Almost topological fractional charge pumping in a non-interacting quantum dot

Masahiro Hasegawa,1 Étienne Jussiau,2 and Robert S. Whitney2

1Institute for Solid State Physics, The University of Tokyo, Kashiwa, Chiba 277-8581, Japan
2Laboratoire de Physique et Modélisation des Milieux Condensés, Université Grenoble Alpes and CNRS, BP 166, 38042 Grenoble, France.
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Topological fractional charge pumping is seen in models of strongly-correlated or topological systems. We show that similar pumping occurs in a much less exotic system; a single-level non-interacting quantum dot, when driving the dot-reservoir coupling from weak to strong coupling. The pumped charge averaged over many cycles is quantized at a fraction of an electron per cycle, determined by the ratio of Lamb shift to level-broadening; this ratio is imposed by the reservoir band-structure. For uniform density of states, half an electron is pumped per cycle. We call this pumping almost topological, because the pumping’s Berry curvature is sharply peaked in the parameter space. Hence, so long as the pumping contour does not touch the peak, the pumped charge depends only on how many times the contour winds around the peak (up to exponentially small corrections). In one limit the Berry curvature becomes a delta-function, and the fractional pumping is entirely topological. Our results show that quantization of the average pumped charge is not proof of exotic physics, such as fractional particles, without other supporting evidence.

I. INTRODUCTION

Since the seminal work of Thouless on quantum pumping,1 there have been many pumping and turnstile protocols discussed in nanoscale systems2–12 and cold atom experiments13,14. In recent years there has been great interest in exotic systems which exhibit topological pumping of fractional charges, meaning that any two driving contours with the same topology will drive the same fractional charge. Such fractional charge pumping has been found in models of Coulomb-blockaded quantum dots,15–19 topological insulators,20,21 systems with fractional quantum Hall physics,22,23 fermionic gases with short range interactions,24 fractional levitons,25 and the Bose-Hubbard model.26 These models have either strong interaction effects or non-trivial topological properties (non-zero Chern numbers, or similar). This makes us ask if either are necessary; can a non-interacting topologically-trivial system also exhibit fractional pumping of a topological nature?

This work considers a non-interacting single-level quantum dot at low temperatures, and shows that fractional pumping of an almost topological nature is possible. This pumping becomes entirely topological in one specific limit. This fractional pumping occurs when we drive the dot-reservoir couplings, $K_L$ and $K_R$, from weak to strong-coupling and back around the pumping cycle, while the dot-level is held fixed at energy $\epsilon_d$, which is above the reservoir’s electro-chemical potential, $\mu$. Here “strong” coupling means that it induces a level broadening larger than $(\epsilon_d - \mu)$, so the dot level spreads across the electrochemical potential. The pumped charge is given by the integral over the Berry curvature inside the contour, which is sharply peaked and decays exponentially away from the peak. Formally, the pumping would only be topological if this peak was a Dirac $\delta$-function. Here the peak has a finite extent, so we refer to the pumping as almost topological, because it depends only on how many times the contour winds around the peak — up to exponentially small corrections — for any pumping contour that does not impinge on the peak. Half an electron is pumped per cycle, if the reservoirs have a uniform density of states (and so impose no Lamb shift of the quantum dot). However, in general the fraction of an electron pumped per cycle (between zero and one) is given by the ratio of the Lamb shift imposed by the reservoirs to...
the level-broadening. This ratio is entirely determined by the reservoirs density of states, which is imposed by their band-structure.

Earlier works on pumping of dot-reservoir coupling — with direct driving of the dot-level[22] a Lamb shift induced by the reservoir band-structure[23,24] Coulomb blockade effects[25,26] or non-adiabatic driving[27] — did not investigate large level-broadening, and so did not find the quantized pumping of fractional charges.

It is crucial to note that we consider the average charge per cycle, showing that it is quantized at a fraction of an electron. There are no fractionally charged quasi-particles in our non-interacting system, so we expect that there is a certain probability that n electrons are pumped (for integer n = 0, ±1, ±2, · · · ) in any given cycle. Yet these probabilities are such that the average over many cycles will reveal itself as a fraction per cycle. This makes it clear that a measurement of a topological average charge per cycle at a fractional value is not a proof of the existence of fractionally charged quasi-particles. One needs have other supporting evidence.

A. Organisation of this work

Sec. II introduces our model Hamiltonian. Sec. IIIA briefly explains our main result about almost topological fractional pumping. Sec. IV shows this is half an electron per cycle for readers familiar with scattering theory (others can skip this section). Sec. V points out that the pumping can not be understood simply as changes in height — with direct driving of the dot-level, or non-adiabatic driving — did not investigate large level-broadening, and so did not find the quantized pumping of fractional charges.

We refer to Γ_i(ω, t) as level broadening, and to Λ_i(ω, t) as a Lamb shift. This is a slight abuse of terminology, but it is justified by the dot’s local density of states being Γ(ω)/[(ω − ϵ_d − Λ(ω))^2 + Γ^2(ω)]. So if Γ and Λ are ω-independent, then they are the level-broadening and Lamb shift, respectively. We simply keep this terminology for cases where Γ and Λ have an ω-dependence.

In what follows, our results will be simplest if K_i is written in terms of the dimensionless coupling X_i, which measures the level-broadening in units of the distance of the dot level from the electrochemical potential.

\[ X_i = \frac{\rho(\mu) K_i}{2(\epsilon_d - \mu)} \]

where \( \rho(\mu) \) is the density of states at the electrochemical potential, and the factor of two makes formulas compact.

We drive the gate-voltages \( V_i \), not the couplings \( K_i \), so we need a relation between them. Typically, the dot is coupled to reservoir \( i \) through tunnel-barriers of height \( E_i \) and width \( L_i \), so \( K_i \sim \exp[-\kappa_i] \) with \( \kappa_i = \sqrt{2mE_i(V_i) L_i(E_i)}/\hbar \). For large \( L_i \) and \( E_i \), even small changes in \( V_i \) will make large percentage changes in \( X_i \), so we can expand up to linear order in \( V_i \) about \( X_i = 1 \). Since electrons are negatively charged, this gives

\[ X_i = \exp[\alpha_i V_i] \]

with \( \alpha_i = -d\kappa_i/dV_i > 0 \), where \( V_i = 0 \) is chosen to coincide with \( X_i = 1 \). We mainly work with Eq. [7], but the almost topological fractional pumping also holds for \( X_i = \exp[f_i(V_i)] \) for any \( f_i(V_i) \) which is very positive for \( V_i \rightarrow \infty \), and is very negative for \( V_i \rightarrow -\infty \) (see the end of Sec. VIIB). This covers many physical systems.

III. ALMOST TOPOLOGICAL PUMPING OF A FRACTION OF AN ELECTRON PER CYCLE

Let us now briefly overview our main results, with the detailed calculations postponed to Sec. VIIA. Firstly, for a dot coupled to reservoirs without band-structure, there
is a topological pumping at half an electron per cycle. Secondly, one can choose the reservoir load-structure to ensure the pumped charge is topologically quantized at an arbitrary fraction of an electron per cycle.

A. Half an electron per cycle

Here we consider a situation where the reservoir density of states is energy independent (ω-independent), which is known as the wide band limit, and so ρ(ω) = ρ. Then the reservoir induces a level-broadening of the quantum dot’s energy level, but induces no Lamb shift: A(ω; K) = 0 in Eq. (5). Our calculations (using scattering theory in section IV or Keldysh theory in section V) show that this control of the level broadening allows the pumping of half an electron per cycle in the low temperature limit.

The dot level is taken to above the reservoir’s electro-chemical potential, (εd − µ) > 0, and the pumping cycle is taken to be cycle 1 in Fig. 2b,c, with neither εd nor µ change during the pumping cycle. The basic physical process, sketched in Fig. 2 is the following:

(a) Loading (segment 1a in Fig. 1): The dot starts weakly coupled to the reservoirs (VL and VR very negative) so the dots level broadening is much less than (εd − µ), as a result the dot’s occupation is negligible. The coupling to reservoir L is increased (VL increased), so that the reservoir wavefunctions spread into the dot (as in Fig. 2a), as the dot state hybridizes with reservoir states. The dot thus absorbs a charge of ∆Qload. Once the level broadening is much more than (εd − µ), one reaches the limit where half the broadened level is below the reservoir’s Fermi energy. In this limit, there is half an electron in the dot, ∆Qload → ½; in other words a 50% chance of finding the dot level occupied.

(b) Moving (segment 1b in Fig. 1): The coupling to reservoir L is slowly reduced to zero, while that to reservoir R is slowly increased to its maximum value (VL reduced and VR increased), in such a way that the sum of the two couplings remains constant. Thus, the wavefunctions of reservoir R spread more into the dot, while those of reservoir L spread less into the dot. The occupation of the dot remains the same, but the dot state’s hybridization moves from reservoir L to reservoir R.

(c) Unloading (segment 1c in Fig. 1): The coupling to R is reduced (VR reduced) so the level broadening again becomes much less than (εd − µ). As a result the dot level empties into reservoir R, as the reservoir wavefunctions spread into the dot become negligible, and one returns the dot to its initial state. This cycle transfers a charge of ∆Q from reservoir L to reservoir R, with ∆Q ≠ ∆Qload. When the coupling is large enough that the level-broadening in step 1b is much more than (εd − µ), then ∆Q → ∆Qload → 1/2.

B. Almost topological to entirely topological

The charge pumped per cycle is said to be topological when it is the same for any pumping cycles of the gate voltages that have the same topology. We will show this is the case for the cycle of VL and VR outlined above under certain conditions, and up to exponentially small corrections; so we call it “almost topological” pumping.

To see what this means, one must write the charge pumped into R as an integral over the surface in the VL-VR plane enclosed by the pumping cycle C,

\[ \Delta Q_R = e \int_C dV_L dV_R \ \Pi_R [V_L, V_R] \]  

(8)

Then one calculates \( \Pi_R [V_L, V_R] \), which is known as the Berry curvature, for the pumping. If one finds that this Berry curvature is a Dirac δ-function, then the pumping is entirely topological; the pumped charge only depends on how many times the pumping contour winds around the δ-function. Here, our central result, Eq. (40), is that
the Berry curvature is not a $\delta$-function, but it is strongly peaked with an exponential decay away from the peak, see Fig. 3. Then we call the pumping *almost topological*, because it depends only on the contours topology (how many times it winds around the peak) if the contour stays away from the peak, and if we neglect the exponentially small corrections coming from the tail of the peak. Thus contours 1 and 2 in Fig. 1b pump the same charge (up to exponentially small corrections) because they both have the same topology — each winds once around the peak.

Fig. 3b shows the peak for reservoirs with uniform density of states. The integral over this peak is 1/2, so the contours in Fig. 3b will thus pump the charge
\[ \Delta Q_{\text{quantized}} = \frac{e}{2}. \]  

In the limit of thick tunnel barriers, $L \to \infty$, one sees that $\alpha_\ell$ in Eq. (6) also goes to infinity. Then the Berry curvature peak becomes a Dirac $\delta$-function in the $V_L-V_R$ plane. This means that the pumping will become *entirely topological*. However, for $L \to \infty$, the tunnel coupling is exponentially small, so we require exponentially small temperatures, so $\epsilon_0 - \mu$ can be as small as the couplings, to ensure we can make $X_L$ and $X_R$ of order one, so the pumping contour can enclose the $\delta$-function peak.

C. Different fractions of an electron per cycle

Let us now consider reservoirs with a non-uniform density of states, so $\rho(\omega)$ depends on $\omega$. In this case, the Lamb shift in Eq. (5) is non-zero, so the dot-reservoir coupling induces a shift of the dot-level as well as a broadening. Sec. VII will use Keldysh theory to show that the almost topological pumping is quantized at a fraction of an electron (between 0 and 1), which is given by the ratio of the Lamb shift to the level-broadening. We define $\lambda$ as the following dimensionless measure of this ratio at $\omega = \mu$,
\[ \lambda = 2\Lambda(\mu,t)/\Gamma(\mu,t), \]  

where the factor of 2 in the definition is to make our results compact. We will show that the almost topological charge that is pumped by the cycle described in Sec. III A above is
\[ \Delta Q_{\text{quantized}} = \frac{e}{\pi} \left[ \frac{\pi}{2} - \arctan(\lambda) - \frac{\lambda}{1 + \lambda^2} \right]. \]  

Hence, for this pumping cycle, $\Delta Q_{\text{quantized}}$ is a monotonically decaying function of $\lambda$, and it take values between $e$ and 0. More precisely, $\Delta Q_{\text{quantized}}$ equals $\left[ 1 - 2/(3\pi \lambda^2) \right] e$ for $\lambda \ll -1$, equals $e/2$ at $\lambda = 0$, and equals $2e/(3\pi \lambda^2)$ for $\lambda \gg 1$.

Crucially, $\lambda$ is entirely determined by the reservoir band-structure since Eqs. (4), (5) and (10) mean that
\[ \lambda = \frac{2}{\rho(\mu)} \int \frac{d\varepsilon}{\mu - \varepsilon}, \]  

so it is independent of $K_L$, $K_R$ and $t$. Hence, any given reservoir band-structure will have a given $\lambda$, and hence a given quantized fraction of an electron pumped per cycle. By choosing a suitable reservoir band-structure, one can choose that fraction.

D. Requirements for experimental observation

There are four requirements for observing this quantized pumping of a fraction of an electron per cycle.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{Plots of the Berry curvature, $\Pi_{\ell R}(V_L, V_R)$, for the dot-reservoir coupling in Eq. (3). This is given by Eq. (17) for $\lambda = 0$ and by Eq. (40) for arbitrary $\lambda$. It is always a sharp peak, but the volume under the peak is highly $\lambda$-dependent, and given by Eq. (11). Contour 1 from Fig. 1b is also shown.}
\end{figure}
The first requirement is a quantum dot which mimics the Hamiltonian in Eq. (1) which neglects electron-electron interactions on the dot. The simplest experimental implementation of Eq. (1) is an interacting quantum dot (described by an Anderson impurity Hamiltonian) in a large enough magnetic field that the dot’s spin-state with higher energy is always empty, which makes the on-dot interaction term negligible.

The second requirement is that $k_B T$ is much smaller than $(\epsilon_d - \mu)$, larger temperatures will destroy the quantization. At the same time $(\epsilon_d - \mu)$ should be small enough that we can make the dot-reservoir coupling $K \gg (\epsilon_d - \mu) / \rho(\mu)$, which means the required value of $T$ depends on how strongly the dot can be coupled to the reservoirs.

The third requirement is related to the fact that the charge pumping is probabilistic, with only the average charge being quantized. This probabilistic nature of the pumping is typical whenever there is part of the pumping cycle in which the dot is coupled to both reservoirs at the same time (segment 1b of the cycle). Thus in any given cycle $n = 0, \pm 1, \pm 2, \cdots$ electrons might flow. Central limit theorem tells us that averaging over many cycles will give an answer that will converge to the quantized fraction that we predict.

The fourth requirement is due to our assumption that the level-broadening and Lamb shift (if present) go exponentially, as above Eq. (7). Hence any effect of the capacitive coupling to gates L and R will act much like the Lamb shift. Therefore, this coupling goes linearly in $\alpha_i V_i$, while the level-broadening and Lamb shift (if present) go exponentially, as above Eq. (7). Hence any effect of the capacitive coupling on $\epsilon_d$ will become negligible compared to the broadening at large $\alpha_i V_i$.

IV. SCATTERING THEORY

The central calculation in this work uses the Keldysh technique, however as a warm up exercise, we can use the scattering theory of quantum pumping for the special case where the reservoirs have uniform density of states. Readers interested in the Keldysh calculation of the general case can skip this section.

The scattering matrix of a single-level dot (see e.g. Refs. 31 and 32) at energy $\mu$ is

$$
\begin{pmatrix}
S_{LL} & S_{RL} \\
S_{LR} & S_{RR}
\end{pmatrix} = I - \frac{i}{\mu - \epsilon_d + i \frac{1}{2} \Gamma} \left( \frac{\Gamma_L}{\sqrt{\Gamma_L \Gamma_R}} \sqrt{\Gamma_L \Gamma_R} \right),
$$

(13)

where $I$ is a 2-by-2 unit matrix. The scattering theory for pumping of $K_L$ and $K_R$ around the contour $C$, gives the charge pumped per cycle into reservoir $R$ as the integral over the surface enclosed by $C$,

$$
\Delta Q_R = e \int_C dK_L dK_R \Pi_R(K)
$$

(14)

where the Berry curvature $\Pi_R(K)$ for our system is

$$
\Pi_R(K) = \frac{1}{\pi} \text{Im} \left[ \frac{\partial S^*_{RL}}{\partial K_L} \frac{\partial S_{RL}}{\partial K_R} + \frac{\partial S^*_{RR}}{\partial K_L} \frac{\partial S_{RR}}{\partial K_R} \right].
$$

(15)

Substituting in Eq. (13) and using Eq. (6), one find that the zero-temperature result for pumped charge per cycle (in units of $e$) is given by the dimensionless integral

$$
\frac{\Delta Q_R}{e} = \frac{2}{\pi} \int_{C'} dX_L dX_R \frac{X}{[1 + X^2]^2},
$$

(16)

where $X = X_L + X_R$. The surface of integration $C'$ is given by the contour $C$ in Eq. (14) rescaled using Eq. (6).

One can show that $\Delta Q_L = -\Delta Q_R$. Now we cast this result in terms of the gate-voltages that control the couplings. Using Eq. (7), $\Delta Q_R$ is given by the integral

$$
\frac{\Delta Q_R}{e} = \frac{2 \alpha_l \alpha_R}{\pi} \frac{e^{\alpha_l V_L} e^{\alpha_R V_R} (e^{\alpha_l V_L} + e^{\alpha_R V_R})}{[1 + (e^{\alpha_l V_L} + e^{\alpha_R V_R})^2]^2}.
$$

(17)

This is shown in Fig. 3b. The crucial feature is that $\Delta Q_R / e$ is highly peaked at small $\alpha_i V_i$ and decays exponentially with increasing $\alpha_i V_i$ (for both $i = L$ and $i = R$). Hence it fulfills the conditions for almost topological pumping discussed in section 5.1.3.

To find the charge pumped by a contour that encloses the above peak once, we take contour 1 in Fig. 1b, whose segment 1b is chosen such that $\exp[\alpha_l V_L] + \exp[\alpha_R V_R] = 1$. We then go back to Eq. (16), for which this contour maps via Eq. (7) to the triangular contour shown in Fig. 1c. The contour $C'$ is the triangle defined by $(X_L, X_R)$ going from $(0, 0) \rightarrow (X_{\text{max}}, 0) \rightarrow (0, X_{\text{max}}) \rightarrow (0, 0)$, where $X_{\text{max}} = \rho K_{\text{max}} / [2(\epsilon_d - \mu)]$. We write

$$
\int_{C'} dX_L dX_R (\cdots) = \frac{1}{2} \int_0^{X_{\text{max}}} dX \int_{-X}^{X} dY (\cdots),
$$

(18)

where $Y = X_L - X_R$. Then

$$
\frac{\Delta Q_R}{e} = \frac{1}{\pi} \left[ \arctan[X_{\text{max}}] - \frac{X_{\text{max}}}{1 + X_{\text{max}}^2} \right]
$$

(19)

for the above triangular contour. This goes to 1/2 for large $X_{\text{max}}$, which corresponds to encircling the peak in Eq. (17). Hence, for uniform reservoir density of states, the pumping is quantized at half an electron per cycle.

We do not know of a scattering theory for non-uniform reservoir density of states, so we use the Keldysh formalism to treat such cases in sections 5.1.7.
V. COMPARISON WITH DOT OCCUPATION

One might naively guess that the pump is simply due to filling the dot state from L in the “loading” part of the cycle, and then emptying it into R in the “unloading” part of the cycle. If this were the case, the charge transferred from L to R would simply equal the charge loaded into the dot, $\Delta Q_{\text{load}}$. We show here that this is not the case; there is no simple relation between the pumped charge and $\Delta Q_{\text{load}}$.

We are varying the coupling around the cycle given above on a very slow timescale. Thus during the “moving” part of the cycle, electrons are continuously tunnelling in and out of the dot from L and R (and tunnelling though the dot from L to R). They do not have enough energy to stay in the dot, but the uncertainty principle means they can enter and leave the dot in a time of order $\hbar/(\epsilon_{i} - \mu)$. So there is no reason to assume the pumped charge is related to the dot occupation. Indeed, the occupation of the dot at low temperatures, see e.g. Ref. [33] is

$$n(K) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\Gamma(\omega; K)}{[\omega - \epsilon_{i} - \Lambda(\omega; K)]^{2} + \frac{1}{4} \Gamma(\omega; K)^{2}}.$$

For a uniform density of states $\Lambda(\omega) = 0$, the integrand is a Lorentzian, and so $n(K) = \arctan |X| / \pi$. Then

$$\Delta Q_{\text{load}} = e \left[ n(K_{\text{max}}) - n(0) \right] = e \arctan \frac{X_{\text{max}}}{\pi}.$$

From Eq. [19], we see the pumped charge is smaller than $\Delta Q_{\text{load}}$ by a factor of $\Delta Q' = eX_{\text{max}}/\pi(1 + X_{\text{max}}^{2})$, which vanishes when $X_{\text{max}} \rightarrow \infty$. This means that the “moving” part of the pumping cycle in section IIIA involves a small flow, $\Delta Q'$, from the R to L through the dot (the dashed arrows in the Fig. 2).

For non-uniform density of states, $\Delta Q_{\text{load}}$ depends on the $\omega$-dependence of the level broadening and the Lamb shift for all $\omega \leq \mu$. In contrast, we will see in section VI that the pumped charge depends only on their values at $\omega = \mu$. Thus in general $\Delta Q$ and $\Delta Q_{\text{load}}$ will not be related in any way, although both will be between 0 and $e$. Either can be larger, so $\Delta Q'$ can be of either sign. Indeed, two different set-ups can have the same $\Delta Q$ and different $\Delta Q_{\text{load}}$, or vice-versa.

VI. KELDYSH FORMALISM

The dot’s occupation and current into reservoir $i$ at time $t$ are[34,35]

$$n(t) = \langle d^{i}(t)d(t) \rangle = -i G^{<}(t,t),$$

$$J_{i}(t) = -e \frac{d}{dt} \sum_{k} \langle c^{i}_{k}(t)c_{k}(t) \rangle$$

$$= e \int_{C} dt_{1} \left[ G^{R}(t_{1}, t) \Sigma_{i}^{<}(t_{1}, t) + G^{<}(t_{1}, t) \Sigma_{i}^{R}(t_{1}, t) - \Sigma_{i}^{<}(t_{1}, t) G^{A}(t_{1}, t) - \Sigma_{i}^{R}(t_{1}, t) G^{<}(t_{1}, t) \right].$$

respectively, in terms of the Keldysh Green’s functions in Appendix A. We will derive the pumped charge for a large driving contour $C$, by summing the contributions from all infinitesimal circular contours inside it $\{C_{n}\}$ (see Fig. 4), as was done in scattering theory by Ref. [6].

The infinitesimal contour $C_{n}$ satisfies

$$K(t) = \left( \begin{array}{c} K_{L}(t) \\ K_{R}(t) \end{array} \right) = K_{n,0} + \delta K \left( \frac{\cos[\Omega t]}{\sin[\Omega t]} \right),$$

where $\Omega$ is a pumping frequency, $\delta K$ is an infinitesimally small amplitude of driving around the time-independent point $K_{n,0}$. Under this infinitesimal driving, the time-dependent charge current into reservoir $i$ is

$$J_{i}(t) = \mathcal{G}_{i}^{1}(\Omega; K_{n,0}) \delta K \cos(\Omega t) + \mathcal{G}_{i}^{2}(\Omega; K_{n,0}) \delta K \sin(\Omega t).$$

This is given in terms of Keldysh Green’s functions in Appendix A and it only depends on the time differences $(t - t_{1})$ because it is evaluated for $\delta K = 0$. We assume the condition for adiabatic driving:

$$\Omega \ll \tau^{-1}.$$

where $\tau$ is the typical time for electrons in the dot to relax. Then it is sufficient to take the dynamic conductance at leading order in the pumping frequency $\Omega$; $\mathcal{G}_{i}^{1}(\Omega; K) = A_{i}^{1}(K) \Omega + O(\Omega^{2})$. Substituting this into Eq. (24), and integrating $\Omega dt$ from 0 to $2\pi$, we find the charge pumped per cycle on the infinitesimal contour $C_{n}$ is

$$\delta Q_{i,n} = \int_{C_{n}} A_{i}^{1}(K) dK_{L} + A_{i}^{2}(K) dK_{R}.$$

Summing all infinitesimal contours inside the large contour $C$, gives charge pumped per cycle around $C$ as

$$\Delta Q_{i} = \int_{C} dK \cdot A_{i}(K)$$

where we define the Berry connection as the vector $A_{i}(K) = (A_{i}^{1}(K), A_{i}^{2}(K))$. Re-writing this in terms
of a surface integral using Stokes theorem, we get
\[
\Delta Q_i = \int_C dS \cdot \Pi_i(K),
\]
(28)
where \( \Pi_i(K) = \nabla_K \times A_i(K) \) is the Berry curvature. This integral is over the surface which is inside the pumping contour \( C \). As this surface is the \( K_x-K_y \) plane, only the component of \( \Pi_i(K) \) perpendicular to this plane contributes; we call this component
\[
\Pi_i(K) = \frac{d}{dK_L} [A_i^R(K)] - \frac{d}{dK_R} [A_i^L(K)].
\]
(29)
we will calculate this for our model in the next section.

We end this derivation with an adiabaticity condition for the large contour \( C \). Given Eq. (25) for the infinitesimal circular contours, adiabaticity on \( C \) requires
\[
|dK/dt| \ll \Delta K_{\text{typical}}/\tau,
\]
(30)
where \( \Delta K_{\text{typical}} \) is the typical scale of the contour (see Fig. [1]). We find that the Berry curvature in the \((\alpha_L,\alpha_R)\)-plane
\[
\Pi_{\alpha_L} = \frac{e}{4\pi} \int d\omega \left( \frac{1}{1 + e^{(\omega - \mu)/T}} \right)^{-1} \left( \frac{\mu - \epsilon_d - \Lambda(\alpha)L; K)}{\left( \frac{1}{2} \Gamma(\alpha)L; K) \right)^2} \right)^2.
\]
(37)
This depends on the sum of the couplings, \( K = (K_L + K_R) \), but not the difference \( (K_L - K_R) \).

VII. RESULTS FOR OUR MODEL.

For the Hamiltonian in Eq. (1), we find that the Berry connection in Eq. (27) contains two terms
\[
A_i(K) = A_i^\text{broad}(K) + A_i^\text{shift}(K)
\]
(31)
because \( A_i(K) \) involves a derivative with respect to \( K \), and that derivative can act on the level-broadening (giving \( A_i^\text{broad} \)) or the Lamb shift (giving \( A_i^\text{shift} \)). If there is no Lamb shift then \( A_i^\text{shift}(K) = 0 \), while if the Lamb shift is much greater than the level-broadening, then Eq. (31) is dominated by \( A_i^\text{shift}(K) \). The Keldysh calculations outlined in Appendix A give
\[
[A_i^\text{broad}(K)]_j = \int \frac{d\omega}{2\pi} \left[ (B^2 - \frac{1}{2} A^2) \frac{f \Lambda}{f \Lambda} \right],
\]
(32)
\[
[A_i^\text{shift}(K)]_j = \int \frac{d\omega}{2\pi} \left[ 2BA f \Lambda - \frac{1}{2} A^2 \frac{f \Gamma_i}{f \Gamma_i} \right],
\]
(33)
where \( i \) and \( j \) are \( L \) or \( R \), and \( f = [1 + e^{(\omega - \mu)/T}]^{-1} \) is the Fermi function. The primed denotes the partial derivative with respect to \( \omega \). The quantities \( \Lambda_i \) and \( \Gamma_i \) are given in Eqs. (25), while \( A = 2 \text{Im}(G^A) \) and \( B = \text{Re}(G^A) \), with \( G^A(\omega) \) given in Eq. (A6).

Turning to the Berry curvature in Eq. (29), we see it contains two derivatives (with respect to \( K_j \), because \( [A_i(K)]_j \) contained one. Hence \( \Pi_i(K) \) contains three terms; a “broad-broad” term due to both derivatives acting on the broadening, a “shift-shift” term due to both derivatives acting the Lamb shift, and a “shift-broad” term with one derivative on each of them. The “shift-shift” term turns out to be zero, showing that the Lamb shift alone is not enough to do pumping. Intuitively, this can be understood as the Lamb shift only moving the dot level, which is not enough to do pumping. Hence
\[
\Pi_R(K) = \Pi_R\text{broad-broad}(K) + \Pi_R\text{shift-shift}(K),
\]
(34)
and \( \Pi_L(K) = -\Pi_R(K) \), with
\[
\Pi_R\text{broad-broad}(K) = \int \frac{d\omega}{4\pi} f'A B \frac{\Gamma^2(\omega, K)}{K^2},
\]
(35)
\[
\Pi_R\text{shift-shift}(K) = \int \frac{d\omega}{4\pi} f' A^2 \frac{\Gamma(\omega, K) \Lambda(\omega, K)}{K^2},
\]
(36)
where we have used the fact that \( \Lambda \) and \( \Gamma \) are proportional to \( K = K_L + K_R \). A bit more algebra gives
\[
\Pi_R(K) = \frac{e}{2} \int d\omega \frac{\rho^2(\omega) \Gamma(\omega, K) \partial f/\partial \omega}{\left[ \mu - \epsilon_d - \Lambda(\omega; K) \right]^2 + \left[ \frac{1}{2} \Gamma(\omega; K) \right]^2}.
\]
(37)

A. Low temperature pumping

In the limit of small temperature, we can make the approximation \( \partial f/\partial \omega = -\delta(\omega - \mu) \) in Eq. (37). To justify this approximation one needs the other terms in the integrand of Eq. (37) to vary little over the window of \( \omega \) given by \( \mu \pm k_B T \). Then, the Berry curvature is
\[
\Pi_R(K) = \frac{e}{4\pi} \frac{(\epsilon_d - \mu)^2 (\mu - \Lambda(\omega; K))}{\left[ \mu - \epsilon_d - \Lambda(\mu; K) \right]^2 + \left[ \frac{1}{2} \Gamma(\mu; K) \right]^2}.
\]
(38)
Writing this in terms of \( \lambda \) in Eq. (10), the low-temperature result for pumped charge per cycle (in units of \( e \)) is given by the dimensionless integral
\[
\frac{\Delta Q_R}{e} = \frac{2}{\pi} \int_{C^*} dX_L dX_R \frac{X}{(1 + \lambda X)^2 + X^2}.
\]
(39)
where \( X_L \) defined in Eq. (9) with \( \rho \) being \( \rho(\mu) \), and \( X = X_L + X_R \). The surface of integration \( C^* \) is given by the contour \( C \) in Eq. (14) rescaled using Eq. (6).

As explained in Sec. II, we control gate-voltages \( V_i \), in experiments. By substituting Eq. (7) into Eq. (39), we find the Berry curvature in the \((V_L, V_R)\)-plane
\[
\Pi_R(V_L, V_R) = \frac{2}{\pi} \int_{V_L} dX_L dX_R \frac{x}{(1 + \lambda X)^2 + X^2} \bigg|_{x = x_L + x_R}.
\]
(40)
shown in Fig. 3. This is our central result, because the fractional and topological nature of the pumping both follow from it, as we now show.

Eq. (40) has a peak at small $|\alpha_1V_1|$, and decays exponentially as $|V_1|$ grows. Hence, any pumping contour that encloses the peak without encroaching on it will give the same pumped charge per cycle (up to exponentially small corrections), ensuring quantized pumping.

To calculate the charge pumped by such a cycle, we return to Eq. (39) and consider a triangular contour $C'$ explained above Eq. (18). Eq. (7) means that for large $X_{\text{max}}$ this triangular contour corresponds to contour 1 in Fig. 1b, that encloses the peak in $\Pi_R(V_L,V_R)$. We transform to $X$ and $Y$ as in Eq. (18), then

$$\Delta Q_R = \frac{1}{e} \left[ \frac{\pi}{2} - \arctan \left( \frac{X_{\text{max}}(1 + \lambda X_{\text{max}})}{1 + X_{\text{max}}^2 + \lambda X_{\text{max}}^2} \right) \right]$$

(41)

see Fig. 3. It reduces to Eq. (19) for $\lambda = 0$, since $\arctan(x) + \arctan(1/x) = \text{sgn}(x)\pi/2$. We take $X_{\text{max}} \to \infty$ to get the pumping for a contour that corresponds to one enclosing the peak of Eq. (40); this gives Eq. (11).

Hence, we have our main conclusions; the pumping is almost topological, and pumps a fraction of an electron (between 0 and 1) given by the value of $\lambda$, which is determined purely by the reservoir’s band-structure.

To generalise to a voltage dependence of the form below Eq. (7), we substitute it into Eq. (39). Then Eq. (40) changes, but it remains strongly peaked with exponentially small tails. This ensures that there is still almost topological pumping. Further more, the faction pumped per cycle is the same for any voltage dependence, since it was calculated directly from Eq. (39).

VIII. ADIABATICITY AND BAND GAPS

To know how slow the pumping needs to be to ensure adiabaticity for a given system, one needs to find its $1/\tau$, to put into Eq. (30). This is simple when the reservoirs have uniform density of states, since there the dot state decays exponentially and the decay rate gives $1/\tau$ in Eq. (30). For systems with non-uniform density of states we can place a lower bound on $1/\tau$ by saying that $1/\tau \geq K\rho_{\text{min}}$, where $\rho_{\text{min}}$ is the minimal value of the density of states.

However, this poses a problem for reservoirs with band-gaps, as the density of states vanishes in the band-gap, so the above lower-bound does not allow us to say when the pumping is slow enough to be considered adiabatic. To investigate this further we consider the case where the electro-chemical potential is near a band-edge in the reservoir, so the reservoir’s density of states is

$$\rho(\omega) = \left\{ \begin{array}{ll} \rho_0 \left( \omega/\omega_c \right)^s e^{-\omega/\omega_c} & \omega > 0 \\ 0 & \omega < 0 \end{array} \right.$$  (42)

where, without loss of generality, we measure energy $\omega$ from the band-edge. Then the level-broadening is $\Gamma_i = K_i\rho(\omega)$, and the Lamb shift is (see e.g. Ref. [33],

$$\lambda = -2\Gamma (1+s) \text{ Re } [(-1)^s \Gamma (-s,-\mu/\omega_c)].$$  (43)

Fig. 6 plots this, and shows that a suitable choice of $s$ and $\mu/\omega_c$ will give almost any desired value of $\lambda$.

It has long been known that this model exhibits an infinite-lifetime bound-state, see Refs. [44] and [45] for reviews. Electron dynamics in various time-dependent versions of this model have been studied; particularly the decay of an initially prepared dot state, the response to switching on a bias, or the response to periodic driving. For $s > 0$, this bound-state appears when the coupling exceeds a critical value, $K_c = \epsilon/\Gamma[s]$. This state has $\tau = \infty$, so pumping never satisfies the adiabaticity condition in Eq. (30) when $K > K_c$. Intriguingly, the Berry curvature in Eq. (37) does not contain $K_c$; it is a smooth function across this line of critical coupling $K = (K_L + K_R) = K_c$. However, the Berry curvature in Eq. (37) ceases to have a physical meaning when one crosses the line of critical coupling, because non-adiabatic contributions dominate beyond this line ($K > K_c$), no matter how slow the pumping is.
For \( K < K_c \), it is difficult to determine the dot’s decay rate, \( 1/\tau \), because it contains terms with an oscillatory powerlaw decay, for which there is no unique way to define \( 1/\tau \). Fig. 7 is an attempt to give a feeling for how \( 1/\tau \) depends on the coupling. The red data points are the inverse time for the dot occupation to decay to threshold (using the method reviewed in Ref. [33] that we set at 2\% of its initial value, i.e. we plot the \( 1/t_i \) that satisfies

\[
\left|\frac{n(t_i) - n_0}{n_\infty - n_0}\right| = 2\%.
\]

The solid curve is the time taken to reach this threshold, if one approximates the decay to an exponential at the rate given by the imaginary part of the energy of the resonance (i.e. neglecting all powerlaw or oscillatory components in the decay). This approximation captures the overall magnitude of the true decay, but misses the sharp drop in \( 1/t_i \) as \( K \to K_c \). This sharp drop shown by the data points indicates that the timescale to decay diverges as \( K \) approaches \( K_c \). This makes it extremely difficult to pump slowly enough to be adiabatic, if \( K \) is close to \( K_c \).

The “error bars” on the data in Fig. 7 are not numerical uncertainties in \( t_i \) (such uncertainties are about the size of the red dots). They indicate the period of the oscillations in the decay, which is maximal for \( K \approx K_c/2 \). A small change in the system parameters (e.g. a change of \( \omega_c \) or \( \mu \)) would shift the phase of the oscillations, thereby shifting where the oscillating decay crosses the threshold to a different place on the vertical “error bar”. Hence, we can expect a change in system parameters to induce a large change in \( 1/t_i \) when \( K \approx K_c/2 \), while the change will be modest for \( K \ll K_c \) and \( K \approx K_c \).

IX. CONCLUSIONS

We show that a system without exotic physics (non-interacting single-level quantum dot at low temperature) can exhibit an almost topological pumping of a fraction of an electron per cycle, when averaged over many cycles. We call it “almost” topological because the pumped charge depending only on the number of times the contour winds around the peak in the Berry curvature, shown in Fig. 3, under the conditions that (i) the contour does not touch the peak, and (ii) we neglect the exponentially small corrections coming from the tail of the peak. Sec. [13] mentions a specific limit in which the pumping is entirely topological. The fraction pumped (between zero and one electron) is determined by the ratio of the Lamb shift to the level-broadening. This ratio is imposed by the reservoir band-structure, which can be chosen to give the desired fraction. Uniform reservoir density of states give the quantized pumping of half an electron per cycle. We emphasize that it is the average charge pumped per cycle that is (almost) topological and fractional. Each cycle has a finite probability that \( n \) electrons are pumped for \( n = 0, \pm 1, \pm 2, \ldots \); the quantized fraction is only revealed by averaging over many cycles.

Our results show that observing topological fractional pumping on average is not enough to prove exotic physics such as the pumping of individual fractionally charged quasi-particles; one would need other supporting evidence. Noise measurements could provide this, but that would require theoretical modelling of the noise properties for both our model and the more exotic cases.

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Contributions: MH derived all results using Keldysh, in discussion with EJ and RW. EJ reproduced the wideband results with scattering theory, and wrote the code for Fig. 7. MH and RW wrote the manuscript.

Appendix A: Keldysh Green’s functions

The quantum dot’s Green’s function are defined as

\[
G^A(t_1, t_2) = i\Theta(t_2 - t_1) \langle [d(t_1), d^\dagger(t_2)]_+ \rangle, \quad (A1)
\]

\[
G^R(t_1, t_2) = -i\Theta(t_1 - t_2) \langle [d(t_1), d^\dagger(t_2)]_+ \rangle, \quad (A2)
\]

\[
G^< (t_1, t_2) = i \langle d^\dagger(t_2) d(t_1) \rangle, \quad (A3)
\]

where \( \Theta(t) \) is a Heaviside function and \( [\cdot, \cdot]_+ \) is an anti-commutator. Their algebraic form is given by Dyson’s
\[ G^<(t_1, t_2) = \int dt_3 dt_4 \, G^R(t_1, t_3) \Sigma^<(t_3, t_4) G^A(t_4, t_2), \]

for \( \kappa = R, A, \) and

\[ G^\kappa(t_1, t_2) = g^\kappa(t_1, t_2) + \int dt_3 dt_4 \, g^\kappa(t_1, t_3) \Sigma^\kappa(t_3, t_4) G^\kappa(t_4, t_2), \]

Here \( g^{A/R}(t_1, t_2) \) are Green’s function of electron of the isolated quantum dot, and \( \Sigma^\kappa(t_1, t_2) \) are the one-particle-irreducible self-energy for \( \kappa = A, R, <, \)

\[ \Sigma_i^\kappa(t_1, t_2) = \sum_k \gamma_i(t_1) g^\kappa_{ik}(t_1, t_2) \gamma_i(t_2), \quad (A4) \]

with \( \Sigma^\kappa = \Sigma^R + \Sigma^A. \) Here \( g^\kappa_{ik}(t_1, t_2) \) is a Green’s function of electrons in the isolated electron reservoirs,

\[
\begin{align*}
g^{A}_{ik}(t_1, t_2) &= i \Theta(t_2 - t_1) \langle [c_{ik}(t_1), c_{ik}^+(t_2)] \rangle_{\gamma_i = 0}, \\
g^{R}_{ik}(t_1, t_2) &= -i \Theta(t_1 - t_2) \langle [c_{ik}(t_1), c_{ik}^+(t_2)] \rangle_{\gamma_i = 0}, \\
g^{R}_{ik}(t_1, t_2) &= i \langle c_{ik}^+(t_2) c_{ik}(t_1) \rangle_{\gamma_i = 0}. \quad (A5)
\end{align*}
\]

The dynamic conductance in Eq. (25) is

\[
S'_i(t_1) = \frac{e^2}{2} \left\{ G^R |\Sigma^R G^R \Sigma^<| + G^R |\Sigma^A G^<| - G^R |\Sigma^< G^R \Sigma^A| G^A \right\}.
\]

where we define \( AB |CD = [AB](t_1, t_2) |CD(t_1, t_2) \) with \( [AB](t_1, t_2) = \int dt_3 A(t_1, t_3) B(t_3, t_2). \)

One can Fourier transform \( G^{A/R}(t_1, t_2) \) to get

\[ G^A(\omega) = \left[ G^R(\omega) \right]^{-1} = \left( \omega - \epsilon_d - \Lambda(\omega) + i \frac{1}{2} \Gamma \right)^{-1}. \quad (A6) \]
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