BOUNDS OF INCIDENCES BETWEEN POINTS AND
ALGEBRAIC CURVES

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Abstract. We prove new bounds on incidence between points and high degree algebraic curves in various fields. The key ingredient is a new “trivial bound” which is valid for all fields. Then we use polynomial partition arguments to get “global” bounds on $\mathbb{R}$ and (following [ST12]) $\mathbb{C}$. The results on $\mathbb{R}$ improves the bound given by [PS98] and generalize the result of [STJ83] to the case of algebraic curves, while the results on $\mathbb{C}$ generalizes [ST12] in the same way and thus almost recover the complex Szemerédi-Trotter theorem [Tóth03].

1. Introduction

The classical Szemerédi-Trotter theorem [STJ83] says that for a finite set $L$ of lines and a finite set $P$ of points in $\mathbb{R}^2$, the number of their incidences is $\lesssim |P|^\frac{2}{3}|L|^\frac{2}{3} + |P| + |L|$. Since [STJ83], there have been several generalizations. In [PS98], Pach and Sharir generalized this result to the case of simple curves that have “$k$ degrees of freedom and multiplicity-type $C$”. They proved

**Theorem 1.1.** For a finite set $L$ of simple curves and a finite set $P$ of points in $\mathbb{R}^2$, suppose $\exists C, k > 0$ s.t. (i) for any $k$ points there are at most $C$ curves of $L$ passing through all of them, and (ii) any pair of curves in $L$ intersect at most $C$ points. Then the number of their incidences $|\mathcal{I}(P, L)| := |\{(p, l) \in P \times L, p \in l\}|$ satisfies

$$|\mathcal{I}(P, L)| \lesssim_{C, k} |P|^\frac{k}{2k-1}|L|^\frac{2k-2}{2k-1} + |P| + |L|.$$  

The main result of this paper will be an improvement of theorem 1.1 in the case of high degree algebraic curves. Let $L$ be a finite set of algebraic curves of degree $\leq d$ in $\mathbb{R}^2$, any two of which do not have common components. Let $P$ be a finite set of points. By Bézout’s theorem, for any $d^2 + 1$ points there is at most one curve in $L$ that passes through all of them, so in the language above they have “$d^2 + 1$ degrees of freedom and multiplicity-type $d^2$”. However, one may wonder whether this $d^2 + 1$ is misleading as a definition of “degree of freedom” since the moduli space of degree $d$ curves is of dim $A$ where $A = (d+2)\frac{d}{2} - 1$ and $A \leq d^2 + 1$ in general. In this way, we can say that “generically”, $A$ points determine a degree $d$ curve. This suggests that theorem 1.1 may still holds when we substitute $A$ instead of $d^2 + 1$ into the place of
Indeed, we prove this is possible: by Bézout and a counting argument we are able to reduce $k = d^2 + 1$ to $k = A$ and thus obtain:

**Theorem 1.2.** Let $d$ be a positive integer, $A = (\begin{array}{c} d+2 \\ 2 \end{array}) - 1$, $L$ be a finite set of algebraic curves in $\mathbb{R}^2$ of degree $\leq d$ among which there are no repeated components, and $P$ be a finite set of points in $\mathbb{R}^2$. Then

$$|I(P, L)| \lesssim_d |P|^{\frac{A}{2^A-1}} |L|^{\frac{2A-2}{2^A-1}} + |P| + |L|.$$  

Hence comparing with the general result in [PS98], we get an improvement for $d \geq 3$. For $d = 1$ or 2, $A = d^2 + 1$.

We can also generalize theorem 1.2 to some “algebraic subset” setting. For this we recall:

**Definition 1.3.** For $d > 0$ and a fixed underlying field $\mathbb{F}$, denote by $S_d$ the set of all algebraic equations in $\mathbb{F}^2$ with degree $\leq d$. There is a natural correspondence between $S_d$ and $\mathbb{F}^A$:

$$a_{ij}x^iy^j = 0 \mapsto [a_{ij}].$$

By abuse of notation, we identify any equation $L \in S_d$ with the corresponding algebraic curve. We call some subset of $S_d$ an algebraic set (variety) if its image under the above correspondence is an algebraic set (variety).

For an algebraic subset in $\mathbb{F}^A$, we say it is of dimension $\leq k$ if it is the restriction of an algebraic subset in $\mathbb{F}^A$ and the latter is of dimension $\leq k$. Here $\mathbb{F}$ is the algebraic closure of $\mathbb{F}$.

Now we can generalize theorem 1.2 to the following:

**Theorem 1.4.** Suppose $d$ is a positive integer, $A = (\begin{array}{c} d+2 \\ 2 \end{array}) - 1$, $\mathcal{M} \subset S_d \simeq \mathbb{F}^A$ is an algebraic subset of dimension $\leq k$. Suppose $P$ is a finite set of points in $\mathbb{F}^2$, $L \subset \mathcal{M}$ satisfying that no two curves in $L$ have a same component. Then

$$|I(P, L)| \lesssim_{\mathcal{M}} |P|^{\frac{k}{2^{k-1}}} |L|^{\frac{2k-2}{2^{k-1}}} + |P| + |L|.$$

where $\lesssim_{\mathcal{M}}$ stands for $\lesssim_{d, \dim \mathcal{M}, \deg \mathcal{M}}$, being a convention we will use throughout the paper.

It is useful to make this generalization when we consider families of curves with special properties like parabolas, circles, etc.

The proof of theorem 1.2 will be a standard application of the “polynomial method” (see for example [Dvi09], [GK10]) with the following new “trivial bound”. For the “polynomial partitioning” method we will adopt, we refer the reader to [KMS12] which is a good summary of using this method to solve discrete geometry problems.
Lemma 1.5 ("Trivial bound"). Let $\mathbb{F}$ be a field, $d$ be a positive integer, $A = \left(\frac{d+2}{2}\right) - 1$, $L$ be a finite set of algebraic curves in $\mathbb{F}^2$ of degree $\leq d$ among which there are no repeated components, and $P$ be a finite set of points in $\mathbb{F}^2$. Then

\begin{equation}
|I(P, L)| \lesssim_d |P|^A + |L|.
\end{equation}

And indeed, theorem 1.4 as its generalization, mainly relies on a similar but improved trivial bound by taking advantage of $\mathcal{M}$.

Remark 1.6. Lemma 1.5 is a general one. It holds for all fields, including finite fields, $\mathbb{R}$ and $\mathbb{C}$. However we will see a huge difference in "global" incidence theorems (like theorem 1.2) we can get out of it. Indeed, the polynomial method is easy for $\mathbb{R}$, a bit harder for $\mathbb{C}$ and has great difficulty with $\mathbb{F}_p$. Besides theorem 1.2 we obtained, we will use two sections to deal with $\mathbb{F}_p$ and $\mathbb{C}$.

For $\mathbb{F}_p$, in the Szemerédi-Trotter setting, it is not hard to show that for $\leq N$ points and $\leq N$ lines in $\mathbb{F}_p^2$, the number of incidences is $O(N^3)$. Bourgain, Katz and Tao [BKT04] proved that when $N = p^\alpha, 0 < \alpha < 2$, the incidence number is $O_\alpha(N^{\frac{3}{2}-\varepsilon})$. We generalize the former to a "reasonable" bound in the higher degree setting, but could not show a generalization of the latter and raise it as a conjecture.

For $\mathbb{C}$, Tóth [Tóth03] proved the Szemerédi-Trotter theorem. Also, Solymosi and Tao [ST12] proved a "cheap" version with an additional $\varepsilon$ on one exponent with less effort by the polynomial method. We will generalize this to a "cheap" version of a "reasonable" incidence theorem of points and high degree curves in $\mathbb{C}^2$ by following the argument of [ST12] closely. We didn’t manage to remove the $\varepsilon$ and a technical "transverse" assumption in the bound and leave the corresponding question open.

For $\mathbb{F}_p$ and $\mathbb{C}$, the generalization process to subvarieties is very similar to that in $\mathbb{R}$ and we will state the most general result we obtain.

For instance, we state one of the complex incidence theorem mentioned above. It is a "cheap" complex counterpart of theorem 1.2.

Theorem 1.7. Let $d$ be a positive integer, $A = \left(\frac{d+2}{2}\right) - 1$, $L$ be a finite set of algebraic curves in $\mathbb{C}^2$ of degree $\leq d$ satisfying that any two curves don’t have common components and intersect transversally at their smooth points and $P$ be a finite set of points in $\mathbb{C}^2$. Then

\begin{equation}
|I(P, L)| \lesssim_{\varepsilon,d} |P|^A |L|^{\frac{2A-2}{2A-1}} + |P| + |L|
\end{equation}

and

\begin{equation}
|I(P, L)| \lesssim_{\varepsilon,d} |P|^A |L|^{\frac{2A-2}{2A-1}+\varepsilon} + |P| + |L|.
\end{equation}
Acknowledgements. We would like to thank Larry Guth for encouraging us working on this problem and suggesting we try doing the two essential things in this paper: improving the “trivial bound” and working on algebraic subsets.

2. Proof of the main theorem

In this section we prove theorem 1.2. First we prove the “trivial bound” lemma 1.5.

2.1. The trivial bound. Recall the notion of Hilbert function [EGH96].

Definition 2.1. Under the natural correspondence between $S_d$ and $\mathbb{F}P^A$:

$$a_{ij}x^iy^j = 0 \mapsto [a_{ij}],$$

each point $p = (x_0, y_0)$ corresponds to a linear condition, i.e. a hyperplane $H_p$ in $\mathbb{F}P^A$.

$$(2.1) \quad H_p := \{[a_{ij}] \in \mathbb{F}P^A | a_{ij}x_0^iy_0^j = 0\}.$$

For a given finite set of points $\Gamma = \{x_1, \ldots, x_\gamma\}$, let $h_d(\Gamma)$ be its Hilbert function, defined as the number of independent conditions that $\Gamma$ impose on $S_d$, i.e. the codimension of $\cap H_{x_i}$.

Notice that $h_d(\Gamma) \leq A$, and the equality holds if and only if there is at most one degree $d$ curve passing through $\Gamma$. For further discussion of Hilbert function, see [EGH96].

Proof of lemma 1.5. For any line $l \in L$, let $N_l = \mathcal{I}(l, P)$ be the set of points of $P$ that are on $l$. Obviously,

$$(2.2) \quad |\mathcal{I}(P, L)| = \sum_{l \in L} |N_l| \leq d^2|L| + \sum_{l \in L: |N_l| \geq d^2+1} |N_l|.$$

Thus it suffices to control the contribution of lines passing through $\geq d^2+1$ points. Suppose $l \in L$ has $\geq d^2+1$ points of $P$ on it. Then by linear algebra and the fact that $S_d$ has dimension $A$, we deduce that for each $d^2 + 1$-tuple $\Gamma$ in $N_l$, we can choose an $A$-tuple $\Gamma'$ out of $\Gamma$ such that $h_d(\Gamma') = h_d(\Gamma)$. If there is another curve $l' \in L$ passing through these $A$ points, then since $\Gamma'$ and $\Gamma$ generate the same set of conditions, $l'$ passes through all points of $\Gamma$. Thus $l$ and $l'$ have $d^2+1$ points and hence a component in common, which is a contradiction. This means $l$ is the only curve in $L$ passing through the $A$-tuple $\Gamma'$. And there are at least $\binom{|N_l|}{d^2+1}/\binom{|N_l|-A}{d^2+1} \geq d |N_l|^A \geq |N_l|$ such $A$-tuples in $l$ (s.t. each tuple actually determines $l$). Taking into account that there are $\leq |P|^A A$-tuples in total, we deduce

$$(2.3) \quad |\mathcal{I}(P, L)| \leq d^2|L| + \sum_{l \in L: |N_l| \geq d^2+1} |N_l| \lesssim d |P|^A + L.$$
2.2. Polynomial partitioning. As is mentioned in the introduction, we use a standard polynomial partitioning technique to conclude the proof of theorem 1.2. First we recall the following polynomial partitioning lemma (see for example theorem 4.1 in [GK10]):

**Proposition 2.2.** Let $P$ be a finite set of points in $\mathbb{R}^m$, and let $D$ be a positive integer. Then there exists a nonzero polynomial $Q$ of degree at most $D$ and a decomposition

$$ \mathbb{R}^m = \{Q = 0\} \cup U_1 \cup \cdots \cup U_M $$

into the hypersurface $\{Q = 0\}$ and a collection $U_1, \ldots, U_M$ of open sets (which will be called cells) bounded by $\{Q = 0\}$, such that $M \sim_m D^m$ and that each cell $U_i$ contains $O_m(\left| P \right|/D^m)$ points.

**Proof of theorem 1.2.** Apply proposition 2.2, we find a polynomial $Q$ of degree $\leq D$ that partitions $\mathbb{R}^2$ into $M$ cells:

$$ \mathbb{R}^2 = \{Q = 0\} \cup U_1 \cup \cdots \cup U_M $$

where $M \sim D^2$, and $U_1, \ldots, U_M$ are connected components of $\mathbb{R}^2$ cut off by $\{Q = 0\}$. Let $P_i = U_i \cap P$, $L_i$ be the set of curves that have non-empty intersection with $U_i$. Then $|P_i| = O(|P|/D^2)$. By Bézout we see every curve meets $\{Q = 0\}$ at no more than $O_d(D)$ components. By Harnack’s curve theorem [Har76], the curve itself has $O_d(1)$ connected components. Thus every curve can only meet $O_d(D)$ cells. Summing over $L$, we obtain $\sum |L_i| \leq D|L|$. 

Without loss of generality we may assume every curve is irreducible (i.e. the restriction of an irreducible plane complex curve to $\mathbb{R}^2$). Let $P_{\text{cell}}$ denotes the points of $P$ in cells, and $P_{\text{alg}}$ denotes those on $\{Q = 0\}$. Let $L_{\text{alg}}$ denotes those curve that lies in $\{Q = 0\}$ and $L_{\text{cell}}$ be the union of other curves (so they intersect $Q$ at a union of points by the irreducibility assumption). We deduce by lemma 1.5 and Hölder:

$$ |I(P, L)| = |I(P_{\text{cell}}, L_{\text{cell}})| + |I(P_{\text{alg}}, L_{\text{cell}})| + |I(P_{\text{alg}}, L_{\text{alg}})| \lesssim_d \sum_i(|P_i|^A + |L_i|) + D|L_{\text{cell}}| + |I(P_{\text{alg}}, L_{\text{alg}})|,$$

(2.4)

$$ \lesssim_d |P|^A D^{-2(A-1)} + D|L| + |I(P_{\text{alg}}, L_{\text{alg}})|. $$

Note that we may at the beginning assume $|P|^{1/2} \leq |L| \leq |P|^A$, or (1.2) already holds by lemma 1.5 or another trivial bound $|I(P, L)| \leq |P| + |L|^2$. In this case, we may choose $D \sim_d |P|^A D^{-2(A-1)}$ and $D \leq |L|/2$. Then the first two terms on RHS of (2.4) are $\lesssim_d |P|^{A/2} \left| \frac{1}{2A-1} \right| L \frac{1}{2A-1}$ and $D \leq |L|/2$. Since $|L_{\text{alg}}| \leq D \leq \frac{|L|}{2}$, we can perform a dyadic induction (on $|L|$) and control the third term. This proves (1.2).
3. Generalization

We prove theorem 1.4 in this section. By the argument of §2.2, it suffices to show the following “trivial bound”.

**Lemma 3.1.** If we have the conditions in theorem 1.4, then

\[(3.1) \quad |I(P,L)| \lesssim |P|^k + |L|.
\]

**Proof.** Without loss of generality we assume \(F\) is algebraically closed and \(M\) is a subvariety. If \(l \in L\) passes through a set of points \(\Gamma = \{x_1, \ldots, x_\gamma\}\), then \(l\) must lie on the intersection of all \(H_x\) and \(M\). Comparing with the proof of lemma 1.5, we only need to prove that for every \((d^2+1)\)-tuple \(\Gamma\), there is a \(\Gamma' \subseteq \Gamma\) such that \(|\Gamma'| = k\) and \(\cap_{p \in \Gamma'} H_p \cap M\) is a finite set of points whose cardinality is \(O_M(1)\). This follows from the following two propositions.

**Proposition 3.2.** (see [Har77], Ch I, Exercise 1.8) If \(V\) is an \(r\)-dimensional variety in \(\mathbb{F}^d\), and \(P: \mathbb{F}^d \to \mathbb{F}\) is a polynomial which is not identically zero on \(V\), then every component of \(V \cap \{P = 0\}\) has dimension \(r - 1\).

**Proposition 3.3.** (see [FFFF84], 2.3) Let \(V_1, \ldots, V_s\) be subvarieties of \(\mathbb{F}P^N\), and let \(Z_1, \ldots, Z_r\) be the irreducible components of \(V_1 \cap \cdots \cap V_r\). Then

\[
\sum_{i=1}^{r} \deg(Z_i) \leq \prod_{j=1}^{s} \deg(V_j)
\]

Indeed, by iterating Proposition 3.2, we can choose \(\Gamma'\), such that \(|\Gamma'| = k\), and \(\cap_{p \in \Gamma'} H_p \cap M\) has dimension 0. By proposition 3.3, the cardinality of \(\cap_{p \in \Gamma'} H_p \cap M\) is controlled by a constant depending on \(k\) and \(\deg M\), thus controlled by \(O_M(1)\).

\(\square\)

4. A complex theorem with \(\epsilon\)

In this section, we follow the approach of [ST12] to sketch a proof of theorem 1.7. For further details, readers may check [ST12]. The idea is to use a polynomial of degree \(O_{d,\epsilon}(1)\) to partition \(P\). With an \(\epsilon\) of room on the exponent, one can perform an induction on \(|P|\), which controls \(I(P_{cell}, L_{cell})\). For \(I(P_{alg}, L)\), since \(P_{alg}\) is in a polynomial of bounded degree, we can do a dimension reduction and control it.

**Proof of theorem 1.7.** In our proof, \(C\) will be an absolute constant depending only on \(d\) and \(\epsilon\) which may vary from place to place. \(C_0, C_1\) and \(C_2\) will be positive constants to be chosen later: \(C_0, C_1 > 2\) are sufficiently large depending on \(d\) and \(\epsilon\), and \(C_2\) will be sufficiently large depending on \(C_1, C_0, d\) and \(\epsilon\).
We do an induction on \(|P|\). Suppose for \(|P'| \leq \frac{|P|}{2}\) and \(|L'| \leq |L|\), we already have (induction hypothesis)

\[
|\mathcal{I}(P', L')| \leq C_2|P'|^{A \frac{1}{2A-1}}|L'|^{2A-2} + C_0(|P'| + |L'|).
\]

(4.1)

Our goal is to prove

\[
|\mathcal{I}(P, L)| \leq C_2|P|^{A \frac{1}{2A-1}}|L|^{2A-2} + C_0(|P| + |L|).
\]

(4.2)

We apply proposition 2.2 to \(D = C_1\) on \(\mathbb{C}^2 \approx \mathbb{R}^4\) and obtain a partition:

\[
\mathbb{R}^4 = \{Q = 0\} \cup U_1 \cup \cdots \cup U_M.
\]

(4.3)

Here \(Q : \mathbb{R}^4 \to \mathbb{R}\) has degree at most \(C_1\), \(M \sim C_1^4\) and \(|P| = |P \cap U_i| = O\left(\frac{|P|}{C_1}\right) \leq \frac{|P|}{2}\). We denote \(L_i\) to be the set of curves in \(L\) with nonempty intersection with \(U_i\). Thus by induction hypothesis,

\[
|\mathcal{I}(P_i, L_i)| \leq C_2|P_i|^{A \frac{1}{2A-1}}|L_i|^{2A-2} + C_0(|P_i| + |L_i|)
\]

(4.4)

For \(l\) belonging to some \(L_i\), we apply a generalization of a classical result in real algebraic geometry which implies the number of connected components (4.7) was explained in the real setting and \((4.8)\) is lemma 1.5:

\[
\sum_{i=1}^{M} |L_i| \leq CC_1^2|L|
\]

(4.5)

Adding up \(|\mathcal{I}(P_i, L_i)|\) and applying Hölder, we get

\[
|\mathcal{I}(P_{\text{cell}}, L_{\text{cell}})| = \sum_{i=1}^{M} |\mathcal{I}(P_i, L_i)|
\]

\[
\leq C_2 C_1^{-4A \frac{1}{2A-1}} |P|^{A \frac{1}{2A-1}} + C_0(|P| + |L|)
\]

(4.6)

Now we recall the two trivial bounds (4.7) was explained in the real setting and

\[
|\mathcal{I}(P, L)| \lesssim_d |L|^2 + |P|.
\]

(4.7)

\[
|\mathcal{I}(P, L)| \lesssim_d |P|^4 + |L|.
\]

(4.8)
Thus we may assume that $|P|^\frac{1}{2} \lesssim d |L| \lesssim d |P|^4$, otherwise $|\mathcal{I}(P, L)| \lesssim d |P| + |L|$ and it suffices to choose $C_0$ bigger than the implicit constant. With this assumption,

\begin{equation}
|\mathcal{I}(P_{\text{cell}}, L_{\text{cell}})| \leq C(C_1^{-\varepsilon}C_2 + C_0(|P|^{-\varepsilon} + C_1^2|P|^{-\varepsilon}))|P|^{\frac{A}{2A-2}+\varepsilon}|L|^{\frac{2A-2}{2A-1}}.
\end{equation}

It’s easy to see that if we can prove

\begin{equation}
|\mathcal{I}(P_{\text{alg}}, L)| \lesssim C_1 |P|^{\frac{A}{2A-1}}|L|^{\frac{2A-2}{2A-1}} + |P| + |L|, 
\end{equation}

we can combine it with (4.9) and a careful choice of $C_0, C_1$ and $C_2$ successfully will give us the desired (4.2). Since the case for $|P| = 1$ is trivial, we can conclude the proof of (1.5). For (1.6), the argument is similar and is omitted.

Finally, (4.10) is a special case of the following proposition 4.1 where $r = 3, D = C_1, \Sigma = \{Q\}$.

**Proposition 4.1.** Let $P$ and $L$ be as in theorem [1.7], $0 \leq r < 4$, and let $\Sigma$ be a subvariety in $\mathbb{C}^2 \simeq \mathbb{R}^4$ of (real) dimension $\leq r$ and degree $\leq D$. Then

\begin{equation}
|\mathcal{I}(P \cap \Sigma, L)| \lesssim D |P|^{\frac{A}{2A-1}}|L|^{\frac{2A-2}{2A-1}} + |P| + |L|.
\end{equation}

**Proof.** We do an induction on $r$. For $r = 0$, $\Sigma$ is a single point, the inequality trivially holds.

Now suppose that $1 \leq r < 3$, and that the claim has already been shown for smaller values of $r$. We assume $P \subseteq \Sigma$ without loss of generality.

By an algebraic geometry result (see for example corollary 4.5 in [ST12]), we can decompose $\Sigma$ into smooth points on subvarieties:

$$\Sigma = \Sigma^{\text{smooth}} \cup \Sigma^{\text{smooth}}_i.$$ 

Here $\Sigma_i$ are subvarieties in $\Sigma$ of dimension $\leq r - 1$ and degree $O_D(1)$, and the number of $\Sigma_i$ is at most $O_D(1)$. Thus by the induction hypothesis, we only need to control $|\mathcal{I}(P \cap \Sigma^{\text{smooth}}, L)|$.

If $l_1, l_2 \in \Sigma$ intersect at $p \in \Sigma^{\text{smooth}}$, by considering the tangent space and the transverse assumption, we know $p$ is a singular point of one of the two curves. But each curve has $O_d(1)$ singular points. Thus

\begin{equation}
|\mathcal{I}(P \cap \Sigma^{\text{smooth}}, L_{\text{alg}})| \leq |P| + O_d(1)|L|.
\end{equation}

Hence it suffices to control $|\mathcal{I}(P \cap \Sigma^{\text{smooth}}, L_{\text{cell}})|$. If $l$ does not lie in $\Sigma$, then again by corollary 4.5 of [ST12], $l \cap \Sigma = \cup_{j=0}^{J(0)}$ for some $J(l) \leq J = O_D(1)$, where for each
1 \leq j \leq J(l), l_j is an algebraic variety of (real) dimension \leq 1 and of degree \text{OD}(1). We obtain
\begin{equation}
|\mathcal{I}(P \cap \Sigma^{\text{smooth}}, L_{\text{cell}})| \leq \sum_{l,j \leq J(l)} |\mathcal{I}_{l,j}|
\end{equation}

where \( \mathcal{I}_{l,j} := \{p \in P : p \in l_j\} \). Notice that for each \( l_j \), if it is not the union of \( \text{OD}(1) \) points, then it belongs to a unique \( l \) because the intersection of \( l \in L \) and \( l' \in L \) has dimension 0. Now a generic projection from \( \mathbb{R}^4 \) to \( \mathbb{R}^2 \) will enable us to use an argument exactly the same as we did in the proof of theorem \( \text{1.2} \). We use the trivial bound given by lemma \( \text{1.5} \) for original curves in \( L \) and then perform the polynomial partitioning argument.

\[ \square \]

If we apply lemma \( \text{3.1} \) instead and do the same reasoning, we can generalize theorem \( \text{1.7} \) to the following theorem for algebraic subsets.

**Theorem 4.2.** For underlying field \( \mathbb{C} \), let \( d \) be a positive integer, \( A = \left( \frac{d+2}{2} \right) - 1 \), \( \mathcal{M} \) be an algebraic subset of \( S_d \) of dimension \( \leq k \), \( L \) be a finite set of algebraic curves in \( \mathcal{M} \) of degree \( \leq d \) with no common components s.t. they intersect transversally at their smooth points, and \( P \) be a finite set of points in \( \mathbb{C}^2 \). Then
\begin{equation}
|\mathcal{I}(P, L)| \lesssim_{\epsilon, \mathcal{M}} |P|^\frac{k}{2k-1-\epsilon} |L|^\frac{2k-2}{2k-1} + |P| + |L|
\end{equation}
and
\begin{equation}
|\mathcal{I}(P, L)| \lesssim_{\epsilon, \mathcal{M}} |P|^\frac{k}{2k-1-\epsilon} |L|^\frac{2k-2}{2k-1} + |P| + |L|.
\end{equation}

5. The case of \( \mathbb{F}_p \)

Let \( p \) be a prime. We devote this short section to incidence theorems over \( \mathbb{F}_p \) we can obtain. Since an analogue of polynomial partition is not known, we have some apparent new difficulties and the results are far from satisfactory. In fact, all the results we obtain here is valid for any underlying field. But for \( \mathbb{F}_p \) we don’t know a way to improve them.

**Theorem 5.1** (Trivial incidence bound over an arbitrary field). Let \( k \) be a non-negative integer, \( d > 0 \) and \( \mathbb{F} \) be a field. Suppose \( \mathcal{M} \subseteq S_d \) is an algebraic set of dimension \( \leq k \). Let \( P \) be a set of points on \( \mathbb{F}^2 \) and \( L \) be a set of curves in \( \mathcal{M} \) with no repeated components. Then the number \( \mathcal{I}(P, L) \) of incidences between the points in \( P \) and the curves in \( L \) satisfies the following:
\begin{equation}
\mathcal{I}(P, L) \lesssim_\mathcal{M} \min\{|P|^\frac{1}{2} |L| + |P|, |P||L|^{\frac{1}{2k-1}} + |L|\}.
\end{equation}
Proof. We shall prove

\[ I(P, L) \lesssim M |P||L|^\frac{k-1}{k} + |L| \]

and the other bound can be derived in a similar and easier way.

Let \( c \geq d^2 + 1 \) be a positive integer parameter we shall choose later. The contribution to \( I(P, L) \) from lines that passing through \( \leq c \) points is obviously \( \leq c|L| \). When a line \( l \in L \) passes through \( n > c \) points, from the proof of lemma 1.5 and lemma 3.1 we can deduce that it passes through \( \gtrsim M n^k \) “good \( k \)-tuples”. Here by being good we mean that there are only \( O_M(1) \) curves in \( M \) passing through the \( k \)-tuple. Since the number of good \( k \)-tuples is \( \leq |P|^k \), a simple counting argument shows the contribution to \( I(P, L) \) from all \( l \in L \) passing through \( > c \) points is \( \lesssim S |P|^k \). Now take \( c = \lceil \frac{|P|}{|L|^\frac{1}{k}} \rceil + d^2 + 1 \), we obtain (5.2).

□

This theorem has a corollary if we make the two terms on the RHS of (5.1) have the same magnitude.

Corollary 5.2. In theorem 5.1, if \( |P| \leq N, |L| \leq N^{\frac{k}{2}} \) then

\[ I(P, L) \lesssim_M N^{\frac{k+1}{2}}. \]

We already saw improvements to the above corollary in theorem 1.4 and theorem 4.2 when \( \mathbb{F} = \mathbb{R} \) or \( \mathbb{C} \) where have better upper bounds \( N^{\frac{k^2}{2k-1}} \) and \( N^{\frac{k^2}{2k-1} + \epsilon} \) by polynomial partitioning, respectively. On the other hand, when the field \( \mathbb{F} = \mathbb{F}_p \) and \( N = p^\alpha, 0 < \alpha < 2 \), whether we can improve this corollary becomes a subtle and interesting problem. Note that \( \alpha < 2 \) is necessary in general: For example, if we take \( M = S_d, P = \mathbb{F}_p \mathbb{P}^2 \) and \( L = M \), the bound in corollary 5.2 becomes sharp. In the case of points and lines, an improvement of corollary 5.2 is indeed possible by sum-product estimates over finite fields [BKT04]. In the general case we thus have the following reasonable-seeming conjecture.

Conjecture 5.3. For any \( 0 < \alpha < 2 \), there exists an \( \epsilon = \epsilon(\alpha) > 0 \) such that: In theorem 5.1, if \( k = \mathbb{F}_p \) for \( p \) prime, \( |P| \leq N, |L| \leq N^{\frac{k}{2}}, 0 < N < p^\alpha \), then

\[ I(P, L) \lesssim_M N^{\frac{k+1}{2} - \epsilon}. \]

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