Klein-Gordon-Maxwell System
in a bounded domain *

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Abstract

This paper is concerned with the Klein-Gordon-Maxwell system in a bounded spatial domain. We discuss the existence of standing waves $\psi = u(x)e^{-i\omega t}$ in equilibrium with a purely electrostatic field $E = -\nabla \phi(x)$. We assume an homogeneous Dirichlet boundary condition on $u$ and an inhomogeneous Neumann boundary condition on $\phi$. In the “linear” case we characterize the existence of nontrivial solutions for small boundary data. With a suitable nonlinear perturbation in the matter equation, we get the existence of infinitely many solutions.

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1 Introduction

Many recent papers show the application of global variational methods to the study of the interaction between matter and electromagnetic fields. A typical example is given by the Klein-Gordon-Maxwell (KGM for short) system.

We consider a matter field $\psi$, whose free Lagrangian density is given by

$$L_0 = \frac{1}{2} \left( |\partial_t \psi|^2 - |\nabla \psi|^2 - m^2 |\psi|^2 \right),$$

(1)

with $m > 0$. The field is charged and in equilibrium with its own electromagnetic field $(E, B)$, represented by means of the gauge potentials $(A, \phi)$,

$$E = - (\nabla \phi + \partial_t A),$$
$$B = \nabla \times A.$$

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Abelian gauge theories provide a model for the interaction; formally we replace the ordinary derivatives \((\partial_t, \nabla)\) in (1) with the so-called gauge covariant derivatives
\[
(\partial_t + iq\phi, \nabla - iqA),
\]
where \(q\) is a nonzero coupling constant (see e.g. [1]). Moreover, we add the Lagrangian density associated with the electromagnetic field
\[
L_1 = \frac{1}{8\pi} \left( |E|^2 - |B|^2 \right).
\]
The KGM system is given by the Euler-Lagrange equations corresponding to the total Lagrangian density
\[
L = L_0(\psi, A, \phi) + L_1(A, \phi).
\]
The study of the KGM system is carried out for special classes of solutions (and for suitable classes of lower order nonlinear perturbation in \(L_0\)). In this paper we consider
\[
\begin{align*}
\psi &= u(x)e^{-i\omega t}, \\
\phi &= \phi(x), \\
A &= 0,
\end{align*}
\]
that is a standing wave in equilibrium with a purely electrostatic field
\[
\begin{align*}
E &= -\nabla \phi(x), \\
B &= 0.
\end{align*}
\]
Under this ansatz, the KGM system reduces to
\[
\begin{align*}
-\Delta u - (q\phi - \omega)^2 u + m^2 u &= 0, \\
\Delta \phi &= 4\pi q (q\phi - \omega) u^2,
\end{align*}
\]
(see [2] or [3] where the complete set of equations has been deducted).

We shall study (2) in a bounded domain \(\Omega \subset \mathbb{R}^3\) with smooth boundary \(\partial\Omega\). The unknowns are the real functions \(u\) and \(\phi\) defined on \(\Omega\) and the frequency \(\omega \in \mathbb{R}\). Throughout the paper we assume the following boundary conditions
\[
\begin{align*}
u(x) &= 0, \\
\frac{\partial \phi}{\partial n}(x) &= h(x).
\end{align*}
\]
The problem (2) has a variational structure and we apply global variational methods.

First we notice that the system is symmetric with respect to \(u\), that is, the pair \((u, \phi)\) is a solution if and only if \((-u, \phi)\) is a solution.

Moreover, due to the Neumann condition (3b), the existence of solutions is independent on the frequency \(\omega\). Indeed the pair \((u, \phi)\) is a solution of (2) if and only if the pair \((u, \phi - \omega/q)\) is a solution of the following problem
\[
\begin{align*}
-\Delta u - q^2 \phi^2 u + m^2 u &= 0 \quad \text{in } \Omega, \\
\Delta \phi &= 4\pi q^2 \phi u^2 \quad \text{in } \Omega,
\end{align*}
\]
with the same boundary conditions (3). In other words, for any \( \omega \in \mathbb{R} \), the existence of a standing wave \( \psi = u(x)e^{-i\omega t} \) in equilibrium with a purely electrostatic field is equivalent to the existence of a static matter field \( u(x) \), in equilibrium with the same electric field. So we focus our attention on the problem (4).

The boundary datum \( h \) plays a key role. If \( h = 0 \), then it is easy to see that the system (4)-(3) have only the solutions \( u = 0, \phi = \text{const} \). If \( \int_{\partial \Omega} h \, d\sigma = 0 \), then (4)-(3) has infinitely many solutions corresponding to \( u = 0 \). Such solutions have the form \( u = 0, \phi = \chi + \text{const} \) (see Lemma 2.1 below, where \( \chi \) is introduced) and we call them trivial. In this case we are interested in finding nontrivial solutions (i.e. solutions with \( u \neq 0 \)).

On the other hand, it is well known that the Neumann condition gives rise to a necessary condition for the existence of solutions of the boundary value problem. In our case, from (4b)-(3b), we get

\[
4\pi q^2 \int_{\Omega} \phi u^2 \, dx = \int_{\partial \Omega} h \, d\sigma.
\]

Hence, whenever \( \int_{\partial \Omega} h \, d\sigma \neq 0 \), solutions of (4)-(3), if any, are nontrivial.

The following theorem characterizes the existence of nontrivial solutions for small boundary data.

**Theorem 1.1.** If \( \| h \|_{H^{1/2}(\partial \Omega)} \) is sufficiently small (with respect to \( m/q \)), then the problem (4)-(3) has nontrivial solutions \( (u, \phi) \in H^1(\Omega) \times H^1(\Omega) \) if and only if

\[
\int_{\partial \Omega} h \, d\sigma \neq 0.
\]

We point out that the Lagrangian density \( L \) contains only the potential \( W(|\psi|) = m^2|\psi|^2/2 \), which gives a positive energy (see the discussion about the energy in [4]). Hence the solutions found in Theorem 1.1 are relevant from the physical point of view.

Theorem 1.1 shows that, if \( q \) is sufficiently small, (4)-(3) has only the trivial solutions if and only if \( \int_{\partial \Omega} h \, d\sigma = 0 \). The same result holds true if \( q = 0 \) (uncoupled system). It is immediately seen that, in the uncoupled case, if \( \int_{\partial \Omega} h \, d\sigma \neq 0 \), then there exist no solutions at all.

Our second result is concerned with a nonlinear lower order perturbation in (4a). So we study the following system

\[
\begin{align*}
-\Delta u - q^2 \phi^2 u + m^2 u = g(x, u) & \quad \text{in } \Omega, \\
\Delta \phi = 4\pi q^2 \phi u^2 & \quad \text{in } \Omega,
\end{align*}
\]

again with the boundary conditions (3). The nonlinear term \( g \) is usually interpreted as a self-interaction among many particles in the same field \( \psi \).

We assume \( g \in C^1(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}) \) and

\( (g1) \) \( \exists a_1, a_2 \geq 0, \exists p \in (2, 6) \) such that

\[
|g(x, t)| \leq a_1 + a_2 |t|^{p-1};
\]

\( (g2) \) \( g(x, t) = o(|t|) \) as \( t \to 0 \) uniformly in \( x \);
$\exists s \in (2, p]$ and $r \geq 0$ such that for every $|t| \geq r$:

$$0 < sG(x, t) \leq tg(x, t),$$

where

$$G(x, t) = \int_0^t g(x, \tau) \, d\tau.$$

**Remark 1.2.** A typical nonlinearity $g$ satisfying $(g_1) - (g_3)$ is $g(x, t) = |t|^{p-2}t$, with $p \in (2, 6)$.

**Theorem 1.3.** Let $g$ satisfy $(g_1) - (g_3)$.

a) If $h \in H^{1/2}(\partial\Omega)$ is sufficiently small (with respect to $m/q$) and satisfies

$$\int_{\partial\Omega} h \, d\sigma = 0,$$

then the problem (3) has a nontrivial solution $(u, \phi) \in H^1_0(\Omega) \times H^1(\Omega)$.

b) If $g$ is odd, then, for every $h \in H^{1/2}(\partial\Omega)$ which satisfies (6), problem (5) has infinitely many solutions $(u_i, \phi_i) \in H^1_0(\Omega) \times H^1(\Omega)$, $i \in \mathbb{N}$, such that

$$\int_\Omega |\nabla u_i|^2 \, dx \to +\infty,$$

whereas the set $\{\phi_i\}$ is uniformly bounded in $H^1(\Omega) \cap L^\infty(\Omega)$.

The present paper has been motivated by some results about the system (6) in the case $\Omega = \mathbb{R}^3$. To the best of our knowledge, our results are the first ones in the case of a bounded domain. Under Dirichlet boundary conditions on both $u$ and $\phi$, the existence results for (4) and (5) are analogous and simpler (see [5]).

About the system (2) in $\mathbb{R}^3$, Theorem 1.1 in [6] shows that there exists only the trivial solution.

In the case of a lower order nonlinear perturbation (problem (5)), the pioneering result contained in [2] has been generalized in several papers: see [3], [7], [8]. Related results on analogous systems are contained in [9], [10].

A different class of solutions for the KGM system is introduced in the papers [3] and [7], where the authors show the existence of magnetostatic and electromagnetostatic solutions (3-dimensional vortices).

From the physical point of view, the case of a positive lower order term

$$W(|\psi|) = \frac{1}{2} m^2 |\psi|^2 - G(x, |\psi|)$$

is more relevant. This case is dealt with in some very recent papers ([4], [11], [12]).

Finally, we recall that global variational methods have been used also in the study of Schroedinger-Maxwell systems (see e.g. [5], [13], [14], [15], [16], [17]).
2 Functional setting

The first step to study problems (4) and (5) is to reduce to homogeneous boundary conditions. For the sake of simplicity, up to a simple rescaling, we can omit the constant $4\pi$.

**Lemma 2.1.** For every $h \in H^{1/2}(\partial\Omega)$, let

$$\kappa = \frac{1}{|\Omega|} \int_{\partial\Omega} h \, d\sigma.$$ 

Then, there exists a unique $\chi \in H^2(\Omega)$ solution of

$$\begin{cases}
\Delta \chi = \kappa & \text{in } \Omega, \\
\frac{\partial \chi}{\partial n}(x) = h(x) & \text{on } \partial\Omega, \\
\int_{\Omega} \chi \, dx = 0.
\end{cases} \quad (7)$$

**Remark 2.2.** It is well known that the solution of (7) satisfies

$$\|\chi\|_{H^2(\Omega)} \leq c \left( \|\kappa\|_2 + \|h\|_{H^{1/2}(\partial\Omega)} \right)$$

where $c$ is a positive constant. So we obtain

$$\|\chi\|_{\infty} \leq c_1 \|h\|_{H^{1/2}(\partial\Omega)}.$$

If we set

$$\varphi = \phi - \chi, \quad (8)$$

then (4) becomes

$$\begin{cases}
-\Delta u - q^2(\varphi + \chi)^2 u + m^2u = 0 & \text{in } \Omega, \\
\Delta \varphi = q^2(\varphi + \chi) u^2 - \kappa & \text{in } \Omega, \\
u(x) = 0 & \text{on } \partial\Omega, \\
\frac{\partial \varphi}{\partial n}(x) = 0 & \text{on } \partial\Omega.
\end{cases} \quad (9)$$

Let us consider on $H^1_0(\Omega)$ the norm $\|\nabla u\|_2$ and on $H^1(\Omega)$

$$\|\varphi\| = \left( \|\nabla \varphi\|_2^2 + |ar{\varphi}|^2 \right)^{1/2},$$

where $\bar{\varphi}$ denotes the average of a function $\varphi$ on $\Omega$, i.e.

$$\bar{\varphi} = \frac{1}{|\Omega|} \int_{\Omega} \varphi \, dx.$$ 

Standard computations show that the solutions of (9) are critical points of the $C^1$ functional

$$F(u, \varphi) = \frac{1}{2} \|\nabla u\|_2^2 + \frac{1}{2} \int_{\Omega} \left[ m^2 - q^2(\varphi + \chi)^2 \right] u^2 \, dx - \frac{1}{2} \|\nabla \varphi\|_2^2 - \kappa |\Omega| \bar{\varphi},$$

defined in $H^1_0(\Omega) \times H^1(\Omega)$. Unfortunately it is strongly unbounded. We adapt a reduction argument introduced in [14]. Let

$$\Lambda = H^1_0(\Omega) \setminus \{0\}.$$
Lemma 2.3. For every $u \in \Lambda$ and $\rho \in L^{6/5}(\Omega)$ there exists a unique $\varphi \in H^1(\Omega)$ solution of

$$
\begin{cases}
-\Delta \varphi + q^2 \varphi u^2 = \rho & \text{in } \Omega, \\
\frac{\partial \varphi}{\partial n}(x) = 0 & \text{on } \partial\Omega.
\end{cases}
$$

Proof. Let $u \in \Lambda$ and $\rho \in L^{6/5}(\Omega)$ be fixed. We shall apply the Lax-Milgram Lemma.

We consider the bilinear form

$$a(\varphi, \zeta) = \int_{\Omega} \nabla \varphi \nabla \zeta \, dx + q^2 \int_{\Omega} \varphi \zeta u^2 \, dx$$

on $H^1(\Omega)$. By the Hölder and Sobolev inequalities, we get

$$a(\varphi, \zeta) \leq \|\nabla \varphi\|_2 \|\nabla \zeta\|_2 + q^2 \|\varphi\|_3 \|\zeta\|_3 \|u\|_6^2$$

and so $a$ is continuous. Moreover,

$$\lim_{\|\varphi\| \to +\infty} a(\varphi, \varphi) = +\infty.$$

Indeed, if $\|\varphi\| \to +\infty$, we distinguish two cases.

1. If $\|\nabla \varphi\|_2 \to +\infty$, then

$$a(\varphi, \varphi) \geq \|\nabla \varphi\|_2^2 \to +\infty.$$

2. If $\|\nabla \varphi\|_2$ is bounded, then $|\bar{\varphi}| \to +\infty$. By the Poincaré-Wirtinger inequality

$$\|\varphi - \bar{\varphi}\|_6 \leq c_2 \|\nabla \varphi\|_2,$$

also $\|\varphi - \bar{\varphi}\|_2$ is bounded. Then we consider $\varphi = (\varphi - \bar{\varphi}) + \bar{\varphi}$ and obtain

$$a(\varphi, \varphi) \geq q^2 |\bar{\varphi}|^2 \|u\|_2^2 - 2q^2 |\bar{\varphi}| \|\varphi - \bar{\varphi}\|_2 \|u\|_4^2 \to +\infty.$$

By standard arguments, we deduce that the bilinear form $a$ is coercive in $H^1(\Omega)$.

On the other hand, by the Sobolev imbedding, we can consider the linear and continuous map

$$\zeta \in H^1(\Omega) \mapsto \int_{\Omega} \rho \zeta \, dx \in \mathbb{R}.$$

The Lax-Milgram Lemma gives the assertion. \(\Box\)

So our reduction argument is based on the following result.

Proposition 2.4. For every $u \in \Lambda$ there exists a unique $\varphi_u \in H^1(\Omega)$ solution of

$$
\begin{cases}
\Delta \varphi = q^2 (\varphi + \chi) u^2 - \kappa & \text{in } \Omega, \\
\frac{\partial \varphi}{\partial n}(x) = 0 & \text{on } \partial\Omega.
\end{cases}
$$

Hence the set

$$\{(u, \varphi) \in \Lambda \times H^1(\Omega) \mid F'_\varphi(u, \varphi) = 0\}$$

coincides with the graph of the map $u \in \Lambda \mapsto \varphi_u \in H^1(\Omega)$. 

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Proposition 2.5. The map \( u \in \Lambda \mapsto \varphi_u \in H^1(\Omega) \) is \( C^1 \).

Proof. Since the graph of the map \( u \mapsto \varphi_u \) is given by (11), we refer to the Implicit Function Theorem.

Straightforward calculations show that for every \( \xi, \eta \in H^1(\Omega) \) and \( w \in H^1_0(\Omega) \)

\[
F''_{\varphi\varphi}(u, \varphi)[\xi, \eta] = -\int_{\Omega} \nabla \xi \nabla \eta \, dx - q^2 \int_{\Omega} u^2 \xi \eta \, dx,
\]

\[
F''_{\varphi u}(u, \varphi)[w, \eta] = -2q^2 \int_{\Omega} (\varphi + \chi) uw \eta \, dx.
\]

Then it is easy to see that \( F''_{\varphi\varphi} \) and \( F''_{\varphi u} \) are continuous.

On the other hand we have already seen that, for every \( (u, \varphi) \in \Lambda \times H^1(\Omega) \), the operator associated to \( F''_{\varphi\varphi}(u, \varphi) \) is invertible (Lemma 2.3). Hence the claim immediately follows.

We can define on \( \Lambda \) the reduced functional

\[
J(u) = F(u, \varphi_u).
\]

It is \( C^1 \) and it is easy to see that \( (u, \varphi) \in \Lambda \times H^1(\Omega) \) is a critical point of \( F \) if and only if \( u \) is a critical point of \( J \) and \( \varphi = \varphi_u \). So, to get nontrivial solutions of (14), we look for critical points of the functional \( J \).

With the same change of variable (5), problem (5) becomes

\[
\begin{array}{ll}
-\Delta u - q^2 (\varphi + \chi)^2 u + m^2 u - g(x, u) = 0 & \text{in } \Omega, \\
\Delta \varphi = q^2 (\varphi + \chi) u^2 - \kappa & \text{in } \Omega, \\
u(x) = 0 & \text{on } \partial \Omega, \\
\frac{\partial \varphi}{\partial n}(x) = 0 & \text{on } \partial \Omega. 
\end{array}
\]

The solutions of (12) are the critical points of the \( C^1 \)-functional

\[
F_g(u, \varphi) = F(u, \varphi) - \int_{\Omega} G(x, u) \, dx
\]

and, as above, we can consider the reduced \( C^1 \)-functional

\[
J_g(u) = F_g(u, \varphi_u).
\]

To get nontrivial solution of (12) we look for critical points of \( J_g \).

3 Behavior of \( \varphi_u \)

By Lemma 2.3 for every \( u \in \Lambda \), problem

\[
\begin{array}{ll}
\Delta \xi - q^2 \xi u^2 = q^2 \chi u^2 & \text{in } \Omega, \\
\frac{\partial \xi}{\partial n} = 0 & \text{on } \partial \Omega 
\end{array}
\]

has a unique solution \( \xi_u \in H^1(\Omega) \).
Analogously, for every $u \in \Lambda$, problem
\[
\begin{align*}
\Delta \eta - q^2 \eta u^2 &= -\kappa & \text{in } \Omega, \\
\frac{\partial \eta}{\partial n} &= 0 & \text{on } \partial \Omega.
\end{align*}
\] (15)
has a unique solution $\eta_u \in H^1(\Omega)$.

Of course, since the solution of (10) is unique, we have
\[
\varphi_u = \xi_u + \eta_u.
\] (16)

**Lemma 3.1 (Properties of $\xi_u$).** For every $u \in \Lambda$,
\[
\int_{\Omega} \xi_u \chi^2 \, dx \leq 0
\] (17)
and
\[
-\max \chi \leq \xi_u \leq -\min \chi
\] (18)
a.e. in $\Omega$.

**Proof.** Multiplying (14) by $\xi_u$ and integrating on $\Omega$, we get immediately (17).

Moreover, if $\xi_u$ is the solution of (14), then $\xi_u + \min \chi$ is the unique solution of
\[
\begin{align*}
\Delta \xi &= q^2 \left[ \xi + (\chi - \min \chi) \right] u^2 & \text{in } \Omega, \\
\frac{\partial \xi}{\partial n} (x) &= 0 & \text{on } \partial \Omega
\end{align*}
\] and minimizes the functional
\[
f (\xi) = \frac{1}{2} \int_{\Omega} |\nabla \xi|^2 \, dx + \frac{q^2}{2} \int_{\Omega} \xi^2 u^2 \, dx + q^2 \int_{\Omega} (\chi - \min \chi) u^2 \xi \, dx
\]
on $H^1(\Omega)$. On the other hand
\[
f (-|\xi_u + \min \chi|) \leq f (\xi_u + \min \chi)
\]
and so
\[
\xi_u + \min \chi = -|\xi_u + \min \chi|,
\]
a.e. in $\Omega$. Hence $\xi_u \leq -\min \chi$, a.e. in $\Omega$.

Analogously, $\xi_u + \max \chi$ is the unique solution of
\[
\begin{align*}
\Delta \xi &= q^2 \left[ \xi + (\chi - \max \chi) \right] u^2 & \text{in } \Omega, \\
\frac{\partial \xi}{\partial n} (x) &= 0 & \text{on } \partial \Omega
\end{align*}
\] and, arguing as before, we get $\xi_u \geq -\max \chi$ a.e. in $\Omega$. \qed

**Corollary 3.2.** For every $u \in \Lambda$,
\[
\| \xi_u \|_\infty \leq \| \chi \|_\infty ,
\] (19)
\[
\| \nabla \xi_u \|_2 \leq \| \nabla \chi \|_2 .
\] (20)
Proof. The inequality (19) easily follows from (18). By (14), \( \xi_u \) satisfies
\[
\int_\Omega \nabla \xi_u \nabla w \, dx + q^2 \int_\Omega (\xi_u + \chi) u^2 w \, dx = 0
\]
for any \( w \in H^1(\Omega) \). For \( w = \xi_u + \chi \) we get
\[
\|\nabla \xi_u\|_2^2 + \int_\Omega \nabla \xi_u \nabla \chi \, dx + q^2 \int_\Omega (\xi_u + \chi)^2 u^2 \, dx = 0
\]
from which one deduces (20).

\[\square\]

Remark 3.3. We point out that, if \( \kappa = 0 \), then \( \varphi_u = \xi_u \). Therefore (17) and (20) become uniform estimates on \( \varphi_u \in H^1(\Omega) \cap L^\infty(\Omega) \) and give rise to estimates on the old variable
\[
\phi = \varphi_u + \chi = \xi_u + \chi.
\]
In other words, if \( \int_{\partial \Omega} h \, d\sigma = 0 \), the solutions \( \phi \) of (17a)-(17b) are uniformly bounded with respect to \( u \neq 0 \). From (18) we deduce also a more precise estimate
\[
\|\phi\|_\infty = \|\xi_u + \chi\|_\infty \leq \max \chi - \min \chi.
\]

Lemma 3.4 (Properties of \( \eta_u \)). For every \( u \in \Lambda \),
\[
\|\eta_u\|_2 \geq \frac{|\kappa| |\Omega|}{q^2 \|u\|_4^2}, \quad (21)
\]
\[\kappa \eta_u \geq 0 \quad (22)\]
a.e. in \( \Omega \) and
\[
\|\nabla \eta_u\|_2 \leq c_1 |\bar{\eta}_u| \|u\|_4^2. \quad (23)
\]

Proof. Let \( u \in \Lambda \) be fixed. If \( \kappa = 0 \), the lemma is trivial. So we suppose \( \kappa \neq 0 \).

By integrating the equation in (15) on \( \Omega \) we get
\[
q^2 \int_\Omega \eta_u u^2 \, dx = \kappa |\Omega|,
\]
from which we deduce (21).

Moreover, since the unique solution \( \eta_u \) of (15) is the minimizer of
\[
f^*(\eta) = \frac{1}{2} \int_\Omega |\nabla \eta|^2 \, dx + \frac{q^2}{2} \int_\Omega \eta^2 u^2 \, dx - \kappa |\Omega| |\bar{\eta}|,
\]
with analogous arguments to those used in the proof of (18), we have that:

- if \( \kappa < 0 \), then \( \eta_u \leq 0 \) a.e. in \( \Omega \);
- if \( \kappa > 0 \), then \( \eta_u \geq 0 \) a.e. in \( \Omega \).

Finally, multiplying the equation in (15) by \( \eta_u - \bar{\eta}_u \) and integrating, we get
\[
- \|\nabla \eta_u\|_2^2 - q^2 \int_\Omega \eta_u (\eta_u - \bar{\eta}_u) u^2 \, dx = 0
\]
from which
\[ \| \nabla \eta u \|^2 + q^2 \int_{\Omega} (\eta u - \bar{\eta} u)^2 \, dx = -\bar{\eta} u \int_{\Omega} (\eta u - \bar{\eta} u) u^2 \, dx. \]

Then, by the Hölder and Poincaré-Wirtinger inequalities, we obtain
\[ \| \nabla \eta u \|^2 \leq |\bar{\eta} u| \| \eta u - \bar{\eta} u \| \| u \|^4 \leq c_1 |\bar{\eta} u| \| \nabla \eta u \| \| u \|^4 \]
which implies (23). \[ \square \]

Finally we have the following relation between \( \xi u \) and \( \eta u \).

**Lemma 3.5.** For every \( u \in \Lambda \),
\[ q^2 \int_{\Omega} \chi \eta u^2 \, dx = -\kappa |\Omega| \bar{\xi} u. \] (24)

**Proof.** Fixed \( u \in \Lambda \), multiplying the equation of (14) by \( \eta u \) and integrating on \( \Omega \), we get
\[ -\int_{\Omega} \nabla \xi u \nabla \eta u \, dx - q^2 \int_{\Omega} \xi u \eta u \, dx = q^2 \int_{\Omega} \chi \eta u^2 \, dx. \]

Multiplying the equation of (15) by \( \xi u \) and integrating on \( \Omega \), we obtain
\[ -\int_{\Omega} \nabla \xi u \nabla \eta u \, dx - q^2 \int_{\Omega} \xi u \eta u \, dx = -\kappa |\Omega| \bar{\xi} u. \]

The claim immediately follows. \[ \square \]

## 4 Proof of Theorem 1.1

Taking into account Remark 2.2, in this section we assume that \( \| h \|_{H^{1/2}(\partial \Omega)} \) is sufficiently small in order to get
\[ \| \chi \|_{\infty} \leq m/q, \]
hence
\[ m^2 - q^2 \chi^2 \geq 0. \] (25)

### 4.1 Existence of nontrivial solutions

In this subsection we assume that \( \int_{\partial \Omega} h \, d\sigma \neq 0 \).

We give the explicit expression of the functional \( J(u) = F(u, \varphi_u) \). If \( u \in \Lambda \), multiplying (10) by \( \varphi_u \) and integrating on \( \Omega \), we have
\[ -\| \nabla \varphi_u \|^2 = q^2 \int_{\Omega} \varphi_u (\varphi_u + \chi) u^2 \, dx - \kappa |\Omega| \bar{\varphi}_u. \]

Then, taking into account (10) and (24), we obtain
\[ J(u) = \frac{1}{2} \| \nabla u \|^2 + \frac{1}{2} \int_{\Omega} \left( m^2 - q^2 \chi^2 \right) \varphi_u (\varphi_u + \chi) u^2 \, dx - \frac{q^2}{2} \int_{\Omega} \xi u \chi u^2 \, dx + \kappa |\Omega| \bar{\xi}_u + \frac{\kappa |\Omega|}{2} \bar{\eta}_u. \] (26)
Moreover, for every \( v \in H^1_0(\Omega) \),

\[
(J'(u), v) = \langle F'(u, \varphi_u), v \rangle = \int_{\Omega} \nabla u \nabla v \, dx + \int_{\Omega} \left[ m^2 - q^2 (\varphi_u + \chi)^2 \right] uv \, dx.
\]  

(27)

**Proposition 4.1.** The functional \( J \) has the following properties:

(a) \( \lim_{u \to 0} J(u) = +\infty \),

(b) \( J \) is coercive,

(c) \( J \) is bounded from below.

**Proof.** Assume \( u \to 0 \). Since the first four terms in (26) are bounded from below, we study the last term. By (22),

\[
\frac{\kappa |\Omega|}{2} \bar{\eta}_u \geq 0.
\]  

(28)

We claim that \( |\bar{\eta}_u| \to +\infty \).

Arguing by contradiction, assume that there exists a sequence \( u_n \to 0 \) such that \( \{\bar{\eta}_n\} \) is bounded (where we mean \( \eta_n = \eta_{u_n} \)). Hence, by (23), we have \( \|\nabla \eta_n\|_2 \to 0 \). Then, using the Poincaré-Wirtinger inequality, we deduce that \( \{\eta_n\} \) is bounded. On the other hand (21) yields

\[
\lim_n \|\eta_n\|_2 = +\infty,
\]

so we get a contradiction and (a) is proved.

By (17), (25) and (28), we obtain

\[
J(u) \geq \frac{1}{2} \|\nabla u\|_2^2 + \kappa |\Omega| \bar{\xi}_u.
\]

Then, by (19), we deduce (b) and (c). \( \square \)

**Proposition 4.2.** The functional \( J \) satisfies the Palais-Smale condition on \( \Lambda \), i.e. every sequence \( \{u_n\} \subset \Lambda \) such that \( \{J(u_n)\} \) is bounded and \( J'(u_n) \to 0 \), admits a converging subsequence in \( \Lambda \).

**Proof.** Let \( \{u_n\} \subset \Lambda \) be a Palais-Smale sequence, i.e.

\[
\{J(u_n)\} \text{ bounded} \tag{29}
\]

and

\[
J'(u_n) \to 0.
\]

From (29) and (b) of Proposition 4.1, we deduce that \( \{u_n\} \) is bounded, hence it converges weakly to \( u \in H^1_0(\Omega) \). It remains to prove that the convergence is strong and that \( u \neq 0 \). As before, for the sake of simplicity, we set \( \varphi_n = \varphi_{u_n} \), \( \xi_n = \xi_{u_n} \) and \( \eta_n = \eta_{u_n} \).

By (27) and (16), we have

\[
\Delta u_n = m^2 u_n - q^2 (\xi_n + \eta_n + \chi)^2 u_n - J'(u_n). \tag{30}
\]
So it is sufficient to prove that the right hand side of (30) is bounded in $H^{-1}(\Omega)$. Since $u_n \rightharpoonup u$ and $J'(u_n) \to 0$, we have only to study $\{(\zeta_n + \eta_n + \chi)^2 u_n\}$.

From (29) we deduce that $\{\kappa |\Omega| \eta_n/2\}$ is bounded, the same being true for the first four terms in $J(u_n)$. Then, using (29), we conclude that $\{\eta_n\}$ is bounded, as well as $\{\zeta_n\}$ by (19). The claim easily follows.

Finally (a) of Proposition 4.1 and (29) show that $u$ cannot be zero. The proof is thereby complete.

Using again (a) of Proposition 4.1, we can see that the sublevels of $J$ are complete. Then, by a standard tool in critical point theory (Deformation Lemma, see e.g. [18]), we conclude that the minimum of $J$ is achieved.

4.2 The only if part

In this subsection we show that if $\int_{\partial \Omega} h d\sigma = 0$, then problem (9) has only trivial solutions.

Let $(u, \varphi)$ be a solution of (9) with $\kappa = 0$. By the first equation we have

$$\|\nabla u\|_2^2 - q^2 \int_{\Omega} (\varphi + \chi)^2 u^2 \, dx + m^2 \|u\|_2^2 = 0. \tag{31}$$

By the second equation we have

$$- \|\nabla \varphi\|_2^2 - q^2 \int_{\Omega} u^2 \varphi^2 \, dx = q^2 \int_{\Omega} \chi u^2 \, dx. \tag{32}$$

Then, substituting $\int_{\Omega} \chi \varphi u^2 \, dx$ in (31), we obtain

$$\|\nabla u\|_2^2 + q^2 \int_{\Omega} u^2 \varphi^2 \, dx + \int_{\Omega} (m^2 - q^2 \chi^2) u^2 \, dx + 2 \|\nabla \varphi\|_2^2 = 0.$$

Therefore, taking into account (25), we deduce $u = 0$.

5 Proof of Theorem 1.3

In this section we assume $\kappa = 0$, so we have

$$\varphi_u = \xi_u. \tag{33}$$

Since $\varphi_u$ satisfies (32), substituting in (13) we find, for every $u \neq 0$,

$$J_g(u) = \frac{1}{2} \|\nabla u\|_2^2 + \frac{m^2}{2} \int_{\Omega} u^2 \, dx - \frac{q^2}{2} \int_{\Omega} \chi (\varphi_u + \chi) u^2 \, dx - \int_{\Omega} G(x,u) \, dx$$

and

$$\langle J'_g(u), v \rangle = \int_{\Omega} \nabla u \nabla v \, dx + \int_{\Omega} \left[ m^2 - q^2 (\varphi_u + \chi)^2 \right] uv \, dx - \int_{\Omega} g(x,u) v \, dx \tag{34}$$

for $v \in H^1_0(\Omega)$.

About the nonlinear term, we recall that $(g_1) - (g_3)$ imply that:
(G₁) for every \( \varepsilon > 0 \) there exists \( A \geq 0 \) such that for every \( t \in \mathbb{R} \)
\[
|G(x,t)| \leq \frac{\varepsilon}{2}t^2 + A|t|^p;
\]

(G₂) there exist two constants \( b_1, b_2 > 0 \) such that for every \( t \in \mathbb{R} \)
\[
G(x,t) \geq b_1|t|^s - b_2.
\]

This time the functional has not a singularity in 0, but it can be extended according to the following proposition.

**Proposition 5.1.** If we set \( J_g(0) = 0 \), then the functional \( J_g \) is \( C^1 \) on \( H^1_0(\Omega) \) with \( J_g'(0) = 0 \).

**Proof.** From (19) and (33) we deduce
\[
\left| \int_{\Omega} \chi(\varphi u + \chi) u^2 \, dx \right| \leq 2\|\chi\|_\infty^2 \|u\|_2^2.
\]
Then it is easy to see that
\[
\lim_{u \to 0} J_g(u) = 0,
\]
hence \( J_g \) is continuous on \( H^1_0(\Omega) \).

Using again (35) and (G₁), we obtain
\[
\lim_{u \to 0} \frac{J_g(u)}{\|\nabla u\|_2} = 0,
\]
which, joint with \( J_g(0) = 0 \), implies that \( J_g \) is differentiable in 0 and \( J_g'(0) = 0 \).

Finally, we have that \( J_g' \) is continuous in 0. Indeed, from (34) we get
\[
\left| \langle J_g'(u), v \rangle \right| \leq \|\nabla u\|_2 \|\nabla v\|_2 + \left( 4q^2 \|\chi\|_\infty^2 + m^2 \right) \|u\|_2 \|v\|_2 + \int_{\Omega} |g(x,u)v| \, dx.
\]
Then, using the hypotheses on \( g \),
\[
\lim_{u \to 0} \|J_g'(u)\| = \lim_{u \to 0} \|\nabla v\|_2 \sup_{\|\nabla v\|_2 = 1} \left| \langle J_g'(u), v \rangle \right| = 0.
\]

**Proposition 5.2.** The functional \( J_g \) satisfies the Palais-Smale condition on \( H^1_0(\Omega) \).

**Proof.** Let \( \{u_n\} \subset H^1_0(\Omega) \) such that
\[
|J_g(u_n)| \leq c \quad (36)
\]
\[
J_g'(u_n) \to 0. \quad (37)
\]

As before, we set \( \varphi_n = \varphi_{u_n} \) and we use \( c_i \) to denote suitable positive constants. By (36)
\[
\frac{1}{2} \|\nabla u_n\|_2^2 \leq c + \int_{\Omega} G(x, u_n) \, dx + \frac{q^2}{2} \int_{\Omega} \chi(\varphi_n + \chi) u_n^2 \, dx + \frac{m^2}{2} \|u_n\|_2^2
\leq c_1 + \frac{1}{s} \int_{\{x \in \Omega : |\varphi_n(x)| \geq r\}} g(x, u_n) u_n \, dx + c_2 \|u_n\|_2^2
\leq c_3 + \frac{1}{s} \int_{\Omega} g(x, u_n) u_n \, dx + c_2 \|u_n\|_2^2. \quad (38)
\]
On the other hand, by (34) and (37),
\[ |\langle J'_g (u_n), u_n \rangle| = \left| \frac{m^2}{2} u_n J''(u_n) - q^2 (\varphi_n + \chi)^2 u_n^2 - \varphi_n + \chi \right| \]
and so
\[ \int_{\Omega} g (x, u_n) \, dx \leq c_4 \| \nabla u_n \|_2 + \| \nabla u_n \|_2^2 + m^2 \| u_n \|_2^2 - q^2 \int_{\Omega} (\varphi_n + \chi)^2 u_n^2 \, dx \]
Hence, substituting (39) in (38) we easily find
\[ \frac{s - 2}{2s} \| \nabla u_n \|_2^2 \leq c_4 \| \nabla u_n \|_2 + \| \nabla u_n \|_2^2 + \| |\nabla u_n| \|_2^2 + m^2 \frac{2}{2s} \| u_n \|_2^2. \]
Now we claim that \( \{u_n\} \) is bounded in \( H^1_0 (\Omega) \). Otherwise by (40)
\[ \| u_n \|_2^2 \geq c_5 \| \nabla u_n \|_2^2 - c_6 \| \nabla u_n \|_2 - c_7 \]
and, for \( n \) sufficiently large, we have
\[ \| u_n \|_2^2 \geq c_8 \| \nabla u_n \|_2^2 \rightarrow +\infty. \]
So, using (G2) and (40), we deduce
\[ J_g (u_n) = \frac{1}{2} \| \nabla u_n \|_2^2 + \frac{m^2}{2} \| u_n \|_2^2 - q^2 \int_{\Omega} (\varphi_n + \chi)^2 u_n^2 \, dx - \int_{\Omega} G (x, u_n) \, dx \]
\[ \leq \frac{1}{2} \| \nabla u_n \|_2^2 + c_9 \| u_n \|_2^2 - b_1 \| |u_n| \|_{\infty}^s + b_2 |\Omega| \]
\[ \leq c_{10} \| u_n \|_2^2 - c_{11} \| u_n \|_{\infty}^s + b_2 |\Omega| \rightarrow -\infty, \]
which contradicts (36). So \( \{u_n\} \) is bounded and, up to a subsequence,
\[ u_n \rightharpoonup u \text{ in } H^1_0 (\Omega). \]
We have to prove that the convergence is strong. We know that
\[ \Delta u_n = m^2 u_n - q^2 (\varphi_n + \chi)^2 u_n - g (x, u_n) - J'_g (u_n). \]
The sequences \( \{J'(u_n)\}, \{u_n\} \) and \( \{g (x, u_n)\} \) are bounded. Finally, by Corollary 3.2 \( \{\varphi_n + \chi\} \) is bounded in \( L^\infty (\Omega) \), then \( \{\varphi_n + \chi\}^2 u_n \) is bounded in \( L^2 (\Omega) \). Therefore the right hand side of (41) is a bounded sequence in \( H^{-1} (\Omega) \).
By standard arguments the proof is complete.
Finally we notice that, by (G2),
\[ J_g (u) \leq \frac{1}{2} \| \nabla u \|_2^2 + \left( q^2 \| \chi \|_{\infty}^2 + \frac{m^2}{2} \right) \| u \|_2^2 - \int_{\Omega} G (x, u) \, dx \]
\[ \leq \frac{1}{2} \| \nabla u \|_2^2 + \left( q^2 \| \chi \|_{\infty}^2 + \frac{m^2}{2} \right) \| u \|_2^2 - b_1 \| |u| \|_{\infty}^s + b_2 |\Omega|. \]
Hence, if \( V \) is a finite dimensional subspace of \( H^1_0 (\Omega) \), then
\[ \lim_{\| \nabla u \|_2 \rightarrow +\infty} J_g (u) = -\infty. \]
5.1 Proof of (a)

Let \( \{ \lambda_j \} \) denote the sequence of the eigenvalues of \(-\Delta\) with Dirichlet boundary conditions. Taking into account Remark 2.2, assume that

\[
q^2 \| \chi \|_\infty^2 < \lambda_1 + m^2.
\]

From (33), (17) and (G1) we deduce

\[
J_g(u) \geq \frac{1}{2} \left[ \| \nabla u \|_2^2 + \left( m^2 - q^2 \| \chi \|_\infty^2 \right) \| u \|_2^2 \right] - \frac{\varepsilon}{2} \| u \|_2^2 - A \| u \|_p^p
\]

\[
\geq \frac{\lambda_1 + m^2 - q^2 \| \chi \|_\infty^2 - \varepsilon}{2\lambda_1} \| \nabla u \|_2^2 - A' \| \nabla u \|_2^p,
\]

with \( A, A' > 0 \) depending on \( \varepsilon > 0 \). Choosing \( \varepsilon \) sufficiently small, we deduce

\[
J_g(u) \geq c \| \nabla u \|_2^2 - A' \| \nabla u \|_2^p
\]

with \( c > 0 \). Hence \( J_g \) has a strict local minimum in 0.

Taking into account (42), the classical Mountain Pass Theorem of Ambrosetti-Rabinowitz applies (see e.g. [18]) and we deduce the existence of a nontrivial solution.

5.2 Proof of (b)

Since \( g \) is odd, the functional \( J_g \) is even and we use the \( \mathbb{Z}_2 \)-Mountain Pass Theorem as stated in [18].

**Theorem 5.3.** Let \( E \) be an infinite dimensional Banach space and let \( I \in C^1(E, \mathbb{R}) \) be even, satisfy the Palais-Smale condition and \( I(0) = 0 \). If \( E = V \oplus X \), where \( V \) is finite dimensional and \( J \) satisfies

1. there are constants \( \rho, \alpha > 0 \) such that \( I|_{\partial B_\rho \cap X} \geq \alpha \), and

2. for each finite dimensional subspace \( \tilde{E} \subset E \), there is an \( R = R(\tilde{E}) \) such that \( I \leq 0 \) on \( E \setminus B_{R(\tilde{E})} \),

then \( I \) possesses an unbounded sequence of critical values.

Taking into account (42), in order to apply Theorem 5.3 we have to prove the geometrical property stated in (1).

We distinguish two cases:

(a) If \( q^2 \| \chi \|_\infty^2 - m^2 < \lambda_1 \) then, using the same estimates given in the previous subsection, Theorem 5.3 applies with \( V = \{ 0 \} \).

(b) If \( \lambda_1 \leq q^2 \| \chi \|_\infty^2 - m^2 \), we set

\[
k = \min \left\{ j \in \mathbb{N} : q^2 \| \chi \|_\infty^2 - m^2 < \lambda_j \right\},
\]

and we consider

\[
V = \bigoplus_{j=1}^{k-1} M_j, \quad X = V^\perp = \bigoplus_{j=k}^{+\infty} M_j.
\]
where $M_j$ is the finite dimensional eigenspace corresponding to $\lambda_j$.

Since

$$\lambda_k = \min \left\{ \frac{\|\nabla v\|_2^2}{\|v\|_2^2} : v \in X, v \neq 0 \right\},$$

for every $u \in X$ we have

$$J_g(u) \geq \frac{\lambda_k + m^2 - q^2\|\chi\|_\infty^2}{2\lambda_k} \|\nabla u\|_2^2 - \int_\Omega G(x, u) dx.$$  

Similar estimates to those used in the previous case show that $J$ is strictly positive on a sphere in $X$.

In both cases we get the existence of infinitely many critical points $\{u_i\}$ such that

$$J_g(u_i) \rightarrow +\infty.$$  

Remark 3.3 gives the uniform estimate on $\{\varphi_{u_i}\}$. Finally we notice that, by (G1),

$$J_g(u_i) = \frac{1}{2} \|\nabla u_i\|_2^2 + \frac{m^2}{2} \|u_i\|_2^2 - \frac{q^2}{2} \int_\Omega \chi (\varphi_i + \chi) u_i^2 dx - \int_\Omega G(x, u_i) \ dx$$

$$\leq c_1 \|\nabla u_i\|_2^2 + c_2 \|\nabla u_i\|_2^2.$$  

Hence $\|\nabla u_i\|_2 \rightarrow +\infty$ and this completes the proof.

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