Variational approach versus accessible soliton approximation in nonlocal, nonlinear media

Branislav N Aleksić¹,², Najdan B Aleksić¹,², Milan S Petrović¹,³, Aleksandra I Strinić¹,² and Milivoj R Belić¹

¹Texas A&M University at Qatar, PO Box 23874, Doha, Qatar
²Institute of Physics, University of Belgrade, PO Box 68, 11080 Belgrade, Serbia
³Institute of Physics, PO Box 57, 11001 Belgrade, Serbia

E-mail: branislav.aleksic@qatar.tamu.edu

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Abstract
We discuss differences between the variational approach to solitons and the accessible soliton approximation in a highly nonlocal, nonlinear medium. We compare results of both approximations by considering the same system of equations in the same spatial region, under the same boundary conditions. We also compare these approximations with the numerical solution of the equations. We find that the variational highly nonlocal approximation provides more accurate results and, as such, is a more appropriate solution than the accessible soliton approximation. The accessible soliton model offers a radical simplification in the treatment of highly nonlocal, nonlinear media, with easy comprehension and convenient parallels to a quantum harmonic oscillator, however, with a hefty price tag: a systematic numerical discrepancy of up to 100% with the numerical results.

Keywords: variational approach, accessible soliton, nonlocal nonlinear media, numerical simulation

1. Introduction

Self-localized wave packets that propagate in a nonlinear medium without changing their structure are known as optical spatial solitons [1]. Their existence is a consequence of the robust balance between dispersion and nonlinearity, between diffraction and nonlinearity, or between all three in the propagation of spatiotemporal solitons or light bullets. An important characteristic of many nonlinear media is their nonlocality, i.e., the fact that the characteristic size of the response of the medium is wider than the size of the excitation itself. Strong nonlocality is of special interest because it is observed in many media. For example, in nematic liquid crystals (NLCs), both experimental and theoretical studies demonstrated that the nonlinearity is highly nonlocal [2–4].

In 1997, Snyder and Mitchell introduced a model of nonlinearity whose response is highly nonlocal [5]—in fact, infinitely nonlocal. They proposed an elegant theoretical model, intimately connected with the linear harmonic oscillator, that describes complex, soliton-like dynamics (collisions, interactions, and deformations) in simple terms, even in two and three dimensions. Because of the simplicity of the theory, they coined the term ‘accessible solitons’ (AS) for these optical spatial solitons waves. But straightforward application of the AS theory, even in nonlinear media with almost infinite range of nonlocality, inevitably led to additional problems [6–8] because there exists no physical medium without boundaries and without noise.

To include interactions between solitons within boundaries, as well as the impact of the finite size of the sample, we developed a variational approach (VA) to solitons in nonlinear media with long-range nonlocality, such as NLCs [9]. Other nonlinear systems of interest include thermal nonlinear materials, photorefractive crystals, and Bose-Einstein condensates [10, 11]. Starting from a convenient ansatz, this approach delivers a stationary solution for the beam amplitude and width, as well as the period of small oscillations about the stationary state. It provides for natural explanation...
of oscillations seen when, e.g., noise is included into the nonlocal nonlinear models. The noise is inevitable in any real physical system and causes a regular oscillation of soliton parameters with the period well predicted by our VA calculus [9]. It may even destroy solitons [12]. Even though our VA results were corroborated by numerics and experiments, they still attracted a published comment from other researchers [13, 14]. We have further investigated the destructive influence of noise on the shape-invariant solitons in highly nonlocal NLCs in [12].

In this study of VA and AS approximations to the fundamental soliton solutions in a (2+1)-dimensional highly nonlocal medium, we adopt the following model of coupled equations (1), (9) in dimensionless form:

\[
\begin{align*}
2i\frac{\partial E}{\partial z} + \Delta E + \theta E &= 0 \quad (1) \\
2\delta \theta + |E|^2 &= a \theta \quad (2)
\end{align*}
\]

with zero boundary conditions on the border of a square transverse region \(|x| \leqslant d\) and \(|y| \leqslant d\). Here, \(z\) is the propagation direction and \(\delta\) the transverse Laplacian. The system of equations of interest consists of the nonlinear Schrödinger equation for the propagation of the optical field \(E\) and the diffusion equation for the nonlocal response of the medium \(\theta\). This is a fairly general model for the nonlinear optical media with a diffusive nonlocality widely used in the literature [1, 6, 9]. In the local limit, the first term in equation (2) can be neglected and the model reduces to the Schrödinger equation with the Kerr nonlinearity. In the opposite limit, the third term in equation (2) can be neglected and the highly nonlocal model is reached. Since we are interested in the strong nonlocality, we will omit in our analysis the term on the right side of equation (2).

2. Variational approach

In this approach, to derive equations describing evolution of an approximate field beam, a Lagrangian density is introduced, corresponding to equations (1), (2):

\[
\mathcal{L} = i \left( \frac{\partial E^*}{\partial z} E - \frac{\partial E}{\partial z} E^* \right) + |VE|^2 + |V\theta|^2 - \theta |E|^2
\]

Thus, the problem is reformulated into a variational problem:

\[
\delta \int \int \mathcal{L} dxdydz = 0
\]

whose solution is equivalent to equations (1), (2). To obtain evolution equations for an approximate field in the highly nonlocal region, an ansatz is introduced in the form of a Gaussian beam for the field [9]:

\[
E = A \exp \left[ -\frac{\rho^2}{2R^2} + iCr^2 + i\psi \right]
\]

in which \(A\) is the amplitude, \(R\) is the beam width, \(C\) is the wave front curvature along the transverse radial coordinate, and \(\psi\) is the phase shift. Variational optimization of these beam parameters will lead to the most appropriate VA solution of the problem. Likewise, a trial function for the nonlocal response of the medium is introduced, in the form

\[
\theta = B \left[ Ei \left( -\frac{r^2}{T^2} - \ln \left( \frac{r^2}{d^2} \right) \right) \right]
\]

which is characterized by the amplitude \(B\) and the width \(T\). Here, \(Ei\) is the exponential integral function. Note that \(T\) does not represent the total width of \(\theta\). The form of \(\theta\) corresponds to a radially symmetric solution of equation (2), with zero boundary conditions on a circle of radius \(d \gg R\) (the limit of a thick cell) [15]. We take this expression as an approximate solution on a square sample.

Let \(\delta = \max \left\{ R^2/d^2 \right\} \ll 1\); then the averaged Lagrangian \(L = \int \int \mathcal{L} dxdy\) is given by:

\[
L = 2P \psi^* + 2P R^2 \left( C^* + 2(C^2) \right) + \frac{P}{R^2}
\]

\[
+ 4\pi B^2 \ln \left( \frac{e^{d^2}}{2T^2} \right) - PB \ln \left( \frac{e^{d^2}}{R^2 + T^2} \right) + O(\delta)
\]

where \(\gamma\) is Euler’s constant and the prime denotes the derivative with respect to \(z\). In the process of optimization from the averaged Lagrangian, one obtains four ordinary differential equations (ODEs):

\[
\frac{dP}{dz} = 0
\]

\[
C = \frac{1}{2R} \frac{dR}{dz}
\]

\[
\frac{d^2R}{dz^2} = \frac{1}{R^3} - \frac{P}{16\pi R^2} + O(\delta)
\]

and two algebraic relations: \(T = R + O(\delta)\) and \(B = P/8\pi + O(\delta)\). The beam power \(P = \int \int \mathcal{L} dxdy \propto \pi R^2\) is conserved according to equation (8). The system of equations (8)–(10) describes the dynamics of the beam around a stationary state.

In the stationary state \((dR/dz = dC/dz = 0)\), we find the equilibrium beam width \(R_0\) as a function of the beam power only:

\[
R_{0VA} = 4 \sqrt{\frac{\pi}{P}} + O(\delta)
\]

From relations (6, 11), we also find the maximum value of \(\theta\):

\[
\theta_{\max} = \frac{P}{8\pi} \ln \left( \frac{e^{d^2}}{16\pi P} \right) + O(\delta)
\]
and the propagation constant $\mu = (d\psi/dz)_0$ can be written as:

$$\mu = \frac{P}{16\pi} \ln \left( \frac{e^{-1/2}}{32\pi d^2 P} \right) + O(\delta) \quad (14)$$

It should perhaps be mentioned that the integral quantity $\Theta = 2\pi \int_0^\infty \theta dr = P\left( d^2 - R^2 \right)/8 + O(\delta) \approx Pd^2/8$, which is proportional to the power, is also conserved.

The period of small oscillations of the perturbation around the equilibrium position ($R = R_0$, $C = 0$) is given by the following relation:

$$\Lambda_{VA} = \frac{16\sqrt{2}\pi^2}{P}. \quad (15)$$

Relations (12)–(15) completely define the VA approximate solution in the highly nonlocal case. It remains to do the same for the AS approximation and then to compare the two.

3. Accessible soliton approximation

In the AS approximation, the basic assumptions are that the shape of the nonlocal response of the medium is a parabolic function of the transverse distance:

$$\theta = \theta_0 - \theta_2 r^2 \quad (16)$$

and that the shape function of the field $E$ is still a Gaussian, given by equation (5). The only refractive index ‘seen’ by the beam is that confined near its propagation axis [5].

The parameters of the trial function (5) are now given by the following equations:

$$\frac{dA}{dz} = -2AC \quad (17)$$

$$C = -\frac{1}{2R} \frac{dR}{dz} \quad (18)$$

$$\frac{d^2R}{dz^2} = -\frac{1}{R^3} + \theta_2 R \quad (19)$$

$$\frac{dy}{dz} = -\frac{1}{R^2} + \theta_0 \frac{A_0}{2} \quad (20)$$

and equation (16), which exactly satisfy equation (1).

The parameter $\theta_0$ is only a phase shift and, as such, is quite arbitrary. On the other hand, the value of $\theta_2$ is much more important; it is determined from equation (2). By replacing (5) and (16) in equation (2), in the limit $r \to 0$, one obtains $\theta_2 = A_0^2/8$ [7]. Then, equation (19) becomes:

$$\frac{d^2R}{dz^2} = -\frac{1}{R^3} + \frac{P}{8\pi R} \quad (21)$$

The equilibrium width $R_0$ in the AS approximation is:

$$R_{AS} = \frac{8\sqrt{2}\pi}{P} \quad (22)$$

and it is $\sqrt{2}$ times less than in the VA approximation at a same power (see figure 1). The corresponding stationary amplitudes in both approximations are presented in figure 2.

Because of the relation $P = \pi A_0^2 R_0^2$, the equilibrium amplitude $A_0$ in the AS approximation is $\sqrt{2}$ times greater than the one in the VA approximation at the same power.

The period of small oscillations of the width perturbation around the equilibrium can be obtained from equation (21):

$$\Lambda_{AS} = \frac{8\sqrt{2}\pi^2}{P} \quad (23)$$

The period in the AS approximation is 2 times less than that in the VA approximation at the same power (see figure 3). Thus, the values of the beam parameters for AS are systematically off the values for the VA approximation, which, on the other hand, happen to be very close to the full numerical solution for the same values of parameters.

Another useful approximation to AS is based on the solution of equation (2) when the parameter $\theta_2$ is independent of $z$. In contrast to equation (19), in which $\theta_2$ may depend on
Figure 3. Period of small oscillations as a function of the beam power in both approximations.

The equation for $R$ has an exact oscillatory solution [16]:

$$ R = R_0 \sqrt{\cos^2(\pi z / A) + \frac{P_0}{P} \sin^2(\pi z / A)} $$

from which the solutions for $C$ and $\psi$ immediately follow:

$$ C = \frac{\pi}{\lambda} \left(\frac{P_0/P - 1}{P_0/P + 1}\right) \sin \left(\frac{2\pi z}{\lambda}\right) $$

$$ \psi = -\arctan \left(\sqrt{\frac{P_0}{P}} \tan \left(\frac{\pi z}{\lambda}\right)\right) $$

Here, $R_0$ and $P_0$ are the width and the power of the AS solution, respectively. When $P = P_0$, one obtains stationary AS; otherwise, the approximate solution oscillates. The quantity $\lambda = \lambda_{AS2} \sqrt{P_0/P}$ represents the period of harmonic oscillations around the equilibrium (soliton) state, while $\lambda_{AS2}$ is the period of small oscillations of the width perturbation:

$$ \lambda_{AS2} \equiv \frac{8\sqrt{2}\pi^2}{P} = \pi R_0^2 $$

which is the same as before. Thus, in the AS approximation, one obtains nice dependencies in the closed form, but they are of little benefit in view of the large discrepancy with the VA and the numerical solution to the full problem.

4. Conclusions

In conclusion, we have discussed the differences between the VA and AS approximate solutions to the propagation of solitons in highly nonlocal, nonlinear media. The AS model provides a radical simplification and allows for an elegant description, but has a limited practical relevance, mainly because of the competition between the nonlocality and the finite size of the sample. The VA solution is not so simple, but works very well in the limited region of large nonlocality. We have found that the AS approximation can differ up to two times, when compared to the more realistic VA approximation and the numerical solution.

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