SIGMA-MODEL SOLUTIONS AND INTERSECTING P-BRANES RELATED TO LIE ALGEBRAS

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A family of Majumdar-Papapetrou type solutions in $\sigma$-model of $p$-brane origin is obtained for all direct sums of finite-dimensional simple Lie algebras. Several examples of $p$-brane dyonic configurations in $D = 10$ (IIA) and $D = 11$ supergravities corresponding to the Lie algebra $A_2$ are considered.
1 Introduction

In this paper we give explicit relations for the Majumdar-Papapetrou type solutions from [2], corresponding to direct sums of simple finite-dimensional Lie algebras (e.g. to semisimple Lie algebras). These solutions appear in the \(\sigma\)-model of \(p\)-brane origin [3]–[6] and are governed by a set of harmonic functions.

For \(p\)-brane configurations the obtained solutions describe bound states of \(p\)-brane extremal configurations with non-standard intersection rules (for non-extremal case see [22]). The standard ”orthogonal” solutions (see [3]–[13] and refs. therein) correspond to the Lie algebras \(A_1 \oplus \ldots \oplus A_1\).

Here we also consider several examples of the solutions in \(D = 10\) IIA supergravity and \(D = 11\) supergravity. These dyonic solutions correspond to the Lie algebra \(A_2 = \text{sl}(3)\) and may be interesting as possible applications in MQCD [14]–[21].

2 Majumdar-Papapetrou solutions in \(\sigma\)-model

We consider the \(\sigma\)-model governed by the action

\[
S_{\sigma} = \int d^{d_0}x \sqrt{|g^0|} \left\{ R[g^0] - \hat{G}_{AB} g^{0\mu\nu} \partial_\mu \sigma^A \partial_\nu \sigma^B - \sum \varepsilon_s e^{-2U^s_{\lambda} \sigma^A} g^{0\mu\nu} \partial_\mu \Phi^s \partial_\nu \Phi^s \right\},
\]

where \(g^0 = g^0_{\mu\nu}(x)dx^\mu \otimes dx^\nu\) is a metric on \(d_0\)-dimensional manifold \(M_0\), \(\sigma = (\sigma^A) \in \mathbb{R}^N\) and \(\Phi = (\Phi^s, s \in S)\) is a set of scalar fields (\(S \neq \emptyset\)), \((\hat{G}_{AB})\) is a non-degenerate symmetric matrix, \(U^s = (U^s_{\lambda}) \in \mathbb{R}^N\) are vectors and \(\varepsilon_s = \pm 1\), \(s \in S\). The scalar product is defined as follows

\[
(U, U') = \hat{G}^{AB} U_A U'_B,
\]

for \(U, U' \in \mathbb{R}^N\), where \((\hat{G}^{AB}) = (\hat{G}_{AB})^{-1}\).

Let the set \(S\) be a union of \(k\) non-intersecting (non-empty) subsets \(S_1, \ldots, S_k\):

\[
S = S_1 \sqcup \ldots \sqcup S_k,
\]

\(S_i \neq \emptyset, i = 1, \ldots, k\), and

\[
(U^s, U^{s'}) = 0
\]

for all \(s \in S_i, s' \in S_j, i \neq j; i, j = 1, \ldots, k\). According to (2.4) the set of vectors \((U^s, s \in S)\) has a block-orthogonal structure with respect to the scalar product (2.2), i.e. it is splitted into \(k\) mutually-orthogonal blocks \((U^s, s \in S_i), i = 1, \ldots, k\).

Equations of motion corresponding to (2.1) have the following form

\[
R_{\mu\nu}[g^0] = \hat{G}_{AB} \partial_\mu \sigma^A \partial_\nu \sigma^B + \sum \varepsilon_s e^{-2U^s_{\lambda} \sigma^A} \partial_\mu \Phi^s \partial_\nu \Phi^s,
\]

\[
\hat{G}_{AB} \Delta[g^0] \sigma^B + \sum \varepsilon_s U^s_{\lambda} e^{-2U^s_{\epsilon} \sigma^C} g^{0\mu\nu} \partial_\mu \Phi^s \partial_\nu \Phi^s = 0,
\]

\[
\partial_\mu \left( \sqrt{|g^0|} g^{0\mu\nu} e^{-2U_{\lambda} \sigma^A} \partial_\nu \Phi^s \right) = 0,
\]
Let $(M_0, g^0)$ be Ricci-flat
\[ R_{\mu\nu}[g^0] = 0. \tag{2.8} \]
Then the field configuration
\[ g^0, \quad \sigma^A = \sum_{s \in S} \varepsilon_s U^s A \nu^2_s \ln H_s, \quad \Phi^s = \frac{\nu_s}{H_s}, \tag{2.9} \]
satisfies the field equations (2.5)–(2.7) with \( V = 0 \) if (real) numbers \( \nu_s \) obey the relations
\[ \sum_{s' \in S} (U^s, U^s') \varepsilon_{s'} \nu^2_{s'} = -1 \tag{2.10} \]
s, \( s \in S \), functions \( H_s > 0 \) are harmonic, i.e. \( \Delta[g^0] H_s = 0 \), \( s \in S \), and \( H_s \) are coinciding inside blocks: \( H_s = H_{s'} \) for \( s, s' \in S_i \), \( i = 1, \ldots, k \).
In special orthogonal case, when any block contains only one vector (i.e. all \( |S_i| = 1 \)) the Proposition 1 coincides with that of [5]. In a general case vectors inside each block \( S_i \) are not orthogonal. The solution under consideration depends on \( k \) independent harmonic functions corresponding to \( k \) blocks. For a given set of vectors \( (U^s, s \in S) \) the maximal number \( k \) arises for irreducible block-orthogonal decomposition (2.3), (2.4), when any block \( (U^s, s \in S_i) \) can not be splitted into two mutually-orthogonal subblocks.

### 3 Solutions related to Lie algebras

Here we put
\[ (U^s, U^s) \neq 0 \tag{3.1} \]
for all \( s \in S \) and introduce the quasi-Cartan matrix \( A = (A^{s's'}) \)
\[ A^{s's'} = \frac{2(U^s, U'^s)}{(U^s, U^s)}, \tag{3.2} \]
\( s, s' \in S \). From (2.4) we get a block-diagonal structure of \( A \):
\[ A = \text{diag}(A^{(1)}, \ldots, A^{(k)}), \tag{3.3} \]
where \( A^{(i)} = (A^{s's'}, s, s' \in S_i), i = 1, \ldots, k \). Here the set \( S \) is ordered, \( S_1 < \ldots < S_k \), and the order in \( S_i \) is inherited by the order in \( S \).
For \( \det A^{(i)} \neq 0 \) relation (2.10) may be rewritten in the equivalent form
\[ \varepsilon_s \nu^2_s (U^s, U^s) = -2 \sum_{s' \in S} A^{(i)}_{s's'} \]
s, \( s \in S_i \), where \( (A^{(i)}_{s's'}) = A_{(i)}^{-1} \). Thus, eq. (2.10) may be resolved in terms of \( \nu_s \) for certain \( \varepsilon_s = \pm 1 \), \( s \in S_i \).
Let $A$ be a generalized Cartan matrix $[24, 25]$. In this case
\[
A^{ss'} \in -\mathbb{Z}_+ \equiv \{0, -1, -2, \ldots\}
\]
for $s \neq s'$ and $A$ generates generalized symmetrizable Kac-Moody algebra $[24, 25]$. Here there are three possibilities for $A(i)$: a) $\det A(i) > 0$, b) $\det A(i) < 0$ and c) $\det A(i) = 0$. For $\det A(i) \neq 0$ the corresponding Kac-Moody algebra is simple, since $A(i)$ is indecomposable $[25]$. In this paper we consider only the case $\det A(i) \neq 0$.

### 3.1 Simple finite-dimensional Lie algebras

Let $A(i)$ be a Cartan matrix of a simple finite-dimensional Lie algebra $L_i$ and hence $\det A(i) > 0$, $A^{ss'} \in \{0, -1, -2, -3\}$, $s \neq s'$, $s \in S_i$, $i = 1, \ldots, k$. In this case the matrix $A$ from (3.3) is the Cartan matrix for the Lie algebra
\[
L = \bigoplus_{j=1}^{k} L_j.
\]
When $\dim L_j \neq 1$, for all $j = 1, \ldots, k$, $L$ is semisimple.

The elements of inverse matrix $A_{(i)}^{-1}$ are positive (see Ch.7 in [25]) and hence we get from (3.4)
\[
\varepsilon_s(U^s, U^s) < 0,
\]
$s \in S_i$.

In summary $[25]$, there are four infinite series of simple Lie algebras, which are denoted by
\[
A_r \ (r \geq 1), \quad B_r \ (r \geq 3), \quad C_r \ (r \geq 2), \quad D_r \ (r \geq 4),
\]
and in addition five isolated cases, which are called
\[
E_6, \quad E_7, \quad E_8, \quad G_2, \quad F_4.
\]
In all cases the subscript denotes the rank of the algebra. The algebras in the infinite series of simple Lie algebras are called the classical (Lie) algebras. They are isomorphic to the matrix algebras
\[
A_r \cong \mathfrak{sl}(r+1), \quad B_r \cong \mathfrak{so}(2r+1), \quad C_r \cong \mathfrak{sp}(r), \quad D_r \cong \mathfrak{so}(2r).
\]
The five isolated cases are referred to as the exceptional Lie algebras.

For the simple Lie algebras of type $A_r$, $D_r$, $E_6$, $E_7$ and $E_8$, all root have the same length, and any two nodes of Dynkin diagram are connected by at most one line. In the other cases there are roots of two different lengths, the length of the long roots being $\sqrt{2}$ times the length of the short roots for $B_r$, $C_r$ and $F_4$, and $\sqrt{3}$ times for $G_2$, respectively.

Now, let us consider the solutions in the sigma-model related to the simple Lie algebras considered above.

$A_r$ series. Let $A(i)$ be $r \times r$ Cartan matrix for the Lie algebra $A_r = \mathfrak{sl}(r+1)$, $r \geq 1$. This matrix is described graphically by the Dynkin diagram pictured on Fig.1.
Fig. 1. Dynkin diagram for $A_r$ Lie algebra

(For $s \neq s'$, $A_{ss'}^{(i)} = -1$ if nodes $s$ and $s'$ are connected by a line on the diagram and $A_{ss'}^{(i)} = 0$ otherwise). Using the relation for the inverse matrix $A_{ss'}^{(i)} = (A_{ss'}^{(i)})^{-1}$ (see Sect. 7.5 in [23])

$$A_{ss'}^{(i)} = \frac{1}{r + 1} \min(s, s')[r + 1 - \max(s, s')]$$

(3.11)

we may rewrite (3.4) as follows

$$\varepsilon_s \nu^2_s(U^s, U^s) = \begin{cases} s(s - 2r - 1) & \text{for } s \neq r, \\ -\frac{s}{2}(r + 1) & \text{for } s = r. \end{cases}$$

(3.12)

$s \in \{1, \ldots, r\} = S_i$.

$B_r$ and $C_r$ series. Dynkin diagrams for these cases are pictured on Fig. 2.

In these cases we have the following formulas for inverse matrices $A_{ss'}^{(i)} = (A_{ss'}^{(i)})^{-1}$:

$$A_{ss'}^{(i)} = \begin{cases} \min(s, s') & \text{for } s \neq r, \\ \frac{1}{2}s' & \text{for } s = r \end{cases}, \quad A_{ss'}^{(i)} = \begin{cases} \min(s, s') & \text{for } s' \neq r, \\ \frac{1}{2}s & \text{for } s' = r. \end{cases}$$

(3.13)

and relation (3.4) takes the form

$$\varepsilon_s \nu^2_s(U^s, U^s) = \begin{cases} s(s - 2r - 1) & \text{for } s \neq r, \\ -\frac{s}{2}(r + 1) & \text{for } s = r, \end{cases}$$

(3.14)

for $B_r$ and $C_r$ series respectively, $s \in \{1, \ldots, r\} = S_i$.

$D_r$ series. We have the following Dynkin diagram for this case (Fig. 3):

Fig. 2. Dynkin diagrams for $B_r$ and $C_r$ Lie algebras

Fig. 3. Dynkin diagram for $D_r$ Lie algebra
and formula for the inverse matrix $A^{-1}_{(i)} = (A^{(i)}_{ss'})$ [23]:

$$A^{(i)}_{ss'} = \begin{cases} 
\min(s, s') & \text{for } s, s' \notin \{r, r-1\}, \\
\frac{1}{2}s & \text{for } s \notin \{r, r-1\}, s' \in \{r, r-1\}, \\
\frac{1}{2}s' & \text{for } s \in \{r, r-1\}, s' \notin \{r, r-1\}, \\
\frac{1}{4}r & \text{for } s = s' = r \text{ or } s = s' = r-1, \\
\frac{1}{4}(r-2) & \text{for } s = r, s' = r-1 \text{ or vice versa.}
\end{cases} \quad (3.15)$$

The relation (3.4) in this case reads

$$\varepsilon s \nu_s^2(U^s, U^s) = \begin{cases} 
s(s - 2r + 1) & \text{for } s \notin \{r, r-1\}, \\
-\frac{1}{2}(r-1) & \text{for } s \in \{r, r-1\};
\end{cases} \quad (3.16)$$

$s \in \{1, \ldots, r\} = S_i$.

Let us consider the exceptional Lie algebras. Dynkin diagrams of these algebras are pictured on Fig.4.

![Dynkin diagrams for $E_6$, $E_7$, $E_8$, $F_4$ and $G_2$ Lie algebras respectively](image)

Relation (3.4) for these algebras takes the following form

$$-\frac{1}{2} \varepsilon_s \nu_s^2(U^s, U^s) = \begin{cases} 
8, 15, 21, 15, 8, 11 & \text{for } E_6, \ s = 1, \ldots, 6; \\
17, 33, 48, \frac{75}{2}, 26, \frac{27}{2}, \frac{49}{2} & \text{for } E_7, \ s = 1, \ldots, 7; \\
29, 57, 84, 110, 135, 91, 46, 68 & \text{for } E_8, \ s = 1, \ldots, 8; \\
11, 21, 15, 8 & \text{for } F_4, \ s = 1, \ldots, 4; \\
5, 3 & \text{for } G_2, \ s = 1, 2.
\end{cases} \quad (3.17)$$

### 3.2 Hyperbolic Kac-Moody algebras

Let $\det A_{(i)} < 0$. Among irreducible symmetrizable matrices satisfying (3.5) there exists a large subclass of Cartan matrices, corresponding to infinite-dimensional simple hyperbolic generalized Kac-Moody (KM) algebras of ranks $r = 2, \ldots, 10$ [24, 25].

**Example.** Let $A_{(i)}$ be a Cartan matrix corresponding to $E_{10}$ hyperbolic KM algebra with the Dynkin diagram pictured on Fig.5.
Fig. 5. Dynkin diagram for $E_{10}$ hyperbolic KM algebra

Using the explicit form for $(-A_{(i)}^{-1})$ we get from (3.4)

$$\frac{1}{2} \varepsilon_s(U^s, U^s) \nu_s^2 = 30, 61, 93, 126, 160, 195, 231, 153, 76, 115$$

(3.18)

for $s = 1, 2, \ldots, 10$ respectively.

In this example for hyperbolic algebra the following relation is satisfied:

$$\varepsilon_s(U^s, U^s) > 0,$$

(3.19)

$s \in S_i$. This relation is valid in a general case, since $(-A_{(i)}^{-1})_{ss'} \leq 0$, $s, s' \in S$, for any hyperbolic algebra. An example of a solution corresponding to the hyperbolic Lie algebra $F_3$ was considered in [26].

4 Solutions with intersecting $p$-branes

4.1 The model

Now we consider a multidimensional gravitational model governed by the action [5]

$$S = \int d^Dz \sqrt{|g|} \left\{ R[g] - h_{\alpha\beta} g^{MN} \partial_M \varphi^\alpha \partial_N \varphi^\beta - \sum_{a \in \Delta} \frac{\theta_a}{n_a!} \exp[2\lambda_a(\varphi)](F^a)^2 \right\}$$

(4.1)

where $g = g_{MN} dz^M \otimes dz^N$ is the metric, $\varphi = (\varphi^\alpha) \in \mathbb{R}^l$ is a vector of scalar fields, $(h_{\alpha\beta})$ is a non-degenerate symmetric $l \times l$ matrix $(l \in \mathbb{N})$, $\theta_a = \pm 1$, $F^a = dA^a$ is an $n_a$-form $(n_a \geq 1)$, $\lambda_a$ is a 1-form on $\mathbb{R}^l$: $\lambda_a(\varphi) = \lambda_{a\alpha} \varphi^\alpha$, $a \in \Delta$, $\alpha = 1, \ldots, l$. Here $\Delta$ is some finite set.

We consider the manifold $M = M_0 \times M_1 \times \ldots \times M_n$, with the metric

$$g = e^{2\gamma(x)} g^0 + \sum_{i=1}^n e^{2\phi^i(x)} g^i$$

(4.2)

where $g^0 = g_{\mu\nu}(x) dx^\mu \otimes dx^\nu$ is a metric on the manifold $M_0$, and $g^i = g_{m_i n_i}(y_i) dy_i^{m_i} \otimes dy_i^{n_i}$ is a metric on the Ricci-flat manifold $M_i$.

Any manifold $M_\nu$ is supposed to be oriented and connected and $d_\nu \equiv \dim M_\nu$, $\nu = 0, \ldots, n$. Let

$$\tau_i \equiv \sqrt{|g^i(y_i)|} dy_i^{d_i} \wedge \ldots \wedge dy_i^{d_i}, \quad \varepsilon(i) \equiv \text{sign}(\det(g^i_{m_i n_i})) = \pm 1$$

(4.3)
denote the volume $d_i$-form and signature parameter respectively, $i = 1, \ldots, n$. Let
$\Omega = \Omega_n$ be a set of all subsets of $\{1, \ldots, n\}$, $|\Omega| = 2^n$. For any $I = \{i_1, \ldots, i_k\} \in \Omega$, $i_1 < \ldots < i_k$, we denote

$$\tau(I) \equiv \tau_{i_1} \wedge \ldots \wedge \tau_{i_k}, \quad d(I) \equiv \sum_{i \in I} d_i, \quad \varepsilon(I) \equiv \prod_{i \in I} \varepsilon(i).$$  \tag{4.4}$$

We also put $\tau(\emptyset) = \varepsilon(\emptyset) = 1$ and $d(\emptyset) = 0$.

For fields of forms we consider the following composite electromagnetic ansatz

$$F^a = \sum_{I \in \Omega_{a,e}} F^{(a,e,I)} + \sum_{J \in \Omega_{a,m}} F^{(a,m,J)}$$  \tag{4.5}$$

where

$$F^{(a,e,I)} = d\Phi^{(a,e,I)} \wedge \tau(I),$$
$$F^{(a,m,J)} = e^{-2\lambda_a(\phi)} \ast (d\Phi^{(a,m,J)} \wedge \tau(J))$$ \tag{4.6}$$

are elementary forms of electric and magnetic types respectively, $a \in \triangle$, $I \in \Omega_{a,e}$, $J \in \Omega_{a,m}$ and $\Omega_{a,e} \subset \Omega$, $\Omega_{a,m} \subset \Omega$. In (4.7) $\ast = \ast[g]$ is the Hodge operator on $(M, g)$.

For scalar functions we put $\phi^a = \phi^a(x)$, $\Phi^s = \Phi^s(x)$, $s \in S$, i.e. they depend only on coordinates of $M_0$. Here and below

$$S = S_e \sqcup S_m, \quad S_v = \bigcup_{a \in \triangle} \{a\} \times \{v\} \times \Omega_{a,v},$$  \tag{4.8}$$

$v = e, m$.

Due to (4.6) and (4.7)

$$d(I) = n_a - 1, \quad d(J) = D - n_a - 1,$$  \tag{4.9}$$

for $I \in \Omega_{a,e}$, $J \in \Omega_{a,m}$.

Let $d_0 \neq 2$ and

$$\gamma = \gamma_0(\phi) \equiv \frac{1}{2 - d_0} \sum_{j=1}^n d_j \phi^j,$$  \tag{4.10}$$

i.e. the generalized harmonic gauge is used.

Now we impose restrictions on sets $\Omega_{a,v}$. These restrictions guarantee the block-diagonal structure of a stress-energy tensor (like for the metric) and the existence of the $\sigma$-model representation [5].

We denote $w_1 \equiv \{i | i \in \{1, \ldots, n\}, \quad d_i = 1\}$, and $n_1 = |w_1|$ (i.e. $n_1$ is the number of 1-dimensional spaces among $M_i$, $i = 1, \ldots, n$).

**Restriction 1.** Let 1a) $n_1 \leq 1$ or 1b) $n_1 \geq 2$ and for any $a \in \triangle$, $v \in \{e, m\}$, $i, j \in w_1$, $i < j$, there are no $I, J \in \Omega_{a,v}$ such that $i \in I$, $j \in J$ and $I \setminus \{i\} = J \setminus \{j\}$.

**Restriction 2** (only for $d_0 = 1, 3$). Let 2a) $n_1 = 0$ or 2b) $n_1 \geq 1$ and for any $a \in \triangle$, $i \in w_1$ there are no $I \in \Omega_{a,m}$, $J \in \Omega_{a,e}$ such that $\bar{I} = \{i\} \sqcup J$ for $d_0 = 1$ and $J = \{i\} \sqcup \bar{I}$ for $d_0 = 3$. Here and in what follows

$$\bar{I} \equiv \{1, \ldots, n\} \setminus I.$$  \tag{4.11}$$
It was proved in [5] that equations of motion for the model (4.1) and the Bianchi identities:

\[ dF_s = 0, \quad s \in S_m, \]

for fields from (4.2)–(4.10), when Restrictions 1 and 2 are imposed, are equivalent to equations of motion for the \( \sigma \)-model (2.1) with \( (\sigma^A) = (\phi^i, \varphi^a) \), the index set \( S \) from (4.8), the target space metric \( (\hat{G}_{AB}) = \text{diag}(G_{ij}, h_{\alpha\beta}) \), with

\[ G_{ij} = d_i \delta_{ij} + \frac{d_i d_j}{d_0 - 2}, \quad (4.12) \]

vectors

\[ (U_A^s) = (d_i \delta_{is}, -\chi_s \lambda_{a_is}), \quad (4.13) \]

where \( s = (a_s, v_s, I_s) \), \( \chi_s = +1, -1 \) for \( v_s = e, m \) respectively,

\[ \delta_{II} = \sum_{j \in I} \delta_{ij} \quad (4.14) \]

is the indicator of \( i \) belonging to \( I \): \( \delta_{II} = 1 \) for \( i \in I \) and \( \delta_{II} = 0 \) otherwise; and

\[ \varepsilon_s = (-\varepsilon[g])^{(1-\chi_s)/2} \varepsilon(I_s) \theta_{a_is}, \quad (4.15) \]

\( s \in S, \varepsilon[g] \equiv \text{sign det}(g_{MN}) \).

**General intersection rules.** Scalar products (2.2) for vectors \( U_s \) were calculated in [5]

\[ (U^s, U^{s'}) = d(I_s \cap I_{s'}) + \frac{d(I_s) d(I_{s'})}{2 - D} + \chi_s \chi_{s'} \lambda_{a_is} \lambda_{a_{is'}} h^{\alpha\beta} \equiv B^{ss'}, \quad (4.16) \]

where \( (h^{\alpha\beta}) = (h_{\alpha\beta})^{-1} ; \quad s = (a_s, v_s, I_s) \) and \( s' = (a_{s'}, v_{s'}, I_{s'}) \) belong to \( S \). Let us put \( (U^s, U^s) \neq 0, \quad s \in S \). Then, we obtain the general intersection rule formulas

\[ d(I_s \cap I_{s'}) = \frac{d(I_s) d(I_{s'})}{D - 2} - \chi_s \chi_{s'} \lambda_{a_is} \lambda_{a_{is'}} h^{\alpha\beta} + \frac{1}{2} (U^{s'}, U^{s'}) A^{ss'}, \quad (4.17) \]

\( s \neq s' \), where \( (A^{ss'}) \) is the quasi-Cartan matrix (3.2) (see also (6.32) from [22]).

### 4.2 Exact solutions

Applying the results from Sect. 2 to our multidimensional model (4.1), when vectors \( (U^s, s \in S) \) obey the block-orthogonal decomposition (2.3), (2.4) with scalar products defined in (4.16) we get the exact solution:

\[ g = U \left\{ g^0 + \sum_{i=1}^n U_i g^i \right\}, \quad (4.18) \]

\[ U = \left( \prod_{s \in S} H_s^{2d(I_s) \varepsilon_{a_is} \nu_{s}^2} \right)^{1/(2-D)}, \quad U_i = \prod_{s \in S} H_s^{2 \varepsilon_{a_is} \nu_{s}^2 \delta_{i_s}}, \quad (4.19) \]

\[ \varphi^a = - \sum_{s \in S} \lambda_{a_is} \chi_s \varepsilon_{a_is} \nu_{s}^2 \ln H_s, \quad F^a = \sum_{s \in S} F^a \delta_{a_is}, \quad (4.20) \]
where
\[ F^s = \nu_s dH_s^{-1} \wedge \tau(I_s), \text{ for } v_s = e, \quad (4.21) \]
\[ F^s = \nu_s (\ast_0 dH_s) \wedge \tau(\bar{I}_s), \text{ for } v_s = m, \quad (4.22) \]
\( H_s \) are harmonic functions on \((M_0, g^0)\) coinciding inside blocks of matrix \((B^{ss'})\) from (4.16) \((H_s = H_s', s, s' \in S_j, j = 1, \ldots, k, \text{ see } (2.3), (2.4))\) and relations
\[ \sum_{s' \in S} B^{ss'} \varepsilon_{s'} \nu_s^2 = -1 \quad (4.23) \]
on the matrix \((B^{ss'})\), parameters \( \varepsilon_s \) (1.13) and \( \nu_s \) are imposed, \( s \in S, i = 1, \ldots, n; \alpha = 1, \ldots, l \). Here \( \lambda^a_\alpha = h^{\alpha\beta} \lambda_a \), \( \ast_0 = \ast [g^0] \) is the Hodge operator on \((M_0, g^0)\) and \( \bar{I} \) is defined in (1.11).

In deriving the solution we used as in [5] the relations for contravariant components of \( U^s \)-vectors:
\[ U^{si} = \delta_{I_s} - \frac{d(I_s)}{D - 2}, \quad U^{s\alpha} = -\chi^s \lambda^\alpha_{a s}, \quad (4.24) \]
\( s = (a_s, v_s, I_s) \).

**Remark 1.** The solution is also valid for \( d_0 = 2 \), if Restriction 2 is replaced by Restriction 2* below. It may be proved using a more general form of the sigma-model representation (see Remark 2 in [4]).

**Restriction 2* (for \( d_0 = 2 \)).** For any \( a \in \triangle \) there are no \( I \in \Omega_{a,m}, J \in \Omega_{a,e} \) such that \( \bar{I} = J \) and for \( n_1 \geq 2, i, j \in w_1, i \neq j \), there are no \( I \in \Omega_{a,m}, J \in \Omega_{a,e} \) such that \( i \in I, j \in J, I \setminus \{i\} = J \setminus \{j\} \).

**Remark 2.** Our terminology: "orthogonal", "block-orthogonal" (intersections), refers to the scalar products of \( U \)-vectors (2.2) but not to the orientation of \( p \)-branes in the multidimensional space-time with the metric (4.2). The metric (4.2) is block-diagonal, hence the \( p \)-branes are orthogonal or parallel with respect to the metric (4.2) even if their intersections rules are "non-orthogonal" ones. (Here we do not consider "branes at angles").

## 5 Some examples

Here we consider several examples of the solutions corresponding to the Lie algebra \( A_2 \).

### 5.1 IIA supergravity

Let us consider the \( D = 10 \) IIA supergravity with the bosonic part of the action
\[ S = \int d^{10}z \sqrt{|g|} \left\{ R[g] - (\partial \varphi)^2 - \sum_{a=2}^4 e^{2\lambda_a \varphi} F_a^2 \right\} - \frac{1}{2} \int F_4 \wedge F_4 \wedge A_2, \quad (5.1) \]
where \( F_a = dA_{a-1} + \delta_{a4} A_1 \wedge F_3 \) is an \( a \)-form and
\[ \lambda_3 = -2\lambda_4, \quad \lambda_2 = 3\lambda_4, \quad \lambda_4^2 = \frac{1}{8}, \quad (5.2) \]
The dimensions of \( p \)-brane worldsheets are
\[
d(I) = \begin{cases} 
1, 2, 3 & \text{electric case} \\
7, 6, 5 & \text{magnetic case}
\end{cases}
\]
(5.3)
for \( a = 2, 3, 4 \) respectively.

We consider here the sector corresponding to \( a = 3, 4 \) describing electric \( p \)-branes: fundamental string (FS), D2-brane and magnetic \( p \)-branes: NS5- and D4-branes.

We get \((U^s, U^s) = 2\) for all \( s \). The solutions with \( A_2 \) intersection rules corresponding to relations
\[
1 \cap 5 = 2 \cap 5 = 2 \cap 4 = 1 \\
5 \cap 4 = 5 \cap 5 = 3, \quad 4 \cap 4 = 2
\]
(5.4)
are valid in the ”truncated case” (without Chern-Simons terms) and in a general case (5.1) as well. Here \((p_1 \cap p_2 = d) \leftrightarrow (d(I) = p_1 + 1, d(J) = p_2 + 1, d(I \cap J) = d)\).

Let us consider the solution describing the electromagnetic dyon with intersections
\[
1 \cap 5 = 2 \cap 4 = 1.
\]
(5.6)
Here the solution reads
\[
g = H^2 g^0 - H^{-2} dt \otimes dt + g^1 + g^2, \\
F_a = \nu_1 dH^{-1} \wedge dt \wedge \tau_1 + \nu_2 (\ast_0 dH) \wedge \tau_1,
\]
(5.7)
(5.8)
where \( H \) is harmonic function on \((M_0, g^0)\), \( d_0 = 3, \nu_1^2 = \nu_2^2 = 1, d_1 = a - 2, d_2 = 8 - a, a = 3, 4 \). The signature restrictions are the following: \( \varepsilon_1 = +1, \varepsilon_2 = -\varepsilon[g] = +1 \).

For \( g^0 = \sum_{\mu=1}^{3} dx^\mu \otimes dx^\mu \) and
\[
H = 1 + \sum_{i=1}^{N} \frac{q_i}{|x - x_i|},
\]
(5.9)
the 4-dimensional section of the metric (5.7) coincides with the standard Majumdar-Papapetrou solution [1] describing \( N \) extremal charged black holes with horizons at points \( x_i \), and charges \( q_i > 0, i = 1, \ldots, N \). Our solution (5.7), (5.8) with \( H \) from (5.9) describes \( N \) extremal \( p \)-brane black holes formed by dyons. Any dyon contains one electric ”brane” and one magnetic ”brane” with equal charge densities.

5.2 \( D = 11 \) supergravity

For the \( D = 11 \) supergravity with the bosonic part of the action
\[
S = \int d^{11}z \sqrt{|g|} \left( R[g] - \frac{F_4^2}{4!} \right) + c \int F_4 \wedge F_4 \wedge A_3
\]
(5.10)
we get the \( A_2 \)-solutions with the intersections
\[
2 \cap 5 = 1, \quad 5 \cap 5 = 3.
\]
(5.11)
The electromagnetic dyon with the intersection \( 2 \cap 5 = 1 \) is given by relations (5.4) and (5.8) with \( a = 4, d_1 = 3, d_2 = 5 \). Other relations are unchanged.
6 Discussions

Here we obtained explicit formulas for the solution from \[2\] corresponding to direct sums of finite-dimensional simple Lie algebras. These solutions for $p$-brane case have nonstandard intersection rules, defined by Cartan matrices of the corresponding Lie algebras and describe bound state configurations of $p$-branes.

Here we presented several examples of such dyons corresponding to the $A_2$ Lie algebra in $D = 10$ IIA and $D = 11$ supergravities. We note that the usual solutions in IIA supergravity corresponding to the Lie algebras $A_1 \oplus A_1$ (D4- and NS5-branes) play a rather important role in MQCD \[14]–\[21\]. It is also interesting to consider the possible applications of the obtained solutions in MQCD.

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References

[1] S.D. Majumdar, *Phys. Rev.* 72, 930 (1947); A. Papapetrou, *Proc. R. Irish Acad.* A51, 191 (1947).

[2] V.D. Ivashchuk and V.N. Melnikov, Majumdar-Papapetrou Type Solutions in Sigma-model and Intersecting p-branes, `hep-th/9802121` submitted to *Class. and Quant. Grav.*

[3] V.D. Ivashchuk and V.N. Melnikov, Intersecting p-brane Solutions in Multidimensional Gravity and M-theory, *Gravitation and Cosmology* 2, No 4, 204 (1996); `hep-th/9612089`.

[4] V.D. Ivashchuk and V.N. Melnikov, *Phys. Lett.* B 403 23 (1997).

[5] V.D. Ivashchuk and V.N. Melnikov, Sigma-model for the Generalized Composite p-branes, *Class. Quant. Grav.* 14 (11), 3001 (1997); `hep-th/9705036`.

[6] V.D. Ivashchuk, V.N. Melnikov and M. Rainer, Multidimensional $\sigma$-models with Composite Electric p-branes, `gr-qc/9705003`; *Gravitation and Cosmology* 4, No. 1 (13), 73 (1998).

[7] K.S. Stelle, Lectures on Supergravity p-branes, `hep-th/9701088`.

[8] G. Papadopoulos and P.K. Townsend, *Phys. Lett.* B 380 273 (1996).

[9] A.A. Tseytlin, *Nucl. Phys.* B 475 149 (1996); `hep-th/9604033`.

[10] J.P. Gauntlett, D.A. Kastor, and J. Traschen, *Nucl. Phys.* B 478 544 (1996); `hep-th/9604179`.

[11] I.Ya. Aref’eva and A.I. Volovich, *Class. Quant. Grav.* 14 (11), 2990 (1997); `hep-th/9611026`.

[12] I.Ya. Aref’eva and O.A. Rytchkov, Incidence Matrix Description of Intersecting p-brane Solutions, *Preprint* SMI-25-96; `hep-th/9612236`.

12
[13] R. Argurio, F. Englert and L. Hourant, *Phys. Lett.* B 398 2991 (1997); [hep-th/9701042].

[14] S. Elitzur, A. Giveon and D. Kutasov, Branes and $N = 1$ Duality in String Theory, *Phys. Lett.* B 400 (1997) 269.

[15] J. Brodie and A. Hanany, Type IIA Superstrings, Chiral Symmetry, and $N = 1$ 4D Gauge Theory Dualities, *Nucl. Phys.* B 506 (1997) 157; [hep-th/9704043].

[16] A. Brandhuber, J. Sonnenschein, S. Theisen and S. Yankielowicz, Brane Configurations and 4D Field Theory Dualities, *Nucl. Phys.* B 502 (1997) 125; [hep-th/9704044].

[17] E. Witten, Branes and the Dynamics of QCD, *Nucl. Phys.* B 507 (1997) 658; [hep-th/9706109].

[18] K. Hori, H. Ooguri and Y. Oz, Strong Coupling Dynamics of Four-Dimensional $N = 1$ Gauge Theory from M-Theory Five-brane, [hep-th/9706082].

[19] A. Hanany, M. Strassler and A. Zaffaroni, Confinement and Strings in MQCD, [hep-th/9707244].

[20] Jan de Boer, Kentaro Hori, Hirosi Ooguri and Yaron Oz, Kahler Potential and Higher Derivative Terms from M-Theory Five-brane, [hep-th/9711142].

[21] A. Brandhuber, N. Itzaki, V. Kaplunovsky, J. Sonnenschein and S. Yankielowicz, Comments on the M-Theory Approach to $N = 1$ SQCD and Brane Dynamics, *Phys. Lett.* B 410 (1997) 27; [hep-th/9706127].

[22] V.D. Ivashchuk and V.N. Melnikov, Multidimensional Classical and Quantum Cosmology with Intersecting p-branes, [hep-th/9708157]. *J. Math. Phys.*, 39, 2866 (1998).

[23] K.A. Bronnikov, Block-orthogonal Brane systems, Black Holes and Wormholes, [hep-th/9710207]. *Gravitation and Cosmology* 4, No 1(13), 49 (1998).

[24] V.G. Kac, Infinite-dimensional Lie Algebras (Cambridge University Press, Cambridge, 1990).

[25] J. Fuchs and C. Schweigert, Symmetries, Lie algebras and Representations. (Cambridge University Press, Cambridge, 1997).

[26] V.D. Ivashchuk, S.-W. Kim and V.N. Melnikov, Hyperbolic Kac-Moody Algebra from Intersecting p-Branes, [hep-th/9803006].