Algebras of Ehresmann semigroups and categories

Itamar Stein*
†
Department of Mathematics
Bar Ilan University
52900 Ramat Gan
Israel

Abstract

E-Ehresmann semigroups are a commonly studied generalization of inverse semigroups. They are closely related to Ehresmann categories in the same way that inverse semigroups are related to inductive groupoids. We prove that under some finiteness condition, the semigroup algebra of an E-Ehresmann semigroup is isomorphic to the category algebra of the corresponding Ehresmann category. This generalizes a result of Steinberg who proved this isomorphism for inverse semigroups and inductive groupoids and a result of Guo and Chen who proved it for ample semigroups. We also characterize E-Ehresmann semigroups whose corresponding Ehresmann category is an EI-category and give some natural examples.

1 Introduction

A semigroup $S$ is called inverse if every element $a \in S$ has a unique inverse, that is, a unique $b \in S$ such that $aba = a$ and $bab = b$. Inverse semigroups are fundamental in semigroup theory and have many unique and important

---

*email: Steinita@gmail.com

†This paper is part of the author’s PHD thesis, being carried out under the supervision of Prof. Stuart Margolis. The author’s research was supported by Grant No. 2012080 from the United States-Israel Binational Science Foundation (BSF) and by the Israeli Ministry of Science, Technology and Space.
properties. For instance, they are ordered with respect to a natural partial order and their idempotents form a semilattice. Another important fact is their close relation with inductive groupoids. More precisely, the Ehresmann-Schein-Nambooripad theorem [13] states that the category of all inverse semigroups is isomorphic to the category of all inductive groupoids. For an extensive study of inverse semigroups see [14]. There are several generalizations of inverse semigroups that keep some of their good properties. In this paper we discuss a generalization called $E$-Ehresmann semigroups. Let $E$ be a subsemilattice of $S$. Define two equivalence relations $\tilde{R}_E$ and $\tilde{L}_E$ in the following way. $a \tilde{R}_E b$ if $a$ and $b$ have precisely the same set of left identities from $E$ and likewise $a \tilde{L}_E b$ if they have the same set of right identities from $E$. Assume that every $\tilde{R}_E$ and $\tilde{L}_E$ class contains precisely one idempotent, denoted $a^+$ and $a^*$ respectively. $S$ is called $E$-Ehresmann if $\tilde{R}_E$ is a left congruence and $\tilde{L}_E$ is a right congruence, or equivalently (see [5] Lemma 4.1), if the two identities $(ab)^+ = (ab)^+$ and $(a^*b)^* = (ab)^*$ hold for every $a, b \in S$. If $S$ is regular and $E = E(S)$ is the set of all idempotents of $S$ then being an $E$-Ehresmann semigroup is equivalent to being an inverse semigroup. There is also a notion of an Ehresmann category which is a generalization of an inductive groupoid. The Ehresmann-Schein-Nambooripad theorem generalizes well to $E$-Ehresmann semigroups and Ehresmann categories. Lawson proved [12] that the category of all $E$-Ehresmann semigroups is isomorphic to the category of all Ehresmann categories. In this paper we discuss algebras of these objects over some commutative unital ring $K$. In recent years, algebras of semigroups related to Ehresmann semigroups have been studied by a number of authors, see [8, 9, 11].

Steinberg [20] proved that if $S$ is an inverse semigroup where $E(S)$ is finite then its algebra is isomorphic to the algebra of the corresponding inductive groupoid. Guo and Chen [8] generalized this isomorphism to the case of finite ample semigroups. These are the $E$-Ehresmann semigroups such that $E = E(S)$, every $R^*$ and $L^*$ class contains an idempotent (see [5] for the definition of the equivalence relations $R^*$ and $L^*$) and the two ample conditions $ae = (ae)^+ a$ and $ea = a(ea)^+$ hold for every $a \in S$ and $e \in E(S)$. An important example of an $E$-Ehresmann semigroup is the monoid $\mathcal{PT}_n$ of all partial functions on an $n$-element set where $E$ is the semilattice of all partial identity functions. The author proved [19] that the algebra of $\mathcal{PT}_n$ is isomorphic to the algebra of the category of all surjections between subsets of an $n$-element set. The
category of surjections is in fact, the Ehresmann category associated to $\mathcal{PT}_n$ as an $E$-Ehresmann semigroup. This result has led to some new results on the representation theory of $\mathcal{PT}_n$. In this paper we generalize all these results and prove that if the subsemilattice $E \subseteq S$ is principally finite (that is, any principal down ideal is finite) then the semigroup algebra $\mathbb{K}S$ (over any commutative unital ring $\mathbb{K}$) is isomorphic to the category algebra $\mathbb{K}C$ of the corresponding Ehresmann category. In section §4 we give some examples and simple corollaries of this isomorphism.

In order to apply this isomorphism to study the semigroup algebra (and representations) of some $E$-Ehresmann semigroup, one need to understand the algebra of the corresponding Ehresmann category. Hence it is natural to consider Ehresmann categories whose algebras are well understood to some extent. EI-categories, which are categories where every endomorphism is an isomorphism, are such a family. If $\mathbb{K}$ is an algebraically closed field of good characteristic then there is a way to describe the Jacobson radical of the algebra of a finite EI-category $C$ ([13, Proposition 4.6]) and its ordinary quiver ([15, Theorem 4.7] or [16, Theorem 6.13]). In section §5.1 we characterize $E$-Ehresmann semigroups whose corresponding Ehresmann category is an EI-category. We give two natural families of such semigroups: $(2, 1, 1)$-subalgebras of $\mathcal{PT}_n$ and $E(S)$-Ehresmann semigroups. Let $S$ be a finite $E$-Ehresmann semigroup whose corresponding category $C$ is an EI-category and let $\mathbb{K}$ be a field such that the orders of the endomorphism groups of all objects in $C$ are invertible in $\mathbb{K}$. In section §5.2 we prove that in this case the maximal semisimple image of $\mathbb{K}S$ can be "seen" inside the semigroup $S$ itself. More precisely, we will prove that the inverse subsemigroup of $S$ that contains all elements corresponding to isomorphisms in $C$ spans an algebra which is isomorphic to the maximal semisimple image of $\mathbb{K}S$.

2 Preliminaries

2.1 $E$-Ehresmann semigroups

We denote as usual by $\mathcal{R}$, $\mathcal{L}$, $\mathcal{D}$ and $\mathcal{H}$ the Green’s relations on a semigroup. We assume that the reader is familiar with Green’s relations and other semigroup basics that can be found in [10]. Recall that a semilattice is a commutative semigroup of idempotents, or equivalently, a poset such that any two elements...
have a meet. Let $S$ be a semigroup. Denote by $E(S)$ its set of idempotents and choose some $E \subseteq E(S)$ such that $E$ is a subsemilattice of $S$. We define equivalence relations $\tilde{R}_E$ and $\tilde{L}_E$ on $S$ by

$$a \tilde{R}_E b \iff (\forall e \in E \ ea = a \iff eb = b)$$

and

$$a \tilde{L}_E b \iff (\forall e \in E \ ae = a \iff be = b).$$

We also define $\tilde{H}_E = \tilde{R}_E \cap \tilde{L}_E$. It is easy to see that $R \subseteq \tilde{R}_E$, $L \subseteq \tilde{L}_E$ and $H \subseteq \tilde{H}_E$.

**Definition 2.1.** A semigroup $S$ with a distinguished semilattice $E \subseteq E(S)$ is called left $E$-Ehresmann if the following two conditions hold.

1. Every $\tilde{R}_E$ class contains precisely one idempotent from $E$.
2. $\tilde{R}_E$ is a left congruence.

**Remark 2.2.** It is easy to see that an $\tilde{R}_E$ class cannot contain more than one idempotent from $E$ so Condition 1 can be replaced by the requirement that every $\tilde{R}_E$ class contains at least one idempotent from $E$.

In any semigroup that satisfies Condition 1 we denote by $a^+$ the unique idempotent from $E$ in the $\tilde{R}_E$ class of $a$. Note that $a^+$ is the unique minimal element $e$ of the semilattice $E$ that satisfies $ea = a$.

Note that if $S$ is a finite monoid, Condition 1 is equivalent to the requirement that $1 \in E$. Indeed, 1 is the only left identity of itself so Condition 1 implies that $1 \in E$. On the other hand, assume $1 \in E$ and take $e$ to be the product of all idempotents of $E$ which are left identity for $a$. This product is not empty since $1 \in E$. It is clear that $e \in E$ and $e \tilde{R}_E a$ so Condition 1 holds.

Condition 2 of Definition 2.1 has the following equivalent characterization (for proof see [5, Lemma 4.1]).

**Lemma 2.3.** Let $S$ be a semigroup with a distinguished semilattice $E \subseteq E(S)$ such that Condition 1 of Definition 2.1 holds. Then $\tilde{R}_E$ is a left congruence if and only if $(ab)^+ = (ab^+)^+$ for every $a, b \in S$.

Dually, we can consider semigroups for which every $\tilde{L}_E$ class contains a unique idempotent. We denote the unique idempotent in the $\tilde{L}_E$ class of $a$ by $a^*$. 

4
Such semigroup is called right $E$-Ehresmann if $\tilde{L}_E$ is a right congruence, or equivalently if $(ab)^* = (a^*b)^*$ for every $a, b \in S$.

**Definition 2.4.** A semigroup $S$ with a distinguished semilattice $E \subseteq E(S)$ is called $E$-Ehresmann if it is both left and right $E$-Ehresmann.

Note that any inverse semigroup $S$ is an $E$-Ehresmann semigroup when one choose $E = E(S)$. In this case $a^+ = aa^{-1}$ and $a^* = a^{-1}a$.

As may be hinted by Lemma 2.3, $E$-Ehresmann semigroups form a variety of bi-unary semigroups, that is, semigroups with two addition binary operations. The proof of the following proposition can be found in [6, Lemma 2.2] and the discussion following it.

**Proposition 2.5.** $E$-Ehresmann semigroups form precisely the variety of $(2, 1, 1)$-algebras (where $+$ and $*$ are the unary operations) subject to the identities:

\[
\begin{align*}
x^+x &= x, \quad (x^+y^+)^+ = x^+y^+, \quad x^+y^+ = y^+x^+, \quad x^+(xy)^+ = (xy)^+, \quad (xy)^+ = (xy)^+ \quad &
\text{for all } x, y, \in S, \\
x^*x &= x, \quad (x^*y^*)^* = x^*y^*, \quad x^*y^* = y^*x^*, \quad (xy)^*y^* = (xy)^*y^* = (x^*y)^* \\
x(yz) &= (xy)z, \quad (x^+)^* = x^+, \quad (x^*)^+ = x^*.
\end{align*}
\]

One of the advantages of the varietal point of view is that one does not need to mention the set $E$ as it is the image of the unary operations:

$$E = \{a^* \mid a \in S\} = \{a^+ \mid a \in S\}.$$

Let $S$ be an inverse semigroup. It is well known that $S$ affords a natural partial order defined by $a \leq b$ if and only if $a = aa^{-1}b$, or equivalently, $a = ba^{-1}a$. In the general case of $E$-Ehresmann semigroups this partial order splits into right and left versions. We say that $a \leq_r b$ if and only if $a = a^+b$. Dually, $a \leq_l b$ if and only if $a = ba^*$.

**Proposition 2.6 (Section 7).**

1. $\leq_r$ and $\leq_l$ are indeed partial orders on $S$.

2. $a \leq_r b$ if and only if $a = eb$ for some $e \in E$. Dually, $a \leq_l b$ if and only if $a = be$ for some $e \in E$. 

5
2.2 Ehresmann Categories

All categories in this paper will be small, that is, their morphisms form a set. Hence we can regard a category $C$, as a set of objects, denoted $C^0$ and a set of morphisms, denoted $C^1$. We will identify an object $e \in C^0$ with its identity morphism $1_e$ so we can regard $C^0$ as a subset of $C^1$. We denote the domain and range of a morphism $x \in C^1$ by $d(x)$ and $r(x)$ respectively. Recall that the multiplication $x \cdot y$ of two morphisms is defined if and only if $r(x) = d(y)$. We also denote the fact that $r(x) = d(y)$ by $\exists x \cdot y$. Note that in this paper we multiply morphisms (and functions) from left to right. Recall that a groupoid is a category where every morphism is invertible.

**Definition 2.7.** A category $C$ equipped with a partial order $\leq$ on its morphisms is called a category with order if the following hold.

(CO1) If $x \leq y$ then $d(x) \leq d(y)$ and $r(x) \leq r(y)$.

(CO2) If $x \leq y$, $u \leq v$, $\exists x \cdot u$ and $\exists y \cdot v$ then $x \cdot u \leq y \cdot v$.

(CO3) If $x \leq y$, $d(x) = d(y)$ and $r(x) = r(y)$ then $x = y$.

**Definition 2.8.** A category $C$ equipped with two partial orders on morphisms $\leq_r, \leq_l$ is called an Ehresmann category if the following hold:

(EC1) $C$ equipped with $\leq_r$ (respectively, $\leq_l$) is a category with order.

(EC2) If $x \in C^1$ and $e \in C^0$ with $e \leq_r d(x)$ then there exists a unique restriction $(e \mid x) \in C^1$ satisfying $d((e \mid x)) = e$ and $(e \mid x) \leq_r x$.

(EC3) If $x \in C^1$ and $e \in C^0$ with $e \leq_l r(x)$ then there exists a unique co-restriction $(x \mid e) \in C^1$ satisfying $r((x \mid e)) = e$ and $(x \mid e) \leq_l x$.

(EC4) For $e, f \in C^0$ we have $e \leq_r f$ if and only if $e \leq_l f$.

(EC5) $C^0$ is a semilattice with respect to $\leq_r$ (or $\leq_l$, since they are equal on $C^0$ by (EC4)).

(EC6) $\leq_r \circ \leq_l = \leq_l \circ \leq_r$.

(EC7) If $x \leq_r y$ and $f \in C^0$ then $(x \mid r(x) \wedge f) \leq_r (y \mid r(y) \wedge f)$.

(EC8) If $x \leq_l y$ and $f \in C^0$ then $(d(x) \wedge f \mid x) \leq_l (d(y) \wedge f \mid y)$.
Remark 2.9. Note that for every morphism \( x \) of an Ehresmann category we have 
\( (x \mid r(x)) = x = (d(x) \mid x) \).

From every \( E \)-Ehresmann semigroup \( S \) we can construct an Ehresmann category 
\( C(S) = C \) in the following way. The object set of \( C(S) \) is the set \( E \) and 
morphisms of \( C(S) \) are in one-to-one correspondence with elements of \( S \). For 
every \( a \in S \) we associate a morphism \( C(a) \in C^1 \) such that \( d(C(a)) = a^+ \) and 
\( r(C(a)) = a^* \). If \( \exists C(a) \cdot C(b) \) then \( C(a) \cdot C(b) = C(ab) \). Finally \( C(a) \leq_r C(b) \) 
\( (C(a) \leq_l C(b)) \) whenever \( a \leq_r b \) (respectively, \( a \leq_l b \)) according to the partial 
order of \( S \) defined above.

**Proposition 2.10 ([12] Proposition 4.1).** \( C(S) \) constructed as above equipped 
with \( \leq_r, \leq_l \) is indeed an Ehresmann category.

The other direction is also possible. Given an Ehresmann category \( C \) we can 
construct an \( E \)-Ehresmann semigroup \( S(C) = S \) in the following way. The 
elements of \( S \) are in one-to-one correspondence with morphisms of \( C \), for every 
\( x \in C^1 \) we associate an element \( S(x) \in S \). The distinguished semilattice is 
\( E = \{ S(x) \mid x \text{ is an identity morphism} \} \). Note that \( \leq_r = \leq_l \) on \( E \) so we can 
denote the common meet operation on \( E \) simply by \( \wedge \). The multiplication of \( S \) 
is defined by

\[
S(x) \cdot S(y) = S((x \mid r(x) \wedge d(y)) \cdot (r(x) \wedge d(y) \mid y)). \tag{1}
\]

**Remark 2.11.** Note that if \( \exists x \cdot y \) then \( S(x) \cdot S(y) = S(xy) \).

**Proposition 2.12 ([12] Theorem 4.21]).** \( S(C) \) constructed above is indeed an 
\( E \)-Ehresmann semigroup where for every \( x \in C^1 \) we have \( (S(x))^+ = S(d(x)) \) 
and \( (S(x))^* = S(r(x)) \).

The functions \( C \) and \( S \) are actually functors, moreover, they are isomorphisms of 
categories. In order to state this theorem accurately we need another definition.

**Definition 2.13.** A functor \( F : C \to D \) between two Ehresmann categories is 
called inductive if the following hold:

1. For every \( x, y \in C^1 \) we have that \( x \leq_r y \) implies \( F(x) \leq_r F(y) \) and \( x \leq_l y \) 
   implies \( F(x) \leq_l F(y) \).
2. \( F(e \wedge f) = F(e) \wedge F(f) \) for every \( e, f \in C^0 \).
In the following theorem, by a homomorphism of Ehresmann semigroups we mean a \((2,1,1)\)-algebra homomorphism, that is, a function that preserves also the unary operations.

**Theorem 2.14 ([12] Theorem 4.24]).** The category of all \(E\)-Ehresmann semigroups and homomorphisms is isomorphic to the category of all Ehresmann categories and inductive functors. The isomorphism being given by the functors \(S\) and \(C\) defined above.

**Remark 2.15.** We neglect the description of the operation of \(S\) and \(C\) on morphisms since it will be inessential in the sequel.

Let \(S\) be an \(E\)-Ehresmann semigroup and let \(C = C(S)\) be the associated Ehresmann category (hence \(S = S(C)\) by Theorem 2.14). Some points about the correspondence between \(S\) and \(C\) are worth mentioning. We will continue to denote by \(C(a)\) the morphism in \(C\) associated to some \(a \in S\) and likewise \(S(x)\) is the element of \(S\) associated to some \(x \in C^1\). In particular, \(S(C(a)) = a\) and \(C(S(x)) = x\). Two partial orders denoted by \(\leq_r\) were defined above, one on \(S\) and one on \(C^1\). Since \(a \leq_r b\) if and only if \(C(a) \leq_r C(b)\) we can identify these partial orders so the identical notation is justified. A dual remark holds for \(\leq_l\). The next lemma identifies the elements of \(S\) corresponding to restriction and co-restriction.

**Lemma 2.16.** Let \(a \in S\) and \(e \in E\) then

\[
C(ea) = (C(ea^+) | C(a))
\]

\[
C(ea) = (C(a) | C(ea^+)).
\]

**Proof.** It is clear that \(ea \leq_r a\) so \(C(ea) \leq_r C(a)\). Moreover, \((ea)^+ = (ea^+)^+ = ea^+\). So \(d(C(ea)) = C(ea^+)\). By \([EC2]\) \((C(ea^+) | C(a))\) is the unique morphism with these two properties so the desired equality follows. The proof for \(ae\) is similar.

### 2.3 Möbius functions

Let \((X, \leq)\) be a locally finite poset and let \(\mathbb{K}\) be a commutative unital ring. Recall that locally finite means that all the intervals \([x,y] = \{z | x \leq z \leq y\}\) are finite. We view \(\leq\) as a set of ordered pairs. The Möbius function of \(\leq\) is a
function $\mu : \leq \to \mathbb{K}$ that can be defined in the following recursive way:

$$\mu(x, x) = 1$$

$$\mu(x, y) = - \sum_{x \leq z < y} \mu(x, z)$$

**Theorem 2.17** (Möbius inversion theorem). Let $G$ be an abelian group and let $f, g : X \to G$ be functions such that

$$g(x) = \sum_{y \leq x} f(y)$$

then

$$f(x) = \sum_{y \leq x} \mu(y, x)g(y).$$

More on Möbius functions can be found in [18, Chapter 3].

### 3 Isomorphism of algebras

Throughout this section we let $S$ denote an $E$-Ehresmann semigroup and $C$ denote the corresponding Ehresmann category. For the sake of simplicity, we set $\leq_{\mathbb{S}} \leq_{\mathbb{S}_{\mathbb{E}}}$. From now on we assume that for any $e \in E$ the set $\{f \in E \mid f \leq e\}$ is finite. This clearly implies that for every $a \in S$ the set $\{b \in S \mid b \leq a\}$ is finite. In this section we will prove that the semigroup algebra of $S$ over a commutative ring $\mathbb{K}$ with identity is isomorphic to the algebra of $C$ over $\mathbb{K}$. This result is a generalization of [20, Theorem 4.2] where it was proved for inverse semigroups and inductive groupoids and of [8, Theorem 4.2] where it was proved for ample semigroups. This also generalizes [19, Proposition 3.2] where this isomorphism was proved for the special case $S = \mathcal{PT}_n$ (actually $\mathcal{PT}_n^{op}$, since there the composition is from right to left). We start by recalling the definition of an algebra of a semigroup or a category.

**Definition 3.1.** Let $S$ be a semigroup. The semigroup algebra $\mathbb{K}S$ is the free $\mathbb{K}$-module with basis the elements of the semigroup. In other words, as a set $\mathbb{K}S$ is all the formal linear combinations

$$\{k_1s_1 + \ldots + k_ns_n \mid k_i \in \mathbb{K}, s_i \in S\}$$

with multiplication being linear extension of the semigroup multiplication.
Definition 3.2. Let $C$ be a category. The category algebra $\mathbb{K}C$ is the free $\mathbb{K}$-module with basis the morphisms of the category. In other words, as a set $\mathbb{K}C$ is all formal linear combinations

$$\{k_1 x_1 + \ldots + k_n x_n \mid k_i \in \mathbb{K}, x_i \in C^1\}$$

with multiplication being linear extension of

$$x \cdot y = \begin{cases} xy & \exists x \cdot y \\ 0 & \text{otherwise.} \end{cases}$$

Remark 3.3. We use the word "algebra" in two different meaning in this paper, in the sense of ring theory as in the above definitions and in the sense of universal algebra. However, when we use it in the later sense we always mention the signature: $(2,1,1)$-algebras, $(2,1)$-subalgebras etc. Hence no ambiguity should arise.

Theorem 3.4. Let $S$ be an $E$-Ehresmann semigroup and denote $C = C(S)$. Then $\mathbb{K}S$ is isomorphic to $\mathbb{K}C$. Explicit isomorphisms $\varphi : \mathbb{K}S \to \mathbb{K}C$, $\psi : \mathbb{K}C \to \mathbb{K}S$ are defined (on basis elements) by

$$\varphi(a) = \sum_{b \leq a} C(b)$$

$$\psi(x) = \sum_{y \leq x} \mu(y,x) S(y)$$

where $\mu$ is the Möbius function of the poset $\leq$.

Note that by our assumption the number of $b \in S$ such that $b \leq a$ is finite so the summations in Theorem 3.4 are also finite. Hence, $\varphi$ and $\psi$ are well defined.

Proof. The proof that $\varphi$ and $\psi$ are bijectives is identical to what is done in [20].

$$\psi(\varphi(a)) = \psi(\sum_{b \leq a} C(b)) = \sum_{b \leq a} \psi(C(b))$$

$$= \sum_{b \leq a} \sum_{c \leq b} \mu(c,b) S(C(c)) = \sum_{c \leq a} c \sum_{c \leq b \leq a} \mu(c,b)$$

$$= \sum_{c \leq a} c \delta(c,a) = a$$
\[ \varphi \psi (x) = \varphi \left( \sum_{y \leq x} \mu(y, x)S(y) \right) = \sum_{y \leq x} \mu(y, x)\varphi(S(y)) = C(S(x)) = x \]

where the third equality follows from the Möbius inversion theorem and the definition of \( \varphi \). Hence, \( \varphi \) and \( \psi \) are bijectives. We now prove that \( \varphi \) is a homomorphism. Let \( a, b \in S \), we have to prove that

\[ \sum_{c \leq ab} C(c) = \left( \sum_{a' \leq a} C(a') \right) \left( \sum_{b' \leq b} C(b') \right). \tag{2} \]

**Case 1.** First assume that \( \exists C(a) \cdot C(b) \), that is, \( r(C(a)) = d(C(b)) \) (or equivalently, \( a^* = b^+ \)). In this case we can set \( x = C(a) \) and \( y = C(b) \) and then \( C(ab) = C(a)C(b) = xy \). So we can write Equation (2) as

\[ \sum_{z \leq xy} z = \left( \sum_{x' \leq x} x' \right) \left( \sum_{y' \leq y} y' \right). \tag{3} \]

According to [CO2] if \( \exists x' \cdot y' \) then \( x' y' \leq xy \). Hence, any element on the right hand side of Equation (3) is less than or equal to \( xy \). So we have only to show that any \( z \) such that \( z \leq xy \) appears on the right hand side once. First, note that \( z = (d(z) \mid xy) \) according to the uniqueness of restriction (part of [EC2]). We can choose \( x' = (d(z) \mid x) \) and \( y' = (r(x') \mid y) \). Clearly, since \( d(y') = r(x') \) we have that \( \exists x' \cdot y \). Moreover by [CO2] \( x' \cdot y' \leq xy \) and \( d(x'y') = d(x') = d(z) \) hence by uniqueness of restriction we have that \( x' \cdot y' = (d(z) \mid xy) = z \). This proves that \( z \) appears in the right hand side of Equation (3).

Now assume that \( x' \cdot y' = z \) for some \( x' \leq x \) and \( y' \leq y \). Then we must have \( d(x') = d(z) \) so by uniqueness of restriction \( x' = (d(z) \mid x) \).

Now, since \( \exists x' \cdot y' \) we must have that \( d(y') = r(x') \) so again by uniqueness of restriction \( y' = (r(x') \mid y) \). So \( z \) appears only once on the right hand side of Equation (3) and this finishes this case.

**Case 2.** Assume \( r(C(a)) \neq d(C(b)) \) (or equivalently, \( a^* \neq b^+ \)). Define \( \tilde{a} = ab^+ \) and \( \tilde{b} = a^* b \). Note that

\[ \tilde{a} \tilde{b} = ab^+ a^* b = aa^* b^+ b = ab \]
so we have
\[ \sum_{c \leq \tilde{a} \tilde{b}} C(c) = \sum_{c \leq \tilde{a} \tilde{b}} C(c). \]

By Lemma 2.16
\[
C(\tilde{a}) = (C(a) \mid C(a^* b^+)) = (C(a) \mid r(C(a)) \land d(C(b)))
\]
and
\[
C(\tilde{b}) = (C(a^* b^+) \mid C(b)) = (r(C(a)) \land d(C(b)) \mid C(b))
\]
so clearly \( \exists C(\tilde{a}) \cdot C(\tilde{b}) \). Case I implies that
\[
\sum_{c \leq \tilde{a} \tilde{b}} C(c) = (\sum_{a' \leq \tilde{a}} C(a'))(\sum_{b' \leq \tilde{b}} C(b')).
\]

Now, all that is left to show is that
\[
(\sum_{a' \leq \tilde{a}} C(a'))(\sum_{b' \leq \tilde{b}} C(b')) = (\sum_{a' \leq a} C(a'))(\sum_{b' \leq b} C(b')).
\] (4)

We can set again \( x = C(a) \), \( \tilde{x} = C(\tilde{a}) \), \( y = C(b) \) and \( \tilde{y} = C(\tilde{b}) \) so Equation (4) can be written as
\[
(\sum_{x' \leq x} x')(\sum_{y' \leq y} y') = (\sum_{x' \leq x} x')(\sum_{y' \leq y} y').
\] (5)

We will show that a multiplication \( x' \cdot y' \) on the right hand side of Equation (5) equals 0 unless \( x' \leq \tilde{x} \) and \( y' \leq \tilde{y} \). Take \( x' \leq x \) such that \( x' \notin \tilde{x} \) and assume that there is a \( y' \leq y \) such that \( \exists x' \cdot y' \), that is, \( r(x') = d(y') \). Since \( y' \leq y \) we have \( r(x') = d(y') \leq d(y) \) by (CO1). Now, by (EC7) (choosing \( f = d(y) \)) we have that
\[
(x' \mid r(x') \land d(y)) \leq (x \mid r(x) \land d(y))
\]

but note that \( (x \mid r(x) \land d(y)) = \tilde{x} \) and \( r(x') \land d(y) = r(x') \) so we get
\[
x' = (x' \mid r(x')) \leq \tilde{x}
\]
a contradiction. Similarly, take \( y' \leq y \) such that \( y' \notin \tilde{y} \) and assume
that there is an \(x' \leq x\) such that \(\exists x' \cdot y', \) that is, \(r(x') = d(y')\). Again, since \(r(x') \leq r(x)\) we have that \(d(y') \leq r(x)\) and clearly \(d(y') \leq d(y)\) hence \(d(y') \leq r(x) \wedge d(y) = d(\tilde{y})\). By \((\text{EC2})\) there exists a restriction \((d(y') \mid \tilde{y})\). But \((d(y') \mid \tilde{y}) \leq \tilde{y} \leq y\) so by the uniqueness of restriction \((d(y') \mid \tilde{y}) = y', \) hence \(y' \leq \tilde{y},\) a contradiction. This finishes the proof.

\[\]

Remark 3.5. Note that Theorem 3.4 can be proved, \textit{mutatis mutandis}, using \(\leq l\) instead of \(\leq r\).

Corollary 3.6. Let \(S\) be an \(E\)-Ehresmann semigroup such that \(E\) is finite, then \(\mathbb{K}S\) is a unital algebra.

Proof. The isomorphic category algebra \(KC\) has the identity element \(\sum_{e \in E} C(e)\).

4 Examples

In the following examples \(C\) will always be the Ehresmann category associated to the \(E\)-Ehresmann semigroup being discussed.

Example 4.1. Let \(M\) be a monoid and take \(E = \{1\}\). It is easy to check that \(M\) is an \(E\)-Ehresmann semigroup. It is easy to see that if we think of \(M\) as a category with one object in the usual way we get precisely \(C\). The fact that \(\mathbb{K}M\) is isomorphic to \(\mathbb{K}C\) is trivial but true.

Example 4.2. Let \(S\) be an inverse semigroup such that \(E(S)\) is finite. If we take \(E = E(S)\) our isomorphism is precisely [20 Theorem 4.2]. If \(S\) is a finite ample semigroup our isomorphism is precisely [8 Theorem 4.2].

Example 4.3. Let \(S = \mathcal{PT}_n\) be the monoid of all partial functions on an \(n\)-element set and take \(E = \{1_A \mid A \subseteq \{1 \ldots n\}\}\) to be the semilattice of all the partial identities. It can be checked that \(\mathcal{PT}_n\) is an \(E\)-Ehresmann semigroup where for every \(t \in \mathcal{PT}_n\) we have \(t^+ = 1_{\text{dom}(t)}\) and \(t^* = 1_{\text{im}(t)}\). The corresponding Ehresmann category is the category of all onto (total) functions between subsets of an \(n\)-element set. Our isomorphism is then precisely [19 Proposition 3.2].
Example 4.4. Let $S = B_n$ be the monoid of all relations on an $n$-element set and take again $E$ to be the semilattice of all the partial identities. Again, $B_n$ is an $E$-Ehresmann semigroup where for every $t \in B_n$ we have $t^+ = 1_{\text{dom}(t)}$ and $t^* = 1_{\text{im}(t)}$. The associated category $C$ has the subsets of $\{1, \ldots, n\}$ as objects and for every $a \in B_n$ there is a corresponding morphism $C(a)$ from $\text{dom}(a)$ to $\text{im}(a)$. This is the category of bi-surjective relations on the subsets of an $n$-element set $X$. That is, the objects are all subsets of $X$ and a morphism from $Y$ to $Z$ are all subsets $R$ of $Y \times Z$ such that both projections of $R$ to $Y$ and $Z$ respectively are onto functions. This is a subcategory of the category applied in [2] to find the dimensions of the simple modules of $B_n$.

Example 4.5. Let $S = [Y, M_\alpha, \varphi_{\alpha, \beta}]$ be a strong semilattice of monoids (where $Y$ is a finite semilattice). If we take $E = \{1_\alpha \in M_\alpha \mid \alpha \in Y\} \cong Y$ then it is proved in [3, Examples 2.5.11-12] that $S$ is an $E$-Ehresmann semigroup. In this case, the objects of $C$ are in one-to-one correspondence with elements of $Y$. Every $a \in M_\alpha$ corresponds to an endomorphism $C(a)$ of $\alpha$. Note that all the morphisms in $C$ are endomorphisms.

Corollary 4.6. If $S = [Y, M_\alpha, \varphi_{\alpha, \beta}]$ is a strong semilattice of monoids with a finite $Y$ then $\mathbb{K}S$ is isomorphic to $\prod_{\alpha \in Y} \mathbb{K}M_\alpha$.

5 Ehresmann EI-categories

5.1 Characterization and examples

Recall that if $C$ is a category an endomorphism is a morphism of $C$ from some object $e \in C^0$ to $e$. A category is called an EI-category if every endomorphism is an isomorphism, or in other words, if every endomorphism monoid is a group. Algebras of EI-categories are better understood than general category algebras. Assume $\mathbb{K}$ is an algebraically closed field and $C$ is a finite EI-category where the orders of the endomorphism groups of its objects are invertible in $\mathbb{K}$. There is a way to describe the Jacobson radical of $\mathbb{K}C$ ([15 Proposition 4.6]) and its ordinary quiver ([15 Theorem 4.7] or [16 Theorem 6.13]). By Theorem 3.4 we then know how to compute the Jacobson radical and ordinary quiver of an $E$-Ehresmann semigroup if the corresponding Ehresmann category is an EI-category. In this section we will characterize such semigroups. We start with understanding elements corresponding to left, right and two-sided invertible morphisms. Recall that if $a \in S$ then $d(C(a)) = C(a^+)$ and $r(C(a)) = C(a^*)$. 

14
Lemma 5.1. $C(a)$ is right invertible if and only if $a R a^+$ and $a$ has an inverse $b$ such that $b L a^+$ and $a^* = b^+$. A dual statement holds for left invertibility.

Proof. Assume that $C(b)$ is a right inverse for $C(a)$, then clearly $d(C(a)) = r(C(b))$ and $r(C(a)) = d(C(b))$ so $a^+ = b^*$ and $a^* = b^+$. Since $C(ab) = C(a)C(b) = d(C(a)) = C(a^+)$ we know that

$$ab = a^+$$

and clearly

$$a^+a = a, \quad ba^+ = bb^* = b.$$ 

Hence

$$a R a^+, \quad b L a^+$$

so $b$ is an inverse of $a$ as required.

In the other direction, we have that $\exists C(a) \cdot C(b)$ since $a^* = b^+$. Now $C(a) \cdot C(b) = C(ab) = C(a^+)$ since $b$ is an inverse of $a$ with $a R a^+ L b$. This finishes the proof since the dual case is similar.

Remark 5.2. The requirement $a R a^+ (a L a^*)$ is not enough for right (respectively, left) invertibility. For instance take $S = \mathcal{PT}_2$ and $E = \{1_A \mid A \subseteq \{1, 2\}\}$ as above. Choose $a$ to be the (total) constant transformation with image $\{1\}$.

$$a = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}.$$ 

It is easy to check that $C(a)$ is not left invertible in $C$ but it is $L$-equivalent to

$$a^* = \begin{pmatrix} 1 & 2 \\ 1 & \emptyset \end{pmatrix}.$$ 

However, we have the following two sided version.

Lemma 5.3. $C(a)$ is invertible in $C$ if and only if $a R a^+$ and $a L a^*$.

Proof. If $C(a)$ is left and right invertible then Lemma 5.1 implies that $a R a^+$ and $a L a^*$. In the other direction take $b$ to be the inverse of $a$ such that $b R a^*$ and $b L a^+$ and again the result follows from Lemma 5.1.
Remark 5.4. The set of all elements \( a \in S \) such that \( a \mathcal{R} a^+ \) and \( a \mathcal{L} a^* \) appears in [13, Section 3] in a more general context and it is denoted \( \text{Reg}_E(S) \). We will see later (Lemma 5.15) that this is an inverse subsemigroup of \( S \).

Corollary 5.5. \( C(e), C(f) \in C^0 \) are isomorphic objects if and only if \( e \mathcal{D} f \).

Proof. If \( e \mathcal{D} f \) take \( a \in \mathcal{R}_e \cap \mathcal{L}_f \) and \( C(a) \) is an isomorphism between \( C(e) \) and \( C(f) \). On the other hand, if \( C(a) \) with \( d(C(a)) = C(e) \) and \( r(C(a)) = C(f) \) is an isomorphism then \( a \mathcal{R} e \) and \( a \mathcal{L} f \) so \( e \mathcal{D} f \). \( \square \)

Corollary 5.6. \( C \) is an EI-category if and only if \( a^+ = a^* \) implies that \( a \) is a group element. In other words, \( C \) is an EI-category if and only if the \( \tilde{H} \)-class of \( e \) equals its \( H \)-class for every \( e \in E \).

Proof. Clear from Lemma 5.3. \( \square \)

We also note the following necessary condition for \( C \) to be an EI-category.

Lemma 5.7. If \( C \) is an EI-category then \( E \) is a maximal semilattice in \( S \).

Proof. Assume that there is some \( f \in E(S) \setminus E \) which commutes with every \( e \in E \). It follows that \( f^+ = f^* \) so the morphism \( C(f) \) is an element in some endomorphism group of \( C \). This is a contradiction since \( C(f)C(f) = C(f) \) and groups have no non-identity idempotents. \( \square \)

Note that the condition of Lemma 5.7 is not sufficient. For instance, take the monoid of all binary relations \( B_n \) which is \( E \)-Ehresmann as mentioned above (Example 4.4). It is easy to check that the set of all partial identities is a maximal semilattice but the corresponding category \( C \) is not an EI-category.

It is also worth mentioning the groupoid case.

Corollary 5.8. \( C \) is a groupoid if and only if \( S \) is an inverse semigroup and \( E = E(S) \).

Proof. Assume that \( C \) is a groupoid. Let \( a, b \in S \) such that \( a \mathcal{R}_E b \). By Lemma 5.3,

\[
a \mathcal{R} a^+ = b^+ \mathcal{R}
\]

hence \( \mathcal{R} = \mathcal{R}_E \) and this implies that any \( \mathcal{R} \) class contains precisely one idempotent. A similar observation is true for \( \mathcal{L} \) classes. Hence \( S \) is inverse and \( E(S) \) is a semilattice. By Lemma 5.7 \( E \) is a maximal semilattice so \( E = E(S) \) as required. The other direction is clear from Lemma 5.3. \( \square \)
Remark 5.9. Note that in this case $C$ is the inductive groupoid corresponding to $S$.

We now give some examples of $E$-Ehresmann semigroups whose corresponding Ehresmann category is an EI-category.

**Example 5.10.** Take $S = \mathcal{PT}_n$ which is $E$-Ehresmann as mentioned in Example 4.3. The corresponding Ehresmann category $E_n$ is the category of all onto functions between subsets of an $n$-element set. $E_n$ is an EI-category since the endomorphism monoid of an object $A$ is the group $S_A$ of all permutations of elements of $A$. Every $(2,1,1)$-subalgebra of $\mathcal{PT}_n$ is also an $E$-Ehresmann semigroup (since $E$-Ehresmann semigroups form a variety). The corresponding Ehresmann category $C$ is a subcategory of $E_n$. So every endomorphism monoid of $C$ is a submonoid of some endomorphism group in $E_n$. Since every submonoid of a finite group is a group, $C$ is also an EI-category. Several well-known examples of such subsemigroups include: order-preserving partial functions (or more generally, order-preserving partial functions with respect to some partial order on $\{1, \ldots, n\}$), weakly-decreasing partial functions, order-preserving and weakly-decreasing partial functions (also known as the partial Catalan monoid), and orientation-preserving partial functions. In [19, Section 5] the author has used the corresponding EI-category to describe the ordinary quiver of the algebras of some of these semigroups. Another remark is worth mentioning. $\mathcal{PT}_n$ satisfies the identity $xy^+ = (xy^+)^+x$. Left $E$-Ehresmann semigroups that satisfy this identity are called left restriction. It is well known [5, Corollary 6.3] that left restriction semigroups are precisely the $(2,1)$-subalgebras of $\mathcal{PT}_n$ where the unary operation is $^+$. Clearly, every $(2,1,1)$-subalgebra of $\mathcal{PT}_n$ is also left restriction.

**Example 5.11.** Let $S$ be a finite $E(S)$-Ehresmann semigroup, i.e., an $E$-Ehresmann semigroup with $E = E(S)$. Let $a \in S$ be an element such that $a^+ = a^*$ and $C(a)$ is an idempotent of the endomorphism monoid of $C(a^+)$. Since $C(a) = C(a)C(a) = C(aa)$ we know that $a \in E(S) = E$ so $a = a^+$. Hence any idempotent of an endomorphism monoid is the identity morphism of the object. A finite monoid with only one idempotent is a group hence the corresponding category is an EI-category. In particular this class contains all finite adequate semigroups and hence all finite ample semigroups which are the semigroups considered in [8].

**Example 5.12.** Let $T_n$ be the monoid of all (total) functions on an $n$-element
set. Denote by \( \text{id} \) the identity function and by \( k \) the constant functions that sends every element to \( k \). Define \( S \) to be the subsemigroup of \( T_2 \times T_2^{\text{op}} \) containing the six elements

\[
(1, 1), (2, 1), (1, 2), (2, 2), (1, \text{id}), (\text{id}, 1).
\]

Note that

\[
B = \{(1, 1), (2, 1), (1, 2), (2, 2)\}
\]

forms a rectangular band and that \((1, \text{id}) ((\text{id}, 1))\) is a left (respectively, right) identity for elements in \( B \). Choose \( E = \{(1, 1), (1, \text{id}), (\text{id}, 1)\} \), which is clearly a subsemilattice. It is easy to check that every \( \overline{R}_E \) and \( \overline{L}_E \) class contains one element of \( E \) and the identities \( (xy)^+ = (xy)^+ \) and \( (xy)^* = (x^*y)^* \) hold so this is an \( E \)-Ehresmann semigroup. The corresponding Ehresmann category (which is clearly an EI-category) is given in the following drawing:

Note that this example is not included in the previous ones. Definitely, \( E \neq S = E(S) \). Moreover, \( S \) is not left or right restriction since

\[
(2, 2)(1, \text{id}) = (1, 2) \neq (2, 2) = (1, \text{id})(2, 2) = ((2, 2)(1, \text{id}))^+(2, 2)
\]

and

\[
(\text{id}, 1)(2, 2) = (2, 1) \neq (2, 2) = (2, 2)(\text{id}, 1) = (2, 2)((\text{id}, 1)(2, 2))^*.
\]

Hence, \( S \) is not a \((2, 1, 1)\)-subalgebra of \( \mathcal{PT}_n \) (or \( \mathcal{PT}_n^{\text{op}} \)). The semigroup \( S \) provides an example of an \( E \)-Ehresmann semigroup whose corresponding Ehresmann category is an EI-category but there is an idempotent \((2, 2) \in S \) which is not \( R \) equivalent to \((2, 2)^+ \) and not \( L \) equivalent to \((2, 2)^* \).
5.2 The maximal semisimple image

Let $S$ be a finite semigroup whose idempotents $E(S)$ commute and let $\mathbb{K}$ be a field whose characteristic does not divide the order of any maximal subgroup of $S$. Denote by $R(S)$ the set of regular elements of $S$. It is known that $R(S)$ is an inverse subsemigroup of $S$. Steinberg proved [21, Section 8.1] that the algebra $\mathbb{K}R(S)$ is isomorphic to the maximal semisimple image of $\mathbb{K}S$. In this section we prove a similar result for finite $E$-Ehresmann semigroups whose corresponding Ehresmann category is an EI-category. More precisely, let $S$ be a finite $E$-Ehresmann semigroup with a corresponding finite Ehresmann category $C$ such that $C$ is an EI-category. Assume that $\mathbb{K}$ is a field such that the orders of the endomorphism groups of all objects in $C$ are invertible in $\mathbb{K}$. Recall that we denote by $\text{Reg}_E(S)$ (see Remark 5.4) the set of elements of $S$ corresponding to isomorphisms in $C$. We will prove that $\text{Reg}_E(S)$ is an inverse subsemigroup of $S$ and $\mathbb{K}\text{Reg}_E(S)$ is isomorphic to the maximal semisimple image of $\mathbb{K}S$.

We start by recalling some elementary definitions and facts about algebras. Recall that a non-zero module is called simple if it has no non-trivial submodules. A module is called semisimple if it is a direct sum of simple modules. An algebra $A$ is called semisimple if all its modules are semisimple. The Jacobson radical of an algebra $A$ denoted $\text{Rad} A$ is the intersection of all its left maximal ideals. Its importance comes from the fact that if $A$ is finite dimensional then $A/\text{Rad}(A)$ is the maximal semisimple image of $A$. Other fundamental facts about algebras can be found in the first chapter of [1].

Let $A$ be a finite dimensional associative algebra. It might be the case that the maximal semisimple image $A/\text{Rad} A$ is isomorphic to a subalgebra of $A$. For instance, if $A/\text{Rad} A$ is a separable algebra then the Wedderburn-Malcev theorem [4, Theorem 72.19] assures that $A/\text{Rad} A$ is isomorphic to a subalgebra of $A$. However, if $A$ is an algebra of a category or a semigroup, it is usually not the case that an isomorphic copy of $A/\text{Rad} A$ is spanned by some subcategory or a subsemigroup. However, if $A$ an EI-category algebra the situation is better.

**Proposition 5.13.** [15, Proposition 4.6] Let $C$ be a finite EI-category and let $\mathbb{K}$ be a field such that the orders of the endomorphism groups of all objects in $C$ are invertible in $\mathbb{K}$. The radical $\text{Rad}(\mathbb{K}C)$ is spanned by all the non-invertible morphisms of $C$.

**Corollary 5.14.** Let $C$ and $\mathbb{K}$ be as above. The subalgebra of $\mathbb{K}C$ spanned by all the invertible morphisms is isomorphic to the maximal semisimple image
\[ \mathbb{K}C/\text{Rad}(\mathbb{K}C). \]

We now want to “translate” this result to semigroup language using Theorem 3.4. Recall that we denote by \( \text{Reg}_E(S) \) the elements of \( S \) that correspond to invertible morphisms in \( C \). According to Lemma 5.3, \( \text{Reg}_E(S) = \{ a \in S \mid a^+ \mathcal{R} a \mathcal{L} a^* \} \).

A natural question is whether this set forms a subsemigroup. This question is also considered in other contexts in [13, 7]. The next lemma gives an affirmative answer in the case of \( E \)-Ehresmann semigroups.

**Lemma 5.15.** Let \( S \) be an \( E \)-Ehresmann semigroup. Then the set \( \text{Reg}_E(S) \) is an inverse subsemigroup of \( S \).

**Proof.** We first prove that \( \text{Reg}_E(S) \) forms a subsemigroup\(^1\). Let \( a \) and \( b \) be two elements of \( \text{Reg}_E(S) \). We will prove only \( ab \mathcal{R} (ab)^+ \) because the proof that \( ab \mathcal{L} (ab)^+ \) is similar. For every \( x \in \text{Reg}_E(S) \) it is convenient to denote by \( x^{-1} \) the unique inverse of \( x \) such that \( xx^{-1} = x^+ \) and \( x^{-1}x = x^* \). Clearly,

\[
(ab)^+ab = ab.
\]

so we need only to prove that \( ab \) is \( \mathcal{R} \)-above \( (ab)^+ \). Now, note that

\[
(a^{-1}(ab)^+)^+a^{-1}(ab)^+ = (a^{-1}ab)^+a^{-1}(ab)^+
\]

\[
= (a^*b)^+a^{-1}(ab)^+ = a^*b^+a^{-1}(ab)^+.
\]

Multiplying by \( a \) on the left we get

\[
a^+(ab)^+ = ab^+a^{-1}(ab)^+ = abb^{-1}a^{-1}(ab)^+.
\]

But since \( (ab)^+ \) is the least element from \( E \) which is left identity for \( ab \) and since

\[
a^+ab = ab
\]

we have that

\[
(ab)^+ = a^+(ab)^+ = abb^{-1}a^{-1}(ab)^+.
\]

\(^1\text{This part of the proof is actually due to Michael Kinyon who proved it using the automated theorem prover called Prover9. For general information on Prover9 see [17].}\)
This implies that $abR(ab)^+$ so $\text{Reg}_E(S)$ is indeed a subsemigroup. It is left to show that $\text{Reg}_E(S)$ is an inverse subsemigroup. Clearly, $E \subseteq \text{Reg}_E(S)$ and all the elements of $\text{Reg}_E(S)$ are regular. Moreover, any idempotent $f \in \text{Reg}_E(S)$ must be in $E$. Indeed, if $f^+Rf \not{\subseteq} f^*$ then $f^*f^+Df$ by the Clifford-Miller theorem [10, Proposition 2.3.7] and it follows that $f^*$ and $f^+$ commute only if $f^* = f^+$. In this case $f \not{\subseteq} f^+$ so $f = f^+ \in E$. So $\text{Reg}_E(S)$ is regular and its set of idempotents is the semilattice $E$, hence it is an inverse semigroup as required.

**Corollary 5.16.** $\text{Reg}_E(S)$ is a down ideal with respect to $\leq_r$ and $\leq_l$.

**Proof.** If $a \in \text{Reg}_E(s)$ and $b \leq_r a$ ($b \leq_l a$) then $b = ea$ ($b = ae$) for some $e \in E$. Lemma 5.13 then implies that $b \in \text{Reg}_E(S)$. \hfill \Box

**Proposition 5.17.** Let $S$ be a finite $E$-Ehresmann semigroup whose corresponding Ehresmann category $C$ is an EI-category. Then $\mathbb{K}\text{Reg}_E(S)$ is isomorphic to $\mathbb{K}S/\text{Rad}(\mathbb{K}S)$.

**Proof.** Denote by $B$ the subvector space of $\mathbb{K}C$ spanned by all the invertible morphisms. Clearly, $B$ is a subalgebra of dimension $|\text{Reg}_E(S)|$, since the product in $\mathbb{K}C$ of two invertible morphisms in $C$ is either another invertible morphism or $0$. Denote by $\psi$ the isomorphism $\psi : \mathbb{K}C \rightarrow \mathbb{K}S$ as in Theorem 3.4. Since

$$\psi(C(a)) = \sum_{b \leq_r a} \mu(b, a)b$$

and since $\text{Reg}_E(S)$ is a down ideal with respect to $\leq_r$ by Corollary 5.16 we have that $\psi(B) \subseteq \mathbb{K}\text{Reg}_E(S)$. But since they have the same (finite) dimension we must have $\psi(B) = \mathbb{K}\text{Reg}_E(S)$. Now, by Corollary 5.14 $B$ is isomorphic to the maximal semisimple image of $\mathbb{K}C$ so $\psi(B) = \mathbb{K}\text{Reg}_E(S)$ is isomorphic to the maximal semisimple image $\mathbb{K}S/\text{Rad}(\mathbb{K}S)$ of $\mathbb{K}S$ as required. \hfill \Box

**Acknowledgements:** The author is grateful to Michael Kinyon for proving that $\text{Reg}_E(S)$ is a subsemigroup (the main part of Lemma 5.13). The author also thanks the referee for his/her helpful comments.
References

[1] Ibrahim Assem, Daniel Simson, and Andrzej Skowroński. Elements of the representation theory of associative algebras. Vol. 1, volume 65 of London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 2006. Techniques of representation theory.

[2] Serge Bouc and Jacques Thévenaz. The representation theory of finite sets and correspondences. arXiv preprint arXiv:1510.03034, 2015.

[3] Claire Cornock. Restriction semigroups: structure, varieties and presentations. PhD thesis, University of York, 2011.

[4] Charles W Curtis and Irving Reiner. Representation theory of finite groups and associative algebras, volume 356. American Mathematical Soc., 1966.

[5] Victoria Gould. Notes on restriction semigroups and related structures; formerly (weakly) left E-ample semigroups, 2010.

[6] Victoria Gould. Restriction and Ehresmann semigroups. In Proceedings of the International Conference on Algebra 2010, pages 265–288. World Sci. Publ., Hackensack, NJ, 2012.

[7] Victoria Gould and Rida-e Zenab. Semigroups with inverse skeletons and Zappa-Szép products. Categories and General Algebraic Structures with Applications, 1(1):59–89, 2013.

[8] Xiaojiang Guo and Lin Chen. Semigroup algebras of finite ample semigroups. Proc. Roy. Soc. Edinburgh Sect. A, 142(2):371–389, 2012.

[9] Xiaojiang Guo and K. P. Shum. Algebras of ample semigroups each of whose $J^*$-class contains a finite number of idempotents. Southeast Asian Bull. Math., 39(3):377–405, 2015.

[10] John M. Howie. Fundamentals of semigroup theory, volume 12 of London Mathematical Society Monographs. New Series. The Clarendon Press Oxford University Press, New York, 1995. Oxford Science Publications.

[11] Yingdan Ji and Yanfeng Luo. Locally adequate semigroup algebras. Open Math., 14:29–48, 2016.

[12] M. V. Lawson. Semigroups and ordered categories. I. The reduced case. J. Algebra, 141(2):422–462, 1991.
[13] Mark V. Lawson. Rees matrix semigroups. *Proc. Edinburgh Math. Soc. (2)*, 33(1):23–37, 1990.

[14] Mark V. Lawson. *Inverse semigroups*. World Scientific Publishing Co., Inc., River Edge, NJ, 1998. The theory of partial symmetries.

[15] Liping Li. A characterization of finite EI categories with hereditary category algebras. *J. Algebra*, 345:213–241, 2011.

[16] Stuart Margolis and Benjamin Steinberg. Quivers of monoids with basic algebras. *Compos. Math.*, 148(5):1516–1560, 2012.

[17] W. McCune. Prover9 and mace4. http://www.cs.unm.edu/~mccune/prover9/, 2005–2010.

[18] Richard P. Stanley. *Enumerative combinatorics. Vol. 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1997. With a foreword by Gian-Carlo Rota, Corrected reprint of the 1986 original.

[19] Itamar Stein. The representation theory of the monoid of all partial functions on a set and related monoids as EI-category algebras. *J. Algebra*, 450:549–569, 2016.

[20] Benjamin Steinberg. Möbius functions and semigroup representation theory. *J. Combin. Theory Ser. A*, 113(5):866–881, 2006.

[21] Benjamin Steinberg. Möbius functions and semigroup representation theory. II. Character formulas and multiplicities. *Adv. Math.*, 217(4):1521–1557, 2008.
Erratum to: Algebras of Ehresmann semigroups and categories

Itamar Stein

Department of Mathematics
Bar Ilan University
Israel
steinita@gmail.com

Shoufeng Wang discovered an error in the main theorem of the author’s Semigroup Forum article ‘Algebras of Ehresmann semigroups and categories’. Wang observed that the function we suggest as an isomorphism is not a homomorphism unless the semigroup being discussed is left restriction. In order to fix our mistake we will add this assumption. Note that our revised result is still a generalization of earlier work of Guo and Chen, the author, and Steinberg.

1 A correction to the main theorem of [3]

Shoufeng Wang [6] has observed that the proof of [3] Theorem 3.4 does not hold without the additional assumption that the semigroup is left restriction. This Erratum shows how this assumption yields a valid result, and examines the consequences. Theorem 1.5 below, which is the revision of [3] Theorem 3.4, nevertheless generalizes [5] Theorem 4.2, [2] Theorem 4.2 and [4] Proposition 3.2.

We assume the reader is familiar with [3] and in particular with the definition of an E-Ehresmann semigroup; for undefined terms the reader should consult [3].

We start by giving a counterexample to the original claim in [3] Theorem 3.4. Let $S$ is an $E$-Ehresmann semigroup with $\leq = \leq_r$ a principally finite poset, and
let $C = C(S)$ be the corresponding Ehresmann category. We do not know whether $K_S$ is always isomorphic to $K_C$, but Shoufeng Wang observed that the function $\varphi : K_S \to K_C$ defined by

$$\varphi(a) = \sum_{b \leq a} C(b)$$

is, in general, not a homomorphism, as can be seen in the following example.

**Example 1.1.** Choose $S = B_2$, the monoid of all binary relations on the set $\{1, 2\}$. As mentioned in [3, Example 4.4], $B_2$ is an $E$-Ehresmann semigroup where $E = \{\text{id}, \{(1, 1)\}, \{(2, 2)\}, \emptyset\}$ is the set of partial identities. For every $a \in B_2$, $a^+(a^*)$ is the identity function on the domain (respectively, image) of $a$. Let $C = C(B_2)$ be the corresponding Ehresmann category. It is easy to see that $\varphi : KB_2 \to KC$ is not a homomorphism. Choose $a = \{(1, 1), (1, 2)\}$ and $b = \{(1, 1)\}$ so $ab = \{(1, 1)\}$. Note that in $B_2$, $\leq$ is domain restriction, so

$$\varphi(b) = \varphi(ab) = \varphi(\{(1, 1)\}) = C(\{(1, 1)\}) + C(\emptyset)$$

and

$$\varphi(a) = C(\{(1, 1), (1, 2)\}) + C(\emptyset).$$

Recall that in the category algebra the multiplication $xy$ equals 0 unless $r(x) = d(y)$. Hence

$$\varphi(a)\varphi(b) = (C(\{(1, 2), (1, 1)\}) + C(\emptyset)) (C(\{(1, 1)\}) + C(\emptyset)) = C(\emptyset)$$

so indeed

$$\varphi(ab) \neq \varphi(a)\varphi(b).$$

In order to fix this problem we will have to add the requirement of being left (or right) restriction.

**Definition 1.2.** Let $S$ be a left $E$-Ehresmann semigroup. $S$ is called **left restriction** if

$$ae = (ae)^+a$$

for every $a \in S$ and $e \in E$. Dually, a right $E$-Ehresmann semigroup $S$ is called **right restriction** if

$$ea = a(ea)^+$$
for every $a \in S$ and $e \in E$. If $S$ is an $E$- Ehresmann semigroup which is both left and right restriction, then it is called a 
restriction semigroup.

**Example 1.3.** Every inverse semigroup is a restriction semigroup. It is well-known that the monoid $\mathcal{P}T_n$ of all partial transformations on an $n$-element set is left restriction but not right-restriction. However, the monoid $B_n$ of all binary relations on an $n$-element set is neither left nor right restriction.

**Lemma 1.4.** If $S$ is an $E$-Ehresmann semigroup which is also left restriction (right restriction), then $\leq \subseteq \leq_r$ (respectively, $\leq_r \subseteq \leq$).

*Proof.* Assume $b \leq_l a$. By [3, Proposition 2.6] there exists an $e \in E$ such that $b = ae = (ae)^+a$ but $(ae)^+ \in E$ so $b \leq_r a$ as well. The other case is dual. $\square$

We can now give a correct version of [3, Theorem 3.4], under the additional assumption of being left restriction.

**Theorem 1.5.** Let $S$ be an $E$-Ehresmann and left restriction semigroup and let $C = C(S)$. Then $\mathbb{K}S$ is isomorphic to $\mathbb{K}C$. Explicit isomorphisms $\varphi : \mathbb{K}S \rightarrow \mathbb{K}C$, $\psi : \mathbb{K}C \rightarrow \mathbb{K}S$ are defined (on basis elements) by

\[
\varphi(a) = \sum_{b \leq a} C(b), \quad \psi(x) = \sum_{y \leq x} \mu(y, x)S(y)
\]

where $\mu$ is the M"obius function of the poset $\leq$.

**Remark 1.6.** Theorem 1.5 can be proved, mutatis mutandis, for $E$-Ehresmann and right restriction semigroups using $\leq_r$ instead of $\leq_l$.

*Proof of Theorem 1.5.* The proof that $\varphi$ and $\psi$ are bijections is identical to that in [3], as is Case 1 of the proof that $\varphi$ is a homomorphism.

For Case 2, we assume $r(C(a)) \neq d(C(b))$ (or equivalently, $a^+ \neq b^+$). Define $\tilde{a} = ab^+$ and $\tilde{b} = a^*b$. Proceeding as in [3], it remains to show that

\[
\left( \sum_{a' \leq a} C(a') \right) \left( \sum_{b' \leq b} C(b') \right) = \left( \sum_{a' \leq \tilde{a}} C(a') \right) \left( \sum_{b' \leq \tilde{b}} C(b') \right).
\]

(1)

We set $x = C(a)$, $\tilde{x} = C(\tilde{a})$, $y = C(b)$ and $\tilde{y} = C(\tilde{b})$ so Equation (1) can be written as

\[
\left( \sum_{x' \leq \tilde{x}} x' \right) \left( \sum_{y' \leq \tilde{y}} y' \right) = \left( \sum_{x' \leq x} x' \right) \left( \sum_{y' \leq y} y' \right).
\]

(2)
Now we need the new assumption of left restriction. First note that \( \hat{y} \leq y \) and \( \hat{x} \leq x \) since they are restriction and co-restriction respectively. \( S \) is left restriction so \( \hat{x} \leq x \) by Lemma 1.4. Hence, every element on the left hand side of Equation (2) appears also on the right hand side. What is left to show is that a multiplication \( x' \cdot y' \) on the right hand side of Equation (2) equals 0 unless \( x' \leq \hat{x} \) and \( y' \leq \hat{y} \). Take \( x' \leq x \) such that \( x' \not\leq \hat{x} \) and assume that there is a \( y' \leq y \) such that \( \exists x' \cdot y' \), that is, \( r(x') = d(y') \). Since \( y' \leq y \) we have \( r(x') = d(y') \leq d(y) \) by [3, CO1]. Now, by [3, EC7] (choosing \( f = d(y) \)) we have that

\[
(x' | r(x') \land d(y)) \leq (x | r(x) \land d(y))
\]

but note that \((x | r(x) \land d(y)) = \hat{x}\) and \((x' \land d(y) = r(x')\) so we get

\[x' = (x' | r(x')) \leq \hat{x}\]

a contradiction. Similarly, take \( y' \leq y \) such that \( y' \not\leq \hat{y} \) and assume that there is an \( x' \leq x \) such that \( \exists x' \cdot y' \), that is, \( r(x') = d(y') \). Again, since \( r(x') \leq r(x) \) we have that \( d(y') \leq r(x) \) and clearly \( d(y') \leq d(y) \) hence \( d(y') \leq r(x) \land d(y) = d(\hat{y}) \). By [3, EC2] there exists a restriction \((d(y') | \hat{y})\). But \((d(y') | \hat{y}) \leq \hat{y} \leq y\) so by the uniqueness of restriction \((d(y') | \hat{y}) = y'\), hence \( y' \leq \hat{y}, \) a contradiction. This finishes the proof. □

2 Consequences for the rest of [3]

[3, Corollary 3.6] should be reformulated as:

**Corollary 2.1.** Let \( S \) be an \( E \)- Ehresmann and left (or right) restriction semigroup such that \( E \) is finite, then \( \mathbb{K}S \) is a unital algebra.

[3, Example 4.4] should be deleted because our weakened theorem no longer applies. The rest of the examples in [3, Section 4] remain valid. In particular, note that the monoid \( \mathcal{P}T_n \) of [3, Example 4.3] is left restriction and that [1, 2.5.11-2.5.12] proves that every strong semilattice of monoids ([3, Example 4.5]) is a restriction semigroup.

The results of [3, Subsection 5.1] remain valid, but one can draw conclusions regarding the algebras of the Ehresmann semigroups only if they are left (or right) restriction. For instance, it can be done for \((2,1,1)\) subalgebras of \( \mathcal{P}T_n \) ([3, Example 5.10]) and finite ample semigroups (part of [3, Example 5.11]).
Likewise, the results of [3, Subsection 5.2] are now valid only under the additional assumption of left or right restriction. [3, Proposition 5.17] should be reformulated as:

**Proposition 2.2.** Let \( S \) be a finite \( E \)-Ehresmann and left (or right) restriction semigroup whose corresponding Ehresmann category \( C \) is an EI-category. Then \( \mathbb{K} \text{Reg}_E(S) \) is isomorphic to \( \mathbb{K} S/\text{Rad}(\mathbb{K} S) \).

The proof is as given in [3].

**References**

[1] Claire Cornock. *Restriction semigroups: structure, varieties and presentations*. PhD thesis, University of York, 2011.

[2] Xiaojiang Guo and Lin Chen. Semigroup algebras of finite ample semigroups. *Proc. Roy. Soc. Edinburgh Sect. A*, 142(2):371–389, 2012.

[3] Itamar Stein. Algebras of Ehresmann semigroups and categories. *Semigroup Forum*. doi:10.1007/s00233-016-9838-1, to appear.

[4] Itamar Stein. The representation theory of the monoid of all partial functions on a set and related monoids as EI-category algebras. *J. Algebra*, 450:549–569, 2016.

[5] Benjamin Steinberg. Möbius functions and semigroup representation theory. *J. Combin. Theory Ser. A*, 113(5):866–881, 2006.

[6] Shoufeng Wang. *Private Communication*. 