On the index of a vector field at an isolated singularity

W.Ebeling and S.M.Gusein-Zade

Dedicated to V.I.Arnold.

Abstract. We consider manifolds with isolated singularities, i.e., topological spaces which are manifolds (say, $C^\infty$–) outside discrete subsets (sets of singular points). For (germs of) manifolds with, so called, cone–like singularities, a notion of the index of an isolated singular point of a vector field is introduced. There is given a formula for the index of a gradient vector field on a (real) isolated complete intersection singularity. The formula is in terms of signatures of certain quadratic forms on the corresponding spaces of thimbles.

Introduction

An isolated singular point of a vector field on $\mathbb{R}^n$ or on an $n$-dimensional smooth manifold has a natural integer invariant — the index. The formula of Eisenbud, Levine and Khimshiashvili ($[8], [11]$) expresses the index of an (algebraically) isolated singular point of a vector field as the signature of a quadratic form on a local algebra associated with the singular point. For a singular point of a gradient vector field there is a formula in terms of signatures of certain quadratic forms defined by the action of the complex conjugation on the corresponding Milnor lattice ($[10], [14]$).

We define a generalisation of the notion of the index of an isolated singular point of a vector field on a manifold with isolated cone-like singularities in such a way that the Poincaré–Hopf theorem (the sum of indices of singular points of a vector field on a closed manifold is equal to its Euler characteristic) holds. It seems that this notion (though very natural) cannot be found in the literature in an explicit form. In particular, the index is defined for vector fields on a germ of a real algebraic variety with an isolated singularity. We give a generalisation of a formula from $[10]$ for the gradient vector field on an isolated complete intersection singularity. For that we define (in a somewhat formal way) the notion of the variation operator for a complete intersection singularity. We show that this operator is an invariant of the singularity.

1991 Mathematics Subject Classification. Primary 57R25, 32S30.
Partially supported by INTAS–96–0713, DFG 436 RUS 17/147/96, and RFBR 96–15–96043.
1. Basic definitions

A manifold with isolated singularities is a topological space $M$ which has the structure of a smooth (say, $C^\infty$-) manifold outside of a discrete set $S$ (the set of singular points of $M$). A diffeomorphism between two such manifolds is a homeomorphism which sends the set of singular points onto the set of singular points and is a diffeomorphism outside of them. We say that $M$ has a cone-like singularity at a (singular) point $P \in S$ if there exists a neighbourhood of the point $P$ diffeomorphic to the cone over a smooth manifold $W_P$ ($W_P$ is called the link of the point $P$). In what follows we assume all manifolds to have only cone-like singularities.

A (smooth or continuous) vector field on a manifold $M$ with isolated singularities is a (smooth or continuous) vector field on the set $M \setminus S$ of regular points of $M$.

The set of singular points $S_X$ of a vector field $X$ on a (singular) manifold $M$ is the union of the set of usual singular points of $X$ on $M \setminus S$ (i.e., points at which $X$ tends to zero) and of the set $S$ of singular points of $M$ itself.

For an isolated usual singular point $P$ of a vector field $X$ there is defined its index $\text{ind}_P X$ (the degree of the map $X/\|X\|: \partial B \to S^{n-1}$ of the boundary of a small ball $B$ centred at the point $P$ in a coordinate neighbourhood of $P$; $n = \dim M$). If the manifold $M$ is closed and has no singularities ($S = \emptyset$) and the vector field $X$ on $M$ has only isolated singularities, then

\[ \sum_{P \in S_X} \text{ind}_P X = \chi(M) \]

($\chi(M)$ is the Euler characteristic of $M$).

Let $(M, P)$ be a cone-like singularity (i.e., a germ of a manifold with such a singular point) and let $X$ be a vector field defined on an open neighbourhood $U$ of the point $P$. Suppose that $X$ has no singular points on $U \setminus \{P\}$. Let $V$ be a closed cone-like neighbourhood of $P$ in $U$ ($V \cong CW_P, V \subset U$). On the cone $CW_P = (I \times W_P)/(\{0\} \times W_P)$ ($I = [0,1]$) there is defined a natural vector field $\partial/\partial t$ ($t$ is the coordinate on $I$). Let $X_{rad}$ be the corresponding vector field on $V$. Let $\bar{X}$ be a smooth vector field on $U$ which coincides with $X$ near the boundary $\partial U$ of the neighbourhood $U$ and with $X_{rad}$ on $V$ and has only isolated singular points.

**Definition 1.1.** The index $\text{ind}_P X$ of the vector field $X$ at the point $P$ is equal to

\[ 1 + \sum_{Q \in \bar{S}_X \setminus \{P\}} \text{ind}_Q \bar{X} \]

(the sum is over all singular points $Q$ of $\bar{X}$ except $P$ itself).

For a cone-like singularity at a point $P \in S$, the link $W_P$ and thus the cone structure of a neighbourhood are, generally speaking, not well-defined (cones over different manifolds may be locally diffeomorphic). However it is not difficult to show that the index $\text{ind}_P X$ does not depend on the choice of a cone structure on a neighbourhood and on the choice of the vector field $\bar{X}$.

**Example 1.2.** The index of the “radial” vector field $X_{rad}$ is equal to 1. The index of the vector field $(-X_{rad})$ is equal to $1 - \chi(W_P)$ where $W_P$ is the link of the singular point $P$.

**Proposition 1.3.** For a vector field $X$ with isolated singular points on a closed manifold $M$ with isolated singularities, the relation (1.1) holds.
ON THE INDEX OF A VECTOR FIELD AT AN ISOLATED SINGULARITY

Definition 1.4. One says that a singular point $P$ of a manifold $M$ (locally diffeomorphic to the cone $CW_P$ over a manifold $W_P$) is smoothable if $W_P$ is the boundary of a smooth compact manifold.

The class of smoothable singularities includes, in particular, the class of (real) isolated complete intersection singularities. For such a singularity, there is a distinguished cone–like structure on its neighbourhood.

Let $(M, P)$ be a smoothable singularity (i.e., a germ of a manifold with such a singular point) and let $X$ be a vector field on $(M, P)$ with an isolated singular point at $P$. Let $V = CW_P$ be a closed cone–like neighbourhood of the point $P$; $X$ is supposed to have no singular points on $V \setminus \{P\}$. Let the link $W_P$ of the point $P$ be the boundary of a compact manifold $\tilde{V}_P$. Using a smoothing one can consider the union $\tilde{V}_P \cup W_P$ of $\tilde{V}_P$ and $W_P \times [1/2, 1]$ with the natural identification of $\partial \tilde{V}_P = W_P$ with $W_P \times \{1/2\}$ as a smooth manifold (with the boundary $W_P \times \{1\}$). The restriction of the vector field $X$ to $W_P \times [1/2, 1] \subset CW_P$ can be extended to a smooth vector field $\tilde{X}$ on $\tilde{V}_P \cup W_P$ with isolated singular points.

Proposition 1.5. The index $\text{ind}_P X$ of the vector field $X$ at the point $P$ is equal to

$$\sum_{Q \in S_{\tilde{X}}} \text{ind}_Q \tilde{X} - \chi(\tilde{V}_P) + 1$$

(the sum is over all singular points of $\tilde{X}$ on $\tilde{V}_P$).

Remark 1.6. In [13], [9] there was defined a notion of the index of a vector field at an isolated singular point of a complex variety (satisfying some conditions). That definition does not coincide with the one given here. These definitions differ by the Euler characteristic of the smoothing of the singularity of the variety. One can say that the index of [13], [9] depends on the Euler characteristic of a smoothing and thus is well-defined only for a singularity with well-defined topological type of a smoothing (at least with well-defined Euler characteristic of it). It is valid, e.g., for complex isolated complete intersection singularities.

A closely related notion has been discussed in [6]. That notion can be considered as a relative version of the index defined here. After a previous version of this paper had been submitted and put on the Duke preprint server as alg-geom/9710008, the authors’ attention was drawn to the preprint [4], where a somewhat more general notion is defined, which coincides with the index considered here for real analytic varieties with isolated singularities.

A generic (smooth or continuous) vector field on a (singular) analytic variety has zeroes only at isolated points. Thus it is desirable to have a definition of the index of such a point. One can use the following definition.

Let $(V, 0) \subset (\mathbb{R}^N, 0)$ be a germ of a real algebraic variety and let $X$ be a continuous vector field on $(V, 0)$ (i.e., the restriction of a continuous vector field on $(\mathbb{R}^N, 0)$ tangent to $V$ at each point) which has an isolated zero at the origin (in $V$). Let $S = \{\Xi\}$ be a semianalytic Whitney stratification of $V$ such that its only zero-dimensional stratum $\Xi^0$ consists of the origin. Let $\Xi$ be a stratum of the stratification $S$ and let $Q$ be a point of $\Xi$. A neighbourhood of the point $Q$ in $V$ is diffeomorphic to the direct product of a linear space $\mathbb{R}^k$ (the dimension $k$ of which is equal to the dimension of the stratum $\Xi$) and the cone $CW_Q$ over a compact
singular analytic variety \( W_Q \). (A diffeomorphism between two stratified spaces is a homeomorphism which is a diffeomorphism on each stratum.) In particular a neighbourhood \( U(0) \) of the origin is diffeomorphic to the cone \( CW_0 \) over a singular variety \( W_0 \). It is not difficult to show that there exists a (continuous) vector field \( \tilde{X} \) on \( (V, 0) \) such that:

1. the vector field \( \tilde{X} \) is defined on the neighbourhood \( U(0) \cong CW_0 \) of the origin;
2. \( \tilde{X} \) coincides with the vector field \( X \) in a neighbourhood of the base \( \{1\} \times W_0 \) of the cone \( CW_0 \);
3. the vector field \( \tilde{X} \) has only a finite number of zeroes;
4. each point \( Q \in U(0) \) with \( \tilde{X}(Q) = 0 \) has a neighbourhood diffeomorphic to \( (\mathbb{R}^k, 0) \times CW_Q \) in which \( \tilde{X}(y, z) \) \((y \in \mathbb{R}^k, z \in CW_q)\) is of the form \( Y(y) + Z_{rad}(z) \), where \( Y \) is a germ of a vector field on \( (\mathbb{R}^k, 0) \) with an isolated singular point at the origin, \( Z_{rad} \) is the radial vector field on the cone \( CW_Q \).

Let \( S_{\tilde{X}} \) be the set of zeroes of the vector field \( \tilde{X} \) (including the origin). For a point \( Q \in S_{\tilde{X}} \), let \( \text{ind}(Q) := \text{ind}_0 Y \), where \( Y \) is the vector field on \( (\mathbb{R}^k, 0) \) described above. We define \( \text{ind}(0) \) to be equal to 1 (in this case \( k = 0 \)).

**Definition 1.7.** \( \text{ind}_{(V, 0)} X = \sum_{Q \in S_{\tilde{X}}} \text{ind}(Q) \).

2. On the topology of isolated complete intersection singularities

Let \((V, 0) \subset (\mathbb{C}^{n+p}, 0) \) be an \((n - 1)\)-dimensional isolated complete intersection singularity (abbreviated icis in the sequel) defined by a germ of an analytic mapping

\[ f = (f_1, \ldots, f_{p+1}) : (\mathbb{C}^{n+p}, 0) \to (\mathbb{C}^{p+1}, 0). \]

(We use somewhat strange notations for the dimension and the number of equations in order to be consistent with the notations in Section 3.) For \( \delta > 0 \), let \( B_\delta \) be the ball of radius \( \delta \) around the origin in \( \mathbb{C}^{n+p} \). For \( \delta > 0 \) small enough and for a generic \( t \in \mathbb{C}^{p+1} \) with \( 0 < \|t\| < < \delta \), the set

\[ V_t = f^{-1}(t) \cap B_\delta \]

is a manifold with boundary and is called a Milnor fibre of the icis \((V, 0)\) (or of the germ \( f \)). The diffeomorphism type of \( V_t \) does not depend on \( t \). The manifold \( V_t \) is homotopy equivalent to the bouquet of \( \mu \) spheres of dimension \((n - 1)\), where \( \mu \) is the Milnor number of the icis \((V, 0)\).

For \( t \in \mathbb{C}^{p+1}, 0 \leq i \leq p \) we define

\[ (V^{(i)}, 0) := (\{x \in B_\delta : f_1(x) = \ldots = f_{p-i+1}(x) = 0\}, 0), \]

\[ V^{(i)}_t := \{x \in B_\delta : f_j(x) = t_j, 1 \leq j \leq p - i + 1\} \]

and we set \((V^{(p+1)}, 0) := (\mathbb{C}^{n+p}, 0) \). We assume that \((f_1, \ldots, f_{p+1})\) is a system of functions such that for \( 0 \leq i \leq p \) the germ \((V^{(i)}, 0)\) is an \((n + i - 1)\)-dimensional icis. For any \( t \in \mathbb{C}^{p+1} \) with \( 0 < |t_1| < < |t_2| < < \ldots < < |t_{p+1}| < < \delta \), the set \( V^{(i)}_t \) is the Milnor fibre of the icis \((V^{(i)}, 0)\). Here the condition \( 0 < |t_1| < < |t_2| < < \ldots < < |t_{p+1}| \) means that \( t_1, \ldots, t_{p+1} \) have to be chosen in such a way that for each
and skew-symmetric if \( n \) is odd and skew-symmetric if \( n \) is even. The form on \( \hat{H}^{(i)} \) is symmetric if \( n \) is even and skew-symmetric if \( n \) is odd. The form on \( \hat{H}^{(i)} \) is symmetric if \( n + i \) is odd and skew-symmetric if \( n + i \) is even.

Denote by \( \hat{H}^* \) the dual module of \( \hat{H} = H_n(\nu', V) \) and let \( \langle \cdot, \cdot \rangle : \hat{H}^* \times \hat{H} \to \mathbb{Z} \) be the Kronecker pairing. We want to define a variation operator \( \text{Var} : \hat{H}^* \to \hat{H} \) or rather its inverse \( \text{Var}^{-1} : \hat{H} \to \hat{H}^* \). For this purpose we need the notion of a distinguished basis of thimbles.

Let \( \tilde{f}_{p+1} : V'_t \to \mathbb{C} \) be a generic perturbation of the restriction of the function \( f_{p+1} \) to \( V'_t \) which has only non-degenerate critical points with different critical values \( z_1, \ldots, z_\nu \) (\( \nu = \nu_0 \)). Let \( z_0 \) be a non-critical value of \( \tilde{f}_{p+1} \) with \( \|z_0\| > \|z_j\| \) for \( j = 1, \ldots, \nu \). The level set \( \{ x \in V'_t : \tilde{f}_{p+1}(x) = z_0 \} \) is diffeomorphic to the Milnor fibre \( V_t \) of the icis \((V, 0)\). Let \( u_j, j = 1, \ldots, \nu \), be non-self-intersecting paths connecting the critical values \( z_j \) with the non-critical value \( z_0 \) in such a way that they lie inside the disc \( D_{\|z_0\|} = \{ z \in \mathbb{C} : \|z\| \leq \|z_0\| \} \) and every two of them intersect each other only at the point \( z_0 \). We suppose that the paths \( u_j \) (and correspondingly the critical values \( z_j \)) are numbered clockwise according to the order in which they arrive at \( z_0 \) starting from the boundary of the disc \( D_{\|z_0\|} \). Each path \( u_j \) defines up to orientation a thimble \( \hat{\delta}_j \) in the relative homology group \( \hat{H} \). The system \( \{ \hat{\delta}_1, \ldots, \hat{\delta}_\nu \} \) is a basis of \( \hat{H} \). A basis obtained in this way is called distinguished. The self-intersection number of a thimble \( \hat{\delta} \) is equal to

\[
\langle \hat{\delta}, \hat{\delta} \rangle = (-1)^{n(n-1)/2}(1 + (-1)^{n-1}).
\]
The Picard-Lefschetz transformation $h_\delta : \hat{H} \to \hat{H}$ corresponding to the thimble $\hat{\delta}$ is given by (cf. 7)

$$h_\delta(y) = y + (-1)^{n(n+1)/2}(y, \hat{\delta}) \hat{\delta} \quad \text{for } y \in \hat{H}.\]

Going once around the disc $D_{\|z_0\|}$ in the positive direction (counterclockwise) along the boundary induces an automorphism of $\hat{H}$, the (classical) monodromy operator $h_\alpha$. If $\{\hat{\delta}_1, \ldots, \hat{\delta}_\nu\}$ is a distinguished basis of $\hat{H}$, then the monodromy operator is given by

$$h_\alpha = h_{\hat{\delta}_1} \circ h_{\hat{\delta}_2} \circ \cdots \circ h_{\hat{\delta}_\nu}.$$

**Definition 2.1.** Let $\{\hat{\delta}_1, \ldots, \hat{\delta}_\nu\}$ be a distinguished basis of thimbles of $\hat{H}$ and let $\{\nabla_1', \ldots, \nabla_\nu'\}$ be the corresponding dual basis of $\hat{H}^*$. The linear operator $\text{Var}^{-1} : \hat{H} \to \hat{H}^*$ (inverse of the variation operator) is defined by

$$\text{Var}^{-1}(\hat{\delta}_i) = (-1)^{n(n+1)/2}\nabla_i - \sum_{j<i} (\hat{\delta}_i, \hat{\delta}_j) \nabla_j.$$

**Proposition 2.2.** The definition of the operator $\text{Var}^{-1}$ does not depend on the choice of the distinguished basis.

**Proof.** Any two distinguished bases of thimbles can be transformed into each other by the braid group transformations $\alpha_j$, $j = 1, \ldots, \nu - 1$, and by changes of orientations (see, e.g., [3, 7]). Here the operation $\alpha_j$ is defined as follows:

$$\alpha_j(\hat{\delta}_1, \ldots, \hat{\delta}_\nu) = (\hat{\delta}'_1, \ldots, \hat{\delta}'_\nu)$$

where $\hat{\delta}'_i = h_{\hat{\delta}_j}(\hat{\delta}_{j+1}) = \hat{\delta}_{j+1} + (-1)^{n(n+1)/2}(\hat{\delta}_{j+1}, \hat{\delta}_j) \hat{\delta}_j$, $\hat{\delta}'_{j+1} = \hat{\delta}_j$, and $\hat{\delta}'_i = \hat{\delta}_i$ for $i \neq j, j + 1$.

It is easily seen that the definition of $\text{Var}^{-1}$ is invariant under a change of orientation. Therefore it suffices to show that the definition of $\text{Var}^{-1}$ is invariant under the transformation $\alpha_j$. One easily computes:

\[
\begin{align*}
\langle \hat{\delta}'_r, \hat{\delta}'_s \rangle &= \langle \hat{\delta}_r, \hat{\delta}_s \rangle \quad \text{for } 1 \leq r, s \leq \nu, \ r, s \neq j, j + 1, \\
\langle \hat{\delta}'_j, \hat{\delta}'_{j+1} \rangle &= -\langle \hat{\delta}_j, \hat{\delta}_{j+1} \rangle, \\
\langle \hat{\delta}'_r, \hat{\delta}'_j \rangle &= \langle \hat{\delta}_r, \hat{\delta}_{j+1} \rangle + (-1)^{n(n+1)/2}\langle \hat{\delta}_{j+1}, \hat{\delta}_j \rangle \hat{\delta}_j, \\
\langle \hat{\delta}'_r, \hat{\delta}'_{j+1} \rangle &= \langle \hat{\delta}_r, \hat{\delta}_j \rangle \quad \text{for } r \neq j, j + 1.
\end{align*}
\]

Let $(\nabla'_1, \ldots, \nabla'_\nu)$ be the dual basis corresponding to $(\hat{\delta}_1', \ldots, \hat{\delta}_\nu')$. Then

$$\nabla'_{j+1} = \nabla'_j$$

$$\nabla_j = \nabla'_{j+1} + (-1)^{n(n+1)/2}\langle \hat{\delta}_{j+1}, \hat{\delta}_j \rangle \nabla'_j.$$
The corresponding formula for \( i < j \)

\[
\text{Var}^{-1}(\hat{\delta}_j) = \text{Var}^{-1}(\hat{\delta}_{j+1}) + (-1)^{n(n+1)/2}(\hat{\delta}_{j+1}, \hat{\delta}_j)\text{Var}^{-1}(\hat{\delta}_j)
\]

\[
= (-1)^{n(n+1)/2} \nabla_{j+1} - \sum_{k<j+1} (\hat{\delta}_{j+1}, \hat{\delta}_k) \nabla_k
\]

\[
\quad + (-1)^{n(n+1)/2}(\hat{\delta}_{j+1}, \hat{\delta}_j)(((-1)^{n(n+1)/2} \nabla_j - \sum_{k<j} (\hat{\delta}_j, \hat{\delta}_k) \nabla_k)
\]

\[
= (-1)^{n(n+1)/2} \nabla_{j+1}
\]

\[
- \sum_{k<j} ((\hat{\delta}_{j+1}, \hat{\delta}_k) + (-1)^{n(n+1)/2}(\hat{\delta}_{j+1}, \hat{\delta}_j)(\hat{\delta}_j, \hat{\delta}_k)) \nabla_k
\]

\[
= (-1)^{n(n+1)/2} \nabla_j - \sum_{k<j} (\hat{\delta}_j, \hat{\delta}_k) \nabla_k,
\]

For \( i > j + 1 \) we have

\[
\text{Var}^{-1}(\hat{\delta}_i) = (-1)^{n(n+1)/2} \nabla_i - \sum_{k<i} (\hat{\delta}_i, \hat{\delta}_k) \nabla_k
\]

\[
= (-1)^{n(n+1)/2} \nabla_i - \sum_{k<i, k \neq j+1} (\hat{\delta}_i, \hat{\delta}_k) \nabla_k - (\hat{\delta}_i, \hat{\delta}_{j+1}) \nabla_{j+1} - (\hat{\delta}_i, \hat{\delta}_j) \nabla_j
\]

\[
= (-1)^{n(n+1)/2} \nabla_i - \sum_{k<i, k \neq j+1} (\hat{\delta}_i, \hat{\delta}_k) \nabla_k
\]

\[
- (\hat{\delta}_i, \hat{\delta}_j) \nabla_j + (-1)^{n(n+1)/2}(\hat{\delta}_{j+1}, \hat{\delta}_j)(\hat{\delta}_j, \hat{\delta}_j) \nabla_j
\]

\[
- (\hat{\delta}_i, \hat{\delta}_{j+1}) \nabla_{j+1} - (-1)^{n(n+1)/2}(\hat{\delta}_{j+1}, \hat{\delta}_j)(\hat{\delta}_i, \hat{\delta}_{j+1}) \nabla_j
\]

\[
= (-1)^{n(n+1)/2} \nabla_i - \sum_{k<i} (\hat{\delta}_i, \hat{\delta}_k) \nabla_k
\]

The corresponding formula for \( i < j \) is obvious. 

**Remark 2.3.** There is an interesting problem to give an invariant (topological) definition of the variation operator.

Let \( S : \hat{H} \to \hat{H}^* \) be the mapping defined by the intersection form on \( \hat{H} \):

\[
(Sx, y) = \langle x, y \rangle, \quad x, y \in \hat{H}.
\]

The mapping \( \text{Var}^{-1} \) is defined in such a way that one has the equality \( S = -\text{Var}^{-1} + (-1)^n(\text{Var}^{-1})^T \) where \((\text{Var}^{-1})^T \) means the transpose operator \((\text{Var}^{-1})^T : \hat{H}^* \to \hat{H}^*\).
3. The index of the gradient vector field on an isolated complete intersection singularity

Let \( (V', 0) = \{ f_1 = f_2 = \ldots = f_p = 0 \} \subset (\mathbb{C}^{n+p}, 0) \) be a real \( n \)-dimensional icis (it means that the function germs \( f_i : (\mathbb{C}^{n+p}, 0) \to (\mathbb{C}, 0) \) are real). We assume that its real part \( V' \cap \mathbb{R}^{n+p} \) does not coincide with the origin (and thus is \( n \)-dimensional). Let \( g = f_{p+1} : (\mathbb{C}^{n+p}, 0) \to (\mathbb{C}, 0) \) be a germ of a real analytic function such that its restriction to \( V' \setminus \{ 0 \} \) has no critical points. A Riemannian metric on \( \mathbb{R}^{n+p} \) determines the gradient vector field \( X = \nabla g \) of the restriction of the function \( g \) to \( (V' \cap \mathbb{R}^{n+p}) \setminus \{ 0 \} \). This vector field has no singular points on a punctured neighbourhood of the origin in \( V' \cap \mathbb{R}^{n+p} \). Since the space of Riemannian metrics is connected, the index \( \text{ind}_0 X \) of the gradient vector field doesn’t depend on the choice of a metric. In the case \( p = 0 \) (and thus \( V' = \mathbb{C}^n \)) the index of the gradient vector field of a function germ \( g \) can be expressed in terms of the action of the complex conjugation on the Milnor lattice of the singularity \( g \) \( (10, 14) \). We give a generalisation of such a formula for icis.

Let \( 0 < \varepsilon_1 < \varepsilon_2 < \ldots < \varepsilon_{p+1} \) be real and small enough, let \( s = (s_1, \ldots, s_{p+1}) \) with \( s_i = \pm 1 \). For \( 0 \leq i \leq p \), let \( \hat{H}^{(i)} = H_{sz}^{(i)} \) be the corresponding space of thimbles: \( \hat{H}^{(i)} = H_{n+1}(V^{(i)}_{sz} \setminus V^{(i)}_{sz}) \) for \( 0 \leq i \leq p - 1 \) (\( s_\varepsilon = (s_1 \varepsilon_1, s_2 \varepsilon_2, \ldots, s_{p+1} \varepsilon_{p+1}) \)); see Section 2 for \( i = p \). Let \( \sigma_s^{(i)} \) be the action of the complex conjugation on the space \( \hat{H}^{(i)} \), let \( \text{Var}_s^{-1} : \hat{H}^{(i)} \to (\hat{H}^{(i)})^* \) be the inverse of the corresponding variation operator. The operator \( \text{Var}_s^{-1} \sigma_s^{(i)} \) acts from the space \( \hat{H}^{(i)} \) to its dual \( (\hat{H}^{(i)})^* \) and thus defines a bilinear form on \( \hat{H}^{(i)} \).

Theorem 3.1. The bilinear forms \( \text{Var}_s^{-1} \sigma_s^{(i)} \) are symmetric and non-degenerate, and we have

\[
\text{ind}_0 \nabla g = s_{p+1}^p (-1)^{\frac{n(n+1)}{2}} \text{sgn} \text{Var}_s^{-1} \sigma_s + \sum_{i=1}^{p} (-1)^{\frac{n(i)(n+1)}{2}} \text{sgn} \text{Var}_s^{-1} \sigma_s^{(i)}. \tag{3.1}
\]

Corollary 3.2. The right-hand side of the equation (3.1) does not depend on \( s = (s_1, \ldots, s_{p+1}) \).
Proof. Let us consider the restriction of the function $f_i$ to the manifold $V_{s_{p-i+2}}$. It may have degenerate critical points. Let $\tilde{f}_i : V_{s_{p-i+2}} \to \mathbb{C}$ be its real morfification (i.e., a perturbation of $f_i$ which is a Morse function on $V_{s_{p-i+2}}$ and maps its real part $V_{s_{p-i+2}} \cap \mathbb{R}^{n+p}$ to $\mathbb{R}$). For $c \in \mathbb{R}$, let $M_c^{(i)} = \{ x \in V_{s_{p-i+2}} \cap \mathbb{R}^{n+p} : \tilde{f}_i \leq c \}$. The topological space $M_c^{(i)}$ is homotopy equivalent to $V_{s_{p-i+2}} \cap \mathbb{R}^{n+p}$ or to $V_{s_{p-i+1}} \cap \mathbb{R}^{n+p}$ for $c$ greater than or less than all the critical values of $\tilde{f}_i$ respectively. The standard arguments of Morse theory give

$$\chi(V_{s_{p-i+2}} \cap \mathbb{R}^{n+p}) = \chi(V_{s_{p-i+1}} \cap \mathbb{R}^{n+p}) + \sum_{Q \in S_{\text{grad } f_i}} \text{ind}_Q \text{grad } \tilde{f}_i.$$ 

Applying the same reasonings to the function $-\tilde{f}_i$ one has

$$\chi(V_{s_{p-i+2}} \cap \mathbb{R}^{n+p}) = \chi(V_{s_{p-i+1}} \cap \mathbb{R}^{n+p}) + (-1)^{n+p-i+1} \sum_{Q \in S_{\text{grad } f_i}} \text{ind}_Q \text{grad } \tilde{f}_i.$$ 

Thus

$$\chi(V_{s_{p-i+2}} \cap \mathbb{R}^{n+p}) = \chi(V_{s_{p-i+1}} \cap \mathbb{R}^{n+p}) + (-s_i)^{n+p-i+1} \sum_{Q \in S_{\text{grad } f_i}} \text{ind}_Q \text{grad } \tilde{f}_i.$$ 

(cf. [2, Lemma in §2]). From Proposition 1 one has

$$\text{ind } \text{grad } g = \sum_{Q \in S_{\text{grad } f_{p+1}}} \text{ind}_Q \text{grad } \tilde{f}_{p+1} - \chi(V_{s_{p}} \cap \mathbb{R}^{n+p}) + 1 = \sum_{Q \in S_{\text{grad } f_{p+1}}} \text{ind}_Q \text{grad } \tilde{f}_{p+1} + (-s_p)^{(n+1)} \sum_{Q \in S_{\text{grad } f_{p}}} \text{ind}_Q \text{grad } \tilde{f}_{p} - \chi(V_{s_{p+1}} \cap \mathbb{R}^{n+p}) + 1 = \ldots = \sum_{Q \in S_{\text{grad } f_{p+1}}} \text{ind}_Q \text{grad } \tilde{f}_{p+1} + \sum_{i=1}^{p} (-s_{p-i+1})^{(n+1)} \sum_{Q \in S_{\text{grad } f_{p-i+1}}} \text{ind}_Q \text{grad } \tilde{f}_{p-i+1}.$$ 

Now Theorem 3.3 follows from the following statement. 

**Theorem 3.3.**

$$\sum_{Q \in S_{\text{grad } f_{p-i+1}}} \text{ind}_Q \text{grad } \tilde{f}_{p-i+1} = (s_{p-i+1})^{n+i}(-1)^{(n+i)(n+i+1)/2} \text{sgn Var}_{s_{p-i+1}}^{(i)}.$$ 

**Proof.** Let us suppose that $s_{p-i+1} = 1$. The case $s_{p-i+1} = -1$ can be reduced to this by multiplying the function $f_{p-i+1}$ (and the function $\tilde{f}_{p-i+1}$) by $(-1)$. Let $\bar{s} = (s_1, \ldots, s_{p-i+1}, s_{p-i+2}, \ldots, s_{p+1})$. Without loss of generality we can suppose that all critical values of the function $\tilde{f}_{p-i+1}$ lie inside the circle $\{ z : \|z\| \leq \frac{s_{p-i+1}}{2} \}$ and have different real parts (except, of course, values at complex conjugate points). Let us identify the space $H_{\bar{s}}^{(i)}$ with the space $H_{\bar{s}}^{(i)}$ using a path which connects $-\varepsilon_{p-i+1}$ with $+\varepsilon_{p-i+1}$ in the upper half plane outside the circle $\{ z : \|z\| < \frac{\varepsilon_{p-i+1}}{2} \}$ (e.g., the half circle $\{ z : \|z\| = \varepsilon_{p-i+1} \}$). This identification permits to consider
\(\sigma_s^{(i)}\) and \(\sigma_s^{(i)}\) as operators on the space \(\hat{H}^{(i)} = \hat{H}_s^{(i)}\). Just as in [10] the classical monodromy operator \(h_{s_{\epsilon}}^{(i)}: \hat{H}^{(i)} \to \hat{H}^{(i)}\) can be represented in the form
\[
h_{s_{\epsilon}}^{(i)} = \sigma_s^{(i)}\sigma_s^{(i)}.
\]
A distinguished basis of the space \(\hat{H}^{(i)}\) is defined by a system of paths connecting the critical values of the function \(f_{p-i+1}\) with the non-critical value \(\epsilon_{p-i+1}\). Let us choose the following system of paths (cf. Fig. 1). The paths from real critical values go vertically upwards up to the boundary of the circle \(\{z : \|z\| \leq \frac{\epsilon_{p-i+1}}{2}\}\). The paths from complex conjugate critical values go vertically (upwards or downwards to the real axis and then go vertically upwards to the boundary of the circle \(\{z : \|z\| \leq \frac{\epsilon_{p-i+1}}{2}\}\) avoiding from the right side a neighborhood of the critical value with positive imaginary part. From the boundary of the circle \(\{z : \|z\| \leq \frac{\epsilon_{p-i+1}}{2}\}\) all the paths go to the non-critical value \(\epsilon_{p-i+1}\) in the upper half plane (see Fig. 1). The cycles are ordered in the usual way which in this case means that they follow each other in the order of decreasing real parts of the corresponding critical values; the vanishing cycle corresponding to the critical value with negative imaginary part precedes that with the positive one.

In the sequel we shall consider the matrices of the operators \(\sigma_s^{(i)}\), \(\sigma_s^{(i)}\), \(\text{Var}_{i}^{-1}\), etc. as block matrices with blocks of size \(1 \times 1, 1 \times 2, 2 \times 1,\) and \(2 \times 2\) corresponding to real critical values and to pairs of complex conjugate critical values of the function \(f_{p-i+1}\). The matrix of the operator \(\sigma_s^{(i)}\) is an upper triangular block matrix. Its diagonal entry corresponding to a real critical value is equal to \((-1)^m\) where \(m\) is the Morse index of the critical point. A diagonal block of size \(2 \times 2\) corresponding to a pair of complex conjugate critical values is equal to \[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}.
\]
The matrix of the operator \(\sigma_s^{(i)}\) is a lower triangular block matrix (we do not need a precise description of its diagonal blocks). The matrix of the operator \(\text{Var}_{i}^{-1}\) is upper triangular with diagonal entries equal to \((-1)^{(n+i)(n+i+1)/2}\) (the
dual of the space $\tilde{H}^{(i)}$ is endowed with the basis dual to the one of $\tilde{H}^{(i)}$. One has $h^{(i)} = \sigma^{(i)}$, $\tilde{\sigma}^{(i)} = (-1)^{n+i} \text{Var}_1(\text{Var}_i)^T$. Thus $\text{Var}_i^{-1} \sigma^{(i)} = (-1)^{n+i}(\text{Var}_i^{-1})^T \tilde{\sigma}^{(i)}$. The matrices $\text{Var}_i^{-1} \sigma^{(i)}$ and $(\text{Var}_i^{-1})^T \tilde{\sigma}^{(i)}$ are upper triangular and lower triangular respectively. Thus the matrix $\text{Var}_i^{-1} \sigma^{(i)}$ is in fact block diagonal with the diagonal entry $(-1)^{[(n+i)(n+i+1)/2]+m}$ corresponding to a real critical point of the function $\tilde{f}_{p-i+1}$ ($m$ is the Morse index) and with the diagonal block of the form

$$(-1)^{\frac{(n+i)(n+i+1)}{2}} \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}$$

corresponding to a pair of complex conjugate critical points (up to a sign $a$ is the intersection number of the corresponding cycles). This description implies Theorem 3.3. □

The formula (3.1) expresses the index of a gradient vector field in terms of bilinear forms on the spaces of thimbles. Actually each second summand of it can be expressed in terms of bilinear forms on the corresponding spaces of vanishing cycles. Let $\Sigma_2^{(i)}$ be the quadratic form on the space $H_{n+i-1}^{(i)}(V^{(i)}_{\infty})$ of vanishing cycles defined by $\Sigma_2^{(i)}(x, y) = \langle \sigma^{(i)} x, y \rangle$. As above, let $\tilde{s} = (s_1, \ldots, s_{p-i}, -1, s_{p-i+2}, \ldots, s_{p+1})$.

**Theorem 3.4.** For $n + i$ odd

$$\sum_{Q \in S_{\text{grad}} \tilde{f}_{p-i+1}} \text{ind}_{\text{grad}} \tilde{f}_{p-i+1} = s_{p-i+1} (n+i+1)/2 - \text{sgn} \Sigma_2^{(i)} - \text{sgn} \Sigma_2^{(i)}/2.$$

**Proof.** It is essentially the same as in [14]. One has to notice that the kernel of the natural (boundary) homomorphism $\tilde{H}^{(i)}_{\infty} \to H_{n+i-1}^{(i)}(V^{(i)}_{\infty})$ is contained in the kernel of the quadratic form $\Sigma_2^{(i)}$ and thus $\text{sgn} \Sigma_2^{(i)}$ coincides with the signature of the form $\langle \sigma^{(i)} \cdot, \cdot \rangle$ on the space $H_{\infty}^{(i)}$. □

**Remark 3.5.** For $n + i$ even, it is not possible to express the number

$$\sum_{Q \in S_{\text{grad}} \tilde{f}_{p-i+1}} \text{ind}_{\text{grad}} \tilde{f}_{p-i+1}$$

in terms of invariants defined by the space of vanishing cycles. It can be understood from the following example. Let $n = 2$, $p = 1$, $f_1 = x_1^2 + x_2^2 - x_3^2$, $f_2 = x_3$, $i = 0$. The discussed sum is different for $s_1 = 1$ and for $s_1 = -1$ (i.e., for $t_1 = s_1 \in \mathbb{Z}$ positive or negative). However the line $\ell = \{t_1 = 0\}$ is not in the bifurcation set for vanishing cycles: it doesn’t lie in the discriminant of the map $(f_1, f_2) : (\mathbb{C}^3, 0) \to (\mathbb{C}^2, 0)$. On the other hand the discriminant of the map $f_1 : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ coincides with the origin $0 \in \mathbb{C}$ and thus the line $\ell$ is in the bifurcation set for thimbles.

**Remark 3.6.** It seems that a relation between complex conjugation and the monodromy operator (similar to (3.2)) was first used in [8]. In [8] it was written in an explicit way. In [12] a similar relation was used to find some properties of the Euler characteristics of links of complete intersection varieties. This paper partially inspired our work.
References

[1] A’Campo, N. [1975] Le groupe de monodromie du déploiement des singularités isolées de courbes planes I, Math. Ann. 213, 1–32.

[2] Arnold, V.I. [1978] The index of a singular point of a vector field, the inequalities of Petrovskii-Oleinik and the mixed Hodge structures, Funct. Anal. and Appl. 12, 1–11.

[3] Arnold, V.I., Gusein-Zade, S.M., Varchenko, A.N. [1988] Singularities of Differentiable Maps, Vol. II, Birkhäuser, Boston–Basel–Berlin.

[4] Aguilar, M.A., Seade, J.A., Verjovsky, A. [1997] Indices of vector fields and topological invariants of real analytic singularities. Preprint.

[5] Bourbaki, N. [1968] Groupes et Algèbres de Lie. Chaps. 4, 5 et 6, Hermann, Paris.

[6] Bonatti, Ch., Gómez-Mont, X. [1994] The index of holomorphic vector fields on singular varieties I, Astérisque 222, 9–35.

[7] Ebeling, W. [1987] The Monodromy Groups of Isolated Singularities of Complete Intersections, Lecture Notes in Mathematics, Vol. 1293, Springer-Verlag, Berlin etc.

[8] Eisenbud, D., Levine, H. [1977] An algebraic formula for the degree of a $C^\infty$ map germ, Ann. Math. 106, 19–38.

[9] Gómez-Mont, X., Seade, J., Verjovsky, A. [1991] The index of a holomorphic flow with an isolated singularity, Math. Ann. 291, 737–751.

[10] Gusein-Zade, S.M. [1984] The index of a singular point of a gradient vector field, Funct. Anal. and Appl. 18, 6–10.

[11] Khimshiashvili, G.N. [1977] On the local degree of a smooth map, Comm. Acad. Sci. Georgian SSR. 85, no. 2, 309–311 (in Russian).

[12] McCrory, C., Parusiński, A. [1997] Complex monodromy and the topology of real algebraic sets, Compositio Math. 106, 211–233.

[13] Seade, J. [1976] The index of a vector field on a complex analytic surface with singularities, Contemp. Math. 37, 253–262.

[14] Varchenko, A.N. [1985] Local residue and the intersection form in vanishing cohomology, Izv. Akad. Nauk SSSR Ser. Mat. 49, no. 1, 52–54 (in Russian; English translation in: Math. USSR Izvestiya 26 (1986), 31–52).

Institut für Mathematik, Universität Hannover, Postfach 6009, D-30060 Hannover, Germany, e-mail: ebeling@math.uni-hannover.de

Department of Mathematics and Mechanics, Moscow State University, Moscow, 119899, Russia, e-mail: sabir@ium.ips.ras.ru