The Number of Locally $p$-stable Functions on $Q_n$

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Abstract
A Boolean function $f : V(G) \to \{-1, 1\}$ on the vertex set of a graph $G$ is locally $p$-stable if for every vertex $v$ the proportion of neighbours $w$ of $v$ with $f(v) = f(w)$ is exactly $p$. This notion was introduced by Gross and Grupel in [1] while studying the scenery reconstruction problem. They give an exponential type lower bound for the number of isomorphism classes of locally $p$-stable functions when $G = Q_n$ is the $n$-dimensional Boolean hypercube and ask for more precise estimates. In this paper we provide such estimates by improving the lower bound to a double exponential type lower bound and finding a matching upper bound. We also show that for a fixed $k$ and increasing $n$, the number of isomorphism classes of locally $(1 - k/n)$-stable functions on $Q_n$ is eventually constant. The proofs use the Fourier decomposition of functions on the Boolean hypercube.

1 Introduction

Let $G$ be a graph with vertex set $V(G)$. By a Boolean function on $G$ we mean a function $f : V(G) \to \{-1, 1\}$. Motivated by the scenery reconstruction problem, Gross and Grupel, in [1], define a locally $p$-stable function on $G$ to be a Boolean function $f$ on $G$ such that for every vertex $v \in V(G)$ we have

$$\frac{|\{w \in \Gamma(v) : f(v) = f(w)\}|}{d(v)} = p,$$

where $\Gamma(v)$ denotes the neighbourhood of $v$ in $G$ and $d(v) = |\Gamma(v)|$ is the degree of $v$. They say that two Boolean functions $f$ and $g$ on $G$ are isomorphic if there is an automorphism $\phi$ of $G$ such that $f = g \circ \phi$. They show that the scenery reconstruction problem on the $n$-dimensional Boolean hypercube is impossible for $n \geq 4$ by constructing two non-isomorphic locally $p$-stable functions and noting that the scenery processes for these functions have exactly the same distribution.

Let us now restrict ourselves to the case when $G = Q_n$ is the $n$-dimensional Boolean hypercube. It will be more convenient for us to work with a re-parametrised definition of locally $p$-stable functions.

Definition 1. A k-function is a Boolean function $f$ on $Q_n$ such that for every vertex $v \in V(G)$ we have

$$|\{w \in \Gamma(v) : f(v) \neq f(w)\}| = k.$$

Note that a k-function is precisely a locally $p$-stable function on $Q_n$ with $p = 1 - k/n$. Combinatorially, a $k$-function corresponds to a partition of $Q_n$ into two parts such that every vertex has precisely $k$ neighbours in the opposite part.

It will also be more convenient for us to work with an extended notion of isomorphism. We say that two real-valued functions $f$ and $g$ on $Q_n$ are isomorphic if there is an automorphism $\phi$ of $Q_n$ and a sign $\epsilon \in \{-1, 1\}$ such that $f = \epsilon g \circ \phi$. Note that if $f$ and $g$ are isomorphic then $f$ is a $k$-function if and only if $g$ is. The number of isomorphism classes of $k$-functions changes by at most a factor of 2 when passing from our definition of isomorphism to that of Gross and Grupel.

Let us now introduce some notation. For integers $0 \leq k \leq n$, let $F(n,k)$ denote the number of $k$-functions on $Q_n$ and let $G(n,k)$ denote the number of isomorphism classes of $k$-functions on $Q_n$. We are mainly interested
in $G(n,k)$, but will need $F(n,k)$ in the proofs.

In [1], Gross and Grupel obtain a lower bound of the form

$$G(n,k) = 2^\Omega(\sqrt{n})$$

for $n \geq 2k - 2$ and ask for more precise estimates (Question 5.12.).

In this paper, we provide such estimates:

**Theorem 1.** Let $0 \leq k \leq n$ be integers. Then

$$2^{2m+o(1)} \leq G(n,k) \leq 2^{2m+O(\log_2 m)},$$

where $m = \min(k,n-k)$.

We also prove the following theorem, which is a key ingredient in the proof of Theorem 1:

**Theorem 2.** Let $k \geq 0$ be an integer. Then the sequence $(G(n,k))_{n=k}^{\infty}$ is increasing and eventually constant. Moreover, denoting by $n(k)$ the first value of $n$ after which the sequence is constant, we have

$$3 \cdot 2^{k-1} - 2 \leq n(k) \leq 4.394 \cdot 2^k.$$

The paper is organised as follows. In Section 2 we introduce some definitions and notation, describe the automorphisms of $Q_n$ and recall some basic facts about Fourier analysis on the Boolean hypercube. In Section 3 we prove Theorems 1 and 2.

### 2 Preliminaries

In this section we introduce some definitions and notation, describe the automorphisms of $Q_n$ and recall some basic facts about Fourier analysis on the Boolean hypercube.

#### 2.1 The Boolean hypercube

We first introduce some definitions and notation that we will need later on. It will be convenient to think of $Q_n$ as having vertex set $V(Q_n) = \{-1,1\}^n$. The edge set $E(Q_n)$ is the set of pairs of vectors differing in precisely one entry. We will sometimes write $Q_S$, where $S$ is a finite set, for the Boolean hypercube indexed by $S$ (so $V(Q_n) = \{-1,1\}^S$ and $E(Q_n)$ is as before). One can think of $Q_n$ as $Q_{[n]}$, where $[n] = \{1,2,3,\ldots,n\}$. We will write vectors $x \in Q_n$ as $x = (x_1,x_2,x_3,\ldots,x_n)$, so that $x_i \in \{-1,1\}$ for all $i \in [n]$.

For each $x \in Q_n$, we write $\Gamma(x) = \{y \in Q_n : xy \in E(Q_n)\}$ for the neighbourhood of $x$. Let $f$ be a real-valued function on $Q_S$. We say that an index $i \in S$ is irrelevant if the value of $f(x)$ does not depend on the value of $x_i$. Otherwise, we say that $i$ is relevant. Given a finite set $T \supseteq S$, we can think of $f$ as a function on $Q_T$ for which all the indices in $T \setminus S$ are irrelevant. Conversely, a real-valued function on $Q_T$ for which all the indices in $T \setminus S$ are irrelevant can be thought of as a function on $Q_S$.

We now describe the automorphisms of $Q_n$. For each $\alpha \in Q_n$ there is an automorphism of $Q_n$, which we will also denote by $\alpha$, given by $\alpha(x)_i = \alpha_i x_i$ for all $x \in Q_n$ and $i \in [n]$. Let $S_n$ be the set of permutations of $[n]$. For each $\sigma \in S_n$ there is an automorphism of $Q_n$, which we will also denote by $\sigma$, given by $\sigma(x)_i = x_{\sigma(i)}$ for all $x \in Q_n$ and $i \in [n]$. It is well known that any automorphism $\phi$ of $Q_n$ can be written uniquely as $\phi = \alpha \circ \sigma$ with $\alpha \in Q_n$ and $\sigma \in S_n$.

In particular, there are $2^n n!$ automorphisms of $Q_n$. Hence, since there are 2 signs, every isomorphism class of $k$-functions on $Q_n$ has size at least 1 and at most $2^{n+1} n!$, which gives the following lemma.

**Lemma 1.** Let $0 \leq k \leq n$ be integers. Then

$$\frac{F(n,k)}{2^{n+1} n!} \leq G(n,k) \leq F(n,k).$$

We will also need the following easy lemma later on.
Lemma 2. Let $0 \leq m \leq n$ be integers and let $f$ and $g$ be real-valued functions on $Q_m$. Then $f$ and $g$ are isomorphic when thought of as functions on $Q_m$ if and only if they are isomorphic when thought of as functions on $Q_n$.

Proof. Suppose $f$ and $g$ are isomorphic when thought of as functions on $Q_m$, say $f = \epsilon g \circ \alpha \circ \sigma$, where $\epsilon \in \{-1, 1\}$, $\alpha \in Q_m$ and $\sigma \in S_m$. Let $\beta \in Q_n$ and $\tau \in S_n$ be given by

$$
\beta_i = \begin{cases} 
\alpha_i & \text{for } i \in [m] \\
1 & \text{for } i \notin [m]
\end{cases} \quad \text{and} \quad \tau(i) = \begin{cases} 
\sigma(i) & \text{for } i \in [m] \\
i & \text{for } i \notin [m]
\end{cases}.
$$

Then, when $f$ and $g$ are thought of as functions on $Q_n$, we have $f = \epsilon g \circ \beta \circ \tau$, so $f$ and $g$ are isomorphic when thought of as functions on $Q_n$. 

Suppose $f$ and $g$ are isomorphic when thought of as functions on $Q_n$, say $f = \epsilon g \circ \alpha \circ \sigma$, where $\epsilon \in \{-1, 1\}$, $\alpha \in Q_n$ and $\sigma \in S_n$. Let $\beta \in Q_n$ be given by $\beta_i = \alpha_i$ for all $i \in [m]$. Let $S, T \subseteq [n]$ be the sets of relevant indices of $f$ and $g$, respectively. Then, by considering the set of relevant indices of $f = \epsilon g \circ \alpha \circ \sigma$, we see that $\sigma(T) = S$. Let $\tau \in S_n$ be any permutation such that $\tau(i) = \sigma(i)$ for all $i \in T$. Then, when $f$ and $g$ are thought of as functions on $Q_n$, we have $f = \epsilon g \circ \beta \circ \tau$, so $f$ and $g$ are isomorphic when thought of as functions on $Q_n$. $
$

2.2 Fourier analysis on the Boolean hypercube

We now recall some basic facts about Fourier analysis on the Boolean hypercube. For a comprehensive treatment see [2]. Let $V$ be the vector space of real-valued functions on $Q_n$. For each subset $S \subseteq [n]$, let $\chi_S \in V$ be the function given by

$$
\chi_S(x) = \prod_{i \in S} x_i.
$$

The $\chi_S$ form a basis of $V$, so any $f \in V$ can be written uniquely as

$$
f = \sum_{S \subseteq [n]} \hat{f}(S) \chi_S,
$$

where the $\hat{f}(S)$ are the Fourier coefficients of $f$. The function $\hat{f}$ mapping each $S \subseteq [n]$ to its Fourier coefficient $\hat{f}(S)$ is known as the Fourier transform of $f$ and this decomposition is known as the Fourier decomposition.

For each integer $0 \leq k \leq n$, let $V_k$ be the subspace of $V$ spanned by the $\chi_S$ with $|S| = k$. We will need the following two basic facts later on, so we state them here as separate lemmas.

Lemma 3 (Parseval’s Theorem in [2]). Let $f$ be a Boolean function on $Q_n$. Then

$$
\sum_{S \subseteq [n]} \hat{f}(S)^2 = 1.
$$

Lemma 4 (Exercise 1.11(b) in [2]). Let $f \in V_k$ be a Boolean function, where $k \geq 1$. Then $\hat{f}$ is $\frac{1}{2^{n-k}}$-valued.

3 Results

In this section we prove Theorems 1 and 2. An outline of the proof is as follows. We first obtain a criterion for a Boolean function to be a $k$-function in terms of its Fourier decomposition which will be used throughout the rest of the paper. We then prove Theorem 2. Next, we prove a symmetry of $F(n, k)$ and $G(n, k)$ which explains the appearance of $m = \min(k, n-k)$ in Theorem 1.

We then show how to obtain a $(k+1)$-function on $Q_{n+2}$ given a pair of $k$-functions on $Q_n$, which is the key to proving the lower bound in Theorem 3. Next, we introduce a new function which counts the number of ways of writing a non-negative integer as a sum of squares and obtain an upper bound for $F(n, k)$ in terms of this function. We then prove an upper bound for this new function. Finally, we put all our previous results together to prove Theorem 4.
3.1 Criterion for a Boolean function to be a $k$-function

In [1], Gross and Grupel show that a Boolean function $f$ on $Q_n$ is an $n/2$-function if and only if $f \in V_{n/2}$ (Proposition 3.5.). The following lemma uses the same argument to generalise this result.

**Lemma 5.** A Boolean function $f$ on $Q_n$ is a $k$-function if and only if $f \in V_k$.

**Proof.** Consider the linear map $\alpha : V \to V$ given by

$$(\alpha f)(x) = \sum_{y \in F(x)} f(y).$$

We claim that a Boolean function $f$ on $Q_n$ is a $k$-function if and only if $\alpha f = (n - 2k)f$. To see this, for each $x \in Q_n$, let $k(x) = |\{y \in \Gamma(x) : f(x) \neq f(y)\}|$. Then

$$(\alpha f)(x) = \sum_{y \in F(x)} f(y) = (n - k(x))f(x) + k(x)(-f(x)) = (n - 2k(x))f(x).$$

Then, by definition, $f$ is a $k$-function if and only if $k(x) = k$ for all $x \in Q_n$, i.e. if and only if $\alpha f = (n - 2k)f$.

For each $S \subseteq [n]$, since $\chi_S$ is an $|S|$-function, we thus have $\alpha \chi_S = (n - 2|S|)\chi_S$. So the Fourier basis diagonalises $\alpha$. Hence, $f$ is a $k$-function if and only if $\alpha f = (n - 2k)f$, which happens if and only if $f \in V_k$.

Throughout the rest of the paper we will view Lemma 5 as the definition of a $k$-function.

3.2 Proof of Theorem 2

We now prove Theorem 2.

**Theorem 2.** Let $k \geq 0$ be an integer. Then the sequence $(F(n,k))_{n=k}^{\infty}$ is increasing and eventually constant. Moreover, denoting by $n(k)$ the first value of $n$ after which the sequence is constant, we have

$$3 \cdot 2^{k-1} - 2 \leq n(k) \leq 4.394 \cdot 2^{k}.$$

**Proof.** Gross and Grupel note in [1] that given integers $n \geq m \geq k$ we can think of a $k$-function on $Q_m$ as a $k$-function on $Q_n$ for which all the indices $m < i \leq n$ are irrelevant (Observation 4.4.). Combining this observation with Lemma 2 we obtain that the sequences $(F(n,k))_{n=k}^{\infty}$ and $(G(n,k))_{n=k}^{\infty}$ are increasing.

Moreover, in light of this observation and Lemma 2 a moment’s thought shows that for all integers $N \geq k$ the following two statements are equivalent:

- $G(n,k) = G(N,k)$ for all integers $n \geq N$.
- Every $k$-function has at most $N$ relevant indices.

Wellens proved in [3] that every $k$-function has at most $4.394 \cdot 2^k$ relevant indices (Theorem 1.1.). Hence $(G(n,k))_{n=k}^{\infty}$ is eventually constant and $n(k) \leq 4.394 \cdot 2^k$. In [1], Chiarelli, Hatami and Saks recursively construct $k$-functions with $3 \cdot 2^{k-1} - 2$ relevant indices (Theorem 3.1.). Hence $n(k) \geq 3 \cdot 2^{k-1} - 2$.

3.3 Symmetry of $F(n,k)$ and $G(n,k)$

We now prove a symmetry of $F(n,k)$ and $G(n,k)$.

**Lemma 6.** Let $0 \leq k \leq n$ be integers. Then $F(n,k) = F(n,n-k)$ and $G(n,k) = G(n,n-k)$.

**Proof.** Define a linear map $\beta : V \to V$ by $\beta f = \chi_{[n]} f$. Since $\chi_{[n]}^2 = 1$, $\beta^2 = id$, where $id$ is the identity function on $V$. We have $\beta \chi_S = \chi_{[n]} \chi_S = \chi_{S'}$ for all $S \subseteq [n]$. Hence $\beta$ swaps $V_k$ and $V_{n-k}$. Note that $\beta f$ is a Boolean function if and only if $f$ is. Hence, by Lemma 5 $\beta$ induces a bijection between $k$-functions and $(n-k)$-functions on $Q_n$. So $F(n,k) = F(n,n-k)$.

To show that $G(n,k) = G(n,n-k)$ it suffices to check that $\beta$ respects isomorphisms. Let $f, g \in V$ be isomorphic, say $f = \epsilon g \circ \phi$, where $\epsilon \in \{-1,1\}$ and $\phi \in Aut(Q_n)$. Then

$$\beta f = \beta (\epsilon g \circ \phi) = \epsilon \chi_{[n]} (g \circ \phi) = \epsilon (\chi_{[n]} \circ \phi^{-1} \circ \phi)(g \circ \phi) = \epsilon ((\chi_{[n]} \circ \phi^{-1})g) \circ \phi .$$
3.4 Obtaining a $(k + 1)$-function on $Q_{n+2}$ from a pair of $k$-functions on $Q_n$

The following lemma is the key to proving the lower bound in Theorem 1. A similar construction was used in [4] by Chiarelli, Hatami and Saks to recursively construct $k$-functions with $3 \cdot 2^{k-1} - 2$ relevant indices.

**Lemma 7.** Let $f$ and $g$ be $k$-functions on $Q_n$. Then

$$h = \left( \frac{f + g}{2} \right) x_{n+1} + \left( \frac{f - g}{2} \right) x_{n+2}$$

is a $(k + 1)$-function on $Q_{n+2}$.

**Proof.** By Lemma 5 we need to check that $h$ is a Boolean function in $V_{k+1}$. By considering the four possible values for the pair $(x_{n+1}, x_{n+2})$, we see that the values obtained by $h$ are those obtained by $\pm f$ and $\pm g$. Since $f$ and $g$ are Boolean functions, so is $h$. Since $f$ and $g$ are in $V_k$, so are $(f + g)/2$ and $(f - g)/2$. Hence $((f + g)/2) x_{n+1}$ and $((f - g)/2) x_{n+2}$ are in $V_{k+1}$, since $(f + g)/2$ and $(f - g)/2$ are functions on $Q_n$. Hence $h \in V_{k+1}$.

**Corollary 1.** Let $0 \leq k \leq n$ be integers. Then $F(n + 2, k + 1) \geq F(n, k)^2$.

**Proof.** In Lemma 7 distinct pairs $(f, g)$ give distinct $h$.

**Corollary 2.** Let $k \geq 0$ be an integer. Then $F(2k, k) \geq 2^{2^k}$.

**Proof.** This follows from $F(0, 0) = 2$ and iterating Corollary 1 with $n = 2k$.

Lemma 7 gives a way of constructing a $(k + 1)$-function given a pair of $k$-functions. One might ask whether every $(k + 1)$-function arises in this way. It turns out this is not the case. We give an example of a 4-function which cannot be obtained from a pair of 3-functions in this way. Note that the $h$ in Lemma 7 is “covered” by the indices $n + 1$ and $n + 2$, in the sense that for all $S \subseteq [n + 2]$ with $\hat{h}(S) \neq 0$, either $n + 1 \in S$ or $n + 2 \in S$. Hence it is sufficient to construct a 4-function $h$ which cannot be covered by two indices.

We have 1-functions $x_1$ and $x_2$, so by Lemma 7 we have a 2-function

$$f(x_1, x_2, x_3, x_4) = \left( \frac{x_1 + x_2}{2} \right) x_3 + \left( \frac{x_1 - x_2}{2} \right) x_4 = x_1 x_3 + x_2 x_3 + x_1 x_4 - x_2 x_4.$$

Let $g_1, g_2, g_3$ and $g_4$ be copies of $f$ with disjoint relevant indices. Let

$$h = f(g_1, g_2, g_3, g_4) = g_1 g_3 + g_2 g_3 + g_1 g_4 - g_2 g_4.$$

Then it is easy to check that $h$ is a 4-function with 64 non-zero terms in its Fourier decomposition and that for every relevant index $i$ there are precisely 16 sets $S$ with $i \in S$ and $\hat{h}(S) \neq 0$. Hence $h$ cannot be covered by 2 indices.
3.5 Relation between $F(n,k)$ and $S(q,t)$

We now introduce a new function, $S(q,t)$, and prove an upper bound for $F(n,k)$ in terms of $S(q,t)$. For integers $q,t \geq 0$, let $S(q,t)$ denote the number of $x \in \mathbb{Z}^t$ such that

$$\sum_{i=1}^{t} x_i^2 = q.$$ 

We then have the following lemma.

**Lemma 8.** Let $1 \leq k \leq n$ be integers. Then $F(n,k) \leq S\left(4^{k-1}, \binom{n}{k}\right)$.

**Proof.** Let $f$ be a $k$-function on $Q_n$. By Lemma 4 and Lemma 5, $\hat{f}(S) = \frac{xS}{2^{|S|}}$, where $x_S \in \mathbb{Z}$, for all $S \subseteq [n]$ with $|S| = k$. Note that $\hat{f}(S) = 0$ for $S \subseteq [n]$ with $|S| \neq k$ by Lemma 5. By Lemma 5,

$$\sum_{S \in \binom{n}{k}} x_S^2 = 4^{k-1}.$$ 

Distinct $f$ give distinct $x \in \mathbb{Z}^{\binom{n}{k}'}$, so the result follows. \[\square\]

3.6 An upper bound for $S(q,t)$

The function $S(q,t)$ has been studied in number theory, where it is denoted by $r_t(q)$. The author searched the literature but was only able to find estimates in the regime where $q$ is fixed and $t$ is large, whereas for our purposes we need to consider the regime where both $q$ and $t$ are large and $t$ is much larger than $q$. When $t$ is much larger than $q$, most of the $x_i$ have to be 0, so the size of $S(q,t)$ is governed less by the number theory and more by the combinatorics of choosing which $x_i$ are non-zero. We have the following upper bound for $S(q,t)$.

**Lemma 9.** For all integers $t \geq q \geq 0$, we have

$$S(q,t) \leq 2^{q \log_2 t + O(q \log_2 q)}.$$ 

**Proof.** We first prove an upper bound for $S(q,t)$ for all integers $q,t \geq 0$. If $x \in \mathbb{Z}^t$ is such that $\sum_{i=1}^{t} x_i^2 = q$, then $|x_i| \leq \sqrt{q}$ for all $i \in [t]$, so there are at most $2\sqrt{q} + 1$ possibilities for each $x_i$. Hence $S(q,t) \leq (2\sqrt{q} + 1)^t$. Now suppose $t \geq q \geq 0$ are integers. For each $x \in \mathbb{Z}^t$ with $\sum_{i=1}^{t} x_i^2 = q$, the set $\{i \in [t] : x_i \neq 0\}$ has size at most $q$, so we can pick a subset of $[t]$ of size $q$ containing it. Then there are $\binom{t}{q}$ such subsets and for each subset there are at most $S(q,q)$ different $x \in \mathbb{Z}^t$ with $\sum_{i=1}^{t} x_i^2 = q$ for which that subset is picked, so $S(q,t) \leq \binom{t}{q} S(q,q)$.

Combining these two bounds, we have

$$S(q,t) \leq \binom{t}{q} S(q,q) \leq \binom{t}{q} (2\sqrt{q} + 1)^q = 2^{q \log_2 t + O(q \log_2 q)}.$$ 

For the last inequality, note that we have

$$\frac{t^q}{q^q} \leq \binom{t}{q} \leq \frac{t^q}{q!}$$

for all integers $t \geq q \geq 0$ and hence

$$\binom{t}{q} = 2^{q \log_2 t + O(q \log_2 q)}.$$ 

By considering $x \in \mathbb{Z}^t$ with $x_i \in \{-1,1\}$ for $q$ different $i \in [t]$ and $x_i = 0$ for all other $i$, we have

$$S(q,t) \geq \binom{t}{q} 2^q = 2^{q \log_2 t + O(q \log_2 q)}$$

for all integers $t \geq q \geq 0$. Hence the bound in Lemma 9 is tight. \[\square\]
3.7 Proof of Theorem

We now put all our previous results together to prove Theorem.

**Theorem.** Let \( 0 \leq k \leq n \) be integers. Then

\[
2^{2^m + o(1)} \leq G(n, k) \leq 2^{2^m + O(\log_2 m)},
\]

where \( m = \min(k, n - k) \).

**Proof.** By Lemma \[\text{6}\], we may assume that \( m = k \), i.e. that \( n \geq 2k \). We first prove the lower bound. We have

\[
G(n, k) \geq G(2k, k) \geq \frac{F(2k, k)}{2^{2k+1}(2k)!} \geq \frac{2^{2^k}}{2^{2k+1}(2k)!} = 2^{2^k + o(1)}.
\]

(by Corollary \[\text{2}\])

We now prove the upper bound. We have

\[
G(n, k) \leq G(n(k), k) \leq S \left( 4^k - 1, \binom{n(k)}{k} \right) \leq 2^{2^k + O(\log_2 k)}.
\]

(by Lemma \[\text{8}\] and Theorem \[\text{2}\])

\[\square\]

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