Many–body Dynamics of D0–Branes

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Abstract

We show that the growth of the size with the number of partons holds in a Thomas–Fermi analysis of the threshold bound state of D0–branes. Our results sharpen the evidence that for a fixed value of the eleven dimensional radius the partonic velocities can be made arbitrarily small as one approaches the large N limit.

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1 Introduction

In this paper we will study the semi–classical behavior of a nonrelativistic gas of D0–brane clusters. D0–branes are point-like solitons carrying the quantum numbers of the first massive Kaluza–Klein modes of the eleven dimensional supergravity multiplet. They are conjectured to be the fundamental degrees of freedom of M theory in the infinite momentum frame and have a ten dimensional rest mass \( m = \hbar / R \), where \( R \) is the radius of the eleventh dimension. The large \( N \) limit of the supersymmetric quantum mechanics of D0–branes is as yet poorly understood. Our work is a first step in this direction. The D0–brane clusters are bound by the attractive potential arising from their two body interactions forming a threshold bound state we will refer to as the “D–atom”. The universal part of this attractive interaction is velocity dependent and its leading behavior is of the form \( v^4/r^7 \).

For a large atom, the majority of the D–clusters have comparatively large quantum numbers so that we can use the semiclassical approximation to describe the D–atom. Since our WKB analysis only requires the universal interaction between D–clusters we will assume, with no loss of generality, that they carry no net spin but will allow for a possible degeneracy of \( n_s \) D0–branes per cluster. Such a cluster is a D–parton carrying \( n_s/R \) units of eleven dimensional momentum. In what follows, we will estimate the growth of the size of the D–atom with the total number \( N \) of D–partons. The consistency of our analysis requires that the de Broglie wavelength of a parton with nonrelativistic mass \( m \) moving in the effective potential \( \phi(r) \) of the remaining partons must vary only slightly over the characteristic scale of the system \([1]\), i.e.,

\[
\frac{m \hbar}{p^3} |\phi'(r)| \ll 1 \quad ,
\]

where \( \lambda(r) = (2\pi \hbar)/p(r) \) is the de Broglie wavelength, and \( r \) is the nine dimensional radial coordinate. The WKB approximation assumes \( p^2 \sim \phi \), and is thus invalid in the vicinity of the turning points of the effective potential. Let us scale distances by the eleven dimensional Planck length \( l_p \), with \( r \rightarrow l_p r \). Expressed in these units, the WKB condition requires that for a potential of the form \( \phi \sim r^{-\alpha} \), we have \( r^{\alpha-2} \ll 1 \).

The spatial momentum of the D–atom is equipartitioned among its \( N \) constituents in the semiclassical approximation. Thus, the D–partons populate phase space uniformly with one parton per phase space cell filling all available states up to some maximum momentum \( p(r) \). This procedure is independent of the statistics of the
D–partons. The total number of quantum states in a spatial volume element $dV$ is given by
\[ \rho(r)dV = \frac{32\pi^4 n_s}{945(2\pi\hbar)^9} p^9 dV, \tag{2} \]
where $(32\pi^4/945)|p|^9$ is the volume of a spherical ball in 9d momentum space.

The growth of the size of the threshold bound state with the number of D–partons has been estimated in a mean field approximation in [2]. If the atom occupies a box of rescaled size $L^9$ with constant number density $\rho_0$, i.e., $N = \rho_0 L^9$, one finds the scaling behavior:
\[ L \sim N^{1/9}, \tag{3} \]
which is also in accordance with the holographic bound of one parton per spatial cell of Planck size area [3]. We will show that the holographic growth of the size of the threshold bound state with the number of partons continues to hold when the partonic interactions are included, at least in the WKB approximation, thus sharpening the conjecture of Matrix theory.

Let us use natural units setting $\hbar=\epsilon=1$. The energy of any individual D–parton is $E=\frac{1}{2} Rp^2 - \phi(r)$. Since this energy is assumed to be bounded from above we can write down a self–consistent relationship between the density and the potential within any spatial volume element of the D–atom:
\[ \rho(r) = \frac{n_s}{945^2\pi^2} \left( \frac{2}{R} \right)^{9/2} \left( \phi - \phi_{\text{max}} \right)^{9/2}, \tag{4} \]
where $\phi_{\text{max}}$ is the value of the potential at the boundary of the atom which we can set equal to zero. In a conventional 3d atom the electrostatic potential $\Phi$ satisfies the Poisson equation. Substituting for the electron number density, we get the Thomas–Fermi equation for the self–consistent potential in a large atom [4]:
\[ \Delta \Phi \sim \Phi^{5/2}. \tag{5} \]

In the case of the D–atom, the self–consistent potential is dominated by the velocity dependent gravitational interactions of D–partons. We will use the analogous Thomas–Fermi equations for the D–atom to estimate the growth of its size with the number of D–partons. In contrast with conventional atoms where $E \sim N^{4/3}$ the nine dimensional D–atom displays holographic behavior: the D–parton energy density decreases as the number of partons increases.
2 The effective potential

We can infer the general form of the effective potential of D–partons from a dimensional analysis of the two–body interactions of D0–branes. Neglecting accelerations and time derivatives, the relative motion of a pair of nonrelativistic D0–branes in Minkowskian spacetime is obtained by expanding in powers of the field strength the DBI action \[ S = \tau_0 \int dt \text{ Tr det} \left( \eta_{\mu\nu} + 2\pi\alpha' F_{\mu\nu} \right)^{1/2} + \text{fermions} \], where we have neglected commutator terms for a configuration of well–separated branes. The D0–brane tension is given by \[ \tau_0 = \frac{1}{\sqrt{\alpha' g_s}} \], and we have used the usual relations between the Dp–brane tension \( \tau_p \), the radius of the eleventh dimension \( R \), and the dimensionless string coupling \( g_s \):

\[ R = g_s^{2/3} l_p, \quad \alpha' = g_s^{-2/3} l_p^2, \quad \tau_p = (2\pi)^{-p} (l_p)^{-(p+1)} g_s^{(p-2)/4}. \] (7)

The one–loop approximation to this action is given by the supersymmetric matrix quantum mechanics of \( SU(N) \) matrices, \( \phi(t) = \phi(t)^a \sigma^a \):

\[ S_0 = \frac{1}{2g} \dot{\phi}_i^2 + \frac{i}{2} \dot{\psi}^a \psi^a - \frac{1}{4g} [\phi_i \times \phi_j]^2 + \frac{i}{2} \epsilon_{abc} \phi_i^a \psi^b \gamma^c \psi^c, \] (8)

reducing to the supersymmetric dynamics of an ordinary coordinate world line, \( X(t) = (2\pi \alpha')^{-1} \phi_{rel}(t) \), when the branes are well–separated as in the semiclassical regime.

By a rescaling of the fermions, the action can be expressed as a loop expansion in the dimensionful parameter \( g = \frac{(2\pi)^2 l_p^3}{g_s^2} \) which plays the role of \( \hbar \) in the semiclassical analysis. The leading terms in the expansion \( S = \sum_n g^{(1-n)} S_n \), namely

\[ S = g \int dt \left[ (\partial_t X_i)^2 + i\psi \partial_t \psi \right] + O(g^0), \] (9)

determine the length dimensions of the fields:

\[ [X] = -1, \quad [\partial_t] = -1, \quad [\psi] = -3/2, \] (10)

consistent with the supersymmetry variations

\[ \delta X^i = i\epsilon \gamma^i \psi, \quad \delta \psi = (\dot{X}^i \gamma^i) \epsilon. \] (11)

It is amusing that the velocity dependence of the universal interaction between D0–branes follows from this dimensional analysis when the leading term is a Coulomb
potential with a $1/r^7$ fall-off in nine spatial dimensions. Since $[S_0] = -4$ the velocity dependence must be of the form $\sim v^4$. We will set the coefficient in the partonic interaction Hamiltonian equal to $al_p^9/R^3$, where $a$ is an undetermined numerical coefficient proportional to the known value for ordinary D0–branes [9]. Thus,

$$\phi(r; v) = \frac{al_p^9}{R^3} \frac{|v(r) - v(r')|^4}{|r - r'|^7}. \quad (12)$$

The semiclassical estimate for the growth of the size of the D–atom is independent of the precise value of this coefficient.

3 The Thomas–Fermi scale

We can now write down the nonrelativistic energy density describing the relative motion of a pair of D–partons with number density and velocity $(\rho(r), v(r))$ and $(\rho(r'), v(r'))$. Integrating over the spatial volume gives the Hamiltonian:

$$H = \frac{1}{2R} \int v^2(r) \rho(r) dV - \frac{al_p^9}{R^3} \int \rho(r) \frac{|v(r) - v(r')|^4}{|r - r'|^7} \rho(r') dV dV'. \quad (13)$$

The dimensionful parameters in this Hamiltonian can be absorbed by the rescaling:

$$r \rightarrow l_p r, \quad v \rightarrow \frac{R}{l_p} v. \quad (14)$$

This scaling renders energy dimensionless and can be motivated by the form of the space–time uncertainty relation in string theory with D–branes [10]:

$$\Delta X \Delta T \geq \alpha', \quad (15)$$

where $\alpha' = l_p^3/R$. The velocity correlations are easily computed in the semiclassical regime. We have

$$<v(r)> = <v(r')> = 0, \quad <v^{2n}(r)> = \frac{9}{9+2n} v^{2n}, \quad <(v(r) \cdot v(r'))^2> = \frac{9}{121} v^2 v'^2, \quad (16)$$

where $v^2(r)$ is the maximum allowed velocity in the volume element $dV(r)$. In the semiclassical regime this velocity is self–consistently related to the number density through the phase space relation

$$\rho(r) = C l_p^{-9} v(r)^9. \quad (17)$$
\[ C = \frac{n_s}{945 \cdot 2^4 \pi}. \]

Performing the average over velocities we obtain

\[
< |\mathbf{v}(r) - \mathbf{v}(r')|^4 > = \frac{9}{13} \left( v^4(r) + v^4(r') + \frac{26}{11} v^2(r) v^2(r') \right).
\] (18)

Substitution into the Hamiltonian yields velocity dependent nonlocal interactions. Such interactions can be rewritten in the form of a local field theory by introducing three auxiliary scalar fields, \( \phi_i \), with equations of motion:

\[
\Delta \phi_1(r) + v^9(r) = 0 \quad \text{(19)}
\]
\[
\Delta \phi_2(r) + \frac{26}{11} v^{11}(r) = 0 \quad \text{(20)}
\]
\[
\Delta \phi_3(r) + v^{13}(r) = 0. \quad \text{(21)}
\]

In simplifying these equations we have rescaled \( r \to \sqrt{\frac{13}{44aC}} \). Varying the Hamiltonian with respect to the velocity yields an algebraic equation relating the maximum kinetic energy density of the D–partons to the potential energy density stored in the three auxiliary potential fields, \( U_i = -\phi_i(r) \):

\[
11 v^2 = 13 v^4 \phi_1 + 11 v^2 \phi_2 + 9 \phi_3.
\] (22)

As boundary conditions on the parton density we require that it be normalizable at spatial infinity with a power law fall–off faster than a Coulomb potential. Within the bulk of the atom, i.e., at the origin, the density can be assumed to approach a (positive) constant with small power law corrections. We have also found solutions to the differential equations with a slow power law decay for which this constant vanishes. The detailed analysis of the differential equations together with the algebraic constraint equation is given in the appendix.

It is easy to see that partonic interactions will induce corrections to the naive scaling behavior \( L \sim N^{1/9} \) following from the relation \( \rho = \rho_0 = NL^{-9} \). Consider the power law solutions \( v^2 \sim r^{-4/11}, \rho \sim r^{-18/11} \). At distances of order the size of the atom, \( r \sim L \), we get an improved estimate for \( \rho \), from which we infer the scaling \( L \sim N^{11/81} \). Restoring the factors of \( l_p \), the dimensionful size grows as

\[
L \sim N^{11/81} l_p, \quad \text{(23)}
\]

for the growth of the size of the D–atom with the number of partons. Unlike conventional atoms where \( N \) plays the role of the nuclear charge \( Z \), and \( E \sim Z^{4/3} \), the mean kinetic energy and number density of the D–atom decrease upon increasing the
number of D–partons. We can verify that the estimated atomic size lies within the WKB regime. For the above mentioned solution, the self–consistent potential \( \phi \) scales as \( r^{-4/11} \). At the scale of the atomic size, \( L \), using eq.(23) we find the scaling behavior \( L^{\alpha - 2} \sim N^{-2/9} \) and the WKB bound is readily satisfied.

The space–time uncertainty relation motivates another analogy drawn from atomic physics. Consider the Heisenberg relation for a highly excited single electron atom. The uncertainty \( \Delta X \) scales as \( n^2 \), the principal quantum number, so that \( \Delta X \Delta P \sim n \hbar \). Let us derive an analogous result from the uncertainty relations in string theory. Assuming the scaling behavior \( L \sim N^{11/81}, \ v \sim N^{-2/81} \), with \( v \Delta T = \Delta X \), from eq.(15):

\[
\Delta X \Delta T \geq \frac{l_{\rho}^3}{R},
\]

we get the scaling inequality:

\[
L^2 / v \sim (l_{\rho}^3 / R) N^{8/27},
\]

with \( N \) playing the role of the “principal quantum number”.

4 Spin dependent forces

We have neglected spin dependent forces in the Thomas-Fermi analysis. In this section we will show that in the semi-classical regime where the Thomas-Fermi approximation applies, all relevant spin-dependent terms in the two body interaction can indeed be ignored.

As in sect. 2, if we rescale all fields so that there is an overall coupling \( g \) in front of the action, the one-loop term of the effective Lagrangian for the relative motion of two D0-branes will be independent of \( g \) and will have dimension \(-1\). The supersymmetric partners of the interaction \( v^4 / r^7 \) will be spin-dependent interactions at the one-loop level. In addition to the dimensional analysis, there is another rule that all the terms must observe. One may assign a quantum number 0 to \( X^i \), 1/2 to \( \psi \), \(-1/2\) to \( \epsilon \) and 1 to \( \partial_i \), consistent with supersymmetry and the form of the action. Thus the quantum number \( N = N_\theta + 1/2 N_f \) is the same for the same super-multiplet. This number is 4 for \( v^4 / r^7 \). Together with dimensional analysis, this rule restricts the possible terms which can appear in the same supermultiplet as \( v^4 / r^7 \). Schematically, we list the allowed terms as follows:
\[(\partial_t)^3(\psi^2/r^5), \ (\partial_t)^2(\psi^4/r^7), \ (\partial_t)(\psi^6/r^9), \ \psi^8/r^{11}].\]

Now, the spin-flip does not occur without interactions, so that for the purposes of estimating the contribution of spin dependent interactions one can ignore terms containing derivatives of fermions. Thus, the relevant terms remaining from the above list are

\[v^3\psi^2/r^8, \ v^2\psi^4/r^9, \ v\psi^6/r^{10}, \ \psi^8/r^{11},\]

where the first term can be interpreted as a spin-orbital interaction, the second term as a dipole-dipole interaction, the third term as both, and the last term contains a possible quadrupole-quadrupole interaction. The magnitude of both the spin-dependent terms and the spin-independent term \(v^4/r^7\) can be expressed succinctly in the form \((vr)^n/r^{11}\), where the spin factors can be dropped since they are independent of \(r\) at the characteristic scale of the large \(N\) bound state. Namely, \((vL)^9 \sim \rho L^9 \sim N\), so that \((vL)^n/L^{11} \sim N^{n/9}/L^{11}\). It follows that this term grows with \(n\), the largest value being \(n = 4\), corresponding to the \(v^4/r^7\) interaction. Of course, we could improve upon this estimate by taking into account our previous Thomas-Fermi analysis.

## 5 Conclusions

We have shown that the holographic growth of the size of the threshold bound state with the number of partons \([2]\) continues to hold for interacting D–partons, at least within the semiclassical approximation. We have thus sharpened the evidence for one of the key conjectures of Matrix theory, namely, that with \(R\) fixed as \(N \to \infty\) the velocities can be made arbitrarily small at large \(N\). We find that the qualitative behavior of the mean field analysis survives partonic interactions. This raises the interesting question of whether it is possible to compute the amplitude for the scattering of two D–atoms within the semiclassical regime. This would enable a direct test of longitudinal boost invariance within the semiclassical regime which, while it misses the tail of the distribution, is sensitive to the bulk of the kinematical phase space explored in the scattering. Boost invariance requires physics to only depend on the ratio of the sizes of the two bound states. It would be remarkable to understand how the holographic growth of particles can avoid contradicting longitudinal boost invariance in this picture.
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6 Appendix

We wish to find solutions to the nonlinear differential equation

$$\Delta \phi = -l \phi^\delta ,$$

of the form $\phi(r) = Ar^a f(r)$ with $\delta$ a positive rational number, where $f(r)$ is regular either at the origin or at infinity. Applying Frobenius’ method we can show that such solutions exist if and only if the leading power $a$ takes the values 0, $2 - d$, or $-\frac{2}{(\delta - 1)}$. Substituting for the radial part of the $d$–dimensional Laplacian, $\Delta_r = r^{1-d} \partial_r (r^{d-1} \partial_r)$, the function $f(r)$ can be shown to satisfy the equation:

$$r^2 f''(r) + (2a + d - 1) r f'(r) + a(a + d - 2) f(r) = -l A^\delta r^{a(\delta - 1) + 2} (f(r))^\delta .$$

Let us expand $f(r)$ in the Taylor series $f(r) = 1 + C^{(1)} r^{b^{(1)}} + \cdots$, where the $b^{(i)} \geq 0$ in the limit $r \to 0$, and $b^{(i)} < 0$ in the opposing limit $r \to \infty$. With $a(a + d - 2) = 0$, and $\alpha \equiv 2a + d - 1$, we have a solution in the form of a recursion relation for the coefficients $C^{(n)}$:

$$b^{(n)} = n(a(\delta - 1) + 2), \quad C^{(n)} = \frac{-(A)^{\delta - 1}}{b^{(n)}} \sum_{m \geq 1} \frac{\delta!}{(\delta - m)! m!} \sum_{n_i = 1}^{n - m} C^{(n_1)} \cdots C^{(n_m)} ,$$

where $\sum_{i=1}^m n_i = n - 1$. The boundary condition corresponds to the fields approaching a constant in the neighborhood of the origin, i.e., $a = 0$. At infinity we can have a Coulomb potential with $a = 2 - d$. In addition, there exist solutions to the differential equation with a slow power law decay when $a = -2/(\delta - 2)$. In nine dimensions such a solution to the differential equation can have complex coefficients.

The Thomas–Fermi equations for the D–atom take a similar form, with $\Delta \phi_i = -l_i \phi_i^\delta$, and $\phi_0 \equiv \nu^2$ a positive definite function satisfying the constraint:

$$13 \phi_0^2 \phi_1 + 11 \phi_0 (\phi_2 - 1) + 9 \phi_3 = 0 .$$
Similarly, we can expand the velocity field in the neighborhood of the origin:

$$\phi_0 = A_0 \left( 1 - \frac{24}{9} \frac{A_0^{(11/2)}}{11(1-A_2) - 26A_1A_0} r^2 + \cdots \right), \quad (30)$$

with the relationship between coefficients of the auxiliary potentials $13A_1A_0^2 + 11(A_2 - 1)A_0 + 9A_3 = 0$. The velocity distribution can have a negative slope in the neighborhood of the origin.

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