We analyze the possible soft breaking of $\mathcal{N}=2$ supersymmetric Yang-Mills theory with and without matter flavour preserving the analyticity properties of the Seiberg-Witten solution. We present the formalism for an arbitrary gauge group and obtain an exact expression for the effective potential. We describe in detail the onset of the confinement description and the vacuum structure for the pure $SU(2)$ Yang-Mills case and also some general features in the $SU(N)$ case. A general mass formula is obtained, as well as explicit results for the mass spectrum in the $SU(2)$ case.

1 Introduction and Conclusions.

In two remarkable papers\cite{Seiberg94,Seiberg95}, Seiberg and Witten obtained exact information on the dynamics of $\mathcal{N}=2$ supersymmetric gauge theories in four dimensions with gauge group $SU(2)$ and $N_f \leq 4$ flavour multiplets. Their work was extended to other groups in\cite{Moore95,Moore96}. One of the crucial advantages of using $\mathcal{N}=2$ supersymmetry is that the low-energy effective action in the Coulomb phase up to two derivatives (i.e. the Kähler potential, the superpotential and the gauge kinetic function in $\mathcal{N}=1$ superspace language) are determined in terms of a single holomorphic function called the prepotential\cite{Ghilencea94}. In references\cite{Seiberg94,Seiberg95}, the exact prepotential was determined using some plausible assumptions and many consistency conditions. For $SU(2)$ the solution is neatly presented by associating to each case an elliptic curve together with a meromorphic differential of the second kind whose periods completely determine the prepotential. For other gauge groups\cite{Moore95,Moore96}, the solution is again presented in terms of the period integrals of a meromorphic differential on a Riemann surface whose genus is the rank of the group considered. It was also shown in\cite{Seiberg94,Seiberg95} that by soft breaking $\mathcal{N}=2$ down to $\mathcal{N}=1$ (by adding a mass term for the adjoint $N=1$ chiral multiplet in the $\mathcal{N}=2$ vector multiplet) confinement follows due to monopole condensation\cite{Seiberg95}.

For $\mathcal{N}=1$ theories exact results have also been obtained\cite{Ghilencea94} using the holomorphy properties of the superpotential and the gauge kinetic function,

\textsuperscript{a}Based on a talk delivered by L. A.-G. at the Conference in honour of C. Itzykson “The Mathematical Beauty of Physics”.

\textsuperscript{a}
culminating in Seiberg’s non-abelian duality conjecture\textsuperscript{10}.

With all this new exact information it is also tempting to obtain exact information about ordinary QCD. The obvious problem encountered is supersymmetry breaking. A useful avenue to explore is soft supersymmetry breaking. The structure of soft supersymmetry breaking in $N = 1$ theories has been known for some time\textsuperscript{11}. In\textsuperscript{12}, soft breaking terms are used to explore $N = 1$ supersymmetric QCD (SQCD) with gauge group $SU(N_c)$ and $N_f$ flavours of quarks, and to extrapolate the exact results concerning the superpotential and the phase structure of these theories in the absence of supersymmetry. This leads to expected and unexpected predictions for non-supersymmetric theories which may eventually be accessible to lattice computations. In some cases however for instance when $N_f \geq N_c$ it is known in the supersymmetric case that the origin of moduli space is singular, and therefore some of the assumptions made about the Kähler potential for meson and baryon operators are probably too strong. Since the methods of\textsuperscript{12} provide us with the effective action up to two derivatives, the kinetic and potential term for all low-energy fields are under control, and therefore in this paper we prefer to explore in which way we can softly break $N = 2$ SQCD directly to $N = 0$ while at the same time preserving the analyticity properties of the Seiberg-Witten solution. This is a very strong constraint and there is, essentially, only one way to accomplish this task: we make the dynamical scale $\Lambda$ of the $N = 2$ theory a function of an $N = 2$ vector multiplet which is then frozen to become a spurion whose $F$ and $D$-components break softly $N = 2$ down to $N = 0$. If we want to interpret physically the spurion, one can recall the string derivation of the Seiberg-Witten solution in\textsuperscript{14}, based on type II-heterotic duality. In the field theory limit in the heterotic side (in order to decouple string and gravity loops) the natural scaling is taken to be $Me^{iS} = \Lambda$, where $M$ is the Planck mass, $S$ is the dilaton (in the low-energy theory $S = \theta/2\pi + 4\pi i/g^2$, with $g$ the gauge coupling constant and $\theta$ the CP-violating phase), and $\Lambda$ the dynamical scale of the gauge theory which is kept fixed while $M \to \infty$ and $iS \to \infty$. Since the dilaton sits in a vector multiplet of $N = 2$ when the heterotic string is compactified on $K3 \times T_2$, this is precisely the field we want to make into a spurion, and procedure is compatible with the Seiberg-Witten monodromies. In this way we obtain a theory at $N = 0$ with a more restricted structure that those used in\textsuperscript{12}.

As soon as the soft breaking terms are turned on monopole condensation appears, and we get a unique ground state (near the massless monopole point of\textsuperscript{13}). Furthermore, in the Higgs region we can compute the effective potential, and we can verify that this potential drives the theory towards the region where condensation takes place. When the supersymmetry breaking parameter
is increased, the minimum displaces to the right along the real $u$-axis. At the same time, the region in the $u$-plane in which the monopole condensate is energetically-favoured expands. Near the massless dyon point of $\mathcal{H}$, we find that dyon condensation is energetically favourable but, unlike monopole condensation, it is not sufficiently-strong an effect to lead to another minimum of the effective potential. Eventually, when the soft supersymmetry breaking parameter is made sufficiently large, the regions where monopole and dyon condensation are favoured begin to overlap. At this point, it is clear that our methods break down, and new physics is needed to describe the dynamics of these mutually-nonlocal degrees of freedom.

One advantage of this method of using the dilaton spurion to softly break supersymmetry from $N = 2$ to $N = 0$ is its universality. It works for any gauge group and any number of massive or massless quarks. We work out the general structure of soft breaking by the dilaton spurion in an arbitrary gauge group paying special attention to the monodromies and the properties of the spurion couplings, and we find the general features of the vacuum structure for the case of $SU(N)$. We also study the evolution of the mass eigenvalues in the case of the $SU(2)$ and also show in general that with this soft breaking procedure there is a general sum rule satisfied by the masses of all the multiplets.

The organization of this paper is as follows: In section one we present the general formalism for the breaking of supersymmetry due to a dilaton spurion for a general gauge group, and we study the symplectic transformations of the various quantities involved. The results agree with the general structure derived in [14] concerning the modification of the symplectic transformations of special geometry in the presence of background $N = 2$ vector superfields. In section three we study the effective potential and vacuum structure. In section four we particularize the formalism to the case of $SU(2)$ where the analysis can be made more explicit. In section five we analyze in some detail the case for $SU(N)$ without hypermultiplets. Finally in section six we present a general mass sum rule for the general case, and also obtain explicit results of the masses in the $SU(2)$ case. It is clear that for the moment we cannot take the supersymmetry decoupling limit due to the fact that as the supersymmetry breaking parameter increases we find that regions where mutually non-local operators acquire vacuum expectation values overlap. This raises the fascinating issue that in order to reach the real QCD limit we have to understand the dynamics of the Argyres-Douglas phases [17].
2 Breaking $N = 2$ with a dilaton spurion: general gauge group

In this section we present the generalization of the procedure introduced in \cite{2} to $N = 2$ Yang-Mills theories with a general gauge group $G$ of rank $r$ and massless matter hypermultiplets.

The low energy theory description of the Coulomb phase \cite{1} involves $r$ abelian $N = 2$ vector superfields $A^i$, $i = 1, \ldots, r$ corresponding to the unbroken gauge group $U(1)^r$. The holomorphic prepotential $F(A^i, \Lambda)$ depends on the $r$ superfields $A^i$ and the dynamically generated scale of the theory, $\Lambda$. The low energy effective lagrangian takes the form (in $N = 1$ notation)\cite{1}:

$$L = \frac{1}{4\pi} \text{Im} \left[ \int d^4 \theta \frac{\partial F}{\partial A^i} \overline{\Lambda} + \frac{1}{2} \int d^2 \theta \frac{\partial^2 F}{\partial A^i \partial A^j} \mathcal{W}^i \mathcal{W}^j \right]. \quad (2.1)$$

We define the dual variables, as in the $SU(2)$ case, by

$$a_{D,i} \equiv \frac{\partial F}{\partial a^i}. \quad (2.2)$$

The Kähler potential and effective couplings associated to (2.1) are:

$$K(a, \bar{a}) = \frac{1}{4\pi} \text{Im} a_{D,i} \bar{a}^i, \quad \tau_{ij} = \frac{\partial^2 F}{\partial a^i \partial a^j}. \quad (2.3)$$

and the metric of the moduli space is given accordingly by:

$$(ds)^2 = \text{Im} \frac{\partial^2 F}{\partial a^i \partial a^j} da^i d\bar{a}^j. \quad (2.4)$$

We introduce now a complex space $\mathbb{C}^{2r}$ with elements of the form

$$v = \left( \begin{array}{c} a_{D,i} \\ a^i \end{array} \right). \quad (2.5)$$

The metric (2.4) can then be written as

$$(ds)^2 = -\frac{i}{2} \sum_i (da_{D,i} d\bar{a}^i - a_{D,i} da^i)$$

$$= -\frac{i}{2} \left( da_{D,i} \quad d\bar{a}^i \right) \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \left( \begin{array}{c} d\bar{a}^i \\ da_{D,i} \end{array} \right), \quad (2.6)$$

4
which shows that the transformations of $v$ preserving the form of the metric are matrices $\Gamma \in Sp(2r, \mathbb{Z})$. They verify $\Gamma^T \Omega \Gamma = \Omega$, where $\Omega$ is the $2r \times 2r$ matrix appearing in (2.6), and can be written as:

$$
\begin{pmatrix}
A & B \\
C & D \\
\end{pmatrix}
$$

(2.7)

where the $r \times r$ matrices $A, B, C, D$ satisfy:

$$
A^T D - C^T B = 1_r, \quad A^T C = C^T A, \quad B^T D = D^T B.
$$

(2.8)

The vector $v$ transforms then as:

$$
\begin{pmatrix}
a_D \\
a \\
\end{pmatrix}
\rightarrow \Gamma 
\begin{pmatrix}
a_D \\
a \\
\end{pmatrix} = 
\begin{pmatrix}
Aa_D + Ba \\
Ca_D + Da \\
\end{pmatrix}.
$$

(2.9)

From this we can obtain the modular transformation properties of the prepotential $F(a^i)$ (see [19]). Since

$$\frac{\partial F_\Gamma}{\partial a^k} = \frac{\partial a_{\hat{p}}^i}{\partial a^k} \frac{\partial F}{\partial a_{\hat{p}}^i} = \left( C^{\hat{p}q} \tau_{pq} + D^i_k \right) \left( A^j_{\hat{i}} a_{D,j} + B_{ij} a^i \right)$$

$$= (D^T B)_{kj} a^j + (D^T A)^j_k \frac{\partial F}{\partial a^j} + (C^T B)^j_p \frac{\partial a_{D,p}}{\partial a^k} a^i$$

$$+ (C^T A)^j_p \frac{\partial a_{D,p}}{\partial a^k} a_{D,j},$$

(2.10)

using the properties (2.8) of the symplectic matrices we can integrate (2.10) to obtain:

$$F_\Gamma = F + \frac{1}{2} a^k (D^T B)_{kj} a^j + \frac{1}{2} a_{D,k} (C^T A)^j_p a_{D,j}$$

$$+ a^k (B^T C)^j_k a_{D,j}.$$

(2.11)

Starting with (2.11) we can prove that the quantity $F - 1/2 \sum_i a^i a_{D,i}$ is a monodromy invariant, and evaluating it asymptotically, one obtains the relation [18],[19],[20]:

$$F - \frac{1}{2} \sum_i a^i a_{D,i} = -4\pi i b_1 u,$$

(2.12)

where $b_1$ is the coefficient of the one-loop beta function (for $SU(N_c)$ with $N_f$ hypermultiplets in the fundamental representation, $b_1 = (2N_c - N_f)/16\pi^2$)
and \( u = \langle \text{Tr} \phi^2 \rangle \). With the normalization for the electric charge used in \( \ref{eq:charge} \) and \( \ref{eq:charge2} \), the r.h.s. of \( \ref{eq:2.12} \) is \(-2\pi i b_1 u\).

As in the \( SU(2) \) case, presented in \( \ref{eq:SU2} \), we break \( N = 2 \) supersymmetry down to \( N = 0 \) by making the dynamical scale \( \Lambda \) a function of a background vector superfield \( S, \Lambda = e^{iS} \). This must be done in such a way that \( s, s_D = \partial F/\partial s \) be monodromy invariant. To see this, we will derive a series of relations analogous to the ones in the \( SU(2) \) case \( \ref{eq:SU2} \), starting with the following expression for the prepotential in terms of local coordinates:

\[
F = \sum_{ij} a^i a^j f_{ij}(a'/\Lambda),
\]

(2.13)

where we take \( f_{ij} = f_{ji} \). We define now a \( (r+1) \times (r+1) \) matrix of couplings including the dilaton spurion \( a^0 = s \):

\[
\tau_{\alpha \beta} = \partial^2 F/\partial a^\alpha \partial a^\beta.
\]

(2.14)

Greek indices \( \alpha, \beta \) go from 0 to \( r \), and latin indices \( i, j \) from 1 to \( r \). We obtain:

\[
a_{D,k} = 2 \sum_i a^i f_{ik} + \frac{1}{\Lambda} \sum_{ij} a^i a^j f_{ij,k},
\]

\[
\tau_{ij} = 2 f_{ij} + \frac{2}{\Lambda} \sum_k a^k (f_{ik,j} + f_{jk,i}) + \frac{1}{\Lambda^2} a^k a^l f_{kl,ij},
\]

\[
\tau_{0i} = -\frac{i}{\Lambda} \sum_{jk} a^j a^k (2 f_{ij,k} + f_{jk,i}) - \frac{i}{\Lambda^2} \sum_{jkl} a^j a^k a^l f_{jk,li},
\]

\[
\tau_{00} = -\frac{1}{\Lambda} \sum_{ijk} a^i a^j a^k f_{ij,k} - \frac{1}{\Lambda^2} \sum_{ijkl} a^i a^j a^k a^l f_{ij,kl},
\]

(2.15)

and the dual spurion field is given by:

\[
s_D = \frac{\partial F}{\partial s} = -\frac{i}{\Lambda} \sum_{ijk} a^i a^j f_{ij,k}
\]

(2.16)

The equations (2.15) and (2.16) give the useful relations:

\[
\tau_{0i} = i(a_{D,i} - \sum_j a^j \tau_{ji}), \quad \frac{\partial \tau_{0i}}{\partial a^k} = -i \sum_j a^j \frac{\partial \tau_{ij}}{\partial a^k}.
\]
\[
\frac{\partial \tau_{00}}{\partial a^k} = i\tau_{0k} - \sum_{ij} a^i a^j \frac{\partial \tau_{ij}}{\partial a^k}.
\] (2.17)

Using now (2.12) one can prove that \( s_D \) is a monodromy invariant,
\[
\frac{\partial F}{\partial s} = i (2F - \sum_i a^i a_{D,i}) = 8\pi b_1 u
\] (2.18)
and from (2.17) and (2.18) we get
\[
\tau_{0i} = 8\pi b_1 \frac{\partial u}{\partial a^i},
\]
\[
\tau_{00} = 8\pi ib_1 \left( 2u - \sum_i a^i \frac{\partial u}{\partial a^i} \right)
\] (2.19)

Now we will present the transformation rules of the gauge couplings \( \tau_{ij} \) under a monodromy matrix \( \Gamma \) in \( Sp(2r,\mathbb{Z}) \). In terms of the local coordinates \( a^i = C^{ip} a_{D,p}(a^j, s) + D^i_{a^p} \) we have the couplings
\[
\tau^\Gamma_{\alpha\beta} = \frac{\partial^2 F}{\partial a^\alpha_\Gamma \partial a^\beta_\Gamma}.
\] (2.20)

The change of coordinates is given by the matrix:
\[
\begin{pmatrix}
\frac{\partial a^i_\Gamma}{\partial a^j} & \frac{\partial a^i_\Gamma}{\partial s} \\
\frac{\partial a^j_\Gamma}{\partial a^i} & \frac{\partial a^j_\Gamma}{\partial s}
\end{pmatrix} = \begin{pmatrix}
C^{ip} \tau_{pj} + D^i_{a^p} & C^{ip} \tau_{0p} \\
0 & 1
\end{pmatrix},
\] (2.21)
with inverse
\[
\begin{pmatrix}
\frac{\partial a^i}{\partial a^j_\Gamma} & \frac{\partial a^i}{\partial s} \\
\frac{\partial a^j}{\partial a^i_\Gamma} & \frac{\partial a^j}{\partial s}
\end{pmatrix} = \begin{pmatrix}
\left( (C_T + D)^{-1} \right)_j^i & -\left( (C_T + D)^{-1} \right)_j^k C^{kp} \tau_{00} \\
0 & 1
\end{pmatrix}.
\] (2.22)

Therefore we have:
\[
\left( \frac{\partial}{\partial a^i_\Gamma} \right)_{\Gamma-\text{basis}} = \left( (C_T + D)^{-1} \right)_j^i \frac{\partial}{\partial a^j}. 
\]
\[
\left( \frac{\partial}{\partial s} \right)_{\Gamma - \text{basis}} = \frac{\partial}{\partial s} - \left[ (C\tau + D)^{-1}C\tau \right]_0^i \frac{\partial}{\partial \mu_i}; \quad (2.23)
\]

which lead to the transformation rules for the couplings:

\[
\tau^\Gamma_{ij} = \left( A\tau + B \right) \left( C\tau + D \right)^{-1}_{ij}, \quad \tau^\Gamma_{0i} = \tau_{0j} \left( (C\tau + D)^{-1} \right)_i^j,
\]

\[
\tau^\Gamma_{00} = \tau_{00} - \tau_{0i} \left[ (C\tau + D)^{-1}C\tau \right]_0^i. \quad (2.24)
\]

3 Effective potential and vacuum structure

In this section we will obtain, starting from the formalism developed in the previous section, the effective potential in the Coulomb phase of the softly broken \( N = 2 \) theory, for a general group of rank \( r \).

To break \( N = 2 \) down to \( N = 0 \) we freeze the spurion superfield to a constant. The lowest component is fixed by the scale \( \Lambda \), and we only turn on the auxiliary \( F^0 \) (\( i.e. \) we take \( D^0 = 0 \)). We must include in the effective lagrangian \( r + 1 \) vector multiplets, where \( r \) is the rank of the gauge group:

\[
A^\alpha = (A^0, A^I), \quad I = 1, \cdots, r. \quad (3.1)
\]

There are submanifolds in the moduli space where extra states become massless and we must include them in the effective lagrangian. They are BPS states corresponding to monopoles or dyons, so we introduce \( n_H \) hypermultiplets near these submanifolds in the low energy description:

\[
(M_i, \tilde{M}_i), \quad i = 1, \cdots, n_H \quad (3.2)
\]

We suppose that these BPS states are mutually local, hence we can find a symplectic transformation such that they have \( U(1)' \) charges \( (q^I_i, -\bar{q}^I_i) \) with respect to the \( I \)-th \( U(1) \) (we follow the \( N = 1 \) notation). The full \( N = 2 \) effective lagrangian contains two terms:

\[
\mathcal{L} = \mathcal{L}_{VM} + \mathcal{L}_{HM}, \quad (3.3)
\]

where \( \mathcal{L}_{VM} \) is given in (2.1), and

\[
\mathcal{L}_{HM} = \sum_i \int d^4\theta (M_i^* e^{2q^I_i V^{(i)}} M_i + \tilde{M}_i^* e^{-2q^I_i V^{(i)}} \tilde{M}_i) + \sum_{I,i} \left( \int d^2\theta \sqrt{2} A^I q^I_i M_i \tilde{M}_i + \text{h.c.} \right) \quad (3.4)
\]

8
The terms in \( (3.3) \) contributing to the effective potential are
\[
V = b_{I,J} F^I T^J + b_{0}\left(F^0 T^0 + T^0 F^0\right) + b_{00}|F^0|^2 \\
+ \frac{1}{2} b_{I,J} D^I D^J + D^I q^J(|m_i|^2 - |\bar{m}_i|^2) + |F_{m_i}|^2 + |F_{\bar{m}_i}|^2 \\
+ \sqrt{2}(F^I q^J_i m_i \bar{m}_i + a^I q^J_i m_i F_{\bar{m}_i} + a^I q^J_i \bar{m}_i F_{m_i} + \text{h.c.}),
\]

(3.5)

where all repeated indices are summed and \( b_{\alpha\beta} = \text{Im} \tau_{\alpha\beta}/4\pi \). We eliminate the auxiliary fields and obtain:
\[
D^I = -(b^{-1})^{IJ} q^J_i (|m_i|^2 - |\bar{m}_i|^2), \\
F^I = -(b^{-1})^{IJ} b_{0J} F^0 - \sqrt{2}(b^{-1})^{IJ} q^J_i \bar{m}_i, \\
F_{m_i} = -\sqrt{2} m_i q^I_i \bar{m}_i, \quad F_{\bar{m}_i} = -\sqrt{2} \bar{m}_i q^I_i m_i.
\]

(3.6)

We denote \( (q_i, \bar{q}_j) = \sum_{IJ} q^I_i (b^{-1})^{IJ} q^J_j, \ (q_i, b_0) = \sum_{IJ} q^I_i (b^{-1})^{IJ} b_{0J}, \ a \cdot q_i = \sum_{I} a^I q_i^I \). Substituting in \( (3.5) \) we obtain:
\[
V = \frac{1}{2} \sum_{ij} (q_i, q_j)(|m_i|^2 - |\bar{m}_i|^2)(|m_j|^2 - |\bar{m}_j|^2) + 2 \sum_{ij} (q_i, q_j)m_i \bar{m}_i m_j \bar{m}_j \\
+ 2 \sum_i |a \cdot q_i|^2(|m_i|^2 + |\bar{m}_i|^2) + \sqrt{2} \sum_i (q_i, b_0)\left(F^0 m_i \bar{m}_i + \bar{F}^0 \bar{m}_i m_i\right) \\
- |F^0|^2 \frac{\text{det}b_{\alpha\beta}}{\text{det}b_{IJ}},
\]

(3.7)

where \( \frac{\text{det}b_{\alpha\beta}}{\text{det}b_{IJ}} = b_{00} - b_{0J}(b^{-1})^{IJ} b_{0J} \) is the cosmological term. This term in the potential is a monodromy invariant. To prove this it is sufficient to prove invariance under the generators of the symplectic group \( Sp(2r, \mathbb{Z}) \):
\[
\begin{pmatrix}
A & 0 \\
0 & (A^T)^{-1}
\end{pmatrix}, \quad A \in GL(r, \mathbb{Z}),
\]
\[
T_\theta = \begin{pmatrix} 1 & \theta \\ 0 & 1 \end{pmatrix}, \quad \theta_{ij} \in \mathbb{Z}, \quad \Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

(3.8)

Invariance under \( T_\theta \) and the matrix involving only \( A \) is obvious, and for \( \Omega \) one can check it easily.

The vacuum structure is determined by the minima of \( (3.7) \). As in \( (3.9) \), we first minimize with respect to \( m_i, \bar{m}_i \):
\[
\frac{\partial V}{\partial m_i} = \sum_j (q_i, q_j)(|m_j|^2 - |\bar{m}_j|^2)m_i + 2|a \cdot q_i|^2 m_i \\
+ 2 \sum_j (q_i, q_j)m_j \bar{m}_j m_i + \sqrt{2} F^0 (q_i, b_0) \bar{m}_i = 0,
\]

(3.9)
\[ \frac{\partial V}{\partial m_i} = \sum_j (q_i, q_j)(-|m_j|^2 + |\tilde{m}_j|^2)m_i + 2|a \cdot q_i|^2 \tilde{m}_i \\
+ 2 \sum_j (q_i, q_j)m_j \tilde{m}_j m_i + \sqrt{2} F_0^0 (q_i, b_0) \tilde{m}_i = 0. \] (3.10)

Multiplying (3.9) by \( \tilde{m}_i \), (3.10) by \( \tilde{m}_i \) and subtracting, we get
\[ \sum_j (q_i, q_j)(|m_j|^2 - |\tilde{m}_j|^2)(|m_i|^2 + |\tilde{m}_i|^2) + 2|a \cdot q_i|^2(|m_i|^2 - |\tilde{m}_i|^2) = 0. \] (3.11)

Suppose now that, for some indices \( i \in I \), \(|m_i|^2 + |\tilde{m}_i|^2 > 0\). Multiplying (3.11) by \(|m_i|^2 - |\tilde{m}_i|^2\) and summing over \( i \) we obtain
\[ \sum_{ij} (q_i, q_j)(|m_i|^2 - |\tilde{m}_i|^2)(|m_j|^2 - |\tilde{m}_j|^2) = -\sum_{i \in I} \frac{2|a \cdot q_i|^2}{|m_i|^2 + |\tilde{m}_i|^2}(|m_i|^2 - |\tilde{m}_i|^2)^2. \] (3.12)

The matrix \((b^{-1})^{ij}\) is positive definite, and if the charge vectors \( q^I \) are linearly independent it follows that the matrix \((q_i, q_j)\) is positive definite too. Then the l.h.s. of (3.12) is \( \geq 0 \) while the r.h.s. is \( \leq 0 \). The only way for this equation to be consistent is if
\[ |m_i| = |\tilde{m}_i|, \quad i = 1, \cdots, n_H. \] (3.13)

In this case we can write the equation (3.9), after absorbing the phase of \( F_0^0 = f_0 e^{i\gamma} \) in \( \tilde{m}_i \), as:
\[ 2|a \cdot q_i|^2 m_i + 2 \sum_j (q_i, q_j)m_j \tilde{m}_j \tilde{m}_i + \sqrt{2} f_0 (q_i, b_0) \tilde{m}_i = 0. \] (3.14)

Multiplying by \( \tilde{m}_i \) and summing over \( i \), we obtain
\[ 2 \sum_i |a \cdot q_i|^2 |m_i|^2 + \sqrt{2} f_0 \sum_i (q_i, b_0) \tilde{m}_i \tilde{m}_i = -2 \sum_{ij} (q_i, q_j)m_j \tilde{m}_j \tilde{m}_i, \] (3.15)

hence \( \sqrt{2} f_0 \sum_i (q_i, b_0) \tilde{m}_i \tilde{m}_i \) is real. We can insert in (3.7) and get the following expression for the effective potential:
\[ V = -f_0^2 \frac{\det b_{\alpha \beta}}{\det b_{IJ}} - 2 \sum_{ij} (q_i, q_j)m_j \tilde{m}_j \tilde{m}_i. \] (3.16)

If (3.13) holds, we can fix the gauge in the \( U(1) \) factors and write
\[ m_i = \rho_i, \quad \tilde{m}_i = \rho_i e^{i\phi_i} \] (3.17)
and (3.14) reads:

$$\rho_{i}^{2} \left( |a \cdot q_{i}|^{2} + \sum_{j} (q_{i}, q_{j}) \rho_{j}^{2} e^{i(\phi_{j} - \phi_{i})} + \frac{f_{0}(q_{i}, b_{0})}{\sqrt{2}} e^{-i\phi_{i}} \right) = 0. \quad (3.18)$$

Apart from the trivial solution \(\rho_{i} = 0\), we have:

$$|a \cdot q_{i}|^{2} + \sum_{j} (q_{i}, q_{j}) \rho_{j}^{2} e^{i(\phi_{j} - \phi_{i})} + \frac{f_{0}(q_{i}, b_{0})}{\sqrt{2}} e^{-i\phi_{i}} = 0 \quad (3.19)$$

and we can have a monopole (or dyon) VEV in some regions of the moduli space. Notice that for groups of rank \(r > 1\) there is a coupling between the different \(U(1)\) factors and one needs a numerical study of the equation above once the values of the charges \(q_{I}^{i}\) are known. In addition, the moduli space is in that case very complicated and explicit solutions for the prepotential and gauge couplings of the \(N = 2\) theory are difficult to find. However we still can have some qualitative information in many cases under some mild assumptions, as we will see.

4 Vacuum structure of the \(SU(2)\) Yang-Mills theory

4.1 The Seiberg-Witten Solution

In\cite{Seiberg:1994rs} Seiberg and Witten obtained the structure of the quantum moduli space of the \(N = 2\) \(SU(2)\) Yang-Mills theory and also the exact solution for the prepotential \(F\) including all the non-perturbative corrections. Some of the properties of this solution are:

i) The moduli space \(\mathcal{M}_{u}\) is parametrized by \(u = \langle \text{Tr} \phi^{2} \rangle\) and can be understood as the complex \(u\)-plane. The \(SU(2)\) symmetry is never restored, and the theory stays in the Coulomb phase throughout the moduli space.

ii) \(\mathcal{M}_{u}\) has a symmetry \(u \rightarrow -u\) (the non-anomalous subset of the \(U(1)_{R}\) group), and at the points \(u = \Lambda^{2}, -\Lambda^{2}\) singularities in the holomorphic prepotential \(F\) develop. Physically they correspond respectively to a massless monopole and dyon with charges \((q_{e}, q_{m}) = (0, 1), (-1, 1)\). Hence near \(u = \Lambda^{2}, -\Lambda^{2}\) the correct effective action should include together with the photon vector multiplet monopole or dyon hypermultiplets.

iii) The vector \(\ell v = (a_{D}, a)\) defines a flat \(SL_{2}(\mathbb{Z})\) vector bundle over the moduli space \(\mathcal{M}_{u}\). Its properties are determined by the singularities and the monodromies around them. Since \(\partial^{2} F/\partial a_{2}\) or \(\partial a_{D}/\partial a\) is the coupling constant, these data are obtained from the \(\beta\)-function in the three patches: large-\(u\), the Higgs phase, the monopole and the dyon regions. From the BPS mass
the mass of a BPS state of charge \((q_e, q_m)\) (with \(q_e, q_m\) coprime for the charge to be stable) is:

\[
M = \sqrt{2|q_e a + q_m a_D|}.
\] (4.1)

If at some point \(u_0\) in \(M_a\), \(M(u_0) = 0\), the monodromy around this point is given by:

\[
\left( \begin{array}{c} a_D \\ a \end{array} \right) \rightarrow M(q_e, q_m) \left( \begin{array}{c} a_D \\ a \end{array} \right),
\] (4.2)

\[
M(q_e, q_m) = \left( \begin{array}{cc} 1 + 2q_e q_m & 2q_e^2 \\ -2q_m^2 & 1 - 2q_e q_m \end{array} \right).
\] (4.3)

Also for large \(u\), \(F\) is dominated by the perturbative one loop contribution, obtained from the one loop \(\beta\)-function:

\[
F_{1\text{-loop}}(a) = \frac{i}{2\pi} a^2 \ln \frac{a^2}{\Lambda}.
\] (4.4)

Hence we also have monodromy at infinity. The three generators of the monodromy are therefore:

\[
M_\infty = \left( \begin{array}{cc} -1 & 2 \\ 0 & -1 \end{array} \right), \quad M_\Lambda = \left( \begin{array}{cc} 1 & 0 \\ -2 & 1 \end{array} \right), \quad M_{-\Lambda} = \left( \begin{array}{cc} -1 & 2 \\ -2 & 3 \end{array} \right);
\] (4.5)

and they satisfy:

\[
M_\infty = M_\Lambda M_{-\Lambda}.
\] (4.6)

These matrices generate the subgroup \(\Gamma_2 \subset SL_2\mathbb{Z}\) of \(2 \times 2\) matrices congruent to the unit matrix modulo 2.

We learn from (4.1)-(4.3) that in the Higgs, monopole and dyon patches, the natural independent variables to use are respectively \(a^{(h)} = a\), \(a^{(m)} = -a_D\), \(a^{(d)} = a_D - a\). Thus in each patch we have a different prepotential:

\[
F^{(h)}(a), \quad F^{(m)}(a^{(m)}), \quad F^{(d)}(a^{(d)}).
\] (4.7)

iv) The explicit form of \(a(u), a_D(u)\) is given in terms of the periods of a meromorphic differential of the second kind on a genus one surface described by the equation:

\[
y^2 = (x^2 - \Lambda^4)(x - u),
\] (4.8)

describing the double covering of the plane branched at \(\pm \Lambda^2, u, \infty\). We choose the cuts \(\{-\Lambda^2, \Lambda^2\}, \{u, \infty\}\). The correctly normalized meromorphic 1-form is:

\[
\lambda = \Lambda \frac{\sqrt{2} \int dx \sqrt{x - u/\Lambda^2}}{\sqrt{x^2 - 1}}.
\] (4.9)
Then:

\[ a(u) = \Lambda \frac{\sqrt{2}}{\pi} \int_{-1}^{1} \frac{dt \sqrt{u/\Lambda^2 - t}}{\sqrt{1 - t^2}}; \]  
(4.10)

\[ a_D(u) = \Lambda \frac{\sqrt{2}}{\pi} \int_{1}^{u/\Lambda^2} \frac{dt \sqrt{u/\Lambda^2 - t}}{\sqrt{1 - t^2}}. \]  
(4.11)

Using the hypergeometric representation of the elliptic functions:

\[ K(k) = \frac{\pi}{2} F(1/2, 1/2, 1; k^2); \quad K'(k) = K(k'); \]

\[ E(k) = \frac{\pi}{2} F(-1/2, 1/2, 1; k^2); \quad E'(k) = E(k'), \quad k'^2 + k^2 = 1, \]

we obtain:

\[ k^2 = \frac{2}{1 + u/\Lambda^2}, \quad k'^2 = \frac{u - \Lambda^2}{u + \Lambda^2}, \]

\[ a(u) = \frac{4\Lambda}{\pi k} E(k), \quad a_D(u) = \frac{4\Lambda}{i\pi} \frac{E'(k) - K'(k)}{k}. \]

(4.13)

Using the elliptic function identities:

\[ \frac{dE}{dk} = \frac{E - K}{k}, \quad \frac{dK}{dk} = \frac{1}{kk'}(E - k'^2K), \]

\[ \frac{dE'}{dk} = -\frac{k}{kk'}(E' - K'), \quad \frac{dK'}{dk} = -\frac{1}{kk'}(E' - k^2K'), \]

the coupling constant becomes:

\[ \tau_{11} = \frac{\partial a_D}{\partial a} = \frac{da_D/dk}{da/dk} = \frac{iK'}{K}, \]

(4.17)

which is indeed the period matrix of the curve (4.8).

4.2 Vacuum structure of the softly broken SU(2) theory

When we softly break the \( N = 2 \) SU(2) Yang-Mills theory we obtain an effective potential including the couplings \( \tau_{01} \) and \( \tau_{00} \). In the normalization of \( \tau_{01} \), and with \( b_1 = 1/4\pi^2 \), the spurion-induced couplings are

\[ \tau_{01} = \frac{2}{\pi} \frac{\partial u}{\partial a}, \quad \tau_{00} = \frac{2i}{\pi} (2u - a \frac{\partial u}{\partial a}). \]

(4.18)
The monodromy transformations of the couplings (2.24) have a simple expression in the $SU(2)$ case:

$$
\tau_{11}^\Gamma = \frac{\alpha \tau_{11} + \beta}{\gamma \tau_{11} + \delta}, \quad \tau_{01}^\Gamma = \frac{\tau_{01}}{\gamma \tau_{11} + \delta},
$$

$$
\tau_{00}^\Gamma = \tau_{00} - \frac{\gamma \tau_{01}}{\gamma \tau_{11} + \delta}.
$$

From the exact Seiberg-Witten solution (4.10), (4.11) and the previous equations we can compute the couplings $\tau_{ij}$ in the Higgs and monopole region.

i) Higgs region:

$$
a^{(h)} = \frac{4 \Lambda}{\pi} E' - \frac{K'}{k}, \quad \tau_{11}^{(h)} = -\frac{8i\Lambda^2}{\pi} \left(\frac{E - K}{k^2 K} + \frac{1}{2}\right),
$$

ii) Monopole region:

$$
a^{(m)} = \frac{4 \Lambda}{\pi k} E(k), \quad \tau_{11}^{(m)} = \frac{i K'}{K^2}, \quad \tau_{00}^{(m)} = \frac{8i\Lambda^2}{\pi} \left(\frac{E'}{k^2 K'} - \frac{1}{2}\right).
$$

In the analysis of the effective potential (3.7) we must first minimize with respect to the monopole (or dyon) field. For $r = 1$ the equation for the VEV (3.19) is

$$
\rho^2 + b_{11}|a|^2 + \frac{b_{01}e^{-i\phi}f_0}{\sqrt{2}} = 0,
$$

and the last term must be real so $e^{-i\phi} = \epsilon = \pm 1$. The charge is $q = 1$ in the $SU(2)$ Yang-Mills theory, both in the monopole and in the dyon regions. Apart from the solution $\rho = 0$ we can have

$$
\rho^2 = -b_{11}|a|^2 - \frac{b_{01}\epsilon f_0}{\sqrt{2}} > 0.
$$

Note that $b_{11} = \frac{1}{4\pi} \text{Im} \ 	au_{11}$ is always positive, and therefore (4.22) determines a region in the $u$-plane where the monopoles acquire a VEV. Depending on the sign of $b_{01}$ we choose the sign of $\epsilon$. In fact we can replace (4.23) by:

$$
\rho^2 = -b_{11}|a|^2 + \frac{1}{\sqrt{2}}|b_{01}|f_0 > 0
$$

(4.24)
and $f_0$ is always measured in units of $\Lambda$. Thus for the numerical plots we set $\Lambda = 1$. From (3.16) we get the effective potential:

$$V = \frac{2}{b_{11}} \rho^4 - \frac{\det b}{b_{11}} f_0^2$$

(4.25)

This is good news. It implies that the region where the monopoles acquire a VEV is energetically favored, and we have a first order phase transition to confinement. Depending on the sign of $b_{01}$, $m$ and $\tilde{m}$ are either aligned or antialigned. The $SU(2)_R$ symmetry of $N = 2$ supersymmetry is broken by the explicit off-diagonal term $b_{01} m \tilde{m} / b_{11}$ in (3.7) and by the VEV $\rho \neq 0$.

Where $\rho^2 \rightarrow 0$, the potential maps smoothly onto the potential for the Higgs region,

$$V^{(h)} = -\frac{\det b^{(h)}}{b_{11}^{(h)}} f_0^2,$$

(4.26)

where, we recall, $\det b / b_{11}$ is monodromy-invariant. In the monopole region, a nonzero monopole VEV is favoured, and the effective potential is given by (4.25) and written in terms of magnetic variables:

$$V^{(m)} = -\frac{2}{b_{11}^{(m)}} \rho^4 - \frac{\det b^{(m)}}{b_{11}^{(m)}} f_0^2$$

(4.27)

where $b^{(h)}$, $b^{(m)}$ are given in (4.20), (4.21).

In the Higgs region, the effective potential is given by (4.26) and we plot it in fig. 1. It has no minimum outside the monopole region near $u = \Lambda^2$ (where, as we shall see, the energy can be further lowered by giving the monopoles a VEV). One sees that the shape of the potential makes the fields roll towards the
monopole region. In fig. 2, we plot slices of the potential $V^{(h)}$ along the real $u$-axis and parallel to the imaginary $u$-axis with $\text{Re}(u) = \Lambda^2$. For comparison, we also plot $V^{(m)}$. Note that they agree in the Higgs region (where the monopole VEV vanishes), and that $V^{(m)}$ lowers the energy (and smooths out the cusp in $V^{(h)}$ at $u = \Lambda^2$) in the monopole region.

Next we look at the monopole region (4.24). \(a\) (i.e. \(a^{(m)}\)) is a good coordinate in this region vanishing at $u = \Lambda^2$. As soon as $f_0$ is turned on monopole condensation and confinement occur. In figs. 3,4 we plot $\rho^2$ in the $u$-plane for values of $f_0 = 0.1\Lambda$, $0.3\Lambda$; and in figs. 5,6 the effective potential (4.27) for the same values of the supersymmetry breaking parameter $f_0$.

One can see that the minimum is stable and that the size of the monopole VEV is $\sim f_0$. There are two features worth noticing. The first is that the absolute minimum occurs along the real $u$-axis. This is seen numerically and also as a consequence of the reality properties of the elliptic functions. Second, as $f_0$ is increased, the region where (4.24) holds becomes wider. This is seen in
Figure 5: Effective potential (4.27) for $f_0 = 0.1\Lambda$.

Figure 6: Effective potential (4.27) for $f_0 = 0.3\Lambda$.

Figure 7: Plot of $\rho^2$ along the real $u$-axis, for $f_0/\Lambda = (\text{from bottom to top})$ 0.1, 0.3, 0.5, 1.0.

Figure 8: $V^{(m)}/f_0^2$ along the real $u$-axis for $f_0 = 0.1\Lambda$ (top), 0.5\Lambda (middle) and $\Lambda$ (bottom).
fig. 7, where $\rho^2$ is plotted along the real $u$-axis as a function of $f_0$. Accordingly, the minimum of the effective potential moves to the right along the real $u$-axis, as one can see in fig. 8, where $V^{(m)}/f_0^2$ is plotted for three increasing values of $f_0$ (we have divided by $f_0^2$ to fit the three potentials on the same graph).

Finally, we turn to the dyon region. To understand what happens in the dyon region, we study the transformation rules of the $\tau_{ij}$ couplings under the residual $Z_8 \subset U(1)_R$ symmetry whose generator acts on the $u$-plane as $u \mapsto -u$. The reason why we need to analyze in general the behavior under $Z_8$ is because the representation we have chosen for the Seiberg-Witten solution in sections 2, 3 is well adapted to study the monopole region. Naively applying them to the dyon region, we may encounter some discontinuities due to the position of the cuts. Outside the curve of marginal stability one can write the prepotential as:

$$ F = \frac{i}{2\pi a^2} \log \frac{a^2}{\Lambda^2} + a^2 \sum_{k \geq 1} c_k \left( \frac{\Lambda}{a} \right)^{4k}. \quad (4.28) $$

If $\omega = e^{2\pi i/8}$ is the generator of the $Z_8$ symmetry, it is easy to show that the couplings $\tau_{ij}$ transform according to:

$$ a \mapsto ia, \quad a_D \mapsto i(a_D - a), \quad \tau_{11} \mapsto \tau_{11} - 1, \quad \tau_{01} \mapsto i\tau_{01}, \quad \tau_{00} \mapsto -\tau_{00}. \quad (4.29) $$

So the relation between the dyon and monopole variables is:

$$ a^{(d)}(u) = ia^{(m)}(-u), \quad a_D^{(d)}(u) = i\left(e_D^{(m)}(-u) - a^{(m)}(-u)\right), \quad (4.30) $$

$$ \tau_{11}^{(d)}(u) = \tau_{11}^{(m)}(-u) - 1, \quad \tau_{01}^{(d)}(u) = i\tau_{01}^{(m)}(-u), \quad \tau_{00}^{(d)}(u) = -\tau_{00}^{(m)}(-u), $$

with $a_D^{(d)} = -a_D$. Using the expressions for the monopole couplings in (4.21), which are well-behaved near $u = \Lambda^2$, we obtain expressions for the dyon couplings which are well-behaved near $u = -\Lambda^2$. The analysis of (4.24) changes crucially once these rules are implemented. Near the monopole region $a^{(m)} \sim i(u - \Lambda^2)$, hence $\tau_{01}^{(m)} \sim i$ is purely imaginary. In (4.27) although

---

There is one more aspect of the $Z_8$ transformation rules worth noticing. If we implement these rules we find that the condensate moves to the dyon region, and one might be tempted to conclude that with this choice it is the dyon that condenses. This is not the case. Using the one-loop $\beta$-function, we know that $\Lambda^4 \sim \exp\left(-\frac{\pi^2}{16} + i\theta\right)$. The action of $Z_8$ amounts to the change $\Lambda \mapsto i\Lambda$ or what is the same, $\theta \mapsto \theta + 2\pi$. Using the relation found in (4.21) when we make this change the massless state at $u = -\Lambda^2$ (before supersymmetry breaking) has zero electric charge, while the state at $u = \Lambda^2$ acquires charge one. Thus we find again a monopole condensate, in a way consistent with the $Z_2$-symmetry.
Figure 9: Dyon expectation value $\rho_{(d)}^2$ for $f_0 = 0.3\Lambda$ on the $u$-plane.

Figure 10: Dyon expectation value $\rho_{(d)}^2$ for $f_0 = \Lambda$ on the $u$-plane.

Figure 11: Plot of $V^{(h)}(u)$ (top) and $V^{(d)}(u)$ (bottom) versus $\text{Im}(u)$ for $\text{Re}(u) = -\Lambda^2$ and $f_0 = \Lambda$.

$b_{11}$ diverges at $u = \Lambda^2$ the divergence is cancelled by the vanishing of $a^{(m)}$ at the same point. Since $\text{Im}\tau^{(m)} > 0$ as soon as $f_0 \neq 0$ the monopoles condense. Using (4.30), however, we see that $a^{(d)} \sim (u + \Lambda^2)$ with a real coefficient. Thus $\text{Im}\tau^{(d)} = 0$ at $u = -\Lambda^2$ and we conclude from (6.21) that the dyon condensate vanishes along the real $u$-axis. Nevertheless, a dyon condensate is energetically favoured in a pair of complex-conjugate regions in the $u$-plane centered about $u = -\Lambda^2$. We plot $\rho_{(d)}^2$, for two different values of $f_0$ in figs. 9,10.

Unlike the monopole VEV, the magnitude of the dyon VEV is tiny on the scale of $V^{(h)}$. It therefore makes an all-but-negligible contribution to the effective potential (fig. 11). In particular, $V^{(d)}$ does not have a minimum in the dyon region. The only minimum of the full effective potential is the one we previously found in the monopole region.

As we have already noted, the monopole region (in which $\rho_{(m)}^2 \neq 0$) expands as $f_0$ is increased. Eventually, for $f_0 \sim 1.3\Lambda$, it reaches the dyon region (in which $\rho_{(d)}^2 \neq 0$). At this point, it is clear that our whole approximation of
including *just* the monopole field (or *just* the dyon field) in the effective action breaks down.

What are the other limitations of our approximations? First, we have neglected certain soft supersymmetry breaking terms which arise when we derive the soft breaking terms from spontaneously broken $N = 2$ supergravity. These additional terms scale to zero in the rigid limit, that is, they are suppressed by powers of $\log \frac{\Lambda}{M_{Pl}}$ or $\frac{\Lambda}{M_{Pl}}$ and, for our purposes are negligible. We have also neglected higher-spinor-derivative corrections to the Seiberg-Witten effective action. These clearly cannot affect the vacuum structure in the supersymmetric limit. They also, *by definition* must be supersymmetric; otherwise they lead to explicitly hard supersymmetry breaking terms, which is an entirely different matter from the soft supersymmetry breaking we are considering. Nevertheless, once supersymmetry is broken, they can, in principle, lead to corrections to the scalar potential suppressed by higher powers of $f_{0}^{2}/\Lambda^{2}$. For the moderate values of $f_{0}$ that we are considering, these corrections are numerically rather small, and do not affect the qualitative features of the solutions we have found. *A priori*, if the higher spinor derivative terms in the Seiberg-Witten effective action were known, we could systematically improve our approximations by going to higher order in $f_{0}^{2}/\Lambda^{2}$.

However, the fundamental obstacle to pushing our approximation to larger values of the soft supersymmetry breaking parameters would remain. The mutual non-locality of the monopoles and dyons leads to our inability to calculate the effective potential where the monopole and dyon regions overlap. Since this is, at least initially, far from the monopole vacuum, we expect that the monopole vacuum persists, at least as metastable minimum, even beyond the critical value of $f_{0}$. But we do not know when (or if) a new, lower minimum develops once the monopole and dyon regions overlap. If a new vacuum does appear there, then we would have a first order phase transition to this new confining phase. This raises the exciting possibility that the correct description of the QCD vacuum requires the introduction of mutually non-local monopoles and dyons. Phases of this nature have been shown to arise in the $N = 2$ moduli space for gauge group $SU(3)$. Perhaps the way to approach the true QCD vacuum in the correct phase is to start with one of these $N = 2$-superconformal field theories and turn on a relevant, soft supersymmetry-breaking perturbation.

---

\[\text{An explicit realization of this phase transition due to the overlapping of monopole and dyon regions occurs in the softly broken } SU(2) \text{ theory with one massless hypermultiplet.}\]
5 Vacuum structure of the $SU(N)$ Yang-Mills theory

The moduli space of vacua of the $N = 2$ $SU(N)$ Yang-Mills can be parametrized in a gauge-invariant way by the elementary symmetric polynomials $s_l, l = 2, \cdots, N$ in the eigenvalues of $\langle \phi \rangle, \phi_i$. The vacuum structure of the theory is associated to the hyperelliptic curve

$$y^2 = P(x)^2 - \Lambda^{2N},$$

$$P(x) = \frac{1}{2} \det(x - \langle \phi \rangle) = \frac{1}{2} \prod_i (x - \phi_i),$$

(5.1)

where $\Lambda$ is the dynamical scale of the $SU(N)$ theory and $P(x)$ can be written in terms of the variables $s_l$ as $P(x) = 1/2 \sum_l (-1)^l s_l x^{N-l}$. Once the hyperelliptic curve is known, one can compute in principle the metric on the moduli space and the exact quantum prepotential, but explicit solutions are difficult to find (they have been obtained in $\text{SU}(3)$ for the $SU(3)$ case). But as in the $SU(2)$ case one expects that the minima of the effective potential for the $SU(N)$ theory are near the $N = 1$ points (at least for a small supersymmetry breaking parameter). The physics of the $N = 1$ points in $SU(N)$ theories has a much simpler description because it involves only small regions of the moduli space, and has been studied in $\text{SU}(3)$. The $N = 1$ points correspond to points in the moduli space where $N - 1$ monopoles coupling to each $U(1)$ become massless simultaneously. From the point of view of the hyperelliptic curve this corresponds to a simultaneous degeneration of the $N - 1$ $a$-cycles, associated to monopoles. This means in turn that the polynomial $P(x)^2 - \Lambda^{2N}$ must have $N - 1$ double zeros and two single zeros. If we set $\Lambda = 1$, this can be achieved with the Chebyshev polynomials

$$P(x) = \cos\left(\frac{N \arccos x}{2}\right),$$

(5.2)

and the corresponding eigenvalues are $\phi_i = 2 \cos \pi (i - \frac{1}{2})/N$. The other $N - 1$ points, corresponding to the simultaneous condensation of $N - 1$ mutually local dyons, are obtained with the action of the anomaly-free discrete subgroup $\mathbb{Z}_{4N} \subset U(1)_R$. One can perturb slightly the curve (5.2) to obtain the effective lagrangian (or equivalently, the prepotential) at lowest order. What is found is that, in terms of the dual monopole variables $a_{D,I}$, the $U(1)$ factors are decoupled and $\tau_{I,J} \sim \delta_{I,J} \tau_I$. Near the $N = 1$ point where $N - 1$ monopoles become massless one can then simplify the equation (3.19) for the monopole VEVs, because $q^I = \delta^I_I, (b^{-1})^{IJ} = \delta^{IJ} b^{-1}_{I}$. The equation reduces then to $r = N - 1$ $SU(2)$-like equations, and in particular the phase factors $e^{-i \phi_i}$ must...
be real. We then set $e^{-i\phi_I} = \epsilon_I$, $\epsilon_I = \pm 1$. The VEVs are determined by:

$$\rho_I^2 = -b_I |a_{D,I}|^2 - \frac{f_0 b_{0I} \epsilon_I}{\sqrt{2}}, \quad I = 1, \cdots, r. \tag{5.3}$$

The effective potential \[3.16\] reads:

$$V = -f_0^2 \left( b_{00} - \sum_I \frac{b_{0I}^2}{b_I} \right) - 2 \sum_I \frac{1}{b_I} \rho_I^4. \tag{5.4}$$

The quantities that control, at least qualitatively, the vacuum structure of the theory, are $b_{0I}$ and $b_{00}$. If $b_{0I} \neq 0$ at the $N = 1$ points, we have a monopole VEV for $\rho_I$ around this point. If $b_{0I} = 0$, we still can have a VEV, as it happens in the $SU(2)$ case in the dyon region, but we expect that it will be too tiny to produce a local minimum. When one has monopole condensation at one of these $N = 1$ points in all the $U(1)$ factors, the value of the potential at this point is given by

$$V = -f_0^2 b_{00}, \tag{5.5}$$

and if the local minimum is very near to the $N = 1$ point, we can compare the energy of the different $N = 1$ points according to (5.5) and determine the true vacuum of the theory. Hence, to have a qualitative picture of the vacuum structure, and if we suppose that the minima of the effective potential will be located near the $N = 1$ points, we only need to evaluate $b_{0I}, b_{00}$ at these points. This can be done using the explicit solution in 5 and the expressions \[2.19\].

To obtain the correct normalization of the constant appearing in (2.18) we can evaluate $\sum_I a_{D,I} da/du - ada_{D,I}/du$ in the $N = 1$ points, obtaining the constant value $4\pi i b_1$. The value of the quadratic Casimir at the $N = 1$ point described by (5.2) is

$$u = \langle \text{Tr} \phi^2 \rangle = 4 \sum_{i=1}^N \cos^2 \frac{\pi i - 1/2}{N} = 2N, \tag{5.6}$$

and the values at the other $N = 1$ points are given by the action of $\mathbb{Z}_N$ ($u$ has charge 4 under $U(1)_R$): $u^{(k)} = 2\omega^k N$, $\omega = e^{\pi i/2N}$ with $k = 0, \cdots, N - 1$. To compute $\tau_{0I}$ we must also compute $\partial u/\partial a_{D,I}$. Using the results of 5.2, we have:

$$\frac{\partial u}{\partial a_{D,I}} = -4i \sin \frac{\pi I}{N}, \tag{5.7}$$

and using $b_1 = 2N/16\pi^2$, we obtain

$$\tau_{0I} = 4\pi b_1 \frac{\partial u}{\partial a_{D,I}} = -\frac{2Ni}{\pi} \sin \frac{\pi I}{N}. \tag{5.8}$$
At the \( N = 1 \) point where \( N - 1 \) monopoles condense, \( a_{D,I} = 0 \), therefore

\[
\tau_{00} = 8\pi i u = \frac{2i}{\pi} N^2. \tag{5.9}
\]

Equation (5.8) indicates that monopoles condense at this point in all the \( U(1) \) factors, but with different VEVs. This is a consequence the spontaneous breaking of the \( S_N \) symmetry permuting the \( U(1) \) factors.

To study the other \( N = 1 \) points we must implement the \( Z_N \) symmetry in the \( u \)-plane. The local coordinates \( a^{(k)}_I \) vanishing at these points are given by a \( Sp(2r,\mathbb{Z}) \) transformation acting on the coordinates \( a_I, a_{D,I} \) around the monopole point. The \( Z_N \) symmetry implies that

\[
\frac{\partial u}{\partial a^{(k)}_I}(u^{(k)}) = \omega^{2k} \frac{\partial u}{\partial a_{D,I}}(u^{(0)}), \tag{5.10}
\]

and then we get

\[
b^{(k)}_{0I} = \frac{1}{4\pi} \text{Im}\tau^{(k)}_{0I} = -\frac{N}{2\pi^2} \cos \frac{\pi k}{N} \sin \frac{\pi I}{N},
\]

\[
b^{(k)}_{00} = \frac{1}{4\pi} \text{Im}\tau^{(k)}_{00} = \frac{1}{2} \left( \frac{N}{\pi} \right)^2 \frac{2\pi k}{N}. \tag{5.11}
\]

The first equation tells us that generically we will have dyon condensation at all the \( N = 1 \) points, and the second equation together with (5.5) implies that the condensate of \( N - 1 \) monopoles at \( u = 2N \) is energetically favoured, and then it will be the true vacuum of the theory. Notice that the \( Z_N \) symmetry works in such a way that the size of the condensate, given by \( |\cos \frac{2\pi k}{N}| \), corresponds to an energy given by \( -\cos \frac{2\pi k}{N} \): as one should expect, the bigger the condensate the smaller its energy. In fact, for \( N \) even the \( N = 1 \) point corresponding to \( k = N/2 \) has no condensation. In this case the energy is still given by (5.5), as the effective potential equals the cosmological term with \( b_{0I} = 0 \), and is the biggest one.

6 Mass formula in softly broken \( N = 2 \) theories

6.1 A general mass formula

In some cases the mass spectrum of a softly broken supersymmetric theory is such that the graded trace of the square of the mass matrix is zero as it happens in supersymmetric theories. We will see in this section that this is also the case when we softly break \( N = 2 \) supersymmetry with a dilaton spurion.
We will then compute the trace of the squared mass matrix which arises from the effective lagrangian (3.3), once the supersymmetry breaking parameter is turned on. The fermionic content of the theory is as follows: we have fermions $\psi^I, \lambda^I$ coming from the $N = 2$ vector multiplet $A^I$ (in $N = 1$ language, $\psi^I$ comes from the $N = 1$ chiral multiplet and $\lambda^I$ from the $N = 1$ vector multiplet). We also have “monopolinos” $\psi_{m_i}, \psi_{\tilde{m}_i}$ from the $n_H$ matter hypermultiplets. To obtain the fermion mass matrix, we just look for fermion bilinears in (3.3). From the gauge kinetic part and the Kähler potential in $\mathcal{L}_{VM}$ we obtain:

$$\frac{i}{16\pi} F^a \partial_a \tau_{IJ} \lambda^I \lambda^J + \frac{i}{16\pi} F^a \partial_a \tau_{IJ} \psi^I \psi^J. \quad (6.12)$$

where $F^0 = f_0$ and the auxiliary fields $F^I$ are given in (3.6). From the kinetic term and the superpotential in $\mathcal{L}_{HM}$ we get:

$$i\sqrt{2} \sum_i q_i \cdot \lambda (m_i \psi_{m_i} - \bar{m}_i \psi_{\tilde{m}_i})$$

$$- \sqrt{2} \sum_i \left( a \cdot q_i \psi_{m_i} \psi_{m_i} - q_i \cdot \psi \psi_{m_i} m_i + q_i \cdot \psi \psi_{\tilde{m}_i} \tilde{m}_i \right) \quad (6.13)$$

If we order the fermions as $(\lambda, \psi, \psi_{m_i}, \psi_{\tilde{m}_i})$ and denote

$$\mu^{IJ} = iF^a \partial_a \tau_{IJ}/4\pi,$$

$$\hat{\mu}^{IJ} = iF^a \partial_a \tau_{IJ}/4\pi,$$

the “bare” fermionic mass matrix reads:

$$M_{1/2} = \left( \begin{array}{cccc}
\mu/2 & 0 & i\sqrt{2} q_i \bar{m}_i & -i\sqrt{2} q_i \bar{m}_i \\
0 & \hat{\mu}/2 & -\sqrt{2} q_i m_i & -\sqrt{2} q_i m_i \\
i\sqrt{2} q_i \bar{m}_i & -\sqrt{2} q_i m_i & 0 & -\sqrt{2} a \cdot q_i \\
-\sqrt{2} q_i \bar{m}_i & -\sqrt{2} q_i m_i & -\sqrt{2} a \cdot q_i & 0
\end{array} \right), \quad (6.14)$$

but we must take into account the wave function renormalization for the fermions $\lambda^I, \psi^I$ and consider

$$\mathcal{M}_{1/2} = Z M_{1/2} Z, \quad Z = \left( \begin{array}{cccc}
b^{-1/2} & 0 & 0 & 0 \\
0 & b^{-1/2} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array} \right) \quad (6.15)$$

The trace of the squared fermionic matrix can be easily computed:

$$\text{Tr} \mathcal{M}_{1/2} \mathcal{M}_{1/2}^\dagger = \frac{1}{4} \text{Tr} [\mu b^{-1} \bar{\mu} b^{-1} + \hat{\mu} b^{-1} \bar{\hat{\mu}} b^{-1}]$$

$$+ 4 \sum_i |a \cdot q_i|^2 + 8 \sum_i (q_i, q_i)(|m_i|^2 + |\bar{m}_i|^2). \quad (6.16)$$
The scalars in the model are the monopole fields \( m_i \), \( \bar{m}_i \) and the lowest components of the \( N = 1 \) chiral superfields in the \( A' \), \( a' \). To compute the trace of the scalar mass matrix we need

\[
\frac{\partial^2 V}{\partial m_i \partial m_i} = \sum_i (q_i, q_i)(|m_i|^2 - |ar{m}_i|^2) + (q_i, q_i)(|m_i|^2 + 2|\bar{m}_i|^2) + 2|a \cdot q_i|^2,
\]

\[
\frac{\partial^2 V}{\partial \bar{m}_i \partial \bar{m}_i} = -\sum_i (q_i, q_i)(|m_i|^2 - |ar{m}_i|^2) + (q_i, q_i)(2|m_i|^2 + |ar{m}_i|^2) + 2|a \cdot q_i|^2,
\]

\[
\frac{\partial^2 V}{\partial a' \partial \bar{a}'} = \sum_{k,l} (q_k, q_l)\left(|m_k|^2 + |ar{m}_k|^2\right) + \sum_i |a \cdot q_i|^2 + 2\sum_{k,i} (b_{ij} - 1)\frac{\partial^2 V}{\partial a' \partial \bar{a}'}.
\]

(6.17)

In the last expression we used that, due to the holomorphy of the couplings \( \tau_{\alpha\beta}, \partial^2_{\alpha\beta} b_{\alpha\beta} = 0 \). If we assume that we are in the conditions of section 2, at the minimum we have \( |m_i| = |ar{m}_i| \), and the trace of the squared scalar matrix is

\[
\text{Tr}M_0^2 = 6 \sum_i (q_i, q_i)(|m_i|^2 + |ar{m}_i|^2) + 8 \sum_i |a \cdot q_i|^2 + 2(b_{ij} - 1)\frac{\partial^2 V}{\partial a' \partial \bar{a}'}.
\]

(6.18)

where we have included the wave function renormalization for the scalars \( a' \).

The mass of the dual photon is given by the monopole VEV through the magnetic Higgs mechanism:

\[
\text{Tr}M_0^2 = 2 \sum_i (q_i, q_i)(|m_i|^2 + |ar{m}_i|^2).
\]

(6.19)

Taking into account all these contributions, the graded trace of the squared matrix is:

\[
\sum_j (-1)^{2j}(2j + 1)\text{Tr}M_j^2 = \frac{1}{2}\text{Tr}[\mu b^{-1}\bar{b}\mu^{-1} + \bar{\mu} b^{-1}\mu^{-1}] + 2f_0^2\text{Tr}b^{-1}\partial\partial(b_0, b_0) + 4 \sum_{k, l} \text{Tr}b^{-1}\partial\partial(q_k, q_l)m_k\bar{m}_k\bar{m}_l
\]

\[
+ 2\sqrt{2} \sum_k \text{Tr}b^{-1}\partial\partial(q_k, b_0)f_0(m_k\bar{m}_k + \bar{m}_k\bar{m}_k).
\]

(6.20)
To see that this is zero, we write the bilinears in the monopole fields in terms of the auxiliary fields $F^I, \tilde{F}^I$, using (3.6):

$$
\sum_i q_i^I \overline{m_i} \tilde{m}_i = -\frac{1}{\sqrt{2}}(b_{IJ} F^J + b_{0I} f_0).
$$

(6.21)

Then we can group the terms in (6.20) depending on the number of $F^I, \tilde{F}^I$, and check that they cancel separately. For instance, for the terms with two auxiliaries, we have from the first term in (6.20):

$$
-2(F^I \tilde{F}^J + \tilde{F}^I F^J) \partial_I b_{MN}(b^{-1})^{NP} \partial_J b_{PQ}(b^{-1})^{QM}
$$

(6.22)

and from the third term

$$
2F^I \tilde{F}^J \partial_M b_{JN}(b^{-1})^{NP} \partial_Q b_{P1}(b^{-1})^{QM} + 2F^I \tilde{F}^J \partial_M b_{P1}(b^{-1})^{NP} \partial_Q b_{JN}(b^{-1})^{QM}.
$$

(6.23)

Taking into account the holomorphy of the couplings and the Kähler geometry, we have $\partial_M b_{P1} = \partial_1 b_{MP}$, $\partial_Q b_{JN} = \partial_N b_{QJ}$, so (6.22) and (6.23) add up to zero. With a little more algebra one can verify that the terms with one $F^I$ (and their conjugates with $\tilde{F}^I$) and without any auxiliaries add up to zero too. The result is then:

$$
\sum_j (-1)^{2j}(2j + 1) \text{Tr} M_j^2 = 0.
$$

(6.24)

6.2 Mass spectrum in the SU(2) case

In the SU(2) case we can obtain much more information about the mass matrix and also determine its eigenvalues. First we consider the fermion mass matrix. Taking into account that at the minimum of the effective potential $m = \overline{m} = \rho, \tilde{m} = \epsilon m$, we can introduce the linear combination:

$$
\eta_\pm = \frac{1}{\sqrt{2}}(\psi_m \pm \epsilon \psi_{\tilde{m}}).
$$

(6.25)

With respect to the new fermion fields $(\lambda, \eta_+, \psi, \eta_-)$, the bare fermion mass matrix reads:

$$
M_{1/2} = \begin{pmatrix}
\frac{1}{2} \mu & -2\epsilon \rho & 0 & 0 \\
-2\epsilon \rho & \sqrt{2} \epsilon a & 0 & 0 \\
0 & 0 & \frac{1}{2} \mu & 2i \rho \\
0 & 0 & 2i \rho & -\sqrt{2} \epsilon a
\end{pmatrix},
$$

(6.26)
Notice that, in the $SU(2)$ case, the auxiliary field $F$ is real and $\mu = \hat{\mu}$. $\mathcal{M}_{1/2}^\dagger \mathcal{M}_{1/2}$ can be easily diagonalized. From (6.26) it is easy to see that the squared fermion mass matrix is block-diagonal with the same $2 \times 2$ matrix in both entries:

$$
\begin{pmatrix}
  b_{11}^{-2} \mu \bar{\mu} /4 + 4 b_{11}^{-1} \rho^2 & -e b_{11}^{-3/2} \mu \rho + 2 \sqrt{2} \alpha \rho \\
  -e b_{11}^{-3/2} \mu \rho + 2 \sqrt{2} \alpha \rho & 4 b_{11}^{-1} \rho^2 + 2 |a|^2
\end{pmatrix}.
$$

Hence there are two different eigenvalues doubly degenerated. In terms of the determinant and trace of (6.27),

$$
\alpha = (m_1^F)^2 + (m_2^F)^2 = \frac{1}{4 b_{11}^2} \mu \bar{\mu} + 2 |a|^2 + \frac{8 a^2}{b_{11}^2},
$$
$$
\beta = \frac{1}{b_{11}^2} 4 \rho^2 + \frac{4 a \mu}{\sqrt{2}},
$$

the eigenvalues are:

$$(m_{1,2}^F)^2 = \frac{\alpha}{2} \pm \frac{1}{2} \sqrt{\alpha^2 - 4 \beta}.\quad (6.29)$$

The computation of the scalar mass matrix is more lengthy. First we must compute the second derivatives of the effective potential, evaluated at the minimum. To obtain more simple expressions, we can use the identities (2.19) to express all the derivatives of the couplings in terms only of $\partial b_{11} / \partial a$, $\partial^2 b_{11} / \partial a^2$. The results are:

$$
\frac{\partial^2 V}{\partial m \partial m} = \frac{3}{b_{11}^2} \rho^2 + 2 |a|^2, \quad \frac{\partial^2 V}{\partial m \partial \bar{m}} = \frac{\partial^2 V}{\partial \bar{m} \partial m} = \frac{\partial^2 V}{\partial m^2} = \frac{1}{b_{11}^2} \rho^2,
$$
$$
\frac{\partial^2 V}{\partial m \partial a} = \frac{\partial^2 V}{\partial a \partial m} = \frac{\partial^2 V}{\partial m \partial \bar{a}} = \frac{\partial^2 V}{\partial \bar{m} \partial a} = \frac{\partial^2 V}{\partial m \partial a} = \frac{\partial^2 V}{\partial \bar{m} \partial a},
$$
$$
\frac{\partial^2 V}{\partial \bar{a} \partial a} = \frac{\partial^2 V}{\partial a \partial \bar{a}} = \frac{4 \rho^2 + \frac{1}{2 b_{11}^2} \mu \bar{\mu}}{2},\quad (6.30)
$$
and the rest of the derivatives are obtained through complex conjugation. In the last line we used the result of the previous section. To obtain the bosonic mass matrix we must take into account the wave-function renormalization of the $a, \bar{a}$ variables, as in (6.18). Its eigenvalues are as follows: we have a zero eigenvalue corresponding to the Goldstone boson of the spontaneously broken $U(1)$ symmetry. There is also an eigenvalue with degeneracy two given by:

$$2\left(\frac{\partial^2 V}{\partial \tilde{m} \partial m} - \frac{\partial^2 V}{\partial m^2}\right) = -\frac{2\sqrt{2}\epsilon}{b_{11}} f_0 b_{01}. \quad (6.31)$$

Notice that this is always positive if we have a non-zero VEV for $\rho$. The other three eigenvalues are best obtained numerically, as they are the solutions to a third-degree algebraic equation.

As an application of these general results, we can plot the mass spectrum as a function of the supersymmetry breaking parameter $f_0$ in the $SU(2)$ Yang-Mills case, where the minimum corresponds to the monopole region and $\epsilon = -1$. We have only to compute the derivatives of the magnetic coupling, with the result:

$$\frac{\partial^2 \tau_{11}^{(m)}}{\partial \tilde{a}_1(m)} = \frac{\pi^2}{8} \frac{k}{k'^2 K'3}, \quad \frac{\partial^2 \tau_{11}^{(m)}}{\partial \tilde{a}_1(m)^2} = -\frac{\pi i}{32} \frac{k^2}{k'^4 K'^4} \left(k'^2 - k^2 + 3E'^4\right). \quad (6.32)$$

These derivatives diverge at the monopole singularity $u = 1$, and we may think that this can give some kind of singular behaviour there. In fact this is not so. The position of the minimum, $u_0$, behaves almost linearly with respect to $f_0$, $u_0 - 1 \sim f_0$, and this guarantees that the behaviour very near to $u = 1$ (corresponding to a very small $f_0$) is perfectly smooth, as one can see in the figures. In fig. 12 we plot the fermion masses (6.29) (top and bottom)
and the photon mass given in (6.13) (middle). In fig. 13 we plot the masses of the scalars, where the second one from the top corresponds to the doubly degenerated eigenvalue (6.31).

Acknowledgments

One of us (L. A.-G.) would like to thank J.M. Drouffe and J.B. Zuber for the opportunity to present this work at the conference in honour of C. Itzykson “The Mathematical Beauty of Physics”. We would also like to thank J. Distler and C. Kounnas for an enjoyable collaboration.

References

1. N. Seiberg and E. Witten, Nucl. Phys. B426 (1994) 19, hep-th/9407087.
2. N. Seiberg and E. Witten, Nucl. Phys. B431 (1994) 484, hep-th/9408099.
3. A. Klemm, W. Lerche, S. Theisen and S. Yankielowicz, Phys. Lett. B344 (1995) 169, hep-th/9411048.
4. A.Klemm, W.Lerche, S.Theisen and S.Yankielowicz, Phys. Lett. B358 (1995) 73, hep-th/9504102.
5. A. Hanany and Y. Oz, Nucl. Phys. B452 (1995) 283, hep-th/9505072.
6. P.C. Argyres, M.R. Plesser and A.D. Shapere, Phys. Rev. Lett. 75 (1995) 1699, hep-th/9505100.
7. B. de Wit and A. Van Proeyen, Nucl. Phys. B245 (1984) 89; see also P. Fré and P. Soriani, “The N = 2 Wonderland”, World Scientific, 1995, for a complete set of references.
8. G. ’t Hooft, 1976, in “High Energy Physics”, edited by A. Zichichi, Palermo, 1976;
S. Mandelstam, Phys. Rep. C23 (1976) 245.
9. I. Affleck, M. Dine and N. Seiberg, Nucl. Phys. B241 (1984) 493; B256 (1985) 557;
D. Amati, G.C. Rossi, G. Veneziano, Nucl. Phys. B249 (1985) 1; D.
Amati, K. Konishi, Y. Meurice, G.C. Rossi and G. Veneziano, Phys. Rep. 162 (1988) 169;
T.R. Taylor, G. Veneziano and S. Yankielowicz, Nucl. Phys. B218 (1982); G. Veneziano and S. Yankielowicz, Phys. Lett. 113B (1982) 231;
N. Seiberg, Phys. Lett. B318 (1993) 469, hep-ph/9309335; Phys. Rev. D49 (1994) 6857, hep-th/9402044;
K. Intriligator, R. Leigh and N. Seiberg, Phys. Rev. D50 (1994) 1052, hep-th/9403198;
K. Intriligator, Phys. Lett. B336 (1994) 409, hep-th/9407106;
K. Intriligator and N. Seiberg, Nucl. Phys. B431 (1994) 551, hep-th/9408155.
10. N. Seiberg, Nucl. Phys. B435 (1995) 129, hep-th/9411149;
   P.C. Argyres, M.R. Plesser, N. Seiberg and E. Witten, Nucl. Phys. B461 (1996) 71, hep-th/9511154.
11. L. Girardello and M.T. Grisaru, Nucl. Phys. B194 (1982) 65.
12. O. Aharony, J. Sonnenschein, M.E. Peskin and S. Yankielowicz, Phys. Rev. D52 (1995) 6157, hep-th/9507013.
13. N. Evans, S.D.H. Hsu and M. Schwetz, Phys. Lett. B355 (1995) 475, hep-th/9503186;
   N. Evans, S.D.H. Hsu, M. Schwetz, S.B. Selipsky, Nucl. Phys. B456 (1995) 205, hep-th/9508002.
14. S. Kachru and C. Vafa, Nucl. Phys. B450 (1995) 69, hep-th/9505103.
15. S. Kachru, A. Klemm, W. Lerche, P. Mayr, and C. Vafa, Nucl. Phys. B459 (1996) 537, hep-th/9508153.
16. B. de Wit, hep-th/9602060.
17. P.C. Argyres and M. Douglas, Nucl. Phys. B448 (1995) 166, hep-th/9505062.
18. M. Matone, Phys. Lett. B357 (1995) 342, hep-th/9506102.
19. J. Sonnenschein, S. Theisen and S. Yankielowicz, Phys. Lett. B367 (1996) 145, hep-th/9510129.
20. T. Eguchi and S.-K. Yang, Mod. Phys. Lett. A11 (1996) 131, hep-th/9510183.
21. L. Alvarez-Gaumé, J. Distler, C. Kounnas and M. Mariño, hep-th/9604004.
22. M.K. Prasad and C.M. Sommerfield, Phys. Rev. Lett. 35 (1975) 760;
   E.B. Bogomolny, Sov. J. Nucl. Phys. 24 (1976) 449.
23. E. Witten and D. Olive, Phys. Lett. B78 (1978) 97.
24. I.S. Gradsteyn and I.M. Ryzhik, “Tables of series, products and integrals”, Academic Press.
25. E. Witten, Phys. Lett. B86 (1979) 283.
26. L. Álvarez-Gaumé and M. Mariño, to appear.
27. S. Ferrara, L. Girardello and F. Palumbo, Phys. Rev. D20 (1979) 403.