Nilpotent Spinor Symmetry with Interacting Spin 3/2 Field

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Abstract.

Consistent interactions of spin 3/2 field that realize a nilpotent spinorial symmetry are presented. Based on our previous results on purely bosonic non-Abelian tensor with consistent interactions, we present a new system for interacting spin 3/2 field that realizes the nilpotent fermionic symmetry.

Keywords: Spin 3/2 Field, Consistent Interactions, Nilpotent Spinor Charge, Non-Abelian Tensors.

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INTRODUCTION

We have recently constructed a system of interacting non-Abelian tensor for purely bosonic consistent interactions [1]. Encouraged by this result, we generalized the basic structure to a fermionic system, in particular with a spin 3/2 field [2], which is different from the well-known supergravity case. We found that there are indeed consistent interactions of spin 3/2 field, realizing the nilpotent spinor algebra \( \{ Q_\alpha^I, Q_\beta^J \} = 0 \).

(I) NON-ABELIAN TENSOR (BOSONIC CASE)

The problem with constructing a system with non-Abelian tensor with consistent interactions can be described as follows: Consider the tensor \( B_{\mu\nu}^I \) with the adjoint index \( I \). The problem is that its naïve field strength \( G_{\mu\nu\rho}^I \equiv 3D_{[\mu}B_{\nu\rho]}^I \) is not invariant under the tensorial gauge transformation \( \delta_\lambda B_{\mu\nu}^I = 2D_{[\mu}\lambda_{\nu]}^I \), because \( \delta_\lambda G_{\mu\nu\rho}^I = +3f^{JKL}F_{[\mu\nu}^J\Lambda_{\rho]}^K \neq 0 \).

A clue to solving this problem lies in the dimensional reduction technique of Scherk and Schwarz [3]. In the generalized dimensional reduction in [3], the original coordinates \((\tilde{x}^\mu)\) in \( D+E \) space-time dimensions are reduced into \((x^\mu, y^\alpha)\), where \( \mu, \nu, \cdots = 0, 1, \cdots, D-1 \) for the \( D \)-dimensions, and \( \alpha, \beta, \cdots = 1, 2, \cdots, E \) for the compact \( E \)-dimensional extra-coordinates. For example, there arises the field strength component

\[
G_{\mu\nu\rho\alpha} = 3D_{[\mu}B_{\nu\rho]}\alpha + 6F_{[\mu\nu}^\beta B_{\rho]}\alpha\beta
\]

1 Talk delivered at SUSY06, the 14th International Conference on Supersymmetry and the Unification of Fundamental Interactions, Irvine, CA, June 2006. Preprint #: CSULB-PA-06-3
as the field strength of the tensor $B_{\mu\nu\alpha}$ with both the $D$- and $E$-dimensional coordinates. The important point is the involvement of the Chern-Simons terms at the end, different from the above-mentioned naively-constructed field strength.

In [1], we mimicked these structures for dimensional reductions in [3] to solve the non-Abelian tensor problem. We introduce the set of fields $(B_{\mu\nu}^I, C_{\mu}^{IJ}, K^{IJK}; A_{\mu}^I)$ and their field strengths:

\[ F_{\mu\nu} \equiv 2\partial_{[\mu}A_{\nu]}^I + f^{IJK}A_{\mu}^JA_{\nu}^K, \]
\[ G_{\mu\nu\rho}^I \equiv +3D_{[\mu}B_{\nu\rho]}^I - 3F_{[\mu\nu}^JC_{\rho]}^{IJI}, \]
\[ H_{\mu\nu}^{IJ} \equiv +2D_{[\mu}C_{\nu]}^{IJ} + F_{\mu\nu}^K K^{IJK} + f^{IJK}B_{\mu\nu}^K, \]
\[ L_{\mu}^{IJK} \equiv D_{\mu}K^{IJK} - 3f^{IJK}L_{[\mu}L_{\nu]}^{,1}. \]

Note that the total number of space-time indices and adjoint indices in the field strengths $G, H$ and $L$ is always four just as in [3]. These field strengths are now invariant under the tensorial $\Lambda$-gauge transformations:

\[ \delta_{\Lambda}B_{\mu\nu}^I = 2D_{[\mu}A_{\nu]}^I - F_{\mu\nu}^J A_{\lambda}^{JI}, \quad \delta_{\Lambda}C_{\mu}^{IJ} = D_{\mu}A_{\lambda}^{IJ} - f^{IJK}\Lambda_{\mu}^K, \]
\[ \delta_{\Lambda}K^{IJK} = 3f^{IJK}L_{[\mu}L_{\nu]}^{,1}, \quad \delta_{\Lambda}A_{\mu}^I = 0. \]

In other words, we succeeded in defining invariant field strengths that are invariant under the $\Lambda$-tensorial transformations.

(II) FERMIONIC NILPOTENT SYMMETRY

The algebra realizing the nilpotent fermionic symmetry is dictated by the three (anti)commutators:

\[ \{Q_{\alpha}^I, Q_{\beta}^J\} = 0, \]
\[ [T^I, Q_{\alpha}^J] = +f^{IJK}Q_{\alpha}^K, \]
\[ [T^I, T^J] = +f^{IJK}T^K, \]

where $I, J, \cdots = 1, 2, \cdots, g$ are for the adjoint representation of an arbitrary gauge group $G$, while $\alpha, \beta, \cdots$ are for the Majorana spinors components in an arbitrary space-time dimensions $D$. As the first anti-commutator in (4) shows, our spinor charges are nilpotent. The last equation in (4) is nothing but the usual non-Abelian generator commutations, while the second one implies that our spinor charges $Q_{\alpha}^I$ are in the adjoint representations.

The required set of fields can be fixed by mimicking the previous bosonic non-Abelian tensors. The correspondence between the bosonic and fermionic systems is

Bosonic Case $\longleftrightarrow$ Fermionic Case

$(B_{\mu\nu}^I, C_{\mu}^{IJ}, K^{IJK}; A_{\mu}^I) \leftrightarrow (\psi_{\mu\alpha}^I, \chi_{\alpha}^{IJ}; A_{\mu}^I)$

Note that the total number of the space-time bosonic, fermionic and adjoint indices is always three in the present fermionic system, similarly to the previous bosonic case. The
correspondence between potentials and field strengths in the present fermionic system is tabulated as

\[
\begin{array}{c|c}
\text{Potentials} & \text{Field Strengths} \\
\hline
\psi_\mu^{\alpha I} & \mathcal{R}_{\mu \nu}^{\alpha I} = 2D_{[\mu} \psi_{\nu]}^{\alpha I} + F_{\mu \nu}^{\alpha I} \\
\chi^{\alpha I} & L_{\mu \alpha}^{\alpha I} = D_{\mu} \chi^{\alpha I} + f_{\mu \alpha}^{IJK} \psi_\mu^{\alpha K} \\
A_\mu^I & F_{\mu \nu}^{I} = 2\partial_{[\mu} A_{\nu]}^{I} + f_{\mu \nu}^{IJK} A_\mu^J A_\nu^K \\
\end{array}
\]

(6)

All the terms accompanying the leading gradient terms are interpreted as generalized Chern-Simons terms.

Our fermionic symmetry transformation rule is now

\[
\begin{align*}
\delta Q \psi_\mu^I &= D_\mu \epsilon^I \\
\delta Q \chi^{I} &= -f^{IJK} \chi^J K = -Q^{IJK} f_{KLM} \epsilon^M \\
\end{align*}
\]

(7)

We can confirm that the commutator \([\delta Q(e_1), \delta Q(e_2)]\) vanishes, consistently with the anti-commutator in (4). Amazingly and amusingly, the field strengths \(\mathcal{R}\) and \(L\) are both invariant under the fermionic symmetry:

\[
\delta Q \mathcal{R}_{\mu \nu}^{I} = 0, \quad \delta Q L_{\mu}^{I} = 0.
\]

(8)

The usual gauge transformation is \(\delta_T A_\mu^I = D_\mu \alpha^I, \ \delta_T \psi_\mu^I = -f^{IJK} \alpha^J \psi_\mu^K, \ \delta_T \chi^{I} = -2f^{IJK} \chi^J K, \) under which the field strengths \(\mathcal{R}\) and \(L\) are both covariant, as desired: \(\delta_T \mathcal{R}_{\mu \nu}^{I} = -2f^{IJK} \alpha^J \mathcal{R}_{\mu \nu}^{K}, \ \delta_T L_{\mu}^{I} = -2f^{IJK} \alpha^K L_{\mu}^{I}.\)

Now that we have established the invariant field strengths \(\mathcal{R}\) and \(L\), it is straightforward to construct invariant actions under both \(\delta_Q\) and usual gauge transformations \(\delta_T\).

Our classical total action \(I_T \equiv I_1 + I_2 + I_3\) consists of the typical invariant actions

\[
\begin{align*}
I_1 &= \int d^D x \left[ + \frac{1}{4} a_0^{-1} f^{IJK} (\mathcal{R}_{\mu \rho}^{I} \gamma^\nu \gamma^\rho \mathcal{R}_{\nu, \rho}^{K}) \right] \\
I_2 &= \int d^D x \left[ + \frac{1}{4} (\bar{\psi}_\mu \gamma^{\nu \rho} \mathcal{R}_{\nu, \rho}^{I}) - \frac{1}{4} a_0^{-1} f^{IJK} (\mathcal{R}_{\mu \rho}^{I} \gamma^\nu \gamma^\rho L_{\mu}^{K} F_\nu L) \right], \\
I_3 &= \int d^D x \left[ + \frac{1}{4} (F_{\mu \rho}^{I})^2 \right] \left( f^{IJK} f_{KLM} = a_0 \delta^{IJ} \right), \\
\end{align*}
\]

(9)

As noted in the bosonic case, the invariance of the field strengths is closely related to the consistency of all the field equations. This can be seen from the \(\psi_\mu\) and \(\chi\)-field equations:

\[
\frac{\partial I_T}{\partial \psi_\mu} = + \frac{1}{2} \gamma^{\mu \rho \sigma} \mathcal{R}_{\rho \sigma}^I - \frac{1}{4} f^{IJK} \sigma J F_{\rho \sigma} + \frac{1}{4} Q^{IJK} \gamma^{\mu \rho \sigma} \chi^{KL} F_{\rho \sigma} J \equiv 0, \\
\]

2 The symbol \(\equiv\) is for a field equation. Relevantly, the symbol \(\neq\) is for an equation under question.
\[ \frac{\delta I_T}{\delta \chi_{IJ}} = P^{IJ, KL} \gamma^L L^L_{\mu} \frac{1}{4} a^{L}_{0} f^{LJK} \gamma^{\mu \rho \sigma} L^L_{\mu} F_{\rho \sigma} L^{K} + \frac{1}{4} a^{L}_{0} f^{LJK} \gamma^{\mu \rho \sigma} L^L_{\mu} F_{\rho \sigma} L^{K} = 0. \]  

(10)

The question now is whether the \( \psi_{\mu} \)-field equation in (10) satisfies the consistency equation \( D_{\mu} \left( \frac{\delta I_T}{\delta \psi_{\mu}^I} \right) \equiv 0 \). Fortunately, our system has a good answer to this question, thanks to the identity:

\[ D_{\mu} \left( \frac{\delta I_T}{\delta \psi_{\mu}^I} \right) + f^{LJK} \left( \frac{\delta I_T}{\delta \chi_{IJ}} \right) \equiv 0. \]  

(11)

Up to this stage, no field equation has been used. Note also that this identity is nothing but the invariance of the action \( \delta Q_I T = 0 \). Now, the first term in (11) vanishes, as the necessary condition of the \( \chi \)-field equation. This feature is also associated with the \( F\chi \)-term and \( \psi \)-linear term in the \( \mathcal{R} \) and \( L \)-field strengths.

In passing we note that we have also performed the quantization of the system [2] by confirming the BRST invariance.

(III) CONCLUSIONS

We have established consistent interactions for spin 3/2 (vector spinor) field \( \psi_{\mu}^{\alpha I} \) carrying spinorial and adjoint indices \( \alpha \) and \( I \). The main breakthrough is the discovery of the right definition of the field strength \( \mathcal{R}_{\mu \nu}^{I} \) for \( \psi_{\mu}^{I} \) inspired by the success for purely bosonic non-Abelian tensors [1], which in turn was inspired by the generalized dimensional reduction by Scherk-Schwarz [3]. The field content of our system is \( (\psi_{\mu}^{\alpha I}, \chi_{\alpha}^{IJ}; A_{\mu}^{I}) \), resembling the purely bosonic non-Abelian tensor case \( (B_{\mu}^{I}, C_{\mu}^{I}; K^{III}; A_{\mu}^{I}) \). Similarly to the purely bosonic non-Abelian tensors [1], our field strengths \( \mathcal{R}_{\mu \nu}^{I} \) for \( \psi_{\mu}^{I} \) contains peculiar generalized Chern-Simons terms.

These successful results indicate that we are on the right track for the formulation of consistent interactions for spin 3/2 fields. We have given the first non-trivial interacting model for the nilpotent spinor charges satisfying \( \{ Q_{\alpha}^{I}, Q_{\beta}^{I} \} = 0 \) for any arbitrary space-time dimensions \( D \) for arbitrary non-Abelian gauge group \( G \).

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