HAMILTONIAN FORMULATION OF GENERAL RELATIVITY
IN THE TELEPARALLEL GEOMETRY

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Abstract

We establish the Hamiltonian formulation of the teleparallel equivalent of general relativity, without fixing the time gauge condition, by rigorously performing the Legendre transform. The time gauge condition, previously considered, restricts the teleparallel geometry to the three-dimensional spacelike hypersurface. Geometrically, the teleparallel geometry is now extended to the four-dimensional space-time. The resulting Hamiltonian formulation is structurally different from the standard ADM formulation in many aspects, the main one being that the dynamics is now governed by the Hamiltonian constraint $H_0$ and a set of primary constraints. The vector constraint $H_i$ is derived from the Hamiltonian constraint. The vanishing of the latter implies the vanishing of the vector constraint.

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I. Introduction

Hamiltonian formulations, when consistently established, not only guarantee that field quantities have a well defined time evolution, but also allow us to understand physical theories from a different perspective. We have learned from the work of Arnowitt, Deser and Misner (ADM)\(^1\) that the Hamiltonian analysis of Einstein’s general relativity reveals the intrinsic structure of the theory: the time evolution of field quantities is determined by the Hamiltonian and vector constraints. Thus four of the ten Einstein’s equations acquire a prominent status in the Hamiltonian framework. Ultimately this is an essential feature for the canonical approach to the quantum theory of gravity.

It is the case in general relativity that two distinct Lagrangian formulations that yield Einstein’s equations lead to completely different Hamiltonian constructions. An important example in this respect is the reformulation of the ordinary variational principle, based on the Hilbert-Einstein action, in terms of self-dual connections that define Ashtekar variables\(^2\). Under a Palatini type variation of the action integral constructed out of these field quantities one obtains precisely Einstein’s equations. Interesting features of this approach reside in the Hamiltonian domain.

Einstein’s general relativity can also be reformulated in the context of the teleparallel (Weitzenböck) geometry\(^3\). In this geometrical setting the dynamical field quantities correspond to orthonormal tetrad fields \(e^a_{\mu}\) (\(a, \mu\) are SO(3,1) and space-time indices, respectively). These fields allow the construction of the Lagrangian density of the teleparallel equivalent of general relativity (TEGR) \(^4\), \(^5\), \(^6\), \(^7\), \(^8\), \(^9\), \(^10\), \(^11\), \(^12\), which offers an alternative geometrical framework for Einstein’s equations. The Lagrangian density for the tetrad field in the TEGR is given by a sum of quadratic terms in the torsion tensor \(T^a_{\mu\nu} = \partial_\mu e^a_\nu - \partial_\nu e^a_\mu\), which is related to the anti-symmetric part of Cartan’s connection...
\[ \Gamma^\lambda_{\mu
u} = e^a_{\lambda} \partial_\mu e_{a\nu}. \] The curvature tensor constructed out of the latter vanishes identically. This connection defines a space with teleparallelism, or absolute parallelism[13].

In a space-time with an underlying tetrad field two vectors at distant points are called parallel[4] if they have identical components with respect to the local tetrads at the points considered. Thus consider a vector field \( V^\mu(x) \). At the point \( x^\lambda \) its tetrad components are given by \( V^a(x) = e^a_{\mu}(x)V^\mu(x) \). For the tetrad components \( V^a(x + dx) \) it is easy to show that \( V^a(x + dx) = V^a(x) + DV^a(x) \), where \( DV^a(x) = e^a_{\mu}(\nabla_\lambda V^\mu)dx^\lambda \). The covariant derivative \( \nabla \) is constructed out of Cartan’s connection \( \Gamma^\lambda_{\mu
u} = e^a_{\lambda} \partial_\mu e_{a\nu} \). Therefore the vanishing of such covariant derivative defines a condition for absolute parallelism in space-time. Hence in the teleparallel geometry tetrad fields transform under the global SO(3,1) group. Teleparallel geometry is less restrictive than Riemannian geometry[14]. For a given Riemannian geometry there are many ways to construct the teleparallel geometry, since one Riemannian geometry corresponds to a whole equivalence class of teleparallel geometries.

In the framework of the TEGR it is possible to make definite statements about the energy and momentum of the gravitational field. This fact constitutes the major motivation for considering this theory. In the 3+1 formulation of the TEGR[12], and by imposing Schwinger’s time gauge condition[13], we find that the Hamiltonian and vector constraints contain each one a divergence in the form of scalar and vector densities, respectively, that can be identified with the energy and momentum densities of the gravitational field[16].

In this paper we carry out the Hamiltonian formulation of the TEGR without imposing the time gauge condition, by rigorously performing the Legendre transform. We have not found it necessary to establish a 3+1 decomposition for the tetrad field. We only assume \( g^{00} \neq 0 \), a condition that ensures that \( t = constant \) hypersurfaces are spacelike. The Lagrange multipliers are given by the zero components of the tetrads, \( e_{a0} \). The constraints
corresponding to the Hamiltonian \((H_0)\) and vector \((H_i)\) constraints are obtained in the form \(C^a = 0\). The dynamical evolution of the field quantities is completely determined by \(H_0\) and by a set of primary constraints \(\Gamma^{ik} \) and \(\Gamma^k\), as we will show. The surprising feature is that if \(H_0 = 0\) in the subspace of the phase space determined by \(\Gamma^{ik} = \Gamma^k = 0\), then it follows that \(H_i = 0\). As we will see, \(H_i\) can be obtained from the very definition of \(H_0\). Furthermore by calculating Poisson brackets we show that the constraints constitute a first class set. Hence the theory is well defined regarding time evolution.

As a consequence of this analysis, we arrive at a scalar density that transforms as a four-vector in the SO(3,1) space, again arising in the expression of the constraints of the theory, and whose zero component is related to the energy of the gravitational field. In analogy with previous investigations, we interpret the constraint equations \(C^a = 0\) as energy-momentum equations for the gravitational field.

The analysis developed here is similar to that developed in Ref. [17], in which the Hamiltonian formulation of the TEGR in null surfaces was established. The 3+1 formulation of the TEGR has already been considered in Ref. [10]. There are several differences between the latter and the present analysis. The investigation in Ref. [10] has not pointed out neither the emergence of the scalar densities mentioned above nor the relationship between \(H_0\) and \(H_i\). Our approach is different and allowed us to proceed further in the understanding of the constraint structure of the theory.

Notation: spacetime indices \(\mu, \nu, ...\) and SO(3,1) indices \(a, b, ...\) run from 0 to 3. Time and space indices are indicated according to \(\mu = 0, i, \ a = (0), (i)\). The tetrad field \(e^a_\mu\) yields the definition of the torsion tensor: \(T^a_{\mu \nu} = \partial_\mu e^a_\nu - \partial_\nu e^a_\mu\). The flat, Minkowski spacetime metric is fixed by \(\eta_{ab} = e_{a \mu} e_{b \nu} g^{\mu \nu} = (-+++).\)
II. Lagrangian formulation

In order to carry out the 3+1 decomposition we need a first order differential formulation of the Lagrangian density of the TEGR. For this purpose we introduce an auxiliary field quantity $\phi_{abc} = -\phi_{acb}$ that will be related to the torsion tensor. The first order differential Lagrangian formulation in empty space-time reads

$$L(e, \phi^a) = k e \Lambda^{abc} (\phi_{abc} - 2T_{abc}) ,$$

where $T_{abc} = e^b_\mu e^c_\nu T_{a\mu\nu}$. $\Lambda^{abc}$ is defined by

$$\Lambda^{abc} = \frac{1}{4} (\phi^{abc} + \phi^{bac} - \phi^{cab}) + \frac{1}{2} (\eta^{ac} \phi^{b} - \eta^{ab} \phi^{c}) ,$$

and $\phi_b = \phi^a_{\ ab}$. The Lagrangian density (1) is invariant under coordinate and global SO(3,1) transformations.

Variation of the action constructed out of (1) with respect to $\phi_{abc}$ yields an equation that can be reduced to $\phi_{abc} = T_{abc}$. This equation can be split into two equations:

$$\phi_{a0k} = T_{a0k} = \partial_0 e_{ak} - \partial_k e_{a0} , \quad (3a)$$

$$\phi_{aik} = T_{aik} = \partial_i e_{ak} - \partial_k e_{ai} . \quad (3b)$$

The variation of the action integral with respect to $e_{a\mu}$ yields the field equation

$$\frac{\delta L}{\delta e^{a\mu}} = e_a \lambda \epsilon_{b\mu} \partial_{b} (e \Sigma^{b\lambda\nu}) - e \left( \Sigma^{b\nu}_{a} T_{b\nu\mu} - \frac{1}{4} \epsilon_{a\mu} T_{bcd} \Sigma^{bcd} \right) = 0 . \quad (4)$$

The tensor $\Sigma^{abc}$ is defined in terms of $T^{abc}$ exactly like $\Lambda^{abc}$ in terms of $\phi^{abc}$. By explicit calculations[12] it is verified that these equations are equivalent to Einstein’s equations in tetrad form:
\[ \frac{\delta L}{\delta e_{a\mu}} \equiv \frac{1}{2} e \left\{ R_{a\mu}(e) - \frac{1}{2} e_{a\mu} R(e) \right\} . \]

We note finally that by substituting (3a,b) into (1) the Lagrangian density reduces to

\[ L(e_{a\mu}) = -k e \Sigma^{abc} T_{abc} = -k e \left( \frac{1}{4} T^{abc} T_{abc} + \frac{1}{2} T^{abc} T_{bac} - T^a T_a \right) . \]

### III. Legendre transform and the 3+1 decomposition

The Hamiltonian density will be obtained by the standard prescription \( L = p \dot{q} - H_0 \) and by properly identifying primary constraints. We have not found it necessary to establish any kind of 3+1 decomposition for the tetrad fields. Therefore in the following both \( e_{a\mu} \) and \( g_{\mu\nu} \) are space-time fields. We will follow here the procedure presented in [17].

Lagrangian density (1) can be expressed as

\[ L(e, \phi) = -4k e \Lambda^{a0k} \dot{e}_k + 4k e \Lambda^{a0k} \partial_k e_{a0} - 2k e \Lambda^{aij} T_{aij} + ke \Lambda^{abc} \phi_{abc} , \quad (5) \]

where the dot indicates time derivative, and \( \Lambda^{a0k} = \Lambda^{abc} e_b^0 e_c^k \), \( \Lambda^{aij} = \Lambda^{abc} e_b^i e_c^j \).

Therefore the momentum canonically conjugated to \( e_{ak} \) is given by

\[ \Pi^{ak} = -4k e \Lambda^{a0k} , \quad (6) \]

In terms of (6) expression (5) reads

\[ L = \Pi^{ak} \dot{e}_{ak} - \Pi^{ak} \partial_k e_{a0} - 2k e \Lambda^{aij} T_{aij} + ke \Lambda^{abc} \phi_{abc} \]

\[ = \Pi^{ak} \dot{e}_{ak} - \Pi^{ak} \partial_k e_{a0} - ke \Lambda^{aij} (2T_{aij} - \phi_{aij}) + 2k e \Lambda^{a0k} \phi_{a0k} . \quad (7) \]
The last term on the right hand side of equation (7) is identified as $2k e \Lambda a_{0k} \phi_{a0k} = -\frac{1}{2} \Pi^a k \phi_{a0k}$.

The Hamiltonian formulation is established once we rewrite the Lagrangian density (7) in terms of $e_{ak}$, $\Pi^a k$ and further nondynamical field quantities. It is carried out in two steps. First, we take into account equation (3b) in (7) so that half of the auxiliary fields, $\phi_{aij}$, are eliminated from the Lagrangian by means of the identification

$$\phi_{aij} = T_{aij}.$$ 

As a consequence we have

$$-k e \Lambda^{aij} (2T_{aij} - \phi_{aij}) = -k e \Lambda^{aij} T_{aij}$$

$$= -k e \left( \frac{1}{4} g^{im} g^{nj} T^a_{mn} T_{aij} + \frac{1}{2} g^{nj} T^i_{mn} T^m_{ij} - g^{jk} T^i_{ji} T^m_{nk} \right)$$

$$+ k e \left( -\frac{1}{2} g^{ik} g^{j0} \phi^a_{0k} T_{aij} - \frac{1}{2} g^{jk} \phi^i_{0k} T^0_{ij} + \frac{1}{2} g^{0j} \phi^i_{0k} T^k_{ij} + g^{jk} \phi^0_{0i} T^i_{jk} \right).$$

The last five terms of the expression above may be rewritten as

$$-\frac{1}{2} k e \phi_{a0k} \left[ g^{0i} g^{kj} T^a_{ij} - e^{ai} (g^{0j} T^k_{ij} - g^{0j} T^0_{ij}) + 2 (e^{ak} g^{0i} - e^{a0} g^{ki}) T^j_{ji} \right].$$

Therefore we have

$$L(e_{ak}, \Pi^a k, e_{a0}, \phi_{a0k}) = \Pi^a k \dot{e}_{ak} + e_{a0} \partial_k \Pi^a k - \partial_k (e_{a0} \Pi^a k)$$

$$-k e \left( \frac{1}{4} g^{im} g^{nj} T^a_{mn} T_{aij} + \frac{1}{2} g^{nj} T^i_{mn} T^m_{ij} - g^{jk} T^i_{ji} T^m_{nk} \right)$$

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\[-\frac{1}{2} \phi_{a0k} \left\{ \Pi^{ak} + k e \left[ g^{0i} g^{kj} T^a_{ij} - e^{ai} (g^{0j} T^k_{ij} - g^{kj} T^0_{ij}) + 2 (e^{ak} g^{0i} - e^{a0} g^{ki}) T^j_{ji} \right] \right\} \]  

(8)

The second step consists of expressing the remaining auxiliary field quantities, the “velocities” \( \phi_{a0k} \), in terms of the momenta \( \Pi^{ak} \). This is the nontrivial step of the Legendre transform.

We need to consider the full expression of \( \Pi^{ak} \). It is given by equation (6),

\[
\Pi^{ak} = k e \left\{ g^{00} (-g^{kj} \phi^a_{0j} - e^{aj} \phi^k_{0j} + 2 e^{ak} \phi^j_{0j}) \right. \\
+ g^{0k} (g^{0j} \phi^a_{0j} + e^{aj} \phi^0_{0j}) + e^{a0} (g^{0j} \phi^k_{0j} + g^{kj} \phi^0_{0j}) - 2 (e^{a0} g^{0k} \phi^j_{0j} + e^{ak} g^{0j} \phi^0_{0j}) \\
- g^{0i} g^{kj} T^a_{ij} + e^{0i} (g^{0j} T^k_{ij} - g^{kj} T^0_{ij}) - 2 (g^{0i} e^{0k} - g^{ik} e^{a0}) T^j_{ji} \left\} , \right.
\]

(9)

where we have already identified \( \phi_{aij} = T_{aij} \). Denoting \( (..) \) and \( [..] \) as the symmetric and anti-symmetric parts of field quantities, respectively, we decompose \( \Pi^{ak} \) into irreducible components:

\[
\Pi^{ak} = e^a \, \Pi^{(ik)} + e^a \, \Pi^{[ik]} + e^a_{0} \, \Pi^{0k} ,
\]

(10)

where

\[
\Pi^{(ik)} = k e \left\{ g^{00} (-g^{kj} \phi^i_{0j} - g^{ij} \phi^0_{0j} + 2 g^{ik} \phi^j_{0j}) + g^{0k} (g^{0j} \phi^i_{0j} + g^{ij} \phi^0_{0j} - g^{0i} \phi^j_{0j}) \\
+ g^{0i} (g^{0j} \phi^k_{0j} + g^{kj} \phi^0_{0j} - g^{0k} \phi^j_{0j}) - 2 g^{ik} g^{0j} \phi^0_{0j} + \Delta^{ik} \right\} ,
\]

(11a)
\[ \Delta^{ik} = -g^{0m}(g^{kj}T^i_{mj} + g^{ij}T^k_{mj} - 2g^{ik}T^j_{mj}) - (g^{km}g^{0i} + g^{im}g^{0k})T^j_{mj}, \quad (11b) \]

\[ \Pi^{[ik]} = ke \left\{ -g^{im}g^{kj}T^0_{mj} + (g^{im}g^{0k} - g^{km}g^{0i})T^j_{mj} \right\}, \quad (12) \]

\[ \Pi^0_k = -2ke \left( g^{kj}g^{0i}T^0_{ij} - g^{0k}g^{0i}T^j_{ij} + g^{00}g^{ik}T^j_{ij} \right). \quad (13) \]

The crucial point in this analysis is that only the symmetrical components \( \Pi^{(ij)} \) depend on the “velocities” \( \phi_{a0k} \). The other six components, \( \Pi^{[ij]} \) and \( \Pi^0_k \) depend solely on \( T_{aij} \). Therefore we can express only six of the “velocity” fields \( \phi_{a0k} \) in terms of the components \( \Pi^{(ij)} \). With the purpose of finding out which components of \( \phi_{a0k} \) can be inverted in terms of the momenta we decompose \( \phi_{a0k} \) identically as

\[ \phi^a_{\ 0j} = e^{ai}\psi_{ij} + e^{ai}\sigma_{ij} + e^{a0}\lambda_j, \quad (14) \]

where \( \psi_{ij} = \frac{1}{2}(\phi_{i0j} + \phi_{j0i}) \), \( \sigma_{ij} = \frac{1}{2}(\phi_{i0j} - \phi_{j0i}) \), \( \lambda_j = \phi_{00j} \), and \( \phi_{\mu0j} = e^{a\mu}\phi_{a0j} \) (like \( \phi_{abc} \), the components \( \psi_{ij}, \sigma_{ij} \) and \( \lambda_j \) are also auxiliary field quantities). Next we substitute (14) in (11a). By defining

\[ P^{ik} = \frac{1}{ke}\Pi^{(ik)} - \Delta^{ik}, \quad (15) \]

we find that \( P^{ik} \) depends only on \( \psi_{ij} \):

\[ P^{ik} = -2g^{00}(g^{im}g^{kj}\psi_{mj} - g^{ik}\psi) \]

\[ + 2(g^{0i}g^{kj}g^{0j} + g^{0k}g^{im}g^{0j})\psi_{mj} - 2(g^{ik}g^{00}g^{jij} + g^{0i}g^{0k}\psi), \quad (16) \]
where $\psi = g^{mn}\psi_{mn}$.

We can now invert $\psi_{mj}$ in terms of $P^{ik}$. After a number of manipulations we arrive at

$$\psi_{mj} = -\frac{1}{2g^{00}}\left(g_{im}g_{kj}P^{ik} - \frac{1}{2}g_{mj}P\right), \quad (17)$$

where $P = g_{ik}P^{ik}$.

At last we need to rewrite the third line of the Lagrangian density (8) in terms of canonical variables. By making use of (9), (14) and (17) we can rewrite

$$-\frac{1}{2}\phi_{a0k}\left\{\Pi^{ak} + ke\left[g^{0i}g^{kj}T^a_{ij} - e_{ai}(g^{0j}T^k_{ij} - g^{kj}T^0_{ij}) + 2(e_{ak}g^{0i} - e_{a0}g^{ki})T^j_{ji}\right]\right\}$$

in the form

$$\frac{1}{4g^{00}}ke\left(g_{ik}g_{jl}P^{ij}P^{kl} - \frac{1}{2}p^2\right).$$

Thus we finally obtain the primary Hamiltonian density $H_0 = \Pi^{ak}\dot{e}_{ak} - L$,

$$H_0(e_{ak}, \Pi^{ak}, e_{a0}) = -e_{a0}\partial_k\Pi^{ak} - \frac{1}{4g^{00}}ke\left(g_{ik}g_{jl}P^{ij}P^{kl} - \frac{1}{2}p^2\right)$$

$$+ke\left(\frac{1}{4}g^{im}g^{nj}T^a_{mn}T_{aij} + \frac{1}{2}g^{nj}T^i_{mn}T^m_{ij} - g^{ik}T^j_{ji}T^m_{nk}\right). \quad (18)$$

We may now write the total Hamiltonian density. For this purpose we have to identify the primary constraints. They are given by expressions (12) and (13), which represent relations between $e_{ak}$ and the momenta $\Pi^{ak}$. Thus we define

$$\Gamma^{ik} = -\Gamma^{ki} = \Pi^{[ik]} - ke\left\{-g^{im}g^{kj}T^0_{mj} + (g^{im}g^{ok} - g^{km}g^{0i})T^j_{mj}\right\}, \quad (19)$$
\[ \Gamma^k = \Pi^{0k} + 2k e \left( g^{kj} g^{0i} T^0_{ij} - g^{0k} g^{0i} T^j_{ij} + g^{0i} g^{ik} T^j_{ij} \right). \]  

Therefore the total Hamiltonian density is given by

\[ H(e_{ak}, \Pi^{ak}, e_{a0}, \alpha_{ik}, \beta_k) = H_0 + \alpha_{ik} \Gamma^{ik} + \beta_k \Gamma^k + \partial_k (e_{a0} \Pi^{ak}), \]  

where \( \alpha_{ik} \) and \( \beta_k \) are Lagrange multipliers.

**IV. Secondary constraints**

Since the momenta \( \{\Pi^{a0}\} \) vanish identically they also constitute primary constraints that induce the secondary constraints

\[ C^a \equiv \frac{\delta H}{\delta e_{a0}} = 0. \]  

In order to obtain the expression of \( C^a \) we have only to vary \( H_0 \) with respect to \( e_{a0} \), because variations of \( \Gamma^{ik} \) and \( \Gamma^k \) with respect to \( e_{a0} \) yield the constraints themselves:

\[ \frac{\delta \Gamma^{ik}}{\delta e_{a0}} = -\frac{1}{2} (e^{ai} \Gamma^k - e^{ak} \Gamma^i), \]  

\[ \frac{\delta \Gamma^k}{\delta e_{a0}} = -e^{a0} \Gamma^k. \]  

In (23a,b) we have made use of variations like

\[ \delta e^{b\mu} / \delta e_{a0} = -e^{a\mu} e^{b0}. \]  

In the process of obtaining \( C^a \) we need the variation of \( P^{ij} \) with respect to \( e_{a0} \). It reads

\[ \frac{\delta P^{ij}}{\delta e_{a0}} = -e^{a0} P^{ij} + \gamma^{aij}, \]  

with \( \gamma^{aij} \) defined by
\[ \gamma_{ij} = -\frac{1}{2k e} (e^{a i} \Gamma^j + e^{a j} \Gamma^i) - e^{a k} \left[ g^{00} (g^{j m} T^i_{k m} + g^{i m} T^j_{k m} + 2g^{i j} T^m_{m k}) + g^0 m (g^{0 j} T^i_{m k} + g^{0 i} T^j_{m k}) \right. \]
\[ \left. \quad + 2g^0 j g^0 i T^m_{m k} + (g^j m g^0 i + g^i m g^0 j - 2g^{i j} g^0 m) T^0_{m k} \right]. \] (24)

Note that \( \gamma_{ij} \) satisfies \( e^{a 0} \gamma_{aij} = 0. \)

After a long calculation we arrive at the expression of \( C^a \):

\[ C^a = -\partial_k \Pi^a k + e^{a 0} \left[ -\frac{1}{4g^{00}} k e \left( g^{i k} g_{j l} P^i j P^{k l} - \frac{1}{2} P^2 \right) \right. \]
\[ \left. + k e \left( \frac{1}{4} g^{i m} g^{n j} T^b_{m n} T^{b i j} + \frac{1}{2} g^{n j} T^i m T^{m}_{i j} - g^{i k} T^m_{m i} T^m_{n k} \right) \right] \]
\[ - \frac{1}{2g^{00}} k e \left( g^{i k} g_{j l} \gamma_{aij} P^{k l} - \frac{1}{2} g^{i j} \gamma_{aij} P \right) - k e e^{a i} \left( g^{0 m} g^{n j} T^b_{i j} T^{b m n} \right. \]
\[ \left. + g^{n j} T^0_{m n} T^m_{i j} + g^{0 i} T^m_{m j} T^m_{n i} - 2g^{0 k} T^m_{m k} T^m_{n i} - 2g^{i k} T^0_{i j} T^m_{n k} \right). \] (25)

Inspite of the fact that expression above is somehow intricate, we immediately notice that

\[ e^{a 0} C^a = H_0. \] (26)

Therefore the total Hamiltonian becomes

\[ H(e^{a k}, \Pi^a k, e^{a 0}, \alpha_{ik}, \beta_k) = e^{a 0} C^a + \alpha_{ik} \Gamma^{ik} + \beta_k \Gamma^k + \partial_k (e^{a 0} \Pi^a k). \] (27)

We observe that \( \{e_{a 0}\} \) arise as Lagrange multipliers (see equation (50) ahead).
Before closing this section we remark that the Hamiltonian formulation described here is different from that developed in Ref. [10], the difference residing in the definition of the canonical momentum. In the latter reference the canonical momentum is not defined by taking the variation of $L$ with respect to $\dot{e}_{ak}$. Instead, it is defined by

$$\pi^a_k = \frac{\delta L}{\delta (N^\perp T^a_{\perp k})} = \frac{\delta L}{\delta (T^a_{0k} - N^i T^a_{ik})},$$

where $N^\perp$ and $N^i$ are the usual lapse and shift functions. As a consequence, three of the six primary constraints of Ref. [10] are different from the corresponding constraints obtained here. The expression of the components $\tau^{[ik]}$ and $\tau^\perp_k$ of Ref. [10], equivalent to $\Pi^{[ik]}$ and $\Pi^{0k}$, respectively, given by (12) and (13), read in our notation

$$\tau^{[ik]} = -e \left\{ g^{im} g^{kj} T^0_{ij} + N^j (g^{im} g^{0k} - g^{km} g^{0i}) T^0_{mj} \right\},$$

$$\tau^\perp_k = \frac{1}{2k} N^\perp \Pi^{0k}.$$

The Hamiltonian and vector constraints of the above mentioned reference are parametrized in terms of the lapse and shift functions. In the present work we have parametrized the set of four constraints according to equation (26), and identified the Lagrange multipliers as $e_{a0}$. The final expression of $C^a$ acquires the total divergence $-\partial_k \Pi^{ak}$. This divergence is different from the one that appears in the expression of the total Hamiltonian density of gravitational fields for asymptotically flat space-times, either in the metric[18] or in the tetrad formulation (see, for example, Eq. (3.17) of Ref. [10] or Eq. (27) above; it is possible to show that the latter expressions are exactly the same field quantities). We finally notice that the constraint algebra to be presented in the coming section has not been evaluated in Ref. [10].
V. Simplification of the constraints and Poisson brackets

The first two terms of the expression of $C^a$ yield the primary Hamiltonian in the form $e^{a0}H_0$. This fact can be easily verified by expressing the first term of (25) as

$$-\partial_k \Pi^{ak} = e^{a0}(-e_{b0}\partial_k \Pi^{bk}) + e^{aj}(-e_{bj}\partial_k \Pi^{bk}) .$$

The second term considered above is the collection of terms in (25) multiplied by $e^{a0}$. Substituting definitions (11b) and (24) for $\Delta^{ij}$ and $\gamma^{aij}$, respectively, into (25) we obtain after a long calculation a simplified form for $C^a$,

$$C^a = e^{a0} H_0 + e^{ai} F_i ,$$

with the following definitions:

$$F_i = H_i + \Gamma^m T_{0mi} + \Gamma^l T_{lmi} + \frac{1}{2g^{00}}(g_{ik}g_{jl}P_{kl} - \frac{1}{2}g_{ij}P)\Gamma^j ,$$

$$H_i = -e_{bi}\partial_k \Pi^{bk} - \Pi^{bk}T_{bki} .$$

We denote $H_0$ the Hamiltonian constraint. $H_i$ is the vector constraint. It amounts to a SO(3,1) version of the vector constraint of Ref. [12]. The true constraints of the theory are $C^a$, $\Gamma^{ik}$ and $\Gamma^k$. Dispensing with the surface term the total Hamiltonian reads

$$H = e_{a0}C^a + \alpha_{ik}\Gamma^{ik} + \beta_k\Gamma^k .$$

The Poisson bracket between two quantities $F$ and $G$ is defined by

$$\{F, G\} = \int d^3x \left( \frac{\delta F}{\delta e_{ai}(x)} \frac{\delta G}{\delta \Pi^{ai}(x)} - \frac{\delta F}{\delta \Pi^{a1}(x)} \frac{\delta G}{\delta e_{ai}(x)} \right) ,$$

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by means of which we can write down the evolution equations. The first set of Hamilton’s equations is given by

\[
\dot{e}_{aj}(x) = \{e_{aj}(x), H\} = \int d^3y \frac{\delta}{\delta \Pi^{aj}(x)} \left( H_0(y) + \alpha_{ik}(y) \Gamma^{ik}(y) + \beta_{k}(y) \Gamma^{k}(y) \right),
\]

(32)

where \( H \) is the total Hamiltonian. This equation can be worked to yield

\[
T_{a0j} = -\frac{1}{2g^{00}} e^k_a (g_{ik} g_{jm} P^{im} - \frac{1}{2} g_{kj} P) + e^i_a \alpha_{ij} + e^0_a \beta_j,
\]

(33)

from which we obtain

\[
\frac{1}{2}(T_{a0j} + T_{j0i}) = \psi_{ij} = -\frac{1}{2g^{00}} (g_{ik} g_{mj} P^{km} - \frac{1}{2} g_{ij} P),
\]

(34a)

\[
\frac{1}{2}(T_{a0j} - T_{j0i}) = \sigma_{ij} = \alpha_{ij},
\]

(34b)

\[
T_{00j} = \lambda_j = \beta_j,
\]

(34c)

according to the definitions in equation (14). Thus the Lagrange multipliers in (31) acquire a well defined meaning. Expression (34a) is in total agreement with (17). Consequently we can obtain an expression for \( \Pi^{(ij)} \) in terms of velocities via equations (15) and (16).

The dynamical evolution of the field quantities is completed with Hamilton’s equations for \( \Pi^{(ij)} \),

\[
\dot{\Pi}^{(ij)}(x) = \{\Pi^{(ij)}(x), H\} = \int d^3y \left( \frac{\delta \Pi^{(ij)}(x)}{\delta e_{ak}(y)} \frac{\delta H(y)}{\delta \Pi^{ak}(y)} - \frac{\delta \Pi^{(ij)}(x)}{\delta \Pi^{ak}(y)} \frac{\delta H(y)}{\delta e_{ak}(y)} \right),
\]

(35)

together with
\[ \Gamma^{ik} = \Gamma^k = 0. \tag{36} \]

The calculations of the Poisson brackets between these constraints are exceedingly complicated. Here we will just present the results. Instead of considering \( C^a(x) \) in the calculations below, we found it more appropriate to consider \( H_0(x) \) and \( H_i(x) \). The constraint algebra is given by

\[ \{ H_0(x), H_0(y) \} = 0, \tag{37} \]

\[ \{ H_0(x), H_i(y) \} = -H_0(x) \frac{\partial}{\partial y^i} \delta(x - y) \]

\[ -H_0 \epsilon^{a0} \partial_i \epsilon_{a0} \delta(x - y) - F_j \epsilon^{a3} \partial_i \epsilon_{a0} \delta(x - y), \tag{38} \]

\[ \{ H_j(x), H_k(y) \} = -H_k(x) \frac{\partial}{\partial x^j} \delta(x - y) - H_j(y) \frac{\partial}{\partial y^k} \delta(x - y), \tag{39} \]

\[ \{ \Gamma^i(x), \Gamma^j(y) \} = 0, \tag{40} \]

\[ \{ \Gamma^{ij}(x), \Gamma^k(y) \} = (g^{0j}\Gamma^{ki} - g^{0i}\Gamma^{kj}) \delta(x - y), \tag{41} \]

\[ \{ \Gamma^{ij}(x), \Gamma^{kl}(y) \} = \frac{1}{2} \left( g^{il}\Gamma^{jk} + g^{jk}\Gamma^{il} - g^{ik}\Gamma^{jl} - g^{jl}\Gamma^{ik} \right) \delta(x - y), \tag{42} \]

\[ \{ H_0(x), \Gamma^{ij}(y) \} = \frac{1}{2g^{mn}} P^{kl} \left( \frac{1}{2} g_{kl} g_{mn} - g_{km} g_{nl} \right) \left( g^{mi}\Gamma^{nj} - g^{mj}\Gamma^{ni} \right) + \]

15
\[+\frac{1}{2}\left(\Gamma^{nj}e^{ai} - \Gamma^{ni}e^{aj}\right)\partial_n e_{a_0}\right]\delta(x - y), \quad (43)\]

\[
\{H_0(x), \Gamma^i(y)\} = \left[g^{0i}_0H_0 + \frac{1}{g^{00}_0}P^{kl}\left(\frac{1}{2}g^{kl}g^{jm} - g^{kj}g^{ml}\right)g^{0j}_0\Gamma^m\right.

\[+\left(\Gamma^{ni}e^{a_0} + \Gamma^{n}e^{ai}\right)\partial_ne_{a_0} + \frac{1}{2}\Gamma^{nm}T^i_{nm}\]

\[+2\partial_n\Gamma^{ni} + g^{in}\left(H_n - \Gamma^jT_{0nj} - \Gamma^{mj}T_{mnj}\right)\right]\delta(x - y)

\[+\Gamma^{ni}(x)\frac{\partial}{\partial x^n}\delta(x - y), \quad (44)\]

\[
\{H_i(x), \Gamma^j(y)\} = \delta^j_i\Gamma^n(y)\frac{\partial}{\partial y^n}\delta(x - y) + \Gamma^j(x)\frac{\partial}{\partial x^i}\delta(x - y) - \Gamma^j e^{a_0}\partial_i e_{a_0}\delta(x - y), \quad (45)\]

\[
\{H_k(x), \Gamma^{ij}(y)\} = \Gamma^{ij}(x)\frac{\partial}{\partial x^k}\delta(x - y) + \left(\delta^j_k\Gamma^{ni}(y) - \delta^i_k\Gamma^{nj}(y)\right)\frac{\partial}{\partial x^n}\delta(x - y)

\[+\frac{1}{2}\left(e^{aj}(x)\Gamma^i(x) - e^{ai}(x)\Gamma^j(x)\right)\frac{\partial}{\partial x^k}e_{a_0}(x)\delta(x - y). \quad (46)\]

It is clear from the constraint algebra above that \(H_0, H_i, \Gamma^{ik}\) and \(\Gamma^k\) constitute a set of first class constraints. Now it is easy to conclude that \(C^a, \Gamma^{ik}\) and \(\Gamma^k\) also constitute a first class set. By means of equation (28) we have \(\{C^a(x), C^b(y)\} = e^{a0}(x)\{H_0(x), H_0(y)\}e^{b0}(y) + H_0(x)\{e^{a0}(x), H_0(y)\}e^{b0}(y) + \cdots\). On the right hand side of this Poisson bracket as well as of the brackets \(\{C^a(x), \Gamma^{ik}(y)\}\) and \(\{C^a(x), \Gamma^k(y)\}\) there will always appear a combination of the constraints \(H_0 = e_{a_0}C^a, \Gamma^{ik}, \Gamma^k\) and
\[ H_i = e_{ai} C^a - \Gamma^m T_{omi} - \Gamma^m T_{lmi} - \frac{1}{2g_{00}} (g_{ik} g_{jl} P^{kl} + \frac{1}{2} g_{ij} P) \Gamma^j. \] (47)

The expression above follows from equation (29). All constraints of the theory are first class, and therefore the theory is well defined regarding time evolution.

The Hamiltonian density (31) determines the time evolution of any field quantity \( f(x) \):

\[ \dot{f}(x) = \left. \int d^3 y \{ f(x), H(y) \} \right|_{\Gamma^{ik}=\Gamma^{ik}=0}. \] (48)

Physical quantities take values in the subspace of the phase space \( P_{\Gamma} \) defined by (36). In this subspace the constraints \( C^a \) become

\[ C^a = e^{a0} H_0 + e^{ai} H_i. \] (49)

Restricting considerations to \( P_{\Gamma} \) we note that if \( H_0 \) vanishes, then \( e_{a0} C^a \) also vanishes. Since \( \{e_{a0}\} \) are arbitrary, it follows that \( C^a = 0 \). In order to arrive at this conclusion we note that the constraints \( C^a \) are independent of \( e_{a0} \). From the orthogonality relation \( e_{a\mu} e^{a\lambda} = \delta^\lambda_\mu \) we obtain \( \delta e^{b\mu}/\delta e_{a0} = -\delta e^{a\mu} e^{b0} \). Using this variational relation and equations (22) and (49) it is possible to show that

\[ \frac{\delta C^a}{\delta e_{b0}} = \frac{\delta}{\delta e_{b0}} \left( e^{a0} H_0 + e^{ai} H_i \right) = -e^{b0} e^{a0} H_0 + e^{a0} \frac{\delta H_0}{\delta e_{b0}} - e^{bi} e^{a0} H_i \]

\[ = -e^{b0} e^{a0} H_0 + e^{a0} (e^{b0} H_0 + e^{bi} H_i) - e^{bi} e^{a0} H_i = 0. \] (50)

\( H_i \) does not depend explicitly or implicitly on \( e_{a0} \). We remark that by taking the variation with respect to \( e_{b0} \) of both sides of equation (26), \( H_0 = e_{a0} C^a \), we arrive at

\[ C^b = C^b + e_{a0} \frac{\delta C^a}{\delta e_{b0}}, \]
from what follows the general result \( e_{a0}(\delta C^a/\delta e_{b0}) = 0 \). Taking into account the arbitrariness of \( e_{a0} \) in the latter equation we are led to equation (50).

Therefore the vanishing of the Hamiltonian constraint \( H_0 \) implies the vanishing of \( C^a \), and ultimately of the vector constraint \( H_i \). Moreover we observe from (47) and (49) that \( H_i \) can be obtained from \( H_0 \) in \( \mathbf{P} \Gamma \) according to

\[
\frac{\delta}{\delta e_{a0}} H_0 = e_{ai} C^a = H_i .
\]

Thus \( H_i \) is derived from \( H_0 \). In the complete phase space the vanishing of \( H_i \) is a consequence of the vanishing of \( H_0, \Gamma^{ik} \) and \( \Gamma^k \).

Finally we would like to remark that the Hamiltonian formulation of the theory can be described more succinctly in terms of the constraints \( H_0, \Gamma^{ik} \) and \( \Gamma^k \), by the Hamiltonian density in the form

\[
H(e_{ak}, \Pi^{ak}, e_{a0}, \alpha_{ik}, \beta_k) = H_0 + \alpha_{ik} \Gamma^{ik} + \beta_k \Gamma^k .
\]

The Poisson brackets between these constraints are given by equations (37), (40-44). They constitute a first class set except for the fact that on the right hand side of (44) there appears the constraint \( H_i \). However it poses no problem for the consistency of the constraints provided \( H_0, \Gamma^{ik} \) and \( \Gamma^k \) are taken to vanish at the initial time \( t = t_0 \). Let \( \phi(x^i, t) \) represent any of the latter constraints. At the initial time we have \( \phi(x^i, t_0) = 0 \). At \( t_0 + \delta t \) we find \( \phi(x^i, t_0 + \delta t) = \phi(x^i, t_0) + \dot{\phi}(x^i, t_0) \delta t \) such that \( \dot{\phi}(x^i, t_0) = \{\phi(x^i, t_0), \mathbf{H}\} \). Since the vanishing of \( H_i \) at an instant of time is a consequence of the vanishing of \( H_0, \Gamma^{ik} \) and \( \Gamma^k \) at the same time, the consistency of the constraints is guaranteed at any \( t > t_0 \).
VI. Discussion

The Weitzenböck space-time allows a consistent description of the Hamiltonian formulation of the gravitational field. Although the underlying geometry is not Riemannian, the Lagrangian field equations (4) assure that the theory determined by (1) is equivalent to Einstein’s general relativity. To our knowledge there does not exist any impediment based on experimental facts that rules out the teleparallel geometry in favour of the Riemannian geometry for the description of the physical space-time. The natural geometrical setting for teleparallel gravity is the teleparallel geometry. The Hamiltonian formulation of the TEGR in the Riemannian geometry, with local SO(3,1) symmetry, requires the introduction of a large number of field variables that renders an intricate constraint structure\[19].

We have shown that the vector constraint $H_i$ can be obtained from the Hamiltonian constraint $H_0$ by means of a functional derivative of $H_0$, making use of the orthogonality properties of the tetrads in the reduced phase space $P_\Gamma$. However, it is an independent constraint. In contrast, in the ADM formulation the Hamiltonian and vector constraints are not mutually related, and in practice one has to consider both constraints for the dynamical evolution via Hamilton equations.

The number of degrees of freedom may be counted as the total number of canonical variables, $e_{ak}$ and $\Pi^{ak}$, minus twice the number of first class constraints. Therefore we have $24 - 20 = 4$ degrees of freedom in the phase space, as expected. Since the constraints $\Gamma^{ik}$ and $\Gamma^k$ are first class they act on $e_{ak}$, and $\Pi^{ak}$ and generate symmetry transformations. In particular, for $e_{a\mu}$ we have

$$
\delta e_{ak}(x) = \int d^3 z \left[ \varepsilon_{ij}(z) \{ e_{ak}(x), \Gamma^{ij}(z) \} + \varepsilon_j(z) \{ e_{ak}(x), \Gamma^j(z) \} \right]
$$
\[
\int d^3z \left[ \varepsilon_{ij}(z) \frac{\delta \Gamma^{ij}(z)}{\delta \Pi^{ak}(x)} + \varepsilon_j(z) \frac{\delta \Gamma^j(z)}{\delta \Pi^{ak}(x)} \right] = \varepsilon_{ik} e_i^a + \varepsilon_k e_a^0,
\]

where \( \varepsilon_{ij}(x) = -\varepsilon_{ji}(x) \) and \( \varepsilon_j(x) \) are infinitesimal parameters. Note that these transformations do not act on \( e_{ab} \). This issue has not been completely analyzed. The physical implications of these symmetries to the theory are currently under investigation.

In the analysis of a theory described by a Lagrangian density similar to (1), Møller pointed out that some supplementary conditions on the tetrads are needed. He suggested these conditions to arise from suitable boundary conditions for the field equations, possibly in the form of an anti-symmetric tensor. These supplementary conditions would uniquely determine a tetrad lattice, apart from a constant rotation of the tetrads in the lattice. The problem of consistently defining these supplementary conditions is likely to be related to the symmetry transformation determined by (53).

The Hamiltonian density (52) determines the time evolution of field quantities via equation (48), and in particular of the metric tensor \( g_{ij} \) of three-dimensional spacelike hypersurfaces. This property might simplify approaches to a canonical, nonperturbative quantization of gravity provided we manage to construct the reduced phase space determined by (36).

After implementing the primary constraints via equations (36), the first term of \( C^a \) is given by \(-\partial_i \Pi^{ai}\), with \( \Pi^{ai} \) defined by (9). From our previous experience (cf. ref. [16]) we are led to conclude that this term is related to energy and momentum of the gravitational field. In the present case we also interpret equations \( C^a = 0 \) as energy-momentum equations for the gravitational field. According to this interpretation, the integral form of the constraint equation \( C^{(0)} = 0 \) can be written in the form \( E - \mathcal{H} = 0 \). Integration of \(-\partial_i \Pi^{ai}\) over the whole three-dimensional space yields the ADM energy. A complete analysis of this issue will be presented elsewhere.
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