THE HUNEKE–WIEGAND CONJECTURE AND MIDDLE TERMS OF ALMOST SPLIT SEQUENCES

TOSHINORI KOBAYASHI

Abstract. Let \( R \) be a Gorenstein local domain of dimension one. We show that a nonfree maximal Cohen–Macaulay \( R \)-module \( M \) possessing more than one nonfree indecomposable summand in the middle term of the almost split sequence ending in \( M \) has a nonvanishing self extension. In other words, we show that the Huneke–Wiegand conjecture is affirmative for such \( R \)-modules \( M \).

1. Introduction

In this paper, we study the following conjecture of Huneke and Wiegand; see \[10\] the discussion following the proof of 5.2.

Conjecture 1.1 (Huneke and Wiegand \[10\]). Let \( R \) be a Gorenstein local domain of dimension one. Let \( M \) be a maximal Cohen–Macaulay \( R \)-module. If \( M \otimes_R \text{Hom}_R(M, R) \) is torsion-free, then \( M \) is free.

Huneke and Wiegand \[10\] showed that this conjecture is true for hypersurfaces. Many other partial answers are known \[1, 2, 4, 5, 8, 12\], but, the conjecture is still open in general. Let \( R \) be a Gorenstein local domain of dimension one. A finitely generated \( R \)-module is torsion-free if and only if it is reflexive if and only if it is maximal Cohen–Macaulay. Therefore Conjecture 1.1 implies the Auslander–Reiten conjecture for Gorenstein local domains (\[3\] Proposition 5.10). Assume that \( M \) is a torsion-free \( R \)-module. Then it is remarkable that the torsion-freeness of \( M \otimes_R \text{Hom}_R(M, R) \) is equivalent to saying that \( \text{Ext}^1_R(M, M) \) is zero; see \[9\] Theorem 5.9.

The main result of this paper is the following.

Theorem 1.2. Let \((R, m)\) be a Gorenstein local domain of dimension one. Let \( M \) be a nonfree indecomposable torsion-free \( R \)-module. Assume that the number of indecomposable summand in the middle term of the Auslander–Reiten sequence ending in \( M \) is greater than one. Then one has \( \text{Ext}^1_R(M, M) \neq 0 \). Hence, Conjecture 1.1 holds true for \( M \).

Remark that Roy \[12\] showed that for one-dimensional graded complete intersections \( R \) satisfying some condition on the \( a \)-invariant, the assertion of Theorem 1.2 holds. Our result is local (not graded), and we do not assume that the ring is a complete intersection.

In section 2, we give some preliminaries. In section 3, the proof of Theorem 1.2 is given.

2. Irreducible Homomorphisms and Almost Split Sequences

In this section, we prove lemmas needed to prove the main theorem. In the rest of this paper, let \((R, m)\) be a commutative Gorenstein henselian local ring, and all modules are finitely generated, unless otherwise stated. We denote by \( \text{CM}_0(R) \) the category of maximal Cohen–Macaulay \( R \)-modules \( M \) such that \( M_p \) is \( R_p \)-free for any nonmaximal prime ideal \( p \) of \( R \). For

2010 Mathematics Subject Classification. 13C60, 13H10, 16G70.

Key words and phrases. Huneke–Wiegand conjecture, almost split sequence, Cohen–Macaulay ring, Gorenstein ring, maximal Cohen–Macaulay module.

The author was partly supported JSPS Grant-in-Aid for JSPS Fellows 18J20660.
an $R$-module $M$, $\Omega M$ (resp. $\Omega^i M$) denotes the first (resp. $i$-th) syzygy module in the minimal free resolution of $M$.

For $R$-modules $M$ and $N$, let $\text{Hom}_R(M, N)$ denote the quotient of $\text{Hom}(M, N)$ by the set of homomorphisms from $M$ to $N$ factoring through a free $R$-module. Since $R$ is Gorenstein, the stable category $\text{CM}_0(R)$ of $\text{CM}_0(R)$ is a triangulated category. Its morphism set is equal to the stable homset $\text{Hom}_R(-, -)$ and its shift functor is the functor taking $\Omega$; see [4, Chapter 1] for instance. Hence we obtain the following lemma.

**Lemma 2.1.** Let $M, N$ be $R$-modules in $\text{CM}_0(R)$. Then we have the following isomorphisms.

1. $\text{Hom}_R(\Omega M, N) \cong \text{Ext}_R^1(M, N)$,
2. $\text{Ext}_R^1(M, N) \cong \text{Ext}_R^1(\Omega M, \Omega N)$,
3. $\text{Hom}_R(M, N) \cong \text{Hom}_R(\Omega M, \Omega N)$.

On the set $\text{Hom}_R(M, N)$, we also use the following lemma.

**Lemma 2.2.** Let $M, N$ be $R$-modules having no free summands and $f : M \to N$ be a homomorphism factoring through a free $R$-module. Then the image $\text{Im } f$ of $f$ is contained in $mN$.

**Proof.** Write $f = hg$ where $g : M \to F$ and $h : F \to N$ are homomorphisms with a free $R$-module $F$. Since $M$ has no free summands, $\text{Im } g$ is contained in $mF$. Hence $\text{Im } f \subseteq (h(mF)) \subseteq mN$. $
$

Recall that a homomorphism $f : X \to Y$ of $R$-modules is said to be **irreducible** if it is neither a split monomorphism nor a split epimorphism, and for any pair of morphisms $g$ and $h$ such that $f = gh$, either $g$ is a split epimorphism or $h$ a split monomorphism.

**Lemma 2.3.** Let $M, N$ be $R$-modules having no free summands and $f, g : M \to N$ be homomorphisms. Assume that $g$ factors through a free $R$-module. Then

1. $f$ is an isomorphism if and only if so is $f + g$,
2. $f$ is a split epimorphism if and only if so is $f + g$,
3. $f$ is a split monomorphism if and only if so is $f + g$.
4. $f$ is irreducible if and only if so is $f + g$.

**Proof.** We only need to show one direction; we can view $f$ as $(f + g) - g$.

1. Assume that $f$ is an isomorphism with an inverse homomorphism $h : N \to M$. Then the composite homomorphisms $gh$ factor through some free $R$-modules. It follows from Lemma 2.2 that there are inclusions $\text{Im } gh \subseteq mM$. By Nakayama’s lemma, we see that $(f + g)h$ is a surjective endomorphism of $M$, and hence are automorphisms. Since $h$ is an isomorphism, it follows that $f + g$ is an isomorphism.

2. Assume that there exists a homomorphism $s : N \to M$ such that $fs = \text{id}_N$. We may apply (1) to the homomorphism $fs + gs$ to see that $(f + g)s$ is also an isomorphism. This means that $f + g$ is a split epimorphism. The item (3) can be checked in the same way.

3. Assume that $f$ is irreducible. According to the previous part, $f + g$ is neither a split monomorphism nor a split epimorphism. By the assumption, $g$ is a composite $ba$ of homomorphisms $a : M \to F$ and $b : F \to N$ with a free $R$-module $F$. If there is a factorization $f + g = dc$ for some homomorphisms $c : M \to X$ and $d : X \to N$, then they induce a decomposition $M \xrightarrow{[a, c]} F \oplus X \xrightarrow{[-b, d]} N$ of $f$. By the irreducibility of $f$, either $[a, c]$ is a split monomorphism or $[-b, d]$ is a split epimorphism. In the former case, we can take a homomorphism $[p, q] : F \oplus X \to N$ such that the composite $pa + qc = [p, q] \circ [a, c]$ is equal to the identity map of $N$. Using (1), $qc$ is also an isomorphism. This yields that $c$ is a split monomorphism. In the latter case, we can see that $d$ is a split epimorphism by similar arguments. Thus we conclude that $f + g$ is an irreducible homomorphism.

Let $M$ be a nonfree indecomposable module in $\text{CM}_0(R)$. Then there exists an almost split sequence ending in $M$. Namely, there is a nonsplit short exact sequence

$$0 \to \tau M \xrightarrow{f} E_M \xrightarrow{g} M \to 0$$
Lemma 2.4. Let $M$ be a nonfree indecomposable module in $\text{CM}_0(R)$. Consider the almost split sequences
\[ 0 \to \tau M \xrightarrow{f} E_M \xrightarrow{g} M \to 0, \quad 0 \to \tau(\Omega M) \to E_{\Omega M} \to \Omega M \to 0 \]
ending in $M$ and $\Omega M$. Then $\Omega(E_M)$ is isomorphic to $E_{\Omega M}$ up to free summands.

Proof. By horseshoe lemma, there exists a short exact sequence $s: 0 \to \Omega(\tau M) \xrightarrow{f'} \Omega E_M \oplus P \xrightarrow{g'} \Omega M \to 0$ with some free $R$-module $P$. Here, the class $g' \in \text{Hom}_R(\Omega E_M, \Omega M)$ of $g'$ coincides with the image $\Omega(g)$ of the class $g$ of $g$ under the isomorphism $\Omega: \text{Hom}_R(E_M, M) \to \text{Hom}_R(\Omega E_M, \Omega M)$ in Lemma 2.4. We want to show that the sequence $s$ is an almost split sequence ending in $\Omega M$. By Lemma 2.3 (2), we see that $g'$ is a split epimorphism if and only if $g'h = \text{id}$ for some $h$ in the category $\text{CM}_0(R)$. In view of the equivalence $\Omega: \text{CM}_0(R) \to \text{CM}_0(R)$, $g'$ as well as $g$ is not a split surjection. This means that $s$ is not a split exact sequence.

We fix a homomorphism $h': X \to \Omega M$ which is not a split epimorphism. We can use the equivalence $\Omega: \text{CM}_0(R) \to \text{CM}_0(R)$ again to obtain an equality $h' = g'p + rq$ with some homomorphism $p: X \to \Omega E_M$, $q: X \to F$, $r: F \to \Omega M$, where $F$ is a free module. As $g'$ is an epimorphism and $F$ is free, $r$ factors through $g$. This shows that $h' = g't$ for some $t: X \to \Omega E_M$. Consequently, $s$ is an almost split sequence ending in $\Omega M$.

Consider the almost split sequence
\[ 0 \to \tau(M) \to E_M \to M \to 0 \]
ended in $M$. We define a number $\alpha(M)$ to be the number of nonfree indecomposable summands of $E_M$.

Lemma 2.5. Let $M$ be a nonfree indecomposable module in $\text{CM}_0(R)$. Then $\alpha(M) = \alpha(\Omega^i M)$ for all $i \geq 0$.

Proof. This is a direct consequence of Lemma 2.4.

The following two lemmas play key roles in the next section. See [11] Lemma 4.1.8] for details of the lemma below.

Lemma 2.6. Let $f: M \to N$ be an irreducible homomorphism such that $M$ and $N$ are indecomposable in $\text{CM}_0(R)$. Assume that $\dim R = 1$. Then $f$ is either injective or surjective.

Recall that an $R$-module $M$ has constant rank $n$ if one has an isomorphism $M_p \cong R_p^{\oplus n}$ for all associated primes $p$ of $R$.

Lemma 2.7. Let $M, N$ be nonfree indecomposable modules in $\text{CM}_0(R)$ having same constant rank. Let $f: M \to N$ be an irreducible monomorphism. Assume that $\dim R = 1$. Then Coker $f$ is isomorphic to $R/m$.

Proof. By the assumption that $f$ is an irreducible monomorphism, $f$ is not surjective. Hence we can take a maximal proper submodule $X$ of $N$ containing $\text{Im } f$. Remark that the quotient $N/X$ is isomorphic to $R/m$ and hence $X$ and $N$ has same constant rank. Since $\dim R = 1$, $X$ is an $R$-module contained in $\text{CM}_0(R)$. Thus we have a factorization $M \to X \to N$ of $f$ in $\text{CM}_0(R)$. By the irreducibility of $f$, it follows that either $M \to X$ is a split monomorphism or $X \to N$ is a split epimorphism. As $X$ is proper submodule of $N$, the later case cannot occur. Therefore, we obtain a split monomorphism $g: M \to X$. Then, by the equalities $\text{rank } M = \text{rank } N = \text{rank } X$, $g$ is an isomorphism. This implies the desired isomorphisms $\text{Coker } f \cong N/X \cong R/m$. ■
3. Proof of the Main Theorem

In this section, we give a proof of Theorem 1.2.

Proof of Theorem 1.2. Since $R$ is a Gorenstein local ring of dimension one, $\tau(N)$ is isomorphic to $\Omega N$ for any nonfree indecomposable $R$-module $N$ in $\mathcal{CM}_0(R)$. We assume that $M$ is a nonfree indecomposable $R$-module in $\mathcal{CM}_0(R)$ satisfying $\text{Ext}_R^1(M, M) = 0$ and want to show that $\alpha(M) = 1$. We see from Lemma 2.1 that the isomorphisms

$$\text{Ext}_R^i(\Omega^{i+1}M, \Omega^{i+1}M) \cong \text{Ext}_R^i(M, M) = 0$$

hold for all $i \geq 0$. If $E_{\Omega M}$ has a free summand, then $\tau(\Omega^i M) = \Omega^{i+1}M$ has an irreducible homomorphism into $R$. Hence $\Omega^{i+1}M$ is a direct summand of the maximal ideal $m$. Since $R$ is a domain, this means that $\Omega^{i+1}M$ is isomorphic to $m$. It follows that $\text{Ext}_R^1(m, m)$ is zero, and so $R$ should be regular. Therefore, we may assume that $E_{\Omega M}$ has no free summands for all $i \geq 0$. By Lemma 2.5, it is enough to show that $\alpha(\Omega^i M) = 1$ for some $i \geq 0$. Thus by replacing $M$ with $\Omega^i M$, we may assume that rank $M$ is minimal in the set $\{\text{rank } \Omega^i M \mid i \geq 0\}$.

Decompose $E_M = E_1 \oplus \cdots \oplus E_n$ as a direct sum of indecomposable modules and consider the almost split sequence

$$0 \to \Omega M \xrightarrow{f = (f_1, \ldots, f_n)} E_1 \oplus \cdots \oplus E_n \xrightarrow{g = (g_1, \ldots, g_n)} M \to 0$$

dended in $M$, where $f_p : \Omega M \to E_p$ and $g_p : E_p \to N$ are irreducible homomorphisms and $n = \alpha(M)$. Lemma 2.6 guarantees that each of $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$ is either injective or surjective.

Claim 1. There is a number $p$ such that $f_p$ is injective.

Proof of Claim 1. Suppose that all of the $f_1, \ldots, f_n$ are surjective. Then we get equalities $\text{Im } g = \sum_i \text{Im } g_i = \sum_i \text{Im } g_i f_i$. Since $\text{Hom}_R(\Omega M, M) = 0$ (Lemma 2.1), it follows from Lemma 2.2 that $\text{Im } g_i f_i \subseteq m M$ for all $i = 1, \ldots, n$. This yields that $\text{Im } g \subseteq m M$, which contradicts to $g$ is surjective.

Claim 2. If there is a number $p$ such that $f_p$ is injective and $g_p$ is surjective, then $\alpha(M) = 1$.

Proof of Claim 2. Suppose that $f_p$ is injective and $g_p$ is surjective. Since $\text{Hom}_R(\Omega M, M) = 0$, there is a free $R$-module $F$ and homomorphisms $a : \Omega M \to F$ and $b : F \to M$ such that $g_p f_p = ba$. Since $F$ is free and $g_p$ is surjective, we have a factorization $b = g_p c$ with some homomorphism $c : F \to E_p$. So we get an equality $g_p (f_p - ca) = 0$. In particular, $f_p - ca$ factors through the kernel $\text{Ker } g_p$ of $g_p$, i.e. $f_p - ca = ed$ with a homomorphism $d : \Omega M \to \text{Ker } g_p$ and the natural inclusion $e : \text{Ker } g_p \to E_p$. By Lemma 2.3 (2), the homomorphism $f_p - ca : M \to E_p$ is also irreducible. Hence either $e$ is a split epimorphism or $d$ is a split monomorphism. In the former case, the equality $\text{Ker } g_p = E_p$ follows. It means that the map $g_p$ is zero. This is a contradiction to the irreducibility of $g_p$. So it follows that $d$ is a split monomorphism. Then one has rank $\Omega M \leq \text{rank } \text{Ker } g_p = \text{rank } E_p - \text{rank } M$. This forces that $n = 1$. □

By Claim 1, we already have an integer $p$ such that $f_p$ is a monomorphism. If $g_p$ is surjective, then by Claim 2 it follows that $\alpha(M) = 1$. Therefore, we may suppose that $g_p$ is injective. Then the inequalities rank $\Omega M \leq \text{rank } E_p \leq \text{rank } M$ hold. By the minimality of rank $M$, we have rank $\Omega M = \text{rank } E_p = \text{rank } M$. In this case, we see isomorphisms $\text{Coker } f_p \cong R/m \cong \text{Coker } g_p$ by Lemma 2.7. Therefore, equalities $\ell(M/\text{Im } (f_p g_p)) = \ell(\text{Coker } f_p) + \ell(\text{Coker } g_p) = 2$ hold (here, $\ell(X)$ denotes the length for an $R$-module $X$). By Lemma 2.2, $\text{Im } (f_p g_p) \subseteq m M$. So it follows that $\ell(M/mM) \leq 2$. In other words, $M$ is generated by two elements as an $R$-module. Since $M$ is nonfree, one has rank $M = 1$ and $\text{Hom}_R(M, R) \cong \Omega M$. As rank $\Omega M = \text{rank } M = 1$, we can apply the same argument above for $\Omega M$ to see that $\Omega M$ is also generated by two elements. Then by [7, Theorem 3.2], one can see that $\text{Ext}_R^1(M, M) \neq 0$, a contradiction. □
References

[1] O. Celikbas, Vanishing of Tor over complete intersections, *J. Commut. Algebra* **3** (2011), 169–206.
[2] O. Celikbas; S. Goto; R. Takahashi; N. Taniguchi, On the ideal case of a conjecture of Huneke and Wiegand, to appear in *Proc. Edinb. Math. Soc. (2)*.
[3] O. Celikbas; R. Takahashi, Auslander–Reiten conjecture and Auslander–Reiten duality, *J. Algebra* **382** (2013), 100–114.
[4] P. A. García-Sánchez; M. J. Leamer, Huneke-Wiegand conjecture for complete intersection numerical semigroup rings, *J. Algebra* **391** (2013), 114–124.
[5] S. Goto; R. Takahashi; N. Taniguchi; H. Le Truong, Huneke-Wiegand conjecture and change of rings, *J. Algebra* **422** (2015), 33–52.
[6] D. Happel, Triangulated categories in the representation theory of finite-dimensional algebras, London Mathematical Society Lecture Note Series **119**, Cambridge University Press, Cambridge, 1988. x+208 pp.
[7] K. Herzinger, The number of generators for an ideal and its dual in a numerical semigroup, *Comm. Algebra* **27** (1999), 4673–4690.
[8] C. Huneke; S. Iyengar; R. Wiegand, Rigid Ideals in Gorenstein Rings of Dimension One. *Acta Mathematica Vietnamica* (2018), 1–19.
[9] C. Huneke; D. A. Jorgensen, Symmetry in the vanishing of Ext over Gorenstein rings, *Math. Scand.* **93** (2003), 161–184.
[10] C. Huneke; R. Wiegand, Tensor products of modules and the rigidity of Tor, *Math. Ann.* **299** (1994), no. 3, 449–476.
[11] R. Roy, Auslander–Reiten Sequences over Gorenstein Rings of Dimension One (2018). Dissertations - ALL. 873. https://surface.syr.edu/etd/873
[12] R. Roy, Graded AR Sequences and the Huneke-Wiegand Conjecture, *arXiv:1808.06600*.
[13] Y. Yoshino, Cohen-Macaulay modules over Cohen–Macaulay rings, London Mathematical Society Lecture Note Series, 146, Cambridge University Press, Cambridge, 1990.

Graduate School of Mathematics, Nagoya University, Furocho, Chikusaku, Nagoya, Aichi 464-8602, Japan

*E-mail address*: m16021z@math.nagoya-u.ac.jp