A Mixed Anti-preferential and Preferential Attachment Model of Graph Evolution

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Abstract

We analyze a dynamic random undirected graph in which newly added vertices are connected to those already present in the graph either using, with probability $p$, an anti-preferential attachment mechanism or, with probability $1-p$, a preferential attachment mechanism. We derive the asymptotic degree distribution in the general case and study the asymptotic behaviour of the expected degree process in the general and that of the degree process in the pure anti-preferential attachment case.

1 Introduction

Many real-world networks exhibit degree distributions with power-law tails and the pure preferential attachment mechanism has been heavily advocated as a possible explanation of this fact. From a modeling perspective, classical preferential attachment random graph models implicitly assume that all vertices behave in a similar way, meaning that all of them tend to connect to vertices of high degree. However, recently there has been interest in random graphs models relaxing the classical preferential attachment hypothesis but still exhibiting similar degree distributions. In those models, newly added vertices are allowed to attach their edges by means of a mixture of uniform and preferential attachment mechanisms (see Cooper and Frieze \cite{2003}, and Pachon et al. \cite{2018}).

In this paper we study a dynamic random undirected graph model in which every newly added vertex is connected to $m \in \mathbb{N}^* = \{1, 2, \ldots\}$ random older vertices, either using an anti-preferential attachment or a preferential attachment mechanism, the former meaning that newly added vertices are more likely to connect to older low-degree vertices. More precisely, at each discrete-time a new vertex is added to the graph. With probability $p \in [0, 1]$ it selects the $m$ target vertices according to a suitable anti-preferential attachment rule (see the definition of the model in Section \textsuperscript{2}), and with probability $1-p$ using a classical preferential attachment mechanism. Notice that the pure anti-preferential attachment and the pure preferential attachment case can be recovered by setting $p = 1$ and $p = 0$, respectively. The simultaneous presence of a random fraction of the vertices chosen by the above mentioned mechanisms reflects in the change of the exponent of the degree distribution of the model. Specifically, the asymptotic degree distribution of such random graph has a right tail decaying as a power-law with exponent $(p - 3)/(1 - p)$, $p \in [0, 1)$.

Note that the mixed preferential attachment–anti-preferential attachment model (mixed PA-APA model in the following) which is analyzed in this paper shares some similarities with the so-called preferential attachment models with choice considered for instance in Krapivsky and Redner \cite{2014}, Malyshkin and Paquette \cite{2015} and Haslegrave and Jordan \cite{2016}. Indeed, in the model analyzed in the present paper, each vertex has the possibility to “choose” in which way to attach its $m$ edges.

The structure of the paper is the following: Section \textsuperscript{2} describes the model of interest and suggests some generalizations. We start the analysis in Section \textsuperscript{2.1} where we analyze the asymptotic degree distribution which turns out to have a power-law tails with exponent $(p - 3)/(1 - p)$ (see Theorem \textsuperscript{2.1} and Remark \textsuperscript{2.5}). This shows that even graphs growing with a mixed mechanism in which the
Remark 2.1

The anti-preferential attachment regime is dominant, have power-law degree distributions. Furthermore, by tuning the value of $p$, every exponent in $(-\infty, -3]$ can be generated. This fact also suggests that the scale-free nature of the graph is robust to important changes in the attachment mechanism. In other words, a graph growing in large part according to an anti-preferential attachment mechanism still shows a power-law degree distribution if we let an infinitesimal possibility to the preferential attachment mechanism when building connections between vertices. The subsequent Sections 2.2 and 2.3 concern the asymptotic behaviour of the degree process for each given vertex. First, it is established (Theorem 2.3, Section 2.2) that the expected degree process is controlled by $t^{(1-p)/2}\ln t$, and then the almost sure convergence for the degree process is investigated in the special case of pure anti-preferential attachment (Section 2.3).

2 Mixed PA-APA random graph

Let $m \in \mathbb{N}^*$, $t \in \mathbb{N}$. The model we investigate produces a graph sequence that we denote by $\{G_t: t \in \mathbb{N}\}$ and which, for every time $t$, yields a graph of $t$ vertices and $mt$ edges. For $t = 0$, $G_0$ is the empty graph and for $t \geq 1$ let us denote the vertices in $G_t$ by $v_1, \ldots, v_t$. Let $F_0$ be the trivial sigma-algebra and denote $F_t$ the $\sigma$-algebra generated by the graph process up to time $t \geq 1$; more precisely,

$$F_t = \sigma(\{G_s: 1 \leq s \leq t\}), \quad t \geq 1.$$  \hspace{1cm} (2.1)

For every $i \in \mathbb{N}^*$ we denote by $d(v_i, t)$ the degree of vertex $v_i$ at time $t \geq i$. We set $d(v_i, t) = 0$ if $t < i$. The random graph process $(G_t)_{t \geq 1}$ evolves as follows:

Let $(Y_i)_{i \geq 2}$ be a sequence of i.i.d. Bernoulli random variables of parameter $p \in [0, 1]$ independent of $(G_t)_{t \geq 1}$. Let $G_1$ be a graph consisting of a single vertex $v_1$ with $m$ self-loops. For every $t \in \mathbb{N}^*$, to construct $G_{t+1}$ from $G_t$, add a new vertex $v_{t+1}$ and then add $m$ edges between $v_{t+1}$ and vertices of $G_t$ (therefore vertices added at times $t \geq 2$ do not have self-loops). The $m$ target vertices in $G_t$ are chosen according to the following procedure, which admits multiple edges between distinct vertices.

- If $Y_{t+1} = 0$ (which happens with probability $1 - p$) we select $m$ random vertices $W_{t+1}^1, \ldots, W_{t+1}^m$ from $G_t$ according to the preferential attachment mechanism

$$\mathbb{P}(W_{t+1}^r = v_i | F_t) = \frac{d(v_i, t)}{2mt}, \quad 1 \leq i \leq t,$$  \hspace{1cm} (2.2)

independently for each $r \in \{1, \ldots, m\}$.

- If $Y_{t+1} = 1$ (which happens with probability $p$) we select $m$ random vertices $W_{t+1}^1, \ldots, W_{t+1}^m$ from $G_t$ according to the anti-preferential attachment mechanism

$$\mathbb{P}(W_{t+1}^r = v_i | F_t) = \frac{2mt + 1 - d(v_i, t)}{t(2mt + 1 - 2m)}, \quad 1 \leq i \leq t,$$  \hspace{1cm} (2.3)

independently for each $r \in \{1, \ldots, m\}$.

Remark 2.1. The process $(Y_t)_{t \geq 2}$ encodes the information concerning the attachment mechanism chosen by vertices when added to the graph: at each time $t$ we add a vertex $v_{t+1}$ to $G_t$, we generate an independent Bernoulli r.v. $Y_{t+1}$ and:

- if $Y_{t+1} = 1$ then we attach $m$ edges between $v_{t+1}$ and vertices of $G_t$, selected according to the anti-preferential attachment mechanism (2.3);

- if $Y_{t+1} = 0$ then we attach $m$ edges between $v_{t+1}$ and vertices of $G_t$, selected according to the preferential attachment mechanism (2.2).

This is the most basic example in which the process $(Y_t)_{t \geq 2}$ models the way in which the two possible regimes coexist in the process’ dynamics. More general cases taking into account different characteristics could be further considered. For instance a dependence structure in the choice process $Y_t$ could model the case in which the newly added vertex regime is somehow affected by the previous choices. Furthermore, formulas (2.2) and (2.3) are both special cases of a general attachment rule in which the target vertices are chosen according to the probabilities

$$\mathbb{P}(W_{t+1}^r = v_i | F_t) = h(d(v_i, t))/\sum_{j=1}^t h(d(v_j, t)), \quad 1 \leq i \leq t, \quad r \in \{1, \ldots, m\},$$

where $h$ is strictly positive and bounded.
2.1 Degree distribution

We adapt a proof technique from [Hou et al. 2011] to determine the asymptotics of the expected proportion of vertices of a given degree. We start by introducing some notation. For $m \leq k \leq 2mt$, and $m, t \in \mathbb{N}^*$, we define:

- $H(k, t + 1) = p \frac{2mt + 1 - k}{2mt + 1 + 2mt} + (1 - p) \frac{k}{2mt}$, the unconditional attachment probability with any rule to a given vertex with degree $k$ at time $t$;
- $K(k, t + 1) = 1 - H(k, t + 1)$;
- $N_k(t) = \sum_{j=1}^t 1_{\{d(v_j, t) = k\}}$, the number of vertices of $G_t$ with degree $k$;
- $P(k, t) = \mathbb{E}[N_k(t)/t]$, the expected proportion of vertices of $G_t$ with degree $k$;
- $Q(k, t) = \frac{1}{t} \sum_{j=2}^t \mathbb{P}(d(v_j, t) = k)$, $t \geq 2$;
- $f(v, k, t) = \mathbb{P}(d(v, t) = k, d(v, s) \neq k \, \forall s = i, \ldots, t - 1)$, and $f(v, k, i) = \delta_{km}$, where $i \geq 2$, $m \leq k \leq (t - i + 1)m$, $t \geq i + 1$, and $\delta_{km}$ is the Kronecker’s delta. The function $f$ is the probability that vertex $v_i$ has degree $k$ for the first time at time $t$.

Henceforth we adopt the conventions that empty products equal unity and empty sums equal zero.

Next three remarks and Lemma 2.1 are needed to prove Proposition 2.1, which constitutes the main result of this section.

Remark 2.2. The limit $\lim_{t \to \infty} P(k, t)$ exists if and only if $\lim_{t \to \infty} Q(k, t)$ exists, in which case the two limits coincide.

Remark 2.3. Recalling that the degree process $(d(v_i, t))_{t \geq i}$ is non decreasing for every given vertex $v_i$, for $m \leq k \leq (t - i + 1)m$, $t \geq i$, and $i \geq 2$,

$$f(v_i, k, t) = \sum_{j=1}^{\lceil (k-m)/m \rceil} \binom{m}{j} H(k - j, t - 1)^j K(k - j, t - 1)^{m-j} \mathbb{P}(d(v_i, t - 1) = k - j).$$  (2.4)

Remark 2.4. for $m \leq k \leq (t - i + 1)m$, $t \geq i$, $i \geq 2$,

$$\mathbb{P}(d(v_i, t) = k) = \sum_{s=1}^i f(v_i, k, s) \prod_{j=s}^{t-1} K(k, j + 1)^m.  \quad \text{(2.5)}$$

Lemma 2.1 (Stolz–Cesaro’s theorem). Let $(x_n)_{n \in \mathbb{N}}$, and $(y_n)_{n \in \mathbb{N}}$ be two sequences of real numbers. Suppose that $y_n > 0$ and $y_n < y_{n+1}$ for every $n \in \mathbb{N}^*$. Further, assume that $y_n \to \infty$ as $n \to \infty$. If the following limit exists:

$$\lim_{n \to \infty} \frac{x_{n+1} - x_n}{y_{n+1} - y_n} = l,$$  \quad \text{(2.6)}

then also the limit $\lim_{n \to \infty} x_n/y_n$ exists and is equal to $l$.

Theorem 2.1. Let $m \in \mathbb{N}^*$ and write $P(k) = \lim_{t \to \infty} P(k, t)$. If $p \in [0, 1)$,

$$P(k) = \frac{2}{2 + m + mp} \cdot \frac{\xi(k)}{\xi(m)}, \quad k \geq m,$$  \quad \text{(2.7)}

with $\xi(k) = \Gamma \left( k + 2m \frac{p}{1-p} \right) / \Gamma \left( k + 1 + \frac{2(m+1)}{1-p} \right)$ and where $\Gamma$ is the Euler’s gamma function.

If $p = 1$,

$$P(k) = \frac{1}{m+1} \left( 1 - \frac{1}{m+1} \right)^{k-m}, \quad k \geq m.$$  \quad \text{(2.8)}

Proof. Let $p \in [0, 1]$. We start by proving that $P(m)$ exists and is equal to $2/(2 + m + mp)$ if $p \in [0, 1)$ and to $1/(m + 1)$ if $p = 1$. Then we extend the result to all $k > m$.

For $k = m$, note that $\mathbb{P}(d(v_i, t) = m) = 1$, $t \geq 2$, as $v_i$ is connected to $m$ vertices when it is added to $G_{t-1}$. Moreover, $\mathbb{P}(d(v_i, t) = m) = \mathbb{P}(d(v_i, t - 1) = m)K(m, t)^m$, $t \geq 3$, as a vertex with degree $m$
at time $t - 1$ keeps its degree unchanged at time $t$ provided it receives none of the $m$ edges attached to the newborn vertex $v_t$. Now observing that, for $t \geq 3$,

$$Q(m, t) = \frac{1}{t - 1} \sum_{j=2}^{t-1} P(d(v_j, t - 1) = m) K(m, t)^m + \frac{1}{t - 1},$$

we obtain the following recurrence equation for $Q(m, t)$:

$$Q(m, t) = \frac{t - 2}{t - 1} K(m, t)^m Q(m, t - 1) + \frac{1}{t - 1}, \quad t \geq 3.$$  \hspace{1cm} (2.10)

Iterating on $t$ it is immediate to show that the solution of \hspace{1cm} (2.10)\hspace{1cm} is

$$Q(m, t) = \frac{1}{t - 1} \prod_{j=2}^{t-1} K(m, j + 1)^m \left[ 1 + \sum_{h=2}^{t-1} \prod_{k=2}^{h} K(m, k + 1)^{-m} \right], \quad t \geq 3.$$  \hspace{1cm} (2.11)

Next we define, for $t \geq 3$, the following numerical sequences:

$$x_t = 1 + \sum_{h=2}^{t-1} \prod_{k=2}^{h} K(m, k + 1)^{-m}, \quad y_t = (t - 1) \prod_{j=2}^{t-1} K(m, j + 1)^{-m},$$

so that $Q(m, t) = x_t / y_t$. Observe that $(y_t)_{t \geq 3}$ is a positive strictly increasing sequence with $y_t \to \infty$ as $t \to \infty$. Moreover,

$$\frac{x_{t+1} - x_t}{y_{t+1} - y_t} = \frac{1}{t - (t - 1) K(m, t + 1)^m}, \quad t \geq 3.$$  \hspace{1cm} (2.13)

We substitute the explicit expression for function $K$ and rewrite the above ratio as

$$\frac{x_{t+1} - x_t}{y_{t+1} - y_t} = \left( mp \frac{2m + 1/t - m}{2m + 1/t - 2m/t} + (1 - p) \frac{m}{2} + 1 + o(1/t) \right)^{-1}.$$  \hspace{1cm} (2.14)

Now, Lemma \hspace{1cm} (2.1)\hspace{1cm} and Remark \hspace{1cm} (2.2)\hspace{1cm} immediately yield

$$\lim_{t \to \infty} P(m, t) = \lim_{t \to \infty} Q(m, t) = \lim_{t \to \infty} \frac{x_{t+1} - x_t}{y_{t+1} - y_t} = \frac{2}{2 + m + mp},$$

which reduces to $\lim_{t \to \infty} P(m, t) = 1/(1 + m)$ if $p = 1$.

For $k > m$, $t \geq 2$, we proceed as in the following. First, note that $\mathbb{P}(d(v_1, t) = k) = 0$ if $k < 2m$ or $k > (t + 1)m$, and for $i \geq 2$, $\mathbb{P}(d(v_i, t) = k) = 0$ if $k > (t - i + 1)m$. Observe that

$$P(k, t) = \frac{1}{t} \sum_{j=2}^{t} \mathbb{P}(d(v_j, t) = k) + \frac{1}{t} \mathbb{P}(d(v_1, t) = k),$$

which, by using Remarks \hspace{1cm} (2.3)\hspace{1cm} and \hspace{1cm} (2.4)\hspace{1cm} can be written as

$$P(k, t) = \frac{1}{t} \sum_{l=1}^{t} \left[ \sum_{j=2}^{t} \left( \sum_{s=1}^{(k-m)^m} G(l, s) \mathbb{P}(d(v_j, s - 1) = k - l) \prod_{h=s}^{l-1} K(k, h + 1)^m \right) + \frac{1}{t} \mathbb{P}(d(v_1, t) = k) \right],$$

where $G(l, s) = \binom{m}{l} H(k - l, s)^{l} K(k - l, s)^{m-l}$. By exchanging the order of summations we obtain

$$P(k, t) = \sum_{l=1}^{(k-m)^m} \frac{1}{t} \sum_{j=2}^{t} \left[ \sum_{s=1}^{(k-m)^m} G(l, s) \mathbb{P}(d(v_j, s - 1) = k - l) \prod_{h=s}^{l-1} K(k, h + 1)^m \right] + \frac{1}{t} \mathbb{P}(d(v_1, t) = k)$$

\hspace{1cm} (2.17)
Wishing to apply Lemma 2.1, we define the following numerical sequences (\( z_{t,l} \)) for each \( t \geq 1 \), and \( (w_t)_{t \geq 2} \):

\[
z_{t,l} = \sum_{s=2}^{t} G(l, s)(s-2)Q(k-l, s-1) \prod_{r=1}^{s-1} K(k, r+1)^{-m}, \quad l \geq 1, \tag{2.20}
\]

\[
w_t = t \prod_{h=1}^{t-1} K(k, h+1)^{-m}. \tag{2.21}
\]

Notice that \((w_t)_{t} \) is strictly positive and strictly increasing towards infinity. Moreover, by Lemma 2.2.

\[
P(k) = \sum_{l=1}^{(k-m)\wedge m} \lim_{t \to \infty} \frac{z_{t+1,l} - z_t}{w_{t+1} - w_t} = \sum_{l=1}^{(k-m)\wedge m} \lim_{t \to \infty} \frac{G(l, t+1)(t-1)}{t+1 - tK(k, t+1)^m} Q(k-l, t). \tag{2.22}
\]

Taking into account Remark 2.2, we derive the limit value of each term of the latter sum. Specifically, for \( l = 1 \) we have

\[
L_{k,1}(t) = \frac{mp(2m+1/t-k/t+1/2)}{1+mp} + o(1/t) \xrightarrow{t \to \infty} 2mp + (1-p)(k-1). \tag{2.23}
\]

Similarly, it is straightforward to see that \( \lim_{t \to \infty} L_{k,l}(t) = 0 \) for each \( l \geq 2 \).

Hence, for \( k > m \),

\[
P(k) = \frac{2mp + (1-p)(k-1)}{2 + 2mp + (1-p)k} P(k - 1). \tag{2.24}
\]

If \( p = 1 \), by iterating backwards \((k-m)\) times, we obtain

\[
P(k) = P(m) \left( \frac{m}{m+1} \right)^{k-m-m} = \frac{1}{m+1} \left( \frac{m}{m+1} \right)^{k-m}, \tag{2.25}
\]

proving the second part of the statement. If \( p \in [0,1) \), rewriting (2.22) as

\[
P(k) = \frac{k + \frac{p(2m+1)-1}{1-p}}{k + \frac{p(2m+1)-1}{1-p}} P(k - 1), \tag{2.26}
\]

and again iterating backward \((k-m)\) times we get

\[
P(k) = P(m) \frac{\Gamma \left( m + 1 + \frac{2(m+1)}{1-p} \right) \Gamma \left( k + 1 + \frac{p(2m+1)-1}{1-p} \right)}{\Gamma \left( m + 1 + \frac{p(2m+1)-1}{1-p} \right) \Gamma \left( k + 1 + \frac{2(m+1)}{1-p} \right)}. \tag{2.27}
\]
Remark 2.6 (Convergence in probability of $W$). We move now to analyze the rate of divergence of the expected degree of a given vertex. Let $c_t = \frac{2ml + 1}{l(2ml + 1 - 2m)}$, $e_t = \frac{m}{l(2ml + 1 - 2m)}$, $l \in \mathbb{N}^*$.

Proposition 2.1. Let $m \in \mathbb{N}^*$, $p \in [0, 1]$. Then for every $t \geq i$,

$$
\mathbb{E}[d(v_i, t)] = (1 + \delta_{i1})m \prod_{l=i}^{t-1} C(l, p, m) + \sum_{l=i}^{t-1} m p c_l \prod_{h=l+1}^{t-1} C(h, p, m),
$$

where $C(l, p, m) = (1 + \frac{k-1}{2l} - p e_l)$ and where $\delta_{ij}$ is the Kronecker’s delta.

Proof of Proposition 2.1. Let us proceed by induction. We will only prove the result for $i \geq 2$ as the case $i = 1$ can be proved easily. Note first that for $t = i$ we have $\mathbb{E}[d(v_i, i)] = (1 + \delta_{i1})m$. Let us now suppose that (2.31) holds for some $t > i$. Since the increment $d(v_i, t + 1) - d(v_i, t)$, conditional on $\mathcal{F}_t$, follows a binomial distribution with parameters $m$ and $H(d(v_i, t), t + 1)$ we have

$$
\mathbb{E}[d(v_i, t + 1) | \mathcal{F}_t] = d(v_i, t) + \mathbb{E}[d(v_i, t + 1) - d(v_i, t) | \mathcal{F}_t] = d(v_i, t) \left( 1 + (1 - p) \frac{1}{2t} - pe_t \right) + pm c_l.
$$

Taking expectations on both sides and using the inductive hypothesis we obtain

$$
\mathbb{E}[d(v_i, t + 1)] = \left[ m \prod_{l=i}^{t-1} C(l, p, m) + \sum_{l=i}^{t-1} m p c_l \prod_{h=l+1}^{t-1} C(h, p, m) \right] \times \left( 1 + (1 - p) \frac{1}{2t} - pe_t \right) + pm c_t
$$

$$
= m \prod_{l=i}^{t} C(l, p, m) + \sum_{l=i}^{t} m p c_l \prod_{h=l+1}^{t} C(h, p, m),
$$

as required. \qed
Remark 2.7. Let \( t \geq i \). Notice that when \( p = 0 \) (i.e. in the pure preferential attachment model) the expected degree (2.31) reduces to

\[
\mathbb{E}[d(v_i, t)] = (1 + \delta_i)m \frac{\Gamma(i)\Gamma(t + 1/2)}{\Gamma(i + 1/2)\Gamma(t)}.
\] (2.34)

Exploiting the fact that \( \Gamma(t + 1/2)/\Gamma(t) = \sqrt{t}(1 + O(1/t)) \), it is immediate to see that, for every \( i \in \mathbb{N}^* \),

\[
\lim_{t \to \infty} \mathbb{E} \left( \frac{d(v_i, t) \Gamma(i + 1/2)}{\sqrt{t} \Gamma(i)} \right) = (1 + \delta_i)m.
\] (2.35)

Therefore, in the preferential attachment case, the expected degree of any given vertex grows as \( \sqrt{t} \) (see e.g. van der Hofstad [2017], Chapter 8). On the other hand, when \( p = 1 \) (i.e. in the pure anti-preferential attachment model), from (2.31),

\[
\mathbb{E}[d(v_i, t)] = (1 + \delta_i)m \sum_{l=1}^{t-1} \frac{1}{(1 + c_l) \prod_{h=l+1}^{t-1} (1 - e_h)}.
\] (2.36)

Next theorem and Remark 2.8 show that, in the pure anti-preferential regime, the expected degree of a vertex grows as \( \ln t \).

Theorem 2.2. Let \( m \in \mathbb{N}^* \) and \( i \geq 2 \). In the pure anti-preferential attachment model \((p = 1)\) we have that

\[
\lim_{t \to \infty} \mathbb{E} \left( \frac{d(v_i, t)}{\sum_{l=1}^{t-1} \ln (1 + c_l) \prod_{h=l+1}^{t-1} (1 - e_h)} \right) = m.
\] (2.37)

Proof. Observe that for every \( t \geq i + 1 \) we have

\[
\left| \mathbb{E} \left[ \frac{d(v_i, t)}{\sum_{l=1}^{t-1} \ln (1 + c_l) \prod_{h=l+1}^{t-1} (1 - e_h)} \right] - m \right| \leq \left| \frac{\sum_{l=1}^{t-1} \ln (1 + c_l) \prod_{h=l+1}^{t-1} (1 - e_h)}{\sum_{l=1}^{t-1} \ln (1 + c_l) \prod_{h=l+1}^{t-1} (1 - e_h)} \right|^{-1} \left( m \prod_{l=i}^{t-1} (1 - e_l) - m \right).
\] (2.38)

It is not difficult to see that the first term of the right-hand side of (2.38) vanishes as \( t \to \infty \). Indeed,

\[
0 \leq \frac{m \prod_{l=i}^{t-1} (1 - e_l)}{\sum_{l=1}^{t-1} \ln (1 + c_l) \prod_{h=l+1}^{t-1} (1 - e_h)} \leq \frac{m}{\sum_{l=1}^{t-1} \ln (1 + c_l)} \leq \frac{m}{\sum_{l=1}^{t-1} \ln (1 + 1/l)}.
\] (2.39)

which goes to zero for \( t \to \infty \) as \( \sum_{l=1}^{t-1} \ln (1 + 1/l) = \ln t - \sum_{l=1}^{t-1} \ln (1 + 1/l) \). In order to prove the limit (2.37) there remain to show that the second term in the right-hand side of (2.38) vanishes as well. To see this, observe that the infinite product \( a = \prod_{l=1}^{\infty} (1 - e_l) \) is convergent as the related series \( \sum_{l=1}^{\infty} e_l \) converges by Raabe’s test. Moreover, \( 0 < a < 1 \) as \( 0 < (1 - e_l) < 1 \) for each \( l > 1 \). Consequently,

\[
\lim_{t \to \infty} \sum_{l=i}^{t-1} \ln (1 + c_l) \prod_{h=l+1}^{t-1} (1 - e_h) \geq \lim_{t \to \infty} \prod_{h=1}^{t-1} (1 - e_h) \left[ \ln(t) - \sum_{l=1}^{t-1} \ln \left( 1 + \frac{1}{l} \right) \right] = \infty
\] (2.40)

Now

\[
\left| \frac{\sum_{l=i}^{t-1} m c_l \prod_{h=l+1}^{t-1} (1 - e_h)}{\sum_{l=1}^{t-1} \ln (1 + c_l) \prod_{h=l+1}^{t-1} (1 - e_h)} - m \right| = \left| \frac{m \prod_{l=i}^{t-1} (1 - e_l)}{\sum_{l=1}^{t-1} \ln (1 + c_l) \prod_{h=l+1}^{t-1} (1 - e_h)} \right|.
\] (2.41)
Recall that for \( x \in (0, 1] \), expanding the logarithm around zero, we can write
\[
x - \ln(1 + x) = \frac{x^2}{2(1 + \eta)^2}
\]
where \( \eta \in (0, x) \). Noticing that, for \( t \geq i \), we have \( c_i < 1 \), and we obtain \( 0 < c_i - \ln(1 + c_i) < c_i^2/2 \). Hence
\[
\frac{m}{\sum_{i=1}^{t-1} \ln (1 + c_i) \prod_{h=i+1}^{t-1} (1 - e_h)} = \frac{(m/2) \sum_{i=1}^{t-1} \ln (1 + c_i) \prod_{h=i+1}^{t-1} (1 - e_h)}{\sum_{i=1}^{t-1} \ln (1 + c_i) \prod_{h=i+1}^{t-1} (1 - e_h)} \leq \frac{(m/2) \sum_{i=1}^{t-1} c_i^2}{\sum_{i=1}^{t-1} \ln (1 + c_i) \prod_{h=i+1}^{t-1} (1 - e_h)} \cdot \frac{\sum_{i=1}^{t-1} c_i^2}{\sum_{i=1}^{t-1} \ln (1 + c_i) \prod_{h=i+1}^{t-1} (1 - e_h)}.
\]
Finally, recalling (2.40) and since \( \sum_{i=1}^{\infty} c_i^2 < \infty \), it follows that
\[
\lim_{t \to \infty} \frac{\sum_{i=1}^{t-1} m c_i \prod_{h=i+1}^{t-1} (1 - e_h)}{\sum_{i=1}^{t-1} \ln (1 + c_i) \prod_{h=i+1}^{t-1} (1 - e_h)} = m = 0.
\]
\( \square \)

**Remark 2.8.** To understand better the rate of divergence of the degree process, observe that the denominator of (2.37) has the following limiting behaviour. Since
\[
\prod_{h=i}^{t-1} (1 - e_h) \sum_{l=i}^{t-1} \ln (1 + c_l) \leq \sum_{l=i}^{t-1} \ln (1 + c_l) \prod_{h=l+1}^{t-1} (1 - e_h) \leq \sum_{l=i}^{t-1} \ln (1 + c_l),
\]
we have for every \( t \geq i + 1 \),
\[
0 < \prod_{h=i}^{t-1} (1 - e_h) \leq \frac{\sum_{l=i}^{t-1} \ln (1 + c_l) \prod_{h=l+1}^{t-1} (1 - e_h)}{\sum_{l=i}^{t-1} \ln (1 + c_l)} < 1.
\]
Thus
\[
0 < \liminf_{t \to \infty} \frac{\sum_{l=i}^{t-1} \ln (1 + c_l) \prod_{h=l+1}^{t-1} (1 - e_h)}{\sum_{l=i}^{t-1} \ln (1 + c_l)} \leq \limsup_{t \to \infty} \frac{\sum_{l=i}^{t-1} \ln (1 + c_l) \prod_{h=l+1}^{t-1} (1 - e_h)}{\sum_{l=i}^{t-1} \ln (1 + c_l)} \leq 1,
\]
so that
\[
\sum_{l=i}^{t-1} \ln (1 + c_l) \prod_{h=l+1}^{t-1} (1 - e_h) = \Theta \left( \sum_{l=i}^{t-1} \ln (1 + c_l) \right) = \Theta(\ln t),
\]
as \( \sum_{l=i}^{t-1} \ln (1 + 1/l) = \ln t - \sum_{l=i}^{t-1} \ln (1 + 1/l) \).

Therefore, from Remarks 2.7 and 2.8 in the classical preferential attachment model \( (p = 0) \) the expected degree of any given vertex grows as \( \sqrt{t} \), whereas in the pure anti-preferential attachment case \( (p = 1) \) it grows as \( \ln t \). Next result tells us that, in the mixed PA-APA model \( (p \in [0, 1]) \), the growth of the expected degree is controlled by \( t^{(1-p)/2} \ln t \).

**Theorem 2.3.** In the mixed PA-APA model of parameter \( p \in [0, 1] \) we have that \( \mathbb{E}[d(v_t)] = O(t^{(1-p)/2} \ln t), \) \( i \geq 2 \).

**Proof.** First, note that
\[
\prod_{l=i}^{t-1} C(l, p, m) \leq \exp \left( \sum_{l=i}^{t-1} \ln \left( 1 + \frac{1 - p}{2l} \right) \right) \leq t^{1-p} \exp \left( \frac{1-p}{2} \ln t + \frac{1-p}{2} \sum_{l=i}^{t-1} \frac{1}{l} \right)
\]
We claim that \(\sum_{i=1}^{t-1} \frac{1}{i \ln t} \leq \frac{1}{p} \left( 1 - \frac{1}{\ln t} \right) \) = \(O\left(t^{(1-p)/2}\right)\).

Now, let \(l_0\) be such that \(l > (1 - p)^{-1} - (2m)^{-1}\) for all \(l \geq l_0\). For \(t \geq \max(l_0 + 1, i + 1)\),

\[
E[d(v_i, t)] = m \prod_{l=i}^{t-1} C(l, p, m) + \sum_{i=1}^{t-1} mp c_i \prod_{h=l+1}^{t-1} C(h, p, m)
\leq O\left(t^{(1-p)/2}\right) + \left( \sum_{i=1}^{t-1} C(h, p, m) \right)^{t-1} \sum_{i=1}^{t-1} mp c_i
= O\left(t^{(1-p)/2}\right) + O\left(t^{(1-p)/2} \ln(t^p)\right) = O\left(t^{(1-p)/2} \ln(t^p)\right).
\]

Remark 2.9. Observe that when \(p = 0\) then \(E[d(v_i, t)] = O(\sqrt{t})\), whereas when \(p = 1\) we have that \(E[d(v_i, t)] = O(\ln t)\), in agreement with Remark 2.7 and Theorem 2.3.

2.3 Almost sure convergence of the degree process

Next we discuss the almost sure convergence of \(d(v_i, t)\) when \(p \in \{0, 1\}\). Concerning the case \(p = 0\), that is in the pure preferential attachment case, it is well known (see e.g. van der Hofstad [2017], Chapter 8) that \(d(v_i, t) \rightarrow \eta_i\), where, for each given \(i\), \(\eta_i\) is an almost surely positive random variable with finite mean. In other terms, the degree of any given vertex \(v_i\) grows as \(\sqrt{t}\) (i.e. \(d(v_i, t) = O(\sqrt{t})\) as \(t \to \infty\)). If \(p = 1\), that is for the pure anti-preferential case we have the following results.

First we show that the degree of each vertex grows much slower than \(\ln^s t\) for every \(s > 1\) (i.e. \(d(v_i, t) = o(\ln^s t)\) as \(t \to \infty\)), and hence much slower than in the usual preferential attachment setting.

Proposition 2.2. Let \(p = 1\) (anti-preferential case) and let \(m, i \in \mathbb{N}^*\). Then, for every fixed \(s > 1\),

\[
\lim_{t \to \infty} \frac{d(v_i, t)}{\sum_{i=1}^{t-1} \ln (1 + c_i)} = 0, \quad \text{a.s.} \quad (2.51)
\]

Proof. Let us fix \(s > 1\). For \(t \geq i + 1\) define the stochastic process

\[
S_{i,s}(t) = \frac{d(v_i, t)}{\left(\sum_{i=1}^{t-1} \ln (1 + c_i)\right)^s} + \sum_{k=t}^{\infty} \frac{c_k}{\left(\sum_{i=1}^{k-1} \ln (1 + c_i)\right)^s}.
\]

We claim \((S_{i,s}(t))_{t \geq i + 1}\) is a non-negative supermartingale relative to \((\mathcal{F}_t)_{t \geq i + 1}\). To see this, first observe that \(c_k/\ln^s k = O(1/(k \ln^s k))\) and \(\sum_{i=1}^{k-1} \ln (1 + c_i) = O(\ln k)\). Now, since \(s > 1\),

\[
\sum_{k=t}^{\infty} \frac{2^k}{(\ln 2^k)^s} = \frac{1}{\ln 2^s} \sum_{k=1}^{\infty} \frac{1}{k^s} < \infty,
\]

and hence, according to the Cauchy condensation test we conclude that \(\sum_{k=i}^{\infty} 1/(k \ln^s k) < \infty\). Therefore, the series in \(2.52\) is convergent. Since \(E[d(v_i, t)] < \infty\) it follows that \(E[S_{i,s}(t)] < \infty\) for every \(t \geq i + 1\) as well. Clearly, \(S_{i,s}(t)\) is \(\mathcal{F}_t\)-measurable for every \(t \geq i + 1\). Finally, recalling \(2.32\) and \(p = 1\),

\[
E[S_{i,s}(t + 1)|\mathcal{F}_t] = \left(\sum_{i=1}^{t} \ln (1 + c_i)\right)^{-s} E[d(v_i, t + 1)|\mathcal{F}_t] + \sum_{k=t+1}^{\infty} \frac{c_k}{\left(\sum_{i=1}^{k-1} \ln (1 + c_i)\right)^s} (2.54)
\]
Then, for every $i$, this and (2.55) immediately yield

$$X_i \approx \frac{d(v_i, t)}{\left(\sum_{t=1}^{t-1} \ln (1 + c_i)\right)^s} \xrightarrow{a.s.} X_{i,s} \text{ as } t \to \infty. \quad (2.55)$$

Let now $s_0 \in (1, s)$. Since $(\sum_{t=1}^{t=1} \ln (1 + c_i))^{-s_0} \xrightarrow{t \to \infty} 0$, it follows that almost surely

$$\frac{d(v_i, t)}{\left(\sum_{t=1}^{t-1} \ln (1 + c_i)\right)^{s_0}} \xrightarrow{a.s.} X_{i,s_0} \cdot 0 \quad \text{as } t \to \infty. \quad (2.56)$$

This and (2.56) immediately yield $X_{i,s} = 0$ almost surely for every $s > 1$, as claimed.

Next proposition shows that in the pure anti-preferential attachment case (i.e. when $\eta = 1$) the growth of $d(v_i, t)$ for any given $i$ is controlled by $\ln t$.

**Proposition 2.3.** Let $m, i \in \mathbb{N}^*$. In the pure anti-preferential attachment model we have that

$$\limsup_{t \to \infty} \frac{d(v_i, t)}{\sum_{t=1}^{t-1} \ln (1 + c_i) \prod_{h=t+1}^{t} (1 - e_h)} < \infty, \quad \text{a.s.} \quad (2.57)$$

**Proof.** First, for $t \geq i+1$ define $Z_i(t) = \exp(d(v_i, t))$. Clearly $\mathbb{E}[Z_i(t)] \leq \exp(2mt)$, for every $t \geq i+1$. Moreover,

$$\mathbb{E}[Z_i(t+1)|\mathcal{F}_t] = \exp(d(v_i, t))\mathbb{E}[\exp(d(v_i, t+1) - d(v_i, t))|\mathcal{F}_t]$$

$$= Z_i(t) \left(1 - \frac{2mt + 1 - d(v_i, t)}{t[2mt + 1 - 2m]} + \frac{2mt + 1 - d(v_i, t)}{t[2mt + 1 - 2m]} \right)^m$$

$$\leq Z_i(t) (1 + (e - 1)c_i)^m.$$  

The second equality in (2.58) follows from the fact that

$$d(v_i, t+1) - d(v_i, t)|\mathcal{F}_t \sim \text{Bin} \left(m, \frac{2mt + 1 - d(v_i, t)}{t[2mt + 1 - 2m]} \right). \quad (2.59)$$

Then, for every $t \geq i+1$ define the positive semimartingale

$$W_i(t) = \frac{Z_i(t)}{\prod_{t=1}^{t-1} (1 + (e - 1)c_i)^m} \quad (2.60)$$

and call $\eta_i$ the random variable with finite mean to which it converges almost surely. We have

$$\frac{d(v_i, t)}{\sum_{t=1}^{t-1} \ln (1 + c_i) \prod_{h=t+1}^{t} (1 - e_h)} \xrightarrow{a.s.} \frac{\eta_i}{\prod_{t=1}^{t-1} (1 + (e - 1)c_i)^m} \quad (2.61)$$
we readily obtain

\[
\frac{\ln(W_i(t))}{\sum_{t=1}^{t-1} \ln(1+c_t) \prod_{h=t+1}^{t-1} (1-e_h)} + m \left[ \sum_{t=1}^{t-1} \ln(1+c_t) \prod_{h=t+1}^{t-1} (1-e_h) \right]^{-1} \sum_{t=1}^{t-1} \ln(1+(e-1)c_t)
\]

\[
\leq \frac{\ln(W_i(t))}{\sum_{t=1}^{t-1} \ln(1+c_t) \prod_{h=t+1}^{t-1} (1-e_h)} + m \left[ \prod_{t=1}^{t-1} (1-e_t) \right]^{-1} \sum_{t=1}^{t-1} \ln(1+(e-1)c_t)
\]

for every \( t \geq i + 1 \). Hence, by (2.40) and recalling that \( \ln x \leq x - 1 \) for every \( x > 0 \) and that, as \( t \to \infty \), \( W_i(t) \xrightarrow{a.s.} \eta_i \in [0, \infty) \), we obtain

\[
\limsup_{t \to \infty} \frac{d(v_i, t)}{\sum_{t=1}^{t-1} \ln(1+c_t) \prod_{h=t+1}^{t-1} (1-e_h)} \leq mC(e-1) < \infty \quad \text{a.s.,} \quad (2.62)
\]

where \( C = \lim_{t \to \infty} \left[ \prod_{t=1}^{t-1} (1-e_h) \right]^{-1} < \infty \).

To gain more insight on the rate of growth of the degree process we show with the following proposition that the degree of any given vertex is bounded from above by \( \kappa \gamma(t) \ln t \) where \( \kappa > 1 \) is a constant and \( \gamma(t) \) is a positive function diverging to infinity such that \( \sum_{j=1}^{t}(j \log j)^{-1}/\gamma(t) \to 0 \).

**Proposition 2.4.** Let \( p = 1 \), \( i > 1 \), and \( \gamma(t) \) be a positive function such that \( \gamma(t) \to \infty \) as \( t \to \infty \) and \( \sum_{j=1}^{t}(j \log j)^{-1}/\gamma(t) \to 0 \). Then \( d(v_i, t) \leq \kappa \gamma(t) \ln t \) with high probability as \( t \to \infty \), where \( \kappa > 1 \) is a constant.

**Proof.** By Markov inequality, note first that,

\[
\mathbb{P} \left( \frac{d(v_i, t)}{\ln t} - \mathbb{E} \left[ \frac{d(v_i, t)}{\ln t} \right] \geq \gamma(t) \right) \leq \exp \left[ -\gamma(t) - \mathbb{E} \left[ \frac{d(v_i, t)}{\ln t} \right] \right] \mathbb{E} \left[ \exp \left( \frac{d(v_i, t)}{\ln t} \right) \right]. \quad (2.63)
\]

The last factor can be written as

\[
\mathbb{E} \left[ \exp \left( \frac{d(v_i, t)}{\ln t} \right) \right] = \mathbb{E} \left[ \exp \left( \frac{d(v_i, t - 1)}{\ln(t - 1)} \right) \right] \mathbb{E} \left[ \exp \left( \frac{d(v_i, t) - d(v_i, t - 1)}{\ln t} \right) \right] \mathbb{E} \left[ \exp \left( \frac{d(v_i, t - 1) - d(v_i, t - 1)}{\ln(t - 1)} \right) \right] \mathbb{E} \left[ \exp \left( \frac{d(v_i, t - 1) - d(v_i, t - 1)}{\ln(t - 1)} \right) \right] \left( 1 + \frac{2m(t - 1) - 1 - d(v_i, t - 1)\left( e^{\frac{1}{2m}} - 1 \right) m}{(t - 1)2m(t - 1) + 1 - 2m} \right)^m \]

which, by iteration gives

\[
\mathbb{E} \left[ \exp \left( \frac{d(v_i, t)}{\ln t} \right) \right] \leq \exp \left[ \sum_{j=i+1}^{t-1} m c_j \left( e^{\frac{1}{m(j+1)}} - 1 \right) \right] \left( \frac{m}{\ln t} \right) \]

\[
= K \cdot \exp \left[ \sum_{j=i_0+1}^{t-1} m c_j \left( e^{\frac{1}{m(j+1)}} - 1 \right) \right], \quad (2.65)
\]

where \( i_0 = \min \{ h \in \mathbb{N}^* : 1/\ln(h+1) < 1 \} \). Now, since \( e^c - 1 \leq c + c^2 \), \( \forall c \in (0, 1) \) and therefore that

\[
e^{\frac{1}{m(j+1)}} - 1 \leq \frac{1}{\ln(j+1)} + \frac{1}{\ln^2(j+1)} \leq \frac{2}{\ln(j+1)}, \quad (2.66)
\]

we readily obtain

\[
\mathbb{E} \left[ \exp \left( \frac{d(v_i, t)}{\ln t} \right) \right] \leq K \cdot \exp \left[ \sum_{j=i_0+1}^{t-1} m c_j \frac{2}{\ln(j+1)} \right]. \quad (2.67)
\]
Hence,
\[
P \left( \frac{d(v_i, t)}{\ln t} - \mathbb{E} \left[ \frac{d(v_i, t)}{\ln t} \right] \geq \gamma(t) \right) \leq K \exp \left[ -\gamma(t) - \mathbb{E} \left[ \frac{d(v_i, t)}{\ln t} \right] + \sum_{j=0+1}^{t-1} m c_j \frac{2}{\ln(j+1)} \right] \overset{t \to \infty}{\longrightarrow} 0.
\]

Recalling Theorem 2.2 and Remark 2.8, for \( t \) large enough, we write
\[
\mathbb{E} \left[ \frac{d(v_i, t)}{\ln t} \right] < 2m \sum_{l=1}^{t-1} \left( \ln (1 + c_l) \prod_{h=l+1}^{t-1} (1 - e_h) \ln t \right) \leq 2m \tilde{\kappa},
\]
where \( \tilde{\kappa} \) is a suitable constant. Hence, for \( \kappa > 1 \),
\[
P \left( d(v_i, t) \geq \kappa \gamma(t) \ln t \right) \leq P \left( d(v_i, t) \geq 2m \tilde{\kappa} \ln t + \gamma(t) \ln t \right)
\]
\[
\leq P \left( d(v_i, t) \geq \left( \mathbb{E} \left[ \frac{d(v_i, t)}{\ln t} \right] + \gamma(t) \right) \ln t \right).
\]

By using (2.68) we conclude the proof.

Remark 2.10. Notice that when \( \gamma(t) \) grows faster than \( \ln^\alpha t \), \( \alpha > 0 \), Proposition 2.4 is weaker than Proposition 2.2. As an example of application in the non-trivial case one might consider \( \gamma(t) = (\ln \ln t)^\beta \), \( \beta \geq 2 \).

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