fact, can be constructed in any $\mathfrak{sl}(m,n)$ for any $m \neq n$.

As is well-known, the solutions to the CYBE on a Lie (super)algebra give us the semiclassical limits of quantum $R$-matrices (solutions to the QYBE) on the associated Lie (super)group. Hence, an understanding of the solutions of the CYBE is essential to the theory of super quantum groups that we would like to develop. So far the results mentioned above show clearly that there will be some intrinsically new structures that need to be considered, although perhaps a mere change in perspective will suffice to see these as natural extensions of the original non-graded results.

1.4. Why Study Dynamical Structures? Quantum $R$-matrices provide us with examples of quantum groups. The standard quantum $R$-matrix is the quantization of the standard $r$-matrix obtained from the trivial Belavin-Drinfeld triple. The natural problem of quantizing all of the Belavin-Drinfeld type $r$-matrices was solved completely first by Etingof and Kazhdan in [23]. However, their method was very abstract, so the explicit constructions by Etingof, Schedler and Schiffmann in [25] were a most welcome development. This latter method in fact works for the dynamical $r$-matrices, i.e. the solutions of the more general classical dynamical Yang-Baxter equation (CDYBE). The associated quantum objects in this dynamical setting are the dynamical quantum groups (or dynamical quantum groupoids, to be more precise), and the associated Hopf objects are in fact Hopf algebroids.

\[^2\]Here, and elsewhere in the paper, the notation $r^{12}$ stands for the sum $\sum a_i \otimes b_i \otimes 1$ where $r = \sum a_i \otimes b_i$. Similarly, $r^{13} = \sum a_i \otimes 1 \otimes b_i$ and $r^{23} = \sum 1 \otimes a_i \otimes b_i$.

\[^3\]This comment is due to I. Penkov.
In this paper, we concentrate on the super solutions of the dynamical Yang-Baxter equation and the super versions of the notions of dynamical quantum groups and Hopf algebroids. Naturally we expect that understanding these will help us in extending the quantization result from [25] cited above to obtain a graded analogue. It should be noted here that the method of Etingof and Kazhdan has already been generalized to the super setting, in [33]. This is a very important development. However, due to the less constructive and more abstract nature of [23] and [33], we think that following the dynamical route can still provide us with additional valuable insight.

2. Dynamical $r$-matrices and the Classical Dynamical Yang-Baxter Equation

2.1. Historical Overview. Even before the classical dynamical Yang-Baxter equation (CDYBE) first appeared, the quantum dynamical Yang-Baxter equation (QDYBE) had been considered by Gervais and Neveu in [35]. Their motivation had been purely physical. Indeed, they were studying monodromy matrices in Liouville theory.

In order to talk about the introduction of the CDYBE, we need to briefly mention conformal field theory. (For more information on the following material, we refer the reader to [22, 49]). The correlation functions in Wess-Zumino-Witten (WZW) conformal field theory can be constructed out of holomorphic sections of certain vector bundles; these sections are known as conformal blocks. The conformal blocks on $\mathbb{P}^1$ for the WZW conformal field theory for a simple Lie algebra $g$ satisfy the Knizhnik-Zamolodchikov (KZ) equations, which were introduced in [50], and generalized in [15]. KZ equations play an important role in representation theory and mathematical physics. For a system of KZ equations to have a solution, the necessary and sufficient consistency condition coincides with the classical Yang-Baxter equation ([15, 22]).

The conformal blocks for an arbitrary elliptic curve satisfy a system of differential equations, a modification of the KZ equations, known as the Knizhnik-Zamolodchikov-Bernard (KZB) equations ([6]). The classical Yang-Baxter equation is no longer the required consistency condition for the KZB equations. However, it turns out that a modification will work.

Felder introduced the CDYBE formally in [31] and [32]. In fact, it had been appearing in the mathematical physics literature in disguise for a while. (See [3], for instance, where some of its basic trigonometric solutions were studied in detail). In [31] and [32], Felder showed that the CDYBE was the consistency condition for the KZB equations. He also related it, in a way analogous to the non-dynamical case, to the quantum dynamical Yang-Baxter equation (QDYBE).
At this point, it may be worth mentioning that, in this context, the label *dynamical* refers to the fact that the relevant equation is now a differential rather than an algebraic equation, and so it may remind us of a dynamical system.

The classical and quantum dynamical Yang-Baxter equations, as well as their various solutions, have been extensively studied. In the late 1990s, Etingof and Varchenko started to develop the classification of solutions of these equations, and together with Schiffmann they completed a classification theory for the classical case under certain assumptions ([26, 28, 67]). Their classification results resemble the classification results of Belavin and Drinfeld ([4, 5]) in the non-dynamical case.

As Drinfeld pointed out in [18], solutions of the classical Yang-Baxter equation have a natural geometrical interpretation as Poisson-Lie structures on the corresponding Lie groups. Similarly, solutions of the CDYBE define Poisson-Lie groupoid structures. See [28, 30, 47, 48, 55, 71] for research in this direction.

By now, the theory of the classical and quantum dynamical Yang-Baxter equations and their solutions has many applications, including some to integrable systems and representation theory. For a recent survey of results and such applications, one can refer to [21, 27]. The text [24] provides a very readable introduction to the topic as well.

### 2.2. Definitions.

Let \( g \) be a simple Lie algebra and \( h \subset g \) a Cartan subalgebra. The *classical dynamical Yang-Baxter equation* for a meromorphic function \( r : h^* \to g \otimes g \) is

\[
\text{Alt}(dr) + [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0;
\]

where the differential of \( r \) itself is considered as a meromorphic function:

\[
dr : h^* \to g \otimes g \otimes g, \quad \lambda \mapsto \sum_i x_i \otimes \frac{\partial r}{\partial x_i}(\lambda).
\]

Here \( \{x_i\} \) is a basis for \( h \), but it is also regarded as a linear system of coordinates on \( h^* \).

Using this we have:

\[
\text{Alt}(dr) = \sum_i x_i^{(1)} \left( \frac{\partial r}{\partial x_i} \right)^{(23)} + \sum_i x_i^{(2)} \left( \frac{\partial r}{\partial x_i} \right)^{(31)} + \sum_i x_i^{(3)} \left( \frac{\partial r}{\partial x_i} \right)^{(12)}.
\]

Therefore, we can rewrite the classical dynamical Yang-Baxter equation as follows:

\[
\sum_i x_i^{(1)} \left( \frac{\partial r}{\partial x_i} \right)^{(23)} + \sum_i x_i^{(2)} \left( \frac{\partial r}{\partial x_i} \right)^{(31)} + \sum_i x_i^{(3)} \left( \frac{\partial r}{\partial x_i} \right)^{(12)} + [r, r] = 0.
\]
A meromorphic function $r : \mathfrak{h}^* \to \mathfrak{g} \otimes \mathfrak{g}$ is called a dynamical $r$-matrix associated to $\mathfrak{h}$ if it is a solution to the classical dynamical Yang-Baxter equation (1) and it satisfies both the zero weight condition and the generalized unitarity condition. These two conditions are described as follows:

1. The zero weight condition for a meromorphic function $r : \mathfrak{h}^* \to \mathfrak{g} \otimes \mathfrak{g}$ is:

\[ [h \otimes 1 + 1 \otimes h, r(\lambda)] = 0 \quad \text{for all} \quad h \in \mathfrak{h}, \lambda \in \mathfrak{h}^*. \]

2. Generalized unitarity\(^4\) with coupling constant $\epsilon$ for a meromorphic function $r : \mathfrak{h}^* \to \mathfrak{g} \otimes \mathfrak{g}$ is:

\[ r(\lambda) + T(r)(\lambda) = \epsilon \Omega. \]

Here $\Omega$ is the Casimir element, i.e. the element of $\mathfrak{g} \otimes \mathfrak{g}$ corresponding to the Killing form. $T$ is the standard twist map on the second tensor power of any vector space: $T(a \otimes b) = b \otimes a$.

### 2.3. Classification Results.

Below is a brief survey of recent results related to the classification of the solutions of the classical dynamical Yang-Baxter equation.

The first classification results about the solutions of the classical dynamical Yang-Baxter equation can be found in [28]. There, Etingof and Varchenko proved two theorems that result in a full classification of all dynamical $r$-matrices satisfying the zero weight condition:

**Etingof - Varchenko Theorem 1.** (1) Let $X$ be a subset of the set of roots $\Delta$ of a simple Lie algebra $\mathfrak{g}$ with nondegenerate Killing form $\langle \cdot, \cdot \rangle$ such that:

(a) If $\alpha, \beta \in X$ and $\alpha + \beta$ is a root, then $\alpha + \beta \in X$, and

(b) If $\alpha \in X$, then $-\alpha \in X$.

Let $\nu \in \mathfrak{h}^*$, and let $D = \sum_{i<j} D_{ij} dx_i \wedge dx_j$ be a closed meromorphic $2$-form on $\mathfrak{h}^*$. If we set $D_{ij} = -D_{ji}$ for $i \geq j$, then the meromorphic function:

\[ r(\lambda) = \sum_{i,j=1}^{N} D_{ij}(\lambda)x_i \otimes x_j + \sum_{\alpha \in X} \frac{1}{(\alpha, \lambda - \nu)} e_\alpha \otimes e_{-\alpha} \]

is a dynamical $r$-matrix with zero weight and zero coupling constant.

\(^4\)This is a generalization of the unitarity condition which corresponds to $\epsilon = 0$. If $R$ is a quantization of $r$, then the statement $RT(R) = 1$ can be considered as a quantization of the statement $r + T(r) = 0$, and hence the term unitarity. At this point, it may be useful to note that, in this paper, we do not focus on the actual process of quantization, but we always keep in mind the relationship between the classical and the quantum pictures. This allows us to make such interpretations as in this footnote, while still keeping this paper at a reasonable length.
(2) Any dynamical r-matrix with zero weight and zero coupling constant is of this form.

Etingof - Varchenko Theorem 2. (1) Let $\mathfrak{g}$ be a simple Lie algebra with nondegenerate Killing form $(\cdot, \cdot)$. Let $\Delta$ be the set of roots of $\mathfrak{g}$, and fix a subset $X$ of the set $\Gamma$ of simple positive roots. Let $\overline{X}$ be the intersection of the linear span of $X$ with $\Delta$. Pick a $\nu \in \mathfrak{h}^*$, and define:

$$
\varphi_\alpha = \begin{cases} 
(\epsilon/2) \coth ((\epsilon/2) (\alpha, \lambda - \nu)) & \text{if } \alpha \in \overline{X} \\
(\epsilon/2) & \text{if } \alpha \notin \overline{X}, \text{ positive} \\
-(\epsilon/2) & \text{if } \alpha \notin \overline{X}, \text{ negative}
\end{cases}
$$

Let $D = \sum_{i<j} D_{ij} dx_i \wedge dx_j$ be a closed meromorphic 2-form on $\mathfrak{h}^*$. If we set $D_{ij} = -D_{ji}$ for $i \geq j$, then the meromorphic function:

$$
r(\lambda) = \sum_{i<j} D_{ij}(\lambda) x_i \otimes x_j + \frac{\epsilon}{2} \Omega + \sum_{\alpha \in \Delta} \varphi_\alpha e_\alpha \otimes e_{-\alpha}
$$

is a dynamical r-matrix with zero weight and nonzero coupling constant $\epsilon$.

(2) Any classical dynamical r-matrix with zero weight and nonzero coupling constant $\epsilon$ is of this form.

These are Theorems 3.2, and 3.10, respectively in [28]. We will provide super analogs of these theorems in §3.2.

A more general theory for classifying dynamical r-matrices, with no need for the zero weight condition, has been developed by Schiffmann in [67]. Below is a brief summary of his results. Later in §3.3 we will discuss their super versions.

The general setup of [67] is as follows: Let $\mathfrak{g}$ be a simple Lie algebra with nondegenerate invariant bilinear form $(\cdot, \cdot)$, $\mathfrak{l} \subset \mathfrak{g}$ a commutative subalgebra containing a regular semisimple element on which $(\cdot, \cdot)$ is nondegenerate, $\mathfrak{h}$ the Cartan subalgebra containing $\mathfrak{l}$, and $\mathfrak{h}_0$ the orthogonal complement of $\mathfrak{l}$ in $\mathfrak{h}$. Denote by $\Omega_{00}$ the $\mathfrak{h}_0$–part of the Casimir element $\Omega$ of $\mathfrak{g}$. Fix a set $\Gamma$ of simple roots.

In this setup, Schiffmann starts by introducing the notion of a a generalized Belavin-Drinfeld triple, which is defined as a triple $(\Gamma_1, \Gamma_2, \tau)$ where both $\Gamma_i$ are subsets of the set $\Gamma$ of simple roots, and $\tau : \Gamma_1 \rightarrow \Gamma_2$ is a norm-preserving bijection. Then, an $\mathfrak{l}$–graded generalized Belavin-Drinfeld triple is one where $\tau$ preserves the decomposition of $\mathfrak{g}$ into $\mathfrak{l}$–weight spaces. Given a generalized Belavin-Drinfeld triple $(\Gamma_1, \Gamma_2, \tau)$, denote by $\Gamma_3$ the largest subset of $\Gamma_1 \cap \Gamma_2$ that is stable under $\tau$. Clearly $\tau$ can be extended to an isomorphism $\mathfrak{gl}_1 \rightarrow \mathfrak{gl}_2$ once we fix a set of Weyl-Chevalley generators.

---

5 This condition about the existence of a regular element was later shown to be redundant ([21]).
If we fix basis vectors \( \{e_\alpha\} \) for the non-Cartan part of \( \mathfrak{g} \), as is done in the standard Belavin-Drinfeld construction, then for any \( \lambda \in \mathfrak{l}^\ast \), we can define a map \( K(\lambda) : n_+(\Gamma_1) \to n_+(\Gamma_2) \) as follows:

\[
K(\lambda)(e_\alpha) = \frac{1}{2} e_\alpha + e^{-(\alpha,\lambda)} \frac{\tau}{1 - e^{-(\alpha,\lambda)} \tau}(e_\alpha)
\]

\[
= \frac{1}{2} e_\alpha + \sum_{n>0} e^{-n(\alpha,\lambda)} \tau^n(e_\alpha).
\]

Note that the sum above is finite as long as \( \alpha \notin \Gamma_3 \). If \( \alpha \in \Gamma_3 \) and \( \tau|_{\Gamma_3} = id_{\Gamma_3} \), then we have:

\[
K(\lambda)(e_\alpha) = \frac{1}{2} \coth \left( \frac{1}{2}(\alpha,\lambda) \right) e_\alpha.
\]

Given a meromorphic map \( r : \mathfrak{h}^* \to (\mathfrak{g} \otimes \mathfrak{g})^\mathfrak{h} \), we consider the following transformations:

1. \( r(\lambda) \mapsto r(\lambda) + \sum_{i<j} C_{ij}(\lambda)x_i \wedge x_j \), where \( \sum_{i<j} C_{ij}(\lambda) d\lambda_i \wedge d\lambda_j \) is a closed meromorphic 2-form,
2. \( r(\lambda) \mapsto r(\lambda - \nu) \), where \( \nu \in \mathfrak{h}^* \),
3. \( r(\lambda) \mapsto (A \otimes A) r(A^* \lambda) \), where \( A \) is an element of the Weyl group of \( \mathfrak{g} \).

We will call these transformations gauge transformations. Two dynamical r-matrices that can be obtained from one another via a sequence of gauge transformations are called gauge-equivalent.

The main result of [67] is the following:

**Schiffmann Theorem.** (1) Any dynamical r-matrix will be gauge-equivalent to a dynamical r-matrix \( r \) satisfying:

\[
r(\lambda) - T(r)(\lambda) \in \mathfrak{l}^\perp \otimes \mathfrak{l}^\perp = \left( \bigoplus_{\alpha \neq 0} \mathfrak{g}_\alpha \otimes \mathfrak{h}_0 \right) \otimes \left( \bigoplus_{\alpha \neq 0} \mathfrak{g}_\alpha \otimes \mathfrak{h}_0 \right)
\]

\[
(3)
\]

Here, once again, \( T \) is the usual twist, mapping \( a \otimes b \) to \( b \otimes a \).

(2) Let \( (\Gamma_1, \Gamma_2, \tau) \) be an \( l \)-graded generalized Belavin-Drinfeld triple. If \( r_{00} \in \mathfrak{h}_0 \otimes \mathfrak{h}_0 \) satisfies:

\[
(\tau(\alpha) \otimes 1)r_{00} + (1 \otimes \alpha)r_{00} = \frac{1}{2}((\alpha + \tau(\alpha)) \otimes 1)\Omega_{00}
\]

for any \( \alpha \in \Gamma_1 \), then:

\[
r(\lambda) = \frac{1}{2} \Omega + r_{00} + \sum_{\alpha \in \Gamma_1 \cap \Delta^+} K(\lambda)(e_\alpha) \wedge e_{-\alpha} + \sum_{\alpha \in \Delta^+, \alpha \notin \Gamma_1} \frac{1}{2} e_\alpha \wedge e_{-\alpha}
\]

is a dynamical r-matrix satisfying Equation (3).
Any dynamical $r$-matrix satisfying Equation (3) is of the above form for a suitable triangular decomposition of $g$.

**Remark.** Note that when $l = h$ we get Etingof - Varchenko Theorem 2. The two subsets $\Gamma_1, \Gamma_2$ are the subset $X$ of the mentioned theorem, and the two statements are equal when we set $\epsilon = 1$. The isometry $\tau$ is the identity on $X$. The zero-weight condition is equivalent to the associated generalized Belavin-Drinfeld triple being $l$-graded.

**Remark.** If $\Gamma_3 = \emptyset$, then $(\Gamma_1, \Gamma_2, \tau)$ is a Belavin-Drinfeld triple. If $\tau$ preserves the grading of the $l$-weight space decomposition of $g$, then $l$ has to be orthogonal to the coroots corresponding to the roots in the span of $(\Gamma_1 \cup \Gamma_2)$.

If $l = \{0\}$, then $l^\ast = \{0\}$, and we get constant (non-dynamical) $r$-matrices, i.e. solutions of the CYBE. The statement in this case can be seen to be equivalent to the Belavin-Drinfeld Theorem.

### 3. Dynamical $r$-matrices in the Super Setting

We can now concentrate on the super versions of the above constructions.

**3.1. Definitions.** Let $g$ be a simple Lie superalgebra with nondegenerate Killing form $(\cdot, \cdot)$. Let $h \subset g$ be a Cartan subsuperalgebra, and let $\Delta \subset h^\ast$ be the set of roots associated to $h$. Fix a set of simple roots $\Gamma$ or equivalently a Borel $b$. The classical dynamical Yang-Baxter equation for a meromorphic function $r : h^\ast \to g \otimes g$ will be:

$$\text{Alt}_s(dr) + [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0$$

(4)

The differential of $r$ will be defined as above as:

$$dr : h^\ast \rightarrow g \otimes g \otimes g \quad \lambda \rightarrow \sum_i x_i \otimes \frac{\partial r}{\partial x_i}(\lambda)^i$$

Here $\{x_i\}$ is a basis for $h$ so all $x_i$ are even. Recall that $\text{Alt}_s : g^\otimes 3 \to g^\otimes 3$ is given on homogeneous elements by:

$$\text{Alt}_s(a \otimes b \otimes c) = a \otimes b \otimes c + (-1)^{|a||b|+|c|}b \otimes c \otimes a + (-1)^{|c||a|+|b|}c \otimes a \otimes b,$$

In view of all this, we can see that, for $r = \sum_i R_{i(1)} \otimes R_{i(2)}$:

$$\text{Alt}_s(dr) = \sum_i x_i^{(1)} \left( \frac{\partial r}{\partial x_i} \right)^{(23)} + \sum_i x_i^{(2)} \left( \frac{\partial r}{\partial x_i} \right)^{(31)} + \sum_i (-1)^{|R_{i(1)}||R_{i(2)}|} x_i^{(3)} \left( \frac{\partial r}{\partial x_i} \right)^{(12)}.$$
For the moment, we will say that a meromorphic function \( r : \mathfrak{h}^* \to \mathfrak{g} \otimes \mathfrak{g} \) is a super dynamical \( r \)-matrix with coupling constant \( \epsilon \) if it is a solution to Equation (4) and satisfies the generalized unitarity condition:

\[
 r(\lambda) + T_s(r)(\lambda) = \epsilon \Omega, \tag{5}
\]

where \( \Omega \) is the Casimir element, i.e. the element of \( \mathfrak{g} \otimes \mathfrak{g} \) corresponding to the Killing form. Here, \( T_s \) is the super twist map, defined on the second tensor power of any given super vector space by: \( T_s(a \otimes b) = (-1)^{|a||b|}b \otimes a \).

The (constant) standard \( r \)-matrix and its super twist are two easy examples of super dynamical \( r \)-matrices. The (constant) non-standard \( r \)-matrices can also be viewed as simple cases of super dynamical \( r \)-matrices. Other simple but non-constant examples can be constructed by the theorems that will be presented in the following subsection. We refer the reader to [45] for more examples and discussion.

### 3.2. Super Dynamical \( r \)-matrices with Zero Weight

A super dynamical \( r \)-matrix \( r \) is said to satisfy the zero weight condition if:

\[
 [h \otimes 1 + 1 \otimes h, r(\lambda)] = 0 \text{ for all } h \in \mathfrak{h}, \lambda \in \mathfrak{h}^*.
\]

In [45], the author proved the super versions of Theorem 3.2 and Theorem 3.10 from [28], which are, respectively, The Etingof-Varchenko Theorem 1 and Etingof-Varchenko Theorem 2 of Section 2.3. The proofs in [45] have clear similarities to the respective proofs in [28]. These results basically extend the full classification results of dynamical \( r \)-matrices with zero weight to the super case.

Here are the two theorems:

**Theorem 3.1.** (1) Let \( X \) be a subset of the set of roots \( \Delta \) of a simple Lie superalgebra \( \mathfrak{g} \) with nondegenerate Killing form \((\cdot, \cdot)\) such that:

(a) If \( \alpha, \beta \in X \) and \( \alpha + \beta \) is a root, then \( \alpha + \beta \in X \), and

(b) If \( \alpha \in X \), then \( -\alpha \in X \).

Let \( \nu \in \mathfrak{h}^* \), and let \( D = \sum_{i<j} D_{ij} dx_i \wedge dx_j \) be a closed meromorphic 2–form on \( \mathfrak{h}^* \). If we set \( D_{ij} = -D_{ji} \) for \( i \geq j \), then the meromorphic function:

\[
 r(\lambda) = \sum_{i,j=1}^N D_{ij}(\lambda)x_i \otimes x_j + \sum_{\alpha \in X} \frac{(-1)^{|\alpha|}(e_{\alpha}, e_{-\alpha})}{(\alpha, \lambda - \nu)}e_{\alpha} \otimes e_{-\alpha}
\]

is a super dynamical \( r \)-matrix with zero weight and zero coupling constant.

(2) Any super dynamical \( r \)-matrix with zero weight and zero coupling constant is of this form.

---

6See [33,3] for a broader definition.
Theorem 3.2. (1) Let $\mathfrak{g}$ be a simple Lie superalgebra with nondegenerate Killing form $(\cdot, \cdot)$. Let $X$ be a subset of the set of roots $\Delta$ of $\mathfrak{g}$ satisfying conditions (a) and (b) of Theorem 3.1. Pick $\nu \in \mathfrak{h}^*$, and define:

$$\varphi_\alpha = \begin{cases} 
\left(\frac{\epsilon}{2}\right) \coth \left( (-1)^{|\alpha|} \frac{\epsilon}{2} \right) & \text{if } \alpha \in X \\
(\pm \frac{\epsilon}{2}) & \text{if } \alpha \notin X, \text{ negative} \\
\mp (-1)^{|\alpha|} \frac{\epsilon}{2} & \text{if } \alpha \notin X, \text{ positive}
\end{cases}$$

Let $D = \sum_{i<j} D_{ij} dx_i \wedge dx_j$ be a closed meromorphic 2-form on $\mathfrak{h}^*$. If we set $D_{ij} = -D_{ji}$ for $i \geq j$, then the meromorphic function:

$$r(\lambda) = \sum_{i,j=1}^N D_{ij}(\lambda) x_i \otimes x_j + \frac{\epsilon}{2} \Omega + \sum_{\alpha \in \Delta} \varphi_\alpha e_\alpha \otimes e_{-\alpha}$$

is a super dynamical $r$-matrix with zero weight and nonzero coupling constant $\epsilon$.

(2) Any super dynamical $r$-matrix with zero weight and nonzero coupling constant $\epsilon$ is of this form.

Remark. Note that, if we take the limit as $\epsilon \to 0$, then the statement of Theorem 3.2 reduces to the statement of Theorem 3.1.

3.3. Generalizing Schiffmann’s Classification. Here we discuss an extension of the full classification result of Schiffmann. The material in this subsection is new, although it is natural and expected after [45]. We start with the necessary terminology, analogous to the non-graded case.

As in the beginning of Section 3, let $\mathfrak{g}$ be a simple Lie superalgebra with nondegenerate Killing form $(\cdot, \cdot)$. Let $\mathfrak{l} \subset \mathfrak{g}$ be a commutative subsuperalgebra, $\mathfrak{h} \subset \mathfrak{g}$ the Cartan subsuperalgebra containing $\mathfrak{l}$, $\Delta \subset \mathfrak{h}^*$ the set of roots associated to $\mathfrak{h}$, $\Gamma$ a fixed set of simple roots and $\mathfrak{b}$ the associated Borel subsuperalgebra. In this setting we can generalize our definition of super dynamical $r$-matrices to include all meromorphic solutions of Equation (4) where the differential $dr$ is given by:

$$dr : \mathfrak{l}^* \rightarrow \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$$

$$\lambda \mapsto \sum_i x_i \otimes \frac{\partial r}{\partial x_i}(\lambda)$$

where, this time, $\{x_i\}$ is a basis for $\mathfrak{l} \subset \mathfrak{h}$. More precisely, we will say that a meromorphic function $r : \mathfrak{l}^* \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ is a super dynamical $r$-matrix with coupling constant $\epsilon$ if it is a solution to Equation (4) and satisfies the generalized unitarity condition given by Equation (5).

A super dynamical $r$-matrix $r : \mathfrak{l}^* \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ is said to be $\mathfrak{l}$-invariant if:

$$[l \otimes 1 + 1 \otimes l, r(\lambda)] = 0 \text{ for all } l \in \mathfrak{l}, \lambda \in \mathfrak{l}^*.$$ 

The zero weight condition of Subsection 3.2 is easily seen to be equivalent to $\mathfrak{h}$-invariance in our new terminology.
We will say that two super dynamical r-matrices are \textit{gauge-equivalent} if they can be obtained from one another via \textit{gauge-transformations}, see \cite{22,23}.

Let us start with a result about gauge equivalence classes. The non-graded version of the following lemma is proved as the first part of the main classification theorem in Schiffmann’s work, \cite{67}. The same proof will work here with no changes, as only the even component of \(g\) is involved when determining the gauge transformations to be used.

\textbf{Lemma 3.1.} \textit{Any} \(l^{-}\)-invariant super dynamical \(r\)-matrix with coupling constant \(1\) is gauge-equivalent to an \(l^{-}\)-invariant super dynamical \(r\)-matrix \(r^*: \mathfrak{l}^* \to \mathfrak{g} \otimes \mathfrak{g}\) satisfying:

\begin{equation}
\tau(\lambda) - T_s(\tau)(\lambda) \in l^\perp \otimes l^\perp = \left( \bigoplus_{\alpha \neq 0} \mathfrak{g}_\alpha \right) \otimes \left( \bigoplus_{\alpha \neq 0} \mathfrak{g}_\alpha \right)
\end{equation}

where \(\mathfrak{h}_0 \subset \mathfrak{h}\) is “the” complement of \(l\) in \(\mathfrak{h}\). \(T_s\), as before, is the super twist, mapping any homogeneous \(a \otimes b\) to \((-1)^{|a||b|} b \otimes a\).

\textbf{Remark.} The orthogonal complement of a subset \(l\) of the Cartan subalgebra of a Lie superalgebra may or may not intersect the subset \(l\) trivially. However, after making certain choices, we can always find a subset \(h_0 \subset \mathfrak{h}\) such that \(h_0 \oplus l = \mathfrak{h}\). More specifically, we pick a set \(\Gamma\) of simple roots for \(\mathfrak{g}\) and an appropriate basis \(\{h_\alpha \in \mathfrak{h} : \alpha \in \Gamma\}\) for \(\mathfrak{h}\) such that for some subset \(A\) of \(\Gamma\) we have:

\begin{equation}
l = \bigoplus_{\alpha \in \mathcal{A}} \mathfrak{C}h_\alpha.
\end{equation}

Then we set \(h_0 = \bigoplus_{\alpha \in \Gamma \setminus A}\mathfrak{C}h_\alpha\). This is the complement we use in the above statement.

Let us say that a triple \((\Gamma_1, \Gamma_2, \tau)\) is an \textit{admissible triple} if:

\begin{enumerate}
\item \(\Gamma_1, \Gamma_2 \subset \Gamma\), and
\item \(\tau : \Gamma_1 \to \Gamma_2\) is a grading preserving isometry.
\end{enumerate}

We will say that an admissible triple is \textit{l-graded} if, in addition, it preserves the decomposition of \(\mathfrak{g}\) into \(l\)-weight spaces. Let \((\Gamma_1, \Gamma_2, \tau)\) be an \(l\)-graded admissible triple.

Denote by \(\Gamma_3\) the largest subset of the intersection \(\Gamma_1 \cap \Gamma_2\) which is stable under \(\tau\), and define:

\[\overline{\Gamma}_1 = \Gamma_1 \setminus \Gamma_3, \quad \overline{\Gamma}_2 = \Gamma_2 \setminus \Gamma_3.\]

Then, the triple \((\overline{\Gamma}_1, \overline{\Gamma}_2, \tau)\) is admissible in the sense of \cite{43} and will give us, through the constructions there, a solution to the CYBE on \(\mathfrak{g}\).
For each choice of a set of “Chevalley” generators, we can extend $\tau$ to two (Lie superalgebra) isomorphisms:

$$\tau_{1,2} : \Gamma_1 \rightarrow \Gamma_2 \quad \text{and} \quad \tau_3 : \Gamma_3 \rightarrow \Gamma_3.$$ 

We can and will denote both of these maps by $\tau$.

If we fix a basis $\{e_\alpha : \alpha \in \Gamma\}$ for the non-Cartan part of $\mathfrak{g}$ and introduce the $A_\alpha$ notation as done before in §3.2, then, for any $\lambda \in \Gamma^*$, we can define a map $K_\lambda : \mathfrak{n}_+^{\Gamma_1} \rightarrow \mathfrak{n}_+^{\Gamma_2}$ as follows:

$$K_\lambda(e_\alpha) = \frac{1}{2} e_\alpha + \frac{1}{1 - e^{\frac{A_\alpha(\alpha,\lambda)}{2}}(\alpha,\lambda)} e_{-A_\alpha(\alpha,\lambda)}(\alpha,\lambda) \tau(e_\alpha)$$

$$= \frac{1}{2} e_\alpha + \sum_{n \geq 0} e^{-nA_\alpha(\alpha,\lambda)} e_\alpha.$$

Note that the sum above is finite as long as $\alpha \notin \Gamma_3$. If $\alpha \in \Gamma_3$ and $\tau|_{\Gamma_3} = \text{id}_{\Gamma_3}$, then we have:

$$K_\lambda(e_\alpha) = \frac{1}{2} \coth \left( \frac{A_\alpha}{2} (\alpha, \lambda) \right) e_\alpha.$$

Let $r_{00} \in \mathfrak{h}_0 \otimes \mathfrak{h}_0$ satisfy:

$$(\tau(\alpha) \otimes 1)r_{00} + (1 \otimes \alpha)r_{00} = \frac{1}{2}((\alpha + \tau(\alpha)) \otimes 1)\Omega_{00}$$

for any $\alpha \in \Gamma_1$, where we denote by $\Omega_{00}$ the $\mathfrak{h}_0 \otimes \mathfrak{h}_0$ part of the Casimir element $\Omega$. Then some computation shows in fact that:

$$r(\lambda) = \frac{1}{2} \Omega + r_{00} + \sum_{\alpha \in \Gamma_1 \cap \Delta^+} K(\lambda)(e_\alpha) \wedge e_{-\alpha} + \sum_{\alpha \in \Delta^+, \alpha \notin \Gamma_1} \frac{1}{2} e_\alpha \wedge e_{-\alpha}$$

is a dynamical r-matrix satisfying Equation (6). Summarizing the above, we get:

**Theorem 3.3.** Let $(\Gamma_1, \Gamma_2, \tau)$ be an $l$-graded admissible triple and let $r_{00} \in \mathfrak{h}_0 \otimes \mathfrak{h}_0$ satisfy:

$$(\tau(\alpha) \otimes 1)r_{00} + (1 \otimes \alpha)r_{00} = \frac{1}{2}((\alpha + \tau(\alpha)) \otimes 1)\Omega_{00}$$

for any $\alpha \in \Gamma_1$. Then:

$$r(\lambda) = \frac{1}{2} \Omega + r_{00} + \sum_{\alpha \in \Gamma_1 \cap \Delta^+} K(\lambda)(e_\alpha) \wedge e_{-\alpha} + \sum_{\alpha \in \Delta^+, \alpha \notin \Gamma_1} \frac{1}{2} e_\alpha \wedge e_{-\alpha}$$

is a dynamical r-matrix satisfying Equation (6).

\[\footnote{For simple Lie superalgebras with non-degenerate Killing form, it is always possible to find a basis \{E_\alpha, F_\alpha, H_\alpha : \alpha \in \Gamma\} which satisfies the super versions of the usual commutation and Serre relations one expects from a set of Chevalley generators in the non-graded case. See, for instance, \([38, 52]\).} \]
Together with Lemma 3.1, Theorem 3.3 gives us the super version of the constructive part of Schiffmann’s result. Hence, we have a very nice way to construct super dynamical $r$-matrices which generalizes naturally the non-graded theory. Note that, as in the non-graded case, this result agrees with and extends the constructive part of the zero weight results of §3.2. More precisely, if we let $l = h$, $\Gamma_1 = \Gamma_2 = X$, $\tau = id$ and $\epsilon = 1$, then Theorems 3.2 and 3.3 coincide.

However, superizing the classifying part of the non-graded theory will be a lot more involved. The $sl(2,1)$ example constructed in [43] and studied in detail in [44] once again turns out to be an issue. It can easily be seen that for $\Gamma_3 = 0$ and $l = 0$, Theorem 3.3 reduces to the construction theorem of [43], and the construction described above gives us a non-skewsymmetric solution to the CYBE. The particular $sl(2,1)$ example which does not fit the framework of [43] therefore will not fit this new framework.

4. General Theory of Dynamical Quantum Groups

Here we review the general theory of dynamical quantum groups. In order to keep the paper at a readable length, we only consider the development of the theory utilizing the notion of Hopf algebroids. Nevertheless, we should note that there are alternative approaches. For instance, one can use quasi-Hopf structures (i.e. Hopf-like structures obtained by weakening the coassociativity condition on the coproduct), which were first introduced by Drinfeld in [19]. Such an approach was initiated in [2], and its implications were investigated in great depth; see, for instance, [1, 20]. It turns out that there is a natural relationship between quasi-Hopf algebras and Hopf algebroids [73]. In fact, the reader interested in details on using different generalizations of Hopf algebras in the context of dynamical quantum groups may find our survey [46] of some value. However, in this note, we will not be too worried about only using the Hopf algebroid approach without any further explanation.

4.1. Groupoids, Bialgebroids and Hopf Algebroids. It is well-known that quantum groups are actually Hopf algebras. It turns out that, in the context of dynamical quantum groups, considering Hopf algebroids as the analogous objects proves quite fruitful. Therefore, we will start with a basic discussion of groupoids, bialgebroids, Hopf algebroids, and quantum groupoids.

For our purposes, a groupoid over a set $X$ is a set $G$ together with the following structure maps:

1. A pair of maps $s, t : G \to X$, respectively called the source and the target.
(2) A product $m$, i.e. a partial function $m : G \times G \to G$ satisfying the following two properties:
(a) $t(m(g,h)) = t(g)$, $s(m(g,h)) = s(h)$ whenever $m(g,h)$ is defined;
(b) $m$ is associative: $m(m(g,h), k) = m(g, m(h,k))$ whenever the relevant terms are defined.

(3) An embedding $\epsilon : X \to G$ called the identity section such that

$m(\epsilon(t(g)), g) = g = m(g, \epsilon(s(g)))$ for all $g \in G$.

(4) An inversion map $i : G \to G$ such that

$m(i(g), g) = \epsilon(s(g))$ and $m(g, i(g)) = \epsilon(t(g))$ for all $g \in G$.

Conventionally one writes $gh$ instead of $m(g,h)$ whenever the latter is defined. Also, $i(g)$ can be denoted by $g^{-1}$. Then we can rewrite the above conditions in a form which makes more transparent the similarities and differences of a groupoid from a group. For instance, we wish to have inverses for all elements of $G$, but the multiplication is only partially defined.

There are other ways to define a groupoid. One very elegant way is to view it as a particular type of category $(X, G)$ with $X$ making up the set of objects, such that the morphisms (elements of $G$) are all invertible. In this more category-theoretic setup, the notion of a group corresponds to the particular case when $X$ has only one object, and this single object is the domain and range for all morphisms (the group elements), just like the set of morphisms of a category with only one object corresponds, in general, to the concept of a monoid. We choose not to consider this categorical description. However, if we draw some schematic figures to represent the structure maps defined above, then we can clearly see how the categorical notion may be derived easily. This may also help interpret, in graph-theoretical terms, the particular names source and target used for the two structure maps in the above definition.

Groupoids were first introduced in 1926, and since then, found applications in differential topology and geometry, algebraic geometry and algebraic topology, and analysis. A very friendly introduction to groupoids with many examples from various areas of mathematics can be found in [72]. For more rigorous accounts one may refer to the bibliography there. We took our definition from [13, Part VI]. Another good reference on groupoids is [62].

It is possible to define Lie groupoids or Poisson groupoids by stipulating the presence of a Lie or Poisson structure on a given groupoid, but since we will not need precise definitions here, we will only mention a few references for the interested reader. For instance, [10] provides a systematic introduction to Lie groupoids, Lie algebroids and symplectic groupoids. Poisson groupoids were first introduced in [71], and they have been studied ever since. The recent monograph [53] on dynamical Poisson groupoids studies in great detail the particular Poisson groupoids most relevant to this note.
A possible connection of Poisson groupoids to quantum groupoids analogous to the connection of Poisson groups to quantum groups was conjectured in [59]. The first full description of a quantum groupoid as an object which should simultaneously be generalizing quantum groups and groupoid algebras was given in [57]. This latter work utilized certain commutativity assumptions, and eventually a new and broader approach, in [54], provided us with a description without such constraints.

Almost simultaneously, the notion of a Hopf algebroid was being developed. Commutative Hopf algebroids were first studied in [65]. There were similar descriptions in [57]. In [54], the definitions were slightly modified in order to include noncommutative cases. However, the clear consensus on the definition of a bialgebroid, first given in [69], was not easy to come for the definition of a Hopf algebroid. The antipode suggested in [54] was not universally accepted, and various other formulations followed. See, for instance, [10]. A comparative study of these various antipodes may be found in [7].

Another parallel development was the introduction of weak Hopf algebroids in [8, 9, 61]. These are the Hopf-algebra-like structures one obtains when one drops the requirement that the comultiplication be unit preserving. They were introduced with a view toward applications to operator algebras, but even from the beginning, their appeal as a means of generalizing quantum groups was recognized. Eventually, Böhm and Szlachányi showed that weak Hopf algebras with bijective antipodes are Hopf algebroids, see [10]. The reader can also refer to [60] for a detailed overview of weak Hopf algebras, their relationship to various generalizations of the idea of quantum groups and in particular to “dynamical deformations of quantum groups.” The survey [46] provides a basic comparative study, in the context of possible generalizations of Hopf algebras.

Here are the definitions we use in this paper, mainly following [7] and [10]: A bialgebroid should be the natural extension of the notion of a bialgebra to the world of groupoids. This will imply that a bialgebroid is no longer an algebra, but a bimodule over a non-commutative ring. More specifically, a left bialgebroid $A_L$ is given by the following data:

1. Two associative unital rings: the total ring $A$ and the base field $L$.
2. Two ring homomorphisms: the source $s_L : L \to A$ and the target $t_L : L^{op} \to A$ such that the images of $L$ in $A$ commute, making $A$ an $L-L$ bimodule denoted by $L_A$.
3. Two maps $\gamma_L : A \to A_L \otimes L_A$ and $\pi_L : A \to L$ making the triple $(L_A, \gamma_L, \pi_L)$ a comonoid in the category of $L-L$ bimodules.

---

8This reference also includes a brief but interesting philosophical discussion of quantization.
The source and target maps \( s_L \) and \( t_L \) may be used to define four commuting actions of \( L \) on \( A \); these in turn give us in an obvious way the new bimodules \( L^A \), \( LA_L \), and \( LA^L \).

Similarly we can define a right bialgebroid \( A_R \) using the following data:

1. Two associative unital rings: the total ring \( A \) and the base field \( R \).
2. Two ring homomorphisms: the source \( s_R : R \to A \) and the target \( t_R : R^{op} \to A \) such that the images of \( R \) in \( A \) commute, making \( A \) an \( R - R \) bimodule denoted by \( R^A \).
3. Two maps \( \gamma_R : A \to A^{op} \otimes RA \) and \( \pi_R : A \to R \) making the triple \((R^A, \gamma_R, \pi_R)\) a comonoid in the category of \( R - R \) bimodules.

As in the case of left bialgebroids, we can define three other bimodule structures on \( A \) using the source and the target, and denote them by \( R^A_R \), \( R^A_R \), and \( R^A_R \). These bimodule structures and the two notions of bialgebroids are related as expected. For instance, if \( A_L = (A, L, s_L, t_L, \gamma_L, \pi_L) \) is a left bialgebroid, then its co-opposite is again a left bialgebroid: \((A_L)^{op} = (A, L^{op}, t_L, s_L, \gamma_L^{op}, \pi_L)\), where \( \gamma_L^{op} \) is defined as \( T \circ \gamma_L \).

The opposite \((A_L)^{op}\) defined by the data \((A^{op}, L, t_L, s_L, \gamma_L, \pi_L)\) is a right bialgebroid. For more on bialgebroids, we refer the reader to [7].

We will take our definition for a Hopf algebroid from [10]. In particular, to define a Hopf algebroid, we will need two associative unital rings \( A \) and \( L \), and set \( R = L^{op} \). We will consider a left bialgebroid structure \( A_L = (A, L, s_L, t_L, \gamma_L, \pi_L) \) and a right bialgebroid structure \( A_R = (A, R, s_R, t_R, \gamma_R, \pi_R) \) associated to this pair of rings. We will require that \( s_L(L) = t_R(R) \) and \( t_L(L) = s_R(R) \) as subrings of \( A \), and:

\[
(\gamma_L \otimes \text{id}_A) \circ \gamma_R = (\text{id}_A \otimes \gamma_R) \circ \gamma_L,
\]

\[
(\gamma_R \otimes \text{id}_A) \circ \gamma_L = (\text{id}_A \otimes \gamma_L) \circ \gamma_R.
\]

The last ingredient is the antipode. This will be a bijection \( S : A \to A \) that will satisfy:

\[
S(t_L(l)at_L(l')) = s_L(l')S(a)s_L(l),
\]

\[
S(t_R(r')at_R(r)) = s_R(r')S(a)s_R(r')
\]

for all \( l, l' \in L \), \( r, r' \in R \), and \( a \in A \). In other words, we require \( S \) to be a twisted isomorphism simultaneously of bimodules \( LA_L \to L^A_L \) and of bimodules \( R^A_R \to RA^R \).

Our final constraint on the antipode \( S \) is as follows:

\[
S(a_{(1)})a_{(2)} = s_R \circ \pi_R(a),
\]

\[
a^{(1)}S(a^{(2)}) = s_L \circ \pi_L(a)
\]

---

9Here, as before, \( T \) is the usual (non-graded) twist, mapping \( a \otimes b \) to \( b \otimes a \).
for any \( a \in A \). The subscripts and the superscripts on \( a \) come from a generalized version of the famous Sweedler notation, which we use to define the two maps \( \gamma_L \) and \( \gamma_R \):

\[
\begin{align*}
\gamma_L(a) &= a_{(1)} \otimes a_{(2)} \in A_L \otimes L A \\
\gamma_R(a) &= a^{(1)} \otimes a^{(2)} \in A_R \otimes R A.
\end{align*}
\]

In this setup, then, we will say that the triple \( A = (A_L, A_R, S) \) is a Hopf algebroid.\(^{10}\) It is easy to see how this symmetric definition, in terms of two bialgebroids and a bijection called the antipode, is analogous to the definition of a Hopf algebra from two bialgebras and a bijective map called an antipode. However, it is not nearly as easy to see why this is the appropriate definition. We accept the definition given above without further analysis, and leave the readers to follow the discussion on this issue on their own. (A good place to start may be the comparative study of Böhm in \(^7\)).

Now, for us, a quantum groupoid will be a particular type of Hopf algebroid, just as a quantum group is a particular type of Hopf algebra. Quantum groups are Hopf algebras obtained from deformations of commutative or cocommutative Hopf algebras. The semiclassical limits of these deformations are the so-called Hopf-Poisson algebras; associated to these latter structures are the Poisson-Lie groups. Hence, a quantum groupoid should be a Hopf algebroid which is a “deformation” of a “nice Hopf algebroid” in such a way that in the semi-classical limit we should get an algebraic structure associated to a Poisson-Lie groupoid.

We will end this section with the rather intuitive discussion above, and not attempt to come up with a full accurate definition for the notion of quantum groupoid. Our main reason for this is the fact that just as there are various definitions for Hopf algebroids, there are some differing notions of quantum groupoids. For instance, in \(^60\), the term quantum groupoid is used almost interchangeably with the term weak Hopf algebra. (Also see \(^74\) for another different approach to quantum groupoids). However, unlike in the case of Hopf algebroids, we can do without a precise definition at this stage, because there is nevertheless a consensus on the definition of dynamical quantum groups.\(^{11}\) We will describe and study those in more detail in the next subsection.

---

\(^{10}\) We should note here that some of the above information in our definition is redundant. In fact one can start with a left bialgebroid \( A_L = (A, s_L, t_L, \gamma_L, \pi_L) \) and an anti-isomorphism \( S \) of the total ring \( A \) satisfying certain conditions, and from here can reconstruct a right bialgebroid \( A_R \) using the same total ring such that the triple \( (A_L, A_R, S) \) is a Hopf algebroid. See \(^7\) for more details.

\(^{11}\) These structures should technically be called dynamical quantum groupoids, as they are certain types of quantum groupoids related to the quantum dynamical Yang-Baxter equation (QDYBE). However, the term dynamical quantum group is more common in the literature.
4.2. Dynamical Quantum Groups. Here we summarize the current theory of dynamical quantum groups. We begin with the basic definitions and then provide the necessary connections with the algebraic terms from the previous subsection. We mainly follow the theory as developed in [29] and summarized in [27], and explain how the earlier discussions of §4.1 are compatible with it.

In order to define dynamical quantum groups, we start with the notion of an $H$-algebra, where $H$ is taken to be a commutative, cocommutative, finitely generated Hopf algebra over $\mathbb{C}$. Let $G = \text{Spec } H$ be the corresponding commutative affine algebraic group. Assuming that $G$ is connected, let $M_G$ denote the field of meromorphic functions on $G$. Then we will say that an associative unital $\mathbb{C}$-algebra $A$ is an $H$-algebra if it has a $G$-bigrading $A = \bigoplus_{\alpha, \beta \in G} A_{\alpha, \beta}$ (called the weight decomposition), and two algebra embeddings $\mu_l, \mu_r : M_G \to A_0$ (called the left and right moment maps respectively) such that we have:

$$\mu_l(f(\lambda))a = a\mu_l(f(\lambda + \alpha)),$$
$$\mu_r(f(\lambda))a = a\mu_r(f(\lambda + \beta)),$$

for any $a \in A_{\alpha, \beta}$ and $f \in M_G$. With a slight change of perspective, we can summarize the above by saying that the ordered quadruple $(A, H, \mu_l, \mu_r)$ is an $H$-algebra.

Next we define the algebra $D_G$ of difference operators on $M_G$. In other words, $D_G$ consists of all operators $M_G \to M_G$ of the form $\sum_{i=1}^n f_i(\lambda)\sigma_{\beta_i}$ where $f_i \in M_G$ and for $\beta \in G$, $\sigma_{\beta}$ is the automorphism of $M_G$ given by $\sigma_{\beta}(f)(\lambda) = f(\lambda + \beta)$. Note that, using these $\sigma_{\alpha}, \sigma_{\beta}$, we can rewrite the conditions on the left and right moment maps given above as follows:

$$\mu_l(f)a = a\mu_l(\sigma_{\alpha}f), \hspace{1cm} \mu_r(f)a = a\mu_r(\sigma_{\beta}f)$$

Now, $D_G$ itself is an $H$-algebra where the bigrading is given by: $f\sigma_{-\alpha} \in (D_G)_{\alpha,\alpha}$. Both moment maps are taken to be the natural embedding of $M_G$ in $D_G$. In fact, it can be shown that $D_G$ behaves very much like the unit object in the category of $H$-algebras in the following sense: Given any $H$-algebra $A$, there are canonical $H$-algebra isomorphisms $A \cong A \otimes D_G \cong D_G \otimes A$ where $a \in A_{\alpha,\beta}$ is mapped to $a \otimes \sigma_{-\beta}$ and $\sigma_{-\alpha} \otimes a$, respectively, see [51] for details.

We can now give the definition for an $H$-bialgebroid.

\[\text{At this point, we should remark that the antipode as defined in [29] contains small inconsistencies, which were noted and modified in [51]. Therefore, it is more accurate to say that we will follow [29] up to some corrections made in [51]. This comment will become clearer in the following discussion.}\]

\[\text{These structure maps generalize certain maps which are interchangeably called moment maps, momentum maps, or momentum mappings in the literature. For purely typographical reasons, we will be using the shortest phrase among these three candidates.}\]
For this, we need a coassociative coproduct, that is, a homomorphism of $H$-algebras $\Delta : A \to A \otimes A$ which satisfies:

$$(\Delta \otimes id_A) \circ \Delta = (id_A \otimes \Delta) \circ \Delta.$$ 

We also need a counit, that is, a homomorphism of $H$-algebras $\epsilon : A \to D_G$ which satisfies the counit axiom:

$$(\epsilon \otimes id_A) \circ \Delta = (id_A \otimes \epsilon) \circ \Delta = id_A.$$ 

In short, we say that the ordered quintuple $A_H = (A, H, \mu_l, \mu_r, \Delta, \epsilon)$ is an $H$-bialgebroid if $(A, H, \mu_l, \mu_r)$ is an $H$-algebra, and $\Delta$ and $\epsilon$ satisfy the required conditions above.

To obtain an $H$-Hopf algebroid, we only need to add to the bag an $H$-antipode, i.e. an antiautomorphism $S : A \to A$ satisfying the following technical condition:

$$a_{(1)} S(a_{(2)}) = \mu_l(\epsilon(a) 1),$$

$$S(a_{(1)}) a_{(2)} = \mu_r(\sigma_\alpha(\epsilon(a) 1)),$$

where $a \in A_{\alpha, \beta}$ and we used the Sweedler notation associated to $\Delta(a)$. $\epsilon(a) 1$ is the result of applying the difference operator $\epsilon(a) \in D_G$ to the constant function 1 $\in M_G$.\(^{14}\) With this last ingredient we can finally define an $H$-Hopf algebroid: It is an $H$-bialgebroid with an $H$-antipode.\(^{15}\)

At this point, it is natural to ask how these structures relate to the structures we defined in the previous subsection. A brief comparison will in fact show that an $H$-bialgebroid as defined in this subsection is going to be a (left) bialgebroid in the sense of §4.1. More specifically, an $H$-bialgebroid $A_H = (A, H, \mu_l, \mu_r, \Delta, \epsilon)$ is a special type of a (left) bialgebroid where the total ring is $A$ and the base field is $L = M_G$. The source and the target maps for the left bialgebroid are determined by the two quantum moment maps $\mu_l$ and $\mu_r$. The coproduct $\Delta$ and the counit $\epsilon$ are, respectively, the two maps $\gamma_L$ and $\pi_L$ making $A$ into a comonoid in the category of $L - L$ bimodules. Similarly adding an $H$-antipode to an $H$-bialgebroid to obtain an $H$-Hopf algebroid is seen to be equivalent to the addition of the antipode to get a Hopf algebroid in §4.1. The $\sigma_\alpha$ will pop up in the third antipode equation while we use the $H$-structure to define a right bialgebroid structure.

\(^{14}\)Note that since $\epsilon$ is an $H$-algebra homomorphism, it preserves the bigrading and so $\epsilon(A_{\alpha, \beta}) = 0$ unless $\alpha = \beta$. Therefore, in the second equation, we could have written $\sigma_\beta$ instead of $\sigma_\alpha$.

\(^{15}\)In \cite{29}, when the antipode for an $H$-bialgebroid is defined, the term $\sigma_\alpha$ is missing from the second equation. However, the correction term is necessary for making the rest of the arguments follow consistently; see \cite{51} for more details. Also, it may be interesting to note that the authors of \cite{51} choose to make a much weaker definition, and then, they prove that one can still get all of the desired properties of an antipode (e.g. uniqueness, antiautomorphism).
How do $H$-Hopf algebroids come up in the realm of quantum groups? To answer this question, we first need to talk about the quantum dynamical Yang-Baxter equation (QDYBE).

Let $\mathfrak{h}$ be a finite-dimensional abelian Lie algebra, and let $V$ be a semisimple $\mathfrak{h}$-module. Then, for a meromorphic function $R : \mathfrak{h}^* \to \text{End}_\mathfrak{h}(V \otimes V)$, the QDYBE is the following equation in $V \otimes V \otimes V$:

$$R^{12}(\lambda - h^{(3)}) R^{13}(\lambda) R^{23}(\lambda - h^{(1)}) = R^{23}(\lambda) R^{13}(\lambda - h^{(2)}) R^{12}(\lambda).$$  \hspace{1cm} (7)

Here, $h^{(i)}$, $i = 1, 2, 3$, is to be replaced by $\mu$ if $\mu$ is the weight of the $i$th tensor component. An invertible solution $R$ of the QDYBE is called a (quantum) dynamical $R$-matrix. Now, certain “nice” dynamical $R$-matrices, in particular the ones satisfying the so-called Hecke condition, may be used to define $H$-Hopf algebroids, where $H$ is the universal enveloping algebra of $\mathfrak{h}$.

A dynamical $R$-matrix $R$ is said to satisfy the Hecke condition, with parameter $q \in \mathbb{C}^*$ if the eigenvalues of $TR$ are 1 on the weight subspaces of $V \otimes V$ of the type $V_a \otimes V_a$ and 1, $q$ on the weight subspaces of the type $(V_a \otimes V_b) \oplus (V_b \otimes V_a)$. As noted in [27], the Hecke condition is a quantum version of Equation (2), the generalized unitarity condition. In particular, if a continuous family of dynamical $R$-matrices of the form $R = 1 - \gamma r + O(\gamma^2)$ satisfies the Hecke condition with parameter $q = 1$, then the semiclassical limit $r$ satisfies the unitarity condition $r + T(r) = 0$.

We do not go into more detail here and refer the reader to [29, 27] for more information. Also see the next section where we discuss superized versions of the QDYBE, dynamical $R$-matrices and the Hecke condition.

Finally we are in a position to define dynamical quantum groups: These are going to be those $H$-Hopf algebroids which can be obtained from dynamical $R$-matrices satisfying the Hecke condition. Clearly they are Hopf algebroids in the sense of §4.1 and are related to the QDYBE, as expected.

4.3. The Categorical Picture. To understand some of the constructions above, and to be able to extend them to the super case, it is imperative that we spend some time on their category-theoretic foundations. The relevant framework was first developed in [29]. In our presentation, we will follow [27].

Recall that an algebra $A$ that is also a coalgebra is a bialgebra precisely when the category of modules over $A$ is a monoidal category with the action of $A$ on any tensor product of modules being induced by the comultiplication. In this sense, bialgebras are algebraic counterparts of monoidal categories. In a similar sense, Hopf algebras are the algebraic counterparts of rigid monoidal categories. In order to talk about quasitriangular structures, $R$-matrices and quantum groups, we need to think further of braided monoidal categories.
More generally, if \( B \) is a braided monoidal category, \( V \) a symmetric tensor category and \( F : B \to V \) a tensor functor, then using any object \( X \) of \( B \), we can construct a \( V \)-automorphism of \( F(X) \otimes F(X) \) which satisfies the QYBE.\(^{16}\) See, for example, \([14, 26]\) for more on monoidal and braided monoidal categories and their relevance to the theory of quantum groups.

In order to talk about the dynamical Yang-Baxter equation and its solutions, we will again need a braided monoidal category \( B \) and a tensor category \( V \). However, this time \( V \) will not typically be a category of modules over a bialgebra or Hopf algebra. Instead, we will consider the so-called dynamical representations of \( H \)-bialgebroids and \( H \)-Hopf algebroids. Below is the definition for this special type of representation, following [27]:

A \textit{dynamical representation} of an \( H \)-algebra \( A \) is a pair \((W, \pi_W)\), where \( W \) is a diagonalizable \( H \)-module and \( \pi_W : A \to D_{G,W} \) is a homomorphism of \( H \)-algebras. Here, as in §4.2, \( H \) is taken to be a commutative, cocommutative, finitely generated Hopf algebra over \( \mathbb{C} \), \( G = \text{Spec } H \) is the corresponding commutative affine algebraic group (assumed to be connected), \( M_G \) is the field of meromorphic functions on \( G \), and \( D_G \) is the algebra of difference operators on \( M_G \). In this setup, we define \( D_{G,W} \) to be the algebra of all difference operators on \( G \) with coefficients in \( \text{End}_G(W) \): \( D_{G,W} = \bigoplus_\alpha D_{G,W}^\alpha \), where \( D_{G,W}^\alpha \subset \text{Hom}_G(W, W \otimes D_G) \) is the space of all difference operators on \( G \) with coefficients in \( \text{End}_G(W) \) that have weight \( \alpha \in G \) with respect to the action of \( H \) in \( W \).

Next, we need to define tensor products of dynamical representations in order to obtain the right kind of category. For instance, following [27, 29], we let the tensor product \( \hat{V} \otimes W \) of two dynamical representations \( V \) and \( W \) of an \( H \)-bialgebroid \( A \) to be the usual tensor product \( V \otimes W \) in the category of vector spaces, and define the map \( \pi_{V \otimes W} : A \to D_{G,V \otimes W} \) by setting \( \pi_{V \otimes W}(a) = \theta_{V,W} \circ (\pi_V \otimes \pi_W) \circ \Delta(a) \). Here \( \theta_{V,W} \) is the natural embedding, as an \( H \)-algebra, of \( D_{G,V} \otimes D_{G,W} \) into \( D_{G,V \otimes W} \) given by:

\[
\sigma_\beta \otimes \sigma_\delta \mapsto (f \hat{\otimes} g) \sigma_\delta
\]

where \( \sigma_\beta \) is the automorphism of \( M_G \) (embedded naturally inside \( D_G \)), given by \( \sigma_\beta(f)(\lambda) = f(\lambda + \beta) \) and \( \sigma_\delta \) is defined in a similar manner. The function \( f \hat{\otimes} g \) is determined by:

\[
f \hat{\otimes} g(\lambda)(v \otimes w) = f(\lambda - \mu)(v) \otimes g(\lambda)(w)
\]

if \( g(\lambda)(w) \) has weight \( \mu \). This definition of tensor products makes the category of dynamical representations of an \( H \)-bialgebroid a tensor category. Following [29], we can also define left and right duals for dynamical representations of \( H \)-Hopf algebroids, but we will not do so here.

\(^{16}\)Some readers may be more familiar with the braid equation, which is a close relative of the QYBE. It was indeed the connection between these two equations that allowed the construction of link invariants from the theory of quantum groups.
Now if $V_d$ is the tensor category of dynamical representations of an $H$-bialgebroid or an $H$-Hopf algebroid, and if we are given a tensor functor $F : B \to V_d$, then using any object $X$ of $B$, we can once again construct a $V_d$-automorphism $R = R(B, F, X) \circ F(X) \otimes F(X)$. It turns out that this new automorphism does not satisfy the quantum Yang-Baxter equation but instead the quantum dynamical Yang-Baxter equation (QDYE). Conversely, given a dynamical $R$-matrix $R$, we can find appropriate categories $B$ and $V_d$, a tensor functor $F : B \to V_d$ and a particular object $X$ of $B$ such that $R$ is the $V_d$-automorphism associated to the triple $(B, F, X)$ in the same manner as above, see [27].

5. Dynamical Quantum Groups in the Super Setting

We now begin our study of the dynamical quantum super groups. Once again we emphasize that we will exclusively follow the Hopf algebroid approach for dynamical quantum groups. We should note that a superization of the quasi-Hopf algebra approach for dynamical quantum groups was already initiated, in [36, 37].

5.1. Dynamical Quantum Groups - The Super Story. In order to get the correct definitions for the super analogues of the notions of Hopf algebroids and dynamical quantum groups, we now concentrate on the structures described in §4.2 and §4.3 and superize them systematically. Most of the superization will be straightforward, but we still wish to state our definitions explicitly as often as possible. Some repetition, therefore, will be unavoidable, but hopefully, this will help the reader follow the paper with more ease.

We start with the superization of the constructions in §4.2. In particular, we start with the notion of an $H$-*superalgebra*, where $H$ is a commutative, cocommutative, finitely generated super Hopf algebra over $\mathbb{C}$ [13]. Set $G = \text{Spec } H$, and assuming that $G$ is connected, let $M_G$ denote the field of meromorphic functions on $G$. Then we will say that an associative unital $\mathbb{C}$-superalgebra $A$ is an $H$-*superalgebra* if it has a $G$-bigrading $A = \bigoplus_{\alpha, \beta \in G} A_{\alpha, \beta}$ (called the *weight decomposition*) compatible with the $\mathbb{Z}/2\mathbb{Z}$-grading [19] and two superalgebra embeddings $\mu_l, \mu_r : M_G \to A_{0,0}$ with:

---

17 Incidentally, an earlier result related to dynamical quantum super groups which is definitely worth mentioning can be found in [40]. Here, the author works in the framework of the linear quantum groups and computes several solutions to the QDYBE in both the non-graded and the graded cases. As we develop our theory based on Hopf structures, it will be interesting to see how our methods relate to those used in [40] and compare results.

18 Super Hopf algebras are special examples of Hopf algebras in the braided monoidal category of Yetter-Drinfeld modules, see [12, 50, 54] for details and important results on Yetter-Drinfeld modules.

19 In other words, we have: $A_{\alpha, \beta} = (A_{\alpha, \beta})_\pi \oplus (A_{\alpha, \beta})_\tau$, where $(A_{\alpha, \beta})_\pi = A_{\alpha, \beta} \cap A_\pi$. 

---
\[
\begin{align*}
\mu_l(f(\lambda))a &= a\mu_l(f(\lambda + \alpha)), \\
\mu_r(f(\lambda))a &= a\mu_r(f(\lambda + \beta)),
\end{align*}
\]
for any \(a \in A_{\alpha,\beta}\) and \(f \in M_G\). (\(\mu_l, \mu_r\) are called the \textit{left} and \textit{right moment maps}). With a slight change of perspective, we can summarize the above by saying that the ordered quadruple \((A, H, \mu_l, \mu_r)\) is an \(H\)-superalgebra. We note that if \(H\) is trivial, i.e. \(H = \mathbb{C}\), then an \(H\)-superalgebra is merely a superalgebra.

Next we define the algebra \(D_G\) precisely the same way that we defined it in \(\S 4.2\). Once again, there is a natural embedding of \(M_G\) in \(D_G\). We can then give the definition for an \(H\)-superbialgebra. As expected, for this, we only need a \textit{coproduct}, i.e. a homomorphism of \(H\)-superalgebras \(\Delta : A \rightarrow A \otimes A\) and a \textit{counit}, i.e. a homomorphism of \(H\)-superalgebras \(\epsilon : A \rightarrow D_G\). We require that \(\Delta\) be coassociative, i.e. \((\Delta \otimes \text{id}_A) \circ \Delta = (\text{id}_A \otimes \Delta) \circ \Delta\), and that the counit \(\epsilon\) satisfy the \textit{counit axiom}:

\[
(\epsilon \otimes \text{id}_A) \circ \Delta = (\text{id}_A \otimes \epsilon) \circ \Delta = \text{id}_A.
\]

In short, we say that \(A_H = (A, H, \mu_l, \mu_r, \Delta, \epsilon)\) is an \textit{\(H\)-superbialgebra} if \((A, H, \mu_l, \mu_r)\) is an \(H\)-superalgebra, and \(\Delta\) and \(\epsilon\) satisfy the required conditions above. We note that if \(H\) is trivial, i.e. \(H = \mathbb{C}\), then an \(H\)-superbialgebra is merely a superbialgebra.

To obtain an \(H\)-Hopf superalgebra, we only need to add to the above an \textit{\(H\)-antipode}, i.e. an antiisomorphism \(S : A \rightarrow A\) satisfying the technical condition:

\[
\begin{align*}
\alpha(1)S(\alpha(2)) &= \mu_l(\epsilon(\alpha)1), \\
S(\alpha(1))\alpha(2) &= \mu_r(\sigma_\alpha \epsilon(\alpha)1),
\end{align*}
\]

where \(a \in A_{\alpha,\beta}\), \(\epsilon(\alpha)1\) is the result of applying the difference operator \(\epsilon(\alpha) \in D_G\) to \(1 \in M_G\) and we used the Sweedler notation associated to \(\Delta(\alpha)\). With this last ingredient, we can finally define an \textit{\(H\)-Hopf superalgebra}: It is an \(H\)-superbialgebra with an \(H\)-antipode. Once again in the case when \(H = \mathbb{C}\), an \(H\)-Hopf superalgebra is only a super Hopf algebra.\(^{20}\)

\(^{20}\)Here, we may mention the idea of \textit{bosonization}, which was introduced by Radford in \(\cite{63}\). For another exposition, the reader may look at \(\cite{56}\). The connection between Hopf and super Hopf structures may be viewed as a special case of bosonization; see \(\cite{34}\) Sec. 7. The main idea of bosonization with respect to super Hopf algebras is as follows: For many practical purposes we can just as well work with regular Hopf algebras instead of super Hopf algebras. More precisely, the tensor category of representations of an ordinary Hopf algebra \(H\) with a grouplike element \(g\) with \(g^2 = 1\) is equivalent to the tensor category of representations of a related super Hopf algebra \(H_s\) with a grouplike odd element \(g_s\) with \(g_s^2 = 1\), and the correspondence between such pairs \((H, g)\) and \((H_s, g_s)\) is one-to-one. The significance and the implications of this categorical equivalence for our context need further investigation.
Now, we are ready to move on to the superization of the constructions of §4.3. We first define super dynamical representations: A super dynamical representation of an $H$-superalgebra $A$ should be a pair $(W, \pi_W)$, where $W$ is a diagonalizable $H$-module and $\pi_W : A \to D_{G,W}$ is a homomorphism of $H$-superalgebras. Here, as before, $H$ is taken to be a commutative, cocommutative, finitely generated Hopf algebra over $\mathbb{C}$. $G$, $M_G$, $D_G$, $D_{G,W}$ and $D_{G,W}^0$ are defined as above. Note that, this time, $W$ lives in the category of super vector spaces.

Next, we will follow §4.3 and superize the construction of tensor products of dynamical representations in order to obtain the right kind of category. In particular, we will let the tensor product $V \otimes W$ of two dynamical representations $V$ and $W$ of an $H$-superalgebra $A$ to be the usual tensor product $V \otimes W$ in the category of super vector spaces, and we will define the map $\pi_{V\otimes W} : A \to D_{G,V\otimes W}$ by setting $\pi_{V\otimes W}(a) = \theta_{V,W} \circ (\pi_V \otimes \pi_W) \circ \Delta(a)$. Here $\theta_{V,W}$ is the natural embedding, as a $H$-superalgebra, of $D_{G,V} \otimes D_{G,W}$ into $D_{G,V \otimes W}$ given by:

$$f \sigma_\beta \otimes g \sigma_\delta \mapsto (f \hat{\otimes} g) \sigma_\delta$$

where $\sigma_\beta$ is the automorphism of $M_G$ (embedded naturally inside $D_G$), given by $\sigma_\beta(f)(\lambda) = f(\lambda + \beta)$ and $\sigma_\delta$ is defined in a similar manner. The function $f \hat{\otimes} g$ is determined by:

$$f \hat{\otimes} g(\lambda)(v \otimes w) = f(\lambda - \mu)(v) \otimes g(\lambda)(w)$$

if $g(\lambda)(w)$ has weight $\mu$. This construction will give us a tensor category.

Let $V_d$ be the tensor category of super dynamical representations of an $H$-superalgebra or an $H$-Hopf superalgebra, and assume that we are given a tensor functor $F : \mathcal{B} \to V_d$. Then using any object $X$ of $\mathcal{B}$, we can construct a $V_d$-automorphism $R_s = R_s(\mathcal{B}, F, X)$ of $F(X) \otimes F(X)$ by setting $R_s = T_s F(\beta_{X,Y})$, where $\beta_{X,Y} : X \otimes_B Y \to Y \otimes_B X$ is the braiding of the category $\mathcal{B}$ and $T_s$ is the usual super twist in the category of super vector spaces. It is a simple exercise to show that this $R_s$ satisfies the quantum dynamical Yang-Baxter equation (QDYBE) and hence is a super dynamical $R$-matrix. Conversely it can be shown that given a super dynamical $R$-matrix $R_s$, i.e. a solution $R_s$ to the QDYBE, we can find appropriate categories $\mathcal{B}$ and $V_d$, a tensor functor $F : \mathcal{B} \to V_d$ and a particular object $X$ of $\mathcal{B}$ such that $R_s$ is the $V_d$-automorphism associated to the triple $(\mathcal{B}, F, X)$ in the same manner as in §4.3; this is a straight-forward extension of Theorem 3.1 of [29].

So far, we see that almost all our constructions are straight-forward superizations of those from the non-graded theory.

---

21If we want rigidity for our category, we need to construct left and right duals, and this can be done, with the additional hypothesis that $A$ be an $H$-Hopf superalgebra. We will once again choose not to go into the duality problem.
Finally we are in a position to define super dynamical quantum groups: These are going to be those \( H \)-Hopf superalgebras which can be obtained from super dynamical \( R \)-matrices satisfying the super Hecke condition. We will discuss this condition and the associated construction in more detail in the next subsection.

We end this subsection with a note about our choice of terminology: In this paper we preferred to use the terms \( H \)-superbialgebra, \( H \)-Hopf superalgebra, and super dynamical quantum group instead of the perfectly acceptable alternatives \( H \)-superbialgebroid, \( H \)-Hopf superalgebroid, and super dynamical quantum groupoid in order to minimize the number of terms which end with \(-oid\). A plausible argument for the second set of terms could be that these signify much more clearly that they are special types of superbialgebroids, Hopf superalgebroids or super quantum groupoids, but since we did not actually define these latter terms, we feel comfortable with our choices.

5.2. Super Dynamical \( R \)-matrices. Here we present some constructions related to super dynamical quantum groups. We start with a discussion of the QDYBE and its solutions in the super setting. We follow closely [27, 29].

We restrict ourselves to the following setup: Let \( \mathfrak{h} \) be the standard Cartan subsuperalgebra of \( \mathfrak{gl}(m, n) \). This is the set of all diagonal matrices from \( \mathfrak{gl}(m, n) \). Let \( V \) be a semisimple \( \mathfrak{h} \)-module. Recall that the quantum dynamical Yang-Baxter equation is:

\[
R^{12}(\lambda - h^{(3)}) R^{13}(\lambda) R^{23}(\lambda - h^{(1)}) = R^{23}(\lambda) R^{13}(\lambda - h^{(2)}) R^{12}(\lambda). \tag{7}
\]

We will say that a solution \( R : \mathfrak{h}^* \to \text{End}_\mathfrak{h}(V \otimes V) \) to the QDYBE is a super dynamical \( R \)-matrix if it is an invertible meromorphic function. For simplicity, we will only consider even solutions.

Rescaling the QDYBE by \( \lambda \mapsto \lambda / \gamma \) we get the quantum dynamical Yang-Baxter equation with step \( \gamma \):

\[
R^{12}(\lambda - \gamma h^{(3)}) R^{13}(\lambda) R^{23}(\lambda - \gamma h^{(1)}) = R^{23}(\lambda) R^{13}(\lambda - \gamma h^{(2)}) R^{12}(\lambda).
\]

Invertible solutions of the QDYBE with step \( \gamma \) will also be called super dynamical \( R \)-matrices.

If \( R : \mathfrak{h}^* \to \text{End}_\mathfrak{h}(V \otimes V)[[\gamma]] \) is a continuous family of \( \mathfrak{h} \)-invariant meromorphic functions of the form \( R = 1 - \gamma r + O(\gamma^2) \) satisfying the QDYBE with step \( \gamma \), then a simple computation will show that \( r \) is an \( \mathfrak{h} \)-invariant super dynamical \( r \)-matrix satisfying the CDYBE. This follows from a straightforward superization of Proposition 3.1 of [27]. We naturally call \( r \) the semiclassical limit of \( R \) and \( R \) a quantization of \( r \).
Now we consider the case when \( V = V_{\mathfrak{h}} \oplus V_{\mathfrak{T}} \) is the standard vector representation of \( \mathfrak{h} \). In other words, \( V \) is a super vector space with even dimension \( m \) (i.e. \( \dim_{\mathbb{C}}(V_{\mathfrak{h}}) = m \)) and odd dimension \( n \) (i.e. \( \dim_{\mathbb{C}}(V_{\mathfrak{T}}) = n \)).

Let \( \{ h_{\mathfrak{h},1}, \ldots, h_{\mathfrak{h},m}, h_{\mathfrak{T},1}, \ldots, h_{\mathfrak{T},n} \} \) be the standard basis for \( \mathfrak{h} \) and let \( \lambda_{\mathfrak{h},1}, \ldots, \lambda_{\mathfrak{h},m}, \lambda_{\mathfrak{T},1}, \ldots, \lambda_{\mathfrak{T},n} \) be the corresponding coordinate functions on \( \mathfrak{h}^* \). Note that each \( h_{\mathfrak{h},i} \) will be even as all of \( \mathfrak{h} \) lies inside \( \mathfrak{gl}(m,n)_{\mathfrak{T}} \). Let \( V_{\mathfrak{h},i} \), \( s = 0,1 \), be the one-dimensional weight subspaces of \( V \) of weight \( \omega_{\mathfrak{h},i} \) such that \( (\omega_{\mathfrak{h},i}, h_{\mathfrak{T},j}) = \delta_{s,t} \delta_{i,j} \). Then, the tensor product \( V \otimes V \) will have weight subspaces of the form \( V_{\mathfrak{h},i} \otimes V_{\mathfrak{h},i} \) and \( (V_{\mathfrak{h},i} \otimes V_{\mathfrak{T},j}) \oplus (V_{\mathfrak{T},j} \otimes V_{\mathfrak{h},i}) \).

We will say that a super dynamical \( R \)-matrix \( R \) satisfies the super Hecke condition\(^{\text{22}} \), with parameter \( q \in \mathbb{C}^* \) if the eigenvalues of \( T_s R \) are 1 on \( V_{\mathfrak{h},i} \otimes V_{\mathfrak{h},i} \) and \( 1, (-1)^st \) on \( (V_{\mathfrak{h},i} \otimes V_{\mathfrak{T},j}) \oplus (V_{\mathfrak{T},j} \otimes V_{\mathfrak{h},i}) \). Note that here \( T_s \) stands for the matrix form of the super twist, but since \( R \) is assumed to be even, it may be replaced simply by the permutation matrix.

We will say that a super dynamical \( R \)-matrix has zero weight if we have:

\[
[R^{ij}(\lambda), h \otimes 1 + 1 \otimes h] = 0
\]  
(8)

for all \( i, j = 1, 2, 3, i \neq j \) and \( h \in \mathfrak{h}, \lambda \in \mathfrak{h}^* \). As in the case of super dynamical \( r \)-matrices in Subsection 3.3, the zero weight condition, i.e. Equation (8), is equivalent to \( \mathfrak{h} \)-invariance.

We can easily see that a zero-weight super dynamical \( R \)-matrix \( R(\lambda) \) satisfying the super Hecke condition with parameter \( q \) has to be of the form:

\[
R(\lambda) = \sum_a E_{aa} \otimes E_{aa} + \sum_{a \neq b} \alpha_{ab}(\lambda) E_{aa} \otimes E_{bb} + \sum_{a \neq b} \beta_{ab}(\lambda) E_{ab} \otimes E_{ba},
\]

where \( E_{ab} \) is the elementary matrix corresponding to the \( (a, b) \)th matrix entry, and \( \alpha_{ab} \) and \( \beta_{ab} \) are certain meromorphic functions \( \mathfrak{h}^* \rightarrow \mathbb{C} \).

In [4,2] we noted in passing that dynamical \( R \)-matrices satisfying the (non-graded) Hecke condition can be used to define \( H \)-Hopf algebroids. Now we will explicitly superize the relevant constructions from [27, 29] in order to see how we can obtain an \( H \)-Hopf superalgebra from a super dynamical \( R \)-matrix satisfying the super Hecke condition.

For this, we start with our commutative Lie superalgebra \( \mathfrak{h} \), the standard Cartan subsuperalgebra of \( \mathfrak{gl}(m,n) \). Recall that this happens to be the set of all diagonal matrices from \( \mathfrak{gl}(m,n) \). Once again, we let \( V = V_{\mathfrak{h}} \oplus V_{\mathfrak{T}} \) be a semisimple \( \mathfrak{h} \)-module (living in the category of super vector spaces).

---

\(^{\text{22}}\) Analogous to the non-graded case, the super Hecke condition is a quantum version of Equation (8), the generalized unitarity condition. In particular, if a continuous family of super dynamical \( R \)-matrices of the form \( R = 1 - \gamma r + O(\gamma^2) \) satisfies the super Hecke condition with parameter \( q = 1 \), then the semiclassical limit \( r \) satisfies the unitarity condition \( r + T_s(r) = 0 \).
Set $\dim_{\mathbb{C}}(V_\mathfrak{p}) = M$ and $\dim_{\mathbb{C}}(V_\mathfrak{q}) = N$. Pick an ordered homogeneous basis for $V$: $\{v_1, \ldots, v_{M+N}\} = \{v_{\mathfrak{p},1}, \ldots, v_{\mathfrak{p},M}, v_{\mathfrak{q},1}, \ldots, v_{\mathfrak{q},N}\}$ with $v_{\mathfrak{p},i} \in V_\mathfrak{p}, p = 0, 1$.

Now let $R: \mathfrak{h}^* \to \text{End}_{\mathfrak{h}}(V \otimes V)$ be a meromorphic function satisfying the zero weight condition such that $R(\lambda)$ is invertible for generic $\lambda$. We denote by $H$ the super Hopf algebra associated to $\mathfrak{h}$; in other words, $H$ is the commutative cocommutative super Hopf algebra which is the universal enveloping superalgebra of $\mathfrak{h}$. Clearly $V$ is also an $H$-module. Let $G, M_G$ and $D_G$ be defined as before. In other words, $G = \text{Spec}(H), M_G$ is the field of meromorphic functions on $G$, and $D_G$ is the algebra of all difference operators on $M_G$.

We first define an $H$-superbialgebra $A_H(R) = (A_R, H, \mu^R_\ell, \mu^R_r, \Delta^R, \epsilon^R)$ associated to $R$. For this, we need an $H$-superalgebra $(A_R, H, \mu^R_\ell, \mu^R_r)$ and two maps $\Delta^R$ and $\epsilon^R$ satisfying the required conditions from Subsection 5.1. To describe $A_R$, we start with the superalgebra generated freely by $M_G \otimes M_G$. We follow [29] in their notations and denote the first copy of $M_G$ by the superscript $^{(1)}$ and the second by $^{(2)}$: If $f \in M_G$ is any function, then we denote the image of $f$ in the first (respectively, the second) copy of $M_G$ by $f^{(1)}(\lambda)$ (respectively, by $f^{(2)}(\lambda)$).

Next we put in some additional generators $L_{i,j}, L_{i,j}^{-1}$, for $i, j$ between 1 and $M + N$. Intuitively, each $L_{i,j}$ corresponds to the $(i,j)$th matrix entry of a particular operator:

$$L = \sum_{a,b=1}^{M+N} E_{ab} \otimes L_{a,b},$$

and each $L_{i,j}^{-1}$ corresponds to the $(i,j)$th matrix entry of the inverse operator $L^{-1}$. The parity of these generators is determined once a homogeneous basis for $V$ is picked. Hence, $L_{i,j}$ and $L_{i,j}^{-1}$ will be even if $(1 \leq i, j \leq M)$ or $(M+1 \leq i, j \leq M+N)$, and they will be odd otherwise.

Now we will define $A_R$ to be the quotient of this superalgebra by several relations. First we will encode the invertibility of $L$ by requiring that:

$$LL^{-1} = L^{-1}L = 1$$

and enforce the QDYBE by requiring:

$$R^{12}(\lambda^{(1)})L^{13}L^{23} = L^{23}L^{13}R^{12}(\lambda^{(2)}):$$

Note that in the equation above, we are using the normal order notation “:.” In other words, if we define the matrix entries of $R(\lambda)$ by:

$$R(\lambda)(v_a \otimes v_b) = \sum_{c,d=1}^{M+N} P_{c,d}^{ab}(\lambda)v_c \otimes v_d,$$

we have:

$$R(\lambda)(v_a \otimes v_b) = \sum_{c,d=1}^{M+N} P_{c,d}^{ab}(\lambda)v_c \otimes v_d.$$
then the above version of the QDYBE may be rewritten as follows:

\[ \sum_{x,y=1}^{M+N} R_{ac}^{bd}(\lambda^{(1)}) L_{x,b} L_{y,d} = \sum_{x,y=1}^{M+N} R_{xy}^{bd}(\lambda^{(2)}) L_{c,y} L_{d,x}. \]

We also require the following relations:

\[
\begin{align*}
  f(\lambda^{(1)}) L_{\alpha,\beta} &= L_{\alpha,\beta} f(\lambda^{(1)} + \alpha) \\
  f(\lambda^{(2)}) L_{\alpha,\beta} &= L_{\alpha,\beta} f(\lambda^{(2)} + \beta)
\end{align*}
\]

along with

\[ [f(\lambda^{(1)}), g(\lambda^{(2)})] = 0. \]

Here, \( L_{\alpha,\beta}, \alpha, \beta \in G, \) are the weight components of \( L \) with respect to the \( G \)-bigrading on \( \text{End}(V) \) corresponding to the \( G \)-weight decomposition \( V = \oplus_{\alpha \in G} V_{\alpha} \).

The last few relations make more sense once we define the moment maps \( \mu_{l}^{R} \) and \( \mu_{r}^{R} : M_{G} \to (A_{R}, 0, 0) : \)

\[ \mu_{l}^{R}(f) = f(\lambda^{(1)}) \quad \text{and} \quad \mu_{r}^{R}(f) = f(\lambda^{(2)}). \]

We also require that \( \mu_{l}^{R} \) and \( \mu_{r}^{R} \) will preserve the \( \mathbb{Z}/2\mathbb{Z} \)-grading. Now we can see that the two relations about \( f(\lambda^{(i)}) L_{\alpha,\beta} \) are merely the defining relations of the moment maps.

At this point, it is clear that the quadruple \((A_{R}, H, \mu_{r}^{R}, \mu_{l}^{R})\) is an \( H \)-superalgebra. To obtain the \( H \)-superbialgebra that we want, we only need to define a suitable coproduct \( \Delta_{R} : A_{R} \to A_{R} \otimes A_{R} \) and an appropriate counit \( \epsilon_{R} : A_{R} \to D_{G} \). The definitions from \cite{27, 29} will work for us:

\[ \Delta_{R}(L) = L^{12} L^{13}, \quad \Delta_{R}(L^{-1}) = (L^{-1})^{13}(L^{-1})^{12}, \]

and:

\[ \epsilon_{R}(L_{\alpha,\beta}) = \delta_{\alpha,\beta} id_{V_{\alpha}} \otimes \sigma^{-1}_{\alpha}, \quad \epsilon_{R}((L^{-1})_{\alpha,\beta}) = \delta_{\alpha,\beta} id_{V_{\alpha}} \otimes \sigma^{-1}_{\alpha}. \]

The arguments in the proofs of Propositions 4.2 and 4.3 from \cite{29} can then be superized in a straight-forward manner to show that the defining relations of \( A_{R} \) are invariant under \( \Delta_{R} \) and are annihilated by \( \epsilon_{R} \), that the counit satisfies the counit axiom, and finally that \( A_{H}(R) = (A_{R}, H, \mu_{l}^{R}, \mu_{r}^{R}, \Delta_{R}, \epsilon_{R}) \) is an \( H \)-superbialgebra.

Now to get an \( H \)-Hopf superalgebra, we need an appropriate \( H \)-antipode. Once again we follow the arguments of \cite{29} and see that they are easily generalized to the super case. The proofs of Propositions 4.4, 4.5 and 4.6 from \cite{29} carry over almost directly. We can show that, if we start with a continuous family of super dynamical \( R \)-matrices \( R_{\gamma}(\lambda) \) with step \( \gamma \) such that \( R_{0} = 1, \)

\[ \text{Equivalently we can extract the weight decomposition of } L \text{ from the natural } \mathfrak{h} \text{-bigrading on } \text{End}(V) \text{ corresponding to the weight decomposition } V = \oplus_{\alpha \in \mathfrak{h}^{\ast}} V_{\alpha} \text{ if we prefer to think only in terms of the Lie superalgebra } \mathfrak{h} \text{ and its associates.} \]
then there is a unique map $S_R$ on $A_R$ which sends $L$ to $L^{-1}$ and makes $A_R$ into an $H$-Hopf superalgebra $A_H(R) = (A_R, H, \mu^R_I, \mu^R_R, \Delta_R, \epsilon_R, S_R)$.

In the above constructions, we did not use the QDYBE. In fact dropping the QDYBE, or more accurately Equation (9), would still give us an $H$-superbialgebra. However, we want more than just an $H$-superbialgebra. In [27], Etingof and Schiffmann show that the QDYBE is a sufficient condition to ensure the existence of at least one dynamical representation of $A_H(R)$. Their argument carries over to the super case. If $R$ is a super dynamical $R$-matrix, then we can obtain a super dynamical representation $(V, \pi_V)$ of $A_H(R)$ on $V$ by first defining a map $\pi^0_V: A_R \to \text{Hom}(V, V \otimes M_G)$ via $\pi^0_V(\lambda) = R(\lambda)$, and, then, taking $\pi_V$ as the unique extension of $\pi^0_V$.

We end this section with a short remark about the super Hecke condition. Notice that we also did not make use of the super Hecke condition in the construction above. In the non-graded case, when we restrict the types of algebras that can be constructed by the above technique, the Hecke condition comes up naturally. In fact, it is shown explicitly in [27] that, if the (non-graded) dynamical $R$-matrix satisfies the (non-graded) Hecke condition, then the associated algebra $A_R$ may be viewed as a deformation of a function algebra of a matrix group. We leave the discussion of the superization of this result to the next section.

6. Open Questions

In this paper, we aimed to discuss both what has been accomplished so far in the theory of super quantum groups and the path still lying ahead. Our approach definitely favored Hopf algebroids as the main framework, and the emphasis was clearly on the dynamical picture.

We now end this paper with a brief discussion of the results presented and the questions that still need to be answered.

As we have seen, many constructions related to dynamical quantum groups can naturally be extended to the super case. Many definitions are the expected super analogues of the non-graded ones. We have presented the super versions of the major classification results in the classical picture, proving explicitly that the constructions still make sense. For the quantum picture, we have only just begun: we made the necessary definitions and extended only the most basic of the constructions.

Below we list some remarks and questions that we think may be relevant for the further development of the theory of super quantum groups. This list may also be viewed as our plan of action for the near future:

(1). In §5.2 we proposed a superization of the Hecke condition that we think best generalizes the non-graded version by considering it as the right
way to quantize Equation (3), the generalized unitarity condition. A good test to see whether we made the correct definition would be to check if the superalgebra $A_R$ associated to a super dynamical $R$-matrix satisfying the super Hecke condition may be viewed as a deformation superalgebra of the appropriate type. In fact, the relevant results from [29] (Theorems 6.1, 6.2, 6.3 and Proposition 6.1) seem, at a first glance, amenable to straightforward superization, and we expect that this should not be hard to verify.

(2). In [29], Etingof and Varchenko classify all (non-graded) dynamical $R$-matrices that satisfy the Hecke condition. If the super Hecke condition is correct, then a natural next step would be to extend this result to the super case.

(3). There is a nice path to a partial solution of the quantization problem in the super setting. In [29], all (non-graded) zero-weight dynamical $r$-matrices (with any coupling constant) are explicitly quantized. We have seen in [32] that the classification paradigm for the zero-weight dynamical $r$-matrices in the non-graded case can be superized in a natural manner. Therefore, one could expect that the solution of the quantization problem for the zero-weight super dynamical $r$-matrices will be a natural superization of the solution of the non-graded problem.

(4). The general problem of quantization remains open. The extension to the graded world of the main quantization result from [25], the general constructive quantization of all classical dynamical $r$-matrices which fit Schiffmann’s classification (2.3) is very important.

(5). The complications mentioned in [13] which made the Belavin-Drinfeld classification results hard to superize are still there in the dynamical case. These still need to be addressed. Hence, the classification problem is still open as well.

References

[1] Arnaudon, D., Buffenoir, E., Ragoucy, E., Roche, Ph.: “Universal solutions of quantum dynamical Yang-Baxter equations”, Lett. Math. Phys. 44 (1998), no. 3, pp.201–214.
[2] Babelon, O., Bernard, D., Billey, E.; “A quasi-Hopf algebra interpretation of quantum 3-j and 6-j symbols and difference equations”, Phys. Lett. B 375 (1996), no. 1-4, pp.89–97.
[3] Balog, J., Dabrowski, L., Fehér, L.; “Classical $r$-matrix and exchange algebra in WZNW and Toda theories”, Phys. Lett. B 244 (1990), no. 2, pp.227–234.
[4] Belavin, A. A., Drinfeld, V. G.; “Solutions of the Classical Yang-Baxter Equation and Simple Lie Algebras”, Funct. Anal. Appl. 16 (1982), pp.159–180.
[5] Belavin, A. A., Drinfeld, V. G.; “Triangle Equation and Simple Lie Algebras”, Soviet Scientific Reviews Sect. C 4 (1984), pp.93–165.
[6] Bernard, D.; “On the Wess-Zumino-Witten Models on the Torus”, Nucl. Phys. B 303 (1988), pp.77–93.
[7] Böhm, G.; "An Alternative Notion of Hopf Algebroid", in: Hopf Algebras in Noncommutative Geometry and Physics, Lecture Notes in Pure and Appl. Math. 239, Dekker, 2005, pp.31–53.

[8] Böhm, G., Nill, F., Szlachá nyi, K.; "Weak Hopf Algebras I. Integral Theory and C*-Structure", J. Algebra 221 (1999), pp.385–438.

[9] Böhm, G., Szlachá nyi, K.; "A Coassociative C*-Quantum Group with Nonintegral Dimensions", Lett. Math. Phys. 38 (1996), no.4, pp.437–456.

[10] Böhm, G., Szlachá nyi, K.; "Hopf Algebroids with Bijective Antipodes: Axioms Integrals and Duals", J. Algebra 274 (2004), no.2, pp.708–750.

[11] Brown, K. A., Goodearl, K. R.; Lectures on Algebraic Quantum Groups, Birkhäuser, 2002.

[12] Caenepeel, S., Guedenon, T.; “Semisimplicity of the categories of Yetter-Drinfeld modules and long dimodules”, Comm. Algebra 32 (2004), no. 7, pp.2767–2781.

[13] Canas da Silva, A., Weinstein, A.; Geometric Models for Noncommutative Algebras, (Berkeley Mathematics Lecture Notes 10), American Mathematical Society, Providence, RI; Berkeley Center for Pure and Applied Mathematics, Berkeley, CA, 1999.

[14] Chari, V., Pressley, A.; A Guide to Quantum Groups, Cambridge University Press, 1995.

[15] Cherednik, I. V.; “Generalized Braid Groups and Local r-matrix Systems”, Soviet Math. Dokl. 307 (1990), pp.43–47.

[16] Coste, A., Dazord, P., Weinstein, A.; “Groupoides symplectiques (French) [Symplectic groupoids], Publications du Département de Mathématiques, Nouvelle Série. A, Vol. 2, i–ii, 1–62, Publ. Dp. Math. Nouvelle Sr. A, 87-2, Univ. Claude-Bernard, Lyon, 1987.

[17] Deligne, P., Freed, D. S.; “Sign manifesto”; Quantum Fields and Strings: A Course for Mathematicians, Vol. 1, 2; Eds. Pierre Deligne, Pavel Etingof, Daniel S. Freed, Lisa C. Jeffrey, David Kazhdan, John W. Morgan, David R. Morrison and Edward Witten; Amer. Math. Soc. Providence, RI, 1999, pp.357–363.

[18] Drinfeld, V. G.; “Hamiltonian Structures on Lie Groups, Lie Bialgebras and the Geometric Meaning of the Classical Yang-Baxter Equations”, Soviet Math. Dokl. 27 (1983), pp.68–71.

[19] Drinfeld, V. G.; “Quasi-Hopf algebras”, Leningrad Math. J. 1 (1990), no. 6, pp.1419–1457.

[20] Enriquez, B., Felder, G.; “Elliptic quantum groups E_{r,\alpha}(sl_2) and quasi-Hopf algebras”, Comm. Math. Phys. 195 (1998), no. 3, pp.–689.

[21] Etingof, P.; “On the Dynamical Yang-Baxter Equation”, Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002), Higher Ed. Press, 2002, pp.555–570.

[22] Etingof, P., Frenkel, I., Kirillov, A. Jr.; Lectures on Representation Theory and Knizhnik-Zamolodchikov Equations, Mathematical Surveys and Monographs 58, American Mathematical Society, 1998.

[23] Etingof, P., Kazhdan, D.; “Quantization of Lie Bialgebras I”, Selecta Math. 2 (1996), no.1, pp.1–41.

[24] Etingof, P., Latour, F.; The Dynamical Yang-Baxter Equation, Representation Theory, and Quantum Integrable Systems; Oxford University Press, 2005

[25] Etingof, P., Scheufer, T., Schiffmann, O.; “Explicit quantization of dynamical r-matrices for finite dimensional semisimple Lie algebras”; J. Amer. Math. Soc. 13 (2000), no.3, pp.595–609.

[26] Etingof, P., Schiffmann, O.; Lectures on Quantum Groups, International Press, 1998.

[27] Etingof, P., Schiffmann, O.; “Lectures on the Dynamical Yang-Baxter Equations”; Quantum groups and Lie theory (Durham, 1999). London Math. Soc. Lecture Note Ser., 290, Cambridge Univ. Press, Cambridge, 2001’ pp.89–129.
[28] Etingof, P., Varchenko, A.; “Geometry and Classification of Solutions of the Classical Dynamical Yang-Baxter Equation”, Comm. Math. Phys. 192 (1998), no.1, pp.77–120.

[29] Etingof, P., Varchenko, A.; “Solutions of the Quantum Dynamical Yang-Baxter Equation and Dynamical Quantum Groups”, Comm. Math. Phys. 196 (1998), no.3, pp.591–640.

[30] Felder, G.; “Conformal Field Theory and Integrable Systems associated to Elliptic Curves”, Proceedings of the International Congress of Mathematicians (Zurich 1994), Birkhäuser, 1994, pp.1247–1255.

[31] Felder, G.; “Elliptic Quantum Groups”, Proceedings of the X1th International Congress of Mathematical Physics (Paris, 1994), International Press, 1995, pp.211–218.

[32] Gelaki, S.; “On the classification of finite-dimensional triangular Hopf algebras”, in: New Directions in Hopf Algebras, S. Montgomery and H-J. Schneider eds., Math. Sci. Cambridge Univ. Press, Cambridge, 2002, pp.69–116.

[33] Grozman, P., Leites, D. A.; “Defining Relations for Classical Lie Superalgebras with Cartan Matrix”, Czechoslovak J. Phys. 51 (2001), no. 1, pp.1–21.

[34] Kac, V. G.; “Lie Superalgebras”, Advances in Mathematics 26 (1977), pp.8–96.

[35] Karolinsky, E., Stolin, A.; “Classical Dynamical r-matrices, Poisson Homogeneous Spaces and Lagrangian Subalgebras”, Lett. Math. Phys. 60 (2002), pp.257–274.

[36] Karolinsky, E., Stolin, A.; “Classical Dynamical r-matrices and Poisson Homogeneous Spaces”, in: Multiple Facets of Quantization and Supersymmetry, M. Olshanetsky and A. Vainshtein eds., World Scientific, 2002, pp.252–266.

[37] Kassel, C.; Quantum Groups, (Graduate Texts in Mathematics 155) Springer-Verlag, New York, 1995.

[38] Knizhnik, V., Zamolodchikov, A.; “Current Algebra and the Wess-Zumino Model in Two Dimensions”, Nucl. Phys. B 247 (1984), pp.83–103.

[39] Kac, V. G.; “On Hopf Algebras and Their Generalizations”, submitted. (preliminary version available at [arXiv:math.QA/0703441](http://arXiv.org/)).
[52] Leites, D. A., Serganova, V.; “Defining Relations for Classical Lie Superalgebras I - Superalgebras with Cartan Matrix or Dynkin-type Diagram”, Topological and Geometrical Methods in Field Theory (Turku 1991), World Scientific, 1992, pp.194–201.

[53] Li, L-C., Parmentier, S.; On Dynamical Poisson Groupoids, I., Mem. Amer. Math. Soc. 174 (2005), no.824, vi+72pp.

[54] Lu, J-H.; “Hopf Algebroids and Quantum Groupoids”, Int. J. Math. 7 (1996), no.1, pp.47–70.

[55] Lu, J-H.; “Classical Dynamical r-matrices and Homogeneous Poisson Structures on $G/H$ and $K/T$”, Comm. Math. Phys. 212 (2000), pp.337–370.

[56] Majid, S.; Foundations of Quantum Group Theory, Cambridge University Press, Cambridge, 1995.

[57] Maltsiniotis, G.; ”Groupoides quantiques” (French. English summary) [Quantum groupoids], C. R. Acad. Sci. Paris S. I Math. 314 (1992), no.4, pp.249–252.

[58] Manin, Yu.; Gauge Field Theory and Complex Geometry; Fundamentals of Mathematical Sciences 289, Springer-Verlag, Berlin, 1997.

[59] Mayer, M. E.; ”From Poisson Groupoids to Quantum Groupoids and Back”, in: Differential Geometric Methods in Theoretical Physics (Rapallo, 1990), Lecture Notes in Phys. 375, Springer-Verlag, Berlin, 1991, pp.143–154.

[60] Nikshych, D., Vainerman, L.; ”Finite Quantum Groupoids and Their Applications”, in: New Directions in Hopf Algebras, MSRI Publications Vol. 43, 2002, pp.211–262.

[61] Nill, F.; ”Axioms of Weak Bialgebras”, e-arXiv preprint [arXiv:math.QA/9805104]

[62] Paterson, A. L. T.; Groupoids, inverse semigroups, and their operator algebras; Progress in Mathematics 170, Birkhäuser Boston, Inc., Boston, MA, 1999.

[63] Radford, D.; “The structure of Hopf algebras with a projection; J. Algebra 92 (1985), pp.322–347.

[64] Radford, D., Towber, J.; “Yetter–Drinfeld categories associated to an arbitrary bialgebra”, J. Pure Appl. Algebra 87 (1993), pp.259–279.

[65] Ravenel, D.; Complex Cobordism and Stable Homotopy Groups of Spheres, Pure and Applied Mathematics 121, Academic Press, Inc., Orlando, FL, 1986.

[66] Scheunert, M.; The Theory of Lie Superalgebras: An Introduction, (Lecture Notes in Mathematics 716), Springer-Verlag, 1979.

[67] Schiffmann, O.; “On Classification of Dynamical r-matrices”, Math. Res. Lett. 5 (1998), pp.13–30.

[68] Sweedler, M.; Hopf Algebras, Mathematics Lecture Note Series, W. A. Benjamin, Inc., New York, 1969.

[69] Takeuchi, M.; ”Groups of Algebras over $A \otimes \overline{A}$, J. Math. Soc. Japan 29 (1977), no.3, pp.459–492.

[70] Varadarajan, V.; Supersymmetry for Mathematicians: An Introduction, (Courant Lecture Notes in Mathematics 11) New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2004.

[71] Weinstein, A.; “Cosisotropic Calculus and Poisson Groupoids”, J. Math. Soc. Japan 40 (1988), pp.705–727.

[72] Weinstein, A.; “Groupoids: Unifying Internal and External Symmetry. A Tour through Some Examples”, Notices Amer. Math. Soc. 43 (1996), no.7, pp.744–752.

[73] Xu, P.; “Quantum groupoids associated to universal dynamical R-matrices”, C. R. Acad. Sci. Paris S. I Math. 328 (1999), no. 4, pp.327–332.

[74] Xu, P.; “Quantum Groupoids”, Comm. Math. Phys. 216 (2001), pp.539–581.

Department of Mathematics, Pomona College, Claremont, CA 91711

E-mail address: gizem.karaali@pomona.edu